A novel method of solution for the fluid-loaded plate

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We study the equations governing a fluid-loaded plate. We first reformulate these equations as a system of two equations, one of which is an explicit non-local equation for the wave height and the velocity potential on the free surface. We then concentrate on the linearized equations and show that the problems formulated either on the full or the half-line can be solved by employing the unified approach to boundary value problems introduced by one of the authors in the late 1990s. The problem on the full line was analysed by Crighton & Oswell (Crighton & Oswell 1991 Phil. Trans. R. Soc. Lond. A 335, 557–592 (doi:10.1098/rsta.1991.0060)) using a combination of Laplace and Fourier transforms. The new approach avoids the technical difficulty of the \textit{a priori} assumption that the amplitude of the plate is in \( L^1_{q1}(\mathbb{R}^+) \) and furthermore yields a simpler solution representation that immediately implies that the problem is well-posed. For the problem on the half-line, a similar analysis yields a solution representation, which, however, involves two unknown functions. The main difficulty with the half-line problem is the characterization of these two functions. By employing the so-called global relation, we show that the two functions can be obtained via the solution of a complex-valued integral equation of the convolution type. This equation can be solved in a closed form using the Laplace transform. By prescribing the initial data \( \eta_0 \) to be in \( H^5_{(6)}(\mathbb{R}^+) \), or equivalently four times continuously differentiable with sufficient decay at infinity, we show that the solution depends continuously on the initial data, and, hence, the problem is well-posed.

Keywords: analysis of PDEs; well-posedness; boundary value problem

1. Introduction

This paper is concerned with the Cauchy problems associated with the dynamics of an elastic plate, either infinite or semi-infinite, driven from below by a mean flow. Both mathematical problems are formulated as a (linearized) free boundary value problem. The potential for the irrotational fluid driving the plate is harmonic, and we have a kinematic boundary condition on the surface of the plate, which is unknown \textit{a priori}. We use a new approach introduced by Fokas (2008) to convert this free boundary value problem into a 1-parameter family of integro-differential equations, referred to as the global relation. It is the analysis of this global relation that allows us to solve the problem, on both the full and the half-lines.

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In the past, there have been many studies involving the interaction of a fluid with mean horizontal velocity and an elastic plate. However, these papers have focused largely on questions of stability (Crighton & Oswell 1991), propagation of positive and negative energy waves (Peake 2004) and far-field approximations. The results in these papers rely heavily on the use of both Fourier and Laplace transforms, and some standard Wiener–Hopf techniques. Our study differs considerably in two ways: (i) this paper is largely theoretical, concerning rigorous questions of well-posedness and (ii) we use a new approach to boundary value problems that makes it possible to solve the much more subtle problem on the half-line.

Using the global relation, we are able to solve the initial-value problem on the full line and immediately determine the function spaces on which the Cauchy problem is well-posed. The problem on the half-line proves to be more subtle. Using similar analysis to that of the problem on the full line, we obtain an explicit expression for the solution in terms of two unknown functions. These unknown functions are related to the curvature of the plate at the hinge. By using analyticity properties of the terms appearing in the global relation, as well as some straightforward asymptotic estimates, we are able to show that these unknown functions can be expressed in terms of the solution to a Volterra integral equation of the first kind. These equations are notoriously ill-posed in standard Sobolev spaces, and as such, identifying appropriate function spaces for the initial data so that the underlying problem remains well-posed is a non-trivial matter.

(a) The governing equations

The problem we study concerns the motion of a (semi) infinite elastic plate lying in the plane \( y = 0 \), driven from below by a uniform flow with velocity \( U \) in the \( x \)-direction. We denote the amplitude of the plate by \( \eta(x, t) \), and use \( \phi(x, y, t) \) to denote the potential function for the perturbation from the mean flow. The surface of the plate is described by \( S_\eta \):

\[
S_\eta = \{(x, y) \in \mathbb{R}^2 : y = \eta(x, t)\}.
\]

We use \( N_\Sigma = (-\eta_x, 1) \) to denote the upward normal to \( S_\eta \) and \( \Omega_\eta \) to denote the region occupied by the fluid in \( y < \eta \). In the case of the half-line problem, we have \( \eta \equiv 0 \) in \( x \leq 0 \). The evolution of the plate is governed by the beam equation and the kinematic boundary condition

\[
\eta_{tt} + \eta_{xxxx} = p \quad \text{on } S_\eta
\]

and

\[
(U + \phi_x, \phi_y) \cdot N_\Sigma = \eta_t \quad \text{on } S_\eta,
\]

where \( p = p(x, y, t) \) denotes the pressure of the fluid in \( \Omega_\eta \). In addition, the Bernoulli condition is valid on the plate

\[
\phi_t + U \phi_x + \frac{1}{2} \phi_x^2 + \frac{1}{2} \phi_y^2 + p = 0 \quad \text{on } S_\eta.
\]

We assume the fluid is incompressible so that the potential is harmonic in \( \Omega_\eta \)

\[
\phi_{xx} + \phi_{yy} = 0 \quad \text{in } \Omega_\eta.
\]
We can eliminate the pressure term using equations (1.1) and (1.3), so the governing equations for the amplitude $\eta$ and the potential $\phi$ become

\begin{align*}
\phi_{xx} + \phi_{yy} &= 0 \quad \text{in} \ \Omega_\eta, \\
\eta_{tt} + \eta_{xxxx} + \phi_t + U\phi_x + \frac{1}{2}\phi_x^2 + \frac{1}{2}\phi_y^2 &= 0 \quad \text{on} \ S_\eta
\end{align*} \quad (1.5a, b)

and

\[(U + \phi_x, \phi_y) \cdot N_S = \eta_t \quad \text{on} \ S_\eta. \quad (1.5c)\]

In addition, we assume $|\nabla \phi| \to 0$ on $\partial \Omega_\eta \setminus S_\eta$.

Let $\varphi$ denote the potential evaluated on the free surface, i.e.

$$\varphi(x, t) = \phi(x, \eta(x, t), t). \quad (1.6)$$

An application of the chain rule gives

$$\varphi_x = \phi_x + \eta_x \phi_y \quad (1.7a)$$

and

$$\varphi_t = \phi_t + \eta_t \phi_y. \quad (1.7b)$$

Equations (1.5c) and (1.7) constitute a non-singular set of equations for \{\phi_x, \phi_y, \phi_t\} on the surface $S_\eta$. Solving these equations, we find

\begin{align*}
(1 + \eta_x^2)\phi_x &= \varphi_x - \eta_x (\eta_t + U\eta_x), \\
(1 + \eta_x^2)\phi_y &= \varphi_x \eta_x + \eta_t + U\eta_x
\end{align*} \quad (1.8a, b)

and

\[(1 + \eta_x^2)\phi_t = (1 + \eta_x^2)\varphi_t - \eta_t (\eta_t + U\eta_x + \eta_x \varphi_t). \quad (1.8c)\]

We can now write (1.5b) in terms of $(\eta, \varphi)$

$$\eta_{tt} + \eta_{xxxx} + \varphi_t - \frac{1}{2}\eta_t^2 - \frac{1}{2}U^2 + \frac{(U + \varphi_x - \eta_x \eta_t)^2}{2(1 + \eta_x^2)} = 0. \quad (1.9)$$

In what follows, we reduce equations (1.5a) and (1.5c) into one non-local equation in the coordinates $(\eta, \varphi)$.

