Abstract
A generating function of the single Hurwitz numbers of the Riemann sphere \( \mathbb{C}P^1 \) is a tau function of the lattice KP hierarchy. The associated Lax operator \( L \) turns out to be expressed as \( L = e^{s} \), where \( L \) is a difference-differential operator of the form \( L = \partial s - ve^{-\partial t} \). \( L \) satisfies a set of Lax equations that form a continuum version of the Bogoyavlensky–Itoh (aka hungry Lotka–Volterra) hierarchies. Emergence of this underlying integrable structure is further explained in the language of generalized string equations for the Lax and Orlov–Schulman operators of the 2D Toda hierarchy. This leads to logarithmic string equations, which are confirmed with the help of a factorization problem of operators.

Keywords: Hurwitz numbers, Toda hierarchy, Volterra lattice, Bogoyavlensky–Itoh equation, string equation, factorization problem

1. Introduction
The Hurwitz numbers of the Riemann sphere \( \mathbb{C}P^1 \) count inequivalent finite ramified coverings of \( \mathbb{C}P^1 \) by compact Riemann surfaces. Okounkov considered a particular set of Hurwitz numbers, called the double Hurwitz numbers, and observed that a generating function of the double Hurwitz numbers is a tau function of the 2D Toda hierarchy [1]. This generating function can be specialized to a generating function of the single Hurwitz numbers. The specialized generating function becomes a tau function of the KP hierarchy [2–4]. The integrable structures of the double Hurwitz numbers are further studied from various aspects [5–9].

We now reconsider the single Hurwitz numbers, and point out that a more fundamental integrable hierarchy is hidden therein. This integrable hierarchy is a large-\( p \) (or continuum) limit [10, 11] of the \( p \)-step Bogoyavlensky–Itoh (aka hungry Lotka–Volterra) hierarchy [12, 13]. The lowest equation of this hierarchy is a 2D Toda-like field equation. Okounkov and Pandharipande remarked, in the course of studies on the Gromov–Witten theory of \( \mathbb{C}P^1 \), that
a generating function of the single Hurwitz numbers satisfies this Toda-like equation [14]. We show how this equation emerges in the Lax formalism of the 2D Toda hierarchy.

The ‘continuum’ version of the Bogoyavlensky–Ioh hierarchies has a Lax operator of the somewhat unusual form \( \mathfrak{L} = \partial_s - v e^{-\theta} \), where \( v \) is a function of \( s \) and time variables. This is a linear combination of the differential operator \( \partial_s \) and the shift operator \( e^{-\theta} \). As our derivation shows, its exponential \( e^\mathfrak{L} \) coincides with the first Lax operator \( L \) of the 2D Toda hierarchy. In other words, \( \mathfrak{L} \) is equal to \( \log L \). Emergence of the logarithm of a Lax operator is not a quite new circumstance. Such operators are used for the construction of variants of the Toda lattice [15, 16].

The second half of this paper is devoted to a more systematic explanation of these observations in the language of \textit{generalized string equations}. Generalized string equations are algebraic equations satisfied by the Lax and Orlov–Schulman operators \( L, L, M, M \) of the 2D Toda hierarchy [17]. Generalized string equations for the tau function of the double Hurwitz numbers are presented in our previous work [6]. We show that those equations can be converted to a logarithmic form that contains \( \log L \) and \( \log \bar{L} \) rather than \( L \) and \( \bar{L} \). The Lax operator \( \mathfrak{L} \) of the continuous Bogoyavlensky–Ioh hierarchy can be readily derived from these \textit{logarithmic string equations}.

2. Generating functions of Hurwitz numbers

The (disconnected) Hurwitz numbers \( H_d(\mu^{(1)}, \ldots, \mu^{(r)}) \) of \( \mathbb{C}P^1 \) are defined by the sums

\[
H_d(\mu^{(1)}, \ldots, \mu^{(r)}) = \sum_{[\pi]} \frac{1}{|\text{Aut}(\pi)|}
\]

over all topological types \([\pi]\) of \( d \)-fold coverings \( \pi : C \rightarrow \mathbb{C}P^1 \) of \( \mathbb{C}P^1 \) by possibly disconnected compact Riemann surfaces \( C \) with the ramification profile \((\mu^{(1)}, \ldots, \mu^{(r)})\). The coverings are assumed to be ramified over \( r \) points, say, \( P_1, \ldots, P_r \), of \( \mathbb{C}P^1 \). \( \mu^{(j)} \) are partitions of \( d \), i.e., \( \mu^{(j)} = (\mu_1^{(j)}, \mu_2^{(j)}, \ldots) \), \( |\mu^{(j)}| = \sum_i \mu_i^{(j)} = d \), and the \( i \)th part of \( \mu_i^{(j)} \) of \( \mu^{(j)} \) represent the order of cyclic ramification at the \( i \)th point of \( \pi^{-1}(P_j) \). \(|\text{Aut}(\pi)|\) denotes the number of covering automorphisms of \( \pi \).

