Excited Baryons in Large $N_c$ QCD

Matching the $1/N_c$ expansion to quark models using the permutation group $S_N$

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Abstract. We show how to match quark models to the $1/N_c$ expansion of QCD. As an example we discuss in detail the mass operator of orbitally excited baryons and match it to the one-gluon exchange and the one-boson exchange variants of the quark model. The matching procedure is very general and makes use of the transformation properties of states and operators under the permutation group acting on the orbital and spin-flavor degrees of freedom of $N$ quarks.

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1 Introduction

Quark models provide a simple and intuitive picture for the physics of ground state baryons and their excitations. These models constitute important assets in the toolbox of the hadronic physicist. It is of great relevance to sharpen up tools that have proven to be useful and put them on a sounder ground, establishing the connection with the underlying theory of the strong interactions. Hopefully this will shed light on why these simple models work, something that can also hint at how to improve them.

The large $N_c$ limit of QCD provides a leading order picture for hadrons which can be corrected systematically including $1/N_c$ corrections [1]. This program can be realized in terms of a quark operator expansion, which gives rise to a physical picture similar to the one of the phenomenological quark models, but is closer connected to QCD. In this context quark models gain additional significance. Here we want to show how it is possible to match these quark models to the more general $1/N_c$ expansion of QCD. It turns out that classifying the transformation properties of states and operators under permutations using the symmetric group $S_N$ is the key to perform this matching explicitly [2]. For $N_c = 3$ the symmetric group $S_3$ has been used previously in Ref. [3] to compare the predictions of two versions of the quark model, one based on one-gluon exchange [4] and one based on one-boson exchange [5]. Here we generalize this to arbitrary $N_c$ and compare both models with the more general $1/N_c$ expansion.

Before getting into the details of the matching procedure it is useful to recall a few general points that make the large number of colors limit interesting and useful:

– Although in the large $N_c$ limit the number of degrees of freedom increases, the physics simplifies.

– The $1/N_c$ expansion is the only candidate for a perturbative expansion of QCD at all energies.

– In the $N_c \to \infty$ limit baryons fall into irreducible representations of the contracted spin-flavor algebra $SU(2n_f)/S_2$, also known as $K$-symmetry, that relates properties of states in different multiplets of flavor symmetry.

– The breaking of spin-flavor symmetry can be studied order by order in $1/N_c$ as an operator expansion.

The successful applications of the $1/N_c$ expansion to the study of ground state baryons make the excited baryons especially interesting because they provide a wider testing ground for the $1/N_c$ expansion. It is important to stress that already at leading order in the large $N_c$ limit it is possible to obtain significant qualitative insights into the structure of excited baryons, among which we would like to highlight the following:

– The three towers [6] [7] [8] predicted by $K$-symmetry for the $L = 1$ negative parity $N^*$ baryons, labeled by $K = 0, 1, 2$ with $K$ related to the isospin $I$ and spin $J$ of the $N^* \pm s$ by $I + J \geq K \geq |I - J|$.

– The vanishing of the strong decay width $\Gamma(N^*_1 \to \omega N\pi)_{S}$ for $N^*_1$ in the $K = 0$ tower, which provides a natural explanation for the relative suppresion of pion decays for the $N^*(1535)$ [6] [8] [9].
The order $O(N_c^0)$ mass splitting of the $SU(3)$ singlets $\Lambda(1405)$ - $\Lambda(1520)$ in the \([70, 1^-]\) multiplet \([10]\).

The $1/N_c$ expansion has also been applied to orbitally excited baryons up to subleading order in $1/N_c$ \([11, 12, 13, 14, 15]\). The construction of the states and operators for excited states presented in Ref. \([12]\) has been inspired by the quark model picture of the excited states and makes use of the decomposition of the spin-flavor states into “core” and “excited” quark subsystems. A formal justification of this in terms of a symmetry argument can be obtained considering the permutation group $S_N$ \([2]\). In several recent papers \([16, 17]\) the validity of the usual approach based on the core+excited quark decomposition has been questioned, and this symmetry argument settles the objections raised in these works.

