Accretion disks around a static black hole in $f(R)$ gravity

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Abstract: We provide a description of a thin accretion disc for a static spherically symmetric black holes in $f(R)$ gravity. In this regard, we first study the horizons of black holes in $f(R)$ gravity. The equation of motion and effective potential are also computed which eventually leads to possible existence of innermost circular orbits of accretion disc. We derive the specific energy, specific angular momentum and angular velocity of the particles moving in circular orbits also. A comparative study of various parameters are also presented. The locations of the event horizon, cosmological horizon, innermost and outermost stable circular orbits are also pointed out.
## Contents

1 Introduction ......................................................... 1

2 The structure of spacetime in $f(R)$ gravity .................. 2
   2.1 The Structure of Horizons ................................. 3
   2.2 Equations of motion and effective potential .......... 5

3 Thin accretion disk .............................................. 6

4 Concluding remarks ............................................... 12

## 1 Introduction

Since the last decades, people are trying to solve the the problems like the accelerated expansion of the universe [1, 2] and the dark matter origin [3] with the help of modified theories of gravity. There are various modified theory of gravity. $f(R)$ theory of gravity is one of them [4]. This is because general relativity (GR) agrees to gravity with high accuracy when the curvature is small [5], however for very large values of the curvature there does not exist any such evidence. In this regard, black holes are the ideal places to look for modified GR [6]. The features of static and spherically symmetric black hole solutions in $f(R)$ gravity theories are discussed [7]. Recently, a linear perturbations of a Kerr black hole in $f(R)$ gravity using the Newman-Penrose formalism are studied in Ref. [8].

It is expected that, at large distances, the geometry of the space-time in $f(R)$ modified gravity models must be different from that of standard GR. Therefore, one needs to develop a method that could observationally distinguish and test the possible deviations from GR. The study of accretion disks around compact objects could be one of the possible methods. It is well-known that most of the astrophysical objects like black holes grow in mass due to accretion. Some observations suggest that around black holes, there exist gas clouds together with an associated accretion disc [9]. The first comprehensive study of mass accretion around rotating black holes in GR was made in [10]. After that general relativistic model of thin accretion disk was developed under the assumption that the disk is in a steady-state (i.e. the mass accretion rate is constant in time and does not depend of the radius of the disk) [11–13]. The radiation properties of the thin accretion disks had been given in [14, 15]. In fact, the descriptions of the accretion disk around wormholes, non-rotating/rotating quark, boson/fermion stars, brane world black holes, naked singularities and $f(R)$ modified gravity models have been given in Refs. [16–33].
In order to study the parameters specifying the thin accretion disc for the static spherically symmetric black hole in $f(R)$ gravity, we first write the metric and the action for the system. From the metric function, we analyze the horizon of the black hole. Here we obtain that existence of event horizon varies according to different dimensionless parameters. For instance, for certain values of these parameters there exist and inner black hole event horizon and outer cosmological horizon; in spite of that in certain cases there exists only single black hole event horizon. In certain cases, the event and cosmological horizons collapse and naked singularities occur. Furthermore, we evaluate equations of motion and effective potential of the system where we note that energy and the angular momentum together with the cosmological constant and a real parameter describe the shape of the orbit.

It is well-known that the accretion disk around the black hole appears due to the particles revolving in a circular orbit. Therefore, we compute the specific energy, the specific angular momentum and the angular velocity of the particles moving in circular orbits for the the static spherically symmetric black hole in $f(R)$ gravity. From the necessary condition for existence of innermost circular orbits of accretion disc we obtain an expression for dimensionless cosmological constant $\tilde{\Lambda}$ as a function of the radial coordinate of the circular orbits. Here, we do comparative analysis of $\tilde{\Lambda}$ corresponding to different values of dimensionless real parameter and find that the value of dimensionless cosmological constant increases with larger values of real parameter. We also analyze $\tilde{\Lambda}$ as a function of the radial co-ordinate of the event horizon and as a function of the radial coordinate of the innermost stable circular orbits. Moreover, we plot the effective potential with the location of innermost circular orbit where the locations of the innermost stable circular orbits are indicated. We provide two tables specifying locations of the event and cosmological horizons, and of the innermost and outermost stable circular orbits in a static black hole in $f(R)$ gravity and locations of the event horizon, and of the innermost stable circular orbit in a static black hole in $f(R)$ gravity, respectively.

