Comments Concerning the CFT Description of Small Objects in AdS

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Abstract

In this paper we resolve a contradiction posed in a recent paper by Horowitz and Hubeny. The contradiction concerns the way small objects in AdS space are described in the holographic dual CFT description.
1 The Apparent Contradiction

According to the Holographic Principle [1, 2] objects deep in the interior of a spatial region should have a description in terms of a holographic theory that in some sense lives on the region’s boundary. A concrete realization of this idea has been given by the AdS/CFT duality [3, 4, 5]. It has become a subject of active investigation to find out exactly how particular objects in AdS space are represented in the corresponding conformal field theory.

There are two apparently contradictory claims in the literature concerning this question. According to [6, 7] the field theoretic representation of an object or event far from the AdS boundary is through nonlocal operators such as Wilson loops whose degree of nonlocality increases as the object recedes from the boundary. This follows from the usual UV/IR correspondence [8].

In apparent contradiction with this view, Horowitz and Hubeny in an extremely interesting paper [9] have presented evidence that local operators of very high dimensionality contain information about the size and shape of small objects. The resolution of this conflict will be seen to lie in an ambiguity in what we mean when we say that a bulk quantity is represented by a certain field theoretic quantity.

We will begin by very briefly describing the results of Horowitz and Hubeny. These authors consider objects in $AdS_5 \times S_5$ which are much smaller than the AdS radius of curvature and which are localized at a point on the 5-sphere. In addition they are also localized near the center of AdS in appropriately chosen coordinates. Using the gravitational dual theory they find that the expectation values of certain scalar operators of dimension $n = l + 4$ are of order

$$\langle \Phi_n \rangle \sim e^{c(1 - \rho^{-1})}$$

where $\rho$ is the size of the object in units of the AdS radius and $c$ is a numerical constant which we will ignore. It follows from eq. (1.1) that for $l > \rho^{-1}$ the signal is appreciable and that by examining the $l$ dependence the size of the object can be determined. This apparently contradicts [6, 7].

Our conventions for describing AdS space are as follows: Global coordinates for AdS can be defined so that the metric is given by

$$ds^2 = \frac{R^2}{(1 - r^2)^2} \left\{ (1 + r^2)^2 dt^2 - 4 dr^2 - 4 r^2 d\Omega^2 \right\}$$

By small we do not mean microscopic. The objects we have in mind may be macroscopic and classical as long as they are much smaller than the AdS radius.
where the radial coordinate runs from 0 to 1. Here, \( R \) is the radius of curvature of the space and \( d\Omega^2 \) represents the metric of a unit sphere. In the case of \( AdS_5 \) the sphere is a 3-sphere.

These coordinates are especially useful when trying to recover infinite flat space in the limit \( R \to \infty \). Indeed the \( AdS \) space as defined above behaves in many respects like a finite cavity of size \( R \) with a reflecting boundary at \( r = 1 \). We will refer to the metric in eq. (1.2) as cavity coordinates.

Another coordinate system which is particularly useful when studying the properties of the CFT in flat space is given by

\[
d s^2 = \frac{R^2}{y^2} (d\tilde{t}^2 - dx^2 - dy^2)
\]

(1.3)

where \( \tilde{t}, x \) labels 4-dimensional Minkowski space and \( y \) is the 5th direction perpendicular to \( x \). At time \( \tilde{t} = 0 \), the center of AdS can be taken to be the point \( x = 0, y = 1 \) in these co-ordinates.

The quantum field theory lives on the 4-dimensional boundary whose metric we take to be either

\[
d s_b^2 = dt^2 - d\Omega^2
\]

(1.4)

in the case of cavity coordinates, or

\[
d s_b^2 = d\tilde{t}^2 - dx^2
\]

(1.5)

for the flat space representation.

As in ref. [8] we will be thinking of the boundary quantum field theory in Wilsonian terms. Thus we imagine the boundary field theory to be defined in terms of a bare set of degrees of freedom at some very small coordinate length scale \( \delta_0 \). The details of the cutoff are not important but an example to keep in mind is the Hamiltonian lattice cutoff sometimes used to study QCD. In that case \( \delta_0 \) would be the spatial lattice constant. We assume \( \delta_0 \) is much smaller than any other length scale we will encounter.

According to the UV/IR correspondence, a connection exists between the UV regulator length scale \( \delta_0 \) of the QFT and an IR cutoff of the bulk theory. The bulk cutoff is implemented by replacing the \( AdS \) boundary at \( r = 1 \) or \( y = 0 \) with the surface \( r = 1 - \delta_0 \) in global \( AdS \) or \( y = \delta_0 \) in the Poincare patch.