Okounkov’s ‘double Hurwitz numbers’ are Hurwitz numbers of the form \( H_d(\mu, \mu, 1^{d-2}, \ldots, 1^{d-2}) \), where \( \mu \) and \( \bar{\mu} \) are arbitrary partitions of \( d \) and \( 1^{d-2} \) denotes the partition \((2, 1, \ldots, 1)\). Let us use two sets of variables \( x = (x_1, x_2, \ldots) \), \( \bar{x} = (x_1, x_2, \ldots) \) and two parameters \( \beta, Q \) to construct the following generating function of the double Hurwitz numbers:

\[
z(x, \bar{x}) = \sum_{\beta} \sum_{Q} \sum_{(\mu, \bar{\mu})} H_d(\mu, \bar{\mu}, 1^{d-2}, \ldots, 1^{d-2}) \frac{\beta^{|\mu|}}{r!} Q^r P_{\mu} P_{\bar{\mu}}.
\]

(1)

\( P_{\mu}, \quad \mu = (\mu_1, \mu_2, \ldots) \), and \( P_{\bar{\mu}}, \quad \bar{\mu} = (\bar{\mu}_1, \bar{\mu}_2, \ldots) \), are the products \( P_{\mu} = P_{\mu_1} P_{\mu_2} \cdots, \)

\( P_{\bar{\mu}} = \bar{P}_{\mu_1} \bar{P}_{\mu_2} \cdots \) of the power sums \( P_k = \sum_i x_i^k \), \( \bar{P}_k = \sum_i \bar{x}_i^k \). As pointed out by Okounkov [1], one can rewrite this generating function as

\[
z(x, \bar{x}) = \sum_{\lambda \in \mathcal{P}} e^{\beta \kappa(\lambda)/2} Q^{\lambda/2} s_\lambda(x) s_\lambda(\bar{x}),
\]

(2)

where \( \mathcal{P} \) denotes the set of all partitions \( \lambda = (\lambda_1, \lambda_2, \ldots) \) of arbitrary length, \( \kappa(\lambda) \) is defined as
\[ \kappa(\lambda) = \sum_{i \geq 1} \lambda_i (\lambda_i - 2i + 1), \]

and \( s_{\lambda}(x) \) and \( s_{\lambda}(\bar{x}) \) are the Schur functions in the sense of Macdonald’s book [18].

\[ z(x, \bar{x}) \] corresponds to the tau function

\[ Z(t, \bar{t}) = \sum_{\lambda \in \mathcal{P}} e^{\beta \kappa(\lambda)/2} Q^{\lambda} S_{\lambda}(t) S_{\lambda}(-\bar{t}) \]  

of the 2-component KP hierarchy with time variables \( t = (t_1, t_2, \ldots) \) and \( \bar{t} = (\bar{t}_1, \bar{t}_2, \ldots) \) by the transformations

\[ t_k = p_k/k, \quad \bar{t}_k = -p_k/k \]

of the variables [1]. \( S_{\lambda}(t) \)'s are defined by the determinant formula

\[ S_{\lambda}(t) = \det (S_{\lambda_{i+j}}(t))_{i,j=1}^N, \]

where \( N \) is chosen to be greater than or equal to the length of \( \lambda \). \( S_n(t) \)'s are defined by the generating function

\[ \sum_{n=0}^{\infty} S_n(t) z^n = \exp \left( \sum_{k=1}^{\infty} t_k z^k \right). \]

Specializing \( Z(t, \bar{t}) \) to \( \bar{t} = (-c, 0, 0, \ldots) \) yields a generating function of the ‘single Hurwitz numbers’ \( H_d(\mu, 1^{d-2}, \ldots, 1^{d-2}) \) [1]. Note that \( S_{\lambda}(-\bar{t}) \) thereby turns into the special value

\[ S_{\lambda}(c, 0, 0, \ldots) = \frac{\dim \lambda}{|\lambda|!} c^{\lambda}, \]

where \( \dim \lambda \) denotes the dimension of the irreducible representation of the symmetric group determined by \( \lambda \). In the following, the one-dimensional subspace \( \bar{t} = (-c, 0, 0, \ldots) \) of the \( t \)-flows is referred to as the single Hurwitz sector.

