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On a class of spaces of skew-symmetric forms related to Hamiltonian systems of conservation laws

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Abstract. It was shown in Ferapontov et al. (Lett Math Phys 108(6):1525–1550, 2018) that the classification of n-component systems of conservation laws possessing a third-order Hamiltonian structure reduces to the following algebraic problem: classify n-planes \( H \) in \( \bigwedge^2 (V_{n+2}) \) such that the induced map \( \text{Sym}^2 H \to \bigwedge^4 V_{n+2} \) has 1-dimensional kernel generated by a non-degenerate quadratic form on \( H^* \). This problem is trivial for \( n = 2, 3 \) and apparently wild for \( n \geq 5 \). In this paper we address the most interesting borderline case \( n = 4 \). We prove that the variety \( \mathcal{V} \) parametrizing those 4-planes \( H \) is an irreducible 38-dimensional \( PGL(V_6) \)-invariant subvariety of the Grassmannian \( G(4, \bigwedge^2 V_6) \). With every \( H \in \mathcal{V} \) we associate a characteristic cubic surface \( S_H \subset P H \), the locus of rank 4 two-forms in \( H \). We demonstrate that the induced characteristic map \( \sigma : \mathcal{V}/PGL(V_6) \to \mathcal{M}_c \), where \( \mathcal{M}_c \) denotes the moduli space of cubic surfaces in \( \mathbb{P}^3 \), is dominant, hence generically finite. Based on Manivel and Mezzetti (Manuscr Math 117:319–331, 2005), a complete classification of 4-planes \( H \in \mathcal{V} \) with the reducible characteristic surface \( S_H \) is given.

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1. Introduction

Our problem originates from the geometric theory of systems of conservation laws,

\[ u_i^t = [V^i(u)]_x, \quad (1) \]

\( i = 1, \ldots, n \). PDEs of this type appear in a wide range of applications in continuum mechanics and mathematical physics, see e.g. [14]. A geometric counterpart of system (1) is a line congruence \((n\)-parameter family of lines in projective space \(\mathbb{P}^{n+1}\)) specified by the equations

\[ y^i = u^i y^{n+1} + V^i(u) y^{n+2}, \quad (2) \]

here \(y^i\) are the homogeneous coordinates in \(\mathbb{P}^{n+1}\) and \(u^i\) are the parameters. In the case \(n = 2\) we obtain a two-parameter family, or a classical congruence of lines in \(\mathbb{P}^3\). Since 19th century the theory of congruences has been one of the most popular chapters of projective differential geometry. It was observed in [1,2] that all standard concepts of the theory of conservation laws such as rarefaction curves, shock curves, linear degeneracy, reciprocal transformations, etc, acquire a simple intuitive interpretation in the language of the projective theory of congruences. Algebro-geometric aspects of the correspondence \((1) \leftrightarrow (2)\) were thoroughly investigated in [6,7].

Particularly interesting examples of systems (1) arise in the context of equations of associativity of 2D topological field theory (WDVV equations) [2,9,13]. Such systems can be represented in Hamiltonian form

\[ u_i^t = P^{ij} \frac{\delta H}{\delta u^j}, \quad (3) \]

where \(P^{ij}\) is a third-order Hamiltonian operator of special type and \(H\) is a (nonlocal) Hamiltonian, see [10] for further details. It was shown in [10] that if system (1) possesses Hamiltonian representation (3), then the associated congruence (2) must necessarily be linear, that is, defined by \(n\) linear equations in the Plücker coordinates. Explicitly, congruence (2) must satisfy the relations

\[ tr YA^i = 0, \quad (4) \]

\( i = 1, \ldots, n \), where \(Y\) is the \((n + 2) \times (n + 2)\) skew-symmetric matrix formed by \(2 \times 2\) minors (Plücker coordinates) of the \(2 \times (n + 2)\) matrix

\[ \left( \begin{array}{ccc} u^1 & \ldots & u^n \\ V^1 & \ldots & V^n \end{array} \right), \quad (5) \]
and $A^i$ are constant $(n+2) \times (n+2)$ skew-symmetric matrices. Note that relations (4) can be viewed as a linear system for the fluxes $V^i$ of system (1). Furthermore, viewed as 2-forms, $A^i$ must satisfy an additional relation of the form

$$\varphi_{ij} A^i \wedge A^j = 0$$

(6)

where the matrix $\varphi$ is symmetric and non-degenerate.

Introducing an $n$-plane $H = \text{span}(A^i)$ in $\wedge^2 V_{n+2}$, interpreting relation (6) as the kernel of the natural map $\text{Sym}^2 H \rightarrow \wedge^4 V_{n+2}$, and $\varphi$ as a non-degenerate quadratic form on $H^*$, we arrive at the algebraic problem formulated in the abstract. In this paper we will concentrate on the particularly interesting case $n = 4$.

Our main results are summarised as follows.

- In Theorem 2.2 of Sect. 2.4 we prove that the variety $\mathcal{V}$ parametrizing those 4-planes $H$ is an irreducible 38-dimensional $\text{PGL}(V_6)$-invariant subvariety of the Grassmannian $G(4, \wedge^2 V_6)$. In Sect. 2.5, with every $H \in \mathcal{V}$ we associate a characteristic cubic surface $S_H \subset \mathbb{P} H$, the locus of rank 4 two-forms in $H$. In Theorem 3.2 of Sect. 3 we demonstrate that the induced characteristic map $\sigma : \mathcal{V}/\text{PGL}(V_6) \rightarrow \mathcal{M}_c$, where $\mathcal{M}_c$ denotes the moduli space of cubic surfaces in $\mathbb{P}^3$, is dominant, hence generically finite.

- Particularly interesting examples of 4-planes $H \in \mathcal{V}$ are discussed in Sect. 3. These include 4-planes containing a semisimple 3-plane (Sect. 3.2) and 4-planes associated with Pfaffian representations of Cayley’s ruled cubic surface (Sect. 3.4).

- Based on [12], a complete classification of 4-planes $H \in \mathcal{V}$ with the reducible characteristic surface $S_H$ is given in Sect. 4.

- A remarkable example of a system of conservation laws (1) is discussed in Sect. 5. It corresponds to one of the reducible cases of Sect. 4 and is conjectured to be a unique integrable example.

2. Problem and strategy

2.1. The problem

As outlined in the introduction, we would like to classify the four-planes $H$ in $\wedge^2 V_6$ such that

(1) the induced map $\text{Sym}^2 H \rightarrow \wedge^4 V_6 \simeq \wedge^2 V_6^*$ defined as the composition

$$\text{Sym}^2 H \hookrightarrow \text{Sym}^2(\wedge^2 V_6) \rightarrow \wedge^4 V_6$$

has rank exactly nine (where the rightmost map is just the wedge product),

(2) its kernel is generated by a non-degenerate quadratic form on $H^*$.

The variety $\mathcal{V}$ parametrizing those four-planes is a $\text{PGL}(V_6)$-invariant subvariety of the Grassmannian $G(4, \wedge^2 V_6)$, whose dimension is 44. The condition that $\text{Sym}^2 H \rightarrow \wedge^4 V_6$ be of rank at most nine describes a closed subvariety $\mathcal{V}^*$ of the Grassmannian (given locally by the vanishing of a collection of ten by ten mi-
nors). The expected codimension of $\mathcal{V}^*$ is $15 - 10 + 1 = 6$ (recall that a morphism $\varphi : E \to F$ between vector bundles of respective ranks $e \leq f$ on an algebraic variety $X$ is expected to have rank at most $e - 1$ on a closed subvariety $Y$ of codimension $f - e + 1$, where "expected" means that this is indeed what happens under suitable transversality assumptions, see e.g. [4]; in general, one can only asserts that the dimension of each irreducible component of $Y$ is at least the expected one).
So the expected dimension of $\mathcal{V}^*$ is $38 = \dim PGL(V_6) + 3$.

Then $\mathcal{V}$ is defined in $\mathcal{V}^*$ by two open conditions, namely that the rank is exactly nine, and that the kernel is spanned by a non-degenerate quadratic form. So $\mathcal{V}$ must be open inside $\mathcal{V}^*$, but not necessarily dense since it is not clear that $\mathcal{V}$ is irreducible. To be precise, $\mathcal{V}$ must be a union of dense open subsets of irreducible components of $\mathcal{V}^*$. Moreover, all these components have dimension at least 38.

2.2. Ranks

Elements of $\wedge^2 V_6$ are skew-symmetric forms on $V_6^*$, for which we have the usual notion of rank. For convenience we will also use the following terminology.

**Definition 2.1.** Let $H$ be a subspace of $\wedge^2 V_6$. Let $\theta$ be an element of $\text{Sym}^2 H$.

1. The $q$-rank of $\theta$ is the rank of the corresponding quadratic form on $H^*$;
2. The rank of $\theta$ is the rank of its image in $\wedge^4 V_6 \simeq \wedge^2 V_6^*$, considered as a skew-symmetric form on $V_6$.

