Nonlinear equation for anomalous diffusion: unified power-law and stretched exponential exact solution

L. C. Malacarne, R. S. Mendes, I. T. Pedron
Departamento de Física, Universidade Estadual de Maringá, Avenida Colombo 5790, 87020-900, Maringá-PR, Brazil
E. K. Lenzi
Centro Brasileiro de Pesquisas Físicas, R. Dr. Xavier Sigaud 150, 22290-180 Rio de Janeiro-RJ, Brazil
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The nonlinear diffusion equation \( \frac{\partial \rho}{\partial t} = D \Delta \rho^\nu \) is analyzed here, where \( \Delta = \frac{1}{r^{d-1}} \frac{\partial}{\partial r} r^{d-1-\theta} \frac{\partial}{\partial r} \) and \( d, \theta \) and \( \nu \) are real parameters. This equation unifies the anomalous diffusion equation on fractals \( (\nu = 1) \) and the spherical anomalous diffusion for porous media \( (\theta = 0) \). Exact point-source solution is obtained, enabling us to describe a large class of subdiffusion \( (\theta > (1 - \nu)d) \), normal diffusion \( (\theta = (1 - \nu)d) \) and superdiffusion \( (\theta < (1 - \nu)d) \). Furthermore, a thermostatistical basis for this solution is given from the maximum entropic principle applied to the Tsallis entropy.

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One of the most ubiquitous processes in nature is the diffusive one. In this context, anomalous diffusion has awakened great interest nowadays, in particular in a variety of physical applications. A representative set of such applications of current interest is surface growth \[1\], diffusion on fractals \[3\], subrecoil laser cooling \[4\], CTAB micelles dissolved in salted water \[5\], two dimensional rotating flow \[6\] and anomalous diffusion at liquid surfaces \[7\]. The anomalous diffusive process is commonly characterized from the mean-square displacement time dependence, \( \langle r^2 \rangle \propto t^\sigma \), with \( \sigma \neq 1 \), i.e., we have superdiffusion for \( \sigma > 1 \) and subdiffusion for \( \sigma < 1 \). For a system which presents anomalous spreading, it is generally associated to a non-gaussian space-time distribution, like power-law or stretched exponential. In this framework, it is desirable to incorporate, in a unified way, these two behaviours, since it enables us to describe a wide class of diffusive processes. The present work is dedicated to giving such unified description.

Power-law or stretched exponential distributions arise naturally from generalizations of the \( d \)-dimensional diffusion equation

\( \frac{\partial \rho}{\partial t} = D \Delta \rho, \quad (1) \)

with \( \rho = \rho(\bar{x}, t), \quad \bar{x} = (x_1, x_2, \ldots, x_d), \quad \Delta = \sum_{n=1}^{d} \frac{\partial^2}{\partial x_n^2}, \) and \( D \) being the diffusion coefficient. The nonlinear equation

\( \frac{\partial \rho}{\partial t} = D \Delta \rho^\nu \quad (2) \)

is just one of these generalizations, where \( \nu \) is a real parameter. Eq. \( (2) \) has been employed to model diffusion in porous medium (see Ref. \[1\] and references therein) and in connection with generalized Tsallis statistics \[8\]. Another important kind of anomalous diffusion, in a tridimensional space, is related to turbulent diffusion in the atmosphere and is usually described by \[1\]

\( \frac{\partial \rho}{\partial t} = \nabla \cdot (K \nabla \rho), \quad (3) \)

where \( K \propto r^{4/3} (r = |\bar{x}|) \). In a more general case, we consider Eq. \( (3) \) in a \( d \)-dimensional space with \( K \propto r^{-\theta} \), where \( \theta \) is a real parameter. Thus, \( \nabla \cdot (K \nabla \rho) \) is proportional to \( r^{-(d-1)} \frac{\partial}{\partial r} r^{d-1-\theta} \frac{\partial}{\partial r} + A/r^{2-\theta} \) (\( A \) is an operator depending on the angular variables), and consequently \( \Delta = r^{-(d-1)} \frac{\partial}{\partial r} r^{d-1-\theta} \frac{\partial}{\partial r} \) is the radial part to be considered in the study of the spherical symmetrical solutions of Eq. \( (3) \). In this context, when \( d \) is interpreted as fractal dimension in an embedding \( N \)-dimensional space, the equation

\( \frac{\partial \rho}{\partial t} = D \Delta \rho \quad (4) \)

has been used to study diffusion on fractals \[12\].

