RANDOM ATTRACTORS FOR STOCHASTIC PARABOLIC EQUATIONS WITH ADDITIVE NOISE IN WEIGHTED SPACES

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Abstract. In this paper, we establish the existence of random attractors for stochastic parabolic equations driven by additive noise as well as deterministic non-autonomous forcing terms in weighted Lebesgue spaces $L^r_\delta(O)$, where $1 < r < \infty$, $\delta$ is the distance from $x$ to the boundary. The nonlinearity $f(x,u)$ of equation depending on the spatial variable does not have the bound on the derivative in $u$, and then causes critical exponent. In both subcritical and critical cases, we get the well-posedness and dissipativeness of the problem under consideration and, by smoothing property of heat semigroup in weighted space, the asymptotical compactness of random dynamical system corresponding to the original system.

1. Introduction. Consider the following stochastic parabolic equation with additive noise

$$
\begin{aligned}
&du - \Delta u dt = (f(x,u) + g(x,t))dt + \sum_{j=1}^{m} h_j d\omega_j, \quad t > \tau, \quad x \in O, \\
&u = 0, \quad t > \tau, \quad x \in \partial O, \\
&u(x,\tau) = u_\tau, \quad \tau \in \mathbb{R}, \quad x \in O,
\end{aligned}
$$

(1.1)

where $O$ is the bounded domain in $\mathbb{R}^N$ with smooth $C^2$ boundary $\partial O$, $h_j \in L^\eta_\delta(O) \cap W^{2,r}_\delta(O)$ (see definition below), $\eta > r$ as in (3.11), $j = 1, \cdots, m$, and $\{\omega_j\}_{j=1}^m$ are independent two-sided real-valued Wiener processes on a probability space which will be specified in Section 3. The nonlinearity $f(x,u)$ and deterministic non-autonomous forcing term $g(x,t)$ satisfy the following assumptions:

$(P_1)$ There exist $\rho > 1$ and $C > 0$ such that

$$
| f(x,u) - f(x,v) | \leq C \| a(x) \| u - v \| (| u |^{\rho-1} + | v |^{\rho-1} + 1).
$$

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\( (P_2) \) \( g(x, t) \in L^\gamma_{\text{loc}}(\mathbb{R}; L^r_\delta(O)), \) i.e.,
\[
\int_{t_1}^{t_2} \| g(x, s) \|^\gamma_{L^r_\delta} ds < +\infty, \quad \forall \, [t_1, t_2] \subset \mathbb{R},
\]
where \( 1 < \gamma, r < \infty, \) and \( \frac{1}{\gamma} < \min\{\frac{1}{2}, 1 - \frac{1}{r}\}. \)

Based on the wide applications such as describing the random phenomena in physics, chemistry, biology and control theory, the stochastic evolution equations have been extensively investigated in recent decades, see, e.g., [13, 19, 32]. One of the most important problems in studying stochastic partial differential equation (SPDE) is to understand the asymptotic behavior of them. When the SPDE possesses the random forcing, for example, additive white noise term, the theory on the existence of attractor for deterministic infinite-dimensional dynamical systems, see, e.g., [3, 12, 20, 27, 28], can not be used directly. In fact, there is no chance that bounded subsets of the state space remain invariant for SPDE. To overcome this difficulty, H. Cruel and F. Flandoli in [14, 15] extended the concept of attractor for deterministic system to random system, and introduced the random attractor (or pullback attractor) which attracts the random set in state space in the sense of pullback. In this situation, one can discover compact invariant sets which are not fixed, but they depend on chance, and they move with time. Afterwards, in order to capture the structure of deterministic non-autonomous dynamical systems, pullback attractor is also used to study the long-time behavior of them, see, e.g., [8, 10]. For other related studies, we refer reader to, e.g., [5, 7, 11, 30, 31, 33].

In this paper, we study the dynamics of SPDE (1.1) in weighted Lebesgue spaces \( L^r_\delta(O). \) There are some inspirations in investigating this topic:

Firstly, when study SPDE (1.1) in \( L^r_\delta(O), \) comparing with the results in [4, 7, 15, 16, 29, 32, 34] we can establish the random attractor for (1.1) in weak topological spaces. Moreover, since \( \delta \) tends to zero as \( x \) close to the boundary of \( O, \) the solution of SPDE (1.1) in \( L^r_\delta(O) \) can grow more rapidly near the boundary than in normal Lebesgue space \( L^r(O). \)

Secondly, comparing with nonlinearity \( f \) in [4, 5, 7, 15, 16, 29, 30, 32, 33, 34], we remove the assumption on the derivative of \( f \) with respect to \( u, \) that is, we do not assume its bounded on \( \frac{\partial f(x, u)}{\partial u} \):
\[
\frac{\partial f(x, u)}{\partial u} \leq c, \quad c > 0. \tag{1.2}
\]
The lack of bound on \( \frac{\partial f(x, u)}{\partial u} \) brings about the critical exponent on \( f. \) In the subcritical case, by the smoothing property of the Dirichlet heat semigroup in weighted space, we get the existence of random attractor for (1.1). When nonlinearity \( f \) has the critical growth, as it is pointed in the deterministic case in [2, 6, 9, 18, 24], the uniform existence time of solutions of (1.1) is only on the compact set of the initial data space. In order to overcome this obstacle, we introduce the almost critical nonlinearity as in [9, 23] to get the uniform existence time of solutions on any bounded set of initial data space, and then get the existence of random attractor for (1.1).

Thirdly, for the fixed growth exponents \( \rho \) and \( \beta \) \((a(x) \text{ belongs to } L^\beta_\delta(O))\) of \( f(x, u), \) there is the new result on the existence of random dynamical system (RDS) generated by SPDE (1.1) in weighted Lebesgue spaces \( L^r_\delta(O). \) Under the assumptions \((P_1)-(P_2)\), if \( r \geq (\rho - 1)(\frac{2}{N} - \frac{1}{\beta})^{-1} \) (equivalently, \( \frac{1}{\beta} + \frac{\alpha - 1}{r} \leq \frac{2}{N} \)) and \( r \geq 1, \) we know from Remark 4 that the problem (1.1) generates RDS in \( L^r(O); \) and if
\( r \geq (\rho - 1)(\frac{2}{N + 1} - \frac{1}{\beta})^{-1} \) (equivalently, \( \frac{1}{\beta} + \frac{\epsilon - 1}{r} \leq \frac{2}{N + 1} \)) and \( r \geq 1 \), it follows from Theorem 3.1-3.2 that the problem (1.1) generates RDS in \( L^r_\delta(O) \). As it is pointed in Remark 3.2 (a) in [18], \( L^r(\O) \subset L^r_\delta(O) \) and \( L^r_\delta(O) \subset L^{r-\epsilon}(\O) \) for \( \epsilon > 0 \), but \( L^r_\delta(O) \not\subset L^{r-\epsilon}(\O) \). Therefore, there is a “gap” existence interval for \( r \):

\[
(\rho - 1)(\frac{2}{N+1} - \frac{1}{\beta})^{-1} \leq r < \max\{2, 2(\rho - 1)(\frac{2}{N} - \frac{1}{\beta})^{-1}\},
\]

that is, (1.1) generates RDS in \( L^r_\delta(O) \) but not in \( L^{r-\epsilon}(\O) \) for \( \epsilon \) small enough. For other study of elliptic and parabolic problems in \( L^r_\delta(O) \), we refer reader to, e.g., [18, 26].

Fourthly, by using the smoothing property of the Dirichlet heat semigroup in weighted space \( L^r_\delta(O) \), there are some advantages in study SPDE (1.1) with bounded domain. On the one hand, since the nonlinearity \( f(x,u) \) of (1.1) depends on the spatial variable, we can get the regularity of solution and existence of random attractor without assumption on the derivative of \( f \) with respect to \( x \) as in [4, 16, 29, 30, 32, 33, 34]:

\[
| \frac{\partial f}{\partial x}(x,u) | \leq \psi_3(x).
\]

The reason for this is that we need not take the inner product of corresponding converted equation with \(-\Delta v\) in \( L^2(O) \), see details, e.g., in the proof of Lemma 4.5 in [4]. In this situation, the assumption on \( f \) can be relaxed further in some sense. On the other hand, it is convenient to establish random attractor for SPDE (1.1) in general Lebesgue space \( L^r_\delta(O) \).

Finally, comparing with the previous works, e.g., in [4, 16, 32, 34], we relax the assumptions on nonlinearity \( f(x,u) \), that is, \( f(x,u) \) does not necessarily satisfy (1.2)-(1.3) and

\[
| f(x,u) | \leq \alpha_2 | u |^{p-2} + \psi_2(x),
\]

where \( \alpha_2 \) is a positive constant, and consider the more general case which only satisfies \((P_2)_\delta \). Recently, authors in [16] consider the existence of \( D \)-uniform attractor for (1.1) in \( L^2(O) \), but the translation compact property for the deterministic non-autonomous forcing term \( g(x,t) \) in \( L^{p,\infty}_{loc}(\mathbb{R}, L^2(O)) \) (translation bounded in \( L^2_{loc}(\mathbb{R}, L^2(O)) \)) is needed. Under the assumption (4.3) which \( g(x,t) \) does not necessarily be translation compact in \( L^{p,\infty}_{loc}(\mathbb{R}, X) \), we establish the existence of \( D \)-random attractor in \( L^r_\delta(O) \). Comparing with the known works in obtaining the existence of random attractors for stochastic dynamical systems, we establish the existence of \( D \)-random attractor for (1.1) in general initial data space \( L^r_\delta(O) \), \( 1 < r < +\infty \); and in getting the \( D \)-pullback asymptotically compact property for them, we use the difference method of smoothing property of heat semigroups in weighted spaces.

The paper is organized as follows: In the next section, we recall some fundamental results on random attractor for stochastic dynamical systems, and heat semigroup theory on weighted space; in Section 3, we show that (1.1) generates a continuous RDS in \( L^r_\delta(O) \); in Section 4 and 5, we establish the existence of random attractor for (1.1) with critical and subcritical cases in \( L^2_\delta(O) \) and \( L^r_\delta(O) \), respectively.

For the rest of paper, we denote by \( C \) the generic positive constant which may changes its value from line to line or even in the same line.
2. Preliminaries. In this section, we recall some concepts and existence results on random attractors for stochastic dynamical systems and the working space for (1.1).

2.1. Random dynamical system

Let \((X, d)\) be a complete separable metric space with Borel \(\sigma\)-algebra \(\mathcal{B}(X)\), \(\Omega_1\) be a nonempty set, and \((\Omega_2, \mathcal{F}_2, P)\) be a probability space. For the following concepts, the reader is also referred to, e.g., [4, 5, 17, 29] for more details.

Definition 2.1. \((\Omega_1, (\theta_{t, t})_{t \in \mathbb{R}})\) is called a parametric dynamical system if for every \(t \in \mathbb{R}\), \(\theta_{t, t} : \Omega_1 \to \Omega_1\) is a mapping, \(\theta_{1, 0}\) is the identity on \(\Omega_1\), and \(\theta_{1, s+t} = \theta_{1, t} \circ \theta_{1, s}\) for all \(t, s \in \mathbb{R}\).