Note that a four-plane $H$ in $\mathcal{V}$ cannot contain any form $\omega$ of rank two, since such a form verifies $\omega \wedge \omega = 0$, hence produces a degenerate element (of q-rank one) in the kernel of the map $\text{Sym}^2 H \to \wedge^4 V_6$. As a consequence, $H$ contains elements of rank six: it was proved in [12] that the linear spaces whose non-zero elements all have rank four have dimension at most three, and such three-planes were classified.

2.3. Birational involution

Recall that the Pfaffian quadratic map

$$\wedge^2 V_6 \longrightarrow \wedge^4 V_6$$
$$\omega \mapsto \omega \wedge \omega$$

descends to a birational map $pf : \mathbb{P}(\wedge^2 V_6) \to \mathbb{P}(\wedge^4 V_6)$ which is essentially an involution. Indeed, if we identify $\wedge^4 V_6$ to $\wedge^2 V_6^*$, hence $\wedge^4 V_6^*$ to $\wedge^2 V_6$, we get a map

$$\wedge^4 V_6 \longrightarrow \wedge^2 V_6$$
$$\omega^* \mapsto \omega^* \wedge \omega^*,$$

that descends to $pf^* : \mathbb{P}(\wedge^4 V_6) \to \mathbb{P}(\wedge^2 V_6)$. Since for $\omega^* = \omega \wedge \omega$ we have $\omega^* \wedge \omega^* = Pf(\omega)\omega$, where the Pfaffian $Pf(\omega) = \omega \wedge \omega \wedge \omega$ up to some constant,
pf* and pf are inverse birational maps. In particular if ω has rank six, we can recover it (up to scalar) from ω ∧ ω which has also rank six. But note that if ω has rank four (resp. two), then ω ∧ ω has rank two (resp. zero). In particular, ω ∧ ω never has rank four.

2.4. Irreducibility

Theorem 2.2. V is irreducible of dimension 38.

Proof. Consider a four-plane H that belongs to V, and denote by q a generator of the kernel of the map Sym2H → ∧4V6. As a quadratic form on H*, the tensor q is non degenerate, hence identifies H* with H, yielding a quadratic form q* on H. As we observed, a generic ω in H has rank six and q-rank four. Let L ⊂ H be its orthogonal complement with respect to q*. Then we can write, up to scalar,

\[ q = ω^2 - Ω, \quad \text{with} \quad Ω ∈ Sym^2L. \]

Note that the map Sym2L → L ∧ L is injective, hence an isomorphism, because its kernel is contained in the kernel of the map Sym2H → ∧4V6, which by hypothesis does not contain any non zero element of q-rank three or less. So we can conclude that Ω = ω ∧ ω belongs to Sym2L ≃ L ∧ L. Moreover, since ω has rank six, Ω also has rank six, and therefore Ω ∧ Ω is a non zero multiple of ω. In particular we can recover H from L and Ω ∈ Sym2L.

