Preserving algebraic structures on exact ∞-categories with the K-theory functor

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Abstract

The purpose of this paper is to establish a new perspective on the K-theory of exact ∞-categories. We show that if the definition of K-theory is slightly modified, one can interpret the K-theory of an exact ∞-category as a stable ∞-category, and not as a spectrum. Since spectra are stable ∞-categories with a single object, this new perspective strictly generalizes the classical viewpoint of K-theory. My formalism encompasses all the information about the K-theory of ring spectra into a single statement: $K : Sp \to Sp$.

As an example of the relative simplicity of calculations in my formalism, in this paper, we compute the K-theory of a ∞-operad of modules, and show that it must be equivalent to a ∞-operad of modules itself! This calculation provides a sufficient condition for the conclusion of a question asked by Carlsson to hold. In addition, we use this computation to state a generic property of the K-theory of the sphere spectrum (which is an open problem). We conclude this paper by proving a derived counterpart of the Morita context in stable model categories (hitherto unknown), which can be used to compare different exact ∞-categories via their K-theories, since it is an invariant.

Motivation and Objectives

Algebraic K-theory is a universal additive and localizing invariant of ring spectra and stable ∞-categories ([9]) which takes values in spectra. Though it satisfies such a universal property, it is extremely difficult to compute; for example, what is $K_1(S)$? The goal of this
paper is to help simplify K-theoretic computations by establishing the foundations of a new interpretation of the K-theory of an exact ∞-category as a stable ∞-category, by slightly modifying its usual definition. In other words, We will view K-theory as a functor $K : \mathcal{E}xact_{\infty} \to \mathcal{C}at^{\mathcal{E}x}_{\infty}$ from the ∞-category $\mathcal{E}xact_{\infty}$ of exact ∞-categories to the ∞-category $\mathcal{C}at^{\mathcal{E}x}_{\infty}$ of stable ∞-categories. Viewed in this way, one can encompass all the information about the ordinary K-theory of ring spectra into the following statement: $K : Sp \to S\mathcal{P}$. To illustrate that our new definition generalizes ordinary K-theory, we compute the K-theory of an ∞-operad of modules, and prove an analogue of Elmendorf and Mandell’s results ([13]) on Carlsson’s question (refer to the appendix).

Notation and Terminology

The notation used in this paper is the same as that in [30] and [31]. In particular, we use the following (somewhat nonstandard) notation/terminology:

1. The word ∞-category is used to denote a simplicial set $X$ such that all inner horns in $X$ have fillers.

2. The word K-theory is used to denote algebraic K-theory.

3. If $\mathcal{C}^\otimes$ and $\mathcal{O}^\otimes$ are ∞-operads and $A$ is an $\mathcal{O}$-algebra object of $\mathcal{C}^\otimes$, we write $\text{Mod}^\otimes_A(\mathcal{C}^\otimes)$ to denote the underlying ∞-category of the ∞-operad of $\mathcal{O}$-modules over $A$.

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The reader should note that Barwick had already begun to interpret the K-theory of an exact ∞-category as a complete Segal space, but did not prove that the smallest ∞-category containing the essential image of the K-theory functor was a subcategory of the ∞-category of stable ∞-categories.
1 Waldhausen and Exact $\infty$-categories

Recall that if $\mathcal{C}$ is an exact category, then Quillen’s Q-construction assigns a new category $Q(\mathcal{C})$ to $\mathcal{C}$. The K-theory space of $\mathcal{C}$ is defined as the topological space $\mathcal{C} \mapsto \Omega |NQ\mathcal{C}|$, and the K-groups $K_i(\mathcal{C})$ are the homotopy groups $\pi_i(\Omega |NQ\mathcal{C}|) = \pi_{i+1} |NQ\mathcal{C}|$. It is natural to ask whether this has an analogue in homotopy theory, for a nice notion of exact $\infty$-categories, and indeed, such a construction exists. The Quillen Q-construction of an exact $\infty$-category $\mathcal{C}$ arises as the stable $\infty$-category of “ambigressive” maps from the opposite of the twisted arrow $\infty$-category of $\Delta^n$ to $\mathcal{C}$.

Barwick recently described such a construction (in fact, he did this only two years ago); in this subsection, we will provide a brief review of the constructions in his papers [1, 2, 3], which are the main references here.

1.1 Waldhausen $\infty$-categories

We will now recall the notion of (co)Waldhausen $\infty$-categories (see [1]).

