A FUCHSIAN VIEWPOINT ON THE WEAK NULL CONDITION

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ABSTRACT. We analyze systems of semilinear wave equations in 3+1 dimensions whose associated asymptotic equation admit bounded solutions for suitably small choices of initial data. Under this special case of the weak null condition, which we refer to as the bounded weak null condition, we prove the existence of solutions to these systems of wave equations on neighborhoods of spatial infinity under a small initial data assumption. Existence is established using the Fuchsian method. This method involves transforming the wave equations into a Fuchsian equation defined on a bounded spacetime region. The existence of solutions to the Fuchsian equation then follows from an application of the existence theory developed in [11]. This, in turn, yields, by construction, solutions to the original system of wave equations on a neighborhood of spatial infinity.

1. Introduction

In this article, we establish global existence results for systems of semilinear wave equations in 3+1 dimensions that satisfy a weak null condition. Specifically, the class of semilinear wave equations that we consider are of the form

\[ g^{\alpha\beta} \nabla_{\alpha} \tilde{\nabla}_{\beta} \tilde{u}^K = \hat{a}_{IJ}^{K\alpha\beta} \nabla_{\alpha} \tilde{u}^I \tilde{\nabla}_{\beta} \tilde{u}^J \]  

(1.1)

where the \( u^I \), \( 1 \leq I \leq N \), are a collection of scalar fields, the \( \hat{a}_{IJ}^K \), \( 1 \leq I, J, K \leq N \), are prescribed smooth (2,0)-tensors fields on \( \mathbb{R}^4 \), and \( \tilde{\nabla} \) is the Levi-Civita connection of the Minkowski metric \( \tilde{g} = \tilde{g}_{\alpha\beta} \tilde{d}x^\alpha \otimes \tilde{d}x^\beta \) on \( \mathbb{R}^4 \). We find it convenient to work throughout this article primarily in spherical coordinates

\[ (\tilde{x}^\mu) = (\tilde{x}^0, \tilde{x}^1, \tilde{x}^2, \tilde{x}^3) = (\tilde{t}, \tilde{r}, \theta, \phi) \]

in which the Minkowski metric is given by

\[ \tilde{g} = -d\tilde{t} \otimes d\tilde{t} + d\tilde{r} \otimes d\tilde{r} + \tilde{r}^2 \hat{g} \]  

(1.2)

where

\[ \hat{g} = d\theta \otimes d\theta + \sin^2(\theta) d\phi \otimes d\phi \]  

(1.3)

is the canonical metric on the 2-sphere \( S^2 \). For simplicity, we assume for the remainder of the article that the tensor fields \( \hat{a}_{IJ}^K \) are covariantly constant, i.e. \( \nabla \hat{a}_{IJ}^K = 0 \), which is equivalent to the condition that the components of \( \hat{a}_{IJ}^K \) in a Cartesian coordinate system (\( \tilde{x}^\mu \)) are constants, that is, \( \hat{a}_{IJ}^K = \hat{a}_{IJ}^{K\alpha\beta} \hat{\partial}_\alpha \otimes \hat{\partial}_\beta \) for some set of constant coefficients \( \hat{a}_{IJ}^{K\alpha\beta} \).

In order to define the weak null condition that we will consider in this article, we first introduce the out-going null one-form \( L = -d\tilde{t} + d\tilde{r} \) and set

\[ \tilde{b}_{IJ}^K := \hat{a}_{IJ}^{K\mu\nu} \tilde{L}_\mu \tilde{L}_\nu = \hat{a}_{IJ}^{K00} - \hat{a}_{IJ}^{K10} = \hat{a}_{IJ}^{K11} + \hat{a}_{IJ}^{K11}. \]  

(1.4)

As we show below, see (2.12), the \( \tilde{b}_{IJ}^K \) define smooth functions on \( S^2 \). We use these functions to define the asymptotic equation associated to (1.1) by

\[ (2 - t) \partial_t \xi = \frac{1}{t} Q(\xi) \]  

(1.5)

where \( \xi = (\xi^K) \) and

\[ Q(\xi) = (Q^K(\xi)) := -2\chi(\rho) \rho^m \tilde{b}_{IJ}^K \xi^I \xi^J. \]  

(1.6)

In this equation, \( t \) and \( \rho \) are coordinates that arise from a compactification of a neighborhood of spatial infinity, see Section 2.1 and equation (3.33) for details, while \( \chi(\rho) \) is a smooth cut-off function. Furthermore, the time coordinate \( t \) is chosen so that 0 < \( t \leq 1 \) and \( t = 0 \) corresponds to future null-infinity. We remark that this type of equation was first introduced by Hörmander [29, 30] to analyze the blow-up time for wave equations that do not satisfy the null condition of Klainerman [38], which in our notation is defined by the vanishing of the \( \tilde{b}_{IJ}^K \).

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1See Appendix A for our indexing conventions.

2This is certainly not necessary, and it is straightforward to verify that all the results of this article can be generalized to allow non-covariantly constant tensors \( \hat{a}_{IJ}^K \) provided that they satisfy suitable asymptotics.
The weak null condition, which was first introduced in [42], is a growth condition on solutions of the asymptotic equation, namely that solutions of (1.5) satisfy a bound of the form $\left| \xi(t) \right| \leq t^{-C} \epsilon$ for some fixed constant $C > 0$ and initial data at $t = 1$ satisfying $|\xi(1)| \leq \epsilon \leq \epsilon_0$ for $\epsilon_0 > 0$ sufficiently small. It is still an open conjecture, even in the semilinear setting, to determine if the weak null condition is enough to ensure the global existence of solutions under a suitable small initial data assumption. In this article, we will consider the following restricted weak null condition, which includes the classical null condition as a special case:

**Definition 1.1.** The asymptotic equation is said to satisfy the bounded weak null condition if there exist constants $R_0 > 0$ and $C > 0$ such that solutions of the asymptotic initial value problem (IVP)

$$
(2 - t)\partial_t \xi = \frac{1}{t} Q(\xi),
$$

$$
\left| \xi \right|_{t=1} = \xi_0,
$$

exist for $t \in (0, 1]$ and are bounded by $\sup_{0 < \xi \leq 1} |\xi(t)| \leq C$ for all initial data $\xi_0$ satisfying $|\xi_0| < R_0$.

We remark here that Keir [35] has analyzed systems of quasilinear wave equations with quadratic semilinear terms under a slightly stronger assumption that requires, in addition to the boundedness assumption, a stability condition on solutions to the asymptotic equation. Under these conditions, Kerr was able to establish, using a generalization of the p-weighted energy method of Dafermos and Rodnianski [16] that was developed in [34], the global existence of solutions to the future of a truncated outgoing characteristic hypersurface under a suitable small initial data assumption. In particular, his results imply that semilinear systems of wave equations of the form (1.1) whose asymptotic equations satisfy his boundedness and stability condition admit solution on spacetime regions of the form \{ $(\tilde{t}, \tilde{r})| \tilde{t} > \max\{0, \tilde{r} - r_0\}, \tilde{r} \geq 0 \} \times S^2$ for suitably small initial data that is prescribed on the truncated null-cone \{ $(\tilde{t}, \tilde{r})| \tilde{t} = \max\{0, \tilde{r} - r_0\}, \tilde{r} \geq 0 \} \times S^2$.

In light of Kerr’s results, we will restrict our attention to establishing the existence of solutions to (1.1) on neighborhoods of spatial infinity of the form

$$
\tilde{M}_{r_0} = \{ (\tilde{t}, \tilde{r})| 0 < \tilde{t} < \tilde{r} - 1/r_0, 1/r_0 < \tilde{r} < \infty \} \times S^2
$$

where $r_0 > 0$ is a positive constant and initial data is prescribed on the hypersurface

$$
\tilde{\Sigma}_{r_0} = \{ (\tilde{t}, \tilde{r})| \tilde{t} = 0, 1/r_0 < \tilde{r} < \infty \} \times S^2.
$$

This will compliment Kerr’s results, at least in the semilinear setting, by establishing the existence of solutions on regions not covered by his existence results. More important, in our opinion, is that we establish these global existence results using a new method, called the Fuchsian method, that we believe will be prove useful, more generally, for the analysis of nonlinear wave equations. Informally, the main existence result of this article, see Corollary 4.3 for the precise version, can be stated as follows:

**Theorem 1.2.** Suppose $z > 0$ and the asymptotic equation (1.5) associated to (1.1) satisfies the bounded weak null condition. Then there exists a $r_0 > 0$ such that for suitably small initial data $\tilde{\psi}^K_r$, $\tilde{u}^K$ defined on $\tilde{\Sigma}_{r_0}$, which does not have to be compactly supported, there exists a unique classical solution $\tilde{u}^K \in C^2(M_{r_0})$ to the initial value problem

$$
\tilde{g}^{\alpha\beta}\tilde{\nabla}_\alpha \tilde{\nabla}_\beta \tilde{u}^K = \tilde{a}_{IJ}^{\alpha\beta}\tilde{\nabla}_\alpha \tilde{u}^I \tilde{\nabla}_\beta \tilde{u}^J \quad \text{in} \quad \tilde{M}_{r_0},
$$

$$(\tilde{u}^K, \partial_i \tilde{u}^K) = (\tilde{\psi}^K_r, \tilde{\psi}^K_r) \quad \text{in} \quad \tilde{\Sigma}_{r_0},
$$

that satisfies the pointwise bound

$$
|\tilde{u}^K| \lesssim \frac{\tilde{r}}{\tilde{r}^2 - \tilde{t}^2} \left( 1 - \frac{\tilde{t}}{\tilde{r}} \right)^{1-z} \quad \text{in} \quad \tilde{M}_{r_0}.
$$

We note that further $L^2$ and pointwise bounds for $\tilde{u}^K$ and its derivatives are easily determined from Corollary 4.3.

1.1. Semilinear wave equations satisfying the bounded weak null condition. In [35], Kerr showed that systems of wave equations of the form (1.1) with

$$
\tilde{a}_{IJ}^{K\alpha\beta} = \tilde{I}^{KL} \tilde{C}_{LJ} \delta^\alpha_0 \delta^\beta_0,
$$

where $\tilde{I}^{KL}$ is a constant, positive definite, symmetric matrix and the $\tilde{C}_{LJ}$ are any constants satisfying

$$
\tilde{C}_{LJ} = -\tilde{C}_{LJ},
$$

satisfy the bounded weak null condition. In this case, the asymptotic equation can be written as

$$
(2 - t)\partial_t \xi = \frac{1}{t} Q(\xi),
$$

$$
\left| \xi \right|_{t=1} = \xi_0,
$$

where $\xi_0$ is a sufficiently small initial data, and the bounded weak null condition is satisfied with $R_0 = 1$ and $C = \tilde{C}_{LJ}$. In this case, the existence of solutions to the asymptotic equation follows from Theorem 1.2.
have associated asymptotic equations that satisfy the bounded weak null condition. This is, in fact, easy to verify since the choice (1.11) leads, by (1.4)-(1.6), to the associated asymptotic equation
\[
(2 - t)\partial_t \xi^K = -\frac{2}{t} \chi(\rho) \rho^m \tilde{I}^{KL} \tilde{C}_{LJK} \xi^J. \tag{1.13}
\]
Introducing the inner-product \((\xi|\eta) = \tilde{I}_{JJ} \xi^J \eta^J\), where \((\tilde{I}_{JJ}) = (\tilde{I}^{JJ})^{-1}\), and contracting (1.13) with \(\tilde{I}_{LK} \xi^K\), we get
\[
(2 - t)(\xi|\partial_t \xi) = -\frac{2}{t} \chi(\rho) \rho^m \tilde{I}_{LK} \tilde{I}^{KL} \tilde{C}_{MJJ} \xi^L \xi^J \xi^J = -\frac{2}{t} \chi(\rho) \rho^m \tilde{C}_{LJK} \xi^J \xi^J \xi^J. \tag{1.14}
\]
But this implies \(\partial_t ((\xi|\xi)) = 0\), and so, we conclude that any solution of the asymptotic IVP (1.7)-(1.7) exists for all \(t \in (0,1]\) and satisfies \((\xi(t))|\xi(t)) = (\xi|\xi)\). Letting \(|\cdot|\) denote the Euclidean norm, we then have that
\[
\frac{1}{\sqrt{C}} |\cdot| \leq \sqrt{(t)} \leq \sqrt{C} |\cdot| \text{ for some constant } C > 0,
\]
and consequently, by the above inequality, we arrive at the bound \(\sup_{0 < t \leq 1} |\xi(t)| \leq C |\xi|\), which verifies that the bounded weak null condition is fulfilled.

The calculation (1.14) also shows that this class of semilinear equations satisfies the structural condition from [33] called Condition H. Because of this, the global existence results established in from [33] apply and yield the existence of global solutions to (1.1) on the region \(t > 0\) for suitably small initial data with compact support. We further note that due to the compact support of the initial data, the results of [33] can, in fact, be deduced as a special case of the global existence theory developed in [35], but do not apply to the situation we are considering in this article because we allow for non-compact initial data in addition to a less restrictive weak null condition.

1.2. Prior and related works. Early global existence results for nonlinear wave equations in \(3 + 1\) dimensions that violate the null condition but satisfy the weak null condition were established for quasilinear wave equations in [1, 41], systems of semilinear wave equations in [2], and the Einstein equations in wave coordinates in [43, 44]. More recent results can be found in [33], which we discussed above, for semilinear equations, and in the articles [12, 18, 28] and, as we discussed above, in [34, 35] for quasilinear equations.

1.3. The Fuchsian method. Singular systems of hyperbolic equations that can be expressed in the form
\[
B^0(t,u)\partial_t u + B^i(t,u)\nabla_i u = \frac{1}{t} B(t,u) u + F(t,u) \tag{1.15}
\]
are said to be Fuchsian. Traditionally, these systems have been viewed as singular initial value problems (SIVP) where asymptotic data is prescribed at the singular time \(t = 0\) and then (1.15) is used to evolve the asymptotic data away from the singular time to construct solutions on time intervals \(t \in (0,T_0]\) for some, possibly small, \(T_0 > 0\). The SIVP for Fuchsian equations has been studied by many mathematicians and has found a wide array of applications, for example, see [6, 13, 14, 17, 27, 31, 32, 37, 51, 52, 54] for applications in the analytic setting and [3, 4, 5, 8, 9, 10, 15, 36, 53, 55, 56] in the class of Sobolev regularity.

While the SIVP approach for establishing the existence of solutions to (1.15) is useful for certain applications, there are many situations where the global initial value problem (GIVP) for the Fuchsian system (1.15) is the relevant problem to solve. In this case, initial data is prescribed at some finite time \(t = T_0 > 0\), and the problem becomes to establish the existence of solutions to (1.15) all the way up to the singular time at \(t = 0\), that is, for \(t \in (0,T_0]\). The flavour of this problem is that of a global existence problem and the study of such problems was initiated by the first author in [49]. In that article, existence results were established for the Fuchsian GIVP, which were then used to deduce the future nonlinear stability of perturbations of Friedmann-Lemaître-Robertson-Walker (FLRW) solutions to the Einstein-Euler equations with a positive cosmological constant and a linear equation of state \(p = K \rho\) for \(0 < K \leq 1/3\). This result represents the first instance of a new method for establishing the global existence of solutions to systems of hyperbolic equations that we now refer to as the Fuchsian method. Specifically, the goal of the Fuchsian method is to transform a given system of hyperbolic equations into a GIVP for a suitable Fuchsian system, and then to deduce the existence of solutions to the Fuchsian GIVP from general existence theorems such as those established in [49].

