Effects of linear redshift space distortions and perturbation theory on BAOs: a 3D spherical analysis

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ABSTRACT

The baryon acoustic oscillations (BAOs) are features in the matter power spectrum on scales of the order of 100–150 $h^{-1}$ Mpc that promise to be a powerful tool to constrain and test cosmological models. The BAO have attracted such attention that future upcoming surveys have been designed with the BAO at the forefront of the primary science goals. Recent studies have advocated the use of a spherical Fourier–Bessel (sFB) expansion for future wide-field surveys that cover both wide and deep regions of the sky necessitating the simultaneous treatment of the spherical sky geometry as well as the extended radial coverage. Ignoring the possible effects of growth, which is not expected to be significant at low redshifts, we present an extended analysis of the BAO’s using the sFB formalism by taking into account the role of non-linearities and linear redshift distortions in the oscillations observed in the galaxy power spectrum. The sFB power spectrum has both radial and tangential dependence and it has been shown that in the limit that we approach a deep survey, the sFB power spectrum is purely radial and collapses to the Cartesian Fourier power spectrum. This radialization of information is shown to hold even in the presence of redshift space distortions and one-loop corrections to the galaxy power spectrum albeit with modified tangential and radial dependence. As per previous studies, we find that the introduction of non-linearities leads to a damping of the oscillations in the matter power spectrum.

Key words: methods: analytical – methods: numerical – methods: analytical – large-scale structure of Universe.

1 INTRODUCTION

Observations of the cosmic microwave background (CMB) and large-scale structure (LSS) will carry complementary cosmological information. While all-sky CMB observations, such as NASA’s WMAP\textsuperscript{1} or ESA’s Planck\textsuperscript{2} experiments, primarily probe the distribution of matter and radiation at redshift $z = 1300$, large-scale surveys such as ESA’s Euclid\textsuperscript{3} or the Square Kilometre Array (SKA)\textsuperscript{4} will provide a window at lower redshifts of the order of $z \approx 0–2$. The study of LSS appears to be a promising candidate in the study of the influence and role of the dark sectors in the standard model of cosmology. One particular phenomena of interest are the baryon acoustic oscillations (BAOs) that manifest themselves in the matter power spectrum of galaxy clusters on cosmological scales of the order of 100 $h^{-1}$ Mpc. These oscillations in the matter power spectrum are generated just before recombination through the interplay between a coupled photon–baryon fluid and gravitationally interacting dark matter (Peebles & Yu 1970; Sunyaev & Zeldovich 1970; Seo & Eisenstein 2003, 2007; Eisenstein et al. 2005).

The scale of the peaks and oscillatory features of the BAOs promises to be an important cosmological tool that acts as a standard ruler from which we can investigate and constrain dark energy parameters (see Amendola, Quercellini & Giallongo 2005; Eisenstein et al. 2005; Dolney, Jain & Takada 2006; Wang & Mukherjee 2006 for a small selection or representative literature), neutrino masses (Goobar et al. 2006), modified theories of gravitation (Alam & Sahni 2006; Lazkoz, Maartens & Majerotto 2006) and deviations from the standard model of cosmology (García-Bellido & Haugboelle 2008, 2009; February, Clarkson & Maartens 2013)). Significant attention has been devoted to the BAOs and they were first detected with Sloan Digital Sky Survey (SDSS\textsuperscript{5}) data (Eisenstein et al. 2005; Adelman-McCarthy et al. 2008) and in subsequent surveys (Colless et al. 2003; Percival et al. 2007a).

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The BAOs have been studied using standard Fourier space decompositions (Seo & Eisenstein 2003, 2007), real-space analysis (Eisenstein et al. 2005; Slosar et al. 2009; Xu et al. 2010; Juszkiewicz, Hellwing & van de Weygaert 2013), in 2D spherical harmonics defined on thin spherical shells (Dolney et al. 2006), but also in the sFB expansion (Rassat & Refregier 2012). It is important to note that different frameworks will make use of different information and will therefore have different constraining power for different cosmological parameters emphasizing the complementarity of mixed studies (Rassat et al. 2008). Previous studies, having predominantly focused on projected 2D surveys, have discarded radial information by projecting galaxy positions into tomographic redshift bins; however, such a loss of information could be avoided by adopting a full 3D description (e.g. Asorey et al. 2012).

Upcoming LSS surveys will provide cover for both large and deep areas of the sky and this will necessitate a formalism that can provide a simultaneous treatment of both the spherical sky geometry as well as an extended radial coverage. A natural basis for such a survey is provided by the sFB decomposition (see Fisher et al. 1995; Heavens & Taylor 1995; Percival et al. 2004; Castro, Heavens & Kitching 2005; Erdogdu et al. 2006; Abram, Reimberg & Xavier 2010; Leistedt et al. 2012; Shapiro, Crittenden & Percival 2011; Asorey et al. 2012; Lanusse, Rassat & Starck 2012; Rassat & Refregier 2012 for an incomplete selection of literature on the subject). In this prescription, we expand a 3D tracer field, such as the galaxy density contrast, using the radial (k) and tangential (i.e. along the surface of a sphere) (r) dependence.

The galaxy matter power spectrum is conventionally modelled using cosmological perturbation theory (PT). The linear order results will be valid at large scales where non-linear (NL) growth of structure under gravitational instability can be neglected. At smaller scales, it is no longer possible to neglect the NL growth of structure and we need to incorporate higher order corrections to the matter power spectrum. There are a number of different approaches currently in the literature to tackle this problem and we will present a more detailed description later on. NL galaxy clustering bias arises from an NL mapping between the underlying matter density field and observed collapsed objects (e.g. galaxies or dark matter haloes) and galaxy bias is, in essence, an isocurvature perturbation. Current literature has investigated more detailed prescriptions for galaxy bias such as the effects of primordial non-Gaussianity, scale dependence or non-local bias. Another form of non-linearity arises from redshift space distortion (RSD) generated through the internal motion of galaxies within haloes. This effect is known as the Finger-of-God (FoG) effect (Jackson 1972) and is distinct from the linear RSD considered in this paper (Kaiser 1987). It is also possible to investigate the role of non-Gaussian initial conditions, such as those generated in various inflationary models, and how this propagates NL corrections through to the growth of structure. The signatures of non-Gaussianity in these models will be distinctly different (e.g. a modified bispectrum) to the signatures of non-Gaussianity in models that have Gaussian initial conditions and are allowed to undergo gravitational collapse.

Throughout this paper, we will follow the construction outlined in Rassat & Refregier (2012) and generalize the method to study the role of RSD and the NL evolution of density perturbations. Previous investigations have used standard perturbation theory (SPT), galaxy bias models and Lagrangian perturbation theory (LPT) to characterize the role of various NL corrections to the BAO signal using the 3D Fourier power spectrum \( P(k) \) (Jeong & Komatsu 2006; Nishimichi et al. 2007; Nomura, Yamamoto & Nishimichi 2008; Jeong & Komatsu 2009; Nomura et al. 2009). These NL corrections can be reassessed within the sFB framework to aid our understanding of how real world effects can impact the radialization of information.

Recent work (Asorey et al. 2012) utilizing the sFB formalism has focused on how to recover the full 3D clustering information including RSD from 2D tomography using the angular auto- and cross-spectra of different redshift bins. Traditionally, RSD measurements have been made through spectroscopic redshift surveys such as the 2dF Galaxy Redshift Survey (Colless et al. 2003) and the SDSS (York et al. 2000) with photometric surveys often being neglected because of the loss of RSD through photometric redshift errors. Upcoming surveys, spectroscopic and photometric, such as the Dark Energy Survey, 6 Euclid, SKA, Physics of the Accelerating Universe Survey 7 (Benitez et al. 2009), Large Synoptic Survey Telescope 8 or the Panoramic Survey Telescope and Rapid Response System 9 offer the possibility of investigating the BAO and RSD through angular or projected clustering measurements (Benitez et al. 2009; Crocce et al. 2010; Nock, Percival & Ross 2010; Gaztanaga et al. 2011; Laureijs et al. 2011; Ross et al. 2011).

As RSD and distortions arising from an incorrect assumption for the underlying geometry are similar (Alcock & Paczynski 1979), the analyses of RSD using 3D data have to be used in conjunction with geometrical constraints (Samushia et al. 2011). As approaches based purely on angular correlation functions do not depend on the background cosmological model, the angular clustering measures will be considerably simpler. The sFB is something of a mid-point between these two approaches and will, in general, be sensitive to the choice of fiducial concordance cosmology. This paper is organized as follows. In Section 2, we discuss the sFB expansion. In Section 3, we outline the effect of linear RSD and Section 4 is devoted to issues related to realistic surveys. In Section 5, we consider perturbative corrections to linear real-space results and consider the structure of the sFB spectra. Results are discussed in Section 6 and conclusions presented in Section 7. Discussions about finite size of the survey and discrete sFB transforms are detailed in the appendices.

Throughout we will adopt the WMAP7 cosmological parameters (Komatsu et al. 2011): \( h = 0.7, \Omega_b h^2 = 0.0226, \Omega_c h^2 = 0.112, \Omega_{\lambda} = 0.725, \sigma_8 = 0.816 \).

2 SPHERICAL FOURIER–BESSEL (sFB) EXPANSION

2.1 Theory

Spherical coordinates are a natural choice for the analysis of cosmological data as they can, by an appropriate choice of basis, be used to place an observer at the origin of the analysis. Upcoming wide-field BAO surveys will provide both large and deep coverage of the sky, and we therefore require a simultaneous treatment of the extended radial coverage and spherical sky geometry. For this problem, the sFB expansion is a natural basis for the analysis of random fields in such a survey.

