COMPUTATION OF HECKE EIGENVALUES (MOD $p$) VIA QUATERNIONS

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Abstract. In a 1987 letter [SL96], Serre proves that the systems of Hecke eigenvalues arising from mod $p$ modular forms (of fixed level $\Gamma(N)$ coprime to $p$, and any weight $k$) are the same as those arising from functions $\Omega(N) \to \overline{\mathbb{F}}_p$, where $\Omega(N)$ is some double quotient of $D^\times(\mathbb{A}_f)$ and $D$ is the unique quaternion algebra over $\mathbb{Q}$ ramified at $\{p, \infty\}$. We present an algorithm which then computes these Hecke eigenvalues on the quaternion side in a combinatorial manner.

1. Introduction

The study of Hecke eigenvalues originated from Ramanujan’s $\Delta$-function

$$\Delta(q) = q \prod_{n=1}^\infty (1 - q^n)^{24},$$

a weight 12 and level 1 Hecke eigenform about which Ramanujan made a number of conjectures that have motivated much of the theory of modular forms over the past century. More recently, it is known [Del69] that a normalised Hecke eigenform $f = \sum a_n q^n$ in $S_k(\Gamma_0(N); \epsilon)$ gives rise to a mod $p$ Galois representation $\rho : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \text{GL}_2(\overline{\mathbb{F}}_p)$. This representation is unramified away from $pN$ and is characterised by the following trace and determinant of Frobenius data (mapped appropriately into $\overline{\mathbb{F}}_p$):

$$\text{tr} (\rho(\text{Frob}_\ell)) = a_\ell, \quad \text{det} (\rho(\text{Frob}_\ell)) = \ell^{k-1} \epsilon(\ell).$$

These representations play a central role in modern number theory, most notably in the conjectures of Serre. As such, the mod $p$ Hecke eigenvalues are objects of great interest, and so one may wish to enumerate these systems of Hecke eigenvalues as a source of examples.

At present, there already exist algorithms for computing these Hecke eigenvalues, for example using modular symbols in connection with the Eichler-Shimura theorem [Wie06]. Our approach will instead make use of the following theorem of Serre (where we have restricted his result to a fixed level $\Gamma(N)$) [SL96], that tells us that one could instead compute these Hecke eigenvalues by working with a particular quaternion algebra. The computation of the Hecke eigenvalues then becomes a combinatorial one, which is perhaps a more elementary approach.

**Theorem 1.1.** Let $D$ be the unique quaternion algebra over $\mathbb{Q}$ ramified at $\{p, \infty\}$. The systems of Hecke eigenvalues $(a_\ell)$ (with $a_\ell \in \overline{\mathbb{F}}_p$, $\ell \nmid pN$ and fixed $N \geq 3$ coprime to $p$) coming from the modular forms (mod $p$) of level $\Gamma(N)$, are the same as those coming from the functions

$$(\Omega(N) := U(N) \backslash D^\times(\mathbb{A}_f)/D^\times(\mathbb{Q})) \to \overline{\mathbb{F}}_p.$$
We remark that in this theorem, we do not necessarily realise all the weight \( k \) systems of eigenvalues from the modular form side as weight \( (k \mod p^2 - 1) \) systems of eigenvalues on the quaternion side, only those arising from weight \( k \) eigenforms not divisible by the Hasse invariant. They will still appear on the quaternion side when ranging over all weights.

**Notation 1.2.** We need to explain some of this notation:

- Let \( \mathcal{O} \) be any maximal order of \( D \).
- Let \( D_\ell = D \otimes \mathbb{Q}_\ell \) and \( \mathcal{O}_\ell = \mathcal{O} \otimes \mathbb{Z}_\ell \) a maximal order, for any prime \( \ell \). So \( \mathcal{O}_\ell \cong M_2(\mathbb{Z}_\ell) \) for \( \ell \neq p \).
- Let \( D^\times(\mathbb{A}_f) \) denote the finite part of the adelic points of \( D^\times \), in other words the restricted product \( \prod_\ell D^\times_\ell \) with respect to the subgroups \( \mathcal{O}^\times_\ell \).
- Let \( \pi \in \mathcal{O} \) be a uniformiser of \( \mathcal{O}_p \).
- Let \( \mathcal{O}_p^\times(1) \) be the kernel of reduction mod \( \pi \), \( \mathcal{O}_p^\times \to \mathbb{F}_p^\times \).
- For \( \ell \neq p \), let \( \mathcal{O}_\ell^\times(N) \) be the subgroup of \( \mathcal{O}_\ell^\times \cong \text{GL}_2(\mathbb{Z}_\ell) \) consisting of elements congruent to 1 mod \( \mathfrak{f}(\ell\mathcal{O}_\ell) \), where \( \ell\mathcal{O}_\ell \) is the highest power of \( \ell \) dividing \( N \).
- Let \( U(N) = \mathcal{O}_\ell^\times(1) \times \prod_{\ell \neq p} \mathcal{O}_\ell^\times(N) \), an open subgroup of \( D^\times(\mathbb{A}_f) \).
- For \( (x_\ell) \in D^\times(\mathbb{A}_f) \), we will denote by \([x_\ell]\) the image in \( \Omega(N) \).
- By a weight \( k \) function \( f : \Omega(N) \to \mathbb{F}_p \), we mean a function which satisfies \( f(\mu \cdot [x_\ell]) = \mu^{-k} f([x_\ell]) \), where \( \mu \in \mathcal{O}_p^\times / \mathcal{O}_p^\times(1) \cong \mathbb{F}_p^\times \) acts on \([x_\ell]\) by multiplication in the \( p \)-place.

Note that elements of \( \Omega(N) \) correspond to isomorphism classes of invertible left \( \mathcal{O} \)-ideals \( I \) with \( \pi N \)-structure, meaning a basis for \( I/\pi NI \) as an \( \mathcal{O}/\pi N \mathcal{O} \)-module. Explicitly, an adelic point \((x_\ell)\) corresponds to an ideal \( I \) with \( I_\ell = \mathcal{O}_\ell x_\ell \) for all \( \ell \). The \( \pi N \)-structure is then given by the reduction modulo \( \pi NI \) of any element \( x \in I \) satisfying the congruences

\[ x \equiv x_\ell \mod \pi NI_\ell \]

for all \( \ell \) (this congruence is vacuous for any \( \ell \nmid pN \), so such \( x \) exists by the Chinese remainder theorem). We then quotient \( D^\times(\mathbb{A}_f) \) by \( U(N) \) and \( D^\times(\mathbb{Q}) \) exactly to get the desired bijection.

For a prime \( \ell_0 \nmid pN \), the Hecke operator \( T_{\ell_0} \) on this space of functions \( \Omega(N) \to \mathbb{F}_p \) is given by

\[ T_{\ell_0} f([x_\ell]) = \ell_0^{-1} \sum_i f(g_i \cdot [x_\ell]) \]

for \( \text{GL}_2(\mathbb{Z}_{\ell_0}) (\begin{smallmatrix} 1 & 0 \\ \ell_0 & 1 \end{smallmatrix}) \text{GL}_2(\mathbb{Z}_{\ell_0}) = \bigcup \text{GL}_2(\mathbb{Z}_{\ell_0}) g_i \). Here \( g_i \cdot [x_{\ell_0}] \) means we pick a representative \((x_\ell) \in D^\times(\mathbb{A}_f)\) of \([x_\ell]\), multiply this in the \( \ell_0 \)-place by the matrix \( g_i \) (under an identification of \( \mathcal{O}_{\ell_0} \cong M_2(\mathbb{Z}_{\ell_0}) \) and hence of \( D_{\ell_0} \cong M_2(\mathbb{Q}_{\ell_0}) \)), and then take the image in \( \Omega(N) \). Each individual \( g_i \cdot [x_\ell] \) is not well defined in \([x_\ell]\), but \( T_{\ell_0} \) is well defined, provided we pick the same representative \((x_\ell) \in D^\times(\mathbb{A}_f)\) for each multiplication. This Hecke module structure has been studied before, as in the likes of [Koh01].

The algorithm we present computes a matrix for the Hecke operator \( T_{\ell_0} \) on the space of (weight \( k \)) functions \( \Omega(N) \to \mathbb{F}_p \), where \( \ell_0, p \) and \( N \) are pairwise coprime. Note that we allow the cases \( N = 1, 2 \) as the definitions on
the quaternion side still make sense here. We begin by computing an illustrative example for the case \( p = 11 \), and then generalise this to weight \( k = 0 \) and level \( N = 1 \) for any \( p \). Note that in this case the matrices we compute are exactly Brandt matrices. This is then extended to higher weight and level, essentially by keeping track of the \( \pi N \)-structure.

We remark that a similar computation has been performed by Pizer in [Piz80]. This has since been applied in, for example, [SW05] and [CS01]. Pizer was interested in computing the subspace of cusp forms on \( \Gamma_0(N) \) generated by theta series, and the Hecke operators on this subspace. The algorithm involves computing certain Brandt matrix series. The main difference between our approach and Pizer’s is that we incorporate level \( N \) structure through coset representatives of \( O_{\ell}(\mathbb{Z}/\ell v(\mathcal{N})\mathbb{Z}) \). See also [Dem05] and [Dem07] for a similar approach in the context of Hilbert modular forms (the corresponding description of Hilbert modular forms on the quaternion side is more involved than for modular forms). Pizer instead works with orders of level \( N \), which allows them to work purely with the quaternion algebras, without having to write down explicit isomorphisms of the form \( O_{\ell} \cong M_2(\mathbb{Z}_{\ell}) \), something that we must do (see Section 3). On the other hand, the use of matrices perhaps makes the contribution from the level structure more visible. The tradeoff is then between computing these isomorphisms with the Hensel-like argument involved, and computing orders of level \( N \). A related point of difference is that in Pizer’s argument one must compute the left ideal classes of an order of level \( N \), whereas we only need to compute these for our maximal order \( O \), because our level structure is already captured in \( O_{\ell}^\times(\mathcal{N}) \cong GL_2(\mathbb{Z}/\ell v(\mathcal{N})\mathbb{Z}) \), which is very concrete. Because a maximal order has minimal discriminant, the corresponding Minkowski bound is lower, and so the computation of these ideal classes in our case may be (somewhat) faster.

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2. An Example

**Example 2.1.** Take \( p \) to be 11. The quaternion algebra \( D \) is then \( D = \left( \frac{-1,-11}{\mathbb{Q}} \right) \). In this example we will work with level \( N = 1 \) as doing so greatly reduces the size of \( \Omega(\mathcal{N}) \). We will then compute the Hecke eigenvalues on the space of functions on \( \Omega := U \backslash D^\times(\mathbb{A}_f)/D^\times(\mathbb{Q}) \), where \( U := O_p^\times \times \prod_{l \neq p} O_l^\times \) is not quite \( U(1) \) as we replace \( O_p^\times \) with \( O_p^\times(1) \). Note that this is the same as computing the Hecke eigenvalues on \( \Omega(1) \) of weight \( k \equiv 0 \mod p^2 - 1 \). Indeed, the modularity condition tells us that these are functions on \( \Omega(1) \) that are invariant under the action of \( O_p^\times \), and so can be identified with functions on \( \Omega \). The computation in this section is based on notes of Buzzard [Buz].

Firstly, let’s understand the set \( \Omega \). By the same argument that \( \Omega(\mathcal{N}) \) corresponds to (isomorphism classes of) invertible left \( O \)-ideals \( I \) with \( \pi \mathcal{N} \)-structure, we see that \( \Omega \) corresponds just to the invertible left \( O \)-ideals...
Running the following in MAGMA [BCP97], we compute a maximal order $O$ of $D$, and then find that its left ideal class set has order 2, whose elements we call $I_1$ and $I_2$. So $\Omega$ has two elements. We can also compute integer bases for $O$, $I_1$, and $I_2$.

\[
\begin{align*}
\text{>> } & \text{D := QuaternionAlgebra< RationalField() | -1, -11>; } \\
\text{>> } & \text{O := MaximalOrder(D); } \\
\text{>> } & \text{Basis(O); } \\
\text{>> } & \text{Classes := LeftIdealClasses(O); } \\
\text{>> } & \text{#Classes; } \\
\text{>> } & \text{I1 := Classes[1]; } \\
\text{>> } & \text{Basis(I1); } \\
\text{>> } & \text{I2 := Classes[2]; } \\
\text{>> } & \text{Basis(I2); }
\end{align*}
\]

which outputs:

\[
\begin{align*}
&\begin{bmatrix} 1, i, 1/2*i + 1/2*k, 1/2 + 1/2*j \end{bmatrix} \\
&2 \\
&\begin{bmatrix} 1, -i, -1/2*i - 1/2*k, 1/2 - 1/2*j \end{bmatrix} \\
&\begin{bmatrix} 2, -2*i, 1 - 3/2*i - 1/2*k, 1/2 - i - 1/2*j \end{bmatrix}
\end{align*}
\]

We now want to rewrite $I_1$ and $I_2$ as adelic points in $\Omega$. For any invertible left $O$-ideal $I$, $I \otimes \mathbb{Z}_\ell$ is a principal $O_\ell$-ideal, generated by any nonzero element $\alpha_\ell$ whose reduced norm has minimal $\ell$-adic valuation. To see this, we refer to Corollary 16.6.12 in Voight [Voi21] that any invertible semi-order (lattice that contains 1 and has reduced norm equal to the ring $R = \mathbb{Z}_\ell$) is an order, so that $(I \otimes \mathbb{Z}_\ell)\alpha_\ell^{-1}$ is an invertible semi-order, which must then by $O_\ell$. So if we are given a $\mathbb{Z}$-basis for $I$, then $I \otimes \mathbb{Z}_\ell$ is generated by a basis element whose reduced norm has minimal $\ell$-adic valuation. Computing the reduced norms of the given basis elements of $I_1$ and $I_2$ we get $[1,1,3,3]$ and $[4,4,6,4]$ respectively. So we see that $I_1 = O$ (which was obvious anyway) and $I_2 \otimes \mathbb{Z}_\ell = O \otimes \mathbb{Z}_\ell$ for all $\ell \neq 2$. Moreover, $I_2 \otimes \mathbb{Z}_2 = O \otimes \mathbb{Z}_2 \cdot (1 - \frac{3}{2}i - \frac{1}{2}ij)$ as 6 has minimal 2-adic valuation. Thus $I_1$ corresponds to $w^1 := [1,1,\ldots] \in \Omega$ and $I_2$ corresponds to $w^2 := [1 - \frac{3}{2}i - \frac{1}{2}ij, 1,1,\ldots] \in \Omega$ (we use superscripts here to avoid overloading the subscripts, which we want to use for the places). An obvious choice of basis for the vector space of $\mathbb{F}_p$ valued functions on $\Omega$ are the characteristic functions $\mathbb{1}_{w^1}$ and $\mathbb{1}_{w^2}$.

