Abstract

We apply the method of QCD sum rules in the presence of the external electromagnetic $F_{\mu\nu}$ field to the problem of the electromagnetic decay of the various vector mesons, such as $\rho \rightarrow \pi + \gamma$, $K^* \rightarrow K + \gamma$, and $\eta' \rightarrow \rho + \gamma$. The induced condensates obtained previously from the study of baryon magnetic moments are adopted, thereby ensuring the parameter-free nature of the present calculation. Further consistency is reinforced by invoking the various QCD sum rules for the meson masses. The numerical results on the various radiative decays agree very well with the experimental data.

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Quantum chromodynamics (QCD) is believed to be the underlying theory of strong
interactions. However, little is known concerning the structure of a meson (for which the naive quark-antiquark picture is at most only the leading approximation), since QCD is nonperturbative at the typical hadronic scale. In the absence of the first principle calculation, the electromagnetic decay of a vector meson, such as $\rho \to \pi \gamma$, may be treated via an effective Lagrangian \[1, 2\]. In the vector meson dominance model (VDM) \[1\], an effective Lagrangian with small flavor $SU(3)$ symmetry breaking was used while in chiral perturbation theory \[2\] the Bardeen-subtracted Wess-Zumino action, together with symmetry-breaking corrections $F_K/F_\pi \approx F_\eta/F_\pi \approx 1.2$ and a given $\omega$-$\phi$ mixing parameter $\epsilon$, was employed to predict radiative vector-meson decays. As pointed out in \[2\], VDM may work to the level of within 20% for the flavor $SU(2)$ sector but is expected to be considerably worse for flavor $SU(3)$. Another approach, which is the focus of the present paper, is made possible via the method of QCD sum rules \[3\] in which the nonperturbative QCD physics is incorporated systematically as power corrections in the short-distance operator product expansion (OPE). Results from QCD sum rules will certainly serve as another useful reference point concerning radiative decays of vector mesons.

To be specific, we consider the electromagnetic decay of a vector meson in the flavor $SU(3)$ sector, i.e. in the sector of (u, d, s). Using $\rho \to \pi + \gamma$ as the illustrative example, we may describe the decay process by the effective Lagrangian,\n
$$L_{\rho\pi\gamma} = e \frac{g_{\rho\pi\gamma}}{m_\rho} \epsilon_{\mu\nu\lambda\delta} \partial^\nu A^\mu A^\lambda \partial^\delta \pi, \quad (1)$$

where $A_\mu$ is the electromagnetic field. The corresponding partial width is\n
$$\Gamma(\rho \to \pi \gamma) = \frac{\alpha g_{\rho\pi\gamma}^2 P_{\text{cm}}^3}{3 m_\rho^2}, \quad (2)$$

with $P_{\text{cm}}$ is the pion momentum in the CM frame (i.e. in the rest frame of the vector meson).

In the original work of Shifman, Vainshtein, and Zakharov \[3\], the nonperturbative QCD effects are incorporated through the various non-zero condensates in the nontrivial QCD vacuum. Ioffe and Smilga \[4\] and, independently, Balitsky and Yung \[5\] studied
baryon magnetic moments by extending the method of QCD sum rules in the presence of the external electromagnetic field. It turns out that the QCD sum rule method as obtained in [4] or [5] can readily be applied to the problem of the electromagnetic decay of the various vector mesons, such as $\rho \to \pi + \gamma$, $K^* \to K + \gamma$, and $\eta' \to \rho + \gamma$. As a result, we wish to pursue along this direction a little further, hoping to obtain results which may help to clarify certain issues in relation to the effective Lagrangian approach [1, 2].

We recall [4, 5] that, in the method of QCD sum rules, the two-point correlation function $\Pi_\mu(p)$ in the presence of the constant electromagnetic field $F_{\mu\nu}$ may be introduced as follows,

$$\Pi_\mu(p) = i \int d^4xe^{ipx} \langle 0 | T[j_\rho_\mu(x), j_\pi(0)] | 0 \rangle_F ,$$

where we introduce the interpolating fields,

$$j_\rho_\mu(x) = \bar{u}^a(x) \gamma_\mu d^a(x) , \quad j_\pi(x) = \bar{u}^a(x) i\gamma_5 d^a(x) ,$$

with $a$ the color index. As in the QCD sum rule analyses for nucleon properties, we may introduce the overlap amplitude (coupling) of the meson current to the (physical) meson:

$$\langle 0 | j_\rho_\mu | \rho \rangle = \lambda_\rho \epsilon_\mu ,$$

$$\langle 0 | j_\pi | \pi \rangle = \lambda_\pi ,$$

where $\epsilon_\mu$ is the polarization 4-vector of the $\rho$ meson.

