Meromorphic Modular Forms with Rational Cycle Integrals

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We study rationality properties of geodesic cycle integrals of meromorphic modular forms associated to positive definite binary quadratic forms. In particular, we obtain finite rational formulas for the cycle integrals of suitable linear combinations of these meromorphic modular forms.

1 Introduction

One of the fundamental results in the classical theory of modular forms is the fact that the vector spaces of modular forms are spanned by forms with rational Fourier coefficients. Besides that, there are other natural rational structures on these spaces, for example, coming from the rationality of periods or cycle integrals of modular forms. This was first shown by Kohnen and Zagier in [21], where they proved the rationality of the even periods of the cusp forms

\[
f_{k,D}(z) := \frac{|D|^{k-\frac{1}{2}}}{\pi} \sum_{Q \in \mathbb{Q}_D} Q(z,1)^{-k} \tag{1.1}
\]
of weight $2k$ for $\Gamma(1) = \text{SL}_2(\mathbb{Z})$, for $k \geq 2$ and all discriminants $D > 0$. Here, the sum runs over the set $Q_D$ of all integral binary quadratic forms of discriminant $D$. These cusp forms were introduced by Zagier while investigating the Doi–Naganuma lift in [27], and they played a prominent role in the explicit description of the Shimura–Shintani correspondence in [20]. The aforementioned rationality result of Kohnen and Zagier was generalized to Fuchsian groups of the 1st kind by Katok [19]. Periods and cycle integrals of other types of modular forms, such as weakly holomorphic modular forms, harmonic Maass forms, or meromorphic modular forms, have been the object of active research over the past years, see for example [3, 4, 10, 11, 14].

If we allow negative discriminants $D < 0$ in (1.1) and restrict the summation to positive definite forms $Q \in Q_D$, then we obtain meromorphic modular forms $f_{k,D}$ of weight $2k$ for $\Gamma(1)$ with poles of order $k$ at the CM points of discriminant $D$. These forms recently attracted some attention, starting with the work of Bengoechea [2] on the rationality properties of their Fourier coefficients. Their regularized inner products and connections to locally harmonic Maass forms were investigated by Bringmann et al. [9] and the 1st author [22]. Furthermore, Zemel [29] used them to prove a higher-dimensional analogue of the Gross–Kohnen–Zagier theorem [17], which hints at a deeper geometric meaning of the meromorphic $f_{k,D}$. Recently, Alfes-Neumann et al. in [1] established modularity properties of the generating series of traces of cycle integrals of $f_{k,D}$ and used this to show the rationality of suitable linear combinations of these traces.

It is natural to ask whether the individual cycle integrals of the meromorphic modular forms $f_{k,D}$ for $D < 0$ have nice rationality properties too. For an indefinite integral binary quadratic form $A = [a, b, c]$ of non-square discriminant the cycle integral of $f_{k,D}$ along the closed geodesic corresponding to $A$ is defined by

$$C(f_{k,D}, A) := \int_{\Gamma(1)_A \backslash S_A} f_{k,D}(z)A(z, 1)^{k-1}dz,$$

where

$$S_A := \{z \in \mathbb{H} : a|z|^2 + b\Re(z) + c = 0\}$$

is a semi-circle centered at the real line and $\Gamma(1)_A$ denotes the stabilizer of $A$ in $\Gamma(1)$. Note that, due to the modularity of $f_{k,D}$, the cycle integral depends only on the $\Gamma(1)$-equivalence class of $A$. If $f_{k,D}$ has a pole on $S_A$, the cycle integral can be defined as a Cauchy principal value, see Section 3.5. Numerical integration yields the following
approximations for $k \in \{2, 4, 6\}$ and $D = -3$.

\[
\begin{array}{|c|c|c|c|c|c|c|}
\hline
A & [1, 1, -1] & [1, 0, -2] & [1, 1, -3] & [1, 1, -4] & [1, 1, -5] & [1, 0, -6] \\
\hline
C(f_{2,-3}, A) & 4 & 8 & 12 & 28 & 10 & 16 \\
C(f_{4,-3}, A) & 20 & 48 & 92 & 452 & 170 & 288 \\
C(f_{6,-3}, A) & 142.36448 & 411.27103 & 1049.99067 & 12351.27103 & 5635.65417 & 8944.31786 \\
\hline
\end{array}
\]

It seems that the cycle integrals of $f_{2,-3}$ and $f_{4,-3}$ are integers, but there is little reason to believe that the cycle integrals of $f_{6,-3}$ are rational numbers.

The main aim of the present work is to investigate the rationality of the cycle integrals of $f_{k,D}$ for $D < 0$. As we will see, the failure of rationality of these cycle integrals is due to the existence of cusp forms of weight $2k$. In particular, we have to take certain linear combinations of cycle integrals of a fixed $f_{k,D}$ or a fixed cycle integral of linear combinations of forms $f_{k,D}$ to obtain convenient rationality results. We also treat forms of higher level $\Gamma_0(N)$, as well as the case $k = 1$. We remark that our results generalize the rationality results of [1] in several aspects, using a very different proof.

2 Statement of Results

Let $N$ and $k$ be positive integers and let $\Gamma = \Gamma_0(N)$. For any $D \in \mathbb{Z}$ the group $\Gamma$ acts on the set $Q_D$ of (positive definite if $D < 0$) integral binary quadratic forms $Q = [a, b, c]$ of discriminant $D = b^2 - 4ac$ with $N \mid a$, with finitely many orbits if $D \neq 0$. We write $[Q_0]$ for the $\Gamma$-class of $Q_0 \in Q_D$. For $D \neq 0$ and $k \geq 2$ we define the associated function

\[
f_{k,Q_0}(z) := \frac{|D|^{k-\frac{1}{2}}}{\pi} \sum_{Q \in [Q_0]} Q(z, 1)^{-k}
\]

on $\mathbb{H}$. For $k = 1$ the function $f_{1,Q_0}(z)$ is defined using Hecke’s trick, see Section 3.3. Throughout, we let $A \in Q_D$ denote an indefinite quadratic form of non-square discriminant $D > 0$, and $P \in Q_d$ a positive definite quadratic form of discriminant $d < 0$. Then, $f_{k,A}$ is a cusp form of weight $2k$ for $\Gamma$, and $f_{k,P}$ is a meromorphic modular form of weight $2k$ for $\Gamma$ that has poles of order $k$ at the CM points $\tau_Q \in \mathbb{H}$ (defined by $Q(\tau_Q, 1) = 0$) for $Q \in [P]$.

Our explicit formulas for the cycle integrals of $f_{k,P}$ will be given in terms of the following function. For $k \geq 2$, an indefinite quadratic form $A \in Q_D$ of non-square discriminant $D > 0$, and $\tau \in \mathbb{H}$ not lying on any of the semi-circles $S_Q$ for $Q \in [A]$, we
define the function

\[ P_{k;A}(\tau) := D^{k-\frac{1}{2}} \frac{(-1)^k \xi_{\Gamma;A}(k) + \xi_{\Gamma,-A}(k)}{2^{k-2}(2k-1) \Im(\tau)^{k-1}} + 2 \left(-i\sqrt{D}\right)^{k-1} \sum_{Q=[a,b,c] \subset [A]} \sgn(a)P_{k-1} \left(\frac{i(a|\tau|^2 + b \Re(\tau) + c)}{\Im(\tau)\sqrt{D}}\right), \]

(2.1)

where the zeta function \( \xi_{\Gamma;A}(s) \) is defined in (3.4), \( P_{k-1} \) denotes the usual Legendre polynomial, and \( \text{Int}(S_Q) \) denotes the bounded component of \( \mathbb{H} \setminus S_Q \). For \( k = 1 \) the function \( P_{1;A}(\tau) \) is defined analogously, but the 1st line has to be omitted. If \( \tau \in \mathbb{H} \) does lie on one of the semi-circles \( S_Q \) for \( Q \in [A] \), we define the value of \( P_{k;A} \) at \( \tau \) by the average value

\[ P_{k;A}(\tau) := \lim_{\varepsilon \to 0} \frac{1}{2} (P_{k;A}(\tau + i\varepsilon) + P_{k;A}(\tau - i\varepsilon)). \]

Note that the sum in the 2nd line of (2.1) is finite and \( P_{k;A}(\tau) \) has discontinuities along the semi-circles \( S_Q \) for \( Q \in [A] \). From the properties of \( \xi_{\Gamma;A}(s) \) given in Section 3.4 it easily follows that the special values

\[ |d|^{k-\frac{1}{2}} P_{k;A}(\tau_P) \]

at CM points \( \tau_P \in \mathbb{H} \) associated to positive definite forms \( P \in Q_d \) are rational numbers.

Our 1st rationality result concerns linear combinations of cycle integrals of a fixed \( f_{k,p} \).

**Theorem 2.1.** Let \( Q \) be a finite family of indefinite quadratic forms of non-square positive discriminants and \( a_A \in \mathbb{Z} \) for \( A \in Q \) such that \( \sum_{A \in Q} a_A f_{k,A} = 0 \) in \( S_{2k}(\Gamma) \). Furthermore, let \( P \in Q_d \) be a positive definite quadratic form of discriminant \( d \). Then, we have the formula

\[ \sum_{A \in Q} a_A C(f_{k,p}, A) = \frac{|d|^{k-\frac{1}{2}}}{|P|} \sum_{A \in Q} a_A P_{k;A}(\tau_P), \]

where \( \Gamma_P \) is the stabilizer of \( P \) in \( \Gamma/[\pm 1] \). In particular, this linear combination of cycle integrals is a rational number whose denominator is bounded only in \( k \) and \( N \).

