Near-Ideal Quantum Efficiency with a Floquet Mode Traveling Wave Parametric Amplifier

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Broadband quantum-limited amplifiers would advance applications in quantum information processing, metrology, and astronomy. However, conventional traveling-wave parametric amplifiers (TWPA) support broadband amplification at the cost of increased added noise. In this work, we develop and apply a multi-mode, quantum input-output theory to quantitatively identify the sidebands as a primary noise mechanism in all conventional TWPA. We then propose an adiabatic Floquet mode scheme that effectively eliminates the sideband-induced noise and subsequently overcomes the trade-off between quantum efficiency (QE) and bandwidth. Crucially, Floquet mode TWPA also strongly suppress the nonlinear forward-backward wave coupling and are therefore genuinely directional. Floquet mode TWPA can thus be directly integrated on-chip without isolators, making near-perfect measurement efficiency possible. The proposed Floquet scheme is also widely applicable to other platforms such as kinetic inductance traveling-wave amplifiers and optical parametric amplifiers.

Faithful amplification and detection of weak signals are of central importance to various research areas in fundamental and applied sciences, ranging from the study of celestial objects in radio astronomy and metrology [1, 2], dark matter detection in cosmology [3, 4], exploration of novel light-matter interactions in atomic physics [5], to superconducting [6–8] and semiconductor [9, 10] qubit readout in quantum information processing. In circuit quantum electrodynamics (cQED), near-quantum-limited amplifiers enable fast, high fidelity readout and have helped achieve numerous scientific advances, such as the observation [11] and reversal [12] of quantum jumps, the “break-even” point in quantum error correction [13–15], and quantum supremacy/advantage [16]. Josephson traveling wave parametric amplifiers (JTWPAs) [17–19] have helped achieve numerous scientific advances, such as the observation [11] and reversal [12] of quantum jumps, the “break-even” point in quantum error correction [13–15], and quantum supremacy/advantage [16].

We propose a new, Floquet mode JTWP scheme that effectively eliminates the sideband-induced noise and subsequently overcomes the trade-off between quantum efficiency (QE) and bandwidth. Crucially, Floquet mode TWPA also strongly suppress the nonlinear forward-backward wave coupling and are therefore genuinely directional. Floquet mode TWPA can thus be directly integrated on-chip without isolators, making near-perfect measurement efficiency possible. The proposed Floquet scheme is also widely applicable to other platforms such as kinetic inductance traveling-wave amplifiers and optical parametric amplifiers.

MULTI-MODE DYNAMICS

The uncertainty principle of quantum mechanics requires any linear phase-preserving amplifier to add at...
least $\sim 1/2$ quantum of noise referred to the input at high gain [40–42]. For an ideal two-mode parametric amplifier, the idler, described by the creation operator $\hat{a}^\dagger(\omega_i)$, acts as the coherent “reservoir” that injects the minimal added noise to the signal $\hat{a}(\omega_j)$ to preserve the bosonic commutator relations at the output. At a power gain $G \geq 1$ in linear units, the ideal quantum efficiency is commonly defined as [26]

$$\eta_{\text{ideal}}(G) = \frac{\left|\Delta \hat{a}_{\text{in}}(\omega_j)\right|^2}{\left|\Delta \hat{a}_{\text{out}}(\omega_j)\right|^2/G} = \frac{1}{2 - 1/G},$$

in which $\hat{a}_{\text{in(out)}}(\omega_j)$ is the input (output) annihilation operator of the signal, and $\Delta \hat{a} \equiv \left\langle (\hat{a}, \hat{a}^\dagger) \right\rangle / 2 - (\hat{a}^\dagger) \hat{a}$ is the mean-square fluctuation of operator $\hat{a}$ [41]. It is worth emphasizing that $\eta_{\text{ideal}}$ corresponds to the standard quantum limit even though it is only $\approx 50\%$ at high gain $G$. No information is lost in the process [42] because the idler is not correlated with any other unmeasured degrees of freedom. Such ideal two-mode amplifiers, albeit halving the measurement strength, can preserve information perfectly as illustrated in Fig. 1a.

Practical TWPA models, however, are intrinsically multimode because their large bandwidth allows for a spectrum of sidebands to propagate simultaneously. Sidebands have been a longstanding problem for all types of wideband TWPA models [43, 44], as their couplings to the signal and idler lead to additional noise and information leakage, as illustrated in Fig. 1b. To suppress the noise of the downstream amplifiers in the measurement chain, the preamplifier needs to provide a signal gain $G \gg 1$ by applying a strong pump tone, which unavoidably induces sideband couplings in all TWPA models, including our proposed Floquet mode amplifiers illustrated in Fig. 1c. Floquet mode amplifiers can still reach the quantum limit by coherently recovering the leaked information back into the signal and idler modes.

The various two-pump-photon parametric interactions responsible for the coherent information leakage in a typical degenerate four-wave mixing (4WM) TWPA can be conveniently visualized using the two-sided frequency spectrum in Fig. 2b. In this representation, the idler mode has a defined negative frequency of $\omega_{-1} = \omega_1 - 2\omega_p \approx -\omega_1$ in recognition of the relation $\hat{a}(\omega) \approx \hat{a}^\dagger(\omega)$. Similarly, the frequencies of the gray-colored sideband modes $\hat{a}_n = \hat{a}(\omega_n)$ are specified by $\omega_n = \omega_p + 2n\omega_p$, in which the integer mode index $n$ can be either negative or positive. Following this convention, 4WM processes couple only adjacent modes that are spaced by $2\omega_p$, whereas six-wave-mixing (6WM) processes can instead couple modes separated by up to two spacings and so forth. Frequency conversion (FC) and parametric amplification (PA) processes can be distinguished by whether the frequencies of their two interacting modes possess the same or opposite signs, respectively. Analogously to a qubit, information can coherently leak out from the “computational subspace” of the signal and the idler in the yellow-shaded region of Fig. 2b into the sidebands, in ad-
dition to incoherent losses from radiation or dissipation. The effects of sidebands on phase matching and hence gain dynamics have been studied in KITs and JTWPAs [46–48], but the connection between sidebands and the noise performance was not recognized, potentially due to the lack of a systematic, rigorous quantum framework.

To quantify the effects of both sideband leakage and propagation loss on the QE of TWPA, we develop a multi-mode, quantum input-output theory framework, as illustrated in Fig. 2a. The circuit design exemplified in the figure is similar to but more general than that of a typical resonantly phase-matched JTWPA [17, 18], as it allows the device to be inhomogeneous and the circuit parameters to have a spatial dependence. The propagation loss is modelled quantum-mechanically with a series of distributed, lossless, and semi-infinite transmission line ports. Dissipations and their associated fluctuations can then be cast as the coherent scattering into and from these transmission lines, whose frequency-dependent scattering parameters are set by the loss rate $\gamma(\omega)$ [49–51]. Furthermore, we extend the beamsplitter model such that our model can now work for any generic second-order equation of motion and properly account for the loss and interactions among the forward and backward modes without taking the slowly-varying envelope approximation (see Supplementary Information for more details). In addition, our model can correctly account for impedance mismatch at the boundaries, insertion loss, and nonlinear processes from arbitrary orders of pump nonlinearities (4WM, 6WM, ...).

