BCS-BEC crossover-like phenomena driven by quantum-size effects in quasi-one-dimensional fermionic condensates

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Quantum confinement is known to influence fermionic condensates, resulting in quantum-size oscillations of superfluid/superconducting properties. Here we show that the impact of quantum-size effects is even more dramatic. Under realistic conditions, a significant phase-space reconfiguration induced by quantum-size effects opens a quasi-molecule channel in the fermionic pairing so that the condensed pairs exhibit features typical of a molecular state. As an illustration we consider a quasi-one-dimensional fermionic condensate, as realized, e.g., in cigar-shaped atomic Fermi gases or superconducting quantum wires. In this case the transverse quantization of the particle motion favors pairing through a coherent superposition of quantum channels that are formed due to the grouping of single-particle levels into a series of well distinguished subbands. Whenever the bottom of a subband approaches the Fermi level, the longitudinal spatial distribution of fermions in a condensed pair becomes strongly localized within the corresponding quantum channel. The fermionic pairs in this channel resemble molecules with bosonic character.

In 1960’s Blatt and Thompson calculated the energy gap of an ultrathin superconducting slab as function of its thickness and found a series of pronounced peaks. They called these peaks *shape superconducting resonances*. The physics behind this result is usually understood as follows. The spectrum describing the electron motion in the direction perpendicular to the film is quantized whereas a quasi-free electron motion is assumed in the direction parallel to the film. Due to the transverse quantization, the conduction band splits up into a series of single-particle subbands. The lower edges (bottoms) of such subbands are determined by the perpendicular discrete electron levels and, so, move in energy with changing thickness $d$ (scaling as $\sim 1/d^2$). Unlike the subbands, the Fermi level exhibits a significantly less pronounced size-dependent shift. So, the bottoms of the subbands pass through the Fermi surface one by one when increasing the thickness. Each time when the bottom of a subband approaches the Fermi surface, the density of single-particle states in the vicinity of the Fermi level increases, resulting in an enhancement of basic superconducting quantities, e.g., the critical temperature, the order parameter and the excitation gap. Thus, the formation of single-particle subbands due to quantum confinement manifests itself through *quantum-size oscillations* of the superconducting properties driven by a series of shape superconducting resonances (below they are also referred to as quantum-size
superconducting/superfluid resonances or size-dependent superconducting/superfluid resonances).

For several decades after the paper by Blatt and Thompson, only few experimental groups reported possible signatures of quantum-size oscillations in superconducting films \(^3\)–\(^5\). However, poor sample purity and significantly limited size control prevented definite conclusions. Blatt and Thompson also argued (see Refs. \(^6\)–\(^8\)) that similar physics could be expected for nucleon pairing in atomic nuclei as a consequence of the shell structure of the single-particle spectrum. This expectation was found to be in agreement with experimental data \(^9\) and, for the next 40 years, atomic nuclei were the only system where quantum-size effects on the BCS fermionic pairing were investigated both experimentally and theoretically (for more details, see Ref. \(^10\) and references therein). Advances in microstructuring of superconductors in the 90’s renewed interest in quantum-size effects in superconducting systems. The work of Blatt and Thompson was extended by Bianconi, Perali and coworkers \(^11\), \(^12\) to quantum striped superconductors, where a sizeable amplification of the critical temperature and excitation gap was found in the presence of a shape superconducting resonance. Recent developments in nanofabrication have resulted in high-quality superconducting nanosystems, and the quantum-size oscillations of the critical superconducting temperature \(T_c\) were eventually observed in single-crystalline atomically uniform in thickness Pb nanofilms \(^13\)–\(^15\).

Furthermore, the results of a numerical self-consistent solution of the Bogoliubov-de Gennes (BdG) equations in Ref. \(^16\) showed that the quantum-size superconducting resonances were responsible for a systematic thickness-dependent shift-up of \(T_c\) found in high-quality aluminum and tin superconducting nanowires \(^17\)–\(^20\). Very recently quantum-size oscillations of the excitation gap were also reported for tin superconducting nanograins \(^21\).

Cooling of trapped fermionic atoms down to ultra-low temperatures such that the atomic gas is Fermi degenerate (see, e.g., Ref. \(^22\)), resulted in another system promising for the investigation of quantum-size effects in fermionic condensates. In the case of trapped atomic Fermi gases the single-particle levels can be tuned by laser light or magnetic fields through, e.g., changing the spatial dimensions of the trap. In particular, for a pancake- or cigar-shaped geometry the perpendicular trapping frequency \(\omega_\perp\) is much larger than the parallel one \(\omega_\parallel\). In this case single-particle states form well distinguished subbands and the physical picture turns out to be similar to that in superconducting metallic nanowires and nanofilms, i.e., a quantum-size superfluid resonance can be expected when the bottom of a single-particle...
subband is located in the vicinity of the Fermi level. In particular, quantum-size oscillations of the critical temperature and order parameter were recently calculated for a superfluid Fermi gas confined in a quasi-two-dimensional (quasi-2D) harmonic trap. We note that fermionic condensates are now attainable even in the tight-confinement regime with only one perpendicular single-particle level below the Fermi energy. Recently, experimental results on the first indications on quantum-size oscillations of the cloud-size aspect ratio in a harmonically trapped quasi-2D Fermi gas became available. Maintaining the atomic cloud at the lowest attainable temperature and at a low number of atoms, the group of Vale was able to detect the filling of the individual perpendicular levels, following in a quantitative way the dimensional crossover from 2D to 3D in a $^6\text{Li}$ Fermi gas, while keeping the 3D character of the two-particle scattering. This work clearly demonstrates that the experimental conditions to explore such a dimensional crossover and the accompanying quantum-size effects are presently achievable with ultracold atoms.

In this paper, we consider a quasi-1D fermionic condensate and show that the impact of the formation of single-particle subbands is more dramatic than simply resulting in oscillations of $T_c$ or the excitation energy gap. Significant reconfiguration of the phase space due to grouping of single-particle levels into a series of subbands (here quasi-1D means that we deal with more than one 1D subband of single-particle states) results in opening of a quasi-molecule channel in the fermionic pairing each time when the bottom of a single-particle subband approaches the Fermi surface. Due to a ”depletion” of the longitudinal Fermi motion in such a subband the longitudinal spatial distribution of fermions inside a condensed fermionic pair squeezes and becomes strongly localized. In this case the system behaves similar to the well-known crossover from the Bardeen-Cooper-Schrieffer (BCS) pairs to the Bose-Einstein condensation (BEC) of quasi-molecules driven through the Feshbach resonance in ultracold Fermi gases (see, e.g., Ref. 22). However, very different from previous case, the effect of interest is now controlled by the quantization of the perpendicular particle motion. By changing the transverse size of the system, one can change the energetic positions of the transverse single-particle discrete levels (bottoms of the single-particle subbands) with respect to the chemical potential $\mu$. This leads to a significant redistribution of the kinetic energy between the transverse and longitudinal degrees of freedom in a subband and, so, to the above mentioned ”depletion” of the longitudinal Fermi motion when the bottom of this subband approaches $\mu$. Below, for illustration, we investigate a cigar-shaped...
atomic Fermi gas and a superconducting quantum wire. Similar results are also expected for quasi-2D fermionic condensates.

I. HARMONICALLY TRAPPED QUASI-1D SUPERFLUID FERMI GAS

We begin with the BCS self-consistent equation for the spin-singlet s-wave pairing gap \( \Delta_k \) of a 3D Fermi gas with an attractive interaction, i.e.,

\[
\Delta_k = -\sum_{k'} V_{kk'} \Delta_{k'} \tanh(\beta E_{k'/2}) \left[ \frac{1}{2E_{k'}} - \frac{1}{2T_{k'}} \right],
\]

(1)

where \( T_k = \hbar^2 k^2 / 2M \) is the single-particle dispersion, \( E_k = \sqrt{(T_k - \mu)^2 + \Delta_k^2} \) stands for the quasiparticle spectrum, \( \mu \) is the chemical potential and \( \beta \) is the inverse temperature \( (\beta = 1/k_B T) \). The standard scattering length regularization is introduced in the gap equation (1) to avoid ultraviolet divergency in 3D. The interaction matrix element in Eq. (1) can be written in the form

\[
V_{kk'} = -g \int d^3r |\varphi_k(r)|^2 |\varphi_{k'}(r)|^2,
\]

(2)

where \( g = 4\pi \hbar^2 |a| / M \), with \( a < 0 \) the 3D s-wave scattering length, and \( \varphi_k(r) \) is the single-particle wave function that is proportional to \( e^{ikr} \) (plane waves) in 3D.

When switching to a Fermi gas confined in a trap, the particle momentum label in Eqs. (1) and (2) should be replaced by an index for discrete energy levels because the relevant wave functions are not plane waves but solutions of the one-particle Schrödinger equation for the corresponding confined geometry. In this section we consider harmonically trapped fermions in an axially symmetric confining potential \( M \left( \frac{\omega_\perp^2 \rho^2 + \omega_\parallel^2 z^2}{2} \right) \) (with \( \omega_\perp \gg \omega_\parallel \)), where cylindrical coordinates are invoked. Due to the axial symmetry the relevant quantum numbers can be chosen as \( n = 0, 1, \ldots \), the radial quantum number; \( m = 0, \pm 1, \pm 2, \ldots \), the azimuthal quantum number; and \( j = 0, 1, \ldots \), the quantum number associated with the parallel (longitudinal) motion. For the aspect ratio of the trap we have \( l_\parallel / l_\perp \gg 1 \) (with \( l_\parallel = \sqrt{\hbar / M \omega_\parallel} \) and \( l_\perp = \sqrt{\hbar / M \omega_\perp} \)) and, so, using the 3D pseudopotential \( g = 4\pi \hbar^2 |a| / M \) seems problematic due to the possible 1D character of the two-particle scattering. However, the aspect ratio is not actually the quantity that controls the character of scattering in a cigar-shaped many-particle system. The analysis of the two-particle scattering in an axially confined geometry, see, e.g., Ref. 27, shows that the scattering rapidly becomes three dimensional when the number of contributing subbands (perpendicular levels) is more
than 1. For parameters considered in this work this number varies from 6 to > 20. Thus we are far beyond the regime of the effectively 1D scattering and, so, any issue related to the confinement-induced Feshbach resonance in a 1D wave guide and the formation of confinement-induced molecules in a 1D Fermi gas (see, e.g., Refs. 22 and 28–31) are beyond the scope of the present work. To avoid any confusion, we would like to note once again that quasi-1D in the present work means that more than one transverse single-particle level contribute to the basic physical quantities.

