Classical and Quantum Singularities of Levi-Civita Spacetimes with and without a Positive Cosmological Constant

D. A. Konkowski*
Department of Mathematics
U.S. Naval Academy
Annapolis, Maryland, 21402 USA

Cassidi Reese
Department of Physics
U.S. Naval Academy
Annapolis, Maryland, 21402 USA

T.M. Helliwell†and C. Wieland
Department of Physics
Harvey Mudd College
Claremont, California, 91711 USA

November 11, 2018

*email address: dak@usna.edu
†email address: T.Helliwell@HMC.edu
Abstract

Levi-Civita spacetimes have classical naked singularities. They also have quantum singularities. Quantum singularities in general relativistic spacetimes are determined by the behavior of quantum test particles. A static spacetime is said to be quantum mechanically singular if the spatial portion of the wave operator is not essentially self-adjoint on a $C^\infty_0$ domain in $L^2$, a Hilbert space of square integrable functions. Here we summarize how Weyl’s limit point-limit circle criterion can be used to determine whether a wave operator is essentially self-adjoint and how this test can then be applied to scalar wave packets in Levi-Civita spacetimes with and without a cosmological constant to help elucidate the physical properties of these spacetimes.

1 Introduction

Although spacetime singularities have been studied since general relativity was first introduced, they are still not well understood [1, 2, 3, 4]. There is still debate as to whether such singularities will exist in a unified theory (see, e.g., [5]). To begin understanding the effects of singularities in quantum gravity, it is important to probe classical singularities with quantum test wave packets and classical test fields [6, 7]. This is such an analysis for certain cylindrically symmetric spacetimes.

This conference proceeding starts with a review of the different properties of classical and quantum singularities. Next is a review of some mathematics, the Weyl limit point-limit circle criterion, that simplifies the evaluation of quantum singularities. The classical and quantum singularity structure of ordinary Levi-Civita spacetimes (without a cosmological constant) is then summarized. This is followed by a discussion of the classical and quantum singularity structure of Levi-Civita spacetimes with a positive cosmological constant. The paper ends with conclusions and a discussion of an area of further interest.

This conference proceeding is based in part on a paper by D.A. Konkowsk, T.M. Helliwell and C. Wieland [8] and in part on a senior thesis by C. Reese [9].

2 Singularities in General Relativity

2.1 Classical Singularities

In general relativity, a maximal spacetime is considered to be classically singular if the world line of a classical test particle ends after some finite proper time so that further evolution of the particle is not well defined [3]. This is usually summarized by saying that the maximal spacetime has incomplete geodesics and/or incomplete curves of bounded acceleration.

A classification scheme devised by Ellis and Schmidt [4] divides classical singularities into three types: quasiregular, nonscalar curvature and scalar curvature. Quasiregular singularities are the mildest classical singularities and are
associated with a topological obstruction such as the apex of a cone \([4]\). Even though an observer’s world line would end at the singularity in some finite proper time, the observer would not see physical quantities diverge. Nonscalar curvature singularities are associated with finite curvature scalars but infinite tidal forces for particles that encounter it. This is due to the fact that curves exist through each point arbitrarily close to the nonscalar curvature singularity such that observers moving on these curves experience perfectly regular tidal forces. Scalar curvature singularities are the strongest of the classical singularities. They are associated with infinite curvature scalars such as at the center of a black hole or the beginning of a Big Bang cosmology. Physical quantities such as energy density and tidal forces diverge in the frames of all observers who approach these singularities.

The three types of classical singularities can be expressed mathematically. A singular point is defined as the endpoint of an incomplete geodesics or incomplete curve of bounded acceleration in a maximal spacetime. A singular point \(q\) is a quasiregular singularity if all components of the Riemann tensor are bounded at that point. If some components are not bounded, there exists a curvature singularity. A nonscalar curvature singularity exists if all scalars constructed from the metric, the totally antisymmetric tensor and the Riemann tensor tend to a finite limit. However, if any of these scalars are unbounded \(q\) is a scalar curvature singularity.

### 2.2 Quantum Singularities

To decide whether a spacetime is quantum mechanically singular we will use the criterion proposed by Horowitz and Marolf \([7]\) following early work by Wald \([6]\) and Kay and Studer \([21]\). They call a spacetime quantum mechanically nonsingular if the evolution of a test wave packet in the spacetime is uniquely determined by the initial wave packet, without having to put arbitrary boundary conditions at the classical singularity. Their construction is restricted to static spacetimes.

