Markov Traces on the Funar Algebra

S. Yu. Orevkov

1 IMT, Université Paul Sabatier, Toulouse, France. E-mail: stepan.orevkov@math.univ-toulouse.fr
2 Steklov Mathematical Institute, Moscow, Russia

Received: 2 June 2012 / Accepted: 15 January 2016
Published online: 7 May 2016 – © Springer-Verlag Berlin Heidelberg 2016

1. Introduction

Let $B_n$ be the braid group with $n$ strings and $\sigma_1, \ldots, \sigma_{n-1}$ its standard generators. Let $k$ be a commutative ring with $1 \neq 0$. Given $\alpha, \beta \in k$, we define the $k$-algebra $K_n = K_n(\alpha, \beta) = K_n(\alpha, \beta; k)$ as the quotient of the group algebra $kB_n$ by the relations

\[ \sigma_1^3 - \alpha \sigma_1^2 + \beta \sigma_1 - 1 = 0 \]

and

\[
\begin{align*}
y \bar{y} y &= 2\alpha - \beta^2 - (x + y) - (\alpha^2 - \beta)(\bar{x} + \bar{y}) + \beta(xy + yx) + \alpha(\bar{x}\bar{y} + y\bar{x} + \bar{x} y + \bar{y}x) \\
&\quad + (\alpha \beta - 1)(\bar{x}\bar{y} + \bar{y}\bar{x}) - \alpha xyx - (\bar{x}yx + x\bar{y}x + xy\bar{x}) - \beta(\bar{x}\bar{y}x + x\bar{y}x) \\
&\quad + (\alpha - \beta^2)\bar{x}\bar{y}\bar{x},
\end{align*}
\]

where $x, \bar{x}, y, \bar{y}$ in (2) stand for $\sigma_1, \sigma_1^{-1}, \sigma_2, \sigma_2^{-1}$ respectively. Up to a change of the sign of $\beta$ (for the sake of symmetricity), our definition of $K_n$ is equivalent to the definition given by Bellingeri and Funar [1]. Our relation (2) is much shorter than the corresponding relation in [1] (see [1; (2) and Table 1]) because we use $\sigma_i^{-1}$ instead of $\sigma_i^2$. Multiplying (2) by $\sigma_1$ from the left or from the right, and simplifying the result using (1) and the braid group relations, we obtain

\[
\begin{align*}
\bar{x} \bar{y} \bar{x} &= 2\beta - \alpha^2 - (\bar{x} + \bar{y}) - (\beta^2 - \alpha)(x + y) + \alpha(\bar{x}\bar{y} + \bar{y}\bar{x}) + \beta(\bar{x}y + \bar{y}x + x\bar{y} + y\bar{x}) \\
&\quad + (\alpha \beta - 1)(xy + yx) - \beta \bar{x}\bar{y}\bar{x} - (x\bar{y}\bar{x} + \bar{x}y\bar{x} + \bar{x}\bar{y}x) - \alpha(xy\bar{x} + \bar{x}yx) \\
&\quad + (\beta - \alpha^2)xyx.
\end{align*}
\]

Note that (3) is obtained from (2) by swapping $x \leftrightarrow \bar{x}, y \leftrightarrow \bar{y}, \alpha \leftrightarrow \beta$.

Using (1) – (3) together with the braid relations, it is easy to see that $K_n$ are finitely generated $k$-modules. Following [2], we denote the image of $\sigma_i$ in $K_n$ by $s_i$. 
Set $K_\infty = \lim K_n$. Contrary to the case of Hecke or BMW algebras, the morphisms

$K_n \to K_{n+1}$ induced by the standard embeddings $B_n \subset B_{n+1}$ are not injective in general. We say that $t : K_\infty \otimes k[u, v] \to M$ is a Markov trace on $K_\infty$ if $M$ is a $k[u, v]$-module and $t$ is a morphism of $k[u, v]$-modules such that $t(xy) = t(yx), t(xs_n) = ut(x), t(xs_n^{-1}) = vt(x), x, y \in K_n, n = 1, 2, \ldots$.

It is claimed in [1,3] that a nontrivial Markov trace is constructed on $K_n$. About 2004–2005 I indicated a gap in the proof of its well-definedness (see Remark 2.11 below). As it is explained in [2], the gap was really serious: formally, the main result of [3] is wrong though the number of them is much bigger than in [1,3] and the algorithm of computation is much more complicated. Theoretically, this approach allows to compute the universal Markov trace on $K_\infty$, i.e., the projection of $K_\infty(\alpha, \beta; \mathbb{Z}[\alpha, \beta, u, v])$ onto its quotient by the submodule $\bar{R}$ generated by

$$xy - yx, \ xsn - ux, \ xs_n^{-1} - vx, \ x, y \in K_n, n = 1, 2, \ldots$$

The volume of computations is huge, so we performed them only in some cases. In particular, we found $K_\infty(0, 0; A)/\bar{R} = A/I$ where $A = \mathbb{Z}[u, v], I = (16, 4u^2 + 4v, 4u^2 + 4u, u^3 + v^3 + uv - 3)$. Note, that it was checked in [2] that $K_5(0, 0; A)/\bar{R}_5 = A/I$ where $\bar{R}_5$ is the submodule generated by the elements of $K_5$ of the form (4).

In a sense, the results of the present paper can be divided into two independent parts: the “theoretical part” (Theorem 2.4 which provides an algorithm for computing the ideal $I$) and the “computational part” (Corollaries 2.5, 2.6 which present the results of computer-aided computations according to this algorithm, and Sect. 4 where we discuss some properties of the obtained link invariants). The explicit form of the coefficients in the right hand side of (2) and (3) is not really used in the “theoretical part”. Theorem 2.4 can be applied to a quotient of $kB_\infty$ by (1) together with any two relations of the form

$$\sigma_2\sigma_1^{-1}\sigma_2 = \sum_{i=1}^{21} \gamma_i X_i, \quad \sigma_2^{-1}\sigma_1\sigma_2^{-1} = \sum_{i=1}^{21} \gamma_i' X_i, \quad \gamma_i, \gamma_i' \in k,$$

where $X_1, \ldots, X_{21}$ are all the reduced words in $\sigma_1^{\pm 1}, \sigma_2^{\pm 1}$ (including the empty word 1) that do not contain any subword of the form $\sigma_1^{\pm 2}$ or $\sigma_2^{\pm 1}\sigma_1^{\pm 1}\sigma_2^{\pm 1}$. So, if we consider $\alpha, \beta$ and all the $\gamma_i, \gamma_i'$ in (5) as indeterminates and compute the ideal $I$ described in Theorem 2.4, then we obtain the universal Markov trace on a cubic Hecke algebra that can be specialized to both Funar and BMW algebras. However, the required computations seem to exceed the capacity of any computer. On the other hand, if one chooses a random specialization of the coefficients $\gamma_i, \gamma_i'$ in (5), then the quotient $A/I$ might well be trivial.

So, the explicit form of (2) and (3) is important for the “computational part” of the present paper.

2. Definitions and Statement of Results

2.1. $K$-reductions. Let $F_n^+$ be the free monoid on generators $x_1^{\pm 1}, \ldots, x_{n-1}^{\pm 1}$ (the set of all not necessarily reduced words in $x_1^{\pm 1}$) and $F_\infty^+ = \bigcup F_n^+$. We denote the empty word by 1. Let $kF_n^+$ and $kF_\infty^+$ be the corresponding free associative algebras over $k$ (as $k$-modules, they are freely generated by $F_n^+$ and by $F_\infty^+$ respectively).

We call basic replacements the pairs $(U, V)$ with $U \in F_\infty^+, V \in kF_\infty^+$ (which we denote by $U \to V$) from the following list:
Markov Traces on the Funar Algebra 373

Let \( \cdots \) Then Part (d) of Proposition 2.1 can be stated as follows: each element of \( X \) is an auxiliary weight \( \pi \) distinct elements of \( x_i \) denote \( x_i \). An element of \( F \) is \( \epsilon \)-reduced. The following statement is easy and we omit its proof.

An elementary \( K \)-reduction of a monomial is \( A U B \rightarrow AVB \) where \( A U B \in F^+ \) and \( U \rightarrow V \) is a basic replacement. An elementary \( K \)-reduction of an element of \( k F^+_{\infty} \) is \( \sum_{j=1}^m c_j W_j \rightarrow c_1 W_1' + \sum_{j=2}^m c_j W_j \) where \( c_1, \ldots, c_m \in k \), \( W_1, \ldots, W_m \) are pairwise distinct elements of \( F^+_{\infty} \), and \( W_1 \rightarrow W_1' \) is an elementary \( K \)-reduction of a monomial.

An element of \( F^+_{\infty} \) (resp. of \( k F^+_{\infty} \)) is \( K \)-reduced if no \( K \)-reduction can be applied to it. We denote the set of such elements by \( F^+_{\text{red}} \) (resp. \( k F^+_{\infty} \)). We set also \( F^+_{\text{red}} = F^+_{\infty} \cap F^+_{\text{red}} \) and \( k F^+_{\text{red}} = k F^+_{\infty} \cap k F^+_{\text{red}} \). Then \( k F^+_{\text{red}} \) is a submodule (not a subalgebra) of \( k F^+_{\infty} \). We denote \( \pi : k F^+_{\infty} \rightarrow K_{\infty} \) and \( \pi_n : k F^+_n \rightarrow K_n \) the morphisms of \( k \)-algebras induced by \( x_i \rightarrow s_i \).

We say that an element \( X \) of \( F^+_{\infty} \) is \( \text{almost} \) \( K \)-reduced if there exists a sequence \( X = X_1 \rightarrow X_2 \rightarrow \cdots \rightarrow X_m \) of elementary \( K \)-reductions of type (viii) such that \( X_m \) is \( K \)-reduced.

For \( X = x_{i_1}^{e_1} \cdots x_{i_m}^{e_m} \in F^+_{\infty} \), \( e_j = \pm 1 \), we define the weight \( \text{wt} \ X = \sum j \ i_j \) and the auxiliary weight \( \text{wt}' \ X = \sum j \ i_j \). It is clear that the set of all monomials of a given weight is finite. For \( X \in k F^+_{\infty} \) we set \( \text{wt} \ X = \max \ i_j \ X_i \) if \( X = \sum_i c_i X_i \) with \( c_i \in k \), \( c_i \neq 0 \), and \( X_1, X_2, \ldots \) pairwise distinct elements of \( F^+_{\infty} \).

The following statement is easy and we omit its proof.

**Proposition 2.1.** (a) If \( X \rightarrow X' \) is an elementary \( K \)-reduction, then \( \pi(X) = \pi(X') \) and \( \text{wt} \ X \geq \text{wt} \ X' \). If, moreover, \( X \) is a monomial, then \( \text{wt} \ X = \text{wt} \ X' \) if and only if \( X \rightarrow X' \) is a \( K \)-reduction of type (viii) and in this case we have \( \text{wt}'(X) < \text{wt}'(X') \).

(b) \( \pi(F^+_{\text{red}}) \) generates \( K_{\infty} \) as a \( k \)-module.

(c) \( k F^+_{\text{red}} \) is a free \( k \)-module and \( F^+_{\text{red}} \) is a free base of \( k F^+_{\infty} \).

(d) \( F^+_{\infty} \) is the set of all words \( X_1 X_2 \cdots X_m \) where \( X_v = x_{i_v}^{v+1} x_{i_v-1}^{v+1} \cdots x_{j_v}^{v+1} \), \( i_v \geq j_v \) \((1 \leq v \leq m) \), \( i_1 < \cdots < i_m \), and all the signs are mutually independent.

(e) (Proven in [3]) \( \pi_3 \) is an isomorphism of \( k \)-modules \( k F^+_{3} \) and \( K_3 \).

**Remark 2.2.** Let

\[ S_{i,j} = \{ x_i^{\pm 1} x_{i-1}^{\pm 1} \cdots x_j^{\pm 1} \} \quad \text{and} \quad S_i = \{ 1 \} \cup S_{i,i} \cup S_{i,i-1} \cup \cdots \cup S_{i,1}. \tag{6} \]

Then Part (d) of Proposition 2.1 can be stated as follows: each element of \( F^+_{\text{red}} \) can be represented in a unique way as a product \( X_1 X_2 \cdots X_{n-1} \) with \( X_i \in S_i \). Since \(|S_i| = 1 + 2 + \cdots + 2^i = 2^{i+1} - 1 \), we obtain \(|F^+_{\text{red}}| = \prod_{i=1}^n (2^i - 1) \), in particular,

\[ |F^+_{2}^{\text{red}}| = 3, \quad |F^+_{3}^{\text{red}}| = 3 \cdot 7 = 21, \]
\[ |F^+_{4}^{\text{red}}| = 3 \cdot 7 \cdot 15 = 315, \quad |F^+_{5}^{\text{red}}| = 3 \cdot 7 \cdot 15 \cdot 31 = 9765. \]
Remark 2.3. In basic replacements (vii), it is enough to consider only words $W$ belonging to $S_{i-1}$ (see (6) for the definition of $S_{i-1}$).

We define a $k$-linear mapping $\mathbf{r} : kF^+ \to kF_{\infty}^{\text{red}}$ as follows. For each $X \in F^+_n$, we fix an arbitrary sequence of elementary $K$-reductions $X = X_1 \to X_2 \to \cdots \to X_m \in kF_{\infty}^{\text{red}}$ and we set $\mathbf{r}(X) = X_m$. Then we extend the mapping to $kF^+_\infty$ by linearity.

