A calculation of the bulk viscosity in SU(3) gluodynamics

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We perform a lattice Monte-Carlo calculation of the trace-anomaly two-point function at finite temperature in the SU(3) gauge theory. We obtain the long-distance properties of the correlator in the continuum limit and extract the bulk viscosity \( \zeta \) via a Kubo formula. Unlike the tensor correlator relevant to the shear viscosity, the scalar correlator depends strongly on temperature. If \( s \) is the entropy density, we find that \( \zeta/s \) becomes rapidly small at high \( T \), \( \zeta/s < 0.15 \) at 1.65\( T_c \), and \( \zeta/s < 0.015 \) at 3.2\( T_c \). However \( \zeta/s \) rises dramatically just above \( T_c \), with \( 0.5 < \zeta/s < 2.0 \) at 1.02\( T_c \).

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Given the success of the hydrodynamical description [1] of high-energy heavy ion reactions, it is of primary interest to compute the shear and bulk viscosities of the quark-gluon plasma in the region \( T_c \leq T \leq 3T_c \) relevant to the RHIC and the forthcoming LHC experiments.

Phenomenology combined with hydrodynamics calculations point to a very small shear viscosity to entropy density ratio, \( \eta/s < 0.2 \) [2], and viscous calculations find even smaller values [3]. The main uncertainty in these determinations is the sensitivity to the initial conditions of the hydro-regime. A recent lattice calculation [4] of high-energy heavy ion reactions, it is of primary interest to compute the shear and bulk viscosities of the quark-gluon plasma in the region \( T_c \leq T \leq 3T_c \) relevant to the RHIC and the forthcoming LHC experiments.

In this Letter we carry out a lattice calculation of the bulk viscosity, \( \zeta \), in the SU(3) gauge theory, reusing much of the technology developed in [4]. The calculation relies on a Kubo formula and on the reconstruction of the real-time spectral function [6, 7, 8] from the Euclidean trace-anomaly correlator. This is known to be a numerically hard problem, in particular some assumptions have to be made on the spectral function in order to extract the viscosity — typically, an assumption of smoothness. In this respect, the bulk viscosity is a more favorable case than the shear viscosity at high temperatures, because the spectral function is rigorously known to be smooth at \( T \to \infty \). This is related to the fact that the bulk viscosity vanishes for any conformal field theory, and non-Abelian gauge theories become conformal at high energies by becoming free.

It has recently been argued [9] based on an exact sum rule, lattice data on \( \epsilon - 3P \) and a simple model for the spectral function, that the bulk viscosity rises sharply close to the (first order) phase transition. This is remarkable, because a gas of structureless point particles has negligible bulk viscosity both in the non-relativistic and the extreme-relativistic limits [10], and the bulk viscosity is often neglected for that reason even when the shear viscosity is taken into account. Furthermore, the leading-order perturbative QCD result [11] for \( \zeta/s \) is numerically very small: to within 10% for 0.06 < \( \alpha_s \) < 0.3,

\[
\zeta/s = 0.020 \alpha_s^2 \quad (N_f = 0).
\]

It satisfies the relation [10] \( \zeta \approx 15 \eta / [3 - v_s^2] \), \( v_s \) being the velocity of sound in the medium. Instead, AdS/CFT calculations that incorporate deviations from conformality at strong coupling find \( \zeta \propto \eta / [3 - v_s^2] \) [12]. These formulas suggest that the dynamics responsible for the bulk viscosity away from conformality, \( v_s^2 < \frac{1}{3} \), is different at strong and weak coupling.

We will show that \( \zeta/s \) depends strongly on temperature: it is phenomenologically negligible above 2\( T_c \), while it becomes O(1) just above \( T_c \). We will also acquire some knowledge of the spectral function from first principles they may help interpret this striking behavior.

**Methodology**

The explicit form of the Euclidean energy-momentum tensor \( T_{\mu\nu} \) can be found for instance in [13]; we write \( T_{00} = \mathcal{T}_{00} + \frac{1}{3} \theta \), where \( \theta \) is the trace anomaly. With \( L_0 = 1/T \) the inverse temperature, we consider the two-point functions \( (0 < x_0 < L_0) \)

\[
C_\theta(x_0) = L_0^3 \int d^3x \langle \theta(0) \theta(x_0, x) \rangle,
\]

\[
C_0(x_0) = L_0^3 \int d^3x \langle \mathcal{T}_{00}(0) \mathcal{T}_{00}(x_0, x) \rangle,
\]

\[
C_{\theta 0}(x_0) = L_0^3 \int d^3x \langle \mathcal{T}_{00}(0) \theta(x_0, x) \rangle.
\]

