I. INTRODUCTION

Like from an incomplete puzzle, we assemble reality from fragments of information incoming to us from the outside world. This coarse-grained grasping of reality is mostly sufficient for successful and safe orientation in our environment. Barring wrong interpretation of reality, the exception to this rule may occur in situations when the partial information available to us carries no signatures of a global property, the knowledge of which is crucial for our correct decision.

There is a parallel with quantum world here. Namely, the wave function contains all available information about a state of a quantum system, but for many tasks we do not need to know it completely. However, unlike in the classical world, the “fragments” of the wave function may not carry traces of the global property which is important for the particular task, yet our knowledge gained from its parts can still be sufficient.

The states with that remarkable property share similarity with entangled states as both exhibit a counterintuitive relationship between the whole and its parts. It is not surprising then, that examples of states with a global property which can be inferred from parts lacking the property were nonlocal correlations and in particular multipartite entanglement. Out of the many flavours of multipartite entanglement, the main attention naturally fell on its strongest form, the genuine multipartite entanglement, which is behind the multipartite tests of quantum nonlocality, complex behaviour of strongly correlated systems, certain models of quantum computing and increased precision of quantum measurements.

So far, only examples of qubit states carrying genuine multipartite entanglement which can be verified solely from separable two-qubit reduced states (marginals) were found and demonstrated experimentally. In all these cases, the set of marginals used to certify the entanglement comprised all two-qubit marginals. Interestingly, genuine multipartite entanglement can be detected even from a smaller set of separable marginals. Indeed, multiqubit states can be found possessing all two-qubit marginals separable, whose genuine multipartite entanglement can be inferred only from the so called minimal set of two-qubit marginals. The minimal set covers any part of the entire system and it contains only marginals between nearest-neighbours, which guarantees that knowledge of the set suffices to confirm global entanglement. In geometric terms, if we represent parts of the global state as vertices of a graph and the bipartite marginals as its edges, then the minimal set corresponds to a tree-like graph. States with genuine multipartite entanglement which can be confirmed using only the elements of the minimal set where found for all configurations of up to six qubits using the iterative numerical search algorithm combining the machinery of entanglement witnesses with the tools of semi-definite programming. The best example obtained was three-qubit with the lowest witness mean being roughly three times smaller than the witness mean for the scenario in which all two-qubit marginals are known. Moreover, the difference is even more pronounced compared to other theoretically predicted witness means of already successfully implemented multipartite entanglement witnesses. This indicates the complexity of the possible experimental demonstration of the studied effect.

In this paper we take a different approach to the prob-
lem by seeking states with the investigated property in the realm of Gaussian states \[22, 23\]. More precisely, we look for Gaussian states with all two-mode marginals separable and whose genuine multipartite entanglement can be proved only from the minimal set of the marginals. For this purpose, we use the methods of Gaussian multipartite entanglement witnesses \[24\] to assemble a Gaussian analog of the search algorithm \[9\]. By running the algorithm we then find examples of the studied states for all configurations of up to four modes (see Fig. 1). Our simplest examples involve only three modes similarly to the simplest known qubit example which consists of three qubits \[8\]. The three-mode example gives, for the Gaussian analog of the genuine multipartite entanglement witness mean, a value which is roughly of the same size as the theoretically predicted values \[24\] for already realized similar Gaussian multimode entanglement experiments \[25\]. Further, the required squeezing is less than one third of a vacuum unit. Given the promising role of Gaussian states in the current problem, we also propose an arable and whose genuine multipartite entanglement can be proved only from the minimal set of the marginals. For this purpose, we use the methods of Gaussian multipartite entanglement witnesses \[24\] to assemble a Gaussian analog of the search algorithm \[9\]. By running the algorithm we then find examples of the studied states for all configurations of up to four modes (see Fig. 1). Our simplest examples involve only three modes similarly to the simplest known qubit example which consists of three qubits \[8\]. The three-mode example gives, for the Gaussian analog of the genuine multipartite entanglement witness mean, a value which is roughly of the same size as the theoretically predicted values \[24\] for already realized similar Gaussian multimode entanglement experiments \[25\]. Further, the required squeezing is less than one third of a vacuum unit. Given the promising role of Gaussian states in the current problem, we also propose a linear-optical circuit for preparation of the three-mode state, which is based on interference of three correlatively displaced squeezed beams on three beam splitters. Our results reveal that a minimal set of overlapping separable marginals may suffice to reveal genuine multipartite entanglement also in Gaussian scenario. Besides, they indicate that Gaussian continuous variables represent a promising alternative platform for experimental demonstration of the studied property of genuine multipartite entanglement.

a) \[
\begin{array}{ccc}
A & \gamma_{AB} & B \\
& \gamma_{BC} & C \\
\end{array}
\]

b) \[
\begin{array}{ccc}
A & \gamma_{AB} & B \\
& \gamma_{BC} & C \\
& & \gamma_{CD} \\
\end{array}
\]

c) \[
\begin{array}{ccc}
D & & \\
& \gamma_{BD} & \\
A & \gamma_{AB} & B \\
& & \gamma_{BC} \\
\end{array}
\]

FIG. 1. Graphical representation of minimal sets of marginal CMs used to detect genuine multipartite entanglement for three and four modes. See text for details.

II. GAUSSIAN STATES

The scene of our considerations is the set of Gaussian states of systems with infinite-dimensional Hilbert space, which we shall call modes in what follows. A collection of \(N\) modes \(A_j, j = 1, 2, \ldots, N\), can be characterized by a vector \(\xi = (x_{A_j}, p_{A_j}, \ldots, x_{A_N}, p_{A_N})^T\) of position and momentum quadratures \(x_{A_j}\) and \(p_{A_j}\), respectively, which obey the canonical commutation rules \([\xi_j, \xi_k] = i(\Omega_N)_{jk}\) with \(\Omega_N = \otimes^N_j i\sigma_y\), where \(\sigma_y\) is the Pauli-\(y\) matrix. Gaussian states are defined as states with a Gaussian-shaped phase-space Wigner function. An \(N\)-mode Gaussian state \(\rho\) is thus fully described by a \(2N \times 1\) vector \(\langle \xi \rangle = \text{Tr}[\rho\xi]\) of first moments and by a \(2N \times 2N\) covariance matrix (CM) \(\gamma\) with entries \((\gamma)_{jk} = \langle \xi_j \xi_k + \xi_k \xi_j \rangle - 2\langle \xi_j \rangle \langle \xi_k \rangle\). The first moments can be nullified by local displacements and thus they are irrelevant as far as the correlation properties investigated here are concerned. For this reason we set them to zero from now on.

