Monopole Floer homology for codimension-3 Riemannian foliations

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Abstract

In this paper, we give a systematic study of Seiberg-Witten theory on a closed oriented manifold $M$ with a codimension-3 oriented Riemannian foliation $F$. Under a certain topological condition, we construct the basic Seiberg-Witten invariant and the monopole Floer homologies $\tilde{HM}(M, F, s; \Gamma)$, $\hat{HM}(M, F, s; \Gamma)$, $\tilde{HM}(M, F, s; \Gamma)$, for each transverse spin$^c$ structure $s$, where $\Gamma$ is a complete local system. We will show that these homologies are independent of the bundle-like metrics and generic perturbations. To define the basic Seiberg-Witten invariant for manifolds with codimension 3 Riemannian foliations, we do not need the taut condition. The main difference between the basic monopole Floer homologies and the ones on manifolds is the necessity to use the Novikov ring on basic monopole Floer homologies.

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1 Introduction

The interaction between geometry and partial differential equation in dimension 4 is a theme which runs through a great deal of work by many mathematicians on gauge theory over the past decades. In particular, the Seiberg–Witten theory’s development is one of the main motivation for the study of differential topology and low dimensional manifolds. Since the foundational paper [28] by Witten, a lot of work has been done to apply this theory to various aspects of 3 and 4-dimensional manifolds. Seiberg-Witten theory can be generalized for studying the orbifolds (see Baldridge’s work [4] for the extension to 3-orbifolds). This article lays the groundwork for the case in which the higher-dimensional manifold admits a Riemannian foliation of codimension 3. From the viewpoint of analysis, gauge theory is closely related to the study of (nonlinear) Fredholm operator and the index of its linearized operator. To extending the framework of gauge theory to the manifold with Riemannian foliation, a natural idea is to study the transverse (nonlinear) elliptic operator on Riemannian foliation. For instance, the compactness of the basic Seiberg-Witten moduli space for manifolds with codimension 4 Riemannian foliations is showed by Kordyukov, Lejmi and Weber [12]. The author gives a construction of basic cohomotopy Seiberg-Witten invariant for codimension 4 Riemannian foliation [15]. In the same paper, the author gives an application to the basic index of the basic Dirac operator.

The theme of this article is to generalize the well-known constructions of Seiberg–Witten theory in 3-manifolds to the manifolds with codimension 3 Riemannian foliations. It is worth to note that in codimension 3 case, we do not need the taut condition to define the basic Seiberg-Witten invariant, which is quite different to the codimension 4 case. For any closed oriented 3-manifold $M$ with a spin$^c$ structure $s$, Kroheimer and Mrowka gave the construction of monopole Floer homologies $\overline{HM}(M, s)$, $\widehat{HM}(M, s)$, $\tilde{HM}(M, s)$ in their celebrated book [13]. The main purpose of this paper is to construct the monopole Floer homologies for the manifold with a codimension 3 Riemannian foliation $(M, F)$ satisfying a certain condition. The idea is to apply the arguments of the non-exact perturbation of the monopole Floer homologies to the case of Riemannian foliation. The following theorem could summarize the result of this paper.

**Theorem 1.1** Let $(M, F)$ be an oriented closed manifold with codimension 3 oriented taut Riemannian foliation $F$ and admit a transverse spin$^c$ structure $s$. Suppose that $H^1_b(M) \cap H^1(M, \mathbb{Z}) \subset H^1(M)$ is a lattice of $H^1_b(M)$. Then, using a bundle-like metric $g$, a generic perturbation $\eta$ and the Novikov ring $\Gamma$, we construct the basic monopole Floer homologies

$$\overline{HM}(M, F, s, g, \eta; \Gamma), \, \widehat{HM}(M, F, s, g, \eta; \Gamma), \, \tilde{HM}(M, F, s, g, \eta; \Gamma),$$

Moreover, these homologies are independent of the bundle-like metrics and the generic perturbations, which are denoted by

$$\overline{HM}(M, F, s; \Gamma), \, \widehat{HM}(M, F, s; \Gamma), \, \tilde{HM}(M, F, s; \Gamma).$$

Some notations and terms will be defined in Section 7. The necessity of using the Novikov ring to construct the homologies $\overline{HM}(M, F, s)$, $\widehat{HM}(M, F, s)$, $\tilde{HM}(M, F, s)$ is reflected in defining the partial operator of the Floer complex.

The structure of this paper is as follows: in Section 2, we review some notions and necessary properties about the Riemannian foliation; in Section 3, some analysis properties for some transverse equations will be given, which are necessary for the later sections; in Section 4, we construct the basic Seiberg-Witten invariant for manifold with codimension 3 Riemannian foliation; in Section 5, we construct the basic Chern-Simons-Dirac functional and give some properties of it; in Section 6, the gluing theorem for the basic moduli spaces will be proved, which is essential to construct the basic monopole Floer homologies; in Section 7, the proof of the above theorem will be given; in Section 8, we construct the monopole Floer homologies for a certain kind of 3 orbifolds, and we give some examples and a method to construct the Riemannian foliation satisfying the assumption of the above theorem.
Acknowledgement: The author warmly thanks Mikio Furuta for his long time invaluable help in both mathematics and life. The author is grateful to Kim. A. Frøyshov for his helpful discussion on Floer homology and Ken. Richardson for the discussion on Riemannian foliation. The research is partially sponsored by the FMSP of The University of Tokyo and The Fundamental Research Funds for the Central Universities No. 2021CDJQY-009.

2 Preliminary

In this section, we review some results about the geometry of Riemannian foliations and foliated bundle. Let $M$ be a closed oriented $n$ dimensional manifold with rank $p$ foliation $F$, and let $Q = TM/F$ be the quotient bundle. We denote the codimension of this foliation by $q = n - p$. For more details of this section, we give a reference [24].

Definition 2.1 A Riemannian metric $g_Q$ on $Q$ is said to be bundle-like, if it holds that

$$L_X g_Q = 0,$$

for any $X \in \Gamma(F)$. We say $(M, F)$ is a Riemannian foliation, if $Q$ admits a bundle-like metric.

Given a metric $g$ on $TM$, $Q$ is identified with the orthogonal complement to $F^\perp$ by $g$. In turn, $Q$ inherits a metric $g_{F^\perp}$, where $g_{F^\perp} = g|_{F^\perp}$. We have the following equivalence,

a metric $g$ of $TM$ corresponds a triple $(g_F, \pi_F, g_Q)$,

where $g_F = g|_F$ and $\pi_F$ is the projection $TM \to F$. A Riemannian metric $g$ on $TM$ is said to be bundle-like, if the induced metric $g_{F^\perp}$ is bundle-like. By the work of Reinhart [19], it is known that the bundle-like metric can be locally written as $g = \sum_{i,j} g_{ij}(x,y) \omega^i \otimes \omega^j + \sum_{k,l} g_{k,l}(y) dy^k \otimes dy^l$, where $(x, y)$ is in the foliated chart of $M$ and $\omega^i = dx^i + a^i_s(x,y) dy^s$. In this paper, we assume that $(M, F)$ is a manifold with Riemannian foliation without specific mention. Let $\pi_Q : TM \to Q$ be the canonical projection. We define a connection $\nabla^T$ on $Q$, by

$$\nabla^T_X s = \begin{cases} 
\pi_Q([X, Z_s]) & X \in \Gamma(F), \\
\pi_Q(\nabla^g_X Z_s) & X \in \Gamma(F^\perp), 
\end{cases}$$

for any section $s \in \Gamma(Q)$, where $Z_s \in \Gamma(TM)$ is a lift of $s$, i.e. $\pi_Q(Z_s) = s$ and $\nabla^g$ denotes the Levi-Civita connection of $g$. We call $\nabla^T$ transverse Levi-Civita connection. If $(M, F)$ is a Riemannian foliation, then by the Koszul-formula [24, Theorem 5.9], it is clear that $\nabla^T$ is uniquely determined by $g_Q$. Moreover, one can verify that it is torsion free and metric-compatible, whose leafwise restriction coincides with the Bott-connection. We set $R^T$ as the curvature of this connection. Similarly, we define the transverse Ricci curvature and scalar curvature by

$$Ric^T(Y) = \sum_{i=1}^q R^T(Y, e_i)e_i, \quad Scal^T = \sum_{i=1}^q g_Q(Ric^T(e_i), e_i),$$

where $\{e_i\}$ is a local orthonormal frame of $Q$. We define the basic forms as follows:

$$\Omega^b_i(M) = \{\omega \in \Omega^r(M) \mid \iota_X(\omega) = 0, \quad L_X(\omega) = 0, \quad \forall X \in \Gamma(F)\}.$$  

By the work of Alvarez L´opez [2], the following $L^2$ orthogonal decomposition holds for the forms on $M$, i.e.

$$\Omega(M) = \Omega_b(M) \oplus \Omega^b(M),$$

with respect to the $C^\infty$-Fréchet topology.
The mean curvature vector field is defined by Definition 2.2 on, we introduce the mean curvature field and mean curvature for \( m \).

The mean curvature form is related to the volume of the foliation, by the following proposition.

Choosing a local orthonormal basis \( \{e_i\}_{1 \leq i \leq p} \) of \( F \), we define the character form \( \chi_F \) of the foliation by, \( \chi_F(Y_1, \ldots, Y_p) = \det(g_F(e_i, Y_j))_{1 \leq i, j \leq p} \), for any section \( Y_1, \ldots, Y_p \in \Gamma(TM) \). By the metric \( g_Q(g_{F^i}) \), we have the basic Hodge-star operator,

\[
\hat{\ast} : \bigwedge^r Q^* \to \bigwedge^{p-r} Q^*.
\]

The basic Hodge-star operator is related to the usual Hodge-star operator by the formula \( \hat{\ast} \alpha = (-1)^{(q-r)\dim(F)} \ast (\alpha \wedge \chi_F) \). Moreover, we have \( \hat{\ast} : \Omega^r_b(M) \to \Omega^{p-r}_b(M) \) and the volume density formula, \( d\text{vol}_M = d\text{vol}_Q \wedge \chi_F \). For a section \( \alpha \in \Omega^r_b(M) \), we define its \( L^2 \) norm by

\[
\|\alpha\|_{L^2}^2 = \int_M \alpha \wedge \hat{\ast} \alpha \wedge \chi_F.
\]

We set \( d_b \) as the restriction of \( d \) to the basic forms, the complex \( d_b : \Omega^r_b(M) \to \Omega^{r+1}_b(M) \) is a subcomplex of the deRham complex, whose cohomology is called basic cohomology, and denoted by \( H^*_{b}(M) \). It is known that \( H^*_{b}(M) \subset H^*(M) \). We denote by \( b^*_\alpha = \dim H^*_{b}(M) \). Before going on, we introduce the mean curvature field and mean curvature form.

**Definition 2.2** The mean curvature vector field is defined by \( \tau = \sum_{i=1}^{\dim F} \pi_Q(\nabla^Q_{\xi_i} \xi_i) \in \Gamma(Q) \), where \( \{\xi_i\} \) is a local orthonormal basis of \( F \). Let \( \kappa \in \Gamma(Q^*) \) be the dual to \( \tau \) via the metric \( g_Q \).

The mean curvature form is related to the volume of the foliation, by the following proposition.

**Proposition 2.3** (Rummler [21]) For any metric \( g \) on \( TM \), it holds that

\[
d\chi_F = -\kappa \wedge \chi_F + \phi_0,
\]

where \( \phi_0 \) belongs to \( F^2 \Omega^{p+1} = \{ \omega \in \Omega^{p+1}(M) | \Omega_1, \ldots, \Omega_p, \omega = 0, \text{ for any } \Omega_1, \ldots, \Omega_p \in \Gamma(F) \} \).

This implies that \( w \wedge \phi_0 = 0 \) for any \( w \in \Gamma(\wedge^{q-1} Q^*) \).

By the decomposition, we have that \( \kappa = \kappa_b + \kappa_0 \) for a bundle-like metric \( g \), where \( \kappa_b \in \Omega^*_{\infty}(M) \) and \( \kappa_0, \omega_b \) \( L^2 = 0 \) for any basic one form \( \omega_b \). Dominguez [8] shows that any Riemannian foliation \( F \) carries a bundle-like metrics, i.e. having basic mean curvature form \( \kappa = \kappa_b \). The form \( \kappa_b \) is called the basic mean curvature form. It is known that \( d\kappa_b = 0 \), and the cohomology class \( [\kappa_b] \) is independent of any bundle-like metric [2].

**Definition 2.4** We say a foliation is taut, if there is a metric on \( M \) such that \( \kappa = 0 \), i.e. all leaves are minimal submanifolds.

In this paper, a bundle-like metric is called to be taut, if the induced mean curvature form vanishes. For a fixed Riemannian foliation \( F \), the taut condition has a topological obstruction.

**Proposition 2.5** (Alvarez López [2]) Let \( F \) be a Riemannian foliation on a closed manifold. Then, \( F \) is taut if and only if the class \( [\kappa_b] \) is trivial. Furthermore, when \( F \) is transversely oriented the foliation is taut if and only if \( H^*_{\infty}(M) \neq 0 \).

By [24, Page 99], the Poincare duality holds for the basic cohomologies under the taut condition, i.e. \( H^*_{b}(M) \cong H^{p-*}_b(M) \).

**Proposition 2.6** (Tondeur [24, Theorem 7.18]) Let \( d_b \) denote the restriction of \( d \) on the basic forms. Then, the \( L^2 \)-formal adjoint of \( d_b \) is \( d_b = (-1)^{(q+1)+1} \ast (d_b - \kappa_b \wedge) \ast \).

We define the basic Laplacian operator by \( \Delta_b = d_b \delta_b + \delta_b d_b \).

Now, we review the definitions of foliated vector bundle and basic connections.

**Definition 2.7** A principal bundle \( P \to M \) is called foliated, if it is equipped with a lifted foliation \( F_P \) invariant under the structure group action, such that it is transversal to the tangent space to the fiber and \( F_P \) projects isomorphically onto \( F \). We say a vector bundle \( E \to M \) is foliated, if its principal bundle \( P_E \) is foliated.
Definition 2.8 A connection $\omega$ of the foliated principal bundle $P$ is called adapted, if the horizontal distribution associated to this connection contains the foliation $\mathcal{F}_P$. A covariant connection on a foliated vector bundle is called adapted, if its associated connection on the principal bundle is. We say an adapted connection $\omega$ is called basic, if it is a Lie algebra valued basic form. Similarly, an adapted covariant connection is called basic, if its principal connection is.

Using an adapted connection, we define the basic sections by

$$\Gamma_b(E) = \{ s \in \Gamma(E) \mid \nabla_X s \equiv 0, \text{ for all } X \in \Gamma(F) \},$$

where $\nabla$ is an adapted connection. It is known that the space of basic sections is independent of the choice of the adapted connection.

Definition 2.9 A transverse Clifford module $E$ is a complex vector bundle over $M$ equipped with a hermitian metric satisfying the following properties:

1. $E$ is a bundle of $\text{Cl}(Q)$-modules, and the Clifford action $\text{Cl}(Q)$ on $E$ is skew-symmetry, i.e.

$$ (s \cdot \psi_1, \psi_2) + (\psi_1, s \cdot \psi_2) = 0,$$

for any $s \in \Gamma(Q)$ and $\psi_1, \psi_2 \in \Gamma(E)$;

2. $E$ admits a basic metric-compatible connection, and this connection is compatible with the Clifford action.

We say that $(M, F)$ admits a transverse spin$^c$ structure, if $Q$ is spin$^c$ and the associated spinor bundle $S$ is a transverse Clifford module over $(M, F)$.

Definition 2.10 Let $E$ be a transverse Clifford module $E$ over $(M, F)$. Fixing a basic connection $\nabla^E$, we define the transverse Dirac operator $\mathcal{D}_b$ by $\mathcal{D}_b^T = \sum_{i=1}^q e_i \cdot \nabla^E_i$ action on $\Gamma(E)$, where $\{e_i\}$ is a local orthonormal basis of $Q$.

Note that $\mathcal{D}_b^T$ is not formally self-adjoint in general, whose adjoint operator is $\mathcal{D}_b^{T,*} = \mathcal{D}_b^T - \tau_b$. We set $\mathcal{D}_b^T = \mathcal{D}_b^T - \frac{1}{2} \tau_b$, which is called basic Dirac operator. This basic Dirac operator is a formally self-adjoint operator and maps the basic sections $\Gamma_b(E) = \{ s \in \Gamma(M, E) \mid \nabla_X s \equiv 0, \text{ for any } X \in \Gamma(F) \}$ to itself.

Let $E$ be a foliated vector bundle on $M$ equipped with a basic Hermitian structure and a compatible basic connection $\nabla^E$. We define the basic $|||_{L^k_b}$-norm by

$$||u||_{L^k_b} = \sum_{j=0}^k \left( \int_M |(\nabla^E)^j u|^p d\text{vol}_M \right)^{\frac{1}{p}},$$

for any $u \in \Gamma_b(E)$. Let $L^k_b$ be the completion of $\Gamma_b(E)$ with respect to this norm. One has the similar Sobolev embedding and Sobolev multiplication properties for basic sections, which are shown in [12, Theorem 9, 10, 11].

Definition 2.11 Let $E_1$ and $E_2$ be two foliated vector bundles over $M$ with compatible basic connections. A differential operator $L : \Gamma(E_1) \rightarrow \Gamma(E_2)$, is called basic, if in any foliated chart $(x, y) \in U \times V$ with distinguished local trivialization of $E_1$ and $E_2$, then one locally writes

$$L|_{U \times V} = \sum_{\alpha} a_\alpha(y) \frac{\partial^{\alpha}}{\partial x_1^{\alpha_1} \cdots \partial x_q^{\alpha_q}}.$$

A basic differential operator $L$ defined as the above, is said to be transverse elliptic, if its transverse symbol is an isomorphism away from the 0-section, i.e. $\sigma(p, y)$ is an isomorphism for any $p \in M$ and non-zero $y \in Q^*_p$.

For any transverse elliptic operator, we have the regularity estimate [12, Theorem 12].
3 Analysis of basic forms

Throughout this section, let \((M,F)\) be a closed oriented manifold with a taut Riemannian foliation. We give some basic tools for analysis on Sobolev space of basic sections. The goal of this section is to prepare for the analysis on the moduli space of the later sections. Firstly, we recall the unique continuation property on Hilbert space(see Kroheimer and Mrowka [13, Chpater 7, 14]).

