HOOK-CO NTENT FORMULA USING EXCITED YOUNG DIAGRAMS

ANATOL N. KIRILLOV AND TRAVIS SCRIMSHAW

Abstract. We construct a hook-content formula and its $q$-analog using excited Young diagrams analogous to Naruse’s hook-length formula for skew shapes. Furthermore, we show that our hook-content formula has a simple factorization and give some conjectures and questions related to its $q$-analog.

1. Introduction

The hook-length formula for the number of standard Young tableaux of skew shape $\lambda/\mu$

$$f_{\lambda/\mu} := |\lambda/\mu|! \sum_{D \in \mathcal{E}(\lambda/\mu)} \prod_{d \in \lambda \setminus D} \frac{1}{h(d)}, \quad (1.1)$$

where $\mathcal{E}(\lambda/\mu)$ is the set of excited Young diagrams [Kre05, IN09] and $h(d)$ is the hook length of $d$ in $\lambda$, was discovered by Naruse [Nar14] from his study of the equivariant cohomology of the Grassmannian. Combinatorial proofs of Equation (1.1) have also been given in [Kon18, MPP18]. When $\mu = \emptyset$, Equation (1.1) reduces to the classical hook-length formula for standard tableaux first proven by Frame, Robinson, and Thrall [FRT54] and has since seen numerous proofs (see, e.g., [Ban08, MPP18, Sag90] and references therein).

In [MPP18], a $q$-analog of Equation (1.1) was given as

$$s_{\lambda/\mu}(1, q, q^2, \ldots) = \sum_{D \in \mathcal{E}(\lambda/\mu)} \prod_{(i,j) \in \lambda \setminus D} \frac{q^{\lambda'_i-i}}{1-q^{h(i,j)}}, \quad (1.2)$$

where the left hand side is the principal specialization of the (skew) Schur function and $\lambda'$ is the conjugate partition to $\lambda$. When taking $\mu = \emptyset$, we obtain the $q$-analog of the hook-length formula due to Stanley [Sta71]:

$$s_{\lambda}(1, q, q^2, \ldots) = q^{b(\lambda)} \prod_{d \in \lambda/\mu} \frac{1}{1-q^{h(d)}}, \quad (1.3)$$

where $b(\lambda) = \sum_{i=1}^{|\lambda|} (i-1)\lambda_i$. After removing the $q^{b(\lambda)}$ factor, Equation (1.3) is equal to the number of reverse plane partitions graded by their size, where a combinatorial proof is given by the Hillman–Grassl correspondence [HG76].

To count the number of semistandard Young tableaux of shape $\lambda$ and maximum entry $n$, we instead use the hook-content formula with its natural $q$-analog given by

$$s_{\lambda}(1, q, \ldots, q^{n-1}, 0, 0, \ldots) = q^{b(\lambda)} \prod_{d \in \lambda} \frac{[n+c(d)]_q}{[h(u)]_q}, \quad (1.4)$$

where $[x]_q = \frac{1-q^x}{1-q}$ is the natural $q$-analog of $x$ (see, e.g., [Sta99, Thm 7.21.2]) and $c(d)$ is the content of $d$. Indeed, we see that when taking the limit $q \to 1$, we
obtain a formula for the number of semistandard Young tableaux of shape $\lambda$ and maximum entry $n$.

The goal of this note is to examine a natural hook-content generalization of Naruse’s hook-length formula by combining Equation (1.1) and Equation (1.4). We show that the result has a simple factorization as a product of $q$-integers of binomials in $n$. Our result gives rise to many interesting conjectures and questions related to our formula, the natural $q$-analogue of $f^{\lambda/\mu}$, and results related to representation theory. In particular, we note that our formula (when $q \to 1$) does not count the number of semistandard skew tableaux of shape $\lambda/\mu$. Thus, finding a combinatorial formula (in particular using excited Young diagrams) for the principal specializations of skew Schur functions

$$s_{\lambda/\mu}(1, q, \ldots, q^{n-1}, 0, 0, \ldots)$$

remains an open problem. Yet, our results might aid in understanding the relationship between excited Young diagrams and the representation theory of the symmetric group $S_n$ and/or $\mathfrak{gl}_n$ as

$$s_{\lambda/\mu} = \sum_{\nu} c^\lambda_{\mu, \nu} s_\nu, \quad f^{\lambda/\mu} = \sum_{\nu} c^\lambda_{\mu, \nu} f_\nu,$$

where $c^\lambda_{\mu, \nu}$ are the Littlewood-Richardson coefficients.

