Time, $E_8$, and the Standard Model

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Abstract

Based upon the unique and simple starting point of the continuous flow of time a physical theory is derived through an analysis of the elementary arithmetic composition and symmetries of this one-dimensional progression. We describe how the explicit development of the theory leads to a prediction of the unique and largest exceptional Lie group $E_8$ as the full ‘symmetry of time’, and hence as the unification group for the physical theory. This proposal results from the identification of a series of esoteric properties of the Standard Model of particle physics from a series of intermediate augmentations in the ‘multi-dimensional form of time’. These physical properties derive from the breaking of the full symmetry of time through the necessary interposition of an external 4-dimensional spacetime arena, itself constructed from a 4-dimensional form of time, as the background to all observations. The basic conceptual picture is presented together with reviews of a number of references regarding $E_8$ structures which may provide a significant guide in pursuing the goal of converging upon a complete unified theory.

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1 Introduction

The conception of a physical theory presented here rests upon the general presumption that all empirical phenomena are infused within the passage of time. On parametrising this temporal continuum with a single real variable, while utilising the elementary arithmetic structure of the real number system, infinitesimal intervals of time can be expressed in terms of a composition of a multi-dimensional set of real variables within well-defined constraints. In this manner the original linear flow of time can be simultaneously manifested in terms of quadratic structures, with a direct interpretation as underpinning a Euclidean spatial framework – as associated for example with a local inertial frame in general relativity. In turn higher-dimensional structures based on cubic or higher-order forms of time, with residual parameters and symmetries over and above those required to construct a 4-dimensional Lorentzian manifold, give rise to matter fields in spacetime, as will be described in detail in this paper.

A principle aim of this project is to see how far this very simple idea can be developed in terms of making contact with the empirical world, and with existing successful physical theories, and the degree of explanatory power that can be achieved in this way. Progress for the theory over a range of topics from connections with the Standard Model of particle physics to the standard model of cosmology, together with the relation to non-Abelian Kaluza-Klein theory and quantum field theory, has been collectively presented in the lengthy tome [1]. The means of obtaining the Standard Model connections are summarised in [2], while the comparisons and contrasts between this theory originating from the one dimension of time and Kaluza-Klein theories constructed with extra dimensions of space are further emphasised in [3].

Building upon the above references, and in particular ([1] chapters 6–9, [2]), we again set out to motivate the basic conception of the theory and here describe in detail how its development leads to consideration of E$_8$ as the ultimate unification group and the manner in which this proposal may meet criteria of testability. To this end a significant degree of contact is made with existing work in the mathematical physics literature regarding the structure and properties of E$_8$. Correspondingly we first review the relevant literature in the following section, with emphasis upon the larger exceptional Lie groups more generally and their relation with the four division algebras. In particular, in subsection 2.3 we describe how some of these structures can be associated with a notion of ‘generalised spacetime’.

In section 3, by contrast, we set out the motivation and argument for founding a theory upon one dimension of time only, with references to historical developments in physics and mathematics. The straightforward technical means by which a physical theory can be constructed from this single dimension is described via a minimal example in subsection 4.1. The development of the theory through subsequent natural extensions of the multi-dimensional form of time leads directly to significant contact with the Standard Model, as will be presented in subsections 4.2 and 4.3.

The progression of the theory towards larger symmetries of time culminates in section 5 in a prediction of E$_8$ as the full unification group. This observation is justified in terms of the need to converge upon a complete description of the Standard Model symmetry and particle multiplet structure and through connections made with various studies regarding E$_8$ as reviewed in section 2. This discussion is continued in section 6 where we also summarise the other areas in which the theory has been developed, with
the aim of converging upon a complete unified theory more generally. The overall status of the theory will be further assessed in the concluding section. While successes that have been achieved mark a proof of principle for the basic idea of the theory, we also consider its potential predictive power.

In recent decades there has been much interest in theories based on extra spatial dimensions and also in unification schemes employing the exceptional Lie algebras. These two areas are drawn together in the framework presented here through a change of emphasis in constructing the theory from the one dimension of time only. While some empirical success has already been achieved, and directions for further progress can be identified, one of the main strengths of the theory lies in the unique simplicity of founding the theory on this single dimension of time, as will be a central theme running through this paper. In summary the main aims of this paper are to:

- Motivate the conceptual basis for the theory in describing how it is possible to build a full physical theory from the one dimension of time,
- Demonstrate the explanatory power of the theory in uncovering a series of properties of the Standard Model with minimal redundancy, and
- Justify the proposed culmination of this progression in $E_8$ as the full symmetry of time and consider the possible form this might take.

The final stage is ultimately presented in the manner of a puzzle that remains to be solved. With this goal in mind in the following section we begin by considering some of the mathematical structures involving the exceptional Lie group $E_8$ that may provide essential input.

# 2 Review of Selected Studies in $E_8$

## 2.1 Unification in Physics – Unique in Mathematics

Several applications of $E_8$ in physical theories and the status of $E_8$ as a mathematical structure itself will be reviewed in this subsection. In the latter case, as the largest exceptional Lie algebra with rich symmetry properties, $E_8$ occupies a unique position in mathematics. First though we consider a well known proposal for a fundamental role for $E_8$ in physics in a branch of superstring theory – itself conceived as a framework incorporating a quantum theory of gravity (see for example [4]). Heterotic string theory combines 10-dimensional superstring theory with the original 26-dimensional bosonic theory, with the additional 16 dimensions compactified on a torus. This torus can be defined by the root lattice of the rank-16 Lie algebra of either $SO(32)$ or $E_8 \times E_8$ in order to obtain a consistent theory free from anomalies.

(In this paper we generally employ upper-case letters for classical Lie groups such as $SO(32)$, with lower-case denoting the corresponding Lie algebra such as $so(32)$; while upper-case is used for both the exceptional Lie groups and their algebras, such as $E_8$, with the distinction taken from the context).

With the further six extra dimensions of the 10-dimensional superstring also being compactified over an external 4-dimensional spacetime, the case with the local...
gauge group $E_8 \times E_8$ emerging at low energies describes a gauge field theory comfortably able to accommodate the internal gauge symmetry $SU(3) \times SU(2) \times U(1)$ of the Standard Model. That is, the $E_8 \times E_8$ symmetry can be considered as an elaborate example of a ‘Grand Unified Theory’ (GUT) within this component of the string theory. Within this setting it is possible to obtain string vacua containing three generations of quarks and leptons of the Standard Model, with additional matter multiplets in ‘hidden sectors’ of the theory which are not empirically observed (see for example [5]). However, one of the challenges for string theory, which is further exacerbated when subsumed into M-theory, is in handling the vast number of different solutions admitted by the equations. The corresponding lack of a unique vacuum solution is referred to as the ‘landscape problem’, while the difficulty in addressing it can be used to motivate a ‘multiverse’ interpretation of the theory.

In fact the Lie group $E_8$ itself is comfortably large enough to contain as a subgroup the Standard Model gauge group $SU(3) \times SU(2) \times U(1)$ together with the external Lorentz symmetry $SO^+(1,3)$, and hence on its own has the potential to be utilised by a theory seeking, beyond the ambition of a GUT, to unify the internal gauge forces together with gravity through a single symmetry group. At the level of the Lie algebra structure alone, and motivated in part by a notion of ‘mathematical beauty’, this is the approach adopted in [6]. However, while components associated with the external gravitational field and internal gauge fields as well as three generations of ‘quarks’ and ‘leptons’ are identified in the $E_8$ root lattice the second and third generations of these ‘fermion’ states lack the appropriate external and internal symmetry properties other than through a ‘graviweak’ $SO(8) \subset E_8$ ‘triality’ transformation with respect to the first ([6] subsections 2.2.3 and 2.4.2).

The impossibility of amending this discrepancy with the Standard Model, while keeping strictly within the goal of embedding these structures within the $E_8$ Lie algebra, owing to the insufficient number of non-compact generators for any real form of $E_8$ is described in [7]. Nevertheless, the fact that structures resembling the Standard Model can be identified for the exceptional Lie algebras (including $E_6$ and $E_7$ GUT models dating from the 1970s [8, 9]), together with the observation that $E_7$ and $E_8$ are large enough to incorporate the external Lorentz group alongside the Standard Model gauge group, is suggestive. A further exploration of some of these mathematical structures and tentative connections with physics is seen for example in [10], in which 4-dimensional spacetime itself is proposed to emerge through fundamental interactions which in turn can be defined in terms of the structure of the $E_8$ Lie algebra.

A review of the more purely mathematical properties of $E_8$, dating from the inception of this 248-dimensional Lie algebra by Wilhelm Killing in 1887 through to much more recent developments, can be found in [11]. Here we briefly consider some of these algebraic structures.

For the rank-8 Lie algebra $E_8$ there are eight independent Casimir operators, that is elements that are defined by the centre of the universal enveloping algebra of $E_8$ and hence commute with all elements of the Lie algebra. These operators are of order 2, 8, 12, 14, 18, 20, 24 and 30 in the Lie algebra generators, where the first of these is the quadratic invariant defined by the Killing metric. The second of these is an eighth order invariant tensor as explicitly constructed for the compact real form $E_8(-248)$, via the adjoint and spinor representations of the maximal subalgebra $so(16) \subset E_8(-248)$, in ([12] equation 2.3) and also described in [13, 14, 15]. These references imply such
an octic invariant can be constructed for any of the three real forms of $E_8$, namely $E_8(-248)$ as well as the two non-compact forms $E_8(-24)$ and $E_8(8)$. For each case, in being composed of the 248 elements of a real $E_8$ Lie algebra, the tensor invariant further implies the existence of an eighth-order real-valued polynomial function of 248 real variables that is invariant under the adjoint action of $E_8$. In fact invariant homogeneous polynomials over $\mathbb{R}^{248}$ could in principle be identified corresponding to each of the eight Casimir invariants and for each of the three real forms of $E_8$.

One of the challenges in studying the Lie group $E_8$ is the unique feature of lacking a non-trivial representation smaller than the adjoint representation in 248 dimensions, which describes $E_8$ in terms of a set of symmetries acting upon its own Lie algebra. However, as for other exceptional Lie algebras, a description of $E_8$ is possible in terms of the octonions $\mathbb{O}$, the largest division algebra, as presented towards the end of [16]. One such construction is in terms of the final entry in the $4 \times 4$ Freudenthal-Tits ‘magic square’ of Lie algebras. Each of the 16 entries in the magic square can be denoted $M(\mathbb{K}, \mathbb{K}')$, where $\mathbb{K}$ and $\mathbb{K}'$ each denote a division algebra $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ or $\mathbb{O}$, and involves a Jordan algebra $h_3 \mathbb{K}'$ of $3 \times 3$ Hermitian matrices over $\mathbb{K}'$ (see for example [16] section 4.3, [17]).

The largest Lie algebra constructed in this way is $E_8$ which occupies a unique position as the fourth row and fourth column entry $M(\mathbb{O}, \mathbb{O})$ of the magic square. Different real forms of $E_8$ can be obtained by employing the ‘split’ octonions $\mathbb{O}_s$; for example with $\mathbb{K} = \mathbb{O}_s$ the real Lie algebra corresponding to $M(\mathbb{O}_s, \mathbb{O})$ is $E_8(-24)$ while for the doubly-split magic square the entry $M(\mathbb{O}_s, \mathbb{O}_s)$ yields $E_8(8)$. This formulation is not explicitly utilised in this paper. However we shall make significant reference to the realisation of $E_8(-24)$ described below in subsection 2.3, with potential connections to physics, which also employs in a central role the octonions and the exceptional Jordan algebra $h_3 \mathbb{O}$, the basic properties of which we hence review in the following subsection.

2.2 The Octonions and the Exceptional Jordan Algebra

For any two elements $a, b \in \mathbb{K}$ of any one of the four normed division algebras $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ or $\mathbb{O}$ (of real dimension 1, 2, 4 and 8 respectively) the relation:

$$|ab| = |a||b|$$

(1)

holds. Here $|a| = (a\bar{a})^{\frac{1}{2}}$ is the norm of $a \in \mathbb{K}$, where the conjugate $\bar{a}$ is obtained by negating each of the imaginary components of $a$. Each element $a \neq 0$ also has a unique multiplicative inverse $a^{-1} = \bar{a}/|a|^2$ (see for example [16], [1] section 6.2). These algebras hence naturally describe symmetry operations since any $a \in \mathbb{K}$ with $|a| = 1$ acting by multiplication on any $b \in \mathbb{K}$ leaves the norm of the latter invariant, that is $|b| \rightarrow |ab| = |b|$ by equation 1, and has an inverse operation.

Unlike the real $\mathbb{R}$ and complex $\mathbb{C}$ numbers the quaternions $\mathbb{H}$ form a non-commutative algebra and the octonions $\mathbb{O}$ are also non-associative, hence multiplication in the latter case can not directly represent the actions of a symmetry group. This non-associativity does not however prohibit the octonions from playing an important role in describing symmetries for a physical theory (see for example [18, 19, 20], the references in [21] for further early studies, and [22, 23, 24]).
Indeed Lie algebras, with elements being the infinitesimal generators of the finite symmetry transformations of a Lie group, themselves provide a very familiar example of non-associative algebras that are of central importance in physics. For the case of employing the octonion algebra as considered here the corresponding non-associativity is apparent at the level of compositions of finite symmetry transformations. In fact the algebraic properties of the octonions allow a high degree of symmetry to be described, as noted for the relation between octonion composition and the Lie group $\text{SO}(7)$ in ([1] section 6.2). However owing to the non-associativity care is needed in the algebraic manipulation of these operations, and in particular many of the general results of group theory, which rely on the group axiom of associativity, cannot be directly applied.

As described for ([1] equation 6.6, [2] equation 21) here we adopt the notation of [24, 25, 26, 27] for many of the conventions regarding the octonions and the exceptional Jordan algebra, with for example an octonion element $a \in \mathbb{O}$ written as:

$$a = a_1 + a_2 i + a_3 j + a_4 k + a_5 \overline{k} + a_6 \overline{j} + a_7 \overline{i} + a_8 l$$

(2)

Here $a_1, \ldots, a_8 \in \mathbb{R}$ are eight real components and $\{i, j, k, \overline{i}, \overline{j}, \overline{k}, l\}$ are the seven imaginary octonion units with $i^2 = j^2 = \ldots = \overline{i}^2 = l^2 = -1$. These seven units are mutually anticommuting, with the notation motivated by products such as $il = -\overline{l}i$. The full multiplication table is given for example in ([24] figure 2, [1] figure 6.1) with the non-associativity exhibited by products such as $(ij)l = -i(jl) = +l\overline{i}$.

The exceptional Jordan algebra $h_3 \mathbb{O}$ of $3 \times 3$ Hermitian matrices over the octonions is of particular significance for this paper, and here we review some of the structures relating to this algebra (see also for example [28, 29]). For any two elements $\mathcal{X}, \mathcal{Y} \in h_3 \mathbb{O}$ the Jordan product is defined by:

$$\mathcal{X} \circ \mathcal{Y} = \frac{1}{2}(\mathcal{X}\mathcal{Y} + \mathcal{Y}\mathcal{X}) \in h_3 \mathbb{O}$$

(3)

where $\mathcal{X}\mathcal{Y}$ denotes the usual $3 \times 3$ matrix multiplication. As for any Jordan algebra the product $\mathcal{X} \circ \mathcal{Y}$ is commutative but not in general associative. In turn the Freudenthal product for elements of $h_3 \mathbb{O}$ can be defined by the commutative composition:

$$\mathcal{X} \times \mathcal{Y} = \mathcal{X} \circ \mathcal{Y} - \frac{1}{2}(\text{tr}(\mathcal{X})\mathcal{Y} + \text{tr}(\mathcal{Y})\mathcal{X}) + \frac{1}{2}(\text{tr}(\mathcal{X})\text{tr}(\mathcal{Y}) - \text{tr}(\mathcal{X} \circ \mathcal{Y}))\mathbf{1}_3 \in h_3 \mathbb{O}$$

(4)

where $\text{tr}(\mathcal{X})$ is the trace of the matrix $\mathcal{X} \in h_3 \mathbb{O}$ and $\mathbf{1}_3$ is the unit $3 \times 3$ matrix. In the final term an inner product on the space $h_3 \mathbb{O}$ is employed, that is the bilinear map denoted and defined by:

$$(\mathcal{X}, \mathcal{Y}) = \text{tr}(\mathcal{X} \circ \mathcal{Y}) \in \mathbb{R}$$

(5)

This map is invariant under $F_4$ transformations, that is under the automorphism group of the exceptional Jordan algebra. We adopt the notation for the components of $\mathcal{X}, \mathcal{Y} \in h_3 \mathbb{O}$ from ([1] equation 9.25), which in turn for $\mathcal{X} \in h_3 \mathbb{O}$ was adopted from [26], and write:

$$\mathcal{X} = \begin{pmatrix} p & \tilde{a} & c \\ a & m & \tilde{b} \\ \tilde{c} & b & n \end{pmatrix}, \quad \mathcal{Y} = \begin{pmatrix} P & \overline{A} & C \\ A & M & \overline{B} \\ \overline{C} & B & N \end{pmatrix}$$

(6)
with \( p, m, n \in \mathbb{R} \) and \( a, b, c \in \mathbb{O} \) for \( \mathcal{X} \) and similarly for their upper-case counterparts in \( \mathcal{Y} \). The inner product of equation 5 can then be written out explicitly as:

\[
\langle \mathcal{X}, \mathcal{Y} \rangle = pP + mM + nN + 2\langle a, A \rangle + 2\langle b, B \rangle + 2\langle c, C \rangle \tag{7}
\]

with \( \langle a, A \rangle = \frac{1}{2}(a\bar{A} + A\bar{a}) = \text{Re}(a\bar{A}) = \sum_{h=1}^{8} a_h A_h \). \( \tag{8} \)

Here equation 8 defines an inner product on the octonion algebra itself, where ‘Re’ denotes the real part of an octonion and the \( a_h \in \mathbb{R} \) are the eight components of \( a \in \mathbb{O} \) of equation 2, with \( A_h \in \mathbb{R} \) similarly the components of \( A \in \mathbb{O} \).

The inner product on the Jordan algebra of equation 5 can be used in conjunction with the Freudenthal product of equation 4 to define a cubic form for any three elements \( \mathcal{X}, \mathcal{Y}, \mathcal{Z} \in h_3\mathbb{O} \) as:

\[
\langle \mathcal{X}, \mathcal{Y}, \mathcal{Z} \rangle := \langle \mathcal{X}, (\mathcal{Y} \times \mathcal{Z}) \rangle = \text{tr}(\mathcal{X} \circ (\mathcal{Y} \times \mathcal{Z})) \in \mathbb{R} \tag{9}
\]

which is totally symmetric in the three arguments. This trilinear form is invariant under transformations of the Lie group \( E_6 \) upon the space \( h_3\mathbb{O} \). The quadratic adjoint map \( \mathcal{X} \rightarrow \mathcal{X}^\sharp \) can also be defined via the Freudenthal product, with:

\[
\mathcal{X}^\sharp = \mathcal{X} \times \mathcal{X} \tag{10}
\]

or explicitly:

\[
\mathcal{X}^\sharp = \begin{pmatrix}
mn - |b|^2 & cb - n\bar{a} & \bar{a}\bar{b} - mc \\
\bar{b}\bar{c} - na & pm - |c|^2 & ac - pb \\
ba - mc & \bar{c}\bar{a} - pb & pm - |a|^2
\end{pmatrix} \in h_3\mathbb{O} \tag{11}
\]

from the components of \( \mathcal{X} \) in equation 6. On setting \( \mathcal{X} = \mathcal{Y} = \mathcal{Z} \) in equation 9 a cubic norm or determinant can be defined for \( \mathcal{X} \in h_3\mathbb{O} \) as:

\[
\det(\mathcal{X}) = \frac{1}{3}\text{tr}(\mathcal{X}^3) = \frac{1}{3}\text{tr}(\mathcal{X} \circ (\mathcal{X} \times \mathcal{X})) = \frac{1}{3}\text{tr}(\mathcal{X} \circ \mathcal{X}^\sharp) \tag{12}
\]

\[
= pmn - p|b|^2 - m|c|^2 - n|a|^2 + 2\text{Re}(\bar{a}\bar{b}\bar{c}) \tag{13}
\]

which is of course also invariant under \( E_6 \). This \( E_6 \) symmetry of \( \det(\mathcal{X}) \) is explicitly constructed in [24, 25, 26, 27].

Alternatively the above structures can be motivated by beginning with the definition of \( \det(\mathcal{X}) \) from equation 13 which exhibits the \( E_6 \) symmetry. Equation 13 has a very similar form to the usual definition of the determinant for a \( 3 \times 3 \) matrix, with the final term adapted for the non-associativity of the octonions. This determinant \( \det(\mathcal{X}) \) can be used to define the quadratic adjoint of \( \mathcal{X} \) as the ‘cross product’ \( \mathcal{X}^2 = \mathcal{X} \times \mathcal{X} \) on \( h_3\mathbb{O} \) through equation 12 ([16] section 3.4), with the adjoint identity \( (\mathcal{X}^2)^2 = \det(\mathcal{X})\mathcal{X} \) also satisfied ([23] equation 9.8). Equation 13 can also be written as:

\[
\det(\mathcal{X}) = \frac{1}{3}\text{tr}(\mathcal{X}^3) - \frac{1}{2}\text{tr}(\mathcal{X}^2)\text{tr}(\mathcal{X}) + \frac{1}{6}\text{tr}(\mathcal{X})^3 \tag{14}
\]

while \( \mathcal{X}^2 = \mathcal{X}^2 - \text{tr}(\mathcal{X})\mathcal{X} + \frac{1}{2}(\text{tr}(\mathcal{X})^2 - \text{tr}(\mathcal{X}^2))1_3 \) \( \tag{15} \)
in terms of trace functions, consistent with equation 12. The quadratic adjoint $X^{\sharp}$ in equation 12, with $\det(X) = \frac{1}{3}(X, X^{\sharp})$, plays a similar role to the classical adjoint of a matrix as used in the standard method of constructing the determinant of the matrix, as can be seen from equations 7 and 11. The ‘cross product’ defined in this way can be linearised to define:

$$X \times Y = \frac{1}{2}[(X + Y)^{\sharp} - X^{\sharp} - Y^{\sharp}]$$  \hspace{1cm} (16)$$

which is equivalent to the Freudenthal product of equation 4, as can be seen via equation 15. Similarly the determinant or cubic norm $\det(X) = \frac{1}{3}(X, X, X)$ can be linearised to define a cubic form, that is:

$$(X, Y, Z) = 3 \times \frac{1}{6} \left[ \det(X + Y + Z) - \det(X + Y) - \det(Y + Z) - \det(Z + X) \\
+ \det(X) + \det(Y) + \det(Z) \right]$$  \hspace{1cm} (17)$$

which is manifestly symmetric in $X, Y, Z \in h_3 \mathbb{O}$ and equivalent to the cubic form defined in equation 9. Hence the role of the Jordan product and Freudenthal product in equation 9 can be justified in this way, considering the determinant $\det(X)$ defined in equation 13 to be of primary significance.

In general matrix determinants exhibit the property that for any elements $a, b \in M$ belonging to a particular $m \times m$ matrix algebra $M$ the product satisfies:

$$\det(ab) = \det(a) \det(b)$$  \hspace{1cm} (18)$$

This property of matrix algebras is analogous to that of the normed division algebras described for equation 1 in the opening of this subsection, and hence both types of algebra can naturally be applied to describe symmetries. In the case of the $3 \times 3$ matrices of $h_3 \mathbb{O}$ over the octonions both kinds of algebraic structure are used together in equation 13. The actions of the real Lie group $E_{6(-26)} \equiv SL(3, \mathbb{O})$ preserve $\det(X)$ for any $X \in h_3 \mathbb{O}$, as described explicitly in [24, 25, 26, 27] and extensively used in [1, 2], and as will be of central importance in this paper.

### 2.3 Generalised Spacetime and E₈ Symmetry

The space $h_3 \mathbb{O}$ can be considered a ‘generalised spacetime’ as a natural extension from 4-dimensional Minkowski spacetime, as described for example in [30, 31] in which such ‘spacetimes’ are defined as having coordinates parametrised by the elements of a Jordan algebra. We first note that Lorentz transformations on the 4-vector $v_4 = (v^0, v^1, v^2, v^3) \in \mathbb{R}^4$ can be represented by the double cover group $SL(2, \mathbb{C})$ acting on the $2 \times 2$ matrices:

$$h = v^0 \sigma^0 + v^1 \sigma^1 + v^2 \sigma^2 + v^3 \sigma^3 = \begin{pmatrix} v^0 + v^3 & v^1 - v^2 i \\ v^1 + v^2 i & v^0 - v^3 \end{pmatrix} \in h_2 \mathbb{C}$$  \hspace{1cm} (19)$$

with $\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ together with the three Pauli matrices $\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ (see for example [2] equations 17 and 18). The $S \in SL(2, \mathbb{C})$ matrix actions
$h \rightarrow ShS^\dagger$ leave invariant $\det(h) = v_4 \cdot v_4 = |v_4|^2$, where $v_4 \cdot v_4 = \eta_{ab} v^a v^b$ is the Lorentz inner product (with the conventional summation over repeated indices used throughout this paper, here for $a, b = 0, 1, 2, 3$ and with $\eta = \text{diag}(+1, -1, -1, -1)$ being the Minkowski metric).

On generalising from $2 \times 2$ Hermitian matrices over $\mathbb{C}$ the space $h_2\mathbb{O}$ of $2 \times 2$ Hermitian matrices over $\mathbb{O}$, with ten real components, can represent 10-dimensional spacetime. The quadratic norm $\det(X)$, with $X \in h_2\mathbb{O}$, is preserved by actions of the group $\text{SL}(2, \mathbb{O})$ as the double cover of the 10-dimensional Lorentz group $\text{SO}^+(1, 9)$ [32]. This space can be further extended to $3 \times 3$ Hermitian matrices over $\mathbb{O}$, with the group $E_6(-26) \equiv \text{SL}(3, \mathbb{O})$ acting on the space $h_3\mathbb{O}$ preserving the cubic norm $\det(X)$, for any $X \in h_3\mathbb{O}$. This latter symmetry was described in the previous subsection for equation 13 and can be constructed in terms of a composition of three interlocking $\text{SL}(2, \mathbb{O})$ subgroup actions [24, 25, 26, 27]. This natural progression justifies consideration of the space $h_3\mathbb{O}$ as a generalised spacetime.