Under the single-pump approximation, the multi-mode system can be linearized around a strong, classic pump with a dimensionless amplitude $I_{pn}(x) = I_p(x)/I_0(x)$, where $I_p(x)$ and $I_0(x)$ are the pump current and the junction critical current at $x$, respectively. In the continuum limit ($|k_{s,x}| \ll 1$), the quantum spatial dynamic equations in the frequency-domain can be derived from the Heisenberg equations and written in block matrix form as:

$$\frac{d}{dx} \begin{pmatrix} \tilde{A}^+(x) \\ \tilde{A}^-(x) \end{pmatrix} = \begin{pmatrix} H_0(x) & \mathbf{0}_m \\ \mathbf{0}_m & \mathbf{0}_m \end{pmatrix} \begin{pmatrix} \tilde{A}^+(x) \\ \tilde{A}^-(x) \end{pmatrix} + \begin{pmatrix} -\Gamma(x) & \mathbf{0}_m \\ \mathbf{0}_m & \mathbf{0}_m \end{pmatrix} \begin{pmatrix} \tilde{A}^0(x) \\ \tilde{A}^0(x) \end{pmatrix} + \begin{pmatrix} \mathbf{0}_m \\ \mathbf{0}_m \end{pmatrix} \begin{pmatrix} \tilde{F}^+(x) \\ \tilde{F}^-(x) \end{pmatrix},$$

where $\tilde{A}^\pm(x) = |\cdots, \tilde{A}_n^\pm(x), \cdots|^T$ are the forward and backward propagating operator vectors, $\tilde{F}^\pm(x)$ are similarly the forward and backward noise operator vectors, $\Gamma(x) = \text{diag}(\gamma(\omega_1), \cdots)$ is the diagonal loss rate matrix, and $\mathbf{0}_m$ is the $m \times m$ zero matrix, with $m$ being the number of frequency modes considered. The field ladder operators $\tilde{A}^\pm_n(x)$ satisfy the commutation relations [45]

$$[\tilde{A}^+_n(x), \tilde{A}^-_{n'}(x')] = \text{sgn}(\omega_{n-n'}) \delta_{n,n'} \delta_{x-x'},$$

in which the superscripts $d_1, d_2 \in \{+, -\}$ denote the forward(+) or backward(-) propagating modes, and $\text{sgn}(\omega)$ is the sign function. The noise operators $\tilde{F}^\pm_n(x)$ follow a similar commutation relation as Eq. (3).

$$\begin{pmatrix} \tilde{A}^+_{n1}(x), \tilde{A}^-_{n2}(x') \end{pmatrix} = \text{sgn}(\omega_{n-n'}) \delta_{1,2} \delta_{x-x'},$$

By solving Eq. (2) and applying the proper boundary conditions at $x = 0$ and $x = L$, we can relate the input and output bosonic modes $\tilde{A}_\text{in}$ and $\tilde{A}_\text{out}$ in the transmission lines ports terminating the TWPA by (see Methods)

$$\tilde{A}_\text{out} = \tilde{S}_0 \tilde{A}_\text{in} + \int_0^L dx \left( \tilde{S}_n(x) \tilde{F}^{1/2}(x) \right),$$

in which $L$ is the device length in unit cells, and $\tilde{S}_0$ and $\tilde{S}_n(x)$ are the $2m \times 2m$ multi-mode and noise scattering matrices that capture the effects of sideband couplings and dissipation on the output signal, respectively. Together they are sufficient to calculate the full quantum statistics of the output modes. The quantum efficiency $\eta$ of a general multi-mode TWPA with dissipation can thus be calculated from $\tilde{S}_0$ and $\tilde{S}_n(x)$ as

$$\eta(G) = \eta(|\tilde{S}_0, x|^2) = \frac{\left| \Delta \tilde{A}_0(\omega) \right|^2}{\left| \Delta \tilde{A}_\text{out}(\omega) \right|^2} \frac{\left| \tilde{S}_n, x \right|^2}{|\tilde{S}_0, x|^2} \int_0^L dx \left( |\tilde{A}_n(x)|^2 \int_0^L dx \left( |\tilde{F}(x)|^2 \right) \right),$$

in which $\tilde{A}_k$ denotes the $k$-th operator of the vector $\tilde{A} \in \{ \tilde{A}_\text{in}, \tilde{F}(x) \}$, $s$ is the index of the forward signal mode $\tilde{A}_n^+$, and $|\tilde{S}_0, x|_s^2$ is the signal power gain $G$. Notice that Eq. (5) can be mapped to the usual form $\eta = (1/2)/((1/2) + A)$ with Cave’s added noise number $A = \sum_k |\tilde{S}_n, x|_k^2 = \int_0^L dx \left( |\tilde{F}(x)|^2 \right) dx/|\tilde{S}_0, x|_s^2 - 1$, in the case when all input and environmental states have the minimum vacuum noise of $|\Delta \tilde{A}_n(x)|^2 = |\Delta \tilde{F}(x)|^2 = 1/2$.

Figure 2d pictorializes the dynamics of a multi-mode TWPA from the perspective of added noise using information from the scattering matrices $\tilde{S}_0$ and $\tilde{S}_n(x)$. A “water flow” from source mode $i$ to target mode $j$ can be equally interpreted as the operator or noise composition of input mode $i$ in the output mode $j$. The noise of component $i$ in output $j$ is proportional to the width of the $i \rightarrow j$ path. The quantum efficiency $\eta$ therefore corresponds to the weight (width) of the input signal noise in the total output signal noise (width). In the case of an ideal two-mode parametric amplifier, the output signal would have all of its in-flow (noise) coming from only the input signal and idler, signifying quantum-limited noise performance. In contrast, as is shown for a practical TWPA, an additional, non-negligible portion of the output signal noise instead comes from the input sideband modes and the reservoir of fluctuation operators, degrading the QE.

THE FLOQUET BASIS PICTURE

We now numerically simulate and visualize the spatial dynamics inside a JTWPA whose circuit parameters are similar to [18] except we assume $\tan \theta = 0$ here (see Table I in Methods). In Fig. 3a, a unit of forward signal $\tilde{A}_n^+$ is injected from $x = 0$ with $I_{pn}(x) = 0.52$, which is chosen to produce $\sim 20$ dB gain in $L = 2037$ cells. Although still being amplified, the forward signal and idler are continuously being converted to the sidebands $\tilde{A}_n^+$ and $\tilde{A}_n^-$, leading to the sideband amplification and information leakage. Figure 3b plots the noise decomposition of the amplified output signal $\tilde{A}_n^+(x = L)$ in the $50 \Omega$ output port as a function of $I_{pn}$. The device length is fixed at $L = 2037$ such that the signal gain increases nearly
monotonically with $I_{pn}$ before the onset of parametric oscillations. From Eq. (5), the QE $\eta$ can be usefully interpreted as the ratio of the amount of noise from the original signal $\hat{a}_0^+$, to the total output signal noise. QE therefore maps exactly to the signal contribution (in solid blue) in Fig. 3b, which decreases with the $I_{pn}$ due to an elevated sideband contribution. This quantitatively accounts for the unknown QE reduction in [18] and provides numerical evidence that the sidebands are indeed a significant noise source in JTWPAs.

Floquet theory [52] provides invaluable insights into the noise performance of TWPAs. Floquet modes are a set of solutions for a periodically driven system that forms a complete, orthonormal basis. Each Floquet mode can be expressed as $Q_n(x) = e^{i\alpha(x)}$, where $\alpha(x)$ is spatially-periodic and $r_\alpha$ is the complex Floquet characteristic exponent of Floquet mode $\alpha$. For a homogeneous TWP A described by Eq. (2), the transfer matrix of Eq. (2) can be written in the form of $\Pi(x) = P(x)\exp(xQ)$ according to the Floquet theorem, where $P(x)$ has the same periodicity as $\Re(x)$. We can thus transform the frequency basis vector $\tilde{A}_\alpha(x)$ into the Floquet mode basis $\hat{Q}(x)$ via (see Supplementary Information)

$$\hat{Q}(x) = (P(x)V)^{-1}\left(\tilde{A}_\alpha(x)\right),$$

where $V$ is the orthonormal basis of matrix $Q$ such that $Q = V\Lambda V^{-1}$ with $\Lambda = \text{diag}(\ldots, r_\alpha, \ldots)$ containing the Floquet exponents. Figure 3c shows the exact same spatial evolution as in Fig. 3a but in the Floquet mode basis. A unit of forward input signal $\hat{a}_0^+ (0)$ is injected and projected into a collection of Floquet modes, each of which then propagates separately with a distinct complex propagation constant $r_\alpha$. Notably, there is only one amplifying and one de-amplifying Floquet mode $Q_a$ and $Q_d$, which can be understood as the anti-squeezing and squeezing quadratures of the signal-idler like mode as shown in Fig. 3d. All the other sideband-like Floquet modes, colored in gray, remain constant in space (see Supplementary Information).

