TWO-CARDINAL IDEAL OPERATORS AND INDESCRIBABILITY

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Abstract. A well-known version of Rowbottom’s theorem for supercompactness ultrafilters leads naturally to notions of two-cardinal Ramseyness and corresponding normal ideals introduced herein. Generalizing results of Baumgartner, Feng and the first author, from the cardinal setting to the two-cardinal setting, we study hierarchies associated with a particular version of two-cardinal Ramseyness and a strong version of two-cardinal ineffability, as well as the relationships between these hierarchies and a natural notion of transfinite two-cardinal indescribability.

1. Introduction

One version of Ramsey’s famous combinatorial theorem states that for every function \( f : [\omega]^2 \to 2 \) there is an infinite set \( H \subseteq \omega \) such that \( H \) is homogeneous for \( f \), in the sense that \( f \upharpoonright [H]^2 \) is constant. Since the work of Erdős, Hajnal, Tarski, Rado and others [19, 20, 21, 22], it has been well-known that certain generalizations of Ramsey’s theorem to uncountable sets necessarily involve large cardinals. For example, we say that \( \kappa > \omega \) is an ineffable cardinal if for every function \( f : [\kappa]^2 \to \kappa \) with \( f(a) < \min(a) \) for all \( a \in [\kappa]^2 \), there is an \( H \subseteq \kappa \) that is stationary in \( \kappa \) and homogeneous for \( f \). Similarly, we say that \( \kappa > \omega \) is a Ramsey cardinal if for every function \( f : [\kappa]<\omega \to \kappa \) with \( f(a) < \min(a) \) for all \( a \in [\kappa]<\omega \), there is a set \( H \subseteq \kappa \) of size \( \kappa \) that is homogeneous for \( f \), that is \( f \upharpoonright [H]^n \) is constant for each \( n < \omega \). The notions of ineffability and Ramsey of cardinals leads naturally to the following definitions of the ineffability ideal operator \( I \) and the Ramsey ideal operator \( R \).

Suppose \( \kappa \) is a regular cardinal and \( I \) is an ideal on \( \kappa \). We let \( I^+ = \{ X \subseteq \kappa \mid X \notin I \} \) be the corresponding collection of \( I \)-positive sets and \( I^* = \{ X \subseteq \kappa \mid \kappa \setminus X \in I \} \) be the filter which is dual to \( I \). We define new ideals \( I(I) \) and \( R(I) \) as follows. A set \( X \subseteq \kappa \) is not in \( I(I) \) if and only if for every function \( f : [X]^2 \to \kappa \) with \( f(a) < \min(a) \) for all \( a \in [X]^2 \), there is a set \( H \in P(X) \cap I^+ \) such that \( f \upharpoonright [H]^2 \) is constant. Similarly, a set \( X \subseteq \kappa \) is not in \( R(I) \) if and only if for every function \( f : [X]<\omega \to \kappa \) with \( f(a) < \min(a) \) for all \( a \in [X]<\omega \), there is a set \( H \in P(X) \cap I^+ \) such that \( f \upharpoonright [H]^n \) is constant for all \( n < \omega \). It follows from the work of Baumgartner [7] that if \( I \supseteq [\kappa]^\kappa \) then \( I(I) \) is a normal ideal. The corresponding result for \( R(I) \) also holds, and is due to Feng [23].

By repeatedly applying the ideal operators \( I \) and \( R \) to various ideals, one is led naturally to consider the ineffability hierarchy [8] and the Ramsey hierarchy [23].
That is, if $\kappa$ is regular, $I$ is an ideal on $\kappa$ and $O \in \{I, R\}$, we inductively define new ideals by letting

$$
O^0(I) = I, \\
O^{\alpha+1}(I) = O(O^\alpha(I)), \text{ and} \\
O^\alpha(I) = \bigcup_{\beta<\alpha} O^\beta(I) \text{ when } \alpha \text{ is a limit.}
$$

We say that $\kappa$ is $\alpha$-ineffable if and only if the ideal $I^\alpha(\text{NS}_\kappa)$ is nontrivial, that is, $I^\alpha(\text{NS}_\kappa) \neq P(\kappa)$. Similarly, $\kappa$ is $\alpha$-Ramsey if and only if the ideal $R^\alpha([\kappa]^{<\kappa})$ is nontrivial.

The hierarchies of $\alpha$-ineffable and $\alpha$-Ramsey cardinals, and their relationship with various notions of indescribability have been extensively studied by Baumgartner, Feng, as well as the first author and Peter Holy. Although there is an extensive literature on two-cardinal combinatorial properties involving various notions of subtlety and ineffability, much less is known about two-cardinal analogues of Ramsey properties.

In this article, we introduce a well-behaved two-cardinal version of the Ramsey operator and generalize many results from the literature to our two-cardinal Ramsey operator as well as to a two-cardinal ineffable operator previously studied by Kamo.

2. Two-cardinal ideal operators associated to ineffability and partition properties

2.1. Stationarity, strong stationarity and strong normality. Suppose $\kappa$ is regular and $A$ is a set of ordinals with $\kappa \leq |A|$. We write $P_\kappa A$ or $[A]^{<\kappa}$ to denote the collection of subsets of $A$ of cardinality less than $\kappa$. A set $S \subseteq P_\kappa A$ is unbounded in $P_\kappa A$ if for every $x \in P_\kappa A$ there is a $y \in S$ with $x \subseteq y$. It is easy to see that the collection

$$
I_{\kappa, A} = \{X \subseteq P_\kappa A \mid X \text{ is not unbounded}\}
$$

is a nontrivial ideal on $P_\kappa A$. Moreover, $I^+_{\kappa, A}$ is the set of unbounded subsets of $P_\kappa A$ and the filter generated by the collection $\{\hat{x} \mid x \in P_\kappa A\}$, where $\hat{x} = \{y \in P_\kappa A : x \subseteq y\}$, equals the filter $I^*_{\kappa, A}$ dual to $I_{\kappa, A}$. Also notice that because $\kappa$ is assumed to be regular, for any $\gamma < \kappa$ and any sequence $(X_\alpha : \alpha < \gamma)$ with $A_\alpha \in I_{\kappa, A}$ for $\alpha < \gamma$, we have $\bigcup_{\alpha<\gamma} A_\alpha \in I_{\kappa, A}$.

Jech defined two-cardinal notions of closed unboundedness and stationarity as follows. A set $C \subseteq P_\kappa A$ is closed if whenever $\{c_\alpha : \alpha < \gamma\}$ is a $\subseteq$-increasing chain in $C$ of length less than $\kappa$, it follows that $\bigcup_{\alpha<\gamma} c_\alpha \in C$. A set $C \subseteq P_\kappa A$ is club in $P_\kappa A$ if it is closed and unbounded in $P_\kappa A$, and a set $S \subseteq P_\kappa A$ is stationary in $P_\kappa A$ if $S \cap C \neq \emptyset$ for all clubs $C$ in $P_\kappa A$. Jech showed that when $\kappa$ is regular the set

$$
\text{NS}_{\kappa, A} = \{X \subseteq P_\kappa A \mid X \text{ is nonstationary}\}
$$

is a nontrivial normal ideal on $P_\kappa A$, meaning that for every $S \in \text{NS}^+_\kappa A$ and every function $f : S \rightarrow \lambda$, with $f(x) \in x$ for all $x \in S$, there is a $T \subseteq S$ which is stationary in $P_\kappa A$ such that $f \upharpoonright T$ is constant. It is easy to see that $\text{NS}^+_\kappa A$ is the filter generated by the club subsets of $P_\kappa A$. 
The ideals $I_{\kappa,A}$ and $\text{NSS}_{\kappa,A}$ are defined using the ordering $(P_\kappa A, \subseteq)$. When $\kappa$ is inaccessible it is often advantageous to work with a different ordering. If $x \in P_\kappa A$ we let $\kappa = |x \cap \kappa|$. For $x, y \in P_\kappa A$ we define $x \prec y$ if and only if $x \subseteq y$ and $|x| < |y \cap \kappa|$ (equivalently $x \in P_\kappa A$). See [28, Section 3] for an introductory discussion of $\prec$ and the notion of strong normality, which we also define below. Notice that if $\kappa$ is inaccessible, a set $S \subseteq P_\kappa A$ is unbounded if and only if for every $x \in P_\kappa A$ there is a $y \in S$ with $x \prec y$. Since we will focus on the case in which $\kappa$ is inaccessible, we don’t lose anything by working with $I_{\kappa,\lambda}$ rather than its $\prec$ counterpart.

We say that a set $C \subseteq P_\kappa A$ is a weak club in $P_\kappa A$ if $C$ is $\prec$-unbounded in $P_\kappa A$ and whenever $C \cap P_\kappa x \in I_{\kappa,x}^*$, for some $x \in P_\kappa A$, we have $x \in C$. It is straightforward to see that when $C \subseteq P_\kappa A$ is a weak club in $P_\kappa A$ there is a function $f : P_\kappa A \to P_\kappa A$ such that the set

$$C_f := \{x \in P_\kappa A \mid x \cap \kappa \neq \emptyset \wedge \forall \psi \in f^* P_\kappa x \subseteq P_\kappa x\}$$

is contained in $C$. Furthermore, for any such function $f$, the set $C_f$ is a weak club subset of $P_\kappa A$.

For $S \subseteq P_\kappa A$, a function $f : S \to P_\kappa A$ is said to be $\prec$-regressive on $S$ if $f(x) \prec x$ for all $x \in S$. An ideal $I$ on $P_\kappa A$ is strongly normal if for all $S \in I^+$ and all $\prec$-regressive functions $f : S \to P_\kappa A$ there is a $T \in P(S) \cap I^+$ such that $f$ is constant on $T$. It is easy to see that an ideal $I$ on $P_\kappa A$ is strongly normal if and only if for every sequence $\vec{A} = \langle A_x \mid x \in P_\kappa A \rangle$ with $A_x \in I^*$ for all $x \in P_\kappa A$, the $\prec$-diagonal intersection

$$\triangleleft \{A_x \mid x \in P_\kappa A\} = \{y \in P_\kappa A \mid y \in \bigcap_{x \prec y} A_x\}$$

is in $I^*$. A set $S \subseteq P_\kappa A$ is strongly stationary if $S \cap C \neq \emptyset$ for all weak clubs $C \subseteq P_\kappa A$. Let us note that this definition of strongly stationary differs slightly from that used elsewhere in the literature, however, it is equivalent to the corresponding notions used in [12, 14] when $\kappa$ is a Mahlo cardinal (see [18, Fact 2.1]). The non-strongly stationary ideal on $P_\kappa A$ is the collection

$$\text{NSS}_{\kappa,A} = \{X \subseteq P_\kappa A \mid X \text{ is not strongly stationary}\}.$$  

When $\kappa$ is Mahlo, it follows that $\text{NSS}_{\kappa,A} \subseteq \text{NSS}_{\kappa,A}$ and that $\text{NSS}_{\kappa,A}$ is the minimal strongly normal ideal (see [12, Section 6] or [14, Corollary 2.3]). See [28, Section 3] for information on the relationship between $\text{NSS}_{\kappa,\lambda}$ and $\text{NSS}_{\kappa,\theta}$ when $\theta = \lambda^{\kappa^+}$.