We introduce a homogeneous 3D random field \( \Psi(\hat{\Omega}, r) \) with \( \hat{\Omega} \) defining a position on the surface of a sphere and \( r \) denoting the comoving radial distance. The eigenfunctions of the Laplacian operators are constructed from products of the spherical Bessel
functions of the first kind $j_i(\kappa r)$ and spherical harmonics $Y_{\ell m}(\hat{\Omega})$ with eigenvalues of $-\kappa^2$ for a two-sphere. Assuming a flat background Universe, the sFB decomposition of our random field (Binney & Quinn 1991; Fisher et al. 1994, 1995; Heavens & Taylor 1995; Castro et al. 2005) is given by

$$\Psi(\hat{\Omega}, r) = \frac{1}{\sqrt{2\pi}} \int d\kappa \sum_{\ell m} \Psi_{\ell m}(k) j_{\ell}(k r) Y_{\ell m}(\hat{\Omega}),$$

and the corresponding inverse relation given by

$$\Psi_{\ell m}(k) = \frac{1}{\sqrt{2\pi}} \int d^3r \Psi(r) j_{\ell}(k r) Y_{\ell m}^*(\hat{\Omega}).$$

In our notation, $\{\ell m\}$ are quantum numbers and $k$ represents the wavenumber.\(^{10}\)

Note that the 3D harmonic coefficients $\Psi_{\ell m}(k)$ are a function of the radial wavenumber $k$. This decomposition can be viewed as the spherical polar analogy to the conventional Cartesian Fourier decomposition defined by

$$\Psi(r) = \frac{1}{k^{3/2}} \int d^3k \Psi(k) e^{i k \cdot r},$$

$$\Psi(k) = \frac{1}{k^{3/2}} \int d^3x \Psi(x) e^{-i k \cdot x}.$$  

The Fourier power spectrum, $P_{\Psi \Psi}$, is defined as the two-point correlation function of the Fourier coefficients $\Psi(k)$:

$$\langle \Psi(k) \Psi^*(k') \rangle = (2\pi)^3 P_{\Psi \Psi}(k) \delta^3(k - k').$$

Similarly, we can define a 3D sFB power spectrum, $C_{\ell}(k)$, of our random field by calculating the two-point correlation function of the 3D harmonic coefficients:

$$\langle \Psi_{\ell m}(k) \Psi_{\ell' m'}^*(k') \rangle = C_{\ell}(k) \delta^{10}(k - k') \delta^3_{\ell \ell'} \delta^3_{m m'}.$$  

It is possible to relate the Fourier coefficients $\Psi(k)$ with their sFB analogue $\Psi_{\ell m}(k)$ through the following expression:

$$\Psi_{\ell m}(k) = i^{\ell} k^{\ell\ell} \int d\Omega_k \Psi(k) Y_{\ell m}(\hat{\Omega}_k),$$

where the angular position of the wavevector $k$ in Fourier space is denoted by the unit vector $\hat{\Omega}_k(\theta_k, \phi_k)$. The Rayleigh-expansion of a plane wave is particularly useful in connecting the spherical harmonic definition with the 3D Cartesian expression. The second expression we present here is derived by differentiating the first and will be used in the derivation of RSD:

$$e^{i k \cdot r} = 4\pi \sum_{\ell m} i^{\ell} \frac{\delta(\ell)}{\ell \ell} \frac{\delta(m)}{m m} j_0(0) Y_{\ell m}(\hat{\Omega})Y_{\ell m}(\hat{\Omega});$$

$$i(\hat{\Omega}_k \cdot \hat{\Omega})e^{i k \cdot r} = 4\pi \sum_{\ell m} i^{\ell} j_0(0) Y_{\ell m}(\hat{\Omega}_k)Y_{\ell m}(\hat{\Omega}).$$

In general, the radial eigenfunctions are ultraspherical Bessel functions but they can be approximated by spherical Bessel functions when the curvature of the Universe is small (e.g. Zaldarriaga, Seljak & Bertschinger 1999). Throughout this paper, we will use $j_0(x)$ and $j_1(x)$ to denote the first and second derivatives of $j_0(x)$ with respect to its argument $x$. The expressions for the first and second derivatives are given in equations (B2) and (B3). Imposing a finite boundary condition on the radial direction will result in a discret sampling of the $k$ modes. This will be discussed in more detail later.

\(^{10}\) We follow the same conventions as (Castro et al. 2005; Leistedt et al. 2012; Rassat & Refregier 2012) but have made the substitutions $f(r) \rightarrow \Psi(r)$ and $W_{\ell}(k_1, k_2) \rightarrow I_{\ell 0}^{(0)}(k_1, k_2)$.

### 2.2 Finite surveys

In order to consider realistic cosmological random fields, such as the galaxy density contrast, we need to take into account the partial observation effects arising from finite survey volumes. Concise discussions of this point are given in Rassat & Refregier 2012 and Asorey et al. 2012, and as such we will not devote much time to this point referring the reader to the given references.

The selection function simply denotes the probability of including a galaxy within a given survey. An observed random field $\Psi_{\text{obs}}(r)$ can be related to an underlying 3D random field through a survey-dependent radial selection function $\phi(r)$ that modulates the underlying field:

$$\Psi_{\text{obs}}(r) = \phi(r) \Psi(r).$$

It is possible to introduce an analogous tangential selection function but we will, as per Rassat & Refregier (2012), neglect this possibility assuming that we have full-sky coverage. The resulting sFB power spectrum is given by

$$C_{\ell, \text{obs}}(k_1, k_2) = \left( \frac{2}{\pi} \right)^2 \int k^2 dk' I_{\ell 0}^{(0)}(k_1, k') I_{\ell 0}^{(0)}(k_2, k') P_{\text{obs}}(k'),$$

where the modified window function is given by

$$I_{\ell 0}^{(0)}(k_1, k_2) = \int dr r^2 \phi(r) j_0(k_1 r) j_0(k_2 r).$$

The sFB power spectrum tends to rapidly decay as we move away from the diagonal $k = k'$ and it will often be much more useful to focus purely on the diagonal contribution $C_{\ell 0}^{(0)}(k, k)$.

### 3 REDSHIFT SPACE DISTORTIONS

The measured distribution of galaxies is not without limits though as various systematic and survey dependent errors become more important. In practice, the observed galaxy redshift distributions are distorted due to the peculiar velocity of each galaxy. The anisotropies generated by the peculiar velocities are known as redshift space distortions. Although this distortion of the measured redshifts will necessarily complicate the cosmological interpretation of the spectroscopic galaxy surveys, RSD are currently one of the most optimistic probes for the measurement of the growth rate of structure formation and, as a result, an interesting probe of models for dark energy and modified theories of gravity.

The effect of RSD on the matter power spectrum can be split into two effects: the Kaiser effect and the FoG effect. The Kaiser effect corresponds to the coherent distortion of the peculiar velocity along the line of sight with an amplitude controlled by the growth rate parameter, leading to an enhancement of the power spectrum amplitude at small $k$ (Kaiser 1987). The FoG effect arises due to the random distribution of peculiar velocities leading to an incoherent contribution in which dephasing occurs and the clustering amplitude is suppressed (Jackson 1972). It is thought that the suppression of the amplitude is particularly important around the size of halo-forming regions, i.e. at large $k$ (Taruya, Nishimichi & Saito 2010).

For an isotropic structure in linear theory, the Kaiser effect means that an observer will measure more power in the radial direction than in the transverse modes. The amplitude of this distortion is modulated by the distortion parameter:

$$\beta = \frac{f(\Omega_0)}{b(z)} = \frac{1}{b(z)} \frac{\ln D(a)}{\ln a} \approx \Omega_0^{1/2} \frac{a}{b(z)}.$$
where
\[
\Omega_m(a) = \Omega_{m,0} \frac{H_0^2}{a^2 H^2(a)}
\] (14)
such that \(a\) is the scalefactor, \(H(a)\) is the Hubble parameter, \(H_0\) is the Hubble parameter at present time and \(D(z)\) the linear growth factor for which \(f(z) \equiv \ln D/d\ln a\). In this parametrization, \(\gamma\) is directly related to our theory of gravitation such that General Relativity predicts \(\gamma \approx 0.55\) and \(\Omega_m\) is the usual mass density parameter (Wang & Steinhardt 1998; Linder 2005). This means that RSD can be used to probe the growth of structure, the galaxy clustering bias function \(b(z)\) as well as probing dark energy and modified theories of gravity (Guzzo et al. 2008). Measuring the growth rate from RSD is a non-trivial procedure and a detailed understanding of systematic errors is crucial in order to disentangle different theories of gravity or dark energy (e.g., de la Torre & Guzzo 2012). Euclid aims to constrain the growth-rate parameter to the per cent-level but incomplete modelling of RSD introduces systematics of the order of 10–15 per cent (Banerjis et al. 2008; Taruya et al. 2010; Okumura & Jing 2011; Bianchi et al. 2012; de la Torre & Guzzo 2012). This makes the study of RSD in the sFB formalism all the more timely. In the next section, we will outline some of the basic ingredients that are used in modelling RSD in Fourier space before constructing the analogous results in the sFB formalism.

### 3.1 RSD in Fourier space

Before presenting the RSD in the sFB formalism, we briefly review some of the key results from modelling RSD in Fourier space and the appropriate limitations that are adopted in the model.