At $G \gg 1$, the amplifying Floquet mode $\hat{Q}_a$ dominates the gain and noise performance of a TWP A. Figure 3d plots the frequency mode decomposition of $\hat{Q}_a$ as a function of $I_{pn}$. The bifurcation point of the signal and idler components at $I_{pn} \approx 0.02$ marks the transition of the system from the region of stability to instability (amplification), whose exact position is dependent upon the phase mismatch. The gain coefficient $g_a = \text{Re}\{r_a\}$ of $Q_a$ (Fig. 3c) increases monotonically with $I_{pn}$ and $Q_a$ mixes in more sideband modes, in particular $\hat{a}_\alpha$ and $\hat{a}_{-\alpha}$. The remarkable resemblance between Fig. 3b and Fig. 3d in the region of large signal gain shows the usefulness of the Floquet basis in understanding the TWP A noise performance. An increased sideband weight in $Q_a$ means that a larger portion of sideband vacuum fluctuations incident upon a TWP A would be projected into the amplifying Floquet mode, and then subsequently generate more noise power at the signal and idler frequencies. The dependence of the $Q_a$ mode mixture on $I_{pn}$ also sheds lights on the experimental observation that the peak SNR improvement often does not coincide with the largest signal gain as a function of pump power [18].

Figure 3e plots in three dimension the complex Floquet character exponents $r_a$ as a function of $I_{pn}$ (x-axis). All but the amplifying and de-amplifying Floquet modes stay within the plane of $g_a = \text{Re}\{r_a\} = 0$ throughout and are stable, whereas the gain coefficient magnitudes of $Q_a$ and $Q_d$ increase with $I_{pn}$ as expected. It is important to point out that $r_a$ and $r_d$, the complex eigenvalues of $Q_a$ and $Q_d$ respectively, do not ever intersect with those of any other Floquet modes at all values of $I_{pn}$, as made clear in the inset of Fig. 3e. The existence of a gap between $r_a/r_d$ and the rest of the spectrum will be crucial to the adia-
batic Floquet scheme introduced in the following section. In addition, the bifurcation between $r_s$ and $r_d$ themselves on the complex plane can be interpreted as a consequence of the standard quantum limit for phase preserving amplifiers: both quadratures of the signal/idler cannot be noiselessly amplified at the same time.

**THE ADIABATIC FLOQUET MODE SCHEME**

We now propose an adiabatic spatial Floquet mode scheme that can effectively eliminate the aforementioned sideband-induced noise and approach the standard quantum limit. The principle idea behind Floquet mode TWPAs is that they operate as effective two-“Floquet mode” amplifiers, as information is exclusively encoded in the signal/idler modes $\hat{a}_0^\pm$ and $\hat{a}_1^\pm$ in the linear input/output to the instantaneous $\hat{Q}_d(x)$ and $\hat{Q}_l(x)$. Near the center of the amplifier, $I_{pn}(x)$ is adiabatically ramped up to $I_{pn}(x)$ ramps down to near zero again in the end. As a result, from the view outside of the device, the signal and idler modes are effectively decoupled from the various sidebands, and this scheme can therefore approach quantum-limited noise performance.

In practice, we can tailor the desired spatial profile of $I_{pn}(x)$ from a constant input pump current $I_p$ by instead varying the junction critical current $I_0(x)$ and ground capacitance $C_g(x)$ as shown in Fig. 4b. In Fig. 4e-j we compare the spatial dynamics of a conventional homogeneous critical current design ($e,g,i$) with our proposed Floquet scheme ($f,h,j$). Both designs use a slightly reduced cutoff frequency $f_c = \omega_c/(2\pi) \sim 65$ GHz with two junctions in series in one unit cell to reduce the physical device length (700 and 2000 unit cells, respectively) necessary to achieve $\sim 25$ dB gain (see Table II in Methods for circuit parameters). For comparison, the JTWA in [18] has a slightly higher cutoff $f_c = 73$ GHz and 2037 cells. Whereas the conventional homogeneous design has a constant drive amplitude of $I_{pn}(x) = I_{pn0} = 0.6$ (Fig. 4e), the instantaneous junction critical current $I_0(x)$ in the adiabatic Floquet scheme construct a Gaussian profile of $I_{pn}(x) = I_{pn0} \exp(-(x - \mu)^2/(2\sigma^2))$ (Fig. 4f), which centers at $\mu = L/2 = 1000$ and has a full-width-half-maximum of FWHM = $2\sqrt{2\ln 2}\sigma = 0.62L$. This practical choice of FWHM leads to a small but non-zero $I_{pn}(x) \approx 0.1$ at the boundaries, but this still results in nearly ideal quantum efficiency as it is close to the bifurcation point ($I_{pn} \approx 0.05$). Furthermore, here the minimum and maximum junction currents required to achieve an overall dynamic range $\approx -100$ dBm are around 3.5 $\mu$A and 21.2 $\mu$A, both of which can be readily fabricated with Lecocq-style junctions [53] or in a niobium trilayer pro-
cess [54], demonstrating the practicality and robustness of our scheme.

Figure 4g,h shows, respectively, the internal field profiles of the homogeneous and Floquet scheme in the frequency basis when a forward input signal $\hat{a}_F^+$ is injected at $x = 0^-$. While the signal and idlers are amplified by $\sim 25 \text{dB}$ in both schemes, the Floquet scheme efficiently suppresses the sidebands and backward modes. Figure 4i, j shows the system response of both schemes when instead only the sideband vacuum fluctuation $\hat{a}_S^+(0^+)$ is injected. Whereas the conventional homogeneous design generates a significant amount of added noise at the signal and idler frequencies, the Floquet scheme minimizes the coupling from $\hat{a}_S^+$ to $\hat{Q}_a$ and thus suppresses the sideband-induced noise by several orders of magnitude, thereby attaining near-ideal quantum-limited noise performance.

To clearly distinguish the noise performance of different near-quantum-limited amplifiers, we define the amplifier inefficiency as

$$\bar{\eta} = 1 - \frac{\eta}{\eta_{\text{ideal}}},$$

(7)

which signifies the relative difference in the resulting output SNR between a realistic and an ideal phase-preserving amplifier at the same power gain. An ideal preserving amplifier by definition will therefore have an inefficiency of $\bar{\eta} = 0$, denoting the standard quantum limit. In Fig. 4c,d we plot respectively the simulated gain and quantum inefficiency spectrum of the proposed Floquet scheme and the conventional homogeneous design. We also include the simulated performance of the experimental device from Ref. [18] at similar gain level for comparison. Our multi-mode quantum model predicts a quantum inefficiency $\bar{\eta} = 0.13$ or $\eta/\eta_{\text{ideal}} = 87\%$ for the experimental device assuming no dielectric loss, which is in good agreement with the experimentally extracted value of $\eta/\eta_{\text{ideal}} = 85\%$ in Ref. [18]. This suggests that our multi-mode quantum model is able to accurately predict and identify the previously unknown experimentally measured noise mechanism as the sideband-induced noise. For both conventional homogeneous schemes, the quantum inefficiency are still on the order of $10^{-1}$ away from the standard quantum limit due to the additional sideband-induced noise, although a design with a lower cutoff frequency of 65 GHz (blue curves in Fig. 4c,d) shows a slight improvement. Notably, sideband effects also manifest themselves in the visible oscillations on the quantum inefficiency or noise spectrum of the conventional homogeneous schemes. Such oscillations in the amplifier added noise have been observed in experiments [28, 38].