**Definition 1.1.1.** Let $\mathcal{C}$ and $\mathcal{C}_*$ be $\infty$-categories. The pair $(\mathcal{C}, \mathcal{C}_*)$ is said to be a Waldhausen $\infty$-category if the following conditions are satisfied:

1. $\mathcal{C}$ is pointed.
2. The morphism $0 \to X$ is in $\mathcal{C}_*$ for any object $X$.
3. Any diagram $Y \leftarrow X \to X'$ with the map $X \to Y$ in the $\infty$-category $\mathcal{C}_*$ can be extended into a pushout $Y'' = X' \coprod_X Y$.
4. If $Y = X' \times_X Y$ is a pushout square, with the map $X \to Y$ in $\mathcal{C}_*$, then the map $X' \to Y''$ is in $\mathcal{C}_*$ as well.

A coWaldhausen $\infty$-category is a pair $(\mathcal{C}, \mathcal{C}_*)$ such that the pair $(\mathcal{C}^{op}, \mathcal{C}_*^{op})$ is a Waldhausen $\infty$-category, and a biWaldhausen $\infty$-category is a triple $(\mathcal{C}, \mathcal{C}_*, \mathcal{C}_*)$ such that $(\mathcal{C}, \mathcal{C}_*)$ is a Waldhausen $\infty$-category and $(\mathcal{C}, \mathcal{C}_*)$ is a coWaldhausen $\infty$-category.
1.2 Exact $\infty$-categories

The notion of exact $\infty$-categories is a combination of the notions of stable $\infty$-categories and (co)Waldhausen $\infty$-categories. In order to make both these structures compatible, one needs the notion of ambigressive pullbacks:

**Definition 1.2.1.** A pullback $X = X' \times_{Y'} Y$ in a biWaldhausen $\infty$-category $(\mathcal{C}, \mathcal{C}_\bullet, \mathcal{C}^\bullet)$ is said to be ambigressive if the maps $X' \rightarrow Y'$ and $Y \rightarrow Y'$ are in $\mathcal{C}_\bullet$ and $\mathcal{C}^\bullet$, respectively.

The usual notion of an exact $\infty$-category involves a weaker condition than the one we impose here, but many interesting examples that are of interest arise in this fashion.

**Definition 1.2.2.** Let $(\mathcal{C}, \mathcal{C}_\bullet, \mathcal{C}^\bullet)$ be a biWaldhausen $\infty$-category. We say that $\mathcal{C}$ is exact if the following conditions hold:

1. $\mathcal{C}$ is a stable $\infty$-category.

2. A square in $\mathcal{C}$ is an ambigressive pullback if and only if it is an ambigressive pushout.

Exact $\infty$-categories arrange themselves into an $\infty$-category $\text{Exact}$ of exact $\infty$-categories.

**Remark 1.2.3.** The terminology used here is slightly abusive since the exactness of a $\infty$-category $\mathcal{C}$ depends on the choice of the $\infty$-categories $\mathcal{C}_\bullet$ and $\mathcal{C}^\bullet$.

**Example 1.2.4.** One of the simplest examples of an exact $\infty$-category is a triple of stable $\infty$-categories $(\mathcal{C}, \mathcal{C}_\bullet = \mathcal{C}, \mathcal{C}^\bullet = \mathcal{C})$; in fact, any stable $\infty$-category is an exact $\infty$-category which arises in this fashion. This example justifies our strengthening of the definition of an exact $\infty$-category. Another example is the nerve of a stable exact model category.

**Example 1.2.5.** The $\infty$-category $\text{Mod}_A^G(\mathcal{C})^\otimes$ is stable, and hence can be regarded as an exact $\infty$-category.

\footnote{The classical definition only requires that $\mathcal{C}$ is additive.}
Definition 1.2.6. A functor from the twisted arrow $\infty$-category $\text{TwArr}(\Delta^n)$ to an exact $\infty$-category $\mathcal{C}$ is said to be ambigressive if for integers $0 \leq i \leq k \leq l \leq j \leq n$, the square

$$
\begin{array}{ccc}
(i,j) & \longrightarrow & (k,j) \\
\downarrow & & \downarrow \\
(i,l) & \longrightarrow & (k,l)
\end{array}
$$

is sent to an ambigressive pullback.

2 The Barwick-Quillen Q-construction

2.1 Definitions

Construction 2.1.1. Let $X$ be a simplicial set. Using the popular notation for marked simplicial sets, we will let $\mathcal{R}_*(X) : \Delta^{op} \to \text{Set}_\Delta$ denote the functor that takes the finite ordinal $[n]$ to $\text{Map}^\sh(\text{TwArr}(\Delta^n)^{op,\flat}, X)$, and for $\mathcal{C}$ an exact $\infty$-category, we will allow $Q_n(\mathcal{C})$ to be the subset of $\mathcal{R}_n(\mathcal{C})$ spanned by the ambigressive functors $\text{TwArr}(\Delta^n)^{op} \to \mathcal{C}$. The $\infty$-category $Q_*(\mathcal{C})$ is defined as the simplicial set whose $n$-simplices are the vertices of $Q_n(\mathcal{C})$.

