BLOW–UP FOR THE WAVE EQUATION WITH HYPERBOLIC DYNAMICAL BOUNDARY CONDITIONS, INTERIOR AND BOUNDARY NONLINEAR DAMPING AND SOURCES

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Abstract. The aim of this paper is to give global nonexistence and blow–up results for the problem

\begin{align*}
\begin{cases}
    u_{tt} - \Delta u + P(x, u_t) = f(x, u) & \text{in } (0, \infty) \times \Omega, \\
    u = 0 & \text{on } (0, \infty) \times \Gamma_0, \\
    u_{tt} + \partial_\nu u - \Delta_\Gamma u + Q(x, u_t) = g(x, u) & \text{on } (0, \infty) \times \Gamma_1, \\
    u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x) & \text{in } \Omega,
\end{cases}
\end{align*}

where $\Omega$ is a bounded open $C^1$ subset of $\mathbb{R}^N$, $N \geq 2$, $\Gamma = \partial \Omega$, $(\Gamma_0, \Gamma_1)$ is a partition of $\Gamma$, $\Delta_\Gamma$ denotes the Laplace–Beltrami operator on $\Gamma$, $\nu$ is the outward normal to $\Omega$, and the terms $P$ and $Q$ represent nonlinear damping terms, while $f$ and $g$ are nonlinear source terms. These results complement the analysis of the problem given by the author in two recent papers, dealing with local and global existence, uniqueness and well–posedness.

1. Introduction and main results.

1.1. Presentation of the problem and literature overview. We deal with the evolution problem consisting of the wave equation posed in a bounded regular open subset of $\mathbb{R}^N$, supplied with a second order dynamical boundary condition of hyperbolic type, in presence of interior and/or boundary damping terms and sources. More precisely we consider the initial –and–boundary value problem

\begin{align*}
\begin{cases}
    u_{tt} - \Delta u + P(x, u_t) = f(x, u) & \text{in } (0, \infty) \times \Omega, \\
    u = 0 & \text{on } (0, \infty) \times \Gamma_0, \\
    u_{tt} + \partial_\nu u - \Delta_\Gamma u + Q(x, u_t) = g(x, u) & \text{on } (0, \infty) \times \Gamma_1, \\
    u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x) & \text{in } \Omega,
\end{cases}
\end{align*}

(1)

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where $\Omega$ is a bounded $C^1$ open subset of $\mathbb{R}^N$, with $N \geq 2$. We denote $\Gamma = \partial \Omega$ and we assume that $\Gamma = \Gamma_0 \cup \Gamma_1$, $\Gamma_0 \cap \Gamma_1 = \emptyset$, that $\Gamma_1 \neq \emptyset$ is relatively open in $\Gamma$ and, denoting by $\sigma$ the standard Lebesgue hypersurface measure on $\Gamma$, that $\sigma(\Gamma_0 \cap \Gamma_1) = 0$. These properties of $\Omega$, $\Gamma_0$ and $\Gamma_1$ will be assumed, without further comments, throughout the paper. Moreover $u = u(t, x)$, $t \geq 0$, $x \in \Omega$, $\Delta = \Delta_x$ denotes the Laplace operator respect to the space variable, while $\Delta_\Gamma$ denotes the Laplace–Beltrami operator on $\Gamma$ and $\nu$ is the outward normal to $\Omega$.

The terms $P$ and $Q$ represent nonlinear damping terms, i.e. $P(x, v)v \geq 0$, $Q(x, v)v \geq 0$, the cases $P \equiv 0$ and $Q \equiv 0$ being specifically allowed, while $f$ and $g$ represent nonlinear source terms. The specific assumptions on them will be introduced later on.

Problems with kinetic boundary conditions, that is boundary conditions involving $u_{tt}$ on $\Gamma$, or on a part of it, naturally arise in several physical applications. A one dimensional model was studied by several authors to describe transversal small oscillations of an elastic rod with a tip mass on one endpoint, while the other one is pinched. See [3, 19, 20, 31, 42, 41, 44] and also [43] were a piezoelectric stack actuator is modeled.

A two dimensional model introduced in [29] deals with a vibrating membrane of surface density $\mu$, subject to a tension $T$, both taken constant and normalized here for simplicity. If $u(t, x)$, $x \in \Omega \subset \mathbb{R}^2$ denotes the vertical displacement from the rest state, then (after a standard linear approximation) $u$ satisfies the wave equation $u_{tt} - \Delta u = 0$, $(t, x) \in \mathbb{R} \times \Omega$. Now suppose that a part $\Gamma_0$ of the boundary is pinched, while the other part $\Gamma_1$ carries a constant linear mass density $m > 0$ and it is subject to a linear tension $\tau$. A practical example of this situation is given by a drumhead with a hole in the interior having a thick border, as common in bass drums. One linearly approximates the force exerted by the membrane on the boundary with $-\partial_\nu u$. The boundary condition thus reads as $mu_{tt} + \partial_\nu u - \tau \Delta_\Gamma u = 0$. In the quoted paper the case $\tau = 0$ was studied (when $\Gamma_0 = \emptyset$), while here we consider the more realistic case $\tau > 0$, with $\tau$ and $m$ normalized for simplicity, and we also allow $\Gamma_0$ to be nonempty. We would like to mention that this model belongs to a more general class of models of Lagrangian type involving boundary energies, as introduced for example in [24].

A three dimensional model involving kinetic dynamical boundary conditions comes out from [27], where a gas undergoing small irrotational perturbations from rest in a domain $\Omega \subset \mathbb{R}^3$ is considered. Normalizing the constant speed of propagation, the velocity potential $\phi$ of the gas (i.e. $-\nabla \phi$ is the particle velocity) satisfies the wave equation $\phi_{tt} - \Delta \phi = 0$ in $\mathbb{R} \times \Omega$. Each point $x \in \partial \Omega$ is assumed to react to the excess pressure of the acoustic wave like a resistive harmonic oscillator or spring, that is the boundary is assumed to be locally reacting (see [45, pp. 259–264]). The normal displacement $\delta$ of the boundary into the domain then satisfies $m\delta_{tt} + d\delta_t + k\delta + \rho \phi_t = 0$, where $\rho > 0$ is the fluid density and $m, d, k \in C(\partial \Omega)$, $m, k > 0$, $d \geq 0$. When the boundary is nonporous one has $\delta_t = \partial_\nu \phi$ on $\mathbb{R} \times \partial \Omega$, so the boundary condition reads as $m\delta_{tt} + d\partial_\nu \phi + k\phi + \rho \phi_t = 0$. The particular case $m = k$ and $d = \rho$ (see [27, Theorem 2]) one proves that $\phi_{\Gamma} = \delta$, so the boundary condition reads as $m\phi_{tt} + d\partial_\nu \phi + k\phi + \rho \phi_t = 0$, on $\mathbb{R} \times \partial \Omega$. Now, if one consider the case in which the boundary is not locally reacting, as in [7], one adds a Laplace–Beltrami term so getting a dynamical boundary condition like in (1). See [47] where this case was studied in detail.
Several papers in the literature deal with the wave equation with kinetic boundary conditions. This fact is even more evident if one takes into account that, plugging the equation in (1) into the boundary condition, we can rewrite it as

$$\Delta u + \partial_t u - \Delta_p u + Q(x, u_t) + P(x, u_t) = f(x, u) + g(x, u).$$

Such a condition is usually called a generalized Wentzell boundary condition, at least when nonlinear perturbations are not present. We refer to [21, 22, 40, 46, 57, 64, 65, 68]. All of them deal either with the case $\tau = 0$ or with linear problems.

Here we shall consider this type of kinetic boundary condition in connection with nonlinear boundary damping and source terms. These terms have been considered by several authors, but mainly in connection with first order dynamical boundary conditions. See [4, 5, 9, 11, 12, 15, 16, 17, 25, 38, 59, 69]. The competition between interior damping and source terms is methodologically related to the competition between boundary damping and source and it possesses a large literature as well. See [6, 28, 39, 48, 49, 52, 58].

Local and global existence, continuation, uniqueness and Hadamard well–posedness for problem (1) has been studied by the author in the recent papers [62, 63] (see also [61] for a preliminary study of a particular case). In [62] a blow–up result was also given when $P$ and $Q$ are linear in $u_t$.

Moreover a linear problem strongly related to (1) has also been recently studied in [26, 35], and another one in the recent paper [66], dealing with holography, a main theme in theoretical high energy physics and quantum gravity. See also [34].

The aim of the present paper is to discuss the optimality of the global existence result in [63] by giving some complementary global nonexistence and blow–up results for solutions of (1) when $P$ and $Q$ are possible nonlinear in $u_t$, a case which remained open in [62].

To simplify the presentation of our main results we shall restrict, in this section, to a parameters–dependent family of model problems, which catches the essential features of (1), as long as the alternative between global existence and nonexistence for arbitrary initial data is concerned.

### 1.2. A family of model problems.

We shall deal with

$$\begin{cases}
u_{tt} - \Delta u + \alpha (a|u_t|^{\overline{m}-2}u_t + |u_t|^{m-2}u_t) = \gamma |u|^{p-2}u & \text{in } (0, \infty) \times \Omega, \\
u = 0 & \text{on } (0, \infty) \times \Gamma_0, \\
u_{tt} + \partial_t u - \Delta_r u + \beta (b|u_t|^{\overline{\mu}-2}u_t + |u_t|^{\mu-2}u_t) = \delta |u|^{q-2}u & \text{in } (0, \infty) \times \Gamma_1, \\
(0, x) = u_0(x), & u_t(0, x) = u_1(x) \\
\text{in } \Omega, \\
\end{cases}$$

where $a, b, \alpha, \beta, \gamma, \delta, \overline{m}, m, \overline{\mu}, \mu, p, q$ are real number verifying

$$a, b, \alpha, \beta, \gamma, \delta \geq 0, \quad 1 < \overline{m} \leq m, \quad 1 < \overline{\mu} \leq \mu, \quad p, q \geq 2.$$  

The terms $a|u_t|^{\overline{m}-2}u_t$ and $b|u_t|^{\overline{\mu}-2}u_t$ are present only for modeling purpose and they need a suitable handling, but their possible vanishing is not relevant in the subsequent discussion, so the reader can take $a = b = 0$ in the sequel.

By the contrary the possible vanishing of each parameter among $\alpha$, $\beta$, $\gamma$ and $\delta$ individuates a different model problem in the family, which is then constituted by sixteen (!) different model problems. The unitary treatment of them was a characteristic feature of [62, 63] but, when dealing with the alternative between global existence and nonexistence it is useful to introduce a classification. In doing it we shall use the standard terminology widely adopted in the literature when dealing with a strongly methodologically related family of model problems, i.e.
(taking \( a = b = 0 \) for simplicity)

\[
\begin{cases}
    u_{tt} - \Delta u + \alpha |u_t|^{m-2} u_t = \gamma |u|^{p-2} u & \text{in } (0, \infty) \times \Omega, \\
    u = 0 & \text{on } (0, \infty) \times \Gamma_0, \\
    \partial_n u + u + \beta |u_t|^{\mu-2} u_t = \delta |u|^{q-2} u & \text{on } (0, \infty) \times \Gamma_1, \\
    u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x) & \text{in } \overline{\Omega},
\end{cases}
\]

(4)

where \( \alpha, \beta, \gamma, \delta, \tilde{m}, \tilde{\mu}, \mu, \rho, \sigma \) are as before. We shall often refer to literature concerning (4).

Our primary classification concerns the presence of the interior source \( |u|^{p-2}u \) and of the boundary source \( |u|^{q-2}u \) (the constants \( \alpha, \beta, \gamma, \delta \) could be normalized when positive), so defining the following four classes of model problems:

A) sourceless, when \( \gamma = \delta = 0 \);
B) with boundary source, when \( \gamma = 0 < \delta \);
C) with interior source, when \( \delta = 0 < \gamma \);
D) with interior and boundary sources, when \( \gamma, \delta > 0 \).

Clearly each class includes four different model problems according to the possible presence of interior and boundary damping terms. We shall set this one to be our secondary classification:

a) undamped, when \( \alpha = \beta = 0 \);

b) with boundary damping, when \( \alpha = 0 < \beta \);

c) with interior damping, when \( \beta = 0 < \alpha \);

d) with interior and boundary damping, when \( \alpha, \beta > 0 \).

When referring to a specific model we shall sometimes use the two letters classification given by previous lists. For example Cb) stands for the problem with interior source and boundary damping.

Source terms are usually classified in the literature concerning (4) (see [11, 12]), and also in [63], according to the relation occurring between their growth and the critical exponents \( r_\Omega \) and \( r_\Gamma \) of the Sobolev Embeddings of \( H^1(\Omega) \) and \( H^1(\Gamma) \) into the corresponding Lebesgue spaces, i.e.

\[
    r_\Omega = \begin{cases} 
        \frac{2N}{N-2} & \text{if } N \geq 3, \\
        \infty & \text{if } N = 2,
    \end{cases}
\]

\[
    r_\Gamma = \begin{cases} 
        \frac{2(N-1)}{N-3} & \text{if } N \geq 4, \\
        \infty & \text{if } N = 2, 3.
    \end{cases}
\]

(5)

In particular the source \( \gamma |u|^{p-2}u \) is:

(i) subcritical if \( \gamma = 0 \) or \( 2 \leq p \leq 1 + r_\Omega/2 \), when the Nemitskii operator \( \gamma |u|^{p-2}u \) is locally Lipschitz from \( H^1(\Omega) \) into \( L^2(\Omega) \);

(ii) supercritical if \( \gamma > 0 \) and \( 1 + r_\Omega/2 < p \leq r_\Omega \), when \( \gamma |u|^{p-2}u \) is no longer locally Lipschitz from \( H^1(\Omega) \) into \( L^2(\Omega) \) but it still possesses a potential energy in \( H^1(\Omega) \);

(iii) super–supercritical if \( \gamma > 0 \) and \( p > r_\Omega \), when \( \gamma |u|^{p-2}u \) has no potentials in \( H^1(\Omega) \).

The analogous classification is made for \( \delta |u|^{q-2}u \) depending on \( \delta, \rho \) and \( r_\Gamma \).

In [62] we studied well–posedness of (2) when both sources are subcritical, while in [63] this condition was relaxed. In the sequel we shall deal with weak solutions of (2), already introduced in the quoted papers. They are solutions in a suitable distribution sense and enjoy ”good properties”, see Definition 3.2 and Lemma 3.3.
An essential ingredient in their definition is that
\[ p \leq \begin{cases} 1 + r_\alpha/2 & \text{if } \gamma > 0, \alpha = 0, \\ 1 + r_\alpha/m & \text{if } \gamma > 0, \alpha > 0, \end{cases} \quad q \leq \begin{cases} 1 + r_\gamma/2 & \text{if } \delta > 0, \beta = 0, \\ 1 + r_\gamma/m' & \text{if } \delta > 0, \beta > 0, \end{cases} \] (6)
where, for any \( \rho \in [1, \infty] \) we denote by \( \rho' \) its Hölder conjugate of \( \rho \), i.e. \( 1/\rho + 1/\rho' = 1 \), and
\[ m = \max\{2, m\}, \quad m' = \max\{2, \mu\}. \] (7)
The notation (7) will be consistently used throughout the paper.

Solutions of (2) in the sense of distributions may be considered also when (6) does not hold. Beside the lack of an available local existence theory, any discussion on the life-span of these solutions looks to be out of reach.

1.3. Known results. In the paper we shall identify \( L^\rho(\Gamma_1) \), \( \rho \in [1, \infty] \), with its isometric image in \( L^{\rho}(\Gamma) \), that is
\[ L^\rho(\Gamma_1) = \{ u \in L^\rho(\Gamma) : u = 0 \ \text{a.e. on } \Gamma_0 \}. \] (8)
We shall denote by \( u_\Gamma \) the trace on \( \Gamma \) of \( u \in H^1(\Omega) \). We introduce the Hilbert spaces \( H^0 = L^2(\Omega) \times L^2(\Gamma_1) \) and
\[ H^1 = \{ (u, v) \in H^1(\Omega) \times H^1(\Gamma) : v = u_\Gamma, v = 0 \ \text{a.e. on } \Gamma_0 \}, \] (9)
with the standard product norm. For the sake of simplicity we shall identify, when useful, \( H^1 \) with its isomorphic counterpart \( \{ u \in H^1(\Omega) : u_\Gamma \in H^1(\Gamma) \cap L^2(\Gamma_1) \} \), through the identification \( (u, u_\Gamma) \mapsto u \), so we shall write, without further mention, \( u \in H^1(\Omega) \) for functions defined on \( \Omega \). Moreover we shall drop the notation \( u_\Gamma \), when useful, so we shall write \( \|u\|_{2,\Gamma} \), \( \int_\Gamma u \), and so on, for \( u \in H^1 \).