This correlator is represented by a spectral function,

\[
C_\theta(x_0) = L_0^5 \int_0^\infty \frac{\rho_\theta(\omega) \cosh \omega \left( \frac{L_0}{2} - x_0 \right)}{\sinh \frac{\omega}{2}} \, d\omega,
\]

in terms of which the bulk viscosity is given by [13]

\[
\zeta(T) = \frac{\pi}{9} \left. \frac{d\rho_\theta}{d\omega} \right|_{\omega=0}.
\]
Important properties of $\rho_0$ are its positivity, $\rho_0(\omega)/\omega \geq 0$ and parity, $\rho_0(-\omega) = -\rho_0(\omega)$. At tree level, $C_{\rho_0}^0(x_0) = 484d^4/9\pi^4\alpha_s^2 N^2 [\tau(\tau - \pi^2/60)]$, with $\tau$, $d_A$ and $f(\tau)$ as in [4], and

$$
\rho_{\theta,1}^0(\omega) = \frac{A_{1,1}}{7/2} \frac{N\omega^4}{\omega L_0}, \quad A_{1,1} = d_A \left( \frac{11\alpha_s N}{3(4\pi)^2} \right)^2.
$$

By contrast with the shear viscosity case [4], the absence of a $\delta$-function at the origin corresponds to the fact that the theory is conformal at tree-level and hence the bulk viscosity vanishes.

Because $\int d^4x \langle T_{\theta,0}(x)|O(0)\rangle = T^2 \partial_T \langle O \rangle_T$ for any local operator $O$ and $x_0 \neq 0$, the two other correlators are related to $C_\theta$ by

$$
C_\theta(x_0) = \frac{1}{16} C_\theta(x_0) + b_\theta(T),
$$

$$
C_{\theta,0}(x_0) = -\frac{1}{4} C_\theta(x_0) + b_\theta(T),
$$

$$
b_\theta(T) = \frac{1}{2} T^{-3} \partial_T (\epsilon - 3P) + 3T^{-3}s,
$$

$$
b_{\theta,0}(T) = T^{-3} \partial_T (\epsilon - 3P),
$$

i.e. they only differ by an $x_0$-independent shift. In perturbation theory, $b_\theta(T) = \frac{11\alpha_s}{12} (N\alpha_s)^2 + \mathcal{O}(\alpha_s^3)$ and $b_{\theta,0}(T)$ is also easily extracted from [14]. This completes the perturbative result for the Euclidean correlators up to $\mathcal{O}(\alpha_s^3)$.

We finally introduce the following moments of the spectral function $(n = 0, 1, \ldots)$:

$$
\langle \omega^{2n} \rangle \equiv \int_0^\infty d\omega \frac{\omega^{2n} \rho_0(\omega)}{\sin \omega L_0/2} = \frac{d^{2n}C_\theta}{dx_0^{2n}} \bigg|_{x_0=L_0/2}.
$$

The latter equality implies that they are directly accessible to lattice calculations. For a spectral function that grows like $\omega^4$ at large frequencies, it is natural to quote the pure numbers $\frac{N}{4\pi(4\pi)^{n/2}} \langle \omega^{2n} \rangle / T^{2n}$.

**Numerical results**

We simulate the isotropic Wilson action with a two-level algorithm [14], as in [4], and employ the ‘bare-plaquette’ discretization of $T_{\theta,0}$ and $\theta$ as in [18]. We use the data of [13] to determine the value of $T_{\theta,0}$ and to compute the lattice beta-function, and use $T_c r_0 = 0.751(1)$ [16] to obtain $T/T_c$.

We present the $C_\theta$ correlator, computed on $L_0/a = 8$ lattices in the temperature range $T_c < T < 3T_c$, in Fig. 1. It has been ‘improved’ by the same technique as in [4], so that cutoff effects are $\mathcal{O}(g_0^2 a^2)$. The deviations from conformality are significantly larger than those seen in the tensor channel relevant to the shear viscosity [4]. As $T$ approaches $T_c$ from above, they become very large.

We investigated finite-volume effects at 1.65 and 1.24$T_c$ by varying $LT$ from 3.5 to 5 and 2.5 to 3.67 respectively; no statistically significant dependence was observed.