The paper is presented as following. In section 2, we discuss the structure of horizons of the static spherically symmetric black hole in $f(R)$ gravity and derive equations of motion together with effective potential. The specific energy, specific angular momentum and the angular velocity of the particles moving in circular orbits are calculated in section 3. Finally, we conclude our work with future remarks in section 4.

2 The structure of spacetime in $f(R)$ gravity

In this section, we recapitulate metric structure of the static spherically symmetric black hole in $f(R)$ gravity. Let us begin by writing a generic action for $f(R)$ gravity in four dimensional spacetime as

$$ S = \frac{1}{2k} \int d^4x \sqrt{-g} f(R) + S_m, \quad (2.1) $$

where $k$ is Einstein’s constant, $R$ is the Ricci scalar and $S_m$ is the matter part. The variation of above action with respect to the metric leads to the following field equations:

$$ f'(R)R_{\mu\nu} - \frac{1}{2} f(R) g_{\mu\nu} - (\nabla_\mu \nabla_\nu - g_{\mu\nu} \Box) f'(R) = k T_{\mu\nu}, \quad (2.2) $$
in which $\Box = \nabla_{\alpha} \nabla^{\alpha}$ and $f'(R) = \frac{df(R)}{dR}$. The line element representing a 4-dimensional static black hole in $f(R)$ gravity is given by \[34\]

$$ds^2 = -A(r)dt^2 + A(r)^{-1}dr^2 + r^2(d\theta^2 + \sin^2 \theta d\varphi^2), \quad (2.3)$$

with metric function

$$g(r) = 1 - \frac{2M}{r} + \beta r - \frac{1}{3}\Lambda r^2, \quad (2.4)$$

where $M$ is the mass, $\beta$ is a real constant and $\Lambda$ is the cosmological constant \[34, 35\]. Here one should mention that non-zero cosmological constant are crucial for various phenomenon of static and spherically symmetric black holes \[36, 37\]. The notion of static radius is crucial here as it can represent a natural boundary of gravitationally bound systems in an expanding universe governed by a cosmological constant, as clearly demonstrated in a variety of situations \[38–43\].

2.1 The Structure of Horizons

The horizon of the spacetime given by equation (2.3) can be determined by setting the condition $g_{00}(r) = 0$, i.e.

$$1 - \frac{2M}{r} + \beta r - \frac{1}{3}\Lambda r^2 = 0. \quad (2.5)$$

Now, we define following dimensionless quantities \[35\]:

$$\tilde{r} = \frac{r}{M}, \quad \tilde{\Lambda} = \Lambda M^2, \quad \tilde{\beta} = \beta M, \quad (2.6)$$

which leads to the equation (2.5) to the following form:

$$\tilde{\Lambda}\tilde{r}^3 - \tilde{\beta}\tilde{r}^2 - \tilde{r} + 2 = 0. \quad (2.7)$$

From Fig. 1, corresponding to $\tilde{\beta} = 0.01$, there exist an inner black hole event horizon and an

![Figure 1: $\tilde{\Lambda}$ vs. $\tilde{r}$ for $\tilde{\beta} = 0.01$](image)

outer cosmological horizon for $\tilde{\Lambda} \in (0, 0.04069)$. However, there exists only single black hole
event horizon for $\tilde{\Lambda} \leq 0$. The event and cosmological horizons collapse for $\tilde{\Lambda} = 0.04069$ and for higher values naked singularities occur and hence, no black holes are possible. Consequently, the trajectories should be studied for $\tilde{\Lambda} \in (-\infty, 0.04069]$.