Remark 2.1.2. The simplicial set $Q_*(\mathcal{C})$ constructed above is a complete Segal spaces, i.e., an $\infty$-category; see [2, Definition 3.8].

One can now define the $K$-theory of an exact $\infty$-category. The reader should note that our definition is slightly modified, and indeed, showing that our definition is more advantageous than the well-known definition is purpose of this paper.

Definition 2.1.3. The $K$-theory of $\mathcal{C}$ is the $\infty$-category $\Omega^3 Q_*(\mathcal{C})$.

Remark 2.1.4. The usual definition of the $K$-theory of an exact $\infty$-category $\mathcal{C}$ is the geometric realization $\Omega|Q_*(\mathcal{C})|$; since we have not included the geometric realization in our definition, we can use the powerful theory of simplicial sets to study $K$-theory.

2.2 Stability of the Barwick-Quillen Q-construction

Theorem 2.2.1. There is an equivalence $\Omega Q_*(\mathcal{C}) \simeq Q_*(\mathcal{C})$, and the $\infty$-category $Q_*(\mathcal{C})$ admits finite limits.
Proof. We will prove the equivalent statement that $Q_*(\mathcal{C})$ is stable, which implies that $Q_*(\mathcal{C})$ admits finite limits and that $\Omega Q_*(\mathcal{C}) \simeq Q_*(\mathcal{C})$ by [30, Corollary 1.4.2.27].

Let $R_*(\mathcal{C})$ denote the $\infty$-category whose $n$-simplices are the vertices of $R_n(\mathcal{C})$. By [30, Proposition 1.1.3.1], $R_*(\mathcal{C})$ is stable and the $\infty$-category $Q_*(\mathcal{C})$ is pointed. Formal nonsense shows that $Q_*(\mathcal{C})$ admits cofibers; thus, in order to show that $Q_*(\mathcal{C})$ is a stable subcategory of $R_*(\mathcal{C})$, it now suffices to observe that $\mathcal{C}$ is stable (i.e. that $\Sigma : \mathcal{C} \to \mathcal{C}$ is an equivalence of $\infty$-categories), so that $Q_*(\mathcal{C})$ is stable under translations. □

Remark 2.2.2. Rephrased differently, the K-theory construction induces a functor $Q_* : \text{Exact}_\infty \to \text{Cat}_\infty^{\text{Ex}}$.

Notation 2.2.3. We will henceforth write $Q_*(\mathcal{C})$ instead of $Q_*(\mathcal{C})$ to emphasize that $Q_*(\mathcal{C}) = \Omega Q_*(\mathcal{C})$ is equivalent to $Q_*(\mathcal{C})$, i.e., that $Q_*(\mathcal{C})$ is a stable $\infty$-category.

Remark 2.2.4. This shift in perspective implies that one should think of the K-theory $Q_*(\mathcal{C})$ not as a spectrum, but rather as a stable $\infty$-category.$^4$ The purpose of this paper is to illustrate that this point of view allows for a simpler way to approach problems.

Example 2.2.5. To illustrate how calculations can be simplified, we will rephrase Carlsson’s question (refer to the appendix) in $\infty$-categorical terms:

Question 2.2.6. When is the K-theory of a symmetric monoidal exact $\infty$-category $\mathcal{D}$ equivalent to the $\infty$-operad of modules over an algebra over the symmetric monoidal stable $\infty$-category $Q_*(\mathcal{C}^{\otimes})$ for some symmetric monoidal exact $\infty$-category $\mathcal{C}^{\otimes}$?

Theorem 3.1.2 provides a sufficient condition on $\mathcal{D}^{\otimes}$ for the conclusion of this question to hold.

3 Computational aspects of K-theory

The shift in perspective described above brings with it a wave of ideas. In this section, we will illustrate the power of this new method

$^4$In fact, the K-theory of a symmetric monoidal exact $\infty$-category yields a symmetric monoidal stable $\infty$-category, much like how the K-theory of a symmetric monoidal category is an $E_{\infty}$-ring, up to “ring completion”.
of thinking by proving that the K-theory functor preserves module objects, to answer Question [2.2.6]. We apply this result to develop a homotopical derived Morita theory.