Here we are going to propose the equation

\( \frac{\partial \rho}{\partial t} = \nabla \cdot (K \nabla \rho^\nu), \quad (5) \)

as unification of Eqs. \( (3) \) and \( (4) \). In fact, Eq. \( (5) \) reduces to the correlated anomalous diffusion \( (\theta = 0) \) if \( K = D \), and to the generalized Richardson equation \( (\theta = 1) \) if \( \nu = 1 \). The present study is mainly addressed to the point-source solution of Eq. \( (5) \), because it contains, as particular cases, \( a(n) \) (asymptotic) power-law and a stretched exponential. In this way, we focus our attention on the radial equation

\( \frac{\partial \rho}{\partial t} = D \Delta \rho^\nu. \quad (6) \)

Using this equation instead of Eq. \( (5) \) enables us to analyze cases with noninteger \( d \), so we can relate \( d \) with
a fractal dimension. Therefore, in the following discussion, we are going to consider $d$ as a nonnegative real parameter.

In order to motivate the ansatz to obtain an exact time-dependent solution for Eq. (3), we recall the corresponding solutions for Eqs. (1), (2), and (3). The time-dependent point-source solution for Eq. (3) is

$$
\rho(r, t) = \frac{\rho_0}{(4\pi D t)^{d/2}} \exp \left( -\frac{r^2}{4Dt} \right),
$$

(7)

where the normalization $\Omega_d \int_0^\infty \rho(r, t)r^{d-1}dr = \rho_0$ and the $d$-dimensional solid angle $\Omega_d \equiv 2\pi^{d/2}/\Gamma(d/2)$ have been used. From Eq. (3) we can easily obtain the Einstein formula for the Brownian montion, i.e., $\langle \rho^2 \rangle = 2dDt$.

The analogous solution for Eq. (3) is 

$$
\rho(r, t) = [1 - (1 - q) \beta_1(t) r^2]^{1/2} / Z_1(t),
$$

(8)

where $q = 2 - \nu$, $Z_1(t) \propto t^{2d/(2d+1-q)}$ and $\beta_1(t) \propto t^{-2/(2d+1-q)}$. It is important to emphasize the short or long tailed behaviour of the mean-square displacement, leading to $\langle \rho^2 \rangle \propto t^{2/(2d+1-q)}$. Again compared with the usual diffusion, $q = 1$, we have a superdiffusion (subdiffusion) for $q > 1$ ($q < 1$).

The fundamental solution of Eq. (3) is the stretched exponential

$$
\rho(r, t) = \exp \left( -\beta_2(t) r^{q+2} \right) / Z_2(t),
$$

(9)

which $Z_2(t) \propto t^{q/(q+2)}$ and $\beta_2(t) \propto t^{-1}$, presenting a short (long) tailed behavior for $\theta > 0$ ($\theta < 0$). Furthermore, the mean-square displacement behaviour is $\langle \rho^2 \rangle \propto t^{2/(\theta+2)}$. Thus, for $\theta > 0$ ($\theta < 0$) we have subdiffusive (superdiffusive) regime.

