Exact solutions in gravity with a sigma model source

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Abstract

We consider a $D$-dimensional model of gravity with non-linear “scalar fields” as a matter source. The model is defined on the product manifold $M$, which contains $n$ Einstein factor spaces. General cosmological type solutions to the field equations are obtained when $n-1$ factor spaces are Ricci-flat, e.g. when one space $M_1$ of dimension $d_1 > 1$ has nonzero scalar curvature. The solutions are defined up to solutions to geodesic equations corresponding to a sigma model target space. Several examples of sigma models are presented. A subclass of spherically symmetric solutions is studied and a restricted version of “no-hair theorem” for black holes is proved. For the case $d_1 = 2$ a subclass of latent soliton solutions is singled out.

1 Introduction

A lot of gravitational models (including scalar-tensor theories and supergravity ones), when required solutions have enough space-time symmetries, can be reduced to an effective model with “scalar fields” governed by a sigma model action, see \cite{1, 2, 3, 4, 5, 6, 7, 8} and references therein.

Here we consider a gravitational model governed by a Lagrangian in $D$ dimensions

$$\mathcal{L} = R[g] - h_{ab}(\varphi) g^{MN} \partial_M \varphi^a \partial_N \varphi^b,$$  

(1.1)

where $g$ is a metric and non-linear “scalar fields” $\varphi^a$ come to the Lagrangian in a sigma model form with a certain target space metric $h$ assumed. In higher dimensional gravitational theories the above model first appeared in connection with spontaneous compactification of the extra dimensions \cite{9, 10}.

The Lagrangian (1.1) (e.g. with $h_{ab} = const$) describes the truncated sectors of various $3 < D \leq 10$ supergravity theories in the Einstein frame \cite{11}. Usually these theories contain form-fields (fluxes) in addition to massless scalar fields and Chern-Simons (CS) terms. In this sense, the Lagrangian (1.1) for $D > 3$ (e.g. with $h_{ab} = const$) matches zero flux (and

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CS) limit. For $D = 3$ the Lagrangians of such types are generic ones when dimensional reductions of (bosonic sectors of) supergravity models are considered, see [1,2,6,7] and refs. therein. In higher dimensional gravitational theories the above model first appeared in connection with spontaneous compactification of extra dimensions [9,10].

Here we deal with cosmological type solutions defined on a product of $n$ Einstein spaces (e.g. Ricci-flat ones). The integrable cosmological configurations were studied in numerous papers, see [14,15] (without scalar fields), [16,17,18,19] (with one scalar field), [20,21,22] etc. However, the authors of these papers restricted their attention to a linear sigma model for which components $h_{ab}$ are constant. Here we will study the solutions for $h_{ab}(\phi)$ with arbitrary dependence on $\phi^a$ (e.g. for a non-linear sigma model).

## 2 The model

We start by considering the following action

$$S = \frac{1}{2\kappa^2} \int_M d^Dx \sqrt{|g|} \left\{ R[g] - h_{ab}(\phi)g^{MN} \partial_M \phi^a \partial_N \phi^b \right\} + S_{YGH},$$

(2.1)

where $\kappa^2$ is a $D$-dimensional gravitational coupling, $g = g_{MN}dx^M \otimes dx^N$ is a metric defined on a manifold $M$, $\varphi : M \rightarrow M_\varphi$ is a smooth sigma model map and $M_\varphi$ is a $l$-dimensional manifold (target space) equipped with the metric $h = h_{ab}(\phi)dx^a \otimes dx^b$ ($\varphi^a$ are coordinates on $M_\varphi$). Here $S_{YGH}$ is the standard York-Gibbons-Hawking boundary term [12,13].

The field equations for the action (2.1) read as follows

$$R_{MN} - \frac{1}{2}g_{MN}R = T_{MN},$$

(2.2)

$$\frac{1}{\sqrt{|g|}} \partial_M (g^{MN} \sqrt{|g|}h_{ab}(\phi)\partial_N \phi^b) - \frac{1}{2} \frac{\partial h_{bc}(\phi)}{\partial \varphi^a} \partial_K \varphi^b \partial_L \varphi^c g^{KL} = 0,$$

(2.3)

where

$$T_{MN} = h_{ab}(\phi)\partial_M \varphi^a \partial_N \varphi^b - \frac{1}{2} h_{ab}(\phi)g_{MN} \partial_K \varphi^a \partial_L \varphi^b g^{KL}$$

(2.4)

is the stress-energy tensor.

Here we consider a cosmological type ansatz for the metric and “scalar fields”

$$g = we^{2\gamma(u)}du \otimes du + \sum_{i=1}^n e^{2\beta_i(u)}g_i,$$

(2.5)

$$\varphi^a = \varphi^a(u),$$

(2.6)

where $w = \pm 1$, $i = 1, \ldots, n$.

The metric is defined on the manifold
\[ M = \mathbb{R}_* \times M_1 \times \ldots \times M_n, \quad (2.7) \]

where \( \mathbb{R}_* = (u_-, u_+) \) and any factor space \( M_i \) is a \( d_i \)-dimensional Einstein manifold with the metric \( g^i \) obeying

\[ R_{m,n_i}[g^i] = \xi_i g_{m,n_i}^i, \quad (2.8) \]

\( i = 1, \ldots, n \).

To find solutions for the equations \((2.2)-(2.3)\) seems to be complicated due to the non-linear structure of the Einstein equations and intricacy having scalar fields. However it may be shown that the field equations for the model \((2.1)\) with the metric and "scalar fields" from \((2.5)\) and \((2.6)\) are equivalent to the Lagrange equations corresponding to the Lagrangian of the one-dimensional \((n + l)\)-component modified \( \sigma \)-model

\[ L = \frac{1}{2} \mathcal{N}^{-1} [G_{ij} \dot{\beta}^i \dot{\beta}^j + h_{ab}(\varphi) \dot{\varphi}^a \dot{\varphi}^b] - \mathcal{N} V_\xi, \quad (2.9) \]

Here \( \mathcal{N} = \exp(\gamma - \gamma_0) > 0 \) is a modified lapse function, \( \gamma_0 = \sum_{i=1}^n d_i \beta^i \),

\[ G_{ij} = d_i \delta_{ij} - d_i d_j, \quad (2.10) \]

\( i, j = 1, \ldots, n \), are components of the gravitational part of the minisuperspace metric \([23]\) and

\[ V_\xi = \frac{w}{2} \sum_{i=1}^n \xi_i d_i e^{-2\beta^i + 2\gamma_0} \quad (2.11) \]

is the potential. Here and in what follows \( \dot{A} = \frac{dA}{du} \).

For the constant \( h_{ab}(\varphi) = h_{ab} \) the reduction to the sigma model was proved (for more general setup) in \([3]\). We note that here \( h_{ab}(\varphi) \) can be interpreted as a scalar part of the total target space metric.