### (b) The non-local formulation

One of the key ingredients in our approach for the analysis of both the nonlinear and linear problems is the employment of the so-called global relation (Fokas 2008). The global relation is a direct consequence of the fact that the Laplace equation is equivalent to the following equation:

$$\partial_x(e^{-ikx+ky}(\phi_x - i\phi_y)) + \partial_y(e^{-ikx+ky}(\phi_y + i\phi_x)) = 0 \quad \text{in} \ \Omega_\eta. \quad (1.10)$$

Integrating equation (1.10) over $\Omega_\eta$ and employing the divergence theorem gives

$$\int_{S_\eta} e^{-ikx+ky}(\phi_x - i\phi_y, \phi_x + i\phi_y) \cdot N_S \, dx$$

$$+ \lim_{h \to \infty} \int_{y=-h}^{y=h} e^{-ikx}(\phi_x - i\phi_y, \phi_y + i\phi_x) \cdot N_h \, dx = 0, \quad (1.11)$$
where \( N_h = (0, -1) \). By making the restriction \( k > 0 \), the second integral in equation (1.11) vanishes. Using the kinematic boundary condition (1.5c) and equation (1.7a), the integral equation (1.11) becomes:

\[
\int_{\mathbb{R}} e^{-ikx + k\eta}(\eta_t + U\eta_x + i\varphi_x) \, dx = 0, \quad k > 0.
\]  

(1.12)

We have thus obtained the following result.

**Proposition 1.1.** The solution to the boundary value problem specified in equation (1.5) is determined by the pair of real-valued functions \( \eta(x, t), \varphi(x, t) \) that satisfy

\[
\int_{\mathbb{R}} e^{-ikx + k\eta}(\eta_t + U\eta_x + i\varphi_x) \, dx = 0, \quad k > 0,
\]  

(1.13a)

and

\[
\eta_{tt} + \eta_{xxxx} + \varphi_t - \frac{1}{2} \eta_t^2 - \frac{1}{2} U^2 + \frac{(U + \varphi_x - \eta_x \eta_t)^2}{2(1 + \eta_t^2)} = 0,
\]  

(1.13b)

where \( \varphi = \phi(x, \eta(x, t), t) \) is the potential on the surface \( S_\eta \) and \( \eta(x, t) \) is the amplitude of the plate.

From this stage onwards, we restrict attention to the linearized problem. For the analysis of the nonlinear problem for a collection of fluid-loaded membranes, we refer the reader to Ashton (2009).

Dropping higher order terms, equations (1.13) become

\[
\hat{\eta}_t + ikU\hat{\eta} - k\hat{\varphi} = 0, \quad k > 0,
\]  

(1.14a)

and

\[
\eta_{tt} + \eta_{xxxx} + \varphi_t + U\varphi_x = 0,
\]  

(1.14b)

where the hat denotes the usual Fourier transform.

(c) The cauchy problems

We will study the initial-boundary value problem (IBVP) corresponding to the equations (1.14) on both the full and the half-line. We first work on a formal level to derive explicit solutions to the underlying problem, then prove what properties these solutions have. Frequently we will refer to the standard Sobolev space \( H^{s}_{dv} (X) \), where \( X \) is either the real or full line and \( H^{s}_{dv} (X) = W^{s, 2}_{dv} (X) \)

\[
W^{k,p}_{dv} (X) \overset{\text{def}}{=} \{ \partial^\alpha f \in L^p_{dv} (X), \ 0 \leq |\alpha| \leq k\}.
\]

where \( L^p_{dv} (X) \) is the usual space of Lebesgue integrable functions with norm

\[
\left( \int_X |f|^p \, dv \right)^{1/p}.
\]

Taking into consideration that the plate is assumed to be thin, it is not obvious that there exists a solution with \( \eta \in L^2_{dx} (X) \) because disturbances in the plate could propagate at arbitrarily high speeds. In other words, even if \( \eta(x, 0) \in C^\infty_c (X) \), i.e. the initial data are smooth with compact support in \( X \), it does not immediately follow that \( \eta \in L^2_{dx} (X) \). However, as we shall see,
the problem with such a restrictive function class does admit a solution. Indeed, it will become apparent that rapid oscillations facilitate the prescribed integrability conditions.

(i) The full line

The governing equations corresponding to the Cauchy problem for the fluid-loaded plate on the full line are given by

\[ \eta_{tt} + \eta_{xxxx} + \varphi_t + U \varphi_x = 0, \quad x \in \mathbb{R}, \quad 0 < t < T, \quad (1.15a) \]

\[ \hat{\eta}_t + i k U \hat{\eta} - k \hat{\varphi} = 0, \quad 0 < k < \infty, \quad 0 < t < T, \quad (1.15b) \]

\[ \eta(x, 0) = \eta_0(x), \quad x \in \mathbb{R}, \quad (1.15c) \]

and

\[ \varphi_t(x, 0) + U \varphi_x(x, 0) = 0, \quad x \in \mathbb{R}, \quad (1.15d) \]

where \( \eta_0 \in H^1_{dx}(\mathbb{R}) \) is the initial profile of the plate.