Any CM \(\gamma\) reflects the uncertainty principle by satisfying the inequality

\[\gamma + i\Omega_N \geq 0,\]

which is not only a necessary but also a sufficient condition for a real symmetric \(2N \times 2N\) matrix \(\gamma\) to be a CM of a physical quantum state \[20\]. Besides, a CM also carries complete information about the separability properties of the corresponding Gaussian state. Recall first, that a quantum state \(\rho_{jk}\) of two subsystems \(j\) and \(k\) is separable if it can be expressed as a convex mixture of product states \(\rho_{jk}^{\text{sep}} = \sum_i p_i \rho_j^{(i)} \otimes \rho_k^{(i)}\), where \(\rho_j^{(i)}\) and \(\rho_k^{(i)}\) are local states of subsystems \(j\) and \(k\), respectively. If the state cannot be written in this form it is called entangled. Separability of a two-mode Gaussian state \(\rho_{jk}\) can be ascertained by the positive partial transposition (PPT) criterion \[17, 23, 27\]. On the CM level the partial transposition operation \(T_j\) with respect to mode \(j\) transforms the CM \(\gamma_{jk}\) of the state as \(\gamma_{jk}^{(T_j)} = (\sigma_z \otimes \mathbb{1}) \gamma_{jk} (\sigma_z \otimes \mathbb{1})\), where \(\sigma_z\) is the Pauli-\(z\) matrix and \(\mathbb{1}\) is the \(2 \times 2\) identity matrix. The PPT criterion then says \[25\], that the state \(\rho_{jk}\) is separable if and only if (iff) the matrix \(\gamma_{jk}^{(T_j)}\) is a physical CM, i.e., iff

\[\gamma_{jk}^{(T_j)} + i\Omega_2 \geq 0.\]

The PPT criterion is a sufficient condition for separability only for two-mode \[26\] and \(1 \times M\)-mode \[28\] Gaussian states. For systems where each party holds more than one mode, one has to use a more powerful criterion \[28\] according to which an \(N\)-mode Gaussian state with CM \(\gamma\) consisting of an \(l\)-mode subsystem \(A = A_1 A_2 \ldots A_l\) and an \((N - l)\)-mode subsystem \(B = A_{l+1} A_{l+2} \ldots A_N\), is separable iff there are CMs \(\gamma_A\) and \(\gamma_B\) of the subsystems such, that

\[\gamma - \gamma_A \otimes \gamma_B \geq 0.\]
gramme (SDP) \cite{24}:
\[
\begin{align*}
\text{minimize} & \quad (-x_e) \\
\text{subject to} & \quad \gamma - \gamma_A \oplus \gamma_B \geq 0, \\
& \quad \gamma_A \oplus \gamma_B + (1 + x_e)\Omega_N \geq 0.
\end{align*}
\]
If there is an optimal solution \( x_e \geq 0 \), then CM \( \gamma \) describes a separable state because there exist CMs \( \gamma_A \) and \( \gamma_B \) such that the separability criterion \cite{13} is satisfied. If, on the other hand, \( x_e \leq 0 \), then the state with CM \( \gamma \) is entangled.

### III. GAUSSIAN ENTANGLEMENT WITNESSES

In practice, one needs most often to certify the presence of entanglement in a given state rather than to show that it is separable. However, many entanglement criteria, including the PPT criterion or criterion \cite{3} require knowledge of the entire quantum state and thus they are not economical as far as the number of measurements is concerned. This also implies that the criteria cannot be used in cases when we have access only to a part of the investigated state. Nevertheless, it is still possible to detect entanglement provided that we have some a priori information about the state. Namely, one can prove the presence of entanglement by measuring the so-called entanglement witnesses \cite{17, 18}, which requires fewer measurements compared to the measurement of the whole quantum state \cite{20}.

#### A. Bipartite entanglement witnesses

For a bipartite state an entanglement witness is a Hermitian operator with a non-negative average for all separable states and a negative average on at least one entangled state. However, the task of finding an entanglement witnesses for density matrices of continuous-variable modes is often hardly tractable owing to their infinite dimension. A much more simple option, which is particularly suitable for Gaussian states, is to seek entanglement witnesses for CMs \cite{24}. Such a witness is, for an \( N \)-mode state, represented by a \( 2N \times 2N \) real, symmetric and positive-semidefinite matrix \( Z \), which satisfies the following conditions:

\[ (i) \quad \text{Tr}[\gamma Z] \geq 1, \quad \text{for all separable } \gamma, \]
\[ (ii) \quad \text{Tr}[\gamma Z] < 1, \quad \text{for some entangled } \gamma. \]

Entanglement detection by means of matrix \( Z \) possesses several advantages. First, the expression \( \text{Tr}[\gamma Z] \) is a linear function of second moments and therefore it can be measured by local homodyne detections followed by a suitable processing of the output photocurrents. More importantly, the expression also typically contains only some elements of CM \( \gamma \) and thus it requires fewer measurements than one needs to measure the entire CM. Another advantage of using the matrix \( Z \) is that for a given CM \( \gamma \) it can be found numerically by solving the dual to program \cite{4} \cite{24}:
\[
\begin{align*}
\text{minimize} & \quad \text{Tr}[\gamma X_{\text{re}}^\text{1}], \\
\text{subject to} & \quad X_{\text{1,2}}^\text{bd, re} = X_{\text{2,1}}^\text{bd, re}, \quad X_1 \geq 0, \quad X_2 \geq 0, \\
& \quad \text{Tr}[\gamma X_{\text{1,2}}] = -1.
\end{align*}
\]
Here \( X_j, j = 1, 2 \), are \( 2N \times 2N \) Hermitian matrices, the symbol \( X_{\text{re}} \) stands for the real part of the matrix \( X_{\text{j}} \), and \( X^{\text{bd}} = X_{\text{jA}} \oplus X_{\text{jB}} \), where \( X_{\text{jA}} \) and \( X_{\text{jB}} \) are diagonal blocks of the matrix \( X_j \) corresponding to subsystems \( A \) and \( B \), respectively.

It can be shown \cite{24}, that for every feasible solution \( X_1 \oplus X_2 \), the matrix \( X_{\text{1,2}}^\text{re} \) satisfies
\[ \text{Tr}[\gamma X_{\text{1,2}}^\text{re}] \geq 1 \]
for every CM \( \gamma \) of a separable state. Further, if \( \gamma \) is a CM of an entangled state, then
\[ \text{Tr}[\gamma X_{\text{1,2}}^\text{re}] < 1. \]
This implies, that the real matrix \( X_{\text{1,2}}^\text{re} \) is an entanglement witness which is, in addition, optimal in the sense that it yields the minimal value of \( \text{Tr}[\gamma Z] \) out of all possible witnesses \( Z \). Needless to say, by adding more constraints into the SDP \cite{4}, one can seek witnesses with a special structure. Below we will see, for instance, that one can seek witnesses which are ‘blind’ to certain parts of CM \( \gamma \).

#### B. Genuine multipartite entanglement witnesses

Bipartite entanglement is just one particular kind of entanglement. In multipartite systems consisting of \( N > 2 \) subsystems one can investigate also multipartite entanglement, which occurs among more than two groups of subsystems. In general, it is possible to split all subsystems into disjoint subsets and analyze entanglement with respect to the \( k \)-partite split. We say that a state is \( k \)-separable if it is fully separable with respect to the \( k \)-partite split, i.e., if it can be expressed as a convex mixture of product states with respect to the split. Otherwise, it is called entangled with respect to the split. This allows us to classify multipartite states according to their separability properties with respect to all possible \( k \)-partite splits for all possible \( k \) \cite{10, 30}. At the top of the hierarchy, there are fully inseparable states which are not separable with respect to any \( k \)-partite split. Nevertheless, even fully inseparable states in general do not carry the strongest form of multipartite entanglement. Namely, some of them can be created by convex mixing of some \( k \)-separable states \cite{31} and thus their preparation does not require a collective operation on all subsystems as we would expect from truly multipartite entangled states. For this reason, the concept of genuine \( N \)-partite entangled states was introduced as states that cannot be
expressed as a convex mixture of some \( k \)-separable states for any \( k \geq 2 \) \[20\]. Note, that any \( k \)-separable state with \( k > 2 \) is also \( 2 \)-separable. Consequently, a set of states that can be expressed as a convex mixture of some \( k \)-separable states is a subset of the set of states that can be expressed as a convex mixture of some \( 2 \)-separable states, which are fittingly called biseparable states. This reveals that for the presence of genuine multipartite entanglement in a given quantum state it is sufficient to show that it is not biseparable.