Definition 2.2. \((\Omega_2, \mathcal{F}_2, P, (\theta_{t, t})_{t \in \mathbb{R}})\) is called a metric dynamical system if \(\theta_{t, t} : \mathbb{R} \times \Omega_2 \to \Omega_2\) is \((\mathcal{B}(\mathbb{R}) \times \mathcal{F}_2, \mathcal{B}(X))\)-measurable, \(\theta_{2, 0}\) is the identity on \(\Omega_2\), \(\theta_{2, s+t} = \theta_{2, t} \circ \theta_{2, s}\) for all \(s, t \in \mathbb{R}\), and \(\theta_{2, P} = P\) for all \(t \in \mathbb{R}\).

Definition 2.3. A continuous random dynamical system (RDS) on \(X\) over a parametric dynamical system \((\Omega_1, (\theta_{t, t})_{t \in \mathbb{R}})\) and a metric dynamical system \((\Omega_2, \mathcal{F}_2, P, (\theta_{t, t})_{t \in \mathbb{R}})\) is a mapping

\[
\Phi : \mathbb{R}^+ \times \Omega_1 \times \Omega_2 \times X \to X, \quad (t, \omega_1, \omega_2, x) \to \Phi(t, \omega_1, \omega_2, x),
\]

and satisfies, for all \(\omega_1 \in \Omega_1\) and \(P\)-a.e. \(\omega_2 \in \Omega_2\):

(i) \(\Phi(t, \omega_1, \cdot, \cdot): \mathbb{R}^+ \times \Omega_2 \times X \to X\) is \((\mathcal{B}(\mathbb{R}) \times \mathcal{F}_2, \mathcal{B}(X))\)-measurable;

(ii) \(\Phi(0, \omega_1, \omega_2, \cdot)\) is the identity on \(X\);

(iii) \(\Phi(t+s, \omega_1, \omega_2, \cdot) = \Phi(t, \theta_1 \omega_1, \omega_2, \cdot) \circ \Phi(s, \omega_1, \omega_2, \cdot)\) for all \(t, s \in \mathbb{R}^+\); and

(iv) \(\Phi(t, \omega_1, \omega_2, \cdot) : X \to X\) is continuous.

In this paper, we always assume that \(\Phi\) is a continuous RDS over \((\Omega_1, (\theta_{t, t})_{t \in \mathbb{R}})\) and \((\Omega_2, \mathcal{F}_2, P, (\theta_{t, t})_{t \in \mathbb{R}})\). Denote by \(\mathcal{D}\) a collection of some families of nonempty subsets of \(X\):

\[
\mathcal{D} = \{ D = \{ D(\omega_1, \omega_2) \subseteq X : D(\omega_1, \omega_2) \neq \emptyset, \omega_1 \in \Omega_1, \omega_2 \in \Omega_2 \} \}.
\]

Definition 2.4. A family of nonempty subsets \(D = \{ D(\omega_1, \omega_2) : \omega_1 \in \Omega_1, \omega_2 \in \Omega_2 \}\) of \(X\) is called tempered with respect to \((\Omega_1(\theta_{t, t})_{t \in \mathbb{R}})\) and \((\Omega_2, \mathcal{F}_2, P, (\theta_{t, t})_{t \in \mathbb{R}})\) if for \(\omega_1 \in \Omega_1\) and \(P\)-a.e. \(\omega_2 \in \Omega_2\),

\[
\lim_{t \to \infty} e^{-ct} d(D(\theta_{1, -t} \omega_1, \omega_2)) = 0, \quad \text{for all } c > 0,
\]

where \(d(D) = \sup_{x \in D} \| x \|_X\).

Definition 2.5. Let \(\mathcal{D}\) be a collection of some families of nonempty subsets of \(X\) and \(K = \{ K(\omega_1, \omega_2) : \omega_1 \in \Omega_1, \omega_2 \in \Omega_2 \} \subseteq \mathcal{D}\). Then \(K\) is called a \(\mathcal{D}\)-pullback absorbing set for \(\Phi\) if for all \(\omega_1 \in \Omega_1\) and \(P\)-a.e. \(\omega_2 \in \Omega_2\), and for every \(B \in \mathcal{D}\), there exists \(T = T(B, \omega_1, \omega_2) > 0\) such that

\[
\Phi(t, \theta_{1, -t} \omega_1, \theta_{2, -t} \omega_2, B(\theta_{1, -t} \omega_1, \theta_{2, -t} \omega_2)) \subseteq K(\omega_1, \omega_2), \quad \text{for all } t \geq T.
\]

If, in addition, for all \(\omega_1 \in \Omega_1\) and \(P\)-a.e. \(\omega_2 \in \Omega_2\), \(K(\omega_1, \omega_2)\) is closed nonempty subset of \(X\), and \(\omega_2 \in \Omega_2 \to d(x, K(\omega_1, \omega_2))\) is \((\mathcal{F}_2, \mathcal{B}(\mathbb{R}))\)-measurable for every fixed \(x \in X\) and \(\omega_1 \in \Omega_1\), then \(K\) is called a closed measurable \(\mathcal{D}\)-pullback absorbing set for \(\Phi\).

Definition 2.6. Let \(\mathcal{D}\) be a collection of some families of nonempty subsets of \(X\). Then \(\Phi\) is said to be \(\mathcal{D}\)-pullback asymptotically compact in \(X\) if for all \(\omega_1 \in \Omega_1\) and \(P\)-a.e. \(\omega_2 \in \Omega_2\), \(\{ \Phi(t_n, \theta_{1, -t_n} \omega_1, \theta_{2, -t_n} \omega_2, x_n) \}_{n=1}^{\infty}\) has a convergent subsequence
in $X$ whenever $t_n \to \infty$, and $x_n \in B(\theta_{1,-t_n}\omega_1, \theta_{2,-t_n}\omega_2)$ with \( B(\omega_1, \omega_2) : \omega_1 \in \Omega_1, \omega_2 \in \Omega_2 \) \in \mathcal{D}.

**Definition 2.7.** Let $\mathcal{D}$ be a collection of some families of nonempty subsets of $X$. Then \( \mathcal{A} = \{ A(\omega_1, \omega_2) : \omega_1 \in \Omega_1, \omega_2 \in \Omega_2 \} \) is called a $\mathcal{D}$-random attractor (or $\mathcal{D}$-pullback attractor) for $\Phi$ if the following conditions are satisfied, for all $\omega_1 \in \Omega_1$ and $P$-a.e. $\omega_2 \in \Omega_2$,

(i) $A(\omega_1, \omega_2)$ is compact, and $\omega_2 \to d(x, A(\omega_1, \omega_2))$ is measurable for every $x \in X$;

(ii) $\{ A(\omega_1, \omega_2) : \omega_1 \in \Omega_1, \omega_2 \in \Omega_2 \}$ is invariant, that is,

$$
\Phi(t, \omega_1, \omega_2, A(\omega_1, \omega_2)) = A(\theta_{1,t}\omega_1, \theta_{2,t}\omega_2) \quad \text{for all } t \geq 0;
$$

(iii) $\mathcal{A}$ attracts every set in $\mathcal{D}$, that is, for every $B = \{ B(\omega_1, \omega_2) : \omega_1 \in \Omega_1, \omega_2 \in \Omega_2 \}$ \in $\mathcal{D}$,

$$
\lim_{t \to \infty} d(\Phi(t, \omega_1, \theta_{2,-t}\omega_2, B(\theta_{1,-t}\omega_1, \theta_{2,-t}\omega_2)), A(\omega_1, \omega_2)) = 0,
$$

where $d$ is the Hausdorff semi-metric given by $d(Y, Z) = \sup_{y \in Y} \inf_{z \in Z} || y - z ||_X$ for any $Y, Z \subset X$.

In order to guarantee $\mathcal{D}$-random attractor $\mathcal{A}$ belonging to $\mathcal{D}$ and being unique, and obtain a sufficient and necessary condition for the existence of $\mathcal{A}$, we introduce the following concept.

**Definition 2.8.** A collection $\mathcal{D}$ of some families of nonempty subsets of $X$ is said to be neighborhood closed if for each $D = \{ D(\omega_1, \omega_2) : \omega_1 \in \Omega_1, \omega_2 \in \Omega_2 \} \in \mathcal{D}$, there exists a positive number $\epsilon$ depending on $D$ such that the family

$$
\{ B(\omega_1, \omega_2) : B(\omega_1, \omega_2) \text{ is a nonempty subset of } N_\epsilon(D(\omega_1, \omega_2)), \forall \omega_1 \in \Omega_1, \omega_2 \in \Omega_2 \}
$$

also belongs to $\mathcal{D}$.

The following concept can be used to describe the structure of $\mathcal{D}$-random attractor $\mathcal{A}$. A mapping $\psi : \mathbb{R} \times \Omega_1 \times \Omega_2 \to X$ is called a $\mathcal{D}$-complete orbit of $\Phi$ in $\mathcal{D}$ if for every $\tau \in \mathbb{R}$, $t \geq 0$, $\omega_1 \in \Omega_1$ and $P$-a.e. $\omega_2 \in \Omega_2$, there holds

$$
\Phi(t, \theta_{1,t}\omega_1, \theta_{2,t}\omega_2, \psi(\tau, \omega_1, \omega_2)) = \psi(t + \tau, \omega_1, \omega_2),
$$

and there exists $D = \{ D(\omega_1, \omega_2) : \omega_1 \in \Omega_1, \omega_2 \in \Omega_2 \} \in \mathcal{D}$ such that $\psi(t, \omega_1, \omega_2)$ belongs to $D(\theta_{1,t}\omega_1, \theta_{2,t}\omega_2)$ for every $t \in \mathbb{R}$, $\omega_1 \in \Omega_1$ and $P$-a.e. $\omega_2 \in \Omega_2$. The following results on existence and uniqueness of $\mathcal{D}$-random attractor for $\Phi$ can be found in [29].

