Minimal polygons with fixed lattice width

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Abstract

We classify the unimodular equivalence classes of inclusion-minimal polygons with a certain fixed lattice width. As a corollary, we find a sharp upper bound on the number of lattice points of these minimal polygons.

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1 Introduction and definitions

Let \( \Delta \subset \mathbb{R}^2 \) be a non-empty lattice polygon, i.e. the convex hull of a finite number of lattice points in \( \mathbb{Z}^2 \), and consider a lattice direction \( v \in \mathbb{Z}^2 \), i.e. a non-zero primitive vector. The lattice width of \( \Delta \) in the direction \( v \) is

\[
\text{lw}_v(\Delta) = \max_{P \in \Delta} \langle P, v \rangle - \min_{P \in \Delta} \langle P, v \rangle.
\]

The lattice width of \( \Delta \) is defined as \( \text{lw}(\Delta) = \min_v \text{lw}_v(\Delta) \). Throughout this paper we will assume that \( \Delta \) is two-dimensional, hence \( \text{lw}(\Delta) > 0 \). A lattice direction \( v \) that satisfies \( \text{lw}_v(\Delta) = \text{lw}(\Delta) \) is called a lattice width direction of \( \Delta \).

Two lattice polygons \( \Delta \) and \( \Delta' \) are called (unimodularly) equivalent if and only if there exists a unimodular transformation \( \varphi \), i.e. a map of the form

\[
\varphi : \mathbb{R}^2 \to \mathbb{R}^2 : x \mapsto Ax + b, \quad \text{where} \quad A \in \text{GL}_2(\mathbb{Z}), \; b \in \mathbb{Z}^2,
\]

such that \( \varphi(\Delta) = \Delta' \). Equivalent lattice polygons have the same lattice width.

The lattice width of a polygon can be seen as a specific instance of the more general notion of lattice size, which was introduced in [3].

Definition 1.1. Let \( X \subset \mathbb{R}^2 \) be a subset with positive Jordan measure. Then the lattice size \( \text{ls}_X(\Delta) \) of a non-empty lattice polygon \( \Delta \) is the smallest \( d \in \mathbb{Z}_{\geq 0} \) for which there exists a unimodular transformation \( \varphi \) such that \( \varphi(\Delta) \subset dX \).
Note that \( \text{lw}(\Delta) = \text{ls}_X(\Delta) \), where \( X = \mathbb{R} \times [0, 1] \).

This paper is concerned with polygons \( \Delta \) that are \textit{minimal} in the following sense: \( \text{lw}(\Delta') < \text{lw}(\Delta) \) for each lattice polygon \( \Delta' \subsetneq \Delta \). Equivalently, a two-dimensional polygon \( \Delta \) is minimal if and only if for each vertex \( P \) of \( \Delta \), we have that \( \text{lw}(\Delta_P) < \text{lw}(\Delta) \), where \( \Delta_P := \text{conv}(\Delta \cap \mathbb{Z}^2) \setminus \{P\} \).

Our main result is a complete classification of minimal polygons up to unimodular equivalence, see Theorem 2.4. As a corollary, we provide a sharp upper bound on the number of lattice points of these minimal polygons. First, we show in Lemma 2.3 that each minimal polygon \( \Delta \) satisfies \( \text{ls}_{\square}(\Delta) = \text{lw}(\Delta) \), where \( \square = \text{conv}\{(0,0), (1,0), (1,1), (0,1)\} \). The latter can also be proven using results on lattice width directions of interior lattice polygons (see [4, Lemma 5.3]), but we choose to keep the paper self-contained and have provided a different proof. Moreover, we use the technical Lemma 2.2 in the proofs of both Lemma 2.3 and Theorem 2.4.

In the joint paper [4] with Castryck and Demeyer, we study the Betti table of the toric surface \( \text{Tor}(\Delta) \subset \mathbb{P}(\Delta \cap \mathbb{Z}^2)^{-1} \) for lattice polygons \( \Delta \). In particular, we present a lower bound for the length of the linear strand of this Betti table in terms of \( \text{lw}(\Delta) \), which we conjecture to be sharp. For showing this conjecture for polygons of a fixed lattice width, it essentially suffices to prove it for the minimal polygons (see [4, Corollary 5.2]). Hence, Theorem 2.4 allows us to check the conjecture using a computer algebra system.