According to Horowitz and Marolf, a static spacetime is quantum mechanically singular if the spatial portion of the Klein-Gordon wave operator is not essentially self-adjoint \([10]\). An operator, \(L\), is called self-adjoint if

(i) \(L = L^\dagger\)

(ii) \(\text{Dom}(L) = \text{Dom}(L^\dagger)\)

where \(L^\dagger\) is the adjoint of \(L\). An operator is essentially self-adjoint if (i) is met and (ii) can be met by expanding the domain of the operator or its adjoint so that it is true \([10]\).

A relativistic scalar quantum particle with mass \(M\) can be described by the positive frequency solution to the Klein-Gordon equation

\[
\frac{\partial^2 \Psi}{\partial t^2} = -A \Psi
\]  

(1)
in a static spacetime where the spatial operator

\[ A = -VD^i(VD_i) + V^2M^2 \tag{2} \]

with \( V = -\xi^\nu\xi_\nu \). Here \( \xi^\nu \) is the timelike Killing field and \( D_i \) is the spatial covariant derivative on the static slice \( \Sigma \). The Hilbert space is \( L^2(\Sigma) \), the space of square integrable functions on \( \Sigma \).

If we initially define the domain of \( A \) to be \( C_0^\infty(\Sigma) \), \( A \) is real, positive, symmetric operator and self-adjoint extensions always exist \[10\]. If there is only a single, unique extension \( A_E \), then \( A \) is essentially self-adjoint. In this case, the Klein-Gordon equation for a free scalar particle takes the form \[7\]:

\[ i\frac{d\Psi}{dt} = A_E^{1/2}\Psi \tag{3} \]

with

\[ \Psi(t) = \exp(-it(A_E)^{1/2})\Psi(0). \tag{4} \]

These equations are ambiguous if \( A \) is not essentially self-adjoint. This fact led Horowitz and Marolf to define quantum mechanically singular spacetimes as those in which \( A \) is not essentially self-adjoint. Examples are considered by Horowitz and Marolf \[7\], Kay and Studer \[21\], Helliwell and Konkowski \[12\], and Helliwell, Konkowski and Arndt \[11\].

3 Mathematical Background

A particularly convenient way to establish essential self-adjointness in the spatial operator of the Klein-Gordon equation is to use the concepts of limit circle and limit point behavior.\(^1\) The approach is as follows. The Klein-Gordon equation for the spacetimes considered in this paper can be separated in the coordinates \( t, r, \theta, z \). Only the radial equation is non-trivial. With changes in both dependent and independent variables, the radial equation can be written as a one-dimensional Schrödinger equation

\[ H\Psi(x) = E\Psi(x) \tag{5} \]

where \( x \in (0, \infty) \) and the operator \( H = -d^2/dx^2 + V(x) \).

Definition 1 The potential \( V(x) \) is in the limit circle case at \( x = 0 \) if for some, and therefore for all \( E \), all solutions of equation (5) are square integrable at zero. If \( V(x) \) is not in the limit circle case, it is in the limit point case.

A similar definition pertains to \( x = \infty \). The potential \( V(x) \) is in the limit circle case at \( x = \infty \) if all solutions of equation (5) are square integrable at infinity; otherwise, \( V(x) \) is in the limit point case at infinity.

\(^1\)This section is based on Appendix to X.1 in Reed and Simon \[10\].
There are of course two linearly independent solutions of the Schrödinger equation for given $E$. If $V(x)$ is in the limit circle case at zero, both solutions are $L^2$ at zero, so all linear combinations are $L^2$ as well. We would therefore need a boundary condition at $x = 0$ to establish a unique solution. If $V(x)$ is in the limit point case, the $L^2$ requirement eliminates one of the solutions, leaving a unique solution without the need of establishing a boundary condition at $x = 0$. This is the whole idea of testing for quantum singularities; there is no singularity if the solution is unique, as it is in the limit point case. The critical theorem is due to Weyl [10, 13].

**Theorem 1 (The Weyl limit point-limit circle criterion.)** If $V(x)$ is a continuous real-valued function on $(0, \infty)$, then $H = -d^2/dx^2 + V(x)$ is essentially self-adjoint on $C_0^\infty(0, \infty)$ if and only if $V(x)$ is in the limit point case at both zero and infinity.

