Hyperbolic Superspaces and Super-Riemann Surfaces

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Abstract: In this paper, we will generalize some results in Manin’s paper (Invent Math 104:223–243, 1991) to the supergeometric setting. More precisely, viewing $\mathbb{C}^{1|1}$ as the boundary of the hyperbolic superspace $\mathcal{H}^{3|2}$, we reexpress the super-Green functions on the supersphere $\hat{\mathbb{C}}^{1|1}$ and the supertorus $T^{1|1}$ by some data derived from the supergeodesics in $\mathcal{H}^{3|2}$.

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1. Introduction

Let $K$ be a number field, and $\mathcal{O}_K$ be the ring of integers of $K$. The choice of a model $X_{\mathcal{O}_K}$ of a smooth algebraic curve over $K$ defines an arithmetic surface over $\text{Spec}(\mathcal{O}_K)$. A closed vertical fiber of $X_{\mathcal{O}_K}$ over a prime $p \in \mathcal{O}_K$ is given by $X_p$, the reduction mod $p$ of the model. The completion $\hat{X}_{\mathcal{O}_K}$ of $X_{\mathcal{O}_K}$ is achieved by adding to $\text{Spec}(\mathcal{O}_K)$ the archimedean places represented by the set of all embeddings $\alpha : K \hookrightarrow \mathbb{C}$. The Arakelov divisors on the completion $\hat{X}_{\mathcal{O}_K}$ are defined by the divisors on $X_{\mathcal{O}_K}$ and by the formal real combinations of the closed vertical fibers at infinity. However, Arakelov geometry does not provide an explicit description of these fibers, and it prescribes instead a Hermitian metric on each Riemann surface $X_{\mathbb{C}}$ for each archimedean prime $\alpha$. The Hermitian geometry on each $X_{\mathbb{C}}$ is sufficient to develop an intersection theory on the completed model. For example, the intersection indices of divisors on the fibers at infinity are obtained via Green functions on $X_{\mathbb{C}}$ [2].

The missing structure in Arakelov’s theory is the analog at the arithmetic infinity of the reductions modulo powers of $p$ of the closed fibers of $X_{\mathcal{O}_K}$. Manin [1] planned
to enrich Arakelov’s metric structure by realizing such missing structure. Inspired by Mumford’s $p$-adic uniformization theory of algebraic curve [3], he suggested to construct a differential-geometric object—a certain hyperbolic 3-manifold—playing the role of a model at infinity. Roughly speaking, by choice of a Schottky uniformization, the Riemann surface $X_{\mathbb{C}}$ is the boundary at the infinity of a hyperbolic 3-manifold described as the quotient of the hyperbolic 3-space $\mathbb{H}^3$ by the action of the Schottky group $\Gamma$. Furthermore, Manin corroborated his suggestion by interpreting Arakelov Green functions in terms of geodesic configurations on this space.

After Manin’s pioneering work, there are several generalizations toward different aspects. The followings are some interesting progresses on this topic.

– Werner generalized Manin’s approach to higher dimensional cases [4]. Namely, she interpreted certain Archimedean (non-Archimedean) Arakelov intersection numbers of linear cycles on $\mathbb{P}^{n-1}$ with Riemannian symmetric space associated to $SL(n, \mathbb{C})$ (Bruhat–Tits building associated to $PGL(n)$).

– Although when Manin’s work was published, one did not yet discover various novel dualities in string theory, from the physical point of view, Manin’s perspective has a heavy flavor of AdS/CFT-correspondence since Green functions are related to the quantum correlation functions of boundary CFT [5] and geodesic configurations are related to the classical bulk gravity with the cosmology constant. In the paper [6], Manin and Marcolli considered certain hyperbolic 3-manifolds as analytic continuations of the known Lorentzian signature black holes (e.g. BTZ black hole, Krasnov black hole), and they demonstrated that the expressions for Green functions in terms of the geodesic configurations on the above hyperbolic 3-manifolds can be nicely interpreted in the spirit of AdS/CFT-correspondence.

– In the paper [7], Consani and Marcolli consider the case of an arithmetic surface over $\text{Spec}(\mathcal{O}_K)$ with the fibers of genus $g \geq 2$. They defined a cohomology of the cone of the local monodromy $N$ at arithmetic infinity which is related to Delinger’s archimedean cohomology and regularized determinants [8]. And by Connes’ theory of spectral triples, they established a connection between such cohomology and the dual graph of the fiber at infinity in terms of the infinite tangle of bounded geodesics in the hyperbolic 3-manifold.

In this paper, we will generalize Manin’s results to the supergeometric setting. The motivation is that in the original AdS/CFT-correspondence coming from string theory, supersymmetries are the necessary ingredients both appearing in the boundary and the bulk theories. Firstly, let us briefly collect some basic materials on geometry of supermanifolds. More details can be found in [9–11].

I. Supermanifolds.
There are two approaches to define supermanifolds:

– Algebro-geometric definition: a supermanifold $M^{p|q}$ of dimension $p|q$ is a pair $(M, \mathcal{O}_M)$ consisting of a (Hausdorff and second countable) topological space $M$ together with a sheaf $\mathcal{O}_M$ of commutative superalgebras with unity, such that
  
  – There exists an open cover $\{U_\alpha\}$ of $M$, where for each $\alpha$, $\mathcal{O}_M(U_\alpha) \simeq C^\infty(U_\alpha) \otimes \Lambda(\mathbb{R}^q)$,
  
  – If $N_M$ is the sheaf of nilpotents of $\mathcal{O}_M$, then $(M, \mathcal{O}_M/N_M)$ is isomorphic to $(M, C^\infty(M))$.

– Differential geometric definition: let $\mathbb{R}^{p|q}$ be a $p|q$-dimensional superspace over the real Grassmann algebra $\Lambda^\infty_{\mathbb{R}}$, and endow $\mathbb{R}^{p|q}$ with the (non-Hausdorff) De Witt topology, then a supermanifold $M^{p|q}$ is obtained by gluing open sets of $\mathbb{R}^{p|q}$ via
superdiffeomorphisms. Denote by $\epsilon_0$ the projection map onto the zero-order part of the Grassmann algebra, and define an equivalence relation $\sim$ on the supermanifold: for $x, y \in M^{p|q}$, $x \sim y$ iff $\epsilon_0(x) = \epsilon_0(y)$, then the space $M = M^{p|q}/\sim$ is a $p$-dimensional $C^\infty$ manifold, called the body of $M^{p|q}$.

The two categories of supermanifolds defined by the above two different manners respectively are essentially equivalent. However, the latter one is more geometric, so it is convenient to talk about the local coordinate $\{X_A\}_{A=1,\ldots,p; p+1,\ldots,p+q} = (x_1, \ldots, x_p; \theta_1, \ldots, \theta_q)$ of $M^{p|q}$ with $x_1, \ldots, x_p$ valued in the even part $(\Lambda^\infty_{\mathbb{R}})_0$ of the Grassmann algebra $\Lambda^\infty_{\mathbb{R}}$ and $\theta_1, \ldots, \theta_q$ valued in the odd part $(\Lambda^\infty_{\mathbb{R}})_1$. Hence it is accepted commonly by the physicists. Therefore, we will adopt the second approach throughout this paper.

**II. Super-Riemann geometry.** One can define the tangent sheaf $T_{M^{p|q}}$ and the cotangent sheaf $\Omega^1_{M^{p|q}}$ over a supermanifold $M^{p|q}$. The supermetric on $M^{p|q}$ is a graded-symmetric even non-degenerate $\mathcal{O}_M$-linear morphism of sheaves $(\cdot, \cdot) : T_{M^{p|q}} \times T_{M^{p|q}} \to \mathcal{O}_M$. Switching to of differential geometric point of view, one writes the supermetric as $g = dX_A(A\otimes B)BdX$ in terms of the local coordinates with $BdX = (-1)^{|B|}dX_B$. Then the corresponding super-Levi-Civita connection $\nabla^g$ is given by the super-Christoffel symbols

$$\Gamma^C_{AB} = \frac{1}{2}(-1)^{|D|}g^{CD} \left[Dg_{AB} + (-1)^{|A|}|B|Dg_{AB}, A - (-1)^{|D|(|A|+|B|)}ABg_{BD}\right].$$

A supercurve $\Upsilon : \mathbb{R}^{1|1} \to M^{p|q}$ with parameter $(u; \gamma)$ is called a super-geodesic with respect to $\nabla^g$ if and only if

$$\frac{\nabla^g}{du}(\Upsilon_u\partial_u) = 0,$$

which is locally equivalent to the system of differential equations

$$\frac{d^2}{du^2} \Upsilon^*(X_A) + \sum_{B,C} \frac{d}{du} \Upsilon^*(X_B) \frac{d}{du} \Upsilon^*(X_C) \Upsilon^* \left(\Gamma^A_{BC}\right) = 0$$

for any $A$. Super-Riemann curvature, super-Ricci curvature and super scalar curvature can be also easily generalized to supermanifolds by formulas

$$R^D_{ABC} = -\Gamma^D_{ABC} + (-1)^{|B|(|A|+|E|)} \Gamma^D_{EB} \Gamma^E_{AC} + (-1)^{|B||C|} \Gamma^D_{AC} B - (-1)^{|C||A|+|B|+|E|} \Gamma^D_{EC} \Gamma_{AB},$$

$$R_{AC} = (-1)^{|B|(|A|+1)} R^B_{ABC},$$

$$R = R_{AB}g^{AB}.$$ 

**Definition 1.** 1. A supermanifold $M^{p|q}$ is called a (positive/negative/ zero) Einstein supermanifold if it admits a supermetric $g = dX_A(A\otimes B)BdX$ such that the corresponding super-Ricci curvature satisfies the condition

$$R_{AB} = c_{AB}$$

for a (positive/ negative/ zero) real number $c$. 

2. A supermanifold $M^{p|q}$ is called a (positive/negative/zero) Bosonic supermanifold if it admits a supermetric $g = dX_A (A, B) dX_B$ such that the corresponding super scalar curvature satisfies the condition that $e_0(R)$ is a (positive/negative/zero) real constant.

Next, in Sect. 2 we will consider the minimal supergeometric extension of some classical low dimensional hyperbolic manifolds, in particular, the supermetrics and super-volume forms invariant under certain Lie supergroups are constructed. In Sect. 3, we will define the super-Green functions for super-Riemann surfaces, then viewing $\mathbb{C}^{1|1}$ as the boundary of the hyperbolic superspace $\mathcal{H}^{3|2}$, we reexpress the super-Green functions on the supersphere $\hat{\mathbb{C}}^{1|1}$ and the supertorus $T^{1|1}$ by some data derived from the supergeodesics in $\mathcal{H}^{3|2}$. The main results of this paper are summarized as follows.