To compute the Hecke operator $T_{\ell_0}$, we will need to work with matrices at the $\ell_0$-place. Let’s compute $T_2$ and $T_3$ with respect to the basis $\{\mathbb{1}_{w^1}, \mathbb{1}_{w^2}\}$. We will need isomorphisms $O \otimes \mathbb{Z}_2 \cong M_2(\mathbb{Z}_2)$ and $O \otimes \mathbb{Z}_3 \cong M_2(\mathbb{Z}_3)$. 
In general, we have an isomorphism

\[ D \otimes \mathbb{Q}_\ell \cong M_2(\mathbb{Q}_\ell) \]

\[ i \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \]

\[ j \mapsto \begin{pmatrix} x & y \\ y & -x \end{pmatrix} \]

for \( x, y \in \mathbb{Q}_\ell \) such that \( x^2 + y^2 = -11 \), when \( \ell \neq 11 \). For \( M_2(\mathbb{Z}_3) \), note that \( \mathcal{O} \otimes \mathbb{Z}_3 \) has \( \mathbb{Z}_3 \)-basis \( \{1, i, j, ij\} \), by using the basis for \( \mathcal{O} \) computed above and noticing that \( 2 \) is invertible in \( \mathbb{Z}_3 \). So for any \( x, y \in \mathbb{Z}_3 \) with \( x^2 + y^2 = -11 \), the above isomorphism restricts to an injection \( \mathcal{O} \otimes \mathbb{Z}_3 \hookrightarrow M_2(\mathbb{Z}_3) \), which must in fact be an isomorphism by maximality. We could take \( (x, y) = (\sqrt{-11}, 0) \) for \( \sqrt{-11} \) a root of \( x^2 + 11 = 0 \) in \( \mathbb{Z}_3 \), choosing for example the root which is \( 1 \mod 3 \) by Hensel’s lemma.

For \( M_2(\mathbb{Z}_2) \) we need to be a little more careful because we need \( 1, i, \frac{1}{2}i + \frac{1}{2}j, \frac{1}{2} + \frac{1}{2}j \) to all map to elements in \( M_2(\mathbb{Z}_2) \), rather than just \( 1, i, j, ij \). If we take \( (x, y) = (\sqrt{-15}, 2) \) for \( \sqrt{-15} \) a root of \( x^2 + 15 = 0 \) in \( \mathbb{Z}_2 \) (taking for example the root congruent to \( 1 \mod 4 \) by Hensel’s lemma), then we see that \( \frac{1}{2} + \frac{1}{2}j \) maps to \( \begin{pmatrix} \frac{1 + \sqrt{15}}{2} & 1 \\ \frac{1 - \sqrt{15}}{2} & 1 \end{pmatrix} \), which is in \( M_2(\mathbb{Z}_2) \). This gives us our desired isomorphism \( \mathcal{O} \otimes \mathbb{Z}_2 \cong M_2(\mathbb{Z}_2) \). Note that if we took instead \( (x, y) = (2, \sqrt{-15}) \), we would then map \( \frac{1}{2} + \frac{1}{2}j \) to \( \begin{pmatrix} \frac{1}{2} + \frac{\sqrt{15}}{2} \\ -\frac{1}{2} \end{pmatrix} \), which is not in \( M_2(\mathbb{Z}_2) \).

We now begin our computation of \( \mathcal{O}_2 \), starting with the value of \( T_2(1_{w^1})(w^1) \). By definition,

\[ 2T_2(1_{w^1})(w^1) = 1_{w^1} \left( \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \cdot [1, 1, \ldots] \right) + 1_{w^1} \left( \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \cdot [1, 1, \ldots] \right) + 1_{w^1} \left( \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \cdot [1, 1, \ldots] \right) \]

so we reduce to checking, for example, whether \( \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \cdot [1, 1, \ldots] = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \cdot [1, 1, \ldots] \) is the same as \( w^1 = [1, 1, \ldots] \) or \( w^2 = [1 - \frac{3}{2}i - \frac{1}{2}ij, 1, 1, \ldots] \) as an element of \( \Omega \).

The condition that \( [\begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}, 1, 1, \ldots] = [1, 1, \ldots] \) in \( \Omega \) is equivalent to saying that the \( \mathcal{O} \)-ideal \( J \) is principal, where \( J \) is defined by \( J \otimes \mathbb{Z}_2 = (\mathcal{O} \otimes \mathbb{Z}_2) \cdot \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \) and \( J \otimes \mathbb{Z}_\ell = \mathcal{O} \otimes \mathbb{Z}_\ell \) for all \( \ell \neq 2 \). If this was the case, then we see that \( J \) must be generated by an element \( \alpha \) of \( \mathcal{O} \) (since \( J \otimes \mathbb{Z}_\ell \subset \mathcal{O} \otimes \mathbb{Z}_\ell \) for all \( \ell \)) of reduced norm \( 2 = \det(\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}) \). We have previously computed a basis \( \{1, i, \frac{1}{2}i + \frac{1}{2}j, \frac{1}{2} + \frac{1}{2}j\} \) for \( \mathcal{O} \). Writing \( \alpha = t + x \cdot i + y \cdot (\frac{1}{2}i + \frac{1}{2}j) + z \cdot (\frac{1}{2} + \frac{1}{2}j) \), for \( t, x, y, z \in \mathbb{Z} \), we compute

\[ \text{nr}(\alpha) = (t + \frac{1}{2}z)^2 + (x + \frac{1}{2}y)^2 + \frac{11}{4}y^2 + \frac{11}{4}z^2. \]

For this to be equal to 2, we must have \( y = z = 0 \) and \( t^2 = x^2 = 1 \). So \( J \) must be \( \mathcal{O} \cdot (1 \pm i) \). This satisfies \( J \otimes \mathbb{Z}_\ell = \mathcal{O} \otimes \mathbb{Z}_\ell \) for \( \ell \neq 2 \), so we only need to check whether this works at 2. In other words, we need to check if

\[ M_2(\mathbb{Z}_2) \cdot \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} = M_2(\mathbb{Z}_2) \cdot \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}. \]
We can rewrite this as $O \otimes \alpha$ as we would expect. The isomorphism $O \otimes Z_2 \cong M_2(Z_2)$ sends $1 + i \cdot 1$ to $(1 - 1 1)$, and $1 - i \cdot 1$ to $(1 - 1 1)$. Equivalently, we need to check if

$$
\begin{pmatrix}
1 & -1 \\
1 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
0 & 2
\end{pmatrix}^{-1}
\in \text{GL}_2(Z_2)
$$

or

$$
\begin{pmatrix}
1 & 1 \\
-1 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 1 \\
0 & 2
\end{pmatrix}^{-1}
\in \text{GL}_2(Z_2).
$$

We see that neither is true. So we have computed one of the terms in $T_2(\mathbb{I}_{w^1})(w^1)$, namely

$$
\mathbb{I}_{w^1}
\begin{pmatrix}
1 & 0 \\
0 & 2
\end{pmatrix}
\cdot [1, 1, \ldots] = 0.
$$

As a sanity check, we verify that $[(\begin{smallmatrix} 1 & 0 \\ 0 & 2 \end{smallmatrix}), 1, 1, \ldots] = [1 - \frac{3}{2}i - \frac{1}{2}ij, 1, \ldots] = w^2 \in \Omega$, so that

$$
\mathbb{I}_{w^2}
\begin{pmatrix}
1 & 0 \\
0 & 2
\end{pmatrix}
\cdot [1, 1, \ldots] = 1
$$

as we would expect. The isomorphism $O \otimes Z_2 \cong M_2(Z_2)$ sends $1 - \frac{3}{2}i - \frac{1}{2}ij$ to $\left(\begin{smallmatrix} 2 & \frac{3 - \sqrt{15}}{2} \\ -3 - \sqrt{15} & 0 \end{smallmatrix}\right)$. Let $\mathcal{J}$ be the $O$-ideal generated by the adelic point $((\begin{smallmatrix} 1 & 0 \\ 0 & 2 \end{smallmatrix}), 1, 1, \ldots)$, and $\mathcal{I} = I_2$ the ideal generated by $(1 - \frac{3}{2}i - \frac{1}{2}ij, 1, \ldots)$.

Then the condition that they correspond to the same element of $\Omega$ is equivalent to the existence of some $\alpha \in D^\times(Q)$ such that $\mathcal{J} \subset \mathcal{I} \alpha$. Checking this locally, this means $\alpha \in (O \otimes Z_\ell)\times$ for $\ell \neq 2$, and at $\ell = 2$ we have

$$
\left(\begin{array}{cc}
2 & \frac{3 - \sqrt{15}}{2} \\
-3 - \sqrt{15} & 0
\end{array}\right) \alpha
\left(\begin{array}{cc}
1 & 0 \\
0 & 2
\end{array}\right)^{-1}
\in \text{GL}_2(Z_2).
$$

We can rewrite this as

$$
\alpha \in
\left(\begin{array}{cc}
2 & \frac{3 - \sqrt{15}}{2} \\
-3 - \sqrt{15} & 0
\end{array}\right)^{-1}
\text{GL}_2(Z_2)
\left(\begin{array}{cc}
1 & 0 \\
0 & 2
\end{array}\right)
$$

so we see that $2\alpha \in M_2(Z_2)$, by inverting the matrix and clearing the denominator. It follows that if $\alpha$ exists, we must have $2\alpha \in O$, and $\text{nrd}(\alpha) = 1$, by checking locally. There are finitely many possibilities for $2\alpha$, namely $\pm 2, \pm 2i, \pm i \pm \frac{1}{2} \pm \frac{1}{2}i, \pm 1 \pm \frac{1}{2}i \pm \frac{1}{2}j$. We need to check if any of these satisfies (1) when we replace $\alpha$ with the corresponding matrix in $\frac{1}{2}M_2(Z_2)$. A computation shows that one can take $\alpha = \frac{1}{2}(i + \frac{1}{2}(1 + j))$, where we need that actually $\sqrt{-15} \equiv 1 \mod 8$ for our choice of square root.

Returning to our computation of $T_2(\mathbb{I}_{w^1})(w^1)$, we need to compute the remaining terms $\mathbb{I}_{w^1}
\left(\begin{smallmatrix} 1 & 1 \\ 0 & 2 \end{smallmatrix}\right) \cdot [1, 1, \ldots]$ and $\mathbb{I}_{w^1}
\left(\begin{smallmatrix} 2 & 0 \\ 0 & 1 \end{smallmatrix}\right) \cdot [1, 1, \ldots]$. By the same argument as above, it suffices to check whether

$$
\begin{pmatrix}
1 & -1 \\
1 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 1 \\
0 & 2
\end{pmatrix}^{-1}
\in \text{GL}_2(Z_2)
$$

or

$$
\begin{pmatrix}
1 & 1 \\
-1 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 1 \\
0 & 2
\end{pmatrix}^{-1}
\in \text{GL}_2(Z_2)
$$

for the first term, and whether

$$
\begin{pmatrix}
1 & -1 \\
1 & 1
\end{pmatrix}
\begin{pmatrix}
2 & 0 \\
0 & 1
\end{pmatrix}^{-1}
\in \text{GL}_2(Z_2)
$$

or

$$
\begin{pmatrix}
1 & 1 \\
-1 & 1
\end{pmatrix}
\begin{pmatrix}
2 & 0 \\
0 & 1
\end{pmatrix}^{-1}
\in \text{GL}_2(Z_2)
$$

for the second term.
for the second. The first two are true and the last two are not. So we deduce that
\[
\mathbb{1}_{w^1} \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \cdot [1, 1, \ldots] = 1 \quad \text{and} \quad \mathbb{1}_{w^1} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \cdot [1, 1, \ldots] = 0.
\]
Therefore
\[
2T_2(\mathbb{1}_{w^1})(w^1) = 0 + 1 + 0 = 1.
\]
We can also deduce that \(2T_2(\mathbb{1}_{w^2})(w^1) = 3 - 1 = 2.\)

One similarly shows that \(2T_2(\mathbb{1}_{w^1})(w^2) = 3\) and so \(2T_2(\mathbb{1}_{w^2})(w^2) = 0.\) The required computation is to show that the \(\mathcal{O}\)-ideals generated by \(\begin{pmatrix} 2 & \frac{3 - \sqrt{15}}{2} \\ -3 - \frac{\sqrt{15}}{2} & 0 \end{pmatrix}, 1, 1, \ldots\), for \(g = (\frac{1}{0}, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2})\) and \((\frac{1}{0}, \frac{1}{0})\), are principal \(\mathcal{O}\)-ideals. The generators can be taken to be \(\alpha = -i + \frac{1}{2}, i + \frac{1}{2}, -\frac{1}{2}j\) and 2 respectively.

