THE SYMMETRIC SQUARE OF THE THETA DIVISOR IN GENUS 4

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INTRODUCTION

Let \((X, \Theta)\) be a principally polarized abelian variety (ppav) over \(\mathbb{C}\) of dimension \(g \geq 2\) with a symmetric theta divisor \(\Theta\). The intersections

\[ Y_x = \Theta_x \cap \Theta_{-x}, \]

where \(\Theta_x = \Theta + x\) denotes the translate of \(\Theta\) by \(x \in X(\mathbb{C})\), have been studied in relation with Torelli’s theorem [Deb1] and the Schottky problem [Deb3]. We examine the variation of \(\mathbb{Q}\)-Hodge structure on \(H^\bullet(Y_x, \mathbb{Q})\) for \(x \in X(\mathbb{C})\) varying. For example, if \(\Theta\) is smooth, \(Y_x\) is smooth of dimension \(g - 2\) for generic \(x \in X(\mathbb{C})\). Then by the weak Lefschetz theorem the cohomology of \(Y_x\) is obtained by restriction from the cohomology of \(X\) except for the middle cohomology degree. So we only need to consider the quotient

\[ H = H^{g-2}(Y_x, \mathbb{Q}) / H^{g-2}(X, \mathbb{Q}). \]

The involution \(\sigma = -id_X\) acts on \(Y_x\) and thereby induces an eigenspace decomposition \(H = H_+ \oplus H_-\). The eigenspaces \(H_{\pm}\) are the fibres of two variations \(\mathcal{V}_{\pm}\) of \(\mathbb{Q}\)-Hodge structures which we study in section 1.

Conjecture A. If \(\Theta\) smooth, then \(\mathcal{V}_{\pm}\) are simple.

Viewed as Hodge modules in the sense of [Sa2], the variations \(\mathcal{V}_{\pm}\) of Hodge structures have two underlying perverse sheaves \(\delta_{\pm}\) on \(X\), see section 3. Thus conjecture A would follow from

Conjecture B. If \(\Theta\) smooth, the perverse sheaves \(\delta_{\pm}\) are simple.

Up to a skyscraper sheaf, we construct \(\delta_{\pm}\) in section 3.4 as the symmetric resp. alternating square of the perverse intersection cohomology sheaf \(\delta_{\Theta}\) in the tensor category introduced in [We2]. This category is equivalent to the category \(\text{Rep}(G)\) of algebraic representations of some (in general unknown) algebraic group \(G = G(X, \Theta)\). So conjecture B would be a consequence of
Conjecture C. If \( \Theta \) is smooth, then
\[
G(X, \Theta) = \begin{cases} 
\text{SO}(g!, \mathbb{C}) & \text{for } g \text{ odd}, \\
\text{Sp}(g!, \mathbb{C}) & \text{for } g \text{ even},
\end{cases}
\]
and \( \delta_\Theta \) corresponds to the standard representation of this group.

A motivation for this is given in section 4.3. In fact, via an analysis of small representations \([KrW1]\) and a Mackey argument, we will see in section 11 that conjectures B and C are equivalent.

Since every ppav of dimension \( g \leq 3 \) with a smooth theta divisor is a Jacobian, the above conjectures are true in these cases \([We]\). The first new case appears for \( g = 4 \). If we drop the assumption that \( \Theta \) is smooth, this is also the first non-trivial case for the Schottky problem (viz. to characterize the Jacobians among all ppav’s). In section 5 we discuss the relationship between the Schottky problem and our conjectures in this case.

The main result of this paper is the proof of conjecture C for \( g = 4 \). In the proof given in sections 9 and 10 we consider a degeneration of \((X, \Theta)\) into the Jacobian variety of a generic curve of genus 4. This provides a restriction functor \( \rho : \text{Rep}(G(X, \Theta)) \rightarrow \text{Rep}(G_\Psi) \), where \( G_\Psi \subset G(X, \Theta) \) is an algebraic subgroup defined via the formalism of the nearby cycles \( \Psi \) as outlined in section 6. To prove the conjecture we compare \( \rho \) with another functor \( \text{MT} \) from \( \text{Rep}(G(X, \Theta)) \) to the category of representations of the Mumford-Tate group of \( \Theta \). For the study of this second functor the main input will be the knowledge of the primitive cohomology of \( \Theta \) which we express in section 2 in terms of the intermediate Jacobian of a cubic threefold as in \([Do]\) and \([Iz]\), using results of \([Co]\), \([CM]\) and \([IvS]\).

1. Variations of Hodge structures (conjecture A)

In this section we assume that \( \Theta \) is smooth, and we study the variations \( V_\pm \) of \( \mathbb{Q} \)-Hodge structures and their Hodge decomposition for \( g \leq 4 \).

1.1. Smoothness of \( Y_x \). For generic \( x \in X(\mathbb{C}) \) the translates \( \Theta_x \) and \( \Theta_{-x} \) intersect each other transversely and hence \( Y_x \) is smooth.

Proof. \( \Theta \) is defined by the zero locus of the Riemann theta function \( \theta(z) = \theta(\tau, z) \) on the universal covering \( p : \mathbb{C}^g \rightarrow \mathbb{C}^g/(\mathbb{Z}^g + \tau \mathbb{Z}^g) \cong X(\mathbb{C}) \). For smooth \( \Theta \) the gradient \( \theta'(z) \) is non-zero for all \( z \in \mathbb{C}^g \) with \( \theta(z) = 0 \). If our claim were false, we could find a non-empty analytic open subset \( V \subset \mathbb{C}^g \) and a complex analytic map \( s : V \rightarrow \mathbb{C}^g \) such that \( \Theta_{p(z)} \) and \( \Theta_{-p(z)} \) intersect non-transversely.
in $s(z)$ for all $z \in V$, i.e. such that

\begin{equation}
\theta(s(z) + z) = \theta(s(z) - z) = 0, 
\end{equation}

\begin{equation}
\theta'(s(z) + z) = \lambda(z) \cdot \theta'(s(z) - z)
\end{equation}

for all $z \in V$. With the $g \times g$ unit matrix $E$ and the Jacobian matrix $(Ds)(z)$ of $s$ at $z$, taking gradients of (1.1.1) implies

\begin{equation}
(E + (Ds)(z)) \cdot \theta'(s(z) + z) = 0,
\end{equation}

\begin{equation}
(E - (Ds)(z)) \cdot \theta'(s(z) - z) = 0.
\end{equation}

Now multiply (1.1.4) by $\lambda(z)$. If we plug in (1.1.2) and add (1.1.3), we get $\theta'(s(z) + z) = 0$ contradicting the smoothness of $\Theta$. \hfill \Box

1.2. Variation in a family. Over some Zariski-open dense $U \subset X$ there exists a smooth proper family $\pi : Y_U \to U$ with fibres $\pi^{-1}(2x) \cong Y_x$ and an étale involution $\sigma : Y_U \to Y_U$ with $\sigma|_{\pi^{-1}(2x)} \cong (-id_X)|_{Y_x}$ for $2x$ in $U(\mathbb{C})$.

To construct such a family, let $\pi : \Theta \times \Theta \to X$ be the addition map. Projecting from $\pi^{-1}(2x) \subset \Theta \times \Theta$ onto the first factor and then translating by $x$, we get an isomorphism $\varphi : \pi^{-1}(2x) \cong \Theta_x \cap \Theta_{-x}$ such that via $\varphi$ the involution $\sigma : \Theta \times \Theta \to \Theta \times \Theta$, $(t_1, t_2) \mapsto (t_2, t_1)$ becomes identified with $(-id_X)|_{\Theta_x \cap \Theta_{-x}}$. By (1.1) the fibres of $\pi$ are generically smooth. So there is an open dense $U \subset X$ such that for $Y_U = \pi^{-1}(U)$ the restriction $\pi = \pi|_{Y_U}$ is smooth, and for sufficiently small $U$ the involution $\sigma$ will be étale on $Y_U$.

1.3. Definition of $\mathcal{V}_ \pm$. For all $\nu$, the higher direct images $R^\nu \pi_*(Q_{Y_U})$ are variations of $\mathbb{Q}$-Hodge structures [PS, cor. 10.32]. It is easy to construct a constant subvariation $H^\nu \to R^\nu \pi_*(Q_{Y_U})$ such that at every point $2x \in U(\mathbb{C})$, the fibre $H^\nu \subset R^\nu \pi_*(Q_{Y_U})_{2x} = H^\nu(Y_x, \mathbb{Q})$ is the pull-back of the cohomology of $X$ to $Y_x$. In particular $H^\nu \cong H^\nu(Y_x, \mathbb{Q})$ for all $\nu \neq g - 2$. We put

$$
\mathcal{V} = R^{g-2} \pi_*(Q_{Y_U})/H^{g-2}
$$

and define $\mathcal{V}_\pm$ as the eigenspace of $\sigma^*$ for the eigenvalue $\pm(-1)^{g-1}$, i.e.

$$
\mathcal{V}_\pm = \ker(\sigma^* \mp (-1)^{g-1} id_\nu).
$$

The reason for this choice of signs will become clear later (see 3.4).

1.4. Small-dimensional cases. In this section we fix a point $x$ in $X(\mathbb{C})$ such that $2x \in U(\mathbb{C})$. We use the notation $Y = Y_x$, and we define $Y^+$ as the quotient $Y/\langle \sigma \rangle$ of $Y$ by the involution $\sigma = -id_X|_Y$.

The case $g = 2$. Here $Y$ consists of 2 points, $Y^+$ is a single point, $\mathcal{V}_- = 0$ and $\mathcal{V}_+ = \mathbb{Q}_U$ is the constant variation with Hodge degree $(0,0)$.

The case $g = 3$. This case has been studied in [Re], [Kr1]. Here $(X, \Theta)$ is the Jacobian of a smooth curve $C$ and $Y^+ \to Y$ is an étale double cover of
smooth curves with $Y$ of genus 7. Now to any étale double cover of curves one may associate a ppav, its Prym variety [MuI] [BL ch. 12]. It turns out that for generic $(X, \Theta)$ the Prym variety $P$ of the cover $Y \to Y^+$ is isomorphic to $(X, \Theta)$. Every étale double cover with this Prym variety arises like this for some $x$. Furthermore, the coverings for two points $x_1, x_2$ are isomorphic iff $x_1 = \pm x_2$. Hence the étale double covers with given Prym variety $(X, \Theta)$ are parametrized by an open dense subset $W$ of the Kummer variety $X/(\pm 1)$. Points outside $W$ parametrize degenerate double covers.

By construction of Prym varieties, the Jacobian variety $JY$ is isogenous to $P \times JY^+$, hence $H^1(Y, \mathbb{C}) \cong H^1(X, \mathbb{C}) \oplus H^1(Y^+, \mathbb{C})$. Thus $\mathcal{V}_- = 0$, and $\mathcal{V}_+$ is a variation of Hodge structures of abelian type whose fibres $H^1(Y^+, \mathbb{Q}) = \mathbb{Q}^8$ have Hodge degrees $(1, 0)$ and $(0, 1)$.

The case $g = 4$. Here $Y \to Y^+$ is an étale covering of smooth surfaces. We claim that $\mathcal{V}_+$ has rank 52, with Hodge degrees $(2, 0)$ and $(0, 2)$ of rank 11 and Hodge degree $(1, 1)$ of rank 30, whereas $\mathcal{V}_-$ is a variation of Hodge structures of rank 6 purely concentrated in Hodge degree $(1, 1)$.

In particular, by the Lefschetz $(1, 1)$-theorem the fibers of $\mathcal{V}_+$ are spanned by six cycles on $Y^+$ which generate $H^{1,1}(Y^+, \mathbb{C})/H^{1,1}(X, \mathbb{C})$. So $\mathcal{V}_+$ has an underlying finite monodromy group $\Gamma = \Gamma(X, \Theta)$ with a six-dimensional faithful representation. Computing $H^2(X, \mathbb{Z})$ and $H^2(Y^+, \mathbb{Z})$ and using a classification of lattices with small discriminant, one sees that the fibres of $\mathcal{V}_-$ have the underlying Néron-Severi lattice $E_6(-1)$, so $\Gamma$ is a subgroup of $\text{Aut}(E_6) = W(E_6) \times \{\pm 1\}$. From the intersection configuration of the 27 Prym-embedded curves in $Y$ of [Z sect. 4.3], one then deduces that the projection $W(E_6) \times \{\pm 1\} \to W(E_6)$ maps $\Gamma$ isomorphically onto a subgroup of $W(E_6)$. To get a lower bound on $\Gamma$ consider a degeneration of $(X, \Theta)$ into a Jacobian variety $(X_0, \Theta_0)$. Let $\Gamma_0 = \Gamma(X_0, \Theta_0)$ be the monodromy group underlying the analog of $\mathcal{V}_-$ on $(X_0, \Theta_0)$, then $\Gamma_0$ is a subquotient of $\Gamma$. In the Jacobian case the configuration of the 27 Prym curves of loc. cit. is no longer symmetric; precisely 12 of them are smooth. They come in 6 pairs of curves which are interchanged by the involution $\pm id_{X_0}$, and the associated 6 cycles are permuted by the monodromy operation of $\Gamma_0$. From this one deduces that $\Gamma_0$ contains the alternating group $A_5$. Altogether one then concludes that $\Gamma$ must be the Weyl group $W(E_6)$ or its finite simple subgroup of index 2. For the details see [Kr2].

Proof of the claim about the Hodge decomposition of $\mathcal{V}_\pm$. The Hodge numbers of the fibres of $\mathcal{V}_\pm$ are $h^{p,q}(Y) - h^{p,q}(Y^+)$ resp. $h^{p,q}(Y^+) - h^{p,q}(X)$ for $p + q = 2$, so it suffices to check the following table:

| $Y$ | $h^{2,0} = h^{0,2}$ | $h^{1,1}$ | $h^{1,0}$ = $h^{0,1}$ |
|-----|---------------------|-----------|---------------------|
| $Y^+$ | 17 | 52 | 4 |
| $Y$ | 6 | 22 | 0 |
The last column of this table follows from $H^1(Y, \mathbb{Q}) \cong H^1(X, \mathbb{Q})$. To check the remaining two columns, put $L_x = \mathcal{O}_X(\Theta_x)$. Since $\mathcal{T}_X = (\mathcal{O}_X)^{\oplus 4}$ and hence $c_1(T_X|_Y) = 1$, the exact sequences $0 \to \mathcal{T}_{\Theta_4}|_Y \to \mathcal{T}_X|_Y \to L_x|_Y \to 0$ and $0 \to \mathcal{T}_Y \to \mathcal{T}_{\Theta_4}|_Y \to \mathcal{L}_{-x}|_Y \to 0$ show $1 = c_1(T_Y)(1 + c_1(L_{-x}|_Y)t)(1 + c_1(L_x|_Y)t)$. Since $L_x$ and $\mathcal{L}_{-x}$ are numerically equivalent to $L = \mathcal{O}_X(\Theta)$, this gives $c_2(T_Y) = 3c_2(L|_Y)$. Then $\chi(Y) = \deg_Y c_2(T_Y) = 3 \deg_Y c_2^2(L)|_Y = 3 \deg_X c_1^4(L) = 3 \cdot 4! = 72$ by the Gauss-Bonnet and Poincaré formulæ. It follows that $h^2(Y) = 86$ since $h^0(Y) = h^4(Y) = 1$ and $h^1(Y) = h^3(Y) = 8$ by the hard Lefschetz theorem. Furthermore, from $c_1(T_Y) = -2c_1(L|_Y)$ we get $\deg_Y c_2^2(T_Y) = 4 \deg_Y c_2^2(L)|_Y = 96$, hence $1 - 4 + h^{0,2}(Y) = \chi(O_Y) = (c_2^2(T_Y) + c_2(T_Y))/12 = 14$ by Hirzebruch-Riemann-Roch. So $h^{0,2}(Y) = 17$. Finally, since $Y \to Y^+$ is an étale double cover, $\chi(Y^+) = \chi(Y)/2 = 36$ and $\chi(O_{Y^+}) = \chi(O_Y)/2 = 7$. Using that $h^0(Y^+) = h^4(Y^+) = 1$ and $h^1(Y^+) = h^3(Y^+) = 0$ since $\sigma$ acts by $(-1)^k$ on $H^k(X, \mathbb{C})$, it follows that $h^2(Y^+) = 34$ and $h^{0,2}(Y^+) = 6$.

2. The Mumford-Tate group MT($\Theta$) in Genus 4

One of the key ingredients to the proof of conjecture \[Del1, sect. 2.1\], that giving a $\mathbb{Q}$-Hodge structure is tantamount to giving a finite-dimensional vector space $V$ over $\mathbb{Q}$ and a homomorphism $h : \mathbb{S} = \text{Res}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m,\mathbb{C}}) \to \text{Gl}(V)_{\mathbb{R}}$ of real algebraic groups whose composite with the weight cocharacter of $\mathbb{S}$ is defined over $\mathbb{Q}$. The Mumford-Tate group MT($V$)
is defined as the smallest algebraic subgroup of \( \text{Gl}(V) \) over \( \mathbb{Q} \) through which \( h \) factors. The \( \mathbb{Q} \)-Hodge substructures of any tensor power of \( V \) are precisely its \( \text{MT}(V) \)-stable subspaces. The Mumford-Tate group of polarized \( \mathbb{Q} \)-Hodge structures is reductive [Dei 4, prop. 3.6]; this in particular applies to \( \text{MT}(X) := \text{MT}(H^\bullet(X, \mathbb{Q})) \) for smooth projective varieties \( X \) over \( \mathbb{C} \).