Lemma 3.1 (c.f. [13, Lemma 7.1.3]) Let \(z : [t_1,t_2] \to H\) be a solution to the equation

\[
\frac{d}{dt}z(t) + L(t)z(t) = f(t),
\]

where \(L(t)\) is a first-order transverse elliptic formal self-adjoint operator, \(H\) is a Hilbert space and \(f(t)\) is an element of \(H\) with \(\|f(t)\| \leq C\|z(t)\|\) for some constant \(C\). If \(z(t)\) is zero at one-point, then it vanishes identically.

On the finite cylinder \(Z = [a,b] \times M\), we have the following trace theorem.

Theorem 3.2 (c.f. [27, Theorem B10]) Set \(Z = [a,b] \times M\) and \(1 \leq p < \infty\) and \(n = \text{codim}(F)\).

In the case \(p < n+1\) we assume \(1 \leq q \leq \frac{(n+1)p-n}{1+n-p}\), and in the case \(p \geq 1+n\) we assume \(1 \leq q < \infty\). Then, the basic trace theorem holds for all basic functions of \(Z\), i.e. \(L^p(Z) \to L^q(\partial Z)\).

By the same idea of the [11, Appendix B], we have the following theorem.

Theorem 3.3 For \(j > \frac{1}{2}\), there is a continuous restriction map between the Sobolev basic sections,

\[
r : L^j(Z,E) \to L^{j-\frac{1}{2}}(\partial Z,E_0),
\]

where \(E\) is the pull-back foliated bundle of \(E_0 \to M\).

Consider the equation

\[
\begin{cases}
\Delta_b u = f & \text{on } Z \\
\frac{\partial u}{\partial \nu} = g & \text{on } \partial Z,
\end{cases}
\]

where \(\nu\) denotes the unit normal vector field along the boundary \(\partial Z\), with the condition

\[
\int_Z f + \int_{\partial Z} g = 0.
\]

Recall that we have that \(\Delta_b u = \Delta u\) for any basic function \(u\)(see [24, Page 86]). Similar to the property of the Laplacian equation with Neumann boundary condition, the following theorems hold.

Theorem 3.4 For any basic function \(u\) on \(Z\), it holds that

\[
\|u\|_{L^2_{k+2}} \leq C(\|\Delta_b u\|_{L^2_k} + \|u\|_{L^2_{k+1/2}} + \|u\|_{L^2_{k+1}}).
\]

Furthermore, if on the boundary \(u\) satisfies \(\frac{\partial u}{\partial \nu}\big|_{\partial Z} = 0\), then one has that

\[
\|u\|_{L^2_{k+2}} \leq C(\|\Delta_b u\|_{L^2_k} + \|u\|_{L^2_{k+1}}).
\]

Proof Since \(\Delta_b u = \Delta u\) for the basic function \(u\), the proof follows by the standard theory, c.f. [23, Formula 7.37 ] for the first formula and [23, Formula 7.34] for the second one.
Theorem 3.5 Let $f \in L^2_k(Z)$ and $g \in L^2_{k+1/2}(\partial Z)$ be basic functions. If the formula $\int_Z f + \int_{\partial Z} g = 0$ holds, then there is a solution to the equation
\[
\begin{cases}
\Delta_b u = f & Z \\
\frac{\partial u}{\partial \nu} = g & \partial Z.
\end{cases}
\]

Proof The idea is similar to the one of [27, Theorem 3.1]. We choose an element $v \in L^2_{k+2}(Z)$ such that $\frac{\partial v}{\partial \nu} \big|_{\partial Z} = g \big|_{\partial Z}$ can be realized by letting $v = \phi(t)g$ for some smooth function $\phi(t)$ with support near the boundary and near each the boundary we have $\phi(t) = t - a$ for $t \in [a, a + \epsilon)$ and $\phi(t) = t - b$ for $t \in (b - \epsilon, b]$. Now we have
\[
\int_Z (f - \Delta_b v) = \int_Z f + \int_{\partial Z} \frac{\partial v}{\partial \nu} = 0.
\]
Thus, by the Theorem 3.4 and the Rellich embedding, there exists a solution $u_1 \in L^2_{k+2}(Z)$ to the Neumann problem with $f$ replaced by $f - \Delta_b v$. The solution is given by $u = u_1 + v$.

Theorem 3.6 On $Z = [a, b] \times M$, if the basic one-form $a \in \Omega^1_b(Z)$ satisfies the condition $a(\nu) = 0$ on $\partial Z$, then the following holds
\[
\int_Z (\nabla Tr a, \nabla Tr a) + \text{Ric}^T(a, a) = \int_Z (d_b a, d_b a) + (\delta_b a, \delta_b a).
\]
(1)
The proof is similar to the classical formula, which is stated by the following proposition.

Proposition 3.7 (Jung [14]) The formula
\[
\Delta_b \alpha = (\nabla^T)^* \nabla^T \alpha + \text{Ric}^T(\alpha),
\]
(2) holds, for any basic one-form $\alpha$ on $(M, F)$ and $(I \times M, F)$.

Recall that $Z = I \times M$, where $I \subset (-\infty, \infty)$ is a compact interval, we establish the following lemmas.

Lemma 3.8 We have that $H^r_b(Z)$ is dual to $H^{m+1-r}_b(Z)$, where $m = \text{codim}(F)$ in $M$ and $H^1_{b,c}(Z)$ denotes the basic deRham cohomology which is vanishing at the ends.

Proof Let $e$ be a 1-form on $I$ with integral 1, which is vanishing at the ends. Define the map
\[
e_* : \Omega^*_b(M) \to \Omega^{*+1}_b(Z),
\]
by
\[\alpha \mapsto \alpha \wedge e.\]
We set a map $\pi_* : \Omega^{*}_b(Z) \to \Omega^{*+1}_b(M)$ as the integration along the $I$-direction. Similar to the arguments of [5, Proposition 4.6], the induced cohomology map $e_* : H^{*+1}_b(M) \to H^{*+1}_b(Z)$ is an isomorphism, whose inverse is the induced cohomology map of $\pi_*$. Since $H^*_b(M)$ is dual to $H^{m-r}_b(M)$, we proved the lemma.

Lemma 3.9 Under the above conditions, one has the isomorphism
\[
H^1_b(Z) \cong H^1_b(M).
\]
Proof It is clear that \( \pi_Z^* : H^1_b(M) \to H^1_b(Z) \) is an injective, where \( \pi_Z \) denotes the canonical projection \( Z \to M \). We set the map \( i : M \to Z \), as \( p \mapsto (p, t_1) \), where \( t_1 \) is the left endpoint. Assume that there is an element \([\omega]\) \( H^1_b(Z) \) such that \( i^*([\omega]) = 0 \), i.e. \( i^*\omega = d_b f \) for some basic function on \( M \). Setting \( \omega' = \omega - d_b \pi_Z f \), we rewrite it as
\[
\omega' = \alpha + f'dt.
\]
The condition \( d\omega' = 0 \) implies that
\[
\delta \alpha = 0, \quad \dot{\alpha} = df'.
\]
Hence, it holds that \( \alpha|_{(t_1)} \times M = 0 \), and \( \alpha(p, t) = \int_{t_1}^t df'(s) ds = d \int_{t_1}^t f'(s) ds \), i.e. \( [\omega'] = [\omega] = 0 \). This implies the isomorphism \( H^1_b(Z) \cong H^1_b(M) \).

We finish this section with the Coulomb gauge fixing property: For any non-trivial homotopy map \( u : Z \to S^1 \), by Theorem 3.5, we can find an element \( v \) of this homotopy class satisfying the equation
\[
\begin{cases}
\delta_b(v^{-1}d_b v) = 0 & \text{in } Z, \\
dv(v) = 0 & \text{on } \partial Z.
\end{cases}
\]
(3)
Notice that for any basic one form \( \alpha \) we have that \( \delta_b \alpha = \delta \alpha \). Let \( \Gamma^1_b \) be the lattice of \( H^1_b(M) \cap H^1(M, Z) \). \( \Gamma^1_b \) also corresponds to a lattice of \( H^1_b(Z) \cap H^1(Z, Z) \). Choose a basis \( \{a_i\} \) of this lattice, by pairing we have a basis \( \{\beta_i\} \) of \( H^3_{b, \iota}(Z) \), which is dual to \( \{a_i\} \).

4 Basic Seiberg-Witten invariant on codimension 3 foliation

In this section, we define the basic Seiberg-Witten invariant on manifolds with a codimension 3 foliation under a certain condition.

4.1 Basic Seiberg-Witten equations on codimension 3 foliation

In this subsection, we focus on the manifolds with a foliation satisfying the following assumption.

Assumption 4.1 Let \((M, F)\) be an oriented closed manifold with codimension 3 oriented Riemannian foliation \( F \) and admits a transverse spin\(^c\) structure \( s \). Suppose that \( H^1_b(M) \cap H^1(M, Z) \subset H^1(M) \) is a lattice of \( H^1_b(M) \).

Let \( \mathcal{A}_b(s) \) be the space of basic spin\(^c\) connections. We define the basic Seiberg-Witten equations for manifolds with a codimension-3 Riemannian foliation by
\[
\begin{cases}
\mathcal{D}_{b, A} \Psi = 0, \\
\frac{1}{2} F^c_{A'} - \mathcal{Q}(\Psi) = 0,
\end{cases}
\]
(4)
for \((A, \Psi) \in \mathcal{A}_b(s) \times \Gamma_b(S)\). Here, we identify the traceless endomorphism of the spinor bundle with the imaginary valued cotangent bundle (we use \( \mathcal{Q}(\Psi) \) instead of \( \rho^{-1}(\Psi \Psi^*)_0 \) in the book [13, Formula 4.4]), \( A^c \) denotes the connection on the determinate bundle of \( S \), see [13, Notation 1.2.1] and \( \mathcal{D}_{b, A} \) denotes the basic Dirac operator twisted with the basic connection \( A \). The basic gauge group
\[\mathcal{G}_b = \{ u : M \to U(1) \mid L_X u \equiv 0, \text{ for all } X \in \Gamma(F) \},\]
acts on \( \mathcal{C}_b(s) = \mathcal{A}_b(s) \times \Gamma_b(S) \) as (see [13, Formula 4.5]):
\[u : (A, \Psi) \mapsto (A - u^{-1} du, u \Psi).\]
To construct the basic Seiberg-Witten invariant, we need to consider the moduli space, which is defined as below.
Definition 4.2 The moduli space $M_g(M, F, s)$ of the basic Seiberg-Witten equations on $(M, F, g, s)$ is the space of the solutions to the above Seiberg-Witten equations modulo the gauge transformation group $\Gamma_b$. The moduli space $M^*_g(M, F, s)$ is the irreducible part of $M_g(M, F, s)$, i.e. the spinor field part is not identically zero.

Similar to the manifold case, we consider the following complex
\[
L^2_{2+k}(\Omega^0_b(M, i\mathbb{R})) \xrightarrow{G(A, \Psi)} L^2_{1+k}(\Omega^0_b(M, i\mathbb{R})) \oplus L^2_{1+k}(\Gamma_b(S)) \xrightarrow{L(A, \Psi)} L^2_{1+k}(\Omega^1_b(M, i\mathbb{R})) \oplus L^2_{1+k}(\Gamma_b(S)),
\]
where $G(A, \Psi)f = (-d_b f, f \Psi)$ and $L(A, \Psi)(a, \Phi) = (-\frac{1}{2}da + q(\Phi, \Phi), \nabla_{b, A} \Phi + a \Phi)$. One has that $L \cdot G = 0$. To read off the virtual dimension of the moduli space, it is convenient to consider the form operator (see [25, Formula 2.7]),
\[
Q(A, \Psi) : L^2_{2+k}(\Omega^0_b(M, i\mathbb{R})) \oplus L^2_{2+k}(\Omega^0_b(M, i\mathbb{R})) \oplus L^2_{2+k}(\Gamma_b(S)) \rightarrow L^2_{1+k}(\Omega^1_b(M, i\mathbb{R})) \oplus L^2_{1+k}(\Omega^1_b(M, i\mathbb{R})) \oplus L^2_{1+k}(\Gamma_b(S)),
\]
where $G^*(A, \Psi)$ is the formal self-adjoint of $G(A, \Psi)$. To show the smoothness of the moduli space, we need to perturb the above equations: Fixing a basic perturbation $\eta \in i\Omega^1_b(M)$, we denote the moduli space of the perturbed basic Seiberg-Witten equations by $M_{g, \eta}(M, F, s)$, i.e. the space of the solutions to equations
\[
\begin{align*}
\nabla_{b, A} \Psi &= 0, \\
\frac{1}{2}F_{A'} - q(\Psi) &= \eta.
\end{align*}
\]
modulo the gauge action.

Definition 4.3 We say that $[A, \Psi] \in M_{g, \eta}(M, F, s)$ is non-degenerate if
\[
\ker(L(A, \Psi)) / \text{Im}G(A, \Psi) = 0.
\]

Proposition 4.4 If $H^1_b(M) \cap H^1(M, \mathbb{Z})$ is a lattice of $H^1_b(M)$, then for a generic perturbation the irreducible moduli space of basic Seiberg-Witten equations is a compact manifold with formal dimension zero.

Proof To show that moduli space is a compact and smooth manifold, we can repeat the similar arguments of [25, Lemma 2.2.3, Lemma 2.2.6, Theorem 2.2.8]. To prove that the formal dimension of the moduli space is zero, we calculate the index the operator (5). Recall that the operator (5) is a compact perturbation of the transverse elliptic operator
\[
\begin{pmatrix}
\delta_b & -d_b \\
-d_b & 0 \\
0 & 0 & \nabla_{b, A}
\end{pmatrix}.
\]
However it is not formal self-adjoint in general. It is clear that the operator
\[
\begin{pmatrix}
\frac{1}{2}(\delta_b + \frac{1}{2}\kappa) & -d_b & 0 \\
-d_b & 0 & 0 \\
0 & 0 & \nabla_{b, A}
\end{pmatrix}
\]
is a formal self-adjoint operator and the difference
\[
\begin{pmatrix}
\frac{1}{2}d_b & -d_b & 0 \\
-d_b & 0 & 0 \\
0 & 0 & \nabla_{b, A}
\end{pmatrix} - \begin{pmatrix}
\frac{1}{2}(\delta_b + \frac{1}{2}\kappa) & -d_b & 0 \\
-d_b & 0 & 0 \\
0 & 0 & \nabla_{b, A}
\end{pmatrix}
\]
is a compact operator, hence they have the same index zero. To equip an orientation of the moduli space, we just need to equip an orientation of the determinant line bundle of (5).
The above proposition implies that the determine line bundle of the operator (5) is trivial over the moduli space $\mathcal{M}_{g,n}^*(M, F, s)$. Hence, there is a natural orientation for the moduli space. The basic Seiberg-Witten invariant $SW_{g,n}(M, F, s)$ on $(M, F)$ is defined by the signed counting of the moduli space $\mathcal{M}_{g,n}^*(M, F, s)$.

### 4.2 Basic Seiberg-Witten invariant on codimension 3 foliation

In this subsection, we show how the basic Seiberg-Witten invariant depends on the bundle-like metrics and basic perturbations. We review the notion of the reducible solution. Let $(A, \Psi)$ be a solution to the basic Seiberg-Witten equations (4). When $\Psi = 0$, the basic Seiberg-Witten equations reduce to a single equation, $\frac{i}{2} F_A = \eta$. If $\eta = 0$, then the reducible class is identified with the moduli space of the flat basic $U(1)$-connection of $\det(s)$. We denote the first Chern class of $\det(s)$ by $c_1(s)$.

**Lemma 4.5** The equation $\frac{i}{2} F_A = \eta$ has a solution if and only if $d_0 \ast \eta = 0$ and $\pi i [c_1(s)] = [\ast \eta]$ in $iH^2_1(M)$. In particular, if $H^1_1(M) \cap H^1(M, \mathbb{Z})$ is a lattice of $H^1_1(M)$, then the set of reducible solutions modulo gauge action is isomorphic to $H^1_1(M)/(H^1_1(M) \cap H^1(M, \mathbb{Z}))$.

**Proof** Obviously, $[\frac{i}{2} F_A] = [\ast \eta]$ is a necessary condition to solving the equation. Conversely, suppose that $\pi i [c_1(s)] = [\ast \eta]$. We fix a basic connection $A_0$ such that $[\frac{i}{2} F_{A_0}] = [\ast \eta]$. It suffices to get a basic one-form $a$ such that $da = \ast \eta - \frac{i}{2} F_{A_0}$. Since $\ast \eta - \frac{i}{2} F_{A_0}$ is an exact basic form, such one-form $a$ always exists. By choosing one solution $a_0$ of the above equation, one represents all the others as

$$a_0 + \text{closed one form}.$$  

Any two solutions $a_1$ and $a_2$ are equivalent, if and only if $a_1 = a_2 + u^{-1} du$ for some $u \in G_0$. \hfill $\Box$

**Theorem 4.6** Let $(M, F)$ satisfy the Assumption 4.1. When $b_2^1 > 1$, then the basic Seiberg-Witten invariant is well-defined, i.e. it is independent of the generic choice of the basic perturbations and bundle-like metrics.

