GENERALIZED LOOSE EDGE FACTORIZATION THEOREMS

BERND SCHÖBER

Abstract. We extend a factorization theorem by Gwoździewicz and Hejmej from the ring of formal power series to any complete regular local ring $R$. More precisely, let $f \in R$ and assume that its Newton polyhedron has a loose edge such that the initial formal of $f$ along the latter is a product of two coprime polynomials, where one of them is not divided by any variable. Then this provides a factorization of $f$ in $R$. As a consequence we obtain a factorization theorem for Weierstraß polynomials with coefficients in $R$, which generalizes an earlier result by Rond and the author.

1. Introduction

The goal of this article is a generalization of a factorization result by Gwoździewicz and Hejmej from $\mathbb{K}[[x_1, \ldots, x_n]]$ to any (complete) regular local ring. (See the results below). We reduce their proof to the essence which is the fact that $\mathbb{K}[[x_1, \ldots, x_n]]$ is complete and provide a new viewpoint via projections of the Newton polyhedron. Note that $R$ may even have mixed characteristics.

Let $(R, m, \mathbb{K} = R/m)$ be a regular local ring (not necessarily complete) with regular system of parameters $(x) = (x_1, \ldots, x_n)$. We denote by $\hat{R}$ the $m$-adic completion of $R$. (If $R$ is complete, we have $\hat{R} = R$). For an element $f \in R$, one can define the notion of a Newton polyhedron $NP(f) \subset \mathbb{R}_{\geq 0}^n$. The latter is a closed convex set with the property $NP(f) + \mathbb{R}_{\geq 0}^n = NP(f)$ and coming from the set of exponents of an expansion of $f$. A loose edge of $NP(f)$ is a compact face of dimension one, say $E \subset NP(f)$, that is not contained in any compact face of $NP(f)$ of dimension $\geq 2$. (In fact, we study loose edges in the slightly more general setting of $F$-subsets later). Associated to such an edge, we have the initial form in $E(f)$ of $f$ (determined by those terms of an expansion contributing to the edge in the polyhedron) which lies in a graded ring $\text{gr}_E(R)$ that is isomorphic to a polynomial ring $\mathbb{K}[X_1, \ldots, X_n]$. (In the remaining introduction, we use this identification without mentioning). For more details on these objects, we refer to sections 2 and 3.

Theorem 1.1. Let $R$ be a regular local ring. Let $f \in R$ be an element in $R$ such that the Newton polyhedron $NP(f)$ has a loose edge $E$. Suppose that the initial form in $E(f)$ of $f$ along $E$ is a product of two coprime polynomials $G$ and $H$, where $G$ is not divided by any variable. Then there exist elements $\hat{g}, \hat{h} \in \hat{R}$ in the completion of $R$ such that

$$f = \hat{g} \cdot \hat{h} \quad \text{in} \ \hat{R}$$

and $\text{in}_{E_1}(\hat{g}) = G$ and $\text{in}_{E_2}(\hat{h}) = H$, for certain faces $E_1, E_2$ of dimension at most one such that $E = E_1 + E_2$.

We cannot avoid to pass to the completion since the construction of the elements $\hat{g}$ and $\hat{h}$ is not necessarily finite and hence may lead to an infinite series.

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Corollary 1.2. Assume that the Newton polyhedron of \( f \in R \) has a loose edge and at least three vertices. Then \( f \) is not irreducible in \( \hat{R} \).

Corollary 1.3. Assume that the Newton polyhedron of \( f \in R \) has a loose edge \( \mathcal{E} \). If \( f \) is irreducible in \( \hat{R} \), then \( \mathcal{E} \) is the only compact edge of \( \text{NP}(f) \) and

\[
in_{\mathcal{E}}(f) = \epsilon \cdot P^k,
\]

where \( \epsilon \in K^\times \) is a unit and \( P \in K[X_1, \ldots, X_n] \) is an irreducible polynomial. Moreover, if the residue field \( K \) is algebraically closed, \( P = X^\alpha + \lambda X^\beta \), for some \( \lambda \in K^\times \) and \( \alpha - \beta \in \mathbb{Z}^n \) is a primitive lattice vector.

An edge \( \mathcal{E} \) is called descendant if it is parallel to a vector \( \delta = (\delta_1, \ldots, \delta_n) \in \mathbb{R}^n \) such that \( \delta_i \geq 0 \), for \( 1 \leq i \leq n - 1 \), and \( \delta_n < 0 \).

Theorem 1.4. Let \( f \in R[z] \). Assume that the Newton polyhedron \( \text{NP}(f) \) has a descendant, loose edge \( \mathcal{E} \). If \( \text{in}_{\mathcal{E}}(f) \) is a product of two coprime polynomials \( G, H \in K[X_1, \ldots, X_n, Z] \), where \( G \) is monic with respect to \( Z \), then there exist \( \hat{g}, \hat{h} \in \hat{R}[z] \) such that

\[
f = \hat{g} \cdot \hat{h} \quad \text{in} \quad \hat{R}[z]
\]

where \( \hat{g} \in \hat{R}[z] \) is monic, \( \text{in}_{\mathcal{E}}(\hat{g}) = G \) and \( \text{in}_{\mathcal{E}}(\hat{h}) = H \), for certain faces \( \mathcal{E}_1, \mathcal{E}_2 \) of dimension at most one such that \( \mathcal{E} = \mathcal{E}_1 + \mathcal{E}_2 \).

In Theorems 1.1 and 1.4, \( \mathcal{E}_1 \) is a compact edge of \( \text{NP}(\hat{g}) \) parallel to \( \mathcal{E} \) and \( \mathcal{E}_2 \) is either a compact edge of \( \text{NP}(\hat{h}) \) parallel to \( \mathcal{E} \) or a vertex.

