GENERALIZED WEYL MODULES AND DEMAZURE SUBMODULES OF LEVEL-ZERO EXTREMAL WEIGHT MODULES.

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Abstract. We study a relationship between the graded characters of generalized Weyl modules $W_{w\lambda}$, $w \in W$, over the positive part of the affine Lie algebra and those of specific quotients $V^-_{w}(\lambda)/X^-_{w}(\lambda)$, $w \in W$, of the Demazure submodules $V^-_{w}(\lambda)$ of the extremal weight modules $V(\lambda)$ over the quantum affine algebra, where $W$ is the finite Weyl group and $\lambda$ is a dominant weight.

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1. Introduction

Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra over $\mathbb{C}$, and $\mathfrak{g}_{aff}$ the untwisted affine Lie algebra associated to $\mathfrak{g}$. In [OS], Orr-Shimozono gave a formula for the specialization at $t = 0$ of nonsymmetric Macdonald polynomials in terms of quantum alcove paths; they also gave a similar formula for the specialization at $t = \infty$. Recently, for a dominant weight $\lambda$ for $\mathfrak{g}$ and an arbitrary element $w$ of the finite Weyl group $W$, Feigin-Makedonskyi [FM] introduced a generalized Weyl module, denoted by $W_{w\lambda}$, over the positive part $\mathfrak{n}_{aff}$ of the affine Lie algebra $\mathfrak{g}_{aff}$, and proved that its graded character equals a certain graded character $C_{\lambda}^{(w\lambda)}$ of the set $\mathcal{QB}(w; t(w\lambda))$ of quantum alcove paths with starting point $w$ and directions given by a reduced decomposition of $t(w\lambda)$, where $w_0$ is the longest element of $W$, and $t(w\lambda)$ is an element of the extended affine Weyl group giving the translation by $w_0\lambda$.

Soon afterward, in [NNS], we proved that the graded character of the specific quotient module $V^-_{w}(\lambda)/X^-_{w}(\lambda)$ of $V^-_{w}(\lambda)$ is identical to the graded character $\text{gch}_{\lambda} \mathcal{QLS}(\lambda)$ of the set $\mathcal{QLS}(\lambda)$ of quantum Lakshmibai-Seshadri (QLS) paths of shape $\lambda$; here, $V(\lambda)$ is the extremal weight module of extremal weight $\lambda$ over the quantum affine algebra $U_{\lambda}(\mathfrak{g}_{aff})$, and $V^-_{w}(\lambda) \subset V(\lambda)$ is the Demazure submodule over the negative part $U^-_{\lambda}(\mathfrak{g}_{aff})$ of $U_{\lambda}(\mathfrak{g}_{aff})$. In particular, in the case $w = w_0$, this graded character was shown to be the specialization $E_{w_0\lambda}(q, \infty)$ of the nonsymmetric Macdonald polynomial $E_{w_0\lambda}(q, t)$ at $t = \infty$.

Also, in [LNSSS3], it is proved that for an arbitrary $w \in W$, the specialization $E_{w\lambda}(q, 0)$ at $t = 0$ is identical to the graded character of a specific quotient $U^+_{w}(\lambda)$ of the Demazure submodule $V^+_{w}(\lambda) \subset V(\lambda)$. Here we remark that for $w \in W$, $U^+_{w}(\lambda) \subset U^+_{w_0}(\lambda) = V^+_{w_0}(\lambda)/X^+_{w_0}(\lambda)$ in the notation of [NS2]; in contrast, there is no inclusion relation between $U^+_{w_0}(\lambda)$ and $V^+_{w}(\lambda)/X^+_{w}(\lambda)$, or equivalently, between $U^-_{w}(\lambda)$ and $V^-_{w}(\lambda)/X^-_{w}(\lambda)$.

The purpose of this paper is to reveal the relationship between the graded character of the generalized Weyl module $W_{w\lambda}$ over $\mathfrak{n}_{aff}$ and that of the quotient module $V^-_{w}(\lambda)/X^-_{w}(\lambda)$ of $V^-_{w}(\lambda)$ for an arbitrary $w \in W$. More precisely, we prove the following.

Theorem 1 (= Theorem 5.1.3). Let $\lambda$ be a dominant weight, and $w \in W$. Then, there holds the following equality:

$$w_0 \left( \text{gch}_{\lambda} W_{w\lambda} \right) = \text{gch}_{\lambda} \left( V^-_{w_ow\lambda}(\lambda)/X^-_{w_ow\lambda}(\lambda) \right).$$
Here, denotes the involution on \( \mathbb{Q}(q)[P] \) given by: \( q \mapsto q^{-1} \), \( e^\mu = e^\mu \) for \( \mu \in P \), and the \( \mathbb{Q}(q) \)-linear Weyl group action on \( \mathbb{Q}(q)[P] \) is given by: \( w \cdot e^\mu := e^{w\mu} \) for \( w \in W \) and \( \mu \in P \), where \( P \) is the weight lattice for \( g \).

In the case \( w = e \), Lenart-Naito-Sagaki-Schilling-Shimozono [LNSSS2] constructed a bijection \( \Xi : QB(e; t(w_o \lambda)) \rightarrow QLS(\lambda) \) between quantum alcove paths and QLS paths in order to prove that the graded character \( gch_{QLS(\lambda)} \) of QLS(\( \lambda \)) is identical to the specialization \( E_{w_0 \lambda}(q,0) \) at \( t = 0 \) of the nonsymmetric Macdonald polynomial \( E_{w_0 \lambda}(q,t) \). In this paper, we generalize their construction to an arbitrary \( w \in W \), and prove Theorem [1] by means of the constructed bijection.

This paper is organized as follows. In Section 2, we recall some basic facts about the quantum Bruhat graph. In Section 3, we review the definitions of quantum alcove paths and their graded characters. In Section 4, we recall the definition of QLS paths, and then define some variants of their graded characters. In Section 5, we give a bijection between the two sets \( QB(w; t(w_o \lambda)) \) and \( QLS(\lambda) \) that preserves weights and degrees. Using this bijection, we finally prove that \( w_o(C^{(w_o \lambda)}_w) = gch_{w_o w w_o} QLS(\lambda) \), and hence Theorem [1].

2. (Parabolic) Quantum Bruhat Graph

Let \( g \) be a finite-dimensional simple Lie algebra over \( \mathbb{C} \), \( I \) the vertex set for the Dynkin diagram of \( g \), and \( \{ \alpha_i \}_{i \in I} \) (resp., \( \{ \alpha_i^\vee \}_{i \in I} \)) the set of simple roots (resp., coroots) of \( g \). Then \( \mathfrak{h} = \bigoplus_{i \in I} C\alpha_i^\vee \) is a Cartan subalgebra of \( g \), with \( \mathfrak{h}^* = \bigoplus_{i \in I} C\alpha_i \) the dual space of \( \mathfrak{h} \), and \( \mathfrak{h}_R^* = \bigoplus_{i \in I} R\alpha_i \) its real form; the canonical pairing between \( \mathfrak{h} \) and \( \mathfrak{h}^* \) is denoted by \( \langle \cdot, \cdot \rangle : \mathfrak{h}^* \times \mathfrak{h} \rightarrow \mathbb{C} \). Let \( Q = \sum_{i \in I} \mathbb{Z}\alpha_i \subset \mathfrak{h}_R^* \) denote the root lattice, \( Q^\vee = \sum_{i \in I} \mathbb{Z}\alpha_i^\vee \subset \mathfrak{h}_R^* \) the coroot lattice, and \( P = \sum_{i \in I} \mathbb{Z}\alpha_i \subset \mathfrak{h}_R \) the weight lattice of \( g \), where the \( \varpi_i, i \in I \), are the fundamental weights for \( g \), i.e., \( \langle \varpi_i, \alpha_j^\vee \rangle = \delta_{ij} \) for \( i, j \in I \); we set \( P^+ := \sum_{i \in I} \mathbb{Z}_{\geq 0} \varpi_i \), and call an element \( \lambda \) of \( P^+ \) a dominant (integral) weight. Let us denote by \( \Delta \) the set of roots, and by \( \Delta^+ \) (resp., \( \Delta^- \)) the set of positive (resp., negative) roots. Also, let \( W := \langle s_i \mid i \in I \rangle \) be the Weyl group of \( g \), where \( s_i, i \in I \), are the simple reflections acting on \( \mathfrak{h}^* \) and on \( \mathfrak{h} \) as follows:

\[
\begin{align*}
  s_i \nu &= \nu - \langle \nu, \alpha_i^\vee \rangle \alpha_i, \quad \nu \in \mathfrak{h}^*, \\
  s_i h &= h - \langle \alpha_i, h \rangle \alpha_i^\vee, \quad h \in \mathfrak{h};
\end{align*}
\]

we denote the identity element and the longest element of \( W \) by \( e \) and \( w_o \), respectively. If \( \alpha \in \Delta \) is written as \( \alpha = w_o \alpha_i \) for \( w \in W \) and \( i \in I \), then its coroot \( \alpha^\vee \) is \( w_o \alpha_i^\vee \); we often identify \( s_\alpha \) with \( s_{w_o \alpha} \). For \( u \in W \), the length of \( u \) is denoted by \( \ell(u) \), which coincides with the cardinality of the set \( \Delta^+ \cap u^{-1} \Delta^- \).

Definition 2.1 (BFP, Definition 6.1]). The quantum Bruhat graph, denoted by \( QBG(W) \), is the directed graph with vertex set \( W \) whose directed edges are labeled by positive roots as follows. For \( u, v \in W \), and \( \beta \in \Delta^+ \), an arrow \( u \xrightarrow{\beta} v \) is a directed edge of \( QBG(W) \) if the following hold:

1. \( v = us_\beta \), and
2. either (2a): \( \ell(v) = \ell(u) + 1 \) or (2b): \( \ell(v) = \ell(u) - 2 \langle \rho, \beta^\vee \rangle + 1 \),

where \( \rho := \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha \). A directed edge satisfying (2a) (resp., (2b)) is called a Bruhat (resp., quantum) edge.

