MAXIMUM PACKINGS
OF THE \( \lambda \)-FOLD COMPLETE 3-UNIFORM HYPERGRAPH
WITH LOOSE 3-CYCLES

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Abstract. It is known that the 3-uniform loose 3-cycle decomposes the complete 3-uniform hypergraph of order \( v \) if and only if \( v \equiv 0, 1, \) or \( 2 \) (mod 9). For all positive integers \( \lambda \) and \( v \), we find a maximum packing with loose 3-cycles of the \( \lambda \)-fold complete 3-uniform hypergraph of order \( v \). We show that, if \( v \geq 6 \), such a packing has a leave of two or fewer edges.

Keywords: maximum packing, \( \lambda \)-fold complete 3-uniform hypergraph, loose 3-cycle.

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1. INTRODUCTION

A hypergraph \( H \) consists of a finite nonempty set \( V \) of vertices and a finite collection \( E = \{e_1, e_2, \ldots, e_m\} \) of nonempty subsets of \( V \) called hyperedges or simply edges. For a given hypergraph \( H \), we use \( V(H) \) and \( E(H) \) to denote the vertex set and the edge set (or multiset) of \( H \), respectively. We call \( |V(H)| \) and \( |E(H)| \) the order and size of \( H \), respectively. The degree of a vertex \( v \in V(H) \) is the number of edges in \( E(H) \) that contain \( v \). A hypergraph \( H \) is simple if no edge appears more than once in \( E(H) \). If for each \( e \in E(H) \) we have \( |e| = t \), then \( H \) is said to be \( t \)-uniform. Thus \( t \)-uniform hypergraphs are generalizations of the concept of a graph (where \( t = 2 \)). Graphs with repeated edges are often called multigraphs. If \( H \) is a simple hypergraph and \( \lambda \) is a positive integer, then \( \lambda \)-fold \( H \), denoted \( \lambda H \), is the multi-hypergraph obtained from \( H \) by repeating each edge exactly \( \lambda \) times. The hypergraph with vertex set \( V \) and edge set the set of all \( t \)-element subsets of \( V \) is called the complete \( t \)-uniform hypergraph on \( V \) and is denoted by \( K^t_V \). If \( v = |V| \), then \( \lambda K^t_V \) is called the \( \lambda \)-fold complete \( t \)-uniform hypergraph of order \( v \) and is used to denote any hypergraph isomorphic...
to $\lambda K^t_v$. When $t = 2$, we will use $\lambda K_v^t$ in place of $\lambda K_v^{(2)}$. Similarly, if $\lambda = 1$, then we will use $K_v^t$ in place of $\lambda K_v^{(t)}$. If $H'$ is a subhypergraph of $H$, then $H \setminus H'$ denotes the hypergraph obtained from $H$ by deleting the edges of $H'$. We may refer to $H \setminus H'$ as the hypergraph $H$ with a hole $H'$. The vertices in $H'$ may be referred to as the vertices in the hole.

A commonly studied problem in combinatorics concerns decompositions of graphs or multigraphs into edge-disjoint subgraphs. A decomposition of a multigraph $M$ is a set $\Delta = \{G_1, G_2, \ldots, G_s\}$ of subgraphs of $M$ such that $\{E(G_1), E(G_2), \ldots, E(G_s)\}$ is a partition of $E(M)$. If each element of $\Delta$ is isomorphic to a fixed graph $G$, then $\Delta$ is called a $G$-decomposition of $M$. If $L$ is a subgraph of $M$ and $\Delta$ is a $G$-decomposition of $M \setminus L$, then $\Delta$ is called a $G$-packing of $M$ with leave $L$. Such a $G$-packing is maximum if no other possible $G$-packing of $M$ has a leave of a smaller size than that of $L$. Clearly, if $|E(L)| < |E(G)|$, then the $G$-packing is maximum. Moreover, a $G$-decomposition of $M$ can be viewed as a maximum $G$-packing with an empty leave.

A $G$-decomposition of $\lambda K_v$ is also known as a $G$-design of order $v$ and index $\lambda$. A $K_k$-design of order $v$ and index $\lambda$ is usually known as a $2-(v, k, \lambda)$ design or as a balanced incomplete block design of index $\lambda$ or a $(v, k, \lambda)$-BIBD. The problem of determining all $v$ for which there exists a $G$-design of order $v$ is of special interest (see [1] for a survey).

The notion of decompositions of graphs naturally extends to hypergraphs. A decomposition of a hypergraph $M$ is a set $\Delta = \{H_1, H_2, \ldots, H_s\}$ of subhypergraphs of $M$ such that $\{E(H_1), E(H_2), \ldots, E(H_s)\}$ is a partition of $E(M)$. Any element of $\Delta$ isomorphic to a fixed hypergraph $H$ is called an $H$-block. If all elements of $\Delta$ are $H$-blocks, then $\Delta$ is called an $H$-decomposition of $M$. If $L$ is a subgraph of $M$ and $\Delta$ is an $H$-decomposition of $M \setminus L$, then $\Delta$ is called an $H$-packing of $M$ with leave $L$, where we again define such a packing to be maximum if $L$ has the fewest edges possible. An $H$-decomposition of $\lambda K_v^t$ is called an $H$-design of order $v$ and index $\lambda$. The problem of determining all $v$ for which there exists an $H$-design of order $v$ and index $\lambda$ is called the $\lambda$-fold spectrum problem for $H$-designs.

A $K_k^t$-design of order $v$ and index $\lambda$ is a generalization of 2-$(v, k, \lambda)$ designs and is known as a $t$-$(v, k, \lambda)$ design or simply as a $t$-design. A summary of results on $t$-designs appears in [15]. A $t$-$(v, k, 1)$ design is also known as a Steiner system and is denoted by $S(t, v, k)$ (see [8] for a summary of results on Steiner systems). Keevash [14] has recently shown that for all $t$ and $k$ the obvious necessary conditions for the existence of an $S(t, k, v)$-design are sufficient for sufficiently large values of $v$. Similar results were obtained by Glock, Kühn, Lo, and Osthus [9,10] and extended to include the corresponding asymptotic results for $H$-designs of order $v$ for all uniform hypergraphs $H$. These results for $t$-uniform hypergraphs mirror the celebrated results of Wilson [23] for graphs. Although these asymptotic results assure the existence of $H$-designs for sufficiently large values of $v$ for any uniform hypergraph $H$, the spectrum problem has been settled for very few hypergraphs of uniformity larger than $2$.