By numerically solving Eqs. (1) (with $k \rightarrow \nu = \{n, m, j\}$), we self-consistently calculate for a given temperature a set of pairing gaps $\Delta_\nu$. Then, the critical temperature $T_c$ can be found as the temperature above which only the trivial ($\Delta_\nu = 0$) solution to the problem exists. To obtain more information, e.g., concerning the spatial distribution of the pair condensate and the fermionic pairing correlations, we should take into account that Eq. (1) follows from the Bogoliubov-de Gennes equations provided that the particle-like and hole-like wave functions $u_\nu(r)$ and $v_\nu(r)$ are approximated as (see, e.g., Refs. 32,33)

$$u_\nu(r) = U_\nu \varphi_\nu(r), \quad v_\nu(r) = V_\nu \varphi_\nu(r),$$

with ($U_\nu$ and $V_\nu$ are taken real)

$$U_\nu^2 = \frac{1}{2} \left( 1 + \frac{T_\nu - \mu}{E_\nu} \right), \quad V_\nu^2 = \frac{1}{2} \left( 1 - \frac{T_\nu - \mu}{E_\nu} \right),$$

and $\varphi_\nu(r) = \vartheta_{nm}(\rho, \varphi) \chi_j(z)$, where $\vartheta_{nm}(\rho, \varphi)$ and $\chi_j(z)$ are the eigenfunctions of the 2D (isotropic) and 1D harmonic oscillators, respectively. This approximation accounts for the pairing of time reversed states, and, as known since the pioneering paper by Anderson$^{34}$, such a simplification yields quite reasonable results in the presence of time reversal symmetry. Then, based on the Bogoliubov transformation of the field operators $\hat{\psi}_\uparrow(r)$ and $\hat{\psi}_\downarrow(r)$ to the quasiparticle quantum amplitudes $\gamma_{\nu,\uparrow}$ and $\gamma_{\nu,\downarrow}$, i.e.,

$$\hat{\psi}_\uparrow(r) = \sum_\nu \left[ u_\nu(r) \gamma_{\nu,\uparrow} - v_\nu^*(r) \gamma_{\nu,\downarrow}^\dagger \right], \quad \hat{\psi}_\downarrow(r) = \sum_\nu \left[ u_\nu^*(r) \gamma_{\nu,\downarrow} + v_\nu(r) \gamma_{\nu,\uparrow}^\dagger \right],$$

one can study the superfluid pair correlations and the spatial distribution of the pair condensate. In particular, introducing the Cooper-pair “wave function” $\Psi(r, r') = \langle \hat{\psi}_\uparrow(r) \hat{\psi}_\downarrow(r) \rangle$, we find

$$\Psi(r, r') = \sum_{nm} \Psi_{nm}(r, r'), \quad \Psi_{nm}(r, r') = \vartheta_{nm}(\rho, \varphi) \vartheta_{nm}^*(\rho', \varphi') \psi_{nm}(z, z'),$$

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with
\[
\psi_{nm}(z, z') = \sum_j \chi_j(z) \chi_j^*(z') \Delta_{nmj} \tanh\left(\frac{\beta E_{nmj}/2}{2E_{nmj}} - \frac{1}{2T_{nmj}}\right),
\]
where \(T_{nmj} = \hbar \omega_\perp (1 + 2n + |m| + \hbar \omega_\parallel (j + 1/2))\). It is worth noting that the regularization term \(1/(2T_{nmj})\) appears in the parenthesis of Eq. (5) as a simple extension to spatially nonuniform systems of the regularization used in Eq. (1). It is important to note that we have verified that different possible regularization schemes lead to insignificant differences in the physical quantities, preserving the basic conclusions of the present work, see a more detailed discussion in the end of this section. The single-particle states with the same \(n\) and \(m\) form a quasi-continuum. Therefore, it is natural to introduce single-particle subbands denoted by \((n, m)\) (see Fig. 1 in Supplementary Information) and the subband dependent fermionic-pair ”wave function” defined by Eq. (1). We remark that treating \(\Psi(r, r')\) as the wave function of a condensed pair of fermions goes back to the classical papers by Gor’kov\(^{35}\) and Bogoliubov\(^{36}\) and is directly related to the conventional interpretation of the order parameter \(\Delta(r) = g \Psi(r, r)\) as the center-of-mass Cooper-pair wave function. However, there exist also other ways to introduce the fermionic-pair wave function (see, e.g., Ref. \(^{37}\) and \(^{38}\)). Nevertheless, all these variants were shown\(^{37}\) to result in the bulk Cooper-pair radius being proportional to the ratio of the Fermi velocity to the energy gap. In the present paper we are interested in the longitudinal Cooper-pair size in a quasi-1D fermionic condensate which can be calculated as
\[
\xi_0 = \left[ \frac{\langle \psi_{nm}(z - z')^2 \rangle}{\langle \psi_{nm} \rangle} \right]^{1/2},
\]
with \(\Psi\) given by Eq. (1). Below we deal also with the subband-dependent longitudinal size of condensed fermionic pairs \(\xi_{0}^{(nm)}\) which is controlled by \(\psi_{nm}(z, z')\) and calculated from Eq. (6) with \(\Psi(r, r')\) replaced by \(\psi_{nm}(z, z')\), i.e., \(\xi_{0}^{(nm)} = \left[ \langle \psi_{nm}(z - z')^2 \rangle / \langle \psi_{nm} \rangle \right]^{1/2}\).

For a numerical solution of Eq. (1) (with \(k \to \nu = \{n, m, j\}\)), we consider a quasi-1D harmonically trapped mixture of \(^{6}\text{Li}\) atoms in the two lowest spin states, \(|F, m_F\rangle = |1/2, 1/2\rangle\) and \(|1/2, -1/2\rangle\). The pair interaction between these states can be significantly modified by an external magnetic field by means of the formation of a broad Feshbach resonance\(^{39}\). In particular, typical values of the \(s\)-wave scattering length \(a\) at the BCS side of this Feshbach resonance varies\(^{39}\) from \(-250\) to \(-100\) nm dependent on the magnetic field \(B\) (theoretically \(a\) goes to \(-\infty\) when approaching the point of the Feshbach resonance \(B = 0.83\, \text{kG}\)). In Fig. 1 our numerical results are shown for \(a = -140, -180\) and \(-210\, \text{nm}\) [panels (a,b),...
(c,d) and (e,f), respectively]. The chemical potential and the longitudinal frequency are kept constant, i.e., $\mu = h \cdot 24\text{kHz}$ and $\hbar \omega_{\parallel} = 0.01\mu$ ($\omega_{\parallel}/2\pi = 240\text{Hz}$), while the trapping frequency in the perpendicular direction varies in such a way that discrete single-particle levels for the perpendicular single-particle motion pass through the Fermi surface when the ratio $\mu/\hbar \omega_{\perp}$ reaches 2, 3, 4 etc. In our cigar-shaped confining geometry $\omega_{\parallel}$ is taken much smaller than $\omega_{\perp} \sim \mu/\hbar$. The particular value of $\omega_{\parallel}$ in this regime appears to be of no importance (changes less than a few percent were found for $\omega_{\parallel} = 2\pi \cdot 300\text{Hz}$ and $2\pi \cdot 480\text{Hz}$). The chemical potential is chosen such that experimentally accessible values of the particle density and the ratio $T_c/T_F$ are obtained, with $T_F$ denoting the Fermi temperature (for more details, see the discussion at the end of this section).

Figure 1 (a) shows our numerical results for $T_c$ (in units of $\mu/k_B$) as a function of $\mu/\hbar \omega_{\perp}$ calculated for $a = -140\text{nm}$. As seen, each time when a discrete transverse level crosses $\mu$, $T_c$ is enhanced, i.e., a superfluid quantum-size resonance occurs. As a result, $T_c$ oscillates with changing $\omega_{\perp}$. These oscillations are damped: their amplitude is reduced when $\omega_{\perp}$ decreases ($l_{\perp}$ increases). Such a decay is typical of quantum-size oscillations of the basic physical quantities in quasi-1D and quasi-2D fermionic condensates\textsuperscript{14-16,25}. The reason is that the number of single-particle subbands making a contribution increases with $l_{\perp}$. In particular, for $\mu/\hbar \omega_{\perp} = 2$ there are six subbands that are responsible for the formation of 99% of the condensate [subbands $(n,m) = (0,0), (0,\pm1), (0,\pm2)$ and $(1,0)$, major contribution is due to $(n,m) = (0,\pm1)$]. For $\mu/\hbar \omega_{\perp} = 4$ there are already 21 contributing subbands. The larger this number is, the less pronounced is the increase in the relevant density of states when a new perpendicular level crosses $\mu$. Then, quantum-size resonances are weakened and the corresponding oscillations are finally washed out. Another important feature of such oscillations is that their amplitude is also reduced when the pair interaction is enhanced (i.e., $|a|$ increases). A stronger pair interaction results in larger pairing gaps. In this case contributions of the levels even far below or above $\mu$ become of importance. As a result, passages of perpendicular levels through the Fermi surface have less pronounced and rather smoothed effects. This is illustrated by Figs. 1 (c) and (e), where $k_B T_c/\mu$ is given versus $\mu/\hbar \omega_{\perp}$ for $a = -180\text{nm}$ and $a = -210\text{nm}$. Note that in the present work $T_c$ is evaluated at the mean-field level. On the other hand, it is well known that the mean field critical temperature through the BCS-BEC crossover (driven by a change in the coupling $k_F a$) corresponds to a characteristic temperature of the pair formation and pseudogap opening, while
the critical temperature for the superfluid transition is renormalized by pair fluctuations. In the simplest approach, the renormalized critical temperature can be evaluated by using a non self-consistent t-matrix approach, as shown in Refs. 23 and 24. For our choice of parameters we have $-1.4 < (k_Fa)^{-1} < -0.9$ (with $k_F$ the Fermi wavevector in the center of the trap, see the discussion at the end of this section). These values of $k_Fa$ are between the BCS and the crossover regimes, where the suppression of the critical temperature due to the pair fluctuations is not larger than 16%, thanks also to the fact that in trapped systems the fluctuation region shrinks with respect to the homogeneous case. Therefore, the amplitude and period of the oscillations of $T_c$ induced by quantum confinement as shown in Fig. 1 will not be importantly modified by fluctuations.

