Indeterminacy relations in random dynamics

Piotr Garbaczewski

Institute of Physics, University of Opole, 45-052 Opole, Poland

We analyze various uncertainty measures for spatial diffusion processes. In this manifestly non-quantum setting, we focus on the existence issue of complementary pairs whose joint dispersion measure has strictly positive lower bound.

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I. CONCEPTUAL BACKGROUND

We take inspiration from the classic meta-theorem in harmonic analysis: “a non-zero function and its Fourier transform cannot both be sharply localized”. This statement lies at the core of standard quantum-mechanical position-momentum indeterminacy relationship in $L^2(\mathbb{R}^n)$, but its range of validity extends to time-frequency indeterminacy measures which are employed in the classical signal analysis: Fourier transform is here a key element.

We look for a fairly distant analogue of the above “no simultaneous sharp localization” statement in the theory of spatial diffusion processes (Wiener process and Smoluchowski processes as examples) whose density functions belong to $L^1(\mathbb{R})$ and there is no notion of momentum observable, nor any physically digestible notion of momentum.

Nonetheless, in this non-quantum setting, we shall address the existence issue for pairs of complementary dispersion measures that mimic the previously mentioned meta-relationship and so preclude an arbitrarily sharp localization for both. The peculiar point of our analysis will be a disregard of any Fourier transform input.

Let us consider continuous probability densities on the real line, with or without an explicit time-dependence: $\rho \in L^1(R); \int_R \rho(x) \, dx = 1$. Our minimal demand is that the first and second moments of each density are finite, so that we can introduce a two-parameter family $\rho_{\alpha,\sigma}(x)$, labeled by the mean value $\langle x \rangle = \int x \rho(x) \, dx = \alpha \in R$ and the standard deviation (here, square root of the variance) $\sigma \in \mathbb{R}^+$, $\sigma^2 = \langle (x - \langle x \rangle)^2 \rangle$.

We assume that $\rho(x)$ admits suitable differentiability properties and impose the natural boundary data at finite or infinite (employed in below) integration boundaries.

Let there be given a one-parameter $\alpha$-family of densities whose mean square deviation
value is fixed at \(\sigma\). By introducing the mean value:

\[
\langle [\sigma^2 \nabla \ln \rho + (x - \langle x \rangle)]^2 \rangle \geq 0
\]

we readily arrive at an inequality

\[
\mathcal{F}(\rho) = \langle (\nabla \ln \rho)^2 \rangle = \int \frac{(\nabla \rho)^2}{\rho} \, dx \geq \frac{1}{\sigma^2}
\]

in which a minimum of \(\mathcal{F}\) is achieved if and only if \(\rho\) is a \(\sigma\)-Gaussian, compare e.g.\(^2,3\). The functional \(\mathcal{F}(\rho)\) is often named the Fisher information associated with \(\rho\).

The above Eq. (2) actually associates a primordial "momentum-position" indeterminacy relationship (here, devoid of any quantum connotations) with the probability distributions under consideration. Namely, let \(D\) be a positive diffusion constant with dimensions of \(\hbar/2m\) or \(k_B T/m \beta\), c. f.\(^6\). We define an auxiliary (named osmotic) velocity field \(u = u(x) = D \nabla \ln \rho\). There holds:

\[
\Delta x \cdot \Delta u \geq D
\]

which correlates the position variance \(\Delta x = \langle [x - \langle x \rangle]^2 \rangle^{1/2}\) with the osmotic velocity variance \(\Delta u = \langle [u - \langle u \rangle]^2 \rangle^{1/2}\). In the above, \(mD\) may be interpreted as the lower bound for the joint position-momentum dispersion measure; at least on formal grounds, \(m\Delta u\) carries a dimension of a physical momentum variable.

This property extends to time-dependent situations and is known to be respected by diffusion-type processes\(^4\). Its primary version for the free Brownian motion has been found by R. Fürth\(^3\).

Given \(\rho(x)\) and a suitable function \(f(x)\), we can readily generalize the previous arguments. Let us introduce notions of a variance and covariance (here, directly borrowed from the random variable analysis\(^5\)) for \(x\) and \(f(x)\). By means of the Schwarz inequality, we get:

\[
\langle [x - \langle x \rangle]^2 \rangle \cdot \langle [f - \langle f \rangle]^2 \rangle \geq \left( \langle [x - \langle x \rangle] \cdot [f - \langle f \rangle] \rangle \right)^2,
\]

hence, accordingly

\[
Var(x) \cdot Var(f) \geq Cov^2(x, f).
\]

We note that for an osmotic velocity field \(u(x)\), we have \(\langle u \rangle = 0\) and \(\langle x \cdot u \rangle = -D\). Therefore

\[
Var(x) \cdot Var(u) \geq Cov^2(x, u) = D^2,
\]

with a dispersion bound \(D^2\), as anticipated in Eq. (3).

