Local 2-separators

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Abstract

How can sparse graph theory be extended to large networks, where algorithms whose running time is estimated using the number of vertices are not good enough? I address this question by introducing 'Local Separators' of graphs. Applications include:

1. A unique decomposition theorem for graphs along their local 2-separators analogous to the 2-separator theorem;
2. an exact characterisation of graphs with no bounded subdivision of a wheel;
3. an analogue of the tangle-tree theorem of Robertson and Seymour, where the decomposition-tree is replaced by a general graph.

1 Introduction

One of the big challenges in Graph Theory today is to develop methods and algorithms to study sparse large networks; that is, graphs where the number of edges is about linear in the number of vertices, and the number of vertices is so large that algorithms whose running time is estimated in terms of the vertex number are not good enough. Important contributions that provide partial results towards this big aim include the following.

1. Bejamini-Schramm limits of graphs. Bejamini and Schramm introduced a notion of convergence of sequences of graphs that is based on neighbourhoods of vertices of bounded radius in [6]. Applications of these methods include: testing for minor closed properties [7] by Benjamini, Schramm and Shapira or the proof of recurrence of planar graph limits by Gurel-Gurevich and Nachmias [19].
2. **From Graphons to Graphexes.** Graphons have turned out to be a useful tool to study dense large networks \([25, 26]\). Motivated by these successes, analogues for sparse graph limits are proposed in \([8, 10, 23]\).

3. **Graph Clustering.** The spectrum of the adjacency matrix of a graph can be used to identify large clusters, see the surveys \([36]\) or \([33]\).

4. **Nowhere dense classes of graphs.** In their book \([27]\), De Mendez and Nešetřil systematically study a whole zoo of classes of sparse graphs and the relation between these classes.

5. **Refining tree-decompositions techniques.** Empirical results by Adcock, Sullivan and Mahoney suggest that some large networks do have tree-like structure \([1]\). In \([2]\), these authors say that: ‘Clearly, there is a need to develop Tree-Decompositions heuristics that are better-suited for the properties of realistic informatics graphs’. And they set the challenge to develop methods that combine the local and global structure of graphs using tree-decompositions methods.

Much of sparse graph theory – in particular of graph minor theory – is built upon the notion of a separator, which allows to cut graphs into smaller pieces, solve the relevant problems there and then stick together these partial solutions to global solutions. These methods include: tree-decompositions \([30]\), the 2-separator theorem and the block-cutvertex theorem, Seymour’s decomposition theorem for regular matroids \([35]\), as well as clique sums and rank width decompositions \([28]\). Understanding the relevant decomposition methods properly is fundamental to recent breakthroughs such as the Graph Minor Theorem \([31]\) or the Strong Perfect Graph Theorem \([32]\). As whether a given vertex set is separating depends on each vertex individually. So in the context of large networks it is unfeasible to test whether a set of vertices is separating. We believe that in order to extend such methods from sparse graphs to large networks, it is key to answer the following question. What are local separators of large networks?

Here, we answer this question. Indeed, we provide an example demonstrating that the naive definition of local separators misses key properties of separators. Then we introduce local separators of graphs that lack this defect. Our new methods have the following applications.

A) A unique decomposition theorem for graphs along their local 2-separators analogous to the 2-separator theorem;
B) an exact characterisation of graphs with no bounded subdivision of a wheel. This connects to direction (4) outlined above;

C) an analogue of the tangle-tree theorem of Robertson and Seymour, where the decomposition-tree is replaced by a general graph. This connects to direction (5).

![Figure 1: The graph $C_6 \boxtimes K_1$.](image)

**Example 1.1.** What is the structure of the graph in Figure 1? According to the 2-separator theorem, this graph is 3-connected and hence a basic graph that cannot be decomposed further. In this paper, however, we consider finer decompositions and according to our main theorem, the structure of this graph is: a family of complete graphs $K_4$ glued together in a cyclic way.

**Our results.** The 2-separator theorem\(^1\) (in the strong form of Cunningham and Edmonds [11]) says that every 2-connected graph has a unique minimal tree-decomposition of adhesion two all of whose torsos are 3-connected or cycles. We work with the natural extension of ‘tree-decompositions’ where the decomposition-tree is replaced by an arbitrary graph. We refer to them as ‘graph-decompositions’.

Addressing the challenge set by Adcock, Sullivan and Mahoney, our main result is the following local strengthening of the 2-separator theorem.

**Theorem 1.2.** For every $r \in \mathbb{N} \cup \{\infty\}$, every connected $r$-locally 2-connected graph $G$ has a graph-decomposition of adhesion two and locality $r$ such that all its torsos are $r$-locally 3-connected or cycles of length at most $r$.

Moreover, the separators of this graph-decomposition are the $r$-local 2-separators of $G$ that do not cross any other $r$-local 2-separator.

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\(^1\)See [11, Section 4] for an overview of the history of the 2-separator theorem, see also [35]. An alternative formulation of this theorem in terms of ‘2-sums’ is given in [Section 2].
A key step in the proof of Theorem 1.2 is the following result, which seems to be of independent interest. This can be seen as a local analogue of the fact that any 2-connected graph that is not 3-connected in which any 2-separator is crossed is a cycle.

**Theorem 1.3.** Let $G$ be a connected graph that is $r$-locally 2-connected. Assume that every $r$-local 2-separator of $G$ is crossed by an $r$-local 2-separator. Then $G$ is $r$-locally 3-connected or a cycle of length at most $r$.

Beyond applications (A) to (C) mentioned above, this research includes the following applications.

D) An algorithmic advantage of our main theorem is that the parallel running time of the corresponding algorithm does not depend on the number of vertices of the graph but just on the local structure\(^2\) and we expect that our novel tool will be useful to study large networks. Indeed, a consequence of Theorem 1.2 is that one can pick the local 2-separators greedily, and all maximal graph-decompositions constructed in that way are essentially the same; in the sense that they contain the minimal graph-decomposition and additionally only have a few insignificant local 2-separators within cycles of length at most $r$.

E) Covers are important tools in Topology \cite{22} and Group Theory \cite{34, 3}. For covers of graphs, we refer the reader to the book \cite{18} or the recent survey \cite{24}. Recent work includes \cite{4}, \cite{5}, \cite{13} and \cite{15}. The universal cover of a connected graph $G$ is always a tree and covers $G$. The $r$-local universal cover, which is obtained by relaxing all cycles not generated by cycles of length at most $r$, is covered by the universal cover but covers $G$ (so if nontrivial it is an infinite graph). Our $r$-local 2-separator theorem lifts to the $r$-local universal cover of $G$, characterising the torsos of the 2-separator theorem of the cover as being the torsos of the $r$-local 2-separator theorem of $G$.

F) Local tree-decompositions are considered in \cite{16} and \cite{14}. Here (and in the follow-up work \cite{9} for arbitrary local separators), we offer tools to unify such collections of local tree-decompositions to a single graph-decomposition displaying the global structure of the graph.

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\(^2\)Indeed, we prove that for any pair of crossing local 2-separators there must be a cycle of length at most $r$ through their vertices.
G) Tree-decompositions have been used to study Cayley graphs of groups and other highly symmetric objects [20, 21]. However, these tools were most helpful for infinite groups as finite groups do not look like trees (roughly speaking). The graph-decompositions we construct here are invariant under the group of automorphisms and we expect that they can be used as a combinatorial tool to study geometric properties of finite groups.

The remainder of this paper is structured as follows. In Section 2 we give an alternative formulation of Theorem 1.2, and start explaining basic concepts, which we continue in Section 3 and Section 4.

In Section 5 we prove an interesting special case of our main result (the parts of the proof that are not needed in our proof of Theorem 1.2 are put into Appendix A). Before proving Theorem 1.3 in Section 7, we do some preparation in Section 6.

In Section 8 we prove Theorem 8.11 which implies Theorem 2.1, a variant of Theorem 1.2. Graph-decompositions are introduced in Section 9, and we conclude this section by deducing Theorem 1.2 from Theorem 8.11. Finally, in Section 10 we discuss directions for further research.

We invite all readers to look at Appendix B before reading the rest of the paper. Indeed, in there we prove a local strengthening of the block-cutvertex theorem. As this is a straightforward exercise, it is not part of the paper. However, we believe it helps to digest the rest of the paper.

2 Constructive perspective

In this section we give an alternative formulation for Theorem 1.2 and define some basic notions for this paper.

The 2-separator theorem can be stated in the decomposition version (as we did in the Introduction) as well as as the ‘constructive version’. For technical reasons we find it easier to work with the constructive version in the proofs and we will deduce the decomposition version in Section 9. We start by explaining the constructive version in this section.

We recall the classical 2-separator theorem in the constructive version in full detail. This theorem has two aspects, the existential statement (which is the easy bit), and the uniqueness statement. The existential statement says that every 2-connected graph $G$ can be constructed from 3-connected graphs and cycles via 2-sums. Clearly, 2-sums commute. Hence this sum is uniquely determined by the set of those summands that are basic; that is, they do not arise as a 2-sum of other summands. We refer to the set of basic
summands as a *decomposition* for $G$. We say that one decomposition for $G$

is *coarser* (or smarter) than another if it has the same set of 3-connected
graphs and its cycles can be built from cycles of the other decomposition
(via the implicitly defined 2-sums). The *uniqueness statement* says that
there is a decomposition for $G$ with the universal property that it is coarser
than any other decomposition for $G$.

In analogy to 2-sums, we introduce the notion of *$r$-local 2-sum*. This
notion includes the usual 2-sums operation but additionally one is allowed
to glue along edges of the same graph – as long as they have distance at
least $r$ (roughly speaking). We also introduce local 1-separators and local
2-separators and essentially\footnote{See Section 4 below for the complete definition.}
define that a graph is *locally 2-connected* if it has no local 1-separator; and ‘locally 3-connected’ is defined analogously. All
these terms carry the parameter ‘$r$’ that measures how local this is (when the
precise value of the parameter is not clear from the context, we shall write
‘$r$-local’ in place of just ‘local’). The constructive version of\footnote{See Section 4 below for the complete definition.}
Theorem 1.2 is the following.

**Theorem 2.1.** Every $r$-locally 2-connected graph can be constructed via
$r$-local 2-sums from $r$-locally 3-connected graphs and cycles of length $\leq r$.

There is such an $r$-local decomposition with the universal property that
it is coarser than any other $r$-local decomposition for $G$.

**Remark 2.2.** As for the classical 2-separator theorem, our local 2-separator
theorem has two parts; the first sentence gives the existential statement and
the second is the uniqueness statement. The uniqueness statement is more
difficult to prove.

We continue by defining some of the basic notions for this paper rigor-
ously. How do we define local cutvertices? Roughly, a vertex should be a
local cutvertex if the ball around it gets disconnected after its removal. But
which definition of ball do we take? Do we take the definition where we
allow edges in the ball joining two vertices of maximum distance or not?
Answer: we take both definitions, formalised as follows.

**Definition 2.3.** Given a graph $G$ with a vertex $v$ and an integer $s$, the *ball*
of radius $s$ around the vertex $v$ is the induced subgraph of $G$, whose vertices
are those of distance at most $s$ from $v$ and without all edges joining two
vertices of distance precisely $s$. Similarly, given a half-integer $s + \frac{1}{2}$, the *ball*
of radius $s + \frac{1}{2}$ around the vertex $v$ is the induced subgraph of $G$, whose
vertices are those of distance at most $s$ from $v$. We denote the ball of radius $s$ around $v$ by $B_s(v)$. Below we will often consider the graph $B_s(v) - v$, to which we refer as a punctured ball. Given a parameter $r \in \mathbb{N} \cup \{\infty\}$, a vertex $v$ is an $r$-local cutvertex if it separates the ball of radius $r/2$ around $v$; formally: $B_{r/2}(v) - v$ is disconnected.

Informally speaking, the ‘2-sums operation’ on graphs can be seen as the inverse operation of cutting along 2-separators and taking torsos. In the following we will introduce a local version of the ‘2-sums operation’ on graphs.

Given a family of weighted graphs $(G_i | i \in [n])$ and a set of weighted directed edges $e_i$ of $G_i$, the local 2-sum of this family is the graph obtained from the disjoint union of the set of graphs $\{G_i | i \in [n]\}$ by identifying the start-vertices of the edges $e_i$, and the terminal vertices of the edges $e_i$, and then deleting all edges $e_i$. For this local 2-sum to be valid, it must further satisfy the following condition for each $i \in [n]$. For each $i \in [n]$, we denote by $\gamma_i$ the length of the shortest path between the endvertices of the edge $e_i$ in the graph $G_i - e_i$. By $\delta_i$ we denote the minimum of the values $\gamma_j$ for $j \neq i$. We now further require that the length of the edge $e_i$ is equal to $\delta_i$.

**Remark 2.4.** We stress that the graphs $G_i$ just form a family, so some of them may coincide, but the edges $e_i$ form a set, so they must all be distinct. In the disjoint union of the set of graphs $G_i$ we only have one copy for every graph, no matter how often it appears in the family.

Often, we will not explicitly specify a direction of the edges $e_i$ but assume it is given implicitly by the context or just take an arbitrary choice.

We say that a local 2-sum is $r$-local if any pair consisting of two starting-vertices or two terminal vertices, respectively, of edges $e_i$ and $e_j$ that live in the same host graph $G_i = G_j$ have distance at least $r + 1$.

**Remark 2.5.** In this paper some graphs will have integer length assigned to their edges. Such graphs can be transformed to usual graph by replacing each edge by a path of its length. The lemmas here are proved for usual graphs, and they apply to graphs with edge length via the construction explained above.

### 3 A lemma on generating cycles

**Lemma 3.1.** Given a parameter $r \in \mathbb{N}$, all cycles of the graph $B_{r/2}(v)$ are generated by the cycles of length at most $r$. 
Proof. We prove this lemma by induction in the slightly stronger form with ‘cycle’ replaced by ‘eulerian subgraph’. The cases \( r = 0, 1, 2 \) are trivially true and we start our induction with the case \( r = 3 \). Here a graph \( G = B_{r/2}(v) \) has a spanning tree that is a star. All its fundamental circuits have length three – or one if they are loops. Hence the cycles of length at most 3 generate all cycles in this case.

For induction, assume that we have already shown the statement for graphs \( G' \) of the form \( B_{(r-2)/2}(v) \). Let \( G = B_{r/2}(v) \) be a ball of radius \( r/2 \) around \( v \). Let \( o \) be an eulerian subgraph of \( G \). We obtain the graph \( G' \) from \( G \) by contracting all edges incident with the vertex \( v \). Here we stress that edges between two neighbours of the vertex \( v \) are contracted to loops.

Now we apply the induction hypothesis to the graph \( G' \) and the eulerian subgraph \( o' \) of \( G' \) induced by \( o \); that is, \( o' \) is obtained from \( o \) by deleting the edges incident with the vertex \( v \). By induction, there are eulerian subgraphs of the graph \( G' \) of length at most \( r - 2 \) that generate \( o' \) in \( G' \). As all eulerian graphs are edge-disjoint unions of cycles, the eulerian subgraph \( o' \) is also generated by cycles of the graph \( G' \) of length at most \( r - 2 \). Each of these cycles is the edge set of a cycle in \( G \) or else it is a path between two neighbours of the vertex \( v \). In either case it can be extended to a cycle of \( G \) by adding at most two edges incident with \( v \). Hence the sum of all these cycles minus the eulerian subgraph \( o \) of \( G \) is trivial except possibly at the edges incident with \( v \).