Remark 2.2. The quantum Bruhat graph defined above is a “right-hand” version, while the one defined in BFP is a “left-hand” version. Note that results of BFP used in this paper (such as Proposition 2.4) are unaffected by this difference (cf. [Pol]).
For a directed edge \( u \xrightarrow{\beta} v \) of \( \text{QBG}(W) \), we set
\[
\text{wt}(u \to v) := \begin{cases} 
0 & \text{if } u \xrightarrow{\beta} v \text{ is a Bruhat edge,} \\
\beta^\vee & \text{if } u \xrightarrow{\beta} v \text{ is a quantum edge.}
\end{cases}
\]

Also, for \( u, v \in W \), we take a shortest directed path \( u = x_0 \xrightarrow{\gamma_1} x_1 \xrightarrow{\gamma_2} \cdots \xrightarrow{\gamma_r} x_r = v \) in \( \text{QBG}(W) \), and set
\[
\text{wt}(u \Rightarrow v) := \text{wt}(x_0 \to x_1) + \cdots + \text{wt}(x_{r-1} \to x_r) \in Q^\vee;
\]
we know from [Po] Lemma 1(2), (3)] that this definition does not depend on the choice of a shortest directed path from \( u \) to \( v \) in \( \text{QBG}(W) \). For a dominant weight \( \lambda \in P^+ \), we set \( \text{wt}_*(u \Rightarrow v) := \langle \lambda, \text{wt}(u \Rightarrow v) \rangle \), and call it the \( \lambda \)-weight of a directed path from \( u \) to \( v \) in \( \text{QBG}(W) \).

**Lemma 2.3 ([NNS] Lemma 2.1.3).** If \( x \xrightarrow{\beta} y \) is a Bruhat (resp., quantum) edge of \( \text{QBG}(W) \), then \( y w_0 \xrightarrow{w_0 \beta} x w_0 \) is also a Bruhat (resp., quantum) edge of \( \text{QBG}(W) \).

Let \( w \in W \). We take (and fix) reduced expressions \( w = s_{i_1} \cdots s_{i_p} \) and \( w_0 w^{-1} = s_{i_0} \cdots s_{i_0} \); note that \( w_0 = s_{i_0} \cdots s_{i_0} s_{i_1} \cdots s_{i_p} \) is also a reduced expression for the longest element \( w_0 \). Now, we set
\[
(2.1) \quad \beta_k := s_{i_p} \cdots s_{i_{k+1}} \alpha_{i_k}, \quad -q \leq k \leq p;
\]
we have \( \{\beta_{-q}, \ldots, \beta_0, \ldots, \beta_p\} = \Delta^+ \). Then we define a total order \( \prec \) on \( \Delta^+ \) by:
\[
(2.2) \quad \beta_{-q} \prec \beta_{-q+1} \prec \cdots \prec \beta_p;
\]
note that this total order is a reflection order; see Remark 5.2.1.

**Proposition 2.4 ([BFP] Theorem 6.4]).** Let \( u, v \in W \).

(1) There exists a unique directed path from \( u \) to \( v \) in \( \text{QBG}(W) \) for which the edge labels are strictly increasing (resp., strictly decreasing) in the total order \( \prec \) above.

(2) The unique label-increasing (resp., label-decreasing) path
\[
u = u_0 \xrightarrow{\gamma_1} u_1 \xrightarrow{\gamma_2} \cdots \xrightarrow{\gamma_r} u_r = v
\]
from \( u \) to \( v \) in \( \text{QBG}(W) \) is a shortest directed path from \( u \) to \( v \). Moreover, it is lexicographically minimal (resp., lexicographically maximal) among all shortest directed paths from \( u \) to \( v \); that is, for an arbitrary shortest directed path
\[
u = u_0' \xrightarrow{\gamma_1'} u_1' \xrightarrow{\gamma_2'} \cdots \xrightarrow{\gamma_r'} u_r' = v
\]
from \( u \) to \( v \) in \( \text{QBG}(W) \), there exists some \( 1 \leq j \leq r \) such that \( \gamma_j \prec \gamma_j' \) (resp., \( \gamma_j > \gamma_j' \)), and \( \gamma_k = \gamma_k' \) for \( 1 \leq k \leq j - 1 \).

For a subset \( S \subseteq I \), we set \( W_S := \langle s_i \mid i \in S \rangle \); notice that \( S \) may be the empty set \( \emptyset \). We denote the longest element of \( W_S \) by \( w_0(S) \). Also, we set \( \Delta_S := Q_S \cap \Delta \), where \( Q_S := \sum_{i \in S} Z \alpha_i \), and \( \Delta^+_S := \Delta_S \cap \Delta^+ \), \( \Delta^-_S := \Delta_S \cap \Delta^- \). For \( w \in W \), we denote by \([w]\) the minimal-length coset representative for the coset \( w W_S \) in \( W/W_S \), and for a subset \( X \subset W \), we set \([X]:=[\{w]\mid w \in X]\subset W_S \), where \( W^S := [W] \) is the set of minimal-length coset representatives for the cosets in \( W/W_S \).
Definition 2.5 (LNSSS1 Section 4.3]). The parabolic quantum Bruhat graph, denoted by \( \text{QBG}(W^S) \), is the directed graph with vertex set \( W^S \) whose directed edges are labeled by positive roots in \( \Delta_+ \setminus \Delta^+_Z \) as follows. For \( u, v \in W^S \), and \( \beta \in \Delta_+ \setminus \Delta^+_Z \), an arrow \( u \rightarrow v \) is a directed edge of \( \text{QBG}(W^S) \) if the following hold:

1. \( v = [u \beta] \) and
2. either (2a): \( \ell(v) = \ell(u) + 1 \) or (2b): \( \ell(v) = \ell(u) - 2(\rho_S, \beta^\vee) + 1 \),

where \( \rho_S = \frac{1}{2} \sum_{\alpha \in \Delta_+^S} \alpha \). A directed edge satisfying (2a) (resp., (2b)) is called a Bruhat (resp., quantum) edge.

For a directed edge \( u \rightarrow v \) in \( \text{QBG}(W^S) \), we set

\[
\text{wt}^S(u \rightarrow v) := \begin{cases} 
0 & \text{if } u \rightarrow v \text{ is a Bruhat edge,} \\
\beta^\vee & \text{if } u \rightarrow v \text{ is a quantum edge.}
\end{cases}
\]

Also, for \( u, v \in W^S \), we take a shortest directed path \( p : u = x_0 \rightarrow x_1 \rightarrow \cdots \rightarrow x_r = v \) in \( \text{QBG}(W^S) \) (such a directed path always exists by [LNSSS1 Lemma 6.12]), and set

\[
\text{wt}^S(p) := \text{wt}^S(x_0 \rightarrow x_1) + \cdots + \text{wt}^S(x_{r-1} \rightarrow x_r) \in Q^\vee;
\]

we know from [LNSSS1 Proposition 8.1] that if \( q \) is another shortest directed path from \( u \) to \( v \) in \( \text{QBG}(W^S) \), then \( \text{wt}^S(p) - \text{wt}^S(q) \in Q^\vee := \sum_{i \in I} \mathbb{Z}_{\geq 0} \alpha_i^\vee \).

Now, for a dominant weight \( \lambda \in P^+ \), we set

\[
S = S_\lambda := \{ i \in I \mid \langle \lambda, \alpha_i^\vee \rangle = 0 \}.
\]

By the remark just above, for \( u, v \in W^S \), the value \( \langle \lambda, \text{wt}^S(p) \rangle \) does not depend on the choice of a shortest directed path \( p \) from \( u \) to \( v \) in \( \text{QBG}(W^S) \); this value is called the \( \lambda \)-weight of a directed path from \( u \) to \( v \) in \( \text{QBG}(W^S) \). Moreover, we know from [LNSSS2 Lemma 7.2] that the value \( \langle \lambda, \text{wt}^S(p) \rangle \) is equal to the value \( \text{wt}_\lambda(x \Rightarrow y) = \langle \lambda, \text{wt}(x \Rightarrow y) \rangle \) for all \( x \in uW^S \) and \( y \in vW^S \). In view of this fact, for \( u, v \in W^S \), we also write \( \text{wt}_\lambda(u \Rightarrow v) \) for the value \( \langle \lambda, \text{wt}^S(p) \rangle \) by abuse of notation; hence in this notation, we have

\[
\text{wt}_\lambda(x \Rightarrow y) = \text{wt}_\lambda([x] \Rightarrow [y])
\]

for all \( x, y \in W \).

Definition 2.6 (LNSSS2 Section 3.2]). Let \( \lambda \in P^+ \) be a dominant weight and \( \sigma \in \mathbb{Q} \setminus [0,1] \), and set \( S = S_\lambda \). We denote by \( \text{QBG}_{\sigma \lambda}(W) \) (resp., \( \text{QBG}_{\sigma \lambda}(W^S) \)) the subgraph of \( \text{QBG}(W) \) (resp., \( \text{QBG}(W^S) \)) with the same vertex set but having only the edges: \( u \rightarrow v \) with \( \sigma \langle \lambda, \beta^\vee \rangle \in \mathbb{Z} \).

Lemma 2.7 (LNSSS2 Lemma 6.1]). Let \( \sigma \in \mathbb{Q} \setminus [0,1] \); notice that \( \sigma \) may be 1. If \( u \rightarrow v \) is a directed edge of \( \text{QBG}_{\sigma \lambda}(W) \), then there exists a directed path from \( [u] \) to \( [v] \) in \( \text{QBG}_{\sigma \lambda}(W^S) \).

Also, for \( u, v \in W \), let \( \ell(u \Rightarrow v) \) denote the length of a shortest directed path in \( \text{QBG}(W) \) from \( u \) to \( v \). For \( w \in W \), following [BFP], we define the \( w \)-tilted Bruhat order \( \preceq_w \) on \( W \) as follows: for \( u, v \in W \),

\[
u \preceq_w v \overset{\text{def}}{\iff} \ell(v \Rightarrow w) = \ell(v \Rightarrow u) + \ell(u \Rightarrow w);
\]

the \( w \)-tilted Bruhat order on \( W \) is a partial order with the unique minimal element \( w \).

Lemma 2.8 (LNSSS1 Theorem 7.1], [LNSSS2 Lemma 6.5]). Let \( u, v \in W^S \), and \( w \in W^S \).
3. Quantum Alcove paths and their graded characters

In this section, we recall from [OS, Sections 4 and 5] and [FM, Section 1] the definition and some of the properties of the graded characters of quantum alcove paths.

Let $\mathfrak{g}$ denote the finite-dimensional simple Lie algebra whose root datum is dual to that of $\mathfrak{g}$; the set of simple roots is $\{\alpha_i\}_{i \in I} \subset \mathfrak{h}$, and the set of simple coroots is $\{\alpha_i^\vee\}_{i \in I} \subset \mathfrak{h}^*$. We denote the set of roots of $\mathfrak{g}$ by $\Delta = \{\alpha \in \Delta \mid \alpha \neq 0\}$, and the set of positive (resp., negative) roots by $\Delta^+$ (resp., $\Delta^-$).