2.2. Two-cardinal ideal operators associated to ineffability and partition properties. Kamo [20] studied several ideal operators associated to notions of two-cardinal ineffability and partition properties introduced by Jech [26]. While the results of Jech [26], Menas [35], Magidor [33] and others focus on ineffability and partition properties defined using the ordering $(P_\kappa A, \subseteq)$, Kamo introduced similar notions defined using $(P_\kappa A, \prec)$. Let us review the relevant definitions.

Given a set $S \subseteq P_\kappa A$, a sequence $\vec{S} = \langle S_x \mid x \in S \rangle$ is called an $(S, \subseteq)$-list if $S_x \subseteq x$ for all $x \in S$, and $\vec{S}$ is called an $(S, \prec)$-list if $S_x \subseteq P_\kappa x$ for all $x \in S$. If $\vec{S}$ is an $(S, \subseteq)$-list, we say that $H \subseteq S$ is homogeneous for $\vec{S}$ if whenever $x, y \in H \cap S$ and $x \subseteq y$ we have $S_x = S_y \cap x$. If $\vec{S}$ is an $(S, \succ)$-list, we say that $H \subseteq S$ is homogeneous for $\vec{S}$ if whenever $x, y \in H \cap S$ and $x \prec y$ we have $S_x = S_y \cap P_\kappa x$.

In the following definitions we will consider 2-colorings of sets of the form

$$|S|^2 = \{(x, y) \mid x, y \in S \land x \prec y\}$$
where $<$ is some ordering on $P_\kappa A$. Here we want to consider colorings of sets of $\prec$-increasing ordered pairs because, in the cases we are interested in, namely $\prec \in \{\subseteq, \prec\}$, it is often the case that large homogeneous sets for colorings of unordered pairs do not exist; see the paragraph after Definition 4.5 in [26] for details. Given a function $f : [S]^2_\omega \to 2$ we say that $H \subseteq S$ is homogeneous for $f$ if $f \upharpoonright [H]^2_\omega$ is a constant function.

**Definition 2.1.** Suppose $I$ is an ideal on $P_\kappa A$ and $\prec$ is some ordering on $P_\kappa A$. We define new ideals $\mathcal{I}_\prec(I)$, $\mathcal{I}_\leq(I)$ and $\mathcal{P}_{\operatorname{art}_\prec}(I)$ on $P_\kappa A$ as follows.

1. $\mathcal{I}_\prec(I)$ is the ideal on $P_\kappa A$ such that $S \in \mathcal{I}_\prec(I)$ if and only if every $(S, \subset)$-list has a homogeneous set $H \subseteq S$ in $I^+$.  
2. $\mathcal{I}_\leq(I)$ is the ideal on $P_\kappa A$ such that $S \in \mathcal{I}_\leq(I)$ if and only if every $(S, \prec)$-list has a homogeneous set $H \subseteq S$ in $I^+$.  
3. $\mathcal{P}_{\operatorname{art}_\prec}(I)$ is the ideal on $P_\kappa A$ such that $S \in \mathcal{P}_{\operatorname{art}_\prec}(I)$ if and only if every function $f : [S]^2_\omega \to 2$ has a homogeneous set $H \subseteq S$ in $I^+$.

It is not too difficult to see that when $I$ is a normal ideal on $P_\kappa A$, it follows that $\mathcal{I}_\prec(I)$ and $\mathcal{P}_{\operatorname{art}_\prec}(I)$ are normal ideals and $\mathcal{I}_\leq(I)$ and $\mathcal{P}_{\operatorname{art}_\prec}(I)$ are strongly normal. Furthermore, if $I \supseteq I_{\kappa, A}$ is any ideal on $P_\kappa A$, then $\mathcal{I}_\prec(I)$ is strongly normal (see the paragraph after Definition 3.1 in [29]). For more details and related results see [29, Section 3].

Note that, in Jech’s terminology, $\kappa$ is $\lambda$-ineffable if and only if $\mathcal{I}_\prec(\text{NS}_{\kappa, \lambda})$ is a nontrivial ideal. Similarly, $S \subseteq P_\kappa A$ is an ineffable subset of $P_\kappa A$ if and only if $S \in \mathcal{I}(\text{NS}_{\kappa, A})^+$. We say that $S$ has the $\prec$-partition property if $S \in \mathcal{P}_{\operatorname{art}_\prec}(\text{NS}_{\kappa, A})^+$. Johnsson [27] showed that when $\text{cf}(\lambda) \geq \kappa$, a set $S \subseteq P_\kappa A$ is ineffable if and only if it has the $\prec$-partition property; hence $\mathcal{I}_\prec(\text{NS}_{\kappa, A}) = \mathcal{P}_{\operatorname{art}_\prec}(\text{NS}_{\kappa, A})$ (see [3, Fact 1.13]). The relationships between the operators $\mathcal{I}_\prec$, $\mathcal{I}_\leq$, $\mathcal{P}_{\operatorname{art}_\prec}$ and $\mathcal{P}_{\operatorname{art}_\leq}$ have been further explored by Kamo [29] as well as Abe and Usuba [3].

### 2.3. A two-cardinal ideal operators associated to Ramseyness

Suppose $\kappa$ is regular and $A$ is a set of ordinals with $\kappa \leq |A|$. Suppose $S \subseteq P_\kappa A$. Given a tuple $\overline{x} = (x_1, \ldots, x_n) \in S^n$, with $x_1 \prec \cdots \prec x_n$, we will identify $\overline{x}$ with the $\prec$-increasing enumeration of its entries. Given $S \subseteq P_\kappa A$ and $n < \omega$, we let $$[S]^n_\preceq = \{(x_1, \ldots, x_n) \in S^n \mid x_1 \prec \cdots \prec x_n\}$$ and $$[S]^{\omega}_\preceq = \bigcup_{n < \omega} [S]^n_\preceq.$$ 

A function $f : [S]^{\omega}_\preceq \to P_\kappa A$ is called $\prec$-regressive if $f(x_1, \ldots, x_n) \prec x_1$ for all $(x_1, \ldots, x_n) \in [S]^{\omega}_\preceq$.

The following is a straightforward generalization of a standard fact about supercompactness ultrafilters which is used to prove that supercompact Prikry forcing satisfies the Prikry property (see [24, Section 1.4]).

**Proposition 2.2.** Suppose $U$ is a $\kappa$-complete normal fine ultrafilter on $P_\kappa \lambda$ and $f : [P_\kappa \lambda]^{\omega}_\preceq \to P_\kappa \lambda$ is a $\prec$-regressive function. Then there is a set $H \in U$ which is homogeneous for $f$, meaning that $f \upharpoonright [H]^{\omega}_\preceq$ is constant for all $n < \omega$.

**Proof.** It suffices to show that for each $n \in \omega \setminus \{0\}$ there is an $H_n \in U$ such that $f \upharpoonright [H_n]^{\omega}_\preceq$ is constant, because then $H = \bigcap_{n < \omega} H_n \in U$ will be the desired homogeneous set for $f$.  

We will prove by induction on \( n \) that for every \( \prec \)-regressive \( F : [P_\kappa \lambda]^n \to P_\kappa \lambda \) there is \( H \in U \) such that \( F \upharpoonright [H]_\prec^n \) is constant. This holds for \( n = 1 \) by the strong normality of \( U \). Suppose its true for \( n \). Let \( F : [P_\kappa \lambda]^{n+1} \to P_\kappa \lambda \) be \( \prec \)-regressive. For each \( x \in P_\kappa \lambda \) define \( F_x : [P_\kappa \lambda]^n \to P_\kappa \lambda \) by

\[
F_x(x_1, \ldots, x_n) = \begin{cases} 
F(x_1, \ldots, x_n) & \text{if } x \prec x_1 \\
0 & \text{o.w.}
\end{cases}
\]

Let \( C_x = \{ y \in P_\kappa \lambda \mid x \prec y \} \). Then \( C_x \) is club and for all \( (x_1, \ldots, x_n) \in [C_x]^n \) we have \( F_x(x_1, \ldots, x_n) = F(x_1, \ldots, x_n) \). By our inductive hypothesis, it follows that for each \( x \in P_\kappa \lambda \) there is \( H_x^\prime \in U \) such that \( F_x \upharpoonright [H_x^\prime]_\prec^n \) is constant. Furthermore, \( H_x = H_x^\prime \cap C_x \in U \), and thus \( F_x \upharpoonright [H_x]_\prec^n \) is constant. For \( (x_1, \ldots, x_n) \in [H_x]_\prec^n \) we let \( i_x = F_x(x_1, \ldots, x_n) = F(x_1, \ldots, x_n) \) denote this constant value and note that \( i_x \prec x \). Now let

\[
H^\prime = \bigtriangleup_\prec \{ H_x \mid x \in P_\kappa \lambda \} = \{ y \in P_\kappa \lambda \mid y \in \bigcap_{z \prec y} H_z \}.
\]

We have \( H^\prime \in U \) by the strong normality of \( U \). If \( (x_1, \ldots, x_n) \in [H^\prime]^{n+1}_\prec \) then \( (x_1, \ldots, x_n) \in [H_x]_\prec^n \) and thus \( F(x_1, \ldots, x_n) = F_x(x_1, \ldots, x_n) = i_x \). Since \( x \mapsto i_x \) is regressive on \( H^\prime \in U \) there is \( H \subseteq H^\prime \in U \) such that for all \( x \in H \) we have \( i_x = i \), where \( i \) is some finite element of \( P_\kappa \lambda \). Thus, if \( (x_0, \ldots, x_n) \in [H]^{n+1}_\prec \) then \( F(x_0, \ldots, x_n) = i_{x_0} = i \).

The previous result motivates the following definitions, which resemble characterizations of the one-cardinal Ramsey operator studied in \([8, 15, 17, 23, 36]\).

