TWO-WEIGHTED NORM INEQUALITIES FOR THE DOUBLE HARDY TRANSFORMS AND STRONG FRACTIONAL MAXIMAL FUNCTIONS IN VARIABLE EXPONENT LEBESGUE SPACES

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Abstract. Two–weight norm estimates for the double Hardy transforms and strong fractional maximal functions are established in variable exponent Lebesgue spaces. Derived conditions are simultaneously necessary and sufficient in the case when the exponent of the right–hand side space is constant.

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1 INTRODUCTION

Our goal is to establish two–weight criteria for the double Hardy transforms and strong fractional maximal functions in variable exponent Lebesgue spaces. Our interest in these problems is stipulated by the following circumstances: by the need in various applications to the boundary value problems in PDE and the fact that strong maximal operator, unlike of the Hardy-Littlewood maximal function is bounded in \( L^{p(x)} \) space if and only if \( p(x) \equiv \text{const} \) (see [25]). As we shall see below the similar phenomenon occurs also for strong fractional maximal operators.

Let us recall some well–known results for the classical Lebesgue spaces (see e.g, [27], [28]).

The celebrated classical Hardy inequality states:

**Theorem A.** Let \( p \) be constant satisfying the condition \( 1 < p < \infty \) and let \( f \) be a measurable, nonnegative function in \((0, \infty)\). Then

\[
\left( \int_0^\infty \left( \frac{1}{x} \int_0^x f(y)dy \right)^\frac{1}{p} dx \right)^\frac{1}{p} \leq \frac{p}{p-1} \left( \int_0^\infty f^p(x)dx \right)^\frac{1}{p}.
\]

Two-weighted boundedness criteria for the Hardy transform

\[
(\mathcal{H}_1 f)(x) = \int_0^x f(y)dy,
\]

reads as follows:

**Theorem B.** Let \( p \) and \( q \) be constants satisfying the condition \( 1 < p \leq q < \infty \). Suppose that \( u \) and \( v \) are weight functions on \( \mathbb{R}_+ \). Then each of the following conditions are necessary
and sufficient for the inequality
\[
\left( \int_0^\infty \left( \int_0^x f(t) dt \right)^q v(x) dx \right)^{\frac{1}{q}} \leq C \left( \int_0^\infty f^p(x) w(x) dx \right)^{\frac{1}{p}}
\]
(1)
to hold for all positive and measurable functions on \( \mathbb{R}_+ \):

a) The Muckenhoupt condition,
\[
A_M := \sup_{x>0} \left( \int_x^\infty v(t) dt \right)^{\frac{1}{q}} \left( \int_0^x \left( \int_0^1 w(t)^{1/p'} dt \right)^{\frac{1}{p'}} \right)^{\frac{1}{p}} < \infty.
\]
Moreover, the best constant \( C \) in (1.1) can be estimated as follows:
\[
A_M \leq C \leq \left( 1 + \frac{q}{p'} \right)^{\frac{1}{q}} \left( 1 + \frac{p}{q} \right)^{\frac{1}{p'}} A_M.
\]

b) The condition of L. E Persson and V. D. Stepanov,
\[
A_{PS} := \sup_{x>0} W(x)^{-\frac{1}{p}} \left( \int_0^x v(t) W(t)^q dt \right)^{\frac{1}{q}} < \infty, \quad W(x) := \int_0^x w(t)^{1-p'} dt.
\]
Moreover, the best constant \( C \) in (1.1) satisfies the following estimates:
\[
A_{PS} \leq C \leq p' A_{PS}.
\]

In 1984 E. Sawyer [38] found a characterization of two-weight inequality in terms of three independent conditions for the double Hardy transform
\[
(H_2 f)(x, y) = \int_0^x \int_0^y f(t, \tau) dtd\tau.
\]

The following statements gives two-weight criteria in terms of just one condition when the weight on the right-hand side is a product of two weights of single variables (see [32], [43], [19], Ch.1):

**Theorem C.** Let \( p \) and \( q \) be constants such that \( 1 < p \leq q < \infty \) and let \( w(x, y) = w_1(x) w_2(y) \). Then the operator \( H_2 \) is bounded from \( L_w^p \) to \( L_v^q \) \( (1 < p \leq q < \infty) \) if and only if the Muckenhoupt's type condition
\[
\sup_{y_1, y_2 > 0} \left( \int_{y_1}^{\infty} \int_{y_2}^{\infty} v(x_1, x_2) dx_1 dx_2 \right)^{\frac{1}{q}} \left( \int_0^{y_1} \int_0^{y_2} w(x_1, x_2)^{1-p'} dx_1 dx_2 \right)^{\frac{1}{p'}} := A_1 < \infty
\]
is fulfilled.

It should be emphasized that from the results regarding the two-weight problem derived in this paper, as a corollary, we deduce trace inequality criteria for the double Hardy transform when the exponent of the initial Lebesgue space is a constant. Another remarkable corollary is
that there exists a variable exponent $p(x)$ for which the double average operator is bounded in $L^{p(x)}$.

In the paper [17] the authors established trace inequality criteria for the strong fractional maximal operator

$$(M_{\alpha,\beta} f)(x,y) := \sup_{I \times J \ni (x,y)} \int_{I \times J} \frac{1}{|I|^{1-\alpha}|J|^{1-\beta}} \int_{I \times J} |f(t,\tau)| dtd\tau, \quad 0 < \alpha, \beta < 1,$$

in constant exponent Lebesgue spaces. In particular, the next statement holds:

**Theorem D ([17])**. Let $p, q, \alpha$ and $\beta$ be constants satisfying the conditions $1 < p < q < \infty$ and let $0 < \alpha, \beta < 1/p$. Then the following statements are equivalent:

(i) $M_{\alpha,\beta}$ is bounded from $L^p(\mathbb{R}^2)$ to $L^q(\mathbb{R}^2)$;

(ii)

$$B_5 := \sup_{I,J} \left( \int_{I \times J} v(x,y) dxdy \right)^{1/q} \left| I \right|^{(p-1)/p} \left| J \right|^{(p-1)/p} < \infty,$$

where $I$ and $J$ are arbitrary bounded intervals in $\mathbb{R}$.

Exploring the two-weight problem for the strong fractional maximal function of variable order, in particular, we prove an analog of Theorem D in variable exponent Lebesgue spaces when the exponent of the initial Lebesgue space is constant.

Let $p$ be a non-negative measurable function on $\mathbb{R}^n$. Suppose that $E$ is a measurable subset of $\mathbb{R}^n$. In the sequel we will use the following notation:

$$p_-(E) := \inf_E p; \quad p_+(E) := \sup_E p; \quad p_- := p_-(\mathbb{R}^n); \quad p_+ := p_+(\mathbb{R}^n).$$

Let $\Omega$ be an open set in $\mathbb{R}$ and let the following condition holds for $p: \Omega \to \mathbb{R}$:

$$1 \leq p_-(\Omega) \leq p(t) \leq p_+(\Omega) < \infty, \quad t \in \mathbb{R}^n.$$

By $L^{p(\cdot)}(\Omega)$ we denote the Banach space of measurable functions $f: \Omega \to \mathbb{R}$ such that

$$\|f\|_{L^{p(\cdot)}(\Omega)} := \|f\|_{p(\cdot)} = \inf \left\{ \lambda > 0 : \int_{\Omega} \left| \frac{f(x)}{\lambda} \right|^{p(x)} dx \leq 1 \right\} < \infty.$$
2 STRONG FRACTIONAL MAXIMAL FUNCTIONS IN $L^{p(\cdot)}$ SPACES

Let
\[ (M_{\alpha(\cdot),\beta(\cdot)}^{s} f)(x, y) = \sup_{Q \ni y} \sup_{J \ni x} |Q|^{\frac{\alpha(x)-1}{n}} |J|^{\frac{\beta(y)-1}{n}} \int_{Q \times J} |f(t, \tau)| dt d\tau, \quad (x, y) \in \mathbb{R}^n \times \mathbb{R}^m, \]

where $\alpha$ and $\beta$ are measurable functions on $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively satisfying the conditions: $0 < \alpha_- \leq \alpha_+ < n$, $0 < \beta_- \leq \beta_+ < n$ and the supremum is taken over all cubes $Q \ni x$ and $J \ni y$ respectively in $\mathbb{R}^n$ and $\mathbb{R}^m$.

If $\alpha(\cdot) \equiv \beta(\cdot)$, then we denote $M_{\alpha(\cdot),\beta(\cdot)}^{s}$ by $M_{\alpha(\cdot)}^{s}$. Further, if $\alpha \equiv \text{const}$, then we use the symbol $M_{\alpha}^{s}$ instead of $M_{\alpha(\cdot)}^{s}$.

