ON THE REGULARITY OF THE HANKEL DETERMINANT SEQUENCE OF THE CHARACTERISTIC SEQUENCE OF POWERS

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Abstract. For any sequences \( u = \{u(n)\}_{n \geq 0}, v = \{v(n)\}_{n \geq 0} \), we define \( uv := \{u(n)v(n)\}_{n \geq 0} \) and \( u + v := \{u(n) + v(n)\}_{n \geq 0} \). Let \( f_i(x) (0 \leq i < k) \) be sequence polynomials whose coefficients are integer sequences. We say an integer sequence \( u = \{u(n)\}_{n \geq 0} \) is a polynomial generated sequence if

\[
\{u(kn + i)\}_{n \geq 0} = f_i(u), \quad (0 \leq i < k).
\]

In this paper, we study the polynomial generated sequences. Assume \( k \geq 2 \) and \( f_i(x) = a_i x + b_i, (0 \leq i < k) \). If \( a_i \) are \( k \)-automatic and \( b_i \) are \( k \)-regular for \( 0 \leq i < k \), then we prove that the corresponding polynomial generated sequences are \( k \)-regular. As a application, we prove that the Hankel determinant sequence \( \{\det(p(i+j))_{i,j=0}^{n-1}\}_{n \geq 0} \) is \( 2 \)-regular, where \( \{p(n)\}_{n \geq 0} = 0110100010000 \cdots \) is the characteristic sequence of powers 2. Moreover, we give a answer of Cigler’s conjecture about the Hankel determinants.

1. Introduction

To introduce our motivation for the problem in this paper, we first recall some basic definitions of automatic and regular sequences.

1.1. Automatic and regular sequences. We say a sequence \( u = \{u(n)\}_{n \geq 0} \) with values in a finite set \( k \)-automatic if, informally speaking, \( u(n) \) is a finite-state function of the base-\( k \) expansion of \( n \). This is equivalent to the fact that the \( k \)-kernel \( K_k(u) \) is a finite set \( \mathbb{N}_0 \), where the \( k \)-kernel is a collection of subsequences

\[
K_k(u) = \{ \{u(kn + j)\}_{n \geq 0} : 0 \leq j < k^i \}.
\]

While all \( k \)-automatic sequences are defined over finite alphabets, Allouche and Shallit introduced a wider class of \( k \)-regular sequences that are allowed to take values in a Noetherian ring \( R \). A sequence is \( k \)-regular if the module generated by its \( k \)-kernel is finitely generated. In this paper, unless otherwise stated, the sequences we consider are integer sequences and assume the underlying ring is \( \mathbb{Z} \). More precisely, we say that an integer sequence \( \{u(n)\}_{n \geq 0} \) is \( k \)-regular if every sequence of its \( k \)-kernel is a \( \mathbb{Z} \)-linear combination of a finite set. That is to say, there exist a finite number of integer sequences \( \{a_1(n)\}_{n \geq 0}, \{a_2(n)\}_{n \geq 0}, \cdots, \{a_N(n)\}_{n \geq 0} \) such that for any \( i \geq 0, 0 \leq j < k^i \), there exist \( c_1, c_2, \cdots, c_N \in \mathbb{Z} \) such that

\[
u(k^i n + j) = \sum_{i=1}^{N} c_i a_i(n), \quad (n \geq 0).
\]

The \( k \)-regular sequences play the same role for integer valued sequences as the \( k \)-automatic sequences play for sequences over a finite alphabet. More relations between the \( k \)-regular and the \( k \)-automatic sequences can be found in [1] [2] [11].

1.2. Polynomial generated sequences. For any sequences \( u = \{u(n)\}_{n \geq 0} \) and \( v = \{v(n)\}_{n \geq 0} \). We define addition and multiplication as follows:

- \( u + v := \{u(n) + v(n)\}_{n \geq 0}, \)
- \( u \cdot v := \{u(n)v(n)\}_{n \geq 0}. \)

Let \( \mathcal{R} \) denote the set of all integer sequences. Then \((\mathcal{R}, +, \cdot)\) forms a commutative ring. Similarly, \( \mathcal{R}[X] \), the set of polynomials in the indeterminate \( x \) over \( \mathcal{R} \), is the set of all expressions of the form

\[
a_0 + a_1 x + \cdots + a_m x^m.
\]

Each element of \( \mathcal{R}[X] \) is call a sequence polynomial. If \( a_m \) is a nonzero sequence, then \( m \) is called the degree of \( f \). If \( a_i = \{a_i\}_{n \geq 0} \) are constant sequences, then the sequence polynomial is called to be a constant sequence polynomial, and denoted briefly by \( a_0 + a_1 x + \cdots + a_m x^m \).

A sequence polynomial can be considered as a map from \( \mathcal{R} \) to \( \mathcal{R} \). Let \( u \) be an integer sequence and \( f(x) = a_0 + a_1 x + \cdots + a_m x^m \) be a sequence polynomial with degree \( m \) in \( \mathcal{R}[X] \). Then the image of \( u \)
under the map $f(x)$ is the integer sequence $f(u) = a_0 + a_1u + \cdots + a_mu^m$. In particular, if $f(x) = a_0 + a_1x$, then $f(u)$ is called to be a linear polynomial of $u$.

**Definition 1.** Given an integer sequence $u = \{u(n)\}_{n \geq 0}$. If there exists an integer $k \geq 1$ and sequence polynomials $f_i(x) \in \mathbb{R}[X] \ (0 \leq i < k)$ such that

$$\{u(kn + i)\}_{n \geq 0} = f_i(u),$$

then we say that $u$ is a polynomial generated sequence. The set of polynomials $\{f_i(x) \in \mathbb{R}[X] : 0 \leq i < k\}$ is called to be a generated polynomial system. The set of the sequences generated by polynomials $\{f_i(x) \in \mathbb{R}[X] : 0 \leq i < k\}$ is denoted by $\mathcal{G}(f_0, f_1, \cdots, f_{k-1})$.

Assume $f_i(x) = a_ix + b_i$, where $a_i = \{a_i(n)\}_{n \geq 0}$, $b_i = \{b_i(n)\}_{n \geq 0}$ for $0 \leq i < k$. If $u = \{u(n)\}_{n \geq 0} \in \mathcal{G}(f_0, f_1, \cdots, f_{k-1})$, then, for $0 \leq i < k, n \geq 0$,

$$u(kn + i) = a_i(n)u(n) + b_i(n).$$

Hence, we have

$$u(0) = \frac{b_0(0)}{1 - a_0(0)},$$

$$u(i) = a_i(0)u(0) + b_i(0) \text{ for } 1 \leq i < k,$$

$$\cdots \cdots \cdots$$

Note from above that $u(n)$ are determined by $a_i(n)$ and $b_i(n)$ for $0 \leq i < k, n \geq 0$. Moreover, if $u(0)$ is integer, then $u$ is an integer sequence.