**Definition 2.3.** Suppose \( \kappa \) is regular, \( A \) is a set of ordinals with \( \kappa \leq |A| \) and \( I \supseteq I_{\kappa,A} \) is an ideal on \( P_\kappa A \). We define ideals \( \text{Ram}(I) \) and \( \text{Ram}_\prec(I) \) on \( P_\kappa A \) as follows.

1. \( \text{Ram}(I) \) is the ideal on \( P_\kappa A \) such that \( S \in \text{Ram}(I)^+ \) if and only if every function \( f : [S]^{\leq \omega}_\prec \to 2 \) has a homogeneous set \( H \subseteq S \in I^+ \), meaning that \( f \upharpoonright [H]^n \) is constant for all \( n < \omega \).

2. \( \text{Ram}_\prec(I) \) is the ideal on \( P_\kappa A \) such that \( S \in \text{Ram}_\prec(I)^+ \) if and only if every \( \prec \)-regressive function \( f : [S]^{\leq \omega}_\prec \to P_\kappa A \) has a homogeneous set \( H \subseteq S \in I^+ \), meaning that \( f \upharpoonright [H]^n \) is constant for all \( n < \omega \).

It is easy to verify that \( \text{Ram}(I) \subseteq \text{Ram}_\prec(I) \), but unfortunately not much else is known about \( \text{Ram}(I) \). For example, generalizing from the one-cardinal case, one would like to show that Definition 2.3(1) and Definition 2.3(2) are equivalent when \( I \supseteq \text{NS}_{\kappa,A} \) (see [15]); however, it remains open whether this can be done. In what follows we will focus on \( \text{Ram}_\prec(I) \) rather than \( \text{Ram}(I) \).

Let us prove that \( \text{Ram}_\prec(I) \) is a strongly normal ideal on \( P_\kappa A \) whenever \( I \) is a strongly normal ideal on \( P_\kappa A \).

**Theorem 2.4.** Suppose \( \kappa \) is regular, \( A \) is a set of ordinals with \( \kappa \leq |A| \) and \( I \) is an ideal on \( P_\kappa A \). Then \( \text{Ram}_\prec(I) \) is a strongly normal ideal on \( P_\kappa A \).

**Proof.** Suppose \( S \in \text{Ram}_\prec(I)^+ \) and \( h : S \to P_\kappa A \) is \( \prec \)-regressive. Define \( f : [S]^{\leq \omega}_\prec \to P_\kappa A \) by letting \( f(\{x\}) = h(x) \) for all \( x \in S \) and \( f(x) = 0 \) whenever \( |x| \neq 1 \). Notice that \( f \) is \( \prec \)-regressive on \([S]^{\leq \omega}_\prec \) and hence there is a set \( H \in P(S) \cap I^+ \) such
that \( f \upharpoonright [H]^n \) is constant for all \( n < \omega \). It follows that \( h \upharpoonright H \) is constant and hence \( \mathcal{R} \text{am}_\omega(I) \) is strongly normal. \( \square \)

Since the non-strongly stationary ideal \( \text{NSS}_{\kappa,A} \) is the minimal strongly normal ideal on \( P_{\kappa,A} \) \cite{12}, we easily obtain the following corollary.

**Corollary 2.5.** Suppose \( \kappa \) is regular, \( A \) is a set of ordinals with \( \kappa \leq |A| \) and \( I \supseteq I_{\kappa,A} \) is an ideal on \( P_{\kappa,A} \). Then \( S \in \mathcal{R} \text{am}_\omega(I)^+ \) if and only if \( S \cap C \in \mathcal{R} \text{am}_\omega(A)^+ \) for all weak clubs \( C \in P_{\kappa,A} \).

Feng \cite{23} Theorem 2.3] gave a characterization of the one-cardinal Ramsey operator in terms of \( (\omega,S) \)-sequences. We would like to generalize this characterization to the two-cardinal \( \mathcal{R} \text{am}_\omega \). Given \( S \subseteq P_{\kappa,A} \), an \( (\omega,S,\triangleleft) \)-list is a function \( S : [S]^\omega \rightarrow P(P_{\kappa,A}) \) such that \( S(x_1, \ldots, x_n) \subseteq P_{\kappa_x}x_1 \) for all \( (x_1, \ldots, x_n) \in [S]^\omega \). We say that a set \( H \subseteq S \) is homogeneous for \( S \) if for all \( n < \omega \) and all \( (x_1, \ldots, x_n) \) and \( (y_1, \ldots, y_n) \) in \( [H]^\omega \) with \( x_1 \triangleleft y_1 \) we have \( S(y_1, \ldots, y_n) \cap P_{\kappa_x}x_1 = S(x_1, \ldots, x_n) \).

**Proposition 2.6.** Suppose \( \kappa \) is a cardinal, \( A \) is a set of ordinals with \( \kappa \leq |A| \), \( I \supseteq I_{\kappa,A} \) is an ideal on \( P_{\kappa,A} \) and \( S \subseteq P_{\kappa,A} \). The following are equivalent.

1. \( S \in \mathcal{R} \text{am}_{\kappa,A}(I)^+ \)
2. For every \( (\omega,S,\triangleleft) \)-list \( S : [S]^\omega \rightarrow P(P_{\kappa,A}) \) there is an \( H \in P(S) \cap I^+ \) which is homogeneous for \( S \).
3. For any \( (\omega,S,\triangleleft) \)-list \( \tilde{S} : [S]^\omega \rightarrow P(P_{\kappa,A}) \) there is an \( H \in P(S) \cap I^+ \) and a sequence \( \langle S_n \mid 1 < n < \omega \rangle \) of subsets of \( P_{\kappa,A} \) such that for all \( n \), for all \( (x_1, \ldots, x_n) \in [H]^\omega \), we have \( \tilde{S}(x_1, \ldots, x_n) = S_n \cap P_{\kappa_x}x_1 \).

**Proof.** It is easy to see that (2) and (3) are equivalent. Let us show that (1) and (2) are equivalent.

For (1) implies (2), suppose \( \tilde{S} \) is an \( (\omega,S,\triangleleft) \)-list. Since \( \mathcal{C} = \{ x \in P_{\kappa,A} \mid x \cap \kappa \) is a limit ordinal\} is club in \( P_{\kappa,A} \), we can assume that \( S \subseteq C \). Define \( g : [S]^\omega \rightarrow P(P_{\kappa,A}) \) such that for all \( (x_1, \ldots, x_{2n}) \in [S]^\omega \), setting \( a = (x_1, \ldots, x_n) \) and \( b = (x_{n+1}, \ldots, x_{2n}) \) we have \( g(x_1, \ldots, x_{2n}) = \emptyset \) if \( \tilde{S}(a) = \tilde{S}(b) \cap P_{\kappa_x}x_1 \), and \( g(x_1, \ldots, x_{2n}) = z \cup \{ x_2 \} \) where \( z \) is some element of \( \tilde{S}(a) \triangleleft \tilde{S}(b) \cap P_{\kappa_x}x_1 \) if \( \tilde{S}(a) \neq \tilde{S}(b) \cap P_{\kappa_x}x_1 \). Notice that since \( x_1 \in S \subseteq C \), it follows that, in the second case above \( z \cup \{ x_2 \} \subseteq P_{\kappa_x}x_1 \), and thus \( g \) is \( \triangleleft \)-regressive. Let \( H \in P(S) \cap I^+ \) be homogeneous for \( g \).

Let us show that if \( a, b \in [H]^\omega \) are such that \( a \triangleleft b \) then \( \tilde{S}(a) = \tilde{S}(b) \cap P_{\kappa_x}x_1 \). Suppose \( \tilde{S}(a) \neq \tilde{S}(b) \cap P_{\kappa_x}x_1 \). Then \( g(a \triangleleft b) = z \cup \{ x_2 \} \) where \( z \in \tilde{S}(a) \triangle \tilde{S}(b) \cap P_{\kappa_x}x_1 \). Without loss of generality, say \( z \in \tilde{S}(a) \setminus \tilde{S}(b) \). Let \( c \in [H]^\omega \) be such that \( b \triangleleft c \). Then, by the homogeneity of \( H \), we have \( g(b \triangleleft c) = z \cup \{ x_2 \} \), and thus by definition of \( g \) we have \( z = \tilde{S}(b) \triangle \tilde{S}(c) \cap P_{\kappa_x}x_1 \). Since \( z \notin \tilde{S}(b) \) we have \( z \in \tilde{S}(a) \cap \tilde{S}(c) \), which implies \( z \notin \tilde{S}(a) \triangle \tilde{S}(c) \cap P_{\kappa_x}x_1 \). However, by homogeneity of \( H \), it follows that \( g(a \triangleleft c) = z \cup \{ x_2 \} \) \( \triangle \), and thus \( z \in \tilde{S}(a) \triangle \tilde{S}(b) \cap P_{\kappa_x}x_1 \), a contradiction.  

\footnote{We use \( z \cup \{ x_2 \} \) in the second case so that the value of \( g(x_1, \ldots, x_{2n}) \), where \( (x_1, \ldots, x_{2n}) \in [S]^\omega \), can be used to determine which case \( (x_1, \ldots, x_{2n}) \) fall into.}
Now, let $a, b \in [H]^n_\kappa$ be such that $a_1 \preceq b_1$. Choose $c \in [H]^n_\kappa$ with $a_n, b_n \prec c_1$. Then $\vec{S}(a) = \vec{S}(c) \cap a_1$ and $\vec{S}(b) = \vec{S}(c) \cap P_{\kappa_3} b_1$, which implies

$$\vec{S}(b) \cap P_{\kappa_4} a_1 = (\vec{S}(c) \cap P_{\kappa_3} b_1) \cap P_{\kappa_4} a_1 = \vec{S}(c) \cap P_{\kappa_4} a_1 = \vec{S}(a).$$

For (2) implies (1), let $f : [S]^{<\omega}_\prec \to P_\kappa A$ be a $\prec$-regressive function and let $C \subseteq P_\kappa A$ be a weak club. We define an $(\omega, S \cap C, \prec)$-sequence $\vec{S}$ as follows. For each $a \in [S \cap C]^{<\omega}_\prec$ let $\vec{S}(a) = \{f(a)\} \subseteq P_{\kappa_4} a_1$. Let $H \in P(S \cap C) \cap I^+$ be homogeneous for $\vec{S}$. Then $H$ is also homogeneous for $f$. \hfill □

Next we demonstrate that the nontriviality of the ideal $\text{Ram}_\prec(I)$ naturally leads to the existence of nonlinear sets of indiscernibles for certain structures in countable languages.

