Probabilities, Tensors and Qubits

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Abstract

In the paper is discussed complete probabilistic description of quantum systems with application to multiqubit quantum computations. In simplest case it is a set of probabilities of transitions to some fixed set of states. The probabilities in the set may be represented linearly via coefficients of density matrix and it is very similar with description using mixed states, but also may give some alternative view on specific properties of quantum circuits due to possibility of direct comparison with classical statistical paradigm.

1 Introduction

Let us consider \( n \)-dimensional Hilbert space \( \mathcal{H}_n \) and set of \( N \) fixed unit vectors \( |v_\alpha\rangle \in \mathcal{H}_n \) denoted as \( S\{v_\alpha : \alpha = 1, \ldots, N\} \) or \( S_N(\mathcal{H}_n) \). Then for quantum system in arbitrary state \( |\psi\rangle \in \mathcal{H}_n \) there are defined \( N \) coefficients:

\[
 p_\alpha \equiv |\langle v_\alpha | \psi \rangle|^2.  \tag{1}
\]

The \( p_\alpha \) is probability to find \( |\psi\rangle \) in state \( |v_\alpha\rangle \) due to measurement described by projector:

\[
 \Pi_\alpha \equiv \Pi(v_\alpha) \equiv |v_\alpha\rangle\langle v_\alpha|. \tag{2}
\]

For \textit{mixed} state with density matrix \( \rho \) it is possible to use instead of Eq. (1):

\[
 p_\alpha \equiv \langle v_\alpha| \rho |v_\alpha\rangle = \text{Tr}(\rho \Pi_\alpha). \tag{3}
\]

For pure state with \( \rho = |\psi\rangle\langle \psi| \) the Eq. (3) coincides with Eq. (1).

Due to Eq. (3) probabilities of transitions \( p_\alpha(\rho) \) are simply linear functions of density matrix and for properly chosen set of vectors \( v_\alpha \) using only the coefficients \( p_\alpha \) may be equivalent to description by density matrices. For example in the paper is shown that for such a set any quantum channel can be described as linear transformation of \( N \)-dimensional vector of probabilities, i.e. simply as \( N \times N \) matrix. For \( n \)-qubit system it is possible to find a simple set with \( 6^n \) vectors, then the probabilities can be ordered in a tensor with \( n \) indexes, but only \( 4^n \) components are linearly independent. It may be also described by \( 4^n \) special linearly independent parameters ordered in 4-dimensional real tensor with \( n \) indexes. A natural and symmetric description of any quantum circuits as linear transformations of these tensors is discussed below in this paper.

2 General description

Let us consider \( N \) probabilities \( p_\alpha \), Eq. (3) as formal vector \( p = (p_1, \ldots, p_N) \). Let \( H(n) \) is space of Hermitian \( n \times n \) matrices \( H \in H(n) \), \( H^{ik} = H^{ki} \) and \( H_{\text{ph}}(n) \) is “physical” subspace of nonnegative definite matrices with trace one:

\[
 H_{\text{ph}}(n) = \{ \rho : \rho \in H(n), \ Tr \rho = 1, \ \langle v|\rho|v\rangle \geq 0, \ \forall \ v \}. \tag{4}
\]
then \( S_N \) via Eq. (3) produces linear maps \( L_S: H_{\rho}(n) \to \mathbb{R}_{\geq 0}^N \) or \( L_S: H(n) \to \mathbb{R}^N \). Here \( H(n) \) is linear space described by \( n^2 \) real parameters and \( H_{\rho}(n) \) is subspace of physically possible density matrices \( \rho \in H_{\rho}(n) \), \( L_S: \rho \mapsto p \).

Let us call the set \( S_N(H_n) \) **representative** if exists right inverse of \( L_S \), i.e., a linear operator \( L_S: \mathbb{R}^N \to H(n) \) with property:

\[
L_S \circ L_S = I, \quad \text{i.e.,} \quad L_S(L_SH) = H, \quad \forall H \in H(n).
\]

where symbol "\( \circ \)" is used for composition of operators (left one acts first).

The space \( H(n) \) is described by \( n^2 \) real parameters and for **minimal representative** set with \( n^2 \) vectors \( S_{n^2}(H_n) \) the \( L_S \) is left and right inverse (or, simply, inverse: \( L_S = L_S^{-1} \)) and together with Eq. (3) it is possible to write:

\[
\tilde{L}_S \circ L_S = I, \quad \text{i.e.,} \quad L_S(\tilde{L}_S p) = p, \quad \forall p \in \mathbb{R}^{n^2},
\]

but if \( n^2 > N \) the map \( L_S \circ \tilde{L}_S \) is not identity, it is projector on some \( n^2 \)-dimensional space of linear maps \( L_S^2 \) with property \( L_S^2(v) = 0 \), \( \forall v \in V \) (here \( \theta \) is zero Hermitian matrix) and so instead of \( L_S \) it is possible to use any other \( L'_S = L_S + L_S^\perp \). Due to such property results of alternative methods of construction of \( L_S \) may differ on \( L_S^2 \) if set \( S \) is not minimal.

For physical space of density matrices Eq. (4) due to condition \( \text{Tr} \rho = 1 \), it is possible to express one of component using other with some affine relation. It is possible to introduce some **affine minimal** set with \( n^2 - 1 \) elements, but instead of linear equation Eq. (3) we will have affine equation:

\[
\rho = \tilde{L}_S(L_S\rho) + \tilde{R}_0, \quad \forall \rho \in H_{\rho}(n).
\]

Let us call a representative set \( S_N \) **complete**, if it can be expressed as union of few orthonormal bases of \( H_n \), i.e., for any \( v_\alpha \in L_S \) it is possible to find other \( n - 1 \) vectors from \( L_S \) to form an orthogonal basis for \( H_n \).

Let us call the complete set \( S_N \) **almost perfect**, if it can be expressed as disjoint union of few orthonormal bases of \( H_n \) and **perfect**, if for any \( v_\alpha \in L_S \) there is unique choice of \( n - 1 \) vectors from \( L_S \) to form the orthogonal basis for \( H_n \).

The definitions of complete, almost perfect and perfect sets are useful for consideration of elementary measurements with \( n \) possible outcomes.

It is clear from definition that **perfect** set is also **almost perfect**. The example of **almost perfect** set that is **not perfect**, is one of main themes of this paper. It is complete set for \( m \geq 2 \) qubits introduced in Sec. 3. The set is almost perfect, but not perfect, due to counterexample in Sec. 5.2 (page 13). It will be shown below (page 14) that almost perfect set may not be minimal.