In the study of graph decompositions, a fair amount of the focus has been on $G$-decompositions of $K_v$ where $G$ is a graph with a relatively small number of edges (see [1] and [5] for known results). Some authors have investigated the corresponding
problem for 3-uniform hypergraphs. For example, in [4], the 1-fold spectrum problem is settled for all 3-uniform hypergraphs on 4 or fewer vertices. More recently, the 1-fold spectrum problem was settled in [6] for all 3-uniform hypergraphs with at most 6 vertices and at most 3 edges. In [6], they also settle the 1-fold spectrum problem for the 3-uniform hypergraph of order 6 whose edges form the lines of the Pasch configuration. Authors have also considered $H$-designs where $H$ is a 3-uniform hypergraph whose edge set is defined by the faces of a regular polyhedron. Let $T$, $O$, and $I$ denote the tetrahedron, the octahedron, and the icosahedron hypergraphs, respectively. The hypergraph $T$ is the same as $K_{4}^{(3)}$, and its spectrum was settled in 1960 by Hanani [11]. In another paper [12], Hanani settled the spectrum problem for $O$-designs and gave necessary conditions for the existence of $I$-designs. The 1-fold spectrum problem is also settled for a type of 3-uniform hyperstars which is part of a larger class of hypergraphs known as delta-systems. For a positive integer $m$, let $S_{m}^{(3)}$ denote the 3-uniform hypergraph of size $m$ which consists of one vertex of degree $m$ and $2m$ vertices of degree one. Necessary and sufficient conditions for the existence of $S_{m}^{(3)}$-decompositions of $K_{v}^{(3)}$ are given in [21] for $m \in [4,6]$ and for all $m$ in [18]. Some results on maximum $S_{m}^{(3)}$-packings of $K_{v}^{(3)}$ are given in [19]. Perhaps the best known general result on decompositions of complete $t$-uniform hypergraphs is Baranyai’s result [3] on the existence of $1$-factorizations of $K_{mt}^{(t)}$ for all positive integers $m$. There are, however, several articles on decompositions of complete $t$-uniform hypergraphs (see [2] and [20]) and of $t$-uniform $t$-partite hypergraphs (see [16] and [22]) into variations on the concept of a Hamilton cycle. There are also several results on decompositions of 3-uniform hypergraphs into structures known as Berge cycles with a given number of edges (see for example [13] and [17]). We note however that the Berge cycles in these decompositions are not required to be isomorphic.

In this paper we are interested in maximum $H$-packings of $\lambda K_{v}^{(3)}$, where $H$ is a 3-uniform loose 3-cycle. For integer $m \geq 3$, a 3-uniform loose $m$-cycle, denoted $LC_{m}^{(3)}$, is a 3-uniform hypergraph with vertex set $\{v_{1}, v_{2}, \ldots, v_{2m}\}$ and edge set $\{\{v_{2i-1}, v_{2i}, v_{2i+1}\} : 1 \leq i \leq m - 1\} \cup \{v_{2m-1}, v_{2m}, v_{1}\}$. Thus $LC_{3}^{(3)}$ has vertex set $\{v_{1}, v_{2}, v_{3}, v_{4}, v_{5}, v_{6}\}$ and edge set $\{\{v_{1}, v_{2}, v_{3}\}, \{v_{3}, v_{4}, v_{5}\}, \{v_{5}, v_{6}, v_{1}\}\}$ for which we use $H[v_{1}, v_{2}, v_{3}, v_{4}, v_{5}, v_{6}]$ to denote (see Figure 1).

![Fig. 1. The 3-uniform loose 3-cycle, $LC_{3}^{(3)}$, denoted by $H[v_{1}, v_{2}, v_{3}, v_{4}, v_{5}, v_{6}]$.](image-url)
Since $LC_3^{(3)}$ has 3 edges and 6 vertices, it is one of the hypergraphs covered in the decomposition results by Bryant, Herke, Maenhaut, and Wannasit in [6]. It is shown in [6] that there exists an $LC_3^{(3)}$-decomposition of $K_v^{(3)}$ if and only if $v \equiv 0, 1, \text{ or } 2 \pmod{9}$. Similarly, it is shown in [7] that there exists an $LC_4^{(3)}$-decomposition of $K_v^{(3)}$ if and only if $v \equiv 0, 1, 2, 4, \text{ or } 6 \pmod{8}$ and $v \notin \{4, 6\}$. Here we focus on maximum $LC_3^{(3)}$-packings of $\lambda K_v^{(3)}$ and show that if $\lambda$ and $v \geq 6$ are positive integers, then there exists a maximum $LC_3^{(3)}$-packing of $\lambda K_v^{(3)}$ where the leave has two or fewer edges.

1.1. ADDITIONAL NOTATION AND TERMINOLOGY

If $a$ and $b$ are integers with $a \leq b$, we define $[a,b]$ to be $\{a,a+1,\ldots,b\}$. We next define some notation for certain types of 3-uniform hypergraphs.

Let $U_1, U_2, U_3$ be pairwise disjoint sets. The hypergraph with vertex set $U_1 \cup U_2 \cup U_3$ and edge set consisting of all 3-element sets having exactly one vertex in each of $U_1, U_2, U_3$ is denoted by $K_{U_1,U_2,U_3}^{(3)}$. The hypergraph with vertex set $U_1 \cup U_2$ and edge set consisting of all 3-element sets having at most 2 vertices in each of $U_1, U_2$ is denoted by $L_{U_1,U_2}^{(3)}$. If $|U_i| = u_i$ for $i \in \{1, 2, 3\}$, we may use $K_{u_1,u_2,u_3}^{(3)}$ or $L_{u_1,u_2}^{(3)}$ to denote any hypergraph that is isomorphic to $K_{U_1,U_2,U_3}^{(3)}$ or $L_{U_1,U_2}^{(3)}$, respectively. From a hypergraph decomposition perspective, we note that if $U_1, U_1', U_2, U_3$ are pairwise vertex disjoint, then

$$E(K_{U_1,U_2,U_3}^{(3)}) = E(K_{U_1',U_2',U_3}^{(3)}) \cup E(K_{U_1',U_2,U_3}^{(3)}).$$

Thus, for any positive integer $x$, it is simple to see that $K_{u_1,u_2,u_3}^{(3)}$ decomposes $K_{u_1+r,u_2,u_3}^{(3)}$ and, in general, $K_{u_1+r,x,u_3}^{(3)}$ decomposes into one copy of $K_{u_1,u_2,u_3}^{(3)}$ and one copy of $K_{x,u_2,u_3}^{(3)}$.

2. MAIN CONSTRUCTIONS

The constructions in this section are dependent on many small examples. These examples are given in the last section. Throughout, we will often identify a hypergraph (e.g., a leave in a packing) with its edge set only. Since the hypergraphs presented here do not contain isolated vertices, this will uniquely define them.

We begin by proving a lemma that is fundamental to our constructions.

**Lemma 2.1.** Let $n \geq 1$, $x \geq 0$, and $r \geq 0$ be integers and let $v = nx + r$. There exists a decomposition of $K_v^{(3)}$ into:

(i) 1 copy of $K_{n+r}^{(3)}$,
(ii) $x - 1$ copies of $K_{n+r}^{(3)} \setminus K_r^{(3)}$ (these are isomorphic to $K_{n+r}^{(3)}$ if $r \in [0, 2]$),
(iii) $\binom{x}{2}$ copies of $K_{r}^{(3)} \cup L_{n}^{(3)}$ (here $K_{r}^{(3)}$ is empty if $r = 0$), and
(iv) $\binom{x}{3}$ copies of $K_{n}^{(3)}$.
Proof. If $x \in \{0, 1\}$, the decomposition is trivial. Thus we may assume that $x \geq 2$.