**Theorem 1.** (= Proposition 8, Proposition 9 and Proposition 13)

1. Viewing the super-Riemann sphere $\hat{\mathbb{C}}^{1|1}$ as the boundary of $\mathcal{H}^{3|2} \cup \{\infty\}$, the super-Green function on $\hat{\mathbb{C}}^{1|1}$ can be reexpressed as

$$G_{P_1}(Z, \Theta) = \log \left( \frac{1}{\cosh d_Q(P_1, P_2)} - \frac{1}{2 \cosh d_Q(P_1, P_2)} \left( 1 - \frac{1}{\cosh^2 d_Q(P_1, P_2)} \right) \Theta \bar{\Theta} \right) = \log \frac{1}{\cosh d(\hat{Q}, P_{12})},$$

where $P_1 = (0, 0)$, $P_2 = (Z, \Theta)$ are points in $\hat{\mathbb{C}}^{1|1}$, $Q = (0, 0, 0; 0, 0)$ is a point in $\mathcal{H}^{3|2}$, and $P_{12}, \hat{Q}$ are also points in $\mathcal{H}^{3|2}$ determined by $P_1, P_2, Q$.

2. Viewing the supertorus $T^{1|1}$ as the boundary of $\mathcal{H}^{3|2}/\bar{\Gamma}$, where $\bar{\Gamma}$ denotes the super-Poincaré extension $\bar{\Gamma}$ of the supertranslation group, the super-Green function on $T^{1|1}$ with supermoduli $(\tau; \delta)$ can be expressed as

$$G(Z, \Theta) = \frac{1}{2} d(P_0, \hat{Q}) B_2 \left( \frac{d(P_0, \hat{P})}{d(P_0, \hat{Q})} \right) + d(P_0, \hat{P}_-)$$

$$+ \sum_{n=1}^{\infty} \left( d(P_0, \Omega^n) + d(P_0, \Omega^n_-) \right) + \frac{4\pi^2}{d(P_0, \hat{Q})} \Theta \bar{\Theta}$$

$$= \frac{1}{2} d(\bar{P}_0, \tilde{Q}) B_2 \left( \frac{d(\bar{P}_0, \bar{P})}{d(\bar{P}_0, \tilde{Q})} \right) + d(\bar{P}_0, \bar{P}_-)$$

$$+ \sum_{n=1}^{\infty} \left( d(\bar{P}_0, \tilde{Q}^n) + d(\bar{P}_0, \tilde{Q}^n_-) \right) + \frac{4\pi^2}{d(\bar{P}_0, \tilde{Q})} \Theta \bar{\Theta},$$

where $\Omega = (q, \Theta)$, $P_- = (1 - \rho; \Theta)$, $\Omega^n = (1 - q^n \rho, \Theta)$ and $\Omega^n_- = (1 - q^n \rho^{-1}, \Theta)$ are all points lying on the boundary of $\mathcal{H}^{3|2}$ with $q = e^{2\pi i (\tau + \Theta \delta)}$.

Some more related questions are worthy to be further studied. For example,

- We should consider the boundary supermanifold with the body as a higher genus Riemann surface, and consider the bulk supermanifold with the body as an another type of hyperbolic 3-manifold in Thurston’s eight geometries [12];
- We should add more odd degree of freedoms beyond the minimal supergeometric extension;
– We should give an interpretation of these Manin-type formulas from the viewpoint of AdS/CFT-correspondence with supersymmetries as done in [6];
– We should consider similar problems for supergeometries of non-Archimedean version, and consider the relation with $p$-adic AdS/CFT-correspondence [13];
– we should consider the mathematical aspects of AdS/CFT-correspondence inspired by the Hyperbolic/Arakelov geometry correspondence. In particular, we should provide an exact expression as the initial step;
– We should ask if the Manin’s model at the arithmetic infinity is replaced by P. Scholze’s perfectoid version of hyperbolic 3-manifolds [14], how does the story go?

2. Hyperbolic Superspaces

Let us consider the superspace $\mathbb{R}^{p+1|2q}$ endowed with a supermetric

$$ds^2 = -dx_0^2 + dz_1^2 + \cdots + dx_p^2 + d\theta_1 d\theta_2 + \cdots + d\theta_{2q-1} d\theta_{2q}$$

written in terms of matrix as

$$g = \left( \begin{array}{cc} \eta_{1,p} & 0 \\ 0 & J_q \end{array} \right)$$

for $\eta_{1,p} = \left( \begin{array}{cc} -1 & 0 \\ 0 & \text{Id}_{p} \end{array} \right)$, $J_q = \frac{1}{2} \left( \begin{array}{cccc} 0 & 1 & & \\ & -1 & 0 & \\ & & \ddots & \\ & & & 0 & 1 \\ & & & & -1 \\ & & & & & 0 \end{array} \right)$. The Lie supergroup

$$OSp(1, p|2q, \mathbb{R}) = \{ X \in \text{SMat}(p+1|2q, \mathbb{R}) : X^{st} g X = g \},$$

where expressing $X = \left( \begin{array}{cc} A_{(0)} & B_{(1)} \\ C_{(1)} & D_{(0)} \end{array} \right)$ in terms of the even parts labelled by subscript (0) and the odd parts labelled by (1), the supertranspose $X^{st}$ is given by $X^{st} = \left( \begin{array}{cc} A_{(0)}^t & C_{(1)}^t \\ -B_{(1)}^t & D_{(0)}^t \end{array} \right)$. A $p|2q$-dimensional hyperbolic total superspace $\mathbb{H}^{p|2q}$ is defined as

$$\mathbb{H}^{p|2q} = \left\{ H := \left( \begin{array}{c} x_0 \\ \vdots \\ x_p \\ \theta_1 \\ \vdots \\ \theta_{2q} \end{array} \right) \in \mathbb{R}^{p+1|2q} : H^{t} g H = -1 \right\}.$$

The supergroup $OSp(1, p|2q, \mathbb{R})$ acts obviously on $(\mathbb{H}^{p|2q}, ds^2)$.

Proposition 1. $\mathbb{H}^{p|2q}$ can be identified with $OSp(1, p|2, \mathbb{R}) / OSp(p|2, \mathbb{R})$. 
Proof. We consider a supervector $H = \left( \begin{array}{c} x_0 \\ \theta \end{array} \right) \in \mathbb{H}^{p|2}$ for $X = \left( \begin{array}{c} x \\ \theta \end{array} \right)$ and $\theta = \left( \begin{array}{c} \theta_1 \\ \theta_2 \end{array} \right)$, then $x_0^2 = 1 + \vec{x} \cdot \vec{x} + \theta_1 \theta_2$, where $\vec{x} \cdot \vec{x} = \sum_{i=1}^{p} x_i^2$, and we consider a supermatrix $X = \left( \begin{array}{cc} A(0) & B(1) \\ C(1) & D(0) \end{array} \right)$, where

$$A(0) = \left( \begin{array}{cc} x_0 & \vec{x}^t \\ \vec{x} & \text{Id}_p + \frac{\vec{\theta}}{1 + \vec{x}^t} \end{array} \right),$$

$$B(1) = \frac{1}{2} \left( \begin{array}{ccc} \frac{x_1}{1 + x_0} \theta_1 & \theta_2 \\ \frac{x_1}{1 + x_0} \theta_1 & \frac{x_1}{1 + x_0} \theta_2 \\ \vdots & \vdots \\ \frac{x_p}{1 + x_0} \theta_1 & \frac{x_p}{1 + x_0} \theta_2 \end{array} \right),$$

$$C(1) = \left( \begin{array}{cccc} \frac{x_1}{1 + x_0} \theta_1 & \cdots & \frac{x_p}{1 + x_0} \theta_1 \\ \frac{x_1}{1 + x_0} \theta_2 & \cdots & \frac{x_p}{1 + x_0} \theta_2 \end{array} \right),$$

$$D(0) = \left( \begin{array}{cc} 0 & -1 + \frac{1}{2} \theta_1 \theta_2 \\ 1 - \frac{1}{2} \theta_1 \theta_2 & 0 \end{array} \right).$$

One can check that $X \in OSP(1, p|2, \mathbb{R})$, and it transforms the supervector $H_0 = \left( \begin{array}{c} 1 \\ 0 \end{array} \right)$ into $H$. The isotropy group of $H_0$ is exactly the supergroup $OSP(p|2, \mathbb{R})$ by the natural embedding. The claim follows. $\square$

From now on, we work only with two odd dimensions, and we call such setup the \textit{minimal supergeometric extension}. Assume the zero-order part $\epsilon_0(x_0)$ of $x_0$ is positive, then $x_0$ has an inverse

$$x_0^{-1} = \frac{1}{\epsilon_0(x_0)} \left[ 1 - \frac{x_0 - \epsilon_0(x_0)}{\epsilon_0(x_0)} + \left( \frac{x_0 - \epsilon_0(x_0)}{\epsilon_0(x_0)} \right)^2 - \cdots \right]$$

in the Grassmann algebra $\Lambda_\mathbb{R}^\infty$, hence we define the map

$$\alpha : \left( \begin{array}{c} x_0 \\ \vec{x} \\ \theta \end{array} \right) \mapsto \left( \begin{array}{c} x'_0 \\ \vec{x}' \\ \theta' \end{array} \right) = \left( \begin{array}{c} x_0^{-1} \vec{x} \\ \theta \end{array} \right)$$
such that $\sum_{i=0}^{p} (x'_i)^2 + \theta'_1 \theta'_2 = 1$. Then since $\sum_{i=0}^{p} (\epsilon'_0 (x'_i))^2 = 1$ and $\epsilon'_0 (x'_0) > 0$, $x'_p + 1$ also has an inverse $(x'_p + 1)^{-1}$, hence we define the map

$$\beta : \begin{pmatrix} \frac{x'_0}{\theta'} \\ \vdots \\ \frac{x'_p}{\theta'} \end{pmatrix} \mapsto \begin{pmatrix} (x'_p + 1)^{-1} x'_0 \\ \vdots \\ (x'_p + 1)^{-1} x'_{p-1} \\ 1 \\ (x'_p + 1)^{-1} \theta' \end{pmatrix}.$$  

There are two supermetrics expressed in terms of the coordinates of supermanifolds as follows

$$ds^2 = \frac{(dx'_0)^2 + \cdots + (dx'_p)^2 + d\theta'_1 d\theta'_2}{(x'_0)^2},$$

$$ds^{2''} = \frac{(dx''_0)^2 + \cdots + (dx''_{p-1})^2 + d\theta''_1 d\theta''_2}{(x''_0)^2}$$

such that $\alpha^*(ds^2) = ds^2$ and $\beta^*(ds^{2''}) = ds^2$.

