The Trace Problem for Toeplitz Matrices and Operators and its Impact in Probability*

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Abstract: The trace approximation problem for Toeplitz matrices and its applications to stationary processes dates back to the classic book by Grenander and Szegö, *Toeplitz forms and their applications*. It has then been extensively studied in the literature.

In this paper we provide a survey and unified treatment of the trace approximation problem both for Toeplitz matrices and for operators and describe applications to discrete- and continuous-time stationary processes.

The trace approximation problem serves indeed as a tool to study many probabilistic and statistical topics for stationary models. These include central and non-central limit theorems and large deviations of Toeplitz type random quadratic functionals, parametric and nonparametric estimation, prediction of the future value based on the observed past of the process, etc.

We review and summarize the known results concerning the trace approximation problem, prove some new results, and provide a number of applications to discrete- and continuous-time stationary time series models with various types of memory structures, such as long memory, anti-persistent and short memory.

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1. Introduction

Toeplitz matrices and operators, which have great independent interest and a wide range of applications in different fields of science (economics, engineering, finance, hydrology, physics, signal processing, etc.), arise naturally in the context of stationary processes. This is because the covariance matrix (resp. operator) of a discrete-time (resp. continuous-time) stationary process is a truncated Toeplitz matrix (resp. operator) generated by the spectral density of that process, and vice versa, any non-negative summable function generates a Toeplitz matrix (resp. operator), which can be considered as a spectral density of some discrete-time (resp. continuous-time) stationary process, and therefore the corresponding truncated Toeplitz matrix (resp. operator) will be the covariance matrix (resp. operator) of that process.

Truncated Toeplitz matrices and operators are of particular importance, and serve as tools, to study many topics in the spectral and statistical analysis of discrete- and continuous-time stationary processes, such as central and non-central limit theorems and large deviations of Toeplitz type random quadratic forms and functionals, estimation of the spectral parameters and functionals, asymptotic expansions of the estimators, hypotheses testing about the spectrum,
prediction of the future value based on the observed past of the process, etc.
(see, e.g., [1] - [7], [9, 11, 12, 14, 15], [17] - [66], and references therein).

The present paper is devoted to the problem of approximation of the traces of
products of truncated Toeplitz matrices and operators generated by integrable
real symmetric functions defined on the unit circle (resp. on the real line). We discuss estimation of the corresponding errors, and describe applications to discrete- and continuous-time stationary time series models with various
types of memory structures (long-memory, anti-persistent and short-memory).

The paper contains a number of new theorems both for Toeplitz matrices and
operators, which are stated in Sections 2 and 3, respectively. Those in Section 3
involving Toeplitz operators are proved in Section 5. The corresponding theo-
rems of Section 2 involving Toeplitz matrices can be proved in a similar way
and hence are omitted. Section 4 concerns applications and also contains some
new results. These are proved within the section.

The paper is organized as follows. In the remainder of this section we state
the trace approximation problem and describe the statistical model. In Section 2 we
discuss the trace problem for Toeplitz matrices. Section 3 considers the
same problem for Toeplitz operators. Section 4 is devoted to some applications
of the trace problem to discrete- and continuous-time stationary processes. In
Section 5 we outline the proofs of Theorems 3.1 - 3.4. An appendix contains the
proofs of technical lemmas.

1.1. The Trace Approximation Problem

The problem of approximating traces of products of truncated Toeplitz matrices
and operators can be stated as follows.

Let \( \mathcal{H} = \{ h_1, h_2, \ldots, h_m \} \) be a collection of integrable real symmetric functions defined on the domain \( \Lambda \), where \( \Lambda = \mathbb{R} := (-\infty, \infty) \) or \( \Lambda = \mathbb{T} := (-\pi, \pi] \),
and let \( A_T(h_k) \) denote either the \( T \)-truncated Toeplitz operator, or the \( T \times T \)
Toeplitz matrix generated by function \( h_k \) (the corresponding definitions are
given below). Further, let

\[
\tau := \{ \tau_k : \tau_k \in \{-1, 1\}, \ k = 1, m \}
\]

be a given sequence of \( \pm 1 \)'s. Define

\[
S_{\mathcal{A}, \mathcal{H}, \tau}(T) := \frac{1}{T} \mathsf{tr} \left[ \prod_{k=1}^{m} \{ A_T(h_k) \}^{\tau_k} \right],
\]

where \( \mathsf{tr}[A] \) stands for the trace of \( A \),

\[
M_{\mathcal{A}, \mathcal{H}, \tau} := (2\pi)^{m-1} \int_{\Lambda} \prod_{k=1}^{m} [h_k(\lambda)]^{\tau_k} d\lambda,
\]

and

\[
\Delta_{\mathcal{A}, \mathcal{H}, \tau}(T) := |S_{\mathcal{A}, \mathcal{H}, \tau}(T) - M_{\mathcal{A}, \mathcal{H}, \tau}|.
\]
The problem is to approximate $S_{A,H,\tau}(T)$ by $M_{\Lambda,H,\tau}$ and estimate the error rate for $\Delta_{A,H,\tau}(T)$ as $T \to \infty$. More precisely, for a given sequence $\tau = \{\tau_k \in \{-1,1\}, k = 1, m\}$ find conditions on functions $\{h_k(\lambda), k = 1, m\}$ such that:

Problem (A) : $\Delta_{A,H,\tau}(T) = o(1)$ as $T \to \infty$, or
Problem (B) : $\Delta_{A,H,\tau}(T) = O(T^{-\gamma})$, $\gamma > 0$, as $T \to \infty$.

The trace approximation problem goes back to the classical monograph by Grenander and Szegö [37], and has been extensively studied in the literature (see, e.g., Kac [46], Rosenblatt [54], [55], Ibragimov [42], Taniguchi [58], Giraitis and Surgailis [34], Ginovyan [21], Taniguchi and Kakizawa [60], Lieberman and Phillips [48], Giraitis et al. [33], Ginovyan and Sahakyan [29]-[32], and references therein).

In this paper we review and summarize the known results concerning Problems (A) and (B), prove some new results, as well as provide a number of applications to discrete- and continuous-time stationary time series models that have various types of memory structures (short-, intermediate-, and long-memory).

We focus on the following special case which is important from an application viewpoint, and is commonly discussed in the literature: $m = 2\nu$, $\tau_k = 1$, $k = 1, m$ (or $\tau_k = (-1)^k$, $k = 1, m$), and

\begin{align*}
    h_1(\lambda) &= h_3(\lambda) = \cdots = h_{2\nu-1}(\lambda) := f(\lambda) \\
    h_2(\lambda) &= h_4(\lambda) = \cdots = h_{2\nu}(\lambda) := g(\lambda). 
\end{align*} 

(1.5)

Throughout the paper the letters $C$ and $c$, with or without index, are used to denote positive constants, the values of which can vary from line to line. Also, all functions defined on $\mathbb{T}$ are assumed to be $2\pi$-periodic and periodically extended to $\mathbb{R}$.

1.2. The Model: Short, Intermediate and Long Memory Processes.

Let $\{X(u), u \in \mathbb{U}\}$ be a centered, real-valued, continuous-time or discrete-time second-order stationary process with covariance function $r(u)$, possessing a spectral density function $f(\lambda)$, $\lambda \in \Lambda$, that is,

\[ E[X(u)] = 0, \quad r(u) = E[X(t+u)X(t)], \quad u, t \in \mathbb{U}, \]

and $r(u)$ and $f(\lambda)$ are connected by the Fourier integral:

\[ r(u) = \int_{\Lambda} \exp\{i\lambda u\} f(\lambda) d\lambda, \quad u \in \mathbb{U}. \]

(1.6)

Thus, the covariance function $r(u)$ and the spectral density function $f(\lambda)$ are equivalent specifications of second order properties for a stationary process $X(u)$.

The time domain $\mathbb{U}$ is the real line $\mathbb{R}$ in the continuous-time case, and the set of integers $\mathbb{Z}$ in the discrete-time case. The frequency domain $\Lambda$ is $\mathbb{R}$ in the
continuous-time case, and $\Lambda = \mathbb{T} = (-\pi, \pi]$ in the discrete-time case. In the continuous-time case the process $X(u)$ is also assumed measurable and mean-square continuous: $\mathbb{E}[X(t) - X(s)]^2 \to 0$ as $t \to s$.

The statistical and spectral analysis of stationary processes requires two types of conditions on the spectral density $f(\lambda)$. The first type controls the singularities of $f(\lambda)$, and involves the dependence (or memory) structure of the process, while the second type - controls the smoothness of $f(\lambda)$.

**Dependence (memory) structure of the model.** We will distinguish the following types of stationary models:

(a) short memory (or short-range dependent),

(b) long memory (or long-range dependent),

(c) intermediate memory (or anti-persistent).

The memory structure of a stationary process is essentially a measure of the dependence between all the variables in the process, considering the effect of all correlations simultaneously. Traditionally memory structure has been defined in the time domain in terms of decay rates of long-lag autocorrelations, or in the frequency domain in terms of rates of explosion of low frequency spectra (see, e.g., Beran [7], Guegan [38], Robinson [52], and references therein).

It is convenient to characterize the memory structure in terms of the spectral density function.

**Short-memory models.** Much of statistical inference is concerned with short-memory stationary models, where the spectral density $f(\lambda)$ of the model is bounded away from zero and infinity, that is, there are constants $C_1$ and $C_2$ such that

$$0 < C_1 \leq f(\lambda) \leq C_2 < \infty.$$ 

A typical short memory model example is the stationary Autoregressive Moving Average (ARMA)($p,q$) process whose covariance $r(u)$ is exponentially bounded:

$$|r(k)| \leq Cr^{-k}, \quad k = 1, 2, \ldots; \quad 0 < C < \infty; \quad 0 < r < 1,$$

and the spectral density is a rational function.

**Discrete-time long-memory and anti-persistent models.** Data in many fields of science (economics, finance, hydrology, etc.), however, is well modeled by a stationary process with unbounded or vanishing spectral density (see, e.g., Beran [7], Guegan [38], Palma [50], Taqqu [61] and references therein).

A long-memory model is defined to be a stationary process with unbounded spectral density, and an anti-persistent model – a stationary process with vanishing spectral density.

In the discrete context, a basic long-memory model is the Autoregressive Fractionally Integrated Moving Average (ARFIMA)($0,d,0$) process $X(t)$ defined by

$$(1 - B)^dX(t) = \varepsilon(t), \quad 0 < d < 1/2,$$
Many applied problems, is an ARFIMA(\(p,d,q\)) process with spectral density \(f(\lambda) = |1 - e^{i\lambda}|^{-2d} = (2\sin(\lambda/2))^{-2d}\), \(0 < \lambda \leq \pi\), \(0 < d < 1/2\).

A typical example of an anti-persistent model is the ARFIMA(0,\(d,0\)) process \(X(t)\) with spectral density \(f(\lambda) = |1 - e^{i\lambda}|^{-2d}\) with \(d < 0\).

Note that the condition \(d < 1/2\) ensures that \(\int_{-\pi}^{\pi} f(\lambda)d\lambda < \infty\), implying that the process \(X(t)\) is well defined because \(E[|X(t)|^2] = \int_{-\pi}^{\pi} f(\lambda)d\lambda\).

Data can also occur in the form of a realization of a "mixed" short-long-intermediate-memory stationary process \(X(t)\) with spectral density

\[
f(\lambda) = f_I(\lambda)f_L(\lambda)f_S(\lambda),
\]

where \(f_I(\lambda)\), \(f_L(\lambda)\) and \(f_S(\lambda)\) are the intermediate, long- and short-memory components, respectively. A well-known example of such process, which appears in many applied problems, is an ARFIMA(\(p, d, q\)) process with spectral density

\[
f(\lambda) = |1 - e^{i\lambda}|^{-2d}h(\lambda), \quad d < 1/2,
\]

where \(h(\lambda)\) is the spectral density of an ARMA(p,\(q\)) process. Observe that for \(0 < d < 1/2\) the model \(X(t)\) specified by (1.8) displays long-memory, for \(d < 0\) - intermediate-memory, and for \(d = 0\) - short-memory. For \(d \geq 1/2\) the function \(f(\lambda)\) is not integrable, and thus it cannot represent a spectral density of a stationary process. Also, if \(d \leq -1\), then the series \(X(t)\) is not invertible in a sense that it cannot be used to recover a white noise \(\varepsilon(t)\) by passing \(X(t)\) through a linear filter (see \([9, 11]\)).

The ARFIMA(p,\(d, q\)) processes, first introduced by Granger and Joyeux \([36]\), and Hosking \([41]\), became very popular due to their ability in providing a good characterization of the long-run properties of many economic and financial time series. They are also very useful for modeling multivariate time series, since they are able to capture a larger number of long term equilibrium relations among economic variables than the traditional multivariate ARIMA models (see, e.g., Henry and Zaffaroni \([40]\) for a survey on this topic).

Another important long-memory model is the fractional Gaussian noise (fGn). To define fGn first consider the fractional Brownian motion (fBm) \(\{B(t) := B_H(t), t \in \mathbb{R}\}\) with Hurst index \(H\), \(0 < H < 1\), defined to be a centered Gaussian \(H\)-self-similar process having stationary increments, that is, \(B_H(t)\) satisfies the following conditions:

(a) \(\mathbb{E}[B(t)] = 0\);
(b) \(B(at) \overset{d}{=} a^{2H}B(t)\) for any \(a > 0\) and \(t \in \mathbb{R}\);
(c) \(B(t + u) - B(u) \overset{d}{=} B(t) - B(0)\) for any \(t, u \in \mathbb{R}\);
(d) the covariance function is given by

\[
\text{Cov}(B(s), B(t)) = \frac{\sigma^2}{2} \left[ |t|^{2H} - |s|^{2H} - |t - s|^{2H} \right],
\]
where $\sigma^2_0 = \text{Var} B(1)$, and where the symbol $\overset{d}{=} \equiv$ stands for equality of the finite-dimensional distributions. Then the increment process
\[ \{X(k) := B_H(k + 1) - B_H(k), k \in \mathbb{Z}\}, \]
called fractional Gaussian noise (fGn), is a discrete-time centered Gaussian stationary process with covariance function
\[ r(t) = \sigma^2_0 \left\lfloor |k + 1|^{2H} - |k|^{2H} - |k - 1|^{2H} \right\rfloor, \quad k \in \mathbb{Z} \quad (1.9) \]
and spectral density function
\[ f(\lambda) = c|1 - e^{i\lambda}|^2 \sum_{k=-\infty}^{\infty} |\lambda + 2\pi k|^{-(2H+1)}, \quad -\pi \leq \lambda \leq \pi, \quad (1.10) \]
where $c$ is a positive constant.

It follows from (1.10) that $f(\lambda) \sim c|\lambda|^{1-2H}$ as $\lambda \to 0$, that is, $f(\lambda)$ blows up if $H > 1/2$ and tends to zero if $H < 1/2$. Also, comparing (1.7) and (1.10), we observe that, up to a constant, the spectral density of fGn has the same behavior at the origin as ARFIMA$(0,d,0)$ with $d = H - 1/2$.

Thus, the fGn $\{X(k), k \in \mathbb{Z}\}$ has long-memory if $1/2 < H < 1$ and is anti-percipient if $0 < H < 1/2$. The variables $X(k)$’s are independent if $H = 1/2$. For more details we refer to Samorodnisky and Taqqu [57] and Taqqu [61].

**Continuous-time long-memory and anti-persistent models.** In the continuous context, a basic process which has commonly been used to model long-range dependence is fractional Brownian motion (fBm) $B_H$ with Hurst index $H$, defined above. It can be regarded as Gaussian process having a spectral density:
\[ f(\lambda) = c|\lambda|^{-(2H+1)}, \quad c > 0, \quad 0 < H < 1, \quad \lambda \in \mathbb{R}, \quad (1.11) \]
where (1.11) can be understood in a generalized sense since the fBm $B_H$ is a nonstationary process (see, e.g., Anh et al. [3] and Gao et al. [20]).

A proper stationary model in lieu of fBm is the fractional Riesz-Bessel motion (fRbm), introduced in Anh et al. [2], and defined as a continuous-time Gaussian process $X(t)$ with spectral density
\[ f(\lambda) = c|\lambda|^{-2\alpha}(1 + \lambda^2)^{-\beta}, \quad \lambda \in \mathbb{R}, \quad 0 < c < \infty, \quad 0 < \alpha < 1, \quad \beta > 0. \quad (1.12) \]
The exponent $\alpha$ determines the long-range dependence, while the exponent $\beta$ indicates the second-order intermittency of the process (see, e.g., [3, 20]).