Hence, if the sequence polynomials $f_i(x) \in \mathbb{R}[X] \ (0 \leq i < k)$ are degree 1, we always assume $b_0(0) = N(1 - a_0(0))$ for some integer $N$, and we define $u(0) = 0$ if $a_0(0) = 1$ in this paper. In this case, the set $\mathcal{G}(f_0, f_1, \cdots, f_{k-1})$ always exists and has exactly one sequence $u = \{u(n)\}_{n \geq 0}$, and we denote the polynomial generated sequence $u$ by $\mathcal{G}(f_0, f_1, \cdots, f_{k-1})$ briefly.

**Example 1.**

1. The polynomial generated sequence $\mathcal{G}(x, -x + 1) = \{t(n)\}_{n \geq 0}$ is the famous Thue-Morse sequence.

2. Assume $k \geq 2$. The polynomial generated sequence

$$\mathcal{G}(x, x + 1, \cdots, x + k - 1) = \{s_k(n)\}_{n \geq 0}$$

is a $k$-regular sequence, where $s_k(n)$ is the sum of the digits in the base-$k$ representation of $n$.

So, given a generated polynomial system $\{f_i(x) \in \mathbb{R}[X] : 0 \leq i < k\}$, where the polynomials are degree 1, we can obtain a set of integer sequences $\mathcal{G}(f_0, f_1, \cdots, f_{k-1})$. Then, a problem appears. What can be said about properties of $\mathcal{G}(f_0, f_1, \cdots, f_{k-1})$?

If $f_i(x)$ are linear polynomials for $0 \leq i < k$ with $k \geq 2$, then it is easy to check that $\mathcal{G}(f_0, f_1, \cdots, f_{k-1})$ is a $k$-regular sequence. Moreover, if $f_i(x) = a_ix + b_i$, for $0 \leq i < k$, where $a_i$ are integers and $b_i$ are $k$-regular, then $\mathcal{G}(f_0, f_1, \cdots, f_{k-1})$ is also $k$-regular. The following theorem gives a general result.

**Theorem 1.** Assume $k \geq 2$ and $f_i(x) = a_ix + b_i$, for $0 \leq i < k$. If $a_i$ are $k$-automatic and $b_i$ are $k$-regular, then the polynomial generated sequence $\mathcal{G}(f_0, f_1, \cdots, f_{k-1})$ is $k$-regular.

Let $u = \{u(n)\}_{n \geq 0}$ be a sequence, then we define the shift map $S(u)$ to be the sequence $\{u(n+1)\}_{n \geq 0}$. Similarly, we have $S^k(u) = u(k)u(k+1)u(k+2) \cdots$ for $k \geq 0$. Let $f = a_0 + a_1x + \cdots + a_mx^m$ be a sequence polynomial. Then, we define the composition $f \circ S(u)$, to be the sequence $f(S(u))$.

**Definition 2.** Given an integer sequence $u = \{u(n)\}_{n \geq 0}$. If there exist integers $k \geq 1, N \geq 0$, polynomials $f_i(x) \in \mathbb{R}[X]$ and shifts $S_i \in \{S^j : j \geq 0\}$ for $0 \leq i < k$ such that

$$u(kn + i) = f_i \circ S_i(u)(n), \ (n \geq N)$$

then we say that $u$ is a polynomial generated sequence with shift.

The following theorem tells us that the shift do not change the regularity of the polynomial generated sequence.

**Theorem 2.** Assume $k \geq 2$ and $f_i(x) = a_ix + b_i$, for $0 \leq i < k$. If $a_i$ are $k$-automatic and $b_i$ are $k$-regular, then the polynomial generated sequences with shift are $k$-regular.

In particular, we have
Corollary 1. Let \( \{a_i(n)\}_{n \geq 0} \) and \( \{b_i(n)\}_{n \geq 0} \) be 2-automatic sequences for \( 0 \leq i < 2 \). Assume \( u = \{u(n)\}_{n \geq 0} \) is a sequence defined by
\[
\begin{align*}
u(2n) &= a_0(n)u(n) + b_0(n), & \nu(2n+1) &= a_1(n)u(n+1) + b_1(n).
\end{align*}
\]
Then the sequence \( u \) is 2-regular.

### 1.3. Hankel determinants of the characteristic sequence of the power of 2

Let \( u = \{u(n)\}_{n \geq 0} \) be a sequence of real numbers. For every integer \( k \geq 0 \), define a Hankel matrix \( u_{m,n}^k \) of order \( m \times n \) associated with \( u \) as follows:
\[
u_{m,n}^k = \begin{pmatrix}
u(k) & \nu(k+1) & \cdots & \nu(k+n-1) \\
\nu(k+1) & \nu(k+2) & \cdots & \nu(k+n) \\
\vdots & \vdots & \ddots & \vdots \\
\nu(k+m-1) & \nu(k+m) & \cdots & \nu(k+m+n-2) \\
\end{pmatrix}.
\]

Note that the rows of \( u_{m,n}^k \) are made up of successive length-\( n \) “windows” into the sequence \( u \). If \( m = n \), we always use the symbols \( u_{n}^k \) and \( |u_n^k| \) to stand for the \( n \)-order Hankel matrix and \( n \)-order Hankel determinant respectively.

Hankel determinants associated with a sequence play an important role in the study of the moment problem, Padé approximation, and the combinatorial properties of sequence \([5, 6, 14, 15, 16]\). Given a \( k \)-automatic integer sequence \( u = \{u(n)\}_{n \geq 0} \), we obtain a sequence of Hankel determinants \( |u_n^k| \). Note that the Hankel determinants \( |u_n^k| \) are determined by the block \( u(m)u(m+1) \cdots u(m+2n) \) for any fixed \( n \geq 1 \). And the block sequence \( \{u(m)u(m+1) \cdots u(m+2n)\}_{m \geq 0} \) is \( k \)-automatic. Hence, the determinant sequence \( \{|u_n^m|\}_{m \geq 0} \) is \( k \)-automatic, please see \([4]\).

There are some results about the automaticity of the Hankel determinant sequences. Allouche, Peyrière, Wen and Wen first studied the Hankel determinant of the Thue-Morse sequence \( t \) in \([4]\). They proved that the sequences \( \{|u_n^m|((\text{mod} 2))\}_{n \geq 0} \) are 2-automatic. In the same way, Wen, Wu \([13]\) and Guo, Wen \([13]\) respectively studied the the Hankel determinants of the Cantor sequence \( c \) and the differences of Thue-Morse \( \Delta^k(t) \). They proved that the sequences \( \{|e_n^m|((\text{mod} 3))\}_{n \geq 0} \) are 3-automatic, and the the sequences \( \{\Delta^k(t)^m_n\}_{n \geq 0} \) are 2-automatic.