In contrast, the Floquet mode TWPA is able to both produce high gain and attain near ideal QE over a large bandwidth of 6.5 GHz (after excluding the bandgap due to the phase-matching resonators), well exceeding an octave in 2000 unit cells. The vanishingly small quantum inefficiency of the adiabatic Floquet design is a direct consequence of the effective decoupling of the signal and idler from the sidebands. The quantum inefficiency $\bar{\eta}$ of the Floquet mode TWPA is shown in Fig. 3i to be smaller than $10^{-3}$ over the full amplifying bandwidth, which is orders of magnitude closer to the quantum limit and can be practically realized.

FIG. 5. The potential of on-chip integrating Floquet mode TWPA. a, The quantum inefficiency $\bar{\eta} = 1 - \eta/\eta_{\text{ideal}}$ of the three amplifier designs in Fig. 4 as a function of the dielectric loss tangent $\tan \delta$. b, The nonlinear impedance of the signal as a function of frequency. c, Signal reflection as a function of frequency, where the gray curve corresponds to the signal reflection of the Floquet scheme with nonlinear forward-backward coupling excluded from the equations of motion.

**DIELECTRIC LOSS, DIRECTIONALITY, AND ON-CHIP INTEGRATION**

We now discuss the non-ideality of finite dielectric loss with Fig. 5a, in which the quantum inefficiency $\bar{\eta}$ at 6 GHz of the three designs in Fig. 4 is computed as a function of the loss tangent $\tan \delta$ with all other conditions fixed. The left and right vertical gray dashed lines correspond to $\tan \delta = 3.4 \times 10^{-3}$ of the SiO$_2$ capacitors in [18, 54] and $\tan \delta = 10^{-6}$ of a typical qubit fabrication process [55, 56], respectively. With SiO$_2$ capacitors, the calculated quantum inefficiency of the homogeneous TWPA in [18] increases to $\bar{\eta}_{\text{loss}} = 0.20$ from its lossless value $\bar{\eta}_{\text{lossless}} = 0.13$, again consistent with the characterized intrinsic quantum inefficiency of $\eta_{\text{loss}} = 0.25$ in Ref. [18]. The $\bar{\eta}$ of the Floquet scheme rapidly diminishes with a smaller $\tan \delta$ and eventually approaches $\sim 10^{-4}$, which is limited by the small impedance mismatch and the finite ramp rate of $f_{\text{pn}}(x)$ in Fig. 4f. Floquet mode JTWPAs fabricated with a typical qubit fabrication process are predicted to have a quantum efficiency on the level of $\eta/\eta_{\text{ideal}} > 99.9\%$ ($\bar{\eta} < 10^{-5}$), demonstrating the practicality of our proposed Floquet scheme.

Finally, we discuss the directionality and the prospect of directly integrating a Floquet mode TWPA on-chip. In a typical superconducting quantum experiment setup, the preamplifier (JPA or JTWPAs) in the measurement chain is only indirectly connected to the device under test via a commercial isolator or circulator to prevent reflections from dephasing the qubits or causing parametric oscillations in the amplifier. Such insertion loss occurring before the preamplifier will degrade measurement efficiency appreciably, and a directional, integrated quantum-limited preamplifier is therefore essential for approaching near-perfect full-chain measurement efficiency. While TWPA are in principle directional, existing TWPA cannot fulfill this promise due to their non-negligible reflections, as also evidenced in Fig. 4c. It is worth noting that for well impedance matched amplifiers, the major obstacle is in fact the nonlinear forward-backward mode coupling, which is properly captured by the off-diagonal block matrices $\mathbf{K}_{12}(x)$ and $\mathbf{K}_{21}(x)$ constituting $\mathbf{K}(x)$ in Eq. (2). In Fig. 5b,c we compare the nonlinear impedance and the signal reflection
spectrum of both the conventional homogeneous scheme and the Floquet mode scheme. We observe that the signal reflection in the conventional homogeneous scheme is significantly worse than the Floquet scheme even at near-identical, near-ideal impedance matching conditions. In contrast, the Floquet mode TWPA minimizes the nonlinear coupling contribution due to the adiabatic Floquet mode transformation and achieves $<-25$ dB reflection over the entire amplifying bandwidth. To support the claim that the signal reflection of a Floquet mode TWPA is near-ideal and limited by impedance mismatch at the boundaries, we simulate the Floquet scheme again using the exact same configurations but manually turning off the nonlinear forward-backward couplings by setting $K_{12}(x) = K_{21}(x) = 0$, which is plotted in gray in Fig. 5c. Indeed, the signal reflection of this “nonlinearly forward-backward decoupled” hypothetical Floquet scheme is almost identical to the actual Floquet scheme (red) as expected.

In conclusion, we have proposed an adiabatic Floquet mode scheme that allows for both high gain and near-ideal quantum efficiency over a large instantaneous bandwidth. In the qQED platform, we show in calculations that a Floquet mode JTWA can achieve $>20$ dB gain, $1 - \eta/\eta_{\text{ideal}} < 10^{-3}$, and $<-20$ dB reflection over $6.5$ GHz of instantaneous bandwidth, using a fabrication process with $\delta \approx 10^{-6}$, typical of qubit fabrication. Crucially, the proposed Floquet mode TWPAs are directional and can thus be directly integrated on-chip, potentially leading to near-perfect full-chain measurement efficiency. We expect this general Floquet mode scheme to have far-reaching applications on amplifier in various platforms and pave the way for scalable, fault-tolerant quantum computing.

CONCLUSION

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\[ H = E_{J0} \int dx \left( -\mu(x) \cos(\phi_x) + \frac{\mu(x) \beta_0^2}{2} \frac{\phi_x^2}{\omega_c^2} + \frac{\nu(x) \phi_x^2}{2} \right) + \frac{\gamma(x)}{2} \left( \phi_x - \psi \right)^2 + C_r \phi_x^2 + \psi^2 \frac{C_r}{2}\omega_c^2. \]

where \( \mu(x) = E_J(x)/E_{J0} \), \( \nu(x) = C_g(x)/C_{g0} \), \( \beta_0 \) is the reduced flux quantum. In the continuum limit, the system Hamiltonian, as detailed in the Supplementary Information, can then be expressed as

\[ H = E_{J0} \int dx \left( -\mu(x) \cos(\phi_x) + \frac{\mu(x) \beta_0^2}{2} \frac{\phi_x^2}{\omega_c^2} + \frac{\nu(x) \phi_x^2}{2} \right) + \frac{\gamma(x)}{2} \left( \phi_x - \psi \right)^2 + C_r \phi_x^2 + \psi^2 \frac{C_r}{2}\omega_c^2. \]

where \( \mu(x) = E_J(x)/E_{J0} \), \( \nu(x) = C_g(x)/C_{g0} \), \( \beta_0 \) is the reduced flux quantum. In the continuum limit, the system

\[ H = E_{J0} \int dx \left( -\mu(x) \cos(\phi_x) + \frac{\mu(x) \beta_0^2}{2} \frac{\phi_x^2}{\omega_c^2} + \frac{\nu(x) \phi_x^2}{2} \right) + \frac{\gamma(x)}{2} \left( \phi_x - \psi \right)^2 + C_r \phi_x^2 + \psi^2 \frac{C_r}{2}\omega_c^2. \]
We make the stiff-pump approximation such that \( \hat{\phi}(x) \to \phi_p(x) + \hat{\phi}(x) \) and \( \phi_p(x) \) is a classical number that is solved independently from the signals and sidebands. Moreover, we neglect the generation of the pump higher harmonics \( 3\omega_p, 5\omega_p, \cdots \), and solve for the fundamental frequency pump consistently in the form of \( d\phi_p(x)/dx = \hat{A}_{px}(x) \), \( \hat{k}(x) = \hat{\phi}(x) \). It should be pointed out that although here we neglect the higher harmonics of the pump, higher order nonlinear processes \( 4\omega_m, 6\omega_m, \cdots \) from the higher order nonlinearities mediated by the fundamental frequency pump are all treated. We can thus linearize Equation (10) linearized around the stiff pump (see Supplementary Information for details) and then transform from the flux basis \( \hat{\phi}(x) = [\cdots, \hat{\phi}(\omega_n), \cdots]^T \) into the field ladder operator basis \( \hat{A}^\pm(x) \) in the frequency domain using