**Proof** To show that the basic Seiberg-Witten invariant is independent of the basic perturbations and bundle-like metrics, we apply the similar proof in codimension 4 case [17, Chapter 5]. Here we give a sketch of the proof. We denote by $\mathcal{N} = \{a \in \Omega^1_0 | d_0 \ast a = 0\}$, which is identified with the space of closed basic two-forms. Set

$$\mathcal{W}_s = \mathcal{W}_s(g) = \{ \eta \in \mathcal{N} | [\ast \eta] = \pi ic_1(s) \}.$$  

Notice that $\mathcal{W}_s$ is a codimension $b_2^1$ affine subspace of $\mathcal{N}$. When $b_2^1 > 1$, $\mathcal{W}_s$ is of codimension two or more. One can choose a generic path $\eta_s$ connecting these two perturbations $\eta_1$ and $\eta_2$, such that for each $s \in [1, 2]$, $\eta_s \in \mathcal{N} \setminus \mathcal{W}_s$. This completes the proof. \hfill $\Box$

At the end of this subsection, we show the dependence on the basic perturbations and bundle-like metrics for the basic Seiberg-Witten invariant, when $b_2^1 = 1$. Choose an orientation of the one dimensional space $iH^2_1(M)$. There exists an unique unit $g$-harmonic basic two form $\omega$, i.e. $\|\omega\|_F = 1$. The wall $\mathcal{W}_s$ is identified with the solutions to the linear equation

$$[\ast \eta - \pi ic_1(s), \omega] = 0.$$  

We set $\mathcal{N}^{\pm} = \{ \eta \in \mathcal{N} | \pm [\ast \eta - \pi ic_1(s), \omega] > 0 \}$. Consider a family of $(g_t, \eta_t)_{t \in [-1, 1]}$ crossing the wall transversely once at $t = 0$, such that $g_t$ is locally constant near $-1$, 0 and 1. The one-parameter family moduli space

$$\mathcal{M}_{[-1, 1]} = \bigcup_{t \in [-1, 1]} \mathcal{M}_{g_t, \eta_t}(s)$$  

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has reducible solutions at \( t = 0 \). Under the Assumption 4.1, the reducible space \( \mathcal{M}^0 = \{(A, 0)| \frac{1}{i} F_A = \ddbar{\eta}^0 + d_\theta \alpha \}/G_0 \) is identified with the torus \( T^6_b \), where \( b_0^k \geq 1 \). We decompose the connection \( A \) as \( A = A_0 + \alpha + a \), where \( A_0 \) is a connection such that \( \frac{1}{i} F^-_A = \ddbar{\eta}^0 \) and \( a \) is a harmonic one-form.

**Lemma 4.7** Suppose that \( \mathcal{M}^0 \) is parameterized by \( a \in H^1_b(M)/H^1(M) \cap H^1(M, \mathbb{Z}) \) with the decomposition \( A = A_0 + \alpha + a \). If \( \ker(\mathcal{D}_{b,A}) = 0 \), then there is no irreducible solution connection \( A \) in \( \mathcal{M}[-1,1] \).

**Proof** The idea is similar to [25, Lemma 2.3.7]. Let \((A_t, \Psi_t)\) be a family of solution to the perturbed basic Seiberg-Witten equations with \((A_t, \Psi_t)|_{t=0} = (A, 0)\). Near \((A, 0)\) we write \((A_t, \Psi_t) = (A + a_t, \Psi_t)\). We let \((a_t, \Psi_t)\) satisfy

\[
\begin{align*}
\delta_\theta a_t &= 0 \\
\ddbar{b} a_t &= q(\Psi(t)) \\
\mathcal{D}_{b,A}(\Psi_t) + a_t \Psi_t &= 0
\end{align*}
\]

near \( t = 0 \). Locally, we write \( \Psi_t = \sum_{i \geq 1} t^i \Psi_i \) and \( a_t = \sum_{i \geq 1} t^i a_i \). Differentiating the third equation with respect to \( t \), gives the result. \( \square \)

**Definition 4.8** A reducible solution is called regular, if \( \ker \mathcal{D}_{b,A} = 0 \).

Therefore, by the above lemma we can perturb the equation such that regular part \( \mathcal{M}^0_{\text{reg}} \) of the reducible solutions is isolated from the irreducible solution in \( \mathcal{M}[-1,1] \).

In particular, when the foliation is taut(\( H^1_b(M) \equiv H^1(M) \)), we can find a generic perturbation such that there are only finitely many points in \( \mathcal{M}^0 \) meeting the irreducible solutions in \( \mathcal{M}[-1,1] \) and the kernel of the associated twisted Dirac operator \( \mathcal{D}_{b,\omega+a} \) is of dimension 1. Following the same arguments as in [25, Proposition 2.3.8], yields the following proposition.

**Proposition 4.9** Let \((M,F)\) satisfy the Assumption 4.1, and \( F \) is taut. Suppose that \((g_{-1}, \eta_{-1})\) and \((g_1, \eta_1)\) belong to the different parts which are separated by \( \text{afmW's} \), choosing an orientation of \( H^1_b(M) \). Then, we have the formula

\[
SW_{g_{-1}, \eta_{-1}}(M, F, s) - SW_{g_{-1}, \eta_{-1}}(M, F, s) = SF(\mathcal{D}_{A(\theta)}),
\]

where \( A(\theta) \) defines a connection joining \( A_{-1}, A_1 \) and the \( SF(\mathcal{D}_{A(\theta)}) \) denotes the spectral flow of the corresponding basic Dirac operator.

**Proof** We choose a family of metrics and perturbations \((g_t, \eta_t), t \in [-1,1]\) such that it crosses the set \( \mathcal{R} = \{(g_0, \eta)| \eta = \eta^0 + \ddbar{\eta^0} + \delta_\theta \alpha, [\eta^0] = \pi_1(c_1(s)), \eta \text{ is harmonic}\} \at t = 0 \) with finite singular points in \( \mathcal{M}^0 \). Similar to [25, Proposition 2.3.8], the difference \( SW_{g_{-1}, \eta_{1}}(M, F, s) - SW_{g_{-1}, \eta_{1}}(M, F, s) \) is equal to the spectral flow of the twisted basic Dirac operator along \( \mathcal{M}^0 \), which proves the proposition. \( \square \)

Summarizing the above arguments, yields the following results.

**Theorem 4.10** Suppose that \((M, F)\) satisfies the Assumption 4.1. Then for each transverse spin* structure \( s \) and for a generic bundle-like metric \( g \), we define the basic Seiberg-Witten invariant \( SW_{g,\eta}(M, F, s) \) by the signed counting of the moduli space \( \mathcal{M}_{g,\eta}^0(M, F, s) \). Moreover, we have the properties:

- If \( b_0^k > 1 \), then for a generic bundle-like metric and perturbation the basic Seiberg-Witten moduli space is a smooth compact manifold, and \( SW_{g,\eta}(M, F, s) \) is generically independent of the choice of bundle-like metrics and perturbations.
- If \( b_0^k = 1 \), then \( SW_{g,\eta}(M, F, s) \) depends only on the component of \( H^2_0(M) \setminus \pi c_1(s) \).
5 Chern-Simons-Dirac functional and moduli space for foliation

The purpose of this section is to give the preparation to show the compactness for the moduli space.

5.1 Chern-Simons-Dirac functional for foliation

Throughout this subsection, the following assumption holds for $(M,F)$.

**Assumption 5.1** Let $(M,F)$ be an oriented closed manifold with codimension 3 oriented Riemannian foliation $F$ and admits a transverse spin$^c$ structure $s$. Suppose that $H^1_b(M) \cap H^1(M,\mathbb{Z}) \subset H^1(M)$ is a lattice of $H^1_b(M)$ and $F$ is taut.

Fixing a bundle-like metric and a transverse spin$^c$ structure $s$, we define the basic Chern-Simons-Dirac functional over $M$ by

$$L(A,\Psi) = -\frac{1}{8} \int_M \left( A' - A_0^b \right) \wedge (F_{A'} + F_{A_0^b}) \wedge \chi_F + \frac{1}{2} \int_M (\Psi, D_A \Psi) d\text{vol}_M,$$

for any $(A,\Psi) \in \mathcal{A}_b(s) \times \Gamma_b(S)$.

The formal gradient of the Chern-Simons-Dirac functional is given by the following lemma.

**Lemma 5.2** It holds that

$$\text{grad}L(A,\Psi) = \left( \frac{1}{2} \ast (F_{A'} + \frac{1}{2} (A' - A_0^b) \wedge \kappa_b) - q(\Psi), D_{b,A} \Psi \right).$$

Note that the gradient is not gauge-invariant in general.

**Proof** Choosing a variation $(A + ta,\Psi + t\Phi) \in \mathcal{A}_b(s) \times \Gamma_b(S)$, one deduces that

$$\left. \partial_t(L(A + ta,\Psi + t\Phi)) \right|_{t=0}$$

$$= -\frac{1}{8} \int_M \left( 2a \wedge (F_{A'} + F_{A_0^b}) + (A' - A_0^b) \wedge 2da \right) \wedge \chi_F$$

$$+ \frac{1}{2} \int_M \left( \langle \Phi, D_A \Phi \rangle + \langle D_A \Psi, \Phi \rangle + \langle \Psi, a \cdot \Phi \rangle \right) d\text{vol}_Q \wedge \chi_F$$

$$= \int_M a \wedge \left( -\frac{1}{2} F_{A'} - \frac{1}{2} (A' - A_0^b) \wedge \kappa_b + \ast q(\Psi) \right) \wedge \chi_F$$

$$+ \int_M \text{Re}(D_A \Psi, \Phi) d\text{vol}_Q \wedge \chi_F$$

$$= \int_M \left( a - \frac{1}{2} \ast (F_{A'} + \frac{1}{2} (A' - A_0^b) \wedge \kappa_b) - q(\Psi) \right) d\text{vol}_M + \int_M \text{Re}(D_A \Psi, \Phi) d\text{vol}_M,$$

where we used Rummler formula (2.3) to deduce the second identity. \(\square\)

We say $(A,\Psi)$ is a critical point of $L$, if its gradient vanishes at $(A,\Psi)$. For any gauge action $u \in \mathcal{G}_b(M,S^1)$, we have

$$L(u(A,\Psi)) - L(A,\Psi) = \frac{1}{2} \int_M u^{-1} du \wedge (F_{A'}) \wedge \chi_F.$$

In general, the above term does depend on the choice of the representation of the cohomology class $[u] = [\frac{1}{2\pi} u^{-1} du]$. When $F$ is taut, the critical points of the basic Chern-Simons-Dirac-functional coincide with the solutions of the basic Seiberg-Witten equations (4). We consider the gradient flow to the Chern-Simons-Dirac functional

$$\frac{d}{dt} (A(t), \Psi(t)) = -\text{grad}L(A(t), \Psi(t))$$
for a path \((A(t), \Psi(t))\) of configuration space \(C_b(M, F; s) = A_b(M) \times \Gamma_b(S)\). Let \(\mathcal{M}(\alpha, \beta)\) denote the moduli space of trajectories connecting the critical points up to gauge, i.e. the solutions to the basic Seiberg-Witten equations

\[
\begin{cases}
\frac{1}{2} F_{\alpha}^+ = g(\Psi), \\
\partial_{\alpha}^+ \Psi = 0,
\end{cases}
\]

on \(\mathbb{R} \times M\) modulo the gauge action, such that \([A(t), \Psi(t)] \to [\alpha] as t \to -\infty and [A(t), \Psi(t)] \to [\beta] as t \to \infty\), where \([\alpha], [\beta] denote the gauge equivalence classes of the critical points. The components of this space have different dimensions corresponding to the different lifts of \([\alpha], [\beta]\). This is a manifestation of the fact that the quotient space \(B_b(M, F; s) = C_b(M, F; s)/G_b\) may have non-trivial fundamental group. We have a decomposition

\[
\mathcal{M}(\alpha, \beta) = \bigcup_{z \in \pi_1([\alpha], [\beta])} \mathcal{M}_z([\alpha], [\beta])
\]

as the union over the moduli spaces in a given relative homotopy class, where \(\pi_1([\alpha], [\beta])\) denotes all the homotopy classes of paths joining \([\alpha]\) and \([\beta]\). Given two critical points \(\alpha, \beta \in C(M, F; s)\), we define the quantity

\[
gr(\alpha, \beta) \in \mathbb{Z}
\]

by the spectral flow of the Hessian operator (5) of a path connecting them. This is well defined because the spectral flow is invariant under homotopy and the configuration space \(C(M, F; s)\) is simply connected. Moreover, such a number computes the formal dimension of the moduli space of trajectories connecting \(\alpha\) and \(\beta\) by the path \(z\), i.e. \(\dim \mathcal{M}_z([\alpha], [\beta])\).

### 5.2 Compactification of the moduli space

In this subsection, we give a compactification of the moduli space on the cylinder. The original idea is using the energy functional, which was introduced by Kroheimer and Mrowka [13, Chapter 5]. Here we gave a foliated version of their work. Recall that \(Z = I \times M, I \subset (-\infty, \infty)\), and \(\alpha, \beta\) are the critical points of the L. Denoting by \(\mathcal{M}(\alpha, \beta)\) the moduli spaces of trajectories in \(B_b(M, F; s)\) and \(\mathcal{M}(\alpha, \beta)\) the unparameterized moduli space. We show that by adding broken trajectories it can be compactified, which is denoted by \(\mathcal{M}^+(\alpha, \beta)\).

**Lemma 5.3** Let \((Z = I \times M, F)\) be a compact taut Riemannian foliation with a taut bundle-like product metric \(g\). For a basic one-form \(\alpha\) on \(Z\), satisfying the boundary condition \(\langle \alpha, \nu \rangle = 0\), where \(\nu\) denotes the outward unit vector field. If there exists a constant \(C_0\) such that \(\int_Z \beta_i \wedge \alpha \wedge \chi_F \in [-C_0, C_0]\) for each \(\{\beta_i\}\), where \(\{\beta_i\}\) is a basis of \(H^3_{t, c}(Z)\). Then, there are constants \(C_1, C_2\) such that

\[
\|\alpha\|^2_{L^2(Z)} \leq C_1 \int_Z |\delta_0 \alpha|^2 + |d_0 \alpha|^2 + C_2.
\]

**Proof** Recall that by Theorem 3.6, we have

\[
\int_Z |\nabla^T \alpha| + \text{Ric}^T(\alpha, \alpha) = \int_Z |\delta_0 \alpha|^2 + |d_0 \alpha|^2.
\]

Because of the product metric on this finite cylinder, \(\text{Ric}^T\) has a uniform bound. Moreover, by the \(L^2\)-bound of \(\alpha\), the estimate follows from the proof of [13, Lemma 5.1.2].

Recall that \((M, F)\) is a closed oriented taut Riemannian foliation with codimension 3, and it admits a transverse spin\(^*\) structure. For the spinor bundle \(S_Z = S^+ \oplus S^-\) on \(Z = I \times M\), we take \(S_Z = S \oplus S\). For the Clifford multiplication \(\rho_Z : TZ \to Hom(S_Z, S_Z)\), we take

\[
\rho_Z(\partial_t) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \rho_Z(v) = \begin{pmatrix} 0 & -\rho(v)^* \\ \rho(v) & 0 \end{pmatrix},
\]

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for \( v \in Q \). A time-dependent spin\(_c\) connection \( B \) on \( S \) gives a spin\(_c\) connection \( A \) on \( S_Z \), whose \( t \) component is an ordinary differentiation, i.e.

\[
\nabla_A = \frac{d}{dt} + \nabla_B.
\]

We call the connection \( A \) in temporal gauge. With a basic connection \( A \) as above, we have the basic Dirac operator

\[
D^+_b,A : \Gamma_b(S^+) \rightarrow \Gamma_b(S^-), \quad D^+_b,A = \frac{d}{dt} + \nabla_B.
\]

For a general basic connection \( A \), we write

\[
A = B + (c dt),
\]

where \( c \) is a basic function. The corresponding basic Seiberg-Witten equations for \((A, \Psi)\) are written as

\[
\begin{align*}
\frac{d}{dt} B - dc &= -(\frac{1}{2} \bar{\chi} F_B - q(\Psi)), \\
\frac{d}{dt} \Psi + c \Psi &= -\nabla_B \Psi.
\end{align*}
\]

(7)

We define the analytic energy by

\[
\mathcal{E}^a(A, \Psi) = 2(L(t_1) - L(t_2)) + \int_Z |\dot{\gamma}(t) + \text{grad}(L)(A, \Psi)|^2,
\]

(8)

where \( \gamma = (A, \Psi) \). By the transverse Weitzenböck formula \([10]\) and similar arguments of \([13, \text{Section 4.5}]\), one has that

\[
\mathcal{E}^a(A, \Psi) = \int_Z (|\nabla_A \Psi|^2 + \frac{\text{Scal}}{4} |\Psi|^2 + \frac{1}{4} |\Psi|^4) + \frac{1}{4} \int_Z |F_A|^2.
\]

The topological energy \( \mathcal{E}^{\text{top}} \) is defined by the twice of drop of the basic Chern-Simons-Dirac functional on the cylinder, i.e.

\[
\mathcal{E}^{\text{top}}(A, \Psi) = 2(L(t_1) - L(t_2)).
\]

For convenience, we denote the Seiberg-Witten map by \( \mathcal{S} \).

**Theorem 5.4** Let \( \gamma_n \in C(Z) \) be a sequence of solutions to the basic Seiberg-Witten equations on manifold with the codimension 4 foliation, \( \mathcal{S}(\gamma) = 0 \), on \( Z = [t_1, t_2] \times M \). Suppose that

\[
L(\gamma_n(t_1)) - L(\gamma_n(t_2)) \leq C.
\]

Then, there is a sequence of gauge transformations \( u_n \in \mathcal{G}(Z) \), such that, after passing to a subsequence, the sequence of transformed solutions \( \{u_n \gamma_n\} \) converges in the \( C^\infty \) topology on \([t_1', t_2'] \times M\) for a smaller interval \([t_1', t_2']\) in the interior of \([t_1, t_2]\).

**Proof** The formula (8) implies that there is a uniform bound for the analytic energy. We take a gauge-fixing action by Theorem 3.4. The arguments of \([13, \text{Theorem 5.1.1}]\) show that there are uniform bounds for the following terms

\[
\|\Phi_n\|_{L^4} \leq C, \quad \|F_{A_n}\|_{L^2} \leq C.
\]

For each \( a_n = A_n - A_0 \), there is a gauge action \( u_{a_n} \) to make \( a_n^1 = u_{a_n}(a_n) \) satisfy the Coulomb-Neumann condition (see \([13, \text{Page 102}]\)). Hence, for each \( a_n \) we have a gauge action \( v_n \) satisfying the equations (3) such that there is a uniform constant \( C \) (independent of \( n \)) to make the following estimate holds:

\[
\int_Z \beta_i \wedge a_n^1 \wedge \chi F \in [-C, C],
\]

(9)
where \( \{\beta_i\}_{1 \leq i \leq b} \) is dual basic to the lattice \( H^1_0(Z) \cap H^1(Z;\mathbb{Z}) \). This implies that there is a uniform \( L^2 \)-bound on \((A_n - A_0, \Psi_n)\) up to gauge. Let \((A, \Psi)\) be the weak limit of \((A_n, \Psi_n)\). One has that
\[
\sup E^n(A_n, \Psi_n) \geq E^n(A, \Psi).
\]
The drop of the Chern-Simons-Dirac functional is bounded above, which implies that up to subsequence the following holds
\[
E^n(A_n, \Psi_n) \to E^n(A, \Psi).
\]
Therefore, the sequence \((A_n - A_0, \Psi_n)\) converges in \( L^2 \) up to a subsequence. The remainder argument is parallel to [13, Page 107-108].