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Let \(|1\rangle, \ldots, |n\rangle\) is a basis of \( H_n \) and \( \rho_{kl} \) are components of density matrix \( \rho = \sum_{k,l=1}^n \rho_{kl}|k\rangle \langle l| \) and the map \( L_S: \rho \mapsto p \) is defined as above by Eq. (3). Then the definition of \( \tilde{L}_S \) using Eq. (4) corresponds to existence of \( n^2N \) complex coefficients \( c_{kl}^\alpha \) (some of them may be zero and the choice is unique only for minimal set), where \( k,l = 1, \ldots, n \) and \( \alpha = 1, \ldots, N \) with property:

\[
\rho_{kl} = \sum_{\alpha=1}^N c_{kl}^\alpha p_\alpha \quad \text{(there is no summation on } k,l \text{)}.
\]
Let us introduce a basis on space $\mathbb{C}^{n \times n}$ of all complex matrices. Elements of the basis are $E_{kl} = |k\rangle \langle l|$, i.e., matrices with only one unit in position $(k, l)$: $(E_{kl})_{ij} = \delta_{ki}\delta_{lj}$. Let us show, that a set is representative and Eq. (3) is true for some coefficients $c_{kl}^a$ if and only if it is possible to express all $n^2$ elements of the basis as linear combinations of projectors $\Pi_n$:

$$E_{lk} = |l\rangle \langle k| = \sum_{\alpha=1}^{N} c_{kl}^\alpha \Pi_n \quad \text{(there is no summation on } k, l). \quad \text{(9)}$$

First, Eq. (3) is direct sequence of Eq. (1) because:

$$\sum_{\alpha=1}^{N} c_{kl}^\alpha p_{\alpha} = \sum_{\alpha=1}^{N} c_{kl}^\alpha \Tr(\rho \Pi_n) = \Tr \left(\rho \sum_{\alpha=1}^{N} c_{kl}^\alpha \Pi_n \right) = \Tr(\rho |l\rangle \langle k|) = \langle k | \rho | l \rangle = \rho_{kl}$$

Now let us show that for representative set always exists some decomposition $\tilde{S}$. It maybe not obvious, the Eq. (3) looks too general, because we considered basis of whole matrix algebra $\mathbb{C}^{n \times n}$ with $n^2$ complex (i.e. $2n^2$ real) parameters instead of space of Hermitian matrices $H(n)$ described by only $n^2$ real parameters.

The condition for representative set is existence of map $\tilde{S}$, such that composition $L_S \circ \tilde{S}$ is identity $H(n) \rightarrow H(n)$ in diagram:

$$H(n) \xrightarrow{L_S} \mathbb{R}^N \xrightarrow{\tilde{S}} H(n), \quad L_S \circ \tilde{S} = \mathbb{I}. \quad \text{(10)}$$

It is necessary and enough to satisfy the identity for whole basis of $H(n)$ with $n^2$ elements. It is possible if $N \geq n^2$ and in this case image of $L_S$ must be some $n^2$-dimensional subspace $\mathbb{V}^{n^2} \subset \mathbb{R}^N$.

On the other hand, the Eq. (3) is equivalent with extension of the $\tilde{S}$, $L_S$ and identity of $L_S \circ \tilde{S}$ on whole space $\mathbb{C}^{n \times n}$:

$$\mathbb{C}^{n \times n} \xrightarrow{L_S^C} \mathbb{C}^N \xrightarrow{\tilde{S}^C} \mathbb{C}^{n \times n}, \quad L_S^C \circ \tilde{S}^C = \mathbb{I}. \quad \text{(11)}$$

But let us show that identity of $L_S \circ \tilde{S}$ for basis elements $H(n)$ described in Eq. (10) is enough for identity of the map for basis of $\mathbb{C}^{n \times n}$, i.e., the Eq. (11) is not more general.

A basis of $H(n)$ is $(n(n+1)/2)$ matrices

$$H_{kl}^+ = (E_{kl} + E_{lk})/2, \quad k \geq l \quad \text{(12)}$$

(i.e. $H_{kk}^+ = H_{kk}$) and $(n(n-1)/2)$ matrices

$$H_{kl}^- = i(E_{kl} - E_{lk})/2, \quad k > l. \quad \text{(13)}$$

But if some linear map $\mathbb{C}^{n \times n} \rightarrow \mathbb{C}^{n \times n}$ is identity on these elements, it is also identity on linear combinations, but any matrix of basis $E_{kl}$ of $\mathbb{C}^{n \times n}$ may be expressed via $H_{kl}^+$ and $H_{kl}^-$:

$$E_{kl} = \begin{cases} H_{kl}^+ - iH_{kl}^-, & k > l \\ H_{kk}^+, & k = l \\ H_{lk}^+ + iH_{lk}^-, & k < l \end{cases} \quad \text{(14)}$$

So $L_S^C \circ \tilde{S}^C$ is identity and Eq. (3) is necessary and enough, it is simply expression for $L_S^C$ written in basis $E_{lk}$.

The following proposition summarizes the consideration above.

Three properties are equivalent:
1. Set $S(v_\alpha)$ is representative ($\tilde{L}_S$ exists).

2. Any complex matrix may be represented as linear combination of projectors $\Pi(v_\alpha)$ with complex coefficients — it is enough to show for any basis, for example $E_{kl}$, see Eq. (3).

3. Any Hermitian matrix may be represented as linear combination of projectors $\Pi(v_\alpha)$ with real coefficients — it is enough to show for any basis, for example $H_{kl}^\pm$ above.

For minimal representative set the construction of $\tilde{L}_S$ (item 1 in the proposition above) and both decompositions (items 2 and 3) are unique.

Now it is possible to show that (almost) perfect set may not be minimal. For $n$-dimensional Hilbert space minimal set must have only $n^2$ elements. If it could be (almost) perfect set, it can be considered as disjoint union of $n$ bases, for example $E_{kl}$, see Eq. (9).

The complex basis $E_{kl}$ is “tr$_{AB}$-orthonormal”, i.e., orthonormal with respect to scalar product on space of matrices defined as

$$\langle A, B \rangle_* \equiv \text{Tr}(AB^*) .$$

Due to it there is simple relation between expression for the basis Eq. (1) and an expansion Eq. (3) of $\tilde{L}$ for representative set. For Hermitian matrices Eq. (15) is close related with Eq. (3), because two different scalar products coincide:

$$\langle A, B \rangle \equiv \text{Tr}(AB) = \text{Tr}(AB^*) \equiv \langle A, B \rangle_* .$$

The basis of Hermitian matrices defined by Eq. (12) and Eq. (13) is tr$_{AB}$-orthonormal, but it is possible to use standard Gram-Schmidt procedure of orthogonalization for given norm (cf. Ref. [6]). An example of tr$_{AB}$-orthonormal Hermitian basis for qubits with dimension of Hilbert space is $2^n$ will be discussed later.

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Other application of $E_{kl}$ is following theorem.

**Composition Theorem:** Let $\mathcal{H}_n^\dagger$ and $\mathcal{H}_m^\dagger$ are two Hilbert spaces with dimensions $n$ and $m$ and $S_N \{v_\alpha \in \mathcal{H}_n^\dagger\}$ and $S_M \{u_\beta \in \mathcal{H}_m^\dagger\}$ are representative (complete, almost perfect) sets with $N$ and $M$ vectors. Then set of $NM$ vectors $S_{NM} \{v_\alpha \otimes u_\beta \in \mathcal{H}_n^\dagger \otimes \mathcal{H}_m^\dagger\}$ is representative (complete, almost perfect) set for composite system.