\[
\hat{\phi}(x) \to i \begin{vmatrix} 2\nu_0 \mathcal{W}^{-1/2}(x) ||\mathcal{W}||^{-1/2}(\hat{A}^+(x) + \hat{A}^-(x)) \end{vmatrix},
\]

\[
\hat{\phi}_s(x) \to \begin{vmatrix} 2\nu_0 \mathcal{L}(x) \mathcal{Z}^{-1/2}(x) ||\mathcal{W}||^{-1/2}(\hat{A}^+(x) - \hat{A}^-(x)) \end{vmatrix},
\]

in which \( \mathcal{W} = \text{diag}(\cdots, \omega_n, \cdots), \mathcal{L}(x), \mathcal{C}(x), \) and \( \mathcal{Z}(x) = \text{diag}(\cdots, \sqrt{\mu_n(x)/C_n(x)}, \cdots) \) are the \( 2m \times 2m \) frequency, inductance, capacitance and nonlinear impedance matrices whose detailed forms can be found in the Supplementary Information. Finally, the lossless, second-order spatial equation of motion in the flux basis can now be rewritten in the ladder operator basis as

\[
\frac{d}{dx} \begin{pmatrix} \hat{A}^+(x) \\ \hat{A}^-(x) \end{pmatrix} = \mathbb{K}_0(x) \begin{pmatrix} \hat{A}^+(x) \\ \hat{A}^-(x) \end{pmatrix} = \begin{pmatrix} \mathbb{K}_{11}(x) & \mathbb{K}_{12}(x) \\ \mathbb{K}_{21}(x) & \mathbb{K}_{22}(x) \end{pmatrix} \begin{pmatrix} \hat{A}^+(x) \\ \hat{A}^-(x) \end{pmatrix},
\]

in which

\[
\mathbb{K}_{11}(x) = -\mathbb{K}_{22}(x) = \frac{i}{2} \left( ||\mathcal{W}||^{-1/2}(x) ||\mathcal{L}||^{-1/2}(x) \right) + \mathcal{Z}^{1/2}(x) \mathcal{C}^{1/2}(x)
\]

\[
\mathbb{K}_{12}(x) = \mathbb{K}_{21}^*(x) = \frac{i}{2} \left( ||\mathcal{W}||^{-1/2}(x) ||\mathcal{L}||^{-1/2}(x) \right) + \mathcal{Z}^{1/2}(x) \mathcal{C}^{1/2}(x)
\]

are the \( m \times m \) multi-mode coupling matrices that describe the forward-forward, backward-backward, forward-backward, and backward-forward interactions.

**Quantum Loss Model**

As illustrated in Fig. 2a, dielectric losses can be modeled quantum-mechanically using a series of lossless transmission line ports, whose frequency-dependent scattering parameters are determined by the loss rate \( \Gamma(x) \). Similar to the time-domain Langevin equations, the effect of dissipation and its associated fluctuation on both the forward and backward waves can thus be incorporated into the lossless spatial equation of motion Eq. (12) to get Eq. (2) in the main text. The phase factors in front of \( \hat{F}^\pm(x) \) are arbitrary and are chosen to be 1 for convenience [50], as they do not affect the quantum statistics of the outputs.

**Boundary Conditions and Input-Output Theory**

Assuming the linear transmission lines ports at \( x = 0 \) and \( x = L \) to be semi-infinite and have inductance and capacitance per unit length of \( l_1 \) and \( c_1 \), we can write the Lagrangian of the extended system as

\[
L_{\text{full}} = E_{\text{in}} \int_0^L dx \left( \mu(x) \cos(\phi(x)) + \frac{\beta}{2} \phi_0^2 + \frac{\nu(x)}{2} \phi_0^2 + \frac{\gamma(x)}{2} (\phi(t) - \phi_0)^2 \right) + \int_0^L dx \left( \frac{\phi_0^2}{2l_t} + \frac{\tilde{c}_1 \phi_0^2}{2} \right),
\]

in which \( \tilde{l}_1 = (l_1 C_0/E_{\text{in}}) \) and \( \tilde{c}_1 = (c_1 C_0) \) are the dimensionless inductance and capacitance parameters of the transmission line ports, and the extended Lagrangian \( L_{\text{full}} \) is piece-wise smooth. The continuity of flux \( \phi(x) \) and the Lagrange’s equations at the boundaries \( x = 0 \) and \( x = L \) yield the boundary conditions

\[
\phi(x = 0^-) = \phi(x = 0^+), \quad \phi(x = L^-) = \phi(x = L^+),
\]

\[
\phi_s(x = 0^-) / \tilde{l}_1 = \beta_0 \phi(x = 0^+) + \mu(x) \sin(\phi(x = 0^+) + \sigma(x = 0^+) \), \quad \phi_s(x = L^+) / \tilde{l}_1 = \beta_0 \phi(x = L^-) + \mu(x) \sin(\phi(x = L^-) + \sigma(x = L^-) \),
\]

which can be interpreted as the flux(voltage) and current continuity conditions at the boundaries. Performing the stiff-pump-approximation, going into the frequency basis and applying again the transformations in Eq. (11), we obtain the linearized ladder operator boundary conditions

\[
\begin{pmatrix} \hat{A}^+(0^+) \\ \hat{A}^-(0+) \end{pmatrix} = \begin{pmatrix} BC_{11}(0^+) & BC_{12}(0+) \\ BC_{21}(0+) & BC_{22}(0+) \end{pmatrix} \begin{pmatrix} \hat{A}^+(0^-) \\ \hat{A}^-(0^-) \end{pmatrix},
\]

\[
\begin{pmatrix} \hat{A}^+(L^-) \\ \hat{A}^-(L^-) \end{pmatrix} = \begin{pmatrix} BC_{11}(L^-) & BC_{12}(L^-) \\ BC_{21}(L^-) & BC_{22}(L^-) \end{pmatrix} \begin{pmatrix} \hat{A}^+(L^+) \\ \hat{A}^-(L^+) \end{pmatrix},
\]

where the diagonal and off-diagonal matrices are

\[
\begin{pmatrix} BC_{11}(x) = BC_{22}(x) = 1 + 1/\sqrt{Z(x)Z_0} \sqrt{Z(x)Z_0} \end{pmatrix},
\]

\[
\begin{pmatrix} BC_{12}(x) = BC_{21}(x) = 1/\sqrt{Z(x)Z_0} \sqrt{Z(x)Z_0} \end{pmatrix},
\]

in which \( Z_0 = \sqrt{l_1/\tilde{l}_1} \) is the characteristic impedance of the input/output transmission line, and we use the notation \( \sqrt{Z(x)} = Z(x)^{1/2} \) and \( 1/\sqrt{Z(x)} = Z(x)^{-1/2} \).

To formulate the input-output theory, we denote the input and output operator vectors as

\[
\hat{A}_\text{in} = \begin{pmatrix} \hat{A}_{1}\text{in} \\ \hat{A}_{m}\text{in} \end{pmatrix} = \begin{pmatrix} \hat{a}_n^+(0^-) \\ \hat{a}_n^-(0^-) \end{pmatrix}, \quad \hat{A}_\text{out} = \begin{pmatrix} \hat{A}_{1}\text{out} \\ \hat{A}_{m}\text{out} \end{pmatrix} = \begin{pmatrix} \hat{a}_n^+(L^+) \\ \hat{a}_n^-(L^+) \end{pmatrix}.
\]

Equation (2) and Eq. (16) constitute a two-point boundary value problem and can therefore be numerically solved to obtain the input-output relation in Eq. (4), with \( S_0 \) being the solution to the sourceless system(i.e., Eq. (12) without the last term \( \hat{F}^\pm(x) \) and \( S_n(x) \) being
the Green’s matrix solution to the system driven by a single point source $\vec{F}^\pm(x)$ and satisfying the boundary conditions Eq. (16). In the lossless model where $S_n(x) = O_{2m}$ is zero, the numerically solved $S_0$ is a unitary matrix as expected. One can also check that in the full loss model, the numerically solved scattering matrices $S_0$ and $S_n(x)$ together conserve the bosonic commutation relations at the output.

