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On the Solvability of the Discrete $s(x, \cdot)$-Laplacian Problems on Simple, Connected, Undirected, Weighted, and Finite Graphs

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Abstract: We use the finite dimensional monotonicity methods in order to investigate problems connected with the discrete $s(x, \cdot)$-Laplacian on simple, connected, undirected, weighted, and finite graphs with nonlinearities given in a non-potential form. Positive solutions are also considered.

Keywords: simple graph; connected graph; undirected graph; weighted graph; finite graph; $s(x, \cdot)$-Laplacian on a graph; monotonicity methods; positive solutions

1. Introduction

In this work, we are concerned with the existence and uniqueness of results to discrete, nonlinear equations governed by the variable Laplacian and considered on a finite connected graph. The methods which we apply pertain to the finite dimensional version of the monotonicity theory and therefore rely on coercivity, monotonicity, and continuity of the relevant operators which may contain non-potential terms. Contrary to what is known about problems with variable exponents in the discrete setting (see for example [1,2]), the analysis related to the coercivity of the difference operator must be performed with some care. This is due to the structure of the domain (compare with [3,4] where problems with constant exponent are considered). Another problem appears with the proper definition of the weak solution, which must take into account both the graph structure of the domain and also the fact that the exponent is now not constant. Problems arising here require analysis related to some symmetry. Having overcome the problems mentioned, one may resort to checking the existence of solutions to abstract tools known from the applied functional analysis. In this work, we have decided to employ the monotonicity theory, but one expects that other approaches would also be applicable. In addition, we indicate that a variational approach can be employed in case the given problem is potential. However, the growth assumptions would, in such a case, be similar to those used in the monotonicity approach, and this is why we do not directly formulate such results. When checking the solvability of problems on graphs one is also interested in some well-posedness and stability, structural stability including the following.

Let $G = (V, E, \omega)$ be a simple, connected, undirected, weighted, and finite graph. Set $V$ is a non-empty set of vertices, and $E \subset [V]^2$ is a set consisting of all graph edges, i.e., unordered pairs elements of $V$. Function $\omega : V \times V \to (0, +\infty)$ is a weight on a graph $G$, i.e., we assume that

(i) $\forall x, y \in V : \omega(x, y) = \omega(y, x),$
(ii) $\forall x, y \in V : \omega(x, y) = 0 \iff \{x, y\} \notin E.$

Let $f : V \times \mathbb{R} \times \mathbb{R}^{|V|} \to \mathbb{R}$ be a continuous function, $q : V \to (0, +\infty), p \in [2; +\infty), u \in \mathbb{R} \to \mathbb{R}, s : V \times V \to [2, +\infty];$ moreover, we assume that $s(x, y) = s(y, x)$ for all $x, y \in V.$ Then we consider the following problem:

$$-\Delta_{s(x, \cdot)} u(x) + q(x)|u(x)|^{p-1}u(x) = f(x, u(x), \nabla_{s(x, \cdot)} u(x)), \quad (1)$$
for $x \in V$. Our goal is to find sufficient conditions for the existence and uniqueness of the solution to (1) in the space

$$A := \{ u : V \to \mathbb{R} \}.$$ 

Note that $A$ is a $|V|$-dimensional Euclidean space with the norm given by

$$\|u\| := \left( \sum_{x \in V} |u(x)|^2 \right)^{\frac{1}{2}}. \quad (2)$$

Now we define two operators, which are necessary in order to properly understand our problem.

**Definition 1.** Let $u : V \to \mathbb{R}$ and $s : V \times V \to [2, +\infty)$.

(i) The $s(x, \cdot)$-gradient of the function $u$ in point $x \in V$, $\nabla_{s(x, \cdot)} u : A \to \mathbb{R}^{|V|}$, is defined by

$$\nabla_{s(x, \cdot)} u(x) := \left( D_{s(x,y)} u(x) \right)_{y \in V}, \quad (3)$$

where

$$D_{s(x,y)} u(x) := |u(y) - u(x)|^{s(y,x) - 2} (u(y) - u(x)) \sqrt{\omega(x,y)}, \quad (4)$$

for all $x \in V$, is called the $s(\cdot, \cdot)$-directional derivative of the function $u$ in the direction $y \in V$.

In the case where $s(\cdot, \cdot) \equiv 2$, we write $\nabla$.

(ii) The discrete $s(x, \cdot)$-Laplacian of the function $u$ at point $x \in V$, $\Delta_{s(x, \cdot)} u : A \to \mathbb{R}$, is defined by

$$\Delta_{s(x, \cdot)} u(x) := \sum_{y \in V} |u(y) - u(x)|^{s(y,x) - 2} (u(y) - u(x)) \omega(x,y)$$

$$= \sum_{y \in V} D_{s(x,y)} u(x) \sqrt{\omega(x,y)}. \quad (5)$$

If $s(\cdot, \cdot) \equiv 2$ we write $\Delta$.