The effect of a peculiar velocity \(v\) is to distort the apparent comoving position \(s_i\) of a galaxy from its true comoving position \(r\):
\[
s = r + \frac{v_i(r)\hat{n}}{aH(a)} = r + f\phi(r)\hat{n},
\] (15)
where \(f\) is the linear growth rate, \(\hat{n}\) is a vector lying parallel to an observer’s line of sight and \(v_i\) is the component of the velocity parallel to the line of sight. The resulting redshift space density field \(\delta_i(s)\) is obtained by imposing mass conservation, \([1 + \delta_i(s)]d^3s = [1 + \delta_i(r)]d^3r\), which results in the following:
\[
[1 + \delta_i(s)] = [1 + \delta_i(r)] \left| \frac{d^3s}{d^3r} \right|^{-1}.
\] (16)

To simplify the analysis, we can adopt the distant observer approximation in which we neglect the curvature of the sky and the Jacobian reduces to a term relating only to the line of sight
\[
\frac{\partial s_i}{\partial r} = 1 + f\phi',
\] (17)
where a prime denotes differentiation with respect to the line of sight, i.e. parallel to \(\hat{n}\):
\[
\phi'(r) = \delta_1 \left[ \frac{v_i}{f a H(a)} \right].
\] (18)

The redshift space density contrast can be rewritten as
\[
\delta_i(s) = \frac{\delta_i(r) - f\phi'(r)}{1 + f\phi'(r)}.
\] (19)

Assuming an irrotational velocity field with a velocity divergence field \(\theta(r) = \nabla \cdot \mathbf{v}(r)\), we obtain the following useful relationship, \(\phi(r) = - (\nabla^{-1} \theta(r))'\). In Fourier space, these equations simplify as \(\phi'(k) = - \mu^2 \theta(k)\), where we have made use of the fact that \((\nabla^{-1})^{-1} = (k_i/k)^2 = \mu^2\). In our notation, \(k_i\) denotes the modes parallel to the line of sight and \(k_j\) denotes modes perpendicular to the line of sight, where \(k^2 = k_1^2 + k_2^2\) (Scoccimarro, Couchman & Frieman 1999). The redshift space density field can be written as
\[
\delta_i(k, \mu) = \int \frac{d^3s}{(2\pi)^3} e^{-ik\cdot s} \delta_i(s)
\] (20)
and the corresponding power spectrum as
\[
P_s(k, \mu) = \int \frac{d^3r}{(2\pi)^3} e^{-ik\cdot r} \left[ e^{-i\mu(\phi(r) - \phi(r'))} \right] \left[ \delta(r) + f\mu^2\theta(r) \right] \left[ \delta(r') + f\mu^2\theta(r') \right]
\] (21)

This prescription for the Fourier power spectrum has been constructed in the plane-parallel or distant observer approximation. The terms in the square brackets represent the conventional Kaiser effect as described earlier. The exponential prefactor corresponds to the small-scale velocity dispersion and relates to the FoG effect described earlier. A simplified phenomenological power spectrum was derived by Scoccimarro (2004) by assuming that the exponential prefactor may be separated from the ensemble average
\[
P_s(k, \mu) = e^{-i(k\mu\sigma_v)^2} \left[ P_{\delta\delta}(k) + 2f\mu^2P_{\delta\theta}(k) + f^2\mu^4P_{\theta\theta}(k) \right],
\] (22)
where \(\sigma_v\) is a velocity dispersion defined in Scoccimarro (2004). In the linear regime, we have \(P_{\delta\delta} = P_{\delta\theta} = P_{\theta\theta}\), and the velocity dispersion prefactor tends towards zero. In such a limit, we simply recover the linear result of Kaiser (1987):
\[
P_s(k, \mu) = \left[ 1 + 2f\mu^2 + f^2\mu^4 \right] P_{\delta\delta}(k).
\] (23)

Such a limit corresponds to making a number of approximations. For example, we require that the velocity gradient is sufficiently small, the density and velocity perturbations must be accurately described by the linear continuity equations, the real-space density perturbations are well described by the linear results, i.e. \(\delta(r) \ll 1\), such that higher order contributions are suppressed, and we also require that the small-scale velocity dispersion tends towards zero and may be neglected. Such approximations appear to hold on the largest scales and a lot of distortion features are well modelled by this approximation. It is however known that this theory breaks down as we approach the quasi-linear and NL regimes. The result of Scoccimarro (2004) makes certain approximations about the separability of the exponential prefactor which neglects possible coupling terms between the velocity and density fields. A lot of effort has been invested in constructing NL models for RSD and upcoming surveys should prove to be a fruitful testing ground for many of these models (Hivon et al. 1995; Scoccimarro et al. 1999; Scoccimarro 2004; Crocce & Scoccimarro 2008; Matsubara 2008a,b; 2011; Taruya et al. 2009, 2010; Okumura, Taruya & Matsubara 2011; Sato & Matsubara 2011; de la Torre & Guzzo 2012; Umeh et al. 2012). We construct the RSD in the sFB formalism by first working to the linear Kaiser result and exploring the phenomenology of such an extension.
3.2 RSD in sFB space

As previously mentioned, the effect a peculiar velocity, or a departure from the Hubble flow, \( v(r) \) at \( r \) is to introduce a distortion to the galaxy positions in the redshift space \( s \):

\[
s(r) = r + v(r) \cdot \hat{\Omega}.
\]

(24)

We denote the harmonics of a field \( \Psi(r) \) when convolved with a selection function, \( \phi(s) \), by \( \tilde{\Psi}_{lm}(k) \). These harmonics take into account the RSD:

\[
\tilde{\Psi}_{lm}(k) = \sqrt{\frac{2}{\pi}} \int \delta^3 ds \, \hat{\Omega} \phi(s) \Psi(r) \, j_l(kr) \, Y_{lm}(s). \quad (25)
\]

The Fourier transform of the linearized Euler equation can be used to relate the Fourier transform of the density contrast, \( \delta(k) \), to that of the peculiar velocity field \( v(r) \):

\[
v(k) = -i \delta k^2 \frac{\tilde{b}(k)}{k^2}.
\]

(26)

where \( b \) is the linear bias parameter. Following the procedure outlined in Heavens & Taylor (1995), we can establish a series expansion in \( b \) such that the lowest order coefficients \( \tilde{\Psi}_{lm}^{(0)}(k) \) are obtained by neglecting the RSD:

\[
\tilde{\Psi}_{lm}(k) = \tilde{\Psi}_{lm}^{(0)}(k) + \tilde{\Psi}_{lm}^{(1)}(k) + \cdots.
\]

(27)

\[
\tilde{\Psi}_{lm}^{(0)}(k) = \sqrt{\frac{2}{\pi}} \int_0^\infty k \, dk' \, \Psi_{lm}(k') \, i_l(k') \, j_l(k);\]

(28)

\[
\tilde{\Psi}_{lm}^{(1)}(k) = \sqrt{\frac{2}{\pi}} \int_0^\infty k \, dk' \, \Psi_{lm}(k') \, l_l^{(1)}(k', k).
\]

(29)

The kernels \( l_l^{(1)}(k', k) \) and \( i_l^{(0)}(k', k) \) define the convolution and are dependent on the choice of selection function. Note that \( l_l^{(1)}(k', k) \) is simply the window function we encountered previously in equation (12). The kernels can be shown to be:

\[
l_l^{(1)}(k', k) = \int dk'' \, r^2(\psi) \, j_l(kr) \, j_l(k' r')\]

(30)

\[
l_l^{(1)}(k', k) = \frac{\delta}{k} \int dk'' \, k \, \frac{d}{dr} (\phi(r) j_l(kr)) \, j'_l(k' r').
\]

(31)

The lowest order corrections due to RSD are therefore encapsulated in \( \tilde{\Psi}_{lm}^{(1)}(k) \). We can define a set of power spectra by using these harmonic coefficients:

\[
\langle \tilde{\Psi}_{lm}^{(0)}(k') \tilde{\Psi}_{lm}^{(0)*}(k') \rangle = C_l^{(0)}(k', k) \delta_{1D}(k - k') \delta_{\text{ann}}.
\]

(32)

\[
\langle \tilde{\Psi}_{lm}^{(0)}(k) \tilde{\Psi}_{lm}^{(0)*}(k') \rangle = C_l^{(0)}(k, k) \delta_{1D} \delta_{\text{ann}}.
\]

(33)

We can construct a generalized power spectrum by using the common structure between equations (28) and (29):

\[
C_l^{(0)}(k_1, k_2) = \left( \frac{2}{\pi} \right)^2 \int k^2 \, dk' \, l_l^{(1)}(k_1, k') \, l_l^{(1)}(k_2, k') \, P_{sd}(k').
\]

(34)

The total redshifted power spectrum will be given by a sum of the various contributions:

\[
\tilde{C}_l(k_1, k_2) = C_l^{(0)}(k_1, k_2) + 2 \, C_l^{(0)}(k_1, k_2) + C_l^{(11)}(k_1, k_2).
\]

(35)

If we ignore the effects introduced by the selection function, i.e. set \( \phi(r) = 1 \), then we recover the result for the unredshifted contributions (Heavens & Taylor 1995; Fisher et al. 1995; Castro et al. 2005):

\[
C_l^{(0)}(k, k) = P_{sd}(k).
\]

(36)

These expressions hold for surveys with all-sky coverage. In the presence of homogeneity and isotropy the 3D power spectrum will be independent of radial wavenumber \( \ell \). The introduction of a sky mask breaks isotropy and introduces additional mode–mode couplings, the analysis will be generalized to this case in the next section. In the above equations, we neglect a number of additional NL terms including General Relativistic corrections, velocity terms and lensing terms. It is also possible to adopt a full NL approach to RSD where the NL spectrum has significantly more complicated angular structure than in linear theory Shaw & Lewis (2008). The RSD information will be dependent on the relative clustering amplitude of the transverse modes and the radial modes, Asorey et al. (2012). Our ability to recover information and the extent to which the information radializes will naturally depend on the geometry of the survey and which modes we are able to include.

3.3 BAO wiggles only

The BAOs can be isolated by constructing a ratio between the observed matter power spectrum \( P_{\text{obs}}^M(k) \) and a theoretical matter power spectrum \( P_{\text{obs}}^{\text{NL}}(k) \) constructed from a zero-baryon (or no-wiggle) transfer function in which the oscillations do not show up (Eisenstein et al. 2005). Using these two power spectra, the ratio \( R^B(k) \) will reduce the dynamic range and isolates the oscillatory features of the BAOs:

\[
R^B(k) = \frac{P^B(k)}{P^\text{nn}(k)}.
\]

(37)

This ratio is clearly defined for the Fourier space power spectrum but an appropriate generalization to the sFB formalism may be constructed by calculating the ratio of the angular power spectra defined in equation (11), with the matter power spectrum \( C_l^B(k) \) to the angular power spectrum with the zero-Baryon power spectrum \( C_l^{\text{nn}}(k) \) (Rassat & Refregier 2012):

\[
R_l^C(k) = \frac{C_l^B(k)}{C_l^{\text{nn}}(k)}.
\]

(38)

It is important to note that the characterization method (i.e. how we choose to construct our ratio) can affect the characteristic scale of the BAOs when we take into account NL effects. This means that care has to be taken when comparing results that implement different methods (Rassat et al. 2008). As an example we could construct our ratio by using the no-wiggles transfer function of Eisenstein et al. (2005) or adopt an interpolation scheme to construct a smooth parametric curve (Blake et al. 2006; Percival et al. 2007a; Seo & Eisenstein 2007). A different choice of smoothed matter power spectra, cosmological parameters, growth history or similar can impact the phenomenological behaviour of the underlying physics (e.g. location of BAO peaks). Other methods for characterizing the acoustic oscillation scales can be found, for example, in Percival et al. 2007b and Nishimichi et al. 2007.