When all \( M_i \) have finite volumes the substitution of \((2.5)\) and \((2.6)\) into the action \((2.1)\) gives us the following relation

\[ S = \mu \int L du \quad (2.12) \]

where \( L \) is defined in \((2.9)\), \( \mu = -\frac{w}{\kappa^2} \prod_{i=1}^n V_i \), and \( V_i = \int_{M_i} d^d y \left( \sqrt{\det(g_{m,n_i}^i)} \right) \) is the volume of \( M_i \), \( i = 1, \ldots, n \).

The relation \((2.12)\) can be derived using the following expression for the scalar curvature

\[ R = -w e^{-2\gamma} \left( 2\dot{\gamma}_0 - 2\dot{\gamma}_0 \dot{\gamma} + \dot{\gamma}_0^2 + \sum_{i=1}^n d_i \left( \dot{\beta}^i \right)^2 \right) + \sum_{i=1}^n e^{-2\beta^i} R[g^i], \quad (2.13) \]

where \( R[g^i] = \xi_i d_i \) is the scalar curvature corresponding to the \( M_i \)-manifold.

To obtain \((2.12)\) from \((2.1)\) (for compact \( M_i \)) one should extract the total derivative term in \((2.13)\) which is cancelled by the York-Gibbons-Hawking boundary term.
We write the Lagrange equations for (2.9) and then put $N = 1$, or equivalently $\gamma = \gamma_0$, i.e. when $u$ is a harmonic variable, as in [24]. We get

$$G_{ij} \ddot{\beta}^j + w \sum_{j=1}^n \xi_j d_j (-\delta_i^j + d_i) e^{-2\beta^j + 2\gamma_0} = 0,$$

(2.14)

$i = 1, \ldots, n$,

$$\frac{d(h_{ab}(\varphi) \dot{\varphi}^b)}{du} - \frac{1}{2} \frac{\partial h_{ab}(\varphi)}{\partial \varphi^a} \dot{\varphi}^c \dot{\varphi}^b = 0,$$

(2.15)

$a = 1, \ldots, l$, and

$$\frac{1}{2} G_{ij} \ddot{\beta}^i \ddot{\beta}^j + \frac{1}{2} h_{ab}(\varphi) \dot{\varphi}^a \dot{\varphi}^b + V_{\xi} = 0.$$

(2.16)

In fact, equations (2.14) are nothing else but Lagrange equations corresponding to the Lagrangian

$$L_\beta = \frac{1}{2} G_{ij} \dot{\beta}^i \dot{\beta}^j - V_{\xi}$$

(2.17)

with the energy integral of motion

$$E_\beta = \frac{1}{2} G_{ij} \dot{\beta}^i \dot{\beta}^j + V_{\xi}.$$

(2.18)

Likewise (2.14), the equations (2.15) are Lagrange equations corresponding to the Lagrangian

$$L_\varphi = \frac{1}{2} h_{ab}(\varphi) \dot{\varphi}^a \dot{\varphi}^b$$

(2.19)

with the energy integral of motion

$$E_\varphi = \frac{1}{2} h_{ab}(\varphi) \dot{\varphi}^a \dot{\varphi}^b.$$

(2.20)

Equations (2.15) are equivalent to geodesic equations corresponding to the metric $h$.

The relation (2.16) is the energy constraint

$$E = E_\beta + E_\varphi = 0,$$

(2.21)

coming from $\partial L/\partial \dot{N} = 0$ (for $N = 1$).

Equations (2.14) can be rewritten in a equivalent form

$$\ddot{\beta}^i - w \xi_i e^{-2\beta^i + 2\gamma_0} = 0,$$

(2.22)

$i = 1, \ldots, n$. These expressions may be obtained from (2.14) by using the inverse matrix $(G^{ij}) = (G_{ij})^{-1}$:

$$G^{ij} = \frac{\delta^{ij}}{d_i} + \frac{1}{2 - D}$$

(2.23)

and the following relations for $u^{(k)}$-vectors:

$$u^{(k)}_i = \frac{\delta^{k}_i}{d_i}.$$

(2.24)
In what follows we will use the following formulae

$$ (u^{(k)}, u^{(k)}) = G^{ij} u_i^{(k)} u_j^{(k)} = \frac{1}{d_k} - 1, \quad (2.25) $$

$k = 1, \ldots, n$.

Hence the problem of finding the cosmological type solutions for the model (2.1) (with $u$ being harmonic variable) is reduced to solving the equations of motion for the Lagrangians $L_\beta$ and $L_\varphi$ with the energy constraint (2.21) imposed.

**Geodesics for a flat metric $h$.**

For the constant $h_{ab}(\varphi) = h_{ab}$ eqs. (2.15) read

$$ \ddot{\varphi}^a = 0, \quad (2.26) $$

or, equivalently,

$$ \varphi^a = v_\varphi^a u + \varphi_0^a, \quad (2.27) $$

where $v_\varphi^a$ and $\varphi_0^a$ are integration constants, $a = 1, \ldots, l$.

The energy for scalar fields (2.20) takes the form

$$ E_\varphi = \frac{1}{2} h_{ab} v_\varphi^a v_\varphi^b. \quad (2.28) $$

More examples of geodesic solutions will be given in Section 4.

### 3 Cosmological type solutions

In this section we deal with certain examples of cosmological type solutions with the metric and “scalar fields” from (2.5) and (2.6), respectively.

#### 3.1 Solutions with $n$ Ricci-flat spaces

Here we focus on the solutions for the case when all factor spaces $M_i$ are Ricci-flat:

$$ \text{Ric}[g^i] = 0, \quad (3.1) $$

$i = 1, \ldots, n$.

Due to (3.1) the potential $V_\xi$ is equal to zero and the equations of motion (2.22) for $\beta^i$ now become

$$ \ddot{\beta}^i = 0, \quad (3.2) $$

$i = 1, \ldots, n$.

Integration of the equations (3.2) yields

$$ \beta^i = v^i u + \beta_0^i, \quad \gamma_0 = \sum_{i=1}^n d_i (v^i u + \beta_0^i), \quad (3.3) $$
where the parameters $\nu^i$ and $\beta^i_0$ are integration constants and the energy (2.18) takes the form

$$E_\beta = \frac{1}{2} G_{ij} \nu^i \nu^j, \quad (3.4)$$

where the minisuperspace metric $G_{ij}$ is given by (2.10).

The metric reads

$$g = w \exp [2 \sum_{i=1}^n d_i (\nu^i u + \beta^i_0)] du \otimes du + \sum_{i=1}^n \exp [2(\nu^i u + \beta^i_0)] g^i. \quad (3.5)$$

The “scalar fields” obey eqs. (2.15) with the energy constraint

$$E_\phi = \frac{1}{2} h_{ab}(\phi) \dot{\phi}^a \dot{\phi}^b = -\frac{1}{2} G_{ij} \nu^i \nu^j. \quad (3.6)$$

In a special case of one (non-fantom) scalar field ($h_{11} = 1$) and $w = -1$, this solution was obtained in [16]. For solutions with several scalar fields and constant $h_{ab}$ see [20, 21].