There are two different large \( N \) limits of the QFT that have two different purposes. The first is the ‘t Hooft limit

\[
N \to \infty
\]
From the bulk point of view this is the limit of classical gravity or classical string theory in an $AdS$ space of fixed radius in string units;

\[ R = (g_s N)^{1/4} l_s. \]  

In this limit the ratio of the size of a physical object to the $AdS$ radius is fixed.

The limit of interest for analyzing the holographic principle as defined in [1][2] is a different one [10][11]. For this purpose we take

\[ N \rightarrow \infty, \quad g_s = \text{constant}. \]  

In this limit the $AdS$ radius becomes much larger than the size of any physical object. This is the limit discussed in [3] where it was claimed that objects at $r = 0$ or $y = 1$ should be represented by Wilson loops of roughly unit size.

2 What it Means to Represent

One reason for confusion is that different people may mean different things when they say a certain field theory quantity represents a corresponding bulk quantity. In order to resolve the paradox raised by the Horowitz-Hubeny result we need to have a clear understanding of what it means for a particular set of observables in the CFT to describe a particular set of circumstances in the $AdS$ space.

The ability to find observables (hermitian operators) in the field theory to represent physical quantities in the bulk theory follows from the assumption that the Hilbert space of bulk states is the same as that of the boundary field theory. We argue that faithfully representing a given bulk quantity $\alpha$ by a field theory quantity $A$ should mean more than just requiring a correspondence between their expectation values. Ideally we would like the probability distribution for $\alpha$ and $A$ to be the same. For example, a faithful representation of a highly classical bulk quantity such as the size of a macroscopic object should involve a field theory quantity with very small fluctuation.

Let us suppose that the quantum state of the system determines a probability distribution $P(\alpha)$ centered at $\alpha_0$ with a width $\Delta(\alpha)$. Now consider a second state characterized by a second distribution $P'(\alpha)$ centered at $\alpha'_0$. These two states are clearly distinguishable if
the two distributions do not overlap. In particular two different macroscopic classical configurations of $\alpha$ should have negligible overlap in their probabilities. A minimum condition for $A$ to faithfully represent $\alpha$ is that two probability distributions for $A$ will not overlap if the two corresponding distributions do not overlap for $\alpha$. In other words two configurations which disagree on the value of $\alpha$ must be represented by probability distributions in $A$ which are almost orthogonal. We will regard this to be a minimal requirement for a faithful representation of a bulk variable by a corresponding holographic variable.

3 The Resolution

Let us now ask whether the high dimension operators $\Phi_n$ are a faithful representation of the size of an object at the center of $AdS$ space. From what we have said in the previous section the question comes down to whether or not the probability distributions for the $\Phi$’s are orthogonal or almost orthogonal for two classically distinguishable values of the size $\rho$. We emphasize again that we are working in a Wilsonian framework where it is assumed that the field theory is defined by a concrete regularized system.

The $\Phi_n$’s are defined to have vanishing vacuum expectation values. We must also specify a convention for normalizing them. We follow the same convention as in [9], namely the two point function $\langle \Phi_n(x)\Phi_n(x') \rangle$ is of order one at unit coordinate separation. Now consider the width of the probability distribution for $\Phi_n$, in other words the fluctuation $\Delta$ in $\Phi_n$:

$$\Delta^2 = \langle \Phi_n^2 \rangle.$$  

(3.1)

Obviously if the difference in expectation values of $\Phi_n$ for two distinct configurations is much less than $\Delta$ then these variables do not faithfully represent the variables they were intended to describe.

The scalar fields $\Phi_n$ are given by

$$\Phi_n = Tr F^2 X^l$$  

(3.2)

where $X$ stands for the six fundamental scalars of maximally supersymmetric $SU(N)$ Super Yang-Mills Theory. The trace is over the adjoint representation of $SU(N)$ and $X^l$ represents a polynomial of order $l$ in the $X$’s. The dimension of $\Phi_n$ is $n = l + 4$. The operators are normal ordered meaning that their vacuum expectation value has been subtracted out. They are normalized so that their two point function at unit coordinate separation is of order one.
Now consider the fluctuation in $\Phi_n$. This is given by the square root of the connected two point function at vanishing separation. In the continuum theory this fluctuation will be divergent. In the Wilsonian cutoff theory the fluctuation will be of order $\delta_0^{-n}$ which we assume is extremely large $^2$. Thus unless these operators are somehow further regulated the fluctuation is divergent.

This is true for any local operator $\Phi$. It means that $\Phi$ can not faithfully describe anything. A measurement of $\Phi$ gives completely random results in any state. This point was made forcefully in a famous paper by Bohr and Rosenfeld in the earliest days of quantum field theory. According to Bohr and Rosenfeld the correct observables for a quantum field theory are what we would today call "regulated" fields. This entails introducing a regulator scale $\delta$ chosen to be much larger than the cutoff scale $\delta_0$. The observables are defined by some form of smearing or point-splitting of the composite operators $\Phi$. This will be discussed further in the next section.

