Partitions of $\mathbb{Z}_n$ into Arithmetic Progressions

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Abstract

We introduce the notion of arithmetic progression blocks or AP-blocks of $\mathbb{Z}_n$, which can be represented as sequences of the form $(x, x + m, x + 2m, \ldots, x + (i - 1)m) \pmod{n}$. Then we consider the problem of partitioning $\mathbb{Z}_n$ into AP-blocks for a given difference $m$. We show that subject to a technical condition, the number of partitions of $\mathbb{Z}_n$ into $m$-AP-blocks of a given type is independent of $m$. When we restrict our attention to blocks of sizes one or two, we are led to a combinatorial interpretation of a formula recently derived by Mansour and Sun as a generalization of the Kaplansky numbers. These numbers have also occurred as the coefficients in Waring’s formula for symmetric functions.

Keywords: Kaplansky number, cycle dissection, $m$-AP-partition, separation algorithm.

AMS Classification: 05A05, 05A15

1 Introduction

Let $\mathbb{Z}_n$ be the cyclic group of order $n$ whose elements are written as 1, 2, ..., $n$. Intuitively, we assume that the elements 1, 2, ..., $n$ are placed clockwise on a cycle. Thus $\mathbb{Z}_n$ can be viewed as an $n$-cycle, more specifically, a directed cycle. In his study of the ménages problem, Kaplansky [7] has shown that the number of ways of choosing $k$ elements from $\mathbb{Z}_n$ such that no two elements differ by one modulo $n$ (see also Braudli [1], Comtet [3], Riordan [14], Ryser [15] and Stanley [16, Lemma 2.3.4]) equals

$$\frac{n}{n-k} \binom{n-k}{k}.$$  \hspace{1cm} (1.1)

Moreover, Kaplansky [8] considered the following generalization. Assume that $n \geq pk + 1$. Then the number of $k$-subsets $\{x_1, x_2, \ldots, x_k\}$ of $\mathbb{Z}_n$ such that

$$x_i - x_j \notin \{1, 2, \ldots, p\} \hspace{1cm} (1.2)$$
for any pair \((x_i, x_j)\) of distinct elements, is given by

\[
\frac{n}{n - pk} \binom{n - pk}{k}.
\]  

Here we clarify the meaning of the notation (1.2). Given two elements \(x\) and \(y\) of \(\mathbb{Z}_n\), \(x - y\) may be considered as the distance from \(y\) to \(x\) on the directed cycle \(\mathbb{Z}_n\). Therefore, (1.2) says that the distance from any element \(x_i\) to any other element \(x_j\) on the directed cycle \(\mathbb{Z}_n\) is at least \(p + 1\).

From a different perspective, Konvalina [10] studied the number of \(k\)-subsets \(\{x_1, x_2, \ldots, x_k\}\) such that no two elements \(x_i\) and \(x_j\) are “uni-separated”, namely \(x_i - x_j \neq 2\) for all \(x_i\) and \(x_j\). Remarkably, Konvalina discovered that the answer is also given by the Kaplansky number (1.1) for \(n \geq 2k + 1\). Other generalizations and related questions have been investigated by Hwang [5], Hwang, Korner and Wei [6], Munarini and Salvi [12], Prodinger [13] and Kirschenhofer and Prodinger [9]. Recently, Mansour and Sun [11] obtained the following unification of the formulas of Kaplansky and Konvalina.

**Theorem 1.1.** Assume that \(m, p, k \geq 1\) and \(n \geq mpk + 1\). The number of \(k\)-subsets \(\{x_1, x_2, \ldots, x_k\}\) of \(\mathbb{Z}_n\) such that

\[
x_i - x_j \notin \{m, 2m, \ldots, pm\}
\]  

for any pair \((x_i, x_j)\), is given by the formula (1.3), and is independent of \(m\).