Let $V_0, V_1, \ldots, V_x$ be pairwise disjoint sets of vertices with $|V_0| = r$, $|V_1| = |V_2| = \ldots = |V_x| = n$ and let $V = V_0 \cup V_1 \cup \ldots \cup V_x$. Then, $K^{(3)}_V$ can be viewed as the (edge-disjoint) union

$$K^{(3)}_{V_1 \cup V_0} \cup \bigcup_{2 \leq i \leq x} \left( K^{(3)}_{V_i \cup V_0} \setminus K^{(3)}_{V_0} \right) \cup \bigcup_{1 \leq i < j \leq x} \left( K^{(3)}_{V_i \cup V_j} \cup L^{(3)}_{V_i, V_j} \right) \cup \bigcup_{1 \leq i < j < k \leq x} \left( K^{(3)}_{V_i \cup V_j \cup V_k} \right)$$

Thus the result follows.

If $LC^{(3)}_3$ decomposes $K^{(3)}_v$, then we must have $3 \mid \binom{v}{3}$ and hence $18 \mid v(v-1)(v-2)$. Therefore we have $v \equiv 0, 1, \text{ or } 2 \pmod{9}$. In [6], it is shown that these necessary conditions are sufficient. Although a proof of Theorem 2.2 is given in [6], we include a proof here for the sake of completeness.

**Theorem 2.2.** There exists an $LC^{(3)}_3$-decomposition of $K_v$ if and only if $v \equiv 0, 1, \text{ or } 2 \pmod{9}$.

Proof. Let $v = 9x + r$, where $r \in [0, 2]$. If $x = 0$, the result is vacuously true. If $x = 1$, we give $LC^{(3)}_3$-decompositions of $K^{(3)}_9$ in Example 3.1, of $K^{(3)}_{10}$ in Example 3.2, and of $K^{(3)}_{11}$ in Example 3.3. Thus we may assume that $x \geq 2$. By Lemma 2.1, it suffices to give $LC^{(3)}_3$-decompositions of $K^{(3)}_v$ of $K^{(3)}_{9+r}$, which is isomorphic to $K^{(3)}_9$, since $r \in [0, 2]$, of $K^{(3)}_{9,9} \cup L^{(3)}_{9,9}$, and of $K^{(3)}_{9,9,9}$. A decomposition of $K^{(3)}_{9,9} \cup L^{(3)}_{9,9}$ is given in Example 3.7, and a decomposition of $L^{(3)}_{9,9}$ is given in Example 3.6. Since $K^{(3)}_{2,3,3}$ decomposes $K^{(3)}_{9,9}$ and $K^{(3)}_{1,3,3}$ decomposes $K^{(3)}_{9,9,9}$, and since $LC^{(3)}_3$-decompositions of $K^{(3)}_{3,3,3}$ of $K^{(3)}_{9,9,9}$ are given in Examples 3.4 and 3.5, we have that $LC^{(3)}_3$ decomposes both $K^{(3)}_{2,9,9}$ and $K^{(3)}_{9,9,9}$. Thus the result follows.

Next, we give our main result on maximum $LC^{(3)}_3$-packings of $K^{(3)}_v$.

**Theorem 2.3.** If $v \geq 6$ is an integer, then there exists a maximum $LC^{(3)}_3$-packing of $K^{(3)}_v$ where the leave has two or fewer edges.

Proof. If $v \equiv 0, 1, \text{ or } 2 \pmod{9}$, then the result follows from the $LC^{(3)}_3$-decomposition result in Theorem 2.2, which translates to a maximum $LC^{(3)}_3$-packing with an empty leave. If $v \in [6, 8]$, a maximum $LC^{(3)}_3$-packing of $K^{(3)}_v$ with a two edge leave is given in Examples 3.13–3.15. Hence, we need only consider when $v = 9x + r$ where $x \geq 1$ and $r \in [3, 8]$. By Lemma 2.1 it suffices to find

(i) a maximum $LC^{(3)}_3$-packing of $K^{(3)}_{9+r}$ with a leave consisting of two or fewer edges and

(ii) $LC^{(3)}_3$-decompositions of $K^{(3)}_{9+r} \setminus K^{(3)}_9$, $K^{(3)}_{9,9} \cup L^{(3)}_{9,9}$, and $K^{(3)}_{9,9,9}$.

We note that an $LC^{(3)}_3$-decomposition of $K^{(3)}_{12} \setminus K^{(3)}_3$ is equivalent an $LC^{(3)}_3$-packing of $K^{(3)}_{12}$ with a leave consisting of the single edge in the hole, which is given
in Example 3.16. Also, for \( r \geq 3 \), it is simple to see that \( K^{(3)}_{r,9,9} \) is decomposable into copies of \( K^{(3)}_{2,3,3} \) and \( K^{(3)}_{3,3,3} \). Maximum \( LC_3^{(3)} \)-packings (with leaves of two or fewer edges) of \( K^{(3)}_{9+r} \), for \( r \in [3,8] \), are given in Examples 3.16–3.21. Similarly, \( LC_3^{(3)} \)-decompositions of \( K^{(3)}_{9+r} \setminus K^{(3)}_{r} \), for \( r \in [4,8] \), are given in Examples 3.8–3.12. Finally, an \( LC_3^{(3)} \)-decomposition of \( L^{(3)}_{9,9} \) is given in Example 3.6, and \( LC_3^{(3)} \)-decompositions of \( K^{(3)}_{2,3,3} \) and of \( K^{(3)}_{3,3,3} \) are given in Examples 3.4 and 3.5, respectively.

Next, we give a lemma on maximum \( LC_3^{(3)} \)-packings of \( 2K^{(3)}_v \) for \( v \in [6,17] \).

**Lemma 2.4.** If \( v \in [6,17] \), then there exists a maximum \( LC_3^{(3)} \)-packing of \( 2K^{(3)}_v \) where the leave has two or fewer edges.

**Proof.** Let \( V(2K^{(3)}_v) = Z_v \). If \( v \in [9,11] \), there exists an \( LC_3^{(3)} \)-decomposition of \( K^{(3)}_v \) and hence of \( 2K^{(3)}_v \). Next, if \( v \in [6,8] \cup [15,17] \), let \( \Delta_1 \) be a maximum \( LC_3^{(3)} \)-packing of \( K^{(3)}_v \) where the leave has edge set \( \{0,1,2\}, \{2,3,4\} \) (which exists by Examples 3.13–3.15 and Examples 3.19–3.21) and let \( \Delta_2 \) be another maximum \( LC_3^{(3)} \)-packing of \( K^{(3)}_v \) where the leave has edge set \( \{4,5,0\}, \{0,1,2\} \). Then \( \Delta_1 \cup \Delta_2 \cup \{H[0,1,2,3,4,5]\} \) is a maximum \( LC_3^{(3)} \)-packing of \( 2K_v^{(3)} \) where \( \{0,1,2\} \) is the only edge in the leave. Finally, if \( v \in [12,14] \), let \( \Delta_1 \) be a maximum packing of \( K^{(3)}_v \) where \( \{0,1,2\} \) is the only edge in the leave (which exists by Examples 3.16–3.18) and let \( \Delta_2 \) be a maximum \( LC_3^{(3)} \)-packing of \( K^{(3)}_v \) where \( \{2,3,4\} \) is the only edge in the leave. Then \( \Delta_1 \cup \Delta_2 \) is a maximum \( LC_3^{(3)} \)-packing of \( 2K^{(3)}_v \) where the leave has edge set \( \{0,1,2\}, \{2,3,4\} \).