3.1 K-theory of modules

The motivation for our calculation comes from Gunnar Carlsson’s question as rephrased in ∞-categorical terms as Question [2.2.6]. We will begin by stating a preliminary lemma:

Lemma 3.1.1. Let Mod_A^0(\mathcal{C})^\otimes \n denote Mod_A^0(\cdots \cdot Mod_A^0(\mathcal{C})^\otimes \cdots)^\otimes iterated n times. Then for n \geq 1, there is an equivalence Mod_A^0(\mathcal{C})^\otimes \n \simeq Mod_A^0(\mathcal{C})^\otimes, where we abuse notation slightly by identifying an object A with itself under the functor \mathcal{C} \to Mod_A^0(\mathcal{C})^\otimes.

Proof. The proof is induction on [30, Corollary 3.4.1.9]. □

We are now ready to state the result of our main calculation:

Theorem 3.1.2. There is an equivalence Q_*(Mod_A^0(\mathcal{C})^\otimes) \simeq Mod_A^0(\mathcal{Q}_*(\mathcal{C}))^\otimes, where we again abuse notation slightly by using the same symbol (here A) for the image of an object (here A) under the functor K : \mathcal{C} \to Q_*(\mathcal{C}).

Remark 3.1.3. In other words, a sufficient condition for the K-theory of a symmetric monoidal exact ∞-category \mathcal{D} to be equivalent to an ∞-operad of modules over an algebra over the \mathcal{Q}_*(\mathcal{C})^\otimes is that \mathcal{D} should be Mod_A^0(\mathcal{C})^\otimes itself.

Proof. We will establish a contradiction by assuming that there is no map Mod_A^0(\mathcal{Q}_*(\mathcal{C}))^\otimes \to Q_*(Mod_A^0(\mathcal{C})^\otimes) that is an equivalence.

Under this assumption, let \alpha : Mod_A^0(\mathcal{Q}_*(\mathcal{C}))^\otimes \to Q_*(Mod_A^0(\mathcal{C})^\otimes) be a map of ∞-categories. There is an induced map \beta : Mod_A^0(\mathcal{Q}_*(\mathcal{C}))^\otimes \to Mod_A^0(\mathcal{Q}_*(\mathcal{C}))^\otimes, and for map of ∞-categories Mod_A^0(\mathcal{C})\otimes \to \mathcal{C} inducing a map Mod_A^0(\mathcal{Q}_*(\mathcal{C}))^\otimes \to Mod_A^0(\mathcal{Q}_*(\mathcal{C}))^\otimes, there is a natural map of ∞-categories from Mod_A^0(\mathcal{Q}_*(\mathcal{C}))^\otimes to Mod_A^0(\mathcal{Q}_*(\mathcal{C}))^\otimes given by the composition Mod_A^0(\mathcal{Q}_*(\mathcal{C}))^\otimes \to Mod_A^0(\mathcal{Q}_*(\mathcal{C}))^\otimes.

Any map from Mod_A^0(\mathcal{Q}_*(\mathcal{Q}_*(\mathcal{C}))^\otimes to itself arises via such a composition (because any such map must arise from a map Mod_A^0(\mathcal{Q}_*(\mathcal{C}))^\otimes \to Mod_A^0(\mathcal{Q}_*(\mathcal{C}))^\otimes defined on Mod_A^0(\mathcal{Q}_*(\mathcal{C}))^\otimes, following which one can restrict to Mod_A^0(\mathcal{Q}_*(\mathcal{C}))^\otimes, or a map from Mod_A^0(\mathcal{Q}_*(\mathcal{C}))^\otimes defined only on Mod_A^0(\mathcal{Q}_*(\mathcal{C}))^\otimes). Since the map Mod_A^0(\mathcal{Q}_*(\mathcal{C}))^\otimes \to Mod_A^0(\mathcal{Q}_*(\mathcal{C}))^\otimes is therefore an equivalence, we conclude that Mod_A^0(\mathcal{Q}_*(\mathcal{C}))^\otimes is an equivalence.
\[ \text{Mod}_A^0(\text{Q}_*(\text{Mod}_A^0(\mathcal{C}^\otimes)))^\otimes \text{ is never an equivalence, there should not be a map } \text{Mod}_A^0(\text{Q}_*(\text{Mod}_A^0(\mathcal{C}^\otimes)))^\otimes \to \text{Mod}_A^0(\text{Q}_*(\text{Mod}_A^0(\mathcal{C}^\otimes)))^\otimes \text{ that is an equivalence. This is obviously false. Hence there exists at least one map } \text{Mod}_A^0(\text{Q}_*(\mathcal{C}))^\otimes \to \text{Q}_*(\text{Mod}_A^0(\mathcal{C}^\otimes)) \text{ that is an equivalence of } \infty\text{-categories.} \]

3.2 Example: The K-theory of the sphere spectrum

Remark 3.2.1. Theorem 3.1.2 allows us to interpret the K-theory \( K(R) \) of a ring spectrum \( R \) as the spectrum \( S \) such that \( \text{Q}_*(\text{Mod}_R(\text{Sp})) \simeq \text{Mod}_X(\text{Q}_*(\text{Sp})) \), where the last equivalence follows from our main calculation. In other words, our formalism strictly generalizes the classical perspective on K-theory as a spectrum.

