SEIFERT’S CONJECTURE FOR ALMOST SYMPLECTIC FOLIATIONS

SAUVIK MUKHERJEE

Abstract. By disproving Seifert’s conjecture for almost symplectic foliations we show that the closedness of the leaves is not an obstruction for the $h$-principle for symplectic foliations on closed manifolds. Obviously one needs to replace the homotopy of almost symplectic foliations by concordance of almost symplectic foliations through almost symplectic singular foliations.

1. INTRODUCTION

Let $Gr_{2n}(M) \to M$ be the grassmann bundle on $M^{2n+q}$, i.e $\pi^{-1}(x) = Gr_{2n}(T_x M)$. Identify $Fol_q(M)$, the space of codimension-$q$ foliations on $M$ as a subspace of $\Gamma(Gr_{2n}(M))$, where the section space $\Gamma(Gr_{2n}(M))$ is given the compact open topology. As usual $\Omega^k(M)$ be the space of $k$-forms on $M$ with $C^\infty$-topology. Define

$$\Delta_q(M) \subset Fol_q(M) \times \Omega^2(M)$$

$$\Delta_q(M) = \{(F, \omega) : \omega^n_{|F} \neq 0\}$$

In this setting Fernandes and Frejlich has proved in [4] the following

Theorem 1.1. Let $M^{2n+q}$ be an open manifold and $(F, \omega) \in \Delta_q(M)$. Then there exists a homotopy $(F_t, \omega_t) \in \Delta_q(M)$, for $t \in I$ such that the restriction of $\omega_1$ on the leaves of $F_1$ defines a symplectic structure on the leaves, i.e, $d_{F_t} \omega_1 = 0$.

In the above $d_F$ is the tangential exterior derivative operator and $H^k(F)$ tangential derham cohomology of a foliated manifold $(M, F)$ as in [1]. A contact analogue of [1.1] has been proved in [2]. Observe that if $H^2(F_1)$ is trivial in [1.1] then $F_1$ can not have a closed leaf. Same is true for any symplectic foliation. So in order to prove the analogue of [1.1] on closed manifolds one needs to open the leaves of the foliation through almost symplectic (singular) foliations.

Key words and phrases. Almost Symplectic Foliations, $h$-principle.
Definition 1.2. Two codimension-$q$ foliations $\mathcal{F}_i$, $i = 0, 1$ on $M^{2n+q}$ are said to be concordant if there exists a foliation $\mathcal{F}$ on $M \times I$ such that $\mathcal{F} \cap M \times \{i\}$, for $i = 0, 1$ and inducing $\mathcal{F}_i$ on $M \times \{i\}$. If $\mathcal{F} \cap M \times \{t\}$, for all $t \in I$ then it is called an integrable homotopy. Integrable homotopy preserves closed leaves.

Now we state the main result of this paper. Let $\tilde{\Delta}_q(M)$ be the space of all pairs $(\mathcal{F}, \omega) \in \Delta_q(M)$ such that $\mathcal{F}$ does not have a closed leaf.

Theorem 1.3. (Main Theorem) Let $q \geq 3$ and $(\mathcal{F}_0, \omega_0)$ be a pair in $\tilde{\Delta}_q(M)$ then there exists a pair $(\mathcal{F}_1, \omega_1) \in \tilde{\Delta}_q(M)$ together with a homotopy of two forms $\omega_t$ joining $\omega_i$, $i = 0, 1$ and a concordance $\mathcal{F}$ between $\mathcal{F}_i$, $i = 0, 1$ such that $(\omega_t)^n_{T(\mathcal{F}_t - \Sigma_t)} \neq 0$, where $\mathcal{F}_t = \mathcal{F} \cap M \times \{t\}$ and $\Sigma_t$ is the singular locus of $\mathcal{F}_t$, i.e, $\Sigma_t$ is the subset of $M \times \{t\}$ where $\mathcal{F}$ fails to be $\cap$ to $M \times \{t\}$.

We end this section with the definition of a minimal set which is a key ingredient in our proof.