Now we extend our results to maximum \( LC_3^{(3)} \)-packings of \( 2K^{(3)}_v \) in general.

**Theorem 2.5.** If \( v \geq 6 \), then there exists a maximum \( LC_3^{(3)} \)-packing of \( 2K^{(3)}_v \) where the leave has two or fewer edges.

**Proof.** If \( v \equiv 0, 1, \text{ or } 2 \pmod{9} \), then the result follows from Theorem 2.2, which translates to a maximum \( LC_3^{(3)} \)-packing with an empty leave. If \( v \in [6,8] \), a maximum \( LC_3^{(3)} \)-packing of \( 2K^{(3)}_v \) with a one edge leave is given in Lemma 2.4. Hence, we need only consider when \( v \equiv r \pmod{9}, r \geq 3, v \geq 12 \). Let \( v = 9x + r \) where \( x \geq 1 \) and \( r \in [3,8] \). By Lemma 2.1 it suffices to find

1. a maximum \( LC_3^{(3)} \)-packing of \( 2K^{(3)}_{9+r} \) with a leave consisting of two or fewer edges and
2. \( LC_3^{(3)} \)-decompositions of \( 2K^{(3)}_{9+r} \setminus K^{(3)}_r, 2K^{(3)}_{r,9,9} \cup 2L^{(3)}_9, \text{ and } 2K^{(3)}_{9,9,9} \).

But since \( LC_3^{(3)} \) decomposes \( K^{(3)}_{9+r} \setminus K^{(3)}_r, K^{(3)}_{r,9,9} \cup L^{(3)}_9, \text{ and } K^{(3)}_{9,9,9} \) (see argument in proof of Theorem 2.3), \( LC_3^{(3)} \) decomposes the 2-fold versions of these hypergraphs. Maximum \( LC_3^{(3)} \)-packings (with leaves of two or fewer edges) of \( 2K^{(3)}_{9+r} \), for \( r \in [3,8] \), are given in Lemma 2.4. The result now follows.

Next, we give a lemma on LC\((3)\)-decompositions of \(3\)\(K_v(3)\) for \(v \in [6, 17]\).

**Lemma 2.6.** If \(v \in [6, 17]\), then there exists an LC\((3)\)-decomposition of \(3\)\(K_v(3)\).

**Proof.** Let \(V\left(3\right)K_v(3)\) = \(Z_v\). If \(v \in [9, 11]\), there exists an LC\((3)\)-decomposition of \(K_v(3)\) and hence of \(3\)\(K_v(3)\). Next, if \(v \in [6, 8] \cup [15, 17]\), let \(\Delta_1\) be a maximum packing of \(K_v(3)\) where the leave has edge set \(\{\{2, 3, 4\}, \{4, 5, 0\}\}\) (which exists by Examples 3.13–3.15 and Examples 3.19–3.21) and let \(\Delta_2\) be a maximum LC\((3)\)-packing of \(3\)\(K_v(3)\) where \(\{0, 1, 2\}\) is the only edge in the leave (which exists by Lemma 2.4). Then \(\Delta_1 \cup \Delta_2 \cup \{H[0, 1, 2, 3, 4, 5]\}\) is an LC\((3)\)-decomposition of \(3\)\(K_v(3)\). Finally, if \(v \in [12, 14]\), let \(\Delta_1\) be a maximum packing of \(K_v(3)\) where \(\{4, 5, 0\}\) is the only edge in the leave (which exists by Examples 3.16–3.18) and let \(\Delta_2\) be a maximum LC\((3)\)-packing of \(3\)\(K_v(3)\) where the leave has edge set \(\{\{0, 1, 2\}, \{2, 3, 4\}\}\) (which exists by Lemma 2.4). Then \(\Delta_1 \cup \Delta_2 \cup \{H[0, 1, 2, 3, 4, 5]\}\) is an LC\((3)\)-decomposition of \(3\)\(K_v(3)\).

It is simple to see that if there is an LC\((3)\)-decomposition of \(\lambda\)\(K_v(3)\), then we must have \(v \not\in [3, 5]\) and either \(\lambda \equiv 0 \pmod{3}\) or \(v \equiv 0, 1, 2 \pmod{9}\). Thus, in light of Theorem 2.2 and Lemmas 2.1 and 2.6 and because \(3\)\(K_v(3)\) decomposes \(3\)\(K_v(3)\) for all positive integers \(k\), we have the following obvious corollary.

**Corollary 2.7.** Let \(\lambda\) and \(v \not\in [3, 5]\) be positive integers. There exists an LC\((3)\)-decomposition of \(\lambda\)\(K_v(3)\) if and only if \(\lambda \equiv 0 \pmod{3}\) or \(v \equiv 0, 1, 2 \pmod{9}\).

Finally we give our general main result.

**Theorem 2.8.** If \(\lambda\) and \(v \not\in [3, 5]\) are positive integers, then there exists a maximum LC\((3)\)-packing of \(\lambda\)\(K_v(3)\) where the leave has two or fewer edges.

**Proof.** If \(\lambda \in \{1, 2\}\), the result follows from Theorems 2.3 and 2.5. If \(\lambda \equiv 0 \pmod{3}\), the result follows from Corollary 2.7. Suppose \(\lambda \geq 4\) and let \(\lambda = 3b + r\) for integers \(b \geq 1\) and \(r \in \{1, 2\}\). We can view \(\lambda\)\(K_v(3)\) as the edge disjoint union of \(3b\)\(K_v(3)\) and \(\lambda\)\(K_v(3)\). An LC\((3)\)-decomposition of \(3b\)\(K_v(3)\) exists by Corollary 2.7 and a maximum LC\((3)\)-packing of \(\lambda\)\(K_v(3)\) where the leave has two or fewer edges follows from Theorems 2.3 and 2.5. Thus the result follows.

3. SMALL EXAMPLES

**Example 3.1.** Let
\[
V\left(3\right)K_v(3) = \mathbb{Z}_7 \cup \{\infty_1, \infty_2\}
\]
and let
\[
B = \{H[0, 1, 2, 4, 6, 3], H[\infty_1, 0, 1, \infty_2, 2, 4], H[0, 1, 3, \infty_2, 6, 4], H[\infty_1, 0, 3, 1, \infty_2, 2]\}.
\]
Then an LC\((3)\)-decomposition of \(3\)\(K_v(3)\) consists of the LC\((3)\)-blocks in \(B\) under the action of the map \(\infty_i \mapsto \infty_i\) and \(j \mapsto j + 1 \pmod{7}\).
Example 3.2. Let
\[ V(K_{10}^{(3)}) = \mathbb{Z}_{10} \]
and let
\[ B = \{ H[0, 2, 1, 3, 4, 9], H[0, 7, 1, 2, 4, 5], H[0, 4, 2, 5, 7, 6], H[0, 6, 2, 4, 7, 3] \}. \]
Then an \( LC_{3}^{(3)} \)-decomposition of \( K_{10}^{(3)} \) consists of the \( LC_{3}^{(3)} \)-blocks in \( B \) under the action of the map \( j \mapsto j + 1 \pmod{10} \).

