HYPERELLIPTIC CURVES OVER $\mathbb{F}_q$ AND GAUSSIAN HYPERGEOMETRIC SERIES

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Abstract. Let $d \geq 2$ be an integer. Denote by $E_d$ and $E'_d$ the hyperelliptic curves over $\mathbb{F}_q$ given by

$E_d : y^2 = x^d + ax + b$ and $E'_d : y^2 = x^d + ax^{d-1} + b$,

respectively. We explicitly find the number of $\mathbb{F}_q$-points on $E_d$ and $E'_d$ in terms of special values of $dF_{d-1}$ and $2(d-1)F_{d-2}$ Gaussian hypergeometric series with characters of orders $d-1, d, 2(d-1), 2d$, and $2d(d-1)$ as parameters. This gives a solution to a problem posed by Ken Ono [16, p. 204] on special values of $n+1F_n$ Gaussian hypergeometric series for $n > 2$. We also show that the results of Lennon [13] and the authors [4] on trace of Frobenius of elliptic curves follow from the main results.

1. Introduction and statement of results

In [9], Greene introduced the notion of hypergeometric functions over finite fields or Gaussian hypergeometric series which are analogous to the classical hypergeometric series. Since then, many interesting relations between special values of these functions and the number of $\mathbb{F}_p$-points on certain varieties have been obtained. For example, Koike [11], Fuselier [8], Lennon [13, 14], and the authors [4, 5] gave formulas for the number of $\mathbb{F}_q$-points on elliptic curves in terms of special values of $2F_1$ Gaussian hypergeometric series. Ono [15] expressed the trace of Frobenius of the Clausen family of elliptic curves in terms of a $3F_2$ Gaussian hypergeometric series. Also, Vega [17] and the authors [2, 3] studied this problem for certain families of more general algebraic curves.

In all the known results connecting Gaussian hypergeometric series and algebraic curves, expressions are obtained in terms of $2F_1$ and $3F_2$ Gaussian hypergeometric series. Hence, the task remained to find similar results for $n+1F_n$ Gaussian hypergeometric series with $n \geq 3$. Ahlgren and Ono studied this problem and deduced the value of a $4F_3$ hypergeometric series at 1 over $\mathbb{F}_p$ in terms of representations of $4p$ as a sum of four squares using the fact that the Calabi-Yau threefold is modular [1]. For $n > 3$, the non-trivial values of $n+1F_n$ Gaussian hypergeometric series have been difficult to obtain, and this problem was also mentioned by Ono [16, p. 204]. Recently, the authors [9] found expressions for the number of zeros of the polynomial $x^d + ax + b$ over a finite field $\mathbb{F}_q$, in terms of special values of $dF_{d-1}$ and $d-1F_{d-2}$ Gaussian hypergeometric series with characters of orders $d-1$ and $d$ as parameters under the condition that $q \equiv 1 \pmod{d(d-1)}$, where $d \geq 2$. The first

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curves

trivial characters on \( F \)
d even, odd powers of a character of degree 2(\( d \) values of \( d \) and \( i \) respectively.

(1.3)

\[ \hat{\chi} \] of multiplicative characters

in terms of special values of \( \chi \) and \( \chi \) respectively.

We also show that the results of Lennon [13] and the authors [4] on trace of Frobenius of elliptic curves follow from the main results.

We begin with some preliminary definitions needed to state our results. Let \( q = p^f \) be a power of an odd prime \( p \), and let \( F_q \) be the finite field of \( q \) elements. Let \( \hat{\chi} \) denote the group of multiplicative characters \( \chi \) on \( \hat{\chi} \). We extend each character \( \chi \in \hat{\chi} \) to all of \( \hat{\chi} \) by setting \( \chi(0) := 0 \). If \( A \) and \( B \) are two characters on \( \hat{\chi} \), then \( (A)_B \) is defined by

(1.1)

\[ (A)_B := \frac{B(-1)}{q} J(A, B) = \frac{B(-1)}{q} \sum_{x \in \hat{\chi}} A(x) B(1-x), \]

where \( J(A, B) \) denotes the usual Jacobi sum and \( B \) is the inverse of \( B \).

Recall the definition of the Gaussian hypergeometric series over \( F_q \) first defined by Greene in [9]. Let \( n \) be a positive integer. For characters \( A_0, A_1, \ldots, A_n \) and \( B_1, B_2, \ldots, B_n \) on \( \hat{\chi} \), the Gaussian hypergeometric series \( n+1 \binom{F_n}{\chi} \) is defined to be

(1.2)

\[ n+1 \binom{F_n}{\chi} := \frac{n}{q-1} \sum_{\chi} \binom{A_0}{\chi} \binom{A_1}{\chi} \cdots \binom{A_n}{\chi} \chi(x), \]

where the sum is over all characters \( \chi \) on \( \hat{\chi} \).

Throughout the paper, for \( d \geq 2 \) and \( a, b \neq 0 \), we consider the hyperelliptic curves \( E_d \) and \( E'_d \) over \( \hat{\chi} \) given by

(1.3)

\[ E_d : y^2 = x^d + ax + b \]

and

(1.4)

\[ E'_d : y^2 = x^d + ax^{d-1} + b, \]

respectively.