A very important question in K-theory is the computation of the K-theory of the sphere spectrum, \( S \). This observation allows us to state a “universality” result on the K-theory of the sphere spectrum \( S \):

Corollary 3.2.2. Let \( R \) be an arbitrary ring spectrum. The K-theory spectrum \( K(S) \) of the sphere spectrum is the spectrum such that the K-theory spectrum \( K(R) \) of any other ring spectrum \( R \) is a \( K(S) \)-module.

Proof. There is an inclusion \( \text{Q}_*(\text{Mod}_S(\text{Sp})) \simeq \text{Mod}_{K(S)}(\text{Q}_*(\text{Sp})) \hookrightarrow \text{Mod}_{K(S)}(\text{Sp}) \) as \( \text{Q}_*: \text{Sp} \to \text{Sp} \).

3.3 Homotopical derived Morita theory

One of the accomplishments of our new perspective is creation of something which we call homotopical derived Morita theory. This theory allows for the comparison of the homotopy categories of different K-theories of modules. The homotopy category \( \text{HoQ}_*(\text{Mod}_A^0(\mathcal{C}^\otimes)) \) is what we call the “derived category of an \( \mathcal{O} \)-algebra \( A \) over the \( \infty \)-operad \( \mathcal{C}^\otimes \)”.

It makes sense to question why we should use K-theory for this purpose.

Remark 3.3.1. Not only is K-theory the universal additive and localizing invariant (see \[\text{[1]}\]), but it also nicely compares to the derived category \( \mathcal{D}(R) \) of a ring \( R \) (see \[\text{[1]}\] for the definition of the derived category of an algebra over an operad):
Remark 3.3.2. It makes sense to thus define the derived category of an $O$-algebra object $A$ in a symmetric monoidal $\infty$-category $C^\otimes$ as the homotopy category $\text{Ho}_{Q^*}(\text{Mod}_{O^A}(C)^\otimes)$, and not as the homotopy category of $\text{Mod}_{O^A}(C)^\otimes$ since we only want to study the algebraic structures on $\text{Mod}_{O^A}(C)^\otimes$ and nothing else. Theorem 3.1.2 asserts that this job is accomplished by K-theory.

As in ordinary derived Morita theory, we must ask the following question:

**Question 3.3.3.** Suppose $A$ is an $O$-algebra object in a symmetric monoidal $\infty$-category $C^\otimes$ and $A'$ an $O'$-algebra object in another symmetric monoidal $\infty$-category $C'^\otimes$. When are the derived categories $\text{Ho}_{Q^*}(\text{Mod}_{O^A}(C)^\otimes)$ and $\text{Ho}_{Q^*}(\text{Mod}_{O'^A}(C')^\otimes)$ triangulated equivalent?

The answer is in fact quite simple, and to answer it, we will introduce the structure of a saturated relative category on $\text{Ho}_{Q^*}(\text{Mod}_{O^A}(C)^\otimes)$.

**Construction 3.3.4.** A morphism $f : A \to B$ in $\text{Ho}_{Q^*}(\text{Mod}_{O^A}(C)^\otimes)$ is called a weak equivalence if there is a distinguished triangle $A \to B \to C \to A$ such that their composition is the identity.

**Lemma 3.3.5.** $\text{Ho}_{Q^*}(\text{Mod}_{O^A}(C)^\otimes)$ is a saturated relative category with the class of weak equivalences introduced above.

**Theorem 3.3.6.** Let $F : Q^*(\text{Mod}_{O^A}(C)^\otimes) \to Q^*(\text{Mod}_{O'^A}(C')^\otimes)$ be a functor between symmetric monoidal stable $\infty$-categories. Then the following statements are equivalent:

1. $F$ is an equivalence of $\infty$-categories.

2. $\text{Ho}F$ is a triangulated equivalence (i.e., $\text{Ho}F$ is an equivalence of categories that takes distinguished triangles to distinguished triangles).

Both of these equivalent statements imply the following condition:
(3) \(\text{Ho} F\) preserves weak equivalences.

All of these statements are equivalent if there are only a finite number of distinguished triangles in \(\text{Ho} \mathcal{Q}_* (\text{Mod}^{O'}_A (C')^\otimes)\).

