Vacuum energy in a noncommutative-geometry setting

Peter K.F. Kuhfittig*
*Department of Mathematics, Milwaukee School of Engineering, Milwaukee, Wisconsin 53202-3109, USA

Abstract
It has recently been proposed that vacuum energy is zero in spite of the quantum-field fluctuations that occur everywhere, even at absolute zero. The implication is that dark energy must have a different origin, unrelated to vacuum energy. The proposal is based on the use of a local Lorentz frame in a non-gravitational field; this also results in a stress-energy tensor of the vacuum that is a perfect fluid with equation of state $p_{\text{vac}} = -\rho_{\text{vac}}$. It is noted in this paper that noncommutative geometry, an offshoot of string theory, yields the same equation of state without being confined to a local Lorentz frame. As a result, the noncommutative-geometry background is able to account for the accelerated expansion and hence for dark energy. So vacuum energy can only be zero if dark energy is indeed unrelated to vacuum energy.

1 Introduction

In a recent paper, Gregory Ryskin [1] proposed that vacuum energy is zero, even though the vacuum is filled with short-lived virtual particle-antiparticle pairs due to the fluctuations of quantum fields. These fluctuations, even at absolute zero, have nonzero energy. Ryskin uses a local Lorentz frame in a non-gravitational field, as a result of which the stress-energy tensor of the vacuum is that of a perfect fluid with equation of state

$$p_{\text{vac}} = -\rho_{\text{vac}}.$$ (1)

According to Ref. [1], this equation is a direct consequence of relativistic invariance. Moreover, the relativistic invariance dictates the form of the stress-energy tensor in terms of the fundamental metric tensor: $-\rho_{\text{vac}} g_{\mu\nu}$, where $\rho_{\text{vac}}$ is the vacuum energy density [2]. Since this tensor vanishes in a particular frame of reference, it must vanish in all. The point is that if the vacuum energy is indeed zero, then dark energy, first discovered in 1998, must have a different origin, unrelated to vacuum energy.

It is proposed in this paper that given a noncommutative-geometry background, we arrive at the same equation of state, even though we are no longer confined to a local Lorentz frame. This enables us to use noncommutative geometry to account for the accelerated expansion on both the local and cosmological scales. The two viewpoints turn out to be consistent with each other.

*kuhfitti@msoe.edu
2 Noncommutative geometry

Our first task is to review the basic ideas in noncommutative geometry, an area that is based on a certain outcome of string theory, namely, that coordinates may become noncommuting operators on a $D$-brane [3, 4]. This statement refers to the commutator $[x^\mu, x^\nu] = i \theta^{\mu \nu}$, where $\theta^{\mu \nu}$ is an antisymmetric matrix. Moreover, noncommutativity replaces point-like structures by smeared objects, thereby elimination the divergences that normally occur in general relativity [5, 6, 7]. A natural way to accomplish the smearing effect is to use a Gaussian distribution of minimal length $\sqrt{\beta}$ [8, 9]. Alternatively, one may assume that the energy density of the static and spherically symmetric and particle-like gravitational source has the form [10, 11]

$$\rho(r) = \frac{\mu \sqrt{\beta}}{\pi^2 (r^2 + \beta)^2}. \quad (2)$$

This form can be interpreted to mean that the mass $\mu$ of the particle is diffused throughout the region of linear dimension $\sqrt{\beta}$ due to the uncertainty. Eq. (2) leads to the mass distribution

$$M(r) = \int_0^r 4\pi(r')^2 \rho(r') \, dr' = \frac{2M}{\pi} \left( \tan^{-1} \frac{r}{\sqrt{\beta}} - \frac{r \sqrt{\beta}}{r^2 + \beta} \right), \quad (3)$$

where $M$ is now the total mass of the source.