Remark 1.2. Of course, the question of classifying minimal polytopes can also be asked in higher dimensions. For instance, it can be shown that each three-dimensional minimal polytope \( \Delta \subset \mathbb{R}^3 \) with \( \text{lw}(\Delta) = 1 \) is equivalent to a tetrahedron of the form

\[
\text{conv}\{(0,0,0), (1,0,0), (0,1,0), (1,y,z)\}
\]

with \( 1 \leq y \leq z \) and \( \gcd(y,z) = 1 \). These include the Reeve tetrahedrons (where \( y = 1 \)). For comparison, there is only one minimal polygon with lattice width one up to equivalence, namely the standard simplex \( \text{conv}\{(0,0), (1,0), (0,1)\} \).

In all dimensions \( k \geq 2 \), among the minimal polytopes we will find back the so-called \textit{empty lattice simplices} \( \Delta \subset \mathbb{R}^k \), i.e. convex hulls of \( k + 1 \) lattice points without interior lattice points. If \( k \geq 4 \), not all empty lattice simplices have lattice width 1. For more information, see [1, 5, 7].

2 The classification of minimal polygons

Throughout this section, we will use the notations from Section 1. The following result appears already in [2, Remark following Lemma 5.2], but can be proven in a shorter way.
Lemma 2.1. Let $\Delta \subset \mathbb{R}^2$ be a lattice polygon with $\text{lw}(\Delta) = d$. If $\Delta$ has two linearly independent lattice width directions $v, w \in \mathbb{Z}^2$, then $\text{ls}(\Delta) = d$.

Proof. If $v$ and $w$ do not form a $\mathbb{Z}$-basis of $\mathbb{Z}^2$, we take a primitive vector $u \in \text{conv}\{(0,0), v, w\}$ such that $v$ and $u$ form a $\mathbb{Z}$-basis. Let $Q, Q'$ be lattice points of $\Delta$ such that $\langle Q', u \rangle - \langle Q, u \rangle = \text{lw}_u(\Delta)$. Write $u = \lambda v + \mu w$ with $0 < \lambda, \mu$ and $\lambda + \mu \leq 1$. Now

$$d \leq \text{lw}_u(\Delta) = \langle Q', (\lambda v + \mu w) \rangle - \langle Q, (\lambda v + \mu w) \rangle \leq |\lambda| \text{lw}_v(\Delta) + |\mu| \text{lw}_w(\Delta) \leq d,$$

so $\text{lw}_u(\Delta) = d$. After applying a unimodular transformation, we may assume that $u = (0,1)$ and $v = (1,0)$, and that $\Delta$ fits into $d\square$, hence $\text{ls}(\Delta) = d$. \hfill $\square$

Lemma 2.2. Let $\Delta$ be a lattice polygon with $\text{lw}(\Delta) = d > 0$. Let $P$ be a vertex of $\Delta$ and $v \in \mathbb{Z}^2$ be a primitive vector. If $\text{lw}_v(\Delta_P) < d$ and $\text{lw}_v(\Delta_P) < \text{lw}_v(\Delta) - 1$, then $\Delta$ is equivalent to $\Upsilon_{d-1} := \text{conv}\{(0,0), (1,d), (d,1)\}$.

Proof. Since $\text{lw}_v(\Delta_P) < \text{lw}_v(\Delta) - 1$, we have that either

$$\min_{Q \in \Delta_P} \langle v, Q \rangle > \langle v, P \rangle + 1 \quad \text{or} \quad \max_{Q \in \Delta_P} \langle v, Q \rangle < \langle v, P \rangle - 1.$$ 

By replacing $v$ by $-v$, we may assume that we are in the first case. Moreover, we may choose $v$ such that the difference $\min_{Q \in \Delta_P} \langle v, Q \rangle - \langle v, P \rangle$ is minimal but greater than 1, and such that $\text{lw}_v(\Delta_P) < d$.

We apply a unimodular transformation so that $P = (0,0)$ and $v = (0,1)$. Let $y_m$ (resp. $y_M$) be the smallest (resp. greatest) $y$-coordinate occurring in $\Delta_P$. Note that $y_m = \min_{Q \in \Delta_P} \langle v, Q \rangle$ and $y_M = \max_{Q \in \Delta_P} \langle v, Q \rangle$, hence $y_m > 1$ and $y_M - y_m < d$.