We consider the untwisted affinization of the root datum of $\mathfrak{g}$. Let us denote by $\tilde{\Delta}_{aff}$ the set of all real roots, and by $\tilde{\Delta}_{aff}^+$ (resp., $\tilde{\Delta}_{aff}^-$) the set of all positive (resp., negative) real roots. Then we have $\tilde{\Delta}_{aff} = \{\alpha^\vee + a\tilde{\delta} \mid \alpha \in \Delta, a \in \mathbb{Z}\}$, with $\tilde{\delta}$ the (primitive) null root. We set $\alpha_i^\vee := \tilde{\delta} - \varphi_i^\vee$, where $\varphi \in \Delta$ denotes the highest short root, and set $I_{aff} := I \cup \{0\}$. Then, $\varpi_{i \in I_{aff}}$ is the set of simple roots. Also, for $\beta \in \mathfrak{h} \oplus \mathbb{C}\tilde{\delta}$, we define $\deg(\beta) \in \mathbb{C}$ and $\overline{\beta} \in \mathfrak{h}$ by:

$$\beta = \overline{\beta} + \deg(\beta)\tilde{\delta}.\quad(3.1)$$

We denote the Weyl group of $\mathfrak{g}$ by $W$; we identify $\tilde{W}$ and $W$ through the identification of the simple reflections of the same index $i \in I$. For $\nu \in \mathfrak{h}^*$, let $t(\nu)$ denote the translation in $\mathfrak{h}^*$: $t(\nu)\gamma = \gamma + \nu$ for $\gamma \in \mathfrak{h}^*$. The corresponding affine Weyl group and the extended affine Weyl group are defined by $\tilde{W}_{aff} := t(Q) \times W$ and $\tilde{W}_{ext} := t(P) \times W$, respectively. Also, we define $s_0 : \mathfrak{h}^* \to \mathfrak{h}^*$ by $\nu \mapsto \nu - \langle\nu, \varphi_i^\vee \rangle \varphi_i$. Then, $\tilde{W}_{aff} = \langle s_i \mid i \in I_{aff} \rangle$; note that $s_0 = t(\varphi)s_\varphi$.

The extended affine Weyl group $\tilde{W}_{ext}$ acts on $\mathfrak{h} \oplus \mathbb{C}\tilde{\delta}$ as linear transformations, and on $\mathfrak{h}^*$ as affine transformations: for $v \in \mathfrak{h}$, $t(\nu) \in t(P)$,

$$vt(\nu)(\overline{\beta} + r\tilde{\delta}) = v\overline{\beta} + (r - \langle\nu, \overline{\beta}\rangle)\tilde{\delta}, \quad \overline{\beta} \in \mathfrak{h}, r \in \mathbb{C},$$

$$vt(\nu)\gamma = vv + v\gamma, \quad \gamma \in \mathfrak{h}^*.$$  

An element $u \in \tilde{W}_{ext}$ can be written as

$$u = t(wt(u))dir(u),\quad(3.2)$$

with $wt(u) \in P$ and $dir(u) \in W$, according to the decomposition $\tilde{W}_{ext} = t(P) \times W$. For $w \in \tilde{W}_{ext}$, we denote the length of $w$ by $\ell(w)$, which equals $\#\left(\tilde{\Delta}_{aff}^+ \cap w^{-1}\tilde{\Delta}_{aff}^-\right)$. Also, we set $\Omega := \{w \in \tilde{W}_{ext} \mid \ell(w) = 0\}$.

Let $u \in \tilde{W}_{ext}$, and denote by $m_\mu \in \tilde{W}_{ext}$ the shortest element in the coset $\tilde{W} = t(P) \times W$. We take a reduced expression $m_\mu = us_{\ell_1} \cdots s_{\ell_L} \in \tilde{W}_{ext} = \Omega \times \tilde{W}_{aff}$, where $u \in \Omega$ and $\ell_1, \ldots, \ell_L \in I_{aff}$.

**Remark 3.1** ([M, Section 2.4]). For a dominant weight $\lambda \in \mathbb{P}^+$, we have

$$m_{w_{\circ} \lambda} = t(w_{\circ}\lambda).\quad(3.3)$$
Let $w \in W$. For each $J = \{j_1 < j_2 < j_3 < \cdots < j_r\} \subset \{1, \ldots, L\}$, we define an alcove path $p_J = \left( \omega m_\mu = z_0, z_1, \ldots, z_r; \tilde{\beta}_{j_1}, \ldots, \tilde{\beta}_{j_r} \right)$ as follows: we set $\tilde{\beta}_k := s_{\ell_k} \cdots s_{\ell_{k+1}} \alpha_k^\vee \in \Delta^+_\mathfrak{aff}$ for $1 \leq k \leq L$, and set

\[
\begin{align*}
z_0 &= \omega m_\mu, \\
z_1 &= \omega m_\mu s_{\tilde{\beta}_{j_1}}, \\
z_2 &= \omega m_\mu s_{\tilde{\beta}_{j_1}} s_{\tilde{\beta}_{j_2}}, \\
&\vdots \\
z_r &= \omega m_\mu s_{\tilde{\beta}_{j_1}} \cdots s_{\tilde{\beta}_{j_r}}.
\end{align*}
\]

Also, following [OS] Section 3.3, we set $B(w; m_\mu) := \{ p_J \mid J \subset \{1, \ldots, L\} \}$ and $\text{end}(p_J) := z_r \in \tilde{W}_{\text{ext}}$. Then we define $QB(w; m_\mu)$ to be the following subset of $B(w; m_\mu)$:

\[
\begin{align*}
\left\{ p_J \in B(w; m_\mu) \mid \text{dir}(z_i) \rightarrow \text{dir}(z_{i+1}) \text{ is a directed edge of } QBG(W), \ 0 \leq i \leq r - 1 \right\};
\end{align*}
\]

an element of $QB(w; m_\mu)$ is called a quantum alcove path with starting point $w$ and directions given by a reduced decomposition of $m_\mu$.

Remark 3.2 ([M (2.4.7)]). If $j \in \{1, \ldots, L\}$, then $- (\tilde{\beta}_j)^\vee \in \Delta^+$.

For $p_J \in QB(w; m_\mu)$, we define $qwt(p_J)$ as follows. Let $J^- \subset J$ denote the set of those indices $j_i \in J$ for which $\text{dir}(z_{i-1}) \rightarrow \text{dir}(z_i)$ is a quantum edge of $QBG(W)$. Then we set

\[
qwt(p_J) := \sum_{j \in J^-} \tilde{\beta}_j.
\]

Now, following [FM] Definition 1.9, let $C_w^{m_\mu}$ denote the graded character

\[
\sum_{p_J \in QB(w; m_\mu)} q^{\deg(qwt(p_J))} e^{wt(\text{end}(p_J))}
\]

of $QB(w; m_\mu)$.

Remark 3.3 ([OS]; see also [FM]). For $\mu \in P$, we denote by $E_\mu(q,t)$ the nonsymmetric Macdonald polynomial, and by $E_\mu(q,0)$ (resp., $E_\mu(q,\infty)$) its specialization $\lim_{t \to 0} E_\mu(q,t)$ (resp., $\lim_{t \to \infty} E_\mu(q,t)$) at $t = 0$ (resp., $t = \infty$).

(1) For the special case $w = e$, it holds that

\[
E_\mu(q,0) = C_e^{m_\mu}.
\]

(2) For the special case $w = w_0$, it holds that

\[
E_\mu(q^{-1},\infty) = w_0 C_{w_0}^{m_\mu};
\]

namely,

\[
E_\mu(q^{-1},\infty) = \sum_{p_J \in QB(w_0; m_\mu)} q^{\deg(qwt(p_J))} e^{w_0 wt(\text{end}(p_J))}.
\]

Let $\mathfrak{g}_{\text{aff}}$ denote the affine Lie algebra associated to $\mathfrak{g}$, and let $\mathfrak{g}_{\text{aff}} = \mathfrak{n}_{\text{aff}} \oplus \mathfrak{h}_{\text{aff}} \oplus \mathfrak{n}^-$ be its triangular decomposition.
Remark 3.4. We should warn the reader that the root datum of the affine Lie algebra $\mathfrak{g}_{aff}$ is not necessarily dual to that of the untwisted affine Lie algebra associated to $\mathfrak{g}$ whose set of real roots is $\Delta_{aff}$, though the root datum of $\tilde{\mathfrak{g}}$ is dual to that of $\mathfrak{g}$. In particular, for the index $0 \in I_{aff}$, the simple coroot $\alpha_0^\vee = c - \theta^\vee$, where $\theta \in \Delta^+$ is the highest root of $\mathfrak{g}$ and $c$ is the canonical central element of $\mathfrak{g}_{aff}$, does not agree with the simple root $\alpha_0^\vee = \delta - \varphi^\vee$ (see the beginning of this section).

Definition 3.5 ([FM] Definition 2.1]). Let $\lambda$ be a dominant weight, and $w \in W$. Then the generalized Weyl module $W_{ww_0}^\lambda$ is the cyclic $\mathfrak{n}_{aff}$-module with a generator $v$ and following relations:

\[
\begin{aligned}
(h \otimes t^k)v &= 0 \text{ for all } h \in \mathfrak{h}, k > 0, \\
(f_\alpha \otimes t)v &= 0 \text{ for } \alpha \in w\Delta^- \cap \Delta^-, \\
(e_\alpha \otimes 1)v &= 0 \text{ for } \alpha \in w\Delta^- \cap \Delta^+, \\
(f_{ww_0} \otimes t)^{-\langle w_\lambda, \alpha^\vee \rangle}v &= 0 \text{ for } \alpha \in \Delta^+ \cap w^{-1}\Delta^-, \\
(e_{ww_0} \otimes 1)^{-\langle w_\lambda, \alpha^\vee \rangle}v &= 0 \text{ for } \alpha \in \Delta^+ \cap w^{-1}\Delta^+,
\end{aligned}
\]

where $e_\alpha, f_\alpha, \alpha \in \Delta^+$, denote the Chevalley generators.