Embedding the system in the constant $F_{\mu\nu}$ field and introducing intermediate states we express the correlation function at the hadronic level as follows,

$$\Pi_\mu(p) = \lambda_\rho \lambda_\pi \frac{g_{\rho\pi\gamma}}{2m_\rho} \frac{1}{p^2 - m_\rho^2} \frac{1}{p^2 - m_\pi^2} \epsilon_{\alpha\beta\mu\sigma} F^{\alpha\beta} p^\sigma + \Pi^{\rho\to\pi\gamma}_\mu(p) + \cdots .$$

The $\rho\pi\gamma$ coupling is defined through the relation:

$$\langle \pi'(p') | j_{\lambda\rho\gamma}^{em} | \rho_\mu^m(p) \rangle \equiv -i \delta^{tm} \epsilon^{\rho\pi\gamma} \frac{g_{\rho\pi\gamma}}{m_\rho} \epsilon^\lambda \epsilon_{\mu\nu} p_\nu q_\sigma K(q^2) ,$$
where $K(q^2)$ is the form factor normalized such that $K(0) = 1$. The superscripts $l$ and $m$ are isospin indices. Note that, in Eq. (7), we write down explicitly the leading $\rho \to \pi \gamma$ contribution, label the next transition term by $\Pi^{* \to \pi \gamma}(\rho)$, and denote the continuum contribution simply by the ellipsis.

The external field $F_{\mu \nu}$ may induce changes in the physical vacuum and modify the propagation of quarks. Up to dimension six ($d \leq 6$), we may introduce three induced condensates as follows.

\begin{align}
\langle 0 | \overline{q} \sigma_{\mu \nu} q | 0 \rangle_{F_{\mu \nu}} &= e_q e \chi F_{\mu \nu} \langle 0 | \overline{q} q | 0 \rangle, \tag{9} \\
g_c \langle 0 | \overline{q} \gamma^5 \gamma^n G^n_{\mu \nu} q | 0 \rangle_{F_{\mu \nu}} &= e_q e \kappa F_{\mu \nu} \langle 0 | \overline{q} q | 0 \rangle, \tag{10} \\
g_c \epsilon_{\mu \nu \lambda \sigma} \langle 0 | \overline{q} \gamma^5 \gamma^n G^n_{\lambda \sigma} q | 0 \rangle_{F_{\mu \nu}} &= i e_q e \xi F_{\mu \nu} \langle 0 | \overline{q} q | 0 \rangle, \tag{11}
\end{align}

where $q$ refers mainly to $u$ or $d$ (with suitable modifications for the $s$ quark), $e$ is the unit charge, $e_u = \frac{2}{3}$, and $e_d = -\frac{1}{3}$. In studying baryon magnetic moments, Ioffe and Smilga \cite{4} obtained the nucleon anomalous magnetic moments $\mu_p = 3.0$ and $\mu_n = -2.0$ (±10%) with the quark condensate susceptibilities $\chi \approx -8 \text{ GeV}^2$ and $\kappa$, $\xi$ quite small. Balitsky and Yung \cite{5} adopted the one-pole approximation to estimate the susceptibilities and obtained,

$$\chi = -3.3 \text{ GeV}^{-2}, \quad \kappa = 0.22, \quad \xi = -0.44.$$ \hspace{1cm} (12)

Subsequently Belyaev and Kogan \cite{6} extended the calculation and obtained an improved estimate $\chi = -5.7 \text{ GeV}^{-2}$ using the two-pole approximation. Chiu et al. \cite{7} also estimated the susceptibilities with the two-pole model and obtained

$$\chi = -4.4 \text{ GeV}^{-2}, \quad \kappa = 0.4, \quad \xi = -0.8.$$ \hspace{1cm} (13)

Furthermore, Chiu and co-workers \cite{7} re-analysed the various sum rules by treating $\chi$, $\kappa$, and $\xi$ as free parameters to provide an overall fit to the observed baryon magnetic moments. The optimal values which they obtained are given by

$$\chi = -3 \text{ GeV}^{-2}, \quad \kappa = 0.75, \quad \xi = -1.5.$$ \hspace{1cm} (14)
We note that in these analyses the susceptibility values are consistent with one another except the earliest result $\chi = -8 \text{ GeV}^{-2}$ in [4], which is considerably larger (in magnitude). In what follows, we shall adopt the condensate parameters $\chi = -3.5 \text{ GeV}^{-2}$, $2\kappa + \xi \approx 0$ with $\kappa$, $\xi$ quite small, which represent the average of the latter values discussed above.