In the spirit of the original approach of Kaplansky, Mansour and Sun first solved the enumeration problem of choosing \(k\)-subset from an \(n\)-set with elements lying on a line. They established a recurrence relation, and solved the equation by computing the residues of some Laurent series. The case for an \(n\)-cycle can be reduced to the case for a line. They raised the question of finding a combinatorial proof of their formula. Guo [4] found a proof by using number theoretic properties and Rothe’s identity:

\[
\sum_{k=0}^{n} \frac{xy}{(x + k)(y + (n-k))} \binom{x + k}{k} \binom{y + (n-k)}{n-k} = \frac{x + y}{x + y + nz} \binom{x + y + nz}{n}.
\]

This paper is motivated by the question of Mansour and Sun. We introduce the notion of arithmetic progression blocks or AP-blocks of \(\mathbb{Z}_n\). A sequence of the form

\[
(x, x + m, x + 2m, \ldots, x + (i-1)m) \pmod{n}
\]

is called an AP-block, or an \(m\)-AP-block, of length \(i\) and of difference \(m\). Then we consider partitions of \(\mathbb{Z}_n\) into \(m\)-AP-blocks \(B_1, B_2, \ldots, B_k\) of the same difference \(m\). The type of such a partition is referred to as the type of the multisets of the sizes of the blocks. Our main result shows that subject to a technical condition, the number
of partitions of $\mathbb{Z}_n$ into $m$-AP-blocks of a given type is independent of $m$ and is equal to the multinomial coefficient.

This paper is organized as follows. In Section 2, we give a review of the cycle dissections and make a connection between the Kaplansky numbers and the cyclic multinomial coefficients. We present the main result in Section 3, that is, subject to a technical condition, the number of partitions of $\mathbb{Z}_n$ into $m$-AP-blocks of a given type equals the multinomial coefficient and does not depend on $m$. We present a separation algorithm which leads to a bijection between $m$-AP-partitions and $m'$-AP-partitions of $\mathbb{Z}_n$. The correspondence between $m$-AP-partitions and cycle dissections ($m' = 1$) implies the main result Theorem 3.2. For the type $1^{n-(p+1)k}(p+1)^k$ we are led to a combinatorial proof which answers the question of Mansour and Sun.

2 Cycle Dissections

In their combinatorial study of Waring’s formula on symmetric functions, Chen, Lih and Yeh \[2\] introduced the notion of cycle dissections. Recall that a dissection of an $n$-cycle is a partition of the cycle into blocks, which can be viewed by putting cutting bars on some edges of the cycle. Note that there at least one bar to cut a cycle into straight segments. A dissection of an $n$-cycle is said of type $1^{k_1}2^{k_2} \cdots n^{k_n}$ if there are $k_i$ blocks of $i$ elements in it. For instance, Figure 1 gives a 20-cycle dissection of type $1^82^33^2$.

![Figure 1: A 20-cycle dissection of type $1^82^33^2$.](image)

The following lemma is due to Chen-Lih-Yeh \[2\] Lemma 3.1].

**Lemma 2.1.** For an $n$-cycle, the number of dissections of type $1^{k_1}2^{k_2} \cdots n^{k_n}$ is given by the cyclic multinomial coefficients:

$$ \frac{n}{k_1 + \cdots + k_n} \binom{k_1 + \cdots + k_n}{k_1, \ldots, k_n}. \quad (2.1) $$
This lemma is easy to prove. Given a dissection, one may pick up any segment as a distinguished segment. This can be done in 
\[ k_1 + k_2 + \cdots + k_n \] 
ways. On the other hand, any of the \( n \) elements can serve as the first element of the distinguished segment.

Consider a cycle dissection of type \( 1^{n-(p+1)}k(p+1)^k \). The set of the first elements of each segment of length \( p + 1 \) corresponds a \( k \)-subset of \( \mathbb{Z}_n \) satisfying (1.2). Thus the cyclic multinomial coefficient of type \( 1^{n-(p+1)}k(p+1)^k \) reduces to (1.3) and particularly the cyclic multinomial coefficient of type \( 1^{n-2k2^k} \) reduces to the Kaplansky number (1.1).