The concept of biseparability carries over straightforwardly to CMs of \( N \)-mode Gaussian states. For this purpose, let us collect modes \( A_j \), \( j = 1, 2, \ldots, N \), into the set \( \mathcal{N} = \{A_1, A_2, \ldots, A_N\} \) and let \( \mathcal{I} = \{1, 2, \ldots, N\} \) be its index set. Next, consider a nonempty proper index subset \( \mathcal{J}_k = \{i_1, i_2, \ldots, i_l\} \) of \( 0 < l < N \) elements of the index set \( \mathcal{I} \) and let \( \bar{\mathcal{J}}_k = \mathcal{I} \setminus \mathcal{J}_k \) denotes its complement containing the remaining \( N - l \) elements of \( \mathcal{I} \). This allows us to split the set \( \mathcal{N} \equiv K = 2^{N - 1} - 1 \) into different inequivalent 2-partitions, called as bipartitions in what follows, \( \pi(k) \equiv \mathcal{M}_{\mathcal{J}_k} \mid \mathcal{M}_{\bar{\mathcal{J}}_k} \), \( k = 1, 2, \ldots, K \), where \( \mathcal{M}_j = \{A_{i_1}, A_{i_2}, \ldots, A_{i_l}\} \) and \( \mathcal{M}_{\bar{\mathcal{J}}_k} = \mathcal{N} \setminus \mathcal{M}_j \).

Moving to the criterion of biseparability one can show \[24\], that an \( N \)-mode Gaussian state with CM \( \gamma \) is biseparable iff there exist bipartitions \( \pi(k) \) and CMs \( \gamma_{\pi(k)} \) which are block diagonal with respect to the bipartition \( \pi(k) \), and probabilities \( \lambda_k \) such that

\[
\gamma - \sum_{k=1}^{K} \lambda_k \gamma_{\pi(k)} \geq 0. \tag{9}
\]

Similarly as bipartite separability can be decided by solving the SDP \[4\], biseparability embodied by condition \[9\] can also be shown to be decided by solving an SDP \[24\]. Analogously, just like an optimal witness of bipartite entanglement can be obtained by solving the dual problem \[\delta\] of the former SDP, the optimal witness of genuine \( N \)-partite entanglement can be found by solving the dual problem of the corresponding SDP \[24\]. Recall first, that the witness of genuine \( N \)-partite entanglement is represented by a \( 2N \times 2N \) real, symmetric, and positive-semidefinite matrix \( Z \) satisfying conditions \[24\]

\[
\begin{align*}
(i) \, & \text{Tr}[\gamma Z] \geq 1, \quad \text{for all biseparable } \gamma, \\
(ii) \, & \text{Tr}[\gamma Z] < 1, \quad \text{for some entangled } \gamma. \tag{10}
\end{align*}
\]

For a given CM \( \gamma \) the witness can be found by solving the following dual problem \[24\]:

\[
\begin{align*}
\text{minimize} \quad & \text{Tr}[\gamma X^\text{re}] - 1 \\
\text{subject to} \quad & X^\text{re,\,bd,\,}\pi(k) = X^\text{re,\,bd,\,}\pi(k) \quad \text{for all } \quad k = 1, \ldots, K, \\
& \text{Tr}[\iota_\Omega N X_{k+1}] + X_{K+2} - X_{K+3} + X_{K+3+k} = 0, \quad \text{for all } \quad k = 1, \ldots, K, \\
& X_{K+2} - X_{K+3} = 1.
\end{align*} \tag{11}
\]

Here, the minimization is preformed over Hermitian positive-semidefinite \( [2N(K+1) + 2 + K] \)-dimensional block-diagonal matrix

\[
X = \bigoplus_{j=1}^{2K+3} X_j, \tag{12}
\]

with \( X_j, \ j = 1, 2, \ldots, K + 1 \) being \( 2N \times 2N \) Hermitian matrices and \( X_j, \ j = K+2, K+3, \ldots, 2K+3 \) being \( 1 \times 1 \) Hermitian matrices, i.e., real numbers. Further, the \( k \)-th equation \( X^\text{re,\,bd,\,}\pi(k) = X^\text{re,\,bd,\,}\pi(k) \) imposes a constraint on diagonal blocks of the matrices \( X_1 \) and \( X_{k+1} \) written in the block form with respect to the bipartition \( \pi(k) \). More precisely, let us express the matrix \( X_1 \) in the block-form with respect to the \( N \)-partite split \( A_1 | A_2 | \ldots | A_N \),

\[
X_1 = \begin{pmatrix}
(X_{11}) & (X_{12}) & \cdots & (X_{1N}) \\
(X_{21}) & (X_{22}) & \cdots & (X_{2N}) \\
\vdots & \vdots & \ddots & \vdots \\
(X_{N1}) & (X_{N2}) & \cdots & (X_{NN})
\end{pmatrix}, \tag{13}
\]

where \( (X_{ij})_{mn} \) is a \( 2 \times 2 \) block. Then, the matrix \( X^\text{bd,\,}\pi(k) \) is of the same block form with the \( 2 \times 2 \) blocks given by

\[
(X^\text{bd,\,}\pi(k))_{mn} = \begin{cases}
(X_{ij})_{mn}, & \text{if } m, n \in J_k \text{ or } \bar{J}_k; \\
0, & \text{otherwise},
\end{cases} \tag{14}
\]

where \( 0 \) is the \( 2 \times 2 \) zero matrix. For relevant cases \( N = 3 \) and \( N = 4 \) discussed in this paper an explicit form of the matrices \( X^\text{bd,\,}\pi(k) \) can be found in Appendix A.

According to the results of Ref. \[24\], for every feasible solution \( X \) of the dual program \[11\] the matrix \( X^\text{re}_1 \) is an optimal genuine multipartite entanglement witness.

### C. Blind genuine multipartite entanglement witnesses

The witness obtained by solving the programme \[11\] acts on the entire CM \( \gamma \) and therefore enables us to certify genuine multipartite entanglement provided that all elements of the CM are known. Viewed from a different perspective, it is equivalent to witnessing the entanglement from all two-mode marginal CMs, because they completely determine the global CM. In this respect, the
domain of Gaussian states differs from the qubit case, where the knowledge of all two-qubit marginals is not generally equivalent to the knowledge of the whole density matrix. To make the task on inference of genuine multipartite entanglement from marginals in Gaussian scenario meaningful, we thus have to work only with a proper subset of the set of all two-mode marginal CMs. In what follows, we utilize the so-called minimal sets of bipartite marginals, which were introduced recently in Ref. [9] to solve the task for qubits. Obviously, a necessary condition for the set to allow detection of global entanglement is that it contains all modes and that one cannot divide it into a subset and its complement without having a common mode. Among all such sets a particularly important role play further irreducible sets containing a minimum possible number of two-mode marginals.