We also need to define the perturbation, which is similar to [13, Section 10, 11]. Let \( V_k(Z) \) be the \( L^2 \)-completion of \( \Omega^+(Z) \oplus \Gamma(Z, S^-) \).

**Definition 5.5 (c.f. [13, Definition 10.5.1])** A perturbation \( q \) is called \( k \)-tame, if it is a formal gradient of a continuous \( G_k(M) \)-invariant function on \( C(M) \) and satisfies the following properties:

1. the corresponding codimension 4 perturbation \( \hat{q} \) defines a smooth section \( \hat{q} : C_k(Z) \to V_k(Z) \) (see [13, Formula 10.2]), where \( Z = I \times M \);
2. for each \( i \in [1, k] \), the codimension 4 perturbation \( \hat{q} \) defines a continuous section \( \hat{q} : C_j(Z) \to V_j(Z) \);
3. the derivative
\[
D\hat{q} : C_k(Z) \to \text{Hom}(TC_k(Z), V_k(Z))
\]
extends to a map
\[
D\hat{q} : C_k(Z) \to \text{Hom}(TC_j(Z), V_j(Z)),
\]
for \( j \in [-k, k] \);
4. we have the estimate
\[
\|q(A, \Psi)\|_{L^2} \leq C(\|\Psi\|_{L^2} + 1),
\]
for some constant \( C \) and each \((A, \Psi) \in C_k(M)\);
5. for any reference connection \( A_0 \), we have
\[
\|\hat{q}(A, \Psi)\|_{L^1, A} \leq f_1(\|A - A_0, \Psi\|_{L^2, A_0}),
\]
where \( f_1 \) is a real function and \((A, \Psi) \in C_k(Z)\);
6. \( q \) defines a \( C^1 \)-section \( C_1(M) \to T_0 \).

We say \( q \) is tame, if it is \( k \)-tame for all \( k \geq 2 \).

Using the Theorem 5.4 and the same idea of [13, Proposition 16.2.1], one establishes the following proposition.

**Proposition 5.6 (c.f. [13, Proposition-16.2.1])** For any \( C > 0 \), there are only finitely many \([\alpha], [\beta]\) and pathes \( z \) with \( E^\top_q(z) \leq C \), such that the space \( \mathcal{M}^+_q([\alpha], [\beta]) \) is non-empty. Furthermore, each \( \mathcal{M}^+_q([\alpha], [\beta]) \) is compact.

## 6 Compactness of moduli space

The purpose of this section is for the preparation to construct the monopole Floer homologies in the next section.
6.1 Gluing trajectories

In this subsection we show the gluing theorem in gauge version. We set the blow-up configuration space of \((M, F)\) as the space of triples, see \([13, \text{Chapter 9}]\)

\[ C^*_k(M, F, s) = \{(A, s, \phi) | (A, \phi) \in C_k(M), \ s \in \mathbb{R}^{\geq 0}, \ ||\phi||_{L^2} = 1\}. \]

We define the quotient space \(B^*_k(M, F, s) = C^*_k(M, F, s, \mathfrak{g}_h + \mathfrak{l})\), which is a Hilbert manifold with boundary. The basic Seiberg-Witten equations naturally extend to the equations

\[
\begin{aligned}
\hat{D}_{b,A}\phi &= 0 \\
\frac{1}{2}g_F A &= s^2 q(\phi).
\end{aligned}
\]

We define

\[ \tilde{C}^*_k(Z, F, s) \subset A_k(Z, s) \times L^2_k(I, \mathbb{R}) \times L^2_k(S^+) \]

to be the subset consisting of triples \((A, s, \phi)\) with \(b(\phi(t))_{L^2(M)} = 1\) for each \(t \in I\). We have an involution map \(i : \tilde{C}^*_k(Z, F, s) \rightarrow \tilde{C}^*_k(Z, F, s)\) defined by \((A, s, \phi) \mapsto (A, -s, \phi)\). Similarly, we define the \(\tau\)-module for the configuration space for \((Z, F)\), where \(Z = I \times M\)

\[ C^*_k(Z, F, s) \subset A_k(Z, s) \times L^2_k(I, \mathbb{R}) \times L^2_k(S^+) \]

to be the subset consisting of triples \((A, s, \phi)\) with

\[ s(t) \geq 0, \ ||\phi(t)||_{L^2(M)} = 1, \]

for each \(t \in I\). There is a well-defined map

\[ \pi : C^*_k(M, F, s) \rightarrow C_k(M, F, s), \ (A, s, \phi) \mapsto (A, s\phi). \]

By this map, any vector field on \(C_k(M, F, s)\) lifts to a vector field on \(C^*_k(M, F, s)\). In order to get the transversality condition, we need to add a perturbation \(p\) as defined in (5.5) on \(\text{grad}(L)\). The sum \(\text{grad}(L) + p\) is a gauge invariant and gives rise to a vector field \(v^*_q\),

\[ v^*_q : B^*_{k+1/2}(M, F, s) \rightarrow T_{k-1/2}(M), \]

where \(T_{k-1/2}(M)\) denotes the \(L^2_{k-1/2}\)-completion of the tangent bundle of \(B_{k+1/2}^*(M, F, s)\). \(\text{grad^a}(L)\) is defined as follows

\[ \text{grad^a}(L)(A, r, \psi) = \left(\frac{1}{2}g_F A - r^2 q(\psi), \Lambda(A, r, \psi)r, \hat{D}_A \psi - \Lambda(A, r, \psi)\psi\right), \]

where \(\Lambda(A, r, \psi) = \langle \psi, \hat{D}_A \psi \rangle_{L^2}\). A trajectory \(\gamma(t) = (A(t), r(t), \psi(t))\) is a solution to the equations

\[
\begin{aligned}
\frac{\, d\!}{\, dt} A &= -(\frac{1}{2} A + F_A - r^2 q(\psi)), \\
\frac{\, d\!}{\, dt} r &= -\Lambda(A, r, \psi)r, \\
\frac{\, d\!}{\, dt} \psi &= -(\hat{D}_A \psi - \Lambda(A, r, \psi)\psi).
\end{aligned}
\]  

We call the perturbation \(q\) admissible, if all critical points of \(v^*_q\) are nondegenerate and moduli spaces of the flow lines connecting them are regular(see [13, Definition 22.1.1]). We categorize the set \(C\) of critical points in \(B^*_k(M, F, s)\) into the disjoint union of three subsets:

- \(C^*\), the set of irreducible points;
- \(C^\circ\), the set of reducible boundary stable(where the spinor part locates on the positive eigenspace part) critical points;
- \(C^\circ\), the set of reducible boundary unstable(where the spinor part locates on the negative eigenspace part) critical points.
We set
\[ M([a], [b]) = \bigcup_{z \in \pi_1([a], [b])} M_z([a], [b]) \]
as the union over the moduli spaces in a given relative homotopy class, where \( \pi_1([a], [b]) \) denotes the homotopy class of path connecting the two critical points in the quotient space. We recall two notions which are given in [13, Page 261].

**Definition 6.1** We say that a moduli space \( M([a], [b]) \) is boundary-obstructed, if \([a], [b] \) are reducible, \([a] \in C^s \) and \([b] \in C^u \).

**Definition 6.2** When the moduli space \( M_z([a], [b]) \) is not boundary-obstructed, we say that \( \gamma \) is regular, if the linearized Seiberg-Witten map \( Q_\gamma \) along \( \gamma \) is surjective for \([\gamma] \in M_z([a], [b]) \). In the boundary-obstructed case, we say \( \gamma \) is regular, if the linearized Seiberg-Witten map restriction along the boundary \( \partial Q_\gamma \) is surjective for \([\gamma] \in M_z([a], [b]) \). We say that \( M_z([a], [b]) \) is regular if its elements are all regular.

We topologize the space of unparameterized broken trajectories as follows[13, Page 276]. Choose an element \([\gamma] = ([\gamma_1], \cdots, [\gamma_n]) \in \tilde{M}_z^+([a], [b]) \), where \([\gamma_i] \in \tilde{M}_z((a_{i-1}, a_i]) \) is represented by a trajectory \([\gamma_i] \in M_z((a_{i-1}, a_i]) \).

Let \( U_i \in B_{k,loc}^1(Z, F, s) = C_{k,loc}^1(Z, F, s) / G_{k+1,loc} \) be an open neighborhood of \([\gamma_i] \) and \( T \in \mathbb{R}^+ \) be a positive number. We define \( \Omega = \Omega(U_1, \cdots, U_n, T) \) to be the subset of \( M_z([a][b]) \) consisting of unparameterized broken trajectories \( \hat{\delta} = ([\hat{\delta}_1], \cdots, [\hat{\delta}_m]) \) satisfying the following condition: there exists a map \((i, s) : \{1, \cdots, n \} \rightarrow \{1, \cdots, m \} \times \mathbb{R} \)
such that
\begin{itemize}
  \item \([\tau_{s(i)}][\delta_{s(i)}]) \in U_i, \)
  \item if \( 1 \leq i_1 < i_2 \leq n \), then either \( \epsilon(i_1) < \epsilon(i_2) \) or \( \epsilon(i_1) = \epsilon(i_2) \) and \( s(i_1) + T \leq s(i_2) \).
\end{itemize}

To prove the compactness theorem for the blow-up model, we need to get a bound of \( \Lambda \) of the moduli space. For any \( C > 0 \) and any \([a], [b] \) with energy \( E_q \leq C \) for which \( M^+([a], [b]) \) is non-empty, the space of broken trajectories \([\gamma] \in M^+([a], [b]) \) with energy \( E_q \leq C \) is compact. For a trajectory \( \gamma^+ \in M([a], [b]), \) we define \( K(\gamma^+) \) to be the total variation of \( \Lambda_q \) by (see [13, Section 16.3])
\[ K(\gamma^+) = \int_{\mathbb{R}} \left| \frac{d\Lambda_q(\gamma^+)}{dt} \right| dt. \]

Set
\[ K^+(\gamma^+) = \int_{\mathbb{R}} \left( \frac{d\Lambda_q(\gamma^+)}{dt} \right)^+ dt. \]

Proposition 5.6 gives a bound on the number of components for which the blow-down is non-constant. To get the energy bound, we need the proposition below.

**Proposition 6.3 (c.f. [13, Proposition 16.1.4])** The space \( \tilde{M}^+([a], [b]) \) of broken trajectories with topology energy \( E_q(\gamma) \leq C \) is compact.

Set two spaces \( Z^T \) and \( Z^\infty \) as \( Z^T = [-T, T] \times M \) and \( Z^\infty = (\mathbb{R}^\infty \times M) \) respectively. Let \( a \) be a critical point on \( C_k(M, F, s) \), we write \( \gamma_a \) as a translation-invariant solution on \( Z^T \) or \( Z^\infty \) in temporal gauge. We assume that \( a = (a_0, \Phi_0) \) is non-degenerate by choosing a generic perturbation. We define the quotient space
\[ B_k(Z^\infty, [a]) = C_k(Z^\infty, [a]) / G_{k+1}(Z^\infty), \]
where \( C_b(Z^\infty, [a]) = \{ \gamma \in C_{b, \text{loc}}(Z^\infty, F; s) \} \gamma - \gamma_a \in L^2_{k, b, A_0} \} \) and \( G_{k+1} = \{ u \in \Gamma_b(Z^\infty, S^1) \} u \in L^2_{k+1, \text{loc}}, 1 - u \in L^2_{k+1} \}. \) Let 
\[
K_{s,a}(M)
\]
be the \( L^2 \)-completion of the complement \( K_a \) to the gauge orbit, where \( K_a \) is the orthogonal complement to the gauge-orbit. Similarly, we denote \( K^- \) by the blow-up model of \( K \). Similar to \cite[Proposition 9.3.4]{13}, the proposition below holds.

**Proposition 6.4** Let \( J_{k, \gamma} \) be the image of \( d_\ast : L^2_{k+1, b}(M, iR) \to T_{k, \gamma}, \) via \( \xi \mapsto (-d\xi, \xi F_0) \), where \( T_{k, \gamma} \) denotes the tangent space at \( \gamma \). As \( \gamma \) varies over \( C^2_k(M) \), we define \( K_{k, \gamma} \) to be the subspace of \( T_{k, \gamma} \), which is orthogonal to \( J_{k, \gamma} \) with respect to the \( L^2 \)-inner product. Then, we have the decomposition
\[
T_{k, \gamma} = J_{k, \gamma} \oplus K_{k, \gamma}.
\]

**Proof** We denote by \( \gamma = (A_0, F_0) \). By doing integral by part, we define \( K_{k, \gamma} = \{ (\Phi, \Phi') \mid \Phi, \Phi' = 0 \} \). It is clear that \( K_{k, \gamma} \) is orthogonal to \( J_{k, \gamma} \). We need to show that
\[
T_{k, \gamma} = J_{k, \gamma} \oplus K_{k, \gamma}.
\]
It is sufficient to show that for any \( (\alpha, \phi) \), there is a unique solution to the equation
\[
\delta_k(\alpha, \phi) + d_\ast(\xi) = 0,
\]
which is equivalent to
\[
\Delta_k \xi + |\Phi_0|^2 \xi = c,
\]
where \( c = G_\ast(a, \phi) \). Since \( \Phi_0 \) is non-zero, this equation has a unique solution by Theorem 3.5. \( \square \)

Following the arguments of \cite[Proposition 9.3.5, 9.4.1]{13}, we can show the similar decompositions for \( \sigma \)-model and \( \tau \)-model.

By doing the integral part with the taut condition, one has that the slice
\[
S_{k,a}(Z^T) \subset C_{k,b}(Z^T)
\]
can be represented by
\[
S_{k,a}(Z^T) = \{ (A_0 + a, \Phi) \mid \delta_k a + iR_e(i\Phi, \Phi) = 0, (a, n) \mid_{\partial Z^T} = 0 \},
\]
where \( \gamma_a = (A_0, F_0) \). Similarly, we define the slices \( S^c_{k,a} \) and \( S^\tau_{k,a} \) (see \cite[Page 144, Page 147]{13}). We can run the arguments in \cite[Section 18.4]{13} in the foliation case. Here we give a sketch. The boundary of \( Z^T \) is \( M \sqcup M \), we have the restriction map
\[
r : C^\tau_{k,b}(Z^T) \to C^\tau_{k,b}(M \sqcup M) \times L^2_{k-\frac{1}{2}, b}(M \sqcup M, iR),
\]
where the second component is defined by the normal component of the basic connection \( A \) at the boundary and \( M \) is a copy of \( M \) with the reversing orientation by reversing the orientation of \( Q \). For the non-degeneracy of the Hessian operator (5) at \( a \), we have the decomposition \( K_{k-\frac{1}{2}}^\gamma |_a = K^+ \oplus K^- \). Let \( H_\hat{M}^+ \) and \( H_\hat{M}^- \) be two subspaces defined by
\[
H_\hat{M}^+ = \{ 0 \} \oplus K^- \oplus L^2_{k-\frac{1}{2}, b}(M, iR),
\]
\[
H_\hat{M}^- = \{ 0 \} \oplus K^+ \oplus L^2_{k-\frac{1}{2}, b}(M, iR),
\]
We define \( H = H_\hat{M}^+ \oplus H_\hat{M}^- \) and define \( \Pi_M = \Pi_M^+ \oplus \Pi_M^- \) by the projection to the space \( K^- \oplus K^+ \), i.e.
\[
\Pi_M : T_{k-\frac{1}{2}} |_a (M \sqcup M) \oplus L^2_{k-\frac{1}{2}}(M \sqcup M, iR) \to K^- \oplus K^+.
\]
For \( \gamma \) in a small neighborhood of \( \gamma_a \in C_{k,b}(Z, F, s) \) on \( Z = Z^T \) or \( Z^\infty \), we consider the equations
\[
\left\{
\begin{array}{l}
\sigma_\gamma = 0, \\
\gamma \in S^\tau_{k,a}(Z), \\
(\Pi_M \ast r)(\gamma) = h,
\end{array}
\right.
\]
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where \( h \in H \). We write the equations as
\[
\begin{aligned}
(Q_{\gamma_a} + \alpha)\gamma &= 0, \\
(\Pi_M \circ r)\gamma &= h,
\end{aligned}
\]
where \( Q_{\gamma_a} \) is defined by \( D_{\gamma_a}^+ \mathfrak{g} \) and \( \alpha \) denotes the remainder terms. We write \( Q_{\gamma_a} = \frac{d}{dt} + L_b \), let \( H^\pm_L \) be the spectral subspaces of \( L_b \) in \( L^2_\mathbb{F}(M) \). By Proposition 6.6, the linear map
\[
(Q_{\gamma_a}, \Pi_L \circ r)
\]
is an isomorphism, where \( \Pi_L^- \) is the spectral projection with kernel \( H^+_L \). Let \( K \) denote the kernel of \( Q_{\gamma_a} \), the domain can be decomposed as \( C \oplus K \). We rewrite the above operator as
\[
\begin{pmatrix}
Q_{\gamma_a}|_C & 0 \\
0 & (\Pi_L \circ r)|_K
\end{pmatrix}.
\]

The isomorphism of two components on diagonal implies that the matrix is an invertible operator. Thus, we verified the abstract hypothesis [13, Hypothesis 18.3.1]. By the definition of the tame perturbation, the abstract hypothesis [13, Hypothesis 18.3.3] follows. Setting \( \mathcal{K} = \mathcal{K}^+ \oplus \mathcal{K}^- \), there is an \( \eta > 0 \) and two maps from the \( B_\eta(\mathcal{K}) \) to the slices parameterizing the subsets of the set of solutions, i.e.
\[
u(T, \cdot) : B_\eta(\mathcal{K}) \rightarrow S^*_{k,\gamma_a}(Z^T) \cap \mathfrak{g}^{-1}q(0), \quad \nu(\infty, \cdot) : B_\eta(\mathcal{K}) \rightarrow S^*_{k,\gamma_a}(Z^\infty) \cap \mathfrak{g}^{-1}q(0),
\]
for some positive number \( \eta > 0 \). By the parallel arguments of the proof of [13, Theorem 18.2.1], one has the proposition below.