\section{Partitions of \( \mathbb{Z}_n \) into Arithmetic Progressions}

In this section, we present the main result of this paper, namely, a formula for the number of partitions of \( \mathbb{Z}_n \) into \( m \)-AP-blocks of a given type. The proof is based on a separation algorithm to transform an \( m \)-AP-partition to an \( m' \)-AP-partition.

We begin with some concepts. First, \( \mathbb{Z}_n \) is considered as a directed cycle. An arithmetic progression block, or an AP-block of \( \mathbb{Z}_n \), is defined to be a sequence of elements of \( \mathbb{Z}_n \) of the following form

\[ B = (x, x + m, x + 2m, \ldots, x + (i - 1)m) \mod n, \]

where \( m \) is called the \textit{difference} and \( i \) is called the \textit{length} of \( B \). An AP-block of difference \( m \) is called an \( m \)-AP-block. If \( B \) contains only one element, then it is called a \textit{singleton}. The first element \( x \) is called the \textit{head} of \( B \). An \( m \)-AP-partition, or a partition of \( \mathbb{Z}_n \) into \( m \)-AP-blocks, is a set of \( m \)-AP-blocks of \( \mathbb{Z}_n \) whose underlying sets form a partition of \( \mathbb{Z}_n \). For example,

\[ (7, 9, 11), (8), (10, 12), (1), (2, 4, 6), (3), (5) \]  

(3.1)

is a 2-AP-partition of \( \mathbb{Z}_{12} \) with four singletons and three non-singleton heads 7, 10 and 2.

It should be noted that different AP-blocks may correspond to the same underlying set. For example, \( (1, 3) \) and \( (3, 1) \) are regarded as different AP-blocks of \( \mathbb{Z}_4 \), but they have the same underlying set \( \{1, 3\} \). On the other hand, as will be seen in Proposition 3.1, it often happens that an AP-block is uniquely determined by its underlying set. For example, given the difference \( m = 3 \), the AP-block \( (12, 15, 2, 5, 8) \) of \( \mathbb{Z}_{16} \) is uniquely determined by the underlying set \( \{2, 5, 8, 12, 15\} \) since there is only one way to order these five elements to form an arithmetic progression of difference 3 modulo 16.

For an \( m \)-AP-partition \( \pi \), the \textit{type} of \( \pi \) is defined by the type of the multisets of the sizes of the blocks. Usually, we use the notation \( 1^{k_1}2^{k_2}\cdots n^{k_n} \) to denote a type for which there are \( k_1 \) blocks of size one, \( k_2 \) blocks of size two, etc. However, for the sake of presentation, we find it more convenient to ignore the zero exponents and express a
type in the form $i_{k_1}^1 i_{k_2}^2 \cdots i_{k_r}^r$, where $1 \leq i_1 < i_2 < \cdots < i_r$ and all $k_j \geq 1$. For example, the AP-partition (3.1) is of type $1^4 2^1 3^2$.

Throughout this paper, we restrict our attention to $m$-AP-partitions with at least one singleton block and also at least one non-singleton block, namely, $i_1 = 1$ and $r \geq 2$ in the above notation of types. Here is the aforementioned condition:

$$\left\lceil \frac{k_1}{k_2 + \cdots + k_r} \right\rceil \geq (m - 1)(i_r - 1),$$

(3.2)

where the notation $\lceil x \rceil$ for a real number $x$ stands for the smallest integer that is larger than or equal to $x$. Obviously, the condition (3.2) holds for $m = 1$. For $m \geq 2$, (3.2) is equivalent to the relation

$$k_1 \geq (k_2 + \cdots + k_r)[(m - 1)(i_r - 1) - 1] + 1.$$  

(3.3)

We prefer the form (3.2) for a reason that will become clear in the combinatorial argument in the proof of Theorem 3.2. In fact on an $n$-cycle dissection, the $\sum_{j=2}^r k_j$ non-singleton heads divide the $k_1$ singletons into $\sum_{j=2}^r k_j$ segments. By virtue of the pigeonhole principle, there exists a segment containing at least $(m - 1)(i_r - 1)$ singletons.