A more convenient pictorial representation of such minimal sets was put forward in Ref. [9] in the form on an unlabeled tree [10], which is a special form of an undirected connected graph containing no cycles. Recall, that a graph is a pair \( G = (V, E) \) of a set \( V = \{ 1, 2, \ldots, N \} \) of vertices and a set \( E \subseteq K \equiv \{ (u, v) | (u, v) \in V^2 \land u \neq v \} \) of edges [32]. In our case a vertex \( j \) of the graph represents mode \( A_j \), whereas the edge connecting adjacent vertices \( j \) and \( k \) represents marginal CM \( \gamma_{A_j A_k} \). By definition, the minimal set contains two-mode marginal CMs corresponding to the edges of the respective tree denoted as \( T = (V, E') \). A closed formula for the number of non-isomorphic trees with \( N \) vertices is not known, yet for small \( N \) it can be found in Ref. [23]. In particular, all trees for the three-mode case \((N = 3)\) and the four-mode case \((N = 4)\) are depicted in Fig. 1 where we performed the following identification \( A \equiv A_1, B \equiv A_2, C \equiv A_3, \) and \( D \equiv A_4 \).

Ignorance of some sectors of CM \( \gamma \) requires to impose some additional constraints onto the structure of the witness \( X_{\gamma}^{re} \) being the solution of SDP (11). Specifically, as the respective tree is connected, the minimal set contains all single-mode CMs as well as \( 2 \times 2 \) blocks of correlations between the modes corresponding to the endpoints of the edges of the tree \( T \). The part of the CM \( \gamma \) which we do not know is therefore given by all \( 2 \times 2 \) off-diagonal blocks of correlations between pairs of modes carried by the marginal two-mode CMs contained in the complement of the minimal set. The elements of the complement correspond to the edges in the complement graph \( \overline{T} = (V, K \backslash E') \), i.e., to the edges which have to be added to the original tree \( T \) to form the whole graph. Since for a given \( N \) the complete graph contains \( \left( \begin{array}{c} N \\ 2 \end{array} \right) \) edges and the tree \( T \) contains exactly \( N - 1 \) edges [22], the number of unknown blocks of correlations is equal to \( L = (N - 1)(N - 2)/2 \). Further, as \( \mathrm{Tr}[\gamma X_{\gamma}^{re}] = \sum_{j<k} (\gamma)_{jk} (X_{\gamma}^{re})_{jk} \), in order for the witness \( X_{\gamma}^{re} \) not to act on the unknown blocks of CM \( \gamma \), its blocks in places of the unknown blocks have to vanish. More precisely, if we express the witness \( X_{\gamma}^{re} \) in the block form with respect to \( N \)-partite split \( A_1 | A_2 | \ldots | A_N \) similar to Eq. (13), its \( 2 \times 2 \) off-diagonal blocks have to satisfy the following set of \( L \) equations:

\[
(X_{\gamma}^{re})_{mn} = 0, \quad \text{if } \{ m, n \} \in K \backslash E',
\]

which have to be added to the SDP (11) as additional constraints. For \( N = 3 \) and the tree in Fig. 1a, the constraint reads explicitly as

\[
(X_{\gamma}^{re})_{13} = 0.
\]

Likewise, in the case \( N = 4 \) and for the linear tree in Fig. 1b), the constraints are

\[
(X_{\gamma}^{re})_{13} = (X_{\gamma}^{re})_{14} = (X_{\gamma}^{re})_{24} = 0,
\]

whereas for the ‘t’-shaped tree in Fig. 1c) one gets the constraints of the following form:

\[
(X_{\gamma}^{re})_{13} = (X_{\gamma}^{re})_{14} = (X_{\gamma}^{re})_{34} = 0.
\]

IV. SEARCH ALGORITHM

The goal of the present paper is to find an example of a Gaussian state with all two-mode marginals separable and whose genuine multipartite entanglement can be verified solely from the minimal set of two-mode marginals. Recently, multiqubit examples of such states have been found [9] using a two-step algorithm proposed in Ref. [8]. Here, we employ the following Gaussian analog of the algorithm:

Step 0: Generate a random pure Gaussian state with CM \( \gamma_0 \) which has, for simplicity, no \( x - p \) correlations.

Step 1: For CM \( \gamma_0 \), find a witness \( X_{\gamma_0}^{re} \) by solving numerically the SDP (11) supplemented with the constraints (15), which we shall call as the SDP 1. Note, that the SDP 1 can be solved by modifying the freely available routine [24] in Matlab by adding the constraints (15) into it.

Step 2: Find a CM \( \gamma \) that gives the least value of \( \mathrm{Tr}[\gamma X_{\gamma}^{re}] \) for the witness \( X_{\gamma}^{re} \) from the first step under the constraint that the CM possesses all two-mode marginals separable. Again, the search can be accomplished by solving the following SDP:

\[
\begin{align*}
\minimize_{\gamma} \quad &\mathrm{Tr}[\gamma X_{\gamma}^{re}] \\
\text{subject to} \quad &\gamma + i\Omega_N \geq 0, \\
&\gamma_{jk}(T_j) + i\Omega_2 \geq 0, \quad \text{for all } j \neq k = 1, \ldots, N, \\
&\gamma_{j1-2k} = (\gamma)_{j2,2k-1} = 0, \quad j, k = 1, \ldots, N,
\end{align*}
\]

which is called as SDP 2 from now. Here, we carry out the minimization over all real symmetric \( 2N \times 2N \) matrices \( \gamma \). The first constraint guarantees that the matrix \( \gamma \) is a CM of a physical quantum state, whereas the second constraint assures that all its two-mode marginal CMs \( \gamma_{jk} \) are separable. Finally, due to the third constraint we perform minimization only over matrices \( \gamma \) which do not contain any \( x - p \) correlations.
By putting the obtained solution from Step 2 as an input to Step 1 we can iteratively seek the CM with the desired properties. In the next section we give explicit examples of such CMs for all three-mode and four-mode minimal sets.

V. RESULTS

A. Three modes

First, we did a numerical search of a three-mode example of the investigated effect. Running SDP 1 and SDP 2 successively for 10 iterations for \( N = 3 \), we found several examples of states with all two-mode marginals separable and whose genuine three-mode entanglement can be verified solely from the nearest neighbour marginal CMs \( \gamma_{AB} \) and \( \gamma_{BC} \) (see Fig. 1a)). The CMs typically exhibited large diagonal entries and required high squeezing for preparation. To get experimentally easier accessible CM, we therefore added another two constraints to the SDP 2 [19]. First, we limited the diagonal elements of the CM to lie within the range [1, 10] and second, we also constrained the smallest eigenvalue of the sought CM \( \gamma \) to be above 0.2. The best CM we got in this way giving the least value of \( \text{Tr}[\gamma Z] \) reads after the rounding to two decimal places as

\[
\gamma_3 = \begin{pmatrix}
1.34 & 0 & -0.35 & 0 & -0.82 & 0 \\
0 & 10.00 & 0 & 8.45 & 0 & 1.87 \\
-0.35 & 0 & 7.80 & 0 & -8.05 & 0 \\
0 & 8.45 & 0 & 7.92 & 0 & 2.09 \\
-0.82 & 0 & -8.05 & 0 & 10.00 & 0 \\
0 & 1.87 & 0 & 2.09 & 0 & 1.62
\end{pmatrix},
\]

and by running the SDP 1 for the rounded CM \( \gamma_3 \) we got \( \text{Tr}[\gamma_3 Z_3] - 1 \approx -0.143 \). The corresponding witness, which is blind to the correlations between a pair of modes \( (A, C) \), is after rounding to three decimal places given by

\[
Z_3 = 10^{-2} \begin{pmatrix}
6.8 & 0 & -0.4 & 0 & 0 & 0 \\
0 & 34.3 & 0 & -39.5 & 0 & 0 \\
-0.4 & 0 & 25.1 & 0 & 20.9 & 0 \\
0 & -39.5 & 0 & 46.1 & 0 & -2.0 \\
0 & 0 & 20.9 & 0 & 17.5 & 0 \\
0 & 0 & 0 & -2.0 & 0 & 6.6
\end{pmatrix}.
\]

The separability of all marginals is evidenced by Tab. I.