Notice that the process $X(t)$, specified by (1.12), is stationary if $0 < \alpha < 1/2$ and is non-stationary with stationary increments if $1/2 \leq \alpha < 1$. Observe also that the spectral density (1.12) behaves as $O(|\lambda|^{-2\alpha})$ as $|\lambda| \to 0$ and as $O(|\lambda|^{-2(\alpha+\beta)})$ as $|\lambda| \to \infty$. Thus, under the conditions $0 < \alpha < 1/2$, $\beta > 0$ and $\alpha + \beta > 1/2$, the function $f(\lambda)$ in (1.12) is well-defined for both $|\lambda| \to 0$ and $|\lambda| \to \infty$ due to the presence of the component $(1 + \lambda^2)^{-\beta}$, $\beta > 0$, which is the Fourier transform of the Bessel potential.

Comparing (1.11) and (1.12), we observe that the spectral density of fBm is the limiting case as $\beta \to 0$ that of fRbm with Hurst index $H = \alpha - 1/2$. 

1.3. A Link Between Stationary Processes and the Trace Problem.

As it was mentioned above, Toeplitz matrices and operators arise naturally in the theory of stationary processes, and serve as tools, to study many topics of the spectral and statistical analysis of discrete- and continuous-time stationary processes.

To understand the relevance of the trace approximation problem to stationary processes, consider a question concerning the asymptotic distribution (as \( T \to \infty \)) of the following Toeplitz type quadratic functionals of a Gaussian stationary process \( \{X(u), u \in U\} \) with spectral density \( f(\lambda) \), \( \lambda \in \Lambda \) and covariance function \( r(t) := \hat{f}(t), t \in U \) (here \( U \) and \( \Lambda \) are as in Section 1.2):

\[
Q_T := \begin{cases} 
\int_0^T \int_0^T \hat{g}(t-s)X(t)X(s) \, dt \, ds & \text{in the continuous-time case} \\
\sum_{k=1}^T \sum_{j=1}^T \hat{g}(k-j)X(k)X(j) & \text{in the discrete-time case},
\end{cases}
\]

(1.13)

where \( \hat{g}(t) = \int_{\Lambda} e^{i\lambda t} g(\lambda) \, d\lambda \), \( t \in U \) is the Fourier transform of some real, even, integrable function \( g(\lambda), \lambda \in \Lambda \). We will refer \( g(\lambda) \) as a generating function for the functional \( Q_T \).

The limit distributions of the functionals (1.13) are completely determined by the spectral density \( f(\lambda) \) and the generating function \( g(\lambda) \), and depending on their properties the limit distributions can be either Gaussian (that is, \( Q_T \) with an appropriate normalization obeys central limit theorem), or non-Gaussian. The following two questions arise naturally:

a) Under what conditions on \( f(\lambda) \) and \( g(\lambda) \) will the limits be Gaussian?

b) Describe the limit distributions, if they are non-Gaussian.

These questions will be discussed in detail in Section 4.2.

Let \( A_T(f) \) be the covariance matrix (or operator) of the process \( \{X(u), u \in U\} \), that is, \( A_T(f) \) denote either the \( T \times T \) Toeplitz matrix, or the \( T \)-truncated Toeplitz operator generated by the spectral density \( f \), and let \( A_T(g) \) denote either the \( T \times T \) Toeplitz matrix, or the \( T \)-truncated Toeplitz operator generated by the function \( g \).

Our study of the asymptotic distribution of the quadratic functionals (1.13) is based on the following well-known results (see, e.g., [37, 42]):

1. The quadratic functional \( Q_T \) in (1.13) has the same distribution as the sum \( \sum_{k=1}^\infty \lambda_k^2 \xi_k^2 \) (\( \sum_{k=1}^T \lambda_k^2 \xi_k^2 \) in the discrete-time case), where \( \{\xi_k, k \geq 1\} \) are independent \( N(0, 1) \) Gaussian random variables and \( \{\lambda_k, k \geq 1\} \) are the eigenvalues of the operator \( A_T(f)A_T(g) \). (Observe that the sets of non-zero eigenvalues of the operators \( A_T(f)A_T(g) \), \( A_T(g)A_T(f) \) and \( A_T^{1/2}(f)A_T(g)A_T^{1/2}(f) \) coincide, where \( A_T^{1/2}(f) \) denotes the positive definite square root of \( A_T(f) \)).

2. The characteristic function \( \varphi(t) \) of \( Q_T \) is given by

\[
\varphi(t) = \prod_{k=1}^\infty |1 - 2it\lambda_k|^{-1/2},
\]

(1.14)
3. The $k$–th order cumulant $\chi_k(\cdot)$ of $Q_T$ is given by

$$\chi_k(Q_T) = 2^{k-1}(k-1)! \sum_{j=1}^{\infty} \lambda_j^k = 2^{k-1}(k-1)! \text{tr} [A_T(f)A_T(g)]^k$$  \hspace{1cm} (1.15)

Thus, to describe the asymptotic distributions of the quadratic functionals (1.13), we have to control the corresponding traces of the products of Toeplitz matrices(or operators), yielding the trace approximation problem with generating functions specified by (1.5).

2. The Trace Problem for Toeplitz Matrices

Let $f(\lambda)$ be an integrable real symmetric function defined on $T = (-\pi, \pi]$. For $T = 1, 2, \ldots$ denote by $B_T(f)$ the $(T \times T)$ Toeplitz matrix generated by function $f$, that is,

$$B_T(f) := \| \hat{f}(s-t) \|_{s,t=1,T} = \begin{pmatrix}
\hat{f}(0) & \hat{f}(-1) & \ldots & \hat{f}(1-T) \\
\hat{f}(1) & \hat{f}(0) & \ldots & \hat{f}(2-T) \\
\vdots & \vdots & \ddots & \vdots \\
\hat{f}(T-1) & \hat{f}(T-2) & \ldots & \hat{f}(0)
\end{pmatrix} ,$$  \hspace{1cm} (2.1)

where

$$\hat{f}(t) = \int_{\pi} e^{i\lambda t} f(\lambda) d\lambda, \quad t \in \mathbb{Z},$$  \hspace{1cm} (2.2)

are the Fourier coefficients of $f$.

Observe that

$$\frac{1}{T} \text{tr} [B_T(f)] = \frac{1}{T} T \hat{f}(0) = \int_{-\pi}^{\pi} f(\lambda) d\lambda.$$  \hspace{1cm} (2.3)

What happens when the matrix $B_T(f)$ is replaced by a product of Toeplitz matrices? Observe that the product of Toeplitz matrices is not a Toeplitz matrix.

The idea is to approximate the trace of the product of Toeplitz matrices by the trace of a Toeplitz matrix generated by the product of the generating functions. More precisely, let $\mathcal{H} = \{ h_1, h_2, \ldots, h_m \}$ be a collection of integrable real symmetric functions defined on $T$. Define

$$S_{B,\mathcal{H}}(T) := \frac{1}{T} \text{tr} \left[ \prod_{i=1}^{m} B_T(h_i) \right] , \quad M_{T,\mathcal{H}} := (2\pi)^{m-1} \int_{-\pi}^{\pi} \left[ \prod_{i=1}^{m} h_i(\lambda) \right] d\lambda ,$$  \hspace{1cm} (2.4)

and let

$$\Delta(T) := \Delta_{B,T,\mathcal{H}}(T) = |S_{B,\mathcal{H}}(T) - M_{T,\mathcal{H}}| .$$  \hspace{1cm} (2.5)

Observe that by (2.3)

$$M_{T,\mathcal{H}} = (2\pi)^{m-1} \int_{-\pi}^{\pi} \left[ \prod_{i=1}^{m} h_i(\lambda) \right] d\lambda = \frac{1}{T} \text{tr} \left[ B_T \left( \prod_{i=1}^{m} h_i(\lambda) \right) \right] .$$  \hspace{1cm} (2.6)
How well is $S_{B,H}(T)$ approximated by $M_{T,H}$? What is the rate of convergence to zero of approximation error $\Delta_{B,T,H}(T)$ as $T \to \infty$? These are Problems (A) and (B).

2.1. Problem (A) for Toeplitz Matrices

Recall that Problem (A) involves finding conditions on the functions $h_1(\lambda), h_2(\lambda), \ldots, h_m(\lambda)$ in (2.4) such that $\Delta_{B,T,H}(T) = o(1)$ as $T \to \infty$.

In Theorem 2.1 and Remark 2.2 we summarize the results concerning Problems (A) for Toeplitz matrices in the case where the exponents $\tau_k$, $k = 1, \ldots, m$ (see (1.2)) are all equal to 1 as in (2.4).

Theorem 2.1. Let $\Delta_{B,T,H}(T)$ be as in (2.5). Each of the following conditions is sufficient for

$$\Delta_{B,T,H}(T) = o(1) \quad \text{as} \quad T \to \infty. \quad (2.7)$$

(A1) $h_i \in L^{p_i}(\mathbb{T}), \; 1 \leq p_i \leq \infty, \; i = 1, m$, with $1/p_1 + \ldots + 1/p_m \leq 1$.

(A2) The function $\varphi(u)$ given by

$$\varphi(u) := \int_{-\pi}^{\pi} h_1(\lambda)h_2(\lambda - u_1)h_3(\lambda - u_2)\cdots h_m(\lambda - u_{m-1}) d\lambda, \quad (2.8)$$

where $u = (u_1, u_2, \ldots, u_{m-1}) \in \mathbb{R}^{m-1}$, belongs to $L^{m-2}(\mathbb{T}^{m-1})$ and is continuous at $0 = (0, 0, \ldots, 0)$.

Remark 2.1. Assertion (A1) was proved by Avram [4]. For the special case $p_i = \infty, \; i = 1, m$, that is, when all $h_i$ are bounded functions, it was first established by Grenander and Szegő ([37], Sec. 7.4). For $m = 4; \; p_1 = p_3 = 2; \; p_2 = p_4 = \infty$, (A1) was proved by Ibragimov [42] and Rosenblatt [54].

Assertion (A2), for $m = 4, \; h_1 = h_3 := f$ and $h_2 = h_4 := g$ was proved in Ginovyan and Sahakyan [29].

Remark 2.2. For the special case $m = 4, \; h_1 = h_3 := f$ and $h_2 = h_4 := g$, in Giraitis and Surgailis [34] (see also Giraitis et al. [33]), and in Ginovyan and Sahakyan [29], it was proved that the following conditions are also sufficient for (2.7):

(A3) (Giraitis and Surgailis [34]). $f \in L^2(\mathbb{T}), \; g \in L^2(\mathbb{T}), \; fg \in L^2(\mathbb{T})$ and

$$\int_{-\pi}^{\pi} f^2(\lambda)g^2(\lambda - \mu) d\lambda \longrightarrow \int_{-\pi}^{\pi} f^2(\lambda)g^2(\lambda) d\lambda \quad \text{as} \quad \mu \to 0.$$ 

(A4) (Ginovyan and Sahakyan [29]). The functions $f$ and $g$ satisfy

$$f(\lambda) \leq |\lambda|^{-\alpha}L_1(\lambda) \quad \text{and} \quad |g(\lambda)| \leq |\lambda|^{-\beta}L_2(\lambda) \quad \text{for} \quad \lambda \in [-\pi, \pi],$$

for some $\alpha < 1, \; \beta < 1$ with $\alpha + \beta \leq 1/2$, and $L_i \in SV(\mathbb{R}), \; \lambda^{-\alpha-\beta}L_i(\lambda) \in L^2(\mathbb{T}), \; i = 1, 2$, where $SV(\mathbb{R})$ is the class of slowly varying at zero functions $u(\lambda), \; \lambda \in \mathbb{R}$, namely $\lim_{\lambda \to 0} \frac{u(\lambda\alpha)}{\lambda^\alpha} = 1$ for all $\alpha > 0$, satisfying also $u(\lambda) \in L^\infty(\mathbb{R}), \lim_{\lambda \to 0} u(\lambda) = 0, \; u(\lambda) = u(-\lambda)$ and $0 < u(\lambda) < u(\mu)$ for $0 < \lambda < \mu$.  

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Remark 2.3. Case (A4), with $\alpha + \beta < 1/2$, was first obtained by Fox and Taqqu [17].

Remark 2.4. It would be of interest to extend the results of (A3) and (A4) to arbitrary $m > 4$.

We now consider the case when the product in (1.2) involves also inverse matrices, that is, $\tau_k = (-1)^k$, $k = 1, m$. We assume that $m = 2\nu$, and the functions from the collection $\mathcal{H} = \{h_1, h_2, \ldots, h_m\}$ that involve Toeplitz matrices are continuous at all $\lambda \in T \setminus \{0\}$, $f^{-1}_i(\lambda)$ are continuous at all $\lambda \in T$, and $f_i(\lambda) = O\left(|\lambda|^{-\alpha-\delta}\right)$ as $\lambda \to 0$ for all $\delta > 0$ and $i = 1, \nu$.

(C1) $f_i(\lambda)$, $i = 1, \nu$, are nonnegative and continuous at all $\lambda \in T \setminus \{0\}$, $f^{-1}_i(\lambda)$ are continuous at all $\lambda \in T$, and $f_i(\lambda) = O\left(|\lambda|^{-\alpha-\delta}\right)$ as $\lambda \to 0$ for all $\delta > 0$ and $i = 1, \nu$.

(C2) $\frac{\partial}{\partial \lambda} f^{-1}_i(\lambda)$ and $\frac{\partial^2}{\partial \lambda^2} f^{-1}_i(\lambda)$ are continuous at all $\lambda \in T \setminus \{0\}$, and for all $\delta > 0$ and $i = 1, \nu$,

\[
\frac{\partial^k}{\partial \lambda^k} f^{-1}_i(\lambda) = O\left(|\lambda|^{-\alpha-k-\delta}\right) \quad \text{for} \quad k = 0, 1.
\]

(C3) $g_i(\lambda)$ are continuous at all $\lambda \in T \setminus \{0\}$ and $g_i(\lambda) = O\left(|\lambda|^{-\beta-\delta}\right)$ as $\lambda \to 0$ for all $\delta > 0$ and $i = 1, \nu$.

Then

\[
\Delta I_{B,T,H}(T) = o(1) \quad \text{as} \quad T \to \infty.
\]

2.2. Problem (B) for Toeplitz Matrices

Recall that Problem (B) involves finding conditions on the functions $h_1(\lambda)$, $h_2(\lambda)$, $h_m(\lambda)$ in (2.4) such that $\Delta_{B,T,H}(T) = O(T^{-\gamma})$ as $T \to \infty$ for some $\gamma > 0$.

In Theorem 2.3 below we summarize the results concerning Problem (B) for Toeplitz matrices in the case where $\tau_k = 1$, $k = 1, m$. First we introduce some classes of functions (see, e.g., [13, 58, 60]). Recall that $T = (-\pi, \pi]$ and denote

\[
\mathcal{F}_1(T) := \left\{ f \in L^1(T) : \sum_{k=-\infty}^{\infty} |k||\hat{f}(k)| < \infty \right\},
\]

(2.12)
where \( \hat{f}(k) = \int_T e^{i\lambda k} f(\lambda) \, d\lambda \), \( \hat{f}(-k) = \hat{f}(k) \), and

\[
\mathcal{F}_2(\mathbb{T}) = \mathcal{F}_{\text{ARMA}}(\mathbb{T}) := \left\{ f : f(\lambda) = \frac{\sigma^2}{2\pi} \left| \frac{A(e^{i\lambda})}{B(e^{i\lambda})} \right|^2 \right\},
\]

(2.13)

where \( 0 < \sigma^2 < \infty \), \( A(z) := \sum_{k=0}^q a_k z^k \) (\( p \in \mathbb{N} \)) and \( B(z) := \sum_{k=0}^p b_k z^k \) (\( q \in \mathbb{N} \)) are both bounded away from zero for \( |z| \leq 1 \).

**Remark 2.5.** The following implications were established in [58]:

(a) If \( f_1, f_2 \in \mathcal{F}_1(\mathbb{T}) \), then \( f_1 f_2 \in \mathcal{F}_1(\mathbb{T}) \).

(b) If \( f \in \mathcal{F}_2(\mathbb{T}) \), then \( f \in \mathcal{F}_1(\mathbb{T}) \) and \( f^{-1} \in \mathcal{F}_2(\mathbb{T}) \).

For \( \psi \in L^p(\mathbb{T}) \), \( 1 \leq p \leq \infty \), let \( \omega_p(\psi, \delta) \) denote the \( L^p \)–modulus of continuity of \( \psi \):

\[
\omega_p(\psi, \delta) := \sup_{0 < h \leq \delta} \| \psi(\cdot + h) - \psi(\cdot) \|_p, \quad \delta > 0.
\]

**Definition 2.1.** Given numbers \( 0 < \gamma \leq 1 \) and \( 1 \leq p \leq \infty \), we denote by \( \text{Lip}(\mathbb{T} : p, \gamma) \) the \( L^p \)-Lipschitz class of functions defined on \( \mathbb{T} \) (see, e.g., [13]):

\[
\text{Lip}(\mathbb{T} : p, \gamma) = \{ \psi(\lambda) \in L^p(\mathbb{T}) : \omega_p(\psi; \delta) = O(\delta^\gamma), \quad \delta \to 0 \}.
\]

Observe that if \( \psi \in \text{Lip}(p, \gamma) \), then there exists a constant \( C \) such that \( \omega_p(\psi; \delta) \leq C \delta^\gamma \) for all \( \delta > 0 \).