In summary, enjoying the same supergeometry of the hyperbolic superspace $H^{p|2}$, we have three models with supermetrics listed as follows, whose bodies are the corresponding models of usual $p$-dimensional hyperbolic space. The isometry group of the supermetrics on $H^{p|2}$ is the subgroup of $OSp(1, p|2, \mathbb{R})$, denoted by $OSp_0(1, p|2, \mathbb{R})$. 


| Model                      | Definition                                                                 | Metric                              |
|----------------------------|---------------------------------------------------------------------------|-------------------------------------|
| Super hyperboloid model    | $\mathcal{H}^{p|2} = \begin{cases} H := \begin{pmatrix} x_0 \\ \vdots \\ x_p \\ \theta_1 \\ \theta_2 \end{pmatrix} & \in \mathbb{R}^{p+1|2} : H^t g H = -1, \epsilon_0(x_0) > 0 \end{cases}$ | $ds^2 = -dx_0^2 + dx_1^2 + \cdots + dx_p^2 + d\theta_1 d\theta_2$ |
| Super semi-sphere model    | $\mathcal{H}^{p|2} = \begin{cases} H := \begin{pmatrix} x_0 \\ \vdots \\ x_p \\ \theta_1 \\ \theta_2 \end{pmatrix} & \in \mathbb{R}^{p+1|2} : H^t \left( \begin{pmatrix} \text{Id}_{p+1} & 0 \\ 0 & J_1 \end{pmatrix} \right) H = 1, \epsilon_0(x_0) > 0 \end{cases}$ | $ds^2 = \frac{dx_0^2 + dx_1^2 + \cdots + dx_p^2 + d\theta_1 d\theta_2}{x_0^2}$ |
| Upper half superspace model| $\mathcal{H}^{p|2} = \begin{cases} (x_0, \cdots, x_{p-1}|\theta_1, \theta_2) & \in \mathbb{R}^{p|2} : \epsilon_0(x_0) > 0 \end{cases}$ | $ds^2 = \frac{dx_0^2 + dx_1^2 + \cdots + dx_{p-1}^2 + d\theta_1 d\theta_2}{x_0^2}$ |
We will focus on the cases of \( p = 2, 3 \), which admit richer structures. Firstly, when \( p = 2 \), \( \mathcal{H}^{2|2} \) (the upper half superplane model) can be made into a 1|1-dimensional complex supermanifold (and redenoted by \( \mathcal{C}H^{1|1} \)) by introducing the complex coordinates \((Z; \Theta)\) for \( Z = ix_0 + x_1, \Theta = i\theta_1 + \theta_2 \) with complex conjugates \( \bar{Z} = -ix_0 + x_1, \bar{\Theta} = i\theta_2 + \theta_1 \) satisfying the rules: \( \bullet + \bullet = \bullet + \bullet, \bullet = \bullet \bar{\bullet} \). The superconformal changes of coordinates are given by the supermatrix
\[
\Gamma = \begin{pmatrix}
a & b & ab - \beta a \\
c & e & ae - \beta c \\
\alpha & \beta & 1 + \beta \alpha 
\end{pmatrix} \in OSp(1|2, \mathbb{C}),
\]
with \( ae - bc = 1 + \alpha \beta \), via the super-Möbius transformations
\[
Z \mapsto Z' = \frac{aZ + b + (ab - \beta a) \Theta}{cZ + e + (ae - \beta c) \Theta}, \quad \Theta \mapsto \Theta' = \frac{\alpha Z + \beta + (1 - \alpha \beta) \Theta}{cZ + e + (ae - \beta c) \Theta}.
\]

The following proposition collects some important properties of \( \mathcal{C}H^{1|1} \) with respect to the super-Möbius transformations. For the convenience of the readers, we give a brief proof of these properties, more details can be found in [15–18].

**Proposition 2.** 1. Assume \( \Gamma \in OSp(1|2, \mathbb{R}) \), and let
\[
Y = \text{Im}(Z) + \frac{1}{2} \Theta \bar{\Theta} = x_0 - \theta_1 \theta_2,
\]
then under the super-Möbius transformations,
\[
Y' = |F_\Gamma(Z, \Theta)|^2 Y,
\]
for
\[
F_\Gamma(Z, \Theta) = \frac{1}{cZ + e + (ae - \beta c) \Theta}.
\]
Hence \( \epsilon_0(Y') = \frac{1}{|\epsilon_0(c)\epsilon_0(z) + \epsilon_0(e)|^2} \epsilon_0(Y) > 0 \).

2. The supermetric
\[
ds^2 = \frac{dx_0^2 + dx_1^2 - 2(\theta_1 d\theta_2 - \theta_2 d\theta_1) dx_0 + 2(\theta_2 d\theta_2 + \theta_1 d\theta_1) dx_1 + 4(t - 2\theta_1 \theta_2) d\theta_1 d\theta_2}{x_0^2 - 2x_0 \theta_1 \theta_2}, \quad (2.1)
\]
is invariant under any super-Möbius transformation \( \Gamma \in OSp(1|2, \mathbb{R}) \), and gives rise to an \( OSp(1|2, \mathbb{R}) \)-invariant super-volume form
\[
dSVol = \frac{1}{2} \left( \frac{1}{x_0} + \frac{\theta_1 \theta_2}{x_0^2} \right) dx_0 dx_1 d\theta_1 d\theta_2.
\]
Moreover, this supermetric makes \( \mathcal{H}^{2|2} \) into a negative Einstein supermanifold.
Proof. The supermetric (2.1) is associated to the Hermitian supermetric

\[ d\bar{X}_A (A\bar{H}_B) \bar{B} dX = -2 d\bar{X}_A \left( \frac{\partial^2}{\partial X_A \partial X_{\bar{B}}} \log Y \right) \bar{B} dX \]

\[ = -2 (-1) ([X_A|+|X_{\bar{B}}|] |X_A| |X_{\bar{B}}|) \left( \frac{\partial^2}{\partial X_A \partial X_{\bar{B}}} \log Y \right) d\bar{X}_A dX_{\bar{B}} \]

\[ = \frac{1}{Y^2} \left( dZ d\bar{Z} - i \Theta_1 dZ \Theta - i \Theta d\Theta d\bar{Z} - (2Y + \Theta \Theta) d\Theta d\bar{\Theta} \right), \]

where \( \{X_A\} = \{Z, \Theta\}, \{X_{\bar{A}}\} = \{\bar{Z}, \bar{\Theta}\} \) and \( \bar{B} dX = (-1)^{|\bar{B}|} dX_{\bar{B}}. \) It has super-Ricci curvature \[ 19 \]

\[ R_{A\bar{B}} = - \frac{\partial^2}{\partial X_A \partial X_{\bar{B}}} \left( \log \text{Sdet} (A\bar{H}_B) \right) = -A H_{\bar{B}}, \]

where

\[ \text{Sdet} (A\bar{H}_B) = \text{Sdet} \left( \begin{array}{cccc} 0 & \frac{1}{2Y^2} & 0 & \frac{i \Theta}{2Y^2} \\ \frac{1}{2Y^2} & 0 & \frac{i \bar{\Theta}}{2Y^2} & 0 \\ 0 & \frac{i \bar{\Theta}}{2Y^2} & 0 & 2Y + \Theta \bar{\Theta} \\ \frac{i \Theta}{2Y^2} & 0 & -2Y - \Theta \bar{\Theta} & 0 \end{array} \right) \]

\[ = - \frac{1}{4Y^2}. \]

The super-volume form \( d\text{SVol} = \sqrt{|\text{Sdet}(A\bar{H}_B)|} dZ d\bar{Z} d\Theta d\bar{\Theta} = \frac{dZ d\bar{Z} d\Theta d\bar{\Theta}}{2Y} \) is \( OSp(1|2, \mathbb{R}) \)-invariant.

To show the \( OSp(1|2, \mathbb{R}) \)-invariance of the given supermetric, we firstly note that the super-Möbius transformations are generated by the following transformations

\[ T_1 : (Z, \Theta) \mapsto (aZ + b, \Theta), \]

\[ T_2 : (Z, \Theta) \mapsto \left( -\frac{1}{Z}, \frac{\Theta}{Z} \right), \]

\[ T_3 : (Z, \Theta) \mapsto (Z - a' Z \Theta, \Theta + a Z), \]

\[ T_4 : (Z, \Theta) \mapsto (Z - \beta \Theta, \Theta + \beta). \]

Therefore, we only need to check the invariance under \( T_i, i = 1, \ldots, 4 \), which can be done easily. \( \square \)

Next we consider the case of \( p = 3. \) One takes

\[ \mathcal{H}^{3|2} = \{(x, y, t; \theta_1, \theta_2) : x, y, t \in (\Lambda_{\mathbb{R}}^\infty)_0, \epsilon_0(t) > 0, \theta_1, \theta_2 \in (\Lambda_{\mathbb{R}}^\infty)_1 \} \]

as the subspace of

\[ \mathbb{C}\mathcal{H}^{2|2} : \{(Z, T; \Theta_1, \Theta_2 : Z, T \in (\Lambda_{\mathbb{C}}^\infty)_0, \Theta_1, \Theta_2 \in (\Lambda_{\mathbb{C}}^\infty)_1, \epsilon_0(\text{Im}(T)) > 0 \}, \]
and then one introduces
\[\tilde{Z} = x + iy + jt\] (or \(\tilde{Z} = Z + jT\)),
\[\tilde{\Theta} = j\theta_1 + \theta_2\] (or \(\tilde{\Theta} = j\Theta_1 + \Theta_2\)),
\[\tilde{Y} = - j \frac{\tilde{Z} - \tilde{Z}}{4} - \frac{\tilde{Z} - \tilde{Z}}{4} j + \frac{1}{2} \tilde{\Theta} \tilde{\Theta},\]

where the imaginary unit \(j\) satisfies \(j^2 = -1, \ i j + j i = 0\). The \(OSp(1|2, \mathbb{C})\)-transformations on \(\mathbb{C}\mathcal{H}^{3|2}\) can be obtained by the super-Poincaré extension of those on \(\mathcal{H}^{2|2}\). More precisely, replacing \(Z\) by \(\tilde{Z}\) and \(\Theta\) by \(\tilde{\Theta}\) in the previous super-Möbius transformations, we get the transformations

\[Z \mapsto \frac{(a \tilde{Z} + b + (\alpha \beta - \beta \alpha) \Theta_2)(c \tilde{Z} + e + (\alpha \epsilon - \beta \epsilon) \Theta_1) + (c \tilde{Z} + e + (\alpha \epsilon - \beta \epsilon) \Theta_1)^2}{(a \tilde{Z} + b + (\alpha \beta - \beta \alpha) \Theta_2) + (c \tilde{Z} + e + (\alpha \epsilon - \beta \epsilon) \Theta_1)^2},\]
\[T \mapsto \frac{(a \tilde{Z} + b + (\alpha \beta - \beta \alpha) \Theta_1)(c \tilde{Z} + e + (\alpha \epsilon - \beta \epsilon) \Theta_1) + (c \tilde{Z} + e + (\alpha \epsilon - \beta \epsilon) \Theta_1)^2}{(a \tilde{Z} + b + (\alpha \beta - \beta \alpha) \Theta_1) + (c \tilde{Z} + e + (\alpha \epsilon - \beta \epsilon) \Theta_1)^2},\]
\[\Theta_1 \mapsto \frac{c \tilde{Z} + e + (\alpha \epsilon - \beta \epsilon) \Theta_1 + (c \tilde{Z} + e + (\alpha \epsilon - \beta \epsilon) \Theta_1)^2}{(a \tilde{Z} + b + (\alpha \beta - \beta \alpha) \Theta_1) + (c \tilde{Z} + e + (\alpha \epsilon - \beta \epsilon) \Theta_1)^2},\]
\[\Theta_2 \mapsto \frac{(a \tilde{Z} + b + (1 - \alpha \beta) \Theta_2)(c \tilde{Z} + e + (\alpha \epsilon - \beta \epsilon) \Theta_2) + (c \tilde{Z} + e + (\alpha \epsilon - \beta \epsilon) \Theta_2)^2}{(a \tilde{Z} + b + (1 - \alpha \beta) \Theta_2) + (c \tilde{Z} + e + (\alpha \epsilon - \beta \epsilon) \Theta_2)^2}.\]

The element \(\Gamma \in OSp(1|2, \mathbb{C})\) preserving \(\mathcal{H}^{3|2}\) is called an \(\mathbb{R}\)-element. The maximal super subgroup of \(OSp(1|2, \mathbb{C})\) consisting of the \(\mathbb{R}\)-elements is denoted by \(\mathcal{H}\).