As for $h_3\mathbb{O}$ the set of elements of $h_2\mathbb{C}$ is closed under the symmetric anticommutator product of equation 3 and in fact also forms a Jordan algebra. As described in [30] the ‘conformal group’ $\text{SU}(2, 2)$ (the double cover of $\text{SO}(2, 4)$) associated with the space $h_2\mathbb{C}$ is defined as leaving invariant the light cone:

$$\det(h - k) = 0$$

for the separation between $h, k \in h_2\mathbb{C}$. The corresponding Lie algebra $g = \text{su}(2, 2) \equiv \text{so}(2, 4)$ possesses a 3-graded structure:

$$g = g^{-1} \oplus g^0 \oplus g^+$$

of dimension: $15 = 4 + (6 + 1) + 4$

with elements: $U_k, S_{kl}, \tilde{U}_k$

labelled by $k, l \in h_2\mathbb{C}$. The generators $U_k$ correspond to translations and the $\tilde{U}_k$ to conformal transformations, while the set $S_{kl}$ is composed of the six generators of the Lorentz group together with a dilation generator. These act upon an element $h \in h_2\mathbb{C}$ as ([30] equation 8):

$$U_k(h) = k, \quad S_{kl}(h) = \{k, l, h\}, \quad \tilde{U}_k(h) = -\frac{1}{2}\{h, k, h\}$$

where: $\{h, k, l\} = (h \cdot k) l + (l \cdot k) h - (h \cdot l) k$

is the Jordan triple product on $h_2\mathbb{C}$. Here $h \cdot k = \frac{1}{2}(\text{tr}(h)\text{tr}(k) - \text{tr}(h \circ k))$ is the Lorentz inner product between the 4-vectors associated with these elements of $h_2\mathbb{C}$ via equation 19. This Jordan triple product can be defined in terms of the Jordan product ([30] equation 3) in a form similar to equation 28 below. The triple product is also utilised in the $\text{su}(2, 2) \equiv \text{so}(2, 4)$ Lie bracket ([30] equation 9):

$$[U_k, \tilde{U}_l] = S_{kl}, \quad [S_{kl}, S_{lm}] = S_{(k, l, h)m} - S_{(l, k, m)h}, \quad [S_{kl}, U_h] = \tilde{U}_{(l, k, h)}$$

Similarly the conformal group for 10-dimensional spacetime $h_2\mathbb{O}$ can be identified as $\text{SO}(2, 10)$ (see for example [33]). However here we consider the Lorentz symmetry $\text{SO}^+(1, 9)$ acting, via the double cover $\text{SL}(2, \mathbb{O})$, on the 10-dimensional vector space.
h_2O and augment this space by an SO^+(1, 9) scalar \( n \in \mathbb{R} \). The product \( n \det(X) \)
for any \( X \in h_2O \), is then invariant under both SO^+(1, 9) and a dilation symmetry
deﬁned with reciprocal scaling actions on \( n \) and \( \det(X) \). Upon further extension by
the components of an SO^+(1, 9) spinor \( \theta \in \mathbb{R}^{16} \) a cubic form can be constructed ([31]
equation 63):

\[
V(n, X, \theta) = n \det(X) - 2X \cdot (\theta \theta^\dagger)
\]  

(25)

where here the inner product is again deﬁned by \( X \cdot Y = \frac{1}{4}(\text{tr}(X)\text{tr}(Y) - \text{tr}(X \circ Y)) \),
now for \( X, Y \in h_2O \), (as adopted from ([25] equations 38 and 39, [26] section 3.3)
with the sign adapted for Lorentz metric signature \((+1, -1, \ldots, -1)\) when interpreted
as a 10-dimensional spacetime inner product). The dilation, which also acts on the
\( \theta \) component such that the cubic form of equation 25 is invariant, is a component
of the full invariance group which is found to be the Lie group \( E_6(-26) \). In fact the
components \((n, X, \theta)\) contain the 27 real parameters of an element of \( X \in h_3O \),
that is such an element \( X \) of equation 6 can be written as ([1] equation 6.26, [26] section 3.3):

\[
X = \begin{pmatrix}
p & \bar{a} & c \\
a & m & \bar{b} \\
\bar{c} & b & n
\end{pmatrix}
\equiv \begin{pmatrix}
X \\
\theta^\dagger \\
n
\end{pmatrix}
\in h_3O
\]  

(26)

with \( p, m, n \in \mathbb{R}, a, b, c \in O \), \( X \in h_2O \) and \( \theta \in O^2 \). Further, we note that the cubic form of equation 25 is identical to that of equation 13 ([1] equations 6.27 and
6.28 respectively), that is \( V(n, X, \theta) = \det(X) \) through the correspondence of
equation 26, constructed here explicitly as an extension of the quadratic form \( \det(X) \)
on 10-dimensional spacetime.

The construction of the conformal group of the generalised spacetime \( h_3O \) is
analogous to that for \( h_2C \) in equation 21. It can also be described by a Lie algebra \( g \)
with a 3-graded structure, here with:

\[
g = g^{-1} \oplus g^0 \oplus g^{+1}
\]

of dimension: 133 = 27 + (78 + 1) + 27

with elements: \( U_Y, S_{YZ}, \tilde{U}_Y \)

(27)

now labelled by \( Y, Z \in h_3O \). The generators \( U_Y, S_{YZ} \) and \( \tilde{U}_Y \) have similar interpre-
tations as those described for the corresponding actions of equation 21, here with the
\( S_{YZ} \) collectively describing the \( \det(X) \) preserving group \( E_6(-26) \) together with a further
dilation. Explicitly, the actions of these 133 generators upon \( X \in h_3O \) are given by
equation 22 with \( h, k, l \to X, Y, Z \), where here the Jordan triple product for elements
\( X, Y, Z \in h_3O \) is of the form:

\[
\{X, Y, Z\} = (X \circ \tilde{Y}) \circ Z + (Z \circ \tilde{Y}) \circ X - (X \circ Z) \circ \tilde{Y}
\]  

(28)

in place of equation 23, where \( \tilde{Y} \) denotes a conjugation on \( h_3O \) ([30] appendix A).
Similarly the Lie bracket relations for \( U_Y, S_{YZ} \) and \( \tilde{U}_Y \) are closely analogous to those
of equation 24. Collectively the 133 elements of equation 27 generate a symmetry
group identified as the exceptional Lie group $E_{7(-25)}$ which, as the conformal group for $h_3 \mathbb{O}$, leaves invariant the generalised cubic ‘light cone’ separation:

$$\det(\mathcal{X} - \mathcal{Y}) = 0 \quad (29)$$

for $\mathcal{X}, \mathcal{Y} \in h_3 \mathbb{O}$. This is a generalisation from equation 20 with $E_{7(-25)}$ acting via a non-linear realisation on the components of the 27-dimensional space $h_3 \mathbb{O}$. The $E_{7(-25)}$ conformal group for $h_3 \mathbb{O}$ is also contrasted with $SO(2,10)$ as the conformal group for $h_2 \mathbb{O}$ in ([33] section 6).

Combining the $E_{6(-26)}$ cubic invariant $\det(\mathcal{X})$ with an appropriate scalar singlet $\alpha \in \mathbb{R}$ the quartic product $\alpha \det(\mathcal{X})$ is invariant under $E_{6(-26)}$ and the additional dilation action. Extending via further components $\mathcal{Y} \in h_3 \mathbb{O}$ and $\beta \in \mathbb{R}$ the quartic norm:

$$q(x) = -2[\alpha \beta - (\mathcal{X}, \mathcal{Y})]^2 - 8[\alpha \det(\mathcal{X}) + \beta \det(\mathcal{Y}) - (\mathcal{X}^2, \mathcal{Y}^2)] \quad (30)$$

can be constructed using the inner product of equations 5 and 7 and the quadratic adjoint of equations 10 and 11 ([1] section 9.2, with the sign convention for $q(x)$ adopted for example from [23, 29, 34]). The argument $x$ of equation 30 belongs to the ‘Freudenthal triple system’, denoted $F(h_3 \mathbb{O})$, an element of which can be written in ‘$2 \times 2$ matrix’ form:

$$x = \begin{pmatrix} \alpha & \mathcal{X} \\ \mathcal{Y} & \beta \end{pmatrix} \quad (31)$$

with $\mathcal{X}, \mathcal{Y} \in h_3 \mathbb{O}$ of equation 6 and $\alpha, \beta \in \mathbb{R}$. With 56 real components $F(h_3 \mathbb{O})$ forms a natural space for the smallest non-trivial representation of the Lie group $E_7$. In fact the quartic norm $q(x)$ of equation 30 is invariant under a full set of $E_{7(-25)}$ linear transformations ([1] equations 9.29–9.32) on the 56-dimensional space $F(h_3 \mathbb{O})$.

A non-degenerate bilinear antisymmetric form can also be defined on the space $F(h_3 \mathbb{O})$, on introducing a second element $y = \begin{pmatrix} \mathcal{W} \\ \mathcal{Z} \end{pmatrix} \in F(h_3 \mathbb{O})$ with $\mathcal{W}, \mathcal{Z} \in h_3 \mathbb{O}$ and $\gamma, \delta \in \mathbb{R}$, by:

$$\{x, y\} = \alpha \delta - \beta \gamma + (\mathcal{X}, \mathcal{Z}) - (\mathcal{Y}, \mathcal{W}) \in \mathbb{R} \quad (32)$$

that is also invariant under the $E_{7(-25)}$ transformations. Further, a symmetric four-linear form can be defined by the linearisation of the quartic norm of equation 30, that is for $x, y, z, w \in F(h_3 \mathbb{O})$:

$$q(x, y, z, w) := \frac{1}{24} \left[ q(x + y + z + w) + q(x + y + z) + q(x + y + w) + q(x + z + w) + q(x + z + w) + q(y + z + w) - q(x) - q(y) - q(z) - q(w) \right] \quad (33)$$

such that $q(x, x, x, x) = q(x)$. In turn a symmetric trilinear product $T$ can be defined uniquely via equations 32 and 33 such that (see for example [29] equation 35, following [35]):

$$\{T(x, y, z), w\} = q(x, y, z, w) \quad (34)$$
for any $x, y, z, w \in F(h_3 \mathbb{O})$. This is the triple product $T(x, y, z) \in F(h_3 \mathbb{O})$ from which the Freudenthal triple system takes its name. This ternary product is presented explicitly in ([36] section 3.3) as:

$$T(x, y, z) = \frac{2}{9} \left[ (x \wedge y)z + (y \wedge z)x + (z \wedge x)y \right]$$  \hspace{1cm} (35)$$

with the Freudenthal product $x \wedge y$ between elements of $F(h_3 \mathbb{O})$ also defined by ([36] equations 3.9 and 3.10).

An alternative definition for the triple product $T(x, y, z)$ for any elements $x, y, z \in F(h_3 \mathbb{O})$ is given for example in ([31] section 3.2), following the convention of [37], which is not symmetric in the three arguments. In turn a four-linear form $q(x, y, z, w)$ can be defined in equation 34 via this axiomatically introduced ternary product $T$ and the bilinear form of equation 32. This latter definition of $q(x, y, z, w)$ is not symmetric in the four arguments, but can be symmetrised as described for ([38] equations 4.1.19 and 4.1.20). However here we adopt the former definition of $q(x, y, z, w)$ from equation 33 and interpret equation 34 as the definition of $T(x, y, z)$, since here the quartic norm $q(x)$ of equation 30 is of central importance.

While the conformal action of $E_7$, with the generator composition of equation 27, extends the $E_6$ action on the space $h_3 \mathbb{O}$, a ‘quasiconformal’ extension of the $E_7$ action on the space $F(h_3 \mathbb{O})$ can also be constructed [30, 31]. A quasiconformal realisation is generated by a 5-graded structure for which the spaces $g^{\pm 2}$ are each one-dimensional, here with (see also [38, 39]):

$$g = g^{-2} \oplus g^{-1} \oplus g^0 \oplus g^{+1} \oplus g^{+2}$$

of dimension: $248 = 1 + 56 + (133 + 1) + 56 + 1$

with elements: $K_\rho, U_y, S_{yz}, \bar{U}_y, \bar{K}_\rho$  \hspace{1cm} (36)$$

labelled by $y, z \in F(h_3 \mathbb{O})$ and $\rho \in \mathbb{R}$. The elements $S_{yz}$ include the 133 generators of $E_7$ together with a dilation $\Delta$; $U_y$ and $\bar{U}_y$ are analogous to the corresponding generators described for equations 21 and 27; while $K_\rho$ and $\bar{K}_\rho$ are each one-dimensional and together with $\Delta$ form a distinguished 3-element closed sl$(2, \mathbb{R})$ subalgebra ([39] section 3).

Collectively the 248 elements of equation 36 form an $E_8$ Lie algebra and generate the real form $E_8(-24)$ of this largest exceptional Lie group acting via a non-linear realisation on the ‘extended Freudenthal triple system’, denoted $eF(h_3 \mathbb{O})$. This is a 57-dimensional space with elements such as $e = (x, \tau) \in eF(h_3 \mathbb{O})$, with $x \in F(h_3 \mathbb{O})$ and $\tau \in \mathbb{R}$ associated respectively with the grade +1 and grade +2 subspaces in equation 36 [30, 31, 34, 38]. Both the quartic norm $q(x)$ of equation 30 and the extra real parameter $\tau \in \mathbb{R}$ are invariant under the $E_7(-25) \subset E_8(-24)$ subgroup. The quartic symplectic distance between any two elements $e = (x, \tau), f = (y, \kappa) \in eF(h_3 \mathbb{O})$ is defined by ([34] equation 61):

$$d(e, f) = q(x - y) - (\tau - \kappa + \{x, y\})^2$$  \hspace{1cm} (37)$$

with the full set of $E_8(-24)$ actions on the space $eF(h_3 \mathbb{O})$ leaving invariant the generalised light cone:

$$d(e, f) = 0$$  \hspace{1cm} (38)$$
This ‘quartic light cone’ is hence a further generalisation from the cubic light cone of equation 29 and the quadratic light cone of equation 20. While a non-zero symplectic distance $d(e, f)$ may be transformed up to an overall factor by the generators of $E_{8(-24)}$, the invariance of the 57-dimensional light cone in equation 38 under the full group $E_{8(-24)}$ justifies the term ‘quasiconformal’ realisation. For example the dilation $\Delta$ scales the components in terms such as $g(x-y), \tau - \kappa$ and $\{x, y\}$ in equation 37 in the appropriate proportions such that equation 38 is invariant.

The transformations of the 248 generators of equation 36, generalising from equation 22 and here acting on any element $e = (x, \tau) \in E(h_3 \mathbb{O})$, are presented explicitly in ([34] equation 63, following [30] equation 29) and include for example $\tilde{K}_\rho(e)$ with the non-linear actions:

$$\tilde{K}_\rho(x) = -\frac{1}{6} \rho T(x, x, x) + \rho x \tau, \quad \tilde{K}_\rho(\tau) = \frac{1}{6} \rho \{T(x, x, x), x\} + 2 \rho \tau^2 \quad (39)$$

Care is needed for differences in notation and the consistency of the definitions involving the asymmetric quadratic form and ternary product as discussed following equation 34 in the construction of this non-linear realisation of $E_8$. The full $E_8$ Lie algebra bracket itself is listed for example in [30, 39].

Different real forms of $E_8$ can also be described in this way. Employing the above analysis for the Freudenthal triple system $F(h_3 \mathbb{O}_s)$, defined over the split octonions $\mathbb{O}_s$, leads to the real form $E_8(8)$ as the corresponding quasiconformal group [30, 38]. This group contains $E_7(7) \subset E_8(8)$ as the conformal subgroup and in turn $E_6(6)$ as the reduced structure group of $h_3 \mathbb{O}_s$. On the other hand the real form of interest here $E_8(-24)$, with the subgroup chain $E_8(-24) \subset E_7(-25) \subset E_8(-24)$, is obtained on employing the non-split octonion algebra $\mathbb{O}$ and the Freudenthal triple system $F(h_3 \mathbb{O})$. This difference is analogous to that discussed for the ‘magic square’ towards the end of subsection 2.1 for which the real form $E_8(-24)$ or $E_8(8)$ obtained depends on the choice of $K' = \mathbb{O}$ or $K' = \mathbb{O}_s$ respectively. On the physics side, applications of the real forms of the exceptional Lie algebras discussed above in supergravity theories are described in [30, 31, 38, 39].

The conclusion of most relevance for the physical theory to be considered in this paper is that in progressing from the Lorentz symmetry of 4-dimensional spacetime, represented by the double cover $SL(2, \mathbb{C})$ acting on the space $h_2 \mathbb{C}$ as described in the opening of this subsection, via a sequence of generalised spacetimes with norm or generalised light cone preserving symmetries, we are led ultimately to the largest exceptional Lie group $E_8$, which leaves equation 38 invariant. This distinguished role for $E_8$ adds to the unique properties of this group reviewed in subsection 2.1, here with a tentative connection to physics through the notion of a ‘generalised spacetime’. By contrast, in the following section we motivate the conception of a ‘general form of time’, and then in section 4 we shall propose that certain structures described in this subsection might rather be interpreted as multi-dimensional temporal forms, leading in section 5 to the proposal that this progression may lead to $E_8(-24)$ as the full symmetry of time.
The General Form of Time

The notion of an observer drifting through space in a spacecraft with the engines turned off, or located within a freely falling lift near the surface of the Earth, following a world line parametrised by a real proper time variable $s$ recording the progression along a trajectory in 4-dimensional spacetime, with an apparent local absence of any force of gravity, is central to Einstein’s theory of general relativity. This idea is encapsulated in the ‘equivalence principle’ (see for example [1] section 3.4), the strong form of which states that at any location in spacetime, in the limit of arbitrarily small spacetime volumes, local inertial coordinates $(x^0, x^1, x^2, x^3)$ can be constructed within which special relativity holds for all laws of physics other than gravity. The proper time interval $\delta s$, for $\delta s \to 0$ in such a local inertial frame, can be expressed with a local Minkowski metric as:

$$\delta s^2 = (\delta x^0)^2 - (\delta x^1)^2 - (\delta x^2)^2 - (\delta x^3)^2$$

with the coordinates transforming under a local Lorentz symmetry that leaves $\delta s$ invariant. With the local gravitational force vanishing, as seen from any other location nearby objects within the same inertial frame (which in practice extends to a very good approximation over finite spacetime volumes such as the vicinity of the spacecraft or interior of the lift in the above examples) will appear to ‘fall’ together (such as a person and a ball in the freely falling spacecraft or lift), hence following a preferred extended path in spacetime independent of the constitution of the falling objects. This implies that the global properties of the gravitational field can be ascribed to the global structure of spacetime itself, as formulated via the curvature tensor and Einstein’s field equation in the theory of general relativity.

For the theory presented in this paper we begin with an even simpler structure than a local inertial frame in spacetime, as expressed by the coordinate intervals $(\delta x^0, \delta x^1, \delta x^2, \delta x^3)$ on the right-hand side of equation 40, and take the irreducible element of the theory to be simply the interval of proper time $\delta s$ on the left-hand side of that equation. Having stripped this structure down to one dimension of time only a theory can be constructed which in some sense generalises general relativity for the notion of an observer ‘drifting through’ a higher-dimensional parameter space, over and above 4-dimensional spacetime, as an expression of and deriving directly from the one-dimensional temporal progression of the observer, as we describe in the following.

While in Newtonian physics time ($x^0$) and space ($x^1, x^2, x^3$) are independent, in Einstein’s relativity they are drawn together in a 4-dimensional spacetime element through the Lorentz invariant proper time interval $\delta s$ in equation 40. Here, in placing the emphasis upon this one-dimensional flow of time, rather than a specific spacetime structure, we aim to combine 4-dimensional spacetime together with ‘extra dimensions’ collectively in a higher-dimensional structure with a full symmetry (augmenting the Lorentz group) acting upon a general form of time. A simple generalisation of equation 40 can be written, balancing the order of the infinitesimal elements on each side, as the homogeneous $p^{th}$-order polynomial:

$$\delta s^p = \alpha_{abc...} \delta x^a \delta x^b \delta x^c \ldots$$

for any integer power $p \geq 1$ and each $\alpha_{abc...} = -1, 0 \text{ or } 1$ with indices $a, b, c = 1, \ldots, n$ for an $n$-parameter space. This general expression naturally contains equation 40 as a
particular quadratic case for $p = 2$, that is:

$$\delta s^2 = \eta_{ab} \delta x^a \delta x^b$$ (42)

with $4 \times 4$ Minkowski metric $\eta = \text{diag}(+1, -1, -1, -1)$ and here with the index convention $a, b = 0, 1, 2, 3$. This 4-dimensional spacetime form can be embedded in higher-dimensional homogeneous polynomials in the form of equation 41 which are not restricted to the quadratic structure of extra spatial dimensions. In fact, since we do not ‘see’ the extra dimensions there is no compelling reason for them to be artificially limited to quadratic extensions of equation 42 with a higher-dimensional Minkowski metric and corresponding local Euclidean properties for the additional ‘spatial’ components. Cubic and higher-order polynomial forms are equally permitted given the emphasis upon generalising the form of the proper time interval on the left-hand side of these equations.

This approach is hence distinct from, and more general than, the class of models based purely on extra spatial dimensions as initiated by Kaluza and Klein [40, 41], as described in [3]. As for Kaluza-Klein models matter fields and physical structures in 4-dimensional spacetime will be associated with the extra-dimensional components, however here the properties directly deriving from these components will differ from Kaluza-Klein theory owing to the more general form of equation 41. The ultimate goal will then be to assess the degree to which derived properties of the resulting matter fields match empirically observed phenomena for the theory presented here based upon the general form of time.

We note that given we have expressed an interval of time $\delta s$ as a homogeneous polynomial in equation 41 in principle a similar decomposition could also be applied to any of the parameter intervals $\delta x^a$. However if we were to substitute for example $(\delta x^1)^q = \beta_{ijk...} \delta y^i \delta y^j \delta y^k \ldots$ as a $q^{\text{th}}$-order homogeneous polynomial, having a structure analogous to equation 41, into equation 41 itself we could express the latter equation in the form $\delta s^{(pq)} = \alpha_{abc...} (\delta x^a)^q (\delta x^b)^q (\delta x^c)^q \ldots$, again balancing the order of the infinitesimal elements. However this new $(pq)^{\text{th}}$-order homogeneous polynomial is of the same form as equation 41 with $p \to pq$, except with a more restrictive structure, and hence such particular cases are already implicitly incorporated. We shall primarily be interested in forms of time with a high degree of symmetry, avoiding preferred components such as $\delta x^1$ above that might be distinguished in this way, and hence equation 41 can be consistently adopted as the most general form of time.

The conceptual basis of the theory is also described in [1, 2, 3] where it is also noted that in order to avoid dealing directly with infinitesimal quantities such as $\delta s$ and $\delta x^a$ we define the differentials $v^a = \frac{dx^a}{ds} = \frac{\delta x^a}{\delta s} |_{\delta s \to 0}$. Hence upon dividing both sides of equation 41 by $\delta s^p$ and taking the limit $\delta s \to 0$, we have:

$$\delta s^p = \alpha_{abc...} \delta x^a \delta x^b \delta x^c \ldots$$

$$\Rightarrow \quad 1 = \alpha_{abc...} \quad \text{written as:} \quad L(v_n) = 1$$ (43)

where $L(v_n) = \alpha_{abc...} v^a v^b v^c \ldots$ is a homogeneous polynomial in the generally finite components $(v^1, v^2, \ldots, v^n)$ of the $n$-dimensional vector $v_n \in \mathbb{R}^n$. Technically, the $\delta x^a$ and $\delta s$ are considered to be ‘infinitesimals of the same order’ in defining the components $v^a = \frac{\delta x^a}{\delta s} |_{\delta s \to 0}$, which is reasonable since the $x^a$ express the flow of time $s$.  

15
itself, with \( s = x^1 \) for the trivial one-dimensional case. The simple general expression of equation 43 for the multi-dimensional form of time, and its symmetries, will provide the basis for a full unified field theory.

Such a physical theory is obtained initially on applying a local translation symmetry of equation 43 to a substructure of four components \( \delta x^a \) exhibiting the quadratic form on the right-hand side of equation 42 in order to construct a local inertial frame in 4-dimensional spacetime, as will be described in subsection 4.1. This preferential treatment of four components is necessary to identify an external spacetime within which all observations and experiments are framed and to provide the necessary background for any physical structures to be ‘seen’ at all. In providing the mathematical framework through which physical structures can be observed in space as well as time, the full symmetry of the general form of time of equation 43 is necessarily broken, as described for example in ([3] subsection 2.3) and in the follow section of this paper, completing the basic conceptual picture upon which a full physical theory can be developed.

Historically the most successful physical theories, including Newtonian mechanics, Maxwell’s equations, the Dirac equation, quantum theory and general relativity, have in common a notion of a continuous flow of time that can be parametrised by a real number \( s \in \mathbb{R} \). It is this feature alone that we have taken as our starting point. Founding the theory purely on the notion of a continuous progression in time, through which all observations of the physical world are made, marks a minimal and conservative basis for a physical theory. On taking the infinitesimal limit the general expression for the multi-dimensional form of time of equation 41 or 43 follows directly from the basic arithmetic composition of the real line. It is through these defining characteristics of the real numbers, as representing the one-dimensional continuum of time, that arithmetic forms together with their associated symmetries can be identified which in principle describe both the properties of external spacetime together with the matter fields it contains, all carried simultaneously within the flow of time itself.

The mathematical structure of the theory hence originates from the real number parametrisation of the continuous flow of time that permeates all of our experiments in, and observations of, the world around us. With the structure of spacetime itself deriving from the symmetries of forms of time, and the properties of matter deriving from multi-dimensional temporal forms over and above that needed to describe 4-dimensional spacetime, in principle the conceptual basis of the theory presents an opportunity to account for the apparently ‘unreasonable effectiveness of mathematics in the natural sciences’ [42]. That is, the mathematical origins of the theory are anchored in the simple structure of the one-dimensional progression of time, through which we necessarily observe the world, that can be identically expressed in the multi-dimensional form of equation 43, through which the physical properties of matter are proposed to derive directly.

An historical precedent for founding a general mathematical structure on the notion of the temporal continuum can be found in the work of William Rowan Hamilton. These ideas were influenced by the relation between time and continuous progression that had been so successfully employed in Newton’s method of fluxions in underpinning Newtonian mechanics, and were being considered by Hamilton around the same time that he was developing his own more general formulation of mechanics. In the introduction to his lengthy paper of 1837 [43] Hamilton contrasts three general
approaches to the study of algebra; namely practical, symbolic and theoretical. In seeking theoretical clarification for the ‘science of algebra’ Hamilton proposed that the notion of time might provide such a basis, complementing the relation between the ‘science of geometry’ and the concept of space. Seemingly influenced also by the ideas of the philosopher Immanuel Kant half a century earlier, regarding the a priori necessity of the forms of space and time as innate structures through which the world is perceived [44], Hamilton comments ([43] in the ‘General Introductory Remarks’, with the upper-case of the original):

The notion or intuition of ORDER IN TIME is not less but more deep-seated in the human mind, than the notion or intuition of ORDER IN SPACE; and a mathematical Science may be founded on the former, as pure and as demonstrative as the science founded on the latter. There is something mysterious and transcendent involved in the idea of Time; but there is also something definite and clear: and while Metaphysicians meditate on the one, Mathematicians may reason from the other.