**Definition 2.7.** If $\mathcal{M}$ is a structure in a countable language and $P_\kappa A \subseteq \mathcal{M}$ we say that $H \subseteq P_\kappa A$ is a set of indiscernibles for $\mathcal{M}$ if for every $n < \omega$ and for all $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in [H]^n_\kappa$ we have

$$\mathcal{M} \models \varphi(x_1, \ldots, x_n) \text{ if and only if } \mathcal{M} \models \varphi(y_1, \ldots, y_n)$$

for all first-order $\varphi$ in the language of $\mathcal{M}$ with exactly $n$ free variables.

**Proposition 2.8.** Suppose $\kappa$ is a cardinal, $A$ is a set of ordinals with $\kappa \leq |A|$ and $I \supseteq I_{\kappa, A}$ is an ideal on $P_\kappa A$. If every function $f : [P_\kappa A]^{<\omega}_\prec \to 2$ has a homogeneous set in $I^+$ then every structure $\mathcal{M}$ in a countable language with $P_\kappa A \subseteq \mathcal{M}$ has a set of indiscernibles $H \in I^+$.

**Proof.** Using an argument similar to that of [31 Proposition 7.14(c)], it is easy to see that our assumption implies that for every $\gamma < \kappa$ every function $f : [P_\kappa A]^{<\omega}_\prec \to \gamma$ has a homogeneous set $H \in I^+$. Let $\mathcal{M}$ be a structure in a countable language with $P_\kappa A \subseteq \mathcal{M}$ and let $\Phi_n$ denote the collection of all first order formulas in the language of $\mathcal{M}$ with exactly $n$ free variables. Define a function $f$ with domain $[P_\kappa A]^{<\omega}_\prec$ by letting

$$f(x_1, \ldots, x_n) = \{\varphi \in \Phi_n \mid \mathcal{M} \models \varphi(x_1, \ldots, x_n)\}.$$ 

Since $|P(\Phi_n)| < \kappa$, $f$ has a homogeneous set $H \in I^+$. It is easy to verify that $H$ is a set of indiscernibles for $\mathcal{M}$. \hfill □

3. Generalizing results of Baumgartner and Feng

The ineffability hierarchy and the Ramsey hierarchy, which were introduced by Baumgartner [8] and Feng [23] respectively, can be obtained by iterating the associated ideal operators. In the present section we consider two-cardinal versions of these hierarchies and investigate their relationship with a notion of transfinite two-cardinal indescribability which generalizes previously studied notions [36, 4, 14].
3.1. **Transfinite two-cardinal indescribability.** Let us now generalize a notion of transfinite indescribability introduced in [4], and further utilized in [5, 16] to the two-cardinal context.

For the reader’s convenience, let us discuss the notion of $\Pi^1_\xi$ formula introduced in [4]. Recall that a formula of second-order logic is $\Pi^1_0$, or equivalently $\Sigma^1_0$, if it does not have any second-order quantifiers, but it may have finitely-many first-order quantifiers and finitely-many first and second-order free variables. Bagaria inductively defined the notion of $\Pi^1_\xi$ formula for any ordinal $\xi$ as follows. A formula is $\Sigma^1_{\xi+1}$ if it is of the form

$$\exists X_0 \cdots \exists X_k \varphi(X_0, \ldots, X_k)$$

where $\varphi$ is $\Pi^1_\xi$, and a formula is $\Pi^1_{\xi+1}$ if it is of the form

$$\forall X_0 \cdots \forall X_k \varphi(X_0, \ldots, X_k)$$

where $\varphi$ is $\Sigma^1_\xi$. If $\xi$ is a limit ordinal, we say that a formula is $\Pi^1_\xi$ if it is of the form

$$\bigwedge_{\zeta < \xi} \varphi_\zeta$$

where $\varphi_\zeta$ is $\Pi^1_\zeta$ for all $\zeta < \xi$ and the infinite conjunction has only finitely-many free second-order variables. We say that a formula is $\Sigma^1_\xi$ if it is of the form

$$\bigvee_{\zeta < \xi} \varphi_\zeta$$

where $\varphi_\zeta$ is $\Sigma^1_\zeta$ for all $\zeta < \xi$ and the infinite disjunction has only finitely-many free second-order variables.

Suppose $\kappa$ is a cardinal and $A$ is a set of ordinals with $\kappa \leq |A|$. We define a two-cardinal version of the cumulative hierarchy up to $\kappa$ as follows:

$$V_0(\kappa, A) = A,$$

$$V_{\alpha+1}(\kappa, A) = P_\kappa(V_\alpha(\kappa, A)) \cup V_\alpha(\kappa, A)$$

and

$$V_\alpha(\kappa, A) = \bigcup_{\beta < \alpha} V_\beta(\kappa, A) \text{ for } \alpha \text{ a limit.}$$

Clearly $V_\kappa \subseteq V_\kappa(\kappa, A)$ and if $A$ is transitive then so is $V_\alpha(\kappa, A)$ for all $\alpha \leq \kappa$. See [10, Section 4] for a discussion of the restricted axioms of ZFC satisfied by $V_\kappa(\kappa, \lambda)$ when $\kappa$ is inaccessible.

**Definition 3.1.** Suppose $\kappa$ is a regular cardinal and $A$ is a set of ordinals with $\kappa \leq |A|$. We say that a set $S \subseteq P_\kappa A$ is $\Pi^1_\xi$-indescribable in $P_\kappa A$ if whenever $(V_\kappa(\kappa, A), \in, R) \models \varphi$ where $k < \omega$, $R \subseteq V_\kappa(\kappa, A)$ and $\varphi$ is a $\Pi^1_\xi$ sentence, there is an $x \in S$ such that

$$x \cap \kappa = |x \cap \kappa| \text{ and } (V_\kappa(\kappa_x, x), \in, R \cap V_\kappa(\kappa_x, x)) \models \varphi.$$  

We define the $\Pi^1_\xi$-indescribability ideal on $P_\kappa A$ to be the collection

$$\Pi^1_\xi(\kappa, A) = \{X \subseteq P_\kappa A \mid X \text{ is not } \Pi^1_\xi\text{-indescribable in } P_\kappa A\}.$$  

---

2Let us note that another notion of transfinite indescribability defined in terms of games was introduced by Welch and Sharpe [36].
Let us note that the first author proved [14] that $\Pi^1_0(\kappa, A) = \text{NSS}_{\kappa, A}$. For notational convenience we let $\Pi^1_{-1}(\kappa, A) = \mathcal{I}_{\kappa, A}$.

Abe proved [1 Lemma 4.1] that $\Pi^1_n(\kappa, A)$ is a strongly normal ideal on $P_\kappa A$ for $n < \omega$ (see [14] for some additional characterizations of $\Pi^1_n(\kappa, A)$). A straightforward generalization of the argument for [4 Proposition 4.4] establishes the following.

**Proposition 3.2.** Suppose $\kappa$ is a regular cardinal and $A$ is a set of ordinals with $\kappa \leq |A|$. Then $\Pi^1_\kappa(\kappa, A)$ is a strongly normal ideal on $P_\kappa A$.

3.2. **Iterating two-cardinal ideal operators.** Given an ideal $I$ on $P_\kappa A$ and an ideal operator $O \in \{\mathcal{I}_\kappa, \mathcal{L}_\kappa, \text{Part}_\kappa, \text{Ram}, \mathcal{Ram}_\kappa\}$ we inductively define a sequence of ideals $O^\alpha(I)$ on $P_\kappa A$ by letting

\[
O^0(I) = I \\
O^{\alpha+1}(I) = O(O^\alpha(I)) \\
O^\alpha(I) = \bigcup_{\beta < \alpha} O^\beta(I) \text{ when } \alpha \text{ is a limit.}
\]

Ideals of the form $\mathcal{I}^\omega_\kappa(\text{NSS}_{\kappa, A})$, $\text{Part}_\kappa^\omega(\text{NSS}_{\kappa, A})$, $\mathcal{I}^\omega_\kappa(\text{NSS}_{\kappa, A})$ and $\text{Part}_\kappa^\omega(\text{NSS}_{\kappa, A})$ were studied by Kamo [29]. In the remainder of the paper we prove several results involving the ideals $O^\alpha(\Pi^1_\kappa(\kappa, A))$ for $\xi < \kappa$, $\alpha < |A|^+$ and $O \in \{\mathcal{I}_\kappa, \text{Ram}_\kappa\}$. For example, recall that Baumgartner proved that the ineffable ideal on a cardinal is equal to the ideal generated by the subtle ideal and the $\Pi^1_\kappa$-indescribability ideal. Generalizing this, we will show that in many cases these ideals $O^\alpha(\Pi^1_\kappa(\kappa, A))$, for $O \in \{\mathcal{I}_\kappa, \text{Ram}_\kappa\}$, can be obtained as the ideal generated by pair of smaller sub-ideals. We will also prove several hierarchy results. For example, it is easy to see that $\beta < \alpha$ implies $O^\beta(I) \subseteq O^\alpha(I)$. We will show that when the ideals involved are nontrivial it follows that $\beta < \alpha < |A|^+$ implies $O^\beta(\Pi^1_\kappa(\kappa, A)) \not\subseteq O^\alpha(\Pi^1_\kappa(\kappa, A))$. We will also show that as $\alpha$ increases, the large cardinal notions associated to $O^\alpha(\Pi^1_\kappa(\kappa, A))$ increase in consistency strength.

3.3. **Generating ideals.** The following lemma is due to Abe [1 Theorem D] in the case $\beta = 1$. Versions of this lemma in the one-cardinal case were first established by Baumgartner [7 Lemma 7.1] and later by the first author [15 Lemma 2.20] as well as the first author and Peter Holy [17].

**Lemma 3.3.** Suppose $S \subseteq P_\kappa A$, $0 < \beta < \kappa$ and for every $(S, \prec)$-list $\bar{S} = \langle S_x \mid x \in S \rangle$ there is a $B \in \bigcap_{\xi < \beta} \Pi^1_\kappa(\kappa, A)^+$ such that $B$ is homogeneous for $\bar{S}$. Then $S$ is a $\Pi^1_{\beta+1}$-indescribable subset of $P_\kappa A$.

**Proof.** Since $\kappa$ is Mahlo, there is a bijection $b : V_\kappa(\kappa, A) \to P_\kappa A$. By [1 Lemma 1.3(4)], the set

\[C_b = \{x \in P_\kappa A \mid b(V_\kappa(\kappa, x)) = P_{\kappa_x} x\}\]

is a weak club in $P_\kappa A$.