The regulated fluctuation in $\Phi_n$ is of order

$$\Delta_n \sim \delta^{-(l+4)}.$$  \hspace{1cm} (3.3)

This follows from dimensional analysis and the fact that $\Phi_n$ has mass dimension $n = l + 4$.

Evidently from eq. (1.1), the condition that the expectation value of $\Phi_n$ is larger than the fluctuation is

$$e^{\epsilon \rho l} \geq \delta^{-(l+4)}.$$ \hspace{1cm} (3.4)

Since $\rho$ is defined to be the size of an object measured in units of $R$, it will vanish in the limit $R \to \infty$. Thus we find that the inequality is satisfied only if

$$\delta \sim 1.$$ \hspace{1cm} (3.5)

In other words the operators $\Phi_n$ must not only be regulated but the regulator scale has to be comparable to the coordinate distance of the object from the $AdS$ boundary. The meaning of this is clear. For an operator to faithfully represent a property of a small object near the center of $AdS$, it must be non-local as described in $^{[6]}[7]$.

4 Regulating $\Phi$

Granted that we must regulate the operators $\Phi$, the question arises as to exactly how to do so. We will begin with an implicit construction. The problem with the unregulated

$^2$This is true for the expectation value in any generic low energy state since the expectation value will be dominated by high energy intermediate states.
operators is that they have large matrix elements connecting two very high energy states. In calculating their fluctuation most of the contribution comes from these high energy intermediate states. We can easily regulate the operators by simply throwing away the matrix elements between states whose energies differ by more than $\delta^{-1}$. Equivalently we can integrate the operators over time with a smooth test function with support over a time interval $\sim \delta$. By solving the equations of motion we can express the regulated operator in terms of operators at a fixed time. The result will be spatially nonlocal over a scale $\delta$. As we have seen in the previous section, $\delta$ must be $\sim 1$ so that the regulated operator is nonlocal over the entire boundary sphere.

On the other hand smearing an operator over time will not change its expectation value in a time independent configuration. Thus, for such configurations, the local operator and its nonlocal counterpart have the same expectation value. This accounts for the results in [3]. However, for time dependent configurations such as those described in [6][7] only the nonlocal operator faithfully represents the relevant instantaneous property of a small object near $r = 0$.

The field theory of interest in this paper is a gauge theory in which all fundamental fields are in the adjoint representation. If the bare theory is a Hamiltonian lattice gauge theory, then any operator at a fixed time can be expressed in terms of generalized Wilson loops in which the Wilson loop is “decorated” with insertions of adjoint fields. The regulated operators will be expressible as linear superpositions of such Wilson loops of size $\delta$. Horowitz and Hubeny provide important information on how to decorate the Wilson loop in order to describe particular features of small objects.

Another important example concerns the “precursors” described in [6][7]. Suppose that an event takes place near the center of $AdS$ which results in the emission of a wave propagating towards the boundary. Bulk causality ensures that all local supergravity fields evaluated within a neighborhood of the boundary will retain their original expectation values until the wave itself arrives at the boundary. Therefore, for a period of time of order one, all local QFT operators corresponding to the bulk supergravity fields will retain their original expectation values carrying no information about the wave. At some time, when the wave arrives at the boundary ($t = 0$), some local operators in the QFT will begin to oscillate. According to the AdS/CFT correspondence, their expectation value at $t = 0$ will be given by the boundary data of the wave. Furthermore, their expectation value will be insensitive to the $R \to \infty$ limit and proportional to the amplitude of the wave. Thus, in regulating the local operators at $t = 0$ so as to keep the signal bigger than
their fluctuations, we only need to introduce a cutoff of order the width of the wave pulse. Now to find the non-local “precursors” describing the wave at an earlier time, when say the wave is at co-ordinate distance $\delta$ from the boundary, we use the equations of motion to express the regulated local operators at $t = 0$ in terms of operators at $t = -\delta$. Then, the “precursors” will be spatially non-local over a scale $\delta$. The results of [7] suggest that the resulting non-local operator will involve superpositions of Wilson loops of size $\delta$.

A point worth mentioning involves the possibility of constructing operators with small fluctuation by spatially averaging $\Phi$ over $\Omega$. It is not hard to see that this diminishes its fluctuation by a factor $\delta^{3/2}$. This would have no important effect on our conclusion.

Finally we want to emphasize that expectation values are not the observables of a system. The observables representing the results of measurements have uncertainties. A correct representation of a variable should not only represent its expectation value but also its entire probability distribution. The wild fluctuation of local fields makes them bad representations of weakly fluctuating positions and sizes of macroscopic objects.

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