ASYMPTOTIC MEAN VALUE PROPERTIES OF META-AND PANHARMONIC FUNCTIONS

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We study asymptotic mean value properties, their converse, and some related results for solutions (metaharmonic functions) to the m-dimensional Helmholtz equation and solutions (panharmonic functions) to the modified m-dimensional Helmholtz equation. Some of these properties have no analogues for harmonic functions. Bibliography: 13 titles.

1 Introduction. The Main Theorem

Mean value properties of harmonic functions (solutions to the Laplace equation; we refer to [1] for the origin of the term harmonic), as well as various versions of converse assertions are well known (cf. the survey articles [2] and [3]). On the other hand, analogous properties of solutions to some other simple partial differential equations are studied less thoroughly and, it may be said, fragmentary. Only recently the mean value property for balls was obtained [4] for solutions to the m-dimensional Helmholtz equation

$$\nabla^2 u + \lambda^2 u = 0, \quad \lambda \in \mathbb{R} \setminus \{0\}. \tag{1.1}$$

Here and below, $\nabla = (\partial_1, \ldots, \partial_m)$ denotes the gradient operator in $\mathbb{R}^m$ and $\partial_i = \partial / \partial x_i$.

In what follows, we use the term metaharmonic function for an abbreviation of solution to the Helmholtz equation. This term was introduced by Vekua [5] (the English translation of [5] can be found in the monograph [6]). We also use the term panharmonic function proposed by Duffin [7] as a convenient equivalent expression of solution to the modified Helmholtz equation

$$\nabla^2 u - \mu^2 u = 0, \quad \mu \in \mathbb{R} \setminus \{0\}. \tag{1.2}$$

The amended coefficient is used to distinguish this equation from (1.1). Indeed, these equations arise in different areas of mathematical physics: (1.1) is usually considered as the reduced wave equation in which $\lambda$ is a wave number, whereas the physical meaning of $\mu$ in (1.2) is completely different. The reason is that its most important application is in the theory of nuclear forces developed by Yukawa [8] (in [7], this equation is referred to as the Yukawa equation).

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In this paper, we continue the study of mean value properties of meta- and panharmonic functions initiated in the author’s papers [4] and [9]. The goal of this paper is to extend a result for harmonic functions dates back to the classical theorems due to Blaschke [10], Priwaloff [11], and Zaremba [12] (cf. also the discussion in [2, Section 9]).

We introduce the notation. For a point \( x = (x_1, \ldots, x_m) \in \mathbb{R}^m, m \geq 2 \), we denote by \( B_r(x) = \{ y : |y - x| < r \} \) the open ball of radius \( r \) centered at \( x \). Such a ball is called admissible with respect to a domain \( D \subset \mathbb{R}^m \) if \( B_r(x) \subset D \), and \( \partial B_r(x) \) is called an admissible sphere. If \( D \) has a finite Lebesgue measure and a function \( f \) is integrable over \( D \), then

\[
M^\ast(f, D) = \frac{1}{|D|} \int_D f(x) \, dx
\]

is its volume mean value over \( D \). Here and below, \( |D| \) denotes the volume (area if \( D \subset \mathbb{R}^2 \)) of \( D \) and \( |B_r| = \omega_m r^m \) is the volume of \( B_r \), where \( \omega_m = 2 \pi^{m/2} / [m \Gamma(m/2)] \) is the volume of unit ball. As usual, \( \Gamma \) denotes the Gamma function. If \( u \in C^0(D) \), the mean value over the admissible sphere \( \partial B_r(x) \subset D \) is expressed by

\[
M^\circ(f, \partial B_r(x)) = \frac{1}{|\partial B_r|} \int_{\partial B_r(x)} f(y) \, dS_y,
\]

where \( |\partial B_r| = m \omega_m r^{m-1} \) and \( dS \) is the surface area measure.

**Theorem 1.1** (Blaschke, Priwaloff, Zaremba). Let \( D \) be a domain in \( \mathbb{R}^m, m \geq 2 \), and let \( u \in C^0(D) \). Then \( u \) is harmonic in \( D \) if and only if

\[
\lim_{r \to +0} r^{-2} [M^\ast(u, B_r(x)) - u(x)] = 0 \quad \forall \ x \in D.
\]

The assertion also holds with \( M^\ast(u, B_r(x)) \) changed to \( M^\circ(u, \partial B_r(x)) \).