The following theorem can be used to establish the limit circle-limit point behavior at infinity [10].

**Theorem 2 (Theorem X.8 of Reed and Simon [10].)** If $V(x)$ is continuous and real-valued on $(0, \infty)$, then $V(x)$ is in the limit point case at infinity if there exists a positive differentiable function $M(x)$ so that

1. $V(x) \geq -M(x)$
2. $\int_1^\infty [M(x)]^{-1/2} dx = \infty$
3. $M'(x)/M^{3/2}(x)$ is bounded near $\infty$.

Then $V(x)$ is in the limit point case (complete) at $\infty$.

A sufficient choice of the $M(x)$ function for our purposes is the power law function $M(x) = cx^2$ where $c > 0$. Then (ii) and (iii) are satisfied, so if $V(x) \geq -cx^2$, $V(x)$ is in the limit point case at infinity.

A theorem useful near zero is the following.

**Theorem 3 (Theorem X.10 of Reed and Simon [10].)** Let $V(x)$ be continuous and positive near zero. If $V(x) \geq \frac{3}{4}x^{-2}$ near zero then $V(x)$ is in the limit point case. If for some $\epsilon > 0$, $V(x) \leq (\frac{3}{4} - \epsilon)x^{-2}$ near zero, then $V(x)$ is in the limit circle case.

These results can now be used to help test for quantum singularities in the Levi-Civita spacetimes.

### 4 Ordinary Levi-Civita Spacetimes

The metric for an ordinary Levi-Civita (LC) spacetime [14] (no cosmological constant) has the form

\[
 ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2
\]
\[ ds^2 = -r^{4\sigma} dt^2 + r^{8\sigma^2+4\sigma} (dr^2 + dz^2) + \frac{r^{2-4\sigma}}{C^2} d\theta^2 \]  

(6)

where \( \sigma \) and \( C \) are real numbers (\( C > 0 \)) and the coordinates are cylindrical with the usual ranges. For some parameter values one can interpret the Levi-Civita spacetime as the spacetime of an “infinite line mass”. In fact, after some controversy in the literature (see, e.g. [15], [16], [17]), the following interpretations have become somewhat accepted: \( \sigma = 0, 1/2 \) locally flat; \( \sigma = 0, C = 1 \) Minkowski spacetime; \( \sigma = 0, C \neq 1 \) cosmic string spacetime; \( 0 < \sigma < 1/2 \) “infinite line mass” spacetime (modeled by a scalar curvature singularity at \( r = 0 \)); \( \sigma = 1/2 \) Minkowski spacetime in accelerated coordinates (planar source).

The following discussions of the classical and quantum singularities in LC spacetimes is based on the paper by Konkowski, Helliwell and Wieland [8].

### 4.1 Classical Singularities

Computation of the Kretschmann scalar

\[ R_{\mu\nu\sigma\tau} R^{\mu\nu\sigma\tau} = \frac{64\sigma^2(2\sigma - 1)^2}{(4\sigma^2 - 2\sigma + 1)^3} \frac{1}{r^4} \]  

(7)

shows that \( r = 0 \) is a scalar curvature singularity for all \( \sigma \) except \( \sigma = 0, 1/2 \). It can be shown that all 14 scalar polynomial invariants in the curvature have the same properties. In fact, the spacetimes with \( \sigma = 0, 1/2 \) are each flat.

If \( \sigma = 0 \), the metric

\[ ds^2 = -dt^2 + dr^2 + dz^2 + \frac{r^2}{C^2} d\theta^2. \]  

(8)

If \( C = 1 \), this is simply Minkowski spacetime in cylindrical coordinates. If \( C \neq 1 \), equation (8) is the metric for an idealized cosmic string. There is a quasiregular (“disclination”) singularity at \( r = 0 \) (see, e.g., [11] for a discussion). This is a topological singularity, not a curvature singularity, and the parameter \( C \) describes a topological property of the spacetime, its deficit angle.