For example in the AP-partition (3.1), the three non-singleton heads divide the four singletons into three segments and therefore there exists one segment containing at least $2$ singletons. In this particular partition it is the path from $2$ to $7$ that contains two singletons $3$ and $5$, see the right cycle in Figure 2.  

**Proposition 3.1.** Under the condition (3.2), an $m$-AP-block is not uniquely determined by its underlying set if and only if $n = i_r m$ and it is of length $i_r$.

**Proof.** Let $n = i_r m$. Consider the AP-blocks,

$$B_j = (x + jm, x + (j + 1)m, \ldots, x + (j + i_r - 1)m) \pmod{n}, \quad 0 \leq j \leq i_r - 1.$$  

It is easy to see that these AP-blocks $B_j$ ($j = 0, 1, \ldots, i_r - 1$) have the same underlying set

$$\{x, x + m, \ldots, x + (i_r - 1)m\}.$$  

Conversely, suppose that there is an $m$-AP-block $B$ of length $i_s$ which is not uniquely determined by its underlying set. We may assume that there exists another AP-block $B'$ having the same underlying set as $B$. Thus the difference between $B$ and $B'$ lies only in the order of their elements as a sequence. It follows that $n = i_s m$ for some $2 \leq s \leq r$. If $m = 1$, then $n = i_s$ which yields $s = r = 1$, a contradiction. So we may assume that $m \geq 2$ and $2 \leq s \leq r - 1$. Hence $i_s \leq i_{r-1} \leq i_r - 1$, and so

$$k_1 + \sum_{j=2}^r k_j i_j = n = i_s m \leq (i_r - 1)m.$$  

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In view of the condition (3.3), we deduce that

\[(i_r - 1)m - \sum_{j=2}^{r} k_j i_j \geq k_1 \geq [(m - 1)(i_r - 1) - 1] \sum_{j=2}^{r} k_j + 1\]

which can be rewritten as

\[1 + \sum_{j=2}^{r-1} k_j i_j + (i_r - 1)m \left( \sum_{j=2}^{r} k_j - 1 \right) \leq i_r \sum_{j=2}^{r-1} k_j.\]

Clearly,

\[\sum_{j=2}^{r} k_j - 1 \geq \sum_{j=2}^{r-1} k_j,\]

so \((i_r - 1)m < i_r\) and thus \(i_r < m/(m - 1) \leq 2\) which implies \(i_r = 1\), a contradiction. Thus we conclude that \(s = r\). This completes the proof.

For example, the AP-partition (3.1) is uniquely determined by its underlying partition:

\[\{7, 9, 11\}, \{8\}, \{10, 12\}, \{1\}, \{2, 4, 6\}, \{3\}, \{5\}.\]

We are now ready to present the main result of this paper.

**Theorem 3.2.** Given a type 

\[1^{k_1} i_2^{k_2} \cdots i_r^{k_r}\]

satisfying the condition (3.2), the number of \(m\)-AP-partitions of \(\mathbb{Z}_n\) does not depend on \(m\), and is equal to the cyclic multinomial coefficient

\[\frac{n}{k_1 + \cdots + k_r} \binom{k_1 + \cdots + k_r}{k_1, \ldots, k_r}.\] \hspace{1cm} (3.4)

In fact, Theorem 3.2 reduces to Theorem 1.1 when we specialize the type to \(1^{n-(p+1)} (p+1)^k\). In this case the condition (3.2) becomes \(n \geq km^p + 1\). The heads of the \(k\) AP-blocks of length \(p+1\) satisfy the condition (1.4). Conversely, any \(k\)-subset of \(\mathbb{Z}_n\) satisfying (1.4) determines an \(m\)-AP-partition of the given type. The cyclic multinomial coefficient (3.4) agrees with the formula (1.3) of Theorem 1.1. For example, given the type \(1^42^13^2\) and difference 2, the AP-partition (3.1) is determined by the selection of \(\{7, 10, 2\}\) as heads from \(\mathbb{Z}_{12}\).