From Fig. 2, corresponding to $\tilde{\beta} = 0.1$, there exist an inner black hole event horizon and an outer cosmological horizon for $\tilde{\Lambda} \in (0, 0.07226)$. However, there exists only single black hole event horizon for $\tilde{\Lambda} \leq 0$. The event and cosmological horizons collapse for $\tilde{\Lambda} = 0.07226$ and for higher values naked singularities occur and hence, no black holes are possible. Consequently, the trajectories should be studied for $\tilde{\Lambda} \in (-\infty, 0.07226]$.

From Fig. 3, corresponding to $\tilde{\beta} = 0.99$, there exist an inner black hole event horizon and an outer cosmological horizon for $\tilde{\Lambda} \in (0, 0.5259)$. However, there exists only single black hole event horizon for $\tilde{\Lambda} \leq 0$. The event and cosmological horizons collapse for $\tilde{\Lambda} = 0.5259$ and for higher values naked singularities occur and hence, no black holes are possible. Consequently,
the trajectories should be studied for $\tilde{\Lambda} \in (-\infty, 0.5259]$.

![Graphs of $\tilde{\Lambda}$ vs $r$ for different values of $\tilde{\beta}$]

(A) Closeup of (a)

Figure 4: Plot of $\tilde{\Lambda}$ as a function of the radial coordinate of the event horizon for different values of $\tilde{\beta}$.

A comparative analysis of these plots are given in Fig. 4 which reflects that as long as the values of $\tilde{\beta}$ reduces the peak value of $\tilde{\Lambda}$ becomes smaller.

2.2 Equations of motion and effective potential

In order to study the dynamics of the system, we evaluate equations of motion and effective potential. In order to do so, we first write the Lagrangian $\mathcal{L}$ for a point particle in the spacetime characterized by metric (2.3) as:

$$\mathcal{L} = \frac{1}{2} g_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} = \frac{1}{2} \epsilon.$$  

(2.8)

Here, $\epsilon$ takes value 1 for massive particle and 0 for photon (massless). In the equatorial plane, conserved energy $E$ and angular momentum $L$ read

$$E = g_{tt} \frac{dt}{ds} = \left(1 - \frac{2m}{r} + \beta r - \frac{1}{3} \Lambda r^2 \right) \frac{dt}{ds},$$  

(2.9)

$$L = g_{\varphi\varphi} \frac{d\varphi}{ds} = r^2 \frac{d\varphi}{ds}.$$  

(2.10)

By considering above equations, the geodesic equations corresponding to massive particles can be obtained as [35]

$$\left( \frac{dr}{ds} \right)^2 = E^2 - \left(1 - \frac{2m}{r} + \beta r - \frac{1}{3} \Lambda r^2 \right) \left(1 + \frac{L^2}{r^2} \right),$$  

(2.11)

$$\left( \frac{dr}{d\varphi} \right)^2 = \frac{r^4}{L^2} \left[ E^2 - \left(1 - \frac{2m}{r} + \beta r - \frac{1}{3} \Lambda r^2 \right) \left(1 + \frac{L^2}{r^2} \right) \right] ,$$  

(2.12)

$$\left( \frac{dr}{dt} \right)^2 = \frac{1}{E^2} \left(1 - \frac{2m}{r} + \beta r - \frac{1}{3} \Lambda r^2 \right)^2 \left[ E^2 - \left(1 - \frac{2m}{r} + \beta r - \frac{1}{3} \Lambda r^2 \right) \left(1 + \frac{L^2}{r^2} \right) \right].$$  

(2.13)
Eqs. (2.11), (2.12) and (2.13) provide us a complete description of the dynamics. However, Eq. (2.11) reflects the expression of an effective potential

\[ V_{\text{eff}} = \left(1 - \frac{2m}{r} + \beta r - \frac{1}{3} \Lambda r^2 \right) \left(1 + \frac{L^2}{r^2} \right). \]  

Using Eqs. (2.6) and (2.14), we have

\[ V_{\text{eff}} = \left(1 - \frac{2\tilde{r}}{\tilde{r}} + \tilde{\beta} \tilde{r} - \frac{1}{3} \tilde{\Lambda} \tilde{r}^2 \right) \left(1 + \frac{\tilde{L}^2}{\tilde{r}^2} \right), \]  

where \( \tilde{L} = L/M \). Here, we note that energy and the angular momentum together with the cosmological constant and \( \beta \) describe the shape of the orbit.