Define the cone $C_k := \{\lambda(k,1) + \mu(k+1,1)|\lambda, \mu \geq 0\}$. Since

$$\Delta \subset (\mathbb{R} \times \mathbb{R}_{>0}) \cup \{P\} = \bigcup_{k \in \mathbb{Z}} C_k$$

and $y_m > 1$, the polygon $\Delta$ is contained in a cone $C_k$ for some $k \in \mathbb{Z}$. Using the unimodular transformation $(x, y) \mapsto (x - ky, y)$, we may assume that $k = 0$, i.e. $\Delta \subseteq C_0 = \{\lambda(0,1) + \mu(1,1)|\lambda, \mu \geq 0\}$. In fact, we then have that

$$\Delta \subseteq \text{conv}\{(0,0), (1,y_M), (y_M-1,y_M)\}.$$ 

If $y_m = 2$, we have $y_M = (y_M - y_m) + 2 \leq d+1$. The strict inequality $y_M < d+1$ is impossible as the horizontal width $\text{lw}_{(1,0)}(\Delta)$ would be less than $d$. So we have that $y_M = d+1$ and $\Delta \subseteq \Delta' = \text{conv}\{(0,0), (1,d+1), (d,d+1)\}$. Since $\text{lw}(\Delta') < d$ for $Q \in \{(1,d+1), (d,d+1)\}$, we must have $\Delta = \Delta'$. This is equivalent to $\Upsilon_{d-1}$ via $(x, y) \mapsto (x, y-x)$.

From now on, assume that $y_m > 2$. Then $(1,2) \notin \Delta$ which means that either

$$\Delta \subseteq \{\lambda(0,1) + \mu(1,2)|\lambda, \mu \geq 0\} \quad \text{or} \quad \Delta \subseteq \{\lambda(1,2) + \mu(1,1)|\lambda, \mu \geq 0\}.$$
We can reduce to the latter case using the transformation \((x, y) \mapsto (y - x, y)\). In fact, we can keep subdividing this cone until we find a cone \(C\) containing \(\Delta\) that does not contain any lattice point with \(y\)-coordinate in \(\{1, \ldots, y_m - 1\}\). Let \(\ell \in \mathbb{Z}\) be such that \(C\) passes in between \((\ell - 1, y_m - 1)\) and \((\ell, y_m - 1)\). Then

\[
\Delta \subseteq \text{conv}\{(\ell, y_m - 1), (\ell, y_M), (\ell + y_M - y_m + 1, y_M)\}.
\]

If \(x_m\) (resp. \(x_M\)) is the smallest (resp. greatest) \(x\)-coordinate occurring in a lattice point of \(\Delta\), then \(2 \leq \ell \leq x_m < y_m\) and \(x_M \leq \ell + y_M - y_m\), so \(x_M - x_m \leq y_M - y_m < d\). But this means that \(\text{lw}(\Delta) = d > 0\), then \(\text{lsw}(\Delta) = 1\) and

\[
1 < \min_{Q \in \Delta} \langle (1, 0), Q \rangle < y_m = \min_{Q \in \Delta} \langle v, Q \rangle
\]

contradicting the minimality of \(v\). \(\square\)

**Lemma 2.3.** If \(\Delta \subset \mathbb{R}^2\) is a non-empty minimal lattice polygon with \(\text{lw}(\Delta) = d > 0\), then \(\text{lsw}(\Delta) = d\).

**Proof.** By Lemma 2.1, we only have to show that there are two linearly independent lattice width directions. Suppose that \(v\) is a lattice width direction and that \(Q, Q' \in \Delta \cap \mathbb{Z}^2\) such that \(\langle Q, v \rangle - \langle Q', v \rangle = d\). Now let \(P\) be a vertex of \(\Delta\) different from \(Q, Q'\). By minimality of \(\Delta\), we have that \(\text{lw}(\Delta) = d\). That means there exists a direction \(w\) such that \(\text{lw}_w(\Delta) < d\). Because \(Q\) and \(Q'\) are still in \(\Delta\), \(w\) cannot be \(v\) or \(-v\), so \(w\) must be linearly independent of \(v\). If \(\text{lw}_w(\Delta) = 1\), we are done. If \(\text{lw}_w(\Delta) > d\), then by Lemma 2.2 \(\Delta\) is equivalent to \(\Upsilon_{d-1} \subset d\). \(\square\)

**Theorem 2.4.** Let \(\Delta \subset \mathbb{R}^2\) be a non-empty minimal lattice polygon with \(\text{lw}(\Delta) = d\). Then \(\Delta\) is equivalent to a minimal polygon of one of the following forms:

\((T1)\) \(\text{conv}\{(0, 0), (d, y), (x, d)\}\) where \(x, y \in \{0, \ldots, d\}\) satisfy \(x + y \leq d\);

\((T2)\) \(\text{conv}\{(x_1, 0), (d, y_2), (x_2, d), (0, y_1)\}\) where \(x_1, x_2, y_1, y_2 \in \{1, \ldots, d - 1\}\) satisfy \(\max(x_2, y_2) \geq \min(x_1, y_1)\) and \(\max(d - x_2, y_1) \geq \min(d - x_1, y_2)\);

\((T3)\) \(\text{conv}\{(0, 0), (\ell, 0), (d, y + d - \ell), (x + \ell, d), (z, z + d - \ell)\}\) with \(\ell \in \{2, \ldots, d - 2\}\), \(x \in \{1, \ldots, d - \ell - 1\}\), \(y, z \in \{1, \ldots, \ell - 1\}\);

\((T4)\) \(\text{conv}\{(0, 0), (z' + \ell, z'), (d, y + d - \ell), (x + \ell, d), (z, z + d - \ell)\}\) with \(\ell \in \{2, \ldots, d - 2\}\), \(y, z \in \{1, \ldots, \ell - 1\}\), \(x, z' \in \{1, \ldots, d - \ell - 1\}\);

\((T5)\) \(\text{conv}\{(x_1, 0), (z_2 + \ell, z_2), (d, d - \ell + y_2), (x_2 + \ell, d), (z_1, z_1 + d - \ell), (0, y_1)\}\) with \(\ell \in \{2, \ldots, d - 2\}\), \(x_1, y_2, z_1 \in \{1, \ldots, \ell - 1\}\), \(x_2, y_1, z_2 \in \{1, \ldots, d - \ell - 1\}\);
Remark 2.5. See Figure 1 for a picture of the five types. The minimal polygons appearing in the types (T3), (T4) and (T5) are inscribed in the hexagon

\[ H_I := \text{conv}\{(0,0), (\ell,0), (d,d-\ell), (d,d), (\ell,d), (0,d-\ell)\}. \]

This is also the case for the triangles of type (T1) with \((x,y) \in \{(d,0), (0,d)\}\) (where we allow \(\ell \in \{0,d\}\)) and for the quadrangles of type (T2) with \(\max(d-x_2,y_1) = \min(d-x_1,y_2)\).

\[ \text{Figure 1: The five types in the classification} \]

Proof. If \(d = 0\), then \(\Delta\) consists of a single point and it is of shape \((T1)\). So assume \(d \geq 1\). Because of Lemma 2.3, we may assume that \(\Delta \subset d\mathcal{D} = [0,d] \times [0,d]\). Moreover, we may assume that \(\Delta \not\sim \Upsilon_{d-1}\) since \(\Upsilon_{d-1}\) is of type \((T1)\). Let \(P\) be any vertex of \(\Delta\). By Lemma 2.2, if \(\text{lw}_v(\Delta_P) < d\) for some primitive vector \(v \in \mathbb{Z}^2\), then \(\text{lw}_v(\Delta_P) = d-1\) and \(\text{lw}_v(\Delta) = d\), hence \(v\) is a lattice width direction.

By minimality, we know that there always exists a lattice direction \(v\) satisfying \(\text{lw}_v(\Delta_P) < d\). We claim that we can always take \(v \in \{(0,1), (1,0), (1,1), (1,-1)\}\). Indeed, suppose that \(v = (v_x,v_y) \in \mathbb{Z}^2\) satisfies

\[ \{v,-v\} \cap \{(0,1),(1,0),(1,1),(1,-1)\} = \emptyset \quad \text{and} \quad \text{lw}_v(\Delta_P) < d. \]