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AUTHOR CONTRIBUTIONS

K.P.O. and K.P. proposed the adiabatic Floquet mode scheme. K.P. and K.P.O. formulated the multi-mode, quantum input-output theory framework. K.P. and M.N. developed the field ladder operator basis model. K.P. and J.W. developed the second-order quantum loss model. K.P. and K.P.O. developed the numerical simulation codes. K.P., M.N., G.D.C., and Y.Y. prepared the figures for the manuscript. K.P. wrote the manuscript with input from all coauthors. K.P.O. supervised the entire scope of the project.

ADDITIONAL INFORMATION

Supplementary information accompanying this paper can be found online. The simulation data and code are available from the corresponding author on reasonable request. Correspondence and requests for materials should be addressed to K.P.O.

COMPETING INTERESTS

The authors declare no competing interests.
Supplementary Material for “Near-Ideal Quantum Efficiency with a Floquet Mode Traveling Wave Parametric Amplifier”

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SUPPLEMENTARY MATERIAL SECTION 1: MULTI-MODE QUANTUM INPUT-OUTPUT THEORY

Lagrangian and Hamiltonnian

FIG. S1: Generic unit cell design of a resonantly phase-matched JTWPA. The subscript $n$ denotes the n-th unit cell and its circuit element values can vary from cell to cell.

We consider the unit cell design of a generic resonantly phase-matched Josephson traveling-wave parametric amplifiers (JTWPAs) shown in Fig. S1. It is similar to but more general than those in [1, 2], as we allow the circuit parameters to have an arbitrary spatial dependence denoted with the subscript $n$. The circuit Lagrangian can be expressed as

$$L = \sum_{n} \left[ E_{J,n} \cos \left( \frac{\Phi_{n+1} - \Phi_n}{\phi_0} \right) + C_{J,n} \frac{d\Phi_{n+1}}{dt} - \frac{d\Phi_n}{dt} \right]^2 + C_{g,n} \left( \frac{d\Phi_n}{dt} \right)^2 + C_{c,n} \left( \frac{d\Phi_{r,n}}{dt} \right)^2 + C_{r,n} \left( \Phi_{r,n} \right)^2 - \Phi_{r,n}^2 L_{r,n} \right],$$

where in the last step we take the continuum approximation $\Phi_{n+1} - \Phi_n \approx a \Phi_x(x)|_{x=na}$ and $\sum_n a \approx \int dx/a$, assuming that the unit cell length $a$ is much smaller than the characteristic wavelength of the system. Here, we also use the subscript notation $\Phi_{x(t)} \equiv \partial \Phi / \partial x(t)$ to denote partial derivatives with respect to $x(t)$. To simplify notations, we introduce the normalized units and dimensionless variables

$$\omega_c = \frac{1}{\sqrt{L_{J0}C_{g0}}}, \quad \tilde{\phi} = \frac{\Phi}{\phi_0}, \quad \tilde{\psi} = \frac{\Phi_r}{\phi_0}, \quad \tilde{\omega} = \frac{2\omega_c}{\omega_c}, \quad \tilde{t} = \omega_c t = t/\tau_c, \quad \tilde{k} = k \cdot a, \quad \tilde{x} = x/a,$$

where $\phi_0 = \Phi_0/(2\pi)$ is the reduced flux quantum, and $L_{J0} = \phi_0^2/E_{J0}$ is the reference junction inductance of choice (For Floquet mode TWPAs, we choose the reference to be at the center where the effective drive amplitude is maximum).
The Lagrangian in Eq. (S1) can now be equivalently expressed with the dimensionless variables \( \tilde{\omega}, \tilde{x}, \tilde{\phi}(\tilde{x}, \tilde{t}), \tilde{\psi}(\tilde{x}, \tilde{t}) \) as

\[
L = E_{J0} \int_0^L dx \left( \mu(x) \cos(\tilde{\phi}_x) + \frac{\mu(x) \beta}{2} \tilde{\phi}_x^2 + \frac{\nu(x)}{2} \tilde{\phi}_t^2 + \frac{\gamma(x)}{2} (\tilde{\phi}_t - \tilde{\psi}_t)^2 + \frac{\tilde{C}_r}{2} \tilde{\psi}_t^2 - \frac{\tilde{\psi}_t^2}{2 L_r} \right),
\]

(S3)

in which the dependence of \( \tilde{\phi}, \tilde{\psi} \) on \( \tilde{x}, \tilde{t} \) is implicitly assumed, and

\[
\mu(x) = \frac{E_J(x)}{E_{J0}} = \frac{L_{J0}}{L_J(x)}, \quad \nu(x) = \frac{C_\psi(x)}{C_{\psi 0}}, \quad \beta = \frac{C_{J0}}{C_{J0}}, \quad \gamma(x) = \frac{C_\phi(x)}{C_{\phi 0}}, \quad \tilde{C}_r = \frac{C_r}{C_{\psi 0}}, \quad \tilde{L}_r = \frac{L_{r0}}{L_{r0}}
\]

(S4)

are the dimensionless parameters describing the spatial profile of the circuit elements. Unless otherwise noted, we will work entirely in the normalized unit from now on and omit all tildes for brevity.

Following the same procedure in [3], we identify the dimensionless node fluxes \( \phi(x, t) \) and \( \psi(x, t) \) as canonical coordinates, and the corresponding canonical momenta \( \pi_\phi(x, t) \) and \( \pi_\psi(x, t) \) are therefore

\[
\pi_\phi(x, t) = \frac{\delta L}{\delta \phi_t} = E_{J0} (\nu(x) \phi_t - \beta \frac{d}{dx} (\mu(x) \phi_{xt}) + \gamma(x)(\phi_t - \psi_t))
\]

\[
\pi_\psi(x, t) = \frac{\delta L}{\delta \psi_t} = E_{J0} (\tilde{C}_r \psi_t - \gamma(x)(\psi_t - \phi_t)).
\]

(S5)

Applying the Legendre transform, we arrive at the Hamiltonian

\[
H = \int_0^L dx \left( \pi_\phi \phi_t + \pi_\psi \psi_t - L \right)
\]

(S6)

\[
= E_{J0} \int_0^L dx \left( -\mu(x) \cos(\phi_x) - \mu(x) \beta \phi_{xxt} \phi_t - \frac{\mu(x) \beta}{2} \phi_{xt}^2 + \frac{\nu(x)}{2} \phi_t^2 + \frac{\gamma(x)}{2} (\phi_t - \psi_t)^2 + \frac{\tilde{C}_r}{2} \psi_t^2 + \frac{\psi_t^2}{2 L_r} \right) + \text{boundary terms},
\]

(S7)

where in the last step we perform integration by parts on the term \(-\beta \phi_{xxt} \phi_t\) and produce the additional constant boundary terms, which will be dropped from now on.