This suggests to consider the diagram

```
\[ \mathbb{P}(Sym^2U) \xrightarrow{p_2} \mathcal{W} \xrightarrow{p} \mathcal{V} \subset G(4, ∧^2V_6), \]
\[ G(3, ∧^2V_6) \xleftarrow{T} \]
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our notations being the following. First, T is the open subset of G(3, ∧^2V_6) parametrizing the three-planes L such that

1. the map Sym2L → ∧4V6 is injective,
2. the general element in L ∧ L has rank six.

Second, U denotes the tautological rank three vector bundle on G(3, ∧^2V_6) and \( \mathcal{W} \) is the open subset of \( \mathbb{P}(Sym^2U) \) parametrizing pairs (L, [Ω]), where L is a three-plane and Ω a non-zero element of Sym2L, such that

1. L belongs to T,
2. Ω ∈ Sym2L ≃ L ∧ L has maximal rank and q-rank,
3. (ω ∧ L) ∩ (L ∧ L) = 0, where \( ω := Ω ∧ Ω \).

The map \( \pi \) from \( \mathcal{W} \) to \( \mathcal{V} \) sends the pair (L, [Ω]) to \( H = ⟨L, ω⟩ \). This is well-defined because

- ω cannot belong to L: otherwise, we would have ω ∧ ω = Pf(Ω)Ω, yielding the relation \( ω^2 = Pf(Ω)Ω \) in Sym2L, and then Ω would have q-rank one, a contradiction;
• the kernel of the map $\text{Sym}^2 H \to \wedge^4 V_6$ is generated by $q = \omega^2 - Pf(\Omega)\Omega$, which is non degenerate since $\Omega$ has q-rank three: indeed, if there was another element in this kernel, we could write it, after subtracting a suitable multiple of $q$ if necessary, as $q' = \omega\lambda - \Phi$ for some $\lambda$ in $L$ and some $\Phi$ in $\text{Sym}^2 L \simeq L \wedge L$; this would yield a relation $\omega \wedge \lambda = \Phi$ in $\wedge^4 V_6$, contradicting the hypothesis that $(\omega \wedge L) \cap (L \wedge L) = 0$.

By the remarks at the beginning of this proof, $\pi$ is surjective, so the irreducibility of $V$ will be a consequence of the irreducibility of $W$. But the latter is obviously an open subset of the projective bundle $\mathbb{P}(\text{Sym}^2 U)$, hence certainly irreducible, of dimension $3 \times 12 + 5 = 41$. Finally, the fibers of $\pi$ are open subsets in three-dimensional projective spaces, and therefore the dimension of $V$ is $41 - 3 = 38$. ⊓⊔

2.5. Characteristic surfaces

For $H \in \mathcal{V} \subset G(4, \wedge^2 V_6)$, the locus of rank four two-forms defines a cubic surface $S_H \subset \mathbb{P}H$, that we call the characteristic surface. This surface is the intersection of $\mathbb{P}H$ with the Pfaffian hypersurface $Pf \subset \mathbb{P}\wedge^2 V_6$. Recall that the singular locus of this hypersurface is the Grassmannian $G(2, V_6)$, the locus of two-forms of rank two. Although $\mathbb{P}H$ does not meet this Grassmannian when $H$ belongs to $\mathcal{V}$, will we see later on that $S_H$ can be badly singular, and even reducible. Nevertheless, we have a natural map

$$\sigma : \mathcal{V}/\text{PGL}(V_6) \rightarrow \mathcal{M}_c,$$

where $\mathcal{M}_c$ denotes the moduli space of cubic surfaces in $\mathbb{P}^3$. Both spaces are four-dimensional, and we will see later on that $\sigma$ is dominant, hence generically finite.

Note that, by construction, $H \in \mathcal{V}$ defines not only the cubic surface $S_H \subset \mathbb{P}H$, but also a Pfaffian representation of this surface. This is a classical topic. For example, any cubic surface (possibly singular) admits a Pfaffian representation [5, Proposition 7.6]. Moreover, a generic such representation can be determined by five points in general position on the surface [5, Example 7.4]. A simple algorithm for that is presented in [15].

The family of Pfaffian representations of a smooth cubic surface is five-dimensional. In comparison, there are only finitely many determinantal representations. They are in bijection with the 72 linear systems of twisted cubics on the surface, or the 72 sixes of non-incident lines (this was already known in the 19th century, see [5, Corollary 6.4]).

3. Semisimple three-planes

3.1. A grading of $\mathfrak{e}_7$

Although the classification of orbits in $G(4, \wedge^2 V_6)$ is not known (a priori a wild problem), the classification of orbits in $G(3, \wedge^2 V_6)$ is known, see, e.g., [11]. This
is because $V_3 \otimes \wedge^2 V_6$ is part of a $\mathbb{Z}_3$-grading of the exceptional Lie algebra $\mathfrak{e}_7$:

$$\mathfrak{e}_7 \cong \mathfrak{sl}(V_3) \times \mathfrak{sl}(V_6) \oplus V_3 \otimes \wedge^2 V_6 \oplus V_3^* \otimes \wedge^2 V_6^*.$$ 

Then it makes sense to talk about semisimple or nilpotent elements in $V_3 \otimes \wedge^2 V_6$, by considering them as elements of $\mathfrak{e}_7$. The set of semisimple elements is the union of the Cartan subspaces, which are all equivalent under the action of $GL(V_3) \times GL(V_6)$. In order to exhibit one of those, we fix a basis $v_1, v_2, v_3$ of $V_3$, and a basis $e_1, \ldots, e_6$ of $V_6$. We let $e_{ij} = e_i \wedge e_j \in \wedge^2 V_6$.

**Proposition 3.1.** The subspace $C$ generated by the following three vectors is a Cartan subspace of $V_3 \otimes \wedge^2 V_6$:

$$c_1 = v_1 \otimes e_{12} + v_2 \otimes e_{34} + v_3 \otimes e_{56},$$
$$c_2 = v_1 \otimes e_{36} + v_2 \otimes e_{52} + v_3 \otimes e_{14},$$
$$c_3 = v_1 \otimes e_{54} + v_2 \otimes e_{16} + v_3 \otimes e_{32}.$$

We say that a three-plane $L$ in $\wedge^2 V_6$ is semisimple if it admits a basis $\omega_1, \omega_2, \omega_3$ such that $v_1 \otimes \omega_1 + v_2 \otimes \omega_2 + v_3 \otimes \omega_3$ is semisimple in $V_3 \otimes \wedge^2 V_6$. By the previous proposition, this is the case if and only if there exists a basis $e_1, \ldots, e_6$ of $V_6$, and coefficients $a, b, c$, not all zero, such that

$$\omega_1 = ae_{12} + be_{36} + ce_{54},$$
$$\omega_2 = ae_{34} + be_{52} + ce_{16},$$
$$\omega_3 = ae_{56} + be_{14} + ce_{32}.$$

### 3.2. Four-planes containing a semisimple three-plane

We would like to understand four-planes $H \in \mathcal{V}$ that contain a semisimple three-plane $L$. Necessarily, the map $\text{Sym}^2 L \to \wedge^4 V_6$ must be injective. Note that

$$\frac{1}{2} \omega_1^2 = bce_{3654} + ace_{1254} + abe_{1236},$$
$$\frac{1}{2} \omega_2^2 = bce_{1652} + ace_{1634} + abe_{3452},$$
$$\frac{1}{2} \omega_3^2 = bce_{1432} + ace_{3256} + abe_{1456},$$
$$-\omega_2 \omega_3 = a^2 e_{3654} + b^2 e_{1254} + c^2 e_{1236},$$
$$-\omega_1 \omega_3 = a^2 e_{1652} + b^2 e_{1634} + c^2 e_{3452},$$
$$-\omega_2 \omega_3 = a^2 e_{1432} + b^2 e_{3256} + c^2 e_{1456},$$

so this injectivity condition is equivalent to the fact that the vectors $(a^2, b^2, c^2)$ and $(bc, ac, ab)$ are not collinear (that is, they are linearly independent).

Now we would like to complete $L$ with a vector $\omega_0 \notin L$ such that the four-plane $H$ they span belongs to $\mathcal{V}$. We can choose $\omega_0$ such that it generates the orthogonal to $L$ in $H$ with respect to its non degenerate quadratic form (indeed, generically we may suppose that the restriction of this quadratic form to $L$ is non-degenerate). This implies that $\omega_0 \wedge \omega_0$ belongs to $L \wedge L$. 
Now observe that we can decompose $V_6 = V_+ \oplus V_-$, where $V_+ = \langle e_1, e_3, e_5 \rangle$ and $V_- = \langle e_2, e_4, e_6 \rangle$. With respect to this decomposition, we have

$$\wedge^2 V_6 = \wedge^2 V_+ \oplus (V_+ \otimes V_-) \oplus \wedge^2 V_-,$$

and $L \subset V_+ \otimes V_-$. Let us decompose accordingly $\omega_0 = \omega_+ + \Omega + \omega_-$. We get

$$\frac{1}{2} \omega_0^2 = \omega_+ \Omega + \left( \omega_+ \omega_- + \frac{1}{2} \Omega^2 \right) + \Omega \omega_-,$$

with respect to the decomposition

$$\wedge^4 V_6 = (\wedge^3 V_+ \otimes V_-) \oplus (\wedge^2 V_+ \otimes \wedge^2 V_-) \oplus (V_- \otimes \wedge^3 V_+).$$

Of course $L \wedge L$ is contained in $\wedge^2 V_+ \otimes \wedge^2 V_-$, so we need in particular that $\omega_+ \Omega = \Omega \omega_- = 0$.