Let us prove the theorem for representative sets, because for complete sets it is trivial implication. For a proof it is enough to use condition Eq. (3) for $\mathcal{H}_n^\dagger$ and $\mathcal{H}_m^\dagger$:

$$|l_1\rangle\langle k_1| = \sum_{\alpha=1}^N \epsilon_{k_1l_1}^{(1)a} \Pi(v_\alpha), \quad |l_2\rangle\langle k_2| = \sum_{\beta=1}^M \epsilon_{k_2l_2}^{(2)\beta} \Pi(u_\beta),$$

(\text{where } |l_1\rangle, |k_1\rangle, |v_\alpha\rangle \in \mathcal{H}(1) \text{ and } |l_2\rangle, |k_2\rangle, |u_\beta\rangle \in \mathcal{H}(2)) \text{ together with properties:}

$$\Pi(v \otimes u) = \Pi(v) \otimes \Pi(u), \quad |l_1\rangle |k_1\rangle \langle k_2| = |l_1\rangle \langle k_1| \otimes |l_2\rangle \langle k_2| .$$
So for \( n^2m^2 \) elements of basis in space of all matrices \( \mathbb{C}^{nm \times nm} \cong \mathbb{C}^{n \times n} \otimes \mathbb{C}^{m \times m} \) we have:

\[
|l_1l_2\rangle\langle k_1k_2| = \sum_{\alpha=1}^{N} \sum_{\beta=1}^{M} c_{k_1l_1}^{(1)\alpha} c_{k_2l_2}^{(2)\beta} \Pi(v_\alpha \otimes u_\beta).
\] (17)

The Eq. (17) corresponds to Eq. (9) for \( nm \)-dimensional Hilbert space \( \mathcal{H}_{nm}(1,2) = \mathcal{H}_n(1) \otimes \mathcal{H}_m(2) \) with representative set \( S_{NM}(\mathcal{H}_{nm}(1,2)) \) with \( NM \) vectors, if to use “compound indexes” like \( k_1k_2 \leftrightarrow (k_1-1)m + k_2 \) and \( \alpha\beta \leftrightarrow (\alpha-1)M + \beta \):

\[
E_{l_1l_2} \otimes k_1k_2 = \sum_{\alpha\beta=1}^{NM} c_{\alpha\beta}^{l_1l_2} \Pi_{\alpha\beta},
\] (18)

where \( c_{\alpha\beta}^{l_1l_2} = c_{k_1l_1}^{(1)\alpha} c_{k_2l_2}^{(2)\beta} \) and \( \Pi_{\alpha\beta} = \Pi(v_\alpha \otimes u_\beta) \).

A proof that tensor product of complete (almost perfect) sets is complete (almost perfect) is directly implied because tensor product of two bases is basis. Tensor product of perfect set is almost perfect, but not necessary perfect, it follows from counterexample in Sec. 5.2.

It is also possible to suggest proof without complexification, because tensor product of Hermitian matrices is also Hermitian, but it is not discussed here due to rather technical points. Such approach may be more clear from consideration with Pauli matrices in Sec. 5.

3 Some applications of representative sets of projectors

The Eq. (8) shows that for representative set using of vector \( p \) is equivalent to description of quantum system by density matrix.

**Example 1:** For any unitary operator \( U \) on \( \mathcal{H} \), \( |\psi'\rangle = U|\psi\rangle \) and

\[
\rho' = U\rho U^* \tag{19}
\]

there exists \( N \times N \) matrix \( A_U \), \( p' = A_U p \), where \( p' = p_{\psi'} \) (here \( A_U \) does not depend on \( |\psi\rangle \)). Really, the operator \( U \) induces a linear transformation \( U \otimes U^* \) Eq. (19) on space of Hermitian matrices. Let us denote it as \( U^{1,1} : \rho \mapsto \rho' \). Then \( A_U = L_S \circ U^{1,1} \circ L_S \) as follows from diagram 1:

\[
\begin{array}{c}
p \quad \xrightarrow{A_U} \quad p' \\
\downarrow L_S \quad \quad \downarrow L_S \\
\rho \quad \xrightarrow{U^{1,1}} \quad \rho'
\end{array}
\] (20)

Let us consider construction of matrix \( A_U \) more directly. Components of the vector \( p' \) are written as:

\[
p'_\alpha = \text{Tr}(U\rho U^* \Pi_\alpha) = \text{Tr}(\rho U^* \Pi_\alpha U)
\] (21)

and because due to third item of proposition in Sec. 3 any unitary matrix can be represented as linear combination of \( \Pi_\alpha \), we have

\[
U^* \Pi_\alpha U = \sum_{\beta=1}^{N} A^\alpha_\beta \Pi_\beta,
\] (22)

\footnote{See also footnote [3] below on page 4}
where for given $\alpha, A_1^\alpha, \ldots, A_N^\alpha$ are real coefficients of decomposition of Hermitian matrix $U^* \Pi_\alpha U$. The coefficients may be unique only if $N = n^2$ and $\Pi_\beta$ are linearly independent, i.e., for minimal representative set. On the other hand, $A_\alpha^\beta$ are components of desired matrix $A_U$, $p' = A_U p$, because

$$p'_{\alpha} = \text{Tr}(\rho \sum_{\beta=1}^N A_\alpha^\beta \Pi_\beta) = \sum_{\beta=1}^N A_\alpha^\beta \text{Tr}(\rho \Pi_\beta) = \sum_{\beta=1}^N A_\alpha^\beta p_\beta. \quad (23)$$

If set is not minimal, the decomposition is not unique, for example it is possible to use only $n^2$ linearly independent projectors $\Pi_\alpha$ between $N > n^2$ and already this choice is not unique. There is a problem with calculation of coefficients $A_\alpha^\beta$ even for minimal set, but it is convenient for set of matrix orthogonal in respect of some norm. The complex extension with matrices $E_{kl}$ orthogonal in norm Eq. (15) was discussed already and for system $(n = 2^m)$ of $m$-qubits basis of Hermitian matrix obtained from $4^m$ different products of Pauli matrices orthogonal in both norms Eq. (16) and Eq. (15) is used below Eq. (19).

Let us suggest, that we choose some basis of $n^2$ Hermitian matrices $H_K$ and express all matrices $\Pi_\alpha$,

$$\Pi_\alpha = \sum_{K=1}^{n^2} h^K_\alpha H_K. \quad (24)$$

The coefficients $h^K_\alpha$ are unique and if basis $H_K$ is $\text{tr}_{AB}$-orthonormal

$$\text{Tr}(H_J H_K) = \delta_{JK} \quad (25)$$

then the coefficients may be expressed as

$$h^K_\alpha = \text{Tr}(\Pi_\alpha H_K) \quad (26)$$

Let us introduce new parameters:

$$\tilde{p}_K = \text{Tr}(\rho H_K). \quad (27)$$

Then it is possible to express all $N \geq n^2$ probabilities $p_\alpha$ using the parameters:

$$p_\alpha = \text{Tr}(\rho \Pi_\alpha) = \text{Tr}(\rho \sum_{K=1}^{n^2} h^K_\alpha H_K) = \sum_{K=1}^{n^2} h^K_\alpha \text{Tr}(\rho H_K) = \sum_{K=1}^{n^2} h^K_\alpha \tilde{p}_K. \quad (28)$$

If set is not minimal, coefficients $\tilde{h}_K^\alpha$ of inverse transformation:

$$H_K = \sum_{\alpha=1}^N \tilde{h}_K^\alpha \Pi_\alpha, \quad (29)$$

and

$$\tilde{p}_K = \sum_{\alpha=1}^N \tilde{h}_K^\alpha p_\alpha \quad (30)$$

are not unique.