We would like to emphasize that the formula on the right-hand side can be evaluated exactly, giving the precise rational value of the linear combination of cycle integrals on the left-hand side.
The proof of Theorem 2.1 uses the fact that the cycle integral $C(f_{k,P}, A)$ equals the special value at the CM point $\tau_P$ of (the iterated derivative of) a so-called locally harmonic Maass form $F_{1-k,A}(\tau)$, see Corollary 4.3. This function was introduced by Bringmann et al. in [7]. They showed that $F_{1-k,A}$ can be decomposed into a sum of a certain local polynomial (whose iterated derivative is $P_{k,A}$) and holomorphic and non-holomorphic Eichler integrals of the cusp form $f_{k,A}$. Taking suitable linear combinations as in the theorem, one can achieve that the Eichler integrals cancel out, which yields the formula in Theorem 2.1. We refer to Section 5 for the details of the proof.

**Example 2.2.** Let $N = 1$. Since there are no nontrivial cusp forms of weight less than 12 or weight 14 for $\Gamma(1)$, the functions $f_{k,A}$ for $k \leq 5$ and $k = 7$ vanish identically for every indefinite quadratic form $A$. Thus, it follows from Theorem 2.1 that the cycle integrals $C(f_{k,P}, A)$ are rational for $k \leq 5$ and $k = 7$ for every choice of $P$ and $A$. This explains the rationality of the cycle integrals of $f_{2,-3}$ and $f_{4,-3}$ that we observed in the introduction. In contrast, we have seen in the introduction that the cycle integrals $C(f_{6,-3}, A)$ do not seem to be rational. This corresponds to the fact that $f_{6,A}$ is a cusp form of weight 12 that usually does not vanish identically. However, using results of [28], one can prove the relations

$$
2f_{6,[1,1,-1]} + f_{6,[1,0,-2]} = 11f_{6,[1,1,-1]} + f_{6,[1,1,-3]} = f_{6,[1,0,-1]} - f_{6,[1,1,-4]} = 0.
$$

Now, Theorem 2.1 asserts that for any positive definite quadratic form $P$, the corresponding linear combinations

$$
2C(f_{6,P}, [1, 1, -1]) + C(f_{6,P}, [1, 0, -2]),
$$

$$
11C(f_{6,P}, [1, 1, -1]) + C(f_{6,P}, [1, 1, -3]),
$$

$$
C(f_{6,P}, [1, 0, -2]) - C(f_{6,P}, [1, 1, -4]),
$$

of cycle integrals of $f_{6,P}$ are rational numbers. For example, for $f_{6,[1,1,1]} = f_{6,-3}$ we have

$$
2C(f_{6,-3}, [1, 1, -1]) + C(f_{6,-3}, [1, 0, -2]) = 696,
$$

$$
11C(f_{6,-3}, [1, 1, -1]) + C(f_{6,-3}, [1, 1, -3]) = 2616,
$$

$$
C(f_{6,-3}, [1, 0, -2]) - C(f_{6,-3}, [1, 1, -4]) = -11940.
$$
Next, we consider cycle integrals of certain linear combinations of forms $f_{k,P}$ over a single geodesic. Following [16], we call a sequence $\lambda = (\lambda_m)_{m=1}^{\infty} \subset \mathbb{Z}$ of integers a relation for $S_{2k}(\Gamma)$ if

1. $\lambda_m = 0$ for almost all $m$,
2. $\sum_{m=1}^{\infty} \lambda_m c_f(m) = 0$ for every cusp form $f(z) = \sum_{m=1}^{\infty} c_f(m) q^m \in S_{2k}(\Gamma)$, and
3. $\lambda_m = 0$ whenever $(m,N) > 0$.

For a meromorphic function $f$ on $\mathbb{H}$ that transforms like a modular form of weight $2k$ for $\Gamma$ we define its Hecke translate corresponding to a relation $\lambda$ by

$$f|T_\lambda := \sum_{m=1}^{\infty} \lambda_m f|T_m,$$

where $T_m$ denotes the usual $m$-th Hecke operator of level $N$, see (6.1). Bengoechea [2] showed that the Fourier coefficients of $f_{k,P}|T_\lambda$ are algebraic multiples of $\pi^{k-1}$ for every positive definite quadratic form $P$ and every relation $\lambda$ for $S_{2k}(\Gamma)$. We obtain a rationality result for the cycle integrals of $f_{k,P}|T_\lambda$.

**Theorem 2.3.** Let $A \in \mathcal{Q}_D$ be an indefinite quadratic form of non-square discriminant $D$. Furthermore, let $P \in \mathcal{Q}_d$ be a positive definite quadratic form of discriminant $d$ and let $\lambda = (\lambda_m)$ be a relation for $S_{2k}(\Gamma)$. Then, we have the formula

$$C \left( f_{k,P}|T_\lambda, A \right) = \frac{|d|^{k-1}}{|\Gamma_p|} \sum_{m \geq 1} \lambda_m m^{k-1} \sum_{\alpha \delta = m} \sum_{\beta (\text{mod } \delta)} P_{k,A} \left( \frac{\alpha \tau_p + \beta}{\delta} \right).$$

In particular, the cycle integrals of $f_{k,P}|T_\lambda$ are rational numbers whose denominators are bounded only in $k$ and $N$.

The idea of the proof is similar as for Theorem 2.1. See Section 6 for the details.

**Example 2.4.** Let $N = 1$. For $k = 6$, we have the relation $\lambda = (24, 1, 0, 0, \ldots)$ for $S_{12}$. By Theorem 2.3 applied to $f_{6,-3} = f_{6,[1,1,1]}$, the function

$$f_{6,-3}|T_\lambda = f_{6,-3}|T_2 + 24f_{6,-3} = f_{6,-12} - 8f_{6,-3}$$
has rational cycle integrals. Here, we used the action of $T_p$ on $f_{k,D}$ as stated in [2]. Indeed, we have

| $A$   | $[1,1,-1]$ | $[1,0,-2]$ | $[1,1,-4]$ | $[1,0,-6]$ | $[1,1,-7]$ | $[1,1,-8]$ |
|-------|------------|------------|------------|------------|------------|------------|
| $C(f_{6,-3}|T_\lambda,A)$ | 5952 | 44112 | 1128096 | 1186056 | 2349504 | 4070304 |

Similarly, for $f_{6,-7} = f_{6,[1,1,2]}$, we have

$$f_{6,-7}|T_\lambda = f_{6,-7}|T_2 + 24f_{6,-7} = f_{6,-28} + 56f_{6,-7}$$

and

| $A$ | $[1,1,-1]$ | $[1,0,-3]$ | $[1,1,-3]$ | $[1,1,-4]$ | $[1,1,-5]$ | $[1,0,-6]$ |
|-----|------------|------------|------------|------------|------------|------------|
| $C(f_{6,-7}|T_\lambda,A)$ | 228704 | 2728656 | 7282240 | 17047968 | 15937488 | 26668656 |

Finally, we consider cycle integrals of linear combinations of the forms $f_{k,D}$ and their twisted analogues $f_{k,\Delta,\delta}$, which we define now. For simplicity, we now assume that $N$ is odd and square-free. Let $k \geq 1$, let $\Delta$ be a discriminant with $(-1)^k \Delta > 0$, and let $\delta$ be a fundamental discriminant with $(-1)^k \delta < 0$, such that $\delta$ is a square modulo $4N$. Let $\chi_\delta$ be the generalized genus character on $\mathbb{Q}/\Delta_\delta$ as defined in [17]. For $k \geq 1$ we define the twisted function

$$f_{k,\Delta,\delta}(z) := \sum_{P \in \mathbb{Q}/\Delta_\delta/\Gamma} \chi_\delta(P) f_{k,P}(z).$$

Then, $f_{k,\Delta,\delta}$ is a meromorphic modular form of weight $2k$ for $\Gamma$. Suppose that

$$F(\tau) = \sum_{m \gg -\infty} c_F(m) q^m$$

is a weakly holomorphic modular form of weight $\frac{3}{2} - k$ for $\Gamma_0(4N)$ satisfying the Kohnen plus space condition, such that the Fourier coefficients $c_F(m)$ are rational for all $m < 0$. We will show in Proposition 7.1 that the Fourier coefficients of the meromorphic modular form

$$\sum_{(-1)^k \Delta > 0} c_F(-|\Delta|) f_{k,\Delta,\delta}(z)$$

(2.2)

are algebraic multiples of $\pi^{k-1}$. Based on extensive numerical experiments, we arrived at the following conjecture.
Conjecture 2.5. The function in (2.2) has rational cycle integrals if it has no poles on the cycle.

It seems that the methods used to prove Theorem 2.1 and Theorem 2.3 are not suitable to prove the conjecture. In particular, numerical computations suggest that the cycle integrals of the function in (2.2) cannot be expressed in a simple way in terms of the functions \( \mathcal{P}_{k,A} \). However, we are able to prove the conjecture in the case \( k = 1 \), using different methods.

Theorem 2.6. Conjecture 2.5 is true for \( k = 1 \).

The proof relies on the fact that the function \( \pi if_{1,\Delta_A}(z)dz \) is the canonical differential of the 3rd kind for its residue divisor on the compactified modular curve \( X_0(N) \). Together with a rationality criterion of Scholl [25] for such differentials we obtain Theorem 2.6. We refer to Section 7 for the proof. We remark that, unfortunately, the proof does not yield finite rational formulas for the cycle integrals of the linear combination (2.2).