\[ Z(t, \bar{t}) \] can be extended to depend on a discrete variable \( s \in \mathbb{Z} \) as

\[ Z(s, t, \bar{t}) = \sum_{\lambda \in \mathcal{P}} e^{\beta (\kappa(\lambda) + 2s|\lambda| + (4s^3 - s)/12) / 2} Q^{\lambda} S_{\lambda(1)}(t) S_{\lambda(-\bar{t})}. \]

The deformations

\[ \kappa(\lambda) \to \kappa(\lambda) + 2s|\lambda| + (4s^3 - s)/12, \quad |\lambda| \to |\lambda| + s(s + 1)/2 \]

of \( \kappa(\lambda) \) and \( |\lambda| \) stem from the matrix elements of operators \( K \) and \( L_0 \) in a fermionic Fock space. \( Z(s, t, \bar{t}) \) thereby turns out to be a tau function of the 2D Toda hierarchy [1, 6]. Its specialization \( Z(s, t, -c, 0, 0, \ldots) \) to the single Hurwitz sector is a tau function of the lattice KP (aka modified KP) hierarchy.

3. Lax equations in single Hurwitz sector

Our consideration is now focussed on the single Hurwitz sector \( t = (\bar{t}_1, 0, 0, \ldots) \). It is convenient to reorganize the expression (5) of \( Z(s, t, \bar{t}) \) as

\[ \text{This generating function is slightly modified from our previous definition [6].} \]
\[ Z(s, t, \bar{t}) = e^{\beta(4s^2 - t)/4} \frac{Q(s)}{Q(t+1)/2} \tilde{Z}(s, t, \bar{t}), \]  

(6)

where

\[ \tilde{Z}(s, t, \bar{t}) = \sum_{\lambda \in \mathbb{P}} e^{\beta \lambda(\lambda)/2} (Qe^{\beta S}) |\lambda| S_{\lambda}(t) S_{\lambda}(-\bar{t}). \]

By the formula (4) of the special value of the Schur functions, \( \tilde{Z}(s, t, \bar{t}) \) in the single Hurwitz sector can be expressed as

\[ \tilde{Z}(s, t, \bar{t}, 0, 0, \ldots) = \sum_{\lambda \in \mathbb{P}} \dim \frac{\beta}{|\lambda|!} e^{\beta \lambda(\lambda)/2} (-Qe^{\beta S_{\lambda}}) |\lambda| S_{\lambda}(t). \]  

(7)

Let \( Z(s, t, \bar{t}) \) and \( \tilde{Z}(s, t, \bar{t}) \) denote these specializations of \( Z(s, t, \bar{t}) \) and \( \tilde{Z}(s, t, \bar{t}) \).

The foregoing expression of \( Z(s, t, \bar{t}) \) and its specialization \( Z(s, t, \bar{t}) \) suggests the extension of the range of \( s \) from \( \mathbb{Z} \) to \( \mathbb{R} \). In such an interpretation, \( Z(s, t, \bar{t}) \) satisfies the differential equation

\[ \frac{\partial \tilde{Z}(s, t, \bar{t})}{\partial s} = \beta t_1 \frac{\partial \tilde{Z}(s, t, \bar{t})}{\partial \bar{t}} \]  

(8)

because this function depends on \( s \) and \( t_1 \) in such a form as \( e^{\beta t_1} \). Let us consider implications of this fact.

Let \( \Psi(s, t, \bar{t}, z) \) denotes the Baker–Akhiezer function

\[ \Psi(s, t, \bar{t}, z) = \frac{Z(s-1, t - [z^{-1}], \bar{t})}{Z(s-1, t, \bar{t})} e^{\xi(t, z)}, \]

\[ [x] = (x, x^2/2, \ldots, x^k/k, \ldots), \quad \xi(t, z) = \sum_{k=1}^{\infty} t_k z^k. \]

This function satisfies the auxiliary linear equations

\[ (\partial_k - B_k) \Psi = 0, \quad k = 1, 2, \ldots, \quad (\partial_{t_1} - \bar{u}_0 e^{-\partial}) \Psi = 0 \]  