We proceed by induction on $\beta$. The base case in which $\beta = 1$ is handled by [1 Theorem D]. The successor case is similar to [1 Theorem D]; we provide details for the reader’s convenience.

Suppose $\beta = \eta + 1$ is a successor ordinal. To show that $S$ is $\Pi^1_{\beta+1}$-indescribable in $P_\kappa A$ it suffices to show that $T = S \cap C_b$ is $\Pi^1_{\beta+1}$-indescribable in $P_\kappa A$. Since $\mathcal{I}_\kappa(\bigcup_{\xi < \beta} \Pi^1_\kappa(\kappa, A))$ is a strongly normal ideal on $P_\kappa A$, our assumption that every
(\(S, \prec\))-list has a homogeneous set in \(P(S) \cap \bigcap_{\xi < \beta} \Pi^1_\xi(\kappa, A)^+\) implies that every 
\((T, \prec\))-list has a homogeneous set in \(P(T) \cap \bigcap_{\xi < \beta} \Pi^1_\xi(\kappa, A)^+\).

Suppose \(R \subseteq V_\kappa(\kappa, A)\) and suppose \(\varphi\) is a \(\Pi^1_\beta+2\) sentence of the form \(\forall X \exists Y \psi\) 
where \(\psi\) is \(\Pi^1_\alpha\) such that 
\[(V_\kappa(\kappa, A), \in, R) \models \forall X \exists Y \psi.\] 
\tag{1}

For contradiction, assume that for each \(x \in T\) we have 
\[(V_{\kappa_x}(\kappa_x, x), \in, R \cap V_{\kappa_x}(\kappa_x, x)) \models \exists X \forall Y \neg \psi.\] 
\tag{2}

For each \(x \in T\) let \(A_x \subseteq V_{\kappa_x}(\kappa_x, x)\) witness \((2)\). Then \(\bar{T} = \langle b[A_x] \mid x \in T \rangle\) is a \((T, \prec\) list, and so by our assumption on \(S\), there is a \(B \in P(T) \cap \bigcap_{\xi < \beta} \Pi^1_\xi(\kappa, A)^+\) homogeneous for \(\bar{T}\). Let \(X^* = \bigcup_{x \in B} A_x\), then by \((1)\), there is a \(Y^* \subseteq V_\kappa(\kappa, A)\) such that 
\[(V_\kappa(\kappa, A), \in, R, X^*, Y^*) \models \psi.\]

Since \(B\) is \(\Pi^1_\beta\)-indescribable and \(\psi\) is a \(\Pi^1_1\) sentence, there is some \(x \in B\) such that 
\[(V_{\kappa_x}(\kappa_x, x), \in, R \cap V_{\kappa_x}(\kappa_x, x), X^* \cap V_{\kappa_x}(\kappa_x, x), Y^* \cap V_{\kappa_x}(\kappa_x, x)) \models \psi.\]

By the homogeneity of \(B\) we have \(X^* \cap V_{\kappa_x}(\kappa_x, x) = A_x\), which contradicts \((2)\).

Next suppose \(\beta\) is a limit ordinal. As before, it suffices to show that \(T = S \cap C_\beta\) 
is \(\Pi^1_\beta+1\)-indescribable. To this end, let \(R \subseteq V_\kappa(\kappa, A)\) and let \(\forall X \bigwedge_{\xi < \beta} \varphi_\xi\) be a \(\Pi^1_\beta+1\) sentence with 
\[(V_\kappa(\kappa, A), \in, R) \models \forall X \bigwedge_{\xi < \beta} \varphi_\xi.\]

For contradiction, suppose that for each \(x \in T\) there is an \(A_x \subseteq V_{\kappa_x}(\kappa_x, x)\) such that 
\[(V_{\kappa_x}(\kappa_x, x), \in, R \cap V_{\kappa_x}(\kappa_x, x), A_x) \models \bigwedge_{\xi < \beta} \neg \varphi_\xi.\] 
\tag{3}

Then \(\bar{T} = \langle b[A_x] \mid x \in T \rangle\) is a \((T, \prec\)-list, hence there is a \(B \in P(T) \cap \bigcap_{\xi < \beta} \Pi^1_\xi(\kappa, A)^+\) homogeneous for \(\bar{T}\). Let \(X^* = \bigcup_{x \in B} A_x\). Then for some \(\xi < \beta\) we have 
\[(V_\kappa(\kappa, A), \in, R, X^*) \models \varphi_\xi,\]

and thus there is an \(x \in B\) such that 
\[(V_{\kappa_x}(\kappa_x, x), \in, R \cap V_{\kappa_x}(\kappa_x, x), X^* \cap V_{\kappa_x}(\kappa_x, x)) \models \varphi_\xi,\]

but this contradicts \((3)\) since \(X^* \cap V_{\kappa_x}(\kappa_x, x) = A_x\). \(\square\)

Next we will show that ideals of the form \(\mathcal{I}_\prec(\Pi^1_\xi(\kappa, A))\) can be obtained as ideals 
generated by a pair of sub ideals, and furthermore, this leads to a characterization of the nontriviality of these ideals. For this result we will need the following two-cardinal notion of subtlety studied by Abe \cite{abe} Definition 2.3.

**Definition 3.4.** Suppose \(\kappa\) is regular and \(\kappa \leq |A|\). A set \(S \subseteq P_\kappa A\) is strongly subtle if for every \((S, \prec\)-list \(\bar{S} = (S_x \subseteq P_\kappa x \mid x \in S)\) and every \(C \in NSS_{\kappa, A}\) there exists \(y, z \in S \cap C\) with \(y \prec z\) and \(S_y = S_z \cap P_\kappa y\). We let 
\[NSS_{\kappa, A} = \{X \subseteq P_\kappa A \mid X\ is \not\ strongly \ subtle\}.\]

Among other things, Abe proved \cite{abe} Proposition 2.5(1) that \(NSS_{\kappa, A}\) is a strongly normal ideal on \(P_\kappa A\).
Theorem 3.5. For all $n < \omega$, we have
\[
\mathcal{I}_x(\Pi_n^1(\kappa, A)) = \text{NSSub}_{\kappa,A} \cup \Pi_{n+2}^1(\kappa, A)\]
Furthermore, it is not the case the $P_\kappa A \notin \mathcal{I}_x(\Pi_n^1(\kappa, A))$ is equivalent to $P_\kappa A \notin \text{NSSub}_{\kappa,A}$ and $P_\kappa A \notin \Pi_{n+2}^1(\kappa, A)$, because if $\kappa$ is the least cardinal such that there is an $A$ with $\kappa \subseteq A$, $\kappa \leq |A|$ and $P_\kappa A \notin \mathcal{I}_x(\Pi_n^1(\kappa, A))$ then there is an $x \in P_\kappa A$ such that $P_\kappa x$ is strongly subtle and $\Pi_{n+2}^1$-indescribable and yet $P_\kappa x \in \mathcal{I}_x(\Pi_n^1(\kappa, x))$.

Proof. Let $I = \text{NSSub}_{\kappa,A} \cup \Pi_{n+2}^1(\kappa, A)$. We show that $S \in \mathcal{I}_x(\Pi_n^1(\kappa, A))^+$ if and only if $S \in I^+$.

Suppose $S \in \mathcal{I}_x(\Pi_n^1(\kappa, A))^+$. To show $S \in I^+$, it suffices to show $S$ is strongly subtle and $\Pi_{n+2}^1$-indescribable. Clearly $S$ is strongly subtle, and by Lemma 3.3 we know $S$ is $\Pi_{n+2}^1$-indescribable. Thus $S \in I^+$. Conversely, suppose $S \in I^+$. For the sake of contradiction, suppose $S \in \mathcal{I}_x(\Pi_n^1(\kappa, A))$. Then there is an $(S, \prec)$-list $\vec{S} = \langle S_x \mid x \in S \rangle$ such that every homogeneous set for $\vec{S}$ is in the ideal $\Pi_n^1(\kappa, A)$. This is expressible by a $\Pi_{n+2}^1$-sentence $\varphi$ over $(V_\kappa(\kappa), e, \vec{S})$. Thus it follows that the set
\[
C = \{ x \in P_\kappa A \mid (V_{\kappa_x}(\kappa_x, x), e, \vec{S} \cap V_{\kappa_x}(\kappa_x, x)) \models \varphi \} = \{ x \in P_\kappa A \mid \text{every hom. set for } \vec{S} \restriction (P_\kappa x \cap S) \text{ is in } \Pi_n^1(\kappa_x, x) \}
\]
is in the filter $\Pi_{n+2}^1(\kappa, A)^*$. Since $S \in I^+$, it follows that $S$ is not equal to the union of a non-strongly subtle set and a non-$\Pi_{n+2}^1$-indescribable set. Since $S = (S \cap C) \cup (S \setminus C)$ and $S \cap C \subseteq \Pi_{n+2}^1(\kappa, A)$, it follows that $S \cap C$ must be strongly subtle. As a direct consequence of [2 Theorem B], there is some $x \in S \cap C$ for which there is an $H \subseteq S \cap C \cap P_\kappa x$ which is $\Pi_n^1$-indescribable in $P_\kappa x$ and homogeneous for $\vec{S}$. This contradicts $x \in C$.

For the remaining statement, let $\kappa$ be the least cardinal such that there is an $A$ with $P_\kappa A \notin \mathcal{I}_x(\Pi_n^1(\kappa, A))$. We show there are many $x \in P_\kappa A$ for which $P_\kappa x$ is both subtle and $\Pi_{n+2}^1$-indescribable. The fact that $P_\kappa A$ is strongly subtle can be expressed by a $\Pi_1$-sentence $\varphi$ over $V_\kappa(\kappa, A)$ and thus the set
\[
C = \{ x \in P_\kappa A \mid (V_{\kappa_x}(\kappa_x, x), e) \models \varphi \} = \{ x \in P_\kappa A \mid P_\kappa x \text{ is strongly subtle} \}
\]
is in the filter $\Pi_1^1(\kappa, A)^* \subseteq \Pi_{n+2}^1(\kappa, A)^*$. Furthermore, by [2 Lemma 3.8] the set
\[
H = \{ x \in P_\kappa A \mid P_\kappa x \text{ is } \Pi_{n+2}^1 \text{-indescribable} \}
\]
is in the filter $\text{NSSub}_{\kappa,A}^*$. Since $\mathcal{I}_x(\Pi_n^1(\kappa, A)) \supseteq \text{NSSub}_{\kappa,A} \cup \Pi_{n+2}^1(\kappa, A)$, it follows that $C \cap H$ is in the filter $\mathcal{I}_x(\Pi_n^1(\kappa, A))^*$.