Based on the above results for $T_c$, one can also anticipate similar oscillations of the longitudinal BCS coherence length $\xi_0$. These oscillations are shown (red solid circles) in Figs. 1(b), (d) and (f) for $a = -140, -180, \text{ and } -210 \text{ nm}$, respectively. Naive expectation based on the ordinary BCS picture suggests that $\xi_0 \propto \hbar v_F/(k_B T_c)$, with the Fermi velocity $v_F = \hbar k_F/m_e$. However, this expectation does not match our numerical results. As seen from Fig. 1(a), the coherence length $\xi_0$ drops by a factor of 2.7 when $\mu/\hbar \omega_\perp$ increases from 1.8 to 2.1. While according to the above naive estimation, $\xi_0$ is expected to fall only by about 40% (the critical temperature increases by about 50% with almost negligible change of $v_F$). Similar difference can be found between panels (d) and (f). The reason for this difference is the formation of the multi-subband structure. For illustrative purposes, the subband-dependent coherence length $\xi_0^{(nm)}$ is also shown in Figs. 1(b), (d) and (f) for $(n, m) = (0, 0), (0, \pm 1), (0, \pm 2)$ and $(0, \pm 3)$. At $\mu = 2\hbar \omega_\perp$ the bottoms of the two degenerate subbands with $(n, m) = (0, \pm 1)$ pass through the Fermi surface and the corresponding quantum-size resonance develops (a subband whose bottom is in the vicinity of the Fermi surface is referred to as resonant). At this point subbands $(0, \pm 1)$ make the major contribution to the basic physical quantities. In particular, their common contribution to the integral of $|\Psi(r_1, r_2)|^2$ is about 70 - 75%. In other words, we may state that almost 70 -75% of the fermionic pairs come from these subbands. As a result, being close to $\xi_0^{(0,0)}$ at $\mu < 1.8 \hbar \omega_\perp$, $\xi_0$ changes its trend abruptly and approaches $\xi_0^{(0,1)} = \xi_0^{(0,-1)}$ when the perpendicular level $2\hbar \omega_\perp$ crosses $\mu$. For $a = -140 \text{ nm}$, panel (b), this change is more dramatic because of a more pronounced difference between $\xi_0^{(0,0)}$ and $\xi_0^{(0,\pm 1)}$. When $|a|$ increases, $\xi_0^{(0,0)}$ and $\xi_0^{(0,\pm 1)}$ become closer to one another and, as a result, the effect is weakened. A similar weakening also occurs for upper-level resonances,
i.e., associated with the perpendicular levels $3\hbar\omega_\perp$, $4\hbar\omega_\perp$, etc. This is mostly because the inter-subband energy spacing is reduced ($\hbar\omega_\perp$ decreases) so that the difference between the subband-dependent lengths becomes less and less pronounced. It is worth noting that the above naive estimation of the longitudinal Cooper-pair size, i.e., $\propto \hbar v_F/(k_B T_c)$, is not totally irrelevant: it yields reasonable results for $\xi^{(nm)}_0$ in a subband with the bottom far below $\mu$. For instance, from Fig. II(b) one finds that $\xi^{(0,0)}_0$ reduces by about 30% when $\mu/\hbar\omega_\perp$ increases from 1.8 to 2.1, which is close to the drop of 40% following from the above simplified estimation. In addition, one can compare $\xi^{(0,0)}_0$ at $\mu/\hbar\omega_\perp = 1.8$ (below the resonance point $\mu/\hbar\omega_\perp = 2.0$) for different panels (different interaction strength) in Fig. II. As seen, this quantity scales approximately as the inverse critical temperature, which agrees with the estimate $\hbar v_F/(k_B T_c)$. Now, let us check what happens with a subband whose bottom is in the vicinity of $\mu$. Here the naive estimation of the longitudinal Cooper-pair size is no longer in agreement with our numerical results. In particular, $\xi^{(0,\pm 1)}_0$ taken at $\mu = 2\hbar\omega_\perp$ does not scale as $1/T_c$ when changing $a$ (the bottoms of the subbands with $(n,m) = (0, \pm 1)$ crosses the Fermi level at $\mu = 2\hbar\omega_\perp$). Here a reasonable agreement can be achieved when assuming a less sensitive scaling, e.g., $1/\sqrt{T_c}$. The same occurs for $\xi^{(0,\pm 2)}_0$ and $\xi^{(0,\pm 3)}_0$ at $\mu = 3\hbar\omega_\perp$ and $4\hbar\omega_\perp$, respectively. However, scaling $1/\sqrt{T_c}$ does not match at all when the bottom of a subband goes above the Fermi surface. As seen from panels (b), (d) and (f), $\xi_{nm}$’s for $(n,m) = (0, \pm 1), (0, \pm 2)$ and $(0, \pm 3)$ are close to one another for $\mu < 2\hbar\omega_\perp$ and practically do not change with $a$. It is worth noting that $\xi^{(1,0)}_0$ and $\xi^{(0,\pm 2)}_0$ are not exactly the same in spite of the fact that subbands $(0, \pm 2)$ and $(1,0)$ are degenerate. This difference is within several percent and appears due to a difference in the relevant interaction matrix elements, which results in $\psi_{1,0}(\rho, \varphi) \neq \psi_{0,\pm 2}(\rho, \varphi)$. A similar difference appears between $\xi^{(0,\pm 3)}_0$ and $\xi^{(1,\pm 1)}_0$.

To go in more detail about the subband-dependent fermionic pairing, we consider how $\psi_{nm}(z, z')$ decays with increasing $|z - z'|$ (the characteristic length for this decay is $\xi^{(nm)}_0$) for different energetic positions of the bottom of the corresponding single-particle subband with respect to $\mu$. Figures 2(a), (b) and (c) show $\psi_{nm}(0, z)$ [given in units of $\psi_{nm}(0, 0)$] as a function of $z$ for $(n,m) = (0,0), (0, \pm 1)$ and $(0, \pm 2)$, respectively (here $a = -140\text{ nm}$). Curves for $\mu/\hbar\omega_\perp = 1.8, 2.2, 2.8$ and $3.1$ are given in each panel. For all these values of $\mu/\hbar\omega_\perp$ the bottom of single-particle subband $(0,0)$ [see panel (a)] is situated far below the Fermi surface and, as a result, $\psi_{0,0}(0, z)$ exhibits features typical for loosely bound Cooper
pairs (see, e.g., Ref. 37), i.e., we observe an exponentially decaying curve with superimposed fast oscillations with period $2\pi \kappa_F^{(0,0)}$, where $k_F^{(n,m)} = \sqrt{\frac{2M}{\hbar^2} [\mu - \hbar \omega_\perp (1 + 2n + |m|)]}$ is the subband-dependent longitudinal Fermi wavevector. Unlike Fig. 2(a), Figs. 2(b) and (c) show results that significantly differ from the typical BCS behavior [except of $\mu/\hbar \omega_\perp = 2.8$ and 3.1 in panel (b), where the bottoms of the subbands with $(n, m) = (0, \pm 1)$ goes below the Fermi surface]. These results resemble the behavior of a wave function of a real bound state in a two-body problem unless the bottom of the corresponding subband goes below the Fermi level, see $\mu/\hbar \omega_\perp = 1.8$ for $(n, m) = (0, \pm 1)$ in panel (b) and $\mu/\hbar \omega_\perp = 1.8, 2.2$ and 2.8 for $(n, m) = (0, \pm 2)$ in panel (c). We note that for the particle densities in the center of the trap $n_p \approx 2 \times 10^{12} \text{ - } 5 \times 10^{12} \text{ cm}^{-3}$ (this corresponds to our choice of the chemical potential, see details below) the mean inter-particle distance $1/n_p^{1/3}$ varies in the range 0.6 - 0.8 $\mu$m. As $l|| \approx 2.5 \mu$m, than $\xi_{0}^{(0,\pm 1)}$ is less than the mean inter-particle distance for $\mu/\hbar \omega_\perp = 1.8$ [see Fig. 2(b)], and the same is true for $\xi_{0}^{(0,\pm 2)}$ when $\mu/\hbar \omega_\perp \leq 2.8$ [see Fig. 2(c)]. Hence, we observe a universal behavior (not depending on particular quantum numbers) of $\psi_{nm}(0, z)$ that exhibits a strong localization when the perpendicular level $\hbar \omega_\perp (1 + 2n + |m|)$ crosses the Fermi surface. Such a localization of the pair distribution is to a great extent similar to the behavior of the fermionic pair-wave function at the ordinary BCS-BEC crossover driven by changing the pair interaction strength. What is the reason for the longitudinal localization of the pair distribution in our case? When the bottom of a single-particle subband is situated far below the Fermi level, the ratio of the interaction energy to the longitudinal kinetic energy in such a subband is small. As a result, we obtain the ordinary BCS picture for the longitudinal distribution of fermions in a condensed fermionic pair. However, when the bottom approaches the Fermi level, the longitudinal Fermi motion is depleted, resulting in a significant decrease of the ratio of the longitudinal kinetic energy to the interaction energy and, as a consequence, in the longitudinal squeezing of a condensed fermionic pair. The depletion of the longitudinal Fermi motion in a subband with the bottom close to $\mu$ can be seen from Figs. 2(b): the period of oscillations in $\psi_{0,\pm 1}(0, z)$ decreases when passing from $\mu = 2.2 \hbar \omega_\perp$ to $3.1 \hbar \omega_\perp$. Here the question can arise if a subband with the bottom above the Fermi level can contribute to the pair condensate. For an ideal Fermi gas such a subband is not populated. However, this is not the case in the presence of superfluid/superconducting correlations that smoothen the Fermi surface. If the energy spacing between the bottom of a subband and the Fermi surface is about or less than the relevant pairing gap, this subband
still contributes. In the opposite case its role is diminished and becomes negligible with an increase of the above spacing. For instance, as seen from Figs. 1(b), (d) and (f) the subbands with \((n, m) = (0, \pm 1)\) make a significant contribution to the coherent properties even at \(\mu = 1.9\hbar\omega_\perp\). However, their contribution becomes negligible for \(\mu < 1.8\hbar\omega_\perp\), where the total longitudinal length \(\xi_0\) approaches \(\xi_0^{(0,0)}\). It is also of importance to note that the ratio of the interaction energy to the total kinetic energy is not affected by the perpendicular quantization, i.e., it remains small enough even when the bottom of one of the single-particle subbands approaches the Fermi surface. The effect of interest is due to a redistribution of the kinetic energy between the parallel and perpendicular degrees of freedom in the subband whose bottom approaches \(\mu\).