It follows that \(2T_2(\mathbb{1}_{w^1}) = \mathbb{1}_{w^1} + 3\mathbb{1}_{w^2}\) and \(2T_2(\mathbb{1}_{w^2}) = 2\mathbb{1}_{w^1}\), so \(T_2\) has matrix \(\frac{1}{2}(\cdot \frac{2}{3}, 0)\) with respect to the basis \(\{1_{w^1}, 1_{w^2}\}\). The eigenvalues are \(\lambda = -\frac{-2}{2}, \frac{3}{2} \in \mathbb{F}_{11}\).

The Hecke operator \(T_3\) can be computed in a similar way. We give one example evaluating the summand
\[
\mathbb{1}_{w^2} \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \cdot \begin{pmatrix} 2 & \frac{3 - \sqrt{15}}{2} \\ -3 - \frac{\sqrt{15}}{2} & 0 \end{pmatrix} \cdot [1, 1, \ldots] = \mathbb{1}_{w^2} \begin{pmatrix} 2 & \frac{3 - \sqrt{15}}{2} \\ -3 - \frac{\sqrt{15}}{2} & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \cdot [1, 1, \ldots]
\]
of \(3T_3(\mathbb{1}_{w^2})(w^2)\). One could do this by checking whether the \(\mathcal{O}\)-ideal \(\mathcal{J}\) generated by the adelic point in the square brackets is principal. If it is, then this value of \(\mathbb{1}_{w^2}\) is 0. This only works because \(\Omega\) has two elements.

In general, we need to check if \(\mathcal{J}\) is in the same left ideal class as the \(\mathcal{O}\)-ideal \(\mathcal{I} = I_2\) generated by the adelic point \(\begin{pmatrix} 2 \\ -3 - \frac{\sqrt{15}}{2} \\ \frac{3 - \sqrt{15}}{2} \\ 0 \end{pmatrix}, 1, 1, \ldots\). So we need to check if there exists \(\alpha \in D^\times\) such that \(\mathcal{J} = \mathcal{I}\alpha\). Checking this locally, we need (for the two isomorphisms \(D \otimes \mathbb{Q}_2 \cong \mathbb{M}_2(\mathbb{Q}_2)\) and \(D \otimes \mathbb{Q}_3 \cong \mathbb{M}_2(\mathbb{Q}_3)\) specified)
\[
(2) \quad \begin{pmatrix} 2 & \frac{3 - \sqrt{15}}{2} \\ -3 - \frac{\sqrt{15}}{2} & 0 \end{pmatrix} \alpha \begin{pmatrix} 2 & \frac{3 - \sqrt{15}}{2} \\ -3 - \frac{\sqrt{15}}{2} & 0 \end{pmatrix}^{-1} \in \text{GL}_2(\mathbb{Z}_2)
\]
and
\[
(3) \quad \alpha \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}^{-1} \in \text{GL}_2(\mathbb{Z}_3)
\]
and \(\alpha \in (\mathcal{O} \otimes \mathbb{Z}_\ell)^\times\) for \(\ell \neq 2, 3\). The first two conditions imply that \(2\alpha \in \mathcal{O} \otimes \mathbb{Z}_2\) and \(\alpha \in \mathcal{O} \otimes \mathbb{Z}_3\). So \(2\alpha \in \mathcal{O}\), and \(\text{nr}(\alpha) = 3\). There are finitely many possibilities for \(2\alpha\).

Before we go through all the possibilities for \(2\alpha\) and check whether they satisfy the two equations, we remark that for equation (2) we only really need to work with \(2\alpha\) and the matrix \(\begin{pmatrix} 2 & \frac{3 - \sqrt{15}}{2} \\ -3 - \frac{\sqrt{15}}{2} & 0 \end{pmatrix}\) modulo some power of 2 (and for equation (3) we only need \(2\alpha\) modulo some power of 3). Indeed, using the fact that our choice of \(\sqrt{-15}\) is congruent to 1 mod 8, we have
\[
\begin{pmatrix} 2 & \frac{3 - \sqrt{15}}{2} \\ -3 - \frac{\sqrt{15}}{2} & 0 \end{pmatrix} \equiv \begin{pmatrix} 2 & 1 \\ 2 & 0 \end{pmatrix} \mod 4.
\]
Suppose $2\alpha$ has matrix $(a_2 b_2) \in M_2(\mathbb{Z}_2)$. Then equation (2) tells us (after inverting the matrix) that we need
\[
\begin{pmatrix}
2 & \frac{3-\sqrt{-15}}{2} \\
-\frac{3+\sqrt{-15}}{2} & 0
\end{pmatrix}
\begin{pmatrix}
a_2 \\
c_2
\end{pmatrix}
= \begin{pmatrix}
0 \\
\frac{3+\sqrt{-15}}{2}
\end{pmatrix}
\in 4M_2(\mathbb{Z}_2).
\]

Note that the norm condition on $\alpha$ will then guarantee that we land in $4\text{GL}_2(\mathbb{Z}_2)$ not just $4M_2(\mathbb{Z}_2)$. Working modulo 4, we need
\[
\begin{pmatrix}
2 & 1 \\
2 & 0
\end{pmatrix}
\begin{pmatrix}
a_2 \\
c_2
\end{pmatrix}
\begin{pmatrix}
0 & -1 \\
2 & 2
\end{pmatrix}
\equiv 0 \mod 4.
\]

Expanding this out, this is equivalent to
\[
(4) \quad 2 \mid a_2, \quad 4 \mid c_2, \quad 2 \mid d_2.
\]

Similarly, if $2\alpha$ has matrix $(a_3 b_3) \in M_2(\mathbb{Z}_3)$, then equation (3) is equivalent to
\[
(5) \quad 3 \mid b_3, \quad 3 \mid d_3.
\]

So now we look through $2\alpha \in \mathcal{O}$ of reduced norm 12. Write
\[
2\alpha = t \cdot 1 + x \cdot i + y \cdot \frac{1}{2}(i + ij) + z \cdot \frac{1}{2}(1 + j)
\]
for $t, x, y, z \in \mathbb{Z}$. We also compute the images of the basis elements 1, $i$, $\frac{1}{2}(i + ij), \frac{1}{2}(1 + j)$ in $M_2(\mathbb{Z}_2)$ and $M_2(\mathbb{Z}_3)$, then reduce them modulo 4 and 3 respectively:
\[
i \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mod 4 \quad i \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mod 3
\]
\[
\frac{1}{2}(i + ij) \mapsto \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix} \mod 4 \quad \frac{1}{2}(i + ij) \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \mod 3.
\]
\[
\frac{1}{2}(1 + j) \mapsto \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \mod 4 \quad \frac{1}{2}(1 + j) \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mod 3.
\]

It follows that
\[
2\alpha \mapsto \begin{pmatrix} t - y + z & -x + z \\ x + y + z & t + y \end{pmatrix} \mod 4 \quad 2\alpha \mapsto \begin{pmatrix} t + z & -x \\ x + y & t \end{pmatrix} \mod 3.
\]

Hence equations (4) and (5) tell us that there exists $2\alpha \in \mathcal{O}$ of reduced norm 12 satisfying equations (2) and (3), if and only if we can find $t, x, y, z \in \mathbb{Z}$ such that
\[
2 \mid t - y + z, t + y, \quad 4 \mid x + y + z, \quad 3 \mid x, t, \quad (t + \frac{1}{2}z)^2 + (x + \frac{1}{2}y)^2 + \frac{11}{4}y^2 + \frac{11}{4}z^2 = 12
\]
where the last equation tells us the reduced norm is 12. We can check that there are no such solutions. For example, we see that $z$ must be even, and then from the norm condition we see that $|z| \leq 2$. Trying $z = 2$, the norm condition tells us that we must have $(t, x, y) = (0, 0, 0), (-2, 0, 0)$ or $(-1, \pm 1, 0)$, none of which satisfy the congruence conditions. For $z = -2$ we similarly must have $(t, x, y) = (0, 0, 0), (2, 0, 0)$ or $(1, \pm 1, 0)$ which also does not work. And finally for $z = 0$ we need $t^2 + (x + \frac{1}{2}y)^2 + \frac{11}{4}y^2 = 12$. Since $3 \mid t$, in one case we have $t^2 = 9$, which yields a solution for $y$ and $z$. But any other case will not yield a solution.
and then in \((x + \frac{1}{2}y)^2 + \frac{11}{4}y^2 = 3\) there are no solutions with \(4 \mid x + y\) and \(3 \mid x\). Otherwise we have \(t = 0\) and 
\((x + \frac{1}{2}y)^2 + \frac{11}{4}y^2 = 12\) with \(2 \mid y\), \(4 \mid x + y\) and \(3 \mid x\), which has no solutions, checking \(y = 0, \pm 2\).

We deduce that

\[
\mathbb{I}_{w^2} \left( \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \right) \cdot \left[ \begin{pmatrix} 2 & -\sqrt{15} \\ -\sqrt{15} & 0 \end{pmatrix}, 1, 1, \ldots \right] = 0.
\]

Performing the remaining calculations, using the relevant mod 3 and 4 matrices we have already written down above, we find that \(T_3\) has matrix \(\frac{1}{4}(\begin{smallmatrix} 2 & 2 \\ 2 & 3 \end{smallmatrix})\) with respect to the basis \(\{\mathbb{I}_{w^1}, \mathbb{I}_{w^2}\}\). We compute the eigenvalues to be \(-\frac{1}{3}, \frac{2}{3} \in \mathbb{F}_{11}\).

Comparing the matrices for \(T_2\) and \(T_3\), they are simultaneously diagonalisable (as expected) with eigenvectors \(\left( \begin{smallmatrix} 2 \\ -3 \\ 1 \end{smallmatrix} \right)\) and \(\left( \begin{smallmatrix} 1 \\ 1 \end{smallmatrix} \right)\). The corresponding eigenvalues are \((a_2, a_3) = \left( \frac{2}{2}, \frac{1}{3} \right) = (-1, -4)\) and \((a_2, a_3) = \left( \frac{2}{2}, \frac{3}{3} \right) = (7, 5)\), viewing \(a_2, a_3\) as elements of \(\mathbb{F}_{11}\). In view of Serre’s Theorem 1.1, one might wonder what mod 11 modular forms give rise to these eigenvalues. The first comes from \(\theta^3(\Delta) = q - q^2 - 4q^3 + \ldots \mod 11\), which has minimal weight filtration 120 (meaning that 120 is the lowest weight at which a mod 11 modular form has this \(q\)-expansion), and the second from the Hasse invariant \(E_{10}\).

3. Explicit isomorphisms \(O \otimes \mathbb{Z}_\ell \cong M_2(\mathbb{Z}_\ell)\)

We now work with any prime \(p\). Let \(D\) be the unique quaternion algebra over \(\mathbb{Q}\) ramified exactly at \(\{p, \infty\}\). It turns out that one can take \(D \cong \left( \frac{-1 \pm p}{q} \right)\) and \(\left( \frac{-2 \pm p}{q} \right)\) when respectively, \(p \equiv 3 \mod 4\) (or \(p = 2\)), and \(p \equiv 5 \mod 8\). In the remaining case \(p \equiv 1 \mod 8\), we can take \(D \cong \left( \frac{-5 - p}{q} \right)\) for any prime \(r \equiv 3 \mod 4\) with \(\left( \frac{r}{p} \right) = -1\). This can be verified by computations with the Hilbert symbol as in [Voi21] Chapter 12. We will write \(D = \left( \frac{-r \pm p}{q} \right)\) for an appropriate \(\epsilon\).

In the example of the previous section, we wrote down explicit isomorphisms \(O \otimes \mathbb{Z}_\ell \cong M_2(\mathbb{Z}_\ell)\). In general this is tricky to do, complicated by the fact that \(O\) might properly contain the order with \(\mathbb{Z}\)-basis \(\{1, i, j, ij\}\) - we can have nontrivial denominators. See also [Voi13] for an algorithm for computing such isomorphisms, which arose as a byproduct of other more involved algorithms. Let \(O\) have \(\mathbb{Z}\)-basis \(\{s^1, s^2, s^3, s^4\}\). What we need to do is find matrices \(A, B \in M_2(\mathbb{Z}_\ell)\) (corresponding to \(i, j\)) such that

\[
A^2 = -\epsilon, \quad B^2 = -p, \quad AB = -BA,
\]

and such that the matrices corresponding to \(s^1, s^2, s^3, s^4\), which a priori have entries in \(\mathbb{Q}_\ell\), actually have entries in \(\mathbb{Z}_\ell\). This latter condition just imposes some congruences on the entries of \(A\) and \(B\). For example, in the case \(p = 11\) using the given basis for \(O\), we also require that

\[
\frac{1}{2}(A + AB) \in M_2(\mathbb{Z}_\ell) \quad \text{and} \quad \frac{1}{2}(1 + B) \in M_2(\mathbb{Z}_\ell)
\]

which only imposes congruence conditions when \(\ell = 2\). If we can do this in the general case, then this gives us an isomorphism \(D \otimes \mathbb{Q}_\ell \cong M_2(\mathbb{Q}_\ell)\), which restricts to an injection \(O \otimes \mathbb{Z}_\ell \hookrightarrow M_2(\mathbb{Z}_\ell)\). But \(O \otimes \mathbb{Z}_\ell\) is a maximal order in \(D \otimes \mathbb{Q}_\ell\), so this injection must actually be our desired isomorphism.
The main observation is that, for our purposes, we really only need the matrices corresponding to the $s^1, s^2, s^3, s^4$ modulo some sufficiently large power of $\ell$. Consequently, we only need to compute $A$ and $B$ modulo some $\ell^n$. This was hinted at in the computation of $T_3$ in the example above. We will describe a formula for a sufficiently large $n_\ell$ later. So we need to search for $A_0$ and $B_0$ in $M_2(\mathbb{Z}/\ell^n\mathbb{Z})$ for some $n_\ell$, which satisfy

$$A_0^2 = -\epsilon, \quad B_0^2 = -p, \quad A_0B_0 = -B_0A_0 \mod \ell^n$$

and the congruence conditions imposed by the basis of $\mathcal{O}$ (we take $n_\ell$ sufficiently large so that these congruences can be viewed as congruences modulo $\ell^n$). We know that such $A_0$ and $B_0$ exist because we know that there exists an isomorphism $\mathcal{O} \otimes \mathbb{Z}_\ell \cong M_2(\mathbb{Z}_\ell)$. Since $i, j \in \mathcal{O}$ have reduced trace 0, we can furthermore assume that $A_0$ and $B_0$ are trace-free. Then such a solution can be found by a finite enumeration of all matrices in $M_2(\mathbb{Z}/\ell^n\mathbb{Z})$ (a more efficient method would be to use the lemma below repeatedly). The claim then is that these solutions can be lifted to our desired $A$ and $B$ (although we do not need to write down $A$ and $B$, just know that $A_0$ and $B_0$ lift). Note that $A$ and $B$ will automatically satisfy the congruence conditions from the basis of $\mathcal{O}$ because we are lifting from $A_0$ and $B_0$. This is the content of the following lemma (for $\ell \neq 2$).