Note that the cyclic multinomial coefficient (3.4) has occurred in Lemma 2.1. Indeed, Lemma 1 is the special case of Theorem 3.2 for \(m = 1\). We proceed to describe an algorithm, called the *separation algorithm*, to transform \(m\)-AP-partitions to \(m'\)-AP-partitions of the same type \(T = i_1^{k_1} i_2^{k_2} \cdots i_r^{k_r}\), assuming the following condition holds:

\[\left[ \frac{k_1}{k_2 + \cdots + k_r} \right] \geq (\max\{m, m'\} - 1)(i_r - 1).\] \hspace{1cm} (3.5)
The separation algorithm enables us to verify Theorem 3.2. We will state our algorithm for \( m \)-AP-partitions and \( m' \)-AP-partitions, instead of restricting \( m' \) to one, because it is more convenient to present the proof by exchanging the role of \( m \) and \( m' \).

Given a type \( T = 1^{k_1}2^{k_2} \cdots r^{k_r} \), let \( \mathcal{P}_m \) be the set of \( m \)-AP-partitions of type \( T \). To prove Theorem 3.2, it suffices to show that there is a bijection between \( \mathcal{P}_m \) and \( \mathcal{P}_m' \) under the condition (3.5).

Let \( \pi \in \mathcal{P}_m \). Denote by \( H(\pi) \) the set of heads in \( \pi \). For each head \( h \) of \( \pi \), we consider the nearest non-singleton head in the counterclockwise direction, denoted \( h^* \). Then we denote by \( g(h) \) the number of singletons lying on the path from \( h^* \) to \( h \) under the convention that \( h \) is not counted by \( g(h) \). For example, for the AP-partition \( \pi' \) on the right of Figure 2 we have \( H(\pi') = \{1, 2, 3, 5, 7, 8, 10\} \), \( g(1) = g(3) = g(8) = 0 \), \( g(2) = g(5) = g(10) = 1 \) and \( g(7) = 2 \). The values \( g(h) \) will be needed in the separation algorithm.

\[
(7, 8, 9), (10), (11, 12), (1), (2, 3, 4), (5), (6) \quad (7, 9, 11), (8), (10, 12), (1), (2, 4, 6), (3), (5)
\]

Figure 2: The algorithms \( \psi \) and \( \varphi \) for \( T = 1^42^13^2, m = 1 \) and \( m' = 2 \).

**The Separation Algorithm.** Let \( \pi \) be an \( m \)-AP-partition of type \( T \). As the first step, we choose a head \( h_1 \) of \( \pi \), called the starting point, such that \( g(h_1) \) is the maximum. Then we impose a linear order on the elements of \( \mathbb{Z}_n \) with respect to the choice of \( h_1 \):

\[
h_1 < h_1 + 1 < h_1 + 2 < \cdots < h_1 - 1 \quad (\text{mod } n). \tag{3.6}
\]

In accordance with the above order, we denote the heads of \( \pi \) by \( h_1 < h_2 < \cdots < h_t \), where \( t = \sum_{i=1}^{r} k_i \). The \( m \)-AP-block of \( \pi \) with head \( h_i \) is denoted by \( B_i \). Let \( l_i \) be the length of \( B_i \), and so \( \sum_{i=1}^{t} l_i = n \).