Remark 3.6. The previous theorem can be generalized to ideals of the form $T_\alpha^2(\Pi_n^1(\kappa, A))$, where $\alpha, \xi < \kappa$ as is done in [15] and [17]. For example, to obtain $T_\alpha^2(\Pi_n^1(\kappa, A))$ as the ideal generated by two proper sub-ideals, one must replace the strongly subtle ideal $\text{NSSub}_{\kappa,A}$ with an ideal defined using a pre-operator. The details are left to the interested reader.

Next, in order to prove a version of Theorem 3.5 for $\text{Ram}_x(\Pi_n^1(\kappa, A))$, we introduce another new large cardinal notion and an associated ideal. The following

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3Given a collection $A \subseteq P(X)$, where $X$ is some set, we write $\mathcal{A}$ to denote the ideal on $X$ generated by $A$. The set $X$ will be clear from the context.
definition can be viewed as a generalization of the notion of pre-Ramseyness introduced in \[8\] and later studied in \[13, 15, 17, 23\].

**Definition 3.7.** Suppose \(\kappa\) is regular and \(\kappa \leq |A|\). Further suppose that \(\bar{I} = \langle I_x \mid x \in P_\kappa A\rangle\) is a function such that for each \(x \in P_\kappa A\), \(I_x\) is an ideal on \(P_\kappa x\). We define an ideal \(\text{Ram}_{\prec}^\preceq(\bar{I})\) on \(P_\kappa A\) by letting \(S \in \text{Ram}_{\prec}^\preceq(\bar{I})^+\) if and only if for every \(\prec\)-regressive function \(f: [S]^{\omega} \rightarrow P_\kappa A\) and every \(C \in \text{NSS}_{\kappa,A}\) there is some \(x \in S \cap C\) such that there is an \(H \in P(S \cap C \cap P_\kappa x) \cap I_x^+\) homogeneous for \(f\).

In the case where the ideals listed by the function \(\bar{I}\) have a uniform definition, we will often use the notation \(\text{Ram}_{\prec}^\preceq(\bar{I}) = \text{Ram}_{\prec}^\preceq(I)\), where \(I_{\kappa,A}\) is the relevant ideal on \(P_\kappa A\). For example, if \(\bar{I} = \langle \text{NSS}_{\kappa,x} \mid x \in P_\kappa A\rangle\), when we write \(\text{Ram}_{\prec}^\preceq(\text{NSS}_{\kappa,A})\) we mean \(\text{Ram}_{\prec}^\preceq(I)\).

**Theorem 3.8.** For all \(n < \omega\),

\[
\text{Ram}_{\prec}^\preceq(\Pi_n^1(\kappa, A)) = \overline{\text{Ram}_{\prec}^\preceq(\Pi_n^1(\kappa, A))} \cup \Pi_{n+2}^1(\kappa, A).
\]

Furthermore, it is not the case that \(P_\kappa A \notin \text{Ram}_{\prec}^\preceq(\Pi_n^1(\kappa, A))\) is equivalent to \(P_\kappa A \notin \overline{\text{Ram}_{\prec}^\preceq(\Pi_n^1(\kappa, A))}\) and \(P_\kappa A \notin \Pi_{n+2}^1(\kappa, A)\), because if \(\kappa\) is the least cardinal such that there is an \(A\) with \(\kappa \leq |A|\), \(\kappa \leq |A|\) and \(P_\kappa A \notin \text{Ram}_{\prec}^\preceq(\Pi_n^1(\kappa, A))\), then there is an \(x \in P_\kappa A\) such that the ideals \(\text{Ram}_{\prec}^\preceq(\Pi_n^1(\kappa, x))\) and \(\Pi_n^1(\kappa, x)\) are nontrivial and yet \(P_\kappa x \in \text{Ram}_{\prec}^\preceq(\Pi_n^1(\kappa, x))\).

The proof of Theorem 3.8 is similar to that of Theorem 3.5, the only difference being that one must work with regressive functions or \((\omega, S, \prec)\)-lists instead of \((S, \prec, \kappa)\)-lists. For similar results in the one-cardinal context see \[15, 17\].

### 3.4. Hierarchy results

In this section we prove several hierarchy results concerning ideals of the form \(I_\alpha(\Pi_\xi^0(\kappa, A))\) and \(\text{Ram}_{\prec}^\preceq(\Pi_\xi^0(\kappa, A))\), where \(\kappa\) is regular, \(A\) is a set of ordinals with \(\kappa \leq |A|\), \(\xi < \kappa\) and \(\alpha < |A|^+\). In order to handle cases in which \(\alpha > \kappa\), let us briefly outline some important properties of canonical functions that we will require (see \[24\] Section 2.6 and \[6\] Section 2).

Given ordinal valued functions \(f\) and \(g\) with domain \(P_\kappa A\) we write \(f \sim g\) if and only if \(\{x \in P_\kappa A \mid f(x) = g(x)\}\) contains a club, and similarly for \(f \leq g\) and \(f < g\). It is easy to see that \(\sim\) is an equivalence relation, \(\leq\) is transitive and reflexive and that \(\prec\) is transitive and well-founded. For each \(f\) we let \(\|f\|\) be the rank of \(f\) with respect to \(\prec\). We say that such a function \(f\) is *canonical* if and only if for every \(g\), \(\|f\| \leq \|g\|\) implies \(f \leq g\); in other words, \(f\) is canonical if it is minimal in the \(\leq\) ordering among all ordinal-valued functions on \(P_\kappa A\) of the same rank. Notice that when \(f\) is canonical, \(\|f\| < \|g\|\) easily implies that \(f < g\).

**Lemma 3.9.** Suppose \(\kappa\) is a regular uncountable cardinal and \(A\) is a set of ordinals with \(\kappa \leq |A|\). There is a sequence \(\langle f_\alpha \mid \alpha < |A|^+\rangle\) of ordinal-valued functions defined on \(P_\kappa A\) such that for all \(\alpha < |A|^+\) it follows that

1. \(f_\alpha\) is a canonical function with rank \(\alpha\),
2. whenever \(x \in P_\kappa A\) is such that \(x \cap \kappa\) is regular and uncountable we have \(f_\alpha \mid P_{x \cap \kappa} x\) is canonical on \(P_{x \cap \kappa} x\) of rank \(f_\alpha(x)\) and
3. the set \(\{x \in P_\kappa A \mid f_\alpha(x) < |x|^+\}\) is club in \(P_\kappa A\).

The proof of Lemma 3.9 is standard and is left to the reader. For example, Baldwin established the existence of a sequence \(\langle f_\alpha \mid \alpha < |A|^+\rangle\) satisfying 3.9(1).
Lemma 3.12. Suppose $P$ on $A$ for all. Suppose $S$ is a set of ordinals with $\kappa \leq |A|$, $\xi < \kappa$, $\alpha < |A|^+$ and $\mathcal{O} \in \{\mathcal{I}_\preceq, \mathcal{R}am_\prec\}$. If $S \in \mathcal{O}^\circ(\Pi^1_\xi(\kappa, A))^+$ and for each $x \in S$ we have a set $S_x \in O^f(x)(\Pi^1_\xi(\kappa, x))^+$, then $\bigcup_{x \in S} S_x \in \mathcal{O}^\circ(\Pi^1_\xi(\kappa, A))^+$.

Proof. We provide a proof for the case in which $\mathcal{O} = \mathcal{R}$; the case in which $\mathcal{O} = \mathcal{I}$ is essentially the same, only one must replace regressive functions by lists.

Suppose $\alpha = 0$. Suppose $S \in \Pi^1_\xi(\kappa, A)^+$ and for each $x \in S$ we have $S_x \in \Pi^1_\xi(\kappa, x)^+$. We must show that $\bigcup_{x \in S} S_x \in \Pi^1_\xi(\kappa, A)^+$. Fix $R \subseteq V_\chi(\kappa, A)$ and let $\varphi$ be a $\Pi^1_\xi$ sentence such that $(V_\chi(\kappa, A), \in, R) \models \varphi$. Since $S \in \Pi^1_\xi(\kappa, A)^+$, there is an $x \in S$ such that $(V_\chi_\kappa(\kappa, x), \in, R \cap V_\chi_\kappa(\kappa, x)) \models \varphi$. Now since $S_x \in \Pi^1_\xi(\kappa, x)^+$, there is a $y \in S_x$ such that $(V_\chi_\kappa(\kappa, y), \in, R \cap V_\chi_\kappa(\kappa, y)) \models \varphi$. Hence $\bigcup_{x \in S} S_x \in \Pi^1_\xi(\kappa, A)^+$.

Now, suppose $\alpha = \eta + 1 > 0$ is a successor ordinal and the result holds for $\eta$. Fix a $\prec$-regressive function $f : \bigcup_{x \in S} S_x \subseteq \omega \to P_\kappa A$. Fix a club $C_0 \subseteq P_\kappa A$ such that $x \in C_0$ implies $f_\alpha(x) = f_\alpha(x) + 1$. By assumption, for each $x \in S \cap C_0$ we have $S_x \in \mathcal{R}am^{f_\alpha(x)+1}(\Pi^1_\xi(\kappa, x))^+$, and thus there is a set $H_x \in P(S) \cap \mathcal{R}am^{f_\alpha(x)}(\Pi^1_\xi(\kappa, x))^+$ homogeneous for $f \upharpoonright [S_x]^{<\omega} \to P_\kappa A$. Since $S \in \mathcal{R}am^{\omega}(\Pi^1_\xi(\kappa, A))^+$, it easily follows that the $(S, \prec)$-sequence $\vec{H} = \langle H_x \upharpoonright x \in S \rangle$ has a homogeneous set $H \in P(S) \cap \mathcal{R}am^{\omega}(\Pi^1_\xi(\kappa, A))^+$ (just extend the $(S, \prec)$-sequence to any $(\omega, S, \prec)$-sequence). By our inductive hypothesis, $\bigcup_{x \in H} H_x \in \mathcal{R}am^{\omega}(\Pi^1_\xi(\kappa, A))^+$. Now it is easy to verify that $\bigcup_{x \in H} H_x$ is homogeneous for $f$.

If $\alpha$ is a limit ordinal and the result holds for ordinals less than $\alpha$, it is easy to verify that the result holds for $\alpha$ using the fact that $\mathcal{R}am^{\omega}(\Pi^1_\xi(\kappa, A)) = \bigcup_{\eta < \alpha} \mathcal{R}am^{\omega}(\Pi^1_\xi(\kappa, A))$.