2.1. The Hodge structure \( B \). Let \( \Lambda^i \) denote the \( i \)-th exterior power of the standard representation of \( \text{Sp}(2g, \mathbb{Q}) \). For every ppav \( (X, \Theta) \) outside a meager subset of \( \mathcal{A}_g \) then by [BL, prop. 17.3.2]

\[
\text{MT}(X) = \text{MT}(X)_{sc} = \text{Sp}(2g, \mathbb{Q}) \quad \text{and} \quad H^i(X, \mathbb{Q}) = \Lambda^i.
\]

If \( \Theta \) is smooth, the weak Lefschetz theorem shows that for some \( \mathbb{Q} \)-Hodge structure \( B \) and \( 0 \leq |i| \leq g - 1 \)

\[
H^{g-1+i}(\Theta, \mathbb{Q}) = \begin{cases} 
\Lambda^{g-1-|i|} & \text{for } i \neq 0, \\
\Lambda^{g-1} \oplus B & \text{for } i = 0,
\end{cases}
\]

in particular \( \text{MT}(\Theta) = \text{MT}(H^\bullet(X, \mathbb{Q}) \oplus B) \). Now it is well-known that for any \( \mathbb{Q} \)-Hodge structures \( V_1, V_2 \) and \( V = V_1 \oplus V_2 \) we have a closed embedding \( \iota : \text{MT}(V) \hookrightarrow \text{MT}(V_1) \times \text{MT}(V_2) \). The image of \( \iota \) surjects onto each of the two factors. If the Mumford-Tate groups are reductive, this surjectivity carries over to \( \text{MT}(\_\_)_{sc} \), and the kernel of the induced map \( \iota_{sc} : \text{MT}(V)_{sc} \to \text{MT}(V_1)_{sc} \times \text{MT}(V_2)_{sc} \) is contained in the kernel of the natural map \( \text{MT}(V)_{sc} \to \text{MT}(V) \). Hence we have a commutative diagram

\[
\begin{array}{ccc}
\text{MT}(\Theta)_{sc} & \longrightarrow & \text{MT}(B)_{sc} \\
\downarrow \iota_{sc} & & \downarrow \text{id} \\
\text{Sp}(2g, \mathbb{Q}) \times \text{MT}(B)_{sc} & \longrightarrow & \text{MT}(B)_{sc}
\end{array}
\]

with \( \ker(\iota_{sc}) \subseteq \ker(\text{MT}(\Theta)_{sc} \to \text{MT}(\Theta)) \).

**Lemma.** To prove the theorem, it suffices to show that for every ppav \( (X, \Theta) \) outside a meager subset of \( \mathcal{A}_4 \), the Hodge structure \( B \) is simple.

**Proof.** Indeed, the simplicity of \( B \) is equivalent to \( \text{End}_{\text{MT}(B)}(B) = \mathbb{Q} \). By [Ri, th. 1] this is equivalent to \( \text{MT}(B) = \text{MT}(B)_{sc} = \text{Sp}(10, \mathbb{C}) \), since \( B \) is a Hodge structure of abelian type (see section 2.2).

Now the projection \( \text{MT}(\Theta)_{sc} \to \text{MT}(X) = \text{Sp}(8, \mathbb{C}) \) shows that there is a reductive group \( G \) with \( \text{MT}(\Theta)_{sc} = \text{Sp}(8, \mathbb{C}) \times G \). If \( B \) is simple, it follows that \( B = B_1 \boxtimes B_2 \) with irreducible representations \( B_1 \) of \( \text{Sp}(8, \mathbb{C}) \) and \( B_2 \) of \( G \). If \( B_1 \) were non-trivial, \( \dim(B_1) = 10 \) would imply \( \dim(B_1) \in \{2, 5, 10\} \) which is impossible [AEV]. Hence \( B_1 \) is the trivial representation, \( B_2 \) is the standard representation of \( \text{MT}(B) = \text{Sp}(10, \mathbb{C}) \), and one easily deduces that \( \iota_{sc} \) is an
isomorphism.

The rest of this section is devoted to the proof of the theorem. By the last lemma it suffices to show that for every ppav \((X, \Theta)\) outside a meager subset of \(A_4\) the \(\mathbb{Q}\)-Hodge structure \(B\) is simple. Notice that \(\text{MT}(B) \subseteq \text{Sp}(10, \mathbb{C})\) in any case \([Go, B.62]\), with equality holding iff \(B\) is simple. Since the Mumford-Tate group of a variation of Hodge structures is constant outside the complement of some meager subset and only becomes smaller on this meager subset, it therefore suffices to prove the simplicity of \(B\) for a single ppav \((X, \Theta)\) in \(A_4\).

2.2. Intermediate Jacobians of cubic threefolds. For any smooth cubic threefold \(T \subset \mathbb{P}^4_{\mathbb{C}}\) one has the intermediate Jacobian

\[
JT = H^{2,1}(T^*)/H_3(T, \mathbb{Z})
\]

which by \([CG]\) is a simple ppav of dimension 5 and determines \(T\) uniquely up to isomorphism. Let \(\mathcal{T} \subset A_5\) be the closure of the locus of all these intermediate Jacobians. In \([Do]\) and \([Iz]\) a smooth cubic threefold \(T = T_{(X, \Theta)}\) has been associated to each \((X, \Theta)\) in a Zariski-open dense subset \(A_4^0 \subset A_4\). This gives a map \(\varphi : A_4^0 \to \mathcal{T}\) which is generically finite and dominant by \([Do]\) birationality of \(\chi\) in thm. 5.2 (2)]. By Chevalley’s theorem the image of \(\varphi\) contains a Zariski-open dense subset of \(\mathcal{T}\).

For \((X, \Theta) \in A_4^0\) and for the corresponding \(\mathbb{Q}\)-Hodge structure \(B\) as in 2.1 the arguments of \([IvS]\) imply

\[
B = H^1(JT_{(X, \Theta)}; \mathbb{Q})(-1).
\]

If for abelian varieties \(A\) we put \(\text{End}^0(A) = \text{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}\), a basic property of Mumford-Tate groups \([Go, B.60]\) hence shows

\[
\text{End}_{\text{MT}(B)}(B) = \text{End}^0(JT_{(X, \Theta)}).
\]

Thus \(B\) is a simple \(\mathbb{Q}\)-Hodge structure if and only if \(\text{End}^0(JT_{(X, \Theta)}) = \mathbb{Q}\). In what follows we prove the latter property for suitable \((X, \Theta)\) in \(A_4^0\) by a degeneration argument. Using a Collino family, this will be reduced to the study of an extension of a generic Jacobian variety by a torus.

2.3. Collino’s family. As in \([Co\text{, part II}]\) and \([CM]\) p. 44-45] one finds a group scheme \(B \to S\) over a non-empty Zariski-open subset \(S \subset \mathbb{P}^1_{\mathbb{C}}\) and a point \(s \in S(\mathbb{C})\) such that

- for all \(t \neq s\) one has \(B_t \cong JT_{(X_t, \Theta_t)}\) for some \((X_t, \Theta_t) \in A_4^0\),
- the special fibre \(B_s\) is an extension

\[
E : 0 \to \mathbb{G}_m \to B_s \to JC \to 0
\]

for some general curve \(C\) of genus 4. As recalled in lemma 5.1 below, the theta divisor of its Jacobian \(JC\) has two singular points \(\pm e\). The class
of the extension $E$ in $\text{Ext}(JC, \mathbb{G}_m)$ is mapped under the isomorphism $\text{Ext}(JC, \mathbb{G}_m) \xrightarrow{\sim} \text{Pic}^0(JC) \xrightarrow{\sim} JC$ [Se 16.VII] to the point $2e$ or $-2e$ depending on the choice of $e$. Since $C$ is general, we know that $JC$ is not in $\theta_{null, A}$. Hence $e \neq -e$ by lemma 5.1, so the extension $E$ is non-trivial.

To prove our theorem we will show $\text{End}^0(\mathcal{B}_t) = \mathbb{Q}$ for all $t$ outside a meager subset of $S(C)$. Over $S^* = S \setminus \{s\}$ the family $\mathcal{B} \to S$ restricts to an abelian scheme $\mathcal{B}^* \to S^*$. Hence by [Del2, prop. 7.5] and [Del1, 4.1.3.2] the restriction map $\text{End}^0(\mathcal{B}^*/S^*) \to \text{End}^0(\mathcal{B}_t)$ is an isomorphism for all but countably many $t \in S^*(C)$ (this is why we had to exclude a meager subset of ppav's in the theorem). So it suffices to show $\text{End}(\mathcal{B}^*/S^*) = \mathbb{Z}$, and this is equivalent to the claim in 2.6 below.

2.4. Endomorphisms of the special fibre. We first show that for general choice of $C$ one has

$$\text{End}(\mathcal{B}_s) = \mathbb{Z}.$$

**Proof.** Every $\psi \in \text{End}(\mathcal{B}_s)$ preserves the toric part $\mathbb{G}_m \subset \mathcal{B}_s$ and induces an endomorphism $\psi_{JC}$ of $JC = \mathcal{B}_s/\mathbb{G}_m$. So we have a ring homomorphism

$$(-)_{JC} : \text{End}(\mathcal{B}_s) \to \text{End}(JC), \quad \psi \mapsto \psi_{JC}.$$

For general $C$ we know $\text{End}(JC) = \mathbb{Z}$, and then $(-)_{JC}$ is surjective because its image contains $1 = (\text{id}_{\mathcal{B}_s})_{JC}$. Now suppose $\psi \in \text{End}(\mathcal{B}_s)$ and $\psi_{JC} = 0$, i.e. $\psi$ factors over $\mathbb{G}_m \subset \mathcal{B}_s$. Then $\psi|_{\mathbb{G}_m}$ is a character $z \mapsto z^n$ of $\mathbb{G}_m$ for some $n \in \mathbb{Z}$. If $\psi \neq 0$, we must have $n \neq 0$. However, the image of the restriction map

$$\text{res} : \text{Hom}(\mathcal{B}_s, \mathbb{G}_m) \to \text{Hom}(\mathbb{G}_m, \mathbb{G}_m) = \mathbb{Z}$$

contains $n = \text{res}(\psi)$, so the image of the differential $d$ of the $\text{Ext}$-sequence

$$\cdots \to \mathbb{Z} = \text{Hom}(\mathbb{G}_m, \mathbb{G}_m) \xrightarrow{d} \text{Ext}(JC, \mathbb{G}_m) \to \cdots$$

is a quotient of $\mathbb{Z}/n\mathbb{Z}$. In particular, the image of $d$ is an $n$-torsion group. But by construction of the $\text{Ext}$-sequence, $d(\text{id}_{\mathbb{G}_m})$ is the class of the extension defining $\mathcal{B}_s$. Hence $\pm 2e \neq 0$ is an $n$-torsion point of $JC$. For $C$ varying this contradicts lemma 2.5 below. Hence $(-)_{JC}$ is an isomorphism. \qed

2.5. Torsion points. Let $\mathcal{M}_4$ be the moduli space of smooth curves of genus 4. Over some Zariski-open dense subset $\mathcal{U}$ of $\mathcal{M}_4$, the map $C \mapsto \pm 2e$ defines a section of the universal Jacobian variety, and this section is not the zero section $0_U$.

**Lemma.** If $\mathcal{U} \subset \mathcal{M}_4$ is a Zariski-open dense subset and $\Sigma \neq 0_\mathcal{U}$ is a section of the universal Jacobian variety $\pi : \mathcal{X} \to \mathcal{U}$, then $\Sigma$ defines a non-torsion point in all fibres of $\pi$ over the complement of a meager subset of $\mathcal{U}$.
Proof. For \( m \in \mathbb{Z} \), let \( Z_m \) denote the zero locus of \( m \cdot \Sigma \). Then \( Z_m \) is a Zariski-closed subset of \( U \), and we must show it is not all of \( U \). If it were for some \( m \geq 0 \), then for minimal such \( m \) the section \( \Sigma \) would define a section over some Zariski-open dense subset of \( \mathcal{M}_4 \) to the map \( m\mathcal{M}_4 \to \mathcal{M}_4 \), where \( m\mathcal{M}_4 \) is the moduli space of cyclic étale covers of precise order \( m \) of curves of genus 4. Then \( m\mathcal{M}_4 \) would be reducible, contradicting \[BF\]. □

2.6. Endomorphisms of the generic fibre. For the generic point \( \eta \) of \( S \) we now claim
\[ \text{End}(\mathcal{B}_\eta) = \mathbb{Z}. \]

Proof. Let \( \mathcal{N} \) be the Néron model \( \mathcal{N} \) of \( \mathcal{B}_\eta \) over \( S \). Its universal property gives an \( S \)-morphism \( \mathcal{B} \to \mathcal{N} \). Since this morphism induces an isomorphism of the generic fibre and since \( \mathcal{B}_s \) is semi-abelian, by \[BLR\, \text{prop. 7.4.3}\] it induces an isomorphism of \( \mathcal{B} \) onto the connected component \( \mathcal{N}^0 \subset \mathcal{N} \). In section 2.4 we have shown \( \text{End}(\mathcal{B}_s) = \mathbb{Z} \). Now consider the composite ring homomorphism
\[ \varphi : \text{End}^0(\mathcal{B}_\eta) \to \text{End}^0(\mathcal{N}_\eta) \to \text{End}^0(\mathcal{N}/S) \xrightarrow{\text{res}} \text{End}^0(\mathcal{B}_s) = \mathbb{Q}, \]
where the first isomorphism comes from the identification \( \mathcal{B}_\eta \to \mathcal{N}_\eta \), the second isomorphism is due to the universal property of \( \mathcal{N} \), and the third map \( \text{res} \) denotes restriction to the fibre \( \mathcal{B}_s = (\mathcal{N}^0)_s \). The image of \( \varphi \) contains \( 1 = \varphi(\text{id}_{\mathcal{B}_\eta}) \), so \( \varphi \) is surjective. On the other hand, \( \mathcal{B}_\eta \) is a simple abelian variety; indeed, all \( \mathcal{B}_t \) with \( t \neq s \) are intermediate Jacobians of smooth cubic threefolds, hence simple by \[CG\]. Therefore \( \text{End}^0(\mathcal{B}_\eta) \) is a skew field, and since \( \varphi \) is a surjective ring homomorphism, it follows that \( \ker(\varphi) = 0 \). Thus \( \text{End}^0(\mathcal{B}_\eta) = \mathbb{Q} \), and our claim follows. □

2.7. Higher genus. For the rest of this section we drop our assumption that \( g = 4 \). It seems likely that for general \((X, \Theta)\) the cohomology
\[ H^{g-1}(\Theta, \mathbb{Q})/H^{g-1}(X, \mathbb{Q}) \]
decomposes into at least \( \left\lfloor \frac{g-1}{2} \right\rfloor \) simple \( \mathbb{Q} \)-Hodge substructures. To illustrate this let us consider the case of Jacobians. Let \( C \) be a general curve of genus \( g \geq 3 \). For \( \nu \in \mathbb{Z} \) let \( \Lambda^\nu \) be the \( \nu \)-th exterior power of the standard representation of \( \text{MT}(\Theta_{JC})_{sc} = \text{Sp}(2g, \mathbb{Q}) \), and consider \( \mathbb{H}^\nu(JC, \delta_{\Theta_{JC}}) \) as a representation of the group \( \langle \sigma^* \rangle \times \text{MT}(\Theta_{JC})_{sc} \) for the involution \( \sigma = -\text{id}_{JC} \). Let \( \text{sgn} \) be the nontrivial character of \( \langle \sigma^* \rangle \).

Lemma. For \( |\nu| \leq g-1 \) define \( n(\nu) = g - 1 - |\nu| \). Then
\[ \mathbb{H}^\nu(JC, \delta_{\Theta_{JC}}) = \bigoplus_{\mu=0}^{\lfloor \frac{n(\nu)}{2} \rfloor} \text{sgn}^{n(\nu)+\mu} \otimes \Lambda^{n(\nu)-2\mu}. \]
Thus for smooth origin \cite[sect. 6.2.4]{BBD} and only depend on the semisimple tensor category $\lambda$ normal, \cite{KW}. For a closed subvariety $i$ of $X$, let $\sigma = IC_Z[d] = i_* j_*(k_U[d]) \in \text{Perv}(X)$ for $d = \dim(Z)$ be the perverse intersection cohomology sheaf of $Z$, and put $\lambda = i_* j_*(k_U[d])$. If $Z$ is normal, $\lambda_Z = k_Z[d]$ by \cite[lemma 1, p. 5]{Well}. Both $\delta_Z$ and $\lambda_Z$ are of geometric origin \cite[sect. 6.2.4]{BBD} and only depend on $Z$ but not on the choice of $U$. Thus for smooth $Z$ we get $\delta_Z = \lambda_Z = k_Z[d]$.

3.3. The semisimple tensor category $\mathcal{P}(X)$. It has been shown in \cite[cor. 6, p. 36]{Well} that every simple perverse sheaf $\delta \in \text{Perv}(X)$ of geometric origin with $\mathbb{H}^\bullet(X, \delta) \neq 0$ is a rigid object in $(D^b(X, k), \ast)$ with dual object $\delta^\vee = (\sim 1_X)^\ast(D(\delta))$, where $D$ denotes the Verdier dual. Using this we want to construct as in \cite{We2} a semisimple $k$-linear rigid abelian tensor category $\mathcal{P}(X)$ whose objects are represented by semisimple perverse sheaves. Note that the convolution of perverse sheaves is in general not perverse.

