EXCEPTIONAL CYCLES FOR PERFECT COMPLEXES
OVER GENTLE ALGEBRAS

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Abstract. Exceptional cycles in a triangulated category $T$ with Serre duality, introduced by N. Broomhead, D. Pauksztello, and D. Ploog, have a notable impact on the global structure of $T$. In this paper we show that if $T$ is homotopy-like, then any exceptional 1-cycle is indecomposable and at the mouth; and any object in an exceptional $n$-cycle with $n \geq 3$ is at the mouth. Let $A$ be an indecomposable gentle $k$-algebra with $A \neq k$. The Hom spaces between string complexes at the mouth are explicitly determined. The main result classifies “almost all” the exceptional cycles in $K^b(A\text{-proj})$, using characteristic components and their AG-invariants, except those exceptional 1-cycles which are band complexes. Namely, the mouth of a characteristic component $C$ of $K^b(A\text{-proj})$ forms a unique exceptional cycle in $C$, up to an equivalent relation $\approx$; if the quiver of $A$ is not of type $A_3$, this gives all the exceptional $n$-cycle in $K^b(A\text{-proj})$ with $n \geq 2$, up to $\approx$; and a string complex is an exceptional 1-cycle if and only if it is at the mouth of a characteristic component with AG-invariant $(1, m)$. However, a band complex at the mouth is possibly not an exceptional 1-cycle.

Keywords: exceptional cycle, gentle algebra, homotopy-like triangulated category, Auslander-Reiten triangle, at the mouth, string complex, characteristic component, AG-invariant.

1. Introduction

1.1. An exceptional cycle in a triangulated category $T$ with Serre duality has been introduced by N. Broomhead, D. Pauksztello, and D. Ploog [BPP]. It is a generalization of a spherical object (see e.g. [ST], [HKP1]), provides an invariant of triangle-equivalences, and closely relates to the global structure of $T$. Its importance also lies in the fact that an exceptional cycle induces an autoequivalences of $T$ ([BPP, Theorem 4.5]), which is a generalization of tubular mutation in [M].

Gentle algebras, introduced by I. Assem and A. Skowronski [AS], have related to different topics in mathematics. It is closed under derived equivalence by J. Schröer and A. Zimmermann [SZ], and exactly the class of finite-dimensional algebras $A$ such that the repetitive algebras $\hat{A}$ are special biserial ([AS], [PS]). It appears in D. Vossieck’s classification of algebras with discrete derived categories ([V]), and in cluster tilted algebras (see e.g. [ABCP], [AG]). The combinatorial description of Auslander-Reiten triangles of $K^b(A\text{-proj})$ has been given by G. Bobiński [B]. Recently, a geometric derived realization and complete derived invariants of gentle algebras are given in [OPS], [APS] and [O].

The aim of this paper is to study exceptional cycles in an indecomposable homotopy-like triangulated category with Serre duality, and to determine all the exceptional cycles in homotopy category $K^b(A\text{-proj})$, where $A$ is an indecomposable finite-dimensional gentle algebra.

1.2. Throughout, $k$ is an algebraically closed field, $T$ is a $k$-linear Hom-finite Krull-Schmidt triangulated category with Serre functor (if no otherwise stated). Let $S : T \to T$ be the right Serre functor.
So, for objects $X$ and $Y$ in $\mathcal{T}$, there is a $k$-linear isomorphism $\Hom_\mathcal{T}(X, Y) \cong \Hom_\mathcal{T}(Y, S(X))^*$, which is functorial in $X$ and $Y$, where $-^*$ is the $k$-dual $\Hom_k(-, k)$. Then $S$ is a triangle-equivalence ([Boc, Appendix]) and $\mathcal{T}$ has Auslander-Reiten triangles with $S = \tau[1]$ on objects, where $\tau$ is the Auslander-Reiten translate (see [RV, Theorem I. 2.4]).

An indecomposable object of $\mathcal{T}$ is said to be at the mouth, if the middle term of the Auslander-Reiten triangle ending at it is indecomposable, or equivalently, the middle term of the Auslander-Reiten triangle starting from it is indecomposable.

A triangulated category $\mathcal{T}$ will be said to be homotopy-like, if $M \not\cong M[i]$ for any indecomposable object $M \in \mathcal{T}$ and for all $i \neq 0$. It is clear that $K^b(\mathcal{B})$ is homotopy-like, where $\mathcal{B}$ is an arbitrary additive subcategory of an abelian category, which is closed under direct summands. So $K^b(A\text{-proj})$ is homotopy-like. On the other hand, if $\Lambda$ is a finite-dimensional self-injective algebra admitting a $K_0$ of $(\text{injective, respectively})$ $A$-$\Lambda$-modules. The Nakayama functor $(\Hom_\Lambda(-, A))^*$ induces componentwisely an equivalence $S : K^b(A\text{-proj}) \cong K^b(A\text{-inj})$, so that for $P^* \in K^b(A\text{-proj})$ and $X \in D^b(A)$ there is a $k$-linear functorial isomorphism in both arguments ([H1, p.37]):

$$\Hom_{D^b(A)}(P, X) \cong \Hom_{D^b(A)}(X, S(P))^*.$$ 

Since $A$ is a Gorenstein algebra ([GR]), $K^b(A\text{-proj}) = K^b(A\text{-inj})$ in $D^b(A)$ and $S$ is the Serre functor of $K^b(A\text{-proj})$ ([H2]). So $K^b(A\text{-proj})$ is a $k$-linear, Hom-finite Krull-Schmidt triangulated category with Serre functor $S$, and having Auslander-Reiten triangles.

1.3. Throughout, $A$ is an indecomposable finite-dimensional gentle $k$-algebra. Let $A\text{-mod}$ be the category of finitely generated left $A$-modules, $D^b(A)$ the bounded derived category of $A$-mod, and $K^b(A\text{-proj})$ ($K^b(A\text{-inj})$, respectively) the bounded homotopy category of finitely generated projective (injective, respectively) $A$-modules. The Nakayama functor $(\Hom_\Lambda(-, A))^*$ induces componentwisely an equivalence $S : K^b(A\text{-proj}) \cong K^b(A\text{-inj})$, so that for $P^* \in K^b(A\text{-proj})$ and $X \in D^b(A)$ there is a $k$-linear functorial isomorphism in both arguments ([H1, p.37]):

$$\Hom_{D^b(A)}(P, X) \cong \Hom_{D^b(A)}(X, S(P))^*.$$ 

Since $A$ is a Gorenstein algebra ([GR]), $K^b(A\text{-proj}) = K^b(A\text{-inj})$ in $D^b(A)$ and $S$ is the Serre functor of $K^b(A\text{-proj})$ ([H2]). So $K^b(A\text{-proj})$ is a $k$-linear, Hom-finite Krull-Schmidt triangulated category with Serre functor $S$, and having Auslander-Reiten triangles.

1.4. Let $d$ be an integer. An object $E \in \mathcal{T}$ is a $d$-Calabi-Yau object if $S(E) \cong E[d]$. In general $d$-Calabi-Yau objects are not closed under taking direct summands (see [CZ]). For objects $X, Y \in \mathcal{T}$, let $\Hom^*(X, Y)$ be the complex of $k$-vector spaces with $\Hom^i(X, Y) = \Hom_\mathcal{T}(X, Y[i])$ and with zero differentials. Thus $\Hom^*(X, Y) = \bigoplus_i \Hom_\mathcal{T}(X, Y[i])[-i]$.

By definition, an exceptional 1-cycle in $\mathcal{T}$ is a $d$-Calabi-Yau object $E$ for some integer $d$, such that $\Hom^*(E, E) \cong k \oplus k[-d]$. It is also called a spherical object for example in [HKP1] and [HKP2]. Here we use the name of an exceptional 1-cycle in [BPP], for the unification of an exceptional $n$-cycle with $n \geq 1$. Note that the terminology “a spherical object” has (slight) different meanings in the literatures, see for example [ST], [KYZ], [CP]. For the definition of an exceptional $n$-cycle with $n \geq 2$ we refer to Subsection 2.1.

Let $E$ be an exceptional 1-cycle which is $d$-Calabi-Yau. Then $d$ is unique. If $d \neq 0$, or $d = 0$ and $\text{End}_\mathcal{T}(E) \cong k[x]/(x^2)$ as algebras, then $E$ is indecomposable; if $d = 0$ and $\text{End}_\mathcal{T}(E) \cong k \times k$ as algebras, then $E$ is decomposable. However we have

**Theorem 1.1.** Let $\mathcal{T}$ be an indecomposable $k$-linear Hom-finite Krull-Schmidt triangulated category with Serre functor. Assume that $\mathcal{T}$ is homotopy-like. Then

1. Any exceptional 1-cycle is indecomposable, and at the mouth.
2. Any object in an exceptional $n$-cycle with $n \geq 3$ is at the mouth.
Remark (1) If $T$ is not indecomposable, then an exceptional 1-cycle may be decomposable and not at the mouth. For example, $(k, k) = (k, 0) \oplus (0, k)$ is an exceptional 1-cycle in $\mathcal{D}^b(k) \times \mathcal{D}^b(k)$, and it is not at the mouth.

(2) An object in an exceptional 2-cycle may be not at the mouth. For example, let $A$ be the path algebra of the quiver $1 \to 2 \to 3$. Then $\mathcal{D}^b(A)$ has an exceptional 2-cycle $(P(2), I(2))$, where $P(2)$ (respectively, $I(2)$) is the indecomposable projective (respectively, injective) $A$-module at vertex 2. However, $P(2)$ and $I(2)$ are not at the mouth of $\mathcal{D}^b(A)$.

The proof of Theorem 1.1 will be given in Section 3. The main ideas in the proof are to use a special non-zero non-isomorphism from an indecomposable object $M$ to $S(M)$, constructed in [RV] (see Lemma 3.1 below), and the mapping cone of the composition of morphisms in a homotopy cartesian square (see Lemmas 3.3 and 3.4 below).

1.5. To determine the exceptional cycles in $K^b(A\text{-proj})$, we need the notion of a characteristic component of $K^b(A\text{-proj})$, to determine its shape, and to introduce its AG-invariant.

An indecomposable object of $K^b(A\text{-proj})$ is either a string complex or a band complex (see [BM]; also [B]). The description of Auslander-Reiten triangles of $K^b(A\text{-proj})$ ([BR Main Theorem]) shows that a connected component $C$ of $K^b(A\text{-proj})$ either consists of string complexes, or consists of band complexes. It will be called a characteristic component, if $C$ contains a string complex at the mouth (thus $C$ consists of string complexes). By some results in [XZ], [Sch], [V], and [BR], $C$ is of the form $ZA_n$ ($n \geq 2$), $ZA_\infty$, $ZA_\infty/(\tau^n)$ ($n \geq 1$).

Since up to shift there are only finitely many string complexes at the mouth, by the shape of $C$, there exists a unique pair $(n, m)$ of integers, such that $\tau^n X \cong X[m - n]$, for any indecomposable object $X$ at the mouth of $C$. This pair $(n, m)$ will be called Alaminos-Geiss invariant (or AG invariant in short) of $C$. For details see Section 4.

In [AAG] a characteristic component and its invariant have been defined for $\hat{A}\text{-mod}$, where $\hat{A}$ is the repetitive algebra. Since $A$ is Gorenstein, the Happel embedding $K^b(A\text{-proj}) \hookrightarrow \hat{A}\text{-mod}$ preserves the Auslander-Reiten components ([HKR Corollary 5.3]). Thus, a characteristic component and its AG invariant here coincide with the ones in [AAG] (but we include the component of type $ZA_n$).

1.6. Since objects in an exceptional $n$-cycle in $K^b(A\text{-proj})$ are at the mouth (with a unique exception in the case $n = 2$), the dimension of the Hom spaces between string complexes at the mouth will play a central role in determining exceptional cycles in $K^b(A\text{-proj})$. This is given as follows.

**Theorem 1.2.** Let $A$ be an indecomposable finite-dimensional gentle algebra, $M$ and $M'$ string complexes at the mouth. Then

$$\dim_k \text{Hom}_{K^b(A\text{-proj})}(M, M') = \begin{cases} 2, & \text{if } M' \cong S(M) \cong M, \\ 1, & \text{if } M' \text{ is isomorphic to one of } S(M) \text{ and } M, \\ & \text{but } M' \text{ is not isomorphic to the other}, \\ 0, & \text{if } M' \not\cong S(M) \text{ and } M' \not\cong M. \end{cases}$$
In particular,

$$\dim_k \text{End}(M) = \dim_k \text{Hom}_{K^b(A\text{-proj})}(M, S(M)) = \begin{cases} 2, & \text{if } M \cong S(M); \\ 1, & \text{if } M \not\cong S(M). \end{cases}$$

**Corollary 1.3.** Let $A$ be an indecomposable finite-dimensional gentle algebra, $C$ and $C'$ different characteristic components of $K^b(A\text{-proj})$, up to shift. Then $\text{Hom}^*(X, Y) = 0$ for $X \in C$ and $Y \in C'$.

The proof of Theorem 1.2 and of Corollary 1.3 will be given in Section 5. The main tools used in the proof are Lemma 3.1, the combinatorial description of morphisms between indecomposable objects in $D^b(A)$, given by K. K. Arnesen, R. Laking and D. Pauksztello in [ALP] (see Subsection 5.1), and the bijections between the set of permitted threads and the set of forbidden threads, given in [AAG] (see also [BB]. See Subsection 2.5).

1.7. The following main result classifies all the exceptional $n$-cycle in $K^b(A\text{-proj})$ with $n \geq 2$. It turns out that such an exceptional cycle is exactly a truncation of the $\tau$-orbit of any indecomposable object at the mouth of a characteristic component, with few exceptions. For the equivalent relation $\approx$ on the set of exceptional cycles we refer to Subsection 2.2.