The \( \sigma = 1/2 \) metric

\[ ds^2 = -r^2 dt^2 + dr^2 + dz^2 + \frac{1}{C^2} d\theta^2 \]  

(9)

is also flat but its interpretation is more difficult [13] [16] [17]. This metric can be transformed to Minkowski coordinates

\[ ds^2 = -d\tilde{t}^2 + d\tilde{x}^2 + d\tilde{y}^2 + d\tilde{z}^2 \]  

(10)

where \( \tilde{t} = r \sinh t, \tilde{x} = r \cosh t, \tilde{y} = \theta/C, \) and \( \tilde{z} = z \) and the \( \tilde{y} \) coordinate can now range from \(-\infty \) to \( \infty \). This is flat spacetime described from the point of view of an accelerating frame of reference. This seems to support an interpretation of the \( \sigma = 1/2 \) case as a planar source producing flat spacetime described by a uniformly accelerating observer [15] [18]. In other words, one can interpret it
as the spacetime of a gravitational field produced by an infinite planar sheet of positive mass density.

4.2 Quantum Singularities

The analysis in [8] uses Weyl’s limit point-limit circle criterion to determine essential self-adjointness of the spatial portion of the Klein-Gordon wave operator on a $C_0^\infty(\Sigma)$ domain in $L^2(\Sigma)$, a Hilbert space of square integrable functions. The conclusions will now be summarized.

If $\sigma$ is neither zero nor one-half, the Klein-Gordon operator is not essentially self-adjoint, so all $\sigma \neq 0$, $\sigma \neq 1/2$ Levi-Civita spacetimes are quantum mechanically singular as well as being classically singular with scalar curvature singularities.

If $\sigma = 0$ and $C = 1$, the spacetime is simply Minkowski space. One of the two solutions of the radial Klein-Gordon equation can be rejected because it diverges at a regular point ($r = 0$) of the spacetime. The operator is therefore quantum mechanically nonsingular (a well known fact, repeated here for completeness).

If $\sigma = 0$ and $C \neq 1$, the spacetime is the conical spacetime corresponding to an idealized cosmic string. The cosmic string spacetimes are quantum mechanically singular for azimuthal quantum number $m$ such that $|m|C < 1$ and nonsingular if $|m|C \geq 1$. If arbitrary values of $m$ are allowed, these spacetimes are quantum mechanically singular in agreement with earlier results [12]. These spacetimes are also classically singular with a quasiregular (“disclination”) singularity at $r = 0$.

If $\sigma = 1/2$ the classical spacetime is flat and without a classical singularity. This spacetime is also quantum mechanically nonsingular. The Weyl limit point-limit circle techniques used in [8] emphasize the flatness of the spacetime and support a description given in [15] of this spacetime as one given by a cylinder whose radius has tended to infinity.

For the Levi-Civita spacetimes, all that are classically singular are also generically quantum mechanically singular, and all that are classically nonsingular ($\sigma = 0$, $C = 1$, and $\sigma = 1/2$) are also quantum mechanically nonsingular. The classically and quantum-mechanically nonsingular spacetimes correspond to isolated values of $\sigma$, so that (for example) even though the spacetime $\sigma = 0$, $C = 1$ is nonsingular, the spacetimes with $\sigma \to 0$, $C = 1$ are singular. The only discrepancy between classical and quantum singularities are for the $\sigma = 0$, $C \neq 1$ modes with $|m|C \geq 1$, which produce no quantum singularity in a classically singular spacetime. The physical reason is that the wavefunction for large values of $m$ in a flat space with a quasiregular singularity at $r = 0$ is unable to detect the presence of the singularity because of a repulsive centrifugal potential.
5 Levi-Civita Spacetimes with Positive Cosmological Constant

Levi-Civita spacetimes with a cosmological constant (LCC) have been studied by de Silva et al. [19]. Originally placed into the field equations as a constant to ensure a static universe, the possibility of a positive cosmological constant is gaining much popularity as it could account for the acceleration of the expansion of the universe as supported by recent astronomical observations of type Ia supernovae [20]. In general, the metric for LCC spacetimes can be written as

\[ ds^2 = -Q(r)^{2/3}(P(r)^{-2(4\sigma^2-8\sigma+1)/3\Lambda})dt^2 + P(r)^{2(8\sigma^2-4\sigma-1)/3\Lambda}dz^2 + C^{-1}P(r)^{-4(2\sigma^2+2\sigma-1)/3\Lambda}d\phi^2 + dr^2 \]  

where \( \{x^\mu\} = \{t, r, \theta, \phi\} \) are the usual cylindrical coordinates with the usual ranges and \( A = 4\sigma^2 - 2\sigma + 1 \). The constant \( C \) is related to angle defects and the constant \( \sigma \) is related to mass per unit length. The functions \( P(r) \) and \( Q(r) \) are defined as