Theorems 1.1, 1.4 and Corollaries 1.2, 1.3 generalize Theorems 1.1, 1.4 and Corollaries 1.2, 1.3 of [GHe], where the case \( R = K[[x_1, \ldots, x_n]] \) is considered. The key step to transfer the proofs of [GHe] into the more general setting is to lift an element \( G \in \text{gr}_\mathcal{E}(R) \) of the graded ring to an element \( g \in R \). (Note that \( g \) is not unique in general. Besides that, we provide a different perspective on the refinement of the grading of \( \text{gr}_\mathcal{E}(R) \), by considering the projection of \( \mathbb{R}_{\geq 0}^n \) along the vector \( \delta \in \mathbb{R}^n \) determined by the direction of the edge \( \mathcal{E} \). In particular, we show that \( \mathcal{E} \) being loose implies that the projection of the Newton polyhedron along \( \delta \) has exactly one vertex corresponding to \( \mathcal{E} \) (Lemma 3.10). In contrast to [GHe], we formulate convex geometry results that are used to study Newton polyhedra with loose edges in a more general variant in terms of \( F \)-subsets.

In [GHe] section 3, one may find other known results for which our results can be considered as generalizations. In particular, Theorem 1.4 is some kind of generalization of a result by Rond and the author [RS], where \( R = K[[x_1, \ldots, x_n]] \), for any field \( K \) and \( f \in R[z] \) is a Weierstraß polynomial of degree \( d \) such that the projection of \( \text{NP}(f) \) along an edge containing \( (0, \ldots, 0, d) \in \mathbb{R}_{\geq 0}^{n+1} \) has exactly one vertex. This type irreducibility criterion is very useful in the study of quasi-ordinary hypersurfaces (see [ACLM] or [MS]). Therefore, our main results open interesting new directions in the context of constructing Teissier’s overweight deformations [T] following the philosophy of [MS].

Throughout the article, we use multi-index notation: \( x^A := x_1^{a_1} \cdots x_n^{a_n} \) for some \( A = (A_1, \ldots, A_n) \in \mathbb{Z}_{\geq 0}^n \).

2. Newton Polyhedron and Graded Rings

We provide the definitions of the Newton polyhedron and the initial form along a face of the Newton polyhedron.

Let \((R, m, K = R/m)\) be a regular local ring (not necessarily complete) and let \((x) = (x_1, \ldots, x_n)\) be a regular system of parameters for \( R \). We consider \( f \in R \setminus \{0\} \).
Since $R$ is Noetherian and since the map $R \subset \hat{R}$ is faithfully flat, $f$ has a finite expansion

$$f = \sum A \rho_A x^A, \quad \text{for } \rho_A \in R^x \cup \{0\}.$$ 

The *Newton polyhedron* $\text{NP}(f) := \text{NP}(f, x)$ of $f$ is defined as the smallest closed convex subset of $\mathbb{R}^n_{\geq 0}$ containing all points of the set

$$\{ A \in \mathbb{Z}^n_{\geq 0} \mid \rho_A \neq 0 \} + \mathbb{R}^n_{\geq 0}.$$ 

A linear form $L = L_\lambda : \mathbb{R}^n \to \mathbb{R}$ is a map defined by

$$L(v) := \lambda_1 v_1 + \ldots + \lambda_n v_n = \langle \lambda, v \rangle,$$

for $v = (v_1, \ldots v_n) \in \mathbb{R}^n$ and some fixed $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n_{\geq 0}$. Given $L$, we define

$$\Delta(L) := \{ v \in \mathbb{R}^n_{\geq 0} \mid L(v) \leq 1 \}.$$ 

If $\lambda \in \mathbb{Q}^n_{\geq 0}$, then $L$ is called rational. If $\lambda \in \mathbb{R}^n_{\geq 0}$, then we say that $L$ is positive.

A closed convex subset $\Delta \subset \mathbb{R}^n_{\geq 0}$ such that $\Delta + \mathbb{R}
_{\geq 0} = \Delta$ is called a $F$-subset of $\mathbb{R}^n_{\geq 0}$, see [H] p. 260. We extend this notion by calling a closed convex subset $\Delta \subset \mathbb{R}^n$ a $\hat{F}$-subset if $\Delta + \mathbb{R}^n_{\geq 0} = \Delta$. Clearly, $\text{NP}(f)$ is an example of a $F$-subset.

**Definition 2.1.** Let $\Delta \subset \mathbb{R}^n_{\geq 0}$ be a $F$-subset of $\mathbb{R}^n$. A convex subset $\mathcal{F} \subset \Delta$ is called a face of $\Delta$ if there exists a linear form $L$ such that

$$\Delta \cap \Delta(L) = \mathcal{F}.$$ 

If $L$ is positive, then $\mathcal{F}$ defines a compact face. A vertex of $\Delta$ is a compact face $v \in \Delta$ of dimension zero. An edge of $\Delta$ is a compact face $E \subset \Delta$ of dimension one.