**Theorem 1.4.** Let $A = kQ/I$ be a finite-dimensional gentle algebra with $A \neq k$, where $Q$ is a finite connected quiver such that the underlying graph of $Q$ is not of type $A_3$.

(1) Let $C$ be a characteristic component of $K^b(A\text{-proj})$ with $\text{AG}$-invariant $(n, m)$, and $X$ an indecomposable object at the mouth of $C$. Then

$$(X, \tau X, \cdots, \tau^{n-1}X)$$

is the unique exceptional cycle in $C$, up to the equivalent relation $\approx$.

(2) Any exception $n$-cycle in $K^b(A\text{-proj})$ with $n \geq 2$ is given in (1), up to $\approx$.

(3) Any object in an exceptional cycle in $K^b(A\text{-proj})$ is indecomposable and at the mouth.

A string complex $E$ is an exceptional 1-cycle if and only if $E$ is at the mouth of a characteristic component of $\text{AG}$-invariant $(1,d)$. In this case $E$ is a $d$-Calabi-Yau object.

If a band complex $E$ is an exceptional 1-cycle, then $E$ is at the mouth of a homogenous tube.

Note that N. Broomhead, D. Pauksztello, and D. Ploog [BPP 5.1] have pointed out the assertion (1) for the gentle algebras $\Lambda(r, n, m)$ with $n > r$: these are exactly derived-discrete algebras of finite global dimension which is not of Dynkin type, up to derived equivalence.

**Remark** (1) If $A = k$, then $K^b(A\text{-proj}) = D^b(k)$ has no mouths, and $(k, k)$ is the unique exceptional cycle in $K^b(A\text{-proj})$, up to $\approx$. So, Theorem 1.4(1) does not hold for $A = k$.

(2) If $A = kQ/I$ is a gentle algebra such that the underlying graph of $Q$ is of type $A_3$, then $K^b(A\text{-proj}) \cong D^b(k(1 \to 2 \to 3))$, and $(P(3), I(3) = P(1), I(1), S(2))$ and $(P(2), I(2))$ are all the exceptional cycles in $D^b(k(1 \to 2 \to 3))$, up to $\approx$. So, Theorem 1.4(1) and (2) do not hold in this case.

(3) A band complex at the mouth of a homogeneous tube is not necessarily an exceptional 1-cycle. See Example 7.2.
The proof of Theorem 1.4 will be given in Section 6. The main tools used in the proof are Theorem 1.2, Lemma 6.2, and Theorem 1.1.

2. Preliminaries

2.1. Exceptional cycles.

**Definition 2.1.** (BPP) An exceptional n-cycle in \( T \) with \( n \geq 2 \) is a sequence \( (E_1, \cdots, E_n) \) of objects satisfying the following conditions:

(E1) \( \text{Hom}^*(E_i, E_i) \cong k \) for each \( i \);

(E2) there are integers \( m_i \) such that \( S(E_i) \cong E_{i+1}[m_i] \) for each \( i \), where \( E_{n+1} := E_1 \);

(E3) \( \text{Hom}^*(E_i, E_j) = 0 \), unless \( j = i \) or \( j = i + 1 \). (This condition vanishes if \( n = 2 \).)

The sequence \( (m_1, \cdots, m_n) \) of integers in the definition is unique, and we will call \( (E_1, \cdots, E_n) \) an exceptional cycle with respect to \( (m_1, \cdots, m_n) \). It is clear that (see e.g. [GZ, Lemma 2.2]) a sequence \( (E_1, \cdots, E_n) \) of objects in \( T \) with \( n \geq 2 \) is an exceptional cycle if and only if it satisfies the condition (E1′), (E2), and (E3′), where

(E1′) \( \text{Hom}^*(E_1, E_1) \cong k \);

(E3′) If \( n \geq 3 \), then \( \text{Hom}^*(E_1, E_j) = 0 \) for \( 3 \leq j \leq n \).

Each object in an exceptional n-cycle with \( n \geq 2 \) is indecomposable. If \( (E_1, \cdots, E_n) \) is an exceptional cycle, then

\[
(E_2, \cdots, E_n, E_1), \quad (S(E_1), \cdots, S(E_n)), \quad (\tau E_1, \cdots, \tau E_n)
\]

are also exceptional cycles. If \( (E_1, \cdots, E_n) \) is an exceptional cycle, with respect to \( (m_1, \cdots, m_n) \), then for arbitrary integers \( t_1, \cdots, t_n \),

\[
(E_1[t_1], \cdots, E_n[t_n])
\]

is an exceptional cycle, with respect to \( (m_1 + t_1 - t_2, \cdots, m_n + t_n - t_{n+1}) \), where \( t_{n+1} = t_1 \).

2.2. Equivalent relation \( \cong \). Consider the set \( \mathcal{E}_n \) of all the exceptional n-cycles in \( T \) with \( n \geq 1 \). For \( (E_1, \cdots, E_n) \in \mathcal{E}_n \) and \( (E'_1, \cdots, E'_n) \in \mathcal{E}_n \), define \( (E_1, \cdots, E_n) \approx (E'_1, \cdots, E'_n) \) if and only if they are up to shift independently at each position and up to rotation, i.e., there are integers \( t_1, \cdots, t_n, s \) with \( 0 \leq s \leq n - 1 \), such that

\[
(E'_1, \cdots, E'_n) = (E_{\sigma(1)}[t_1], \cdots, E_{\sigma(n)}[t_n])
\]

where \( \sigma \) is the cyclic permutation \( (1 \ 2 \ \cdots \ n) \). Then \( \approx \) is an equivalent relation on \( \mathcal{E}_n \).

For \( (E_1, \cdots, E_n) \in \mathcal{E}_n \), by (E2) one has \( (\tau E_1, \cdots, \tau E_n) \approx (S(E_1), \cdots, S(E_n)) \approx (E_1, \cdots, E_n) \).

The following fact is convenient for later use. See [GZ, Lemma 2.1].

**Lemma 2.2.** (1) If \( (E_1, \cdots, E_n) \) and \( (E'_1, \cdots, E'_m) \) are exceptional cycles in \( T \) with \( E'_i \cong E_1[t] \) for some integers \( t \) and \( i \), then \( n = m \) and \( (E_1, \cdots, E_n) \approx (E'_1, \cdots, E'_m) \).

Thus each indecomposable object occurs in at most one exceptional n-cycle with \( n \geq 2 \), up to the equivalent relation \( \cong \).
In particular, an exceptional cycle can not be enlarged, i.e., if \((E_1, \cdots, E_n)\) is an exceptional \(n\)-cycle with \(n \geq 1\), then there are no exceptional cycles of the form \((E_1, \cdots, E_n, F_1, \cdots, F_t)\) with \(t \geq 1\). Thus, an exceptional cycle can not be shorten such that it is again an exceptional cycle.

(2) Assume that \((E_1, \cdots, E_n)\) is an exceptional cycle with \(n \geq 3\). Then \(E_i \not\equiv E_i[t]\) for all \(2 \leq i \leq n\) and for all \(t \in \mathbb{Z}\).

2.3. Gentle algebras. Let \(Q = (Q_0, Q_1, s, e)\) be a finite connected quiver. Write the conjunction of paths from right to left. A bound quiver \((Q, I)\) algebra \(A = kQ/I\) (see e.g. [ASS]) is a gentle algebra (see [AS]), if the following conditions are satisfied:

- (G1) for each vertex \(x\), there are at most two arrows starting from \(x\), and at most two arrows ending at \(x\);
- (G2) for each arrow \(\alpha\), there is at most one arrow \(\beta\) with \(\beta \alpha \notin I\), and at most one arrow \(\gamma\) with \(\alpha \gamma \notin I\);
- (G3) for each arrow \(\alpha\), there is at most one arrow \(\beta\) with \(\beta \alpha \in I\), and at most one arrow \(\gamma\) with \(\alpha \gamma \in I\);
- (G4) the ideal \(I\) is generated by paths of length 2.

A gentle algebra is possibly infinite dimensional. We only consider finite-dimensional gentle algebra. We need the following equivalent definition of a gentle algebra (see [BR], [B]). A bound quiver algebra \(A = kQ/I\) is gentle, where \(I\) is generated by paths of length 2, if there are maps \(s', e' : Q_1 \to \{1, -1\}\) such that

- (i) if \(\alpha \in Q_1\) and \(\beta \in Q_1\) start at the same vertex with \(\alpha \neq \beta\), then \(s'(\alpha) = -s'(\beta)\);
- (ii) if \(\alpha \in Q_1\) and \(\beta \in Q_1\) end at the same vertex with \(\alpha \neq \beta\), then \(e'(\alpha) = -e'(\beta)\);
- (iii) if \(\alpha \in Q_1\) ends at the vertex where \(\beta \in Q_1\) starts and \(\beta \alpha \notin I\), then \(s'(\beta) = -e'(\alpha)\);
- (iv) if \(\alpha \in Q_1\) ends at the vertex where \(\beta \in Q_1\) starts and \(\beta \alpha \in I\), then \(s'(\beta) = e'(\alpha)\).

In this case, for a path \(p = \alpha_n \cdots \alpha_1\) with each \(\alpha_i \in Q_1\), we define \(s'(p) := s'(\alpha_1)\) and \(e'(p) := e'(\alpha_n)\).

2.4. Permitted (forbidden, respectively) threads of a gentle algebra. Let \(A = kQ/I\) be a gentle algebra. A path \(p\) is a permitted path if \(p \notin Q_0\) and \(p \notin I\). Following [AAG], a non-trivial permitted thread \(p\) is a permitted path, such that \(\alpha \alpha = 0 = \alpha p\) for all \(\alpha \in Q_1\).

A trivial permitted thread is a vertex \(x\) such that

- (i) there is at most one arrow starting from \(x\), and at most one arrow ending at \(x\);
- (ii) if \(\beta\) is an arrow ending at \(x\) and \(\gamma\) is an arrow starting from \(x\), then \(\gamma \beta \notin I\).

Denote by \(P_A\) the set of permitted threads (whatever they are non-trivial or trivial).

A path \(p = \alpha_n \cdots \alpha_1\) with each \(\alpha_i \in Q_1\) is a forbidden path, if \(n \geq 1\) and \(\alpha_1, \cdots, \alpha_n\) are pairwise different, such that \(\alpha_{i+1} \alpha_i \in I\) for \(1 \leq i \leq n - 1\). A non-trivial forbidden thread \(p\) is a forbidden path, if for all \(\alpha \in Q_1\), neither \(\alpha p\) nor \(p \alpha\) is a forbidden path.

A trivial forbidden thread is a vertex \(x\), such that

- (i) there is at most one arrow starting from \(x\), and at most one arrow ending at \(x\);
- (ii) if \(\beta\) is an arrow ending at \(x\) and \(\gamma\) is an arrow starting from \(x\), then \(\gamma \beta \in I\).

Denote by \(F_A\) the set of forbidden threads (whatever they are non-trivial or trivial).
Extend the maps $s'$ and $e'$ to trivial permitted (forbidden, respectively) threads as follows. We write a vertex $x$ as $1_x$. For a trivial permitted thread $x \in Q_0$, since $Q$ is connected, there is either $\gamma \in Q_1$ with $s(\gamma) = x$, or $\beta \in Q_1$ with $e(\beta) = x$. Define
\[ s'(1_x) = -e'(1_x) = -s'(\gamma), \quad \text{or} \quad s'(1_x) = -e'(1_x) = e'(\beta). \]
For a trivial forbidden thread $y \in Q_0$, there is either $\gamma \in Q_1$ with $s(\gamma) = y$, or $\beta \in Q_1$ with $e(\beta) = y$. Define
\[ s'(1_y) = e'(1_y) = -s'(\gamma), \quad \text{or} \quad s'(1_y) = e'(1_y) = -e'(\beta). \]

2.5. **Bijection between $\mathcal{P}_A$ and $\mathcal{F}_A$.** Avella-Alaminos and Geiss [AAG] (see also [BB]) observed that there are bijections between $\mathcal{P}_A$ and $\mathcal{F}_A$. For each permitted thread $v$, there is a unique forbidden thread $w$ such that
\[ e(w) = e(v), \quad e'(w) = -e'(v). \]
This defines a bijection $\Phi_1 : \mathcal{P}_A \to \mathcal{F}_A$, $v \mapsto w$. Also, for each forbidden thread $w$, there is a unique permitted thread $v$ such that $s(w) = s(v)$, $s'(w) = -s'(v)$.
This defines a bijection $\Phi_2 : \mathcal{F}_A \to \mathcal{P}_A$, $w \mapsto v$. Note that $\Phi_2 \Phi_1 : \mathcal{P}_A \to \mathcal{P}_A$ is not necessarily the identity map.

2.6. **Homotopy strings and homotopy bands.** For each $\alpha \in Q_1$, define its formal inverse $\alpha^{-1}$ such that $s(\alpha^{-1}) = e(\alpha)$, $e(\alpha^{-1}) = s(\alpha)$ and $(\alpha^{-1})^{-1} = \alpha$. For path $p = \alpha_n \cdots \alpha_1$ with each $\alpha_i \in Q_1$, define $p^{-1} := \alpha_1^{-1} \cdots \alpha_n^{-1}$. Define
\[ s(p^{-1}) := e(p), \quad e(p^{-1}) := s(p), \quad s'(p^{-1}) := e'(p), \quad e'(p^{-1}) := s'(p). \]

A *homotopy letter* $w$ is either a permitted path $p$ (in this case, $w$ is said to be *direct*), or the formal inverse $p^{-1}$ of $p$ (in this case, $w$ is *inverse*). The composition $ww'$ of homotopy letters is defined if $e(w') = s(w)$ and if the following conditions are satisfied:

(i) if both $w$ and $w'$ are direct, or, if both $w$ and $w'$ are inverse, then $e'(w') = s'(w)$;
(ii) if one of $w$ and $w'$ is direct and the other is inverse, then $e'(w') = -s'(w)$.