Let us take the case $n = m = 1$ and consider the operator $M_{\alpha}^{s}$ with constant parameter $\alpha$:
\[ (M_{\alpha}^{s} f)(x, y) = \sup_{I \ni y} |I \times J|^{\alpha-1} \int_{I \times J} |f|, \quad (x, y) \in \mathbb{R}^2, \]

where the supremum is taken over all bounded intervals $I \ni x$ and $J \ni y$, $0 < \alpha < 1$.

**Theorem.** Let $1 < p_- \leq p_+ < \infty$ and let $0 \leq \alpha < \frac{1}{p_-}$. We set $q(x) = \frac{p(x)}{1-\alpha\cdot p(x)}$. Then $M_{\alpha}^{s}$ is bounded from $L^{p(\cdot)}(\mathbb{R}^2)$ to $L^{q(\cdot)}(\mathbb{R}^2)$ if and only if $p \equiv \text{const}$.

**Proof.** Sufficiency is obvious using iterating process of one-dimensional $L^p \to L^q$ boundedness of the one–dimensional fractional maximal operator
\[ (M_{\alpha} f)(x) = \sup_{I \ni \mathbb{R}} \frac{1}{|I|^{1-\alpha}} \int_{I} |f(t)| dt, \quad 0 \leq \alpha < 1. \]

**Necessity.** We follow T. Kopaliani [25] which proved the theorem for $\alpha = 0$. First we see that if $M_{\alpha}^{s}$ is bounded from $L^{p(\cdot)}(\mathbb{R}^2)$ to $L^{q(\cdot)}(\mathbb{R}^2)$, then
\[ \sup_{R} A_{R} := \sup_{R} \frac{1}{|R|^{1-\alpha}} \|\chi_{R}\|_{L^{p(\cdot)}} \|\chi_{R}\|_{L^{q(\cdot)}} < \infty, \]

where the supremum is taken over all rectangles $R$ in $\mathbb{R}^2$.

Indeed, let $\|f\|_{L^{p(\cdot)}(\mathbb{R}^2)} \leq 1$. Then for every rectangle $R$ we have
\[ c \geq \|M_{\alpha}^{s} f\|_{L^{q(\cdot)}(\mathbb{R}^2)} \geq \|M_{\alpha}^{s} f\|_{L^{q(\cdot)}(R)} \geq \|\chi_{R}\|_{L^{q(\cdot)}} |R|^{\alpha-1} \int_{R} |f(t, \tau)| dt d\tau. \]

Taking now the supremum with respect to $f$, $\|f\|_{L^{p(\cdot)}} \leq 1$, we find that
\[ |R|^{\alpha-1} \|\chi_{R}\|_{L^{q(\cdot)}} \|\chi_{R}\|_{L^{q(\cdot)}} \leq c \]

for all $R \subset \mathbb{R}^2$.

Further, suppose the contrary: $p$ is not constant, i. e.
\[ \inf_{\mathbb{R}^2} p(t) < \sup_{\mathbb{R}^2} p(t). \]

By Luzin’s theorem, there is a family of pointwise disjoint sets $F_i$ satisfying the conditions:
(i) $|\mathbb{R}^2 \setminus \cup_{j} F_j| = 0$;
Let $\tau$ be a fixed point. We first prove the following lemma:

For every fixed $i$, all points of $F_i$ are points of density with respect to the basis consisting of all open rectangles in $\mathbb{R}^2$.

We can find a pair of the type $((x_0, y_1), (x_0, y_2))$ or $((x_1, y_0), (x_2, y_0))$ from $\cup F_i$ such that $p(x_0, y_1) \neq p(x_0, y_2)$ or $p(x_1, y_0) \neq p(x_2, y_0)$. Without loss of generality, assume that this pair is $((x_0, y_1), (x_0, y_2))$ such that $(x_0, y_1) \in F_1$ and $(x_0, y_2) \in F_2$, $y_1 < y_2$.

Let $0 < \varepsilon < 1$ be fixed number. Then there is a number $\delta > 0$ such that for any rectangles $Q_1 \ni (x_0, y_1)$ and $Q_2 \ni (x_0, y_2)$ with diameters less than $\delta$, the following inequalities hold:

$$|Q_1 \cap F_1| > (1 - \varepsilon)|Q_1|, \quad |Q_2 \cap F_2| > (1 - \varepsilon)|Q_2|,$$

where $c_1$ and $c_2$ are some constants.

Let $Q_{1,\tau}$ and $Q_{2,\tau}$ be rectangles with properties (1.1) and (1.2) with the forms $(x_0 - \tau, x_0 + \tau) \times (a, b)$ and $(x_0 - \tau, x_0 + \tau) \times (c, d)$ respectively, where $a < b < c < d$.

Observe now that the following embeddings hold:

$$L^{q(\cdot)}(Q_{2,\tau}) \hookrightarrow L^{q_{Q_2}}(Q_{2,\tau})$$

$$L^{p(\cdot)}(Q_{1,\tau}) \hookrightarrow L^{(p_{Q_1})'}(Q_{1,\tau}),$$

where $q_{Q_2} = \inf_{Q_2} q = \frac{p_{Q_2}}{1 - p_{Q_2} - p_{Q_1}}$, $(p_{Q_1})' = \frac{p_{Q_2}}{p_{Q_1} - 1}$. Further, for the rectangle $Q_{\tau} = (x_0 - \tau, x_0 + \tau) \times (a, d)$, we have that

$$A_{\tau} := \frac{1}{|Q_{\tau}|^{1 - \alpha}} \| \chi_{Q_{\tau}} \|_{L^{p(\cdot)}} \| \chi_{Q_{\tau}} \|_{L^{p(\cdot)}}$$

$$\geq \frac{1}{[2\tau(d - a)]^{1 - \alpha}} \| \chi_{Q_{2,\tau} \cap F_2} \|_{L^{p(\cdot)}} \| \chi_{Q_{1,\tau} \cap F_1} \|_{L^{p(\cdot)}}$$

$$\geq \frac{\alpha}{[2\tau(d - a)]^{1 - \alpha}} \left[ 2\tau(d - c) \right]^{\frac{1}{q_{Q_2}}} \left[ 2\tau(b - a) \right]^{\frac{1}{p_{Q_1}}}$$

$$= c\tau^{\alpha - 1 + \frac{1}{q_{Q_2}} + 1 - \frac{1}{p_{Q_1}}} = c\tau^{\alpha - \left[ \frac{1}{p_{Q_1}} + \frac{1}{q_{Q_2}} \right]} \to \infty \text{ as } \tau \to 0 \text{ because } \alpha - \frac{1}{p_{Q_1}} - \frac{1}{q_{Q_2}} = \alpha - \frac{1}{p_{Q_1}} - \frac{1}{p_{Q_2}} - \alpha < 0$$

(recall that $a$, $b$, $c$ and $d$ are fixed).

This contradicts the condition $\sup_{R} A_{\tau} < \infty$.

\[ \square \]

3. Double Hardy Transform in $L^p(\cdot)$ spaces

Let

$$(\mathcal{H}_2 f)(x, y) = \int_0^x \int_0^{y/\tau} f(t, \tau) dt d\tau, \quad (x, y) \in \mathbb{R}_+^2.$$ 

First we prove the following lemma:
Lemma 3.1. Let $p$ be a constant satisfying the condition $1 < p < \infty$. Suppose that $0 < b \leq \infty$. Let $\rho$ be an almost everywhere positive function on $[0, b)$. Then there is a positive constant $c$ such that for all $f \in L^p_\rho([0, b))$, $f \geq 0$, the inequality

$$
\int_0^b \left( \frac{1}{\lambda([0, x])} \int_0^x f(t) dt \right)^p \lambda(x) dx \leq C \int_0^b (f(x) \rho(x))^p dx
$$

holds, where $\lambda(x) = \rho^{-p'}(x)$ and $\lambda([0, x]) := \int_0^x \lambda(t) dt$.

Proof. It is enough to show that (see e.g. [31], Chapter 1) the condition

$$
\sup_{0 < t < b} \left( \int_t^b \lambda([0, x])^{-p} \lambda(x) dx \right) \left( \int_0^t \lambda(x) dx \right)^{p-1} < \infty
$$

is satisfied.