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2. Preliminaries

A partition is a weakly decreasing sequence of positive integers. We equate a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ with a set of cells \{(i, j) \mid 1 \leq j \leq \ell, 1 \leq i \leq \lambda_j\} via the Young diagram of $\lambda$. We will consider our Young diagrams using English convention. For a partition $\mu \subseteq \lambda$, we form the skew partition $\lambda/\mu$ as the set of cells $\lambda \setminus \mu$. More generally, we call any finite set of cells $D \subseteq \mathbb{Z}^2$ a diagram. The size of a diagram $|D|$ is the number of cells in $D$.

Let $\lambda' = (\lambda'_1, \lambda'_2, \ldots, \lambda'_m) = \{(j, i) \mid (i, j) \in \lambda\}$, where $m = \lambda_1$, be the conjugate partition to $\lambda$. Let

$$c(d) := j - i, \quad h(d) := \lambda_i - j + \lambda'_j - i + 1,$$

be the content and hook length, respectively, of a cell $d \in \lambda$. Recall that the content of a cell $d$ is the diagonal the cell lies on and the hook length is the number of boxes in the row and column to the right and below, respectively, $d$, including also $d$ (i.e., the size of the largest hook shape whose corner is at $d$).

Let $\lambda/\mu$ be a skew partition with $|\lambda/\mu| = n$. A standard tableau of (skew) shape $\lambda/\mu$ is a bijection $T: \lambda/\mu \to \{1, \ldots, n\}$ such that every row (resp. column) is increasing when read left to right (resp. top to bottom). Let $f^{\lambda/\mu}$ denote the number of standard tableau of shape $\lambda/\mu$. A semistandard tableau of (skew) shape $\lambda/\mu$ is a function $T: \lambda/\mu \to \mathbb{Z}_{\geq 0}$ such that rows are weakly increasing and columns are strictly increasing. Let $\text{SST}^n(\lambda/\mu)$ denote the set of semistandard Young tableaux of shape $\lambda/\mu$ with maximum entry $n$, and we simply write $\text{SST}(\lambda/\mu)$ when $n = \infty$. We will simply write $\lambda$ for $\lambda/\mu$ when $\mu = \emptyset$.

Following [IN09], define an elementary excitation on a diagram $D$ to take a cell $(i, j) \in D$ such that $(i+1, j), (i, j+1), (i+1, j+1) \not\in D$ and forming a new diagram
by \((D \setminus \{(i, j)\}) \cup \{(i + 1, j + 1)\}\). Pictorially, an elementary excitation moves the cell in \((i, j)\) (locally) as

\[
\begin{array}{c|c|c}
& & \\
& & \\
& & \\
\end{array}
\rightarrow
\begin{array}{c|c|c}
& & \\
& & \\
& & \\
\end{array}
\]

Define the set of excited Young diagrams \(\mathcal{E}(\lambda/\mu)\) to be all diagrams obtained from \(\mu\) using a sequence of elementary excitations such that the resulting diagram is contained inside \(\lambda\).

### 3. Hook-Content Formula Using Excited Young Diagrams

Let \([n]_q! = [n]_q[n - 1]_q \cdots [1]_q\) denote the \(q\)-factorial. We define

\[
f_q^{\lambda/\mu} := [[\lambda/\mu]]_q! \sum_{D \in \mathcal{E}(\lambda/\mu)} \prod_{d \in \lambda \setminus D} \frac{1}{[h(d)]_q}
\]

as the natural \(q\)-analog of \(f^{\lambda/\mu}\). Note that \(\lim_{q \to 1} f_q^{\lambda/\mu} = f^{\lambda/\mu}\) by Equation (1.1).

**Theorem 3.1.** Let \(\mu \subseteq \lambda\). We have

\[
H_{\lambda/\mu}(n; q) := [[\lambda/\mu]]_q! \sum_{D \in \mathcal{E}(\lambda/\mu)} \prod_{d \in \lambda \setminus D} \frac{1 - q^{n+c(d)}}{1 - q^{h(d)}} = f_q^{\lambda/\mu} \prod_{d \in \lambda/\mu} [n + c(d)]_q.
\]

**Proof.** We first note that

\[
C_{\lambda/\mu}(q) := \prod_{d \in \lambda \setminus D} [n + c(d)]_q
\]

does not depend on the choice of excited Young diagram \(D \in \mathcal{E}(\lambda/\mu)\) as an elementary excitation moves a box along a diagonal \(j - i\), which does not change its content. Thus, we take \(C_{\lambda/\mu}(q)\) to be with \(D = \mu\). Hence, we have

\[
H_{\lambda/\mu}(n; q) = \frac{1}{[[\lambda/\mu]]_q!} \sum_{D \in \mathcal{E}(\lambda/\mu)} \prod_{d \in \lambda \setminus D} \frac{1 - q^{n+c(d)}}{1 - q^{h(d)}} = \frac{1}{[[\lambda/\mu]]_q!} \sum_{D \in \mathcal{E}(\lambda/\mu)} \prod_{d \in \lambda \setminus D} \frac{[n + c(d)]_q}{[h(d)]_q} = C_{\lambda/\mu}(q) f_q^{\lambda/\mu}
\]
as desired. \(\square\)