Figure 2 shows the continuum extrapolation of the first two moments of $\rho_0$ from lattices $L_0/a = 6, 8, 10, 12$. The scaling region for $\langle \omega^0 \rangle$ starts at $L_0/a = 6$, while for $\langle \omega^2 \rangle$ it presumably only starts at $L_0/a = 8$ or 10. We also find $s/T^3 = 4.46(6)$ and $5.48(7)$ at $T = 1.24$ and $1.65T_c$ in the continuum.

On the lattice, the relations $[8, 11]$ are only satisfied up to $\mathcal{O}(a^2)$ corrections. They can be used to over-constrain the continuum extrapolation of the $C_\theta$ correlator. It turns out that the direct measurement of $C_\theta$ is more accurate than the estimators based on relations $[8, 9]$ by at least a factor five, and that $C_\theta$ has large discretization errors; we have nonetheless checked that the data is consistent with relations $[8, 11]$ in the continuum limit. While the spectral representation $[5]$ is in principle only valid in that limit, smaller lattice spacings will be necessary to extrapolate the whole correlator to the continuum.

We thus proceed to reconstruct the spectral function at the smallest lattice spacing, based on $N$ points of
The correlator at abscissas \( \{x_0^{(i)} > 2a\} \), with (strictly positive-definite) covariance matrix \( S_{\alpha\beta} \). We write \( \rho(\omega) = m(\omega) [1 + a(\omega)] \) where \( m(\omega) > 0 \) has the high-frequency behavior of Eq. 7. In practice we have used \( m(\omega) = A \omega^2/[\tanh(\frac{\omega}{2}L_0) \tanh^2(\omega L_0)] \) where \( c \) is a tunable parameter typically set to \( \frac{1}{2} \). We build an estimator of \( a(\omega), \hat{a}(\omega) = \sum_{i=1}^{N} u_i(\omega) \), using the same basis \( \{u_\ell\} \) as in [4]. Upon discretization of the \( \omega \) variable, \( M(x_0, \omega) = K(x_0, \omega) m(\omega) \) becomes an \( N \times N \), \( M_{ij} \), and the functions \( \{u_\ell\}_{\ell=1}^{N} \) are the columns of the matrix \( U \) in the singular-value decomposition of \( M \) = \( UVV^T \). The positivity of \( \rho(\omega) = m(\omega)[1 + \hat{a}(\omega)] \) provides an important a posteriori consistency check. The spectral functions obtained in this way from \( L_0/a = 12 \) lattices are shown on Fig. 3 for which \( N = 4 \).

Solving for the coefficients \( c_\ell \) results in \( \hat{a}(\omega) = \sum_{i=1}^{N} \Delta C_i q_i(\omega) \), with \( \Delta C_i = C(x_0^{(i)}) - \int d\omega M_i(\omega) \) and \( q_i(\omega) = V_d \omega_i^{-1} u_i(\omega) \), which is related to the genuine spectral function by

\[
\hat{a}(\omega) = \int_0^\infty d\omega' \delta(\omega, \omega') a(\omega'), \quad (13) \\
\hat{\delta}(\omega, \omega') = \sum_{i=1}^{N} q_i(\omega) M_i(\omega'). \quad (14)
\]

The resolution function \( \hat{\delta}(0, \omega) \) is plotted in Fig. 3. Clearly it has the effect of smoothing \( a(\omega) \) over a region \( \Delta \omega \approx 5T \) around the origin; note that \( a(\omega) \xrightarrow{\omega \to \infty} 0 \).

In solving inverse problems, it is often found necessary, when \( N \) becomes large, to introduce a 'regulator', whose role is to prevent the reconstruction from becoming unstable, at the cost of making the resolution function \( \delta(\omega, \omega') \) less sharply peaked around \( \omega' = \omega \). One way to implement this in our method is to replace the \( q_i(\omega) \) by \( [wV^2 + \lambda V^2]^{-1} u_i(\omega) \), where \( \lambda \) is typically chosen so that \( \rho \) yields a \( \chi^2/d.o.f. \) of order unity. We have however not found it necessary to make \( \lambda \neq 0 \) in any of the cases presented here.

The value of \( \hat{\rho}/\omega|_{\omega=0} \) and its statistical error, together with the resolution function, form an assumption-free representation of the information on \( \zeta \) contained in the lattice data. In order to quote a systematic error on \( \zeta \), some modelling of the spectral function is necessary.