On the other hand, the correlation function $\Pi_\mu(p)$ may also be evaluated at the quark level.

$$\Pi_\mu(p) = -i \int d^4xe^{ipx} \text{Tr}[iS_{ab}^\mu(x)\gamma_\mu iS_{\bar{d}b}(-x)i\gamma_5],$$

where $iS_{ab}^\mu(x)$ is the coordinate-space propagator in the presence of the $F_{\mu\nu}$ field [4, 5].

Equating the correlation function obtained both at the quark level (l.h.s.) and at the hadron level (r.h.s.), performing the Borel transform on both the sides (to accentuate the contribution from the leading $\rho \rightarrow \pi\gamma$ contribution), we finally obtain the QCD sum rule for $g_{\rho\pi\gamma}$:

$$(e_u + e_d)a_q \frac{1}{24\pi^2} \left[-3\chi + (2 + 2\kappa + \xi)\frac{1}{M_B^2}\right] + \cdots = \frac{g_{\rho\pi\gamma}}{2m_\rho} \frac{\lambda_\rho^2}{m_\rho^2 - m_\pi^2} \left(e^{-\frac{m_\rho^2}{M_B^2}} - e^{-\frac{m_\pi^2}{M_B^2}}\right) + \Pi^*_{\rho \rightarrow \pi\gamma}(M),$$

where $a_q \equiv -(2\pi)^2\langle 0|\bar{q}q|0\rangle$ and $M_B$ is the Borel mass (with $M_B^2$ playing essentially the role of $p^2$). Note that in this sum rule (16) the dominant contribution comes from the induced quark condensate characterized by the susceptibility $\chi$. The contribution from $\rho^* \rightarrow \pi\gamma$ is suppressed by the factor $(\frac{m_\rho}{m_\pi})^3$ and so is negligible to within 10% (which is the typical accuracy of the QCD sum rule approach). The sum rule (16) is stable within the working interval $M_B^2 \sim m_\rho^2$.

We proceed to note that in the framework of QCD sum rules the overlap amplitude $\lambda_\rho$ can be determined in a self-consistent manner by making use of the mass sum rules for the $\rho$ and $\pi$ mesons [6, 8],

$$\lambda_\rho^2 = \frac{1}{\pi^2} e^{\frac{m_\rho^2}{M_B^2}} \left\{ \frac{1}{4} \left(1 + \frac{\alpha_s}{\pi}\right) M_B^4 E_1 - \frac{b}{48} + \frac{\alpha_s}{\pi} \frac{14}{81} a_q^2 \frac{1}{M_B^2}\right\},$$

$$\lambda_\pi^2 = \frac{1}{\pi^2} e^{\frac{m_\pi^2}{M_B^2}} \left\{ \frac{3}{8} \left(1 + \frac{11}{3} \frac{\alpha_s}{\pi}\right) M_B^4 E_1 + \frac{b}{32} + \frac{\alpha_s}{\pi} \frac{7}{27} a_q^2 \frac{1}{M_B^2}\right\},$$