3.4 Results: RSD

In Fig. 1, we compare \( C_l(k) \) against a linear redshift space power spectrum, \( P_{\ell}(k) \), spectra for \( \ell = 5 \) and 50 at two given surveys corresponding to \( r = 100 \) and 1400 h\(^{-1}\)Mpc. In this plot, the ratios are constructed by considering the distributions of the appropriate spectra. The following ratios have been used:

\[
R_l^C(k) = \frac{C_l^{\text{RSD, Lin B}}(k)}{C_l^{\text{RSD, Lin B}}(k)}.
\]

(39)
Linear RSD, PT and BAOs: 3D Sph. analysis

Figure 1. Slice in $l$-space showing $R_{C}^C(k)$ for a wide and shallow survey of $r_0 = 100 \ h^{-1} \ Mpc$ at $\ell = 5$ (first panel) and $\ell = 50$ (second panel) and for a wide and deep survey of $r_0 = 1400 \ h^{-1} \ Mpc$ at $\ell = 5$ (third panel) and $\ell = 50$ (fourth panel). The blue line denotes the $C(00)$ term, the purple line the sFB spectra incorporating RSD and the red line shows the Fourier space power spectra. In the linear regime, the linear prefactors for RSD in the Fourier power spectra cancel and the results correspond to the unredshifted Fourier space power spectra.

Figure 2. Ratio $R_{C}^C(k)$ of sFB spectrum with and without the physical effects of baryons in $(\ell, k)$ phase space for a wide and shallow survey of $r_0 = 100 \ h^{-1} \ Mpc$ using a Gaussian selection function but with the inclusion of RSDs.

Figure 3. Ratio $R_{C}^C(k)$ of sFB spectrum with and without the physical effects of baryons in $(\ell, k)$ phase space for a wide and deep survey of $r_0 = 1400 \ h^{-1} \ Mpc$ using a Gaussian selection function but with the inclusion of RSDs.

In Fig. 1, the blue line corresponds to equation (39), the purple line to equation (40) and the red line to equation (41). Figs 2, 3 and 4 correspond to equation (39).

The redshift space Fourier power spectrum is simply the result derived in Kaiser (1987) and corresponds to

\[ P_s(k, \mu) = \left[ 1 + 2\mu^2 + \mu^4 f^2 \right] P(k). \]  

In this linear limit, the redshift space ratio $R_s(k)$ tends to the real-space ratio $R(k)$ as the linear prefactors corresponding to the redshift space corrections cancel. It is apparent that in Fig. 1 the sFB spectra...
Figure 4. Ratio $R^C(k)$ of sFB spectrum for a wide and shallow survey of $r_0 = 100 h^{-1}$ Mpc using a Gaussian selection function but with the inclusion of RSD. Here, we have reduced the dynamic range to highlight the impact that RSDs have on the BAOs. This plot is equivalent to Fig. 2. Compare to the unredshifted results of Fig. 7. RSD suppress the power at lower $\ell$ and $k$ modes and smear the wiggles in the $k$ direction. Power in the first peak is reduced as per Fig. 1 but the amplitudes level at higher $\ell$.

are damped relative to the power spectra. This arises due to mode-mixing contributions inherent when working with the sFB formalism. The unredshifted contributions are constructed from products of Bessel functions that form an orthogonal basis and there is no radial mode-mixing. When introducing RSD the higher order terms are decomposed with respect to products involving derivatives of the spherical Bessel functions which does not form a perfectly orthogonal set of basis functions. As a result of RSD, off-diagonal elements will be generated and there is now coupling between modes. This radial mode-mixing is an intrinsic geometrical artefact of RSD on large scales and carries a distinctive damping signature (Heavens & Taylor 1995; Zaroubi & Hoffman 1996; Shapiro et al. 2011). Such a mode-mixing term is not present in the Kaiser analysis where the basis functions are plane waves which have well-behaved derivatives that maintain the orthogonality of the basis. In the deep survey limit, it is seen that the redshift space sFB spectra do tend towards their Fourier spectra counterparts in terms of the shape, amplitude and phase albeit with the presence of the distinctive damping generated by mode-mixing which is predominantly seen at small scales and hence large $k$.

The effects of RSD can be seen in Figs 2 and 3 in comparison to the equivalent configurations without the presence of RSD in Figs 5 and 6. A lower dynamical range comparison is presented in Figs 4 and 7 to enhance the impact that RSD have on the BAOs. Note the enhanced power at low-$\ell$ and -$k$ as well as some level of fuzziness introduced by the mode mixing. The peak amplitudes are damped at low-$\ell$ and all the features can be seen in the corresponding slice plots of Fig. 1. In a future paper, we will consider the hierarchy of multipole moments in Fourier space RSD and how measures constructed from the multipole moments can be related to RSD in the sFB formalism.

4 REALISTIC SURVEYS

The results that have been discussed above are somewhat idealized in the sense that we assume all-sky coverage with no noise. In realistic surveys, we will often need to take into account the presence of a mask (relating to partial-sky coverage) and noise. If the noise is inhomogeneous, we will be presented with a further complication. For partial-sky coverage, we find mode–mode couplings in the harmonic domain that result in the individual masked harmonics being described by a linear combination of our idealized all-sky harmonics. We do not discuss the role of partial-sky coverage in much detail but do present results generalizing our formalism to include a survey mask.

4.1 Partial-sky coverage and mode mixing

Large-scale surveys do not, generally, have full-sky coverage. Instead, the information regarding sky coverage is encapsulated in a mask $\chi(\Omega)$ which is unity for areas covered in the survey and zero...
distance from photometric redshifts, \( r_i \), and the true distance \( r \) in terms of the conditional probability:

\[
\Psi_{lm}(k) = \sqrt{\frac{2}{\pi}} \int d^3 \tilde{r} \int r p(r | \tilde{r}) \Psi(r) k j_i(k \tilde{r}) Y^*_{lm}(\tilde{\Omega}).
\]

(47)

Such a Gaussian error leads to photometric redshift smoothing.

### 4.3 Error estimate

The signal to noise for individual modes for a given power-spectrum can be expressed as

\[
\frac{\delta C_i(k, k)}{C_i(k, k)} = \sqrt{\frac{2}{2\ell + 1}} \left( 1 + \frac{1}{\bar{n} C_i(k, k)} \right),
\]

(48)

where \( \bar{n} \) is the average number density of galaxies and the second term represents the leading order shot-noise contribution. For our results, we take \( \bar{n} = 10^{-3} \, h^3 \text{Mpc}^{-3} \).

### 5 NL CORRECTIONS

The role of NL gravitational clustering can be investigated in the sFB formalism by incorporating higher order corrections to the power spectrum as described in PT. The approach we adopt here is SPT, also known as Eulerian perturbation theory, which provides a rigorous framework from which we can investigate the structure of the sFB spectra in a fully analytic manner (Vishniac 1983; Fry 1984; Goroff et al. 1986; Suto & Sasaki 1991; Makino, Sasaki & Suto 1992; Jain & Bertschinger 1994; Scoccimarro & Frieman 1996). SPT is one of the most straightforward approaches to studies beyond linear theory and is based on a series solution to the hydrodynamical fluid equations in powers of an initial density or velocity field. The NL clustering of matter arises from mode–mode couplings of density fluctuations and velocity divergence as seen from the Fourier space equations. The role of PT in the NL evolution of the BAO in the power spectrum has been previously investigated (for an incomplete selection of references please see Jeong & Komatsu 2006; Nishimichi et al. 2007; Nomura et al. 2008, 2009; Taruya et al. 2009, 2010). In this paper, we generalize these investigations to the sFB approach. The redshift of the Fourier space power spectra was taken to be \( z \sim 0.2 \) and the effects of growth have not been analysed in detail. For small surveys, the growth does not seem to have significant effects.

### 5.1 Standard perturbation theory

Consider the hydrodynamic equations of motion for density perturbations \( \delta \) such that our coming coordinates are denoted by \( x \) and the conformal time by \( \eta \):

\[
\delta(x, \eta) + \nabla \cdot [(1 + \delta(x, \eta))v(x, \eta)] = 0,
\]

(49)

\[
v'(x, \eta) + [v(x, \eta) \cdot \nabla] v(x, \eta) + H(\eta) v(x, \eta) = -\nabla \phi(x, \eta),
\]

(50)

\[
\nabla^2 \phi(x, \eta) = \frac{3}{2} H^2(\eta) \delta(x, \eta),
\]

(51)

where a prime denotes the derivative with respect to the conformal time and \( H = a' / a \). The rotational mode of the peculiar velocity \( v \) is a decaying solution in an expanding universe and can be neglected in this approach. We introduce a scalar field describing the velocity divergence:

\[
\Theta(x, \eta) = \nabla \cdot v(x, \eta).
\]

(52)
In our discussion, we will focus on a description of the density perturbations and Fourier decompose the above equations to set up and solve a system of integro-diﬀerential equations. The Fourier decomposition of the perturbations are deﬁned by
\[
\delta(x, \eta) = \int \frac{dk}{(2\pi)^3} \Theta(k, \eta)e^{i k \cdot x},
\]
\[
\Theta(x, \eta) = \int \frac{dk}{(2\pi)^3} \Theta(k, \eta)e^{-i k \cdot x}.
\]