As these edges form a tree plus parallel edges and loops and the sum is an eulerian subgraph, the sum is equal to a sum of cycles of length two or one. Hence we have shown that the eulerian subgraph \( o \) is generated by cycles of length at most \( r \). This completes the induction step.

Remark 3.2. We remark that the bound \( r \) in Lemma 3.1 is sharp as can be seen by considering graph \( G \) that are equal to cycles of length \( r \).

4 Explorer neighbourhood

In this section we define local 2-separators and explain the motivation behind our definition.

The notion of local 1-separators has been explained above. But how should one define local 2-separators? The first thing is that perhaps one only might want to consider pairs of vertices as local 2-separators if they have bounded distance between them. Indeed, otherwise if they were separating we would rather like to think about them as each being a local 1-separator. Okay, so we have a pair \((v, w)\) of vertices of bounded distance
that separates their neighbourhood. But how do we define their neighbourhood precisely? Something that looks almost right is just picking one of the vertices arbitrarily and taking a ball around them. More precisely, one could require that $B_{r/2}(v) - v - w$ is disconnected for some parameter $r$. However, it could be that when we swap $v$ and $w$ then it switches from disconnected to connected. So perhaps the next attempt would be to take $B_{r/2}(v) \cup B_{r/2}(w) - v - w$; just to make it symmetric in $v$ and $w$. Below we will refer to this long expression as the punctured double-ball.

The disadvantages of this definition, although almost correct, are more subtle. The main reason is perhaps that with that definition our proofs do not seem to work, as important structural properties are simply not true. Indeed, with this definition Lemma 6.9 (Corner Lemma), does not work. This lemma is a natural generalisation of a lemma for usual separators, and we believe that any natural notion of local separators should have this property. The reason why that lemma is not true in this case is that the double-ball $B_{r/2}(v) \cup B_{r/2}(w)$ may contain cycles that are composed of a path from the ball $B_{r/2}(v)$ and from $B_{r/2}(w)$ but are not a cycle of either of these two balls, see Figure 2.

Informally speaking, the definition we take is similar to the double ball $B_{r/2}(v) \cup B_{r/2}(w)$ and actually agrees with it up to distance $r - d$, where $d$ is the distance between the vertices $v$ and $w$ – but towards the boundary it ‘gets more fuzzy’. We will call our notion of neighbourhood ‘explorer-neighbourhood’ and think about it as follows: imagine two explorers discovering the graph starting from the vertices $v$ and $w$ with the goal of separately discovering the graph and at the end combining their maps of the

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure2.png}
\caption{The balls $B_{r/2}(v)$ and $B_{r/2}(w)$ are marked by grey stripes, in rising and falling patterns, respectively. Two paths between the vertices $x$ and $y$, one from either ball, form a cycle that is contained in neither ball.}
\end{figure}
balls $B_{r/2}(v)$ and $B_{r/2}(w)$ into a single map. First they discover all shortest paths between the vertices $v$ and $w$ together and put them on the common map. We refer to the set of vertices on these paths as the core. Then they return to their respective starting vertices and start exploring the graph from there up to distance $r/2$. On their map they mark each vertex by the set of shortest paths to that vertex from the core (within their respective balls). There may be vertices with distance $r/2$ from the core that have distance at most $r/2$ to the vertex $v$ but a larger distance to the vertex $w$. Such vertices are only discovered by the explorer based at $v$. There may also be vertices $u$ discovered by both explorers. However they might be so far away that some explorer sees some shortest paths to it that the other one does not see. In this case there will be two copies of that vertex in the explorer-neighbourhood, while there is only one copy in the double ball $B_{r/2}(v) \cup B_{r/2}(w)$.

Now we give a formal definition of the explorer-neighbourhood of parameter $r$ with explorers based at the vertices $v$ and $w$. The core is the set of all vertices on shortest paths between the vertices $v$ and $w$. We take a copy of the ball $B_{r/2}(v)$ where we label a vertex $u$ with the set of shortest paths from the core to $u$ contained in the ball $B_{r/2}(v)$. Similarly, we take a copy of the ball $B_{r/2}(w)$ where we label a vertex $u$ with the set of shortest paths from the core to $u$ contained in the ball $B_{r/2}(w)$. Now we take the union of these two labelled balls – with the convention that two vertices are only identical if they have the same label (note that the same vertex $x$ of $G$ could be in both balls but with different labels, see Figure 3). In this case there would be two copies of that vertex in the union. In such a case the union would not be a subgraph of the original graph). We denote the explorer neighbourhood by $E_{r}(v, w)$. This completes the definition of explorer neighbourhood.

**Lemma 4.1.** Given two vertices $a_1$ and $a_2$ of distance at most $r/2$, vertices on shortest paths between $a_1$ and $a_2$ and their neighbours have unique copies in $E_{r}(a_1, a_2)$.

In particular, neighbours of the vertices $a_1$ and $a_2$ have unique copies in $E_{r}(a_1, a_2)$.

**Proof.** By definition vertices on shortest paths between $a_1$ and $a_2$ have unique copies in $E_{r}(a_1, a_2)$. Let $x$ be a neighbour of such a vertex. Clearly the vertex $x$ is in at least one of the balls $B_{r/2}(a_1)$ and $B_{r/2}(a_2)$, and so has at least one copy in the explorer-neighbourhood. So it remains to prove uniqueness. For that assume that $x$ is contained in both balls $B_{r/2}(a_1)$ and $B_{r/2}(a_2)$. Then both balls contain all edges from $x$ to the core. So all shortest paths from $x$ to the core within either ball agree – or else $x$ is in the
Figure 3: On the right we depicted the explorer-neighbourhood $\text{Ex}_r(v, w)$ of the graph on the left. Here the grey paths all have length equal to $(r/2) - 2$. The core is just the path of length four between $v$ and $w$. The cycle of length $r$ is still a cycle in $\text{Ex}_r(v, w)$ since as a cycle it is included in both $B_{r/2}(v)$ and $B_{r/2}(w)$, see Lemma 4.2 for details. The cycle of length $r + 2$ is not contained in one of the balls $B_{r/2}(v)$ or $B_{r/2}(w)$ and hence some of its vertices get two copies in $\text{Ex}_r(v, w)$. Indeed, the vertex $x$ has distance at most $r$ from both vertices $v$ and $w$. Still it has the two copies $x_1$ and $x_2$ in the explorer-neighbourhood.

Let $o$ be a cycle (or more generally a closed walk) of length at most $r$ containing vertices $a_1$ and $a_2$. Vertices of $o$ have unique copies in $\text{Ex}_r(a_1, a_2)$.

**Proof.** Let $o$ be a closed walk as in the statement of the lemma, and let $x$ be an arbitrary vertex on $o$. Let $S$ be a shortest path from $x$ to the core in the underlying graph (not just some subballs). We will show that $S$ is completely included in both balls $B_{r/2}(a_1)$ and $B_{r/2}(a_2)$. By symmetry, it suffices to show that $S$ is completely included in $B_{r/2}(a_1)$.

For any pair of vertices of the set $\{a_1, a_2, x\}$, pick a shortest path between these vertices. Let $o'$ be the closed walk obtained by concatenating these three paths. Let $y$ be the endvertex of the path $S$ on the core. We can pick, and we do pick, the shortest path between $a_1$ and $a_2$ so that it contains the vertex $y$. Hence the vertex $y$ is on the closed walk $o'$. As the closed walk $o$ also contains the vertices $a_1$, $a_2$ and $x$, its length is at least that of the closed walk $o'$; that is, the closed walk $o'$ has length at most $r$.

Let $o''$ be the closed walk obtained by concatenating a shortest path
from $a_1$ to $x$, the path $S$ and a shortest path from the vertex $y$ to $a_1$. Such a closed walk $o''$ can be obtained from the closed walk $o'$ by replacing a subwalk from $x$ via $a_2$ to $y$ by the path $S$. As $S$ is a shortest path between its endvertices, the length of $o''$ is at most that of $o'$; and thus at most $r$. Hence the closed walk $o''$ is completely contained within the ball $B_{r/2}(a_1)$ around $a_1$. Thus the shortest path $S$ is contained in that ball. As $S$ was chosen arbitrarily, every shortest path from $x$ to the core is included in the ball $B_{r/2}(a_1)$. By symmetry, the same is true for $‘a_2’$ in place of $‘a_1’$. Thus $x$ has a unique copy in the explorer-neighbourhood $Ex_r(a_1, a_2)$. 

The balls $B_{r/2}(v)$ and $B_{r/2}(w)$ are embedded within the explorer-neighbourhood by construction. We refer to these embedded balls as $\iota(B_{r/2}(v))'$ and $\iota(B_{r/2}(w))'$, or simply $B_{r/2}(v)'$ and $B_{r/2}(w)'$ if the embedding map $\iota$ is clear from the context.

**Lemma 4.3.** Every cycle $o$ of the explorer neighbourhood $Ex_r(v, w)$ is generated from the cycles of the embedded balls $B_{r/2}(v)'$ and $B_{r/2}(w)'$.

**Proof.** Each vertex of the cycle $o$ is a vertex of $B_{r/2}(v)'$ or $B_{r/2}(w)'$. We mark it with the respective vertex; and if it is in both, we mark it with both vertices $v$ and $w$. For each vertex $x$ on the cycle $o$ marked by a vertex $y \in \{v, w\}$, we pick a shortest path from $x$ to the core within the ball $B_{r/2}(y)'$. If a vertex is marked with $v$ and $w$ by the definition of explorer-neighbourhood, we can assume, and we do assume, that we picked the same path for $y = v$ and $y = w$.

Now for each edge $e$ we construct a closed walk $o_e$ as follows. Start with $e$ and the two paths chosen at either endvertex of $e$, then join their endvertices in the core by a path within the core (which is connected by construction). Since for each edge $e$ of $o$, there is a mark $y \in \{v, w\}$ that is present at both endvertices of edge $e$, the closed walk $o_e$ is contained in $B_{r/2}(v)'$ or $B_{r/2}(w)'$.

Our aim is to generate the cycle $o$ from cycles of $B_{r/2}(v)'$ and $B_{r/2}(w)'$. For that we first add to $o$ the sum of all the cycles $o_e$ ranging over all $e \in o$ (taken over the binary field $F_2$). This sum takes only non-zero entries at edges of the core. As the core is a subset of $B_{r/2}(v)' \cap B_{r/2}(w)'$, the remainder is generated from the common cycles of $B_{r/2}(v)'$ and $B_{r/2}(w)'$. 

A set $\{v, w\}$ is an $r$-local 2-separator if the punctured explorer-neighbourhood $Ex_r(v, w) - v - w$ is disconnected; and the vertices $v$ and $w$ have distance at most $r/2$. 

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A connected graph is r-locally 2-connected if it does not have an r-local cutvertex and it has a cycle of length at most r. So there are no r-locally 2-connected graphs for r < 3. A graph is r-locally 2-connected if all its components are r-locally 2-connected.

A connected r-locally 2-connected graph is r-locally 3-connected if it does not have an r-local 2-separator and it has at least four vertices. A graph is r-locally 3-connected if it r-locally 2-connected and all its components are r-locally 3-connected.

Example 4.4. A cycle of length r + 1 is not r-locally 2-connected and has no r-local 2-separator. Hence it is reasonable to add the assumption of ‘local 2-connectivity’ in the definition of ‘local 3-connectivity’.

In a sense the next lemma says that local 2-components sitting at a local 2-separator are local (in that they contain a short path between the neighbours of the two separating vertices).

Lemma 4.5 (Local 2-Connectivity Lemma). Let \( \{v, w\} \) be an r-local 2-separator in an r-locally 2-connected graph \( G \). For every connected component \( k \) of the punctured explorer-neighbourhood \( Ex_r(v, w) - v - w \), there is a cycle \( o' \) of length at most r containing the vertices \( v \) and \( w \), and \( o' \) contains a vertex of the component \( k \) and of a different component of \( Ex_r(v, w) - v - w \).

Proof. Let \( k = k_1 \) be an arbitrary component of the punctured explorer-neighbourhood \( Ex_r(v, w) - v - w \), and let \( k_2 \) be the union of all other components of the punctured explorer-neighbourhood \( Ex_r(v, w) - v - w \), which is nonempty as \( \{v, w\} \) is a local 2-separator. If one component of \( Ex_r(v, w) - v - w \) had only one of the vertices \( v \) and \( w \) in its neighbourhood, then that vertex would be a local cutvertex. However, this is not possible as \( G \) is r-locally 2-connected by assumption. Hence all components of \( Ex_r(v, w) - v - w \) have both vertices \( v \) and \( w \) in their neighbourhood. In particular, the vertex \( v \) is adjacent to vertices of \( k_1 \) and \( k_2 \).

Let \( x_i \) be an arbitrary neighbour of the vertex \( v \) in \( k_i \) (for \( i = 1, 2 \)). As the graph \( G \) is r-locally 2-connected, the vertex \( v \) is not a cutvertex of the ball \( B_{r/2}(v) \). So there is a path \( P \) included in that ball from \( x_1 \) to \( x_2 \) avoiding \( v \). Let \( o \) be the cycle obtained from \( P \) by adding the vertex \( v \). By Lemma 3.1, the cycle \( o \) is generated from cycles of the ball \( B_{r/2}(v) \) of length at most r. Consider the set \( C \) of these cycles that contain the vertex \( v \). As \( o \) has precisely one edge to \( k_1 \) incident with \( v \), there must be a cycle \( o' \) in \( C \) that contains an odd number of edges to \( k_1 \) incident with \( v \). As the cycle \( o' \) has maximum degree two, it contains precisely one edge to \( k_1 \) incident with
v. The other edge of \( o' \) incident with \( v \) has its other endvertex in \( k_2 \). This completes the proof. 

**Remark 4.6.** The bound \( r \) for the cycle \( o' \) in Lemma 4.5 is best possible as can be seen by considering graphs that are a single cycle of length \( r \).

**Remark 4.7.** The notion of the explorer-neighbourhood is crucial in the proof of Lemma 6.12 (Projection Lemma) and Lemma 6.9 (Corner Lemma) below. This is explained in detail in Remark 6.18 and Remark 6.7 below.

**Remark 4.8.** Above we said the explorer-neighbourhood and the double-ball ‘almost lead’ to the same notion of local 2-separator. This can be quantified as follows. If the punctured explorer-neighbourhood is connected, then so is the punctured double ball. If the punctured double ball of radius \( r/2 \) around two vertices of distance at most \( d \) is connected, then the punctured explorer-neighbourhood of radius \( (r/2) + d \) is connected.

### 5 The existential statement of the local 2-separator theorem

In this section, we prove the lemmas necessary to deduce the existential statement of the local 2-separator theorem; that is, the first sentence of Theorem 2.1.