Table 1. Minimal eigenvalue \( \varepsilon_{jk} \equiv \min\{\text{eig}[\gamma_{3,jk} + i\Omega_3]\} \).

\[
\begin{array}{|c|c|c|c|}
\hline
jk & AB & AC & BC \\
\hline
\varepsilon_{jk} & 0.002 & 0.849 & 0.004 \\
\hline
\end{array}
\]

Inspection of Tab. I reveals that all eigenvalues are strictly positive and therefore all three two-mode marginal states are separable by PPT criterion as required.

The present result can be compared with the results for qubits derived in Ref. [9]. Note first that the value of \( \text{Tr}[\gamma_3 Z_3] - 1 \approx -0.143 \) found here, for the simplest three-mode state, is slightly larger than the theoretical value of \(-0.103\) for the same quantity of the comparable effect of Gaussian bound entanglement [24, 28], which was already observed experimentally [23]. On the other hand, the best qubit mean of \( \text{Tr}[\rho W] = -6.58 \cdot 10^{-3} \) obtained for the three-qubit state [8] is approximately three times smaller than the best theoretical witness mean of \(-1.98 \cdot 10^{-2}\) for the case when all two-qubit marginals are known [8], which was recently demonstrated in [15]. Recall further, that in the qubit scenario the noise tolerance is 5% [8]. For comparison, the produced state with CM \( \gamma_3 \) also tolerates the addition of a small amount of thermal noise, i.e., the CM \( \gamma_p = \gamma_3 + \rho \mathbb{I} \) exhibits the effect for up to \( p \approx 0.1 \), yet the value is the same as one would get for the successfully demonstrated Gaussian bound entanglement [24].

All these facts indicate the domain of Gaussian states to be a more promising platform for the near-future experimental demonstration of the analyzed effect. Therefore, in the next section we present a linear-optical scheme for preparation of a close approximation of the state with CM \( \gamma_3 \). However, before doing so, we first construct also four-mode states carrying the investigated property.

B. Four modes

Next, we extended the search of example CMs to four modes. In this case there are two different minimal sets of marginals corresponding to the linear tree and the ‘t’-shaped tree displayed in Figs. 1b) and c), respectively. Through the same procedure as for the three-mode case, we found CMs with the desired properties for both the minimal sets, which are given explicitly below.

1. Linear tree

First, we considered the minimal set of marginals given by the CMs \( \gamma_{AB} \), \( \gamma_{BC} \), \( \gamma_{CD} \), corresponding to the edges in the linear tree in Fig. 1b). This was reflected by inclusion of the constraints (17) into our search algorithm. By running the algorithm for 10 iterations, we produced several four-mode CMs with the desired properties. The best such CM is
The optimal witness $Z^{(1)}_4$, which is blind to correlations of modes $(A,C),(A,D)$ and $(B,D)$, gives the value of $\text{Tr}[\gamma^{(1)}_4 Z^{(1)}_4] - 1 \doteq -0.069$ and it can be found in Appendix 3. The separability of all marginals can be confirmed again by the PPT criterion (2) which is captured in Appendix B. The separability of all marginals can be confirmed again by the PPT criterion (2) which is captured in Appendix B. Once again, the separability of the marginals can be verified via the PPT criterion. The results are summarized in Table II.

| $jk$ | $\varepsilon^{(1)}_{jk}$ | AB | AC | AD | BC | BD | CD |
|------|-----------------|----|----|----|----|----|----|
| 0.005 | 0.347 | 0.213 | 0.004 | 0.987 | 0.224 |

As all entries in the second row of the Tab. III are strictly positive, all two-mode marginal CMs of CM $\gamma^{(1)}_4$ are separable as required. Note further, that the effect is roughly half that of the three-mode case, which makes its experimental demonstration a bigger challenge.

2. ‘t’-shaped tree

Finally, we looked for states whose genuine four-mode entanglement can be witnessed from its nearest-neighbour marginals as per the graph in Fig. 1c). This corresponds to the ‘t’-shaped tree for which the minimal set comprise marginal CMs $\gamma_{AB}, \gamma_{BC}$ and $\gamma_{BD}$, and the witness then fulfils the constraints (18). The best example CM found reads as

$$\gamma^{(2)}_4 = \begin{pmatrix} 5.23 & 0 & 0.45 & 0 & -0.02 & 0 & -2.43 & 0 \\ 0 & 1.16 & 0 & 3.00 & 0 & 1.15 & 0 & 0.51 \\ 0.45 & 0 & 3.35 & 0 & 0.91 & 0 & -5.20 & 0 \\ 0 & 3.00 & 0 & 10.00 & 0 & 3.52 & 0 & 2.06 \\ -0.02 & 0 & 0.91 & 0 & 4.09 & 0 & -2.97 & 0 \\ 0 & 1.15 & 0 & 3.52 & 0 & 1.62 & 0 & 0.62 \\ -2.43 & 0 & -5.20 & 0 & -2.97 & 0 & 10.00 & 0 \\ 0 & 0.51 & 0 & 2.06 & 0 & 0.62 & 0 & 1.49 \end{pmatrix}.$$  

The corresponding optimal witness $Z^{(2)}_4$ is blind to intermodal correlations of pairs of modes $(A,C),(A,D)$ and $(C,D)$. It gives the value of $\text{Tr}[\gamma^{(2)}_4 Z^{(2)}_4] - 1 \doteq -0.068$ and its explicit form can be found in Appendix B. Once again, the separability of the marginals can be verified via the PPT criterion. The results are summarized in Table III.

| $jk$ | $\varepsilon^{(2)}_{jk}$ | AB | AC | AD | BC | BD | CD |
|------|-----------------|----|----|----|----|----|----|
| 0.0481 | 0.0032 | 0.5256 | 0.1103 | 0.0001 | 0.5489 |

VI. EXPERIMENTAL SCHEME

In the previous section we have seen that the investigated effect is strongest in the three-mode case. For this reason, we now derive a linear-optical scheme for preparing a Gaussian state with the three-mode CM $\gamma_3$. The scheme is depicted in Fig. 2.

The scheme follows from Williamson’s symplectic di-
agonalisation of a CM \[\gamma_3\], the Bloch-Messiah decomposition of a symplectic matrix \[S\] and the decomposition of an orthogonal symplectic matrix into an array of beam splitters and phase-shifters \[\gamma_3\]. More precisely, according to Williamson’s theorem \[35\] for any CM \(\gamma\) there is a symplectic transformation \(S\) which brings the CM to the normal form,

\[
S\gamma S^T = \bigoplus_{i=1}^{N} \nu_i \mathbb{I} \equiv W,
\]

where \(\nu_1, \nu_2, \ldots, \nu_N \geq 1\) are the so-called symplectic eigenvalues of CM \(\gamma\). In particular, \(\nu_1 = \nu_2 = \ldots = \nu_N = 1\) if the state is pure. Consequently, making use of the symplectic transformation \(S \equiv S^{-1}\), one can write \(\gamma = SWS^T\). The symplectic eigenvalues are the magnitudes of the eigenvalues of the matrix \(i\Omega\gamma\) \[39\] and for CM \(\gamma_3\) they are written in Tab. IV. The corresponding symplectic matrix \(S\) can be found numerically either using a method of Ref. \[40\] or a method of Ref. \[41\].