(9)

of the positive and first negative flows in the 2D Toda hierarchy. \( B_k \)'s are difference operators of the form

\[ B_k = (t^k)_\geq 0 = e^{k\partial} + b_{k1} e^{(k-1)\partial} + \cdots + b_{kk}. \]

\( L \) is the first Lax operator

\[ L = e^{\partial} + u_1 + u_2 e^{-\partial} + \cdots \]

of the 2D Toda hierarchy, and \( (\ldots )_\geq 0 \) means extracting the non-negative powers of \( e^{\partial} \). \( \bar{u}_0 \) is the function

\[ \bar{u}_0 = \frac{Z(s, t, \bar{t})Z(s - 2, t, \bar{t})}{Z(s-1, t, \bar{t})^2} \]

that arises in the leading part of the second Lax operator \( \bar{L} \) as

\[ \bar{L}^{-1} = \bar{u}_0 e^{-\partial} + \bar{u}_1 + \bar{u}_2 e^{\partial} + \cdots. \]

The equation with respect to \( \bar{t}_1 \) in (9) can be converted to a bilinear differential equation for \( Z(s, t, \bar{t}) \). One can rewrite \( \bar{t}_1 \)-derivatives therein to \( s \)-derivatives with the aid of (8). After some algebra, this bilinear differential equation turns into the linear equation.
\[(\partial_s - \beta \bar{t}_1 \bar{u}_0 e^{-\beta_1}) \Psi = (\log z) \Psi \quad (10)\]

for \(\Psi(s, t, \bar{t}_1)\).

We thus encounter a new Lax operator of the form
\[
\mathcal{L} = \partial_s - v e^{-\beta_1}, \quad v = \beta \bar{t}_1 \bar{u}_0. \quad (11)
\]

As a consequence of (10) and the other equations of (9), \(\mathcal{L}\) satisfies the Lax equations
\[
\frac{\partial \mathcal{L}}{\partial t_k} = [B_k, \mathcal{L}], \quad k = 1, 2, \ldots \quad (12)
\]

Moreover, since (10) implies the exponentiated equation \(e^\mathcal{L} \Psi = z \Psi\) and \(L\) satisfies the equation \(L \Psi = z \Psi\), one can conclude that
\[
e^\mathcal{L} = L. \quad (13)
\]

The reduced Lax operator \(\mathcal{L}\) thus turns out to be the logarithm of \(L\). \(L\), in turn, satisfies the Lax equations
\[
\frac{\partial L}{\partial t_k} = [B_k, L]
\]

of the lattice KP hierarchy.

Let us examine the lowest equation of (12):
\[
[\partial_t - e^{\beta_1} - u_1, \partial_s - v e^{-\beta_1}] = 0. \quad (14)
\]

Upon substituting
\[
u_1(s) = \frac{\partial \phi(s)}{\partial \bar{t}_1}, \quad v(s) = e^{\phi(s)-\phi(s-1)}, \quad \phi(s) = \phi(s, t, \bar{t}_1),
\]

this equation turns into the Toda-like field equation
\[
\frac{\partial^2 \phi(s)}{\partial t_1 \partial s} + e^{\phi(s+1)-\phi(s)} - e^{\phi(s)-\phi(s-1)} = 0. \quad (15)
\]

This is exactly the continuum version of the Bogoyavlensky–Itoh equations [10, 11]. We can thus reproduce the remark of Okounkov and Pandharipande [14].

4. Logarithmic string equations

Let us return to the double Hurwitz numbers, and consider the associated Lax operators \(L, \bar{L}\) and the Orlov–Schulman operators
\[
M = \sum_{k=1}^{\infty} k \bar{t}_k L^k + s + \sum_{n=1}^{\infty} v_n \bar{L}^{-n},
\]
\[
\bar{M} = -\sum_{k=1}^{\infty} k \bar{t}_k \bar{L}^{-k} + s + \sum_{n=1}^{\infty} \bar{v}_n \bar{L}^{n}
\]

of the full 2D Toda hierarchy [17]. These operators are defined as
\[ L = W e^{\beta L^{-1}}, \quad \bar{L} = \bar{W} e^{\beta \bar{L}^{-1}}, \]
\[ M = W \left( s + \sum_{k=1}^{\infty} k \bar{\lambda} e^{k \beta k} \right) W^{-1}, \quad \bar{M} = \bar{W} \left( s - \sum_{k=1}^{\infty} k \lambda e^{-k \beta k} \right) \bar{W}^{-1} \]