**Theorem 2.3.** Assume that \( \tau_k = 1 \), \( k = \overline{1,m} \), \( \mathcal{H} = \{ h_1, h_2, \ldots, h_m \} \), and let \( \Delta_{B,T;\mathcal{H}}(T) \) be as in (2.5). The following assertions hold:

(B1) If \( h_i \in \mathcal{F}_1(\mathbb{T}) \), \( i = \overline{1,m} \), then

\[
\Delta_{B,T;\mathcal{H}}(T) = O(T^{-1}) \quad \text{as} \quad T \to \infty.
\]

(B2) If the functions \( h_i(\lambda), i = \overline{1,m} \), have uniformly bounded derivatives on \( \mathbb{T} := (-\pi, \pi] \), then for any \( \epsilon > 0 \)

\[
\Delta_{B,T;\mathcal{H}}(T) = O(T^{-1+\epsilon}) \quad \text{as} \quad T \to \infty.
\]

(B3) Assume that the function \( \varphi(u) \) given by (2.8) with some constants \( C > 0 \) and \( \gamma \in (0,1] \) satisfies

\[
|\varphi(u) - \varphi(0)| \leq C|u|^\gamma, \quad u = (u_1, u_2, \ldots, u_{m-1}) \in \mathbb{T}^{m-1},
\]

where \( 0 = (0,0,\ldots,0) \) and \( |u| = |u_1| + |u_2| + \cdots + |u_{m-1}| \). Then for any \( \epsilon > 0 \)

\[
\Delta_{B,T;\mathcal{H}}(T) = O\left(T^{-\gamma+\epsilon}\right) \quad \text{as} \quad T \to \infty.
\]

(B4) Let \( h_i(\lambda) \in \text{Lip}(\mathbb{T} : p_i, \gamma) \), \( p_i > 1 \), \( i = 1, 2, \ldots, m \), \( 1/p_1 + \cdots + 1/p_m \leq 1 \) and \( \gamma \in (0,1] \). Then (2.14) holds for any \( \epsilon > 0 \).
(B5) Let \( h_i(\lambda), i = 1, 2, \ldots, m, \) be differentiable functions defined on \( T \setminus \{0\}, \) such that for some constants \( C_{1i} > 0, C_{2i} > 0, \) and \( \alpha_i, \) \( i = 1, 2, \ldots, m, \) satisfying \( 0 < \alpha_i < 1, \) \( \alpha := \sum_{i=1}^{m} \alpha_i < 1 \)

\[
|h_i(\lambda)| \leq C_{1i}|\lambda|^{-\alpha_i}, \quad |h'_i(\lambda)| \leq C_{2i}|\lambda|^{-(\alpha_i + 1)}, \quad \lambda \in T \setminus \{0\}, \quad i = 1, m.
\]

Then (2.14) holds for any \( \varepsilon > 0 \) with

\[
\gamma = \frac{1}{m}(1 - \alpha).
\]  

Remark 2.6. Assertion (B1) was proved in Taniguchi [58] (see, also, [60]). Assertion (B2), which is weaker than (B1), but under weaker conditions than those in (B1), was proved in Lieberman and Phillips [48]. Assertions (B3)–(B5) for \( m = 4 \) were proved in Ginovyan and Sahakyan [32].

Remark 2.7. It is easy to see that under the conditions of (B2) we have \( h_i \in \text{Lip}(T; p, 1) \) for any \( i = 1, 2, \ldots, m \) and \( p \geq 1. \) Hence (B4) implies (B2).

Example 2.1. Let \( h_i(\lambda) = |\lambda|^{-\alpha_i}, \lambda \in [-\pi, \pi], i = 1, 2, \ldots, m, \) with \( 0 < \alpha_i < 1 \) and \( \alpha := \sum_{i=1}^{m} \alpha_i < 1. \) It is easy to see that the conditions of (B5) are satisfied, and hence we have (2.14) with \( \gamma \) as in (2.15).

The next results (cf. Ginovyan [25]) show that for special case \( m = 2 \) the rates in Theorem 2.3 (B4) and (B5) can be substantially improved.

Theorem 2.4. Let \( h_i(\lambda) \in \text{Lip}(T; p_i, \gamma_i), \) with \( p_i > 1, 1/p_1 + 1/p_2 = 1 \) and \( \gamma_i \in (0, 1], \) \( i = 1, 2, \) and let

\[
\Delta_{2,B}(T) := \left| \frac{1}{T} \text{tr}[B_T(h_1)B_T(h_2)] - 2\pi \int_T h_1(\lambda)h_2(\lambda) d\lambda \right|.
\]

Then

\[
\Delta_{2,B}(T) = \begin{cases} 
O(T^{-(\gamma_1 + \gamma_2)}), & \text{if } \gamma_1 + \gamma_2 < 1 \\
O(T^{-1} \ln T), & \text{if } \gamma_1 + \gamma_2 = 1 \\
O(T^{-1}), & \text{if } \gamma_1 + \gamma_2 > 1.
\end{cases}
\]

Theorem 2.5. Assume that the functions \( h_i(\lambda), i = 1, 2, \) satisfy the conditions of Theorem 2.3 (B5) with \( m = 2. \) Then

\[
\Delta_{2,B}(T) = O \left( T^{-1+\alpha_1+\alpha_2} \right) \quad \text{as} \quad T \rightarrow \infty.
\]  

The next result, due to Taniguchi [58] (see, also, [60]), concerns the case when the product in (1.2) involves also inverse matrices, that is, \( \tau_k = (-1)^k, k = 1, m. \)

Theorem 2.6. Let \( S_B(H), M_{1,T,\mathcal{H}} \) and \( \Delta I_{B,T,\mathcal{H}}(T) \) be as in (2.9), and let \( F_1(T) \) and \( F_2(T) \) be as in (2.12) and (2.13), respectively. If \( f_i \in F_2 \) and \( g_i \in F_1(T), i = 1, \ldots, m, \) then

\[
\Delta I_{B,T,\mathcal{H}}(T) = O(T^{-1}) \quad \text{as} \quad T \rightarrow \infty.
\]
3. The Trace Problem for Toeplitz Operators

In this section we consider Problems (A) and (B) for Toeplitz operators, that is, in the case where the generating functions are defined on the real line. Again, Problem (A) involves $o(1)$ approximation and Problem (B) involves $O(T^{-\gamma})$ approximation with $\gamma > 0$. The theorems in this section are proved in Section 5.

Let $f(\lambda)$ be an integrable real symmetric function defined on $\mathbb{R}$. The analogue of the Fourier coefficients $\hat{f}(k)$ in (2.2) is the Fourier transform $\hat{f}(t)$ of $f(\lambda)$:

$$\hat{f}(t) = \int_{-\infty}^{+\infty} e^{it\lambda} f(\lambda) d\lambda, \quad t \in \mathbb{R}.$$  

(3.1)

The $\hat{f}$ in (3.1) will play the role of kernel in an integral operator.

Given $T > 0$ and an integrable real symmetric function $f(\lambda)$ defined on $\mathbb{R}$, the $T$-truncated Toeplitz operator generated by $f(\lambda)$, denoted by $W_T(f)$, is defined by the following equation (see, e.g., [30, 37, 42]):

$$[W_T(f)u](t) = \int_{0}^{T} \hat{f}(t-s)u(s)ds, \quad u(s) \in L^2[0,T],$$  

(3.2)

where $\hat{f}$ is as in (3.1).

It follows from (3.1), (3.2) and the formula for traces of integral operators (see, e.g., [35], p. 114) that

$$\text{tr} [W_T(f)] = \int_{0}^{T} \hat{f}(t-t)dt = T \hat{f}(0) = T \int_{-\infty}^{+\infty} f(\lambda) d\lambda.$$  

(3.3)

We pose the same question as in the case of Toeplitz matrices: what happens when the single operator $W_T(f)$ is replaced by a product of such operators? Observe that the product of Toeplitz operators again is not a Toeplitz operator.

The approach is similar to that of Toeplitz matrices - to approximate the trace of the product of Toeplitz operators by the trace of a Toeplitz operator generated by the product of generating functions. More precisely, let $\mathcal{H} = \{h_1, h_2, \ldots, h_m\}$ be a collection of integrable real symmetric functions defined on $\mathbb{R}$. Define

$$S_{W,\mathcal{H}}(T) := \frac{1}{T} \text{tr} \left[ \prod_{i=1}^{m} W_T(h_i) \right], \quad M_{\mathbb{R},\mathcal{H}} := (2\pi)^{m-1} \int_{-\infty}^{+\infty} \left[ \prod_{i=1}^{m} h_i(\lambda) \right] d\lambda,$$  

(3.4)

and let

$$\Delta(T) := \Delta_{W,\mathbb{R},\mathcal{H}}(T) = |S_{W,\mathcal{H}}(T) - M_{\mathbb{R},\mathcal{H}}|.$$  

(3.5)

Observe that by (3.3),

$$M_{\mathbb{R},\mathcal{H}} = (2\pi)^{m-1} \int_{-\infty}^{+\infty} \left[ \prod_{i=1}^{m} h_i(\lambda) \right] d\lambda = \frac{1}{T} \text{tr} \left[ W_T \left( \prod_{i=1}^{m} h_i(\lambda) \right) \right].$$  

(3.6)

How well is $S_{W,\mathcal{H}}(T)$ approximated by $M_{\mathbb{R},\mathcal{H}}$? What is the rate of convergence to zero of approximation error $\Delta_{W,\mathbb{R},\mathcal{H}}(T)$ as $T \to \infty$? These are Problems (A) and (B) in this case.
3.1. Problem (A) for Toeplitz Operators

In Theorem 3.1 and Remark 3.1 we summarize the results concerning Problems (A) for Toeplitz operators in the case where \( \tau_k = 1, \ k = 1, m \).

**Theorem 3.1.** Let \( \Delta(T) := \Delta_{W,R,H}(T) \) be as in (3.5). Each of the following conditions is sufficient for
\[
\Delta(T) = o(1) \quad \text{as} \quad T \to \infty. \tag{3.7}
\]

(A1) \( h_i \in L^1(\mathbb{R}) \cap L^{p_i}(\mathbb{R}), \ p_i > 1, \ i = 1, m, \) with \( 1/p_1 + \ldots + 1/p_m \leq 1 \).

(A2) The function \( \varphi(u) \) defined by
\[
\varphi(u) := \int_{-\infty}^{+\infty} h_1(\lambda)h_2(\lambda-u_1)h_3(\lambda-u_2)\cdots h_m(\lambda-u_{m-1}) \, d\lambda, \tag{3.8}
\]
where \( u = (u_1, u_2, \ldots, u_{m-1}) \in \mathbb{R}^{m-1} \), belongs to \( L^{m-2}(\mathbb{R}^{m-1}) \) and is continuous at \( 0 = (0,0,\ldots,0) \in \mathbb{R}^{m-1} \).

Remark 3.1. For the special case \( m = 4, \ h_1 = h_3 := f \) and \( h_2 = h_4 := g \), Ginovyan and Sahakyan [30] prove that the following conditions are also sufficient for (3.7):

(A3) \( f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}), \ g \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}), \ f g \in L^2(\mathbb{R}) \) and
\[
\int_{-\infty}^{+\infty} f^2(\lambda)g^2(\lambda-\mu) \, d\lambda \longrightarrow \int_{-\infty}^{+\infty} f^2(\lambda)g^2(\lambda) \, d\lambda \quad \text{as} \quad \mu \to 0.
\]

(A4) The functions \( f \) and \( g \) are integrable on \( \mathbb{R} \), bounded on \( \mathbb{R} \setminus (-\pi, \pi) \), and satisfy
\[
f(\lambda) \leq |\lambda|^{-\alpha} L_1(\lambda) \quad \text{and} \quad |g(\lambda)| \leq |\lambda|^{-\beta} L_2(\lambda) \quad \text{for} \quad \lambda \in [-\pi, \pi],
\]
for some \( \alpha < 1, \ \beta < 1 \) with \( \alpha + \beta \leq 1/2 \), and \( L_i \in SV(\mathbb{R}), \ \lambda^{-(\alpha+\beta)} L_i(\lambda) \in L^2(\mathbb{T}), \ i = 1, 2 \), where \( SV(\mathbb{R}) \) is the class of slowly varying at zero functions \( u(\lambda), \ \lambda \in \mathbb{R}, \) satisfying \( u(\lambda) \in L^\infty(\mathbb{R}), \lim_{\lambda \to 0} u(\lambda) = 0, \ u(\lambda) = u(-\lambda) \) and \( 0 < u(\lambda) < u(\mu) \) for \( 0 < \lambda < \mu \).

Remark 3.2. It would be of interest to extend the results of (A3) and (A4) to arbitrary \( m > 4 \).

3.2. Problem (B) for Toeplitz Operators

In Theorem 3.2 below we summarize the results concerning Problems (B) for Toeplitz operators in the case where \( \tau_k = 1, \ k = 1, m \). Let
\[
\mathcal{F}_1(\mathbb{R}) := \left\{ f \in L^1(\mathbb{R}) : \int_{-\infty}^{+\infty} |t| |\hat{f}(t)| \, dt < \infty \right\}, \tag{3.9}
\]
where \( \hat{f}(t) = \int_{\mathbb{R}} e^{i\lambda t} f(\lambda) \, d\lambda, \ \hat{f}(-t) = \hat{f}(t) \).

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For $\psi \in L^p(\mathbb{R})$, $1 \leq p \leq \infty$ let $\omega_p(\psi, \delta)$ denote the $L^p$–modulus of continuity of $\psi$:

$$
\omega_p(\psi, \delta) := \sup_{0 < h \leq \delta} \|\psi(\cdot + h) - \psi(\cdot)\|_p, \quad \delta > 0.
$$

Given numbers $0 < \gamma \leq 1$ and $1 \leq p \leq \infty$, we denote by Lip($\mathbb{R}; p, \gamma$) the $L^p$-Lipschitz class of functions defined on $\mathbb{R}$ (see, e.g., [13]):

$$
\text{Lip}(\mathbb{R}; p, \gamma) = \{\psi(\lambda) \in L^p(\mathbb{R}) : \omega_p(\psi; \delta) = O(\delta^\gamma) \text{ as } \delta \to 0\}.
$$

Theorem 3.2 is the continuous version of Theorem 2.3.

**Theorem 3.2.** Let $\mathcal{H} = \{h_1, h_2, \ldots, h_m\}$, and $\Delta_{W,R,\mathcal{H}}(T)$ and $\varphi(u)$ be as in (3.5) and (3.8), respectively. The following assertions hold:

**B1** If $h_i \in F_1(\mathbb{R})$, $i = 1, m$, then

$$
\Delta_{W,R,\mathcal{H}}(T) = O(T^{-1}) \text{ as } T \to \infty.
$$

**B2** Assume that $\varphi(u) \in L^\infty(\mathbb{R}^{m-1})$ and with some constants $C > 0$ and $\gamma \in (0, 1]$

$$
|\varphi(u) - \varphi(0)| \leq C|u|^\gamma, \quad u = (u_1, u_2, \ldots, u_{m-1}) \in \mathbb{R}^{m-1},
$$

where $0 = (0, 0, \ldots, 0)$ and $|u| = |u_1| + |u_2| + \cdots + |u_{m-1}|$. Then for any $\varepsilon > 0$

$$
\Delta_{W,R,\mathcal{H}}(T) = O\left(T^{-\gamma + \varepsilon}\right) \text{ as } T \to \infty.
$$

**B3** Let $h_i(\lambda) \in \text{Lip}(\mathbb{R}; p_i, \gamma)$, $i = 1, 2, \ldots, m$, $1/p_1 + \cdots + 1/p_m \leq 1$ and $\gamma \in (0, 1]$. Then (3.12) holds for any $\varepsilon > 0$.

