Symmetrization and enhancement of the continuous Morlet transform

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Abstract

The continuous wavelet transform using the Morlet basis may be symmetrized by using an appropriate normalization factor. The cone-of-influence is addressed through a renormalization of the wavelet based on power. The power spectral density may be deconvolved with the wavelet response matrix to produce an enhanced wavelet spectrum. The enhanced transform provides maximum resolution of the harmonic content of a signal.

1 Introduction

The continuous wavelet transform using the Morlet basis [1, 2] has become quite popular for data analysis [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14]. There is some variety in the literature as to the assignment of the normalization factors, and we propose a rearrangement so as to produce a symmetric forward and inverse transform pair. The cone-of-influence is addressed through a renormalization of the wavelet amplitude which keeps its power constant for a given scale. The renormalized power spectral density may then be enhanced by deconvolution with the wavelet response matrix, yielding the maximum spectral resolution of the harmonic content. We conclude by discussing the utility of these algorithms and point out a recent application.

The lack of a quantitative power spectral density has long hampered wider adoption of the continuous wavelet transform for data analysis. A mathematical engineer wants more than just a pretty picture—being able to give a numerical estimate to the power carried within a particular frequency band is of practical importance, and the use of a wavelet rather than Fourier transform allows that estimate to be time dependent. When the data is limited in duration, interesting features may be located in that region where wavelet truncation has become a significant effect. By renormalizing the power, the useful range may be extended.
beyond the cone-of-influence, and perfect reconstruction holds for all but the edge-most locations.

The breadth in scale of the wavelet response to a pure signal tone is a consequence of its localization in phase space, as the spectral and temporal resolutions are inversely related. Again, what a mathematical engineer wants is a precise identification of the frequency spectrum, in which features are maximally resolved. By treating the continuous wavelet transform as a theoretical apparatus acting upon a signal, one may essentially calibrate the device given some basic assumptions on the form of the signal components. Minimizing the discrepancy between the continuous instant wavelet power and the convolution of the calibration matrix with the estimate then yields the sharpest resolution of the time-varying harmonic content of a signal.

2 Normalization and central frequency

We first discuss the symmetrization of the transform and its effect on the central frequency employed. One may write the usual Morlet wavelet \[6, 13\] at scale \(s = 1/f_s = 2\pi/\omega_s\) and offset \(t\) with unity sample rate \(\Delta t' = 1\) using the parameter \(\eta \equiv (t' - t)/s\) as the product of a scale dependent constant \(C\), a normalized window \(\Phi\), and a normalized wave \(\Theta\),

\[
\psi_{s,t}^0(t') \equiv C_s^0 \Phi_{s,t}^0(t') \Theta_{s,t}^0(t') = \pi^{-1/4} s^{-1/2} e^{-\eta^2/2} e^{i\omega_1 \eta},
\]

where \(\omega_1 \approx 2\pi\) is the central frequency of the mother wavelet at unity scale and zero offset, \(\psi_{1,0}^0(t') = \pi^{-1/4} e^{-t'^2/2} e^{i\omega_1 t'}\). The window has an extent of \(-s\chi\) to \(s\chi\) and the wavelet a length of \(N_{t'} = 2s\chi + 1\), where the parameter \(\chi = 6\) defines the resolution width. The mother wavelet is normalized to unit energy \(\int_{-\infty}^{\infty} |\tilde{\psi}_{0,0}^0(\omega)|^2 d\omega = 1\) so that the Fourier transform of \(\psi^0(t'/s)\) is \(\tilde{\psi}^0(s\omega) = \sqrt{2\pi s} \tilde{\psi}_{0,0}^0(s\omega)\). Pulling over from the denominator of the inverse transform a factor of the scale \(s\) and including a factor \(\sqrt{2}\) which represents the transform response at negative scales gives a normalization \(C_s = \sqrt{2} C_s^0/s\) which produces a transform with some very desirable properties,

\[
\psi_{s,t}(t') = \sqrt{2} \pi^{-1/4} s^{-3/2} e^{-\eta^2/2} e^{i\omega_1 \eta}.
\]

The mother wavelet now has a norm of 2, which we interpret as including the response at negative scales to negative frequencies, and that of a scaled wavelet is now \(2/s^2\). With appeal to the photon, the energy of a localized wave is proportional to its frequency \(E_\nu \propto \nu = s^{-1}_\nu\), thus its power (energy per time) should be proportional to its energy over its period, \(P_\nu \propto s^{-2}_\nu\).