On the right hand side of Formula (3) there appears a $|V|$-dimensional vector that is indexed by $y \in V$. We would like to underline that to the best of our knowledge, the discrete $s(x, \cdot)$-Laplacian on finite graphs have not been considered yet, which means that several auxiliary tools that are known on a graph setting must be worked out in detail. This means that we had to investigate the problem in a detailed manner which involves derivation of some auxiliary tools. The results in the literature cover only the case of the $p$-Laplacian and $p(x)$-Laplacian on graphs, where, however, other methods are applied and monotonicity approaches are not used. Thus, our results are also new in the context of constant $p$. In case $s(\cdot, \cdot)$ is a constant function, we may recover the well-known $p$-Laplacian on a graph setting for which there are some existence and multiplicity results (see [5]).

The graph $s(x, \cdot)$-Laplacian correspond to anisotropic boundary value problems which on the other hand serve as mathematical models used in elastic mechanics [6] or image restoration [7]. Variational continuous anisotropic problems have been started by Fan and Zhang in [8]. A very extensive study contained in [9] provides many tools and methods used in the area of anisotropic problems. The general reference for discrete problems is [10], whereas for the background in graph theory we refer to [11]. The existence of positive solutions follows the pattern employed for problems with constant $p$.

The paper is organized as follows. We start with a summation-by-parts formula and some inequalities that are used further on. The summation-by-parts formula is used in derivation of a notion of a weak solution, whereas relevant inequalities are used in order to apply monotonicity techniques. However, these may useful also when applying other nonlinear analysis tools. Next, existence and uniqueness is concerned. Finally the question of the existence of positive solutions is undertaken.
2. Preliminary and Auxiliary Results

For the weak formulation of problem (1), we need the following lemma, which is in fact a graph variant of summation by parts

Lemma 1. For any pair of functions \( u, v : V \to \mathbb{R} \), if \( s(x, y) = s(y, x) \) for all \( x, y \in V \), we have

\[
-2 \sum_{x \in V} (\Delta_{s(x,)} u(x)) v(x) = \sum_{x \in V} (\nabla_{s(x,)} u(x), \nabla v(x)),
\]

where \( \langle \cdot, \cdot \rangle \) is a Euclidean scalar product in \( \mathbb{R}^{|V|} \).

Proof. Let \( u, v : V \to \mathbb{R} \). By Formulas (3)–(5) we have

\[
\sum_{x \in V} (\nabla_{s(x,)} u(x), \nabla v(x)) = \sum_{x,y \in V} |u(y) - u(x)|^{s(x,y)-2}(u(y) - u(x))\omega(x,y)(v(y) - v(x))
\]

\[
= \sum_{x,y \in V} |u(y) - u(x)|^{s(x,y)-2}(u(y) - u(x))\omega(x,y)v(y)
\]

\[
- \sum_{x,y \in V} |u(y) - u(x)|^{s(x,y)-2}(u(y) - u(x))\omega(x,y)v(x)
\]

\[
= \sum_{x,y \in V} |u(x) - u(y)|^{s(y,x)-2}(u(x) - u(y))\omega(y,x)v(x)
\]

\[
- \sum_{x \in V} (\Delta_{s(x,)} u(x)) v(x)
\]

\[
= - \sum_{x,y \in V} |u(y) - u(x)|^{s(x,y)-2}(u(y) - u(x))\omega(x,y)v(x)
\]

\[
- \sum_{x \in V} (\Delta_{s(x,)} u(x)) v(x)
\]

\[
= - 2 \sum_{x \in V} (\Delta_{s(x,)} u(x)) v(x).
\]

We want to emphasize that the symmetry of function \( s \) is obligatory here, because we need equality \( |u(x) - u(y)|^{s(y,x)-2} = |u(y) - u(x)|^{s(x,y)-2} \). \( \square \)

Next, we will derive a weak formulation of problem (1). Multiplying Equation (1) by some \( v \in A \), and then by taking summation on both sides and using Lemma 1, we get

\[
\frac{1}{2} \sum_{x \in V} (\nabla_{s(x,)} u(x), \nabla v(x)) + \sum_{x \in V} q(x)|u(x)|^p u(x)v(x)
\]

\[
= \sum_{x \in V} v(x)f(x, u(x), \nabla_{s(x,)} u(x)).
\]

If the above equation is fulfilled for all \( v \in A \), then it is equivalent to having Equation (1) for all \( x \in V \). To prove that, for each \( x_0 \in V \), we take \( v = \delta_{x_0} \), where

\[
\delta_{x_0}(x) = \begin{cases} 
1, & \text{when } x = x_0, \\
0, & \text{when } x \neq x_0.
\end{cases}
\]

This implies (1) for all \( x \in V \).