The advantage of the Fuchsian method is that solving the Fuchsian GIVP is technically much simpler compared to establishing global existence results for the original system of hyperbolic equations. The most difficult part of applying the Fuchsian method is finding a suitable set of variables and coordinates needed to bring the original system of hyperbolic equations into the required Fuchsian form, and this problem is typically more geometric-algebraic than analytic. In recent years, the GIVP existence theory for Fuchsian systems has been further developed in the articles [46, 45, 11, 20], and these existence results and those from [49] have been used to deduce the global existence of solutions for a number of different systems of hyperbolic equations in the articles [11, 20, 40, 45, 46, 47, 50, 58].
1.4. Outlook and future work. The results of this article can be generalized and extended in a number of ways, which we describe briefly here. First, the hierarchical weak null condition defined in [34], which is general enough to encompass all known results with the exception of those from [33, 35] and this article, can be easily handled using the Fuchsian method. Indeed, by introducing a re-scaling of the form $V^K = t^{\mu_K} V^K$ for a suitable choices of the constants $\mu_K$, it is straightforward, assuming the hierarchical weak null condition, to bring the system (3.34) into a Fuchsian form in terms of the variables $V^K$ that satisfies all the hypotheses of Theorem 3.8 of [11]. An application of this theorem then yields the existence of solutions to the semilinear system of wave equations (1.1) on $M_{r_0}$ for suitable initial data specified on $\Sigma_{r_0}$. We further note that all of the results of this article also hold in a neighborhood of timelike infinity. This follows from replacing the cylinder at spatial infinity construction from Section 2.1 by an analogous cylinder at temporal infinity construction. With this change, the arguments in this article go through in a similar fashion. In particular, this type of result in conjunction with the results of the current article can be combined to establish a global existence result for (1.1) on the regions $t > 0$ for initial data specified on the constant time hyperspace $t = 0$ that, importantly, does not have to be compactly supported. We will report on these extensions in a separate article.

We further note that systems of quasilinear wave equations can also be handled via the Fuchsian method. However, this requires an extension of the Fuchsian GIVP existence theory from [11] to allow for spatial manifolds with boundaries and the use of boundary weighted Sobolev spaces. This work will be reported on in a separate article that is currently in preparation. Finally, we note these types of weighted results are also of interest in the semilinear setting because they allow for more general choices of initial data.

1.5. Overview. To prove the main result of this article, we use the Fuchsian method. This involves a sequence of steps where we transform the wave equation (1.1) on the non-compact domain $\bar{M}_{r_0}$ into a Fuchsian equation on a compact domain that is in a form that allows us to directly apply the Fuchsian GIVP existence theory from [11], and thereby, establish the existence of solutions on $\bar{M}_{r_0}$ to semilinear wave equations satisfying the bounded weak null condition.

The derivation of the Fuchsian equation begins in Section 2.1 where we map the domain $\bar{M}$, see (2.1), onto the cylinder at spatial infinity $(0, 2) \times (0, \infty) \times S^2$, which compactifies the outgoing null-rays. We then use this mapping in Section 2.3 to push-forward the wave equation (1.1) on $\bar{M}_{r_0}$. This yields the conformal wave equation (2.19) (see also (2.22)) defined on the manifold with boundary $M_{r_0}$ (see (2.23)) whose closure is now compact. We then proceed, in Section 3.1, to write the conformal wave equation in the first order form (3.13) using the variables $V^K = (V^K_j)$ defined by (3.5) and (3.11). This choice of variables is motivated by the structure of the most singular terms appearing in the conformal wave equation (see (3.4)) and the requirement that the resulting first order system be symmetric hyperbolic. The first order equation (3.13) is then replaced, in Section 3.2, by the extended system (3.39). The point of the extended system is twofold. First, it is defined on an extended spacetime region of the form $(0, 1) \times S$ where $S$ is now a closed manifold, which, unlike $M_{r_0}$, is needed to apply the existence theory developed in [11], and second, its solutions yield solutions to (3.13) on $M_{r_0} \subset (0, 1) \times S$ that are independent of the particular form of the initial data on the “unphysical” part of the initial hypersurface $\{1\} \times S$ given by $\{1\} \times S \setminus \Sigma_{r_0}$. Here, $\Sigma_{r_0}$ is the spatial hypersurface where the initial data is prescribed. The upshot of this is that we lose nothing by working with the extended system (3.39) rather than the first order form of the conformal wave equation given by (3.13). Next, we differentiate the extended system (3.39) to obtain the system (3.59), which can be viewed as an evolution equation for the variables $W^K_j = t^\kappa (\mathcal{D}_j V^K)$, and we use the flow of the asymptotic equation (1.5) to change variables from $V_0$ to a new variable $Y$ defined by (3.67)-(3.66). The point of this change of variables is that it removes the most singular term from the “time” component of the extended system (3.39), which results in the evolution equation (3.68). In Section 3.5, we then apply the projector $\mathcal{P}$ to the extended system (3.39) to obtain an equation for $X^K = t^{-\nu} P V^K$ given by (3.85), and we combine the three systems (3.39), (3.59) and (3.85) into the single Fuchsian system (3.88) to obtain an evolution system for $Z = (W^K, X^K, Y^K)$. It is then shown in Section 3.6 that, under the flow assumptions from Section 3.4.1, the Fuchsian system (3.88) satisfies, for a suitable choice of the parameters $\kappa, \nu$, all the assumptions needed to apply the Fuchsian GIVP existence theory from [11]. Applying Theorem 3.8. from [11] then yields the GIVP result for the Fuchsian system (3.88) that is stated in Theorem 4.1. This, in turn, yields, by construction, a small initial data global existence result for the original system of wave equations (1.1). Finally, we easily derive Corollary 4.3 from Theorem 4.1 using Proposition 3.2, which is the main result of the article and is stated informally as Theorem 1.2 in the Introduction.

2. Conformal wave equations near spatial infinity

2.1. The cylinder at spatial infinity. The first step in transforming the system of semilinear wave equations (1.1) into Fuchsian form is to compactify the neighborhoods of spatial infinity defined by (1.9). To this end, we
let
\[ \tilde{M} = \{ (\tilde{t}, \tilde{r}) \in (-\infty, \infty) \times (0, \infty) \mid -\tilde{t}^2 + \tilde{r}^2 > 0 \} \times S^2 \] \hspace{1cm} (2.1)
and follow [21], see also [11, §4.1.1], by mapping \( \tilde{M} \) to
\[ M = (0, 2) \times (0, \infty) \times S^2 \]
using the diffeomorphism
\[ \psi : \tilde{M} \to M : (\tilde{t}, \tilde{r}, \theta, \phi) \mapsto (t, \theta, \phi) = \left( 1 - \frac{\tilde{t}}{\tilde{r}}, \frac{\tilde{r}}{-\tilde{t}^2 + \tilde{r}^2}, \theta, \phi \right), \] \hspace{1cm} (2.2)
where we label the coordinates on \( M \) by
\[ (x^\mu) = (x^0, x^1, x^2, x^3) = (t, r, \theta, \phi). \]
The inverse of the mapping (2.2) is readily obtained by solving
\[ t = 1 - \frac{\tilde{t}}{r}, \quad r = \frac{\tilde{r}}{-\tilde{t}^2 + \tilde{r}^2}, \] \hspace{1cm} (2.3)
for \((\tilde{t}, \tilde{r})\) to get
\[ \tilde{t} = \frac{1 - t}{rt(2 - t)}, \quad \tilde{r} = \frac{1}{rt(2 - t)}. \] \hspace{1cm} (2.4)

The importance of the diffeomorphism (2.2) is that it defines a compactification of null-rays in \( \tilde{M} \). Decomposing the boundary of \( M \) as
\[ \partial M = \mathcal{I}^+ \cup \partial M \cup \mathcal{I}^-, \]
where
\[ \mathcal{I}^+ = \{ 0 \} \times (0, \infty) \times S^2, \quad \mathcal{I}^- = \{ 2 \} \times (0, \infty) \times S^2 \quad \text{and} \quad \partial M = [0, 2] \times \{ 0 \} \times S^2, \]
the boundary components \( \mathcal{I}^\pm \) correspond to portions of (+) future and (−) past null-infinity, respectively, while \( \partial M \) corresponds to spatial infinity. Furthermore, the spacelike hypersurface \( \{ 1 \} \times (0, \infty) \times S^2 \) in \( M \) corresponds to the constant time hypersurface \( \tilde{t} = 0 \) in Minkowski spacetime.

Remark 2.1. The method used above to obtain the Lorentzian manifold \((M, \bar{g})\) from \((\tilde{M}, \bar{g})\) is an example of the cylinder at spatial infinity construction that was first introduced by Friedrich in [25] as a tool to analyze the behavior of his conformal version of the Einstein field equations, see [22, 23, 24], near spatial infinity. For further applications of this construction to linear wave and spin-2 equations on Minkowski spacetime, see the articles [7, 19, 21, 26, 48].

2.2. Expansion formulas for the tensor components \( \bar{a}_{ij}^{\alpha \beta} \). Before proceeding with the transformation to Fuchsian form, we first derive an expansion formula for the tensor components \( \bar{a}_{ij}^{\alpha \beta} \) that will play an important role in the calculations that follow. To start, we compute the Jacobian matrix
\[ (J^\alpha_{\mu}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sin(\theta) \cos(\phi) & \sin(\theta) \sin(\phi) & \cos(\theta) \\ 0 & \cos(\theta) \cos(\phi) & \cos(\theta) \sin(\phi) & -\sin(\theta) \\ 0 & \csc(\theta) \sin(\phi) & 0 & 0 \end{pmatrix} \] \hspace{1cm} (2.5)
from the change of variables \((\tilde{x}^\mu) = (\tilde{t}, \tilde{r} \cos(\phi) \sin(\theta), \tilde{r} \sin(\phi) \sin(\theta), \tilde{r} \cos(\theta))\) from spherical to Cartesian coordinates. Using this and the tensorial transformation law
\[ \bar{a}_{ij}^{\alpha \beta} = J^{\alpha}_{\mu} \bar{a}_{ij}^{\mu \nu} J_{\nu}^{\beta}, \] \hspace{1cm} (2.6)
we can expand the components (2.6) in powers of \( \tilde{r} \) as
\[ \bar{a}_{ij}^{\alpha \beta} = \frac{1}{\tilde{r}^2} c_{ij}^{\alpha \beta} + \frac{1}{\tilde{r}} d_{ij}^{\alpha \beta} + e_{ij}^{\alpha \beta} \] \hspace{1cm} (2.7)
where the expansions coefficients can be used to define the following geometric objects (see Appendix A for our indexing conventions) on \( S^2 \):

(a) smooth functions \( c_{ij}^{Kq}, d_{ij}^{Kq}, e_{ij}^{Kq} \) and \( f_{ij}^{Kq} \),
(b) smooth vector fields \( e_{ij}^{Kq}, e_{ij}^{Kq}, d_{ij}^{Kq}, d_{ij}^{Kq}, e_{ij}^{Kq}, e_{ij}^{Kq} \) and \( f_{ij}^{Kq}, f_{ij}^{Kq} \),
(c) and smooth (2,0)-tensor fields \( e_{ij}^{Kq}, d_{ij}^{Kq} \) and \( f_{ij}^{Kq} \).
The only terms of the expansion (2.7) that we will need to consider in any detail are the $\tilde{c}_{i,j}^{K\alpha\beta}$. Now, it can be easily verified that the non-vanishing $\tilde{c}_{i,j}^{K\alpha\beta}$ are given by

$$\tilde{c}_{i,j}^{K00} = \dot{a}_{ij}^{K00},$$

$$\tilde{c}_{i,j}^{K01} = \sin(\theta)(\dot{a}_{ij}^{K01} \cos(\phi) + \ddot{a}_{ij}^{K02} \sin(\phi)) + \ddot{a}_{ij}^{K03} \cos(\theta),$$

$$\tilde{c}_{i,j}^{K10} = \sin(\theta)(\dot{a}_{ij}^{K10} \cos(\phi) + \ddot{a}_{ij}^{K20} \sin(\phi)) + \ddot{a}_{ij}^{K30} \cos(\theta)$$

and

$$\tilde{c}_{i,j}^{K11} = \sin^2(\theta) \left( \dot{a}_{ij}^{K11} \cos^2(\phi) + (\dot{a}_{ij}^{K12} + \ddot{a}_{ij}^{K21}) \sin(\phi) \cos(\phi) + \ddot{a}_{ij}^{K22} \sin^2(\phi) \right)$$

$$+ \sin(\theta) \cos(\theta) \left( (\dot{a}_{ij}^{K13} + \ddot{a}_{ij}^{K31}) \cos(\phi) + (\dot{a}_{ij}^{K23} + \ddot{a}_{ij}^{K32}) \sin(\phi) \right) + \ddot{a}_{ij}^{K33} \cos^2(\theta).$$

Furthermore, with the help of (2.5) and (2.6), we find via a straightforward calculation that the $\tilde{b}_{ij}^{K}$, which are defined by (1.4), can be expressed as

$$\tilde{b}_{ij}^{K} = \dot{a}_{ij}^{K00} - \sin(\theta)(\dot{a}_{ij}^{K01} \cos(\phi) + \ddot{a}_{ij}^{K02} \sin(\phi)) - \ddot{a}_{ij}^{K03} \cos(\theta) - \sin(\theta)(\dot{a}_{ij}^{K10} \cos(\phi) + \ddot{a}_{ij}^{K20} \sin(\phi))$$

$$+ \sin^2(\theta) \left( \dot{a}_{ij}^{K11} \cos^2(\phi) + (\dot{a}_{ij}^{K12} + \ddot{a}_{ij}^{K21}) \sin(\phi) \cos(\phi) + \ddot{a}_{ij}^{K22} \sin^2(\phi) \right)$$

$$+ \sin(\theta) \cos(\theta) \left( (\dot{a}_{ij}^{K13} + \ddot{a}_{ij}^{K31}) \cos(\phi) + (\dot{a}_{ij}^{K23} + \ddot{a}_{ij}^{K32}) \sin(\phi) \right) - \ddot{a}_{ij}^{K30} \cos(\theta) + \ddot{a}_{ij}^{K33} \cos^2(\theta).$$

2.3. The conformal wave equation. Letting

$$\tilde{g} = \psi^2 \bar{g}$$

denote the push-forward of the Minkowski metric (1.2) from $\bar{M}$ to $M$ using the map (2.2), we find after a routine calculation that

$$\tilde{g} = \Omega^2 g$$

where

$$\Omega = \frac{1}{r(2-t)^t}$$

and

$$g = -dt \otimes dt + \frac{1-t}{r} (dr \otimes dr + dr \otimes dt) + \frac{(2-t)t}{r^2} dr \otimes dr + \tilde{g}.$$  

Using the map (2.2) to push-forward the wave equations (1.1) yields the system of wave equations

$$\tilde{g}^{\alpha\beta} \tilde{\nabla}_\alpha \tilde{\nabla}_\beta \tilde{u}^K = \dot{a}_{ij}^{K\alpha\beta} \tilde{\nabla}_\alpha \tilde{u}^I \tilde{\nabla}_\beta \tilde{u}^J$$

where $\tilde{\nabla}_\alpha$ is the Levi-Civita connection of the metric $\tilde{g}_{\alpha\beta}$,

$$\tilde{u}^K = \psi \bar{u}^K$$

and

$$\dot{a}_{ij}^{K\alpha\beta} = \psi \dot{a}_{ij}^{K\alpha\beta}.$$  

Since $M = \psi(\bar{M})$, it is clear that the original system of semilinear wave equations (1.1) on $\bar{M}$ is completely equivalent to (2.16) on $M$.