**B4** Let $h_i(\lambda)$, $i = 1, 2, \ldots, m$, be differentiable functions defined on $\mathbb{R} \setminus \{0\}$, such that for some constants $C_i > 0$ and $\sigma_i > 0$, $\delta_i > 1$, $i = 1, 2, \ldots, m$ with $\sigma := \sum_{i=1}^m \sigma_i < 1$

$$
|h_i(\lambda)| \leq \begin{cases} C_i|\lambda|^{-\sigma_i} & \text{if } |\lambda| \leq 1, \\ C_i|\lambda|^{-\delta_i} & \text{if } |\lambda| > 1, \end{cases}
$$

for all $i = 1, 2, \ldots, m$. Then for any $\varepsilon > 0$

$$
\Delta_{W,R,\mathcal{H}}(T) = O\left(T^{-\gamma + \varepsilon}\right) \text{ as } T \to \infty
$$

with

$$
\gamma = \frac{1}{m}(1 - \sigma).
$$

The next results (see [25]), which are continuous versions of Theorems 2.4 and 2.5, respectively, show that for the special case $m = 2$, the rates in Theorem 3.2 (B3) and (B4) can be substantially improved.
Theorem 3.3. Let \( h_i(\lambda) \in \text{Lip}(\mathbb{R}; p_i, \gamma_i) \) with \( p_i > 1, \ 1/p_1 + 1/p_2 = 1 \) and \( \gamma_i \in (0, 1], \ i = 1, 2, \) and let
\[
\Delta_{2,W}(T) := \left| \frac{1}{T} \text{tr}[W_T(h_1)W_T(h_2)] - 2\pi \int_{\mathbb{R}} h_1(\lambda)h_2(\lambda) \, d\lambda \right|.
\]
Then
\[
\Delta_{2,W}(T) = \begin{cases} 
O(T^{-(\gamma_1 + \gamma_2)}), & \text{if } \gamma_1 + \gamma_2 < 1 \\
O(T^{-1} \ln T), & \text{if } \gamma_1 + \gamma_2 = 1 \\
O(T^{-1}), & \text{if } \gamma_1 + \gamma_2 > 1.
\end{cases}
\]

Theorem 3.4. Assume that the functions \( h_i(\lambda), i = 1, 2, \) satisfy the conditions of Theorem 3.2 (B4) with \( m = 2 \). Then
\[
\Delta_{2,W}(T) = O \left( T^{-1+(\sigma_1 + \sigma_2)} \right) \quad \text{as } T \to \infty. \tag{3.16}
\]

Remark 3.3. It would be of interest to prove the continuous analogs of Dahlhaus theorem (Theorem 2.2) and Taniguchi theorem (Theorem 2.6) for Toeplitz operators.

4. Applications to Stationary Processes

In this section we provide some applications of the trace problem to discrete- and continuous-time stationary processes: ARFIMA and Fractional Riesz-Bessel motions; central and non-central limit theorems, Berry-Esseen bounds, and large deviations for Toeplitz quadratic forms and functionals.

4.1. Applications to ARFIMA time series and fractional Riesz-Bessel motions

In this subsection we apply the results of Sections 2 and 3 to the important special cases where the generating functions are spectral densities of a discrete-time ARFIMA(0,d,0) stationary processes or continuous-time stationary fractional Riesz-Bessel motions.

We use the following notation: \( m = 2\nu; \)
\[
h_1(\lambda) = h_3(\lambda) = \cdots = h_{2\nu-1}(\lambda) := f_1(\lambda) \\
h_2(\lambda) = h_4(\lambda) = \cdots = h_{2\nu}(\lambda) := f_2(\lambda);
\]
and
\[
S_{\nu,A}(T) = \frac{1}{T} \text{tr}[A_T(f_1)A_T(f_2)]^{\nu}, \tag{4.1}
\]
\[
\Delta_{\nu,A}(T) := \left| S_{\nu,A}(T) - (2\pi)^{2\nu-1} \int_{\Lambda} [f_1(\lambda)f_2(\lambda)]^{\nu} \, d\lambda \right|, \tag{4.2}
\]
where either \( A_T(f_i) = B_T(f_i) \) and \( \Lambda = \mathbb{T} \) or \( A_T(f_i) = W_T(f_i) \) and \( \Lambda = \mathbb{R}, \ i = 1, 2. \)
4.1.1. Applications to ARFIMA time series

The next theorem gives an error bound for $\Delta_{2,B}(T)$ in the case where the corresponding Toeplitz matrices are generated by spectral densities of two discrete-time ARFIMA($0,d,0$) stationary processes.

**Theorem 4.1.** Let $f_i(\lambda), i = 1, 2,$ be the spectral density functions of two ARFIMA($0,d,0$) stationary processes defined as

$$f_i(\lambda) = \frac{\sigma^2_i}{2\pi} |1 - e^{i\lambda}|^{-2d_i}, \quad i = 1, 2$$

with $0 < \sigma^2_i < \infty$ and $0 < d_i < 1/2$. Then under $d := d_1 + d_2 < 1/(2\nu), \nu \in \mathbb{N}$, for any $\varepsilon > 0$,

$$\Delta_{\nu,B}(T) = O(T^{-\gamma + \varepsilon}) \quad \text{as} \quad T \to \infty$$

with

$$\gamma = \frac{1}{2\nu} - (d_1 + d_2).$$

**Proof.** Assuming that $\lambda \in (0, \pi]$ (the case $\lambda \in [-\pi, 0)$ is treated similarly), and taking into account $|1 - e^{i\lambda}| = 2\sin(\lambda/2)$, we have for $i = 1, 2$

$$f_i(\lambda) = \frac{\sigma^2_i}{2\pi} \cdot 2^{-2d_i} \left[ \sin \frac{\lambda}{2} \right]^{-2d_i},$$

$$f_i'(\lambda) = \frac{\sigma^2_i}{2\pi} \cdot \left[ -2d_i 2^{-2d_i-1} \left( \sin \frac{\lambda}{2} \right)^{-2d_i-1} \cos \frac{\lambda}{2} \right].$$

It is clear that the conditions of Theorem 2.3 (B5) are satisfied with $\alpha_i = 2d_i$ and $C_{1i} = C_{2i} = \sigma^2_i$, $i = 1, 2$, and the result follows.

The next theorem, which was proved in Lieberman and Phillips [48], gives an explicit second-order asymptotic expansion for $S_{1,B}(T)$ in the case where the Toeplitz matrices are generated by the spectral densities given by (4.3), and shows that in this special case a second-order asymptotic expansion successfully removes the singularity and delivers a substantially improved approximation.

**Theorem 4.2.** Let $f_i(\lambda), i = 1, 2,$ be the spectral density functions of two ARFIMA($0,d,0$) stationary processes defined by (4.3) with $0 < \sigma^2_i < \infty$ and $0 < d_i < 1/2, i = 1, 2$. Then under $d := d_1 + d_2 < 1/2$

$$S_{1,B}(T) = \frac{1}{T} \text{tr}[B_T(f_1)B_T(f_2)]$$

$$= 2\pi \int_{-\pi}^{\pi} f_1(\lambda) f_2(\lambda) d\lambda - \frac{C(d_1,d_2)}{T^{1-2d}} + o \left( \frac{1}{T^{1-2d}} \right)$$

as $T \to \infty$, where

$$C(d_1,d_2) = \frac{2\sigma^2_1 \sigma^2_2 \pi^2}{\cos(\pi d_1) \cos(\pi d_2) \Gamma(2d_1) \Gamma(2d_2)} \cdot \frac{1}{2d(1-2d)}.$$
Remark 4.1. The asymptotic relation (4.7) in Lieberman and Phillips [48] was established by direct calculations using the explicit forms of functions \( f_i \) given by (4.3). On the other hand, as it follows from (4.6), the functions \( f_i \) for \( i = 1, 2 \) satisfy conditions of Theorem 2.5 with \( \alpha_i = 2d_i \) for \( i = 1, 2 \), and hence (4.7) is a special case of Theorem 2.5.

4.1.2. Applications to fractional Riesz-Bessel motions

Now we assume that the underlying model is a continuous-time stationary process specified by a fractional Riesz-Bessel motion. The following result is an immediate consequence of Theorem 3.1 (A1).

**Theorem 4.3.** Let \( f_1(\lambda) = f(\lambda) \) be the spectral density of a fractional Riesz-Bessel motion defined by (1.12), and let \( f_2(\lambda) = g(\lambda) \) be an integrable real symmetric function on \( \mathbb{R} \). If for some \( p, q \geq 1 \) with \( 1/p + 1/q \leq 1/\nu \) \( (\nu \in \mathbb{N}) \) we have \( g(\lambda) \in L^q(\mathbb{R}) \) and \( 0 < \alpha < 1/(2p) \), \( \alpha + \beta > 1/2 \), then

\[
\Delta_{\nu,W}(T) = o(1) \quad \text{as} \quad T \to \infty.
\]

**Theorem 4.4.** Let \( f_1(\lambda) = f(\lambda) \) be as in (1.12) with \( 0 < \alpha < 1/(2p) \) and \( \alpha + \beta > 1/2 \), and let \( f_2(\lambda) = g(\lambda) \) be a real symmetric function from the class \( \text{Lip}(q, 1/p - 2\alpha) \) with \( 1/p + 1/q \leq 1/\nu \), \( \nu \in \mathbb{N} \). Then for any \( \varepsilon > 0 \)

\[
\Delta_{\nu,W}(T) = O(T^{-\gamma+\varepsilon}) \quad \text{as} \quad T \to \infty.
\]

with \( \gamma = 1/p - 2\alpha \).

**Proof.** The result follows from Theorem 3.2 (B3) and the following lemma, which is proved in the Appendix.

**Lemma 4.1.** Let \( p > 1 \), \( 0 < \sigma < 1/p \) and let \( f(\lambda) \) be a differentiable function defined on \( \mathbb{R} \setminus \{0\} \), such that for some constant \( C > 0 \)

\[
|f(\lambda)| \leq \begin{cases} C|\lambda|^{-\sigma} & \text{if} \quad |\lambda| \leq 1, \\ C|\lambda|^{-\delta} & \text{if} \quad |\lambda| > 1, \end{cases} \quad |f'(\lambda)| \leq \begin{cases} C|\lambda|^{-\sigma-1} & \text{if} \quad |\lambda| \leq 1, \\ C|\lambda|^{-\delta-1} & \text{if} \quad |\lambda| > 1. \end{cases}
\]

Then \( f \in \text{Lip}(p, 1/p - \sigma) \).

Now to prove (4.9) observe that by (1.12),

\[
f'(\lambda) = \frac{2\alpha + 2(\alpha + \beta)\lambda^2}{\lambda^{2\alpha+1}(1+\lambda)^{\beta+1}}.
\]

It follows from (1.12) and (4.11) that the functions \( f \) and \( f' \) satisfy conditions (4.10) with \( \sigma = 2\alpha \) and \( \delta = 2\alpha + 2\beta \). Hence by Lemma 4.1, \( f(\lambda) \in \text{Lip}(p, 1/p - 2\alpha) \). Therefore the result follows from Theorem 3.2 (B3).
Theorem 4.5. Let $f_i(\lambda)$, $i = 1, 2$, be the spectral density functions of two fractional Riesz-Bessel motions defined as

$$f_i(\lambda) = \frac{C_i}{|\lambda|^{2\alpha_i}(1 + \lambda^2)^{\beta_i}}, \quad 0 < \alpha_i < 1/2, \quad \alpha_i + \beta_i > 1/2, \quad i = 1, 2. \quad (4.12)$$

If $\alpha_1 + \alpha_2 < 1/(2\nu)$, then (4.9) holds for any $\varepsilon > 0$ with

$$\gamma = \frac{1}{2\nu} - (\alpha_1 + \alpha_2). \quad (4.13)$$

Proof. It follows from (1.12) and (4.11) that the functions $f_1$ and $f_2$ satisfy conditions of Theorem 3.2 (B4) with $\sigma_i = 2\alpha_i$ and $\delta_i = 2\alpha_i + 2\beta_i$, $i = 1, 2$, and the result follows.

The next theorem, which is a continuous version of Theorem 4.2, contains an explicit second-order asymptotic expansion for $S_{1,W}(T)$ in the case where the Toeplitz operators are generated by the spectral densities given by (4.12), and shows that in this special case a second-order asymptotic expansion successfully removes the singularity and delivers a substantially improved approximation.

Theorem 4.6. Let $f_i(\lambda)$, $i = 1, 2$, be as in (4.12). Then under $\alpha := \alpha_1 + \alpha_2 < 1/2$

$$S_{1,W}(T) := \frac{1}{T} \text{tr}[W_T(f_1)W_T(f_2)]= 2\pi \int_{-\infty}^{+\infty} f_1(\lambda)f_2(\lambda) d\lambda - \frac{C(\alpha_1, \alpha_2)}{T^{1-2\alpha}} + o\left(\frac{1}{T^{1-2\alpha}}\right) \quad (4.14)$$

as $T \to \infty$, where

$$C(\alpha_1, \alpha_2) = \frac{2C_1C_2\pi^2}{\cos(\pi\alpha_1) \cos(\pi\alpha_2) \Gamma(2\alpha_1) \Gamma(2\alpha_2)} \cdot \frac{1}{2\alpha(1-2\alpha)}. \quad (4.15)$$

The proof is based on the following lemma, which contains an asymptotic formula for the covariance function of a fRBM process. It is proved in the Appendix.

Lemma 4.2. Let $f(\lambda)$ be as in (1.12) with $0 < \alpha < 1/2$ and $\beta > 1/2$, and let $r(t) := \hat{f}(t)$ be the Fourier transform of $f(\lambda)$. Then

$$r(t) = t^{2\alpha-1} \cdot \frac{\pi C}{\cos(\pi\alpha) \Gamma(2\alpha)} \cdot (1 + o(1)) \quad \text{as} \quad t \to \infty. \quad (4.16)$$

Remark 4.2. Taking into account the reflection formula $\Gamma(2\alpha)\Gamma(1-2\alpha) = \pi/\sin(2\pi\alpha)$, the asymptotic relation (4.16) can be written in the following form

$$r(t) = Ct^{2\alpha-1} \sin(\pi\alpha) \Gamma(1-2\alpha) \cdot (1 + o(1)) \quad \text{as} \quad t \to \infty. \quad (4.17)$$
Proof of Theorem 4.6. By Lemma 5.2 and Parseval-Plancherel theorem we have
\[ S_{T,1} = \int_{-T}^{T} \left( 1 - \frac{|t|}{T} \right) r_1(t)r_2(t) \, dt = 2\pi \int_{-\infty}^{+\infty} f_1(\lambda)f_2(\lambda) \, d\lambda - I_1 - I_2, \quad (4.18) \]
where
\[ I_1 = \int_{|t|>T} r_1(t)r_2(t) \, dt \quad (4.19) \]
and
\[ I_2 = \frac{1}{T} \int_{-T}^{T} |t|r_1(t)r_2(t) \, dt. \quad (4.20) \]
Hence, for \( \alpha := \alpha_1 + \alpha_2 < 1/2 \) from Lemma 4.2 and (4.19), we have as \( T \to \infty \)
\[ I_1 = \int_{|t|>T} r_1(t)r_2(t) \, dt = 2 \int_{|t|>T} r_1(t)r_2(t) \, dt \]
\[ = \frac{2C_1C_2\pi^2}{\cos(\pi\alpha_1)\cos(\pi\alpha_2)\Gamma(2\alpha_1)\Gamma(2\alpha_2)} \int_{t>T} t^{2(\alpha-1)} (1 + o(1)) \]
\[ = \frac{2C_1C_2\pi^2}{\cos(\pi\alpha_1)\cos(\pi\alpha_2)\Gamma(2\alpha_1)\Gamma(2\alpha_2)} \frac{1}{2\alpha} \cdot T^{2\alpha-1} (1 + o(1)). \quad (4.21) \]
For \( I_2 \), from Lemma 4.2 and (4.20), we have as \( T \to \infty \) and \( \alpha := \alpha_1 + \alpha_2 < 1/2 \)
\[ I_2 = \frac{1}{T} \int_{-T}^{T} |t|r_1(t)r_2(t) \, dt = \frac{2}{T} \int_{0}^{T} t r_1(t)r_2(t) \, dt \]
\[ = \frac{2}{T} \cdot \frac{C_1C_2\pi^2}{\cos(\pi\alpha_1)\cos(\pi\alpha_2)\Gamma(2\alpha_1)\Gamma(2\alpha_2)} \int_{0}^{T} t^{2\alpha-1} \, dt (1 + o(1)) \]
\[ = \frac{2C_1C_2\pi^2}{\cos(\pi\alpha_1)\cos(\pi\alpha_2)\Gamma(2\alpha_1)\Gamma(2\alpha_2)} \frac{1}{2\alpha} \cdot T^{2\alpha-1} (1 + o(1)). \quad (4.22) \]
From (4.18), (4.21) and (4.22) the result follows. Theorem 4.6 is proved.