The scalar curvature for the metric (3.5) reads (see (2.13))

$$R[g] = -w \left( G_{ij} \nu^i \nu^j \right) e^{-2\gamma_0}. \quad (3.7)$$

In what follows we use a parameter

$$\Sigma = \Sigma(\nu) \equiv \sum_{i=1}^n d_i \nu^i \quad (3.8)$$

to classify the solutions.

Non-special Kasner-like solutions.

We shall first consider the non-special case when $\Sigma(\nu) \neq 0$.

Let us define a “synchronous” variable

$$\tau = \frac{1}{|\Sigma(\nu)|} \exp \left[ \sum_{j=1}^n (\nu^j u + \beta^j_0) d_j \right] \quad (3.9)$$

obeying $e^{2\gamma_0(\beta)} du^2 = d\tau^2$.

We introduce new parameters:

$$\alpha^i = \nu^i / \Sigma(\nu), \quad (3.10)$$

$i = 1, \ldots, n$, and

$$E_\varphi = E_\phi / (\Sigma(\nu))^2. \quad (3.11)$$

Then the metric reads

$$g = w d\tau \otimes \tau + \sum_{i=1}^n c_i^2 \tau^{2\alpha^i} g^i, \quad (3.12)$$
\[ \tau > 0. \quad "Scalar \ fields" \ are \ solutions \ to \ equations \ of \ motion \ (see \ (2.15)) \]
\[
\frac{d}{d\tau} \left[ \tau h_{ab}(\varphi) \frac{d\varphi^b}{d\tau} \right] - \frac{1}{2} \tau \partial_{\varphi^n} h_{ab}(\varphi) \frac{d\varphi^c}{d\tau} \frac{d\varphi^b}{d\tau} = 0, \quad (3.13)
\]

\(a = 1, \ldots, l\). The parameters (3.10) obey the Kasner-like conditions

\[
\sum_{i=1}^{n} d_i \alpha_i = 1, \quad (3.14)
\]
\[
\sum_{i=1}^{n} d_i (\alpha_i)^2 = 1 - 2E_{\varphi}, \quad (3.15)
\]

where

\[
2E_{\varphi} = \tau^2 h_{ab}(\varphi) \frac{d\varphi^a}{d\tau} \frac{d\varphi^b}{d\tau}, \quad (3.16)
\]

is the integral of motion for eqs. (3.13).

In (3.12) \(c_i\) are constants

\[
c_i = |\Sigma|^{d_i} \exp \left[ \beta_i^0 - \alpha_i \sum_{j=1}^{n} \beta_j d_j \right], \quad (3.17)
\]

\(i = 1, \ldots, n\), obeying \(\prod_{i=1}^{n} c_i^{d_i} = |\Sigma(v)|\).

**Flat** \(h\). For the special case of a flat target space metric \(h_{ab}(\varphi) = h_{ab}\) we get

\[
\varphi^a = \alpha^a_{\varphi} \ln \tau + \varphi_0^a, \quad (3.18)
\]

where \(\varphi_0^a\) are constants, \(a = 1, \ldots, l\), and

\[
E_{\varphi} = \frac{1}{2} h_{ab} \alpha^a_{\varphi} \alpha^b_{\varphi}. \quad (3.19)
\]

The scalar curvature (3.7) reads in this case

\[
R[g] = 2wE_{\varphi} \tau^{-2}. \quad (3.20)
\]

It diverges for \(\tau \to 0\) if \(E_{\varphi} \neq 0\). Hence all solutions with \(E_{\varphi} \neq 0\) are singular.

Let \(E_{\varphi} = 0\). For Milne-type sets of parameters, i.e. when \(d_i = 1\) and \(\alpha_i = 1\) for some \(i\) (\(\alpha_i = 0\) for all \(j \neq i\)) the metric is regular, when either (i) \(g^{(i)} = -wdy_i \otimes dy_i\), \(M_i = \mathbb{R} \) \((-\infty < y^i < +\infty)\), or (ii) \(g^{(i)} = wdy_i \otimes dy_i\), \(M_i\) is a circle of length \(L_i\) \((0 < y^i < L_i)\) and \(c_i L_i = 2\pi\) (i.e. when the cone singularity is absent).

For \(E_{\varphi} = 0\) the solutions with non-Milne-type sets of the Kasner parameters are singular (at least) if the Riemann tensor squared for metrics \(g^i\) obeys: \(R_{m,n,p,q} R^{m,n,p,q}[g^i] \geq C_i\), for some \(C_i\). (For \(d_i = 1, 2, 3\) \(R_{m,n,p,q}[g^i] = 0\), e.g. due to \(R_{m,n}[g^i] = 0\) for \(d_i = 2, 3\).) This is valid since the Riemann tensor squared for the metric \(g\) is divergent at \(\tau = 0\) in this case, see [26].
Special (steady state) solutions.
Now we consider the special case when \( \Sigma(v) = 0 \). Due to (3.6) we obtain

\[
E_{\varphi} = -\frac{1}{2} \sum_{i=1}^{n} d_i (v^i)^2 \leq 0. \tag{3.21}
\]

We get in this case \( \gamma_0 = \sum_{i=1}^{n} d_i \beta_0^i = \text{const} \) and hence the scalar curvature

\[
R[g] = 2wE_{\varphi}e^{-2\gamma_0} \tag{3.22}
\]

and the volume scale factor \( v = e^{\gamma_0} \) are constants.

The “synchronous” variable is proportional to \( u \) (\( \tau = e^{\gamma_0}u \)).
Hence, we obtained a restriction for the energy \( E_{\varphi} \leq 0 \). For \( E_{\varphi} = 0 \) all \( v^i = 0 \) and we are led to a static Ricci-flat solution.

This possibility occurs if the target space metric \( h \) is not positive-definite (e.g. there are phantom scalar fields for flat \( h \)). The solutions of such type with one (phantom) scalar field were considered in \( \text{[16]} \). (They are called as steady state solutions, see \( \text{[16]} \).)

Solutions with acceleration. Let \( d_1 = 3 \) and \( M_1 = \mathbb{R}^3 \). The factor space \( M_1 \) may be considered as describing our space. In both cases there exist subclasses of solutions describing accelerated expansion of our space.

Indeed, for Kasner-like solutions with \( w = -1 \) one could make a replacement \( \tau \mapsto \tau_0 - \tau \) where \( \tau_0 \) is a constant (corresponding to the so-called “big rip”). For such replacement the scale factor of \( M_1 \) reads

\[
a_1(\tau) = c_1(\tau_0 - \tau)^{\alpha_1}, \tag{3.23}
\]

For \( \alpha_1 < 0 \) we get an accelerated expansion of “our space” \( M_1 \).