**Proposition 6.5** There is an \( \eta_0 \), such that all \( \eta < \eta_0 \), there is a number \( \eta' \), independent of \( T \), such that:

1. the map
   \[
   \mu : \{ \gamma \in S^*_{k,\gamma_a}(Z^T) \mid \| \gamma - \gamma_a \|_{L^2_k} \leq \eta \} \rightarrow \mathcal{B}^+_k(Z^T)
   \]
is a diffeomorphism onto its image, where \( \mathcal{B}^*_k(Z^T) \) denotes the quotient of \( C^*_k(Z^T) \) by the gauge action;

2. the image of the above map contains all gauge-equivalent classes \( [\gamma] \in \mathcal{B}^*_k(Z^T) \) represented by the elements \( \gamma \) satisfying
   \[
   \| \gamma - \gamma_a \|_{L^2_k} \leq \eta'.
   \]

By the similar arguments, we have the following propositions for the foliated case.

**Proposition 6.6** (c.f. [13, Proposition 17.2.7]) Let \( Z = (-\infty, 0] \times M \) and \( D_0 : C^\infty(Z, E) \rightarrow L^2(Z, E) \) be a transverse elliptic operator of the form
\[
D_0 = \frac{d}{dt} + L_0,
\]
where \( L_0 : C^\infty(M, E_0) \rightarrow C^\infty(M, E_0) \) is a transverse self-adjoint elliptic operator on \( M \). Suppose that the spectrum of \( L_0 \) does not contain zero. Then, the operator
\[
D_0 \oplus (\Pi_0 \circ r) : L^2_j(Z, E) \rightarrow L^2_{j-1}(Z, E_0) \oplus (H^-_0 \cap L^2_j(M, E_0))
\]
is an isomorphism for all \( j \geq 1 \), where \( \Pi_0 \) denotes the projection to the negative eigen-vector part of \( L_0 \). Moreover, it holds that \( H^-_0 \cap L^2_j(M, E_0) = \text{Im}(r |_{\text{ker}(D_0)}) \).
The proof only needs the parametrix patching and regularity. Let \( Z = I \times M \) be a closed finite cylinder, suppose that \( I = I_1 \cup I_2 \) with \( I_1 \cap I_2 = \emptyset \). We denote by \( Z = Z_1 \cup Z_2 \), where \( Z_1 = I_1 \times M \) and \( Z_2 = I_2 \times M \). Let \( D : C^\infty(Z, E) \to L^2(Z, E) \) be a transverse elliptic operator of the form
\[
D = \frac{d}{dt} + L_0 + h(t),
\]
where \( L_0 \) is a self-adjoint operator on \( M \), \( h : L^2(Z, E) \to L^2(Z, E) \) is a bounded operator. We set \( D_1, D_2 \) as the restriction of these operators to the two subcylinders respectively, and set
\[
H^j_{-\frac{1}{2}} \subset L^2_{j-\frac{1}{2}}(\{0\} \times M, E_0)
\]
as the image of the \( \ker(D_j) \) under the restriction map
\[
r_j : L^2_j(Z, E) \to L^2_{j-\frac{1}{2}}(\{0\} \times M, E_0).
\]
Denoting by \( D_0 = -\frac{d}{dt} + L_0 \). We have the following lemma.

**Proposition 6.7 (c.f. [13, Proposition 17.2.8])** Suppose that \( D : L^2_j(Z) \to L^2_{j-1}(Z) \) is surjective for \( 2 \leq j \). Then, we get the decomposition
\[
L^2_{j-\frac{1}{2}}(\{0\} \times M, E_0) = H^1_{j-\frac{1}{2}} + H^2_{j-\frac{1}{1}}.
\]
Conversely, if the above formula holds and \( D_1, D_2 \) are surjective, then \( D \) is surjective.

Let \( Z \) be a finite cylinder. We set \( \tilde{M} = \{[\gamma] \in \tilde{B}^+_k(Z) | \beta_k(\gamma) = 0 \} \). We have the following theorem.

**Theorem 6.8 (c.f. [13, Theorem 17.3.1])** The subspace \( \tilde{M}(Z) \subset \tilde{B}^+_k(Z) \) is a closed Hilbert submanifold. The subset \( M(Z) \) is a Hilbert submanifold with boundary, i.e. it is identified with the quotient of \( \tilde{M}(Z) \) by the involution \( \iota \).

Let \( [\gamma] \in \tilde{M}(Z) \) and let \( \bar{a} \) and \( a \) be the restrictions of \( \gamma \) to the two boundary components. We have the restriction maps
\[
R_+ : \tilde{M}(Z) \to B^+_k(M), \quad R_- : \tilde{M}(Z) \to B^+_k(M).
\]

**Theorem 6.9 (c.f. [13, Theorem 17.3.2])** Let \( \gamma, a \) and \( \bar{a} \) be as above, and let \( \Pi : K^+_k(\tilde{M}) \oplus K^+_k(M) \to K^+_k(\tilde{M}) \oplus K^+_k(M) \) be the projection with kernel \( K^+_k(M) = 1_{\tilde{M}(M)} \). Then, the two composition maps \( \Pi (DR_- R_+ \gamma) \) and \( (1 - \Pi)(DR_- R_+) \) are Fredholm and compact respectively, where \( DR_- \) and \( DR_+ \) denote the derivatives of \( R_- \) and \( R_+ \) respectively.

We can prove the foliated version of [13, Lemma 16.5.3, Proposition 16.5.2, Proposition 16.5.5]. By Lemma 3.1, Proposition 6.7, Theorem 6.8 and Theorem 6.9, there is no difficulty to apply similar arguments of [13, Section 19.1, Section 19.2, Section 19.3 and Section 19.4], one establishes the following theorems.

**Theorem 6.10 (c.f. [13, Theorem 19.5.4])** Suppose that the moduli space \( \bar{M}'([a], [b]) \) is \( d \)-dimensional and contains irreducible trajectories, such that the moduli space \( M'([a], [b]) \) is a \( (d-1) \)-dimensional space stratified by manifolds (see [13, Definition 16.5.1]). Let \( M' \subset M'([a], [b]) \) be any component of the codimension-1 stratum. Then along \( M' \), the moduli space is either a \( C^0 \)-manifold with boundary, or has a codimension-1 \( \delta \)-structure in the sense of [13, Definition 19.5.3]. The latter occurs only when \( M' \) consists of 3-component broken trajectories, with the middle component boundary-ordered.
6.2 Finite result on moduli space

In this subsection, we will give some properties for the compactified moduli space, which are necessary to construct the basic monopole Floer homologies without using the Novikov ring.

Recall that a reducible critical point \( a \) corresponds to a pair \((\alpha, \lambda)\), where \( \alpha = (B, 0) = \pi(a) \) is a critical point in \( C_s(M, F, s) \), and \( \lambda \) is an element of the spectrum of \( D_{B,q} \). For such \( a \), we define \( \iota(a) \) (see [13, Formula 16.2]) by

\[
\iota(a) = \begin{cases} 
|\text{Spec}(D_{B,q}) \cap [0, \lambda]|, & \lambda > 0, \\
1/2 - |\text{Spec}(D_{B,q}) \cap [0, \lambda]|, & \lambda < 0.
\end{cases}
\]

We denote by \( M^\text{red}([a], [b]) \subset M([a], [b]) \) the subspace consisting of all the reducible trajectories. For the moduli space of reducible trajectories, we have simple structure, i.e. it is always a manifold without boundary. For its dimension, we have the formula (see [13, Formula 16.9])

\[
\dim(M^\text{red}([a], [b])) = g_rz([a], [b]) = gr_z([a], [b]) - o[a] + o[b],
\]

where \( o[a] = 0 \) when \( a \in C^s \) and \( o[a] = 1 \) when \( a \in C^u \). For an irreducible critical point \( a \), we set \( \iota(a) = 0 \). If \([a]\) and \([a']\) are two critical points whose images under \( \pi \) equal to the same critical point \([\alpha]\) \( \in B_k(M) \), then we have the following identity

\[
gr_{z_\alpha}([a], [a']) = 2(\iota(a) - \iota(a'))
\]

for a trivial homotopy class \( z_\alpha \) (see [13, Formula 16.3]).

**Lemma 6.11** Suppose that all moduli spaces are regular and there is positive number \( C_0 > 0 \) such that

\[
\mathcal{E}^\text{top}_q(z_u) + C_0 gr_{z_u}([a], [a]) = 0,
\]

where \( z_u \) is the closed loop joins \( a \) to \( ua \) for any \( u \in G_0(M) \). Then, there exists a constant \( C \) such that for every \([a], [b]\) an each broken trajectory \([\gamma]\) \( \in M^+([a], [b]) \), we have the energy bound

\[
\mathcal{E}^\text{top}_q(\gamma) \leq C + C(\iota([a]) - \iota([b])).
\]

**Proof** The idea is similar to [13, Lemma 16.4.4]. Let \([\gamma] = ([\gamma_1], \ldots, [\gamma_l])\) be a broken trajectory in \( M^+_s([a], [b]) \) with \([\gamma_i] \in M_z([a_{i-1}], [a_i]) \). The space \( M_z([a_{i-1}], [a_i]) \) is non-empty, and it is manifold of dimension 1, possibly with boundary. We have that \( \dim(M_z([a_{i-1}], [a_i])) \) is either \( gr_z([a_{i-1}], [a_i]) - 1 \) or \( gr_z([a_{i-1}], [a_i]) \). In either case, \( gr_z([a_{i-1}], [a_i]) \geq 0 \). By adding the grading, it holds that

\[
gr_z([a_0], [a_l]) \geq 0.
\]

The energy \( \mathcal{E}^\text{top}_q(\gamma) \) is equal to \( \mathcal{E}^\text{top}_q(z) \), defined by the twice of the change in \( L \) along any path \( \zeta \) in \( C^0(M, F, s) \) whose image \( \zeta \in B^0(M, F, s) \) belongs to the class \( z \in \pi_1(B^0(M, F, s), [a], [b]) \). By the condition, we have that the quantity \( \mathcal{E}^\text{top}_q(w) + C_0 gr_{z_\omega}([a], [a]) \) depends only on \([a]\) and \([b]\) not on the homotopy class. This implies that the term

\[
\mathcal{E}^\text{top}_q(w) + C_0 (gr_{z_\omega}([a], [a]) - 2\iota(a) + 2\iota(b))
\]

depends only on the critical points \([\alpha] = [\pi a] \) and \([\beta] = [\pi b] \). Since there are only finitely many critical points in \( B(M, F, s) \), there is a constant \( C \) such that this quantity is at most \( C \), which proves the lemma.

**Remark:** In particular when \( b^1_0 = 0 \), the above lemma automatically holds.

In the 3-manifold case, i.e. \( F = 0 \), Kroheimer and Mrowka consider the quantity

\[
\mathcal{E}^\text{top}_q(z) + 4\pi^2 gr_z([a], [b]),
\]

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Proof is non-empty.  

where $z$ is a homotopy class connecting $[a], [b]$. When $b^1 > 0$, the difference of the above quantity for two different homotopy classes is the class of a closed loop whose lift to the configuration space joins $a$ to $ua$, for some gauge action $u$. By the Atiyah-Singer theory on closed oriented manifolds, the difference is zero. For the basic Dirac operator $D_A$, Brüning, F. W. Kamber, K. Richardson gave an expression for its index [6]. They showed that

$$\text{Ind}(D) = \int_{\bar{M}_0/F} A_{0,b}[d\bar{x}] + \sum_{j=1}^{r} \beta(M_j),$$

$$\beta(M_j) = \frac{1}{2} \sum_{\tau} \frac{1}{n_{\tau} \text{rank}(W_\tau)} (-\eta(D_{j}^{S^+ \tau}) + h(D_{j}^{S^+ \tau})) \int_{\bar{M}_j/F} A^{\tau}_{j,b}(x)[d\bar{x}],$$

where the integrands $A_{0,b}, A^{\tau}_{j,b}(x)$ are similar to the Atiyah-Singer integrands and notations are explained in their paper. Here we pose a question.

**Question:** For $b^1 > 0$, under what topological condition, there is a constant $C_0$, such that for any non-degenerate critical point $\mathbf{s}$, we have

$$C_0 \text{gr}(a, ua) - \int_M [F_A] \wedge [u] \wedge \chi_F = 0,$$

where $A$ is the connection component of $a$ and $u \in \mathcal{G}_d(M)$.

By Lemma 6.11, we have the following proposition, which is analog to [13, Proposition 16.4.1].

**Proposition 6.12** Suppose all the moduli spaces $\bar{M}_z([a], [b])$ are regular and (12) holds. Then, there are finitely many homotopy classes $z$ for which space $\bar{M}^+([a], [b])$ is non-empty.

**Proposition 6.13** Suppose (12) holds and the moduli space $\bar{M}_z([a], [b])$ is regular. If $c_1(s) = 0 \in H^2(M)$, then for a given $[a]$ and $d \geq 0$, there are finitely many pairs $([b], [z])$ such that the moduli space $\bar{M}^+([a], [b])$ is non-empty and of dimension $d$. If $c_1(s) \neq 0 \in H^2(M)$, then for a generic perturbation there are only finitely many triples $([a], [b], z)$ for which the moduli space $\bar{M}^+([a], [b])$ is non-empty.

**Proof** The idea is no different to [13, Proposition 16.4.3], here we just give a sketch of the proof to the case $c_1(s) = 0$. The functional $L$ descends to a well-defined function on $\mathcal{B}^\infty(M, F, s)$, which is pulled back from $\mathcal{B}(M, F, s)$. Since the image of critical points in $\mathcal{B}(M, F, s)$ is finite, $L$ takes finitely many values, the energy $E^\text{top}_q$ of a trajectory is the twice of the drop of $L$, so there is a uniform bound on the energy of all solutions. The expression (13) depends on $[\pi a]$, $[\pi b]$, which takes only finitely many values. By the condition that the dimension is bounded and $[a]$ is fixed, we have that $\iota([b])$ is uniform bounded, which leaves finitely many choices. 

\[ \square \]

**7 Basic monopole Floer homologies on manifold with codimension 3 foliation**

In this section, we show the main result of this paper, i.e. to construct the basic monopole Floer homologies.

**7.1 Basic Seiberg-Witten Floer homology for $b^1 > 1$**

In this subsection, we assume that $(M, F)$ is an oriented closed taut Riemannian foliation admitting a transverse spin$^c$ structure and whose basic first deRham cohomology is nontrivial. Note that $\mathcal{B}(M, F, s)$ is not simply connected in general, which implies that the index of critical points $\text{gr}([a], [b]) \in \mathbb{Z}$

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might not be well defined. However we can still define relative gradings. On the other hand, the components of the moduli space $\mathcal{M}([a],[b])$, trajectories connecting the critical points mod the gauge action, might have different dimensions corresponding to the different lifts of $[a]$ and $[b]$, where $[a]$ and $[b]$ are the gauge equivalence classes of the critical points. Recall that we decompose the space of trajectories $\mathcal{M}([a],[b]) = \bigcup_{\pi_1([a],[b])} \mathcal{M}_2([a],[b])$ as the union over the moduli spaces of different relative homotopy classes, where $\pi_1([a],[b])$ denotes the homotopy class of path connecting the two critical points in the quotient space.

For one critical point $[a] \in \mathcal{B}(M,F,s)$, we might have different lifts in $\mathcal{C}(M,F,s)$, say $\alpha$ and $\beta\alpha$, we can measure their spectral by the following index,

$$gr(\alpha, \beta\alpha) = Ind(\mathcal{D}^+_\alpha),$$

where $Ind(\mathcal{D}^+_\alpha)$ denotes the index of the basic Dirac operator on the product space $(M \times S^1, F)$.

**Proposition 7.1** The index $Ind(\mathcal{D}^+_\alpha)$ defined above lifts to a homomorphism

$$Ind : \pi_0(\mathcal{G}) \rightarrow \mathbb{Z}.$$

**Proof** We need to show that for different critical points the index is unchanged, since it is clear to see that for the same homotopy class, the index is well-defined. For another critical point $\beta$, the connection difference is a one-form, which is a compact operator. Hence the index lifts to a homomorphism. \hfill \square

We define

$$d(s) = \gcd(Ind : \pi_0(\mathcal{G}) \rightarrow \mathbb{Z}).$$

For two distinct irreducible critical points $\alpha$ and $\beta$, we denote by $\mathcal{M}^i([\alpha],[\beta])$ the dimension $i$ component of $\mathcal{M}([\alpha],[\beta])$. Let $\mathcal{M}([\alpha],[\beta])$ be the unparameterized space of $\mathcal{M}([\alpha],[\beta])$, i.e. $\mathcal{M}([\alpha],[\beta]) = \mathcal{M}([\alpha],[\beta]) / \mathbb{R}$. At the irreducible critical points, the slice decomposition of Proposition 6.4 holds. By Theorem 6.10, we have the following proposition.

**Proposition 7.2** (c.f. [26, Corollary 3.1.24]) Suppose that $[a_0], [a_2]$ are two irreducible critical points with the relative index $gr([a_2],[a_0]) = 2 \mod d(s)$. Then the boundary of $\mathcal{M}^2([a_0],[a_2])$ consists of union

$$\bigcup_{[a_1] \in \text{Crit}} \mathcal{M}^1([a_0],[a_1]) \times \mathcal{M}^1([a_1],[a_2]),$$

where $a_1$ runs over critical points with $gr([a_1],[a_0]) = 1 \mod d(s)$ and $\text{Crit}$ denotes the set of irreducible critical points in $\mathcal{B}(M,F,s)$.

We define the relative Floer complex $C(M)$ by the complex generated by the irreducible critical points of Chern-Simon-Dirac functional with grading given by the relative indices $\mathbb{Z}_{d(s)}$ or $\mathbb{Z}$, i.e.

$$C(M) = \bigoplus_{\alpha \in \text{Crit}} \mathbb{Z}_{2\alpha}.$$

The boundary operator of the complex is defined by

$$\partial : C(M) \rightarrow C(M), \quad \partial([\alpha]) = \sum_{[b]} 2\mathcal{M}^1([\alpha],[b]),$$

where $2\mathcal{M}^1([\alpha],[b]) \in \mathbb{Z}_2$ denotes the signed number of points in $\mathcal{M}^1([\alpha],[b]) \mod 2$.