A non-trivial *homotopy string* is a sequence of consecutive composable homotopy letters $w = w_n \cdots w_1$. By the definition, a non-trivial permitted (forbidden, respectively) thread is a non-trivial homotopy string.

For $x \in Q_0$ and $\varepsilon \in \{1, -1\}$, define two trivial homotopy strings $1_{x,1}$ and $1_{x,-1}$, with $(1_{x,\varepsilon}) := x =: e(1_{x,\varepsilon})$ and $s'(1_{x,\varepsilon}) = \varepsilon$, $e'(1_{x,\varepsilon}) = -\varepsilon$. By the definition, a trivial permitted (forbidden, respectively) thread can be regarded a trivial homotopy string.

Put $(1_{x,-\varepsilon})^{-1} = 1_{x,\varepsilon}$. For the composition of (non-trivial or trivial) homotopy strings we refer to [E] Section 3. Extend the maps $s, e, s', e'$ to homotopy strings as
\[ s(w) = s(w_1), \quad e(w) = e(w_n), \quad s'(w) = s'(w_1), \quad e'(w) = e'(w_n). \]
The *degree* $\deg(w)$ of a homotopy string $w$ is the number of direct homotopy letters in $w$ minus the number of inverse homotopy letter in $w$. For examples, in the path algebra of the quiver
\[ \begin{array}{c} \alpha \\ \downarrow \gamma \downarrow \beta \end{array} \]
the degree of the homotopy string $\gamma^{-1}\beta\alpha$ is 0, rather than 1; and in the algebra given by the same quiver with relation $\beta\alpha$, the degree of the homotopy string $\gamma^{-1}\beta\alpha$ is 1, rather than 0.

A non-trivial homotopy string $w = w_n \cdots w_1$ is a homotopy band if the conditions are satisfied:

(i) $\deg(w) = 0$ and $s(w) = e(w)$;
(ii) one of $w_n$ and $w_1$ is direct and the other is inverse; and
(iii) $w$ is not a proper power of a homotopy string.

2.7. String complexes and band complexes. Bobiński [B] has introduced a string complex and a band complex in $K^b(A\text{-proj})$ for a gentle algebra $A$ (see also [BM] for $D^b(A)$; also [AAG] and [BR]). For a homotopy string $w$ and an integer $m$, there is an associated string complex $P_{m,w}$. Also, for a homotopy band $w$, an integer $m$, and an indecomposable automorphism $\mu$ of a finite-dimensional vector space, there is an associated band complex $P_{m,w,\mu}$. For details we refer to [B] Section 3] and [BM, 4.1]. Note that for different integers $m$ and $m'$, $P_{m',w}$ is a shift of $P_{m,w}$, and $P_{m',w,\mu}$ is a shift of $P_{m,w,\mu}$.

**Theorem 2.3.** ([B], [BM]) Let $A$ be a gentle algebra. Then up to isomorphism any indecomposable object in $K^b(A\text{-proj})$ is either a string complex $P_{m,w}$, or a band complex $P_{m,w,\mu}$.

A string complex given by a forbidden thread is of the following important property.

**Lemma 2.4.** ([B], Corollary 6.3) Let $A$ be a gentle algebra, $w$ a homotopy string, and $m$ an integer. Then the string complex $P_{m,w}$ is at the mouth if and only if $w$ is a forbidden thread.

2.8. Auslander-Reiten triangles in $K^b(A\text{-proj})$. For each homotopy string $w$, in order to describe the Auslander-Reiten triangle involving the string complex $P_{m,w}$, Bobiński [B] defines $w^+$, $w_+$ and $w_+$, and integer $m'(w)$, $m''(w)$, where $w^+$ and $w_+$ are either homotopy strings or 0, and $w^+_+$ is a homotopy string. Also, for an indecomposable automorphism $\mu$ of a finite-dimensional non-zero $k$-vector space $V$, one can define $\mu_1$ and $\mu_2$, which are indecomposable automorphisms of the associated finite-dimensional $k$-vector spaces. See [B] for details.

**Theorem 2.5.** ([B] Main theorem) Let $A$ be an indecomposable finite-dimensional gentle $k$-algebra with $A \neq k$, $m$ an integer.

1. Let $w$ be a homotopy string. Then there is an Auslander-Reiten triangle in $K^b(A\text{-proj})$:

$$P_{m,w} \rightarrow P_{m+m'(w),w_+} \oplus P_{m,w^+} \rightarrow P_{m+m''(w),w^+_+} \rightarrow P_{m,w}[1]$$

consisting of string complexes, where if $w_+ = 0$ then $P_{m+m'(w),w_+} = 0$, and if $w^+ = 0$ then $P_{m,w^+} = 0$.

2. Let $w$ be a homotopy band, and $\mu$ an indecomposable automorphism of a finite-dimensional vector space. Then there is an Auslander-Reiten triangle in $K^b(A\text{-proj})$:

$$P_{m,w,\mu} \rightarrow P_{m,w,\mu_1} \oplus P_{m,w,\mu_2} \rightarrow P_{m,w,\mu} \rightarrow P_{m,w,\mu}[1]$$

consisting of band complexes, where if $\mu_1 = 0$ then $P_{m,w,\mu_1} = 0$. In particular, a band complex is in a homogeneous tube.

We stress that in Theorem 2.5 (1) the middle can not be zero, i.e., the situation $w^+ = 0 = w_+$ can not occur (since $A \neq k$ by our assumption, this follows from Lemmas 3.1 and 3.6), thus the middle can not be zero. The same remark on Theorem 2.5 (2).
Note that band complex $P_{m,w,\mu}$ is at the mouth if and only if $\mu$ is an indecomposable automorphism of $1$-dimensional vector space (i.e. $\mu \in k^*$). By Theorem 2.5 the number of indecomposable direct summands of the middle terms of an Auslander-Reiten triangle is $1$ or $2$; and by Theorems 2.5 and 2.9 a component of the Auslander-Reiten quiver of $K^b(A\text{-proj})$ consists of either string complexes, or band complexes.

2.9. Mapping cones of irreducible maps between string complexes. The following fact is known in [BCS, p.38]. In $\Lambda\text{-mod}$ of artin algebra $\Lambda$, there is a corresponding result (see [Br]).

**Lemma 2.6.** Let $f$ be an irreducible map between string complexes in $K^b(A\text{-proj})$. Then $\text{Cone}(f)$ is at the mouth.

3. Proof of Theorem 1.1

3.1. The following observation is essentially due to I. Reiten and M. Van den Bergh [RV].

**Lemma 3.1.** Let $T$ be an indecomposable $k$-linear Hom-finite Krull-Schmidt triangulated category with Serre functor. Suppose that $T$ is homotopy-like and $T \not\cong D^b(k)$. Then for any indecomposable object $M$ of $T$, there are no Auslander-Reiten triangles of the form $\tau M \rightarrow 0 \rightarrow M \rightarrow S(M)$; and there is a non-zero morphism $f : M \rightarrow S(M)$ which is not an isomorphism.

**Remark.** In $D^b(k)$ any non-zero morphism $k \rightarrow S(k) = k$ is an isomorphism, and $k[-1] \rightarrow 0 \rightarrow k \rightarrow k$ is an Auslander-Reiten triangle.

**Proof of Lemma 3.1** Otherwise, assume that $\tau M \rightarrow 0 \rightarrow M \rightarrow S(M)$ is an Auslander-Reiten triangle. Suppose that $g : X \rightarrow M$ is an arbitrary morphism with $X$ indecomposable and $X \not\cong M$. Then $g$ factors through $0$, thus $g = 0$. So $\text{Hom}(X,M) = 0$. Since $h : M \rightarrow S(M)$ is an isomorphism, $\tau M = S(M)[-1] \cong M[-1]$. So $M \rightarrow 0 \rightarrow M[1] \rightarrow M[1]$ is an Auslander-Reiten triangle. Similarly, $\text{Hom}(M,Y) = 0$ for an arbitrary indecomposable object $Y$ with $Y \not\cong M$.

Let $\langle M \rangle$ be the smallest triangulated subcategory of $T$ containing $M$. Since by assumption $M \not\cong M[i]$ for each $i \neq 0$, it follows that $\text{Hom}(M,M[i]) = 0 = \text{Hom}(M[i],M)$ for all $i \neq 0$. Since $\tau(M) \rightarrow 0 \rightarrow M \rightarrow S(M)$ is an Auslander-Reiten triangle, any non-isomorphism $M \rightarrow M$ factors through $0$, i.e., $\text{End}(M)$ is a field, and hence $\text{End}(M) = k$, since $k$ is algebraically closed. All together one has

$$\text{Hom}(M[i],M[j]) = 0, \quad \forall i \neq j, \quad \text{End}(M[i]) = k, \quad \forall i \in \mathbb{Z}.$$  

By the construction of $\langle M \rangle$ (see [Ro, 3.1]) one sees that $\langle M \rangle$ is exactly the full subcategory of $T$ consisting of finite direct sums of objects of the form $M[n]$ with $n \in \mathbb{Z}$. Consider the functor $D^b(k) \rightarrow \langle M \rangle$ given by

$$k[n] \rightarrow M[n], \quad \lambda \text{Id}_{k[n]} \rightarrow \lambda \text{Id}_{M[n]}, \quad \forall n \in \mathbb{Z}, \quad \lambda \in k.$$  

This gives the triangle-equivalence $D^b(k) \cong \langle M \rangle$.

Let $T'$ be the smallest triangulated subcategory of $T$ containing all the indecomposable objects which are not isomorphic to $M[n]$ for all $n \in \mathbb{Z}$. Then $\text{Hom}(\langle M \rangle, T') = 0 = \text{Hom}(T',\langle M \rangle)$.

Since by assumption $T$ is Krull-Schmidt, it follows that any object of $T$ is a direct sum $M' \oplus X$ with $M' \in \langle M \rangle$ and $X \in T'$. Thus $T \cong \langle M \rangle \times T'$. Since by assumption $T$ is indecomposable, $T \cong \langle M \rangle \cong D^b(k)$, which contradicts the assumption. This proves the first assertion.
Now we show the second assertion. Since \( \text{End}(M) \) is a local algebra and \( k \) is algebraically closed, \( \text{End}(M)/\text{radEnd}(M) \cong k \). Put \( \theta : \text{End}(M) \to k \) to be the canonical surjective map with \( \text{Ker}\theta = \text{radEnd}(M) \). Then \( \theta \in (\text{End}(M))^\ast \). Consider the \( k \)-linear isomorphism \( \eta : \text{Hom}_T(M, S(M)) \to (\text{End}(M))^\ast \), and let \( f : M \to S(M) \) be the non-zero morphism such that \( \eta(f) = \theta \). Embedding \( f \) into a distinguished triangle

\[
\tau M \to Z \to M \to \tau S(M) = (\tau M)[1].
\]

By the proof of [RV, Theorem I.2.4], it is an Auslander-Reiten triangle. Thus \( f \) is not an isomorphism (otherwise \( \tau M \to 0 \to M \to \tau S(M) \) is an Auslander-Reiten triangle). \( \square \)

3.2. Applying Lemma 3.1 one can prove

**Lemma 3.2.** Let \( \mathcal{T} \) be an indecomposable \( k \)-linear Hom-finite Krull-Schmidt triangulated category with Serre functor. Assume that \( \mathcal{T} \) is homotopy-like. Then any exceptional 1-cycle in \( \mathcal{T} \) is indecomposable.

**Proof.** Since \( D^b(k) \) has no exceptional 1-cycles, one may assume that \( \mathcal{T} \not\cong D^b(k) \).

Assume that \( E \cong E_1 \oplus E_2 \) is an exceptional 1-cycle with \( E_1 \neq 0 \neq E_2 \). Then \( E \) has to be a 0-Calabi-Yau object, so \( S(E) \cong E \) and \( \text{Hom}^\ast(E, E) \cong k \oplus k \). In particular, \( \text{Hom}_T(E_1, E_1) \cong k \cong \text{Hom}_T(E_2, E_2) \) and \( \text{Hom}_T(E_1, E_2) = 0 \). It follows that \( E_1 \) and \( E_2 \) are indecomposable. Since \( \mathcal{T} \) is Krull-Schmidt and \( E_1 \oplus E_2 \cong S(E_1) \oplus S(E_2) \), either \( E_1 \cong S(E_1) \) or \( E_2 \cong S(E_1) \).

If \( E_1 \cong S(E_1) \), then by Lemma 3.1 there exists a non-isomorphism \( 0 \neq f : E_1 \to S(E_1) \cong E_1 \).

This contradicts \( \text{Hom}_T(E_1, E_1) \cong k \).

If \( E_2 \cong S(E_1) \), then one gets a contradiction \( k \cong \text{Hom}_T(E_1, E_1) \cong \text{Hom}_T(E_1, S(E_1)) \cong \text{Hom}_T(E_1, E_2) = 0 \). This completes the proof. \( \square \)

3.3. The following general result in triangulated category is a consequence of A. Neemann [N, Lemma 1.4.4].

**Lemma 3.3.** Let \( \mathcal{T} \) be a triangulated category, and \( X \xrightarrow{(f_1)} Y_1 \oplus Y_2 \xrightarrow{(g_1, -g_2)} Z \to X[1] \) a distinguished triangle in \( \mathcal{T} \). Then \( \text{Cone}(f_1) \cong \text{Cone}(g_2) \), \( \text{Cone}(f_2) \cong \text{Cone}(g_1) \), \( \text{Cone}(g_1 f_1) \cong \text{Cone}(f_1) \oplus \text{Cone}(g_1) \).