Next, we observe that the Ricci scalar curvature of $\tilde{g}_{\alpha\beta}$ vanishes by virtue of $\tilde{g}_{\alpha\beta}$ being the push-forward of the Minkowski metric. Furthermore, a straightforward calculation using (2.15) shows that the Ricci scalar of the metric $g_{\alpha\beta}$ also vanishes. Consequently, it follows from the formulas (B.5)-(B.6) and (B.8)-(B.9), with $n = 4$, from Appendix B that the system of wave equations (2.16) transform under the conformal transformation (2.13) into

$$\tilde{g}^{\alpha\beta} \tilde{\nabla}_\alpha \tilde{\nabla}_\beta \tilde{u}^K = f^K$$

where $\nabla$ is the Levi-Civita connection of $g$,

$$\tilde{u}^K = rt(2-t)u^K$$

and

$$f^K = \dot{a}_{ij}^{K\mu\nu} \left( \frac{1}{rt(2-t)} \nabla_\mu u^I \nabla_\nu u^J + \frac{1}{(rt(2-t))^2} (\nabla_\mu (rt(2-t))u^I \nabla_\nu u^J + \nabla_\nu u^I \nabla_\mu (rt(2-t))u^J) + \frac{1}{(rt(2-t))^3} \nabla_\mu (rt(2-t)) \nabla_\nu (rt(2-t))u^I u^J \right).$$  

We will refer to this system as the conformal wave equations.
A routine computation involving (2.15) then shows that conformal wave equations (2.19) can be expressed as
\[ (-2 + t)\partial_t^2 u^K + r^2\partial_r^2 u^K + 2r(1 - t)\partial_r\partial_t u^K + g^{\Lambda\Sigma}\nabla_{\Lambda}\nabla_{\Sigma} u^K + 2(t - 1)\partial_t u^K = f^K \]
where $\nabla_{\Lambda}$ is the Levi-Civita connection of the metric (1.3) on $\mathbb{S}^2$.

For the remainder of the article, we will focus on solving the conformal wave equations (2.22) on neighborhoods of spatial infinity of the form
\[ M_{r_0} = \{(t, r) \in (1, 0) \times (0, r_0) | t > 2 - r_0/r \} \times \mathbb{S}^2 \subset M, \quad r_0 > 0, \]
where initial data is prescribed on the spacelike hypersurface
\[ \Sigma_{r_0} = \{1\} \times (0, r_0) \times \mathbb{S}^2 \]
that forms the “top” of the domain $M_{r_0}$. Noting that
\[ \psi(M_{r_0}) = M_{r_0} \text{ and } \psi(\Sigma_{r_0}) = \Sigma_{r_0} \]
by (1.9) and (1.10), we conclude that any solution of the conformal wave equations on $M_{r_0}$ with initial data prescribed on $\Sigma_{r_0}$ corresponds uniquely to a solution of the semilinear wave equations (1.1) on $M_{r_0}$ with initial data prescribed on $\Sigma_{r_0}$.

### 2.4. Expansion formulas for the tensor components $\tilde{a}^{K\alpha\beta}_{ij}$

We now turn to deriving expansion formulas for the tensor components $\tilde{a}^{K\alpha\beta}_{ij}$, defined by (2.18), that will determine their behavior in the limit $t \searrow 0$. These results will be essential for transforming the conformal wave equations (2.22) into Fuchsian form, which will be carried out in the following section.

Now, from (2.2) and (2.18), we find, after a routine calculation, that
\begin{align*}
\tilde{a}^{K00}_{ij} &= 4t^2 - 2\tilde{\delta}_{ij} + t^2 \tilde{\delta}^{K00}_{ij}, \\
\tilde{a}^{K01}_{ij} &= -4t^2 \tilde{\delta}_{ij}^{K01} + t^2 \tilde{\delta}^{K01}_{ij}, \\
\tilde{a}^{K10}_{ij} &= -4t^2 \tilde{\delta}_{ij}^{K10} + t^2 \tilde{\delta}^{K10}_{ij}, \\
\tilde{a}^{K11}_{ij} &= 4t^4 - 4t^2 \tilde{\delta}_{ij}^{K11} + t^2 \tilde{\delta}^{K11}_{ij}, \\
\tilde{a}^{K0A}_{ij} &= -2t^2(\tilde{a}^{K0A}_{ij} - \tilde{a}^{K1A}_{ij}) \circ \psi^{-1} + t^2 \tilde{\delta}^{2K0A}_{ij} \circ \psi^{-1} + t^3 \tilde{a}^{K1A}_{ij} \circ \psi^{-1}, \\
\tilde{a}^{K\Sigma0}_{ij} &= -2t^2(\tilde{a}^{K\Sigma0}_{ij} - \tilde{a}^{K\Sigma1}_{ij}) \circ \psi^{-1} + t^2 (\tilde{a}^{K\Sigma0}_{ij} - \tilde{a}^{K\Sigma1}_{ij}) \circ \psi^{-1} + t^3 \tilde{a}^{K\Sigma1}_{ij} \circ \psi^{-1}, \\
\tilde{a}^{K\Sigma1}_{ij} &= 2t^2 (\tilde{a}^{K\Sigma0}_{ij} - \tilde{a}^{K\Sigma1}_{ij}) \circ \psi^{-1} - 2t^2 (\tilde{a}^{K\Sigma0}_{ij} - \tilde{a}^{K\Sigma1}_{ij}) \circ \psi^{-1} - t^3 \tilde{a}^{K\Sigma1}_{ij} \circ \psi^{-1},
\end{align*}
and
\[ \tilde{a}^{K\Sigma\Lambda}_{ij} = \tilde{a}^{K\Sigma\Lambda}_{ij} \circ \psi^{-1}, \]
where
\begin{align*}
\tilde{\delta}^{K00}_{ij} &= (\tilde{d}^{K00}_{ij} - \tilde{d}^{K01}_{ij}^{K01} + \tilde{d}^{K10}_{ij}^{K10} + \tilde{d}^{K11}_{ij}^{K11}) \circ \psi^{-1}, \\
\tilde{\delta}^{K01}_{ij} &= -4((\tilde{a}_{ij}^{K00} - 2\tilde{\delta}_{ij}^{K01} + 3\tilde{\delta}_{ij}^{K11})) \circ \psi^{-1} + (\tilde{a}_{ij}^{K00} - 5\tilde{\delta}_{ij}^{K01} - 5\tilde{\delta}_{ij}^{K10} + 13\tilde{\delta}_{ij}^{K11}) \circ \psi^{-1} + t^2 (\tilde{a}_{ij}^{K01} + \tilde{a}_{ij}^{K10} - 6\tilde{\delta}_{ij}^{K11}) \circ \psi^{-1} + t^3 \tilde{a}_{ij}^{K11} \circ \psi^{-1}, \\
\tilde{\delta}^{K10}_{ij} &= 2(3\tilde{a}_{ij}^{K00} - 3\tilde{\delta}_{ij}^{K01} - 5\tilde{\delta}_{ij}^{K10} + 5\tilde{\delta}_{ij}^{K11}) \circ \psi^{-1} - 2t ((\tilde{a}_{ij}^{K00} - 2\tilde{\delta}_{ij}^{K01} - 4\tilde{\delta}_{ij}^{K10} + 5\tilde{\delta}_{ij}^{K11})) \circ \psi^{-1} - t^2 (\tilde{a}_{ij}^{K01} + 2\tilde{\delta}_{ij}^{K10} - 5\tilde{\delta}_{ij}^{K11}) \circ \psi^{-1} - t^3 \tilde{a}_{ij}^{K11} \circ \psi^{-1}, \\
\tilde{\delta}^{K11}_{ij} &= 2(3\tilde{a}_{ij}^{K00} - 5\tilde{\delta}_{ij}^{K01} - 3\tilde{\delta}_{ij}^{K10} + 5\tilde{\delta}_{ij}^{K11}) \circ \psi^{-1} - 2t ((\tilde{a}_{ij}^{K00} - 4\tilde{\delta}_{ij}^{K01} - 2\tilde{\delta}_{ij}^{K10} + 5\tilde{\delta}_{ij}^{K11})) \circ \psi^{-1} - t^2 (2\tilde{a}_{ij}^{K01} + \tilde{a}_{ij}^{K10} - 5\tilde{\delta}_{ij}^{K11}) \circ \psi^{-1} - t^3 \tilde{a}_{ij}^{K11} \circ \psi^{-1}, \\
\end{align*}
and
\[ \tilde{\delta}^{K11}_{ij} = 2(2\tilde{a}_{ij}^{K00} - 3\tilde{\delta}_{ij}^{K01} - 3\tilde{\delta}_{ij}^{K10} + 4\tilde{\delta}_{ij}^{K11}) \circ \psi^{-1} + 2t ((\tilde{a}_{ij}^{K01} + \tilde{a}_{ij}^{K10} - 2\tilde{\delta}_{ij}^{K11}) \circ \psi^{-1} + t^2 \tilde{a}_{ij}^{K11} \circ \psi^{-1}. \]

We further observe from (2.2), (2.4), (2.7)-(2.12) and (2.34) that
\begin{align*}
\tilde{a}^{K\eta}_{ij} \circ \psi^{-1} &= \tilde{a}^{K\eta}_{ij} + tr(2 - t)\tilde{a}^{K\eta}_{ij} + t^2 r^2 (2 - t)^2 \tilde{a}^{K\eta}_{ij}, \\
\tilde{a}^{K\gamma}_{ij} \circ \psi^{-1} &= tr(2 - t)\tilde{a}^{K\gamma}_{ij} + t^2 r^2 (2 - t)^2 \tilde{a}^{K\gamma}_{ij},
\end{align*}
respectively. It is worthwhile noting that system (3.1) becomes
\[ (2 - t)\partial_t U^K_0 - \frac{2(1 - t)}{t} r\partial_r U^K_0 - \frac{1}{t^2} r^2 \partial_r U^K_1 - \frac{1}{t^2} \delta^{\Lambda\Sigma} \nabla_\Lambda U^K_0 = -\frac{1}{t^2} U^K_1 + U^K_0 - f^K, \] (3.2)
while the evolution equations for the variables \( U^K_0, U^K_\Lambda \) and \( U^K_4 \) are easily computed to be
\[ \partial_t U^K_1 = \frac{1}{t} r r_0 U^K_0 + \frac{1}{2t} U^K_1, \quad \partial_t U^K_\Lambda = \frac{1}{t^2} \nabla_\Lambda U^K_0 + \frac{1}{2t} U^K_\Lambda \quad \text{and} \quad \partial_t U^K_4 = \frac{1}{2t} U^K_4 + \frac{1}{t^2} U^K_0, \] (3.3)
respectively. It is worthwhile noting that system (3.2)-(3.3) is in symmetric hyperbolic form.

To proceed, we use the first order variables (3.1) to write \( \nabla_\mu u^I \) as
\[ \nabla_\mu u^I = t \partial_t (t^{-\frac{1}{2}} U^K_0 \delta_\mu^I + r^{-1} U^K_1 \delta_\mu^I + U^K_\Lambda \delta_\mu^I). \]

Using this, we then observe that the three main groups of terms from (2.21) can be expressed in terms of the first order variables as
\[ \frac{1}{r t^{(2 - t)^2}} \partial_{(2 - t)^2}^{\mu\nu} \nabla_\mu (r t^{2 - t}) u^I \nabla_\nu u^J = \frac{1}{2} \partial_t \left[ \left( 2 r (1 - t) \tilde{a}^{K00}_{IJJ} + 2 t (2 - t) \tilde{a}^{K10}_{IJJ} \right) U^K_0 U^K_1 \right. \]
\[ + \left. 2 t \partial_t \tilde{a}^{K00}_{IJJ} U^K_1 U^K_0 + \frac{1}{r} \left( \partial_t \tilde{a}^{K00}_{IJJ} U^K_0 U^K_1 + t \partial_t \tilde{a}^{K00}_{IJJ} U^K_1 U^K_0 \right) \right] \]
\[ = \frac{1}{2} \partial_t \left[ \left( 2 r (1 - t) \tilde{a}^{K00}_{IJJ} + 2 t (2 - t) \tilde{a}^{K10}_{IJJ} \right) U^K_0 U^K_1 \right. \]
\[ + \left. 2 t \partial_t \tilde{a}^{K00}_{IJJ} U^K_1 U^K_0 + \frac{1}{r} \left( \partial_t \tilde{a}^{K00}_{IJJ} U^K_0 U^K_1 + t \partial_t \tilde{a}^{K00}_{IJJ} U^K_1 U^K_0 \right) \right]. \]

and
\[ \frac{1}{(2 - t)^2} \partial_{(2 - t)^2}^{\mu\nu} \nabla_\mu (r t^{2 - t}) u^I \nabla_\nu (r t^{2 - t}) u^J = \frac{1}{2} \partial_t \left[ \left( 4 r^2 (1 - t)^2 \tilde{a}^{K00}_{IJJ} \right) \right. \]
\[ + 2 t r (1 - t) \tilde{a}^{K00}_{IJJ} U^K_0 U^K_1 + \left. \frac{1}{r} \left( \partial_t \tilde{a}^{K00}_{IJJ} U^K_0 U^K_1 + t \partial_t \tilde{a}^{K00}_{IJJ} U^K_1 U^K_0 \right) \right]. \]

With the help of these results, it is then not difficult to verify, using (2.12), (2.25)-(2.33) and (2.39)-(2.42), that the nonlinear term (2.21) becomes
\[ -f^K = -\frac{1}{2} t^2 b^K_{IJ} V^K_0 U^K_0 + \frac{1}{t} \left[ r f^K_{IJJ} V^K_0 U^K_1 + t f^K_{IJJ} U^K_1 U^K_0 \right. \]
\[ + \left. \frac{1}{r} \left( \partial_t f^K_{IJJ} V^K_0 U^K_1 + t \partial_t f^K_{IJJ} U^K_1 U^K_0 \right) \right]. \]
when written in terms of the first order variables, where \( \{f_{ij}^K(t,r), g_{ij}^K(t,r), h_{ij}^K(t,r)\} \) and \( \{f_{ij}^{K\Sigma}(t,r)\} \) are collections of smooth scalar, vector, and \((2,0)\)-tensor fields, respectively, on \( S^2 \) that depend smoothly on \((t,r) \in \mathbb{R} \times \mathbb{R} \), and we have set
\[
V_0^K = U_0^K - \frac{1}{t^2} U_1^K. \tag{3.5}
\]

The expansion (3.4) motivates us to replace the first order variable \( U_0^K \) with \( V_0^K \). Doing so, we see via a routine computation involving (3.2) and (3.3) that \( V_0^K \) evolves according to
\[
(2 - t)\partial_t V_0^K + r\partial_r V_0^K - \frac{1}{t^2} g^{\Lambda\Omega} \nabla_{\Lambda} U_{\Sigma}^K = V_0^K - f^K. \tag{3.6}
\]

One difficulty with this change of variables is the system of evolution equations (3.3) and (3.6) for the first order variables \( V_0^K, U_1^K, U_\Lambda^K \) and \( U_4^K \) is no longer symmetric hyperbolic. To restore the symmetry, we use the identity \( \nabla_{\Lambda} U_1 = r\partial_r U_\Lambda \) to write (3.3) as
\[
\begin{align*}
\partial_t U_1^K &= \frac{q}{t} r\partial_r U_1^K + \frac{1}{t^2} r\partial_r V_0^K + \frac{1}{2t^2} U_1^K, \\
\partial_t U_\Lambda^K &= \frac{q}{t} r\partial_r U_\Lambda^K + \frac{1}{t^2} \nabla_{\Lambda} U_0^K + \frac{q + 1}{t^2} \nabla_{\Lambda} U_1^K + \frac{1}{2t^2} U_\Lambda^K, \\
\partial_t U_4^K &= \frac{1}{2t} U_4 + \frac{1}{t^4} U_1^K + \frac{1}{t^2} V_0^K,
\end{align*}
\tag{3.7-3.9}
\]

where
\[
q = \frac{-1 + 2t^2 - t^3}{1 + 4t - 4t^2 + t^3}. \tag{3.10}
\]