**Lemma 3.1.** Suppose we have trace-free matrices $A_0, B_0 \in M_2(\mathbb{Z}/\ell^m\mathbb{Z})$ for $\ell \neq p$ an odd prime and $m \geq 2$, and suppose they satisfy

$$A_0^2 = -\epsilon \mod \ell^m, \quad B_0^2 = -p \mod \ell^m, \quad A_0B_0 = -B_0A_0 \mod \ell^m.$$  

Then we can find trace-free matrices $A_1, B_1 \in M_2(\mathbb{Z}/\ell^{m+1}\mathbb{Z})$ satisfying

$$A_1^2 = -\epsilon \mod \ell^{m+1}, \quad B_1^2 = -p \mod \ell^{m+1}, \quad A_1B_1 = -B_1A_1 \mod \ell^{m+1}$$

with $A_1 \equiv A_0 \mod \ell^m$ and $B_1 \equiv B_0 \mod \ell^m$. It follows that we can lift to desired matrices $A, B \in M_2(\mathbb{Z}_\ell)$ by induction.

**Proof.** Our proof involves writing out all the matrix entries and multiplying them together. The requirement that the matrices be trace-free simplify our calculations. Let

$$A_0 = \begin{pmatrix} a_1 & a_2 \\ a_3 & -a_1 \end{pmatrix} \mod \ell^m, \quad B_0 = \begin{pmatrix} b_1 & b_2 \\ b_3 & -b_1 \end{pmatrix} \mod \ell^m.$$  

Lifting these to (trace-free) matrices modulo $\ell^{m+1}$, we will write

$$A_1 = A_0 + \ell^m X \mod \ell^{m+1}, \quad B_1 = B_0 + \ell^m Y \mod \ell^{m+1}$$

for trace-free matrices

$$X = \begin{pmatrix} x_1 & x_2 \\ x_3 & -x_1 \end{pmatrix} \mod \ell, \quad Y = \begin{pmatrix} y_1 & y_2 \\ y_3 & -y_1 \end{pmatrix} \mod \ell.$$  

A useful calculation is the following: if $P = \begin{pmatrix} p_1 & p_2 \\ p_3 & -p_1 \end{pmatrix}$ and $Q = \begin{pmatrix} q_1 & q_2 \\ q_3 & -q_1 \end{pmatrix}$ are trace-free matrices, then

$$PQ + QP = \begin{pmatrix} 2p_1q_1 + p_2q_3 + p_3q_2 & 0 \\ 0 & 2p_1q_1 + p_2q_3 + p_3q_2 \end{pmatrix}.$$
It follows that if we lift $A_0$ and $B_0$ to any trace-free matrices modulo $\ell^{m+1}$, also denoted $A_0$ and $B_0$, we have that

$$A_0^2 = -\epsilon + \ell^m(*) \mod \ell^{m+1}, \quad B_0^2 = -p + \ell^m(*) \mod \ell^{m+1}, \quad A_0B_0 + B_0A_0 = \ell^m(*) \mod \ell^{m+1}$$

where the $(*)$ are all scalar matrices modulo $\ell$. The conditions on $X$ and $Y$ coming from those on $A_1$ and $B_1$ are the following:

$$2a_1x_1 + a_2x_3 + a_3x_2 = (*) \mod \ell, \quad 2b_1y_1 + b_2y_3 + b_3y_2 = (*) \mod \ell$$

$$(2a_1y_1 + a_2y_3 + a_3y_2) + (2b_1x_1 + b_2x_3 + b_3x_2) = (*) \mod \ell$$

for some scalars $(*) \mod \ell$. We can rewrite this as

$$\begin{pmatrix} 2a_1 & a_3 & a_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2b_1 & b_3 & b_2 \\ 2b_1 & b_3 & b_2 & 2a_1 & a_3 & a_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ y_1 \\ y_2 \\ y_3 \end{pmatrix} = (*) \mod \ell$$

for some arbitrary vector $(*) \in \mathbb{F}_\ell^3$. This is always possible if the vectors $\begin{pmatrix} 2a_1 & a_3 & a_2 \end{pmatrix} \mod \ell$ and $\begin{pmatrix} 2b_1 & b_3 & b_2 \end{pmatrix} \mod \ell$ are linearly independent. Since $\ell \neq 2$, linear dependence is equivalent to the existence of a nonzero scalar $\lambda$ such that $A_0 = \lambda B_0 \mod \ell$, or that either $A_0$ or $B_0$ is 0 mod $\ell$. The latter two cases are not possible because otherwise $A_0^2 = 0 \mod \ell^2$ or $B_0^2 = 0 \mod \ell^2$. Since $m \geq 2$, this means that $\ell^2$ divides $\epsilon$ or $p$, which does not happen (we had to use $\ell^2$ in case $\epsilon = r$ and $\ell = r$). And in the first case, $A_0B_0 + B_0A_0 = 0 \mod \ell$ tells us that $A_0^2 = B_0^2 = 0 \mod \ell$, which is false since $\ell \neq p$. Hence we can always lift $A_0, B_0 \mod \ell^m$ to $A_1, B_1 \mod \ell^{m+1}$. \hfill \qed

3.1. The case $\ell = 2$. When $\ell = 2$, the argument of Lemma 3.1 does not work, analogous to the difficulty of using a naive Hensel’s lemma for finding square roots in $\mathbb{Z}_2$. We will make use of a generalised Hensel’s lemma for multiple variables, which can be found as Theorem 3.3 in Conrad’s notes [Con], specialised to the case of $\mathbb{Q}_2$ with its usual absolute value $|\cdot|$.

**Theorem 3.2.** Let $f = (f_1, f_2, \ldots, f_d) \in \mathbb{Z}_2[X_1, X_2, \ldots, X_d]^d$ and $a = (a_1, \ldots, a_d) \in \mathbb{Z}_2^d$ satisfy

$$\|f(a)\| < |J_f(a)|^2$$

where $J_f$ is the Jacobian of $f$ - the determinant of its derivative matrix - and the norm of a vector is defined to be the maximum of the absolute values of its entries. Then there is a unique $a \in \mathbb{Z}_2^d$ such that $f(a) = 0$ and $|a - a| < |J_f(a)|$.

To see how to apply this, we are looking for matrices $A, B \in \mathbb{M}_2(\mathbb{Z}_2)$ satisfying

$$A^2 = -\epsilon, \quad B^2 = -p, \quad AB = -BA,$$
which also satisfy certain congruence conditions mod 4 (the highest power of 2 possibly dividing denominators in $s^1, s^2, s^3, s^4$ - see Proposition 3.7 to follow). Because we know $\mathcal{O} \otimes \mathbb{Z}_2 \cong M_2(\mathbb{Z}_2)$, we know that such $A$ and $B$ exist, so we can find such a solution modulo some power of 2 bigger than 4 and try to lift to $\mathbb{Z}_2$. If $A = \left( \begin{array}{c} a_1 \\ a_2 \\ a_3 \\ -a_1 \end{array} \right)$ and $B = \left( \begin{array}{c} b_1 \\ b_2 \\ b_3 \\ -b_1 \end{array} \right)$ with entries in $\mathbb{Z}_2$, then the conditions

$$A^2 = -\epsilon, \quad B^2 = -p, \quad AB = -BA,$$

are equivalent to

$$a_1^2 + a_2 a_3 = -\epsilon, \quad b_1^2 + b_2 b_3 = -p, \quad 2a_1 b_1 + a_2 b_3 + a_3 b_2 = 0.$$

**Lemma 3.3.** Suppose we have $a, b, c, x, y, z \in \mathbb{Z}$ satisfying

$$a^2 + bc = -\epsilon \mod 128, \quad x^2 + yz = -p \mod 128, \quad 2ax + bz + cy = 0 \mod 128.$$

Then these are congruent to some $a_1, a_2, a_3, b_1, b_2, b_3 \in \mathbb{Z}_2$ modulo 128 respectively, satisfying equation (6).

**Proof.** We have 3 equations and 6 variables, so to apply Theorem 3.2 we need to fix 3 variables. For example, if we fixed $(b, c, x)$ and considered the polynomials

$$f_1(X_1, X_2, X_3) = X_1^2 + bc + \epsilon, \quad f_2(X_1, X_2, X_3) = x^2 + X_2 X_3 + p, \quad f_3(X_1, X_2, X_3) = 2X_1 + bX_3 + cX_2$$

then we know that $f(a, y, z) = 0 \mod 128$. Note 128 = $2^7$. The derivative matrix is

$$(Df)(X) = \begin{pmatrix} 2X_1 & 0 & 0 \\ 0 & X_3 & X_2 \\ 2x & c & b \end{pmatrix}.$$ 

In order to lift to a solution $a_1, a_2, a_3, b_1, b_2, b_3 \in \mathbb{Z}_2$, we need $16 = 2^4$ to not divide

$$J_f(a, y, z) = 2a(bz - cy).$$

Whether this holds depends on the $a, b, c, x, y, z$ we were given. If this does not hold for our particular $a, b, c, x, y, z$, then we could instead fix some of the other variables, replacing $f$ with some $g$, and check whether 16 divides the new value of $J_g$. This means that the proof reduces to an analysis of several cases depending on the parities of the $a, b, c, x, y, z$.

**Notation 3.4.** Let $f_{r,s,t}$ denote the length 3 vector of polynomials in 3 variables obtained by fixing the variables other than $r, s, t \in \{a, b, c, x, y, z\}$. So for example we considered the case $(r, s, t) = (a, y, z)$ above.

We compute the following Jacobians:

$$\begin{pmatrix} 2X_1 & 0 & c \\ 0 & 2X_2 & 0 \\ 2X_2 & 2X_1 & z \end{pmatrix}$$

$$J_{f_{a,x,b}}(a, x, b) = 4x(ax - cx).$$
(9) \((Df_{a,x,c})(X) = \begin{pmatrix} 2X_1 & 0 & b \\ 0 & 2X_2 & 0 \\ 2X_2 & 2X_1 & y \end{pmatrix}\) \(J_{f_{a,x,c}}(a,x,c) = 4x(ay - bx)\).

(10) \((Df_{a,x,y})(X) = \begin{pmatrix} 2X_1 & 0 & 0 \\ 0 & 2X_2 & z \\ 2X_2 & 2X_1 & c \end{pmatrix}\) \(J_{f_{a,x,y}}(a,x,y) = 4a(cx - az)\).

(11) \((Df_{a,x,z})(X) = \begin{pmatrix} 2X_1 & 0 & 0 \\ 0 & 2X_2 & y \\ 2X_2 & 2X_1 & b \end{pmatrix}\) \(J_{f_{a,x,z}}(a,x,z) = 4a(bx - ay)\).

(12) \((Df_{b,y,z})(X) = \begin{pmatrix} c & 0 & 0 \\ 0 & X_3 & X_2 \\ X_3 & c & X_1 \end{pmatrix}\) \(J_{f_{b,y,z}}(b,y,z) = c(bz - cy)\).

Now we split into several cases. Firstly, assume \(p, \epsilon \neq 2\) are odd. We will frequently use from equation (7) that \(bz + cy\) is divisible by \(2ax\) mod 128, and in particular is even.

(i) \(a, x\) both even. From equation (7), we see that \(b, c, y, z\) are all odd. But \(4 \nmid bz + cy\) and so \(4 \nmid bz - cy\).

So using (12), we can lift using the fact that \(16 \nmid c(bz - cy)\).

(ii) \(a\) odd and \(x\) even. We see that \(y, z\) are odd and \(bc\) is even. Then use (10) and the fact that \(16 \nmid 4a(cx - az)\).

(iii) \(a\) even and \(x\) odd. We see that \(b, c\) are odd and \(yz\) is even. Then use (8) and the fact that \(16 \nmid 4x(az - cx)\).

(iv) \(a, x\) both odd. Then \(a^2 \equiv x^2 \equiv 1\) mod 4. From the way we defined \(\epsilon\), we have that \(p \equiv -\epsilon\) mod 4 (when neither is equal to 2). By symmetry, suppose \(p \equiv 1\) mod 4. Then we see that we must have \(yz \equiv 2\) mod 4. By symmetry, suppose \(y\) is odd and \(z\) is even but not divisible by 4. Then since \(bz + cy\) is even, we must have that \(c\) is even. If we then have \(b\) even, use (11), and \(16 \nmid 4a(bx - ay)\). If instead \(b\) is odd, then because \(4 \nmid z\) we know \(bz \equiv 2\) mod 4. But \(a\) and \(x\) are odd, so \(4 \nmid 2ax\) and therefore \(4 \nmid bz + cy\). It follows that \(c\) is divisible by 4. Then we use (10), where \(4a(cx - az)\) is divisible by 8 but not by 16. The other cases are symmetric.