(ii) The semi-infinite line

In the case of the semi-infinite line, we impose extra conditions at the hinge that govern the curvature of the plate

\[ \eta_{tt} + \eta_{xxxx} + \varphi_t + U \varphi_x = 0, \quad 0 < x < \infty, \quad 0 < t < T, \quad (1.16a) \]

\[ \hat{\eta}_t + i k U \hat{\eta} - k \hat{\varphi} = 0, \quad 0 < k < \infty, \quad 0 < t < T, \quad (1.16b) \]

\[ \eta(x, 0) = \eta_0(x), \quad 0 < x < \infty, \quad (1.16c) \]

\[ \varphi_t(x, 0) + U \varphi_x(x, 0) = 0, \quad 0 < x < \infty, \quad (1.16d) \]

and

\[ \eta(0, t) = \eta_x(0, t) = 0, \quad 0 < t < T, \quad (1.16e) \]

where the hat denotes the Fourier transform on the full line. In this case, we assume \( \eta_0 \in H^3_{dx}(\mathbb{R}^+) \) as well as \( \eta_0^{(p)}(0) = 0, \quad p = 0, 1, 2, 3 \). We denote the corresponding function space \( H^3_{(\eta)}(\mathbb{R}^+) \subset H^3_{dx}(\mathbb{R}^+) \) whose elements have zero derivatives at the hinge up to and including the third derivative. Given \( f \in H^s_{(\eta)}(\mathbb{R}^+) \) \( (s \in \mathbb{Z}^+) \), note that the extension operator

\[ E : H^s_{(\eta)}(\mathbb{R}^+) \to H^s_{dx}(\mathbb{R}) : f \mapsto Ef = \begin{cases} f, & x > 0, \\ 0, & x \leq 0, \end{cases} \]

is continuous, i.e. \( \|Ef\|_{H^s(\mathbb{R})} \leq C_s \|f\|_{H^s_{(\eta)}(\mathbb{R}^+)} \), hence we may regard \( \eta_0 \in H^3_{dx}(\mathbb{R}) \) with \( \supp \eta_0 \subset \mathbb{R}^+ \).

Equations (1.15) are solved in §2. By employing the unified approach for analysing IBVPs introduced by Fokas (2008), we find expressions for \( \hat{\eta} \) and \( \hat{\varphi} \). The issue of well-posedness can then be addressed: we demonstrate that the solution to the IBVP in equation (1.15) depends continuously on the initial data, with respect to the \( L^2_{dx}(\mathbb{R}) \) topology, and so the problem is well-posed in the Hadamard sense.

In §§3 and 4, we concentrate on the problem on the half-line described by equations (1.16). The analysis is similar, but there are two main differences. (i) Using an analytic continuation argument, we show that \( k \), instead of being

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restricted to be positive, satisfies the less stringent condition that it lies within the fourth quadrant of the complex plane. (ii) The solution representation involves the two unknown functions $\eta_{xx}(0, t)$ and $\eta_{xxx}(0, t)$.

By utilizing the extra freedom in $k$ (see (i)) and employing the approach of Fokas (2008), we can determine the two unknown functions. Using an appropriate function space, we show that the two unknown functions can be determined uniquely in terms of the given data. In contrast to the problem on the full line, we find that for $\eta_0 \in H^s_{0, \delta}(\mathbb{R}^+)$, $s < 5$, the solution does not necessarily depend continuously on the initial data, and in this sense, is ill-posed in $L^2_{dx}(\mathbb{R}^+)$; however, for $s \geq 5$, the problem is well-posed.

2. The infinite line

We use the Fourier transform pair

$$\eta(x, t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ikx} \hat{\eta}(k, t) \, dk \quad \text{and} \quad \hat{\eta}(k, t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ikx} \eta(x, t) \, dx, \quad (2.1)$$

where the integrals are understood in the Lebesgue sense and, as such, equalities that follow from inversion theorems are to be understood almost everywhere. Our assumptions about $\eta(x, t)$ imply that $\hat{\eta}$ is well defined. Similarly, we define the Fourier transform pair for the potential $\varphi(x, t)$

$$\varphi(x, t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ikx} \hat{\varphi}(k, t) \, dk \quad \text{and} \quad \hat{\varphi}(k, t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ikx} \varphi(x, t) \, dx. \quad (2.2)$$

The reality of $\eta$ implies that the knowledge of $\hat{\eta}$ for $k > 0$ is sufficient for the reconstruction of $\eta$

$$\int_{\mathbb{R}} e^{ikx} \hat{\eta}(k, t) \, dk = \int_{\mathbb{R}^+} (e^{ikx} \hat{\eta}(k, t) + e^{-ikx} \bar{\hat{\eta}}(k, t)) \, dk. \quad (2.3)$$

The following proposition provides an equivalent initial-value problem to equation (1.15) for $\hat{\eta}$, from which we can reconstruct $\varphi(x, t)$ and $\eta(x, t)$.

**Proposition 2.1.** The IBVP stated in equation (1.15) is equivalent to the following initial-value problem in the spectral (Fourier) space:

$$\left(1 + \frac{1}{k}\right) \hat{\eta}_{tt} + 2iU \hat{\eta}_t + (k^4 - U^2k)\hat{\eta} = 0, \quad k > 0, \quad 0 < t < T, \quad (2.4a)$$

$$\frac{1}{k} \hat{\varphi}_{tt}(k, 0) + 2iU \hat{\varphi}_t(k, 0) - kU^2 \hat{\eta}(k, 0) = 0, \quad k > 0, \quad (2.4b)$$

and

$$\hat{\eta}(k, 0) = \hat{\eta}_0(k), \quad k > 0. \quad (2.4c)$$

For the sake of brevity, we refer the reader to the proof of a similar result in proposition 3.2. We next introduce some useful notation that shall be used throughout.
Definition 2.2. The functions \( \{c_+(k), c_-(k), \alpha(k)\} \) are defined as follows:

\[
c_\pm(k) \overset{\text{def}}{=} \pm \left( \frac{\omega_\pm^2 - k^4}{\omega_-^2 - \omega_+^2} \right) \hat{\eta}_0(k) \quad \text{and} \quad \alpha(k) \overset{\text{def}}{=} \frac{1}{i(\omega_- - \omega_+)}, \tag{2.5}
\]

where \( \omega_\pm \) are defined as the roots of the dispersion relation \( D(\omega, k) = 0 \)

\[
D(\omega, k) \overset{\text{def}}{=} -\left(1 + \frac{1}{k}\right) \omega^2 + 2U\omega + (k^4 - U^2k). \tag{2.6}
\]

The roots are given explicitly by

\[
\omega_\pm = \frac{U \pm k^2(1 + 1/k - U^2/k^3)^{1/2}}{1 + 1/k}. \tag{2.7}
\]

Employing this notation, we construct the solution to equation (2.4) and hence (1.15).

Proposition 2.3. The solution to the IBVP posed in equation (1.15) is given by

\[
\eta(x, t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}^+} (e^{ikx} \hat{\eta}(k, t) + e^{-ikx} \hat{\eta}(k, t)) \, dk
\]

and

\[
\varphi(x, t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}^+} (e^{ikx} \hat{\varphi}(k, t) + e^{-ikx} \hat{\varphi}(k, t)) \, dk.
\]

The function \( \hat{\eta}(k, t) \) is defined by

\[
\hat{\eta}(k, t) = c_+(k)e^{-i\omega_+(k)t} + c_-(k)e^{-i\omega_-(k)t}, \quad k > 0, \tag{2.8}
\]

where \( c_\pm \) and \( \omega_\pm \) are defined in equations (2.5) and (2.7), and \( \hat{\varphi}(k, t) \) defined by equation (1.15b).