Here, we point out that if \( \{u(n)\}_{n \geq 0} \) is \( k \)-regular over \( \mathbb{Z} \), then \( \{u(n)(\text{mod} m)\}_{n \geq 0} \) is \( k \)-automatic for any \( m \geq 1 \). But the converse does not hold. For example, the sequence \( \{2^n(\text{mod} m)\}_{n \geq 0} \) is \( k \)-automatic for any \( m \geq 1, k \geq 2 \), but the sequence \( \{2^n\}_{n \geq 0} \) is not \( k \)-regular for any \( k \geq 2 \), please see \([4]\). Hence, although there are many sequences are either \( k \)-automatic or \( k \)-regular in \([11]\), it is often quite challenging to determine the automaticity of the Hankel determinant sequences \( \{|u_n^m|\}_{n \geq 0} \).

In this paper, we consider the Hankel determinants of the characteristic sequence of powers \( 2 \)
\[
p = \{p(n)\}_{n \geq 0} = 0110100010000 \cdots,
\]
where \( p(n) = 1 \) if \( n = 2^k \) for some \( k \geq 0 \) and \( p(n) = 0 \) otherwise.

Let \( d(m,n) \) denote the \( |p_m^n| \) for \( m \geq n \geq 0 \). Using polynomial generated sequences, we prove that

**Theorem 3.** The sequence \( \{d(0,n)\}_{n \geq 0} \) is 2-regular; The sequences \( \{d(2k,n)\}_{n \geq 0} \) are \( 2 \)-automatic for all \( k \geq 1 \); The sequences \( \{d(2k+1,n)\}_{n \geq 0} \) are periodic for all \( k \geq 0 \).

The sequence \( p \) is a \( 2 \)-automatic sequence, more about the sequence \( p \), please see \([7]\). Let \( C_n = \frac{1}{n+1} \binom{2n}{n} \) be a Catalan number, then \( C_n(\text{mod} 2) = p(n+1) \) for all \( n \geq 0 \). Recently, using permutation, Cigler \([8]\) studied the Hankel determinants of the sequence \( \{C_n(\text{mod} 2)\}_{n \geq 0} = \{p(n+1)\}_{n \geq 0} \). By computer experiments, they have a conjecture which have been proved by us. We state it as follows.

**Theorem 4 (Cigler \([8]\)).** Assume \( m \) is an integer with \( 2^k < m \leq 2^{k+1} \) for some \( k \geq 1 \).

1. If \( m = 2r+1 \), then \( d(m,2^{k+1}n) = 1 \) and \( d(m,2^{k+1}n-m+1) = (-1)^r \),
2. If \( m = 2r \), then \( d(m,2^{k+1}n) = d(2,2^{k+1}n) \) and \( d(m,2^{k+1}n-m+1) = (-1)^{n+r}d(2,2^{k+1}n-m+1) \), where \( \{e_r\}_{r \geq 1} \) is a sequence over \( \{0,1\} \) defined by
\[
e_1 = 1, \quad e_{2r} = (e_r + r) \mod 2, \quad e_{2r+1} = e_{r+1}, \quad (r \geq 1).
\]

The paper is organized as follows. In Section \([2]\) we recall some notation briefly. In Section \([3]\) we give a proof of Theorem \([1]\) and Theorem \([2]\). In the last section, we study the Hankel determinants \( d(m,n) \) and answer this conjecture by a precise recurrence formula of the determinants \( d(m,n) \) for \( m \geq 0, n \geq 1 \). At last, we give a proof of Theorem \([8]\).
2. Preliminaries

In this section, we briefly recall some notation and theorems. More notation, please see \cite{3}.

In this paper, we always denote the set of non-negative integers by \( \mathbb{N} \) and denote the set of integers by \( \mathbb{Z} \). For any two sets \( A, B \) and number \( c \), define \( A + B = \{ a + b : a \in A, b \in B \} \), \( AB = \{ ab : a \in A, b \in B \} \) and \( cA = \{ ca : a \in A \} \). We define \( A^0 = 1 \) and \( A^1 = AA^{-1} \), \( A^{-1} = \emptyset \) for any \( i \geq 1 \). Assume \( f \) is a map defined on \( A \), then the image of \( A \) under \( f \) is denoted by \( f(A) \), i.e., \( f(A) = \{ f(a) : a \in A \} \). In particular, if \( f(A) \subseteq A \), then we say that the set \( A \) is invariant under the map \( f \).

We always call a finite set \( \Sigma \) alphabet and its elements letters. A word is made up of letters by the operation of concatenation. We denote the set of all finite words by \( \Sigma^* \), including empty word \( \epsilon \). Together with the operation of concatenation, \( \Sigma^* \) forms a free monoid. An infinite sequence, denote by \( u = \{ u(n) \}_{n \geq 0} = u(0)u(1)u(2) \cdots \), is a map from \( \mathbb{N} \) to \( \Sigma \). The set of all infinite sequences over \( \Sigma \) is denoted by \( \Sigma^\mathbb{N} \).

Among the infinite sequences, \( k \)-automatic and \( k \)-regular sequences satisfy a variety of useful properties. We recall some results from \cite{1, 2}.

**Theorem 5.** (\cite{1}, Theorem 2.3) A sequence is \( k \)-regular and takes on only finitely many values if and only if it is \( k \)-automatic.

**Theorem 6.** (\cite{1}, Theorem 2.5) Let \( \{ u(n) \}_{n \geq 0} \) and \( \{ v(n) \}_{n \geq 0} \) be \( k \)-regular sequences. Then so are \( \{ u(n) + v(n) \}_{n \geq 0} \), \( \{ u(n)v(n) \}_{n \geq 0} \) and \( \{ cu(n) \}_{n \geq 0} \) for any \( c \).

**Theorem 7.** (\cite{1}, Theorem 2.10) Let \( \{ u(n) \}_{n \geq 0} \) be a \( k \)-regular sequence with values in \( \mathbb{C} \), the set of complex numbers. Then there exists a constant \( c \) such that \( u(n) = O(n^c) \).