**Proof.** The map \(F\) is an equivalence of \(\infty\)-categories if and only if \(\text{Ho} F\) is. Since all equivalences of \(\infty\)-categories are exact, the functor \(\text{Ho} F\) preserves distinguished triangles. It therefore is a triangulated equivalence, i.e., it preserves weak equivalences. To prove the theorem, we must show that all these statements are equivalent if the number of distinguished triangles in \(\text{Ho} \mathcal{Q}_* (\text{Mod}^{O'}_A (C')^\otimes)\) is finite.

If \(\text{Ho} F\) is a triangulated equivalence, then it preserves weak equivalences. We must now prove that if \(\text{Ho} F\) preserves weak equivalences, then it is a triangulated equivalence. Consider a triangle \(X \to Y \to Z \to X\). If \(X \simeq \tilde{X}, Y \simeq \tilde{Y}, \) and \(Z \simeq \tilde{Z},\) then it can be split into a larger distinguished triangle:

\[
\begin{array}{c}
\tilde{X} \\
\downarrow \\
\tilde{Y} \\
\downarrow \\
\tilde{Z} \\
\downarrow \\
\tilde{X}
\end{array}
\]

One can again split the triangle \(\tilde{X} \to \tilde{Z} \to \tilde{Y} \to \tilde{X}\) into:

\[
\begin{array}{c}
\tilde{Y} \\
\downarrow \\
\tilde{Y}' \\
\downarrow \\
\tilde{Z} \\
\downarrow \\
\tilde{X}'
\end{array}
\]

After mapping the distinguished triangles to \(\text{Ho} \mathcal{Q}_* (\text{Mod}^{O'}_A (C')^\otimes)\), we observe that we will always obtain a weak equivalence after some finite number of iterations of this procedure, since the number of distinguished triangles is assumed to be finite and the class of weak equivalences is a strict subclass of the class of distinguished triangles. If weak equivalences are preserved and the map \(\text{Ho} F\) is an equivalence of categories, we can “de-iterate” this procedure to recover all distinguished triangles. Thus \(\text{Ho} F\) is a triangulated equivalence if it preserves weak equivalences and the number of distinguished triangles in \(\text{Ho} \mathcal{Q}_* (\text{Mod}^{O'}_A (C')^\otimes)\) is finite.

**Remark 3.3.7.** This is a comparison theorem for the K-theories of
exact ∞-categories if we let $\mathcal{O}^\circ = \mathcal{O}^{\circ,\prime} = E_0^\circ$. In that case, for $\mathcal{C}^\circ$ and $\mathcal{C}^{\circ,\prime}$ exact ∞-categories, the stable ∞-categories $Q_* (\mathcal{C}^\circ)$ and $Q_* (\mathcal{C}^{\circ,\prime})$ are equivalent if the homotopy categories $\text{Ho} Q_* (\mathcal{C}^\circ)$ and $\text{Ho} Q_* (\mathcal{C}^{\circ,\prime})$ are triangulated equivalent.

4 Future Directions

In this final section, we will briefly introduce some vaguely phrased future directions stemming from the ideas presented in this paper.

4.1 (Co)Flatness for stable ∞-categories

Let us recall the “classical derived” Morita context:

**Theorem 4.1.1.** Let $R$ and $S$ be symmetric ring spectra. The following conditions are equivalent:

1. There is a Quillen equivalence $\text{Mod}_R \simeq Q \text{Mod}_S$ of model categories.

2. The model category $\text{Mod}_S$ has a small bifibrant generator $C$ such that $\text{End}_{\text{Sp}} (C)$ is equivalent to $R$.

The following condition implies the above two conditions:

3. There is an equivalence of categories $\text{HoMod}_R \to \text{HoMod}_S$ given by a derived smash product functor.

All three conditions are equivalent if $(R$ or $) S$ is flat as a symmetric spectrum.

Comparing Theorem 3.3.6 to the above theorem suggests that the condition that there should only be a finite number of distinguished triangles in $\text{Ho} Q_* (\text{Mod}_A^\circ (\mathcal{C}^{\circ,\prime}))$ should be an analogue of “(co)flatness” for stable ∞-categories.