Finally we consider the case when either \(p\) or \(\epsilon\) is 2 (we cannot have both). Firstly, consider \(\epsilon = 2\), so \(p \equiv 5\) mod 8.
(i) $a, x$ both even. We see that $y, z$ are odd. Since $a^2 + bc = -2 \mod 128$ and $a$ is even, exactly one of $b, c$ is even. But then $2 \nmid bz + cy$ is a contradiction.

(ii) $a$ odd and $x$ even. We see that $b, c, y, z$ are all odd. We also have from $2ax \mid bz + cy$ that $4 \mid bz + cy$, and therefore $4 \nmid bz - cy$. Then use (12) and the fact that $16 \nmid c(bz - cy)$.

(iii) $a$ even and $x$ odd. We see that $bc$ and $yz$ are even. Because $a^2 + bc = -2 \mod 128$ and $a$ is even, exactly one of $b, c$ is even. Similarly, $x^2 + yz \equiv -1 \mod 4$ and $x$ is odd, so exactly one of $y, z$ is even. We also know that $bz + cy$ is even. Thus either $b, y$ are even and $c, z$ are odd, or $b, y$ are odd and $c, z$ are even. In the first case use (8) and in the second case use (9).

(iv) $a, x$ both odd. We see that $b, c$ are odd and $yz$ is even. Again we have that exactly one of $y, z$ is even. If $y$ is even use (9) and if $z$ is even use (8).

The only property of $p$ that we use is that $p \equiv 1 \mod 4$. If we took $p = 2$, then $\epsilon = 1$ is also 1 mod 4. So this case is symmetric to the above.

The upshot of all this work is the following:

**Corollary 3.5.** Let $\ell$ be any prime, and let $n_\ell$ be any integer at least 2, where we also ask that $n_2 \geq 7$. Suppose we can find trace-free matrices $A_0, B_0 \in M_2(\mathbb{Z}/\ell^{n_\ell}\mathbb{Z})$ satisfying

$$A_0^2 = -\epsilon, \quad B_0^2 = -p, \quad A_0B_0 = -B_0A_0 \mod \ell^{n_\ell}.$$  

Then these lift to matrices $A, B \in M_2(\mathbb{Z}_\ell)$ with $A \equiv A_0 \mod \ell^{n_\ell}$ and $B \equiv B_0 \mod \ell^{n_\ell}$ such that

$$A^2 = -\epsilon, \quad B^2 = -p, \quad AB = -BA.$$

So suppose we can find for $n_\ell \geq 2$, or $n_2 \geq 7$ (for example by exhaustion of finitely many matrices in $M_2(\mathbb{Z}/\ell^{n_\ell}\mathbb{Z})$, or working through the Hensel argument for $\ell \neq 2$) such matrices $A_0$ and $B_0$ which satisfy the congruence conditions determined by the basis of $O$. In other words, if we mapped $i \mapsto A_0$ and $j \mapsto B_0$, then the induced map on $s^1, s^2, s^3, s^4$ sends them to well defined matrices modulo $\ell^{n_\ell - 2}$ (the 2 accounts for denominators when writing the $s^1, s^2, s^3, s^4$ in terms of $i, j$ - see below). Then $A_0$ and $B_0$ lift to $A, B \in M_2(\mathbb{Z}_\ell)$ such that the map determined by $i \mapsto A$ and $j \mapsto B$ gives an isomorphism $O \otimes \mathbb{Z}_\ell \cong M_2(\mathbb{Z}_\ell)$.

**Remark 3.6.** The following Proposition in Pizer’s paper [Piz80] (Proposition 5.2) gives explicit $\mathbb{Z}$-bases for a maximal order of our quaternion algebra $D$. We see in particular that the only primes dividing a denominator are 2 and $r$ (when $p \equiv 1 \mod 8$ and $\epsilon = r$), with power at most $2^2$ and $r$.

**Proposition 3.7.** A maximal order of $D$ is given by the $\mathbb{Z}$-basis:

$$\frac{1}{2}(1 + i + j + ij), i, j, ij \quad \text{if } p = 2$$

$$\frac{1}{2}(1 + j), \frac{1}{2}(i + ij), j, ij \quad \text{if } p \equiv 3 \mod 4$$

$$\frac{1}{2}(1 + j + ij), \frac{1}{4}(i + 2j + ij), j, ij \quad \text{if } p \equiv 5 \mod 8$$

$$\frac{1}{2}(1 + j), \frac{1}{2}(i + ij), \frac{1}{p}(j + a \cdot ij), ij \quad \text{if } p \equiv 1 \mod 8$$
where $a$ is some integer such that $r \mid a^2p + 1$.

This means we can explicitly write down the congruences we want our matrices $A_0$ and $B_0$ in the above Corollary to satisfy. We know that these matrices exist for $\ell \neq p$ by the ramification properties of the quaternion algebra $D$.

**Condition 3.8.** We want to find matrices $A_0, B_0 \in M_2(\mathbb{Z}/\ell^n\mathbb{Z})$, for $n_2 \geq 2$ and $n_2 \geq 7$ satisfying

$$A_0^2 = -\epsilon, \quad B_0^2 = -p, \quad A_0B_0 = -B_0A_0 \mod \ell^n,$$

and also

\[
2 \mid 1 + A_0 + B_0 + A_0B_0 \quad \text{if } p = 2 \\
2 \mid 1 + B_0, \quad 2 \mid A_0 + A_0B_0 \quad \text{if } p \equiv 3 \mod 4 \\
2 \mid 1 + B_0 + A_0B_0, \quad 4 \mid A_0 + 2B_0 + A_0B_0 \quad \text{if } p \equiv 5 \mod 8 \\
2 \mid 1 + B_0, \quad 2 \mid A_0 + A_0B_0, \quad r \mid B_0 + a \cdot A_0B_0 \quad \text{if } p \equiv 1 \mod 8
\]

where we remind ourselves that when $p \equiv 1 \mod 8$, we need $\epsilon = r$ is a prime congruent to 3 mod 4 with \((\frac{p}{r}) = -1\), and $a$ is some integer such that $r \mid a^2p + 1$, which we choose in writing down the basis in Proposition 3.7. Note that these last four congruences are vacuous unless $\ell = 2$ or $r$.

4. Weight 0 mod $p^2 - 1$ and level 1, for general $p$

We now work with any prime $p$. Let $D$ be the unique quaternion algebra over $\mathbb{Q}$ ramified exactly at $\{p, \infty\}$, $\mathcal{O}$ a maximal order, and let $\Omega := U \setminus D^\times(A_f)/D^\times(\mathbb{Q})$, where $\Omega := \mathcal{O}_p^\times \times \prod_{\ell \neq p} \mathcal{O}_\ell^\times$. We will compute the Hecke operator $T_{\ell_0}$ on $\Omega$, which will give us the weight 0 mod $p^2 - 1$ eigenvalues on $\Omega(1)$. The argument in the example above largely generalises to this case. Recall the definition of $T_{\ell_0}$.

**Definition 4.1.** For a prime $\ell_0 \neq p$, the Hecke operator $T_{\ell_0}$ on the space of functions $\Omega \to \mathbb{F}_p$ is given by

$$T_{\ell_0}f([x_\ell]) = \ell_0^{-1} \sum_i f(g_i \cdot [x_\ell])$$

for $GL_2(\mathbb{Z}_{\ell_0})(\begin{smallmatrix} 1 & 0 \\ 0 & \ell_0 \end{smallmatrix})GL_2(\mathbb{Z}_{\ell_0}) = \bigcup GL_2(\mathbb{Z}_{\ell_0})g_i$. Recall that $g_i \cdot [x_\ell]$ means we pick a representative $(x_\ell) \in D^\times(A_f)$ of $[x_\ell]$, multiply this in the $\ell_0$-place by the matrix $g_i$ (under an identification of $\mathcal{O}_{\ell_0} \cong M_2(\mathbb{Z}_{\ell_0})$ and hence of $D_{\ell_0} \cong M_2(\mathbb{Q}_{\ell_0})$), and then take the image in $\Omega$.

We explain part of this using the matrices of the previous section. Suppose we had some $w^i = [w^i_2, w^i_3, \ldots] \in \Omega$ and $w^j = [w^j_2, w^j_3, \ldots] \in \Omega$ corresponding to left ideal classes of $\mathcal{O}$ (so almost all $w^i_2, w^i_3$ can be taken to be 1), and we wanted to know if $g_k \cdot w^j = w^i \in \Omega$, for some $g_k$ appearing in $T_{\ell_0}$. Denote by $\mathcal{J}$ the left $\mathcal{O}$-ideal with local generators $\{w^j_2, w^j_3, \ldots, g_k \cdot w^j_\ell, \ldots\}$, where we view $g_k \in M_2(\mathbb{Z}_{\ell_0})$ as an element of $\mathcal{O}_{\ell_0}$. Denote by $\mathcal{I}$ the left $\mathcal{O}$-ideal with local generators $\{w^i_2, w^i_3, \ldots\}$. Then we need to check whether there exists $\alpha \in D^\times(\mathbb{Q})$ such
that $J = I\alpha$ (the question of determining when two $O$-ideals are in the same ideal class, for $O$ an Eichler order, has been studied in [KV12], but in our case we have local generators for the ideals, meaning that a basis for an ideal is not obvious). This means that for all $\ell \neq \ell_0$, we require

$$w_i^j\alpha(w_i^j)^{-1} \in \mathcal{O}_\ell^\times$$

and also

$$w_i^j\alpha(w_i^j)^{-1}g_k^{-1} \in \mathcal{O}_{\ell_0}^\times.$$

We deduce the following conditions on $\alpha$: let $V_{i,j}$ be the set of primes $\ell$ at which at least one of $w_i^j$ and $w_i^j$ is not 1, throwing out $p$ and $\ell_0$. For all $\ell \notin V_{i,j} \cup \{p, \ell_0\}$, equation (13) is equivalent to

$$\alpha \in \mathcal{O}_\ell^\times \quad \ell \notin V_{i,j} \cup \{p, \ell_0\}.$$  

For $\ell = p$, because $D$ is ramified at $p$, equation (13) is equivalent to

$$v_p(nrd(\alpha)) = v_p(nrd(w_i^j)) - v_p(nrd(w_i^j)),$$

where $v_p$ denotes the usual $p$-adic valuation on $\mathbb{Z}$. For $\ell \in V_{i,j}$, equation (13) tells us that

$$v_\ell(nrd(\alpha)) = v_\ell(nrd(w_i^j)) - v_\ell(nrd(w_i^j)) \quad \ell \in V_{i,j}.$$  

Additionally, since $\alpha \in (w_i^j)^{-1} \cdot \mathcal{O}_\ell^\times \cdot w_i^j$, and because $nrd(w_i^j)$ has $\ell$-adic valuation at most $m_\ell$ by definition, we see that

$$\ell^{m_\ell} \alpha \in \mathcal{O}_\ell \cong M_2(\mathbb{Z}_\ell) \quad \ell \in V_{i,j}.$$  

A similar argument using equation (14) shows that

$$v_\ell_0(nrd(\alpha)) = v_\ell_0(nrd(w_i^j)) - v_\ell(nrd(w_i^j)) + 1$$

and

$$\ell_0^{m_{\ell_0} + 1} \alpha \in \mathcal{O}_{\ell_0} \cong M_2(\mathbb{Z}_{\ell_0}).$$  

Combining equations (15) to (20), we see that we can compute rational numbers

$$M = \ell_0 \prod_{\ell \in V_{i,j} \cup \{p, \ell_0\}} \ell^{m_\ell}$$

and

$$K = \ell_0 \prod_{\ell \in V_{i,j} \cup \{p, \ell_0\}} \ell^{v_\ell(nrd(w_i^j)) - v_\ell(nrd(w_i^j))}$$

such that

$$M\alpha \in \mathcal{O} \quad \text{and} \quad nrd(\alpha) = K.$$
Note that we can rewrite, for \( \ell \neq p, \ell_0 \), the condition \( w_\ell^i \alpha (w_\ell^j)^{-1} \in \mathcal{O}_\ell \) (being a unit is then guaranteed by the norm condition) as saying

\[
w_\ell^i \cdot M \alpha \cdot w_\ell^j \in \ell^{m_\ell + v_\ell (\mathrm{nr}(w_\ell^j))} \mathcal{O}_\ell,
\]

where \( w_\ell^i \) denotes the standard involution in \( D \otimes \mathbb{Q}_\ell \). We can do the same for \( \ell_0 \), using \( g_k = \mathrm{adj}(g_k) \) the adjugate matrix of \( g_k \), which gives an extra factor of \( \ell_0 \). Since \( m_\ell + v_\ell (\mathrm{nr}(w_\ell^j)) \leq 2m_\ell \), to check this condition we only really need \( w_\ell^i, M \alpha \) and \( w_\ell^j \) modulo \( \ell^{2m_\ell} \). Similarly, we only really need \( w_{\ell_0}^i, M \alpha \) and \( w_{\ell_0}^j \) modulo \( \ell_0^{2m_{\ell_0} + 2} \). This means we can make use of the matrices computed in the previous section to rephrase our conditions on \( \alpha \). Let \( s^1, s^2, s^3, s^4 \) be an integer basis for \( \mathcal{O} \), taking for instance the basis of Proposition 3.7. Since \( M \alpha \in \mathcal{O} \), we can write

\[
M \alpha = t \cdot s^1 + x \cdot s^2 + y \cdot s^3 + z \cdot s^4
\]

for variables \( t, x, y, z \) which are to take values in \( \mathbb{Z} \). Let \( \ell \) be a prime in \( \mathcal{V}_{i,j} \cup \{ \ell_0 \} \). We can identify \( M \alpha \mod \ell^{2m_\ell + 2} \) with a matrix \( \left( \begin{array}{cc} a_\ell & b_\ell \\ c_\ell & -a_\ell \end{array} \right) \in M_2(\mathbb{Z}/\ell^{2m_\ell} \mathbb{Z}) \), where \( a_\ell, b_\ell, c_\ell \) are \( (\mathbb{Z}/\ell^{2m_\ell}) \)-linear functions in \( t, x, y, z \) which can be computed. We can also, for any \( \ell \neq p \), identify \( w_\ell^i \) and \( w_\ell^j \) with matrices \( W_\ell^i, W_\ell^j \in M_2(\mathbb{Z}/\ell^{2m_\ell + 2} \mathbb{Z}) \).