Let us also introduce a notation used throughout the paper, namely

\[
s^- := \min_{x,y \in V} s(x, y), \quad s^+ := \max_{x,y \in V} s(x, y),
\]

\[
q^+ := \max_{x \in V} q(x), \quad \omega^+ := \max_{x,y \in V} \omega(x, y).
\]

For the case of our problem, we need several inequalities known from [12]. We provide the suitable proofs due to many subtle differences. These differences arise from the fact of how
we treat the boundary of the graph. This is in contrast to [12] where the boundary plays its role, and therefore there appear several restrictions with which we do not need to cope.

**Lemma 2.** On the space A the following inequalities hold.

(i) For every \( u \in A \) and for every \( m \geq 0 \) we have

\[
\sum_{x \in V} |u(x)|^m \leq |V| \|u\|^m.
\]

(ii) For every \( u \in A \) and for every \( m \geq 2 \) we have

\[
\sum_{x, y \in V} |u(y) - u(x)|^m \leq 2^m |V|^2 \|u\|^m.
\]

(iii) Let \( s^+ \geq 2 \) for every \( u \in A \), and we have

\[
\sum_{x, y \in V} |u(y) - u(x)|^{s(x, y) } \omega(x, y) \leq \omega^+ 2^{s^+} |V|^2 \|u\|^{s^+} + \omega^+ |V|^2.
\]

(iv) For every \( u \in A \) and for every \( m \geq 2 \) we have

\[
\sum_{x \in V} |u(x)|^m \geq |V|^{2-m} \|u\|^m.
\]

**Proof.** To see (i), note that for any fixed \( x \in V \) we have

\[
|u(x)|^2 \leq \sum_{y \in V} |u(y)|^2,
\]

so

\[
|u(x)| \leq \|u\|.
\]

Therefore for every \( m \geq 0 \) we obtain

\[
\sum_{x \in V} |u(x)|^m \leq |V| \|u\|^m.
\]

Relation (ii) is obtain from the triangle inequality and from inequality (i), namely

\[
\sum_{x, y \in V} |u(y) - u(x)|^m \leq \sum_{x, y \in V} (|u(y)| + |u(x)|)^m
\]

\[
\leq \sum_{x, y \in V} (2\|u\|)^m = 2^m |V|^2 \|u\|^m.
\]

At first with inequality (iii), the most important thing is to divide sum with respect to the 1, to effectively use values of \( s^- \) and \( s^+ \).
The sum of the differences in squared norms is bounded by double sums of absolute differences:

$$
\sum_{x,y \in V} |u(y) - u(x)|^{s(x,y)} \omega(x, y) \leq \omega^+ \sum_{x,y \in V} |u(y) - u(x)|^{s(x,y)}
$$

$$
= \omega^+ \sum_{\{x,y \in V: |u(y) - u(x)| \leq 1\}} |u(y) - u(x)|^{s(x,y)}
+ \omega^+ \sum_{\{x,y \in V: |u(y) - u(x)| > 1\}} |u(y) - u(x)|^{s(x,y)}
$$

$$
\leq \omega^+ \sum_{\{x,y \in V: |u(y) - u(x)| \leq 1\}} |u(y) - u(x)|^s
+ \omega^+ \sum_{\{x,y \in V: |u(y) - u(x)| > 1\}} |u(y) - u(x)|^s
$$

$$
= \omega^+ \sum_{\{x,y \in V: |u(y) - u(x)| \leq 1\}} \left(|u(y) - u(x)|^s - |u(y) - u(x)|^s\right)
+ \omega^+ \sum_{\{x,y \in V: |u(y) - u(x)| > 1\}} |u(y) - u(x)|^s.
$$

Now we can use inequality (ii) to get the following:

$$
\sum_{x,y \in V} |u(y) - u(x)|^s \leq 2^s |V|^2 \|u\|^s.
$$

Moreover,

$$
\sum_{\{x,y \in V: |u(y) - u(x)| \leq 1\}} \left(|u(y) - u(x)|^s - |u(y) - u(x)|^s\right) \leq \sum_{\{x,y \in V: |u(y) - u(x)| \leq 1\}} 1 \leq |V|^2.
$$

Consequently, we finally have

$$
\sum_{x,y \in V} |u(y) - u(x)|^{s(x,y)} \omega(x, y) \leq \omega^+ 2^s |V|^2 \|u\|^s + \omega^+ |V|^2.
$$

We will show that (iv) holds. First, we will use discrete Hölder’s inequality with $p = \frac{m}{2}$ and $q = \frac{m}{m-2}$ for $m > 2$

$$
\sum_{x \in V} 1 \cdot |u(x)|^2 \leq \left(\sum_{x \in V} 1^{\frac{m}{m-2}}\right)^{\frac{m-2}{m}} \left(\sum_{x \in V} |u(x)|^m\right)^{\frac{1}{m}}.
$$