A positive linear form $L : \mathbb{R}^n \to \mathbb{R}$, induces a monomial valuation $\nu_L$ on $R$ via

$$\nu_L(\rho x^A) := L(A), \quad \text{for } \rho \in R^x, \ A \in \mathbb{Z}^n_{\geq 0}.$$ 

For $f = \sum A \rho_A x^A \in R \setminus \{0\}$ as before, we have

$$\nu_L(f) = \min \{ L(A) \mid A \in \mathbb{Z}^n_{\geq 0} : \rho_A \in R^x \}.$$ 

**Definition 2.2.** Let $(R, m, K)$ be as before and let $L : \mathbb{R}^n \to \mathbb{R}$ be a positive linear form. The graded ring of $R$ associated to $L$ is defined as

$$\text{gr}_L(R) := \bigoplus_{\rho \in \mathbb{R}^n_{\geq 0}} \mathcal{P}_a/\mathcal{P}_a^+, \quad \text{where } \mathcal{P}_a := \{ f \in R \mid \nu_L(f) \geq a \} \text{ and } \mathcal{P}_a^+ := \{ f \in R \mid \nu_L(f) > a \}.$$ 

Let $f = \sum A \rho_A x^A \in R$ be as before. The $L$-initial form of $f$ is defined as

$$\text{in}_L(f) := \text{in}_L(f)_x := \sum_{A : L(A) = \nu_L(f)} \mathcal{P}_a X^A \in \mathcal{P}_{\nu_L(f)}/\mathcal{P}_{\nu_L(f)}^+ \subset \text{gr}_L(R),$$

where $\mathcal{P}_a := \rho_A \mod m \in K$ and $(X) = (X_1, \ldots, X_n)$ denotes the images of $(x)$ in $\text{gr}_L(R)$.

Since $L$ takes only values in a discrete subset of $\mathbb{R}$, the set $\{ a \in \mathbb{R}_{\geq 0} \mid \mathcal{P}_a/\mathcal{P}_a^+ \neq 0 \}$ is a discrete subset of $\mathbb{R}$. We observe that $\text{in}_L(f)$ is weighted homogenous of degree $\nu_L(f)$ with respect to the weights on $(x)$ given by $L$. Since $L$ is positive, we have

$$\text{gr}_L(R) \cong K[X_1, \ldots, X_n].$$

**Definition 2.3.** Let $f = \sum A \rho_A x^A \in R$ be as before. Let $\mathcal{F} \subset \text{NP}(f)$ be a compact face of the Newton polyhedron and let $L_\mathcal{F} : \mathbb{R}^n \to \mathbb{R}$ be a positive linear form determining $\mathcal{F}$. The initial form of $f$ along $\mathcal{F}$ is defined as the $L_\mathcal{F}$-initial form of $f$,

$$\text{in}_\mathcal{F}(f) := \text{in}_{L_\mathcal{F}}(f) \in \text{gr}_\mathcal{F}(R) := \text{gr}_{L_\mathcal{F}}(R) \cong K[X_1, \ldots, X_n].$$

Without loss of generality, we can choose $L_\mathcal{F}$ rational.
3. Loose Edges and Projected Polyhedra

We recall the notion of a loose edge and some of their properties proven in [GHe]. Furthermore, we provide a different viewpoint via a suitable projection of a given \( F \)-subset. Even though [GHe] considers only the case \( R = \mathbb{K}[x_1, \ldots, x_n] \), the proofs apply in our more general setting since the statements are either on the convex geometry of a \( F \)-subset or on the properties of the graded ring \( \text{gr}_L(R) \cong \mathbb{K}[X_1, \ldots, X_n] \), for some positive linear form \( L \).

**Definition 3.1.** Let \( \Delta \subset \mathbb{R}_{\geq 0}^n \) be a \( F \)-subset of \( \mathbb{R}_{\geq 0}^n \). A loose edge of \( \Delta \) is a compact edge \( E \subset \Delta \) that is not contained in any compact face of \( \Delta \) of dimension \( \geq 2 \).

**Lemma 3.2.** Let \( \Delta \subset \mathbb{R}_{\geq 0}^n \) be a \( F \)-subset with a loose edge \( E \subset \Delta \) that has ends \( \alpha, \beta \in \mathbb{R}_{\geq 0}^n \). Let \( L : \mathbb{R}^n \to \mathbb{R} \) be a linear form such that \( L(\alpha) = L(\beta) \). For every \( \gamma \in \Delta \), we have \( L(\gamma) \geq L(\alpha) \).

The same proof as for [GHe] Lemma 2.1 applies. In fact, this is also a corollary from Lemma 3.10 below.

The crucial point in the previous result is that \( L \) is a linear form that is not necessarily positive, see the example below. For positive linear forms the statement is true for any compact face of \( \Delta \).

**Example 3.3.** Let \( \Delta \subset \mathbb{R}_{\geq 0}^3 \) be the \( F \)-subset defined given by the three vertices \( \alpha = (1, 0, 0), \beta = (0, 1, 0) \), and \( \gamma = (0, 0, 1) \). Consider the linear form \( L : \mathbb{R}^3 \to \mathbb{R} \) with \( L(v_1, v_2, v_3) = v_1 + v_2 \). Clearly, the edge \( E \) with ends \( \alpha \) and \( \beta \) is not loose. We observe that \( L(\alpha) = L(\beta) = 1 > 0 = L(\gamma) \).

**Lemma 3.4.** Let \( \Delta \subset \mathbb{R}_{\geq 0}^n \) be a \( F \)-subset with a loose edge \( E \subset \Delta \) that has ends \( \alpha = (\alpha_1, \ldots, \alpha_n), \beta = (\beta_1, \ldots, \beta_n) \in \mathbb{R}_{\geq 0}^n \). If \( \min\{\alpha_1, \beta_1\} = \ldots = \min\{\alpha_n, \beta_n\} = 0 \), then \( \alpha \) and \( \beta \) are the only vertices of \( \Delta \).

The same proof as for [GHe] Lemma 2.2 applies. This can also be deduced from Lemma 3.10.

A \( F \)-subset \( \Delta \subset \mathbb{R}_{\geq 0}^m, m \in \mathbb{Z}_+ \), is called orthant if it has exactly one vertex, i.e., if \( \Delta = v + \mathbb{R}_{\geq 0}^m \), for some \( v \in \mathbb{R}_{\geq 0}^m \). This notion plays an important role in [RS].