We shall also use the main phase space for problem (2), that is
\[ \mathcal{H} = H^1 \times H^0, \quad \text{with the standard norm } \|(u, v)\|_\mathcal{H}^2 = \|u\|_{H^0}^2 + \|v\|_{H^1}^2. \] (10)
As a particular case of [63, Theorems 1.1 and 1.2] (see Theorems 6.1–6.2 below when \( p \leq r_\alpha \) and \( q \leq r_\gamma \)) the following results hold. When (3), (6) hold and
\[ p < 1 + r_\alpha/m \quad \text{when } N \geq 5, \gamma > 0, \quad m > r_\alpha, \]
\[ q < 1 + r_\gamma/m' \quad \text{when } N \geq 6, \delta > 0, \quad \mu > r_\gamma, \] (11)
for all \( U_0 := (u_0, u_1) \in \mathcal{H} \) such that
\[ u_0 \in L^{r_\alpha/(p-2)}(\Omega_{\alpha/2}) \quad \text{if } N = 3, 4, \gamma > 0, \quad p = 1 + r_\alpha/m', \quad m > r_\alpha, \]
\[ u_0|\Gamma \in L^{r_\gamma/(q-2)}(\Omega_{\gamma/2}) \quad \text{if } N = 4, 5, \delta > 0, \quad q = 1 + r_\gamma/m', \quad \mu > r_\gamma, \] (12)
problem (2) possesses a maximal weak solution \( u \in C([0, T_{\max}); H^1) \cap C^1([0, T_{\max}); H^0) \) for some \( T_{\max} \in (0, \infty) \). In the sequel we shall denote \( U = (u, u') \in C([0, T_{\max}); \mathcal{H}) \).

It is worth observing that (11) and (12) can be disregarded when \( p \leq r_\alpha \) and \( q \leq r_\gamma \), since \( m > r_\alpha \) and \( \mu > r_\gamma \) respectively yield \( p = 1 + r_\alpha/m' > r_\alpha \) and \( q = 1 + r_\gamma/m' > r_\gamma \).

Moreover, if
\[ p \leq 1 + r_\alpha/2 \text{ when } N \geq 5, \gamma > 0, \quad \text{and } q \leq 1 + r_\gamma/2 \text{ when } N \geq 6, \quad \delta > 0, \] (13)
then the previously found solution is unique.

---

1The sets in the planes \( (p, m) \) and \( (q, \mu) \), for which (6) holds, corresponding to the classification above, are illustrated in dimensions \( N = 2, 3, 4 \) in Figure 1. Clearly the two sets are respectively relevant only when \( \gamma > 0 \) and \( \delta > 0 \), and (6) can be disregarded when \( N = 2 \).
Beside the local theory described above, in [63, Theorem 1.5 and Remarks 1.1, 1.3] also existence of global solutions for arbitrary initial data was proved, provided the parameters satisfy a further restriction.

**Theorem 1.1 (Global existence).** Let (3),(6),(11) hold and

\[
p \leq \begin{cases} 
2 & \text{if } \gamma > 0, \alpha = 0, \\
\frac{2}{m} & \text{if } \gamma > 0, \alpha > 0,
\end{cases} \quad q \leq \begin{cases} 
2 & \text{if } \delta > 0, \beta = 0, \\
\frac{2}{\mu} & \text{if } \delta > 0, \beta > 0.
\end{cases}
\]  

(14)

Then for any \( U_0 \in \mathcal{H} \) such that \( u_0 \in L^p(\Omega) \) when \( \gamma > 0 \) and \( p > r_\gamma \), \( u_0|_{\Gamma} \in L^q(\Gamma_1) \) when \( \delta > 0 \) and \( q > r_\delta \), problem (2) has a global weak solution, which is unique when also (13) holds.

**Remark 1.** The sets in the planes \((p,m)\) and \((q,\mu)\), for which (6) and (14) hold, and those for which (6) holds while (14) does not, depending on the vanishing of \( \alpha \) and \( \beta \), are illustrated in dimensions \( N = 2, 3, 4 \) in Figure 1. As shown (when
\( N \leq 4 \) in it, when (6) holds and one has

\[
\gamma > 0 \quad \text{and} \quad p > \begin{cases} 2 & \text{if } \alpha = 0, \\ \frac{2}{\overline{m}} & \text{if } \alpha > 0, \end{cases}
\]

i.e. when the first half of assumption (14) does not hold, one necessarily has \( \overline{m}, p < r_\alpha \). Indeed, if \( \alpha = 0 \) then \( p \leq 1 + r_\alpha / 2 < r_\alpha \) by (6), and we can freely choose \( m = 2 \), while if \( \alpha > 0 \) then by (6) and (15) one has \( \overline{m} < 1 + r_\alpha / \overline{m} \), i.e. \( \overline{m}^2 - (r_\alpha + 1)\overline{m} + r_\alpha > 0 \), so \( \overline{m} < r_\alpha \) and a further application of (6) yields \( p \leq 1 + r_\alpha / \overline{m} \), i.e. \( m < 2 \), while if \( \alpha > 0 \) then by (6) and (15) one has \( m < 1 + \frac{r_\alpha}{m'} \), i.e. \( m^2 - (r_\alpha + 1)m + r_\alpha > 0 \), so \( m < r_\alpha \) and a further application of (6) yields \( p \leq 1 + \frac{r_\alpha}{m'} < 1 + \frac{r_\alpha}{r_\alpha'} = r_\alpha \).

The same arguments show that then (6) holds and one has

\[
\delta > 0 \quad \text{and} \quad q > \begin{cases} 2 & \text{if } \beta = 0, \\ \frac{2}{\mu'} & \text{if } \beta > 0, \end{cases}
\]

one necessarily has \( \mu, q < r_\Gamma \).

The optimality of assumption (14) was already discussed in [62] when both damping terms are linear, i.e. when

\[
a = b = 0, \quad \alpha = 0 \quad \text{or} \quad m = 2, \quad \beta = 0 \quad \text{or} \quad \mu = 2.
\]

In this case, which includes the sourceless class, by (6) both sources are subcritical, so (13) holds, and assumption (14) trivializes to

\[
p = 2 \quad \text{when } \gamma > 0, \quad q = 2 \quad \text{when } \delta > 0.
\]

As a particular case of [62, Theorem 1.5] the following result holds.

**Theorem 1.2 (Blow–up for linear damping).** Let (3), (6), (17) hold, and

\[
(\gamma, \delta) \neq (0, 0), \quad p > 2 \quad \text{when } \gamma > 0, \quad q > 2 \quad \text{when } \delta > 0.
\]

Then, for any \( U_0 \in \mathcal{H} \) such that

\[
\mathcal{E}(U_0) := \frac{1}{2} \| u_1 \|^2_{\mathcal{H}^2} + \frac{1}{2} \int_{\Omega} |\nabla u_0|^2 \, dx + \frac{1}{2} \int_{\Gamma_1} |\nabla_{\Gamma} u_0|^2 \, d\sigma
\]

\[
- \frac{\gamma}{p} \int_{\Omega} |u_0|^p \, dx - \frac{\delta}{q} \int_{\Gamma_1} |u_0|^q \, d\sigma < 0,
\]

the unique maximal weak solution of (2) has \( T_{\text{max}} < \infty \). Moreover

\[
\lim_{t \to T_{\text{max}}} \| U(t) \|_{\mathcal{H}} = \lim_{t \to T_{\text{max}}} \int_{\Omega} |u(t)|^p \, dx + \int_{\Gamma_1} |u(t)|^q \, d\sigma = \infty,
\]

where we can take \( p = 2 \) when \( \gamma = 0 \) and \( q = 2 \) when \( \delta = 0 \).

The relations between the parameters ranges (18) and (19), which respectively yield global existence for (almost) all data and blow–up for suitable data (which trivially exist), is clearest when separately considering the previously introduced model classes A–D). This comparison is made explicit, for the readers’ convenience, in Table 1. In it \( \checkmark \) stands for no assumptions and \( \times \) for class exclusion.

**Table 1: (18) vs. (19) for the model classes A–D).**

| Case (16) | A) \( \gamma = 0, \delta = 0 \) | B) \( \gamma = 0, \delta > 0 \) | C) \( \gamma > 0, \delta = 0 \) | D) \( \gamma > 0, \delta > 0 \) |
|----------|-----------------|-----------------|-----------------|-----------------|
| (18)     | \( \checkmark \) | \( q = 2 \)     | \( p = 2 \)     | \( p = 2, q = 2 \) |
| (19)     | \( \times \)    | \( q > 2 \)     | \( p > 2 \)     | \( p > 2, q > 2 \) |
Theorem 1.1, and consequently also Theorem 1.2, is sharp in the classes A–C), while in the class D) the combined answer given by them is incomplete. Indeed, when \( p = 2, q > 2 \) and when \( p > 2, q = 2 \) no information is given. This easy case exhibits two difficulties in the analysis of the class D) which will persists in the general case:

- even if a source is superlinear, the linearity of the other one may inhibit global nonexistence arguments,
- when the growth of one source dominated the growth of the corresponding damping term, but the opposite domination holds for the other couple, the solutions behavior may remain undetermined.

1.4. Main results. We shall present our main results for (2) by distinguishing among the previously introduces model classes A–D). Clearly when (2) is sourceless Theorem 1.2 assures global existence for (almost) all data, so the class A) needs no further attention.

We start by making some remarks on class B), when only a boundary source is present. This class is covered by Theorem 1.2 when the damping is linear, but the situation is quite different in the nonlinear case. Indeed it is possible to find some blow–up results in the literature concerning class B) for the related family (4), such as [32, 33, 67]. Unfortunately all the proofs of this type of results are, to the author’s knowledge, adaptations of the classical arguments in [28, 39] and are, at some point, problematic. Referring to the quoted papers, in [67, p. 868] the authors treats the norms \( \| \cdot \|_{L^p(\Omega)} \) and \( \| \cdot \|_{L^p(\Gamma_1)} \) as equivalent, while in [32, p. 333] (the same argument being used in [33]) the author implicitly uses boundedness of the function \( t \mapsto \| u(t) \|_{L^2(\Omega)} \) while proving the finite time blow–up of an auxiliary functional, which in turns yields finite time blow–up of \( t \mapsto \| u(t) \|_{L^2(\Omega)} \). To the author’s understanding the arguments in [28, 39] cannot be adapted to wave equation with boundary nonlinear damping and sources and, since this is the state of the arts, is still a challenging open problem to prove blow–up results in class B) for problem (4). In the present paper we shall not deal with this class for (2).

Our first main result concerns the model class C).

**Theorem 1.3 (Global nonexistence and blow–up with interior source).** Let (3), (6) hold, \( \gamma > 0, \delta = 0 \) and

\[
P > \begin{cases} 
2 & \text{if } \alpha = 0, \\
\frac{2}{\mu} & \text{if } \alpha > 0,
\end{cases} \quad \mu < 1 + p/2 \quad \text{when } \beta > 0. \tag{22}
\]

Then, for any \( U_0 \in \mathcal{H} \) such that

\[
\mathcal{E}(U_0) = \frac{1}{2} \| u_1 \|_{H^0}^2 + \frac{1}{2} \int_{\Omega} |\nabla u_0|^2 dx + \frac{1}{2} \int_{\Gamma_1} |\nabla_\Gamma u_0|^2 d\sigma - \frac{2}{p} \int_{\Omega} |u_0|^p dx < 0, \tag{23}
\]

and any maximal weak solution of (2) in \([0, T_{\max})\), one has \( T_{\max} < \infty \) and

\[
\lim_{t \to T_{\max}} \| U(t) \|_{\mathcal{H}} = \lim_{t \to T_{\max}} \int_{\Omega} |u(t)|^p dx + \int_{\Gamma_1} |u(t)|^2 d\sigma = \infty. \tag{24}
\]

Finally, when \( N \leq 4 \) or \( p \leq 1 + \frac{r_1}{2} \), we can replace \( \lim_{t \to T_{\max}} \) with \( \lim_{t \to T_{\max}} \) in (24).

**Remark 2.** As already pointed out in Remark 1, by (22) we necessarily have \( \mu, p < r_1 \). Since \( \delta = 0 \), in Theorem 1.3 super–supercritical sources are not considered, so ruling out conditions (11)–(12) from local existence theory. We also point out that (24) holds for all possible maximal weak solutions of (2), also when uniqueness is unknown.
The relation between the parameter ranges (14) and (22), which respectively yield global existence for (almost) all data and blow-up for suitable data (which trivially exist) is clearest when separately considering the model problems Ca–Cd), as we do in Table 2 for the reader’s convenience.

| Parameter Ranges | Table 2: (14) vs. (22) for the model problems Ca–Cd). |
|------------------|-----------------------------------------------------|
| $\gamma > 0 = \delta$ | $\alpha = \beta = 0$ | $\alpha = 0 < \beta$ | $\alpha > 0 = \beta$ | $\alpha, \beta > 0$ |
| (14) | $p = 2$ | $p = 2$ | $p \leq \overline{m}$ | $p \leq \overline{m}$ |
| (22) | $p > 2$ | $p > 2, \overline{m} < 1 + p/2$ | $p > \overline{m}$ | $p > \overline{m}, \overline{m} < 1 + p/2$ |

It makes clear that Theorem 1.1, and consequently also Theorem 1.3, is sharp for the model problems Ca) and Cc), i.e. when boundary source and damping are not present in (2). By the contrary, when dealing with the model problems Cb) and Cd) and

$$p > \begin{cases} 2 & \text{if } \alpha = 0, \\ \overline{m} & \text{if } \alpha > 0, \end{cases} \quad \overline{m} \geq 1 + p/2, \quad (25)$$

Theorems 1.1 and 1.3 give no information. When (25) holds the growth of the interior source dominates the one of the corresponding damping term, while the boundary damping term has no a homologous counterpart and, consequently, can be controlled only by the transversal influence of the interior source. This type of transversal control was already pointed out for the class C) of the related family (4) (without the term $u$ on $\Gamma_1$) in [60] and subsequently improved in [25], where the exact assumption $\mu < 1 + p/2$ appears. As to the author’s knowledge such an assumption has been skipped only when $N = 1$ and $\Omega$ is a suitably large interval (see [23]).

Our second main result concerns the model class D).

**Theorem 1.4 (Global nonexistence and blow–up with two sources).**

Let (3), (6) hold, $\gamma, \delta > 0$ and

$$p > \begin{cases} 2 & \text{if } \alpha = 0, \\ \overline{m} & \text{if } \alpha > 0, \end{cases} \quad q > 2, \quad \overline{m} < \max\{q, 1 + p/2\} \quad \text{when } \beta > 0. \quad (26)$$

Then, for any $U_0 \in \mathcal{H}$ such that (20) holds and any maximal weak solution $^2$ of (2) in $[0, T_{\text{max}})$, one has $T_{\text{max}} < \infty$ and

$$\lim_{t \to T_{\text{max}}} \|U(t)\|_{\mathcal{H}} = \lim_{t \to T_{\text{max}}} \int_{\Omega} |u(t)|^p dx + \int_{\Gamma_1} |u(t)|^q d\sigma = \infty. \quad (27)$$

Finally, when $N \leq 4$, or $N = 5$ and $p \leq 1 + r_0/2 = 8/3$, or $N \geq 6$, $p \leq 1 + r_0/2$, $q \leq 1 + r_1/2$, we can replace $\lim_{t \to T_{\text{max}}}$ with $\lim_{t \to T_{\text{max}}}$ in (27).

**Remark 3.** As already pointed out in Remark 1, by (6) and (26) we necessarily have $\overline{m}, p < r_0$. By (6) and (26) it also follows that $\overline{m}, q < r_1$. Indeed, when $\beta = 0$, by (6) we have $q \leq 1 + r_1/2 < r_1$ and we can freely choose $\mu = 2$, while when $\beta > 0$ by (26) either $\overline{m} < q$ or $\overline{m} < 1 + p/2$. In the first case, see also Figure 1, by (6) one has $\overline{m} < 1 + r_1/\overline{m}$, i.e. $\overline{m}^2 - (r_1 + 1)\overline{m} + r_1 < 0$, so $\overline{m} < r_1$, and consequently, using (6) again, $q \leq 1 + r_1/\overline{m} < 1 + r_1/r_1 = r_1$. In the second case, since $p < r_1$, we have $\overline{m} < 1 + r_1/2$. But, by (5), $1 + r_1/2 \leq r_1$ for all $N \geq 2$, so $\overline{m} < r_1$ and, as in

---

$^2$at least one of them exists
the previous case, \( q < r \). Hence, also in Theorem 1.4 super–supercritical sources are not considered, so ruling out conditions (11)–(12) from local existence theory. Also in this case (27) holds for all possible maximal weak solutions of (2), also when uniqueness is unknown.

Clearly (26) does not exhaust all possible parameters values for which (14) does not hold in class D). Also for this class it is useful to separately considering the model problems Da–Dd). This comparison is made, for the reader’s convenience, in Table 3.

Table 3: (14) vs (26) for the model problems Da–Dd).

| \( \gamma, \delta > 0 \) | \( \alpha = \beta = 0 \) | \( \alpha = 0 < \beta \) | \( \alpha > 0 = \beta \) | \( \alpha, \beta > 0 \) |
|-----------------|-----------------|-----------------|-----------------|-----------------|
| (14)            | \( p = 2, \)   | \( p = 2, \)   | \( p \leq \mu, \) | \( p \leq \mu, \) |
|                 | \( q = 2 \)    | \( q \leq \mu \) | \( q = 2 \)    | \( q \leq \mu \) |
| (26)            | \( p > 2, \)   | \( p > 2, \)   | \( p > \mu, \) | \( p > \mu, \) |
|                 | \( q > 2 \)    | \( q > 2 \)    | \( q > \mu \)  | \( q > \mu \)  |

It clearly shows the following facts. In models Da) and Dc) Theorem 1.4 predicts finite time blow–up of solutions only when both inequalities in (14) do not hold. So no information is given in the following two cases:

i) when the linear boundary source inhibits global nonexistence arguments, i.e. when

\[
p > \begin{cases} 
    2 & \text{if } \alpha = 0, \\
    \mu & \text{if } \alpha > 0,
\end{cases} \quad q = 2;
\]

ii) when the growth of the interior damping dominates the one of the interior source, but the boundary source has no a damping homologous counterpart, i.e. when

\[
p \leq \begin{cases} 
    2 & \text{if } \alpha = 0, \\
    \mu & \text{if } \alpha > 0,
\end{cases} \quad q > 2.
\]

These two cases exactly correspond to the difficulties \( \bullet \) and \( \bullet \bullet \) (which show up at their top level for these models) already emphasized for class D) when damping terms are linear.