2.2. Markov traces. Let $A = k[u, v]$ and $AK_n = K_n(\alpha, \beta; A)$. Let $M = M(\alpha, \beta; k)$ be the quotient of $AK_\infty$ by the relations (4) and let $t : AK_\infty \to M$ be the quotient map. We call $t$ the universal Markov trace on $K_\infty$ over $k$. It is indeed universal in the sense that any Markov trace on $K_\infty(\alpha, \beta; A)$ with values in an $A$-module $M'$ is $f \circ t$ for some $f \in \text{Hom}_A(M, M')$.

We define $A$-linear mappings $\tau_n : AF_n^+ \to AF_{n-1}^{\text{red}}$ called Markov reductions as follows. By Proposition 2.1(d), we have $F_n^{\text{red}} \subset F_{n-1}^{\text{red}} \cup (F_{n-1}^{\text{red}}x_{n-1}^{-1}F_{n-1}^{\text{red}})$. So, we set $\tau_n(X) = X$, $\tau_n(Xx_{n-1}^{-1}Y) = \mathbf{r}(XY)$, and $\tau_n(Xx_{n-1}^{-1}Y) = \mathbf{r}(XY)$ for $X, Y \in F_{n-1}^{\text{red}}$ and then we extend $\tau_n$ to $AF_n^{\text{red}}$ by linearity and to $AF_n^+$ by setting $\tau_n(X) = \tau_n(\mathbf{r}(X))$. Finally, we define $\tau : F_{\infty}^{\text{red}} \to AF_1^+$ is $A$ by setting $\tau(X) = \tau_\circ \cdots \circ \tau_n(X)$ for $X \in AF_n^+$.

By definition of $t$ and $\tau$, we have $t(\pi(X)) = t(\pi(\tau(X)))$, thus $M = t(K_\infty)$ is generated by $t(1)$. Let $I = I(\alpha, \beta; k)$ be the annihilator of $M$. Thus we have $M \cong A/I$.

2.3. Statement of the main result. Let $sh^n : AF_{\infty}^+ \to AF_{\infty}^+$, $n \in \mathbb{Z}$, be the $A$-algebra endomorphism (the $n$-shift) induced by

$$sh^n x_i = \begin{cases} x_{i+n}, & i + n > 0, \\ 0, & i + n \leq 0. \end{cases}$$

We set $sh = sh^1$.

For $X \in F^+_3$, we define $\rho_X \in \text{End}_A(AF_{4}^{\text{red}})$ by setting $\rho_X(Y) = \tau_5(XshY)$. Let $J_4 = J_4(\alpha, \beta; k)$ be the minimal submodule of $AF_4^{\text{red}}$ satisfying the following properties (recall that the sets $S_{i,j}$ and $S_i$ are defined in (6)):

(J1) $\mathbf{r}(r(X_3X_2X_1)) - r(X_3r(X_2X_1)) \in J_4$ for any $X_j \in sh^{-j}S_j \!\backslash\!\{1\}$, $j = 1, 2, 3$;

(J2) $\rho_X(J_4) \subset J_4$ for any $X \in S_4$.

In a similar way we define a module $L$. Let $N = AF_2^{\text{red}} \otimes_A AF_2^{\text{red}}$. We define $A$-linear mappings $\tau_N : N \to N$ and $\rho_\delta : N \to N$, $\delta = (\delta_1, \delta_2) \in \{-1, 0, 1\}^2$, by setting for any $Y = x_1^{\varepsilon_1} \otimes x_1^{\varepsilon_2}$ ($\varepsilon_1, \varepsilon_2 \in \{-1, 0, 1\}$)

$$\tau_N(Y) = \tau(x_1^{\varepsilon_1}x_1^{\varepsilon_2}), \quad \rho_\delta(Y) = x_1^{\delta_1} \otimes \tau_3(x_2^{\varepsilon_1}x_1^{\delta_2}x_2^{\varepsilon_2})$$

and we define $L$ as the minimal submodule of $N$ satisfying the conditions:

(L1) $\tau_3(x_2^{\varepsilon_1}x_1^{\varepsilon_2}x_2^{\varepsilon_3}) \otimes x_1^{\varepsilon_4} - x_1^{\varepsilon_2} \otimes \tau_3(x_2^{\varepsilon_1}x_1^{\varepsilon_2}x_2^{\varepsilon_3}) \in L$ for any $\varepsilon_1, \varepsilon_3 \in \{-1, 1\}$ and for any $\varepsilon_2, \varepsilon_4 \in \{-1, 0, 1\}$;

(L2) $\rho_\delta(L) \subset L$ for any $\delta \in \{-1, 0, 1\}^2$.

Theorem 2.4 (Main Theorem). $I = \tau(J_4) + \tau_N(L)$.

The theorem is proved in Sect. 3 (see Proposition 3.2 for the inclusion “$\supset$” and Proposition 3.7 for the reverse inclusion).
This result allows (at least theoretically) to compute $I$. Indeed, we start with the $A$-module $J_4^{(0)}$ generated by the elements in (J1) and we set $J_4^{(i+1)} = \sum_{X \in S_k} \rho_X(J_4^{(i)})$. Then the Gröbner bases $G^{(i)}$ of the modules $J_4^{(i)}$ can be computed recursively using the fact that $J_4^{(i+1)}$ is the module generated by $\bigcup_{X \in S_k} \rho_X(G^{(i)})$. So, we construct an increasing sequence of submodules $J_4^{(0)} \subset J_4^{(1)} \subset \ldots$. Since the subring of $A$ generated by $\alpha, \beta, u, v$ is noetherian, there exists $m_0$ such that $J_4^{(m_0)} = J_4^{(m_0+1)} = \ldots (m_0$ is determined by the condition $G^{(m_0)} = G^{(m_0+1)}$). The module $L$ can be computed in a similar way as the limit of $L^{(0)} \subset L^{(1)} \subset \ldots$ where $L^{(0)}$ is generated by the elements in (L1) and $L^{(i+1)} = \sum_{\beta} \rho_{\beta}(L^{(i)})$.

Performing in practice this computation for $\alpha = \beta = 0$, $k = \mathbb{Z}$ (the case considered in [2,3]) and in some other special cases, we obtain the following results. To compute the Gröbner bases, we used Singular 3-1-3 and Macaulay2 software; see details on the web page [6].

**Corollary 2.5.** $I(0, 0; \mathbb{Z}) = (16, 4u^2 + 4v, 4v^2 + 4u, u^3 + v^3 + uv - 3)$.

**Corollary 2.6.** Let $k = k[\alpha]$ for a ring $k$ specified below and let $I = I(\alpha, 0, k)$. Let $G$ be the reduced Gröbner base of $I$ with respect to the lexicographic order such that $v > u > \alpha$ (the Gröbner bases in (e) – (g) also correspond to this order). Let

$$
\begin{align*}
f_1 &= \gamma_1 \gamma_2 \gamma_3, \quad \text{where} \quad \gamma_1 = \alpha^3 + 8, \quad \gamma_2 = 2\alpha^3 + 1, \quad \gamma_3 = 3\alpha^3 + 8, \\
f_2 &= \gamma_1 \gamma_3 (u - \alpha), \\
f_3 &= \gamma_3 (6u^3 - 3\alpha^2 u + \alpha^3 + 2), \\
f_4 &= 336u^4 - 792\alpha u^3 + 12(15\alpha^3 + 106)\alpha^2 u^2 + 6(141\alpha^3 + 544)u \\
&\quad - 114\alpha^7 - 1405\alpha^6 - 3152\alpha, \\
f_5 &= 288u + 336\alpha^2 u^3 + 72(3\alpha^3 + 28)u^2 - 48(9\alpha^3 + 44)\alpha u - 6\alpha^8 + 53\alpha^5 + 472\alpha^2.
\end{align*}
$$

(a) If $k = \mathbb{F}_2$, then $G = \{a^4, a^2(u^3 + 1 + a^2u), a^2(v + u^2), v^3 + u^3 + uv + 1 + \alpha(uv^2 + v + auv)\}$, hence $\dim_k(A/I) = \infty$ and the base of $A/I$ over $k$ is $\{a^lu^mv | l \leq 1, m \leq 2\} \cup \{a^lu^mv | 2 \leq l \leq 3, m \leq 2\}.$

(b) If $k = \mathbb{F}_3$, then $G = \{\alpha^3 - 1, (u^2 - \alpha^2)(u^2 - \alpha u - \alpha^2), v + u^2\}$, hence $\dim_k(A/I) = 12.$

(c) If $k = \mathbb{Q}$ or $k = \mathbb{F}_p$ for a prime $p$ in the range $5 \leq p \leq 599$, $p \neq 37$ (conjecturally, for any prime $p \not\in \{2, 3, 37\}$), then $I = (f_1, \ldots, f_5)$. We have $G = \{f_1, \ldots, f_5\}$ and hence $\dim_k(A/I) = 24$ except the case $k = \mathbb{F}_7$ where we have $G = \{f_1, f_2, u^3 + 2uv(\gamma_1 u + \gamma_3 \alpha) + 3\alpha^6 - 3\alpha^3, f_3\}$ and hence $\dim_k(A/I) = 21.$

(d) If $k = \mathbb{F}_{37}$, then we have $I = (f_1, f_2, f_3, (u + 7\alpha)f_4, f_5 - 14\alpha f_4)$ and $G = \{f_1, f_2, f_3, f_4, f_5\}$ where

$$
\begin{align*}
f_{4,37} &= u^5 + 2\alpha u^4 + 7\alpha^2 u^3 - 9(\alpha^3 - 1)u^2 + (6\alpha^3 + 2)\alpha u - 12\alpha^8 + 9\alpha^5, \\
f_{5,37} &= v - 4\alpha u^4 + 15\alpha^2 u^3 - (14\alpha^3 + 16)u^2 - (\alpha^3 + 18)\alpha u - 4\alpha^8 + 2\alpha^5 - 2\alpha^2
\end{align*}
$$

and hence $\dim_k(A/I) = 27.$

(e) If $k = \mathbb{Z}/32\mathbb{Z}$, then $\{8\alpha^5, 4\alpha^4, 8\alpha^5 + 8\alpha^2, 16\alpha u^3 + 16\alpha^2 u^2 + 4\alpha^3 u + 2\alpha^4, 8\alpha^2 u^3 - 12\alpha^3 u + 8\alpha^2, 2\alpha^3 u + 16u^3 + 16\alpha u + 6\alpha^3 + 16, 8\alpha^4 u^3 + 8\alpha^3 u + 4\alpha^5 u^2 + 16\alpha^2 u^2 + \alpha^4 + 8\alpha, 16u^4 - 8\alpha u^3 + 8\alpha^2 u^2 - 6\alpha^3 u + \alpha^4 + 16\alpha, 4\alpha^2 u^4 + 7\alpha^3 u^3 - 8\alpha^3 - 2\alpha^4 u^2 - 4\alpha^2 u - 9\alpha^3 - 8, 8\alpha u^6 + 16\alpha^5 + \alpha^4 u^2 + 8\alpha^2 u + 6\alpha^3, \alpha^3 u^5 - 8u^5 - 8\alpha u^4 - 4\alpha^2 u^3 -
$$
5a^3u^2 + 8u^2 - 13a^4u + 16au - 4a^2, 4au^6 - 9a^2u^4 + 8u^4 + 3a^3u + 8u - 3a^4 + 4a, 16v + 8au^4 + 12a^2u^3 + 16u^2 + 3a^4u - 12a^2, 8au + 10a^4u^2 - 8a^2u + 16a^2u + 2a^2 + 16, 4a^2v - 12a^2u^2 + 12a^3u - 13a^4 - 8a, \alpha^3v - 8v + \alpha^3u^2 - 8u^2, 2a^2uv + 8v + 4au^4 - 12a^2u^3 - 4a^4u^2 - 8u^2 + 13a^4u + 12au - 2a^2, 4au^3v + 12av - 12au^5 + 13a^3u^3 + 8a^2u - 10a^4u^2 + 12a^2u + 3a^3 - 8, 4v^2 - 2a^2uv + 3a^2v + \alpha^2u^2 + 4u - 2a, 2a^2v + 15a^2uv - 4v + 15a^2u^2 - 4u^2 + 2au + 15a^2, \alpha^2v^2 + 4uv - 2a - 4a^3u^2 + 15a^4u^2 - 2au^2 - 7a^2u + 13a^3 - 4, v^3 - 15auv^2 - 3uv - 7av + u^3 + 6a^2u + \alpha^2u + 2a^3 - 15}\) is a Gröbner base of \(I\).

(f) If \(k = \mathbb{Z}/3\mathbb{Z}\) for \(r \leq 6\) (conjecturally, for any \(r\)), then \(I = (F)\) for \(F = \{f_1, f_2, f_3, 2u^4 + au^3 - \alpha^2u^2 + 2u - \alpha, v + u^3\}\). If, moreover, \(r \geq 2\), then \(F \cup \{3^{r-1}(\alpha^3 - 1)\}\) is a Gröbner base of \(I\), hence \(|A/I| = 3^{12r}\). By (b), this implies that \(A/I\) is a free \(k\)-module of rank 12.

(g) If \(k = \mathbb{Z}/37p\mathbb{Z}\) for a prime \(p \leq 67\), \(p \not\in \{2, 3, 7\}\) (conjecturally, for any prime \(p \not\in \{2, 3, 7\}\)), then \(\{f_1, f_2, f_3, 37f_4, (u + 7a) f_4, f_5 - 14a f_4\}\) is a Gröbner base of \(I\).