The parameters $\tilde{p}_K$ can be considered as a $n^2$-dimensional vector $\tilde{p}$, but now transformation $p' = A_U p$ always unique and can be calculated directly, let

$$p'_K = \text{Tr}(U \rho U^* H_K) = \text{Tr}(\rho U^* H_K U), \quad (31)$$
it is possible to use decomposition

\[ U^* H K U = \sum_{j=1}^{n^2} \tilde{A}_j^H J, \]  

(32)

and so

\[ \tilde{p}_K = \sum_{j=1}^{n^2} \tilde{A}_j^H \tilde{p}_j, \]  

(33)

where matrix \( \tilde{A}_U \) is expressed as:

\[(\tilde{A}_U)_J^K = \text{Tr}(H_J U^* H_K U). \]  

(34)

Initial matrix \( A_U \) can be expressed from \( \tilde{A}_U \) using linear maps \( \tilde{h}_K^\alpha \) and \( \tilde{h}_\alpha^K \) between \( p_\alpha \) and \( \tilde{p}_K \). If set is not minimal, \( A_U \) depends on \( \tilde{h}_K^\alpha \) and is not unique. It is unique only restriction of \( A_U \) on \( n^2 \)-dimensional linear subspace \( V \) discussed above. It is possible also directly work with \( \tilde{p}_K \), because all probabilities can be expressed using the parameters and expressions Eq. (28) where \( h_K^\alpha \) are always unique. The \( \tilde{p}_K \) can be considered as “coordinates” on \( V \).

The idea to write transformations of state of quantum system in terms of probabilities (weights) was also suggested in relation with some other problems in [2, 3, 4], but authors found linear transformation only for \( n = 2 \) and work only with pure states.

**Example 2:** The representation may be even more convenient for general quantum channel [1]:

\[ \rho' = \sum_k V_k \rho V_k^*, \quad \sum_k V_k V_k^* = \mathbb{I}. \]  

(35)

In this case linear transformation[1] \( V : \rho' \mapsto \rho \) is defined by some set \( \{V_k\} \) and the set is even not unique, but because the diagram Eq. (20) is valid for any linear map \( H(n) \rightarrow H(n) \), any quantum channel can be expressed using only one matrix \( A_V = \sum_k A_{V_k} \), \( p' = A_V p \). For minimal set matrix \( A_V \) is unique, otherwise unique only restriction of \( A_V \) on \( n^2 \)-dimensional linear subspace \( V \).

Applications of parameters \( \tilde{p} \) here is also justified. It is directly followed from consideration of previous example. A matrix of map \( \tilde{A}_V : \tilde{p} \mapsto \tilde{p}' \) is also always unique.

\footnote{Initial version of present paper (May 2000) appears as positive answer on question if the maps like \( A_U \) may be linear for \( n > 2 \). It should be mentioned also, that for pure state due to polynomial relations between coefficients of density matrix like \( \rho_{ki}\rho_{mn} = \rho_{km}\rho_{nl} \) it may be possible to express some probabilities in minimal set from the others by some nonlinear functions and so use nonlinear transformations with lesser amount of vectors \( v_\alpha \) instead of \( L_\alpha \) and \( A_U \), but it is not discussed here. On the other hand, even for linear \( A_U \) the existence of inverse map \( L_\alpha \) for \( L_\alpha \) is too strong requirement. If to look only for the linear map \( A_U \), it is enough to consider condition that diagram Eq. (20) is commutative. In this case it is reasonable to consider irreducible representations of tensor product \( U(n) \times U^*(n) \). It can be shown, that such product is sum of two irreps with dimensions are \( n^2 - 1 \) and \( 1 \). The second one is simply scalar representation corresponding to constancy of density matrix trace and so splitting off the one-dimensional irrep may be explained without applications of theory of group representations. Due to such situation, instead of commutativity of diagram Eq. (20) here was used straightforward, but less general suggestion about existence of \( L_\alpha \), especially because the map is anyway necessary for other applications discussed in this paper.

\footnote{Sometimes it is called “superoperator.”}
Example 3: Any expressions for probabilities can be found using only elements of vector \( p \) (cf. Ref. [4]). For any vector \( v \in H \) there is linear map \( B_v: \mathbb{R}^N \to \mathbb{R} \), defined as
\[
\langle v|\rho|v \rangle = B_v \cdot p = \langle v|\bar{L}_S p|v \rangle
\]  
(\( B_v \) depends only on \( v \), not \( |\psi\rangle \)). Let \( p_\psi \) corresponds to some pure state \( \psi \), \( p_\psi = L_S(|\psi\rangle \langle \psi|) \) by definition of \( L_S \) via Eq. (14), Eq. (15), then Eq. (22) can be rewritten as \( |\langle v|\psi\rangle|^2 = B_v p_\psi \). It is probability of transition. Let \( p_n = L_S(|v\rangle \langle v|) \), then because \( |\langle v|\psi\rangle|^2 = Tr([v|\psi\rangle \langle \psi|v]\rangle) = Tr(\bar{L}_S(p_n)\bar{L}_S(p_\psi)) \) is some bilinear map \( \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R} \), due to general theorem of linear algebra, it is scalar product \( \langle p_n, p_\psi \rangle_S = |\langle v|\psi\rangle|^2 \) defined by some matrix \( G_S \):
\[
(p_n, p_\psi)_S = \sum_{\alpha,\beta=1}^N G_{\alpha\beta} p_\alpha p_\beta.
\]  
(37)
So it is possible to define some metric \( G_S \) on \( \mathbb{R}^N \) with property that probability of transition for pure states is scalar product in this metric.

Example 4: It is possible to calculate average value of any Hermitian operator \( X \) for some observable: \( \bar{X}^\psi = \langle \psi|X|\psi \rangle = Tr(\bar{L}_S(p_\psi)X) \) (cf. Refs. [2], [3], §8.1.1) for pure state. For mixed state it is possible to use formula \( \bar{X} = Tr(\rho X) = Tr(\bar{L}_S(p_\rho)X) \) and if to represent a Hermitian operator \( X \) as some vector \( p_X \equiv L_S(X) \in \mathbb{R}^N \), then average value of operator \( X \) for some mixed state \( \rho \) is expressed via the same metric \( G_S \) Eq. (37) introduced in example above
\[
\bar{X}^\rho = (p_X, p_\rho)_S.
\]  
(38)

4 Example of representative and complete sets

Let us consider example with Hilbert space \( H_n \) and minimal representative set with \( N = n^2 \) elements. The \( n^2 \) vectors are expressed using \( n \) basis vectors \( |\alpha\rangle \) as following three families of vectors:
- \( n \) basis vectors itself: \( |v_\alpha^*\rangle = |\alpha\rangle \),
- \( \frac{n(n-1)}{2} \) vectors: \( |v_{\alpha\beta}^x\rangle = \frac{1}{\sqrt{2}} (|\alpha\rangle + |\beta\rangle) \), \( \alpha < \beta \),
- \( \frac{n(n-1)}{2} \) vectors: \( |v_{\alpha\beta}^y\rangle = \frac{1}{\sqrt{2}} (|\alpha\rangle + i|\beta\rangle) \), \( \alpha < \beta \).

The set is known also due to application for construction of POVM [1].