Returning now to Ref. [8], we examine the equation of state (1) from a different perspective. Suppose we start with the stress-energy tensor $T_{\alpha \beta}$ and the covariant conservation equation $T_{\alpha ; \beta} = 0$. If $\beta = r$, we obtain

$$\frac{\partial}{\partial r} T^r_r = -\frac{1}{2} g^{tt} \frac{\partial g_{tt}}{\partial r} (T^r_r - T^t_t) - g^{\theta \theta} \frac{\partial g_{\theta \theta}}{\partial r} (T^r_r - T^\theta_\theta) \quad (4)$$

in terms of the stress-energy tensor and the fundamental metric tensor. According to Ref. [8], to preserve the property $g_{tt} = g_{rr}^{-1}$, we require that $T^r_r = T^t_t = -\rho(r)$, while

$$T^\theta_\theta = -\rho(r) - \frac{r}{2} \frac{\partial \rho(r)}{\partial r}. \quad (5)$$

The main conclusion is that a massive structureless point is replaced by a self-gravitating droplet of anisotropic fluid of density $\rho$, yielding the radial pressure

$$p_r(r) = -\rho(r) \quad (6)$$

and the tangential pressure

$$p_\perp(r) = -\rho(r) - \frac{r}{2} \frac{\partial \rho(r)}{\partial r}. \quad (7)$$

It is also noted that on physical grounds, the radial pressure is needed to prevent the collapse to a matter point.

In seeking a connection to Eq. (1), we return to Ref. [8] once again: it is emphasized that noncommutative geometry is an intrinsic property of spacetime and does not depend
on any particular properties such as curvature. Moreover, the effects of noncommutativity can be described as follows: keep the standard form of the Einstein tensor on the left-hand side of the field equations and insert the modified stress-energy tensor as a source on the right-hand side. This leads to the important conclusion that the length scales need not be microscopic. So unlike Eq. (1), we are no longer restricted to a local Lorentz frame and can retain Eq. (6). As a result, we can use Eqs. (2) and (7) to determine

\[ p_\perp(r) = -\rho(r) - \frac{r}{2} \frac{\partial \rho(r)}{\partial r} = \rho_\perp(r) + \frac{2\mu r^2 \sqrt{\beta}}{\pi^2 (r^2 + \beta)^3}. \] (8)

So for larger \( r \), we have

\[ p_\perp(r) \approx p_\perp(r). \] (9)

We can therefore take the equation of state to be

\[ p(r) = -\rho(r) \] (10)

since the pressure is isotropic.

### 3 The accelerating expansion

#### 3.1 Large scale

Because of its isotropic nature, Eq. (10) fits into a more general cosmological setting. (We will return to the local frame in the next subsection.) In this setting, we are dealing with a barotropic equation of state \( p = \omega \rho \), where \( \omega \) is a constant. Now, by Eq. (10), \( \omega = -1 \), which corresponds to Einstein’s cosmological constant, possibly the best model for dark energy. These observations are consistent with the accelerated expansion \( \ddot{a}(t) > 0 \) in the Friedmann equation

\[ \frac{\ddot{a}(t)}{a(t)} = -\frac{4\pi}{3} (\rho + 3p). \] (11)

By Eq. (10), it now follows directly that

\[ -\frac{4\pi}{3} (\rho - 3\rho) > 0. \] (12)

#### 3.2 Small scale

While Eq. (12) deals with a global scale, our starting point, Eq. (1), assumes a local frame. In this subsection, we return to a microscopic scale to show that the results are consistent with those above.

First we recall that a quantum fluctuation is the temporary appearance of energetic particles out of empty space, as allowed by the uncertainty principle. Although transient fluctuations, they exhibit some of the characteristics of ordinary particles. So both Eqs. (2) and (3) can be applied. Also, since the Universe is a 3-sphere, any point can be chosen for the origin.
Consider now the following metric in Schwarzschild coordinates [12]:

\[ ds^2 = -e^{\Phi(r)} dt^2 + \frac{dr^2}{1 - \frac{2M(r)}{r}} + r^2 (d\theta^2 + \sin^2 \theta \, d\phi^2). \tag{13} \]