Including also the orbital degrees of freedom, the complete permutation symmetry is $S_N \subset S_N^{orb} \times S_N^{sp-f}$, the diagonal subgroup of the permutations acting on both orbital and spin-flavor degrees of freedom. Of course, although $S_N$ is a good symmetry of the quark model Hamiltonian, $S_N^{sp-f}$ is not, and mixing between different irreps can occur in general (configuration mixing).

We consider here in some detail the states transforming in the $[N-1, 1]$ $(MS)$ irrep of $S_N^{sp-f}$. The matrix elements $(MS|\hat{O}|MS)$ of any operator $\hat{O}$ on mixed symmetric spin-flavor states $MS$ can transform as:

$$MS \otimes MS = S \oplus MS \oplus A' \oplus E.$$ (1)

These irreps of $S_N$ are identified by a partition $\{n_i\}$ corresponding to a Young diagram with $n_1, n_2, \cdots$ boxes in each row, see Table 1 for notation and typical Young diagrams for $N = 5$.

The construction of the states with correct permutation symmetry under $S_N$ is described in Sec. 2. As an example of the decomposition Eq. (1) we show in detail in Sec. 3 the explicit decomposition of the spin-spin two-body quark-quark interaction for the one-gluon exchange model, into irreps of $S_N^{sp} \times S_N^{sp-f}$. Finally, the derivation of the mass operator and the matching onto the $1/N_c$ expansion are presented in Sec. 4. A complete discussion can be found in Ref. [2].

2 The $MS$ states and their relation to CCGL

A basis for the $MS$ spin-flavor wave function can be constructed using the method of the Young operators (for further details and examples for $N = 5$ see Ref. [2]). It can be chosen as the set of $N - 1$ wave functions, with $k = 2, 3, \cdots, N$

$$\phi_k = |q_k\rangle \otimes |im\alpha\rangle_{N-1} - |q_1\rangle \otimes |im\alpha\rangle_{N-1}$$ (2)

where $|q_k\rangle$ denotes the spin-flavor state of the quark $k$, and $|im\alpha\rangle_{N-1}$ denotes the spin-flavor state of the subset of $N - 1$ quarks (‘core’) obtained by removing quark $k$ from the $N$ quarks. The latter states are symmetric under any permutation of the $N - 1$ quarks. The states $\phi_i$ are not orthogonal, and have the scalar products

$$\langle \phi_i | \phi_j \rangle = S_{ij}, \quad S_{ij} = \begin{cases} 2, & i = j \\ 1, & i \neq j \end{cases}$$ (3)

The basis states $\phi_k$ have the following transformation properties under the action of the transpositions $P_{ij}$ (exchange of the quarks $i,j$)

$$P_{ij} \phi_k = \begin{cases} -\phi_k, & j = k \\ \phi_k - \phi_j, & i \neq j \end{cases}$$ (4)

$$P_{ik} \phi_k = 0 \quad \text{if} \quad i,j \neq k,1$$ (5)

$$P_{ik} \phi_k = P_{kl} \phi_k = \phi_i \quad \text{if} \quad i \neq 1.$$ (6)

These transformation relations can be obtained defining a basis for the $MS$ irrep of $S_N$ in a general way that is valid for states and operators, namely, as $\phi_k = (P_{1k} - 1)\phi$ with any $\phi$ satisfying $\phi = P_{il} \phi$ for $i,j = 2, \cdots, N$.