The equations of motion can be decomposed as follows:
\[
\delta'(x, \eta) + \Theta(k, \eta) = -\int d^3k_1 \int d^3k_2 \delta^{(3)}(k_1 + k_2 - k) \times \frac{k \cdot k_1}{k_1^2} \Theta(k_1, \eta) \delta(k_2, \eta),
\]
\[
\Theta'(k, \eta) = \frac{3}{2} \mathcal{H}(\eta) \delta(k, \eta).
\]

In order to solve these coupled integro-diﬀerential equations, we introduce a perturbative expansion of our variables:
\[
\delta(k, \eta) = \sum_{n=1}^{\infty} a^n(\eta) \delta_n(k),
\]
\[
\Theta(k, \eta) = \mathcal{H}(\eta) \sum_{n=1}^{\infty} a^n(\eta) \Theta_n(k).
\]

The general nth order solutions are given by
\[
\delta_n(k) = \int d^3q_1 \cdots \int d^3q_n \delta^{(3)} \left( \sum_{i=1}^{n} q_i - k \right) \times F_n(q_1, \ldots, q_n) \Pi_{i=1}^{n} \delta(q_i),
\]
\[
\Theta_n(k) = -\int d^3q_1 \cdots \int d^3q_n \delta^{(3)} \left( \sum_{i=1}^{n} q_i - k \right) \times G_n(q_1, \ldots, q_n) \Pi_{i=1}^{n} \delta(q_i),
\]
where the kernels \( F_n(q_1, \ldots, q_n) \) and \( G_n(q_1, \ldots, q_n) \) are given by (Jain & Bertschinger 1994)
\[
F_n(q_1, \ldots, q_n) = \sum_{m=1}^{n-1} \frac{G_n(q_1, \ldots, q_m)}{(2n + 3)(n - 1)!} \times \left[ (1 + 2n) \frac{k \cdot k_1}{k_1^2} F_{m-1}(q_{m+1}, \ldots, q_n) + \frac{k^2(k_1 \cdot k_2)}{k_1^2k_2^2} G_{m-1}(q_{m+1}, \ldots, q_n) \right],
\]
\[
G_n(q_1, \ldots, q_n) = \sum_{m=1}^{n-1} \frac{G_n(q_1, \ldots, q_m)}{(2n + 3)(n - 1)!} \times \left[ 3 \frac{k \cdot k_1}{k_1^2} F_{m-1}(q_{m+1}, \ldots, q_n) + \frac{k^2(k_1 \cdot k_2)}{k_1^2k_2^2} G_{m-1}(q_{m+1}, \ldots, q_n) \right].
\]

The kernel \( F_n(q_1, \ldots, q_n) \) is not symmetric under permutations of the argument \( q_1, \ldots, q_n \) and must be symmetrized:
\[
F_n^{(s)} = \frac{1}{n!} \sum_{\text{Permutations}} F_n(q_1, \ldots, q_n).
\]

As an example, the second-order symmetrized solution is given by
\[
F_2^{(s)}(k_1, k_2) = \frac{5}{7} + \frac{2(k_1 \cdot k_2)^2}{k_1^2k_2^2} + \frac{2}{7} \left( \frac{1}{k_1^2} + \frac{1}{k_2^2} \right) \cdot
\]

The corresponding second-order matter power spectrum represents the linear matter power spectrum plus the additional higher order corrections. This calculation is made under the assumption that the first-order density perturbations \( \delta_1(k) \) constitute a Gaussian random ﬁeld. The power spectrum up to second order is given by
\[
P_{\text{SPT}}(k, z) = D^2(z) P_{\text{lin}}(k) + D^2(z) P_2(k),
\]
where \( P_{\text{lin}} \) is the conventional linear matter power spectrum and the second-order correction are given by
\[
P_2(k) = P_{22}(k) + 2 P_{13}(k).
\]

These terms correspond to the contributions to the four-point correlation function from the (2,2)-order and the (1,3)-order cross-correlations. The explicit form of these terms are given by
\[
P_{22}(k) = 2 \int d^3q P_{\text{lin}}(k - q) \left[ F_2(q, k - q) \right]^2,
\]
\[
P_{13}(k) = 3 P_{\text{lin}}(q) \int d^3q P_{\text{lin}}(q) F_1(q, -q, k),
\]
and the full equations are presented in Appendix C.

It should be noted that the analytical predictions arising from SPT will eventually break down as the NL terms become dominant over the linear theory predictions. Jeong & Komatsu (2006) demonstrated that one-loop SPT was able to ﬁt N-body simulations to greater than 1 per cent accuracy when the maximum wavenumber \( k_{\text{per cent}} \) satisﬁes (Taruya et al. 2009)
\[
\frac{k_{\text{per cent}}}{6\pi^2} \int_0^{k_{\text{per cent}}} dq P_{\text{lin}}(q; z) = C,
\]
where \( C = 0.18 \) in SPT. SPT theory relies on a straightforward expansion of the set of cosmological hydrodynamical equations and the approach has been repeatedly noted as being insuﬃciently accurate to model and describe the BAOs (Jeong & Komatsu 2006; Carlson, White & Padmanabhan 2009; Nishimichi et al. 2009; Taruya et al. 2009; Taruya et al. 2010). In particular, the amplitude of SPT predicts a monotonical increase with wavenumber that overestimates the amplitude (Fig. 8) with respect to N-body simulations (Taruya et al. 2009). This is also seen in the full \((k, \ell)\) space spectra in Figs 9 and 10.

### 5.2 Results: SPT

In Figs 9 and 10, we have divided the NL power spectrum by a linear no-baryon power spectrum when constructing the ratio \( R_{\ell}^{(s)} \) highlighting the scale dependence introduced by mode coupling. An alternative possibility would be to divide the NL power spectrum \( P_{\text{NL}} \) by a power spectrum constructed from smoothing the NL spectrum \( P_{\text{NL}}^{\text{smooth}} \), that removes the scale dependence and allows for a more detailed comparison of PT predictions against numerical simulations. We construct the ratios as follows:
\[
R_{\ell}^{\text{NL}/\text{SPT}(k)} = \frac{C_{\text{NL}/\text{SPT}, B}(k)}{C_{\ell}^{\text{Lin,ab}}(k)}
\]

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Figure 8. Slice in $l$-space showing $R_C^\ell(k)$ for $\ell = 5$ (top panels) and $\ell = 50$ (bottom panels) in a wide and shallow survey of $r_0 = 100 \, h^{-1} \text{Mpc}$ (left-hand panels) as well as for a deep survey of $r_0 = 1400 \, h^{-1} \text{Mpc}$ (right-hand panels). The solid blue line represents the linear angular spectra, the solid purple line the NL one-loop SPT angular spectra and the dashed line the NL one-loop SPT power spectrum. SPT consistently overestimates the linear power spectrum in the large-$k$ limit and it is well known that SPT works well at high-$z$ and large scales.

Figure 9. Ratio $R_C^\ell(k)$ of sFB spectrum with and without the physical effects of baryons in $(\ell, k)$ phase space for a wide and shallow survey of $r_0 = 100 \, h^{-1} \text{Mpc}$ using a Gaussian selection function but with the inclusion of NL features as calculated in SPT.

Figure 10. Ratio $R_C^\ell(k)$ of sFB spectrum with and without the physical effects of baryons in $(\ell, k)$ phase space for a wide and deep survey of $r_0 = 1400 \, h^{-1} \text{Mpc}$ using a Gaussian selection function but with the inclusion of NL features as calculated in SPT.

$R_C^{\ell, \text{Lin}}(k) = \frac{C_\ell^{\text{Lin}, B}(k)}{C_\ell^{\text{Lin}}(k)}$ \hspace{1cm} (71)

$R_P^{\text{NL}/\text{SPT}}(k) = \frac{P_{\text{NL}/\text{SPT}, B}(k)}{P^{\text{Lin}, \text{SPT}}(k)}$. \hspace{1cm} (72)

In Fig. 8, the blue spectra corresponds to equation (70), the purple spectra to equation (71) and the red spectra to equation (72). These spectra do not incorporate RSD. In Figs 9 and 10, the ratio equation (70) is used.

5.3 Lagrangian perturbation theory

LPT (Matsubara 2008a) provides a description of the formation of structure by relating the Eulerian coordinates, $x$, to comoving.
coordinates, \( q \), through the displacement field \( \Psi(q, t) \):

\[
x(q, t) = q + \Psi(q, t).
\]

With the assumption that the initial density field is sufficiently uniform, the Eulerian density field \( \rho(x) \) will satisfy the continuity relation \( \rho(x) \partial_x^3 x = \tilde{\rho} \partial_q^3 q \), where we have denoted the mean density in comoving coordinates by \( \tilde{\rho} \). The fraction densities will then be given by

\[
\delta(x) = \int d^3 q \, \delta^3 \left[ x - q - \Psi(q) \right] - 1,
\]

\[
\delta(k) = \int d^3 q \, e^{-ik\cdot q} \left[ e^{-ik\cdot \Psi(q)} - 1 \right].
\]

Assuming a pressureless self-gravitating Newtonian fluid in an expanding \( \Lambda \)CDM universe, the equations of motion for the displacement field are given by (Matsubara 2008a)

\[
\frac{d^2}{dt^2} \Psi + 2H \frac{d}{dt} \Psi = -\nabla_x \phi [q + \Psi(q)].
\]

where \( \phi \) is the gravitational potential as determined by Poisson’s equation: \( \nabla^2 \phi(x) = 4\pi G \rho a^2 \delta(x) \). LPT proceeds by performing a perturbative series expansion of the displacement field:

\[
\Psi = \Psi^{(1)} + \Psi^{(2)} + \cdots
\]

\[
\Psi^{(N)} = \mathcal{O} \left( \left[ \Psi^{(1)} \right]^N \right). \tag{78}
\]

The perturbative terms in the series expansion can be written schematically as

\[
\Psi^{(n)}(p) = \frac{i}{n!} D^n(t) \int \frac{d^3 p_1}{(2\pi)^3} \frac{d^3 p_n}{(2\pi)^3} \delta^3 \left( \sum_{j=1}^n p_j - p \right)
\times L^{(n)}(p_1, \ldots, p_n) \delta_0(p_1) \cdots \delta_0(p_n).
\]