We can regard the generalized Weyl module $W_{ww_0}^\lambda$ as $\mathfrak{n}_{aff} \oplus \mathfrak{h}$-module by: $hv = \langle w_\lambda, h \rangle v, h \in \mathfrak{h}$; hence the module $W_{ww_0}^\lambda$ is a $\mathfrak{h}$-weighted module. Also, the module $W_{ww_0}^\lambda$ has a grading defined by the conditions: $\deg(v) = 0$, and the operator of the form $x \otimes t^k \in \mathfrak{n}_{aff}$ increases a degree by $k$. Thus, the graded character of the generalized Weyl module $W_{ww_0}^\lambda$ is defined by:

\[
gch W_{ww_0}^\lambda := \sum \dim (W_{ww_0}^\lambda[\gamma, k]) q^k e^{\gamma},
\]

where $W_{ww_0}^\lambda[\gamma, k]$ denotes the subspace of $W_{ww_0}^\lambda$ of degree $k$ and $\mathfrak{h}$-weight $\gamma$. Feigin-Makedonskyi proved that the graded character $gch W_{ww_0}^\lambda$ of $W_{ww_0}^\lambda$ is identical to the graded character $C^w_{ww_0} = C_{ww_0}^{m_{ww_0}}$ (see [FM] Theorem 2.21) for details).

4. Quantum Lakshmibai-Seshadri paths and some variants of their graded characters

4.1. Quantum Lakshmibai-Seshadri paths.

Definition 4.1.1 ([LNSSS2] Definition 3.1]). Let $\lambda \in P^+$ be a dominant weight, and set $S := S_\lambda = \{i \in I \mid \langle \lambda, \alpha_i^\vee \rangle = 0\}$. A pair $\eta = (w_1, w_2, \ldots, w_s; \sigma_0, \sigma_1, \ldots, \sigma_s)$ of a sequence $w_1, \ldots, w_s$ of elements in $W^S$ such that $w_k \neq w_{k+1}$ for $1 \leq k \leq s-1$ and an increasing sequence $0 = \sigma_0 < \cdots < \sigma_s = 1$ of rational numbers, is called a quantum Lakshmibai-Seshadri (QLS) path of shape $\lambda$ if

(C) for every $1 \leq i \leq s-1$, there exists a directed path from $w_{i+1}$ to $w_i$ in $QBG_{\sigma_i}(W^S)$.

Let $QLS(\lambda)$ denote the set of all QLS paths of shape $\lambda$.

Remark 4.1.2. As in [LNSSS4] Definition 3.2.2 and Theorem 4.1.1], condition (C) can be replaced by:

(C') for every $1 \leq i \leq s-1$, there exists a shortest directed path in $QBG(W^S)$ from $w_{i+1}$ to $w_i$ that is also a directed path in $QBG_{\sigma_i}(W^S)$.

The set $QLS(\lambda)$ provides a realization of the crystal basis of a particular quantum Weyl module $W_\nu(\lambda)$ over $U'_\nu(\mathfrak{g}_{aff})$, where $U'_\nu(\mathfrak{g}_{aff})$ denotes the quantum affine algebra without the
degree operator. (see [LNSSS4 Theorem 4.1.1], [NS1 Theorem 3.2], [N] Remark 2.15]). Moreover, $QLS(\lambda) \cong \bigotimes_{i \in I} QLS(\varpi_i)^{m_i}$ as $U'_\varphi(\mathfrak{g}_{\text{aff}})$-crystals, where $\lambda = \sum_{i \in I} m_i \varpi_i$; in particular, $\#QLS(\lambda) = \prod_{i \in I} (\#QLS(\varpi_i))^{m_i}$.

4.2. Some variants of graded characters. Let $\lambda = \sum_{i \in I} m_i \varpi_i \in P^+$, $m_i \in \mathbb{Z}_{\geq 0}$, be a dominant weight, and $w \in W$. For $\eta = (w_1, \ldots, w_s; \sigma_0, \ldots, \sigma_s) \in QLS(\lambda)$, we set

$$
wt(\eta) := \sum_{i=1}^{s} (\sigma_i - \sigma_{i-1}) w_i \lambda \in P,
$$

$$
\text{Deg}^*(\eta) := \sum_{i=1}^{s-1} (1 - \sigma_i) \text{wt}_\lambda(w_{i+1} \Rightarrow w_i),
$$

$$
\text{Deg}_s(\eta) := \sum_{i=1}^{s-1} \sigma_i \text{wt}_\lambda(w_{i+1} \Rightarrow w_i),
$$

$$
\text{Deg}^w(\eta) := \text{Deg}^*(\eta) + \text{wt}_\lambda(w_1 \Rightarrow w_0) = \sum_{i=0}^{s-1} (1 - \sigma_i) \text{wt}_\lambda(w_{i+1} \Rightarrow w_i),
$$

$$
\text{Deg}_w(\eta) := \text{Deg}_s(\eta) + \text{wt}(w_{s+1} \Rightarrow w_s) = \sum_{i=1}^{s} \sigma_i \text{wt}(w_{i+1} \Rightarrow w_i),
$$

where we set $w_0 := w$ and $w_{s+1} := w$. Note that by Remark 4.1.2 we have $\sigma_i \text{wt}_\lambda(w_{i+1} \Rightarrow w_i) \in \mathbb{Z}_{\geq 0}$ for $1 \leq i \leq s - 1$. Hence it follows that $\text{Deg}^*(\eta), \text{Deg}_s(\eta), \text{Deg}^w(\eta), \text{Deg}_w(\eta) \in \mathbb{Z}_{\geq 0}$; notice that $\sigma_0 = 0$, and $\sigma_s = 1$.

Also, we set

$$
\text{gch}^w QLS(\lambda) := \sum_{\eta \in QLS(\lambda)} q^{-\text{Deg}^w(\eta)} e^{\text{wt}(\eta)},
$$

$$
\text{gch}_w QLS(\lambda) := \sum_{\eta \in QLS(\lambda)} q^{-\text{Deg}_w(\eta)} e^{\text{wt}(\eta)}.
$$

Let $V(\lambda)$ be the extremal weight module of extremal weight $\lambda$ over the quantum affine algebra $U_q(\mathfrak{g}_{\text{aff}})$, and by $V^-_w(\lambda) \subset V(\lambda)$ the Demazure submodule over the negative part $U^-_q(\mathfrak{g}_{\text{aff}})$ of $U_q(\mathfrak{g}_{\text{aff}})$; see [NS2] for details.

In [NNS], we proved that $\text{gch}^w V^-_w(\lambda)$ of the Demazure submodule $V^-_w(\lambda)$ of $V(\lambda)$ is identical to

$$
\left( \prod_{i \in I} \prod_{r=1}^{m_i} (1 - q^{-r}) \right)^{-1} \text{gch}^w QLS(\lambda).
$$

Moreover, the graded character of a specific (finite-dimensional) quotient module $V^-_w(\lambda)/X^-_w(\lambda)$ of $V^-_w(\lambda)$ over $U^-_q(\mathfrak{g}_{\text{aff}})$ is identical to $\text{gch}_w QLS(\lambda)$; see [NNS] (5.7) for the definition of the quotient module $V^-_w(\lambda)/X^-_w(\lambda)$.

Remark 4.2.1. Let $\lambda \in P^+$ be a dominant weight. Then we know the following:

1. if $w = e$, then $\text{gch}_w QLS(\lambda) = E_{w_0,\lambda}(q^{-1}, 0)$ ([LNSSS2 Lemma 7.7 and Theorem 7.9]);
2. if $w = w_0$, then $\text{gch}_w QLS(\lambda) = E_{w_0,\lambda}(q, \infty)$ ([NNS] Theorem 3.2.7]),

where $E_{w_0,\lambda}(q, 0)$ and $E_{w_0,\lambda}(q, \infty)$ denote the specializations of the nonsymmetric Macdonald polynomial $E_{w_0,\lambda}(q, t)$ at $t = 0$ and $t = \infty$, respectively. Here we remark that for $\eta \in QLS(\lambda)$, $\text{Deg}^*(\eta)$ and $\text{Deg}_w(\eta)$ are equal to $-\text{Deg}(\eta)$ in [LNSSS2 Theorem 4.6] and $-\text{Deg}_w(\eta)$ in [NNS Section 3.2], respectively.
4.3. Lusztig involution. Let $\lambda \in P^+$ be a dominant weight, and set $S := S_\lambda = \{ i \in I \mid \langle \lambda, \alpha_i^\vee \rangle = 0 \}$. In this subsection, we state the relationship between the graded characters $\text{gch}_w QLS(\lambda)$ and $\text{gch}^w QLS(\lambda)$ for $w \in W$.

For $\eta = (\omega_1, \ldots, \omega_s; \sigma_0, \ldots, \sigma_s) \in QLS(\lambda)$, we define $S(\eta)$ to be $([\omega_0 \omega_s], \ldots, [\omega_0 \omega_1]; 1 - \sigma_s, \ldots, 1 - \sigma_0)$. The following follows from \cite[Section 4.5]{LNSSS2}.

Lemma 4.3.1. 
(1) Let $w_1, w_2 \in W$. Then, $\text{wt}_\lambda(w_0 w_1 \Rightarrow w_0 w_2) = \text{wt}_\lambda(w_2 \Rightarrow w_1)$.
(2) Let $\eta = (\omega_1, \ldots, \omega_s; \sigma_0, \ldots, \sigma_s) \in QLS(\lambda)$. Then we have
   (a) $S(\eta) \in QLS(\lambda)$;
   (b) $\text{wt}(S(\eta)) = w_0 \text{wt}(\eta)$;
   (c) $\text{Deg}_w(S(\eta)) = \text{Deg}^{w_0 w}(\eta)$.

Proof. Parts (1), (2a), and (2b) are proved in \cite[Section 4.5]{LNSSS2}. Let us prove part (2c). It follows from \cite[Corollary 4.8]{LNSSS2} that
\[
\text{Deg}_w(S(\eta)) = \text{Deg}^*(\eta).
\]
Also, by part (1), we have
\[
\text{wt}_\lambda(w \Rightarrow [w_0 \omega_s]) = \text{wt}_\lambda(w \Rightarrow w_0 \omega_s) = \text{wt}_\lambda(w_0 \Rightarrow w_0 w) \quad \text{(by equation (2.3))}.
\]
From these, we see that
\[
\text{Deg}_w(S(\eta)) = \text{Deg}_w(S(\eta)) + \text{wt}_\lambda(w \Rightarrow w_0 \omega_s) = \text{Deg}^*(\eta) + \text{wt}_\lambda(w_s \Rightarrow w_0 w) = \text{Deg}^{w_0 w}(\eta),
\]

as desired. This proves the lemma. \qed

The operator $S$ above is an involution on $QLS(\lambda)$, called the Lusztig involution. By using Lemma 4.3.1, we deduce that
\[
\text{gch}_w QLS(\lambda) = w_0 (\text{gch}^{w_0 w} QLS(\lambda)) = \sum_{\eta \in QLS(\lambda)} q^{-\text{Deg}^{w_0 w}(\eta)} e^{w_0 \text{wt}(\eta)}
\]
for $w \in W$.