Then we use simple transformations to get our inequality

$$
\|u\|^2 \leq |V|^{\frac{m-2}{m}} \left(\sum_{x \in V} |u(x)|^m\right)^{\frac{2}{m}},
$$

$$
\|u\| \leq |V|^{\frac{m-2}{2m}} \left(\sum_{x \in V} |u(x)|^m\right)^{\frac{1}{m}}
$$

and finally

$$
\sum_{x \in V} |u(x)|^m \geq |V|^{\frac{2m}{2m}} \|u\|^m.
$$
In the end, we notice that for \( m = 2 \) we get
\[
\sum_{x \in V} |u(x)|^2 \geq |V|^0 |u|^2
\]
\[
\|u\|^2 \geq 1 : \|u\|^2,
\]
so it works also for \( m = 2 \). \( \square \)

**Lemma 3** (see [13] p. 3). Let \( n \in \mathbb{N}, a, b \in \mathbb{R}^n \) and \( p \geq 2 \). Then we have the following inequality
\[
\left\langle \|a\|^{p-2} a - \|b\|^{p-2} b, a - b \right\rangle \geq c_p \|a - b\|,
\]
where \( c_p = \frac{2}{p(2^{p-1} - 1)} \); moreover \( \langle \cdot, \cdot \rangle \) denotes an inner scalar product in \( \mathbb{R}^n \) and \( \|\cdot\| \) denotes a Euclidean norm in \( \mathbb{R}^n \).

The methods which we employ rely on monotonicity notions (see [14]). By \( A \) we denote the finite dimensional real Banach space, by \( A^* \) its dual, \( \langle \cdot, \cdot \rangle \) denotes the duality pairing between \( A^* \) and \( A \), i.e., the scalar product.

**Definition 2.** Operator \( T : A \to A^* \) is called
(i) bounded if it maps bounded sets into bounded sets,
(ii) coercive if
\[
\lim_{\|u\|_A \to +\infty} \frac{\langle T(u), u \rangle}{\|u\|_A} \to +\infty,
\]
(iii) monotone if
\[
\langle T(u) - T(v), u - v \rangle \geq 0,
\]
for all \( u, v \in A \), and (iv) strictly monotone if
\[
\langle T(u) - T(v), u - v \rangle > 0,
\]
for any \( u \neq v \).

**Theorem 1.** Let \( T : A \to A^* \) be continuous and coercive. Then \( T \) is surjective, i.e., for any \( f \in A^* \), there is at least one solution to the equation \( Tu = f \). If \( A \) is additionally strictly monotone, then \( A \) is a homeomorphism.

3. Existence and Uniqueness of Solutions

In order to use the monotonicity tools, we introduce the following operator \( T : A \to A^* \)
\[
\langle T(u), v \rangle := \langle T_1(u), v \rangle + \langle T_2(u), v \rangle - \langle T_3(u), v \rangle,
\]
where
\[
\langle T_1(u), v \rangle := \frac{1}{2} \sum_{x \in V} \langle \nabla_{s(x,)} u(x), \nabla v(x) \rangle,
\]
\[
\langle T_2(u), v \rangle := \sum_{x \in V} q(x)|u(x)|^p u(x)v(x),
\]
\[
\langle T_3(u), v \rangle := \sum_{x \in V} p(x)f(x, u(x), \nabla_{s(x)} u(x)).
\]
We observe that when if for \( u \in A \) it follows that
\[
\langle T(u), v \rangle = 0 \text{ for all } v \in A
\]
then $u$ is a solution to (1).

**Lemma 4.** (i) If for all $u \in \mathbb{R}$ and for all $v \in \mathbb{R}^{|V|}$ we have

$$uf(x, u, v) \leq |u|^{\alpha_1} + \beta_1,$$

(7)

where $\alpha_1 < p + 2$, $\beta_1 \in \mathbb{R}$, then operator $T$ is coercive, and (ii) if for all $u \in \mathbb{R}$ and for all $v \in \mathbb{R}^{|V|}$, we have

$$uf(x, u, v) \geq |u|^{\alpha_2} + \beta_2,$$

(8)

where $\alpha_2 > \max(s^+, p + 2)$, $\beta_2 \in \mathbb{R}$, and then operator $-T$ is coercive.