The weight vector \( p \) with \( N = n^2 \) real components is composed as set of numbers from three different families:
\[
\begin{align*}
p^x_\alpha &= \langle \alpha|\rho|\alpha \rangle = \rho_{\alpha\alpha} \\
p^x_{\alpha\beta} &= \frac{1}{2}(|\alpha\rangle + \langle \beta|) \rho (|\alpha\rangle + |\beta\rangle) = \frac{1}{2}(\rho_{\alpha\alpha} + \rho_{\beta\beta} + (\rho_{\alpha\beta} + \rho_{\beta\alpha})) \\
p^y_{\alpha\beta} &= \frac{1}{2}(|\alpha\rangle - i\langle \beta|) \rho (|\alpha\rangle + i|\beta\rangle) = \frac{1}{2}(\rho_{\alpha\alpha} + \rho_{\beta\beta} + i(\rho_{\alpha\beta} - \rho_{\beta\alpha}))
\end{align*}
\]

So \( \rho_{\alpha\alpha} = p^x_\alpha \) and \( (\rho_{\alpha\beta} + \rho_{\beta\alpha})/2 = p^x_{\alpha\beta} - \frac{1}{2}(p^x_{\alpha\beta} + p^y_{\alpha\beta}) \), \( i(\rho_{\alpha\beta} + \rho_{\beta\alpha})/2 = p^y_{\alpha\beta} - \frac{1}{2}(p^x_{\alpha\beta} + p^y_{\alpha\beta}) \) for \( \alpha < \beta \) and it is possible to write for arbitrary matrix:
\[
\rho_{\alpha\beta} = \begin{cases} 
p^x_\alpha; & \alpha = \beta \\
p^x_{\alpha\beta} - i p^y_{\alpha\beta} - \frac{1}{4}(p^x_{\alpha\beta} + p^y_{\alpha\beta}); & \alpha < \beta \\
p^y_{\alpha\beta} + i p^y_{\alpha\beta} - \frac{1}{4}(p^x_{\alpha\beta} + p^y_{\alpha\beta}); & \alpha > \beta
\end{cases}
\]  
(39)
or for Hermitian matrix it is convenient also to use expressions with \( n^2 \) real parameters for real and imaginary components of elements \( \rho_{\alpha \beta} \) (here \( \alpha < \beta \)):

\[
\rho_{\alpha \alpha} = p_{\alpha}^z, \\
\Re \rho_{\alpha \beta} = \Re \rho_{\beta \alpha} = p_{\alpha}^z - \frac{1}{2}(p_{\alpha}^x + p_{\beta}^x), \\
\Im \rho_{\alpha \beta} = -\Im \rho_{\beta \alpha} = p_{\alpha}^y - \frac{1}{2}(p_{\alpha}^x + p_{\beta}^x).
\]

The Eq. (39) is direct construction of linear map \( \hat{L} \) discussed above and so \( n^2 \) vectors \( v_{\alpha \beta}^{x,y,z} \) are representative set.

But the set is not complete. It is possible to add two other families of vectors for completion:

\[
\frac{n(n-1)}{2} \text{ vectors: } |v_{\alpha \beta}^{x}⟩ = \frac{1}{\sqrt{2}}(|\alpha⟩ - |\beta⟩), \alpha < \beta,
\]

\[
\frac{n(n-1)}{2} \text{ vectors: } |v_{\alpha \beta}^{y}⟩ = \frac{1}{\sqrt{2}}(|\alpha⟩ - i|\beta⟩), \alpha < \beta.
\]

5 Complete sets for quantum circuits

5.1 One qubit

The complete set described in the previous example has \( 2n^2 - n \) elements and for case of qubit \( n = 2 \) with basis \( |0⟩, |1⟩ \in \mathcal{H}_2 \) it is especially simple and symmetric perfect set with six elements:

\[
|0^x⟩ \equiv |v_{12}^x⟩ = |0⟩, \quad |1^z⟩ \equiv |v_{12}^z⟩ = |1⟩,
\]

\[
|0^y⟩ \equiv |v_{12}^y⟩ = \frac{1}{\sqrt{2}}(|0⟩ + |1⟩), \quad |1^x⟩ \equiv |v_{12}^x⟩ = \frac{1}{\sqrt{2}}(|0⟩ - |1⟩),
\]

\[
|0^y⟩ \equiv |v_{12}^y⟩ = \frac{1}{\sqrt{2}}(|0⟩ + i|1⟩), \quad |1^y⟩ \equiv |v_{12}^y⟩ = \frac{1}{\sqrt{2}}(|0⟩ - i|1⟩).
\]

It is simple to check that they are eigenvectors of Pauli matrices with eigenvalues \( \pm 1 \):

\[
\sigma^\mu |\Theta^\mu⟩ = \lambda_\Theta |\Theta^\mu⟩ \quad \text{(no summation),} \quad \lambda_\Theta = (-1)^\Theta,
\]

where \( \Theta = 0, 1 \) and \( \mu \in \{ x, y, z \} \). The Eq. (40) can be simply checked using expression for projectors:

\[
\Pi_0^\mu = \Pi(\Theta^\mu) = |\Theta^\mu⟩⟨\Theta^\mu| = \frac{1}{2}(I + \lambda_\Theta \sigma^\mu),
\]

\[
\sigma^\mu = \lambda_\Theta (2\Pi_0^\mu - I) = 2\Pi_0^\mu - I = I - 2\Pi_1^\mu = \Pi_0^\mu - \Pi_1^\mu, \quad \Pi_0^\mu + \Pi_1^\mu = I,
\]

then

\[
\sigma^\mu |\Theta^\mu⟩ = \lambda_\Theta (2\Pi_0^\mu - I)|\Theta^\mu⟩ = \lambda_\Theta (2\Pi_0^\mu|\Theta^\mu⟩ - |\Theta^\mu⟩) = \lambda_\Theta (2|\Theta^\mu⟩ - |\Theta^\mu⟩) = \lambda_\Theta |\Theta^\mu⟩.
\]

Because operator of spin can be expressed\(^4\) as \( \frac{1}{2} \sigma^\mu \) the Eq. (40) shows that \( |0^\mu⟩ \) and \( |1^\mu⟩ \) correspond to spin \( +\frac{1}{2} \) or \( -\frac{1}{2} \) respectively for measurements in three orthogonal directions \( \mu \in \{ x, y, z \} \).

Six equations for probabilities \( p_0^\mu, \ldots, p_3^\mu \) are:

\[
p_0^\mu = ⟨\Theta^\mu | \rho | \Theta^\mu⟩ = \text{Tr}(\rho \Pi_0^\mu) = \text{Tr}(\rho \frac{1}{2}(I + \lambda_\Theta \sigma^\mu)) = \frac{1}{2}(1 + \lambda_\Theta \text{Tr}(\rho \sigma^\mu)).
\]

Let us now use together with \( \mu \in \{ x, y, z \} \) numerical indexes \( \mu = 1, 2, 3 \) and also \( \mu, \nu = 0, \ldots, 3 \):

\[
\sigma^0 \equiv I, \quad \sigma^1 \equiv \sigma^x, \quad \sigma^2 \equiv \sigma^y, \quad \sigma^3 \equiv \sigma^z.
\]

\(^4\) in Planck’s units
It is a basis of space of Hermitian matrices:

\[ H = \sum_{\nu=0}^{3} a_{\nu} \sigma^{\nu}, \quad H \in H(2), \quad a_{\nu} \in \mathbb{R}. \]  \hspace{1cm} (44)

Because of a property of the Pauli matrices:

\[ \text{Tr}(\sigma^{\mu} \sigma^{\nu}) = 2\delta_{\mu\nu} \]  \hspace{1cm} (45)

they form the tr_{AB}-orthogonal basis and it is simple to find the coefficients \( a_{\nu} \) in Eq. (44):

\[ a_{\nu} = \frac{1}{2} \text{Tr}(H \sigma^{\nu}). \]  \hspace{1cm} (46)

For density matrix it is always \( a_{0} = \frac{1}{2} \), but here is used all four indexes due to application to composition below. Let us denote \( \tilde{p}_{\mu} = p_{\mu}^{0} - p_{1}^{\mu} \) and \( \tilde{p}_{0} = p_{0}^{0} + p_{1}^{0} = 1, \forall \mu = 1, 2, 3 \). Then from Eq. (43) follows:

\[ \tilde{p}_{\nu} = \text{Tr}(\rho \sigma^{\nu}), \quad \nu = 0, \ldots, 3. \]  \hspace{1cm} (47)

Up to insignificant multiplier \( \frac{1}{2} \), \( \tilde{p}_{\nu} \) correspond to parameters \( a_{\nu} \) just defined in Eq. (46) or \( \tilde{p}_{K} \) introduced earlier in Eq. (27). The density matrix can be expressed as

\[ \rho = \frac{1}{2} \sum_{\nu=0}^{3} \tilde{p}_{\nu} \sigma^{\nu}, \]  \hspace{1cm} (48)

due to Eq. (44) and probabilities \( p_{\mu}^{0} \) can be expressed as

\[ p_{\mu}^{0} = \frac{1}{2}(\tilde{p}_{0} + \lambda_{8} \tilde{p}_{\mu}), \quad \mu = 1, 2, 3, \quad \lambda = 0, 1, \quad \tilde{p}_{0} = 1 \]  \hspace{1cm} (49)

due to Eq. (43).