In the previous paragraphs we considered the impact of the multi-subband structure on the off-diagonal longitudinal behavior of \(\Psi(r, r')\). It is of interest to check what happens with the diagonal function \(\Psi(r, r)\) that controls the order parameter \(\Delta(r) = g\Psi(r, r)\) and, so, the spatial distribution of the fermionic condensate. In Fig. 3 the contour plots of \(\Delta(r, z)\) (calculated for \(T = 0\) and given in units of \(\mu\)) are shown together with the partial contributions of the relevant subbands at \(a = -140\) nm for \(\mu/\hbar\omega_\perp = 1.9, 2.1\) and 2.8. The left contour plot of each panel shows the order parameter whereas the two right contour plots depicts the partial contributions to \(\Delta(r)\) of the subbands with \((n, m) = (0, 0)\) [lower] and \((n, m) = (0, \pm 1)\) [upper]. At \(\mu/\hbar\omega_\perp = 1.8\), see panel (a), the quantum-size resonance associated with the two degenerating subbands \((0, \pm 1)\) begins to develop. So, the longitudinal distribution of the order parameter is mainly determined by the states with \((n, m) = (0, 0)\). As a result, the distribution of the condensate distribution is rather extended in the \(z\) direction, i.e., approximately from \(-11l_\parallel\) to \(11l_\parallel\). Two points of enhancement of the order parameter can be seen next to the left and right edges of the longitudinal condensate distribution: at \(z/l_\parallel = -9.3\) and 9.3, respectively. These local enhancements are signatures of the BCS character of the pairing in the quantum channel \((0, 0)\): their \(z\)-coordinates are solutions of the equation \(M\omega_\perp z^2/2 = \hbar^2(k_F^{(0,0)})^2/2M\). Unlike \((n, m) = (0, 0)\), subbands with \((0, \pm 1)\) produce a contribution localized around \(z = 0\), which is in agreement with the quasi-molecule character of the fermionic pairing in these subbands. In panel (a) such a contribution does not have a significant effect on the longitudinal distribution of \(\Delta(r)\). However, when the bottoms of subbands \((0, \pm 1)\) cross \(\mu\), the corresponding shape resonance develops and the condensate distribution acquires a clear bimodal character, as seen from panel (b). Here the main con-
tribution to the pairing correlations comes from the states with \((n,m) = (0, \pm 1)\), which gives rise to a significant enhancement of the order parameter around \(z = 0\), i.e. from \(z = -3l|\) to \(3l|\). Such a localization along the \(z\) direction disappears when the perpendicular level \(2\hbar\omega_\perp\) goes significantly below \(\mu\), as seen from Fig. 3(c). Here we arrive at the BCS picture of the fermionic pairing in subbands \((0, \pm 1)\), similar to that of \((n,m) = (0, 0)\).

Next we discuss important physical characteristics such as \(T_c/T_F\) and \(k_F|a|\) that are directly related to the problem of cooling the Fermi gas\(^{22}\). Typical values of \(k_F|a|\) reached at the BCS side of the Feshbach resonance are down to \(\approx 1\) whereas the corresponding values of \(T_c/T_F\) fall into the interval 0.05 - 0.2 (see Ref. 22). To check if our choice of \(\mu\) is consistent with these experimental results, we need to estimate \(k_F\), the Fermi wavevector in the center of the trap. The simplest way to estimate \(k_F\) is to extract its value from the difference \(\mu - \hbar\omega_\perp\), i.e., to use \(k_F = \sqrt{\frac{2M}{\hbar^2}(\mu - \hbar\omega_\perp)}\). Such an estimate tells us that \(k_F|a|\) varies from 0.56 to 0.7 at \(a = -140\) nm when \(\mu/\hbar\omega_\perp\) increases from 2 to 4. At the same time, as follows from Fig. 1(a), \(T_c/T_F\) \(\left(T_F = \hbar^2k_F^2/2M\right)\) varies from 0.06 down to 0.03 (see also Fig. 2 in Supplementary Information). For \(a = -210\) nm we have \(k_F|a| = 0.84\) and 1.05 at \(\mu/\hbar\omega_\perp = 2\) and 4, respectively. Here \(T_c/T_F\) goes from 0.14 down to 0.07 (Supplementary Information, Fig. 2). A more accurate estimate of \(k_F\) can be found when calculating the position dependent particle density \(n_p(\rho, z)\) (see the relevant formulas in Supplementary Information). Contour plots of \(n_p(\rho, z)\) [calculated at zero temperature for \(a = -140\) nm and given in units of \(10^{12} \text{cm}^{-3}\)] are shown in Fig. 4 for \(\mu/\hbar\omega_\perp = 1.7\) (a), 2.1 (b), 2.8 (c), 3.1 (d) and 3.7 (e). In general, from Fig. 4 one can extract particle densities close to those of the simplified estimate. Indeed, based on \(k_F = \sqrt{\frac{2M}{\hbar^2}(\mu - \hbar\omega_\perp)}\), one finds that the mean particle density is about \(2 \times 10^{12} \text{cm}^{-3}\) at \(\mu/\hbar\omega_\perp = 2\) and increases to \(4 \times 10^{12} \text{cm}^{-3}\) when \(\hbar\omega_\perp\) approaches 4. From Fig. 4 it follows that the mean density in the center of the trap is about \(4 - 5 \times 10^{12} \text{cm}^{-3}\). However, it does not show any systematic increase with decreasing \(\omega_\perp\). As follows from Fig. 4, \(n_p(\rho, z)\) in the center of the trap oscillates with changing \(\omega_\perp\). Such oscillations are due to an interplay of the relevant single-particle subbands whose number changes by one each time when \(\mu/\hbar\omega_\perp\) approaches an integer. Now, taking \(n_p = 4.5 \times 10^{12} \text{cm}^{-3}\) in the center of the trap and using \(a = -210\) nm, we obtain that \(k_F|a| = 1.07\) and \(T_c/T_F\) varies between 0.07 and 0.09 (see Fig. 3, Supplementary Information). Thus, both ways of estimating \(k_F|a|\) and \(T_c/T_F\) give experimentally attainable values. We remark that typical densities of ultracold trapped alkali-metal gases are about
$10^{12} - 10^{15}$ cm$^{-3}$ (see Ref. 22). We would also like to note that the quantum-size oscillations of the coherence length are much more pronounced for smaller couplings, i.e., for $k_F|a| \ll 1$. However, for such couplings the ratio $T_c/T_F$ becomes significantly below its experimentally attainable lower bounds 0.05 - 0.2.

We also note that small quantum-size oscillations in $n_p(\rho, z)$ are both qualitatively and quantitatively different from those observed in the fermionic condensate and, so, can in no way explain the qualitative changes in the pairing correlations discussed above. Although the common origin of oscillations in both single- and many-body characteristics is the formation of the subband structure, the superfluid quantum-size oscillations cannot be understood by looking at the single-particle density $n_p(\rho, z)$. For example, when the system goes through the first resonance, i.e., $1.7 < \mu/\hbar\omega_\perp < 2.8$, the single-particle density in the center of the trap changes by about 20%, see Fig. 4. Based on the homogeneous BCS theory, one can estimate that the corresponding change in the BCS coherence length is only about 7%, which is far smaller than observed in our calculations for $a = -140$ nm.

Another interesting quantity to discuss is the product $k_F\xi_0$ that is used to check the evolution of the fermionic condensate at the BCS-BEC crossover. According to the paper 40, the intermediate region of a BCS-BEC crossover is approached when $1/\pi < k_F\xi_0 < 2\pi$. This criterion corresponds, for a uniform 3D system, to the domain $-1 < (k_Fa)^{-1} < 1$, as discussed in Ref. 23. The upper boundary can be rewritten as $\xi_0 = \lambda_F$. When $\xi_0 < \lambda_F$, a condensed fermionic pair has an average size of the order of or less than the interparticle spacing. Such a pair acquires a local character: it weakly, or even does not, overlap with other fermionic pairs. The lower boundary $k_F\xi_0 = 1/\pi$ sets the BEC regime (i.e., for $k_F\xi_0 < 1/\pi$) with the formation of stable point-like molecules. Let us now consider how the product $k_F\xi_0$ changes with $\mu/\hbar\omega_\perp$ in a quasi-1D fermionic condensate. The most interesting is the case of $a = -140$ nm. Here $k_F\xi_0$ is well above $2\pi$ except for the points of the size-dependent drops of $\xi_0$, where this product reaches values between 4 and 6 (see Supplementary Information). It means that each time when the bottom of a single-particle subband crosses $\mu$, we approach the intermediate region of our BCS-BEC-like crossover driven by quantum-size effects. As for the subband-dependent length $\xi_0^{(nm)}$, it is always well below $2\pi$ in the vicinity of the corresponding quantum-size resonance. For instance, $k_F\xi_0^{(0,\pm1)}$ (taken at $\mu/\hbar\omega_\perp = 2$) is about 2.0 - 2.5 (see Supplementary Information). This is of importance because, e.g., for $\mu/\hbar\omega_\perp = 2$ about 70 - 75% of the condensed pairs come from the subbands with $(n, m) = (0, \pm1)$,
as already mentioned above. Note that for the subband-dependent coherence length it looks more natural to check the product $k_F^{(n,m)}\xi_0^{(nm)}$ that gives significantly smaller values as compared to $k_F\xi_0^{(nm)}$. However, in the vicinity of a quantum-size resonance, $k_F^{(nm)}$ associated with the resonant subband significantly underestimates the mean longitudinal single-particle momentum due to the neglect of the contribution from pairing correlations. Moreover, $k_F^{(nm)}$ is not defined when the bottom of a subband is positioned above $\mu$. So, a proper generalization of the criterion $1/\pi < k_F\xi_0 < 2\pi$ for the subband-dependent BCS-BEC crossover can be of interest.