\[ P(r) = \frac{2}{\sqrt{3\Lambda}} \tan\left(\frac{\sqrt{3\Lambda}r}{2}\right), \quad Q(r) = \frac{1}{\sqrt{3\Lambda}} \sin\left(\sqrt{3\Lambda}r\right) \]  

where \( \Lambda \) is the cosmological constant. Here we will only consider the cases where \( \Lambda > 0 \) to correspond to the apparent physical reality of an expanding and accelerating universe. From equations (11) and (12), it can be shown that as \( r \to 0 \), \( Q(r) \approx r \), \( P(r) \approx r \) and ordinary Levi-Civita spacetime is regained, the results of whose study [8] were summarized in Section 4 and will not be discussed further here.

However, equation (11) is also singular on the hypersurface \( r = r_g = \pi/\alpha \), where \( \alpha \equiv \sqrt{3|\Lambda|} \). As \( r \to r_g \), \( Q(r) \approx R \) and \( P(r) \approx R^{-1} \), where \( R \equiv r - r_g \). The approximate metric is

\[ ds^2 \approx -R^{4(4\sigma^2-5\sigma+1)/3A}dt^2 + R^{-4(2\sigma^2-\sigma-1)/3A}dz^2 + C^{-2}R^{2(8\sigma^2+2\sigma-1)/3A}d\phi^2 + dR^2 \]  

where \( R \approx 0 \). For the sake of mathematical convenience, the approximation in equation (13) will be considered exact throughout the rest of this paper. Although this is not valid away from the hypersurface, the behavior of interest occurs near \( R \approx 0 \), where the equality holds true. In this case, the spacetime described by equation (13) generically contains a classical timelike scalar curvature singularity at \( R = 0 \). In some cases we will consider \( R = 0 \) to be the symmetry axis rather than \( r = 0 \) [19]. Those cases will be specially noted.

The following discussion of the classical and quantum singularities in LCC spacetimes is based on the senior thesis of C. Reese [9].
5.1 Classical Singularities

Computation of the Kretschmann scalar

\[ R_{\mu\nu\sigma\tau}R^{\mu\nu\sigma\tau} = \frac{64(\sigma - 1)^2(2\sigma + 1)^2(4\sigma - 1)^2}{27A^3R^4} \]  

(14)

shows that \( R = 0 \) is a scalar curvature singularity for all \( \sigma \) except \( \sigma = -1/2, 1/4, 1 \). It can be shown that all 14 scalar polynomial invariants in the curvature have the same properties. In fact, the spacetimes with \( \sigma = -1/2, 1/4, 1 \) are each flat.

If \( \sigma = -1/2 \), the metric

\[ ds^2 = -R^2 dt^2 + dR^2 + dz^2 + C^{-2} d\phi^2. \]  

(15)

This solution is flat and has planar symmetry. It is similar to the \( \sigma = 1/2 \) case for the ordinary LC spacetime; that is, it is flat spacetime described from the point of view of an accelerating reference frame.

If \( \sigma = 1/4 \), the metric

\[ ds^2 = -dt^2 + dR^2 + R^2 dz^2 + C^{-2} d\phi^2. \]  

(16)

This solution is flat. It resembles Minkowski spacetime in cylindrical coordinates with the variable \( z \) acting as the angular variable. Applying the coordinate transformation \( \bar{\phi} = C^{-1}\phi \), where \( -\infty < \bar{\phi} < \infty \), and \( x = R\cos(z) \) and \( y = R\sin(z) \), equation (16) becomes

\[ ds^2 = -dt^2 + dx^2 + dy^2 + d\bar{\phi}^2. \]  

(17)

which is ordinary Minkowski spacetime. Thus \( R = 0 \) is simply a coordinate singularity in this case.

If \( \sigma = 1 \), the metric

\[ ds^2 = -dt^2 + dR^2 + dz^2 + C^{-2} d\phi^2. \]  

(18)

If \( C = 1 \), this is simply Minkowski spacetime in cylindrical coordinates. If \( C \neq 1 \), equation (18) is the metric for an idealized cosmic string (assuming we let \( R = 0 \) be the symmetry axis of the spacetime). There is a quasiregular (“disclination”) singularity at \( R = 0 \) (see, e.g., \[11\] for a discussion). This is a topological singularity, not a curvature singularity, and the parameter \( C \) describes a topological property of the spacetime, its deficit angle. This case is similar to the \( \sigma = 0 \) case for the ordinary LC spacetime.