**Theorem 2.9.** Let $\mathcal{D}$ be a neighborhood closed collection of some families of nonempty subsets of $X$, and $\Phi$ a continuous RDS on $X$ over $(\Omega_1, (\theta_{1,t})_{t \in \mathbb{R}})$ and $(\Omega_2, \mathcal{F}_2, P, (\theta_{2,t})_{t \in \mathbb{R}})$. Then $\Phi$ has a $\mathcal{D}$-random attractor $\mathcal{A}$ in $\mathcal{D}$ if and only if $\Phi$ is $\mathcal{D}$-pullback asymptotically compact in $X$ and $\Phi$ has a closed measurable $\mathcal{D}$-pullback absorbing set $K$ in $\mathcal{D}$. The $\mathcal{D}$-random attractor $\mathcal{A}$ is unique and given by, for each $\omega_1 \in \Omega_1$ and $P$-a.e. $\omega_2 \in \Omega_2$,

$$
\mathcal{A}(\omega_1, \omega_2) = \bigcap_{\tau \geq 0} \bigcup_{t \geq \tau} \Phi(t, \theta_{1,-t}\omega_1, \theta_{2,-t}\omega_2, K(\theta_{1,-t}\omega_1, \theta_{2,-t}\omega_2)) = \{ \psi(0, \omega_1, \omega_2) : \psi \text{ is a } \mathcal{D}\text{-complete orbit of } \Phi \}.
$$

2.2. Weighted Lebesgue space

Let $\mathcal{O}$ be a smooth ($C^2$) bounded domain in $\mathbb{R}^N$. For $1 \leq r < \infty$, define the weighted Lebesgue spaces $L^r_\delta(\mathcal{O})$ by

$$
L^r_\delta(\mathcal{O}) := L^r(\mathcal{O}; \delta(x)dx),
$$
where $\delta = \delta(x) = \text{dist}(x, \partial \mathcal{O})$. Evidently, $L^r_\delta(\mathcal{O})$ is a Banach space endowed with the norm
$$
\| u \|_{L^r_\delta} = \left( \int_{\mathcal{O}} | u |^r \delta(x) dx \right)^{\frac{1}{r}}.
$$

When $\mathcal{O}$ has a smooth $C^2$ boundary, it is well-known that there exist constants $c_1, c_2 > 0$ such that
$$
c_1 \phi_1(x) \leq \delta(x) \leq c_2 \phi_1(x),
$$
where $\phi_1(x)$ is the first eigenfunction associated to the first eigenvalue $\lambda_1 > 0$ of $-\Delta$ in $H^1_0(\mathcal{O})$, normalized by $\int_{\mathcal{O}} \phi_1(x) dx = 1$. In this situation, $L^r_\delta(\mathcal{O})$ endowed with the norm $\left( \int_{\mathcal{O}} | u |^r \phi_1(x) dx \right)^{\frac{1}{r}}$ for $u \in L^r_\delta(\mathcal{O})$ equals to $L^r(\mathcal{O})$ and two norms of them are equivalent. It is also clear that $L^\infty_\delta = L^\infty = L^\infty$.

Denote by $W^{1,p}_\delta(\mathcal{O})$, $l \geq 0$, $p \geq 1$, the weighted Sobolev spaces with $D^l u \in L^p_\delta(\mathcal{O})$ for $|i| \leq l$, and denote by $W^{r,p}_{0,\delta}(\mathcal{O})$, $p \geq 1$, the closure of $C^\infty(\mathcal{O})$ in the space $W^{r,p}_\delta(\mathcal{O})$, which is equipped with the norm $\left( \sum_{|i|=l} \| D^i u \|_{L^p_\delta} \right)^{\frac{1}{p}}$. Let $S(t)$ be Dirichlet heat semigroup, by interpolation theory and Lemma 2.3 in [22], we have the following result:

**Theorem 2.10.** Let $1 \leq p \leq q \leq \infty$, $0 \leq r \leq s < \infty$, $S(t)$ be a heat semigroup on $W^{r,p}_{0,\delta}(\mathcal{O})$. Then there exists a constant $C > 0$ such that
$$
\| S(t) u \|_{W^{r,s}_{0,\delta}} \leq C t^{-\frac{r}{s}} \| u \|_{W^{r,p}_{0,\delta}},
$$
for all $t > 0$ and $u \in W^{r,p}_{0,\delta}(\mathcal{O})$.

3. **Random dynamical system for (1.1).** Here we show that there is a continuous random dynamical system on $L^2_\delta(\mathcal{O})$ generated by the SPDE (1.1).

Let $\Omega_1 = \mathbb{R}$, define a family of operators $\{\theta_{1,t}\}_{t \in \mathbb{R}}$ by
$$
\theta_{1,t}(h) = h + t, \quad \text{for all } t, h \in \mathbb{R}. \tag{3.1}
$$

In the following, we consider the probability space $(\Omega_2, \mathcal{F}_2, P)$, where
$$
\Omega_2 = \{ \omega_2 = (\omega^1_2, \omega^2_2, \cdots, \omega^m_2) \in C(\mathbb{R}; \mathbb{R}^m) : \omega_2(0) = 0 \},
$$
$\mathcal{F}_2$ is the Borel $\sigma$-algebra induced by the compact-open topology of $\Omega_2$, and $P$ is the corresponding Wiener measure on $(\Omega_2, \mathcal{F}_2)$. Then we can identify $\omega_2$ with
$$
W_2(t) = (w^1_2(t), w^2_2(t), \cdots, w^m_2(t)) = \omega_2(t), \quad \text{for all } t \in \mathbb{R}.
$$

Define the time shift by
$$
\theta_{2,t} = \omega_2(\cdot + t) - \omega_2(t), \quad \omega_2 \in \Omega_2, \quad t \in \mathbb{R}. \tag{3.2}
$$

Then $(\Omega_2, \mathcal{F}_2, P, (\theta_{2,t})_{t \in \mathbb{R}})$ is a metric dynamical system.

In order to deal with the random forcing term of (1.1), we consider the one-dimensional Ornstein-Uhlenbeck equation
$$
dz_j + z_j dt = dw^j_2(t), \quad j = 1, 2, \cdots, m. \tag{3.3}
$$

By Itô integral and its integration by parts formula, see, e.g., [25], one can obtain that the solution of (3.3) is given by
$$
z_j(t) = z_j(\theta_{2,t} \omega^j_2) = - \int_{-\infty}^0 e^s(\theta_{2,t} \omega^j_2)(s) ds, \quad t \in \mathbb{R}. \tag{3.4}
$$
Note that the random variable $z_j(\omega_2)$ is tempered and $z_j(\theta_2,\omega_2)$ is $P$-a.e. continuous. Therefore, by Proposition 4.3.3 in [1], one can get that there exists a tempered function $q(\omega_2) > 0$ such that

$$\sum_{j=1}^{m} \left| z_j(\omega_2) \right|^r + \left| z_j(\omega_2) \right|^q \leq q(\omega_2),$$

(3.5)

where $q(\omega_2)$ satisfies, for $P$-a.e. $\omega_2 \in \Omega_2$,

$$q(\theta_2,\omega_2) \leq e^{\frac{2}{\beta}} \| q(\omega_2) \|, \quad t \in \mathbb{R},$$

(3.6)

where $0 < \kappa < \lambda_1$.

It follows from (3.5)-(3.6) that, for $P$-a.e. $\omega_2 \in \Omega_2$,

$$\sum_{j=1}^{m} \left| z_j(\omega_2) \right|^r + \left| z_j(\theta_2,\omega_2) \right|^q \leq e^{\frac{2}{\beta}} \| q(\omega_2) \|, \quad t \in \mathbb{R}.$$  

(3.7)

Let $z(\theta_2,\omega_2) = \sum_{j=1}^{m} h_j z_j(\theta_2,\omega_2)$, then (3.3) yields

$$dz + zdt = \sum_{j=1}^{m} h_j d\omega_2.$$  

Let $v(t) = u(t) - z(\theta_2,\omega_2)$, where $u$ is the solution of (1.1). Then we can obtain that $v$ satisfies

$$\frac{\partial v}{\partial t} - \Delta v = f(x, v + z(\theta_2,\omega_2)) + g(x, t) + \Delta z(\theta_2,\omega_2) + z(\theta_2,\omega_2)$$

(3.8)

with initial data

$$v(\tau, x) = v_\tau = u_\tau - z(\theta_2,\omega_2).$$

(3.9)

As the proof of Theorem 3.1 in [22], we can get the well-posedness of (3.8)-(3.9) depending on the random parameter:

**Theorem 3.1.** Suppose that (P1)-(P2) hold, and let $a(x) \in L^\beta_\delta(\mathcal{O})$ with $1 < \beta \leq \infty$, and $\frac{1}{\beta} + \frac{1}{r} < \frac{2}{N+1}$ (resp. $\frac{1}{\beta} + \frac{1}{r} = \frac{2}{N+1}$), $r > 1$. Then for $P$-a.e. $\omega_2 \in \Omega_2$, if $v_\tau \in L^\delta_\beta(\mathcal{O})$, there exists a $T = T(v_\tau)$ such that there is a solution

$$v(t) \in C([\tau, T]; L^\delta_\beta(\mathcal{O})) \bigcap C((\tau, T]; L^2_\delta(\mathcal{O})) \bigcap C((\tau, T]; W^{1,m(\eta)}_0(\mathcal{O}))$$

(3.10)

of (3.8)-(3.9), where $\eta$ satisfies

$$\frac{1}{r} - \frac{2}{(N+1)\rho} < \frac{1}{\eta} < \min\left\{ \frac{1}{\rho}(1 - \frac{1}{\beta}), 1, \frac{1}{\rho - 1}\left\{ \frac{2}{N+1} - \frac{1}{\beta} \right\} \right\},$$

(3.11)

and $m(\eta) > 2$. This solution depends continuously on the initial data and is unique in the class

$$C([\tau, T]; L^\delta_\beta(\mathcal{O})) \bigcap L^\infty((\tau, T]; L^2_\delta(\mathcal{O})).$$

(3.12)

Moreover, there exists a constant $C > 0$ such that

$$(t - \tau)^{\frac{N+1}{2}(\frac{1}{r} - \frac{1}{\eta})} \| v \|_{L^2_\delta} \leq C.$$  

(3.13)
Furthermore, for any bounded set (resp. compact set) $K$ in $L_0^3(\mathcal{O})$, there is a uniform time $T = T(K)$ such that, for any $v_\tau \in K$, the solution of (3.8)-(3.9) exists on $[\tau, T]$.

Remark 1. Since we reduce the regularity on deterministic forcing term $g(x,t)$, we can not obtain the higher regularity of solution of (3.8)-(3.9) as in [22].

We know from Theorem 3.1 that in critical case, the uniform existence time of solutions of (3.8)-(3.9) is only on the compact set of initial data spaces. This fact is also pointed in Theorem 1 in [6], Theorem 1 in [2], and Theorem 5 in [18] in studying different types of evolution equations. In order to get the uniform existence time of solutions on any bounded set in initial data spaces for (3.8)-(3.9) with critical nonlinearity, we impose the following assumption on $f(x,u)$ as in [9, 23]:

$$(P_3)$$ Suppose that $f(x,u)$ satisfies $f(x,u) = b(x)f_1(u)$ with $b(x) \in L_0^3(\mathcal{O})$, $\beta > 1$, and $f_1(s) \in C^1(\mathbb{R})$ such that

$$\lim_{|s| \to \infty} \frac{f'_1(s)}{s^{\beta-1}} = 0, \quad \rho > 1.$$  

Remark 2. The nonlinearity satisfying above growth condition is close to critical in this situation, we call it the almost critical nonlinearity.

Theorem 3.2. Suppose that $(P_3)-(P_1)$ hold, and let $\frac{1}{\beta} + \frac{\rho-1}{2} = \frac{2}{N+1}$. Then for $P$-a.e. $\omega_2 \in \Omega_2$, if $v_\tau \in L_0^3(\mathcal{O})$, there exists a $T = T(v_\tau)$ such that there is a solution $v(t)$ of (3.8)-(3.9) belonging to (3.10). This solution depends continuously on the initial data, is unique in the class (3.12) and satisfies (3.13).

Furthermore, for any bounded set $K$ in $L_0^3(\mathcal{O})$, there is a uniform time $T = T(K)$ such that, for any $v_\tau \in K$, the solution of (3.8)-(3.9) exists on $[\tau, T]$.