Several remarks are in order about the ultraviolet regularization used in our calculations. As mentioned above, to arrive at an ultraviolet finite approach, we adopt the regularization by subtracting $1/(2T_{nmj})$ in Eqs. (1) (taken with $k \rightarrow \nu = \{n, m, j\}$), (4) and (5), which is a simple and straightforward extension of the regularization for homogeneous superfluid systems to our spatially nonuniform case. In fact, our procedure is a simplified version of a comprehensive scheme for spatially nonuniform systems suggested and investigated in Ref. 41. Such a simplification is very similar to the ultraviolet regularization used in Ref. 43. We stress that our basic conclusions are not sensitive to the specific form of the ultraviolet regularization. In particular, almost the same results are obtained with a simple cut-off procedure, i.e., the term $-1/(2T_{nmj})$ is removed from Eqs. (1), (4) and (5) and the summation in the relevant expressions runs over the single-particle states with $|T_{nmj} - \mu| < \mu$, see Fig. 4 in Supplementary Information. A similar ultraviolet cut-off is often used in papers dealing with superfluidity in nuclei and uniform atomic condensates, see, e.g., refs. 25, 42, 44.

Concluding this section, we note that the longitudinal shrinking of condensed fermionic pairs driven by quantum-size effects in a cigar-shaped fermionic condensate can be investigated experimentally via rf-spectroscopy. Another possibility is to exploit the presence of a coherent superposition of the subband-dependent fermionic condensates. When a shape resonance develops, one of them has a molecule-like character and the others are of the BCS type. Such fermionic condensates will evolve in a different manner when switching off the longitudinal confinement. Supplemented by measurements of the quantum-size oscillations of the basic quantities, i.e., $T_c$ and relevant pairing gaps (the gaps can be probed by the rf-spectroscopy), such experiments could give useful information about the quantum-size driven reconstruction of the fermionic pairing in a confined condensate.
II. QUANTUM SUPERCONDUCTING NANOWIRES

The effect in question is universal and can also be expected for high-quality superconducting nanowires, when disorder is small enough and, so, scattering on imperfections do not shadow the formation of single-electron subbands due to the transverse quantization of electron motion. For conventional superconducting materials, e.g., Al or Sn, the bulk (zero-temperature) excitation gap $\Delta_{\text{bulk}}$ varies from $45 \sim 0.1 \text{ to } 1.0 \text{ meV}$ and, so, the inter-subband energy spacing $\delta \approx \frac{h^2}{2m_e} \pi^2 \frac{d^2}{2}$ (with $d$ the nanowire diameter) is of the same order or larger when $d < 20 - 40 \text{ nm}$. For these diameters quantum-size effects on the superconducting properties become of importance. In particular, recently found width-dependent systematic shift of $T_c$ to upper values in superconducting aluminum/tin nanowires\textsuperscript{17,19,20} have been attributed to these quantum-size effects\textsuperscript{16}. We remark that at present the narrowest fabricated superconducting nanowires are aluminum samples with width down to 8 - 11 nm (see Ref. 20).

All the formulas for the superconducting condensate in a quantum wire are similar to those of the previous section. However the relevant single-electron wave functions will be different. The electron motion is not quantized in the $z$ direction (we deal with periodic boundary conditions with unit cell of length $L$) and, so, for $\psi_i(z, z') = \psi_i(z - z')$ we have (at $T = 0$)

$$\psi_i(z - z') = \frac{1}{L} \sum_k \frac{\Delta_i}{2E_{ik}} e^{ik(z-z')} = \frac{1}{\pi} \int_{k_1}^{k_2} \frac{dk}{2E_{ik}} e^{ik(z-z')},$$

(7)

where $i$ stands for the set of quantum numbers controlling the motion perpendicular to the nanowire; $\Delta_i$ (chosen real) is the subband pairing gap that does not depend on the longitudinal wavevector $k$ (due to periodic boundary conditions in the $z$-direction); and $E_i = \sqrt{\varepsilon_{ik}^2 + \Delta_i^2}$, where $\varepsilon_{ik}$ is the single-electron energy measured from the chemical potential, i.e., $\varepsilon_{ik} = \varepsilon_i + \frac{h^2k^2}{2m_e} - \mu$ with $\varepsilon_i$ the discrete perpendicular single-electron level. As seen from Eq. (7), there is no regularization term similar to that of Eq. (5). But instead, the sum in Eq. (7) runs only over the single-particle states in the Debye window, i.e., with $|\varepsilon_{ik}| < h\omega_D$, where $\omega_D$ is the Debye frequency appearing in the problem due to the phonon mediated attraction between electrons. This is why the lower $k_1$ and upper $k_2$ cut-off momenta appear in the integral in Eq. (7). For $k = k_1$, $\varepsilon_{ik}$ crosses the lower boundary of the Debye window, i.e., $k_1$ is a nonnegative solution of $\varepsilon_{ik} = -h\omega_D$. This is when the subband bottom $\varepsilon_i$ is
situated below $\mu - \hbar \omega_D$. When the subband bottom goes above $\mu - \hbar \omega_D$, $k_1$ is set to zero. In turn, $k_2$ is a nonnegative solution of $\varepsilon_{ik} = \hbar \omega_D$ provided that the subband bottom is below the upper boundary of the Debye window. When $\varepsilon_i > \mu + \hbar \omega_D$, we get $k_{\text{up}} = 0$ and, so, such a subband does not contribute to the superconducting characteristics. A major contribution to the integral in Eq. (7) comes from the states in the vicinity of the minimum of $|\varepsilon_{ik}|$ so that in most interesting cases one can simply put $k_1 = 0$ and $k_2 \to +\infty$ (the resulting integral is perfectly convergent). It means that the presence of the cut-off does not significantly influence the longitudinal properties of a quasi-1D superconducting condensate and, in turn, our conclusions. Based on Eq. (7), one can calculate $\Psi(r, r')$ from Eq. (4), where $n,m$ are replaced by $i$ and $\vartheta_i$ is no longer the eigenfunction of the 2D isotropic harmonic oscillator but the single-electron wave function corresponding to the transverse quantum number $i$.

When inserting $k_1 = 0$ and $k_2 = +\infty$ into the integral in Eq. (7), the decay length of $\psi_{nm}(z - z')$ can be calculated analytically by using the contour integration in the complex plane (this length is proportional to the longitudinal subband-dependent coherence length $\xi_0^{(i)}$). The integrand in Eq. (7) has four singular points (the square-root branch points) with the same absolute value for the imaginary part. For, say, positive $z - z'$, the contour should be closed in the upper half plane and distorted to encircle the cut between the two upper singular points having the same imaginary part $[m_e(\sqrt{\mu_i^2 + \Delta_i^2} - \mu_i)]^{1/2}/\hbar$, where $\mu_i = \mu - \varepsilon_i$ is the chemical potential measured from the subband bottom. This imaginary part controls the decay of $\psi_i(z - z')$ with increasing $z - z'$, so that we obtain

$$\xi_0^{(i)} \propto \frac{\hbar}{\sqrt{m_e}} \left[ \sqrt{\mu_i^2 + \Delta_i^2} - \mu_i \right]^{-1/2}. \quad (8)$$

When $\mu_i/\Delta_i \gg 1$, i.e., the ratio of the longitudinal kinetic energy to the interaction energy in the corresponding subband is large enough, we arrive at the conventional result for the BCS coherence length, i.e., $\xi_0^{(i)} \propto \hbar v_i/\Delta_i$, with $v_i = \sqrt{2\mu_i/m_e}$ the longitudinal subband-dependent Fermi velocity. This is for a subband with the bottom far below the Fermi surface so that $v_i \approx v_F$, with $v_F = \sqrt{2\mu/m_e}$. As $\Delta_i \propto T_c$, we get $\xi_0^{(i)} \propto \hbar v_F/T_c$, which is in agreement with our results for $\xi_0^{(0,0)}$ of a cigar-shaped fermionic condensate in Fig. II. At the point of a quantum-size superconducting resonance, when $\mu_i \to 0$, Eq. (8) reduces to a completely different expression, i.e., $\xi_0^{(i)} \approx \hbar/(m_e \Delta_i)^{1/2}$. This is also fully consistent with our numerical results given in Fig. II (see the discussion in the previous section about
for \((n, m) = (1, 0), (2, 0)\) and \((3, 0)\) at points \(\mu/h\omega_\perp = 2, 3\) and 4, respectively). When \(\mu_i < 0\) and \(|\mu_i| \gg \Delta_i\), one obtains \(\xi_0^{(i)} \propto \hbar/(2m_e|\mu_i|)^{1/2}\). Here the dependence of \(\xi_0^{(i)}\) on the subband pairing gap \(\Delta_i\) (and, so, on \(T_c\)) disappears entirely. The same was found in the previous section for \(\xi_0^{(3,0)}\) for \(\mu/h\omega_\perp < 3.5\), see Fig. 1.