For the orbital wave functions we adopt a Hartree representation, in terms of one-body wave functions $\varphi_i^o(r)$ for the ground state orbitals, and $\varphi_i^o(r)$ for the orbitally excited quark. The index $m = \pm 1, 0$ denotes the projection of the orbital angular momentum $L$ along the $z$ axis. The Young operator basis for the orbital wave functions is given by (with $k = 2, 3, \cdots, N$)

$$\chi^m_k(r_1, \cdots, r_N) = (P_{1k} - 1)\varphi^o_{r_1}(r_1)\Pi^N_{i\neq 1}\varphi^o_{r_i}(r_i).$$ (7)

and have the same transformations under transpositions as the spin-flavor basis functions Eqs. (3-6).

The complete spin-flavor-orbital wave function of a baryon $B$ with mixed-symmetric spin-flavor symmetry is written as the $MS \times MS \rightarrow S$ inner product of the two basis wave functions, for the orbital and spin-flavor components, respectively

$$|B\rangle = \sum_{k,l=2}^N \phi_k \chi^m_k M_{kl}$$ (8)

The matrix of coefficients $M_{ij}$ are the Clebsch-Gordan coefficients for the $MS \times MS \rightarrow S$ inner product of two irreps of $S_N$. They can be determined by requiring that the state Eq. (3) is left invariant under the action of transpositions acting simultaneously on the spin-flavor and orbital components. In the $MS$ basis defined by the transformations Eqs. (2) the matrix $M_{ij}$ is

$$\hat{M} = \begin{pmatrix}
1 & -\frac{1}{N-1} & -\frac{1}{N-1} & \cdots & -\frac{1}{N-1} \\
-\frac{1}{N-1} & 1 & -\frac{1}{N-1} & \cdots & -\frac{1}{N-1} \\
-\frac{1}{N-1} & -\frac{1}{N-1} & 1 & \cdots & -\frac{1}{N-1} \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
-\frac{1}{N-1} & -\frac{1}{N-1} & -\frac{1}{N-1} & \cdots & 1
\end{pmatrix}$$ (9)

Since any permutation can be represented as a product of transpositions, the state given in Eq. (3) transforms indeed in the $S$ irrep of the overall $S_N$ group.

The state Eq. (3) can be related to the $MS$ states constructed in Ref. [12], and commonly used in the literature in the context of the $1/N_c$ expansion. These states
Table 1. Irreps of the permutation group $S_N$ used in the text. The Young diagrams shown correspond to $N = 5$.

| irrep | $S$ | $MS$ | $E$ | $A'$ |
|-------|-----|------|-----|------|
| partition: | $[N]$ | $[N - 1, 1]$ | $[N - 2, 2]$ | $[N - 2, 1, 1]$ |
| dim : | 1 | $N - 1$ | $N(N - 3)/2$ | $(N - 1)(N - 2)/2$ |

are constructed as tensor products of an ‘excited’ quark whose identity is fixed as quark 1, with a symmetric ‘core’ of $N - 1$ quarks. We will refer to these states as CCGL states. With this convention, the wave function used in Ref. [12] has the form

$$|CCGL⟩ = Φ(SI) \varphi^m_p(r_1)P_i^N j_{i=2}^N \varphi_s(r_i)$$

(10)

where $Φ(SI)$ denotes the spin-flavor component, and the remainder is the orbital wave function in Hartree form. By construction, the spin-flavor wave function $Φ(SI)$ transforms in the $MS$ irrep of $SU(4)$, and thus also like $MS$ under $S_N$. It is symmetric under any exchange of the core quarks, $P_i \Phi(SI) = Φ(SI)$ for $i, j \neq 1$. These two properties identify it uniquely in terms of the $MS$ basis functions defined in Eq. (2) as $Φ(SI) = \sqrt{N(N - 1)} \sum_{k=2}^N φ_k$.

The normalization factor is chosen such that the state is properly normalized symmetric state

$$|CCGL⟩ \rightarrow \frac{1}{\sqrt{N}} \sum_{i=1}^N P_i |CCGL⟩ = \frac{1}{N\sqrt{N - 1}} \sum_{i=2}^N \sum_{j=2}^N (P_i φ_j)χ_i = -\frac{N - 1}{N} |B⟩,$$

(11)

where the terms with different $i$ in the sum are orthogonal states. This gives the relation between the CCGL state and the symmetric state constructed above in Eq. (5).