In this paper we give two explicit formulas for the number of \( \hat{\chi} \)-points on \( E_d \) in terms of special values of \( d F_{d-1} \) and \( d-1 F_{d-2} \) Gaussian hypergeometric series with characters of orders \( d, 2(d-1) \), and \( 2d(d-1) \) as parameters. In case of \( d \) is even, odd powers of a character of degree \( 2(d-1) \) appear in the 2nd row of a \( d F_{d-1} \) Gaussian hypergeometric series; whereas even powers of the same character appear in the 2nd row of a \( d-1 F_{d-2} \) Gaussian hypergeometric series in case \( d \) is odd. The main results are stated below. The notation \( \phi \) and \( \varepsilon \) are reserved for quadratic and trivial characters on \( \hat{\chi} \), respectively. We also fix a generator \( T \) for the cyclic group of multiplicative characters \( \hat{\chi} \).
Theorem 1.1. Let \( q = p^e, p > 0 \) be a prime, and let \( N_d \) denote the number of \( \mathbb{F}_q \)-points on \( E_d \). If \( d \geq 2 \) is even and \( q \equiv 1 \pmod{2d(d-1)} \), then
\[
N_d = q + \phi(b) + q^d \phi(b(d - 1))
\]
\[
	imes dF_{d-1}\left( \phi, \varepsilon, \chi, \chi^2, \ldots, \chi^{d-2}, \chi^{d+2}, \ldots, \chi^{d-1} | \alpha \right),
\]
where \( \chi \) and \( \psi \) are characters of order \( d \) and \( 2(d - 1) \), respectively; and \( \alpha = \frac{d}{a}\left(\frac{bd}{a(d-1)}\right)^{-1} \).

Theorem 1.2. Let \( q = p^e, p > 0 \) be a prime, and let \( N_d \) denote the number of \( \mathbb{F}_q \)-points on \( E_d \). If \( d \geq 3 \) is odd and \( q \equiv 1 \pmod{2d(d-1)} \), then
\[
N_d = q + \phi(b) - \phi(b(d - 1))T^{\frac{q-1}{2}} (-1)
\]
\[
+ q^\frac{d-1}{2} \phi(-b(d - 1))T^{\frac{q-1}{4}} (-1)T^{\frac{q-1}{2d-1}} \left( \frac{1}{\alpha} \right) \times
\]
\[
d-F_{d-2}\left( \xi^{d-2}, \xi^{3d-4}, \ldots, \xi^{d^2-3d+1}, \xi^{d^2-3d-1}, \ldots, \xi^{2d^2-5d+2} | -\alpha \right),
\]
where \( \psi \) and \( \xi \) are characters of order \( 2(d - 1) \) and \( 2d(d - 1) \), respectively; and \( \alpha = \frac{d}{a}\left(\frac{db}{a(d-1)}\right)^{-1} \).

We now state similar results for the curve \( E'_d: y^2 = x^d + ax^{d-1} + b \). If \( d \) is odd, then we obtain a formula for the number of \( \mathbb{F}_q \)-points on \( E'_d \) under the condition that \( q \equiv 1 \pmod{d(d-1)} \).

Theorem 1.3. Let \( q = p^e, p > 0 \) be a prime, and let \( N'_d \) denote the number of \( \mathbb{F}_q \)-points on \( E'_d \). If \( d \geq 2 \) is even and \( q \equiv 1 \pmod{2d(d-1)} \), then
\[
N'_d = q + \phi(b) + q^d \phi((d - 1)) \times
\]
\[
dF_{d-1}\left( \phi, \varepsilon, \chi, \chi^2, \ldots, \chi^{d-2}, \chi^{d+2}, \ldots, \chi^{d-1} | \beta \right),
\]
where \( \chi \) and \( \psi \) are characters of order \( d \) and \( 2(d - 1) \), respectively; and \( \beta = \frac{bd^d}{a^d(d-1)^{d-1}} \).

Theorem 1.4. Let \( q = p^e, p > 0 \) be a prime, and let \( N'_d \) denote the number of \( \mathbb{F}_q \)-points on \( E'_d \). If \( d \geq 3 \) is odd and \( q \equiv 1 \pmod{d(d-1)} \), then
\[
N'_d = q + q^\frac{d-1}{2} \phi(-ad) \times
\]
\[
d-F_{d-2}\left( \eta, \eta^2, \eta^3, \ldots, \eta^{d-2}, \eta^{d+2}, \ldots, \eta^{2d-3}, \eta^{2d-1}, \varepsilon | -\beta \right),
\]
where \( \eta \) and \( \rho \) are characters of order \( 2d \) and \( (d - 1) \), respectively; and \( \beta = \frac{bd^d}{a^d(d-1)^{d-1}} \).

From Theorem 1.1 and Theorem 1.3, we have the following result.
Corollary 1.5. Let $d \geq 2$ be even and $a, b \in \mathbb{F}_q^\times$. For $q \equiv 1(\text{mod } 2d(d - 1))$, the hyperelliptic curves $y^2 = x^d + ax + b$ and $y^2 = x^d + ax^{d-1} + b$ have equal number of $\mathbb{F}_q$-points if $b^{d-2} \equiv 1$ and $b$ is a quadratic residue in $\mathbb{F}_q$.

We now give two examples to show how the above theorems are applied in specific values of $d$.

Example 1.6. Let $d = 4$. Let $a, b \in \mathbb{F}_q^\times$ and $q \equiv 1(\text{mod } 24)$. Also, let $\chi_4$ and $\chi_6$ be characters of order 4 and 6, respectively. Then from Theorem 1.1, we deduce that
\[
\# \{ (x, y) \in \mathbb{F}_q^2 : y^2 = x^4 + ax + b \} = q + \phi(b) + q^2 \phi(3b) \cdot 4F_3 \left( \begin{array}{c} \phi, \varepsilon, \chi_4, \chi_6^3, \chi_6^6 \varepsilon \mid \frac{256b^6}{27a^4} \end{array} \right).
\]

From Theorem 1.3, we deduce that
\[
\# \{ (x, y) \in \mathbb{F}_q^2 : y^2 = x^4 + ax^3 + b \} = q + \phi(b) + q^2 \phi(3) \cdot 4F_3 \left( \begin{array}{c} \phi, \varepsilon, \chi_4, \chi_6^3, \chi_6^6 \varepsilon \mid \frac{256b}{27a^4} \end{array} \right).
\]