To check that this condition holds observe that

$$
\int_t^b \lambda([0, x])^{-p} \lambda(x) dx = \int_t^b \left( \int_0^x \lambda(\tau) d\tau \right)^{-p} \lambda(x) dx
$$

$$
= \frac{1}{1-p} \int_t^b d \left( \int_0^x \lambda(\tau) d\tau \right)^{1-p} = \frac{1}{p-1} \left[ \left( \int_0^t \lambda(\tau) d\tau \right)^{1-p} - \left( \int_0^b \lambda(\tau) d\tau \right)^{1-p} \right]
$$

$$
\leq \frac{1}{p-1} \left[ \left( \int_0^t \lambda(\tau) d\tau \right)^{1-p} + \left( \int_0^b \lambda(\tau) d\tau \right)^{1-p} \right].
$$

Now the result follows easily.

\[ \square \]

Theorem 3.1. Let $p$ be constant and let $1 < p \leq q_- \leq q_+ < \infty$. Suppose that $v$ and $w$ be weights on $\mathbb{R}_+^2$ with $w(x, y) = w_1(x) w_2(y)$ for some one-dimensional weights $w_1$ and $w_2$. Then $H_2$ is bounded from $L^p_v(\mathbb{R}_+^2)$ to $L^{q_+}(\mathbb{R}_+^2)$ if and only if

$$
B := \sup_{a, b > 0} \| v(\chi_{J_{ab}}^\infty) \|_{L^{q_+}(\mathbb{R}_+^2)} \left\| \frac{1}{w} \cdot \chi_{J_{ab}^0} \right\|_{L^{p'}(\mathbb{R}_+^2)} < \infty,
$$

where $J_{ab}^\infty = [a, \infty) \times [b, \infty)$ and $J_{ab}^0 = [0, a) \times [0, b)$.

Proof. Necessity follows by the standard way taking the test function

$$
f(x, y) = \left( \int_0^a \int_0^b w^{-p'}(x,y) dx dy \right) \chi_{[0,a] \times [0,b]}(x, y)
$$

in the two-weight inequality and do simple estimates.
**Sufficiency.** Suppose that \( f \geq 0 \) and \( \|f\|_{L^p_b(\mathbb{R}^2)} \leq 1 \). Let \( \{x_k\} \) and \( \{y_j\} \) be sequences of positive numbers chosen so that

\[
\int_0^{x_k} w_1^{-p'} = 2^k, \quad \int_0^{y_j} w_2^{-p'} = 2^j. \tag{3.1}
\]

Without loss of generality assume that \( \int_0^\infty w_1^{-p'} = \int_0^\infty w_2^{-p'} = \infty \). Then \([0, \infty) = \bigcup_{k} [x_k, x_{k+1}) = \bigcup_{j} [y_j, y_{j+1})\). On the other hand, \( \mathbb{R}^2_+ = \bigcup_{k,j} [x_k, x_{k+1}) \times [y_j, y_{j+1}) \). It is easy to see that equalities (3.1) imply:

\[
\int_{x_k}^{x_{k+1}} w_1^{-p'} = 2^k, \quad \int_{y_j}^{y_{j+1}} w_2^{-p'} = 2^j. \tag{3.2}
\]

Denote: \([x_k, x_{k+1}) =: E_k\), \([y_j, y_{j+1}) =: F_j\).

Let us choose \( r \) so that \( p \leq r \leq q_- \). Then

\[
\|v(H_2f)\|^{r}_{L^{q}(\mathbb{R}^2_+)} = \|v(H_2f)\|_{{L^{q}(\mathbb{R}^2_+)}^{r}} \\
\leq c \sup_{\|h\|_{L^{(q)/(r')'}} \leq 1} \iint_{\mathbb{R}^2_+} (v(x, y))^r (H_2f(x, y))^r h(x, y) dxdy.
\]

Further, taking (3.1) and (3.2) into account we have that

\[
\iint_{\mathbb{R}^2_+} (v(x, y))^r (H_2f)^r (x, y) h(x, y) dxdy \\
= \sum_{k,j} \left[ \int_{x_k}^{x_{k+1}} \int_{y_j}^{y_{j+1}} v^r(x, y) h(x, y) dxdy \right] \left[ \int_0^{x_k} \int_0^{y_j} f \right]^r \\
\leq \sum_{k,j} \|v^r(\cdot)\|_{L^{(q)/(r')}(E_k \times F_j)} \|h\|_{L^{(q)/(r')}(\mathbb{R}^2_+)} \left[ \int_0^{x_k} \int_0^{y_j} f \right]^r \\
\leq \sum_{k,j} \|v\|^r_{L^{(q)}(E_k \times F_j)} \left[ \int_0^{x_k} \int_0^{y_j} f \right]^r \\
\leq B^r \sum_{k,j} \|w_1^{-1}\|_{L^{q'}([0,x_k])}^{-r} \|w_2^{-1}\|_{L^{q'}([0,y_j])}^{-r} \left[ \int_0^{x_k} \int_0^{y_j} f \right]^r \\
= c_r B^r \sum_{k,j} \left[ \int_{x_{k+1}}^{x_{k+2}} \int_{y_{j+1}}^{y_{j+2}} (w_1(x)w_2(y))^{-p'} dxdy \right]^r \left[ \frac{1}{\sigma_1(E_k)\sigma_2(F_j)} \int_0^{x_k} \int_0^{y_j} f \right]^r.
\]
(where $\sigma_1(E_k) := \int w_1^{-p'}$, $\sigma_2(F_j) := \int w_2^{-p'}$)

$$\leq c B^r \sum_{k,j} \left( \int \int_{x_{k+1} y_{j+1}} (w_1(x)w_2(y))^{-p'} \left[ \frac{1}{\sigma_1([0, x_{k+2}])} \frac{x_{k+1} y_{j+1}}{x_k y_j} \int \int f \right] dx dy \right)^{r/p}$$

$$\leq c B^r \sum_{k,j} \left( \int \int_{x_{k+1} y_{j+1}} \left[ w_1(x)w_2(y) \right]^{-p'} \left[ \frac{1}{\sigma_1([0, x])} \frac{x y}{x_k y_j} \int \int f \right] dx dy \right)^{r/p}$$

$$\leq c B^r \left[ \int \int \left[ w_1(x)w_2(y) \right]^{-p'} \left[ \frac{1}{\sigma_1([0, x])} \frac{x y}{x_k y_j} \int \int f \right] dx dy \right]^{r/p} .$$

Observe now that Lemma 3.1 implies the inequality

$$\int_{\mathbb{R}^+} \left[ \frac{1}{\sigma([0, x])] \int f \right]^p d\sigma(x) \leq c \int_{\mathbb{R}^+} (f(x)w(x))^p dx. \quad (3.3)$$

By using inequality (3.3) twice together with Fubini’s theorem we find that

$$\left[ \int \int w^{-p'}(x,y) \left[ \frac{1}{\sigma([0, x] \times [0, y])} \int \int f \right] dx dy \right]^{r/p}$$

$$\leq c \left[ \int \int [f(x,y)]^p (w(x,y))^p dx dy \right]^{r/p} \leq c .$$

**Corollary 3.1.** (Trace inequality) Let $1 < p \leq q_- \leq q_+ < \infty$ and let $v$ be a.e. positive function on $\mathbb{R}_+^2$. Then $\mathcal{H}_2$ is bounded from $L^p(\mathbb{R}_+^2)$ to $L^q_v(\mathbb{R}_+^2)$ if and only if

$$\sup_{a,b>0} \| v \chi_{ab} \|_{L^q_v(\mathbb{R}_+^2)} (ab)^{1/p} < \infty .$$

**Definition 3.1.** Let $\Omega$ be an open set in $\mathbb{R}^n$. We say that the exponent function $p(\cdot) \in \mathcal{P}(\Omega)$ if there is a constant $0 < \delta < 1$ such that

$$\int_{\Omega} \delta^{\frac{p(x)}{p_+} - 1} dx < +\infty .$$

Further, we say that $p(\cdot) \in \mathcal{P}_\infty(\Omega)$ if

$$|p(x) - p(y)| \leq \frac{c}{\ln(e + |x|)}$$
for all $x, y \in \Omega$ with $|y| \geq |x|$.

**Corollary 3.2.** Let $1 < p_- \leq q_- \leq q_+ < \infty$ with $p_+ < \infty$. Let $v$ and $w$ be a.e. positive functions on $\mathbb{R}^2$ with $w(x, y) = w_1(x)w_2(y)$. Suppose that $p \in P(\mathbb{R}^2_+)$. If

$$\sup_{a, b > 0} \left\| v \chi_{J_{ab}} \right\|_{L^p(\mathbb{R}^2_+)} \left\| \frac{1}{w} \chi_{J_{ab}} \right\|_{L^{p(q'-2)}(\mathbb{R}^2_+)} < \infty,$$

then $\mathcal{H}_2$ is bounded from $L^p_w(\mathbb{R}^2_+)$ to $L^q_v(\mathbb{R}^2_+)$.  