The simple analytical structure of Eq. (8) highlights a strong similarity of our results with the BCS-BEC crossover driven by a change of the strength of the pair interaction: \(\mu_i\) and \(\Delta_i\) in Eq. (8) can simply be replaced by the chemical potential \(\mu\) and the gap \(\Delta\) to reproduce the evolution of the size of the condensed fermionic pairs through the region of the crossover driven by the Feshbach resonance (see, e.g., Ref. 22). However, there is also an important difference: in our case a trend similar to a BCS-BEC crossover is found for the resonant single-particle subband whose bottom is close to the Fermi surface. Other relevant subbands remain in the ordinary BCS regime. Thus, as already mentioned above, at the point of a quantum-size resonance the pair condensate is governed by a coherent superposition of the quasi-molecule quantum channel with a set of ordinary BCS channels. Resonant channels significantly contribute to the condensation energy. For example, this is about 70\% and 50\% for a superfluid cigar-shaped Fermi gas at \(\mu/h\omega_\perp = 2\) and 3, respectively (see the previous section). For superconducting cylindrical nanowires with diameters less than 8 – 10 nm the contribution of the resonant subbands at a quantum-size superconducting resonance is typically about 60 – 70\%. This makes it possible to conclude that sufficiently narrow superconducting nanowires at the resonant points are mainly governed by the quasi-molecule channel of the fermionic condensate. As an illustration, Figs. 5(a) and (b) show contour plots of the off-diagonal superconducting order parameter \(g\Psi(\rho, \varphi, z; \rho, \varphi, z')\) (the transverse coordinates of two electrons are taken the same) for a cylindrical superconducting nanowire at diameters \(d = 3.85\) nm [panel (a) is for the case of a quantum-size superconducting resonance] and \(d = 4.08\) nm [panel (b) is for the non-resonant case]. Such a representation is very convenient because it provides us with information of not only the longitudinal correlations but, in addition, on the (diagonal) order parameter \(\Delta(\rho) = g\Psi(\rho, \rho)\). Unlike the previous case of the cigar-shaped trap, the order parameter does not depend on \(z\) due to periodic boundary conditions in the \(z\) direction. As seen from Figs. 5(a) and (b), the longitudinal distribution of electrons in a condensed pair significantly shrinks at \(d = 3.85\) nm as compared to \(d = 4.08\) nm. This is because at \(d = 3.85\) nm the bottoms of the two single-electron subbands with the quantum numbers \(i = (n, m) = (5, \pm 1)\) and \((1, \pm 11)\) are
situated in the vicinity of the Fermi surface (due to the cylindrical shape the relevant transverse quantum numbers are again $n$ and $m$, the radial and azimuthal numbers, respectively). The corresponding quantum-size superconducting resonance is responsible for a significant enhancement of the order parameter $\Delta(\rho)$ as compared to its bulk value (for our parameters it is $\Delta_B = 0.25\,\text{nm}$, see Supplementary Information) and for the main contribution of these subbands to its averaged value (about 70%). When increasing the diameter, these subbands shift down in energy, the corresponding resonance decays and, so, one can see a rather extended distribution along the $z$ direction at $d = 4.08\,\text{nm}$, see panel (b). Fig. (5)(c) demonstrates the longitudinal-decay length of $\Delta(\rho, z - z')$ calculated from our results by a numerical exponential fit for $\rho/R = 0.18$. As seen, the variation of the length (it is proportional to the longitudinal pair size $\xi_0$) for diameters from $d = 3.85\,\text{nm}$ to $4.08\,\text{nm}$ is two-three orders of magnitude. It is significantly more pronounced than the drops in $\xi_0$ found in the previous section for an ultracold quasi-1D Fermi gas. The reason is as follows. From our analytical results it follows that the resonance-driven decrease of the longitudinal coherence length is governed by a factor $\approx \hbar k_F/\sqrt{m\Delta}$, where $\Delta$ is the typical energy gap; $m$ is the relevant particle mass (e.g., $m = m_e$ for electrons and $m = M$ for lithium atoms); and $k_F$ can be estimated as $\sqrt{2M/(\mu - \hbar \omega_\perp)}$ for a cigar-shaped Fermi gas whereas for a metallic nanowire we have $k_F = \sqrt{2m_e/\hbar^2}$. Then, one can find that the reason for the significantly more pronounced drop of the longitudinal BCS coherence length in the superconducting nanowire is the value of $k_F$ which is more than three orders of magnitude larger in metals. The effect of an increase of the typical excitation gap in metals by seven orders of magnitude as compared to a trapped atomic condensate is strongly weakened by the $10^4$-drop of the particle mass and, in addition, by the presence of the product $m\Delta$ under the square root. For instance, one can simply compare $\xi_0^{(i)} = \hbar v_i/\Delta_i$ with $\xi_0^{(i)} = \hbar/(m_e\Delta_i)^{1/2}$ for typical metallic parameters of weak-coupling superconductors: $v_i \sim 1 \times 10^6 - 2 \times 10^6\,\text{m/s}$ and $\Delta_i \approx 0.2\,\text{meV}$ (see,e.g., Ref. 45). This allows one to find that $\xi_0^{(i)}$ drops at a resonant point by two-three orders of magnitude, from the micron to the nanometer scale, i.e., down to $\xi_0^{(i)} \sim 20\,\text{nm}$. Then, taking account of the fact that the diagonal order parameter $\Delta(\rho) = \Delta(\rho, z - z')|_{z - z'} = 0$ is 3 – 4 times enhanced as compared to bulk at $d = 3.85\,\text{nm}$, one can eventually obtain a drop shown in Fig. (5)(c). We note that when moving below diameter 3.85 nm, the longitudinal decay-length first increases with decreasing $d$ but then, when reaching $d = 3.82\,\text{nm}$, it begins to drop. This is the effect of
another quantum-size resonance situated at $d = 3.72 \text{ nm}$ (see Ref. 32). For diameters larger than 3.85 nm, the next quantum-size resonance develops at $d = 4.2 \text{ nm}$.

We would like to remark that the above results for superconducting nanowires were calculated in the clean limit. In the presence of surface imperfections and disorder of real metallic nanowires the quantum-size oscillations of the longitudinal coherence length will be smoothed. However, such giant drops of $\xi_0$ can hardly be completely washed out (see discussion in Supplementary Information). Yet, it will be difficult to observe details of oscillations of the BCS coherence length due to fluctuations of the confinement dimensions of real samples. To capture the details, it is more promising to use materials with lower charge-carrier densities (e.g., semimetals and doped semiconductors). Such materials have larger $\lambda_F$ that controls the number of the occupied transverse modes and the width of the superconducting resonances (this width can be defined as a maximal variation of the nanowire diameter with no significant effect on a resonant enhancement). We can expect that for $\lambda_F \gg 1 \text{ nm}$ smoothing of quantum-size oscillations due to width fluctuations will be less significant.

III. CONCLUSION

Since the classical paper by Cooper\textsuperscript{48} it is well-known that the configuration of the phase space available for scattering of time-reversed fermions plays a crucial role for the formation of condensed fermionic pairs. Indeed, only a strong enough attractive interaction between fermions with opposite spin in 3D is able to produce a two-body bound state in vacuum. However, when scattering of fermions is influenced by the exclusion of a filled Fermi sea, i.e., the available phase space is restricted by exclusion of the single-particle states inside the Fermi sea, we arrive at the Cooper instability resulting in the formation of weakly bound in-medium pairs of fermions for arbitrary strength of the attractive interaction. Restricting the phase space by removing the filled Fermi sea, one actually removes long range contributions to the Cooper-pair wave function, which, say, “encourage” fermions to form in-medium bound states. Our present results show that the additional reconfiguration of the phase space due to quantum confinement, such that the band of single-particle states splits up into a series of subbands, can further modify the scenario of pairing. The formation of multiple subbands due to quantum confinement results in a significant redistribution of the
kinetic energy between confined and unconfined directions. In particular, for a quasi-1D fermionic condensate the subband-dependent ratio of the longitudinal kinetic energy to the interaction energy drops down to almost zero when the bottom of a subband crosses the Fermi level (this is rather similar to superconducting semimetals with the chemical potential driven by superconducting correlations below the bottom of the conduction band\cite{47}). The longitudinal Fermi motion of fermions in such a subband is depleted, which results in a drop of the longitudinal size of condensed fermionic pairs. So, each time when the bottom of a single-particle subband passes through the Fermi level, a quantum-size resonance develops and the superfluid/superconducting system exhibits trends similar to the well-known BCS-BEC crossover driven through the Feshbach resonance in ultracold Fermi gases in 3D traps (see, e.g., Ref.\cite{22}). In this case condensed fermionic pairs behave, to a great extent, similar to condensed boson-molecules. For instance, this is reflected in a clear bimodal spatial distribution of a harmonically trapped quasi-1D fermionic condensate. A contribution governed by the quasi-molecule channel associated with the subband whose bottom is situated in the vicinity of the Fermi surface is strongly localized in the center of a trap. Other quantum channels are due to single-particle subbands with bottoms far below the Fermi surface. They are in the BCS regime and yield an extended longitudinal distribution of the fermionic condensate typical of Cooper pairs. When the number of relevant single-particle subbands increases, the role of quantum-size effects is diminished and, finally, all related phenomena disappear. One can expect that the impact of the subband formation on superfluid/superconducting properties is of importance when the inter-subband energy spacing is larger than or close to $\approx k_B T_c$. We remark that similar physics can be expected for quasi-2D fermionic condensates.

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Supplementary information

Appendix A: Quasi-1D superfluid Fermi gas

Our investigation is focused on the quasi-1D Fermi gas, where the single-particle energies $T_{nmj}$ (with $j$ corresponding to the longitudinal motion) with the same $n$ and $m$ form a quasi-continuum. So, it is convenient to introduce single-particle subbands $(n,m)$ as seen from Fig. 6. In panel (a) we show $T_{nmj}$ versus the quantum number $j$ for $\mu/\hbar\omega_\perp = 2$. In this case the bottom of subband $(0,0)$ is located far below the chemical potential $\mu$ whereas the bottom of two degenerate subbands $(0,\pm 1)$ almost touches $\mu$ (it is higher only by $\hbar\omega_\parallel/2 = 0.005\mu$, where we take $\mu = \hbar \cdot 24$ kHz, see the discussion about the chosen parameters in the article). When decreasing $\omega_\perp$ (and keeping $\mu$ the same), more and more new subbands crosses the chemical potential. In particular, Fig. 6(b) illustrates the case of $\mu/\hbar\omega_\perp = 3$. Here the bottoms of subbands $(0,0)$ and $(0,\pm 1)$ are below $\mu$. The bottoms of the three degenerate subbands with $(n,m) = (0,\pm 2)$ and $(1,0)$ almost touch $\mu$, which results in the appearance of a new quantum-size resonance.

When discussing in the article important physical parameters of the superfluid fermionic condensate such as $T_c/T_F$, $k_F|a|$, and $k_F\xi_0$, we invoked two different ways to estimate $k_F$ in the center of the trap. The simple one involves the chemical potential $\mu$ measured from the lowest single-particle level $\hbar\omega_\perp + \frac{1}{2}\hbar\omega_\parallel \approx \hbar\omega_\perp$ and is based on the relation $k_F = \sqrt{\frac{2M}{\hbar^2}(\mu - \hbar\omega_\perp)}$. This estimation is good enough when a trap is almost isotropic, i.e., the trapping frequencies are close to each other in all directions. For the quasi-1D case, when $\omega_\perp \gg \omega_\parallel$, the reliability of the above estimate can suffer from the anisotropy in the distribution of the kinetic energy in single-particle subbands. For instance, when the subband bottom is situated in the vicinity of $\mu$, the longitudinal kinetic energy is suppressed as compared to the kinetic energy of the motion perpendicular to the system. While the subband bottom goes far below $\mu$, the above anisotropy disappears in favor of another: now
the longitudinal motion prevails. So, the resulting values of \( k_F \) and \( T_F = \hbar^2 k_F^2 / 2m_e \) will be dependent on whether \( k_F \) is estimated from the longitudinal kinetic energy or it is extracted from the energy of the perpendicular motion. Another issue of uncertainties is the choice of the relevant subband to extract \( k_F \). Notice that the above simple estimate is based on the longitudinal kinetic energy of the lowest single-electron subband.