Example 1.7. Let $d = 5$ and $a, b \in \mathbb{F}_q^\times$. Then from Theorem 1.2 for $q \equiv 1(\text{mod } 40)$, we deduce that
\[
\# \{ (x, y) \in \mathbb{F}_q^2 : y^2 = x^5 + ax + b \} = q + \phi(b) - \phi(-b)T^{\frac{1}{256}}(-1) + q^2 \phi(-b)T^{\frac{1}{256}}(-1)T^{\frac{1}{256}} \left( -\frac{256a^5}{3125b^4} \right) \times 4F_3 \left( \begin{array}{cccc} \xi^3, & \xi^{11}, & \xi^{19}, & \xi^{27} \\ \psi^2, & \psi^4, & \psi^6 \end{array} \mid -\frac{3125b^4}{256a^5} \right),
\]
where $\xi$ and $\psi$ are characters of order 8 and 40, respectively. Again from Theorem 1.2 for $q \equiv 1(\text{mod } 20)$, we deduce that
\[
\# \{ (x, y) \in \mathbb{F}_q^2 : y^2 = x^5 + ax^4 + b \} = q + q^2 \phi(-5a) \cdot 4F_3 \left( \begin{array}{cccc} \eta, & \eta^3, & \eta^7, & \eta^9 \\ \chi_4, & \chi_4^3, & \varepsilon \end{array} \mid -\frac{3125b}{256a^2} \right),
\]
where $\chi_4$ and $\eta$ are characters of order 4 and 10, respectively.

2. Preliminaries

In this section, we recall some results which we will use to prove our main results. We start defining the additive character $\theta : \mathbb{F}_q \to \mathbb{C}^\times$ by
\[
\theta(\alpha) = \zeta^{\text{tr}(\alpha)}
\]
where $\zeta = e^{2\pi i/p}$ and $\text{tr} : \mathbb{F}_q \to \mathbb{F}_p$ is the trace map given by
\[
\text{tr}(\alpha) = \alpha + \alpha^p + \alpha^{p^2} + \cdots + \alpha^{p^{e-1}}.
\]
For $A \in \widehat{\mathbb{F}_q^\times}$, the Gauss sum is defined by
\[
G(A) := \sum_{x \in \mathbb{F}_q} A(x)\zeta^{\text{tr}(x)} = \sum_{x \in \mathbb{F}_q} A(x)\theta(x).
\]
Recall that $T$ denotes a fixed generator of the cyclic group $\mathbb{F}_q^\times$. We denote by $G_m$ the Gauss sum $G(T^m)$. The following lemma provides a formula for the multiplicative inverse of a Gauss sum.

**Lemma 2.1.** ([9 Eqn. 1.12]). If $k \in \mathbb{Z}$ and $T^k \neq \varepsilon$, then

$$G_k G_{-k} = qT^k(-1).$$

The following lemma gives a relationship between Gauss and Jacobi sums.

**Lemma 2.2.** ([9 Eqn. 1.14]). If $T^{m-n} \neq \varepsilon$, then

$$G_m G_{-n} = q\left(\frac{T^m}{T^n}\right) G_{m-n}T^n(-1) = J(T^m, T^{-n})G_{m-n}.$$

We now state the orthogonality relations for multiplicative characters.

**Lemma 2.3.** ([10 Chapter 8]). We have

1. $$\sum_{x \in \mathbb{F}_q} T^n(x) = \begin{cases} q - 1 & \text{if } T^n = \varepsilon; \\ 0 & \text{if } T^n \neq \varepsilon. \end{cases}$$

2. $$\sum_{n=0}^{q-2} T^n(x) = \begin{cases} q - 1 & \text{if } x = 1; \\ 0 & \text{if } x \neq 1. \end{cases}$$

Using orthogonality, we can write $\theta$ in terms of Gauss sums as given in the following lemma.

**Lemma 2.4.** ([8 Lemma 2.2]). For all $\alpha \in \mathbb{F}_q^\times$,

$$\theta(\alpha) = \frac{1}{q-1} \sum_{m=0}^{q-2} G_{-m}T^m(\alpha).$$

**Theorem 2.5.** ([12 Davenport-Hasse Relation]). Let $m$ be a positive integer and let $q = p^e$ be a prime power such that $q \equiv 1 \pmod{m}$. For multiplicative characters $\chi, \psi \in \mathbb{F}_q^\times$, we have

$$\prod_{\chi^m = 1} G(\chi \psi) = -G(\psi^m)\psi(m^{-m}) \prod_{\chi^m = 1} G(\chi).$$

We have the following two special cases of Davenport-Hasse relation. For details, see [9].

**Corollary 2.6.** Let $d$ be a positive integer, $l \in \mathbb{Z}$, $q = p^e \equiv 1 \pmod{d}$, and $t \in \{1, -1\}$.

- If $d > 1$ is odd, then

$$G_l G_{l+1} \cdots G_{l+\frac{(d-1)(q-1)}{2}} = q^{\frac{d-1}{2}} T^{\frac{(d-1)(d+1)(q-1)}{8d}} (-1)^{T^{-l}(d^l)} G_{ld}.$$

- If $d$ is even, then

$$G_l G_{l+1} \cdots G_{l+\frac{(d-1)(q-1)}{2}} = q^{\frac{d-2}{2}} G_{d-l} T^{\frac{(d-2)(q-1)}{4d}} (-1)^{T^{-l}(d^l)} G_{ld}.$$
3. Proof of Theorem 1.1 and Theorem 1.2

For \( d \geq 2 \), the hyperelliptic curve \( E_d \) defined over \( \mathbb{F}_q \) is given by

\[
E_d : y^2 = x^d + ax + b.
\]

Then the number of points on \( E_d \) over \( \mathbb{F}_q \) is given by

\[
N_d = \# \{(x, y) \in \mathbb{F}_q^2 : P(x, y) = 0 \},
\]

where

\[
P(x, y) = x^d + ax + b - y^2.
\]