**Proof.** Recall that (see [4]) if $p \in P(\mathbb{R}^2_+)$, then $L^p(\mathbb{R}^2_+) \hookrightarrow L^p_w(\mathbb{R}^2_+)$. Now Theorem 3.1 completes the proof. \hfill \Box

**Corollary 3.3.** Let $1 < p_- \leq q_- \leq q_+ < \infty$ with $p_+ < \infty$ and $p_\infty = p_-$. Assume that $p \in P(\mathbb{R}^2_+)$. Suppose that $v$ and $w$ are a.e. positive functions on $\mathbb{R}^2_+$ with $w(x, y) = w_2(x)w_2(y)$. If (3.4) holds, then $\mathcal{H}_2$ is bounded from $L^p_w(\mathbb{R}^2_+)$ to $L^q_v(\mathbb{R}^2)$. Let us now consider the operator $\mathcal{H}_2$ on a rectangle $J := [0, a_0] \times [0, b_0]$. In the sequel the following notation will be used:

$$J_{ab}^0 := [0, a] \times [0, b], \quad J_{ab}^1 := [a, a_0] \times [b, b_0].$$

The arguments used in the proof of Theorem 3.1 enable us to formulate the next statements:

**Theorem 3.2.** Let $1 < p_-(J) \leq q_-(J) \leq q_+(J) < \infty$ with $p_+(J) < \infty$. Suppose that $v$ and $w$ are a.e. positive functions on $J$ with $w(x, y) = w_1(x)w_2(y)$ for some one-dimensional weights $w_1$ and $w_2$. If

$$\sup_{0 < a \leq a_0} \left\| v \chi_{J_{ab}^1} \right\|_{L^p(J)} \left\| w^{-1} \chi_{J_{ab}^0} \right\|_{L^{p(q'-2)}(\mathbb{R}^2_+)} < \infty,$$

then $\mathcal{H}_2$ is bounded from $L^p_w(J)$ to $L^q_v(J)$.

**Corollary 3.4.** Let $1 < p_-(J) \leq q_-(J) \leq q_+(J) < \infty$ with $p_-(J) = p(0)$ and $p_+ < \infty$. Let $v$ be a.e. positive function on $J$. Then $\mathcal{H}_2$ is bounded from $L^p(J)$ to $L^q_v(J)$ if

$$\sup_{0 < a \leq a_0} \left\| v \chi_{J_{ab}^1} \right\|_{L^p(J)} \left\| \chi_{J_{ab}^0} \right\|_{L^{p(q')}(\mathbb{R}^2_+)} < \infty.$$

**Corollary 3.5.** There is non-constant exponent $p$ on $[0, 2]^2$ such that the double average operator

$$(Af)(x, y) = \frac{1}{xy} \int_0^x \int_0^y f(t, \tau)dt d\tau$$

is bounded in $L^p([0, 2]^2)$. \n
**Proof.** Let $p$ be defined as follows:

$$p(x, y) = \begin{cases} 3, & (x, y) \in [1, 2]^2; \\ 2, & (x, y) \in [0, 2]^2 \setminus [1, 2]^2. \end{cases}$$

It is clear that $p(0, 0) = p_- = 2$. Also, it is easy to check that

$$\sup_{0 < a, b \leq 2} \left\| (xy)^{-1} \chi_{[a, 2] \times [b, 2]}(x, y) \right\|_{L^p(\mathbb{R}^2_+)} \left\| \chi_{[0, 2]}(ab)^{-1} \right\|_{L^{p(0, 0)}} < \infty.$$  

Corollary 3.4 completes the proof. \hfill \Box
4 TWO-WEIGHT ESTIMATES FOR STRONG FRACTIONAL MAXIMAL FUNCTION IN $L^p(\cdot)$ SPACES

In order to establish two-weight estimates for strong fractional maximal function of variable order we need the next Carleson-Hörmander’s embedding theorem regarding dyadic intervals.

A weight function $\rho$ is said to be satisfying the dyadic reverse doubling condition ($\rho \in RD^d(\mathbb{R})$) if for any two dyadic intervals $I$ and $I'$ with $I \subset I'$, $|I| = \frac{|I'|}{2}$ the inequality

$$\rho(I') \leq b \rho(I)$$

holds with some constant $b > 1$.

**Theorem E** ([42], [39], Lemma 3.10). Let $p$ and $q$ be constants satisfying the condition $1 < p < q < \infty$ and let $\rho$ be a weight function on $\mathbb{R}$ such that $\rho^{1-p'}$ satisfies the dyadic reverse doubling condition. Let $\{c_I\}$ be a sequence of non–negative numbers corresponding to dyadic intervals $I$ in $\mathbb{R}$. Then the following two statements are equivalent:

(i) There is a positive constant $C$ such that

$$\sum_{I \in D} c_I \left( \frac{1}{|I|} \int_I g(x)dx \right)^q \leq c \left( \int_{\mathbb{R}} g(x)^p \rho(x)dx \right)^{q/p}$$

for all nonnegative $g \in L^p_{\rho}(\mathbb{R})$;

(ii) There is a positive constant $C_1$ such that

$$c_I \leq C_1 |I|^q \left( \int_I \rho(x)^{1-p'}dx \right)^{-q/p'}$$

for all $I \in D$.

This result yields the following corollary.

**Corollary A.** Let $p$ and $q$ be constants satisfying the condition $1 < p < q < \infty$ and let $\rho$ be a weight function on $\mathbb{R}$ such that $\rho^{1-p'}$ satisfies the dyadic reverse doubling condition. Then the Carleson-Hörmander inequality

$$\sum_{I \in D} \left( \int_I \rho^{1-p'}(x)dx \right)^{-q/p'} \left( \int_I f(x)dx \right)^q \leq c \left( \int_{\mathbb{R}} f^p(x) \rho(x)dx \right)^{q/p}$$

holds for all nonnegative $f \in L^p_{\rho}(\mathbb{R})$.

Recall that (see Section 2)

$$\left( M^S_{\alpha(\cdot),\beta(\cdot)} f \right)(x, y) = \sup_{Q \in \mathbb{E}_x, J \in \mathbb{E}_y} |Q|^{\frac{\alpha(x)}{n} - 1} |J|^{\frac{\alpha(y)}{m} - 1} \int_{Q \times J} |f|,$$

where $(x, y) \in \mathbb{R}^n \times \mathbb{R}^m$ and $Q$ and $J$ are cubes in $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. For simplicity we take $n = m = 1$ and consider the strong maximal operator

$$(M^S_{\alpha(\cdot),\beta(\cdot)} f)(x, y) = \sup_{I \in \mathbb{E}_x, J \in \mathbb{E}_y} |I|^\alpha(x) - 1 |J|^{\beta(y) - 1} \int_{I \times J} |f|, \ (x, y) \in \mathbb{R}^2,$$
where $0 < \alpha_- \leq \alpha_+ < 1$, $0 < \beta_- \leq \beta_+ < 1$ and $I$ and $J$ are intervals in $\mathbb{R}$.

Together with the operator $M^S_{\alpha(\cdot), \beta(\cdot)}$ we are interested in the dyadic strong fractional maximal operator

$$
\left( M^S_{\alpha(\cdot), \beta(\cdot)} f \right)(x, y) = \sup_{I \ni x, J \ni y} \left| I \right|^{\alpha(x)-1} \left| J \right|^{\beta(y)-1} \int_{I \times J} |f(t, \tau)| dt d\tau, \quad (x, y) \in \mathbb{R}^2,
$$

where $I$ and $J$ belong to the dyadic lattice $D(\mathbb{R})$ of $\mathbb{R}$.

The Fefferman-Stein Type Inequalities. Criteria for the Trace Inequality

We start by the Fefferman-Stein type inequality. The original inequality for fractional maximal operator defined on cubes in $L^p$ spaces with constant $p$ was derived by E. T. Sawyer.