### 3 Thin accretion disk

The accretion disk appears due to the particles revolving in a circular orbit around a compact object such as a black hole. The specific energy \( \tilde{E} \), the specific angular momentum \( \tilde{L} \) and the angular velocity \( \Omega \), of the particles moving in circular orbits can be calculated, respectively, by

\[ \tilde{E} = \frac{g_{\mu \nu}}{\sqrt{g_{\mu \nu} - g_{\phi \phi} \Omega^2}} = -\frac{\sqrt{2}}{3} \left( \tilde{\Lambda} \tilde{r}^3 - 3 \tilde{\beta} \tilde{r}^2 - 3 \tilde{r} + 6 \right), \]  
\[ \tilde{L} = \frac{g_{\phi \phi} \Omega}{\sqrt{g_{\mu \nu} - g_{\phi \phi} \Omega^2}} = \tilde{r} \sqrt{\frac{-2 \tilde{\Lambda} \tilde{r}^3 - 3 \tilde{\beta} \tilde{r}^2 - 6}{3(\tilde{\beta} \tilde{r}^2 + 2 \tilde{r} - 6)}}, \]  
\[ \Omega = \sqrt{\frac{g_{\mu \nu, \phi}}{g_{\phi \phi, r}}} = \sqrt{\frac{-2 \tilde{\Lambda} \tilde{r}^3 - 3 \tilde{\beta} \tilde{r}^2 - 6}{6 \tilde{r}^3}}. \]

Using Eqs. (2.14) and (3.2), we have

\[ \frac{d^2 V_{\text{eff}}}{dr^2} = -\frac{2}{3} \frac{3 \tilde{\Lambda} \tilde{\beta} \tilde{r}^5 - 3 \tilde{\beta}^2 \tilde{r}^4 + 8 \tilde{\Lambda} \tilde{r}^4 - 30 \tilde{\Lambda} \tilde{r}^3 - 9 \tilde{\beta} \tilde{r}^3 + 36 \tilde{\beta} \tilde{r}^2 - 6 \tilde{r} + 36}{\tilde{r}^3 \left( \tilde{\beta} \tilde{r}^2 + 2 \tilde{r} - 6 \right)}. \]  

Note that the necessary condition for existence of innermost circular orbits is \( \frac{d^2 V_{\text{eff}}}{dr^2} = 0 \) and the signs of \( \frac{d^2 V_{\text{eff}}}{dr^2} \) show the stability of orbits. Thus, by equating Eq. (3.4) to zero, we obtain the \( \tilde{\Lambda} \) as a function of the radial coordinate of the circular orbits as following:

\[ \tilde{\Lambda} = 3 \frac{\tilde{\beta}^2 \tilde{r}^4 + 3 \tilde{\beta} \tilde{r}^3 - 12 \tilde{\beta} \tilde{r}^2 + 2 \tilde{r} - 12}{\tilde{r}^3 \left( 3 \tilde{\beta} \tilde{r}^2 + 8 \tilde{r} - 30 \right)}. \]

Plot of the this equation is shown in Fig. 5. From the plot, it is evident that the values of \( \tilde{\Lambda} \) is an increasing function of \( \tilde{\beta} \). Also, plot of \( \tilde{\Lambda} \) as a function of the radial co-ordinate of the event
horizon and of \( \tilde{\Lambda} \) as a function of the radial coordinate of the innermost stable circular orbits is shown in Fig. 6. Moreover, plots of effective potential (Eq. 2.15) with the location of innermost circular orbit are shown in Figs. (7, 8) for \( \Lambda > 0 \) and \( \Lambda < 0 \) respectively. The dots indicate the location of the innermost stable circular orbits. Locations of the event and cosmological horizons, and of the innermost and outermost stable circular orbits in a static black hole in \( f(R) \) gravity are shown in Tables. (1, 2) for \( \Lambda > 0 \) and \( \Lambda < 0 \) respectively.