**Lemma 7.3** $\frac{d^2}{dx^2} = 0$.

**Proof** By definition we have that $\partial^2([\alpha]) = \sum_{[b]} 2\mathcal{M}^1([\alpha],[b])2\mathcal{M}^1([b],[c])([c])$, where $[b]$ runs over the irreducible critical points with relative index 1. By the above proposition, it is known that each term $2\mathcal{M}^1([\alpha],[b])2\mathcal{M}^1([b],[c])([c])$ is the sum of the number of oriented boundary points of a compact 1-dimensional manifold, which is zero. \hfill \square
We define the basic Seiberg-Witten Floer homology as \( HF(M,F,s,\eta,g) = \ker(\partial)/\text{Im}(\partial) \), which is \( \mathbb{Z}_{d(s)} \)-relative grading (or \( \mathbb{Z} \)-grading).

**Proposition 7.4** Suppose that \((M,F)\) satisfies the Assumption 5.1. Then, for \( b_1^s > 1 \), we have that the relative Floer homology is independent of the taut bundle-like metrics and perturbations. We denote the basic Seiberg-Witten Floer homology group by \( HF(M,F,s) \).

**Proof** For a bundle-like metric \( g \), it corresponds to a triple

\[
g \leftrightarrow (g_F,g_Q,s),
\]

where \( g_F \) is the leafwise restriction, \( s \) corresponds to the decomposition

\[
s : Q \to TM, \pi_Q \circ s = \text{Id}_Q
\]

and \( g_Q \) is the transverse restriction. It is clear to see that the domain \( A_b \times \Gamma_b(M,S) \) and the Seiberg-Witten equations (4) are independent of the leafwise metric \( g_F \) and the decomposition \( s \). For two distinct leafwise metrics \( g_F \) and \( g_F' \) with the same \((p,l)\), the two Sobolev spaces \( L^2_p \) and \( L^2_p' \) are mutually equivalent to each other. Therefore, we have that the basic Seiberg-Witten Floer homology groups are invariant under the leafwise metric. For two distinct decompositions \( s \) and \( s' \), we can apply the same argument, as the character form \( \chi_F \) only depends on the leafwise metric \( g_F \) and the decomposition \( s \).

The remaining part is to verify that the Floer homology group is independent of the generic basic perturbation and metric \( g_Q \). The idea is exactly the same as Floer’s original proof [9].

\[\square\]

### 7.2 Basic monopole Floer homologies for \( b_1^s = 0 \)

The purpose of this subsection is to construct the basic monopole Floer homologies and show that they are independent of the perturbations and taut bundle-like metrics in a special case \( b_1^s = 0 \). We define the basic monopole Floer homologies \( \overline{HM}_*(M,F,s,F) \), \( \tilde{H}_M(M,F,s,F) \) and \( \tilde{H}M_*(M,F,s,F) \) by the homologies of the chain complexes freely generated by \( \mathcal{C} = C^s \cup C^u \), \( \mathcal{C} = C^s \cup C^u \) respectively (see [13, Section 22]), where \( F = \mathbb{Z}_2 \). The differentials on them are given in components as

\[
\partial = \begin{pmatrix} \partial_{s}^o & \partial_{s}^u \end{pmatrix}, \quad \tilde{\partial} = \begin{pmatrix} \partial_{u}^o & \partial_{u}^u & \partial_{u}^o \partial_{s}^o & \partial_{u}^o \partial_{s}^u & \partial_{u}^u \partial_{s}^o & \partial_{u}^u \partial_{s}^u \end{pmatrix}, \quad \hat{\partial} = \begin{pmatrix} \partial_{u}^o & \partial_{u}^u & \partial_{u}^o \partial_{s}^o & \partial_{u}^o \partial_{s}^u & \partial_{u}^u \partial_{s}^o & \partial_{u}^u \partial_{s}^u \end{pmatrix}.
\]

The linear maps

\[
\partial^o : C^o \to C^o, \quad \partial^u : C^o \to C^u,
\]

\[
\partial^u : C^u \to C^o, \quad \partial^u : C^u \to C^u
\]

are defined by the formula

\[
\partial^o[a] = \sum_{[b] \in C^o} \tilde{\mathcal{M}}([a],[b]) [b], \quad [a] \in C^o,
\]

where \( \tilde{\mathcal{M}}([a],[b]) \in \mathbb{F} \) is the signed counting number, the other three are defined similarly. By considering the number \( \tilde{\mathcal{M}}^{red}([a],[b]) \), we similarly define the linear maps

\[
\partial^o : C^s \to C^s, \quad \partial^u : C^s \to C^s,
\]

\[
\partial^u : C^u \to C^o, \quad \partial^u : C^u \to C^u.
\]

When \( b_1^s = 0 \), it is clear that for a given \([a]\), there are finitely many pairs \(([b],z)\) such that the moduli space \( M_z([a],[b]) \) is non-empty and of dimension 1.
Proposition 7.5 (c.f. [13, Proposition 22.1.4])

$$\bar{\partial}^2 = 0, \quad \partial^2 = 0, \quad \bar{\partial}^2 = 0.$$  

Proof The proof is by showing that $\bar{\partial}^2 = 0$, which is the same as the blow-down case (Lemma 7.3), and the following identities:

1. $\partial^2 \partial^2 + \partial^2 \bar{\partial}^2 + \bar{\partial}^2 \partial^2 = 0$;
2. $\partial^2 \partial^2 + \partial^2 \bar{\partial}^2 + \bar{\partial}^2 \partial^2 = 0$;
3. $\partial^2 \partial^2 + \partial^2 \bar{\partial}^2 + \bar{\partial}^2 \partial^2 = 0$;
4. $\partial^2 \partial^2 + \partial^2 \bar{\partial}^2 + \bar{\partial}^2 \partial^2 = 0$.

Each of the four formulas is proved by considering a moduli space $\tilde{M}_+([a], [b])$ of dimension 1. By Theorem 6.10, we can run the similar arguments of the proof in [13, Proposition 22.1.4].

We give a grading for these homologies. Let $\mathcal{P}$ be the space of the perturbations. We define $\mathcal{J}$ by the quotient of $\mathcal{B}^\circ (M, F, a) \times \mathcal{P} \times \mathbb{Z}/\sim$, where the equivalent relation $\sim$ is defined as follows (see [13, Section 22.3]): for any two elements $([a], q_1, m), ([b], q_2, n) \in \mathcal{B}^\circ (M, F, a) \times \mathcal{P} \times \mathbb{Z}$, let $\zeta$ be a path joining $[a]$ and $[b]$ and $\mathfrak{p}$ be a path of perturbation joining $q_1$ and $q_2$. We have a Fredholm operator $P_{\zeta, \mathfrak{p}}$ as defined on (11), we say that $([a], q_1, m) \sim ([b], q_2, n)$, if there is a path $\zeta$ such that

$$\text{Ind}(P_{\zeta, \mathfrak{p}}) = n - m.$$ 

The map $([a], q, m) \mapsto ([a], q, m + 1)$ descends to $\mathcal{J}$, and raises to an action of $\mathbb{Z}$.

Note that the above construction of the index set $\mathcal{J}$ is also available when $b_k^1 > 0$. Let $q$ be a fixed admissible perturbation, for a critical point $[a]$, we define its grading by

$$gr([a]) = ([a], q, 0)/\sim \in \mathcal{J}.$$ 

For reducible critical points, we define the modified grading by

$$\bar{gr}([a]) = \begin{cases} 
gr([a]) & [a] \in C^s \\
gr([b]) - 1 & [a] \in C^u.
\end{cases}$$

We show that the basic monopole Floer homologies is independent of the generic perturbations and bundle-like metrics. Let $W = [0, 1] \times M$ and $W^* = (-\infty, 0] \times M \cup W \cup [1, \infty) \times M$. To tell the distinguish, we denote the left boundary $\{0\} \times M$ with metric by $Y_-$ and perturbation and the right boundary $\{1\} \times M$ with another metric and perturbation by $Y_+$. We consider the moduli space $M([a], W^*; [b])$, as defined in [13, Section 25]. Using broken trajectories, we denote its compactification by $M^+([a], W^*; [b])$.

Fix a positive integer $d_0$, we consider a pair $([a], [b])$ for which the moduli space $M([a], W^*; [b])$ or $M^+([a], W^*; [b])$ has dimension $d_0$ at most. To prove the independence of the metrics, we need to define the homomorphism maps by the trivial cobordism, which are given by counting the number of solutions in the zero-dimensional moduli spaces. We define linear operators,

$$m^s_o : C^s_o(Y_-) \to C^s_o(Y_+), \quad m^u_o : C^u_o(Y_-) \to C^u_o(Y_+)$$

by

$$m^s_o(-) = \sum_{[a] \in C^s_o(Y_-)} \sum_{[b] \in C^s_o(Y_+)} \sharp M([a], W^*, [b]),$$

for the first one and by the similar formulas for the other three. Similarly, we define operators on the reducible part of the Floer complexes: we have an operator

$$\bar{m} : \bar{C}_+(Y_-) \to \bar{C}_+(Y_+)$$
Proposition 7.6 The operators $\tilde{m}$, $\hat{m}$ and $\check{m}$ satisfy the identities:

\[
\begin{cases}
\check{\partial}(Y_+)\tilde{m}_{-+} = \check{m}_{-+}(\check{\partial}(Y_-)), \\
\check{\partial}(Y_+)\hat{m}_{-+} = \hat{m}_{-+}(\check{\partial}(Y_-)), \\
\check{\partial}(Y_+)\check{m}_{-+} = \check{m}_{-+}(\check{\partial}(Y_-)).
\end{cases}
\]

In particular, we give rise to the operators

\[
\begin{cases}
\check{m}_{-+} : \check{HM}_*(Y_-) \to \check{HM}_*(Y_+) \\
\hat{m}_{-+} : \hat{HM}_*(Y_-) \to \hat{HM}_*(Y_+) \\
\tilde{m}_{-+} : \tilde{HM}_*(Y_-) \to \tilde{HM}_*(Y_+). 
\end{cases}
\]

Moreover, the above operators only depend on the data of $Y_-$ and $Y_+$.

Note that since we focus on the zero dimension part, the proof is much easier than [13, Proposition 25.3.8].

Repeat the same argument in [13, Section 26.1]. We have the composition law below for the cobordisms.

Proposition 7.7 (c.f. [13, Proposition 26.1.2]) Let $(M, F)$ satisfy the Assumption 5.1. Fix a transverse spin$^c$ structure. Let $Y_-$, $Y_0$, $Y_+$ be three data of handle-like metrics and basic perturbations, and let $W_{-0}$ be the cobordism from $Y_-$ to $Y_0$, $W_{0+}$ be the cobordism from $Y_0$ to $Y_+$ and $W_{-+}$ be the composition of the $W_{-0}$ and $W_{0+}$. Suppose that $m_{-0}$, $m_{0+}$ and $m_{-+}$ are the operators in Proposition 7.6. Then we have that

\[ m_{0+}m_{-0} = m_{-+}. \]

The above proposition implies the corollary below.

Corollary 7.8 The monopole Floer homologies are independent of the generic choice of the perturbation and the handle-like metric, which are denoted by $\check{HM}_*(M, F; s; F)$, $\hat{HM}_*(M, F; s; \mathbb{F})$ and $\tilde{HM}_*(M, F; s; \mathbb{F})$. 

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Proposition 7.9 (c.f. [13, Proposition 22.2.1]) Let \((M, F)\) satisfy the Assumption 5.1. Then, there is an exact sequence

\[
\ldots \overline{HM}_*(M, F; s; F) \xrightarrow{i_*} \overline{HM}_*(M, F; s; F) \xrightarrow{j_*} \overline{HM}_*(M, F; F) \xrightarrow{p_*} \overline{HM}_*(M, F; s; F) \xrightarrow{i_*} \ldots
\]

in which the maps \(i_*, j_*\) and \(p_*\) arise from the chain-maps

\[
i : \check{C} \to \check{C}, \quad j : \check{C} \to \check{C}, \quad p : \check{C} \to \check{C},
\]

which are defined by

\[
i = \begin{pmatrix} 0 & -\partial^u_s \\
1 & -\partial^u_s \end{pmatrix}, \quad j = \begin{pmatrix} 1 & 0 \\
0 & -\partial^u_s \end{pmatrix}, \quad p = \begin{pmatrix} \partial^o_s & \partial^u_s \\
0 & 1 \end{pmatrix}.
\]

Here \(i\) and \(j\) are genuine chain maps, however \(p\) is an anti chain map, i.e. \(p\partial + \partial p = 0\).

We review the completion of graded groups, c.f. [13, Definition 3.1.3]. Let \(G_\ast\) be an abelian group graded by the set \(J\) equipped with a \(\mathbb{Z}\)-action. Let \(O_a(a \in A)\) be the set of free \(\mathbb{Z}\)-orbits in \(J\) and fix an element \(j_a \in O_a\) for each \(a\). Consider the subgroups

\[
G_\ast[n] = \bigoplus_{a} \bigoplus_{m \geq n} G_{j_a - m},
\]

which form a decreasing filtration of \(G_\ast\). We define the negative completion of \(G_\ast\) as the topological group \(G_- \supset G_\ast\) obtained by completing with respect to this filtration. We define the negative completions

\[
\overline{HM}_*(-(M, F; s; F)), \overline{HM}_*(M, F; s; F), \overline{HM}_*(M, F; F),
\]

of the basic monopole Floer homologies defined as the above. If we want to consider all transverse \(\text{spin}^c\) structures at the same time, we need to consider the completed basic monopole Floer homology

\[
\overline{HM}_*(M, F; F) = \bigoplus_s \overline{HM}_*(M, F; s; F).
\]

We make similar definitions for \(\overline{HM}_*(M, F; s; F)\) and \(\overline{HM}_*(M, F; F)\).

Remark: Even though what we did is to construct the basic monopole Floer homology groups with \(\mathbb{Z}_2\) coefficient, as argued in [13, Section 20.4-20.5], one can define the basic monopole Floer homology with integer coefficient or any commutative ring.

7.3 Basic monopole Floer homologies for \(b^1_F > 0\)

In this subsection, we construct the basic monopole Floer homologies for \(b^1_F > 0\) with the Novikov ring. We recall the notion of local system \(\Gamma\) over a topological space \(X\).

Definition 7.10 A local system on a topological space \(X\), is a system to distribute abelian groups \(\{\Gamma_a\}\) for each point \(a \in X\), such that for each relative homotopy class of paths \(z\) from \(a\) to \(b\), there is an isomorphism

\[
\Gamma(z) : \Gamma_a \to \Gamma_b
\]

satisfying the composition law for the composition of two paths.

We review the classical results of [13, Section 22, 29, 30]. We choose \(\Gamma\) to be a local system of abelian groups on \(B_\ast^c(M, F, s)\), such that to each point \([a] \in B_\ast^c(M, F, s)\) there is an associated group \(\Gamma[a]\) and to each homotopy class \(z\) of the paths from \([a]\) to \([b]\), there is an associated isomorphism \(\Gamma(z) : \Gamma[a] \to \Gamma[b]\).
By using the above local system $\Gamma$, the boundary maps are well-defined. For instance, we consider

$$C^\infty(Y, s, c, \Gamma) = \bigoplus_{[a]} \Gamma[a],$$

where $[a]$ denotes the irreducible critical point. We define the partial by

$$\partial^\omega_o = \sum_{[a]} \sum_{[b]} \sum_z \sum_{\gamma} \in M_z([a], [b]) \otimes \Gamma(z),$$

where the sum is over all the moduli space $M_z([a], [b])$ with dimension 1 and $[b]$ denotes the irreducible critical point. The contribution for a given pair of critical points takes the form

$$\sum_z n_z \Gamma(z),$$

(14)

where $z$ runs through all relative homotopy classes satisfying the conditions $gr_z([a], [b]) = 1$. Before proceeding, we review some definitions and notions which are given in [13, Section 30].

**Definition 7.11 (c.f. [13, Definition 30.2.1])** Let $\mathcal{E}_4^{top}$ be a corresponding perturbation of the topological energy. A subset $S \subset \pi_1([a], [b])$ is called $c$-finite, where $\pi_1([a], [b])$ denotes the homotopy classes of paths joining $[a]$ and $[b]$ in $B^m(M, F, \mathfrak{s})$, if the following conditions are satisfied:

- for all $C$, $S \cap \{z|\mathcal{E}_4^{top}(z) \leq C\}$ is finite;
- there exists $d \geq 0$ such that $|gr_z([a], [b])| \leq d$ for all $z \in S$.

We consider a local system of complete topological abelian groups $\Gamma$ on $B^m(M, F, \mathfrak{s})$, i.e. each $\Gamma[a]$ is a complete topological group and each homomorphism $\Gamma(z) : \Gamma[a] \rightarrow \Gamma[b]$ is continuous. Assume that $0 \in \Gamma[a]$ has a neighborhood basis consisting of subgroups, such that $\Gamma[a]$ is a complete filtered group, which is filtered by the open subgroups. Let $\text{Hom}(\Gamma[a], \Gamma[b])$ be the group of continuous homomorphisms, equipped with the compact-open topology. A neighborhood basis for $0$ in $\text{Hom}(\Gamma[a], \Gamma[b])$ consists of subgroups

$$\Omega(N, V) = \{k : \Gamma[a] \rightarrow \Gamma[b]|k(N) \subset V\},$$

where $N$ runs over all precompact subsets of $\Gamma[a]$ and $V$ runs all open subgroups of $\Gamma[b]$. Note that a subset $N \subset \Gamma[a]$ is precompact if and only if $(N + U)/U$ is finite for all open subgroups $U$ of $\Gamma[a]$.

**Definition 7.12** A countable series $\sum_{k \in K} k$ of $\text{Hom}(\Gamma[a], \Gamma[b])$ is said to be equicontinuous, if for each open subgroup $U \subset \Gamma[b]$, there exists an open subgroup $V$ such that $k(V) \subset U$ for each $k \in K$.

**Definition 7.13 (c.f. [13, Definition 30.2.2])** A local system of complete filtered abelian groups $\Gamma$ is called $c$-complete, if it satisfies the following properties for each $[a]$, $[b]$:

- for any $c$-finite set $S \subset \pi_1(B^m, [a], [b])$, the set $\{\Gamma(z)|z \in S\} \subset \text{Hom}(\Gamma[a], \Gamma[b])$ is equicontinuous;
- for any $c$-finite set $S \subset \pi_1(B^m, [a], [b])$, $\Gamma(z)$ converges to zero as $z$ runs through $S$ in the compact-open topology.