Given a graph \( G \) with an \( r \)-local 2-separator \( \{v_0, v_1\} \), the graph obtained from \( G \) by \( r \)-locally cutting \( \{v_0, v_1\} \) is defined as follows. Let \( X \) be the set of connected components of the punctured explorer-neighbourhood \( \text{Ex}_r(v_0, v_1) - v_0 - v_1 \). We now replace in the graph \( G \) the vertices \( v_0 \) and \( v_1 \) each by one copy for every element of \( X \). Here a copy of \( v_i \) labelled by some \( x \in X \) inherits an edge from \( v_i \) if the other endvertex of that edge is a vertex of the component \( x \). We refer to the newly added vertices as the *slices* of the vertices \( v_1 \) or \( v_2 \), respectively. We additionally add a weighted edge between any two slices for the same \( x \in X \). Its weight is given by the minimum length of a path between \( v_0 \) and \( v_1 \) in the explorer-neighbourhood \( \text{Ex}_r(v_0, v_1) \) with the component \( x \) removed. It follows that all but one of these weights are always the same. We refer to these additional edges as *torso edges*. If the vertices \( v_0 \) and \( v_1 \) share an edge \( e \) in \( G \), we add a new connected component consisting of the edge \( e \) and one edge in parallel to \( e \). This other edge is a torso edge and its length is the minimum length of a path between \( v_0 \) and \( v_1 \) in the explorer-neighbourhood \( \text{Ex}_r(v_0, v_1) \). This completes the definition of local cutting.
Example 5.1. All edges incident with the vertices $v_0$ or $v_1$ are inherited by a unique slice except for possibly an edge between $v_0$ and $v_1$, which is in this artificial component of size two. If an edge between $v_0$ and $v_1$ in $G$ is a shortest path between these vertices, its length is the length of all torso edges.

Observation 5.2. Slices of the same vertex have distance at least $r + 1$.

Proof. If $r$ is even, the balls of radius $r/2$ around different slices do not overlap. If $r$ is odd, the ball of radius $\frac{r-1}{2}$ around different slices do not overlap, and there is no edge between these balls. \hfill \Box

Lemma 5.3. Let $G'$ be obtained from a graph $G$ by $r$-locally cutting a local 2-separator $\{v, w\}$. Then $G$ can be obtained from $G'$ by $r$-local sums.

Proof. The family of graphs for the local sum consists of copies of the graph $G'$, one copy for each component of the punctured explorer-neighbourhood $\text{Ex}_r(v, w) - v - w$, together with the artificial component of size two if $vw$ is an edge of $G$. The gluing edges are the torso edges.

It follows directly from the definitions of local cutting and local sums that the graph $G$ is equal to the graph obtained from $G'$ by applying the local 2-sum as described above. This local sum is $r$-local by Observation 5.2. \hfill \Box

Lemma 5.4. Let $G'$ be a graph obtained from an $r$-locally 2-connected graph $G$ by $r$-locally cutting a local 2-separator. Then the graph $G'$ is $r$-locally 2-connected.

Proof. By Lemma 4.5 (Local 2-Connectivity Lemma), every connected component of the graph $G'$ contains a cycle has at most $r$. So it remains to show that there are no $r$-local cutvertices. Let $v$ be an arbitrary vertex of the graph $G'$. We distinguish two cases.

Case 1: the vertex $v$ is a slice. We denote the local 2-separator of $G$ at which we cut by $\{a, b\}$. We may assume, and we do assume, that the vertex $v$ is a slice of the vertex $a$. Let $X$ denote the set of components of the punctured explorer-neighbourhood $\text{Ex}_r(a, b) - a - b$. Recall that the ball $B_{r/2}(a)$ of radius $r/2$ around $a$ in the graph $G$ is a subgraph of the explorer-neighbourhood $\text{Ex}_r(a, b)$. We let $H_x$ be intersection of the punctured ball $B_{r/2}(a) - a$ with some component $x \in X$. Note that $b$ has distance at most $r/2$ from $a$ by Lemma 4.5 (Local 2-Connectivity Lemma). The punctured ball $B_{r/2}(a) - a$ is obtained by taking the union of the graphs $H_x$ and adding the vertex $b$. As this punctured ball is connected by assumption, all graph $H_x$ must have the vertex $b$ in their neighbourhood and all graphs $H_x + b$
must be connected. The punctured ball $B_{r/2}(v) - v$ around $v$ is $H_y + b$, where $y$ is the component of $\text{Ex}_r(a,b) - a - b$ that belongs to the slice $v$. So $B_{r/2}(v) - v$ is connected. Thus the vertex $v$ is not an $r$-local cutvertex. This completes Case 1.

**Case 2:** the vertex $v$ is not a slice. Then the vertex $v$ is a vertex of the graph $G$.

Suppose for a contradiction that the punctured ball $B_{r/2}(v) - v$ around $v$ of radius $r/2$ in the graph $G'$ is disconnected. Let $w'_1$ and $w'_2$ be two arbitrary neighbours of $v$ in $G'$ in different components of that punctured ball. Let $w_1$ and $w_2$ be the vertices of the graph $G$ from which the vertices $w'_1$ and $w'_2$ are slices of or that are equal to them, respectively. Then the vertices $w_1$ and $w_2$ are adjacent to the vertex $v$ in the graph $G$ by the definition of local cutting. As the punctured ball $B_{r/2}(v) - v$ of radius $r/2$ around the vertex $v$ in the graph $G$ is connected by assumption, there is a path $P$ within that punctured ball from $w_1$ to $w_2$. This path together with the vertex $v$ is a cycle $o$ within that ball. So by Lemma 3.1 this cycle is generated by cycles within that ball of length at most $r$.

Let $W'$ be the set of neighbours of the vertex $v$ in the graph $G'$ in the component of the punctured ball containing the vertex $w'_1$. Let $W$ be the set of vertices of the graph $G$ that are equal to vertices in $W'$ or that have slices in the set $W'$. By $E(W)$ we denote the set of edges in the graph $G$ from $v$ to a vertex in $W$.

By construction the cycle $o$ contains precisely one edge from the set $E(W)$. Hence there must be a cycle $\hat{o}$ from the generating set that contains an odd number of edges from $E(W)$. As $\hat{o}$ has maximum degree two, it contains precisely one edge from the set $E(W)$.

We denote the local 2-separator of $G$ at which we locally cut by $\{a, b\}$.

**Case 2A:** the cycle $\hat{o}$ does not contain any of the vertices $a$ or $b$. Then $\hat{o} - v$ is a path in the graph $G'$ from a vertex of $W'$ to a neighbour of the vertex $v$ outside $W'$. This is a contradiction to the assumption that the punctured ball is disconnected. This completes this case.

**Case 2B:** the cycle $\hat{o}$ contains one of the vertices $a$ or $b$, say $a$. As the cycle $\hat{o}$ has length bounded by $r$, it is also a cycle of the explorer-neighbourhood $\text{Ex}_r(a,b)$. So it must traverse the 2-separator $\{a, b\}$ of the explorer-neighbourhood evenly. If it does not traverse it at all, then $\hat{o}$ is a cycle of $G'$ and we argue as in Case 2A. Otherwise, one obtains a cycle in $G'$ by replacing the subpath of $\hat{o}$ between $a$ and $b$ that does not contain the vertex $v$ by a torso edge. Also here one can argue similarly as in Case 2A. This completes Case 2, and hence the whole proof. □
Remark 5.5. In Appendix A we give an alternative proof of the first sentence of Theorem 2.1 that only relies on lemmas of the paper proved up to this point. We encourage the reader to look at this proof next.

6 Properties of local 2-separators

In this section we prove some lemmas that are used in our proof of Theorem 1.3 and Theorem 1.2.

We say that a path $P$ traverses a separator $X$ oddly if $P$ contains an odd number of edges that have precisely one endvertex in $X$.

Lemma 6.1. Let $X$ be a separator in a graph $G$ with precisely two components. And let $P$ be a path between vertices in different components of $G \setminus X$. Then $P$ traverses the set $X$ oddly.

Proof: by induction on the length of the path $P$. \qed

A cut is the set of edges between a bipartition of the vertex set. The bipartition classes are referred to as the sides of the cut.

Lemma 6.2. Let $Y$ be a cut in a graph $G$. Then the endvertices of a path $P$ are on the same side of $Y$ if and only if $P$ intersects $Y$ evenly.

Proof: by induction on the length of the path $P$. \qed

Given an $r$-local 2-separator $\{v, w\}$ and a pair of vertices $(a, b)$ of the explorer-neighbourhood $\text{Ex}_r(v, w)$, we say that $(a, b)$ pre-crosses $\{v, w\}$ if $a$ and $b$ are in different components of the punctured explorer-neighbourhood $\text{Ex}_r(v, w) - v - w$. And $(a, b)$ crosses the $r$-local 2-separator $\{v, w\}$ if it pre-crosses it and there is a cycle of length at most $r$ through $a$ and $b$ in the explorer-neighbourhood that contains $v$ or $w$.

We say that a pair $(a, b)$ of (distinct) vertices of $G$ crosses a local 2-separator $\{v, w\}$ of $G$ if there exist copies $a'$ and $b'$ of $a$ and $b$ in the explorer-neighbourhood $\text{Ex}_r(v, w)$, respectively, so that $(a', b')$ crosses $\{v, w\}$.

Remark 6.3. If $\{a, b\}$ is a local separator, then the existence of a cycle $o$ of length at most $r$ through $a$ and $b$ is guaranteed by Lemma 4.5 (Local 2-Connectivity Lemma). Hence ‘crossing’ essentially means ‘pre-crossing’ plus the crossing vertices are ‘near’ to the local separator. Phrasing being ‘near’ in terms of this cycle seems particularly natural in view of Lemma 6.4 and Lemma 6.5 below.
Given two disjoint sets $A_1$ and $A_2$, we say that a cyclic ordering alternates between $A_1$ and $A_2$ if it has even length and each element of the cyclic ordering in $A_i$ has its two neighbours in $A_{i+1}$ (for $i \in \mathbb{F}_2$).

An alternating cycle is a cycle $o$ together with two local 2-separators $\{a_1, a_2\}$ and $\{b_1, b_2\}$ such that the order in which these four vertices appear on the cycle $o$ alternates between the two local separators (ie, it is $a_1b_1a_2b_2$ or its reverse $a_1b_2a_2b_1$). Below sometimes it will be more convenient to refer to this situation by saying that the cycles $o$ alternates between the local 2-separators $\{a_1, a_2\}$ and $\{b_1, b_2\}$, see Figure 4.

![Figure 4: An alternating cycle.](image)

The ‘nearness’ condition in the definition of ‘crossing’ is equivalent to the following stronger property.

**Lemma 6.4** (Alternating Cycle Lemma). Assume an $r$-local 2-separator $\{a_1, a_2\}$ crosses an $r$-local 2-separator $\{b_1, b_2\}$. Then there is an alternating cycle of length at most $r$ of $G$ alternating between the local 2-separators $\{a_1, a_2\}$ and $\{b_1, b_2\}$.

Moreover, any cycle of $G$ of length at most $r$ through the vertices $a_1$ and $a_2$ containing a vertex $b_i$ (for $i = 1, 2$) alternates between $\{a_1, a_2\}$ and $\{b_1, b_2\}$.

**Proof.** By assumption, there is a cycle $o$ of length at most $r$ through copies of the vertices $a_1$ and $a_2$ that contains $b_1$ or $b_2$, say $b_1$. So $o$ is a cycle of $\operatorname{Ex}_r(b_1, b_2)$. As $o$ contains vertices of different components of the graph $\operatorname{Ex}_r(b_1, b_2) - b_1 - b_2$, it traverses the separator $\{b_1, b_2\}$ at least once. As $o$ is a cycle, it has to traverse the separator at least a second time by Lemma 6.1.
(indeed, apply the lemma to the two paths between \(a_1\) and \(a_2\)). Hence the cycle \(o\) must contain the vertex \(b_2\). Hence by Lemma 4.2 also the vertices \(a_1\) and \(a_2\) have unique copies in \(\text{Ex}_r(b_1, b_2)\). To simplify notation, for the rest of this proof we will suppress a bijection between the vertices \(a_i\) and their copies in \(\text{Ex}_r(b_1, b_2)\).

Recall that by the definition of crossing all the four vertices \(a_1, a_2, b_1\) and \(b_2\) are distinct. Let \(P\) be the subpath of the cycle \(o\) between the vertices \(a_1\) and \(a_2\) that avoids the vertex \(b_1\). As the vertices \(a_1\) and \(a_2\) are in different connected components of the punctured explorer-neighbourhood \(\text{Ex}_r(b_1, b_2) - b_1 - b_2\), the path \(P\) must contain the vertex \(b_2\). Thus the cycle \(o\) alternates between the local 2-separators \(\{a_1, a_2\}\) and \(\{b_1, b_2\}\).

To see the ’Moreover’-part, first note that by Lemma 4.2 the vertices \(a_1\) and \(a_2\) have unique copies in \(\text{Ex}_r(b_1, b_2)\). Hence the cycle from the ’Moreover’-part is a candidate for the cycle \(o\), and so the above proof applies.

The next lemma essentially says that crossing is a symmetric relation on \(r\)-local 2-separators.

**Lemma 6.5.** Let \(G\) be an \(r\)-locally 2-connected graph with two \(r\)-local 2-separators \(\{a_1, a_2\}\) and \(\{b_1, b_2\}\). If \(\{a_1, a_2\}\) crosses \(\{b_1, b_2\}\), then \(\{b_1, b_2\}\) crosses \(\{a_1, a_2\}\).

Furthermore the punctured explorer-neighbourhood \(\text{Ex}_r(a_1, a_2) - a_1 - a_2\) has precisely two components.

**Proof.** By Lemma 6.4 (Alternating Cycle Lemma), there is a cycle \(o\) of length at most \(r\) alternating between the local separators \(\{a_1, a_2\}\) and \(\{b_1, b_2\}\). This cycle \(o\) is contained in each of the four balls around the vertices \(a_1, a_2, b_1\) and \(b_2\). In particular, the explorer-neighbourhoods \(\text{Ex}_r(a_1, a_2)\) and \(\text{Ex}_r(b_1, b_2)\) contain unique copies of each of \(a_1, a_2, b_1\) and \(b_2\) by Lemma 4.2.

Hence it (is unambiguous to say and it) remains to show that \(\{b_1, b_2\}\) pre-crosses \(\{a_1, a_2\}\). By Lemma 4.5 (Local 2-Connectivity Lemma), there is a cycle \(o'\) of length at most \(r\) through \(a_1\) and \(a_2\) that contains vertices of different components of \(\text{Ex}_r(a_1, a_2) - a_1 - a_2\).

**Sublemma 6.6.** One of the vertices \(b_1\) or \(b_2\) is a vertex of the cycle \(o'\).

**Proof.** As the cycle \(o'\) has length at most \(r\), it contains a path \(P_1\) between \(a_1\) and \(a_2\) of length at most \(r/2\). As the cycle \(o\) has length at most \(r\), it contains a path \(P_2\) between \(a_1\) and \(a_2\) of length at most \(r/2\). Let \(o''\) be the closed walk obtained from concatenating the paths \(P_1\) and \(P_2\). As the cycle \(o\) is alternating, it contains a vertex \(b_i\) (for \(i = 1, 2\)). Hence the closed walk
" is contained in the ball of radius \( r/2 \) around \( b_1 \) and hence the path \( P_1 \) is in the explorer-neighbourhood \( \text{Ex}_r(b_1, b_2) \). As \( a_1 \) and \( a_2 \) are in different components of the punctured explorer-neighbourhood, the path \( P_1 \) must contain one of the vertices \( b_1 \) or \( b_2 \).