Remark 4.3. (a) As in Remark 4.1, the rate in (4.14) can be obtained from Theorem 3.4.
(b) We analyze the behavior of the approximations as \( \alpha_1, \alpha_2 \to 1/4 \). First observe that the first-order asymptotic formula has a pole when \( \alpha := \alpha_1 + \alpha_2 = 1/2 \). In particular, denoting \( \beta := \beta_1 + \beta_2 \), and using the change of variable \( \lambda^2 = u \), we have
\[ 2\pi \int_{-\infty}^{+\infty} f_1(\lambda)f_2(\lambda) \, d\lambda = 2\pi C_1C_2 \int_{-\infty}^{+\infty} \frac{1}{|\lambda|^{2\alpha}(1 + \lambda^2)^{\beta}} \, d\lambda \]
\[ = 2\pi C_1C_2 \int_{0}^{\infty} \frac{u^{-1/2-\alpha}}{(1 + u)^{\beta}} \, du. \quad (4.23) \]
Applying the formula (see, e.g., [16])
\[ \int_{0}^{\infty} \frac{u^{m-1}}{(1 + u)^{m+n}} \, du = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}, \quad m > 0, \, n > 0, \quad (4.24) \]
with \( m = 1/2 - \alpha \) and \( n = \beta - m = \alpha + \beta - 1/2 \), from (4.23) we find
\[
2\pi C_1 C_2 \int_{-\infty}^{+\infty} f_1(\lambda) f_2(\lambda) \, d\lambda = 2\pi C_1 C_2 \frac{\Gamma(1/2 - \alpha)\Gamma(\alpha + \beta - 1/2)}{\Gamma(\beta)}.
\] (4.25)

Then, using the Laurent expansion of the gamma function \( \Gamma(1/2 - \alpha) \) around the pole \( \alpha = 1/2 \), from (4.25) we obtain: as \( \alpha = \alpha_1 + \alpha_2 \to 1/2 \)
\[
2\pi \int_{-\infty}^{+\infty} f_1(\lambda) f_2(\lambda) \, d\lambda = \frac{4\pi C_1 C_2}{1 - 2\alpha} + O(1).
\] (4.26)

However, the asymptotic behavior of the second-order term \( C(\alpha_1, \alpha_2) \) (see (4.15)), as \( \alpha_1, \alpha_2 \to 1/4 \) is readily seen to be
\[
C(\alpha_1, \alpha_2) = \frac{2\pi^2 C_1 C_2}{\cos(\pi \alpha_1) \cos(\pi \alpha_2) \Gamma(2\alpha_1) \Gamma(2\alpha_2)} \cdot \frac{1}{2\alpha(1 - 2\alpha)}
\] (4.27)

Thus, the pole in the first-order approximation is removed by the second-order approximation, so that the approximation
\[
2\pi \int_{-\infty}^{+\infty} f_1(\lambda) f_2(\lambda) \, d\lambda - \frac{C(\alpha_1, \alpha_2)}{\Gamma^{1-2(\alpha_1+\alpha_2)}}
\] (4.28)

is bounded as \( \alpha_1, \alpha_2 \to 1/4 \).

This good behavior explains why the second-order approximation produces a good approximation that does uniformly well over \( \alpha_1, \alpha_2 \in [0, 1/4] \), including the limits of the domain.

(c) The second-order equivalence holds along an arbitrary ray for which \( \alpha = \alpha_1 + \alpha_2 \to 1/2 \). Indeed, let \( \alpha_0^k \in [0, 1/2] \) be any fixed number such that \( \alpha_1 \to \alpha_0^k \) and \( \alpha = \alpha_1 + \alpha_2 \to 1/2 \), then the representation (4.26) for the first-order asymptotic term continues to apply.

On the other hand, using the reflection formula \( \Gamma(z)\Gamma(1-z) = \pi / \sin(\pi z) \) with \( z = 2\alpha_0^k \), we have from (4.15) as \( \alpha_1 \to \alpha_0^k \) and \( \alpha = \alpha_1 + \alpha_2 \to 1/2 \)
\[
C(\alpha_1, \alpha_2) = \frac{2\pi^2 C_1 C_2}{\cos(\alpha_1 \pi) \cos(\alpha_2 \pi) \Gamma(2\alpha_1) \Gamma(2\alpha_2)} \cdot \frac{1}{2\alpha(1 - 2\alpha)}
\] (4.28)

This good behavior explains why the second-order approximation produces a good approximation that does uniformly well over \( \alpha_1, \alpha_2 \in [0, 1/4] \), including the limits of the domain.

(c) The second-order equivalence holds along an arbitrary ray for which \( \alpha = \alpha_1 + \alpha_2 \to 1/2 \). Indeed, let \( \alpha_0^k \in [0, 1/2] \) be any fixed number such that \( \alpha_1 \to \alpha_0^k \) and \( \alpha = \alpha_1 + \alpha_2 \to 1/2 \), then the representation (4.26) for the first-order asymptotic term continues to apply.

On the other hand, using the reflection formula \( \Gamma(z)\Gamma(1-z) = \pi / \sin(\pi z) \) with \( z = 2\alpha_0^k \), we have from (4.15) as \( \alpha_1 \to \alpha_0^k \) and \( \alpha = \alpha_1 + \alpha_2 \to 1/2 \)
and again the second-order equivalence holds. Thus, in this special case, the second-order asymptotic expansion removes the singularity in the first-order approximation, and provides a substantially improved approximation to the original functional.

4.2. Limit Theorems for Toeplitz Quadratic Functionals

In this section we examine the limit behavior of quadratic forms and functionals of discrete- and continuous-time stationary Gaussian processes with possibly long-range dependence. The matrix and the operator that characterize the quadratic form and functional are Toeplitz.

Let \( \{X(u), u \in \mathbb{U}\} \) be a centered real-valued Gaussian stationary process with spectral density \( f(\lambda), \lambda \in \Lambda \) and covariance function \( r(t) := \hat{f}(t), t \in \mathbb{U} \), where \( \mathbb{U} \) and \( \Lambda \) are as in Section 1.2. We are interested in the asymptotic distribution (as \( T \to \infty \)) of the following Toeplitz type quadratic functionals of the process \( X(u) \):

\[
Q_T := \begin{cases} 
\int_0^T \int_0^T \hat{g}(t - s)X(t)X(s) \, dt \, ds & \text{in the continuous-time case} \\
\sum_{k=1}^{T} \sum_{j=1}^{T} \hat{g}(k - j)X(k)X(j) & \text{in the discrete-time case},
\end{cases}
\]

where

\[
\hat{g}(t) = \int_{\Lambda} \hat{g}(\lambda) \, d\lambda, \quad t \in \mathbb{U}
\]

is the Fourier transform of some real, even, integrable function \( g(\lambda), \lambda \in \Lambda \). We will refer \( g(\lambda) \) as a generating function for the functional \( Q_T \). In the discrete-time case the functions \( f(\lambda) \) and \( g(\lambda) \) are assumed to be 2\( \pi \)-periodic and periodically extended to \( \mathbb{R} \).

The limit distributions of the functionals (4.29) are completely determined by the spectral density \( f(\lambda) \) and the generating function \( g(\lambda) \), and depending on their properties the limit distributions can be either Gaussian (i.e., \( Q_T \) with an appropriate normalization obeys central limit theorem), or non-Gaussian. The following two questions arise naturally:

a) Under what conditions on \( f(\lambda) \) and \( g(\lambda) \) will the limits be Gaussian?

b) Describe the limit distributions, if they are non-Gaussian.

4.2.1. Central limit theorems for Toeplitz quadratic functionals

We first discuss the question a), that is, finding conditions on the spectral density \( f(\lambda) \) and the generating function \( g(\lambda) \) under which the functional \( Q_T \), defined by (4.29), obeys central limit theorem. This question goes back to the classical monograph by Grenander and Szegö [37], where the problem was considered for discrete time processes, as an application
of the authors’ theory of the asymptotic behavior of the trace of products of truncated Toeplitz matrices.

Later the problem a) was studied by Ibragimov [42] and Rosenblatt [54], in connection with the statistical estimation of the spectral \((F(\lambda))\) and covariance \((r(t))\) functions, respectively. Since 1986, there has been a renewed interest in both questions a) and b), related to the statistical inferences for long-memory processes (see, e.g., Avram [4], Fox and Taqqu [17], Giraitis and Surgailis [34], Giraitis et al. [33], Terrin and Taqqu [66], Taniguchi [59], Taniguchi and Kakizawa [60], Ginovyan and Sahakyan [29], and references therein). In particular, Avram [4], Fox and Taqqu [17], Giraitis and Surgailis [34], Ginovyan and Sahakyan [29] have obtained sufficient conditions for the quadratic form \(Q_T\) to obey the central limit theorem (CLT), when the model \(X(t)\) is a discrete-time process.

For continuous time processes the question a) was studied in Ibragimov [42], Ginovyan [22, 24], and Ginovyan and Sahakyan [30].

Let \(Q_T\) be as in (4.29). We will use the following notation: By \(\tilde{Q}_T\) we denote the standard normalized quadratic functional:

\[
\tilde{Q}_T = \frac{1}{\sqrt{T}} (Q_T - EQ_T).
\]

The notation

\[
\tilde{Q}_T \iff N(0, \sigma^2) \text{ as } T \to \infty
\]

will mean that the distribution of the random variable \(\tilde{Q}_T\) tends (as \(T \to \infty\)) to the centered normal distribution with variance \(\sigma^2\).

Our study of the asymptotic distribution of the quadratic functionals (4.29) is based on the following representation of the \(k\)-th order cumulant \(\chi_k(\cdot)\) of \(\tilde{Q}_T\), which follows from (1.15) (see, also, [37, 42]):

\[
\chi_k(\tilde{Q}_T) = \begin{cases} 
0, & \text{for } k = 1 \\
\displaystyle T^{-k/2}2^{k-1}(k-1)! \text{tr } [A_T(f)A_T(g)]^k, & \text{for } k \geq 2,
\end{cases}
\]

where \(A_T(f)\) and \(A_T(g)\) denote either the \(T\)-truncated Toeplitz operators (for continuous-time case), or the \(T \times T\) Toeplitz matrices (for discrete-time case) generated by the functions \(f\) and \(g\) respectively, and \(\text{tr}[A]\) stands for the trace of an operator \(A\).

The next result contains sufficient conditions in terms of \(f(\lambda)\) and \(g(\lambda)\) ensuring central limit theorems for standard normalized quadratic functionals \(\tilde{Q}_T\) both for discrete- and continuous-time processes.

Below we assume that \(f, g \in L^1(\Lambda)\), and with no loss of generality, that \(g \geq 0\). Also, we set

\[
\sigma_0^2 = 16\pi^3 \int_{\Lambda} f^2(\lambda)g^2(\lambda) d\lambda.
\]

As usual \(\Lambda = \mathbb{T} = (-\pi, \pi]\) or \(\Lambda = \mathbb{R} = (-\infty, \infty)\). The following theorem includes both discrete-time (\(\Lambda = (-\pi, \pi]\)) and continuous-time (\(\Lambda = (-\infty, \infty)\)).

\[
\sigma_0^2 = 16\pi^3 \int_{\Lambda} f^2(\lambda)g^2(\lambda) d\lambda.
\]
Theorem 4.7. Each of the following conditions is sufficient for
\[ \tilde{Q}_T \subseteq N(0, \sigma_0^2) \quad \text{as} \quad T \to \infty, \] (4.35)
with \( \sigma_0^2 \) given by (4.34).

(A) \( f \cdot g \in L^1(\Lambda) \cap L^2(\Lambda) \) and
\[ \chi_2(\tilde{Q}_T) := \frac{2}{T} \text{tr}[B_T(f)B_T(g)]^2 \to \sigma_0^2 < \infty. \] (4.36)

(B) The function
\[ \varphi(u) := \varphi(u_1, u_2, u_3) = \int_{\Lambda} f(\lambda)g(\lambda - u_1)f(\lambda - u_2)g(\lambda - u_3) d\lambda \] (4.37)
belongs to \( L^2(\Lambda^3) \) and is continuous at \( 0 = (0, 0, 0) \).

(C) \( f \in L^1(\Lambda) \cap L^p(\Lambda) \) \( (p \geq 2) \) and \( g \in L^1(\Lambda) \cap L^q(\Lambda) \) \( (q \geq 2) \) with
\[ \frac{1}{p} + \frac{1}{q} \leq \frac{1}{2}. \]

(D) \( f \in L^1(\Lambda) \cap L^2(\Lambda), \; g \in L^1(\Lambda) \cap L^2(\Lambda), \; fg \in L^2(\Lambda) \) and
\[ \int_{\Lambda} f^2(\lambda)g^2(\lambda - \mu) d\lambda \to \int_{\Lambda} f^2(\lambda)g^2(\lambda) d\lambda \quad \text{as} \quad \mu \to 0. \]

(E) The spectral density \( f(\lambda) \) and the generating function \( g(\lambda) \) satisfy
\[ f(\lambda) \leq |\lambda|^{-\alpha}L_1(\lambda), \; |g(\lambda)| \leq |\lambda|^{-\beta}L_2(\lambda), \; \lambda \in \Lambda, \]
for some \( \alpha < 1, \; \beta < 1 \) with
\[ \alpha + \beta \leq 1/2 \quad \text{and} \quad L_i \in SV(\mathbb{R}), \; \lambda^{-(\alpha + \beta)}L_i(\lambda) \in L^2(\Lambda), \; i = 1, 2, \]
where \( SV(\mathbb{R}) \) is the class of slowly varying at zero functions \( u(\lambda), \lambda \in \mathbb{R}, \)
satisfying \( u(\lambda) \in L^\infty(\mathbb{R}), \lim_{\lambda \to 0} u(\lambda) = 0, \; u(\lambda) = u(-\lambda) \) and \( 0 < u(\lambda) < u(\mu) \) for \( 0 < \lambda < \mu. \)

In the continuous-time case, we also assume that the functions \( f(\lambda) \) and \( g(\lambda) \) are bounded on \( \mathbb{R} \setminus (-\pi, \pi). \)

Remark 4.4. For discrete-time case: assertions (A) and (D) were proved in Giraitis and Surgailis [34] (see also Giraitis et al. [33]); assertions (B) and (E) were proved in Ginovyan and Sahakyan [29]; assertion (E) with \( \alpha + \beta < 1/2 \) was first obtained by Fox and Taqqu [17]; assertion (C) for \( p = q = \infty \) was first established by Grenander and Szegö [37], Sec. 11.7, while the case \( p = 2, \; q = \infty \) was proved by Ibragimov [42] and Rosenblatt [54], in the general discrete-time case assertion (D) was proved by Avram [4].

For continuous-time case assertions (A) – (E) were proved in Ginovyan [24] and Ginovyan and Sahakyan [30].
Remark 4.5. Assertion (A) implies assertions (B) – (E). Assertion (B) implies assertions (C) and (D). On the other hand, for functions \( f(\lambda) = \lambda^{-3/4} \) and \( g(\lambda) = \lambda^{3/4} \) satisfying the conditions of (E), the function \( \varphi(t_1, t_2, t_3) \) is not defined for \( t_2 = 0, t_1 \neq 0, t_3 \neq 0 \), showing that assertion (B) generally does not imply assertion (E) (see Ginovyan and Sahakyan [29]).

Remark 4.6. Examples of spectral density \( f(\lambda) \) and generating function \( g(\lambda) \) satisfying the conditions of Theorem 4.7 (E) provide the functions

\[
f(\lambda) = |\lambda|^{-\alpha} |\ln|\lambda||^{-\gamma} \quad \text{and} \quad g(\lambda) = |\lambda|^{-\beta} |\ln|\lambda||^{-\gamma},
\]

where \( \alpha < 1, \beta < 1, \alpha + \beta \leq 1/2 \) and \( \gamma > 1/2 \) (see Ginovyan and Sahakyan [29]).

Remark 4.7. The functions \( f(\lambda) \) and \( g(\lambda) \) in Theorem 4.7 (E) have singularities at the point \( \lambda = 0 \), and are bounded in any neighborhood of this point. It can be shown that the choice of the point \( \lambda = 0 \) is not essential, and instead it can be taken to be any point \( \lambda_0 \in [-\pi, \pi] \). Using the properties of the products of Toeplitz matrices and operators it can be shown that Theorem 4.7 (E) remains valid when \( f(\lambda) \) and \( g(\lambda) \) have singularities of the form (4.38) at the same finite number of points of the segment \([-\pi, \pi]\) (see Ginovyan and Sahakyan [29]).

Remark 4.8. The next proposition shows that the condition of positiveness and finiteness of asymptotic variance of quadratic form \( Q_T \) is not sufficient for \( Q_T \) to obey CLT as it was conjectured in Giraitis and Surgailis [34], and Ginovyan [23].

Proposition 4.1. There exist a spectral density \( f(\lambda) \) and a generating function \( g(\lambda) \) such that

\[
0 < \int_{-\pi}^{\pi} f^2(\lambda) g^2(\lambda) \, d\lambda < \infty
\]

and

\[
\lim_{T \to \infty} \sup_{n \to \infty} \chi^2(Q_T) = \lim_{n \to \infty} \frac{1}{2} \text{tr}(B_T(f)B_T(g))^2 = \infty,
\]

that is, the condition (4.39) does not guarantee convergence in (4.36).