**Proof.** In the first case, we want to find estimate for the operators $T_1, T_2$ from the bottom. From the definition of $T_1$ we have

$$\langle T_1(u), u \rangle = \frac{1}{2} \sum_{x,y \in V} |u(y) - u(x)|^{p(x,y)} \omega(x, y) \geq 0.$$

The second estimate by (iv) from Lemma 2

$$\langle T_2(u), u \rangle = \sum_{x \in V} q(x)|u(x)|^{p+2} \geq q^{-} \sum_{x \in V} |u(x)|^{p+2} \geq q^{-} |V|^{\frac{p}{2}} \|u\|^{p+2}.$$

Hence we have

$$\langle T_1(u), u \rangle + \langle T_2(u), u \rangle \geq q^{-} |V|^{\frac{p}{2}} \|u\|^{p+2}.$$

Moreover, from (7) we obtain

$$\langle T_3(u), u \rangle = \sum_{x \in V} u(x)f(x, u(x), \nabla s(x, )u(x)) \leq |V|\|u\|^{\alpha_1} + |V|\beta_1.$$

Finally, we see that

$$\langle T(u), u \rangle \geq q^{-} |V|^{\frac{p}{2}} \|u\|^{p+2} - |V|\|u\|^{\alpha_1} - |V|\beta_1$$

because $\alpha_1 < p + 2$ operators $T_1$ and $T_2$ dominate, and it is easy to see that operator $T$ is coercive.

In the second case, we want to find an estimate for the operators $T_1, T_2$ from the above. First, estimate by (iii) from Lemma 2

$$\langle T_1(u), u \rangle = \frac{1}{2} \sum_{x,y \in V} |u(y) - u(x)|^{s(x,y)} \omega(x, y) \leq$$

$$\leq \omega^+ 2^{s^+ -1} |V|^2 \|u\|^{s^+} + \frac{1}{2} \omega^+ |V|^2.$$

Secondly, estimate by (i) from Lemma 2

$$\langle T_2(u), u \rangle = \sum_{x \in V} q(x)|u(x)|^{p+2} \leq q^+ |V|\|u\|^{p+2}.$$

Now we see that

$$\langle T_1(u), u \rangle + \langle T_2(u), u \rangle \leq a\|u\|^{\max(s^+, p+2)} + b,$$

where

$$a = \omega^+ 2^{s^+ -1} |V|^2 + q^+ |V|.$$
Observe that
\[ b = \frac{1}{2} \omega^+ |V|^2. \]

Moreover, from (8) we have
\[ \langle T_3(u), u \rangle = \sum_{x \in V} u(x)f(x, u(x), \nabla_x u(x)) \geq \sum_{x \in V} (|u(x)|^{\alpha_2} + \beta_2) \]
\[ = \sum_{x \in V} |u(x)|^{\alpha_2} + |V| \beta_2 \geq |V|^{\frac{2-\alpha_2}{2}} \|u\|^{\alpha_2} + |V| \beta_2. \]

Finally, we see that
\[ \langle -T(u), u \rangle \geq |V|^{\frac{2-\alpha_2}{2}} \|u\|^{\alpha_2} + |V| \beta_2 - \alpha_1 \|u\|^\max(s^+, p+2) - b. \]

Because \( \alpha_2 > \max(s^+, p + 2) \) operator \( T_3 \) dominates, and it is easy to see that operator \(-T\) is coercive.

Now we give an example to illustrate condition (7).

**Example 1.** Let \( f : V \times \mathbb{R} \times \mathbb{R}^{\#V} \to \mathbb{R} \) and \( p \geq 2. \) Put
\[ f(x, t_1, t_2) = t_1 \sin t_1 - t_1^2 t_2 e^{-x} + \frac{t_1}{1 + t_1^2}. \]

Observe that
\[ t_1 f(x, t_1, t_2) = t_1^2 \sin t_1 - t_1^2 t_2 e^{-x} + \frac{t_1^2}{1 + t_1^2} \]
and
\[ t_1 f(x, t_1, t_2) \leq |t_1|^2 + 1, \]
so condition (7) is satisfied with \( \alpha_1 = 2 \) (\( \alpha_1 < p + 2 \)) and \( \beta_1 = 1. \)

To illustrate condition (8) let us take the following example.

**Example 2.** Let \( f : V \times \mathbb{R} \times \mathbb{R}^{\#V} \to \mathbb{R}, s : V \times V \to [2, 10) \) and \( p \in [2; 8]. \) Put
\[ f(x, t_1, t_2) = t_1^p (\cos t_1 + 2) + t_1 t_2^2 \arccot x - \frac{3 t_1^2}{1 + t_1^2}. \]

Observe that
\[ t_1 f(x, t_1, t_2) = t_1^{10} (\cos t_1 + 2) + t_1^2 t_2^2 \arccot x - \frac{3 t_1^2}{1 + t_1^2} \]
and
\[ t_1 f(x, t_1, t_2) \geq |t_1|^{10} - 3, \]
so condition (7) is satisfied with \( \alpha_2 = 10 \) (\( \alpha_2 > \max(s^+, p + 2) \)) and \( \beta_1 = -3. \)

**Theorem 2.** If assumption (7) or (8) holds, then problem (1) has at least one solution.

**Proof.** Lemma 3 implies that \( T \) or \(-T\) is coercive. Moreover \( T \) and \(-T\) are continuous, so the assumptions of Theorem 1 are fulfilled. This means that there exists a solution to problem (6) that is equivalent to the existence of the solution to problem (1). 