Using the elementary identity from [10]

\[
\sum_{z \in \mathbb{F}_q} \theta(zP(x, y)) = \begin{cases} 
q & \text{if } P(x, y) = 0; \\
0 & \text{if } P(x, y) \neq 0,
\end{cases}
\]

we obtain that

\[
q \cdot N_d = \sum_{x, y, z \in \mathbb{F}_q} \theta(zP(x, y))
\]

\[
= q^2 + \sum_{z \in \mathbb{F}_q^*} \theta(zb) + \sum_{y \in \mathbb{F}_q^*} \theta(zy^2) + \sum_{x \in \mathbb{F}_q^*} \theta(zx^d)\theta(zax)
\]

\[
+ \sum_{x \in \mathbb{F}_q^*} \theta(zb)\theta(zx^d)\theta(zax)\theta(-zy^2)
\]

(3.1)

\[
:= q^2 + A + B + C + D.
\]

We use Lemma 2.4 and Lemma 2.3 repeatedly in each term of (3.1) to simplify and express in terms of Gauss sums. We obtain

\[
A = \frac{1}{q-1} \sum_{z \in \mathbb{F}_q^*} G_m T^l(z) = \frac{1}{q-1} \sum_{l=0}^{q-2} G_m T^l(b) \sum_{z \in \mathbb{F}_q^*} T^l(z) = -1.
\]

Similarly,

\[
B = \frac{1}{(q-1)^2} \sum_{l,m=0}^{q-2} G_m G_{-m} T^l(b) T^m(-1) \sum_{z \in \mathbb{F}_q^*} T^{l+m}(z) \sum_{y \in \mathbb{F}_q^*} T^2m(y).
\]

This term is nonzero only if \( m = 0 \) or \( n = \frac{q-1}{2} \) and \( l = -m \). Thus the fact \( G_0 = -1 \) and Lemma 2.1 yield

\[
B = 1 + q\phi(b).
\]

Expanding the next term, we have

\[
C = \frac{1}{(q-1)^3} \sum_{l,m,n=0}^{q-2} G_m G_{-m} G_{-n} T^l(b) T^n(a) \sum_{z \in \mathbb{F}_q^*} T^{l+m+n}(z) \sum_{x \in \mathbb{F}_q^*} T^{md+n}(x).
\]
Finally,

$$D = \frac{1}{(q-1)^4} \sum_{l,m,n,k=0}^{q-2} G_{-l} G_{-m} G_{-n} G_{-k} T^l(b) T^n(a) T^k(-1) \times$$

$$\sum_{z \in \mathbb{F}_q^*} T^{l+m+n+k}(z) \sum_{x \in \mathbb{F}_q^*} T^{md+n}(x) \sum_{y \in \mathbb{F}_q^*} T^{2k}(y).$$

The innermost sum of $D$ is nonzero only if $k = 0$ or $\frac{q-1}{2}$. For $k = 0$, we obtain the term equal to $-C$. Further for $k = \frac{q-1}{2}$, we denote the term by $D_{\frac{q-1}{2}}$ given as

$$D_{\frac{q-1}{2}} = \frac{\phi(-1) G_{\frac{q-1}{2}}}{(q-1)^3} \sum_{l,m,n=0}^{q-2} G_{-l} G_{-m} G_{-n} T^l(b) T^n(a) \times$$

$$\sum_{z \in \mathbb{F}_q^*} T^{l+m+n+\frac{q-1}{2}}(z) \sum_{x \in \mathbb{F}_q^*} T^{md+n}(x).$$

Here, the term $D_{\frac{q-1}{2}}$ is zero unless $n = -md$ and $l = (d-1)\{m + \frac{q-1}{2(d-1)}\}$. Then using Lemma 2.3 we have

$$D_{\frac{q-1}{2}} = \frac{\phi(-1) G_{\frac{q-1}{2}}}{(q-1)} \sum_{m=0}^{q-2} G_{(d-1)(-m-\frac{q-1}{2(d-1)})} G_{-m} G_{dm} T^m \left( \frac{b^{d-1}}{a^d} \right).$$

Thus, combining values of $A, B, C, D$ all together in (3.1), we have

$$q \cdot N_d = q^2 + q\phi(b) + D_{\frac{q-1}{2}}.$$

If $d \geq 2$ is an even integer, from (2.2) and (2.3), we have

$$G_{dm} = \frac{G_{m} G_{m+\frac{q-1}{2}} G_{m+\frac{2(q-1)}{4}} \cdots G_{m+(d-1)(q-1)}}{q^{\frac{d-2}{2}} G_{\frac{q-1}{2}} T^{\frac{(d-2)(q-1)}{8(d-1)}} (-1) T^{-m}(d^d)}$$