In model problems Db) and Dd) Theorem 1.4 predicts blow–up of solutions when both inequalities in (14) do not hold, that is when (15)–(16) hold, but not only in this case. Indeed it also partially cover the case in which (15) holds while (16) does not, this part exactly corresponding to the one covered in models Cb) and Cd).

It is worth observing that the parameter restriction (26) in the literature concerning class D) for the family (4), see [10, 13, 37], represents the state of the art.

Finally we remark that (26) is the exact complementary of (14), so Theorems 1.3–1.4 are both sharp, in the “diagonal” case \( \alpha = \beta, \gamma = \delta, m = \mu \) and \( p = q \).

Theorems 1.3–1.4 are suitably recombined particular cases of our more general blow–up results Theorems 6.3 and 6.4, which are essentially based on our main global nonexistence results Theorem 4.1 and 5.1. Their proofs both rely on suitable non–trivial adaptations of the techniques in [10, 13, 25, 37, 28, 39, 60].
1.5. **Organization of the paper.** The sequel of the paper is organized as follows:

i) in Section 2 we give some background on the functional spaces used and on a linear version of problem (1);

ii) Section 3 is devoted to give our main assumptions, to introduce weak solutions of (1) and to some preliminary results;

iii) in Section 4 we state and prove our first main global nonexistence result for problem (1), dealing with the case when two sources are present in it;

iv) in Section 5 we state and prove our second main global nonexistence result for problem (1), dealing with the case in which $g$ may vanish;

v) Section 6 is devoted to recall the local theory from [63], to give our main blow-up results and to show how Theorems 1.3–1.4 follow from them.

2. **Background.**

2.1. **Notation.** We shall adopt the standard notation for (real) Lebesgue and Sobolev spaces in $\Omega$ (see [1]) and $\Gamma$ (see [30]). Moreover $\| \cdot \|_\rho := \| \cdot \|_{L^\rho(\Omega)}$ and $\| \cdot \|_{\rho,\Gamma'} := \| \cdot \|_{L^\rho(\Gamma')} \| \rho \in [1, \infty)$ and $\Gamma' \subseteq \Gamma$ measurable.

Given any Banach space $X$ we shall denote by $(\cdot, \cdot)_X$ the duality product between $X$ and its dual $X'$, and we shall use the standard notation for $X$–valued Bochner–Lebesgue and Bochner–Sobolev spaces in a real interval. Moreover $L(X, Y)$ will denote the class of linear bounded operators from $X$ to another Banach space $Y$.

Given $\alpha \in L^\infty(\Omega)$, $\beta \in L^\infty(\Gamma_1)$, $\alpha, \beta \geq 0$, $-\infty \leq c < d \leq \infty$ and $\rho \in [1, \infty)$ we shall respectively denote by $\lambda^\alpha$, $\lambda^\beta$, $\lambda^\beta_\rho$ the measures respectively defined in $\Omega$, $\mathbb{R} \times \Omega$, $\Gamma_1$, $\mathbb{R} \times \Gamma_1$, by $d\lambda^\alpha = \alpha \, dx$, $d\lambda^\beta = \beta \, ds$, $d\lambda^\beta_\rho = \rho \, c \, ds$, and by $L^\rho(\Omega; \lambda^\alpha)$, $L^\rho((c,d) \times \Omega; \lambda^\alpha)$, $L^\rho(\Gamma_1; \lambda^\beta)$, $L^\rho((c,d) \times \Gamma_1; \lambda^\beta_\rho)$ the corresponding Lebesgue spaces. The equivalence classes with respect to $\lambda^\alpha$ and $\lambda^\beta$ a.e. equivalences will be denoted by $[\cdot]_{\lambda^\alpha}$, those with respect to $\lambda^\beta$ and $\lambda^\beta_\rho$ equivalences by $[\cdot]_{\beta}$. By the density of $C_c((c,d) \times \Omega)$ and $C_c((c,d) \times \Gamma_1)$, respectively in $L^\rho((c,d) \times \Omega; \lambda^\alpha)$ and $L^\rho((c,d) \times \Gamma_1; \lambda^\beta_\rho)$, see [51, Theorem 2.18 p. 48 and Theorem 3.14 p. 68], one can prove, as in the standard case, that

$$L^\rho((c,d) \times \Omega, \lambda^\alpha) \simeq L^\rho((c,d) \times \Omega; \lambda^\alpha),$$

$$L^\rho((c,d) \times \Gamma_1, \lambda^\beta) \simeq L^\rho((c,d) \times \Gamma_1; \lambda^\beta_\rho).$$

(28)

We recall some well–known preliminaries on the Riemannian gradient on the $C^1$ compact manifold $\Gamma$, referring to [54] for more details and proofs in the smooth setting and to [53] for a general background on differential geometry on $C^1$ manifolds. The interested reader may also see [47] in the non–compact case. We denote by $(\cdot, \cdot)_\Gamma$ the metric inherited from $\mathbb{R}^N$, given in local coordinates $(y_1, \ldots, y_{N-1})$ by $(g^{ij})_{i,j=1,\ldots,N-1}$, and $| \cdot |^2_\Gamma = (\cdot, \cdot)_\Gamma$. We denote by $ds$ the natural volume element on $\Gamma$, given by $\sqrt{\det(g_{ij})} \, dy_1 \wedge \ldots \wedge dy_{N-1}$. The Riemannian gradient is given in local coordinates by $\nabla_\Gamma u = g^{ij} \partial_i u \, \partial_j$ for all $u \in H^1(\Gamma)$, where $(g^{ij}) = (g_{ij})^{-1}$. It is well–known, see [36, 47, 54] that the norm $\| u \|_{H^1(\Gamma)} = \| u \|_{2, \Gamma}^2 + \| \nabla_\Gamma u \|_{2, \Gamma}^2$, where $\| \nabla_\Gamma u \|_{2, \Gamma}^2 := \int_\Gamma |\nabla_\Gamma u|^2_\Gamma$, is equivalent in $H^1(\Gamma)$ to the standard one.

2.2. **Functional setting.** Given $\alpha \in L^\infty(\Omega)$, $\beta \in L^\infty(\Gamma_1)$, $\alpha, \beta \geq 0$ and $\rho \in [2, \infty)$ we recall the reflexive spaces introduced in [62], that is

$$L^{2,\rho}_\alpha(\Omega) = \{ u \in L^2(\Omega) : \alpha^{1/\rho} u \in L^\rho(\Omega) \}, \quad \| \cdot \|_{L^{2,\rho}_\alpha(\Omega)} = \| \cdot \|_2 + \| \alpha^{1/\rho} \cdot \|_\rho,$$

$$L^{2,\rho}_\beta(\Gamma_1) = \{ u \in L^2(\Gamma_1) : \beta^{1/\rho} u \in L^\rho(\Gamma_1) \}, \quad \| \cdot \|_{L^{2,\rho}_\beta(\Gamma_1)} = \| \cdot \|_2, \Gamma_1 + \| \beta^{1/\rho} \cdot \|_{\rho, \Gamma_1}.$$
as well as the trivial embeddings and boundedness properties

\[ L^2_\alpha(\Omega) \hookrightarrow L^2(\Omega), \quad [\cdot]_\alpha \in \mathcal{L}(L^2_\alpha(\Omega), L^2(\Omega, \lambda_\alpha)) \]

\[ L^2_\beta(\Gamma_1) \hookrightarrow L^2(\Gamma_1), \quad [\cdot]_\beta \in \mathcal{L}(L^2_\beta(\Gamma_1), L^2(\Gamma_1, \lambda_\beta)). \]

(29)

All operators in (29) trivially have dense range (see [62]), so by applying [18, Theorem 5.11–3, Chapter 5, p. 280], or [14, Corollary 2.18 p. 45], we have the embeddings

\[ [\cdot]' \hookrightarrow [L^2_\alpha(\Omega)]', \quad [L^2(\Omega, \lambda_\alpha)]' \hookrightarrow [L^2_\alpha(\Omega)]' \]

\[ [L^2(\Gamma_1)]' \hookrightarrow [L^2_\beta(\Gamma_1)]', \quad [L^\rho(\Gamma_1, \lambda_\beta)]' \hookrightarrow [L^2_\beta(\Gamma_1)]'. \]

(30)

As usual we shall identify \([L^2(\Omega, \lambda_\alpha)]' \simeq L^2(\Omega)\) and \([L^2(\Gamma_1)]' \simeq L^2(\Gamma_1)\). These identifications, essentially made in the distribution sense, make impossible to identify \([L^\rho(\Omega, \lambda_\alpha)]'\) with \([L^\rho(\Omega, \lambda_\alpha)]\), the same remark applying to measures \(\lambda_\alpha', \lambda_\beta\) and \(\lambda'_\beta\).

We shall identify all spaces in (30) with the corresponding subspaces of \([L^2_\alpha(\Omega)]'\) or \([L^2_\beta(\Gamma_1)]'\).

For any \(\xi \in L^\rho(\Omega, \lambda_\alpha)\), even if \(\xi\) is not a.e. well-defined in \(\Omega\), since it takes arbitrary values in the possibly large set where \(\alpha\) vanishes, the function \(\alpha\xi\) is well-defined a.e. in it, and actually \(\alpha\xi \in L^1(\Omega)\). Moreover, by the form of the Riesz isomorphism in \(L^\rho(\Omega, \lambda_\alpha)\), we can represent \([L^\rho(\Omega, \lambda_\alpha)]'\) as \(\{\alpha\xi, \xi \in L^\rho(\Omega, \lambda_\alpha)\}\).

Using the same arguments on \(\Gamma_1\) we have that for any \(\eta \in L^{\rho'}(\Gamma_1, \lambda_\beta)\) we have \(\beta\eta \in L^1(\Gamma_1)\) and the following identifications hold

\[ [L^\rho(\Omega, \lambda_\alpha)]' \simeq \{\alpha\xi, \xi \in L^\rho(\Omega, \lambda_\alpha)\}, \quad [L^\rho(\Gamma_1, \lambda_\beta)]' \simeq \{\beta\eta, \eta \in L^{\rho'}(\Gamma_1, \lambda_\beta)\}. \]

(31)

In the sequel we shall also use, for any \(\alpha \in L^\infty(\Omega), \beta \in L^\infty(\Gamma_1), \alpha, \beta \geq 0, -\infty < c < d \leq \infty\) and \(\rho \in [2, \infty)\), the spaces \(L^\rho(c, d; L^2_\alpha(\Omega))\) and \(L^\rho(c, d; L^2_\beta(\Gamma_1))\).

Trivially, by (28), (29) and (30),

\[ [\cdot]_\alpha \in \mathcal{L}(L^\rho(c, d; L^2_\alpha(\Omega)), L^\rho(c, d; L^\rho(\Omega, \lambda_\alpha))) \]

\[ [\cdot]_\beta \in \mathcal{L}(L^\rho(c, d; L^2_\beta(\Gamma_1)), L^\rho(c, d; L^\rho(\Gamma_1, \lambda_\beta))) \]

\[ L^\rho((c, d) \times \Omega, \lambda'_\alpha) \hookrightarrow L^\rho(c, d; [L^2_\alpha(\Omega)]') \]

\[ L^\rho((c, d) \times \Gamma_1, \lambda'_\beta) \hookrightarrow L^\rho(c, d; [L^2_\beta(\Gamma_1)]'). \]

(32)

In the sequel we shall treat the embeddings in last two lines of (32) as identifications. The same arguments used before to get (31) allows us to make the further identifications

\[ [L^\rho(c, d; L^\rho(\Gamma_1, \lambda_\beta))]' \simeq \{\beta\eta, \eta \in L^{\rho'}(c, d; L^\rho(\Gamma_1, \lambda_\beta))\}, \]

(33)

where, when \(-\infty < c < d \leq \infty, \alpha\xi \in L^1((c, d) \times \Omega)\) and \(\beta\eta \in L^1((c, d) \times \Gamma_1)\).

Next, given \(\rho, \theta \in [2, \infty)\) and \(-\infty \leq c < d \leq \infty\) we introduce the space

\[ L^2_{\alpha,\theta}(c, d) = L^\rho(c, d; L^2_{\alpha,\theta}(\Omega)) \times L^\theta(c, d; L^2_{\beta,\theta}(\Gamma_1)), \]

(34)

together with its right–local version

\[ L^2_{\alpha,\theta,\text{loc}}(c, d) = L^\rho_{\text{loc}}([c, d); L^2_{\alpha,\theta}(\Omega)) \times L^\theta_{\text{loc}}([c, d); L^2_{\beta,\theta}(\Gamma_1)). \]

(35)

We respectively endow the Hilbert spaces \(H^0\) and \(H^1\) introduced in § 1.3 with the standard inner product given, for \(w_i = (u_i, v_i) \in H^0, i = 1, 2\), by

\[ (w_1, w_2)_{H^0} = \int_\Omega u_1 u_2 \, dx + \int_{\Gamma_1} v_1 v_2 \, d\sigma, \]

(36)
and with the inner product
\[(u, v)_{H^1} = \int_\Omega \nabla u \nabla v \, dx + \int_{\Gamma_1} (\nabla u, \nabla v)_{\Gamma} \, d\sigma + \int_{\Gamma_1} uv \, d\sigma, \quad u, v \in H^1. \tag{37}\]

Its associated norm \(\| \cdot \|_{H^1} = (\cdot, \cdot)_{H^1}^{1/2}\) is equivalent to the standard one inherited from the product. We also introduce, for any \(\alpha \in L^\infty(\Omega), \beta \in L^\infty(\Gamma_1), \alpha, \beta \geq 0, \rho, \theta \in [2, \infty),\) the Banach space
\[H^{1, \rho, \theta}_{\alpha, \beta} = H^1 \cap [L^\rho_\alpha(\Omega) \times L^\theta_\beta(\Gamma_1)], \quad \| \cdot \|_{H^{1, \rho, \theta}_{\alpha, \beta}} = \| \cdot \|_{H^1} + \| \cdot \|_{L^\rho_\alpha(\Omega) \times L^\theta_\beta(\Gamma_1)}, \tag{38}\]
and
\[H^{1, \rho, \theta} = H^{1, \rho, \theta}_{1, 1} = H^1 \cap [L^\rho(\Omega) \times L^\theta(\Gamma_1)]. \tag{39}\]

Trivially
\[H^{1, \rho, \theta} \hookrightarrow H^{1, \rho, \theta}_{\alpha, \beta} \hookrightarrow H^1. \tag{40}\]

Finally, beside the main phase space \(\mathcal{H}\) introduced in (10), we also introduce the auxiliary phase spaces
\[\mathcal{H}^{\rho, \theta} = H^{1, \rho, \theta} \times H^0, \tag{41}\]
which trivially does not coincide with \(\mathcal{H}\) only when \(\rho > r_0\) or \(\theta > r_\Gamma\). Although in our main result we shall not consider super–supercritical sources, for which these spaces are needed, we shall use them when introducing weak solutions of (1). They can be useful in further studies.

### 2.3. Weak solutions for a linear version of (1)

We consider the linear evolution boundary value problem

\[
\begin{aligned}
\begin{cases}
\frac{u_{tt}}{\alpha} - \Delta u = \xi & \text{in } (0, T) \times \Omega, \\
u = 0 & \text{on } (0, T) \times \Gamma_0, \\
\frac{u_{tt}}{\alpha} + \partial_\nu u - \Delta_\Gamma u = \eta & \text{on } (0, T) \times \Gamma_1,
\end{cases}
\end{aligned}
\]  

where \(0 < T < \infty\) and \(\xi = \xi(t, x), \eta = \eta(t, x)\) are given forcing terms of the form

\[
\begin{aligned}
\begin{cases}
\xi = \xi_1 + \alpha \xi_2, & \xi_1 \in L^1(0, T; L^2(\Omega)), \quad \xi_2 \in L^{\rho'}(0, T; L^{\theta'}(\Omega, \lambda_\alpha)), \\
\eta = \eta_1 + \beta \eta_2, & \eta_1 \in L^1(0, T; L^2(\Gamma_1)), \quad \eta_2 \in L^{\rho'}(0, T; L^{\theta'}(\Gamma_1, \lambda_\beta)),
\end{cases}
\end{aligned}
\]  

where \(\alpha \in L^\infty(\Omega), \beta \in L^\infty(\Gamma_1), \alpha, \beta \geq 0\) and \(\rho, \theta \in [2, \infty)\). Hence \(\xi \in L^1((0, T) \times \Omega), \eta \in L^1((0, T) \times \Gamma_1)\) and, by (33) and (34),

\[
\begin{aligned}
\xi \in L^1(0, T; [L^2(\alpha)]'), & \quad \eta \in L^1(0, T; [L^2(\beta)]'),
\end{aligned}
\]  

so that the following definition makes sense.