Analogous consideration may be carried out for special (steady state) solutions. For \( w = -1, \ d_1 = 3 \) and \( v^1 > 0 \) we get an accelerated expansion of 3-dimensional factor space \( M_1 \).

3.2 Solutions with one curved Einstein space and \( n - 1 \) Ricci flat spaces

Here we put

\[
\text{Ric}[g^1] = \xi_1 g^1, \quad \xi_1 \neq 0, \quad \text{Ric}[g^i] = 0, \quad i > 1, \tag{3.24}
\]

i.e. the first space \( (M_1, g^1) \) is an Einstein space of non-zero scalar curvature and other spaces \( (M_i, g^i) \) are Ricci-flat.

The Lagrangian (2.17) reads in this case

\[
L_\beta = \frac{1}{2} G_{ij} \dot{\beta}^i \dot{\beta}^j - \frac{w}{2} \xi_1 d_1 \exp (-2\beta^1 + 2\gamma_0), \tag{3.25}
\]

where \(-\beta^1 + \gamma_0 = u^{(1)}_i \beta^i \) and \( u^{(1)}_i = -\delta^1_i + d_i \).
The Lagrange equations corresponding to the Lagrangian (3.25) are integrated in Appendix. The solution reads

\begin{align*}
\beta^1 &= \frac{1}{1 - d_1} \ln |f| + v^1 u + \beta^1_0, \\
\beta^i &= v^i u + \beta^i_0, \quad i > 1,
\end{align*}

where \( \beta^i_0, v^i \) are constants obeying

\begin{equation}
\begin{aligned}
v^1 &= \sum_{i=1}^n v^i d_i, \\
\beta^1_0 &= \sum_{i=1}^n \beta^i_0 d_i.
\end{aligned}
\end{equation}

The function \( f \) is following

\begin{equation}
f = \begin{cases} 
B \sinh(\sqrt{C} (u - u_0)), & C > 0, \quad w\xi_1 > 0; \\
|\xi_1 (d_1 - 1)|^{1/2} (u - u_0), & C = 0, \quad w\xi_1 > 0; \\
B \sin(\sqrt{-C} (u - u_0)), & C < 0, \quad w\xi_1 > 0; \\
B \cosh(\sqrt{C} (u - u_0)), & C > 0, \quad w\xi_1 < 0.
\end{cases}
\end{equation}

where \( u_0 \) and \( C \) are constants and

\begin{equation}
B = \sqrt{\frac{\xi_1 (d_1 - 1)}{|C|}}.
\end{equation}

For \( \gamma_0 \) we get

\begin{equation}
\gamma_0 = \beta^1 - \ln |f|.
\end{equation}

The energy integral of motion \( E_\beta \) corresponding to \( L_\beta \) reads (see Appendix)

\begin{equation}
E_\beta = \frac{Cd_1}{2(1 - d_1)} + \frac{1}{2} G_{ij} v^i v^j.
\end{equation}

Using (3.26), (3.27) and (3.31) we are led to the relation for the metric

\begin{equation}
g = |f|^{\frac{2d_1}{1 - d_1}} \exp [2(v^1 u + \beta^1_0)] \left( wdu \otimes du + f^2 g^1 \right) + \sum_{i=2}^n \exp [2(v^i u + \beta^i_0)] g^i.
\end{equation}

The “scalar fields” obey eqs. (2.15) with the energy constraint

\begin{equation}
E_\varphi = \frac{1}{2} h_{ab}(\varphi) \dot{\varphi}^a \dot{\varphi}^b = \frac{Cd_1}{2(d_1 - 1)} - \frac{1}{2} G_{ij} v^i v^j.
\end{equation}

Here the constraints (3.28) on \( \beta^i_0, v^i \) should be kept in mind and the function \( f \) is defined in (3.29).

Relations (3.28) are equivalent to the following ones
\[ v^1 = \frac{1}{1 - d_1} \sum_{i=2}^{n} v^i d_i, \quad \beta_0 = \frac{1}{1 - d_1} \sum_{i=2}^{n} \beta_0^i d_i. \]  \hspace{1cm} (3.35)

Using (3.28) and (3.35) we exclude \( v^1 \) and obtain

\[
G_{ij} v^i v^j = \sum_{i=1}^{n} d_i (v^i)^2 - \left( \sum_{i=1}^{n} d_i v^i \right)^2 = \frac{1}{d_1 - 1} \left( \sum_{i=1}^{n} d_i v^i \right)^2 + \sum_{i=2}^{n} d_i (v^i)^2 \geq 0 \]  \hspace{1cm} (3.36)

and hence the relation (3.34) can be rewritten as follows

\[
\frac{1}{d_1 - 1} \left( \sum_{i=2}^{n} d_i v^i \right)^2 + \sum_{i=2}^{n} d_i (v^i)^2 = \frac{Cd_1}{(d_1 - 1)} - 2E_\varphi \geq 0. \]  \hspace{1cm} (3.37)

In a special case of one (non-fantom) scalar field \( h_{11} = 1 \) and \( w = -1 \) this solution was obtained earlier in [17, 18], see also [19]. The solutions with several scalar fields and constant \( h_{ab} \) were presented in [20, 21]. We note that four-dimensional solutions with one scalar field (and electromagnetic one) were obtained earlier in [24].

### 4 Examples of geodesic solutions

In this section we consider three examples of solutions to geodesic equations corresponding to the metric \( h \) that may be used for the cosmological type solutions above.

#### 4.1 Two-dimensional sphere.

Let \( h \) be a metric on a two-dimensional sphere \( S^2 \)

\[
h = d\theta \otimes d\theta + \sin^2 \theta d\varphi \otimes d\varphi, \]  \hspace{1cm} (4.1)

where \( 0 < \varphi < 2\pi \) and \( 0 < \theta < \pi \). The simplest solution to geodesic equations (2.15) for the metric reads

\[
\varphi = \omega u, \quad \theta = 0, \]  \hspace{1cm} (4.2)

where \( \omega \) is constant. Here \( E_\varphi = \frac{1}{2} \omega^2 \). The general solution to geodesic equations may be obtained by a proper isometry \( SO(3) \) - transformation of the solution from (4.2).

Here and in what follows the first relation in (4.2) should be understood as modulo \( 2\pi \).