We can observe that the operators \( T_1, T_2, T_3 \) have the following properties.
Lemma 5. (i) Operator $T_1$ is monotone, (ii) Operator $T_2$ is strictly monotone, (iii) if we assume that the continuous function $f$ does not depend on $\nabla_{s(x,v)}$, so $f : V \times \mathbb{R} \to \mathbb{R}$ and satisfies

$$
(u - v)(f(x, u) - f(x, v)) \leq 0,
$$

for all $x \in V$ and all $u, v \in \mathbb{R}$, then operator $T_3$ is monotone.

**Proof.**

1. By direct calculations and by Lemma 3, we get

$$
\langle T_1(u) - T_1(v), u - v \rangle
\geq \sum_{x,y \in V} \frac{2}{s(x,y)(2^{s(x,y)} - 1)} : |(u(y) - u(x)) - (v(y) - v(x))| \geq 0.
$$

2. By direct calculations and by Lemma 3, we also have

$$
\langle T_2(u) - T_2(v), u - v \rangle
\geq \sum_{x \in V} \frac{2q(x)}{p(2^{p-1} - 1)} : |u(x) - v(x)|^p > 0, \text{ for all } u \neq v.
$$

3. By assumption (9)

$$
\langle T_3(u) - T_3(v), u - v \rangle
\geq \sum_{x \in V} (u(x) - v(x))(f(x, u(x)) - f(x, v(x))) \leq 0,
$$

so $-T_3$ is monotone.

\square

**Theorem 3.** If assumption (7) or (8) holds, and moreover assumption (9) is fulfilled, then problem (1) has a unique solution.

**Proof.** By direct calculations and by Lemma 3,

$$
\langle T_1(u) - T_1(v), u - v \rangle
\geq \sum_{x,y \in V} \frac{2}{s(x,y)(2^{s(x,y)} - 1)} : |(u(y) - u(x)) - (v(y) - v(x))| \geq 0.
$$

By direct calculations and by Lemma 3,

$$
\langle T_2(u) - T_2(v), u - v \rangle
\geq \sum_{x \in V} \frac{2q(x)}{p(2^{p-1} - 1)} : |u(x) - v(x)|^p > 0, \text{ for all } u \neq v.
$$
By assumption (9),
\[
\langle T_3(u) - T_3(v), u - v \rangle = \sum_{x \in V} (u(x) - v(x))(f(x, u(x)) - f(x, v(x))) \leq 0,
\]
operator $-T_3$ is monotone.  

Now we give an example to illustrate the above theorem.

**Example 3.** Let $f : V \times \mathbb{R} \to \mathbb{R}$ and $p > 2$. Put
\[
f(x, t) = -t^3 e^{-2x}.
\]
Observe that
\[
tf(x, t) = -t^4 e^{-2x},
\]
and
\[
tf(x, t) \leq |t|^4,
\]
so condition (7) is satisfied for all $x \in V$, and all $t \in \mathbb{R}$ with $\alpha_1 = 4$ ($\alpha_1 < p + 2$) and $\beta_1 = 0$. Note also that
\[
(t_1 - t_2)(f(x, t_1) - f(x, t_2)) = (t_1 - t_2)(-t_1^3 e^{-2x} + t_2^3 e^{-2x}) = -e^{-2x}(t_1 - t_2)^4 \leq 0
\]
for all $x \in V$ and all $t_1, t_2 \in \mathbb{R}$, so condition (9) is satisfied. The assumptions of Theorem 3 are fulfilled, so problem (1) has a unique solution with any function $q : V \to (0, +\infty)$.

As far as we know, Theorem 2 cannot be obtained by a variational method because the function $f$ depends on $\nabla s(x, \cdot)u(x)$. On the other hand, the assertion from Theorem 3 can be obtained by variational methods, with small changes in assumptions.

4. Positive Solutions

In this section, we will look for positive solutions to problem (1). By a positive solution to problem (1), we mean the solution, which has only positive values on $V$. Positive solutions to (1) are again investigated in the space $A$, considered with the norm (2).