Generically, $\Omega$ has rank three, and then the previous equations imply $\omega_+ = \omega_- = 0$. This means that $H \subset V_+ \otimes V_-$. Then $H$ belongs to $\mathcal{W}$, since the image of $\text{Sym}^2 H$ is contained in $\wedge^2 V_+ \otimes \wedge^2 V_-$, which has dimension nine. One can check that a generic such $H$ does belong to $\mathcal{V}$. In fact it suffices to produce an explicit example; here is one:

$$\omega_1 = e_{34} - e_{56},$$
$$\omega_2 = e_{52} - e_{14},$$
$$\omega_3 = e_{16} - e_{32},$$
$$\omega_0 = e_{12} + e_{14} + e_{32} + e_{36} + e_{54} - e_{56}.$$

The relation is

$$\omega_0^2 + \omega_1^2 - \omega_2^2 + \omega_3^2 + \omega_1 \omega_2 + \omega_1 \omega_3 - \omega_2 \omega_3 = 0,$$

which is non-degenerate.

When $H \subset V_+ \otimes V_-$, by choosing basis of these spaces we get a $3 \times 3$ matrix $M$ of linear forms in four variables; the corresponding skew-symmetric matrix of size six is just

$$\begin{pmatrix} 0 & M \\ -M & 0 \end{pmatrix},$$

whose Pfaffian is just $\det(M)$. So the characteristic surface $S_H$ is given by the determinant of the matrix $M$. Moreover, the quadratic relation in $\text{Sym}^2 H$ is orthogonal to the image of the map

$$\wedge^2 V_+^* \otimes \wedge^2 V_-^* \longrightarrow \text{Sym}^2 H^*$$

given by the $2 \times 2$ minors of our matrix. In the classical terminology, the quadratic relation in $\text{Sym}^2 H$ is thus nothing else than the Schur quadric associated to the given determinantal representation of the cubic surface $S_H$. When this surface is smooth, it is known that the Schur quadric is unique and smooth (see [8, 9.1.3]). In particular this implies:
Theorem 3.2. The image of the characteristic map \( \sigma : V/\text{PGL}(V_6) \rightarrow M_c \) contains the open subset parametrizing smooth cubic surfaces. In particular \( \sigma \) is dominant, hence generically finite.

Remark. The stabilizer in \( \text{PGL}(V_6) \) of a general point in \( V \) is the copy of \( \mathbb{C}^* \) given by the automorphisms of the form \( t \text{Id}_{V_6} + \text{Id}_{V_6} \). This explains that the quotient \( V/\text{PGL}(V_6) \) has dimension four, rather than \( 3 = 38 - 35 \).

We can also conclude from the previous discussion that in general, our special Pfaffian representations are in fact determinantal. Since \( \sigma \) is generically finite, this implies that \( V/\text{PGL}(V_6) \) is birational to the moduli space of determinantal representations of cubic surfaces. As a consequence, the degree of \( \sigma \) must be 72.

3.3. More examples

Consider the three-plane \( L \) generated by

\[
\omega_0 = e_0 \wedge e_2 + e_1 \wedge e_3, \\
\omega_1 = e_0 \wedge e_4 + e_1 \wedge e_5, \\
\omega_2 = e_0 \wedge e_1 + e_2 \wedge e_5 + e_3 \wedge e_4.
\]

This is triple tritangent plane, in the sense that it intersects the Pfaffian hypersurface along a triple line: the line generated by \( \omega_0 \) and \( \omega_1 \), which is a generic line. Let us complete \( L \) into a four-plane \( H \in V \). For that we consider a general element \( \Omega \) of \( L \wedge L \). For simplicity let us denote \( e_{pqrs} \) by \( e_{ij} \), where \( (pqrsij) \) is a permutation of \( (123456) \) of sign \( \epsilon \). We can then write

\[
\Omega = xf_{23} + yf_{45} + z(f_{25} - f_{34}) + a(f_{04} - f_{15}) + b(f_{02} - f_{13}) + c(f_{01} + f_{25} + f_{34}).
\]

Then we need to compute \( \frac{1}{2} \Omega^2 \). After simplifying a little bit the result by adding a suitable element of \( L \), we get

\[
\omega = (xy - z^2)e_{01} + (yc + a^2)e_{23} + (xc + b^2)e_{45} + \\
+ (yb + za)(e_{02} - e_{13}) + (zb + xa)(e_{04} - e_{15}) + (zc + ab)(e_{34} - e_{25}).
\]

Then we need to check that \( H = \{ \omega, L \} \) belongs to \( V \) when the parameters are general. We know that \( \omega^2 \) is a general element of \( L \wedge L \), so the only thing we need to check is that the intersection of \( L \wedge L \) with \( \omega \wedge L \) is zero. Let \( \Omega_1 = f_{04} + f_{15}, \Omega_2 = f_{02} + f_{13}, \Omega_3 = f_{34} + f_{25} \). Note the space \( \langle \Omega_1, \Omega_2, \Omega_3 \rangle \) is transverse to \( L \wedge L \). Moreover, modulo the latter we have

\[
\omega_0 \omega_0 = -(zc + ab)\Omega_1 - (xc + b^2)\Omega_2 + (zb + xa)\Omega_3, \\
\omega_0 \omega_1 = -(yc + a^2)\Omega_1 - (zc + ab)\Omega_2 - (yb + za)\Omega_3, \\
\omega_0 \omega_2 = -(yb + za)\Omega_1 - (zb + xa)\Omega_2 - (zc + ab)\Omega_3.
\]

In general those three combinations of \( \Omega_1, \Omega_2, \Omega_3 \) are independent, and we are done.

Note that the characteristic surface \( S_H \) cannot be smooth. Indeed, by construction \( L \) is a triple tritangent plane, and therefore \( S_H \) must admit two singular points of type \( A_2 \), or even worse singularities (see e.g. [8, 9.2.2]).
Example 1. Let $z, c$ be non-zero and the other coefficients be zero, so (after factoring out a $z$) $\omega = -ze_{01} - c(e_{25} - e_{34})$. Here the (unique) quadratic relation is

$$\frac{z}{2}\omega^2 + c^2\omega_2 = c(c^2 - z^2)\omega_0\omega_1 - \frac{zc^2}{2}\omega_2^2$$

and is non-degenerate in general. The equation of the surface $S_H$ is

$$Pf(t\omega + x_0\omega_0 + x_1\omega_1 + x_2\omega_2) = zc^2t^3 - c^2t^2x_2 - ztx_2^2 - 2ctx_0x_1 + x_2^3 = 0.$$ 

This surface has two singular points $[0, 0, 1, 0]$ and $[0, 1, 0, 0]$, each yielding an $A_2$ singularity.

Example 2. Let $b = y = 1$ and the other coefficients be zero, so $\omega = e_{02} - e_{13} + e_{45}$. Here $\Omega = f_{02} - f_{13} + f_{45}$, so the (unique) quadratic relation is

$$\frac{1}{2}\omega^2 = \omega_1\omega_2 + \frac{1}{2}\omega_0^2$$

and is non-degenerate. The equation of the characteristic surface $S_H$ is

$$Pf(t\omega + x_0\omega_0 + x_1\omega_1 + x_2\omega_2) = t^3 - tx_0^2 - 2tx_1x_2 + x_3^2 = 0.$$ 

This surface has a unique singularity at $[0, 0, 1, 0]$ and this is an $A_4$ singularity.

3.4. Special Pfaffian representations of Cayley’s ruled surface

It seems difficult to determine precisely what is the image of the characteristic map. We have just found some singular cubic surfaces with $A_2$ and even $A_4$ singularities, and we will consider reducible surfaces in the next section. Among irreducible surfaces, there are two types of non normal ones [8, Theorem 9.2.1], and the most degenerate one is Cayley’s ruled surface, of equation

$$x_0^2x_2 + x_1^2x_3 = 0.$$ 

In this section we discuss the special Pfaffian representations of this surface.

So we consider $H = \langle \omega_0, \omega_1, \omega_2, \omega_3 \rangle$ in $\mathcal{V}$ and suppose that its characteristic surface $S_H$ has equation

$$Pf(x_0\omega_0 + x_1\omega_1 + x_2\omega_2 + x_3\omega_3) = x_0^2x_2 + x_1^2x_3.$$ 

In particular, among the wedge products $\omega_i\omega_j\omega_k$, only $\omega_0^2\omega_2$ and $\omega_2^2\omega_3$ are non-zero. Since $H$ belongs to $\mathcal{V}$, there exists a non-degenerate relation $q = \sum_{i,j} q_{ij}\omega_i\omega_j = 0$. Taking the products with $\omega_k$ for $k = 0, \ldots, 3$ we get $q_{00} = q_{02} = q_{11} = q_{13} = 0$. So the relation $q$ must be a combination of $\omega_0\omega_1, \omega_0\omega_3, \omega_1\omega_2, \omega_2^2, \omega_2\omega_3, \omega_3$.

The pencil $\langle \omega_2, \omega_3 \rangle$ is made of forms of constant rank four, and is therefore of one of the two possible types found in [12]:

$$\omega_2 = e_{02} + e_{13}, \quad \omega_3 = e_{04} + e_{15}$$
for the generic type, while the special type is
\[ \omega_2 = e_2 + e_3, \quad \omega_3 = e_3 + e_4. \]

Note that the pencil \( \langle \omega_2, \omega_3 \rangle \) spans the singular locus of the characteristic surface. In particular, it is uniquely defined by \( H \) and we call it the singular pencil.

Let us decompose the two other two-forms as