We then define new first order variables by setting
\[
V_1^K = 2U_1^K + (2 - t)t^{\frac{3}{2}} V_0^K, \quad V_\Lambda^K = p U_\Lambda^K \quad \text{and} \quad V_4^K = U_4^K, \tag{3.11}
\]

where
\[
p = \sqrt{\frac{1 + 4t - 4t^2 + t^3}{2 - t}}. \tag{3.12}
\]

and observe that the evolution equations (3.6)-(3.9) can be expressed in terms of the variables \( V_0^K, V_1^K, V_\Lambda^K \) and \( V_4^K \) in the following symmetric hyperbolic form:
\[
B^0 \partial_t V^K + \frac{1}{t} B^1 r\partial_r V^K + \frac{1}{t^2} B^\Sigma \nabla_{\Sigma} V^K = \frac{1}{t} B^\Omega V^K + CV^K + F^K \tag{3.13}
\]

where
\[
\begin{align*}
V^K &= (V^K_0) = (V^K_0^1, V^K_1^1, V^K_0^\Lambda, V^K_4^1)^{tr}, \\
B^0 &= \begin{pmatrix}
2 - t & 0 & 0 & 0 \\
0 & 2 - t & 0 & 0 \\
0 & 0 & (2 - t) \delta_\Omega^\Lambda & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}, \\
B^1 &= \begin{pmatrix}
t & 0 & 0 & 0 \\
0 & -(2 - t) & 0 & 0 \\
0 & 0 & (2 - t) q \delta_\Omega^\Lambda & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \\
B^\Sigma &= \begin{pmatrix}
0 & 0 & -\frac{1}{p} g^\Omega^\Lambda & 0 \\
0 & 0 & -(2 - t) \frac{3}{2} g^\Omega^\Lambda & 0 \\
-\frac{1}{p} g^\Sigma_\Omega & -(2 - t) \frac{3}{2} g^\Sigma_\Omega & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \\
B &= \begin{pmatrix}
2 - \frac{t}{2} & 0 & 0 & 0 \\
0 & 2 - \frac{t}{2} & 0 & 0 \\
0 & 0 & 2 - \frac{t}{2} g^\Sigma_\Omega & 0 \\
0 & \frac{1}{2} & 0 & \frac{1}{2}
\end{pmatrix}.
\end{align*}
\tag{3.14-3.18}
where the symmetry is with respect to the inner-product $B$ for all

$$\mathcal{C} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1/2 t^2 & 0 & 0 & 0 \end{pmatrix},$$

$$\mathcal{P} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$F^K = (-f^K - (2 - t) t^2 f^K, 0, 0)^{tr}.$$  

Now, from the definitions (3.15), (3.16), (3.18) and (3.20), it is not difficult to verify that $\mathcal{P}$ is a covariantly constant, time-independent, symmetric projection operator that commutes with $B^0, B^1$ and $\mathcal{B}$, that is,

$$\mathcal{P}^2 = \mathcal{P}, \quad \mathcal{P}^{tr} = \mathcal{P}, \quad \partial_t \mathcal{P} = 0, \quad \partial_r \mathcal{P} = 0, \quad \text{and} \quad \nabla_\Lambda \mathcal{P} = 0$$

and

$$[B^0, \mathcal{P}] = [B^1, \mathcal{P}] = [\mathcal{B}, \mathcal{P}] = 0,$$

where the symmetry is with respect to the inner-product $h(Y, Z) = \delta^\rho Y_\rho Z_\rho + g^\Lambda \Sigma Y_\Lambda Z_\Sigma + Y_4 Z_4.$

Furthermore, it is also not difficult to verify that $B^0$ and $B^1$ and $B^\Sigma Y_\Sigma$ are symmetric with respect to (3.24) and that $B^0$ satisfies

$$h(Y, Y) \leq h(Y, B^0 Y)$$

for all $Y = (Y_2)$ and $0 < t \leq 1$, which in particular, implies that the system (3.13) is symmetric hyperbolic.

Using (3.18) and (3.24), we observe, with the help of Young’s inequality (i.e. $|ab| \leq \frac{\epsilon}{2} a^2 + \frac{1}{2 \epsilon} b^2$), that

$$h(Y, B^0 Y) = 2 Y_0^2 + \frac{2 - t}{2} Y_1^2 + Y_4 Y_4 + \frac{2 - t}{2} g^{\Lambda \Sigma} Y_\Lambda Y_\Sigma + \frac{1}{2} Y_4^2$$

$$\geq 2 Y_0^2 + \frac{2 - t - \epsilon}{2} Y_1^2 + \frac{2 - t}{2} g^{\Lambda \Sigma} Y_\Lambda Y_\Sigma + \frac{1}{2} \left(1 - \frac{1}{\epsilon} \right) Y_4^2.$$

Choosing $\epsilon = \frac{1}{2} (1 - t - \sqrt{5 - 2t + t^2})$, we then have

$$h(Y, B^0 Y) \geq 2 Y_0^2 + \frac{1}{4} (3 - t + \sqrt{5 - 2t + t^2}) Y_1^2 + \frac{2 - t}{2} g^{\Lambda \Sigma} Y_\Lambda Y_\Sigma + \frac{1}{4} (3 - t + \sqrt{5 - 2t + t^2}) Y_4^2$$

$$\geq 2 Y_0^2 + \frac{2 - t}{2} (Y_1^2 + g^{\Lambda \Sigma} Y_\Lambda Y_\Sigma + Y_4^2)$$

$$\geq \frac{1}{2} (2 - t) Y_0^2 + (2 - t) Y_1^2 + (2 - t) g^{\Lambda \Sigma} Y_\Lambda Y_\Sigma + Y_4^2$$

which together with (3.15) and (3.24) allows us to conclude that

$$h(Y, B^0 Y) \leq 2 h(Y, B^0 Y)$$

for all $Y = (Y_2)$ and $0 < t \leq 1$.

Next, from (3.5) and (3.11), we get

$$t^2 U_0^K = \frac{1}{2} (V_1^K + t V_0^K), \quad U_1^K = \frac{1}{2} (V_1^K - (2 - t) t^2 V_0^K), \quad U_\Lambda^K = \frac{1}{\mathcal{P}^{tr}} V_\Lambda^K \quad \text{and} \quad U_4^K = V_4^K.$$

Using these along with (3.4) and (3.20) allows us to expand (3.21) as

$$F^K = -\frac{2}{t^2} t^2 r V_0 t V_0^K e_0 + G^K$$

where

$$G^K = G_0^K (t^2, t, r, V, V) + \frac{1}{t^2} G_1^K (t^2, t, r, V, \mathcal{P} V) + \frac{1}{t} G_2^K (t^2, t, r, \mathcal{P} V, \mathcal{P} V),$$

$$e_0 = (\delta x^0) = (1 \ 0 \ 0 \ 0)^{tr},$$

$$V = (V^t) = (V_2^t),$$

and the $G_a^K (r, t, r, Y, Z), a = 0, 1, 2,$ are smooth bilinear maps with $G_2^K$ satisfying

$$\mathcal{P} G_2^K = 0.$$
Remark 3.1. Here, we are using the term smooth bilinear map to mean a map of the form
\[ H^K(\tau, t, r, Y, Z) = H^{Kpq}_{ij}(\tau, t, r)Y^i J^j_p + H^{Kp\Lambda}_{ij}(\tau, t, r)Y^i \zeta^j_\Lambda + H^{K\Sigma\Lambda}_{ij}(\tau, t, r)Y^i \zeta^j_\Lambda \]
where \( H^{Kpq}_{ij}(\tau, t, r) \), \( H^{Kp\Lambda}_{ij}(\tau, t, r) \), and \( H^{K\Sigma\Lambda}_{ij}(\tau, t, r) \) are collections of smooth scalar, vector, and (2,0)-tensor fields on \( \mathbb{S}^2 \) that depend smoothly on the parameters \( (\tau, t, r) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} \).

For the subsequent analysis, it will be advantageous to introduce a change of radial coordinate via
\[ r = \rho^m, \quad m \in \mathbb{Z}_{\geq 1}. \] (3.33)
Using the transformation law \( r \partial_r = \frac{d^2}{dr^2} \partial \rho = \frac{d^2}{dr^2} \partial \rho \), we can express the system (3.13) as
\[ B^0 \partial_t V^K + \frac{1}{t^m} B^1 \partial_\rho V^K + \frac{1}{t^2} B^2 \nabla_\Sigma V^K = \frac{1}{t} B^P V^K + C V^K + F^K \] (3.34)
where now
\[ F^K = -\frac{2}{t^m} B^0 \rho^m V^0_0 V^0_0 e_0 + G^K \] (3.35)
and
\[ G^K = G^K_0 (t^\frac{1}{2}, t, \rho^m, V, V) + \frac{1}{t^2} G^1_K (t^\frac{1}{2}, t, \rho^m, V, \nabla V) + \frac{1}{t} G^2_K (t^\frac{1}{2}, t, \rho^m, V, \nabla V, \nabla V). \] (3.36)
It is also clear that the neighborhood of infinity \( M_{\rho_0} \) and the initial data hypersurface \( \Sigma_{\rho_0} \), see (2.23) and (2.24), can be expressed in terms of \( \rho \) as
\[ M_{\rho_0} = \{(t, \rho) \in (1, 0) \times (0, \rho_0) \mid t > 2 - \rho_0 / \rho^m \} \times \mathbb{S}^2, \quad \rho_0 = (r_0)^{\frac{1}{2}}, \] (3.37)
and
\[ \Sigma_{\rho_0} = \{1\} \times (0, \rho_0) \times \mathbb{S}^2, \] (3.38)
respectively.

3.2. The extended system. Rather than solving (3.34) on \( M_{\rho_0} \), we will instead solve an extended version of this system on the extended spacetime \((0, 1) \times \mathcal{S}\) where
\[ \mathcal{S} = T \times \mathbb{S}^2 \]
and \( T \) is the 1-dimensional torus obtained from identifying the end points of the interval \([-3\rho_0, 3\rho_0]\). Initial data will be prescribed on the hypersurface \( \{1\} \times \mathcal{S} \).

To define the extended system, we let \( \hat{\chi}(\rho) \) denote a smooth cut-off function satisfying \( \hat{\chi} \geq 0 \), \( \hat{\chi}|_{[-1,1]} = 1 \) and \( \text{supp}(\hat{\chi}) \subset (-2, 2) \), and use it to define the smooth cut-off function
\[ \chi(\rho) = \hat{\chi}(\rho / \rho_0) \]
on \( T \), which is easily seen to satisfy \( \chi \geq 0 \), \( \chi|_{[-\rho_0, \rho_0]} = 1 \) and \( \text{supp}(\chi) \subset (-2\rho_0, 2\rho_0) \). With the help of this cut-off function, we then define the extended system by
\[ B^0 \partial_t V^K + \frac{1}{t^m} B^1 \partial_\rho V^K + \frac{1}{t^2} B^2 \nabla_\Sigma V^K = \frac{1}{t} B^P V^K + C V^K + \mathcal{F}^K \] (3.39)
where
\[ \mathcal{F}^K = \frac{1}{t} Q^K e_0 + G^K, \] (3.40)
\[ Q^K = -2\hat{\chi}^K (t^{\frac{1}{2}}, t, \chi(\rho) \rho^m V^0_0 V^0_0), \] (3.41)
\[ G^K = G^K_0 (t^{\frac{1}{2}}, t, \chi(\rho) \rho^m, V, V), \] (3.42)
\[ G^K_0 = G^K_0 (t^\frac{1}{2}, t, \chi(\rho) \rho^m, V, V), \] (3.43)
\[ G^K_1 = G^K_1 (t^\frac{1}{2}, t, \chi(\rho) \rho^m, V, \nabla V), \] (3.44)
\[ G^K_2 = G^K_2 (t^\frac{1}{2}, t, \chi(\rho) \rho^m, V, \nabla V, \nabla V) \] (3.45)
and
\[ \mathcal{P} G^K_2 = 0. \] (3.46)
By definition, see (3.14), the fields \( V^K \) are time-dependent sections of the vector bundle
\[ \mathcal{V} = \bigcup_{y \in \mathcal{S}} \mathcal{V}_y \]
over \( S \) with fibers \( \mathcal{V}_g = \mathbb{R} \times \mathbb{R} \times T^*_{pr(g)}S^2 \times \mathbb{R} \) where \( pr : S = \mathbb{T} \times S^2 \rightarrow S^2 \) is the canonical projection. We further note that \( (3.24) \) defines an inner-product on \( \mathcal{V} \), and recall that \( B^0, B^1 \) and \( B^\Sigma \xi^\Sigma \) are symmetric with respect to this inner-product. The symmetry of these operators together with the lower bound \( (3.25) \) for \( B^0 \) imply that the extended system \( (3.39) \) is symmetric hyperbolic, a fact that will be essential to our arguments below.

Noting from the definition \( (3.37) \) that the boundary of the region \( M_{r_0} \) can be decomposed as

\[
\partial M_{r_0} = \Sigma_{r_0} \cup \Sigma_{r_0}^+ \cup \Gamma^- \cup \Gamma^+_r
\]

where

\[
\Gamma^- = [0,1] \times \{0\} \times S^2, \quad \Gamma^+_r = \left\{ (t,r) \in [0,1] \times (0,\rho_0) \mid t = 2 - \frac{\rho_0^m}{\rho^m} \right\} \times S^2 \quad \text{and} \quad \Sigma_{r_0}^+ = \{0\} \times \left(0, \frac{\rho_0}{2\pi} \right) \times S^2,
\]

we find that \( n^- = -d\rho \) and \( n^+ = -dt + m\frac{\rho_0^m}{\rho^m} d\rho \) define outward pointing co_normals to \( \Gamma^- \) and \( \Gamma^+_r \), respectively. Furthermore, we have from \((3.15)-(3.17)\) that

\[
\left( n_0^+ B^0 + n_1^+ \frac{\chi^\rho}{m} B^1 + n_2^+ B^\Sigma \right) \bigg|_{\Gamma^-} = 0
\]

and

\[
\left( n_0^+ B^0 + n_1^+ \frac{\chi^\rho}{m} B^1 + n_2^+ B^\Sigma \right) \bigg|_{\Gamma^+_r} = \begin{pmatrix}
-(1-t)(2-t) & 0 & 0 & 0 \\
0 & -(2-t)(3-t) & 0 & 0 \\
0 & 0 & -(2-t)(1-q(2-t))\delta^\Lambda_1 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},
\]

where in deriving this we have used the fact that \( 2-t = \frac{\rho_0^m}{\rho^m} \) on \( \Gamma^+_r \). By \( (3.10) \), we have that \( 1 - q(2-t) \) satisfies \( 1 < 1 - q(2-t) < 3 \) for \( 0 < t < 1 \). From this inequality, \( (3.24), (3.47) \) and \( (3.48) \), we deduce that

\[
h(Y, \left( n_0^+ B^0 + n_1^+ \frac{\chi^\rho}{m} B^1 + n_2^+ B^\Sigma \right) |_{\Gamma^-} Y) \leq 0 \quad \text{and} \quad h(Y, \left( n_0^+ B^0 + n_1^+ \frac{\chi^\rho}{m} B^1 + n_2^+ B^\Sigma \right) |_{\Gamma^+_r} Y) \leq 0
\]

for all \( Y = (Y_T) \). Consequently, by definition, see \([39, \S 4.3]\), the surfaces \( \Gamma^- \) and \( \Gamma^+_r \) are weakly spacelike, and it follows that any solution of the extended system \( (3.39) \) on the extended spacetime \( (0,1) \times S \) will yield by restriction a solution of the system \( (3.34) \) on the region \( (3.37) \) that is uniquely determined by the restriction of the initial data to \( (3.38) \). From this property and the above arguments, we conclude that the existence of solutions to the conformal wave equations \( (2.22) \) on \( M_{r_0} \) can be obtained from solving the initial value problem

\[
B^0 \partial_t V^K + \frac{1}{t} \frac{\chi^\rho}{m} \partial_r V^K + \frac{1}{t^2} B^\Sigma \nabla V^K = \frac{1}{t} B \Omega V^K + CV^K + \mathcal{F}^K \quad \text{in} \quad (0,1) \times S,
\]

\[
V^K = \tilde{V}^K \quad \text{in} \quad \{1\} \times S,
\]

for initial data \( \tilde{V}^K = (\tilde{V}_2^K) \) satisfying the constraints

\[
\nabla_\Lambda \tilde{V}_4^K = \frac{1}{\sqrt{2}} \tilde{V}_4^K \quad \text{and} \quad \rho \frac{\partial_r \tilde{V}_4^K}{m} = \frac{1}{2} (\tilde{V}_4^K - \tilde{V}_0^K) \quad \text{in} \quad \Sigma_{r_0}.
\]