**Theorem 8.** (\cite{2}, Theorem 6) Let \( \{ u(n) \}_{n \geq 0} \) be a sequence with values in a Noetherian ring \( R \). Suppose there exist integers \( k \geq 2, t, r, n_0 \) such that each sequence \( \{ u(k^{j+1}n + j) \}_{n \geq n_0} \) for \( 0 \leq e < k^{j+1} \) is a linear combination of the sequences \( \{ u(k^m n + j) \}_{n \geq n_0} \) with \( 0 \leq i \leq t, 0 \leq j < k^i \) and the sequences \( \{ u(n + p) \}_{n \geq n_0} \) with \( 0 \leq p \leq r \). Then the sequence \( \{ u(n) \}_{n \geq 0} \) is \( k \)-regular.

3. Polynomial generated sequence

In this section, we study the polynomial generated sequences and give a proof of Theorem \cite{1} and Theorem \cite{2}.

Let \( f_i(x) = a_i x + b_i \) be sequence polynomials, where \( a_i = \{ a_i(n) \}_{n \geq 0}, b_i = \{ b_i(n) \}_{n \geq 0} \) for \( 0 \leq i < k \). Let

\[ A = K_k(a_0) \cup K_k(a_1) \cup \cdots \cup K_k(a_{k-1}) \]

and

\[ B = K_k(b_0) \cup K_k(b_1) \cup \cdots \cup K_k(b_{k-1}). \]

To prove Theorem \cite{1}, we need the following lemma.

**Lemma 1.** Let \( S \) be a finite set of integer sequences which take finitely many values, then the set \( \bigcup_{i \geq 0} S^i \) is finitely generated. Moreover, if the sequences of \( S \) take values in \( \{-1, 0, 1\} \), then \( \bigcup_{i \geq 0} S^i \) is a finite set.

**Proof.** Let \( u \) be an integer sequence over the alphabet \( \{ a_0, a_1, \ldots, a_M \} \), we define the characteristic sequences \( \chi_u^i \) for \( 0 \leq j \leq M \), by

\[ \chi_u^i(n) = \begin{cases} 1 & \text{if } u(n) = a_j, \\
0 & \text{otherwise}. \end{cases} \]

Then \( u \) is a \( \mathbb{Z} \)-linear combination of a finite set \( \{ \chi_u^0, \chi_u^1, \ldots, \chi_u^M \} \), i.e.,

\[ u = a_0 \chi_u^0 + a_1 \chi_u^1 + \cdots + a_M \chi_u^M. \]

Note that \( \chi_u^i \) are binary sequences, so we have \( (\chi_u^i)^k = \chi_u^i \) for all \( k \geq 1 \). Hence, \( u^k \) is a \( \mathbb{Z} \)-linear combination of at most \( 2^{M+1} \) sequences which are of the form

\[ (\chi_u^0)^{i_0} (\chi_u^1)^{i_1} \cdots (\chi_u^M)^{i_M}, \]

where \( i_j \in \{ 0, 1 \} \) for \( 0 \leq j \leq M \).

Now, assume \( S = \{ u_0, u_1, \ldots, u_N \} \). Then, for \( i \geq 0 \), we have

\[ S^i = \{ u_0^{n_0} u_1^{n_1} \cdots u_N^{n_N} : n_j \geq 0 \text{ for all } 0 \leq j \leq N \text{ and } n_0 + n_1 + \cdots + n_N = i \}. \]

So, each sequence \( u_i^{n_i} \) of \( S^i \) is a \( \mathbb{Z} \)-linear combination of a finite number of sequences. Hence, the set \( \bigcup_{i \geq 0} S^i \) is finitely generated.
In particular, if a sequence $\mathbf{u}$ takes values in $\{-1, 0, 1\}$, then $\mathbf{u}^k \in \{\mathbf{u}, \mathbf{u}^2\}$ for all $k \geq 1$. Hence, if the sequences of $\mathcal{S}$ take values in $\{-1, 0, 1\}$, then $\mathbf{u}_i^k \in \{\mathbf{u}, \mathbf{u}^2\}$ for all $k \geq 1$ for each $0 \leq i \leq N$. Thus, $\bigcup_{i \geq 0} S^i$ is a finite set.

**Proof of Theorem 1.** Assume $\mathbf{u} = \mathcal{G}(f_0, f_1, \cdots, f_{k-1})$. Then, we have $u(kn + i) = a_i(n)u(n) + b_i(n)$ for $0 \leq i < k, n \geq 0$. Hence,

$$K_k(\mathbf{u}) \subset \bigcup_{i \geq 0} \left( A^i \mathbf{u} + B \sum_{s=0}^{i-1} A^s \right).$$

Since the sequences $a_i = \{a_i(n)\}_{n \geq 0}$ are $k$-automatic and $b_i = \{b_i(n)\}_{n \geq 0}$ are $k$-regular for $0 \leq i < k$, we know that the set $A$ is finite and the set $B$ is finitely generated. Hence, by Lemma 1, $\bigcup_{i \geq 0} A^i \mathbf{u}$ and $\bigcup_{i \geq 0} B(\sum_{s=0}^{i-1} A^s)$ are finitely generated by some finite set. Thus, the $k$-kernel $K_k(\mathbf{u})$ is finitely generated, which implies that the polynomial generated sequence $\mathcal{G}(f_0, f_1, \cdots, f_{k-1})$ is $k$-regular.

Now, using Lemma 1, we prove Theorem 2.

**Proof of Theorem 2.** Let $\mathbf{u} = \{u(n)\}_{n \geq 0}$ be a polynomial generated sequence with polynomials $f_0, f_1, \cdots, f_{k-1}$ and shifts $S_i$ for $0 \leq i < k$. Since $S_i \in \{S^j : j \geq 0\}$, we assume $S_i = S^j$, where $j_i \geq 0$ are integers for $0 \leq i < k$. By definition 2, for any $0 \leq i < k$ and $n \geq N$, we have

$$(3.1) u(kn + i) = a_i(n)u(n) + b_i(n).$$

Note that there are at most $kN$ terms of $\mathbf{u}$ which do not satisfy above formula (3.1). For $0 \leq i < k$, define a ultimately periodic sequence $c_i = \{c_i(n)\}_{n \geq 0}$ by

$$c_i(n) = \begin{cases} u(kn + i) - a_i(n)u(n) - b_i(n), & 0 \leq n < N, \\ 0, & \text{otherwise.} \end{cases}$$