After a unimodular transformation, we may assume that \(0 < v_x < v_y\), hence \((1,1) \in \text{conv}\{(0,0),(1,0),v\}\). Using a similar trick as in Lemma 2.1, we get that \(\text{lw}_{(1,1)}(\Delta_P) < d\), which proves the claim.
Let $\mathcal{V}$ be set consisting of vectors $v \in \{(1,1), (1, -1)\}$ for which there exists a vertex $P$ of $\Delta$ with $\text{lwr}(\Delta P) < d$. If $\mathcal{V} = \{(1,1), (1, -1)\}$, then $\Delta$ has 4 different lattice width directions, namely $(1,0), (0,1), (1,1)$ and $(1,-1)$. By [2, Lemma 5.2(v)] or [5], this means that $\Delta \cong \text{conv}\{(d/2,0),(0,d/2),(d/2,d),(d,d/2)\}$ for some even $d$, hence it is of type $(T2)$. If $\mathcal{V} = \emptyset$, we claim that $\Delta$ is of type $(T1)$ or $(T2)$. Indeed, for every vertex $P$ of $\Delta$, we have that either $\text{lwr}_{(1,0)}(\Delta P)$ or $\text{lwr}_{(0,1)}(\Delta P)$ is smaller than $d$. In particular, this means that there has to be a side of $d\Box$ with $P$ as its only point in $\Delta$. One then easily checks the claim: if $\Delta$ is a triangle, then it will be of type $(T1)$; if it is a quadrangle, then it is of type $(T2)$.

From now on, suppose that $\mathcal{V}$ is not equal to $\emptyset$ or $\{(1,1), (1,-1)\}$, hence $\mathcal{V} = \{(1,1)\}$ or $\mathcal{V} = \{(1,-1)\}$. We can suppose that $\mathcal{V} = \{(1,1)\}$ by using the transformation $(x,y) \mapsto (x,-y)$ if necessary. Hence, for each vertex $P$ of $\Delta$, there is a vector $v \in \{(1,0), (0,1), (1,1)\}$ with $\text{lwr}(\Delta P) < d$. Since $\text{lwr}_{(1,1)}(\Delta) = d$, there exists an integer $\ell \in \{0,\ldots,d\}$ such that $\langle Q, (1,-1) \rangle \in \ell - d, \ell \rangle$ for all $Q \in \Delta$. If $\ell \in \{0,d\}$, then $\Delta$ is a triangle whose vertices are vertices of $d\Box$, so it is of the form $(T1)$. Now assume that $\ell \in \{1,\ldots,d-1\}$, hence $\Delta$ is contained in the hexagon $H_\ell$ from Remark 2.5. Each side of $H_\ell$ contains at least one lattice point of $\Delta$, and if it contains more than one point, it is also an edge of $\Delta$. Otherwise, there would be a vertex $P$ lying on exactly one side of $H_\ell$, while not being the only point of $\Delta$ on that side of $H_\ell$. But then there is no $v \in \{(0,1), (1,0), (1,-1)\}$ with $\text{lwr}(\Delta P) < d$ (as every side of $H_\ell$ contains a point of $\Delta P$), a contradiction.

Denote by $\mathcal{S}$ the set of sides that $\Delta$ and $H_\ell$ have in common. Then $\mathcal{S}$ cannot contain two adjacent sides $S_1, S_2$: otherwise for the vertex $P = S_1 \cap S_2$, each side of $H_\ell$ would have a non-empty intersection with $\Delta P$, contradicting the fact that there is a $v \in \{(0,1), (1,0), (1,-1)\}$ with $\text{lwr}(\Delta P) < d$.

Assume that $|\mathcal{S}| \geq 2$ and take $S_1 = [Q_1, Q_2] \in \mathcal{S}$. Its adjacent sides of $H_\ell$ contain no points of $\Delta$ except from $Q_1$ and $Q_2$. This implies that $\mathcal{S} = \{S_1, S_2\}$ where $S_1, S_2$ are opposite edges of $H_\ell$, and that $\Delta$ is the convex hull of these two edges. Hence $\Delta$ is equivalent to the quadrangle $\text{conv}\{(\ell,0), (d,d-\ell), (\ell,d), (0,d-\ell)\} \subset H_\ell$, which is of type $(T2)$.

If $\mathcal{S}$ consists of a single side $S$, we may assume that $S = [Q_1, Q_2]$ is the bottom edge of $H_\ell$. Let $P_1$ (resp. $P_2$) be the vertex of $\Delta$ on the upper left diagonal side (resp. the right vertical edge) of $H_\ell$. If $P_1$ is also on the top edge of $H_\ell$ (i.e. $P_1 = (\ell,d)$), then $\Delta$ has only four vertices, namely $Q_1, Q_2, P_1, P_2$. Applying the transformation $(x,y) \mapsto (x,-x+y+\ell)$, we end up with a quadrangle of type $(T2)$. By a similar reasoning, if $P_2$ is on the top edge of $H_\ell$ (i.e. $P_2 = (d,d)$), we end up with type $(T2)$. If neither $P_1$ nor $P_2$ are on the top edge of $H_\ell$, then there is a fifth vertex $P_3$ on that top edge, and we are in case $(T3)$.