A more accurate (but more time consuming) procedure to estimate the relevant \( k_F \) in the center of the trap is to calculate the spatial particle distribution \( n_e(\mathbf{r}) \) and, then, to find the mean particle density in the center of the trap. In the mean-field approximation the quantity \( n_e(\mathbf{r}) \) is calculated through the coherence factors \( u_\nu(\mathbf{r}) \) and \( v_\nu(\mathbf{r}) \), see Eq. (3) in the article, i.e.,

\[
\begin{align*}
 n_p(\mathbf{r}) &= 2 \sum_\nu \left[ |v_\nu(\mathbf{r})|^2 (1 - f_\nu) + |u_\nu(\mathbf{r})|^2 f_\nu \right], \quad (A1)
\end{align*}
\]

with \( f_\nu = 1/(e^{\beta E_\nu} + 1) \), the Fermi distribution of the bogolons. We note that for a trap with axial symmetry the density of particles depends only on the radial coordinate \( \rho \) and the longitudinal coordinate \( z \). Based on Eq. (3) of the article, one can rewrite Eq. (A1) as (at \( T = 0 \))

\[
\begin{align*}
 n_p(\rho, z) &= \sum_{nmj} \left( 1 - \frac{T_{nmj}}{E_{nmj}} - \mu \right) |\vartheta_{nm}(\rho, \varphi)|^2 \chi_j^2(z), \quad (A2)
\end{align*}
\]

where \( |\vartheta_{nm}(\rho, \varphi)| \) does not depend on the azimuthal angle \( \varphi \). We note that no regularization similar to subtracting the term \( 1/2T_{nmj} \) in Eq. (5) of the article is now needed. The sum in Eq. (A2) is convergent and, in addition, the regularization used in Eq. (5) of the article assumes the explicit presence of the coupling constant, which is not the case for the position dependent particle density \( n_p(\mathbf{r}) \), see Eq. (A1). A numerical solution of the self-consistency equation (1) of the article provides us with a set of \( \Delta_\nu \)'s that should be inserted into the above relation for \( n_p(\rho, z) \). Our numerical results for \( n_p(\rho, z) \) are shown in Fig. 4 of the article as contour plots. When checking the mean particle density in the center of the trap, one can find that it varies from \( 4 \times 10^{12} \) to \( 5 \times 10^{12} \, \text{cm}^{-3} \). So, when taking \( n_p = 4.5 \times 10^{12} \, \text{cm}^{-3} \), we can find \( k_F = (3\pi^2 n_p)^{1/3} = 5.1 \, \mu \text{m}^{-1} \) and \( \lambda_F = 2\pi/k_F = 1.23 \, \mu \text{m} \). This can be compared with \( k_F = \sqrt{2M/(\mu - \hbar \omega_\perp)} = 3.99 \, \mu \text{m}^{-1} \) and \( 4.88 \, \mu \text{m}^{-1} \) at \( \mu/\hbar \omega_\perp = 2 \) and 4, respectively. Both estimates produce close values for \( k_F \) in the center of the trap but we obtain different trends for the mean density in the center of the trap. According to the simple estimate, \( n_p \) plainly increases with \( \mu/\hbar \omega_\perp \). Whereas \( n_p \) found from the position dependent distribution of the particles slightly oscillates around \( 4.5 \times 10^{12} \, \text{cm}^{-3} \).
Having at our disposal possible values of $k_F$ and using our numerical results given in Fig. 1 of the article, we can find $T_c/T_F$ and $k_F\xi_0$ as shown below in Fig. 7 for the simple estimate and Fig. 8 for the estimate based on $n_e(\rho, z)$.

Concluding this section, we would like to note that our results are not sensitive to the regularization procedure used in Eq. (1) of the article (taken with $k \rightarrow \{n, m, j\}$) in order to remedy the divergent sum (the ultraviolet divergence). For illustration, Fig. 4 shows numerical results for the critical temperature found when the regularization term $1/(2T_{nmj})$ is abandoned in favor of a simple cut-off $|T_{nmj} - \mu| < \mu (\mu \gg \Delta_{nmj})$, i.e., the sum in Eq. (1) is only limited to the single-particle states with energies satisfying this inequality. Note that similar cut-off is often used in papers on ultracold atomic gases, see, e.g., Refs. 25, 41, 44. Though $T_c$ calculated with the cut-off regularization are slightly different as compared to the results given in Fig. 1 of the article, this difference practically disappears when making a small shift in the scattering length, i.e., $a \rightarrow a + 20 \text{ nm}$, see Fig. 9. Thus, a particular way of ultraviolet regularization does not influence our results on quantum-size oscillations of $T_c$, and the same holds for our conclusions about the size-dependent oscillations in the BCS coherence length.

Appendix B: Superconducting metallic nanowires: parameters used in the calculations

Figure 5 of the article shows our numerical results calculated for a cylindrical superconducting aluminum nanowire with the unit cell (periodic boundary conditions) in the $z$ direction $L_z = 50 \mu \text{m}$. The coupling constant $g$ is chosen so that $gN(0) = 0.18$, with $N(0)$ the bulk density of states at the Fermi level per unit volume and per spin projection (this value of $gN(0)$ is typical of aluminum, see, e.g., the textbook45). Effects resulting in the width dependent coupling are beyond the scope of the present investigation (such effects do not influence our main conclusions). The Debye energy is taken the same as in bulk45, i.e., $\hbar \omega_D/k_B = 375 \text{ K}$ ($\hbar \omega_D = 32.3 \text{ meV}$). The electron band mass $m_e$ is set to the free-electron mass in the calculations. The Fermi level is put to 0.9 eV. We note that this is an effective Fermi level which is used together with the parabolic band approximation to correctly reproduce the period of the quantum-size oscillations. It is a parameter that depends on the interplay between the main crystalline directions and the direction of quantum confine-
ment (for a more detail, see discussion in Ref. 32). The above value of the effective Fermi level is justified by a good agreement of theoretical and experimental results for the critical temperature in aluminum nanowires. However, this particular choice is not crucial for our conclusions. We also take into consideration a systematic shift up of the Fermi level with decreasing $d$ for $d < 5 - 6$ nm. Such a shift appears if one keeps the same value of the mean electron density when changing $d$.

1. Surface imperfections

Our consideration of the superconducting nanowires is in the clean limit and assumes mirror reflections from the boundaries. So the question arises to what extent imperfections of real metallic nanowires can influence our results. We focus on recently fabricated high-quality superconducting nanowires, where the superconducting state survives down to diameters of about $8 - 10$ nm without any signatures of suppression of the critical temperature driven by disorder [see refs. 19, 20 about Al nanowires made of strongly coupled grains and paper 17 about Sn single-crystalline nanowires]. Unlike strongly disordered nanosized superconductors, such high-quality superconducting nanowires exhibit a systematic shift-up of the critical temperature with a reduction in their cross-section (50% in Al specimens and 10 - 20% in Sn nanowires). As recently shown 46, it is the quantization of the transverse electron spectrum that shifts $T_c$ up in such nanowires through quantum-size superconducting resonances. Thus, disorder is relatively minor in high-quality superconducting nanowires and the effects of the formation of well distinguished single-electron subbands become of importance.

To go in a more detail about a role of imperfections in quantum superconducting nanowires, we remark that the surface roughness is here a major disorder mechanism. Indeed, in most papers 19, 20, the mean free path is estimated to approximately follow the nanowire width $\ell \sim d$, i.e., elastic scattering on the boundary imperfections controls the electron mean free path. So, we need to clarify whether or not the size-dependent drops of the longitudinal BCS coherence length found in the clean limit will be significantly influenced by the surface roughness. The first thing to do with this is to compare the Cooper-pair size $\xi_0$ with the electron mean free path. Beyond a resonant point $\xi_0$ increases up to microns and, so, we are in the dirty limit, i.e., $\xi_0 \gg \ell$. However, when approaching a point of the quantum-size
superconducting resonance, $\xi_0$ drops down to values close to the nanowire diameter, see, e.g., Fig. 5 in the article (this is true for nanowires with $d < 10 - 20$ nm, for larger diameters the effect is washed out). As the mean free electron path $\ell$ is about $d$, then we obtain $\xi_0 \approx \ell$. So, the effect of disorder on the superconducting correlations is not significant: the suppression factor in Green’s functions is $e^{-r/2\ell} \sim 1$ for $r \sim \xi_0$. Thus, the elastic scattering on the surface imperfections will not significantly alter our conclusions: at points of the size-dependent drops we are close to the clean limit.

The above reasoning assumes that the density of single-electron states at the Fermi level does not change significantly due to surface imperfections. So, the next step is to clarify if the single-electron spectrum is seriously influenced by surface imperfections. A size-dependent drop of the longitudinal fermionic-pair size occurs due to a subband (subbands) whose bottom (bottoms) is situated in the vicinity of the Fermi surface (see the article). For such a subband the longitudinal motion of electrons is suppressed and the most contribution to the single-electron energy comes from the transverse spectrum of electrons. This is exactly the reason for the quantum-size driven BCS-BEC crossover. Another consequence of this redistribution of the kinetic energy between the longitudinal and transverse degrees of freedom is long longitudinal wavelengths of the electrons in a resonant subband. The longitudinal electron energy in a subband whose bottom is situated in the Debye window is about or smaller than the Debye energy. This makes it possible to find that the typical longitudinal wavenumber is smaller than $k < \sqrt{2m_0\omega_D/\hbar}$. So, this simple estimations suggests that the typical longitudinal wavelength of electrons in a resonant subband is larger than 10 nm. This is significantly larger than the characteristic size of the surface imperfections usually estimated as being of the order of dimensions of crystalline unit cell (smaller than 1 nm). Hence, the longitudinal part of the electron spectrum in a resonant subband is stable against surface imperfections. This is not the case for subbands with bottoms positioned far below the Fermi level. Here the electron spectrum can be rather sensitive to the surface roughness and the corresponding density of states can even be suppressed at the Fermi level in the presence of strong disorder. However, the contribution of such single-electron subbands at a resonant point is only of secondary importance: the major effect comes from the resonant subbands.