A simple transformation property of parameters \( \tilde{p}_{\mu} \) (here \( \mu = 1, 2, 3 \)) for unitary 1-qubit gate (cf. example 1 in Sec. III and \[4\]) is related with 2-1 homomorphism \( SO(3) \) and \( SU(2) \) groups — for any matrix \( U \in SU(2) \) there is \( 3 \times 3 \) matrix \( O \in SO(3) \) with property:

\[ U \sigma^{\mu} U^{-1} = \sum_{\nu=1}^{3} O_{\mu\nu} \sigma^{\nu}. \]  \hspace{1cm} (50)

For a map \( A_{U}: \tilde{p}_{\mu} \mapsto \tilde{p}_{\mu}' \) we have:

\[ \tilde{p}_{\mu}' = \text{Tr}(U \rho U^{*} \sigma^{\mu}) = \text{Tr}(\rho U^{*} \sigma^{\mu} U) = \text{Tr}\left(\rho \sum_{\nu=1}^{3} O_{\mu\nu}^{-1} \sigma^{\nu}\right) = \sum_{\nu=1}^{3} O_{\mu\nu}^{-1} \tilde{p}_{\nu}. \]

So for \( \tilde{p}_{\mu} \equiv (\tilde{p}_{1}, \tilde{p}_{2}, \tilde{p}_{3}) \) it is possible simply to write:

\[ \tilde{p}' = O_{U}^{-1} \tilde{p}, \quad O_{U} \in SO(3). \]  \hspace{1cm} (51)

The Eq. (51) shows that one-qubit gates corresponds to 3D rotations of vector \( (\tilde{p}_{1}, \tilde{p}_{2}, \tilde{p}_{3}) \) and parameter \( \tilde{p}_{0} \) is not changing.

Here is used mainly a model with spin-half systems, but an analogy of the parameters Eq. (47), Stokes parameters was also considered in optical models of qubits \[10\].
5.2 Two qubits

Let us apply the composition theorem for system of two qubits. Here we have complete set of 36 = 6² vectors $\{|\Theta_{1}\rangle, |\Theta_{2}\rangle\}$ with $\mu_1, \mu_2 = 1, 2, 3$. The set is almost perfect, it can be considered as disjoint union of nine bases with four elements: 

$$
\{ |0\mu_1\rangle, |0\mu_1\rangle, |1\mu_1\rangle, |1\mu_1\rangle \}
$$

for nine pairs $(\mu_1, \mu_2)$. In example with spin systems, it corresponds to nine possible combinations of measurements of two spins $(S_{\mu_1}, S_{\mu_2})$ with four possible combinations of results, $(\pm \frac{1}{2}, \pm \frac{1}{2})$, i.e., $(\frac{1}{2} \lambda_{\mu_1}, \frac{1}{2} \lambda_{\mu_2})$.

The set is not perfect, for example for vector $|0\mu_1\rangle$ the choice of basis 

$$
\{ |0\mu_1\rangle, |0\mu_1\rangle, |1\mu_1\rangle, |1\mu_1\rangle \}
$$

is not unique, it is simple to check that vectors 

$$
\{ |0\mu_1\rangle, |0\mu_1\rangle, |1\mu_1\rangle, |1\mu_1\rangle \}
$$

are also orthogonal. The example shows that tensor product of two perfect sets may be almost perfect, but not perfect.

It is possible to consider 36 (linearly dependent) probabilities:

$$
p^{\mu_1\mu_2}_{\Theta_1\Theta_2} = Tr(\rho \Theta_{1}) \otimes \Theta_{2}, \quad \mu_1, \mu_2 = 1, 2, 3, \quad \Theta_1, \Theta_2 = 0, 1 \tag{52}
$$

described by 16 parameters:

$$
\tilde{p}_{\nu_1\nu_2} = Tr(\rho \sigma^{\nu_1} \otimes \sigma^{\nu_2}), \quad \nu_1, \nu_2 = 0, 1, 2, 3. \tag{53}
$$

The Hermitian basis with 16 elements $\frac{1}{\sqrt{2}} \sigma^{\nu_1} \otimes \sigma^{\nu_2}$ is $tr_{AB}$-orthonormal (see Sec. 5.3 below) and it is possible to write

$$
\rho = \frac{1}{4} \sum_{\nu_1, \nu_2 = 0}^{3} \tilde{p}_{\nu_1\nu_2} \sigma^{\nu_1} \otimes \sigma^{\nu_2} \tag{54}
$$

The expressions of $p$ via $\tilde{p}$ can be simply calculated

$$
p^{\mu_1\mu_2}_{\Theta_1\Theta_2} = Tr(\rho \Theta_{1}) \otimes \Theta_{2}) = \frac{1}{4} Tr \left( \rho (\sigma^{0} + \lambda_{\Theta_1} \sigma^{1}) \otimes (\sigma^{0} + \lambda_{\Theta_2} \sigma^{2}) \right)
= \frac{1}{4} (\tilde{p}_{00} + \lambda_{\Theta_1} \tilde{p}_{10} + \lambda_{\Theta_2} \tilde{p}_{02} + \lambda_{\Theta_1} \lambda_{\Theta_2} \tilde{p}_{11}). \tag{55}
$$

where $\tilde{p}_{00} = Tr \rho = 1$.

On the other hand, to express $\tilde{p}$ via $p$ it is necessary to use few kind of formulas (here $\mu_1, \mu_2 \neq 0$):

$$
\tilde{p}_{\mu_1\nu_2} = Tr(\rho \Theta_{1}) \otimes \Theta_{2}) = p^{\mu_1\mu_2}_{0} - p^{\mu_1\mu_2}_{01} + p^{\mu_1\mu_2}_{10} - p^{\mu_1\mu_2}_{11},
\tilde{p}_{\nu_1\mu_2} = Tr(\rho \Theta_{1}) \otimes \Theta_{2}) = p^{\mu_1\mu_2}_{0} - p^{\mu_1\mu_2}_{01} + p^{\mu_1\mu_2}_{10} - p^{\mu_1\mu_2}_{11},
\tilde{p}_{\mu_1\nu_1} = Tr(\rho \Theta_{1}) \otimes \Theta_{2}) = p^{\mu_1\mu_2}_{0} + p^{\mu_1\mu_2}_{01} - p^{\mu_1\mu_2}_{10} - p^{\mu_1\mu_2}_{11},
1 = \tilde{p}_{00} = Tr(\rho \Theta_{1}) \otimes \Theta_{2}) = p^{\mu_1\mu_2}_{0} + p^{\mu_1\mu_2}_{01} + p^{\mu_1\mu_2}_{10} + p^{\mu_1\mu_2}_{11} \tag{56}
$$

there only for nine parameters $\tilde{p}_{\mu_1\mu_2}$ with $\mu_1, \mu_2 \neq 0$ it is unique decomposition, for six parameters with one zero index $\tilde{p}_{\mu_10}$ or $\tilde{p}_{0\mu_2}$ there are three different expressions and the last equation shows nine different ways to decompose unit.