Example 3.3. Let
\[ V(K_{11}^{(3)}) = \mathbb{Z}_{11} \]
and let
\[ B = \{ H[0, 8, 2, 6, 9, 1], H[0, 8, 1, 4, 7, 2], H[1, 0, 5, 3, 8, 9], H[0, 7, 1, 10, 5, 6], H[0, 3, 1, 8, 10, 9] \}. \]
Then an \( LC_{3}^{(3)} \)-decomposition of \( K_{11}^{(3)} \) consists of the \( LC_{3}^{(3)} \)-blocks in \( B \) under the action of the map \( j \mapsto j + 1 \pmod{11} \).

Example 3.4. Let
\[ V(K_{2,3,3}^{(3)}) = \mathbb{Z}_{6} \cup \{ \infty_1, \infty_2 \} \]
with the vertex partition \( \{ \{ \infty_1, \infty_2 \}, \{ 0, 2, 4 \}, \{ 1, 3, 5 \} \} \) and let
\[ B = \{ H[\infty_1, 0, 1, \infty_2, 2, 5], H[5, \infty_2, 0, 1, \infty_1, 2, 5] \}. \]
Then an \( LC_{3}^{(3)} \)-decomposition of \( K_{2,3,3}^{(3)} \) consists of the \( LC_{3}^{(3)} \)-blocks in \( B \) under the action of the map \( \infty_1 \mapsto \infty_1 \) and \( j \mapsto j + 2 \pmod{6} \).

Example 3.5. Let
\[ V(K_{3,3,3}^{(3)}) = \mathbb{Z}_{9} \]
with vertex partition \( \{ \{ 0, 3, 6 \}, \{ 1, 4, 7 \}, \{ 2, 5, 8 \} \} \) and let
\[ B = \{ H[7, 3, 2, 1, 0, 5] \}. \]
Then an \( LC_{3}^{(3)} \)-decomposition of \( K_{3,3,3}^{(3)} \) consists of the \( LC_{3}^{(3)} \)-block in \( B \) under the action of the map \( j \mapsto j + 1 \pmod{9} \).

Example 3.6. Let
\[ V(L_{9,9}^{(3)}) = \mathbb{Z}_{18} \]
with vertex partition \( \{ \{ 0, 2, \ldots, 16 \}, \{ 1, 3, \ldots, 17 \} \} \), and let
\[ B = \{ H[0, 16, 1, 4, 15, 2], H[14, 0, 11, 16, 9], H[4, 0, 9, 1, 14, 3], H[9, 0, 1, 6, 17, 2], H[0, 5, 1, 12, 15, 14], H[0, 1, 2, 5, 3, 15], H[0, 1, 15, 10, 4, 13], H[1, 5, 12, 7, 0, 13], H[0, 1, 12, 3, 9, 16], H[10, 2, 17, 8, 1, 0], H[1, 0, 8, 2, 17, 4], H[1, 4, 7, 12, 6, 16] \}. \]
Then an \( LC_{3}^{(3)} \)-decomposition of \( L_{9,9}^{(3)} \) consists of the \( LC_{3}^{(3)} \)-blocks in \( B \) under the action of the map \( j \mapsto j + 1 \) (mod 18).

**Example 3.7.** Let
\[
V \left( L_{9,9}^{(3)} \cup K_{1,9,9}^{(3)} \right) = \mathbb{Z}_{18} \cup \{ \infty \}
\]
with vertex partition \( \{ \{ \infty \}, \{ 0, 2, \ldots, 16 \}, \{ 1, 3, \ldots, 17 \} \} \), and let
\[
B = \{ H[0, 16, 1, 4, 15, 2], H[14, 0, 1, 11, 16, 9], H[4, 0, 9, 1, 14, 3], H[9, 0, 1, 6, 17, 2],
\]
\[
H[0, 5, 1, 12, 15, 14], H[0, 1, 2, \infty, 3, 15], H[0, 1, 15, 10, 4, 13], H[1, 5, 12, 7, 0, 13],
\]
\[
H[0, \infty, 3, 12, 9, 16], H[14, 7, 17, 8, 1, \infty], H[1, \infty, 8, 2, 17, 4], H[1, 0, 7, 12, 6, 16] \},
\]
\[
B' = \{ H[0, \infty, 9, 10, 1, 3], H[1, \infty, 10, 11, 2, 4], H[2, \infty, 11, 12, 3, 5],
\]
\[
H[3, \infty, 12, 13, 4, 6], H[4, \infty, 13, 14, 5, 7], H[5, \infty, 14, 15, 6, 8],
\]
\[
H[6, \infty, 15, 16, 7, 9], H[7, \infty, 16, 17, 8, 10], H[8, \infty, 17, 0, 9, 11],
\]
\[
H[10, 0, 1, 2, 9, 3], H[11, 1, 2, 3, 10, 4], H[12, 2, 3, 4, 11, 5],
\]
\[
H[13, 3, 4, 5, 12, 6], H[14, 4, 5, 6, 13, 7], H[15, 5, 6, 7, 14, 8],
\]
\[
H[16, 6, 7, 8, 15, 9], H[17, 7, 8, 9, 16, 10], H[0, 8, 9, 10, 17, 11],
\]
\[
H[12, 9, 10, 11, 0, 1], H[13, 10, 11, 12, 1, 2], H[14, 11, 12, 13, 2, 3],
\]
\[
H[15, 12, 13, 14, 3, 4], H[16, 13, 14, 15, 4, 5], H[17, 14, 15, 16, 5, 6],
\]
\[
H[0, 15, 16, 17, 6, 7], H[1, 16, 17, 0, 7, 8], H[2, 17, 0, 1, 8, 9] \}.
\]
Then an \( LC_{3}^{(3)} \)-decomposition of \( L_{9,9}^{(3)} \cup K_{1,9,9}^{(3)} \) consists of the \( LC_{3}^{(3)} \)-blocks in \( B \) under the action of the map \( \infty \mapsto \infty \) and \( j \mapsto j + 1 \) (mod 18), along with the \( LC_{3}^{(3)} \)-blocks in \( B' \).