**Definition 2.1.** Let \(\xi\) and \(\eta\) be given by (43). By a weak solution of (42) in \([0, T]\) we mean \(u \in L^\infty(0, T; H^1) \cap W^{1, \infty}(0, T; H^0), u' \in L^{2, \rho, \theta}_{\alpha, \beta}(0, T),\) such that the distribution identity

\[
\int_0^T \left[ -(u', \phi')_{H^0} + \int_\Omega \nabla u \nabla \phi \, dx + \int_{\Gamma_1} (\nabla u, \nabla \phi)_{\Gamma} \, d\sigma - \int_\Omega \xi \phi \, dx - \int_{\Gamma_1} \eta \phi \, d\sigma \right] = 0 \tag{45}\]

holds for all \(\phi \in C_c((0, T); H^1) \cap C^1_c((0, T); H^0) \cap L^{2, \rho, \theta}_{\alpha, \beta}(0, T).\)
Lemma 2.2. Any weak solution $u$ of (42), one has $u' = (u_t, (u_t)_t)$. Since the two components of $u'$, respectively acting in $(0, T) \times \Omega$ and in $(0, T) \times \Gamma_1$, cannot be confused, with a slight abuse we shall denote, for simplicity, $(u_t)_t = u_t$. Hence we shall systematically denote in the paper

$$u' = (u_t, u_t) \quad \text{and} \quad U = (u, u') \in L^\infty(0, T; \mathcal{H}).$$

(46)

We recall [62, Lemma 2.2].

**Lemma 2.2.** Any weak solution $u$ of (42) enjoys the further regularity $U \in C([0, T]; \mathcal{H})$. Moreover it satisfies the following identities:

i) the energy identity

$$\frac{1}{2} \|u\|^2_{\mathcal{H}^0} + \frac{1}{2} \int_\Omega |\nabla u|^2 \, dx + \frac{1}{2} \int_{\Gamma_1} |\nabla u|_{\Gamma_1}^2 \, d\sigma \bigg|_s^t = \int_s^t \left( \int_\Omega \xi u_t \, dx + \int_{\Gamma_1} \eta u_t \, d\sigma \right) \, dr$$

for $0 \leq s \leq t \leq T$

ii) the generalized distribution identity

$$(u', \phi)_{\mathcal{H}^0}^T + \int_0^T \left[ -(u', \phi)'_{\mathcal{H}^0} + \int_\Omega \nabla u \nabla \phi \, dx \right. \left. + \int_{\Gamma_1} (\nabla_\Gamma u, \nabla_\Gamma \phi)_{\Gamma} \, d\sigma - \int_\Omega \xi \phi \, dx - \int_{\Gamma_1} \eta \phi \, d\sigma \right] = 0$$

for all $\phi \in C([0, T]; H^1) \cap C^1([0, T]; H^0) \cap L^{2,q,p}_\alpha,\beta(0, T)$.

3. Preliminaries.

3.1. Main assumptions. With reference to problem (1) we suppose that

(A1) $P$ and $Q$ are Carathéodory functions, respectively in $\Omega \times \mathbb{R}$ and $\Gamma_1 \times \mathbb{R}$, that $P(x, v) \geq 0$ a.e. in $\Omega \times \mathbb{R}$, $Q(x, v) \geq 0$ a.e. on $\Gamma_1 \times \mathbb{R}$, and there are $\alpha \in L^\infty(\Omega), \beta \in L^\infty(\Gamma_1), \alpha, \beta \geq 0$, and $1 < \bar{m} \leq m$, $1 < \bar{\mu} \leq \mu$, $c_m, c_\mu \geq 0$, such that

$$|P(x, v)| \leq c_m \alpha(x) (|v|^\bar{m} - 1 + |v|^{m-1}) \quad \text{for a.a. } x \in \Omega, \forall v \in \mathbb{R};$$

$$|Q(x, v)| \leq c_\mu \beta(x) (|v|^\bar{\mu} - 1 + |v|^{\mu-1}) \quad \text{for a.a. } x \in \Gamma_1, \forall v \in \mathbb{R};$$

(47)

(A2) $f$ and $g$ are Carathéodory functions, respectively in $\Omega \times \mathbb{R}$ and $\Gamma_1 \times \mathbb{R}$, and there $p, q \geq 2, c_p, c_q \geq 0$ such that

$$|f(x, u)| \leq c_p (1 + |u|^{p-1}), \quad \text{for a.a. } x \in \Omega \text{ and all } u \in \mathbb{R},$$

$$|g(x, u)| \leq c_q (1 + |u|^{q-1}), \quad \text{for a.a. } x \in \Gamma_1 \text{ and all } u \in \mathbb{R};$$

(48)

(A3) $p \leq 1 + r_\alpha/2$ or $\text{essinf}_\Omega \alpha > 0$, $q \leq 1 + r_\beta/2$ or $\text{essinf}_{\Gamma_1} \beta > 0$, and

$$2 \leq p \leq 1 + r_\alpha/\bar{m}, \quad 2 \leq q \leq 1 + r_\beta/\bar{\mu},$$

(49)

where $\bar{m}$ and $\bar{\mu}$ are given by (7).

When $P(x, v) = \alpha(x) P_0(v)$ and $Q(x, v) = \beta(x) Q_0(v)$ with $\alpha \in L^\infty(\Omega)$ and $\beta \in L^\infty(\Gamma_1)$, $\alpha, \beta \geq 0$, assumption (A1) trivially holds when $P_0, Q_0 \in C(\mathbb{R}), P_0(v) \geq 0, Q_0(v) \geq 0$, and there are $1 < \bar{m} \leq m, 1 < \bar{\mu} \leq \mu$ such that

$$Q_0(v) = O(|v|^\bar{m} - 1), \quad P_0(v) = O(|v|^\bar{\mu} - 1) \quad \text{as } v \to 0,$$

$$Q_0(v) = O(|v|^{m-1}), \quad P_0(v) = O(|v|^{\mu-1}) \quad \text{as } |v| \to \infty.$$
\[ P(x,v) = f_1(u) := \gamma |u|^{\gamma - 2}u + \tilde{\gamma}|u|^{\tilde{\gamma} - 2}u + \tilde{\gamma}', \]
\[ p(x,u) = g_1(u) := \delta |u|^{\delta - 2}u + \tilde{\delta}|u|^{\tilde{\delta} - 2}u + \tilde{\delta}', \]

provided
\[ a, b, \alpha, \beta \geq 0, \quad \gamma, \tilde{\gamma}, \tilde{\gamma}', \delta, \tilde{\delta}, \tilde{\delta}' \in \mathbb{R}, \]
\[ 1 < \tilde{m} \leq m, \quad 1 < \tilde{\mu} \leq \mu, \quad 2 \leq \tilde{p} \leq p, \quad 2 \leq \tilde{q} \leq q. \]

Moreover trivially assumption (A3) hold true provided
\[ p \leq \begin{cases} 1 + r_{\tilde{\mu}}/2 & \text{if } \gamma \neq 0, \alpha = 0, \\ 1 + r_{\tilde{\mu}}/\tilde{m}' & \text{if } \gamma = 0, \alpha > 0, \end{cases} \quad q \leq \begin{cases} 1 + r_{\tilde{\mu}}/2 & \text{if } \delta \neq 0, \beta = 0, \\ 1 + r_{\tilde{\mu}}/\tilde{p}' & \text{if } \delta = 0, \beta > 0, \end{cases} \]
\[ \tilde{p} \leq \begin{cases} 1 + r_{\tilde{\mu}}/2 & \text{if } \tilde{\gamma} \neq 0, \alpha = 0, \\ 1 + r_{\tilde{\mu}}/\tilde{m}' & \text{if } \tilde{\gamma} = 0, \alpha > 0, \end{cases} \quad \tilde{q} \leq \begin{cases} 1 + r_{\tilde{\mu}}/2 & \text{if } \tilde{\delta} \neq 0, \beta = 0, \\ 1 + r_{\tilde{\mu}}/\tilde{p}' & \text{if } \tilde{\delta} = 0, \beta > 0. \end{cases} \]
3.2. Weak solutions. To define weak solutions of problem (1) we first point out the following easy result, noticing that in the sequel we shall denote by \( \hat{P} \), \( \hat{Q} \), \( \hat{f} \) and \( \hat{g} \) the Nemitskii operators respectively associated to \( P \), \( Q \), \( f \) and \( g \) (see [2, Definition 2.1, p. 15] or [50, Definition 10.57, p. 370]).

**Lemma 3.1.** Let assumptions (A1–3) hold and \( u \in L^\infty(0, T; H^1) \cap W^{1, \infty}(0, T; H^0) \), \( u' \in L^{2,p,q}_\alpha(0, T) \) for some \( 0 < T < \infty \). Then \( \xi = \hat{f}(u) - \hat{P}(u_t) \) and \( \eta = \hat{g}(u_{1T}) - \hat{Q}(u_t) \) are of the form (43), with \( \rho = \overline{m} \) and \( \theta = \overline{\rho} \).

**Proof.** We first remark that classical results on Nemitskii operators (see [2, Theorem 2.2, p. 16]) trivially extend to abstract measure spaces. Hence, by (47), the Nemitskii operator associated to \( P/\alpha \), this function being \( \lambda_\alpha' \) a.e. well defined, is continuous from \( L^\overline{m}((0,T) \times \Omega, \lambda_\alpha') \) to \( L^\overline{m}((0,T) \times \Omega, \lambda_\alpha') \). Since, by (28), (32) and (34) we have \( [u_t]_\alpha \in L^\overline{m}((0,T) \times \Omega, \lambda_\alpha') \), we consequently get \( [P(\cdot, u_t)/\alpha]_\alpha \in L^\overline{m}((0,T) \times \Omega, \lambda_\alpha') \).

Next, when \( p \leq 1 + r_2/2 \), by (48) from \( u \in L^\infty(0, T; H^1(\Omega)) \) and Sobolev Embedding Theorem we get \( \hat{f}(u) \in L^\infty(0, T; L^2(\Omega)) \). When \( p > 1 + r_2/2 \), by assumption (A3) we have \( \text{essinf}_\Omega \alpha > 0 \) so \( L^\overline{m}((0,T) \times \Omega, \lambda_\alpha') = L^\overline{m}((0,T) \times \Omega) \), the norms being equivalent. By (48)–(49) we thus get \( \hat{f}(u) \in L^\infty(0, T; L^m(\Omega)) \). Consequently, as \( 1/\alpha \in L^\infty(\Omega) \), \( \hat{f}(u) = \alpha \xi_2 \), \( \xi_2 \in L^\overline{m}((0,T) \times \Omega, \lambda_\alpha') \). Hence in both cases \( \xi \) is in the form prescribed by (43) with \( \rho = \overline{m} \).

The same arguments show that \( \eta \) is in the form prescribed by (43) with \( \theta = \overline{\rho} \). □

Thanks to Lemmas 2.2 and 3.1 the following definition makes sense.

**Definition 3.2.** Let assumptions (A1–3) hold and \( U_0 = (u_0, u_1) \in \mathcal{H} \). By a weak solution of problem (1) in \([0,T]\), \( 0 < T < \infty \), we mean a weak solution of (42) with

\[
\xi = \hat{f}(u) - \hat{P}(u_t), \quad \eta = \hat{g}(u_{1T}) - \hat{Q}(u_t), \quad \rho = \overline{m} \quad \text{and} \quad \theta = \overline{\rho},
\]

such that \( U(0) = U_0 \). By a weak solution of (1) in \([0,T]\), \( 0 < T \leq \infty \), we mean \( u \in L^\infty_\text{loc}([0,T]; H^1) \) which is a weak solution of (1) in \([0,T']\) for any \( T' \in (0,T) \). Such a solution is called maximal if it has no proper extensions and global if \( T = \infty \).

We now introduce the primitives of \( f \) and \( g \)

\[
F(x,u) = \int_0^u f(x, \tau) \, d\tau, \quad \text{and} \quad G(x,u) = \int_0^u g(x, \tau) \, d\tau,
\]

the potential functional \( J : H^{1,p,q} \to \mathbb{R} \) given by

\[
J(u) = \int_{\Omega} F(\cdot, u) \, dx + \int_{\Gamma_1} G(\cdot, u) \, d\sigma,
\]

and the energy functional \( \mathcal{E} : H^{p,q} \to \mathbb{R} \) given by

\[
\mathcal{E}(u,v) = \frac{1}{2} ||v||^2_{H^p} + \frac{1}{2} \int_{\Omega} |\nabla u|^2 \, dx + \frac{1}{2} \int_{\Gamma_1} |\nabla_{\Gamma} u|^2 \, d\sigma - J(u).
\]

By standard results on Nemitskii and potential operators (see [2, pp. 16–22]) we have \( J \in C^1(H^{1,p,q}) \), with Fréchet derivative \( J' = (\hat{f}, \hat{g}) \), and \( \mathcal{E} \in C^1(H^{p,q}) \).

Weak solutions of (1) enjoy good properties, as shown in the next result.

**Lemma 3.3.** Let assumptions (A1–3) hold and \( u \) be a weak solution of (1) in \([0,T]\). Then
i) $U \in C([0,T];H)$ and the energy identity

$$\frac{1}{2} \|u'\|_{H^0}^2 + \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx + \frac{1}{2} \int_{\Gamma_1} |\nabla u|^2 d\sigma + \int_s^t \left( \int_{\Omega} \hat{P}(u_t) u_\tau dx \right) ds \geq 0$$

holds for all $0 \leq s \leq t < T$;

ii) the generalized distribution identity

$$\int_0^t \left[ -(u', \phi)_H + \int_{\Omega} \nabla u \nabla \phi dx + \int_{\Gamma_1} (\nabla \Gamma_1 u, \nabla \Gamma_1 \phi) d\sigma \right] dt$$

$$+ \int_{\Gamma_1} \hat{Q}(u_t) \phi d\sigma - \int_{\Omega} \hat{f}(u) \phi dx - \int_{\Gamma_1} \hat{g}(u) \phi d\sigma = 0$$

holds for all $t \in [0,T)$ and $\phi \in C([0,T);H^1) \cap C^1([0,T);H^0) \cap L^{2,p,q}(0,T)$.

iii) when $U_0 \in H^{p,q}$ we have $U \in C([0,T],H^{p,q})$ and

$$J(u(t)) - J(u(s)) = \int_s^t \left( \int_{\Omega} \hat{f}(u_t) u_\tau dx + \int_{\Gamma_1} \hat{g}(u_t) u_\tau d\sigma \right) d\tau,$$

$$E(U(t)) - E(U(s)) = \int_s^t \left( \int_{\Omega} \hat{P}(u_t) u_\tau dx + \int_{\Gamma_1} \hat{Q}(u_t) u_\tau d\sigma \right) d\tau = 0$$

for $0 \leq s \leq t < T$.

**Proof.** Trivially i–ii) follow from Definition 3.2 and Lemma 2.2. To prove iii) we take $U_0 \in H^{p,q}$. We first claim that $u \in C([0,T);H^1(\Omega) \cap L^p(\Omega))$. When $p \leq r_\Omega$, by Sobolev Embedding Theorem, there is nothing to prove, so let us take $p > r_\Omega$. By (49) it immediately follows $p > r_\Omega$, so $p^2 > (r_\Omega + 1)p + r_\Omega > 0$, that is $1 + r_\Omega/p < \bar{m}$ and, again by (49), $p < \bar{m}$. Consequently, since by assumption (A3) we have $\text{essinf}_\Omega \alpha > 0$, $u_t \in L^{m_\alpha}_{loc}(0,T;L^p(\Omega))$. Since $u_\Omega \in L^p(\Omega)$ and $u(s) = u_\Omega + \int_0^s u_t(\tau) d\tau$ in $L^2(\Omega)$ for $s \in [0,T)$, we get $u \in W^{1,m}(0,T;L^p(\Omega)) \hookrightarrow C([0,T];L^p(\Omega))$ for all $t \in [0,T)$, proving our claim. The same arguments also show that $u_{\mid \Gamma} \in C([0,T);H^1(\Gamma_1) \cap L^2(\Gamma_1))$, so proving that $U \in C([0,T),H^{p,q})$.