\[ \omega_0 = ae_{01} + e_0 \land f_0 + e_1 \land f_1 + \mu, \quad \omega_1 = be_{01} + e_0 \land g_0 + e_1 \land g_1 + \nu, \quad (7) \]
where \( f_0, f_1, g_0, g_1, \mu, \nu \) do not involve \( e_0, e_1 \).

**Lemma 3.3.** The singular pencil \( \langle \omega_2, \omega_3 \rangle \) must be special.

**Proof.** We proceed by contradiction. Suppose the singular pencil is of generic type, and choose an adapted basis as above.

If \( q_{01} \neq 0 \), we can modify \( \omega_0 \) and \( \omega_1 \) by suitable linear combinations of \( \omega_2 \) and \( \omega_3 \), in such a way that the relation takes the form \( q = q_{01} \omega_0 \omega_1 + Q(\omega_2, \omega_3) \). (The equation of the characteristic surface itself will not change.) In particular, we get the relations
\[ f_0 \land \nu + g_0 \land \mu = f_1 \land \nu + g_1 \land \mu = 0, \quad \mu \land \nu = 0. \quad (8) \]

Suppose that the vectors \( f_0, f_1, g_0, g_1 \) are linearly dependent. Then there is a three-plane \( P \) such that \( f_0, f_1, g_0, g_1 \) belong to \( P \) and \( \mu, \nu \) belong to \( \land^2 P \). But then \( \omega_0 \land \omega_1 = e_{01} \land \theta \) for some \( \theta \) in \( \land^2 P \), which has therefore rank at most two. Now the linear relation \( q \) yields
\[ q_{01} \theta = 2q_{22} e_{23} + 2q_{23} (e_{25} - e_{34}) + 2q_{33} e_{45}. \]
The right-hand side is a form of rank at most two when \( q_{23}^2 = q_{22} q_{33} \). But then \( q \) itself has q-rank at most three, a contradiction.

Suppose now that the vectors \( f_0, f_1, g_0, g_1 \) are linearly independent. Then the first two equations in (8) are verified if and only if we can write \( \mu, \nu \) in the form
\[ \mu = xf_0 \land f_1 + y(f_0 \land g_1 + g_0 \land f_1) + zg_0 \land g_1, \]
\[ \nu = yf_0 \land f_1 + z(f_0 \land g_1 + g_0 \land f_1) + tg_0 \land g_1. \]
Then the relation \( \mu \land \nu = 0 \) amounts to \( xt = yz \). Moreover, the conditions that \( \omega_0^3 = \omega_1^3 = 0 \) amount to \( z = a(xz - y^2) \) and \( y = b(yt - z^2) \). If \( yz \neq 0 \), then \( ab \neq 0 \) and we can write
\[ \omega_0 = \frac{1}{a}(ae_0 - f_1) \land (ae_1 + f_0) + \frac{1}{z}(yf_0 + zg_0) \land (yf_1 + zg_1), \]
\[ \omega_1 = \frac{1}{b}(be_0 - g_1) \land (be_1 + g_0) + \frac{1}{y}(yf_0 + zg_0) \land (yf_1 + zg_1). \]
Then the conditions \( \omega_0^2 \omega_1 = 0 \) and \( \omega_0 \omega_1^2 = 0 \) reduce to \( ay + bz = 0 \). In this case, both triples \( \langle ae_0 - f_1, be_0 - g_1, yf_1 + zg_1 \rangle \) and \( \langle ae_1 + f_0, be_1 + g_0, yf_0 + zg_0 \rangle \) are linearly dependent. This implies that these six vectors generate a space \( Q \subset V_6 \) of
dimension at most four. But then since \( \omega_0 \) and \( \omega_1 \) belong to \( \wedge^2 Q \), the pencil they generate will necessarily contain a two-form of rank two, which is a contradiction.

We also need to consider the case where for example \( y = 0 \), hence also \( bz = 0 \). If \( y = 0 \) and \( b = 0 \), then the condition \( \omega_0 \omega_1 = 0 \) amounts to \( x = 0 \), and then \( z = a(xz - y^2) = 0 \). If \( y = z = 0 \), the conditions that \( \omega_0^2 \omega_1 = 0 \) and \( \omega_0 \omega_1^2 = 0 \) imply that \( x = t = 0 \). But then all the two-forms in \( H \) are of the form \( e_0 \wedge h_0 + e_1 \wedge h_1 \), hence they have rank at most four and the rank will necessarily drop somewhere, again a contradiction.

There remains to consider the case where \( q_{01} = 0 \). In that case we need \( q_{03} q_{12} \) to be non-zero for \( q \) to remain non-degenerate, and we can suppose that \( q_{03} = q_{12} = 1 \). Then the relation \( q \) implies that

\[
\mu \wedge e_4 + v \wedge e_2 = \mu \wedge e_5 + v \wedge e_3 = 0. \tag{9}
\]

On the other hand, the fact that the products of \( \omega_0, \omega_1 \) with \( \omega_2, \omega_2 \omega_3, \omega_3^2 \) vanish imply that we can write \( \mu \) and \( v \) in the form

\[
\mu = xe_{24} + y(e_{25} + e_{34}) + ze_{35}, \quad v = x'e_{24} + y'(e_{25} + e_{34}) + z'e_{35}.
\]

But then (9) readily imply that \( \mu = v = 0 \). We are then again in a situation where all the two-forms in \( H \) are of the form \( e_0 \wedge h_0 + e_1 \wedge h_1 \), which is not possible. \( \square \)

We can therefore suppose that the singular pencil \( \langle \omega_2, \omega_3 \rangle \) is of special type, as given above. Then \( \omega_2^2, \omega_2 \omega_3, \omega_3^2 \) give \( e_{0123}, e_{0124}, e_{0134} \). Let us decompose \( \omega_0 \) and \( \omega_1 \) as in (7). The fact that their products with \( \omega_2^2, \omega_2 \omega_3, \omega_3^2 \) vanish means that \( \mu \) and \( v \) do not involve \( e_5 \).

Suppose that \( q_{01} \neq 0 \), so that the relation \( q \) can be normalized as before to the form \( q = \omega_0 \omega_1 + Q(\omega_2, \omega_3) \). We deduce once again that \( f_0 \wedge v + g_0 \wedge \mu = f_1 \wedge v + g_1 \wedge \mu = 0 \), and moreover that \( f_1 \wedge g_0 - f_0 \wedge g_1 \) does not involve \( e_5 \). In particular we get the relations

\[
f_{05} v + g_{05} \mu = f_{15} v + g_{15} \mu = 0, \quad f_{15} g_0 - g_{05} f_1 - f_{05} g_1 + g_{15} f_0 = 0. \tag{10}
\]

Note that \( \mu \) and \( v \) cannot both be zero, since otherwise all the two-forms in \( H \) are of the form \( e_0 \wedge h_0 + e_1 \wedge h_1 \), which is not possible. Hence the first two relations imply that \( f_{05} g_{15} = f_{15} g_{05} \), and the third one, multiplied by \( f_{05} \), yields

\[
f_{15} (g_{05} f_0 + f_{05} g_0) = f_{05} (g_{05} f_1 + f_{05} g_1).
\]

But then the two-form \( g_{05} \omega_0 + f_{05} \omega_1 \) can be expressed only in terms of \( e_0, e_1 \) and \( g_{05} f_0 + f_{05} g_0 \) (if \( f_{05} \neq 0 \), or \( g_{05} f_1 + f_{05} g_1 \) if \( f_{15} \neq 0 \)), hence has rank at most two: a contradiction!

If \( f_{05} = f_{15} = 0 \), then \( g_{05} \) and \( g_{15} \) cannot both be zero, for otherwise the two-forms in \( H \) would not involve \( e_5 \) at all, and thus the rank would be at most four on \( H \) and would have to drop somewhere. Then the equations (10) imply that \( \mu = 0 \) and that \( f_0 \) and \( f_1 \) are proportional. But then \( \omega_0 \) can be written in terms of three vectors only, and thus has rank two, a contradiction again.
Finally, suppose that \( q_{01} = 0 \), and then normalize as before the relation \( q \) to
\[ q = \omega_0\omega_3 + \omega_1\omega_2 + Q(\omega_2, \omega_3), \]
which is of maximal \( q \)-rank for any \( Q \). The existence of such a relation is equivalent to the equations
\[ \mu \wedge e_3 + v \wedge e_2 = \mu \wedge e_4 + v \wedge e_3 = 0, \quad f_{05} = g_{15} = f_{15} - g_{05} = 0. \tag{11} \]
The fact that the pencil \( \langle \omega_0, \omega_1 \rangle \) contains only forms of rank four is equivalent to the following conditions:
\[ \mu \wedge f_0 = v \wedge g_1 = 0, \quad v \wedge f_0 = \mu \wedge (f_1 - g_0), \quad \mu \wedge g_1 = v \wedge (g_0 - f_1). \tag{12} \]
Moreover we need \( \omega_0^2\omega_3 = \omega_1^2\omega_2 = 0 \), which amount to \( \mu \wedge e_3 = v \wedge e_3 = 0 \), and \( \omega_0\omega_1\omega_2 = \omega_0\omega_1\omega_3 = 0 \), which amount to \( \mu \wedge e_4 = v \wedge e_2 = 0 \). Therefore, there exist two scalars \( m, n \) such that \( \mu = me_{34} \) and \( v = ne_{23} \), and the first two equations of (11) are then verified. Moreover, \( \omega_0^2\omega_2 = \omega_1^2\omega_3 \neq 0 \) if and only if \( m = -n \neq 0 \). The conditions (12) then reduce to
\[ g_{14} = f_{02} = 0, \quad f_{04} = g_{02} - f_{12}, \quad g_{12} = f_{14} - g_{04}. \tag{13} \]