**Example 3.8.** Let
\[
V \left( K_{1,13}^{(3)} \setminus K_{1}^{(3)} \right) = \mathbb{Z}_{9} \cup \{ \infty_{1}, \infty_{2}, \infty_{3}, \infty_{4} \}
\]
with \( \infty_{1}, \ldots, \infty_{4} \) being the vertices in the hole and let
\[
B = \{ H[\infty_{4}, 0, \infty_{2}, 1, 4, 8], H[\infty_{0}, \infty_{3}, 1, 4, 8], H[\infty_{3}, 0, \infty_{4}, 1, 4, 8],
\]
\[
H[\infty_{4}, 0, \infty_{1}, 1, 4, 8], H[\infty_{1}, 0, \infty_{3}, 1, 2, 3], H[\infty_{2}, 0, \infty_{4}, 1, 2, 3],
\]
\[
H[0, \infty_{1}, 3, \infty_{2}, 6, \infty_{3}], H[0, 1, 4, 8, 3, \infty_{4}], H[0, 2, 4, 6, 1, 8] \},
\]
\[
B' = \{ H[0, 2, 8, 5, 3, 6], H[1, 3, 0, 6, 4, 7], H[2, 4, 1, 7, 5, 8], H[5, 2, 3, 6, 4, 7],
\]
\[
H[8, 5, 6, 0, 7, 1], H[6, 2, 8, 4, 1, 3], H[7, 3, 0, 5, 2, 4], H[6, 1, 0, 4, 3, 7],
\]
\[
H[7, 2, 1, 5, 4, 8], H[8, 3, 2, 6, 5, 0], H[0, 7, 1, 8, 2, 3], H[3, 1, 4, 2, 5, 6],
\]
\[
H[6, 4, 7, 5, 8, 0] \}.
\]
Then an \( LC_{3}^{(3)} \)-decomposition of \( K_{1,13}^{(3)} \setminus K_{1}^{(3)} \) consists of the \( LC_{3}^{(3)} \)-blocks in \( B \) under the action of the map \( \infty_{i} \mapsto \infty_{i} \) and \( j \mapsto j + 1 \) (mod 9) along with the \( LC_{3}^{(3)} \)-blocks in \( B' \).
Example 3.9. Let
\[ V(K_{14}^{(3)} \setminus K_9^{(3)}) = \mathbb{Z}_9 \cup \{\infty_1, \infty_2, \infty_3, \infty_4, \infty_5\} \]
with \(\infty_1, \ldots, \infty_5\) being the vertices in the hole and let
\[
B = \{H[\infty_1, 0, \infty_2, 1, 2, 6], H[\infty_2, 0, \infty_3, 1, 2, 6], H[\infty_3, 0, \infty_4, 1, 2, 6],
    H[\infty_4, 0, \infty_5, 1, 2, 6], H[\infty_5, 0, \infty_1, 1, 2, 6],
    H[\infty_1, 0, \infty_3, 1, 3, 6],
    H[\infty_2, 0, \infty_4, 1, 3, 6], H[\infty_3, 0, \infty_5, 1, 3, 6], H[\infty_4, 0, \infty_1, 1, 3, 6],
    H[\infty_5, 0, \infty_2, 1, 3, 6], H[0, 2, 4, 6, 1, 8], H[1, 0, 4, 5, 2, 6]\},
B' = \{H[0, 2, 8, 5, 3, 6], H[1, 3, 0, 6, 4, 7], H[2, 4, 1, 7, 5, 8], H[5, 2, 3, 6, 4, 7],
    H[8, 5, 6, 0, 7, 1], H[6, 2, 8, 4, 1, 3], H[7, 3, 0, 5, 2, 4], H[0, 7, 1, 8, 2, 3],
    H[3, 1, 4, 2, 5, 6], H[6, 4, 7, 5, 8, 0]\}.
\]
Then an \(LC_3^{(3)}\)-decomposition of \(K_{14}^{(3)} \setminus K_9^{(3)}\) consists of the \(LC_3^{(3)}\)-blocks in \(B\) under the action of the map \(\infty_i \mapsto \infty_i\) and \(j \mapsto j + 1 \pmod{9}\) along with the \(LC_3^{(3)}\)-blocks in \(B'\).

Example 3.10. Let
\[ V(K_{15}^{(3)} \setminus K_6^{(3)}) = \mathbb{Z}_9 \cup \{\infty_1, \infty_2, \infty_3, \infty_4, \infty_5, \infty_6\} \]
with \(\infty_1, \ldots, \infty_6\) being the vertices in the hole and let
\[
B = \{H[\infty_1, 0, \infty_2, 1, \infty_3, 2], H[\infty_3, 0, \infty_4, 1, \infty_5, 2], H[\infty_5, 0, \infty_6, 1, \infty_1, 2],
    H[\infty_2, 0, \infty_4, 1, \infty_6, 2], H[\infty_1, 0, 4, \infty_2, 3, 6], H[\infty_2, 0, 4, \infty_3, 3, 6],
    H[\infty_3, 0, 4, \infty_4, 3, 6], H[\infty_4, 0, 4, \infty_5, 3, 6], H[\infty_5, 0, 4, \infty_6, 3, 6],
    H[\infty_6, 0, 4, \infty_1, 3, 6], H[0, 2, \infty_1, 1, \infty_4, 7], H[0, 2, \infty_2, 1, \infty_5, 7],
    H[0, 2, \infty_3, 1, \infty_6, 7], H[0, 2, 4, 6, 1, 8], H[1, 0, 4, 5, 2, 6]\},
B' = \{H[0, 2, 8, 5, 3, 6], H[1, 3, 0, 6, 4, 7], H[2, 4, 1, 7, 5, 8], H[5, 2, 3, 6, 4, 7],
    H[8, 5, 6, 0, 7, 1], H[6, 2, 8, 4, 1, 3], H[7, 3, 0, 5, 2, 4], H[0, 7, 1, 8, 2, 3],
    H[3, 1, 4, 2, 5, 6], H[6, 4, 7, 5, 8, 0]\}.
\]
Then an \(LC_3^{(3)}\)-decomposition of \(K_{15}^{(3)} \setminus K_6^{(3)}\) consists of the \(LC_3^{(3)}\)-blocks in \(B\) under the action of the map \(\infty_i \mapsto \infty_i\) and \(j \mapsto j + 1 \pmod{9}\) along with the \(LC_3^{(3)}\)-blocks in \(B'\).

Example 3.11. Let
\[ V(K_{16}^{(3)} \setminus K_7^{(3)}) = \mathbb{Z}_9 \cup \{\infty_1, \infty_2, \infty_3, \infty_4, \infty_5, \infty_6, \infty_7\} \]
with \(\infty_1, \ldots, \infty_7\) being the vertices in the hole and let
Then an \( LC_3^{(3)} \)-decomposition of \( K_{17}^{(3)} \setminus K_8^{(3)} \) consists of the \( LC_3^{(3)} \)-blocks in \( B \) under the action of the map \( \infty_i \mapsto \infty_i \) and \( j \mapsto j + 1 \) (mod 9) along with the \( LC_3^{(3)} \)-blocks in \( B' \).