Remark 2.8. The Markov trace \(t\) over \(k\) defines an invariant of oriented links \(P(L) = P_{a, b, k}(L) = u^{(1-n-e)/2} v^{(1+n-e)/2} t(b) \in k[u^{\pm 1/2}, v^{\pm 1/2}]/I(\alpha, \beta; k)\) where \(b\) is a representation of a link \(L\) by a braid with \(n\) strings and \(e\) is the sum of exponents of \(b\). It is shown in [2] that \(P_{0, 0; \mathbb{Z}/4\mathbb{Z}}\) is determined by the HOMFLY polynomial (see Sect. 4.5 below). In the first arxiv version of this paper (arxiv:1206.0765v1) it was claimed that \(P_{a, 0; \mathbb{Q}[\alpha]}\) and \(P_{a, 0; \mathbb{F}_{27}[\alpha]}\) detect the chirality of the knot 10\(_{71}\). Unfortunately, this is not so. However, \(P_{a, 0; \mathbb{Z}/17[\alpha]}\) detects the chirality of the knots 10\(_{48}\), 10\(_{41}\) and it distinguishes many other pairs of knots with equal HOMFLY polynomials. In the cases computed so far, the invariants \(P_{a, 0; k}\) do not distinguish any pair of knots up to 11 crossings with equal Kauffman polynomials. It is shown in Proposition 4.1(f) below that \(P_{a, \beta; k}\) never distinguishes mutant links.

Remark 2.9. According to [4; Theorem 1.5], we have \(I(\alpha, \beta; \mathbb{Q}(\alpha, \beta)) = 1\). This is equivalent to say that \(I(\alpha, \beta; \mathbb{Q}(\alpha, \beta))\) contains a nonzero polynomial in \(\alpha\) and \(\beta\) (observe that the same phenomenon takes place in Corollaries 2.5 and 2.6).

Remark 2.10. Our main motivation for studying quotients of cubic Hecke algebras and their Markov traces stems from the Markov-type theorem for transversal links proved in [7]. If \(u, v\) were zero divisors in \(A/I\) then one might possibly obtain transversal links invariants which distinguish isotopic links with the same Bennequin invariants. In the cases computed so far, however, \(u\) and \(v\) are not zero divisors.

We say that \(t \in \text{Hom}_{k[u]}(K) \otimes k[u], M)\) is a semi-Markov trace on \(K\) if \(t(xy) = t(yx)\) and \(t(xn) = ut(x)\) for \(x, y \in K, n > 0\). Due to [7], any semi-Markov trace provides an invariant of transversal links. In [5], the methods of the present paper are adapted for studying the universal semi-Markov trace on \(K\).

Remark 2.11. The main error in [3] (which was repeated also in [1]) is that the modules \(J_4^{(0)}\) and \(L^{(0)}\) were considered instead of \(J_4\) and \(L\).

3. Proof of Main Theorem

The idea of the proof is as follows. Given \(X \in AF_n^+\), the element of \(A\) representing \(\tau(X)\) is computed by successive reductions (i)−(viii), Markov relations, and cyclic permutations. We have to find the minimal possible ideal \(I\) such that the result does not depend on
the order of these operations. It is easy to observe (though not so easy to formalize this observation) that the main sources of the ambiguity are as follows. First, the reduction of a subword \( W = x_3^{e_1}x_2^{e_2}x_1^{e_3}x_3^{e_4}x_2^{e_5}x_3^{e_6} \) can be started either with \( x_3^{e_4}x_2^{e_5}x_3^{e_6} \) or (after commuting \( x_1^{e_3} \) and \( x_3^{e_4} \)) with \( x_3^{e_1}x_2^{e_2}x_3^{e_4} \). Thus \( \tau(J_4^{(0)}) \) should be included in \( I \). Second, the reduction of a word \( x_3^{e_1}x_2^{e_2}x_1^{e_3} \) can be started either by \( x_3^{e_1}x_1^{e_2}x_3^{e_2} \) or (after a cyclic permutation) by \( x_3^{e_3}x_1^{e_1}x_2^{e_2} \). Thus \( \tau_N(L^{(0)}) \) should be included into \( I \).

Let us focus on the first case. So, let \( Y \) be an element of \( J_4^{(0)} \). Then, for any \( X \) and any \( n \), we should have \( \tau(X \text{sh}^n Y) \in I \). In particular, for any \( X \in F_5^+ \), we should have \( \tau(\rho_X(Y)) \in I \), i.e., \( \tau(J_4^{(1)}) \subset I \). By iterating this process, we conclude that \( \tau(J_4) \subset I \). Similarly, \( \tau_N(L) \subset I \). This is the easy part of the proof which is formally exposed in Sect. 3.1.

The difficult part of the proof (formally exposed in Sect. 3.2) consists in checking that any choice of the reduction process leads to the same result modulo \( I \). We use induction on the weight (see the definition of the weight function \( \text{wt} \) in Sect. 2.1). As we pointed out above, there are two main sources of the ambiguity. Again, we discuss here only the first one. So, we have to prove that \( \tau(X \text{sh}^n Y) \in I \) for any \( X \) when \( Y = J_4 \).

By additivity, we may assume that \( X \) is a monomial. If any reduction can be applied to \( X \), then we do it and we use the induction hypothesis. So, we may assume that \( X = AF_{n+4}^+ \). If \( X = X_1X_2 \) where \( X_2 \) commutes with \( \text{sh}^n Y \), then we replace \( X \text{sh}^n Y \) by \( X_2 \text{sh}^n Y \) in \( Y \). Thus we arrive to the case when \( X = X' \text{sh}^{n-1}X_1 \) with \( X' \in F_{n+3}^+ \), \( X_1 \in F_{5}^+ \), and we apply the induction hypothesis to \( X' \text{sh}^{n-1}Y' \) where \( Y' = \rho_{X_1}(Y) \).

3.1. Easy part: \( \tau(J_4) + \tau_N(L) \subset I \). Let \( J_4^{(0)} \subset J_4^{(1)} \subset \ldots \) and \( L^{(0)} \subset L^{(1)} \subset \ldots \) be as defined in Sect. 2.3.

For \( n \geq 4 \) and \( a \in AK_n \), we define \( t_{n,a} \in \text{Hom}_A(F_4^{\text{red}}, A) \) by setting \( t_{n,a}(X) = t(a \pi(\text{sh}^{n-4} X)) \). Similarly, for \( n \geq 1 \) and \( a, b \in AK_n \), we define \( t_{n,a,b} \in \text{Hom}_A(N, A) \) by setting \( t_{n,a,b}(X \otimes Y) = t(\pi(\text{sh}^{n-4} X) a \pi(\text{sh}^{n-1} Y) b) \).

Lemma 3.1. (a) \( J_4 \subset \ker t_{n,a} \) for any \( n \geq 4 \) and any \( a \in K_n \).
(b) \( L \subset \ker t_{n,a,b} \) for any \( n \geq 1 \) and any \( a, b \in K_n \).

Proof. We prove by induction that (a) \( J_4^{(i)} \subset \ker t_{n,a} \) and (b) \( L^{(i)} \subset \ker t_{n,a,b} \). For \( i = 0 \), the statement is evident. Suppose that it is true for \( i - 1 \) and let us prove it for \( i \). Note that we have
\[
t(a \pi(\text{sh}^p \tau_{n-p}(X)) b) = t(a \pi(\text{sh}^p X) b) \quad \text{for} \quad a, b \in K_{n-1}, \quad X \in AF_n^+(7)
\]

(a) It is enough to check that \( \rho_X(Y) \in \ker t_{n,a} \) for any \( Y \in J_4^{(i-1)}, X \in S_4, n \geq 4, a \in K_n \). Indeed,
\[
t_{n,a}(\rho_X(Y)) = t(a \pi(\text{sh}^{n-4} \rho_X(Y))) \quad \text{by definition of} \ t_{n,a}
\]
\[
= t(a \pi(\text{sh}^{n-4} \tau_5(X \text{sh} Y))) \quad \text{by definition of} \ \rho_X
\]
\[
= t(a \pi((\text{sh}^{n-4} X)(\text{sh}^{n-3} Y))) \quad \text{by (7)}
\]
\[
= t_{n+1,a'}(Y) \quad \text{for} \ a' = a \pi(\text{sh}^{n-4} X) \in K_{n+1}
\]
\[
= 0 \quad \text{by the induction hypothesis}
\]
(b) It is enough to check that \( \rho_5(Y) \in \ker t_{n,a,b} \) for any \( Y \in L^{(i-1)}, \) \( \delta = (\delta_1, \delta_2) \in \{-1, 0, 1\}^2, \) \( n \geq 1, \) \( a, b \in K_n. \) Indeed, let \( Y = \sum_j c_j x_1^{\epsilon_1(j)} x_1^{\epsilon_2(j)}. \) Then

\[
t_{n,a,b}(\rho_5(Y)) = t_{n,a,b}\left( \sum_j c_j x_1^{\delta_1} \otimes \tau_3(x_2^{\epsilon_1(j)} x_1^{\delta_2} x_2^{\epsilon_2(j)}) \right)
\]
def. of \( \rho_5 \)
\[
= \sum_j c_j \left( \pi(\sh^{n-1} x_1^{\delta_1}) a \pi(\sh^{n-1} \tau_3(x_2^{\epsilon_1(j)} x_1^{\delta_2} x_2^{\epsilon_2(j)})) b \right) \]
def. of \( t_{n,a,b} \)
\[
= \sum_j c_j \left( \delta_1 \s_{n+1}^{\epsilon_1(j)} \s_n^{\delta_2} \s_{n+1}^{\epsilon_2(j)} b \right) \]
by (7)
\[
= \sum_j c_j \left( \delta_1 \s_n^{\epsilon_1(j)} \s_n^{\delta_2} \s_{n+1}^{\epsilon_2(j)} b \right) 
\]
t\( (xy) = t(yx) \)
\[
= \sum_j c_j \left( \pi(\sh^{n} x_1^{\epsilon_1(j)}) \s_n^{\delta_2} \pi(\sh^{n} x_1^{\epsilon_2(j)} b \s_n^{\delta_1} a) \right)
\]
by induction hypothesis

**Proposition 3.2.** \( \tau(J_4) + \tau_N(L) \subset I. \)

**Proof.** Indeed, by Lemma 3.1, we have \( t(\tau(X)) = t_{4,1}(X) = 0 \) for any \( X \in J_4 \) and \( t(\tau_N(X)) = t_{1,1,1}(X) = 0 \) for any \( X \in L. \) Thus \( \tau(J_4) + \tau_N(L) \subset \ker(t|_A) = I. \) □

### 3.2. Difficult part: \( I \subset \tau(J_4) + \tau_N(L). \)

Let, as above, \( \tilde{R} \) be the submodule of \( K_\infty \) generated by the elements (4). Set \( R = \pi^{-1}(\tilde{R}). \) Then we have \( I = A \cap \tilde{R} = A \cap R. \) Let \( \wt : AF_+^* \rightarrow \mathbb{Z}_{\geq 0} \) be the weight function defined in Sect. 2.1. It defines a filtration on \( AF_+^* \), namely, \( A = AF_+^* \subset AF_+^* \subset AF_+^* \subset \ldots \) where \( AF_+^* = \{X \in AF_+^* | \wt X \leq w \}. \)

We shall work with the following set of generators \( \mathcal{R} = \mathcal{R}_T \cup \mathcal{R}_M \cup \mathcal{R}_N \cup \mathcal{R}_H \) of \( R \) as an \( A \)-module (we set here \( u_+ = u, u_- = v \)):

- \( \mathcal{R}_T = \{XY - YX | X, Y \in F_\infty^+\}, \) trace relations;
- \( \mathcal{R}_M = \{x_{n+1}^\pm X - u_{n}^\pm X | X \in F_\infty^+, n \geq 1\}, \) Markov relations;
- \( \mathcal{R}_N = \{UX - VX | X, U \in F_\infty^+, U \xrightarrow{(i)-(vi)} V\}, \) nonhomogeneous \( K \)-relations;
- \( \mathcal{R}_H = \{UX - VX | X, U \in F_\infty^+, U \xrightarrow{(wi)} V\}, \) homogeneous \( K \)-relations.

Let \( \mathcal{R}_w = \mathcal{R} \cap AF_+^*[w], \) let \( R[w] \) be the \( A \)-submodule of \( R \) generated by \( \mathcal{R}_w, \) and let \( H \) be the submodule generated by \( \mathcal{R}_T \cup \mathcal{R}_H. \) Note, that by Proposition 2.1(a) we have

\[
X \equiv r(X) \equiv \tau_n(X) \equiv \tau(X) \mod R_{[w]} \quad \text{for } X \in AF_\infty^+.
\]

In what follows, a notation like \( X_1 \equiv X_2 \equiv X_3 \equiv \ldots \) means that \( X_i = X_{i+1} \mod R_{[w]} \) and \( \wt X_{i+1} \leq \wt X_i, \) in particular, in this case we always have \( X_1 \equiv X_2 \equiv X_3 \equiv \ldots \mod R_{[w]} \).

**Lemma 3.3.** Let \( Z = X \sh^{n-4} Y \) for \( X \in AF_\infty^+, Y \in J_4 \cap \sh^{4-n} AF_\infty^+, n \geq 1. \) Then \( Z \in R[w] + \tau(J_4) \) where \( w = \wt Z. \)
Proof. We denote $\text{sh}^{n-4} Y$ by $Y_n$. If $X \in AF_m^+$ with $m > n$, then
\[ XY_n \equiv \tau_m(X)Y_n \equiv \tau_{m-1}(\tau_m(X))Y_n \equiv \cdots \equiv \tau_{n+1} \circ \cdots \circ \tau_{m-1} \circ \tau_m(X)Y_n, \]
hence it is enough to prove the statement of the lemma under the additional hypothesis $X \in AF_n^+$. We prove it by induction.