**Remark 3.5.** The main result of [RS] uses the associated polyhedron \( \Delta_P := \Delta(P; x; z) \subset \mathbb{R}_{\geq 0}^d \) of a Weierstraß polynomial \( P = z^d + \sum_{(A,b)} \rho_{A,b} x^A z^b \in \mathbb{K}[x_1, \ldots, x_n][z] \), where \( \rho_{A,b} \in \mathbb{K} \). Here, \( \Delta_P \subset \mathbb{R}_{\geq 0}^n \) is the projection of the Newton polyhedron \( \text{NP}(P) \subset \mathbb{R}_{\geq 0}^{n+1} \) from the distinguished point \( (0, \ldots, 0, d) \) onto the subspace determined by the variables \( (x_1, \ldots, x_n) \). In other words, \( \Delta_P \) is the smallest \( F \)-subset containing all points of the set \( \{ d \cdot \frac{A}{z^b} | \rho_{A,b} \neq 0 \} \). The interesting case in [RS] is when \( \Delta_P \) is orthant. Then the unique vertex corresponds to a descendant edge of \( \text{NP}(P) \) (that is not necessarily loose).

The idea of projecting from the distinguished point corresponding to \( z^d \) comes from resolution of singularities and is used to provide refined information on a given singularity, see [H], [CP2], [CS], [S].

**Setup 3.6.** We fix a \( F \)-subset \( \Delta \subset \mathbb{R}_{\geq 0}^n \) that has a loose edge \( E \subset \Delta \) with ends \( \alpha, \beta \in \Delta \). Let \( L : \mathbb{R}^n \to \mathbb{R} \) be a positive linear form determining the edge \( E \). We define \( \delta := \beta - \alpha \).

Since \( \alpha = (\alpha_1, \ldots, \alpha_n) \neq \beta = (\beta_1, \ldots, \beta_n) \), we may assume without loss of generality \( \beta_n < \alpha_n \). This implies \( \delta_n < 0 \). Note that \( L(\delta) = L(\beta) - L(\alpha) = 0 \) and hence there exists at least one \( i \in \{1, \ldots, n - 1\} \) such that \( \delta_i > 0 \).
Further, let \((R, m, \mathbb{K})\) be a regular local ring with regular system of parameters \((x_1, \ldots, x_n)\). Recall that we denote the images of the latter in \(\text{gr}_L(R)\) by capital letters \((X_1, \ldots, X_n)\) and \(\text{gr}_L(R) \cong \mathbb{K}[X_1, \ldots, X_n]\).

We adapt the idea of projecting a \(F\)-subset \(\Delta \subset \mathbb{R}^n \geq 0\) in a suitable way to some \(\mathbb{R}^{n-1}\). Our goal is to obtain a refinement of the grading \(\text{gr}_L(R) = \bigoplus a \mathcal{P}_a/\mathcal{P}_a^+\). For this, we do not project from a particular point, but along the vector \(\delta\) that is defined by the difference of the ends of the fixed loose edge \(E\).

**Construction 3.7 (Projection in direction \(\delta\)).** Let \(\delta = (\delta_1, \ldots, \delta_n) \in \mathbb{R}^n\) be any vector with \(\delta_n < 0\) and \(\delta_i > 0\), for at least one \(i \in \{1, \ldots, n-1\}\). Let \(v = (v_1, \ldots, v_n) \in \mathbb{R}^n_{\geq 0}\). The projection of \(v\) along \(\delta\) to \(\mathbb{R}^{n-1}\) is given by

\[
\text{pr}_\delta(v) := \left( v_1 - \frac{v_n}{\delta_n} \cdot \delta_1, \ldots, v_{n-1} - \frac{v_n}{\delta_n} \cdot \delta_{n-1} \right) \in \mathbb{R}^{n-1}.
\]

Note that \(v - \frac{v_n}{\delta_n} \cdot \delta = (\text{pr}_\delta(v), 0)\) and \(\text{pr}_\delta(v + u) = \text{pr}_\delta(v) + \text{pr}_\delta(u)\). This provides a map \(\text{pr}_\delta : \mathbb{R}^n_{\geq 0} \rightarrow \mathbb{R}^{n-1}\).

For \(w \in \mathbb{R}^{n-1}\), we define

\[
I_{\delta,w} := \text{pr}_\delta^{-1}(w) \cap \mathbb{Z}^n_{\geq 0} = \{ v \in \mathbb{Z}^n_{\geq 0} \mid \text{pr}_\delta(v) = w \} \subset \mathbb{Z}^n_{\geq 0}.
\]

**Remark 3.8.** (1) The condition \(\delta_n < 0\) and \(\delta_i > 0\), for at least one \(i\), (up to reordering the coordinates) is equivalent to the property that the line generated by \(\delta\) does not intersect \(\mathbb{R}^n_{\geq 0}\) only in the origin, i.e., \((\delta \cdot \mathbb{R}) \cap \mathbb{R}^n_{\geq 0} = \{0\}\).

This is essential to obtain that \(I_{\delta,w}\) is a finite set.

(2) It is possible that \(\text{pr}_\delta(v) \in \mathbb{R}^{n-1} \setminus \mathbb{R}^n_{\geq 0}\). For example, if we consider \(\delta = (-1, 1, -1)\) and \(v = (0, 0, a)\), then \((\text{pr}_\delta(v), 0) = v + a \cdot \delta = (-a, a, 0)\), for every \(a \in \mathbb{R}_{\geq 0}\). We observe that \(\mathbb{R}^2_\delta\) has a non-compact face that is not parallel to a coordinate axis:

\[
\begin{array}{c}
\begin{tikzpicture}
\fill[red!30!white] (0,0) rectangle (3,3);
\draw[thick] (0,0) -- (3,0);
\draw[thick] (0,0) -- (0,3);
\draw[thick] (0,0) -- (3,3);
\end{tikzpicture}
\end{array}
\]

(3) Suppose \(\Delta, \mathcal{E}, \delta\) are as in Setup 3.6. We have \(\text{pr}_\delta(v) \in \mathbb{R}^{n-1}_{\geq 0}\), for all \(v \in \mathbb{R}^n_{\geq 0}\) if and only if \(\mathcal{E}\) is descendant (i.e., \(\delta_n < 0\) and \(\delta_i \geq 0\) for all \(i \in \{1, \ldots, n-1\}\)).