We can perform a similar expansion for both the fractional density and the power spectrum, further details can be found in Matsubara (2008a) and we will just introduce the results for the power spectrum and how it relates to the predictions of SPT. The power spectrum can be written as

\[
P(k) = \int d^3 q \, e^{-ik\cdot q} \left( \left[ e^{-ik\cdot \Psi(q_1) - \Psi(q_2)} \right] - 1 \right).
\]

The two main types of terms that we find in these equations are those terms that depend only on a single position, which are factored out into the first exponential term, and those terms that depend on some separation between positions, as seen in the second exponential term. Using the cumulant expansion theorem, the power spectrum can be written as

\[
P(k) = \exp \left[ -2 \sum_{n=1}^{\infty} \frac{k_{1} \cdot k_{2n}}{(2n)!} A_{12n}^{(2n)} \right]
\times \int d^3 q e^{-ik\cdot q} \left\{ \exp \left[ \sum_{N=2}^{\infty} \frac{k_{1} \cdot k_{N}}{(N)!} B_{1N}^{(N)}(q) \right] - 1 \right\}, \tag{81}
\]

where \( A_{12n}^{(2n)} \) and \( B_{1N}^{(N)} \) are given in Matsubara (2008a). \( A^{(N)} \) relates to the cumulant of a displacement vector at a single position and \( B^{(N)} \) relates to the cumulant of two displacement vectors separated by \( |q| \). Expanding both the \( A^{(N)} \) and the \( B^{(N)} \) terms yields SPT. Matsubara (2008a), however, proposes expanding only the \( B^{(N)} \) terms and leaving the \( A^{(N)} \) terms as an exponential prefactor. The justification for this is that this exponential prefactor will contain infinitely higher order perturbations in terms of SPT and has effectively given a way to resum the infinite series of perturbations found in SPT. Expanding and solving for the \( B^{(N)} \) terms yields the standard LPT results Matsubara (2008a):

\[
P(k) = e^{-i(k/2)^2} \left[ P_{\text{lin}}(k) + P_2(k) + P_{13}^{\text{LPT}}(k) \right]. \tag{82}
\]

The term \( P_{2} \) is identical to its SPT counterpart but the term \( P_{13}^{\text{LPT}} \) is now slightly modified but retains much of the structure found in SPT.

5.4 Results: LPT

In Figs 11 and 12, we again divide the NL power spectrum by a linear no-baryon power spectrum when constructing the ratio \( R_C(k) \). The

![Figure 11. Ratio \( R_C(k) \) of sFB spectrum with and without the physical effects of baryons in \((\ell, k)\) phase space for a wide and shallow survey of \( r_0 = 100 \, h^{-1} \, \text{Mpc} \) using a Gaussian selection function but with the inclusion of NL features as calculated in LPT.](image)

![Figure 12. Ratio \( R_C(k) \) of sFB spectrum with and without the physical effects of baryons in \((\ell, k)\) phase space for a wide and deep survey of \( r_0 = 1400 \, h^{-1} \, \text{Mpc} \) using a Gaussian selection function but with the inclusion of NL features as calculated in LPT.](image)
explicit ratios used are

\[ R_{C,L}^{NL/LPT}(k) = \frac{C_{NL}^{LPT,B}(k)}{C_{Lin,nB}(k)} \]  

(83)

\[ R_{C,L}^{Lin}(k) = \frac{C_{Lin,B}(k)}{C_{Lin,nB}(k)} \]  

(84)

\[ R_{P,NL/LPT}^{NL/LPT}(k) = \frac{P_{NL,LPT,B}(k)}{P_{Lin,nB}(k)}. \]  

(85)

In Fig. 13, the blue spectra corresponds to equation (83), the purple spectra to equation (84) and the red spectra to equation (85). These spectra do not incorporate RSD. In Figs 11 and 12, the ratio equation (83) is used.

The sFB can be seen to mimic the predictions of LPT in consistently underestimating the power at large \( k \) but we also see that the sFB power spectra radialize towards the NL LPT spectra in the limit \( r \to \infty \). This can be seen in Fig. 13 where the NL sFB tends towards the Fourier space power spectrum in amplitude and phase. We have included a comparison to the linear sFB spectra, which we know to radialize to the linear Fourier space spectra. This behaviour is completely expected due to the nature of the sFB formalism and the fact that the resulting angular spectra are still constructed via products of Bessel functions which form an orthogonal set of basis functions. As such we do not observe the types of mode-mixing that are inherent when considering RSD in the sFB formalism. The damping and smearing of the BAOs in this instance is purely from gravitational instability and is encapsulated in the power spectrum [see Padmanabhan et al. (2007) for a discussion of the reconstruction of baryon acoustic oscillations in the LPT formalism]. We also note that the full \((\ell, k)\) plane is an interesting arena for visualizing some of the differences in behaviour between various models for structure formation. This can be seen in the changes to the widths and amplitudes of the BAO wiggles as seen in the plane in Figs 9–12.

As future wide-field surveys will cover both wide and deep regions of the sky we can use the sFB formalism as a tool to distinguish between different models for NL evolution of the matter density field. Interesting questions include, how do different theories affect the distribution of power in the radial and tangential modes? How can the sFB formalism be expanded to compare the RSD results to those as derived from higher order PT? How can we best characterize the sFB spectra and how can we characterize the radialization of information in these higher order models? The analysis and results to these questions will be presented in a forthcoming paper.

6 RESULTS

Following Rassat & Refregier (2012), we construct the quantity \( R_{C,L}^{NL/LPT}(k) \) to isolate the BAOs in the sFB formalism. The matter power spectrum includes the physical effects of baryons leading to the characteristic oscillations as seen in Fourier space (Peebles & Yu 1970; Sunyaev & Zeldovich 1970; Seo & Eisenstein 2003, 2007). In our analysis, we have adopted the zero-Baryon transfer function of Eisenstein et al. (2005) to model the power spectra excluding the physical effects of baryons.

In Fig. 1, we construct slices of constant \( \ell \) through \( R_{C,L}^{NL/LPT}(k) \) to investigate how RSD manifest themselves in the oscillations. Rassat & Refregier (2012) used such slice plots to investigate the radialization of information when varying levels of tangential and radial perturbations from SPT that occurs in LPT.

Figure 13. Slice in \( \ell \)-space showing \( R_{C,L}^{NL/LPT}(k) \) for \( \ell = 5 \) (top panels) and \( \ell = 50 \) (bottom panels) in a wide and shallow survey of \( r_0 = 100 h^{-1} \) Mpc (left-hand panels) and a wide and deep survey of \( r_0 = 1400 h^{-1} \) Mpc (right-hand panels). The solid blue line denotes the linear results, the solid purple line the NL one-loop LPT spectra and the dashed line the NL one-loop LPT spectra. LPT consistently underestimates the power spectrum in the large-\( k \) limit contrasting to the divergence at large-\( k \) in one-loop SPT results. This difference occurs due to the effective resummation of an infinite series of perturbations from SPT that occurs in LPT.
information is included in a survey. The radialization of information can be investigated by noting that in the limit $r_0 \to \infty$ we find
\[
\lim_{r_0 \to \infty} R_\ell^C(k) = R_\ell^P(k) = \frac{p_\ell^B(k)}{p^{\text{NL}}_\ell(k)}, \tag{86}
\]
Using this definition, radialization means that $R_\ell^C(k)$ tends towards $R_\ell^P(k)$ in both phase and amplitude. This occurs as the tangential modes are attenuated due to mode-cancelling along the line of sight (Rassat & Refregier 2012). The radialization can be seen in Figs 1, 8 and 13 as the amplitude and phase of the sFB spectra tend towards those of the Fourier space spectra. Additionally, the BAOs appear to only have a radial ($\ell$) dependence in surveys with a large radial parameter $r_0$, as can be seen by the invariance the BAOs under a varying multipole $\ell$. The addition of RSD does not change this trend drastically, though we do see more prominent radial and tangential dependence in Fig. 1 with the rate at which the BAOs radialize being affected due to mode-mixing that leads to attenuation and peak shifts. The results appear to be in agreement with previous studies with per cent-level shifts in the peaks to smaller $k$ and damping of the amplitude (Nishimichi et al. 2007; Nomura, Yamamoto & Nishimichi 2008; Smith, Scoccimarro & Sheth 2008; Nomura et al. 2009; Taruya et al. 2010). As can be seen in Fig. 1, the BAOs seem to effectively radialize, even in the presence of RSD, at large values of the radius parameter $r_0$ and for higher multipoles $\ell$. Effective radialization simply means that the behaviour (i.e. amplitude and phase of the peaks and troughs) of the sFB spectra with RSD asymptotes towards the Fourier space spectra with RSD under the caveat that the intrinsic mode-mixing causes some smearing of radial information and leads to the distinctive damping features seen at high-$k$. The radialization of information can be linked with the preservation of the orthogonality of the basis functions. In the case of RSD, the appearance of derivatives of spherical Bessel functions guarantees that the basis will not be perfectly orthogonal and we observe mode–mode coupling and the generation of off-diagonal contributions. In higher order PT, the basis functions are still spherical Bessel functions and we observe the radialization as per linear theory. The behaviour of the NL sFB spectra in the full ($\ell, k$) space is naturally different for various descriptions of non-linearity in gravitational collapse.

The BAOs in the sFB formalism will radialize as the survey size, $r_0$, is allowed to increase. This corresponds to the amplitude and the phase of the BAOs tending towards the values as measured in the Fourier space ratio $R_\ell^P(k)$. As noted in Rassat & Refregier (2012), for a wide-field shallow survey the BAOs will have smaller amplitudes and are spread across the ($\ell, k$) space. It was also shown in Rassat & Refregier (2012) that the BAOs appear to radialize before the full sFB spectrum is able to and notably so at large $\ell$ (Figs 5 and 6) and this is one of the key motivations for implementing the sFB formalism. With the addition of RSD, the radialization of information is intrinsically limited due to mode-mixing but a lot of the same phenomenological behaviour can be seen: dependence on radial modes and not on tangential modes at large $r$ and the asymptotic behaviour towards the Fourier space spectra at large $r$.