Moreover, solutions to \( (2.22) \) generated this way are independent of the particular form of the initial data \( \tilde{V} \) on \( (\{1\} \times S) \setminus \Sigma_{r_0} \) and are determined from solutions of the IVP \( (3.49)-(3.50) \) via

\[
a^K(t,r,\theta,\phi) = \frac{1}{t^2} V_4^K(t,r^\pm,\theta,\phi).
\]

Finally, solutions to the semilinear wave equations \( (1.1) \) on \( M_{r_0} \) can then be obtained from \( (3.52) \) using \( (2.17) \) and \( (2.20) \), which yield the explicit formula

\[
a^K(\tilde{t},\tilde{r},\theta,\phi) = \frac{\tilde{r}}{\tilde{r}^2 - \tilde{t}^2} \left(1 - \frac{\tilde{t}^2}{\tilde{r}^2}\right)^{\frac{1}{4}} (1 + \frac{\tilde{t}}{\tilde{r}}) V_4^K(1 - \frac{\tilde{t}}{\tilde{r}} \left(\frac{\tilde{r}}{\tilde{r}^2 - \tilde{t}^2}\right)^{\frac{1}{m}}, \theta, \phi).
\]
3.2.1. Initial data transformations. The relation between the initial data

\[ (\tilde{u}^K, \partial_t \tilde{u}^K) = (\bar{v}^K, \bar{w}^K) \quad \text{in } \Sigma_{r_0} \]  

(3.54)

for the semilinear wave equations (1.1) and the corresponding initial data

\[ (u^K, \partial_t u^K) = (v^K, w^K) \quad \text{in } \Sigma_{r_0} \]

for the conformal wave equations (2.19) is given by

\[ v^K(r, \theta, \phi) = \frac{1}{r} \bar{v}^K \left( \frac{1}{r}, \theta, \phi \right) \quad \text{and} \quad w^K(r, \theta, \phi) = -\frac{1}{r^2} \bar{w}^K \left( \frac{1}{r}, \theta, \phi \right) \]

as can be readily verified with the help of (2.2), (2.17) and (2.20). The initial data for the conformal wave equations, in turn, determines via (3.1), (3.5), (3.11) and (3.33) the following initial data for the system (3.34):

\[ \tilde{V}(\rho, \theta, \phi) = \begin{pmatrix}
\frac{1}{\rho^m} \left[ \frac{1}{\rho^m} \partial_r \bar{v}^K \left( \frac{1}{\rho^m}, \theta, \phi \right) + \bar{v}^K \left( \frac{1}{\rho^m}, \theta, \phi \right) \right] \\
\frac{1}{\rho^m} \left[ \frac{1}{\rho^m} \partial_\theta \bar{v}^K \left( \frac{1}{\rho^m}, \theta, \phi \right) + \bar{v}^K \left( \frac{1}{\rho^m}, \theta, \phi \right) \right] \\
\frac{1}{\rho^m} \left[ \frac{1}{\rho^m} \partial_\phi \bar{v}^K \left( \frac{1}{\rho^m}, \theta, \phi \right) + \bar{v}^K \left( \frac{1}{\rho^m}, \theta, \phi \right) \right]
\end{pmatrix}, \quad (3.55)\]

which, of course, satisfies the constraint (3.51). By the above discussion, we can extend this data in any matter we like to \( \mathcal{S} \) to obtain initial data for the extended system (3.49), and thus, we can choose any initial \( \tilde{V} \) for (3.49) on \( \mathcal{S} \) satisfying

\[ \tilde{V}|_{\Sigma_{r_0}} = \tilde{V} \]  

(3.56)

in order to obtain solutions to (1.1) on \( M_{r_0} \) that are uniquely determined by the initial data (3.54).

3.3. The differentiated system. While the extended system (3.39) is in Fuchsian form, it is not yet in a form that is required in order to apply the Fuchsian GIVP existence theory developed in [11]. To obtain a system that is in the required form, we need to modify (3.39) and complement it with a differentiated version. The differentiated version is obtained by applying the Levi-Civita connection \( \mathcal{D}_j \) of the Riemannian metric\(^3\)

\[ q = q_{ij} dy^i \otimes dy^j := d\rho \otimes d\rho + \mathring{g}, \quad y = (y^i) := (\rho, \theta, \phi), \]  

(3.57)

on \( \mathcal{S} \). Noting that

\[ \mathcal{D}_i = \delta^j_i \partial_j + \delta^j_i \nabla_A, \]  

(3.58)

where we recall that \( \nabla_A \) is the Levi-Civita connection of the metric \( \mathring{g}_{\Sigma} \) on \( S^2 \), we see after a short calculation that applying \( \mathcal{D}_j \) to (3.39) and multiplying the result by \( t^\kappa \), where \( \kappa \geq 0 \) is a constant to be fixed below, yields

\[ B^0 \partial_t W^K_j + \frac{1}{t} B^1 \partial_\rho W^K_j + \frac{1}{t^2} B^2 \nabla_\Sigma W^K_j = \frac{1}{t} (B^p + \kappa B^0) W^K_j + \frac{1}{t} Q^K_j + \mathcal{H}_j^K \]  

(3.59)

where

\[ W^K_j = (W^K_j)^{\Sigma_j} := (t^\kappa \mathcal{D}_j V^K_j), \]  

(3.60)

\[ Q^K_j = -t^\kappa 2 \chi(\rho) \rho m^j_d \mathcal{D}_j (V^K_0 V^K_0) e_0 \]  

(3.61)

and

\[ \mathcal{H}_j^K = C W^K_j + t^\kappa - \frac{1}{2} B^2 \nabla_\Sigma \mathcal{D}_j |V^K_1 - \frac{1}{t} \partial_\rho \left( \frac{\chi(\rho)}{m} B^1 \right) \delta^1 W^K_1 \\
+ t^\kappa \mathcal{D}_j G - t^\kappa - 2 \mathcal{D}_j (b^K_{jI}) \chi(\rho)^I V^K_0 V^K_0 e_0. \]  

(3.62)

It is worthwhile pointing out that the term \( \nabla_\Sigma \mathcal{D}_j |V^K_1 \) does not involve any differentiation since the commutator can be expressed completely in terms of the curvature of the metric \( \mathring{g}_{\Sigma} \).

\(^3\)See Appendix A for our indexing conventions.
3.4. The asymptotic equation. The next step in the derivation of a suitable Fuchsian equation involves modifying the $V_0^K$ component of the extended system (3.39) given by
\[(2 - t)\partial_t V_0^K = -\frac{2}{t} \chi^m b_{ij} V_0^j + V_0^K - \frac{1}{t^\kappa} \frac{\chi p}{m} W_0^{\kappa} + \frac{1}{t^{\kappa + \ell}} \mathbf{g} \mathbf{I} W_\Lambda + \mathbf{g}^K,
\]
where $\mathbf{g}^K = (\mathbf{g}^K)$, in order to remove the singular term $\frac{1}{t^\kappa} Q^K$. We remove this singular term using the flow\(^4\) $\mathcal{F}(t, t_0, y, \xi) = (\mathcal{F}^K(t, t_0, y, \xi))$ of the asymptotic equation (1.5), i.e.
\[(2 - t)\partial_t \mathcal{F}(t, t_0, y, \xi) = \frac{1}{t^\kappa} Q \mathcal{F}(t, t_0, y, \xi),
\]
\[\mathcal{F}(t, t_0, y, \xi) = \xi.
\]
Before proceeding, we note that, for fixed $(t, t_0, y)$, the flow $\mathcal{F}(t, t_0, y, \xi)$ maps $\mathbb{R}^N$ to itself, and consequently, the derivative $D_\xi F(t, t_0, y, \xi)$ defines a linear map from $\mathbb{R}^N$ to itself, or equivalently, a $N \times N$-matrix.

Using the asymptotic flow, we define a new set of variables $Y(t, y) = (Y^K(t, y))$ via
\[V_0(t, y) = \mathcal{F}(t, 1, y, Y(t, y))\]
where
\[V_0 = (V_0^K).
\]
A short calculation involving (3.63) and (3.64) then shows that $Y$ satisfies
\[(2 - t)\partial_t Y = \mathcal{L} \mathcal{G}
\]
where
\[\mathcal{L} = (D_\xi \mathcal{F}(t, 1, y, Y))^{-1} \quad \text{and} \quad \mathcal{G} = \left(V_0^K - \frac{1}{t^\kappa} \frac{\chi p}{m} W_0^K + \frac{1}{t^{\kappa + \ell}} \mathbf{g} \mathbf{I} W_\Lambda + \mathbf{g}^K\right).
\]

3.4.1. Asymptotic flow assumptions. We now assume that the flow $\mathcal{F}(t, t_0, y, \xi) = (\mathcal{F}^K(t, t_0, y, \xi))$ satisfies the following: for any $\mathbb{N} \in \mathbb{Z}_{\geq 0}$, there exist constants $R_0 > 0$, $\epsilon \in [0, 1/10]$ and $C_{k\ell} > 0$, where $k, \ell \in \mathbb{Z}_{\geq 0}$ and $0 \leq k + \ell \leq \mathbb{N}$, and a function $\omega(R)$ satisfying $\lim_{R \to 0} \omega(R) = 0$ such that
\[|\mathcal{F}(t, 1, y, \xi)| \leq \omega(R)
\]
and
\[|D_\xi^k D_\xi^\ell \mathcal{F}(t, 1, y, \xi)| + |D_\xi^k D_\xi^\ell (D_\xi \mathcal{F}(t, 1, y, \xi))^{-1}| \leq \frac{1}{t^\kappa} C_{k\ell}
\]
for all $(t, y, \xi) \in (0, 1] \times \mathcal{S} \times B_R(\mathbb{R}^N)$ and $R \in (0, R_0]$. A direct consequence of this assumption is that for any $\sigma > 0$ the maps $\mathcal{F}$ and $\mathcal{F}$ defined by
\[F(t, y, \xi) = t^{\kappa + \sigma} \mathcal{F}(t, 1, y, \xi) \quad \text{and} \quad \tilde{F}(t, y, \xi) = t^{\kappa + \sigma} (D_\xi \mathcal{F}(t, 1, y, \xi))^{-1},
\]
respectively, satisfy $F \in C^0([0, 1], C^0(\mathcal{S} \times B_R(\mathbb{R}^N), \mathbb{R}))$ and $\tilde{F} \in C^0([0, 1], C^1(\mathcal{S} \times B_R(\mathbb{R}^N), \mathbb{R}^{N \times N}))$. Furthermore, since $\xi = 0$ obviously solves the asymptotic equation (1.5), the flow obviously satisfies $\mathcal{F}(t, t_0, y, 0) = 0$, which in turn, implies that
\[F(t, y, 0) = 0
\]
for all $(t, y) \in [0, 1] \times \mathcal{S}$. We further note if the asymptotic flow assumptions are satisfied for some $\epsilon \in [0, 1/10]$, then they will continue to be satisfied for all $\epsilon \in (\epsilon, 1/10]$. Consequently, by increasing $\epsilon$ slightly, we are free to replace $\sigma + \epsilon$ in (3.72) by $\epsilon$.

Proposition 3.2. Suppose the bounded weak null condition holds (see Definition 1.1). Then there exists a $R_0 \in (0, R_0)$ such that the flow $\mathcal{F}(t, t_0, y, \xi)$ of the asymptotic equation (1.5) satisfies the flow assumptions (3.70)-(3.71) for this choice of $R_0$ and any choice of $\epsilon \in (0, 1/10]$.

Proof. We begin the proof by first establishing the following lemma that gives an effective bound on solutions of the asymptotic equation.

Lemma 3.3. For any $R \in (0, R_0)$, the solutions $\xi$ of the asymptotic IVP (1.7)-(1.8) exist for $t \in (0, 1]$ and satisfies
\[\sup_{0 < t \leq 1} |\xi(t)| \leq \frac{C}{R_0} R
\]
for any choice of initial data that is bounded by $|\xi| \leq R$.

\(^4\)Note that the flow depends on $y = (y^i) = (\rho, \theta, \phi) \in \mathcal{S}$ through the coefficients $\chi^m b_{ij}^K$, which are smooth functions on $\mathcal{S}$. 
Proof. Since $Q(\xi)$, see (1.6), is independent of $t$, we can make the asymptotic equation autonomous through the introduction of the new time variable $\tau = -\frac{1}{2} \ln(2 - t) + \frac{1}{2} \ln(t)$, which maps the time interval $0 < t \leq 1$ to $-\infty < \tau \leq 0$. In terms of this new time variable $\tau$, the asymptotic IVP (1.7)-(1.8) becomes

$$\partial_\tau \xi = Q(\xi),$$

$$\xi|_{\tau=0} = \hat{\xi}. \quad (3.75)$$

By the bounded weak null condition, this IVP admits solutions that are defined for $\tau \in (-\infty, 0]$ and satisfy

$$\sup_{-\infty < \tau \leq 0} |\xi(\tau)| \leq C \quad (3.77)$$

provided that $|\hat{\xi}| < R_0$. Next, we assume that the initial value $\hat{\xi}$ satisfies $|\hat{\xi}| < R$ for some $R \in (0, R_0]$, and we set $\bar{\xi}(\tau) = \frac{1}{t} \xi\left(\frac{\tau}{t}\right)$ where $t = \frac{\tau}{R_0} \in (0, 1]$. Then a quick calculation shows that $\bar{\xi}$ satisfies asymptotic equation (3.75) that where initial value is bounded by $|\bar{\xi}|_{|\tau=0} = \frac{1}{t} |\hat{\xi}| < \frac{R_0}{R} R = R_0$, and consequently, we deduce from (3.77) that $\sup_{-\infty < \tau \leq 0} |\bar{\xi}(\tau)| \leq C$. But this implies that $\sup_{-\infty < \tau \leq 0} |\xi(\tau)| \leq \frac{R_0}{R} R$, and the proof of the lemma is complete. \hfill \Box

Implicitly, the solution $\xi = (\xi^K)$ depends on $y \in S$ and the initial data $\hat{\xi}$. Fixing $\epsilon > 0$ and differentiating the asymptotic equation (1.5) with respect to $y = (y^i)$ shows that

$$\eta^K_i = t^2 \partial_\tau \xi^K_i \quad (3.78)$$

satisfies the differential equation

$$(2 - t) \partial_\tau \eta^K_i = \frac{1}{t} \left[ (2 - t) \epsilon \delta^K_j - 2\chi \rho^K (\tilde{h}^{K}_{ij} + \bar{b}^{K}_{ij}) \xi^{j} \right] \eta^K_i - \frac{1}{t^{1-\epsilon}} \partial_\tau D_i (2\chi \rho^K \bar{b}^{K}_{ij} \xi^{j} \xi^{j}). \quad (3.79)$$

Contracting this equation with $\delta_{LK} \delta^{KI} \eta^K_i$ gives

$$(2 - t) \delta_{LK} \partial_\tau \eta^K_i = \frac{1}{t} \left[ (2 - t) \epsilon \eta^K_i - 2\chi \rho^K (\tilde{h}^{K}_{ij} + \bar{b}^{K}_{ij}) \delta_{LK} \xi^{j} \delta^{KI} \eta^K_i \xi^{j} \right] - \frac{1}{t^{1-\epsilon}} \delta_{LK} \delta^{KI} \partial_\tau D_i (2\chi \rho^K \bar{b}^{K}_{ij} \xi^{j} \xi^{j}). \quad (3.80)$$