To prove a hierarchy result (Theorem 3.13), we need the following.

Lemma 3.11. Suppose $\kappa$ is a regular uncountable cardinal and $A$ is a set of ordinals with $\kappa \leq |A|$. The following properties of canonical functions on $P_\kappa A$ hold.

(a) If $f \leq g$ and $g \leq f$ then $\{x \in P_\kappa A \mid f(x) = g(x)\}$ is in the club filter on $P_\kappa A$.

(b) If $f$ and $g$ are both canonical on $P_\kappa A$ and $\|f\| = \|g\|$ then $f$ and $g$ are equal on a club.

(c) If $f$ is canonical on $P_\kappa A$ and $g(x) = f(x) + 1$ for club-many $x \in P_\kappa A$, then $g$ is canonical and $\|g\| = \|f\| + 1$.

(d) If $\langle f_\gamma \mid \gamma \in A \rangle$ is a sequence of canonical functions on $P_\kappa A$ and $f$ is an ordinal-valued function on $P_\kappa A$ defined by $f(x) = \bigcup_{\eta \in x} f_\eta(x)$, then $f$ is canonical and $\|f\| = \bigcup_{\eta \in A} \|f_\eta\|$.

Lemma 3.12. Suppose $\kappa$ is a regular uncountable cardinal, $A$ is a set of ordinals with $\kappa \leq |A|$, $\xi < \kappa$, $\alpha < |A|^+$ and $\mathcal{O} \in \{\mathcal{I}_\preceq, \mathcal{R}am_\prec\}$. If $P_\kappa A \in \mathcal{O}^\circ(\Pi^1_\xi(\kappa, A))^+$
where \( \alpha < |A|^+ \), then the set
\[
X_\alpha = \{ x \in P_\kappa A \mid P_{\kappa_x} x \in \mathcal{O}^{f_\alpha(x)}(\Pi^1_\xi(\kappa_x, x)) \}
\]
is in \( \mathcal{O}^\alpha(\Pi^1_\xi(\kappa, A))^+ \).

**Proof.** We will prove this for \( \mathcal{O} = \text{Ram}_{< \xi} \); the case in which \( \mathcal{O} = \text{I}_{< \xi} \) is similar.

We follow [23, Theorem 5.2] and proceed by induction on \( \kappa \). We assume the result holds for all cardinals less than \( \kappa \) and prove that it holds for \( \kappa \). If
\[
S = \{ x \in P_\kappa A \mid P_{\kappa_x} x \in \text{Ram}^{f_\alpha(x)}_{< \xi}(\Pi^1_\xi(\kappa_x, x))^+ \}
\]
is in \( \text{Ram}^\alpha_{< \xi}(\Pi^1_\xi(\kappa, A)) \), then \( X_\alpha \in \text{Ram}^\alpha_{< \xi}(\Pi^1_\xi(\kappa, A))^* \) and we are done. So, we assume that \( S \in \text{Ram}^\alpha_{< \xi}(\Pi^1_\xi(\kappa, A))^+ \).

For each \( z \in P_\kappa A \), we let \( \langle f^z_\eta \mid \eta < |z|^+ \rangle \) denote a sequence of canonical functions defined on \( P_{\kappa_x} z \) satisfying conditions analogous to Lemma 3.9(1)-(3). Let \( C_\alpha \subseteq P_\kappa A \) be a club such that for all \( z \in C_\alpha \) the following properties hold:

1. \( z \cap \kappa < \kappa \),
2. \( f_\alpha(z) < |z|^+ \),
3. when \( z \cap \kappa \) is a regular uncountable cardinal we have that \( f_\alpha \restriction P_{\kappa_x} z \) is a canonical function on \( P_{\kappa_x} z \) of rank \( f_\alpha(z) \), and thus \( f_\alpha \restriction P_{\kappa_x} z = f^z_{f_\alpha(z)} \).

Let us show that for each \( z \in C_\alpha \cap S \) with \( \kappa_z > \xi \), the set \( X_\alpha \cap P_{\kappa_x} z \) is in \( \text{Ram}^{f^z_{f_\alpha(z)}}_{< \xi}(\Pi^1_\xi(\kappa_x, z))^+ \); then, by Lemma 3.10, it will follow that \( X_\alpha \in \text{Ram}^\alpha_{< \xi}(\Pi^1_\xi(\kappa, A))^+ \).

Fix \( z \in C_\alpha \cap S \). Notice that \( \kappa_z < \kappa \) and \( f_\alpha(z) < |z|^+ \). Thus, by our inductive hypothesis, the set
\[
\{ x \in P_{\kappa_x} z \mid P_{\kappa_x} x \in \text{Ram}^{f^z_{\alpha(x)}}_{< \xi}(\Pi^1_\xi(\kappa_x, x)) \}
\]
is in \( \text{Ram}^{f^z_{f_\alpha(z)}}_{< \xi}(\Pi^1_\xi(\kappa_x, z))^+ \). But, since \( z \in C_\alpha \) we have \( f^z_{f_\alpha(z)}(x) = f_\alpha(x) \), and thus the set
\[
X_\alpha \cap P_{\kappa_x} z = \{ x \in P_{\kappa_x} z \mid P_{\kappa_x} x \in \text{Ram}^{f_\alpha(x)}_{< \xi}(\Pi^1_\xi(\kappa_x, x)) \}
\]
is in \( \text{Ram}^{f_\alpha(x)}_{< \xi}(\Pi^1_\xi(\kappa_x, z))^+ \).

\( \Box \)

**Theorem 3.13.** Suppose \( \kappa \) is a regular uncountable cardinal, \( A \) is a set of ordinals with \( \kappa \leq |A| \), \( \xi < \kappa \), \( \alpha < |A|^+ \) and \( \mathcal{O} \in \{ \text{I}_{< \xi}, \text{Ram}_{< \xi} \} \). If \( P_\kappa A \in \text{Ram}^{\alpha+1}_{< \xi}(\Pi^1_\xi(\kappa, A))^+ \), then for all \( \beta \leq \alpha \) and all sets \( X \in \text{Ram}^{\beta}_{< \xi}(\Pi^1_\xi(\kappa, A))^+ \), it follows that the set
\[
\{ x \in P_\kappa A \mid X \cap P_{\kappa_x} x \in \text{Ram}^{f_\beta(x)}_{< \xi}(\Pi^1_\xi(\kappa_x, x))^+ \}
\]
is in the filter \( \text{Ram}^{\beta+1}_{< \xi}(\Pi^1_\xi(\kappa, A))^* \).

**Proof.** We give a proof for the case \( \mathcal{O} = \text{Ram}_{< \xi} \). The proof for \( \mathcal{O} = \text{I}_{< \xi} \) is similar, but uses lists instead of \( \prec\)-regressive functions.

Following [23, Theorem 5.3], we proceed by induction on \( \beta \).

Suppose \( \beta = 0 \). The assumption that \( P_\kappa A \in \text{Ram}^{\alpha+1}_{< \xi}(\Pi^1_\xi(\kappa, A))^+ \) implies that \( P_\kappa A \) is \( \Pi^1_{\xi+1} \)-indescribable, and since the fact that \( X \in \text{Ram}^{\alpha}_{< \xi}(\Pi^1_\xi(\kappa, A))^+ = \Pi^1_{\xi}(\kappa, A))^+ \) is expressible by a \( \Pi^1_{\xi+1} \) sentence over \( (V_\kappa(\kappa, A), \in, X) \), it follows the set
\[
\{ x \in P_\kappa A \mid X \cap P_{\kappa_x} x \in \Pi^1_\xi(\kappa_x, x)^+ \}
\]
is in the filter \( \Pi^1_{\xi+1}(\kappa, A)^* \subseteq \text{Ram}_{< \xi}(\Pi^1_\xi(\kappa, A))^* \) (the last containment follows from Lemma 3.3).
Suppose $\beta = \eta + 1$. Let $C_0 \subseteq P_{\kappa}A$ be a club such that $x \in C_0$ implies $f_\beta(x) = f_\eta(x) + 1$. Suppose $X \in \text{Ram}_\xi^\beta(\Pi_\xi^1(\kappa,A))$. By our inductive hypothesis the set

$$\{ x \in C_0 \mid X \cap P_{\kappa}x \in \text{Ram}_\xi^{f_\eta(x)}(\Pi_\xi^1(\kappa_A,x)) \}$$

is in the filter $\text{Ram}_\xi^{\eta+1}(\Pi_\xi^1(\kappa,A))^+$ and is hence also in the filter $\text{Ram}_\xi^{\beta+1}(\Pi_\xi^1(\kappa,A))^+$. 

Now let

$$T = \{ x \in P_{\kappa}A \mid X \cap P_{\kappa}x \in \text{Ram}_\xi^{f_\eta(x)+1}(\Pi_\xi^1(\kappa_A,x)) \}.$$ 

It will suffice to show that $T \in \text{Ram}_\xi^{\beta+1}(\Pi_\xi^1(\kappa,A))$. For a contradiction, suppose $T \notin \text{Ram}_\xi^{\beta+1}(\Pi_\xi^1(\kappa,A))^+$. Then the set

$$Y = \{ x \in C_0 \mid X \cap P_{\kappa}x \in \text{Ram}_\xi^{f_\eta(x)}(\Pi_\xi^1(\kappa_A,x)) \} \cap \text{Ram}_\xi^{f_\eta(x)+1}(\Pi_\xi^1(\kappa_A,x))$$

is in $\text{Ram}_\xi^{\beta+1}(\Pi_\xi^1(\kappa,A))^+$. For each $x \in Y$, let $g_x : [X \cap P_{\kappa}x]^\omega \rightarrow P_{\kappa}x$ be a $\prec$-regressive function with no homogeneous set in $\text{Ram}_\xi^{f_\eta(x)}(\Pi_\xi^1(\kappa_A))^+$. 

Fix a bijection $b : (P_{\kappa}A) \times (\text {Ram}_\xi^{f_\eta(x)}(\Pi_\xi^1(\kappa_A))^+) \rightarrow P_{\kappa}A$ and note that the set

$$C_1 = \{ x \in P_{\kappa}A \mid b[P_{\kappa}x \times P_{\kappa}x] = P_{\kappa}x \}$$

is a weak club in $P_{\kappa}A$. Now let $Z = Y \cap C_1$. For each $x \in Z$ let $Z_x = b[g_x] \subseteq P_{\kappa}x$. This defines a $\kappa$-list $\bar{Z} = \{ Z_x \mid x \in Z \}$. By assumption, there is a set $B \in \text{Ram}_\xi^{f_\eta(\kappa_A)}(\Pi_\xi^1(\kappa_A))^+$ homogeneous for $\bar{Z}$. Let $f = \bigcup \{ g_x \mid x \in B \}$. Then $f$ is $\prec$-regressive on $[X]^\omega$. 