**Theorem 4.1.** Let $1 < p_- \leq p_+ < q_- \leq q_+ < \infty$ and let $\frac{1}{p_-} - \frac{1}{q_+} < \alpha_- \leq \alpha_+ < \frac{1}{p_-}$, $\frac{1}{p_-} - \frac{1}{q_+} < \beta_- \leq \beta_+ < \frac{1}{p_-}$. Then there is a positive constant $c$ such that

$$
\| (M^S_{\alpha(\cdot), \beta(\cdot)} f) v \|_{L^{q_+}(\mathbb{R}^2)} \leq c \| f (\tilde{M}_{\alpha(\cdot), \beta(\cdot)} v) \|_{L^{p_+}(\mathbb{R}^2)},
$$

where

$$
(\tilde{M}_{\alpha(\cdot), \beta(\cdot)} v)(x, y) = \max \{ (\tilde{M}^{(1)}_{\alpha(\cdot), \beta(\cdot)} v)(x, y), (\tilde{M}^{(2)}_{\alpha(\cdot), \beta(\cdot)} v)(x, y) \},
$$

$$
(\tilde{M}^{(1)}_{\alpha(\cdot), \beta(\cdot)} v)(x, y) = \sup_{I \ni x, J \ni y} \left| I \right|^{-\frac{1}{p_-}} \left| J \right|^{\alpha(x)} \| f (\cdot) \|_{L^{q_+}(I \times J)},
$$

$$
(\tilde{M}^{(2)}_{\alpha(\cdot), \beta(\cdot)} v)(x, y) = \sup_{I \ni x, J \ni y} \left| I \right|^{-\frac{1}{q_-}} \left| J \right|^{\beta(y)} \| f (\cdot) \|_{L^{q_+}(I \times J)}.
$$

**Corollary 4.1.** Let $p$ be constant such that $1 < p < q_- \leq q_+ < \infty$ and let $\frac{1}{p} - \frac{1}{q_+} < \alpha_- \leq \alpha_+ < \frac{1}{p}$, $\frac{1}{p} - \frac{1}{q_-} < \beta_- \leq \beta_+ < \frac{1}{p}$. Then the following inequality holds:

$$
\| (M^S_{\alpha(\cdot), \beta(\cdot)} f) v \|_{L^{q_+}(\mathbb{R}^2)} \leq c \| f (\tilde{M}_{\alpha(\cdot), \beta(\cdot)} v) \|_{L^{p_+}(\mathbb{R}^2)},
$$

where

$$
(\tilde{M}_{\alpha(\cdot), \beta(\cdot)} v)(x, y) = \sup_{I \ni x, J \ni y} \left| I \right|^{-\frac{1}{p}} \left| J \right|^{\alpha(x)} \| f (\cdot) \|_{L^{q_+}(I \times J)}.
$$

**Remark 4.1.** Notice that for $\alpha \equiv \text{const}$, $\beta \equiv \text{const}$, the operator $\tilde{M}_{\alpha, \beta}$ has the form

$$
(\tilde{M}_{\alpha, \beta} v)(x, y) = \sup_{I \ni x, J \ni y} \left| I \right|^{\alpha - \frac{1}{p}} \left| J \right|^{\beta - \frac{1}{p}} \| f (\cdot) \|_{L^{q}(I \times J)}.
$$

**Remark 4.1.** If $q = \text{const}$, then

$$
(\tilde{M}_{\alpha(\cdot), \beta(\cdot)} v)(x, y) = \sup_{I \ni x, J \ni y} \left( \int_I \int_J v^q(x, y) |I|^{q_0(x)} |J|^{q_0(y)} dx dy \right)^{\frac{1}{q}}
$$
Corollary 4.2. [Trace inequality] Let $1 < p_+ < q_+ < \infty$ and let $\frac{1}{p_-} - \frac{1}{q_-} < \alpha_- < \frac{1}{p_+} - \frac{1}{q_+} < \beta_- < \frac{1}{p_-} - \frac{1}{q_-} < \beta_+ < \frac{1}{p_+} - \frac{1}{q_+}$. Suppose that the weight function $v$ satisfies the condition
\[
\sup_{I,J \subset \mathbb{R}} \left\| I \right\| |J|^\beta(\cdot) v(\cdot) \left\| L^{\alpha(\cdot),J}(I \times J) I \times J \right\| \frac{1}{p_{I \times J}} < \infty,
\]
where
\[
\overline{p}_{I \times J} = \begin{cases} p_- & \text{if } |I| |J| \leq 1 \\ p_+ & \text{if } |I| |J| > 1 \end{cases}
\]
Then $M^{S}_{\alpha(\cdot),\beta(\cdot)}$ is bounded from $L^p(\mathbb{R}^2)$ to $L^{q(\cdot)}(\mathbb{R}^2)$.

Theorem 4.2. [Criteria for the trace inequality] Let $p$ be constant and $1 < p < q_- \leq q_+ < \infty$. Suppose that $\frac{1}{p} - \frac{1}{q_+} < \alpha_- < \alpha_+ < \frac{1}{p}$, $\frac{1}{p} - \frac{1}{q_+} < \beta_- < \beta_+ < \frac{1}{p}$. Then $M^{S}_{\alpha(\cdot),\beta(\cdot)}$ is bounded from $L^p(\mathbb{R}^2)$ to $L^{q(\cdot)}(\mathbb{R}^2)$ if and only if
\[
\sup_{I,J \subset \mathbb{R}} \left\| I \right\| |J|^\beta(\cdot) v(\cdot) \left\| L^{\alpha(\cdot),J}(I \times J) I \times J \right\| \frac{1}{p_{I \times J}} < \infty.
\]

Theorem 4.3. Let $p$ be constant and $1 < p < q_- \leq q_+ < \infty$. Suppose that $0 < \alpha_- \leq \alpha_+ < 1$, $0 < \beta_- \leq \beta_+ < 1$. Let $v$ and $w$ be weights on $\mathbb{R}^2$ and let $w$ is of product type, i.e. $w(x,y) = w_1(x)w_2(y)$. Then $M^{S}_{\alpha(\cdot),\beta(\cdot)}$ is bounded from $L^p_v(\mathbb{R}^2)$ to $L^{q(\cdot)}_v(\mathbb{R}^2)$ if and only if
\[
\sup_{I,J \subset \mathbb{R}^2} \left\| I \right\| |J|^\beta(\cdot) v(\cdot) \left\| L^{\alpha(\cdot),J}(I \times J) I \times J \right\| \frac{1}{p_{I \times J}} < \infty
\]
provided that $w_1^{-p'}, w_2^{-p'} \in RD(\mathbb{R})$.

Corollary 4.3. Let $1 < p_- < q_- \leq q_+ < \infty$ with $p_+ < \infty$, $0 < \alpha_- \leq \alpha_+ < 1$, $0 < \beta_- \leq \beta_+ < 1$. Suppose that $p \in \mathcal{P}(\mathbb{R}^2)$. Assume that $v$ and $w$ are weight functions on $\mathbb{R}^2$ and that $w(x,y) = w_1(x)w_2(y)$ with $w_1^{-p_-'}, w_2^{-p_-' \in RD(\mathbb{R})}$. If the condition
\[
\sup_{I,J \subset \mathbb{R}^2} \left\| I \right\| |J|^\beta(\cdot) v(\cdot) \left\| L^{\alpha(\cdot),J}(I \times J) I \times J \right\| \frac{1}{p_{I \times J}} < \infty
\]
holds, then $M^{S}_{\alpha(\cdot),\beta(\cdot)}$ is bounded from $L^p_v(\mathbb{R}^2)$ to $L^{q(\cdot)}_v(\mathbb{R}^2)$.

Corollary 4.4. Let $1 < p_- < q_- \leq q_+ < \infty$ with $p_+ < \infty$, $0 < \alpha_- \leq \alpha_+ < 1$, $0 < \beta_- \leq \beta_+ < 1$. Suppose that $p_- = p(\infty)$ and that $p \in \mathcal{P}_\infty(\mathbb{R}^2)$. Suppose that $v$ and $w$ are weights on $\mathbb{R}^2$ and $w(x,y) = w_1(x)w_2(y)$ with $w_1^{-p_-'}, w_2^{-p_-' \in RD(\mathbb{R})}$. Then condition (4.1) guarantees the boundedness of $M^{S}_{\alpha(\cdot),\beta(\cdot)}$, from $L^p_v(\mathbb{R}^2)$ to $L^{q(\cdot)}_v(\mathbb{R}^2)$.

Proofs of the Results

Proof of Theorem 4.1. Recall that by $M^{S,(d)}_{\alpha(\cdot),\beta(\cdot)}$ we denote the dyadic fractional maximal operator. Without loss of generality we can assume that $f \geq 0$ and $f$ is bounded with compact support.

It is obvious that for $(x,y) \in \mathbb{R}^2$ there are dyadic intervals $I(x) \ni x$, $J(y) \ni y$ such that
\[
\frac{2}{|I(x)|^{1-\alpha(x)}|J(y)|^{1-\beta(y)}} \iiint_{I(x) \times J(y)} |f(t,\tau)| dtd\tau > (M^{S,(d)}_{\alpha(\cdot),\beta(\cdot)}f)(x,y).
\]
Let us introduce the set:

\[ F_{I,J} = \{(x, y) \in \mathbb{R}^2 : x \in I, y \in J \} \]

and the latter inequality holds for \( I \) and \( J \).