But $\chi \rho^K$ and $\bar{b}^{K}_{ij}$ are smooth on $S$, and consequently, these functions and their derivatives are bounded on $S$. From this fact and the bound on $\xi$ from Lemma 3.3, we deduce from (3.80) and the Cauchy Schwartz inequality that for any $\sigma \in (0, \epsilon)$ there exists constants $R_0 \in (0, R_0]$ and $C > 0$ such that the energy inequality

$$\frac{(2-t)}{2} \partial_\tau |\eta|^2 \geq \frac{(2-t)}{t} (\epsilon - \sigma) |\eta|^2 - \frac{C}{t^{1-\epsilon}} |\eta|$$

holds for any given $R \in (0, R_0]$ and for all $t \in (0, 1]$. But from this inequality, we see that

$$\partial_\tau |\eta| \geq \frac{1}{t} (\epsilon - \sigma) |\eta| - \frac{C}{t^{1-\epsilon}}.$$

An application of Grönwall’s inequality\footnote{Here, we are using the following form of Grönwall’s inequality: if $x(t)$ satisfies $x'(t) \geq a(t)x(t) - b(t), 0 < t \leq T_0$, then $x(t) \leq x(T_0) e^{-\int_0^t a(t) - b(t)} + \int_0^t e^{-\int_0^\tau a(t) - b(t)} d\tau$ where $A(t) = \int_t^T a(\tau) d\tau$. In particular, we observe from this that if, $x(T_0) \geq 0$ and $a(t) = \frac{h(t)}{t} - b(t), \lambda \in \mathbb{R}$ and $\int_{T_0}^{T_0} \int_t^T a(\tau) b(\tau) d\tau \leq r$, then $x(t) \leq e^{\epsilon(t)} x(T_0) \left( \frac{t}{T_0} \right)^\lambda + e^{2\epsilon(t)} \int_t^T \frac{|h(\tau)|}{T_0^{\lambda}} d\tau$ for $0 < t < T_0$.} then yields

$$|\eta(t)| \leq |\eta(1)| t^{1-\sigma} + t^{1-\sigma} \int_t^1 \frac{C}{t^{1-\sigma}} d\tau = t^{1-\sigma} |\eta(1)| + \frac{1}{\sigma} t^{1-\sigma} (1 - t^{\sigma}).$$

From the definition (3.78) and the fact that $\xi(t) = \mathcal{F}(t, 1, y, \hat{\xi})$, we conclude from the above inequality and (3.74) that there exist constants $C_0, C_{01} > 0$ such that the flow $\mathcal{F}$ satisfies the bounds

$$|\mathcal{F}(t, 1, y, \hat{\xi})| \leq C_0 R \quad \text{and} \quad |D \mathcal{F}(t, 1, y, \hat{\xi})| \leq \frac{1}{t^{\sigma}} C_{01}$$

for all $(t, y, \hat{\xi}) \in (0, 1] \times S \times B_R(\mathbb{R}^N), R \in (0, R_0]$. \hfill \Box
Next, differentiating the asymptotic equation (1.5) with respect to the initial data $\xi$ shows that the derivative

$$D_\xi \xi = \left( \frac{\partial x^K}{\partial x^L} \right)$$

satisfies the equation

$$(2 - t)\partial_t D_\xi \xi = \frac{1}{t} LD_\xi \xi$$

(3.82)

where

$$L = (L^K_j) := -2\chi_{\rho}^{\mu} \left( b_{\rho j}^{K} + b_{\rho j}^{K} \right) \xi^\ell.$$

Furthermore, multiplying (3.82) on the right by $(D_\xi \xi)^{-1}$ yields the equation

$$(2 - t)\partial_t ((D_\xi \xi)^{-1})^{tr} = -\frac{1}{t} L^{tr} ((D_\xi \xi)^{-1})^{tr}$$

(3.83)

for the transpose of $(D_\xi \xi)^{-1}$. Multiplying (3.82) and (3.83) by $t^\nu$, we find that

$$(2 - t)\partial_t (t^\nu D_\xi \xi) = \frac{1}{t} \left( (2 - t)\epsilon + L \right) t^\nu D_\xi \xi$$

and

$$(2 - t)\partial_t (t^\nu (D_\xi \xi)^{-1})^{tr} = \frac{1}{t} \left( (2 - t)\epsilon - L^{tr} \right) (t^\nu (D_\xi \xi)^{-1})^{tr}.$$

Both of the these equations are of the same general form as (3.79), and the same arguments used to derive from (3.79) the bounds (3.81) for $\eta = t^\nu D_\xi \xi$ can be used to obtain similar estimates for $t^\nu D_\xi \xi$ and $(t^\nu (D_\xi \xi)^{-1})^{tr}$. Consequently, shrinking $R_0$ if necessary and arguing as above, we deduce the existence of a constant $C_{10} > 0$ such that the estimate

$$|D_\xi \xi| + |(D_\xi \xi)^{-1}| \leq \frac{1}{t^\sigma} C_{10}$$

holds for $0 < t \leq 1$. From this estimate, we see immediately that

$$|D_\xi \xi \mathcal{F}(t, 1, y, \xi)| + |(D_\xi \xi \mathcal{F}(t, 1, y, \xi))^{-1}| \leq \frac{1}{t^\sigma} C_{10},$$

for all $(t, y, \xi) \in (0, 1] \times S \times B_R(\mathbb{R}^N)$ and $R \in (0, R_0]$.

Finally, by shrinking $R_0$ again if necessary, similar arguments as above can be used to derive, for any fixed $N \in \mathbb{Z}_{\geq 1}$, the bounds

$$|D_\xi^k \mathcal{D}_\xi^l \xi| + |D_\xi^k \mathcal{D}_\xi^l (D_\xi \xi)^{-1}| \leq \frac{1}{t^\sigma} C_{kl}$$

on the higher derivatives for $1 \leq k + l \leq N$. It is then clear from this inequality that the flow bounds

$$|D_\xi^k \mathcal{D}_\xi^l \mathcal{F}(t, 1, y, \xi)| \leq \frac{1}{t^\sigma} C_{lk},$$

hold for all $(t, y, \xi) \in (0, 1] \times S \times B_R(\mathbb{R}^N)$, $2 \leq k + l \leq N$, and $R \in (0, R_0]$. This completes the proof of the proposition. \hfill \square

3.5. **The complete Fuchsian system.** We complete the derivation of the Fuchsian equation by complimenting (3.59) and (3.68) with a third system obtained from applying the projection operator $\mathbb{P}$ to (3.39), which leads to an equation for the variables

$$X^K = \frac{1}{t^\nu} \mathbb{P}V^K,$$

(3.84)

where $\nu \geq 0$ is a constant to be fixed below. Now, a straightforward calculation using (3.20), (3.22)-(3.23), (3.40), (3.42) and (3.60) shows that after multiplying (3.39) by $t^{-\nu} \mathbb{P}$ that $X^K$ satisfies

$$B^K_0 \partial_t X^K + \frac{1}{t} \chi_{\rho}^{\mu} B^K_1 \partial_\mu X^K = \frac{1}{t} (B - \nu B^K_0) X^K + \mathcal{K}^K$$

(3.85)

where

$$\mathcal{K}^K = -\frac{1}{t^{\nu + \epsilon + \nu}} BP \Sigma W^K_\Sigma + \mathbb{P} \left( \frac{1}{t^\nu} \mathbb{P}^V V^K + X^K \right) + \frac{1}{t^\nu} \mathbb{P} \mathcal{G}^0 + \frac{1}{t^{\nu + \epsilon}} \mathbb{P} \mathcal{G}^1$$

(3.86)

and

$$\mathbb{P}^L = \mathbb{I} - \mathbb{P}$$

(3.87)
is the complementary projection operator. We now complete our derivation of the Fuchsian equation, which will be crucial for our existence proof, by collecting (3.59), (3.68) and (3.85) into the following single system:

\[
A^0 \partial_t Z + \frac{1}{t \Lambda} A^1 \partial_t \eta + \frac{1}{t \Lambda^2} \nabla_\Sigma Z = \frac{1}{t} A \Pi Z + \frac{1}{t} Q + J
\]

where

\[
Z = \begin{pmatrix} W^K_j & X^K & Y \end{pmatrix}^T, \\
A^0 = \begin{pmatrix} B^0 & 0 & 0 \\
0 & B^0 & 0 \\
0 & 0 & (2 - t) \mathbb{I} \end{pmatrix}, \\
A^1 = \begin{pmatrix} B^1 \delta_j^L \delta^K_L & 0 & 0 \\
0 & B^1 \delta_j^L & 0 \\
0 & 0 & 0 \end{pmatrix}, \\
A^\Sigma = \begin{pmatrix} B^\Sigma & 0 & 0 \\
0 & B^\Sigma & 0 \\
0 & 0 & 0 \end{pmatrix}, \\
A = \begin{pmatrix} B^\Sigma + \kappa B^0 & 0 & 0 \\
0 & B - \nu B^0 & 0 \\
0 & 0 & 2 \mathbb{I} \end{pmatrix}, \\
\Pi = \begin{pmatrix} \mathbb{I} & 0 & 0 \\
0 & \mathbb{I} & 0 \\
0 & 0 & 0 \end{pmatrix}, \\
Q = (Q^K_j 0 0)^T
\]

and

\[
J = (H^K_j K^K \mathcal{L} \mathcal{G})^T.
\]

### 3.6. Coefficient properties

We now turn to verifying that the system (3.88) satisfies all the assumptions needed to apply the Fuchsian GIVP existence theory from [11].

#### 3.6.1. The projection operator \( \Pi \) and its commutation properties

By construction, the field \( Z \), defined by (3.89), is a time-dependent section of the vector bundle

\[
\mathcal{W} = \bigcup_{y \in S} \mathcal{W}_y
\]

over \( S \) with fibers \( \mathcal{W}_y = \left( T^*_y S \times T^*_y S \times (T^*_y S \otimes T^*_y S^2) \times T^*_y S \right) \times \mathbb{R}^N \times \mathbb{R}^N \) where, as above, \( \text{pr} : S \rightarrow \mathbb{S}^2 \) is the canonical projection and \( \mathbb{V}_y = \mathbb{R} \times \mathbb{R} \times T^*_y S^2 \times \mathbb{R} \). Letting \( \dot{Z} = (\dot{W}^K_j, \dot{X}^K, \dot{Y}) \) and \( Z \) be as defined above by (3.89), we introduce an inner-product on \( \mathcal{W} \) via

\[
h(Z, \dot{Z}) = \delta_{KL} q_j h(W^K_j, \dot{W}^L_j) + \delta_{KL} h(X^K, \dot{X}^L) + \delta_{KL} Y^{KL} \dot{Y}^{KL},
\]

where \( h(\cdot, \cdot) \) is the inner-product defined previously by (3.24). It is then not difficult to verify that this inner-product is compatible, i.e. \( D_j (h(Z, \dot{Z})) = h(D_j Z, \dot{Z}) + h(Z, D_j \dot{Z}) \), with the connection \( D_j \) defined above by (3.58). We further observe from (3.94) that \( \Pi \) defines a projection operator, i.e.

\[
\Pi^2 = \Pi
\]

that is symmetric with respect to the inner-product (3.97). It also follows directly from the definitions (3.90), (3.91) and (3.93) that

\[
[A^0, \Pi] = [A, \Pi] = 0,
\]

\[
\Pi A^1 = A^1 \Pi = A^1, \quad \Pi A^\Sigma = A^\Sigma \Pi,
\]

and

\[
\Pi^\perp A^1 = A^1 \Pi^\perp = \Pi^\perp A^\Sigma \Pi^\perp = A^\Sigma \Pi^\perp = 0,
\]

where

\[
\Pi^\perp = \mathbb{I} - \Pi
\]

is the complementary projection operator.
3.6.2. The operators $A^0$, $A^1$, $A^2\eta_{\Sigma}$ and $A$: Next, we see from (3.25), (3.90) and (3.97) that $A^0$ satisfies
\[
\hat{h}(Z, A^0 Z) = \delta_{KL} q^{ij} h(W^K_i B^0 W^L_j) + \delta_{KL} h(X^K, B^0 X^L) + (2 - t) \delta_{KL} Y^K Y^L
\]
and hence, that
\[
\hat{h}(Z, Z) \leq \hat{h}(Z, A^0 Z). \tag{3.102}
\]
Similar calculations using (3.22)-(3.23), (3.26), (3.90), (3.93) and (3.97) show that
\[
k \hat{h}(Z, A^0 Z) \leq \hat{h}(Z, AZ) \tag{3.103}
\]
provided that $\nu, \kappa \geq 0$ and $\kappa + \nu \leq 1/2$. It is also clear from (3.90)-(3.92) that $A^0$, $A^1$ and $A^2\eta_{\Sigma}$ are symmetric with respect to the inner-product (3.97). Finally, we observe that the inequality
\[
\left| \partial_\rho \left( \frac{\hat{h}(Z)}{m} B^1 \right) \right| \leq \max_{0 \leq t \leq 1} |B^1(t)| ||\partial_\rho (\hat{h}(t))||_{L^\infty(\tau)} \frac{1}{m}
\]
follows easily from (3.16) and (3.57). With the help of this inequality, we deduce from (3.91) that, for any given $\sigma > 0$, there exists an integer $m = m(\sigma) \geq 1$ such that
\[
\left| \partial_\rho \left( \frac{\hat{h}(Z)}{m} B^1 \right) \right| + \left| \partial_\rho \left( \frac{\hat{h}(Z)}{m} A^1 \right) \right| < \sigma \quad \text{in} \ (0, 1) \times \mathcal{S}. \tag{3.104}
\]

3.6.3. The source term $\mathcal{J}$: Using (3.14), (3.20), (3.66)-(3.67), (3.72) and (3.84), we can decompose $V^K$ as
\[
V^K(t, y) = \mathbb{P} V^K(t, y) + \mathbb{P}^1 V^K(t, y), \tag{3.105}
\]
where
\[
\mathbb{P} V^K(t, y) = t^\nu X^K(t, y) \tag{3.106}
\]
and
\[
\mathbb{P}^1 V^K(t, y) = \frac{1}{t}\left( (t^\nu V^K(t, y)) e_0 - \frac{1}{t^\nu} F^K(t, y, Y(t, y)) \right) e_0, \tag{3.107}
\]
while we recall from (3.60) that the derivative $D_j V^K$ is determined by
\[
D_j V^K(t, y) = W^K_j(t, y). \tag{3.108}
\]
We further observe from (3.69) and (3.72) that the map $\mathcal{L}$ can be expressed as
\[
\mathcal{L} = \frac{1}{t^\nu} \hat{F}(t, y, Y(t, y)). \tag{3.109}
\]

Now, setting
\[
X = (X^K),
\]
we can use (3.105)-(3.107) along with (3.43)-(3.44) to write the source term (3.86) as
\[
K^K = \frac{1}{t^{1+\kappa+\nu}} \mathbb{P} B^\Sigma(t, y) p^K_{\Sigma}(y) + \frac{1}{t^{1+\nu}} F^K(t, y, Y(t, y)) \mathbb{P} C(t) e_0 + \mathbb{P} C(t) X^K(t, y)
\]
\[
+ \frac{1}{t^{1+2\nu}} \mathbb{P} G^K_a \left( t^\frac{1}{2}, t, \chi(t) \rho^m, F(t, y, Y(t, y)) e_0, F(t, y, Y(t, y)) e_0 \right)
\]
\[
+ \sum_{a=1}^{2} \left\{ \frac{1}{t^{1+\nu}} \mathbb{P} G^K_a \left( t\frac{1}{2}, t, \chi(t) \rho^m, F(t, y, Y(t, y)) e_0, X(t, y) \right) \right. 
\]
\[
+ \left. \mathbb{P} G^K_a \left( t\frac{1}{2}, t, \chi(t) \rho^m, X(t, y), F(t, y, Y(t, y)) e_0 \right) \right\} + \frac{1}{t^{1+2\nu}} \mathbb{P} G^K_a \left( t\frac{1}{2}, t, \chi(t) \rho^m, X(t, y), X(t, y) \right).
\]