With the properties of the real line \( \mathbb{R}^1 \) considered to represent directly the continuum of time, in developing this idea Hamilton introduced ‘number-couples’ \((a_1, a_2) \in \mathbb{R}^2\) associated with two independent steps in time, termed primary and secondary but not mutually related by succession in the same one-dimensional progression. Basic arithmetic operations \((+, -, \times, \div)\) were then defined between the number-couples in the spirit of ‘an algebra of pure time’ by analogy with such operations for \( \mathbb{R}^1 \) alone, while also being guided by considerations of simplicity. He found the resulting properties to be equivalent to the arithmetic of the complex numbers, elements of which can be written as \(a_1 + a_2i \in \mathbb{C}\). Noting that the imaginary unit \(i = \sqrt{-1}\), denoting an ‘impossible extraction’, can be represented by the real number-couple \((a_1, a_2) = (0, 1) = \sqrt{(-1, 0)}\) provided some of the justification for this approach. The significant observation for the present paper is that Hamilton’s number-couples \((a_1, a_2)\) effectively express time progressing as a two-parameter entity.

Having been unable to construct a system of ‘number-triplets’ in a similar spirit, Hamilton conceived a 4-dimensional algebra, known as the quaternions \( \mathbb{H} \), in 1843 while speculating upon ‘an additional illustration of his view respecting the Science of Pure Time’ [45]. However, with the multiplication of the three quaternion imaginary units \(i, j, k\) being non-commutative and having a natural geometric interpretation, the new algebra immediately became associated more with the properties of space. Hamilton subsequently focussed upon promoting the quaternions for their practical applications in this geometrical sense, rather than with regard to his earlier more theoretical view as an algebra of time.

In the mid-1840s, shortly after Hamilton’s first presentation of the quaternions, Graves and Cayley independently discovered the 8-dimensional octonions \( \mathbb{O} \) as a further generalisation. This latter development was made by considering the symbolic manipulation of imaginary units, of which there are seven for the octonions as described for equation 2 in subsection 2.2. This completed the set of four normed division algebras \(\mathbb{R}, \mathbb{C}, \mathbb{H}\) and \(\mathbb{O}\), which were proven to be uniquely the only such algebras in 1898 by Hurwitz (see for example [16]). In the case of the discovery of the octonions it took over one hundred years of developments in physics until this largest division algebra
began to be taken seriously for possible applications in a scientific theory, as noted in subsection 2.2 and illustrated by the examples of \([18, 19, 20, 21, 22, 23, 24]\).

The point of view adopted in this paper is not to regard the continuum of time as a foundation of a mathematical algebra in isolation but rather as the starting point for the mathematical description of a full physical theory, while maintaining some of the philosophical influences as alluded to for Hamilton above. Such a physical theory can be developed from the basic arithmetic composition of the real numbers \(\mathbb{R}\) as embodying the structure of the continuous flow of time. In contrast to Hamilton’s invention of algebraic rules for composing finite number-couples \((a_1, a_2)\) by analogy with the algebraic properties of one-dimensional time, here we simply write down a direct and exact identity for the general multi-dimensional form of time expressed for the limit of infinitesimal intervals in equation 41. This expression can then be written in terms of the finite components \(v^a = \frac{dx^a}{ds}\) as described for equation 43.

Any of the four division algebras \(K = \mathbb{R}, \mathbb{C}, \mathbb{H}\) or \(\mathbb{O}\) can be of significance for forms of time owing to the norm compatibility of the composition of their elements, as described for equation 1. For example with \(a, b \in K\) and \(v_n = b\) for \(n = 1, 2, 4\) or 8, equation 43 could take the quadratic form \(L(v_n) = |b|^2 = \bar{b}b = 1\) and the mapping \(b \rightarrow ab\) for \(|a| = 1\) then represents a symmetry leaving the form \(L(v_n) = 1\) invariant. As noted in equation 18 at the end of subsection 2.2 this norm-preserving property is shared by the determinant for matrix algebras, and hence we are naturally drawn to consider matrices over the division algebras to identify possible forms for \(L(v_n) = 1\) and the corresponding symmetry. In particular, while being non-associative, compositions involving octonions can incorporate a high degree of symmetry in a compact algebraic form, as noted in subsection 2.2. Hence it is a curious observation that the octonion algebra, which will feature heavily in the ‘symmetry of time’ for the present theory, was discovered historically via a short sequence of developments initiated by Hamilton’s ambitions concerning the ‘algebra of time’.

Considered as a form of time the 4-dimensional quadratic ‘spacetime’ form in equation 42, with Minkowski metric, can be written via equation 43 as \(L(v_4) = 1\). As noted for equation 19 the Lorentz symmetry \(SO^+(1, 3)\) of \(L(v_4) = |v_4|^2 = 1\) can be represented by its double cover \(SL(2, \mathbb{C})\) acting on the space of \(2 \times 2\) matrices \(h_2\mathbb{C}\), via:

\[
L(v_4) = \eta_{ab}v^av^b = \det(h) = 1 \quad \text{with} \quad v_4 \equiv h \in h_2\mathbb{C}
\]  

While the structure of such a metric or the norm of a division algebra is limited to a quadratic form, on employing determinants of matrices higher-order polynomial forms for equation 43 may be introduced. For example the spaces \(h_m\mathbb{K}\) of \(m \times m\) Hermitian matrices over \(\mathbb{K}\) can be considered and in particular, via the space \(h_3\mathbb{C}\) or \(h_2\mathbb{O}\) as described for ([3] equation 95), we can construct the homogeneous cubic form:

\[
L(v_{27}) = \det(\mathcal{X}) = 1 \quad \text{with} \quad v_{27} \equiv \mathcal{X} \in h_3\mathbb{O}
\]  

As noted for equations 13 and 25 this 27-dimensional form has an \(E_6(-26) \equiv SL(3, \mathbb{O})\) symmetry, which can be constructed in terms of the actions of octonion-valued matrices. In turn this cubic form further embeds in the homogeneous quartic form:

\[
L(v_{56}) = q(x) = 1 \quad \text{with} \quad v_{56} \equiv x \in F(h_3\mathbb{O})
\]  

which has an \(E_7(-25)\) symmetry, as described for equation 30. Given the minus signs in equation 30 and the fact we are looking for expressions of the form \(L(v_n) = +1\) a
change of sign \( q(x) \rightarrow -q(x) \), in place of the convention for equation 30, might be more appropriate, but would not change the results to be presented in this paper. We also note that the appearance of integer coefficients differing from \( \pm 1 \) or 0 in equation 30, even after a possible scaling of the components, is compatible with the general form of equation 43 with coefficients \( a_{abc...} = -1, 0 \) or +1 owing to the multiple opportunities for double counting in the index summations over the \( n \) real components.

Hence the spaces \( h_3 O \) and \( F(h_3 O) \) of the above homogeneous polynomial forms of equations 45 and 46, rather than representing structures considered as ‘generalised spacetimes’ as reviewed in the previous subsection, can be interpreted as spaces underlying higher-dimensional \textit{forms of time}, that is equation 43 with \( n = 27 \) and \( n = 56 \) respectively.

Constructed from the original one dimension of time the form of 4-dimensional spacetime in equation 44 is itself embedded as an intermediate form of time of particular significance. This significance arises from the need to identify an arena of space as well as time, with the local geometrical properties described by this 4-dimensional form and its symmetries, within which observations are made. That is, from the philosophical perspective, equation 44 underlies the \textit{a priori} forms of space and time through which the world is perceived. The higher-dimensional forms of time of equations 45 and 46 incorporate this 4-dimensional spacetime \textit{together with} a structure of ‘extra dimensions’. The necessary imposition of an extended 4-dimensional spacetime arena is then central to the breaking of the symmetry of the higher-dimensional forms of time, with the properties of the residual extra dimensions identified as matter fields in spacetime. This symmetry breaking, and the physical structures of matter over the extended spacetime manifold deriving from it, will be explicitly described in the following section.

4 Symmetry Breaking and the Standard Model

In this section we aim to demonstrate how particular features of the Standard Model of particle physics emerge in the present theory for the 27-dimensional form of time of equation 45 and accumulate further for the 56-dimensional form of time of equation 46. These two cases will be presented in subsections 4.2 and 4.3 respectively, summarising and further analysing the progress made in ([1] chapters 6–9, [2]). In the first subsection below we first consider how an extended spacetime itself can be identified from the symmetries of the 4-dimensional form of time of equation 44. We then review a minimal extension from this 4-dimensional spacetime form in order to describe the mechanism of symmetry breaking for this simpler case (following [3] section 2.3), for which, nevertheless, a non-trivial physical structure will be identified.

4.1 Extended Spacetime and \( \text{SL}(3, \mathbb{C}) \) Symmetry

The structure of a local inertial frame, central to general relativity as described in the opening of section 3, can be constructed in a straightforward manner from the arithmetic properties of an interval of time alone. The form of time \( L(v_4) = 1 \) of equation 44 not only possesses an \( \text{SO}^+(1, 3) \) Lorentz symmetry but also an \( \mathbb{R}^4 \) translation
symmetry with:

\[ L(v_4) = (v^0)^2 - (v^1)^2 - (v^2)^2 - (v^3)^2 \]

\[ = \left( \frac{dx^0}{ds} \right)^2 - \left( \frac{dx^1}{ds} \right)^2 - \left( \frac{dx^2}{ds} \right)^2 - \left( \frac{dx^3}{ds} \right)^2 \]

\[ = \left( \frac{d(x^0 + r^0)}{ds} \right)^2 - \left( \frac{d(x^1 + r^1)}{ds} \right)^2 - \left( \frac{d(x^2 + r^2)}{ds} \right)^2 - \left( \frac{d(x^3 + r^3)}{ds} \right)^2 = 1 \quad (47) \]

for any constant \( r_4 = (r^0, r^1, r^2, r^3) \in \mathbb{R}^4 \). These symmetries are pictured in figure 1, which exhibits the basic geometric properties of a local inertial frame of an empty 4-dimensional spacetime. This demonstrates how the structure of an extended spacetime can itself be derived from the one dimension of time alone, given that the time interval \( \delta s \) on the left-hand side of equation 40 and figure 1 is interpreted as the fundamental element of the theory.

Figure 1: A one-dimensional time interval \( \delta s \) expressed in the form of equation 42 or 44 possesses the symmetries described for equation 47 which parametrise a 4-dimensional spacetime volume element \( M_4 \equiv \mathbb{R}^4 \) of arbitrary extension that can be interpreted as describing a matterless vacuum state.

The manner in which 4-dimensional spacetime can ‘pop-up’ out of the one-dimensional flow of time is analogous to the construction of the 3-dimensional creations of origami from the folding of a 2-dimensional sheet of paper. One significant difference is that here spacetime is itself created from the original one-dimensional element, rather than being constructed within a pre-existing space as for the origami analogy. The differing mathematical structure for the two cases is however similarly simple to describe, as expressed through equation 47 and figure 1 here.

The essential point is that the 4-dimensional spacetime element of figure 1 expresses directly the symmetries of \( L(v_4) = 1 \) which in turn derives directly from the equality for \( \delta s \) in equations 40 and 42 without adding anything to the flow of time itself. By further exploring this idea we shall find that the structure of matter in spacetime can also be ‘enfolded’ within higher-dimensional forms of time, as we describe in the following.

The cubic form of time to be considered here represents the simplest and most direct generalisation of the 4-dimensional quadratic form of time in equation 44 to a
higher-order homogeneous polynomial based on the determinant of a matrix. This is achieved by first expressing the Lorentz 4-vector \( v_4 = (v^0, v^1, v^2, v^3) \) as an element \( h \in h_2 \mathbb{C} \) via the Pauli matrices, as described for equation 19 in the opening of subsection 2.3 and utilised in equation 44, and then simply embedding this \( 2 \times 2 \) Hermitian complex matrix inside a \( 3 \times 3 \) Hermitian complex matrix as:

\[
h_2 \mathbb{C} \ni h = \begin{pmatrix} v^0 + v^3 & v^1 - v^2 i \\ v^1 + v^2 i & v^0 - v^3 \end{pmatrix} \rightarrow \begin{pmatrix} h & \psi \\ \psi^\dagger & n \end{pmatrix} \equiv v_9 \in h_3 \mathbb{C} \quad (48)
\]

Here there are five real parameter ‘extra dimensions’, which could be labelled for example by \((v^4, v^5, v^6, v^7, v^8)\) in the form of \( \psi = (v^4 + v^5 i) \in \mathbb{C}^2 \) and \( n = v^8 \in \mathbb{R} \) (with the notation \( n \in \mathbb{R} \) consistent with equation 26 used here). This structure then represents a full 9-dimensional cubic form of time:

\[
L(v_9) = \det(v_9) = 1 \quad \text{with} \quad v_9 \in h_3 \mathbb{C} \quad (49)
\]

which is invariant under an \( \text{SL}(3, \mathbb{C}) \) symmetry, as a direct augmentation of the \( \text{SL}(2, \mathbb{C}) \) symmetry of \( L(v_4) = 1 \) in equation 44. (This is the ‘Lorentz group’ \( \text{SL}(3, \mathbb{C}) \) of ([31] equation 5) with \( h_3 \mathbb{C} \) there considered a ‘generalised spacetime’, while here \( \text{SL}(3, \mathbb{C}) \) is the symmetry of a ‘general form of time’).

In general \( n \)-dimensional translation symmetries, similar to equation 47, can also be identified for any form \( L(v_n) = 1 \) since \( x^a \) for each component \( v^a = dx^a/ds \) implicitly represents any value \( x^a \in \mathbb{R} \) of the real line, as described for example in ([3] figure 1 and equation 5). The necessary identification of an external 4-dimensional spacetime manifold \( M_4 \) can be realised through a preferred choice of four components for which this translation symmetry is explicitly employed. That is, for the case of the full form \( L(v_9) = 1 \) of equation 49 we note that for the subspace vector \( v_1 \equiv h \in h_2 \mathbb{C} \subset h_3 \mathbb{C} \) we can take, explicitly:

\[
L(v_9) = \det \begin{pmatrix} \frac{dx^0}{ds} + \frac{dx^2}{ds} i & \frac{dx^1}{ds} - \frac{dx^2}{ds} i & \frac{dx^4}{ds} + \frac{dx^5}{ds} i \\ \frac{dx^1}{ds} + \frac{dx^2}{ds} i & \frac{dx^0}{ds} - \frac{dx^3}{ds} i & \frac{dx^6}{ds} + \frac{dx^7}{ds} i \\ \frac{dx^4}{ds} - \frac{dx^5}{ds} i & \frac{dx^6}{ds} - \frac{dx^7}{ds} i & \frac{dx^8}{ds} \end{pmatrix} = 1 \quad (50)
\]

for any constant \( r_4 = (v^0, r^1, r^2, r^3) \in \mathbb{R}^4 \), similarly as for equation 47. The identification of \( M_4 \equiv \mathbb{R}^4 \), with \( v_1 \in TM_4 \) on the local tangent space, breaks the full \( \text{SL}(3, \mathbb{C}) \) symmetry of equation 49, as described for ([3] figure 3) and reproduced here in figure 2.

The choice of a preferred external symmetry \( \text{SL}(2, \mathbb{C}) \subset \text{SL}(3, \mathbb{C}) \) acting on the external tangent space \( TM_4 \) reduces the original element \( v_9 \in h_3 \mathbb{C} \) to the Lorentz
Figure 2: (a) The full symmetry $\text{SL}(3, \mathbb{C})$ of $L(v_9) = 1$ over the translational parameter subspace $x \in M_4$ (b) is broken to the subgroup $\text{SL}(2, \mathbb{C}) \times \text{U}(1)$ by the necessary identification of the external 4-dimensional spacetime manifold $M_4$ itself through the translation symmetry of equation 50.

vector $v_4 \in TM_4 (\equiv h \in h_2 \mathbb{C})$, a spinor $\psi \in \mathbb{C}^2$ and a scalar $n \in \mathbb{R}$, as anticipated by the structure of equation 48. A residual internal $\text{U}(1)$ is also identified and we have the symmetry breaking pattern:

$$\begin{align*}
\text{SL}(3, \mathbb{C}) & \rightarrow \quad \text{SL}(2, \mathbb{C}) \times \text{U}(1) \\
v_9 & \rightarrow \begin{cases}
  v_4 & \text{vector} \\
  \psi & \text{spinor} \\
  n & \text{scalar}
\end{cases}
\end{align*}$$

(51)

where the final column entries denote the $\text{U}(1)$ charges with unit normalisation and with $\psi$ hence identified as a ‘charged’ spinor field over $M_4$.

The internal $\text{U}(1)$ symmetry over the base space $M_4$ in figure 2(b) implies a ‘principle bundle’ structure $P \equiv M_4 \times \text{U}(1)$. This leads to a relationship between the curvature of the external spacetime and the internal curvature of a gauge field $A(x)$ associated with the $\text{U}(1)$ gauge symmetry in a manner analogous to and guided by Kaluza-Klein theory (as formulated for example in [46, 47]), which is the main topic of [3] (see also equation 82 here). While both the external and internal symmetry derive from the same unification group, namely $\text{SL}(3, \mathbb{C})$ in figure 2(a), compatibility with the Coleman-Mandula theorem [48] for the quantised theory follows from the absolute symmetry breaking structure in figure 2(b) and equation 51 as described in ([3] subsection 5.3) and as will be reviewed in section 6 of this paper.

Here we note that through both the Kaluza-Klein relation between the external and internal curvature and the subsequent quantisation of the theory the field values over $M_4$ in general vary and hence the translation symmetry of equation 50 is only strictly valid in a vanishingly small spacetime volume. As will also be described in section 6 here the dynamic degrees of freedom of the continuous spacetime geometry...
are not themselves ‘quantised’, and the resulting physical notion of a local inertial frame in the extended spacetime is essentially equivalent to that reviewed in the opening of section 3 as central to general relativity.

The gauge field $A(x)$ associated with the internal symmetry $U(1)$ together with the residual temporal components $\psi(x)$ and $n(x)$ of $v_9 \in h_3 \mathbb{C}$ over $v_4 \in TM_4$, as pictured in figure 2(b), are interpreted as ‘matter fields’ over $M_4$. These fields, and their symmetry properties summarised in equation 51, derive directly from the elementary structure of the theory based on the symmetries and possible arithmetic composition of the continuum of time alone.

For a standard Kaluza-Klein theory, which already begins with the non-trivial structure of a principle fibre bundle $P \equiv M_4 \times G$ over an extended spacetime base manifold $M_4$, a further complication is required in order to introduce spinor fields, for example via a supersymmetric extension of the theory (see for example [49] and the references therein). A suitable Lagrangian may also be proposed to introduce interactions between the various fields by hand.

For the theory developed here from the simple starting point of one dimension of time, even for this minimal model based on $SL(3, \mathbb{C})$ as the symmetry of a cubic form of time, we have identified both an internal $U(1)$ gauge field $A(x)$ and a spinor field $\psi(x)$ together with their mutual interaction, as represented in figure 2(b) and determined by the constraints of the theory. These constraints, such as equation 49 itself, are implied since the theory derives from the very simple structure of one dimension only, and in turn the need to add further restrictions by hand through postulated Lagrangian terms can in principle be avoided. The question is then whether these constraints lead to mathematical structures that are recognisable in comparison with the physical structures deduced from empirical observations.

For the $SL(3, \mathbb{C})$ model described in this subsection, on interpreting the internal $U(1)$ as an electromagnetic gauge symmetry, we obtain a primitive ‘electrodynamics’ with the gauge field $A(x)$ coupled to the spinor field $\psi(x)$. This is in addition to the Kaluza-Klein relationship deduced between the external gravitational curvature and the electromagnetic field curvature as alluded to above, while the neutral scalar field $n(x)$ of equation 51 can in principle be identified as a ‘dark matter’ candidate. We can then ask how these possible physical structures might be augmented as we progress towards natural higher-dimensional extensions for the general form of time in equation 43, as will be described in the following subsections.

4.2 $E_6 \equiv SL(3, \mathbb{O})$ Symmetry

The natural generalisation of the complex numbers through the division algebras $\mathbb{C} \to \mathbb{H} \to \mathbb{O}$ to the octonions and the existence of the cubic norm or ‘determinant’, as described for equation 13, suggests that the form $L(v_9) = 1$ of equation 49, with $v_9 \in h_3 \mathbb{C}$ and an $SL(3, \mathbb{C})$ symmetry, can naturally be extended to the 27-dimensional form of time of equation 45, that is:

$$L(v_{27}) = \det(X) = pmn - p|b|^2 - m|c|^2 - n|a|^2 + 2\text{Re}(\bar{a}\bar{b}\bar{c}) = 1 \quad (52)$$

with $v_{27} \equiv X \in h_3 \mathbb{O}$ of equation 6, and with an $E_6(-26) \equiv SL(3, \mathbb{O})$ symmetry.
As noted in subsection 2.2 this real form of the $E_6$ Lie group symmetry is explicitly constructed in [24, 25, 26, 27] as summarised in ([1] chapter 6) and employed extensively in ([1] chapter 8, [2]). Following those references a basis for the 78-dimensional Lie algebra of $E_{6(-26)}$ can be represented by a linearly independent set of vector fields in the tangent space $T_{h_3\mathbb{C}}$ of the form:

$$
\hat{R} = \begin{pmatrix}
\dot{p} & \dot{a} & \dot{c} \\
\dot{a} & \dot{m} & \dot{b} \\
\dot{c} & \dot{b} & \dot{n}
\end{pmatrix} \in T_{h_3\mathbb{C}}
$$

(53)

Here the notation convention for $X \in h_3\mathbb{C}$ in equation 6 is employed with a ‘dot’, as for $\dot{p}$, denoting a tangent vector component. A complete preferred basis for this Lie algebra is listed in ([26] table A.1) and also described for ([1] table 6.3, [2] table 1), consisting of ‘boosts’ and ‘rotations’ when considered as generators of a generalised Lorentz group.

Each of the 78 tangent vector fields in the form of equation 53 is determined via the derivative of a corresponding finite $E_{6(-26)}$ symmetry transformation preserving $\det(X)$ and is written out explicitly in ([1] tables 6.6 and 6.7, [2] tables 6 and 7). The Lie bracket between any pair of these $E_{6(-26)}$ generators was calculated for the entries of the full $78 \times 78$ algebra commutation table determined for [26], with the various sign and other conventions adopted in [1, 2] oriented through consistency with that full $E_{6(-26)}$ Lie algebra table.

In the context of the theory presented in this paper the breaking of the full $E_{6(-26)}$ symmetry of the 27-dimensional form of time $L(v_{27}) = \det(\mathcal{X}) = 1$ follows from the same argument as described for the $SL(3,\mathbb{C})$ symmetry of $L(v_9) = 1$ in the previous subsection, through the necessary interposition of an intermediate 4-dimensional spacetime form. However, in place of equation 48, here the spacetime components $v_4 = (v^0, v^1, v^2, v^3) \equiv h \in h_2\mathbb{C}$ are embedded within an element of $v_{27} \equiv \mathcal{X} \in h_3\mathbb{C}$ as:

$$
h_2\mathbb{C} \ni h \rightarrow \frac{v^0 + v^3}{v^1 + v^2 l + a(6)} \begin{pmatrix}
v^1 - v^2 l + \bar{a}(6) \\
v^0 - v^3 \\
\bar{c} \\
b \\
n
\end{pmatrix} \equiv \frac{h + a(6)}{\theta^i} \begin{pmatrix}
\theta \\
\theta^i \\
\theta \\
n
\end{pmatrix} \equiv v_{27} \in h_3\mathbb{C}
$$

(54)

With respect to the components of $\mathcal{X} \in h_3\mathbb{C}$ in equation 26 we have substituted $p = v^0 + v^3$ and $m = v^0 - v^3$, while for the component $a \in \mathbb{O}$ with the real subcomponents of equation 2 we have $a_1 = v^1$ and $a_8 = v^2$, that is with the octonion imaginary unit $l$ employed for the $v^2$ external spacetime component, in place of the complex unit $i$ in equation 48, in line with the conventions of the preferred basis for equation 53 (as explained in [1] following equation 6.58). The six remaining imaginary components are denoted by:

$$
a(6) = a_7 \hat{\mathbb{O}} + a_2 i + a_6 \hat{\mathbb{J}} + a_3 j + a_5 \hat{\mathbb{L}} + a_4 k
$$

ordered here for later reference. In the second $3 \times 3$ matrix in equation 54 these components are embedded in the upper-left-hand $2 \times 2$ part as $a(6) \equiv a(6)_{\begin{pmatrix}0 & -1 \\1 & 0\end{pmatrix}}$, consistent
with equation 26. There are now a total of 23 real parameter ‘extra dimensions’ including $a(6)$ together with the sixteen real components of $\theta = \begin{pmatrix} i \\ j \end{pmatrix} \in \mathbb{O}^2$ and again with $n \in \mathbb{R}$.

On identifying a 4-dimensional spacetime manifold $M_4$ the components of $\psi_4 \in TM_4$ projected onto the tangent space transform as a Lorentz 4-vector under an external $\text{SL}(2, \mathbb{C}) \subset \mathbb{E}_6(-26)$ symmetry, similarly as depicted in figure 2(b). The symmetry breaking pattern is now found to incorporate four Weyl spinors, denoted:

$$
\theta_l = \begin{pmatrix} c_1 + c_8 l \\ b_1 - b_8 l \end{pmatrix}, \quad \theta_i = \begin{pmatrix} c_7 \bar{l} + c_2 i \\ -b_7 \bar{l} - b_2 i \end{pmatrix}, \quad \theta_j = \begin{pmatrix} c_6 \bar{l} + c_3 j \\ -b_6 \bar{l} - b_3 j \end{pmatrix}, \quad \theta_k = \begin{pmatrix} c_5 \bar{l} + c_4 k \\ -b_5 \bar{l} - b_4 k \end{pmatrix}
$$

from the components of $\theta = \begin{pmatrix} i \\ j \end{pmatrix} \in \mathbb{O}^2$, and seven scalars corresponding to the components of $a(6)$ and $n$ under the external Lorentz symmetry, as described in detail in ([1] section 8.1) and summarised in ([2] section 5). Each of $\theta_l, \theta_i, \theta_j$ and $\theta_k$ transform in the same way as two-component Weyl spinors under $\text{SL}(2, \mathbb{C})$ and they are taken to be ‘left-handed’ by convention.