The only remaining case is when $\mathcal{S} = \emptyset$, hence each edge of $H_\ell$ contains only one point of $\Delta$. If $H_\ell$ and $\Delta$ have no common vertex, then $\Delta$ is of type $(T5)$. If they share one vertex, we can reduce to type $(T4)$ using a transformation if necessary.
Note that two common vertices of $H_\ell$ and $\Delta$ can never be connected by an edge of $H_\ell$ as that edge would be in $S$, so there are at most three common vertices. If there are three shared vertices, then $\Delta$ is a triangle of type $(T1)$, again using a transformation if necessary. So assume $H_\ell$ and $\Delta$ share two vertices. Together these two points occupy four edges of $H_\ell$ and each of the other two edges of $H_\ell$ (call them $A$ and $B$) contains exactly one vertex of $\Delta$. Take two pairs of opposite sides of $H_\ell$ (so four sides in total) that together contain $A$ and $B$, then they contain all vertices of $\Delta$: since any common vertex of $H_\ell$ and $\Delta$ lies on two sides of $H_\ell$, they cannot lie both on the sides we didn’t choose, as they are parallel. We can find a unimodular transformation mapping these sides into the four sides of $d\Box$, hence $\Delta$ is of type $(T2)$.

**Remark 2.6.** From the classification in Theorem 2.4 one can easily deduce the following result from [8]: $\text{vol}(\Delta) \geq \frac{3}{8} \text{lw}(\Delta)^2$ for each lattice polygon $\Delta \subset \mathbb{R}^2$, and equality holds for minimal polygons of type $(T1)$ with $d$ even and $x = y = \frac{d}{2}$. For odd $d$, this inequality can be sharpened to $\text{vol}(\Delta) \geq \frac{3}{8} \text{lw}(\Delta)^2 + \frac{1}{8}$, and equality holds for minimal polygons of type $(T1)$ with $x = \frac{d-1}{2}$ and $y = \frac{d+1}{2}$.

**Corollary 2.7.** If $\Delta \subset \mathbb{R}^2$ is a non-empty minimal lattice polygon with $\text{lw}(\Delta) = d > 1$, then

$$\sharp(\Delta \cap \mathbb{Z}^2) \leq \max((d-1)^2 + 4, (d+1)(d+2)/2).$$

Moreover, this bound is sharp.

**Proof.** Note that there exist minimal polygons attaining the bound (see Figure 2): the simplex $\text{conv}\{(0,0), (d,0), (0,d)\}$ is of type $(T1)$ and has $(d+1)(d+2)/2$ lattice points, and the quadrangle $\text{conv}\{(1,0), (d,1), (d-1,d), (0,d-1)\}$ is of type $(T2)$ and has $(d-1)^2 + 4$ lattice points.

![Figure 2: Minimal polygons attaining the upper bound](image)

Now let’s show that we indeed have an upper bound. If $\Delta$ is minimal of type $(T2)$, $(T4)$ or $(T5)$, then $\sharp(\Delta \cap \mathbb{Z}^2) \leq (d-1)^2 + 4$, since there are at most 4 lattice points of $\Delta$ on the boundary of $d\Box$ and all the others are in

$$(d\Box)^{\circ} \cap \mathbb{Z}^2 = \{1, \ldots, d-1\} \times \{1, \ldots, d-1\}.$$
This also holds for triangles of type $(T1)$ with $x$ and $y$ non-zero. If $\Delta$ is of type $(T3)$, we obtain the same upper bound $(d-1)^2 + 4$ after applying a unimodular transformation that maps the bottom edge of $\Delta$ to the left upper diagonal edge of $H_\ell$. We are left with triangles of type $(T1)$ with either $x$ or $y$ zero. Assume that $y = 0$ (the case $x = 0$ is similar). Then $\Delta$ has the edge $[(0,0),(d,0)]$ in common with $d\Box$ and its other vertex is $(x,d)$. For each $k \in \{0,\ldots,d\}$, the intersection of $\Delta$ with the horizontal line on height $k$ is a line segment of length $d-k$, hence it contains at most $d-k+1$ lattice points. So in total, $\Delta$ has at most
\[
\sum_{k=0}^{d} (d-k+1) = (d+1)(d+2)/2
\]
lattice points.

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