and

$$G_{(d-1)(-m-\frac{q-1}{2(d-1)})} = \frac{G_{-m} G_{-m-\frac{3(q-1)}{4}} G_{-m-\frac{5(q-1)}{4(d-1)}} \cdots G_{-m-(2d-3)(q-1)}}{q^{\frac{d-2}{2}} T^{\frac{(d-2)(q-1)}{8(d-1)}} (-1) T^{m+\frac{q-1}{2(d-1)}} ((d-1)d-1)}.$$
Using these in (3.2), we obtain
\[
D_{\alpha d} = \frac{\phi(-b(d-1)) T^{\frac{(d-2)(q-1)}{2(d-1)}}}{q^{d-2}(q-1)} \sum_{m=0}^{q-2} \left\{ G_m G_{m+ \frac{2(q-1)}{d-1}} G_{m+ \frac{(d-1)(q-1)}{d(d-1)}} \right\} \\
\times \left\{ G_{m+ \frac{2(q-1)}{d-1}} G_{m- \frac{3(q-1)}{2(d-1)}} \right\} \cdots \left\{ G_{m+ \frac{(d-2)(q-1)}{d} G_{m- \frac{(d-3)(q-1)}{2(d-1)}} \right\} \\
\times \left\{ G_{m+ \frac{2(q-1)}{d-1}} G_{m- \frac{(d-1)(q-1)}{d(d-1)}} \right\} \cdots \left\{ G_{m+ \frac{(d-2)(q-1)}{d} G_{m- \frac{(d-1)(q-1)}{d(d-1)}} \right\} \\
= \phi(-b(d-1)) T^{\frac{(d-2)(q-1)}{2(d-1)}} \sum_{m=0}^{q-2} \left\{ G_m G_{m+ \frac{2(q-1)}{d-1}} G_{m- \frac{3(q-1)}{2(d-1)}} \right\} \\
\cdots \left\{ G_{m+ \frac{2(q-1)}{d-1}} G_{m- \frac{(d-1)(q-1)}{d(d-1)}} \right\} \cdots \left\{ G_{m+ \frac{(d-1)(q-1)}{d} G_{m- \frac{(d-2)(q-1)}{2(d-1)}} \right\} T^{m \left( \frac{d^d b^{d-1}}{(d-1)^{d-1} q^d} \right)}.
\]

Using Lemma 2.2 in each term in bracket, we deduce that
\[
D_{\alpha d} = \frac{q^2 \phi(-b(d-1)) T^{\frac{(d-2)(q-1)}{2(d-1)}}}{(q-1)} \sum_{m=0}^{q-2} \left( T^{m + \frac{d-2}{d}} \right) \left( T^{m + \frac{d-1}{d}} \right) \cdots \left( T^{m + \frac{1}{d}} \right) T^{m \left( \frac{d^d b^{d-1}}{(d-1)^{d-1} q^d} \right)}
\]
\[
\times T^{m \left( \frac{d^d b^{d-1}}{(d-1)^{d-1} q^d} \right)}.
\]

Thus Lemma 2.4 yields
\[
D_{\alpha d} = q^{d+2} \phi(b(d-1))
\]
\[
\times \alpha F_{\alpha d-1} \left( \phi, \psi, \phi, \psi^3, \phi, \psi^4, \cdots, \phi, \psi^{d-3}, \phi, \psi^{d+1}, \cdots, \phi, \psi^{2d-3} \mid \alpha \right),
\]
where \(\alpha = \frac{d}{a} \left( \frac{bd}{a(d-1)} \right)^{d-1} \). Hence (3.3) completes the proof of Theorem 1.4.

Now we prove Theorem 1.5. For \(d > 1\) odd integer, the Davenport-Hasse relations (2.3) and (2.5) yield
\[
G_{dm} = \frac{G_m G_{m+ \frac{2(q-1)}{d-1}} G_{m+ \frac{(d-1)(q-1)}{d(d-1)}}}{q^d T^{\frac{(d-2)(q-1)}{2(d-1)}}} \left( -1 \right) T^{m(d^d)}
\]
and
\[
G_{m- \frac{1}{2(d-1)}} = \frac{G_{m- \frac{1}{2(d-1)}} G_{m- \frac{3(q-1)}{2(d-1)}} G_{m- \frac{(d-3)(q-1)}{2(d-1)}}}{q^d T^{\frac{(d-1)(q-1)}{2q(d-1)}}} \left( -1 \right) T^{m- \frac{1}{m-1}}((d-1)^{d-1}).
\]
We use these relations in (3.2) to obtain
\[
D_{\frac{d-1}{q-1}} = \frac{\phi(-b(d-1))T^{\frac{3d-1}{2d}(q-1)}(-1)}{q^{d-2}(q-1)} \sum_{m=0}^{q-2} \{G_m G_m - \frac{1}{2}(q-1)\} \times \{G_m + \frac{2q-1}{2} G_m - \frac{3(q-1)}{2} \cdots \times \{G_m + \frac{(d-1)(q-1)}{2d} G_m - \frac{(d-2)(q-1)}{2d} \} \times \{G_m + \frac{(d+1)(q-1)}{2d} G_m - \frac{(d-3)(q-1)}{2d} \} \times \cdots \times \{G_m + \frac{(d-1)(q-1)}{2d} G_m - \frac{(2d-3)(q-1)}{2d} \} T^m \left( \frac{d!b^{d-1}}{(d-1)!q^{d-1}} \right),
\]
To eliminate $G_m G_m$, we use the facts that if $m \neq 0$, then $G_m G_m = qT^{m-1}$; and if $m = 0$, then $G_m G_m = 1 = qT^{m-1} - (q-1)$ in appropriate identities above. After that, we rearrange the second term to deduce that
\[
D_{\frac{d-1}{q-1}} = \frac{\phi(-b(d-1))T^{\frac{3d-1}{2d}(q-1)}(-1)}{q^{d-2}(q-1)} \sum_{m=0}^{q-2} \{G_m + \frac{1}{2} G_m - \frac{1}{2} \} \times \{G_m + \frac{2q-1}{2} G_m - \frac{3(q-1)}{2} \cdots \times \{G_m + \frac{(d-1)(q-1)}{2d} G_m - \frac{(d-2)(q-1)}{2d} \} \times \{G_m + \frac{(d+1)(q-1)}{2d} G_m - \frac{(d-3)(q-1)}{2d} \} \times \cdots \times \{G_m + \frac{(d-1)(q-1)}{2d} G_m - \frac{(2d-3)(q-1)}{2d} \} T^m \left( \frac{d!b^{d-1}}{(d-1)!q^{d-1}} \right)
\]
Using Lemma 2.2 in the first term, and then Lemma 2.1 in both terms, we obtain
\[
D_{\frac{d-1}{q-1}} = \frac{q^2 \phi(-b(d-1))T^{\frac{3d-1}{2d}(q-1)}(-1)}{(q-1)} \{G_m + \frac{q-1}{2} G_m - \frac{1}{2} \} \times \{G_m + \frac{(d-1)(q-1)}{2d} G_m - \frac{(d-2)(q-1)}{2d} \} \times \cdots \times \{G_m + \frac{(d-1)(q-1)}{2d} G_m - \frac{(2d-3)(q-1)}{2d} \} T^m \left( \frac{d!b^{d-1}}{(d-1)!q^{d-1}} \right)
\]