In particular, \(\mathbb{R}^{n-1}_{\delta} = \mathbb{R}^{n-1}_{\geq 0}\) in this case.

The previous leads to

**Definition 3.9.** Let \(\delta = (\delta_1, \ldots, \delta_n) \in \mathbb{R}^n\) be any vector with \(\delta_n < 0\) and \(\delta_i > 0\), for at least one \(i \in \{1, \ldots, n-1\}\). A \(F\)-subset \(\Delta \subset \mathbb{R}^{n-1}\) is called \(\delta\)-orthant if it is of the form \(\Delta = w + \mathbb{R}^{n-1}_{\geq 0} + \mathbb{R}^{n-1}_{\geq 0}\), for a unique vertex \(w \in \mathbb{R}^{n-1}\).

Using this notation, we can provide a connection to [RS].
Lemma 3.10. Let $\Delta, E, \delta$ be as in Setup 3.6. Since $E$ is a loose edge, we obtain that the projection $\Delta_\delta \subset \mathbb{R}^{n-1}$ is $\delta$-orthant. In particular, if $E$ is descendant, then $\Delta_\delta \subset \mathbb{R}_{\geq 0}^{n-1}$ is orthant.

In general, the converse statement is not true, i.e., if $\Delta_\delta$ is $\delta$-orthant for some vector $\delta \in \mathbb{R}^n$, then $\delta$ does not necessarily determine a loose edge of $\Delta$.

Proof. The result follows by the same arguments as [RS] Corollary 2.7 iv): Let $w_1 = \text{pr}_\delta(\alpha) = \text{pr}_\delta(\beta)$ be the vertex of $\Delta_\delta$ coming from the projection of the ends of $E$. Suppose $\Delta_\delta$ is not orthant. Then there exists at least one further vertex $w_2 \in \Delta_\delta$, $w_2 \neq w_1$, such that the segment $[w_1, w_2]$ is contained in the boundary of $\Delta_\delta$. Hence, there exists a vertex $\gamma \in \Delta$ with $\text{pr}_\delta(\gamma) = w_2$. Clearly, $\alpha, \beta, \gamma$ are pairwise different and the triangle defined by these three points is a face of $\Delta$. This contradicts the assumption that the edge given by $\alpha$ and $\beta$ is loose. \hfill \Box

Observation 3.11. Let $\Delta, E, L, \delta, R, (x)$ be as in Setup 3.6. Since $\delta$ is given by $E$, we have

$$\text{gr}_L(R) = \bigoplus_{w \in \mathbb{R}^{n-1}} S_{\delta,w},$$

where $S_{\delta,w}$ is the $K$-vector space with basis $B_{\delta,w} := \{X^A \mid A \in I_{\delta,w}\}$. Let us point out that the set $\{w \in \mathbb{R}^{n-1} \mid S_{\delta,w} \neq 0\} \subset \mathbb{R}^{n-1}$ is a discrete subset. Further, for all $w \in \mathbb{R}^{n-1}$ such that $S_{\delta,w} \neq 0$, there exists at least one $v \in \mathbb{Z}_{\geq 0}^n$ with $\text{pr}_\delta(v) = w$.

This is compatible with $\text{gr}_L(R) = \bigoplus_{a \in \mathbb{R}_{\geq 0}} P_a/P_a^+$. For $a \in \mathbb{R}_{\geq 0}$, let $R_a := P_a/P_a^+$, which is the $K$-vector space with basis $B_a := \{X^A \mid A \in \mathbb{Z}_{\geq 0}^n \land L(A) = a\}$, and we define $I_{L,a} := \{w \in \mathbb{R}^{n-1} \mid L(w,0) = a\}$. We have $B_a = \bigcup_{w \in I_{L,a}} B_{\delta,w}$. Note that is a disjoint union and all but finitely many of the appearing $B_{\delta,w}$ are empty. Therefore,

$$R_a = \bigoplus_{w \in I_{L,a}} S_{\delta,w}.$$

We remark that the property $L(\delta) = 0$ is crucial. The following pictures illustrates the compatibility:

The black triangle are all points $v \in \mathbb{R}_{\geq 0}^3$ for which $L(v) = a$, for some fixed positive linear form $L : \mathbb{R}^3 \to \mathbb{R}$ and $a \in \mathbb{R}_+$. The blue dashed lines show the projection lines (from $\mathbb{R}_{\geq 0}^3$ to $\mathbb{R}^2 \times \{0\}$) along a vector $\delta \in \mathbb{R}^3$ with $L(\delta) = 0$ (with $\delta$ descendant on the left and $\delta$ not descendant on the right). The triangle on the right determined by the dotted lines is a subset of $\{v \in \mathbb{R}^3 \mid L(v) = a\}$. As we see, there are $v \in \mathbb{R}_{\geq 0}^2$ such that $\text{pr}_\delta(v) \in \mathbb{R}^2 \setminus \mathbb{R}_{\geq 0}^2$.
In the situation of the previous observation, we have \( L(\alpha) = L(\beta) \) if \( \alpha \) and \( \beta \) denote the end points of the loose edge \( \mathcal{E} \). (Recall Lemma 3.2). Furthermore, we have \( \text{pr}_\delta(\alpha) = \text{pr}_\delta(\beta) =: u \), by construction, and hence \( X^\alpha, X^\beta \in S_{\delta,u} \).