We have that \( g_k \cdot w^j = w^i \in \Omega \) if and only if we can find some \( t, x, y, z \in \mathbb{Z} \) such that the following conditions hold:

\[
\begin{align*}
\text{nr}(M \alpha) = KM^2 \\
W_\ell^i \cdot \left( \begin{array}{cc} a_\ell & b_\ell \\ c_\ell & -a_\ell \end{array} \right) \cdot \mathrm{adj}(W_\ell^j) \in \ell^{m_\ell + v_\ell (\mathrm{det}(W_\ell^j))} \cdot M_2(\mathbb{Z}/\ell^{2m_\ell + 2} \mathbb{Z}) & \quad \text{for } \ell \in \mathcal{V}_{i,j} \\
W_{\ell_0}^i \cdot \left( \begin{array}{cc} a_{\ell_0} & b_{\ell_0} \\ c_{\ell_0} & -a_{\ell_0} \end{array} \right) \cdot \mathrm{adj}(W_{\ell_0}^j) \cdot \mathrm{adj}(g_k) \in \ell_0^{m_{\ell_0} + v_{\ell_0}(\mathrm{det}(W_{\ell_0}^j)) + 2} \cdot M_2(\mathbb{Z}/\ell_0^{2m_{\ell_0} + 2} \mathbb{Z}).
\end{align*}
\]

The first condition tells us that \( t, x, y, z \in \mathbb{Z} \) satisfy some quadratic equation. Because the reduced norm is a positive definite quadratic form in the \( t, x, y, z \) (due to ramification at \( \infty \)), this means that there are only finitely many solutions to the quadratic equation. We can then enumerate them (for example, diagonalising the quadratic form, computing the finitely many solutions with the new basis, and then solving for \( t, x, y, z \)). Once we do so, it remains to check whether they satisfy the last two conditions of \( (21) \). By expanding them out, these conditions can be interpreted as congruence conditions on \( t, x, y, z \mod \ell^{2m_\ell + 2} \) for \( \ell \in \mathcal{V}_{i,j} \cup \{ \ell_0 \} \), or more precisely modulo \( \ell^{m_\ell + v_\ell (\mathrm{det}(W_\ell^j))} \) and \( \ell_0^{m_{\ell_0} + v_{\ell_0}(\mathrm{det}(W_{\ell_0}^j)) + 2} \). This allows us to determine whether \( g_k \cdot w^j = w^i \in \Omega \), and hence compute a matrix representative for the Hecke operator \( T_{\ell_0} \). We are now ready to present the algorithm. This should be read in conjunction with the following Table of Notation. The column for ‘Corresponding Matrices’ refers to matrices generated using the methods of Section 3.
### Table of Notation

| Notation | Definition | Corresponding Matrices |
|----------|------------|------------------------|
| $D = \left( \begin{smallmatrix} -\varepsilon & -p \\ 0 & 0 \end{smallmatrix} \right)$ | Quaternion algebra ramified at $\{p, \infty\}$. | |
| $\mathcal{O}$ | A maximal order of $D$. | |
| $I_1, \ldots, I_h$ | Representatives of the left ideal classes of $\mathcal{O}$. | |
| $\mathcal{B}_1, \ldots, \mathcal{B}_h$ | $\mathbb{Z}$-bases for $I_1, \ldots, I_h$. | |
| $\mathcal{V}$ | The set of primes $\ell$ for which in some $\mathcal{B}_j$ all elements have reduced norm divisible by $\ell$, excluding $p$ and including $\ell_0$. These are the primes at which we need to compute matrices. | |
| $i, j$ | Generators for $D$. | $A_\ell, B_\ell \in M_2(\mathbb{Z}/\ell^n\mathbb{Z})$ for each $\ell \in \mathcal{V}$. |
| $\{s^1, s^2, s^3, s^4\}$ | A $\mathbb{Z}$-basis for $\mathcal{O}$. | $S^1_\ell, S^2_\ell, S^3_\ell, S^4_\ell \in M_2(\mathbb{Z}/(2m\ell+2\mathbb{Z}))$ for each $\ell \in \mathcal{V}$. |
| $w^j = [w^j_2, w^j_3, w^j_5, \ldots] \in \Omega$ | The elements of $\Omega$ corresponding to the $I_j$. The square brackets means the double coset represented by the adelic point $(w^j_2, w^j_3, \ldots)$. From the way we compute this, almost all $w^j_\ell$ will be 1. | $W^j_\ell \in M_2(\mathbb{Z}/(2m\ell+2\mathbb{Z}))$ corresponding to the $w^j_\ell$ for each $\ell \in \mathcal{V}$. |
| $m_\ell, n_\ell = 2m_\ell + 4$ for $\ell \neq 2$, and $n_2 = \max(7, 2m_2 + 4)$ | $m_\ell := \max_j(\nu(\text{nrd}(w^j_\ell)))$. | |
| $g_0, \ldots, g_\ell_0 \in M_2(\mathbb{Z}_{\ell_0})$ | The matrices $(\begin{smallmatrix} 1 & 0 \\ 0 & \ell_0 \end{smallmatrix})$, $(\begin{smallmatrix} 1 & 1 \\ 0 & \ell_0 \end{smallmatrix})$, $(\begin{smallmatrix} 1 & \ell_0 - 1 \\ 0 & \ell_0 \end{smallmatrix})$, $(\begin{smallmatrix} \ell_0 & 0 \\ 0 & 1 \end{smallmatrix})$. | |
| $\mathbb{1}_{w^j}$ | The characteristic function $\Omega \to \mathbb{F}_p$ of the point $w^j \in \Omega$. | |
| $e_{i,j,k} \in \{0, 1\}$ | $e_{i,j,k} = \mathbb{1}_{w^j}(g_k \cdot w^j)$ for $1 \leq i, j \leq h$ and $0 \leq k \leq \ell_0$. The multiplication $g_k \cdot w^j$ means $(w_2^j, w_3^j, \ldots, g_k \cdot w^j_\ell_0, \ldots)$, with multiplication occurring only in the $\ell_0$-th place. | |
| $\mathcal{V}_{i,j} \subset \mathcal{V}$ | The set of primes $\ell$ such that at least one of $w^i_\ell$ and $w^j_\ell$ is not 1, excluding $p$ and $\ell_0$. | |
**Algorithm 4.2.** Input: distinct primes \( p \) and \( \ell_0 \).

Output: a matrix representing the action of the Hecke operator \( T_{\ell_0} \) on the space of all functions \( \Omega \to \mathbb{F}_p \).

(i) Define a quaternion algebra \( D = \left( \frac{-1,-p}{\mathbb{Q}} \right) \) over \( \mathbb{Q} \), ramified exactly at \( \{ p, \infty \} \). Define a maximal order \( \mathcal{O} \) with integer basis given as in Proposition 3.7, for which we denote the basis elements \( \{ s^1, s^2, s^3, s^4 \} \). Compute the left ideal classes \( I_1, \ldots, I_h \) of \( \mathcal{O} \), and bases \( B_1, \ldots, B_h \) for them.

(ii) Compute the points \( w^j = [w^j_1, w^j_2, w^j_3, \ldots] \in \Omega \) corresponding to \( I_j \) for each \( j \) as follows. We take \( w^j_i \) to be any generator of \( I_j \otimes \mathbb{Z}_\ell \) (which we know is principal). To do this, for our basis \( B_j \), compute the reduced norm of each of the four elements and set \( w^j_i \) to be any of these elements whose reduced norm has minimal \( \ell \)-adic valuation. Note that for almost all \( \ell \) this valuation is zero, so we can instead take \( w^j_i = 1 \), and do so when possible.

(iii) Determine the set \( \mathcal{V} \), defined in the Table of Notation. For each \( \ell \in \mathcal{V} \cup \{ p \} \), compute \( m_\ell \) and \( n_\ell \). For each \( \ell \in \mathcal{V} \), compute matrices \( A_\ell, B_\ell \in \text{M}_2(\mathbb{Z}/\ell^{m_\ell} \mathbb{Z}) \) satisfying Condition 3.8. Using these, compute matrices corresponding to the \( s^1, s^2, s^3, s^4 \), which in any case are well defined modulo \( \ell^{2m_\ell+2} \). Denote these by \( S^j_\ell \in \text{M}_2(\mathbb{Z}/\ell^{2m_\ell+2} \mathbb{Z}) \) for each \( \ell \in \mathcal{V} \). By expressing each \( w^j_i \in B_j \) as a \( \mathbb{Z} \)-linear combination of \( s^1, s^2, s^3, s^4 \), we can compute matrices \( W^j_\ell \in \text{M}_2(\mathbb{Z}/\ell^{2m_\ell+2} \mathbb{Z}) \) corresponding to the \( w^j_i \).

**Remark 4.3.** So far we have only mentioned \( \ell_0 \) as that it needed to be added to \( \mathcal{V} \). If one wanted to compute several Hecke operators at once, they could add all the primes for the operators into \( \mathcal{V} \). Then in the above steps we computed all the relevant matrices, to save us repeating the calculations if we were to compute the Hecke operators one by one.

(iv) Let \( \mathbb{1}_{w^1}, \ldots, \mathbb{1}_{w^h} \) be the characteristic functions of the points \( w^1, \ldots, w^h \in \Omega \). This is a basis for the vector space of \( \mathbb{F}_p \)-valued functions on \( \Omega \). Let \( g_0, \ldots, g_{n_\ell_0-1}, g_{n_\ell_0} \in \text{M}_2(\mathbb{Z}_{\ell_0}) \) be the matrices \( \left( \begin{smallmatrix} 1 & 0 \\ 0 & \ell_0 \end{smallmatrix} \right), \left( \begin{smallmatrix} 1 & 1 \\ 0 & \ell_0 \end{smallmatrix} \right), \ldots, \left( \begin{smallmatrix} 1 & n_\ell_0-1 \\ 0 & \ell_0 \end{smallmatrix} \right), \left( \begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} \right) \). Define the quantity \( e_{i,j,k} \) for \( 1 \leq i, j \leq h \) and \( 0 \leq k \leq \ell_0 \) by

\[
e_{i,j,k} = \mathbb{1}_{w^i}(g_k \cdot w^j).
\]

Then by definition we have the formula

\[
T_{\ell_0}(\mathbb{1}_{w^i})(w^j) = \frac{1}{\ell_0} \sum_{k=0}^{\ell_0} e_{i,j,k}.
\]

This is then the \((j,i)\)-th entry of the matrix for \( T_{\ell_0} \) with respect to the basis \( \{ \mathbb{1}_{w^1}, \ldots, \mathbb{1}_{w^h} \} \). Hence it remains to compute this quantity \( e_{i,j,k} \).

(v) Fix \( i, j \). Determine the set \( \mathcal{V}_{i,j} \). Compute the quantities

\[
M = \ell_0 \prod_{\ell \in \mathcal{V}_{i,j} \cup \{ p, \ell_0 \}} \ell^{m_\ell}
\]

and

\[
K = \ell_0 \prod_{\ell \in \mathcal{V}_{i,j} \cup \{ p, \ell_0 \}} \ell^{
u_p(nrd(w_i^j)) - \nu_p(nrd(w_i^j))},
\]
(vi) Check if there exist integers \(t, x, y, z \in \mathbb{Z}\) such that the following conditions hold:

\[
\begin{cases}
\text{nd}(t \cdot s^4 + x \cdot s^2 + y \cdot s^3 + z \cdot s^4) = KM^2 \\
W^j_\ell \cdot \left( \begin{array}{cc}
    a_{\ell} & b_{\ell} \\
    c_{\ell} & -a_{\ell}
\end{array} \right) \cdot \det(W^j_\ell) \in \ell^m + v_\ell(\det(W^j_\ell)) \cdot M_2(\mathbb{Z}/\ell^{2m_\ell+2}\mathbb{Z}) \\
\text{adj}(W^j_\ell) \cdot \det(g_k) \in \ell_0^{m_{\ell_0} + v_{\ell_0}(\det(W^j_{\ell_0})) + 2} \cdot M_2(\mathbb{Z}/\ell^{2m_{\ell_0}+2}\mathbb{Z}).
\end{cases}
\]

where \(\left( \frac{a_{\ell} b_{\ell}}{c_{\ell} - a_{\ell}} \right) \in M_2(\mathbb{Z}/\ell^{2m_\ell+2}\mathbb{Z})\) is the matrix \(t \cdot S_1^x + x \cdot S_2^x + y \cdot S_3^x + z \cdot S_4^x\). If such \(t, x, y, z\) exist, set \(e_{i,j,k}\) to be 1, and otherwise 0. Note that the only dependence on \(k\) is in the last condition.