The analysis of a (mean-subtracted) signal \(y(t) = \sum_k Re(A_k e^{i\omega_k t})\) with duration \(N_t\) proceeds by the symmetric forward and inverse transform pair

\[
\text{CWT}(s,t) = \sum_{t'} \psi_{s,t}(t') y(t'), \quad (3)
\]

\[
\text{ICWT}(t) = \text{Re} \left[ \sum_s \sum_{t'} \psi_{s,t}(t') \text{CWT}(s,t') \Delta s \right], \quad (4)
\]
for $\psi^*(\eta) = \psi(-\eta)$, with perfect reconstruction within the cone-of-influence and quantitative agreement between the estimated power and the sum of the squared amplitudes of the signal components (cf. Equations (6) and (9) by Frick, et al [6]). Use of logarithmic scale spacing requires retention of the factor $\Delta s$. The root-mean-square power spectral density $\text{PSD}(s,t) \equiv |\text{CWT}|^2$ is normalized such that the integrated area of an isolated peak in the instant wavelet power $\text{IWP}_t(s) \equiv \text{PSD}(s,t)$ returns half the square of the amplitude $A_k$ of the signal component, $P_{\text{rms}} = \sum_k A_k^2/2$. The margins of the PSD determine the mean wavelet power $\text{MWP}(s) = N_t^{-1}\sum_t \text{PSD}(s,t)$ and the integrated instant power $\text{IIP}(t) = \sum_s \text{PSD}(s,t)\Delta s$. The analysis of a test signal of duration $N_t = 300$ time units with signal components of unit amplitude and periods of 5, 15, and 50 is shown in Figure 1. The cone-of-influence is marked with a solid line in (a), and the more restrictive cone-of-admissibility denoting the first wavelet truncation at a given scale is marked with a dashed line. A trough appears at the scale of the signal duration $N_t$ in (b), beyond which we identify the extremely low frequency ELF region where $s > N_t$. Apparent is the loss of transform response in (c), where the IIP falls below the rms power $P_{\text{rms}} = 1.5$, and the loss of reconstruction in (d) at the signal edge.

The central frequency given by Torrence and Compo [13] to unify the Fourier period and wavelet scale, $\lambda_1/s_1 = 4\pi/(\omega_1 + \sqrt{2 + \omega_1^2}) = 1$ yielding $\omega_1 = 2\pi - 1/4\pi$, is no longer appropriate for our normalization. Using the same test signal, we consider transforms with central frequencies $2\pi - 1/4\pi$, $2\pi$, and $2\pi + 1/4\pi$ and forward scalings of $s^{-1/2}$, $s^{-1}$, and $s^{-3/2}$ appearing in the CWT. The top row in Figure 2 displays the instant wavelet power for a single central frequency at the center of the transform $t = N_t/2$, and the bottom row
Figure 2: Instant wavelet power (top row) and its gradient at the central peak (bottom row) for forward transform normalizations labelled by column and central frequencies of $2\pi - 1/4\pi$ (dash-dot), $2\pi$ (dashed), and $2\pi + 1/4\pi$ (solid). The test signal has components of unit amplitude and periods 5, 15, and 50. Each peak in (c) has an area of 0.5, which is equal to the rms power of the signal component.

shows its gradient for all three central frequencies in the vicinity of the central signal peak; similar graphs obtain for the other peaks, noting that the forward scaling of $s^{-1}$ in (b) and (e) corresponds to that recently proposed by Liu, et al [15]. Only the transform with scaling $s^{-3/2}$ produces peaks with an integrated area equal to half the sum of squared amplitudes, and we note that the locations of its peaks coincide with the signal periods for the central frequency of $\omega_1 = 2\pi + 1/4\pi$. The response of the symmetrically normalized CWT is that of a theoretical apparatus whose point spread function preserves the area of a Dirac distribution representing the power carried by a pure signal component of infinite duration with constant amplitude and period.

3 Renormalization

We next introduce a renormalization which compensates for the reduction in response outside the cone-of-influence. The cone-of-influence is defined by the $e$-folding time, $t_e = s\sqrt{2}$ for the Morlet wavelet, indicating that region beyond which the response of the CWT is significantly affected by the wavelet truncation, which begins at the cone-of-admissibility. Various algorithms have been proposed for its rectification [5, 6, 8, 16, 12]; however, we have found that algorithms which alter the shape of the analyzing wavelet also affect its frequency response. Thus, we are led to proposing a simple renormalization such that for transform coefficients outside the cone-of-admissibility the wavelet is given a norm of $2/s^2$. For wavelets truncated by either edge of the signal, the window $\Phi_{\tau}$ is shifted by an offset $\tau$ relative to an unshifted window $\Phi_0$ defining the time span $t'$. The length of a truncated wavelet $\psi_{\tau}$ is
Figure 3: RCWT power spectral density (a), mean wavelet power (b), integrated instant renormalized power (c), and reconstruction (d) for a test signal with components of unit amplitude and periods 5, 15, and 50. The MWP for the RAWT is shown in (b) as a dashed line. Reconstruction is maintained until the edge-most data points.