It is possible to introduce six probabilities for description of first subsystem,

$$
p^{\mu}_{\Theta} = Tr(\rho \Theta_{1}) \otimes I, \quad \mu = 1, 2, 3, \quad \Theta = 0, 1 \tag{57}
$$

and six for second one,

$$
p^{\mu}_{\Theta} = Tr(\rho I \otimes \Theta_{1}), \quad \mu = 1, 2, 3, \quad \Theta = 0, 1. \tag{58}
$$

The equations correspond to measurement of only first or second spin respectively for models with two spin-half systems.
Each such probability can be expressed as sum of two probabilities described
by Eq. (52) in three different ways (corresponding to free parameter $\mu_1$ or $\mu_2$):
\begin{align}
\hat{p}_\mu^{(1)} &= \hat{p}_0^{\mu_2} + \hat{p}_1^{\mu_2}, \\
\hat{p}_\mu^{(2)} &= \hat{p}_0^{\mu_1} + \hat{p}_1^{\mu_1}, \\
\forall \mu_1, \mu_2 &= 1, 2, 3.
\end{align}

It is also possible to introduce similar description with parameters $\tilde{p}$, but in
this case it is only change of notation:
\begin{align}
\tilde{p}_\nu^{(1)} &= \text{Tr}(\rho \sigma^\nu \mathbb{1}) = \tilde{p}_{0\nu}, \\
\tilde{p}_\nu^{(2)} &= \text{Tr}(\rho \mathbb{1} \sigma^\nu) = \tilde{p}_{0\nu},
\end{align}

where $\tilde{p}_0^{(j)} = \tilde{p}_{00} = \text{Tr} \rho = 1, \forall j$. It is also possible to use the parameters
Eq. (53) and Eq. (51) to express probabilities Eq. (57) and Eq. (58) respectively:
\begin{align}
\hat{p}_\mu^{(j)} &= \frac{1}{2}(1 + (-1)^j \tilde{p}_\mu^{(j)}), \quad j = 1, 2, \mu = 1, 2, 3, \quad \Theta = 0, 1.
\end{align}

Let us consider situation \textit{without entanglement}, when:
\begin{align}
\rho &= \rho_1 \otimes \rho_2.
\end{align}

(cf. $|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$ for pure states)

Using identities:
\begin{align}
(A \otimes B)(C \otimes D) = (AC) \otimes (BD)
\end{align}
together with
\begin{align}
\text{Tr}(A \otimes B) &= \text{Tr}(A) \text{Tr}(B)
\end{align}
we have for any matrices $A_1, A_2$:
\begin{align}
\text{Tr}((\rho_1 \otimes \rho_2)(A_1 \otimes A_2)) &= \text{Tr}((\rho_1 A_1) \otimes (\rho_2 A_2)) = \text{Tr}(\rho_1 A_1) \text{Tr}(\rho_2 A_2).
\end{align}

So if systems are not entangled, then due to Eq. (63) and Eq. (64) it is
simple to prove expression for “independent probabilities”. Due to definitions
Eq. (52), Eq. (57), and Eq. (58), it is possible to write:
\begin{align}
\hat{p}_0^{\mu_1 \mu_2} &= \text{Tr}(\rho_1 \otimes \rho_2 \Pi_0^{\mu_1} \otimes \Pi_0^{\mu_2}) = \text{Tr}(\rho_1 \Pi_0^{\mu_1}) \text{Tr}(\rho_2 \Pi_0^{\mu_2}), \\
\hat{p}_1^{\mu_1 (1)} &= \text{Tr}(\rho_1 \Pi_0^{\mu_1} \otimes \mathbb{1}) = \text{Tr}(\rho_1 \Pi_0^{\mu_1}) \text{Tr}(\rho_2) = \text{Tr}(\rho_1 \Pi_0^{\mu_1}), \\
\hat{p}_2^{\mu_2 (2)} &= \text{Tr}(\rho_1 \mathbb{1} \otimes \Pi_0^{\mu_2}) = \text{Tr}(\rho_1) \text{Tr}(\rho_2 \Pi_0^{\mu_2}) = \text{Tr}(\rho_2 \Pi_0^{\mu_2}),
\end{align}
and, finally,
\begin{align}
\hat{p}_0^{\mu_1 \mu_2} &= \hat{p}_0^{\mu_1 (1)} \hat{p}_0^{\mu_2 (2)}
\end{align}

Similar expression
\begin{align}
\tilde{p}_{\nu_1 \nu_2} &= \tilde{p}_{\nu_1}^{(1)} \tilde{p}_{\nu_2}^{(2)}
\end{align}
can be proved for definitions Eq. (53), Eq. (60), and Eq. (61).

5.3 Many qubits

For composite system with $m$ qubits density matrix is $\rho \in H_{2^m}$ and it is
possible to use $4^m$ different tensor products of Pauli matrices as a basis:
\begin{align}
H = \sum_{\nu_1 \cdots \nu_m}^{3} a_{\nu_1 \cdots \nu_m} \sigma^{\nu_1} \otimes \cdots \otimes \sigma^{\nu_m}, \quad H \in H(2^m), \quad a_{\nu_1 \cdots \nu_m} \in \mathbb{R}.
\end{align}

Let us denote $\sigma^{\nu_1 \cdots \nu_m} = \sigma^{\nu_1} \otimes \cdots \otimes \sigma^{\nu_m}$. Due to Eq. (62), it is possible to write:
\begin{align}
\text{Tr}(\sigma^{\nu_1 \cdots \nu_m} \sigma^{\mu_1 \cdots \mu_m}) &= 2^m \delta_{\nu_1 \mu_1} \cdots \delta_{\nu_m \mu_m}.
\end{align}
i.e., the Hermitian basis $2^{-m} \sigma_{\nu_1 \ldots \nu_m}$ is $\text{tr}_{AB}$-orthonormal. An expression for coefficients $a_{\nu_1 \ldots \nu_m}$ directly follows from the Eq. (71):

$$a_{\nu_1 \ldots \nu_m} = 2^{-m} \text{Tr}(H \sigma_{\nu_1 \ldots \nu_m}).$$

(71)

So for density matrix $a_{00 \ldots 0} = 2^{-m} \text{Tr} \rho = 2^{-m}$.  

Using composition of $m$ perfect qubit bases with six vectors discussed above we can produce basis with $6^m$ components. It is almost perfect because can be considered as disjoint union of $3^m$ bases with $2^m$ elements: $|\Theta^1 \mu^1 \ldots \Theta^m \mu^m\rangle$ marked by $3^m$ different sets $(\mu_1, \ldots, \mu_m)$. It is similar with description of case $m = 2$ above and the counterexample provided there is enough to show that the set is not perfect for any $m > 1$.