Take for instance \( T = 1.24T_c \), where our best estimate is \( \hat{\zeta}/s = 0.065(17)_{\text{stat}} \). Increasing \( c \) from \( \frac{1}{2} \) to \( \frac{3}{4} \) reduces \( \hat{\zeta}/s \) to 0.010(2) but causes \( \hat{\rho} \) to assume negative values around \( \omega = 10T \) by one standard deviation. One may therefore regard this as a lower bound on \( \zeta/s \). We also infer a conservative upper bound based on the positivity of \( \rho \) [4]. Setting \( \hat{\rho} = \rho_{\text{full}} \). For \( \omega > 20T_c \), and introducing a Breit-Wigner of width \( 2T \) at the origin, we find that \( \zeta/s < 0.37 \) at 90\% confidence level in order for this contribution to the correlator not to exceed \( C_0(L_0/2) \). We obtain similarly \( \zeta/s < 0.15 \) at 1.65\( T_c \).

Bulk viscosity & non-conformality

It is clear from Fig. 3 that for \( T \gtrsim T_c \) the spectral function increases overall and also the spectral weight clusters around the origin. Both features contribute to an increase in the bulk viscosity, and they are robust predictions, since they essentially correspond to the increase of \( \langle u^0 \rangle \) and the decrease of \( \langle w^2 \rangle/(T^2 \langle w^0 \rangle) \). We now study the relation of this effect to non-conformality in more detail. As a measure of (non)-conformality, we use \( \frac{\epsilon - 3P}{\epsilon + P} \), which is easier to compute than the velocity of sound but to which it is related at weak coupling by

\[
1 - 3u_s^2 = \frac{4}{3}(\epsilon - 3P)/(\epsilon + P) \left[ 1 + O(\alpha_s) \right].
\]

We obtain \( \epsilon - 3P \) from \[10\] by quadratic interpolation and compute \( \epsilon + P \) ourselves. In Fig. 4 we show the
ratio $\zeta/s$ obtained on $L_0/a = 8$ lattices, where we have data at five different temperatures, as a function of this measure. We define a renormalized, “$(\epsilon - 3P)$-scheme coupling”, via \( \frac{\epsilon - 3P_0}{\epsilon + P} = \frac{55N^2\alpha_s^2(2\pi T)}{96\pi^2} \), and insert it into Eq. 1. That leads to the curve shown on the same plot. It lies generally below the lattice points and, unsurprisingly, does not account for the sharp rise of $\zeta$ close to $T_c$.

### Conclusion

We have computed the correlation functions of the energy-momentum tensor relevant to the bulk viscosity in the SU(3) pure gauge theory, as well as the entropy density $s$, to high accuracy. Our best estimates of the bulk viscosity, obtained on $L_0/a = 12$ lattices, are

\[
\zeta/s = \begin{cases} 
0.008(7) & (T = 1.65T_c, \ LT = \frac{4}{3}) \\
0.065(17) & (T = 1.24T_c, \ LT = \frac{2}{3})
\end{cases}
\]

where the statistical error is given and the square bracket specifies conservative upper and lower bounds. This constitutes the first successful lattice calculation of the bulk viscosity of an SU($N$) gauge theory. In Eq. 15 the bounds hold in the continuum. Since we now have estimates of both $\eta$ and $\zeta$ at $1.24T_c$, we can ask whether the relation $\zeta \approx 15\eta(1 - v_s^2)$ holds. Using $v_s^2 \approx 0.25$ [16], the formula predicts $\zeta/\eta \approx 0.1$, while our central value is 0.6. Our data thus points to a much larger bulk viscosity than the formula suggests. A non-negligible bulk viscosity is a signature of the interplay between translational and internal degrees of freedom; a small shear viscosity that of a strongly interacting system.

The bulk viscosity is non-vanishing below $1.3T_c$, where deviations from conformality in $\epsilon$ and $P$ are large. In particular we observe a sharp rise in $\zeta$ in an $L_0/a = 8$ simulation just above the first order phase transition,

\[
\zeta/s = 0.73(3) \left[ \frac{2.0}{0.5} \right] (T = 1.02T_c, \ LT = 3).
\]

Looking to ‘ordinary’ substances, it is worth noting that the bulk viscosity of nitrogen remains finite on the liquid-vapor coexistence line [21], while it becomes very large close to the endpoint (for the case of xenon see [22]). Although QCD with physical $u, d, s$ quark masses seems to undergo a crossover rather than a phase transition [10], the rise of $\zeta$ is related to the large rate of increase of $(\epsilon - 3P)/T^4$ [9] and to the clustering of the spectral weight towards $\omega = 0$ (Fig. 2). We therefore also expect an increase in the bulk viscosity around 200MeV in the real world. The effect could manifest itself in heavy ion collisions by an enhanced entropy production in the later stages of the reaction, provided the viscous hydrodynamical description is still valid at that time.

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