In addition to the influence of the surface roughness on the longitudinal electron motion, one should also keep in mind an uncertainty in the transverse energy due to fluctuations in
the nanowire diameter. This uncertainty is able to affect the formation of superconducting resonances when it approaches the Debye energy that controls the selection of the single-electron states making a contribution to the basic superconducting characteristics. The energy spacing between single-electron subbands can be estimated as $\frac{\hbar^2 \pi^2}{2m_e d^2}$. Let us take $d = d_0 + \Delta d$, where $\Delta d$ represents a fluctuation part with values from 0 to 1 nm (this a typical scale for the diameter fluctuations in nanowires with width less than 10-20 nm). The corresponding fluctuation of the inter-subband energy spacing is estimated as $\frac{\hbar^2 \pi^2}{m_e d_0^2} \Delta d$, which can be considered as an uncertainty in the transverse electron energy. For the narrowest high-quality nanowires with $d = 8 - 10$ nm, such an uncertainty is about 1 meV, which is much smaller than the Debye energy for aluminum $\hbar \omega_D \approx 32$ meV. Even for extremely small diameters about 4 nm chosen for a simple illustration in the article (to avoid a discussion of many single-particle subbands responsible for the formation of superconducting resonance at nanowire diameters $\sim 10$ nm), this uncertainty is close to 10 meV (when taking $\Delta d = 1$ nm). For aluminum this is still three times smaller than the Debye energy.

Another reason for broadening of the single-electron levels is the hybridization with electrons of a semiconductor substrate, which is strongly dependent on the fabrication conditions. However, such a hybridization can be expected to be of importance for specimens with width down to a few monolayers like in ultrathin superconducting nanofilms.

Thus, based on the above discussion we can conclude that our results are quite stable to imperfections of real superconducting nanowires. These imperfections will definitely smooth quantum-size oscillations of $\xi_0$ but will hardly avoid the effect of interest.

2. Fluctuations

We work in the mean-field approximation and, so, one more point to discuss is the effect of fluctuations. It is well-known that fluctuations generally play an important role in low-dimensional systems. In superconducting nanowires the main focus is usually on the phase fluctuations of the pair condensate: thermally-activated and quantum phase slips, see, e.g., the paper\textsuperscript{19,49}. These fluctuations lead to a residual resistance remaining below $T_c$ in narrow nanowires, corrupting the superconducting state. Below we argue that the BCS-BEC crossover driven by quantum confinement is still pronounced above the nanowire diameters, where phase fluctuations proliferate.
Effect of thermal fluctuations is usually estimated through the Ginzburg-Levanyuk parameter calculated from the conventional Ginzburg-Landau functional. However, for quantum nanowires of width $5 - 20 \text{nm}$ the conventional Ginzburg-Landau formalism can not be applied due to the breakdown of the translational invariance in the direction perpendicular to the nanowire. A consequence of such a breakdown is that the order parameter strongly varies in the direction perpendicular to the nanowire (see, e.g., Ref. [16]). The characteristic length for its spatial variations in this direction is about the nanowire width $d$, which prevents us from using the conventional Ginzburg-Landau formalism which assumes that the order parameter varies slowly on the scale governed by the zero-temperature coherence length. In addition, the 1D Ginzburg-Levanyuk parameter is not relevant either because a superconducting quantum nanowire is an essentially 3D system with multiple single-electron subbands (more than $10 - 20$ subbands even for $d = 4 \text{nm}$). Such a multichannel system can not be reduced to effectively one-dimensional case, and this is also seen from the fact that the order parameter has a nontrivial transverse profile that changes significantly with a change of $d$. To overcome this problem, one should abandon the conventional Ginzburg-Landau formalism in favor of its multichannel version. However, a significant complication here is to accurately incorporate issues related to the quantum-size driven BCS-BEC crossover (e.g., fluctuating Cooper pairs with a nonzero center-of-mass momentum). A simpler option is to rely upon available experimental results. They suggest that the Ginzburg-Levanyuk parameter is about $0.1 - 0.15$ in superconducting nanowires with diameters $\approx 10 \text{nm}$. In particular, Fig. 1 of the paper [20] (for zero magnetic field) demonstrates that the nanowire resistance falls by two orders of magnitude when temperature reduces from $T = T_c$ to $T = 0.85 T_c$. This is still a rather sharp transition with the thermal broadening of about $\delta T/T_c = 0.1 - 0.15$ and, so, the mean-field treatment looks quite reasonable.

From the same Fig. 1 in the paper [20] one can learn that quantum phase fluctuations in an aluminum nanowire with width $10 \text{nm}$ result in a residual resistance even below $T/T_c = 0.5 - 0.6$. At this nanowire width such a resistance is smaller than $10^{-4}$ in units of the normal resistance. It is however expected that for diameters $\lesssim 10 \text{nm}$ quantum-phase slips proliferate, which leads to a superconductor-to-normal crossover at $d = d_c$, with $d_c \lesssim 10 \text{nm}$ (see, e.g., papers [19, 49]). Recent results of the paper [19, 49] suggest that $d_c \approx 8 \text{nm}$. Yet, it is rather difficult to analyze experimental data for very narrow nanowires because it is not possible to completely rule out weak links as the sources of the residual resistance. Thus,
we can conclude that the best regime to probe any signatures of the BCS-BEC crossover induced by quantum confinement is to investigate the superconducting nanowires with diameters just above $d_c$. We would like to note that the effect of interest is pronounced for $d < 10 \text{ nm}$ and completely washed out only when $d > 20 \text{ nm}$. In addition, the subject of our investigation is the zero-temperature coherence length that is one of important parameters controlling the rate of quantum-phase slips, see, e.g., Ref. 20,49. This makes it possible to expect that our results can be of relevance even for $d < d_c$. 
Figure 1

FIG. 1: Superfluid Fermi gas confined in a cigar-shaped harmonic trap: the critical temperature $T_c$ (in units of $\mu$) and the BCS coherence length $\xi_0$ (in units of $l_\parallel$) versus $\mu/\hbar \omega_\perp$. Panels (a) and (b) for $a = -140$ nm; (c),(d) for $a = -180$ nm; (e) and (f) for $a = -210$ nm. The subband-dependent coherence lengths $\xi_0^{(nm)}$ for $(n,m) = (0,0), (0,\pm 1) (0,\pm 2)$ and $(0,\pm 3)$ are also given to compare with $\xi_0$. 
FIG. 2: Cigar-shaped superfluid Fermi gas: the absolute value of the subband-dependent longitudinal fermionic pair “wave function” $\psi_{nm}(0, z)/\psi(0, 0)$ (i.e., one fermion is fixed at the origin and another is at the point $z$) versus $z/l_{||}$ for $a = -140$ nm at $\mu/\hbar \omega_{\perp} = 1.8, 2.2, 2.8$ and 3.1. Panels (a), (b) and (c) show the results for $(n, m) = (0, 0), (0, \pm 1)$ and $(0, \pm 2)$, respectively.
FIG. 3: (a) Contour plots of the order parameter $\Delta(\rho, z)$ [in units of $\mu$, the left panel] together with the partial contributions to the order parameter of the subbands with $(n, m) = (0, 0)$ [the lower right panel] and $(0, \pm 1)$ [the upper right panel] for a cigar-shaped superfluid Fermi gas at $\mu/\hbar \omega_\perp = 1.9$ at $a = -140$ nm; (b) and (c) show the same but for $\mu/\hbar \omega_\perp = 2.1$ and 2.8.
FIG. 4: Contour plots of the position dependent particle density $n_p(\rho, z)$ (in units of $10^{12} \text{ cm}^{-3}$) for $\mu/\hbar\omega_\perp = 1.7$ (a), $\mu/\hbar\omega_\perp = 2.1$ (b), $\mu/\hbar\perp = 2.8$ (c), $\mu/\hbar\omega_\perp = 3.1$ (d) and $\mu/\hbar\omega_\perp = 3.7$ (e). For all the panels the scattering length is $a = -140 \text{ nm}$.
FIG. 5: Superconducting nanocylinder: (a) contour plot of $g\Psi(\rho, \varphi, z; \rho, \varphi, z')$ (given in units of the bulk gap $\Delta_B = 0.25$ meV) in the presence of the superconducting resonance at $d = 3.85$ nm; (b) the same as in the previous panel but now for the non-resonant diameter $d = 4.08$ nm; (c) the longitudinal decay length for the distribution $\Psi(\rho, \varphi, z; \rho, \varphi, z')$ (it is proportional to $\xi_0$) versus the diameter near the resonance point $d = 3.85$ nm.
FIG. 6: The single-particle energy $T_{nmj}$ versus the longitudinal quantum number $j$ for different subbands $(n,m)$ at $\mu/h\omega_{\perp} = 2$ (a) and $\mu/h\omega_{\perp} = 3$. 
FIG. 7: $T_c/T_F$ and $k_F\xi_0$ as calculated from the results given in Fig. 1 of the article on the basis of the simple estimate of $k_F = \sqrt{\frac{2M}{\hbar^2}(\mu - h\omega_\perp)}$ (with $T_F = \hbar^2 k_F^2 / 2M$). Panels (a), (b) represent $a = -140\,\text{nm}$; (c) and (d) are for $a = -180\,\text{nm}$; (e), (f) show the data for $a = -210\,\text{nm}$. The quantities $k_F\xi_0^{(nm)}$ for $(n,m) = (0,0), (0,\pm1), (0,\pm2)$ and $(0,\pm3)$ are also plotted to compare with $k_F\xi_0$. 

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FIG. 8: The same as in the previous figure but now for $k_F$ extracted from the mean particle density in the center of the trap, i.e., $n_p = 4.5 \times 10^{12} \text{ cm}^{-3}$ ($k_F = (3\pi^2 n_p)^{1/3}$).
FIG. 9: The critical temperature $T_c$ (in units of $\mu$) versus $\mu/\hbar\omega_\perp$ for a cigar-shaped superfluid Fermi gas of $^6Li$: circles are results for $a = -120$ nm found from the BCS-like self-consistency equation with the simple cut-off regularization $|T_{nmj} - \mu| < \mu$; squares represent the previous data from Fig. 1(a) of the article.