Note that Eqs. (3), (8) and (9) can be interpolated if we employ a generalized stretched Gaussian function, i.e., $G_{(q,\lambda)}(x) \equiv \left[ 1 - (1 - q) |x|^{\lambda} \right]^{1/(1-q)}$ or $G_{(q,\lambda)}(x) \equiv 0$ when $1 - (1 - q) |x|^{\lambda} < 0$, with $q$ and $\lambda$ being real parameters. In this direction, our ansatz to solve Eq. (3) is

$$
\rho(r, t) = [1 - (1 - q) \beta(t) r^{\lambda}]^{1/\lambda} / Z(t)
$$

(10)

or $\rho(r, t) = 0$ if $1 - (1 - q) \beta(t) r^{\lambda} < 0$, where $\beta(t)$ and $Z(t)$ are functions to be determined. By using this ansatz in Eq. (3) we verify that $\beta(t)$ and $Z(t)$, with $\lambda = \theta + 2$ and $q = 2 - \nu$, obey the equations

$$
\frac{d\beta(t)}{dt} = D\lambda (2 - q) \beta(t) Z^q(t)
$$

and

$$
\frac{dZ(t)}{dt} = -D \lambda^2 (2 - q) \beta^2(t) Z^{q-1}(t).
$$

(11)

The solutions of these nonlinear differential equations, that lead Eqs. (7), (8) and (9) as limit cases of Eq. (10), are

$$
\beta(t) = A t^{-\lambda/\alpha(q,\lambda)} Z(t) = B t^{-d/\lambda(q,\lambda)},
$$

(12)

where

$$
A = \left\{ \gamma^{q-1} \left[ D\lambda (2 - q)(\lambda + d(1 - q)) \right] \right\}^{\lambda/\alpha(q,\lambda)}
$$

and

$$
B = \left\{ \gamma \left[ D\lambda (2 - q)(\lambda + d(1 - q)) \right]^{q} \right\}^{\lambda/\alpha(q,\lambda)}
$$

(13)

which

$$
\gamma = 2\pi^{d/2} \Gamma \left( \frac{d}{2} \right) \frac{\Gamma \left( \frac{d}{2} - \frac{\theta}{2} \right)}{\lambda \rho_0 \Gamma \left( \frac{d}{2} \right)}
$$

(14)

The normalization condition, $\Omega_d \int_0^\infty \rho(r, t)r^{d-1}dr = \rho_0$, employed in the above calculation can only be satisfied if $\lambda > 0$ and $\lambda + d(1 - q) > 0$. From these conditions over the parameters $q$ and $\lambda$, we verify that the exponents in $\beta(t)$ and $Z(t)$ are respectively negative and positive. In addition to the normalization condition, the restriction $q < 2$ is necessary for $\rho(r, t)$ to be real. In the following, we assume that the parameters obey the above restrictions.

Of course, by setting the appropriate limits of parameters $\theta$ and $\nu$, or equivalently $\lambda$ and $q$, the solutions (7), (8) and (9) are recovered, giving the full expression for $\beta_1(t)$, $Z_1(t)$, $\beta_2(t)$ and $Z_2(t)$.

By using the above solution we can calculate the mean value of $r^\alpha$; it is

$$
\langle r^\alpha \rangle = \frac{\int_0^\infty r^\alpha \rho(r, t)r^{d-1}dr}{\int_0^\infty \rho(r, t)r^{d-1}dr} = C_\alpha t^{\frac{\alpha + d(1 - q)}{\lambda(q,\lambda)}},
$$

(15)

where

$$
C_\alpha = A^{-\frac{\alpha}{q}} \frac{\Gamma \left( \frac{d}{2} + \frac{\theta}{2} \right)}{\Gamma \left( \frac{d}{2} \right)} \frac{\Gamma \left( \frac{d}{2} - \frac{\theta}{2} \right)}{\lambda \rho_0 \Gamma \left( \frac{d}{2} \right)}
$$

(16)

with $A$ given by Eq. (13). When $q < 1$, the mean value $\langle r^\alpha \rangle$ always exist. On the other hand, the existence of $\langle r^\alpha \rangle$ for $q > 1$ imposes a further restriction over the parameters: $\lambda + d(1 - q) > a(q - 1)$.