where
where \( E_1 \equiv 1 - (1 + \frac{S^2_0}{M_B^2}) e^{-\frac{S^2_0}{M_B^2}} \) (with \( S^2_0 = 1.5 \text{GeV}^2 \)) is the factor arising from a quark-level approximation for the continuum, and \( b \equiv (2\pi)^2 \langle 0 | \bar{q}_a G_{\alpha\beta}^n G_{\alpha\beta}^n | 0 \rangle \) is the gluon condensate. It was pointed out in [10] that the instanton contributions may invalidate the usual sum rule techniques for the pseudoscalar current. Nevertheless, it was suggested in [3, 8] that Eq. (18) may still provide a rough estimate of \( \lambda_\pi \) with the experimental pion mass as input (despite the fact that it cannot be used to predict the pion mass). Moreover there are many careful analyses of this correlator which yield very reasonable results for \( \lambda_\pi \) [11, 12, 13]. We therefore adopt the value from the QCD sum rule for \( \lambda_\pi \) with the physical mass \( m_\pi \) as the input. We obtain the estimates \( \lambda_\pi \approx 0.17 \pm 0.03 \text{GeV}^2 \) and \( \lambda_\rho \approx 0.17 \text{GeV}^2 \). An independent phenomenological analysis in [14] yields \( \lambda_\rho = f_\rho m_\rho = 0.17 \text{GeV}^2, \lambda_\pi = f_\pi \frac{m_u^2 + m_d^2}{m_u + m_d} = 0.20 \pm 0.04 \text{GeV}^2 \), values consistent with those from the present QCD sum rule approach.

Using our result on \( \lambda_\rho \) and \( \lambda_\pi \), the observed masses, and the condensate parameters discussed earlier, we finally obtain from the QCD sum rule (16),

\[
g_{\rho \pi \gamma} = 0.59, \tag{19}
\]

which is very close to the value \( g_{\rho \pi \gamma} = 0.58 \) as may be extracted from the partial width for the process \( \rho \to \pi \gamma \).

It is clear that the method described so far for \( \rho \to \pi \gamma \) can be generalized in a straightforward manner to other electromagnetic decays of vector mesons, among which we have considered additional processes as listed in Table I. The working interval for the Borel mass for the corresponding sum rules is \( M_B^2 \sim m_{\text{par}}^2 \), where \( m_{\text{par}} \) is the mass of the parent particle in the decay process. In the case for the meson containing a strange quark or antiquark, we have taken into account the contributions due to the nonzero quark mass \( m_s \approx 150 \text{MeV} \), which in the cases such as \( K^* \to K \gamma \) may give rise to corrections of order of about 20%. We also use \( \langle 0 | \bar{s}s | 0 \rangle \approx 0.8 \times \langle 0 | \bar{u}u | 0 \rangle \) (as suggested by chiral perturbation theory) which is essential for the \( K^* \to K \gamma \) channel.
In Table I, we present the experimental values for the various electromagnetic decays of vector mesons, the predictions from VDM [1], those from the effective Lagrangian approach [2], and what we have obtained from the QCD sum rules (this work). The results shown in this table indicate that the QCD sum rules yield predictions which are in better overall agreement with the experimental values.

TABLE I. Comparison between experimental data [9] and predictions from the Vector Meson Dominance Model (VDM) [1], effective chiral Lagrangian approach [2], and QCD sum rules. The unit is KeV.

| Decay        | Experimental value | VDM Dominance | Effective chiral Lagrangian | QCD sum rules |
|--------------|--------------------|---------------|-----------------------------|---------------|
| $\omega \to \pi \gamma$ | 716           | 888.0         | 580                        | 720            |
| $\omega \to \eta \gamma$ | 7             | 5.4           | 5                           | 7.1            |
| $\rho \to \pi \gamma$ | 68            | 63.0          | 80                         | 65             |
| $\rho \to \eta \gamma$ | 58            | 55.0          | 38                         | 48             |
| $\phi \to \eta \gamma$ | 57            | 71.0          | 68                         | 80             |
| $\phi \to \eta' \gamma$ | < 1.8        | 0.24          | 1                           | 0.67           |
| $K^0* \to K^0 \gamma$ | 116           | 149.0         | 117                        | 102            |
| $K^{+*} \to K^+ \gamma$ | 50            | 68.0          | 29                         | 60             |
| $\eta' \to \rho \gamma$ | 60            | 119.0         | 77                         | 60$^a$         |
| $\eta' \to \omega \gamma$ | 6             | 12.5          | 7                          | 6.1            |
| $\phi \to \pi \gamma$ | 5.8           | 6.1           | 5                          | 5.8$^b$        |

$^a$ Used as the input to determine $\eta$-$\eta'$ mixing angle $\theta$.