5. Relationship between the two graded characters

5.1. Relationship between the graded characters $gch W_{w \lambda}$ and $gch(W_-(\lambda)/X_w(\lambda))$. We define an involution $\overline{\cdot}$ on $\mathbb{Q}(q)$ by $\overline{q} = q^{-1}$, and set $\overline{f} := \sum_{\mu \in P} f_{\mu} e^{i\mu}$ for $f = \sum_{\mu \in P} f_{\mu} e^{i\mu}$ with $f_{\mu} \in \mathbb{Q}(q)$.

Theorem 5.1.1. Let $\lambda \in P^+$ be a dominant weight, and $w \in W$. Then we have
\[
\overline{C_w^{(w_0 \lambda)}} = gch^{w_0 w} QLS(\lambda).
\]
In particular, $\#QB(w, t(w_0 \lambda)) = \#QLS(\lambda)$ (see Proposition 5.3.1).

We will give a proof of Theorem 5.1.1 in Section 5.3. By combining Theorem 5.1.1 and 4.3.1 with $w$ replaced by $w_0 w_\sigma$, we obtain the following theorem.

Theorem 5.1.2. Let $\lambda \in P^+$ be a dominant weight, and $w \in W$. Then we have
\[
w_0 \left(\overline{C_w^{(w_0 \lambda)}}\right) = gch_{w_0 w_\sigma} QLS(\lambda).
\]
Because $C_{w}^{(\omega \lambda)}$ equals the graded character $\text{gch}W_{w\omega \lambda}$ of the generalized Weyl module $W_{w\omega \lambda}$, and $\text{gch}W_{w\omega \omega}$ QLS($\lambda$) equals the graded character $\text{gch}(V_{w\omega \omega}(\lambda)/X_{w\omega \omega}(\lambda))$ of the quotient module $V_{w\omega \omega}(\lambda)/X_{w\omega \omega}(\lambda)$, we obtain the following.

**Theorem 5.1.3.** Let $\lambda \in P^+$ be a dominant weight, and $w \in W$. Then we have
\begin{equation}
(5.1) \quad w_0 \left( \text{gch}W_{w\omega \lambda} \right) = \text{gch}(V_{w\omega \omega}(\lambda)/X_{w\omega \omega}(\lambda)).
\end{equation}

5.2. Reduced expressions for $t(w_0 \lambda)$ and a total order on $\Delta_{\text{aff}}^+ \cap t(w_0 \lambda)^{-1} \Delta_{\text{aff}}^-$. Let $\lambda \in P^+$ be a dominant weight, and set $\lambda_- := w_0 \lambda$, $S := S_\lambda = \{i \in I \mid \langle \lambda, \omega_i \rangle = 0 \}$. For $\mu \in W_\lambda$, we denote by $v(\mu) \in W_S$ the minimal-length coset representative for the coset $\{w \in W \mid w \lambda = \mu \}$ in $W/W_S$; note that $w_0 = v(\lambda_-)w_0(S)$ and $\ell(w_0) = \ell(v(\lambda_-)) + \ell(w_0(S))$, where $w_0(S)$ denotes the longest element of $W_S$.

In this subsection, we recall from [NNS] a particular reduced expression for $m_{\lambda_-} (= t(\lambda_-)$ by (3.3)) with respect to a fixed total order on $\Delta_{\text{aff}}^+ \cap t(w_0 \lambda)^{-1} \Delta_{\text{aff}}^-$, and review some of its properties.

We fix reduced expressions
\begin{align}
(5.2) & \quad v(\lambda_) = s_{i_1} \cdots s_{i_M}, \\
(5.3) & \quad w_0(S) = s_{i_{M+1}} \cdots s_{i_N}
\end{align}
for $v(\lambda_-)$ and $w_0(S)$, respectively. Then
\begin{equation}
(5.4) \quad w_0 = s_{i_1} \cdots s_{i_N}
\end{equation}
is a reduced expression for $w_0$. We set $\beta_j := s_{i_{j+1}} \cdots s_{i_{j+1}} \omega_{i_j}, 1 \leq j \leq N$. Then we have $\Delta^+ \setminus \Delta^+_S = \{\beta_1, \ldots, \beta_M\}$ and $\Delta^+_S = \{\beta_{M+1}, \ldots, \beta_N\}$. We fix a total order on $\Delta^+$ such that
\begin{equation}
(5.5) \quad \beta_1 > \beta_2 > \cdots > \beta_M > \beta_{M+1} > \cdots > \beta_N,
\end{equation}
each $\in \Delta^+ \setminus \Delta^+_S$.

**Remark 5.2.1.** We call the total order $\prec$ above a reflection order on $\Delta^+$; if $\alpha, \beta, \gamma \in \Delta^+$ with $\gamma = \alpha' + \beta'$, then $\alpha < \gamma < \beta$ or $\beta < \gamma < \alpha$.

Now, we define an injective map
\begin{equation}
(5.6) \quad \Phi : \Delta^+_{\text{aff}} \cap t(\lambda_-)^{-1} \Delta_{\text{aff}}^- \rightarrow Q_{\geq 0} \times (\Delta^+ \setminus \Delta^+_S), \quad \beta = \overline{\beta} + \deg(\beta) \overline{\delta} \mapsto \left( \langle \lambda_-, \overline{\beta} \rangle - \deg(\beta), w_0 \overline{\beta}' \right).
\end{equation}

Here we note that $\langle \lambda_-, \overline{\beta} \rangle > 0$, $\langle \lambda_-, \overline{\beta} \rangle - \deg(\beta) \geq 0$, and $w_0 \overline{\beta}' \in \Delta^+ \setminus \Delta^+_S$ since $\langle \lambda_-, \overline{\beta} \rangle = \langle \lambda, w_0 \overline{\beta} \rangle > 0$; recall from [M] (2.4.7) (i) that
\begin{equation}
(5.6) \quad \Delta^+_{\text{aff}} \cap t(\lambda_-)^{-1} \Delta_{\text{aff}}^- = \{a + a \overline{\delta} \mid a \in \Delta^-, a \in \mathbb{Z}, 0 < a \leq \langle \lambda_-, \alpha' \rangle \}.
\end{equation}

Let us consider the lexicographic order $\prec$ on $Q_{\geq 0} \times (\Delta^+ \setminus \Delta^+_S)$ induced by the usual total order on $Q_{\geq 0}$ and the reverse order of the restriction to $\Delta^+ \setminus \Delta^+_S$ of the total order $\prec$ on $\Delta^+$ above; that is, for $(a, \alpha), (b, \beta) \in Q_{\geq 0} \times (\Delta^+ \setminus \Delta^+_S),$
\begin{equation}
(5.7) \quad (a, \alpha) < (b, \beta) \text{ if and only if } a < b, \text{ or } a = b \text{ and } \alpha > \beta.
\end{equation}

Then we denote by $\prec'$ the total order on $\Delta^+_{\text{aff}} \cap t(\lambda_-)^{-1} \Delta_{\text{aff}}^-$ induced by the lexicographic order on $Q_{\geq 0} \times (\Delta^+ \setminus \Delta^+_S)$ through the injective map $\Phi$.

The proof of the following proposition is the same as that of [NNS, Proposition 3.1.8].
Proposition 5.2.2. With the notation and setting above, let us write \( \overline{\Delta_{\text{aff}}}^+ \cap t(\lambda_-)^{-1} \overline{\Delta_{\text{aff}}}^- \) as \( \{ \gamma_1 \prec \cdots \prec \gamma_L \} \). Then, there exists a unique reduced expression \( t(\lambda_-) = us_{\ell_1} \cdots s_{\ell_L} \) for \( t(\lambda_-) \), with \( u \in \Omega \) and \( \{ \ell_1, \ldots, \ell_L \} \subset I_{\text{aff}} \), such that \( \tilde{\beta}_j = \gamma_j \) for \( 1 \leq j \leq L \), where \( \tilde{\beta}_j = s_{\ell_L} \cdots s_{\ell_{j+1}} \alpha_{t_j}^\vee \), \( 1 \leq j \leq L \).

In the following, we use the reduced expression \( t(\lambda_-) = us_{\ell_1} \cdots s_{\ell_L} \) for \( t(\lambda_-) \) given by this proposition. The proof of the following lemma is the same as that of [NNS, Lemma 3.1.10].

Lemma 5.2.3. Keep the notation and setting above. Then, \( us_{\ell_{M+1}} \cdots s_{\ell_L} \) is a reduced expression for \( m_\lambda \). Moreover, if we write \( us_{\ell_k} = s_k u \) for \( 1 \leq k \leq M \), then \( i_k = i_k' \), where \( w_0 = s_{i_1} \cdots s_{i_N} \) is the reduced expression (5.1) for \( w_0 \).

We set \( a_k := \deg(\tilde{\beta}_k) \in \mathbb{Z}_{\geq 0} \) for \( 1 \leq k \leq L \); since \( \overline{\Delta_{\text{aff}}}^+ \cap t(\lambda_-)^{-1} \overline{\Delta_{\text{aff}}}^- = \{ \tilde{\beta}_1, \ldots, \tilde{\beta}_L \} \), we see by (5.4) that \( 0 < a_k \leq \langle \lambda_-, \tilde{\beta}_k \rangle \).

Corollary 5.2.4. For \( 1 \leq k \leq M \), we have \( w_0 \overline{\beta}_k = \beta_k^\vee \), where \( \beta_k := s_{i_N} \cdots s_{i_{k+1}} \alpha_{i_k} \).

Proof. If we set \( \tilde{\beta}_k := us_{\ell_1} \cdots s_{\ell_{k-1}} \alpha_{t_k}^\vee \), \( 1 \leq k \leq M \), then we have

\[
-t(\lambda_-) \tilde{\beta}_k = -(us_{\ell_1} \cdots s_{\ell_L}) (s_{\ell_L} \cdots s_{\ell_{k+1}} \alpha_{t_k}^\vee) = -us_{\ell_1} \cdots s_{\ell_{k-1}} \alpha_{t_k}^\vee
\]

Therefore, we see that

\[
\overline{w_0 \beta_k} = \overline{w_0 (-\beta_k)} = \overline{w_0 (-us_{\ell_1} \cdots s_{\ell_{k-1}} \alpha_{t_k}^\vee)} = \overline{w_0 (-s_{i_1} \cdots s_{i_{k-1}} \alpha_{t_k}^\vee)} = \overline{s_{i_N} \cdots s_{i_{k-1}} \alpha_{t_k}^\vee} = \overline{\beta_k^\vee},
\]

as desired.