Using (3.105)-(3.109) to similarly express the source terms $H^K$ and $\mathcal{L} \mathcal{G}$, see (3.62) and (3.69), in terms of $W^K_j$, $X^K$ and $Y^K$, it is then not difficult, with the help of (3.73), (3.104) and the assumptions $\epsilon, \kappa, \nu \geq 0$, that we can expand the source term (3.96) as
\[
\mathcal{J} = \left( \frac{1}{t^{1+\epsilon}} + \frac{1}{t^{1+\kappa+\nu}} + \frac{1}{t^{1+\kappa+\nu}} \right) \mathcal{J}_0(t, y, Z(t, y)) + \left( \frac{1}{t^{1+\kappa+\nu}} + \frac{1}{t^{1+2\nu}} \right) \mathcal{J}_1(t, y, Z(t, y))
\]
\[
+ \frac{1}{t} \left( \sigma + t^{\frac{1}{2}-\kappa-\nu} + t^{\frac{1}{2}-\nu} + t^{\frac{1}{2}-\kappa-\nu} + t^{2\nu-\epsilon} \right) \mathcal{J}_2(t, y, Z(t, y)) \tag{3.110}
\]
where \( \mathcal{J}_a \in C^0([0, 1], C^k(S \times B_R(W), W)) \), \( a = 0, 1, 2 \), for any fixed \( N \in \mathbb{Z}_{\geq 0} \), and these maps satisfy\(^\text{6}\)

\[
\mathcal{J}_0 = O(Z), \quad \mathcal{J}_1 = O(IZ), \quad \Pi \mathcal{J}_2 = O(IZ) \quad \text{and} \quad \Pi^2 \mathcal{J}_2 = O(IZ \otimes IZ). \qquad (3.111)
\]

To proceed, we choose the constants \( \kappa, \nu \in \mathbb{R}_{>0} \) to satisfy the inequalities

\[
2\epsilon < \kappa < 1 - \epsilon, \quad \kappa + \nu < \frac{1}{2} - \epsilon, \quad \epsilon < 2\nu \quad \text{and} \quad \kappa \leq \frac{1}{3},
\]

which is possible since in the following we assume that the asymptotic assumptions are satisfied for some \( \epsilon \in (0, 1/10) \). For example, if \( \epsilon = 1/11 \), we could choose \( \kappa = 5/22 \) and \( \nu = 1/11 \). Now, it is not difficult to verify that (3.112) implies the inequalities

\[
3\epsilon \leq 1 - \kappa + 2\epsilon, \quad \nu + 2\epsilon \leq 1 - \kappa + 2\epsilon, \quad 0 < 2\nu - \epsilon, \quad 0 < \frac{1}{2} - \kappa - \epsilon, \quad 0 < \frac{1}{2} - \kappa - \nu,
\]

and that, with the help of these inequalities, we can, after suitably redefining the maps \( \mathcal{J}_a \), rewrite (3.110) as

\[
\mathcal{J} = \frac{1}{t^{1-\kappa+2\epsilon}} \mathcal{J}_0(t, y, Z(t, y)) + \frac{1}{t^{1-\kappa+\epsilon}} \mathcal{J}_1(t, y, Z(t, y)) + \frac{1}{t} (\sigma + t^\nu) \mathcal{J}_2(t, y, Z(t, y)) \quad (3.113)
\]

for some suitably small constant \( \tilde{\epsilon} > 0 \). Here, the constant \( \sigma > 0 \) can be chosen as small as we like, and the redefined maps \( \mathcal{J}_a \) have the same smoothness properties as above and satisfy (3.111).

**Remark 3.4.** The point of the expansion (3.113) is that source term \( \mathcal{J} \) satisfies all the assumptions from Section 3.1.(iii) of [11] except for the following:

1. the differentiability of each of the maps \( \mathcal{J}_a \) is finite,
2. and \( \mathcal{J}_2 \) does not satisfy \( \Pi \mathcal{J}_2 = 0 \).

Neither of these exceptions pose any difficulties and are easily dealt with. To see why the first exception is not problematic, we observe from arguments of [11] that all of the results of that paper are valid provided that the order of the differentiability of the source term is greater than \( n/2 + 3 \), where \( n \) is the dimension of the spatial manifold. Since the spatial manifold we are considering, i.e. \( S \), is 3-dimensional and we have established above that the maps \( \mathcal{J}_a \) are \( N \)-times differentiable for any \( N \in \mathbb{Z}_{\geq 0} \), it follows by taking \( N > 3/2 + 3 \) that the finite differentiability is no obstruction to applying the results from [11] to the Fuchsian equation (3.88). In regards to the second exception, we note, since \( \Pi \mathcal{J}_2 = O(IZ) \), that the term \( \frac{1}{t} (\sigma + t^\nu) \Pi \mathcal{J}_2 \) can be absorbed into the term \( \frac{1}{t^s} AIZ \) on the right hand side of the Fuchsian equation (3.88) via a redefinition of the operator \( A \). Due to the factor \( \sigma + t^\nu \), we can ensure, for any choice of \( \tilde{\epsilon} \in (0, \kappa) \), that the redefined matrix \( A \) would satisfy for all \( t \in (0, T_0) \) an inequality of the form (3.103) with \( \kappa \) replaced by \( \tilde{\epsilon} \) provided that \( \sigma \) and \( T_0 \) are chosen sufficiently small. After doing this, the redefined \( \mathcal{J}_2 \) would satisfy \( \Pi \mathcal{J}_2 = 0 \) as required and the source term \( \mathcal{J} \) would satisfy all the assumptions needed to apply the existence theory from [11].

### 3.6.4. The source term \( Q \):

We now analyze the nonlinear term (3.95) (see also (3.61)) in more detail. Recalling that the \( \chi^\mu \omega^I_\nu \) are smooth functions on \( S \), we can, with the help of the product estimate [57, Ch. 13, Prop. 3.7] and Hölder’s inequality, estimate \( Q \) for any \( s \in \mathbb{Z}_{\geq 0} \) by

\[
\| Q \|_{H^s(S)} \lesssim t^s \left( \| D(V_0) V_0 \|_{L^\infty(S)} + \| D(V_0) V_0 \|_{H^{s}(S)} \right)
\]

\[
\lesssim t^s \left( \| V_0 \|_{L^\infty(S)} \| D V_0 \|_{L^\infty(S)} + \| V_0 \|_{L^\infty(S)} \| D V_0 \|_{H^{s}(S)} \right)
\]

\[
\lesssim \| V_0 \|_{L^\infty(S)} \| W \|_{L^\infty(S)} + \| V_0 \|_{L^\infty(S)} \| W \|_{H^{s}(S)} + \| W \|_{L^\infty(S)} \| V_0 \|_{L^\infty(S)}. \quad (3.114)
\]

Next, for \( k \in \mathbb{Z}_{\geq 3/2} \), we let \( C_{Sob} \) denote the constant that appears in the Sobolev inequality [57, Ch. 13, Prop. 2.4.], that is,

\[
\| f \|_{L^\infty(S)} \leq C_{Sob} \| f \|_{H^k(S)}. \quad (3.115)
\]

Then by (3.66), the flow bounds (3.70)-(3.71), and the Sobolev and Hölder inequalities, we see that the inequalities

\[
\| V_0 \|_{L^\infty(S)} + \| V_0 \|_{L^2(S)} \lesssim \omega(R) \quad (3.116)
\]

and

\[
\| V_0 \|_{L^\infty(S)} \lesssim \| V_0 \|_{L^2(S)} + \| D V_0 \|_{L^\infty(S)} \lesssim \omega(R) + \| W \|_{L^\infty(S)}, \quad s \in \mathbb{Z}_{\geq 1}, \quad (3.117)
\]

---

\( ^6 \)Here, we are using are the order notation \( O(\cdot) \) from [11, §2.4] where the maps are finitely rather than infinitely differentiable.
hold for all \( t \in (0, 1] \) and \( \|Y\|_{L^k} \leq R/C_{\text{Sob}} \). Using these estimates, Sobolev’s inequality and the estimate \( \|W\|_{L^\infty(S)} \lesssim \|W\|_{L^2(S)} \), which follows from Hölder’s inequality, we find from setting \( s = 0 \) and \( s = k \in (3.114) \) that
\[
\|Q\|_{L^2(S)} \lesssim \omega(R) \|W\|_{L^2(S)} \lesssim \omega(R) \|\Pi Z\|_{L^2(S)}
\]
and
\[
\|Q\|_{L^k(S)} \lesssim \left( \omega(R) + \|W\|_{L^k(S)} \right) \|W\|_{L^k(S)} \lesssim \left( \omega(R) + R \right) \|\Pi Z\|_{L^k(S)}
\]
for all \( \|Z\|_{L^k(S)} \leq R/C_{\text{Sob}} \). We further observe from (3.94) and (3.61) that
\[
\Pi Q = Q.
\]

Remark 3.5. The importance of the estimates (3.118)-(3.119) and the identity (3.120) is that, by an obvious modification of the proof of Theorem 3.8. in [11], these results show that terms in the energy estimates for the Fuchsian equation (3.88) that arise due to the “bad” singular term \( \frac{1}{\gamma} \omega \Pi Z \) can be controlled using the “good” singular \( \frac{1}{\gamma} \omega \Pi Z \) by choosing \( \omega(R) + R \) sufficiently small, which we can do by choosing \( R \) suitably small since \( \lim_{R \to 0} \omega(R) = 0 \) by assumption.

4. Existence

Theorem 4.1. Suppose \( k \in \mathbb{Z}_{\geq 5} \), \( \rho_0 > 0 \), the asymptotic flow assumptions (3.70)-(3.71) are satisfied for constants \( n \in \mathbb{Z}_{\geq k} \), \( R_0 > 0 \) and \( \epsilon \in (0, 1/10) \), the constants \( \kappa, \nu \in \mathbb{R}_{>0} \) satisfy the inequalities (3.12), and \( \gamma \in (0, \kappa) \). Then there exist constants \( m \in \mathbb{Z}_{\geq 1} \) and \( \delta > 0 \) such that for any \( V = (V^K) \in H^{k+1}(S, V^N) \) satisfying \( \|V\|_{H^{k+1}(S)} \leq \delta \), there exists a unique solution
\[
V = (V^K) \in C^0((0, 1], H^{k+1}(S, V^N)) \cap C^1((0, 1], H^k(S, V^N))
\]
to the GIVP (3.49)-(3.50) for the extended system. Moreover, the following hold:
\( a) \) The solution \( V \) satisfies the bounds
\[
\|V_0(t)\|_{L^\infty(S)} \lesssim 1, \quad \|V_0(t)\|_{H^k(S)} \lesssim \frac{1}{\|t\|}, \quad \|FV(t)\|_{H^k(S)} \lesssim t^\nu
\]
\[
\|DV(t)\|_{H^k(S)} \lesssim \frac{1}{\nu \kappa}, \quad \|FV(t)\|_{H^{k-1}(S)} \lesssim t^{\nu + \kappa - \gamma} \quad \text{and} \quad \|DV(t)\|_{H^{k-1}(S)} \lesssim \frac{1}{\|t\|^\nu}
\]
for \( t \in (0, 1] \). Additionally, there exists an element \( Z^\perp \in H^{k-1}(S, W) \) satisfying \( \Pi Z^\perp \perp 0 = Z^\perp \) such that
\[
\|\Pi Z(t)\|_{H^{k-1}(S)} + \|\Pi^\perp Z(t) - Z^\perp\|_{H^{k-1}(S)} \lesssim t^{\kappa - \gamma}
\]
for \( t \in (0, 1] \) where \( Z \) is determined from \( V \) by (3.89).
\( b) \) If, additionally, the initial data \( V \) is chosen so that the constraint (3.51) is satisfied, then the solution \( V \) determines a unique classical solution \( \tilde{u}^K \in C^2(M_{r_0}) \), with \( r_0 = \rho_0^n \), of the IVP
\[
\bar{g}^{\alpha \beta} \nabla_\alpha \nabla_\beta \tilde{u}^K = \bar{a}^{KJ} \bar{a}^J \bar{u}^I \nabla_\beta \bar{u}^I \quad \text{in} \quad M_{r_0},
\]
\[
(\tilde{u}^K, \partial_\gamma \tilde{u}^K) = (\bar{u}^K, \bar{u}^K) \quad \text{in} \quad \Sigma_{r_0},
\]
where \( \bar{a}^K, \bar{v}^K \) and \( \bar{w}^K \) are determined from \( V \) by (3.53), (3.55) and (3.56). Furthermore, the \( \tilde{u}^K \) satisfy the pointwise bounds
\[
|\tilde{u}^K| \lesssim \frac{\bar{r}}{r^2 - \frac{\rho_0}{4}} \left( 1 - \frac{\rho_0}{r} \right)^{\frac{\kappa - \gamma}{2}} \quad \text{in} \quad M_{r_0}.
\]

Proof.
Existence and uniqueness for the extended system: Having established that the extended system (3.49) is symmetric hyperbolic, we can, since \( k > 3/2 + 1 \) by assumption, appeal to standard local-in-time existence and uniqueness results for symmetric hyperbolic systems, e.g. [57, Ch. 16, Prop. 1.4.], to conclude the existence of a \( t^* \in [0, 1) \), which we take to be maximal, and a unique solution
\[
V = (V^K) \in C^0((t^*, 1], H^{k+1}(S, V^N)) \cap C^1((t^*, 1], H^k(S, V^N))
\]
to the IVP (3.49)-(3.50) for given initial data \( \hat{V} = (\hat{V}^K) \in H^{k+1}(S, V^N) \), where the maximal time \( t^* \) depends on \( \hat{V} \). Next, by (3.66), we have that
\[
V_{t=1} = \hat{V}_0 = (\hat{V}_0^K).
\]
From this, (3.60), (3.84) and (3.89), we see, by choosing the initial data to satisfy \(|\tilde{V}|_{H^{k+1}(S)} < \delta\), that \(|Z(1)|_{H^k(S)} < \tilde{C}\delta\) for some positive constant \(\tilde{C} > 0\) that is independent of \(\delta\). We then fix \(R \in (0, R_0]\) and choose \(\delta\) small enough to satisfy

\[
\delta < \frac{R}{8\tilde{C}C_{Sob}}
\]  

(4.2)

so that

\[
|Z(1)|_{H^k(S)} < \tilde{C}\delta < \frac{R}{8C_{Sob}}.
\]  

(4.3)

For \(Z\) to be well-defined, it is enough for \(Z\) to satisfy

\[
|Z|_{H^k(S)} \leq \frac{R}{2C_{Sob}}.
\]  

(4.4)

This is because this bound will ensure by Sobolev’s inequality (3.115) that

\[
|Y|_{L^\infty} \leq C_{Sob}|Y|_{H^k(S)} \leq C_{Sob}|Z|_{H^k(S)} \leq \frac{R}{2} < R < R_0,
\]

which, by the flow assumptions (3.70)-(3.71), will guarantee that the change of variables (3.66) is well-defined and invertible, and hence that \(Z\) is well-defined by (3.60), (3.84) and (3.89).