Making use of the Bloch-Messiah decomposition \[36\] we further numerically decomposed the symplectic matrix \(S\) into passive transformations \(U\) and \(V\), and an active transformation \(R\), as

\[
S = VRU.
\]

Here, \(U\) and \(V\) are orthogonal and symplectic transformations and \(R = R_A(s_A) \oplus R_B(s_B^{-1}) \oplus R_C(s_C^{-1})\) is the squeezing transformation, where \(R_j(s_j) = \text{diag}(s_j, s_j^{-1})\), \(j = A, B, C\), is the diagonal matrix and the squeezing parameters \(s_j < 1\) may be found in Tab. IV. These transformations are highlighted by the coloured boxes in Fig. 2.

Next, following the method of Refs. \[37\] \[38\] one can decompose the passive transformations \(U\) and \(V\) into an array of three beam splitters as in Fig. 2

\[
U = B_{BC}^{(U)}(T_{BC}) B_{AC}^{(U)}(T_{AC}) B_{AB}^{(U)}(T_{AB}),
\]

\[
V = B_{AB}^{(V)}(\tau_{AB}) B_{AC}^{(V)}(\tau_{AC}) B_{BC}^{(V)}(\tau_{BC}),
\]

where the beam splitter matrices \(B_{jk}^{(U)}(T_{jk})\) and \(B_{jk}^{(V)}(\tau_{jk})\), \(j = AB, AC, BC\), are given explicitly in Appendix C and the beam splitter transmissivities \(T_{jk}\) and \(\tau_{jk}\) can be found in Tab. V.

In the next section, we present an equivalent, yet simpler circuit whose output CM still retains all required properties.

### A. Simplified circuit

The scheme in Fig. 2 offers two simplifications which make its experimental realization easier. First, inspecting Tab. IV one may see that the input states of modes \(A, B, C\) can be approximated by vacuums. Second, the classically correlated state subject to the squeezing transformations can be replaced by correlatively displaced squeezed vacuum states. This follows from the fact that a thermal state at the input of mode \(A\) can be prepared by the displacements \(x_A(0) \rightarrow x_A(0) + t\) and \(p_A(0) \rightarrow p_A(0) + w\) of its position and momentum vacuum quadratures \(x_A(0)\) and \(p_A(0)\), respectively, where \(t\) and \(w\) are uncorrelated

| \(j\) | A     | B     | C        |
|-----|-------|-------|----------|
| \(\nu_j\) | 6.835 | 1.012 | 1.004    |
| \(s_j\)  | 0.396 | 0.851 | 0.478    |

TABLE IV. Symplectic eigenvalues \(\nu_j\) and the squeezing parameters \(s_j\).

| \(j\) | AB | AC | BC |
|-----|----|----|----|
| \(T_{jk}\) | 0.555 | 0.947 | 0.492 |
| \(\tau_{jk}\) | 0.716 | 0.904 | 0.657 |

TABLE V. Amplitude transmissivities \(T_{jk}\) and \(\tau_{jk}\).

TABLE VI. Minimal eigenvalue \(\epsilon'_{jk} \equiv \min\{\text{eig}[S_{\gamma_j}^T + i\Omega]\}\).

| \(j\) | AB | AC | BC |
|-----|----|----|----|
| \(\epsilon_{jk}\) | 0.005 | 0.852 | 0.010 |

Witness \(Z'_j\) can be found in Appendix D.

In the next section, we present an equivalent, yet simpler circuit whose output CM still retains all required properties.
classical zero mean Gaussian random variables with second moments $\langle t^2 \rangle = \langle w^2 \rangle = (\nu_A - 1)/2$. As on the level of quadrature operators the transformations $U$ and $R$ are linear, we can push the displacements through the transformations so that behind the transformation $R$ they attain the following form:

$$x_j \rightarrow x_j + \alpha_j t, \quad p_j \rightarrow p_j + \beta_j w,$$

(25)

$j = A, B, C$, where the parameters $\alpha_j$ and $\beta_j$ after rounding read as in Tab. VII. Further, the first step of the obtained scheme consists of application of a passive transformation $U$ on three vacuum states, which is nothing but a triple of vacuum states and thus the transformation $U$ can be omitted completely. In this way, we arrive at the simplified scheme depicted in Fig. 3.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|}
\hline
$j$ & $A$ & $B$ & $C$ \\
\hline
$\alpha_j$ & 0.2 & 0.7 & 1.3 \\
$\beta_j$ & 1.3 & 0.5 & 0.3 \\
\hline
\end{tabular}
\caption{Parameters $\alpha_j$ and $\beta_j$ of displacements $t, w$.}
\end{table}

FIG. 3. Scheme for preparation of a Gaussian state with CM $\gamma_3$ carrying genuine multipartite entanglement verifiable from nearest-neighbour separable marginals. The input comprises of three vacuum states (red circles). The squeezing transformation $R$ (red box) and the transformation $V$ (green box) are the same as in Fig. 2. The block $D$ (gray box) contains correlated displacements $D_A, D_B$ and $D_C$ (white boxes) given in Eq. (25), where the parameters $\alpha_j$ and $\beta_j$ are in Tab. VII and $\langle t^2 \rangle = \langle w^2 \rangle = (\nu_A - 1)/2$. See text for details.

Using the squeezing parameters and transmissivities, found in the second row of Tabs. IV and V respectively, as well as the displacements in Tab. VII the circuit in Fig. 3 produces a state which is genuinely multipartite entangled and has all marginals separable. Calling the CM of this state $\gamma_3$, the optimal witness for this CM gives $\text{Tr}[\gamma_3 Z_3] = -0.139$. The numerical CM $\gamma_3$ along with the corresponding entanglement witness may be found in Appendix D.

The simplified scheme in Fig. 3 makes experimental demonstration of the investigated effect more viable. Primarily, preparation of squeezed states at the input is easier than implementation of squeezing operations in between beam splitter arrays $U$ and $V$ (compare positions of pink boxes $R$ in Figs. 2 and 3). Further, the largest amount of squeezing, $10 \log_{10}(s_A)^2 = -8 \text{dB}$, is well within the reach of the current technology [12], and what is more, one may decrease the squeezing required at the cost of decreased effect strength. Additionally, the effect is immune to rounding of CMs and some parameters of the circuit components, which indicates, that perfect matching of the setup parameters with the theoretical values is not critical for its demonstration. Finally, as we have already mentioned, the output state tolerates the addition of a small amount of thermal noise, which is, however, of the same size as for the comparably fragile, yet already demonstrated similarly complex setup [25]. The extent to which the relatively low noise tolerance and other imperfections are detrimental to observability of the investigated phenomenon depends on the used experimental platform and will be addressed elsewhere.

VII. CONCLUSIONS

In this paper we extended the concept of genuine multipartite entanglement verifiable from separable marginals to the domain of Gaussian states. We constructed many examples of Gaussian states possessing all two-mode marginals separable and whose genuine multipartite entanglement can be certified solely from the set of nearest-neighbour marginals. Each of the sets is characterized by a connected graph with no cycles, where the vertices represent the modes and the edges the nearest-neighbour marginals. Our examples are numerical and result from an iterative search algorithm relying on construction of a genuine multipartite witness in the space of covariance matrices. Moreover, the witness is ‘blind’ to correlations between modes corresponding to non-adjacent vertices in the respective graph.