Proof. All that is needed is to show that \( \hat{\eta}(k, t) \) satisfies the IBVP in proposition 2.1. This follows routinely by using the definitions and the fact that \( \omega_\pm(k) \) satisfy the dispersion relation given in definition (2.1).

Proposition 2.4. The solution to the Cauchy problem (1.15) is well-posed. The map \( S_t : H^1_{\text{dx}}(\mathbb{R}) \to L^2_{\text{dx}}(\mathbb{R}) \) defined by \( \eta_0 \mapsto \eta \) is continuous.
where equation (2.11) becomes which follows from Parseval’s theorem. Definitions 2.2 and some algebra yield problem. Indeed, the work in Fokas & Papageorgiou (2005) establishes stability which are of a convenient form for the study of the long-time asymptotics of the solution (2.8).

In addition, it is a simple task to evaluate numerically results for solutions of evolution PDEs on the half-line by analysing precisely these types of integrals. In what follows.

Recalling that the classical Fourier transform pair on the half-line is

\[
\hat{\eta}(k) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \eta(x) e^{-ikx} \, dx, \quad \eta(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ikx} \hat{\eta}(k) \, dk.
\]

Clearly, \( \hat{\eta}(k) \) is well defined and analytic for \( k \in \mathcal{D}_3 \cup \mathcal{D}_4 \), where \( \mathcal{D}_i \) represents the \( i \)th quadrant in \( \mathbb{C} \). Similarly,

\[
\hat{\varphi}(k) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \varphi(x) e^{-ikx} \, dx, \quad \varphi(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ikx} \hat{\varphi}(k) \, dk.
\]

These definitions will be used extensively in what follows.

3. The semi-infinite line

The classical Fourier transform pair on the half-line is

\[
\eta(x, t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ikx} \hat{\eta}(k, t) \, dk \quad \text{and} \quad \hat{\eta}(k, t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ikx} \eta(x, t) \, dx.
\]

Clearly, \( \hat{\eta}(k, t) \) is well defined and analytic for \( k \in \mathcal{D}_3 \cup \mathcal{D}_4 \), where \( \mathcal{D}_i \) represents the \( i \)th quadrant in \( \mathbb{C} \). Similarly,

\[
\varphi(x, t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ikx} \hat{\varphi}(k, t) \, dk \quad \text{and} \quad \hat{\varphi}(k, t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ikx} \varphi(x, t) \, dx.
\]

These definitions will be used extensively in what follows.
Proposition 3.1. Let $\eta(x,t)$ and $\varphi(x,t)$ satisfy equation (1.16). Then the half-line Fourier transforms of $\eta(x,t)$ and $\varphi(x,t)$ satisfy the following equation:

$$\hat{\varphi}(k,t) = \frac{\hat{\eta}_t(k,t)}{k} + iU\hat{\eta}(k,t), \quad k \in \mathcal{D}_4.$$  

(3.3)

Proof. Equation (1.16b) can be written as

$$0 = k\hat{\varphi} - \hat{\eta}_t - iUk\hat{\eta} + k \int_{-\infty}^{0} e^{-ikx} \varphi(x,t) \, dx$$

$$= \Phi^+(k) + \Phi^-(k),$$  

(3.4)

where $\Phi^+$ ($\Phi^-$) is analytic in $\mathcal{D}_1$ ($\mathcal{D}_4$), and integration by parts and the Riemann Lebesgue lemma gives $\Phi^\pm = O(1/k)$ at $\infty$. We observe that equation (3.4) defines the jump condition of an elementary Riemann–Hilbert problem with associated contour $\gamma = \mathbb{R}^+$, and the unique solution is $\Phi^+(k) = \Phi^-(k) = 0$. We conclude that for $k > 0$, we have the following relation:

$$\hat{\varphi}(k,t) = \frac{\hat{\eta}_t(k,t)}{k} + iU\hat{\eta}(k,t),$$

and by analytic continuation, this extends to $k \in \mathcal{D}_4$.  

Proposition 3.2. The IBVP stated in equations (1.1)–(1.6) is equivalent to the following problem:

$$\left(1 + \frac{1}{k}\right)\hat{\eta}_{tt} + 2iU\hat{\eta}_t + (k^4 - U^2k)\hat{\eta} = f(k,t), \quad k \in \mathcal{D}_4, \quad 0 < t < T,$$

(3.5)

$$\frac{1}{k}\hat{\eta}_{tt}(k,0) + 2iU\hat{\eta}_t(k,0) - kU^2\hat{\eta}(k,0) = 0, \quad k \in \mathcal{D}_4,$$

(3.6)

and

$$\hat{\eta}(k,0) = \hat{\eta}_0(k), \quad k \in \mathcal{D}_4,$$

(3.7)

where

$$f(k,t) \equiv \eta_{xxx}(0,t) + ik\eta_{xx}(0,t).$$

Proof. We begin by applying the Fourier transform on $\mathbb{R}^+$ to equation (1.16a). Integrating by parts several times gives

$$\hat{\eta}_{tt} + k^4\hat{\eta} + \hat{\varphi}_t + ikU\hat{\varphi} = \eta_{xxx}(0,t) + ik\eta_{xx}(0,t),$$

where again we have set $\varphi(0,0,t) = 0$ without loss of generality. Using the result from proposition 3.1, and the result from differentiating equation (3.3), we find equation (3.5). Next, we apply the Fourier transform to equation (1.16d), which gives

$$\hat{\varphi}_t(k,0) + ikU\hat{\varphi}(k,0) = 0.$$

Using equation (3.3) as well as the time derivative of equation (3.3), we find (3.6). Finally, equation (3.7) follows from the application of the Fourier transform to the initial data in equation (1.16c).  

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Proposition 3.3. The solution to the IBVP presented in proposition 3.2 is given by
\[
\hat{\eta}(k, t) = [c_+(k) + \alpha(k) F_t(\omega_+, k)] e^{-i\omega_+ t} + [c_-(k) - \alpha(k) F_t(\omega_-, k)] e^{-i\omega_- t}, \quad k \in D_4, \ 0 < t < T, \quad (3.8)
\]
where \( c_\pm \) and \( \omega_\pm \) are defined in equations (2.5) and (2.7) and \( F_t[\omega(k), k] \) is defined by
\[
F_t[\omega(k), k] = \int_0^t e^{i\omega(k) \tau} [\eta_{xxx}(0, \tau) + i k \eta_{xx}(0, \tau)] d\tau, \quad k \in \mathbb{C}, \ 0 < t < T. \quad (3.9)
\]
This will be referred to as the global relation (Fokas 1997).