#### 4.2 Two-dimensional de Sitter space.

Now we put \( h \) to be a metric on a two-dimensional de Sitter space \( dS^2 \)

\[
h = -d\chi \otimes d\chi + \cosh^2 \chi d\varphi \otimes d\varphi, \]  \hspace{1cm} (4.3)

where \( 0 < \varphi < 2\pi \) and \( -\infty < \chi < +\infty \). (For a review on the de Sitter space see [25].)
There are three basic solutions to geodesic equations (2.15) in this case

\[ \begin{align*}
\varphi &= \omega u, & \chi &= 0, \\
\chi &= vu, & \varphi &= 0, \\
\tan \varphi &= \sinh \chi = mu,
\end{align*} \]

(4.4) \hspace{1cm} (4.5) \hspace{1cm} (4.6)

where \( \omega, v \) and \( m \) are constants. For the energy we have \( E_\varphi = \frac{1}{2} \omega^2, -\frac{1}{2} v^2 \) and 0, for space-like, time-like and null geodesics, respectively. The general solution to geodesic equations may be obtained by a proper isometry \( SO(1,2) \)-transformation of the solutions from (4.4) - (4.6).

### 4.3 The space with diagonal metric \( h \)

Here we consider a diagonal metric

\[ h = \varepsilon_0 d\varphi \otimes d\varphi + \sum_{k=1}^{l-1} \varepsilon_k A_k^2(\varphi) d\psi^k \otimes d\psi^k, \]

(4.7)

where \( \varepsilon_0 = \pm 1, \varepsilon_k = \pm 1 \) (\( k > 0 \)) and all \( A_k(\varphi) > 0 \) are smooth functions. The Lagrange function for the non-linear sigma model is given by

\[ L_\varphi = \frac{1}{2}[\varepsilon_0 \dot{\varphi}^2 + \sum_{k=1}^{l-1} \varepsilon_k A_k^2(\varphi)(\dot{\psi}^k)^2]. \]

(4.8)

Equations of motion for cyclic variables \( \psi^k \)

\[ \frac{d}{du} \left( \varepsilon_k A_k^2(\varphi) \dot{\psi}^k \right) = 0 \]

(4.9)

yield the following integrals of motion

\[ \varepsilon_k A_k^2(\varphi) \dot{\psi}^k = M_k, \]

(4.10)

\( k = 1, \ldots, l - 1. \)

Another constant of integration is the energy \( E_\varphi \)

\[ E_\varphi = \frac{1}{2}[\varepsilon_0 \dot{\varphi}^2 + \sum_{k=1}^{l-1} \varepsilon_k A_k^2(\varphi)(\dot{\psi}^k)^2] \]

(4.11)

which due to (4.10) reads

\[ E_\varphi = \frac{1}{2}[\varepsilon_0 \dot{\varphi}^2 + \sum_{k=1}^{l-1} \varepsilon_k M_k^2 A_k^{-2}(\varphi)]. \]

(4.12)

This relation implies the following quadrature
\[
\int_{\varphi_0}^{\varphi} \frac{d\tilde{\varphi}}{\sqrt{2 \varepsilon_0 E_{\varphi} - \sum_{k=1}^{l-1} \varepsilon_0 \varepsilon_k M_k^2 A_k^{-2}(\tilde{\varphi})}} = \pm (u - u_0), \quad (4.13)
\]

which implicitly defines the function \( \varphi = \varphi(u) \).

Another quadratures just following from (4.10)

\[
\psi^k - \psi^k_0 = \int_{u_0}^{u} d\bar{u}_\varepsilon M_k A_k^{-2}(\bar{u}), \quad (4.14)
\]

which complete the integration of the geodesic equations for the metric (4.7).

For \( A_k(\varphi) = \exp(\lambda \varphi) \), \( \lambda \neq 0 \), the metric (4.7) may describe either a part of de Sitter space (if \( \varepsilon_0 = -1 \), \( \varepsilon_k = 1 \), \( k > 0 \)) or a part of anti-de Sitter space (if \( \varepsilon_1 = -1 \) \( \varepsilon_r = 1 \), \( r \neq 1 \)). The case \( l = 3 \) is of interest in connection with the so-called AWE hypothesis \([28]\) (based on Damour-Gibbons-Gudlach approach \([29]\)) aimed to describe the dark sector of the Universe. In \([27]\) a four-dimensional de Sitter sigma model coupled to Einstein gravity (e.g. describing the current acceleration of the Universe) was studied.

## 5 Spherically symmetric solutions

In this section we study a subclass of spherically symmetric solutions with the metric (3.33) and “scalar fields” obeying (2.15) and (3.34). Here \( g^1 \) is a canonical metric on the sphere \( S^{d_1} \) and

\[
w = 1, \quad \xi_1 = d_1 - 1, \quad (5.1)
\]

where \( d_1 > 1 \).

Here we assume \( C > 0 \). Then the governing function looks like

\[
f = \frac{1}{\mu} \sinh (\bar{\mu} u) \quad (5.2)
\]

where

\[
\bar{\mu} = \sqrt{C}, \quad \mu = \frac{\bar{\mu}}{d_1 - 1}. \quad (5.3)
\]

For simplicity, we put \( u_0 = 0 \) and \( \beta_i^0 = 0 \) for all \( i \).

By introducing a new radial variable \( R = R(u) \)

\[
\exp [-2\bar{\mu} u] = 1 - \frac{2\mu}{R^{d_1 - 1}} = F(R) = F, \quad (5.4)
\]

with

\[
R > R_0 \equiv (2\mu)^{1/(d_1 - 1)}, \quad (5.5)
\]

the solution for the metric can be rewritten as follows
\[ g = F^{b-1}dR \otimes dR + R^2 F^b g^1 + \sum_{i=2}^{n} F^{a_i} g^i, \quad (5.6) \]

where

\[ b = \frac{1}{d_1 - 1} - \frac{v^1}{\mu}, \quad a_i = -\frac{v^i}{\mu}, \quad 2 \leq i \leq n. \quad (5.7) \]

Here we assign a metric for the time direction putting down \( M_2 = \mathbb{R} \) and \( g^2 = -dt \otimes dt \). Then the metric (5.6) reads

\[ g = -F^{a_2} dt \otimes dt + F^{b-1} dR \otimes dR + R^2 F^b g^1 + \sum_{i=3}^{n} F^{a_i} g^i, \quad (5.8) \]

where due to (3.35), (3.37) the constants \( b \) and \( a_i \) satisfy the relations (see (5.3) and (5.7))

\[ b = \frac{1}{d_1 - 1} \left( 1 - \sum_{i=2}^{n} d_i a_i \right) \quad (5.9) \]

and

\[ \frac{1}{d_1 - 1} \left( \sum_{i=2}^{n} d_i a_i \right)^2 + \sum_{i=2}^{n} d_i a_i^2 = \frac{d_1}{d_1 - 1} - e_\varphi. \quad (5.10) \]

Here we denote

\[ e_\varphi \equiv 2E_\varphi \frac{C}{C} = \frac{h_{ab}(\varphi) \varphi^a \varphi^b}{C} = \frac{1}{\mu^2} h_{ab}(\varphi) \frac{d\varphi^a}{dR} \frac{d\varphi^b}{dR} R^{d_1} F^2. \quad (5.11) \]

The “scalar fields” \( \varphi^a = \varphi^a(R) \) obey the equations of motions

\[ -\frac{d}{dR} \left( F(R) R^{d_1} h_{bc}(\varphi) \frac{d\varphi^c}{dR} \right) + \frac{1}{2} F(R) R^{d_1} h_{ab,c}(\varphi) \frac{d\varphi^a}{dR} \frac{d\varphi^b}{dR} = 0, \quad (5.12) \]

\( c = 1, \ldots, l \), which are equivalent to eqs. (2.15) (see (5.4)).