As a special case of Theorem 3.1 when \(\mu = \emptyset\), Equation (1.4) implies that

\[
s_\lambda(1, q, \ldots, q^{n-1}, 0, 0, \ldots) = q^{b(\lambda) - h(\lambda)} \frac{H_{\lambda}(n; q) [[\lambda]]_q!}{[[\lambda]]_q!}.
\]

**Corollary 3.2.** Let \(\mu \subseteq \lambda\). Then we have

\[
H_{\lambda/\mu}(n; 1) = \frac{1}{[[\lambda/\mu]]_q!} \sum_{D \in \mathcal{E}(\lambda/\mu)} \prod_{d \in \lambda \setminus D} \frac{n + c(d)}{h(d)} = f_q^{\lambda/\mu} \prod_{d \in \lambda/\mu} n + c(d).
\]

**Proof.** This follows from Theorem 3.1 by taking the limit \(q \to 1\) with applying L’Hôpital’s rule and Naruse’s hook-length formula (Equation (1.1)). \(\square\)

We note that we could have proven Corollary 3.2 directly using a similar argument to Theorem 3.1 and Naruse’s hook-length formula. Furthermore, Corollary 3.2 is equivalent to Naruse’s hook-length formula. To simplify our notation, we write \(H_{\lambda/\mu}(n) := H_{\lambda/\mu}(n; 1)\).
Corollary 3.3. Assume Corollary 3.2 holds, then we have
\[ \lim_{n \to \infty} \frac{H_{\lambda/\mu}(n)}{n^{\lambda/\mu}} = f^{\lambda/\mu}. \]

Proof. Note that \((n + c(d))/n \to 1\) as \(n \to \infty\), and the claim follows from Corollary 3.2 and the degree of \(H_{\lambda/\mu}(n)\) (which is a polynomial in \(n\)) is \(|\lambda/\mu|\). \(\square\)

To obtain the classical hook-content formula for \(\lambda\) and \(\mu = \emptyset\), we must divide \(H_{\lambda/\mu}(n)\) by \(|\lambda|!\) as in Equation (3.1). Therefore, we define the polynomial
\[ H_{\lambda/\mu}(n) := \frac{H_{\lambda/\mu}(n)}{|\lambda|!}, \]
and note that \(\overline{H}_{\lambda}(n) = |\text{SST}^n(\lambda)|\) by the hook-content formula.

Example 3.4. The excited Young diagrams \(E(3321/21)\) are

First, we compute
\[ f^{3321/21}_q = q^{10} + q^9 + 3q^8 + 6q^7 + 8q^6 + 8q^5 + 9q^4 + 10q^3 + 5q^2 + 4q + 5. \] (3.2)

Completing the computation and factoring the result, we see that
\[ H_{3321/21}(n; q) = f^{3321/21}_q[n - 3]_q[n - 2]_q[n - 1]_q[n]_q[n + 1]_q[n + 2]_q. \]

We remark that \(f^{3321/21}_q = H_{3321/21}(4; q)/[6]_q!\). By taking \(q \to 1\), we obtain
\[ \overline{H}_{3321/21}(n) = \frac{61}{720} (n - 3)(n - 2)(n - 1)n(n + 1)(n + 2). \] as \(f^{3321/21} = 61.\)

Example 3.5. There are five excited diagrams of type (553,321):

which yields the \(q\)-standard tableau number of
\[ f_q^{553/321} = \frac{(q^6 + q^5 + q^4 + q^3 + q^2 + q + 1) \cdot a(q)}{(q + 1) \cdot (q^4 + q^3 + q^2 + q + 1)}, \] (3.3)
where
\[ a(q) = q^{12} + 2q^{11} + 4q^{10} + 7q^9 + 12q^8 + 14q^7 + 17q^6 + 18q^5 + 18q^4 + 14q^3 + 11q^2 + 7q + 5, \]
and a hook-content formula (and \( q \to 1 \) version) of
\[
H_{553/321}(n; q) = f_{553/321}^{553/321}[n-1]_q[n]_q[n+1]_q[n+2]_q[n+3]_q^2[n+4]_q,
\]
\[
\overline{H}_{553/321}(n) = \frac{91}{5040}(n-1)n(n+1)(n+2)(n+3)^2(n+4).
\]

It is not obvious that \( \overline{H}_{\lambda/\mu}(n) \) is an integer for all integers \( n \geq \ell \), where \( \ell \) is the length of \( \lambda \). However, we have verified this in numerous cases and have the following conjecture.