Replacing $m$ by $m - \frac{d-1}{2(d+1)}$ in the first term, we have

$$D_{d+1} = \frac{d^{d+1}}{(q-1)} \sum_{m=0}^{q-2} \left( T^{m+\frac{(d-2)(q-1)}{2d+1}} \right) \left( T^{m+\frac{2(q-1)}{2d+1}} \right) \cdots \times$$

$$\left( T^{m+\frac{(d^2-3d+5)(q-1)}{2d+1}} \right) \left( T^{m+\frac{(d^2-3d+5)(q-1)}{2d+1}} \right) \times$$

$$\left( T^{m+\frac{(d^2-3d+5)(q-1)}{2d+1}} \right) \left( T^{m+\frac{(d^2-3d+5)(q-1)}{2d+1}} \right) \times$$

$$T^{m} \left( \frac{d^{d+1}b^{d-1}}{(d-1)^{d-1}a^{d}} \right) T^{\frac{q-1}{2(d+1)}} \left( -\frac{(d-1)^{d-1}a^{d}}{d^{d+1}b^{d-1}} \right) - q\phi(-b(d-1))T^{\frac{q-1}{2(d+1)}}(-1)$$

$$- q\phi(-b(d-1))T^{\frac{q-1}{2(d+1)}}(-1) + q^{d+1} \phi(-b(d-1))T^{\frac{q-1}{2(d+1)}}(-1)T^{\frac{q-1}{2(d+1)}}(-1) \times$$

$$d-1 \left( \xi^{d-2}, \xi^{3d-4}, \ldots, \xi^{d-3d+1}, \xi^{d-2d-1}, \ldots, \xi^{2d-3d+2} \psi^{d-3}, \ldots, \psi^{d-2d-4} | -\alpha \right),$$

where $\alpha = \frac{d}{a} \left( \frac{bd}{a(d-1)} \right)^{d-1}$. We complete the proof of Theorem 1.2 by putting the above value of $D_{d+1}$ in (3.3).

Now we show that Theorem 2.1 of Lennon [13] follows from Theorem 1.2.

**Theorem 3.1.** ([13] Thm. 2.1). Let $q = p^r, p > 3$ a prime and $q \equiv 1 \pmod{12}$. Let $E : y^2 = x^3 + ax + b$ be an elliptic curve over $\mathbb{F}_q$ with $j(E) \neq 0, 1728$. Then the trace of the Frobenius map on $E$ can be expressed as

$$a(E(\mathbb{F}_q)) = -q \cdot T^{\frac{q-1}{27}} \left( \frac{a^3}{27} \right) 2F1 \left( T^{\frac{q-1}{12}}, \frac{T^{\frac{5(q-1)}{12}}}{T^{\frac{q-1}{27}}} | -\frac{27b^2}{4a^3} \right).$$

**Proof.** We have $E_3 : y^2 = x^3 + ax + b$ with $a, b \neq 0$. Hence, $j(E_3) \neq 0, 1728$. For $q \equiv 1 \pmod{12}$, from Theorem 1.2 we have

$$N_3 = \# \{ (x, y) \in \mathbb{F}_q : y^2 = x^3 + ax + b \}$$

$$= q + \phi(b) - \phi(-2b)T^{\frac{q-1}{27}}(-1)$$

$$+ q\phi(-2b)T^{\frac{q-1}{27}}(-1)T^{\frac{q-1}{12}} \left( \frac{4a^3}{27b^2} \right) 2F1 \left( \xi, \xi^5 \psi^2 | -\frac{27b^2}{4a^3} \right)$$

$$= q + \phi(b) - \phi(2)\phi(b)T^{\frac{q-1}{27}}(-1)$$

$$+ qT^{\frac{q-1}{27}}(4b^2)T^{\frac{q-1}{27}}(-1)T^{\frac{q-1}{12}} \left( \frac{4a^3}{27b^2} \right) 2F1 \left( T^{\frac{q-1}{12}}, \frac{T^{\frac{5(q-1)}{12}}}{T^{\frac{q-1}{27}}} | -\frac{27b^2}{4a^3} \right).$$

Since $\phi(2) = T^{\frac{q-1}{27}}(-1)$, therefore $\phi(2)\phi(b)T^{\frac{q-1}{27}}(-1) = \phi(b)$. Hence

$$N_3 = q + qT^{\frac{q-1}{27}} \left( \frac{a^3}{27} \right) 2F1 \left( T^{\frac{q-1}{12}}, \frac{T^{\frac{5(q-1)}{12}}}{T^{\frac{q-1}{27}}} | -\frac{27b^2}{4a^3} \right).$$

We complete the proof using the fact that $a(E(\mathbb{F}_q)) = a(E_3(\mathbb{F}_q)) = q - N_3$. □