**Proof.** Up to the $\sigma$-action this is clear as $H^\bullet(JC, \delta_{\Theta JC}) = \Lambda^{g-1}(H^\bullet(JC, \delta_C))$ by the non-hyperelliptic case of \cite[cor. 13(iii) on p. 64 and p. 124]{We1}, but beware that unlike $\delta_{\Theta JC}$, the perverse sheaf $\delta_C$ is not $\sigma$-equivariant! To find the $\sigma$-action on $H^\bullet(JC, \delta_{\Theta JC})$, we use the evaluation map \cite[sect. 4.2]{BBD}

$$H^\bullet(JC, \mathbb{Q}) \otimes \mathbb{Q}[x] \longrightarrow H^\bullet(C^{(g-1)}, \mathbb{Q}) \cong \mathbb{H}^\bullet(JC, \delta_{\Theta JC})[1-g]$$

which is an isomorphism in degrees $\leq g-1$ and $\sigma$-equivariant, where on the left hand side $\sigma$ acts in the usual way on $H^\bullet(JC, \mathbb{Q})$ and on the powers of the variable $x$ it acts by

$$\sigma^\ast(x^\mu) = \sum_{i=0}^\mu (-1)^i \frac{[\Theta_{JC}]^{\mu-i} \otimes x^i}{(\mu-i)!}$$

as observed in prop. 4.3.1 of loc. cit. From this our claim easily follows. \hfill $\square$

3. Convolutions of perverse sheaves (conjecture [B])

In this section we introduce the perverse sheaves $\delta_\pm$ that are used in the formulation of conjecture [B].

3.1. Convolution. Put $k = \mathbb{C}$ or $k = \overline{\mathbb{Q}}_l$ for a fixed prime $l$, and denote by $D^b_c(X, k)$ the triangulated category of bounded constructible complexes of sheaves with coefficients in $k$ as in \cite{KW}. Using the addition $a : X \times X \rightarrow X$, define the convolution of $\gamma_1, \gamma_2 \in D^b_c(X, k)$ to be $\gamma_1 \ast \gamma_2 = Ra_*(\gamma_1 \boxtimes \gamma_2)$. With respect to convolution $D^b_c(X, k)$ is a $k$-linear tensor category \cite[sect. 2.1]{Well} whose unit object is the skyscraper sheaf $\delta_{\{0\}} = 1$ supported in the origin. For hypercohomology $\mathbb{H}^\bullet(X, \gamma_1 \ast \gamma_2) = \mathbb{H}^\bullet(X, \gamma_1) \otimes \mathbb{H}^\bullet(X, \gamma_2)$ by the relative Künneth formula, so $\mathbb{H}^\bullet(X, -)$ is a $\otimes$-functor on $(D^b_c(X, k), \ast)$.

3.2. Let $\text{Perv}(X) \subset D^b_c(X, k)$ be the abelian category of perverse sheaves on $X$ \cite{KW}. For a closed subvariety $i : Z \hookrightarrow X$ and a smooth open dense $j : U \hookrightarrow Z$, let $\delta_Z = IC_Z[d] = i_* j_*(k_U[d]) \in \text{Perv}(X)$ for $d = \dim(Z)$ be the perverse intersection cohomology sheaf of $Z$, and put $\lambda_Z = i_* j_*(k_U[d])$. If $Z$ is normal, $\lambda_Z = k_Z[d]$ by \cite[lemma 1, p. 5]{Well}. Both $\delta_Z$ and $\lambda_Z$ are of geometric origin \cite[sect. 6.2.4]{BBD} and only depend on $Z$ but not on the choice of $U$. Thus for smooth $Z$ we get $\delta_Z = \lambda_Z = k_Z[d]$.

3.3. The semisimple tensor category $\mathcal{P}(X)$. It has been shown in \cite[cor. 6, p. 36]{Well} that every simple perverse sheaf $\delta \in \text{Perv}(X)$ of geometric origin with $\mathbb{H}^\bullet(X, \delta) \neq 0$ is a rigid object in $(D^b_c(X, k), \ast)$ with dual object $\delta^\vee = (\sim 1_X)^\ast(D(\delta))$, where $D$ denotes the Verdier dual. Using this we want to construct as in \cite{We2} a semisimple $k$-linear rigid abelian tensor category $\mathcal{P}(X)$ whose objects are represented by semisimple perverse sheaves. Note that the convolution of perverse sheaves is in general not perverse.
By [KrW2] the perverse sheaves $\delta \in \text{Perv}(X)$ with Euler characteristic $\chi(\delta) = 0$ define a thick subcategory $T(\pi) \subset D^b_k(X, k)$, and a perverse sheaf is in $T(\pi)$ if and only if any of its constituents is in $T(\pi)$. Clearly $T(\pi)$ defines a $\otimes$-ideal in $D^b_k(X, k)$, so the quotient category $D^b(X, k) = D^b_k(X, k)/T(\pi)$ is again a $k$-linear tensor category. All simple objects in it are rigid.

Although the full abelian subcategory $P(\pi) \subset D^b_k(X, k)$ of semisimple perverse sheaves on $X$ is not stable under convolution, it turns out in loc. cit. that its image $\overline{P}(\pi)$ in $\overline{D}^b(X, k)$ indeed is, which in terms of [We2] amounts to saying that every (semisimple) perverse sheaf on $X$ is a multiplier. So $\overline{P}(\pi)$ is a $k$-linear semisimple rigid abelian tensor category under convolution. Similarly, via [We1, lemma 10, p. 36] the mixed perverse sheaves define a $k$-linear rigid abelian tensor category under convolution, and this category contains $\overline{P}(\pi)$ as a full subcategory.

3.4. Definition of $\delta_\pm$. Assume $\Theta$ is normal and hence irreducible [EL]. The convolution square $\delta_{\Theta} \ast \delta_{\Theta}$ contains the unit object $1$ precisely once, since $\delta_{\Theta}$ is a simple self-dual object of $\overline{P}(\pi)$. Furthermore the commutativity constraint $S : \delta_{\Theta} \ast \delta_{\Theta} \cong \delta_{\Theta} \ast \delta_{\Theta}$ of [We1, sect. 2.1] is multiplication by $(-1)^{\kappa - 1}$ on $\mathcal{H}^0(\delta_{\Theta} \ast \delta_{\Theta})_0$. Indeed, one has $\delta_{\Theta} = IC_{\Theta}[g - 1]$. The commutativity constraint for $IC_{\Theta}$ is the identity on $\mathcal{H}^{2g - 2}(IC_{\Theta} \ast IC_{\Theta})_0$. This follows from considering fundamental classes, since we may replace $IC_{\Theta}$ by the constant sheaf as $\Theta$ is normal. The shift by $g - 1$ accounts for the factor $(-1)^{\kappa - 1}$.

This being said, it follows that $1$ lies in the alternating square $\Lambda^2(\delta_{\Theta})$ for even $g$ and in the symmetric square $S^2(\delta_{\Theta})$ for odd $g$. So there are perverse sheaves $\delta_\pm$ without constituents from $T(\pi)$ and complexes $\tau_\pm \in T(\pi)$ such that

$$S^2(\delta_{\Theta}) = \begin{cases} \delta_+ \oplus \tau_- & \text{and} \quad \Lambda^2(\delta_{\Theta}) = \begin{cases} 1 \oplus \delta_- \oplus \tau_- & \text{for} & g \text{ even,} \\ \delta_- \oplus \tau_- & \text{for} & g \text{ odd.} \end{cases} \end{cases}$$

If $\Theta$ is smooth, our construction of the family $\pi : Y_{\pi} \rightarrow U$ in section 1.2 shows that $R\pi_*(k\Theta)|_{2g - 2}$ corresponds to $(\delta_{\Theta} \ast \delta_{\Theta})|_U$. To prove our claim from the introduction that conjecture $\text{B}$ implies conjecture $\text{A}$ let us check that $V_{\pm}[2g - 2]$, as a Hodge modules in the sense of [Sa1] and [Sa2], have the underlying perverse sheaves $\delta_{\pm}|_U$.

Indeed, the commutativity constraint is $S = Ra_* (\phi)$ for the involution $\phi$ of $k_{\Theta}[g - 1] \boxtimes k_{\Theta}[g - 1]$ given by $\phi(s \boxtimes t) = (-1)^{g - 1} \cdot t \boxtimes s$. Thus $S|_U$ is the $\sigma$ of section 1.2 on $R\pi_*(k\Theta_U)$ twisted by $(-1)^{g - 1}$, and $S^2(\delta_{\Theta})|_U$ and $\Lambda^2(\delta_{\Theta})|_U$ are the part of $R\pi_*(k\Theta_U)$ on which $\sigma$ acts by $\pm(-1)^{g - 1}$ respectively. It only remains to notice that $\tau_\pm$ in lemma 3.5 are the constant subvariations of section 1.3 as we will check now.
3.5. The translation-invariant summands $\tau_\pm$ in $\delta_\Theta \ast \delta_\Theta$ can be computed explicitly as follows.

**Lemma.** If $\Theta$ is smooth, then

$$\tau_\pm = \bigoplus_{\mu \text{ odd for } "+" \mu \text{ even for } "-"} H^{g-2-|\mu|}(X, k) \otimes \delta_X[\mu].$$

**Proof.** By semisimplicity we have a decomposition $\tau_\pm = \tau'_\pm \oplus \tau''_\pm$ where $\tau'_\pm$ denote the direct sum of all complex shifts of $\delta_X$ that enter $\tau'_\pm$. In particular then $\mathbb{H}^*(X, \tau''_\pm) = 0$ since every translation-invariant simple perverse sheaf different from $\delta_X$ has vanishing hypercohomology [We1, sect.2.3]. Hence $\tau'_\pm$ can be computed from hypercohomology as follows:

Using the Künneth formula $\mathbb{H}^*(X, \delta_\Theta \ast \delta_\Theta) = \mathbb{H}^*(X, \delta_\Theta) \otimes \mathbb{H}^*(X, \delta_\Theta)$, one then checks that in perverse cohomology degrees $\mu \leq 0$ the complexes $\tau'_\pm$ coincide with the right hand side of the lemma. By the hard Lefschetz theorem the same then also holds in perverse cohomology degrees $\mu > 0$. Finally, using the result for $\tau'_\pm$ for $\mu = 0$, we have the stalk cohomology

$$H^{-g}(\tau'_\pm)_0 = H^{g-2}(X, k) = H^{g-2}(\Theta, k) = \mathcal{H}^{-g}(\delta_\Theta \ast \delta_\Theta)_0 = \mathcal{H}^{-g}(\tau'_\pm \oplus \tau''_\pm)_0,$$

so the non-constant translation-invariant complexes $\tau''_\pm$ must be zero. $\square$

4. Tannakian categories (conjecture C)

We now discuss the construction the algebraic group $G(X, \Theta)$ mentioned in conjecture C, again following [We2] and [KrW2].

4.1. **Definition of** $G(X, \Theta)$. Suppose $\Theta$ is normal. Inside the rigid abelian tensor category $\mathcal{P}(X)$ of section 3.3, we consider the full abelian tensor subcategory $\mathcal{P}(X, \Theta) = \langle \delta_\Theta \rangle$ generated by $\delta_\Theta$. By [KrW2] there exists an affine algebraic group $G = G(X, \Theta)$ over $k$ together with an equivalence

$$\omega : \mathcal{P}(X, \Theta) \xrightarrow{\cong} \text{Rep}(G)$$

of tensor categories, where $\text{Rep}(G)$ denotes the tensor category of algebraic representations of $G$ over $k$. Since $\delta_\Theta$ is a tensor generator of $\mathcal{P}(X, \Theta)$, the action of $G(X, \Theta)$ on $\omega(\delta_\Theta)$ is faithful. Let $P(X, \Theta) \subset D^b_c(X, k)$ denote the full monoidal subcategory above $\mathcal{P}(X, \Theta)$.

4.2. For objects $\gamma \in \mathcal{P}(X, \Theta)$ consider the Euler characteristic $\chi(\gamma) = \sum_{i \in \mathbb{Z}} (-1)^i \dim_k(\mathbb{H}^i(X, \gamma))$. We claim

$$\dim_k(\omega(\gamma)) = \chi(\gamma).$$

Indeed, the composite $\varphi : \mathbf{1} \xrightarrow{\text{coev}} \gamma \ast \gamma' \cong \gamma' \ast \gamma \xrightarrow{\text{ev}} \mathbf{1}$ of coevaluation and evaluation in $\mathcal{P}(X, \Theta)$ satisfies $\omega(\varphi) = d \cdot \text{id}$ for $d = \dim_k(\omega(\gamma))$. Hence
\[ \mathbb{H}^*(X, \varphi) = d \cdot id, \text{ and since } \mathbb{H}^*(\varphi) \text{ is also the composite of coevaluation and evaluation in the category of super vector spaces, we get } d = \chi(\gamma). \]

4.3. On the representation-theoretic side, the fact that the unit object 1 enters \( \Lambda^2(\delta_0) \) resp. \( S^2(\delta_0) \) means that the representation \( \omega(\delta_0) \) respects a symplectic resp. orthogonal bilinear form. So we have the following

**Lemma.** If \( \Theta \) is smooth, then

\[
G(X, \Theta) \subseteq \begin{cases} 
\text{Sp}(g!, k) \text{ for } g \text{ even}, \\
\text{SO}(g!, k) \text{ for } g \text{ odd}.
\end{cases}
\]

**Proof.** By the dimension formula \[4.2\], a Gauss-Bonnet calculation like the one in section \[1.4\] shows \( \dim_k(\omega(\delta_0)) = g! \) provided \( \Theta \) is smooth. Furthermore, in section \[3.4\] we have seen that 1 occurs in \( S^2(\delta_0) \) resp. \( \Lambda^2(\delta_0) \) for \( g \text{ odd} \) resp. even. This proves the lemma with \( \text{O}(g!, \mathbb{C}) \) in place of \( \text{SO}(g!, \mathbb{C}) \). So it only remains to note that by \[KrW2\] the group \( G(X, \Theta) \) does not admit non-trivial characters. \[\square\]

This lemma and the cases \( g = 2, 3, 4 \) motivate conjecture \[C\]. For \( g = 2 \) every ppav \( (X, \Theta) \) with smooth \( \Theta \) is the Jacobian of a hyperelliptic curve; this case is covered by \[We1\], p. 124 and th. 14. For \( g = 3 \) every ppav \( (X, \Theta) \) with smooth \( \Theta \) is the Jacobian of a non-hyperelliptic curve, and \( \omega(\delta_0) \) is the second fundamental representation of \( G(X, \Theta) = \text{Sl}(4, \mathbb{C})/\mu_2 \) by loc. cit. It corresponds to the standard representation of \( \text{SO}(6, \mathbb{C}) \) via the exceptional isomorphism \( \text{Sl}(4, \mathbb{C})/\mu_2 \cong \text{SO}(6, \mathbb{C}) \).

5. THE SCHOTTKY PROBLEM IN GENUS 4

We now relate conjecture \[C\] to the Schottky problem. For this we drop the assumption that \( \Theta \) is smooth, expecting that the ppav’s which are Jacobians of curves are detected by the corresponding \( G(X, \Theta) \). Indeed, for genus \( g = 4 \) this turns out to be true as we show in this section.

5.1. Stratification of the moduli space. Each \( (X, \Theta) \) defines a point in the moduli space \( \mathcal{A}_g \) of ppav’s of dimension \( g \). The locus \( \mathcal{N}_g \subset \mathcal{A}_g \) of ppav’s with singular theta divisor is a divisor which for \( g \geq 4 \) has precisely two irreducible components \[Deb2\]. The first component is the locus \( \theta_{\text{null}, g} \) of all ppav’s that admit a symmetric theta divisor containing a 2-division point with even multiplicity; the second component contains the closure \( \mathcal{J}_g \) of the locus of Jacobian varieties and for \( g = 4 \) is equal to it \[Be\]. Let \( \mathcal{J}_{g, \text{hyp}} \) and \( \mathcal{A}_{g, \text{dec}} \) be the closure of the locus of hyperelliptic Jacobians resp. of decomposable ppav’s in \( \mathcal{A}_g \). Notice that \( \mathcal{A}_{4, \text{dec}} \subset \theta_{\text{null}, 4} \cap \mathcal{J}_4 \).
Lemma. For $g = 4$ the following holds: a) On $\mathcal{A}_4 \setminus \mathcal{N}_4$ the theta divisors are smooth. b) On $\mathcal{J}_4 \setminus (\theta_{null,A} \cap \mathcal{J}_4)$ they have precisely two singularities, both ordinary double points. c) On $\theta_{null,A} \setminus (\theta_{null,A} \cap \mathcal{J}_4)$ they have precisely one singularity, an ordinary double point. d) On $(\theta_{null,A} \cap \mathcal{J}_4) \setminus (\mathcal{J}_{4, hyp} \cup \mathcal{A}_{4, dec})$ they have precisely one singularity, but the Hesse matrix of the Riemann theta function has rank three there. e) On $\mathcal{J}_{4, hyp} \setminus (\mathcal{J}_{4, hyp} \cap \mathcal{A}_{4, dec})$ their singular locus is of dimension one. f) On $\mathcal{A}_{4, dec}$ it is of dimension two.

Proof. It has been shown in [GSM, thm. 10 and cor. 15] that for b) – d) the rank of the Hesse matrices is the given one. In particular, for b) and c) we only have isolated singularities. This also holds for d) by Brill-Noether theory. The number of these isolated singularities can be obtained from the description in [CvdG, sect. 10] and the generic results of [Be] and [Mu2, case b2, p. 55f]. Part e) follows from Brill-Noether theory, and f) is trivial. □

5.2. The Schottky problem in genus 4. As an application of our results we claim that

$$G(X, \Theta) \text{ determines the loci } \mathcal{J}_4 \text{ and } \theta_{null,A} \text{ in } \mathcal{A}_4.$$ 

Indeed, for $(X, \Theta)$ in $\mathcal{A}_4 \setminus \mathcal{N}_4$ we have $G(X, \Theta) = \text{Sp}(24, \mathbb{C})$ by conjecture [C], which will be proven for $g = 4$ in sections 9 and 10 below. By way of contrast, for $(X, \Theta)$ in $\mathcal{J}_4 \setminus (\mathcal{J}_{4, hyp} \cup \mathcal{A}_{4, dec})$ resp. in $\mathcal{J}_{4, hyp} \setminus (\mathcal{A}_{4, dec} \cap \mathcal{J}_{4, hyp})$ we know from [WcI, p. 124] that $G(X, \Theta)$ is $\text{Sl}(6, \mathbb{C})/\mu_3$ resp. $\text{Sp}(6, \mathbb{C})$. Finally, for $(X, \Theta)$ in $\theta_{null,A} \setminus (\mathcal{J}_4 \cap \theta_{null,A})$ we claim that $G = G(X, \Theta)$ is different from the groups occurring above: From remark 10.3 one gets $\chi(\delta_0) = 22$ for $(X, \Theta)$ in $\theta_{null,A} \setminus (\mathcal{J}_4 \cap \theta_{null,A})$, but $\text{Sp}(24, \mathbb{C})$, $\text{Sl}(6, \mathbb{C})/\mu_3$ and $\text{Sp}(6, \mathbb{C})$ do not admit irreducible representations of dimension 22. Presumably $G = \text{Sp}(22, \mathbb{C})$.