We introduce the following notion:
\[
u_+(x) = \max\{u(x), 0\}, \quad u_-(x) = \max\{-u(x), 0\}, \quad \text{for all } x \in V.
\]
It is easy to see that for all $x \in V$ we have
\[
u_+(x) u_-(x) \geq 0,
\]
\[
u_+(x) u_-(x) = 0,
\]
\[
u(x) = u_+(x) - u_-(x),
\]
\[
|\nu(x)| = u_+(x) + u_-(x),
\]
\[
|\nu_+(x)| \leq |\nu(x)|.
\]

If we assume that
\[
f(x, u, v) > 0, \text{ for all } x \in V, u \geq 0 \text{ and all } v \in \mathbb{R}^{\lvert V \rvert},
\]
(10)
then we can define a new problem,
\[ \frac{1}{2} \sum_{x \in V} \langle \nabla s(x, v) u(x), \nabla v(x) \rangle + \sum_{x \in V} q(x) |u(x)|^p u(x) v(x) \]
\[ = \sum_{x \in V} v(x) f(x, u^+(x), \nabla s(x) u(x)) . \]
(11)

The solution to this problem fulfilled (11) for all \( v \in A \). The only difference between (11) and (6) is the right hand side, i.e., the second argument of \( f \) in (11) is \( u^+ \) not \( u(x) \).

Let us formulate an auxiliary result which plays an important role in proving all the existence results in this section. This result shows that any solution to (6) is in fact a positive solution to (1). It may be viewed as a kind of a discrete maximum principle.

**Lemma 6.** Assume that (10) holds, and assume that \( u^* \in A \) is a solution to problem (11). Then \( u^* \) is a positive solution to (1).

**Proof.** A straightforward computation shows that for every \( x, y \in V \) and \( u \in A \), we have the following inequality:
\[ u(x) u_-(x) = (u^+(x) - u_-(x)) u_-(x) \]
\[ = u^+(x) u_-(x) - (u_-(x))^2 = -(u_-(x))^2 \leq 0. \]
(12)

Moreover,
\[ (u(y) - u(x))(u_-(y) - u_-(x)) \leq 0 \]
(13)

holds, because
\[ (u(y) - u(x))(u_-(y) - u_-(x)) \]
\[ = ((u^+(y) - u_+(x)) - (u_-(y) - u_-(x)))(u_-(y) - u_-(x)) \]
\[ = -u^+(y)u_-(x) - u_+(x)u_-(y) - (u_-(y) - u_-(x))^2 \leq 0. \]

Assume that \( u^* \in A \) is a solution to (11). We set \( v = u^* \), and we get
\[ \frac{1}{2} \sum_{x \in V} |u^*(y) - u^* (x)|^p (x, y)^{p(x, y) - 2} (u^*(y) - u^*(x))(u^* (y) - u^*(x)) \omega(x, y) \]
\[ = \sum_{x \in V} f(x, u^*_+(x), \nabla p(x) u^*_+(x))u^*_+(x) - \sum_{x \in V} q(x)|u^*(x)|^p u^*(x)u^*_-(x) . \]
(14)

Because \( f \) and \( q \) are functions with positive values only and because (12) holds, the terms on the right hand side of (14) are together non-negative.

Due to (13), the term on the left hand side of (14) is non-positive. Therefore, Equation (14) holds only if its both sides are equal to zero, which leads to \( u^*_+(x) = 0 \) for all \( x \in V \). Thus, \( u^*_+(x) = u^*_+(x) \) for all \( x \in V \), which means that for \( u^* \) problems (6) and (11) are equivalent, so in fact \( u^* \) is solution of (1).

Moreover, \( u^*(x) \neq 0 \) for all \( x \in V \). Indeed, assume that there exists \( x_0 \in V \) such that \( u^*(x_0) = 0 \). Then, by (1) we have
\[ - \sum_{y \in V \setminus \{x_0\}} |u^*(y)|^p(x_0, y)^{p(x_0, y) - 1} \omega(x_0, y) = f(x_0, 0, \nabla s(x_0) u^*(x_0)). \]

Because the term on the left is non-positive and the term on the right is positive, we have a contradiction. Thus \( u^*(x) \neq 0 \) for all \( x \in V \), it follows that \( u^* \) is a positive solution to (1). \( \square \)

To illustrate assumption (10), take the following example.
Example 4. Let \( f : V \times \mathbb{R} \times \mathbb{R}^{\left| V \right|} \to \mathbb{R} \) and put
\[
f(x, t_1, t_2) = 5t_1 + e^{t_2} + \arccos x.
\]
It is easy to see that
\[
f(x, t_1, t_2) > 0
\]
for all \( x \in V \), for all \( t_1 \geq 0 \) and for all \( t_2 \in \mathbb{R}^{\left| V \right|} \).

Remark 1. We can easily get analogical results for negativity of solutions, by a change of signs in assumption (10), in exactly the same way.

Remark 2. Theorems 2 and 3 now hold when we add condition (10) to its assumptions.