We quantize the system by promoting the variables to operators

\[
\hat{\phi}(x, t) \to \hat{\phi}(x, t), \quad \hat{\psi}(x, t) \to \hat{\psi}(x, t), \quad \pi_\phi(x, t) \to \hat{\pi}_\phi(x, t), \quad \pi_\psi(x, t) \to \hat{\pi}_\psi(x, t),
\]

(S8)

such that they obey the commutation relations

\[
[\hat{\phi}(x, t), \hat{\pi}_\phi(x, t)] = [\hat{\psi}(x, t), \hat{\pi}_\psi(x, t)] = i\hbar,
\]

\[
\hat{\psi}(x, t), \hat{\pi}_\phi(x, t) \quad [\hat{\phi}(x, t), \hat{\pi}_\psi(x, t)] = [\hat{\phi}(x, t), \hat{\psi}(x, t)] = 0.
\]

(S9)

The quantum spatial equation of motion can thus be derived from the Heisenberg equations of motion:

\[
\frac{d\hat{\pi}_\phi(x, t)}{dt} = \frac{1}{i\hbar} [\hat{\pi}_\phi(x, t), H] = E_{J0} \left( \frac{d}{dx} \left( \mu(x) \sin(\hat{\phi}_x) \right) \right) = E_{J0} (\nu(x) \phi_{xt} - \beta \frac{d}{dx} (\mu(x) \phi_{xt}) + \gamma(x)(\phi_{xt} - \psi_{xt}))
\]

\[
\frac{d\hat{\pi}_\psi(x, t)}{dt} = \frac{1}{i\hbar} [\hat{\pi}_\psi(x, t), H] = E_{J0} \left( -\frac{\hat{\psi}}{L_r} \right) = E_{J0} (\tilde{C}_r \psi_{xt} - \gamma(x)(\phi_{xt} - \psi_{xt})),
\]

(S10)

which are exactly the equations of motions presented in the Methods after rearranging terms and dividing both sides by \( E_{J0} \). Furthermore, under the stiff-pump approximation, we make the substitution of \( \phi \to \phi_p + \delta \phi \) to linearize Eq. (S10) around the strong, classical pump amplitude \( d\phi_p/dx = A_{p0}(x) \sin(\omega_p t - \int_0^x dx' k_p(x')) \) and get
\[
\frac{d}{dx} \left( \mu(x) (\cos(\phi_px) \dot{\phi}_x + \beta \dot{\phi}_{xtt}) \right) = \nu(x) \dot{\phi}_{tt} + \gamma(x) (\dot{\phi}_{tt} - \dot{\psi}_{tt}) \\
- \frac{\dot{\psi}}{L_r} = \gamma(x) (\dot{\psi}_{tt} - \dot{\phi}_{tt}) + \tilde{C}_r \dot{\psi}_{tt}
\]
(S11)

Finally, performing Fourier transform on Eq. (S11) and cross-eliminating \( \dot{\psi}(x, \omega) \), we arrive at the single-variable equation of motion in the flux basis

\[
\frac{d}{dx} \left( -\mu(x) \beta \omega^2 \dot{\phi}_x(\omega) + \mu(x) \sum_{n=-\infty}^{\infty} J_{2n}(A_p x_0(x)) \dot{\phi}_x(\omega + 2n \omega_p) e^{-i2n \int_0^x dx' \kappa_p(x')} \right) = -(\nu(x) + \gamma_\alpha(\omega)) \omega^2 \dot{\phi}(\omega),
\]
(S12)

where \( J_{2n}(z) \) is the Bessel function of the first kind of order 2n, and

\[
\alpha_\alpha(\omega) = \frac{1 - \bar{L}_r \bar{C}_r \omega^2}{1 - L_r (\bar{C}_r + \gamma_\alpha) \omega^2} = \frac{1 - \omega^2/\omega_r^2}{1 - \omega^2/\omega_{rt}^2},
\]
(S13)

accounts for the effect of the phase matching resonators and acts as an effective frequency-dependent capacitor. Notice that the cross-elimination is only valid when the frequency \( \omega \) is away from the resonances \( \omega_r, \omega_{rt} \) [3].

For an injected signal at frequency \( \omega_s = \omega_n \), the only frequency components it can couple to are \( \omega_n = \omega_s + 2n\omega_p \), where \( n \) is any integer. In practice however, \( n \) cannot be an arbitrarily small(negative) or large(positive) due to the restrictions of the junction plasma frequency and the transmission line cutoff frequency (from the discreteness of lumped-element transmission lines). We can therefore truncate the number of frequency components coupled to the signal to a finite number \( m = n_{\text{max}} - n_{\text{min}} + 1 \) and define a flux operator vector as

\[
\Phi(x) = [\cdots, \tilde{\phi}_x(\omega_n), \cdots]^T.
\]
(S14)

We can now rewrite Eq. (S13) as a matrix equation in block matrix format:

\[
\frac{d}{dx} \left( -L^{-1}(x) \bar{C}_r \Phi(x) \right) = -C(x) \bar{W}^2 \Phi(x),
\]
(S15)

in which we define the \( m \times m \) block matrices

\[
\bar{W} = \text{diag}(\cdots, \omega_n, \cdots), \quad \text{and} \quad C(x) = \text{diag}(\cdots, \nu(x) + \gamma(x) \alpha_\alpha(\omega_n), \cdots)
\]
(S16)

\[
L^{-1}(x) = -\mu(x) \beta \bar{W}^2 + \mu(x) \begin{pmatrix}
J_0 \theta_0 & J_{-2} \theta_{-2} & J_{-4} \theta_{-4} & \cdots & J_{2n_{\text{max}}} \theta_{2n_{\text{max}}} \\
J_2 \theta_2 & J_0 \theta_0 & J_{-2} \theta_{-2} & \cdots & \\
J_4 \theta_4 & J_2 \theta_2 & J_0 \theta_0 & \cdots & \\
& \vdots & \ddots & \ddots & \\
J_{2n_{\text{max}}} \theta_{2n_{\text{max}}} & J_{2n_{\text{max}}} \theta_{2n_{\text{max}}} & \cdots & J_2 \theta_2 & J_0 \theta_0
\end{pmatrix}
\]
(S18)

The last term in \( L^{-1}(x) \) is a toeplitz matrix, and we use the notations \( J_{2n} \equiv J_{2n}(A_p x_0(x)) \) and \( \theta_{2n} = \theta_{2n}(x) \equiv \exp\left( -i2n \int_0^x dx' \kappa_p(x') \right) \) for readability purposes. Note that as long as no \( \omega_n \) fall in between the resonant bandgap \([\omega_r, \omega_{rt}]\) and the pump current is below the junction critical current, \( \bar{W}, C(x), \) and \( L^{-1}(x) \) are all positive-definite matrices, and therefore the inverse of \( L^{-1}(x) \) or \( \overline{L}(x) \equiv \left( L^{-1}(x) \right)^{-1} \) exists and is well-defined.

We now define the diagonal nonlinear impedance matrix

\[
Z(x) = \text{diag}(\cdots, Z_{n}(x), \cdots) = \text{diag}(\cdots, \sqrt{\frac{L_{nn}}{C_{nn}}}, \cdots),
\]
(S19)
where $L_{nn}$ and $C_{nn}$ denote the n-th diagonal elements of the two matrices. As can be observed later in the boundary condition calculations, the diagonal element $L_{nn}(x)$ indeed represents the effective nonlinear impedance of mode $n$. Finally, applying the transformation

$$
\tilde{\phi}(x) = i \sqrt{\frac{2\phi_0}{h}} W^{-1/2}(x) W^{-1/2}(\tilde{A}^+(x) + \tilde{A}^-(x))
$$

(S20)

$$
\tilde{\phi}_x(x) = -i \sqrt{\frac{2\phi_0}{h}} L(x) W^{-1/2}(x) W^{-1/2}(\tilde{A}^+(x) - \tilde{A}^-(x)),
$$

we arrive at the field ladder operator basis equation of motion

$$
\frac{d}{dx} \begin{pmatrix} \tilde{A}^+(x) \\ \tilde{A}^-(x) \end{pmatrix} = K_0(x) \begin{pmatrix} \tilde{A}^+(x) \\ \tilde{A}^-(x) \end{pmatrix} = \begin{pmatrix} K_{11}(x) & K_{12}(x) \\ K_{21}(x) & K_{22}(x) \end{pmatrix} \begin{pmatrix} \tilde{A}^+(x) \\ \tilde{A}^-(x) \end{pmatrix},
$$

(S21)

in which

$$
K_{11}(x) = -K_{22}(x) = \frac{i}{2} \left( W^{-1/2} W (Z^{-1/2} L Z^{-1/2} + Z^{-1/2} C Z^{1/2} W) \right), \quad \text{and}
$$

$$
K_{12}(x) = K_{21}(x) = \frac{i}{2} \left( W^{-1/2} W (Z^{-1/2} L Z^{-1/2} + Z^{1/2} C Z^{1/2} W) \right). + Z_0(x) Z^{-1}(x).
$$

Notice that here we did not apply the usual slowly-varying envelope approximation (SVEA) to reduce the equation of motion in the flux basis to the first order. Going beyond the SVEA allows us to capture the interactions between the forward and backward modes, model the reflection due to impedance mismatch at the boundaries, and crucially to conserve the bosonic commutation relations without making additional ad-hoc approximations, such as in [4].