Then, for all $n \geq 0$,

$$u(kn + i) = a_i(n)u(n) + b_i(n) + c_i(n), \quad (0 \leq i < k).$$

Let $r = \max\{j_i : 0 \leq i < k\}$ and $Q$ is a fixed number satisfying $Q \geq \frac{k(1+r)}{k-1}$. Define sets

$$\mathcal{U} = \big\{\{u(n + t)\}_{n \geq 0} : 0 \leq t \leq Q\big\},$$

$$\mathcal{A}_Q = \bigcup_{s=0}^{k-1} \big\{a_s(k^i(n + t) + j)\}_{n \geq 0} : 0 \leq j < k^i, 0 \leq t \leq Q\big\},$$

$$\mathcal{B}_Q = \bigcup_{s=0}^{k-1} \big\{b_s(k^i(n + t) + j)\}_{n \geq 0} : 0 \leq j < k^i, 0 \leq t \leq Q\big\},$$

$$\mathcal{C}_Q = \bigcup_{s=0}^{k-1} \big\{c_s(k^i(n + t) + j)\}_{n \geq 0} : 0 \leq j < k^i, 0 \leq t \leq Q\big\}.$$

We claim that,

$$K_k(\mathbf{u}) \subset \bigcup_{i \geq 0} \left( A^i \mathcal{U} + (\mathcal{B}_Q + \mathcal{C}_Q) \sum_{s=0}^{i-1} A^s \right).$$

To prove this claim, we need some maps. For $0 \leq \ell < k$, define $\psi_\ell : \{u(n)\}_{n \geq 0} \rightarrow \{u(kn + \ell)\}_{n \geq 0}$. Then, we only need to show that $\psi_\ell(\mathcal{U}) \subset \mathcal{A}_Q + \mathcal{B}_Q + \mathcal{C}_Q$ and the sets $\mathcal{A}_Q, \mathcal{B}_Q$ and $\mathcal{C}_Q$ are invariant under the maps $\psi_\ell$.

For any $0 \leq t \leq Q$ and $0 \leq \ell < k$, assume $t + \ell = kn + y$, where $0 \leq y < k$. So, by the choice of $Q$ and note that $0 \leq j_y \leq r$, we have

$$0 \leq x \leq x + j_y \leq \frac{t + \ell}{k} + r \leq \frac{k-1+Q}{k} + r \leq Q.$$

Then, we have the following cases.

- $u(kn + \ell + t) = u(k(n + x) + y) = a_y(n + x)u(n + x + y) + b_y(n + x) + c_y(n + x)$, so we have $\psi_\ell(\mathcal{U}) \subset \mathcal{A}_Q + \mathcal{B}_Q + \mathcal{C}_Q$;
- For any $i \geq 0, 0 \leq j < k^i$, we have $a_i(k^i(n + \ell + t) + j) = a_i(k^i(n + x) + y) + j = a_i(k^i(n + x) + k^i'y + j)$. Note that $0 \leq k^i'y + j < k^{i+1}$, we have $\psi_\ell(\mathcal{A}_Q) \subset \mathcal{A}_Q$. In the same way, $\psi_\ell(\mathcal{B}_Q) \subset \mathcal{B}_Q$ and $\psi_\ell(\mathcal{B}_Q) \subset \mathcal{B}_Q$. 

If the sequences $a_i = \{a_i(n)\}_{n \geq 0}$ are $k$-automatic and $b_i = \{b_i(n)\}_{n \geq 0}$ are $k$-regular for $0 \leq i < k$, then the set $\mathcal{A}_Q$ is finite and the set $\mathcal{B}_Q$ is finitely generated. Note that the sets $\mathcal{U}$ and $\mathcal{C}_Q$ are finite, by Lemma 1 there exists a finite set $\mathcal{S}$ such that the sets

$$
\bigcup_{i \geq 0} \left( A'_Q \mathcal{U} + (B'_Q + C'_Q) \sum_{s=0}^{i-1} A'_Q \right)
$$

are $\mathbb{Z}$-linear combination of $\mathcal{S}$. Hence, the $k$-kernel $K_k(\mathcal{U})$ is finitely generated and our theorem follows. □

**Remark 1.** Note from the proof of Theorem 3 that the condition of Definition 1 can be weakened by $u(kn + i) = f_i(u(n))$ $(n \geq N)$ for some integer $N \geq 0$. Denote

$$
\mathcal{G}_N(f_0, f_1, \cdots, f_{k-1}) = \left\{ \{u(n)\}_{n \geq 0} : u(kn + i) = f_i(u(n)), n \geq N \text{ for } 0 \leq i \leq k \right\}.
$$

If $f_i = a_i x + b_i$, where $a_i$ are $k$-automatic and $b_i$ are $k$-regular for $0 \leq i \leq k$, then for every $N \geq 0$, the sequences of $\mathcal{G}_N(f_0, f_1, \cdots, f_{k-1})$ are $k$-regular.

We end this section by some examples. Example 2 implies that the condition that all polynomials are degree 1 in Theorem 1 is necessary. Example 3 shows that if we replace the $k$-automatic condition of the sequence $a_i$, by a $k$-regular condition in Theorem 1, then the polynomial generated sequence maybe not $k$-regular.

**Example 2.** The sequences in $\mathcal{G}(x^2, x + 1)$ are not 2-regular. Assume $\{u(n)\}_{n \geq 0} \in \mathcal{G}(x^2, x + 1)$, and $u(2n) = u(n)^2$, $u(2n + 1) = u(n) + 1$.

If $u(0) = 0$, then $u(3) = 2$. If $u(0) = 1$, then $u(2) = 2$. In either case, $a \in \{2, 3\}$, we have

$$
\frac{\log_2(u(a \cdot 2^k))}{\log_2(a \cdot 2^k)} = \frac{2^k \log_2(u(a))}{k(\log_2 a + 1)} = \frac{2^k}{(1 + \log_2 a)^k} \to \infty, \ (k \to \infty).
$$

Hence, by Theorem 1, the sequences in $\mathcal{G}(x^2, x + 1)$ are not 2-regular.

**Example 3.** The sequence $\mathcal{G}(vx, x + 1)$ is not 2-regular, where $v = \{n\}_{n \geq 0}$ is a 2-regular sequence. Assume $\{u(n)\}_{n \geq 0} \in \mathcal{G}(vx, x + 1)$, then

$$
\frac{\log_2(u(3 \cdot 2^k))}{\log_2(3 \cdot 2^k)} = \frac{k(k - 1)/2 + k \log_2 3 + 1}{k(\log_2 3 + 1)} \to \infty, \ (k \to \infty)
$$

which implies the sequence $\mathcal{G}(vx, x + 1)$ is not 2-regular, by Theorem 1.

**Example 4.** Let $t = \{t(n)\}_{n \geq 0} = 01101001 \cdots$ be the Thue-Morse sequence. Clearly that $t(2n) = t(n), t(2n + 1) = 1 - t(n)$ for all $n \geq 0$. Let $u = \{u(n)\}_{n \geq 0} = \mathcal{G}(tx + 1, x + t)$. Then, for $n \geq 0$,

$$
\begin{align*}
\ u(0) &= u(1) = 1, \\
\ u(4n) &= u(4n + 1) = u(2n) + u(2n + 1) - u(n), \\
\ u(4n + 2) &= -u(2n) + u(n) + 2, \\
\ u(4n + 3) &= u(n) + 1.
\end{align*}
$$

Hence, by Theorem 3, $\mathcal{G}(tx + 1, x + t)$ is a 2-regular sequence.