Proof. We only give the different ingredient in the proof of Theorem 3.1-3.2 in [22]. For $P$-a.e. $\omega_2 \in \Omega_2$, define

$$\Psi(v)(t) = S(t-\tau)v_\tau + \int_{\tau}^{t} S(t-s)\left[f(x,v+\theta_{2,s}\omega_2) + g(x,s) + \Delta z(\theta_{2,s}\omega_2) + \right. \left. v(\theta_{2,s}\omega_2)\right]ds.$$  

Let $$E = \{C(\tau,T); L_0^3(\mathcal{O}) : \lim_{t \to T} (t-\tau)^\alpha v(t) = 0\},$$ and $$X = \{v(t) \in E : \lim_{\tau < t \leq T} (t-\tau)^\alpha \|v(t)\|_{L_0^3} \leq \sigma\},$$

where $\alpha = \frac{N+1}{2}(\frac{1}{\beta} - \frac{1}{\rho})$ such that $\alpha \rho < 1$, $\eta > r$ satisfying (3.11), $0 < \sigma < 1$.

Without loss of generality, we can assume that $0 < T - \tau < 1$. Equip $X$ with the distance $d(v_1,v_2) = \lim_{\tau < t \leq T} (t-\tau)^\alpha \|v_1(t) - v_2(t)\|_{L_0^3}$ for all $v_1,v_2 \in X$.

It follows from $(P_3)$ that for any $\epsilon > 0$, there exists $C_\epsilon > 0$ such that

$$|f(x,u) - f(x,v)| \leq |b(x)| (C_\epsilon + \epsilon |u|^{\rho})$$  

$$|f(x,u) - f(x,v)| \leq |b(x)| (C_\epsilon + \epsilon |u|^{\rho-1} + \epsilon |v|^{\rho-1}).$$

(3.15) (3.16)

By (3.15) we get that

$$(t-\tau)^\alpha \|\Psi(v)(t)\|_{L_0^3} \leq (t-\tau)^\alpha\|S(t-\tau)v_\tau\|_{L_0^3} + (t-\tau)^\alpha \int_{\tau}^{t} \|S(t-s)[f(x,v+\theta_{2,s}\omega_2)] + g(x,s) + \Delta z(\theta_{2,s}\omega_2) + z(\theta_{2,s}\omega_2)\|_{L_0^3} ds$$
\[ \leq (t - \tau)^{\tau} \| S(t - \tau)v_\tau \|_{L^p_t} + C(t - \tau)^{\tau} \int_\tau^t \| S(t - s)b(x)(C_\epsilon + \epsilon | v(s) |^p + \\
\epsilon | z(\theta_{2,\omega_2}) |^p) \|_{L^q_t} ds + (t - \tau)^{\tau} \int_\tau^t \| S(t - s)g(x, s) \|_{L^q_t} ds + (t - \tau)^{\tau} \times \\
\int_\tau^t \| S(t - s)(\Delta z(\theta_{2,\omega_2}) + z(\theta_{2,\omega_2})) \|_{L^q_t} ds. \] 

(3.17)

Now, we estimate every term except first one on the right-hand side of (3.17). Let \( \frac{1}{q} = \frac{1}{p} + \frac{\rho}{\eta} \). By Theorem 2.10 and (3.5)-(3.6) we have

\[ C(t - \tau)^{\tau} \int_\tau^t \| S(t - s)b(x)(C_\epsilon + \epsilon | v(s) |^p + \epsilon | z(\theta_{2,\omega_2}) |^p) \|_{L^q_t} ds \]

\[ \leq C(t - \tau)^{\tau} \int_\tau^t (t - s)^{-\frac{\eta + 1}{\eta} \left( \frac{1}{p} + \frac{\rho}{\eta} \right)} \| b(x)(C_\epsilon + \epsilon | v(s) |^p + \epsilon | z(\theta_{2,\omega_2}) |^p) \|_{L^q_t} ds \]

\[ \leq C(t - \tau)^{\tau} \int_\tau^t \left( t - s \right)^{-\frac{\eta + 1}{\eta} \left( \frac{1}{p} + \frac{\rho}{\eta} \right)} \left( \int_\Omega | b(x) |^{\frac{\eta}{\eta + 1}} G^{1\frac{\eta}{\eta + 1}}(s, \theta(\omega_2)) \right)^{\frac{\eta}{\eta + 1}} (\int_\Omega (C_\epsilon + \epsilon | v(s) |^p + \epsilon | z(\theta_{2,\omega_2}) |^p)^{\frac{\eta}{\eta + 1}} ds \]

\[ \leq C(t - \tau)^{\tau} \| b(x) \|_{L^p_t} \int_\tau^t \left( t - s \right)^{-\frac{\eta + 1}{\eta} \left( \frac{1}{p} + \frac{\rho}{\eta} \right)} (C_\epsilon + \epsilon \| v(s) \|_{L^p_t}) \]

\[ + \epsilon \| z(\theta_{2,\omega_2}) \|_{L^p_t} ds \]

\[ \leq C(t - \tau)^{\tau} \| b(x) \|_{L^p_t} \left[ C_\epsilon \frac{1}{1 - \frac{\eta + 1}{\eta} \left( \frac{1}{p} + \frac{\rho}{\eta} \right)} (t - \tau)^{1 - \frac{\eta + 1}{\eta} \left( \frac{1}{p} + \frac{\rho}{\eta} \right)} + \\
\epsilon ( \sup_{\tau < s \leq T} (s - \tau)^{\tau} \| v(s) \|_{L^p_t}) \int_\tau^t \left( t - s \right)^{-\frac{\eta + 1}{\eta} \left( \frac{1}{p} + \frac{\rho}{\eta} \right)} (s - \tau)^{-\alpha_\rho} ds + \\
\epsilon C \int_\tau^t \left( t - s \right)^{-\frac{\eta + 1}{\eta} \left( \frac{1}{p} + \frac{\rho}{\eta} \right)} q(\theta_{2,\omega_2}) ds \]

\[ \leq \frac{C_\epsilon C \| b(x) \|_{L^p_t} \| L^q_t \|}{1 - \frac{\eta + 1}{\eta} \left( \frac{1}{p} + \frac{\rho}{\eta} \right)} \left( t - \tau \right)^{\frac{\eta + 1}{\eta} \left( \frac{1}{p} + \frac{\rho}{\eta} \right)} + \epsilon C ( \sup_{\tau < s \leq T} (s - \tau)^{\tau} \| v(s) \|_{L^q_t}) \]

\[ \times \int_0^1 (1 - s)^{-\frac{\eta + 1}{\eta} \left( \frac{1}{p} + \frac{\rho}{\eta} \right)} s^{-\alpha_\rho} ds + \epsilon C \| b(x) \|_{L^p_t} \int_\tau^t \left( t - s \right)^{-\frac{\eta + 1}{\eta} \left( \frac{1}{p} + \frac{\rho}{\eta} \right)} \left( 1 - \frac{\eta + 1}{\eta} \left( \frac{1}{p} + \frac{\rho}{\eta} \right) \right) e^{\frac{\eta}{\eta - 1}} |s| \\
\times q(\omega_2) ds \]

\[ \leq \frac{C_\epsilon C \| b(x) \|_{L^p_t} \| L^q_t \|}{1 - \frac{\eta + 1}{\eta} \left( \frac{1}{p} + \frac{\rho}{\eta} \right)} \left( t - \tau \right)^{\frac{\eta + 1}{\eta} \left( \frac{1}{p} + \frac{\rho}{\eta} \right)} + \epsilon C ( \sup_{\tau < s \leq T} (s - \tau)^{\tau} \| v(s) \|_{L^q_t}) \]

\[ \times \left[ \int_0^1 (1 - s)^{-\frac{\eta + 1}{\eta} \left( \frac{1}{p} + \frac{\rho}{\eta} \right)} s^{-\alpha_\rho} ds + \epsilon C \| b(x) \|_{L^p_t} e^{\frac{\eta}{\eta - 1} Q(\tau)} q(\omega_2) \]

\[ \times (t - \tau)^{1 - \frac{\eta + 1}{\eta} \left( \frac{1}{p} + \frac{\rho}{\eta} \right)}, \] 

(3.18)

where \( Q(\tau) = \max\{ | \tau |, | \tau + 1 | \} \). Note that \( \frac{1}{p} < 1 - \frac{1}{p} \), by (3.11) we get that \( \frac{\eta + 1}{\eta} (\frac{1}{p} - \frac{1}{\eta}) < 1 \), where \( \frac{1}{\eta} + \frac{1}{p} = 1 \). Thus, we can obtain that

\[ (t - \tau)^{\tau} \int_\tau^t \| S(t - s)g(x, s) \|_{L^q_t} ds \]
Remark 3. Let \( L \) obtain the well-posedness of (3.8)-(3.9) in \( s > \gamma \) for Remark 4.

Theorem 3.1.

**Theorem 3.2.** The growth condition \( 1 \in X \) be a family of bounded nonempty subsets of \( X \) respectively.

\[ \Phi(\cdot, \cdot, \omega) \text{ is a continuous RDS on } L^p(\Omega). \]

Similarly, we get that \( \Psi(\cdot) \text{ satisfying } \int \Psi(\cdot) \leq \sigma. \)

We can get the similar estimates as (3.8)-(3.10) in [22], together with above inequality we get that \( \Psi(X) \subset X \). The remain proof is similar to that of Theorem 3.1 in [22].

\[ (t-\tau)^{\tilde{a}} \int_{\tau}^{t} \| S(t-s)(\Delta z(\theta_{2,s}, \omega_2) + z(\theta_{2,s}, \omega_2)) \|_{L^2} \, ds \]

\[ \leq (t-\tau)^{\tilde{a}} \int_{\tau}^{t} (t-s)^{\frac{N+1}{4} \left( \frac{1}{p} - \frac{1}{q} \right)} \| \Delta z(\theta_{2,s}, \omega_2) + z(\theta_{2,s}, \omega_2) \|_{L^2} \, ds \]

\[ \leq C(t-\tau)^{\tilde{a}} \int_{\tau}^{t} (t-s)^{\frac{N+1}{4} \left( \frac{1}{p} - \frac{1}{q} \right)} \| \Delta z(\theta_{2,s}, \omega_2) + z(\theta_{2,s}, \omega_2) \|_{L^2} \, ds \]

\[ \leq C e^{\frac{2Q}{\tilde{a}}(r)} q(\omega_2) \int_{\tau}^{t} (t-s)^{\frac{N+1}{4} \left( \frac{1}{p} - \frac{1}{q} \right)} \, ds \]

\[ \leq C e^{\frac{2Q}{\tilde{a}}(r)} q(\omega_2) \int_{\tau}^{t} (t-s)^{\frac{N+1}{4} \left( \frac{1}{p} - \frac{1}{q} \right)} \, ds \]

(3.20)

Now, by (3.17)-(3.20) we can choose \( \epsilon \) small enough and appropriate \( T \) such that

\[ (t-\tau)^{\tilde{a}} \| \Psi(\cdot) \|_{L^p(X)} \leq \sigma. \]

We can get the similar estimates as (3.8)-(3.10) in [22], together with above inequality we get that \( \Psi(X) \subset X \). The remain proof is similar to that of Theorem 3.1 in [22].