5. Introducing weight and level

We now discuss what modifications must be done when computing the Hecke eigenvalues mod \(p\) of general weight and level on the quaternion side. In the previous section we checked whether two elements of \(\Omega\) are the same by checking if they determine isomorphic invertible left \(O\)-ideals. The point is that if we now attach \(\pi N\)-structure to our ideals, we need to determine whether this isomorphism sends one \(\pi N\)-structure to the other. Our aim is to compute the Hecke operators \(T_{\ell_0}\), for \(\ell_0 \nmid pN\), on the space of functions \(\Omega(N) = U(N) \backslash D^\times(\mathbb{A}_f)/D^\times(\mathbb{Q}) \to \overline{\mathbb{F}}_p\). Recall this was given by

\[
T_{\ell_0} f([x_\ell]) = \ell_0^{-1} \sum x f(\ell_0 \cdot [x_\ell])
\]

for \(GL_2(\mathbb{Z}_{\ell_0})(\frac{1}{\ell_0} 0 0 \ell_0)GL_2(\mathbb{Z}_{\ell_0}) = \bigcup GL_2(\mathbb{Z}_{\ell_0})g_i\). If we wanted to isolate the Hecke eigenvalues arising from weight \(k\) mod \(p^2 - 1\), then we need to restrict to the functions which satisfy \(f(\mu \cdot [x_\ell]) = \mu^{-k} f([x_\ell])\), where \(\mu \in O_p^\times / O_p^\times(1) \cong \mathbb{F}_p^\times\) acts on \([x_\ell]\) by multiplication in the \(p\)-place. In our case, we can identify \(O_p^\times / O_p^\times(1) \cong \mathbb{F}_p^\times\) with \(\{s + ti \mid s, t \in \mathbb{F}_p\) not both zero\}, which is closed under multiplication, where \(i\) and \(j\) are the generators of \(D\) (\(j\) can be viewed as a uniformiser \(\pi\) of \(O_p^\times\)).

Firstly, we write down the elements of \(\Omega(N)\). For \(U\) defined as \(O_p^\times \times \prod_{\ell \neq p} O_{\ell}^\times\) previously, we have

\[
U(N) \backslash U \cong \mathbb{F}_p^\times \times \prod_{\ell \neq p} GL_2(\mathbb{Z}/\ell^{v_\ell(N)}\mathbb{Z}) \cong \mathbb{F}_p^\times \times GL_2(\mathbb{Z}/N\mathbb{Z}).
\]

Hence, for our points \(w^1, \ldots, w^h\) of \(\Omega\), we need to multiply on the left by representatives of \(\mathbb{F}_p^\times\) and \(GL_2(\mathbb{Z}/N\mathbb{Z})\) at the appropriate places, in order to determine all the elements of \(\Omega(N)\). Ranging over each choice of \(\mu \in \mathbb{F}_p^\times\) and \(\gamma \in GL_2(\mathbb{Z}/N\mathbb{Z})\), the corresponding \((U(N), D^\times(\mathbb{Q}))\)-double cosets cover \(D^\times(\mathbb{A}_f)\), but need not be distinct. For example, for any triple

\[
\tilde{j} = (j, \mu, \gamma) \in \{1, \ldots, h\} \times \mathbb{F}_p^\times \times GL_2(\mathbb{Z}/N\mathbb{Z}),
\]

if we define \(v^{\tilde{j}}\) as in the Updated Table of Notation below, then \(\tilde{j} = (j, \mu, \gamma)\) and \(\tilde{j}' = (j, -\mu, -\gamma)\) define the same element \(v^{\tilde{j}} = v^{\tilde{j}'}\) of \(\Omega(N)\). This example comes from the fact that \(-1 \in D^\times(\mathbb{Q})\) and \(-1 \in O_{\ell}^\times\) (but
\(-1 \not\in \mathcal{O}_p^\times(1)\) and \(-1 \not\in \mathcal{O}_\ell^\times(N)\) for \(\ell \mid N\), so that
\[
[w_2^j, w_3^j, \ldots, \mu \cdot w_p^j, \ldots, \gamma_\ell \cdot w_1^j, \ldots] = [-w_2^j, -w_3^j, \ldots, -\mu \cdot w_p^j, \ldots, -\gamma_\ell \cdot w_1^j, \ldots]
= [w_2^j, w_3^j, \ldots, -\mu \cdot w_p^j, \ldots, -\gamma_\ell \cdot w_1^j, \ldots].
\]

When interpreting elements of \(\Omega(N)\) in terms of isomorphism classes of invertible left \(\mathcal{O}\)-ideals with \(\pi N\)-structure, this phenomenon is due to automorphisms of the left \(\mathcal{O}\)-ideals, which then shift around the \(\pi N\)-structure; in our example we always have the automorphism given by multiplication by \(-1\), which sends \(\mu \mapsto -\mu\), \(\gamma \mapsto -\gamma\). It is possible to determine these automorphisms. Recall that the points \(w^1, \ldots, w^h \in \Omega\) give local generators for representatives \(I_1, \ldots, I_h\) of the left ideal classes of \(\mathcal{O}\). The automorphisms of \(I_j\) as a left \(\mathcal{O}\)-ideal are precisely given by right multiplication by the units of the right order \(\mathcal{O}_R(I_j)\) of \(I_j\), where we define this as in [Voi21] to be:
\[
\mathcal{O}_R(I_j) := \{\alpha \in D \mid I_j \alpha \subset I_j\}.
\]

Note that \(\mathcal{O}_R(I_j)^\times\) will always be a finite set, once again because the quaternion algebra \(D\) is ramified at infinity.

**Remark 5.1.** In Serre’s letter [SL96], the relationship between Hecke eigenvalues on the modular form and quaternion sides arises as a result of some generalisation of the Deuring correspondence. Classically, this establishes an equivalence of categories between supersingular elliptic curves mod \(p\) under isogenies, and invertible left \(\mathcal{O}\)-ideals under nonzero left \(\mathcal{O}\)-module homomorphisms (for a maximal order \(\mathcal{O}\) of the quaternion algebra \(D\) ramified at \(\{p, \infty\}\)). See Theorem 42.3.2 of [Voi21] for a reference. Then, if a supersingular elliptic curve \(E\) corresponds to an invertible left \(\mathcal{O}\)-ideal \(I\), we see that \(\mathcal{O}_R(I)^\times \cong \text{Aut}(E)\). But the automorphism group of an elliptic curve is well understood. As in Theorem III.10.1 of [Sil09], the automorphism group of \(E\) has order dividing 24, and if \(j(E) \neq 0, 1728\), then \(\text{Aut}(E)\) has order 2 and is just given by \(\pm 1\).

We have seen that in listing the elements of \(\Omega(N)\), we need to identify, for example, \(v^\jmath\) and \(v^\jhat\) when \(\jmath = (j, \mu, \gamma)\) and \(\jhat = (j, -\mu, -\gamma)\), because \(-1 \in \mathcal{O}_R(I_j)^\times\). In general, if \(\zeta \in \mathcal{O}_R(I_j)^\times\), we know
\[
[w_2^\jmath, w_3^\jmath, \ldots, \mu \cdot w_p^\jmath, \ldots, \gamma_\ell \cdot w_1^\jmath, \ldots] = [w_2^\jhat, w_3^\jhat, \ldots, \mu \cdot w_p^\jhat, \ldots, \gamma_\ell \cdot w_1^\jhat, \ldots]
= [w_2^\jhat, w_3^\jhat, \ldots, \mu \phi_j(\zeta) \cdot w_p^\jhat, \ldots, \gamma_\ell \psi_j(\zeta; \ell) \cdot w_1^\jhat, \ldots]
\]
for \(\phi_j(\zeta)\) and \(\psi_j(\zeta; \ell)\) as defined in the Updated Table of Notation. To define this, we use the fact that because \(I_j \zeta = I_j\), we have for any \(\ell\), \(\mathcal{O}_\ell w_1^\jmath \zeta = \mathcal{O}_\ell w_1^\jhat\), and so \(w_1^\jmath \zeta (w_1^\jhat)^{-1} \in \mathcal{O}_\ell^\times\). Let \(\psi_j(\zeta) \in \text{GL}_2(\mathbb{Z}/N\mathbb{Z})\) be the matrix congruent to \(\psi_j(\zeta; \ell) \mod \ell^{\nu(N)}\) for each \(\ell \mid N\).

From this, we are motivated to define the set
\[
\mathcal{S} = \left(\{1, \ldots, h\} \times \mathbb{P}_p^\times \times \text{GL}_2(\mathbb{Z}/N\mathbb{Z})\right) / \sim,
\]
where we identify \((j, \mu, \gamma) \sim (j, \mu \phi_j(\zeta), \gamma \psi_j(\zeta))\) for any \(\zeta \in \mathcal{O}_R(I_j)^\times\). Then we can enumerate the elements of \(\Omega(N)\) as \(\{v^\jmath \mid \jmath \in \mathcal{S}\}\). Note that \(v^\jmath\) reduces to \(w^\jmath\) when we quotient by \(U\) in \(\Omega = U \setminus \Omega(N)\). Once again, an
obvious choice of basis for functions $\Omega(N) \to \mathbb{F}_p$ is given by the characteristic functions $\mathbb{1}_{v^j}$. One is then led to computing the following quantity:

$$e_{i,j,k} = \mathbb{1}_{v^j}(g_k \cdot v^j) \in \{0, 1\}$$

for $i, j \in S$. Then we have the formula

$$T_{\ell_0}(\mathbb{1}_{v^j})(v^j) = \frac{1}{\ell_0} \cdot \sum_{k=0}^{\ell_0} e_{i,j,k}.$$ 

If we index the rows and columns of the matrix for $T_{\ell_0}$, with respect to this basis, by $\vec{i} \in S$, then the $(\vec{j}, \vec{i})$-th entry is this value above. We observe that $e_{i,j,k} = 0$ if $e_{i,j,k} = 0$, for $e_{i,j,k}$ as defined in step (iv) in Algorithm 4.2. This is because if $v^j = g_k \cdot v^j \in \Omega(N)$, then they generate left $O$-ideals with $\pi N$-structure that are isomorphic; in particular the ideals are isomorphic, and so $w^i = g_k \cdot w^j$. One can think of this as replacing $e_{i,j,k}$ in Algorithm 4.2 with a permutation matrix, depending on $i, j, k$, keeping track of the $\pi N$-structure.

So now we see how to compute $e_{i,j,k}$. It is 0 if $e_{i,j,k} = 0$. If $e_{i,j,k} = 1$, then from Algorithm 4.2 we compute some $\alpha \in D^\times(Q)$ such that for all $\ell \neq \ell_0$,

$$w^i_\ell \alpha(w^j_\ell)^{-1} \in O^\times_\ell,$$

and also

$$w^i_{\ell_0} \alpha(w^j_{\ell_0})^{-1} g_k^{-1} \in O^\times_{\ell_0}.$$ 

Pick representatives $(i, \mu, \gamma)$ and $(j, \mu', \gamma')$ for $\vec{i}, \vec{j}$. What we need to check now is whether we also have

$$(\mu \phi_i(\zeta_i)) \cdot w^i_p \cdot \alpha \cdot (w^j_p)^{-1} \cdot (\mu' \phi_j(\zeta_j))^{-1} \in O^\times_p(1)$$

and for $\ell | N$

$$(\gamma \psi_i(\zeta_i; \ell)) \cdot w^i_\ell \cdot \alpha \cdot (w^j_\ell)^{-1} \cdot (\gamma' \psi_j(\zeta_j; \ell))^{-1} \in O^\times_\ell(N)$$

for some $\zeta_i \in O_R(I_i)^\times$ and $\zeta_j \in O_R(I_j)^\times$. In other words, we need to compute $w^i_p \cdot \alpha \cdot (w^j_p)^{-1}$, which we know is in $O^\times_p$ by definition of $\alpha$, and then check if this reduces to $(\mu \phi_i(\zeta_i))^{-1}(\mu' \phi_j(\zeta_j)) \in \mathbb{F}^\times_p$ modulo the uniformiser $\pi$, for some $\zeta_i \in O_R(I_i)^\times$ and $\zeta_j \in O_R(I_j)^\times$. We also need to compute for each $\ell | N$ the terms $w^i_\ell \cdot \alpha \cdot (w^j_\ell)^{-1}$, which we know are in $O^\times_\ell$, and then check whether they reduce to $(\gamma \psi_i(\zeta_i; \ell))^{-1}(\gamma' \psi_j(\zeta_j; \ell)) \in \text{GL}_2(\mathbb{Z}/\ell^{\nu(N)}\mathbb{Z})$ modulo $\ell^{\nu(N)}$, for the same $\zeta_i \in O_R(I_i)^\times$ and $\zeta_j \in O_R(I_j)^\times$. To make sense of this, we need to write down matrices at the primes dividing $N$ as well, so we extend our set $V$ in the Updated Table of Notation (changes from the previous table in bold).