An internal $\text{SU}(3) \times \text{U}(1) \subset \mathbb{E}_6$ symmetry is also identified. The three components of the broken symmetry $\text{SL}(2, \mathbb{C}) \times \text{SU}(3) \times \text{U}(1) \subset \mathbb{E}_6$ are generated by the Lie algebra elements in the form of equation 53 as listed here:

$$
\begin{align*}
\text{SL}(2, \mathbb{C}) : & \quad \{ \hat{B}_L^1, \hat{R}_d^1, \hat{B}_d^1, \hat{B}_u^1, \hat{R}_x^1, \hat{R}_d^1 \} \\
\text{SU}(3) : & \quad \{ \hat{A}_q, \hat{G}_i \} \quad \text{for } q = \{ i, j, k, \bar{l}, \bar{l}, \bar{u}, \bar{u}, l \} \\
\text{U}(1) : & \quad \hat{S}_l^1
\end{align*}
$$

These elements form part of the preferred basis of ([26] table A.1, as reproduced in [1] table 6.3, [2] table 1) where the notation is explained in full. (These generators are listed here explicitly in equation 57 largely as a reference for equation 62 later in this subsection). Any element of any one of the three sets of generators in equation 57 commutes with any element of the other two sets, by the Lie algebra table of [26].

The internal $\text{SU}(3) \times \text{U}(1)$ symmetry is associated with the internal colour $\text{SU}(3)_c$ and electromagnetic $\text{U}(1)_Q$ symmetry of the Standard Model, owing to the transformation properties of the components of $\psi_{27} \equiv \chi \in h_3 \mathbb{O}$ under the broken $\mathbb{E}_6$ symmetry, as described in detail in ([1] section 8.2) and summarised in ([2] section 5), with the results listed here in table 1.

The provisional association of the 4-vector $\psi_4$ with a non-standard Higgs sector of the theory in table 1 will be explained in the following subsection. The interpretation of the scalar $n$ as a dark matter (DM) candidate was alluded to at the end of the previous subsection and will be discussed further in section 6. Here we focus on the direct connections established with the Standard Model.

The Weyl spinor $\theta_l$ of equation 56 transforms as a singlet $\mathbf{1}$ under the internal $\text{SU}(3)_c$ but non-trivially under the internal $\text{U}(1)_Q$. Hence $\theta_l$ (essentially the equivalent of the spinor $\psi$ of equation 51) is provisionally associated with the electron state, and motivates the charge normalisation of 1 for this case. The remaining $\text{U}(1)_Q$ charges listed in table 1 are given by their magnitudes relative to the unit electron charge, with no interpretation of particle as compared with antiparticle state made here. These relative charges are fixed by the structure of the $\mathbb{E}_6$ Lie algebra and in particular the $\text{U}(1)_Q$ generator in equation 57. The three Weyl spinors $\{ \theta_l, \theta_j, \theta_k \}$ in equation 56
The six components of $a(6)$ in equation 55 transform as $\text{SL}(2, \mathbb{C}) \subset E_6$ scalars rather than spinors, but they can be paired up in the set $\{a_{7,2}, a_{6,3}, a_{5,4}\}$ which also transforms as an $\text{SU}(3)_c$ triplet, and owing to their $U(1)_Q$ charges of $\frac{2}{3}$ these components are provisionally correlated with $d$-quarks via their transformation properties under the internal symmetry. The two remaining components $a_{1,8}$ of the octonion element $a$ provide a natural slot to be assigned to the neutrino state owing to the invariance of the $a_{1,8}$ components under the internal $SU(3)_c \times U(1)_Q$ symmetry. The neutrino component $a_1 + a_8 l$ is then related to the electron component $\theta_l = (c_1 + c_8 l, b_1 - b_8 l)$ similarly as the $u$-quark states in the remaining $a(6)$ components are related to the $d$-quark states in the corresponding remaining imaginary components of $\theta = (\theta_l) \in \mathbb{O}^2$, on comparing equations 55 and 56, suggesting doublets of leptons and quarks. However these $a_1 + a_8 l$ components, as for those of $a(6)$, also do not transform via a spinor representation, and in fact the $a_{1,8}$ slot is already ‘occupied’ by two of the four external spacetime components, explicitly with $v_4 = (v^0, v^1, v^2, v^3) = (\frac{1}{2}(p + m), a_1, a_8, \frac{1}{2}(p - m)) \in TM_4$ as described for equation 54; and hence this provisional ‘$\nu$-lepton state’ is bracketed in table 1.

Hence it remains to be explained how the neutrino can be accommodated within this theory and how a neutrino and $u$-quark components transforming as $\text{SL}(2, \mathbb{C})$ spinors can be identified while retaining the internal symmetry properties of table 1. Further, even the $e$-lepton and $d$-quark states in table 1 are only identified here as two-component left-handed Weyl spinors rather than the four-component Dirac spinors of the Standard Model. In addition these patterns will need to be repeated for a full three generations of states. The incompleteness of this picture is further demonstrated by the well known fact that the Lie group $E_6$ is not large enough to contain the subgroup

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|}
\hline
$E_6$ & $\rightarrow$ & $\text{SL}(2, \mathbb{C}) \times \text{SU}(3)_c \times U(1)_Q$ & (B)SM \\
\hline
$v_4$ & vector & 1 & 0 & (Higgs) \\
$\theta_l$ & L-spinor & 1 & 1 & e-lepton \\
$\theta_{i,j,k}$ & L-spinor & 3 & $\frac{1}{3}$ & $d$-quarks \\
$[a_{1,8}]$ & $X$ & 1 & 0 & $\nu$-lepton \\
$a(6)$ & scalar & 3 & $\frac{2}{3}$ & $u$-quarks \\
$n$ & scalar & 1 & 0 & (DM) \\
\hline
\end{tabular}
\caption{Decomposition of the components of $L(v_{27}) = 1$ as the $E_6$ symmetry is broken through the identification of an external spacetime with projected components $v_4 \in TM_4$, augmenting the structure of figure 2 and equation 51. The final column lists the (Beyond the) Standard Model correlations that can be drawn from the symmetry breaking pattern.}
\end{table}
SL(2, \mathbb{C}) \times SU(3) \times SU(2) \times U(1) \text{ representing the external Lorentz symmetry and the full Standard Model internal gauge symmetry together, and hence the full structure of electroweak theory cannot be incorporated here.}

However, some of the properties of the Standard Model electroweak theory can be identified based on SU(2) \times U(1) \subset E_6 subgroups for the E_6 symmetry of \( L(v_{27}) = 1 \) presented in this subsection. These properties include an analogue of the standard symmetry breaking pattern SU(2)_L \times U(1)_Y \rightarrow U(1)_Q \text{ in the projection of } v_4 \in TM_4, \text{ as described in ([1] section 8.3)} \text{ where a ‘mock electroweak theory’ is constructed, in part motivating the link between the 4-vector } v_4 \text{ and the Higgs in table 1, (a link that will be further justified after figure 4). These observations hint that a full electroweak theory might be accommodated in a further augmentation of the theory towards a higher-dimensional form of time with a larger symmetry group.}

In the meantime simply by generalising from the complex numbers \( \mathbb{C} \) in equations 48 and 49 to the octonions \( \mathbb{O} \) in equations 52 and 54, consistent with the form of equation 43, the simple matter fields of equation 51 have been augmented to those of table 1 and established a recognisable foothold in the structures of the Standard Model. That is, in place of the ‘primitive electrodynamics’ described at the end of the previous subsection the matter fields identified here now resemble one generation of Standard Model leptons and quarks.

The inability to fit an SL(2, \mathbb{C}) \times SU(3) \times SU(2) \times U(1) subgroup inside E_6 can be seen by analysis of the Dynkin diagrams for the corresponding complex Lie algebras. In typical unification models the Lie group E_6 is generally considered from the point of view of internal symmetries alone, that is as a ‘Grand Unified Theory’ with SU(3)_c \times SU(2)_L \times U(1)_Y \subset E_6 (see for example [8, 50]). Through the conceptual scheme presented here, as pictured for the SL(3, \mathbb{C}) model in figure 2 and now with the matter fields over \( M_4 \) described in table 1 for the E_6 case, the full symmetry necessarily includes both the external symmetry and the internal symmetry, with the latter identified as far as SU(3)_c \times U(1)_Q for the E_6 symmetry described in this subsection. (For the overall theory this framework is consistent with the Coleman-Mandula theorem due to the absolute nature of the symmetry breaking, as discussed in the previous subsection for figure 2 and further in section 6).

However, Dynkin analysis can also be employed to study the symmetry breaking pattern for SL(2, \mathbb{C}) \times SU(3) \times SU(2) \times U(1) \subset E_6(-26) on the 27-dimensional representation of E_6. Here for the complex Lie algebra we take the 27 weights of the 27 representation of E_6 described by Dynkin labels such as \((1 0 0 0 0 0)\), as listed in ([50] table 11b) and ([51] chapter 27) for example, with the six Dynkin coefficients ordered in correspondence with the six simple roots \( \alpha_i (i = 1, \ldots, 6) \) of the rank-6 E_6 Lie algebra, which in turn are matched with the six nodes of the E_6 Dynkin diagram as shown in figure 3(a).

Dynkin analysis deals with complex Lie algebras and with the ‘complexified’ Lie algebra for the Lorentz algebra so^+(1, 3) \equiv sl(2, \mathbb{C}) being su(2) \oplus su(2) we are hence looking for a symmetry breaking structure of the semi-simple form:

\[
SU(2) \times SU(2) \times SU(3)_c \times U(1)_Q \subset E_6 \tag{58}
\]

in the analysis of the Dynkin diagrams (see also [1] section 7.3). This can be achieved in a number of ways, including that depicted in figure 3(b) with the corresponding
Figure 3: (a) Dynkin diagram for the complex Lie algebra $E_6$ with six simple root labels and (b) a possible breaking pattern that at the level of the corresponding real Lie groups describes $SL(2,\mathbb{C}) \times SU(3) \times U(1) \subset E_6(-26)$.

weight projection matrices:

$$P = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}, \quad Q = \begin{bmatrix}
0 & 1 & 2 & \frac{4}{3} & \frac{2}{3} & 1
\end{bmatrix} \quad (59)$$

Here the $4 \times 6$ matrix $P$ projects each weight of the $27$ representation of $E_6$, such as $(1 \ 0 \ 0 \ 0 \ 0 \ 0)$, onto four Dynkin labels aligned with the rank-4 $SU(2) \times SU(2) \times SU(3)_c$ subgroup, while the $1 \times 6$ matrix $Q$ projects out the $U(1)_Q$ charges; such that the $27$ weights of this $E_6$ representation collectively branch under the symmetry breaking as listed in table 2.

| $E_6$ | Lorentz | $SU(3)_c$ | $U(1)_Q$ | $v_{27} \in h_3 \mathbb{Q}$ |
|-------|----------|-----------|-----------|-----------------------------|
| 27    |          |           |           |                             |
| 4     | vector   | 1         | 0         | $v_4$                       |
| 2     | $L$-spinor | 1         | +1        | $\theta_1$                  |
| 2     | $R$-spinor | 1         | −1        |                             |
| 6     | $L$-spinor | $\mathbf{3}$ | $-\frac{1}{3}$ | $\theta_{i,j,k}$            |
| 6     | $R$-spinor  | $\mathbf{3}$ | $\frac{1}{3}$ |                             |
| 3     | scalar    | $\mathbf{3}$ | $\frac{2}{3}$ | $a(6)$                      |
| 3     | scalar    | $\mathbf{3}$ | $-\frac{2}{3}$ |                             |
| 1     | scalar    | 1         | 0         | $n$                         |

Table 2: Branching of the $27$ representation of $E_6$ under the subgroup of equation 58 via the Dynkin label projection matrices of equation 59. The overall choice for each of $L \leftrightarrow R$, $\mathbf{3} \leftrightarrow \bar{\mathbf{3}}$, and $\pm Q$ is a matter of convention for the Lorentz, $SU(3)_c$ and $U(1)_Q$ representations respectively. The final column lists the best-matching correspondence with the components of table 1.

In comparison with the matrix $P$ of equation 59 a very different and more complicated projection for $E_6$ representations can be constructed from a combination
of the weight projection matrices described in ([50] section 7) via:

\[ P_{20}(SU(4) \subset SU(5)) \circ P_{17}(SU(5) \subset SU(6)) \circ P_{16}(SU(2) \times SU(6) \subset E_6) \] (60)

In this case the subgroup \( SU(3)_c \times U(1)_Q \subset SU(4) \subset E_6 \) is identified as a further projection after the above \( P_{20} \) factor, with an appropriate \( Q^{\text{em}} \) projecting the electromagnetic charge, with the resulting branching structure for the \( 27 \) representation of \( E_6 \) listed in ([50] table 21). While [50] only deals with internal gauge symmetries a further \( SU(2) \subset E_6 \) projection can be identified along with the \( SU(2) \) factor from \( P_{16} \) in equation 60 which combined together can be interpreted as an external Lorentz \( \equiv SU(2) \times SU(2) \) symmetry as described for equation 58. The structure of the full projection for the corresponding complete subgroup of equation 58 applied to the \( 27 \) representation of \( E_6 \) is found to match that of table 2 above. A third calculation, projecting the same subgroup symmetry this time via the extended Dynkin diagram for \( E_6 \) (using the diagram and weights listed in [51] section 27.2), yields again the same result and conclusions.

The Dynkin analysis for the broken symmetry branching of the 27-dimensional representation of \( E_6 \) described in table 2 has a recognisable connection with that obtained through the explicit \( E_6(-26) \) actions on the space \( h_3 \mathbb{O} \) for table 1, as noted by the correspondence with subcomponents of \( v_{27} \in h_3 \mathbb{O} \) identified in the final column of table 2. However there are also significant differences. Formulated in terms of complex Lie algebras the 27 weights for the Dynkin analysis are defined over a complex \( \mathbb{C} \)-valued space. On the other hand each of the 27 components of \( v_{27} \in h_3 \mathbb{O} \) can be considered as a real \( \mathbb{R} \)-valued parameter, while the explicit determinant preserving symmetry transformations of \( E_6(-26) \equiv SL(3, \mathbb{O}) \) acting on the space \( h_3 \mathbb{O} \) involve both the quaternion \( \mathbb{H} \) and octonion \( \mathbb{O} \) algebras!

This difference is seen for example with both left \( L \) and right \( R \) Weyl spinors being listed in table 2 while only left-handed \( L \)-spinors are identified in table 1. This feature arises in the latter case since we have a non-standard representation of the Lorentz group \( SO^+(1, 3) \), via the Clifford algebra \( C(1, 3) \), as effectively embedded in \( 2 \times 2 \) quaternion matrices \( \mathbb{H}(2) \) acting on the spinor space \( \mathbb{H}^2 \subset \mathbb{O}^2 \). As described for ([1] equations 8.10–8.13) this leads to the set of four Weyl spinors of equation 56 of the same handedness. This observation is key for the left-right structure of the theory under the larger \( E_7 \) symmetry, in comparison with the Standard Model, as will be described in the following subsection.

While the structure of complex Lie algebras and Dynkin analysis provides a useful guide the above differences highlight the need to study the breaking of the full symmetry of \( L(v_n) = 1 \) of equation 43 explicitly if the octonion algebra is involved. As noted in subsection 2.2 non-associative octonion composition, while not directly forming a group structure, can describe a high degree of symmetry. This is the case with the construction of the \( E_6(-26) \) symmetry of \( L(v_{27}) = 1 \) of equation 52, and hence the explicit anatomy of this particular structure (following [24, 25, 26, 27]) and symmetry breaking pattern has been analysed in detail in [1, 2] in order to uncover the properties of the resulting matter fields listed in table 1.

Direct analysis of the extended Dynkin diagram for \( E_6 \) shows that in addition to the breaking pattern of figure 3(b) and equation 58 the Lie group \( E_6 \) also contains the larger subgroup:

\[ SU(2) \times SU(2) \times SU(4) \subset E_6 \] (61)
in turn with SU(3) × U(1) ⊂ SU(4) contained as a further possible decomposition. A
similar subgroup structure for E_6 was implied above in the discussion of equation 60.
Taking this as a guide for the breaking of the explicit E_6(−26) action on L(\nu_{27}) = 1,
through the projection of the external \nu_4 \in TM_4 components, this suggests that com-
plementing the external Lorentz symmetry a full internal SU(4) symmetry might also
be identified. This is indeed the case with the following set of 15 elements of the E_6
Lie algebra in the form of equation 53 from the basis of ([26] table A.1, [1] table 6.3
or [2] table 1) generating such an SU(4):

\[
SU(3) : \{ \hat{A}_q, \hat{G}_l \} \quad \text{for } q = \{ i, j, k, kl, jl, \bar{j}l, \bar{l}l \}
\]

\[
U(1) : \begin{cases} \hat{S}_l^1 \quad \text{SU(4)} \quad (62) \\ + : \{ \hat{G}_q + 2\hat{S}_q^1 \} \quad \text{for } q = \{ i, j, k, kl, jl, \bar{j}l \} \end{cases}
\]

With 6 further generators added to the 8 of SU(3) and the one for U(1) in equation 57,
this set of 15 forms a closed su(4) algebra under the Lie bracket commutator described
in [26], while each of the 15 commutes with the 6 external Lorentz generators on the
top line of equation 57.

The action of this SU(4) ⊂ E_6 subgroup on the components of \nu_{27} \in h_3Ω can
be determined by explicit examination of the 15 generators of equation 62 as elements
\hat{R} \in Th_3Ω, equation 53, as listed in ([1] table 6.7 or [2] table 7). Interpreted as an
internal symmetry this SU(4) acts by the fundamental 4-dimensional representation
\mathbf{4} upon the set of four left-handed Weyl spinors \{ \theta_i, \theta_j, \theta_k \} of equation 56, which
would be to mix the e-lepton and d-quark components according to the Standard
Model assignments proposed in table 1. The same su(4) ≃ so(6) internal algebra acts
via the fundamental 6-dimensional representation \mathbf{6} of the rotation group SO(6) on
the set of six scalar components a(6) in equation 55. The remaining 5 = 27 − 16 − 6
real subcomponents listed in table 1, namely of \nu_4 (including the a_{1,8} part) and n, are
invariant under this SU(4) subgroup.

In fact a simple Dynkin analysis shows that the rank-5 subgroup in equation 61
can be also be obtained as SU(2) × SU(2) × SU(4) ⊂ D_5 ≃ SO(10). A real form of the
rank-5 Lie group D_5 can be constructed as SL(2, \mathbb{O}), the double cover of SO^(+)(1,9),
acting as the symmetry of L(\nu_{10}) = det(X) = 1 for \nu_{10} \equiv X \in \mathbb{O}^3 as a quadratic
form of time (that could be interpreted as representing 10-dimensional spacetime, as
described in leading to equation 25). In this case SU(4) is the double cover of an internal
SO(6) which acts on the six Lorentz scalar extra dimensions over 4-dimensional
spacetime as pictured in ([2] figure 1). The same six scalars a(6) remain for the augmenta-
tion to the full E_6 symmetry of the cubic form of time L(\nu_{27}) = 1, as embedded
in equations 26 and 54 and listed in table 1, and again transform as a 6-dimensional
vector under this internal SO(6).

The proposal here is that under a further extension to a larger symmetry for
a yet higher-dimensional form of time L(\nu_n) = 1, with n > 27, the components of
a(6) = (a_1\mathbf{2} + a_2i) + (a_6\mathbf{2} + a_3j) + (a_5\mathbf{2} + a_4k) from equation 55 will effectively pair
up while being subsumed into a set of three Weyl spinors, essentially aligned with the
existing spinors \{ \theta_i, \theta_j, \theta_k \} of equation 56. This will break the SU(4) symmetry to the
SU(3)_c × U(1)_Q of table 1 according to which the components underlying the three
new spinors transform under SU(3)_c as a triplet of u-quarks with U(1)_Q charges of \frac{2}{3}.
In principle with \((u_d)\)-quark doublets identified and a surviving \(U(1)Q\) symmetry component this structure is hence expected to be closely related to the identification of a full electroweak symmetry with the breaking pattern \(SU(2)_L \times U(1)_Y \to U(1)Q\) for the augmented theory. This extension of the theory will clearly also need to account for the left-right asymmetry of weak interactions. While keeping in mind these well known properties of the Standard Model, the general approach here is to seek natural mathematical augmentations to higher-dimensional homogeneous polynomial forms of time, one of which we describe in the following subsection.

### 4.3 \(E_7\) Symmetry

Here we describe a further natural extension of the multi-dimensional form of time, consistent with the general homogeneous polynomial form \(L(v_n) = 1\) introduced in equations 41 and 43. The 27-dimensional cubic form \(L(v_{27}) = \det(\mathcal{X})\) of equations 45 and 52 is invariant under \(E_6(-26)\), providing an explicit description of the smallest non-trivial \(E_6\) representation. With \(v_{27} \equiv \mathcal{X} \in h_3\) this structure is embedded within the 56-dimensional quartic form \(q(x)\), with \(x \in F(h_3)\), as can be seen from the explicit structure of equation 30, which suggests that we can consider:

\[
L(v_{56}) = q(x) = -2[\alpha \beta - (\mathcal{X}, \mathcal{Y})]^2 - 8[\alpha \det(\mathcal{X}) + \beta \det(\mathcal{Y}) - (\mathcal{X}^\sharp, \mathcal{Y}^\sharp)] = 1 \quad (63)
\]

for \(v_{56} \equiv x \in F(h_3)\) as a suitable candidate for a higher-dimensional form of time with an \(E_{7(-25)}\) symmetry, as noted for equation 46.

In fact an element \(x = (\alpha \mathcal{X}, \beta \mathcal{Y}) \in F(h_3)\) of the Freudenthal triple system with 56 real components, as described for equation 31, is composed of two elements \(\mathcal{X}, \mathcal{Y} \in h_3\) of the exceptional Jordan algebra together with two real variables \(\alpha, \beta \in \mathbb{R}\). Collectively this object provides a description of the smallest non-trivial representation of \(E_7\) which is known to branch under the \(E_6 \subset E_7\) subgroup as:

\[
56_{E_7} \to (27 + \overline{27} + 1 + 1)_{E_6} \quad (64)
\]

as described for ([1] equation 9.33, [2] equation 62). Unlike the case for \(E_6\) the Lie group \(E_7\) does not have complex representations. Here the complex 27 representation of \(E_6\), corresponding to \(\mathcal{X} \in h_3\), is combined with the complex conjugate \(\overline{27}\) representation of \(E_6\), corresponding to \(\mathcal{Y} \in h_3\), within the full \(E_7\) symmetry action. This implies a doubling up of the components listed in table 1 with the addition of a corresponding ‘complex conjugate’ set. As noted in the previous subsection, since the octonion algebra is used to describe these symmetry transformations, an explicit analysis is required. Such an analysis is described in ([1] section 9.2, [2] section 6) where each left-handed component under \(SL(2,\mathbb{C}) \subset E_6\) in table 1 is found to be now partnered with a corresponding right-handed component for the augmented \(E_7\) case, as might have been expected on appending the ‘complex conjugate’ \(E_6\) representation.

Hence this natural augmentation from a cubic \(E_6\) to a quartic \(E_7\) form of time leads from the original set of four 2-component left-handed Weyl spinors of equation 56 and table 1 to the identification four 4-component Dirac spinors associated with the \(e\)-lepton and \(d\)-quark states. The internal \(SU(3)_c \times U(1)_Q\) transformation properties identified now also for the right-handed components are identical to those for the left-handed partner components as originally listed in table 1.
A further significant observation can be made relating to this structure. The identification of the external 4-dimensional spacetime through the component \( v_4 \in T M_4 \), described for the 4-, 9- and 27-dimensional forms of time via equations 47, 48 and 54 respectively, now for the case of the 56-dimensional form of time \( L(v_{56}) = 1 \) is necessarily projected out from \textit{either} the \( \mathcal{X} \in h_3 \Omega \subset F(h_3 \Omega) \) component, that is the ‘left-handed’ part of \( x \in F(h_3 \Omega) \), or from the \( \mathcal{Y} \in h_3 \Omega \subset F(h_3 \Omega) \) component, that is the ‘right-handed’ part of \( x \in F(h_3 \Omega) \), but \textit{not from both}. This concrete difference between the left and right-handed sectors of the theory with respect to the identification of the external spacetime manifold \( M_4 \) itself, at this \( E_7 \) level, in principle underlies the manifestation of parity violating phenomena observed in the laboratory.

This latter feature is introduced by hand in the Standard Model via Lagrangian terms, as is typically also the case for many unification schemes. For example in [52] the Standard Model gauge group derives from a local \( G = SU(2)_L \times SU(2)_R \times SU(4) \) symmetry for which the right-handed gauge bosons \( W_R \), associated with the \( SU(2)_R \) component, are \textit{assumed} to have a large mass relative to their left-handed counterparts \( W_L \), in order to be compatible with empirically observed parity violating phenomena. This is arranged by \textit{assigning} appropriate parameters to terms in the Lagrangian for the spontaneous symmetry breaking structure ([52] equations 17–19). The ‘colour \( SU(4) \)’ component of the internal symmetry group \( G \) in equation 65 is analogous to the internal \( SU(4) \) in equation 61, in both cases mixing leptons and quarks, although in the latter equation the \( SU(2) \times SU(2) \) subgroup represents the external Lorentz symmetry rather than a further internal component. In the present theory a further augmentation in the symmetry of time will ultimately be required in order to incorporate a full \( SU(2)_L \times U(1)_Y \) electroweak symmetry, but a natural origin for a left-right asymmetry is clear at the \( E_7 \) level as described above.

We also note that one puzzling feature for the symmetry breaking from the \( E_6 \) level, described in the previous subsection, was the observation that the natural slot to be assigned to neutrino state, that is the \( SU(3)_c \times U(1)_Q \) invariant components \( a_{1,8} \) of table 1, is already occupied by two of the four components of \( v_4 \in T M_4 \) in the external spacetime tangent space. This slot in the left-handed sector \( \mathcal{X} \in h_3 \Omega \) is now \textit{freed up} on projecting the external \( v_4 \in T M_4 \) from the corresponding components of \( \mathcal{Y} \in h_3 \Omega \) instead, which in turn \textit{prohibits} the identification of a neutrino state in the right-handed sector of the theory (the choice of \( \mathcal{X} \) or \( \mathcal{Y} \), as for the naming of ‘left’ or ‘right’, is a matter of convention that does not affect the conclusions). Hence the empirical asymmetry in the handedness of the physical neutrino particle state can in principle be naturally accounted for, while still identifying both left and right-handed components (in \( \mathcal{X} \) and \( \mathcal{Y} \) respectively) for the \( e \)-lepton and \( u \)- and \( d \)-quark states, within the caveat for the ‘\( \nu \)’-lepton and ‘\( u \)’-quark components recalled below.