These equations are nothing more than Euler-Lagrange equations for the action

\[ S_\varphi = -\frac{1}{2} \int dR F(R) R^{d_1} h_{ab}(\varphi) \frac{d\varphi^a}{dR} \frac{d\varphi^b}{dR} = \frac{1}{2} \int du h_{ab}(\varphi) \frac{d\varphi^a}{du} \frac{d\varphi^b}{du}. \quad (5.13) \]

Thus we have obtained a family of spherically symmetric solutions to field equations with a sigma model source which are given by relations (5.8) - (5.11). These solutions are defined up to solutions to “scalar fields equations” (5.12) which are in fact the geodesic equations for the metric \( h \) (2.15) rewritten in the \( R \)-variable.

**Example: the case of constant \( h_{ab} \).** Let us consider a special case when \( h_{ab}(\varphi) = h_{ab} \) are constants. It follows from (2.27) and (5.4) that
\[ \varphi^a = \frac{1}{2} q^a \ln F(R) + \varphi_0^a, \]  
(5.14)

where \( q^a = -\nu^a / \bar{\mu} \) and \( \varphi_0^a \) are integration constants, \( a = 1, \ldots, l \). For the energy parameter we get from (2.28)

\[ e_{\varphi} = h_{ab} q^a q^b. \]  
(5.15)

Due to Hilbert-Einstein field equations (2.2) (with \( T_{MN} \) from (2.3)) we get the following relation for the scalar curvature

\[ R[g] = g^{MN} R_{MN} = g^{MN} T_{MN} / (1 - D/2) = g^{MN} h_{ab}(\varphi) \partial_M \varphi^a \partial_N \varphi^b. \]  
(5.16)

which implies (for the solution under consideration)

\[ R[g] = F^{1-b} h_{ab}(\varphi) d\varphi^a d\varphi^b / dR = e_{\varphi} \bar{\mu}^2 R^{1-b} F^{-1-b}. \]  
(5.17)

Due to (5.16) the Hilbert-Einstein equations (2.2) can be rewritten in an equivalent form

\[ R_{MN} = h_{ab}(\varphi) \partial_M \varphi^a \partial_N \varphi^b. \]  
(5.18)

In what follows we denote

\[ b_0 = \sum_{i=2}^n d_i a_i. \]  
(5.19)

We remind that \( e_{\varphi} \) is defined by (5.11) and obeys the inequality following from (5.10)

\[ e_{\varphi} \leq \frac{d_1}{d_1 - 1}. \]  
(5.20)

**Proposition 1.** For all \( a_i \) and \( e_{\varphi} \) obeying (5.10)

\[ |b_0(a)| \leq \sqrt{(d_1 - 1)(D - d_1 - 1)} \sqrt{\frac{d_1}{d_1 - 1} - e_{\varphi}}. \]  
(5.21)

Maximum and minimum of the function \( b_0(a) \) are attained at the points \( a_+ \) and \( a_- \), respectively, where \( a_\pm = (a_\pm, i) \)

\[ a_{\pm, i} = \pm \sqrt{\frac{(d_1 - 1)}{(D - 2)(D - d_1 - 1)}} \sqrt{\frac{d_1}{d_1 - 1} - e_{\varphi}}, \]  
(5.22)

for \( i = 2, \ldots, n \).

**Proof.** For \( e_{\varphi} = \frac{d_1}{d_1 - 1} \) the proposition is trivial. Let us consider the case \( A = \frac{d_1}{d_1 - 1} - e_{\varphi} > 0 \). We prove the proposition by the method of Lagrange multipliers. We introduce a new variable \( \lambda \) called a Lagrange multiplier, and study the Lagrange function defined by

\[ \bar{b}_0(a, \lambda) = b_0(a) - \lambda(Q(a) - A), \]  
(5.23)

where
\[ Q(a) = \frac{1}{d_1 - 1} \left( \sum_{i=2}^{n} d_i a_i \right)^2 + \sum_{i=2}^{n} d_i a_i^2. \]  

(5.24)

For the points of extremum we get

\[ \frac{\partial b_0}{\partial \lambda} = 0 \iff Q = A, \]  

(5.25)

\[ \frac{\partial b_0}{\partial a_i} = 0 \iff 1 - 2\lambda \left[ \frac{1}{d_1 - 1} \sum_{j=2}^{n} d_j a_j + a_i \right] = 0, \]  

(5.26)

\[ i = 2, \ldots, n \]

It follows from (5.26) that all \( a_i \) are coinciding

\[ a_i = \frac{(d_1 - 1)}{D - 2} \frac{1}{2\lambda} \]  

(5.27)

for \( i = 2, \ldots, n \). The substitution of (5.27) into relation (5.25) gives us two points of extremum of the function \( b_0(a) \) on the \((n - 2)\)-dimensional ellipsoid \( Q(a) = A \) which are given by relations (5.22). The point \( a_+ \) is the point of maximum and \( a_- \) is the point minimum. We get the inequality \( b_0(a_-) \leq b_0(a) \leq b_0(a_+) \) coinciding with (5.21). The proposition is proved.

**Proposition 2.** Let \( e_\varphi \neq 0 \) and

\[ -\frac{d_1(D - 1)}{D - d_1 - 1} < e_\varphi \leq \frac{d_1}{d_1 - 1}. \]  

(5.28)

Then the scalar curvature \( R[g] \) for the metric (5.6) with parameters \( b, a_i \) obeying (5.9) and (5.10) is divergent at \( R = R_0 + 0 \), i.e.

\[ R[g] \rightarrow +\infty \quad \text{for} \quad R \rightarrow R_0. \]  

(5.29)

**Proof.** It follows from the Proposition 2 and the first inequality for the energy parameter \( e_\varphi \) in (5.28) that

\[ |b_0(a)| < d_1, \]  

(5.30)

and hence \((d_1 > 1)\)

\[ 1 - b = \frac{1}{d_1 - 1} (-d_1 + b_0) < 0. \]  

(5.31)

Using the relation for the scalar curvature (5.17), the inequality (5.31) and \( e_\varphi \neq 0 \) we get the divergence of \( R[g] \) for \( R \rightarrow R_0 \). This completes the proof of the proposition.