Table 1: Ratio of mean integrated power to $P_{rms}$.

| PSD | CWT | RCWT |
|-----|-----|------|
| $s_{max}$ | 600 | 300 | 600 | 300 |
| MIP/$P_{rms}$ | 0.95754 | 0.95697 | 1.002 | 1.0004 |

defined to be the lesser of the raw wavelet length or the signal length, $N_T = \min(N_{t'}, N_t)$. The offset $\tau(t)$ is determined from either the center of the signal or the location of the cone-of-admissibility, and the algorithm to keep everything aligned gets a bit complicated: for $\tau' = \max(0, |s\chi| - |N_t|/2)$ and $t'(k) = [-|s\chi|, |s\chi|]$ indexed by $k$ from 1 to $N_{t'}$, if $\tau \leq 0$ then $t' \rightarrow t'[1, \min(N_{t'}, N_t)] + \tau'$, else $t' \rightarrow t'[\max(1, N_{t'} - N_t + 1), N_{t'}] - \tau'$. The end result is simply to truncate either edge of the wavelet as necessary. Then for the amplitude of the truncated wavelet $\psi_{\tau'} = C_s \Phi_{\tau} \Theta$, with $C_s \leftarrow C_{s,t} = C_s \sqrt{2/s^2|\psi_{\tau'}|^2}$ we define the renormalized continuous wavelet transform RCWT.

Using the same test signal as above, in Figure 3 we display the analysis using the RCWT; the reconstruction in (d) is noticeably improved, and the power estimation in (c) is not as affected near the signal edge. The apparent increase in the IIP over the rms value represents we feel an aliasing in time, rather than scale, of the total power, as the mean discrepancy from the rms power $\sum_t (IIP - P_{rms})/N_t = 5.6 \times 10^{-4}$ is small. In Table 1 we display the ratio of the mean instant power $MIP = \sum_t \sum_{s=2}^{s_{max}} \Delta_s PSD/N_t$ to the rms power for the CWT and RCWT, considering also an integration over scale which stops at the signal duration $s_{max} = N_t$ rather than $s_{max} = 2N_t$. 

5
Figure 4: RCWT power spectral density (a), mean wavelet power (b), integrated instant renormalized power (c), and reconstruction (d) for a test signal with components of time-varying periods around 5, 15, and 50 and squared amplitudes of .1, 1, and .5 respectively.

Table 2: Comparison of FFT and RCWT rms power.

|                | ∑ₜ y²/Nₜ | ∑ₜ FFT | ∑ₜ=₂⁻¹ Δₜ MWP | ∑ₜ=₂⁻³ Δₜ MWP |
|----------------|----------|--------|---------------|---------------|
| RCWT           | 0.79793  | 0.79793| 0.80908       | 0.80777       |

As pointed out by Frick et al [6], wavelet truncation also affects the admissibility condition. One commonly subtracts from $\Theta^0 \equiv e^{i\omega_1 \eta}$ the DC component $\psi_1(0) \propto e^{-\omega_1^2/2} \equiv d_c \sim 10^{-9}$ of the mother wavelet so that the zero mean wave becomes $\Theta = \Theta^0 - d_c$. Then, the adaptive wavelet transform is defined by $\Theta = \Theta^0 - d_{s,t}$, where $d_{s,t} \equiv \langle \Theta^0 \rangle = \sum \Theta^0 \Phi^\tau / \sum \Phi^\tau$ is the weighted mean of the remaining wave $\Theta^0$. Normalization as above with $C_{s,t}$ then defines the renormalized adaptive wavelet transform RAWT of Reference [16]. In practice, we have found that the RCWT neglecting admissibility outperforms the RAWT, shown in Figure 3(b) as a dashed line, by a small but noticeable margin; the troughs between peaks are slightly deeper, and the reconstruction is slightly better. The reason, we feel, is that the adaptive admissibility condition alters the shape of the wavelet, hence its frequency response.

The hallmark of wavelet analysis is its ability to track signal components with periods that vary in time. Considering now a test signal of the same duration, for periods 5, 15, and 50 we adjust the squared amplitudes to be 0.1, 1, and 0.5 respectively (rms power of 0.8) and impose independent sinusoidal variation to the periods on the order of the duration. In Figure 4 we display the RCWT analysis of such a signal, and in Figure 5 we compare the FFT power spectrum [17] with the MWP. The height of the peaks in the MWP appears to represent the relationship between the powers of the signal components better than the peaks in the FFT. Also, shown in Table 2 is the agreement between the mean squared amplitude
Figure 5: The FFT power spectrum (dotted) compared to the RCWT mean wavelet power (solid). Locations of the discrete FFT values are marked (*).

of the signal and the sum of the FFT power spectrum, which are quite close numerically to the value of the integrated MWP—the discrepancy decreases with increasing resolution in scale. For comparison, the integrated MWP for the CWT analysis of this signal is 0.77325, which is significantly less than the rms power.