5.2 Quantum Singularities

The existence of quantum singularities is next considered. Applying the Klein-Gordon equation to the LCC metric and assuming a solution of the form, \( \Phi \sim e^{-i\omega t}e^{im\phi}e^{ikz}h(R) \), the following radial equation is obtained:
In order to apply the limit point-limit circle criterion from Section 3, the radial equation must first be changed to a one-dimensional Schrödinger equation form

$$\frac{d^2\Psi}{dx^2} + [E - V(x)]\Psi = 0$$

where $E$ is a kinetic energy term that is constant for the system and $V(x)$ is a potential energy term dependent upon the position. Except for the special case $\sigma = -1/2$, the substitutions $h(R) = x^{-1/2}\Psi(x)$ and $R = (x^2\alpha C)^{1/2}$, where $\alpha \equiv (2\sigma + 1)/3A$ may be used to transform equation (19) to equation (20). If $E = \omega^2\alpha^{-1}C^2$ and

$$V(x) = \frac{m^2C^3}{\alpha}(\alpha C x^2)^{(4\sigma+1)/\alpha A} + \frac{M^2C}{\alpha}(\alpha C x^2)^{-2(4\sigma^2-5\sigma+1)/3A} + \frac{k^2C}{\alpha}(\alpha C x^2)^{4\sigma(\sigma-1)/\alpha A} - \frac{1}{4}x^{-2},$$

the form of equation (20) is obtained.

### 5.2.1 The Generic Case

In order to determine the limit point-limit circle behavior near infinity, Theorem 2 of Section 3 must be applied. A choice of $M(x) = cx^2$ where $c > 0$ satisfies (ii) and (iii). For the potential given by equation (21) as $x \to \infty$, (i) will also be satisfied. Thus, $V(x)$ is always in the limit point case at infinity for all values of $\sigma$, except $\sigma = -1/2$, as the substitutions do not apply in this case.

To study the behavior in the limit $x \to 0$, the important terms in $V(x)$ must be identified. The first and third terms are never more divergent than $x^{-2}$, with the minimum values occurring at $\sigma = 1$ and $\sigma = 1/4$, respectively. The second term diverges as $x^{-1/2}$, with a minimum at $\sigma = -1/2$. The last term always diverges as $x^{-2}$. The three cases of $\sigma$ that make any of the first three terms significant are also those which are classically important as they are the values which lead to a non-divergent Kretschmann scalar. It is interesting to note that these values are also special when determining the quantum mechanical nature of the singularities in the spacetime. Thus, for all values of $\sigma$ except $\sigma = -1/2, 1/4, 1$, equation (20) has the form
\[ \frac{d^2\Psi}{dx^2} + \frac{1}{4x^2}\Psi = 0 \]  
\hspace{1cm} (22)

as \( x \to 0 \). The solutions are \( \Psi_1 = x^{1/2} \) and \( \Psi_2 = x^{1/2} \ln(x) \), both of which are square integrable at \( x = 0 \). Thus according to Theorem 3, the potential is in the limit circle case.

Since \( V(x) \) is not in the limit point case at both zero and infinity, it is not essentially self-adjoint and the spacetime is quantum mechanically singular. Thus, the LCC spacetimes with a classical scalar curvature singularity at \( R = 0 \) are also quantum mechanically singular.

### 5.2.2 Special Cases

Three special cases were found for this metric wherein the curvature scalars do not diverge as \( R \to 0 \). They are \( \sigma = -1/2, 1/4, 1 \). It was determined in Section 4 that the potential is always in the limit point case at infinity, for all \( \sigma \) except \( \sigma = -1/2 \). Thus, only the behavior near zero needs to be studied for the other two special cases \( \sigma = 1/4, 1 \); however, for \( \sigma = -1/2 \), the behavior near both zero and infinity needs to be studied once the new potential is obtained.