![Figure 5](image1.png)

**Figure 5:** Plot of \( \tilde{\Lambda} \) as a function of the radial coordinate of the innermost stable circular orbits for different values of \( \tilde{\beta} \).

![Figure 6](image2.png)

**Figure 6:** Plot of \( \tilde{\Lambda} \) as a function of the radial coordinate of the event horizon (orange line) and \( \tilde{\Lambda} \) as a function of the radial coordinate of the innermost stable circular orbits (blue dashed line) for \( \tilde{\beta} = 0.99 \).
Table 1: Location of the event and cosmological horizons, and of the innermost and outermost stable circular orbits in a static black hole in \( f(R) \) gravity.

| \( \tilde{\Lambda} \) | \( \tilde{\beta} \) | Event horizon radius | Cosmological horizon radius | Innermost stable circular orbit radius | Outermost stable circular orbit radius |
|-----------------|-----------------|-----------------------|-----------------------------|----------------------------------------|---------------------------------------|
|                 | 0               | \( 2 \times \)       | \( \times \)                | 6                                      | \( \times \)                          |
| 0.001           | 1.9960          | \( \times \)         | 5.9001                      | \( \times \)                          |
| 0.01            | 1.9615          | \( \times \)         | 5.3675                      | \( \times \)                          |
| 0.1             | 1.7082          | \( \times \)         | 4.14                        | \( \times \)                          |
| 0.99            | 1.0033          | \( \times \)         | 2.4393                      | \( \times \)                          |
| 0.0003          | 0               | 2.0024                | 56.7077                     | 6.2425                                 | 12.2499                               |
| 0.001           | 1.9984          | 58.4294               | 6.0963                      | 13.7866                               |
| 0.01            | 1.9367          | 76.0250               | 5.4355                      | 38.0848                               |
| 0.1             | 1.7093          | 342.9949              | 4.1518                      | 336.3436                              |
| 0.99            | 1.0034          | 3301.0091             | 2.4395                      | 3300.3357                             |
| 0.00071         | 0               | 2.0057                | 36.4862                     | 7.4038                                | 7.6016                                |
| 0.001           | 2.0016          | 37.2180               | 6.6190                      | 8.9474                                |
| 0.01            | 1.9667          | 44.3863               | 5.5474                      | 17.9842                               |
| 0.1             | 1.7108          | 150.1032              | 4.1598                      | 143.5071                              |
| 0.99            | 1.0035          | 1395.3741             | 2.4397                      | 1394.7007                             |

The flux of the radiant energy over the disk can be calculated from following relation \([44, 45]\)

\[
F(r) = \frac{-\dot{M}_0}{4\pi \sqrt{-g}} \frac{\Omega_r}{(E - \Omega \tilde{L})^2} \int_{r_{ms}}^r (\dot{E} - \Omega \dot{L}) \tilde{L}_r dr. \tag{3.6}
\]