Observe that \( \mathbb{R}^2 = \bigcup_{I, J \in D_C(\mathbb{R})} F_{I,J} \). Also, \( F_{I,J} \subset I \times J \). It might be happen that \( F_{I_1,J_1} \cap F_{I_2,J_2} \neq 0 \) for some different couples of dyadic intervals \((I_1, J_1), (I_2, J_2)\). Let us take a number \( r \) so that \( p_+ < r < q_- \). Then we have

\[
\left\| v^{\alpha(\cdot), \beta(\cdot)} f \right\|_{L^{q_+}(\mathbb{R}^2)}^r = \left\| \left[ v^{\alpha(\cdot), \beta(\cdot)} f \right] \right\|_{L^{q_+}(\mathbb{R}^2)}^r \leq c \sup_{h \in L^{q_+}(\mathbb{R}^2)} \left( \int_{\mathbb{R}^2} h \left[ v^{\alpha(\cdot), \beta(\cdot)} f \right] \right)^r.
\]

Further, using the above-observed arguments, we find that for such \( h \) (we assume that \( \| f^\beta v \|_{L^p(\mathbb{R}^2)} \leq 1 \))

\[
\int_{\mathbb{R}^2} h \left[ v^{\alpha(\cdot), \beta(\cdot)} f \right] \leq c \sum_{I, J \in D(\mathbb{R})} \int_{I \times J} v^r |f(t, \tau)| dt d\tau \leq c \sum_{I, J \in D(\mathbb{R})} \frac{1}{|I| \cdot |J|} \int_{I \times J} |f(t, \tau)| dt d\tau \leq c \left[ \int_{I \times J} \left| \int_{I \times J} |f(t, \tau)| dt d\tau \right|^r \right] = c[S_1 + S_2],
\]

where \( f_1 = f \chi_{\{ f^\beta v \leq 1 \}}, f_2 = f - f_1 \).

Now we estimate \( S_1 \) and \( S_2 \) separately. By Corollary A with \( \rho \equiv 1 \) we have that

\[
S_1 = \sum_{I, J \in D(\mathbb{R})} (|I| \cdot |J|)^{-\frac{r}{p_- - \gamma}} \left( \int_{I \times J} \left| \int_{I} |f_1| |(I| \cdot |J|)^{-\frac{1}{p_-}} \right|^p \right)^r \leq c \sum_{I \in D(\mathbb{R})} |I|^{-\frac{r}{p_- - \gamma}} \sum_{J \in D(\mathbb{R})} |J|^{-\frac{r}{p_- - \gamma}} \left( \int_{I} \left[ \int_{I} f_1 \left( |\tilde{M}^{(1)}_{\alpha(\cdot), \beta(\cdot)} v| \right) \right] \right)^r.
\]

By applying again Corollary A with \( \rho \equiv 1 \) and generalized Minkowski inequality, we get

\[
S_1 \leq c \sum_{I \in D(\mathbb{R})} |I|^{-\frac{r}{p_- - \gamma}} \left( \int_{I} \left( \int_{I} f_1 \left| |\tilde{M}^{(1)}_{\alpha(\cdot), \beta(\cdot)} v| \right|^{p_-} \right)^{-\frac{1}{p_-}} \right)^r \leq c \sum_{I \in D(\mathbb{R})} |I|^{-\frac{r}{p_- - \gamma}} \left( \int_{I} \left( \int_{I} f_1 \left| \tilde{M}^{(1)}_{\alpha(\cdot), \beta(\cdot)} v \right| \right)^{p_-} \right)^{-\frac{1}{p_-}} \right)^r.
\]
\begin{align*}
\leq c \left( \iint_{\mathbb{R}^2} |f_1|^{p_*} \left( \widetilde{M}_{\alpha(\cdot), \beta(\cdot)}^{(1)} v \right) \right)^{\frac{1}{p_*}} \\
\leq c \left( \iint_{\mathbb{R}^2} \left[ f(x, y) \left( \widetilde{M}_{\alpha(\cdot), \beta(\cdot)} v \right)(x, y) \right]^{p(x, y)} \, dx \, dy \right)^{\frac{1}{p_*}} \leq c.
\end{align*}

By the similar arguments we can see that

\[ S_2 \leq c \left( \iint_{\mathbb{R}^2} \left[ f(x, y) \left( \widetilde{M}_{\alpha(\cdot), \beta(\cdot)} v \right)(x, y) \right]^{p(x, y)} \, dx \, dy \right)^{\frac{1}{p_*}} \leq c. \]

Thus we established the desired estimate for the dyadic fractional maximal function.

Now we pass from \( M^{S, (d)}_{\alpha(\cdot), \beta(\cdot)} \) to \( M^S_{\alpha(\cdot), \beta(\cdot)} \).

The following inequality for constant \( \alpha \) and \( \beta \) was proved in [17] but it is true also for variable \( \alpha \) and \( \beta \):

\begin{equation}
\left( M^S_{\alpha(\cdot), \beta(\cdot)}, f \right)(x, y) \leq \frac{C_{\alpha, \beta}}{|R(0, 2^{k+2})|^2} \iint_{R(0, 2^{k+2})^2} S_{t, \tau}(x, y) \, dt \, d\tau, \tag{4.2}
\end{equation}

where

\begin{equation}
\left( M^S_{\alpha(\cdot), \beta(\cdot)}, f \right)(x, y) = \sup_{I \times J \ni (x, y)} |I|^{\alpha(x)-1} |J|^{\beta(y)-1} \iint_{I \times J} |f|, \tag{4.3}
\end{equation}

\begin{equation}
S_{t, \tau}(x, y) = \sup_{I - t \ni (x, y)} |I|^{\alpha(x)-1} |J|^{\beta(y)-1} \iint_{I - t \times J - \tau} |f|, \tag{4.4}
\end{equation}

\[ R(0, r) = \{ t : -r \leq t \leq r \}. \]

Indeed, let \( j \in \mathbb{Z} \) and let \( I \) be an integral such that \( 2^j - 1 < |I| \leq 2^j \). Let \( j \leq k, k \in \mathbb{Z} \). Suppose that \( E \) be the set of those \( t \in R(0, 2^{k+2}) \) for which there is some \( I_1 \in D - t \) with \( |I_1| = 2^{j+1} \) and such that \( I \subset I_1 \). Then (see, e.g., [13], p. 431)

\[ |E| \geq 2^{k+2}, \]

where \( D - t := \{ I - t : I \in D(\mathbb{R}) \} \).

By the similar arguments, for another interval \( J \subset \mathbb{R} \), there is \( i \in \mathbb{Z} \) such that \( 2^i - 1 < |J| \leq 2^i \). Then for \( i \leq k, k \in \mathbb{Z} \) we have that the set \( F \) of those \( t \in R(0, 2^{k+2}) \) for which there is \( J_1 \in D - t \) with \( |J_1| = 2^{i+1} \) and \( J \subset J_1 \) has measure greater than or equal to \( 2^{k+2} \).

To prove (4.2) observe that for and \( (x, y) \in \mathbb{R}^2 \), there are intervals \( Q_1 \) and \( Q_2 \) such that \( Q_1 \ni x, Q_2 \ni y, |Q_1|, |Q_2| \leq 2^k \) and

\[ \frac{2}{|Q_1|^{1-\alpha(x)} |Q_2|^{1-\beta(y)}} \iint_{Q_1 \times Q_2} |f| > (M^S_{\alpha(\cdot), \beta(\cdot)}, f)(x, y). \]

Let \( j \) and \( i \) be integers such that

\[ 2^{j-1} \leq |Q_1| \leq 2^j, \quad 2^{i-1} \leq |Q_2| \leq 2^i. \]
It is obvious that \( j, i \leq k \). Let us define the following sets:

\[
E_1 := \{ t \in R(0, 2^{k+2}) : \exists I \in D(\mathbb{R}) - t, \quad |I| = 2^{j+1}, Q_1 \subset I \}
\]

\[
E_2 := \{ t \in R(0, 2^{k+2}) : \exists J \in D(\mathbb{R}) - t, \quad |J| = 2^{j+1}, Q_2 \subset J \}.
\]

Then using above-observed arguments, we have that \((x \in Q_1 \subset I, y \in Q_2 \subset J)\):

\[
\frac{1}{2}(M^{S_{\alpha},(2k)}_{\alpha,\beta}(x,y)) \leq \frac{1}{|Q_1|^{1-\alpha(x)}|Q_2|^{1-\beta(y)}} \int_{Q_1 \times Q_2} |f| \leq \frac{c_{\alpha,\beta}}{|I|^{1-\alpha(x)}|J|^{1-\beta(y)}} \int_{I \times J} |f| \leq c_{\alpha,\beta}S_{t,\tau}(x,y)
\]

because \( I \in D(\mathbb{R}) - t, J \in D(\mathbb{R}) - t, I \ni x, J \ni y \). Since \(|E_1|, E_2 \geq \frac{|R(0, 2^{k+2})|}{2}\), we have that

\[
(M^{S_{\alpha},(2k)}_{\alpha,\beta}(x,y)) \leq \frac{c}{(E_1 \times E_2)} \int_{E_1 \times E_2} S_{t,\tau}(x,y)dtd\tau \leq \frac{c}{|R(0, 2^{k+2})|^2} \int_{R(0, 2^{k+2})^2} S_{t,\tau}(x,y)dtd\tau.
\]

Inequality (4.2) is proved.