Here, we gave examples for all configurations encompassing three and four modes thus complementing the study of the investigated phenomenon in multi-qubit systems [9]. The three-mode state found by us exhibits the strongest form of the property compared to the four-mode cases and therefore we also proposed a scheme for preparation of the state, which consists of three quadrature squeezers sandwiched between two triples of phase-free beam splitters. Further, we replaced the original scheme by a simpler scheme, which still produces the desired effect, but requires only interference of three squeezed states subjected to correlated displacements on three beam splitters. The squeezing used in the setup is well within the reach of the current technology. Additionally, all relevant properties of the output state remain preserved after contamination by a small amount of thermal noise which gives us a hope that the investigated property of genuine multipartite entanglement could be observed. A successful realization of the proposed setup would mean extension of the experimental analysis of the
phenomenon of emergent genuine multipartite entanglement [8] from qubits and the scenario when all bipartite marginals are known [15], to the realm of Gaussian states and more generic situation when only some bipartite marginals are known.

The impact of the presented results is twofold. On one hand, they point at an alternative approach towards experimental investigation of the remarkable concept of genuine multipartite entanglement verifiable from incomplete sets of separable marginals. On the other hand, they also stimulate theoretical questions concerning the existence of a Gaussian classical analog of the quantum marginal problem [43] or the extendibility of the entanglement marginal problem [44] to Gaussian case. On a more general level, our results contribute to the development of methods of detection of global properties of multipartite quantum systems from partial information.

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COMPETING INTERESTS

The authors declare that there are no competing interests.

AUTHOR CONTRIBUTION

L.M. conceived the theory, J.P., V.N. and O.L. contributed to writing of the code, V.N., O.L., J.P. and L.M. carried out calculations, V.N., L.M. and N.K. wrote the manuscript, N.K. and L.M. supervised the project, all authors discussed the paper.

DATA AVAILABILITY

The data generated or analysed during the current study are available from the corresponding author upon reasonable request.

Appendix A: Block-diagonal matrices in SDP [11]

In this section we give an explicit form of matrices $X^{\text{bd},\pi(k)}_j$ appearing in SDP [11] for $N=3$ and $N=4$.

1. $N=3$

For $N=3$ we have altogether $K=3$ bipartitions $\pi(1)=A|BC$, $\pi(2)=B|AC$ and $\pi(3)=C|AB$, where we have omitted the curly brackets from the lists of elements of the sets $\mathcal{M}_{\mathcal{E}_1}$ and $\mathcal{M}_{\mathcal{E}_k}$ for brevity. The first equality in SDP [11] imposes constraints on certain elements of real parts of $6 \times 6$ Hermitian matrices $X_j$, $j=1,2,3,4$, which are embodied into matrices, $X^{\text{bd},\pi(k)}_j$, given explicitly as

$$X^{\text{bd},\pi(1)}_j = \begin{pmatrix} (X_j)_{11} & \odot & \odot & \odot \\ \odot & (X_j)_{22} & (X_j)_{23} & (X_j)_{24} \\ \odot & (X_j)_{23} & (X_j)_{33} \\ \odot & \odot & \odot & \odot \end{pmatrix},$$

$$X^{\text{bd},\pi(2)}_j = \begin{pmatrix} (X_j)_{11} & \odot & \odot & \odot \\ \odot & (X_j)_{22} & (X_j)_{23} & (X_j)_{24} \\ \odot & (X_j)_{23} & (X_j)_{33} \\ \odot & \odot & \odot & \odot \end{pmatrix},$$

$$X^{\text{bd},\pi(3)}_j = \begin{pmatrix} (X_j)_{11} & (X_j)_{12} & \odot & (X_j)_{14} \\ \odot & \odot & \odot & \odot \\ (X_j)_{12} & (X_j)_{22} & (X_j)_{24} \\ \odot & (X_j)_{22} & (X_j)_{33} & (X_j)_{34} \end{pmatrix},$$

$$X^{\text{bd},\pi(4)}_j = \begin{pmatrix} (X_j)_{11} & (X_j)_{12} & (X_j)_{13} & \odot \\ \odot & \odot & \odot & \odot \\ (X_j)_{12} & (X_j)_{22} & (X_j)_{23} & (X_j)_{24} \\ (X_j)_{13} & (X_j)_{23} & (X_j)_{33} & (X_j)_{34} \end{pmatrix},$$

$$X^{\text{bd},\pi(5)}_j = \begin{pmatrix} (X_j)_{11} & (X_j)_{12} & \odot & \odot \\ \odot & \odot & \odot & \odot \\ (X_j)_{12} & (X_j)_{22} & (X_j)_{23} & (X_j)_{24} \\ \odot & (X_j)_{22} & (X_j)_{33} & (X_j)_{34} \end{pmatrix},$$

2. $N=4$

For $N=4$ there are $K=7$ bipartitions $\pi(1)=A|BCD$, $\pi(2)=B|ACD$, $\pi(3)=C|ABD$, $\pi(4)=D|ABC$, $\pi(5)=AB|CD$, $\pi(6)=AC|BD$ and $\pi(7)=AD|BC$. The matrices $X^{\text{bd},\pi(k)}_j$, $k=1,\ldots,7$, obtained by projection of the matrices $X_j$ onto the block-diagonal form corresponding to bipartition $\pi(k)$ read explicitly as

$$X^{\text{bd},\pi(1)}_j = \begin{pmatrix} (X_j)_{11} & \odot & \odot & \odot \\ \odot & (X_j)_{22} & (X_j)_{23} & (X_j)_{24} \\ \odot & (X_j)_{23} & (X_j)_{33} & (X_j)_{34} \\ \odot & \odot & \odot & \odot \end{pmatrix},$$

$$X^{\text{bd},\pi(2)}_j = \begin{pmatrix} (X_j)_{11} & \odot & \odot & \odot \\ \odot & (X_j)_{22} & (X_j)_{23} & (X_j)_{24} \\ (X_j)_{13} & (X_j)_{23} & (X_j)_{33} & (X_j)_{34} \\ (X_j)_{14} & (X_j)_{24} & (X_j)_{34} \end{pmatrix},$$

$$X^{\text{bd},\pi(3)}_j = \begin{pmatrix} (X_j)_{11} & (X_j)_{12} & \odot & (X_j)_{14} \\ (X_j)_{12} & (X_j)_{22} & (X_j)_{24} & \odot \\ (X_j)_{14} & (X_j)_{24} & (X_j)_{34} & \odot \end{pmatrix},$$

$$X^{\text{bd},\pi(4)}_j = \begin{pmatrix} (X_j)_{11} & (X_j)_{12} & (X_j)_{13} & \odot \\ \odot & \odot & \odot & \odot \\ (X_j)_{12} & (X_j)_{22} & (X_j)_{23} & (X_j)_{24} \\ (X_j)_{13} & (X_j)_{23} & (X_j)_{33} & (X_j)_{34} \end{pmatrix},$$

$$X^{\text{bd},\pi(5)}_j = \begin{pmatrix} (X_j)_{11} & (X_j)_{12} & \odot & \odot \\ \odot & \odot & \odot & \odot \\ (X_j)_{12} & (X_j)_{22} & (X_j)_{23} & (X_j)_{24} \\ \odot & (X_j)_{22} & (X_j)_{33} & (X_j)_{34} \end{pmatrix}. $$
Appendix B: Four-mode numerical examples

We give explicit form of numeric witnesses for the four-mode CMs $\gamma_4^{(1)}$ and $\gamma_4^{(2)}$ detecting genuine multipartite entanglement from minimal sets of two-mode marginal CMs characterized by the linear tree and the ‘t’-shaped tree in Figs. 1 b) and c), respectively.