### 3 OGE and OBE quark model interactions

In this Section we present two representative quark models that will be matched to the $1/N_c$ expansion. We start considering the one-gluon exchange potential (OGE), which follows from a perturbative expansion in $α_s(m_Q)$ in the heavy quark limit $m_Q \gg Λ_{QCD}$.

We consider a Hamiltonian containing a spin-flavor symmetric term $H_0$ (the confining potential and kinetic terms), plus spin-isospin dependent two-body interaction terms $V_{ij}$

$$H = H_0 + g^2 \sum_{i<j} λ^a_i λ^a_j \frac{N_c}{2} V_{ij} \rightarrow H_0 - g^2 N_c \frac{1}{2} \sum_{i<j} V_{ij}$$

(12)

where $λ^a$ are the generators of SU(3) color in the fundamental representation. The second equality holds on color-singlet hadronic states, on which the color interaction $t^a_i t^a_j$ evaluates to the color factor $(N_c + 1)/(2N_c)$. We will restrict ourselves only to color neutral hadronic states.

In the nonrelativistic limit, the two-body interaction $V_{ij}$ contains three terms: the spin-spin interaction ($V_{ss}$), the quadrupole interaction $V_q$ and the spin-orbit terms $V_{so}$. We write these interaction terms following Ref. [3], where $f_{0,1,2}(r_{ij})$ are unspecified functions of the interquark distances

$$V_{ss} = \sum_{i<j=1}^N f_0(r_{ij}) s_i \cdot s_j$$

(13)

$$V_q = \sum_{i<j=1}^N f_2(r_{ij}) \left[3(r_{ij} \cdot s_i)(r_{ij} \cdot s_j) - (s_i \cdot s_j) \right]$$

(14)

$$V_{so} = \sum_{i<j=1}^N f_1(r_{ij}) \left[ (r_{ij} \times p_i) \cdot s_i - (r_{ij} \times p_j) \cdot s_j \right] + 2(r_{ij} \times p_i) \cdot s_j - 2(r_{ij} \times p_j) \cdot s_i$$

(15)

We consider also the mass operator of the orbitally excited baryons in a second model for the quark-quark interaction. In Ref. [5] it was suggested that pion-exchange mediated quark-quark interactions can reproduce better the observed mass spectrum of these states. The physical idea is that at the energy scales of quarks inside a hadron, the appropriate degrees of freedom are quarks, gluons and the Goldstone bosons of the broken chiral group $SU(2)_L \times SU(2)_R \rightarrow SU(2)$ [18]. The exchange of Goldstone bosons changes the short distance form of the quark-quark interactions, and introduces a different spin-flavor structure. The new potentials $V_{ij}$ are obtained from Eqs. (12)-(15), with the replacements $f_{0,1,2}(r_{ij}) \rightarrow g_{0,1,2}(r_{ij}) t^a_i t^a_j$ where the isospin generators are $t^a = \frac{i}{2} \tau^a$, and $g_{0,1,2}(r_{ij})$ are unspecified functions.

The interaction Hamiltonian of the one boson exchange (OBE) model has then the form

$$H = H_0 + g_A^2 \sum_{i<j} \tilde{V}_{ij}$$

(16)

where $g_A$ is a quark-pion coupling which scales like $O(N_c^0)$ with the number of colors $N_c$. 
4. $S_N \rightarrow S^\text{orb}_N \times S^\text{sp-fl}_N$ decomposition of the interaction Hamiltonian