If $n = 1$, then $X \in AF_1^+ = A$ and $Y \in J_4 \cap \text{sh}^3 AF_\infty^+ = J_4 \cap A \subset \tau(J_4)$, so the statement is trivial.

Suppose that $n \geq 2$, the statement is true for $n - 1$, and let us prove it for $n$. By linearity, it is enough to consider the case when $X \in F_n^\text{red}$ and since $X \equiv \text{r}(X)$, we may assume that $X \in F_n^\text{red}$. Let $X = X_1X_2 \cdots X_{n-1}, X_i \in S_t$ (see Remark 2.2). We have $X_{n-1} = (\text{sh}^{n-5} X'_4)X''_{n-5}$ with $X'_4 \in S_3 \cap \text{sh}^{5-n} AF_\infty^+$ and $X''_{n-5} \in S_{n-5}$ (we assume here that $S_t = \{1\}$ when $i \leq 0$). Note that $Y_n$ may involve only $x_{n-4+i}$, $i = 1, 2, 3$, whereas $X''_{n-5}$ may involve only $x_{i \pm 1}$, $i \leq n - 5$, hence they commute. Therefore, denoting $X_1 \cdots X_n \times_{n-2}$, we obtain
\[ Z = X''_{n-2}(\text{sh}^{n-5} X'_4)X''_{n-5}(\text{sh}^{n-4} Y) \equiv X''_{n-2}(\text{sh}^{n-5} X'_4)(\text{sh}^{n-4} Y)X''_{n-5} \]
\[ \equiv X''_{n-5}X''_{n-2}(\text{sh}^{n-5} X'_4)(\text{sh}^{n-4} Y) = X''_{n-5}X''_{n-2} \text{sh}^{n-5}(X'_4 \text{sh} Y) \]
\[ \equiv X''_{n-5}X''_{n-2} \text{sh}^{n-5}(\tau_5(X'_4 \text{sh} Y)) = X' \text{sh}^{n-5} Y' \]
where $X' = X''_{n-5}X''_{n-2} \in AF_1^+$ and $Y' = \tau_5(X'_4 \text{sh} Y) = \rho(X'_4 \text{sh} Y) \in J_4$.

To complete the proof, it remains to check that $Y' \in \text{sh}^{5-n} AF_\infty^+$. Indeed, we have $X'_4 \in \text{sh}^{5-n} AF_\infty^+, Y \in \text{sh}^{4-n} AF_\infty^+$, hence $Y \in \text{sh}^{5-n} AF_\infty^+$ and we obtain $X'_4 \text{sh} Y \in \text{sh}^{5-n} AF_\infty^+$ whence $Y' = \tau_5(X'_4 \text{sh} Y) \in \text{sh}^{5-n} AF_\infty^+$. $\square$

The next lemma is similar. For $n \geq 1$ and $X_1, X_2 \in AF_n^+$ we define $\varphi_{n,X_1,X_2} \in \text{Hom}_A(N, AF_n^+)$ by setting $\varphi_{n,X_1,X_2}(Y_1 \otimes Y_2) = X_1(\text{sh}^{n-1} Y_1)X_2(\text{sh}^{n-1} Y_2)$.

Lemma 3.4. Let $Z = \varphi_{n,X_1,X_2}(Y)$ for $n \geq 1$, $X_1, X_2 \in AF_n^+$, $Y \in L$. Then $Z \in R_{[w]} + \tau_N(L)$ where $w = \text{wt} Z$.

Proof. It is enough to consider the case when $X_1, X_2 \in F_n^\text{red}$. Then there exist $X'_1, X''_1 \in F_{n-1}^\text{red}$ and $\delta_i \in \{-1, 0, 1\}$ such that $X_i = X'_i x_{i-1} X''_i$ ($i = 1, 2$). Let
\[ Y = \sum_j c_j x_{1}^{\delta_1(j)} \otimes x_2^{\delta_2(j)} \tag{9} \]

Then we have
\[ Z = \sum_j c_j X_1 x_{n-1}^{\delta_1(j)} X_2 x_{n-1}^{\delta_2(j)} = \sum_j c_j X'_1 x_{n-1}^{\delta_1(j)} X'_2 x_{n-1}^{\delta_2(j)} \]
\[ \equiv \sum_j c_j X'_2 X'_1 x_{n-1}^{\delta_1(j)} X'_2 x_{n-1}^{\delta_2(j)} \]
\[ \equiv \sum_j c_j X'_2 X'_1 x_{n-1}^{\delta_1(j)} X'_2 x_{n-1}^{\delta_2(j)} \text{sh}^{n-2}(x_{n-1}^{\delta_1(j)} x_{n-1}^{\delta_2(j)}) \]
\[ \equiv \sum_j c_j X'_2 X'_1 x_{n-1}^{\delta_1(j)} X'_2 x_{n-1}^{\delta_2(j)} \tilde{\tau}_3(x_{n-1}^{\delta_1(j)} x_{n-1}^{\delta_2(j)}) = \varphi_{n-1, \tilde{X}_1, \tilde{X}_2}(\tilde{Y}) \]
where $\tilde{X}_1 = X'_2 X'_1$, $\tilde{X}_2 = X'_1 X'_2$, $\tilde{Y} = \rho_3(Y)$. So, we have $Z \equiv \tilde{Z} = \varphi_{n-1, \tilde{X}_1, \tilde{X}_2}(\tilde{Y})$ where $\tilde{X}_1, \tilde{X}_2 \in AF_{n-1}^+$, $\tilde{Y} \in L$. 

Markov Traces on the Fünar Algebra 379
Thus, by induction we reduce the problem to the case \( n = 1 \). In this case we have \( X_1, X_2 \in AF^+_1 = A \), hence, for \( Y \) as in (9), we have \( Z = \varphi_{1, X_1, X_2}(Y) = X_1X_2 \sum c_j x_1^{e_{1(j)}} x_2^{e_{2(j)}} \), hence \( Z \equiv \tau_2(Z) = \tau_N(Y) \in \tau_N(L) \). \( \square \)

The next statement can be considered as an improvement of the Pentagon Lemma from [3].

**Lemma 3.5** (Pentagon Lemma). Let \( Z_1, Z_2 \in \mathcal{R}_N \cup \mathcal{R}_M \) and \( Z_1 - Z_2 \in H + AF^+_{[w-1]} \) where \( w = \text{wt } Z_1 = \text{wt } Z_2 \). Then \( Z_1 - Z_2 \in H + R_{[w-1]} + \tau(J_4) + \tau_N(L) \).

**Proof.** Let \( X_i \in F^*_\infty \) be the leading monomial of \( Z_i \), \( i = 1, 2 \), i.e., \( \text{wt } X_i = \text{wt } Z_i \) and \( \text{wt}(Z_i - X_i) \leq w - 1 \). Then \( X_1 - X_2 \in H \), hence there exists a sequence of words \( X_1 = W_1, \ldots, W_m = X_2 \) such that \( W_{i+1} \) is obtained from \( W_i \) either by a cyclic permutation or by exchanging two consecutive commuting letters. By definition of \( \mathcal{R}_M \) and \( \mathcal{R}_N \) we have \( X_i = U_iX'_i \) and \( Z_i = (U_i - V_i)X'_i, i = 1, 2 \), where \( U_i \rightarrow V_i \) is an elementary \( K \)-reduction of types (i)–(vi) if \( Z_i \in \mathcal{R}_N \) and \( U_i = x_i^{\pm 1}, V_i = u \pm, X'_i \in F^*_n \) if \( Z_i \in \mathcal{R}_M \).

Following [1,3], we represent such sequences \( W_1, \ldots, W_m \) by diagrams. A diagram is a union of mutually transversal curves in the cylinder \( S^1 \times [0, 1] \), each curve being labeled by a letter \( x_i^{\pm 1} \). In pictures we represent the cylinder by a rectangle whose vertical sides are supposed to be identified, so, the fibers of the projection \( pr_2 : S^1 \times [0, 1] \rightarrow [0, 1] \) will be called horizontal circles. Each curve is monotone, i.e., its projection onto \([0, 1]\) is bijective. We say that a diagram is admissible if two curves labeled by \( x_i^{\pm 1} \) and \( x_j^{\pm 1} \) may cross only if \(|i - j| \geq 2 \). The words \( W_i \) (up to cyclic permutation) are read on horizontal circles.

We say that curves \( \Gamma_1, \ldots, \Gamma_m \) form a bunch of parallel curves or just a bunch if they are pairwise disjoint and all the crossings lying on \( \bigcup \Gamma_i \) can be covered by disks whose intersections with the diagram are as in Fig. 1 up to symmetry.

In our case, the first and the last word of the sequence are \( X_1 \) and \( X_2 \). So, on the boundary of the cylinder we indicate (by a bold line) segments corresponding to \( U_1 \) and \( U_2 \). As in [1,3], a diagram is called interactive if it contains a curve which joins the bold segments. We also say that a curve is active if it meets at least one bold segment.

**Step 1.** If all active curves form a single bunch all whose ends are on the bold segments, then \( Z_1 - Z_2 \in H \).

In this case we have \( U_1 = U_2 \). Let \( V_1 = r(U_1) = \sum c_j W_j, c_j \in A, W_j \in F^*_\infty \). For each \( j \) we consider the diagram obtained from the initial diagram by replacing the bunch of active curves by a bunch of curves labeled by \( W_j \). If a curve crosses the bunch, its label commutes with all letters occurring in \( U_1 \), hence it commutes with all letters in \( W_j \), i.e., the new diagram is admissible and it defines a congruence \( W_j X'_1 \equiv W_j X'_2 \mod H \).

![Fig. 1. Bunch crossing](image-url)
Hence (recall that $X_1 - X_2 \in H$) we have $Z_1 - Z_2 = (X_1 - V_i X'_1) - (X_2 - V_i X'_2) \equiv V_1 X'_2 - V_1 X'_1 = \sum c_j W_j (X'_2 - X'_1) \equiv 0 \mod H.$

**Step 2.** If $Z_1, Z_2 \in \mathcal{R}_M$, then $Z_1 - Z_2 \in H$.

In this case there is only one active curve, so we apply the result of Step 1.

**Step 3.** If the diagram is non-interactive, then $Z_1 - Z_2 \in H + R_{[w-1]}.$

Due to Step 2, we may suppose that $Z_1 \in \mathcal{R}_N$. Then $U_1 = x_n^{\epsilon_1} x_{n-1}^{\epsilon_2} x_n^{\epsilon_3}$ with $\epsilon_1, \epsilon_2 \in \{-1, 1\}$ and $\epsilon_3 \in \{-1, 0, 1\}$.

Let $A$ and $B$ be the points on the lower bold segment that correspond to the letters $x_n^{\epsilon_1}$ and $x_{n-1}^{\epsilon_2}$ of $U_1$ and let $AD$ and $BC$ be the corresponding active curves (see Fig. 2). They cut the cylinder into two halves. Let $Q$ be that half whose side $AB$ is contained in the bold segment (the quadrangle $ABCD$ in Fig. 2).

Let $\Gamma$ be the curve occurring from $U_1$ and labeled by $x_{n-1}^{\epsilon_2}$ if $\epsilon_2 \neq 0$ or a generic monotone curve in $Q$ if $\epsilon_2 = 0$. Let us choose a horizontal circle (the dashed line in Fig. 2) so that all crossings are below it and let us choose points $E$ and $F$ on it so that the segment $EF$ which crosses $\Gamma$ has no other intersections with the diagram. We may suppose that the intersection of the diagram with the upper half-cylinder (above $EF$) is a union of segments of vertical lines.

Let $\Delta$ be the diagram obtained by replacing $AD$ and $BC$ with monotone curves $AED$ and $BFC$ where $ED$, $FC$ are straight line segments and $AE$, $BF$ are curves in $Q$ which are chosen so close to $\Gamma$ that the active curves outcoming from $U_1$ form a bunch in the lower half-cylinder (below $EF$). The label of any curve $\Gamma' \neq \Gamma$ entering $Q$ is not $x_i^{\pm 1}$ with $|n-i| \leq 1$ (indeed, since $\Gamma'$ attains the lower boundary outside the bold segment, it crosses $AD$ or $BC$). Hence $\Delta$ is admissible.

Let $Y$ be the word read from $\Delta$ along the circle $EF$. The bunch of active curves in the lower half-cylinder ensures that $Y = U_1 Y'$ and the result of Step 1 yields

$$Z_1 \equiv (U_1 - r(U_1)) Y' \mod H.$$  \hfill (10)

Now, let us study the upper part of $\Delta$ (above $EF$). All the possible crossings in this part are on $ED$ and $FC$. Hence, up to cyclic permutation, we have $X_2 = x_n^{\epsilon_1} x_3 x_n^{\epsilon_2} x_4 x_n^{\epsilon_3} X_5$ and $Y = U_1 Y' = U_1 X_4 X_3 X_5$ (see Fig. 2). Since the diagram is not interactive, $U_2$ is a subword of one of $X_3$, $X_4$, $X_5$, hence the active curves outcoming from $U_2$ form a bunch and $Y' = Y_1 U_2 Y_2$, i.e., $Y = U_1 Y_1 U_2 Y_2$, $Y_1, Y_2 \in F_\infty^+$. Hence, by Step 1, we have

$$Z_2 \equiv U_1 Y_1 (U_2 - r(U_2)) Y_2 \mod H.$$  \hfill (11)

We have also

$$U_1 Y_1 r(U_2) Y_2 \equiv r(U_1) Y_1 r(U_2) Y_2 \equiv r(U_1) Y_1 U_2 Y_2 \mod R_{[w-1]}.$$
are non-interactive. By Step 3 this implies $U$ of exchanges of commuting letters such that

Combining this with (10) and (11), we obtain

$$Z_1 \equiv (U_1 - r(U_1))Y_1U_2Y_2 \equiv U_1Y_1(U_2 - r(U_2))Y_2 \equiv Z_2 \mod H + R_{[w-1]}.$$

**Step 4.** Consider the open intervals obtained after removing all endpoints of all active curves. If at least one of the words corresponding to these intervals is not almost $K$-reduced (see the definition in Sect. 2.1), then $Z_1 - Z_2 \in H + R_{[w-1]}$.