**Proposition 3.**
1. \(\mathcal{H}^{3|2}\) can be equipped with an \(\mathcal{H}\)-invariant supermetric such that it is a negative Einstein supermanifold.
2. \(\mathcal{H}^{3|2}\) can be equipped with a supermetric such that it is a positive Bosonic supermanifold.
3. \(\frac{1}{r} + \frac{2\theta \theta}{r^2}\) is an \(\mathcal{H}\)-invariant super-volume form on \(\mathcal{H}^{3|2}\).

**Proof.** By super-Poincaré extension described above, the \(\mathcal{H}\)-invariant and negative Einstein supermetric on \(\mathcal{H}^{3|2}\) can be given by

\[ds^2 = \frac{dx^2 + dy^2 + dt^2 - 2(\theta_2 d\theta_1 + \theta_1 d\theta_2) dx + 2(\theta_1 d\theta_2 - \theta_2 d\theta_1) dt - 4(t - 2\theta_1 \theta_2) d\theta_1 d\theta_2}{r^2 - 2t \theta_1 \theta_2}.\]
which gives rise to an $\mathcal{H}$-invariant super-volume form

\[
\sqrt{\det \left( \begin{array}{cccc}
\left( \frac{1}{t^2} + \frac{2t_1^2}{t^3} \right) & 0 & 0 & -t_1^2 \\
0 & \left( \frac{1}{t^2} + \frac{2t_1^2}{t^3} \right) & 0 & -t_1^2 \\
0 & 0 & -\frac{1}{t^2} + \frac{2t_1^2}{t^3} & -t_1^2 \\
-t_1^2 & 0 & -t_1^2 & -2 \\
\end{array} \right)} \ dx dy dt d\theta_1 d\theta_2 \\
= \frac{1}{2} \left( \frac{3t_1^2}{t^3} \right) \ dx dy dt d\theta_1 d\theta_2.
\]

We can also consider the following supermetric

\[
ds^2 = \frac{dx^2 + dy^2 + dt^2 + 2(\theta_2 d\theta_2 + \theta_1 d\theta_1) dx - 2(\theta_1 d\theta_2 - \theta_2 d\theta_1) dt + 4(t - 2t_1^2) d\theta_1 d\theta_2}{t^2 - 2t \theta_1 \theta_2}.
\]

We calculate the corresponding super scalar curvature. In terms of $X_1 = x$, $X_2 = y$, $X_3 = t$, $X_4 = \theta_1$, $X_5 = \theta_2$, the nonzero super-Christoffel symbols are given by

\[
\Gamma^3_{11} = -\frac{1}{t} - \frac{\theta_1 t_1^2}{t^2}, \quad \Gamma^4_{11} = \frac{\theta_1}{t^2}, \quad \Gamma^5_{11} = \frac{\theta_2}{t^2}, \quad \Gamma^1_{13} = \Gamma^4_{13} = -\frac{1}{t} + \frac{\theta_1 t_1^2}{t^2},
\]

\[
\Gamma^1_{14} = \Gamma^4_{14} = \frac{3t_1}{2t}, \quad \Gamma^3_{14} = \frac{3t_1^2}{2t}, \quad \Gamma^1_{14} = \Gamma^4_{14} = \frac{\theta_1}{2t}, \quad \Gamma^5_{14} = \Gamma^4_{14} = -\frac{1}{2t} - \frac{\theta_1 t_1^2}{2t^2},
\]

\[
\Gamma^1_{15} = \Gamma^5_{15} = -\frac{3t_1}{2t}, \quad \Gamma^3_{15} = \frac{3t_1^2}{2t}, \quad \Gamma^4_{15} = \frac{\theta_2}{2t}, \quad \Gamma^5_{15} = \frac{\theta_2}{2t} + \frac{\theta_1 t_1^2}{2t^2},
\]

\[
\Gamma^2_{12} = \Gamma^5_{12} = -\frac{1}{t} + \frac{\theta_1 t_1^2}{t^2}, \quad \Gamma^4_{22} = \frac{\theta_1}{t^2}, \quad \Gamma^5_{22} = \frac{\theta_2}{t^2},
\]

\[
\Gamma^3_{23} = \Gamma^5_{23} = -\frac{1}{t} + \frac{\theta_1 t_1^2}{t^2}, \quad \Gamma^4_{24} = \frac{\theta_2}{t}, \quad \Gamma^5_{25} = \frac{\theta_1}{t}, \quad \Gamma^5_{25} = \frac{\theta_1}{t} + \frac{\theta_2}{t},
\]

\[
\Gamma^3_{33} = -\frac{1}{t} + \frac{\theta_1 t_1^2}{t^2}, \quad \Gamma^4_{33} = \frac{\theta_2}{t^2}, \quad \Gamma^5_{33} = -\frac{\theta_2}{t^2},
\]

\[
\Gamma^1_{34} = \Gamma^3_{43} = -\frac{3t_1}{2t}, \quad \Gamma^3_{34} = \frac{3t_1^2}{2t}, \quad \Gamma^4_{34} = \frac{\theta_2}{2t}, \quad \Gamma^4_{43} = \frac{1}{2t} - \frac{\theta_1 t_1^2}{2t^2},
\]

\[
\Gamma^1_{35} = \Gamma^3_{53} = -\frac{3t_1}{2t}, \quad \Gamma^3_{35} = \frac{3t_1^2}{2t}, \quad \Gamma^4_{35} = \frac{\theta_2}{2t}, \quad \Gamma^5_{53} = \frac{\theta_1}{2t} + \frac{\theta_2}{2t^2},
\]

\[
\Gamma^3_{45} = -\Gamma^5_{45} = 2 - \frac{6t_1}{t}, \quad \Gamma^4_{45} = -\Gamma^4_{54} = \frac{\theta_1}{t}, \quad \Gamma^5_{45} = -\Gamma^5_{54} = \frac{\theta_2}{t}.
\]

Hence the non-vanishing components of the super-Ricci curvature are given by

\[
R_{11} = -\frac{11}{2t^2} - \frac{13t_1^2}{t^3}, \quad R_{13} = R_{31} = \frac{\theta_1 t_1^2}{2t^3}, \quad R_{14} = R_{41} = \frac{3t_1}{2t^2},
\]

\[
R_{15} = R_{51} = \frac{3t_2}{2t^2}, \quad R_{22} = -\frac{5}{t} - \frac{8t_1^2}{t^3}, \quad R_{33} = \frac{7}{2t^2} - \frac{17t_1^2}{t^3},
\]

\[
R_{34} = R_{43} = \frac{\theta_2}{2t^2}, \quad R_{35} = R_{53} = -\frac{\theta_1}{2t^2}, \quad R_{45} = -R_{54} = -\frac{16}{t} + \frac{25t_1^2}{t^2}.
\]

Therefore, the super scalar curvature reads

\[
R = 2 - \frac{27t_1^2}{t},
\]
which means $\mathcal{H}^{[1]}$ can be made into a positive Bosonic supermanifold. □

As the end of this section, we mention another important 3|2-dimensional hyperbolic supermanifold, the supergroup $OSp(1|2, \mathbb{R})$ whose body is the non-compact Lie group $SL(2, \mathbb{R})$ closely related to the BTZ black hole in AdS$_3$ gravity [6,12]. The basis of the corresponding Lie superalgebra $osp(1|2, \mathbb{R})$ are given by three even generators

$$L_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad L_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad L_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and two odd generators

$$Q_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix}.$$

They satisfy the following (anti-)commutative relations:

$$[L_i, L_j] = 2\epsilon_{ijk}n^{kl}L_l,$$

$$[L_i, Q_\alpha] = (\sigma_i)_{\alpha\beta}Q_\beta,$$

$$\{Q_\alpha, Q_\beta\} = 2(C\sigma_i)_{\alpha\beta}L_i,$$

where the indices $i, j, k$ run over 1, 2, 3, and $\alpha, \beta$ run over 1, 2, and $\{\sigma_i\}$ denote the Pauli matrices, i.e. $\sigma_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\sigma_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, C = (\epsilon_{\alpha\beta})$. Let $\text{Str}$ denote the super-Killing form on $osp(1|2, \mathbb{R})$, then

$$\text{Str}(L_i, L_j) = \eta_{ij}, \quad \text{Str}(L_i, Q_\alpha) = 0, \quad \text{Str}(Q_\alpha, Q_\beta) = -2C_{\alpha\beta}.$$

We parameterize the elements of $OSp(1|2, \mathbb{R})$ by

$$g = \exp(\alpha L_2) \exp(\lambda L_3) \exp(\beta L_2) \exp(\theta_1 R_1) \exp(\theta_2 R_2) \quad (2.2)$$

with $\alpha, \beta, \lambda \in (\Lambda^\infty_\mathbb{R})_0, \epsilon_0(\alpha), \epsilon_0(\beta) \in [0, 2\pi), -\infty < \epsilon_0(\lambda) < +\infty$ and $\theta_1, \theta_2 \in (\Lambda^\infty_\mathbb{R})_1$, where $R_{1,2} = \frac{1}{2}(Q_1 \pm Q_2)$.