Notice that there might be infinitely many nonzero terms in the form (14), we set the support as

$$\text{supp}(n) = \{z| n_z \neq 0\}.$$

By the definition of c-complete, we have that

$$\text{supp}(n) \cap \{z|\mathcal{E}_4^{top}(z) \leq C\}$$
is finite. Using the completeness of the local system, we have that the form the form \((14)\) is convergent. Similarly, one can verify that the maps \(\partial, \bar{\partial} \) and \(\partial\) are well-defined. Combining with the equicontinuous property of the local system \(\Gamma\), the proofs of \(\partial^2, \bar{\partial}^2 \) and \(\partial^2\) go through as the non-exact perturbation of 3 manifold case (see \[13, \text{Section 30.2}\]).

We give an example of such a local system, e.g. a Novikov ring \([18]\). We have a homomorphism

\[ E^{top} : \pi_1(B(M,F,s)) \to \mathbb{R}, \ z \mapsto E^{top}(z), \]

where \(E^{top}(z)\) denotes the difference of the Chern-Simons-Dirac functional between the different representatives of the quotient point. Since \(\pi_1(B^\sigma(M,F,s)) \cong \mathbb{Z}^l\), we can choose a basis \(\{z_i\}_{1 \leq i \leq l}\) such that each element \(z\) can be written as \(z = k_1z_1 + \cdots + k_1z_1 + \cdots + k_1z_1\), where \(k_i \in \mathbb{Z}\) for \(i = 1, \cdots, b^1\), and \(E^{top}(z) \geq 0\). Moreover, we may assume that for \(1 \leq i \leq l, E^{top}(z_i) > 0\). It is not hard to see that such a basis \(\{z_i\}_{1 \leq i \leq l}\) is independent of the metric \(g\). Choosing a commutative ring \(R(\mathbb{Z}, \mathbb{Z})\), we define \(R[t, t^{-1}]\) by

\[ R[t, t^{-1}] = \{ \sum_{k \leq i \leq K} r_i t^i \mid \text{only for finitely many } i, r_i \neq 0 \}. \]

For \(k \in \mathbb{Z}\), let \(U_{-k}\) be the \(R\)-module spanned by the generators \(t^i\), satisfying \(i \leq -k\). Using these as open neighborhoods of \(0\), we form the completion \(\overline{R[t, t^{-1}]}\), i.e. each element is of the form

\[ \sum_{i=-\infty}^C r_i t^i. \]

We define a local system by taking at \([a_0]\) to be \(\overline{R[t, t^{-1}]}\), and specifying that for each closed loop \(z\) based at \([a_0]\), the automorphism \(\Gamma(z)\) be the multiplication by \(t^{-k_1 + \cdots + k_1}z_1 + \cdots + k_1z_1 + \cdots + k_1z_1\).

Repeat the parallel arguments of previous subsection, together with Proposition 6.12 and Proposition 6.13, we have that:

**Theorem 7.14** Let \((M,F)\) satisfy the Assumption 5.1. For a complete local system \(\Gamma\), e.g. a Novikov ring, we can construct the basic monopole Floer homologies. Moreover, The monopole Floer homologies are independent of the generic choices of the perturbation and the bundle-like metric, which are denoted by \(\overline{\mathcal{H}}\mathcal{M}_*(M,F,s;\Gamma)\), \(\overline{\mathcal{H}}\mathcal{M}_*(M,F,s;\Gamma)\) and \(\overline{\mathcal{H}}\mathcal{M}_*(M,F,s;\Gamma)\). Moreover, if \((12)\) holds, then for any local system, we have the well-defined basic monopole Floer homologies.

In general, we consider the (non-exact) perturbed basic Chern-Simons-Dirac functional defined as below: given a class \(c \in H^2_b(M)\), we write

\[ L_\omega(A, \Psi) = L(A, \Psi) - \frac{1}{2} \int\limits_M (A^F - A_0^F) \wedge \omega \wedge \chi_F, \]

where \(\omega \in \Omega^2 c\). It is known that a critical point \((A, s, \psi)\) in the blow-up model \(\mathcal{C}^\sigma(M,F,s)\) is defined by

\[
\begin{aligned}
\frac{1}{2}(F_{A^\tau} - \omega) = s^2 q(\psi), \\
\partial_A \psi = 0.
\end{aligned}
\]

The corresponding perturbed equations for \((A, s, \phi) \in C^\sigma(\mathbb{R} \times M)\) are defined by

\[
\begin{aligned}
\frac{1}{2} (F_{A^+} - \omega^+) = s^2 q(\phi), \\
\frac{1}{s} s + \Lambda(A, s, \phi) s = 0, \\
\partial_A^+ \phi - \Lambda(A, s, \phi) \phi = 0.
\end{aligned}
\]

Applying the standard argument of the manifold case, we have the following lemma.
Lemma 7.15 If \(c \neq c_1(s)\), then there are no reducible critical points of \(L_s\).

With the non-exact perturbation, the space of broken trajectories space \(\tilde{\mathcal{M}}^+([a],[b])\) can be defined as the manner of exact perturbation.

Following the same strategy of the construction in Section 6 (or see [13, Section 20-Section 26]), we have the following theorem.

**Theorem 7.16** Let \(\Gamma\) be a complete local system, e.g. a Novikov ring, and \(L_s\) be a non-exact perturbation for the Chern-Simons-Dirac functional defined as above. Then we have the basic monopole Floer homologies

\[
\hat{H}_s(M,F,s,c;\Gamma), \quad \hat{H}_s(M,F,s,c;\Gamma), \quad \hat{M}_s(M,F,s,c;\Gamma),
\]

where \(c \in H^2(M)\). These homologies depend only on the isomorphism class of the spin\(^c\) structure \(s, c\) and \((M,F)\), however they are independent of the bundle-like metrics and the basic perturbations.

At the end of this subsection, we give a necessary condition to avoid the complete local system.

**Theorem 7.17** Let \((M,F,s,c)\) be as above. Let \(g\) be a bundle like metric and \(\chi_F\) be the character form of the foliation. Suppose that there is a constant \(t\) such that the identity holds

\[
- \int_M (c_1(s) - c) \wedge [u] \wedge \chi_F + t \cdot gr(a,ua) = 0,
\]

for a non-degenerate critical point. Then, with any local coefficient \(\Gamma\) we have the basic monopole Floer homologies

\[
\hat{M}_s(M,F,s,c;\Gamma), \quad \hat{H}_s(M,F,s,c;\Gamma), \quad \hat{M}_s(M,F,s,c;\Gamma).
\]

**Proof** Here we give a sketch of the proof. The idea is to show that \(\sum_i n_i \Gamma(z)\) is of finitely many sum, for each \(z \in M_s([a],[a])\), where \(M_s([a],[b])\) is a moduli space of dimension 1 and \([a],[b]\) are two regular critical points. It is sufficient to show a foliated version of [13, Proposition 29.2.1], which is stated as below.

**Proposition 7.18** Let \((M,F,s,c)\) be as above, let \(g\) be a bundle like metric and \(\chi_F\) be the character form of the foliation. Suppose that there is a constant \(t\) such that the identity holds

\[
- \int_M (c_1(s) - c) \wedge [u] \wedge \chi_F + t \cdot gr(a,ua) = 0,
\]

for a non-degenerate critical point. Then, we have the following:

1. When \(t \leq 0\), then for a given \([a]\) and a non-negative integer \(d_0\), there are only finitely many pairs \(([b],z)\) such that the moduli space \(M^+_s([a],[b])\) is non-empty and of dimension at most \(d_0\).

2. When \(t > 0\), then for a given \([a]\), there are only finitely many pairs \(([b],z)\) such that the moduli space \(M^+_s([a],[b])\) is non-empty.

**Proof**

- When \(t > 0\), we repeat the same argument of Proposition 6.13 to get the conclusion.

- When \(t = 0\), and the moduli space \(M_s([a],[b])\) are non-empty. It is known that the image of the critical-points set under the blow-down map \(\pi : \mathcal{B}^s(M,F,s) \to \mathcal{B}(M,F,s)\) is a set of finite points. We may assume that \(\pi([b]) = [\beta]\) for all \(i\). \(L_s\) descends to a single-valued function on \(\mathcal{B}(M,F,s)\), hence the energy of the trajectories in these moduli spaces has an up-bound. For the blow-down case, Proposition 5.2 implies that there are only finitely many choices for the homotopy class of the path \(\pi(z_i)\) in \(\mathcal{B}(M,F,s)\). In addition, the dimension \(d_0\) gives a lower-bound and up-bound for \(\iota([b_i])\), there are only finitely many \([b_i]\).
• When \( t < 0 \), there is a negative number \( t \) such that \( \varepsilon_{\text{top}}^g(z) + t \nu(z) \) is independent of \( z \). To give a bound for the dimension, it suffices to give a bound for \( \nu(z) \). Since \( t < 0 \), we have an above bound for \( \varepsilon_{\text{top}}^g(z) \). Since the dimension is bounded by \( d_0 \) and \( [a] \) is fixed, there are finitely many pairs \((b, z)\) such that \( \nu([b], [z]) \) is bounded above and below, and \( gr_z([a], [b]) \geq 0 \). The energy bound implies that only finitely many moduli spaces which are non-empty, and there are only finitely many critical points in the absence of reducibles, so the conclusion also holds.

\[
\square
\]

It is known that \( gr(a, u\alpha) \) equals to the index of basic Dirac operator on \( M \times S^1 \), by [3]. We rewrite the above formula in Proposition 7.18 as

\[
\int_M (c_1(s) - c) \wedge [u] \wedge \chi_F + t (\int_{\bar{S}_0 \times S^1/F} A_{0,b} |\bar{d}x| + \sum_{j=1}^r \beta(M_j \times S^1)) = 0,
\]

where

\[
\beta(M_j \times S^1) = \frac{1}{2} \sum_{\tau} \frac{1}{n_r \text{rank}(W^\tau)} (-\eta(D_j^{s+,\tau}) + h(D_j^{s+,\tau})) \int_{\bar{S}_1 \times S^1/F} A^{\tau}_{j,b}(x)|\bar{dx}|,
\]

the integrands \( A_{0,b}, A^{\tau}_{j,b}(x) \) are similar to the Atiyah-Singer integrands, \( \bar{M}_b \times S^1 \) is the principal domain of \( M \times S^1 \) and \( M_j \times S^1 \)'s are the finite desingularities of \( M \times S^1 \), more details are explained in the paper [6].

8 Examples

In this section, we will give a family of manifold with foliation satisfying the Assumption 5.1.

8.1 Fibration and orbifold

The easiest model is to consider \( M = Y \times F \), where \( Y \) is a closed oriented 3 manifold and \( F \) is a closed oriented manifold. Given a metric \( g_Y \) and a spin \( c \) structure \( s \) of \( Y \), by pulling back, one has a data \((M, F, \pi^* g_Y \oplus g_F, \pi^* s)\), where \( \pi : M \rightarrow Y \). Such a manifold with foliation \((M, F)\) satisfies the Assumption 5.1. We can generalize the global product model to the local product model, i.e. the fibration over \( Y \).

Let \( Y \) be a closed oriented 3 manifold, and \( M \rightarrow Y \) be a fibration over \( Y \), such that \( M \) is closed and oriented. Fix a metric \( g_Y \) and a spin \( c \) structure \( s \) of \( Y \), via pulling back, we have a bundle like metric and a transverse spin \( c \) structure, still denoted by \( s \). Since the volume form of \( Y \) is closed, by pulling back, one has that \( H^3_b(M) \neq 0 \). We have that \((M, F)\) satisfies the Assumption 5.1, by Proposition 2.5. By the identification between the basic forms(sections) of \( M \) and the forms(sections) of \( Y \), one establishes the proposition below.

Proposition 8.1 Let \((M, F, s)\) be defined as above. Then, the basic monopole Floer homology groups \( \overline{HM}_s(Y, s) \), \( \overline{HM}_s(M, F, s) \), \( \overline{HM}_s(M, F, s) \) are isomorphic to the basic monopole Floer homology groups \( \overline{HM}_s(Y, s) \), \( \overline{HM}_s(Y, s) \), \( \overline{HM}_s(Y, s) \) respectively with any coefficient.

One can generalize the model of fibration over manifold to the model of fibration over orbifold. First, we recall the notion of orbifold, which was first introduced by Satake [22].

Definition 8.2 (c.f. [4]) An \( n \)-dimensional orbifold \( Y \) is a Hausdorff space \(|Y|\) together with an atlas \( \{ (U_i), \{ \phi_i \}, \{ \tilde{U}_i \}, \{ \Gamma_i \} \) with transition maps \( \{ \phi_{ij} \} \), which satisfies

• \( \{ U_i \} \) is locally finite;
• \( \{ U_i \} \) is closed under finite intersections;
• For each \( U_i \), the finite group \( \Gamma_i \) actions smoothly and effectively on a connected open subset \( \tilde{U}_i \subset \mathbb{R}^n \), and there is a homeomorphism \( \phi_i : \tilde{U}_i / \Gamma_i \to U_i \);
• If \( U_i \subset U_j \), then there exists a monomorphism \( f_{ij} : \Gamma_i \to \Gamma_j \) and a smooth embedding \( \phi_{ij} : \tilde{U}_i \to \tilde{U}_j \) such that for any \( g \in \Gamma_i, \ x \in \tilde{U}_i \), we have that \( \phi_{ij}(g \cdot x) = f_{ij}(g) \cdot \phi_{ij}(x) \) and the following diagram commutes:

\[
\begin{array}{ccc}
\tilde{U}_i & \xrightarrow{\phi_{ij}} & \tilde{U}_j \\
\downarrow & & \downarrow \\
\tilde{U}_i / \Gamma_i & \xrightarrow{f_{ij}} & \tilde{U}_j / \Gamma_j \\
\phi_i & & \phi_j \\
U_i & \to & U_j
\end{array}
\]

where \( f_{ij} \) is induced by the monomorphism and the canonical projection.

An \( n \)-dimensional orbifold bundle over \( Y \) is defined in the similar manner.

**Definition 8.3 (c.f. [4])** An orbifold \( E \) is called an orbifold bundle over \( Y \), if there exists a smooth orbifold map \( p : E \to Y \), such that

• there is an atlas \( \{ \{ V_i \}, \{ \tilde{V}_i \}, G_i \} \) of \( E \), satisfying \( V_i = p^{-1}(U_i) \) and \( \tilde{V}_i = \tilde{U}_i \times E_0 \), where \( \{ \{ U_i \}, \{ \phi_i \}, \{ \tilde{U}_i \}, \{ \Gamma_i \} \) is an atlas of \( Y \) and \( E_0 \) is a standard fiber;
• the following diagram commutes

\[
\begin{array}{ccc}
\tilde{U}_i \times E_0 & \xrightarrow{\tilde{p}} & \tilde{U}_i \\
\downarrow & & \downarrow \\
\tilde{V}_i / G_i & \xrightarrow{p} & \tilde{U}_i / \Gamma_i \\
\phi_i & & \phi_i \\
V_i & \to & U_i
\end{array}
\]

where \( \tilde{p} \) is a \( (G_i, \Gamma_i) \)-equivariant map.

When \( G_i \) acts freely, \( E \) becomes a manifold, e.g. the frame bundle of an oriented orbifold(see [1, Theorem 1.3]). Let \( Y \) be an oriented closed 3-orbifold. Suppose the singular set \( \Sigma Y = \{ x \in Y | G_x \neq 1 \} \) is a set of disjoint union of finite circles, where \( G_x \) denotes the isotropy group at \( x \). We rewrite

\[
\Sigma Y = \bigcup_{1 \leq i \leq n} l_i
\]

and each circle \( l_i \) is assigned a positive integer \( \alpha_i \), given by its isotropy group \( \mathbb{Z}_{\alpha_i} \). Let \( D \) be the unit disk and \( \mathbb{Z}_{\alpha_i} \) act on it by rotation. Near each \( l_i \), we have an atlas,

\[
\phi_i : (S^1 \times D, S^1 \times \{ 0 \}) \to (U_i, l_i),
\]

where \( \phi_i \) induces a homeomorphism from \((S^1 \times D)/\mathbb{Z}_{\alpha_i}, S^1 \times \{ 0 \}) \) to \((U_i, l_i) \). It is known that \( TY \) always lifts to an orbifold spin\(^c\)-bundle for such a 3-orbifold. The definition of the Seiberg-Witten invariant can be generalized to 3-orbifold, see Baldridge [4] and Chen [7]. For Seiberg-Witten invariant, we have the following proposition, which is similar to the manifold case.
Proposition 8.4 Let $Y$ be a closed oriented 3-orbifold and $M \to Y$ be a fibration over $Y$. Suppose that $\mathfrak{s}$ is a transverse spin$^c$ structure which comes from the pull-back spin$^c$ structure of $Y$ and $M$ is a closed oriented manifold. Then, we have that basic Seiberg-Witten invariant of $M$ is equal to the Seiberg-Witten invariant of $Y$, for $b^1(Y) > 1$.

Under tensor product, the topological isomorphism classes of orbifold line bundles form a group. We give a local description for each class of such a group. We have an orbifold line bundle over $Y$, which is a trivial line bundle over $Y \setminus \Sigma Y$, and over each $U_i$, it is given by $(S^1 \times D \times \mathbb{C})/\mathbb{Z}_{\alpha_i}$, where $\mathbb{Z}_{\alpha_i}$ action is defined by,

$$a \cdot (t, w, z) \mapsto (t, e^{2\pi i a} w, e^{2\pi i a} z),$$

for each element $a \in \mathbb{Z}_{\alpha_i}$. This bundle is glued together by a transition function $\varphi(t, w) = w$ on the overlap $\partial(S^1 \times D)$. Each $l_i$ generates a line bundle $E_i$. Let $L$ be a line bundle over $Y$. There is a collection of integers $\{\beta_1, \cdots, \beta_n\}$ such that

- $0 \leq \beta_i < \alpha_i$, for each $i = 1, \cdots, n$;
- the bundle $L \otimes E_{i_1}^{-\beta_{i_1}} \cdots E_{i_n}^{-\beta_{i_n}}$ is trivial over each neighborhood of $l_i$.

By forgetting the orbifold structure, it can be naturally identified with a smooth line bundle (denoted by $[L]$) over the smooth manifold $[Y]$. We will list some necessary results of such orbifolds.

Theorem 8.5 (Baldridge [4]) The tangent bundle $TY$ lifts to an orbifold spin$^c$ bundle.