#### Case (1) \( \sigma = -1/2 \)

For this case, new substitutions for \( h(R) \) and \( R \) must be chosen to find the potential, equation (19) becomes

\[ h''(R) + \frac{1}{R} h'(R) + [\omega^2 R^{-2} - m^2 C^2 - k^2 - M^2]h(R) = 0. \]  
\hspace{1cm} (23)

Using the substitutions \( h(R) = \Psi(x) \) and \( R = e^{C x} \), equation (23) transforms to equation (20) with \( E = \omega^2 C^2 \) and

\[ V(x) = m^2 C^4 e^{2Cx} + k^2 C^2 e^{2Cx} + M^2 C^2 e^{2Cx}. \]  
\hspace{1cm} (24)

Now \( R = 0 \) corresponds to \( x \to -\infty \) and \( R \to \infty \) corresponds to \( x \to \infty \). Thus the behavior at \( x = \pm \infty \) needs to be studied, and Theorem 2 to Section 3 must be employed. A choice of \( M(x) = -\beta x^2 \), where \( \beta > 0 \) will satisfy conditions (ii) and (iii). Since all terms in equation (24) are positive, this choice will also satisfy condition (i). Thus, for the case of \( \sigma = -1/2 \), the potential is in the limit point case at \( x = \pm \infty \), corresponding to \( R = 0 \) and \( R = \infty \). The radial operator is thus essentially self-adjoint and we can call the spacetime quantum mechanically nonsingular. This is the planar case that is very similar to the \( \sigma = 1/2 \) case in the ordinary LC spacetime.

#### Case (2) \( \sigma = 1/4 \)

In this case, as \( x \to 0 \), \( V(x) \to (k^2 - 1/4)x^{-2} \), and thus equation (20) has the form
\[
\frac{d^2 \Psi}{dx^2} - \left(\frac{k^2 - 1/4}{x^2}\right) \Psi = 0. \tag{25}
\]

For \( k \neq 0 \), the solutions are \( \Psi_1 = x^{1/2+|k|} \) and \( \Psi_2 = x^{1/2-|k|} \), and for \( k = 0 \), the solutions are \( \Psi_1 = x^{1/2} \) and \( \Psi_2 = x^2 \ln x \). Thus, \( \Psi_1 \) is always square integrable, but \( \Psi_2 \) is only square integrable when \( k = 0 \). However, since \( R = 0 \) is a regular point of the spacetime, solutions which diverge at \( R = 0 \) must be discarded and, therefore, there is only one square integrable \( \Psi \) for each \( k \) value. The radial operator is thus essentially self-adjoint as it must be in a complete Minkowski spacetime, and the \( \sigma = 1/4 \) spacetime is quantum mechanically nonsingular.

**Case(3) \( \sigma = 1 \)**

In this case, as \( x \to 0 \), \( V(x) \to (m^2C^2 - 1/4)x^{-2} \), and thus equation (20) has the form

\[
\frac{d^2 \Psi}{dx^2} - \left(\frac{m^2C^2 - 1/4}{x^2}\right) \Psi = 0. \tag{26}
\]

In a manner similar to the previous case, for \( m \neq 0 \), the solutions are \( \Psi_1 = x^{1/2+|m|C} \) and \( \Psi_2 = x^{1/2-|m|C} \), and for \( m = 0 \), the solutions are \( \Psi_1 = x^{1/2} \) and \( \Psi_2 = x^2 \ln x \). Thus, \( \Psi_1 \) is always square integrable, but \( \Psi_2 \) is only square integrable when \( |m|C < 1 \). However, unlike the previous case, \( R = 0 \) is not a regular point of the spacetime in all cases; it is only a regular point if \( C = 1 \), otherwise it is a quasiregular singular point. Therefore, only in the \( C = 1 \) case can we discard the solutions which diverge at \( R = 0 \) and automatically obtain a quantum mechanically nonsingular spacetime; it is simply a complete Minkowski spacetime. The other cases are more complicated: When \( C \neq 1 \), \( V(x) \) is in the limit point case if and only if \( |m|C \geq 1 \) and in the limit circle case if and only if \( |m|C < 1 \). There is a range of \( m \) in each quasiregular case which cause a quantum singularity to occur as in the \( \sigma = 0 \) LC spacetime. If this range of modes is forbidden the classically singular spacetime is quantum mechanically nonsingular, but, in the generic case, LCC spacetimes with quasiregular singularities are quantum mechanically singular.