by the dressing operators
\[ W = 1 + \sum_{n=1}^{\infty} w_n e^{-n \beta n}, \quad \bar{W} = \sum_{n=0}^{\infty} \bar{w}_n e^{n \beta n}, \]

and satisfy the Lax equations
\[ \frac{\partial L}{\partial t_k} = [B_k, L], \quad \frac{\partial \bar{L}}{\partial \bar{t}_k} = [\bar{B}_k, \bar{L}], \]
\[ \frac{\partial \bar{L}}{\partial t_k} = [B_k, \bar{L}], \quad \frac{\partial L}{\partial \bar{t}_k} = [\bar{B}_k, L], \tag{16} \]

where \( B_k \)'s are the same as those in (9), and \( \bar{B}_k \)'s are difference operators of the form
\[ \bar{B}_k = (L^{-k})_{<0} = \bar{b}_{00} e^{-\beta \lambda}, + \cdots + \bar{b}_{k,k-1} e^{-\beta \lambda}, \]

namely, the operators obtained by extracting the negative powers of \( e^{\beta \lambda} \) from \( \bar{L}^{-k} \).

We observed in our previous work [6] that these operators for the double Hurwitz numbers satisfy the generalized string equations\(^2\)
\[ L = Q e^{\beta M}, \quad \bar{L}^{-1} = Q^{-1} e^{\beta \bar{M}}. \tag{17} \]

We here derive a logarithmic form of these equations, namely, equations for the logarithmic Lax operators.

\[ \log L = W \partial_s W^{-1}, \quad \log \bar{L} = \bar{W} \partial_s \bar{W}^{-1} \]

and the Orlov–Schulman operators.

A clue is the canonical commutation relations
\[ [\log L, M] = 1, \quad [\log L, \bar{M}] = 1. \tag{18} \]

One can use these relations and the Baker–Campbell–Hausdorff formula to rewrite the right sides of (17) as
\[ Q e^{\beta M} \bar{L} = Q e^{\beta M} e^{\log L} = \exp(\beta M + \log \bar{L} - \beta / 2 + \log Q), \]
\[ Q L^{-1} e^{\beta \bar{M}} = Q^{-1} e^{-\log L} e^{\beta \bar{M}} = \exp(- \log L + \beta M - \beta / 2 + \log Q). \]

Equating these results with the logarithm of the left sides of (17) yields the logarithmic string equations
\[ \log L = \beta M + \log \bar{L} - \beta / 2 + \log Q, \]
\[ \log \bar{L} = \log L - \beta M - \beta / 2 - \log Q \tag{19} \]

for \( \log L, \log \bar{L}, M \) and \( \bar{M} \).

This is not a perfect proof of these equations, because taking the logarithm of both sides of (17) can leave ambiguity of integral multiples of \( 2\pi \sqrt{-1} \). Actually, this ambiguity can be

\(^2\)These equations are slightly different from those in the previous work due to modification of the tau function.
resolved by computations of the initial values of the Lax and Orlov–Schulman operators at a particular point of the \((t, \bar{t})\) space. We consider this issue in the next section.

The reduced Lax operator (11) in the single Hurwitz sector can be derived from (19) as well. Note that the first equation of (19) implies the relation

\[
(\log L)_{<0} = (\beta M)_{<0} = -\beta \sum_{k=1}^{\infty} k \bar{t}^k (L^{-k})_{<0}.
\]

In the single Hurwitz sector \(t = (\bar{t}, 0, 0, \ldots)\), this relation reduces to

\[
(\log L)_{<0} = -\beta \bar{t}_1 \bar{u}_0 e^{-\partial},
\]

hence

\[
\log L = \partial_s - \beta \bar{t}_1 \bar{u}_0 e^{-\partial}.
\]  

(20)

This is exactly the reduced Lax operator (11).