To construct functions \( f(\lambda) \) and \( g(\lambda) \) satisfying (4.39) and (4.40), for a fixed \( p \geq 2 \) we choose a number \( q > 1 \) to satisfy \( 1/p + 1/q > 1 \), and for such \( p \) and \( q \) we consider the functions \( f_0(\lambda) \) and \( g_0(\lambda) \) defined by

\[
f_0(\lambda) = \begin{cases} \frac{(2^{s-1})^{1/p}}{2^{s-1}} & \text{if } 2^{-s-1} \leq \lambda \leq 2^{-s}, \ s = 2m, \\ 0, & \text{if } 2^{-s-1} \leq \lambda \leq 2^{-s}, \ s = 2m + 1 \end{cases}
\]

and

\[
g_0(\lambda) = \begin{cases} \frac{(2^{s-1})^{1/q}}{2^{s-1}} & \text{if } 2^{-s-1} \leq \lambda \leq 2^{-s}, \ s = 2m + 1, \\ 0, & \text{if } 2^{-s-1} \leq \lambda \leq 2^{-s}, \ s = 2m, \end{cases}
\]

where \( m \) is a positive integer. For an arbitrary finite positive constant \( C \) we set \( g_\pm(\lambda) = g_0(\lambda) \pm C \). Then the functions \( f = f_0 \) and \( g = g_+ \text{ or } g = g_- \).
satisfy (4.39) and (4.40) (for details we refer to Ginovyan and Sahakyan [29]. Consequently, for these functions the standard normalized quadratic form $Q_T$ does not obey CLT, and it is of interest to describe the limiting non-Gaussian distribution of $Q_T$ in this special case.

### 4.2.2. Non-central Limit Theorems

The problem b) for discrete-time processes, that is, the description of the limit distributions of the quadratic form

$$Q_T := \sum_{k=1}^{T} \sum_{j=1}^{T} \hat{g}(k-j)X(k)X(j), \quad T \in \mathbb{N}$$

(4.43)

if it is non-Gaussian, goes back to the papers by Rosenblatt [53]-[55]. Later this problem was studied in a series of papers by Taqqu, and Terrin and Taqqu (see, e.g., in [61], [64], [65], [66], and references therein). Specifically, suppose that the spectral density $f(\lambda)$ and the generating function $g(\lambda)$ are regularly varying functions at the origin:

$$f(\lambda) = |\lambda|^{-\alpha}L_1(\lambda) \quad \text{and} \quad g(\lambda) = |\lambda|^{-\beta}L_2(\lambda), \quad \alpha < 1, \beta < 1,$$

(4.44)

where $L_1(\lambda)$ and $L_2(\lambda)$ are slowly varying functions at zero, which are bounded on bounded intervals. The conditions $\alpha < 1$ and $\beta < 1$ ensure that the Fourier coefficients of $f$ and $g$ are well defined. When $\alpha > 0$ the model $\{X(t), t \in \mathbb{Z}\}$ exhibits long memory.

It is the sum $\alpha + \beta$ that determines the asymptotic behavior of the quadratic form $Q_T$. If $\alpha + \beta \leq 1/2$, then by Theorem 4.7(E) the standard normalized quadratic form

$$T^{-1/2}(Q_T - EQ_T)$$

converges in distribution to a Gaussian random variable. If $\alpha + \beta > 1/2$, convergence to Gaussian fails.

Consider the embedding of the discrete sequence $\{Q_T, T \in \mathbb{N}\}$ into a continuous-time process $\{Q_T(t), T \in \mathbb{N}, t \in \mathbb{R}\}$ defined by

$$Q_T(t) := \sum_{k=1}^{[Tt]} \sum_{j=1}^{[Tt]} \hat{g}(k-j)X(k)X(j),$$

(4.45)

where $[\cdot]$ stands for the greatest integer. Denote by $Z(\cdot)$ the complex-valued Gaussian random measure defined on the Borel $\sigma$-algebra $\mathcal{B}(\mathbb{R})$, and satisfying $EZ(B) = 0, E|Z(B)|^2 = |B|$, and $Z(-B) = Z(B)$ for any $B \in \mathcal{B}(\mathbb{R})$.

The next result, proved in Terrin and Taqqu [65], describes the non-Gaussian limit distribution of the suitable normalized process $Q_T(t)$. 

Theorem 4.8. Let \( f(\lambda) \) and \( g(\lambda) \) be as in (4.44) with \( \alpha < 1, \beta < 1 \) and slowly varying at zero and bounded on bounded intervals factors \( L_1(\lambda) \) and \( L_2(\lambda) \). Let the process \( Q_T(t) \) be as in (4.45). Then for \( \alpha + \beta > 1/2 \)

\[
\hat{Q}_T(t) := \frac{1}{T^{\alpha+\beta}L_1(1/T)L_2(1/T)} \left( Q_T(t) - E[Q_T(t)] \right) \quad (4.46)
\]

converges (as \( T \to \infty \)) weakly in \( D[0,1] \) to

\[
Q(t) := \int_{\mathbb{R}^2} K_t(x,y) dZ(x)dZ(y), \quad (4.47)
\]

where

\[
K_t(x,y) = |xy|^{-\alpha/2} \int_{\mathbb{R}} \left( \frac{e^{it(x+u)} - 1}{i(x+u)} \cdot \frac{e^{it(y-u)} - 1}{i(y-u)} \right) |u|^{-\beta} du. \quad (4.48)
\]

The double prime in the integral (4.47) indicates that the integration excludes the diagonals \( x = \pm y \).

Remark 4.9. The limiting process in (4.47) is real-valued, non-Gaussian, and satisfies \( EQ(t) = 0 \) and \( EQ^2(t) = \int_{\mathbb{R}^2} |K_t(x,y)|^2 dx dy \). It is self-similar with parameter \( H = \alpha + \beta \in (1/2, 2) \), that is, the processes \( \{Q(at), t \geq 0\} \) and \( \{a^H Q(t), t \geq 0\} \) have the same finite dimensional distributions for all \( a > 0 \).

Remark 4.10. In [53] (see also [55]) Rosenblatt showed that if a discrete-time centered Gaussian process \( X(t) \) has covariance function \( r(t) = (1 + t^2)^{\alpha/2 - 1/2} \) with \( 1/2 < \alpha < 1 \), then the random variable

\[
Q_T := T^{-\alpha} \sum_{k=1}^{T} [X^2(k) - 1]
\]

has a non-Gaussian limiting distribution, and described this distribution in terms of characteristic functions. This is a special case of Theorem 4.8 with \( t = 1, 1/2 < \alpha < 1 \) and \( \beta = 0 \). In [61] (see also [64]) Taqqu extended Rosenblatt’s result by showing that the stochastic process

\[
Q_T(t) := T^{-\alpha} \sum_{k=1}^{[Tt]} [X^2(k) - 1]
\]

converges (as \( T \to \infty \)) weakly to a process (called Rosenblatt process) which has the double Wiener-Itô integral representation

\[
Q(t) := C_\alpha \int_{\mathbb{R}^2} \frac{e^{it(x+y)} - 1}{i(x+y)} |x|^{-\alpha/2} |y|^{-\alpha/2} dZ(x)dZ(y). \quad (4.49)
\]

The distribution of the random variable \( Q(t) \) in (4.49) for \( t = 1 \) is described in Veillette and Taqqu [67].
Remark 4.11. The slowly varying functions $L_1$ and $L_2$ in (4.44) are of importance because they provide a great flexibility in the choice of spectral density $f$ and generating function $g$. Observe that in Theorem 4.8 the functions $L_1$ and $L_2$ influence only the normalization (see (4.46)), but not the limit $Q(t)$. Theorem 4.7(E) shows that in the critical case $\alpha + \beta = 1/2$ the limit distribution of the standard normalized quadratic form $Q_T$ is Gaussian and essentially depends on slowly varying factors $L_1$ and $L_2$.

Note also that the critical case $\alpha + \beta = 1/2$ was partially investigated by Terrin and Taqqu in [66]. Starting from the limiting random variable $Q(1) = Q(1; \alpha, \beta)$, which exists only when $\alpha + \beta > 1/2$, they showed the random variable

$$(\alpha + \beta - 1/2)Q(1; \alpha, \beta)$$

converges in distribution to a Gaussian random variable as $\alpha + \beta$ approaches to $1/2$.

Remark 4.12. For continuous-time processes the problem b) has not been investigated, and it would be of interest to describe the limiting non-Gaussian distribution of the quadratic functional $Q_T$.

4.3. Berry-Esseen Bounds and Large Deviations for Toeplitz Quadratic Functionals

In this section, we briefly discuss Berry-Esséen bounds in CLT and large deviations principle for quadratic functionals both for continuous- and discrete-time Gaussian stationary processes (for more about these topics we refer to [6, 12, 14, 19, 44, 49, 59], and reference therein).

Berry-Esseen Bounds. Let $Q_T$ and $\tilde{Q}_T$ be as in (4.29) and (4.31), respectively. Denote $\hat{Q}_T := \tilde{Q}_T / \sqrt{\text{Var}(\tilde{Q}_T)}$, and let $Z$ be the standard normal random variable: $Z \sim N(0, 1)$. The CLT for $Q_T$ (Theorem 4.7) tells us that $\hat{Q}_T \to Z$ in distribution as $T \to \infty$. The natural next step concerns the closeness between the distribution of $\hat{Q}_T$ and standard normal distribution, which means asking for the rate of convergence in the CLT. Results of this type are known as Berry-Esseen bounds (or asymptotics).

In discrete-time case, for special quadratic functionals, Berry-Esseen bounds were established in Tanoguchi [59], while for continuous-time case, Berry-Esseen-type bounds were obtained in Nourdin and Peccati [49]. The next theorem captures both cases.

Theorem 4.9. Let $\tilde{Q}_T$ be as in (4.31), $\hat{Q}_T := \tilde{Q}_T / \sqrt{\text{Var}(\tilde{Q}_T)}$, and $\Phi(z) = P(Z \leq z)$, where $Z \sim N(0, 1)$. Assume that $f(\lambda) \in L^1(\Lambda) \cap L^p(\Lambda)$ ($p \geq 1$) and $g(\lambda) \in L^1(\Lambda) \cap L^q(\Lambda)$ ($q \geq 1$). The following assertions hold.

1. If $1/p + 1/q \leq 1/4$, then there exists a constant $C = C(f, g) > 0$ such that for all $T > 0$ we have

$$\sup_{z \in \mathbb{R}} |P(\hat{Q}_T \leq z) - \Phi(z)| \leq \frac{C}{\sqrt{T}}. \quad (4.50)$$
(Theorem 4.54 was proved in Nourdin and Peccati [49], by appealing to a general CLT of Section 4.2 (Theorem 4.7), and Stein’s method. The proof, in the discrete-time case, is similar to that of the continuous-time case.

**Large Deviations.** We now present sufficient conditions that ensure large deviations principle (LDP) for Toeplitz type quadratic functionals of stationary Gaussian processes. For more about LDP we refer to Bryc and Dembo [12], Bercu et al. [6], Sato et al. [56], Taniguchi and Kakizawa [60], and references therein.

First observe that large deviation theory can be viewed as an extension of the law of large numbers (LLN). The LLN states that certain probabilities converge to zero, while the large deviation theory focuses on the rate of convergence. Specifically, consider a sequence of random variables \( \{\xi_n, n \geq 1\} \) converging in probability to a real constant \( c \). Let \( \xi_n \) could represent, for instance, the \( n \)-th partial sum of another sequence of random variables: \( \xi_n = \frac{1}{n} \sum_{k=1}^{n} \eta_k \), where the sequence \( \{\eta_k\} \) may be independent identically distributed, or dependent as in an observed stretch of a stochastic process. By the LLN, we have for \( \varepsilon > 0 \)

\[
P\{|\xi_n - m| > \varepsilon\} \to 0 \quad \text{as} \quad n \to \infty. \tag{4.53}
\]

It is often the case that the convergence in (4.53) is exponentially fast, that is,

\[
P\{|\xi_n - m| > \varepsilon\} \approx R(\varepsilon) \exp[-nI(\varepsilon, m)] \quad \text{as} \quad n \to \infty, \tag{4.54}
\]

where \( R(\cdot) = R(\varepsilon, m, n) \) is a slowly varying (relative to an exponential) function of \( n \) and \( I(\varepsilon, m) \) is a positive quantity. Loosely, if (4.54) holds, we say that the sequence \( \{\xi_n\} \) satisfies a large deviations principle. One of the basic problems of the large deviation theory is to determine \( I(\varepsilon, m) \) and \( R(\varepsilon, m, n) \). To be more precise, we recall the definition of Large Deviation Principle (LDP) (see, for instance, [12, 60]).

**Definition 4.1.** Let \( \{\xi_n, n \in \mathbb{Z}\} \) be a sequence of real–valued random variables defined on the probability space \((\Omega, \mathcal{F}, \mathbb{P})\). We say that \( \{\xi_n\} \) satisfies a Large Deviation Principle (LDP) with speed \( a_n \to 0 \) and rate function \( I: \mathbb{R} \to [0, \infty] \), if \( I(x) \) is lower semicontinuous, that is, if \( x_n \to x \) then \( \liminf_{n \to \infty} I(x_n) \geq I(x) \), and

\[
\liminf_{n \to \infty} a_n \log \mathbb{P}\{\xi_n \in A\} \geq - \inf_{x \in A} I(x)
\]
for all open subsets $A \subset \mathbb{R}$, while
\[
\limsup_{n \to \infty} a_n \log \mathbb{P}\{\xi_n \in B\} \leq - \inf_{x \in B} I(x)
\]
for all closed subsets $B \subset \mathbb{R}$. The function $I(x)$ is called a good rate function if its level sets are compact, that is, the set $\{x \in \mathbb{R} : I(x) \leq b\}$ is compact for each $b \in \mathbb{R}$.

Now let $Q_T$ be the Toeplitz type quadratic functionals of a process $X(u)$ defined by (4.29) with spectral density $f(\lambda)$ and generating function $g(\lambda)$.

The next result states sufficient conditions in terms of $f(\lambda)$ and $g(\lambda)$ to ensure that the LDP for normalized quadratic functionals $\{1/2Q_T\}$ holds both for discrete- and continuous-time processes.

**Theorem 4.10.** Assume that $f(\lambda)g(\lambda) \in L^\infty(\Lambda)$. Then the random variable $\{1/2Q_T\}$ satisfies a LDP with the speed $a_T = 1/2$ and a good rate function $I(x)$:
\[
I(x) = \sup_{-\infty < y < 1/(2C)} \{xy - V(f;g;y)\},
\]
where $C = \text{ess sup} f(\lambda)|g(\lambda)|$ and for $y < 1/(2C)$
\[
V(f;g;y) = -\frac{1}{4\pi} \int_\Lambda \log(1 - 2yf(\lambda)g(\lambda))d\lambda.
\]

**Remark 4.14.** In the special case of $g(\lambda) = 1$, Theorem 4.10 was proved by Bryc and Dembo [12], for general case we refer to Bercu et al. [6].

5. **Proof of Theorems 3.1 - 3.4**

We only prove the results concerning Toeplitz operators (Theorems 3.1 - 3.4). The proofs of the corresponding results for Toeplitz matrices are similar. First we state a number of technical lemmas, which are proved in the Appendix.

The following result is known (see, e.g., [30], [33], p. 8).

**Lemma 5.1.** Let $D_T(u)$ be the Dirichlet kernel
\[
D_T(u) = \frac{\sin(Tu/2)}{u/2}.
\]

Then, for any $\delta \in (0, 1)$
\[
|D_T(u)| \leq 2T^\delta |u|^{\delta - 1}, \quad u \in \mathbb{R}.
\]

Denote
\[
G_T(u) := \int_0^T e^{itu}dt = e^{T/2}D_T(u), \quad u \in \mathbb{R},
\]
\[
\Phi_T(u) := \frac{1}{(2\pi)^{m-1}T} \cdot D_T(u_1) \cdots D_T(u_{m-1}) D_T(u_1 + \cdots + u_{m-1}), \quad u \in \mathbb{R},
\]
\[
\Psi(u) := \varphi(u_1, u_1 + u_2, \ldots, u_1 + \cdots + u_{m-1}),
\]
where $\varphi$ is a function.
where \( u = (u_1, \ldots, u_{m-1}) \in \mathbb{R}^{m-1} \) and the function \( \varphi(u) \), corresponding to the collection \( \mathcal{H} = \{h_1, h_2, \ldots, h_m\} \), is defined by (3.8).

The next lemma follows from (3.4) and (5.3) - (5.5) (cf. [30], Lemma 1).

**Lemma 5.2.** Let \( \mathcal{H} = \{h_1, h_2, \ldots, h_m\} \) be a collection of integrable real symmetric functions on \( \mathbb{R} \), \( \hat{h}_k \) be the Fourier transform of function \( h_k \) (\( k = 1, \ldots, m \)), and let \( S(T) := S_{W, \mathcal{H}}(T) \) be as in (3.4). The following equalities hold.