The Hamiltonians $H$ in Eqs. (12), (13) can be decomposed into a sum of terms transforming according to irreps of the permutation group acting on the spin-flavor degrees of freedom $S^\text{sp-fl}_N$. The operators in Eqs. (12), (13) are two-body interactions, of the generic form

$$V = \sum_{1 \leq i < j \leq N} R_{ij} O_{ij}$$ (17)

where $R_{ij}$ acts only on the orbital coordinates of the quarks $i, j$, and $O_{ij}$ acts only on their spin-flavor degrees of freedom. For example, the spin-spin interaction $V_{ss}$ has $R_{ij} = f_0(r_{ij})$ and $O_{ij} = s_i \cdot s_j$. $V$ must be symmetric under any permutation of the $N$ quarks, but the transformation of the spin-flavor and orbital factors separately can be more complicated. We distinguish two possibilities for the transformation of these operators under a transposition of the quarks $i, j$, corresponding to the two irreps of $S_N$: i) symmetric two-body operators $V_{\text{symm}}$: $P_{ij} R_{ij} P_{ij}^{-1} = R_{ij}$ and $P_{ij} O_{ij} P_{ij}^{-1} = O_{ij}$. ii) antisymmetric two-body operators $V_{\text{anti}}$: $P_{ij} R_{ij} P_{ij}^{-1} = -R_{ij}$ and $P_{ij} O_{ij} P_{ij}^{-1} = -O_{ij}$. The spin-spin and quadrupole interactions $V_{ss}, V_q$ are symmetric two-body operators, while the spin-orbit interaction $V_{so}$ contains both symmetric and antisymmetric components.

In general, the $k$–body operators can be classified into irreps of the permutation group of $k$ objects $S_k$. For example, there are three classes of 3-body operators, corresponding to the $S, MS$ and $A$ irreps of $S_3$.

We start by considering the symmetric two-body operators. The set of all spin-flavor operators $O_{ij}$ (and analogous for the orbital operators $R_{ij}$) with $1 \leq i < j \leq N$ form a $\frac{1}{2}N(N - 1)$ dimensional reducible representation of the $S_N$ group, which contains the following irreps

$$\{O_{ij}\} = S \oplus MS \oplus E$$ (18)

We will use as a basis for the operators on the right-hand side the Young operator basis supplemented by the phase convention Eq. (4). For simplicity we neglect the $E$ irrep in the following explicit example.

The $S$ and $MS$ projections are

$$O^S = \sum_{i < j} O_{ij} ,$$ (19)

$$O^{MS}_k = \sum_{2 \leq j \neq k \leq N} (O_{ij} - O_{kj}) , \quad k = 2, 3, \cdots, N$$ (20)

The interaction $V_{\text{symm}}$ is symmetric under $S_N$, and its decomposition under $S_N \subset S^\text{orb}_N \times S^\text{sp-fl}_N$ has the form

$$V_{\text{symm}} = \frac{2 R^S}{N(N - 1)} O^S + \frac{N - 1}{N(N - 2)} \sum_{j, k = 2}^N R^{MS} O^{MS}_k M_{jk}$$ (21)

where the matrix $M_{jk}$ is given in Eq. (9) in explicit form.

The antisymmetric two-body operators $O_{ij}$ form a reducible representation of $S_N$ of dimension $\frac{1}{2}N(N - 1)$, which is decomposed into irreps as

$$\{O_{ij}\} = MS \oplus A'$$ (22)

and will not be considered further here (for a complete discussion see Ref. [2]).

5. Mass operator - OGE interaction

For simplicity we keep only the $S$ and $MS$ operators, and compute their matrix elements on the $|B\rangle$ states constructed in Eq. (8). The matrix elements of the spin-flavor operators on the basis functions $\phi_i$ are given by

$$\langle \phi_j | O^S | \phi_k \rangle = \langle O^S | S_{jk}$$ (23)

$$\langle \phi_j | O^{MS} | \phi_k \rangle = \langle O^{MS} | (1 - \delta_{ji} \delta_{ik})$$ (24)

where $S_{jk}$ was defined in Eq. (3), and $\langle O^S, \langle O^{MS} are reduced matrix elements. The proportionality of the matrix elements to just one reduced matrix element follows from the Wigner-Eckart theorem for the $S_N$ group. The form of the Clebsch-Gordan coefficients is specific to the $MS$ basis used in Eqs. (21), and can be derived by repeated application of Eqs. (4) to the states and operators.