Suppose that the word which is not almost $K$-reduced is a subword $Y$ of $X_2$. Since it is disjoint from the active curves, we can write $X_2 = U_2X'YX''$. The fact that $Y$ is not almost $K$-reduced means that there exists a sequence $Y = Y_0 \rightarrow Y_1 \rightarrow \cdots \rightarrow Y'U_3Y''$ of exchanges of commuting letters such that $U_3$ is the left hand side of an elementary replacement of type (i)–(vi). The fact that $Y$ does not meet any active curve means that the diagrams corresponding to the both chains

$$X_1 \rightarrow \cdots \rightarrow X_2 = U_2X'Y_0X'' \rightarrow U_2X'Y_1X'' \rightarrow \cdots \rightarrow U_2X'(Y'U_3Y'')X'' \equiv Z_1 \equiv Z_2 \mod H + R_{[w-1]}$$

are non-interactive. By Step 3 this implies $Z_1 \equiv Z_3 \equiv Z_2 \mod H + R_{[w-1]}$ where $Z_3 = U_2X'Y(U_3 - r(U_3))Y''X''$.

**Step 5.** Suppose that $Z_1 \in R_N$ and the diagram is interactive but not as in Step 1. Then the active curves are arranged up to symmetry either as in Fig. 3 or as in Fig. 4 where each of the dashed lines may or may not be included into the diagram, $n \geq 1$.

Indeed, we draw the curves adjacent to one of the bold segments and we try all the possibilities to complete the picture to an admissible diagram. It is easy to see that the pictures that could arise this way are the two pictures from the statement of Step 5.

**Step 6.** If the active curves are as in Fig. 3, then $Z_1 - Z_2 \in H + R_{[w-1]} + \tau(J_4)$.

Suppose that the active curves are as in Fig. 3 (the bottom boundary corresponds to $X_1$). Then $U_1 = x_n^{\varepsilon_4}x_{n-1}^{\varepsilon_6}x_n^{\varepsilon_6}$, $U_2 = x_n^{\varepsilon_1}x_{n-1}^{\varepsilon_2}x_n^{\varepsilon_4}$, $X_2 = U_2Yx_n^{\varepsilon_5}x_{n-1}^{\varepsilon_3}x_n^{\varepsilon_6}X_4$ where $\varepsilon_1, \varepsilon_4, \varepsilon_6 = \pm 1$ and $\varepsilon_2, \varepsilon_5 \in \{-1, 0, 1\}$.

We begin as in Step 3. Let $Q$ be the curvilinear quadrangle adjacent to the lower bold segment and bounded by the active $x_n$-curves outcoming from $U_1$. Let $C_1$ be a horizontal circle such that the part of the diagram above $C_1$ is a union of segments of vertical lines. Let $\Gamma$ be either the $(x_{n-1})$-curve outcoming from $U_1$ (if it exists) or just a generic monotone curve in $Q$. Then we push the $x_n$-curves inside the domain $Q$ from its

![Fig. 3. First arrangement of the active curves](image-url)
boundary so that they form (together with $\Gamma$) a bunch below $C_1$, and so that the portions of the pushed curves above $C_1$ are segments of straight lines (see Fig. 5).

Since all curves outgoing from $Y$ cross an $x_n$-curve, $Y$ does not contain $x_{i}^{\pm 1}$ for $n - 1 \leq i \leq n + 1$. By Step 4, we may suppose that $Y$ is almost $K$-reduced, hence $Y$ has at most one occurrence of $x_{i}^{\pm 1}$, i.e., $Y = Y_1 x_{n-2}^{\varepsilon_3} Y_2$ with $\varepsilon_3 \in \{-1, 0, 1\}$ and $Y_1, Y_2$ do not contain $x_{i}^{\pm 1}$ for $n - 2 \leq i \leq n + 1$.

We choose horizontal circles $C_2$ and $C_3$ so that the intersection point of the $x_n$-curve and the $(x_{n-2})$-curve (if it exists) is between them and we modify the diagram as it is shown in Fig. 6. If we apply the result of Step 1 to the part of the diagram which is below $C_2$ and to that which is above $C_3$, we obtain:

$$Z_1 \equiv Y_1 x_{n-1}^{\varepsilon_1} x_n^{\varepsilon_2} x_{n-2}^{\varepsilon_3} (U_1 - r(U_1)) Y_2 X_3 X_4 \mod H\quad \text{below } C_2,$$

$$Z_2 \equiv Y_1 (U_2 - r(U_2)) x_{n-2}^{\varepsilon_3} x_{n-1}^{\varepsilon_4} x_n^{\varepsilon_5} Y_2 X_3 X_4 \mod H. \quad \text{above } C_3.$$

Hence $Z_1 - Z_2 \equiv X'\text{ sh}^{n-3} Y' \mod H$ where $X' = Y_2 X_3 X_4 Y_1$ and

$$Y' = r(x_3^{\varepsilon_1} x_2^{\varepsilon_2} x_3^{\varepsilon_4}) x_1^{\varepsilon_5} x_2^{\varepsilon_6} - x_3^{\varepsilon_1} x_2^{\varepsilon_2} x_1^{\varepsilon_3} r(x_3^{\varepsilon_4} x_2^{\varepsilon_5} x_3^{\varepsilon_6}).$$

If $\varepsilon_2 \neq 0$, then $r(Y') \in J_4$ by Condition (J1) of the definition of $J_4$. Thus, using Lemma 3.3 and observing that $\text{wt}(X'\text{ sh}^{n-3} Y') < w$, we obtain
If $\varepsilon_2 = 0$, then $X' \cup_{\varepsilon_1} \cup_{\varepsilon_2} \cup_{\varepsilon_3} \equiv X' \cup_{\varepsilon_1} \cup_{\varepsilon_3} \cup_{\varepsilon_4} \mod H$ where $r(Y'') \in J_4$, thus

$$Z_1 - Z_2 \equiv X' \cup_{\varepsilon_1} \cup_{\varepsilon_3} \cup_{\varepsilon_4} \mod H + R_{[w-1]} + \tau(J_4).$$

Step 7. If the active curves are as in Fig. 4, then $Z_1 - Z_2 \in H + R_{[w-1]} + \tau_N(L)$.

Again, as in the beginning of Steps 3 and 6, we transform the diagram as in Fig. 7 and we obtain

$$Z_1 \equiv X_3(U_1 - r(U_1))X_4 \cup_{\varepsilon_4} \quad \text{and} \quad Z_2 \equiv X_3 \cup_{\varepsilon_2} X_4(U_2 - r(U_2)) \mod H$$

where $U_1 = x_{n-1} \cup_{\varepsilon_1} \cup_{\varepsilon_2} \cup_{\varepsilon_3} \cup_{\varepsilon_4}, \quad U_2 = x_{n-1} \cup_{\varepsilon_1} \cup_{\varepsilon_3} \cup_{\varepsilon_4}, \quad \varepsilon_1, \varepsilon_3 = \pm 1, \quad \varepsilon_2, \varepsilon_4 \in \{-1, 0, 1\}$. Hence

$$Z_1 - Z_2 \equiv X_3 \cup_{\varepsilon_2} X_4 \cup_{\varepsilon_4}(U_2) - X_3 \cup_{\varepsilon_2} X_4 \cup_{\varepsilon_4}(U_1) \mod H \quad (12)$$

Note that $x_i^\pm_1$ for $n - 1 \leq i \leq n + 1$ does not occur in $X_j$, $j = 3, 4$. Indeed, if it does, then the diagram $x_i^\pm_1$-curve starting at a point from the upper circle corresponding to the letter $x_i^\pm_1$ in $X_j$ cannot attain the opposite side of the cylinder outside the bold segment because it cannot cross the $x_n$-curves. Thus,

$$X_3 \cup_{\varepsilon_2} X_4 \cup_{\varepsilon_4}(U_2) \equiv X_3 \cup_{\varepsilon_2} X_4 \cup_{\varepsilon_4}(U_2) \equiv X_3 \cup_{\varepsilon_2} X_4 \cup_{\varepsilon_4}(U_2) \tau(X_5) \mod H + R_{[w-1]}$$

where $X'_3, \quad X'_4 \in F_{n-1}^+$ and $X_5 \in \cup_{\varepsilon_3} \cup_{\varepsilon_4} F^+$ (the same for other term in (12)). So, replacing, if necessary, $X_j$ by $X'_j$ ($j = 3, 4$), we may assume that $X_3, \quad X_4 \in F_{n-1}^+$. Then we can pass from (12) to

$$Z_1 - Z_2 \equiv X_3 \cup_{\varepsilon_2} X_4 \cup_{\varepsilon_4}(U_2) - X_3 \cup_{\varepsilon_2} X_4 \cup_{\varepsilon_4}(U_1) \mod H + R_{[w-1]}$$

$$= \varphi_{n-1}(X_3, X_4(Y)).$$
where \( Y = x_{1}\, x_{2}\, x_{3}\, x_{4}\, x_{5}\, x_{6}\, x_{7}\, x_{8} = L \). Thus if \( n > 1 \), then the result follows from Lemma 3.4. If \( n = 1 \), then \( \varepsilon_{2} = \varepsilon_{4} = 0 \), \( X_{3} = X_{4} = 1 \), and (12) yields \( Z_{1} - Z_{2} = 0 \). □

**Lemma 3.6.** \( R_{[w]} \cap AF_{[w]}^{+} \subseteq H + R_{[w]} + \tau(J_{4}) + \tau_{N}(L) \)

**Proof.** For \( Z \in R_{[w]} \cap AF_{[w]}^{+} \), let \( m = m(Z) \) be the minimal number such that \( Z = c_{1}Z_{1} + \cdots + c_{m}Z_{m} \mod H + R_{[w]} + \tau(J_{4}) + \tau_{N}(L) \). To prove that \( Z \in H + R_{[w]} + \tau(J_{4}) + \tau_{N}(L) \), we use induction on \( m \). The statement is trivial for \( m = 0 \).

Suppose that \( m > 0 \) and the statement is true for all smaller values of \( m \). Let \( X_{i} \) be the leading monomial of \( Z_{i} \), i.e., \( X_{i} \in F_{\infty}^{+} \) and \( wt(Z_{i} - X_{i}) \). Then \( \sum c_{i}X_{i} = 0 \mod H \). The term \( c_{m}X_{m} \) of this congruence should be cancelled by other terms. Hence there exists \( j < m \) such that \( X_{m} - X_{j} = H \). Then \( c_{m}(Z_{m} - Z_{j}) \in H + R_{[w]} + \tau(J_{4}) + \tau_{N}(L) \) by Lemma 3.5 and \( Z - c_{m}(Z_{m} - Z_{j}) \in H + R_{[w]} + \tau(J_{4}) + \tau_{N}(L) \) by the induction hypothesis. □

**Proposition 3.7.** \( I \subseteq \tau(J_{4}) + \tau_{N}(L) \).

**Proof.** Let \( I' = \tau(J_{4}) + \tau_{N}(L) \). Since \( I = R \cap A \) and \( R = \bigcup R_{[w]} \), it is enough to prove that \( R_{[w]} \cap A \subseteq I' \) for any \( w \). For \( w = 0 \) we have \( R_{[0]} = 0 \), hence \( R_{[0]} \cap A \subseteq I' \).

Suppose that \( R_{[w]} \cap A \subseteq I' \). Let \( Z \in R_{[w]} \cap A \). Since \( R_{[w]} \cap A \subseteq R_{[w]} \cap AF_{[w]}^{+} \), by Lemma 3.6 we have \( Z \in H + R_{[w]} + I' \), i.e., \( Z = Z_{H} + Z' + Z_{0} \) with \( Z_{H} \in H \), \( Z' \in R_{[w]} \), \( Z_{0} \in I' \). Since \( Z_{H} = Z - Z' - Z_{0} \in AF_{[w]}^{+} \) and \( H \) is homogeneous, we have \( Z_{H} \in R_{[w]} \), thus \( Z = Z'' + Z_{0} \) with \( Z'' = Z' + Z_{H} \in R_{[w]} \). Since \( Z'' = Z - Z_{0} \in A \), we have \( Z'' \in R_{[w]} \cap A \) and by the induction hypothesis we obtain \( Z'' \in I' \) whence \( Z = Z'' + Z_{0} \in I' \). Thus \( R_{[w]} \cap A \subseteq I' \). □

Our main Theorem follows from Propositions 3.2 and 3.7.