Example 2.7. We give some numerical examples of Conjecture 2.5. Let \( N = 1 \). If \( k \) is odd, we can pick \( \delta = 1 \), such that there is no twist. The 1st odd \( k \) for which there are nontrivial weight \( 2k \) cusp forms is \( k = 9 \) with \( S_{18} = \mathbb{C}E_6 \). The space \( S_{18} \) is isomorphic to the Kohnen plus space of weight \( 9 + \frac{1}{2} \) under the Shimura correspondence, and the latter space is spanned by the cusp form

\[
q^3 - 2q^4 - 16q^7 + 36q^8 + O(q^{11}).
\]

This implies that there is a weakly holomorphic modular form with principal part \( q^{-4} + 2q^{-3} + O(1) \) in the Kohnen plus space of weight \( \frac{3}{2} - 9 \). In this case, Conjecture 2.5 predicts that the linear combination

\[
g := f_{9,-4} + 2f_{9,-3}
\]

has rational cycle integrals. One can easily see that, since \( k \) is odd, we have \( \mathcal{C}(g,A) = 0 \) whenever the form \( A \) is \( \Gamma(1) \)-equivalent to \(-A\). But for quadratic forms that are not equivalent to their negatives, we obtain numerically

| \( A \)          | 3143284 | 235476 | 4350060 | 116285048 | 255683332 | 254947680 |
|------------------|---------|--------|---------|-----------|-----------|-----------|
| \( \mathcal{C}(g,A) \) |         |        |         |           |           |           |
If \( k \) is even, we have to introduce a twist, since \( \delta < 0 \). Here, we consider \( k = 6 \) and \( \delta = -3 \). The weight \( 6 + \frac{1}{2} \) cusp form corresponding to \( \Delta \in S_{12} \) under the Shimura correspondence is given by

\[
q - 56q^4 + 120q^5 - 240q^8 + 9q^9 + O(q^{12}),
\]

so for example the functions

\[
g_1 := f_{6,4,-3} + 56f_{6,1,-3} \quad \text{and} \quad g_2 := f_{6,5,-3} - 120f_{6,1,-3}
\]

should have rational cycle integrals. Indeed, it is easy to check that the function \( g_1 \) coincides with \( f_{6,-3}|T_\lambda \) from Example 2.4, so it does have rational cycle integrals by Theorem 2.3. In contrast, \( g_2 \) cannot be obtained by acting with Hecke operators, since \(-15\) is square-free. However, for this function we obtain numerically

| \( A \) | \( [1, 1, -1] \) | \( [1, 0, -2] \) | \( [1, 1, -3] \) | \( [1, 1, -4] \) | \( [1, 1, -5] \) | \( [1, 1, -7] \) | \( [1, 1, -8] \) |
|---|---|---|---|---|---|---|---|
| \( C(g_2, A) \) | \(-51012\) | \(-126816\) | \(57876\) | \(-2108352\) | \(134946\) | \(3813312\) | \(-7458750\) |

The work is organized as follows. In Section 3, we introduce the necessary functions and notation. Then, we relate the cycle integrals \( C(f_{k,p, A}) \) to locally harmonic Maass forms in Section 4. The proofs of Theorems 2.1, 2.3, and 2.6 are given in the remaining sections.

### 3 Preliminaries

#### 3.1 Weight 2 Eisenstein series

For \( z = x + iy \in \mathbb{H} \) we define the quasimodular weight 2 Eisenstein series for \( \Gamma = \Gamma_0(N) \) associated to the cusp \( i\infty \) by the conditionally convergent series

\[
E_{2,\Gamma}(z) := 1 + \sum_{c \geq 1} \sum_{d \in \mathbb{Z}} \frac{1}{1|2M, (c d \in \Gamma_\infty \setminus \Gamma)}
\]

where \( \Gamma_\infty := \{ \pm \left( \begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix} \right) : n \in \mathbb{Z} \} \) and \((f|_k \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right))(z) := (cz + d)^{-k} f(\frac{az + b}{cz + d})\) denotes the usual weight \( k \) slash operator. The function \( E_{2,\Gamma} \) has a non-holomorphic modular completion

\[
E_{2,\Gamma}^*(z) := -\frac{3}{\pi [\Gamma(1) : \Gamma|y]} + E_{2,\Gamma}(z)
\]
that has constant term 1 at $i\infty$ and 0 at all other cusps. The following lemma expresses $E_{2,\Gamma}^*$ in terms of the Eisenstein series for the full modular group and follows from Equation (9) on p. 546 of [17].

**Lemma 3.1.** We have

$$E_{2,\Gamma}^*(z) = \prod_{p|N} \left(1 - p^{-2}\right)^{-1} \sum_{d|N} \frac{\mu(d)}{d^2} E_{2,\Gamma(1)}^* \left(\frac{N}{d} z\right)$$

$$= -\frac{3}{\pi [\Gamma(1) : \Gamma]} y + 1 - 24 \prod_{p|N} \left(1 - p^{-2}\right)^{-1} \sum_{d|N} \frac{\mu(d)}{d^2} \sum_{n \geq 1} \sigma\left(\frac{dn}{N}\right) q^n,$$

where $\sigma$ denotes the divisor sum function and we set $\sigma(x) := 0$ for $x \notin \mathbb{Z}$.

### 3.2 Petersson’s Poincaré series

For $z = x + iy, \tau = u + iv \in \mathbb{H}$ and $k \in \mathbb{Z}$ with $k \geq 2$ we define Petersson’s Poincaré series

$$H_k(z, \tau) := \sum_{M \in \Gamma} \left(\frac{(z - \tau)(z - \overline{\tau})}{v}\right)^{-k} M = \sum_{M \in \Gamma} \left(\frac{(z - \tau)(z - \overline{\tau})}{v}\right)^{-k} \left|_{0,\tau}\right. M,$$

which has weight $2k$ in $z$ for $\Gamma$ and weight 0 in $\tau$ for $\Gamma$. Furthermore, it is meromorphic as a function of $z$, and an eigenfunction of the invariant Laplace operator $\Delta_0$ with eigenvalue $k(1 - k)$ as a function of $\tau$ for $\tau$ not lying in the $\Gamma$-orbit of $z$. This Poincaré series was first introduced by Petersson in [24] and has been studied more recently by Bringmann and Kane in [5, 6]. The series does not converge for $k = 1$. However, we can apply Hecke’s trick as in [5, Section 3.1], and define for $\Re(s) > 0$

$$H_{1,s}(z, \tau) := \sum_{M \in \Gamma} \left(\frac{(z - \tau)(z - \overline{\tau})}{v}\right)^{-1} \left(\frac{|z - \tau||z - \overline{\tau}|}{vy}\right)^{-s} \left|_{0,\tau}\right. M.$$

Using Poisson summation together with rather delicate estimates, Bringmann and Kane showed that $H_{1,s}(z, \tau)$ has an analytic continuation $H_{1}^*(z, \tau)$ to $s = 0$. We refer the reader to [5, Theorem 3.1], for the details. It follows from the Fourier expansion of $H_{1}^*(z, \tau)$ given in [5, Lemma 5.4], that the function

$$H_{1}(z, \tau) := H_{1}^*(z, \tau) - 2\pi E_{2,\Gamma}^*(z)$$

is a meromorphic modular form of weight 2 in $z$ and a harmonic Maass form of weight 0 in $\tau$ for $\Gamma$. The function $z \mapsto H_{1}(z, \tau)$ has a simple pole when $z$ is $\Gamma$-conjugate to $\tau$. 
Similarly, we define for \( k \geq 2 \) and \( \ell \in \mathbb{Z} \) the function
\[
H_{k,\ell}(z, \tau) := \sum_{M \in \Gamma} \nu^{k+\ell} \left((z - \tau)^{k-\ell}(z - \overline{\tau})^{-\ell-k}\right) \bigg|_{2k, z}^M,
\]
which has also been studied in [6]. It has weight \( 2k \) in \( z \) and weight \(-2\ell\) in \( \tau \) for \( \Gamma \), and it also behaves nicely under the raising and lowering operators
\[
R_\kappa := 2i\tau \frac{\partial}{\partial \tau} + \kappa \nu^{-1}, \quad L_\kappa := -2iv^2 \frac{\partial}{\partial \tau},
\]
which raise and lower the weight of an automorphic form of weight \( \kappa \) by 2, respectively. The following lemma can be checked by a direct computation.

**Lemma 3.2.** For \( k \geq 2 \) and \( \ell \in \mathbb{Z} \) we have
\[
R_{-2\ell, \tau} \left( H_{k,\ell}(z, \tau) \right) = (k - \ell)H_{k,\ell-1}(z, \tau),
\]
\[
L_{-2\ell, \tau} \left( H_{k,\ell}(z, \tau) \right) = (k + \ell)H_{k,\ell+1}(z, \tau).
\]

We are particularly interested in the function \( H_{k,k-1}(z, \tau) \) (with \( H_{1,0}(z, \tau) := H_1(z, \tau) \)), which has weight \( 2k \) in \( z \) and \( 2 - 2k \) in \( \tau \). It is meromorphic in \( z \) and harmonic in \( \tau \) for \( \tau \) not lying in the \( \Gamma \)-orbit of \( z \), and as a function of \( \tau \) it is bounded at the cusps (and vanishes at \( i\infty \) if \( k = 1 \), compare Lemma 5.4 in [5]). Furthermore, by Lemma 3.2 it is related to \( H_k(z, \tau) \) by
\[
R_{2, \Gamma}^{k-1} \left( H_{k,k-1}(z, \tau) \right) = (k - 1)! H_k(z, \tau),
\]
where \( R_{2, \Gamma}^{k-1} := R_{-2} \circ \cdots \circ R_{-2} \) is an iterated version of the raising operator.

### 3.3 Modular forms associated to quadratic forms

In the Introduction, we defined the function \( f_{k,Q_0} \) associated to an integral binary quadratic form \( Q_0 \) of discriminant \( D \neq 0 \) for \( k \geq 2 \). We briefly explain the definition for \( k = 1 \). For \( s \in \mathbb{C} \) with \( \Im(s) > 1 \) we consider the series
\[
f_{1, Q_0, \delta}(z) := \frac{|D|^{\frac{s+1}{2}}}{2^s \pi} y^s \sum_{Q \in [Q_0]} Q(z, 1)^{-1} |Q(z, 1)|^{-s}.
\]
It converges absolutely and has a holomorphic continuation to \( s = 0 \). If \( Q_0 = A \) is indefinite and \( D \) not a square, then

\[
f_{1,A}(z) := f_{1,A,0}(z)
\]
is a cusp form of weight 2 for \( \Gamma \) (see \([17, p. 517]\)). If \( Q_0 = P \) is positive definite, then it follows from the following lemma and (3.1) that

\[
f_{1,P}(z) := f_{1,P,0}(z) - \frac{2}{|\Gamma_P|} E_{2,1}^e(z)
\]
is a meromorphic modular form of weight 2 for \( \Gamma \).