4. **Proof of Theorem [13] and Theorem 1.4**

We now prove Theorem 1.3 and Theorem 1.4. For $d \geq 2$, the hyperelliptic curve $E'_d$ defined over $\mathbb{F}_q$ is given by

$$E'_d : y^2 = x^d + ax^{d-1} + b,$$
where $a, b \in \mathbb{F}_q^\times$. Then the number of points on $E'_d$ over $\mathbb{F}_q$ is given by

$$N'_q = \#E'_d(\mathbb{F}_q) = \# \{(x, y) \in \mathbb{F}_q^2 : P'(x, y) = 0 \},$$

where

$$P'(x, y) = x^d + ax^{d-1} + b - y^2.$$

As shown earlier, we have

$$q \cdot N'_d = q^2 + \sum_{z \in \mathbb{F}_q^\times} \theta(zb) + \sum_{y, z \in \mathbb{F}_q^\times} \theta(zb) \theta(-zy^2) + \sum_{x, z \in \mathbb{F}_q^\times} \theta(zb) \theta(zax^{d-1})$$

$$+ \sum_{x, y, z \in \mathbb{F}_q^\times} \theta(zb) \theta(zax^{d-1}) \theta(-zy^2)$$

$$:= q^2 + A + B + C + D.$$

Following the same procedure as followed in the proof of Theorem 1.1, we obtain

$$A = -1,$$
$$B = 1 + q\phi(b),$$

and

$$D = -C + D_{a,b}.$$

where

$$D_{a,b} = \frac{\phi(-1)G_{a,b}}{(q - 1)^3} \sum_{l, m, n = 0}^{q-2} G_{-l}G_{-m}G_{-n}T^d(b)T^n(a) \sum_{z \in \mathbb{F}_q^\times} T^{l+m+n+\frac{x-1}{2}}(z)$$

$$\times \sum_{x \in \mathbb{F}_q^\times} T^{md+n(d-1)}(x).$$

Combining all values of $A, B, C$ and $D$ together, we have

$$qN'_d = q^2 + q\phi(b) + D_{a,b}. (4.2)$$

Now let $d$ be even. Then the term $D_{a,b}$ is zero unless $n = -ld$ and $m = (d-1)(l + \frac{a-1}{2(d-1)})$. Then using Lemma 2.3, we have

$$D_{a,b} = \frac{\phi(-1)G_{a,b}}{(q - 1)^3} \sum_{l = 0}^{q-2} G_{-l}G_{(d-1)(-l - \frac{a-1}{2(d-1)})}G_{ld}T^d \left( \frac{b}{a^d} \right).$$

Following the proof of Theorem 1.1 we deduce that

$$D_{a,b} = q^{\frac{d+2}{2}} \phi((d - 1))$$

$$\times a^{F_{d-1}} \left( \phi, \varphi, \chi, \chi^2, \ldots, \chi^{\frac{d-2}{2}}, \chi^{\frac{d+2}{2}}, \psi^3, \ldots, \psi^{d-3}, \psi^{d+1}, \ldots, \psi^{2d-3} | \beta \right),$$

where $\beta = \frac{bd^d}{a^d(d-1)^{d-1}}$. Hence (4.2) completes the proof of Theorem 1.3

For $d > 1$ odd integer, the innermost sums of $D_{a,b}$ are nonzero only if $m = l(d-1)$ and $n = d(-l - \frac{a-1}{2(d-1)})$, at which both are $q - 1$. Therefore, we have

$$D_{a,b} = \frac{\phi(-a)G_{a,b}}{(q - 1)^3} \sum_{l = 0}^{q-2} G_{-l}G_{-(d-1)}G_{ld(l + \frac{a-1}{2(d-1)})}T^d \left( \frac{b}{a^d} \right).$$
The Davenport-Hasse relations \(2.2\) and \(2.3\) yield

\[
G_{-(d-1)l} = \frac{G_{l} G_{-l-q+1} G_{-l-2(q-1)} \cdots G_{-l-(d-2)(q-1)}}{q^{d-2} G_{\frac{q-1}{2}} T^{\frac{(d-3)(q-1)}{8}} (-1) T^l ((d - 1)^{d-1})}
\]

and

\[
G_{d(l+\frac{q-1}{2})} = \frac{G_{l+\frac{q-1}{2}} G_{l+\frac{2(q-1)}{2}} G_{l+\frac{5(q-1)}{2}} \cdots G_{l+(2d-1)(q-1)}}{q^{\frac{d}{2}} T^{\frac{(d-1)(d+1)(q-1)}{2d}} (-1) T^{-l-\frac{q-1}{2}} (d^l)}.
\]

We use these relations in \((1.3)\) to obtain

\[
D_{\frac{q-1}{2}} = \phi(-ad) T^{\frac{(3d-1)(q-1)}{8d}} (-1) \sum_{l=0}^{q-2} \left\{ G_{l+\frac{q-1}{2}} G_{-l} \left( \frac{q-1}{2} \right) - G_{l+\frac{3(q-1)}{2}} G_{-l-\frac{q-1}{2}} \right\} \times \left\{ G_{l+\frac{5(q-1)}{2}} G_{-l-2(q-1)} \right\} \cdots \left\{ G_{l+(d-2)(q-1)} G_{-l-(d-3)(q-1)} \right\} \times \left\{ G_{l+\frac{(d+2)(q-1)}{2}} G_{-l-(d+2)(q-1)} \right\} \cdots \left\{ G_{l+(2d)(q-1)} G_{-l-(2d)(q-1)} \right\} \times G_{l+\frac{(2d-1)(q-1)}{2}} T^l(\beta),
\]

where \(\beta = \frac{d^2 b}{(d-1)^{d-1} a^d}\). We now eliminate the term \(\left\{ G_{l+\frac{d(q-1)}{2}} G_{-l-\frac{(d+1)(q-1)}{2}} \right\}\), which is equal to \(G_{l+\frac{q-1}{2}} G_{-l-\frac{q-1}{2}}\). We use the fact that

\[
G_{l+\frac{q-1}{2}} G_{-l-\frac{q-1}{2}} = \begin{cases} 
q T^l + \frac{q-1}{2} (-1), & \text{if } l \neq \frac{q-1}{2}; \\
q T^l + \frac{q}{2} (-1) - (q-1), & \text{if } l = \frac{q-1}{2}.
\end{cases}
\]