4 Enhancement

With our transform now responding like a theoretical apparatus for measuring a signal’s power spectral density, one may apply the techniques of resolution enhancement common in the analysis of experimental data [18]. First one writes the point spread function as the response matrix $R(s, s')$ defined by the integrated power of a wavelet of scale $s$ convoluted with a signal component of period $\lambda = s'$. For this analysis we take the signal components to be Hann windowed cosine functions for the duration of the wavelet,

$$R(s, s') = \left| \sum_{t'} \psi^{*}_{s,0}(t') \cos(2\pi t'/s') H_s(t') \right|^2,$$

which gave the best performance among the options. We note that here one is making an assumption on the form of the underlying signal elements whose composition represents the original signal. Then, for each IWP in the PSD, the enhanced instant power EIP is the solution to the equation

$$0 = \sum_s \left[ \sum_{s'} R(s, s') EIP_t(s') \Delta s' - IWP_t(s) \right]^2 \Delta s,$$

found in a least-squares sense with non-negativity constraints. Note that it is the redundancy in scale of the CWT which provides the resolution enhancement of the EIP. In practice, we apply the enhancement to the peak PSD using $R \rightarrow 2R$. The effect is to replace broad peaks in the IWP with sharp spikes at the scale of the corresponding signal component, as shown in Figure 6 (a) for the IWP at the midpoint of the duration of the signal in the paragraph above. The reconstructed IWP (dashed) differs slightly from the original IWP (solid), sometimes exceeding it as no constraint has been placed on preserving the norm. In general, one’s wavelet response may extend beyond one’s region of calculation for signal
Figure 6: EIP (spikes) and IWP (solid) at the midpoint of the signal duration. The reconstructed IWP (dashed) slightly exceeds the original IWP. Panel (a) used a tolerance of $10^{-8}$, and panel (b) used a tolerance of $10^{-4}$ to suppress the low frequency components.

periods near either cutoff, and the enhancement procedure is capable of recapturing the lost (uncomputed) power. If one’s application indicates the signal is bandwidth limited to that region well within the cone-of-influence yet far from the Nyquist scale, then enforcement of a norm-preserving constraint during the minimization is suggested. The enhanced power spectral density $\text{EPSD}(s, t)$ is then defined simply as the collection of enhanced instant powers. Interestingly, one may effectively suppress the low frequency contribution by raising the tolerance of the minimization as in panel (b), where the EIP represents only the main signal components.

In Figure 7 we show the EPSD of our most recent test signal, as well as the mean enhanced power MEP and integrated enhanced power IEP, which exceeds the previous IIP by a significant margin for which we have not yet found a correction factor. The variation in scale of the signal periods is well-resolved within the cone-of-influence, and a video scanning through the EIPs is available as an online supplement. Reconstruction from the EPSD is not yet well-defined; however, one may attempt a reconstruction using the phase of the RCWT and the original renormalized basis as shown in (d), which might not be perfect but does faithfully represent the original signal. For signal components within the cone-of-influence, the EPSD provides the maximum resolution in scale available from the RCWT.

5 Conclusions

The utility of these algorithms should be apparent to anyone familiar with one dimensional data analysis and power spectrum estimation. The extension of the renormalization prescription to multi-dimensional wavelet analysis is straightforward; less so for the enhancement procedure. The symmetric normalization adopted here presents a power spectral density
Figure 7: Enhanced power spectral density (a), mean enhanced power (b), integrated enhanced power (c), and reconstruction (d) for a test signal with components of time-varying periods around 5, 15, and 50. The increase in the IEP is attributed to the reconstructed IWP exceeding the original IWP.

which behaves exactly as it should, where its margins may put on a physical basis, and provides perfect reconstruction without the introduction of an arbitrary factor. The mean wavelet power represents the relationship between signal components better than the fast Fourier transform, and the integrated instant power agrees with the rms power of the signal components.

A recent application of algorithms very similar to the ones presented here is found in [8, 16]. In that work, by addressing the power spectral density of the historical sunspot record, a relation is found between the level of solar magnetic activity and the temperature observed in central England. One also may consider its application to signal encoding, manipulation, and compression, providing an alternate basis for reconstruction. For temporally resolved power spectrum estimation, the symmetric wavelet periodogram has become quite a useful tool indeed.

In summary, the continuous wavelet transform may be normalized to account for the response at negative scales, resulting in a symmetric forward and inverse transform pair with perfect reconstruction. It may then be renormalized to account for wavelet truncation by keeping a constant wavelet power across scale, where neglecting the admissibility condition results in better performance, extending the useful range beyond the cone-of-influence. By deconvolution with the wavelet response matrix, the enhanced power spectral density provides the maximum resolution in scale of the harmonic content carried by a signal.
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