Similar expressions apply for the orbital operators $R$, up to an additional complexity introduced by the presence of the magnetic quantum numbers of the initial and final state orbital basis functions $m$. The dependence on $m, m'$ is related to the Lorentz structure of the orbital operator and can be parametrized by the matrix element of a tensor operator which in the present case can be constructed in terms of the angular momentum operator $L$.

Inserting these expressions into Eq. (21) and combining all factors one finds the general expression for a symmetric 2-body operator

$$\langle B | V_{\text{symm}} | B \rangle = \frac{2 \langle O^S | R^S \rangle}{N(N - 1)} + \frac{\langle O^{MS} | R^{MS} \rangle}{N}$$ (25)

where the contributions of operators transforming in the $E$ irrep are neglected.

The reduced matrix elements depend on the precise form of the interaction. We consider for definiteness the spin-spin interaction $V_{ss}$ in some detail. For this case the reduced matrix elements of the symmetric operators are given by

$$\langle R^S \rangle = \frac{1}{2} (N - 1)(N - 2)I_s + (N - 1)I_{\text{dir}} - I_{\text{exc}}$$ (26)

$$\langle O^S \rangle = \langle \Phi(SI) \rangle \left( \frac{1}{2} S^2 - \frac{3}{8} N \langle \Phi(SI) \rangle \right) .$$ (27)

For the ease of comparison with the literature on the $1/N_c$ expansion for excited baryons, we expressed the reduced matrix element of the spin-flavor operator as a matrix element on the state $| \Phi(SI) \rangle$ where the excited quark is
quark no. 1. The reduced matrix elements of the orbital operator are expressed in terms of the 3 overlap integrals over the one-body wave functions

$$\mathcal{I}_s = \int dr_1 dr_2 f_0(r_{12}) |\phi_s(r_1)|^2 |\phi_s(r_2)|^2$$  (28)

$$\mathcal{I}_{\text{dir}} = \int dr_1 dr_2 f_0(r_{12}) |\phi_s(r_1)|^2 |\phi^m_p(r_2)|^2$$

$$\mathcal{I}_{\text{exc}} = \int dr_1 dr_2 f_0(r_{12}) |\phi_s(r_2)|^2 |\phi^m_p(r_1)|^2 |\phi^*_s(r_1)|^2$$

The spin-flavor operator transforming in the $M_S$ irrep is $O^{MS}_k = (s_i - s_k) \cdot S$, and the corresponding orbital operator is $R^{MS}_k = \sum_{j=2,j\neq k}^N f_0(r_{1j}) - f_0(r_{kj})$. Their reduced matrix elements are

$$\langle O^{MS} \rangle = \frac{1}{N-2} \langle \Phi(SI) | S^2 + N S_1 \cdot S_c + \frac{3}{4} N | \Phi(SI) \rangle$$

$$\langle R^{MS} \rangle = (N-2)(\mathcal{I}_{\text{dir}} - \mathcal{I}_s) - 2 \mathcal{I}_{\text{exc}}.$$

We denoted $S_c$, the ‘core’ spin, defined as $S_c = S - s_1$. The reduced matrix element of the orbital operator ($R^{MS}$) was computed by taking representative values of $i, j, k$ in Eq. (24) and evaluating the integrals.