**Proof.** Applying [N, Lemma 1.4.4] to the homotopy cartesian square

\[
\begin{array}{ccc}
X & \xrightarrow{f_1} & Y_1 \\
\downarrow{f_2} & & \downarrow{g_1} \\
Y_2 & \xrightarrow{g_2} & Z
\end{array}
\]

one has \( \text{Cone}(f_1) \cong \text{Cone}(g_2) \) and \( \text{Cone}(f_2) \cong \text{Cone}(g_1) \), such that there is a commutative diagram of distinguished triangles:

\[
\begin{array}{ccc}
X & \xrightarrow{f_2} & Y_2 & \xrightarrow{\text{Cone}(f_2)} & X[1] \\
\downarrow{f_1} & & \downarrow{g_2} & \cong \downarrow{h} & \downarrow{f_1[1]} \\
Y_1 & \xrightarrow{g_1} & Z & \xrightarrow{\text{Cone}(g_1)} & Y_1[1]
\end{array}
\]

Thus \( bh = f_1[1]a \). Applying the octahedral axiom one gets the commutative diagram
Lemma 3.4. The following lemma will play an important role in this paper.

**Lemma 3.3.** Suppose that Cone(\(\tau E\)) is indecomposable and \(g\) is not zero. Otherwise, the distinguished triangle Cone(\(\tau E\)) splits, and hence Cone(\(\tau E\)) \(\cong\) Cone(\(f_1\)) \(\oplus\) Cone(\(g_1\)).

3.4. The following lemma will play an important role in this paper.

**Lemma 3.4.** Let \(\mathcal{T}\) be a \(k\)-linear Hom-finite Krull-Schmidt triangulated category with Serre functor. Suppose that \(\tau E \xrightarrow{(f_1)_{(1)}} Z_1 \oplus Z_2 \xrightarrow{(g_1, -g_2)} E \rightarrow S(E)\) is an Auslander-Reiten triangle of \(\mathcal{T}\) with \(Z_1\) indecomposable and \(Z_2 \neq 0\). If one of the following conditions is satisfied, then \(g_1f_1 : \tau E \rightarrow E\) is not zero.

1. \(\text{Hom}_\mathcal{T}(E, E) \cong k\).

2. \(\text{Hom}_\mathcal{T}(E, E) \cong k \oplus k\) and \(S(E) \cong E\).

**Proof.** According to Lemma 3.3, Cone(\(g_1f_1\)) \(\cong\) Cone(\(f_1\)) \(\oplus\) Cone(\(g_1\)). We claim that \(g_1f_1 : \tau E \rightarrow E\) is not zero. Otherwise, the distinguished triangle Cone(\(g_1f_1\)) \([-1] \rightarrow \tau E \xrightarrow{g_1f_1} E \rightarrow \text{Cone}(f_1) \oplus \text{Cone}(g_1)\) splits, and hence Cone(\(f_1\)) \(\oplus\) Cone(\(g_1\)) \(\cong\) E \(\oplus\) S(E). Note that Cone(\(f_1\)) \(\neq 0\) (otherwise, \(f_1\) is an isomorphism, which contradicts that \(f_1\) is an irreducible map). Similarly, Cone(\(g_1\)) \(\neq 0\). Since \(\mathcal{T}\) is Krull-Schmidt, it follows that either Cone(\(f_1\)) \(\cong E\) or Cone(\(g_1\)) \(\cong E\).

1. Assume that \(\text{Hom}_\mathcal{T}(E, E) \cong k\).

If Cone(\(f_1\)) \(\cong E\), then one has distinguished triangles

\[ \tau E \xrightarrow{f_1} Z_1 \rightarrow E \xrightarrow{\sigma_1} S(E) \]

\[ \tau E \xrightarrow{(f_1)_2} Z_1 \oplus Z_2 \xrightarrow{(g_1, -g_2)} E \xrightarrow{\sigma} S(E). \]

Note that \(\sigma_1\) and \(\sigma\) are linearly independent (otherwise, the two distinguished triangles above are isomorphic, and then \(Z_1 \cong Z_1 \oplus Z_2\), which is absurd). Thus \(\dim_k \text{Hom}_\mathcal{T}(E, S(E)) \geq 2\). This contradicts \(\dim_k \text{Hom}_\mathcal{T}(E, S(E)) = \dim_k \text{Hom}_\mathcal{T}(E, E) = 1\).

If Cone(\(g_1\)) \(\cong E\), then one gets a distinguished triangle \(Z_1 \xrightarrow{g_1} E \xrightarrow{s} E \rightarrow Z_1[1]\). Note that \(s \neq 0\) (otherwise, we get a contradiction \(Z_1 \cong E[-1] \oplus E\)). Thus \(s\) and \(\text{Id}_E\) are linearly independent.
(otherwise, $s$ is an isomorphism, and then one gets a contradiction $Z_1 = 0$). So $\dim_k \text{Hom}_T(E, E) \geq 2$, which contradicts $\text{Hom}_T(E, E) \cong k$.

All together $g_1 f_1 \neq 0$.

(2) Assume that $\text{Hom}_T(E, E) \cong k \oplus k$ and $S(E) \cong E$, say, with isomorphism $h : E \to S(E)$.

If $\text{Cone}(f_1) \cong E$, then as in the proof of (1) one has distinguished triangles

$$
\tau E \xrightarrow{f_1} Z_1 \xrightarrow{\sigma} E \xrightarrow{\sigma} S(E)
$$

such that $h^{-1} \sigma_1$ and $h^{-1} \sigma$ are linearly independent in $\text{radEnd}_T(E)$. Since $\text{End}_T(E)$ is local, it follows that $h \sigma_1$, $h \sigma$, $\text{Id}_E$ are linearly independent in $\text{End}_T(E)$, which contradicts $\dim_k \text{Hom}_T(E, E) = 2$.

If $\text{Cone}(g_1) \cong E$, then as in the proof of (1) one gets a distinguished triangle $Z_1 \xrightarrow{g_1} E \xrightarrow{s} E \to Z_1[1]$, with $s \neq 0$. By the two distinguished triangles

$$
E[-1] \xrightarrow{(f_2 h_{[1]}^1)} Z_1 \xrightarrow{g_1} E \xrightarrow{s} E
$$

one knows that $s$ and $h^{-1} \sigma$ are linearly independent in $\text{radEnd}_T(E)$. Since $\text{End}_T(E)$ is local, it follows that $s$, $h^{-1} \sigma$, $\text{Id}_E$ are linearly independent in $\text{End}_T(E)$, again contradicts $\dim_k \text{Hom}_T(E, E) = 2$.

All together $g_1 f_1 \neq 0$. This completes the proof. \hfill \square

3.5. To prove Theorem 1.1 we first show the following.

**Lemma 3.5.** Let $\mathcal{T}$ be an indecomposable $k$-linear Hom-finite Krull-Schmidt triangulated category with Serre functor. Assume that $\mathcal{T}$ is homotopy-like. Then any exceptional 1-cycle in $\mathcal{T}$ is at the mouth.

*Proof.* Since $\mathcal{D}^b(k)$ has no exceptional 1-cycles, one may assume that $\mathcal{T} \not\cong \mathcal{D}^b(k)$.

Let $E$ be an exceptional 1-cycle in $\mathcal{T}$, which is a $d$-Calabi-Yau object. By Lemma 3.2, $E$ is indecomposable. Assume that $E$ is not at the mouth, i.e., the middle term of the Auslander-Reiten triangle ending at $E$ is either 0, or of the form $Z_1 \oplus Z_2$ with $Z_1$ indecomposable and $Z_2 \neq 0$. However, the first case is impossible, by Lemma 3.1. Thus there is an Auslander-Reiten triangle

$$
\tau E \xrightarrow{(f_1 h_{[1]}^1)} Z_1 \oplus Z_2 \xrightarrow{(g_1, -g_2)} E \xrightarrow{\sigma} S(E)
$$

with $Z_1$ indecomposable and $Z_2 \neq 0$. Then $g_1 f_1 : \tau E \to E$ is not zero. In fact, if $d \neq 0$, then $\text{Hom}_T(E, E) \cong k$, and hence $g_1 f_1 \neq 0$, by Lemma 3.1(1); if $d = 0$, then $\text{Hom}_T(E, E) \cong k \oplus k$ and $S(E) \cong E$, and hence $g_1 f_1 \neq 0$, by Lemma 3.1(2). It is clear that $g_1 f_1$ is not an isomorphism (otherwise, $f_1$ is a splitting monomorphism, which contradicts that $f_1$ is irreducible). Now we divide into 2 cases.

If $d = 1$, then $\tau E \cong S(E)[-1] \cong E$ and $g_1 f_1 \in \text{Hom}_T(E, E) \cong k$. Since $g_1 f_1$ and $\text{Id}_E$ are linearly independent, this contradicts $\text{Hom}_T(E, E) \cong k$. 


If $d \neq 1$, then $\tau E \cong S(E)[-1] \cong E[d - 1]$. Since $1 - d \neq 0$ and $1 - d \neq d$, one has

$$\text{Hom}_\tau(\tau E, E) \cong \text{Hom}_\tau(E[d - 1], E) \cong \text{Hom}_\tau(E, E[1 - d]) = 0$$

where the last equality follows from the definition of an exceptional 1-cycle. Since $0 \neq g_1f_1 \in \text{Hom}_\tau(\tau E, E)$, this contradicts $\text{Hom}_\tau(\tau E, E) = 0$.

This completes the proof. \hfill \Box

3.6. **Proof of Theorem 1.1** (1) This follows from Lemmas 3.2 and 3.5

(2) Let $(E_1, \cdots, E_n)$ be an exceptional cycle in $K^b(\text{A-proj})$ with $n \geq 3$. Since $D^b(k)$ has no exceptional $n$-cycles with $n \geq 3$, one may assume that $\tau \neq D^b(k)$. It suffices to show that $E_1$ is at the mouth. Otherwise, the middle term of the Auslander-Reiten triangle starting at $E_1$ is either 0, or of the form $Z_1 \oplus Z_2$ with $Z_1$ indecomposable and $Z_2 \neq 0$. However, the first case is impossible, by Lemma 3.1. Thus there is an Auslander-Reiten triangle

$$E_1 \xrightarrow{(f_1)} Z_1 \oplus Z_2 \xrightarrow{(g_1, -g_2)} \tau^{-1}E_1 \rightarrow E_1[1]$$

with $Z_1$ indecomposable and $Z_2 \neq 0$. By (E2), $S(E_n) = E_1[-t]$ for some integer $t$, and hence $\tau^{-1}E_1 \cong S^{-1}(E_1)[-1] = E_n[t - 1]$. By Lemma 3.4(1), one has $0 \neq g_1f_1 : E_1 \rightarrow \tau^{-1}E_1$. Thus $\text{Hom}(E_1, E_n(t - 1)) \neq 0$, so $\text{Hom}^*(E_1, E_n) \neq 0$. Since $n \geq 3$, this contradicts the condition (E3). This completes the proof of (2). \hfill \Box

3.7. For later applications of Theorem 1.1 we include the following well-known fact.

**Lemma 3.6.** Let $\Lambda$ be a finite-dimensional algebra. Then $\Lambda$ is indecomposable as an algebra if and only if $K^b(\Lambda\text{-proj})$ is indecomposable as a triangulated category.

*In addition $\Lambda$ is basic, then $K^b(\Lambda\text{-proj}) \cong D^b(k)$ if and only if $\Lambda = k$.*

**Proof.** For convenience, we include a proof of the “only if” parts. Assume that $K^b(\Lambda\text{-proj}) \cong \mathcal{T}_1 \times \mathcal{T}_2$ with $\mathcal{T}_1 \neq 0$ and $\mathcal{T}_2 \neq 0$. Let $P = (P_1, P_2)$ be an indecomposable projective $\Lambda$-module with $P_i \in \mathcal{T}_i$. Since $\text{End}_\Lambda(P) = \text{End}_{K^b(\Lambda\text{-proj})}(P) = \text{End}_{\mathcal{T}_1}(P_1) \times \text{End}_{\mathcal{T}_2}(P_2)$ is a local algebra, it follows that $\text{End}_\Lambda(P)$ is an indecomposable algebra. Thus $P \in \mathcal{T}_1$ or $P \in \mathcal{T}_2$. Since all the indecomposable projective modules generate $K^b(\Lambda\text{-proj})$, it follows that both $\mathcal{T}_1$ and $\mathcal{T}_2$ contains at least one indecomposable projective module. Thus $\Lambda$ is not indecomposable.

Suppose that $\Lambda$ is basic. If $K^b(\Lambda\text{-proj}) \cong D^b(k)$, then $\Lambda$ is indecomposable. Since in $D^b(k)$ there are no nonzero morphism between two non-isomorphic indecomposable objects, $\Lambda$ has only one isoclass of indecomposable projective module $P$ with $\text{End}(P) = k$. Since $\Lambda$ is basic, $\Lambda = P$ and $\Lambda \cong \text{End}(P) = k$. \hfill \Box

4. **Characteristic components of $K^b(\Lambda\text{-proj})$**

Let $\Lambda$ be an indecomposable finite-dimensional gentle $k$-algebra.

4.1. **The shape of a characteristic component.** Recall that a connected component of the Auslander-Reiten quiver of $K^b(\Lambda\text{-proj})$ is a characteristic component, if it contains a string complex at the mouth. To get the shape of a characteristic component, we need the following result due to S. Scherotzke.
Lemma 4.1. ([Sch, Theorem 4.14, Corollary 3.4]; [V, Theorem]) Let $\Lambda$ be a finite-dimensional algebra. If the Auslander-Reiten quiver of $\mathcal{D}^b(\Lambda)$ has a component $\mathbb{Z}\Delta/G$, where $\Delta$ a Dynkin graph, and $G$ is an admissible automorphism group of $\mathbb{Z}\Delta$, then $\mathcal{D}^b(\Lambda)$ is of finite type (i.e., it has only finitely many isoclasses of indecomposable objects, up to shift), and $K^b(\Lambda\text{-proj}) = \mathcal{D}^b(\Lambda) \cong \mathcal{D}^b(k\Delta)$.