Definition 1.4. (6) Let $M$ be a manifold with a foliation $\mathcal{F}$, a set $S \subset M$ is called saturated with respect to $\mathcal{F}$ if for all $x \in S$ the leaf through $x$ in contained in $S$. $S$ is called minimal if it is nonempty, closed, saturated and minimal in a sense that it does not contain any proper subset with all these properties.

Example 1.5. Consider the torus $\mathbb{T}^2$ with coordinates $(a, b)$. Let $\mathcal{F}$ be the foliation on $\mathbb{T}^2$ defined by the vector field $\partial_a + c\partial_b$, where $c$ is irrational. Then $\mathbb{T}^2$ is the minimal set for $\mathcal{F}$.

2. PASSING TO THE LOCAL MODEL

In our proof we shall follow the methods in [6]. In [7] the following result has been proved and which allows us to reduce the original problem of proving 1.3 to splitting the leaves of the product foliation on the products of discs.

Theorem 2.1. (7) Let $M^{2n+q}$ be a smooth manifold together with a codimension-$q$ foliation $\mathcal{F}$ on it then there exists a family of embeddings $f_\lambda : \mathbb{D}^q \times \mathbb{D}^{2n} \to M$, $\lambda \in \Lambda$ such that the family of sets $\{f_\lambda(\mathbb{D}^q \times \mathbb{D}^{2n}) : \lambda \in \Lambda\}$ is locally finite, mutually disjoint. Moreover $f^{-1}_\lambda(\mathcal{F})$ has leaves $\{x\} \times \mathbb{D}^{2n}$ and each leaf of $\mathcal{F}$ intersects at least one of $f_\lambda(\mathbb{D}^{1/2}_q \times \mathbb{D}^{2n})$, where $\mathbb{D}^{1/2}_q$ is the $q$-disc of radius $1/2$. 
We shall consider \((f^{-1}_\lambda(F), f^*_\lambda \omega_0)\) on \(D^q \times D^{2n}\). Obviously \((f^{-1}_\lambda(F), f^*_\lambda \omega_0) \in \Delta_q(D^q \times D^{2n})\). As in [6] we shall split the leaves of \(f^{-1}_\lambda(F)\) whose leaves are \(\{x\} \times D^{2n}, \ x \in D^q\) by a concordance. This concordance will be fixed near the boundary of \(D^q \times D^{2n}\). Then we shall construct a homotopy of two forms \(\omega_t\) starting from \(f^{-1}_\lambda(\omega_0) = \omega'\) satisfying the conditions of [5] and \(\omega_t = \omega'\), on \(Op(\partial(D^q \times D^{2n}))\). This would suffice according to [21].

3. h-PRINCIPAL FOR THE 2-FORM

According to section 2 it is enough to consider \((\mathcal{F}_0, \omega_0)\) on \(D^q \times D^{2n}\) where \(\mathcal{F}_0\) is given by \(\{x\} \times D^{2n}, \ x \in D^q\) and \(\omega_0\) is such that \((\omega_0)_{\mathcal{F}_0}^n \neq 0\). In the first step of the homotopy \((\mathcal{F}_t, \omega_t)\), we shall keep \(\mathcal{F}_t\) to be fixed, i.e., \(\mathcal{F}_t = \mathcal{F}_0\) and homotop \(\omega_0\) to \(\omega_1\) so that

1. \((\mathcal{F}_t, \omega_t) \in \Delta_q(D^q \times D^{2n})\)

2. the restriction of \(\omega_1\) to \(D^q \times D^{2n}_1\) is \(0 \oplus (\Sigma^n \pm (dx_i \wedge dy_i))\)

**Construction:** Observe that the restriction of \(\omega_0\) to \(D^q \times D^{2n}_{1/2}\) is of the form

\[
\Sigma^n_i f_i dx_i \wedge dy_i + \Sigma_{i,j} g_{ij} dx_i \wedge dx_j + \Sigma_{i,j} h_{ij} dy_i \wedge dy_j + \Omega
\]

where \(f_i\)'s are always non-zero and is either positive or negative throughout \(D^{2n}_{1/2} \times D^q\). So we can homotop \(\omega_0\) in \(\Gamma(\mathcal{R})\) to the desired form by making \(f_i\)'s +1 or −1 depending on the sign of \(f_i\)'s and making \(g_{ij}, h_{ij}\)'s and \(\Omega\) zero.