(We are using units in which \( c = G = 1 \).) Here \( M(r) \) is obtained in Eq. (3). If we remain sufficiently close to the origin, we can assume that \( \Phi(r) \approx \) constant. The Einstein field equations are

\[ \rho(r) = \frac{2M'(r)}{8\pi r^2}, \tag{14} \]

\[ p_r(r) = \frac{1}{8\pi} \left[ -\frac{2M}{r^3} + \frac{2\Phi'}{r} \left( 1 - \frac{2M}{r} \right) \right], \tag{15} \]

and

\[ p_t(r) = \frac{1}{8\pi} \left( 1 - \frac{2M}{r} \right) \left[ \Phi'' - \frac{2M'r - 2M}{2r(r - 2M)} \Phi' + \left( \Phi' \right)^2 + \frac{\Phi'}{r} - \frac{2M'r - 2M}{2r^2(r - 2M)} \right]. \tag{16} \]

The conservation law \( T^\alpha_{\beta, \alpha} = 0 \) implies that only Eqs. (14) and (15) are actually needed. Since \( \Phi'(r) \approx 0 \), we now get

\[ \rho(r) = \frac{2M'(r)}{8\pi r^2} \tag{17} \]

and

\[ p_r(r) = -\frac{1}{8\pi} \frac{2M(r)}{r^3}. \tag{18} \]

Retaining our assumption of isotropic pressure, Eqs. (17) and (18) yield, in view of Eqs. (2) and (3),

\[ \rho + 3p = \frac{\mu\sqrt{\beta}}{\pi^2(r^2 + \beta)^2} - \frac{3}{8\pi} \left( \frac{1}{r^3} \right) \frac{4M}{\pi} \left( \tan^{-1} \frac{r}{\sqrt{\beta}} - \frac{r\sqrt{\beta}}{r^2 + \beta} \right) \]

\[ = \frac{1}{\pi^2\beta^{3/2}} \left[ \frac{\mu}{(r^2/\beta + 1)^2} - \frac{3}{2} \frac{M}{(r/\sqrt{\beta})^3} \left( \tan^{-1} \frac{r}{\sqrt{\beta}} - \frac{r/\sqrt{\beta}}{r^2/\beta + 1} \right) \right]. \tag{19} \]

\( M \) is likely to be only slightly larger than \( \mu \), making \( \rho + 3p \) only slightly less than zero at the origin. Fig. 1 shows the expression for \( \rho + 3p \) plotted against \( r/\sqrt{\beta} \); so near the origin (\( r \) close to zero and \( \sqrt{\beta} > 0 \)), \( \rho + 3p < 0 \), a result that is consistent with Eqs. (11) and (12). So both the small-scale and large-scale viewpoints lead to \( \ddot{a}(t) > 0 \), characteristic of dark energy, all attributable to the noncommutative-geometry background.

In Fig. 1, \( \rho + 3p \) is negative near the origin and approaches zero asymptotically. It is interesting to note that if the smearing disappears altogether, then we return to the classical setting: Fig. 1 shows that if \( \sqrt{\beta} \rightarrow 0 \), then \( \rho + 3p < 0 \) for all \( r \), which takes us back to Subsection [3,1]. So the local and global viewpoints complement each other. [That \( \lim_{\sqrt{\beta} \rightarrow 0} (\rho + 3p) < 0 \) also follows from Eq. (19).]
4 Discussion

It is proposed in Ref. [1] that vacuum energy is zero, as a result of which the origin of dark energy must be unrelated to vacuum energy. The equation of state $p_{\text{vac}} = -\rho_{\text{vac}}$ is a direct consequence of relativistic invariance, according to Ref. [1]. It is also a consequence of a noncommutative-geometry background. The latter does not require the local Lorentz frame of the former. The result is that on both the local and cosmological scales, $\rho + 3p < 0$, yielding $\ddot{a}(t) > 0$ in the Friedmann equation. A noncommutative-geometry background, as formulated by Nicolini et al., can therefore account for the accelerated expansion and hence for dark energy. So vacuum energy can only be zero if dark energy is indeed unrelated to vacuum energy.

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