Let us point out that the constructed grading on \( \text{gr}_L(R) \cong \mathbb{K}[X_1, \ldots, X_n] \) given by \( \bigoplus_{w \in \mathbb{Z}^n_{\geq 0}} S_{\delta,w} \) is a variant of the grading \( \bigoplus_{w \in \mathbb{Z}^n_{\geq 0}} R_w \) by [GHe] (which is defined by a certain weight \( \omega \), see loc. cit. before Lemma 2.4). The key in their construction is to choose a particular basis \( \xi_1, \ldots, \xi_n \in \mathbb{Z}^n_{\geq 0} \) of the vector space \( \mathbb{R}^n \) such that the projection along \( \delta \) becomes the projection to the first this \( n - 1 \) coordinates with respect to \( \xi_1, \ldots, \xi_n \) (see loc. cit. Lemma 2.3).

The following two lemmas are the ingredients for the proof of Theorem 1.1. For them, we need to introduce a variant of the set \( M \) defined in [GHe] before Lemma 2.4: Let \( L : \mathbb{R}^n \to \mathbb{R} \) be a positive linear form and let \( \delta \in \mathbb{R}^n \) be a vector with \( \delta_n < 0 \) and \( L(\delta) = 0 \). We define

\[
M_\delta := \text{pr}_\delta(\mathbb{Z}^n_{\geq 0}) \subset \mathbb{R}^{n-1}.
\]

Clearly, for \( w_1, w_2 \in M_\delta \), we have \( w_1 + w_2 \in M_\delta \) and \( \text{dim} S_{\delta,u} > 0 \) implies \( u \in M_\delta \).

Note that \( M_\delta \neq M \) (of [GHe]). In particular, \( \text{dim} S_{\delta,w} \neq 0 \) for every \( w \in M_\delta \).

**Lemma 3.12.** Let \( \Delta, \mathcal{E}, L, \delta, R \) be as in Setup 3.6. Let \( u \in \mathbb{R}^{n-1} \) and \( w \in M_\delta \). Assume that \( S_{\delta,u} \) contains two coprime monomials. Then

\[
\text{dim} S_{\delta,u+w} = \text{dim} S_{\delta,u} + \text{dim} S_{\delta,w} - 1.
\]

The same arguments as in the proof for [GHe] Lemma 2.4 apply: Using that the dimension of \( S_{\delta,u} \) coincides with the number of elements in \( I_{\delta,u} \) (analogously for \( \text{dim} S_{\delta,u} \)), the proof reduces to the combinatorial problem of determining the number of points in \( \mathbb{Z}^n_{\geq 0} \) appearing on the sum of two parallel segments. For more details, we refer to [GHe].

The assumption that \( S_{\delta,u} \) contains two coprime monomials is essential as the following example shows. This is the reason, why we have to impose in Theorem 1.1 that \( G \) is not divided by any variable. Another example for this (in the context of factoring a given element \( f \in R \)) is given in [GHe] Remark 2.6.

**Example 3.13.** Let \( R := \mathbb{K}[x_1, x_2] \), for any field \( \mathbb{K} \). Consider \( \delta = (3, -2) \in \mathbb{R}^2 \). For \( w \in \mathbb{R} \), we have

\[
\text{pr}_\delta^{-1}(w) = \{ v = (v_1, v_2) \in \mathbb{R}^2_{\geq 0} \mid 2v_1 + 3v_2 = 2w \}.
\]

The following picture shows \( \text{pr}_\delta^{-1}(3.5) \) (red), \( \text{pr}_\delta^{-1}(6.5) \) (blue), and \( \text{pr}_\delta^{-1}(10) \) (black), where filled points are lattice points corresponding to elements in \( I_{\delta,w} \).

Thus, \( \text{dim} S_{\delta,3.5} = \#I_{\delta,3.5} = 1 \), \( \text{dim} S_{\delta,6.5} = \#I_{\delta,6.5} = 2 \), and \( \text{dim} S_{\delta,10} = \#I_{\delta,10} = 4 \). In particular, \( \text{dim} S_{\delta,3.5} + \text{dim} S_{\delta,6.5} - 1 = 2 \neq 4 = \text{dim} S_{\delta,10} \). But clearly, \( S_{\delta,6.5} \) does not contain two coprime monomials.
Lemma 3.14. Let $\Delta, \mathcal{E}, L, \delta, R$ be as in Setup 3.6. Let $G \in S_{\delta,u}$ and $H \in S_{\delta,w}$ be coprime polynomials. If $G$ is not divisible by any monomial, then

$$GS_{\delta,u+i} + HS_{\delta,u+i} = S_{\delta,u+w+i}, \quad \text{for every } i \in \mathcal{M}_\delta.$$ 

The same proof as in [GHe] Lemma 2.5 applies: The idea is to show that the sequence

$$0 \to S_{\delta,i} \xrightarrow{\Phi} S_{\delta,u+i} \times S_{\delta,u+i} \xrightarrow{\Psi} S_{\delta,u+w+i} \to 0$$

is exact, where $\Phi(\eta) := (\eta H, -\eta G)$, for $\eta \in S_{\delta,i}$, and $\Psi(\psi, \varphi) := \psi G + \varphi H$, for $(\psi, \varphi) \in S_{\delta,u+i} \times S_{\delta,u+i}$. The non-trivial part is the surjectivity of $\Psi$ which can be deduced using Lemma 3.12. For more details, we refer to [GHe].