**Remark 3.** From (3.19) we get that if we assume that \( g(x,t) \in L^p_{loc}([\tau,T]; L^2_\delta(\Omega)) \) for \( s > \gamma \), then the results in Theorem 3.2 still hold. The same facts are true for Theorem 3.1.

**Remark 4.** If the exponents of \( f(x,u) \) with \( a(x) \) or \( b(x) \) belonging to \( L^p(\Omega) \) satisfy the growth condition \( \frac{1}{p} + \frac{1}{q} - 1 \leq \frac{2}{N} \), as the proof of Theorem 3.1 and 3.2, we can obtain the well-posedness of (3.8)-(3.9) in \( L^p(\Omega) \).

By Theorem 3.1-3.2, we can define a continuous RDS on \( L^p(\Omega) \) over \((\mathbb{R}, (\theta_{t,t})_{t \in \mathbb{R}})\) and \((\Omega_2, \mathcal{F}_2, \mathcal{P}, (\theta_{2,t})_{t \in \mathbb{R}})\) by

\[ \Phi(t, \tau, \omega_2, u_\tau) = u(t, \tau, \theta_{2,-\tau \omega_2, u_\tau}) = v(t, \tau, \theta_{2,-\tau \omega_2, v_\tau}) + z(\theta_{2,\tau \omega_2}), \]

for all \((t, \tau, \omega_2, u_\tau) \in \mathbb{R}^+ \times \mathbb{R} \times \Omega_2 \times L^p_\delta(\Omega)\),

where \( v_\tau = u_\tau - z(\theta_{2,\tau \omega_2}) \), \((\theta_{1,t})_{t \in \mathbb{R}}\) and \((\theta_{2,t})_{t \in \mathbb{R}}\) are given by (3.1) and (3.2), respectively.

**4. Random attractor for (1.1) in \( L_2(\Omega) \).** The random attractor for (1.1) in \( L_2(\Omega) \) is established in this section. Suppose \( D_{X,\nu}^\tau = \{ D(\tau, \omega_2) : \tau \in \mathbb{R}, \omega_2 \in \Omega_2 \} \) be a family of bounded nonempty subsets of \( X \) such that for every \( \tau \in \mathbb{R} \) and \( P\text{-a.e.} \omega_2 \in \Omega_2 \),

\[ \lim_{s \to -\infty} e^{\nu s} \| D(\tau + s, \theta_{2,s}, \omega_2) \|_{X}^k = 0, \]

(4.1)

where \( k, \nu > 0 \). We need the following dissipative condition as in, e.g., [4, 29]:

\[ f(x,s) s \leq -\alpha_1 \| s \|^{p+1} + \alpha_2, \]

(4.2)
where $\alpha_1, \alpha_2 > 0$. For the deterministic non-autonomous forcing term of (1.1), we assume that there exist positive constants $p_1$ and $p_2$ such that
\[
\int_{-\infty}^{\tau} e^{p_1 s} \| g(x, s) \|_{L^2_x}^2 \, ds < +\infty, \quad \forall \, \tau \in \mathbb{R}.
\] (4.3)

**Lemma 4.1.** Assume that one of following conditions holds: (i) $(P_1)$-$(P_2)$ hold with $a(x) \in L^\beta_0(\Omega)$, $1 < \beta \leq \infty$, and $\frac{1}{\beta} + \frac{p_1 - 1}{2} < \frac{2}{N+1}$; (ii) $(P_2)$-$(P_3)$ hold with $b(x) \in L^\beta_0(\Omega)$, $1 < \beta \leq \infty$, and $\frac{1}{\beta} + \frac{p_1 - 1}{2} = \frac{2}{N+1}$. Let $\rho + 1 < \beta$, $h_j \in L^\beta_0(\Omega) \cap L^{|j+1|}_2(\Omega) \cap W^2_2(\Omega)$, $j = 1, 2, \cdots, m$, and (4.2)-(4.3) hold with $p_1 = \kappa$, $p_2 = \gamma$ and $X = L^\beta_0(\Omega)$, where $0 < \kappa < \lambda_1$. Then there exists a random absorbing set $K = \{ k(\omega_1, \omega_2) : \omega_1 \in \mathbb{R}, \omega_2 \in \Omega_2 \} \in \mathcal{D}_{{L^\beta_0, \kappa}}^2$ for the RDS $\Phi(t, \tau, \omega_2, u_t)$ defined on $L^\beta_0(\Omega)$ over $(\mathbb{R}, (\theta, t, \in)_{\in \mathbb{R}})$ and $(\Omega_2, \mathcal{F}_t, P, (\theta, t, \in)_{\in \mathbb{R}})$.

**Proof.** Multiplying (3.8) by $\psi_1$ and then integrating over $\Omega$, we obtain that
\[
\int_\Omega v_1 \psi_1 \, dx - \int_\Omega \Delta v \psi_1 \, dx = \int_\Omega f(x, v + z(\theta_2, \omega_2)) \psi_1 \, dx + \int_\Omega g(x, t) \psi_1 \, dx + \int_\Omega \Delta(z(\theta_2, \omega_2)) \psi_1 \, dx + \int_\Omega z(\theta_2, \omega_2) \psi_1 \, dx.
\] (4.4)

Note that
\[
\int_\Omega v_1 \psi_1 \, dx = \frac{1}{2} d \int_\Omega |v|^2 \psi_1 \, dx,
\] (4.5)
\[
\int_\Omega (- \Delta v) \psi_1 \, dx = \int_\Omega \nabla v \cdot \nabla \psi_1 \, dx + \int_\Omega \nabla v \cdot \nabla \psi_1 \, dx = \int_\Omega | \nabla v|^2 \psi_1 \, dx + \lambda_1 \int_\Omega v^2 \psi_1 \, dx.
\] (4.6)

By (4.2) we get that
\[
\int_\Omega f(x, v + z(\theta_2, \omega_2)) \psi_1 \, dx = \int_\Omega f(x, v + z(\theta_2, \omega_2)) (v + z(\theta_2, \omega_2)) \phi_1 \, dx - \int_\Omega f(x, v + z(\theta_2, \omega_2)) z(\theta_2, \omega_2) \phi_1 \, dx 
\leq -\alpha_1 \int_\Omega | u |^{p+1} \phi_1 \, dx + C \int_\Omega | a(x) | (1 + | u |^p) z(\theta_2, \omega_2) \phi_1 \, dx + \alpha_2.
\] (4.7)

Now we estimate the second term on the right-hand side of (4.7). Using Hölder inequality we find that
\[
\int_\Omega | a(x) | | u |^p z(\theta_2, \omega_2) \phi_1 \, dx 
\leq C \int_\Omega | a(x) | \phi_1 \, dx \frac{1}{\beta} \bigg( \int_\Omega | u |^{p+1} \phi_1 \, dx \bigg)^{\frac{1}{p+1}} \int_\Omega | z(\theta_2, \omega_2) | \phi_1 \, dx = C \int_\Omega | a(x) | \phi_1 \, dx \frac{1}{\beta} \bigg( \int_\Omega | u |^{p+1} \phi_1 \, dx \bigg)^{\frac{1}{p+1}} \int_\Omega | z(\theta_2, \omega_2) | \phi_1 \, dx 
\leq \frac{\alpha_1}{2} \int_\Omega | u |^{p+1} \phi_1 \, dx + C \int_\Omega | a(x) | \phi_1 \, dx \int_\Omega | z(\theta_2, \omega_2) | \phi_1 \, dx 
\leq \frac{\alpha_1}{2} \int_\Omega | u |^{p+1} \phi_1 \, dx + C \int_\Omega | a(x) | \phi_1 \, dx \int_\Omega | z(\theta_2, \omega_2) | \phi_1 \, dx.
\] (4.8)
where we have used the condition $\rho + 1 < \beta$. Since $\frac{\beta}{\beta-1} < \frac{\beta(\rho+1)}{\beta-\rho+1}$, we have that

$$
\int_\mathcal{O} |a(x)| z(\theta_{2,t}\omega_2) |\phi_1| dx \leq (\int_\mathcal{O} |a(x)|^\beta |\phi_1| dx)^{\frac{\beta}{\beta-\rho+1}} (\int_\mathcal{O} |z(\theta_{2,t}\omega_2)|^{\beta-1} |\phi_1| dx)^{\frac{\rho-1}{\beta-\rho+1}}.
$$

(4.9)

Finally, the last three terms on the right-hand side of (4.4) can be estimated by

$$
\int_\mathcal{O} g(x,t)v\phi_1 dx \leq (\int_\mathcal{O} |\Delta z(\theta_{2,t}\omega_2)|^2 |\phi_1| dx)^{\frac{1}{2}} (\int_\mathcal{O} |v|^2 |\phi_1| dx)^{\frac{1}{2}} + (\int_\mathcal{O} |z(\theta_{2,t}\omega_2)|^2 |\phi_1| dx)^{\frac{1}{2}} \times

(\int_\mathcal{O} |v|^2 |\phi_1| dx)^{\frac{1}{2}}
$$

$$
\leq \frac{\lambda_1-\kappa}{4} \int_\mathcal{O} |v|^2 |\phi_1| dx + \frac{2}{\lambda_1-\kappa} \int_\mathcal{O} |\Delta z(\theta_{2,t}\omega_2)|^2 |\phi_1| dx + \frac{2}{\lambda_1-\kappa} \int_\mathcal{O} |z(\theta_{2,t}\omega_2)|^2 |\phi_1| dx.
$$

(4.11)

It follows from (4.4)-(4.11) that

$$
\frac{d}{dt} \|v\|^2_{L^2_{\omega_1}} + 2 \|\nabla v\|^2_{L^2_{\omega_1}} + \kappa \|v\|^2_{L^2_{\omega_1}} + \alpha_1 \|u\|^{\rho+1}_{L^\rho_{\omega_1}} + \frac{2}{\lambda_1-\kappa} \|g(x,t)\|^2_{L^2_{\omega_1}} + C.
$$

(4.12)

Let

$$
M_1(\theta_{2,t}\omega_2) = \|a(x)\|^{\rho+1}_{L^\beta_{\omega_1}} z(\theta_{2,t}\omega_2) |z(\theta_{2,t}\omega_2)|^{\beta-1}_{L^\beta_{\omega_1}} + \|\Delta z(\theta_{2,t}\omega_2)\|^2_{L^2_{\omega_1}} + \|z(\theta_{2,t}\omega_2)\|^2_{L^2_{\omega_1}}.
$$

(4.13)

Since $z(\theta_{2,t}\omega_2) = \sum_{j=1}^m h_j z_j(\theta_{2,t}\omega_2)$ and $h_j \in L^\frac{\beta(\rho+1)}{\beta-\rho+1}(\mathcal{O}) \cap W^2_\delta(\mathcal{O})$, by (3.6) we get that

$$
M_1(\theta_{2,t}\omega_2) \leq Ce^{\frac{\beta}{2}}|t| q(\omega_2), \quad \forall t \in \mathbb{R}.
$$

(4.14)