**Lemma 3.3.** For \( k \geq 2 \) and \( z \) not lying in the \( \Gamma \)-orbit of the CM point \( \tau_P \) we have

\[
f_{k,P}(z) = \frac{2^{k-1}|d|^{\frac{k-1}{2}}}{|\Gamma_P|^{\frac{1}{2}}} H_k(z, \tau_P)
\]
and

\[
f_{1,P,0}(z) = \frac{1}{|\Gamma_P|^{\frac{1}{2}}} H_1^\ast(z, \tau_P).
\]

**Proof.** We have the formula

\[
P(z, 1) = \sqrt{|d|} \frac{(z - \tau_P)(z - \overline{\tau}_P)}{2 \, \text{Im}(\tau_P)}.
\]

Hence, for \( k \geq 2 \) we get

\[
H_k(z, \tau_P) = \sum_{M \in \Gamma} j(M, z)^{-2k} \left( \frac{(Mz - \tau_P)(Mz - \overline{\tau}_P)}{\text{Im}(\tau_P)} \right)^{-k}
\]

\[
= 2^{-k}|d|^{\frac{k}{2}} \sum_{M \in \Gamma} j(M, z)^{-2k} P(M, 1)^{-k} = 2^{1-k}|d|^{\frac{1-k}{2}} |\Gamma_P|^{-\frac{k}{2}} f_{k,P}(z).
\]

For \( k = 1 \) we can show in the same way that \( H_{1,s}(z, \tau_P) \) is a multiple of \( f_{1,P,s}(z) \) and then use analytic continuation. ■

We will also need the Fourier expansion of \( f_{k,P} \). The proof of the following formula is analogous to the proof of Proposition 2.2 from [2] (correcting a sign error), but additionally uses (3.3) and Lemma 3.1 in case that \( k = 1 \).
**Proposition 3.4.** For \( k \geq 2 \) and \( z \in \mathbb{H} \) with \( y > \sqrt{|d|}/2 \) we have the Fourier expansion

\[
f_{k,P}(z) = \sum_{n \geq 1} c_{f,k,P}(n)e^{2\pi inz},
\]

where

\[
c_{f,k,P}(n) = \frac{(-1)^k 2^{k+\frac{1}{2}} \pi^k}{(k-1)!} |d|^{\frac{k}{2} - \frac{1}{4}} n^{k-\frac{1}{2}} \sum_{a \geq 1 \atop N|a} a^{-\frac{1}{2}} S_{a,P}(n) I_{k-\frac{1}{2}} \left( \frac{\pi n \sqrt{|d|}}{a} \right),
\]

with the usual \( I \)-Bessel function and the exponential sum

\[
S_{a,P}(n) := \sum_{b \pmod{2a} \atop b^2 \equiv d \pmod{4a} \atop \left[ a, b, b^2 - d \atop 4a \right] \in [P]} e \left( \frac{n b}{2a} \right).
\]

For \( k = 1 \) the formula is analogous, but we have to add

\[
\frac{12}{|\Gamma_p|} \prod_{p \nmid N} \left(1 - p^{-2}\right)^{-1} \sum_{d \mid N} \frac{\mu(d)}{d^2} \sigma \left( \frac{dn}{N} \right)
\]

to \( c_{f_1,P}(n) \), and we get a constant term \( c_{f_1,P}(0) = -\frac{2}{|\Gamma_p|} \).

**3.4 Zeta functions associated to indefinite quadratic forms**

Let \( A \in \mathcal{O}_D \) be an indefinite quadratic form of non-square discriminant \( D > 0 \). We define the associated zeta function

\[
\zeta_{\Gamma,A}(s) := \sum_{\substack{m \equiv n_0 \pmod{n} \atop A(m,n) > 0}} A(m,n)^{-s} = \sum_{(m,n) \in \mathbb{Z}^2 / \Gamma_A^I \cap \Gamma_A / \Gamma_\infty \atop N|n(N,m)=1 \atop A(m,n) > 0} A(m,n)^{-s}. \tag{3.4}
\]

The series converges absolutely for \( \Re(s) > 1 \) and it only depends on the \( \Gamma \)-equivalence class of \( A \). For \( N = 1 \) and a fundamental discriminant \( D > 0 \) the function \( \zeta(2s)\zeta_{\Gamma(1),A}(s) \) is the usual zeta function of the ideal class in \( \mathbb{Q}(\sqrt{D}) \) associated to \( A \). The above zeta function can also be written as a Dirichlet series

\[
\zeta_{\Gamma,A}(s) = \sum_{a \geq 0 \atop N|a} \frac{n_A(a)}{a^s}, \quad n_A(a) := \# \left\{ b \pmod{2a}, b^2 \equiv D \pmod{4a}, \left[ a, b, \frac{b^2 - D}{4a} \right] \in [A] \right\},
\]

compare [29, Proposition 3 (i)].
Lemma 3.5. For $\Re(s) > 1$ we have

$$
\zeta_{\Gamma, A}(s) = N^{-s} \prod_{p \mid N} (1 - p^{-2s})^{-1} \sum_{d \mid N} \frac{\mu(d)}{d^{2s}} \left[ \Gamma(1)_{A_{N/d}} : \Gamma(0)_{A_{N/d}}(d) \right] \zeta(1, A_{N/d}(s)),
$$

where $A_d := [a/d, b, d/c]$ for $A = [a, b, c]$.

Proof. Let $\zeta_N(s) = \sum_{a=1}^{\infty} (a, N)^{-s}$. We have

$$
\zeta_N(2s) \zeta_{\Gamma, A}(s) = \sum_{a=1}^{\infty} \sum_{(a, N) = 1 \atop (m, n) \in \mathbb{Z}^2 / \Gamma_0(N)^t_A \atop M_n(m, n) = 1 \atop A(m, n) > 0} A(a m, a n)^{-s} = \sum_{(m, n) \in \mathbb{Z}^2 / \Gamma_0(N)^t_A \atop M_n(m, n) = 1 \atop A(m, n) > 0} A(m, n)^{-s}
$$

$$
= \sum_{d \mid N} \mu(d) \sum_{(m, n) \in \mathbb{Z}^2 / \Gamma_0(N)^t_A \atop Nn, d \mid m \atop A(m, n) > 0} A(m, n)^{-s} = \sum_{d \mid N} \frac{\mu(d)}{d^{2s}} \sum_{(m, n) \in \mathbb{Z}^2 / \Gamma_0(N)^t_A \atop \frac{N}{d} \mid n \atop A(m, n) > 0} A(m, n)^{-s}.
$$

If $(a, b)$ runs through $\mathbb{Z}^2 / \Gamma_0(N)^t_A$ then $(m, n) = (a, \frac{N}{d}b)$ runs through $\mathbb{Z}^2 / \Gamma_0(N)^t_A$ with $\frac{N}{d} \mid n$. Hence, we obtain

$$
\sum_{(m, n) \in \mathbb{Z}^2 / \Gamma_0(N)^t_A \atop \frac{N}{d} \mid n, A(m, n) > 0} A(m, n)^{-s} = \sum_{(m, n) \in \mathbb{Z}^2 / \Gamma_0(N)^t_A \atop \frac{N}{d} \mid n, A(m, n) > 0} A(m, \frac{N}{d}n)^{-s}
$$

$$
= \left( \frac{N}{d} \right)^{-s} \sum_{(m, n) \in \mathbb{Z}^2 / \Gamma_0(N)^t_A \atop A(m, \frac{N}{d}n) > 0} A_{N/d}(m, n)^{-s}
$$

$$
= \left[ \Gamma(1)_{A_{N/d}} : \Gamma(0)_{A_{N/d}}(d) \right] N^{-s} d^{2s} \zeta(2s) \zeta_{\Gamma, A_{N/d}}(s).
$$

Using $\frac{\zeta(2s)}{\zeta_N(2s)} = \prod_{p \mid N} (1 - p^{-2s})^{-1}$ we obtain the stated formula. 

The following result concerns the rationality of the special values of $\zeta_{\Gamma, A}(s)$ at positive integers.
Proposition 3.6. The expression

\[ D^{k-\frac{1}{2}} \left( \zeta_{\Gamma,A}(k) + (-1)^k \zeta_{\Gamma,-A}(k) \right) \]

is rational for any \( k \geq 2 \), any non-square discriminant \( D > 0 \), and \( N \in \mathbb{N} \).

Proof. We set \( \widehat{\zeta}_A(s) := \zeta(2s) \zeta_{\Gamma(1),A}(s) \). It is well known that for \( k \geq 2 \) and any non-square discriminant \( D > 0 \) we have the functional equation

\[ D^{k-\frac{1}{2}} \left( \widehat{\zeta}_A(k) + (-1)^k \widehat{\zeta}_A(k) \right) = \frac{2^{2k-1} \pi^{2k}}{(k-1)!^2} \zeta_A(1-k), \]

compare [21, p. 230]. Furthermore, \( \widehat{\zeta}_A(1-k) \) is rational by Theorem 8 in [21]. Dividing by \( \zeta(2k) = (-1)^{k+1} \frac{B_{2k} (2\pi)^{2k}}{(2k)!} \) on both sides, we see that

\[ D^{k-\frac{1}{2}} \left( \zeta_{\Gamma(1),A}(k) + (-1)^k \zeta_{\Gamma(1),-A}(k) \right) \]

is rational, too. Using Lemma 3.5 we obtain the result for all \( N \in \mathbb{N} \). ■

Remark 3.7. It follows from the explicit formula in Theorem 8 of [21] that the denominator of \( \widehat{\zeta}_A(1-k) \) is bounded by a constant only depending on \( k \) but not on \( A \). This formula can also be used to evaluate \( D^{k-\frac{1}{2}} \left( \zeta_{\Gamma,A}(k) + (-1)^k \zeta_{\Gamma,-A}(k) \right) \) explicitly as a rational number.