Proof. Differentiating and using the fundamental theorem of calculus, we find the RHS of equation (3.8) solves equation (3.5) and satisfies equations (3.6)–(3.7).

We have now obtained a solution to the IBVP in equation (1.16), but in terms of the unknown functions \( \eta_{xxx}(0, t) \) and \( \eta_{xx}(0, t) \). These functions will be determined in §4.

Remark 3.4. For a number of IBVPs, it is possible to eliminate the transforms of the unknown boundary values using only algebraic manipulations (Fokas 1997). This approach utilizes the analytic dependence on \( k \), hence it suggests that we should re-parametrize the spectral problem, so that \( \omega \) takes a simpler form. However, there does not exist a rational parametrization \( k = k(t), \ \omega = \omega(t) \), i.e. rational functions \( k(t) \) and \( \omega(t) \) such that \( D(\omega(t), k(t)) = 0 \) for each \( t \in \mathbb{R} \). Indeed, let us seek a rational parametrization to the equation \( P(X, Y; U) = 0 \), where
\[
P(X, Y; U) \overset{\text{def}}{=} XY^2 - X^5 + (Y - UX)^2.
\]
This equation defines an algebraic curve over \( \mathbb{Q} \), and it is possible to show that the genus of this curve, \( g(P) \), is equal to one for all but a few (unphysical) values of the parameter \( U \). This can be done by blowing up the singularity of \( P \) at 0, reducing it to a cubic, then examining the singularities of the resulting curve. It is well known (see Sendra et al. 2007, for example) that an algebraic curve \( P \) has a rational parametrization iff \( g(P) = 0 \), so we may deduce that no rational parametrization exists for the cases of interest.

4. Determination of the unknown boundary values

The solution \( \eta(x, t) \) requires that \( \hat{\eta}(k, t) \) is solved for \( k > 0 \). However, the global relation (3.8) is valid for \( k \) in a much larger domain, namely \( k \in D_4 \). It turns out that this extra freedom allows us to determine the unknown boundary values \( \eta_{xxx}(0, t) \) and \( \eta_{xx}(0, t) \).

Our analysis involves the complex \( k \)-plane, so we must first choose appropriate branches for the functions \( \omega_\pm(k) \). It is clear that the branch points of \( \omega_\pm(k) \) are

\[1\]Here we assume \( U \) is rational, which is without loss of generality, as \( U \) corresponds to a physical constant and can be approximated arbitrarily by rationals.
at $k = 0$ and at the three roots of the cubic equation
\[ k^3 + k^2 - U^2 = 0. \]

As $U$ is real, two of the roots are a complex conjugate pair and the remaining root is real. The loci of these points is shown in Figure 1 with appropriate branch cuts.

**Proposition 4.1.** Let $\hat{h}(k, t)$ satisfy the global relation (3.8) for $k \in D_4$. Then the unknown functions $\eta_{xx}(0, t)$ and $\eta_{xxx}(0, t)$ satisfy the equation
\[ 2\pi \imath \eta_{xx}(0, t) = g(t) - \int_0^t K(t - \tau) \eta_{xxx}(0, \tau) \, d\tau, \tag{4.1} \]
where the functions $g(t)$ and $K(t)$ are defined as follows:
\[ g(t) = \left( \int_{\gamma} \frac{c_-(k)}{\alpha(k)} e^{-i\omega_-(k)t} \, d\mu(k) \right) \quad \text{and} \quad K(t) = \left( \int_{\gamma} e^{-i\omega_-(k)t} \, d\mu(k) \right). \]

The contour $\gamma$ is defined by $\gamma = (-\infty + 1/4, 1/4] \cup [1/4, \infty)$ and $d\mu = (1/k) \, d\omega_-(k)$ defines an analytic measure (Forelli 1963) on $\gamma$.

**Proof.** We evaluate equation (3.8) at $t = T$, and multiply the resulting expression by
\[ \frac{e^{i\omega_-(T-t)}}{k \alpha(k)} \left( \frac{d\omega_-}{dk} \right), \]
and then we integrate with respect to $dk$ along $\gamma$
\[ \int_{\gamma} \hat{h}(k, T) e^{i\omega_-(T-t)} \, d\mu - \int_{\gamma} \left[ \frac{c_+(k)}{\alpha(k)} + F_T(\omega_+, k) \right] e^{-i\omega_+ T} e^{i\omega_-(T-t)} \, d\mu \]
\[ = \int_{\gamma} \left[ \frac{c_-(k)}{\alpha(k)} - F_T(\omega_-, k) \right] e^{-i\omega_- t} \, d\mu. \tag{4.2} \]
The integrand of the integral on the LHS of this equation is bounded and analytic in the domain-bounded $D \subset D_4$ defined by

$$D = \left\{ k \in D_4 : \text{Re } k \geq \frac{1}{4} \right\}.$$ 

Indeed, this is a consequence of the following two facts:

(i) The following estimates hold for $k \to \infty$ in $D$:

$$\frac{c_+ (k)}{k^\alpha (k)} \left( \frac{d \omega_+}{dk} \right) = O \left( \frac{1}{k} \right) \quad \text{and} \quad \hat{\eta} (k, T) \left( \frac{d \omega_-}{dk} \right) = O \left( \frac{1}{k} \right).$$

The determination of these asymptotic estimates uses integration by parts, the Riemann–Lebesgue lemma and the assumptions $\eta_0 \in H_3^3 (\mathbb{R}^+)$ and $\eta_0 (0, T) = \eta_x (0, T) = 0$.

(ii) For $|k|$ sufficiently large, $t > 0$, we have

$$|\exp (-i \omega_+ t)| < \exp \left\{ -\frac{1}{2} |k| [4 k R - 1] t \right\}$$

and

$$|\exp (-i \omega_- t)| < \exp \left\{ \frac{1}{2} |k| [4 k R - 1] t \right\}$$

for $\epsilon > 0$ (see figure 2). The determination of these estimates is based on the asymptotic formulae

$$\omega_+ (k) = + k^2 - \frac{1}{2} k + \left( U + \frac{3}{8} \right) + o(1), \quad k \to \infty,$$

and

$$\omega_- (k) = - k^2 + \frac{1}{2} k + \left( U - \frac{3}{8} \right) + o(1), \quad k \to \infty.$$ 

Thus, for $|k|$ sufficiently large, we have $|\text{Im } \omega_+ (k) - \text{Im } (k^2 - (1/2) k)| < \epsilon$.