Now we consider the special case \( e_\varphi = 0 \). In this case the stress-energy tensor for \( \varphi \) vanishes and the metric \( g \) is Ricci-flat, i.e. \( R_{MN}[g] = 0 \). Thus we are led to a vacuum solution from [34] which was considered in [26]. Recently the solution from [34] was intensively studied in [35].

In what follows we are interested in the case of non-zero gravitational mass, i.e. we put \( a_2 \neq 0 \).
Proposition 3. Let \( g \) be a metric (5.3) with the parameters \( b, a_i \) obeying (5.9), (5.10), \( e_\varphi = 0 \) and \( a_2 \neq 0 \). Let the Riemann tensor squared for any metric \( g^i, i = 3, ..., n \), obey a self-boundededness condition

\[
I_i = R_{mnpq} R^{mnpq}[g^i] \geq C_i,
\]

for some \( C_i \). Then for all sets \( a = (a_2, ..., a_n) \neq (1, 0, ..., 0) \) the Riemann tensor squared for the metric (5.3) is divergent at \( R = R_0 + 0 \), i.e.

\[
R_{MNPQ} R^{MNPQ}[g] \rightarrow +\infty \quad \text{for} \quad R \rightarrow R_0.
\]

The Proposition 3 is a special case of the Proposition 6 from [26]. The Propositions 2 and 3 may be summarized as the following proposition. We note that the condition (5.32) is satisfied for any metric \( g^i \) of Euclidean signature since \( I_i \geq 0 \) in this case. It is valid also for any \( g^i \) of dimension \( d_i = 1 \) and for any Ricci-flat \( g^i \) of dimension \( d_i = 2, 3 \) since in all these cases the Riemann tensor of \( g^i \) is zero and hence \( I_i = 0 \).

Proposition 4. Let \( g \) be a metric (5.3) with the parameters \( b, a_i \) obeying (5.9), (5.10), \( a_2 \neq 0 \) and (5.28). Let the Riemann tensor squared for any metric \( g^i, i = 3, ..., n \), obey a self-boundededness condition (5.32). Then the regular horizon of the metric (5.3) at \( R = R_0 \) takes place if and only if

\[
a_2 - 1 = a_3 = ... = a_n = e_\varphi = 0.
\]

The Proposition 4 may be considered as a restricted version of the “no-hair” theorem for the metric (5.3). For the case of the positive-definite sigma model metric \( h_{ab}(\varphi) \) (i.e. when \( e_\varphi \geq 0 \)) it was proved for more general assumptions in [36] (see Theorem 5 therein).

For the set of parameters from (5.34) we get

\[
g = g_T + \sum_{i=3}^{n} g^i,
\]

where \( g_T = -F dt \otimes dt + F^{-1} dR \otimes dR + R^2 g^1 \) is the metric of the \((2 + d_1)\)-dimensional Tangherlini black hole solution [30].

Remark. Let all \( h_{ab} \) be constant. Then the relation \( e_\varphi = 0 \) reads as \( h_{ab} q^a q^b = 0 \), where the scalar charges \( q^a \) appeared previously for the scalar field solutions (5.14). For the positive- or negative-definite matrix \( h_{ab} \) we get \( q^a = 0 \) for all \( a \). If the matrix \( (h_{ab}) \) is a semi-definite one, i.e. it has a signature \((-,...,-,+,...,+)\) then there exist solutions \( \varphi^a \) (5.14) with two or more non-zero \( q^a \). However these solutions are singular on a horizon. The only regular solutions are trivial ones \( \varphi^a = \varphi^a_0 \). So, we do not obtain non-trivial “scalar hairs” in the case of constant \( h_{ab} \). Analogous situations takes place for the \( dS^2 \) sigma model solution with \( e_\varphi = 0 \) (see (4.6)): \( \varphi = \arctan(\frac{1}{2} q \ln F(R)), \chi = \arcsinh(\frac{1}{2} q \ln F(R)) \) with \( q = -m/\bar{\mu} \neq 0 \). These solutions are also singular at the horizon.

The PPN parameters for \( d_1 = 2 \). Let us consider the case \( d_1 = 2 \). The pure gravitational solution (without scalar fields) was obtained in cite [31] and generalized to the scalar-vacuum case (with one scalar field) in [32], see also [33]. The calculations of PPN parameters for a 4-dimensional part of the metric (5.8) with \( a_2 \neq 0 \).
\[ g^{(4)} = -F^{a_2} dt \otimes dt + F^{b_1} dR \otimes dR + R^2 F^{b_1} g^{1}, \]  
\[ (5.36) \]  
and \( F = 1 - \frac{2\mu}{R} \) give us the following relations \[37, 38\] (see also \[39\])
\[
\beta = 1, \quad \gamma = 1 + \frac{1}{a_2} \sum_{i=3}^{n} d_i a_i. 
\]
\[ (5.37) \]
For
\[
\sum_{i=3}^{n} d_i a_i = 0 \quad (5.38)
\]
and \( a_2 \neq 0 \) we are led to a subclass of solutions with \( \beta = \gamma = 1 \) for the 4-dimensional metric \( (5.36) \). The solution of such type were called in \[39\] (for pure gravity) as latent solitons \[39\]. The four-dimensional section of the latent soliton metric gives the same PPN parameters \( \beta = 1 \) and \( \gamma = 1 \) as the Schwarzschildian metric does, i.e. gravitational experiments lead to the same results for \( g^{(4)} \) and for the metric of the Schwarzschild solution.

6 Conclusions

In this paper we have considered a \( D \)-dimensional model of gravity with non-linear “scalar fields” governed by a sigma model action (as a matter source). The model is defined on the product manifold \( M \), which contains \( n \) Einstein factor spaces \( M_1, \ldots M_n \).

We have obtained general cosmological type solutions corresponding to the field equations in two cases: when either all factor spaces are Ricci-flat or when only one factor space, \( M_1 \), has nonzero scalar curvature. The solutions are defined up to solutions of geodesic equations corresponding to the sigma model target space. We have considered several examples of sigma models, e.g. with \( S^2 \) and \( dS^2 \) target spaces. It is shown that for certain parameters cosmological solutions may describe an accelerated expansion of 3-dimensional factor space.

Here we have also studied a subclass of spherically symmetric solutions with sinh-behaviour of the governing function \( f \) from \( (3.29) \). We have proved a restricted version of the “no-hair theorem” (Proposition 4) when the scalar energy parameter \( e_\phi \) obeys restriction \( (5.28) \) and all factor space metrics \( g^i, i = 3, \ldots , n \), have Euclidean signatures. We have found for \( d_1 = 2 \) a subclass of latent solitonic solutions generalizing those from ref. \[39\].

An open problem here is to generalize the “non-hair” theorem for the case of all types of the governing function \( f \) and all values of the energy parameter \( e_\phi \) as it was done recently for \( D = 4 \) case (with one scalar field) in \[40\].