Note that in Lemma 4.1 $\mathcal{D}^b(\Lambda)$ is not assumed to have Auslander-Reiten triangles (or equivalently, the global dimension of $\Lambda$ is finite. See [H3]).

Proposition 4.2. Let $A$ be an indecomposable finite-dimensional gentle algebra. Then a characteristic component of $K^b(A\text{-proj})$ is one of the following:

$$\mathbb{Z}\tilde{A}_n/(n \geq 2), \quad \mathbb{Z}\tilde{A}_\infty, \quad \mathbb{Z}\tilde{A}_\infty/(\tau^n) (n \geq 1).$$

Proof. Let $C$ be a characteristic component of $K^b(A\text{-proj})$. Then $C$ contains no loops, by J. Xiao and B. Zhu [XZ, Corollary 2.2.3] (i.e., for a $k$-linear Hom-finite indecomposable triangulated category $\mathcal{T}$ with Serre functor, if the Auslander-Reiten quiver of $\mathcal{T}$ contains a loop, then $X \cong X[-1]$ for any object $X \in \mathcal{T}$). One may assume that $C$ has no multiple arrows (otherwise, regarding $C$ as a valued quiver so that it has no multiple arrows). Thus, $C$ is a valued stable translation quiver without loops and multiple arrows. By Theorem 3.4, $\alpha(x) \leq 2$ for each vertex $x \in C$, where $\alpha(x)$ is the number of indecomposable direct summands of the middle term in the Auslander-Reiten triangle starting from $x$. By M. C. R. Butler and C. M. Ringel [BR, p.154] (see also [Rm] and [HPR]), one has $C = \mathbb{Z}\tilde{\Delta}/G$, where the underlying graph $\Delta$ is of type

$$A_n (n \geq 2): \quad 1 \cdots n \quad A_\infty: \quad \circ \circ \circ \circ \cdots$$

$$\tilde{A}_{1,2}: \quad \circ \circ (2,2) \quad A_\infty: \quad \cdots \circ \circ \circ \circ \cdots$$

and $G$ is an admissible automorphism group of $\mathbb{Z}\tilde{\Delta}$. (Since $k$ is algebraically closed, the cases of $C_n, \tilde{A}_{11}, \tilde{C}_n, C_\infty$ can not occur).

If $\Delta = A_n$, then $K^b(A\text{-proj})$ has an Auslander-Reiten component $C = \mathbb{Z}\tilde{A}_n/G$. By the definition of an Auslander-Reiten component of $K^b(A\text{-proj})$ is an Auslander-Reiten component of $\mathcal{D}^b(A) = K^{-b}(A\text{-proj})$. Then by Lemma 4.1, $K^b(A\text{-proj}) = \mathcal{D}^b(A) \cong \mathcal{D}^b(k\tilde{A}_n)$ and $G = \{1\}$.

If $\Delta = A_\infty$, then $G = \{1\}$ or $G = \langle \tau^m \rangle$ for some $m \geq 1$. See [BR, p. 154].

Note that $\Delta$ can not be $\tilde{A}_{1,2}$ or $A_\infty$: otherwise $C$ contains no objects at the mouth.

This completes the proof. \hfill $\square$

4.2. Number of characteristic components. Let $A = kQ/I$ be a finite-dimensional gentle algebra with $Q$ a finite connected quiver. Since $Q$ is a finite quiver, by definition there are only finitely many forbidden paths, and hence only finitely many forbidden threads. By Lemma 2.4, any string complex at the mouth is of the form $P_{m,w}$, where $w$ is a forbidden thread and $m$ is an integer. Since $P_{m,w}$ is a shift of $P_{m,w}$ for different integers $m$ and $m'$, up to shift, there are only finitely many string complexes at the mouths. This shows

Lemma 4.3. Let $A$ be an indecomposable finite-dimensional gentle algebra. Then there are only finitely many string complexes at the mouth, and $K^b(A\text{-proj})$ has only finitely many characteristic components, up to shift.
4.3. The AG-invariant of a characteristic component. Let $C$ be a characteristic component of $K^b(A$-proj). For a string complex $X$ at the mouth of $C$, $\tau^t X$ is again a string complex at the mouth for $t \in \mathbb{Z}$. Since up to shift there are only finitely many string complexes at the mouth, there exists a minimal positive integer $n_X$ such that

$$\tau^n X \cong X[m_X - n_X]$$

for some integer $m_X$. Since $K^b(A$-proj) is homotopy-like, such an integer $m_X$ is unique. We will prove $(n_X, m_X) = (n_Y, m_Y)$ for any two objects $X$ and $Y$ at the mouth of $C$. For this we need

**Fact 4.4.** Let $A$ be an indecomposable finite-dimensional gentle algebra. Assume that $K^b(A$-proj) has a characteristic component $C = \mathbb{Z}A_n$ with $n \geq 2$. Then $K^b(A$-proj) = $\mathcal{D}^b(A) \cong \mathcal{D}^b(kA_n)$, and for any objects $X$ and $Y$ at the mouth of $C$, one has $Y = \tau^t X[s]$ for some integers $t$ and $s$.

**Proof.** Note that $C$ is also an Auslander-Reiten component of $D^b(A)$. By Lemma 4.1 $K^b(A$-proj) = $\mathcal{D}^b(A) \cong \mathcal{D}^b(kA_n)$.

Note that the set of indecomposable objects at the mouth of $C$ is the disjoint union of two $\tau$-orbits of $C$, here $\tau$ is the Auslander-Reiten of $D^b(A)$. Labelling $A_n$ as $1 \rightarrow 2 \rightarrow \cdots \rightarrow n$, the indecomposable projective $kA_n$-module $P(n)$ and the indecomposable injective $kA_n$-module $I(1)$ are at the lower $\tau$-orbit with $\tau^{n-1}I(1) = P(n)$; and the indecomposable injective $kA_n$-module $I(n) = P(1)$ is at the upper $\tau$-orbit. Then $\tau P(n) = S(P(n))[-1] = I(n)[-1]$ and $\tau P(1) = S(P(1))[-1] = I(1)[-1] = \tau^{-1}P(n)[-1]$. Thus $Y = \tau^t X[s]$ for some integers $t$ and $s$. \hfill \Box

**Proposition 4.5.** Let $A$ be an indecomposable finite-dimensional gentle algebra, and $C$ a characteristic component of $K^b(A$-proj). Then $(n_X, m_X) = (n_Y, m_Y)$ for any objects $X$ and $Y$ at the mouth of $C$.

**Proof.** By Proposition 4.2 $C$ is either $\mathbb{Z}A_n$ with $n \geq 2$, or $\mathbb{Z}A_{\infty}$, or $\mathbb{Z}A_{\infty}/(\tau^n)$ with $n \geq 1$. If $C = \mathbb{Z}A_n$ with $n \geq 2$, then $Y = \tau^t X[s]$ for some integers $t$ and $s$, by Lemma 4.3(2). Thus

$$\tau^n Y = \tau^t \tau^n X[s] \cong \tau^t X[m_X - n_X][s] = Y[m_X - n_X].$$

Then $n_Y \leq n_X$, by the minimality of $n_Y$. Similarly, $n_X \leq n_Y$. Thus $n_X = n_Y$, and then $m_X = m_Y$, by the uniqueness of $m_Y$. If $C$ is either $\mathbb{Z}A_{\infty}$ or $\mathbb{Z}A_{\infty}/(\tau^n)$ with $n \geq 1$, then $Y = \tau^t X$ for some integer $t$. By the same argument one has $(n_X, m_X) = (n_Y, m_Y)$.

By Proposition 4.5 for each characteristic component $C$ of $K^b(A$-proj), there exists a unique pair $(n, m)$ of integers, so that $n$ is the minimal positive integer such that $\tau^n X \cong X[m - n]$, for any indecomposable object $X$ at the mouth of $C$. We will call $(n, m)$ the AG-invariant of $C$.

4.4. The height function of a characteristic component. According to Proposition 4.2 for each characteristic component $C$ of $K^b(A$-proj), there is a well-defined height function $h : C \rightarrow \mathbb{N}$ such that $h(X) = 1$ if and only if $X$ is at the mouth. Thus in Theorem 2.5(1), if $h(P_{m, w}) = n$ then $h(P_{m+m'(w), w_+}) = n - 1$, $h(P_{m, w}) = n + 1$, and $h(P_{m+m'(w), w_+}) = n$. 

5. Proof of Theorem 4.2 and Corollary 1.3

A key observation for proving Theorem 4.2 is Lemma 5.2 below. One of the tools in the proof of Lemma 5.2 is a description of morphisms between string complexes in the bounded complex category $C^b(A$-proj) of $A$-proj, via single maps, double maps, and graph maps, introduced in [ALP].
5.1. Morphisms between some indecomposable objects of $K^b(A\text{-proj})$. Let $A$ be a gentle algebra. We use the conventions: We denote by $P_w$ the string complex $P_{m,w}$ (if we do not need to specify $m$); and if $\mu$ is an automorphism of a 1-dimensional vector space, we also denote by $P_w$ the band complex $P_{m,w,\mu}$ (if no confusions caused. In fact, to study exceptional cycles, we only need to consider this special case of $\mu$). In this way some results can be stated for both string complexes and band complexes for $\mu \in k^*$.

We use the unfolded diagram in [ALP] to express $P_w$. This presentation is based on properties of gentle algebras. We write a right multiplication $r_p$ by a path $p$ simply as $p$. Thus, if $f = r_p$ and $g = r_q$ then the composition $gf = r_{pq}$ (sometimes $gf$ is written as $pq$. In this way $gf$ is written $fg$ below). In the following $v$ and $w$ are homotopy strings or homotopy bands.

**Single maps.** Suppose that we are given a chain map between unfolded diagrams

$$
P_v : \cdots \cdot v_L \cdot v_M \cdot v_R \cdots \hspace{1cm} P_w : \cdots \cdot w_L \cdot w_M \cdot w_R \cdots \hspace{1cm} f \hspace{1cm}$$

with permitted path $f$. A chain map in $\text{Hom}_{C^b(A\text{-proj})}(P_v, P_w)$ is a single map if it has only one non-zero component $f$, and $f$ satisfies the following conditions ([ALP, 3.1]):

(L) If $v_L$ is direct, then $v_L f = 0$; and if $w_L$ is inverse, then $f w_L = 0$.
(R) If $v_R$ is inverse, then $v_R f = 0$; and if $w_R$ is direct, then $f w_R = 0$.

Denote the set of single maps $P_v \to P_w$ by $S_{v,w}$.

**Double maps.** Suppose that we are given a chain map between unfolded diagrams:

$$
P_v : \cdots \cdot v_L \cdot v_M \cdot v_R \cdots \hspace{1cm} P_w : \cdots \cdot w_L \cdot w_M \cdot w_R \cdots \hspace{1cm} f_L \hspace{1cm} f_R \hspace{1cm}$$

with permitted paths $f_L$ and $f_R$, such that $v_M f_R = f_L w_M \neq 0$. A map in $\text{Hom}_{C^b(A\text{-proj})}(P_v, P_w)$ is a double map if it has only two consecutive non-zero components $f_L$ and $f_R$, such that $f_L$ satisfies (L) and $f_R$ satisfies (R). See [ALP, 3.3]. Write $D_{v,w}$ for the set of double maps $P_v \to P_w$.

**Graph maps.** Suppose that we are given a chain map between unfolded diagrams

$$
P_v : \cdots \cdot v_L \cdot u_p \cdot u_{p-1} \cdot u_2 \cdot u_1 \cdot v_R \cdots \hspace{1cm} P_w : \cdots \cdot u_p \cdot v_L \cdot u_{p-1} \cdot u_2 \cdot u_1 \cdot w_R \hspace{1cm} f_L \hspace{1cm} f_R \hspace{1cm}$$

where $v_L \neq w_L$ and $v_R \neq w_R$, $f_L$ is either a permitted path or zero, and $f_R$ is either a permitted path or zero. A map in $\text{Hom}_{C^b(A\text{-proj})}(P_v, P_w)$ is a graph map if one of the following conditions (LG1) and (LG2) holds, and one of (RG1) and (RG2) holds, where

(LG1) The arrows $v_L$ and $w_L$ are either both direct or both inverse. In this case, there exists some (scalar multiple of a) permitted path $f_L$ such that the square on the left commutes.

(LG2) The arrows $v_L$ and $w_L$ are neither both direct nor both inverse. In this case, if $v_L$ is non-zero then it is inverse, and if $w_L$ is non-zero then it is direct.

(RG1) The arrows $v_R$ and $w_R$ are either both direct or both inverse. In this case, there exists some
(scalar multiple of a) permitted path $f_R$ such that the square on the right commutes. The arrows $v_R$ and $w_R$ are neither both direct nor both inverse. In this case, if $w_R$ is non-zero then it is inverse, and if $v_R$ is non-zero then it is direct.

Denote the set of graph maps $P_v \to P_w$ by $G_{v,w}$.

**Lemma 5.1.** ([ALP], 4.1) Let $A$ be a gentle algebra. The set $B_{v,w} := S_{v,w} \cup D_{v,w} \cup G_{v,w}$ is a basis of $\text{Hom}_{C^b(A\text{-proj})}(P_v, P_w)$.

Moreover, if both $v$ and $w$ are forbidden threads, then $G_{v,w}$ contains at most one graph map.

**Proof.** We only need to justify the second assertion. Assume that both $v$ and $w$ are forbidden threads, and $f \in G_{v,w}$, i.e., $f$ is a graph map from $P_v$ to $P_w$.

**Case 1.** If $f$ has only one non-zero component, then $f$ is of the form:

\[
\begin{aligned}
P_v & : \quad \cdots \bullet v_L \rightarrow \bullet v_R \rightarrow \cdots \\
P_w & : \quad \cdots \bullet w_L \rightarrow \bullet w_R \rightarrow \cdots
\end{aligned}
\]

Since both $v$ and $w$ are forbidden threads, only (LG2) and (RG2) is possible. Thus we have $v_L = 0 = w_R$, and hence $f = e(v) = s(w)$. Since $v$ and $w$ are given, it follows that such an $f$ is unique.