1. Linear tree

The witness which detects the genuine multipartite entanglement of CM $\gamma_4^{(1)}$ without accessing correlations between pairs of modes (A, C), (A, D) and (B, D) is

$$Z_4^{(1)} = 10^{-2} .$$

$$Z_4^{(1)} = 10^{-2} .$$

2. ‘t’-shaped tree

The witness detecting genuine multipartite entanglement of CM $\gamma_4^{(2)}$, which is ‘blind’ with respect to correlations between the pairs of modes (A, C), (A, D), (C, D), reads as

$$Z_4^{(2)} = 10^{-2} .$$

Appendix C: Beam splitter transformations

In this section we give explicit form of beam splitter matrices appearing in Eq. (24) of the main text,

$$B_{AB}^{(U)}(T_{AB}) = \begin{pmatrix} T_{AB} \mathbb{I} & R_{AB} \mathbb{I} & 0 \\ R_{AB} \mathbb{I} & -T_{AB} \mathbb{I} & 0 \\ 0 & 0 & -\mathbb{I} \end{pmatrix} ,$$

$$B_{AC}^{(U)}(T_{AC}) = \begin{pmatrix} T_{AC} \mathbb{I} & 0 & R_{AC} \mathbb{I} \\ 0 & \mathbb{I} & 0 \\ R_{AC} \mathbb{I} & 0 & -T_{AC} \mathbb{I} \end{pmatrix} ,$$
\[
B_{BC}^{(U)}(T_{BC}) = \begin{pmatrix}
1 & 0 & 0 \\
0 & -T_{BC} & -R_{BC} \\
0 & R_{BC} & -T_{BC}
\end{pmatrix},
\]

\[
B_{AB}^{(V)}(\tau_{AB}) = \begin{pmatrix}
\tau_{AB} & \rho_{AB} & 0 \\
\rho_{AB} & -\tau_{AB} & 0 \\
0 & 0 & 1
\end{pmatrix},
\]

\[
B_{AC}^{(V)}(\tau_{AC}) = \begin{pmatrix}
-\tau_{AC} & 0 & \rho_{AC} \\
0 & 1 & 0 \\
-\rho_{AC} & 0 & -\tau_{AC}
\end{pmatrix},
\]

\[
B_{BC}^{(V)}(\tau_{BC}) = \begin{pmatrix}
1 & 0 & 0 \\
0 & \tau_{BC} & \rho_{BC} \\
0 & \rho_{BC} & -\tau_{BC}
\end{pmatrix},
\]

where the transmissivities \(T_{jk}\) and \(\tau_{jk}\) are given in Tab. \(V\) of the main text, while \(R_{jk} = \sqrt{1 - T_{jk}^2}\) and \(\rho_{jk} = \sqrt{1 - \tau_{jk}^2}\) are the corresponding reflectivities.

**Appendix D: Circuit output covariance matrices**

In this section we present output CMs, witnesses and relevant eigenvalues of linear-optical circuits in Figs. 2 and 3.

1. Circuit in Fig. 2

First, we present the results for the scheme in Fig. 2 with parameters given in Tabs. \(IV\) and \(V\) of the main text. In this case the output CM, rounded to two decimal places, is given by

\[
\gamma_3 = \begin{pmatrix}
1.34 & -0.35 & -0.82 & 0 \\
0 & 10.01 & 0.845 & 1.86 \\
-0.35 & 7.78 & -8.03 & 0 \\
0.845 & 7.92 & 2.08 & 0.82 \\
-0.82 & -8.03 & 9.99 & 0 \\
0 & 1.86 & 2.08 & 1.62
\end{pmatrix}.
\]

The corresponding witness then reads as

\[
Z_3 = 10^{-2} \begin{pmatrix}
6.86 & 0 & -0.45 & 0 & 0 & 0 \\
0 & 34.11 & 0 & -39.31 & 0 & 0 \\
-0.45 & 25.04 & 20.87 & 0 & 0 & 0 \\
0 & -39.31 & 45.92 & 0 & -2.05 & 0 \\
0 & 20.87 & 17.43 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -2.05 & 6.62
\end{pmatrix}
\]

and it gives \(\text{Tr}[\gamma_3^\dagger Z_3^\dagger] - 1 = -0.138\).

Further, the marginals of the CMs are all separable as can be seen in Tab. \(VIII\).

**TABLE VIII. Minimal eigenvalue \(\epsilon_{jk}'\) \(\equiv \min\{\text{eig}[\gamma_{jk}^\dagger] + i\Omega_2]\).**

| \(jk\) | AB | AC | BC |
|-------|----|----|----|
| \(\epsilon_{jk}'\) | 0.005 | 0.852 | 0.010 |

2. Circuit in Fig. 3

In the last section we derive and analyze entanglement properties of the CM \(\gamma_3\) at the output of the circuit in Fig. 3.

Initially, vacuum modes \(A, B\) and \(C\) enter quadrature squeezers with squeezing parameters given in the second row of Tab. \(IV\). Next, they are subject to displacements

\[
x_j \rightarrow x_j + \alpha_j t, \quad p_j \rightarrow p_j + \beta_j w,
\]

where \(t\) and \(w\) are zero mean Gaussian random variables with second moments \(\langle t^2 \rangle = \langle w^2 \rangle = \langle \nu_4 - 1 \rangle / 2\) and where the parameters \(\alpha_j\) and \(\beta_j\) are given in Tab. \(V\). Finally, the three modes interfere on an array of three beam splitters described by the matrix \(V\) in Eq. (24). At the output of the circuit one gets the following CM:

\[
\bar{\gamma}_3 = \begin{pmatrix}
1.39 & 0 & -0.21 & 0 & -1.05 & 0 \\
0 & 9.95 & 0 & 8.26 & 0 & 1.7 \\
-0.21 & 7.36 & 0 & -7.83 & 0 & 1.94 \\
0 & 8.26 & 0 & 7.63 & 0 & 1.94 \\
-1.05 & 0 & -7.83 & 0 & 10.12 & 0 \\
0 & 1.7 & 0 & 1.94 & 0 & 1.59
\end{pmatrix}.
\]

The optimal witness, which gives \(\text{Tr}[\bar{\gamma}_3^\dagger Z_3^\dagger] - 1 = -0.139\), is given by

\[
Z_3 = 10^{-2} \begin{pmatrix}
5.87 & 0 & -0.54 & 0 & 0 & 0 \\
0 & 33.71 & 0 & -39.6 & 0 & 0 \\
-0.54 & 26.22 & 0 & 21.01 & 0 & 0 \\
0 & -39.6 & 0 & 47.1 & 0 & -1.87 \\
0 & 0 & 21.01 & 0 & 16.86 & 0 \\
0 & 0 & 0 & -1.87 & 0 & 6.17
\end{pmatrix}.
\]

All marginals are separable as evidenced by Tab. \(IX\)

**TABLE IX. Minimal eigenvalue \(\epsilon_{jk}\) \(\equiv \min\{\text{eig}[\gamma_{jk}^\dagger] + i\Omega_2]\).**

| \(jk\) | AB | AC | BC |
|-------|----|----|----|
| \(\epsilon_{jk}\) | 0.027 | 0.862 | 0.037 |
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