For \( 1 \leq k \leq L \), we set

\[
d_k := \frac{\langle \lambda_-, \beta_k \rangle - a_k}{\langle \lambda_-, \beta_k \rangle};
\]

here, \( d_k \) is just the first component of \( \Phi(\tilde{\beta}_k) \in \mathbb{Q}_{\geq 0} \times (\Delta^+ \setminus \Delta_0^+) \). Recall that for \( 1 \leq k, j \leq L \), \( \Phi(\tilde{\beta}_k) \prec \Phi(\tilde{\beta}_j) \) if and only if \( k < j \). Therefore, it follows that

\[
0 \leq d_1 \leq \cdots \leq d_L \leq 1.
\]

The following lemma follows from the definition of the map \( \Phi \).

Lemma 5.2.5 (cf. [NNS, Lemma 3.1.12]). If \( 1 \leq k < j \leq L \) and \( d_k = d_j \), then \( w_0 \overline{\beta_k}^\vee \geq w_0 \overline{\beta_j}^\vee \).

5.3. Proof of Theorem 5.1.1. We keep the notation of Section 5.2. In this subsection, in order to prove Theorem 5.1.1 we give a bijection

\[
\Xi_w : QB(w; t(\lambda_-)) \rightarrow \text{QLS}(\lambda)
\]

that preserves weights and degrees for an arbitrary \( w \in W \). The way to construct this bijection is similar to the one for the bijection \( \Xi : \overline{QB}(e, t(\lambda_-)) \rightarrow \text{QLS}(\lambda) \) in [NNS, Section 3.3], where
\( \overline{QB}(e, t(\lambda_-)) \) is a subset of \( B(e; t(\lambda_-)) \) defined as follows:

\[
\{ p_J \in B(e; t(\lambda_-)) \mid \text{dir}(z_{j_i}) \leftarrow - (\beta_{j_i})^\vee \text{dir}(z_{j_i+1}) \text{ is a directed edge of } \overline{QB}(W) \text{ for } j \in J \}.
\]

Here we remark that the defining condition for \( \overline{QB}(e, t(\lambda_-)) \) differs from the one for \( \overline{QB}(w, t(\lambda_-)) \) in that all allowable directed edges are reversed. Also, note that we ignore the directed path \( (5.11) \) below for \( p = 0 \) (which depends on \( w \in W \)) in the construction of the map \( \Xi_w \), while we ignore the path \( (3.17) \) in [NNS] for \( p = 0 \) in the construction of the map \( \Xi \). Accordingly, in order to prove that \( \Xi_w \) is both injective and surjective, we need to check that for each \( w \in W \) and \( \eta \in \text{QLS}(\lambda) \), there exists a unique directed path in \( \overline{QB}(W) \) from some element in the coset \( \kappa(\eta)W_\delta \) to \( uw_0 \) whose edge labels are increasing and lie in \( \Delta^+ \setminus \Delta^+_S \).

**Remark 5.3.1.** Let \( \gamma_1, \gamma_2, \ldots, \gamma_r \in \Delta^+ \cap t(\lambda_-)^{-1} \Delta^-_{\text{aff}} \), and define a sequence \((y_0, y_1, \ldots, y_r)\) by:

\[
y_0 = wt(\lambda_-), \quad y_i = y_{i-1} - \gamma_i \quad \text{for } 1 \leq i \leq r.
\]

Then, the pair of sequences \((y_0, y_1, \ldots, y_r; \gamma_1, \gamma_2, \ldots, \gamma_r)\) is an element of \( \overline{QB}(w; t(\lambda_-)) \) if and only if the following conditions are satisfied:

1. \( \gamma_1 \prec \gamma_2 \prec \cdots \prec \gamma_r \), where the order \( \prec \) is the total order on \( \Delta^+ \cap t(\lambda_-)^{-1} \Delta^-_{\text{aff}} \) introduced in Section 5.2;

2. for \( 1 \leq i \leq r \), \( \text{dir}(y_{i-1}) \leftarrow -(\gamma_i)^\vee \text{dir}(y_i) \) is a directed edge of \( \overline{QB}(W) \).

Let us define a map \( \Xi_w : \overline{QB}(w; t(\lambda_-)) \rightarrow \text{QLS}(\lambda) \) as follows. Let \( p_J \) be an arbitrary element of \( \overline{QB}(w; t(\lambda_-)) \) of the form

\[
p_J = \left( wt(\lambda_-) = z_0, z_1, \ldots, z_r; \tilde{b}_{j_1}, \tilde{b}_{j_2}, \ldots, \tilde{b}_{j_r} \right) \in \overline{QB}(w; t(\lambda_-)),
\]

where \( J = \{ j_1 < \cdots < j_r \} \subset \{ 1, \ldots, L \} \). We set \( x_k := \text{dir}(z_k), 0 \leq k \leq r \). Then, by the definition of \( \overline{QB}(w; t(\lambda_-)) \),

\[
(5.9) \quad w = x_0 \leftarrow -(\beta_{j_1})^\vee \rightarrow x_1 \leftarrow -(\beta_{j_2})^\vee \rightarrow \cdots \leftarrow -(\beta_{j_r})^\vee \rightarrow x_r
\]

is a directed path in \( \overline{QB}(W) \); the equality \( w = x_0 \) follows from the equality \( \text{dir}(t(w_0 \lambda)) = e \). We take \( 0 = u_0 \leq u_1 < \cdots < u_{s-1} < u_s = r \) and \( 0 = \sigma_0 < \sigma_1 < \cdots < \sigma_{s-1} < 1 = \sigma_s \) in such a way that (see (5.8))

\[
(5.10) \quad 0 = d_{j_1} = \cdots = d_{j_{u_0+1}} < d_{j_{u_0+1}+1} = \cdots = d_{j_{u_2}} < \cdots < d_{j_{u_{s-1}+1}} = \cdots = d_{j_r} < 1 = \sigma_s;
\]

note that \( d_{j_1} > 0 \) if and only if \( u_1 = 0 \). We set \( w'_p := x_{u_p} \) for \( 0 \leq p \leq s \). Then, by taking a subsequence of (5.9), we obtain the following directed path in \( \overline{QB}(W) \) for each \( 0 \leq p \leq s-1 \):

\[
w'_p = x_{u_p} \leftarrow -(\beta_{j_{u_p+1}})^\vee \rightarrow x_{u_p+1} \leftarrow -(\beta_{j_{u_p+2}})^\vee \rightarrow \cdots \leftarrow -(\beta_{j_{u_p+1}})^\vee \rightarrow x_{u_p+1} = w'_{p+1}.
\]

Multiplying this directed path on the right by \( w_0 \), we obtain the following directed path in \( \overline{QB}(W) \) for each \( 0 \leq p \leq s-1 \) (see Lemma 2.3):

\[
(5.11) \quad w_p := w'_p w_0 = x_{u_p} w_0 \leftarrow -(\beta_{j_{u_p+1}})^\vee \rightarrow x_{u_p+1} \leftarrow -(\beta_{j_{u_p+1}})^\vee \rightarrow x_{u_p+1} w_0 = w'_{p+1} w_0 =: w_{p+1}.
\]

Note that \( w_0 = w'_0 w_0 = x_0 w_0 = w w_0 \). In addition, the edge labels of this directed path are increasing in the reflection order \( \prec \) on \( \Delta^+ \) given by (5.5) (see Lemma 5.2.3), and lie in
$\Delta^+ \setminus \Delta_+^{S}$; this property is used to prove that the map $\Xi_w$ is injective. Because

$$\sigma_p(\lambda, w_0 \beta_{j_u}) = d_{j_u} \sigma_p(\lambda, w_0 \beta_{j_u}) = \left(\frac{\lambda - \beta_{j_u}}{\lambda - \beta_{j_u}}\right) - a_{j_u} \sigma_p(\lambda, w_0 \beta_{j_u}) = \langle \lambda, \beta_{j_u} \rangle - a_{j_u} \in \mathbb{Z}$$

for $u_p + 1 \leq u \leq u_{p+1}$, $0 \leq p \leq s - 1$, we find that (5.11) is a directed path in $\text{QB}_p^\lambda(W)$ for each $0 \leq p \leq s - 1$. Therefore, by Lemma 2.7, there exists a directed path in $\text{QB}_p^\lambda(W^S)$ from $[w_{p+1}]$ to $[w_{p}]$, where $S := S_\lambda = \{ i \in I \mid \langle \lambda, \alpha_i^\vee \rangle = 0 \}$. Also, we claim that $[w_p] \neq [w_{p+1}]$ for any $1 \leq p \leq s - 1$. Suppose, for a contradiction, that $[w_p] = [w_{p+1}]$ for some $p$. Then, $w_p W_S = w_{p+1} W_S$, and hence

$$\min(w_{p+1} W_S, \leq w_p) = \min(w_p W_S, \leq w_p) = w_p.$$  

Recall that the directed path (5.11) is a path in $\text{QB}_p^\lambda(W)$ from $w_{p+1}$ to $w_p$ whose edge labels are increasing and lie in $\Delta^+ \setminus \Delta_+^{S}$. By Lemma 2.8(1), (2), the directed path (5.11) is a shortest path in $\text{QB}_p^\lambda(W)$ from $w_{p+1} = \min(w_{p+1} W_S, \leq w_p)$ to $w_p$. It follows from (5.13) that $w_{p+1} = \min(w_{p+1} W_S, \leq w_p) = w_p$, and hence the length of the directed path (5.11) is equal to 0. Therefore, we have $\{ j_{u_p+1}, \ldots, j_{u_p+1} \} = \emptyset$, and hence $u_p = u_{p+1}$, which contradicts the fact that $u_p < u_{p+1}$. Thus we obtain

$$\eta := ([w_1], \ldots, [w_s]; \sigma_0, \ldots, \sigma_s) \in \text{QLS}(\lambda).$$

We now define $\Xi_w(p) := \eta$.

In order to prove that the map $\Xi_w : \text{QB}(w; t(\lambda_\cdot)) \rightarrow \text{QLS}(\lambda)$ is bijective, we prove that the map is both injective and surjective; in [NNS Sect. 3.3], we gave the inverse map $\Theta : \text{QLS}(\lambda) \rightarrow \text{QB}(c; t(\lambda_\cdot))$ of the map $\Xi$.