**Case 2.** If $f$ has at least two non-zero component, then $f$ is of the form:

\[
\begin{aligned}
P_v & : \quad \cdots \bullet v_L \rightarrow \bullet v_M \rightarrow \bullet v_R \rightarrow \cdots \\
P_w & : \quad \cdots \bullet v_L \rightarrow \bullet v_M \rightarrow \bullet v_R \rightarrow \cdots
\end{aligned}
\]

Since any two different forbidden threads has no common arrows, it follows that $v = w$ and $f$ is identity.

This completes the proof. \hfill \Box

5.2. We are now in position to state a main lemma for proving Theorem 1.2.

**Lemma 5.2.** Let $v$ and $c$ be forbidden threads, $m$ and $\tilde{m}$ integers, $f : P_{m,v} \to P_{\tilde{m},c}$ a non-zero non-isomorphism in $K^b(A\text{-proj})$. Then

1. The chain map $f$ is of the form:

\[
\begin{aligned}
P_{m,v} & : \quad \cdots \bullet \rightarrow \bullet \rightarrow \cdots \\
P_{\tilde{m},c} & : \quad \cdots \bullet \rightarrow \bullet
\end{aligned}
\]

where $\lambda \in k^\ast$, and $h$ is a permitted thread satisfying

- $e(h) = e(v)$, $e'(h) = -e'(v)$
- $s(h) = s(c)$, $s'(h) = -s'(c)$.

(Thus, the unique nonzero component of $f$ is from the first nonzero component of $P_{m,v}$ on the left hand to the first nonzero component of $P_{\tilde{m},c}$ on the right hand.)

2. If $g : P_{m,v} \to P_{m',w}$ is also a non-zero non-isomorphism in $K^b(A\text{-proj})$, where $w$ is a forbidden thread and $m'$ an integer, then we have $w = c$, $P_{m',w} \cong P_{\tilde{m},c}$ and $g \in kf$. 

Proof. (1) By [ALP, Proposition 4.1] (c.f. Lemma 5.1), \( f \) is a linear combination of single maps, double maps and graph maps. Let \( f_i \) \((1 \leq i \leq n)\) be a basis of \( \text{Hom}_{\mathcal{C}^b(\text{A-proj})}(P_{m,v}, P_{\tilde{m},c}) \), where \( f_1 \) is a graph map, \( f_i \ (i \in I_1) \) are single maps, \( f_i \ (i \in I_2) \) are double maps, and \( \{1, \ldots, n\} = \{1\} \cup I_1 \cup I_2 \). Thus in \( \text{Hom}_{\mathcal{C}^b(\text{A-proj})}(P_{m,v}, P_{\tilde{m},c}) \), \( f = \sum_{i=1}^n \lambda_i f_i \) for each \( \lambda_i \in \mathbb{K} \). But in \( \text{Hom}_{\mathcal{K}^b(\text{A-proj})}(P_{m,v}, P_{\tilde{m},c}) \), some \( f_i \) possibly become zero. We analyse all the \( f_i \) such that \( f_i \neq 0 \) in \( \mathcal{K}^b(\text{A-proj}) \). In the following, we denote \( f_i \) by \( f' \).

**Step 1.** Suppose that \( f' \) is a single map:

\[
P_{m,v} : \quad \cdots \bullet \xrightarrow{v_L} \bullet \xrightarrow{v_R} \cdots \\
P_{\tilde{m},c} : \quad \cdots \bullet \xrightarrow{c_L} \bullet \xrightarrow{c_R} \cdots
\]

Thus \( f' \) is a permitted path. There are 4 cases:

(i): both of \( v \) and \( c \) are trivial forbidden threads;

(ii): \( v \) is a trivial forbidden thread, and \( c \) is a non-trivial forbidden thread;

(iii): \( v \) is a non-trivial forbidden thread, and \( c \) is a trivial forbidden thread;

(iv): both of \( v \) and \( c \) are non-trivial forbidden threads.

We prove that (1) holds in all these 4 cases.

If \( v \) is a trivial forbidden thread, then \( e'(f') = -e'(v) \), \( e(f') = e(v) \). And if there exists an arrow \( \beta \) with \( s(\beta) = e(f') \), then \( \beta f' = 0 \) (otherwise, \( \beta f' \neq 0 \)). This contradicts the assumption that \( v \) is a trivial forbidden thread). Similarly, if \( c \) is a trivial forbidden thread, then \( s'(f') = -s'(c) \), \( s(f') = s(c) \). And if there exists an arrow \( \gamma \) with \( e(\gamma) = s(f') \), then \( f' \gamma = 0 \).

If \( v \) is a non-trivial forbidden thread, then \( v_R \neq 0 \) (otherwise \( v_R = 0 \)). By (L), \( v_L f' = 0 \). This contradicts the assumption that \( v \) is a forbidden thread). **Claim 1:** \( e'(f') = -e'(v_R) \). Otherwise we have \( e'(f') = e'(v_R) \). Since \( v_R \) is an arrow, \( v_R \) is a subletter of \( f' \). Thus either \( f' = v_R \) or \( f' = v_R \alpha \), where \( \alpha \) is a permitted path. If \( f' = v_R \), then \( f' \) is null homotopic as indicated below, which contradicts \( f' \neq 0 \):

\[
P_{m,v} : \quad \cdots \bullet \xrightarrow{v_L} \bullet \xrightarrow{v_R} \cdots \\
P_{\tilde{m},c} : \quad \cdots \bullet \xrightarrow{c_L} \bullet \xrightarrow{c_R} \cdots
\]

If \( f' = v_R \alpha \), then \( f' \) is again null homotopic:

\[
P_{m,v} : \quad \cdots \bullet \xrightarrow{v_L} \bullet \xrightarrow{v_R} \cdots \\
P_{\tilde{m},c} : \quad \cdots \bullet \xrightarrow{c_L} \bullet \xrightarrow{c_R} \cdots
\]

This proves **Claim 1**, i.e., \( e'(f') = -e'(v_R) \). It follows that \( v_L = 0 \) (otherwise \( v_L \neq 0 \)). By (L), \( v_L f' = 0 = v_L v_R \). This contradicts \( e'(f') = -e'(v_R) \). Hence \( e'(f') = e'(v_R) = e'(v) \). **Claim 2:** If there exists an arrow \( \beta \) with \( s(\beta) = e(f') \), then \( \beta f' = 0 \) (otherwise, \( \beta f' \neq 0 \). According to (G2) and (G3) in the definition of a gentle algebra, we have \( \beta v = 0 \). This contradicts the assumption that \( v \) is a forbidden thread).
In conclusion, if \( v \) is a non-trivial forbidden thread, then \( v_R \neq 0 \), \( e'(f') = -e'(v) \), \( e(f') = e(v) \). And if there exists an arrow \( \beta \) with \( s(\beta) = e(f') \), then \( \beta f' = 0 \) (otherwise, \( \beta f' \neq 0 \). This contradicts the assumption that \( v \) is a forbidden thread). Similarly, if \( c \) is a non-trivial forbidden thread, then \( c_R = 0 \), \( s'(f') = -s'(c) \), \( s(f') = s(c) \). And if there exists an arrow \( \gamma \) with \( e(\gamma) = s(f') \) then \( f'\gamma = 0 \).

Putting together, in all the case (i) - (iv), any non-zero single map \( f' : P_{m,v} \to P_{m,c} \) is of form:

\[
P_{m,v} : \quad \cdots \quad P_{m,c} : \quad \cdots
\]

and \( f' \) is a (non-trivial) permitted thread with

\[
e(f') = e(v), \quad e'(f') = -e'(v)
\]

\[
s(f') = s(c), \quad s'(f') = -s'(c).
\]

**Step 2.** Suppose that \( f' \) is a double map:

\[
P_{m,v} : \quad \cdots \quad v_L \quad \quad \quad v_M \quad \quad \quad v_R \quad \cdots
\]

\[
P_{m,c} : \quad \cdots \quad c_L \quad \quad \quad c_M \quad \quad \quad c_R \quad \cdots
\]

where \( v_M f'_R = f'_L c_M \neq 0 \) for permitted paths \( f'_L \) and \( f'_R \). Since there are no commutative relations in a gentle algebra, \( v_M f'_R \) and \( f'_L c_M \) are the same path, and hence there exists a path \( h \) such that \( f'_L = v_M h \) and \( f'_R = h c_M \). Hence \( f' \) is null homotopic:

\[
P_{m,v} : \quad \cdots \quad v_L \quad \quad \quad v_M \quad \quad \quad v_R \quad \cdots
\]

\[
P_{m,c} : \quad \cdots \quad c_L \quad \quad \quad c_M \quad \quad \quad c_R \quad \cdots
\]

This contradiction means that a double map can not appear as \( f' \).

**Step 3.** Suppose that \( f' \) is a graph map, then \( f' \) is either the identity graph map or a non-identity graph map. Here we only consider the case that \( f' \) is a non-identity graph map, since in **Step 4.** we will prove that \( f' \) can not be an identity graph map.

By the definition, two different forbidden threads have no common arrows. It follows from (LG2) and (RG2) that a non-zero non-identity graph map \( f' \) is as follows:

\[
P_{m,v} : \quad \cdots \quad f \quad \cdots
\]

\[
P_{m,c} : \quad \cdots \quad f \quad \cdots
\]

We need to prove that \( f' \) is a trivial permitted thread, and then the relations automatically hold:

\[
e(f') = e(v), \quad e'(f') = -e'(v)
\]

\[
s(f') = s(c), \quad s'(f') = -s'(c).
\]

There are 4 cases:

(i): both of \( v \) and \( c \) are trivial forbidden threads;
(ii): \( v \) is a trivial forbidden thread, and \( c \) is a non-trivial forbidden thread;

(iii): \( v \) is a non-trivial forbidden thread, and \( c \) is a trivial forbidden thread;

(iv): both of \( v \) and \( c \) are non-trivial forbidden threads.

The case (i) can not occur, since \( f' \) is a non-identity graph map.

In the case (ii), since \( v \) is a trivial forbidden thread, it has the following possibilities:
(1) there is exactly one arrow \( \beta \) with \( e(\beta) = v \) and no arrows starting at \( v \);
(2) there is exactly one arrow \( \gamma \) with \( s(\gamma) = v \) and no arrows ending at \( v \);
(3) there is exactly one arrow \( \beta \) with \( e(\beta) = v \), and exactly one arrow \( \gamma \) with \( s(\gamma) = v \), such that \( \gamma \beta = 0 \). Since \( c \) is a non-trivial forbidden thread, the situation (1) can not occur. Also, the situation (3) can not occur (otherwise, \( c\beta = 0 \), this contradicts \( c \) is a non-trivial forbidden thread). Hence there is exactly one arrow \( \gamma \) with \( s(\gamma) = v \) and no arrows ending at \( v \). In this case, \( f' = v \) is a trivial permitted thread.

In the case (iii), we can similarly prove that \( f' \) is a trivial permitted thread.

In the case (iv), since both of \( v \) and \( c \) are non-trivial forbidden threads, there exist exactly one arrow \( \beta \) with \( e(\beta) = e(\beta) = f' \) and exactly one arrow \( \gamma \) with \( s(c) = s(\gamma) = f' \) such that \( \gamma \beta \neq 0 \). Thus \( f' \) is a trivial permitted thread.

In conclusion, any non-identity graph map \( f' \) is of form:

\[
P_{m,v} : \begin{array}{c}
\bullet \\
\overline{f}
\end{array} \rightarrow \begin{array}{c}
\bullet \\
\overline{\cdots}
\end{array}
\]

\[
P_{m,c} : \begin{array}{c}
\bullet \\
\overline{\cdots}
\end{array} \rightarrow \begin{array}{c}
\bullet
\end{array}
\]

and \( f' \) is a trivial permitted thread.

**Step 4.** By the assumption, \( f \) is a non-zero non-isomorphism from \( P_{m,v} \) to \( P_{m,c} \). Write \( f \) as \( f = \lambda_1 f_1 + \sum_{2 \leq i \leq m} \lambda_i f_i \) in \( K^b(A\text{-proj}) \), where all \( \lambda_i \in k \), \( f_i \) is a graph map, \( f_i \) \((2 \leq i \leq m)\) are single maps (note that by **Step 2**, double maps can not appear).

(i) If \( f_1 \) is a non-identity graph map, by **Step 1** and **Step 3**, then each \( f_i \) is a permitted thread and \( e'(f_i) = -e'(v) \) and \( e(f_i) = e(v) \). That is, \( \Phi_1 : P_A \rightarrow F_A, \ f_i \mapsto v \) (c.f. Subsection 2.5) for all \( i \). Since \( \Phi_1 \) is a bijection, it follows that all the \( f_i \) are the same. This is a contradiction, since \( f_1 \) is a trivial permitted thread and \( f_2, \ldots, f_m \) are non-trivial permitted threads. So \( f_1 = f_2 = \cdots = f_m \) can not appear simultaneously as basis elements of \( \text{Hom}_{K^b(A\text{-proj})}(P_{m,v}, P_{m,c}) \). Denoted \( f_1 \) or \( f_2 \) by \( h \) (thus \( h \) is a permitted thread). Hence \( f = \lambda h \) with \( \lambda \in k^* \), and we are done.

(ii) Assume that \( f_1 \) is the identity graph map. As in case (i), all the \( f_i \) for \( 2 \leq i \leq m \) are the same, again denoted by \( h \). Thus \( h \) is a permitted thread and \( e'(h) = -e'(v) \) and \( e(h) = e(v) \). So \( f = \lambda_1 f_1 + \lambda h \) where \( \lambda \in k \). Since \( f_1 \) is the identity, all the morphisms \( f, f_1, h \) can be regarded as elements in \( \text{End}(P_{m,v}) \), such that \( f \) and \( h \) are non-invertible elements. Since \( \text{End}(P_{m,v}) \) is a local algebra, it follows that \( \lambda_1 f_1 \) is also non-invertible element in \( \text{End}(P_{m,v}) \), it follows that \( \lambda_1 = 0 \) and \( f = \lambda h \).