5. Final Comments

With reference to the graph domain setting, some questions and comments arise. We underline here that due to our approach we do not need to take the boundary into account, contrary to what is known from the literature. Should the graph become disjointed, we would obtain two different settings, and in fact two different problems would come under consideration. Toward the multiplicity of solutions, one may employ variational tools and critical point theory (see for example [15] for some tools available).

Now we turn the question of the Hadamard well-posedness of the given problem. To be more precise, let us formulate some background. Let \( Z \) be a metric space and let \((z_n) \subset Z\). We can consider the following family of problems for \( n \in \mathbb{N} \)
\[
-\Delta_{s(x, \cdot)}u(x) + q(x)|u(x)|^{p-1}u(x) = f(x, u(x), \nabla s(x, \cdot)u(x), z_n)
\]
with assumptions (independent on the parameter) leading to the existence and uniqueness of solutions employed in Section 3. By using the continuity of the solution operator in the uniqueness case as follows from the approach suggested in [16], we may obtain the Hadamard well-posedness, which says, roughly speaking, that small deviations from the unique solution return to such solution in the limit. Nevertheless, in case the solution is nonunique, we suggest some continuous dependence on parameters result as well. However, in this case we can approximate some solution with a subsequence only. Moreover, no procedure for how to obtain such subsequence is to be recovered, as suggested by the case of the Galerkin-type approximations in the non-unique case.

Another question concerns the structural stability, i.e., some changes introduced in the graph on which the problem is defined. Such a question is related to the above in case the weights are continuous, and we may proceed as sketched in the above remarks.

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References
1. Galewski, M.; Wieteska, R. Existence and multiplicity of positive solutions for discrete anisotropic equations. *Turk. J. Math.* 2014, 38, 297–310. [CrossRef]
2. Bereanu, C.; Jebelean, P.; Serban, C. Periodic and Neumann problems for discrete \( p(\cdot)-\)Laplacian. *J. Math. Anal. Appl.* 2013, 399, 75–87. [CrossRef]
3. Park, J.H. On a resonance problem with the discrete \( p\)-Laplacian on finite graphs. *Nonlinear Anal.* 2011, 74, 6662–6675. [CrossRef]
4. Park, J.H.; Chung, S.Y. Positive solutions for discrete boundary value problems involving the $p$-Laplace with potential terms. *Comput. Math. Appl.* 2011, 61, 17–29. [CrossRef]

5. Chung, S.-Y.; Park, J.-H. Multiple solutions to discrete boundary value problems for the $p$-Laplacian with potential terms on finite graphs. *Bull. Korean Math. Soc.* 2015, 52, 1517–1533. [CrossRef]

6. Zhikov, V.V. Averaging of functionals of the calculus of variations and elasticity theory. *Math. USSR Izv.* 1987, 29, 33–66. [CrossRef]

7. Chen, Y.; Levine, S.; Rao, M. Variable exponent, linear growth functionals in image processing. *SIAM J. Appl. Math.* 2006, 66, 1383–1406. [CrossRef]

8. Fan, X.L.; Zhang, D. Existence of solutions for $p(x)$-Laplacian Dirichlet problem. *Nonlinear Anal.* 2003, 52, 1843–1852. [CrossRef]

9. Harjulehto, P.; Hästö, P.; Le, U.V.; Nuortio, M. Overview of differential equations with non-standard growth. *Nonlinear Anal.* 2010, 72, 4551–4574. [CrossRef]

10. Agarwal, R.P. *Difference Equations and Inequalities*; Marcel Dekker: New York, NY, USA, 1992.

11. West, D.B. *Introduction to Graph Theory*, 2nd ed.; Prentice Hall: Upper Saddle River, NJ, USA, 2001.

12. Galewski, M.; Wieteska, R. Existence and multiplicity results for boundary value problems connected with the discrete $p(\cdot)$-Laplacian on weighted finite graphs. *Appl. Math. Comput.* 2016, 290, 376–391. [CrossRef]

13. Chabrowski, J. *Variational Methods for Potential Operator Equations*; de Gruyter Studies in Mathematics 24; Walter de Gruyter: Berlin, Germany, 1997; Volume ix, p. 290.

14. Galewski, M. *Basic Monotonicity Methods with Some Applications*; Compact Textbooks in Mathematics; Birkhäuser: Cham, Switzerland, 2021; p. 180.

15. Kristály, A.; Radulescu, V.; Varga, C. *Variational Principles in Mathematical Physics, Geometry and Economics: Qualitative Analysis of Nonlinear Equations and Unilateral Problems*; Encyclopedia of Mathematics (No. 136); Cambridge University Press: Cambridge, UK, 2010.

16. Beldziński, M.; Galewski, M.; Kossowski, I. Dependence on parameters for nonlinear equations—Abstract principles and applications. *Math. Methods Appl. Sci.* 2022, 45, 1668–1686. [CrossRef]