To proceed, we let \(t_* \in (t^*, 0)\) denote the first time such that

\[
|Z(t_*)|_{H^k(S)} = \frac{R}{2C_{Sob}},
\]  

(4.5)

and if there is no such time, then we set \(t_* = t^*\). We note that \(t_*\) is well-defined by (4.2) and (4.3), and we further note from (4.1) and the definition of \(Z\) that

\[
Z \in C^0((t_*, 1], H^k(S, \mathbb{W})) \cap C^1((t_*, 1], H^{k-1}(S, \mathbb{W})).
\]

Now, since \(F(t, 1, y, 0) = 0\) by virtue of \(\xi = 0\) being a solution of the asymptotic equation (1.5), it is not difficult to verify that the symmetric hyperbolic equations (3.49) and (3.88) both admit the trivial solution. Because of (4.3), we can therefore appeal to the Cauchy stability property enjoyed by symmetry hyperbolic equations to conclude, by choosing \(\delta\) small enough, that \(t_*\), where of course \(t_* \geq t^*\), can be made to be as small as we like and that the inequality

\[
\max_{0 \leq t \leq 1} |Z(t)|_{H^k(S)} < 2\tilde{C}\delta < \frac{R}{4C_{Sob}}
\]  

(4.6)

is valid for

\[
t_0 = \min\{2t_*, 1/2\}.
\]

Recalling that we are free to choose the constant \(\sigma > 0\), see (3.104), as small as we like by choosing the constant \(m \in \mathbb{Z}_{\geq 1}\) sufficiently large, we can, for any given \(\sigma_* > 0\), arrange, since \(\tilde{c} > 0\) (see (3.113)), that

\[
\sigma + t\tilde{c} < \sigma_* \quad t \in (0, t_0],
\]  

(4.7)

by choosing \(\delta\) small enough to guarantee that \(t_0\) is sufficiently small to ensure that this inequality holds.

In light of Remarks 3.4 and 3.5, the bounds (3.102), (3.103), (3.104), and (4.7), the relations (3.98)-(3.101), the expansion (3.113), and the estimates (3.118)-(3.119), all taken together, show that if the constants \(m \in \mathbb{Z}_{\geq 1}\) and \(\delta > 0\) are chosen sufficiently large and small, respectively, and the constants \(\kappa, \nu\) are chosen to satisfy (3.112), then the Fuchsian system (3.88), which \(Z\) satisfies, will, after the simple time transformation \(t \mapsto -t\), satisfy all the required assumptions needed to apply the time rescaled version, see [11, §3.4.] and the remark below, of Theorem 3.8. from [11].

Remark 4.2. From the discussion from Section 3.4. of [11] and Section 3.6 of this article, it is not difficult to see that the appropriate rescaling power \(p\), see equation (3.106) in [11], in the current context is

\[
p = \kappa - 2\tilde{c},
\]  

(4.8)

which, we note, by (3.112), satisfies the required bounds \(0 < \kappa - 2\tilde{c} \leq 1\). We further note from Theorem 3.8. from [11], see also [11, §3.4.], that parameter \(\zeta\) defined by equation (3.59) of [11], which is involved in determining the decay of solutions, is, in the current context, determined by

\[
\zeta = \kappa - \varepsilon
\]  

(4.9)

where \(\varepsilon > 0\) can be made as small as we like by choosing the constant \(m\) large enough and the constants \(R, t_0\) small enough to ensure that \(\sigma_*\) and \(|Z|_{H^k(S)}\) are sufficiently small.
We therefore conclude from the proof of Theorem 3.8. from [11] that $Z$, which solves (3.88), satisfies an energy estimate of the form

$$
\|Z(t)\|^2_{H^k(S)} + \int_t^{t_0} \frac{1}{r^2} \|I(Z)\|^2_{H^k(S)} \, dt \leq C^2 E \|Z(t_0)\|^2
$$

(4.10)

for all $t \in (t_*, t_0]$. By Grönwall’s inequality and (4.3), we then have

$$
\sup_{t \in (t_*, t_0)} \|Z(t)\|_{H^k(S)} \leq e^{C \epsilon (t_* - t_0)} \|Z(t_0)\|_{H^k(S)} < e^{C \epsilon (t_* - t_0)} \tilde{C} \delta.
$$

(4.11)

Choosing $\delta$ now, by shrinking it if necessary, to satisfy $\delta < \frac{R}{3 \tilde{C} \chi}$, in addition to (4.2), the bounds (4.6) and (4.11) implies that

$$
\sup_{t \in (t_*, t_1)} \|Z(t)\|_{H^k(S)} < \frac{R}{3 \tilde{C} \chi}.
$$

(4.12)

From this inequality and the definition (4.5) for $t_*$, we conclude that $t_* = t^*$.

Now, from (3.72), (3.73), Sobolev’s inequality, and the Moser estimates (e.g. [57, Ch. 13, Prop. 3.9.]), we see from (3.66) and (3.89) that $V_0$ can be bounded by

$$
\|V_0(t)\|_{H^k(S)} \leq \frac{1}{t^\nu} C(\|Z(t)\|_{H^k(S)} \|Z(t)\|_{H^k(S)}).
$$

(4.13)

for $Z$ satisfying (4.4), while we see from (3.84), (3.89) and (3.94) that $PV(t)$ is bounded by

$$
\|PV(t)\|_{H^k(S)} \leq t^\nu \|I(Z(t))\|_{H^k(S)}, \quad s \in \mathbb{Z}_{\geq 0}.
$$

(4.14)

Since $t_* = t^*$, the estimates (4.12), (4.13) and (4.14) imply that $\|V(t)\|_{H^k(S)}$ is finite for any $t \in (t_*, 0)$. By the maximality of $t^*$ and the continuation principle for symmetric hyperbolic equations, we conclude that $t^* = 0$, which establishes the existence of solutions to the extended IVP (3.49)-(3.50) on the spacetime region $(0, 1) \times S$.

**Uniform bounds for $V$:** From (3.66), (3.89), (3.116), (4.12), (4.13) and (4.14), we see that the estimates

$$
\|V_0(t)\|_{\mathcal{L}_\infty(S)} \lesssim \omega(\delta), \quad \|V_0(t)\|_{H^k(S)} \lesssim \frac{1}{t^\nu} \delta, \quad \|PV(t)\|_{H^k(S)} \lesssim t^\nu \delta \quad \text{and} \quad \|DV(t)\|_{H^k(S)} \lesssim \frac{1}{t^\nu} \delta
$$

hold for $t \in (0, 1]$. Furthermore, in view of the Remark 4.2, see in particular, (4.9), the coefficient properties from Section 3.6, and the fact that $\kappa \in (0, 1/3)$, we conclude from Theorem 3.8. and Section 3.4. of [11] that, for any fixed $x > 0$, there exists, provided that $m$ and $\delta$ are chosen sufficiently large and small respectively, an element $Z^+ \in H^{k-1}(S, W)$ satisfying $\mathcal{P}^+ Z^+ = Z^+$ such that

$$
\|\Pi Z(t)\|_{H^{k-1}(S)} + \|\Pi Z(t) - Z^+\|_{H^{k-1}(S)} \lesssim t^{\nu - \delta}
$$

for $t \in (0, 1]$. With the help of the above inequality, (3.60), (3.89), (3.94) and (4.14), we conclude that $V$ also satisfies

$$
\|PV(t)\|_{H^{k-1}(S)} \lesssim t^{\nu + \kappa - \delta} \quad \text{and} \quad \|DV(t)\|_{H^{k-1}(S)} \lesssim \frac{1}{t^\nu}
$$

(4.15)

for $t \in (0, 1]$.

**Existence for the wave equations (1.1):** Letting $r_0 = \rho_0^m$, we know from the discussion contained in Section 3.2, that if the initial data $V$ is chosen to satisfy the constraints (3.51) on the spacelike hypersurface $\Sigma_{r_0}$, then the solution $V = (V^K_0, V^K_1, V^K_2, V^K_3)$ to the extended system (3.49) determines a classical solution $\tilde{u}^K$ of the semilinear wave equations (1.1) on $M_{r_0}$ via the formula (3.53). Moreover, this solution is uniquely determined by the initial data on $\Sigma_{r_0}$ that is obtained from the restriction of the initial data $V$ to the initial hypersurface $\Sigma_{r_0}$ and the transformation formulas (3.54) and (3.55). To complete the proof, we note from (3.52), Sobolev’s inequality, the decay estimate (4.15), and (2.3) that each $\tilde{u}^K$ satisfies the pointwise bound

$$
|\tilde{u}^K| \lesssim \frac{\tilde{r}}{r^2 - \tilde{r}^2} \left(1 - \frac{\tilde{r}}{\tilde{r}}\right)^{\frac{1}{2} + \nu + \kappa - \delta} \quad \text{in} \quad M_{r_0}.
$$

\[\square\]

**Corollary 4.3.** Suppose $k \in \mathbb{Z}_{\geq 5}, \rho_0 > 0, \zeta > 0$ and the bounded weak null condition (see Definition 1.1) holds. Then there exist constants $m \in \mathbb{Z}_{\geq 1}$ and $\delta > 0$ such that for any $\tilde{V} = (\tilde{V}^K) \in H^{k+1}(S, \mathcal{V}^N)$ satisfying $\|\tilde{V}\|_{H^k(S)} < \delta$, there exists a unique solution

$$
V = (V^K) \in C^0([0, 1], H^{k+1}(S, \mathcal{V}^N)) \cap C^1((0, 1], H^k(S, \mathcal{V}^N))
$$

to the IVP (3.49)-(3.50). Moreover, the following hold:
(a) The solution $V$ satisfies the uniform bounds
\[
\|V_0(t)\|_{L^\infty(S)} \lesssim 1, \quad \|V_0(t)\|_{H^k(S)} + \|DV(t)\|_{H^{k+1}(S)} \lesssim \frac{1}{t^2} \quad \text{and} \quad \|PV(t)\|_{H^k(S)} \lesssim t^{\frac{1}{2} - z}
\]
for $t \in (0, 1]$.

(b) If, additionally, the initial data $\hat{V}$ is chosen so that the constraint (3.51) is satisfied, then the solution $V$ determines a unique classical solution $\bar{u}_K \in C^2(\bar{M}_{r_0})$, with $r_0 = \rho_0^m$, of the IVP
\[
\hat{g}^{\alpha\beta} \hat{\nabla}_\alpha \hat{\nabla}_\beta \bar{u}_K = \hat{a}^{K\alpha\beta} \hat{\nabla}_\alpha \bar{u}^I \hat{\nabla}_\beta \bar{u}^I \quad \text{in} \quad \bar{M}_{r_0},
\]
\[(\bar{u}_K, \partial_t \bar{u}_K) = (\bar{u}^K, \bar{w}^K) \quad \text{in} \quad \Sigma_{r_0},
\]
where $\bar{u}^K$, $\bar{w}^K$ and $\bar{u}_K$ are determined from $V$ by (3.53), (3.55) and (3.56). Furthermore, the $\bar{u}_K$ satisfy the pointwise bounds
\[
|\bar{u}_K| \lesssim \hat{f} \left( \frac{\hat{r}}{\hat{r}_0^2 - \hat{r}^2} \left( 1 - \frac{\hat{r}}{\hat{r}_0} \right)^{1-z} \right) \quad \text{in} \quad \bar{M}_{r_0}.
\]

Proof. By Proposition 3.2, we know that the asymptotic flow satisfies the flow assumptions (3.70)-(3.71) for some $R_0 > 0$ and any $\epsilon \in (0, 1/10]$. Fixing $\epsilon \in (0, 1/11)$, we set $z = \epsilon$, $\nu = \frac{1}{2} - 5\zeta$ and $\kappa = 3\zeta$. It is then not difficult to verify that these choices for $z$, $\nu$ and $\kappa$ satisfy the inequalities (3.112) and $0 < z < \kappa$. The proof now follows directly from Theorem 4.1. \qed

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APPENDIX A. INDEXING CONVENTIONS

Below is a summary of the indexing conventions that are employed throughout this article:

| Alphabet | Examples | Index range | Index quantities |
|------------------|-----------|-------------|------------------|
| Lowercase Greek | $\mu, \nu, \gamma$ | 0, 1, 2, 3 | spacetime coordinate components, e.g. $(x^\mu) = (t, r, \theta, \phi)$ |
| Uppercase Greek | $A, \Sigma, \Omega$ | 2, 3 | spherical coordinate components, e.g. $(x^A) = (\theta, \phi)$ |
| Lowercase Latin | $i, j, k$ | 1, 2, 3 | spatial coordinate components, e.g. $(y^i) = (\rho, \theta, \phi)$ |
| Uppercase Latin | $I, J, K$ | 1 to $N$ | wave equation indexing, e.g. $u^I$ |
| Lowercase Calligraphic | $q, p, r$ | 0, 1 | time and radial coordinate components, e.g. $(x^q) = (t, r)$ |
| Uppercase Calligraphic | $Z, \hat{J}, K$ | 0, 1, 2, 3, 4 | first order wave formulation indexing, e.g. $V_j^\mu$ |

APPENDIX B. CONFORMAL TRANSFORMATIONS

In this section, we recall a number of formulas that govern the transformation laws for geometric objects under a conformal transformation that will be needed for our application to wave equations. Under a conformal transformation of the form
\[
\hat{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}, \tag{B.1}
\]
the Levi-Civita connection $\hat{\nabla}_\mu$ and $\nabla_\mu$ of $\hat{g}_{\mu\nu}$ and $g_{\mu\nu}$, respectively, are related by
\[
\hat{\nabla}_\mu \hat{\omega}_\nu = \nabla_\mu \omega_\nu - C^\lambda_{\mu\nu} \omega_\lambda,
\]
where
\[
C^\lambda_{\mu\nu} = 2\delta^\lambda_{(\mu} \nabla_{\nu)} \ln(\Omega) - g_{\mu\nu} \hat{g}^{\lambda\sigma} \nabla_\sigma \ln(\Omega).
\]
Using this, it can be shown that the wave operator transforms as
\[
\hat{g}^{\mu\nu} \hat{\nabla}_\mu \hat{\nabla}_\nu \hat{u} - \frac{n-2}{4(n-1)} \hat{R} \hat{u} = \Omega^{-1-\frac{2}{n}} \left( g^{\mu\nu} \nabla_\mu \nabla_\nu u - \frac{n-2}{4(n-1)} R u \right) \tag{B.2}
\]
where $\hat{R}$ and $R$ are the Ricci curvature scalars of $\hat{g}$ and $g$, respectively, $n$ is the dimension of spacetime, and
\[
\hat{u} = \Omega^{1-\frac{2}{n}} u. \tag{B.3}
\]
Assuming now that the scalar functions $\bar{u}_K$ satisfy the system of wave equations
\[
\hat{g}^{\mu\nu} \hat{\nabla}_\mu \hat{\nabla}_\nu \bar{u}_K - \frac{n-2}{4(n-1)} \hat{R} \bar{u}_K = \hat{f}^K, \tag{B.4}
\]
it then follows immediately from (B.2) and (B.3) that the scalar functions
\[
u^K = \Omega^{\frac{2}{n}} \bar{u}_K^{-1} \tag{B.5}
\]
satisfy the conformal system of wave equations given by
\[ g^{\mu\nu} \nabla_\mu \nabla_\nu u^K - \frac{n-2}{4(n-1)} R u^K = f^K \] (B.6)
where
\[ f^K = \Omega^{1+\frac{2}{n}} \tilde{f}^K. \] (B.7)
Specializing to source terms \( \tilde{f}^K \) that are quadratic in the derivatives, that is, of the form
\[ \tilde{f}^K = \tilde{\alpha}_{ij} K_{ij} \tilde{u}^i \tilde{u}^j, \] (B.8)
a short calculation using (B.1) and (B.5) shows that the corresponding conformal source \( f^K \), defined by (B.7), is given by
\[ f^K = \tilde{\alpha}^{K \mu \nu}_{ij} \left( \Omega^{3-\frac{2}{n}} \nabla_\mu u^i \nabla_\nu u^j + \left( \frac{n}{2} - 1 \right) \Omega^{1+\frac{2}{n}} \left( \nabla_\mu \Omega^{-1} u^i \nabla_\nu u^j + \nabla_\mu u^i \nabla_\nu \Omega^{-1} u^j \right) \\
+ \left( 1 - \frac{n}{2} \right) \Omega^{3-\frac{2}{n}} \nabla_\mu \Omega^{-1} \nabla_\nu \Omega^{-1} u^j \right). \] (B.9)

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