**4. The Link Invariant \( P_{\alpha, \beta; k} \)**

**4.1. Some general properties.** As we mentioned in Remark 2.8, the Markov trace on \( AK_{\infty} \) defines an invariant of oriented links \( P = P_{\alpha, \beta; k} \) which takes its values in \( \tilde{A}/I \) where \( \tilde{A} = [\mu^{1/2}, v^{1/2}] \). If \( u \) and \( v \) are not zero divisors (this is so in all the cases computed so far; see Remark 2.10), then \( A \) embeds into \( \tilde{A} \).

**Proposition 4.1.** (a) \( P(\text{unknot}) = 1 \).

(b) \( P \) satisfies the skein relation coming from (1):

\[
u^{3/2}P(\downarrow\downarrow\downarrow\downarrow) - \alpha u\nu^{1/2}P(\downarrow\downarrow) + \beta u^{1/2}vP(\downarrow\downarrow) - \nu^{3/2}P(\downarrow\downarrow) = 0
\]

Similarly, (2) and (3) yield skein relations for tangles with 6 endpoints.

(c) \( P(L_{1} \cup L_{2}) = (uv)^{-1/2}P(L_{1})P(L_{2}) \) (disjoint union).

(d) \( P(L_{1} \# L_{2}) = P(L_{1})P(L_{2}) \) (connected sum).

(e) If \( \tilde{L} \) is the mirror image of \( L \), then \( P_{\alpha, \beta}(\tilde{L}) = P_{\beta, \alpha}(L) \).

(f) If \( L_{2} \) is obtained from \( L_{1} \) by a mutation (i.e., a rotation of a tangle with 4 endpoints by 180° such that no endpoint is fixed), then \( P(L_{1}) = P(L_{2}) \).

(g) \( P(L) \equiv 1 \mod (u - 1, v - 1, \alpha - \beta, 2(\alpha - 2)^{2}) \).
(h) If $L$ can be represented by an $n$-braid, then $P(L)$ can be represented by a Laurent polynomial $f \in \tilde{A}$ such that $1 - n \leq \deg_{u,v}(T) \leq 0$ for any monomial $T$ of $f$.

Proof. (a)–(e) and (h). Immediate from the definition of $P$.

(f) Mutant links $L_1$ and $L_2$ can be represented by braids $Z_1 = X \sh^{n-2} Y$ and $Z_2 = X \sh^{n-2}(\sigma_1 Y \sigma_1^{-1})$ respectively where $X \in B_n$, $Y \in B_m$. Hence we have $t(Z_1) = t(\tau_{n+1} \circ \tau_{n+2} \circ \cdots \circ \tau_{n+m-2}(Z_1)) = t(X \sh^{n-2} Y_2)$ where $Y_2 = \tau_3 \circ \cdots \circ \tau_m(Y) \in AB_2$ and, similarly, $t(Z_2) = t(X \sh^{n-2}(\sigma_1 Y_2 \sigma_1^{-1}))$. It remains to note that $Y_2 = \sigma_1 Y_2 \sigma_1^{-1}$ because the group $B_2$ is abelian.

(g) If we set $x = y = \tilde{x} = \tilde{y} = 1$ in (1) and (2), then we obtain identities modulo the ideal $(\alpha - \beta, \ 2(\alpha - 2)^2)$. □

From now on we assume that $k = \k[\alpha, \beta]$ where $\k$ is a commutative ring and each of $\alpha$ and $\beta$ is either zero or transcendent over $\k$. For a monomial in $\alpha, \beta, u, v$ we define its degree modulo 3 (denoted by $\deg_3$) by setting $\deg_3(u) = \deg_3(\alpha) = 1$ and $\deg_3(v) = \deg_3(\beta) = -1$. We denote the $\deg_3$-homogeneous component of $A$ (resp. of $\tilde{A}$) of degree $d$ by $A_d$ (resp. by $\tilde{A}_d$). So, we have $A = A_0 \oplus A_1 \oplus A_2$ and $\tilde{A} = \tilde{A}_0 \oplus \tilde{A}_{1/2} \oplus \cdots \oplus \tilde{A}_{5/2}$. The following fact follows immediately from the definitions.

**Proposition 4.2.** The ideal $I$ is $\deg_3$-homogeneous. $P(L) \in \tilde{A}_0/I$ for any $L$. □

4.2. Normal form of elements of $k[u^{\pm 1}, v^{\pm 1}]/I$. In fact, the square roots of $u$ and $v$ in the definition of the invariant $P$ are needed only for writing the skein relation (13) in a nice form. Otherwise, for a $\mu$-component link $L$ represented by an $n$-braid $X$, we can set

$$P(L) = u^{(p-n-e)/2} v^{(p-n+e)/2} t(X) \quad \text{where} \quad p = \begin{cases} 1, & \mu \text{ is odd,} \\ 0, & \mu \text{ is even} \end{cases}$$

which ensures that $P(L)$ belongs to $k[u^{\pm 1}, v^{\pm 1}]/I$ for any link $L$. Note that Proposition 4.2 still holds for $P(L)$ defined by (14). So, from now on we forget about fractional powers of $u$ and $v$ and we discuss the normal form of elements of the ring $k[u^{\pm 1}, v^{\pm 1}]/I$.

Assume that $k = \k[\alpha, \beta]$ as in the previous subsection. We have a natural identification of $k[u^{\pm 1}, v^{\pm 1}]$ with $\tilde{A}/\tilde{I}$ where $\tilde{A} = A[\tilde{u}, \tilde{v}]$ ($\tilde{u}$ and $\tilde{v}$ are new independent variables) and $\tilde{I} = I + (u\tilde{u} - 1, \tilde{v}\tilde{v} - 1)$. So, to define a normal form in $k[u^{\pm 1}, v^{\pm 1}]$, it is enough to compute a Gröbner base $\tilde{G}$ of $\tilde{I}$.

We assume that the monomial order is chosen so that $\tilde{u}$ and $\tilde{v}$ are greater than any monomial in $\alpha, \beta, u, v$. In this case, the following conditions are equivalent:

1. $A/I$ embeds into $\tilde{A}/\tilde{I} \cong k[u^{\pm 1}, v^{\pm 1}]/I$;
2. $u$ and $v$ are not zero divisors in $A/I$;
3. $\tilde{G}$ contains a Gröbner base of $I$.

Condition (3) holds in all the cases computed so far (see Remark 2.10).
If $k$ is a field, then the normal form in $\tilde{A}/\tilde{I}$ defined by $\tilde{G}$ is evident: it is just a $k$-linear combination of monomials which are not divisible by the leading terms of elements of $\tilde{G}$. In the case when $k$ is $\mathbb{Z}$ or $\mathbb{Z}/m\mathbb{Z}$, the situation is more delicate. For any monomial $T$, we have to fix canonical representatives in $k$ for elements of $k/I(T)$ where $I(T)$ is the ideal of $k$ consisting of the leading coefficients of elements of $\tilde{A}$ whose leading monomial is $T$. When $I(T) \neq 0$, we choose them in $\{0, \ldots, m(T) - 1\}$ where $m(T)$ is the positive generator of $I(T)$. Thus the normal form is a $k$-linear combination of monomials $T$ with coefficients belonging to $\{0, \ldots, m(T) - 1\}$. Note that $m(T)$ is the gcd of the leading coefficients of those elements $G$ whose leading monomials divide $T$.

4.3. The case $k = \mathbb{Q}$ or $\mathbb{F}_p$ and $\beta = 0$.

Proposition 4.3. Let the notation be as in the respective parts of Corollary 2.6 and let $\tilde{G}$ be the reduced Gröbner base of $\tilde{I}$ with respect to the lexicographic order such that $\tilde{v} > \tilde{u} > v > u > \alpha$. Then:

(a) $(k = \mathbb{F}_2)$. $\tilde{G} = G \cup \{u\tilde{u} + 1, v\tilde{u} + 1, \alpha^2(\tilde{u} + u^2 + au), \alpha^2(\tilde{v} + u + \alpha), u^3\tilde{v} + v^2 + u + \alpha(\kappa v + 1 + au^2 + \alpha^2u), \tilde{u}\tilde{v} + \tilde{u}v^2 + 1 + \alpha(\tilde{u} + v + au + \alpha^2))\}$.

(b) $(k = \mathbb{F}_3)$. $\tilde{G} = G \cup \{u\alpha^2u^3 - u^2 + au + \alpha^2, \tilde{v} + \alpha\tilde{u}^3 + \alpha^2u^2 - u\}$.

(c) $(k = \mathbb{Q}$ or $\mathbb{F}_p, p \not\in \{2, 3, 37\})$. $\tilde{G} = G \cup \{f_6, f_7\}$ where

\[
\begin{align*}
    f_6 &= \tilde{u} + \frac{7}{12}\alpha^2u^3 + (\frac{7}{8}\alpha^3 + 6)u^2 - (\frac{9}{8}\alpha^3 + \frac{20}{3})au + \frac{1}{3}\alpha^8 + \frac{163}{8}\alpha^5 + \frac{28}{9}\alpha^2, \\
    f_7 &= \tilde{v} + \frac{35}{6}\alpha u^3 - \left(\frac{69}{32}\alpha^3 + \frac{71}{4}\alpha^2u^2 - \left(\frac{75}{8}\alpha^3 + \frac{155}{3}\alpha^2u + \frac{2867}{144}\alpha^4 + \frac{433}{9}\alpha\right)\right).
\end{align*}
\]

(d) $(k = \mathbb{F}_{37})$. $\tilde{G} = G \cup \{f_{6,37}, f_{7,37}\}$ where

\[
\begin{align*}
    f_{6,37} &= \tilde{u} - 3\alpha u^4 - 6\alpha^2u^3 + (6\alpha^3 - 2)u^2 - 10(\alpha^3 - 1)au - 2\alpha^8 + 15\alpha^5 + 14\alpha^2, \\
    f_{7,37} &= \tilde{v} + 14u^4 + 16au^3 - (12\alpha^3u^2 + 11)\alpha^2u^2 + (12\alpha^3 - 1)u - 12\alpha^7 - 6\alpha^4 - \alpha.
\end{align*}
\]

Thus, in the setting of Corollary 2.6(c) $(k = \mathbb{Q}$ or $\mathbb{F}_p, p \not\in \{2, 3, 37\})$, the normal form of elements of $\tilde{A}/\tilde{I}$ always belongs to $A$ and we have $\tilde{A}/\tilde{I} \cong A/I$, in particular, $\dim_k \tilde{A}/\tilde{I} = 24$ or 21 if $k = \mathbb{F}_7$). By Proposition 4.2, the invariant $P(L)$ takes its values in $\tilde{A}/\tilde{I}$. So, its normal form is a linear combination of the eight monomials indicated in the header line of Table 1 (without $u^3$ in the case $k = \mathbb{F}_7$). The values of $P_{0,0,k[\alpha]}$ for knots up to 9 crossings are presented in Table 2 (the choice between the “right” knots $3_1, 5_1, 5_2, \ldots$ and their mirror images $3_1, \tilde{5}_1, \tilde{5}_2, \ldots$ is done according to the database “The Knot Atlas” http://katlas.org).

The most interesting case is $k = \mathbb{F}_p$ for $p = 37$. Up to now this is the only case when the invariant $P$ distinguishes knots with equal HOMFLY polynomials. In this case the normal form has one more monomial: $\alpha^2u^4$.

4.4. The case $k = \mathbb{Z}, \alpha = \beta = 0$.

Proposition 4.4. Let $k = \mathbb{Z}$ and $\alpha = \beta = 0$. We introduce the monomial order on $\tilde{A}$ by saying that $u^{a_1}v^{b_1}\tilde{u}^{c_1}\tilde{v}^{d_1} > u^{a_2}v^{b_2}\tilde{u}^{c_2}\tilde{v}^{d_2}$ if and only if one of the following conditions holds:

- either $d_1 > d_2$, or $d_1 = d_2$ and $c_1 > c_2$,
- $(d_1, c_1) = (d_2, c_2)$ and $a_1 + b_1 > a_2 + b_2$,
- $(d_1, c_1, a_1 + b_1) = (d_2, c_2, a_2 + b_2)$ and $b_1 > b_2$. 