**Question 4.1.2.** Can this analogy be made precise?

4.2 Stable $(\infty, 2)$-categories

The K-theory of ring spectra is subsumed in the statement that $K : \text{Cat}_E^{\infty} \to \text{Cat}_E^{\infty}$. Similarly, this statement should be subsumed within a K-theory of “stable $(\infty, 2)$-categories”. In this section we will pose a conjecture regarding these stable $(\infty, 2)$-categories. The reader should keep in mind that this is section is very vague, and some proofs may seem incorrect due to the lack of detail.
4.2.1 First Definitions

Definition 4.2.1. Let $\mathcal{C}$ be a $(\infty,2)$-category, i.e., a fibrant object in the model structure on the category $\text{Set}^\infty_\Delta$ of scaled simplicial sets. We will say that $\mathcal{C}$ is stable if the $\infty$-categories $\mathcal{C}[\Delta^k \to \mathcal{C}, k > 1]^{-1}$ and $\mathcal{C}[\Delta^l \to \mathcal{C}, l > 1]^{-1}$ are stable. Here $\mathcal{C}$ is the $\infty$-category whose $n$-simplices are the $(n+1)$-simplices of $\mathcal{C}$.

How does one know whether this is the right definition of stable $(\infty,2)$-categories? We first need to check if the dualizable $(\infty,2)$-categories, i.e., the dualizable objects of the scaled nerve of $"(\infty,2)\text{-categories}"$ (which we will now construct), were stable $(\infty,2)$-categories.

4.2.2 Consistency of the definitions

Proposition 4.2.2. The Cartesian product of scaled simplicial sets defined as $(X, \mathcal{E}) \times (X', \mathcal{E}') = (X \times X', \mathcal{E} \times \mathcal{E}')$ endows $\text{Set}^\infty_\Delta$ with the structure of a monoidal model category.

Proof. It is easy to see that the functor $\times : \text{Set}^\infty_\Delta \times \text{Set}^\infty_\Delta \to \text{Set}^\infty_\Delta$ preserves colimits separately in each variable. We now need to show that if $i : (X, \mathcal{E}) \to (X', \mathcal{E}')$ and $j : (Y, \mathcal{F}) \to (Y', \mathcal{F}')$ are (trivial) cofibrations, then the induced map $i \vee j : (X \times Y, \mathcal{E} \times \mathcal{F}) \coprod_{(X,Y,\mathcal{E},\mathcal{F})} (X \times Y', \mathcal{E} \times \mathcal{F}') \to (X' \times Y, \mathcal{E}' \times \mathcal{F}) \coprod_{(X',Y,\mathcal{E}',\mathcal{F'})} (X' \times Y', \mathcal{E}' \times \mathcal{F}')$ is also a (trivial) cofibration. If $i$ and $j$ are cofibrations, then it is automatically true that $i \vee j$ is, since the condition that a map is a cofibration depends only on the underlying simplicial set, and the Cartesian product makes $\text{Set}^\infty_\Delta$, with the Joyal model structure, into a monoidal model category.

We must now show that if $i$ or $j$ is a trivial cofibration, then so is $i \vee j$. Let us work one variable at a time. Suppose that $i$ is a trivial cofibration and $Y = Y'$. By definition, $\mathcal{C}^\infty[X] \simeq \mathcal{C}^\infty[X']$ and $\mathcal{C}^\infty[Y] \simeq \mathcal{C}^\infty[Y']$. Since $\mathcal{C}^\infty$ is a left adjoint, it preserves colimits, so that there is a pushout diagram $\mathcal{C}^\infty[i \vee j] : \mathcal{C}^\infty[(X \times Y, \mathcal{E} \times \mathcal{F})] \coprod_{\mathcal{C}^\infty[(X,Y,\mathcal{E},\mathcal{F})]} \mathcal{C}^\infty[(X \times Y', \mathcal{E} \times \mathcal{F}')] \to \mathcal{C}^\infty[(X' \times Y, \mathcal{E}' \times \mathcal{F})] \coprod_{\mathcal{C}^\infty[(X',Y,\mathcal{E}',\mathcal{F'})]} \mathcal{C}^\infty[(X' \times Y', \mathcal{E}' \times \mathcal{F}')]$.

It now suffices to show that the Cartesian product preserves weak equivalences; this is precisely the content of [29 Lemma 4.2.6]! Hence, there is a chain of equivalences $\mathcal{C}^\infty[(X \times Y, \mathcal{E} \times \mathcal{F})] \simeq \mathcal{C}^\infty[(X \times Y, \mathcal{E} \times \mathcal{F})] \simeq \mathcal{C}^\infty[(X \times Y', \mathcal{E} \times \mathcal{F}')] \simeq \mathcal{C}^\infty[(X' \times Y, \mathcal{E}' \times \mathcal{F})] \simeq \mathcal{C}^\infty[(X' \times Y', \mathcal{E}' \times \mathcal{F}')]$. This shows that if $i$ and $j$ are trivial cofibrations, so is $i \vee j$; this concludes the proof that the functor $\times : \text{Set}^\infty_\Delta \times \text{Set}^\infty_\Delta \to \text{Set}^\infty_\Delta$ is a left Quillen bifunctor.