In order to adapt the proof of Theorem 1.1 for Theorem 1.4, one needs the following two results.

Lemma 3.15. Let $\Delta, \mathcal{E}, L, \delta, R$ be as in Setup 3.6. Let $G \in S_{\delta,u}$ and $H_j \in S_{\delta,w_j}$, for $u, w_j \in \mathcal{M}_\delta$ and $j \in \{1, 2\}$. Assume that, for every $i \in \mathcal{M}_\delta$,

$$GS_{\delta,w_j+i} + H_j S_{\delta,u+i} = S_{\delta,u+w_j+i}, \quad j \in \{1, 2\}.$$

Then, we have, for every $i \in \mathcal{M}_\delta$,

$$GS_{\delta,w_1+w_2+i} + H_1 H_2 S_{\delta,u+i} = S_{\delta,u+w_1+w_2+i}.$$

The same proof as in [GHe] Lemma 2.7 applies: This is a short computation applying the hypothesis in a clever way. For details, we refer to [GHe].

Lemma 3.16. Let $\Delta, \mathcal{E}, L, \delta, R, (x_1, \ldots, x_n)$ be as in Setup 3.6. Let $G \in S_{\delta,u}$ and $H \in S_{\delta,w}$ be coprime polynomials. If $G$ is monic with respect to $X_n$, then

$$GS_{\delta,w+i} + HS_{\delta,u+i} = S_{\delta,u+w+i}, \quad \text{for every } i \in \mathcal{M}_\delta.$$ 

The same proof as in [GHe] Lemma 2.8 applies (recall also the paragraph before Lemma 2.8 in [GHe]): First, one proves the special case $G = X_n$ and $H \in S_{\delta,w} \cap \mathbb{K}[X_1, \ldots, X_{n-1}]$. The rest follows then by Lemmas 3.14 and 3.15. For more details, we refer to [GHe].

Remark 3.17. In contrast to [RS], we do not project from a distinguished point to $\mathbb{R}^{n-1}$. (One candidate for such a point would be the end point $\beta$ of the loose edge.) The reason for projecting along the vector $\delta$ given by the loose edge is to obtain an appropriate refinement of the grading of $\text{gr}_E(R)$ such that Lemma 3.12 holds which is one of the key ingredients for the proofs.

Let us mention that the projection of $\text{NP}(f)$ along $\delta$ is $\delta$-orthant if and only if the projection of $\text{NP}(f)$ from $\beta$ to $\mathbb{R}^{n-1}_0$ is orthant. Hence, this is another reason why Lemma 3.10 yields a connection to [RS].

4. Proofs

We come to the proofs of the main theorems. The key step that allows to extend the results in [GHe] to any complete regular local ring is the following:

Let $(R, m, \mathbb{K})$ be a regular local ring, still not necessarily complete, with regular system of parameters $(x) = (x_1, \ldots, x_n)$. Let $\Delta \subset \mathbb{R}^n_{>0}$ be a $F$-subset, $\mathcal{E} \subset \Delta$ be a loose edge, and $L : \mathbb{R}^n \to \mathbb{R}$ be a positive linear form defining $\mathcal{E}$. As in Setup 3.6, we introduce $\delta \in \mathbb{R}^n$ with $\delta_n < 0$ and $L(\delta) = 0$.

Let $w \in \mathcal{M}_\delta$ and let $G \in S_{\delta,w} \subset \text{gr}_L(R) \cong \mathbb{K}[X_1, \ldots, X_n]$. We can write $G$ as a finite sum

$$G = \sum_{A \in \mathbb{Z}^n_{>0}} \lambda_A X^A, \quad \text{for } \lambda_A \in \mathbb{K}.$$
Note that $\lambda_A \neq 0$ implies $pr_\delta(A) = w$. For every $A \in \mathbb{Z}_{\geq 0}^n$ with $\lambda_A \neq 0$, we choose $\rho_A \in R^\times$ with the property
\[ \rho_A \equiv \lambda_A \mod m. \]
Otherwise, we set $\rho_A := 0 \in R$. Using this, we define
\[ g := \sum_A \rho_A x^A \in R. \]
Clearly, the image of $g$ in $gr_L(R)$ is $G$. (In fact, we can apply this procedure for any element in $gr_L(R)$, not only for those in $S_{\delta,w}$.) Note that $g$ is not unique and, in particular, $g$ depends on a choice of a system of representatives in $R$ for the residue field $K = R/m$. On the other hand, if $K \subset R$, then we can uniquely choose $\rho_A := \lambda_A$.

Using the above, we adapt the proofs of [GHe] to prove our results. Even though this is straightforward, we believe it is more pedagogical to give the proofs of the theorems. Moreover, we present a slightly different argument using Lemma 3.10.

**Proof of Theorem 1.1.** Let $L : \mathbb{R}^n \to \mathbb{R}$ be a positive linear form defining the edge $\mathcal{E}$. Let
\[ a_0 := \nu_L(f) \quad \text{and} \quad v := pr_\delta(\alpha) = pr_\delta(\beta) \in M_\delta \subset \mathbb{R}^{n-1}, \]
where $\alpha, \beta \in \mathcal{E}$ are the end points of the loose edge $\mathcal{E}$. Without loss of generality, we may assume that $\delta \in \mathbb{R}^n$ fulfills the properties of Setup 3.6.