**Example 3.12.** Let

\[
V\left( K_{17}^{(3)} \setminus K_8^{(3)} \right) = \mathbb{Z}_9 \cup \{ \infty_1, \infty_2, \infty_3, \infty_4, \infty_5, \infty_6, \infty_7, \infty_8 \}
\]

with \( \infty_1, \ldots, \infty_8 \) being the vertices in the hole and let

\[
B = \{ H[\infty_1, 0, \infty_2, 1, \infty_4, 2], H[\infty_2, 0, \infty_3, 1, \infty_5, 2], H[\infty_3, 0, \infty_4, 1, \infty_6, 2], H[\infty_4, 0, \infty_5, 1, \infty_7, 2], H[\infty_5, 0, \infty_6, 1, \infty_1, 2], H[\infty_6, 0, \infty_7, 1, \infty_2, 2], H[\infty_7, 0, \infty_8, 1, \infty_3, 2], H[\infty_8, 0, \infty_1, 1, \infty_4, 3, 6], H[\infty_2, 0, 4, \infty_3, 3, 6], H[\infty_3, 0, 4, \infty_4, 3, 6], H[\infty_4, 0, 4, \infty_5, 3, 6], H[\infty_5, 0, 4, \infty_6, 3, 6], H[\infty_6, 0, 4, \infty_7, 3, 6], H[\infty_7, 0, 4, \infty_8, 3, 6], H[\infty_8, 0, 4, \infty_1, 3, 6], H[\infty_1, 0, 2, 4, \infty_5, 1], H[\infty_2, 0, 2, 4, \infty_6, 1], H[\infty_3, 0, 2, 4, \infty_7, 1], H[\infty_4, 0, 2, 4, \infty_8, 1], H[0, 2, 4, 6, 1, 8], H[1, 0, 4, 5, 2, 6], H[0, 2, 8, 5, 3, 6], H[1, 3, 0, 6, 4, 7], H[2, 4, 1, 7, 5, 8], H[5, 2, 3, 6, 4, 7], H[8, 5, 6, 0, 7, 1], H[6, 2, 8, 4, 1, 3], H[7, 3, 0, 5, 2, 4], H[0, 7, 1, 8, 2, 3], H[3, 1, 4, 2, 5, 6], H[6, 4, 7, 5, 8, 0] \}\.

Then an \( LC_3^{(3)} \)-decomposition of \( K_{17}^{(3)} \setminus K_8^{(3)} \) consists of the \( LC_3^{(3)} \)-blocks in \( B \) under the action of the map \( \infty_i \mapsto \infty_i \) and \( j \mapsto j + 1 \) (mod 9) along with the \( LC_3^{(3)} \)-blocks in \( B' \).

**Example 3.13.** Let

\[
V\left( K_6^{(3)} \right) = \mathbb{Z}_6
\]

and let

\[
B = \{ H[4, 0, 1, 5, 3, 2], H[5, 1, 2, 0, 4, 3], H[4, 5, 0, 3, 2, 1], H[5, 0, 1, 4, 3, 2], H[0, 1, 2, 5, 4, 3], H[1, 2, 3, 0, 5, 4] \}.
\]
Then $B$ is a maximum $LC_3^{(3)}$-packing of $K_6^{(3)}$, where the leave has edge set \{\{0,1,3\}, \{1,2,5\}\}. Note that by renaming the vertices in this packing, any two hyperedges in $K_6^{(3)}$ that intersect in a single vertex can be made into the edge set of the leave of a maximum $LC_3^{(3)}$-packing of $K_6^{(3)}$.

**Example 3.14.** Let

$$V\left(K_7^{(3)}\right) = \mathbb{Z}_7$$

and let

$$B = \{H[0,1,2,4,6,3]\},$$

$$B' = \{H[3,0,1,4,2,5], H[6,3,4,0,5,1], H[1,6,2,0,3,4], H[4,2,5,3,6,0]\}.$$ 

Then a maximum $LC_3^{(3)}$-packing of $K_7^{(3)}$, where the leave has edge set \{\{0,1,5\}, \{0,2,6\}\}, consists of the $LC_3^{(3)}$-blocks in $B$ under the action of the map $j \mapsto j + 1 \pmod{7}$ along with the $LC_3^{(3)}$-blocks in $B'$. Again, we note that by renaming the vertices in this packing, any two hyperedges in $K_7^{(3)}$ that intersect in a single vertex can be made into the edge set of a maximum $LC_3^{(3)}$-packing of $K_7^{(3)}$.

**Example 3.15.** Let

$$V\left(K_8^{(3)}\right) = \mathbb{Z}_8$$

and let

$$B = \{H[6,0,7,2,3,1], H[0,2,6,7,4,1]\},$$

$$B' = \{H[3,0,1,4,2,5], H[6,3,4,7,5,0]\}.$$ 

Then a maximum $LC_3^{(3)}$-packing of $K_8^{(3)}$, where the leave has edge set \{\{1,6,7\}, \{0,2,7\}\}, consists of the $LC_3^{(3)}$-blocks in $B$ under the action of the map $j \mapsto j + 1 \pmod{8}$ along with the $LC_3^{(3)}$-blocks in $B'$. Again, we note that by renaming the vertices in this packing, any two hyperedges in $K_8^{(3)}$ that intersect in a single vertex can be made into the edge set of a maximum $LC_3^{(3)}$-packing of $K_8^{(3)}$.

**Example 3.16.** Let

$$V\left(K_{12}^{(3)}\right) = \mathbb{Z}_{11} \cup \{\infty\}$$

and let

$$B = \{H[0,\infty,1,3,2,4], H[0,\infty,2,8,5,10], H[0,\infty,3,7,4,8], H[0,\infty,4,9,5,2], H[0,\infty,5,10,9,6], H[0,6,2,9,4,3]\},$$

$$B' = \{H[3,0,1,4,2,5], H[6,3,4,7,5,8], H[9,6,7,10,8,0], H[1,10,2,0,3,4], H[4,2,5,3,6,7], H[8,6,9,1,10,0], H[9,7,10,2,0,1]\}.$$
Then a maximum $LC_3^{(3)}$-packing of $K_{13}^{(3)}$, where the leave is the single edge $\{5, 7, 8\}$, consists of the $LC_3^{(3)}$-blocks in $B$ under the action of the map $\infty \mapsto \infty$ and $j \mapsto j + 1 \pmod{11}$ along with the $LC_3^{(3)}$-blocks in $B'$. Note that by renaming the vertices in this packing, any edge in $K_{14}^{(3)}$ can be made into the leave of a maximum $LC_3^{(3)}$-packing of $K_{14}^{(3)}$.

**Example 3.17.** Let

$$V(K_{13}^{(3)}) = \mathbb{Z}_{13}$$

and let

$$B = \{H[0, 3, 9, 12, 1, 11], H[0, 4, 8, 12, 1, 10], H[12, 4, 9, 0, 1, 7], H[12, 5, 8, 0, 1, 6],$$

$$H[7, 10, 4, 11, 0, 1], H[6, 10, 5, 2, 0, 1], H[4, 2, 3, 5, 1, 0]\},$$

$$B' = \{H[3, 0, 1, 4, 2, 5], H[6, 3, 4, 7, 5, 8], H[9, 6, 7, 10, 8, 11], H[12, 9, 10, 0, 11, 1]\}.$$

Then a maximum $LC_3^{(3)}$-packing of $K_{13}^{(3)}$, where the leave is the single edge $\{0, 2, 12\}$, consists of the $LC_3^{(3)}$-blocks in $B$ under the action of the map $j \mapsto j + 1 \pmod{13}$ along with the $LC_3^{(3)}$-blocks in $B'$. Again, we note that by renaming the vertices in this packing, any edge in $K_{13}^{(3)}$ can be made into the leave of a maximum $LC_3^{(3)}$-packing of $K_{13}^{(3)}$.