Let us denote $\Pi^\mu_{\Theta_1 \ldots \Theta_m} \equiv \Pi^\mu_{\Theta_1} \otimes \cdots \otimes \Pi^\mu_{\Theta_m}$. When $6^m$ probabilities introduced by complete (and almost perfect) set $\Pi^\mu_{\Theta_1, \ldots, \Theta_m}$ can be represented as:

$$p_{\Theta_1 \ldots \Theta_m}^\mu = \text{Tr}(\rho \Pi^\mu_{\Theta_1 \ldots \Theta_m}), \quad \mu = 1, 2, 3, \quad \Theta_k = 0, 1, \quad k = 1, \ldots, m,$$  

(72)
described by $4^m$ parameters:

$$\tilde{p}_{\nu_1 \ldots \nu_m} = \text{Tr}(\rho \sigma_{\nu_1 \ldots \nu_m}), \quad \nu_k = 0, 1, 2, 3, \quad k = 1, \ldots, m,$$  

(73)

where $\tilde{p}_{00 \ldots 0} = \text{Tr} \rho = 1$.

In example with spin systems, $p_{\Theta_1 \ldots \Theta_m}^\mu$ correspond to $3^m$ possible combinations of measurements of $m$ spins $(S^\mu_{\Theta_1}, \ldots, S^\mu_{\Theta_m})$ with $2^m$ possible combinations of results $(\frac{1}{2} \lambda_{\Theta_1}, \ldots, \frac{1}{2} \lambda_{\Theta_m})$.

It is possible to introduce six probabilities for each subsystem, similarly with Eq. (77) and Eq. (78),

$$p^\mu_\Theta(k) \equiv \text{Tr}(\rho \Pi^k_{\Theta} \otimes \Pi^\mu_{(1-k)}, \quad \mu = 1, 2, 3, \quad \Theta = 0, 1, \quad k = 1, \ldots, m,$$  

(74)
described by three parameters similarly with Eq. (80) and Eq. (81):

$$\tilde{p}^\mu_\Theta(k) = \text{Tr}(\rho \Pi^k_{\Theta} \otimes \sigma_{\Theta} \otimes \Pi^\mu_{(1-k)}), = \tilde{p}_{0 \Theta 0 \ldots 0 \ldots 0 \ldots 0}^k, \quad k = 1, \ldots, m,$$  

(75)

(\text{where } \tilde{p}_0^k = \tilde{p}_{00 \ldots 0} = \text{Tr} \rho = 1, \forall j). The Eq. (72) again can be used for calculation of probabilities Eq. (74),

$$p^\mu_\Theta(k) = \frac{1}{2}(1 + (-1)^\Theta \tilde{p}^\nu_\Theta(k)), \quad k = 1, \ldots, m \quad \mu = 1, 2, 3, \quad \Theta = 0, 1,$$  

(76)

but analogue of Eq. (72) is too difficult, because it would contain $2^{m-1}$ terms and may be expressed in $3^{m-1}$ different ways. It is yet another example of usefulness of parameters $\tilde{p}$.

Here is also can be considered systems without entanglement, where

$$\rho = \rho_1 \otimes \cdots \otimes \rho_m = \bigotimes_{k=1}^m \rho_k,$$  

(77)

For such states is also can be used analogue of Eq. (77)

$$p_{\Theta_1 \ldots \Theta_m}^\nu = \prod_{k=1}^m p^\nu_\Theta^k(k), \quad \mu_k = 1, 2, 3, \quad \Theta_k = 0, 1, \quad k = 1, \ldots, m,$$  

(78)

and Eq. (88)

$$\tilde{p}_{\nu_1 \ldots \nu_m} = \prod_{k=1}^m \tilde{p}^\nu_\Theta(k), \quad \nu_k = 0, 1, 2, 3, \quad k = 1, \ldots, m, ($$  

(79)
So, systems without entanglement can be described by $6m$ probabilities Eq. (74) instead of $6^m$ Eq. (72) or $3m$ parameters Eq. (75) instead of $4^m - 1$ parameters Eq. (73). For systems with entanglement such simple idea does not work and $6m$ probabilities of Eq. (72) and $4m$ parameters Eq. (73) can be considered as some tensors, $\tilde{p}_\nu \equiv p^{\nu_1 \ldots \nu_m}$ and $\tilde{\rho}_\nu \equiv \tilde{p}_{\nu_1 \ldots \nu_m}$.

Transformations of the tensors due to action of local quantum gates on initial state is also local. If some quantum gate acts only on a few ($l = 1, 2, \ldots; l \ll m$) qubits and can be described as $2^l \times 2^l$ complex matrix, then transformation of tensor $\tilde{p}_\nu$ can be described by $4^l \times 4^l$ real matrix (and $6^l \times 6^l$ matrix for $p^\mu_\Theta$).

It is clear after rewriting of expressions with trace like Eq. (31) already used above, i.e,

$$\tilde{p}'_{\nu_1 \ldots \nu_m} = \text{Tr}(U \rho U^* \sigma^{\nu_1 \ldots \nu_m}) = \text{Tr}(\rho U^* \sigma^{\nu_1 \ldots \nu_m} U)$$

(80)

and so if $U$ acts only on few indexes between $\nu_1, \ldots, \nu_m$, then only the indexes present in matrix $\tilde{A}_U$ of the transformation $\tilde{p}' = \tilde{A}_U \tilde{p}$ or, formally,

$$\tilde{p}'_{\nu_1 \ldots \nu_m} = \sum_{\mu_1, \ldots, \mu_m = 0}^3 \tilde{A}^{(\mu_1 \ldots \mu_m)}_{(\nu_1 \ldots \nu_m)} \tilde{p}_{\mu_1 \ldots \mu_m}$$

(81)

Really it follows directly form expressions for coefficients Eq. (34) that can be rewritten as

$$\tilde{A}^{(\mu_1 \ldots \mu_m)}_{(\nu_1 \ldots \nu_m)} = \text{Tr}(\sigma^{\nu_1 \ldots \nu_m} U^* \sigma^{\mu_1 \ldots \mu_m} U).$$

(82)

For example for two-qubit gate acting on some indexes $j$ and $k$ instead of Eq. (81) for $\tilde{A}_U$ with $4^m$ components it is enough to use only $4^2 = 16$ components and write:

$$\tilde{p}'_{\nu_1 \ldots \nu_j \ldots \nu_k \ldots \nu_m} = \sum_{\mu_j, \mu_k = 0}^3 \tilde{A}^{(\mu_j \mu_k)}_{(\nu_j \nu_k)} \tilde{p}'_{\nu_1 \ldots \mu_j \ldots \mu_k \ldots \nu_m}.$$  

(83)

The similar approach is true for $A_U$ and $p^\mu_\Theta$.

Such probabilistic description of quantum circuits is not more complex, than usual description with pure states, and for description of quantum channels with mixed states it is even more simple, because here is not necessary to use some special decomposition for “superoperators”, they are also may be expressed by one matrix with same size as for “usual” operators.

Here again was rather used model with $m$ spin-half systems, but parameters similar with $\tilde{p}_\nu$ was also used as “multiple beam Stokes parameters” in rather optical framework [10].

6 Conclusion and bibliographical notes

The set of problems discussed in present paper is part of more general task of complete description of quantum systems using some set of observables. The task had a long history and very wide range of different approaches and it is not possible here to review that, but together with references used in main body of the paper it should be mentioned some alternative approaches [3-6] together with references therein. Present paper itself appears initially in rather specific circumstances of discussions related with macroscopic realization of quantum logic [3-6] with later development in more general framework related with quantum computations.
Acknowledgements

Author is grateful to A. Grib, V. Bubovich, R. Zapatrin for discussions and also to many other colleagues for participation in some seminars about year ago devoted to consideration of initial version of present work in A.A.Friedmann Laboratory for Theoretical Physics.

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5See also footnote 2 on page 7.