Now, we are in a position to formulate the main result of the paper.

**Theorem 1.2.** Let \( D \) be a domain in \( \mathbb{R}^m, m \geq 2 \), and let \( u \in C^2(D) \). Then \( u \) is panharmonic in \( D \) if and only if

\[
\lim_{r \to +0} \frac{M^\ast(u, B_r(x)) - u(x)}{r^2} = \frac{\mu^2 u(x)}{2(m+2)} \quad \forall \ x \in D.
\]

The assertion also holds with \( M^\ast(u, B_r(x)) \) changed to \( M^\circ(u, \partial B_r(x)) \) in (1.3) provided that the right-hand side term is \( \mu^2 u(x)/(2m) \). Under the change of \( \mu^2 u(x) \) to \( -\lambda^2 u(x) \), this assertion turns into a necessary and sufficient condition for the metaharmonicity of \( u \).

## 2 Proof of Theorem 1.2. Discussion

The elementary proof of Theorem 1.2 given below is based on the well-known relationship between the Laplacian and asymptotic mean values (cf. [13, Chapter 2, Section 2]). By the Taylor formula,

\[
u(x + y) - u(x) = y \cdot \nabla u(x) + 2^{-1} y \cdot [H_u(x)] y + o(r^2),\]

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for $u \in C^2(D)$ at $x \in D$ as $r \to 0$ if $B_r(x)$ is admissible and $|y| \leq r$; here, $H_u(x)$ denotes the Hessian matrix of $u$ at $x$ and the symbol $\cdot$ stands for the inner product in $\mathbb{R}^m$. Averaging each term of the equality with respect to $y \in B_r(0)$, we have

$$M^\bullet(u, B_r(x)) - u(x) = \frac{1}{2|B_r|} \int_{B_r(0)} y \cdot [H_u(x)]y \, dy + o(r^2)$$

because the mean value of the first order term vanishes. It is straightforward to calculate that

$$\lim_{r \to +0} \frac{M^\bullet(u, B_r(x)) - u(x)}{r^2} = \frac{\nabla^2 u(x)}{2(m + 2)} , \quad x \in D. \tag{2.1}$$

Similarly,

$$\lim_{r \to +0} \frac{M^\circ(u, \partial B_r(x)) - u(x)}{r^2} = \frac{\nabla^2 u(x)}{2m} , \quad x \in D, \tag{2.2}$$

by averaging with respect to $y \in \partial B_r(0)$.

**Proof of Theorem 1.2.** Assume that the equality (1.3) holds. Combining (1.3) with (2.1), we find that $u$ is panharmonic in $D$. Similarly, (2.2) yields an analogous assertion with $M^\circ(u, \partial B_r(x))$ in (1.3) instead of $M^\bullet(u, \partial B_r(x))$.

Let $u$ be panharmonic in $D$. Then the mean value equality $M^\bullet(u, B_r(x)) = a^\bullet(\mu r)u(x)$ holds for every admissible $B_r(x) \subset D$ (cf. [4]). Here,

$$a^\bullet(\mu r) = \Gamma\left(\frac{m}{2} + 1\right) \frac{I_{m/2}(\mu r)}{(\mu r/2)^{m/2}},$$

where $I_\nu$ denotes the modified Bessel function of order $\nu$. Thus, (1.3) is true provided that

$$\lim_{r \to +0} \frac{a^\bullet(\mu r) - 1}{(\mu r)^2} = \frac{1}{2(m + 2)},$$

which follows from the definition of $I_{m/2}$.

To prove the necessity of the condition

$$\lim_{r \to +0} \frac{M^\circ(u, \partial B_r(x)) - u(x)}{r^2} = \frac{\mu^2 u(x)}{2m} \quad \forall \ x \in D \tag{2.3}$$

for the panharmonicity of $u$, it suffices to apply the mean value equality $M^\circ(u, \partial B_r(x)) = a^\circ(\mu r)u(x)$ which holds for panharmonic $u$ provided that $B_r(x) \subset D$ is admissible and

$$a^\circ(\mu r) = \Gamma\left(\frac{m}{2}\right) \frac{I_{(m-2)/2}(\mu r)}{(\mu r/2)^{(m-2)/2}}$$

(cf. [4]). Then (1.3) follows from the equality

$$\lim_{r \to +0} \frac{a^\circ(\mu r) - 1}{(\mu r)^2} = \frac{1}{2m},$$

which is true by the definition of $I_{(m-2)/2}$.