1. \( S(T) = \frac{1}{T} \int_0^T \ldots \int_0^T \hat{h}_1(u_1 - u_2)\hat{h}_2(u_2 - u_3)\ldots\hat{h}_m(u_m - u_1) \)
2. \( S(T) = \frac{1}{T} \int_{\mathbb{R}^m} h_1(u_1)\ldots h_m(u_m)G_T(u_1 - u_2)G_T(u_2 - u_3)\ldots \times G_T(u_m - u_1)du_1\ldots du_m. \)
3. \( S(T) = (2\pi)^{m-1} \int_{\mathbb{R}^{m-1}} \Psi(u)\Phi_T(u)du. \)

For \( m = 3, 4, \ldots \) and \( \delta > 0 \) we denote
\[
E_\delta = \{(u_1, \ldots, u_{m-1}) \in \mathbb{R}^{m-1} : |u_i| \leq \delta, i = 1, \ldots, m-1\}, \quad E_\delta^c = \mathbb{R}^{m-1} \setminus E_\delta
\]
and
\[
p_3 = 2, \quad p(m) = \frac{m-2}{m-3} \quad (m > 3).
\]

**Lemma 5.3.** The kernel \( \Phi_T(u), \ u \in \mathbb{R}^{m-1}, \ m \geq 3 \) possesses the following properties:

a) \( \sup_{T} \int_{\mathbb{R}^{m-1}} |\Phi_T(u)| \, du = C_1 < \infty; \)

b) \( \int_{\mathbb{R}^{m-1}} \Phi_T(u) \, du = 1; \)

c) \( \lim_{T \to \infty} \int_{E_\delta^c} |\Phi_T(u)| \, du = 0 \quad \text{for any} \quad \delta > 0; \)

d) \( \text{for any} \ \delta > 0 \ \text{there exists a constant} \ C_\delta > 0 \ \text{such that} \)
\[
\int_{E_\delta^c} |\Phi_T(u)|^{p(m)} \, du \leq C_\delta \quad \text{for} \quad T > 0,
\]

(5.6)

The proof of the next lemma can be found in [33], p. 161.

**Lemma 5.4.** Let \( 0 < \beta < 1, \ 0 < \alpha < 1, \ \text{and} \ \alpha + \beta > 1 \). Then for any \( y \in \mathbb{R}, \ y \neq 0, \)
\[
\int_{\mathbb{R}} \frac{1}{|x|^\alpha |x+y|^{\beta}} \, dx = \frac{C}{|y|^\alpha + \beta - 1},
\]
where \( C \) is a constant depending on \( \alpha \) and \( \beta \).

Denote \( E = \{(u_1, u_2, \ldots, u_n) \in \mathbb{R}^n : |u_i| \leq 1, \ i = 1, 2, \ldots, n\} \) and let \( E^c = \mathbb{R}^n \setminus E \).
Lemma 5.5. Let \( 0 < \alpha \leq 1 \) and \( \frac{n}{n+1} < \beta < \frac{n+\alpha}{n+1} \). Then
\[
B_i := \int_E \frac{|u_i|^\alpha}{|u_1 \cdots u_n(u_1 + \cdots + u_n)|^\beta} \, du_1 \cdots du_n < \infty, \quad i = 1, \ldots, n. \tag{5.8}
\]

Lemma 5.6. Let \( \frac{n}{n+1} < \beta < 1 \). Then
\[
I := \int_E \frac{1}{|u_1 \cdots u_n(u_1 + \cdots + u_n)|^\beta} \, du_1 \cdots du_n < \infty. \tag{5.9}
\]

Proof of Theorem 3.1. We start with (A2). By Lemma 5.2, 3) and 5.3, b), we have \( \Delta(T) = (2\pi)^{m-1}|R(T)| \), where
\[
R(T) = \int_{\mathbb{R}^3} |\Psi(u) - \Psi(0)| \Phi_T(u) \, du.
\]
It follows from (5.5) that the function \( \Psi(u) \) belongs to \( L^{m-2}(\mathbb{R}^{m-1}) \) and is continuous at \( 0 = (0, \ldots, 0) \in \mathbb{R}^{m-1} \). Hence for any \( \varepsilon > 0 \) we can find \( \delta > 0 \) to satisfy
\[
|\Psi(u) - \Psi(0)| < \frac{\varepsilon}{C_1}, \quad u \in E_\delta, \tag{5.10}
\]
where \( C_1 \) is the constant from Lemma 5.3, a). Consider the decomposition \( \Psi = \Psi_1 + \Psi_2 \) such that
\[
\|\Psi_1\|_{(m-2)} \leq \frac{\varepsilon}{\sqrt{C_\delta}} \quad \text{and} \quad \|\Psi_2\|_\infty < \infty, \tag{5.11}
\]
where \( C_\delta \) is as in Lemma 5.3, d).

Observe that \( \frac{1}{m-2} + \frac{1}{p(m)} = 1 \), where \( p(m) = \frac{m-2}{m-1} \). Hence, applying Lemma 5.3 and (5.10), (5.11) for sufficiently large \( T \) we obtain
\[
|R(T)| \leq \int_{E_\delta} |\Psi(u) - \Psi(0)| |\Phi_T(u)| \, du + C_m \int_{E_\delta} |\Psi_1(u)| |\Phi_T(u)| \, du
+ \int_{E_\delta} |\Psi_2(u) - \Psi(0)| |\Phi_T(u)| \, du \leq \frac{\varepsilon}{C_1} \int_{E_\delta} |\Phi_T(u)| \, du \\
+ \|\Psi_1\|_{(m-2)} \left[ \int_{E_\delta} \Phi_T^{p(m)}(u) \, du \right]^{1/p(m)} + C_2 \int_{E_\delta} |\Phi_T(u)| \, du \leq 3 \varepsilon,
\]
and the result follows.

Proof of (A1). According to Theorem (A2) it is enough to prove that the function
\[
\varphi(u) := \int_{-\infty}^{+\infty} h_1(\lambda)h_2(\lambda - u_1)h_3(\lambda - u_2) \cdots h_m(\lambda - u_{m-1}) \, d\lambda, \tag{5.12}
\]
where \( u = (u_1, \ldots, u_{m-1}) \in \mathbb{R}^{m-1} \), belongs to \( L^{m-2}(\mathbb{R}^{m-1}) \) and is continuous at \( 0 = (0, \ldots, 0) \in \mathbb{R}^{m-1} \), provided that
\[
h_i \in L^1(\mathbb{R}) \bigcap L^{p_i}(\mathbb{R}), \quad 1 \leq p_i \leq \infty, \quad i = 1, \ldots, m, \quad \sum_{i=1}^{m} \frac{1}{p_i} \leq 1. \tag{5.13}
\]
It follows from Hölder inequality, (5.12) and (5.13) that
\[ |\varphi(u)| \leq \prod_{i=1}^{m} \|f_i\|_{L^p_i(\mathbb{R})} < \infty, \quad u \in \mathbb{R}^{m-1}. \]

Hence, \( \varphi \in L^\infty(\mathbb{R}^{m-1}) \). On the other hand, the condition \( h_i \in L^1(\mathbb{R}) \) and (5.12) imply \( \varphi \in L^1(\mathbb{R}^{m-1}) \). Therefore \( \varphi \in L^{m-2}(\mathbb{R}^{m-1}) \).

To prove the continuity of \( \varphi(u) \) at the point \( 0 \) we consider three cases.

**Case 1.** \( p_i < \infty, \quad i = 1, \ldots, m \).

For an arbitrary \( \varepsilon > 0 \) we can find \( \delta > 0 \) satisfying (see (5.13))
\[ \|h_i(\lambda - u) - h_i(\lambda)\|_{L^p_i(\mathbb{R})} \leq \varepsilon, \quad i = 2, \ldots, m, \quad \text{if} \quad |u| \leq \delta. \] (5.14)

We fix \( u = (u_1, \ldots, u_{m-1}) \) with \( |u| < \delta \) and denote
\[ \tilde{h}_i(\lambda) = h_i(\lambda - u_{i-1}) - h_i(\lambda), \quad i = 2, \ldots, m. \]

Then in view of (5.12) we have
\[ \varphi(u) = \int_{\mathbb{R}} h_1(\lambda) \prod_{i=2}^{m-1} (\tilde{h}_i(\lambda) + h_i(\lambda)) \, d\lambda = \varphi(0) + W. \]

It follows from (5.14) that \( \|\tilde{h}_i\|_{p_i} \leq \varepsilon, \quad i = 2, \ldots, m \). Observe that each of the integrals comprising \( W \) contains at least one function \( \tilde{h}_i \) and can be estimated as follows:
\[ \left| \int_{\mathbb{R}} h_1(u) \tilde{h}_2(\lambda) h_3(\lambda) \cdots h_{m-1}(\lambda) \, d\lambda \right| \leq \|h_1\|_{L^{p_1}} \|\tilde{h}_2\|_{L^{p_2}} \|h_3\|_{L^{p_3}} \cdots \|h_{m}\|_{L^{p_m}} \leq C\varepsilon. \]

**Case 2.** \( p_i \leq \infty, \quad i = 1, \ldots, m, \quad \sum_{i=1}^{m} \frac{1}{p_i} < 1 \).

There exist finite numbers \( p'_i < p_i, \quad i = 1, \ldots, m, \) such that \( \sum_{i=1}^{m} 1/p'_i \leq 1 \). Hence according to (5.13) we have \( h_i \in L^{p'}_i, \quad i = 1, \ldots, m \) and \( \varphi \) is continuous at \( 0 \) as in the case 1.

**Case 3.** \( p_i \leq \infty, \quad i = 1, \ldots, m, \quad \sum_{i=1}^{m} \frac{1}{p_i} = 1 \).

First observe that at least one of the numbers \( p_i \) is finite. Suppose, without loss of generality, that \( p_1 < \infty \). For any \( \varepsilon > 0 \) we can find functions \( h'_1, h''_1 \) such that
\[ h_1 = h'_1 + h''_1, \quad h'_1 \in L^\infty(\mathbb{R}), \quad \|h''_1\|_{L^{p_1}} < \varepsilon. \] (5.15)

Therefore
\[ \varphi(u) = \varphi'(u) + \varphi''(u), \]
where the functions \( \varphi' \) and \( \varphi'' \) are defined as \( \varphi \) in (5.12) with \( h_1 \) replaced by \( h'_1 \) and \( h''_1 \), respectively. It follows from (5.15) that \( \varphi' \) is continuous at \( 0 \) (see Case 2), while by Hölder inequality \( \|\varphi''(u)\| \leq C \cdot \varepsilon \). Hence, for sufficiently small \( |u| \)
\[ |\varphi(u) - \varphi(0)| \leq |\varphi'(u) - \varphi'(0)| + |\varphi''(u) - \varphi''(0)| \leq (C + 1)\varepsilon, \]
and the result follows. Theorem 3.1 is proved.

Proof of Theorem 3.2. We start with (B1). First observe that the condition $h_i \in F_1$ implies that

$$\hat{h}_i \in L^1(\mathbb{R}) \quad \text{and} \quad |\hat{h}_i(t)| \leq A, \quad t \in \mathbb{R}, \quad i = 1, 2, \ldots, m \tag{5.16}$$

for some constant $A > 0$. By Lemma 5.2 we have

$$T \cdot S(T) = \int_0^T \ldots \int_0^T \hat{h}_1(u_1 - u_2)\hat{h}_2(u_2 - u_3)\ldots\hat{h}_m(u_m - u_1)du_1 \ldots du_m. \tag{5.17}$$

Making the change of variables

$$u_1 - u_2 = t_1, \quad u_2 - u_3 = t_2, \ldots, u_{m-1} - u_m = t_{m-1},$$

and observing that $t_1 + \ldots + t_{m-1} = u_1 - u_m$, we get (below we use the notation: $t_{m-1} = (t_1, \ldots, t_{m-1})$ and $dt_{m-1} = dt_1 \cdot \ldots \cdot dt_{m-1}$),

$$T \cdot S(T) = \int_0^T \ldots \int_0^{u_m-t_1-\ldots-t_{m-1}} \hat{h}_1(t_1)\ldots\hat{h}_{m-1}(t_{m-1})\hat{h}_m(-t_1 - \ldots - t_{m-1})dt_{m-1}du_m \tag{5.18}$$

where

$$|l(t_{m-1})| = |l(t_1, \ldots, t_{m-1})| \leq 2(|t_1| + \ldots + |t_{m-1}|).$$

On the other hand, by (3.4) and Parseval’s equality we have

$$M : = M_{\mathbb{R}, \mathcal{H}} = (2\pi)^{m-1} \int_{-\infty}^{\infty} \left[ \int_{t=1}^{m} h_i(\lambda) \right] d\lambda \tag{5.19}$$

$$= \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} \hat{h}_1(t_1) \ldots \hat{h}_{m-1}(t_{m-1})\hat{h}_m(-t_1 - \ldots - t_{m-1})dt_{m-1}.$$  

It follows from (3.4), (5.17) and (5.19) that

$$S(T) - M := S_{\mathbb{R}, \mathcal{H}}(T) - M_{\mathbb{R}, \mathcal{H}}$$

$$= -\frac{1}{T} \int_{[-T,T]^{m-1}} \hat{h}_1(t_1) \ldots \hat{h}_{m-1}(t_{m-1})\hat{h}_m(-t_1 - \ldots - t_{m-1})l(t_{m-1})dt_{m-1}$$

$$+ \int_{\mathbb{R}^{m-1} \setminus [-T,T]^{m-1}} \hat{h}_1(t_1) \ldots \hat{h}_{m-1}(t_{m-1})\hat{h}_m(-t_1 - \ldots - t_{m-1})dt_{m-1}$$

$$=: \Delta_T^1 + \Delta_T^2. \tag{5.20}$$
By (5.16), (5.18) and (5.20) we have
\[ |T \cdot \Delta_T^m| \leq 2A \sum_{i=1}^{m-1} \int_{E_{m-1}} |\hat{h}_1(t_1) \cdots \hat{h}_{m-1}(t_{m-1})t_i| dt_{m-1} =: A_1 < \infty, \quad (5.21) \]
since \(h_i \in F_i, i = 1, 2, \ldots, m.\)

Further, observe that
\[ \mathbb{R}^{m-1} \setminus [-T, T]^{m-1} \subset \bigcup_{i=1}^{m} \{(t_1, \ldots, t_{m-1}) \in \mathbb{R}^{m-1} : |t_i| > T \} =: \bigcup_{i=1}^{m} E_i. \]

Hence by (5.16) and (5.20) we have
\[ |T \cdot \Delta_T^m| \leq 2A \sum_{i=1}^{m-1} \int_{E_i} |\hat{h}_1(t_1) \cdots \hat{h}_{m-1}(t_{m-1})t_i| dt_{m-1} =: A_2 < \infty. \quad (5.22) \]

From (5.20) – (5.22) we get (B1).

Proof of (B2). By Lemma 5.2, 3) and Lemma 5.3, b), and (3.5) we have
\[ \Delta(T) = (2\pi)^{m-1} \int_{\mathbb{R}^{m-1}} |\Psi(u) - \Psi(0)| |\Phi_T(u)| du, \quad 0 = (0, \ldots, 0). \quad (5.23) \]

It follows from (3.11) and (5.5) that for \(u = (u_1, \ldots, u_{m-1}) \in \mathbb{R}^{m-1}\)
\[ |\Psi(u) - \Psi(0)| \leq (m-1)C ((|u_1|^{\gamma} + \cdots + |u_{m-1}|^{\gamma})^\beta. \quad (5.24) \]

Let \(\varepsilon \in (0, \gamma).\) Then, applying Lemma 5.1 with \(\delta = \frac{1+\gamma-\varepsilon}{m},\) and using (5.23) and (5.24), we can write
\[ \Delta(T) \leq C_m \int_{E} |\Psi(u) - \Psi(0)| |\Phi_T(u)| du \\
+ \|\Phi\|_\infty C_m \int_{E^c} |\Psi(u) - \Psi(0)| |\Phi_T(u)| du \quad (5.25) \]
\[ \leq \frac{C_m}{T^{1-\delta \beta}} \sum_{i=1}^{m-1} \int_{E} \frac{|u_i|^{\gamma}}{|u_1 \cdots u_{m-1} (u_1 + \cdots + u_{m-1})|^{1-\lambda}} du_1 \cdots du_{m-1} \]
\[ + 2\|\Phi\|_\infty C_m \int_{E^c} \frac{1}{|u_1 \cdots u_{m-1} (u_1 + \cdots + u_{m-1})|^{1-\delta}} du_1 \cdots du_{m-1}, \]

where \(E = \{(u_1, u_2, \ldots, u_{m-1}) \in \mathbb{R}^{m-1} : |u_i| \leq 1, \ i = 1, 2, \ldots, m-1\} \) and \(E^c = \mathbb{R}^{m-1} \setminus E.\) Since
\[ \frac{m-1}{m} < 1 - \delta < \frac{m-1 + \gamma}{m}, \]
we can apply Lemmas 5.5 and 5.6 with \(\alpha = \gamma, \ n = m - 1 \) and \(\beta = 1 - \delta\) to conclude that all the integrals in (5.25) are finite. Since \(1 - m\delta = \gamma - \varepsilon,\) from (5.25) follows the statement (B2).  \(\square\)
Proof of (B3). According to (B2) it is enough to prove that the function
\[
\varphi(u) := \int_{\mathbb{R}} h_1(\lambda) h_2(\lambda - u_1) \cdots h_m(\lambda - u_{m-1}) du,
\]
where \( u = (u_1, \ldots, u_{m-1}) \in \mathbb{R}^{m-1} \), belongs to \( L^\infty(\mathbb{R}^{m-1}) \), and with some positive constant \( C \)
\[
|\varphi(u) - \varphi(0)| \leq C |u|^\gamma, \quad u \in \mathbb{R}^{m-1},
\]
provided that
\[
\|h_i\|_{p_i} \leq C_i |u|^\tau, \quad i = 2, \ldots, m.
\]

It follows from Hölder inequality and (5.28) that
\[
|\varphi(u)| \leq \prod_{i=1}^m \|h_i\|_{p_i} < \infty, \quad u \in \mathbb{R}^{m-1}.
\]

Hence \( \varphi \in L^\infty(\mathbb{R}^{m-1}) \).