The Lie supergroup $OSp(1|2, \mathbb{R})$ can be endowed with the following pseudo-supermetric (where the phrase "pseudo" means that the $(\Lambda^\infty_\mathbb{R})_0$-component gives rise to a pseudo-metric with signature $(-1, 1, 1)$ on the body manifold) invariant under the $OSp(1|2, \mathbb{R})$ left-action and the $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ bi-action

$$ds^2 = (1 + 2\theta_1\theta_2) \left( d\alpha^2 + d\lambda^2 + d\beta^2 + 2 \cosh 2\lambda d\alpha d\beta \right)$$

$$+ (\theta_1 \cosh 2\beta \sinh 2\lambda + \theta_1 \cosh 2\lambda + 2\theta_2 \sinh 2\lambda \sinh 2\beta) d\alpha d\theta_1$$

$$+ \theta_1 d\beta d\theta_1 - (\theta_1 \sinh 2\beta + 2\theta_2 \cosh 2\beta) d\lambda d\theta_1$$

$$+ \theta_2 (\cosh 2\beta \sinh 2\lambda - \cosh 2\lambda) d\alpha d\theta_2 - \theta_2 d\beta d\theta_2$$

$$- \theta_2 \sinh 2\beta \cosh 2\lambda d\theta_2 - (1 - \theta_1\theta_2) d\theta_1 d\theta_2.$$
and the associated $OSp(1|2, \mathbb{R}) \times OSp(1|2, \mathbb{R})$ bi-invariant super-volume form

$$dSVol$$

$$= \sqrt{\left| \begin{vmatrix}
1 + \theta_1 \theta_2 (1 + \sinh^2 2\beta \sinh^2 2\lambda) & -\frac{\theta_1 \theta_2}{2} \sinh 4\beta \sinh 2\lambda (1 + \theta_1 \theta_2) \cosh 2\lambda \\
-\frac{\theta_1 \theta_2}{2} \sinh 4\beta \sinh 2\lambda (1 + \theta_1 \theta_2) \cosh 2\lambda & 1 + \theta_1 \theta_2 (1 + \sinh^2 2\beta) & 0 \\
1 + \theta_1 \theta_2 & 0 & 1 + \theta_1 \theta_2
\end{vmatrix} \right|} \, d\alpha d\beta d\theta_1 d\theta_2$$

$$= 2(1 + 3\theta_1 \theta_2) \sinh 2\lambda d\alpha d\beta d\theta_1 d\theta_2.$$  

Indeed, with the given parametrization (2.2), we have the current

$$g^{-1} dg = e^i L_i + \mathcal{E}^a Q_a,$$

where

$$e^1 = - \cosh 2\beta \sinh 2\lambda d\alpha - \theta_1 \theta_2 (\sinh 2\lambda \cosh 2\beta + \cosh 2\lambda) d\alpha$$

$$+ (1 + \theta_1 \theta_2) \sinh 2\beta d\alpha \cosh 2\lambda \theta_1 d\beta + \frac{\theta_1}{2} d\theta_1 + \frac{\theta_2}{2} d\theta_2,$$

$$e^2 = \cosh 2\lambda d\alpha + \theta_1 \theta_2 (\sinh 2\lambda \cosh 2\beta + \cosh 2\lambda) d\alpha$$

$$- \theta_1 \theta_2 \sinh 2\beta d\alpha + (1 + \theta_1 \theta_2) \cosh 2\lambda \theta_1 d\beta + \frac{\theta_1}{2} d\theta_1 - \frac{\theta_2}{2} d\theta_2,$$

$$e^3 = -(1 + \theta_1 \theta_2) \sinh 2\beta \sinh 2\lambda d\alpha + (1 + \theta_1 \theta_2) \cosh 2\beta d\lambda - \theta_2 \theta_1 d\alpha,$$

$$\mathcal{E}^1 = \left[ \frac{\theta_1 - \theta_2}{2} \sinh 2\lambda (\cosh 2\beta - \sinh 2\beta) + \frac{\theta_1 + \theta_2}{2} \cosh 2\lambda \right] d\alpha$$

$$+ \frac{\theta_1 - \theta_2}{2} (\cosh 2\beta - \sinh 2\beta) d\lambda + \frac{\theta_1 + \theta_2}{2} d\beta + \frac{1 - \theta_1 \theta_2}{2} d\theta_1 + \frac{1}{2} d\theta_2,$$

$$\mathcal{E}^2 = - \left[ \frac{\theta_1 + \theta_2}{2} \sinh 2\lambda (\cosh 2\beta + \sinh 2\beta) + \frac{\theta_1 - \theta_2}{2} \cosh 2\lambda \right] d\alpha$$

$$+ \frac{\theta_1 + \theta_2}{2} (\cosh 2\beta + \sinh 2\beta) d\lambda - \frac{\theta_1 - \theta_2}{2} d\beta + \frac{1 - \theta_1 \theta_2}{2} d\theta_1 - \frac{1}{2} d\theta_2,$$

then the super-Killing form provides our pseudo-supermetric

$$ds^2 = \text{Str}(g^{-1} dg, g^{-1} dg) = -(e^1)^2 + (e^2)^2 + (e^3)^2 + 2\mathcal{E}^1 \mathcal{E}^2 - 2\mathcal{E}^2 \mathcal{E}^1,$$

which obviously has the desired invariance.

The renormalized volume of a hyperbolic manifold is a quantity motivated by the AdS/CFT correspondence and can be computed via certain regularization procedure [20].

**Proposition 4.** With respect to the above pseudo-supermetric, $OSp(1|2, \mathbb{R})$ has the renormalized volume $-24\pi^2$.

**Proof.** The volume of $OSp(1|2, \mathbb{R})$ is calculated as

$$\text{Vol}(OSp(1|2, \mathbb{R})) = \int_{OSp(1|2, \mathbb{R})} dSVol$$

$$= 6 \int_0^{2\pi} d\alpha_0 \int_0^{2\pi} d\beta_0 \int_{-\infty}^{\infty} \sinh 2\lambda_0 d\lambda_0,$$
where $\alpha_0 = \epsilon_0(\alpha)$, $\beta_0 = \epsilon_0(\beta)$, $\lambda_0 = \epsilon_0(\lambda)$. Let $\lambda_0 = \ln \frac{2}{t}$ with $0 < t \leq 2$, then

$$\text{Vol} \left( OSp \left( 1|2, \mathbb{R} \right) \right) = -96\pi^2 \int_0^2 \frac{1}{t} \left( \frac{1}{t} - \frac{t}{4} \right) \left( \frac{1}{t} + \frac{t}{4} \right) \, dt$$

$$= -96\pi^2 \lim_{z \to 0} 2^{z-4} \int_0^2 \left( 1 - \frac{t^4}{16} \right) \left( \frac{t^4}{16} \right)^{\frac{z}{t^4}}$$

$$= -96\pi^2 \lim_{z \to 0} 2^{z-4} \frac{\Gamma(2) \Gamma \left( \frac{z}{4} - \frac{1}{2} \right)}{\Gamma \left( \frac{z}{4} + \frac{3}{2} \right)}$$

$$= -96\pi^2 \frac{\Gamma(2)}{4} = -24\pi^2,$$

which gives the renormalized volume. \square

3. Super-Green Functions and Supergeodesics

A super-Riemann surface $S^{1|1}$ is a complex $1|1$-dimensional supermanifold with the following properties in terms of the local coordinate $(Z; \Theta)$ \cite{11, 18, 21, 22}

- (supercomplex structure) The transition functions are holomorphic: $Z' = F(Z, \Theta)$, $\Theta' = \Psi(Z, \Theta)$,
- (superconformal structure) The differential operator $\mathbb{D} = \frac{\partial}{\partial \Theta} + \Theta \frac{\partial}{\partial Z}$ transforms homogeneously: $\mathbb{D}' \propto \mathbb{D}$.

More explicitly, the general form of transition functions reads

$$Z' = f(Z) + \Theta' \psi \sqrt{\frac{\partial f}{\partial Z}},$$

$$\Theta' = \psi(Z) + \Theta \sqrt{\frac{\partial f}{\partial Z} + \psi \frac{\partial \psi}{\partial Z}}.$$

In other words, the extra structure on super-Riemann surface $S^{1|1}$ is given by a $0|1$-dimensional subbundle $\mathcal{D}$ of the tangent bundle $T_{S^{1|1}}$ such that the following sequence is exact

$$0 \to \mathcal{D} \to T_{S^{1|1}} \to \mathcal{D}^2 \to 0.$$

There are three typical $1|1$-dimensional super-Riemann surfaces with simply-connected bodies:

- The complex superplane $\mathbb{C}^{1|1}$;
- The super-Riemann sphere $\hat{\mathbb{C}}^{1|1}$: covered by two open domains (in the De Witt topology) which are glued by the transition functions
  $$(Z', \Theta') = \left( -\frac{1}{Z}, \frac{\Theta}{Z} \right),$$
- The upper half superplane $\mathbb{CH}^{1|1}$ discussed in the previous section.

The groups of superconformal automorphisms on $\hat{\mathbb{C}}^{1|1}$ and $\mathbb{CH}^{1|1}$ are $OSp(1|2, \mathbb{C})/\{\pm \text{Id}\}$ and $OSp(1|2, \mathbb{R})/\{\pm \text{Id}\}$, respectively.
Proposition 5. Let $S^{1|1}$ be a super-Riemann surface with a compact Riemann surface $S$ of genus $g_S \geq 2$ as the body of $S^{1|1}$. Then the superconformal structure on $S^{1|1}$ produces irreducible representations $\rho : \pi_1(S) \rightarrow SL(2, \mathbb{R})$ of the fundamental group $\pi_1(S)$ of $S$.

Proof. Manin et al. have showed that the superconformal structure on $S^{1|1}$ corresponds to a choice of the theta characteristic on $S$, namely, a line bundle $L$ over $S$ satisfying $L \otimes^2 \cong \Omega^1_S [17]$. Then one can construct a bundle $\hat{E}$ by the following extension

$$0 \rightarrow L^{-1} \rightarrow E \rightarrow L \rightarrow 0.$$  

The Higgs field $\phi \in H^0(S, \text{End}(E) \otimes \Omega^1_S)$ is defined by the composition $\phi : E \rightarrow L \cong L^{-1} \otimes L^2 \subset E \otimes \Omega^1_S$. It is obvious that the line subbundle of $E$ preserved by the Higgs field $\phi$ is exactly $L^{-1}$, and $\deg(L^{-1}) < 0 = \deg(E)$ when $g_S \geq 2$. Hence $(E, \phi)$ is a stable Higgs bundle over $S$, which yields an irreducible representation $\rho : \pi_1(S) \rightarrow GL(2, \mathbb{C})$ by Simpson correspondence [23]. Obviously, the image of this representation lies in the subgroup $SL(2, \mathbb{R})$. □

Inspired by the definition of the classical Arakelov-Green function [2,5,24], we propose the supergeometric version as follows.