Since $X \in \text{Ram}_\xi^{\eta+1}(\Pi_\xi^1(\kappa_A))^+$, there is an $H \in P(X) \cap \text{Ram}_\xi^{\eta}(\Pi_\xi^1(\kappa_A))^+$ homogeneous for $f$. By induction, the set

$$\{ x \in P_{\kappa}A \mid H \cap P_{\kappa}x \in \text{Ram}_\xi^{f_\eta(x)}(\Pi_\xi^1(\kappa_A)) \}$$

is in $\text{Ram}_\xi^{\eta+1}(\Pi_\xi^1(\kappa_A))^+$. Choose $x \in B$ such that $H \cap P_{\kappa}x \in \text{Ram}_\xi^{f_\eta(x)}(\Pi_\xi^1(\kappa_A))^+$. Then $f \restriction [X \cap P_{\kappa}x]^\omega = g_x$ and $H \cap P_{\kappa}x$ is homogeneous for $g_x$, a contradiction.

Now suppose $\beta \leq \alpha$ is a limit ordinal and $X \subseteq P_{\kappa}A$ is in $\text{Ram}_\xi^{\beta}(\Pi_\xi^1(\kappa_A))^+$. If $\eta < \beta$ then $X \in \text{Ram}_\xi^{\eta}(\Pi_\xi^1(\kappa_A))^+$ since $\text{Ram}_\xi^{\beta}(\Pi_\xi^1(\kappa_A))^+ = \bigcup_{\eta < \beta} \text{Ram}_\xi^{\eta}(\Pi_\xi^1(\kappa_A))^+$. Thus, by our inductive hypothesis, for each $\eta < \beta$ the set

$$D_\eta = \{ x \in P_{\kappa}A \mid X \cap P_{\kappa}x \in \text{Ram}_\xi^{f_\eta(x)}(\Pi_\xi^1(\kappa_A))^+ \}$$

is in the filter $\text{Ram}_\xi^{\eta+1}(\Pi_\xi^1(\kappa_A))^+$. Thus, each $D_\eta$ is in the filter $\text{Ram}_\xi^{\beta}(\Pi_\xi^1(\kappa_A))^+$ and thus also in $\text{Ram}_\xi^{\beta+1}(\Pi_\xi^1(\kappa_A))^+$, which is nontrivial and strongly normal. By normality, the set

$$\Delta_{\eta < \beta} D_\eta = \{ x \in P_{\kappa}A \mid x \in \bigcap_{\eta \in \varepsilon} D_\eta \}$$

is in the filter $\text{Ram}_\xi^{\beta+1}(\Pi_\xi^1(\kappa_A))^+$. Applying Lemma 3.13(d) to the sequence $\langle f_\eta \mid \eta < \beta \rangle$ (using a reindexing if necessary), it follows that the function $x \mapsto \bigcup_{\eta \in \varepsilon} f_\eta(x)$ is canonical on $P_{\kappa}A$ of rank $\bigcup_{\eta < \beta} \| f_\eta \| = \beta$. Therefore, the set

$$C = \{ x \in P_{\kappa}A \mid \bigcup_{\eta \in \varepsilon} f_\eta(x) = f_\beta(x) \}$$
is club in $P_\kappa A$. Hence the set $C \cap \triangle_{\eta<\beta} D_\eta$, which is contained in
\[
\{ x \in P_\kappa A \mid X \cap P_\kappa x \in \text{Ram}_{f^\beta(x)}(\prod_1(\kappa_x, x))^+ \},
\]
is in the filter $\text{Ram}_{\beta+1}(\prod_1(\kappa, A))^+$.

\[\square\]

**Corollary 3.14.** Suppose $\kappa$ is a regular uncountable cardinal, $A$ is a set of ordinals with $\kappa \leq |A|$, $\xi < \kappa$, $\alpha < |A|^+$ and $O \in \{ \mathcal{L}_\zeta, \text{Ram}_\zeta \}$. If the ideal $\mathcal{O}^\alpha(\prod_1(\kappa, A))$ is nontrivial then
\[
\mathcal{O}^\alpha(\prod_1(\kappa, A)) \subseteq \mathcal{O}^{\alpha+1}(\prod_1(\kappa, A)).
\]

Next we will generalize a theorem of Baumgartner \[7\] and results of the first author and Peter Holy \[17\], by proving a theorem which establishes, among other things, that the existence of cardinals $\kappa \leq \lambda$ such that $\mathcal{I}_\zeta^2(I_\kappa, \lambda)$ is nontrivial is strictly stronger in consistency strength than the existence of cardinals $\kappa \leq \lambda$ for which $\mathcal{I}_\zeta(\prod_1(\kappa, \lambda))$ is nontrivial for all $\beta < \kappa$. This theorem strengthens Theorem 3.13 in the case where $O = \mathcal{L}_\zeta$. See the comments after the proof of Theorem 3.15 for more information on generalizing Theorem 3.5 to $\text{Ram}_\zeta$.

**Theorem 3.15.** Suppose $\kappa$ is a regular uncountable cardinal, $A$ is a set of ordinals with $\kappa \leq |A|$, $\alpha < |A|^+$ and $S \in \mathcal{I}_\zeta^{\alpha+1}(I_\kappa, A)^+$. Suppose $\bar{S} = \langle S_x \mid x \in S \rangle$ is an $(S, \prec)$-list. Let
\[
Z = \{ x \in S \mid (\exists x \subseteq S \cap P_\kappa x)(\forall \beta < \kappa_\bar{x} X \in \mathcal{I}_\zeta^{f^\beta(x)}(\prod_1(\kappa_x, x))^+) \land (X \cup \{ x \} \text{ is hom. for } \bar{S}) \}
\]
Then $S \setminus Z \in \mathcal{I}_\zeta^{\alpha+1}(I_\kappa, A)$.

**Proof.** We proceed by induction on $\alpha < |A|^+$. The case in which $\alpha = 0$ follows directly from an argument given by Abe \[2\] Lemma 3.8], which is a straightforward generalization of Baumgartner’s \[17\] Theorem 4.1]; the arguments for the successor case and the limit case are similar. Let us provide a proof for the successor case. The interested reader may easily piece together a proof of the limit case by consulting the following successor case and the detailed arguments in \[17\].

Suppose $\alpha = \delta + 1 < |A|^+$ is a successor ordinal, and suppose for a contradiction that $S \setminus Z \in \mathcal{I}_\zeta^{\delta+2}(I_\kappa, A)^+$. By Lemma 3.11 we may let $C$ be a club subset of $P_\kappa A$ such that $x \in C$ implies $f_{\beta+1}(x) = f_{\beta}(x) + 1$. The set
\[
E = \{ x \in S \setminus Z \mid \kappa_x \text{ is inaccessible} \} \cap C
\]
is in $\mathcal{I}_\zeta^{\delta+2}(I_\kappa, A)^+$. For each $x \in E$, let $B_x = \{ y \in S \cap P_\kappa x \mid S_y = S_x \cap P_\kappa y \}$. Since $B_x \cup \{ x \}$ is homogeneous for $\bar{S}$ and $x \in S \setminus Z$, there is an ordinal $\xi_x < \kappa_x$ such that $B_\alpha \in \mathcal{I}_\zeta^{f^\alpha(x)}(\prod_1(\kappa_x, x))$, and hence we may fix a $(B_x, \prec)$-list $\bar{B}_x = \langle b^x_y \mid y \in B_x \rangle$ such that $\bar{B}_x$ has no homogeneous set in $\mathcal{I}_\zeta^{f^\alpha(x)}(\prod_1(\kappa_x, x))^+$.

Since $E \in \mathcal{I}_\zeta^{\delta+2}(I_\kappa, A)^+$, there is an $H \in P(E) \cap \mathcal{I}_\zeta^{\delta+1}(I_\kappa, A)^+$ such that whenever $y \prec x$ and $x, y \in H$ we have $S_y = S_x \cap P_\kappa y$, $B_y = B_x \cap P_\kappa y$ and $\bar{B}^y = \bar{B}_x | y \in B_x$. Let $D = \bigcup_{x \in H} S_x$, $B = \bigcup_{x \in H} B_x$ and $\bar{B} = \bigcup_{x \in H} \bar{B}_x = \langle b^x_y \mid x \in B \rangle$. Since $B = \{ x \in P_\kappa A \mid S_x = D \cap P_\kappa x \}$, it follows that $H \subseteq B$.

Now let $A_0$ be the set of all $x \in H$ such that there is an $X \subseteq P \cap P_\kappa x$ such that
\[
(\forall \xi < \kappa_x X \in \mathcal{I}_\zeta^{f^\xi(x)}(\prod_1(\kappa_x, x))^+) \land (X \cup \{ x \} \text{ is hom. for } \bar{B})
\]
By our inductive hypothesis, $H \setminus A_0 \in \mathcal{I}_{\delta+1}(I_{\kappa,A})$, and hence $A_0 \in \mathcal{I}_{\delta+1}(I_{\kappa,A})^+$. Thus, there is an $x \in A_0$. Since $x \in H$, it follows by homogeneity that $\vec{B} \upharpoonright (H \cap P_{\kappa,x}) = \vec{B}^x \upharpoonright H$. But, by the definition of $A_0$, and since $\xi_x < \kappa_x$, there is some $X \in P(H \cap P_{\kappa,x}) \cap \mathcal{I}_{\delta}(\Pi_{\xi_x}(\kappa_x,x))^+$ which is homogeneous for $\vec{B}^x$, a contradiction. □

At the time of writing this article, the authors did not know whether Theorem 3.15 holds if we replace $\mathcal{I} \prec \mathcal{R}$ with $\mathcal{R} \prec \mathcal{S}$ and $\vec{S}$ with an $(\omega, S, \prec)$-list. In fact, at that time, it was not known whether the corresponding result holds for the single cardinal Ramsey operator. See [17] for a detailed discussion about the problems involved with generalizing Theorem 3.15 to the Ramsey operator in the one-cardinal case. Since the current article was written, the first author, Lambie-Hanson and Zhang proved theorems analogous to Theorem 3.15 for both the single cardinal Ramsey operator and $\mathcal{R}$ (see [18, Section 4]).

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