4. **Application**

In this section, we study the Hankel determinants $d(m, n)$ and obtain the recurrence formulae of the determinants $d(m, n)$ for $m \geq 0, n \geq 1$. By these formulae, we prove the conjecture of Cigler. Moreover, we find that the Hankel determinant sequence $\{d(0, n)\}_{n \geq 0}$ is a polynomial generated sequence with shift. Using Corollary 1 we give a proof of Theorem 3 at last.

Let $p = \{p(n)\}_{n \geq 0}$ be the characteristic sequence of powers 2. Recall that the sequence $\{p(n)\}_{n \geq 0}$ can be generated by the recurrence formula:

$$
p(0) = 0, p(1) = 1, p(2n) = p(n), \ p(2n + 1) = 0, \ (n \geq 1).
$$

Let $d(m, n) = |p_n^m| = \det(p(i + j + m))_{i,j = 0}^{n-1}$ for any $m \geq 0, n \geq 1$, then we have the following lemma which plays important role in this paper.

**Lemma 2.** For any $m \geq 1, n \geq 1$, we have
ON THE REGULARITY OF THE HANKEL DETERMINANT SEQUENCE OF THE CHARACTERISTIC SEQUENCE OF POWERS

(1) \(d(0, 2n) = d(0, n)d(1, n) - d(2, n - 1)d(3, n - 1)\),
(2) \(d(0, 2n + 1) = d(0, n + 1)d(1, n) - d(2, n)d(3, n - 1)\),
(3) \(d(1, 2n) = (-1)^nd^2(1, n)\),
(4) \(d(1, 2n + 1) = (-1)^nd^2(2, n)\),
(5) \(d(2m, 2n) = d(m, n)d(m + 1, n)\),
(6) \(d(2m, 2n + 1) = d(m, n + 1)d(m + 1, n)\),
(7) \(d(2m + 1, 2n) = (-1)^nd^2(m + 1, n)\),
(8) \(d(2m + 1, 2n + 1) = 0\).

Here, we define \(d(2, 0) = (3, 0) = 1\).

Proof. For each \(n\)-order square matrix \(M = (m_{i,j})_{1 \leq i, j \leq n}\), there exists a matrix \(U\) with \(|U| = \pm 1\) such that

\[
UMU^t = \begin{pmatrix}
(m_{2i-1,2j-1})_{1 \leq i, j \leq \mu} & (m_{2i-1,2j})_{1 \leq i \leq \mu, \nu < j} \\
(m_{2i,2j-1})_{1 \leq i \leq \mu, \nu < j} & (m_{2i,2j})_{1 \leq i \leq \nu, \mu < j}
\end{pmatrix},
\]

where \(\mu = \lfloor \frac{1}{2}(n + 1) \rfloor\) and \(\nu = \lfloor \frac{1}{2}n \rfloor\) and \(U^t\) denote the transposed matrix of \(U\).

(1) By Formula (4.1) and (4.2), we have

\[
U^p^0_{2n}U^t = \begin{pmatrix}
p^0_0 & A_{n,n} \\
A_{n,n} & p^0_n
\end{pmatrix}
\]

where \(A_{m,n} = (a_{ij})_{1 \leq i \leq m, 1 \leq j \leq n}\) denote the \(m \times n\) matrix with all entries are zero except \(a_{11} = 1\). Hence,

\[
|p^0_{2n}| = |p^0_0 + A_{n,n}p^0_n| = |p^0_0||p^1_n| - |p^1_{n-1}||p^3_{n-1}|.
\]

(2) By Formula (4.1) and (4.2), we have

\[
|p^0_{2n+1}| = |p^0_{n+1} + A_{n+1,n}p^0_n| = |p^0_{n+1}||p^1_n| - |p^1_{n-1}||p^3_{n-1}|.
\]

(3) By Formula (4.1) and (4.2), we have

\[
|p^0_{2n}| = |A_{n,n}p^0_n + p^1_{n,n}| = (-1)^n|p^1_n|^2.
\]

where \(0_{m,n}\) denote the \(m \times n\) matrix with all entries are zero.

(4) By Formula (4.1) and (4.2), we have

\[
|p^1_{2n+1}| = |A^1_{n+1,n+1}p^1_{n+1,n} + p^0_{n,n+1}| = (-1)^n|p^2_{n+1}|^2.
\]

(5) By Formula (4.1) and (4.2), we have

\[
|p^2_{2n}| = |p^m_m + m_{n,n+1}| = |p^m_m||p^m_{n+1}|.
\]

(6) By Formula (4.1) and (4.2), we have

\[
|p^2_{2n+1}| = |0_{n,n+1} + p^m_{n,n+1}| = (-1)^n|p^m_{n+1}|^2.
\]

(7) By Formula (4.1) and (4.2), we have

\[
|p^2_{2n+1}| = |0_{n,n+1} + p^m_{n,n+1}| = |p^m_{n+1}||p^m_{n+1}|.
\]

(8) By Formula (4.1) and (4.2), we have

\[
|p^2_{2n+1}| = |0_{n+1,n+1} + p^m_{n+1,n+1}| = 0.
\]

Remark 2. Define \(d(m,0) = 1, d(m,-1) = 0\) for all \(m \geq 0\), then Formulae of Lemma 2 hold for \(n \geq 0\).

Proposition 1. For any \(n \geq 1\), \(d(1,n), d(2,n) \in \{-1,1\}, d(m,n) \in \{-1,0,1\} (m \geq 3)\). Moreover, \(d(1,n) = (-1)^{\frac{n}{2}}\).
Proof. We first prove that \( d(1, n), d(2, n) \in \{-1, 1\} \) by induction on \( n \). It is easy to check that \( d(1, 1) = -d(1, 2) = -d(1, 3) = 1, d(2, 1) = d(2, 2) = -d(2, 3) = 1 \). Now, assume that \( d(1, n), d(2, n) \in \{-1, 1\} \) for all \( n < 2^k \) with \( k \geq 1 \). Then, for any \( 2^k \leq n < 2^{k+1} \), there exists an integer \( m < 2^k \) such that \( n = 2m \) or \( n = 2m + 1 \). Moreover,

- \( d(1, n) = d(1, 2m) = (-1)^m d^2(1, m) \in \{-1, 1\} \),
- \( d(1, n) = d(1, 2m + 1) = (-1)^m d^2(2, m) \in \{-1, 1\} \),
- \( d(2, n) = d(1, 2m) = d(1, m)d(2, m) \in \{-1, 1\} \),
- \( d(2, n) = d(1, 2m + 1) \in \{-1, 1\} \).