Definition 2. Let $S^{1|1}$ be a super-Riemann surface with local coordinates $\{X_A\} = (Z; \Theta)$. A triple $(P, g, G_P)$ consisting of a fixed point $P \in S^{1|1}$, a supermetric $g = g_{AB}dX_AdX_B$, and a superfunction $G_P : S^{1|1} \rightarrow (\Lambda^\infty_{\mathbb{R}})_0$ is called a super-Green triple on $S^{1|1}$ if it satisfies the following conditions:

1. $\epsilon_0(G_P(Q)) \geq 0$ for any $Q \in S^{1|1}$,
2. writing $G_P(X) = G_{P0}^0(Z) + (G_{P1}^1(Z)\Theta + \tilde{G}_{P1}^1(Z)\tilde{\Theta}) + G_{P2}(Z)\Theta\tilde{\Theta}$ in the neighborhood centered at $P = (0; 0)$, each nonzero component $G_{P}^i(Z)$, $i \in \{0, 1, 2\}$ has a first order zero for $Z = 0$,
3. $-(-1)^{|X_B|}dX_A(\frac{\partial^2}{\partial X_A \partial \bar{X}_B}) \log G_P(X)dX_B$ coincides with $g$ outside the singular locus of $\log G_P(X)$,
4. $(\epsilon_0(P), \epsilon_0(g), \epsilon_0(G_P(X)))$ provides a classical Green triple on the body of $S^{1|1}$.

In particular, the superfunction $G_P = \log G_P$ is called a super-Green function on $S^{1|1}$ associated with the supermetric $g$.

We first consider the super-Riemann sphere $\hat{C}^{1|1}$.

Proposition 6. $\hat{C}^{1|1}$ can be endowed with a supermetric

$$ds^2 = -\frac{1}{2}(-1)^{|X_B|}dX_A \left(\partial_A \partial_B \log \frac{Z\bar{Z}}{1 + Z\bar{Z} + \Theta\tilde{\Theta}}\right)dX_B$$

$$= \frac{1}{(1 + Z\bar{Z} + \Theta\tilde{\Theta})^2}((1 + \Theta\tilde{\Theta})dZd\bar{Z} - \Theta\bar{Z}d\Theta d\bar{Z} + Z\Theta d\Theta d\bar{Z} + (1 + Z\bar{Z} + 2\Theta\tilde{\Theta})d\Theta d\tilde{\Theta}).$$ (3.1)
and a super-volume form

\[
\text{dSVol} = \sqrt{\det \begin{pmatrix}
0 & \frac{1 + \Theta \bar{\Theta}}{2(1 + Z \bar{Z} + \Theta \bar{\Theta})} & 0 & \frac{\Theta \bar{\Theta}}{2(1 + Z \bar{Z} + \Theta \bar{\Theta})} \\
\frac{1 + \Theta \bar{\Theta}}{2(1 + Z \bar{Z} + \Theta \bar{\Theta})} & 0 & -\frac{Z \bar{\Theta}}{2(1 + Z \bar{Z} + \Theta \bar{\Theta})} & 0 \\
0 & -\frac{Z \bar{\Theta}}{2(1 + Z \bar{Z} + \Theta \bar{\Theta})} & 0 & \frac{1 + Z \bar{Z} + 2 \Theta \bar{\Theta}}{2(1 + Z \bar{Z} + \Theta \bar{\Theta})} \\
\frac{\Theta \bar{\Theta}}{2(1 + Z \bar{Z} + \Theta \bar{\Theta})} & 0 & \frac{1 + Z \bar{Z} + 2 \Theta \bar{\Theta}}{2(1 + Z \bar{Z} + \Theta \bar{\Theta})} & 0
\end{pmatrix}} \cdot \text{d}Z \text{d}\bar{Z} \text{d}\Theta \text{d}\bar{\Theta}.
\]

Proof. We calculate the transformation of each summand with respect to the transition functions of \( \hat{\mathbb{C}}^{1|1} \)

\[
\frac{1 + \Theta \bar{\Theta}}{(1 + Z \bar{Z} + \Theta \bar{\Theta})^2} \text{d}Z \text{d}\bar{Z} \mapsto \frac{1}{(1 + Z \bar{Z} + \Theta \bar{\Theta})^2} \left( 1 + \frac{\Theta \bar{\Theta}}{Z \bar{Z}} \right) \text{d}Z \text{d}\bar{Z},
\]

\[
-\frac{\Theta \bar{\Theta}}{(1 + Z \bar{Z} + \Theta \bar{\Theta})^2} \text{d}Z \text{d}\bar{\Theta} \mapsto \frac{1}{(1 + Z \bar{Z} + \Theta \bar{\Theta})^2} \left( \frac{\Theta}{Z} \text{d}Z \text{d}\bar{\Theta} - \frac{\Theta \bar{\Theta}}{Z \bar{Z}} \text{d}Z \text{d}\bar{Z} \right),
\]

\[
\frac{Z \bar{\Theta}}{(1 + Z \bar{Z} + \Theta \bar{\Theta})^2} \text{d}\Theta \text{d}\bar{Z} \mapsto \frac{1}{(1 + Z \bar{Z} + \Theta \bar{\Theta})^2} \left( -\frac{\bar{\Theta}}{Z} \text{d}\Theta \text{d}\bar{Z} - \frac{\Theta \bar{\Theta}}{Z \bar{Z}} \text{d}\Theta \text{d}\bar{Z} \right),
\]

\[
\frac{1 + Z \bar{Z} + 2 \Theta \bar{\Theta}}{(1 + Z \bar{Z} + \Theta \bar{\Theta})^2} \text{d}\Theta \text{d}\bar{\Theta} \mapsto \frac{1}{(1 + Z \bar{Z} + \Theta \bar{\Theta})^2} \left( (1 + Z \bar{Z} + 2 \Theta \bar{\Theta}) \text{d}\Theta \text{d}\bar{\Theta} - \left( 1 + \frac{1}{Z \bar{Z}} \right) \Theta \bar{\Theta} \text{d}\Theta \text{d}\bar{\Theta} \right)
\]

\[
+ \left( 1 + \frac{1}{Z \bar{Z}} \right) Z \bar{\Theta} \text{d}\Theta \text{d}\bar{Z} + \left( 1 + \frac{1}{Z \bar{Z}} \right) \Theta \bar{\Theta} \text{d}\Theta \text{d}\bar{Z}.
\]

Combining these results, the given metric is globally well-defined on the supersphere.

\[ \square \]

Corollary 1. Let \( P = (0; 0) \in \hat{\mathbb{C}}^{1|1} \), \( g \) is the supermetric given by (3.1), and \( G_P(X) = \sqrt{\frac{ZZ}{1 + ZZ + \Theta \bar{\Theta}}} \), then \( (P, g, G_P) \) forms a super-Green triple on \( \hat{\mathbb{C}}^{1|1} \).

To obtain the supergeometric analog of Manin’s result connecting Green function and geodesics, we need to study the supergeodesics in the hyperbolic superspaces. The supergeodesic in \( \mathbb{C}H^{1|1} \) with respect to the supermetric (2.1) is determined by the following equations [15, 16]

\[
\frac{d^2 Z}{d^2 u} + \frac{i}{Y} \left( \frac{dZ}{du} \right)^2 + \frac{\Theta}{Y} \frac{dZ}{du} \frac{d\Theta}{du} = 0,
\]

\[
\frac{d^2 \Theta}{d^2 u} + \frac{i}{Y} \frac{dZ}{du} \frac{d\Theta}{du} = 0,
\]

and the complex conjugated ones.

Proposition 7. Let \( P_1 = (Z_1, \Theta_1) \), \( P_2 = (Z_2, \Theta_2) \) be two points in \( \mathbb{C}H^{1|1} \), then one can join \( P_1 \) and \( P_2 \) by supergeodesics piecewisely.

Proof. We have the following solutions for the supergeodesic equations:
– Type-I: when $\frac{dZ}{du} = 0$, there is a solution

\[ Z = c, \bar{Z} = \bar{c}, \Theta(u) = \gamma u + \zeta, \bar{\Theta}(u) = \bar{\gamma} u + \bar{\zeta}, \]

for constants $c \in (\Lambda^\infty_0), \gamma, \zeta \in (\Lambda^\infty_0)$,

– Type-II: when $\frac{dZ}{du} \neq 0$, there is a solution

\[ Z(u) = c_1 \{ \text{tanh}(\omega(u + u_0) + i \text{sech}(\omega(u + u_0)) + c_2, \] \[ \Theta(u) = \xi Z(u), \]

\[ \bar{Z}(u) = \bar{Z}(u), \bar{\Theta}(u) = \bar{\Theta}(u), \]

for constants $c_1, c_2, c_3, \omega, u_0 \in (\Lambda^\infty_0)$, and $\xi \in (\Lambda^\infty_0)$.

Firstly, we join $P_1$ and $P'_1 = (Z_1, \xi Z_1)$ for some $\xi \in (\Lambda^\infty_0)$ by virtue of a supergeodesic $G_I$ described by the type-I solution. The same argument as that for the classical geodesics implies that there exists a supergeodesic $G_{II}$ described by the type-II solution such that it is parameterized by $u$ with $Z(u_1) = Z_1, Z(u_2) = Z_2$ and $\epsilon_0(u) \in [\epsilon_0(u_1), \epsilon_0(u_2)]$. Namely, $G_{II}$ joins the points $P'_1$ and $P'_2 = (Z_2, \xi Z_2)$. Finally, we join the points $P'_2$ and $P_2$ via a supergeodesic $G'_I$ described by the type-I solution. □

The above proposition suggests the following definition.

**Definition 3.** The bosonic superdistance $d(P_1, P_2)$ between $P_1$ and $P_2$ is defined by the integral along the supergeodesic $G_{II}$

\[ d(P_1, P_2) = \int_{u_1}^{u_2} \sqrt{\left( \frac{ds}{du} \right)^2} \, du = \omega(u_2 - u_1), \]

or defined by

\[ \cosh d(P_1, P_2) = 1 + \frac{|Z_1 - Z_2|^2}{2\text{Im}(Z_1)\text{Im}(Z_2)}. \]