Further,

\[
D_{t,\tau}^{(q)} := \int_{\mathbb{R}^2} (S_{t,\tau}(x,y))^{q(x,y)}v(x,y)^{q(x,y)}dx\,dy =
\]

\[
= \int_{\mathbb{R}^2} \left( \sup_{I \ni x \ni J, J \ni t \ni \mathbb{R} \times y} |I|^\alpha(x-1)|J|^\beta(y-1) \int_{I \times (J - t)} |f(s, \varepsilon)|ds\,d\varepsilon \right)^{q(x,y)} v(x,y)^{q(x,y)}dx\,dy =
\]

\[
= \int_{\mathbb{R}^2} \left( \sup_{I \ni x \ni J, J \ni t \ni \mathbb{R} \times y} |I|^\alpha(x-1)|J|^\beta(y-1) \int_{I \times J} f(s - t, s - t)ds\,d\varepsilon \right)^{q(x-t,y-\tau)} \times v(x - t, y - \tau)^{q(x-t,y-\tau)}dx\,dy =
\]

\[
= \int_{\mathbb{R}^2} (M^{S_{\alpha}(d)}_{\alpha,\beta}(f(\cdot - t, \cdot - \tau)))^{q(x-t,y-\tau)} v^{q(x-t,y-\tau)}(x - t, y - \tau)dx\,dy.
\]

Observe now that

\[
\frac{1}{p(\cdot - t, \cdot - \tau)} - \frac{1}{(q(\cdot - t, \cdot - \tau))_+} < (\alpha(\cdot - t, \cdot - \tau))_-
\]

\[
\leq (\alpha(\cdot - t, \cdot - \tau))_+ < \frac{1}{(p(\cdot - t, \cdot - \tau))_-}
\]

and since the constants in the estimates of \( \|M^{S_{\alpha}(d)}_{\alpha,\beta}(f)\|_{L^q(\mathbb{R}^2)} \) depend only on \( p_+, q_- \), we have that
because

\[ D_{t,\tau}^{(q)} \leq c \]

Now let us see that equality (5) holds.

First observe that

\[
\widetilde{M}_{\alpha(-t),\beta(-\tau)}^{(1)}(x,y) = \left\{ \lambda > 0 : \int_I \frac{1}{\lambda} \inf \left\{ \frac{\int_J \left( v(t, y) \right) dx \leq 1 \} }{\| v \|_{L^\infty(I \times J)}} \lambda \right\} \right\] = \left( \widetilde{M}_{\alpha(-t),\beta(-\tau)}^{(1)} \right)(x,y).
\]

Analogous estimates hold also for \( \left( \widetilde{M}_{\alpha(-t),\beta(-\tau)}^{(2)} \right)(x,y) \).

Hence

\[ I = \int \int |f(x-t, t-\tau)|^{p(x-t, y-\tau)}(\tilde{M}_{\alpha(-t),\beta(-\tau)}^{(1)}(x-y))^{p(x-t, y-\tau)}(x, y) dx dy = \int \int |f(x,y)|^{p(x,y)}(\tilde{M}_{\alpha(-t),\beta(-\tau)}^{(1)}(x-y))^{p(x,y)}(x, y) dx dy. \]

Thus, we have seen that

\[ D_{t,\tau}^{(q)} \leq c \quad \text{if} \quad \|f(\tilde{M}_{\alpha(-t),\beta(-\tau)}^{(1)}(x, y))\|_{L^p(\mathbb{R}^2)} \leq 1 \iff \|S_{t,\tau}(\cdot, \cdot) v(\cdot, \cdot)\|_{L^p(\mathbb{R}^2)} \leq c \|f(\tilde{M}_{\alpha(-t),\beta(-\tau)}^{(1)}(x, y))\|_{L^p(\mathbb{R}^2)} \iff \sup_{\|h\|_{L^p(\mathbb{R}^2)} \leq 1} \int \int S_{t,\tau}(x, y) v(x, y) h(x, y) dx dy \leq c \|f(\tilde{M}_{\alpha(-t),\beta(-\tau)}^{(1)}(x, y))\|_{L^p(\mathbb{R}^2)}.
\]

For \( h, \|h\|_{L^p(\mathbb{R}^2)} \leq 1 \), we have that (recall that (4.2) holds):

\[ \int \int \left( M_{\alpha(-t),\beta(-\tau)}^{S(2^k)}(x, y) \right) v(x, y) h(x, y) dx dy \leq \frac{C}{k!} \int \int R(0, 2^{k+2}) S_{t,\tau}(x, y) dx dy \]

\[ \leq c |R(0, 2^{k+2})| - \frac{1}{2} \int \int R(0, 2^{k+2}) S_{t,\tau}(x, y) dx dy = \]

\[ \leq c |R(0, 2^{k+2})| - 2 \int \int R(0, 2^{k+2}) S_{t,\tau}(x, y) dx dy = \]
\[
\begin{align*}
&= c|R(0,2^{k+2})|^{-2} \int \int_{R(0,2^{k+2})^2} \left( \int \int_{\mathbb{R}^2} S_{t,\tau(x,y)} v(x,y)h(x,y)dx dy \right) dt d\tau \\
&\leq c\|f(\tilde{M}_{\alpha(\cdot),\beta(\cdot)}v)\|_{L^p(\cdot)(\mathbb{R}^2)}|R(0,2^{k+2})|^{-2} \int \int_{R(0,2^{k+2})^2} dt d\tau = c\|f(\tilde{M}_{\alpha(\cdot),\beta(\cdot)}v)\|_{L^p(\cdot)(\mathbb{R}^2)}.
\end{align*}
\]

Passing now \(k\) to the infinity and taking the supremum with respect to \(h\) in the last inequality, we get the desired result.

Remark 4.3. Observe that for \(p \equiv \text{const}, \alpha \equiv \text{const}, \beta \equiv \text{const}\) and \(v \in L^q(\cdot)(\mathbb{R}^2)\), the estimate

\[
(\tilde{M}_{\alpha(\cdot),\beta(\cdot)}v)(x,y) \leq \sup_{I,J \subset \mathbb{R}^2} |I|^{-\frac{1}{p}}|J|^{-\frac{1}{q}} \left( \int \int_{I \times J} v(x,y)dx dy \right)^{\frac{1}{p} + \frac{1}{q}} =: (M_{\alpha,\beta}v)(x,y)
\]

holds.

Corollary 4.5. Let \(p, \alpha\) and \(\beta\) be constant and let \(\frac{1}{p} - \frac{1}{q_+} < \alpha, \beta < \frac{1}{p}\). Suppose that \(v \in L^q(\cdot)(\mathbb{R}^2)\). Then the inequality

\[
\|v(M_{\alpha(\cdot),\beta(\cdot)}f)\|_{L^q(\cdot)(\mathbb{R}^2)} \leq c\|f(\cdot,\cdot)(\tilde{M}_{\alpha,\beta}(\cdot,\cdot))\|_{L^p(\mathbb{R}^2)}
\]

holds.

Corollary 4.5 follows immediately from Theorem 4.1 and Remark 4.3.

Proof of Corollary 4.2. This proposition will be proved if we show that \((\tilde{M}_{\alpha(\cdot),\beta(\cdot)}v)(x,y) \leq c\) in Theorem 4.1. Indeed, if the condition

\[
A := \sup_{I,J \subset \mathbb{R}} \| |I|^{\alpha(\cdot)}|J|^{\beta(\cdot)}v(\cdot)\|_{L^q(\cdot)(I \times J)}|I \times J|^{-\frac{1}{p+1}} < \infty,
\]

is satisfied, where

\[
\mathcal{P}_{I \times J} = \begin{cases} p_-, & \text{if } |I||J| \leq 1, \\ p_+, & \text{if } |I||J| > 1,
\end{cases}
\]

then

\[
\| |I|^{\alpha(\cdot)}|J|^{\beta(\cdot)}v(\cdot)\|_{L^q(\cdot)(I \times J)}|I \times J|^{-\frac{1}{p_+}} \leq A < \infty
\]

and

\[
\| |I|^{\alpha(\cdot)}|J|^{\beta(\cdot)}v(\cdot)\|_{L^q(\cdot)(I \times J)}|I \times J|^{-\frac{1}{p_-}} \leq A < \infty.
\]

Proof of Theorem 4.3. Let us recall that by the symbol \(M_{\alpha(\cdot),\beta(\cdot)}^{S,d}\) is denoted the dyadic strong fractional maximal operator.

Sufficiency. We use the notation of the proof of Theorem 4.1. First we construct the sets \(F_{I \times J}\).