Hence, \( d(1, n), d(2, n) \in \{-1, 1\} \) for all \( n \geq 1 \).

Now, assume there exists an integer \( k \) such that \( d(m, n) \in \{-1, 0, 1\} \) for all \( m \leq 2^k, n \geq 1 \). We need to prove the conclusion hold for \( m \leq 2^k + 1 \). Since \( m = 2s \) or \( m = 2s + 1 \) for some \( s \leq 2^k \), by Lemma 2 and the hypothesis, we have

- \( d(m, 2n) = d(2s, 2n) = d(s, n)d(s + 1, n) \in \{-1, 0, 1\} \),
- \( d(m, 2n + 1) = d(2s, 2n + 1) = d(s, n + 1)d(s + 1, n) \in \{-1, 0, 1\} \),
- \( d(m, 2n) = d(2s + 1, 2n) = (-1)^s d^2(s + 1, n) \in \{-1, 0, 1\} \),
- \( d(m, 2n + 1) = d(2s + 1, 2n + 1) = 0 \in \{-1, 0, 1\} \).

Thus, \( d(m, n) \in \{-1, 0, 1\} \) for all \( m, n \geq 1 \).

By Lemma 2 it follows that \( d(1, n) = (-1)^{\lfloor \frac{n}{2} \rfloor} \), which completes this proof. \( \square \)

The following two propositions have been proved by Cigler in [8]. Here, we give another proofs of them. Our method which is different from Cigler mainly depends on the recurrence formulae. The first proposition gives a description of the sequence \( \{d(2, n)\}_{n \geq 0} \). The second proposition gives a description of the sequence \( \{d(m, n)\}_{n \geq 0} \) for all \( m \geq 3 \).

**Proposition 2.** If \( 2^k \leq n < 2^{k+1} \) for some integer \( k \geq 2 \), then

\[
d(2, n) = \begin{cases} 
-d(2, n - 2^k) & \text{if } 2^k \leq n < 2^k + 2^{k-1}, \\
-d(2, n - 2^k) & \text{if } 2^k + 2^{k-1} \leq n < 2^{k+1}.
\end{cases}
\]

**Proof.** For \( k = 2 \), the assertions above can be checked directly. Assume the proposition is true for \( k \leq N \). Now, we discuss the case \( k = N + 1 \).

If \( 2^{N+1} \leq n < 2^{N+1} + 2^N \) and \( n = 2^{N+1} + 2m \), then \( 0 \leq m < 2^{N-1} \). By Lemma 2 and the hypothesis, we have \( d(2, 2N + m) = -d(2, m) \) and

\[
d(2, n) = d(2, 2N + 2m) = d(2N + m) = -d(1, m)d(2, m) = -d(2, m).
\]

The other ones can be obtained by the same method. \( \square \)

**Remark 3.** In fact, Proposition 2 gives a generation method of the sequence \( \{d(2, n)\}_{n \geq 0} \). Let \( A_0 = 11, B_0 = 1 - 1, A_n = A_{n-1}B_{n-1}, B_n = A_{n-1}B_{n-1} \) \((n \geq 1)\), where the overbar is shorthand for the morphism that maps 1 to -1 and -1 to 1. Then,

\[
\{d(2, n)\}_{n \geq 0} = \lim_{n \to \infty} A_n = A_0B_0A_0B_0A_0B_0A_0B_0 \cdots = 111 - 1 - 11 - 1 - 1 - 1 - 11 \cdots.
\]

**Proposition 3.** If \( 2^k \leq n < 2^{k+1} \) for some integer \( k \geq 1 \), then

\[
d(m, n) = \begin{cases} 
\pm 1, & \text{if } n \equiv 0 \text{ or } 1 - m \pmod{2^{k+1}}, \\
0, & \text{otherwise.}
\end{cases}
\]

**Proof.** By (7), (8) of Lemma 2 and Proposition 1 we have

\[
d(3, 2n) = (-1)^{\lfloor \frac{n}{2} \rfloor} d^2(2, n) = (-1)^{\lfloor \frac{n}{2} \rfloor} d(3, 2n + 1) = 0.
\]

Then, by (5), (6) of Lemma 2 we have

- \( d(4, 4n) = d(2, 2n)d(3, 2n) = (-1)^n d(2, 2n) \),
- \( d(4, 4n + 1) = d(2, 2n + 1)d(3, 2n) = (-1)^n d(2, 2n + 1) \),
- \( d(4, 4n + 2) = d(2, 2n + 1)d(3, 2n + 1) = 0, \)
- \( d(4, 4n + 3) = d(2, 2n + 2)d(3, 2n + 1) = 0. \)
Hence, $d(3,n) \neq 0 \Leftrightarrow n \equiv 0$ or $2 \pmod{4}, d(4,n) \neq 0 \Leftrightarrow n \equiv 0$ or $1 \pmod{4}$, which implies that the conclusions hold for $k = 1$.

Now, assume the assertions hold for $k \leq N$, we need to prove the case $k = N + 1$. There are following cases to discuss.

- If $2^{N+1} < m < 2^{N+2}$ and $m = 2r$ for some integer $r$, then $2^N < r, r + 1 \leq 2^{N+1}$. By Lemma 2 and the hypothesis, we have
  
  $-d(2r, 2s) \neq 0 \Leftrightarrow d(r, s)d(r + 1, s) \neq 0 \Leftrightarrow s \equiv 0 \pmod{2^{N+1}} \Leftrightarrow 2s \equiv 0 \pmod{2^{N+2}}.