5.1 The reduced matrix element $\langle O_{MS} \rangle$

We present here the details of the derivation of the reduced matrix element of a $MS$ spin-flavor operator as a matrix element on the CCGL state $\Phi(SI)$ with fixed identity of the ‘excited’ quark (such as e.g. Eq. (29)). As explained in Sec. 2 this state is given by

$$\Phi(SI) = \frac{1}{\sqrt{N(N-1)}} \sum_{k=2}^N \phi_k.$$  (30)

The matrix element of an $MS$ operator on our $MS$ basis states is given by the Wigner-Eckart theorem for $S_N$, Eq. (24). Summing over the index of the operator, the matrix element on the $\Phi(SI)$ state is

$$\langle \Phi(SI) | O^{MS}_k | \Phi(SI) \rangle = (N-2) \langle O^{MS} \rangle$$  (31)

which can be used to express $\langle O^{MS} \rangle$ as a matrix element on the CCGL-type spin-flavor state $\Phi(SI)$.

The advantage of taking the sum $\sum_{k=2}^N O^{MS}_k$ is that it singles out the quark no. 1, just as in the state $\Phi(SI)$. An explicit calculation gives

$$\sum_{k=2}^N O^{MS}_k = -S^2 + N s_1 \cdot S_c + \frac{3}{4} N$$

which gives Eq. (29) after combining it with Eq. (31).

### Table 2. Subleading operators

| Coef. | OGE | OBE |
|-------|-----|-----|
| $c_2$ | $-\frac{2^2}{\sqrt{3}} \{3J_{\text{dir}} - J_{\text{dir}}^a\}$ | $-\frac{1}{2} g_A^2 \{3J_{\text{dir}} + J_{\text{dir}}^a\}$ |
| $c_3$ | $0$ | $\frac{2^2}{\sqrt{3}} g_A^2 \{3J_{\text{dir}} + J_{\text{dir}}^a\}$ |
| $c_4$ | $0$ | $0$ |
| $c_5$ | $-\frac{2^2}{\sqrt{3}} \{3J_{\text{dir}} + J_{\text{dir}}^a\}$ | $0$ |
| $c_6$ | $-\frac{2^2}{\sqrt{3}} J_{\text{dir}}$ | $-\frac{1}{2} g_A^2 J_{\text{dir}}$ |
| $c_7$ | $-\frac{2^2}{\sqrt{3}} J_{\text{dir}}$ | $-\frac{1}{2} g_A^2 J_{\text{dir}}$ |
| $c_8$ | $0$ | $0$ |
| $c_9$ | $0$ | $-\frac{1}{2} g_A^2 \{3J_{\text{dir}} - J_{\text{dir}}^a\}$ |
| $c_{10}$ | $0$ | $-\frac{1}{2} g_A^2 J_{\text{dir}}$ |
| $c_{11}$ | $0$ | $0$ |

6 Matching onto the $1/N_c$ expansion

The spin-flavor structure of the one-gluon exchange interaction matches a subset of the operators appearing in the $1/N_c$ expansion of the mass operator for orbitally excited states. Working to order $1/N_c$, the most general set of operators contributing to the mass of these states is

$$\hat{M} = c_1 N_c \mathbf{1} + c_2 L^i s^i + c_3 \frac{3}{2} N_c L^j_{\text{dir}} g_A^a G_{\text{dir}}^a + \sum_{i=4}^8 c_i \mathcal{O}_i,$$  (34)

The terms proportional to $c_{2,3}$ contribute at order $O(N_c^0)$, and the remaining operators proportional to $c_{4-8}$ are of order $1/N_c$. A complete basis of subleading operators can be chosen as in Ref. [12] and is shown in Table 2. The corresponding coefficients that we find matching the quark-quark interactions are given in Table 2 and can be expressed in terms of overlap integrals, where $J, K$ are similar to the ones given in Eqs. (28) and originate from the spin-orbit and tensor interaction respectively.