We also need that \( f_{15} = g_{05} \neq 0 \) (for otherwise \( e_5 \) is not involved at all and the rank has to drop somewhere). If those conditions are satisfied, then the characteristic surface has the correct equation and the relation \( q \) is verified for some \( Q \). The last condition to check, in order to get a four plane \( H \) that belongs to \( \mathcal{V} \), is that this is the only relation in \( \text{Sym}^2 H \).

**Lemma 3.4.** There is no relation other than \( q \).

**Proof.** We grade \( \wedge^4 V_6 \) by the degrees on \( e_0, e_1 \), respectively \( e_2, e_3, e_4, \) and \( e_5 \). For example, \( \omega_0^2, \omega_2\omega_3, \omega_3^2 \) span the space of forms of multidegree \((2, 2, 0)\) generated by \( e_{0123}, e_{0124}, e_{0134} \). We note that among the binomials in the \( \omega_i \), the only ones having terms of multidegree \((1, 1, 1)\) are \( \omega_0^2 \) (which gives \( e_{1345} \)), \( \omega_1^2 \) (which gives \( e_{0235} \)), and \( \omega_0\omega_1 \) (which gives \( e_{0345} + e_{1235} \)); those three terms are independent. Moreover, among the remaining monomials, only \( \omega_0\omega_3, \omega_1\omega_2, \omega_0\omega_2, \omega_1\omega_3 \) have terms of multidegree \((2, 1, 1)\) \((e_{0135} > \) for the first two, \( e_{0125} \) for the third one, \( e_{0145} \) for the last one). This readily implies that the dimension of \( H \wedge H \) is nine. \( \Box \)

Multiplying \( e_0, e_1 \) by \( m \) and rescaling, we may suppose that \( m = 1 \). We get the following result:

**Theorem 3.5.** Suppose that the characteristic surface \( S_H \) of a four plane \( H \) in \( \mathcal{V} \) is Cayley’s ruled cubic surface. Then there exists a basis of \( V_6 \), and coefficients \( A, B, C, D, E \), such that
\[ \omega_0 = e_{15} + e_{34} + Ae_{03} + 2Be_{04} - Be_{12} + Ce_{13} + De_{14}, \]
\[ \omega_1 = e_{05} - e_{23} + Be_{02} + Ee_{03} - De_{04} + 2De_{12} + Ee_{13}, \]
\[ \omega_2 = e_{02} + e_{13}, \quad \omega_3 = e_{03} + e_{14}. \]

Conversely, any such four plane belongs to \( \mathcal{V} \) and its characteristic surface is Cayley’s ruled surface.
In particular, letting $A = B = C = D = E = 0$ we get the following four-plane of skew-symmetric matrices:

$$\begin{pmatrix} 0 & 0 & c & d & 0 & b \\ 0 & 0 & 0 & c & d & a \\ -c & 0 & 0 & -b & 0 & 0 \\ -d & -c & b & 0 & a & 0 \\ 0 & -d & 0 & -a & 0 & 0 \\ -b & -a & 0 & 0 & 0 & 0 \end{pmatrix}$$

The dependence relation is simply $q = \omega_0 \omega_3 + \omega_1 \omega_2 = 0$.

### 3.5. Cones

Our final result in this section shows that the characteristic map is not surjective:

**Proposition 3.6.** For $H \in \mathcal{V}$, the characteristic surface $S_H$ cannot be a cone.

**Proof.** Suppose that $S_H$ is a cone over $\omega_0$. In particular, $\omega_0^3 = 0$, so $\omega_0$ has rank four and there is a unique $V_4 \subset V_6$ such that $\omega_0$ belongs to $\wedge^2 V_4$. Moreover, $\omega_0^2$ is a generator of $\wedge^4 V_4$, and the fact that $\omega_0^2 \omega = 0$ for all $\omega \in H$ is equivalent to the fact that $H \subset V_4 \wedge V_6$. More concretely, we can complete $V_4$ with two independent vectors $e_5, e_6$ such that every $\omega \in H$ can be written in the form

$$\omega = e_5 \wedge \alpha + e_6 \wedge \beta + \theta,$$

where $\alpha, \beta$ belong to $V_4$ and $\theta$ to $\wedge^2 V_4$. We remain with the condition that $\omega_0 \omega^2 = 0$, which is equivalent to $\omega_0 \wedge \alpha \wedge \beta = 0$. Hence the following statement:

**Lemma 3.7.** The characteristic surface $S_H$ is a cone if and only if $\omega_0$ is $H$-symmetric, in the sense that $\omega_0 \wedge \alpha \wedge \beta = 0$ for all $\omega = e_5 \wedge \alpha + e_6 \wedge \beta + \theta \in H$.

A convenient way to use this lemma is to denote $\tilde{\omega}_0(\alpha, \beta) = \omega_0 \wedge \alpha \wedge \beta$, where $\tilde{\omega}_0$ is now a non-degenerate skew-symmetric form on $V_4$ (although defined only up to scalar).

Let us denote by $A$ the subspace of $V_4$ generated by the $\alpha$’s, and by $B$ the subspace generated by the $\beta$’s. The generic situation is the following: $A$ and $B$ are three-dimensional and different; moreover, they come with a map $\phi : A \to B$ which is an isomorphism. Then $C = A \cap B$ and $D = \phi^{-1}(C)$ are two-dimensional subspaces of $A$, meeting along a line $E = C \cap D$ such that $\phi(E) \neq E$. Then $C = \langle E, \phi(E) \rangle$ and $D = \langle E, \phi^{-1}(E) \rangle$, $A = C + D = \langle E, \phi(E), \phi^{-1}(E) \rangle$. We can therefore find a basis $e_1, e_2, e_3, e_4$ of $V_4$ such that $e_1, e_2, e_3$ is a basis of $A$ and $\phi(e_i) = e_{i+1}$ for $1 \leq i \leq 3$. But then the fact that $\omega_0$ is $H$-symmetric readily implies that $\omega_0 = 0$, a contradiction!

Now we examine the non-generic situations. Suppose first that everything goes as above, except that $\phi(E) = E$. After rescaling $e_5$ or $e_6$ we may suppose that $\phi$ is the identity on $E$. Then we can find a basis $e_1, e_2, e_3, e_4$ of $V_4$ such that $C = \langle e_1, e_2 \rangle$, $\phi(e_1) = e_1, \phi(e_2) = e_3$ and $\phi(e_3) = e_4$. Then the fact that $\omega_0$ is $H$-symmetric
On a class of spaces of skew-symmetric

implies that \( \tilde{\omega}_0(e_2, e_3) = \tilde{\omega}_0(e_2, e_4) = \tilde{\omega}_0(e_3, e_4) = 0 \), which still contradicts the non-degeneracy of \( \tilde{\omega}_0 \).

If we suppose that \( C = D \), then this space is preserved by \( \phi \), and we can put \( \phi|_C \) in Jordan form. If it is diagonalizable, with eigenvalues \( \mu \) and \( \nu \), then we can find a basis \( e_1, e_2, e_3, e_4 \) of \( V_4 \) such that \( \phi(e_1) = \mu e_1 \), \( \phi(e_2) = \nu e_2 \) and \( \phi(e_3) = e_4 \). If \( \mu \neq \nu \), then we can replace \( e_5 \) and \( e_6 \) by suitable combinations and reduce to

\[
\omega_1 = e_{15} + \theta_1, \quad \omega_2 = e_{26} + \theta_2, \quad \omega_3 = e_{35} + e_{46} + \theta_3.
\]

Moreover, the \( H \)-symmetry of \( \omega_0 \) implies that \( \tilde{\omega}_0(e_1, e_2) = \tilde{\omega}_0(e_2, e_3) = \tilde{\omega}_0(e_3, e_4) = \tilde{\omega}_0(e_1, e_4) = 0 \), so that \( \omega_0 = a e_{13} + b e_{24} \) for some non-zero scalars \( a \) and \( b \). Then an easy computation shows that the relation \( q \) must be such that

\[
q_{12} = q_{13} = q_{23} = q_{33} = 0.
\]

Moreover, by considering the terms in \( \wedge^3 V_4 \wedge e_5 \) we get the relation \( q_{01} e_1 \omega_0 + q_{03} e_3 \omega_0 + q_{11} e_1 \theta_1 = 0 \). In particular, taking the coefficient of \( e_{234} \) in this relation we get \( q_{03} = 0 \). Finally \( q_{13} = 0 \) for each \( i = 0 \ldots 3 \). But then the relation \( q \) is degenerate, a contradiction!

Now suppose that \( \mu = \nu \), then we can reduce to

\[
\omega_1 = e_{15} + \theta_1, \quad \omega_2 = e_{25} + \theta_2, \quad \omega_3 = e_{35} + e_{46} + \theta_3.
\]

But then the \( H \)-symmetry of \( \omega_0 \) implies that \( \tilde{\omega}_0(e_1, e_4) = \tilde{\omega}_0(e_2, e_4) = \tilde{\omega}_0(e_3, e_4) = 0 \), contradicting the non-degeneracy of \( \omega_0 \).

Finally, suppose that \( \phi|_C \) is not diagonalizable. Then we can reduce to

\[
\omega_1 = e_{15} + \theta_1, \quad \omega_2 = e_{25} + e_{16} + \theta_2, \quad \omega_3 = e_{35} + e_{46} + \theta_3.
\]

The \( H \)-symmetry of \( \omega_0 \) implies that \( \tilde{\omega}_0(e_1, e_2) = \tilde{\omega}_0(e_1, e_4) = \tilde{\omega}_0(e_3, e_4) = 0 \) and \( \tilde{\omega}_0(e_2, e_4) = \tilde{\omega}_0(e_1, e_3) \), so that \( \omega_0 = a(e_{13} + e_{24}) + b e_{14} \), with \( a \neq 0 \). Considering the terms in \( \wedge^2 V_4 \wedge e_6 \), we deduce that \( q \) must be such that \( q_{22} = q_{33} = q_{23} = q_{13} = 0 \). The relation has therefore the form

\[