5. Perspective from factorization problem

The generating function \(Z(s, t, \bar{t})\) is derived from a fermionic formula [1, 6]. The fermionic construction of a tau function can be translated to the matrix factorization problem [19]

\[
\exp \left( \sum_{k=1}^{\infty} t_k \Lambda^k \right) U \exp \left( - \sum_{k=1}^{\infty} \bar{t}_k \Lambda^{-k} \right) = W(t, \bar{t})^{-1} \bar{W}(t, \bar{t}),
\]

(21)

where \(U\) is the \(\mathbb{Z} \times \mathbb{Z}\) matrix representing an element of the Clifford group \(\widehat{\text{GL}}(\infty)\) in the fermionic construction, \(\Lambda^k\)'s are the shift matrices \((\delta_{i+kj})_{i, j \in \mathbb{Z}}\), and \(W(t, \bar{t})\) and \(\bar{W}(t, \bar{t})\) are lower and upper triangular matrices corresponding to the dressing operators. In the case of the double Hurwitz numbers, \(U\) is a matrix of the form

\[
U = e^{\beta (\Delta^{-1/2})^2/2} Q^\Delta,
\]

(22)

where \(\Delta\) is the diagonal matrix \((\delta_{ij})_{i, j \in \mathbb{Z}}\). Note that \(\Lambda\) and \(\Delta\) are matrix representation of the difference operators \(e^{\partial}\) and \(s\) on the lattice \(\mathbb{Z}\).

We now extend this interpretation to the continuum \(\mathbb{R}\). Namely, \(\Lambda\) and \(s\) are understood to be the difference operators

\[
\Lambda = e^{\partial}, \quad \Delta = s
\]

(23)

defined on \(\mathbb{R}\). Equation (21) thereby becomes a factorization problem for difference operators.

Since (22) is a diagonal matrix, the factorization problem can be solved explicitly at the particular point \(t = 0, \quad \bar{t} = -\epsilon = (-c_1, -c_2, \ldots)\), where \(c_k\)'s are arbitrary constants. The solutions, which can be identified with the initial values \(W_0 = W(0, -\epsilon), \quad \bar{W}_0 = \bar{W}(0, -\epsilon)\) of the dressing operators, read

\[
W_0 = e^{\beta (s-1/2)^2/2} Q \exp \left( - \sum_{k=1}^{\infty} c_k e^{\partial k^2} \right) Q^{-s} e^{-\beta (s-1/2)^2/2},
\]

\[
\bar{W}_0 = e^{\beta (s-1/2)^2/2} Q'.
\]

(24)

These expressions of \(W_0\) and \(\bar{W}_0\) enable us to compute the associated initial values.
\[
\log L_0 = L(0, -c) = W_0 \partial_s W_0^{-1}, \quad \log \bar{L}_0 = L(0, -c) = \bar{W}_0 \partial_s \bar{W}_0^{-1},
\]

\[
M_0 = M(0, -c) = W_0 s W_0^{-1}, \quad \bar{M}_0 = \bar{M}(0, -c) = \bar{W}_0 \left( s + \sum_{k=1}^{\infty} k e^{-k\beta} e^{-k\beta s} \right) \bar{W}_0^{-1}
\]

of the logarithmic Lax operators and the Orlov–Schulman operators. The outcome takes the following form:

\[
\log L_0 = \partial_s + \beta s \sum_{k=1}^{\infty} k c_k Q_k e^{-\beta k/2} e^{k\beta s} e^{-k\beta s},
\]

\[
\log \bar{L}_0 = \partial_s - \beta(s - 1/2) - \log Q,
\]

\[
M_0 = \bar{M}_0 = s + \sum_{k=1}^{\infty} k e^{-\beta k(1+k)/2} e^{k\beta s} e^{-k\beta s}.
\]

This implies the algebraic relations

\[
\log L_0 = \beta \bar{M}_0 + \log \bar{L}_0 = \beta/2 + \log Q,
\]

\[
\log L_0 = \log L_0 - \beta M_0 - \beta/2 - \log Q,
\]

namely, the logarithmic string equation (19) are satisfied at the initial time \( t = 0, \bar{t} = -c \). This is enough to conclude that (19) themselves are satisfied, because both sides of these equations solve the same Lax equations as (16), and one can resort to the uniqueness of the initial value problem.

6. Conclusion

The Bogoyavlensky–Itoh hierarchies [10–13] are variants of the well known Volterra lattice. We have shown that the continuum version [10, 11] of these integrable hierarchies underlies the single Hurwitz numbers. In this respect, recent work of Dubrovin et al on cubic Hodge integrals [20, 21] is very interesting. They proved that the Volterra lattice is an integrable structure of cubic Hodge integrals in a special case [20], and conjectured a similar link with variants of the Volterra lattice in more general cases [21]. We can show, by the factorization technique of section 5, that the finite-step version [12, 13] of the Bogoyavlensky–Itoh hierarchies is indeed hidden in those cubic Hodge integrals. This issue will be reported elsewhere.

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