Now we view $\hat{\mathcal{C}}^{11}_{\mathcal{H}}$ as the boundary of $\mathcal{H}^{31}_{\mathcal{C}} \cup \{\infty\}$. For two distinct point $P_1, P_2 \in \hat{\mathcal{C}}^{11}_{\mathcal{H}}$, one has an upper half superplane $\mathcal{C}^{11}_{\mathcal{H} P_1 P_2} = \{(Z, \Theta) : \text{Im}(Z) = i > 0\}$ embedded in $\mathcal{H}^{31}_{\mathcal{C}}$ such that $P_1, P_2$ lie on its boundary. According to Manin’s approach, one joins $P_1$ and $P_2$ piecewisely with supergeodesics in $\mathcal{C}^{11}_{\mathcal{H} P_1 P_2}$, as described in the proof of Proposition 7. In particular, within these supergeodesics, the supergeodesic $G_{II}$ with a chosen constant $\xi$ for odd coordinates is denoted by $\{P_1, P_2\}_{\xi}$. Let $Q$ be another point in $\mathcal{H}^{31}_{\mathcal{C}}$, then one introduces

\[ d_Q(P_1, P_2) = d(Q, P^Q_{12}) \]

where $P^Q_{12} \in \{P_1, P_2\}_{\xi}$ is uniquely determined by the following condition

\[ \epsilon_0(d(Q, P^Q_{12})) = \inf_{P \in \{P_1, P_2\}_{\xi}} \epsilon_0(d(Q, P)). \]
and the bosonic superdistances appearing here are calculated in the upper half superplane $\mathcal{H}^{1|1}_{Q|P_{12}}$ with $Q = Q|_{t=0}$, $P_{12}^Q = P_{12}|_{t=0}$. For two given points $P_1 = (0, 0, 0; 0, 0)$, $P_2 = (x, y, 0; \theta_1, \theta_2)$ lying on the boundary of $\mathcal{H}^{3|2}$ and $Q = (0, 0, 1; 0, 0) \in \mathcal{H}^{3|2}$, we have

$$P_{12}^Q = \left( \frac{x}{2 + x^2 + y^2}, \frac{y}{2 + x^2 + y^2}, \frac{\sqrt{(x^2 + y^2)(1 + x^2 + y^2)}}{2 + x^2 + y^2}, \frac{x\xi}{2 + x^2 + y^2}, \frac{y\xi}{2 + x^2 + y^2} \right).$$

hence

$$\cosh d_Q(P_1, P_2) = 1 + \frac{\left( \frac{|Z|}{2|Z|^2} \right)^2 + \left( 1 - \frac{|Z|\sqrt{1 + |Z|^2}}{2|Z|^2} \right)^2}{2|Z|\sqrt{1 + |Z|^2}}$$

$$= \sqrt{1 + \frac{1}{|Z|^2}},$$

where $Z = x + iy$. Consequently, we arrive at the following proposition.

**Proposition 8.** The super-Green function on $\hat{\mathcal{C}}^{1|1}$ defined as in Corollary 1 can be re-expressed as

$$\mathcal{G}_{P_1}(Z, \Theta) = \log \left( \cosh d_Q(P_1, P_2) - \frac{1}{2 \cosh d_Q(P_1, P_2)} \left( 1 - \frac{1}{\cosh^2 d_Q(P_1, P_2)} \right) \Theta \tilde{\Theta} \right)$$

with $P_2 = (Z; \Theta)$.

The above approach is not sensitive to the odd coordinates. However, we can do some modifications for taking the odd part into account. To achieve that, one introduces the superdistances function $d : \mathcal{H}^{1|1} \times \mathcal{H}^{1|1} \rightarrow (\mathbb{A}_\mathbb{R}^\infty)_0$, which was firstly defined by physicists Uehara and Yasui [16], as follows

$$\cosh d(P_1, P_2) = 1 + \frac{1}{2} R(P_1, P_2) - 2r(P_1, P_2)$$

where $P_1 = (Z_1; \Theta_1)$, $P_2 = (Z_2; \Theta_2) \in \mathcal{H}^{1|1}$, and

$$R(P_1, P_2) = \frac{|Z_1 - Z_2 - \Theta_1 \Theta_2|^2}{Y_1 Y_2},$$

$$r(P_1, P_2) = \frac{2\Theta_1 \Theta_1 + i (\Theta_2 - i \Theta_2) (\Theta_1 + i \Theta_1) + 2\Theta_2 \Theta_2 + i (\Theta_1 - i \Theta_1) (\Theta_2 + i \Theta_2)}{4Y_1} + \frac{\Theta_2 + i \Theta_2}{4Y_2} \left( \Theta_1 + i \Theta_1 \right) \text{Re} (Z_1 - Z_2 - \Theta_1 \Theta_2)$$

for $Y_i = \text{Im}(Z_i) + \frac{\Theta_i \Theta_i}{2}$, $i = 1, 2$. It is easy to see that this superdistance function enjoys the properties

$$d(P_1, P_2) = d(P_2, P_1) = d(\Gamma \cdot P_1, \Gamma \cdot P_2)$$
for any $\Gamma \in OSp(1|2, \mathbb{R})$ [17]. In our setting, the inputs are two given points $P_1 = (0; 0), P_2 = (Z; \Theta)$ lying on the boundary of $\mathcal{H}^{1|1}$. Then in the upper half superplane $\mathbb{C}^{1|1}$, we join $P_1$ and $P_2$ by the supergeodesic $\tilde{G}_{11}$ governed by the following solution

$$Z(u) = \left(\frac{|Z|}{2} - \frac{i\Theta e^{\alpha(u+u_0)}}{4 \cosh \Theta \omega (u+u_0)}\right) \left[ \tan \omega (u+u_0) + i \frac{|Z|}{2} \right],$$

$$\Theta(u) = \frac{\Theta}{2} \left[ 1 + \tanh \omega (u+u_0) + i \frac{|Z|}{2} \right] \Theta(u),$$

and $\tilde{Z}(u) = Z(u)$, with $\epsilon_0(u) \in [-\infty, +\infty]$. The point $\tilde{P}_{12} \in \tilde{G}_{11}$ determined by $\epsilon_0(\tilde{P}_{12}) = \epsilon_0(\tilde{P}_2)$ is given by

$$\tilde{P}_{12} = \left( \frac{|Z|}{2 + |Z|^2} + \frac{\sqrt{1 + |Z|^2}}{2(2 + |Z|^2)} \Theta \Theta + i \frac{|Z|}{2} \left( \frac{1}{2 + |Z|^2} \right) \right),$$

as a point in $\mathbb{C}^{1|1}$, hence the superdistance between $\tilde{Q} = (0, 0, 1 + \frac{2|Z|+\sqrt{1+|Z|^2}}{2|Z||1+|Z|^2| - |Z|\sqrt{1+|Z|^2}} \Theta \Theta; 0, 0)$ and $\tilde{P}$ is given by

$$\cosh \left( \tilde{Q}, \tilde{P}_{12} \right) = 1 + \left( \frac{|Z|}{2 + |Z|^2} + \frac{\sqrt{1 + |Z|^2}}{2(2 + |Z|^2)} \Theta \Theta \right)^2 + \left( \frac{1}{2} - \frac{|Z|\sqrt{1+|Z|^2}}{2(2 + |Z|^2)} \Theta \Theta \right)^2 - \frac{|Z|\sqrt{1+|Z|^2}}{2(2 + |Z|^2)} \Theta \Theta.$$
where the pair \((\tau, \delta) \in (\Lambda^\infty_C)_0 \times (\Lambda^\infty_C)_1\) with \(\epsilon_0(\text{Im}(\tau)) > 0\) is called the supermoduli of \(T^{1|1}\) \cite{25}. The Jacobi theta function on the ordinary torus \(\mathbb{C}/(\mathbb{Z} + \tau \mathbb{Z})\) is given by

\[
\vartheta(Z; \tau) = -i \sum_{n \in \mathbb{Z}} (-1)^n q^{(n+1)^2} e^{(2n+1)\pi i Z} \quad \text{for } \rho = e^{2\pi i Z}, q = e^{\pi i \tau},
\]

and

\[
\vartheta(Z + 1; \tau) = -\vartheta(Z; \tau) = \vartheta(-Z; \tau),
\]

\[
\vartheta(Z + \tau; \tau) = -(\rho q)^{-1} \vartheta(Z; \tau).
\]

As an analog, we have the super-theta function, which was firstly introduced by Rabin and Freund \cite{25,26},

\[
T(Z, \Theta; \tau, \delta) = \vartheta(Z; \tau + \Theta \delta),
\]

then one easily checks the following proposition.

**Proposition 10.** The super-theta function satisfies the properties:

\[
T(Z, \Theta; \tau, \delta) = \vartheta(Z; \tau) + \Theta \delta \vartheta(Z; \tau),
\]

\[
T(Z + 1; \Theta; \tau, \delta) = -T(Z, \Theta; \tau, \delta) = T(-Z, \Theta; \tau, \delta),
\]

\[
T(Z + \tau; \Theta + \delta, \Theta + \delta; \tau, \delta) = -(1 - \pi i \Theta \delta) q^{-1} e^{-2\pi i Z} T(Z, \Theta; \tau, \delta),
\]

\[
T'(0, \Theta; \tau, \delta) = -2\pi q^{1/8} \prod_{n=1}^{\infty} (1 - q^n)^3 \left(1 + \frac{\pi i}{4} \Theta \delta - 6\pi i \sum_{m=1}^{\infty} \frac{mq^n}{1 - q^m \Theta \delta}\right),
\]

where the symbols dot and prime denote the partial derivatives with respect to \(\tau\) and \(Z\), respectively.

Following the classical Arakelov–Green function on the usual torus \cite{2}, we introduce the super-Green function \(G(Z, \Theta)\) on \(T^{1|1}\) as follows

\[
G(Z, \Theta) = \log \left| \frac{T(Z, \Theta; \tau, \delta)}{T'(0, \Theta; \tau, \delta)} \right| - 2\pi \left(\frac{(\text{Im}(Z))^2 + 2\Theta \delta \text{Im}(\tau + \Theta \delta)}{2 \text{Im}(\tau + \Theta \delta)} - \frac{1}{\pi} \sum_{n=1}^{\infty} \log |1 - q^n| - \frac{1}{12} \text{Im}(\tau + \Theta \delta) - \frac{\log 2\pi}{2\pi}\right).
\]

By the same manner of supergeometric extension, one can generalize the classical Faltings invariant on the torus \cite{1,24,27} to a superfunction \(F(\Theta)\) that depends only on the odd coordinate of the supertorus

\[
F(\Theta) = -\frac{1}{2\pi} \log |(\text{Im}(\tau + \Theta \delta))^{6} (T'(0, \Theta; \tau, \delta))^8|\]

\[
= - \frac{3}{\pi} \log |\text{Im}(\tau + \Theta \delta)| - \frac{12}{\pi} \sum_{n=1}^{\infty} \log |1 - q^n| + \text{Im}(\tau + \Theta \delta) - \frac{\log 2\pi}{\pi}.\]
**Proposition 11.** 1. $G(Z, \Theta)$ is invariant under the supertranslations.
2. \( \lim_{(Z, \Theta) \to (0, 0)} \frac{G(Z, \Theta)}{\log |Z|} = 1. \)
3. $G(Z, \Theta)$ can be rewritten as

\[
G(Z, \Theta) = \log |q^\frac{B_2(\frac{\log Z}{\Im(\tau)})}{2} (1 - \rho) \prod_{n=1}^{\infty} (1 - \rho q^n)(1 - \rho^{-1}q^n) |
\]

\[
+ 2\pi \sum_{m=1}^{\infty} \Im \left( \frac{2 - \rho q^m - \rho^{-1}q^m}{(1 - \rho q^m)(1 - \rho^{-1}q^m) \Theta \delta} \right) + \left( \frac{1}{2} \frac{\Im (Z)}{\Im (\tau)}^2 - \frac{1}{12} \right) \Im (\Theta \delta)
\]