which should follow Stefan-Boltzmann law when the disk is supposed to be in thermal equilibrium.
Figure 7: Effective potential for different values of $\tilde{\Lambda}$ and $\tilde{L}$ of black hole in f(R) gravity. The dots indicate the location of the innermost stable circular orbit.
| $\beta$  | $\Lambda$  | Event horizon radius | Innermost stable circular orbit radius |
|---------|-------------|----------------------|----------------------------------------|
| $\beta = 0$ | -0.00071 | 1.994 | 5.669 |
|          | -0.003  | 1.977 | 5.192 |
|          | -0.02   | 1.869 | 4.429 |
|          | -0.1    | 1.595 | 3.988 |
|          | -0.5    | 1.180 | 3.809 |
| $\beta = 0.01$ | -0.00071 | 1.956 | 5.237 |
|          | -0.003  | 1.940 | 4.968 |
|          | -0.02   | 1.841 | 4.363 |
|          | -0.1    | 1.580 | 3.943 |
|          | -0.5    | 1.175 | 3.761 |
| $\beta = 0.1$ | -0.00071 | 1.706 | 4.133 |
|          | -0.003  | 1.697 | 4.094 |
|          | -0.02   | 1.642 | 3.904 |
|          | -0.1    | 1.468 | 3.619 |
|          | -0.5    | 1.137 | 3.420 |
| $\beta = 0.99$ | -0.00071 | 1.003 | 2.439 |
|          | -0.003  | 1.002 | 2.437 |
|          | -0.02   | 0.9967| 2.427 |
|          | -0.1    | 0.9723| 2.387 |
|          | -0.5    | 0.8832| 2.283 |

Table 2: Location of the event horizon, and of the innermost stable circular orbit in a static black hole in f(R) gravity.
Figure 8: Effective potential for different values of \( \Lambda < 0 \) and \( \tilde{L} \) of black hole in f(R) gravity. The dots indicate the location of the innermost stable circular orbit.
4 Concluding remarks

Let us summarize our work here. In order to study the thin accretion disc for the static spherically symmetric black hole in \( f(R) \) gravity, we have given preliminary idea about the metric and the action for such system. We have computed the horizon of the black hole by equating metric function to zero. Furthermore, we have expressed metric function in terms of dimensionless parameters \( \tilde{r}, \tilde{\Lambda} \) and \( \tilde{\beta} \). In order to discuss the horizons, we have plotted \( \tilde{\Lambda} \) with respect to \( \tilde{r} \) for different values of \( \tilde{\beta} \). For \( \tilde{\beta} = 0.01 \) case, we have found an inner black hole event horizon and an outer cosmological horizon for \( \tilde{\Lambda} \in (0, 0.04069) \). In this case, only single black hole event horizon is seen for \( \tilde{\Lambda} \leq 0 \). The event and cosmological horizons collapse for \( \tilde{\Lambda} = 0.04069 \) and for higher values naked singularities occur and hence, no black holes are possible. Consequently, the trajectories should be studied for \( \tilde{\Lambda} \in (-\infty, 0.04069) \]. Similarly, we have studied \( \tilde{\beta} = 0.1 \) and \( \tilde{\beta} = 0.99 \) cases.

In order to discuss a thin accretion disk around the spherically symmetric black hole in \( f(R) \) gravity, we have computed the specific energy, the specific angular momentum and the angular velocity of the particles moving in circular orbits. The equation of motion and effective potential are also computed. From the necessary condition (i.e. the double derivative of effective potential with respect to radial co-ordinate must be zero) for existence of innermost circular orbits of accretion disc, we have computed an expression for dimensionless cosmological constant \( \tilde{\Lambda} \). Here, we have done comparative analysis of \( \tilde{\Lambda} \) as a function of \( \tilde{r} \) corresponding to different values of dimensionless real parameter \( \tilde{\beta} \) and found that the value of dimensionless cosmological constant increases with larger values of real parameter. Furthermore, we have discussed \( \tilde{\Lambda} \) as a function of the radial co-ordinate of the event horizon and \( \tilde{\Lambda} \) as a function of the radial coordinate of the innermost stable circular orbits. Finally, we have studied the effective potential with respect to \( \tilde{r} \) for \( \tilde{\Lambda} \geq 0 \) and different \( \tilde{L} \) where the locations of the innermost stable circular orbits are indicated. The effective potential with respect to \( \tilde{r} \) for \( \tilde{\Lambda} < 0 \) and different \( \tilde{L} \) are also studied, where the locations of the innermost stable circular orbits are indicated. We have provided two tables, one specifying locations of the event and cosmological horizons, and of the innermost and outermost stable circular orbits in a static black hole in \( f(R) \) gravity and other specifying locations of the event horizon, and of the innermost stable circular orbit in a static black hole in \( f(R) \) gravity.

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