$^b$ Used as the input to determine $\phi$-$\omega$ mixing parameter $\epsilon$. 
We note that all parameters have been determined consistently in the framework of QCD sum rules as illustrated before. In particular, the various overlap amplitudes (couplings) are determined from the various mass sum rules: $\lambda_\pi = 0.17 \pm 0.03 \text{GeV}^{-2}$, $\lambda_K = 0.23 \pm 0.03 \text{GeV}^{-2}$, $\lambda_\rho = 0.17 \pm 0.01 \text{GeV}^{-2}$, $\lambda_{K^*} = 0.21 \pm 0.02 \text{GeV}^{-2}$. The above predictions and the other overlap amplitudes given in the following after taking into account the mixing effects are in good agreement with those in [14]. Note that the value for $\lambda_{\eta'}$ is of some specific interest in view of the $U_A(1)$ problem.

In the case of the $\phi$-$\omega$ mixing, we introduce

$$|\omega\rangle = \cos \epsilon |\omega_0\rangle - \sin \epsilon |\phi_0\rangle, \quad |\phi\rangle = \sin \epsilon |\omega_0\rangle + \cos \epsilon |\phi_0\rangle,$$  \hspace{1cm} (20)

and

$$j_\omega = \cos \epsilon j_{\omega_0} - \sin \epsilon j_{\phi_0}, \quad j_\phi = \sin \epsilon j_{\omega_0} + \cos \epsilon j_{\phi_0}.$$  \hspace{1cm} (21)

where

$$|\omega_0\rangle = \frac{1}{\sqrt{2}}(|u\bar{u}\rangle + |d\bar{d}\rangle), \quad |\phi_0\rangle = |s\bar{s}\rangle.$$  \hspace{1cm} (22)

Using the interpolating field in (21), we derive the QCD sum rules for the relevant radiative decays and the mass sum rules for $\omega$ and $\phi$ mesons. We then use the observed value for the width of the decay $\phi \rightarrow \pi \gamma$ to determine the mixing angle. We obtain the mixing parameter $\epsilon = 0.063 \pm 0.005$ and the overlap amplitudes $\lambda_\omega = 0.16 \pm 0.01 \text{GeV}^{-2}$ and $\lambda_\phi = 0.24 \pm 0.02 \text{GeV}^{-2}$. As the Borel mass is varied in the interval $0.6 \text{GeV}^2 \leq M_B^2 \leq 1.1 \text{GeV}^2$, the mixing parameter $\epsilon$ changes from 0.068 to 0.058. Note that, as a consistency check, our result is consistent with the value $\epsilon = 0.079$ as determined from $\phi \rightarrow \rho\pi$ [2].

Furthermore, we use the experimental partial width for the decay process $\eta' \rightarrow \rho\gamma$ as the input in the derived QCD sum rules to extract the $\eta_0 - \eta_8$ mixing angle, which is parametrized as in [13, 9]:

$$|\eta\rangle = \cos \theta |\eta_8\rangle - \sin \theta |\eta_0\rangle, \quad |\eta'\rangle = \sin \theta |\eta_8\rangle + \cos \theta |\eta_0\rangle,$$  \hspace{1cm} (23)
and correspondingly for the interpolating fields,

\[ j_\eta = \cos \theta j_{\eta 8} - \sin \theta j_{\eta 0}, \quad j_{\eta'} = \sin \theta j_{\eta 8} + \cos \theta j_{\eta 0}. \]  

(24)

At the same time we obtain the \( \eta, \eta' \) mass sum rules with the interpolating fields in (24) and use the physical meson masses to calculate the overlap amplitudes. We then use the extracted mixing angle to predict other processes. Our results are \( \lambda_\eta = 0.23 \pm 0.03 \text{GeV}^{-2} \), \( \lambda_{\eta'} = 0.33 \pm 0.05 \text{GeV}^{-2} \), \( \theta = -19^\circ \pm 2^\circ \), a value consistent with that extracted from the decays \( \eta \to \gamma \gamma \) and \( \eta' \to \gamma \gamma \) [15] and in [9]. We find that such value on the \( \eta_0 - \eta_8 \) mixing is essential for the overall agreement with the experimental data.

We wish to emphasize the parameter-free nature of the present calculation, in that all parameters are determined consistently in the framework of QCD sum rules, with a single set of the values for the induced condensates obtained previously from studies of baryon magnetic moments. We note that the uncertainty of the overlap amplitudes (couplings) derived in this way might be as large as 10% in some cases and the method of QCD sum rules, being an operator product expansion used for moderate \( Q^2 \), may have some inherent error of order of 10%. With this in mind, we may conclude that our results as shown in Table I agree very well with the experimental data.

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