In comparison with the simpler case of subsection 4.1 for the breaking of the \( SL(3, \mathbb{C}) \) symmetry of \( L(v_9) = 1 \), as pictured in figure 2, and the intermediate case of subsection 4.2 for the breaking of the \( E_6 \) symmetry of \( L(v_{27}) = 1 \), as described for table 1, the matter fields resulting from the breaking of the \( E_7 \) symmetry of \( L(v_{56}) = 1 \) over the base manifold \( M_4 \) are summarised in figure 4 (see also [1] equation 9.46, [2] equation 66). These primarily include gauge fields associated with the internal gauge symmetry \( SU(3)_c \times U(1)_Q \) in interaction with states associated with the decomposition.
of $v_{56}$ with properties further resembling a generation of Standard Model leptons and quarks. While the $e$-lepton and $d$-quark components are identified as Dirac spinors, the $\nu$'-lepton and $u'$-quark components are listed in quote marks in figure 4 since they lack the appropriate spinor structure and are provisionally identified via the internal symmetry transformation properties alone.

$$L(v_{56}) = q(x) = 1 \text{ symmetry } E_7 \longrightarrow \text{SL}(2, \mathbb{C}) \times \text{SU}(3)_c \times \text{U}(1)_Q$$

(B)SM

\[
\begin{align*}
\begin{cases}
\chi \to X \to Y \to v_4 \equiv x = (\alpha X, \beta Y) \in F(h_3 \mathbb{O}) \\
\text{with } X, Y \in h_3 \mathbb{O} \\
X = (p^a \bar{a})_n, Y = (p^A \bar{A})_M \in h_2 \mathbb{O} \\
\text{and } v_4 \equiv h \in h_2 \mathbb{C}
\end{cases}
\end{align*}
\]

$\Delta$ and $\Lambda$.

Figure 4: The full symmetry $E_7$ of $L(v_{56}) = 1$ broken through the necessary identification of an external 4-dimensional spacetime manifold $M_4$. The projected components $v_4 \in TM_4 (\equiv h \in h_2 \mathbb{C})$ are now embedded in $Y$ similarly as they had been in $X$ in equation 54 (see also equations 6 and 26). The subcomponents $\theta_{\ell,i,j,k}$ are the four right-handed Weyl spinors in the $Y$ components corresponding to the four left-handed Weyl spinors $\theta_{l,i,j,k}$ of $X$ listed in equation 56. The scalar dark matter (DM) candidates and the ‘vector-Higgs’ $v_4$, included in the list of matter fields, exhibit ‘Beyond the Standard Model’ features.

It can also be noted that there is very little redundancy in identifying these Standard Model structures in figure 4. In fact the only residual pieces of $v_{56} \equiv x \in F(h_3 \mathbb{O})$ are the four Lorentz scalar and $\text{SU}(3)_c \times \text{U}(1)_Q$ invariant components $\{\alpha, \beta, n, N\}$, which hence provide a set of candidates to account for the ‘dark sector’ in cosmology, augmenting the $n \in \mathbb{R}$ component alone of figure 2(b) and table 1, as will be discussed further towards the end of section 6.

The Higgs sector of the Standard Model is formulated in terms of suitably parametrised postulated Lagrangian terms introduced to describe the phenomena of
electroweak symmetry breaking without explaining their physical origin. For the present theory the necessary identification of an external spacetime $M_4$ from subcomponents of the full form of time, as pictured for the form $L(v_{56}) = 1$ in figure 4, necessarily breaks the full symmetry of time and hence in principle accounts for the origin of the symmetry breaking phenomena of the properties of matter observed in spacetime. Central to this symmetry breaking structure is the projected 4-vector $v_4 \in TM_4$ itself which is associated with the Standard Model scalar Higgs state as described in ([1] section 8.3). There are three principle arguments for this association:

- The choice of a particular ‘direction’ for $v_4(x) \in TM_4$ for any $x \in M_4$, as pictured in figure 4, when considered globally over the extended spacetime is analogous to the spontaneous symmetry breaking associated with the choice of direction of magnetisation through the alignment of atomic spins in a ferromagnet below the critical temperature (as discussed after [1] figure 13.3). Further, the frame-dependent choice of the four real components of $v_4 \in TM_4$ is analogous to the gauge-dependent choice of four real components for the vacuum value of the Higgs complex doublet field in a ‘Mexican hat’ potential.

- Variation in the magnitude $|v_4(x)|$ on the tangent space of $M_4$ in the projection out of the full form $L(v_n) = 1$ is directly associated with a conformal warping of the spacetime geometry (as described following [1] figure 13.1). This in turn directly generates variations in energy-momentum via the Einstein field equation, interpreted as $-\kappa T^{\mu\nu} := G^{\mu\nu}$, motivating the link between $|v_4|$ and the scalar Higgs as the ‘origin of mass’. (See also equation 85 and the discussion towards the end of section 6).

- As noted in subsection 4.2 at the level of the $E_6$ symmetry of $L(v_{27}) = 1$ a ‘mock electroweak theory’ can be constructed in which natural SU(2) $\times$ U(1) subgroups can be identified which are broken to U(1)$_Q$ through impingement upon the ‘vector-Higgs’ $v_4 \in TM_4$ components ([1] subsections 8.3.1 and 8.3.2). This impingement is proposed to generate the masses for the heavy gauge bosons $W^\pm$ and $Z^0$ in the complete theory, with masses for the leptons and quarks deriving from terms in the expansion of the full form $L(v_n) = 1$ ([1] subsection 8.3.3).

In non-standard models in which the Higgs is not taken to be a fundamental scalar the Higgs might be considered as, or replaced by, a composite of spin-$\frac{1}{2}$ states (see for example [53]). In principle such an approach might be connected with the present theory since spinors can also be composed to form objects transforming as a vector, such as for example $v_4 \in TM_4$. Here such a spinor decomposition of the $v_4 \in TM_4$ part of $Y \in h_3 \mathbb{O}$ in figure 4 could be mirrored by a spinor decomposition of the corresponding 4-vector $\{p, m, a_1, a_3\}$ components in $X \in h_2 \mathbb{O}$, which is required to describe the left-handed neutrino state. More broadly a spinor decomposition of the full set of $X \in h_2 \mathbb{O}$ and $Y \in h_2 \mathbb{O}$ components is required to account also for the Lorentz transformation properties of the left and right-handed $u$-quark states respectively.

This suggests that in the full theory, with a higher-dimensional form $L(v_n) = 1$ beyond $n = 56$, more spinors will be introduced in place of some of the vector and scalar states in figure 4. Such a structure might then be consistent with a combination of the Standard Model and composite Higgs approaches, in a physical theory providing a full explanation for the origin of symmetry breaking as described in this section.
In summary, via a sequence of mathematically natural augmentations of the general form of time through cubic determinants to the quartic norm \( L(v_{56}) = q(x) = 1 \) of equation 63, with an \( E_{7(-25)} \) symmetry broken over the \( M_4 \) base manifold, a series of esoteric properties of the Standard Model have been directly or partially uncovered, as summarised in figure 4, bearing a close resemblance to a generation of Standard Model quarks and leptons with elements of an electroweak symmetry breaking structure also identified.

The rank-7 Lie algebra \( E_7 \) is in fact large enough to incorporate the full rank-6 subgroup \( SL(2, \mathbb{C}) \times SU(3) \times SU(2) \times U(1) \subset E_7 \), in principle including an \( SU(2)_L \times U(1)_Y \) electroweak symmetry. However, empirically such an \( SU(2)_L \) symmetry is required to act upon left-handed doublets \((\nu)_L\) and \((u)_L\), within which each state is a Weyl spinor, while it is still the case at this stage in figure 4 that the ‘\( \nu \)’-lepton and ‘\( u \)’-quark states are identified by their internal \( SU(3)_c \times U(1)_Q \) transformation properties and the corresponding components are not spinors under the external \( SL(2, \mathbb{C}) \), as noted above and similarly as described in the previous subsection following table 1. Hence these electroweak features remain to be identified, and are expected to be closely related to the breaking of an internal \( SU(4) \subset E_6 \subset E_7 \) to an \( SU(3)_c \times U(1)_Q \subset SU(4) \) as discussed at the end of the previous subsection.

In addition, perhaps the most obvious absence in figure 4 is the Standard Model feature of a full three generations of quarks and leptons. Since empirically the \( SU(2)_L \) action also mixes the three generations and the standard scalar Higgs transforms as an \( SU(2)_L \) doublet, all of the further significant symmetry structures and states of the Standard Model remaining to be fully identified over those summarised in figure 4 are mutually correlated, and hence in principle might collectively be sought and identified in a further extension of the theory. This motivates consideration of a further augmentation to a yet higher-dimensional form of time, with a symmetry beyond both \( E_6 \) and \( E_7 \), as we describe in the following section.

5 \( E_8 \) Symmetry and Time

5.1 Completing the Standard Model

The progression of multi-dimensional forms of time, beyond \( L(v_4) = 1 \) of equation 44 with Lorentz symmetry and as described in the previous section, is summarised here in table 3 along with the main Standard Model properties identified in the symmetry breaking over the base manifold \( M_4 \).

We can then ask how a further stage in the progression of forms of time beyond table 3 might be identified, and the extent to which it may account for the further properties of the Standard Model required as described at the end of the previous section. Inspection of table 3 suggests an augmentation of the progression of spaces with \( h_2 \mathbb{C} \rightarrow h_3 \mathbb{C} \rightarrow h_3 \mathbb{O} \rightarrow F(h_3 \mathbb{O}) \rightarrow \mathcal{T} \), where \( \mathcal{T} \) represents the \( n \)-dimensional space for the full form of time upon which a quintic or higher-order polynomial norm denoted \( L(v_n) = Q(t) = 1 \), with \( v_n \equiv t \in \mathcal{T} \), might be defined. In seeking a natural mathematical extension with \( SL(2, \mathbb{C}) \rightarrow SL(3, \mathbb{C}) \rightarrow E_6 \rightarrow E_7 \rightarrow \hat{G} \) from the symmetries of table 3 there is a strong hint that \( Q(t) = 1 \), as the full form of time, may be invariant
under $\hat{G} = E_8$ as the full symmetry of time, uniquely terminating this series in $E_8$ as the largest exceptional Lie group. Ideally having first identified such an $E_8$ symmetry of $L(v_n) = 1$ we might then consider whether the physical implications of the additional structure match the further Standard Model properties sought, completing the picture of known elementary particle states and potentially making predictions for new phenomena.

As described in subsection 2.3 one possible augmentation of the $E_7(-25)$ symmetry of the quartic form $q(x)$ of equation 30, with $x \in F(h_3O)$, is in fact known, namely the ‘quasiconformal’ non-linear realisation of $E_8(-24)$ as the symmetry preserving the ‘light cone separation’ $d(e,f) = 0$ of equation 38, with $e, f \in EF(h_3O)$ elements of the 57-dimensional extended Freudenthal triple system. The quartic symplectic distance $d(e,f)$ between any two points incorporates the norm of an arbitrary element $e = (x, \tau) \in EF(h_3O)$ which can be written, via equation 37 on setting $f = (y, \kappa) = 0$, as:

$$N(e) = d(e,0) = q(x) - \tau^2$$  \hspace{1cm} (66)

The $E_7(-25) \subset E_8(-24)$ subgroup acts as a linear 56-dimensional representation on $x \in F(h_3O)$, leaving $q(x)$ invariant, and upon $\tau \in \mathbb{R}$ as a singlet. Hence the light cone $N(e) = 0$ can be written as:

$$q(x) = \tau^2 \quad \equiv \quad L(v_{56}) = 1$$  \hspace{1cm} (67)

upon a trivial normalisation, with an $E_7(-25)$ symmetry. Hence equation 63, that is the final entry of table 3, can be identified as a substructure of the $E_8(-24)$ symmetry of $d(e,f) = 0$. However this latter structure is clearly not of the form $L(v_n) = 1$ of equation 43, and the question remains regarding whether $E_8$ can describe the symmetry of the hypothetical homogeneous polynomial form $Q(t) = 1$, again incorporating $q(x)$ in some manner but now interpreted as an augmentation to a higher-dimensional form of time, similarly as $q(x)$ itself incorporates the cubic form $\det(X)$ in equation 63.

In particular, the most natural extension might be expected to involve the smallest non-trivial representation of $E_8$, that is of 248 dimensions, and hence the full
form of time can be provisionally written as $L(v_{248}) = Q(t) = 1$, with $v_{248} \equiv t \in T$. While such a form, as an extension of the mathematical structures listed in table 3, has not been explicitly identified in the literature we can consider in general terms whether the further augmentation $E_6 \to E_7 \to E_8$ acting on a space of $27 \to 56 \to 248$ dimensions might in principle accommodate the further Standard Model properties required as outlined at the end of the previous section.

As we noted there for the matter fields identified over the base space $M_4$ in figure 4 the most evident discrepancy at the level of the broken $E_7(-25)$ action on the space $F(h_3)$ is the need to account for three generations of Standard Model quarks and leptons. With this in mind it is suggestive to note that while the smallest non-trivial representation of $E_7$ branches to representations of the subgroup $E_6$ as described in equation 64, the $248$ representation of $E_8$ exhibits the branching pattern under a subgroup containing $E_6$ as:

$$E_8 \supset E_6 \times SU(3) : \quad 248 \to (27, 3) + (\overline{27}, 3) + (78, 1) + (1, 8) \quad (68)$$

Hence at the level of the standard representation structure there is an apparent threefold nature to the embedding of the smallest non-trivial representations of $E_6$, that is the $27$ and $\overline{27}$, within the $248$ of $E_8$. While the embedding of $E_6$ in $E_7$, with a representation structure of equation 64, led to the identification of both the left and right-handed states of Dirac spinors, through the explicit construction summarised in figure 4, in general terms equation 68 suggests that the augmentation from $E_6$ to $E_8$ might in principle also incorporate three generations of such states.

The question then concerns how such a threefold structure might be explicitly realised in a homogeneous polynomial form of time $L(v_{248}) = 1$ in the context of the present theory. In particular we can consider how such a structure might be built up from the $SL(2, \mathbb{C}) \times SU(3)_c \times U(1)_Q \subset E_6(-26) \subset E_7(-25)$ action on $\mathcal{X} \in h_3$ and $Y \in h_3$, as the $27$ and $\overline{27}$ components of $x \in F(h_3)$ of equation 31, as described in subsections 4.2 and 4.3, to a full Standard Model subgroup:

$$SL(2, \mathbb{C}) \times SU(3)_c \times SU(2)_L \times U(1)_Y \subset E_8(-24) \quad (69)$$

acting on a 248-dimensional space $T$ incorporating three generations. If such a space $T$ can be constructed, built up from $F(h_3)$ by following these hints from the Standard Model, the unbroken $E_8$ symmetry action on the homogeneous form $Q(t)$, with $t \in T$, could then be further studied as a mathematical structure of interest in its own right.

As noted in subsection 4.2 it is known from both Dynkin analysis for $E_6$ and explicit study of the $E_6(-26)$ action on the space $h_3$ that the full Lorentz and Standard Model subgroup $SL(2, \mathbb{C}) \times SU(3) \times SU(2) \times U(1)$ cannot be accommodated inside this exceptional Lie group alone. However it was also noted that there are $SU(2) \times U(1) \subset E_6(-26)$ subgroups acting on the components of $h_3$ that exhibit some of the properties sought for standard electroweak theory, with the association between $v_4 \in TM_4$ and the Higgs further motivated in subsection 4.3, as described in detail in ([1] section 8.3). Hence, since the $E_6(-26)$ level can partially account for electroweak properties, a realistic $SU(2)_L \times U(1)_Y \subset E_8(-24)$ might be expected to be composed of generators which straddle the $E_6(-26) \subset E_8(-24)$ subgroup, neither completely contained as a subgroup of $E_6(-26)$ nor fully independent of it. Similarly it is also known that the decomposition of $\mathcal{X} \in h_3$ and $Y \in h_3$ under the subgroup...
\( \text{SL}(2, \mathbb{C}) \subset E_6(-26) \) does not accommodate Weyl spinor components for the candidate \( \nu \)-lepton and \( u \)-quark states, as summarised in table 1 and further discussed for figure 4 for the \( E_7(-25) \) extension.

For the above reasons the hypothetical full physical symmetry breaking pattern \( \text{SL}(2, \mathbb{C}) \times \text{SU}(3)_c \times \text{SU}(2)_L \times \text{U}(1)_Y \subset E_8 \) of equation 69 is generally not expected to be neatly aligned with an \( E_8 \) decomposition of the form of equation 68 containing the \( \text{SL}(2, \mathbb{C}) \times \text{SU}(3)_c \times \text{U}(1)_Q \subset E_6 \) structure already identified. Hence unlike the progression of forms listed in table 3 through to the case \( E_6 \to E_7 \), for which the extension from \( v_{27} \) to \( v_{56} \) matched the component structure of equation 64, the further augmentation to an \( E_8 \) symmetry of \( L(v_{248}) = 1 \) is expected to exhibit a different breaking pattern. That is the final and largest ‘Russian doll’ of the sequence may be somewhat ‘skewed’ with respect to the others, with an \( E_8 \) decomposition and its representations not directly aligned with the intermediate substructures of table 3. The question then concerns the possible construction of the higher-dimensional form itself.

Further, as described in subsection 4.2, standard Dynkin analysis of representations based on complex Lie algebras only serves as a guide here, since the octonion algebra is central to the description of these symmetry structures. Hence here we consider how we might proceed towards the construction of the desired \( E_8 \) symmetry by studying the mathematical patterns that emerge in the explicit algebraic augmentations of the forms of time that have already been identified.

In table 3 the 9-dimensional cubic form \( L(v_9) = 1 \) with \( v_9 \in h_3 \mathbb{C} \) is neatly contained within the 27-dimensional cubic form \( L(v_{27}) = 1 \) as can be seen by restricting the elements \( a, b, c \in \mathbb{O} \) of \( X \in h_3 \mathbb{O} \) in equation 6 to subspaces with \( a, b, c \in \mathbb{C} \) in equation 13. A similar restriction can be applied to the components \( X \in h_2 \mathbb{O} \) and \( \theta \in \mathbb{O}^2 \) in equation 25, through which equation 70 below can be obtained. Equation 25 itself also demonstrates how in place of \( L(v_9) = 1 \) an intermediate 10-dimensional form of time \( L(v_{10}) = \det(X) = 1 \) can be considered, with \( v_{10} \equiv X \in h_2 \mathbb{O} \) embedded within the larger spaces as shown in equation 26 and figure 4. This second route is taken for example in ([2] table 5), with the choice of these two possibilities discussed for ([3] equation 95) and before equation 45 here.

Taking into account this choice of progression we have the following series of augmentations of homogeneous polynomial forms of time based on the extension of spaces and symmetries:

- **SL(2, \mathbb{C})** on \( L(v_4) = \det(h) = 1 \) \( \to \) **SL(3, \mathbb{C})** on \( L(v_9) = \det(v_9) = 1 \)

  \[ v_4 \equiv h \in h_2 \mathbb{C} \to v_9 \in h_3 \mathbb{C} \quad \text{via} \quad 2 \times 2 \to 3 \times 3 \text{ determinant (equation 48)} \]

  \[ \det(h) \to \det(v_9) = n \det(h) - 2h \cdot (\psi \psi^\dagger) \quad (70) \]

  **or**  **SL(2, \mathbb{C})** on \( L(v_4) = \det(h) = 1 \) \( \to \) **SL(2, \mathbb{O})** on \( L(v_{10}) = \det(X) = 1 \)

  \[ v_4 \equiv h \in h_2 \mathbb{C} \to v_{10} \equiv X \in h_2 \mathbb{O} \quad \text{via} \quad \mathbb{C} \to \mathbb{O} \text{ division algebra ([1] section 6.3)} \]

  \[ \det(h) \to \det(X) = \det(h) - |a(6)|^2 \quad (71) \]
• SL(2, \mathbb{O}) on \(L(v_{10}) = \det(X) = 1\) \(\rightarrow\) \(E_{6(-26)} \cong SL(3, \mathbb{O})\) on \(L(v_{27}) = \det(X) = 1\)

\[v_{10} \equiv X \in h_2 \mathbb{O} \rightarrow v_{27} \equiv X \in h_3 \mathbb{O} \text{ via } 2 \times 2 \rightarrow 3 \times 3 \text{ determinant (equation 25)}\]

\[
\det(X) \rightarrow \det(X) = n \det(X) - 2X \cdot (\theta \theta^\dagger)
\] (72)

• \(E_{6(-26)}\) on \(L(v_{27}) = \det(X) = 1\) \(\rightarrow\) \(E_{7(-25)}\) on \(L(v_{56}) = q(x) = 1\)

\[v_{27} \equiv X \in h_3 \mathbb{O} \rightarrow v_{56} \equiv x \in F(h_3 \mathbb{O}) \text{ via cubic } \rightarrow \text{ quartic form (equation 30)}\]

\[
\det(X) \rightarrow q(x) = -2[\alpha \beta - (X, Y)]^2 - 8[\alpha \det(X) + \beta \det(Y) - (X^2, Y^2)]
\] (73)

We note in particular that in progressing from 10 to 27 dimensions with \(h_2 \mathbb{O} \rightarrow h_3 \mathbb{O}\) the expression for \(L(v_{27}) = \det(X) = 1\) with \(E_{6(-26)}\) symmetry contains a single term \(n \det(X)\) in equation 72, within which both \(n\) and \(\det(X)\) are invariant under \(SL(2, \mathbb{O}) \in E_{6(-26)}\). On the other hand in the progression from 27 to 56 dimensions in the following stage the expression for \(L(v_{56}) = q(x) = 1\) with \(E_{7(-25)}\) symmetry contains two terms of a similar form, namely \(\alpha \det(X)\) and \(\beta \det(Y)\) in equation 73, with each of the four factors being invariant under the subgroup \(E_{6(-26)} \subset E_{7(-25)}\). We then conjecture that for a further stage in this progression there may be three terms of the form \(\lambda q(x)\), with \(\lambda \in \mathbb{R}\) and each factor invariant under an \(E_{7(-25)} \subset E_{8(-24)}\) subgroup of the full symmetry \(E_{8(-24)}\) of the proposed full form of time, and hence we consider the extension:

• \(E_{7(-25)}\) on \(L(v_{56}) = q(x) = 1\) \(\rightarrow\) \(E_{8(-24)}\) on \(L(v_{248}) = Q(t) = 1\)

\[v_{56} \equiv x \in F(h_2 \mathbb{O}) \rightarrow v_{248} \equiv t \in T \text{ to hypothetical } \geq \text{ quintic form } Q(t)\]

\[q(x) \rightarrow Q(t) \sim f((\lambda q(x) + \eta q(y) + \sigma q(z)), (x, y, z))
\] (74)

where \(f\) is a function of \(x, y, z \in F(h_3 \mathbb{O})\) and \(\lambda, \eta, \sigma \in \mathbb{R}\). The three parameters \(\lambda, \eta, \sigma\) can be chosen such that they are proportionately scaled by the dilation \(\Delta \in E_8\), described for equation 36, in the same way as \(\tau^{-2}\) from an element \(e = (x, \tau) \in eF(h_3 \mathbb{O})\) in equation 37 for example. While in the latter case the dilation scaling for \(\tau \in \mathbb{R}\) is such that equation 38 is invariant, in the former case terms of the form \(\lambda q(x)\) will then be invariant under the action of \(\Delta \in E_8\). This structure is analogous to, and motivated by, the invariance of \(n \det(X)\) under the dilation subgroup of \(E_6\), as described for equation 25, and the invariance of \(\alpha \det(X)\) under the \(E_7\) dilation ([1] equation 9.30). It is also suggested that the ‘trilinear product’ \((x, y, z)\) may feature in the expression for \(Q(t)\) in equation 74, by analogy with the bilinear products of the form \((X, Y)\) in the expression for \(q(x)\) in equation 73. There may also be further terms involving \(x, y, z\) in combination with \(\lambda, \eta, \sigma\) as well as possible additional parameters.

The progression from \(L(v_{10}) = 1\) to \(L(v_{27}) = 1\) introduced the further component \(\theta \in \mathbb{O}^2\) in equation 72, as explained for equations 25 and 26, which decomposes as the set of Weyl spinors of equation 56 under the external \(SL(2, \mathbb{C}) \subset E_{6(-26)}\) symmetry as reviewed in subsection 4.2. In turn the necessary incorporation of both \(X\) and \(Y \in h_3 \mathbb{O}\) in the extension from \(L(v_{27}) = 1\) to \(L(v_{56}) = 1\) in equation 73 leads to the identification of both left and right-handed spinors, as described in subsection 4.3.
noting this pattern the conjecture here is that an expression involving \(x, y, z \in F(\mathbb{h}_3\mathbb{O})\) as proposed in equation 74 for the full form \(L(\mathbf{v}_{248}) = 1\) will account for three generations of Standard Model spinor states as an explicit expression for the suggestion made following equation 68.

The property of normed division algebras in equation 1 and of matrix determinants in equation 18, described in subsection 2.2, motivated a consideration of these structures for symmetries of the forms of time as introduced in section 3. However the cubic form \(L(\mathbf{v}_{27}) = \det(X) = 1\), while employing both of these structures in equation 13, can also be written in terms of trace functions as equation 14, while the quartic form \(L(\mathbf{v}_{56}) = q(x) = 1\) of equation 63 cannot be expressed as a simple determinant function. On the other hand both of these temporal forms and their symmetries involve the octonion algebra in an essential manner. Similarly here for equation 74 the overall expression for \(L(\mathbf{v}_{248}) = Q(t) = 1\) is not expected to be a determinant function while the determinant of various subcomponents may feature and octonion composition is expected to play a significant role in describing this structure and its symmetry.

The provisional structure of equation 74 contains \(3 \times (56 + 1) = 171\) real variables, well short of the 248 dimensions for the smallest non-trivial representation of \(E_8\). However, in generalising to the higher-dimensional form of time another key feature of the Standard Model required to be identified is a set of components, correlated with \(X \in \mathbb{h}_2\mathbb{O}\) at the \(E_{6(-26)}\) stage and also \(Y \in \mathbb{h}_2\mathbb{O}\) at the \(E_{7(-25)}\) stage, as described for table 1 and figure 4 respectively, which will transform as a set of Weyl spinors under the external \(\text{SL}(2, \mathbb{C}) \subset E_{8(-24)}\) to account fully for the \(\nu\)-lepton and \(u\)-quark states. While \(X\) and \(Y\) are embedded directly in the higher-dimensional structures of equations 72 and 73, via equations 6 and 26, for the further extension the need to open up \(X = \left(\begin{array}{c} p \\ a \\ m \end{array}\right), Y = \left(\begin{array}{c} P \\ A \\ M \end{array}\right) \in \mathbb{h}_2\mathbb{O}\), or more specifically the \(a, A \in \mathbb{O}\ degrees of freedom, to act as spinors will necessarily increase the total number of components involved.