7 CONCLUSION
The BAOs give rise to a characteristic signature in the observed matter power spectrum that acts as a standard ruler. Unfortunately, the observed matter power spectrum is contaminated and complicated by the NL evolution of density perturbations, galaxy clustering bias, RSD and survey specific systematic errors. Additionally, upcoming future surveys will cover both large and deep areas of the sky demanding a formalism that simultaneously treats the both the spherical sky geometry and the extended radial coverage. The sFB basis was proposed as a natural basis for random fields in this geometry. The recent study by Rassat & Refregier (2012) was an initial step into investigating the role of the sFB formalism in the study and analysis of the BAO. This study, however, did not go as far as including higher order contributions to the power spectrum that may impact the radialization of information by introducing, for example, mode–mode couplings. The stability of this radialization of information and the information content of tangential ($\ell$) and radial ($k$) modes for higher order physics is the key topic of interest.

In this paper, we have presented a short treatment of the effects of linear RSD and NL corrections to measurements of BAOs in the sFB expansion. In order to guide this investigation, we have extended the formalism and techniques outlined in Rassat & Refregier (2012) and the appropriate machinery for partial-sky coverage was introduced. In particular, we have been able to use the procedure outlined in Heavens & Taylor (1995) to construct a series expansion solution to model RSD. This solution was used to numerically and analytically investigate the modulation to the angular sFB power spectrum. The qualitative behaviour of these corrections was outlined for surveys with varying levels of radial ($k$ modes) and tangential ($\ell$ modes) information. It was seen that the RSD impact the radialization of information through mode-mixing that generates a distinct signature in the spectra. These RSD were investigated over a range of survey configurations. The mode–mode coupling was related to the presence of derivatives of spherical Bessel functions and was contrasted to the linear Kaiser result in which the basis functions are constructed from plane waves or derivatives of plane waves which simply return a plane wave of the same frequency and preserve orthogonality. This mode–mode coupling can therefore be thought of as a geometrical artefact in the sFB formalism arising from RSD on large scales (Heavens & Taylor 1995; Zaroubi & Hoffman 1996; Shapiro et al. 2011).

Additionally, we considered the structure and form of the sFB spectra when non-linearity arising from gravitational clustering was considered. We primarily investigated one-loop corrections to the matter power spectrum arising from two mainstream models for leading order corrections as given by SPT and LPT. A brief outline of PT methods was given and the basic equations for SPT and LPT introduced. The NL corrections, and how we expect them to be independent of the notion of radialization of information in the BAOs, was numerically investigated. The redshift of the Fourier space power spectrum was taken to be $z \sim 0.2$ and the detailed study of the NL corrections with redshift will be presented elsewhere. These are not thought to be important at low redshifts or shallow surveys where the impact of growth seems negligible.

In this paper, we have neglected other contributions to the power spectrum such as General Relativistic corrections, lensing terms, the role of non-linearities through more detailed studies (see Umeh et al. 2012 for example), more complex treatments of galaxy biasing and more detailed modelling of the dynamical and radiative processes involved in these processes (Guillet, Teyssier & Colombi 2010; Juszkiewicz et al. 2013). In addition, we have not considered the role of systematic errors associated with a given survey. It would be interesting to compare the results from SKA-like configurations and $N$-body simulations but we leave this to a future paper.

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APPENDIX A: SPHERICAL BESSEL FUNCTION

In this section, we quickly outline some of the more useful properties of the spherical Bessel functions that have been used in the derivation of our results. The first important property of spherical Bessel functions is that they obey a well-known orthogonality condition:

$$\int_0^\infty r^2drj_\ell(kr)j_\ell(k'r) = \frac{\pi}{2kk'}\delta(k - k').$$  \hspace{1cm} (A1)
The first derivative of the spherical Bessel function can be expressed using the following recursion relation:

\[ j'_\ell(r) = \frac{1}{2\ell + 1} [j_{\ell+1}(r) - (\ell + 1)j_{\ell}(r)]. \tag{A2} \]

The second- and higher order derivatives are deduced by successive application of the above expression:

\[ j''_\ell(r) = \frac{(2\ell^2 + 2\ell - 1)}{(2\ell + 3)(2\ell + 1)} j'_\ell(r) - \frac{\ell(\ell - 1)}{(2\ell - 1)(2\ell + 1)} j_{\ell-2}(r) \]
\[ - \frac{(\ell + 1)(\ell + 2)}{(2\ell + 1)(2\ell + 3)} j_{\ell+2}(r). \tag{A3} \]

These expressions can be used to simply the kernels \( I^{(1)}_{k,k'}(k_1) \) defined in equation (31) to express mode-mixing due to RSD.

**APPENDIX B: SPHERICAL HARMONICS**

The spherical-harmonics are complete and orthogonal on the surface of the sphere:

\[ \sum_{\ell m} Y_{\ell m}(\hat{\Omega})Y_{\ell' m'}(\hat{\Omega}) = \delta^{2D}(\hat{\Omega} - \hat{\Omega}'); \tag{B1} \]

\[ \int d\Omega \ Y_{\ell m}(\hat{\Omega})Y_{\ell' m'}(\hat{\Omega}) = \delta^{\ell}_\ell \delta^m_{m'}. \tag{B2} \]

The overlap integrals of three spherical harmonics are given by the Gaunt integral which are expressed in terms of 3j symbols (denoted by matrices below):

\[ \int d\Omega Y_{\ell m}(\hat{\Omega})Y_{\ell' m'}(\hat{\Omega})Y_{\ell'' m''}(\hat{\Omega}) = I_{3j} \]
\[ \times \left( \begin{array}{ccc} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{array} \right) \; ; \tag{B3} \]
\[ I_{3j} = \sqrt{(2\ell_1 + 1)(2\ell_2 + 1)(2\ell_3 + 1)} \frac{\pi}{46}. \tag{B4} \]

**APPENDIX C: 3J SYMBOLS**

The following orthogonality properties of 3j symbols were used to simplify various expressions:

\[ \sum_{(m_3)} (2\ell_3 + 1) \left( \begin{array}{ccc} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{array} \right) \left( \begin{array}{ccc} \ell_1 & \ell_2 & \ell \\ m_1 & m_2 & m \end{array} \right) = \delta^m_{m_1} \delta^m_{m_2} ; \tag{C1} \]

\[ \sum_{m_1, m_2} \left( \begin{array}{ccc} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{array} \right) \left( \begin{array}{ccc} \ell_1 & \ell_2 & \ell_3' \\ m_1 & m_2 & m_3' \end{array} \right) = \delta^{m_3}_{m_3'} \delta^{m_3'}_{m_3}. \tag{C2} \]

**APPENDIX D: FINITE SURVEYS AND DISCRETE SPHERICAL BESSEL–FOURIER TRANSFORMATION AND PSEUDO-\( \ell \)S**

### D1 3D scalar fields

Different types of boundary conditions are employed in the literature for finite surveys (Binney & Quinn 1991; Fisher et al. 1995; Heavens & Taylor 1995).

A natural choice for the boundary condition is to assume that the field vanishes at the boundary of the survey \( r = R \) leading to following condition on the radial modes that is determined by the zeros of the spherical Bessel functions \( j_\ell(r) \):

\[ j_\ell(q_{\ell a}) = j_\ell(k_{\ell a} R) = 0; \quad q_{\ell a} = k_{\ell a} R. \tag{D1} \]

The closure relation for spherical harmonics will take the following form:

\[ \int_0^1 dz z^2 j_\ell(k_{\ell a} z) j_\ell(k_{\ell a} z) = \frac{1}{2} [j_{\ell + 1}(q_{\ell a})]^2 \delta_{\ell \ell'} \delta_{m m'} \tag{D2} \]

which, in terms of the radial wavenumber, can be expressed as follows:

\[ \int_0^R dr r^2 k_{\ell a} k_{\ell a'} j_\ell(k_{\ell a} r) j_\ell(k_{\ell a'} r) = \frac{k_{\ell a}^2 [j_{\ell + 1}(q_{\ell a})]^2}{2R^3} \delta_{\ell \ell'} \delta_{m m'}. \tag{D3} \]

The discrete spectrum is determined by the zeros of the spherical Bessel function. The normalization coefficients are given by

\[ \frac{1}{\tau_{\ell a}} = \frac{R^3}{2} k_{\ell a} j_{\ell + 1}(k_{\ell a} R)^2. \tag{D4} \]

The inverse and forward discrete sFB transforms are as follows:

\[ \Psi_{\ell a}(k_{\ell a}) = \tau_{\ell a} \int d^3 r \Psi(r) j_\ell(k r) Y_{\ell m}(\hat{\Omega}); \tag{D5} \]

\[ \Psi(r) = \sum_{\ell a} \tau_{\ell a} \Psi_{\ell a}(k_{\ell a}) Y_{\ell m}(\hat{\Omega}). \tag{D6} \]

The following expression is useful:

\[ \Psi_{\ell a}(k_{\ell a}) = \frac{i^\ell k_{\ell a}}{(2\pi)^{3/2}} \int d\Omega \Psi(k_{\ell a}, \hat{\Omega}) Y_{\ell m}(\hat{\Omega}). \tag{D7} \]

In case of finite survey the 3D power-spectrum samples only discrete radial wavenumbers \( k_{\ell a} \) which is defined by the survey radius \( R \):

\[ \langle \Psi_{\ell a}(k_{\ell a}) \Psi_{\ell a'}(k_{\ell a'}) \rangle = P_{\ell a} \delta_{\ell \ell'} \delta_{m m'} \delta_{m m'}. \tag{D8} \]