Table 1. \( P_{a,0;Q}(K) \) for knots \( K \) up to 9 crossings (\( \tilde{K} \) is the mirror of \( K \))

| \( K \) | Coefficients of \( P_{a,0;Q}(K) \) | Coefficients of \( P_{a,0;Q}(\tilde{K}) \) if differ |
|------|---------------------------------|---------------------------------|
| 01   | 1  \( a^3 \)  \( a^6 \)  \( a^2u \)  \( a^5u \)  \( au^2 \)  \( a^4u^2 \)  \( u^3 \) | 1  \( a^3 \)  \( a^6 \)  \( a^2u \)  \( a^5u \)  \( au^2 \)  \( a^4u^2 \)  \( u^3 \) |
| 31   | \(-20\)  \(-61\)  \(-3\)  \(-44\)  \(-3\)  \(-26\)  \(-27\)  \(-35\)  \(-145\)  \(-899\)  \(-317\)  \(-605\)  \(-275\)  \(-20\)  \(-23\)  \(-56\) |
| 41   | \(-10\)  \(-25\)  \(-43\)  \(-127\)  \(-32\)  \(-10\)  \(-5\)  \(-7\)  \(-786\)  \(-329\)  \(-1565\)  \(-4769\)  \(-659\)  \(-55\)  \(-1\)  \(-791\) |
| 51   | \(22\)  \(-9\)  \(965\)  \(365\)  \(-1221\)  \(-1449\)  \(367\)  \(1\)  \(-375\)  \(-119\)  \(-729\)  \(-1458\)  \(-7776\)  \(972\)  \(-2592\)  \(-72\)  \(41\)  \(-23\)  \(-71\) |
| 61   | \(34\)  \(-9\)  \(-112\)  \(-48\)  \(-31\)  \(-9\)  \(-1\)  \(-13\)  \(-28\)  \(-136\)  \(-179\)  \(-125\)  \(-203\)  \(-11\)  \(-1\)  \(-19\)  \(-81\) |
| 71   | \(-55\)  \(-81\)  \(-160\)  \(-35\)  \(-39\)  \(-9\)  \(-7\)  \(-340\)  \(-29\)  \(-14\)  \(-1\)  \(-1\)  \(-21\)  \(-8\)  \(-1\)  \(-1\)  \(-81\) |
| 81   | \(-94\)  \(-2165\)  \(-701\)  \(-1493\)  \(-1943\)  \(-101\)  \(-35\)  \(-1928\)  \(-2591\)  \(-4099\)  \(-5215\)  \(-6301\)  \(-179\)  \(-665\)  \(-203\)  \(-243\)  \(-243\) |
| 91   | \(-84\)  \(-9\)  \(-36\)  \(-96\)  \(-12\)  \(-32\)  \(-1\)  \(-1\)  \(-182\)  \(-39\)  \(-77\)  \(-2471\)  \(-288\)  \(-774\)  \(-275\)  \(-23\)  \(-71\)  \(-2569\) |

S. Yu. Orevkov
Table 1. continued

| $K$ | $\alpha^3$ | $\alpha^6$ | $\alpha^2 u$ | $\alpha^5 u$ | $\alpha u^2$ | $\alpha^4 u^2$ | $u^3$ |
|-----|------------|------------|--------------|--------------|--------------|--------------|------|
| 83  | 226        | 812        | 1127         | -1307        | -437         | 3            | 1/2  |
| 84  | -98        | -721       | -385         | 2905         | 1327         | -1           | 1/8  |
| 85  | 185        | 2443       | 4015         | -6967        | -7765        | 245          | 203  |
| 86  | -151       | -5915      | -137         | 1745         | 343          | -228         | 4    |
| 87  | -29        | -4489      | -257         | 1117         | 601          | -133         | 27   |
| 88  | 217        | -1895      | -283         | 589          | 25           | -13          | 5    |
| 89  | 53         | 361        | 31           | 14           | 5            | -28          | 1/11 |
| 90  | -707       | -3593      | -3007        | 6283         | 3229         | -193         | 4/9  |
| 91  | -238       | -3049      | -551         | 1613         | 313          | -202         | 1/8  |
| 92  | -4         | 661        | 253          | 288          | 36           | -127         | 2/3  |
| 93  | -296       | -545       | -1073        | -923         | 17           | 35           | 1/8  |
| 94  | -76        | -2183      | -719         | 1511         | 333          | -34          | 1/3  |
| 95  | 53         | 1331       | 1057         | -3605        | -4353        | 1109         | 2/1  |
| 96  | -5         | 377        | 73           | -542         | -53          | 74           | 3/4  |
| 97  | -79        | -91        | 216          | 125          | 29           | -28          | 1/3  |
| 98  | -146       | -2347      | -751         | 1975         | 613          | -58          | 1/3  |
| 99  | -1486      | -10,007    | -547         | 2337         | 643          | -329         | -58  |

$P_{\alpha,0;\mathbb{Q}}(K)$ if differ

| $K$ | $\alpha^3$ | $\alpha^6$ | $\alpha^2 u$ | $\alpha^5 u$ | $\alpha u^2$ | $\alpha^4 u^2$ | $u^3$ |
|-----|------------|------------|--------------|--------------|--------------|--------------|------|
| 83  | 224        | 3245       | 2293         | -5785        | -1723        | 43           | 25   |
| 84  | 243        | 486        | 2592         | 324          | 804          | 3            | 24   |
| 85  | 243        | 486        | 2592         | 324          | 804          | 3            | 24   |
| 86  | -148       | -1735      | -299         | 785          | 293          | -269         | 203  |
| 87  | -729       | -1245      | 243          | 324          | 3            | 12           | 243  |
| 88  | 77         | 1235       | 241          | -907         | -91          | 129          | 131  |
| 89  | 89         | 1621       | 193          | -1012        | -221         | 55           | 3    |
| 90  | 81         | 324        | 216          | 27           | 3           | 1/7          |
| 91  | -10        | 2287       | 119          | -1039        | -53          | 155          | 79   |
| 92  | 71         | 729        | 7776         | 972          | 2592         | 3            | 24   |
| 93  | 2135       | -1750      | -3973        | 3793         | 5059         | -61          | -140 |
| 94  | 729        | 729        | 7776         | 972          | 2592         | 3            | 24   |
| 95  | 18146      | 72,113     | 13,781       | -17,984      | -21,011      | 2449         | 1106  |
| 96  | 2187       | 8748       | 58,32        | 729          | 1944         | 81           | 816  |
| 97  | 914        | -749       | -169         | 259          | 181          | -94          | -101 |
| 98  | 81         | 162        | 216          | 27           | 72           | 9            | 36   |
| 99  | 914        | -749       | -169         | 259          | 181          | -94          | -101 |

Markov Traces on the Funar Algebra
| K   | Coefficients of $P_{\alpha,0;Q}(K)$ | Coefficients of $P_{\alpha,0;\tilde{Q}}(\tilde{K})$ if differ |
|-----|-----------------------------------|---------------------------------------------------|
|     | $\alpha^3$ | $\alpha^6$ | $\alpha^2 u$ | $\alpha^5 u$ | $\alpha u^2$ | $\alpha^4 u^2$ | $\alpha^3$ | $\alpha^6$ | $\alpha^2 u$ | $\alpha^5 u$ | $\alpha u^2$ | $\alpha^4 u^2$ | $\alpha^3$ | $\alpha^6$ | $\alpha^2 u$ | $\alpha^5 u$ | $\alpha u^2$ | $\alpha^4 u^2$ |
| 8 20 | 22     | 799    | 18       | -497.9  | 53       | 81       | 21       | -287.9  | 253     | -673    | -979    | 3175     | 1057     | -11      | -1      | 322     | 81       |
| 8 21 | -52/9  | -428/9 | -19/3    | 416/9   | 79/4    | -147/4  | -75/4    | 119/3   | -71/3   | -6985   | -4057   | 7873     | 5755     | -29/3    | -31/3   | 322     | 243      |
| 9 1  | -265/3 | -10.993/6 | -9047/32 | 28.819/4 | 34.923/32 | -86.14/32 | -8.433/3 | 2940/1  | 230/278 | 510/649 | 23.542/260/498 | -285.37/1108 | 39.74/22.519 | 65/61 | 17.49/243 | 1944/6551 |
| 9 2  | 448/9  | 5303/9  | 157/32   | 79/4    | -210/4  | -177/4   | 553/3    | -4643/3 | 2187/8 | 87/48   | 23.328/2916/166 | 77/66 | 31/3    | 31/3    | 648/729 | 729/8 |
| 9 3  | 134/3  | -736/12 | -599/64  | 14.167/8 | 18.003/4 | -47.41/4 | -46.17/4 | 1841/1  | -76/616 | -211/865 | -281/941/448/729 | 415/435/15.509 | -42.239/73 | 1944/2187 |
| 9 4  | 332/3  | 689/12  | -35/4    | 827/4   | 1203/4  | -45/4    | -137/4   | 707/1   | 32/555 | 161/341 | 140/143/194/719 | -226/657 | 73/62 | 2653/6692 |
| 9 5  | 637/9  | 2606/9  | 191/32   | -13/6   | -49/4    | 232/4    | 69/4     | 196/3   | -653/2187 | -14/83 | -967/4 | 16.951/14.725/33.56 | -740/1960 | -59/483 | 763/7 |
| 9 6  | 31/3   | -2465/12 | -2263/244 | 8147/8 | 10.203/4 | -26/4    | -254/4 | 980/1 | -105/18 | -289/351 | 24/527 | 154/610/145.757/21.332 | -59.483/7637 | -73 | 85/729 |
| 9 7  | 94/3   | -166/32 | -535/4    | 2795/4  | 3759/4   | -104/4   | -501/4   | 455/1   | 49/886 | 62/257 | 206/071/292/507/327 | 541/3334 | 57/17 | 704/9 |
| 9 8  | 172/3  | 1003/64 | 209/32   | 43/4    | 913/4    | -43/4    | 133/4    | 133/1   | 298/6561 | 95/75 | 364/7 | -12.35/4865 | 139/139 | -1253/2187 |
| 9 9  | 243/3  | 486/64  | 2592/32 | 324/4    | 884/4    | 9/4    | 72/8     | 81/1    | 324/6561 | 864/367 | -3288/288 | -1288/23578 | 1944/2187 |
| 9 10 | 98/12  | 263/4   | -3/8     | 732/8   | 1125/4   | -14/8   | -133/5   | 735/1   | 65/669 | 70/606 | 209/935/287 | 731/318/4108 | 3707/6776 |
| 9 11 | 313/9  | 3637/18 | 2471/12 | -646/12 | -239/4    | 54/6    | 567/4    | -392/3 | -282/2187 | 4374/532 | 10.87 | -10.979 | -1438/448 | 1609/1435 |
| 9 12 | 329/27 | 9265/72 | 799/22   | -212/9 | -787/4    | 240/4    | 123/4    | -532/3 | 2729/2916 | 243/243 | 323/27 | -308/505 | -56/56 |
| 9 13 | 67/4   | 499/32 | 39/8     | 326/8   | 555/4    | -740/4   | -339/4   | 420/1   | 61/103 | 314/719 | 15.613/90.109/49.67 | 4096/6827 | 69/44 |
| 9 14 | -199/245 | 347/324 | 119/32   | -868/8 | -559/8    | 223/8    | -322/4   | 119/8 | 2804/8675 | 815/553 | -129/3 | -129/1190 |

**Table 1. continued**
Table 1. continued

| K | Coefficients of $P_{a,0}:Q(K)$ | Coefficients of $P_{a,0}:Q(\bar{K})$ if differ |
|---|-----------------------------|------------------------------------------------|
|   | $\alpha^3$ | $\alpha^6$ | $\alpha^2u$ | $\alpha^4u$ | $\alpha^2u^2$ | $\alpha^4u^2$ | $\alpha^6u^2$ | $\alpha^8u^2$ | $\alpha^{10}u^2$ | $\alpha^{12}u^2$ | $\alpha^{14}u^2$ | $\alpha^{16}u^2$ | $\alpha^{18}u^2$ |
| 915 | -1082 | -1905 | -1675 | 2521 | 344 | -374 | -1157 | 49 | 491 | 3255 | 4681 | -13,429 | -5,119 | 408 | 417 | -1036 |
| 916 | 7 | -201 | -1107 | 3751 | 4683 | -1173 | -561 | 434 | -43018 | -255,323 | -45,233 | 71,738 | 68,693 | -29,651 | -42,163 | 2597 |
| 917 | 313 | 3619 | 2543 | -6395 | -2321 | 58 | 55 | 8 | -392 | -352 | 266 | 332 | 75 | 191 | 729 |
| 918 | 52 | 75 | -279 | 4215 | -224 | 4347 | 3014 | -120 | 1483 | -120 | 1483 | 83000 | -37111 | 275863 | -385519 | -419725 |
| 919 | 26 | 541 | 211 | -595 | -245 | 167 | 179 | -49 | 8561 | -26244 | 69984 | 8748 | 23328 | 81 | 162 | 2187 |
| 920 | 124 | 2153 | 671 | -1487 | -509 | 104 | 57 | -35 | -13120 | 52589 | 39173 | -53933 | -59471 | 10084 | 3 | 24 | 87 |
| 921 | 404 | 12997 | 4669 | -13957 | -5359 | 434 | 111 | -1114 | -3116 | -15901 | -9553 | 15529 | 13483 | -554 | -284 | 553 |
| 922 | -172 | 667 | 79 | -439 | 227 | -9 | -65 | -7 | 729 | 2916 | 7776 | 972 | 2592 | 7 | 54 | 243 |
| 923 | 21 | 12 | -99 | 791 | 1167 | -336 | -303 | 168 | 78434 | 201803 | 39467 | -114556 | -123355 | 5222 | 17129 | 7301 |
| 924 | -94 | 2155 | 277 | -1366 | -607 | 215 | 59 | -721 | -227 | -1619 | -967 | 4675 | 1405 | -61 | -5 | 742 |
| 925 | 233 | 1078 | 173 | -742 | -253 | 67 | 69 | -70 | -1591 | -9443 | -1997 | 2660 | 1603 | -389 | -1361 | -738 |
| 926 | -917 | -3094 | -1391 | 2938 | 2449 | -121 | -319 | 70 | -101 | -3445 | -307 | 838 | 161 | -99 | -103 | 238 |
| 927 | 167 | 676 | 143 | -970 | -427 | 137 | 155 | -406 | -451 | -641 | -1 | 715 | 145 | -41 | -17 | 574 |
| 928 | -188 | -907 | -155 | 706 | 131 | -73 | -19 | 133 | -2951 | -20467 | -10141 | 17197 | 12271 | -181 | -97 | 574 |
| 929 | -38 | 1310 | 521 | -2105 | -415 | 100 | 99 | -959 | -227 | -283 | -625 | 1309 | 943 | -154 | -289 | 112 |
| 930 | -4 | 901 | 1531 | -5023 | -2149 | 167 | 185 | -469 | -62 | 2159 | 1145 | -569 | -743 | -11 | 13 | 385 |
| 931 | -26 | 257 | 245 | -2117 | -923 | 95 | 95 | -371 | -1208 | -4432 | -10057 | 18949 | 13735 | -203 | -437 | 679 |
| $K$ | Coefficients of $P_{α,0;Q}(K)$ | Coefficients of $P_{α,0;Q}(\bar{K})$ if differ |
|-----|-------------------------------|---------------------------------------------|
| 9.32 | $\alpha^3$ | $\alpha^6$ | $\alpha^2 u$ | $\alpha^5 u$ | $\alpha u^2$ | $\alpha^4 u^2$ | $\alpha^3 u^3$ |
| $\frac{9}{9}$ | $-61$ | $-1469$ | $-67$ | $401$ | $79$ | $-154$ | $-20$ | $140$ |
| $\frac{9}{9}$ | $729$ | $2916$ | $1944$ | $243$ | $648$ | $9$ | $18$ | $243$ |
| $\frac{9}{9}$ | $-98$ | $1061$ | $31$ | $-347$ | $-175$ | $3$ | $7$ | $133$ |
| $\frac{9}{9}$ | $19$ | $722$ | $1199$ | $-3107$ | $-1373$ | $89$ | $55$ | $-182$ |
| $\frac{9}{9}$ | $722$ | $992$ | $81$ | $108$ | $108$ | $81$ | $864$ | $238$ |
| $\frac{9}{9}$ | $19$ | $722$ | $1199$ | $-3107$ | $-1373$ | $89$ | $55$ | $-182$ |
| $\frac{9}{9}$ | $243$ | $992$ | $81$ | $108$ | $108$ | $81$ | $864$ | $238$ |
| $\frac{9}{9}$ | $19$ | $722$ | $1199$ | $-3107$ | $-1373$ | $89$ | $55$ | $-182$ |
| $\frac{9}{9}$ | $722$ | $992$ | $81$ | $108$ | $108$ | $81$ | $864$ | $238$ |
| $\frac{9}{9}$ | $19$ | $722$ | $1199$ | $-3107$ | $-1373$ | $89$ | $55$ | $-182$ |
| $\frac{9}{9}$ | $243$ | $992$ | $81$ | $108$ | $108$ | $81$ | $864$ | $238$ |
| $\frac{9}{9}$ | $19$ | $722$ | $1199$ | $-3107$ | $-1373$ | $89$ | $55$ | $-182$ |
Let \( G = \{ 16, 4u^2 + 4v, 4v^2 + 4u, 4uv - 4, v^3 + uv + u^3 - 3 \} \) and
\[
\tilde{G} = G \cup \{ 4\tilde{u} - 4v, 4\tilde{v} - 4u, \tilde{u}\tilde{u} - 1, \tilde{v}\tilde{v} - 1, u^3\tilde{v} - 3\tilde{v} + v^2 + u, \tilde{u}\tilde{v} + u^2\tilde{v} + \tilde{u}v^2 - 3 \}.
\]
Then \( G \) and \( \tilde{G} \) are Gröbner bases of \( I \) and \( \tilde{I} \) respectively.