Using these facts as well as the equation

$$\int_{\gamma} e^{-i \omega_- (k) (t - \tau)} d \omega_- (k) = 2 \pi \delta (t - \tau),$$

where the equality is understood in the distributional sense, we find

$$0 = \int_{\gamma} \frac{c_- (k)}{\alpha (k)} e^{-i \omega_- t} d \mu - \int_{0}^{T} e^{i \omega_- (t - \tau)} \eta_{xxx} (0, \tau) d \mu d \tau - 2 \pi i \eta_{xx} (0, t).$$

Finally, we observe that

$$\int_{\gamma} \int_{t}^{T} e^{-i \omega_- (t - \tau)} \eta_{xxx} (0, \tau) d \tau d \mu = 0,$$

which follows from the fact that $t - \tau < 0$ in the integrand, so the integrand is bounded and analytic on the RHS of $\gamma$ (i.e. in $D$), and then Cauchy’s theorem implies that the above integral vanishes.

$\blacksquare$
Figure 2. Contour plots of $\text{Im} \omega_+$ and $\text{Im} \omega_-$ for two values of $U$. (a) (i) $\text{Im} \omega_+$ for $U = 45$; (ii) $\text{Im} \omega_-$ for $U = 45$. (b) (i) $\text{Im} \omega_+$ for $U = 55$; (ii) $\text{Im} \omega_-$ for $U = 55$.

Lemma 4.2. Let $K(t)$ and $g(t)$ be defined by proposition 4.1 and let $\Sigma \subset \mathbb{R}^+$ be a bounded interval. Then,

(i) the integral operator $K$ defined by

$$Kf \overset{\text{def}}{=} \int_0^t K(t - \tau)f(\tau) \, d\tau, \quad f \in L^2_{dt}(\Sigma),$$

has a weak singularity;

(ii) $g \in L^2_{dt}(\Sigma)$.

Proof. For (i), it is enough to prove

$$|K(t, \tau)| \leq M|t - \tau|^{-\alpha}$$

for some $\alpha \in (0, 1)$. We show this as follows: choose $\delta > 1$ such that the boundary $\partial B_\delta$, where $B_\delta \equiv B_\delta(0)$, lies beyond the branch cut between the complex pair of roots of $k^3 + k^2 - U^2 = 0$. The contribution from within $B_\delta$ is certainly uniformly bounded because $\gamma$ has been chosen to avoid the singularities of the relevant functions. This gives us the following inequality:

$$|K(t, \tau)| < c_0 + \left| \int_{\gamma \setminus (\gamma \cap B_\delta)} e^{-i\omega_-(k)(t-\tau)} \, d\mu \right|.$$
Now split the contour into $\gamma_1$ and $\gamma_2$, which constitute the disjoint elements of $\gamma \setminus (\gamma \cap B_\delta)$: the first is parallel with the negative imaginary axis, and the second is parallel with the positive real axis. Choosing $\delta > 0$ sufficiently large, we can use the result in proposition 4.1 so that

$$\left| \omega_-(k) + k^2 - \frac{1}{2} k - U + \frac{3}{8} \right| < \epsilon$$

on $\gamma_1$ and $\gamma_2$. Noting that the integrand decays exponentially in the first and third quadrants, we deform the $\gamma_1$ and $\gamma_2$ onto the rays $\arg(k) = \pi/4$ and $\arg(k) = -3\pi/4$, respectively, picking up a contribution from the relevant parts of $\partial B_\delta$. These additional contributions will also be uniformly bounded. Denoting the partial rays by $\Gamma_1$ and $\Gamma_2$, respectively, gives

$$\left| \int_{\gamma_1} e^{-i\omega(k)(t-\tau)} \, d\mu \right| < c_1 + \int_{\Gamma_1} |e^{-i\omega_-(k)(t-\tau)}| \, |d\mu|.$$ 

Once again using the result in proposition 4.1, we have

$$\int_{\Gamma_1} |e^{-i\omega_-(k)(t-\tau)}| \, |d\mu| \leq \int_{\delta}^{\infty} e^{\epsilon(t-\tau)} e^{(-r^2+2r)(t-\tau)}(\epsilon + 2) \, dr$$

$$\leq c_2 e^{(\epsilon+1)(t-\tau)} \int_{1}^{\infty} e^{-(r-1)^2(t-\tau)} \, dr$$

$$\leq c_3 \frac{e^{(\epsilon+1)(t-\tau)}}{\sqrt{(t-\tau)}}$$

$$< c_4 |t-\tau|^{-1/2}.$$ 

Using an entirely analogous argument for the contribution from $\Gamma_2$, we find

$$\int_{\Gamma_2} |e^{-i\omega_-(k)(t-\tau)}| \, |d\mu| < c_5 |t-\tau|^{-1/2}.$$ 

Each of the previous bounds implies the following result:

$$|K(t, \tau)| \leq c_6 + c_7 |t-\tau|^{-1/2}$$

$$= (c_6 |t-\tau|^{1/2} + c_7) |t-\tau|^{-1/2}$$

$$< M |t-\tau|^{-1/2}.$$ 

Hence, $K$ has a weak singularity with exponent $\alpha = 1/2$.

For (ii), we proceed as in the previous case by splitting the integral. Noting the asymptotic result in proposition 4.1, for $\delta$ sufficiently large, we have

$$\left| \frac{c_-(k)}{\alpha(k)} \right| < \frac{\epsilon}{|k|}.$$
for \( k \in \gamma \setminus (\gamma \cap B_\delta) \), where \( \epsilon > 0 \). This gives

\[
\left| \int_{\gamma \setminus (\gamma \cap B_\delta)} \frac{c_- (k)}{\alpha (k)} e^{-i\omega_- (k)t} \, d\mu \right| \leq \epsilon \left| \int_{\gamma \setminus (\gamma \cap B_\delta)} \frac{e^{-i\omega_- (k)t}}{|k|} \, d\mu \right|
\]

\[
\leq c_1 \int_2^\infty r^{-1} e^{-(r-1)^2} \, dr
\]

\[
< c_2 |\log t|,
\]

where we have used similar estimates as earlier. Note also the singularities of the integrand lie off \( \gamma \) by construction, so the contribution from the integral along \( \gamma \cap B_\delta \) is uniformly bounded and we have

\[
\left| \int_{\gamma} c_- (k) e^{-i\omega_- (k)t} \, d\mu \right| < c_3 |\log t|.
\]

It then follows that for \( \Sigma \) bounded, we have \( \| g \|_{L^2_{dt} (\Sigma)} < \infty \). ■