By Sublemma 6.6 there is a vertex \( b_i \) on the cycle \( o' \). Thus the cycle \( o' \) is included in the ball \( B_{r/2}(b_i) \) of radius \( r/2 \) around the vertex \( b_i \). Let \( P_{i+1} \) be the subpath of the cycle \( o' \) between the vertices \( a_1 \) and \( a_2 \) not containing the vertex \( b_i \). As \( P_{i+1} \) is a path of the explorer-neighbourhood \( \text{Ex}_r(b_1, b_2) \), and \( a_1 \) and \( a_2 \) are in different components of the punctured explorer-neighbourhood, the vertex \( b_{i+1} \) must be on the path \( P_{i+1} \). Let \( P_i \) be the subpath of \( o' \) between \( a_1 \) and \( a_2 \) containing \( b_i \). As the nonempty connected set \( P_i - a_1 - a_2 \) and \( P_{i+1} - a_1 - a_2 \) contain vertices of different components of the punctured explorer-neighbourhood \( \text{Ex}_r(a_1, a_2) - a_1 - a_2 \), these two connected sets are in different connected components. In particular, the vertices \( b_1 \) and \( b_2 \) are in different connected components of \( \text{Ex}_r(a_1, a_2) - a_1 - a_2 \) that is, the pair \((b_1, b_2)\) pre-crosses the local separator \( \{a_1, a_2\} \). Thus \((b_1, b_2)\) crosses \( \{a_1, a_2\} \).

To see the ‘Furthermore’-part, suppose for a contradiction that there is a component \( k \) of the punctured explorer-neighbourhood \( \text{Ex}_r(a_1, a_2) - a_1 - a_2 \) that does not contain \( b_1 \) or \( b_2 \). Then we could pick the cycle \( o' \) above so that additionally it contains a vertex of \( k \). This is a contradiction as all its vertices aside from \( a_1 \) and \( a_2 \) are in the components containing \( b_1 \) or \( b_2 \).

Remark 6.7. A key-feature of separators in graphs is the ‘Corner Property; that is, for two crossings separators the construction of a new separator sitting at the ‘corner’ of these two in the sense of Lemma 6.9 below. This is a central lemma of the paper and the notion of ‘explorer-neighbourhood’ is key to this lemma. The Corner-Lemma says given two crossing local separators \( \{a_1, a_2\} \) and \( \{b_1, b_2\} \), then \( \{a_1, b_1\} \) is also a local separator – under certain non-triviality conditions.

Intuitively speaking, the reason why this lemma is true is the following. As the Corner-Lemma is true for separators in the classical version, the only reason why a local version could break is essentially if one of the involved vertices, say \( a_1 \), would explore a new path around the local separator \( \{b_1, b_2\} \) to the other component of the punctured explorer-neighbourhood \( \text{Ex}_r(b_1, b_2) - b_1 - b_2 \) that was not known to \( b_1 \) or \( b_2 \). If we used ‘double balls’ instead of our local notion of ‘explorer-neighbourhoods’, this could well happen, see Figure 5 for an example. The intuition now is that \( a_1 \) may well ‘explore’ a new path to the other local component but when the explorers compare their maps, they have given the things different names and so
the explorers do not realise that between them they know a path around. Hence they believe that the corner \{a_1, b_1\} is separating. That is how we think about locally separating: the explorers cannot prove that there is a way round with their local information.

This is somewhat similar to the following situation. Imagine you are running on a graph and at any point in time you can only see your neighbours. If the graph is a cycle, you cannot tell its length, – and you even cannot distinguish it from the 2-way-infinite path \(\mathbb{Z}\).

Now we start setting up some notation for Lemma 6.9 below. Given an \(r\)-local separator \(\{a_1, a_2\}\) crossing an \(r\)-local 2-separator \(\{b_1, b_2\}\), by Lemma 6.5 and Lemma 6.4, \(\{b_1, b_2\}\) crosses \(\{a_1, a_2\}\) and there is a cycle \(o\) of length at most \(r\) alternating between these two local separators. A person of type one is a vertex \(x\) of \(\mathcal{V}(o) - a_1 - a_2 - b_1 - b_2\) that has a copy in the same component of \(\mathcal{E}_r(a_1, a_2) - a_1 - a_2\) as \(b_1\) and a copy in the same component of \(\mathcal{E}_r(b_1, b_2) - b_1 - b_2\) as \(a_1\). A person of type two is a neighbour \(x\) of \(b_1\) outside the cycle \(o\) that has a copy in the same component of \(\mathcal{E}_r(b_1, b_2) - b_1 - b_2\) as \(a_1\). A person is a person of type one or two. We say that a person lives in the corner \{a_1, b_1\} if there exists a person.

Remark 6.8. A person of type one has unique copies in \(\mathcal{E}_r(a_1, a_2)\), \(\mathcal{E}_r(b_1, b_2)\) and \(\mathcal{E}_r(a_1, b_1)\) by Lemma 4.2. A person of type two has unique copies in \(\mathcal{E}_r(b_1, b_2)\) and \(\mathcal{E}_r(a_1, b_1)\) by Lemma 4.1.

Hence to simplify notation, below we suppress a bijection between a person and its unique copies in the explorer-neighbourhoods where copies are unique.

**Lemma 6.9** (Corner Lemma). Assume \(G\) is locally 2-connected. Assume a person \(x\) lives in the corner \{a_1, b_1\}, then \(\{a_1, b_1\}\) is an \(r\)-local 2-separator.

Moreover, \(x\) is in a different component of \(\mathcal{E}_r(a_1, b_1) - a_1 - b_1\) than (copies of) \(a_2\) and \(b_2\).

**Proof.** By Lemma 4.2, the vertices \(a_1, a_2, b_1\) and \(b_2\) have unique copies in \(\mathcal{E}_r(a_1, b_1)\); hence for this proof we suppress a bijection between these vertices and their copies in \(\mathcal{E}_r(a_1, b_1)\). We start by showing the following.

**Sublemma 6.10.** The vertices \(b_1\) and \(b_2\) are in different components of the graph \(\mathcal{E}_r(a_1, b_1) - a_1 - a_2\).

**Proof.** Suppose not for a contradiction. Then there is a path \(P\) of the graph \(\mathcal{E}_r(a_1, b_1) - a_1 - a_2\) from \(b_1\) to \(b_2\).

Let \(W\) be the neighbourhood of the set \(\{a_1, a_2\}\) in the punctured explorer-neighbourhood \(\mathcal{E}_r(a_1, a_2) - a_1 - a_2\) in the component containing \(b_1\). By
Lemma 4.1. Neighbours of $a_1$ or $a_2$ have unique copies in the explorer-neighbourhood $\operatorname{Ex}_r(a_1, a_2)$; hence there is a bijection between the neighbours of $a_1$ and $a_2$ in $G$ and the explorer-neighbourhood. To simplify notation we suppress this map in our notation. And we will simply consider $W$ as a vertex set of the graph $G$, as well. Let $E(W)$ be the set of edges of the graph $G$ from $\{a_1, a_2\}$ to $W$.

By Lemma 6.4 (Alternating Cycle Lemma), there is a cycle of length at most $r$ alternating between the local separators $\{a_1, a_2\}$ and $\{b_1, b_2\}$. Thus it has a subpath $Q$ from $b_1$ to $b_2$ of length at most $r/2$; this subpath contains precisely one of the vertices $a_1$ and $a_2$. This alternating cycle is also a cycle of the explorer-neighbourhood $\operatorname{Ex}_r(a_1, a_2)$ and the path $Q$ has to intersect the set $E(W)$ oddly as it connects vertices in different components.

Let $o$ be the closed walk of the explorer-neighbourhood $\operatorname{Ex}_r(a_1, b_1)$ obtained by concatenating $P$ and $Q$. The path $P$ considered as a closed walk of the graph $G$ contains no vertex $a_i$ and thus does not intersect the edge set $E(W)$. Thus the closed walk $o$, considered as an edge set of the graph $G$ intersects the edge set $E(W)$ in an odd number of edges.

By Lemma 4.3 the closed walk $o$ is generated by cycles of $G$ included in the balls $B_{r/2}(a_1)$ and $B_{r/2}(b_1)$. These cycles are in turn by Lemma 3.1 generated by cycles of length at most $r$ included in these balls. Hence one of the generating cycles has to intersect the edge set $E(W)$ oddly. Call such a cycle $o'$. So the cycle $o'$ contains the vertex $a_1$ or $a_2$. As it has bounded length, it is a cycle of the explorer-neighbourhood $\operatorname{Ex}_r(a_1, a_2)$. This is a contradiction as in the explorer-neighbourhood $\operatorname{Ex}_r(a_1, b_1)$ the cycle $o'$ and the cut $E(W)$ cannot intersect oddly. Thus the vertices $b_1$ and $b_2$ must be in different components of the graph $\operatorname{Ex}_r(a_1, b_1) - a_1 - a_2$.

By exchanging the roles of the ‘$a_i$’ and ‘$b_i$’ in Sublemma 6.10 one obtains the following.

Sublemma 6.11. The vertices $a_1$ and $a_2$ are in different components of the graph $\operatorname{Ex}_r(a_1, b_1) - b_1 - b_2$.

Proof. The proof is analogous to that of Sublemma 6.10.

By $C(a, i)$ we denote the component of $\operatorname{Ex}_r(a_1, b_1) - a_1 - a_2$ containing the vertex $b_i$. By $C(b, i)$ we denote the component of $\operatorname{Ex}_r(a_1, b_1) - b_1 - b_2$ containing the vertex $a_i$.

By assumption there is vertex $x$ of $G$ that is a person living in the corner $\{a_1, b_1\}$. If $x$ is a person of type one, then it is contained in both

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4In fact it intersects this set just once but we will not need that strengthening.
components \( C(a, 1) \) and \( C(b, 1) \) by definition. If \( x \) is a person of type two, then it is contained in the component \( C(a, 1) \) by definition; and in \( C(b, 1) \) as (the unique copy of) \( x \) is adjacent to \( b_1 \). To summarise, in either case the intersection of \( C(a, 1) \) and \( C(b, 1) \) contains the person \( x \).

In particular, the intersection of \( C(a, 1) \) and \( C(b, 1) \) is nonempty and includes a component of \( \text{Ex}_x(a_1, b_1) - a_1 - a_2 - b_1 - b_2 \). Denote such a component containing the person \( x \) by \( k \). As the component \( k \) is included in \( C(a, 1) \), it does not contain any neighbour of the vertex \( b_2 \). By [Sublemma 6.10](#), similarly, \( k \) does not contain any neighbour of the vertex \( a_2 \). Hence \( k \) is also a component of \( \text{Ex}_x(a_1, b_1) - a_1 - b_1 \). As \( k \) does not contain the vertex \( a_2 \), the punctured explorer-neighbourhood \( \text{Ex}_x(a_1, b_1) - a_1 - b_1 \) is disconnected. Thus \( \{a_1, b_1\} \) is an \( r \)-local 2-separator.

The ‘Moreover’-part is clear by construction.

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Given a graph \( G' \) obtained from \( G \) by locally splitting a local 2-separator, by definition there is a bijection between the edges of \( G \) and the edges of \( G' \) that are not torso-edges. To simplify notation, we suppress this bijection from our notation. Let \( b \) be a vertex of \( G \) and let \( b' \) be a vertex of \( G' \) that is equal to \( b \) or a slice thereof. Let \( e \) be an edge that is incident with the vertex \( b \). Then the edge \( e \) is incident with the vertex \( b' \) in \( G' \) or else the vertex \( b' \) must be a slice, and thus is incident with a unique torso edge. The contact of the edge \( e \) at the vertex \( b' \) is the edge \( e \) itself if \( e \) is incident with \( b' \) in \( G' \) or else the contact is the unique torso edge incident with \( b' \).

Given a local 2-separator \( \{b_1, b_2\} \) in a graph \( G \) and two edges \( e \) and \( f \) incident with precisely one of \( b_1 \) or \( b_2 \), we say that \( e \) and \( f \) are separated by \( \{b_1, b_2\} \) if the edges \( e \) and \( f \) have endvertices in different components of the punctured explorer-neighbourhood \( \text{Ex}_x(b_1, b_2) - b_1 - b_2 \).

For a vertex \( x' \) of \( G' \), there is a unique vertex of \( G \) that is equal to \( x' \) or such that \( x' \) is a slice of that vertex. We denote this vertex by \( x \).

**Lemma 6.12** (Projection Lemma). Assume \( G \) is \( r \)-locally 2-connected. For any \( r \)-local 2-separator \( \{b'_1, b'_2\} \) of \( G' \), the set \( \{b_1, b_2\} \) is an \( r \)-local 2-separator of \( G \).

More specifically, edges \( e \) and \( f \) incident with precisely one of \( b_1 \) or \( b_2 \) are separated by \( \{b_1, b_2\} \) in \( G \) if their contacts are separated by \( \{b'_1, b'_2\} \) in \( G' \).

**Proof.** In this proof we will distinguish between the vertices of \( G \) and \( G' \) by adding a dash to the vertices of the graph \( G' \); for example we write \( b'_1 \) when we consider \( b_1 \) as a vertex of \( G' \) and \( b_1 \) when we consider it as a vertex of the graph \( G \).
Let $e$ and $f$ be edges incident with precisely one of $b_1$ or $b_2$ such that their contacts are separated by $\{b'_1, b'_2\}$ in $G'$.

**Sublemma 6.13.** There is at most one torso edge incident with vertices of $\{b'_1, b'_2\}$.

**Proof.** Let $\{a_1, a_2\}$ be a local 2-separator of $G$ such that $G'$ is obtained from $G$ by locally cutting at $\{a_1, a_2\}$. Suppose for a contradiction there are two torso edges incident with vertices of $\{b'_1, b'_2\}$. As each vertex is incident with at most one torso edge, both $b'_1$ and $b'_2$ must be slices.

As $\{b'_1, b'_2\}$ is an $r$-local 2-separator by Lemma 5.4 by Lemma 4.5 the distance between $b'_1$ and $b'_2$ is at most $r/2$. So $b'_1$ and $b'_2$ cannot be slices of the same vertex by Observation 5.2. By symmetry assume that $b'_1$ is a slice of $a_1$ and $b'_2$ is a slice of $a_2$. As there are at least two torso edges, $b'_1$ and $b'_2$ must be slices for different components of $\text{Ex}_r(a_1, a_2) - a_1 - a_2$. As $\{a_1, a_2\}$ is an $r$-local 2-separator of the $r$-locally 2-connected graph $G$, by Lemma 4.5 there is a path of length at most $r/2$ between the vertices $a_1$ and $a_2$ in $G$. So the distance between any two slices of $a_1$ and $a_2$ for the same component of $\text{Ex}_r(a_1, a_2) - a_1 - a_2$ is at most $r/2$ by the definition of local cutting.

Let $a'_1$ be the slice of the vertex $a_1$ for the same component as the slice $b'_2$ of $a_2$. So the distance from $a'_1$ to $b'_2$ is at most $r/2$. So the distance between the distinct slices $a'_1$ and $b'_1$ of $a_1$ is at most $r$. This is a contradiction to Observation 5.2. Hence there is at most one torso edge incident with vertices of $\{b'_1, b'_2\}$.

Let $k'$ be the component of the punctured explorer-neighbourhood $\text{Ex}_r(b'_1, b'_2) - b'_1 - b'_2$ in $G'$ that contains an endvertex of the contact for $e$. Let $W$ be the set of edges of $G'$ with one endvertex in $\{b'_1, b'_2\}$ and the other endvertex in $k'$. By Sublemma 6.13 by exchanging the roles of the edges $e$ and $f$ if necessary, we may assume, and we do assume, that no torso edge incident with a vertex of $\{b'_1, b'_2\}$ has its other endvertex in the component $k'$. Hence the edge set $W$ is also an edge set of the graph $G$.

**Sublemma 6.14.** There is a path $Q$ from $b_1$ to $b_2$ contained in $\text{Ex}_r(b_1, b_2)$ that contains an even number of edges from $W$.