Focussing on \(X \in \mathbb{h}_2\mathbb{O}\) this is studied in detail in ([1] section 9.1) where it is also described how in principle this can be achieved while maintaining the appropriate \(\text{SU}(3)_c \times \text{U}(1)_Q\) transformation properties of the components of \(a \in \mathbb{O}\) listed in table 1 and figure 4 which already match those of the \(\nu\)-lepton and \(u\)-quark states. This will necessarily augment the 171 real components of equation 74 with the aim of converging upon a homogeneous polynomial form \(L(\mathbf{v}_3) = Q(t) = 1\), potentially with \(n = 248\), and the ambition of incorporating the complete set of Standard Model states under the breaking of the full \(E_{8(-24)}\) symmetry over \(M_4\).

As a generalisation from \(q(x) = 1\) this new structure developed from equation 74 should be expected to not only incorporate the necessary spinor states for the \(\nu\)-lepton and \(u\)-quarks from the \(X, Y \in \mathbb{h}_2\mathbb{O}\) components, but to see these associated with the existing \(e\)-lepton and \(d\)-quark spinor states respectively from the corresponding \(\theta \in \mathbb{O}^2\) components in figure 4 as doublets of an \(\text{SU}(2)_L \in E_{8(-24)}\) symmetry, all accommodated within the components of the homogeneous polynomial form \(L(\mathbf{v}_{248}) = Q(t) = 1\) of quintic or higher order.

As noted above in general in extending from \(q(x) = 1\) with \(x \in F(\mathbb{h}_3\mathbb{O})\) and with an \(E_{7(-25)}\) symmetry to the full hypothetical form \(Q(t) = 1\) with \(E_{8(-24)}\) symmetry this form may involve three elements \(x, y, z \in F(\mathbb{h}_3\mathbb{O})\), or a closely related structure. The final term in the provisional expression of equation 74 alludes to a possible role for the triple product \(T(x, y, z)\) as defined in equations 34 and 35 and/or the Jordan
triple product \( \{X, Y, Z\} \) of equation 28. At the latter level of \( X, Y, Z \in h_3 \Omega \), and with \( Q(t) \in \mathbb{R} \) being a scalar quantity, the cubic form \( (X, Y, Z) \) of equation 9 or 17, as a direct generalisation of \( L(v_{27}) = \det(X) = 1 \) from equation 12 or 13, might also feature in a significant way.

As a tentative step towards deducing what the structure of \( Q(t) \) might look like we take three elements \( X, Y, Z \in h_3 \Omega \), with the first two having the components of equation 6 while \( Z \) is assigned corresponding components \( p', m', n' \in \mathbb{R} \) and \( a', b', c' \in \mathbb{O} \). Then we find via substitution into equation 9, and cross-checking with ([28] equation 214), that the cubic form \( (X, Y, Z) \in \mathbb{R} \) can be written out explicitly as the following cubic polynomial function:

\[
(X, Y, Z) = \frac{1}{2} (pMn' + p'mN + Pm'n + Pmn' + Pmn) - (p\langle B, b' \rangle + P\langle b', b \rangle + p'\langle b, B \rangle + m\langle C, c' \rangle + M\langle c', c \rangle + m'(c, C) + n\langle A, a' \rangle + N\langle a', a \rangle + n'(a, A)) + \Re (bCa' + cAb' + ABc' + bcA + ca'B + ab'C)
\]  

(75)

where the octonion inner product \( \langle B, b' \rangle \) is defined in equation 8. This expression can be considered as a generalisation either from equation 7 for the bilinear form \( (X, Y) \) or from equation 13 for the cubic norm \( \det(X) = \frac{1}{2} \langle X, X, X \rangle \). One possible means of introducing Weyl spinors into the \( X = (\rho \, \bar{\alpha} \atop a \, m) \in h_2 \mathbb{O} \) subcomponents of \( X \in h_3 \Omega \) is to express \( X \) in terms of a new 2-component object \( \theta_X = (\lambda \atop i) \in \mathbb{O}^2 \) as the product:

\[
X = \begin{pmatrix} p & \bar{a} \\ a & m \end{pmatrix} = \theta_X \theta_X^\dagger = \begin{pmatrix} \bar{r} \\ r \end{pmatrix} \begin{pmatrix} \bar{s} \\ s \end{pmatrix} = \begin{pmatrix} \bar{r}\bar{s} \\ rs \end{pmatrix} \]

(76)

As described in ([1] section 9.1, see equation 9.3) \( \theta_X \) then itself decomposes into a set of four Weyl spinors under the external \( \text{SL}(2, \mathbb{C}) \subset \text{SL}(2, \mathbb{O}) \subset E_{6(-26)} \subset E_{7(-25)} \) symmetry. This also shows how ‘further dimensions’ are needed to incorporate the new spinors, with the 10 real components of \( X \in h_2 \mathbb{O} \) here replaced by the 16 of \( \theta_X \in \mathbb{O}^2 \). Applying a similar augmentation to the corresponding subcomponents \( Y = (\rho \, \bar{A})_M, Z = (\rho' \, \bar{a'})_{m'} \in h_2 \mathbb{O} \) of \( Y, Z \in h_3 \mathbb{O} \) we have collectively the replaced components:

\[
\begin{align*}
p & \rightarrow (\bar{r}r) & P & \rightarrow (\bar{R}R) & p' & \rightarrow (\bar{r}'r') \\
m & \rightarrow (ss) & M & \rightarrow (SS) & m' & \rightarrow (s's') \\
a & \rightarrow (sr) & A & \rightarrow (SR) & a' & \rightarrow (s'r')
\end{align*}
\]

(77)

where the first column can be read off directly from equation 76.

On applying all of the substitutions of equation 77 in equation 75 it can easily be seen that while the second and fourth rows now contain only 4th-order terms, in the first and third row we have 5th-order terms, and hence we no longer have a homogeneous polynomial expression. However on setting \( X = Y = Z \) for equation 75 the 5th-order terms cancel out. These observations are similar to the case of substituting several spinors, \( \theta_X, \phi_X \ldots \in \mathbb{O}^2 \), via \( X = \theta_X \theta_X^\dagger + \phi_X \phi_X^\dagger + \ldots \) as an extension of equation 76 into...
the expression for $\det(\mathcal{X})$ in equation 13 or 72, which then also contains 4th-order and 5th-order terms, with the latter cancelling out for $X = \theta_X \theta_X^\dagger$ alone, as described for ([1] equations 9.5–9.8).

In principle the aim is then to incorporate structures similar to equation 76 into a generalisation from $q(x), q(y), q(z)$ and $(x, y, z)$ in equation 74, with $x, y, z \in F(h_3 \mathbb{C})$, in a manner such that the augmented terms collect together in $Q(t)$ as a homogeneous polynomial form, with any inhomogeneous terms automatically cancelling out in the full expression, consistent with the underlying basis of equation 43. This full form of time $Q(t)$ may closely involve the octonion triality symmetry in obtaining Weyl spinors under the external $\text{SL}(2, \mathbb{C}) \subset \text{E}_8$ while maintaining the appropriate charges under the internal subgroup $\text{SU}(3)_c \times \text{U}(1)_Q \subset \text{E}_6 \subset \text{E}_7$ of table 1 and figure 4 to describe $\nu$-lepton and $u$-quark states (as also suggested for [1] equations 9.9–9.12). This intrinsic employment of octonions in the symmetry structure again signals that Dynkin analysis can only serve as an approximate guide, as we consider further in the following subsection.

It may be that all components of $\mathbf{v}_{248} \equiv \mathbf{t} \in \mathcal{T}$ transform directly as spinors or scalars under the external $\text{SL}(2, \mathbb{C}) \subset \text{E}_8$ while maintaining the appropriate charges on the left-hand side and in principle with several spinors $\theta, \phi, \psi \in \mathbb{C}^2$ involved in forming $\mathbf{v}_4 \equiv \mathbf{h} = \theta \theta^\dagger + \phi \phi^\dagger + \psi \psi^\dagger \in h_2 \mathbb{C}$. With the $\text{E}_8$ symmetry breaking projection of $\mathbf{v}_4 \in TM_4$ closely related with the phenomena of the Standard Model Higgs mechanism this structure is analogous to composite Higgs models, as noted in the discussion towards the end of subsection 4.3.

As also noted in subsection 4.3 in the Standard Model itself the Higgs mechanism is introduced as an ad hoc phenomenological model to describe and parametrise, but not explain, the empirical observations of electroweak symmetry breaking. In the present theory the origin of the symmetry breaking is in the necessary identification of 4-dimensional spacetime itself, as described for equation 50 and figures 2 and 4, with the components $\mathbf{v}_4 \in TM_4$ projected out of the $\mathbf{v}_{248}$ components, as depicted for the simpler model in figure 2 as well as in figure 4, imply that some of these spinor components, for example the ‘right-handed neutrino’ slots, will necessarily be combined in a composite external 4-vector $\mathbf{v}_4$ (as discussed for [1] equation 9.52). This composition might be similar to that described in equation 76, except with $\mathbf{v}_4 \equiv \mathbf{h} \in h_2 \mathbb{C}$ on the left-hand side and in principle with several spinors $\theta, \phi, \psi \in \mathbb{C}^2$ involved in forming $\mathbf{v}_4 \equiv \mathbf{h} = \theta \theta^\dagger + \phi \phi^\dagger + \psi \psi^\dagger \in h_2 \mathbb{C}$. With the $\text{E}_8$ symmetry breaking projection of $\mathbf{v}_4 \in TM_4$ closely related with the phenomena of the Standard Model Higgs mechanism this structure is analogous to composite Higgs models, as noted in the discussion towards the end of subsection 4.3.

In summary, we have considered the progression from the vacuum case of $L(\mathbf{v}_4) = 1$ over $M_4$ in figure 1 through a series of higher-dimensional forms of time projected over the same base manifold resulting in matter fields resembling a series of properties of the Standard Model. These include the form $L(\mathbf{v}_0) = 1$ with a broken $\text{SL}(3, \mathbb{C})$ symmetry for which an ‘electromagnetic’ gauge field in interaction with a Weyl spinor was identified in figure 2, succeeded by the fractional charges and colour triplets of table 1 for $L(\mathbf{v}_{27}) = 1$ under the broken $\text{E}_6$ symmetry and the left-right
asymmetry in figure 4 for the components of $L(v_{56}) = 1$ under the broken E7 symmetry. In addition to structures resembling one generation of Standard Model leptons and quarks the symmetry breaking mechanism itself exhibits features analogous to the Higgs sector. Given the simplicity of the theory, in deriving from the one dimension of time only, the degree of explanatory power already uncovered up to the level of this E7 symmetry of time is noteworthy.

In turn while a full E8 action is in principle large enough to incorporate three generations and the full phenomena of electroweak symmetry breaking in the projection over $M_4$ it is non-trivial to see in detail how this might arise since we currently lack a specific homogeneous polynomial form $L(v_{248}) = Q(t) = 1$ as a natural extension from table 3. On the other hand the potential existence of such an object, as provisionally described for equation 74, together with the fact that it is not straightforward to see in detail how it might carry further explanatory power in terms of Standard Model structures, implies that the theory is testable in this non-trivial theoretical sense. In the following subsection we consider how the pursuit of this higher-dimensional form of time might further relate to some of the E8 studies reviewed in section 2.

5.2 Further Connections with E8 Studies

The highest-dimensional form of time that we have considered explicitly, $L(v_{56}) = q(x) = 1$ with $x \in F(h_3 \mathbb{O})$ and an E7(-25) symmetry as introduced in subsection 2.3, extends to an E8(-24) realisation on the 57-dimensional space of the extended Freundenthal triple system as a symmetry preserving the light cone separation $d(e, f) = 0$ with $e, f \in eF(h_3 \mathbb{O})$, as also described in subsection 2.3. While the structure of equation 38 arose from considering the spaces $h_3 \mathbb{O}$ and $F(h_3 \mathbb{O})$ as ‘generalised spacetimes’ [30, 31] here we are primarily interested in the ‘general form of time’ [1, 2, 3] as motivated here in section 3 and derived for equation 43. Here 4-dimensional ‘spacetime’ itself is identified through a particular ‘form of time’, as described for equation 47 and figure 1, with the necessary projection of the corresponding components $v_4 \equiv h \in h_2 \mathbb{C}$ out of the full form of time, pictured in figure 4 for $L(v_{56}) = 1$, breaking the full symmetry of time and leading to the empirical properties of the matter fields identified over $M_4$ as presented through section 4.

In the previous subsection we considered the possible construction of a homogeneous polynomial form $L(v_{248}) = 1$ over a 248-dimensional space with an E8 symmetry as a further progression from the forms of time listed in table 3, in principle involving three elements of $F(h_3 \mathbb{O})$ as suggested in equation 74, with the aim of incorporating the complete set of Standard Model states and symmetry structures, including a full three generations of leptons and quarks. We can also ask whether this currently hypothetical E8 action on $L(v_{248}) = Q(t) = 1$ might be related to or built upon the E8 realisation on the 57-dimensional space $eF(h_3 \mathbb{O})$ by in some sense opening up and rearranging the components within the expression $d(e, f) = 0$ in a manner compatible with a homogeneous form $L(v_n) = 1$, potentially with $n = 248$.

This is analogous to the fact that E7(-25) can itself be considered as a conformal group for the space $h_3 \mathbb{O}$ that leaves the cubic light cone $\det(X - Y) = 0$ invariant, as described for equation 29, while also being the symmetry group of the homogeneous quartic form $q(x)$ of equation 30. As noted towards the end of subsection 2.3 employ-
ment of the split octonions leads to a conformal realisation of the real form $E_{7(7)}$ as set of non-linear actions on the space $h_3O$, as described for ([30] equation 47). In addition to leaving a 27-dimensional cubic light cone invariant ([30] equation 45), the same group $E_{7(7)}$ acts via a linear 56-dimensional representation on a quartic invariant ([30] equation 18). An explicit identification of a relationship between these two kinds of symmetry actions as a realisation in $\mathbb{R}^{27}$ and a representation in $\mathbb{R}^{56}$ for the analogous case of $h_3O$ for the non-split octonions and the real form $E_{7(-25)}$ could provide a clue for how the 57-dimensional realisation of $E_{8(-24)}$ acting upon $d(e, f) = 0$ might be unfolded into a possible $E_{8(-24)}$ action leaving invariant a homogeneous polynomial form $Q(t) = 1$ as a 248-dimensional representation. Further, the observation that $L(v_{56}) = q(x) = 1$ with $E_{7(-25)}$ symmetry is embedded within the $E_{8(-24)}$ symmetry of $d(e, f) = 0$, as described for equations 66 and 67, may assist in the identification of this augmented form of time.

The quartic distance $d(e, f)$, defined in equation 37, itself incorporates terms involving $q(x-y)$ and the form $\{x, y\}$ of equation 32, with $x, y \in F(h_3O)$ collectively formed of 114 real components, and hence might already be associated with two generations of fermions in the context of figure 4. From equation 34 both the antisymmetric bilinear form $\{x, y\}$ and the symmetric quartic form $q(x, y, z, w)$ (the linearisation of the quartic norm $q(x)$ in equation 33) are closely related to the symmetric triple product $T(x, y, z)$, and hence they might also relate to the provisional ‘trilinear product’ denoted $(x, y, z)$ in equation 74, presumed to be associated with a full three generations of quarks and leptons.

Terms in the full form of time $L(v_{248}) = Q(t) = 1$, provisionally introduced in equation 74, might also then in principle contain factors such as $\{x, y\}$, as well as $q(x)$ and $\lambda$, each of which is invariant under an $E_{7(-25)}$ subgroup. Again, this is analogous to the $E_{6(-26)} \subset E_{7(-25)}$ invariance of the $\alpha, \beta, \det(X)$ and $\det(Y)$ factors in the expression for $L(v_{56}) = q(x) = 1$ of equation 73. In all cases these structures ultimately need to incorporate a modification of $q(x)$ and $\det(X)$ to open up the $X \in h_3O$ or $a \in O$ subcomponents, for example similarly as described in equation 76, to incorporate further $SL(2, \mathbb{C}) \subset E_{8(-24)}$ spinor states while being compatible with an overall homogeneous polynomial form for the full $L(v_{248}) = Q(t) = 1$ expression.

As noted in the discussion following equation 68 the need to identify the above spinor decompositions implies that any embedding of the full Standard Model structure is not expected to directly match the $E_8 \supset E_6 \times SU(3)$ branching pattern of that equation. It has also been seen from equations 75–77 that augmenting the cubic form $X, Y, Z)$ of equations 9 and 17 to incorporate more spinor components leads to quartic and quintic terms. On the other hand since the terms such as $\lambda q(x)$ in equation 74 are already of quintic order, incorporating spinors in such an expression is expected to lead to terms of at least sixth or seventh order. This raises the question of whether the full form $L(v_{248}) = Q(t) = 1$ might in fact converge upon an eighth order polynomial, that is an octic $E_8$ invariant [12, 13, 14, 15] as reviewed in subsection 2.1 and which might then itself provide a significant guide.

Indeed, as was described in subsection 2.1, in seeking a 248-dimensional form $L(v_{248}) = 1$ the lowest-order invariant homogeneous polynomial incorporating an irreducible representation of $E_8$ beyond the quadratic Killing form is such an eighth order expression. More generally the $E_8$ tensor invariants are defined in terms of the Casimir operators in the 248-dimensional $E_8$ algebra. A similar association is not possible be-
between elements of the 133-dimensional $E_7$ Lie algebra and the 56-dimensional $E_{7(-25)}$ quartic invariant $q(x)$, however a correspondence can still be made between $q(x)$ and an invariant fourth order tensor. The relation between such a quartic invariant tensor and the Freudenthal triple system is indicated for $E_{7(7)}$ acting on $F(h_3 \mathbb{O}_s)$, the case alluded to above, in ([30] section 2.3). For the case of $E_{7(-25)}$ the invariant tensor is explicitly constructed in [31] on employing an $SO(2,10) \subset E_{7(-25)}$ basis. (As noted after equations 24 and 29 the non-compact group $SO(2,10)$ is the conformal group for $h_2 \mathbb{O}$, while $E_{7(-25)}$ is the conformal group for $h_3 \mathbb{O}$). With elements $X^{\alpha a}$ and $\psi^a$ transforming as an $SO(2,10)$ vector and spinor respectively the quartic invariant $I_4$ includes for example the term (see [31] equation 82 for details of the notation):

$$I_4 = \ldots + 2\epsilon_{ab}X^{\mu a}X^{\nu b}\psi^a(CT_{\mu \nu})_{\alpha \beta}\psi^\beta + \ldots$$

(78)

The octic $E_8$ invariant $X_8$ is written out explicitly in ([12] equation 2.3) in an $SO(16) \subset E_8$ basis, where $SO(16)$ is a maximal subgroup of the compact real form of $E_8$. With generators $T^{ab}$ in the adjoint and $\phi^a$ in the spinor representation of $SO(16)$ there are many terms in this case including as an example:

$$X_8 = \ldots - \frac{3}{64}T^{ab}T^{cd}T^{ef}T^{gh}(\phi_\alpha \phi^\alpha)(\phi_\beta \Gamma_{abcdefgh})\phi^\gamma + \ldots$$

(79)

where again the details and notation are described in the reference. The above octic invariant $X_8$ is constructed for the compact real form $E_{8(-24)}$, and hence we require a ‘twisted form’ of this expression to identify an octic invariant for the real form $E_{8(-24)}$. This might be constructed via an $SO(4,12) \subset E_{8(-24)}$ basis, where the non-compact subgroup $SO(4,12)$ is the quasiconformal group associated with 10-dimensional spacetime $h_3 \mathbb{O}$ ([31] equation 7), while $SO(2,12)$, as employed for equation 78 above, is the conformal group of the same space. (While $E_{8(-24)}$ and $E_{7(-25)}$ are the quasiconformal and conformal groups associated with the ‘generalised spacetime’ $h_3 \mathbb{O}$). The Lie algebra itself for the compact real form $E_{8(-24)}$ is presented with respect to the maximal compact subgroup $SO(16)$ in ([12] equation 2.1), with a corresponding decomposition for $E_{8(-24)}$ in the subgroup $SO(4,12)$ basis described for example in ([54] equation 4.3).

While providing no detailed analysis here the suggestion is to consider whether the construction of an octic invariant $X_8$, similar to that of equation 79, for the real form $E_{8(-24)}$ can be seen as an augmentation from the quartic $E_{7(-25)}$ invariant $I_4$ of equation 78, in principle via a subgroup substructure $SO(2,10) \subset SO(4,12)$ within the embedding $E_{7(-25)} \subset E_{8(-24)}$. With the tensor invariant $I_4$ correlated with the 56-dimensional space $F(h_3 \mathbb{O})$ underlying the quartic invariant $q(x)$ under the $E_{7(-25)}$ symmetry in principle it may be possible to identify a parallel relation for the augmented invariant $X_8$ in terms of a 248-dimensional octic polynomial expression explicitly relating to the familiar elements of $\det(X)$, $\det(X)$ and $q(x)$ in the progression of equations 71–73 in leading to an $E_{8(-24)}$ invariant $L(v_{248}) = 1$ as provisionally proposed in equation 74. With $X \in h_2 \mathbb{O}$, $X \in h_3 \mathbb{O}$ and $x \in F(h_3 \mathbb{O})$ care is needed in this translation from the tensor invariants to polynomial forms in $\mathbb{R}^{56}$ and $\mathbb{R}^{248}$ owing to the algebraic properties of the octonions. In the latter case for the $E_{8(-24)}$ symmetry this would ideally involve the $X \in h_2 \mathbb{O}$ parts already expressed in terms of an $SL(2,\mathbb{C}) \subset E_{8(-24)}$ spinor decomposition such as equation 76, and in a manner in principle related to the property of octonion triality as noted in the previous subsection and discussed further below.
The initial aim would then be to identify an explicit structure for an octic $E_{8(-24)}$ invariant over $\mathbb{R}^{248}$ as a candidate for the homogeneous polynomial form $L(v_{248}) = Q(t) = 1$ as a further progression from the forms listed in table 3. Subsequently the embedding of the symmetry breaking structure $SL(2, \mathbb{C}) \times SU(3)_c \times U(1)_Q \subset E_{8(-26)} \subset E_{7(-25)} \subset E_{8(-24)}$ might be used as an initial basis upon which to seek further structures of the Standard Model, subsuming the findings of figure 4, by essentially reading off the symmetry properties of all the physical matter fields deriving from the breaking of $E_{8(-24)}$ over the base space $M_4$.

From the above discussion the full form $L(v_n) = Q(t) = 1$ that we seek is expected to lie somewhere between an unfolding of the $E_{8(-24)}$ realisation on the quartic light cone form $d(e, f) = 0$ of equations 37 and 38, with $e, f \in eF(h_3\mathcal{O})$ and possibly $n \neq 248$, and an octic $E_{8(-24)}$ invariant, with $n = 248$, based on a twisted form of the invariant $X_8$ described for equation 79. An intermediate homogeneous polynomial form, for example of quintic order, would also correspond to a realisation rather than a representation of $E_{8(-24)}$, since the invariants of the 248-dimensional adjoint representation are expected to be homogeneous polynomials of order 2, 8, 12, 14, 18, 20, 24 or 30 as described in subsection 2.1.

A group realisation can be defined as a map from elements of the group into an algebraic structure with isomorphic composition properties, which in general involves non-linear actions (as for several conformal group generator elements of $su(2, 2) \equiv so(2, 4)$ in equation 22) and in principle can act on a space of arbitrary dimension. The $E_{8(-24)}$ Lie algebra realisation on the 57-dimensional space of elements $e = (x, \tau) \in eF(h_3\mathcal{O})$ is expressed through actions such as those of equation 39, which are highly non-linear.

On the other hand a group representation is a map from the group elements into a set of matrices, expressing linear transformations on a space of a specific dimension. The smallest non-trivial irreducible representation of $E_8$ is the 248-dimensional adjoint representation, which can hence be expressed by a subset of the general linear matrices $GL(248, \mathbb{R})$ or $GL(248, \mathbb{C})$ acting on the 248-dimensional vector space of the $E_8$ Lie algebra, as a real or complex space, itself.

However, even for the hypothetical case of a non-linear realisation of $E_{8(-24)}$ on for example a quintic homogeneous form $L(v_n) = Q(t) = 1$, with $n \neq 248$, when projected over $M_4$ the broken symmetry subgroup could still be expressed by a reducible linear representation on the components of $v_n$. Indeed for the $E_{8(-24)}$ realisation on the 57-dimensional space of elements $e = (x, \tau) \in eF(h_3\mathcal{O})$ the $E_{7(-25)} \subset E_{8(-24)}$ subgroup acts via a linear representation on the $x \in F(h_3\mathcal{O})$ components ([1] equations 9.29–9.32) and the $\tau \in \mathbb{R}$ component (as a singlet), with this substructure employed in equation 67 for example. In principle not only $SL(2, \mathbb{C}) \times SU(3)_c \times U(1)_Q \subset E_{7(-25)}$ of figure 4 but a full subgroup $SL(2, \mathbb{C}) \times SU(3)_c \times SU(2)_L \times U(1)_Y \subset E_{8(-24)}$ could act via a linear representation on the multiplets of matter fields over $M_4$, with interaction terms and a charge structure read off for direct comparison with the Standard Model, and with non-linear $E_{8(-24)}$ actions not surviving the symmetry breaking. The rearrangement and augmentation of the light cone $d(e, f) = 0$ into the form of a norm $L(v_n) = Q(t) = 1$ might itself not exhibit the full $E_{8(-24)}$ symmetry of the former structure while still leaving room to identify the full Standard Model in the further symmetry breaking in the necessary identification of the external spacetime $M_4$.