The complete explicit calculation confirms the $N_c$ power counting rules of Ref. [11], in particular the leading order $O(N_c^0)$ contribution to the mass coming from the spin-orbit interaction given by $c_2$ confirms in a direct way the prediction of the breaking of the $SU(4)$ spin-flavor symmetry at leading order in $N_c$. The nonrelativistic quark model with gluon mediated quark interactions displays the same breaking phenomenon.

The two-body quark interactions considered here produce one-, two- and three-body ($\mathcal{O}_{17} = \frac{1}{N_c^2} L^j_{\text{dir}} \{S^i_{\text{dir}}, S^j_{\text{dir}}\}$).
of Ref. [12], which correctly appears at order $O(1/N_c^3)$ effective operators in the $1/N_c$ expansion.

Another important conclusion following from this calculation is that operators with nontrivial permutation symmetry are indeed required by a correct implementation of the $1/N_c$ expansion.

A distinctive prediction of the one-gluon exchange potential is the vanishing of the coefficient $c_3$ of one of the leading order operators. On the other hand, in the one-boson exchange model, the coefficients of the two leading operators can be of natural size. There are predictions for the vanishing of the coefficients of some subleading operators, and also one relation between the coefficients of the leading and subleading operators $c_2 = -c_4$. The Goldstone boson interaction also generates the three-body operator, $O_{17}$ of Ref. [12], which correctly appears at order $O(1/N_c^2)$.

The coefficients of the leading order operators $c_{1,2,3}$ have been determined in [7] from a fit to the masses of the nonstrange $L = 1$ baryons, working at leading order in $1/N_c$. The results depend on the assignment of the observed baryons into the irreps of the contracted symmetry (towers). There are four possible assignments, but only two of them are favored by data. These two assignments give the coefficients

\[
\begin{align*}
\text{assignment 1:} & \quad c_2^{(0)} = 83 \pm 14 \text{ MeV}, & \quad c_3^{(0)} = -188 \pm 28 \text{ MeV} \\
\text{assignment 3:} & \quad c_2^{(0)} = -12 \pm 16 \text{ MeV}, & \quad c_3^{(0)} = 142 \pm 38 \text{ MeV}
\end{align*}
\]

Comparing with the predictions from the gluon mediated quark-quark interactions, we see that there is no evidence in the data for a suppression of the coefficient $c_3$ relative to $c_2$. For both assignments, the coefficient $c_3$ is sizeable, a situation which favors the Goldstone boson exchange model, or at least indicates that some kind of flavor dependent effective interactions cannot be neglected.

7 Conclusions

We presented a general procedure to match quark models to the $1/N_c$ expansion of QCD, using as an example the mass operator for orbitally excited baryons. The transformation properties of the states and operators under the permutation group $S_N$ allow to parameterize our ignorance of the spatial dependence of the wave functions in terms of only a few reduced matrix elements.

This approach is similar to the method used for $N_c = 3$ in Ref. [3] to compare the predictions of nonrelativistic quark models in a form independent of the orbital wave functions. In addition to extending this result to arbitrary $N_c$, we also extract here the spin-flavor structure in an operator form, which allows to make an explicit comparison of two different models before taking the full matrix elements.

The results obtained for the mass operator match precisely the operators appearing in the $1/N_c$ expansion [11], up to new contributions, not considered previously, transforming in the $E$ and $A'$ irreps of $S_N^{2n-n}$. We confirmed by an explicit calculation the $N_c$ power counting rules of Refs. [11, 12]. In particular we proved that the effective spin-orbit interaction is of leading order $O(N_c^0)$.

The transformation properties under $S_N$ also imply that the decomposition into core and excited quark operators used previously in the literature on the subject is necessary to obtain the most general expression for the mass operator.

The two models considered here induce different hierarchies among the effective constants of the general $1/N_c$ expansion. Comparing with existing fits to the masses of nonstrange $L = 1$ excited baryons, we find that flavor dependent interactions cannot be neglected, and may be necessary to supplement the gluon exchange model. This is in line with the chiral quark picture proposed in Ref. [18].

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