in appropriate identities above to obtain

\[
D_{\frac{q-1}{2}} = \phi(ad) T^{\frac{(3d-1)(q-1)}{8d}} (-1) \sum_{l=0}^{q-2} \left\{ G_{l+\frac{q-1}{2}} G_{-l} \left( \frac{q-1}{2} \right) - G_{l+\frac{3(q-1)}{2}} G_{-l-\frac{q-1}{2}} \right\} \times \left\{ G_{l+\frac{5(q-1)}{2}} G_{-l-2(q-1)} \right\} \cdots \left\{ G_{l+(d-2)(q-1)} G_{-l-(d-3)(q-1)} \right\} \times \left\{ G_{l+\frac{(d+2)(q-1)}{2}} G_{-l-(d+2)(q-1)} \right\} \cdots \left\{ G_{l+(2d-2)(q-1)} G_{-l-(2d)(q-1)} \right\} \times \left\{ G_{l+\frac{(2d)(q-1)}{2}} G_{-l-(2d)(q-1)} \right\} T^l(-\beta) - \frac{\phi(-b) T^{\frac{(3d-1)(q-1)}{8d}} (-1)}{q^{d-2}} \times \left\{ G_{l+\frac{(d+1)(q-1)}{2}} G_{-l-\frac{d(q-1)}{2}} \right\} \times \left\{ G_{l+\frac{(d+2)(q-1)}{2}} G_{-l-(d+2)(q-1)} \right\} \cdots \left\{ G_{l+\frac{(2d)(q-1)}{2}} G_{-l-(2d)(q-1)} \right\}.
\]
Theorem 4.1. Let \( D_{\frac{d-1}{2}} \) in the first term, and rearranging both terms, we deduce that

\[
D_{\frac{d-1}{2}} = q^2 \phi(ad) T^{\frac{(d-1)(q-1)}{2d}} (-1) \{ G_{\frac{d-1}{2}} G_{\frac{q-1}{2}} \} \{ G_{\frac{(d-3)(q-1)}{2d(d-1)}} \} \{ G_{\frac{(d-3)(q-1)}{2d(d-1)}} \} \times \{ G_{\frac{(d-5)(q-1)}{2d(d-1)}} G_{\frac{(d-5)(q-1)}{2d(d-1)}} \} \cdots \{ G_{\frac{2(q-1)}{2d(d-1)}} G_{\frac{2(q-1)}{2d(d-1)}} \} \\
\times \sum_{l=0}^{q-2} \left( T^{l+\frac{(d-2)(q-1)}{2d}} T^l \right) \left( T^{l+\frac{(2d-3)(q-1)}{2d}} T^l \right) \left( T^{l+\frac{(2d-2)(q-1)}{2d}} T^l \right) \cdots \left( T^{l+\frac{(d-3)(q-1)}{2d}} T^l \right) \\
\times \frac{T(-\beta) - \phi(-b) T^{\frac{(d-1)(q-1)}{2d}} (-1)}{q^{d-2}} \{ \{ G_{\frac{(d+1)(q-1)}{2d}} G_{\frac{(d+1)(q-1)}{2d}} \} \{ \{ G_{\frac{(d+3)(q-1)}{2d(d-1)}} G_{\frac{(d+3)(q-1)}{2d(d-1)}} \} \cdots \{ G_{\frac{2(q-1)}{2d(d-1)}} G_{\frac{2(q-1)}{2d(d-1)}} \} \{ G_{\frac{q-1}{2}} G_{\frac{q-1}{2}} \}
\]

Finally, using Lemma \([22]\) we have

\[
D_{\frac{d-1}{2}} = -q^2 \phi(b) + q^{\frac{d-1}{2}} \phi(-ad) \\
\times _{d-1} F_{d-2} \left( \eta, \eta^3, \eta^5, \ldots, \eta^{d-2}, \eta^{d+2}, \ldots, \eta^{2d-3}, \eta^{2d-1}, \varepsilon \right) = 0.
\]

We complete the proof of Theorem 4.1 by putting the above value of \( D_{\frac{d-1}{2}} \) in (4.2).

Now we show that Theorem 3.1 of authors [3] follows from Theorem 1.4.

Theorem 4.1. (H Thm. 3.1). Let \( q = p^e, p > 2 \) a prime and \( q \equiv 1 \pmod{6} \). If \( E_1 : y^2 = x^3 + ax^2 + b \) is an elliptic curve, then the trace of the Frobenius on the elliptic curve \( E_1 : y^2 = x^3 + ax^2 + b, a \neq 0 \) is given by

\[
a_q(E_1) = -q T^{\frac{a-1}{2}} (-3a) \ _2F_1 \left( \frac{T^{a-1}}{\varepsilon}, \frac{T^{\frac{2(q-1)}{6}}}{} \right) \varepsilon \left( -\frac{27b}{4a^3} \right),
\]

where \( \varepsilon \) is the trivial character on \( \mathbb{F}_q \).

Proof. Since \( E_1 : y^2 = x^3 + ax^2 + b \) is an elliptic curve, so \( b \neq 0 \). Hence, both \( a \) and \( b \) are non-zero. Putting \( d = 3 \) in Theorem 1.4 for \( q \equiv 1 \pmod{6} \), we deduce that

\[
N_d' = q + q^2 \phi(-3a) \ _2F_1 \left( \frac{\eta}{\varepsilon}, \frac{\eta^5}{\varepsilon} \right) = q + q^2 \phi(-3a) \ _2F_1 \left( \frac{T^{\frac{a-1}{2}}}{\varepsilon}, \frac{T^{\frac{2(q-1)}{6}}}{\varepsilon} \right) \varepsilon \left( -\frac{27b}{4a^3} \right).
\]

We complete the proof using the fact that \( a_q(E_1) = a_q(E_3') = q - N_d' \).

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