6. Tensor functors defined by nearby cycles

To study the tensor categories $\Phi(X)$ or $\Phi(X, \Theta)$ of section 5.3 when $(X, \Theta)$ varies in families, we briefly recall some facts from nearby cycle theory as exposed in [SGA7, exp. XIII-XIV] and [KS]. Let $f : \mathcal{X} \rightarrow S$ be a proper surjective algebraic morphism from a smooth complex algebraic variety $\mathcal{X}$ to a smooth complex algebraic curve $S$. For $s \in S(\mathbb{C})$ we want to relate complexes on the special fibre $\mathcal{X}_s = f^{-1}(s)$ to complexes on the nearby fibres $\mathcal{X}_t = f^{-1}(t)$ for $t$ in a small pointed analytic neighborhood of $s$.

6.1. Analytic nearby cycles. Let $D \subset S(\mathbb{C})$ be a small coordinate disc centered at $s$. The morphism $f : \mathcal{X} \rightarrow S$ induces a proper holomorphic map $\mathcal{X}_D = \mathcal{X} \times_S D \rightarrow D$ of analytic spaces, ditto with $D$ replaced by the pointed disc $D^* = D \setminus \{s\}$ or by the universal covering $\tilde{D}^*$ of $D^*$. So we have a cartesian
Diagram

$\mathcal{X}_s \leftarrow \mathcal{X}_D \xrightarrow{j} \mathcal{X}_{D^*} \xrightarrow{\pi} \mathcal{X}_{D^*}$

$\{s\} \leftarrow D \xrightarrow{\delta} D^*$

where $\pi$ is a covering map and $i$ resp. $j$ are closed resp. open immersions. Let $\tilde{j} = j \circ \pi$. Following [SGA7, exp. XIV] and [KS] sect. 8.6] we consider the functor

$$\Psi : D^b_c(\mathcal{X}, \mathbb{C}) \rightarrow D^b_c(\mathcal{X}_s, \mathbb{C}), \delta \mapsto i^* \tilde{R}\tilde{j}_* \tilde{i}^*(\delta|_{\mathcal{X}_D})$$

of nearby cycles. It factors over a functor $D^b_c(\mathcal{X}_{D^*}, \mathbb{C}) \rightarrow D^b_c(\mathcal{X}_s, \mathbb{C})$ which we also denote by $\Psi$. If $D$ has been chosen sufficiently small, one has an isomorphism

$$(6.1.1) \quad \mathbb{H}^i(\mathcal{X}_s, \Psi(\delta)) \cong \mathbb{H}^i(\mathcal{X}_t, \delta|_{\mathcal{X}_t}) \quad \text{for all} \quad t \in D^*.$$

For $\delta \in D^b_c(\mathcal{X}_D, \mathbb{C})$ the pullback of the morphism $\delta \rightarrow \tilde{j}_* \tilde{i}^*(\delta)$ under $i$ defines a morphism $i^*(\delta) \rightarrow \Psi(\delta)$. As in [KS] eq. 8.6.7] the cone of this morphism defines the functor $\Phi : D^b_c(\mathcal{X}_D, \mathbb{C}) \rightarrow D^b_c(\mathcal{X}_s, \mathbb{C})$ of vanishing cycles, so for every $\delta \in D^b_c(\mathcal{X}_D, \mathbb{C})$ we have a distinguished triangle

$$(6.1.2) \quad i^*(\delta) \rightarrow \Psi(\delta) \rightarrow \Phi(\delta) \rightarrow i^*(\delta)[1]$$

in $D^b_c(\mathcal{X}_s, \mathbb{C})$. Be aware of the various shifting conventions in the literature; e.g. our $\Phi$ would be denoted $\Phi[-1]$ in [KS]. With our conventions, the shifted functors $\Psi[-1]$ and $\Phi[-1]$ commute with Verdier duality [Br] 1.4], and they map $\text{Perv}(\mathcal{X})$ to $\text{Perv}(\mathcal{X}_s)$ [KS, cor. 10.3.13].

6.2. Tensor functoriality. If $\mathcal{X} \rightarrow S$ is an abelian scheme over $S$ and if $a, p_1, p_2 : \mathcal{X} \times S \mathcal{X} \rightarrow \mathcal{X}$ are the relative addition morphism resp. the two projections, we define the relative convolution of $\delta_1, \delta_2 \in D^b_c(\mathcal{X}, \mathbb{C})$ to be $\delta_1 \ast \delta_2 = Ra_* (p_1^!(\delta_1) \otimes p_2^!(\delta_2))$. For all $t \in S(\mathbb{C})$ it restricts on the fibre $\mathcal{X}_t$ to the convolution $\delta_1 \ast \delta_2|_{\mathcal{X}_t} = \delta_1|_{\mathcal{X}_t} \ast \delta_2|_{\mathcal{X}_t}$ as defined in [3.11]. The Küneth isomorphism [KS] dual of ex. II.18] and the compatibility of $\Psi$ with proper maps [KS] ex. VIII.15] implies that $\Psi$ is a tensor functor in the sense that for all $\delta_1, \delta_2$ one has $\Psi(\delta_1 \ast \delta_2) \cong \Psi(\delta_1) \ast \Psi(\delta_2)$.

6.3. Algebraic nearby cycles. Localizing in the point $s \in S(\mathbb{C})$, let us now replace $S$ by the spectrum of a Henselian discrete valuation ring centered at the special point $s$, and denote by $\eta$ its generic point. We then have an algebraic version of the nearby cycles, a functor $\Psi : D^b_c(\mathcal{X}) \rightarrow D^b_c(\mathcal{X}_s \times_s \eta)$ in the sense of [Il] §3.1 and §4]. It factors over a functor $D^b_c(\mathcal{X}_\eta) \rightarrow D^b_c(\mathcal{X}_s \times_s \eta)$ which we also denote by $\Psi$. The properties described in the analytic setting carry over to this case. We have a sequence $\textbf{(6.1.2)}$, and $\Psi$ maps $\text{Perv}(\mathcal{X}_\eta)$ to $\text{Perv}(\mathcal{X}_s)$ and
6.4. **Group-theoretical reformulation.** For an abelian scheme $X \to S$, in the setting of 6.3, let $T$ be a tensor subcategory of $\mathcal{P}^\text{mixed}(X_\eta)$. Denote by $T_\Psi$ be the tensor subcategory of $\mathcal{P}^\text{mixed}(X_s)$ generated by the image $\Psi(T)$. If $T$ is a finitely generated tensor category, so is $T_\Psi$. Then there are algebraic $k$-groups $G$ and $G_\Psi$ such that $T = \text{Rep}(G)$ and $T_\Psi = \text{Rep}(G_\Psi)$, where the right hand sides denote the tensor categories of algebraic representations of $G$ resp. $G_\Psi$.

**Lemma.** The functor $\Psi$ is a $k$-linear $\otimes$-functor $ACU$ and maps perverse sheaves in $T(X_\eta)$ to perverse sheaves in $T(X_s)$. Hence it induces a $k$-linear exact $\otimes$-functor

$$\Psi : \text{Rep}(G) \longrightarrow \text{Rep}(G_\Psi).$$

**Proof.** The algebraic analog of the isomorphism (6.1.1) shows that the functor $\Psi$ maps complexes with vanishing Euler characteristic to complexes with vanishing Euler characteristic. So we get a $\otimes$-functor

$$\mathcal{P}^\text{mixed}(X_\eta) \longrightarrow \mathcal{P}^\text{mixed}(X_s),$$

which immediately implies the assertions. Notice that $\Psi$ maps distinguished triangles to distinguished triangles, hence induces an exact functor. $\square$

6.5. Deligne [Del6, sect. 8] has attached to any Tannaka category $\mathcal{C}$ an $\text{Ind}(\mathcal{C})$-groupscheme $\pi(\mathcal{C})$, called the fundamental group of $\mathcal{C}$. By 8.15 of loc. cit. any $k$-linear exact $\otimes$-functor $\eta : \mathcal{T}_1 \to \mathcal{T}_2$ induces a morphism

$$(6.5.1) \quad \pi(\mathcal{T}_2) \longrightarrow \eta(\pi(\mathcal{T}_1)).$$

Under the weak conditions (2.2.1) and (8.1) of loc. cit. (which are verified for representation categories $\mathcal{T}_i = \text{Rep}(G_i)$ of algebraic groups $G_i$ over an algebraically closed field $k$ of characteristic zero) theorem 8.17 of loc. cit. implies that the functor $\eta$ induces an equivalence of $\mathcal{T}_1$ with the category of objects in $\mathcal{T}_2$ endowed with an action of $\eta(\pi(\mathcal{T}_1))$ such that the natural action of $\pi(\mathcal{T}_2)$ is induced by (6.5.1) above. If $\eta$ is a fiber functor to the tensor category $\mathcal{T}_2 = \text{Vec}_k$ of finite-dimensional vector spaces over $k$, this reduces to the assertion $\mathcal{T}_1 = \text{Rep}(G_1)$ for $G_1 = \eta(\pi(\mathcal{T}_1))$.

6.6. Let $\mathcal{T}_1 = \mathcal{T}$ be a finitely generated tensor subcategory of $\mathcal{P}^\text{mixed}(X_\eta)$ as in section 6.4, and choose a fiber functor $\omega$ of $\mathcal{T}_2 = \mathcal{T}_\Psi$. Then $\omega \circ \Psi$ is a fiber functor of $\mathcal{T}_1 = \mathcal{T}$, since it is exact and therefore faithful by the isomorphism (6.1.1). Hence (6.5.1) applied to the functor $\Psi$ induces a morphism of algebraic $k$-groups

$$G_\Psi = G_2 \longrightarrow G_1 = G.$$
In [DM, p. 118] it is shown that this morphism $G_{\Psi} \to G$ is a closed immersion iff every object $K$ of $T_2 = \text{Rep}(G_{\Psi})$ is isomorphic to a subquotient of an object $\Psi(K')$ for some $K'$ in $T_1 = \text{Rep}(G)$. In our situation this holds by the definition of $T_{\Psi}$, so we get the

**Lemma.** The algebraic $k$-group $G_{\Psi}$ is a closed algebraic $k$-subgroup of $G$, and $\Psi$ can be identified with the restriction functor $\text{Rep}(G) \to \text{Rep}(G_{\Psi})$.

6.7. In dealing with tensor categories of mixed perverse sheaves on abelian varieties $X$ over $k$ one can use the following

**Lemma.** Let $\langle \delta \rangle = \text{Rep}(G)$ be the full tensor subcategory of $\overline{T}_{\text{mixed}}(X)$ generated by a mixed perverse sheaf $\delta$. Then the full tensor subcategory generated by the semisimplification $\delta^{ss}$ of $\delta$ is

$$\langle \delta^{ss} \rangle = \text{Rep}(G^{red})$$

where $G^{red} = G/R_u(G)$ denotes the quotient of $G$ by its unipotent radical.

**Proof.** This is just a statement about the categories of representations of algebraic groups over a field of characteristic zero; see [Kr W2]. □

7. **Local Monodromy**

In the setting of 6.1 let $\delta \in \text{Perv}(\mathcal{X})[−1]$. Then $\Psi(\delta)$ is a perverse sheaf on $\mathcal{X}_s$, and for fixed $t \in D^*$ the action of $\pi_1 = \pi_1(D^*, t)$ on the universal cover $\tilde{D}^*$ induces a monodromy operation on this perverse sheaf. In the algebraic setting of section 6.3 we can proceed similarly, replacing $\pi_1$ by the pro-cyclic local monodromy group $\mathbb{Z}_l(1)$ as in [Il, §3.6], cf. 7.4 below.

7.1. **Unipotent nearby cycles.** Let $T$ be a generator of $\pi_1$ acting on $\Psi(\delta)$ as above. We have a direct sum decomposition

$$\Psi(\delta) = \Psi_1(\delta) \oplus \Psi_{\neq 1}(\delta)$$

where $\Psi_1(\delta) \subset \Psi(\delta)$ denotes the maximal perverse subsheaf on which $T$ acts unipotently [Rei, lemma 1.1]. Similarly $\Phi(\delta) = \Phi_1(\delta) \oplus \Phi_{\neq 1}(\delta)$. We say $\delta$ has unipotent global monodromy if $\mathbb{H}^*(\mathcal{X}_s, \Psi_1(\delta)) = \mathbb{H}^*(\mathcal{X}_s, \Psi(\delta))$. From the Picard-Lefschetz formulas [SGA7, exp. XV, th. 3.4(iii)] one draws the

**Lemma.** If $\mathcal{X}_s$ is regular except for finitely many ordinary double points and if $\dim(\mathcal{X}_s)$ is odd, then $\delta = \delta_{\mathcal{X}}[−1]$ has unipotent global monodromy; more precisely $(T − 1)^2$ acts trivially on $\mathbb{H}^*(\mathcal{X}_s, \Psi(\delta))$.

Returning to the general case, since by definition $T − 1$ acts nilpotently on $\Psi_1(\delta)$, we can define $N = \frac{1}{2\pi i} \log(T)$ : $\Psi_1(\delta) \longrightarrow \Psi_1(\delta)(−1)$. The cone of $N$ in $D^b_c(\mathcal{X}_s, \mathbb{C})$ is given by

$$C(\Psi_1(\delta) \stackrel{N}{\longrightarrow} \Psi_1(\delta)(−1)) = C(\Psi(\delta) \stackrel{T−1}{\longrightarrow} \Psi(\delta)(−1)) = i^*R_{j^*}(\delta[1]).$$
Indeed, the first equality holds because $T - 1$ is an isomorphism on $\Psi_{\neq 1}(\delta)$ whereas on $\Psi_1(\delta)$ its kernel and cokernel coincide with those of $N$ up to a weight shift. For the second equality see [Il, eq. (3.6.2)] and the remarks thereafter. The perversity of $\Psi_1(\delta)$ and the above formula for the cone of $N$ imply that if we define specialization functors by

$$sp(-) = \oplus H^0(i^* Rj_* j^*(-)) \quad \text{and} \quad sp^\dagger(-) = \oplus H^1(i^* Rj_* j^*(-)),$$

we obtain an exact sequence of perverse sheaves on $X$:

$$0 \longrightarrow sp(\delta) \longrightarrow \Psi_1(\delta) \longrightarrow \Psi_1(\delta)(-1) \longrightarrow sp^\dagger(\delta) \longrightarrow 0.$$ 

Since $\Psi$ and hence also $\Psi_1$ preserve distinguished triangles, the functor $sp$ is left exact on perverse sheaves.

7.2. The monodromy filtration on $\Psi_1(\delta)$. As in [Del3, section 1.6] the nilpotent operator $N$ gives rise to a unique finite increasing filtration $F_\bullet$ of $\Psi_1(\delta)$ in $Perv(X)$ such that for all $i$,

- $N(F_i(\Psi_1(\delta))) \subset F_{i-2}(\Psi_1(\delta))(-1)$, and
- $N^i$ induces an isomorphism $Gr_i(\Psi_1(\delta)) \xrightarrow{\cong} Gr_{i-1}(\Psi_1(\delta))(-i)$.

Each $Gr_{-i}(\Psi_1(\delta))$ with $i \geq 0$ has an increasing filtration with composition factors $P_1(\delta), P_{i-2}(\delta)(-1), P_{i-4}(\delta)(-2), \ldots$ where

$$P_i(\delta) := \ker(N : Gr_i(\Psi_1(\delta)) \rightarrow Gr_{i-2}(\Psi_1(\delta))(-1)).$$

In what follows we will represent this situation as in loc. cit. by a triangle

$$\begin{array}{cccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \\
Gr_2(\Psi_1(\delta)) & P_2(\delta)(-2) & \cdots & \cdots & \cdots & \\
Gr_1(\Psi_1(\delta)) & P_1(\delta)(-1) & \cong & N & N & \cdots & \\
Gr_0(\Psi_1(\delta)) & P_0(\delta) & \cong & N & P_2(\delta)(-1) & \cdots & \\
Gr_{-1}(\Psi_1(\delta)) & P_{-1}(\delta) & \cong & N & \cdots & & \\
Gr_{-2}(\Psi_1(\delta)) & & & & P_{-2}(\delta) & & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \\
\end{array}$$

where each line gives the decomposition of the corresponding graded piece. The lower boundary entries $P_0(\delta), P_{-1}(\delta), P_{-2}(\delta), \ldots$ in the triangle are the graded pieces of $sp(\delta) = \ker(N)$, with $P_0(\delta)$ as the top quotient. In the situation of proposition 8.3(a) below, the entries above these lower boundary entries belong to $\Phi_1(\delta)$. 
7.3. The monodromy filtration on $\Psi(\delta)$. For any $\delta \in \operatorname{Perv}(\mathcal{X})[-1]$, the local monodromy theorem \cite[th. 2.1.2]{II} and the Jordan decomposition of the nearby cycles \cite[lemma 4.2]{Rei} show that there is an $a \in \mathbb{N}$ such that $T^a - 1$ is nilpotent on all of $\Psi(\delta)$. Using $N' := \frac{1}{2\pi i} \log(T^a) : \Psi(\delta) \to \Psi(\delta)(-1)$ in place of $N : \Psi_1(\delta) \to \Psi_1(\delta)(-1)$ one can then define a filtration $F^*_{\bullet}$ as in \cite[7.2]{} on all of $\Psi(\delta)$. This filtration does not depend on the choice of the integer $a$ with $T^a - 1$ nilpotent. Even though in general $N'|_{\Psi_1(\delta)} \neq N$, using that $T$ acts unipotently on $\Psi_1(\delta)$ one sees that the kernel and the image of $N'|_{\Psi_1(\delta)}$ are the same as those of $N$, so

$$F^*_{\bullet}(\Psi_1(\delta)) = \Psi_1(\delta) \cap F^*_{\bullet}(\Psi(\delta)).$$

Notice however that $sp(\delta) = \ker(N : \Psi_1(\delta) \to \Psi_1(\delta)(-1))$, as defined in \cite{II} will in general only be a perverse subsheaf of $\ker(N' : \Psi(\delta) \to \Psi(\delta)(-1))$ because $N'$ may have a non-trivial kernel on $\Psi_{\neq 1}(\delta)$. On the other hand, working with $\Psi(\delta)$ instead of $\Psi_1(\delta)$ has the following advantage.