We now aim to construct \( m' \)-AP-blocks \( B'_1, B'_2, \ldots, B'_t \) such that \( B'_i \) has the same number of elements as \( B_i \). We begin with \( B'_1 \) by setting \( h'_1 = h_1 \) and letting \( B'_1 \) be the \( m' \)-AP-block of length \( l_1 \), namely,

\[
B'_1 = (h'_1, h'_1 + m', \ldots, h'_1 + (l_1 - 1)m').
\]
Among the remaining elements, namely, those that are not in $B'_1$, we choose the smallest element with respect to $\equiv_{3,6}$, denoted by $h'_2$, and let $B'_2$ be the $m'$-AP-block of length $l_2$ with head $h'_2$. Repeating the above procedure, as will be justified later, after $t$ steps we obtain an $m'$-AP-partition, denoted $\psi(\pi)$, of type $T$ with blocks $B'_1, B'_2, \ldots, B'_t$.

Figure 2 illustrates the separation algorithm from a 1-AP-partition $\pi'$ of the same type $T = 1^42^13^2$ and vice versa. The solid dots stand for singletons, whereas the other symbols represent different AP-blocks.

We remark that, as indicated by the example, the starting point can never be a singleton. In fact, if $s$ is a singleton and $h$ is a non-singleton head such that all the heads lying on the path from $s$ to $h$ are singletons, then we have the relation $g(h) > g(s)$. Since $g(h_1)$ is maximum, we see that the starting point is always a non-singleton head.

Clearly, it is necessary to demonstrate that the above algorithm $\psi$ is valid, namely, we need to justify that underlying sets of the blocks $B'_1, B'_2, \ldots, B'_t$ are disjoint.

**Proposition 3.3.** The mapping $\psi$ is well-defined, and for any $\pi \in P_m$, we have $\psi(\pi) \in P_{m'}$.

**Proof.** Let $\pi \in P_m$ with AP-blocks $B_1, B_2, \ldots, B_t$. Without loss of generality, we may assume that $h_1, h_2, \ldots, h_t$ are the heads of $B_1, B_2, \ldots, B_t$, where $h_1$ is the starting point for the mapping $\psi$ and $h'_1, h'_2, \ldots, h'_t$ are the corresponding heads generated by $\psi$. Let $l_i$ be the length of $B_i$. Suppose to the contrary that there exist two heads $h_i$ and $h_j$ ($i < j$) such that

$$h'_i + am' \equiv h'_j + bm' \pmod{n},$$

where $0 \leq a \leq l_i - 1$ and $0 \leq b \leq l_j - 1$.

If $a \geq b$, then $0 \leq a - b \leq l_i - 1$ and $h'_j \equiv h'_i + (a - b)m' \pmod{n}$. But the point $h'_i + (a - b)m'$ is in $B'_i$, contradicting the choice of $h'_j$. This yields $a < b$ and thus $0 \leq b - a \leq l_j - 1$.

We claim that the starting point $h_1$ lies on the path from $h'_j$ to $h'_i$. In fact, when the Algorithm $\psi$ is at the $j$-th step to deal with the head $h'_j$, all the points smaller than $h'_i$ lie in one of the blocks $B'_1, B'_2, \ldots, B'_i$. Then we see that $h'_j > h'_i$. Meanwhile, there are $n - l_1 - l_2 - \cdots - l_{j-1} > 0$ points which are not contained in $B'_1, B'_2, \ldots, B'_{j-1}$. But the head $h'_j$ is chosen to be the smallest point not in $B'_1, B'_2, \ldots, B'_{j-1}$, we find that $h'_j$ lies on the path from $h'_i$ to $h_1$.

In addition to $h'_i$ and $h'_j$, we assume that there are $N$ points on the path from $h'_j$ to $h'_i$. Since $h'_i \equiv h'_j + (b - a)m' \pmod{n}$ and $1 \leq b - a \leq l_j - 1$, we obtain $N = (b - a)m' - 1$. On the other hand, at the $j$-th step, in addition to the point $h'_j$, there are at least $l_j - 1$ points not contained in $B'_1, B'_2, \ldots, B'_{j-1}$. Similarly, the choice of $h_1$ and the condition (3.5) yield that the largest $(\max\{m, m'\} - 1)(i - 1)$ heads with respect to the order (3.6) are all singletons by the pigeonhole principle. Therefore, there are at least $(\max\{m, m'\} - 1)(i_r - 1)$ points not contained in $B'_1, B'_2, \ldots, B'_{j-1}$.
It follows that
\[ N \geq (\max\{m, m'\} - 1)(i_r - 1) + (l_j - 1). \]  
(3.7)
Since \( N = (b - a)m' - 1 \) and \( 1 \leq b - a \leq l_j - 1 \), we deduce that
\[ (m' - 1)(i_r - 1) + (l_j - 1) \leq (b - a)m' - 1 \leq (l_j - 1)m' - 1, \]
leading to the contradiction \( l_j > i_r \). This completes the proof. □