**Pump Dynamics**

Throughout the analysis, we have assumed the pump flux to be sinusoidal and is in the form of $d\phi_p/dx = A_{p,0}(x) \sin(\omega_p t - \int_0^x dx' k_p(x'))$. The dimensionless pump current can be expressed with the pump flux as

$$
I_{pn}(x) = \frac{I_p}{I_0(x)} = \frac{I_p}{\mu(x) I_0} = (\sin(\phi_{px,0}) + \beta \phi_{px,xt}) ,
$$

(S23)

where $I_0(x)$ is the junction critical current at $x$, and $I_p$ is the constant physical current (again neglecting the coupling to the higher harmonics of pump). Assuming a single sinusoidal pump at frequency $\omega_p$ and taking the Fourier transform of Eq. (S23), we get

$$
\frac{I_{pn}(x)}{2} e^{i \int_0^x dx' k_p(x')} = \frac{1}{2} \left( 2 J_1(A_{p,0}(x)) - \beta \omega_p^2 A_{p,0}(x) \right) e^{i \int_0^x dx' k_p(x')} .
$$

(S24)

We can thus numerically solve the pump flux amplitude $A_{p,0}(x)$ that is needed to solve the signal dynamics from Eq. (S24), when the dimensionless drive amplitude $I_{pn}(x) = I_p/I_0(x)$ is given as a parameter.

**SUPPLEMENTARY MATERIAL SECTION 2: FLOQUET MODE TWPAS**

**Floquet Theory**

In the case of a homogenous TWPA driven with a constant pump, $A_{p,0}(x) = A_{p,0}$ and $\theta_{2n}(x) = \theta_{2n} = \exp(-i2nk_p x)$. Therefore, the multi-mode coupling matrix is periodic and has a period of $x_T = \pi/k_p$. We can therefore apply the Floquet theory to analyze the system. Denote the unique frequency-basis transfer matrix solution of Eq. (S21) to be $\Pi(x)$, such that
\[ \vec{A}(x) = \begin{pmatrix} \vec{A}^+(x) \\ \vec{A}^-(x) \end{pmatrix} = \Pi(x) \begin{pmatrix} \vec{A}^+(0) \\ \vec{A}^-(0) \end{pmatrix} = \Pi(x) \vec{A}(0), \] (S25)

which is an initial value problem and can be solved numerically. The Floquet theorem states that the \( \Pi(x) \) can be written in the form of [5]

\[ \Pi(x) = \mathbb{P}(x) \exp(xQ), \] (S26)

where the \( 2m \times 2m \) matrix \( \mathbb{P}(x) \) has the same period \( x_T \) as \( \mathbb{K}(x) \), \( \mathbb{P}(0) = \mathbb{I} \) is the identity matrix, and \( Q \) is a constant \( 2m \times 2m \) matrix that can be obtained from the monodromy matrix.

\[ \mathbb{M}_0 = \Pi(x_T) = \exp(xTQ). \] (S27)

Applying the transformation \( \vec{B}(x) = \mathbb{P}^{-1}(x) \vec{A}(x) \), Eq. (S21) can be now rewritten in the form of a constant dynamic matrix

\[ \frac{d}{dx} \vec{B}(x) = Q \vec{B}(x). \] (S28)

With the eigendecomposition of \( Q \) to be \( Q = \mathbb{V} \Lambda \mathbb{V}^{-1} \), where \( \Lambda = \text{diag}(\cdots, r_\alpha, \cdots) \), and the columns of \( \mathbb{V} \) are the corresponding set of normalized eigenvectors, we can therefore transform from the frequency basis \( \vec{A}(x) \) into the Floquet basis \( \vec{Q}(x) \) using

\[ \vec{Q}(x) = \mathbb{V}^{-1} \mathbb{P}^{-1}(x) \vec{A}(x). \] (S29)

We can gain insights of Eq. (S29) by applying Eq. (S25) to it:

\[ \vec{Q}(x) = \mathbb{V}^{-1} \mathbb{P}^{-1}(x) \vec{A}(0) = \mathbb{V}^{-1} \mathbb{P}^{-1}(x) \Pi(x) \vec{A}(0) = \mathbb{V}^{-1} \mathbb{P}^{-1}(x) (\mathbb{P}(x) \exp(xQ)) \vec{A}(0) \\
= \mathbb{V}^{-1} (\mathbb{V} \exp(x\Lambda) \mathbb{V}^{-1}) \vec{A}(0) = \exp(x\Lambda) (\mathbb{I} \mathbb{V}^{-1} \vec{A}(0)) = \exp(x\Lambda) (\mathbb{P}(0) \mathbb{V}^{-1} \vec{A}(0)) \\
= \exp(x\Lambda) \vec{Q}(0), \] (S30)

which shows that Floquet modes \( \vec{Q}(x) \) are decoupled from each other and each propagates with a distinct dynamic factor \( r_\alpha \), which are also often referred to as the Floquet characteristic exponents. Figure S2 shows the spatial dynamics of the system in the Floquet basis, with each subfigure representing the case when a specific Floquet modes is injected at \( x = 0 \). As expected, when only a single Floquet mode is injected, it does not generate or couple to other Floquet modes.

**Frequency Decomposition of Floquet modes**

We can now analyze the Floquet modes using Eq. (S29). In Fig. S3 we plot the frequency mode decomposition of three Floquet modes as a function of the dimensionless drive amplitude \( I_{pn} = I_p/I_0 \). Figure S3a shows the decomposition of the amplifying Floquet mode \( \hat{Q}_a \), which is the same as Fig. 3d in the main text. From the decomposition of the de-amplifying Floquet mode \( \hat{Q}_d \) in Fig. S3b, we see that passing the bifurcation point \( I_{pn} \approx 0.13 \) the magnitude of the frequency mode decomposition for \( \hat{Q}_a \) and \( \hat{Q}_d \) are exactly the same and only differ in the relative phase between the frequency components. Therefore, \( \hat{Q}_a \) and \( \hat{Q}_d \) can be understood as the squeezing and anti-squeezing quadratures mostly composed of the signal and idlers. Figure S3c corresponds to the frequency mode decomposition of a stable Floquet mode which is \( \hat{a}_1^+ \)-like. At high \( I_{pn} \), more signal and idler components are mixed in as expected.

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FIG. S2: **Spatial Dynamics of Floquet modes.** Each figure describes the system response in the Floquet basis when a single Floquet mode is injected at $x = 0$. 
- **a,** The amplifying Floquet mode $\hat{Q}_a$ is injected.
- **b,** The de-amplifying Floquet mode $\hat{Q}_d$ is injected.
- **c,** Any of the other Floquet modes is injected.

FIG. S3: **Frequency mode decomposition of Floquet modes as a function of pump current.**
- **a,** Amplifying Floquet mode $\hat{Q}_a$.
- **b,** De-amplifying Floquet mode $\hat{Q}_d$.
- **c,** A stable Floquet mode that is $\hat{a}_1^\dagger$-like.

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