To prove (62) we introduce the auxiliary exponents

$$m_p = \begin{cases} 2 & \text{if } p \leq 1 + r_\Omega/2, \\ \bar{m} & \text{if } p > \max\{m,1 + r_\Omega/2\}, \\ p & \text{if } 1 + r_\Omega/2 < p \leq \bar{m}, \end{cases}$$

$$\mu_q = \begin{cases} 2 & \text{if } q \leq 1 + r_\Omega/2, \\ \bar{\mu} & \text{if } q > \max\{m,1 + r_\Omega/2\}, \\ q & \text{if } 1 + r_\Omega/2 < q \leq \bar{\mu}, \end{cases}$$

and we claim that $\tilde{f} \in C(H^1(\Omega) \cap L^p(\Omega))$. We first remark that, by standard properties of Nemitskii operators and Sobolev Embedding Theorem, $\tilde{f} \in C(L^p(\Omega);L^q(\Omega))$ and, when $N \geq 3$ so $r_\Omega < \infty$, $\tilde{f} \in C(H^1(\Omega);L^{m'/q}(\Omega))$. We now consider the three cases in (64). When $p \leq 1 + r_\Omega/2$ we have $\tilde{f} \in C(H^1(\Omega);L^2(\Omega)) = C(H^1(\Omega);L^{m'/2}(\Omega))$. When $p > \max\{m,1 + r_\Omega/2\}$ by (49) it follows that $\tilde{f} \in C(H^1(\Omega);L^{m}/(\Omega)) = C(H^1(\Omega);L^{m'/p}(\Omega))$. Finally, when $1 + r_\Omega/2 < p \leq \bar{m}$ we have $\tilde{f} \in C(L^p(\Omega);L^2(\Omega)) = C(L^p(\Omega);L^{m'/p}(\Omega))$, so proving our claim. By the same arguments we get that $\tilde{g} \in C(H^1(\Gamma_1) \cap L^q(\Gamma_1);L^{m'/q}(\Gamma_1))$, and then

$$J' = (\tilde{f}, \tilde{g}) \in C(H^{1,p,q};L^{m'/p}(\Omega) \times L^{m'/q}(\Gamma_1)).$$
Next we remark that, for any \( t \in [0, T) \) we have \( u_t \in L^1(0, t; L^{m_p}(\Omega)) \) and \( u_t \in L^1(0, t; L^{p_s}(\Gamma_1)) \) in all three cases considered in (64), so
\[
 u \in W^{1,1}(0, t; L^{m_p}(\Omega) \times L^{p_s}(\Gamma_1)).
\] (66)

By (65)–(66) we can then apply the abstract version of the classical chain rule proved in [63, Lemma 7.1], by taking
\[
X
\]
By (65)–(66) we can then apply the abstract version of the classical chain rule proved in [63, Lemma 7.1], by taking \( X_1 = H^{1-p,q} \) and \( Y_1 = L^{m_p}(\Omega) \times L^{p_s}(\Gamma_1) \), from which we get that \( J \cdot u \in W^{1,1}(0, t) \) and \( (J \cdot u') = \oint_\Omega f(u)u_t + \oint_{\Gamma_1} \hat{g}(u)u_t \) a.e. in \( (0, t) \). Being \( t \in [0, T) \) arbitrary (62) follows. Finally (63) simply follows by recalling (59) and combining (60) with (63).

3.3. An elementary result. We conclude this section by pointing out the following elementary result, which should be well-known, but for which we do not have a precise reference. We also sketch its proof for the reader’s convenience.

**Lemma 3.4.** Let \( l > 1, c > 0, 0 < T < \infty \) and \( \psi \in W^{1,1}_{\text{loc}}([0, T)) \) be such that
\[
\begin{align*}
\psi' &\geq |\psi|^l - c \quad \text{a.e. in } (0, T) \\
\psi(0) &= \psi_0 > c^{1/l}.
\end{align*}
\] (67)

Then \( T \leq T_m(\psi_0) := \int_{\psi_0}^{\infty} \frac{dr}{r^{1-c}} < \infty \) and \( \psi(t) \to \infty \) as \( t \to T_m(\psi_0)^- \) provided \( T = T_m(\psi_0) \).

**Proof.** We first consider the Cauchy problem \( y' = |y|^l - c, \ y(0) = \psi_0 \). Since \( \psi_0 > c^{1/l} \), by standard ODE’s Theory and separation of variables, it has a unique maximal classical solution \( y \in C^2(\mathbb{R}, T_m(\psi_0)) \) given by \( y(t) = B^{-1}(t) \), where \( B : (c^{1/l}, \infty) \to (-\infty, T_m(\psi_0)) \) is strictly increasing and surjective, so \( y(t) \to \infty \) as \( t \to T_m(\psi_0)^- \). Then, since the standard comparison argument for ODE’s (see for example [55, Chapter 1, Theorem 1.3, p. 27]) trivially extends to generalized solutions, since the function \( y \to |y|^l - c \) is locally Lipschitz continuous, and since \( y' - |y|^l - c \leq \psi' - |\psi|^l - c \) and \( y(0) = \psi(0) \) by (67), by comparison we get \( y \leq \psi \) in \([0, T)\), from which the entire statement follows.

4. Global nonexistence with two sources. In this section we state and prove our first main global nonexistence result for weak solutions of (1) under the following additional specific assumptions, which clearly imply that \( f \neq 0 \) and \( g \neq 0 \).

**(F1)** There are \( \gamma_0 > 0 \) and \( \gamma_1 \geq 0 \) such that
\[
f(x, u)u - 2F(x, u) \geq \gamma_0 |u|^p - \gamma_1 \quad \text{for a.a. } x \in \Omega \text{ and all } u \in \mathbb{R};
\]

**(G1)** there are \( \delta_0 > 0 \) and \( \delta_1 \geq 0 \) such that
\[
g(x, u)u - 2G(x, u) \geq \delta_0 |u|^q - \delta_1 \quad \text{for a.a. } x \in \Gamma_1 \text{ and all } u \in \mathbb{R}.
\]

We now check that, when (A2) holds, assumptions (F1) and (G1) are respectively equivalent to the following ones:

**(F1)’** there are \( \gamma_2 > 0 \) and \( M_f \geq 0 \) such that
\[
f(x, u)u - 2F(x, u) \geq \gamma_2 |u|^p \quad \text{for a.a. } x \in \Omega \text{ and all } |u| \geq M_f;
\]

**(G1)’** there are \( \delta_2 > 0 \) and \( M_g \geq 0 \) such that
\[
g(x, u)u - 2G(x, u) \geq \delta_2 |u|^q \quad \text{for a.a. } x \in \Gamma_1 \text{ and all } |u| \geq M_g.
\]
Indeed, when (F1) holds, we get (F1)' by choosing $\gamma_2 = \gamma_0/2$ and $M_f = (2\gamma_1/\gamma_0)^{1/p}$.

Conversely, when (F1)' holds, then (F1) is trivial when $M_f = 0$. When $M_f > 0$ it also follows by (F1)' by choosing $\gamma_0 = \min\{\gamma_2, c_f/2M_f^p\}$ and $\gamma_1 = 3c_f/2$, where $c_f = 3\gamma_0(M_f + M_f^p)$. Indeed when $|u| \geq M_f$ then the inequality in (F1) holds. When $|u| \leq M_f$, by (48) and (57) one easily gets that $|f(x, u)u - 2F(x, u)| \leq c_f$ for a.a. $x \in \Omega$. Since we also have $\gamma_0|u|^p - \gamma_1 \leq \frac{1}{2}\gamma_0M_f^p - \frac{3}{2}c_f \leq -\gamma_0$ we get (F1).

The equivalence between (G1) and (G1)' is checked by the same arguments.

Hence, when $f(x, u) = f_0(u)$ and $g(x, u) = g_0(u)$, with $f_0, g_0 \in C(\mathbb{R})$ verifying (62), denoting by $F_0$ and $G_0$ their primitives still defined by (57), assumptions (F1) and (G1) respectively reduce to

$$\lim_{|u| \to \infty} \frac{f_0(u)u - 2F_0(u)}{|u|^p} > 0, \quad \text{and} \quad \lim_{|u| \to \infty} \frac{g_0(u)u - 2G_0(u)}{|u|^q} > 0. \quad (68)$$

**Remark 5.** When dealing with the model nonlinearities $f_1$ and $g_1$ defined in (52), conditions (68) respectively hold when

$$\gamma > 0, \quad 2 \leq p < \bar{p}, \quad \text{and} \quad \delta > 0, \quad 2 \leq \bar{q} < q. \quad (69)$$

When restricting to the case $\tilde{\gamma} = \tilde{\gamma}' = \tilde{\delta} = \tilde{\delta}' = 0$ and $\gamma, \delta \geq 0$, as in problem (2), see Remark 4, (69) respectively reduce to

$$\gamma > 0, \quad p > 2, \quad \text{and} \quad \delta > 0, \quad q > 2, \quad (70)$$

so that assumptions (A1–3), (F1) and (G1) hold true for problem (2) provided (3), (6) and (70) hold.

Our first main global nonexistence result is the following one.

**Theorem 4.1.** Let assumptions (A1–3), (F1), (G1) hold, and

$$p > \bar{m}, \quad q > \bar{p}. \quad (71)$$

Then, for any $U_0 \in \mathcal{H}$ such that $\mathcal{E}(U_0) < 0$ problem (1) does not admit global weak solutions.

**Proof.** We first recall that, as explained in Remark 4, by (49) and (71) it follows that $\bar{m}, p < r_1$ and $\bar{p}, q < r_2$, so $\mathcal{H}^{p,q} = \mathcal{H}$ in Lemma 3.3–iii).

The proof is based on a contradiction argument, so let $u$ be a global weak solution of (1) with $\mathcal{E}(U_0) < 0$. We introduce the auxiliary function

$$\mathcal{K}(t) = -\mathcal{E}(U(t)). \quad (72)$$

By Lemma 3.3–iii) and assumption (A1) the function $\mathcal{K}$ belongs to $W_{\text{loc}}^{1,1}([0, \infty))$ and

$$\mathcal{K}'(t) = \int_{\Omega} \tilde{P}(t)u(t)u(t) \, dx + \int_{\Gamma_1} \tilde{Q}(t)u(t)u(t) \, d\sigma \geq 0 \quad \text{for a.a. } t > 0. \quad (73)$$

Consequently, since $\mathcal{E}(U_0) < 0$, by (72) and (59) we have

$$0 < \mathcal{K}_0 := \mathcal{K}(0) \leq \mathcal{K}(t) \leq J(u(t)) \quad \text{for all } t \geq 0. \quad (74)$$

We now remark that, by assumption (A2), one easily gets the existence of $c_p', c_q' > 0$ such that

$$|F(x, u)| \leq c_p'(1 + |u|^p) \quad \text{for a.a. } x \in \Omega \text{ and all } u \in \mathbb{R},$$

$$|G(x, u)| \leq c_q'(1 + |u|^q) \quad \text{for a.a. } x \in \Gamma_1 \text{ and all } u \in \mathbb{R}. \quad (75)$$
By (58), (73) and (74) we thus get
\[ 0 < K_0 \leq \mathcal{K}(t) \leq J_1(u(t)) \quad \text{for all } t \geq 0, \tag{76} \]
where \( J_1 \) is defined by
\[ J_1(u) = c_p' \| u \| + c_p' \| u \|_p + c_p' \| u \|_q, \quad u \in H^1. \tag{77} \]
By Lemma 3.3 we can take \( \phi = u \) as a test function in the generalized distribution identity (61). Omitting in the sequel, for the sake of simplicity, the explicit dependence of \( u \) and \( u' \) on \( t \), using (59) and (72) we get
\[
\frac{d}{dt} \langle u', u \rangle_{H^0} = \| u' \|^2_{H^0} - \| \nabla u \|^2_{2, 1, 1} \\
+ \int_{\Omega} \hat{f}(u)u \, dx + \int_{\Gamma_1} \hat{g}(u)u \, d\sigma - \int_{\Omega} \hat{P}(u_t)u \, dx - \int_{\Gamma_1} \hat{Q}(u_t)u \sigma \\
= 2\| u' \|^2_{H^0} + 2\mathcal{K}(t) + \int_{\Omega} \left[ f(\cdot, u)u - 2F(\cdot, u) \right] \, dx \\
+ \int_{\Gamma_1} [g(\cdot, u)u - 2G(\cdot, u)] d\sigma - \int_{\Omega} \hat{P}(u_t)u \, dx - \int_{\Gamma_1} \hat{Q}(u_t)u \, d\sigma
\tag{78}
\]
for all \( t \geq 0 \). Using assumptions (F1) and (G1) in (78) we obtain
\[
\frac{d}{dt} \langle u', u \rangle_{H^0} \geq 2\| u' \|^2_{H^0} + 2\mathcal{K}(t) + \gamma_0 \| u \|^p + \delta_0 \| u \|^q - \gamma_1 |\Omega| - \delta_1 \sigma(\Gamma_1) \\
- \int_{\Omega} \hat{P}(u_t)u \, dx - \int_{\Gamma_1} \hat{Q}(u_t)u \, d\sigma \quad \text{for all } t \geq 0. \tag{79}
\]
In the sequel we shall introduce several positive constants depending on \( \Omega, \Gamma_1, P, Q, f \) and \( g \), on the various constants appearing in the assumptions, and on the initial data \( U_0 \). Since they are fixed we shall not give further notice of this dependence and we shall denote these constants by \( c_i, i \in \mathbb{N} \). We shall denote positive constants depending also on other objects \( \Upsilon_1, \ldots, \Upsilon_n \) by \( c_i = c_i(\Upsilon_1, \ldots, \Upsilon_n), i \in \mathbb{N} \).

We now preliminarily estimate from above the last two terms in the right-hand side of (79). By (55), Hölder inequality and (73), noticing that both integrals in it are nonnegative, we get
\[
\mathcal{I}_1(t) := \int_{\Omega} \hat{P}(u_t)u \leq \int_{\Omega} |\hat{P}(u_t)||u| \\
\leq c_m' \left[ \int_{\Omega} (\hat{P}(u_t)u_t)^{1/m'} ||u||_m \, dx + \int_{\Omega} (\hat{P}(u_t)u_t)^{1/m'} ||u||_m \, dx \right] \\
\leq c_m' \left\{ [\mathcal{K}'(t)]^{1/m'} ||u||_m + [\mathcal{K}'(t)]^{1/m'} ||u||_m \right\} \quad \text{for all } t \geq 0.
\]
Since \( \tilde{m} \leq m < p \), using (76) and (77), previous estimate yields that
\[
\mathcal{I}_1(t) \leq c_1 \left\{ [\mathcal{K}'(t)]^{1/m'} + [\mathcal{K}'(t)]^{1/m'} \right\} [J_1(u)]^{1/p} \\
= c_1 \left\{ [\mathcal{K}'(t)]^{1/m'} [J_1(u)]^{1/m} + [\mathcal{K}'(t)]^{1/m} [J_1(u)]^{1/m} \right\} \tag{80} \\
\leq c_2 \left\{ [\mathcal{K}'(t)]^{1/m'} [J_1(u)]^{1/m} + [\mathcal{K}'(t)]^{1/m} [J_1(u)]^{1/m} \right\} \quad \text{for all } t \geq 0.
\]
for all \( t \geq 0 \). Setting \( k_1 = 1/m - 1/p \in (0, 1) \), and using (76) again, we get
\[
\mathcal{I}_1(t) \leq c_3 \left\{ [\mathcal{K}'(t)]^{1/m} [J_1(u)]^{1/m} + [\mathcal{K}'(t)]^{1/m} [J_1(u)]^{1/m} \right\} [\mathcal{K}(t)]^{k_1} \quad \text{for all } t \geq 0.
\]
For any $\varepsilon \in (0, 1]$, to be conveniently fixed in the sequel, using weighted Young inequality we then get

$$
\mathcal{I}_1(t) \leq c_3 \left[ (e^{-m'\varepsilon} + e^{-m'\varepsilon}) K'(t) + (e^m + e^m) J_1(u) \right] [K(t)]^{-k_1} \leq c_4 \left[ e^{\hat{m}} J_1(u) + e^{-m'\varepsilon} K'(t) \right] [K(t)]^{-k_1} \quad \text{for all } t \geq 0.
$$

(81)

Using exactly the same arguments and setting $k_2 = 1/\mu - 1/p \in (0, 1)$ we get

$$
\mathcal{I}_2(t) := \int_{\Gamma_1} \hat{Q}(u) u \, d\sigma \leq c_5 \left[ e^{\hat{m}} J_1(u) + e^{-m'\varepsilon} K'(t) \right] [K(t)]^{-k_2} \quad \text{for all } t \geq 0.
$$

(82)

We combine (81) and (82) by setting $k = \min\{k_1, k_2\} \in (0, 1)$ and using (76) again, so getting

$$
\mathcal{I}_1(t) + \mathcal{I}_2(t) \leq c_6 \left[ (e^{m_1} + e^{\hat{m}}) J_1(u) + (e^{-m_1} + e^{-\hat{m}}) K'(t) \right] [K(t)]^{-\kappa} \quad \text{for all } t \geq 0.
$$

(83)

Setting $m_l = \min\{m, \hat{m}\} > 1$ the last estimate is simplified to

$$
\mathcal{I}_1(t) + \mathcal{I}_2(t) \leq c_7 \left[ e^{m_1} J_1(u) + e^{-m_1} K'(t) \right] [K(t)]^{-\kappa} \quad \text{for all } t \geq 0.
$$

(84)

After these preliminary estimates we now introduce the main Lyapunov functional

$$
Z(t) = [K(t)]^{1-k} + \omega(u^*, u)_{H^0}, \quad k \in (0, \overline{k}), \quad \omega > 0,
$$

where $k$ and $\omega$ will be conveniently fixed in the sequel. By (79) and (84)

$$
Z'(t) = (1-k)[K(t)]^{-k} K'(t) + \omega \frac{d}{dt}(u^*, u)_{H^0}
$$

$$
\geq (1-k)[K(t)]^{-k} K'(t) + \omega \left[ 2\|u^*\|_{H^0}^2 + 2K(t) + \gamma_0\|u\|_{L^2}^p + \delta_0\|u\|_{q, \Gamma_1}^q \right] - \gamma_1\|\Omega\| - \delta_1\sigma(\Gamma_1) + \int_{\Gamma_1} \hat{P}(u) u \, dx - \int_{\Gamma_1} \hat{Q}(u) u \, d\sigma \quad \text{for all } t \geq 0.
$$