The problem is that a careful selection of a function \(f(x)\) is necessary, if we expect \(Cov^2(x, f)\) to set a definite (fixed) lower bound for the joint dispersion measure, like \(Cov^2(x, u) = D^2\) does in the above.
II. DYNAMICS

Let us consider spatial diffusion processes in one space dimension, like e.g. standard Smoluchowski processes and their generalizations. Let there be given $\dot{x} = b(x,t) + A(t)$ with $\langle A(s) \rangle = 0$, $\langle A(s)A(s') \rangle = \sqrt{2D} \delta(s - s')$ and the corresponding Fokker-Planck equation for the probability density $\rho \in L^1(R)$ which we analyze under the natural boundary conditions:

$$\partial_t \rho = D \Delta \rho - \nabla \cdot (b \rho) .$$  \hspace{1cm} (7)

We assume the gradient form for the forward drift $b = b(x,t)$ and take $D$ as a diffusion constant with dimensions of $k_B T/m\beta$. By introducing $u(x,t) = D \nabla \ln \rho(x,t)$ we define the current velocity of the process $v(x,t) = b(x,t) - u(x,t)$, in terms of which the continuity equation $\partial_t \rho = -\nabla \cdot (v \rho)$ follows. The diffusion current reads $j = v \rho$.

Automatically, we have an indeterminacy relationship $\text{Var}(x) \cdot \text{Var}(u) \geq \text{Cov}^2(x,u) = D^2$. The corresponding inequality for the current velocity field $\text{Var}(x) \cdot \text{Var}(v) \geq \text{Cov}^2(x,v)$, does not involve any obvious lower bound.

The cumulative identity $\text{Var}(x) \cdot [\text{Var}(u) + \text{Var}(v)] \geq \text{Cov}^2(x,v) + D^2$, reproduced in Ref.\textsuperscript{5}, does not convey any illuminating message about the diffusion process. It cannot be directly inferred from the Fisher functional $\mathcal{F}(\rho)$, if $\rho$ has a non-quantum origin: the major obstacle at this point is that there is no diffusive analogue of the quantum momentum observable.

Let us also mention another attempt\textsuperscript{7} to set an uncertainty principle for general diffusion processes. If adopted to our convention (natural boundary data), in view of $\langle u \rangle = 0$ and $v = b - u$, we have $\langle v \rangle = \langle b \rangle$.

For an arbitrary real constant $C \neq 0$, we obviously have: $[C \cdot (v - \langle v \rangle) + (x - \langle x \rangle)]^2 \geq 0$. The mean value of this auxiliary inequality reads:

$$C^2(\Delta v)^2 + 2C[C \cdot \text{Cov}(x,b) + D] + (\Delta x)^2 \geq 0 .$$  \hspace{1cm} (8)

and is non-negative for all $C$, which enforces a condition

$$[D + \text{Cov}(x,b)]^2 - (\Delta v)^2 \cdot (\Delta x)^2 \leq 0 .$$  \hspace{1cm} (9)

Note that $\text{Cov}(x,v) = D + \text{Cov}(x,b)$, so we have in fact an alternative derivation of the previous indeterminacy relationship $\text{Var}(x) \cdot \text{Var}(v) \geq \text{Cov}^2(x,v)$ for the current velocity field. The problem of the existence (or not) of a lower bound for the joint dispersion measure of $x$ and $v$ has been left untouched.

This observation extends to standard Smoluchowski processes, whose forward drifts are proportional to externally imposed force fields, typically through $b = F/m\beta$. Therefore the position-current velocity dispersion correlation is controlled by $\text{Cov}(x,F)$. For the free
Brownian motion (e.g., the Wiener process) we have \( b = 0 \), and hence \( \text{Cov}(x,v) = D \) is a genuine lower bound.