**Example 3.18.** Let

$$V(K_{14}^{(3)}) = \mathbb{Z}_{14}$$

and let

$$B = \{H[13, 0, 8, 3, 5, 12], H[0, 10, 1, 3, 5, 12], H[0, 8, 2, 3, 4, 13], H[0, 1, 11, 5, 6, 3],$$

$$H[0, 3, 8, 2, 9, 4], H[0, 2, 9, 13, 7, 3], H[0, 3, 10, 5, 8, 4]H[0, 4, 1, 7, 12, 8]\},$$

$$B' = \{H[3, 0, 1, 4, 2, 5], H[6, 3, 4, 7, 5, 8], H[9, 6, 7, 10, 8, 11], H[12, 9, 10, 13, 11, 0],$$

$$H[1, 13, 2, 0, 3, 4], H[4, 2, 5, 3, 6, 7], H[7, 5, 8, 6, 9, 10], H[11, 9, 12, 1, 13, 0],$$

$$H[12, 10, 13, 2, 0, 1]\}.$$

Then a maximum $LC_3^{(3)}$-packing of $K_{14}^{(3)}$, where the leave is the single edge $\{8, 10, 11\}$, consists of the $LC_3^{(3)}$-blocks in $B$ under the action of the map $j \mapsto j + 1 \pmod{14}$ along with the $LC_3^{(3)}$-blocks in $B'$. Again, we note that by renaming the vertices in this packing, any edge in $K_{14}^{(3)}$ can be made into the leave of a maximum $LC_3^{(3)}$-packing of $K_{14}^{(3)}$.

**Example 3.19.** Let

$$V(K_{15}^{(3)}) = \mathbb{Z}_{13} \cup \{\infty_1, \infty_2\}$$
and let

\[ B = \{ H[\infty, 0, \infty_2, 1, 2, 3], H[0, \infty_1, 2\infty_2, 4, 8], H[0, \infty_1, 3\infty_2, 6, 12], \]
\[ H[0, \infty_1, 4\infty_2, 8, 3], H[0, \infty_1, 5\infty_2, 10, 7], H[0, \infty_1, 6\infty_2, 12, 11], \]
\[ H[0, 9, 11, 8, 4, 10], H[0, 5, 2, 4, 8, 6], H[0, 10, 2, 4, 11, 6], \]
\[ H[0, 1, 4, 3, 8, 7], H[0, 1, 10, 9, 5, 6] \}, \]
\[ B' = \{ H[3, 0, 1, 4, 2, 5], H[6, 3, 4, 7, 5, 8], H[9, 6, 7, 10, 8, 11], H[12, 9, 10, 0, 11, 1], \]
\[ H[1, 12, 2, 0, 3, 4], H[4, 2, 5, 3, 6, 7], H[7, 5, 8, 6, 9, 10], H[10, 8, 11, 9, 12, 0] \}. \]

Then a maximum $LC_3^{(3)}$-packing of $K_{16}^{(3)}$, where the leave has edge set
\{ {0, 2, 14}, {2, 4, 5} \}, consists of the $LC_3^{(3)}$-blocks in $B$ under the action of the map
\[ \infty_i \mapsto \infty_i \text{ and } j \mapsto j + 1 \pmod{16} \] along with the $LC_3^{(3)}$-blocks in $B'$. Again, we note that by renaming the vertices in this packing, any two hyperedges in $K_{16}^{(3)}$ that intersect in a single vertex can be made into the edge set the leave of a maximum $LC_3^{(3)}$-packing of $K_{16}^{(3)}$.

**Example 3.20.** Let

\[ V(K_{16}^{(3)}) = \mathbb{Z}_{16} \]

and let

\[ B = \{ H[1, 0, 13, 10, 6, 4], H[0, 5, 10, 6, 11, 4], H[1, 14, 9, 3, 7, 13], \]
\[ H[0, 11, 1, 12, 3, 10], H[0, 10, 2, 14, 7, 6], H[13, 5, 12, 0, 4, 7], \]
\[ H[0, 3, 8, 15, 7, 14], H[15, 5, 14, 8, 6, 2], H[0, 13, 3, 8, 1, 6], \]
\[ H[1, 0, 4, 8, 3, 7], H[0, 14, 12, 10, 11, 13] \}, \]
\[ B' = \{ H[3, 0, 1, 4, 2, 5], H[6, 3, 4, 7, 5, 8], H[9, 6, 7, 10, 8, 11], \]
\[ H[12, 9, 10, 13, 11, 14], H[15, 12, 13, 0, 14, 1], H[1, 15, 2, 0, 3, 4], \]
\[ H[4, 2, 5, 3, 6, 7], H[7, 5, 8, 6, 9, 10], H[10, 8, 11, 9, 12, 13], \]
\[ H[13, 11, 14, 12, 15, 0] \}. \]

Then a maximum $LC_3^{(3)}$-packing of $K_{16}^{(3)}$, where the leave has edge set
\{ {0, 1, 14}, {0, 2, 15} \}, consists of the $LC_3^{(3)}$-blocks in $B$ under the action of the map
\[ j \mapsto j + 1 \pmod{16} \] along with the $LC_3^{(3)}$-blocks in $B'$. Again, we note that by renaming the vertices in this packing, any two hyperedges in $K_{16}^{(3)}$ that intersect in a single vertex can be made into the edge set the leave of a maximum $LC_3^{(3)}$-packing of $K_{16}^{(3)}$. 
Example 3.21. Let

\[ V(K_{17}^{(3)}) = \mathbb{Z}_{17} \]

and let

\[ B = \{ H[0,15,1,16,13,3], H[0,14,1,13,5,11], H[2,13,0,4,11,16], \\
    H[11,16,1,13,0,3], H[0,12,1,11,5,10], H[0,11,1,16,10,3], \\
    H[0,10,1,16,9,4], H[14,6,0,9,1,10], H[0,8,1,6,15,7], \\
    H[15,5,1,7,0,6], H[0,6,1,4,15,5], H[0,15,4,8,5,1], \\
    H[0,15,3,5,4,2] \}, \]

\[ B' = \{ H[3,0,1,4,2,5], H[6,3,4,7,5,8], H[9,6,7,10,8,11], \\
    H[12,9,10,13,11,14], H[15,12,13,16,14,0] \}. \]

Then a maximum \( LC_{3}^{(3)} \)-packing of \( K_{17}^{(3)} \), where the leave has edge set \( \{ \{1,15,16\}, \{0,2,16\} \} \), consists of the \( LC_{3}^{(3)} \)-blocks in \( B \) under the action of the map \( j \mapsto j + 1 \) (mod 17) along with the \( LC_{3}^{(3)} \)-blocks in \( B' \). Again, we note that by renaming the vertices in this packing, any two hyperedges in \( K_{17}^{(3)} \) that intersect in a single vertex can be made into the edge set of the leave of a maximum \( LC_{3}^{(3)} \)-packing of \( K_{17}^{(3)} \).

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Maximum packings of the $\lambda$-fold complete 3-uniform hypergraph with loose 3-cycles

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