We are now ready to write out the algorithm for the general case.
### Updated Table of Notation

| Notation | Definition | Corresponding Matrices |
|----------|------------|------------------------|
| $D = \begin{pmatrix} -\epsilon & -p \\ \frac{q}{\epsilon} \end{pmatrix}$ | Quaternion algebra ramified at $\{p, \infty\}$. | |
| $\mathcal{O}$ | A maximal order of $D$. | |
| $I_1, \ldots, I_h$ | Representatives of the left ideal classes of $\mathcal{O}$. | |
| $B_1, \ldots, B_h$ | $\mathbb{Z}$-bases for $I_1, \ldots, I_h$. | |
| $\mathcal{O}_R(I_1)^\times, \ldots, \mathcal{O}_R(I_h)^\times$ | The units of the right orders of $I_1, \ldots, I_h$. | |
| $\mathcal{V}$ | The set of primes $\ell$ for which in some $B_j$ all elements have reduced norm divisible by $\ell$, excluding $p$, and including $\ell_0$ and all primes dividing $N$. These are the primes at which we need to compute matrices. | |
| $i, j$ | $A_\ell, B_\ell \in M_2(\mathbb{Z}/\ell^m \mathbb{Z})$ for each $\ell \in \mathcal{V}$. | |
| $\{s^1, s^2, s^3, s^4\}$ | $S_{\ell}^1, S_{\ell}^2, S_{\ell}^3, S_{\ell}^4 \in M_2(\mathbb{Z}/\ell^{2m_\ell + v_\ell(N)+2} \mathbb{Z})$ for each $\ell \in \mathcal{V}$. | |
| $w^j = [w_2^j, w_3^j, w_5^j, \ldots] \in \Omega$ | The elements of $\Omega$ corresponding to the $I_j$. The square brackets means the double coset represented by the adelic point $(w_2^j, w_3^j, \ldots)$. From the way we compute this, almost all $w_\ell^j$ will be 1. | $W_\ell^j \in M_2(\mathbb{Z}/\ell^{2m_\ell + v_\ell(N)+2} \mathbb{Z})$ for each $\ell \in \mathcal{V}$. |
| $v_\ell(N)$ | The $\ell$-adic valuation of $N$. | |
| $m_\ell, n_\ell = 2m_\ell + v_\ell(N) + 4$ for $\ell \neq 2$, and $n_2 = \max(7, 2m_2 + v_2(N) + 4)$ | $m_\ell := \max_j(v_j(\text{nrd}(w_\ell^j))))$. | |
| $g_0, \ldots, g_{\ell_0} \in M_2(\mathbb{Z}_{\ell_0})$ | The matrices $\begin{pmatrix} 1 & 0 \\ 0 & \ell_0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ \ell_0^{-1} & 1 \end{pmatrix}, \ldots, \begin{pmatrix} 1 & \ell_0^{-1} \\ 0 & 1 \end{pmatrix}$. | |
| $\mathcal{O}_p^\times/\mathcal{O}_p^\times(1) \cong \mathbb{F}_p^\times$ | Identified with $\{s + ti \mid s, t \in \mathbb{F}_p \text{ not both zero}\}$. | |
| Notation | Definition | Corresponding Matrices |
|----------|------------|------------------------|
| $\phi_j(\zeta) \in \mathbb{F}_{p^2}^\times$ | For $\zeta \in O_R(I_j)^\times$, let $\phi_j(\zeta)$ be the image of $w_p^j \cdot \zeta \cdot (w_p^j)^{-1} \in O_R^\times$ in $\mathbb{F}_{p^2}^\times$. | |
| $\psi_j(\zeta; \ell) \in \text{GL}_2(\mathbb{Z}/\ell^v(N)\mathbb{Z})$ for $\ell \mid N$, $\psi_j(\zeta) \in \text{GL}_2(\mathbb{Z}/N\mathbb{Z})$ | For $\zeta \in O_R(I_j)^\times$, consider $w_p^j \cdot \zeta \cdot (w_p^j)^{-1} \in O_R^\times \cap D$ for all $\ell \mid N$. This lives in $O$, so we can compute the corresponding matrix in $\text{GL}_2(\mathbb{Z}/\ell^v(N)\mathbb{Z})$, and consider the reduction $\psi_j(\zeta; \ell) \in \text{GL}_2(\mathbb{Z}/\ell^v(N)\mathbb{Z})$. Let $\psi_j(\zeta) \in \text{GL}_2(\mathbb{Z}/N\mathbb{Z})$ be the matrix congruent to $\psi_j(\zeta; \ell) \mod \ell^v(N)$ for each $\ell \mid N$. | |
| $S = \left\{ (1, \ldots, h) \times \mathbb{F}_{p^2}^\times \times \text{GL}_2(\mathbb{Z}/N\mathbb{Z}) \right\} / \sim$ | Here we identify $(j, \mu, \gamma) \sim (j, \mu \phi_j(\zeta), \gamma \psi_j(\zeta))$ for any $\zeta \in O_R(I_j)^\times$. | |
| $\vec{v} = [v_2^j, v_3^j, \ldots] \in \Omega(N)$ for $\vec{j} = [(j, \mu, \gamma)] \in S$ | Here $v_\ell^j = \begin{cases} w_\ell^j & \text{if } \ell \neq p \text{ and } \ell \nmid N \\ \mu \cdot w_p^j & \text{if } \ell = p \\ \gamma_\ell \cdot w_\ell^j & \text{if } \ell \mid N \end{cases}$ for $\mu \in \mathbb{F}_{p^2}^\times$ and $\gamma \in \text{GL}_2(\mathbb{Z}/N\mathbb{Z})$ with reduction $\gamma_\ell \mod \ell^v(N)$. To be precise, we should choose lifts of $\mu$ to $O_R^\times$ and $\gamma_\ell$ to $O_R^\times \cong \text{GL}_2(\mathbb{Z}_\ell)$. By construction, $v^j \in \Omega(N)$ is well defined for any choice of representative $(j, \mu, \gamma)$ of $\vec{j} \in S$. |
| $1_{v^j}$ | The characteristic function $\Omega(N) \to \mathbb{F}_p$, of the point $v^j \in \Omega(N)$. | |
| $e_{\vec{i},\vec{j},k} \in \{0, 1\}$ | $e_{\vec{i},\vec{j},k} = 1_{v^j}(g_k \cdot v^j)$ for $\vec{i}, \vec{j} \in S$. | |
| $\mathcal{V}_{i,j} \subset \mathcal{V}$ | The set of primes $\ell$ such that at least one of $w_\ell^i$ and $w_\ell^j$ is not 1, excluding $p$ and $\ell_0$, and including all primes dividing $N$. | |
Algorithm 5.2. Input: a prime $p$ and level $N$ coprime to $p$, together with a prime $\ell_0$ coprime to $pN$.

Output: a matrix representing the action of the Hecke operator $T_{\ell_0}$ on the space of all functions $\Omega(N) \to \mathbb{F}_p$.

(i) Define a quaternion algebra $D = \left( \frac{-1,-1}{\mathbb{Q}} \right)$ over $\mathbb{Q}$, ramified exactly at $\{p, \infty\}$. Define a maximal order $\mathcal{O}$ with integer basis given as in Proposition 3.7, for which we denote the basis elements $\{s^1, s^2, s^3, s^4\}$.

Compute the left ideal classes $I_1, \ldots, I_h$ of $\mathcal{O}$, and bases $B_1, \ldots, B_h$ for them.

(ii) Compute the points $w^j = [w^j_1, w^j_2, w^j_3, \ldots] \in \Omega$ corresponding to $I_j$ for each $j$ as follows. We take $w^j_\ell$ to be any generator of $I_j \otimes \mathbb{Z}_\ell$ (which we know is principal). To do this, for our basis $B_j$, compute the reduced norm of each of the four elements and set $w^j_\ell$ to be any of these elements whose reduced norm has minimal $\ell$-adic valuation. Note that for almost all $\ell$ this valuation is zero, so we can instead take $w^j_\ell = 1$, and do so when possible.

(iii) Determine the set $\mathcal{V}$, defined in the Updated Table of Notation. For each $\ell \in \mathcal{V} \cup \{p\}$, compute $m_\ell$ and $n_\ell$. For each $\ell \in \mathcal{V}$, compute matrices $A_\ell, B_\ell \in M_2(\mathbb{Z}/\ell^{m_\ell}\mathbb{Z})$ satisfying Condition 3.8. Using these, compute matrices corresponding to the $s^1, s^2, s^3, s^4$, which in any case are well defined modulo $\ell^{2m_\ell+n_\ell(N)+2}\mathbb{Z}$. Denote these by $S^j_\ell \in M_2(\mathbb{Z}/\ell^{2m_\ell+n_\ell(N)+2}\mathbb{Z})$ for each $\ell \in \mathcal{V}$. By expressing each $w^j_\ell \in B_j$ as a $\mathbb{Z}$-linear combination of $s^1, s^2, s^3, s^4$, we can compute matrices $W^j_\ell \in M_2(\mathbb{Z}/\ell^{2m_\ell+n_\ell(N)+2}\mathbb{Z})$ corresponding to the $w^j_\ell$.

(iv) Compute each $O_R(I_j)\times$, and for each $\zeta \in O_R(I_j)\times$, compute $\phi_j(\zeta) \in \mathbb{F}_p^x$ and $\psi_j(\zeta; \ell) \in \text{GL}_2(\mathbb{Z}/\ell^{n_\ell(N)}\mathbb{Z})$ for $\ell \mid N$. Determine representatives for $S$. Fix $i, j$. We will compute $e_{i,j,k}$ for $0 \leq k \leq \ell$ and $\vec{i} = (i, \cdot, \cdot)$, $\vec{j} = (j, \cdot, \cdot) \in \mathcal{S}$ among these representatives. Determine the set $\mathcal{V}_{i,j}$.

Compute

$$M = \ell_0 \prod_{\ell \in \mathcal{V}_{i,j} \cup \{p, \ell_0\}} \ell^{m_\ell}$$

and

$$K = \ell_0 \prod_{\ell \in \mathcal{V}_{i,j} \cup \{p, \ell_0\}} \ell^{n_\ell(nrd(w^1_\ell)) - n_\ell(nrd(w^j_\ell))}.$$

(v) We firstly compute $e_{i,j,k}$ as in Algorithm 4.2. Check if there exist integers $t, x, y, z \in \mathbb{Z}$ such that the following conditions hold:

$$\begin{align*}
\text{nrd}(t \cdot s^1 + x \cdot s^2 + y \cdot s^3 + z \cdot s^4) &= KM^2 \\
W^j_\ell \cdot \begin{pmatrix} a_\ell & b_\ell \\ c_\ell & -a_\ell \end{pmatrix} \cdot \text{adj}(W^j_\ell) &\in \ell^{m_\ell+n_\ell(\det(W^j_\ell))} \cdot M_2(\mathbb{Z}/\ell^{2m_\ell+n_\ell(N)+2}\mathbb{Z}) &\text{for } \ell \in \mathcal{V}_{i,j} \\
W^j_\ell \cdot \begin{pmatrix} a_{\ell_0} & b_{\ell_0} \\ c_{\ell_0} & -a_{\ell_0} \end{pmatrix} \cdot \text{adj}(W^j_\ell) \cdot \text{adj}(g_k) &\in \ell_0^{m_{\ell_0}+n_{\ell_0}(\det(W^j_{\ell_0}))+2} \cdot M_2(\mathbb{Z}/\ell_0^{2m_{\ell_0}+2}\mathbb{Z}).
\end{align*}$$

where $\begin{pmatrix} a_\ell & b_\ell \\ c_\ell & -a_\ell \end{pmatrix} \in M_2(\mathbb{Z}/\ell^{2m_\ell+n_\ell(N)+2}\mathbb{Z})$ is the matrix $t \cdot S^j_\ell + x \cdot S^2_\ell + y \cdot S^3_\ell + z \cdot S^4_\ell$. If such $t, x, y, z$ exist, set $e_{i,j,k}$ to be 1, and otherwise 0. Note that the only dependence on $k$ is in the last condition.
(vi) If \(e_{i,j,k} = 0\), set \(e_{i,j,k} = 0\) for all \(i = (i, \cdot, \cdot), j = (j, \cdot, \cdot)\). Otherwise, taking our solution \((t, x, y, z)\) from the above step, compute the matrices

\[
Q_\ell := W_\ell^i \cdot \frac{1}{M} \begin{pmatrix} a_\ell & b_\ell \\ c_\ell & -a_\ell \end{pmatrix} \cdot (W_\ell^j)^{-1} \mod \ell^{\nu(N)} \in \text{GL}_2(\mathbb{Z}/\ell^{\nu(N)}\mathbb{Z})
\]

for \(\ell \mid N\), and

\[
Q_p := w_p^i \cdot \frac{1}{M} (t \cdot s^1 + x \cdot s^2 + y \cdot s^3 + z \cdot s^4) \cdot (w_p^j)^{-1} \in \mathbb{O}_p^\times.
\]

Writing \(Q_p\) in terms of the generators \(i\) and \(j\) of the quaternion algebra \(D\), let \(\overline{Q_p} \in \mathbb{F}_p^\times\) be the reduction modulo \(j\), where we view the elements of \(\mathbb{F}_p^\times\) as \(\{s + ti \mid s, t \in \mathbb{F}_p\text{ not both zero}\}\), which is a group under multiplication.

Then, for representatives \(\vec{i} = (i, \mu, \gamma)\) and \(\vec{j} = (j, \mu', \gamma')\) of \(S\), define

\[
e_{\vec{i}, \vec{j}, k} = \begin{cases} 1 & \text{if there exists } \zeta_i \in O_R(I_i) \times \text{ and } \zeta_j \in O_R(I_j) \times \text{ such that } (\mu \phi_i(\zeta_i))^{-1}(\mu' \phi_j(\zeta_j)) = \overline{Q_p} \in \mathbb{F}_p^\times \\
\text{and } (\gamma \psi_i(\zeta_i; \ell))^{-1}(\gamma' \psi_j(\zeta_j; \ell)) = \overline{Q_\ell} \in \text{GL}_2(\mathbb{Z}/\ell^{\nu(N)}\mathbb{Z}) \text{ for all } \ell \mid N \\
0 & \text{otherwise} \end{cases}
\]

(vii) If we index the rows and columns of the matrix for \(T_{\ell_0}\), with respect to the basis consisting of \(1_v\), by \(\vec{i} \in S\), then the \((\vec{j}, \vec{i})\)-th entry is

\[
\frac{1}{\ell_0} \sum_{k=0}^{\ell_0} e_{\vec{i}, \vec{j}, k}.
\]

This gives us the matrix for \(T_{\ell_0}\).
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