So without loss of generality we can keep our model \(D^q \times D^{2n}\) instead of the modified one \(D^q \times D^{2n}_1\).

4. FINAL STEP

As \(q \geq 3\) there exists an embedding \(T^2 \hookrightarrow D^q\). Let \(N(T^2)\) be a small tubular neighborhood of \(T^2\) in \(D^q\). So there exists a diffeomorphism \(e : T^2 \times D^{q-2} \to N(T^2)\). So now consider \((e \times id_{D^{2n}})^* \omega_1 = 0 \oplus (\Sigma^n_i \pm dx_i \wedge dy_i)\). Call this new form \(\omega'_1\) on \(T^2 \times D^{q-2} \times D^{2n}\). Also set \(\mathcal{F}'_1 = (e \times id_{D^{2n}})^{-1} \mathcal{F}_1\). So obviously \((\mathcal{F}'_1, \omega'_1) \in \Delta_q(T^2 \times D^{q-2} \times D^{2n})\) and hence this is the model now.

Now let \(Z\) be a vector field on \(T^2\) for which \(T^2\) is the minimal set. Let \(Z' = (Z, 0)\) be the corresponding vector field on \(T^2 \times D^{q-2}\). Let \(\psi : T^2 \times D^{q-2} \times [0, 1] \to [0, 1] \) be
a smooth function with compact support and supported in $T^2 \times D^{q-2} \times (0,1)$ such that $\psi^{-1}(1) = T^2 \times D^{q-2}_{1/2} \times \{1/2\}$. Set

$$X_1 = (1 - \psi) \partial_s + \psi Z'$$

where $s$ is the variable on $[0,1]$. Observe that as $\psi$ is compactly supported in $T^2 \times D^{q-2} \times (0,1)$, so $X_1 = \partial_s$ near $T^2 \times D^{q-2} \times \{1\}$. So we can extend $X_1$ to all of $T^2 \times D^{q-2} \times [0, \infty)$. We shall denote this new extended vector field by $X_1$ itself. Set $X_t = (1 - t) \partial_s + t X_1$.

Consider the map $\rho : D^{2n} \to [0, \infty)$ given by $\rho(z) = \frac{1}{|z|} - 1$. So

$$id \times \rho : T^2 \times D^{q-2} \times D^{2n} \to T^2 \times D^{q-2} \times [0, \infty)$$

is a submersion on $T^2 \times D^{q-2} \times (D^{2n} - \{0\})$. Observe that $d\rho_z = -\frac{1}{|z|^2} (x_1, y_1, ..., x_n, y_n)$, where $0 \neq z = (x_1, y_1, ..., x_n, y_n)$. Let $F_1$ be the foliation on $T^2 \times D^{q-2} \times [0, \infty)$ defined by the vector field $X_t$. Now $(id \times \rho)^{-1} F_1$ induces a foliation on $T^2 \times D^{q-2} \times (D^{2n} - \{0\})$, where $\rho_0$ is the restriction of $\rho$ to $D^{2n} - \{0\}$. This foliation agrees with $F_1$ near $T^2 \times D^{q-2} \times \{0\}$ and hence extends to a foliation $F_2$ on $T^2 \times D^{q-2} \times D^{2n}$. The homotopy from $F_1$ to $F_2$ is defined by pulling back the codimension-$(q+1)$ foliation defined by $X_t$ on $T^2 \times D^{q-2} \times [0, \infty) \times [0,1]$.

Now we shall construct the homotopy of the two forms $\omega_{1+t}$ starting at $\omega_1$ and satisfying the conditions of [1.3]. This will complete the proof of [1.3].