**Proposition 3.4.** Given an \( m \)-AP-partition of \( \mathbb{Z}_n \), the separation algorithm \( \psi \) generates the same \( m' \)-AP-partition regardless of the choice of the starting point subject to the maximum property.

**Proof.** Let \( \pi \) be an \( m \)-AP-partition of \( \mathbb{Z}_n \). Suppose that \( u_1, u_2, \ldots, u_s \) \((s \geq 2)\) are all the heads such that \( g(u_1) = g(u_2) = \cdots = g(u_s) \) is the maximum on \( \pi \). Let \( u_1 \) be the starting point and \( u_1 < u_2 < \cdots < u_s \) with respect to (3.6).

It suffices to show that when the Algorithm \( \psi \) processes \( u_i \) \((1 \leq i \leq s)\), the \( m' \)-AP-blocks which have been generated consist of all the elements smaller than \( u_i \). By induction we assume that this statement holds up to \( u_{j-1} \).

Let \( v_q, v_{q-1}, \ldots, v_1, u_j \) be all heads lying on the path \( Q \) from \( u_{j-1} \) to \( u_j \) such that \( u_{j-1} = v_q < v_{q-1} < \cdots < v_1 < u_j \). Let \( B_i \) be the \( m \)-AP-block containing \( v_i \). Let \( l_i \) be the length of \( B_i \) and
\[ B'_i = (v'_i, v'_i + m', \ldots, v'_i + (l_i - 1)m') \]
be the corresponding \( m' \)-AP-blocks generated by the Algorithm \( \psi \). It suffices to show that the path \( Q \) consists of the elements of \( B'_s, B'_{s-1}, \ldots, B'_1 \).

Suppose that \( v_1, v_2, \ldots, v_p \) are all singletons, but \( v_{p+1} \) is not a singleton. Then \( p \leq q - 1 \) since \( u_{j-1} \) is always a non-singleton head. The condition (3.5) yields that
\[ p \geq (\max\{m, m'\} - 1)(i_r - 1). \]

We now wish to show that for any \( 1 \leq i \leq q \), the block \( B_i \) lies entirely on the path \( Q \). If \( i \leq p \), then \( B_i = (v_i) \) is a singleton block lying on \( Q \). Otherwise, we have \( i \geq p + 1 \) and
\[ B_i = (v_i, v_i + m, \ldots, v_i + (l_i - 1)m). \]
But the total number of points between any two consecutive elements of \( B_i \) is
\[ (l_i - 1)(m - 1) \leq (\max\{m, m'\} - 1)(i_r - 1) \leq p. \]
Intuitively, all these points can be fulfilled by the singletons \( v_p, v_{p-1}, \ldots, v_1 \). Since \( u_j > v_1 \), the largest element \( v_i + (l_i - 1)m \) in the block \( B_i \) is smaller than \( u_j \). Hence the block \( B_i \) \((i = 1, 2, \ldots, q)\) lies entirely on \( Q \).
Therefore, the total number of elements in $B_q, B_{q-1}, \ldots, B_1$ equals the length $u_j - u_{j-1}$ of the path $Q$. Since $B'_i$ has the same number of elements as $B_i$, the total number of elements in $B'_q, B'_{q-1}, \ldots, B'_1$ also equals $u_j - u_{j-1}$.