Using the estimate (83) in it and using (76) once again we get

$$
Z'(t) \geq (1-k)[K(t)]^{-k} K'(t) + \omega \left[ 2\|u^*\|_{H^0}^2 + 2K(t) + \gamma_0\|u\|_{L^2}^p + \delta_0\|u\|_{q, \Gamma_1}^q \right] - \gamma_1\|\Omega\| - \delta_1\sigma(\Gamma_1) - c_7\epsilon^{m_1} J_1(u)[K(t)]^{-\kappa} - c_7\epsilon^{-m_1} K'(t)[K(t)]^{-\kappa} \geq (1-k)[K(t)]^{-k} K'(t) + 2\omega\|u^*\|_{H^0}^2 + 2\omega K(t) + \omega\gamma_0\|u\|_{L^2}^p + \omega\delta_0\|u\|_{q, \Gamma_1}^q - \omega\gamma_1\|\Omega\| - \omega\delta_1\sigma(\Gamma_1) - c_7\omega K_0^{-k}\epsilon^{m_1} J_1(u)[K(t)]^{-\kappa} - c_7\omega K_0^{-k}\epsilon^{-m_1} K'(t)[K(t)]^{-\kappa} \quad \text{for all } t \geq 0.
$$

for $t \geq 0$. Hence, setting $C_1 = C_1(k) = c_7 K_0^{-k\kappa}$ and reordering

$$
Z'(t) \geq (1-k - \omega C_1\epsilon^{-m_1})[K(t)]^{-k} K'(t) + 2\omega\|u^*\|_{H^0}^2 + 2\omega K(t) + \omega\gamma_0\|u\|_{L^2}^p + \omega\delta_0\|u\|_{q, \Gamma_1}^q - \omega\gamma_1\|\Omega\| + \omega\delta_1\sigma(\Gamma_1) - \omega C_1\epsilon^{m_1} J_1(u)[K(t)]^{-\kappa}
$$

for all $t \geq 0$. Using (76)–(77) in the last estimate we get

$$
Z'(t) \geq (1-k - \omega C_1\epsilon^{-m_1})[K(t)]^{-k} K'(t) + \omega\|u^*\|_{H^0}^2 + \omega K(t) + \omega\tilde{K}_0
$$

$$
+ \omega\gamma_0\|u\|_{L^2}^p + \omega\delta_0\|u\|_{q, \Gamma_1}^q - \omega\gamma_1\|\Omega\| + \omega\delta_1\sigma(\Gamma_1) - \omega C_1\epsilon^{m_1} K_0^{-k}\epsilon J_1(u)
$$

(85)

$$
= (1-k - \omega C_1\epsilon^{-m_1})[K(t)]^{-k} K'(t) + \omega\|u^*\|_{H^0}^2 + \omega K(t)
$$

$$
\omega\gamma_0\|u\|_{L^2}^p + \omega\delta_0\|u\|_{q, \Gamma_1}^q - \omega\gamma_1\|\Omega\| + \omega\delta_1\sigma(\Gamma_1) - \omega C_1\epsilon^{m_1} K_0^{-k}\epsilon J_1(u)
$$

$$
+ \omega\left[ \gamma_0 - C_1\epsilon^{m_1} K_0^{-k}\epsilon \right] \|u\|_{L^2}^p + \omega\left[ \delta_0 - C_1\epsilon^{m_1} K_0^{-k}\epsilon \right] \|u\|_{q, \Gamma_1}^q
$$

$$
+ \omega\left\{ K_0 - C_1\epsilon^{m_1} K_0^{-k}\{\gamma_0 + \delta_1\sigma(\Gamma_1)\} - \omega c_8 \right\} \quad \text{for all } t \geq 0.
$$
Remembering that $C_1 = C_1(k)$ we now choose $\varepsilon = \varepsilon_0(k) \in (0,1]$ so small that
\[
K_0 - C_1(k)\varepsilon^{m_k}K_0^{-k} \mid \gamma_1 + \delta_0(k) \leq 0,
\]
\[
\gamma_0 - C_1(k)\varepsilon^{m_k}K_0^{-k}c_3' \geq \gamma_0 / 2, \quad \delta_0 - C_1(k)\varepsilon^{m_k}K_0^{-k}c_3' \geq \delta_0 / 2.
\]
With this choice (85) yields
\[
Z'(t) \geq \left[1 - k - \omega C_1(k)\varepsilon^{m_k} \right] K(t)^{k} + \omega||u'||^2_{H^0} + \omega K(t)
\]
\[
+ \frac{1}{2}\omega_\gamma_0 ||u'||^p_p + \frac{1}{2}\omega_\delta_0 ||u||^{q}_{q,1_1} - \omega c_8 \quad \text{for all } t \geq 0.
\]
We now take $\omega \in (0,\omega_1(k))$, where $\omega_1(k) := (1 - k)\varepsilon^{m_k} C_1^{-1}(k)$, so that $1 - k - \omega C_1(k)\varepsilon^{m_k} \leq 0$ and, as $K' \geq 0$, previous estimate yields
\[
Z'(t) \geq \omega \left[ ||u'||^2_{H^0} + K(t) + \frac{1}{2}\omega_\gamma_0 ||u'||^p_p + \frac{1}{2}\omega_\delta_0 ||u||^{q}_{q,1_1} - \omega c_8 \right] \quad \text{for all } t \geq 0.
\]
Hence, taking $c_0 = \min\{1, \gamma_0/(2c_3'), \delta_0/(2c_3')\}$ and set $C_2 = C_2(k) = c_0\omega_0(k)$, $C'_3 = C_3(k) = c_8\omega_0(k)$. In this way we can combine (86)–(87) to get
\[
\begin{align*}
Z'(t) &\geq \omega C_2 \left[ ||u'||^2_{H^0} + K(t) + J_1(u) \right] - \omega c_8, \quad \text{for all } t \geq 0. \quad \text{(86)}
\end{align*}
\]
Since $k < 1$, by (84) there is $\omega_2(k) > 0$ such that for all $\omega \in (0,\omega_2(k))$ we have
\[
\begin{align*}
\Omega &\leq K_0^{-k} - \omega_2(1 - k) \geq \omega_1^{1-k} \omega_2. \quad \text{(87)}
\end{align*}
\]
We then choose $\omega = \omega_1(k) = \min\{\omega_1(k), \omega_2(k)\}$ and set $C_2 = C_2(k) = c_0\omega_0(k)$, $C'_3 = C_3(k) = c_8\omega_0(k)$. In this way we can combine (86)–(87) to get
\[
\begin{align*}
Z'(t) &\geq \omega C_2 \left[ ||u'||^2_{H^0} + K(t) + J_1(u) \right] - C_3, \quad t \geq 0
\end{align*}
\]
\[
Z(0) \geq C_3^{-1-k} \quad \text{(88)}
\]
for all $k \in (0,\bar{k})$.

We are now going to estimate from above $|Z(t)|^l$, where $l = l(k) = 1/(1-k)$ and $k \in (0,\bar{k})$, so that $\bar{l} = \bar{l}(\bar{k})$ where $l = l(\bar{k})$. By (84) we have
\[
|Z(t)|^l \leq \left[ K^{1-k}(t) + \omega_0 ||u'||_{H^0} ||u||_{H^0} \right]^l \leq 2^{l-1} \left[ K(t) + \omega_0^l ||u'||_{H^0}^l ||u||_{H^0}^l \right].
\]
Consequently, when we also take $k < 1/2$, using Young inequality with conjugate exponents $2(1-k)$ and $2(1-k)/(1-2k)$, we have
\[
|Z(t)|^l \leq 2^{l-1} \left[ K(t) + \omega_0^l ||u'||_{H^0}^l ||u||_{H^0}^l \right]. \quad \text{(89)}
\]
We finally choose $k = k_0 := \min\{\bar{k}, 1/2 - 1/p, 1/2 - 1/q\} \in (0,1/2)$, so that
\[
||u||_{\frac{2}{2-k_0}} = \left( ||u||^2 + ||u||_{2,1_1}^2 \right)^{1/2-k_0} \leq 2^{1/2-k_0} \left( ||u||_{2}^{2/(2-k_0)} + ||u||_{2,1_1}^{2/(2-k_0)} \right).
\]
Since $2/(1-2k_0) \leq p$ and $2/(1-2k_0) \leq q$ and $\Omega$ is bounded the previous estimates yields
\[
||u||_{\frac{2}{2-k_0}} \leq c_{10} \left( 1 + ||u||_{p}^p + ||u||_{q,1_1}^q \right). \quad \text{(90)}
\]
Denoting $l_0 = l(k_0)$, by (89)–(90) we thus have
\[
|Z(t)|^{l_0} \leq 2^{l_0-1} \left[ K(t) + \omega_0^{l_0} ||u'||_{H^0}^{l_0} + c_{10} \left( 1 + ||u||_{p}^p + ||u||_{q,1_1}^q \right) \right],
\]
and consequently, by (77), we get
\[
|Z(t)|^{l_0} \leq c_{11} \left[ K(t) + ||u'||_{H^0}^l + J_1(u) \right]. \quad \text{(91)}
\]
By combining the estimates (88) and (91) we get that $Z$ satisfies the assumptions of Lemma (3.4), which gives the required contradiction.
5. **Global nonexistence with interior source.** This section is devoted to our second main global nonexistence result for weak solutions of (1) when the interior source is present in it, while \( g \) may also vanish. In particular we shall keep the specific assumption (F1) on \( f \), while assumption (G1) is replaced by the following one.

(G2) There is \( \bar{q} > 2 \) such that

\[
g(x, u)u \geq \bar{q}G(x, u) \geq 0 \quad \text{for a.a. } x \in \Gamma_1 \text{ and all } u \in \mathbb{R}.
\]

**Remark 6.** When dealing with the model nonlinearity \( g_1 \) defined in (52), assumption (G2) reduces to

\[
\delta, \tilde{\delta} \geq 0, \quad \tilde{\delta}' = 0, \quad 2 < \bar{q} \leq q,
\]

and then, comparing with (69), assumptions (G1) and (G2) are unrelated. Moreover, by Remark 5, the couple of assumptions (F1), (G2) holds for the model nonlinearities \( f_1 \) and \( g_1 \) in (52) when

\[
\gamma > 0, \quad 2 < \bar{p} < p, \quad \text{and} \quad \delta, \tilde{\delta} \geq 0, \quad \tilde{\delta}' = 0, \quad 2 < \bar{q} \leq q. \tag{92}
\]

Consequently, when restricting to the case \( \bar{\gamma} = \bar{\gamma}' = \bar{\delta} = \bar{\delta}' = 0 \) and \( \gamma, \delta \geq 0 \) as in problem (2), see Remark 4, (F1) and (G2) hold provided

\[
\gamma > 0, \quad p > 2, \quad \text{and} \quad \delta \geq 0, \quad q > 2, \tag{93}
\]

so that assumptions (A1–3), (F1) and (G2) hold true for problem (2) provided (3), (6) and (93) hold.

Our second main global nonexistence result is the following one.

**Theorem 5.1.** Let assumptions (A1–3), (F1), (G2) hold, and

\[
p > m, \quad m < 1 + p/2. \tag{94}
\]

Then, for any \( U_0 \in \mathcal{H} \) such that \( \mathcal{E}(U_0) < 0 \) problem (1) does not admit global weak solutions.

**Proof.** We first recall that, as explained in Remark 3, by (49) and (94) it follows that \( m, p < r_\alpha \) and \( m, q < r_\gamma \), so \( \mathcal{H}^{p,q} = \mathcal{H} \) in Lemma 3.3–iii).

The proof is a variant of the one of Theorem 4.1, so we shall keep all notation in it. Also in this case we use a contradiction argument, so let \( u \) be a global weak solution of (1) with \( \mathcal{E}(U_0) < 0 \). As in the quoted proof we introduce the auxiliary function \( \mathcal{K} \) defined in (72) and we get that \( \mathcal{K} \) is increasing in \([0, \infty)\) and formulas (73)–(75) hold true. By (74)–(75), also using (58), we have

\[
0 < \mathcal{K}_0 \leq \mathcal{K}(t) \leq J_2(u(t)) \quad \text{for all } t \geq 0, \tag{95}
\]

where \( J_2 \) is defined by

\[
J_2(u) = c_p'\|\Omega\| + c_p'\|u\|_p^p + \int_{\Gamma_1} G(\cdot, u) d\sigma, \quad u \in H^1. \tag{96}
\]
Also in this case we get the identity (78) and, for any \( \varepsilon \in \left(0, \frac{7 - 2}{2}\right) \) to be fixed in the sequel, using (58) and (72), we rewrite it as
\[
\frac{d}{dt}(u', u)_{H^0} = \frac{4 + \varepsilon}{2}\|u'\|_{H^0}^2 + \frac{\varepsilon}{2}\|\nabla u\|_2^2 + \frac{\varepsilon}{2}\|\nabla t u\|_{L^2}^2 + \gamma_0\|u\|_p^p - \gamma_1|\Omega|
+ \int_\Omega [f(\cdot, u)u - (2 + \varepsilon)F(\cdot, u)] \, dx + \int_{\Gamma_1} [g(\cdot, u)u - (2 + \varepsilon)G(\cdot, u)] \, d\sigma
- \int_\Omega \hat{P}(u_t)u \, dx - \int_{\Gamma_1} \hat{Q}(u_t)u \, d\sigma \quad \text{for all } t \geq 0.
\]
Consequently, using (75) and the assumptions (F1), (G2), as \( \gamma - 2 - \varepsilon > \varepsilon \) we get the estimate
\[
\frac{d}{dt}(u', u)_{H^0} \geq 2\|u'\|_{H^0}^2 + \frac{\varepsilon}{2}\|\nabla u\|_2^2 + 2\mathcal{K}(t) + \gamma_0\|u\|_p^p - \gamma_1|\Omega|
- \varepsilon \int_\Omega F(\cdot, u) \, dx - \int_\Omega \hat{P}(u_t)u \, dx - \int_{\Gamma_1} [g(\cdot, u)u - \hat{G}(\cdot, u)] \, d\sigma
+ (\gamma - 2 - \varepsilon) \int_{\Gamma_1} G(\cdot, u) \, d\sigma - \int_{\Gamma_1} \hat{Q}(u_t)u \, d\sigma
\geq 2\|u'\|_{H^0}^2 + \frac{\varepsilon}{2}\|\nabla u\|_2^2 + 2\mathcal{K}(t) + (\gamma_0 - \varepsilon\gamma'_0)\|u\|_p^p - (\gamma_1 + \varepsilon\gamma'_0)|\Omega|
+ \varepsilon \int_\Omega G(\cdot, u) \, d\sigma - \int_{\Gamma_1} \hat{P}(u_t)u \, dx - \int_{\Gamma_1} \hat{Q}(u_t)u \, d\sigma.
\]
Restricting to \( \varepsilon \in (0, \varepsilon_1) \), where \( \varepsilon_1 := \min\left\{\frac{7 - 2}{2}, \frac{\gamma_0}{\gamma_1}\right\} \) we then get
\[
\frac{d}{dt}(u', u)_{H^0} \geq 2\|u'\|_{H^0}^2 + \frac{\varepsilon}{2}\|\nabla u\|_2^2 + 2\mathcal{K}(t) + \frac{1}{2}\gamma_0\|u\|_p^p - (\gamma_1 + \varepsilon\gamma'_0)|\Omega|
+ \varepsilon \int_{\Gamma_1} G(\cdot, u) \, d\sigma - \int_{\Gamma_1} \hat{P}(u_t)u \, dx - \int_{\Gamma_1} \hat{Q}(u_t)u \, d\sigma.
\]
Also in this case we estimate from below the last two terms in the right–hand side of (97). Since, by (96) and assumption (G2) we have \( \gamma'_0\|u\|_p^p \leq J_2(u) \) we can estimate the term \( \int_{\Omega} \hat{P}(u_t)u \) exactly as in the proof of Theorem 4.1, with \( J_2 \) replacing \( J_1 \). In this way, for \( \varepsilon \in (0, \varepsilon_2) \), where \( \varepsilon_2 = \min\{1, \varepsilon_1\} \), we get
\[
\mathcal{I}_1(t) := \int_{\Omega} \hat{P}(u_t)u \, dx \leq c_4 \left[\varepsilon^{\hat{m}} J_2(u) + \varepsilon^{-\hat{m}'} \mathcal{K}'(t)\right]\left[\mathcal{K}(t)^{-k_1}\right] \quad \text{for all } t \geq 0,
\]
where \( k_1 = 1/m - 1/p \in (0, 1) \).