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Appendix

A Solutions governed by Liouville equation

Here we consider a Toda-like system with the following Lagrangian

$$L = \frac{1}{2} \langle \dot{\beta}, \dot{\beta} \rangle - A \exp (2 \langle b, \beta \rangle), \quad (A.1)$$

where $\beta \in \mathbb{R}^n$, $A \neq 0$, $b \in \mathbb{R}^n$. Here (in Appendix) $\dot{\beta} = \frac{d\beta}{dt}$. The scalar product for vectors belonging to $\mathbb{R}^n$ is defined by

$$\langle v_1, v_2 \rangle = G_{ij} v_1^i v_2^j, \quad (A.2)$$

where $(G_{ij})$ is a non-degenerate symmetric matrix (e.g. given by (2.10)).

The Lagrange equations corresponding to the model (A.1) read (in a condensed vector form)

$$\ddot{\beta} + 2Ab \exp (2 \langle b, \beta \rangle) = 0. \quad (A.3)$$

Let $\langle b, b \rangle \neq 0$.

Eqs. (A.3) is exactly integrable and the solution may be written

$$\beta = \frac{b}{\langle b, b \rangle} q + vt + \beta_0, \quad (A.4)$$

where $\langle b, b \rangle \neq 0$ and $v, \beta_0 \in \mathbb{R}^n$ are constant vectors obeying

$$\langle v, b \rangle = \langle \beta_0, b \rangle = 0. \quad (A.5)$$

The function $q = q(t)$ obeys the Liouville equation

$$\ddot{q} + 2A \langle b, b \rangle e^{2q} = 0. \quad (A.6)$$

The solution to Liouville equation reads

$$q = -\ln|f|, \quad (A.7)$$

where

$$f = \begin{cases} B \sinh(\sqrt{C}(t - t_0)), & C > 0, \quad \bar{A} < 0 \quad ; \\ |2\bar{A}|^{1/2}(t - t_0), & C = 0, \quad \bar{A} < 0 \quad ; \\ B \sin(\sqrt{-C}(t - t_0)), & C < 0, \quad \bar{A} < 0 \quad ; \\ |B| \cosh(\sqrt{C}(t - t_0)), & C > 0, \quad \bar{A} > 0 \quad . \end{cases} \quad (A.8)$$

here we put $\bar{A} = A \langle b, b \rangle$ and

$$B = \sqrt{\frac{2|A\langle b, b \rangle|}{|C|}}. \quad (A.9)$$
The energy function corresponding to the Lagrangian (A.1) reads
\[ \mathcal{E} = \frac{1}{2} \langle \dot{\beta}, \dot{\beta} \rangle + A e^{2 \langle b, \beta \rangle}. \]  
(A.10)

After substitution of (A.4) to (A.10) we obtain
\[ \mathcal{E} = \mathcal{E}_T + \frac{1}{2} \langle v, v \rangle, \]  
(A.11)
where
\[ \mathcal{E}_T = \frac{1}{2} \langle b, b \rangle \dot{q}^2 + A e^{2q}. \]  
(A.12)

Due to (A.7) and (A.8) we get
\[ \mathcal{E}_T = C \frac{1}{2 \langle b, b \rangle} \]  
(A.13)

**Proposition 1A.** For \( \langle b, b \rangle \neq 0 \) all solutions to Lagrange equations (A.3) are covered by the relations (A.4), (A.5), (A.7) and (A.8).

**Proof.**

\( \Rightarrow \). It is obvious that the solutions (A.4), (A.5) with \( q \) from (A.7), (A.8) obey the equations of motion (A.3).

\( \Leftarrow \). Let us show that the relations (A.4), (A.5), (A.7) and (A.8) followed from (A.3).

Let \( q = \langle b, \beta \rangle \), \( y = \beta - \langle bq \rangle / \langle b, b \rangle \). It is obvious that \( \langle b, y \rangle = 0 \). It follows from (A.3) that the equation (A.6) is satisfied identically and
\[ \ddot{y} = 0 \quad \Rightarrow \quad y = vt + \beta_0, \]  
(A.14)
where constant vectors \( v \) and \( \beta_0 \) obey (due to \( \langle b, y \rangle = 0 \))
\[ \langle b, v \rangle = \langle b, \beta_0 \rangle = 0. \]  
(A.15)

Hence
\[ \beta = \frac{bq}{\langle b, b \rangle} + y = \frac{bq}{\langle b, b \rangle} + vt + \beta_0, \]  
(A.16)
where \( q = q(t) \) obeys (A.6) and hence it is given by relations (A.7) and (A.8).

The Proposition 1A is proved.

Let us introduce a dual vector \( u = (u_i) : u_i = G^{ij} b^j \). Then we get \( u(\beta) = u_i \beta^i = \langle b, \beta \rangle \), \( (u, u) = G^{ij} u_i u_j = \langle b, b \rangle \), where \( (G^{ij}) = (G_{ij})^{-1} \), and the solution (A.4) reads
\[ \beta^i = -\frac{u_i}{(u, u)} \ln |f| + v^i t + \beta_0^i, \]  
(A.17)
\( i = 1, \ldots, n \), where \( (u, u) \neq 0 \),
\[ u(v) = u_i v^i = 0, \quad u(\beta_0) = u_i \beta_0^i = 0, \]  
(A.18)

and the function \( f \) is defined in (A.8) with

\[ B = \sqrt{\frac{2|A(u,u)|}{|C|}}, \quad \bar{A} = A(u,u). \]  
(A.19)

For the energy (A.12) we obtain from (A.11) and (A.13)

\[ E = \frac{C}{2(u,u)} + \frac{1}{2} G_{ij} v^i v^j. \]  
(A.20)

**Example.** Let us consider a Lagrange system from Section 3 with parameters: \( A = \frac{w}{2} \xi_1 d_1, \quad u_i = u^{(1)}_i = -\delta^1_i + d_i, \quad d_1 > 1. \) Then due to (2.24) and (2.25) we get \( u^i = -\delta^1_i d_1 \) and \( (u,u) = \frac{1}{d_1} - 1 < 0. \) The solution reads

\[ \beta^i = \frac{\delta^i_1}{1 - d_1} \ln |f| + v^i t + \beta_0^i, \]  
(A.21)

\( i = 1, \ldots, n, \) with constraints

\[ v^1 = \sum_{i=1}^n v^i d_i, \quad \beta_0^1 = \sum_{i=1}^n \beta_0^i d_i \]  
(A.22)

imposed. In (A.8) we should put \( \bar{A} = \frac{w}{2} \xi_1 (1 - d_1) \) and \( B = \sqrt{|\xi_1 (d_1 - 1)|}. \) For the energy we get

\[ E = \frac{Cd_1}{2(1 - d_1)} + \frac{1}{2} G_{ij} v^i v^j. \]  
(A.23)

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