**III. INDETERMINACY MEASURES FOR DIFFUSION PROCESSES**

We begin from a classic observation that, once we set \( b = -2D\nabla\Phi \) with \( \Phi = \Phi(x,t) \), a substitution:

\[
\rho(x,t) \doteq \theta_*(x,t) \exp[-\Phi(x,t)]
\]

with \( \theta_* \) and \( \Phi \) being real functions, converts the Fokker-Planck equation Eq. ([7]) into a generalized diffusion equation for \( \theta_* \):

\[
\partial_t \theta_* = D\Delta \theta_* - \frac{\mathcal{V}(x,t)}{2mD} \theta_*
\]

and its formal time adjoint (admitting also a more familiar interpretation of a consistency condition, i.e. of an indirect definition of the potential \( \mathcal{V} \))

\[
\partial_t \theta = -D\Delta \theta + \frac{\mathcal{V}(x,t)}{2mD} \theta
\]

which is valid for a real function \( \theta(x,t) = \exp[-\Phi(x,t)] \).

Accordingly, we deduce

\[
\frac{\mathcal{V}(x,t)}{2mD} = -\partial_t \Phi + \frac{1}{2} \left( \frac{b^2}{2D} + \nabla \cdot b \right) = -\partial_t \Phi + D[(\nabla \Phi)^2 - \Delta \Phi].
\]

There holds an obvious factorization property for the Fokker-Planck probability density:

\[
\rho(x,t) = \theta(x,t) \cdot \theta_*(x,t).
\]

**Remark 1:** Notice an affinity with a familiar quantum mechanical factorization formula \( \rho = \psi^*\psi \), albeit presently realized in terms of two real functions \( \theta \) and \( \theta^* \), instead of a complex conjugate pair. This issue is elucidated in some detail in the Appendix.

We have:

\[
\mathcal{F} \left( \rho \right) = \frac{1}{D^2} \sigma_u^2 = \int dx (\theta \theta^*) \left[ \frac{\nabla \theta}{\theta} + \frac{\nabla \theta^*}{\theta^*} \right]^2 = 4 \int dx \nabla \theta_\ast \nabla \theta + \int dx (\theta \theta^*) \left[ \frac{\nabla \theta}{\theta} - \frac{\nabla \theta^*}{\theta^*} \right]^2.
\]

Since a continuity equation \( \partial_t \rho = -\nabla j \) is identically fulfilled by

\[
j(x,t) = \rho(x,t)v(x,t) = D(\theta_\ast \nabla \theta - \theta \nabla \theta_\ast)
\]
we obviously get (after an integration by parts and accounting for the generalized diffusion equations):

\[ F(\rho) = 4 \int dx (\nabla \theta)(\nabla \theta^*) + \frac{1}{D^2} \langle v^2 \rangle = -\frac{4}{D} \langle \partial_t \Phi \rangle - \frac{2}{mD^2} \langle \mathcal{V} \rangle + \frac{1}{D^2} \langle v^2 \rangle. \]  \hspace{1cm} (17)

The \( \langle \partial_t \Phi \rangle \) contributions cancel away once we invoke an explicit expression for \( \mathcal{V} \). Therefore, we may simplify our further discussion by assuming that \( \partial_t \Phi = 0 \) identically. This amounts to passing to Smoluchowski diffusion processes.

**Remark 2:** Let us indicate a formal similarity of our reasoning for diffusion-type processes to that developed by J-C. Zambrini\(^8,9\) in the framework of the so-called Euclidean quantum mechanics. In the latter approach, another (Euclidean) version of the uncertainty relations was introduced, based on accepting a skew-adjoint operator \(-2mD\nabla\) as a Euclidean version of a quantum momentum observable \(-i2mD\nabla\). To establish a corresponding Heisenberg indeterminacy relation one needs to accept a hypothesis that \( 0 < \int dx (\nabla \theta)(\nabla \theta^*) = -(1/2mD^2)\mathcal{V} < \infty \) in Eq. (17). This property clearly is not respected by the free Brownian motion where \( \mathcal{V} = 0 \) and, in view of Eq. (13), may surely be violated by drifted Brownian motions.

We have (compare e.g.\(^6\)):

\[ F(\rho = \theta \theta^*) = \frac{2}{mD^2} \langle \frac{mv^2}{2} - \mathcal{V} \rangle \Rightarrow \langle \frac{mv^2}{2} - \frac{mu^2}{2} - \mathcal{V} \rangle = 0. \]  \hspace{1cm} (18)

The variances of osmotic and current velocity fields are correlated as follows

\[ \rho = \theta \theta^* \Rightarrow m^2[(\Delta u)^2 - (\Delta v)^2] = 2m\langle \frac{mv^2}{2} - \mathcal{V} \rangle. \]  \hspace{1cm} (19)

On the left-hand-side of Eq. (19), there appears a difference of variances for the current and osmotic velocity fields. This expression is not necessarily positive definite, unless \( \langle \mathcal{V} \rangle \leq 0 \) for all times.