To decide if the diffusion is anomalous or normal, we consider (13) with $\alpha = 2$. In this way, we have $< r^2 > \propto t^{\theta}$ with $\sigma = 2/[2 + \theta + d(\nu - 1)]$. Thus, the
condition for normal diffusion, $\sigma = 1$, can be satisfied even when $\rho$ does not obey Eq. (5), i.e., $\theta = d(1 - \nu)$ with $\theta \neq 0$ and $\nu \neq 1$. In this case, we can also verify that the anomalous diffusive regime induced by $\theta \neq 0$ is compensated by a convenient one with $\nu \neq 1$. Furthermore, this competition between $\theta$ and $\nu$ values can lead to a subdiffusion ($\sigma < 1$) if $\theta > d(1 - \nu)$ or a superdiffusion ($\sigma > 1$) if $\theta < d(1 - \nu)$. This classification is illustrated in Fig. (b) for $d = 3$.

In the following, we discuss the consequences of the above classification on the $\rho$ shape. From the solution (2), for porous medium, and the solution (3), for diffusion on fractals, we can see that the superdiffusive (subdiffusive) regime is associated to long (short) tail of $\rho(r, t)$ when compared with Gaussian (4). However, this connection is not valid in general. To illustrate the relation between regime of diffusion and tail behaviour of $\rho(r, t)$, we consider Fig. (c). In this figure, we plot $\rho(r, t)$ given by Eq. (4) versus $r$ to some values of $\theta$ and $\nu$, subject to the restriction $\theta = d(1 - \nu)$ (the normal diffusion line indicated in Fig. (b)). In this case we observed short and long tail behaviours compared with the Gaussian one.

![Fig. 1](image1)

**FIG. 1.** The diagram indicates the diffusive regime related to Eq. (3) in terms of its parameters $\theta$ and $\nu$ to $d = 3$. Thus, by using $(r^2) \propto t^{2/(d+2)}(d-\nu)$ from Eq. (13) we have classified the subdiffusive ($\theta > (1-\nu)d$), normal ($\theta = (1-\nu)d$), and superdiffusive ($\theta < (1-\nu)d$) regimes. The forbidden region refers to the region of parameters where $\langle r^2 \rangle$ does not exist (diverges).

To conclude our discussion about Eq. (3) and its radial time-dependent solution (4), we present an entropic basis for this solution. This basis is motivated by the Tsallis generalized statistical mechanics (13-14), where the Tsallis entropy (13) $S_q = (1 - \sum_{j=1}^W p_j^q)/(1-q)$ plays a central role with $\{p_j\}$ being the probabilities for the $W$ states of the system, and $q \in R$ being the Tsallis index (by taking the limit $q \to 1$ we recover the usual entropy $S_1 = -\sum_{j=1}^W p_j \ln p_j$). To understand the entropic basis, for simplicity, let us consider the maximization of $S_q = (1 - \int_{-\infty}^{\infty} \rho(x)^q dx)/(q-1)$ subject to the constraints (3) $\int_{-\infty}^{\infty} \rho(x) dx = 1$ and $\int_{-\infty}^{\infty} |x|^\lambda \rho(x) dx = U_q$. This maximization leads to $\rho(x) = (1-(1-q)/\beta) |x|^\lambda 1/(1-q) Z_q$, where $Z_q = \int_{-\infty}^{\infty} [1 - (1-q)/\beta] |x|^\lambda 1/(1-q) dx$ and $\beta$ is related to the Lagrange multipliers.

![Fig. 2](image2)

**FIG. 2.** By considering the normal diffusion, $\theta = (1-\nu)d$, the shape of $\rho(r, t)$ with $d = 3$ and $t = 5$ is illustrated in three cases: short tail($\theta = -0.6$ and $\nu = 1.2$), Gaussian ($\theta = 0$ and $\nu = 1$), and long tail ($\theta = 0.6$ and $\nu = 0.8$). Inset plot: detail of tail behaviour in the three cases above.

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