To prove (5.27) we fix \( u = (u_1, \ldots, u_{m-1}) \in \mathbb{R}^{m-1} \) and denote
\[
\overline{h}_i(\lambda) = h_i(\lambda - u_{i-1}) - h_i(\lambda), \quad \lambda \in \mathbb{R}, \quad i = 2, \ldots, m.
\]
Since \( h_i \in \text{Lip}(\mathbb{R}; p_i, \gamma) \) we have
\[
\|\overline{h}_i\|_{p_i} \leq C_i |u|^\gamma, \quad i = 2, \ldots, m.
\]

By (5.26) and (5.29),
\[
\varphi(u) = \int_{\mathbb{R}} h_1(\lambda) \prod_{i=2}^m (\overline{h}_i(\lambda) + h_i(\lambda)) d\lambda = \varphi(0) + W.
\]

Each of the \( (2^{m-1} - 1) \) integrals comprising \( W \) contains at least one function \( \overline{f}_i \), and in view of (5.30), can be estimated as follows:
\[
\left| \int_{\mathbb{R}} h_1(\lambda) \overline{f}_2(\lambda) h_3(\lambda) \cdots h_m(\lambda) d\lambda \right|
\leq \|h_1\|_{p_1} \|\overline{f}_2\|_{p_2} \|h_3(u)\|_{p_2} \cdots \|h_m\|_{p_m} \leq C |u|^\gamma.
\]

This completes the proof of (B3).

Proof of (B4). We set
\[
\frac{1}{p_i} := \sigma_i + \frac{1}{m} [1 - (\sigma_1 + \cdots \sigma_m)], \quad i = 1, 2, \ldots, m.
\]
Then
\[
\sum_{i=1}^{m} \frac{1}{p_i} = 1 \quad \text{and} \quad \frac{1}{p_i} - \sigma_i + \frac{1}{m} [1 - (\sigma_1 + \cdots + \sigma_m)] = \gamma > 0, \quad i = 1, 2, \ldots, m,
\]
and
\[
0 < \sigma_i < \frac{1}{p_i} < 1 < \delta_i, \quad i = 1, 2, \ldots, m.
\]
Hence, according to Lemma 4.1, \( h_i \in \text{Lip}(\rho_i, \gamma) \), \( i = 1, 2, \ldots, m \). Applying (B3), we get (3.14) with \( \gamma \) as in (3.15).

Proof of Theorem 3.3. First observe that by Lemma 7 from [31]
\[
S_{2,W}(T) := \frac{1}{T} \text{tr}[W_T(h_1)W_T(h_2)] = 2\pi \int_{\mathbb{R}} \int_{\mathbb{R}} F_T(s-t) h_1(s) h_2(t) \, dt \, ds, \quad (5.31)
\]
where \( F_T(u) \) is the Fejér kernel:
\[
F_T(u) = \frac{1}{2\pi T} \left( \sin \frac{Tu}{2} \right)^2, \quad t \in \mathbb{R}.
\]
Below we use the following properties of \( F_T(u) \) (see, e.g., [13]):
\[
\int_{\mathbb{R}} F_T(u) \, du = 1, \quad (5.32)
\]
\[
\int_{u \geq 1} F_T(u) \, du \leq C T^{-1}, \quad (5.33)
\]
\[
\int_{0}^{1} F_T(u) u^\alpha \, du \leq \begin{cases} 
C T^{-\alpha}, & \text{if } \alpha \leq 1 \\
C T^{-1} \ln T, & \text{if } \alpha = 1 \\
C T^{-1}, & \text{if } \alpha > 1.
\end{cases} \quad (5.34)
\]

Since the function \( F_T(u) \) is even, in view of (5.31) we can write
\[
S_{2,W}(T) = \pi \int_{\mathbb{R}} \int_{\mathbb{R}} F_T(u) [h_1(u+t)h_2(t) + h_1(t)h_2(u+t)] \, dudt. \quad (5.35)
\]
Consequently, taking into account (5.32) and the equality
\[
\int_{\mathbb{R}} h_1(t)h_2(t) \, dt = \int_{\mathbb{R}} h_1(u+t)h_2(u+t) \, dt,
\]
by (5.35) we get
\[
\Delta_{2,W}(T) := \left| \frac{1}{T} \text{tr}[W_T(h_1)W_T(h_2)] - 2\pi \int_{\mathbb{R}} h_1(t)h_2(t) \, dt \right| \quad (5.36)
\]
\[
= \left| \pi \int_{\mathbb{R}} F_T(u) \int_{\mathbb{R}} (h_1(t) - h_1(u+t))(h_2(u+t) - h_2(t)) \, dt \, du \right|.
\]
Using Hölder’s inequality, we find from (5.37)
\[
\Delta_2 W(T) \leq \pi \int_{\mathbb{R}} \mathcal{F}_T(u) \|h_1(u + \cdot) - h_1(\cdot)\|_p \|h_2(u + \cdot) - h_2(\cdot)\|_q du.
\] (5.37)

In view of (5.37) we have
\[
\Delta_2 W(T) \leq C_1 \int_{0}^{1} \mathcal{F}_T(u) |u|^{\gamma_1 + \gamma_2} du + C_2 \|h_1\|_p \|h_2\|_q \int_{u > 1} \mathcal{F}_T(u) du.
\] (5.38)

Therefore, the result follows from (5.33), (5.34) and (5.38).

**Proof of Theorem 3.4.** It follows from Lemma 4.1 that under the assumptions of theorem \( h_i \in \text{Lip}(p_i, 1/p_i - \sigma_i) \), \( i = 1, 2 \). Hence, applying Theorem 3.3 with \( \gamma_i = 1/p_i - \sigma_i \), we obtain (3.16).

6. **Appendix. Proof of Technical Lemmas**

In this section we give proofs of technical lemmas stated and used in Sections 4 and 5.

**Proof of Lemma 4.1.** Let \( h \in (0, 1/2) \) be fixed. Then
\[
\int_{|\lambda| \leq 2h} |f(\lambda + h) - f(\lambda)|^p d\lambda \leq (2C)^p \int_{0}^{3h} |\lambda|^{-p\sigma} d\lambda \leq C_1 h^{1 - p\sigma}. \tag{6.1}
\]

Next, for \( |\lambda| > 2h \) we have \( f(\lambda + h) - f(\lambda) = f'(\xi) \cdot h \) with some \( \xi \in (\lambda, \lambda + h) \).

Hence
\[
\int_{2h < |\lambda| < 1/2} |f(\lambda + h) - f(\lambda)|^p d\lambda \leq C^p h^p \int_{h}^{1/2} |\lambda|^{-p(\sigma + 1)} d\lambda \leq Ch^{1 - p\sigma} \tag{6.2}
\]
and
\[
\int_{1/2 < |\lambda| < \infty} |f(\lambda + h) - f(\lambda)|^p d\lambda \leq C^p h^p \int_{1/2}^{\infty} |\lambda|^{-p(\sigma + 1)} d\lambda + C^p h^p \int_{1}^{\infty} |\lambda|^{-p(\delta + 1)} d\lambda \\
\leq Ch^p \leq Ch^{1 - p\sigma}. \tag{6.3}
\]

From (6.1), (6.2) and (6.3) we get
\[
\|f(\lambda + h) - f(\lambda)\|_p \leq Ch^{1/p - \sigma},
\]
implying \( f \in \text{Lip}(p, 1/p - \sigma) \).

\[\square\]
Proof of Lemma 4.2. We use the technique of [45], where (4.16) was proved for \( \beta = 1 \). Since the underlying process \( X(t) \) is real-valued, we have for \( t > 0 \)
\[
    r(t) := \int_{-\infty}^{+\infty} e^{it\lambda} f(\lambda) \, d\lambda = 2 \int_{0}^{\infty} \frac{C}{\lambda^{2\alpha} (1 + \lambda^2)^{\beta}} \cos(t\lambda) \, d\lambda. \tag{6.4}
\]
Using the change of variable \( \lambda = 1/(tu) \), we obtain
\[
    r(t) = 2C \cdot t^{2\alpha - 1} \int_{0}^{\infty} \left( \frac{(tu)^2}{1 + (tu)^2} \right)^{\beta} u^{2\alpha - 2} \cos(1/u) \, du = 2C \cdot t^{2\alpha - 1} \int_{0}^{\infty} L(tu) k(u) \, du, \tag{6.5}
\]
where
\[
    L(u) = \frac{u^{2\beta}}{(1 + u^2)^{\beta}} \quad \text{and} \quad k(u) = u^{2\alpha - 2} \cos(1/u).
\]
Choose \( \delta > 0 \) such that \( \delta < \min(1 - 2\alpha, 2\alpha) \). Then the improper integrals
\[
    \int_{0+}^{1} u^{-\delta} k(u) \, du \quad \text{and} \quad \int_{1}^{\infty} u^{\delta} k(u) \, du
\]
exist. Therefore, by the Bojanic-Karamata theorem (see [8], Th.4.1.5) we have
\[
    \int_{0}^{\infty} k(u) L(tu) \, du \longrightarrow \int_{0+}^{\infty} k(u) \, du \quad \text{as} \quad t \to \infty. \tag{6.6}
\]
Next, using the change of variable \( 1/u = \nu \) and the formula (see, e.g., [16])
\[
    \int_{0}^{\infty} x^{-p} \cos(mx) \, dx = \frac{\pi m^{p-1}}{2 \cos(p\pi/2)\Gamma(p)}, \quad 0 < p < 1, \quad m > 0,
\]
with \( p = 2\alpha \) and \( m = 1 \), we obtain
\[
    \int_{0+}^{\infty} k(u) \, du = \int_{0}^{\infty} u^{2\alpha - 2} \cos(1/u) \, du = \frac{\pi}{2 \cos(\pi\alpha)\Gamma(2\alpha)}. \tag{6.7}
\]
From (6.4)–(6.7) we get (4.16). \( \square \)

Proof of Lemma 5.3. The proof of properties a) - c) can be found in [5], Lemma 3.2 (see also [30], Lemma 2). To prove d) first observe that for \( T > 0 \)
\[
    \int_{\mathbb{R}} |D_T(u)|^{p(m)} \, du \leq C \cdot T^{p(m) - 1} \quad \text{and} \quad |D_T(u)| \leq C_\delta \quad \text{for} \quad |u| > \delta. \tag{6.8}
\]
For \( u = (u_1, \ldots, u_{m-1}) \in \mathbb{R}^{m-1} \) we have
\[
    \int_{E_{\delta}} \Phi^p_T(m)(u) \, du \leq \int_{|u_1| > \delta} \Phi^p_T(m)(u) \, du + \int_{|u_2| > \delta} \Phi^p_T(m)(u) \, du \\
    + \ldots + \int_{|u_{m-1}| > \delta} \Phi^p_T(m)(u) \, du \\
    =: I_1 + I_2 + \ldots + I_{m-1}. \tag{6.9}
\]
Likewise, we have
\[ I_1 \leq \int_{|u_1| > \delta, |u_2| > \delta / m} \Phi_T^{(m)}(u)du + \int_{|u_1| > \delta, |u_m| > \delta / m} \Phi_T^{(m)}(u)du + \int_{|u_1| > \delta, |u_2| \leq \delta / m, |u_m| \leq \delta / m} \Phi_T^{(m)}(u)du =: I_1^{(2)} + \ldots + I_1^{(m-1)} + I_1^{(m)}. \] (6.10)

According to (6.8)
\[ I_1^{(2)} \leq C_\delta \cdot \frac{1}{T^{p(m)}} \int_{|u_2| > \delta / m} |D_T(u_2)|^{p(m)} \ldots |D_T(u_{m-1})|^{p(m)} \times |D_T(u_1 + \ldots + u_{m-1})|^{p(m)} du_1 du_{m-1} \ldots du_2 \leq C_\delta \cdot \frac{1}{T^{p(m)}} \cdot T^{(p(m)-1)(m-2)} \int_{|u_2| > \delta / m} \frac{1}{|u_2|^{p(m)}} du_2 \leq C_\delta. \] (6.11)

Likewise,
\[ I_1^{(j)} \leq C_\delta, \quad j = 3, \ldots, m - 1. \] (6.12)

Next, observe that in the integral \( I_1^{(m)} \), we have \( |u_1 + \ldots + u_{m-1}| > \delta / m \). Hence by (6.8)
\[ I_1^{(m)} \leq C_\delta \cdot \frac{1}{T^{p(m)}} \int_{|u_1| > \delta} D_T^{(m)}(u_1) \cdots D_T^{(m)}(u_{m-1}) du_2 \cdots du_{m-1} du_1 \leq C_\delta \int_{|u_1| > \delta} \frac{1}{|u_1|^{p(m)}} du_1 \leq C_\delta. \] (6.13)

From (6.9) – (6.13) we obtain (5.6). Lemma 5.3 is proved.

Proof of Lemma 5.5. Using Lemma 5.4 and the notation \( du = du_n du_{n-1} \cdots du_1 \), we can write
\[ B_1 \leq \int_{\{|u_1| \leq 1\}} \frac{1}{|u_1|^{\beta-\alpha}} \int_{\mathbb{R}^{n-2}} \frac{1}{|u_2 \cdots u_{n-1}|^{\beta}} \int_{\mathbb{R}} \frac{1}{|u_n(u_1 + \cdots + u_n)|^{\beta}} du \]
\[ \leq C \int_{\{|u_1| \leq 1\}} \frac{1}{|u_1|^{\beta-\alpha}} \int_{\mathbb{R}^{n-2}} \frac{1}{|u_2 \cdots u_{n-1}|^{\beta} (u_1 + \cdots + u_{n-1})^{2\beta-1}} du_{n-1} \cdots du_1 \]
\[ \leq C^2 \int_{\{|u_1| \leq 1\}} \frac{1}{|u_1|^{\beta-\alpha}} \int_{\mathbb{R}^{n-3}} \frac{1}{|u_2 \cdots u_{n-2}|^{\beta} (u_1 + \cdots + u_{n-2})^{3\beta-2}} du_{n-2} \cdots du_1 \]
\[ I \leq \int_{|u_1| > 1} \cdot \cdot \cdot + \int_{|u_n| > 1} \frac{1}{|u_1 \cdot \cdot \cdot u_n(u_1 + \cdot \cdot \cdot + u_n)^\beta} \, du_1 \cdot \cdot \cdot du_n =: I_1 + \cdot \cdot \cdot + I_n. \]  

Using Lemma 5.4 we get

\[ I_1 \leq \int_{|u_1| > 1} \frac{1}{|u_1|^\beta} \int_{\mathbb{R}^{n-2}} \frac{1}{|u_2 \cdot \cdot \cdot u_{n-1}|^\beta} \int_{\mathbb{R}} \frac{1}{|u_n(u_1 + \cdot \cdot \cdot + u_n)|^\beta} \, du_{n-1} \cdot \cdot \cdot du_1 \]

\[ \leq C \int_{|u_1| > 1} \frac{1}{|u_1|^\beta} \int_{\mathbb{R}^{n-2}} \frac{1}{|u_2 \cdot \cdot \cdot u_{n-1}|^\beta} |u_1 + \cdot \cdot \cdot + u_{n-1}|^{2(\beta - 1)} \, du_{n-1} \cdot \cdot \cdot du_1 \]

\[ \leq \quad \cdot \cdot \cdot \quad \int_{|u_1| > 1} \frac{1}{|u_1|^{(n+1)\beta-n+1}} \, du_1 < \infty. \]

The quantities \( I_2, \ldots, I_n \) can be estimated in the same way, and by (6.14) the result follows.

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