This completes the proof of (1).

(2) According to (1), \( g : P_{m,v} \rightarrow P_{m',v} \) is of the form:
5.3. Proof of Theorem 1.2. Let $M$ and $M'$ be string complexes at the mouth. Thus $A \neq k$.

**Step 1.** Since $A$ is indecomposable and $A \neq k$, it follows from Lemma 3.6 that $K^b(A\text{-proj})$ is an indecomposable and $K^b(A\text{-proj}) \ncong D^b(k)$. According to Lemma 3.7, there exists a non-zero morphism $f : M \to S(M)$ such that $f$ is not an isomorphism.

**Step 2.** If $g \in \text{Hom}_{K^b(A\text{-proj})}(M,S(M))$ is non-zero and non-isomorphism, then $g \in kf$.

Since $M$ is a string complex at the mouth, by Lemma 2.24, $M = P_{m,v}$, where $v$ is a forbidden thread and $m$ is an integer. Similarly, $S(M) = P_{\tilde{m},c}$ for a forbidden thread $c$ and an integer $\tilde{m}$.

Since $f : P_{m,v} \to P_{\tilde{m},c} = S(P_{m,v})$ is a non-zero morphism which is not an isomorphism, it follows from Lemma 2.2 (2) that $g \in kf$.

**Step 3.** If there is an non-zero non-isomorphism $g \in \text{Hom}_{K^b(A\text{-proj})}(M,M')$, then $M' \cong S(M)$ and $g \in kf$.

In fact, write $M' = P_{m',w}$ for a forbidden thread $w$ and an integer $m'$. Since we have already a non-zero non-isomorphism $f : P_{m,v} \to P_{\tilde{m},c} = S(M)$, it follows from Lemma 2.2 (2) that $P_{m',w} \cong S(M)$ and $g \in kf$.

**Step 4.** One has $\dim_k \text{End}(M) = \dim_k \text{Hom}_{K^b(A\text{-proj})}(M,S(M)) = \begin{cases} 1, & \text{if } M \ncong S(M); \\ 2, & \text{if } M \cong S(M). \end{cases}$

In fact, if $M \ncong S(M)$, then any non-zero morphism $g : M \to S(M)$ is of course not an isomorphism, and hence by **Step 2**, $g \in kf$. It follows that $\dim_k \text{Hom}_{K^b(A\text{-proj})}(M,S(M)) = 1$. Thus $\dim_k \text{End}(M) = 1$. If $M \cong S(M)$ with an isomorphism $\sigma : S(M) \to M$, then $0 \neq \sigma f \in \text{radEnd}(M)$. By **Step 3**, any non-zero element in $\text{radEnd}(M)$ is in $kf$, thus $\dim_k \text{radEnd}(M) = 1$. While $\text{End}(M)/\text{radEnd}(M) \cong k$, one has $\dim_k \text{End}(P_{m,v}) = 2.$
Step 5. One has

$$\dim_k \text{Hom}_{K^b(A\text{-proj})}(M, M') = \begin{cases} 1, & \text{if } M' \cong S(M) \text{ and } M' \not\cong M; \\ 2, & \text{if } M' \cong S(M) \cong M; \\ 1, & \text{if } M' \not\cong S(M) \text{ and } M' \cong M; \\ 0, & \text{if } M' \not\cong S(M) \text{ and } M' \not\cong M. \end{cases}$$

In fact, if $M' \cong S(M)$, then the assertion follows from Step 4. If $M' \not\cong S(M)$ and $M' \cong M$, then the assertion again follows from Step 4. If $M' \not\cong S(M)$ and $M' \not\cong M$, then any nonzero morphism (if there exists) in $\text{Hom}_{K^b(A\text{-proj})}(M, M')$ should be an isomorphism, by Step 3. But, since $M' \not\cong M$, it follows that $\text{Hom}_{K^b(A\text{-proj})}(M, M') = 0$.

Now, Theorem 1.2 is just a reformulation of Step 5. □

5.4. Proof of Corollary 1.3. Let $C$ and $C'$ be different characteristic components of $K^b(A\text{-proj})$, up to shift, $X \in C$, and $Y \in C'$. Since $Y$ and $Y' \in C'$ are in the same characteristic component $C'$, up to shift, it suffices to show that $\text{Hom}_{K^b(A\text{-proj})}(X, Y) = 0$ for each $X \in C$ and each $Y \in C'$. Use double induction on the height $h(X)$ and $h(Y)$ (cf., Subsection 4.5). Assume that $h(X) = 1$, i.e., $X$ is at the mouth of $C$. If $h(Y) = 1$, i.e., $Y$ is at the mouth of $C'$, then $Y \not\cong S(X)$ and $Y \not\cong X$. Since $C'$ and $C$ are different characteristic components, up to shift. It follows from Theorem 1.2 that $\text{Hom}_{K^b(A\text{-proj})}(X, Y) = 0$. If $h(Y) = 2$, then by the shape of $C'$ (Proposition 4.2) there is an Auslander-Reiten triangle $\tau Z \rightarrow Y \rightarrow Z \rightarrow S(Z)$ with $h(Z) = h(\tau Z) = 1$. Applying $\text{Hom}_{K^b(A\text{-proj})}(X, -)$ to this Auslander-Reiten triangle one sees $\text{Hom}_{K^b(A\text{-proj})}(X, Y) = 0$.

Suppose that the assertion holds for $h(X) = 1$ and $h(Y) = m (m \geq 2)$. We prove the assertion holds also for $h(Y) = m + 1 \geq 3$. By Proposition 4.2 there is an Auslander-Reiten triangle $\tau Z \rightarrow Y \oplus Y' \rightarrow Z \rightarrow S(Z)$ with $h(Y') = m - 1, h(Z) = h(\tau Z) = m$. Again applying $\text{Hom}_{K^b(A\text{-proj})}(X, -)$ and using the inductive hypothesis one gets $\text{Hom}_{K^b(A\text{-proj})}(X, Y) = 0$.

Thus, we have proved the assertion for $h(X) = 1$. By the same argument one can show the assertion for $h(X) = 2$. Assume that the assertion holds for $h(X) = n (n \geq 2)$. By the similar argument one sees that the assertion holds also for $h(X) = n + 1$. This completes the proof. □

6. Proof of Theorem 1.4

6.1. Exceptional 2-cycles of the form $(E, E)$. We first point out the following fact.

Proposition 6.1. Let $\mathcal{T}$ be an indecomposable $k$-linear Hom-finite Krull-Schmidt triangulated category with Serre functor. Assume that $\mathcal{T}$ is homotopy-like. Then there exists an exceptional 2-cycle of the form $(E, E)$ in $\mathcal{T}$ if and only if $\mathcal{T} \cong \mathcal{D}^b(k)$.

Proof. It is clear that $(k, k)$ is an exceptional cycle in $\mathcal{D}^b(k)$. Assume that $(E, E)$ is an exceptional cycle in $\mathcal{T}$. Then $S(E) \cong E[t]$ for some integer $t$, by (E2). Since $\text{Hom}_\mathcal{T}(E, E[t]) = \text{Hom}_\mathcal{T}(E, S(E)) \cong \text{End}(E) \cong k$, by (E1) one has $t = 0$. Thus there is an isomorphism $\sigma : E \rightarrow S(E)$. We then claim $\mathcal{T} \cong \mathcal{D}^b(k)$. Otherwise, by Lemma 6.1 there is a nonzero non-isomorphism $f : E \rightarrow S(E)$. Since $\sigma$ and $f$ are linearly independent, $\dim_k \text{End}_\mathcal{T}(E) = \dim_k \text{Hom}_\mathcal{T}(E, S(E)) \geq 2$. This contradicts $\text{End}(E) \cong k$. □
6.2. Exceptional 2-cycles. Exceptional 2-cycles in $K^b(A\text{-proj})$ are more complicated than the other cases. This phenomenon is caused by the fact that there is no restriction of the condition (E3).

Lemma 6.2. Let $A = kQ/I$ be a finite-dimensional gentle algebra with $A \neq k$, where $Q$ is a finite connected quiver such that the underlying graph of $Q$ is not of type $A_3$, $(X_1, X_2)$ an exceptional cycle in $K^b(A\text{-proj})$. Then $X_1$ is at the mouth of a characteristic component of $AG$-invariant $(2, m)$, and $(X_1, X_2) \cong (X_1, \tau X_1)$.

Proof. By Lemma 3.6, $K^b(A\text{-proj})$ is indecomposable and $K^b(A\text{-proj}) \ncong D^b(k)$.

First, $X_1$ is a string complex. Otherwise, $X_1$ is a band complex. Then $\tau X_1 = X_1$ (cf. Theorem 2.5(2)), and hence by (E1) one gets a contradiction

$$0 = \text{Hom}(X_1, X_1[1]) = \text{Hom}(X_1, \tau X_1[1]) = \text{Hom}(X_1, S(X_1)) \cong \text{Hom}(X_1, X_1)^*.$$

Second, we show that $X_1$ is at the mouth. Otherwise, the middle term of the Auslander-Reiten triangle starting at $X_1$ is either 0, or of the form $Z_1 \oplus Z_2$ with $Z_1$ and $Z_2$ indecomposable (cf. Theorem 2.4(1)). However, the first case is impossible, by Lemma 3.1. Thus there is an Auslander-Reiten triangle

$$X_1 \xrightarrow{(f_1)} Z_1 \oplus Z_2 \xrightarrow{(g_1, g_2)} \tau^{-1} X_1 \rightarrow X_1[1]$$

with $Z_1$ and $Z_2$ indecomposable. Since $(X_1, X_2)$ is an exceptional cycle, $\text{End}(X_1) \cong k$. By Lemma 3.4(1) one has $0 \neq g_1 f_1 : X_1 \rightarrow \tau^{-1} X_1$. According to Lemma 3.3 there is a distinguished triangle

$$\text{Cone}(f_1)[-1] \oplus \text{Cone}(g_1)[-1] \rightarrow X_1 \xrightarrow{g_1 f_1} \tau^{-1} X_1 \rightarrow \text{Cone}(f_1) \oplus \text{Cone}(g_1).$$

By Lemma 2.6, $\text{Cone}(f_1)$ and $\text{Cone}(g_1)$ are at the mouth. By the assumption $\tau X_1$ is also not at the mouth, thus there is an Auslander-Reiten triangle

$$\tau X_1 \rightarrow Y_1 \oplus Y_2 \rightarrow X_1 \xrightarrow{h} S(X_1)$$

where both $Y_1$ and $Y_2$ are indecomposable. We claim that at most one of $Y_1$ and $Y_2$ is at the mouth.

Otherwise, both $Y_1$ and $Y_2$ are at the mouth. Thus $Y_1 \ncong Y_2$ (otherwise, $Y_1$ is not at the mouth). Since the component $C$ where $X_1$ lies in is a characteristic component, it follows from Proposition 1.22 that $C$ has to be of type $Z\overline{A_3}$. By Lemma 1.1 $D^b(A) \cong D^b(k\overline{A_3})$. Thus, by $[AH]$, the quiver $Q$ of $A$ is just of type $A_3$, which contradicts the assumption.

Thus, at most one of $Y_1$ and $Y_2$ is at the mouth, while both $\text{Cone}(f_1)$ and $\text{Cone}(g_1)$ are at the mouth. It follows that

$$\text{Cone}(h) = Y_1[1] \oplus Y_2[1] \ncong \text{Cone}(g_1 f_1) = \text{Cone}(f_1) \oplus \text{Cone}(g_1).$$

By (E2), $S(X_1) \cong X_2[m_1]$ and $S^{-1}(X_1) \cong X_2[-m_2]$ for some integers $m_1$ and $m_2$. Thus $\tau^{-1} X_1 \cong X_2[1 - m_2]$, and both $g_1 f_1 : X_1 \rightarrow \tau^{-1} X_1$ and $h : X_1 \rightarrow S(X_1)$ are in some components of the complex $\text{Hom}^\bullet(X_1, X_2)$. Note that $g_1 f_1$ and $h$ can not be linear dependent, even if they are in the same components of $\text{Hom}^\bullet(X_1, X_2)$: otherwise, by the distinguished triangles

$$\begin{align*}
X_1 \xrightarrow{h} S(X_1) & \xrightarrow{} Y_1[1] \oplus Y_2[1] \xrightarrow{} X_1[1] \\
X_1 \xrightarrow{g_1 f_1} \tau^{-1} X_1 & \xrightarrow{} \text{Cone}(f_1) \oplus \text{Cone}(g_1) \xrightarrow{} X_1[1]
\end{align*}$$
we get a contradiction \( \text{Cone}(h) \cong \text{Cone}(g_1 f_1) \). Write \( \dim_k \text{Hom}^\bullet(X_1, X_2) \) for \( \sum_{i \in \mathbb{Z}} \dim_k \text{Hom}(X_1, X_2[i]) \). Thus \( \dim_k \text{Hom}^\bullet(X_1, X_2) \geq 2 \), and then
\[
\dim_k \text{Hom}^\bullet(X_1, X_1) = \dim_k \text{Hom}^\bullet(X_1, S(X_1)) = \dim_k \text{Hom}^\bullet(X_1, X_2) \geq 2
\]
which contradicts (E1). This proves that \( X_1 \) is at the mouth.