Lemma 8.6 (Chen [7]) Let $Y$ be defined as above. Then we have that

$$\pi_0(C^\infty(Y, S^1)) \cong H^1([Y], \mathbb{Z}).$$

Proposition 8.7 Let $Y$ be the orbifold as before. Then, we have the following isomorphism

$$H^*(|Y|, \mathbb{R}) \cong H^*_{dR}(Y, \mathbb{R}).$$

Proof We have the fine resolution below for orbifold $Y$,

$$0 \to \mathbb{R} \to A^0 \xrightarrow{d} A^1 \cdots$$

of the constant sheaf $\mathbb{R}$. By the double complex argument, we have the isomorphism

$$\tilde{H}^*(Y, \mathbb{R}) \cong H^*_{dR}(Y, \mathbb{R}),$$

where the first cohomology group is the Čech-cohomology group. Since we can find a finite covering $\{U_i\}$, such that all non-empty intersections of finitely-many sets are contractible, Čech cohomology is isomorphic to the singular cohomology of the of the underlying space $[Y]$. Thus, we have that

$$H^*(|Y|, \mathbb{R}) \cong \tilde{H}^*(Y, \mathbb{R}) \cong H^*_{dR}(Y, \mathbb{R}).$$

For an oriented closed 3-orbifold $Y$ with a metric $g$ and spin$^c$ structure $\mathfrak{s}$ whose determinant line bundle has the Seifert data $(b, \beta_1, \cdots, \beta_n)$, one defines the Chern-Simons-Dirac functional

$$L(A, \Psi) = -\frac{1}{8} \int_Y (A^I - A^I_0) \wedge (F_{A^I} + F_{A^I_0}) + \frac{1}{2} \int_Y (\Psi, \bar{\vartheta}_A \Psi) d\text{vol}_Y,$$

for any $(A, \Psi) \in \mathcal{C}(Y, \mathfrak{s})$. Let $u \in \mathcal{G}(Y)$, we have that

$$L(A, \Psi) - L(u(A, \Psi)) = -\frac{1}{2} \int_Y u^{-1} du \wedge F_{A^I_0} = -2\pi^2 (c_1(\mathfrak{s}), [u]),$$

\[33\]
where \( c_1(s) = \frac{1}{2\pi} F_{A_s} \) and \([u] = \frac{1}{\pi} u^{-1} du\). Similar to the manifold case, we define the critical points of the Chern-Simons-Dirac functional and the blow-up configuration space. By the Proposition 8.7 and the Poincaré duality, it is known that there is a unique second cohomology class \( c(s, Y) \in H^2(Y, \mathbb{R}) \) such that

\[
gr(a, ua) = \langle c_1(s) - c(s, Y), [u] \rangle,
\]

where \( gr(a, ua) \) denotes the grading between \( a \) and \( ua \) for a non-degenerate critical point \( a \in \text{Crit}^c(L) \). Using a complete local system \( \Gamma \), we can construct the monopole Floer homologies for \((Y, s)\).

When \( c(s, Y) \) is propositional to \( c_1(s) \), i.e. there is a real constant \( k \) such that

\[
c(s, Y) = kc_1(s).
\]

Suppose that \( k \neq 1 \), then we can find a real constant \( t \), such that

\[
L_\omega(A, \Psi) - L_\omega(u(A, \Psi) + tgr(a, ua) = 0,
\]

which is equivalent to the formula

\[
-c_1(s) + t(c_1(s) - c(s, Y)) = 0. \tag{17}
\]

**Proposition 8.8** Let \((Y, s)\) be a closed oriented 3-orbifold as above. Suppose that and all the moduli spaces \( M^+_s([a], [b]) \) for the perturbation \( q \) are regular and the formula (17) holds for each non-degenerate critical \( a \) and \((A, \Psi) = \pi(a)\). Then, the following holds:

1. When \( t \leq 0 \), then for a given \([a]\) and a non-negative integer \( d_0 \), there are only finitely many pairs \(([b], z)\) for which the moduli space \( M^{+}_s([a], [b]) \) is non-empty and of dimension at most \( d_0 \).

2. When \( t > 0 \), then for a given \([a]\), there are only finitely many pairs \(([b], z)\) for which the moduli space \( M^{+}_s([a], [b]) \) is non-empty, \(([b], z)\) for which the moduli space \( M^{+}_s([a], [b]) \) is non-empty and has dimension no more than \( d_0 \).

The proof is similar to Proposition 7.18, here we omit it.

The space of broken trajectories \( \tilde{M}^+_s([a], [b]) \) can be identified with the manifold model. This space is still compact for fixed \([a], [b]\) and \( z \) as in [13, Theorem 16.1.3]. We apply the same arguments of [13, Section 20–Section 25] or of the previous section to establish the following theorem.

**Theorem 8.9** Let \( \Gamma \) be any local system of abelian groups on \( \mathcal{B}^e(Y, s) \) and let \((Y, s)\) be a closed oriented 3-orbifold as above. Suppose that the the formula (17) holds for each non-degenerate critical point. Then we construct the basic monopole Floer homologies

\[
\tilde{HM}_s(Y, s; \Gamma), \ 	ilde{HM}_s(Y, s; \Gamma), \ 	ilde{HM}_s(Y, s; \Gamma).
\]

We give an example of such a complete local system, i.e. a Novikov ring [18]. Let \( I \subset \mathbb{R} \) be the set of the image of the homomorphism

\[
\mathcal{E}^{top} : \pi_1(\mathcal{B}^e(Y, s)) \to \mathbb{R}, \ z \mapsto \mathcal{E}^{top}(z),
\]

where \( \mathcal{E}(z) \) denotes the difference of the Chern-Simons-Dirac functional between the different representatives of the quotient point. Set \( \mathbb{F} = \mathbb{Z}_2 \). We define \( \mathbb{F}[I] \) by

\[
\mathbb{F}[I] = \left\{ \sum_{i \in I} r_i t^i \mid \text{only for finitely many } i, \ r_i \neq 0 \right\}.
\]

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For $k \in \mathbb{R}$, let $U_{-k}$ be the $\mathbb{F}$-module spanned by the generators $t^i$, $i \in I$ satisfying $i \leq -k$. Using these as open neighborhoods of 0, we form the completion $\hat{\mathbb{F}}[I]$, i.e. each element is of the form
\[
\sum_{i=-\infty}^{c} r_i t^i.
\]
We define a local system by taking at $[a_0]$ to be $\hat{\mathbb{F}}[I]$, and specifying that for each closed loop $z$ based at $[a_0]$, the automorphism $\Gamma(z)$ be the multiplication by $t^{-\top tr_{\gamma}(z)}$. This is a $c$-complete local system.

Similar to the foliation case of the previous section or to the non-exact perturbation on manifold case, we have the following theorem.

**Theorem 8.10** Let $(Y, s)$ be a 3 orbifold defined as above with a spin$^c$ structure $s$. Then we have the monopole Floer homologies
\[
\overline{HM}_4(Y, s; c, \Gamma), \overline{HM}_4(Y, s, c; \Gamma), \overline{HM}_4(Y, s, c; \Gamma).
\]
where $\Gamma$ is a complete local system. Moreover, these homologies depend only on the isomorphism class of the spin$^c$ structure $s$, c and $Y$, and are independent of the metrics or the perturbations.

We give a necessary condition to avoid the complete local system.

**Theorem 8.11** Let $Y'$ be a closed oriented 3-orbifold defined as above, and $(s, c)$ be as above. Suppose that there is a constant $t$ such that
\[
-(c_1(s) - c) + t(c_1(s) - c(s, Y)) = 0.
\]
Then, for any local system the monopole Floer homologies are well-defined.

### 8.2 Suspension

Another way to construct the foliation is by suspension, here we give two references of this subsection, see [16, Chapter 3.8] and [20]. Let $(Y, g)$ be a closed oriented 3 Riemannian manifold. Suppose that a compact Lie group $G$ acts on $(Y, g)$ isometrically and preserving the orientation of $Y$, and we have a representation
\[
f : \pi_1(X) \to G
\]
such that the closure of $\text{Im}(f)$ is $G$, where $X$ is a closed oriented manifold with fundamental group $\pi_1(X)$. We set $M = X \times Y/f$, where $X$ denotes the universal covering of $X$ and $(x, y) \sim (x[\gamma]^{-1}, f([\gamma])y)$ for $[\gamma] \in \pi_1(X)$. Fixing a point $p = [y_0, x_0] \in M$, its leaf is defined by the set of the form
\[
F_p = \{[x, y_0] \mid x \in \tilde{X}\}.
\]
Since one can find a $G$-invariant volume form over $Y$, lifting back on $M$ it holds that $H^3_b(M, F) \neq 0$, which implies that the foliation is taut by Proposition 2.5. Before preceding, we have the following lemma(see [15]).

**Lemma 8.12** Let $(M, F)$ be defined as above. Then, we have an identification
\[
\pi_0(\text{Map}^G(Y, S^1)) \cong H^1(M, \mathbb{Z}) \cap H^1_b(M),
\]
where $\text{Map}^G(Y, S^1)$ denotes the space of $G$-invariant $S^1$-valued functions.

Suppose there is a $G$-equivariant spin$^c$ structure. Given a $G$-equivariant spinor bundle
\[
S' \to Y,
\]
we construct a foliated spinor bundle $S = \tilde{X} \times S'/f$, where the action of $[\gamma] \in \pi_1(X)$ is defined by $[\gamma](x, s_p) = (x[\gamma]^{-1}, f[\gamma]s_p)$. By [20], it is known that there is an identification
\[
\Gamma^G(Y, S') \cong \Gamma_0(M, S).
\]
Summarizing the above arguments, we have the following proposition(see [15]).
Proposition 8.13 Let \((M, F)\) be a manifold with a foliation constructed as above, and let \(Y\) admit a \(G\)-equivariant spinor bundle. Suppose it holds that \(\text{rank}(\pi_0(\text{Map}^G(Y, S^1))) = b_1^G(Y)\), where \(\text{Map}^G(Y, S^1)\) denotes the set of \(G\)-invariant \(S^1\)-valued functions and \(b_1^G\) denotes the dimension of the first cohomology for the \(G\)-invariant deRham complex. Then \((M, F)\) satisfies the Assumption 5.1.

Remark:

1. The condition that \(\text{rank}(\pi_0(\text{Map}^G(Y, S^1))) = b_1^G(Y)\) is necessary. For example, let \(Y = T^3 = (S^1)^3, G = S^1\) action canonically on the first slot of \(Y\), and \(X = S^1\) with \(f : \pi_1(X) \to S^1\) by sending the generator element of \(\pi_1(X)\) to a dense element of \(S^1\), e.g. \(1 \mapsto e^{i2\pi \theta}\) for some \(\theta \notin \mathbb{Q}\). We have that \(\dim H^1_1(M) = \dim H^1_{1R}(Y) = 3\), and \(H^1_1(M) \cap H^1_1(M, \mathbb{Z}) \cong \pi_0(\{u : M \to S^1\} \text{ such that } L_\xi u \equiv 0, \text{ for any } \xi \in \Gamma(\mathcal{F}))\) \(\cong \mathbb{Z}^2\).

2. When \(G\) is connected, we have that \(b_1^G = b_1\), since any homology cycle \(\sigma\) is homotopic to \(g_*\sigma\), for any \(g \in G\). When \(G\) acts freely, we have that \(\pi_0(\text{Map}^G(Y, S^1)) \cong H^1(Y/G, \mathbb{Z})\), and \(b_1(Y/G) = b_1^G(Y)\).

At the end of this section, we give an explicit example. Let \(Y = SO(3)\), and \(T_1, T_2\) and \(T_3\) be three maximal tori(circles), such that their Lie algebras span the Lie algebra of \(Y\), i.e. \(so(3)\). We choose a closed oriented manifold \(X\) whose fundamental groups is isomorphic to \(\mathbb{Z} \ast \mathbb{Z} \ast \mathbb{Z}\), e.g. \(X = \mathbb{Z}_3 S^1 \times S^k\) with \(k \geq 2\). We can consider a family of representations

\[ f_t : \pi_1(X) \to T_1, T_2, T_3 \]

such that the first component \((1,0,0)\) sends to an element \(\begin{pmatrix} \cos 2\pi t & -\sin 2\pi t \\ \sin 2\pi t & \cos 2\pi t \end{pmatrix} \) of \(T_1\), the second component \((0,1,0)\) sends to an element \(\begin{pmatrix} \cos 2\pi t & \sin 2\pi t \\ -\sin 2\pi t & \cos 2\pi t \end{pmatrix} \) of \(T_2\) and the third component \((0,0,1)\) sends to an element \(\begin{pmatrix} 1 & \cos 2\pi t & -\sin 2\pi t \\ \sin 2\pi t & \cos 2\pi t \end{pmatrix} \) of \(T_3\). We set \(M_t = Y \times \tilde{X}/f_t\), the codimension 3 foliation \(F_t\) on \(M_t\) is defined by letting the leaves be

\[ F_{t,y} = \{[x,y] | x \in \tilde{X}\} \]

We choose a trivial \(SO(3)\)-equivariant spin structure of \(Y\).

- When the group \(G_t\) is a finite group of \(Y\). Since \(Y\) admits a metric of positive scalar curvature, by Proposition 8.1 and [13, Proposition 36.1.3], one deduces that

\[ \overline{M}_*(M_t, F_t) \cong \mathbb{F}[U, U^{-1}], \quad \overline{M}_*(M_t, F_t) \cong \mathbb{F}[U, U^{-1}]/\mathbb{F}[U] \]

- When \(G_t\) is dense in \(M_t\). Since the transverse spin\(^c\) structure \(s\) is trivial, then \(\Gamma_0(S) = \mathbb{C}^2\). By the argument at the beginning of this subsection, we have that \(\Omega_3^1(M_t) \cong \Omega_3^{1, G}(Y) \cong \mathbb{R}^3\). Let \(g_Y\) be a bi-invariant metric of \(Y\) with positive scalar curvature, then the associated

\[ \overline{D}_{g_Y}^3 \]

is an \(SO(3)\) equivariant, which corresponds to a basic Dirac operator \(\overline{D}_0\) with spin connection \(A_0\). Their spectrums have a one-to-one corresponding. Therefore, the solutions of the basic Seiberg-Witten equations (4) corresponds to the solutions of \(SO(3)\)-invariant Seiberg-Witten equations, i.e.

\[
\begin{cases}
\overline{D}_0^3 \psi = 0 \\
\frac{1}{2} *_{g_Y} F_A = g(\psi)
\end{cases}
\]
where \((A, \Psi) \in C^{SO(3)}(Y) = \{A_0 + i\Omega^{1,SO(3)}(Y) \times C^2\}\). Since \(g_Y\) has a positive scalar curvature, it is known that \(\Psi \equiv 0\), i.e. there is no irreducible solution to the basic Seiberg-Witten equations. Consider the reducible solutions, we have that \(dA = 0\) and \(\Psi\) is an eigenvector of \(D_0^A\). Since \(A = A_0 + a\) and \(dA_0 = 0\), this implies that \(a\) is closed. Combining with \(H^{1,G}(Y) = 0\), we have that \(a = df\) for a \(SO(3)\)-invariant function \(f\), which implies that \(f\) is constant and \(a = 0\). Thus, all reducible solutions are eigenvector of \(D_0\). Recall that \(D_0 = \sum_i e^i \nabla_{e^i}'\), where \(\{e_i\}\) is an orthonormal frame of \(TY\); and

\[
\nabla' = d + \frac{1}{2} \sum_{i<j} \omega_{ij} e^i e^j
\]

where \(d\) is the flat connection of \(S'\) and \(\omega_{ij}\) is the Levi-Civita connection associated to \(g_Y\). Since \(D_0\) is independent of the choice of the orthonormal frame, we choose a frame \(\{e_1, e_2, e_3\}\) such that they are generated by the left action of a frame \(\{L_x, L_y, L_z\}\) of \(T_1Y\), where \(L_x = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\), \(L_y = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\) and \(L_z = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\). Since \(g_Y\) is bi-invariant \(\{e_1, e_2, e_3\}\) are left-invariant, we have that

\[
\omega_{12}(e_3) = \frac{1}{2} g_Y([e_2, e_1], e_2) = -\frac{1}{2} g_Y(e_2, e_2) = \frac{1}{2},
\]

\[
\omega_{13}(e_2) = \frac{1}{2} g_Y([e_2, e_1], e_3) = -\frac{1}{2} g_Y(e_3, e_3) = -\frac{1}{2},
\]

\[
\omega_{23}(e_1) = \frac{1}{2} g_Y([e_1, e_2], e_3) = \frac{1}{2} g_Y(e_2, e_2) = \frac{1}{2}.
\]

We inherit the convection \(e^1 \cdot e^2 \cdot e^3 = Id\) from the book [13], by assigning

\[
e^1 \mapsto \begin{pmatrix} i \\ -i \end{pmatrix}, \quad e^2 \mapsto \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad e^3 \mapsto \begin{pmatrix} i \\ i \end{pmatrix}.
\]

For any \(\Psi \in \Gamma^{SO(3)}(Y, S') \cong \mathbb{C}^2\), we have that

\[
D_0 \Psi = \sum_k e^k d\Psi + \sum_{i<j,k} e^k \frac{1}{2} \omega_{ij}(e_k) e^i e^j \Psi = \frac{3}{4} \Psi.
\]

Hence, \(D_0\) acts as a diagonal matrix with eigenvalues \((\frac{3}{4}, \frac{3}{4})\). Summarizing the above arguments, we have that

\[
\overline{HM}_*(M_t, F_t, s) \cong \mathbb{F} \oplus \mathbb{F}, \quad \overline{HM}_*(M_t, F_t, s) \cong \mathbb{F} \oplus \mathbb{F}, \quad \overline{HM}_*(M_t, F_t, s) \cong 0.
\]

**Remark:** Note that the closure of the image of \(f_t\) can not be of one dimensional. Otherwise, let \(H\) be this one-dimensional closed subgroup in \(SO(3)\). It is well known that \(H\) preserves a vector in \(S^2 \subset \mathbb{R}^3\), say \((x, y, z)^t \in S^2\). We have that the group generated by

\[
\begin{pmatrix}
1 & \cos 2\pi t & -\sin 2\pi t \\
\sin 2\pi t & \cos 2\pi t & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

preserves \((x, y, z)^t\), which implies that either \(y = z = 0\) or \(t = 0, 1\). We can apply the same arguments for the other two subgroups. In conclusion we have that either \(x = y = z = 0\) or \(t = 0, 1\), which contradicts to our assumption.
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