Finally, we relate the expression from Proposition 3.6 to the cycle integrals of the Eisenstein series \( E_{2k,\Gamma} \) of weight 2k for the cusp \( i\infty \) of \( \Gamma \), which is normalized such that its Fourier expansion at \( i\infty \) has constant term 1. The following result can be proven by a similar computation as on pp. 240–241 of [21].

Proposition 3.8. Let \( k \geq 2 \) and let \( A \in \mathcal{Q}_D \) be an indefinite quadratic form of non-square discriminant \( D > 0 \). Then,

\[ C(E_{2k,\Gamma},A) = (-1)^k \frac{(k-1)!^2}{(2k-1)!} D^{k-\frac{1}{2}} \left( \zeta_{\Gamma,A}(k) + (-1)^k \zeta_{\Gamma,-A}(k) \right) \]

Although we will not use this formula in the proofs of our main results, we decided to include it since it gives an interesting interpretation of the expression from Proposition 3.6.
3.5 Cycle integrals of meromorphic modular forms

Let $A = [a, b, c] \in \mathbb{Q}_D$ be an indefinite quadratic form of non-square discriminant $D > 0$. Then, the set

$$S_A := \{z \in \mathbb{H} : a|z|^2 + b\Re(z) + c = 0\}$$

is a semi-circle centered at the real line, which is oriented counterclockwise if $a > 0$ and clockwise if $a < 0$. Let $f : \mathbb{H} \to \mathbb{C}$ be a meromorphic function that transforms like a modular form of weight $2k$ for $\Gamma$. If the poles of $f$ do not meet the semi-circle $S_A$, then we define the cycle integral of $f$ along the closed geodesic $c_A = \Gamma_A \setminus S_A$ by

$$C(f, A) := \int_{c_A} f(z)A(z, 1)^{k-1}dz,$$

where $\Gamma_A$ denotes the stabilizer of $A$ in $\Gamma$. It only depends on the $\Gamma$-equivalence class of $A$. If some poles of $f$ do lie on $S_A$, we modify $S_A$ by circumventing these poles and all of their $\Gamma$-translates on small arcs of radius $\varepsilon > 0$ above and below the poles. Thereby, we obtain two paths $S_{A,+}^{\varepsilon}$ and $S_{A,-}^{\varepsilon}$ and corresponding geodesics $c_{A,+}^{\varepsilon}$ and $c_{A,-}^{\varepsilon}$ that avoid the poles of $f$. We define the regularized cycle integral of $f$ along $c_A$ by the Cauchy principal value

$$C(f, A) := \lim_{\varepsilon \to 0} \frac{1}{2} \left( \int_{c_{A,+}^{\varepsilon}} f(z)A(z, 1)^{k-1}dz + \int_{c_{A,-}^{\varepsilon}} f(z)A(z, 1)^{k-1}dz \right). \tag{3.5}$$

Note that, since $f$ is meromorphic, the integrals on the right-hand side are actually independent of $\varepsilon$ for $\varepsilon > 0$ small enough, so the limit exists.

3.6 Maass Poincaré series

Throughout this section, we let $N$ be odd and square-free. One can construct harmonic Maass form of half-integral weight as special values of Maass Poincaré series, see [23], for example. In this way, one obtains for every integer $k \geq 1$ and $n < 0$ with $(-1)^k n \equiv 0, 3 \pmod{4}$ a harmonic Maass form $P_{\frac{3}{2} - k, n}(\tau)$ of weight $\frac{3}{2} - k$ for $\Gamma_0(4N)$ that satisfies the Kohnen plus space condition, whose Fourier expansion at $i\infty$ starts with $q^{-|n|} + O(1)$, and that is bounded at the other cusps.

The holomorphic part of $P_{\frac{3}{2} - k, n}$ has a Fourier expansion of the shape

$$P_{\frac{3}{2} - k, n}^+(\tau) = q^{-|n|} + \sum_{m \geq 0} c_{P_{\frac{3}{2} - k, n}^+}(m)q^m,$$

whose coefficients of positive index are given as follows.
Theorem 3.9 (Theorem 2.1 in [23]). Let $n < 0$ and $m > 0$ with $(-1)^kn, (-1)^km \equiv 0, 3 \pmod{4}$. Then,

$$c^{+}_{\mathcal{P}_{2-k,n}}(m) = -(1)^{\frac{k}{2}} \pi \sqrt{2} \left( \frac{m}{|n|} \right)^{\frac{k}{4} - \frac{1}{2}} \sum_{a > 0 \atop N|a} \frac{K^+((-1)^{k+1}n, (-1)^{k+1}m, a)}{a} I_{k-\frac{1}{2}} \left( \frac{\pi \sqrt{m|n|}}{a} \right),$$

with the half-integral weight Kloosterman sum

$$K^+(m, n, a) := \frac{1 - i}{4} \left( 1 + \frac{4}{a} \right) \sum_{v \pmod{4a}^*} \left( \frac{4a}{v} \right) \left( \frac{-4}{v} \right)^{\frac{1}{2}} e \left( \frac{mv + nv}{4a} \right).$$

Let $\Delta, \delta \in \mathbb{Z}$ be discriminants and assume that $\delta$ is fundamental. For $a, n \in \mathbb{Z}$ we consider the Salié sum

$$S_{a,\Delta,\delta}(n) := \sum_{b \pmod{2a} \atop b^2 \equiv \delta \Delta \pmod{4a}} \chi_\delta \left( \left[ a, b, \frac{b^2 - \delta \Delta}{4a} \right] \right) e \left( \frac{nb}{2a} \right),$$

where $\chi_\delta$ is the generalized genus character of $\mathcal{Q}_{\Delta,\delta}$ as defined in [17]. It is related to the half-integral weight Kloosterman sum by the following formula.

Proposition 3.10 (Proposition 3 in [14]). Let $\Delta, \delta \in \mathbb{Z}$ be discriminants and assume that $\delta$ is fundamental. Then, for $a, n \in \mathbb{Z}$ we have the identity

$$S_{a,\Delta,\delta}(n) = \sum_{m|(n,a)} \left( \frac{\delta}{m} \right) \sqrt{\frac{m}{a}} K^+ \left( \Delta, \frac{n^2}{m}, \delta, \frac{a}{m} \right).$$

4 Locally Harmonic Maass Forms

The key to proving Theorems 2.1 and 2.3 is relating the cycle integrals of $f_{k,\mathcal{P}}$ to certain locally harmonic Maass forms introduced by Bringman et al. in [7] and Hövel in [18]. Namely, for $k \geq 2$, $\tau = u + iv \in \mathbb{H}$, and an indefinite quadratic form $A \in \mathcal{Q}_D$ of non-square discriminant $D > 0$, these are defined by the series

$$F_{1-k,A}(\tau) := \frac{(-1)^k D^\frac{1}{2}}{(2k-2)\pi} \sum_{Q \in [A]} \text{sgn}(Q_\tau) Q(\tau, 1)^{k-1} \psi \left( \frac{Dv^2}{|Q(\tau, 1)|^2} \right),$$

(4.1)
where $Q_{\tau} := \frac{1}{\psi}(a|\tau|^2 + bu + c)$ and

$$\psi(v) := \frac{1}{2} \beta(v; k - \frac{1}{2}, \frac{1}{2}) = \frac{1}{2} \int_0^v t^{k-\frac{3}{2}}(1-t)^{-\frac{1}{2}} dt$$

is a special value of the incomplete $\beta$-function. For $k = 1$ one can define a weight 0 analogue $F_{0,A}$ of (4.1) using the Hecke trick as in [8, 15]. By adding a suitable constant we can normalize $F_{0,A}$ such that it vanishes at $i\infty$.

The function $F_{1-k,A}$ transforms like a modular form of weight $2 - 2k$ for $\Gamma$, is harmonic on $\mathbb{H} \setminus \bigcup_{Q \in [A]} S_Q$ and bounded at the cusps and has discontinuities along the semi-circles $S_Q$ for $Q \in [A]$. Its value at a point $\tau$ lying on $S_Q$ for $Q \in [A]$ is given by the average value

$$F_{1-k,A}(\tau) = \lim_{\varepsilon \to 0} \frac{1}{2}(F_{1-k,A}(\tau + i\varepsilon) + F_{1-k,A}(\tau - i\varepsilon)). \tag{4.2}$$

Furthermore, outside the singularities $F_{1-k,A}$ is related to the cusp form $f_{k,A} \in S_{2k}(\Gamma)$ by the differential equations

$$\xi_{2-2k}(F_{1-k,A}) = (-1)^k D^{\frac{1}{2}-k}_{(2k-2)} f_{k,A},$$

$$D^{2k-1}(F_{1-k,A}) = (-1)^{k+1} D^{\frac{1}{2}-k}_{(2k-2)} (k-1)!^2 f_{k,A},$$

where $\xi_k := 2i v^k \frac{\partial}{\partial \tau}$ and $D := \frac{1}{2\pi i} \frac{\partial}{\partial \tau}$. Note that our normalization of $f_{k,A}$ differs from the one used in [7], which explains the different constants in the above differential equations.