**Remark 4.5.** The monomial order used in Proposition 4.4 can be defined by saying that this is the lexicographic order with \( \tilde{v} > \tilde{u} > w > v' > u' \) under the change of variables \( u = u'w, v = v'w \).

Thus, in the normal form of an element of \( \tilde{A}/\tilde{I} \), the coefficients of the monomials 1, \( u, v \) range in \( \{ 0, \ldots, 15 \} \) (note that \( u \) and \( v \) do not appear in \( P(L) \) by Proposition 4.2), the coefficients of
\[
u^{n+1}, u^n v, u^{n-1} v^2, \tilde{u}^n, v\tilde{u}^n, v^2\tilde{u}^n, \tilde{v}^n, u\tilde{v}^n, u^2\tilde{v}^n, \ n \geq 1,
\]
range in \( \{ 0, 1, 2, 3 \} \), and all the other coefficients vanish. Due to Proposition 4.1(h), this fact implies the following nice property of the normal form of \( P(L) \). Let us define the degree of an element of \( A/I \) represented by \( f \in A \) as
\[
\deg_T = \max_{g \in f+I} \deg_T \text{ where } T \text{ runs over all monomials of } g \text{ and } \deg u^a v^b \tilde{u}^c \tilde{v}^d = a + b - c - d.
\]

**Proposition 4.6.** If \( f(u, v, \tilde{u}, \tilde{v}) \) is the normal form of an element of \( \tilde{A}/\tilde{I} \) of degree \( \leq 2 \), then \( f(v, u, \tilde{v}, \tilde{u}) \) is the normal form of the corresponding element. In particular, if \( \tilde{L} \) is the mirror image of a link \( L \), then the normal form of \( P_{0,0;\mathbb{Z}}(\tilde{L}) \) is obtained from the normal form of \( P_{0,0;\mathbb{Z}}(L) \) by swapping \( u \) with \( v \) and \( \tilde{u} \) with \( \tilde{v} \).

Note that if the degree of an element of \( \tilde{A}/\tilde{I} \) is greater than two, then the swapping of \( u \) and \( v \) can drastically change the normal form. For example, \( u^3 \) is already in its normal form whereas the normal form of \( v^3 \) is \( 3uv + 3u^3 + 3 \).

In Table 2 we give the normal forms of the invariant \( P_{0,0;\mathbb{Z}}(K) \) for all knots \( K \) up to 10 crossings. We see in this table that there are many repetitions. In Table 3, for each \( n = 0, 1, \ldots, 12 \), we give the number of different values (up to exchange of \( u \) and \( v \)) that \( P_{0,0;\mathbb{Z}}(K) \) takes on knots with \( \leq n \) crossings.

**Example 4.7.** Let us compute \( P_{0,0;\mathbb{Z}}(K) \) where \( K \) is the figure-eight knot 4_1. We represent \( K \) by the 3-braid \( X = \tilde{x}y\tilde{y} \) where \( x, \tilde{x}, y, \tilde{y} \) are as in Introduction. Then
\[
t(X) = t(\tilde{x}(-x - y - \tilde{x}\tilde{y} - \tilde{y}\tilde{x} - \tilde{x}y - xy\tilde{x} - x\tilde{y}x - x\tilde{y}x)) \quad \text{by (2)}
\]
\[
= t(-1 - \tilde{x}y - \tilde{x}^2\tilde{y} - \tilde{x}\tilde{y}\tilde{x} - \tilde{x}^2y - \tilde{y}x - y\tilde{x})
\]
\[
= t(-1 - u\tilde{x} - v\tilde{x}^2 - v\tilde{x}^2 - u\tilde{x} - vx - u\tilde{x}) \quad \text{by the Markov relation}
\]
\[
= t(-1 - 3u\tilde{x} - 3vx) \quad \text{since } \tilde{x}^2 = x \text{ by (1)}
\]
\[
= -1 - 3uv - 3uv = -1 - 6uv \quad \text{by the Markov relation}
\]
whence
\[
P(K) = u^{(1-3-0)/2} v^{(1-3+0)/2} t(X) \quad \text{by the definition of } P
\]
\[
= -\tilde{u}\tilde{v} - 6 \quad \text{see } t(X) \text{ computed above}
\]
\[
= 7 + u^2\tilde{v} + \tilde{u}v^2 \quad \text{Gröbner reduction } -\tilde{u}\tilde{v} \rightarrow u^2\tilde{v} + \tilde{u}v^2 - 3
\]
In this subsection we reproduce their arguments in another form, and then we show that
4.5. Relation between $P_{0,0,2}(K)$ and the HOMFLY polynomial. It was discovered by
Cabaness and Marin [2] that $P_{0,0,2}\mathbb{Z}$ is a specialization of the HOMFLY polynomial.
In this subsection we reproduce their arguments in another form, and then we show that
Markov Traces on the Funar Algebra

Proposition 4.10. If \( h(L_1) = h(L_2) \), then \( P_{0,0;\mathbb{Z}}(L_1) - P_{0,0;\mathbb{Z}}(L_2) \in \{0, 8\} \). So, \( P_{0,0;\mathbb{Z}} \) can be thought of as a sort of spin of a certain specialization of the HOMFLY polynomial.

Let \( k = \mathbb{Z}/4\mathbb{Z} \), \( K_\infty = K_\infty(0, 0; k) \), \( A = k[u, v] \), and \( I = I(0, 0; k) \). Let \( \hat{k} = k[j]/(j^3 - 1) \) and let \( \hat{A} = \hat{k}[u, v] = k[j, u, v]/(j^3 - 1) \). Let \( H_\infty \) be the Hecke algebra \( \hat{k}B_\infty/(\sigma_1^2 + j\sigma_1 + j^2) \) (we denote the image of \( \sigma_i \) in \( H_\infty \) by \( g_i \)). A straightforward computation using (1) and (2) shows that the correspondence \( p \) homomorphism \( \{ \}

\[ h = (v^3 + u^3 + uv + 1, u^2 - v, j(uv - 1) + v^2 - u, j(u^2 - v) - u^2 + v, j^2 + ju + j^2). \]

Since \( I = (v^3 + u^3 + uv + 1) \), it follows that we can identify \( A/I \) with the subring of \( M_H \) generated by \( u \) and \( v \). Then we have \( t = t_H \circ p \), i.e., the Markov trace on \( K_\infty(0, 0; \mathbb{Z}/4\mathbb{Z}) \) is determined by the Markov trace on \( H_\infty \), hence \( P_{0,0;\mathbb{Z}/4\mathbb{Z}} \) is determined by the HOMFLY polynomial. For example, if we normalize the latter (we denote it \( h(a, z) \)) by \( h(\text{unknot}) = 1 \), \( ah(\mathcal{X}) - zh(\mathcal{X}) - a^{-1}h(\mathcal{X}) = 0 \), then we have:

**Proposition 4.8.** \( P_{0,0;\mathbb{Z}/4\mathbb{Z}}(u, v) = h(i^2u^1/2v^{-1/2} - i) \) where \( i = \sqrt{-1} \).

If we plug \( a = i^2u^1/2v^{-1/2} \), \( z = -i \) into \( h(a, z) \), we obtain a Laurent polynomial in \( i, j, u^1/2, v^{1/2} \). To get rid of the square roots of \( u, v \), and \( -1 \), we just multiply the result by \( (uv)^{-1/2} \) in the case of a link \( L \) with an even number of components (this corresponds to the normalization (14) of \( P(L) \)). To eliminate \( j \) and to put the result to the canonical form, we reduce it using the Gröbner base \( \tilde{G}_H \) of the ideal \( (j^3 - 1, v + ju + j^2, u\bar{u} - 1, v\bar{v} - 1) \) of the ring \( k[j, u, v, \bar{u}, \bar{v}] \) for the lexicographic monomial order with \( j > \bar{v} > \bar{u} > w > v' > u' \) (\( u = uu', v = vv' \); see Remark 4.5). We have \( \tilde{G}_H = G_H \cup \{ u\bar{u} - 1, \bar{v} - 1, u^3\bar{v} + \bar{v} + v^2 + u, \bar{u}v + u^2\bar{v} + v^2\bar{u} + 1, j(\bar{u} - v) - v^2\bar{u} + 1, j(\bar{v} - u) + u\bar{v} - v \} \).

**Example 4.9.** Let \( K \) be the trefoil knot given by the 2-braid \( \sigma_1^{-3} \). Then \( h(K) = -a^4 + a^2z^2 + 2a^2 \), hence \( P_{0,0;\mathbb{Z}/4\mathbb{Z}}(K) = -j^2u^2\bar{v}^2 - jju\bar{v} \) by Proposition 4.8. The following code for Singular reduces \( P(K) \) to \( u\bar{v}^2 \).

```plaintext
> ring r=(integer,4),(j,v,u,v,u),(lp(3),dp(2));
> ideal I=j^3-1,v+ju+j^2,uU-1,vV-1; reduce(-j^2u^2v^2-juv,vstd(I));
V2u
```

**Proposition 4.10.** If \( h(L_1) = h(L_2) \), then \( P_{0,0;\mathbb{Z}}(L_1) - P_{0,0;\mathbb{Z}}(L_2) \in \{0, 8\} \).

**Proof.** We have shown in Sect. 4.4 that the coefficients of all monomials in the normal form of \( P(L) \) except the constant term are in \( \{0, 1, 2, 3\} \). Thus they are determined by \( h(L) \) due to Proposition 4.8. The constant term is determined mod 8 by the other coefficients due to Proposition 4.1(g). \( \square \)

**Acknowledgements.** I am grateful to Andrey Levin and Alexey Muranov for useful discussions and advice. I am grateful to the referees for valuable remarks.
References

1. Bellingeri, P., Funar, L.: Polynomial invariants of links satisfying cubic skein relations. Asian J. Math. 8, 475–509 (2004)
2. Cabanes, M., Marin, I.: On ternary quotients of cubic Hecke algebras. Commun. Math. Phys. 314, 57–92 (2012)
3. Funar, L.: On cubic Hecke algebras. Commun. Math. Phys. 173, 513–558 (1995)
4. Marin, I.: The cubic Hecke algebra on at most 5 strands. J. Pure Appl. Algebra 216, 2754–2782 (2012)
5. Orevkov, S.Yu.: Cubic Hecke algebras and invariants of transversal links. Doklady Math. 89, 115–118 (2014)
6. Orevkov, S.Yu.: Files related to this paper. http://picard.ups-tlse.fr/~orevkov/fu.html
7. Orevkov, S.Yu., Shevchishin, V.V.: Markov theorem for transversal links. J. Knot Theory Ramif. 12, 905–913 (2003)

Communicated by Y. Kawahigashi