Moreover, it can be shown that the block $B'_i$ also lies entirely on the path $Q$ for any $1 \leq i \leq q$. If $i \leq p$, the block $B'_i = (v'_i)$ is a singleton given by the separation algorithm. Since the total number of elements in $B'_q, B'_{q-1}, \ldots, B'_{i+1}$ is smaller than $u_j - u_{j-1}$ and $v'_i$ is chosen to be the smallest element which is not in $B'_q, B'_{q-1}, \ldots, B'_{i+1}$, we see the relation $v'_i < u_j$. Otherwise, we have $i \geq p + 1$ and the total number of points between any two consecutive elements of $B'_i$ equals

$$(l_i - 1)(m' - 1) \leq (\max\{m, m'\} - 1)(i_r - 1) \leq p.$$ 

Intuitively, all these points can be fulfilled by the singletons $v'_{p}, v'_{p-1}, \ldots, v'_1$. Since $u_j > v'_1$, the largest element $v'_1 + (l_i - 1)m'$ in the block $B'_i$ is smaller than $u_j$. Consequently, the block $B'_i$ lies entirely on $Q$.

In summary, the total number of elements in $B'_q, B'_{q-1}, \ldots, B'_1$ which lie on the path $Q$ coincides with the length of $Q$. Hence the path $Q$ consists of the elements of $B'_s, B'_{s-1}, \ldots, B'_1$. This completes the proof. 

**Theorem 3.5.** Let $T$ be a type as given before. The separation algorithm induces a bijection between $P_m$ and $P_{m'}$ under the condition (3.5).

**Proof.** We may employ the separation algorithm by interchanging the roles of $m$ and $m'$ to construct an $m$-AP-partition from an $m'$-AP-partition, and we denote this map by $\varphi$. We aim to show that $\varphi$ is indeed the inverse map of $\psi$, namely, $\varphi(\psi(\pi)) = \pi$ for any $\pi \in P_m$.

Let $h_1, h_2, \ldots, h_t$ be the heads of $\pi$ for the map $\psi$, where $h_1$ is the starting point. Assume that $\pi$ has AP-blocks $B_1, B_2, \ldots, B_t$ with $h_i$ being the head of $B_i$. Let $l_i$ be the length of $B_i$. By the construction of $\psi$, the generated heads $h'_1 = h_1, h'_2, \ldots, h'_t$ have the order $h'_1 < h'_2 < \cdots < h'_t$ in accordance with $h_1 < h_2 < \cdots < h_t$. It follows that $g(h'_1)$ is the maximum considering all heads of the AP-partition $\psi(\pi)$.

We now apply the map $\varphi$ on the $m'$-AP-partition $\psi(\pi)$ and choose $h'_1$ as the starting point. Let $h''_1, h''_2, \ldots, h''_t$ be the heads generated by $\varphi$ respectively. In light of the construction of $\varphi$, we have $h''_1 = h'_1 = h_1$ and $h''_1 < h''_2 < \cdots < h''_t$. For any $i$, the separation algorithm has the property that the length of the $m$-AP-block in $\varphi(\psi(\pi))$ containing $h''_i$ is $l_i$, which is the length of the $m$-AP-block in $\pi$ containing $h_i$.

Note that both $\varphi(\psi(\pi))$ and $\pi$ are $m$-AP-partitions. They have the same starting point $h''_1 = h_1$ and the same length sequence $(l_1, l_2, \ldots, l_t)$. Thus for any $i = 2, 3, \ldots, t$, the head $h''_i$ is the smallest point which is not contained in the $m$-AP-blocks $B_1, B_2, \ldots, B_{i-1}$, and so does $h_i$. Hence we conclude that $h''_i = h_i$ and $\varphi(\psi(\pi)) = \pi$. This completes the proof.
Acknowledgments. This work was supported by the 973 Project, the PCSIRT Project of the Ministry of Education, the Ministry of Science and Technology, and the National Science Foundation of China.

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