**Conjecture 3.6.** Let \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell) \) be a partition. Let \( n \geq \ell \) be an integer. Then \( \overline{H}_{\lambda/\mu}(n) \in \mathbb{Z}_{\geq 0} \).

Thus, if Conjecture 3.6 is true, a natural question to ask is what does \( \overline{H}_{\lambda/\mu}(n) \) count? A first guess would likely be semistandard skew tableaux of shape \( \lambda/\mu \) and maximum entry \( n \), but this is not the case. Indeed, we have \( \overline{H}_{3321/21}(4) = 61 \), but there are 204 semistandard skew tableaux of shape 3321/21 and maximum entry 4. Therefore, we suggest the following problem.

**Problem 3.7.** Assuming Conjecture 3.6, determine what objects count \( \overline{H}_{\lambda/\mu}(n) \).

We note that the principal specialization \( s_{\lambda/\mu}(1, q, \ldots, q^{n-1}, 0, \ldots) \) was considered in [MPP18, Sec. 8]. Yet this cannot be related to our \( q \)-hook-content formula as they have different \( q \to 1 \) limits as noted above.

We note that \( f_{\lambda/\mu}^{\lambda/\mu} \) (and hence \( H_{\lambda/\mu}(n; q)/[[\lambda/\mu]]_q \) for a fixed integer \( n \in \mathbb{Z}_{\geq 0} \)) is not symmetric nor unimodal as seen in Equation (3.2). In fact, \( f_{\lambda/\mu}^{\lambda/\mu} \) is not always polynomial by Equation (3.3) in contrast to Conjecture 3.6. Furthermore, even when \( f_{\lambda/\mu}^{\lambda/\mu} \in \mathbb{Z}_{\geq 0}[q] \), the value \( H_{\lambda/\mu}(n; q)/[[\lambda/\mu]]_q \) is not always a polynomial for a fixed integer \( n \geq \ell \):

\[
H_{3322/21}(4; q) = \frac{f(q)}{[7]_q!} = \frac{f(q)}{q^4 + q^3 + q^2 + q + 1},
\]
where
\[
f(q) = q^{12} + 2q^{11} + 4q^{10} + 7q^9 + 12q^8 + 14q^7 + 17q^6 + 18q^5 + 18q^4 + 14q^3 + 11q^2 + 7q + 5.
\]
Note also that \( f(q) \) is an irreducible polynomial over \( \mathbb{Q} \). Yet, we do have the following conjectures based on experimental evidence.

**Conjecture 3.8.** Let \( \mu \subset \lambda \) be partitions. We have \( f_{\lambda/\mu}^{\lambda/\mu} = a(q)/b(q) \), where \( a, b \in \mathbb{Z}_{\geq 0}[q] \) such that \( a(-1) \in \mathbb{Z}_{\geq 0} \).

**Conjecture 3.9.** Let \( \mu \subset \lambda \) be partitions. Fix some integer \( n \geq \ell \), where \( \ell \) is the length of \( \lambda \). We have \( H_{\lambda/\mu}(n; q)/[[\lambda/\mu]]_q = a(q)/b(q) \), where \( a, b \in \mathbb{Z}_{\geq 0}[q] \) such that \( a(-1) \in \mathbb{Z}_{\geq 0} \).

Note that \( g \) in both conjectures must be a product of cyclotomic polynomials since the denominator is a product of \( q \)-integers. The examples above also suggests the following problems.

**Problem 3.10.** Determine which partitions \( \mu \subset \lambda \) such that \( f_{\lambda/\mu}^{\lambda/\mu} \in \mathbb{Z}_{\geq 0}[q] \) and also for which \( n \in \mathbb{Z}_{\geq 0} \) such that \( H_{\lambda/\mu}(n; q)/[[\lambda/\mu]]_q \in \mathbb{Z}_{\geq 0}[q] \).

**Problem 3.11.** For which partitions \( \mu \subset \lambda \) the all terms in Naruse’s hook-length formula and its \( q \)-analog are integers and in \( \mathbb{Z}_{\geq 0}[q] \), respectively?
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(A. Kirillov) Research Institute for Mathematical Sciences (RIMS), Kyoto University, Kyoto 606-8502, Japan; The Kavli Institute for the Physics and Mathematics of the Universe (IPMU), 277-8583, Kashiwanoha, Japan; Department of Mathematics, National Research University Higher School of Economics (HES), 7 Vavilova Str., 117312, Moscow, Russia
E-mail address: kirillov@kurims.kyoto-u.ac.jp
URL: http://www.kurims.kyoto-u.ac.jp/~kirillov/

(T. Scrimshaw) School of Mathematics and Physics, The University of Queensland, St. Lucia, QLD 4072, Australia
E-mail address: tcs scrim@gmail.com
URL: https://people.ens.STRU/TravisScrimshaw/