For the case of employing the smallest non-trivial $E_8$ representation the matter
fields will be described by the broken symmetry acting on a 248-dimensional space. Since this smallest non-trivial representation is the adjoint representation of $E_8$, the external Lorentz and internal gauge symmetry properties of the Standard Model identified in this way might be expected to correlate with the structure of the 248-dimensional $E_8$ Lie algebra itself. The possibility of this latter construction is directly and explicitly examined in [6], as reviewed in subsection 2.1, in which all states of the Standard Model together with gravitational field parameters are associated with elements of the root system of the $E_8$ Lie algebra. While a resemblance is identified in [6] between these structures of the $E_8$ Lie algebra and the structure of the Standard Model of particle physics, as noted in subsection 2.1 the three generations of ‘fermions’ are only obtained with respect to mutual $SO(8)$ triality maps and this intrinsic feature cannot be avoided while remaining strictly within this framework [7]. In [6] the required triality maps relate factors of the external Lorentz $SL(2,\mathbb{C}) \equiv SU(2) \times SU(2)$ (for the ‘complexified’ case as noted for equation 58) and an internal $SU(2)_L \times SU(2)_R$ symmetry via the subgroup:

$$SU(2) \times SU(2) \times SU(2)_L \times SU(2)_R \subset SO(8) \subset E_8$$

in the ‘graviweak’ sector of the theory, with one of the $SU(2)_R$ generators associated with hypercharge $Y$. Here the first $SU(2) \times SU(2)$ factor is analogous to the corresponding external symmetry factor in equation 58 or 61 while the internal $SU(2)_L \times SU(2)_R$ is analogous to that in equation 65; combining gravitational and electroweak structures in equation 80.

By contrast, in the conceptual picture of the present work particle states corresponding to Standard Model fermions and the Higgs are associated with components of the representation space $v_{248} \equiv t \in T$ of the form $L(v_{248}) = Q(t) = 1$, assuming the full ($n = 248$)-dimensional case, upon which the subgroup symmetry of equation 69 acts. Hence with the Lorentz symmetry and internal gauge symmetry generated by elements of the real form $E_8(-24)$ only the gauge bosons of the Standard Model, rather than all states, are identified explicitly within the $E_8$ Lie algebra itself. Further, as a development from the $E_6(-26)$ action on $h_3\Omega$ of subsection 4.2 and the $E_7(-25)$ action on $F(h_3\Omega)$ of subsection 4.3, the octonion algebra $\Omega$ is expected to play a central role in the construction of the $E_8(-24)$ symmetry of $L(v_{248}) = Q(t) = 1$.

With the non-associative composition of the octonions able to describe a high degree of symmetry, as noted in subsections 2.2 and 4.2, and employed in the explicit $E_8(-24)$ action on the components of $v_{248}$, standard arguments for associative symmetry groups via the root system of their complex Lie algebras and Dynkin analysis may not directly apply here. The possibility of such a discrepancy was demonstrated in subsection 4.2 in which only Weyl spinors of a single handedness were identified in table 1 for the explicit breaking of the $E_6(-26)$ symmetry on the components of $v_{27} \equiv \mathcal{X} \in h_3\Omega$, while both left and right-handed states are seen for the closely related Dynkin analysis in table 2.

In particular it was noted in the discussion following equation 77 in the previous subsection and after equation 79 above that the properties of octonion triality, associated with an $SO(8) \subset E_6(-26)$ subgroup as described in ([1] section 9.1), may be intimately involved in the identification of $SL(2,\mathbb{C})$ Weyl spinors in a decomposition of the $X,Y \in h_2\Omega$ components to fully account for the $\nu$-lepton and $u$-quark states, building on the structures identified in figure 4. The same, or a
closely related, triality might also be intimately involved in the identification of a full three generations of Standard Model fermions in the components of \( L(v_{248}) = 1 \) under a broken \( SL(2, \mathbb{C}) \times SU(3)_c \times SU(2)_L \times U(1)_Y \subset E_{8(-24)} \) symmetry, effectively untangling the corresponding three ‘triality-related’ generations identified in the \( E_8 \) Lie algebra root lattice in [6] relating to the SO(8) subgroup of equation 80. (The relation between SO(8) triality and the octonion algebra is explored for example in [55] while the SO(8) triality structure employed in ([39] equations 87–90), with SO(8) \( \subset \) SO(2, 12) \( \subset \) E7(−25) \( \subset \) E8(−24) might be significant in relation to the decompositions described following equations 78 and 79 here).

In the present theory the above structures closely relate to the need to identify an electroweak symmetry \( SU(2)_L \times U(1)_Y \subset E_{8(-24)} \), acting on the components of \( L(v_{248}) = 1 \), as would be required to consider triality relations between the SU(2) factors in equation 80. Since both in [6, 54] for \( E_8 \) and here in figure 4 at the \( E_7(-25) \) level one generation of the \( e \)-lepton and \( d \)-quark states is identified, this common feature might provide a handle to help identify the first generation \( \nu \)-lepton and \( u \)-quark spinor states from the \( X, Y \in h_2 \Omega \) components here, potentially correlating with those already identified in [6, 54]. These states can be sought through their connection in \((^{(e)}_L) \) and \((^{(d)}_L)\) doublets under an internal \( SU(2)_L \) symmetry, and may act as a stage on the way to identifying a full three generations of leptons and quarks.

For the full subgroup symmetry of equation 69 the manner in which the action of the electroweak factor \( SU(2)_L \times U(1)_Y \subset E_{8(-24)} \) explicitly augments the \( U(1)_Q \subset E_6(-26) \subset E_7(-25) \) component of table 1 and figure 4 will closely relate to the ‘skew’ in the largest ‘Russian doll’ of \( E_{8(-24)} \) acting upon \( L(v_{248}) = 1 \) with respect to the subsumed structures of table 3, as proposed in the previous subsection. The structures of table 3 would seem to lead more directly to an \( E_8 \) composition incorporating a set of three ‘Jordan pairs’, each of which might here be denoted \((h_3 \Omega, \overline{h}_3 \Omega)\), as described for ([10] figure 1 and the top line of equation 1.1). This structure reflects the \( E_8 \supset E_6 \times SU(3) \) subgroup branching of equation 68 ([10] equation 2.26) and could be interpreted here as describing three generations of the ‘quarks’ and ‘leptons’ listed in figure 4. Similarly as the spaces \( h_3 \Omega \) and \( \overline{h}_3 \Omega \) need to be opened up to correctly describe the full spinor structure, so the breaking of \( E_8 \) is not expected to be directly aligned with this \( E_6 \times SU(3) \) branching.

Further, while the \( E_6 \times SU(3) \subset E_8 \) subgroup branching of equation 68 is apparent in ([6] figure 4) there the Standard Model embedding is more directly aligned within an \( F_4 \times G_2 \subset E_8 \) subgroup (as more apparent in [6] figure 3). That is, in [6] the external gravitational and internal electroweak sector derive from a graviweak SO(8) \( \subset F_4 \), incorporating the subgroup in equation 80, with the internal colour symmetry accommodated via \( SU(3)_c \subset G_2 \). While the latter symmetry can be correlated with the \( SU(3)_c \subset G_2 \subset E_6(-26) \) of table 1 it has been concluded here that a departure from that full \( E_{6(-26)} \) representation structure in an \( E_6(-26) \subset E_7(-25) \subset E_{8(-24)} \) embedding seems to be needed in order to identify the \( \nu \)-lepton and \( u \)-quark spinor states in a decomposition of \( X, Y \in h_2 \Omega \). Hence on the way to equation 69 an intermediate decomposition:

\[
SO(8) \subset F_4 \times SU(3)_c \subset G_2 \subset F_4 \times G_2 \subset E_8
\]

might be sought, with the first factor SO(8) further containing the ‘graviweak’ subgroup of equation 80. Out of this full symmetry the electromagnetic gauge sym-
metry may be obtained as $U(1)_Q \subset SU(2)_L \times U(1)_Y \subset SO(8) \subset F_4 \subset E_8$, incorporating electroweak symmetry breaking, rather than directly through the chain $U(1)_Q \subset SU(4) \subset E_6 \subset E_7$ via equations 61 and 62 as for the intermediate forms of time of table 1 and figure 4. Since $F_4 \times G_2 \subset E_8$ is a maximal subgroup the branching pattern of equation 81 cannot be directly aligned with an intermediate $E_6 \subset E_8$ or $E_7 \subset E_8$ subgroup.

In conclusion, the significant new feature introduced here is that particle states are to be identified from a combination of both the components of the full form of time $L(\mathbf{v}_{248}) = 1$, a homogeneous polynomial form potentially of octic order, together with broken subgroups of an $E_8$ symmetry, constructed through octonion composition, rather than directly in terms of the 248 elements of the $E_8$ root system or from any analysis of the complex $E_8$ Lie algebra itself. In this manner the aim is to identify the full structure of the Standard Model, ironing out the discrepant features found in [6, 54] while avoiding the prohibitive theorems of [7] through this different perspective on the role of $E_8$.

On the mathematical side a possible connection with the ‘magic square’ of Lie algebras, briefly reviewed in subsection 2.1, may shed further light on the role of the octonions and the uniqueness of $E_8$ as employed in this structure. The chain of real forms of exceptional Lie algebras $F_4(-52) \rightarrow E_6(-26) \rightarrow E_7(-25) \rightarrow E_8(-24)$ appears in the final column of the $4 \times 4$ magic square $M(\mathbb{K}, \mathbb{K}')$ involving a split division algebra $\mathbb{K}$ and $3 \times 3$ Hermitian matrices over a non-split division algebra $\mathbb{K}'$ as described in ([17], [33] table 1). In [33] the $E_7(-25)$ Lie algebra entry in this ‘half-split’ magic square is linked to the $E_7(-25)$ group action on the space $F(\mathbb{h}_3 \mathbb{O})$, as the conformal group associated with the $E_6(-26)$ action on $\mathbb{h}_3 \mathbb{O}$ (see also the comment following [31] equation 6).

Similarly the conformal group $SU(2, 2) \equiv SO(2, 4)$ (the former being the double cover of the latter) associated with the Lorentz group in 4-dimensional spacetime $\mathbb{H}_2 \mathbb{C}$, as introduced here in subsection 2.3, is linked with the entry in the $4 \times 4$ ‘half-split’ magic square involving $2 \times 2$ matrices over $\mathbb{H}_s \otimes \mathbb{C}$, where $\mathbb{H}_s$ denotes the split quaternions, as described in ([56] table 1). This table (see also [33] table 2) also contains the groups $SO(2, 10)$ and $SO(4, 12)$, as the $\mathbb{H}_s \otimes \mathbb{O}$ and $\mathbb{O}_s \otimes \mathbb{O}$ entries respectively, which were discussed above for the substructure decomposition of the invariant tensors of equations 78 and 79 respectively. In the literature there are hints that a generalisation of such magic square constructions for $2 \times 2$ matrices might ultimately lead to a geometric interpretation for the exceptional Lie group $E_8$ associated with the final entry of the magic square for $3 \times 3$ matrices (see for example [56] section 7, [57] section 2.3 and chapter 4).

A relation between real forms of Lie algebras obtained from the magic square and the corresponding real forms identified as conformal or quasiconformal symmetries of generalised spacetimes was also noted towards the end of subsection 2.3. As one possible connection with the identification of physical structures, including Standard Model states, discussed in this subsection we note that in ([58] equation 7) the subgroup decomposition $F_4 \times G_2 \subset E_8$ of equation 81 above is described for $E_8$ as the largest group of the magic square. These structures associated with the magic square might then also relate to the potential uniqueness of $E_8(-24)$ for application as the full symmetry of time for the present theory.

With the emphasis on the conceptual picture described in section 3, as devel-
oped through the explicit forms of time of section 4 as summarised in table 3 and leading to the prediction of a central role for the exceptional Lie group \( E_{8(-24)} \) in the previous subsection, while drawing together the various mathematical threads discussed in this subsection, the ambition is to converge upon the full form of time 
\[ L(v_{248}) = Q(t) = 1 \]

invariant under an \( E_{8(-24)} \) symmetry and assess the broader properties of the full physical theory. The underlying basis of the theory, constructed from a single dimension of time, is arguably the simplest possible starting point for any unification scheme. The construction of higher-dimensional manifestations of the general form of time \( L(v_n) = 1 \) has been guided by this underlying simplicity, which ultimately translates into a high degree of symmetry for the proposed full form \( L(v_{248}) = Q(t) = 1 \), exhibiting the structure of \( E_{8(-24)} \) as a real form of the largest exceptional Lie group.

Through this exploration of the structure implicit within the one-dimensional flow of time the physical properties of matter in spacetime might be fully and directly deduced as described in this paper. As noted at the end of the previous subsection the identification of the full form of time \( L(v_{248}) = Q(t) = 1 \) as a natural mathematical structure accounting for these physical properties then presents a non-trivial puzzle. Further possible means towards finding a solution have been suggested in this subsection. In the following section we consider these developments in the context of the broader ambitions of the full theory, and in relation to further aspects of the existing literature reviewed in section 2, aiming towards the goal of collectively tying all of these threads together into a complete theory.

6 Comments on the Overall Physical Theory

In this section we describe how the main topic of this paper, concerning \( E_8 \) as a candidate for the full symmetry of time, relates to the other areas in which the theory has developed as described in [1], with the four main branches of the theory summarised in ([1] figure 15.1). In particular this regards the connections with Kaluza-Klein theory, quantum field theory (QFT) and cosmology, as well as with the Standard Model of particle physics.

The symmetry breaking structure for a higher-dimensional form of time over the projected external 4-dimensional spacetime, as introduced here for the \( SL(3, \mathbb{C}) \) case in subsection 4.1 through equation 50 and figure 2, leads directly to the framework of a principle fibre bundle \( P \equiv M_4 \times G \) as employed by a range of Kaluza-Klein theories (including [46, 47]). This connection is examined in detail in ([1] chapters 2–5, [3]), with the geometric structure of figure 2(b) in this paper analysed more thoroughly for ([3] figure 3(b)). This leads to a relation between the external curvature, described by the components of the Einstein tensor \( G_{\mu\nu} \), and the internal curvature, with components \( F^{\alpha\mu\nu} \), of the form ([3] equation 93):

\[
G_{\mu\nu} = 2\chi(-F^{\alpha\rho}_{\mu}F_{\alpha}^{\rho\nu} - \frac{1}{4}g^{\mu\nu}F_{\rho\sigma}^{\alpha}F_{\alpha}^{\rho\sigma}) : = \kappa T_{\mu\nu}
\]

which also defines the energy-momentum tensor \( T_{\mu\nu} \) for this spacetime geometry solution. Here \( \mu, \nu, \rho, \sigma \) are spacetime indices, \( \alpha \) is a Lie algebra index and the real
parameters $\chi$ and $\kappa$ can both be considered as normalisation constants for the empirical units ultimately employed for gravitation, gauge fields and energy-momentum.

As described in ([3] subsections 2.3 and 5.3) the breaking of the full symmetry group, denoted \( \hat{G} \), of the full form of time \( L(\hat{v}) = 1 \) (where \( \hat{v} = v_{n} \) for the largest \( n \) considered) over the base manifold \( M_{4} \) is absolute. That is the surviving symmetry for the physical theory can be written as ([3] equation 23):

\[
\text{Lorentz} \times G \subset \hat{G}
\]

where the Lorentz group for 4-dimensional spacetime may be replaced by its double cover \( \text{SL}(2, \mathbb{C}) \) and \( G \) is the residual internal symmetry group. For example in equation 51 we have \( \hat{G} = \text{SL}(3, \mathbb{C}) \) and \( G = \text{U}(1) \). With the external Lorentz and internal \( G \) symmetries both originating from the same unification group \( \hat{G} \) such an absolute symmetry breaking is required in order to avoid a trivial \( S \)-matrix for the quantised theory as an implication of the Coleman-Mandula theorem [48], as noted following equation 51 in subsection 4.1 and discussed in ([3] subsection 5.3).

An avoidance of the prohibitive restrictions of the Coleman-Mandula theorem without invoking supersymmetry and in a manner relating to the identification of the external spacetime structure itself can also be found in other theories. For example the gravivweak models described in [59] place gravitational and weak interactions on an equal footing, with the full symmetry of theory manifest in a ‘topological phase’ without a metric in spacetime. Only in the ‘broken phase’ can a residual symmetry of a global Lorentz and a local internal symmetry be identified, compatible with the Coleman-Mandula theorem ([59] section 7).

For the present theory the identification of spacetime \( M_{4} \) itself out of the symmetries of a 4-dimensional substructure of the form \( L(\hat{v}) = 1 \) breaks the full symmetry \( \hat{G} \), again compatible with the restrictions of the Coleman-Mandula theorem in the QFT limit. Here we have considered equation 83 explicitly for the cases of \( \hat{G} = \text{SL}(3, \mathbb{C}) \), \( \hat{G} = \text{E}_{6}(-26) \) and \( \hat{G} = \text{E}_{7}(-25) \) in the three subsections of section 4 and have been led to propose \( \hat{G} = \text{E}_{8}(-24) \) as the full symmetry for the full form of time \( L(v_{24}) = Q(t) = 1 \) in section 5. In all cases this theory is manifestly background-free in that we begin simply with the one dimension of time only, expressed through the general mathematical form of equation 43 with symmetry \( \hat{G} \), and only then break this symmetry through the necessary identification of an external 4-dimensional spacetime, building upon the basic vacuum structure of equation 47 and figure 1, through which the physical theory is constructed.

For the minimal physical model based on \( \hat{G} = \text{SL}(3, \mathbb{C}) \) acting on \( L(v_{9}) = 1 \) in figure 2 both the spinor field \( \psi(x) \) and the gauge field \( A(x) \), associated with the internal \( \text{U}(1) \) symmetry, and their mutual interaction arise naturally through the geometric structures described in subsection 4.1. In principle the postulates of QFT could simply be applied to quantise the gauge and spinor matter fields arising over \( M_{4} \), as deduced not only for equation 51 but also for example in table 1 and figure 4 for the higher-dimensional full forms of time. On the other hand the ambition of the present theory has been to avoid the need for ‘postulates’ and instead at every stage to make the case for deriving the structure of the complete unified theory upwards from the original basic conceptual picture, as has been the case for equations 82 and 83 in leading to connections with Kaluza-Klein theory and with the Standard Model. This leads to
a proposal that all quantum phenomena arise through a degeneracy of solutions for identifying the external spacetime $M_4$ out of the components of the multi-dimensional form of time and its symmetries. That is, the external geometry of equation 82 can be generalised for these multiple possible solutions as expressed by:

$$G^{\mu\nu} = f(A, \hat{v})$$

(84)

where $f(A, \hat{v})$ denotes a rank-2 tensor function of the gauge fields $A(x)$, both Abelian and non-Abelian, and components of the multi-dimensional flow of time $\hat{v}$, including the $\psi(x)$ spinor components (see also the discussion of [1] equation 5.32). Again we emphasise that, rather than beginning with 4-dimensional spacetime containing a set of fields and then applying restrictions associated with a one-dimensional constraining structure, we begin purely with the one-dimensional flow of time through which physical relations such as equation 84 are derived which inherently incorporate constraints owing to the simplicity of this underlying foundation.

The manner in which the many possible solutions for equation 84 can in principle be related to calculations in QFT is described in ([1] chapters 10 and 11), by comparison with the mathematical methods of canonical quantisation. The permitted field exchanges implicitly integrated into equation 84 within the constraints of the theory are summarised and discussed for ([1] equation 11.29), taking the place of Lagrangian terms in a standard QFT. A provisional link between the determination of an event probability based on the degeneracy of equation 84 and the calculation of a process cross-section is made via the optical theorem of QFT as described for ([1] equation 11.46).

As noted in ([3] subsection 5.3) it is proposed that in the quantum field theory limit the spin-statistics theorem will also apply here, accounting for the bosonic nature of the gauge fields and the fermionic nature of the spinor fields, and hence also of the lepton and quark particle states subsequently identified, for example in figure 4. Indeed the phenomena of ‘particle’ entities themselves, as observed in the laboratory to exhibit bosonic or fermionic properties, are also to be accounted for through this ‘quantisation’ of the fields ([1] sections 10.1, 11.3 and 15.2). Hence here supersymmetry is neither required to sidestep the Coleman-Mandula theorem, as noted above, nor to combine bosons and fermions in a consistent unified field theory, and there is no corresponding extension to the Standard Model expected.

More generally, rather than making a direct comparison with any of a range of existing quantisation schemes (such as the canonical or path integral approach) the degeneracy of spacetime solutions can be considered as the source of quantisation in its own right, converging upon the structural form of a standard QFT, and in turn the various theorems listed above, only in the flat spacetime limit, and upon quantum mechanics in the non-relativistic limit.

However with curved spacetime solutions $G^{\mu\nu}(x) \neq 0$ generally implied for equation 84 gravity is fully embraced in this picture right down to the level of high energy physics (HEP) particle interactions. Such laboratory phenomena can be considered as microscopic solutions for general relativity, on a scale at which the intrinsic indeterminacy in the composition of the matter fields underlying a local spacetime geometry in equation 84 becomes evident in the quantum uncertainty exhibited in such experiments (see also the discussion of [1] figure 15.2). In this picture the gravitational
field itself is not quantised, and on the contrary it is gravity in the form of the external spacetime geometry that provides the means through which all other fields are quantised, through the degeneracy on the right-hand side of equation 84. Hence, in principle, the need to apply the postulates of quantum theory can be avoided, while the aim remains of reproducing the successes of QFT calculations in describing HEP phenomena, and of accounting for quantum effects more generally, in a suitable limiting approximation.

Deviations from a flat spacetime background are well beyond the reach of direct observation on the scale of HEP experiments, and gravity is generally completely neglected as an assumption underlying the corresponding QFT calculations. Further, as a pragmatic tool for relating observables the machinery of such QFT calculations says very little about what is actually physically happening at the interaction point in a HEP experiment. On placing the postulates of quantum theory at the heart of the calculation the evolution of the physical system is generally described in terms of a superposition of states in an abstract Hilbert space. In recording an observation at some stage the ‘measurement problem’ is inevitably encountered, involving the grey area through which this superposition apparently ‘collapses’ into a single state registered by the macroscopic apparatus. The thought experiment that places Schrödinger’s hypothetical cat in superposition of an $|\text{alive}\rangle$ and a $|\text{dead}\rangle$ state emphasises the conceptual difficulty in where to draw the line between the ‘quantum’ and ‘classical’ worlds.

Here on the other hand we can begin with the world of classical probabilities on the large scale, based on the relative frequencies of possible outcomes for events taking place in spacetime, such as the result of tossing a coin. As we then zoom in to a microscopic physical state we retain this classical notion of probability, in the sense of being determined by the relative frequency of possible outcomes which, at the level of equation 84, ultimately correspond to the degeneracy of solutions for the local spacetime geometry itself. This would apply for example to the decay of an atomic nucleus that might lie at the centre of a ‘Schrödinger’s cat’ type of experiment. Hence from this perspective there is essentially no discontinuity in the method of probability calculation in terms of the relative frequencies of possible outcomes.

With the gravitational field itself not quantised the spacetime geometry is considered to be completely smooth, with the Einstein field equation of general relativity fully preserved here by simply defining the energy-momentum tensor identically with equation 84, that is:

$$G^{\mu\nu} = f(A, \dot{\vartheta}) =: -\kappa T^{\mu\nu}$$

which also generalises equation 82. While $G^{\mu\nu}(x)$ describes the external spacetime geometry, $T^{\mu\nu}(x)$ can be interpreted as describing the underlying matter field composition, including its quantum properties. The spacetime smoothness is valid down to arbitrarily small scales consistent with the initial conceptual picture motivated in the opening of section 3 in the discussion of equation 40 regarding vanishingly small inertial frames, as also noted in the discussion following equation 51. In combining general relativity and quantum theory in this unified framework essentially it the latter that is subsumed within a generalisation of the former.

While writing three volumes on the foundations, applications and successes of quantum field theory, in the context of the Standard Model of particle physics,
Weinberg notes that ([60] volume I, chapter 12 opening):

It is generally believed today that the realistic theories that we use to describe physics at accessible energies are what are known as 'effective field theories'. . . . these are low-energy approximations to a more fundamental theory that may not be a field theory at all.

It is such a fundamental theory that we are aiming for here with the one-dimensional progression in time alone acting as the progenitor both of matter fields and spacetime itself. The physical properties of these structures derive from the symmetries and components of the multi-dimensional forms of time studied in section 4. These developments now culminate in the proposal that the calculational structures of QFT also arise in the limiting approximation of a flat 4-dimensional spacetime base manifold $M_4$, which is built through equation 84 out of the symmetry breaking structures of the full $E_{8(-24)}$ action on the full form of time $L(v_{248}) = Q(t) = 1$, as proposed in section 5, through a degeneracy of multiple possible solutions.

This building of spacetime itself in a manner intimately related to the structure of $E_8$ may share some of the ambition described in ([10] section 3), although the original motivation and the general conceptual picture is very different. In ([10] section 3) a discrete spacetime structure is proposed to emerge through elementary interactions determined by the $E_8$ Lie algebra structure, while here we are considering a continuous spacetime with a smooth geometry, on the left-hand side of equation 84, arising through the basic arithmetic composition and symmetries of the continuous flow of a single dimension of time directly expressed in the proposed multi-dimensional form $L(v_{248}) = Q(t) = 1$ with a full $E_{8(-24)}$ symmetry. In the context of the quantisation of the theory we can further assess here the degree to which we may be uniquely led to $E_8$, and in particular the non-compact real form $E_{8(-24)}$, as the full symmetry of time.

The chain of vector spaces $h_2^C \rightarrow h_3^C \rightarrow h_3^O \rightarrow F(h_3^O)$ for the multi-dimensional forms of time and their corresponding symmetries was considered in section 4, as summarised in table 3, and led to the predicted full symmetry $E_{8(-24)}$ acting on the yet higher-dimensional space $T$ for the next stage. However, the alternative augmentation of spaces $h_2^C \rightarrow h_3^C \rightarrow h_4^C \rightarrow \ldots h_m^C$ with $m \times m$ Hermitian complex matrices of real dimension $m^2$ up to an arbitrary positive integer $m$ might also have been considered in place of the path followed after equation 44. The corresponding full symmetry $G = SL(m, \mathbb{C})$ for the $m^2$-dimensional homogeneous polynomial form of time:

$$L(v_{m^2}) = \det(v_{m^2}) = 1 \text{ with } v_{m^2} \in h_m^C$$

would imply a breaking pattern over the base manifold $M_4$ corresponding to equation 83 of:

$$SL(2, \mathbb{C}) \times SL(m - 2, \mathbb{C}) \times U(1) \subset SL(m, \mathbb{C})$$

for $m \geq 4$, with the internal symmetry $G = SL(m - 2, \mathbb{C}) \times U(1)$ including a non-compact group factor. The application of matrix structures to describe symmetries of time was initially motivated following equation 18, although it was noted in the discussion following equation 74 that higher-dimensional forms of time do not need to take the form of a matrix determinant. Nevertheless the structure of equation 86 for $m = 2, 3, 4 \ldots$ might be considered a simple and natural progression towards higher-dimensional forms of time.
In quantum field theory the Lagrangian for a model with an internal gauge field generally includes a kinetic energy term of the form:

\[ \mathcal{L} = \frac{1}{4} K_{\alpha \beta} F_{\mu \nu}^\alpha F^{\alpha \mu \nu} \]  

(88)

where \( K_{\alpha \beta} \) is the Killing metric for the Lie algebra of the gauge group and \( F_{\mu \nu}^\alpha \) is the gauge curvature introduced in equation 82. The requirement of positive kinetic energy is ensured for a negative definite Killing form which in turn implies that the gauge group is required to be a product of compact simple groups and U(1) factors (see also for example [60] volume II, section 15.2). Hence a non-compact internal symmetry group cannot feature in a consistent quantum field theory.