q = q_{00} \omega_0^2 + 2q_{01} \omega_0 \omega_1 + 2q_{02} \omega_0 \omega_2 + 2q_{03} \omega_0 \omega_3 + q_{11} \omega_1^2 + 2q_{12} \omega_1 \omega_2 = 0,
\]

which is non-degenerate if and only if \( q_{12} q_{03} \neq 0 \). Let us now consider the projections of this relation on \( \wedge^4 V_4 \), \( \wedge^3 V_4 \wedge e_5 \) and \( \wedge^3 V_4 \wedge e_6 \). The first one will always be satisfied by a suitable choice of \( q_{00} \), since \( \omega_0^2 \) generates \( \wedge^4 V_4 \). The two other projections yield

\[
q_{01} \omega_0 e_1 + q_{02} \omega_0 e_2 + q_{03} \omega_0 e_3 + q_{11} \theta_1 e_1 + q_{12} (\theta_1 e_2 + \theta_2 e_1) = 0,
\]

\[
q_{02} \omega_0 e_1 + q_{03} \omega_0 e_4 + q_{12} \theta_1 e_1 = 0.
\]

These are two equations in \( \wedge^3 V_4 \), which has dimension four. The term in \( e_{234} \) in the first equation yields \(-aq_{03} + q_{12} \theta_1 e_4 = 0 \). The term in \( e_{134} \) in the second equation yields \( aq_{03} + q_{12} \theta_1 e_4 = 0 \). Hence \( aq_{03} = 0 \), a contradiction.

We have considered all the cases for which \( A \) and \( B \) are three-dimensional and distinct. Suppose now that \( A = B \), again a three-dimensional subspace of \( V_4 \), and consider the Jordan type of \( \phi \).

If \( \phi \) has only one Jordan block for each of its eigenvalues, the \( H \)-symmetry of \( \omega_0 \) readily implies that \( \wedge^2 A \wedge \omega_0 = 0 \), which contradicts the non-degeneracy of \( \omega_0 \).
If $\phi$ has a triple eigenvalue, then $H$ itself is contained in a $\mathbb{P}(\wedge^2 V_5)$; but then it would meet $G(2, V_5)$, a contradiction.

If $\phi$ has a double and a simple eigenvalue, then after a suitable change of $e_5$, $e_6$ we may suppose that

$$\omega_1 = e_{15} + \theta_1, \quad \omega_2 = e_{25} + e_{16} + \theta_2, \quad \omega_3 = e_{36} + \theta_3.$$ 

Let us consider the relation $q$. The terms in $\wedge^2 V_4 \wedge e_5$ yield $q_{13} = q_{23}$. The terms in $\wedge^3 V_4 \wedge e_5$ and $\wedge^3 V_4 \wedge e_6$ yield the two equations

$$q_{01} \omega_0 e_1 + q_{02} \omega_0 e_2 + q_{11} \theta_1 e_1 + q_{12} (\theta_1 e_2 + \theta_2 e_1) + q_{22} \theta_2 e_2 = 0,$$

$$q_{03} \omega_0 e_3 + q_{33} \theta_3 e_3 = 0.$$ 

Both equations take values in $\wedge^3 V_4$, which is four-dimensional. In particular the first equation, which has five unknowns, is underdetermined, which means that the relation $q$ will in fact never be unique (up to scalar): a contradiction!

Finally, if $A$ and $B$ are not three-dimensional, and if this remains true for any choice of $e_5$ and $e_6$, then the plane $\langle \omega_1, \omega_2, \omega_3 \rangle$ must contain an element $\omega_4$ in $\mathbb{P}(\wedge^2 V_4)$, and then the pencil $\langle \omega_0, \omega_4 \rangle$ would contain a rank two tensor, a contradiction again. $\square$

4. Reducible case

In this section we discuss the case where the characteristic surface $S_H$ is a reducible cubic. Equivalently, $S_H$ contains a plane $P$. Since $H$ belongs to $\mathcal{V}$, this plane must be made of two-forms of constant rank four. Such planes have been classified in [12]; up to the action of $PGL_6$ there are exactly four different types, which we consider case by case.

Type 1. The first type consists in three-planes contained in $\wedge^2 C^5$; it is represented by the plane $L$ generated by

$$\omega_1 = e_1 \wedge e_4 + e_2 \wedge e_3,$$

$$\omega_2 = e_1 \wedge e_5 + e_2 \wedge e_4,$$

$$\omega_3 = e_2 \wedge e_5 + e_3 \wedge e_4.$$ 

Note that $\text{Sym}^2 L \subset \wedge^4 C^5 \subset \wedge^4 C^6$, and therefore any tensor in $L \wedge L$ has rank two. Suppose there exists $\lambda \in L$ and $\omega \in \wedge^2 C^6$ such that $\omega \wedge \lambda$ belongs to $L \wedge L$. Then necessarily $\omega$ belongs to $\wedge^2 C^5$. But then $H$ must be a four-plane in $\wedge^2 C^5$, in which the variety of tensors of rank at most two has codimension 3. So $H$ necessarily contains some tensors of rank two, a contradiction.

Now suppose that $\omega \wedge \omega$ belongs to $L \wedge L$. Let us decompose $\omega = \phi + \alpha \wedge e_6$. Then $\omega \wedge \omega = \phi \wedge \phi + 2\phi \wedge \alpha \wedge e_6$. In particular we need $\phi \wedge \alpha = 0$. If $\alpha = 0$, then $\omega$ belongs to $\wedge^2 C^5$ and we get a contradiction as before. Otherwise, $\alpha$ must divide $\phi$ and then $\omega$ has rank two, a contradiction again. We conclude that no three-plane of type 1 can be contained in a four-plane of $\mathcal{V}$. 


Type 2. The second type is provided by the plane $L$ generated by

$$
\begin{align*}
\omega_1 &= e_0 \wedge e_4 - e_1 \wedge e_3, \\
\omega_2 &= e_0 \wedge e_5 - e_2 \wedge e_3, \\
\omega_3 &= e_1 \wedge e_5 - e_2 \wedge e_4,
\end{align*}
$$

An easy computation shows that $L$ is stabilized by a subgroup $\text{Stab}_L$ of $GL_6$ isomorphic to $\mathbb{C}^* \times GL_3$. Moreover, the action of $\text{Stab}_L$ on $L$ is equivalent to the action of $\mathbb{C}^* \times GL_3$ on $A_3(\mathbb{C})$, the space of $3 \times 3$ skew-symmetric matrices, given by $(z, A) : X \mapsto zA^t XA$. In particular, the action of $\text{Stab}_L$ on $L$ has exactly two orbits, the origin and its complement; and the action of $\text{Stab}_L$ on $\text{Sym}^2 L$ has exactly four orbits, given by the q-rank.

For $H$ to belong to $\mathcal{V}$, we need $\omega \wedge \omega$ to be an element of $L \wedge L$ of q-rank three. We may suppose this element, up to the action of $\text{Stab}_L$, to be

$$
-\frac{1}{2}(\omega_1^2 + \omega_2^2 + \omega_3^2) = e_{0134} + e_{0235} + e_{1245}.
$$

Then we recover $\omega$ by applying $pf^*$, which yields (up to scalar)

$$
\omega = e_0 \wedge e_3 + e_1 \wedge e_4 + e_2 \wedge e_5.
$$

We also need to consider the case where there exists $\lambda \in L$ such that $\omega \wedge \lambda$ belongs to $L \wedge L$. Up to the action of $\text{Stab}_L$, we may suppose that $\lambda = \omega_1$. Then a straightforward computation shows that (modulo $L$) $\omega$ must be a combination of $e_0 \wedge e_1, e_0 \wedge e_3, e_1 \wedge e_4, e_3 \wedge e_4$. In particular, $\omega$ belongs to $\wedge^2 M$, where $M = \langle e_0, e_1, e_3, e_4 \rangle$. Therefore, the line joining $\omega$ to $\omega_1$, which also belongs to $\wedge^2 M$, will necessarily contain an element of rank two, and we get a contradiction.

The conclusion of this discussion is that up to equivalence, a three-plane $L$ of type two can be uniquely extended to a four-plane $H$ in $\mathcal{V}$, which in a suitable basis is the four-plane of matrices of the form:

$$
\begin{pmatrix}
0 & 0 & 0 & d & a & b \\
0 & 0 & 0 & -a & d & c \\
0 & 0 & 0 & -b & -c & d \\
-d & a & b & 0 & 0 & 0 \\
-a & -d & c & 0 & 0 & 0 \\
-b & -c & -d & 0 & 0 & 0
\end{pmatrix}
$$

The associated cubic surface has equation $d(a^2 + b^2 + c^2 + d^2) = 0$. It is the union of the plane $L$ and a smooth quadric.

Type 3. The third type is provided by the plane $L$ generated by

$$
\begin{align*}
\omega_1 &= e_0 \wedge e_2 + e_1 \wedge e_3, \\
\omega_2 &= e_0 \wedge e_3 + e_1 \wedge e_4, \\
\omega_3 &= e_0 \wedge e_4 + e_1 \wedge e_5,
\end{align*}
$$

In this case, $L \wedge L \cong \text{Sym}^2 L$ is the space of four-forms divisible by $e_0 \wedge e_1$. In particular, $L \wedge L$ does not contain any rank six element, so our general strategy does not apply.
Instead, let us try directly to complete $L$ into a four-plane $H \in \mathcal{V}$, with some two-form $\omega$ such that $\omega \wedge \omega$ belongs to $L \wedge L$. We can decompose $\omega = e_0 \wedge u_0 + e_1 \wedge u_1 + \psi$, where $\psi$ does not involve $e_0$, $e_1$. Then $\omega \wedge \omega$ is divisible by $e_0 \wedge e_1$ if and only if

$$u_0 \wedge \psi = u_1 \wedge \psi = 0 \quad \text{and} \quad \psi \wedge \psi = 0.$$ 