Recall that the non-holomorphic and holomorphic Eichler integrals of a cusp form $f = \sum_{n \geq 1} c_f(n)q^n \in S_{2k}(\Gamma)$ are defined by

$$f^*(\tau) := (-2i)^{1-2k} \int_{-\tau}^{i\infty} \frac{f(z+\tau)}{(z+\tau)^{2k-2}} dz, \quad E_f(\tau) := \sum_{n \geq 1} \frac{c_f(n)}{n^{2k-1}} q^n.$$

They satisfy

$$\xi_{2-2k}(f^*) = f, \quad D^{2k-1}(f^*) = 0, \quad \xi_{2-2k}(E_f) = 0, \quad D^{2k-1}(E_f) = f.$$

The following decomposition of $F_{1-k,A}$ was derived by Bringmann et al. for $N = 1$ and $k \geq 2$ in [7], but the same methods work for all $N \geq 1$ and $k = 1$ (see also [15] for $k = 1$).
Theorem 4.1 (Theorem 7.1 of [7]). For $\tau$ not lying on any of the semi-circles $S_Q$ for $Q \in [A]$ we have

$$F_{1-k,A}(\tau) = P_{1-k,A}(\tau) + (-1)^k \frac{D^\frac{1}{2} - k}{(2k-2)^2} f^A_{k,A}(\tau) + (-1)^{k+1} \frac{D^\frac{1}{2} - k (k-1)^2}{(4\pi)^{2k-1}} \mathcal{E}_{k,A}(\tau),$$

where $P_{1-k,A}(\tau)$ is locally a polynomial of degree at most $2k - 2$. More precisely, it is a polynomial on each connected component of $\mathbb{H} \setminus \bigcup_{Q \in [A]} S_Q$, which is given by

$$P_{1-k,A}(\tau) := c_k(A) + (-1)^{k-1}2^{-2k}D^\frac{1}{2} - k \sum_{Q = [a,b,c] \in [A]} \sum_{\tau \in \text{Int}(S_Q)} \text{sgn}(a)Q(\tau,1)^{k-1},$$

where $c_1(A) := 0$ and

$$c_k(A) := -\frac{\zeta_{\Gamma,A}(k) + (-1)^k \zeta_{\Gamma,-A}(k)}{2^{2k-2}(2k-1)(2k-2)^{k-1}}$$

for $k \geq 2$, and $\text{Int}(S_Q)$ denotes the bounded component of $\mathbb{H} \setminus S_Q$.

The main goal of this section is to show that $F_{1-k,A}(\tau)$ can be written as a cycle integral of Petersson’s Poincaré series $H_{k,k-1}(z,\tau)$.

Theorem 4.2. We have

$$F_{1-k,A}(\tau) = \frac{D^\frac{1}{2} - k}{2\pi} C(H_{k,k-1}(\cdot,\tau),A).$$

If $\tau$ lies on a semi-circle $S_Q$ for $Q \in [A]$ the left-hand side has to interpreted as the average value (4.2), and the cycle integral on the right-hand side is defined as the Cauchy principal value (3.5).

Before we come to the proof of the theorem we state an important corollary, which immediately follows from Theorem 4.2 together with Lemma 3.3 and the identity (3.2).

Corollary 4.3. We have

$$C(f_{k,p},A) = \frac{2^k|d|^{k-1}D^{k-\frac{1}{2}}}{(k-1)! |\Gamma_p|} R_{2-2k}^{k-1} (F_{1-k,\lambda})(\tau_p),$$

where $R_{2-2k}^{k-1}$ denotes the iterated raising operator defined in Section 3.2.
Note that a harmonic function on $\mathbb{H}$ that transforms like a modular form of weight $2 - 2k$ and is bounded at the cusps has to be a constant (and therefore vanishes if $k > 1$). Hence, in order to prove Theorem 4.2 in the case that $\tau$ does not lie on $S_Q$ for $Q \in [A]$ it suffices to show that both sides in the theorem have the same singularities on $\mathbb{H}$ and are bounded at the cusps (and vanish at $i\infty$ if $k = 1$).

We say that a function $f$ has a singularity of type $g$ at a point $\tau_0$ if there exists a neighborhood $U$ of $\tau_0$ such that $f$ and $g$ are defined on a dense subset of $U$ and $f - g$ can be extended to a harmonic function on $U$. For example, Theorem 4.1 shows that the function $F_1 - k \tau, A(\tau)$ has a singularity of type $\left( -1 \right)^{21 - 2k} \sum_{Q = [a, b, c] \in [A]} \sum_{\tau_0 \in S_Q} \frac{\text{sgn}(Q_\tau)Q(\tau, 1)^{k - 1}}{2}$ at each point $\tau_0 \in \mathbb{H}$, which easily follows from the fact that $\tau \in \text{Int}(S_Q)$ is equivalent to $\text{sgn}(a) \text{sgn}(Q_\tau) < 0$.

**Lemma 4.4.** The function $C(H_{k,k-1}, A)$ is harmonic on $\mathbb{H} \setminus \bigcup_{Q \in [A]} S_Q$ and bounded at the cusps. For $k = 1$ it vanishes at $i\infty$. At a point $\tau_0 \in \mathbb{H}$ it has a singularity of type $\left( -1 \right)^{2 - 2k} \sum_{Q = [a, b, c] \in [A]} \sum_{\tau_0 \in S_Q} \frac{\text{sgn}(Q_\tau)Q(\tau, 1)^{k - 1}}{\mathbb{A}}$.

**Proof.** Since the function $\tau \mapsto H_{k,k-1}(z, \tau)$ is harmonic on $\mathbb{H} \setminus \Gamma z$, the function $C(H_{k,k-1}, A)$ is harmonic on $\mathbb{H} \setminus \bigcup_{Q \in [A]} S_Q$. Moreover, $C(H_{k,k-1}, A)$ is bounded at the cusps (and vanishes at $i\infty$ if $k = 1$) because the same is true for $\tau \mapsto H_{k,k-1}(z, \tau)$.

To determine the singularities, we keep $\tau_0 \in \mathbb{H}$ fixed and consider the function

$$G_{\tau_0}(z, \tau) := \sum_{Q \in [A]} \sum_{\tau_0 \in S_Q} \left( \frac{\nu^{2k-1}}{(z - \tau)(z - x)^{2k-1}} \right) M.$$

Note that the sum over $Q \in [A]$ with $\tau_0 \in S_Q$ is finite, and the group $\Gamma Q$ is infinite cyclic. It is not hard to show that the series converges absolutely and locally uniformly for all $k \geq 1$ and is meromorphic in $z$ and harmonic in $\tau$ for $\tau$ not lying in the $\Gamma$-orbit of $z$. We split the cycle integral into

$$C(H_{k,k-1}, A) = C(H_{k,k-1}, \cdot, \tau) - G_{\tau_0}(\cdot, \tau, A) + C(G_{\tau_0}(\cdot, \tau, A)).$$
The function
\[ \tau \mapsto C(H_{k,k-1}(\cdot, \tau) - G_{\tau_0}(\cdot, \tau), A) \]
is harmonic in a neighborhood of \( \tau_0 \). For the 2nd summand we compute for any \( \tau \notin \Gamma_{S_A} \)
\[
C(G_{\tau_0}(\cdot, \tau), A) = \int_{C_A} \sum_{Q \in [A]} \sum_{M \in \Gamma_A} \left( \left( \frac{v^{2k-1}}{(z - \tau)(z - \tau)^{2k-1}} \right)_{2k,z} \right) M A(z, 1)^{k-1} \, dz
\]
\[
= 2 \sum_{Q \in [A]} \int_{S_Q} \frac{v^{2k-1}}{(z - \tau)(z - \tau)^{2k-1}} Q(z, 1)^{k-1} \, dz.
\]
Note that the integrand is meromorphic in \( z \). The integral is oriented counterclockwise if \( a > 0 \) and clockwise if \( a < 0 \). We complete \( S_Q \) to a closed path by adding the horizontal line connecting the two real endpoints \( w < w' \) of \( S_Q \). The function
\[ \tau \mapsto \int_w^{w'} \frac{v^{2k-1}}{(x - \tau)(x - \tau)^{2k-1}} Q(x, 1)^{k-1} \, dx \]
is harmonic on \( \mathbb{H} \), so it does not contribute to the singularity. From the residue theorem we obtain that the integral over the closed path equals 0 if \( \tau \notin \text{Int}(S_Q) \) and
\[
2\pi i \text{sgn}(a) \text{Res}_{z=\tau} \left( \frac{v^{2k-1}}{(z - \tau)(z - \tau)^{2k-1}} Q(z, 1)^{k-1} \right) = (-1)^{k-1} 2^{2k-2} \pi \text{sgn}(a) Q(\tau, 1)^{k-1}
\]
if \( \tau \in \text{Int}(S_Q) \). This yields the claimed singularity. \( \blacksquare \)

**Proof of Theorem 4.2.** By what we have said above, Theorem 4.2 for \( \tau \) not lying on \( S_Q \) for any \( Q \in [A] \) follows from the above lemma. By a similar idea as in the proof of the lemma above we find that for \( \tau \) lying on a semi-circle \( S_Q \) for \( Q \in [A] \) we have
\[
C(H_{k,k-1}(\cdot, \tau), A) = \lim_{\varepsilon \to 0} \frac{1}{2} (C(H_{k,k-1}(z, \tau + i\varepsilon), A) + C(H_{k,k-1}(z, \tau - i\varepsilon), A)),
\]
where the cycle integral integral on the left-hand side is defined as the Cauchy principal value (3.5). This implies that Theorem 4.2 is also true for \( \tau \) lying on \( S_Q \) for some \( Q \in [A] \). \( \blacksquare \)
5 The Proof of Theorem 2.1

By Corollary 4.3 we have the identity

\[
C(f_{k,P}, A) = \frac{2^k |d|^{k-\frac{1}{2}} D^{k-\frac{1}{2}}}{(k-1)! |\Gamma_p|} R_{2-2k}^{k-1} (F_{1-k,A}) (\tau_P). \tag{5.1}
\]

Let \( Q \) be a finite family of indefinite quadratic forms \( A \in Q_D \) of non-square discriminants \( D_A > 0 \), and let \( a_A \in \mathbb{Z} \) for \( A \in Q \) such that \( \sum_{A \in Q} a_A f_{k,A} = 0 \). If we multiply (5.1) by \( a_A \) and sum over \( A \in Q \), and then plug in the splitting of \( F_{1-k,A} \) from Theorem 4.1, we see that the Eichler integrals \( f_{k,A}^* \) and \( E_{f_{k,A}} \) cancel out due to the assumption \( \sum_{A \in Q} a_A f_{k,A} = 0 \). Hence, we obtain

\[
\sum_{A \in Q} a_A C(f_{k,P}, A) = \frac{2^k |d|^{k-\frac{1}{2}}}{(k-1)! |\Gamma_p|} \sum_{A \in Q} a_A D A^{k-\frac{1}{2}} R_{2-2k}^{k-1} (P_{1-k,A}) (\tau_P),
\]

where \( P_{1-k,A} \) is the local polynomial defined in Theorem 4.1. The action of the iterated raising operator on \( P_{1-k,A} \) has been computed in Lemmas 5.3 and 5.4 of [1] and is given as follows.