In addition to finite survey size, surveys often have a mask \( s(\hat{\Omega}) \). The sFB transform of a masked field defines the convolved or pseudo-\( \ell \) harmonics \( \Psi_{\ell a}(k_{\ell a}) \):

\[ \Psi_{\ell a}(k_{\ell a}) = \int_0^\tau \tau_{\ell a} \int_0^R r^2 dr \int d\Omega [s(\hat{\Omega}) \Psi(r) j_\ell(k_{\ell a} r)] Y_{\ell m}(\hat{\Omega}) d\Omega. \tag{D9} \]
The convolved or pseudo-harmonics are expressed in terms of all-sky harmonics \( \Psi_{\ell m}(k_n) \) by the following expression:

\[
\tilde{\Psi}_{\ell m}(k_n) = \sum_{\ell'} \sum_{\ell''} \sum_c \tau_{\ell'\ell''} W(k_{\ell'}, k_{\ell''}) \Psi_{\ell m}(k_{\ell''})
\times s_{\ell''}^c I_{\ell'\ell''} \left( \frac{\ell_m}{m}, \frac{\ell}{m'}, \frac{\ell''}{m''} \right).
\]

The kernel \( W(k_{\ell'}, k_{\ell''}) \) depends on selection function \( \phi(r) \):

\[
W(k_{\ell'}, k_{\ell''}) = \int_0^R r^2 \, dr \, \phi(r) \, j_\ell(k_{\ell''} r). \tag{D11}
\]

The pseudo-C\( \ell \)s (PCLs) constructed from the convolved harmonics are a function of power spectrum of the angular mask \( C_\ell \), normalization coefficients \( \tau_{\ell} \) and the selection function \( \phi \):

\[
\tilde{C}_\ell(k_n) = (\Psi_{\ell m}(k_n) \Psi_{\ell m}^*(k_n))
= \sum_{\ell'} \sum_{\ell''} \sum_c t_{\ell''}^c I_{\ell'\ell''} \left( \frac{\ell}{2\ell + 1}, \frac{\ell}{0}, \frac{\ell''}{0} \right)^2 \times W^2(k_{\ell'}, k_{\ell''}) C_{\ell''}^c.
\tag{D12}
\]

Note that the PCLs \( \tilde{C}_\ell(k_n) \) are linear superposition of the power spectrum of underlying field \( C_\ell(k_n) \). The mixing matrix \( M_{\ell \ell', \ell''} \) is given by:

\[
\tilde{C}_\ell(k_n) = \sum_{\ell''} M_{\ell \ell', \ell''} C_{\ell''}(k_n).
\tag{D13}
\]

where the mixing matrix is given by the following expression:

\[
M_{\ell \ell', \ell''} = \sum_{c} t_{\ell''}^c I_{\ell'\ell''} \left( \frac{\ell}{2\ell + 1}, \frac{\ell}{0}, \frac{\ell''}{0} \right)^2 W^2(k_{\ell'}, k_{\ell''}) C_{\ell''}^c.
\tag{D14}
\]

An unbiased estimate of the 3D power spectra can be written as

\[
C_\ell(k_n) = \sum_{\ell''} M_{\ell \ell', \ell''}^\dagger \tilde{C}_\ell(k_{\ell''}).
\tag{D15}
\]

This is an extension of well-known results for the projected surveys (Hivon et al. 2002). For low sky coverage and small survey volumes the matrix \( M_{\ell \ell', \ell''} \) is expected to be singular and binning of modes may be required.

A different choice of boundary condition is often employed (Fisher et al. 1995):

\[
j_{\ell \ell''}(k_{\ell''} R) = 0; \tag{D16}
\]

The normalization constants in this case are given by

\[
\frac{1}{\tau_{\ell m}} = \frac{R^3}{2} |j_\ell(k_n R)|^2. \tag{D17}
\]

The expressions for the mixing matrix derived above can still be used simply replacing the normalization coefficients \( \tau_{\ell m} \).

For discrete fields such as the galaxy distribution, we can use the PCL approach if we replace the continuous function \( \Psi(r) \) with a sum of delta functions that peak at galaxy positions \( r_j: \Psi(r) = \sum_j \delta^{(3)}(r - r_j) \); here, \( N \) is the number of galaxies. The sFB for such a discrete field is given by \( \Psi_{\ell m}(k) = \sum_{j=1}^N \tau_{\ell m} j_\ell(r_j k) Y_{\ell m}(\Omega_j) \) where the radial and angular position of galaxies are denoted by \( r_j = (r_j, \Omega_j) = (r_j, \theta_j, \phi_j) \).

**APPENDIX E: STANDARD PERTURBATION THEORY**

In the formalism outlined in Section 5, any statistical observable can be computed to arbitrary order. Typically, we are only interested in the second-order corrections to the matter power spectrum though expressions for higher order corrections have been derived. One of the key issues regarding the inclusion is the computational costs required for these higher order corrections in part due to the high dimensionality of the integrals, even after symmetry arguments have been taken into account. The analytic expressions for the first corrections can be analytically derived (Makino et al. 1992):

\[
P_{1s}(k) = \frac{1}{252} \frac{k^3}{4\pi^2} \int_0^\infty dx P_{\ell m}(k) P_{\ell m}(k x) \left[ \frac{12}{x^2} - 158 + 100 x^2 - 42 x^4 + \frac{3}{x^2} (x^2 - 1) \right] \log \left[ \frac{1 + x}{1 - x} \right], \tag{E1}
\]

\[
P_{1s}(k) = \frac{1}{98} \frac{k^3}{4\pi^2} \int_0^\infty dx P_{\ell m}(k) \int_0^1 d\mu P_{\ell m}(k \sqrt{1 + x^2 - 2x\mu}) \times \frac{(3x + 7\mu - 10x\mu^2)}{(1 + x^2 - 2x\mu)^2} \tag{E2}
\]

\[
P_{13}^{\text{PT}}(k) = \frac{1}{252} \frac{k^3}{4\pi^2} P_{\ell m}(k) \int_0^\infty dx P_{\ell m}(k x) \left[ \frac{12}{x^2} + 10 + 100 x^2 - 42 x^4 + \frac{3}{x^2} (x^2 - 1) \right] \log \left[ \frac{1 + x}{1 - x} \right]. \tag{E3}
\]

**APPENDIX F: FLAT-SKY LIMIT**

For surveys that cover large opening angles on the sky, the full sFB expansion detailed above is the most natural and convenient choice. This expansion does, however, break down for small-angle surveys where the signal of interest occurs at high-\( \ell \) modes. In such a situation, the accurate computation of high-\( \ell \) spherical harmonics is cumbersome and computationally expensive. Instead, it is more natural to approximate the spherical harmonics as sums of exponentials corresponding to a 2D Fourier expansion. Essentially, we are replacing the spherical harmonics solutions with a plane-wave approximation valid at high multipoles.

In the flat-sky limit, we expand a 3D field \( \Psi \) at a 3D position \( r = (r, \theta, \phi) \) on the sky using a basis consisting of 2D Fourier modes and radial Bessel functions:

\[
f(r, \theta) = \sqrt{\frac{7}{\pi}} \int k \, dk \int \frac{d^2\ell}{(2\pi)^2} f(k, \ell) j_\ell(k) e^{i\ell\theta}, \tag{F1}
\]

\[
f(k, \ell) = \sqrt{\frac{7}{\pi}} \int r^2 dr \int d^2\theta f(r, \theta) j_\ell(k) e^{-i\ell\theta}, \tag{F2}
\]

where \( \ell \) is a 2D angular wavenumber and \( k \) is a conventional radial wavenumber. We can simplify the analysis by adopting coordinates such that the survey corresponds to small angles around the pole of the spherical coordinates, defined by angles \((\theta, \phi)\) for which, in the limit \( \theta \to 0 \), we can apply a 2D expansion of the plane waves:

\[
e^{i\ell\theta} \simeq \sqrt{\frac{2\pi}{\ell}} \sum_m i^m Y_{\ell m}(\theta) \phi e^{-im\phi}, \tag{F3}
\]
where \( \ell = (\ell \cos \varphi, \ell \sin \varphi) \) and \( \theta = (\theta \cos \varphi, \theta \sin \varphi) \). The correspondence between the 3D flat-sky and 3D full-sky coefficients can be obtained by substituting equation (F3) into equation (F1) and noting that \( \int d^2 \ell = \int \ell \, d\ell \int d\varphi \ell \to \sum \ell \int d\varphi \ell \) in the high-\( \ell \) limit. The correspondence can be shown to be

\[
f_{\ell m}(k) = \sqrt{\frac{2 \pi}{\ell}} \int d\varphi \ell \frac{\ell}{(2\pi)} e^{-i m \varphi} f(k, \ell)
\]  

(F4)

\[
f(k, \ell) = \sqrt{\frac{2 \pi}{\ell}} \sum m i^{-m} f_{\ell m}(k) e^{im \varphi}.
\]  

(F5)

We now extend this analysis to RSD by constructing harmonics of a field \( \Psi(r) \) in the flat-sky limit when convolved with a selection function \( \phi(s) \). These new flat-sky harmonics take into account the RSD much as before:

\[
\tilde{\Psi}(k, \ell) = \sqrt{\frac{2}{\pi}} \int ds \int d^2 \theta k \Psi(r, \theta) [\phi(s) \ell j_\ell(ks)] e^{-i \ell \cdot \theta}.
\]  

(F6)

Following the same perturbative procedure results in a series expansion in \( \beta \), where

\[
\tilde{\Psi}_\ell(k, \ell) = \tilde{\Psi}^{(0)}_\ell(k, \ell) + \tilde{\Psi}^{(1)}_\ell(k, \ell) + \cdots
\]  

(F7)

As before, the \( \tilde{\Psi}^{(0)}_\ell(k, \ell) \) term represents the unredshifted contribution:

\[
\tilde{\Psi}^{(0)}_\ell(k, \ell) = \sqrt{\frac{2}{\pi}} \int d^2 r \psi(k r) \psi(r) k j_\ell(k r) e^{-i \ell \cdot \theta}
\]  

(F8)

\[
\tilde{\Psi}^{(1)}_\ell(k, \ell) = \sqrt{\frac{2}{\pi}} \int d^2 r \psi(k r) \psi(r) k \left\{ [\psi(r) \cdot \theta] \frac{d}{dr} [j_\ell(k r) \psi(r)] \right\} e^{-i \ell \cdot \theta}.
\]  

(F9)