Therefore, we have

$$
\frac{d}{dt} \|v\|^2_{L^2_{\omega_1}} + \kappa \|v\|^2_{L^2_{\omega_1}} \leq CM_1(\theta_{2,t}\omega_2) + \frac{2}{\lambda_1-\kappa} \|g(x,t)\|^2_{L^2_{\omega_1}} + C.
$$

(4.15)

Applying Gronwall’s lemma to (4.15), we get that for all $T \geq 0$,

$$
\|v(t,t-T,\omega_2,v_{t-T})\|^2_{L^2_{\omega_1}} \leq e^{-\kappa T} \|v_{t-T}\|^2_{L^2_{\omega_1}} + C \int_{t-T}^t e^{-\kappa(t-s)} M_1(\theta_{2,s}\omega_2) ds.
$$
Replacing \( \omega_2 \) by \( \theta_{2,-}\omega_2 \) in (4.16) and applying (4.14), we get that for all \( T \geq 0 \),

\[
\| v(t, t - T, \theta_{2,-}\omega_2, v_{t-T}) \|_{L_{x,s}^2}^2 \leq e^{-\kappa T} \| v_{t-T} \|_{L_{x,s}^2}^2 + C \int_{t-T}^{t} e^{-\kappa(t-s)} M_1(\theta_{2,s}\omega_2) ds + \frac{2}{\lambda_1 - \kappa} e^{-\kappa t} \int_{t-T}^{t} e^{\kappa s} \| g(x, s) \|_{L_{x,s}^2}^2 ds + \frac{C}{\kappa} \int_{-\infty}^{t} e^{\kappa s} ds
\]

By (4.1) we get from (4.17) that for every \( T \geq 0 \), there exists \( T_1 = T_B(\omega_1, \omega_2) > 0 \) such that for all \( T \geq T_1 \) and every \( v_{t-T} \in B \), there holds

\[
\| v(t, t - T, \theta_{2,-}\omega_2, v_{t-T}) \|_{L_{x,s}^2}^2 \leq 2\| C e^{-\kappa t} \int_{-\infty}^{t} e^{\kappa s} ds + C(q(\omega_2) + 1) \|_{L_{x,s}^2}^2 ds + C(q(\omega_2) + 1)
\]

\[
\Phi(T, t - T, \theta_{2,-}\omega_2, v_{t-T}) \|_{L_{x,s}^2}^2 \leq C \| u(t, t - T, \theta_{2,-}\omega_2, u_{t-T}) \|_{L_{x,s}^2}^2
\]

\[
= C \| v(t, t - T, \theta_{2,-}\omega_2, u_{t-T}) + z(\omega_2) \|_{L_{x,s}^2}^2
\]

\[
\leq 2C R_1(t, \omega_2) + 2 \| z(\omega_2) \|_{L_{x,s}^2}^2
\]

\[
= R_2(t, \omega_2).
\]

Let \( k(t, \omega_2) = \{ u \in L_{s}^2(\mathcal{O}) : \| u \|_{L_{x,s}^2}^2 \leq R_2(t, \omega_2) \} \) and \( K = \{ k(t, \omega_2) : t \in \mathbb{R}, \omega_2 \in \Omega_2 \} \). Note that \( K \in D_{L_{x,s}^2}^{\delta,\kappa} \) with \( 0 < \kappa < \lambda_1 \). This completes the proof. \( \square \)

In order to get the asymptotic compactness of RDS \( \Phi \), we decompose \( \Phi \) into \( \Phi(t, \tau, \omega_1, \omega_2) = \Phi_1(t, \tau, u_{\tau}) + \Phi_2(t, \tau, \omega_2, u_{\tau}) \), where \( \Phi_1(t, \tau, u_{\tau}) = v_1(t) \) and \( \Phi_2(t, \tau, \omega_2, u_{\tau}) = v_2(t) \) satisfy the following equations, respectively:

\[
\begin{align*}
\frac{\partial v_1}{\partial t} - \Delta v_1 &= 0, \\
v_1 |_{\partial \mathcal{O}} &= 0, \\
v_1(\tau) &= v(\tau),
\end{align*}
\]

(4.20)
and
\[
\begin{aligned}
&\frac{\partial v_2}{\partial t} - \Delta v_2 = f(x, v_1 + v_2 + z(\theta_2, t_2 \omega_2)) + g(x, t) + \Delta z(\theta_2, t \omega_2) + z(\theta_2, t \omega_2), \\
v_2|_{\partial \Omega} = 0, \\
v_2(\tau) = 0.
\end{aligned}
\tag{4.21}
\]

For the solution of (4.20), it is easy to obtain that
\[
\| v_1(t) \|_{L^2_{\alpha,T}}^2 \leq C e^{-\lambda_1(t-\tau)} \| v_{\tau} \|_{L^2_{\alpha,(\tau)}}^2.
\tag{4.22}
\]

For the solution of (4.21), we have

**Lemma 4.2.** Let the assumptions of Lemma 4.1 hold. Then for any $B \in D^{-\frac{N}{2}}_{L^2_{\alpha,T}}$ and $P$-a.e. $\omega_2 \in \Omega_2$, there exists $T_B > 0$ such that for any $T \geq T_B$, the solution of (4.21) satisfies
\[
\| v_2(t, t - T, \theta_2, \tau \omega_2, v_{2,t-\tau}) \|_{W^{s,2}_{0,\delta}} \\
\leq C [e^{-\rho \epsilon (t-T)} \int_{-\infty}^{t} \| g(x, s) \|_{L^2_{\alpha,s}}^p \| \phi \|_{L^2_{\alpha}} + q^p(\omega_2) + 1] + C e^{C \varphi(t)},
\tag{4.23}
\]
where $0 < \epsilon < 2 - (N + 1)(\frac{1}{2} + \frac{\alpha}{\beta} - \frac{1}{2})$, $\eta$ is as in (3.11) and $q(\omega_2)$ is as in (3.6), and $\varphi(t) = \max \{ |t - T|, |t - 2T| \}$, $C$ depends only on $(\epsilon, \eta, T, \beta, \rho)$.

**Proof.** We only give the proof in case (i), and the proof in case (ii) is similar. As the proof of Theorem 3.2, we get that the solution of (4.21) satisfies
\[
v_2(t) = \int_{t-T}^{t} S(t - s)[f(x, v + z(\theta_2,t \omega_2)) + g(x, s) + \Delta z(\theta_2, t \omega_2) + z(\theta_2, t \omega_2)] ds.
\tag{4.24}
\]
Thus, we get that
\[
\| v_2(t) \|_{W^{s,2}_{0,\delta}} \leq \int_{t-T}^{t} \| S(t - s)[f(x, v + z(\theta_2,t \omega_2)) \|_{W^{s,2}_{0,\delta}} ds + \int_{t-T}^{t} \| S(s - s)(g(x, s) + \Delta z(\theta_2, t \omega_2) + z(\theta_2, t \omega_2)) \|_{W^{s,2}_{0,\delta}} ds.
\tag{4.25}
\]
Let $\frac{1}{\alpha} = \frac{1}{\beta} + \frac{\epsilon}{\gamma}$. By (P1) and Theorem 2.10, we get that
\[
\int_{t-T}^{t} \| S(s - s)[f(x, v + z(\theta_2,t \omega_2)) \|_{W^{s,2}_{0,\delta}} ds
\]
\[
\leq C \int_{t-T}^{t} \| S(s - s) | a(x) | (1 + | u |) \|_{W^{s,2}_{0,\delta}} ds
\]
\[
\leq C \int_{t-T}^{t} (s - s)^{-\frac{N+1}{2} \left( \frac{1}{\alpha} - \frac{1}{2} \right)} \| a(x) | (1 + | u |) \|_{L^2_{\alpha}} ds
\]
\[
\leq C \int_{t-T}^{t} (s - s)^{-\frac{N+1}{2} \left( \frac{1}{\alpha} + \frac{\epsilon}{\gamma} - \frac{1}{2} \right)} \| a(x) \|_{L^2_{\alpha}} (1 + \| v + z(\theta_2,t \omega_2) \|_{L^2_{\alpha}}) ds
\]
\[
\leq C \int_{t-T}^{t} (s - s)^{-\frac{N+1}{2} \left( \frac{1}{\alpha} + \frac{\epsilon}{\gamma} - \frac{1}{2} \right)} \| a(x) \|_{L^2_{\alpha}} (1 + \| v \|_{L^2_{\alpha}} + \| z(\theta_2,t \omega_2) \|_{L^2_{\alpha}}) ds
\]
\[
\leq C \int_{t-T}^{t} (s - s)^{-\frac{N+1}{2} \left( \frac{1}{\alpha} + \frac{\epsilon}{\gamma} - \frac{1}{2} \right)} \| a(x) \|_{L^2_{\alpha}} (1 + \| v \|_{L^2_{\alpha}} + M(\theta_2,t \omega_2)) ds,
\tag{4.26}
\]
Replacing (4.18) again, we get that
\[ t \int \| z(\theta_2, \omega_2) \|^2_{L^2_t} \leq t \int \Delta z(\theta_2, \omega_2) \| \nabla \Delta z(\theta_2, \omega_2) \|_{L^2_t} + \| z(\theta_2, \omega_2) \|_{L^2_t} \]
satisfies
\[ M_2(\theta_2, \omega_2) \leq C e^{\frac{T}{2} |t|} q(\omega_2). \] (4.27)

For the second term on the right-hand side of (4.25), using Proposition and Definition 2.1 in [18], we have
\[
\int_{t-T}^t \| S(t-s)(g(x, s) + \Delta z(\theta_2, \omega_2) + z(\theta_2, \omega_2)) \|_{W^{1,2}} ds \\
\leq C \int_{t-T}^t e^{-\lambda_1(t-s)} \| g(x, s) \|_{L^2_s} ds + C \int_{t-T}^t e^{-\lambda_1(t-s)} \| \Delta z(\theta_2, \omega_2) \|_{L^2_s} ds \\
+ z(\theta_2, \omega_2) \| L^2_s ds \\
\leq C \left( \int_{t-T}^t e^{-\left(\lambda_1 - \frac{\gamma}{2}\right)(t-s)} \| g(x, s) \|_{L^2_s} ds \right)^\frac{2}{\gamma} \left( \int_{t-T}^t e^{-\kappa(t-s)} \| z(\theta_2, \omega_2) \|_{L^2_s} ds \right)^\frac{1}{\gamma} \\
+ C \int_{t-T}^t e^{-\lambda_1(t-s)} M_2(\theta_2, \omega_2) ds \\
\leq C\frac{(\gamma - 1)^2}{(\lambda_1 \gamma - \kappa)\gamma} + \frac{1}{\gamma} e^{-\kappa t} \int_{-\infty}^t \| g(x, s) \|_{L^2_s} ds + C \int_{t-T}^t e^{-\lambda_1(t-s)} M_2(\theta_2, \omega_2) ds. \] (4.28)