**Corollary 4.3.** The operator \( K : L^2_{dt} (\Sigma) \to L^2_{dt} (\Sigma) \) is compact.

**Proof.** \( K \) has a weak singularity, and hence it is bounded in \( L^2_{dt} (\Sigma) \). Indeed, take \( f \in L^2_{dt} (\Sigma) \), then the previous result gives

\[
|(Kf)(t)|^2 \leq \left| \int_{\Sigma} |K(t, \tau)\frac{1}{|K(t, \tau)|}|f(\tau)| \, d\tau \right|^2
\]

\[
\leq \left| \int_{\Sigma} |K(t, \tau)| |f(\tau)|^2 \, d\tau \right| \left| \int_{\Sigma} |K(t, \tau)| \, d\tau \right|
\]

\[
\leq M_1 \left| \int_{\Sigma} |K(t, \tau)| |f(\tau)|^2 \, d\tau \right|,
\]

which follows from the Cauchy–Schwarz inequality and the fact that a weak singularity is integrable. Thus,

\[
\| Kf \|_{L^2_{dt} (\Sigma)}^2 \leq M_1 \int_{\Sigma \times \Sigma} |K(t, \tau)| |f(\tau)|^2 \, d\tau \, dt
\]

\[
\leq M_2 \| f \|_{L^2_{dt} (\Sigma)}^2,
\]

where we have used Fubini’s theorem. Thus, standard results (e.g. Kantorovich & Akilov 1964), imply the desired result. ■

**Remark 4.4.** The estimates of the above lemmas are not sufficient to handle the case \( T = \infty \), i.e. \( \Sigma = (0, \infty) \), because in this case the contributions from \( \gamma \cap B_\delta \) fail to decay rapidly for large \( t \). This underlines the fact that instabilities could grow exponentially in time and hence analysis on \( L^2_{dt} (\mathbb{R}^+) \) is inappropriate. The lemma also indicates the importance of the conditions on the curvature of the plate at the hinge because without them we cannot require that \( g \in L^2_{dt} (\mathbb{R}^+) \).

As the integral operator \( K \) is of the convolution type, it follows that equation (4.1) can be solved in a straightforward manner using Laplace’s transform. In this respect, the following estimates, which can be derived using similar methodology to that used in lemma 4.2, are essential.
Lemma 4.5. Given \( g \) and \( K \) as in proposition 4.1, \( \exists \alpha > 0 \) such that \( e^{-\alpha t} g \in L_{dt}^1(\mathbb{R}^+) \) and \( e^{-\alpha t} K \in L_{dt}^1(\mathbb{R}^+) \).

The solution of equation (4.1) is unique provided that \( g \in \text{dom}(K^{-1}) \). The compactness of \( K \) means that \( K^{-1} \) is unbounded and hence necessarily discontinuous on \( L_{dt}^2(\Sigma) \). This fact suggests the existence of a sequence of initial data, \( \{\eta_0^n\} \in H^3(\mathbb{R}^+) \), such that the corresponding sequence \( g_n \in L_{dt}^2(\Sigma) \), defined by

\[
g_n(t) = \left[ \frac{\tilde{c}_-(k)}{\alpha(k)} \right] \hat{\eta}_0^n(k)e^{-i\omega_-(k)t} d\mu,
\]

has the following two properties:

\[
g_n \to 0,
\]

and

\[
\|K^{-1}g_n\|_{L_{dt}^2(\Sigma)} \to M > 0.
\]

Now as \( \hat{\eta}(k, t) \) has non-pathological dependence on \( \eta_{xxx}(0, t) \), and hence \( K^{-1}g \), it would follow that \( \hat{\eta}(k, t) \) would change discontinuously with the initial data. Hence, in this case, the problem would be ill-posed in \( L_{dt}^2(\mathbb{R}^+) \).

We now perform a regularization of the problem by modifying the relevant function spaces so that \( K \) is continuously invertible, which will ensure well-posedness. A key result, central to our argument, is the following well-known lemma.

Lemma 4.6. Let \( X, Y \) be Banach spaces and let \( X_c \subset X \) be compact. Then, if \( T : X_c \to Y \) is continuous and one-to-one, then \( T^{-1} : \mathcal{R}(X_c) \to X_c \) exists and is continuous.

As \( K : L_{dt}^2 \to L_{dt}^2 \) is compact, it is necessarily bounded (continuous), so it follows immediately from lemma 4.6 that if we choose \( X_c \) to be compactly embedded in \( L_{dt}^2(\Sigma) \) such that \( K|_{X_c} \) is one-to-one, then the following map is a homeomorphism:

\[
K|_{X_c} : X_c \to \mathcal{R}(K|_{X_c}),
\]

where \( K|_{X_c} \) denotes the restriction of \( K \) to \( X_c \). We now prove that \( K|_{X_c} \) is one-to-one.

Lemma 4.7. The integral operator \( K|_{X_c} : X_c \to L_{dt}^2(\Sigma) \) is one-to-one.

Proof. It suffices to prove that \( \mathcal{N}(K|_{X_c}) = \{0\} \) because \( K|_{X_c} \) is linear. Suppose \( \theta \in \mathcal{N}(K|_{X_c}) \), so that

\[
K|_{X_c} \theta = 0.
\]

As \( K|_{X_c} \) is of the convolution type, equation (4.3) can be written as

\[
(K \ast \theta)(t) = 0, \quad t \in \Sigma,
\]

where \( K(t) = \int_\gamma e^{-i\omega_-(k)t} d\mu \). Now it follows from Titchmarsh’s convolution theorem that \( \theta = 0 \) almost everywhere in \((0, t_1) \) and \( K = 0 \) almost everywhere in \((0, t_2) \), where \( t_1 + t_2 \geq T \). The operator \( K \) defines an analytic function of \( t \) for \( t \in \Sigma \), and as such the zero set for \( K \) has measure zero. Consequently, \( \theta = 0 \) almost everywhere in \( \Sigma \) and we conclude that \( \mathcal{N}(K|_{X_c}) = \{0\} \), so \( K|_{X_c} \) is one-to-one.