Let us make a guess that \( \Delta u > \Delta v \), in the least locally in time (in a finite time interval). Then, the resulting expression

\[ m^2(\Delta u)^2 = m^2\langle u^2 \rangle = 2m\langle \frac{mv^2}{2} - \mathcal{V} \rangle \geq (\Delta p_u)^2 \geq \frac{m^2D^2}{\sigma^2}, \]  \hspace{1cm} (20)

as we already know, yields a dimensionally acceptable position-momentum indeterminacy relationship for diffusion-type processes,

\[ \Delta x \cdot \Delta p_u \geq mD, \]  \hspace{1cm} (21)
where $\Delta p_u > 0$ may be interpreted as the pertinent "momentum dispersion" measure. For the free Brownian motion we have $\mathcal{V} = 0$ and $v = -u$, hence Eq. (3) is recovered.

Upon making an opposite guess i.e. admit $\Delta v > \Delta u$ (again, at least locally in time), in view of $\mathcal{F} \geq 1/\sigma^2$, we would have

$$m^2(\Delta v)^2 = m^2(\Delta u)^2 + 2m[\langle \mathcal{V} \rangle - \frac{m(v)^2}{2}] \geq (\Delta p_v)^2 \geq \frac{m^2 D^2}{\sigma^2}$$

and thus

$$\Delta x \cdot \Delta p_v \geq mD$$

would ultimately arise.

The above two indeterminacy options (21) and (23) are a consequence of a possibly indefinite sign for a difference $\Delta u - \Delta v$ of standard deviations, in the course of a diffusion process. This sign issue seems to be a local in time property and may not persist in the asymptotic (large time) regime. We shall give an argument towards a non-existence of a fixed positive lower bound for the joint position-current velocity uncertainty measure in the vicinity of an asymptotic stationary solution of the involved Fokker-Planck equation.

The diffusive potential $\mathcal{V}$ is not quite arbitrary and has a pre-determined functional form, Eq. (13). Our general restriction on $\mathcal{V}$ is that it should be a continuous and bounded from below function. In the diffusive case this demand guarantees that $\exp(-tH)$ with $H = -D\Delta + (1/2mD)\mathcal{V}$ is a legitimate dynamical semigroup operator, such that $\theta_s(t = 0) = \theta_{s0} \rightarrow \theta_s(t) = \exp(-Ht)\theta_{s0}$. That would suggest that we may expect $\langle \mathcal{V} \rangle \geq 0$, which however typically is not the case.

We cannot dwell on the general issue of entropy methods in the study of the large time asymptotic for solutions of the diffusion equations. In case of Smoluchowski diffusion processes we may take for granted that they asymptotically approach unique stationary solutions, for which the current velocity $v$ identically vanishes. Then $\Delta v = 0$ as well, while $0 < \text{Var}(x) < \infty$ (e.g. $\Delta x$ stays finite).

In view of Eq. (13), an asymptotic value of the strictly positive Fisher functional $\mathcal{F}$ equals $-(2/mD^2)\langle \mathcal{V} \rangle > 0$. Accordingly, to secure $\mathcal{F} > 0$, an expectation value of $\mathcal{V}$ with respect to the stationary probability density must be negative. Even, under an assumption that $\mathcal{V}$ is bounded from below.

Consequently, in the large time asymptotic we surely have $(\Delta u)^2 > (1/\sigma^2) > (\Delta v)^2$ and $\Delta v \rightarrow 0$, while $\sigma$ has a finite limiting value (an exception is the free Brownian motion when $\sigma$ diverges). The validity of the above argument can be checked by inspection, after invoking an explicit solution for the Ornstein-Uhlenbeck process$^{6,10}$.

Thus, $\Delta x \cdot \Delta p_v \geq mD$ does not hold true in the vicinity of the asymptotic solution. On the contrary, $\Delta x \cdot \Delta p_u \geq mD$ is universally valid.
IV. APPENDIX: QUANTUM INDETERMINACY

Given an $L^2(R)$-normalized function $\psi(x)$. We denote $(F\psi)(p)$ its Fourier transform. The corresponding probability densities follow: $\rho(x) = |\psi(x)|^2$ and $\tilde{\rho}(p) = |(F\psi)(p)|^2$. We introduce the related position and momentum information (differential, e.g. Shannon) entropies:

\[ S(\rho) \equiv S_q = -\langle \ln \rho \rangle = - \int \rho(x) \ln \rho(x) \, dx \] (24)

and

\[ S(\tilde{\rho}) \equiv S_p = -\langle \ln \tilde{\rho} \rangle = - \int \tilde{\rho}(p) \ln \tilde{\rho}(p) \, dp \] (25)

where $S$ denotes the Shannon entropy for a continuous probability distribution. For the sake of clarity, we use dimensionless quantities, although there exists a consistent procedure for handling dimensional quantities in the Shannon entropy definition.