It follows from (4.25)-(4.28) that
\[
\| v_2(t, t-T, \omega_2, v_{2, t-T}) \|_{W^{1,2}_0} \\
\leq C \left( \int_{t-T}^t (t-s)^{\frac{\gamma}{\gamma - 1} - \frac{N+1}{2} \left( 1 + \frac{N+1}{2} - \frac{1}{2} \right)} \| a(x) \|_{L^p_s} (1 + \| v(s, s-T, \omega_2, v_{s-T}) \|_{L^2_s} + M_2(\theta_2, \omega_2)) ds \\
+ C\frac{(\gamma - 1)^2}{(\lambda_1 \gamma - \kappa)\gamma} + \frac{1}{\gamma} e^{-\kappa t} \int_{-\infty}^t \| g(x, s) \|_{L^2_s} ds + C \int_{t-T}^t e^{-\lambda_1(t-s)} M_2(\theta_2, \omega_2) ds. \right) \] (4.29)

Replacing \( \omega_2 \) by \( \theta_{2, -\omega_2} \) in (4.29) and applying (4.18), we have that
\[
\| v_2(t, t-T, \theta_{2, -\omega_2}, v_{2, t-T}) \|_{W^{1,2}_0} \\
\leq C \left( \int_{t-T}^t (t-s)^{\frac{\gamma}{\gamma - 1} - \frac{N+1}{2} \left( 1 + \frac{N+1}{2} - \frac{1}{2} \right)} \| a(x) \|_{L^p_s} (1 + R_1(s, \omega_2)^p + M_2(\theta_{2, -\omega_2})) ds \\
+ C\frac{(\gamma - 1)^2}{(\lambda_1 \gamma - \kappa)\gamma} + \frac{1}{\gamma} e^{-\kappa t} \int_{-\infty}^t \| g(x, s) \|_{L^2_s} ds + C \int_{t-T}^t e^{-\lambda_1(t-s)} M_2(\theta_{2, -\omega_2}) ds. \right) \] (4.30)

Using (4.18) again, we get that
\[
\int_{t-T}^t (t-s)^{\frac{\gamma}{\gamma - 1} - \frac{N+1}{2} \left( 1 + \frac{N+1}{2} - \frac{1}{2} \right)} R_1(s, \omega_2)^p ds \\
\leq C^p \int_{t-T}^t (t-s)^{\frac{\gamma}{\gamma - 1} - \frac{N+1}{2} \left( 1 + \frac{N+1}{2} - \frac{1}{2} \right)} e^{-\kappa p s} \left( \int_{-\infty}^s e^{-\kappa p \tau} \| g(x, \tau) \|^p_{L^2_s} d\tau \right)^p + q(\omega_2)^p \\
+ 1 ds \]
\[ \leq \frac{C T^{1-\frac{1}{2} - \frac{N+1}{2} (\frac{1}{\beta} + \frac{\delta}{r} - \frac{1}{2})}}{1 - \frac{\varepsilon}{2} - \frac{N+1}{2} (\frac{1}{\beta} + \frac{\delta}{r} - \frac{1}{2})} \left( e^{-\kappa \rho (T-t)} \left( \int_{-\infty}^{t} e^{-\kappa \tau} \| g(x, \tau) \|_{L_2^s}^2 \ d\tau \right)^{\rho} + q(\omega_2)^{\rho} + 1 \right). \] 

Note that

\[ \int_{t-T}^{t} (t-s)^{-\frac{1}{2} - \frac{N+1}{2} (\frac{1}{\beta} + \frac{\delta}{r} - \frac{1}{2})} M_2(\theta_{2s-T} \omega_2) ds \]
\[ \leq C \int_{t-T}^{t} (t-s)^{-\frac{1}{2} - \frac{N+1}{2} (\frac{1}{\beta} + \frac{\delta}{r} - \frac{1}{2})} e^{\frac{s}{2} |s-T|} ds \]
\[ \leq \frac{C T^{1-\frac{1}{2} - \frac{N+1}{2} (\frac{1}{\beta} + \frac{\delta}{r} - \frac{1}{2})}}{1 - \frac{\varepsilon}{2} - \frac{N+1}{2} (\frac{1}{\beta} + \frac{\delta}{r} - \frac{1}{2})} e^{\frac{t}{2} \varphi(t)}, \] (4.32)

and

\[ \int_{t-T}^{t} (t-s)^{-\lambda_1(t-s)} M_2(\theta_{2s-T} \omega_2) ds \leq \int_{t-T}^{t} (t-s)^{-\lambda_1(t-s)} e^{\frac{s}{2} |s-T|} q(\omega_2) ds \]
\[ \leq \frac{1}{\lambda_1} e^{\frac{t}{2} \varphi(t)} q(\omega_2). \] (4.33)

By (4.30)-(4.33), we get the results. \[\Box\]

By Lemma 4.1-4.2 and Theorem 2.9, we have the main results of this section:

**Theorem 4.3.** Assume that one of following conditions holds: (i) (P1)-(P2) hold with \( a(x) \in L_\infty^\beta (\mathcal{O}) \), \( 1 < \beta \leq \infty \), and \( \frac{1}{\beta} + \frac{\rho-1}{r} = \frac{2}{N+1} \); (ii) (P2)-(P3) hold with \( b(x) \in L_\infty^\beta (\mathcal{O}) \), \( 1 < \beta \leq \infty \), and \( \frac{1}{\beta} + \frac{\rho-1}{r} = \frac{2}{N+1} \). Let \( \rho + 1 < \beta \), \( h_j \in L_\infty^\beta (\mathcal{O}) \bigcap L_\infty^\beta (\mathcal{O}) \bigcap W_\infty^2 (\mathcal{O}) \), \( j = 1, 2, \cdots, m \), and (4.2)-(4.3) hold with \( p_1 = \kappa \), \( p_2 = \gamma \) and \( X = L_\infty^\beta (\mathcal{O}) \), where \( 0 < \kappa < \lambda_1 \). Then there exists a unique \( D_{\mathcal{O}}^\infty \) random attractor for random dynamical system \( \Phi \) in \( L_\infty^\beta (\mathcal{O}) \).

5. Random attractor for (1.1) in \( L_\infty^\beta (\mathcal{O}) \). In this section, we establish the random attractor for (1.1) in \( L_\infty^\beta (\mathcal{O}) \).

**Lemma 5.1.** Assume that one of following conditions holds: (i) (P1)-(P2) hold with \( a(x) \in L_\infty^\beta (\mathcal{O}) \), \( 1 < \beta \leq \infty \), and \( \frac{1}{\beta} + \frac{\rho-1}{r} = \frac{2}{N+1} \); (ii) (P2)-(P3) hold with \( b(x) \in L_\infty^\beta (\mathcal{O}) \), \( 1 < \beta \leq \infty \), and \( \frac{1}{\beta} + \frac{\rho-1}{r} = \frac{2}{N+1} \). Let \( \rho + 1 < \beta \), \( h_j \in L_\infty^\beta (\mathcal{O}) \bigcap L_\infty^\beta (\mathcal{O}) \bigcap W_\infty^2 (\mathcal{O}) \), \( j = 1, 2, \cdots, m \), and (4.2)-(4.3) hold with \( X = L_\infty^\beta (\mathcal{O}) \), \( p_1 = \kappa \), \( p_2 = \mu = \max \{ \gamma, r \} \). Then there exists a random absorbing set \( K = \{ k(\omega_1, \omega_2) : \omega_1 \in \mathbb{R}, \omega_2 \in \Omega_2 \} \in D_{\mathcal{O}}^\infty \) for the RDS \( \Phi(t, \tau, \omega_2, u_\tau) \) defined on \( L_\infty^\beta (\mathcal{O}) \) over \( (\mathbb{R}, (\theta_{1,t}) \in \mathbb{R}) \) and \( (\Omega_2, F_2, P, (\theta_{2,t}) \in \mathbb{R}) \).

**Proof.** We only give the proof in case (i). Multiplying (3.8) by \( |v|^{-2} \psi_1 \) and integrating over \( \mathcal{O} \), we get that

\[ \int_{\mathcal{O}} v_1 |v|^{-2} \psi_1 dx = \int_{\mathcal{O}} \Delta v |v|^{-2} \psi_1 dx = \int_{\mathcal{O}} f(x, v + \kappa(\theta_2, \omega_2)) |v|^{-2} \psi_1 dx + \int_{\mathcal{O}} g(x, t) |v|^{-2} \psi_1 dx + \int_{\mathcal{O}} (\Delta z(\theta_2, \omega_2) + z(\theta_2, \omega_2)) |v|^{-2} \psi_1 dx. \] (5.1)
Firstly, we have
\[
\int_O v_1 | v |^{-2} v \phi_1 dx = \frac{d}{dt} \int_O v |^r \phi_1 dx, \quad (5.2)
\]
\[
\int_O (-\Delta v) | v |^{-2} v \phi_1 dx = \int_O \nabla v \cdot \nabla (| v |^{-2} v) \phi_1 dx + \int_O \nabla v \cdot \nabla \phi_1 | v |^{-2} v \phi_1 dx
= \frac{4(r-1)}{r} \int_O \nabla | v |^{2} \phi_1 dx + \frac{\lambda_1}{r} \int_O | v |^r \phi_1 dx.
\quad (5.3)
\]
For the first term on the right-hand side of (5.1), by (4.2) we get that
\[
\int_O f(x, v + z(\theta_2, t\omega_2)) | v |^{-2} v \phi_1 dx
= \int_O f(x, v + z(\theta_2, t\omega_2))(v + z(\theta_2, t\omega_2)) | v |^{-2} \phi_1 dx -
\int_O f(x, v + z(\theta_2, t\omega_2))z(\theta_2, t\omega_2) | v |^{-2} \phi_1 dx
\leq -\alpha_1 \int_O | u |^{\rho+1} | v |^{-2} \phi_1 dx + \alpha_2 \int_O | v |^{-2} \phi_1 dx -
\int_O f(x, v + z(\theta_2, t\omega_2))z(\theta_2, t\omega_2) | v |^{-2} \phi_1 dx.
\quad (5.4)
\]
By (P_1), the third term on the right-hand side of (5.4) can be estimated by
\[
\int_O f(x, v + z(\theta_2, t\omega_2))z(\theta_2, t\omega_2) | v |^{-2} \phi_1 dx
\leq C \int_O | a(x) || (1 + | u |^\rho) | z(\theta_2, t\omega_2) || v |^{-2} \phi_1 dx
\leq C \int_O | a(x) || z(\theta_2, t\omega_2) || v |^{-2} \phi_1 dx +
C \int_O | a(x) || u |^\rho || z(\theta_2, t\omega_2) || v |^{-2} \phi_1 dx.
\quad (5.5)
\]
By Hölder inequality and Young inequality, the two terms on the right-hand side of (5.5) can be estimated by
\[
C \int_O | a(x) || u |^\rho || z(\theta_2, t\omega_2) || v |^{-2} \phi_1 dx
\leq C \int_O | a(x) || \phi_1^{\frac{1}{\rho+1}} u |^\rho | v |^{(r-2)\frac{\rho}{\rho+1}} \phi_1^{\frac{2}{\rho+1}} || z(\theta_2, t\omega_2) || \phi_1^{\frac{2}{r(r+1)}}
\times | v |^{\frac{r-2}{r+1}} \phi_1^{\frac{r-2}{r(r+1)}} dx
\leq C \| a(x) \|_{L^\rho_{\phi_1}} \left( \int_O | u |^{\rho+1} | v |^{-2} \phi_1 dx \right)^{\frac{\rho}{\rho+1}} \| v \|_{L^2_{\phi_1}} \| z(\theta_2, t\omega_2) \|_{L^2_{\phi_1} \left( (\rho_1+1) dx \right)}
\leq \frac{\alpha_1}{2} \int_O | u |^{\rho+1} | v |^{-2} \phi_1 dx + C \| a(x) \|^{\frac{\rho+1}{\rho}}_{L^\rho_{\phi_1}} \| v \|^{\rho}_{L^2_{\phi_1}} \| z(\theta_2, t\omega_2) \|^{\frac{\rho+1}{\rho+1}}_{L^2_{\phi_1} \left( (\rho_1+1) dx \right)}
\leq \frac{\alpha_1}{2} \int_O | u |^{\rho+1} | v |^{-2} \phi_1 dx + \frac{\lambda_1 - \kappa}{5r} \| v \|^{\rho}_{L^2_{\phi_1}} + C \| a(x) \|^{\frac{\rho+1}{\rho}}_{L^\rho_{\phi_1}}.
\[ \times \| z(\theta_{2,t},\omega_2) \|_{L^r_{\alpha_1}((\rho+1)\kappa r, r)} \]  