By hypothesis, we have $in_L(f) = G \cdot H \in S_\delta$. We set $G_u := G \in S_{\delta,u}$ and $H_w := H \in S_{\delta,w}$, for $u, w \in M_\delta$. Let $g_u \in R$ (resp. $h_w \in R$) be a lift of $G_u$ (resp. $H_w$), as described before. We define $\phi_1 := g_u \cdot h_w$, which is our first approximation of $f$. For
\[ f_1 := f - \phi_1 = f - g_u \cdot h_w, \quad \text{we have} \quad a_1 := \nu_L(f_1) > a_0. \]
The vertices of $NP(f_1)$ lie in $\mathbb{Z}_{\geq 0}^n$ which implies that the vertices of $NP(f_1)_\delta$ are contained in $M_\delta$. Since the projection $NP(f)_\delta$ is $\delta$-orthant (Lemma 3.10), we get that each vertex of $NP(f_1)_\delta$ is of the form
\[ v + i = u + w + i, \quad \text{for some} \quad i \in M_\delta, i \neq 0. \]
In particular, $in_L(f_1) \in gr_L(R)$ can be written as
\[ in_L(f_1) = \sum_{i \in M_\delta} F_{u+w+i}^{(1)} \quad \text{for} \quad F_{u+w+i}^{(1)} \in S_{\delta,u+w+i}. \]
By assumption $G_u$ is not divisible by a monomial, hence, by Lemma 3.14, we have
\[ G_u S_{\delta,w+i} + H_w S_{\delta,u+i} = S_{\delta,u+w+i}, \quad \text{for every} \quad i \in M_\delta. \]
Thus, for every $i \in M_\delta$ with $L(u + w + i, 0) = a_1$, there are $H_{w+i} \in S_{\delta,w+i}$ and $G_{u+i} \in S_{\delta,u+i}$ such that $F_{u+w+i}^{(1)} = G_u H_{w+i} + H_w G_{u+i}$. We choose $h_{w+i}, g_{u+i} \in R$, as described before and define the second approximation of $f$ by
\[ \phi_2 := \left( g_u + \sum_{i \in M_\delta, i \neq 0} \sum_{L(u+w+i, 0) = a_1} g_{u+i} \right) \left( h_w + \sum_{i \in M_\delta, i \neq 0} \sum_{L(u+w+i, 0) = a_1} h_{w+i} \right) \in R. \]
Note that $\phi_2 = g_u h_w + \sum_{i \neq 0} g_{u+i} h_{w+i} + h_w g_{u+i} + \sum_{i,j \neq 0} g_{u+i} h_{w+j}$ and, by construction, $\nu_L(g_{u+i} h_{w+j}) > a_1$ since $L$ is positive. If we define
\[ f_2 := f - \phi_2, \quad \text{we have} \quad a_2 := \nu_L(f_2) > a_1. \]
We continue the construction and obtain
\[ \hat{g} := \sum_{i \in M_\delta} g_{u+i}, \quad \hat{h} := \sum_{i \in M_\delta} h_{w+i} \in \hat{R}. \]
such that \( f = \hat{g} \cdot \hat{h} \) in the completion \( \widehat{R} \), as desired.

Recall that \( L \) is a positive linear form on \( \mathbb{R}^n \) defining the edge \( \mathcal{E} \). Let \( \mathcal{E}_1 \) (resp. \( \mathcal{E}_2 \)) be the face of the Newton polyhedron of \( \hat{g} \) (resp. \( \hat{h} \)) determined by the same \( L \). We have that \( \text{in}_{\mathcal{E}_1}(\hat{g}) = G \) and \( \text{in}_{\mathcal{E}_2}(\hat{h}) = H \), \( \mathcal{E} = \mathcal{E}_1 + \mathcal{E}_2 \), \( \text{in}_{\mathcal{E}}(f) = \text{in}_{\mathcal{E}_1}(\hat{g}) \cdot \text{in}_{\mathcal{E}_1}(\hat{h}) \), and \( \mathcal{E}_1 \) is parallel to \( \mathcal{E} \). (The latter is also true for \( \mathcal{E}_2 \) if it is not a vertex). \( \square \)

Corollary 1.2 is the same as for \([\text{GHe}]\) Corollary 1.2, except that the reference to loc. cit. Lemma 2.2 has to be replaced by Lemma 3.4 in the present paper.

**Proof of Theorem 1.4.** We follow \([\text{GHe}]\). Using Lemma 3.16 instead of Lemma 3.14 in the proof of Theorem 1.1, we find \( \overline{g}, \overline{h} \in \widehat{R}[[z]] \) such that \( f = \overline{g} \cdot \overline{h}, \) \( \text{in}_{\mathcal{E}_1}(\overline{g}) = G, \) \( \text{in}_{\mathcal{E}_2}(\overline{h}) = H, \) \( \mathcal{E}_1 + \mathcal{E}_2 = \mathcal{E} \).

Since \( \mathcal{E} \) is descendant and \( G \) is monic in \( Z \), the Newton polyhedron of \( \overline{g} \) has a vertex of the form \((0, \ldots, 0, d)\), for some \( d \in \mathbb{Z}^+ \). Hence, the monomial \( \epsilon z^d \) appears in an expansion of \( \overline{g} \), for some unit \( \epsilon \in \widehat{R}[[z]]^\times \). The Weierstraß preparation theorem ([B], Ch. VII, §3, no. 8, Proposition 6, p. 41) implies that there exist a unit \( u \in \widehat{R}[[z]]^\times \) and \( \hat{g} \in \widehat{R}[z] \) such that

\[
\overline{g} = u \hat{g}.
\]

We define \( \hat{h} := u^{-1} \overline{h} \) and obtain that \( f = \hat{g} \cdot \hat{h} \). Since \( f, \hat{g} \in \widehat{R}[z] \), we also have \( \hat{h} \in \widehat{R}[z] \).

The remaining parts of the theorem follow easily. \( \square \)

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**Bernd Schober, Institut für Algebraische Geometrie, Leibniz Universität Hannover, Welfengarten 1, 30167 Hannover, Germany**

**E-mail address:** schober@math.uni-hannover.de