This means that $\psi$ has rank two ($\psi$ cannot be zero, for otherwise $H \wedge H = L \wedge L$) and is divisible by $u_0$ and $u_1$. Note that $u_0$, $u_1$ cannot be dependent, since otherwise $\omega$ would have rank two. So $\psi$ must be a multiple of $u_0 \wedge u_1$ and after a suitable normalization we can suppose that

$$\omega = e_0 \wedge u_0 + e_1 \wedge u_1 + u_0 \wedge u_1.$$

Note that an element in $L$ is of the form $\lambda = e_0 \wedge \ell_0 + e_1 \wedge \ell_1$, and the planes $\langle \ell_0, \ell_1 \rangle$ span a Veronese surface in $G(2, 5)$. It is easy to see that $(\omega \wedge L) \cap (L \wedge L) = 0$ if and only if the plane $\langle u_0, u_1 \rangle$ does not belong to this surface. This ensures that the kernel of the map $\text{Sym}^2 H \to H \wedge H$ is one-dimensional. For $H$ to belong to $\mathcal{V}$, we finally need that a generator of this kernel be of maximal q-rank, which is easily checked to be true in general: take for example $u_0 = e_5$ and $u_1 = e_2$, then the relation is

$$\omega^2 + \omega_1 \omega_3 + \omega_2^2 = 0,$$

which is non-degenerate. The four-plane of skew-symmetric matrices is:

$$\begin{pmatrix} 0 & 0 & b & c & d & a \\ 0 & 0 & a & b & c & d \\ -b & -a & 0 & 0 & 0 & -a \\ -c & -b & 0 & 0 & 0 & 0 \\ -d & -c & 0 & 0 & 0 & 0 \\ -a & -d & a & 0 & 0 & 0 \end{pmatrix}$$

The associated cubic surface has equation $a(bd - c^2) = 0$. A general point in $H$ is of the form $\phi = e_0 \wedge (su_0 + \ell_0) + e_1 \wedge (su_1 + \ell_1) + su_0 \wedge u_1$, and has rank four or less when $s \ell_0 \wedge \ell_1 \wedge u_0 \wedge u_1 = 0$. This implies that the characteristic surface is the union of $L$ and a cone over a conic in $L$. This conic may be singular: consider for example the case where $u_0 = e_5$ and $u_1 = e_3$; we get the following four-plane of skew-symmetric matrices:

$$\begin{pmatrix} 0 & 0 & b & c & d & a \\ 0 & 0 & 0 & a + b & c & d \\ -b & 0 & 0 & 0 & 0 & 0 \\ -c & -a - b & 0 & 0 & 0 & -a \\ -d & -c & 0 & 0 & 0 & 0 \\ -a & -d & a & 0 & 0 & 0 \end{pmatrix}$$

The associated cubic surface has equation $abc = 0$. 

Type 4. The fourth type is represented by the plane \( L \) generated by
\[
\begin{align*}
\omega_1 &= e_0 \wedge e_3 + e_1 \wedge e_2, \\
\omega_2 &= e_0 \wedge e_4 + e_2 \wedge e_3, \\
\omega_3 &= e_0 \wedge e_5 + e_1 \wedge e_3.
\end{align*}
\]
Then \( L \wedge L \simeq \text{Sym}^2 L \) is generated by \( f_{14} + f_{25}, f_{24}, f_{34}, f_{15}, f_{35}, f_{45} \). In particular, an element of \( L \wedge L \) can be written as \( f_4 \wedge g_4 + f_5 \wedge g_5 \), hence never has rank six.

So let us directly try to complete \( L \) into a four-plane \( H \in \mathcal{V} \) with a two-form \( \omega \) such that \( \omega \wedge \omega \) belongs to \( L \wedge L \). Let us decompose
\[
\omega = \psi + u_4 \wedge e_4 + u_5 \wedge e_5 + z e_4 \wedge e_5,
\]
where \( \psi, u_4, u_5 \) do not involve \( e_4, e_5 \). Then we have \[
\frac{1}{2} \omega \wedge \omega = \frac{1}{2} \psi \wedge \psi + \psi \wedge u_4 \wedge e_4 + \psi \wedge u_5 \wedge e_5 + (z \psi - u_4 \wedge u_5) \wedge e_4 \wedge e_5.
\]
For this to belong to \( L \wedge L \), we need that \( z \psi = u_4 \wedge u_5 \). If \( z \neq 0 \), we simply get that \( \omega \wedge \omega = 0 \), which is excluded. If \( z = 0 \), then \( u_4 \) and \( u_5 \) must be collinear (linearly dependent), and after a change of basis we may suppose that \( u_5 = 0 \). Then we need \( \psi \wedge u_4 \) to be a combination of \( e_{012} \) and \( e_{023} \), in which case \( \psi \wedge u_4 \wedge e_4 \) is a linear combination of \( \omega_1 \wedge \omega_2 \) and \( \omega_2 \wedge \omega_2 \). But then the kernel of \( \text{Sym}^2 H \rightarrow H \wedge H \) is generated by a quadratic relation of the form \( \omega^2 = Q(\omega_1, \omega_2) \), which contradicts the non-degeneracy condition.

There remains the possibility that \( (\omega \wedge L) \cap (L \wedge L) \neq 0 \). Another computation leads to the same conclusion.

We summarize this discussion as follows:

**Theorem 4.1.** Let \( H \in \mathcal{V} \) be such that the characteristic surface \( S_H \) contains a plane \( L \). Then either:

1. \( L \) is of type 2 and the residual component of \( S_H \) is a smooth quadric;
2. \( L \) is of type 3 and the residual component of \( S_H \) is a quadratic cone, whose vertex is outside \( L \).

5. **Example: integrable system of conservation laws**

The following 4-component system was obtained in [3] in the classification of non-diagonalisable linearly degenerate systems of conservation laws of Temple’s class whose characteristic speeds are harmonic (have cross-ratio \(-1\)):
\[
\begin{align*}
u_1^t &= u_3^x, \\
u_2^t &= u_4^x, \\
u_3^t &= \left( \frac{u^1 u^2 u^4 + u^3 ((u^1)^2 + (u^4)^2 - (u^2)^2 - 1)}{u^1 u^3 + u^4 u^4} \right)_x, \\
u_4^t &= \left( \frac{u^1 u^2 u^3 + u^4 ((u^1)^2 + (u^4)^2 - (u^3)^2 - 1)}{u^1 u^3 + u^3 u^4} \right)_x.
\end{align*}
\]
This system has a third-order Hamiltonian structure [10] and therefore belongs to the class discussed in this paper. System (14) possesses an important additional property of integrability (which can be defined in many equivalent ways, in particular, by the existence of a suitable Lax pair). To demonstrate the Lax pair, we introduce the $5 \times 5$ skew-symmetric matrix

$$S = \begin{pmatrix}
0 & 0 & u^1 & u^2 & 1 \\
0 & 0 & u^3 & u^4 & 1 \\
-u^1 & -u^3 & 0 & 0 & 0 \\
-u^2 & -u^4 & 0 & 0 & 0 \\
-1 & -1 & 0 & 0 & 0
\end{pmatrix},$$

in terms of which system (14) can be written as a single matrix Hopf equation,

$$S_t = (bS - aS^3)_x,$$

where $a = \frac{1}{u^1u^3 + u^2u^4}$, $b = \frac{(u^1)^2 + (u^2)^2 + 1}{u^1u^3 + u^2u^4}$. The corresponding Lax pair is

$$\psi_x = \lambda S\psi, \quad \psi_t = \lambda (bS - aS^3)\psi,$$

where $\lambda$ is a spectral parameter. In the notation of (1), we have

$$V^1 = u^3, \quad V^2 = u^4,$$

$$V^3 = \frac{u^1u^2u^4 + u^3((u^3)^2 + (u^4)^2 - (u^2)^2 - 1)}{u^1u^3 + u^2u^4}, \quad V^4 = \frac{u^1u^2u^3 + u^4((u^3)^2 + (u^4)^2 - (u^1)^2 - 1)}{u^1u^3 + u^2u^4}.$$

The corresponding congruence (2) is a linear congruence in $\mathbb{P}^5$ specified by four linear relations in the Plücker coordinates (minors of matrix (5) from the Introduction):

$$V^1 - u^3 = 0, \quad V^2 - u^4 = 0,$$

$$u^1V^2 - u^2V^1 + u^3V^4 - u^4V^3 = 0, \quad u^1V^3 - u^3V^1 + u^2V^4 - u^4V^2 + 1 = 0.$$

The corresponding subspace $H$ in $\wedge^2(V_6)$ is the following 4-plane of skew-symmetric matrices:

$$\begin{pmatrix}
0 & c & d & 0 & a & 0 \\
-c & 0 & 0 & d & b & 0 \\
-d & 0 & 0 & c & 0 & a \\
0 & -d & -c & 0 & 0 & b \\
-a & -b & 0 & 0 & 0 & d \\
0 & 0 & -a & -b & -d & 0
\end{pmatrix}$$

The associated cubic surface has equation $d(a^2 + b^2 + c^2 - d^2) = 0$. It is the union of a plane and a smooth quadric. Thus, we have reducible case, Type 2.
6. Concluding remarks

Based on the existing results on Hamiltonian systems of conservation laws, it is tempting to conjecture that example (14) is the unique such system which satisfies the additional property of integrability. The main step in proving this conjecture would be to show that systems whose characteristic cubic surfaces are irreducible, cannot be integrable. Unfortunately, there is no complete list of canonical forms of such systems (to which one would apply any of the existing integrability tests). There is, however, a way to bypass this problem by noting that the property of integrability is preserved ‘in the limit’. Thus, it would be sufficient to prove non-integrability in the ‘most degenerate’ cases with irreducible (possibly, singular) characteristic surfaces. We hope to address these issues elsewhere.

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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