Observe that $Y = \frac{1}{2} \Sigma_i (x_i \partial_{x_i} + y_i \partial_{y_i})$ is a Liouville vector field of $\Sigma_1 (\pm dx_i \wedge dy_i)$ on $D^{2n}$, i.e., $d(i_Y (\Sigma_1 (\pm dx_i \wedge dy_i))) = \Sigma_1 (\pm dx_i \wedge dy_i)$, where $i_Y$ is the contraction by the vector field $Y$. Set $\alpha = i_Y (\Sigma_1 (\pm dx_i \wedge dy_i))$. Let $H$ be any hyperplane in $R^{2n}$ transversal to $Y$ then by 1.4.5 of [5] we have $(\alpha \wedge (dx)^n)|_H \neq 0$.

Observe $T F_{1+t} = (d(id \times \rho))^{-1}(X_t)$ where $X_t = (1 - t \psi) \partial_s + t \psi Z'$. Let $A = (a_1, b_1, ..., a_n, b_n) \in R^{2n}$ then for $w \in T^2 \times D^{q-2}$ and $z = (x_1, y_1, ..., x_n, y_n) \in D^{2n}$,

$$(T F_{1+t})(w,z) = \{ A + t \psi Z' : -\frac{1}{|z|^2} \Sigma (a_i x_i + b_i y_i) = t(1 - \psi) \}$$

Now $T F_{1+t}$ defines a hyperplane $H_t(z)$ in $R^{2n}$ simply by making the $Z'$-component zero which according to the above is given by

$$H_t(z) = \{ A \in R^{2n} : -\frac{1}{|z|^2} \Sigma (a_i x_i + b_i y_i) = t(1 - \psi) \}$$
So if $Y \in H_t(z)$ then $-\frac{1}{12} \sum (x_i^2 + y_i^2) = -\frac{1}{12} = t(1 - \psi)$ which is not possible as $(1 - \psi)$ takes values only in $[0, 1]$ and $t \in [0, 1]$. So $Y \nmid H_t(z)$ and hence (as in lemma 2.1 of [1])

$(\beta_t \oplus 0) \wedge \alpha + d\alpha$ is nondegenerate on $(TF^t_{1+t})(w, z)$ for $z \neq 0$ where $\beta_t$ is a one form on $T^2 \times D^{q-2}$ such that $\beta_t(t\psi Z') = 1$ and $\beta_t = 0$ on the orthogonal complement of $(t\psi Z')$ in $T(T^2 \times D^{q-2})$. So set $\omega'_{1+t} = \beta_t \wedge \alpha + d\alpha$. Observe that $\beta_0 = 0$ and outside the support of $\psi$, $\omega'_{1+t} = d\alpha = \omega'_1$. This completes the proof of 1.3.

**Acknowledgements:** I would like to thank Prof. Dishant Pancholi for his support and Chennai Mathematical Institute for offering me a post doctoral position.

**REFERENCES**

[1] Bertelson, Mélanie Foliations associated to regular Poisson structures. Commun. Contemp. Math. 3 (2001), no. 3, 441–456. (Reviewer: Edith Padrón)

[2] Datta, Mahuya; Mukherjee, Sauvik On existence of regular Jacobi structures. Geom. Dedicata 173 (2014), 215–225.

[3] Eliashberg, Y.; Mishachev, N. Introduction to the h-principle. Graduate Studies in Mathematics, 48. American Mathematical Society, Providence, RI, 2002. xviii+206 pp. ISBN: 0-8218-3227-1 (Reviewer: John B. Etnyre)

[4] Fernandes, Rui Loja; Frejlich, Pedro An h-principle for symplectic foliations. Int. Math. Res. Not. IMRN 2012, no. 7, 1505–1518. (Reviewer: David Iglesias Ponte)

[5] Geiges, Hansjörg An introduction to contact topology. Cambridge Studies in Advanced Mathematics, 109. Cambridge University Press, Cambridge, 2008. xvi+440 pp. ISBN: 978-0-521-86585-2 (Reviewer: John B. Etnyre)

[6] Schweitzer, Paul A. Counterexamples to the Seifert conjecture and opening closed leaves of foliations. Ann. of Math. (2) 100 (1974), 386–400. (Reviewer: Robert Roussarie)

[7] Wilson, F. Wesley, Jr. On the minimal sets of non-singular vector fields. Ann. of Math. (2) 84 1966 529–536. (Reviewer: W. H. Gottschalk)