7.4. Tensor functoriality. All of the above has an analog in the algebraic setting of section \cite{6.3} if $T$ is a topological generator of the local monodromy group $\mathbb{Z}_l(1)$ as in \cite[§3.6]{II}.

**Lemma.** For $\delta \in \operatorname{Perv}(\mathcal{X})$, denote by $Gr^*_{\bullet}(\Psi(\delta)) = \bigoplus_{i \in \mathbb{Z}} Gr^i_{\bullet}(\Psi(\delta))$ the associated graded with respect to the filtration $F^*_{\bullet}$ on $\Psi(\delta)$ as defined above. Then we have an induced functor

$$Gr^*_{\bullet} \circ \Psi : \mathcal{P}(\mathcal{X}) \longrightarrow \mathcal{P}(\mathcal{X}), \quad \delta \mapsto Gr^*_{\bullet}(\Psi(\delta))$$

which is a tensor functor with respect to convolution.

**Proof.** By lemma \cite{6.4}, $\Psi$ induces a tensor functor $\mathcal{P}(\mathcal{X}) \to \mathcal{P}_{\text{mixed}}(\mathcal{X})$ with respect to convolution. Furthermore, if a fixed generator of $\mathbb{Z}_l(1)$ acts on $\delta_i \in \operatorname{Perv}(\mathcal{X})$ via endomorphisms $T_i : \Psi(\delta_i) \to \Psi(\delta_i)$ for $i \in \{1, 2\}$, then this generator acts on $\Psi(\delta_1 + \delta_2) = \Psi(\delta_1) * \Psi(\delta_2)$ via $T_1 * T_2$. Since the tensor subcategory of $\mathcal{P}_{\text{mixed}}(\mathcal{X})$ generated by $\Psi(\delta_1)$ and $\Psi(\delta_2)$ is equivalent to the category of representations of some algebraic group, as in \cite[prop. 1.6.9]{DeKr} one deduces $Gr^i_{\bullet}(\Psi(\delta_1) * \Psi(\delta_2)) = \bigoplus_{i_1 + i_2 = i} Gr^i_{\bullet}(\Psi(\delta_1)) * Gr^i_{\bullet}(\Psi(\delta_2))$. \hfill $\square$

7.5. Weights. Concerning weights we have the following observation due to Gabber \cite[1.19]{Sa2} [\cite[th 5.1.2]{BB}].

**Lemma.** If $\delta$ underlies a pure Hodge module of weight $w$ on $\mathcal{X}$, then each graded piece $Gr^i_{\bullet}(\Psi(\delta))$ is pure of weight $w + i$ and the filtration $F^*_{\bullet}(\Psi(\delta))$ is the weight filtration of $\Psi(\delta)$ up to an index shift. Hence the analogous statement also holds for the filtration $F^*_{\bullet}$ of $\Psi_1(\delta)$.
8. Behavior of $G(X, \Theta)$ under specialization

In this section we study the behavior of the Tannaka group $G(X, \Theta)$ when the ppav $(X, \Theta)$ degenerates. For this we need to control the specialization functor $sp$ as defined in [7.1]

8.1. Alternative description of $sp(\delta)$. Let $\delta \in \text{Perv}(\mathcal{X})[-1]$. Then with notations as in 6.1 we claim that

$$sp(\delta) = i^*(j_!*_! j^* \gamma)[-1]$$

for the perverse sheaf $\gamma := \delta[1]|_{X_0}$. Indeed, a basic property of intermediate extensions [KW, III.5.1(7)] shows that $i^*j_!*_! j^* \gamma = p_{\tau < 0}i^*Rj_!\gamma = sp(\delta)[1]$ since $j$ is affine so that $i^*Rj_!\gamma \in pD^{-1,0}(\mathcal{X}_0)$ by Artin’s vanishing theorem [BBD, th. 4.1.1]. See also the formula for the cone of $N$ in section 7.1.

Proposition. From the formula $sp(\delta) = i^*(j_!*_! j^* \gamma)[-1]$ one draws the following two basic observations.

(a) If $\delta = (j_!*_! j^* \gamma)[-1] = j_!*(\delta[1]|_{X_0})[-1]$, then $sp(\delta) = i^*(\delta)$, so (6.1.2) yields an exact sequence of perverse sheaves

$$0 \rightarrow sp(\delta) \rightarrow \Psi(\delta) \rightarrow \Phi(\delta) \rightarrow 0.$$

The same holds with $\Psi$ and $\Phi$ replaced by $\Psi_1$ and $\Phi_1$ since $T$ acts trivially on $sp(\delta)$.

(b) If $\delta$ underlies a Hodge module of weights $\leq w$, so does $sp(\delta)$. This follows from the permanence properties of weights under pull-back and intermediate extensions.

8.2. In situation (a), the local invariant cycle theorem [BBD, cor. 6.2.9] states that for all $i \in \mathbb{Z}$ the induced morphisms $\mathbb{H}^i(\mathcal{X}_s, sp(\delta)) \rightarrow \mathbb{H}^i(\mathcal{X}_s, \Psi(\delta))$ factor through epimorphisms

$$\mathbb{H}^i(\mathcal{X}_s, sp(\delta)) \rightarrow \mathbb{H}^i(\mathcal{X}_s, \Psi(\delta))^T$$

onto the invariants under the local monodromy group. In (b), if $\delta$ is pure of some weight $w$, we denote by $\overline{sp}(\delta)$ the highest top quotient (of weight $w$) of the weight filtration of $sp(\delta)$.

Example. (i) If $\mathcal{X}$ is smooth over $S$ of relative dimension $d$, then

$$sp(\mathcal{C}_\mathcal{X}[d]) = \mathbb{C}_{\mathcal{X}_s}[d].$$

(ii) If $\mathcal{X}$ is regular of dimension $d + 1$, then $sp(\mathcal{C}_\mathcal{X}[d]) = \mathbb{C}_{\mathcal{X}_s}[d]$. If in addition $\mathcal{X}_s$ is normal and if the only singularities of $\mathcal{X}_s$ are finitely many ordinary double points, then $\overline{sp}(\mathcal{C}_\mathcal{X}[d]) = \mathcal{D}_{\mathcal{X}_s}$.

Proof. Part (i) is obvious. To check the first statement in (ii), note that the regularity of $\mathcal{X}$ implies $j_! j^*(\mathbb{C}_\mathcal{X}[d + 1]) = \mathbb{C}_\mathcal{X}[d + 1]$. For the second statement
in (ii) see lemma 12.1, noting that in the case at hand \( \mathcal{X}_s \) is automatically a
local complete intersection. \( \square \)

8.3. As an immediate corollary, part (i) of the above example applied to a
smooth family of theta divisors gives the following

**Rigidity lemma.** If conjecture 8 holds for a single ppav in \( \mathcal{A}_g \setminus \mathcal{N}_g \), then it
holds for all ppav’s in \( \mathcal{A}_g \setminus \mathcal{N}_g \).

*Proof.* Any two ppav’s in \( \mathcal{A}_g \setminus \mathcal{N}_g \) with corresponding groups \( G_1, G_2 \) can be
connected by a sequence of smooth curves in \( \mathcal{A}_g \setminus \mathcal{N}_g \). These curves give smooth families of ppav’s whose relative theta divisors are smooth over the respective
base curves. From part (i) of the example and section 6.6 we conclude \( G_1 \hookrightarrow G_2 \).
Hence by symmetry \( G_1 = G_2 \). \( \square \)

8.4. Now, for a principally polarized abelian scheme \( (\mathcal{X}, \Theta_{\mathcal{X}}) \) over a discrete
valuation ring \( S \) as in sections 6.3 and 6.4, we write \( \Theta_s = \Theta_{\mathcal{X}_s} \) and \( \Theta_n = \Theta_{\mathcal{X}_n} \).
Then we have the following

**Lemma.** If \( \Theta_{\mathcal{X}} \) and \( \Theta_s \) are normal, then \( \overline{\mathcal{P}}(\delta_{\Theta_n}) \) contains \( \delta_s \). In particular,
then \( G_s = G(\mathcal{X}_s, \Theta_s) \) is a subquotient of \( G = G(\mathcal{X}_n, \Theta_n) \), and the defining
representations of these groups satisfy \( \dim_k(\omega(\delta_{\Theta_s})) \leq \dim_k(\omega(\delta_{\Theta_n})) \).

*Proof.* a) If \( \delta_{\Theta_s} \) is a constituent of \( \overline{\mathcal{P}}(\delta_{\Theta_n}) \), then \( G_s \) is a quotient of the
Tannaka group \( G_\psi \), and lemma 6.6 implies \( G_\psi \hookrightarrow G \). So the second statement
of the lemma follows from the first one. It remains to show that, if \( \Theta_{\mathcal{X}} \) and \( \Theta_s \)
are normal, \( \delta_{\Theta_s} \) is a constituent of \( \overline{\mathcal{P}}(\delta_{\Theta_n}) \).

b) If \( \Theta_{\mathcal{X}} \) is normal, lemma 12.1(ii) gives an exact sequence of perverse sheaves
\( 0 \to \psi_{\Theta_{\mathcal{X}}} \to k_{\Theta_{\mathcal{X}}}[g] \to \delta_{\Theta_{\mathcal{X}}} \to 0 \) where \( \psi_{\Theta_{\mathcal{X}}} \) has weights \( < g \). Recall that
\( i^*\delta_{\Theta_{\mathcal{X}}}[{-1}] = sp(\delta_{\Theta_n}) \) is perverse by proposition 8.1. Hence, if we apply \( i^*[-1] \)
to this sequence, we get an exact sequence of perverse sheaves on \( \mathcal{X}_s \)
\( 0 \to i^*H^0(i^*\psi_{\Theta_{\mathcal{X}}}[{-1}]) \to k_{\Theta_s}[g-1] \to sp(\delta_{\Theta_n}) \to i^*H^1(i^*\psi_{\Theta_{\mathcal{X}}}[{-1}]) \to 0 \),
since \( \Theta_s \) is a local complete intersection and therefore also \( i^*k_{\Theta_{\mathcal{X}}}[g-1] = k_{\Theta_s}[g-1] \) is perverse. Since the restriction functor \( i^* \) preserves upper bounds
on weights, the first term \( i^*H^0(i^*\psi_{\Theta_{\mathcal{X}}}[{-1}]) \) has weights \( < g-1 \).

c) Lemma 12.1(ii) applied to the special fiber gives an exact sequence of perverse sheaves
\( 0 \to \psi_{\Theta_s} \to k_{\Theta_s}[g-1] \to \delta_{\Theta_s} \to 0 \)
where \( \psi_{\Theta_s} \) has weights \( < g-1 \) with pure quotient \( \delta_{\Theta_s} \) of weight \( g-1 \). Thus the
perverse sheaf \( k_{\Theta_s}[g-1] \) admits \( \delta_{\Theta_s} \) as the highest weight quotient of weight \( g-1 \). By weight reasons and the exact sequence of perverse sheaves in b),
this epimorphism factorizes over the quotient perverse sheaf \( sp(\delta_{\Theta_n}) \), because
\( p^H_0(i^*\psi_{\Theta X}[-1]) \) has weights \( < g - 1 \). Again by weight reasons the epimorphism then also factorizes over \( \overline{\mathcal{P}}(\delta_{\Theta_n}) \), i.e. we get an epimorphism \( \overline{\mathcal{P}}(\delta_{\Theta_n}) \rightarrow \delta_{\Theta_n} \). \( \square \)

In passing we remark that, if in the setting of lemma 8.4 the assumption that \( \Theta_s \) is normal is replaced by the assumption that \( \Theta_n \) is smooth, then the argument in b) does imply that \( k_{\Theta_n}[g - 1] \) is a perverse subsheaf of \( sp(\delta_{\Theta_n}) \). Indeed, then \( \psi_{\Theta X} = i_*(\alpha) \) for some perverse sheaf \( \alpha \) supported in the special fiber \( \Theta_s \), so that \( p^H_0(i^*\psi_{\Theta X}[-1]) \) vanishes since \( i^*\psi_{\Theta X} = \alpha \) is perverse.

9. Proof of conjecture \( B \) for \( g = 4 \): Outline

To prove conjectures \( B \) and \( C \), let \((X, \Theta)\) be a ppav of genus 4 with smooth theta divisor \( \Theta \). By the rigidity lemma in section 8.3 and by section 2 we may assume \( MT(\Theta)_{sc} = MT(X) \times \text{Sp}(10, \mathbb{C}) \). We have to show that \( \delta_{\pm} \) are simple objects in the representation category \( \text{Rep}(G(X, \Theta)) = \overline{P}(X, \Theta) \). For this we compare two tensor functors,

(a) the global motivic functor \( MT \) related to the Mumford-Tate group of \( \Theta \), and

(b) a restriction functor induced from an embedding \( G_\Psi \hookrightarrow G = G(X, \Theta) \) of algebraic groups via the theory of vanishing cycles, obtained by a degeneration of \( X \) into the Jacobian \( JC \) of a general curve \( C \) of genus 4.

To describe the first functor recall the realization functor from the abelian category \( \text{MHM}(X) \) of mixed Hodge modules to the abelian category \( \text{Perv}(X) \) of perverse sheaves, and similarly the functor \( D^b(\text{MHM}(X)) \rightarrow D^b_c(X, k) \). We consider the tensor subcategory \( \text{MHM}(X, \Theta) \subset D^b(\text{MHM}(X)) \) over \( P(X, \Theta) \subset D^b_c(X, k) \). The direct image under the structure morphism \( X \rightarrow \text{Spec}(\mathbb{C}) \) induces a tensor functor to the category of \( k \)-linear finite dimensional super representations of the group \( MT(X, \Theta) \),

\[
\text{MT} : \text{MHM}(X, \Theta) \rightarrow \text{sRep}(MT(X, \Theta)).
\]

To define the second functor we let \((X, \Theta)\) degenerate into the Jacobian \( JC \) of a general curve of genus 4. For the details of the construction we refer to section 10.1. We show that the subgroup \( G_\Psi \subset G = G(X, \Theta) \) defined by this degeneration contains the group \( G(JC, \Theta_{JC}) = \text{Sl}(6, \mathbb{C})/\mu_3 \) (see \[\text{We1}\]).

We thus obtain a diagram of \( \otimes \)-functors, where the vertical \( \otimes \)-functor is the composition of the realization functor \( \text{MHM}(X, \Theta) \rightarrow P(X, \Theta) \) and the quotient functor \( P(X, \Theta) \rightarrow \overline{P}(X, \Theta) \):

\[
\begin{array}{ccc}
\text{MHM}(X, \Theta) & \xrightarrow{\text{MT}} & \text{sRep}(MT(X, \Theta)) \\
\downarrow & & \downarrow \\
\overline{P}(X, \Theta) & \xrightarrow{} & \text{Rep}(G) \\
& & \xrightarrow{} \text{Rep}(G(JC, \Theta_{JC}))
\end{array}
\]
Notice that the thereby defined functors $MHX(X, \Theta) \to s\text{Rep}(MT(X, \Theta))$ and $MHX(X, \Theta) \to \text{Rep}(G(JC, \Theta_{JC}))$ are completely unrelated to each other. Since unlike the second functor the first functor is non-trivial on $T(X)$, in order to compare the two functors, we have to carefully keep track of all constituents from $T(X)$. These constituents arise in the tensor square of the generator $\delta = \delta_{\varphi}$ of the tensor category $\text{MHM}(X, \Theta)$, see lemma 3.5. The most important step for the proof, that the two summands $\delta_{\pm}$ of the tensor square of $\delta$ are simple, is the key lemma 10.5 where we construct two large simple subobjects $\gamma_{\pm} \subseteq \delta_{\pm}$. For the proof of this lemma we compare the decomposition of the objects $\delta_{\pm} \in \text{MHM}(X, \Theta)$ in the two representation categories $s\text{Rep}(MT(\Theta))$ and $\text{Rep}(G(JC, \Theta_{JC}))$. In the decomposition of the image of $\delta_{\pm}$ in $s\text{Rep}(MT(X, \Theta))$ there arise many irreducible summands; the complete list can be read off from table 1 in section 13. On the other hand, by [We1] the decomposition of the image of $\delta_{\pm}$ in $\text{Rep}(G(JC, \Theta_{JC}))$ has only few summands. In order to compare these two apparently unrelated decompositions one needs to know the behavior of the Mumford-Tate group for $(X, \Theta)$ degenerating into $(JC, \Theta_{JC})$. Indeed, since the simply connected covering of the Mumford-Tate group $MT(\Theta_{JC})$ for a general curve of genus 4 is isomorphic to the group $\text{Sp}(8, \mathbb{C})$, this behavior is described by a group homomorphism

$MT(JC, \Theta_{JC})_{sc} = \text{Sp}(8, \mathbb{C}) \to \text{Sp}(8, \mathbb{C}) \times \text{Sp}(10, \mathbb{C}) \to MT(X, \Theta)$

where the first homomorphism is the diagonal embedding $(id, \varphi)$ defined in [10.4] and where the second homomorphism is the one from section 2. Finally, in the last step of the proof we also need to consider the action of the involution $\sigma$ on the perverse sheaves involved, and their weights, in order to control the many summands arising from table 1. Using this we show in subsection 10.6 that $\gamma_{\pm} = \delta_{\pm}$.