**Lemma 5.3.2.** The map $\Xi_w : \text{QB}(w; t(\lambda_\cdot)) \rightarrow \text{QLS}(\lambda)$ is injective.

**Proof.** Let $J = \{ j_1, \ldots, j_r \}$ and $K = \{ k_1, \ldots, k_{r'} \}$ be subsets of $\{ 1, \ldots, L \}$ such that $\Xi_w(p) = \Xi_w(p_K) = (v_1, \ldots, v_s; \sigma_0, \ldots, \sigma_s) \in \text{QLS}(\lambda)$. As in (5.10), we set 0 = $u_0 = u_1 < \cdots < u_s = r$ and 0 = $u_0' = u_1' < \cdots < u_{s'} = r'$ in such a way that

$$0 = d_{j_1} = \cdots = d_{j_{u_1}} < d_{j_{u_1}+1} = \cdots = d_{j_{u_2}} < \cdots < d_{j_{u_s-1}+1} = \cdots = d_{j_r} < 1 = \sigma_s,$$

$$0 = d_{k_1} = \cdots = d_{k_{u'_1}} < d_{k_{u'_1}+1} = \cdots = d_{k_{u'_{s-1}+1}} = \cdots = d_{k_{r'}} < 1 = \sigma_{s'}.$$

As in (5.11), we consider the directed paths in $\text{QB}_p^\lambda(W)$

$$w_p \xleftarrow{w_0 \beta_{j_{u+1}}} \cdots \xleftarrow{w_0 \beta_{j_{u+1}}} w_{p+1}, \quad \text{for } 0 \leq p \leq s - 1,$$

$$y_p \xleftarrow{w_0 \beta_{k_{u'+1}}} \cdots \xleftarrow{w_0 \beta_{k_{u'+1}}} y_{p+1}, \quad \text{for } 0 \leq p \leq s - 1;$$

here we note that $w_0 = y_0 = w w_0$, and $[w_p] = [y_p] = v_p$, 1 $\leq p \leq s$.

Now, let 0 $\leq p \leq s - 1$, and assume that $w_p = y_p$ and $u_p = u_p'$. Then both of the directed paths in (5.15) are directed paths from some element in $v_{p+1} W_S$ to $w_p$ in $\text{QB}_p^\lambda(W)$ whose edge labels are increasing and lie in $\Delta^+ \setminus \Delta_+^{S}$. Therefore, $w_{p+1} = y_{p+1} \in v_{p+1} W_S$ and $w_0 (\beta_{j_i}) = w_0 (\beta_{k_i})$ for $u_p + 1 \leq i \leq u_{p+1}$, and $u_{p+1} = u_{p+1}'$ by Lemma 2.8(2); also, $d_{j_i} = d_{k_i} = \sigma_p$ for $u_p + 1 \leq i \leq u_{p+1}$. It follows from the equalities $\beta_j = \beta_j + (1 - d_j) \langle \lambda, \beta_j \rangle$, $\beta_k = \beta_k + (1 - d_k) \langle \lambda, \beta_k \rangle$. \hfill $\Box$
1 \leq j \leq L$, that $\tilde{\beta}_j = \tilde{\beta}_k$, $u_p + 1 \leq i \leq u_{p+1}$, and hence $j_i = k_i$, $u_p + 1 \leq i \leq u_{p+1}$. Thus, by induction on $p$, we deduce that $u_p = u'_p$ for all $0 \leq p \leq s$, and that $j_i = k_i$ for all $u_0 + 1 \leq i \leq u_s$. Consequently, we obtain $r = u_s = u'_s = r'$, and hence $J = \{j_1, \ldots, j_r\} = \{k_1, \ldots, k_r\} = K$. This proves the lemma. 

**Lemma 5.3.3.** The map $\Xi_w : \mathcal{QB}(w; t(\lambda_-)) \to \mathcal{QLS}(\lambda)$ is surjective.

**Proof.** Take an arbitrary element $\eta = (y_1, \ldots, y_s; \tau_0, \ldots, \tau_s) \in \mathcal{QLS}(\lambda)$; we set $y_0 = [ww_0] \in W^S$. We define elements $v_p \in W$, $0 \leq p \leq s$, by: $v_0 = ww_0$, and $v_p = \min(y_pW_S, \leq v_{p-1})$ for $1 \leq p \leq s$.

Because there exists a directed path in $\mathcal{QB}_{\tau_p}(W^S)$ from $y_{p+1}$ to $y_p$ for $1 \leq p \leq s - 1$, it follows from Lemma 2.3, (3) that there exists a unique directed path

$$
(5.16) \quad v_p \leftarrow w_0 \gamma_{p,1} \leftarrow \cdots \leftarrow w_0 \gamma_{p,1} v_{p+1}
$$

in $\mathcal{QB}_{\tau_p}(W)$ from $v_{p+1}$ to $v_p$ whose edge labels $-w_0 \gamma_{p,1}, \ldots, -w_0 \gamma_{p,1}$ are increasing in the reflection order $\lhd$, and lie in $\Delta^+ \setminus \Delta^+_S$ for $1 \leq p \leq s - 1$. We remark that this is also the case for $p = 0$ since $\tau_0 = 0$; if $y_1 = y_0 = [ww_0]$, then we set $t_0 = 0$. Multiplying this directed path on the right by $w_0$, we get the following directed paths by Lemma 2.3

$$
(5.17) \quad v_{p,0} := v_p w_0 \gamma_{p,1} v_{p,1} \gamma_{p,2} \cdots \gamma_{p,s} v_{p+1} w_0 := v_{p,t_p}, \quad 0 \leq p \leq s - 1.
$$

Concatenating these paths for $0 \leq p \leq s - 1$, we obtain the following directed path

$$
(5.18) \quad v_0,0 \gamma_{0,1} \cdots \gamma_{0,s} v_{0,t_0,0} = v_{1,0} \gamma_{1,1} \cdots \gamma_{1,t_1,0} v_{1,t_1} = v_{2,0} \gamma_{2,1} \cdots \gamma_{s-1,t_{s-1}} v_{s-1,t_{s-1}}
$$

in $\mathcal{QB}(W)$. Now, for $0 \leq p \leq s - 1$ and $1 \leq m \leq t_p$, we set $d_{p,m} := \tau_p \in \mathbb{Q} \cap [0,1)$, $a_{p,m} := (1 - d_{p,m})(\lambda_-, \gamma_{p,m})$, and $\tilde{\gamma}_{p,m} := a_{p,m} \hat{\beta} - \gamma_{p,m}$. It follows from 5.6 that $\tilde{\gamma}_{p,m} \in \Delta_{\text{aff}}^+ \cap t(\lambda_-)^{-1} \Delta_{\text{aff}}^-$. 

**Claim 1.**

1. We have

$$
(5.19) \quad \tilde{\gamma}_{0,1} \prec \cdots \prec \tilde{\gamma}_{0,t_0} \prec \tilde{\gamma}_{1,1} \prec \cdots \prec \tilde{\gamma}_{s-1,t_{s-1}},
$$

where $\prec'$ denotes the total order on $\Delta_{\text{aff}}^+ \cap t(\lambda_-)^{-1} \Delta_{\text{aff}}^-$ introduced in Section 5.2; hence we can choose $J' = \{j'_1, \ldots, j'_r\} \subset \{1, \ldots, L\}$ such that

$$
(5.20) \quad (\tilde{\beta}_{j'_1}, \ldots, \tilde{\beta}_{j'_r}) = (\tilde{\gamma}_{0,1}, \ldots, \tilde{\gamma}_{0,t_0}, \tilde{\gamma}_{1,1}, \ldots, \tilde{\gamma}_{s-1,t_{s-1}}).
$$

2. Let $1 \leq k \leq r'$, and take $0 \leq p \leq s - 1$ and $1 \leq m \leq t_p$ such that

$$
(5.21) \quad \left(\tilde{\beta}_{j'_1} \prec' \cdots \prec' \tilde{\beta}_{j'_k}\right) = (\tilde{\gamma}_{0,1} \prec' \cdots \prec' \tilde{\gamma}_{p,m}).
$$

Then, $\text{dir}(z_k) = v_{p,m}$. Moreover, $\text{dir}(z_{k-1}) \xrightarrow{-\tilde{\beta}_{j'_k}} \text{dir}(z_k)$ is a directed edge of $\mathcal{QB}(W)$.

**Proof of Claim 1.**

1. It suffices to show the following:

   (i) For $0 \leq p < s - 1$ and $1 \leq m < t_p$, we have $\tilde{\gamma}_{p,m} \prec' \tilde{\gamma}_{p,m+1}$;

   (ii) For $0 \leq p \leq s - 2$, we have $\tilde{\gamma}_{p,t_p} \prec' \tilde{\gamma}_{p+1,1}$.

2. Because $\frac{(\lambda_-, -\gamma_{p,m}) - a_{p,m}}{(\lambda_-, -\gamma_{p,m+1})} = d_{p,m}$ and $\frac{(\lambda_-, -\gamma_{p,m+1}) - a_{p,m+1}}{(\lambda_-, -\gamma_{p,m+1})} = d_{p,m+1}$, we have

$$
\Phi(\tilde{\gamma}_{p,m}) = (d_{p,m}, -w_0 \gamma_{p,m}),
$$

$$
\Phi(\tilde{\gamma}_{p,m+1}) = (d_{p,m+1}, -w_0 \gamma_{p,m+1}).
$$
Therefore, the first component of $\Phi(\overline{\gamma}_{p,m})$ is equal to that of $\Phi(\overline{\gamma}_{p,m+1})$ since $d_{p,m} = 1 - \tau_p = d_{p,m+1}$. Since $-w_0 \gamma_{p,m} \succ -w_0 \gamma_{p,m+1}$, we deduce that $\Phi(\overline{\gamma}_{p,m}) < \Phi(\overline{\gamma}_{p,m+1})$. This implies that $\overline{\gamma}_{p,m} \not\prec \overline{\gamma}_{p,m+1}$.

(ii) The proof of (ii) is similar to that of (i). The first components of $\Phi(\overline{\gamma}_{p,t_p})$ and $\Phi(\overline{\gamma}_{p+1,1})$ are $d_{p,t_p}$ and $d_{p+1,1}$, respectively. Since $d_{p,t_p} = \tau_p < \tau_{p+1} = d_{p+1,1}$, we have $\Phi(\overline{\gamma}_{p,t_p}) < \Phi(\overline{\gamma}_{p+1,1})$. This implies that $\overline{\gamma}_{p,t_p} \not\prec \overline{\gamma}_{p+1,1}$.