Take \(r\) so that \(p < r < q_-\) and observe that

\[
\|v(M_{\alpha(\cdot),\beta(\cdot)}^{S,d}f)\|_{L^q(\cdot)(\mathbb{R}^2)} \leq c \sup_{h \in L^q(\cdot,\cdot)(\mathbb{R}^2)} \left( \int \int_{\mathbb{R}^2} h[vM_{\alpha(\cdot),\beta(\cdot)}^{S,d}f]^r \right).
\]
Let \( \|f\|_{L^p(R^2)} \leq 1 \). Then for such an \( h \) we have that

\[
S = \iint_{R^2} h[vM^{S,(d)}_{\alpha(\cdot),\beta(\cdot)} f]^r \leq \sum_{I,J \in D(R)} \iint_{F_l,J} h[vM^{\alpha(d)}_{\alpha(\cdot),\beta(\cdot)} f]^r
\]

\[
\leq c \sum_{I,J \in D(R)} \left( \iint_{I \times J} v^r(|I||\alpha(x)|J|\beta(y)|^r h(x,y)dxdy)\left( \frac{1}{|I||J|} \iint_{I \times J} |f(t, \tau)|dtd\tau \right)^r \right)
\]

\[
\leq c \sum_{I,J \in D(R)} \|v(\cdot)|I|^{\alpha(\cdot)}J|^{\beta(\cdot)}\|_{L^r(I \times J)} h|L^{(\cdot)/r}(I \times J)} \left( \frac{1}{|I||J|} \iint_{I \times J} |f(t, \tau)|dtd\tau \right)^r
\]

\[
= c \sum_{I,J \in D(R)} \|v(\cdot)|I|^{\alpha(\cdot)}J|^{\beta(\cdot)}\|_{L^r(I \times J)} \left( \frac{1}{|I||J|} \iint_{I \times J} |f(t, \tau)|dtd\tau \right)^r.
\]

By the condition of theorem we get that

\[
S \leq c \sum_{I,J \in D(R)} \left( \int_I w_1^{-p'} \right)^{-\frac{r}{p'}} \left( \int_J w_2^{-p'} \right)^{-\frac{r}{p'}} \left( \iint_{I \times J} |f| \right)^r.
\]

Applying Corollary A with \( \rho \equiv 1 \) we derive the following estimates:

\[
S \leq c \sum_{J \in D(R)} \left( \int_J w_2^{-p'} \right)^{-\frac{r}{p'}} \left( \iint_{\mathbb{R}} w_1(t)^p \left( \int_J |f(t, \tau)|d\tau \right)^p dt \right)^{\frac{r}{p}}
\]

\[
\leq c \sum_{J \in D(R)} \left( \int_J w_2^{-p'} \right)^{-\frac{r}{p'}} \left( \int_J \left( \int_{\mathbb{R}} w_1^p(t) |f(t, \tau)|d\tau \right)^\frac{1}{p} d\tau \right)^r
\]

\[
\leq c \left( \iint_{\mathbb{R}^2} |f(t, \tau)|^p w^p(t, \tau) dtd\tau \right)^{r/p} \leq c.
\]

Thus we established the desired inequality for the dyadic fractional maximal function.

Now we can pass to the fractional maximal function \( M^S_{\alpha(\cdot),\beta(\cdot)} \) in the same manner as in the proof of Theorem 4.1.

\textit{Necessity} follows easily by taking appropriate test functions in the two–weight inequality. Details are omitted. \( \square \)

\textit{Proof of Corollary 4.3.} The proof is a direct consequence of Theorem 4.3 and the fact that the condition \( p \in \mathcal{P}(\mathbb{R}^2) \) implies the inequality

\[
\|fw\|_{L^p(I \times J)} \leq c\|fw\|_{L^p(\mathbb{R}^2)}.
\]

Corollary 4.4 follows from the fact: \( p \in \mathcal{P}_\infty(\mathbb{R}^2) \Rightarrow p \in \mathcal{P}(\mathbb{R}^2) \), provided that \( p_- = p(\infty) \), and Corollary 4.3.

\textbf{The Case of a Bounded Domain}
Let $\Omega$ be a bounded domain in $\mathbb{R}^2$ and let
\[
(M^{S, R_0}_{\alpha(\cdot), \beta(\cdot)} f)(x, y) = \sup_{R = I \times J, R \supset (x, y)} |I|^{\alpha(x)-1} |J|^{\beta(y)-1} \int_R |f(t, \tau)| dt d\tau
\]
where $(x, y) \in \Omega$. For simplicity assume that $\Omega = R_0$, where $R_0$ is a fixed rectangle in $\mathbb{R}^2$.

Taking the results of the previous subsections into account we can formulate the following statements proofs of which are omitted:

**Theorem 4.4 (Fefferman-Stein type inequality).** Let $1 < p_-(R_0) \leq p_+(R_0) < q_-(R_0) \leq q_+(R_0) < \infty$ and let
\[
\frac{1}{p_-(R_0)} - \frac{1}{p_-(R_0)} < \alpha_-(R_0) \leq \alpha_+(R_0) < \frac{1}{p_-(R_0)},
\]
\[
\frac{1}{p_-(R_0)} - \frac{1}{q_-(R_0)} < \beta_-(R_0) \leq \beta_+(R_0) < \frac{1}{p_-(R_0)}.
\]
Then there is a positive constant $b$ such that the following inequality
\[
\|(M^{S, R_0}_{\alpha(\cdot), \beta(\cdot)} f)\|_{L^{p_+(R_0)}(R_0)} \leq b\|f(M^{R_0}_{\alpha(\cdot), \beta(\cdot)} v)\|_{L^{q_+(R_0)}(R_0)},
\]
holds, where
\[
(\tilde{M}^{(R_0)}_{\alpha(\cdot), \beta(\cdot)} v)(x, y) = \sup_{R = I \times J, R \supset (x, y)} |R|^{-\frac{1}{p_+}} \|v(\cdot) |I|^{\alpha(\cdot)} |J|^{\beta(\cdot)} \|_{L^{q_+(R \cap R_0)}}, \quad (x, y) \in R_0.
\]

**Remark 4.4.** Let $\alpha(x) \equiv \alpha \equiv \text{const}$, $\beta(x) \equiv \beta \equiv \text{const}$ in Theorem 4.4. Then it is easy to see that
\[
(\tilde{M}^{(R_0)}_{\alpha, \beta} v)(x, y) \leq \sup_{R = I \times J, R \supset (x, y)} |I|^{\alpha(x)-1} |J|^{\beta(y)-1} \left( \int_{R \cap R_0} v^{q(x,y)}(x,y) dxdy \right)^{\frac{1}{q(x,y)}}.
\]

**Theorem 4.5 (Trace inequality).** Let $1 < p_-(R_0) \leq p_+(R_0) < q_-(R_0) \leq q_+(R_0) < \infty$ and let
\[
\frac{1}{p_-(R_0)} - \frac{1}{q_+(R_0)} < \alpha_-(R_0) \leq \alpha_+(R_0) < \frac{1}{p_-(R_0)} - \frac{1}{q_+(R_0)} < \beta_-(R_0) \leq \beta_+(R_0) < \frac{1}{p_-(R_0)}.
\]
Suppose that the weight function $v$ on $\mathbb{R}^2$ satisfies the condition
\[
\sup_{R = I \times J} \|I|^{\alpha(\cdot)} |J|^{\beta(\cdot)} v(\cdot) \|_{L^{q_+(R \cap R_0)}} |R|^{-\frac{1}{p_+(R_0)}} < \infty.
\]
Then $M^{S, R_0}_{\alpha(\cdot), \beta(\cdot)}$ is bounded from $L^{p_+(R_0)}(R_0)$ to $L^{q_+(R_0)}(R_0)$.

**Theorem 4.6.** Let $R_0 := I_0 \times J_0$, $1 < p_-(R_0) \leq q_-(R_0) \leq q_+(R_0) < \infty$ with $p_+(R_0) < \infty$, $0 < \alpha_-(R_0) \leq \alpha_+(R_0) < 1$ and $0 < \beta_-(R_0) \leq \beta_+(R_0) < 1$. Suppose that $v$ and $w$ are weights on $R_0$ with $w(x, y) = w_1(x)w_2(y)$, where $w_1 \in RD^{(d)}(I_0)$, $w_2 \in RD^{(d)}(J_0)$. If
\[
\sup_{R \supset R_0} |R|^{-\frac{1}{p_-(R_0)}} \|v(\cdot) |I|^{\alpha(\cdot)} |J|^{\beta(\cdot)} \|_{L^{q_+(R \cap R_0)}} \|w^{-1}\|_{L^{p_-(R_0)}} < \infty,
\]
then $M^{S, R_0}_{\alpha(\cdot), \beta(\cdot)}$ is bounded from $L^{p_+(R_0)}(R_0)$ to $L^{q_+(R_0)}(R_0)$. 
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