Hence \( (X_1, X_2) \cong (X_1, S(X_1)) \cong (X_1, \tau X_1) \), with \( X_1 \) at the mouth of the characteristic component \( C \), say, of AG-invariant \((n, m)\). Finally, we show that \( n = 2 \). In fact,
\[
\tau^2 X_1 = S^2(X_1)[-2] \cong S(X_2[m_1 - 2]) \cong X_1[m_1 + m_2 - 2].
\]
By the definition of the AG-invariant, \( n = 2 \) or \( n = 1 \). We claim that \( n = 2 \). Otherwise, \((1, m)\) is the AG-invariant, and thus \( \tau X_1 \cong X_1 [m - 1] \) and \( S(X_1) \cong X_1[m] \). Then
\[
\text{Hom}_{K^b(A-proj)}(X_1, X_1[m]) \cong \text{Hom}_{K^b(A-proj)}(X_1, S(X_1)) \cong \text{End}(X_1) \neq 0.
\]
By (E1), \( m = 0 \) and \( S(X_1) \cong X_1 \). Then \( \dim_k \text{End}(X_1) = 2 \), by Lemma \[\text{3.3}\] (or by Theorem \[\text{1.2}\]). This contradicts (E1). This completes the proof. \( \square \)

6.3. **Proof of Theorem** \[\text{1.4}\] 

(1) Let \( X \) be an indecomposable object at the mouth of a characteristic component \( C \) with AG-invariant \((n, m)\). Thus, \( n \) is the smallest positive integer and \( m \) is the unique integer, such that \( \tau^n Y \cong Y[m - n] \) for an arbitrary indecomposable object \( Y \) at the mouth of \( C \).

First, assume that \( n \geq 2 \). Then \( S(\tau^i X) \cong \tau^{i-1} X[1] \) for \( 0 \leq i \leq n - 2 \), and \( S(\tau^{n-1} X) \cong \tau^n X[1] \cong X[m - n + 1] \). Thus \( (X, \tau X, \cdots, \tau^{n-1} X) \) satisfies (E2).

To check that \( (X, \tau X, \cdots, \tau^{n-1} X) \) satisfies (E3'), assume \( n \geq 3 \). It suffices to show \( \text{Hom}(X, \tau^i X[j]) = 0 \) for \( 0 \leq i \leq n - 1 \) and for all \( j \in \mathbb{Z} \). Otherwise, since \( \tau^i X[j] \) is at the mouth of characteristic component \( C[j] \), it follows from Theorem \[\text{1.2}\] that either \( \tau^i X[j] \cong X \) or \( \tau^i X[j] \cong S(X) \). In the first case \( \tau^i X \cong X[-j] \). Since \( i \geq 0 \), this contradicts the definition of the AG-invariant. In the second case \( \tau^{-i} X \cong X[1-j] \), a contradiction for the same reason.

Now we prove that \( (X, \tau X, \cdots, \tau^{n-1} X) \) satisfies (E1'). Since \( X \not\cong S(X) \) (otherwise \( \tau X = X[-1] \), which contradicts \( n \geq 2 \)), it follows from Theorem \[\text{1.2}\] that \( \text{End}(X) = k \). Also, \( \text{Hom}(X, X[j]) = 0 \) for \( j \neq 0 \). Otherwise, by Theorem \[\text{1.2}\] either \( X[j] \cong X \) or \( X[j] \cong S(X) \). The first case is impossible, since \( K^b(A-proj) \) is homotopy-like. In the second case, we have \( \tau X \cong X[1-j] \), which contradicts \( n \geq 2 \). This shows \( \text{Hom}^\bullet(X, X) = k \), i.e., the condition (E1') holds.

Thus, if \( n \geq 2 \) then \( (X, \tau X, \cdots, \tau^{n-1} X) \) is an exceptional \( n \)-cycle.

Next, assume \( n = 1 \). Since \((1, m)\) is the AG-invariant of \( C \), \( S(X) = \tau X[1] \cong X[m-n+1] = X[m] \), i.e., \( X \) is an \( m \)-Calabi-Yau object. To say that \( X \) is an exceptional 1-cycle, it remains to show \( \text{Hom}^\bullet(X, X) = k \oplus k[-m] \). We divide into two cases.

If \( m = 0 \), then \( S(X) \cong X \), and then by Theorem \[\text{1.2}\] \( \text{End}(X) = k \oplus k \). For \( j \neq 0 \), we have \( X[j] \not\cong X \) and \( X[j] \not\cong S(X) \), then by Theorem \[\text{1.2}\] \( \text{Hom}^\bullet(X, X[j]) = 0 \). Thus \( \text{Hom}^\bullet(X, X) = k \oplus k \).

If \( m \neq 0 \), then \( X \not\cong X[m] \), i.e., \( X \not\cong S(X) \), and then \( \text{End}(X) = k \) by Theorem \[\text{1.2}\]. Also,
\[
\text{Hom}_{K^b(A-proj)}(X, X[m]) \cong \text{Hom}_{K^b(A-proj)}(X, S(X)) \cong \text{End}(X) = k.
\]
For \( j \neq 0 \) and \( j \neq m \), one has \( X[j] \not\cong X \) and \( X[j] \not\cong X[m] \not\cong S(X) \). It follows from Theorem \[\text{1.2}\] that \( \text{Hom}^\bullet(X, X[j]) = 0 \). Thus \( \text{Hom}^\bullet(X, X) = k \oplus k[-m] \).
This shows that if $n = 1$ then $X$ is an exceptional 1-cycle.

Now, we prove the uniqueness of exceptional cycle in the characteristic component $C$, up to $\approx$. For this, assume that $(E_1, \cdots, E_l)$ is an arbitrary exceptional cycle in $C$ with $l \geq 1$ (thus each $E_i \in C$).

Then $E_1$ is at the mouth: if $l \neq 2$ then this follows from Theorem 1.1(2), and if $l = 2$ then this follows from Lemma 6.2. By what we have proved, $(E_1, \tau E_1, \cdots, \tau^{n-1} E_1)$ is also an exceptional cycle. It follows from Lemma 2.2(1) that $l = n$ and $(E_1, \cdots, E_l) \approx (E_1, \tau E_1, \cdots, \tau^{n-1} E_1)$.

It remains to prove that $(E_1, \tau E_1, \cdots, \tau^{n-1} E_1) \approx (X, \tau X, \cdots, \tau^{n-1} X)$, where $X$ is an arbitrary indecomposable object at the mouth of $C$. By Proposition 4.2 $C$ is one of the form

$$Z\tilde{A}_n^0 \quad (n \geq 2), \quad Z\tilde{A}_\infty, \quad Z\tilde{A}_\infty^0/(\tau^n) \quad (n \geq 1).$$

If $C$ is of the form $Z\tilde{A}_\infty$ or $Z\tilde{A}_\infty^0/(\tau^n) \quad (n \geq 1)$, then $E_1 = \tau^t X$ for some integer $t$, since both $X$ and $E_1$ are at the mouth of $C$. Write $t = qn + r$, where $q$ and $r$ are integers with $0 \leq r \leq n - 1$. Then $E_1 = \tau^t X = \tau^{qn} \tau^t X = \tau^t X[q(m-n)]$, and thus by Lemma 2.2(1) one has

$$(E_1, \tau E_1, \cdots, \tau^{n-1} E_1) \approx (\tau^t X, \tau E_1, \cdots, \tau^{n-1} E_1) \approx (X, \tau X, \cdots, \tau^{n-1} X).$$

If $C$ is of the form $Z\tilde{A}_n$, then by Fact 4.1 $E_1 = \tau^t (X)[s]$ for some integers $t$ and $s$, since both $X$ and $E_1$ are at the mouth of $C$. Again write $t = qn + r$, where $q$ and $r$ are integers with $0 \leq r \leq n - 1$. By the same argument as above one has

$$(E_1, \tau E_1, \cdots, \tau^{n-1} E_1) \approx (\tau^t X, \tau E_1, \cdots, \tau^{n-1} E_1) \approx (X, \tau X, \cdots, \tau^{n-1} X).$$

This completes the proof of assertion (1).

(2) Let $(E_1, \cdots, E_n)$ be an exceptional cycle in $K^b(A$-proj) with $n \geq 2$. We want to prove that there is a characteristic component $C$ of AG-invariant $(n, m)$ such that $E_1$ is at the mouth of $C$ and

$$(E_1, \cdots, E_n) \approx (E_1, \tau E_1, \cdots, \tau^{n-1} E_1).$$

First, $E_1$ is a string complex. Otherwise, $E_1$ is a band complex, and then $\tau E_1 = E_1$ (cf. Theorem 2.4(2)); and hence by (E1) we get a contradiction $0 = \text{Hom}(E_1, E_1[1]) = \text{Hom}(E_1, \tau E_1[1]) = \text{Hom}(E_1, S(E_1)) = \text{Hom}(E_1, E_1)^\tau$.

Second, $E_1$ is at the mouth. In fact, for $n = 2$ this follows from Lemma 6.2 and for $n \geq 3$ this follows from Theorem 1.1(2).

Thus $E_1$ is a string complex at the mouth. By definition $E_1$ is in a characteristic component $C$, say with AG-invariant $(n', m)$. By (1), $(E_1, \tau E_1, \cdots, \tau^{n'-1} E_1)$ is also an exceptional cycle. By Lemma 2.2(1) we have $n' = n$ and $(E_1, \cdots, E_n) \approx (E_1, \tau E_1, \cdots, \tau^{n-1} E_1)$. This completes the proof of assertion (2).

(3) By (1) and (2), any object in exceptional $n$-cycles with $n \geq 2$ is indecomposable and at the mouth. By Lemma 3.6 $K^b(A$-proj) is indecomposable; since $K^b(A$-proj) is homotopy-like, by Theorem 1.1(1), any exceptional 1-cycle in $K^b(A$-proj) is indecomposable and at the mouth.

Let $E$ be a string complex. If $E$ is at the mouth of a characteristic component of AG-invariant $(1, d)$, then by (1), $E$ is an exceptional 1-cycle. Conversely, if $E$ is an exceptional 1-cycle, such that $E$ is $d$-Calabi-Yau, then $E$ is at the mouth. Hence $E$ is in a characteristic component $C$ of $K^b(A$-proj), say with AG-invariant $(n, m)$. Since $S(E) \cong E[d], \tau(E) \cong E[d-1]$. By the definition of AG-invariant one has $n = 1$ and $m = d$.  


If $E$ is a band complex which is an exceptional 1-cycle, then $E$ is at the mouth, and hence $E$ is at the mouth of a homogeneous tube (cf. Theorem 1.3(2)).

7. Examples

Example 7.1. Let $A = k[x]/\langle x^2 \rangle$. Since $A$ is symmetric, $K^b(A\text{-proj})$ is a 0-Calabi-Yau category (in the sense of [K]), and hence each object is 0-Calabi-Yau. The indecomposable objects in $K^b(A\text{-proj})$ are of form $X_i[t]$ with $t \in \mathbb{Z}$ and $l \geq 0$, where $X_i$ is the following complex with each differential given by multiplication by $x$:

$$0 \overset{x}{\longrightarrow} A = X^{-t}_l \overset{x}{\longrightarrow} A \overset{x}{\longrightarrow} \cdots \overset{x}{\longrightarrow} A = X^0_l \overset{x}{\longrightarrow} 0.$$  

The Auslander-Reiten quiver of $K^b(A\text{-proj})$ has a unique component of the type $Z_{A_{\infty}}$, which is a characteristic component with AG invariant $(1,0)$, as given below.

$$\vdots \quad \vdots \quad \vdots$$

$$\cdots \quad X_2[-2] \quad X_2[-1] \quad X_2 \quad \cdots$$

$$\cdots \quad X_1[-1] \quad X_1 \quad X_1[1] \quad \cdots$$

$$\cdots \quad A[-1] \quad A = X_0 \quad A[1] \quad \cdots$$

All the exceptional 1-cycles are $A[t]$ with $t \in \mathbb{Z}$. There are no exceptional $n$-cycle with $n \geq 2$. One can see this directly as follows. If $l \geq 1$, then one constructs a chain map $f : X_l \to X_l[1]$ with all non-zero component being identity. Then $f$ is not null-homotopic. Since $X_l \not\cong X_l[1]$, $f$ can not be an isomorphism, and hence $\text{Hom}^\bullet(X_l, X_l) \neq k$.

Example 7.2. Theorem 1.4(4) assert that a band complex which is an exceptional 1-cycle is at the mouth. However, a band complex at the mouth is not necessarily an exceptional 1-cycle.

1. We take an example from [ALP]. Let $A = kQ/I$, where $Q$ is the quiver

$$\begin{array}{cccc}
1 & 2 & 3 & 4 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
a & b & c & d \\
\downarrow & \downarrow & \downarrow & \downarrow \\
0 & e & 0 & \alpha \beta \\
\downarrow & \downarrow & \downarrow & \downarrow \\
2 & 3 & 4 & 1 \\
\end{array}$$

and $I = \langle ba, cb, ac, ed, fe, df \rangle$. Consider the homotopy band $v = d^{-1}e^{-1}f^{-1}cba$ and the band complex $P_{m,v,\lambda}$, where $m \in \mathbb{Z}$ and $\lambda \in k^*$. Then $P_{m,v,\lambda}$ is at the mouth of a homogeneous tube, and $\text{Hom}^{k}(A\text{-proj})(P_{m,v,\lambda}, P_{m,v,\lambda}[3]) \neq 0$ ([ALP, 5.17]). Thus $P_{m,v,\lambda}$ is not an exceptional 1-cycle.

2. Let $A = k(\begin{array}{cc}
\alpha & 1 \\
\beta & 2 \\
\end{array})$ be the Kronecker algebra. Consider the homotopy band $w = \beta^{-1}\alpha$ and the band complex $P_{w,\lambda}$ with $\lambda \in k^*$: $0 \to P(2) \overset{\alpha + \lambda \beta}{\to} P(1) \to 0$. Then $P_{m,w,\lambda}$ is at the mouth of a homogeneous tube, and $P_{w,\lambda}$ is an exceptional 1-cycle.


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