\[
- \left( \frac{1}{\Im (\tau)} + \frac{(\Im (Z))^2}{4(\Im (\tau))^3} \Theta \delta \right).
\]

where $B_2(y) = y^2 - y + \frac{1}{6}$ is the second Bernoulli polynomial, and the first term on the right-hand side of the equal sign is recognized as the Néron function \cite{2}. 4. The second-order partial derivatives of $G(Z, \Theta)$ outside the singular locus of $G(Z, \Theta)$ read

\[
\frac{-1}{2\pi} \frac{\partial^2 G(Z, \Theta)}{\partial Z \partial \bar{Z}} = \frac{1}{4\Im (\tau)} - \frac{\Im (\Theta \delta)}{4(\Im (\tau))^2} + \frac{\delta \bar{\delta} \Theta \bar{\Theta}}{8(\Im (\tau))^3},
\]

\[
\frac{-1}{2\pi} \frac{\partial^2 G(Z, \Theta)}{\partial Z \partial \bar{\Theta}} = \frac{\Im (Z)}{4(\Im (\tau))^2} \delta \bar{\delta} + \frac{i\Im (Z)}{4(\Im (\tau))^3} \delta \bar{\Theta},
\]

\[
\frac{-1}{2\pi} \frac{\partial^2 G(Z, \Theta)}{\partial \Theta \partial \bar{Z}} = -\frac{\Im (Z)}{4(\Im (\tau))^2} \bar{\delta} + \frac{i\Im (Z)}{4(\Im (\tau))^3} \bar{\delta} \bar{\Theta},
\]

\[
\frac{-1}{2\pi} \frac{\partial^2 G(Z, \Theta)}{\partial \Theta \partial \bar{\Theta}} = -\frac{1}{\Im (\tau)} - \frac{(\Im (Z))^2}{4(\Im (\tau))^3} \delta \bar{\delta}.
\]

**Proof.** (1) It obviously follows from the invariance of the classical Arakelov–Green function.

(2) We only need to note that

\[
\frac{T(Z, \Theta; \tau, \delta)}{T'(0, \Theta; \tau, \delta)} \sim (0, 0) \quad Z.
\]

Hence $G(Z, \Theta) \sim (0, 0) \quad \log |Z|.$

(3) Letting $q = e^{2\pi i (\tau + \Theta \delta)}$, we have the identity

\[
\prod_{n=1}^{\infty} (1 - \rho q^n)(1 - \rho^{-1}q^n) = \frac{-2\pi i \rho \frac{1}{2} (1 - \rho - 1) q^{\frac{1}{2}} \prod_{n=1}^{\infty} (1 - q^n)^3 (1 - \rho q^n)(1 - \rho^{-1}q^n)}{-2\pi i \rho \frac{1}{2} (1 - \rho - 1) q^{\frac{1}{2}} \prod_{n=1}^{\infty} (1 - q^n)^3} \]

\[
= \frac{2\pi i \prod_{n=1}^{\infty} (1 - q^n)^2 T(Z, \Theta; \tau, \delta)}{\rho \frac{1}{2} (1 - \rho - 1) T'(0, \Theta; \tau, \delta)}.
\]
which yields

\[ G(Z, \Theta) = \log |q| \frac{b_z(\frac{\Im(Z)}{\Im(\tau)})}{2} (1 - \rho) \prod_{n=1}^{\infty} (1 - \rho q^n) (1 - \rho^{-1} q^n) | - 2\pi \frac{\Theta \tilde{\Theta}}{\Im(\tau + \Theta \delta)} \]

\[ = \log |(1 - \rho) \prod_{n=1}^{\infty} (1 - \rho q^n) (1 - \rho^{-1} q^n)| - 2\pi \left( \frac{(\Im(Z))^2 + 2\Theta \tilde{\Theta}}{2\Im(\tau + \Theta \delta)} - \frac{\Im(Z)}{2} + \frac{1}{12} \Im(\tau + \Theta \delta) \right). \]

Hence (3) follows.

(4) These second-order partial derivatives can be directly calculated by means of the formula in (3). \(\square\)

From the above proposition, we see that in order for illustrating \(G(Z, \Theta)\) as a super-Green function we also need the following proposition.

**Proposition 12.** The supertorus \(T^{11}\) can be equipped with a supermetric

\[ ds^2 = \frac{1}{\Im(\tau + \Theta \delta)} \left[ \frac{1}{\Im(\tau)} \left( 1 - \frac{\Im(\Theta \delta)}{\Im(\tau)} + \frac{\delta \delta \Theta \tilde{\Theta}}{2(\Im(\tau))^2} \right) dZ d\bar{Z} - \frac{\Im(Z)^2}{\Im(\tau + \Theta \delta)} d\Theta d\bar{\Theta} \right]. \]

**Proof.** By the concentrated expression of the supermetric

\[ ds^2 = \frac{1}{\Im(\tau + \Theta \delta)} \left[ \frac{1}{\Im(\tau)} dZ d\bar{Z} - \frac{\Im(Z)}{\Im(\tau + \Theta \delta)} \delta \delta \Theta \tilde{\Theta} dZ d\bar{\Theta} + \frac{\Im(Z)}{\Im(\tau + \Theta \delta)} d\Theta d\bar{\Theta} \right], \]

one checks the invariance under the transformation \(S\). Indeed, we have

\[ dZ d\bar{Z} \mapsto dZ d\bar{Z} + \delta \delta \Theta \tilde{\Theta} dZ d\bar{\Theta} - \delta \delta \Theta \tilde{\Theta} dZ d\bar{\Theta}. \]

Thus the conclusion follows. \(\square\)

Assume the odd moduli \(\delta\) is valued in \((\Lambda^{\infty}_{\mathbb{R}})_1\), then the super-Poincaré extension \(\tilde{T}\) of the supertranslation group generated by the transformations

\[ \tilde{S} : \ x \mapsto x + 1, \ y \mapsto y, \ t \mapsto t, \ \theta_1 \mapsto \theta_1, \ \theta_2 \mapsto \theta_2, \]

\[ \tilde{T} : \ x + iy \mapsto x + iy + \tau + \theta_2 \delta, \ t \mapsto t + \theta_1 \delta, \ \theta_1 \mapsto \theta_1, \ \theta_2 \mapsto \theta_2 + \delta \]

acts on \(\mathcal{H}^{3|2}\). Hence one regards \(T^{11}\) as the boundary of \(\mathcal{H}^{3|2}/\tilde{T}\), and the latter one is treated as a solid supertorus. To avoid extra requirements on the odd moduli, we adopt
the following approach of extension instead of the super-Poincaré extension. One defines
the supertorus by the equivalence relation \((\rho; \Theta) \sim (q^n \rho; \Theta + n\delta)\) for \(n \in \mathbb{Z}\) which can
be extended to \(\mathcal{H}^{3,2}\) as \((\rho, t; \Theta) \sim (q^n \rho, |q|^n t; \Theta + n\delta)\). Hence one views \(T^{1,1}\) as the
boundary of \(\mathcal{H}^{3,2}\). For two equivalent points \(P_0 = (0, 1; 0)\) and \(Q_0 = (0, |q|; \delta)\)
lying in \(\mathcal{H}^{3,2}\), one calculates the superdistance between \(P\) and \(Q\) in some upper half
superplane \(\mathcal{H}^{1,1}\) as follows

\[
\cosh d(P_0, Q_0) = 1 + \frac{(1 - |q|)^2 - 2\delta \bar{\delta}}{2(|q| + \frac{\delta \bar{\delta}}{2})},
\]

namely, we have

\[
d(P_0, Q_0) = d(P_0, Q_0) + d_1(P_0, Q_0)\delta \bar{\delta} = \log |q| + \frac{1 + |q|}{2|q|(1 - |q|)} \delta \bar{\delta}.
\]

For a point \(P = (\rho, \Theta)\) on the boundary of \(\mathcal{H}^{3,2}\), there is a supergeodesic in \(\mathcal{H}^{1,1}\)
determined by the following equations

\[
\rho(u) = \left|\rho\right| - \frac{i \Theta \Theta e^{\omega(u+u_0)}}{4 \cosh \omega(u+u_0)} \left[\tanh \omega(u + u_0) + i \text{sech} \omega(u + u_0)\right],
\]

\[
\Theta(u) = \frac{\Theta}{2} \left[1 + \tanh \omega(u + u_0) + i \text{sech} \omega(u + u_0)\right],
\]

\[
\bar{\rho}(u) = \frac{\Theta}{2} \left[1 - \tanh \omega(u + u_0) - i \text{sech} \omega(u + u_0)\right],
\]

which joins \(P\) and \(\bar{P} = (\frac{\Theta}{4}, |\rho|; 1 + i \Theta)\). Then the superdistance between \(P_0\) and \(\bar{P}\) is
given by

\[
d(P_0, \bar{P}) = d(P_0, \bar{P}) + d_1(P_0, \bar{P})\Theta \bar{\Theta} = \log |\rho| + \frac{1 + |\rho|}{4|\rho|(1 - |\rho|)} \Theta \bar{\Theta}.
\]

Similarly, for the point \(\bar{P}_0 = (0, 1 - \frac{|\rho|^2 + 1}{4|\rho||(|\rho|^2 - 1)} \Theta \bar{\Theta}; 0) \in \mathcal{H}^{3,2}\), we have

\[
d(\bar{P}_0, \bar{P}) = \log |\rho|.
\]

As a consequence, we arrive at

**Proposition 13.** The super-Green function on the supertorus \(T^{1,1}\) can be expressed as

\[
G(Z, \Theta) = \frac{1}{2} d(P_0, \bar{\Omega}) B_2 \left(\frac{d(P_0, \bar{P})}{d(P_0, \bar{\Omega})}\right) + d(P_0, \bar{P})
\]

\[
+ \sum_{n=1}^{\infty} \left[d(P_0, \bar{\Omega}^n) + d(P_0, \bar{\Omega}^n)\right] + \frac{4\pi^2}{d(P_0, \bar{\Omega})} \Theta \bar{\Theta}
\]

\[
= \frac{1}{2} d(\bar{P}_0, \bar{\Omega}) B_2 \left(\frac{d(\bar{P}_0, \bar{P})}{d(\bar{P}_0, \bar{\Omega})}\right) + d(\bar{P}_0, \bar{P})
\]

\[
+ \sum_{n=1}^{\infty} \left[d(\bar{P}_0, \bar{\Omega}^n) + d(\bar{P}_0, \bar{\Omega}^n)\right] + \frac{4\pi^2}{d(\bar{P}_0, \bar{\Omega})} \Theta \bar{\Theta},
\]

where \(\bar{\Omega} = (q, \Theta), P_- = (1 - \rho; \Theta), \bar{\Omega}^n = (1 - q^n \rho, \Theta)\) and \(\bar{\Omega}^{-n} = (1 - q^n \rho^{-1}, \Theta)\)
are all points lying on the boundary of \(\mathcal{H}^{3,2}\).
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