- $d(2r, 2s + 1) \neq 0 \Leftrightarrow d(r, s + 1)d(r + 1, s) \neq 0 \Leftrightarrow s \equiv -r \pmod{2^{N+1}} \Leftrightarrow 2s + 1 \equiv 1 - 2r \pmod{2^{N+2}}.$

- If $m = 2^{N+2}$ and $m = 2r$, then $r = 2^{N+1}, 2^{N+1} < r + 1 \leq 2^{N+2}$. We have $d(r + 1, s) \neq 0 \Leftrightarrow s \equiv 0$ or $2^{N+1} \pmod{2^{N+2}} \Leftrightarrow s \equiv 0 \pmod{2^{N+2}}$. Hence, by Lemma 2 and the hypothesis, we have
  
  $-d(2r, 2s) \neq 0 \Leftrightarrow d(r, s)d(r + 1, s) \neq 0 \Leftrightarrow s \equiv 0 \pmod{2^{N+1}} \Leftrightarrow 2s \equiv 0 \pmod{2^{N+2}}.$

- $d(2r, 2s + 1) \neq 0 \Leftrightarrow d(r, s + 1)d(r + 1, s) \neq 0 \Leftrightarrow s \equiv 0 \pmod{2^{N+1}} \Leftrightarrow 2s + 1 \equiv 1 - 2r \pmod{2^{N+2}}.$

- If $2^{N+1} < m \leq 2^{N+2}$ and $m = 2r + 1$ for some integer $r$, then $2^N < r + 1 \leq 2^{N+1}$. By Lemma 2 and the hypothesis, we have
  
  $d(2r + 1, 2s) \neq 0 \Leftrightarrow d(r + 1, s) \neq 0 \Leftrightarrow s \equiv 0$ or $-r \pmod{2^{N+1}} \Leftrightarrow 2s \equiv 0$ or $-2r \pmod{2^{N+2}}.$

Thus, if $2^{N+1} < m \leq 2^{N+2}$, then

$d(m, n) \neq 0 \Leftrightarrow n \equiv 0$ or $1 - m \pmod{2^{N+2}},$

which completes the proof. \hfill \square

Now, we give a proof of Theorem 4 which is an answer of Cigler’s conjecture.

Proof of Theorem 4. It is easy to check that the two assertions hold for $k = 1$. Assume the two assertions hold for $k \leq N \ (N \geq 1)$, it suffices to show that the assertions also hold for $k = N + 1$. There are three cases to discuss.

- If $2^{N+1} < m \leq 2^{N+2}$ and $m = 2r + 1$, then, by (7) of Lemma 2, we have
  
  $d(m, 2^{N+2}) = d(2r + 1, 2^{N+2}) = (-1)^{2^{N+1} - 1} = 1.$

- If $2^{N+1} < m \leq 2^{N+2}$ and $m = 4r + 2$, then $2^N < 2r + 1, 2r + 2 \leq 2^{N+1}$. Note that $d(2, 2^{N+2}) = d(2, 2^{N+1} + 1)$. By (5) of Lemma 2 and the hypothesis, we have
  
  $d(m, 2^{N+2}) = d(4r + 2, 2^{N+2}) = d(2r + 1, 2^{N+1})d(2r + 2, 2^{N+2}) = d(2, 2^{N+1} + 1) = d(2, 2^{N+2}).$

- Note that $d(2, 2^{N+4} - 4r - 1) = (-1)^r d(2, 2^{N+1} - 2r - 1)$ and $-2r - 1 = (-1)^{r+1}$. By (6) of Lemma 2 and the hypothesis, we have
  
  $d(m, 2^{N+4} - 4r - 1) = d(2, 2^{N+4} - 4r - 1) = d(2r + 1, 2^{N+1} + 1 - 2r) = (-1)^r (-1)^{r+1} d(2, 2^{N+1} - 2r - 1) = (-1)^{r+1} d(2, 2^{N+1} - 2r - 1).$

- If $2^{N+1} < m \leq 2^{N+2}$ and $m = 4r$, then $2^N < 2r \leq 2^{N+1}, 2r + 1 < 2r + 1 \leq 2^{N+1} + 1$.

- Note that $d(2, 2^{N+2}) = d(2, 2^{N+1})$. By (5) of Lemma 2 and the hypothesis, we have
  
  $d(m, 2^{N+2}) = d(2, 2^{N+2}) = d(2, 2^{N+1} + 1) = d(2, 2^{N+2}).$

- Note that $d(2, 2^{N+4} - 4r + 1) = (-1)^r d(2, 2^{N+1} - 2r) = d(2, 2^{N+1} - 2r + 1)$ and $2r - 1 = (-1)^{r+1}$. By (6) of Lemma 2 and the hypothesis, we have
  
  $d(m, 2^{N+4} - 4r + 1) = d(2, 2^{N+4} - 4r + 1) = d(2, 2^{N+1} - 2r + 1) = d(2, 2^{N+1} - 2r + 1) = (-1)^{r+1} d(2, 2^{N+1} - 2r + 1).$

Hence, our theorem follows. \hfill \square

By Lemma 2, Propositions 1, 2, and Theorem 4, we prove Theorem 5.
Proof of Theorem 3. Let \( a = \{d(1, n)\}_{n \geq 0}, b = \{d(2, n - 1)d(3, n - 1)\}_{n \geq 0} \) and \( c = \{d(2, n)d(3, n - 1)\}_{n \geq 0} \). By Proposition 12, we know that \( \{d(1, n)\}_{n \geq 0} \) is periodic with period 2 and \( \{d(2, n)\}_{n \geq 0} \) is 2-automatic. Hence, by Theorem 6 we know that \( a, b \) and \( c \) are 2-automatic sequences. By Lemma 2 the sequence \( \{d(0, n)\}_{n \geq 0} \) is a polynomial generated sequence with shift, as
\[
d(0, 2n) = a(n)d(0, n) - b(n), \quad d(0, 2n + 1) = a(n)d(0, n + 1) - c(n).
\]
By Corollary 1 the sequence \( \{d(0, n)\}_{n \geq 0} \) is 2-regular.

If \( m \geq 3 \) is odd, then by Proposition 3 and Theorem 4 we know that the sequence \( \{d(m, n)\}_{n \geq 0} \) is periodic. Moreover, \( 2^{k+1} \) is a period if \( 2^k < m \leq 2^{k+1} \). If \( m \geq 3 \) is even, then by Proposition 3 and Theorem 4 we know that the sequence \( \{d(m, n)\}_{n \geq 0} \) is 2-automatic. We completes this proof. \( \square \)

Remark 4. Note from Proposition 3 that \( d(m, n) \in \{-1, 0, 1\} \), we know that
\[
d^k(m, n) \in \{d(m, n), d^2(m, n)\}
\]
for any integer \( k \geq 0 \). By Lemma 2, it is easy to check directly that the 2-kernel of the sequence \( \{d(m, n)\}_{m \geq 1, n \geq 0} \) is finite and the 2-kernel of the sequence \( \{d(m, n)\}_{m \geq 0, n \geq 0} \) is finitely generated.

Hence, the two-dimensional sequence \( \{d(m, n)\}_{n \geq 1, n \geq 0} \) is 2-automatic and \( \{d(m, n)\}_{m \geq 0, n \geq 0} \) is 2-regular. An immediate consequence of a result of Salon in \( 17 \) is that the sequences \( \{d(m, n)\}_{n \geq 0} \) are 2-automatic for all \( m \geq 1 \). More about multidimensional automatic sequences and regular sequences, please see \( 3 \) \( 2 \).

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