The estimate of the last term in the right–hand side of (97) in this case is different. Indeed, as in the estimate of \( \mathcal{I}_1(t) \) we get
\[
\mathcal{I}_2(t) := \int_{\Gamma_1} \hat{Q}(u_t)u \, d\sigma \leq c_\mu \left\{[\mathcal{K}'(t)]^{1/\nu'} \|u\|_{\mu, \Gamma_1} + [\mathcal{K}'(t)]^{1/\hat{\mu}'} \|u\|_{\hat{\mu}, \Gamma_1}\right\}.
\]
Since \( \hat{\mu} \leq \mu \leq \overline{\mu} \) and \( \sigma(\Gamma_1) < \infty \) previous estimate yields
\[
\mathcal{I}_2(t) \leq c_{12} \left\{[\mathcal{K}'(t)]^{1/\nu'} + [\mathcal{K}'(t)]^{1/\hat{\mu}'}\right\} \|u\|_{\mu, \Gamma_1},
\]
To estimate the term \( \|u\|_{\mu, \Gamma_1} \) in (99) we set \( s = \frac{1}{\overline{\mu}} \left(\frac{1}{2} + \frac{p - \mu}{p - 2}\right) \). Since, by (94), \( p > 2(\overline{\mu} - 1) \), we have \( (p - \overline{\mu})/(p - 2) > 1/2 \). Hence, as \( \overline{\mu} \geq 2 \),
\[
\frac{1}{\overline{\mu}} < s < \frac{2(p - \overline{\mu})}{\overline{\mu}(p - 2)} \leq 1.
\]
We can then use the Trace Inequality [30, Theorem 1.5.1.2, p. 37] to estimate
\[ \|u\|_{\pi, g_1^s} \leq c_{13} \|u\|_{W^{s-\frac{s}{2}}} \|u\|_{p}^{\frac{s}{2}}. \]  
(101)

Now, by interpolation theory, see [56, Theorem 4.3.1.2 p. 317 and Remark 2, formula (9), §2.4.2 p. 185], we know that
\[ W^{s, \theta}(\Omega) = B^{s, \theta}_p(\Omega) = (H^1(\Omega), L^p(\Omega))_{1-s, \theta} \]  
(102)

where \((\cdot, \cdot)_{1-s, \theta}\) denotes the real interpolator functor, provided
\[ \frac{1}{\theta} = \frac{s}{2} + \frac{1-s}{p}, \quad \text{i.e.} \quad \theta = \frac{2p}{s(p-2)+2}. \]  
(103)

Now, by (97) and (103) we have \(\Pi < \theta\). Hence, as \(\Omega\) is bounded and \(p > 2\), combining (101) with the interpolation inequality (see [8, Theorem 3.2.2 p. 43]) which follows from (102) we get
\[ \|u\|_{\pi, g_1^s} \leq c_{14} \|u\|_{H^1(\Omega)} \|u\|_{p}^{\frac{s}{2}} \leq c_{15} \left( \|u\|_{p} + \|\nabla u\|_{2} \|u\|_{p}^{\frac{s}{2}} \right). \]

Plugging it into (99) and using (96) and assumption (G2) we get
\[ I_2(t) \leq c_{16} \left\{ (\mathcal{K}(t))^{1/\mu'} + [\mathcal{K}(t)]^{1/\tilde{\mu}'} \right\} \left( \|u\|_{p} + \|\nabla u\|_{2} \|u\|_{p}^{\frac{s}{2}} \right) \]
\[ \leq I_1'(t) + I_2'(t), \]  
(104)

where
\[ I_1'(t) := c_{17} \left\{ [\mathcal{K}(t)]^{1/\mu'} + [\mathcal{K}(t)]^{1/\tilde{\mu}'} \right\} [J_2(u)]^{1/p}, \]
\[ I_2'(t) := c_{17} \left\{ [\mathcal{K}(t)]^{1/\mu'} + [\mathcal{K}(t)]^{1/\tilde{\mu}'} \right\} \|\nabla u\|_{2} [J_2(u)]^{\frac{1}{1+\frac{s}{2}}}. \]

(105)

(106)

We now separately estimate \(I_1'(t)\) and \(I_2'(t)\). We estimate the first one exactly as we estimated the term \(I_1(t)\) in (80), by respectively replacing \(m, \tilde{m}\) and \(g_1, g_2\) with \(\mu, \tilde{\mu}\) and \(J_1, J_2\). In this way, setting \(k_2 = 1/\mu - 1/p \in (0, 1)\), we get the estimate
\[ I_1'(t) \leq c_{18} \left[ \varepsilon^\tilde{\mu} J_2(u) + \varepsilon^{-\tilde{\mu}} \mathcal{K}(t) \right] [\mathcal{K}(t)]^{-k_2} \quad \text{for all} \ t \geq 0. \]  
(107)

To estimate the term \(I_2'(t)\) we first set
\[ k_3 = \frac{1}{\mu} - \left( \frac{s}{2} + \frac{1-s}{p} \right) = \frac{1}{\mu} - \frac{1}{\tilde{\theta}} > 0. \]  
(108)

By (95), (105) and (108), since \(\tilde{\mu} \leq \mu \leq \mu\),
\[ I_2'(t) = c_{17} \left\{ [\mathcal{K}(t)]^{\frac{1}{\mu'}} \|\nabla u\|_{2}^{\frac{1}{\mu'}} [J_2(u)]^{\frac{1}{\mu'}} + [\mathcal{K}(t)]^{\frac{1}{\tilde{\mu}'}} \|\nabla u\|_{2}^{\frac{1}{\tilde{\mu}'}} [J_2(u)]^{-k_3} \right\} \right\} \leq c_{18} \left\{ [\mathcal{K}(t)]^{\frac{1}{\mu'}} \|\nabla u\|_{2}^{\frac{1}{\mu'}} [J_2(u)]^{\frac{1}{\mu'}} + [\mathcal{K}(t)]^{\frac{1}{\tilde{\mu}'}} \|\nabla u\|_{2}^{\frac{1}{\tilde{\mu}'}} [J_2(u)]^{-k_3} \right\} \]  
(109)

By (100) we have \(\frac{s}{2} < \frac{p-\mu}{\mu(p-2)} < \frac{1}{\mu}\), so
\[ \frac{1}{\mu} - \frac{s}{2} > \frac{1}{\mu} - \frac{s}{2} > \frac{1}{\mu} - \frac{s}{2} > 0. \]

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Hence, using triple Young inequality with exponents $\mu'$, $2/s$ and $1/\left(\frac{1}{\mu} - \frac{s}{2}\right)$ we get the estimates

$$\begin{align*}
[K'(t)]^{\frac{1}{2}} \|\nabla u\|_{2}^{2} [J_2(u)]^{\frac{1}{2} - \frac{s}{2}} & = \left[\varepsilon^{2} k' \mu \right]^{\frac{1}{2}} \left(\varepsilon \|\nabla u\|_{2}\right)^{s} \left\{ \varepsilon^{2} [J_2(u)] \right\}^{\frac{1}{2} - \frac{s}{2}} \\
& \leq \varepsilon^{2} J_2(u) + \varepsilon \|\nabla u\|_{2}^{2} + \varepsilon^{-2 \frac{\mu'}{\mu}} K'(t),
\end{align*}$$

(110)

$$\begin{align*}
[K'(t)]^{\frac{1}{2}} \|\nabla u\|_{2}^{2} [J_2(u)]^{\frac{1}{2} - \frac{s}{2}} & \leq \varepsilon^{2} J_2(u) + \varepsilon \|\nabla u\|_{2}^{2} + \varepsilon^{-2 \frac{\mu'}{\mu}} K'(t)
\end{align*}$$

(111)

for all $t \geq 0$. Since $\varepsilon \leq 1$ and $\bar{\mu} \leq \mu$, by (109), (110) and (111)

$$\begin{align*}
I_2(t) & \leq c_{19} \left[ \varepsilon^{2} J_2(u) + \varepsilon \|\nabla u\|_{2}^{2} + \varepsilon^{-2 \frac{\mu'}{\mu}} K'(t) \right] [K(t)]^{-k_3} \text{ for all } t \geq 0.
\end{align*}$$

(112)

To estimate $I_2(t)$ we now remark that, being $p > 2$, using (108), we have $k_2 = \frac{1}{\mu} - \frac{1}{p} \geq \frac{1}{p} - \frac{1}{p} \geq k_3$. By (95) we can combine (104), (107) and (112) to get

$$\begin{align*}
I_2(t) & \leq c_{20} \left[ \varepsilon^{\min(2, \bar{\mu})} J_2(u) + \varepsilon \|\nabla u\|_{2}^{2} + \varepsilon^{-\min\left\{ \bar{\mu}', \frac{2\mu'}{\mu} \right\}} K'(t) \right] [K(t)]^{-k_4}
\end{align*}$$

(113)

for all $t \geq 0$. We then set $k_4 = \min\{k_1, k_3\}$ and, using (95) again we combine (98) and (113) to get

$$\begin{align*}
I_1(t) + I_2(t) & \leq c_{21} \left[ \varepsilon^{\min(2, \bar{\mu}, \bar{m})} J_2(u) + \varepsilon \|\nabla u\|_{2}^{2} + \varepsilon^{-\min\left\{ \bar{\mu}', \bar{m}', \frac{2\mu'}{\mu} \right\}} K'(t) \right] [K(t)]^{-k_4}
\end{align*}$$

for all $t \geq 0$. Hence, using (95)–(96), for any $k \in (0, k_4]$ there is $C_4 = C_4(k)$ such that

$$\begin{align*}
I_1(t) + I_2(t) & \leq c_{22} \varepsilon^{\min(2, \bar{\mu}, \bar{m})} \left( c_p' \Omega + c_p' \|u\|_{p} + \int_{\Gamma_1} G(\cdot, u) \, d\sigma \right) + c_{22} \varepsilon^{2} \|\nabla u\|_{2}^{2} + c_{4} \varepsilon^{-\min\left\{ \bar{\mu}', \bar{m}', \frac{2\mu'}{\mu} \right\}} K'(t) [K(t)]^{-k}
\end{align*}$$

(114)

for all $t \geq 0$. Plugging this estimate into (97) we get

$$\begin{align*}
\frac{d}{dt} \left\langle u', u \right\rangle_{H^0} & \geq 2 \|u\|_{H^0}^{2} + \varepsilon \left( \frac{1}{2} - c_{22} \varepsilon \right) \|\nabla u\|_{2}^{2} + 2 \mathcal{K}(t) \\
& \quad + \left( \frac{\gamma_0}{4} - c_{22} \varepsilon^{\min(2, \bar{\mu}, \bar{m})} c_p' \right) \|u\|_{p}^{2} \\
& \quad + \left( \varepsilon - c_{22} \varepsilon^{\min(2, \bar{\mu}, \bar{m})} \right) \int_{\Gamma_1} G(\cdot, u) \, d\sigma \\\n& \quad - C_{4} \varepsilon^{-\min\left\{ \bar{\mu}', \bar{m}', \frac{2\mu'}{\mu} \right\}} K'(t) [K(t)]^{-k} \\
& \quad - (\gamma_1 + \varepsilon c_p' + c_{22} \varepsilon^{\min(2, \bar{\mu}, \bar{m})} c_p') \|\Omega\| \text{ for all } t \geq 0.
\end{align*}$$

(115)

Since $\bar{m}, \bar{\mu} > 1$ we can finally fix $\varepsilon = \varepsilon_0 \in (0, \varepsilon_2]$ so small that

$$\begin{align*}
\frac{1}{2} - c_{22} \varepsilon & \geq \frac{1}{4}, \\
\frac{\gamma_0}{4} - c_{22} \varepsilon^{\min(2, \bar{\mu}, \bar{m})} c_p' & \geq \frac{\gamma_0}{4}, \\
\varepsilon - c_{22} \varepsilon^{\min(2, \bar{\mu}, \bar{m})} & \geq 0
\end{align*}$$

With this choice, using assumption (G2), from (114) we get

$$\begin{align*}
\frac{d}{dt} \left\langle u', u \right\rangle_{H^0} & \geq 2 \|u\|_{H^0}^{2} + 2 \mathcal{K}(t) + c_{4} \varepsilon \|\nabla u\|_{2}^{2} + c_{4} \varepsilon^{2} \|u\|_{p}^{2} - C_{5} \mathcal{K}'(t) [K(t)]^{-k} - c_{23}
\end{align*}$$

(115)

for all $t \geq 0$, where $C_5 = C_5(k)$. 
We now introduce, as in the proof of Theorem 4.1, the Lyapunov functional \( Z \) given by (84), with \( k \in (0, k_4) \) and \( \omega > 0 \) to be conveniently fixed in the sequel. By (115) we have

\[
Z'(t) = (1 - k)\left[ \mathcal{K}(t) \right]^{-k} \mathcal{K}'(t) + \omega \frac{d}{dt} (u', u)_{H^0} \geq (1 - k - C_5 \omega) \left[ \mathcal{K}(t) \right]^{-k} \mathcal{K}'(t) + 2\omega \|u'\|_{H^0}^2 + 2\omega \mathcal{K}(t) + \frac{\epsilon_0}{16} \|\nabla u\|_2^2 + \frac{\omega_k}{4} \|u\|^p_{p} - \omega C_{23} \quad \text{for all } t \geq 0.
\]

Hence, by taking \( \omega \in (0, \omega_3) \), where \( \omega_3 = \omega_3(k) := (1 - k) - C_5(k)^{-1} \), so that

\[
1 - k - C_5(k)\omega \leq 0,
\]

we have

\[
Z'(t) \geq \omega \left[ 2\|u'\|_{H^0}^2 + 2\mathcal{K}(t) + \frac{\epsilon_0}{4} \|\nabla u\|_2^2 + \frac{\omega_k}{4} \|u\|^p_{p} - C_{23} \right] \geq c_{24} \omega \left[ \|u'\|_{H^0}^2 + \mathcal{K}(t) + \|\nabla u\|_2^2 + \|u\|^p_{p} \right] - C_7, \quad t \geq 0
\]

for all \( k \in (0, k_4) \).

Also in this case we are now going to estimate from above \( |Z(t)| \), where \( l = l(k) = 1/(1 - k) \) and \( k \in (0, k_4) \), so that \( l \in (1, 1/(1 - k_4)) \). Exactly as in the proof of Theorem 4.1, also taking \( k < 1/2 \) we get

\[
|Z(t)| \leq 2^{l-1} \left[ \mathcal{K}(t) + \omega_0^2 \|u'\|_{H^0}^2 + \|u\|_{H^0}^{2/(1-2k)} \right].
\]

We now set \( k_5 = \min\{k_4, 1/2 - 1/p\} \), so that \( k_5 < 1/2 \) and we restrict to \( k \in (0, k_5) \).

Since \( k \leq 1/2 - 1/p \) we also have \( 2(k - 1/2) \leq (p - 2)/2 \) and consequently

\[
\|u\|_{H^0}^{2/(1-2k)} \leq 2^{2(k - 1/2)} \left( \|u\|_{L^2}^{2/k} + \|u\|_{H^1}^{2/k} \right).
\]

Since \( 2/(1 - 2k) \leq p \) and \( \Omega \) is bounded we have

\[
\|u\|_{L^2}^{2/k} \leq c_{25} \left( 1 + \|u\|_p^p \right).
\]

Moreover, by standard trace and interpolation inequalities

\[
\|u\|_{L^2} \leq c_{26} \|u\|_{H^{1/2}(\Omega)} \leq c_{27} \|u\|_{H^{3/4}(\Omega)} \leq c_{28} \|u\|_2^{1/4} \left( \|u\|_2^2 + \|\nabla u\|_2^2 \right)^{3/8} \leq c_{28} \left( \|u\|_2 + \|\nabla u\|_2 \right)^{1/4} \|\nabla u\|_2^{3/4}.
\]

Consequently, since \( \Omega \) is bounded, there is \( C_8 = C_8(k) \) such that

\[
\|u\|_{L^2} \leq c_{29} \left( \|u\|_{L^p} + \|u\|_p \right)^{1/4} \|\nabla u\|_2^{3/4} \leq C_8 \left( \|u\|_{L^p}^{1/p} + \|u\|_p \right)^{1/4} \|\nabla u\|_2^{3/4}.
\]

Since \( 2/(1 - 2k) \leq p \) we have \( \|u\|_{L^2} \leq 1 + \|u\|_p^p \), so by the previous estimate

\[
\|u\|_{L^2}^{1/p} \leq C_8 \left( 1 + \|u\|_p + \|u\|_p^{1/p} \right)^{1/4} \|\nabla u\|_2^{3/4}.
\]

We finally choose \( k = k_{00} \), where \( k_{00} := \min\{k_5, (1/2 - 1/p)/4\} > 0 \). By applying Young inequality with conjugate exponents \( 4(1 - 2k_{00})/(1 - 8k_{00}) \) and \( 4(1 - 2k_{00})/3 \)
we get \( ||u||_{p \infty, \Omega} \leq ||u||_{p \infty, \Omega}^{1 - \frac{2}{n}} + ||\nabla u||_2^2 \). Moreover, since \( k_{00} \leq (1/2 - 1/p)/4 \) we have \( 2/(1 - 8k_{00}) \leq p \), so \( ||u||_{p \infty, \Omega} \leq ||u||_{p \infty, \Omega}^{1 - \frac{2}{n}} + ||\nabla u||_2^2 \leq 1 + ||\nabla u||_2^2 + ||u||_p^p \) and consequently, by (122), taking \( c_{29} = 2C_8(k_{00}), \)
\[
||u||_{p \infty, \Omega}^{1 - \frac{2}{n}} \leq c_{29} (1 + ||u||_p^p + ||\nabla u||_2^2). \tag{123}
\]

By (120)–(121) and (123) we then get \( ||u||_{H_0^1, \Omega}^{1 - \frac{2}{n}} \leq c_{30} (1 + ||u||_p^p + ||\nabla u||_2^2) \) and consequently, by (119), denoting \( l_{00} = l(k_{00}), \)
\[
|Z(t)|_{l_{00}^{\infty}} \leq c_{31} \left( ||u||_{H_0^1, \Omega}^{2} + K(t) + ||\nabla u||_2^2 + ||u||_p^p \right). \tag{124}
\]

By combining the estimates (118) and (124) we get that \( Z \) satisfies the assumptions of Lemma (3.4), which gives the required contradiction. \( \square \)

6. Blow–up results and proofs of Theorems 1.3–1.4. The aim of this section is to combine the local theory in [63] with Theorems 4.1–5.1 to get two blow–up results for weak solutions of (1). As explained in Remarks 1, 3 and in the proofs of Theorems 4.1–5.1, when (71) or (94) holds we necessarily have \( p < r_1 \) and \( q < r_2 \). Hence, for the sake of simplicity, we shall recall the local theory in [63] only when neither \( f \) nor \( g \) is super–supercritical.