12
Every object is cofibrant, and the monoidal structure is closed and symmetric. Hence the Cartesian product endows $\mathbf{Set}^\Delta$ with the structure of a monoidal model category.

**Definition 4.2.3.** Let $\mathbf{Cat}^\Delta_{(\infty,2)}$ be the $\mathbf{Set}^\Delta$-enriched category defined as follows:

1. The objects of $\mathbf{Cat}^\Delta_{(\infty,2)}$ are (small) $(\infty,2)$-categories.
2. If $C$ and $D$ are $(\infty,2)$-categories, then $\text{Map}_{\mathbf{Cat}^\Delta_{(\infty,2)}}(C, D)$ is the largest $(\infty,2)$-category contained in the $(\infty,2)$-category $\text{Fun}(C, D)$ (this is a $(\infty,2)$-category because $\mathbf{Set}^\Delta$ has the closed symmetric monoidal structure described above, so it can be considered to be enriched over itself).

The scaled nerve $N^{\mathbf{sc}}(\mathbf{Cat}^\Delta_{(\infty,2)})$ of $\mathbf{Cat}^\Delta_{(\infty,2)}$ is the $(\infty,2)$-category $\mathbf{Cat}_{(\infty,2)}$ of $(\infty,2)$-categories. This has a symmetric monoidal structure coming from the Cartesian product of scaled simplicial sets.

**Definition 4.2.4.** A dualizable $(\infty,2)$-category is a dualizable object in the symmetric monoidal structure on $\mathbf{Cat}_{(\infty,2)}$.

**Theorem 4.2.5.** Every dualizable $(\infty,2)$-category is a stable $(\infty,2)$-category.

**Proof.** Suppose $C$ is a dualizable $(\infty,2)$-category, with dual $C^\vee$. Then the $(\infty,2)$-categories $C[\Delta^k \rightarrow C, k > 1]^{-1}$ and $C[\Delta^l \rightarrow C, l > 1]^{-1}$ have duals given by $C^\vee[\Delta^k \rightarrow C^\vee, k > 1]^{-1}$ and $C^\vee[\Delta^l \rightarrow C^\vee, l > 1]^{-1}$ since the Cartesian product of simplicial sets is defined “cell-wise”, i.e., $(X \times Y)_n = X_n \times Y_n$. This means that they are stable $(\infty,2)$-categories (since dualizable $(\infty,2)$-categories are examples of stable $(\infty,2)$-categories), and by definition, $C$ is then itself a stable $(\infty,2)$-category.

Since the $(\infty,2)$-category of spectra is naturally a stable $(\infty,2)$-category, it makes sense to conjecture:

**Conjecture 4.2.6.** Let $\mathbf{Cat}^{\mathbf{Ex}}_{(\infty,2)}$ denote the $(\infty,2)$-category of stable $(\infty,2)$-categories and exact functors between them. Then $\mathbf{Cat}^{\mathbf{Ex}}_{(\infty,2)}$ is a stable $(\infty,2)$-category.

We plan to provide a proof of this conjecture (and its natural generalization to higher dimensions) in future papers.
A Carlsson’s Question

Let $X$ be a permutative category and $Y$ a bipermutative category. Then $K$-theories $K(X)$ and $K(Y)$ are ring spectra, but $K(Y)$ has the additional structure of an $\mathbb{E}_\infty$-ring. Since many ring spectra are naturally modules over $\mathbb{E}_\infty$-rings (in fact, any spectrum is a module over an $\mathbb{E}_\infty$-ring, namely the sphere spectrum!), Gunnar Carlsson found it natural to ask the following question:

Question A.0.7. When is $K(X)$ a module spectrum over $K(Y)$?

B Elmendorf and Mandell’s results

Progress on this question was reported by Elmendorf and Mandell in [12], and it was answered as [13, Corollary 9.1.7 and Corollary 9.2.10]. Here we will state their results.

Proposition B.0.8. If $\mathcal{C}$ is a left (resp. right) module over $\mathcal{D}$, then (up to equivalence) $K(\mathcal{C})$ is a left (resp. right) $K(\mathcal{D})$-module.

Proposition B.0.9. If $\mathcal{C}$ is a $\mathcal{D}$-$\mathcal{D}$-bimodule (resp. $\mathcal{D}$-algebra), then (up to equivalence) $K(\mathcal{C})$ is a $K(\mathcal{D})$-$K(\mathcal{D})$-bimodule (resp. $\mathcal{D}$-algebra).

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