### 6 Conclusions and an Area of Further Interest

For the LC and LCC spacetimes, all that are classically singular are also generically quantum mechanically singular, and all that are classically nonsingular are also quantum mechanically nonsingular. The only discrepancy between classical and quantum singularities are (1) for the \( \sigma = 0 \), \( C \neq 1 \) modes with \( |m|C \geq 1 \) in the LC case, and (2) for the \( \sigma = 1 \), \( C \neq 1 \) modes with \( |m|C \geq 1 \) in the LCC case; each of these sets of modes produces no quantum singularity in a classically singular spacetime. The physical reason is that the wavefunction for large values of \( m \) in a flat space with a quasiregular singularity is unable to detect the presence of the singularity because of a repulsive centrifugal potential.
Finally, there is an area of further interest. The importance of the underlying Hilbert space should be considered through a comparison in this instance of the notion of quantum singularity with the notion of wave singularity \cite{22}. In the latter, the Hilbert space is the first Sobolev space $H^1$ rather than the space of square integrable functions $L^2$. Spacetimes that are quantum mechanically singular may be wave regular \cite{22}. We are currently studying LC and LCC spacetimes for wave regularity, and plan to address it, together with a related notion of global hyperbolicity \cite{23}, in an upcoming paper \cite{24}.

7 Acknowledgements

One of us (DAK) was partially funded by NSF grants PHY-9988607 and PHY-02411384 to the U.S. Naval Academy. She also thanks Queen Mary, University of London, where some of this work was carried out.

References

[1] F. Tipler, C. Clarke, and G. Ellis, "Singularities and Horizons: A Review Article," in A. Held General Relativity and Gravitation, Volume 2 (New York: Plenum Press, 1980) 97.

[2] B.K. Berger, "Numerical Approaches to Spacetime Singularities," Living Reviews in Relativity, No. 2002-1.

[3] S.W. Hawking and G.F.R. Ellis, The Large-Scale Structure of Spacetime (Cambridge: Cambridge University Press, 1973).

[4] G.F.R. Ellis and B.G. Schmidt, Gen. Relativ. Grav 8, 915 (1977).

[5] G.T. Horowitz and J. Polchinski, Phys. Review D 66, 103512 (2002).

[6] R.M. Wald, J. Math Phys. 21, 2802 (1980).

[7] G.T. Horowitz and D. Marolf, Phys. Rev. D 52, 5670 (1995).

[8] D.A. Konkowski, T.M. Helliwell and C. Wieland, Class. Quantum Grav. 21, 265 (2004).

[9] C. Reese, "Classical and Quantum Singularities in Levi-Civita Spacetimes with Cosmological Constant," Proceedings of the National Conference on Undergraduate Research (NCUR) 2004.

[10] M. Reed and B. Simon Functional Analysis (New York: Academic Press,1972); M. Reed and B. Simon 1972 Fourier Analysis and Self-Adjointness (New York: Academic Press, 1972)

[11] T.M. Helliwell, D.A. Konkowski and V. Arndt, Gen. Relativ. Grav. 35, 79 (2003).
[12] D.A. Konkowski and T.M. Helliwell, *Gen. Relativ. Grav.* 33, 1131 (2001).
[13] H. Weyl *Math. Ann.* 68 220 (1910).
[14] T. Levi-Civita *Rend. Acc. Lincei* 28 101 (1919).
[15] W.B. Bonnor “The Static Cylinder in General Relativity” in *On Einstein’s Path* ed. A. Harvey (New York: Springer, 1999) 113.
[16] L. Herrera, J. Ruifernández and N.O. Santos *Gen. Relativ. Grav.* 33 515 (2001).
[17] L. Herrera, N.O. Santos, A.F.F. Teixeira, and A.Z. Wang *Class. Quantum Grav.* 18 3847 (2001).
[18] R. Gautreau and R.B. Hoffmann, *Nuovo Cimento B* 61 411 (1969).
[19] M.F.A. de Silva, A. Wang, F.M. Paiva and N.O. Santos, *Phys. Review D* 61, 44003 (2000).
[20] A.G. Riese et al, *Astron. J.* 116, 1009 (1998).
[21] B.S. Kay and U.M. Studer, *Commun. Math. Phys.* 139, 103 (1991).
[22] A. Ishibashi and A. Hosoya, *Phys. Rev. D* 60, 104028 (1999).
[23] J.A. Vickers and J.P. Wilson, *Class. Quantum Grav.* 17 1333 (2000).
[24] T.M. Helliwell, D.A. Konkowski and C. Reese, in preparation.