**Lemma 5.1.** For \( \tau \in \mathbb{H} \setminus \bigcup_{Q \in |A|} S_Q \) we have

\[
R_{2-2k}^{k-1} (P_{1-k,A}) (\tau) = \frac{(k-1)!}{2^k D^{k-\frac{1}{2}}} P_{k,A} (\tau),
\]

where \( P_{k,A} (\tau) \) is the function defined in (2.1).

We arrive at

\[
\sum_{A \in Q} a_A C(f_{k,P}, A) = \frac{|d|^{k-\frac{1}{2}}}{|\Gamma_p|} \sum_{A \in Q} a_A P_{k,A} (\tau_P),
\]

which is the formula from Theorem 2.1. Finally, we show that the right-hand side is rational.

**Lemma 5.2.** For any CM-point \( \tau_P \in \mathbb{H} \) of discriminant \( d < 0 \), we have

\[
|d|^{k-\frac{1}{2}} P_{k,A} (\tau_P) \in \mathbb{Q}.
\]
Proof. If $P = [a, b, c]$ with $a > 0$, then $\tau_P$ is given by

$$
\tau_P = -\frac{b + i\sqrt{|d|}}{2a}.
$$

In particular, $\frac{\sqrt{|d|}}{\iota(\tau_P)}$ and $\sqrt{|d|}Q_{\tau_P}$ are rational. We have seen in Proposition 3.6 that

$$
D^{k-\frac{1}{2}} \left( \xi_{\Gamma, A}(k) + (-1)^k \xi_{\Gamma, -A}(k) \right)
$$

is a rational number for $k \geq 2$ (and this expression does not occur in $P_{k,A}$ for $k = 1$). Moreover, the Legendre polynomial $P_{k-1}$ is odd if $k$ is even and even if $k$ is odd. Hence,

$$
|d|^{k-1} (i\sqrt{D})^{k-1} P_{k-1} \left( \frac{iQ_{\tau_P}}{\sqrt{D}} \right)
$$

is rational. Combining all these facts we see that $|d|^{k-1} P_{k,A}(\tau_P)$ is a rational number. ■

6 The Proof of Theorem 2.3

For $(m, N) = 1$ the $m$-th Hecke operator $T_m$ on a function $f$ transforming like a modular form of weight $2k$ for $\Gamma$ is defined by

$$
f|T_m := m^{k-1} \sum_{M \in \Gamma \backslash M_m(N)} f|_{2kM}, \quad (6.1)
$$

where $M_m(N)$ is the set of integral 2 by 2 matrices of determinant $m$ whose lower left entry is divisible by $N$, and the slash operator is defined by $(f|_{2kM})(z) := \det(M)^{kj(M,z)}^{-2k} f(Mz)$. It acts on the Fourier expansion of a cusp form $f(z) = \sum_{n=1}^{\infty} c_f(n)q^n \in S_{2k}(\Gamma)$ by

$$
(f|T_m)(z) = \sum_{n=1}^{\infty} \sum_{d | (m,n)} d^{2k-1} c_f(mn/d^2)q^n.
$$

In order to show Theorem 2.3, we would like to use the splitting of $F_{1-k,A}$ from Theorem 4.1 and get rid of the Eichler integrals by taking suitable linear combinations. To this end, the following well-known lemma is useful.

Lemma 6.1. If $\lambda$ is a relation for $S_{2k}(\Gamma)$, then $f|T_{\lambda} = 0$ for every $f \in S_{2k}(\Gamma)$. 
Proof. We have

\[
(f|\tau_\lambda)(z) = \sum_{n=1}^{\infty} \lambda_n \sum_{m=1}^{\infty} \sum_{d | (m,n)} d^{2k-1} c_f(mn/d^2) q^m = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \lambda_n \sum_{d | (m,n)} d^{2k-1} c_f(mn/d^2) q^m.
\]

Since the innermost sum is just the n-th coefficient of the cusp form \( f|T_m \), the sum over \( n \) vanishes by the definition of a relation for \( S_{2k}(\Gamma) \).

An important ingredient in the proof of Theorem 2.3 is the fact that Petersson’s Poincaré series \( H_k(z, \tau) \) behaves nicely under the action of Hecke operators.

Lemma 6.2. For \( k \geq 1 \) and \( (m, N) = 1 \) we have

\[
H_k(z, \tau)|z T_m = m^k H_k(z, \tau)|\tau T_m.
\]

Proof. For \( k \geq 2 \) we plug in the definition of \( H_k(z, \tau) \) and \( T_m \) and write

\[
H_k(z, \tau)|z T_m = m^{k-1} \sum_{M \in \mathcal{M}_m(N)} \left( \frac{(z-\tau)(z-\bar{\tau})}{V} \right)^{-k} M.
\]

Now, a short calculation gives

\[
\left( \frac{(z-\tau)(z-\bar{\tau})}{V} \right)^{-k} M = \left( \frac{(z-\tau)(z-\bar{\tau})}{V} \right)^{-k} M',
\]

where \( \begin{pmatrix} a & b \\ c & d \end{pmatrix}' = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \). Since \( M' \) also runs through \( \mathcal{M}_m(N) \) we obtain the stated identity for \( k \geq 2 \). For \( k = 1 \) and \( \Re(s) > 0 \) we compute analogously

\[
H_{1,s}(z, \tau)|z T_m = m H_{1,s}(z, \tau)|\tau T_m.
\]

Using the well-known fact that

\[
E_{2,\Gamma}^s(z)|z T_m = \sigma(m) E_{2,\Gamma}^s(z) = |\Gamma \backslash \mathcal{M}_m(N)| E_{2,\Gamma}^s(z) = m E_{2,\Gamma}^s(z)|\tau T_m
\]

and analytic continuation we also obtain the result for \( k = 1 \). □
We now come to the proof of Theorem 2.3. Using Lemmas 3.3 and 6.2 we compute
\[
C(f_{k,p}|T_m, A) = \frac{2^{k-1}|d|^{k-1}}{|\Gamma_p|} C(H_k(\cdot, \tau_p)|\tau T_m, A) = \frac{2^{k-1}|d|^{k-1}}{|\Gamma_p|} m^k (C(H_k(\cdot, \tau), A)|\tau T_m)(\tau_p).
\]
By (3.2) and Theorem 4.2, we obtain
\[
C(H_k(\cdot, \tau), A)|\tau T_m = \frac{1}{(k-1)!} (R_{2-2k}^{k-1} (C(H_{k-1}(\cdot, \tau), A)))|\tau T_m
= \frac{2\pi D^{k-\frac{1}{2}}}{(k-1)!} (R_{2-2k}^{k-1}(F_{1-k,A}))|\tau T_m
= \frac{2\pi D^{k-\frac{1}{2}}}{(k-1)!} m^{k-1} R_{2-2k}^{k-1} (F_{1-k,A}|\tau T_m).
\]
Since every coset in $\Gamma \backslash M_m(N)$ is represented by a matrix $M$ with $M \infty = i \infty$, we have for any $f \in S_{2k}(\Gamma)$
\[
E_f|T_m = m^{1-2k} E_f|\tau T_m \quad \text{and} \quad f^*|T_m = m^{1-2k} (f|T_m)^*.
\]
This implies
\[
F_{1-k,A}|T_m = P_{1-k,A}|T_m + m^{1-2k} (-1)^k \frac{D^{\frac{1}{2}-k}}{(2k-2)^{k-1}} (f_{k,A}|T_m)^* - m^{1-2k} (-1)^k D^{\frac{1}{2}-k} (k-1)!^2 \frac{1}{(4\pi)^{2k-1}} E_{f_k,A}|T_m.
\]
It follows from Lemma 6.1 that $\sum_{m>0} \lambda_m f_{k,A}|T_m = 0$, and therefore
\[
\sum_{m>0} \lambda_m C(f_{k,p}|T_m, A) = 2^{k-1|d|^{k-1}} D^{k-\frac{1}{2}} \sum_{m>0} \lambda_m m^{2k-1} \left( R_{2-2k}^{k-1} (P_{1-k,A}|T_m) \right)(\tau_p)
= 2^{k-1|d|^{k-1}} D^{k-\frac{1}{2}} \sum_{m>0} \lambda_m m^k \left( R_{2-2k}^{k-1} (P_{1-k,A}|T_m) \right)(\tau_p).
\]
The expression $R_{2-2k}^{k-1} (P_{1-k,A})$ can be rewritten using Lemma 5.1. We plug in the definition of $T_m$ and choose as a system of representatives for $\Gamma \backslash M_m(N)$ the matrices $\begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix}$ with $\alpha, \beta, \delta \in \mathbb{Z}, \alpha > 0, \alpha \delta = m$, and $\beta \pmod{\delta}$. This yields the formula in Theorem 2.3. Note that $\frac{\alpha \tau_p + \beta}{\delta}$ is a CM point of discriminant $\delta^2 d$. Hence, Lemma 5.2 implies that the expression
\[
|d|^{\frac{k-1}{2}} P_{k,A} \left( \frac{\alpha \tau_p + \beta}{\delta} \right)
\]
is rational. This finishes the proof of Theorem 2.3.
7 The Proof of Theorem 2.6

Throughout this section, we assume that \( N \) is odd and square-free. Furthermore, we let \( \Delta \) be a discriminant with \((-1)^k \Delta > 0\) and \( \delta \) a fundamental discriminant with \((-1)^k \delta < 0\) such that \( \delta \) is a square modulo \( 4N \). Finally, let \( F(\tau) = \sum_{m \geq -\infty} c_F(m)q^m \) be a weakly holomorphic modular form of weight \( \frac{3}{2} - k \) for \( \Gamma_0(4N) \) in the Kohnen plus space with rational coefficients \( c_F(m) \) for \( m < 0 \). We first show that the Fourier coefficients of the meromorphic modular form \((2.2)\) are algebraic multiples of \( \pi^{k-1} \).