6.1. Known results. Beside the main assumptions (A1–3) made in § 3.1, to fit with the setting in the quoted paper we shall consider in the sequel nonlinearities satisfying also the following additional structural conditions:

(A4) \( P \) (respectively \( Q \)) is monotone increasing in \( v \) for a.a. \( x \in \Omega \times \mathbb{R} \) \( (x \in \Gamma_1) \), and \( P, Q \) are coercive, that is there are \( c_m, c_m' > 0 \), such that
\[
P(x,v)v \geq c_m'' \alpha(x)|v|^m \quad \text{for a.a. } x \in \Omega, \text{ all } v \in \mathbb{R};
\]
\[
Q(x,v)v \geq c_m'' \beta(x)|v|^m \quad \text{for a.a. } x \in \Gamma_1, \text{ all } v \in \mathbb{R};
\]

(A5) there are constants \( c_p'' \), \( c_q'' \geq 0 \) such that
\[
|f(x,u) - f(x,v)| \leq c_p'' |u - v|(1 + |u|^{p-2} + |v|^{p-2}), \quad \text{for a.a. } x \in \Omega \text{ and all } u,v \in \mathbb{R},
\]
\[
|g(x,u) - g(x,v)| \leq c_q'' |u - v|(1 + |u|^{q-2} + |v|^{q-2}), \quad \text{for a.a. } x \in \Gamma_1 \text{ and all } u,v \in \mathbb{R}.
\]

By combining the remarks made in §3.1 with those in [63, §2.1, pp. 4893–4894] we get the following conclusions. When \( P(x,v) = \alpha(x)P_0(v) \) and \( Q(x,v) = \beta(x)Q_0(v) \), with \( \alpha \in L^\infty(\Omega) \) and \( \beta \in L^\infty(\Gamma_1) \), \( \alpha, \beta \geq 0 \), assumptions (A1) and (A4) trivially hold when \( P_0, Q_0 \in C(\mathbb{R}) \) are monotone increasing and there are \( 1 < m \leq m \), \( 1 < \mu \leq \mu \) such that
\[
\lim_{v \to 0} \frac{|P_0(v)|}{|v|^{m-1}} > 0, \quad \lim_{|v| \to \infty} \frac{|P_0(v)|}{|v|^{m-1}} > 0, \quad \lim_{v \to 0} \frac{|Q_0(v)|}{|v|^{\mu-1}} > 0, \quad \lim_{|v| \to \infty} \frac{|Q_0(v)|}{|v|^{\mu-1}} > 0,
\]
\[
\lim_{v \to 0} \frac{|P_0(v)|}{|v|^{m-1}} < \infty, \quad \lim_{|v| \to \infty} \frac{|P_0(v)|}{|v|^{m-1}} < \infty, \quad \lim_{v \to 0} \frac{|Q_0(v)|}{|v|^{\mu-1}} < \infty, \quad \lim_{|v| \to \infty} \frac{|Q_0(v)|}{|v|^{\mu-1}} < \infty.
\]

Moreover when \( f(x,u) = f_0(u) \) and \( g(x,u) = g_0(u) \) assumptions (A2) and (A5) trivially hold when \( f_0, g_0 \in C_{0,1}^{\infty}(\mathbb{R}) \) and there are \( p, q \geq 2 \) such that
\[
f_0' = O(|u|^{p-2}), \quad g_0' = O(|u|^{q-2}) \quad \text{as } |u| \to \infty.
\]

Finally the statement of assumption (A3) is unchanged when \( P, Q, f \) and \( g \) are as before.
Remark 7. From the previous discussion it is clear that the model nonlinearities in (52) satisfy assumptions (A1–5) provided (53)–(54) hold. Restricting (52) to the case of \( p > q \), we trivially get, see also Remark 4, that the nonlinearities in problem (2), satisfy assumptions (A1–5) provided (3) and (6) hold.

Since the set of assumptions (A1–5) is (slightly) more restrictive that the set of assumptions \([63, (PQ1–3), (FG1–2), (FGQP1)]\), the following result is a particular case of \([63, Corollary 5.2]\).

Theorem 6.1 (Existence and continuation). Let assumptions (A1–5) hold, with \( p \leq r_1 \) and \( q \leq r_2 \). Then for any \( U_0 = (u_0, u_1) \in H \) problem (1) possesses a maximal weak solution \( u \in C([0,T_{max}); H^1) \cap C^1([0,T_{max}); H^0) \) for some \( T_{max} \in (0, \infty) \). Moreover, denoting \( U = (u, u') \), when \( T_{max} < \infty \) we have \( \lim_{t \to T_{max}} \|U(t)\|_H = \infty \).

Remark 8. In \([63, Theorem 5.1 and Corollary 5.2]\) the conclusion that when \( T_{max} < \infty \) we have \( \lim_{t \to T_{max}} \|U(t)\|_H = \infty \) was stated only for the weak maximal solution built there. On the other hand the proof of \([63, Theorem 6.1]\) makes evident that the same conclusion holds true for any weak maximal solution of (1).

Although Theorem 6.1 gives the necessary motivation for studying the behavior of weak solutions of (1) and includes a basic continuation result, a more precise behavior as \( t \to T_{max} \) (when \( T_{max} < \infty \)) is known when also \([63, assumption (FG2)']\), p. 4898] holds true. We recall it here for the reader convenience.

(A6) If \( p > 1 + r_1/2 \) then \( N \leq 4, f(x, \cdot) \in C^2(\mathbb{R}) \) for a.a. \( x \in \Omega \) and there is \( c_p''' \geq 0 \) such that
\[
|f_u(x, u) - f_u(x, v)| \leq c_p'''|u - v|(1 + |u|^{p-3} + |v|^{p-3})
\] (125)
for a.a. \( x \in \Omega \) and all \( u, v \in \mathbb{R} \).

If \( q > 1 + r_2/2 \) then \( N \leq 5, g(x, \cdot) \in C^2(\mathbb{R}) \) for a.a. \( x \in \Gamma_1 \) and there is \( c_q''' \geq 0 \) such that
\[
|g_u(x, u) - g_u(x, v)| \leq c_q'''|u - v|(1 + |u|^{q-3} + |v|^{q-3})
\] (126)
for a.a. \( x \in \Gamma_1 \) and all \( u, v \in \mathbb{R} \).

Remark 9. It is worth noting that, when \( N \leq 4 \), we have \( 1 + r_1/2 > 3, \) so \( p > 3 \) in (125). Similarly \( q > 3 \) in (126).

As showed in \([63, Remark 5.3, p. 4898]\), when \( f(x, u) = f_0(u) \) and \( g(x, u) = g_0(u) \) assumption (A6) reduces to the following one:

if \( p > 1 + r_1/2 \) then \( N \leq 4, f_0 \in C^2(\mathbb{R}) \) and \( f_0''(u) = O(|u|^{p-3}) \) as \( |u| \to \infty \);
if \( q > 1 + r_2/2 \) then \( N \leq 5, g_0 \in C^2(\mathbb{R}) \) and \( g_0''(u) = O(|u|^{q-3}) \) as \( |u| \to \infty \).

Remark 10. From the previous discussion is then clear that the model nonlinearities in (52) satisfy assumption (A6) provided

if \( \gamma \neq 0 \) and \( p > 1 + r_1/2 \) then \( N \leq 4 \) and \( p > 3 \) or \( \gamma = 0 \);
if \( \delta \neq 0 \) and \( q > 1 + r_2/2 \) then \( N \leq 5 \) and \( q > 3 \) or \( \delta = 0 \).

In particular, when \( \gamma = \gamma' = \delta = \delta' = 0 \) and \( \gamma, \delta \geq 0 \) as in problem (2), assumption (A6) holds when (13) does.

Since the set of assumptions (A1–6) is (slightly) more restrictive that the set of assumptions \([63, (PQ1–3), (FG1), (FG2)', (FGQP1)]\), the following result is a particular case of \([63, Theorem 6.2]\).
Theorem 6.2 (Existence, uniqueness, continuation). Let assumptions (A1–6) hold, with \( p \leq r_1 \) and \( q \leq r_2 \). Then for any \( U_0 = (u_0, u_1) \in \mathcal{H} \) the maximal weak solution \( u \) of problem (1) in Theorem 6.1 is unique. Moreover, when \( T_{\max} < \infty \), we have \( \lim_{t \to T_{\max}} \|U(t)\|_{\mathcal{H}} = \infty \).

6.2. Blow–up results and proofs of Theorems 1.3–1.4. Our first blow–up result is an application of Theorem 4.1.

Theorem 6.3 (Blow–up with two sources). Let assumptions (A1–5), (F1), (G1) and (71) hold. Then for any \( U_0 = (u_0, u_1) \in \mathcal{H} \) such that \( E(U_0) < 0 \) and any maximal weak solution of problem (1) one has \( T_{\max} < \infty \) and (27) holds. Moreover when also assumption (A6) holds we can replace \( \lim_{t \to T_{\max}} \) with \( \lim_{t \to T_{\max}} \) in (27).

Proof. By Theorem 6.1 and Remark 8 for any maximal weak solution \( u \) of (1) either \( T_{\max} = \infty \) or \( T_{\max} < \infty \) and \( \lim_{t \to T_{\max}} \|U(t)\|_{\mathcal{H}} = \infty \). Theorem 4.1 allows to exclude the first alternative. To conclude the proof of (27) we now remark that, since \( E(U_0) < 0 \), by the energy identity (60) and assumption (A1) we have \( E(U(t)) < 0 \) for all \( t \in [0, T_{\max}] \), that is

\[
\frac{1}{2} \|u(t)\|_{\mathcal{H}}^2 + \frac{1}{2} \int_{\Omega} |\nabla u(t)|^2 \, dx + \frac{1}{2} \int_{\Gamma_1} |\nabla \Gamma u(t)|^2 \, d\sigma < J(u(t)).
\]

(127)

Hence, by (37), (58) and (75), since \( p, q \geq 2 \), we have

\[
\frac{1}{2} \|U(t)\|_{\mathcal{H}}^2 \leq \frac{1}{2} \|u(t)\|_{E, \Gamma_1}^2 + c_p' |\Omega| + c_q' \sigma(\Gamma_1) + c_p' \|u(t)\|_p^p + c_q' \|u(t)\|_q^q,
\]

\[
\leq c_{32} + c_{33} \left( \|u(t)\|_p^p + \|u(t)\|_q^q \right),
\]

concluding the proof of (27).

\[\square\]

Remark 11. It is useful to combine Remarks 5 and 7 to point out when the model nonlinearities in (52) satisfy the full set of assumptions of Theorem 6.3, but for (A6). Assumptions (A1–5), (F1) and (G1) are satisfied when (53) holds, with \( \gamma, \delta > 0 \), \( \bar{p} < p, \bar{q} < q \),

\[
p \leq \begin{cases} \frac{1 + r_\alpha}{2} & \text{if } \alpha = 0, \\ \frac{1 + r_\alpha}{m'} & \text{if } \alpha > 0, \end{cases} \quad q \leq \begin{cases} \frac{1 + r_\beta}{2} & \text{if } \beta = 0, \\ \frac{1 + r_\beta}{p'} & \text{if } \beta > 0, \end{cases}
\]

\[
\bar{\gamma} = 0 \text{ or } \bar{p} \leq \begin{cases} \frac{1 + r_\alpha}{2} & \text{if } \alpha = 0, \\ \frac{1 + r_\alpha}{m'} & \text{if } \alpha > 0, \end{cases} \quad \bar{\delta} = 0 \text{ or } \bar{q} \leq \begin{cases} \frac{1 + r_\beta}{2} & \text{if } \beta = 0, \\ \frac{1 + r_\beta}{p'} & \text{if } \beta > 0. \end{cases}
\]

(128)

Moreover, in assumption (71) we can take \( m = 2 \) when \( \alpha = 0 \) and \( p = 2 \) when \( \beta = 0 \). Hence, for the model nonlinearities in (52), since \( \gamma, \delta > 0 \), assumption (71) can be rewritten as

\[
p > \begin{cases} \frac{2}{m} & \text{if } \alpha = 0, \\ \frac{2}{m} & \text{if } \alpha > 0, \end{cases} \quad q > \begin{cases} \frac{2}{p} & \text{if } \beta = 0, \\ \frac{2}{p} & \text{if } \beta > 0. \end{cases}
\]

(129)

In particular, when \( \bar{\gamma} = \bar{\gamma}' = \bar{\delta} = \bar{\delta}' = 0 \) and \( \gamma, \delta \geq 0 \) as in problem (2), the full set of assumptions of Theorem 6.3, but for (A6), is satisfied when \( \gamma, \delta \geq 0 \), (3), (6) and (129) hold true. Finally, assumption (A6) can be checked as in Remark 10, and in particular in problem (2) is satisfied when \( N \leq 4 \), or \( N = 5 \) and \( p \leq 1 + r_\alpha/2 = 8/3 \), or \( N \geq 6, p \leq 1 + r_\alpha/2, q \leq 1 + r_\alpha/2 \).

Our second blow–up result is an application of Theorem 5.1.
Theorem 6.4 (Blow–up with interior source). Let assumptions (A1–5), (F1), (G2) and (94) hold. Then for any \( U_0 = (u_0, u_1) \in H \) such that \( E(U_0) < 0 \) and any maximal weak solution of problem (1) one has \( T_{\text{max}} < \infty \) and
\[
\lim_{t \to T_{\text{max}}} \| U(t) \|_H = \lim_{t \to T_{\text{max}}} \| u(t) \|_p + \| u(t) \|_{2, \Gamma_1} + \int_{\Gamma_1} G(\cdot, u(t)) \, d\sigma = \infty.
\] (130)
Moreover when also assumption (A6) holds we can replace \( \lim_{t \to T_{\text{max}}} \) with \( \lim_{t \to T_{\text{max}}} \) in (130).

Proof. By Theorem 6.1 and Remark 8 for any maximal weak solution \( u \) of (1) either \( T_{\text{max}} = \infty \) or \( T_{\text{max}} < \infty \) and \( \lim_{t \to T_{\text{max}}} \| U(t) \|_H = \infty \), and Theorem 5.1 allows to exclude the first alternative. To conclude the proof of (130) we remark that (127) holds also in this case for all \( t \in [0, T_{\text{max}}) \). Hence, by (37), (58) and (75), since \( p \geq 2 \), we have
\[
\frac{1}{2} \| U(t) \|_H^2 < \frac{1}{2} \| u(t) \|_{2, \Gamma_1}^2 + c_p' \| \Omega \| + c_p' \| u(t) \|_p^p + \int_{\Gamma_1} G(\cdot, u(t)) \, d\sigma
\leq c_{34} + c_{35} \left( \| u(t) \|_p + \| u(t) \|_{2, \Gamma_1} + \int_{\Gamma_1} G(\cdot, u(t)) \, d\sigma \right)^p,
\]
concluding the proof of (130). ◻

Remark 12. Also in this case it is useful to combine Remarks 6 and 7 to point out when the model nonlinearities in (52) satisfy the full set of assumptions of Theorem 6.4, but for (A6). Assumptions (A1–5), (F1) and (G2) are satisfied when (53), (69) and (128) hold.

Moreover, in assumption (94) we can take \( m = 2 \) when \( \alpha = 0 \) and \( \mu = 2 \) when \( \beta = 0 \). Hence, for the model nonlinearities in (52), assumption (94) can be rewritten as (22). In particular, when \( \tilde{\gamma} = \tilde{\gamma}' = \tilde{\delta} = \tilde{\delta}' = 0 \) and \( \gamma, \delta \geq 0 \) as in problem (2), the full set of assumptions of Theorem 6.4, but for (A6), is satisfied when \( \gamma > 0, \delta \geq 0, (3), (6) \) and (22) hold true. Finally, assumption (A6) can be checked as in Remark 11.

We can finally prove the main results stated in § 1.

Proof of Theorem 1.3. As seen in Remark 4, problem (2) is a particular case of problem (1), with \( P, Q, f, g \) given by (52) and \( \tilde{\gamma} = \tilde{\gamma}' = \tilde{\delta} = \tilde{\delta}' = 0, \gamma, \delta \geq 0 \). Under the specific assumptions of Theorem 1.3 we have \( \gamma > 0 \) and \( \delta = 0 \), so by Remark 12 the assumptions of Theorem 6.4 are satisfied. By observing that since \( G \equiv 0 \) in this case (130) and (24) are equivalent and applying Theorem 6.4 we then get the result but for the final statement, which follows by Remark 10 and the final statement of Theorem 6.4. ◻

Proof of Theorem 1.4. As in the previous proof problem (2) is a particular case of problem (1), with the same \( P, Q, f, g \). Under the specific assumptions of Theorem 1.4 we have \( \gamma, \delta > 0 \). To get the statement we have to consider two different cases: \( \overline{m} < q \) and \( \overline{m} \geq q \). In the first case assumption (26) coincides with (129). Hence, recalling Remark 129, we can apply Theorem 6.3 and complete the proof. In the second case assumption (26) coincides with assumption (22), so by Remark 12 we can apply Theorem 6.4 and complete the proof. ◻
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