**Proof.** As $G'$ is $r$-locally 2-connected by Lemma 5.4 by Lemma 4.5 there is a cycle $\gamma'$ of length at most $r$ included in $\text{Ex}_r(b'_1, b'_2)$ containing the vertices $b'_1$ and $b'_2$. By Observation 5.2 the cycle $\gamma'$ can contain at most one torso edge. Hence there is a path $Q'$ from $b'_1$ to $b'_2$ included in $\gamma'$ that does not contain any torso edge. As $Q'$ has both its endvertices on the same side of
the cut \( W \), it intersects that cut evenly. The edges of \( Q' \) form a path \( Q \) in the graph \( G \) from \( b_1 \) to \( b_2 \), which is also a path in \( \text{Ex}_r(b_1, b_2) \).

Suppose for a contradiction that the edges \( e \) and \( f \) are not separated by \( \{b_1, b_2\} \) in \( G \); that is, they are incident with vertices of the same component of the punctured explorer-neighbourhood \( \text{Ex}_r(b_1, b_2) - b_1 - b_2 \).

**Sublemma 6.15.** There is a cycle \( o \) of the explorer-neighbourhood \( \text{Ex}_r(b_1, b_2) \) in \( G \) that intersects the set \( W \) oddly.

**Proof.** By assumption, there is a path \( P \) included in the punctured explorer-neighbourhood \( \text{Ex}_r(b_1, b_2) - b_1 - b_2 \) between the endvertices of the edges \( e \) and \( f \) outside \( \{b_1, b_2\} \). Now we extend the path \( P \) to a walk by adding the edges \( e \) and \( f \) at the endvertices of \( P \). The endvertices of this extended walk are in the set \( \{b_1, b_2\} \). This walk intersects, the edge set \( W \) precisely in the edge \( e \). Either this walk is a cycle, or it is a path whose endvertices are \( b_1 \) and \( b_2 \). While we are done immediately in the first case, in the second case we concatenate this path \( ePf \) with a path \( Q \) as in Sublemma 6.14. This way we obtain a closed walk, which includes the desired cycle \( o \).

**Sublemma 6.16.** There is a cycle \( o_1 \) of \( G \) contained in the explorer-neighbourhood \( \text{Ex}_r(b_1, b_2) \) of length bounded by \( r \) that intersects the set \( W \) oddly.

**Proof.** Let \( o \) be a cycle as in Sublemma 6.16. By Lemma 4.3, the cycle \( o \) is generated from cycles of \( o \) that are included within the balls of radius \( r/2 \) around the vertices \( b_1 \) and \( b_2 \). These cycles, in turn by Lemma 3.1 are generated by cycles within the respective balls of length bounded by \( r \). To summarise: the cycle \( o \) is generated over the finite field \( \mathbb{F}_2 \) by cycles of \( G \) contained in the explorer-neighbourhood \( \text{Ex}_r(b_1, b_2) \) of length bounded by \( r \). As the cycle \( o \) intersects the set \( W \) oddly, one of the cycles in the generating set, has to intersect the set \( W \) oddly. We pick such a cycle for \( o_1 \).

**Sublemma 6.17.** There is a cycle \( o_1' \) of \( G' \) included in the explorer-neighbourhood \( \text{Ex}_r(b_1', b_2') \) of length bounded by \( r \) that intersects the set \( W' \) oddly.

**Proof.** Let \( o_1 \) be a cycle as in Sublemma 6.16. First assume the cycle \( o_1 \) does not traverse the local 2-separator \( \{a_1, a_2\} \) (here we say that a cycle traverses a local 2-separator if the cycle traverses this set when being a separator of the explorer-neighbourhood [of parameter \( r \)]). Then the cycle \( o_1 \) of \( G \) is a cycle of \( G' \). So we can take \( o' = o_1 \) and are done. Hence we may assume, and we do assume, that the cycle \( o_1 \) traverses the local 2-separator \( \{a_1, a_2\} \). We remark that as the cycle \( o_1 \) is a cycle of the graph \( G \) – not just of the
explorer-neighbourhood – it cannot contain two copies of a vertex of the graph $G$.

One of the subpaths of $o_1$ from $a_1$ to $a_2$ contains an even number of edges of $W$, the other one an odd number of edges of $W$. Let $P$ be the subpath of $o_1$ from $a_1$ to $a_2$ that contains an odd number of edges of $W$. Then the edges of $P$ form a path of the graph $G'$ from a slice of $a_1$ to a slice of $a_2$. We obtain the cycle $o'$ from $P$ by adding a torso edge, which is not in $W$ by the choice of the component $k'$.

The edge set $W$ is a cut of the explorer-neighbourhood $Ex(b_1', b_2')$, and $o'$ is a cycle of that graph. So they must intersect evenly (as all cuts and cycles do). This is a direct contradiction to Sublemma 6.17. Hence the punctured explorer-neighbourhood $Ex(b_1, b_2) - b_1 - b_2$ in $G$ is disconnected, and so $\{b_1, b_2\}$ is an $r$-local separator of the graph $G$. More specifically, the edges $e$ and $f$ are separated by $\{b_1, b_2\}$.

Remark 6.18. Here we come back to Remark 4.7. Lemma 6.12 (Projection Lemma) would not be true for local separators defined using ‘double-balls’ in place of ‘explorer-neighbourhoods’. An example is given in Figure 5. If one obtained $G'$ from the depicted graph by locally cutting at the local separator $\{b_1, b_2\}$, the corner $\{a_1, b_1\}$ gets a local separator – also for ‘double-balls’. So then the corner $\{a_1, b_1\}$ would be a local separator of $G'$, which does not come from a local separator of $G$.

7 When all local 2-separators are crossed...

In this section we prove Theorem 1.3 which later will be used in the proof of Theorem 2.1.

Proof of Theorem 1.3. Let $\{a_1, a_2\}$ be an $r$-local 2-separator, and let $\{b_1, b_2\}$ be an $r$-local 2-separator that crosses it. By Lemma 6.4 (Alternating Cycle Lemma), there is a cycle $o$ of length at most $r$ alternating between these two local separators. Our aim is to show that the graph $G$ is equal to the cycle $o$. Suppose not for a contradiction. Then there is a vertex outside the cycle $o$. As the graph $G$ is connected, there is a vertex that is outside $o$ and adjacent to a vertex on the cycle $o$. Pick such a vertex and call it $x_2$, and denote one of its neighbours on the cycle $o$ by $x_1$. We distinguish two cases.

Case 1: there is no vertex $y$ on the cycle $o$ such that $\{x_1, y\}$ is an $r$-local 2-separator.
Figure 5: Two crossing local separators. They are highlighted in grey and denoted by \( \{a_1, a_2\} \) and \( \{b_1, b_2\} \). The long red strip joins a neighbour of \( a_1 \) with a neighbour of \( b_1 \). Its length is long enough so that the punctured double-balls \((B_{r/2}(a_1) \cup B_{r/2}(a_2)) - a_1 - a_2\) and \((B_{r/2}(b_1) \cup B_{r/2}(b_2)) - b_1 - b_2\) are disconnected but so short that the punctured double-ball \((B_{r/2}(a_1) \cup B_{r/2}(b_1)) - a_1 - b_1\) is connected.
By $S$ we denote the set of $r$-local 2-separators with both their vertices on the cycle $o$. Given $\{c_1, c_2\} \in S$, by Lemma 4.2 all vertices on the cycle $o$ have a unique copy in the explorer-neighbourhood $Ex_i(c_1, c_2)$. For these vertices we suppress a bijection between them and their unique copies in $Ex_i(c_1, c_2)$ to simplify notation. By $\Gamma(c_1, c_2)$ we denote the set of vertices on the cycle $o$ in the component of $o - c_1 - c_2$ that contains the vertex $x_1$. The size of a local separator $\{c_1, c_2\}$ in $S$ is $|\Gamma(c_1, c_2)|$. The set $S$ is nonempty as $\{a_1, a_2\} \in S$. Pick a local separator $\{w_1, w_2\} \in S$ of minimal size.

**Sublemma 7.1.** In Case 1, no $r$-local 2-separator crosses $\{w_1, w_2\}$.

*Proof.* Suppose for a contradiction there is an $r$-local 2-separator $\{v_1, v_2\}$ that crosses $\{w_1, w_2\}$. By Lemma 6.4 (Alternating Cycle Lemma) there is a cycle $o'$ alternating between $\{v_1, v_2\}$ and $\{w_1, w_2\}$. Hence by Lemma 4.2 the vertices $v_1$, $v_2$, $w_1$ and $w_2$ have unique copies in $Ex_i(v_1, v_2)$; and so in the following we will suppress a bijection between them and their copies in $Ex_i(v_1, v_2)$ from our notation. Let $P'$ be a subpath of $o'$ between $w_1$ and $w_2$ of length at most $r/2$. Let $P$ be a subpath of $o$ between $w_1$ and $w_2$ of length at most $r/2$. Let $o''$ be the closed walk obtained by concatenating $P$ and $P'$. The path $P'$ must contain one of the vertices $v_1$ or $v_2$, say $v_1$. Hence $o''$ is a closed walk through $v_1$ of length at most $r/2$; so it is contained within $B_{r/2}(v_1)$. So the path $P$ is a path of the ball $B_{r/2}(v_1)$. As $w_1$ and $w_2$ are in different components, the path $P$ must contain the vertex $v_1$ or $v_2$. Thus the cycle $o$ contains the vertex $v_1$ or $v_2$. By the ‘Moreover’-part of Lemma 6.4 (Alternating Cycle Lemma) the cycle $o$ alternates between the local separators $\{v_1, v_2\}$ and $\{w_1, w_2\}$.

By Lemma 6.5 each of the punctured explorer-neighbourhoods $Ex_i(v_1, v_2) - v_1 - v_2$ and $Ex_i(w_1, w_2) - w_1 - w_2$ has precisely two components. Hence there is a corner $\{v_i, w_j\}$ with $i, j \in \{1, 2\}$ such that the vertex $x_1$ is a person of type one living in the corner $\{v_i, w_j\}$. Hence by Lemma 6.9 (Corner Lemma), $\{v_i, w_j\}$ is an $r$-local 2-separator. It is in the set $S$. We claim that its size is strictly smaller than that of $\{w_1, w_2\}$. Indeed, by the ‘Moreover’-part of Lemma 6.9 (Corner Lemma) $\Gamma(v_i, w_j)$ only contains those vertices of $o$ on the path between $v_i$ and $w_j$ containing $x_1$. As the vertices $x_1$ and $v_i$ are in the same component of the punctured explorer-neighbourhood $Ex_i(w_1, w_2) - w_1 - w_2$, all vertices of $\Gamma(v_i, w_j)$ are also in $\Gamma(w_1, w_2)$ but that set additionally contains the vertex $v_i$. This is the desired contradiction. Thus no $r$-local 2-separator crosses $\{w_1, w_2\}$. \[\square\]

Sublemma 7.1 contradicts the assumptions of the theorem. Hence the
graph $G$ is a cycle. Having finished Case 1, it remains to treat the following
(which will be somewhat similar).

**Case 2:** not Case 1; that is, there is a vertex $y$ on the cycle $o$ such
that $\{x_1, y\}$ is an $r$-local 2-separator.

By $S$ we denote the set of $r$-local 2-separators with both their vertices
on the cycle $o$ and one of these vertices is equal to the vertex $x_1$. Given
$\{c_1, c_2\} \in S$, by Lemma 4.2 all vertices on the cycle $o$ have a unique copy
in the explorer-neighbourhood $\text{Ex}_r(c_1, c_2)$. Moreover, the vertex $x_2$ has a
unique copy in $\text{Ex}_r(c_1, c_2)$ by Lemma 4.1. For these vertices we suppress
a bijection between them and their unique copies in $\text{Ex}_r(c_1, c_2)$ to simplify
notation. By $\Gamma(c_1, c_2)$ we denote the set of vertices on the cycle $o$ in the
component of the punctured explorer-neighbourhood $\text{Ex}_r(c_1, c_2) - c_1 - c_2$
that contains the vertex $x_2$. The size of a local separator $\{c_1, c_2\}$ in $S$ is
$|\Gamma(c_1, c_2)|$. The set $S$ is nonempty by the assumption of Case 2. Pick a local
separator $\{w_1, w_2\} \in S$ of minimal size.

Arguing the same as in the proof of **Sublemma 7.1** but referring to the
fact that ‘$x_2$ is a person of type two’ instead of ‘$x_1$ is a person of type one’,
one proves the following.

**Sublemma 7.2.** In Case 2, no $r$-local 2-separator crosses $\{w_1, w_2\}$. $\square$

This completes all the cases. Hence in all cases, there is an $r$-local 2-
separator that is not crossed by any other $r$-local 2-separator. This is a
contradiction to the assumptions of this theorem. Hence the graph $G$ must
be equal to the cycle $o$. $\square$

8 The uniqueness statement of the local 2-separator theorem

In this section we prove **Theorem 2.1**. Our first goal is to prove **Lemma 8.3**
below, which can be seen as the ‘inverse’ of **Lemma 6.12** (Projection Lemma).

Let $G'$ be a graph obtained from a graph $G$ by $r$-locally cutting an $r$-local
2-separator $\{a_1, a_2\}$. Let $\{b_1, b_2\}$ be an $r$-local 2-separator of $G$.

**Lemma 8.1.** Assume $G$ is $r$-locally 2-connected. Assume $\{b_1, b_2\}$ is not
crossed by $\{a_1, a_2\}$. Then there is a cycle $o'$ of the graph $G'$ of length at
most $r$ that contains vertices $b'_i$ that are equal to $b_i$ or a slice of $b_i$ (for
$i = 1, 2$).

Moreover, for any $i \in \{1, 2\}$ with $b_i \notin \{a_1, a_2\}$ the two edges of $o'$ incident
with $b'_i$ are separated by $\{b'_1, b'_2\}$.
We remark that the neighbours of $b'_i$ are also neighbours of $b_i$ (unless the edge joining them is a torso-edge), and they have unique copies in $E_x(b_1, b_2)$ by [Lemma 4.1] in this sense the ‘Moreover’-part is unambiguously defined.

**Proof of Lemma 8.1.** As the graph $G$ is $r$-locally 2-connected, we can apply [Lemma 4.5] (Local 2-Connectivity Lemma) to deduce that there is a cycle $o$ of length at most $r$ in $G$ through the vertices $b_1$ and $b_2$ and such that interior vertices of different subpaths of $o$ between $b_1$ and $b_2$ are in different components of the punctured explorer-neighborhood $E_x(b_1, b_2) - b_1 - b_2$.

If the cycle $o$ does not contain any vertex $a_i$, it is a cycle of the graph $G'$ and we are done. So we may assume, and we do assume, that a vertex $a_i$, say $a_1$, is on the cycle $o$. So the cycle $o$ is a cycle of the explorer-neighborhood $E_x(a_1, a_2)$. In there it must traverse the 2-separator $\{a_1, a_2\}$ evenly. If it does not traverse it at all, then $o$ is a cycle of the graph $G'$ and we are done. Otherwise also the vertex $a_2$ is on $o$. As the local separator $\{a_1, a_2\}$ does not cross $\{b_1, b_2\}$, the vertices $a_1$ and $a_2$ must be in the same component of the punctured explorer-neighborhood $E_x(b_1, b_2) - b_1 - b_2$. In particular, the cycle $o$ does not alternate between the local separators $\{a_1, a_2\}$ and $\{b_1, b_2\}$. So there is a subpath $P$ of $o$ between the $a_i$ containing no vertex $b_j$ as an interior vertex. We obtain $o'$ from $o$ by replacing the path $P$ by a torso-edge to obtain a cycle of the graph $G'$. By the definition of the weights of the torso edges the length of $o'$ is at most that of $o$. And $o'$ contains the vertices $b_i$ or slices thereof. This completes the proof except for the ‘Moreover’-part.