**Proposition 7.1.** For \( k \geq 1 \) the meromorphic modular form

\[
\pi^{1-k|\delta|^{\frac{1}{2}}-k} \sum_{(-1)^k \Delta > 0} c_F(-|\Delta|)f_{k,\Delta,\delta}
\]

has rational Fourier coefficients.

For the proof, we write the coefficients of \( f_{k,\Delta,\delta} \) as linear combinations of coefficients of half-integral weight Maass Poincaré series.

**Lemma 7.2.** Let \( k \geq 1 \). For \( n \geq 1 \) we have

\[
c_{f_{k,\Delta,\delta}}(n) = -\left(-1\right)^{\left\lfloor \frac{k}{2} \right\rfloor} \frac{2^k \pi^{k-1}|\delta|^{k-\frac{1}{2}} n^{2k-1}}{(k-1)!} \sum_{m|n} \left( \frac{\delta}{m} \right) m^{-k} c_{P_{\frac{3}{2}-k,|\Delta|}}^{+} \left( \frac{n^2|\delta|}{m^2} \right)
\]

\[
+ \delta_{k=1} 12 \prod_{p|N} \left(1 - p^{-2}\right)^{-1} \sum_{d|N} \mu(d) \frac{dn}{N} \sum_{P \in \mathcal{Q}_{\Delta,\delta}/\Gamma} \frac{\chi_{\delta}(P)}{|P_p|}.
\]

**Proof.** This identity follows from a straightforward calculation using Proposition 3.4, Theorem 3.9 and Proposition 3.10. It could alternatively be derived from the fact that \( f_{k,\Delta,\delta} \) is a theta lift of \( P_{\frac{3}{2}-k,|\Delta|} \), compare [9, 29].

**Proof of Proposition 7.1.** Looking at the formula for \( c_{f_{k,\Delta,\delta}}(n) \) in Lemma 7.2, we see that the 2nd summand on the right-hand side is rational if \( k = \delta = 1 \) and vanishes otherwise. It remains to show that the coefficients of

\[
\sum_{(-1)^k \Delta > 0} c_F(-|\Delta|) \sum_{n \geq 1} n^{2k-1} \sum_{m|n} \left( \frac{\delta}{m} \right) m^{-k} c_{P_{\frac{3}{2}-k,|\Delta|}}^{+} \left( \frac{n^2|\delta|}{m^2} \right) q^n
\]

are rational.
Let $F$ be a weakly holomorphic modular form of weight $\frac{3}{2} - k$. Then, so is the function
\[ \tilde{F}(\tau) := \sum_{m < 0} c_F(m) P_{\frac{3}{2} - k, m}(\tau). \]

If $k > 1$, we have $F = \tilde{F}$ since there are no holomorphic modular forms of negative weight. In particular, since the space of weakly holomorphic modular forms of weight $\frac{3}{2} - k$ has a basis consisting of forms with rational coefficients and the principal part of $F$ is rational, we find that all coefficients of $F$ are rational for $k > 1$. However, for $k = 1$ the functions $F$ and $\tilde{F}$ may differ by a holomorphic modular form. Note that every $P_{\frac{3}{2} - k, m}$ is orthogonal to cusp forms with respect to the regularized Petersson inner product and has rational principal part. Hence, the same is true for $\tilde{F}$. It now follows from Proposition 3.2 in [13] that all Fourier coefficients of $\tilde{F}$ are rational. Now, we see that (7.1) equals
\[ \sum_{n \geq 1} n^{2k - 1} \sum_{m | n} \left( \frac{\delta}{m} \right)^{m - k} c_{\tilde{F}} \left( \frac{n^2 |\delta|}{m^2} \right) q^n, \]
which has rational Fourier coefficients. This finishes the proof.  

We now proceed to the proof of Theorem 2.6. For the rest of this section, we let $k = 1$ and $\delta > 0$ a fundamental discriminant that is a square modulo $4N$. We can assume without loss of generality that the coefficients $c_F(\Delta)$ for $\Delta < 0$ are integers. We consider the differential
\[ \eta_\delta(F) := \pi i \sum_{\Delta < 0} c_F(\Delta) f_{1, \Delta, \delta}(z) dz \]
on $X_0(N)$. For $P \in \mathbb{Q}_{\Delta \delta}$ we have
\[ \text{Res}_{z = \tau_P}(f_{1, \Delta, \delta}(z)) = \frac{\chi_\delta(P)}{\pi i}, \]
so $\eta_\delta(F)$ has simple poles with integral residues. In particular, $\eta_\delta(F)$ is a differential of the 3rd kind on $X_0(N)$.

Following [12], we define the twisted Heegner divisor
\[ Z_\delta(F) := \sum_{\Delta < 0} c_F(\Delta) Z_\delta(\Delta), \quad Z_\delta(\Delta) := \sum_{P \in \mathbb{Q}_{\Delta \delta} \setminus \Gamma} \frac{\chi_\delta(P)}{|\Gamma_P|}[\tau_P], \]
associated to $F$, and the corresponding degree 0 divisor
\[ y_\delta(F) := Z_\delta(F) - \text{deg}(Z_\delta(F)) \cdot [i\infty]. \]

By [12, Lemma 5.1], $y_\delta(F)$ is defined over $\mathbb{Q}(\sqrt{\delta})$. Note that $y_\delta(F)$ is precisely the residue divisor of $\eta_\delta(F)$ on $X_0(N)$. Moreover, we have the following result.
**Lemma 7.3.** The differential $\eta_\delta(F)$ is the canonical differential of the 3rd kind for $y_\delta(F)$, that is, the unique differential of the 3rd kind with residue divisor $y_\delta(F)$ such that

$$\Re\left(\int_\gamma \eta_\delta(F)\right) = 0$$

for all cycles $\gamma \in H_1(X_0(N) \setminus y_\delta(F), \mathbb{Z})$.

**Proof.** One can see from Theorem 4.1 that $F_{0,A}(\tau) \in \mathbb{R}$ for all $\tau \in \mathbb{H}$ not lying on any of the semi-circles $S_Q$ for $Q \in [A]$. It follows from Corollary 4.3 that

$$\Re\left(\int_\gamma \eta_\delta(F)\right) = 0$$

if $\gamma$ is any cycle of the form $c_A$ that does not meet any poles of $\eta_\delta(F)$. It is well known that the group $H_1(X_0(N) \setminus y_\delta(F), \mathbb{Z})$ is generated by these cycles, which yields the result. \[\square\]

The crucial ingredient for the proof of Theorem 2.6 is the following rationality result of Scholl [25] for differentials of the 3rd kind (see also Theorem 3.3 of [12]).

**Theorem 7.4 (Scholl).** Let $D$ be a divisor of degree 0 on $X_0(N)$ defined over a number field $F$. Let $\eta_D$ be the canonical differential of the 3rd kind associated to $D$ and write $\eta_D = 2\pi ifdz$. If all the Fourier coefficients of $f$ are contained in $F$, then some nonzero multiple of $D$ is a principal divisor.

It follows from Proposition 7.1 that the Fourier coefficients of $\frac{1}{2\pi i} \eta_\delta(F)$ are contained in $\mathbb{Q}(\sqrt{\delta})$, which is also the field of definition of the divisor $y_\delta(F)$. In particular, the above criterion of Scholl implies that some non-zero multiple of $y_\delta(F)$, say $m \cdot y_\delta(F)$ for some $m \in \mathbb{Z}$, is the divisor of a meromorphic function $g$ on $X_0(N)$.

Fix some point $z_0 \in \mathbb{H}$ which is not a pole of $\eta_\delta(F)$. For $z \in \mathbb{H}$ not being a pole of $\eta_\delta(F)$ we consider the function

$$\Psi_\delta(F,z) := \exp\left(m \int_{z_0}^{z} \eta_\delta(F)\right),$$

where the integral is over any path from $z_0$ to $z$ in $\mathbb{H}$ avoiding the poles of $\eta_\delta(F)$. Since the residues of $\eta_\delta(F)$ are integers, this does not depend on the choice of the path. Note that $\Psi_\delta(F,z)$ is meromorphic on $\mathbb{H}$ and has the same divisor as $g$. Thus, their quotient is constant on $\mathbb{H}$ and $\Psi_\delta(F,z)$ is $\Gamma'$-invariant.
For any $M \in \Gamma$, we have
\[ \Psi_{\delta}(F, Mz) = \exp \left( m \int_{z_0}^{Mz} \eta_{\delta}(F) \right) = \exp \left( m \int_{z_0}^{Mz_0} \eta_{\delta}(F) \right) \Psi_{\delta}(F, z), \]
so for $\Psi_{\delta}(F, z)$ to be $\Gamma$-invariant, the integral
\[ \frac{m}{2\pi i} \int_{z_0}^{Mz_0} \eta_{\delta}(F) = \frac{m}{2} \int_{z_0}^{Mz_0} \sum_{\Delta < 0} c_F(\Delta)f_{1,\Delta,\delta}(z)dz \]
has to be an integer. If we choose $z_0$ to lie on a geodesic $S_A$ and $M$ to be a generator of $\Gamma_A$, then this implies that the cycle integral of $\sum_{\Delta < 0} c_F(\Delta)f_{1,\Delta,\delta}$ along $A$ is a rational number. This finishes the proof of Theorem 2.6.

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