For the present theory, in place of any Lagrangian terms such as equation 88 constraints such as equation 82 are employed, as discussed following equation 84, and with the energy-momentum tensor defined through the Einstein field equation \(-\kappa T_{\mu \nu} := G_{\mu \nu}\) in the general case as described for equation 85. The aim of converging upon a consistent quantum field theory in the flat spacetime limiting approximation, and with terms similar to that in equation 88 appearing in equation 82, suggests that here also the internal symmetry \( G \) in equation 83 will be required to be a compact group for the overall theory to be consistent, hence disfavouring the higher-dimensional forms described for equations 86 and 87.

This requirement relates to both the manner in which equation 82 itself is derived, as proposed via a perturbed action integral over the base manifold \( M_4 \) in ([3] equation 91), and the specific structure of the breaking of the full symmetry \( \hat{G} \) of the full form of time \( L(\dot{v}) = 1 \), to equation 83, when projected over the same base space \( M_4 \). In the context of the present theory alone compactness of the internal gauge groups may be a requirement for obtaining a consistent solution for the spacetime geometry through equation 84, a result which would then translate into a similar constraint in the QFT limit expressed in terms of energy-momentum via equation 85, in principle equivalent to the condition of positive kinetic energy.

Here we are necessarily employing a non-compact unification group \( \hat{G} \) in equation 83 in order to incorporate the non-compact Lorentz group \( \text{SO}^+(1,3) \) and hence some care is needed to see how the compactness of the internal group component \( G \) might be guaranteed. One observation is that for the chain of spaces for the general form of time \( h_2 \mathbb{O} \rightarrow h_3 \mathbb{O} \rightarrow F(h_3 \mathbb{O}) \rightarrow T \), summarised for equations 71–74, the corresponding full symmetry groups:

\[ \hat{G} = \text{SO}^+(1,9) \rightarrow \text{E}_6(-26) \rightarrow \text{E}_7(-25) \rightarrow \text{E}_8(-24) \]  

(89)

are the ‘most compact’ of the non-compact real forms of the several options available for each complex Lie group in the chain \( \text{SO}(10) \rightarrow \text{E}_6 \rightarrow \text{E}_7 \rightarrow \text{E}_8 \). That is, each of the real groups in equation 89 has the most negative signature of Killing form possible other than the compact real form of the group itself. This observation, together with the fact that the internal symmetry groups identified so far for \( \text{E}_6(-26) \) and \( \text{E}_7(-25) \) are compact (namely \( G = \text{SU}(3)_c \times \text{U}(1)_Q \) in table 1 and figure 4), might in part explain the uniqueness of this series of augmentations to the general form of time, with the symmetries of equation 89, for constructing a complete and consistent physical theory.

As noted in subsections 2.1 and 5.2 the \( \text{E}_8 \) unification framework as strictly proposed in [6] has been shown to be ultimately untenable owing to the lack of a
sufficiently non-compact real form $E_8$ capable of accommodating three generations of Standard Model spinor states [7]. Here in a sense we have the opposite issue, and it is a question whether the real form $E_{8(-24)}$ is sufficiently compact to ensure a compact internal symmetry group $G$ in equation 83. It is proposed then that the absolute symmetry breaking described for that equation can be compatible not only with the Coleman-Mandula theorem, as discussed above, but also with the requirement of compact internal symmetry groups in the QFT limit of the theory. The exact nature of the $E_{8(-24)}$ symmetry breaking will itself depend on the structure of the currently hypothetical full form of time $L(v_{248}) = Q(t) = 1$ as the subcomponents $v_4 \in TM_4$ are projected out over the base manifold $M_4$ for this higher-dimensional generalisation from figures 2 and 4.

More generally, consistency with QFT properties should also include for example the optical theorem and spin-statistics theorem as also alluded to earlier in this section. On the other hand, since we are not applying postulates of quantum theory from the outset it also remains a possibility that non-compact factors of an internal symmetry $G$ may simply be suppressed or filtered out as not giving possible solutions for the external geometry under equation 84, leaving only the compact factors featuring in the physics. This also raises broader questions concerning the interrelations between the various branches of the theory as discussed more generally in ([1] section 15.1).

The requirement of a compact internal symmetry subgroup $G \subset E_{8(-24)}$ for compatibility with the QFT limit can be contrasted with the heterotic branch of string theory, briefly reviewed in subsection 2.1, that is guided inevitably towards the compact group $E_8 \times E_8$ as an internal symmetry in order to obtain a consistent QFT free from anomalies. A fundamental difference in the role of the $E_8$ symmetry is that for the present theory the full symmetry $\hat{G} = E_{8(-24)}$ in equation 83 contains both the external Lorentz symmetry and an internal symmetry $G$, which is large enough to encapsulate the Standard Model gauge groups with little to spare, while for the case of string theory the compact $E_8 \times E_8$ purely internal gauge symmetry can very comfortably incorporate the Standard Model as will be discussed further below.

For many theories a set of quantisation postulates are assumed and applied from the outset. In string theory the dynamical degrees of freedom of strings and membranes in a higher-dimensional spacetime are quantised, with the gravitational field itself incorporated into this scheme of a proposed consistent theory of ‘quantum gravity’. A very different approach to ‘quantisation’ is taken in the present theory, as described in this section with reference to equation 85, with many features of ‘classical general relativity’ fully preserved. It might however also be considered whether the ‘anomaly-free’ properties of $E_8$ might have application for the present theory in the QFT limit, or rather for a subgroup of $E_8$ since only a broken fragment survives as the internal gauge symmetry. This is part of the broader question concerning the emergence of a consistent QFT from the theory presented here, as considered more generally in this section.

Possible connections between the present framework and string theory might be established through shared mathematical structures. For example the exceptional Jordan algebra, with the $E_6$ cubic invariant of equation 9 as reviewed here in subsection 2.2 and central to the present theory, has also played a significant role in string theory (see for example [61]), with further potential links via the mathematical structures described in subsection 2.3 to supergravity as noted for [30, 31, 38, 39] at the
Another common feature is the notion of ‘extra-dimensional’ structures. The significant conceptual difference is that while string theory is formulated from the outset in for example a 10 or 11-dimensional spacetime in this paper we begin with the much simpler structure of one dimension of time only, as motivated in section 3. Form one dimension we are led directly to the multi-dimensional form of time via the simple mathematical identities leading to equation 43. The full symmetry $\hat{G}$ and the full form of time $L(\hat{v}) = 1$ it acts upon collectively give rise to the components of matter fields over a 4-dimensional spacetime $M_4$, which itself is constructed out of substructures of $L(\hat{v}) = 1$ thus breaking the full symmetry.

For string theory the question remains regarding the identification of a 4-dimensional spacetime vacuum solution exhibiting properties corresponding to the Standard Model (see for example [5]) out of a vast landscape of possibilities, as noted in subsection 2.1. The structures of string theory are typically much larger than needed to accommodate the Standard Model alone. For example as a grand unification scheme the $E_8 \times E_8$ gauge group embedded within the structure of heterotic string theory, on employing the subgroup decomposition of equation 68 to one factor of $E_8$ ([4] equations 16.3.5 and 16.3.6), is large enough to describe 36 generations of quarks and leptons ([4] section 16.3).

This is in stark contrast to the present theory where we have converged upon the predicted $E_8(-24)$ symmetry of $L(v_{248}) = Q(t) = 1$, which in principle is just large enough to account for the required three generations, after already directly identifying a series of significant esoteric properties of the Standard Model through the natural augmentations of the forms of time described in section 4 and summarised in table 3. In particular there is very little redundancy in the explicit case of the breaking of the $G = E_7(-25)$ symmetry of $L(v_{56}) = q(x) = 1$ over the base space $M_4$ as depicted in figure 4, which already largely accounts for a single generation with very little in the way of a ‘hidden sector’. The full three generations are proposed to enter via the components of $x, y, z \in F(h_3 \mathcal{O})$ in an expression for the full temporal form $Q(t)$, based upon equation 74, again necessarily with a fairly tight fit, but now with a full $E_8(-24)$ symmetry.

In fact at the $E_7(-25)$ level depicted in figure 4 the only components of the 56-dimensional form of time not closely associated with an aspect of the Standard Model are the four Lorentz scalars $\alpha, \beta, n, N \in \mathbb{R}$. As $SU(3)_c \times U(1)_Q \subset E_7(-25)$ invariants these fields provide possible candidates for the cosmological dark matter or dark energy as alluded to in subsection 4.3 and described in ([1] section 13.1). These components, which potentially make a significant impact upon the large scale structure of the universe, are housed in a complementary sector of the components of the space $F(h_3 \mathcal{O})$ compared with the Standard Model states that are seen in the laboratory, as can be seen from equations 6, 26 and 31 and figure 4. On the other hand the full $E_8(-24)$ symmetry action on the components of the currently hypothetical form $L(v_{248}) = Q(t) = 1$ may be needed to uncover both the full Standard Model and the details of any new physics that may be significant either in the laboratory or on the cosmological scale.

We also note that in addition to the compact internal gauge groups already identified, in general non-compact ‘dilation’ transformations might also play a significant role. For example for the case of the full symmetry $\hat{G} = SL(3, \mathbb{C})$ broken over
the base space $M_4$, as depicted in figure 2, the breaking pattern of equation 51 can be augmented to the subgroup:

$$SL(2, \mathbb{C}) \times U(1) \times D(1) \subset SL(3, \mathbb{C})$$

where $D(1)$ is a dilation action, as described for ([3] equation 25). (An additional dilation `$\times D(1)$' can also be appended to the subgroup on the left-hand side of equation 87 for any $m \geq 4$). Other possible dilations are described in the opening of ([1] section 13.2). These include a similar dilation to that in equation 90 at the $E_6(-26) \equiv SL(3, \mathbb{O})$ level as discussed for equation 25 ([1] equations 8.35, 13.5 and 13.6), corresponding also to the second removed ‘Dynkin dot’ in figure 3(b). A further dilation is included at the $E_7(-25)$ stage as noted for equations 27 and 30 ([1] equation 9.30), and potentially also at the $E_8(-24)$ stage as denoted by $\Delta$ after equation 36 and discussed after equation 74.

While being symmetries of the full form of time $L(\hat{v}) = 1$, none of these dilation transformations represent a physical symmetry of the external spacetime $M_4$ after the symmetry breaking, rather each of them changes the ‘scale’ of the 4-vector $v_4 \in TM_4$ projected out of the full set of $\hat{v}$ components. Hence the subgroup decomposition of equation 90, for this full $L(v_9) = 1$ example, reduces to the subgroup $SL(2, \mathbb{C}) \times U(1) \subset SL(3, \mathbb{C})$ as the physical symmetry of 4-dimensional spacetime as analysed for equation 51.

However a possible role for the dilation transformations in the very early universe is considered in ([1] section 13.2) in which an initially unstable value of the magnitude $|v_4| \approx 0$ for the projection of $v_4 \in TM_4$ out of the full form $L(\hat{v}) = 1$ for cosmic time $t \approx 0$ rapidly increases and converges upon the present day stable value with $v_4 = h_0 > 0$ via a phase transition in the ‘Big Bang’, as sketched in ([1] figure 13.3). At the $E_7(-25)$ level these dilations can also drive any of the scalar invariants $\alpha, \beta, n, N$ to extreme values for $t \to 0$, allowing very different properties for the ‘dark sector’ in the very early universe. For the full theory the Big Bang phase transition then marks the very early stage at which both the dark sector stabilises and the familiar properties of the Standard Model of particle physics first emerge.

In addition to the potentially rapid growth in $|v_4|$ in time marking a significant evolution of the spacetime geometry in the very early universe, small variations in this scalar field about $|v_4(x)| = h_0$ in space and time since the phase transition have been associated with the Standard Model Higgs field, as noted in the second bullet point following figure 4. In this context it is these small variations in the magnitude of $v_4(x) \in TM_4$ projected out of $L(\hat{v}) = 1$ that directly impact upon the local spacetime geometry $G^{\mu\nu}(x)$ of equation 85, through which the energy-momentum $T^{\mu\nu}(x)$ is defined, and hence $\delta |v_4(x)|$ acts as an apparent source of mass.

In turn variations in other components of $\hat{v}$, such as any of $\{\alpha, \beta, n, N\}$, are correlated with variations of $|v_4|$ under the constraint $L(\hat{v}) = 1$, and hence any of the components of $\hat{v}$ can be a constituent of a ‘massive’ field in spacetime $M_4$, as alluded to in the third bullet point after figure 4. Such a field may be ‘dark’ if the underlying components are invariant under the internal gauge symmetry, as for $\{\alpha, \beta, n, N\}$, and hence potentially only apparent on the cosmological scale, or ‘visible’ if the components interact with the gauge fields, as is the case for the Dirac spinor states for example in figure 4, and hence in principle observable in the laboratory. Gauge fields $A(x)$
themselves, such as the electromagnetic field, directly carry energy-momentum via $f(A, \dot{\nu})$ in equation 85 as expressed through the Kaluza-Klein relation of equation 82, and can also become ‘massive’ through impingement on the $v_4 \in TM_4$ components in the symmetry breaking structure, as is proposed for $W^\pm$ and $Z^0$ gauge bosons in the electroweak sector, ([1] section 8.3). The potential of this theory for applications in cosmology more generally is described in ([1] chapters 12 and 13).

The Standard Model has been the main focus of this paper, which has summarised and built upon ([1] chapters 6–9, [2]). Through a combination of arguments, developed through sections 4 and 5 and continued in this section, we are guided to the non-compact real form $\hat{G} = E_{8(-24)}$ as playing at least a highly significant role as the symmetry of time. Further questions regarding the degree of uniqueness of the theory, including the nature of the external geometry of the base space $M_4$ itself, are discussed in [1] section 13.3). A range of potential empirical predictions for the laboratory that may follow from the developments of the theory proposed in this paper will be assessed in the following section. In the meantime, the connections made between the symmetry breaking patterns for higher-dimensional forms of time and the empirical properties of particle states already observed in modern day high energy physics laboratories, as summarised for example in table 1 and figure 4, together with the simplicity of the underlying conceptual picture, the broader developments of the theory discussed in this section and the potential for further progress make the overall case for the basic conception of the theory.

7 Conclusions

In this paper we have described how a physical theory can be derived from the continuum of time expressed as a multi-dimensional form, from the structure of which both an external 4-dimensional spacetime and the matter fields within it can be identified. This has led to the proposal of $E_{8(-24)}$ as the main candidate for the full symmetry of the highest-dimensional form of time supported by an analysis of the Standard Model structures that have already been uncovered for intermediate forms of time and consideration of the remaining particle multiplet properties required.

The conceptual basis and more general development of the theory has been presented in [1, 2, 3], while here we have emphasised the connections with the exceptional Lie group $E_8$ and related literature including [4, 5, 6, 7, 10, 11, 12, 13, 14, 15, 16, 17, 30, 31, 34] as briefly reviewed in section 2. Unlike several other unification schemes the theory here is not initially motivated by a notion of mathematical ‘beauty’ or ‘uniqueness’ but rather follows the clear and simple conceptual picture of constructing a full theory through an analysis of the structures and symmetries of the basic arithmetic composition of the one-dimensional flow of time itself.

The structure of an extended 4-dimensional spacetime and forms of matter collectively derived as a manifestation of this one dimension are then carried implicitly within and simultaneously with the flow of time, without the need to introduce an independent ‘material substratum’ – posited to model empirical observations. This approach can be contrasted with a range of theories and models that aim to account for the properties of matter by initially postulating extra spatial dimensions over and
above 4-dimensional spacetime. Here we take a much simpler starting point and begin with fewer dimensions, namely the one dimension of time alone.

The possibility of constructing a full physical theory from the one dimension of time has been motivated in section 3. There it was noted that the general multidimensional form of time, as derived for equations 41 and 43, contains within it as a particular case the quadratic form of 4-dimensional spacetime of equations 42 and 44. Through the symmetries of $L(v_4) = 1$ the basic geometric structure of an inertial frame can be identified, as described for equation 47 and figure 1. Subsequently in the three subsections of section 4 ‘matter fields’ were obtained from the residual components in the projection of the higher-dimensional forms of time of equations 49, 52 and 63 over the external spacetime $M_4$, with properties determined by the corresponding breaking of the SL$(3, \mathbb{C})$, $E_6(-26)$ and $E_7(-25)$ symmetry respectively.

In this manner a series of Standard Model properties have been identified, including spinor fields with fractional charges in colour triplets and a left-right asymmetry, as summarised in figure 4. These structures resemble one generation of Standard Model quarks and leptons, firmly rooting the theory in the empirical world through this correspondence of physical properties. This empirical success parallels and vindicates the argument that the full mathematical structure of the theory is firmly anchored in the observable world in deriving from the elementary mathematical structure of the real line $\mathbb{R}$ representing the continuous universal flow of time. That is, as had been proposed in section 3, the effectiveness of the theory is not ‘unreasonable’.

In pursuing this theory further the ambition of uncovering the symmetry transformation properties of the complete set of Standard Model states, taking into account the known structures and relations of the exceptional Lie groups, led to the proposal of $E_8(-24)$ as the ultimate unification symmetry in subsection 5.1, with the capacity in principle to incorporate three generations of quarks and leptons. The proposed provisional form of $L(v_{248}) = Q(t) = 1$ in equation 74 is motivated through a combination of the mathematical patterns observed for equations 70–73 as well as the ambitions for the full physical theory. However beneath these considerations the theory rests on the firm and simple foundation of the universal nature of the one-dimensional flow of time, through which the structures of the physical world are directly determined.

This underlying simplicity, which led to the ‘general form of time’ in equation 43, is perhaps analogous to the guide of ‘simplicity and elegance’ [43] employed by Hamilton in deducing the arithmetic rules for his ‘algebra of time’ in the 1830s, as reviewed in section 3. As also noted there the philosophical influences behind Hamilton’s algebraic work, regarding our perception of the world through the necessary forms of space and time, also provide some of the basis for the theory presented in this paper. In more recent decades a notion of ‘mathematical beauty’ has to some degree motivated many developments in theoretical physics, including the perceived elegance of the high degree of symmetry possessed by the structure of the Lie group $E_8$. In addition to this aesthetic appeal the existence of $E_8$ itself might be considered unique mathematically, as discussed in section 2. However here these aspects of $E_8$ are not the original motivation for considering this symmetry group.

Indeed the original conception of the theory, as introduced in section 3 in leading to equation 43, had no immediate connection with the structure of $E_8$ or any of the exceptional Lie groups. The development of a more explicit construction of the theory, as summarised towards the end of section 3, led to consideration of $E_6(-26)$ and
$E_7(-25)$ as symmetries of time, structures which in turn have been consolidated through the connections with the Standard Model described in section 4. This progression has inevitably directed our attention towards $E_8(-24)$ as the potential symmetry of the full form of time as explained in subsection 5.1. This proposal is supported by several other studies of structures relating to $E_8$, both in the physics and the mathematics literature, as described in subsection 5.2 and further in section 6 where the broader developments of the present theory have also been reviewed. The ambition is then to tie together the various strands in the literature regarding $E_8$ by establishing a unique role for this largest exceptional Lie group in the theory presented in this paper, hence relating $E_8$ directly to the fundamental structure of the physical world.

While $E_6$ and $E_7$ symmetries of homogeneous polynomial forms are already well known, equations 45 and 46 respectively, the apparent absence of an explicit augmentation to an $E_8$ symmetry on a possible form of time $L(v_{248}) = 1$ means that the suggestion that $E_8$ itself, as the largest of this series of exceptional Lie groups, might play a significant role as a symmetry of time is in part based on mathematical aesthetics. Further, regarding the construction of the currently hypothetical full form of time $L(v_{248}) = Q(t) = 1$, this project might be guided to some degree by mathematical structures that could be considered ‘beautiful’ or ‘unique’ or ‘natural’ on observing patterns in the progression from equation 70 to 73, elements of which have helped to shape the provisional form of equation 74 as noted above.

If, following $E_6(-26)$ and $E_7(-25)$, a further augmentation to an $E_8(-24)$ symmetry of a homogeneous from $L(v_{248}) = 1$ was already known it would simply be a case of extended the analysis of subsections 4.2 and 4.3 for this higher-dimensional form of time and reading off the symmetry breaking pattern while looking for further correspondence with known Standard Model structure. On the other hand if such an $E_8(-24)$ symmetry cannot be identified prior to this analysis, in practice hints from the Standard Model itself, including the need to open up further spinor states and identify three generations in the extension from figure 4, might be employed to help construct the mathematical form $L(v_{248}) = 1$.

As noted at the end of subsection 5.1, this unification scheme is then testable in the theoretical sense through the prediction of this $E_8(-24)$ symmetry of a homogeneous polynomial form as a further augmentation from the structures of table 3 capable of incorporating a full explanation of the Standard Model symmetry properties, and with little redundancy. From the current progression of symmetries of time through $\text{SL}(2, \mathbb{C}) \rightarrow \text{SL}(3, \mathbb{C}) \rightarrow E_6(-26) \rightarrow E_7(-25)$ the aim of completing the Standard Model structure in a further stage with an $E_8(-24)$ symmetry of a homogeneous form $L(v_{248}) = Q(t) = 1$, subsuming the previous stages and utilising known mathematical properties of $E_8$ such as the quasiconformal symmetry of the quartic light cone of equation 38 or the symmetry of an octic invariant as described for equation 79, at this moment presents us with a non-trivial puzzle, as noted at the end of subsection 5.2.

We have sought ways in which this problem might be approached and have considered what a solution might look like, but it is by no means obvious that it should exist at all. As noted in sections 4 and 5 the octonion algebra is expected to play a key role in the explicit construction of this $E_8$ symmetry and hence care is needed in any direct interpretation of more general studies involving for example Dynkin analysis of complex Lie algebras. However the discussion of subsection 5.2 involving the relevant mathematical structures suggests that the problem of finding a
solution for the explicit action of an $E_{8(-24)}$ symmetry on a concrete expression for $L(v_{248}) = Q(t) = 1$ should be tractable if a solution exists at all. Hence the search for a rigorous solution satisfying the appropriate mathematical and empirical criteria provides a robust theoretical test.

Further, since significant Standard Model properties have already been established up to the $E_{7(-25)}$ stage summarised in figure 4, any new features beyond the Standard Model identified at the $E_{8(-24)}$ level could lead directly and unambiguously to empirical predictions that might be tested in the high energy physics laboratory or through cosmological observations. In particular, three potential areas for ‘new physics’ to arise can be identified as follows:

- The simplest observation is that the rank-8 Lie group $E_8$ is large enough to contain the rank-8 subgroup ([1] equation 9.51):
  \[
  \text{Lorentz} \times \text{SU}(3) \times \text{SU}(2) \times \text{U}(1) \times \text{SU}(2) \times \text{U}(1) \subset E_8
  \]
  Hence in addition to the external Lorentz symmetry and the Standard Model gauge symmetry $\text{SU}(3)_c \times \text{SU}(2)_L \times \text{U}(1)_Y$ (see equation 69) in principle $E_8$ can contain a further internal $\text{SU}(2) \times \text{U}(1)$, implying the possibility of further gauge interactions. The additional $\text{SU}(2)$ might be identified as an ‘$\text{SU}(2)_R$’ factor, analogous to that in equation 65 or 80, while the additional $\text{U}(1)$ factor could be expressed by the dilation $\Delta$ of equation 36 for $E_{8(-24)}$ as the non-compact real form of interest here, with a possible role as discussed following equation 90.

- With variations in the 4-vector magnitude $|v_4|$ associated with the Higgs field, as discussed after figure 4, non-standard physics for the electroweak symmetry breaking sector of the Standard Model might be expected. As noted towards the end of subsection 5.1 under the external $\text{Lorentz} \subset E_{8(-24)}$ symmetry the components of $L(v_{248}) = Q(t) = 1$ may transform as spinors and scalars only. In this case it is the components transforming as ‘right-handed neutrinos’ that are proposed to be subsumed into the necessary composite 4-vector projection $v_4 \in TM_4$ onto the external spacetime out of $L(v_{248}) = 1$. Hence in principle this sector of the theory may also have implications for neutrino physics.

- While under the $E_{7(-25)}$ action on $L(v_{56}) = q(x) = 1$ there are only four components $\{\alpha, \beta, n, N\}$ of $v_{56} \equiv x \in F(h_3 \bar{O})$ that are not directly associated with Standard Model structures in figure 4, in identifying a full three generations of leptons and quarks from the hypothetical $E_{8(-24)}$ action on $L(v_{248}) = Q(t) = 1$, with $v_{248} \equiv t \in T$, there could be as many as $\sim 50$ components of the 248-dimensional space $T$ remaining. In principle these may give rise to new particle phenomena detectable in HEP experiments or have observable consequences on a galactic or cosmological scale as an extended ‘dark sector’.

The specific mathematical structure of an $E_{8(-24)}$ symmetry acting on the full form of time $L(v_{248}) = 1$ and its symmetry breaking pattern over the base manifold $M_4$ will be needed to analyse these potential empirical predictions in detail. A more complete theory, in particular incorporating quantum theory as proposed in section 6, might also be required for precise calculations of the properties of possible new particle states, as well as of the known ones. However one general feature is that there is only
limited room for new physics beyond the Standard Model, with for example no set
of ‘mirror states’ or supersymmetric partners associated with the familiar Standard
Model particles. In this sense the theory at this stage is already ‘falsifiable’ since the
discovery of, for example, supersymmetry states would not be compatible with this
theory without a significant, and internally poorly motivated, modification.

In the meantime the main arguments for the merit of this theory echo the
ambitions outlined in the three bullet points at the end of section 1. The theory is
firmly grounded in the basic notion of the flow of time that infuses all of our experiments
and observations in the world. Given the interest dating over many decades in a wide
range of theories based on extra spatial dimensions, the observation that it is possible
to construct a full physical theory simply from the one dimension of time alone, and
the explanatory power that has already emerged in particular regarding connections
with the Standard Model, underpins the plausibility of this approach. It is possible
to extract significantly more out of this theory than is put in through the simple
underlying assumptions. In addition, the development of the theory points towards a
unique role for $E_8$ as the full symmetry of the full multi-dimensional form of time, as
also emphasised in this paper, highlighting the prospects for further progress and the
potential predictive power of the theory.

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