Analogous necessary and sufficient metaharmonicity conditions are proved in a similar way, but the coefficients $a^\bullet(\lambda r)$ and $a^\circ(\lambda r)$ in the mean value equalities involve the Bessel function $J_\nu$ instead of $I_\nu$ of the same order as above in both cases. \qed
We consider some equalities related to (1.3) and (2.3). Their immediate consequence is the equality

\[
(m + 2) \lim_{r \to +0} \frac{M^\bullet(u, B_r(x)) - u(x)}{r^2} = m \lim_{r \to +0} \frac{M^\circ(u, \partial B_r(x)) - u(x)}{r^2}
\]

which is valid for every \( x \in D \) provided that \( u \) is meta- or panharmonic. This is a rare mean value property shared without any distinction by both classes of functions.

If \( u \) is harmonic in \( D \), then the analogous equality immediately follows from Theorem 1.1:

\[
\lim_{r \to +0} \frac{M^\bullet(u, B_r(x)) - M^\circ(u, \partial B_r(x))}{r^2} = 0 \quad \forall \ x \in D.
\] (2.4)

To the best author's knowledge, the question whether (2.4) implies that \( u \) is harmonic has not been studied yet. Thus, the following assertion complements Theorem 1.1.

**Proposition 2.1.** Let \( D \) be a domain in \( \mathbb{R}^m \), \( m \geq 2 \), and let \( u \in C^2(D) \). Then \( u \) is harmonic in \( D \) provided that the equality (2.4) holds for every \( x \in D \).

**Proof.** We write (2.4) in the form

\[
\lim_{r \to +0} \frac{M^\bullet(u, B_r(x)) - u(x)}{r^2} = \lim_{r \to +0} \frac{M^\circ(u, \partial B_r(x)) - u(x)}{r^2} \quad \forall \ x \in D.
\]

By (2.1) and (2.2), we have

\[
\frac{\nabla^2 u(x)}{m + 2} = \frac{\nabla^2 u(x)}{m} \quad \forall \ x \in D,
\]

which implies that \( u \) is harmonic in \( D \).

Now, we turn to properties that are related to (1.3), but have no analogues for harmonic functions. Since the equalities

\[
u(x) = \frac{M^\bullet(u, B_r(x))}{a^\bullet(\mu r)} = \frac{M^\circ(u, \partial B_r(x))}{a^\circ(\mu r)}
\]

hold for every \( x \in D \) and every admissible \( B_r(x) \) if \( u \) is panharmonic, from (1.3) it follows that

\[
\lim_{r \to +0} \frac{M^\bullet(u, B_r(x)) - u(x)}{r^2} = \frac{\mu^2 M^\bullet(u, B_r(x))}{2(m + 2) a^\bullet(\mu r)}
\]

\[
\lim_{r \to +0} \frac{M^\circ(u, \partial B_r(x)) - u(x)}{r^2} = \frac{\mu^2 M^\circ(u, \partial B_r(x))}{2 m a^\circ(\mu r)}
\]

for these \( x \) and \( r \).

On the other hand, if each of these equalities holds for every \( x \in D \) and every admissible \( B_r(x) \), then \( u \in C^2(D) \) is panharmonic in \( D \). Indeed, from (2.1) it follows that the left-hand side of the first equality is equal to \( \nabla^2 u(x)/2(m + 2) \), whereas letting \( r \to 0 \) on the right-hand side we get \( \mu^2 u(x)/(2(m + 2)) \), which implies that \( u \) is panharmonic in \( D \). The same is obtained by combining the second equality and (2.2).

For metaharmonic functions the two equalities analogous to the last ones contain \(-\lambda^2\) instead of \(\mu^2\) and other functions \(a^\bullet\) and \(a^\circ\), but these equalities also imply that \( u \) is metaharmonic in \( D \).
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