10. Proof of conjecture $B$ for $g = 4$: The details

In order to apply the results of section 2, we give the proof in an analytic framework. For this recall that if $V$ is a complex algebraic variety, $D_c^b(V, k)$ is a full subcategory of the derived category of complexes on the analytic space $V^{an}$ which are constructible for an algebraic stratification [BBD, 6.1.2]. Hence it is a full subcategory of the category of all $\mathbb{C}$-constructible complexes on $V^{an}$ in the sense of [KS]. The number of constituents of a semisimple complex is the same in any of the above triangulated categories in which it lies, so we will denote all of them indifferently by $D_c^b(-, \mathbb{C})$.

10.1. Degeneration into a Jacobian. We first construct the degenerating family of ppav’s to be used in the proof. Let $C$ be a generic smooth algebraic curve of genus $g = 4$ over $\mathbb{C}$ and $(JC, \Theta_{JC})$ its Jacobian variety. Recall from
lemma \[5.1\] that the theta divisor \( \Theta_{JC} \) has precisely two distinct ordinary double points \( \pm e \) as singularities.

**Lemma.** There exists a principally polarized abelian scheme \((X, \Theta_X)\) over a smooth quasi-projective curve \(S\),

\[
\begin{array}{c}
\Theta_X \xrightarrow{\psi} X \\
\downarrow f \\
S
\end{array}
\]

and a point \( s \in S(\mathbb{C}) \) such that

- the fibre in \( s \) is \((X_s, \Theta_s) \cong (JC, \Theta_{JC})\),
- the total space \( X \) and the relative theta divisor \( \Theta_X \) are nonsingular,
- over \( S^* = S \setminus \{s\} \), the structure morphisms \( X_{S^*} = X \times_S S^* \to S^* \) and \( \Theta_{X^*} = \Theta_X \times_S S^* \to S^* \) are smooth.

**Proof.** Write \( JC = \mathbb{C}^4/(\mathbb{Z}^4 + \tau_0 \mathbb{Z}^4) \) for some point \( \tau_0 \) in the Siegel upper half plane \( \mathcal{H}_4 \), and choose lifts \( \pm z_0 \) in \( \mathbb{C}^4 \) of \( \pm e \in JC \). Let \( \pi : \mathcal{H}_4 \to \mathcal{A}_4 \) be the analytic quotient map. Since \( \pm e \) are ordinary double points of \( \Theta_{JC} \), by the heat equation the gradient of the Riemann theta function \( \theta(\tau, z) \) in \( \tau \)-direction does not vanish at \((\tau_0, \pm z_0)\). Since \( \mathcal{A}_4 \) is quasi-projective, we can use a suitable system of regular parameters of the regular local ring at \( \tau_0 \) to construct Zariski-locally a smooth algebraic curve \( S \subset \mathcal{A}_4 \) intersecting \( \mathcal{J}_4 \) transversely in \( s = \pi(\tau_0) \) in a tangent direction along which the gradient of \( \tau \mapsto \theta(\tau, \pm z_0) \) does not vanish at \( \tau_0 \). We define \((X, \Theta_X)\) as the restriction of the universal ppav \((\mathbb{C}^g \times \mathcal{H}_g)/(\mathbb{Z}^g \times \text{Sp}(8, \mathbb{Z})) \to \mathcal{A}_4 \) to \( S \), shrinking \( S \) if required. \( \square \)

### 10.2. Some Notations.

To transfer the constructions of \[3.4\] to the relative situation \[10.1\] consider the pure Hodge module \( \delta = \delta_{\Theta_X}[-1] = \mathbb{C}_{\Theta_X}[3] \). By Gabber’s theorem its relative symmetric resp. alternating convolution square decomposes as \( S^2(\delta) = \delta_+ \oplus \tau_+ \) resp. \( \Lambda^2(\delta) = 1_X \oplus \delta_- \oplus \tau_- \) where \( 1_X \) is a complex supported on the zero section of \( X \) and concentrated in degree zero, where \( \delta_\pm \in D_c^b(\mathcal{X}, \mathbb{C}) \) are semisimple complexes and where as in lemma \[3.5\] one has \( \tau_\pm = \bigoplus_{\mu = \pm (\text{mod } 2)} f^*(R^{3-|\mu|}f_*(\delta_X)) \otimes \delta_X[\mu - 1] \). For each \( t \in S^*(\mathbb{C}) \) the restrictions \( \tau_{\pm}|_{X_t} \) and \( \delta_{\pm}|_{X_t} \) are the complexes that were previously denoted by \( \tau_\pm \) resp. \( \delta_\pm \). To prove conjecture \[3\] it clearly suffices to show that the new \( \delta_\pm \in D_c^b(\mathcal{X}, \mathbb{C}) \) are simple.

We consider equivariant perverse sheaves [KW, section III.15] with respect to the involution \( \sigma = -id_{\mathcal{X}_s} \). Let \( 1_{\sigma \pm} \) be the \( \sigma \)-equivariant skyscraper sheaf \( 1 \) with \( \sigma \) acting by \( \pm 1 \). For \( x \in \mathcal{X}_s \setminus \{0\} \) we denote by \( 1_{\pm x} = t^x_*(1) \oplus t^{-x}_*(1) \) the simple \( \sigma \)-equivariant skyscraper sheaf supported in \( \{\pm x\} \), with \( \sigma \) flipping the two summands.
10.3. Monodromy filtrations. Recall that by construction $\Theta_s = \Theta_{X_s}$ is regular except for two distinct ordinary double points $\pm e$. In particular, then $\delta = \delta_{X_s}[-1]$ has unipotent global monodromy by lemma 7.1. From section 7.2 we deduce that the monodromy filtration diagram of $\Psi_1(\delta)$ is

$$
\begin{array}{c}
\delta_{X_s} \\
1_{\pm e}(-2)
\end{array}
\cong
\begin{array}{c}
N \\
1_{\pm e}(-1)
\end{array}
$$

since $sp(\delta) = \lambda_{\Theta_s}$ by ex. 8.4 and since the weight filtration of $\lambda_{\Theta_s}$ is defined by the exact sequence $0 \to 1_{\pm e}(-1) \to \lambda_{\Theta_s} \to \delta_{\Theta_s} \to 0$ of lemma 12.1.

Remark. The above implies $\chi(\delta_{\Theta_s}) = \chi(\Psi_1(\delta)) - 4 = g! - 4 = 20$. More generally, for any $g \geq 4$ and $r \in \mathbb{N}$, similar arguments show that for a ppav $(X_s, \Theta_s)$ whose theta divisor $\Theta_s$ has precisely $r$ ordinary double points as singularities, one has $\chi(\delta_{\Theta_s}) = g! - 2r$.

Now consider $\Psi_1(\delta_{\pm})$. Clearly $S^2(1_{\pm e}) = 1_{\pm 2e} \oplus 1_{\sigma+}$ and $\Lambda^2(1_{\pm e}) = 1_{\sigma-}$. Furthermore, with notations as in Weil, theorem 14 on p. 123 in loc. cit. and the representation theory of $\text{Sl}(6, \mathbb{C})$ imply that

$$S^2(\delta_{\Theta_s}) = \delta_{3,3} \oplus \delta_{5,1} \quad \text{and} \quad \Lambda^2(\delta_{\Theta_s}) = \delta_{4,2} \oplus \delta_{6,0}.$$ 

In what follows, we write $\delta_{\alpha} = p\delta_{\alpha} \oplus c\delta_{\alpha}$ where $c\delta_{\alpha}$ is a direct sum of complex shifts of $\delta_{X_s}$ and $p\delta_{\alpha}$ has no constituents in $T(X)$. Here $c$ and $p$ stand for the properties of being constant resp. perverse.

Lemma. Up to constituents with vanishing hypercohomology, for $\Psi_1(\delta_{\pm})$ the monodromy filtration diagram is

$$
\begin{array}{c}
(p\delta_{3,3} \oplus p\delta_{5,1} \oplus 1_{\sigma-})(-3) \\
(p\delta_{3,3} \oplus 1_{\pm e})(-2) \\
\delta_{\Theta_s} \ast 1_{\pm e})(-1)
\end{array}
\cong
\begin{array}{c}
N \\
N \\
N
\end{array}
\begin{array}{c}
(1_{\pm 2e} \oplus 1_{\sigma+})(-4) \\
(1_{\pm 2e} \oplus 1_{\sigma+})(-3) \\
(1_{\pm 2e} \oplus 1_{\sigma+})(-2)
\end{array}$$
with graded pieces of weights 4, 5, 6, 7 and 8. For $\Psi_1(\delta_-)$ one has the diagram

\[
\begin{array}{c}
\delta_\Theta \ast (1_{\pm \epsilon} \oplus 1_{\sigma})(-2) \\
\delta_\Theta \ast (1_{\pm \epsilon} \oplus 1_{\sigma})(-1)
\end{array}
\begin{array}{c}
\cong N \\
\cong N
\end{array}
\begin{array}{c}
1_{\sigma}(-4) \\
1_{\sigma}(-3) \\
1_{\sigma}(-2)
\end{array}
\]

with graded pieces of the same weights as above.

Proof. Use lemma 7.4 and the fact that $\Psi_1(\delta_\pm)$ differs from $\Psi(\delta_\pm)$ at most by constituents with vanishing hypercohomology, taking into account that $\delta_\pm$ has unipotent global monodromy. To lift the obtained result from $P(\mathcal{X}_s)$ to $Perv(\mathcal{X}_s)$, note that the complex translates of $\delta_\mathcal{X}_0$ which enter $\Psi_1(S^2(\delta))$ and $\Psi_1(\Lambda^2(\delta))$ can be computed as in 3.5 so no additional such terms occur in $\Psi_1(\delta_\pm)$. \qed

10.4. The degenerate Hodge structure. For a suitable choice of $(\mathcal{X}, \Theta_\mathcal{X})$ in lemma 10.1 and suitable (fixed) $t \in S^*(\mathbb{C})$ we abbreviate $\Theta_t = \Theta_{\mathcal{X}_t}$. Then section 13 shows

$$MT(\Theta_t)_{sc} = MT(\mathcal{X}_t) \times \text{Sp}(10, \mathbb{C}) \quad \text{where} \quad MT(\mathcal{X}_t) = \text{Sp}(8, \mathbb{C}).$$

In section 13 we compute the natural pure Hodge structure on $H^\bullet(\mathcal{X}_t, \delta_\mathcal{X}_t|\mathcal{X}_t)$ as a representation of this group $MT(\Theta_t)_{sc}$. But $H^\bullet(\mathcal{X}_t, \delta_\mathcal{X}_t|\mathcal{X}_t)$ can also be equipped with a different (mixed) Hodge structure, induced under the isomorphism $H^\bullet(\mathcal{X}_t, \delta_\mathcal{X}_t|\mathcal{X}_t) \cong H^\bullet(\mathcal{X}_s, \Psi(\delta_\pm))$ from the Hodge structure on the hypercohomology of the mixed Hodge module $\Psi(\delta_\pm)$.

Let us call this Hodge structure on $H^\bullet(\mathcal{X}_t, \delta_\mathcal{X}_t|\mathcal{X}_t)$ the degenerate Hodge structure. Its subquotient Hodge structures — e.g. its invariants under the monodromy operator $N$ — are no longer representations of $MT(\Theta_t)_{sc}$, but they are representations of $MT(\Theta_s)_{sc} = MT(C) = \text{Sp}(8, \mathbb{C})$.

To obtain the degenerate Hodge structure on $H^\bullet(\mathcal{X}_t, \delta_\mathcal{X}_t|\mathcal{X}_t)$, one easily sees that in table 1 of section 13 the representations of $MT(\mathcal{X}_t) = \text{Sp}(8, \mathbb{C})$ are unchanged (including their weights) when viewed as representations of $MT(\mathcal{X}_s) = \text{Sp}(8, \mathbb{C})$. In particular, all of them are invariant under the monodromy operator $N$. However, for the standard representation $B$ of $\text{Sp}(10, \mathbb{C})$ the situation is different as we will see in the lemma below: The degenerate Hodge structure arises from the natural one by pull-back along a monomorphism

$$\text{Id} \ast \varphi : MT(\Theta_s)_{sc} = \text{Sp}(8, \mathbb{C}) \hookrightarrow MT(\Theta_t)_{sc} = \text{Sp}(8, \mathbb{C}) \times \text{Sp}(10, \mathbb{C})$$
for some embedding $\varphi : \text{Sp}(8, \mathbb{C}) \hookrightarrow \text{Sp}(10, \mathbb{C})$.

Put $\Lambda^i = H^i(\mathcal{X}_s, \mathbb{Q})$ for $i \in \mathbb{N}$. As representations of the group $\text{Sp}(8, \mathbb{C})$ these $\Lambda^i$ are considered as the exterior powers of the standard representation. Notice $\Lambda^0 = (0000), \Lambda^1 = (1000), \Lambda^2 = (0100) \oplus \Lambda^0$ and $\Lambda^3 = (0010) \oplus \Lambda^1$ in the notations of section 13.

**Lemma.** On the quotient $B$ of $H^3(\Theta_s, \mathbb{C}) \cong \mathbb{H}^0(\mathcal{X}_t, \delta_{\Theta_t}) \cong \mathbb{H}^0(\mathcal{X}_s, \Psi(\delta))$ with its degenerate limit Hodge structure, the monodromy operator $N$ has the coinvariants

$$B/B^N = \Lambda^0(-2)$$

which are pure of weight 4, and the weight filtration of the invariants $B^N$ is given by an exact sequence

$$0 \longrightarrow \Lambda^0(-1) \longrightarrow B^N \longrightarrow \Lambda^1(-1) \longrightarrow 0$$

with $\Lambda^1(-1)$ pure of weight 3 and with $\Lambda^0(-1)$ pure of weight 2.

**Proof.** Since $\Phi(\delta) = 1_{\pm \epsilon}(-2)$ and $sp(\delta) = \lambda_{\Theta_s}$ by section 10.3 we get from proposition 8.1(a) an exact sequence

$$0 \longrightarrow \mathbb{H}^0(\mathcal{X}_s, \lambda_{\Theta_s}) \xrightarrow{\alpha} \mathbb{H}^0(\mathcal{X}_t, \delta_{\Theta_t}) \xrightarrow{\beta} \mathbb{H}^0(\mathcal{X}_0, 1_{\pm \epsilon}(-2)) \xrightarrow{\Lambda^3 \oplus \Lambda^1}$$

where $\alpha$ and $\beta$ are morphisms of Hodge structures if the middle term is equipped with the degenerate limit Hodge structure. By the local invariant cycle theorem the image of $\alpha$ is the subspace of $N$-invariant elements. Since $\dim_B(B) = 10$ and $\dim_N(\Lambda^0) + \dim_N(\Lambda^1) = 9$, by a dimension count the claim follows once we can exhibit an exact sequence

$$(10.4.1) \quad 0 \longrightarrow \Lambda^0(-1) \longrightarrow \mathbb{H}^0(\mathcal{X}_s, \lambda_{\Theta_s}) \longrightarrow \Lambda^3 \oplus \Lambda^1(-1) \longrightarrow 0$$

The sequence $(10.4.1)$ is obtained as the short exact sequence

$$(10.4.2) \quad 0 \longrightarrow \ker(\mathbb{H}^0(\nu)) \longrightarrow \mathbb{H}^0(\mathcal{X}_s, \lambda_{\Theta_s}) \longrightarrow \mathbb{H}^0(\mathcal{X}_s, \delta_{\Theta_s}) \longrightarrow 0$$

obtained by splicing up the long exact sequence attached to the short exact sequence $0 \rightarrow 1_{\pm \epsilon}(-1) \rightarrow \lambda_{\Theta_s} \rightarrow \delta_{\Theta_s} \rightarrow 0$ of lemma 12.1 i.e. with $\ker(\mathbb{H}^0(\nu))$ isomorphic to the quotient

$$(10.4.3) \quad 0 \longrightarrow \mathbb{H}^{-1}(\mathcal{X}_s, \lambda_{\Theta_s}) \longrightarrow \mathbb{H}^{-1}(\mathcal{X}_s, \delta_{\Theta_s}) \longrightarrow \ker(\mathbb{H}^0(\nu)) \rightarrow 0.$$ 

The right hand side of $(10.4.2)$ and the middle term of $(10.4.3)$ are given by the cohomology of the smooth curve $C$ via $\mathbb{H}^\bullet(\delta_{\Theta_s}) = \Lambda^0(-1)(\mathbb{H}^\bullet(\delta_C))$, see [Weil].