To see the ‘Moreover’-part, pick $b_i \notin \{a_1, a_2\}$. The two incident edges of the vertex $b'_i$ on $o'$ are not torso edges. By the construction of the cycle $o'$, the two neighbours of $b'_i$ on $o'$ are separated by $\{b'_1, b'_2\}$. □

A lift of a local 2-separator $\{b_1, b_2\}$ is a set $\{b'_1, b'_2\}$ such that each vertex $b'_i$ is equal to $b_i$ or a slice thereof, and such that there is a cycle $o'$ of length at most $r$ containing $b'_1$ and $b'_2$.

**Remark 8.2.** Under the circumstances of [Lemma 8.1] it can be shown that for each local 2-separator $\{b_1, b_2\}$ a lift is uniquely defined – unless $\{b_1, b_2\}$ is identical to $\{a_1, a_2\}$. Indeed, if no $b_i$ is in the set $\{a_1, a_2\}$, this is clear. Otherwise there can be at most one vertex $b_i$ that is in $\{a_1, a_2\}$, say it is $b_1$. A vertex $b'_1$ in a lift has distance at most $r/2$ from $b_2 = b'_2$. As any other slice $x'$ of $b_1$ has distance at least $r + 1$ from $b'_1$ by [Observation 5.2] the vertex $b_2$ has a too large distance from that vertex. So $\{x', b_2\}$ cannot be a lift. Thus we will in the following always refer to ‘the’ lift.
Lemma 8.3 (Lifting Lemma). Assume $G$ is $r$-locally 2-connected. Let \( \{b_1, b_2\} \) be an $r$-local 2-separator of $G$. Assume \( \{b_1, b_2\} \) is not crossed by \( \{a_1, a_2\} \) and they are not identical. Then the lift \( \{b'_1, b'_2\} \) of \( \{b_1, b_2\} \) is an $r$-local 2-separator of $G'$.

More specifically, if edges $e$ and $f$ incident with precisely one of $b_1$ or $b_2$ are separated by \( \{b_1, b_2\} \) in $G$, then their contacts are separated by \( \{b'_1, b'_2\} \) in $G'$.

Proof. The proof strategy is somewhat similar to that of Lemma 6.12 (Projection Lemma). Similarly as in the proof of Lemma 6.12, in this proof we will distinguish between the vertices of $G$ and $G'$ by adding a dash to the vertices of the graph $G'$.

Let $e$ and $f$ be edges incident with precisely one of $b_1$ or $b_2$ that are separated by \( \{b_1, b_2\} \) in $G$.

Sublemma 8.4. There is a single component $k$ of $\text{Ex}_r(b_1, b_2) - b_1 - b_2$ containing an endvertex of every edge incident with precisely one of $b_1$ or $b_2$ whose contact is a torso edge.

Proof. By assumption not both vertices $b_1$ and $b_2$ are in the set \( \{a_1, a_2\} \). As we are done otherwise, we assume that precisely one of the vertices $b_i$ is in \( \{a_1, a_2\} \). By symmetry, we assume that $b_1 = a_1$.

Next, we determine the component of $\text{Ex}_r(a_1, a_2) - a_1 - a_2$ belonging to the slice $b'_1$ of $a_1$. As \( \{b_1, b_2\} \) is an $r$-local 2-separator, by Lemma 4.5 there is a cycle $\sigma$ of length at most $r$ through $b_1$ and $b_2$. By Lemma 4.2, the vertex $b_2$ has a unique copy in $\text{Ex}_r(a_1, a_2)$. So there is a unique slice of the vertex $a_1$ that has distance at most $r/2$ from the vertex $b_2 = b'_2$. This is the slice for the component of $\text{Ex}_r(a_1, a_2) - a_1 - a_2$ containing $b_2$. Denote that component by $k_2$. Hence the vertex $b'_1$ is the slice of the vertex $a_1$ for the component $k_2$.

Next we define the component $k$. As \( \{a_1, a_2\} \) is an $r$-local 2-separator, by Lemma 4.5 there is a cycle of length at most $r$ through $a_1$ and $a_2$. By Lemma 4.2, the vertex $a_2$ has a unique copy in $\text{Ex}_r(b_1, b_2)$. Let $k$ be the component of $\text{Ex}_r(b_1, b_2) - b_1 - b_2$ that contains the vertex $a_2$.

Now let $e$ be an edge of $G$ incident with precisely one of $b_1$ or $b_2$ whose contact is a torso edge. Then $e$ is incident with the vertex $b_1 = a_1$. Let $x$ be the endvertex of the edge $e$ aside from $b_1$. As the contact for $e$ is a torso edge, the vertex $x$ is outside the component $k_2$ of $\text{Ex}_r(a_1, a_2) - a_1 - a_2$. As $G$ is $r$-locally 2-connected, the punctured ball $B_{r/2}(a_1) - a_1$ is connected. So there is a path $P$ from $x$ to $a_2$ within that ball. As the vertex $b_2$ is in a different connected component of $\text{Ex}_r(a_1, a_2) - a_1 - a_2$ than $x$, it is also in
a different connected component of $B_{r/2}(a_1) - a_1 - a_2$ than $x$. So the path $P$ does not contain the vertex $b_2$. Thus $P$ is a path from $x$ to $a_2$ included in the component $k$. So the path $P$ witnesses that the vertex $x$ is in the component $k$ of $Ex_r(b_1, b_2) - b_1 - b_2$.

As the edge $e$ was arbitrary, the component $k$ contains an endvertex of every edge incident with precisely one of $b_1$ or $b_2$ whose contact is a torso edge.

Let $k_1$ be the component of the punctured explorer-neighbourhood $Ex_r(b_1, b_2) - b_1 - b_2$ containing an endvertex of the edge $e$. By exchanging the roles of $e$ and $f$ if necessary, we assume that the component $k_1$ is different from the component $k$ in Sublemma 8.4. Let $W$ be the set of edges of $G$ with one endvertex in $\{b_1, b_2\}$ and the other endvertex in $k_1$. By Sublemma 8.4, the edge set $W$ does not contain a torso edge and hence is an edge set of the graph $G'$ consisting of edges with precisely one endvertex in the set $\{b'_1, b'_2\}$.

Sublemma 8.5. There is a path $Q'$ from $b'_1$ to $b'_2$ contained in $Ex_r(b'_1, b'_2)$ that contains an even number of edges from $W$.

Proof. As $G$ is $r$-locally 2-connected, by Lemma 4.5, there is a cycle $o$ included in $Ex_r(b_1, b_2)$ containing the vertices $b_1$ and $b_2$ of length at most $r$, and it traverses the local separator $\{b_1, b_2\}$. As $\{a_1, a_2\}$ does not cross $\{b_1, b_2\}$, one of the two subpaths of $o$ from $b_1$ to $b_2$ has no vertex $a_i$ as interior vertex. Pick such a path and call it $Q$. As $Q$ has both its endvertices on the same side of the cut $W$, it intersects that cut evenly. The edges of $Q$ form a path $Q'$ in the graph $G'$ from $b'_1$ to $b'_2$, which is also a path in $Ex_r(b'_1, b'_2)$. \[\square\]

Suppose for a contradiction that the contacts for the edges $e$ and $f$ are not separated by $\{b'_1, b'_2\}$ in $G'$; that is, they are incident with vertices of the same component of the punctured explorer-neighbourhood $Ex_r(b'_1, b'_2) - b'_1 - b'_2$.

Sublemma 8.6. There is a cycle $o'$ of the explorer-neighbourhood $Ex_r(b'_1, b'_2)$ in $G'$ that intersects the set $W$ oddly.

Proof. By assumption, there is a path $P'$ included in the punctured explorer-neighbourhood between the endvertices of the contacts for the edges $e$ and $f$ outside $\{b'_1, b'_2\}$. Now we extend the path $P'$ to a walk by adding the contacts for the edges $e$ and $f$ at the two ends. This walk intersects the set...
$W$ precisely in the edge $e$. If we added the same vertex to both ends, we obtained the desired cycle $o'$. Otherwise we obtain a path between the vertices $b'_1$ and $b'_2$ in the explorer-neighbourhood that intersects the set $W$ oddly. Concatenating this path with a path $Q'$ as in Sublemma 8.5 yields a closed walk that includes the desired cycle $o'$.

\textbf{Sublemma 8.7.} There is a cycle $o'_1$ of $G'$ contained in the explorer-neighbourhood $\text{Ex}_r(b'_1, b'_2)$ of length bounded by $r$ that intersects the set $W$ oddly.

\begin{proof}
Let $o'$ be a cycle as in Sublemma 8.6. By Lemma 4.3 the cycle $o'$ is generated from cycles of $o'$ that are included within the balls of radius $r/2$ around the vertices $b'_1$ and $b'_2$. These cycles, in turn by Lemma 3.1 are generated by cycles within the respective balls of length bounded by $r$. To summarise: the cycle $o'_1$ is generated over the finite field $\mathbb{F}_2$ by cycles of $G'$ contained in the explorer-neighbourhood $\text{Ex}_r(b'_1, b'_2)$ of length bounded by $r$. As the cycle $o'_1$ intersects the set $W$ oddly, one of the cycles in the generating set, has to intersect the set $W$ oddly. We pick such a cycle for $o'_1$.
\end{proof}

\textbf{Sublemma 8.8.} There is a cycle $o$ of $G$ included in the explorer-neighbourhood $\text{Ex}_r(b_1, b_2)$ of length bounded by $r$ that intersects the set $W$ oddly.

\begin{proof}
Let $o'_1$ be a cycle as in Sublemma 8.7. We remark that as the cycle $o'_1$ is a cycle of the graph $G'$ – not just of the explorer-neighbourhood – it cannot contain two copies of a vertex of the graph $G'$. As the cycle $o'_1$ has length at most $r$, by Observation 5.2 it contains at most one slice of any vertex of $G$. In particular, the cycle $o'_1$ contains at most one torso edge.

First assume the cycle $o'_1$ does not use any torso edge. Then the edges of the cycle $o'_1$ of $G'$ form a cycle $o$ of $G$. So we are done in this case.

Hence we may assume, and we do assume, that the cycle $o'_1$ contains a unique torso edge. We denote by $Q'$ the subpath of $o'_1$ obtained by removing the torso edge. Then $Q'$ bijects to an edge set $Q$ of the graph $G$. As the torso edge is not in $W$ by construction, we deduce that $Q$ contains an odd number of edges of $W$.

By the definition of local cutting, there is a path $P$ of $G$ between $a_1$ and $a_2$ associated to this torso edge of $o'_1$ such that the cycle $o'_1$ with the torso edge.

\text{Sublemma 8.4} implies that one of the edges $e$ and $f$ is its own contact. By fixing the roles of $e$ and $f$ above in the definition of the set $W$, we ensured that the edge $e$ is its own contact.
edge replaced by $P$ is a cycle of the graph $G$ of the same length as $o'$. We denote that cycle by $o$.

As $Q$ contains an odd number of edges of $W$, it remains to show that the path $P$ contains an even number of edges from the set $W$. Consider the cycle $o$, which has length at most $r$ and contains the vertices $a_1$, $a_2$ and a vertex of $\{b_1, b_2\}$ as it contains an edge of $W$ in the subpath $Q$. By assumption $\{a_1, a_2\}$ does not cross $\{b_1, b_2\}$. So the existence of $o$ implies that $\{a_1, a_2\}$ does not pre-cross $\{b_1, b_2\}$; that is, the vertices $a_1$ and $a_2$ are in the same component of $\text{Ex}_r(b_1, b_2) - b_1 - b_2$. So they are on the same side of the cut $W$. By Lemma 6.1, the path $P$ contains an even number of edges of the cut $W$. Thus the cycle $o$, which is composed of the paths $P$ and $Q$ contains an odd number of edges of $W$. $\square$

The edge set $W$ is a cut of the explorer-neighbourhood $\text{Ex}_r(b_1, b_2)$, and a cycle $o$ as in Sublemma 8.8 is a cycle of that graph. So they must intersect evenly (as all cuts and cycles do). This is a direct contradiction to Sublemma 8.8. Hence the punctured explorer-neighbourhood $\text{Ex}_r(b_1, b_2) - b_1 - b_2$ in $G'$ is disconnected, and so $\{b_1', b_2'\}$ is an $r$-local separator of the graph $G'$. More specifically, the contacts for $e$ and $f$ are separated by $\{b_1', b_2'\}$. $\square$

**Lemma 8.9.** Let $G$ be an $r$-locally 2-connected graph. Let $\{a_1, a_2\}$, $\{b_1, b_2\}$ and $\{c_1, c_2\}$ be $r$-local 2-separators so that $\{a_1, a_2\}$ crosses neither $\{b_1, b_2\}$ nor $\{c_1, c_2\}$. Construct $G'$ from $G$ by $r$-locally cutting $\{a_1, a_2\}$.

Then the lifts of $\{b_1, b_2\}$ and $\{c_1, c_2\}$ in $G'$ if and only if $\{b_1, b_2\}$ and $\{c_1, c_2\}$ cross in $G$.

**Proof.** Assume $\{b_1, b_2\}$ and $\{c_1, c_2\}$ cross in $G$. Then by the ‘Moreover’-part of Lemma 6.4 (Alternating Cycle Lemma), there is a cycle $o$ of length at most $r$ alternating between $\{b_1, b_2\}$ and $\{c_1, c_2\}$ traversing the local separator $\{b_1, b_2\}$ twice (that is, the subpaths of $o$ between $b_1$ and $b_2$ have interior vertices in different components of the punctured explorer-neighbourhood $\text{Ex}_r(b_1, b_2) - b_1 - b_2$).

If the cycle $o$ does not traverse the local separator $\{a_1, a_2\}$ twice, then $o$ is a cycle of the graph $G'$. Then by Lemma 8.3 (Lifting Lemma), $o$ traverses the lift of $\{b_1, b_2\}$ twice. Thus the lifts of $\{b_1, b_2\}$ and $\{c_1, c_2\}$ cross in $G'$.

Hence we may assume, and we do assume, that the cycle $o$ traverses the local separator $\{a_1, a_2\}$ twice. As $\{a_1, a_2\}$ crosses neither $\{b_1, b_2\}$ nor $\{c_1, c_2\}$, any path between the $a_i$ on $o$ must contain an even number of vertices $b_i$ and $c_j$. As the cycle $o$ alternates, one of these path must contain all vertices $b_i$ and $c_j$, and the other none. Let $o'$ be the cycle obtained from $o$.
by replacing the path containing none by a torso edge. The cycle $o'$ contains a lift of $\{b_1, b_2\}$ by construction. As the lift is unique by Remark 8.2, it must contain the lift of $\{b_1, b_2\}$. The cycle $o'$ traverses the lift of $\{b_1, b_2\}$ twice by the ‘More specifically’-part of Lemma 8.3 (Lifting Lemma). Similarly, $o'$ traverses the lift of $\{c_1, c_2\}$ twice. So it alternates between the lifts of $\{b_1, b_2\}$ and $\{c_1, c_2\}$. Thus the lifts of $\{b_1, b_2\}$ and $\{c_1, c_2\}$ cross.