(2) We proceed by induction on $k$. If $\overline{\beta}_{j'_1} = \overline{\gamma}_{0,1}$, i.e., $y_1 \not\in \{ww_0\}$, then we have $\text{dir}(z_1) = \text{dir}(z_0) \overline{s}_{\beta_{j'_1}} = \nu_0 s_{\gamma_{0,1}} = \nu_0$, since $\text{dir}(z_0) = \text{dir}(\text{wt}(\lambda_-)) = w = v_{0,0}$. If $\overline{\beta}_{j'_1} = \overline{\gamma}_{1,1}$, i.e., $y_1 \in \{ww_0\}$ and $t_0 = 0$, then we have $\text{dir}(z_1) = \text{dir}(z_0) \overline{s}_{\beta_{j'_1}} = ws_{\gamma_{1,1}} = v_{1,0} s_{\gamma_{1,1}} = v_1$, since $\text{dir}(z_0) = \text{dir}(\text{wt}(\lambda_-)) = w = v_{1,0}$. Hence the assertion holds in the case $k = 1$.

Assume that $\text{dir}(z_{k-1}) = v_{p,m-1}$ for $0 \leq m \leq t_p$; here we remark that $v_{p,m}$ is the successor of $v_{p,m-1}$ in the directed path (5.17). Therefore, we see that $\text{dir}(z_k) = \text{dir}(z_{k-1}) \overline{s}_{\beta_{j'_k}} = v_{p,m-1} s_{\beta_{j'_k}} = v_{p,m}$ (since $v_{p,m-1} \overrightarrow{\gamma_{p,m}} v_{p,m}$ is a directed edge in (5.17)).

Also, since (5.17) is a directed path in $\text{QBG}(W)$, $\overrightarrow{v_{p,m-1} \gamma_{p,m}} v_{p,m}$ is a directed edge of $\text{QBG}(W)$.

Since $J' = \{j'_1, \ldots, j'_{r'}\} \subset \{1, \ldots, L\}$, we can define an element $p_{J'}$ by $\langle \text{wt}(\lambda_-), z_0, z_1, \ldots, z_{r'}; \overline{\beta}_{j'_1}, \overline{\beta}_{j'_2}, \ldots, \overline{\beta}_{j'_{r'}} \rangle$, where $z_0 = \text{wt}(\lambda_-), z_k = z_{k-1} s_{\beta_{j'_k}}$ for $1 \leq k \leq r'$; it follows from Remark 5.3.1 and Claim 1 that $p_{J'} \in \text{QB}(w; t(\lambda_-))$.

**Claim 2.** $\Xi_w(p_{J'}) = \eta$.

**Proof of Claim 2.** In the following description of $p_{J'}$, we employ the notation $w_p, \sigma_p, w'_p$, and $w_p, 0 \leq p \leq s$, used in the definition of $\Xi_w(p_{J'})$.

For $1 \leq k \leq r'$, if we set $\beta_{j'_k} = \overline{\gamma}_{p,m}$, then we have

$$d_{j'_k} = 1 + \frac{\deg(\overline{\gamma}_{p,m})}{\langle \lambda_-, \overline{\beta}_{j'_k} \rangle} = 1 + \frac{\deg(\overline{\gamma}_{p,m})}{\langle \lambda_-, \overline{\gamma}_{p,m} \rangle} = 1 + \frac{\deg(\overline{\gamma}_{p,m})}{\langle \lambda_-, \overline{\gamma}_{p,m} \rangle} = 1.$$  

Therefore, the sequence (5.10) determined by $p_{J'}$ is (5.18)

$$0 = d_{0,1} = \cdots = d_{0,t_0} < d_{1,1} = \cdots = d_{1,t_1} < \cdots < d_{s-1,1} = \cdots = d_{s-1,t_{s-1}} < 1 = \tau_s = \sigma_s.$$  

Because the sequence (5.18) of rational numbers is just the sequence (5.10) for $\Theta(\eta) = p_{J'}$, we deduce that $w_{p+1} - w_p = t_p$ for $0 \leq p \leq s-1$, $\overline{\beta}_{j'_{p+1}} = \overline{\gamma}_{p,k}$ for $0 \leq p \leq s-1$, $1 \leq k \leq u_{p+1} - u_p$, and $\sigma_p = \tau_p$ for $0 \leq p \leq s$. From these, we see that

$$w'_{p} = \text{dir}(z_{u_p}) = v_{u_{p-1},t_{p-1}} = v_{p,0}$$  

(since $v_{p,m-1} \overrightarrow{\gamma_{p,m}} v_{p,m}$ is a directed edge in (5.17)).

and $w_p = v_{p,0} w_0 = v_p$. Since $\{w_p\} = \{v_p\} = \{y_p\}$, we conclude that $\Xi_w(p_{J'}) = (\{w_1\}, \ldots, \{w_s\}; \sigma_0, \ldots, \sigma_s) = (y_1, \ldots, y_s; \tau_0, \ldots, \tau_s) = \eta$, as desired.

This completes the proof of the lemma.
By Lemmas 5.3.2 and 5.3.3, we obtain the following proposition.

**Proposition 5.3.4.** The map $\Xi_w$ is bijective. In particular, the cardinality of the set $QB(w; t(\lambda-))$ is independent of $w \in W$.

**Remark 5.3.5.** In the proof of Lemma 5.3.3, we showed that for $\eta \in QLS(\lambda)$, there exists a unique $J \subset \{1, \ldots, L\}$ such that $p_J \in QB(w; t(\lambda-))$ and $\Xi_w(p_J) = \eta$; this assignment gives the inverse map $\Xi_w^{-1} : QLS(\lambda) \to QB(w; t(\lambda-))$ of $\Xi_w$.

Recall from (3.1) and (3.2) that $deg(\eta) := deg(QBG(\eta)) = 0$ for $\eta \in W$. From these, we deduce that $deg(qwt(p_J)) = Deg(\Xi_w^{-1}(\eta)) = Deg(\Xi_w(p_J))$ for $\eta \in QLS(\lambda)$. Also, note that if $\delta := deg(\eta) \in \mathbb{R}$, then $deg(qwt(p_J)) = Deg(\Xi_w(p_J))$ for $\eta \in QLS(\lambda)$.

**Proposition 5.3.6.** The bijection $\Xi_w : QB(w; t(\lambda-)) \to QLS(\lambda)$ has the following properties:

1. $wt(end(p_J)) = wt(\Xi_w(p_J))$;
2. $deg(qwt(p_J)) = Deg^{ww_0}(\Xi_w(p_J))$.

**Proof.** We proceed by induction on $\#J$.

If $J = \emptyset$, it is obvious that $deg(qwt(p_J)) = Deg^{ww_0}(\Xi_w(p_J)) = 0$ and $wt(end(p_J)) = wt(\Xi_w(p_J)) = w\lambda_z$, since $\Xi_w(p_J) = \{w\lambda_z; 0, 1\}$.

Let $J = \{j_1 < j_2 < \cdots < j_r\}$, and set $K := J \setminus \{j_r\}$; assume that $\Xi_w(p_K)$ is of the form: $\Xi_w(p_K) = (\{w_1, \ldots, w_s\}; \sigma_0, \ldots, \sigma_s)$. In the following, we employ the notation $w_J, 0 \leq p \leq s$, used in the definition of the map $\Xi_w$. Note that $dir(qwt(p_J)) = w_J w_0$ by the definition of $\Xi_w$. Also, note that if $d_{j_r} = d_{j_{r-1}} = \sigma_{s-1}$, then $\{d_{j_1} \leq \cdots \leq d_{j_{r-1}} \leq d_{j_r}\} = \{d_{j_1} \leq \cdots \leq d_{j_{r-1}} \leq d_{j_r}\}$, and that if $d_{j_r} = d_{j_{r-1}} = \sigma_{s-1}$, then $\{d_{j_1} \leq \cdots \leq d_{j_{r-1}} \leq d_{j_r}\} = \{d_{j_1} \leq \cdots \leq d_{j_{r-1}} < d_{j_r}\}$.

From these, we deduce that

$$\Xi_w(p_J) = \begin{cases} (\{w_1, \ldots, w_{s-1}\}, \{w_s w_{w_0 \beta_{j_r}}^{-1}; \sigma_0, \ldots, \sigma_{s-1}, \sigma_s\}) & \text{if } d_{j_1} = d_{j_{r-1}} = \sigma_{s-1}, \\ (\{w_1, \ldots, w_{s-1}\}, \{w_s, w_{w_0 \beta_{j_r}}^{-1}; \sigma_0, \ldots, \sigma_{s-1}, d_{j_r}, \sigma_s\}) & \text{if } d_{j_1} > d_{j_{r-1}} = \sigma_{s-1}. \end{cases}$$

For the induction step, it suffices to show the following claims.

**Claim 1.**

1. We have
   $$wt(\Xi_w(p_J)) = wt(\Xi_w(p_K)) + a_{j_r} w_s w_0 \left(\overline{-\beta_{j_r}}\right)^\vee.$$
2. We have
   $$Deg^{ww_0}(\Xi_w(p_J)) = Deg^{ww_0}(\Xi_w(p_K)) + \chi \deg(\tilde{\beta}_{j_r}),$$
   where $\chi := 0$ (resp., $\chi := 1$) if $w_s w_{w_0 \beta_{j_r}}^{-1} \to w_s$ is a Bruhat (resp., quantum) edge of $QBG(W)$.

**Claim 2.**

1. We have
   $$wt(end(p_J)) = wt(end(p_K)) + a_{j_r} w_s w_0 \left(\overline{-\beta_{j_r}}\right)^\vee.$$
2. We have
   $$deg(qwt(p_J)) = deg(qwt(p_K)) + \chi \deg(\tilde{\beta}_{j_r}).$$

The proofs of Claims 1 and 2 are the same as those of Claims 1 and 2 in [NNS] Proposition 3.3.6, respectively. This proves the proposition. □
Proof of Theorem 5.1.1. It follows from Propositions 5.3.4 and 5.3.6 that
\[
C_w^{\ell_0} = \sum_{p \in \mathfrak{QB}(\ell_0)} q^{-\deg(qw(t(p)))} e^{\wt(t(p))} = \sum_{p \in \mathfrak{QB}(\ell_0)} q^{-\deg(qw(t(p)))} e^{\wt(t(p))} = \sum_{\eta \in QLS(\lambda)} q^{-\Deg(\ell_0)} e^{\wt(\eta)} = \text{gch}^{\ell_0} \text{QLS}(\lambda),
\]
as desired. \qed

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