Thus

$$\mathbb{H}^0(\mathcal{X}_s, \delta_{\Theta_s}) = \Lambda^3 \oplus \Lambda^1(-1) \quad \text{and} \quad \mathbb{H}^{-1}(\mathcal{X}_s, \delta_{\Theta_s}) = \Lambda^2 \oplus \Lambda^0(-1).$$

To finish the proof it remains to compute the left hand side $\mathbb{H}^{-1}(\mathcal{X}_s, \lambda_{\Theta_s})$ of $(10.4.3)$. For this use the long exact cohomology sequence associated with
the triple \((X_s \setminus \Theta_s, X_s, \Theta_s)\). Since \(H_i^j(X_s \setminus \Theta_s, \mathbb{C}) = 0\) for \(i \leq 3\) by the ampleness of \(\Theta_s \subset X_s\), this implies
\[
\mathbb{H}^{-2}(X_s, \lambda_X) = H^2(X_s, \mathbb{C}) \cong H^2(\Theta_s, \mathbb{C}) = \mathbb{H}^{-1}(X_s, \lambda_{\Theta_s}),
\]
which is isomorphic to \(\Lambda^2\). From this one easily concludes the proof. \(\square\)

10.5. **The simple constituents \(\gamma_{\pm}\) of \(\delta_{\pm}\).** We now exhibit two irreducible constituents \(\gamma_{\pm} \hookrightarrow \delta_{\pm}\) such that \(\overline{sp}(\gamma_{\pm})\) differs from \(\overline{sp}(\delta_{\pm})\) at most by skyscraper sheaves. For \(w_0 \in \mathbb{Z}\) and a mixed perverse sheaf \(\pi\) on \(X_s\), we denote the maximal perverse subsheaf of \(\pi\) of weights \(< w_0\) by \(\pi_{w<w_0}\).

**Key lemma.** There are unique irreducible constituents \(\gamma_{\pm} \hookrightarrow \delta_{\pm}\) in \(\text{Perv}(X_t)\) with \(p\delta_{5,1} \oplus p\delta_{3,3} \hookrightarrow \overline{sp}(\gamma_{\pm})\) and \(p\delta_{4,2} \hookrightarrow \overline{sp}(\gamma_{\mp})\) in \(\text{Perv}(X_s)\). The perverse sheaves \(sp(\gamma_{+})_{w<0}\) and \(sp(\gamma_{-})_{w<0}\) both admit \(\delta_{\Theta_s} \ast 1_{\pm}(1)\) as a quotient.

**Proof.** Recall from lemma 10.3 that \(K = \delta_{\Theta_s} \ast 1_{\pm}(-1)\) is a \(\sigma\)-equivariant simple constituent of \(sp(\delta_{\pm})_{w<0}\). Suppose \(\epsilon_{\pm}\) is a constituent of \(\delta_{\pm}\) for which \(sp(\epsilon_{\pm})\) does not contain \(K\). Then \(sp(\epsilon_{\pm})_{w<0}\) and \(\Phi(\epsilon_{\pm})\) are skyscraper sheaves, hence
\[
\mathbb{H}^{-3}(X_s, \Phi_1(\epsilon_{\pm})) = 0 \quad \text{and} \quad \mathbb{H}^{-2}(X_s, sp(\epsilon_{\pm})) \cong \mathbb{H}^{-2}(X_s, \Psi_1(\epsilon_{\pm})).
\]

By the local invariant cycle theorem we then have an isomorphism
\[
\mathbb{H}^{-2}(X_s, \Psi_1(\epsilon_{\pm})) \cong \mathbb{H}^{-2}(X_s, \epsilon_{\pm}, \lambda_{X_t})^N.
\]

To compute \(\mathbb{H}^{-2}(X_t, \epsilon_{\pm}|_{X_t})^N\) we use the second line of table 1 in section 13 where \(\mathbb{H}^{-2}(X_t, \delta_{\pm}|_{X_t})^N\) is listed. The monodromy operator \(N\) acts non-trivially only on \(B\), and \(B/B_N^t\) and \(B_N^t\) were computed in lemma 10.4.

We claim that the summand \((1000) \otimes B_N^t\) of \(\mathbb{H}^{-2}(X_t, \delta_{\pm}|_{X_t})^N\) is linearly disjoint from \(\mathbb{H}^{-2}(X_t, \epsilon_{\pm}|_{X_t})^N\). Otherwise the summand \((1000) \otimes B_N^t\) would be contained in \(\mathbb{H}^{-2}(X_t, \overline{sp}(\epsilon_{\pm}))\) by global monodromy reasons since \(\epsilon_{\pm}\) are complexes defined globally on \(X\). Namely, \((1000) \otimes B\) would occur in \(\mathbb{H}^*(X_s, \Psi(\epsilon_{\pm})) = \mathbb{H}^*(X_t, \epsilon_{\pm})\) as a representation of the Mumford-Tate group, so by the local invariant cycle theorem \((1000) \otimes B_N^t\) would occur in \(\mathbb{H}^{-2}(X_s, \overline{sp}(\epsilon_{\pm}))\). This is impossible, since \(B_N^t\) is not pure.

By section 13 our claim shows \(\mathbb{H}^{-2}(X_s, \overline{sp}(\epsilon_{\pm})) \subseteq (1010) \oplus (0100)\). Hence again by section 13 the simple perverse sheaves
\[
L = p\delta_{5,1}, p\delta_{4,2} \quad \text{or} \quad p\delta_{3,3}
\]
cannot be constituents of \(\overline{sp}(\epsilon_{\pm})\), since for these \(L\) the representation of \(\text{MT}(X_s) = \text{Sp}(8, \mathbb{C})\) on \(\mathbb{H}^{-2}(X_s, L)\) is not contained in \((1010) \oplus (0100)\) (see line 2 of table 2).

Hence for any \(\sigma\)-equivariant constituent \(\gamma_{\pm}\) of \(\delta_{\pm}\) such that \(\overline{sp}(\gamma_{\pm})\) contains one of the constituents \(L\) above, \(sp(\gamma_{\pm})\) has \(K\) as a constituent. Since from lemma 10.3 we know \(K\) enters with multiplicity one in \(sp(\delta_{\pm})\), it follows that
there are unique $\sigma$-equivariant simple constituents $\gamma_{\pm}$ of $\delta_{\pm}$ containing one of the simple perverse sheaves $L$ above. These satisfy
\[ p_{\delta_{5,1}} \oplus p_{\delta_{3,3}} \hookrightarrow \overline{sp}(\gamma_+) \quad \text{and} \quad p_{\delta_{4,2}} \hookrightarrow \overline{sp}(\gamma_-), \]
which easily concludes the proof. \hfill \Box

10.6. **Excluding skyscraper sheaves.** Now define $\varepsilon_{\pm} \in D^b_c(\mathcal{X}, \mathbb{C})$ by the decomposition $\delta_{\pm} = \gamma_{\pm} \oplus \varepsilon_{\pm}$ of semisimple perverse sheaves. To prove conjecture $[\mathcal{B}]$ we must show $\varepsilon_{\pm} = 0$. From lemmas $[10.5]$ and $[10.3]$ we know
\[ sp_{w<6}(\varepsilon_+) \hookrightarrow (1_{\pm 2e} \oplus 1_{\sigma^+})(-2), \quad \overline{sp}(\varepsilon_+) \hookrightarrow 1_{\sigma^+}(-3), \]
\[ sp_{w<6}(\varepsilon_-) \hookrightarrow 1_{\sigma^-}(-2), \quad \overline{sp}(\varepsilon_-) \hookrightarrow (1_{\pm 2e} \oplus 1_{\sigma^+})(-3). \]
In particular, $sp(\varepsilon_{\pm})$ and hence $\Psi(\varepsilon_{\pm})$ are skyscraper sheaves, so $H^0(\mathcal{X}, \varepsilon_{\pm})$ is a direct sum of trivial representations (0000). A look at table 1 shows that these can only arise from lines 2 and 4 of this table; possible candidates arising from (1000) $\otimes B$ in line 2 are ruled out by global monodromy reasons since $B$ is an irreducible representation of $MT(\Theta_t)$. Hence the irreducibility of $1_{\pm 2e}$ as a $\sigma$-equivariant perverse sheaf implies
\[ sp(\varepsilon_+) = sp_{w<6}(\varepsilon_+) \hookrightarrow 1_{\sigma^+}(-2) \quad \text{and} \quad sp(\varepsilon_-) = \overline{sp}(\varepsilon_-) \hookrightarrow 1_{\sigma^+}(-3). \]

We now claim $sp(\varepsilon_+)$ = 0. Indeed, otherwise $H^0(\mathcal{X}, sp(\varepsilon))$ would be the trivial representation (0000) of weight 4. But the trivial representation could only arise from one of the summands
\[(S^2(B))^N, \quad (0010) \otimes B^N \quad \text{or} \quad (0000),\]
in line 4 of the first column of table 1. The first two of these summands cannot contribute because of global monodromy reasons, and the last one has the wrong weight 6. This proves our claim that $sp(\varepsilon_+)$ = 0. But then also $\Psi(\varepsilon_+)$ = 0 and hence
\[ H^\bullet(\mathcal{X}, \varepsilon_+|_{\mathcal{X}_t}) = H^\bullet(\mathcal{X}, \Psi(\varepsilon_+)) = 0 \]
which easily implies $\varepsilon_+ = 0$ since $\varepsilon_+|_{\mathcal{X}_0} \notin T(\mathcal{X}_t)$.

By the same argument, to show $\varepsilon_- = 0$ it suffices to see that $sp(\varepsilon_-) = 0$. If $sp(\varepsilon_-) \neq 0$, then by what we have seen above
\[ sp(\varepsilon_-) = \overline{sp}(\varepsilon_-) = 1_{\sigma^+}(-3), \]
hence also $\Psi(\varepsilon_-) = 1_{\sigma^+}(-3)$ and therefore $H^\nu(\mathcal{X}, \varepsilon_-|_{\mathcal{X}_t})$ is $\mathbb{C}$ for $\nu = 0$ and zero otherwise. Therefore $\chi(\varepsilon_-|_{\mathcal{X}_t}) = 1$, and by $[\mathcal{K}r\mathcal{W}2]$ then $\varepsilon_-|_{\mathcal{X}_t} = 1$. But this is impossible since $\delta_{\Theta_t} \ast \delta_{\Theta_t}$ contains the unit object $1$ only with multiplicity one. \hfill \Box
11. Equivalence of conjectures \([B]\) and \([C]\)

In this section we supply the proof of our earlier statement that for any \(g\) conjectures \([B]\) and \([C]\) are equivalent. Again let \(k = \mathbb{Q}_l\) or \(k = \mathbb{C}\). Let \(G\) be a reductive group over \(k\) and \(H \hookrightarrow G\) a closed subgroup of finite index. For a representation \(U\) of \(G\), denote by \(R_H^G(U)\) the restriction of \(U\) to \(H\). Similarly, for a representation \(V\) of \(H\), let \(I_H^G\) be the induced representation of \(G\). One then easily proves the following version of Mackey’s lemma.

**Lemma.** For every irreducible representation \(U\) of \(G\) there is a subgroup \(H' \subseteq G\) containing \(H\) and an irreducible representation \(V'\) of \(H'\) such that the restriction \(R_{H'}^H(V')\) is isotypic and \(U \cong I_{H'}^G(V')\).

**Corollary.** If \(U\) is an irreducible representation of \(G\) and \(R_{G^0}^G(U)\) contains the trivial representation, then all constituents of \(R_{G^0}^G(U)\) are trivial.

Using this we now prove the equivalence of conjectures \([B]\) and \([C]\) for all \(g\). We must see that \([B]\) implies \([C]\). If \([B]\) holds, the irreducible representations \(U = \omega(\delta_\Theta)\) and \(W_\pm = \omega(\delta_\Theta)\) of \(G = G(X, \Theta)\) satisfy

\[
S^2(U) = \begin{cases} W_+ & \text{and} \Lambda^2(U) = \begin{cases} W_+ \oplus k & \text{for } g \text{ even,} \\ W_- & \text{for } g \text{ odd.} \end{cases} \\
W_+ \oplus k & \text{for } g \text{ even,} \\ W_- & \text{for } g \text{ odd.} \end{cases}
\]

Then \(U\) cannot be a representation induced from a proper subgroup of \(G\) because otherwise \(S^2(U)\) would contain at least two non-trivial irreducible constituents. Hence by the isotypic case of the lemma, \(R_{G^0}^G(U) = U_0^\otimes n\) for some irreducible super representation \(U_0\) of \(G^0\) and some \(n \in \mathbb{N}\).

If \(n > 1\), then for \(g\) even resp. odd, \(R_{G^0}^G(W_+)\) resp. \(R_{G^0}^G(W_-)\) contains \(U_0 \otimes U_0\). Since by self-duality of \(U_0\) the trivial representation enters \(U_0 \otimes U_0\) and since \(W_+\) and \(W_-\) are irreducible, the corollary would then imply that \(U_0\) were trivial. By faithfulness then \(G^0\) would be trivial, i.e. \(G\) were a finite group. Then we could find \(K \in \text{Perv}(X)\) with \(\omega(K)\) being the regular representation \(R = k[G]\). Since \(R \otimes R = R^m\) for \(m = |G|\), then \(K \ast K \equiv K^\oplus m\) modulo \(T(X)\), so \(\mathbb{H}(X, K) = 0\) for all \(i \neq 0\) by [We1, lemma 5, p. 17], a contradiction since \(\delta_\Theta\) is a direct summand of \(K\) (every representation of \(G\) enters the regular representation \(R\)).

Hence \(n = 1\) and \(R_{G^0}^G(U) = U_0\) is irreducible. Then \(R_{G^0}^G(U)\) contains a unique highest weight vector up to scalars, so the same holds for \(R_{G^0}^G(S^2(U))\). Thus \(R_{G^0}^G(W_+)\) is neither induced nor isotypic in a non-trivial way, hence by the lemma it must be irreducible. It follows that \(W_+\) is irreducible as a super representation of the super Lie algebra \(g\) of \(G\). The classification in [KrW] and the fact that \(\dim(V) = g!\) then imply that

\(a)\) if \(g\) is even, \(G\) is of type \(A_{g-1}\) or of type \(C_{g/2}\);
\(b)\) if \(g\) is odd, \(G\) is of type \(D_{g/2}\),
and that in all these cases \( V \) is the standard representation. On the other hand, we have already observed in section 4.3 that the faithful action of \( G \) on \( V \) preserves a nondegenerate alternating resp. symmetric bilinear form \( \beta : V \times V \to k \), defining an embedding of \( G \) into \( \text{Sp}(V, \beta) \) resp. \( \text{O}(V, \beta) \), for \( g \) even resp. odd. So \( G = \text{Sp}(V, \beta) \) in case (a). In case (b) either \( G = \text{O}(V, \beta) \) or \( G = \text{SO}(V, \beta) \); but the arguments of [KrW2] show that \( G \) does not admit any non-trivial character, so \( G = \text{SO}(V, \beta) \) and we are done. \( \square \)

12. Appendix: Two lemmas on perverse sheaves

Let \( k = \overline{\mathbb{Q}}_l \) or \( k = \mathbb{C} \), and for a variety \( Y \) over an algebraically closed field of characteristic zero, let \( k_Y \in D^b_c(Y, k) \) denote the constant sheaf on \( Y \).

12.1. Weight filtrations. For the computation of nearby cycles we need the following refinement of [We1, lemma 2].

Lemma. (i) If \( Y \) is an irreducible normal local complete intersection of dimension \( d \) with singular locus \( \Sigma \subset Y \), then on \( Y \) we have an exact sequence of perverse sheaves

\[ 0 \to \psi_Y \to k_Y[d] \to \delta_Y \to 0 \]

where \( \psi_Y \) is a mixed perverse sheaf of weights \( < d \) whose support is contained in \( \Sigma \) and \( \delta_Y = \text{IC}_Y[d] \) is a pure perverse sheaf of weight \( d \).

(ii) If furthermore the only singularities of \( Y \) are finitely many ordinary double points \( y_1, \ldots, y_n \), then \( \psi_Y = \bigoplus_{i=1}^n \delta_{y_i}(-\frac{d-1}{2}) \) is a direct sum of skyscraper sheaves for \( d \) odd and \( \psi_Y = 0 \) otherwise.

Proof. (i) This follows from lemma 1 and 2 in [We1]. For the convenience of the reader we sketch the proof. Suppose \( Y \) is irreducible and normal. Then as in loc. cit. there exists a natural morphism \( \nu : \lambda_Y \to \delta_Y \) of sheaf complexes, such that \( H^{-d}(\nu) \) is an isomorphism. For \( Y \) normal \( \lambda_Y = k_Y[d] \), hence \( \lambda_X \) is a complex of weights \( \leq d \) whereas \( \delta_Y = \text{IC}_Y[d] \) is pure of weight \( d \). Since \( \lambda_Y \in pD^{\leq 0}(Y) \) by definition and \( \delta_Y \) is a perverse sheaf, \( \nu \) factorizes over the truncation morphism \( \lambda_Y \to pH^0(\lambda_Y) \). Since \( \delta_Y \) is an irreducible perverse sheaf and \( \nu \) is nontrivial, it is easy to see that the induced morphism \( \mu : pH^0(\lambda_Y) \to \delta_Y \) is nontrivial. Hence \( \mu \) defines an epimorphism in the category of perverse sheaves. So we obtain a distinguished triangle

\[ 0 \to \psi_Y \to k_Y[d] \to \delta_Y \to 0. \]

with \( \psi_Y \in pD^{\leq 0}(Y) \). The long exact sequence of cohomology sheaves for this distinguished triangle implies \( \mathcal{H}^{-\nu}(\psi_Y) \cong \mathcal{H}^{-\nu-1}(\delta_Y) \) for \( \nu \geq -d + 2 \) and \( \mathcal{H}^{-\nu}(\psi_Y) = 0 \) otherwise, and hence \( \psi_Y \) is of weights \( < d \). Finally, if \( Y \) is also a local complete intersection, it follows from [KW, III.6.5] that \( \lambda_Y \) itself is a perverse sheaf.
(ii) Let \( \pi : \tilde{Y} \to Y \) be the blow-up of \( Y \) in \( \Sigma = \{ y_1, \ldots, y_n \} \). Notice that \( \tilde{Y} \) is smooth. Since \( \pi \) restricts to an isomorphism over \( U = Y \setminus \Sigma \), by purity \( R\pi_*(\lambda_Y) = \delta_Y \oplus \gamma \) for some \( \gamma \in D^b_c(Y, k) \) with \( \text{Supp}(\gamma) \subseteq \Sigma \), say \( \gamma = \bigoplus_{i=1}^n \bigoplus_{j \in \mathbb{Z}} (\delta_{y_i}(j - d)/2)[j]^{\otimes m_{ij}} \) with \( m_{ij} \in \mathbb{N}_0 \). Since the \( y_i \) are ordinary double points, the \( Q_i = \pi^{-1}(y_i) \) are smooth quadrics of dimension \( d - 1 \), so by \([\text{SGA7}, \exp. \text{XII, th.} \ 3.3]\)

\[
H^j(Q_i, k) = \begin{cases} 
k(-\frac{1}{2}) & \text{for } j \in \{0, 2, \ldots, 2(d-1)\} \setminus \{d-1\}, \\
(k(-\frac{1}{2}))^{2\delta} & \text{for } j = d - 1, \\
0 & \text{otherwise}, \end{cases}
\]

where \( \delta = 1 \) for \( d \) odd and \( \delta = 0 \) for \( d \) even. Now \( \mathcal{H}^\bullet(R\pi_*(\lambda_Y))_{y_i} = H^\bullet(Q_i, k)[d] \).