Next assume that the lifts $\{b'_1, b'_2\}$ and $\{c'_1, c'_2\}$ of $\{b_1, b_2\}$ and $\{c_1, c_2\}$ cross. Then by the Moreover-part of Lemma 8.4 (Alternating Cycle Lemma) there is a cycle $o'$ of length at most $r$ alternating between $\{b'_1, b'_2\}$ and $\{c'_1, c'_2\}$ traversing $\{b'_1, b'_2\}$ twice. Let $o$ be the cycle obtained from the cycle $o'$ by replacing torso edges by paths of the same length. By Lemma 6.12 (Projection Lemma) the cycle $o$ traverses the local separator $\{b_1, b_2\}$ twice.

As the cycle $o$ alternates between the local separators $\{b_1, b_2\}$ and $\{c_1, c_2\}$, these two local separators cross. □

Sometimes we will omit the term ‘lift’ and simply consider local separators of $G$ as local separators of a graph $G'$ obtained by cutting. The next lemma says that $r$-local cuttings along non-crossing 2-separators commute.

**Lemma 8.10.** Let $G$ be an $r$-locally 2-connected graph with non-crossing $r$-local 2-separators $\{a_1, a_2\}$ and $\{b_1, b_2\}$. Then the graphs obtained from $r$-locally cutting these two local separators in either order are identical.

**Proof.** Let $G'$ be the graph obtained from $G$ by $r$-locally cutting $\{a_1, a_2\}$. Let $G''$ be the graph obtained from $G'$ by $r$-locally cutting (the lift of) $\{b_1, b_2\}$. Let $G_2$ be the graph obtained from $G$ by first cutting $\{b_1, b_2\}$ and then (the lift of) $\{a_1, a_2\}$. Let $\{b'_1, b'_2\}$ be the lift of $\{b_1, b_2\}$ in $G'$. By Lemma 8.3 (Lifting Lemma), the slices of $\{b_1, b_2\}$ in $G''$ and $G_2$ are identical. By symmetry, the same is true for slices of $\{a_1, a_2\}$. Vertices that are not slices are identical by construction. This defines a bijection between the vertices of the graphs $G''$ and $G_2$. It is straightforward to check that this bijection between the vertices extends to a bijection between the edges. □

Given a set $S$ of $r$-local 2-separators of a graph $G$ that pairwise do not cross, we say that a graph $G'$ is obtained from $G$ by $r$-locally cutting $S$ if $G'$ is obtained from $G$ by the following procedure. Pick a linear ordering of the set $S$. Then starting with the graph $G$ we cut along the local separators of $S$ in that linear order. By Lemma 8.3 (Lifting Lemma) and Lemma 8.9 this is well-defined. By Lemma 8.10, changing the linear ordering does not affect the graph obtained by cutting. Hence in the following we shall speak of the graph obtained from $G$ by cutting along $S$.  

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Given a graph $G$, by $\mathcal{N}$ we denote the set of all $r$-local 2-separators of $G$ that are not crossed by any $r$-local 2-separator.

**Theorem 8.11.** Assume $G$ is $r$-locally 2-connected, and let $G'$ be the graph obtained from $G$ by $r$-locally cutting $\mathcal{N}$. Then every connected component of $G'$ is $r$-locally 3-connected or a cycle of length at most $r$.

Let $G''$ be a graph obtained from $G$ by $r$-local cuttings such that all connected components are $r$-locally 3-connected or cycles of length at most $r$. Then in the construction of $G''$ one has to cut at any local separator $X$ or one of its lifts for all $X \in \mathcal{N}$.

**Proof.** Fix a linear ordering of the set $\mathcal{N}$ and let $G_i$ be the graph obtained from $G$ by $r$-locally cutting the first $i$ elements of $\mathcal{N}$. By Lemma 5.4 applied recursively each graph $G_i$ is $r$-locally 2-connected. The graph $G'$ is the last graph $G_i$.

Suppose for a contradiction that some connected component of the graph $G'$ is neither $r$-locally 3-connected nor a cycle of length at most $r$. Then by Theorem 1.3 the graph $G'$ has an $r$-local 2-separator $\{v, w\}$ that is not crossed by any other $r$-local 2-separator of $G'$. By Lemma 6.12 (Projection Lemma) applied recursively, $\{v, w\}$ is also an $r$-local 2-separator of the graph $G$. By the construction of the graph $G'$, the local separator $\{v, w\}$ is not in the set $\mathcal{N}$. Thus there is some $r$-local 2-separator $\{a, b\}$ of $G$ that crosses $\{v, w\}$. By the choice of the set $\mathcal{N}$, the local 2-separators $\{v, w\}$ and $\{a, b\}$ are not crossed by any local separator of $\mathcal{N}$. Hence we can apply Lemma 8.3 (Lifting Lemma) recursively to deduce that $\{a, b\}$ is a local separator of the graph $G'$. Applying Lemma 8.9 recursively yields that $\{v, w\}$ and $\{a, b\}$ are crossing in the graph $G'$. This is a contradiction to the existence of the local separator $\{v, w\}$. Thus every connected component of the graph $G'$ is $r$-locally 3-connected or a cycle of length at most $r$.

Now let $G''$ be a graph obtained from $G$ by $r$-local cuttings such that all connected components are $r$-locally 3-connected or cycles of length at most $r$. Let $(G_i)$ be a sequence of graphs starting with $G_0 = G$ and ending with $G_n = G''$ such that $G_{i+1}$ is obtained from $G_i$ by $r$-local cutting. By Lemma 5.4 applied recursively each graph $G_i$ is $r$-locally 2-connected.

Suppose for a contraction there is an $r$-local 2-separator $\{v, w\}$ of the set $\mathcal{N}$ such that neither it nor any of its lifts is identical to a local separator at which we locally cut to obtain $G_{i+1}$ from $G_i$.

We will show by induction that $\{v, w\}$ does not cross any local separator of any graph $G_i$. Assume we have shown it does not cross any local separator of a graph $G_j$. By Lemma 6.12 (Projection Lemma), all local
2-separators of $G_{j+1}$ are lifts of local 2-separators of $G_j$. Applying this recursively yields that all local 2-separators at which we locally cut are lifts of local 2-separators of the graph $G$. So by Lemma 8.9 $\{v, w\}$ does not cross any local separator of the graphs $G_{j+1}$. This completes the induction step.

Thus the above is also true for the last graph $G_n = G''$, all of whose connected components are $r$-locally 3-connected or cycles of size at most $r$ by assumption. Such graphs do not have an $r$-local 2-separator that is not crossed. Hence $\{v, w\}$ cannot exist. This is a contraction. So for any local separator of $\mathcal{N}$, it or one of its lifts has to appear in the construction of $G''$. □

Proof of Theorem 2.1. This theorem is a direct consequence of Theorem 8.11 □

9 Graph-Decompositions

The purpose of this section is to define graph-decompositions and explain why they can be understood as a generalisation of tree-decompositions with the decomposition-tree replaced by a general graph.

First we need some preparation. Given a graph $G$, a graph $F$ and a family $\mathcal{F}$ of subgraphs of $G$ that are isomorphic to $F$, the graph obtained from $G$ by identifying along $\mathcal{F}$ is the graph obtained from $G$ by identifying all elements of $\mathcal{F}$. Formally, the vertex set of this new graph is the vertex set of $G$ modulo the equivalence relation generated by the relation where two vertices $v_1$ and $v_2$ are related if there are graphs $F_1, F_2 \in \mathcal{F}$ with $v_i \in F_i$ (for $i = 1, 2$) such that after applying the isomorphisms to $F$ the vertices $v_1$ and $v_2$ are equal to the same vertex of $F$. The edges of the identification-graph are the edges of $G$, where the endvertices are the equivalence classes of the original endvertices of $G$ – with the following exception: if two vertices $v$ and $w$ in $F$ are joined by edge, then in the identification graph we keep only one copy of all the edges of $G$ between the clones of $v$ and $w$ in the graphs $F' \in \mathcal{F}$. This completes the definition of gluing. Examples of gluing are given in Figure 6 and Figure 7.

Remark 9.1. Even if the graph $G$ has no loops or parallel edges, graphs obtained from $G$ by identifying along a family may have loops or parallel edges, see Figure 7.

A graph-decomposition consists of a bipartite graph $(B, S)$, where the elements of $B$ are referred to as ‘bags’ and the elements of $S$ are referred to as ‘local separators’. This bipartite graph is referred to as the ‘decomposition-
Figure 6: An example of a gluing. Here the graph $F$ consists of a single vertex and the family $\mathcal{F}$ has three members, which are marked in blue. The graph $G$ before the gluing is depicted on the left. The graph after the gluing is depicted on the right.

Figure 7: An example of a gluing. Here the graph $F$ consists of two vertices joined by an edge and the family $\mathcal{F}$ has two members, which are marked in blue. The graph $G$ before the gluing is depicted on the left. The graph after the gluing is depicted on the right. If the graph $F$ consisted just of the two vertices without the edge, the gluing would be the graph obtained by the graph on the right by adding an edge in parallel to the blue edge.
For each node $x$ of the decomposition graph, there is a graph $G_x$ associated to $x$. Moreover for every edge $e$ of the decomposition graph from a local separator $s$ to a bag $b$, there is a map $\iota_e$ that maps the associated graph $G_s$ to a subgraph of the associated graph $G_b$.

The underlying graph of a graph-decomposition $(G_x | x \in V(B,S))$ is constructed from the disjoint union of the bags $b \in B$ by identifying along all the families given by the copies of the graph $G_s$ for $s \in S$. Formally, for each local separator $s \in S$, its family is $(\iota_e(G_s))$, where this family contains a copy of $G_s$ for every edge $e$ incident with the vertex $s$ in the bipartite graph $(B,S)$. Now we perform the identification for all these families separately. We remark that different orderings in which we perform these identification result in the same graph.

The width of a graph-decomposition $(G_x | x \in V(B,S))$ is the maximal vertex number of a bag $b \in B$—take away one. The adhesion of a graph-decomposition $(G_x | x \in V(B,S))$ is the maximum vertex number of a local separator $s \in S$. This completes the definition of graph-decompositions and related concepts.

**Example 9.2.** Essentially, tree-decompositions are examples of graph-decompositions. Indeed, given a tree-decomposition, one obtains a new tree-decomposition by subdividing every edge once and associating to that new vertex the separator associated to that edge (this separator is given by taking the intersection of the two bags at the endvertices of that edge).

This defines a graph-decomposition whose decomposition-graph is the decomposition-tree of this newly constructed tree-decomposition. Its bags are the original bags of that tree-decomposition and its local separators are separators of the old tree-decomposition; that is, the bags associated to the new vertices of the new tree-decomposition.

The notions of width and adhesion as defined above for graph-decompositions whose decomposition graphs are trees coincide with the standard notions for tree-decompositions when interpreted as graph-decompositions in the way explained above.

**Example 9.3.** The graph on the right of Figure 6 has a graph-decomposition with only one bag which is given by the graph on the left, and only one local separator, which is given by the blue vertex. Its decomposition-graph is the bipartite graph consisting of three edges in parallel.

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6This convention to take away one is common in the literature. Consequently, trees have tree-width one.
Example 9.4. The graph on the right of Figure 7 has a graph-decomposition with only one bag which is given by the graph on the left, and only one local separator, which consists of an edge. Its decomposition-graph is the bipartite graph consisting of two edges in parallel.

A graph-decomposition has locality $r$ if every cycle traversing a local separator of this graph-decomposition oddly has length larger than $r$; here ‘traversing oddly’ is defined as follows.

This definition is slightly technical. For this definition consider graph-decompositions such that for every local separator $s$ its embedding maps $\iota_f$ into bags have disjoint images. Given a graph-decomposition of a graph $G$, we label an edge $e$ of $G$ between a vertex of a local separator $s$ and a vertex outside $s$ by the unique edge $f$ of the graph-decomposition whose map $\iota_f$ maps a vertex of $S$ to an endvertex of $e$.

A traversal of a cycle $o$ of a graph $G$ of a local separator $s$ consists of a maximal nonempty subpath $P$ of $o$ included in $s$ such that the edges of $o$ just before $P$ and just after $P$ are labelled with different edges of the graph-decomposition. We say that a cycle traverses a local separator oddly if the number of traversals is odd.

Example 9.5. In graph-decompositions whose decomposition graph is a tree all cycles traverse evenly and hence their locality is infinite.

Example 9.6. The locality of the graph-decomposition described in Figure 7 is 11, as the shortest cycle traversing oddly has length 12. The locality of the graph-decomposition described in Figure 6 is 9.

There is a correspondence between nested sets of separations and tree-decompositions. A corresponding fact for graph-decompositions is also true. Here we only need the following special case of this correspondence.

Lemma 9.7. Let $G$ be a graph with a set $S$ of non-crossing $r$-local 2-separators. Let $B$ be the set of connected components of the graph obtained from $G$ by $r$-locally cutting along $S$.

Then there is a decomposition graph with bipartition $(B, S)$ of a graph-decomposition of $G$ of adhesion two and locality $r$.

Proof. For every local component of a local separator $s \in S$ we have an edge $e$ in the graph $(B, S)$ from $s$ to the unique bag $b \in B$ containing the

An alternative definition would be to replace ‘traversing oddly’ by ‘traversing effectively zero’, taking additionally orientations of traversals into account. For simplicity we just make the definition with ‘oddly’.
slices of $s$ for that local component. The map $\iota_e$ maps $s$ to this copy of $s$ in $b$. So $(B, S)$ is the decomposition graph of a graph-decomposition of $G$. It has adhesion two as all elements of $S$ have size two. It is $r$-local as all local separators in $S$ are $r$-local.

A torso of a bag $b$ of a graph-decomposition is obtained from $b$ by joining for every map $\iota$ from a local separator $s$ to the bag $b$ any two vertices in the image of $\iota$ by an edge.

Proof of Theorem 1.2. Combine Lemma 9.7 and Theorem 8.11.

10 Outlook

Having finished the proof of the local 2-separator theorem, we outline possible ways in which this theorem could be extended. We continue our investigation of local separators in [9] by proving a local version of the tangle tree theorem.

Richter extended the 2-separator theorem to infinite graphs [29]. We expect that the results of this paper are also true for infinite graphs.

Conjecture 10.1. [Theorem 2.1] is true for infinite graphs.

Another direction, might be to prove a matroidal analogue of our local 2-separator theorem.

Question 10.2. Can you prove a local 2-separator theorem for (representable) matroids that is reminiscent of [Theorem 1.2]?

A natural next step would be to prove a local version of the Grohe-Decomposition-Theorem, which gives a decomposition of a 3-connected graph into ‘quasi 4-connected components’ [17]. We hope that with the methods of this paper one should be able to prove the following.

Local 3-separators are defined analogously to local 2-separators using an explorer-neighbourhood around three vertices, see [9] for details. Given a parameter $s \in \mathbb{N} \cup \{\infty\}$, we say that a graph is $s$-locally quasi 4-connected if for every $s$-local 3-separator all but one of its components (that is, components of the punctured explorer-neighbourhood) contain at most one vertex.

Conjecture 10.3. For every parameter $r \in \mathbb{N} \cup \{\infty\}$ there is a parameter $s$ such that every $s$-locally 3-connected graph $G$ has a graph-decomposition of locality at least $r$ and adhesion at most $3$ such that all its torsos are minors of $G$ that are either $r$-locally quasi-4-connected or a complete graph of order at most $4$.
Remark 10.4. We do not conjecture any relationship between $r$ and $s$, and even $r = s$ may be possible.