We assume both entropies to take finite values. Then, there holds the familiar entropic uncertainty relation:

\[ S_q + S_p \geq (1 + \ln \pi). \] (26)

If following conventions we define the squared standard deviation value for an observable $A$ in a pure state $\psi$ as $(\Delta A)^2 = (\psi, [A - \langle A \rangle]^2 \psi)$ with $\langle A \rangle = (\psi, A\psi)$, then for the position $X$ and momentum $P$ operators we have the following version of the entropic uncertainty relation (here expressed through so-called entropy powers, $\hbar \equiv 1$):

\[ \Delta X \cdot \Delta P \geq \frac{1}{2\pi e} \exp[S(\rho) + S(\tilde{\rho})] \geq \frac{1}{2} \] (27)

which is an alternative version of the entropic uncertainty relation. For Gaussian densities, $(2\pi e)\Delta X \cdot \Delta P = \exp[S(\rho) + S(\tilde{\rho})]$ holds true, but the minimum 1/2 on the right-hand-side of Eq. (27), is not necessarily reached.

Let us notice that in view of properties of the Fourier transform, there is a complete symmetry between the inferred information-theory functionals. After the Fourier transformation, taking into account the entropic uncertainty relation Eq. (26), we arrive at:

\[ 4\tilde{\sigma}^2 \geq 2(e\pi)^{-1} \exp[-2\langle \ln \tilde{\rho} \rangle] \geq (2e\pi) \exp[2\langle \ln \rho \rangle] \geq \sigma^{-2} \] (28)

Let us consider a momentum operator $P$ that is conjugate to the position operator $X$ in the adopted dimensional convention $\hbar \equiv 1$. Setting $P = -i\partial/\partial x$ and presuming that all averages are finite, we get:

\[ \langle P^2 \rangle - \langle P \rangle^2 = (\Delta P)^2 = \tilde{\sigma}^2. \] (29)

The standard indeterminacy relationship $\sigma \cdot \tilde{\sigma} \geq (1/2)$ follows.
In the above, no explicit time-dependence has been indicated, but all derivations go through with any wave-packet solution $\psi(x,t)$ of the Schrödinger equation. The induced dynamics of probability densities may imply the time-evolution of entropies: $S_q(t), S_p(t)$ and thence the dynamics of quantum uncertainty measures $\Delta X(t) = \sigma(t)$ and $\Delta P(t) = \tilde{\sigma}(t)$.

We consider the Schrödinger equation in the form:

$$i\partial_t \psi = -D\Delta \psi + \frac{V}{2mD}\psi. \quad (30)$$

where the potential $V = V(\vec{x}, t)$ (possibly time-dependent) is a continuous (it is useful, if bounded from below) function with dimensions of energy, $D = \hbar/2m$.

By employing the Madelung decomposition:

$$\psi = \rho^{1/2} \exp(is/2D), \quad (31)$$

with the phase function $s = s(x,t)$ defining a (current) velocity field $v = \nabla s$, we readily arrive at the continuity equation

$$\partial_t \rho = -\nabla (v\rho) \quad (32)$$

and the generalized Hamilton-Jacobi equation:

$$\partial_t s + \frac{1}{2}(\nabla s)^2 + (\Omega - Q) = 0 \quad (33)$$

where $\Omega = V/m$ and, after introducing an osmotic velocity field $u(x,t) = D\nabla \ln \rho(x,t)$ we have, compare e.g. our discussion of Section I:

$$Q = 2D^2 \frac{\Delta \rho^{1/2}}{\rho^{1/2}} = \frac{1}{2}u^2 + D \nabla \cdot u. \quad (34)$$

If a quantum mechanical expectation value of the standard Schrödinger Hamiltonian $\hat{H} = -(\hbar^2/2m)\Delta + V$ exists (i.e. is finite$^{12}$),

$$\langle \psi|\hat{H}|\psi \rangle \doteq E < \infty \quad (35)$$

then the unitary quantum dynamics warrants that this value is a constant of the Schrödinger picture evolution:

$$\mathcal{H} = \frac{1}{2}[\langle v^2 \rangle + \langle u^2 \rangle] + \langle \Omega \rangle = -\langle \partial_t s \rangle \doteq \mathcal{E} = \frac{E}{m} = \text{const.}. \quad (36)$$