In particular, \( \mathcal{H}^d(\gamma)_{y_i} = 0 \) and \( \mathcal{H}^{d-2}(\gamma)_{y_i} = k \), so \( m_{i,d} = 0 \) and \( m_{i,d-2} = 1 \). Then \( m_{i,d} = 0 \) and \( m_{i,d} \geq 1 \) for \( j = d - 2, d - 4, \ldots, 4 - d, 2 - d \) by the hard Lefschetz theorem. Another look at stalk cohomology then shows that \( m_{ij} = 1 \) for \( j = d - 2, d - 4, \ldots, 4 - d, 2 - d \) and \( m_{ij} = 0 \) otherwise. Thus

\[
\mathcal{H}^r(\delta_Y)y = \begin{cases} 
k & \text{for } r = -d \text{ and all } y, \\
(k(-\frac{d-1}{2}))^\delta & \text{for } r = -1 \text{ and } y \in \Sigma, \\
0 & \text{otherwise}, \end{cases}
\]

with \( \delta \) as above. On the other hand, \( k_Y[d] = \lambda_Y = \mathcal{H}^{-d}(\delta_Y) = \text{st}_{r \geq -d}(\delta_Y) \) by normality of \( Y \) \([\text{We1, lemma} \ 1]\) and \([\text{KW}, \text{III.5.14}]\). \( \square \)

12.2. Counting IC-constituents. Let \( Y \) be a variety of dimension \( g \) over an algebraically closed field of characteristic zero, and let \( \delta \in \text{Perv}(Y) \).

Lemma. The perverse sheaf \( \delta \) admits \( \delta_Y \) as a constituent iff \( \mathbb{H}^{-g}(Y, \delta) \neq 0 \).

Proof. We know \( \delta \in \text{st}D^{\geq -g}(Y, \mathbb{C}) \), so \( E_{p,q}^\delta = H^p(Y, \mathcal{H}^q(\delta)) \implies \mathbb{H}^{-g}(Y, \delta) = H^0(Y, \mathcal{H}^{-g}(\delta)) \). Let \( j : U \hookrightarrow Y \) be open dense such that \( \mathcal{G} = \delta[-g]|_U \) is locally constant, possibly zero. Then \([\text{KW, III.5.14}]\) gives \( \mathcal{H}^{-g}(\delta) = j_!(\mathcal{G}) \), hence \( H^0(Y, \mathcal{H}^{-g}(\delta)) = H^0(U, \mathcal{G}) \), and this group is zero iff \( \mathcal{G} \) has no constant subsheaf. \( \square \)

13. Appendix: Hypercohomology computations

In this appendix we determine the hypercohomology of some perverse sheaves required in section 10. Let \( (X, \Theta) \) be a general ppav in \( \mathcal{A}_4 \) as in section 2 and define \( \delta_\alpha \in \text{Perv}(X) \) as in 3.3. Let \( JC \) be the Jacobian of a general curve \( C \) of genus \( g = 4 \), and consider the associated perverse sheaves \( \mathcal{L}_\alpha \) on \( JC \) as in section 10.3.

Let us group the hypercohomology of perverse sheaves into packages \( [n]_j \) which are stable under the Lefschetz operator and occur precisely in degrees \( n, n-2, \ldots, 2-n, -n \) for some \( n \in \mathbb{N}_0 \). Denote by \( (a_1, a_2, a_3, a_4) \) the
irreducible representation of $\text{Sp}(8, \mathbb{C})$ with highest weight $a_1 \omega_1 + \cdots + a_4 \omega_4$ for the fundamental dominant weights $\omega_1, \ldots, \omega_4$, and let $B$ be as in section 2 for $(X, \Theta)$. To indicate that a representation enters with multiplicity $m > 1$ we use a superscript $\oplus m$, and we specify the action of $\sigma = -id_X$ with a a subscript $\sigma \pm$.

**Lemma.** The representation of $\text{MT}(\Theta)_{sc} = \text{Sp}(8, \mathbb{C}) \times \text{Sp}(10, \mathbb{C})$ on the Hodge structure $\mathbb{H}^\bullet(X, \delta_\pm)$ is given in table 1, with $\sigma = -id_X$ acting trivially on $B$. Similarly, the representation of $\text{MT}(C) = \text{Sp}(8, \mathbb{C})$ on $\mathbb{H}^\bullet(JC, p_\delta_\alpha)$ for $\alpha \in \{(5, 1), (4, 2), (3, 3)\}$ is given in table 2.

Proof. This has been worked out using the computer algebra systems MAGMA and SAGE. For $\mathbb{H}^\bullet(X, \delta_\pm)$ take the symmetric resp. alternating square of $\mathbb{H}^\bullet(X, \delta_\Theta) = (0000)[3]_t \oplus (1000)[2]_t \oplus (0100)[1]_t \oplus ((0010) \oplus B)) [0]_t$ in the super sense and then subtract $\mathbb{H}^\bullet(X, \tau_\pm)$. Here $\sigma$ acts by $-1$ on $(1000)$ and on $(0010)$ but trivially on $(0000)$, $(0100)$, $(0001)$. To check that $\sigma$ acts trivially on $B$, note that by [BL ex. 4.12(14)] the number of 2-torsion points on $\Theta$ is $2^{g-1}(2^g - 1) = 120$; the Lefschetz fixed point formula for $\sigma$ then implies that $\sigma^*|_B = id_B$. For the last three columns, for $a \geq b > 0$ the Littlewood-Richardson rule in [We1] says that $p_\delta_{a,b}$ is the difference of $p_\delta_{a} \ast p_\delta_{b}$ and $p_\delta_{a+1} \ast p_\delta_{b-1}$ up to complex shifts of $\delta_{JC}$. These complex shifts of $\delta_{JC}$ can be computed as in the proof of lemma 3.5. □
Table 2. Decomposition of the Hodge structures $H^\bullet(JC, p_\delta)$ as representations of $\text{MT}(C) = \text{Sp}(8, \mathbb{C})$

| $\bullet$ | $H^\bullet(JC, p_{\delta_1})$ | $H^\bullet(JC, p_{\delta_2})$ | $H^\bullet(JC, p_{\delta_3})$ |
|-----------|-----------------|-----------------|-----------------|
| $[3]\circ$ | (1000)          | (1000)          | (1000)          |
| $[2]\circ$ | (0000)          | (2000)          | (2000)          |
|           |                 | (0100)$\oplus^2$| (1010)          |
|           |                 | (0000)$\oplus^3$| (0100)$\oplus^2$|
| $[1]\circ$ | (1000)$\oplus^2$| (1100)$\oplus^2$| (1100)$\oplus^3$|
|           | (0000)          | (1000)$\oplus^4$| (1000)          |
|           |                 | (0010)          | (0110)          |
|           |                 | (0100)          | (0110)          |
| $[0]\circ$ | (2000)          | (2000)          | (2000)$\oplus^2$|
|           | (0100)          | (0100)          | (1010)          |
|           | (0000)          | (0200)          | (0200)          |
|           |                 | (0100)$\oplus^3$| (0100)          |
|           |                 | (0000)$\oplus^2$| (0000)$\oplus^2$|

References

[ACGH] E. Arbarello, M. Cornalba, P. A. Griffiths, J. Harris, Geometry of algebraic curves, vol. 1. Grundlehren Math. Wiss. 267, Springer (1985).

[AEV] E. M. Andreev, E. B. Vinberg, A. G. Elashvili, Orbits of greatest dimension in semi-simple linear Lie groups. Funct. Anal. Appl. 1, no. 4 (1967) 257-261.

[AM] A. Andreotti, A. L. Mayer, On period relations for abelian integrals on algebraic curves. Ann. Sc. Norm. Super. Pisa Cl. Sci. 21 (1967) 189-238.

[BB] A. Beilinson, J. Bernstein, A proof of Jantzen conjectures. Adv. Soviet Math. 16 (1993) 1-50.

[BrB] P. Bressler, J.-L. Brylinski, On the singularities of theta divisors on jacobians. J. Algebraic Geom. 7 (1998) 781-796.

[BBD] A. Beilinson, J. Bernstein, P. Deligne, Faisceaux pervers. Astérisque 100 (1982).

[Be] A. Beauville, Prym varieties and the Schottky problem. Invent. Math. 41 (1977) 149-196.

[BF] R. Biggers, M. Fried, Irreducibility of moduli spaces of cyclic unramified covers of genus $g$ curves. Trans. Amer. Math. Soc., vol.295, no. 1 (1986) 59-70.

[BL] C. Birkenhake, H. Lange, Complex abelian varieties. Second, augmented edition. Grundlehren Math. Wiss. 302, Springer (2004).

[BLR] S. Bosch, W. Lütkebohmert, M. Raynaud, Néron models. Ergeb. Math. Grenzgeb. (3. Folge) 21, Springer (1990).

[Br] J. L. Brylinski, Transformations canoniques, dualité projective, théorie de Lefschetz, transformations de Fourier et sommes trigonométriques. In: Géométrie et analyse microlocales. Astérisque 140-141 (1986).

[BW] T. Brzezinski, R. Wisbauer, Corings and comodules. LMS Lecture Note Series 309, Cambridge University Press (2003).
[CG] H. Clemens, P. Griffiths, *The intermediate Jacobian of the cubic threefold*. Ann. of Math. 95 (1972) 281-356.

[CM] A. Collino, J. P. Murre, *The intermediate Jacobian of a cubic threefold with one ordinary double point; an algebraic-geometric approach*. Indag. Math. 40 (1978) 43-71.

[Co] A. Collino, *A cheap proof of the irrationality of most cubic threefolds*. Boll. Unione Math. Ital. (5) 16-B (1979) 451-465.

[CvdG] C. Ciliberto, G. van der Geer, *Andreotti-Mayer loci and the Schottky problem*. Doc. Math. 13 (2008) 453-504.

[Deb1] O. Debarre, *Sur la démonstration de A. Weil du théorème de Torelli pour les courbes*. Compos. Math. 58 (1986) 3-11.

[Deb2] O. Debarre, *Le lieu des variétés abéliennes dont le diviseur thêta est singulier a deux composantes*. Ann. Sci. Éc. Norm. Supér. 25 (1992) 687-708.

[Deb3] O. Debarre, *The Schottky problem – an update*. In: Complex Algebraic Geometry, Math. Sci. Res. Inst. Publ. 28 (1995) 57-64.

[Del1] P. Deligne, *Théorie de Hodge II*. Publ. Math. Inst. Hautes Études Sci. 40 (1972) 5-57.

[Del2] P. Deligne, *La théorie de Weil pour les surfaces K3*. Invent. Math. 15 (1972) 206-226.

[Del3] P. Deligne, *La conjecture de Weil II*. Publ. Math. Inst. Hautes Études Sci. 52 (1980) 137-252.

[Del4] P. Deligne, *Hodge cycles on abelian varieties*. In: Hodge cycles, motives and Shimura varieties. Lecture Notes in Math. 900 (1982) 9-100.

[Del5] P. Deligne, *Catégories tensorielles*. Moscow Math. J. 2 (2002) 227-248.

[Del6] P. Deligne, *Catégories Tannakiennes*. In: The Grothendieck Festschrift, vol. II, Modern Birkhäuser Classics (2007) 111-195.

[DM] P. Deligne, J. S. Milne, *Tannakian categories*. In: Hodge cycles, motives, and Shimura varieties. Lecture Notes in Math. 900 (1981) 101-228.

[Do] R. Donagi, *The fibers of the Prym map*. Contemporary Mathematics 136 (1992), 55-125.

[EL] L. Ein, R. Lazarsfeld, *Singularities of theta divisors and the birational geometry of irregular varieties*. Journal of the AMS, 10, n.1 (1997) 243-258

[GSM] S. Grushevsky, R. Salvati Manni, *Jacobians with a vanishing theta-null in genus 4*. Israel J. Math. 164 (2008) 303-315.

[GMP] M. Goresky, R. MacPherson, *Lefschetz fixed point theorem for intersection cohomology*. Comment. Math. Helv. 60 (1985) 366-391.

[Go] B. B. Gordon, *A survey of the Hodge conjecture for abelian varieties*. Appendix to J. D. Lewis: A survey of the Hodge conjecture. CRM Monogr. Ser. 10, AMS (1999).

[Ho] G. Hochschild, *Note on algebraic Lie algebras*. Proc. Amer. Math. Soc. 29 (1971) 10-16.

[Ill] L. Illusie, *Autour du théorème de monodromie locale*. Astérisque 223 (1994) 9-57.

[IvS] E. Izadi, D. van Straten, *The intermediate jacobians of the theta divisors of four-dimensional principally polarized abelian varieties*, J. Algebraic Geom. 4 (1995) 557-590.

[Iz] E. Izadi, *The geometric structure of $A_4$, the structure of the Prym map, double solids and $\Gamma_0$-divisors*, J. Reine Angew. Math. 462 (1995) 93-158.

[KW] E. Kiehl, R. Weissauer, *Weil conjectures, perverse sheaves and l-adic Fourier transform*. Ergeb. Math. Grenzgeb. (3. Folge) 42, Springer (2000).
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[ Ko ] B. Kostant, *Graded manifolds, graded Lie theory and prequantization*. In: Differential geometrical methods in mathematical physics. Lecture Notes in Math. 570, Springer (1977) 177-306.

[ Kr1 ] T. Krämer, *Brill-Noether sheaves*. Diplomarbeit, Universität Heidelberg (2009).

[ Kr2 ] T. Krämer, *Monodromy of the self-intersections of the theta divisor*. To appear.

[ KrW1 ] T. Krämer, R. Weissauer, *On the tensor square of irreducible representations of reductive Lie superalgebras*. Preprint (2009), arXiv:0910.5212v1.

[ KrW2 ] T. Krämer, R. Weissauer, *Vanishing theorems for constructible sheaves on abelian varieties*. To appear.

[ KS ] M. Kashiwara, P. Schapira, *Sheaves on Manifolds*. Grundlehren Math. Wiss. 292, Springer (1990).

[ Mu1 ] D. Mumford, *Prym varieties I*. In: L. V. Ahlfors et al., Contributions to analysis, Academic Press (1974) 325-350.

[ Mu2 ] D. Mumford, *Curves and their Jacobians*. Reprint as an appendix in the second edition of: The red book of varieties and schemes. Lecture Notes in Math. 1358, Springer (1999).

[ PS ] Ch. A. Peters, J. H. M. Steenbrink, *Mixed Hodge structures*. Ergeb. Math. Grenzgeb. (3. Folge) 52, Springer (2008).

[ Re ] D. Recillas, *A relation between curves of genus 4 and genus 3*. Ph. D. thesis, Brandeis University (1970).

[ Rei ] R. Reich, *Notes on Beilinson’s “How to glue perverse sheaves”*, J. Singularities 1 (2010) 94-115.

[ Ri ] K. A. Ribet, *Hodge classes on certain types of abelian varieties*. Amer. J. Math. 105 (1983) 523-538.

[ Sa1 ] M. Saito, *Modules de Hodge polarisables*. Publ. Res. Inst. Math. Sci., Kyoto Univ. 24 (1988) 849-995.

[ Sa2 ] M. Saito, *Introduction to mixed Hodge modules*. Astérisque no. 179-180 (1989) 145-162.

[ Se ] J.-P. Serre, *Algebraic groups and class fields*. Graduate Texts in Math. 117, Springer (1988).

[ SGA7 ] P. Deligne, N. Katz, *Séminaire de géométrie algébrique du Bois-Marie 1967-1969: Groupes de monodromie en géométrie algébrique (SGA 7, II)*. Lecture Notes in Math. 340, Springer (1973).

[ Va ] V. S. Varadarajan, *Supersymmetry for mathematicians*. Courant Lect. Notes Math. 11, Amer. Math. Soc. (2004).

[ We1 ] R. Weissauer, *Brill-Noether sheaves*. Preprint (2007), arXiv:math/0610923v4.

[ We2 ] R. Weissauer, *Tannakian categories attached to abelian varieties*. In: B. Edixhoven, G. van der Geer, B. Moonen: Modular forms on Schiermonnikoog. Cambridge University Press (2008) 267-274.

[ We3 ] R. Weissauer, *Semisimple algebraic tensor categories*. Preprint (2009), arXiv:0909.1793v2.