A Appendix I: an alternative proof for the existential statement

In this part we give an alternative proof of the first sentence of Theorem 2.1 that only relies on lemmas proved before Section 6.

Basic examples of $r$-locally 2-connected graphs are cycles of length at most $r$ and $r$-locally 3-connected graphs. The following theorem says that all $r$-locally 2-connected graphs can be constructed from these basic graphs by $r$-local 2-sums.

Theorem A.1. Given a parameter $r$, every $r$-locally 2-connected graph $G$ can be obtained by $r$-local 2-sums from $r$-locally 3-connected graphs and cycles of length at most $r$.

Proof. Our strategy to prove this theorem is the following. If the graph $G$ does not have any $r$-local 2-separator, it is $r$-locally 3-connected and we are done. Otherwise, pick an $r$-local 2-separator arbitrarily and $r$-locally cut along that local 2-separator. Then we iterate this procedure.

Formally, this is expressed as follows. Let $G_1 = G$. Assume we already defined $G_i$. If $G_i$ is $r$-locally 3-connected or a cycle of length at most $r$, we stop. Otherwise, pick an $r$-local 2-separator $\{a_i, b_i\}$ arbitrarily, and obtain the graph $G_{i+1}$ by $r$-locally cutting at the local separator $\{a_i, b_i\}$. It is easily proved by induction that each graph $G_i$ is $r$-locally 2-connected using Lemma 5.4.

If this procedure terminates, then by Lemma 5.3 applied recursively, the graph $G$ has the desired decomposition.

Hence all that remains to show is that this procedure stops eventually. For that we introduce parameters that decrease in each cutting operation. Let $\gamma_i$ be the dimension of the cycle space of the graph $G_i$. We abbreviate: $e_i = |E(G_i)|$ and $v_i = |V(G_i)|$. By $k_i$ we denote the number of components of the graph $G_i$. The following identity relating edge number, vertex number and the number of components of a graph is well-known (see for example [12]):

$$e_i = \gamma_i + v_i - k_i \quad (1)$$

Let $\ell$ be the number of torso edges produced at the $i$-th cutting step. We have:
Subtracting Equation 1 from itself with indices ‘\(i+1\)’ and ‘\(i\)’, respectively, and plugging in the above yields:

\[
\gamma_{i+1} = \gamma_i - (\ell - 2) + (k_{i+1} - k_i)
\]  

(2)

**Remark A.2** (Motivation). First we explain heuristically why one expects this process to terminate referring to Equation 2. If \(G_{i+1}\) has more components than \(G_i\), then one expects that each of these new components is smaller than \(G_i\) and hence one could apply induction.

If \(\ell > 2\), then one can apply induction on \(\gamma_i\). Hence it remains to consider the case that \(\ell = 2\) and the graphs \(G_{i+1}\) and \(G_i\) have the same numbers of components; that is, \(k_{i+1} - k_i = 0\). Still we would like to apply induction on \(\gamma_i\) but it might not go down immediately – but it will go down eventually. To see that consider \(\tau_i\), which is defined to be the dimension of the cycles generated by those of length at most \(r\). We will show that \(\tau_{i+1} > \tau_i\) in this case.

As \(\tau_i \leq \gamma_i\), we can make only a bounded number of steps in which \(\gamma_i\) stays constant – as \(\tau_i\) increases in each of these steps. Thus \(\gamma_i\) will decrease eventually and we may apply induction.

Formally, we argue like this. Given a connected graph \(G\), we consider the triple \((\gamma, -\tau, v)\), where \(\gamma\) is the dimension of the cycle space of \(G\), \(\tau\) is the dimension of its cycles generated by cycles of length at most \(r\), and \(v\) is the number of vertices of \(G\). We consider the order on connected graphs given by the lexicographical ordering according to the triples: \((\gamma, -\tau, v)\); that is, if \(\gamma(H) < \gamma(G)\), then \(H < G\). If the \(\gamma\)-values are equal, we compare the values for \(-\tau\). If they are also equal, we compare the vertex-numbers. We refer to this ordering as the *triplex-ordering*.

**Sublemma A.3.** The triplex-ordering is well-founded.

*Proof.* Suppose for a contradiction there is an infinite sequence that is strictly decreasing in the triplex ordering. Then the parameter \(\gamma\) must be eventually constant. As \(\tau \leq \gamma\), there are only boundedly many possible values for \(-\tau\) (after that). Hence this parameter also has to be eventually constant. Then also the vertex number has to be eventually constant. This is a contradiction to the assumption that the sequence is infinite and strictly decreasing.  

\qed
Sublemma A.4. Let $G$ be a connected graph and $G'$ be obtained from $G$ by $r$-locally cutting an $r$-local 2-separator of $G$, then every connected component of $G'$ is strictly smaller than $G$ in the triplex-ordering.

Proof. Case 1: the graph $G'$ is connected. If the number $\ell$ of slices is at least three, then by [Equation 2] $G'$ is strictly smaller in the triplex-ordering. So we may assume, and we do assume, that $\ell = 2$. Then by [Equation 2] $\gamma(G') = \gamma(G)$. Hence it suffices to do the following computation.

Sublemma A.5. $\tau(G') > \tau(G)$.

Proof. We denote by $\mathbb{F}_2^E$ the vector space over the finite field $\mathbb{F}_2$ whose set of coordinates is $E$, the set of edges of the graph $G$. Similarly, we denote by $\mathbb{F}_2^{E'}$ the vector space over the finite field $\mathbb{F}_2$ whose set of coordinates is $E'$, the set of edges of the graph $G'$. We denote by $C$ those vectors generated by the cycles of length at most $r$ in the graph $G$. We denote by $C'$ those vectors generated by the cycles of length at most $r$ in the graph $G'$. The set $E'$ is obtained from the set $E$ by adding the two torso edges. Consider the set $C''$ of vectors $v$ with coordinates in $E$ such that there are assignments to the torso edges so that $v$ extends to a vector in $C'$. Note that $C''$ is a vector space that has the same dimension as $C'$.

Next we will show that $C \subseteq C''$. For that take a cycle $o$ of the graph $G$ of length at most $r$. If its edge set is a cycle of the graph $G'$, the cycle $o$ is also in $C''$. Otherwise, it contains a vertex of the local separator. So it is contained in a ball of radius $r/2$ around that vertex. If it does not contain the other vertex of the local separator, it is also a cycle of $G'$. Otherwise, in $G'$ it consists of two paths joining the slices of that local separator. This edge set is also in the vector space $C''$.

By [Lemma 4.5] (Local 2-Connectivity Lemma) the vector space $C''$ contains a path between two slices, which is not in $C$ (indeed such a path is a subpath of the cycle $o'$ from that lemma). Thus the dimension of the vector space $C''$ is strictly larger than the dimension of the vector space $C$. So the dimension of the vector space $C'$ is strictly larger than the dimension of the vector space $C$.

Hence by [Sublemma A.5] $\tau(G') > \tau(G)$, and so $G'$ is strictly smaller than $G$ in the triplex ordering in this case.

Case 2: the graph $G'$ is disconnected. Let $H'$ be an arbitrary component of the graph $G'$. As each connected component of the graph $G'$ contains a cycle through one of its torso edges by [Lemma 4.5] (Local 2-Connectivity Lemma), by [Equation 2] we have that $\gamma(H') \leq \gamma(G)$. Moreover, if we have
equality, all other components of $G'$ are single cycles (as they don’t have local cutvertices by Lemma 5.4), and $\ell = 2$. So $\bar{\gamma}(H') = \bar{\gamma}(G)$. By Lemma 4.5 (Local 2-Connectivity Lemma) each component of $G'$ has at least one vertex that is not a slice (note that as $\ell = 2$ local cutting does not produce an artificial cycle of length two). Hence $v(H') < v(G)$, and so $H'$ is strictly smaller than $G$ in the triplex-ordering. This completes the proof in Case 2 of this sublemma.

By Sublemma A.4, each connected component of the graph $G_{i+1}$ is strictly smaller than some connected component of the graph $G_i$ in the triplex-ordering. Hence by Sublemma A.3 we may apply induction. Thus this procedure has to stop eventually. Let $\mathcal{G}$ be the set of connected components of the graph $G_i$ where this terminates. As explained above, we can apply Lemma 5.3 recursively to deduce that the graph $G$ is constructed via local 2-sums from the set $\mathcal{G}$.

Remark A.6. It seems to us that Theorem A.1 is also true with a different notion of local 2-separators that is based on double-balls. However, examples such as that given in Figure 5 show that such alternative decompositions cannot be unique. Thus we believe that such a statement would be less applicable than Theorem A.1; see Application (D) in the Introduction for details.

B Appendix II: Block-Cutvertex Graphs

In this section we prove a generalisation of the block-cutvertex theorem allowing for $r$-local cutvertices, which generalise cutvertices (indeed, the $r$-local cutvertices for $r = \infty$ are precisely the cutvertices). See Section 2 for a definition of $r$-local cutvertices and Section 9 for a definition of graph-decompositions.

It seems to us that the most natural generalisation of the block-cutvertex theorem to this context is the following.

**Theorem B.1.** Given $r \in \mathbb{N} \cup \{\infty\}$, every connected graph has a graph-decomposition of adhesion one and locality $r$ such that all its bags are $r$-locally 2-connected or single edges.

**Remark B.2.** The strengthening of Theorem B.1 with ‘bags are $r$-locally 2-connected’ replaced by ‘bags are $r$-locally 2-connected subgraphs’ is not true. An example is given in Figure 6.
As a preparation for the proof of Theorem B.1, we investigate the operation of locally cutting vertices, defined as follows.

Given a parameter $r \geq 1$ and a graph $G$ with a vertex $v$, the graph obtained from $G$ by \textit{r-locally cutting} the vertex $v$ is defined as follows. Let $X$ be the set of connected components of the ball of radius $r$ around $v$ with $v$ removed; formally $X$ is the set of components of the graph $B_{r/2}(v) - v$. Define a new graph from $G$ by replacing the vertex $v$ by one new vertex for each element of the set $X$, where the vertex labelled with $x \in X$ inherits the incidences with those edges incident with $v$ that are incident with a vertex of the connected component $X$. We refer to the new vertices as the \textit{slices} of $v$. This completes the construction of the $r$-local cutting of $G$.

\textbf{Observation B.3.} \textit{Let $G'$ be obtained from $G$ by $r$-locally cutting a vertex $v$ into a set $X$ of new vertices. Then in the graph $G'$, no vertex $x \in X$ is an $r$-local cutvertex.}\hfill \Box

The next lemma says that $r$-local cuttings commute.

\textbf{Lemma B.4.} \textit{Given a graph $G$ with vertices $v$ and $w$, first $r$-locally cutting $v$ and then $w$ results in the same graph as first locally cutting $w$ and then $v$.}\hfill \Box

\textit{Proof.} Consider the graph $G'$ obtained from $G$ by $r$-locally cutting the vertex $v$. We denote the ball of radius $r/2$ around the vertex $w$ in the graph $G$ by $B_{r/2}(w)$, and by $B'_{r/2}(w)$ we denote the ball of radius $r/2$ around the vertex $w$ in the graph $G'$.

In the graphs $G$ and $G'$, the vertex $w$ has the same neighbours. Indeed, if $v$ and $w$ are not adjacent, this is immediate. Otherwise $w$ is adjacent with a unique slice of $v$ in $G'$, and in the following we will suppress a bijection between the vertex $v$ and this particular slice of $v$ – in order to simplify notation. With this notation at hand, we next prove the following.

\textbf{Sublemma B.5.} \textit{Two neighbours $x$ and $y$ of $w$ are in the same connected component of $B_{r/2}(w) - w$ if and only if they are in the same connected component of $B'_{r/2}(w) - w$.}\hfill \Box

\textit{Proof.} If $x$ and $y$ are in the same connected component of $B'_{r/2}(w) - w$, they are joined by a path in that graph and this path is also a path (or a walk) in the graph $B_{r/2}(w) - w$.

Hence conversely assume that $x$ and $y$ are vertices of the same connected component of the ball $B_{r/2}(w) - w$. Let $P$ be a path between these two vertices in the graph $B_{r/2}(w) - w$. Then this path $P$ together with the
vertex \( w \) forms a cycle, which we denote by \( o \). By Lemma 3.1, the cycle \( o \) is generated by cycles of length at most \( r \) in the graph \( B_{r/2}(w) - w \).

If one of these cycles does not include the vertex \( v \), then it is also a cycle in the graph \( B'_{r/2}(w) \). Otherwise, such a cycle is also a cycle completely contained with in the ball \( B_{r/2}(v) \) around \( v \) in \( G \). In particular this cycle witnesses that the two neighbours on that cycle adjacent to \( v \) are in the same connected component of \( B_{r/2}(v) - v \). Thus these two neighbours are neighbours of the same slice of the vertex \( v \) in \( G' \). Hence this cycle is also a cycle in \( G' \) and hence in the ball \( B'_{r/2}(w) \). To summarise, all those cycles of length at most \( r \) that generate \( o \) are cycles in \( B'_{r/2}(w) \). In the ball \( B'_{r/2}(w) \) they generate (the edge set of) \( o \). So \( o \) is an eulerian subgraph in \( B'_{r/2}(w) \), and so a cycle as it cannot have a vertex of degree strictly more than two and it is connected. In particular the vertices \( x \) and \( y \) are in the same connected component of the punctured ball \( B'_{r/2}(w) - w \).

It is a direct consequence of Sublemma B.5 that cutting locally commutes. □

**Lemma B.6.** Let \( G \) be a connected graph. Let \( G' \) be obtained from \( G \) by \( r \)-locally cutting all \( r \)-local cutvertices of \( G \). Then \( G' \) is \( r \)-locally 2-connected.

**Proof.** First we remark that the graph \( G' \) is well-defined by Lemma B.4. Let \( v_1, v_2, \ldots, v_n \) be an enumeration of the vertices of \( G \). Here we stress that we include vertices in this enumeration that are not \( r \)-local cutvertices; and cutting them does not change the graph at all. We may assume by Lemma B.4 that we obtain \( G' \) from \( G \) by first cutting \( v_1 \), then \( v_2 \), etc., so that in the final step we cut the vertex \( v_n \). By Observation B.3 all slices of the vertex \( v_n \) are not \( r \)-local cutvertices. As cutting locally commutes by Lemma B.4 we can argue the same for any other ordering of the vertices of \( G \). Hence no vertex of the graph \( G' \) is an \( r \)-local cutvertex. □

**Proof of Theorem B.1.** Let \( r \in \mathbb{N} \cup \{\infty\} \) be a parameter. Let \( G \) be a connected graph. We construct the graph \( H \) from \( G \) by \( r \)-locally cutting all \( r \)-local cutvertices of \( G \). By Lemma B.4 this is well-defined, and the graph \( H \) is \( r \)-locally 2-connected by Lemma B.6.

Let \( S \) be the set of \( r \)-local cutvertices of \( G \). Let \( B \) be the set of connected components of the graph \( H \). We define a bipartite graph with bipartition \((B, S)\), where we add one edge between an \( r \)-local cutvertex \( s \) of \( G \) to a connected component \( k \) of the graph \( H \) for every slice of \( s \) that is contained in \( k \). The map associated to that edge map the singleton subgraph \( s \) to its corresponding slice.
This defines a graph-decomposition of adhesion one and locality $r$ all of whose bags are $r$-locally 2-connected. It is straightforward to check that the underlying graph of that graph-decomposition is the graph $G$. 

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