Let us notice that $\langle u^2 \rangle = -D\langle \nabla u \rangle$ and therefore:

$$\frac{D^2}{2} \mathcal{F} = \frac{D^2}{2} \int \frac{1}{\rho} \left( \frac{\partial \rho}{\partial x} \right)^2 dx = \int \rho \cdot \frac{u^2}{2} dx = -\langle Q \rangle. \quad (37)$$
We observe that $D^2\mathcal{F}$ stands for the mean square deviation value of a function $u(x,t)$ about its mean value $\langle u \rangle = 0$, whose vanishing is a consequence of the boundary conditions (here, at infinity):

$$\langle [u - \langle u \rangle]^2 \rangle = \sigma_u^2 = (\Delta u)^2 = D^2\mathcal{F}. \quad (38)$$

The mean square deviation of $v(x,t)$ about its mean value $\langle v \rangle$ reads:

$$\langle [v - \langle v \rangle]^2 \rangle = \sigma_v^2 = (\Delta v)^2 = \langle v^2 \rangle - \langle v \rangle^2. \quad (39)$$

It is clear, that with the definition $P = -i(2mD)d/dx$, the mean value of the operator $P$ is related to the mean value of a function $v(x,t)$ (we do not discriminate between technically different implementations of the mean): $\langle P \rangle = m\langle v \rangle$. Accordingly,

$$\tilde{\sigma}^2 = (\Delta P)^2 = \langle P^2 \rangle - \langle P \rangle^2 \quad (40)$$

Moreover, we can directly check that with $\rho = |\psi|^2$ there holds\[11:\]

$$\mathcal{F}(\rho) = \frac{1}{D^2}\sigma_u^2 = \int dx |\psi|^2[\psi'(x)/\psi(x) + \psi^*(x)/\psi^*(x)]^2 =$$

$$4 \int dx \psi^*(x)\psi'(x) + \int dx |\psi(x)|^2[\psi'(x)/\psi(x) - \psi^*(x)/\psi^*(x)]^2 =$$

$$\frac{1}{m^2D^2}(\langle P^2 \rangle - m^2\langle v^2 \rangle) = \frac{1}{m^2D^2}[(\Delta P)^2 - m^2\sigma_v^2]$$

i.e.

$$m^2(\sigma_u^2 + \sigma_v^2) = \tilde{\sigma}^2. \quad (42)$$

It is interesting to notice that $\langle (P - mv) \rangle = 0$ and the corresponding mean square deviation reads: $\langle (P - mv)^2 \rangle = \langle P^2 \rangle - m^2\langle v^2 \rangle = m^2D^2\mathcal{F}$.

By passing to dimensionless quantities in Eqs. \[11\] (e.g. $2mD \equiv 1$), and denoting $p_{cl} = (\arg \psi(x,t))'$ we get:

$$\mathcal{F} = 4[\langle P^2 \rangle - \langle p_{cl}^2 \rangle] = 4[(\Delta P)^2 - (\Delta p_{cl})^2] = 4[\tilde{\sigma}^2 - \tilde{\sigma}_{cl}^2] \quad (43)$$

and therefore:

$$4\tilde{\sigma}^2 \geq 4[\tilde{\sigma}^2 - \tilde{\sigma}_{cl}^2] = \mathcal{F} \geq (2\pi e) \exp[-2S(\rho)] \geq \frac{1}{\tilde{\sigma}^2}. \quad (44)$$

We recall that all "tilde" quantities can be deduced from the once given $\psi$ and its Fourier transform $\tilde{\psi}$.

As a side comment let us add that a direct consequence of the mean energy conservation law Eq. \[36\] are identities: $\langle P^2 \rangle/2m = E - \langle V \rangle$ and

$$\mathcal{F} = \frac{1}{m^2D^2}[\langle P^2 \rangle - m^2\langle v^2 \rangle] = \frac{1}{D^2}[2(\mathcal{E} - \langle \Omega \rangle) - \langle v^2 \rangle] \quad (45)$$
plus a complementary expression for the variance of the momentum observable:

\[(\Delta P)^2 = 2m(E - \langle m(v)^2 + V \rangle).\]  \hspace{1cm} (46)

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