(5.6) 

and

\[
C \int_{\Omega} |a(x)| |z(\theta_{2,t},\omega_2)| v |^{r-2} \phi_1 dx 
\leq C \left( \int_{\Omega} |a(x)|^{\beta} \phi_1 dx \right)^{\frac{1}{\beta}} \left( \int_{\Omega} |v|^{(r-2)\frac{\beta}{2}} \phi_1 dx \right)^{\frac{\beta-2}{2}} \left( \int_{\Omega} |z(\theta_{2,t},\omega_2)|^{\frac{\beta}{2}} \phi_1 dx \right)^{\frac{\beta}{2}},
\]

(5.7)

Finally, we have

\[
\int_{\Omega} g(x,t) |v|^{r-2} v \phi_1 dx \leq \| v \|_{L^r_{\alpha_1}} \| g(x,t) \|_{L^r_{\alpha_1}} 
\leq \frac{\lambda_1 - \kappa}{5r} \| v \|_{L^r_{\alpha_1}} + C \| g(x,t) \|_{L^r_{\alpha_1}},
\]

(5.8)

\[
\int_{\Omega} (\Delta z(\theta_{2,t},\omega_2) + z(\theta_{2,t},\omega_2)) |v|^{r-2} v \phi_1 dx 
\leq \frac{\lambda_1 - \kappa}{5r} \| v \|_{L^r_{\alpha_1}} + C \| \Delta z(\theta_{2,t},\omega_2) \|_{L^r_{\alpha_1}} + C \| z(\theta_{2,t},\omega_2) \|_{L^r_{\alpha_1}},
\]

(5.9)

and

\[
\alpha_2 \int_{\Omega} |v|^{r-2} \phi_1 dx \leq \alpha_2 \left( \int_{\Omega} |v|^{r-2} \phi_1 dx \right)^{\frac{\beta}{\beta-2}} \left( \int_{\Omega} \phi_1 dx \right)^{\frac{\beta-2}{\beta}} \leq \frac{\lambda_1 - \kappa}{5r} \| v \|_{L^r_{\alpha_1}} + C \alpha_2^\frac{r}{\beta}. 
\]

(5.10)

It follows from (5.1)-(5.10) that

\[
\frac{1}{r} \frac{d}{dt} \| v \|_{L^r_{\alpha_1}} + \kappa \frac{1}{r} \| v \|_{L^r_{\alpha_1}} 
\leq - \frac{\alpha_1}{2} \int_{\Omega} \| u \|^{p+1} \| v|^{r-2} \phi_1 dx + C \| a(x) \|_{L^\beta_{\alpha_1}} \| z(\theta_{2,t},\omega_2) \|_{L^r_{\alpha_1}((\rho+1)\kappa r, r)} 
\]

(5.11)

which yields

\[
\frac{d}{dt} \| v \|_{L^r_{\alpha_1}} + \kappa \| v \|_{L^r_{\alpha_1}} \leq C \| g(x,t) \|_{L^r_{\alpha_1}} + CM_3(\theta_{2,t},\omega_2) + C, 
\]

(5.12)

where

\[
M_3(\theta_{2,t},\omega_2) 
= \| a(x) \|_{L^\beta_{\alpha_1}} \| z(\theta_{2,t},\omega_2) \|_{L^r_{\alpha_1}((\rho+1)\kappa r, r)} + \| a(x) \|_{L^\beta_{\alpha_1}} \| z(\theta_{2,t},\omega_2) \|_{L^r_{\alpha_1}((\rho+1)\kappa r, r)} 
+ \| \Delta z(\theta_{2,t},\omega_2) \|_{L^r_{\alpha_1}} + \| z(\theta_{2,t},\omega_2) \|_{L^r_{\alpha_1}}. 
\]
Since \( h_j \in L^r_{\delta} \cap W^{2,r}_\delta (\mathcal{O}) \), \( j = 1, 2, \ldots, m \), by (3.6) we get that
\[
M_3(\theta_{2,t}\omega_2) \leq Ce^{2|t|}q(\omega_2).
\] (5.13)
Therefore, we get from (5.13) that for all \( T > 0 \),
\[
\| v(t, t - T, \omega_2, v_{t-T}) \|_{L^r_{\delta}} \leq e^{-\kappa T} \| v_{t-T} \|_{L^r_{\delta}} + C \int_{t-T}^{t} e^{-\kappa(t-s)} M_3(\theta_{2,s}\omega_2) ds + C \int_{t-T}^{t} e^{-\kappa(t-s)} \| g(x, s) \|_{L^r_{\delta}} ds + \frac{C}{\kappa}.
\] (5.14)
If \( \gamma \geq r \), then
\[
\int_{t-T}^{t} e^{-\kappa(t-s)} \| g(x, s) \|_{L^r_{\delta}} ds \leq C \int_{t-T}^{t} e^{-\kappa(t-s)} (\| g(x, s) \|_{L^r_{\delta}} + 1) ds \leq C (e^{-\kappa t} \int_{-\infty}^{t} e^{\kappa s} ds + 1); \quad (5.15)
\]
and if \( \gamma < r \), then by Remark 3 we get that (3.8)-(3.9) is well-posed in \( L^r_{\delta}(\mathcal{O}) \) and (5.15) still hold. Replacing \( \omega_2 \) in (5.14) by \( \theta_{2,-\omega_2} \) and considering (5.15), and proceeding as the derivation of (4.17), we obtain that for all \( T > 0 \), there holds
\[
\| v(t, t - T, \omega_2, v_{t-T}) \|_{L^r_{\delta}} \leq e^{-\kappa T} \| v_{t-T} \|_{L^r_{\delta}} + \frac{2C}{\kappa}q(\omega_2) + C e^{-\kappa T} \int_{-\infty}^{t} e^{\kappa s} \| g(x, s) \|_{L^r_{\delta}} ds + 1 \quad (5.16)
\]
Thus, for any \( B \in D^{r}_{L^{r}_{\delta}, \kappa} \), \( 0 < \kappa < \lambda_1 \), there exists \( T_2 = T_B(\omega_1, \omega_2) > 0 \) such that for all \( T \geq T_2 \) and for any \( v_{t-T} \in B \), there holds
\[
\Phi(t, t - T, \omega_2, v_{t-T}) \|_{L^r_{\delta}} \leq 2 \left[ \frac{2C}{\kappa} r(\omega_2) + C e^{-\kappa T} \int_{-\infty}^{t} e^{\kappa s} \| g(x, s) \|_{L^r_{\delta}} ds + 1 \right] \]
\[
\leq R_3(t, \omega_2), \quad \forall t \in \mathbb{R}.
\] (5.17)
Therefore, for all \( T > T_2 \), there holds
\[
\| \Phi(t, t - T, \omega_2, v_{t-T}) \|_{L^r_{\delta}} \leq 2CR_3(t, \omega_2) + 2C \| z(\theta_{2,t}\omega_2) \|_{L^r_{\delta}} \]
\[
\leq R_4(t, \omega_2), \quad \forall t \in \mathbb{R}.
\] (5.18)
Now, let \( k(\omega_1, \omega_2) = \{ u \in L^r_{\delta}(\mathcal{O}) : \| u \|_{L^r_{\delta}} \leq R_4(t, \omega_2) \} \), and the proof is completed.  

As the derivation of (4.22), we can get that the solution of (4.20) satisfies
\[
\| v_1(t) \|_{L^r_{\delta}} \leq Ce^{-\lambda_1(t-\tau)} \| v_\tau \|_{L^r_{\delta}}.
\] (5.19)
For the solution of (4.21), we have the following result which the proof is similar to that of Lemma 4.2.

**Lemma 5.2.** Let the assumptions of Lemma 5.1 hold. Then for any \( B \in D^{r}_{L^{r}_{\delta}, \kappa} \) and \( P \)-a.e. \( \omega_2 \in \Omega_2 \), there exists \( T_B > 0 \) such that for any \( T \geq T_B \), the solution of (4.21) satisfies
\[
\| v_2(t, t - T, \theta_{2,-\omega_2}, v_{2,t-T}) \|_{W^{r,r}_{0,\delta}} \leq C e^{-\rho \kappa T} \left( \int_{-\infty}^{t} e^{\kappa s} \| g(x, s) \|^r_{L^r_{\delta}} ds + q(\omega_2)^\rho + 1 \right) + Ce^{C\rho(t)},
\]
where \( C \) and \( C\) are positive constants. 

\[ \]
where $0 < \epsilon < 2 - (N + 1)(\frac{1}{\beta} + \frac{q}{q} - \frac{1}{\rho})$, $\eta$ is as in (3.11), and $q(\omega_2)$ is as in (3.6).

By Lemma 5.1-5.2 and applying Theorem 2.9, we have the main results of this section:

**Theorem 5.3.** Assume that one of following conditions holds: (i) $(P_1)-(P_2)$ hold with $a(x) \in L_3^p(O)$, $1 < \beta \leq \infty$, and $\frac{1}{\beta} + \frac{r-1}{r} < \frac{2}{N+1}$; and (ii) $(P_2)-(P_3)$ hold with $b(x) \in L_3^p(O)$, $1 < \beta \leq \infty$, and $\frac{1}{\beta} + \frac{r-1}{r} = \frac{2}{N+1}$. Let $(\rho + 1)r < 2\beta$, $h_j \in L_3^p(O) \bigcap L_3^{\frac{\beta(r+1)-2r}{2}}(O) \bigcap W_3^{2r}(O)$, $j = 1, 2, \cdots, m$, and (4.2)-(4.3) hold with $X = L_3^p(O)$, $p_1 = \kappa$, where $0 < \kappa < \lambda_1$, and $p_2 = \mu = \max\{\gamma, r\}$. Then there exists a unique $D_{L_3^p}^{\kappa, \omega_2}$-random attractor for random dynamical system $\Phi$ in $L_3^p(O)$.

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