Corollary 4.8. The map \( K|_{X_c} : X_c \to \mathcal{R}(K|_{X_c}) \) is a homeomorphism.
Now recall the classical result owing to Rellich and Kondrachov (Robinson 2001, theorem 5.32), which states that $H^1_{dt}(\Sigma)$ is compactly embedded in $L^2_{dt}(\Sigma)$, so our previous discussion reveals that the map

$$K|_{H^1_{dt}(\Sigma)} : H^1_{dt}(\Sigma) \to \mathcal{R}(K|_{H^1_{dt}(\Sigma)})$$

is continuously invertible. It now suffices to choose $\eta_0$ so that $g \in \mathcal{R}(K|_{H^1_{dt}(\Sigma)})$, where $g$ is defined in proposition 4.1. In fact, we can simply choose $\eta_0$ so that $g \in H^1_{dt}(\Sigma)$ because the Fredholm alternative theorem implies

$$\mathcal{R}(K|_{H^1_{dt}(\Sigma)}) = \mathcal{N}(K^*|_{H^1_{dt}(\Sigma)})^\perp,$$

where the $K^*$ denotes the adjoint operator. As $K^*|_{H^1_{dt}(\Sigma)}$ is also a convolution operator, a similar argument to that used in the proof of lemma 4.7 shows that $\mathcal{N}(K^*|_{H^1_{dt}(\Sigma)}) = \{0\}$, so we conclude that

$$\mathcal{R}(K|_{H^1_{dt}(\Sigma)}) = H^1_{dt}(\Sigma).$$

We can now give sufficient conditions on $\eta_0$, so IBVP in equation (1.16) is well-posed.

**Proposition 4.9.** If $\eta_0 \in H^5_{(5)}(\mathbb{R}^+)$, then $g \in H^1_{dt}(\Sigma)$ and the IBVP (1.16) is well posed, i.e. the solution defined by $\eta_0 \mapsto \eta$ defines a continuous map from $H^5_{(5)}(\mathbb{R}^+)$ to $L^2_{dt}(\mathbb{R}^+)$.  

**Sketch proof.** First we note that for $\eta_0 \in H^5_{(5)}(\mathbb{R}^+)$, we have

$$\left| \frac{c_-(k)}{\alpha(k)} \right| \leq \frac{\epsilon}{|k|^3}$$

for $|k|$ sufficiently large. This follows from estimates similar to those in proposition 4.1. In this case, an application of Cauchy’s theorem gives that $g(0) = 0$ as required, if we impose continuity of $\eta_{xxx}(0, t)$ at $t = 0$. Also, for $\eta_0 \in H^5_{(5)}(\mathbb{R}^+)$, the map defined by

$$H^5_{(5)}(\mathbb{R}^+) \to L^2_{dt}(\Sigma) : \eta_0 \mapsto g$$

is continuous. In addition, using estimates similar to those in the proof of lemma 4.2, we find $g'(t) \in L^2_{dt}(\Sigma)$, hence $g \in H^1_{dt}(\Sigma)$, and it follows from our previous discussion that the solution to the integral equation depends continuously on $g$ with respect to the $L^2_{dt}(\Sigma)$ topology. It follows from estimates similar to those in proposition 2.4 that the solution map $S_t : H^5_{(5)}(\mathbb{R}^+) \to L^2_{dt}(\mathbb{R}^+)$, defined by $\eta_0 \mapsto \eta$, is continuous. Finally, from the standard Sobolev theory, we note the continuous embeddings

$$H^5_{(5)}(\mathbb{R}^+) \hookrightarrow H^5_{dx}(\mathbb{R}) \hookrightarrow C^4_0(\mathbb{R}),$$

so that $\eta_0$ is equivalent to a four times continuously differentiable function with sufficient decay at infinity (up to a set of measure zero).  

The question of whether this estimate is sharp remains open. We can say, however, that the conditions on the initial data are necessary to ensure $g \in H^1_{dt}(\Sigma)$. This can be seen from the following heuristic argument. The conditions
on the initial data are reflected in the bound in the integrand appearing in the definition of \( g \), so that for \( \eta_0 \in H^s_{(s)}(\mathbb{R}^+), s \in \mathbb{Z}^+ \), we have

\[
\frac{|c_-(k)|}{|\alpha(k)|} < \frac{\epsilon}{|k|^{s-2}},
\]

for \( |k| \) sufficiently large. The functions \( g \) and \( g' \) are good away from \( t = 0 \) because the relevant integrals converge uniformly owing to the exponential decay of the integrand (after a suitable deformation of the contour \( \gamma \)). For small \( t \), using analysis similar to that in the proof of lemma 4.2 and a change of variables in the integral, it can be shown that the dominant small \( t \) behaviour of \( g'(t) \) is determined by the integral

\[
t^{(s-5)/2} \int_0^\infty \tau^{-(s-3)/2} e^{-\tau} \, d\tau.
\]

For \( s \geq 5 \), one can make use of standard asymptotic techniques to show that this integral is at worst \( O(|\log t|) \) for small \( t \) and so \( g'(t) \in L^4_{dt}(\Sigma) \) and \( g \in H^1_{dt}(\Sigma) \). However, for \( s < 5 \), the following asymptotic estimate is clear:

\[
|g'(t)|^2 = O(t^{-|s-5|}) \quad \text{as } t \to 0^+,
\]

and we see \( g \notin H^1_{dt}(\Sigma) \). So, for \( s < 5 \), we cannot guarantee the continuity of the map \( K^{-1} : g \mapsto \eta_{xxx}(0, t) \), and the problem is not necessarily well-posed in this case.

As was noted in the proof to proposition 4.9, standard Sobolev embedding results imply the condition that the initial profile belongs to \( H^5_{(5)}(\mathbb{R}^+) \) means

\[
\eta_0 \in C^4(\mathbb{R}) \quad \text{and} \quad \text{supp } \eta_0 \subset \mathbb{R}^+.
\]

Recall equation (1.16a,b) from the governing equations on the half-line

\[
\eta_{tt} + \eta_{xxxx} + \varphi_t + U\varphi_x = 0, \quad 0 < x < \infty, \quad 0 < t < T,
\]

\[
\varphi_t(x, 0) + U\varphi_x(x, 0) = 0, \quad 0 < x < \infty.
\]

If we impose continuity at \( t = 0 \) and use equation (4.8) in equation (4.7), we see

\[
\lim_{t \to 0} (\eta_{tt}(x, t) + \eta_{xxx}(x, t)) = 0.
\]

Again, imposing continuity at \( t = 0 \) and appealing to the condition in equation (4.6), we find that the above limit reduces to

\[
\lim_{t \to 0} \eta_{tt}(x, t) = 0,
\]

i.e. there is no initial acceleration of the plate.

5. Conclusions

This paper has been concerned with the initial-value problem associated with an infinite and a semi-infinite fluid-loaded plate. In particular, we have examined the well-posedness of the problems and determined appropriate function spaces so that the solutions depend continuously on the initial data. In both cases, our approach relied on a non-local reformulation of the equations (1.5).
For the problem on the full line, it was determined in proposition 2.4 that the Cauchy problem was well-posed if the initial data belonged to $H^{1}_{dx}(\mathbb{R})$. Using standard Sobolev embedding results (Robinson 2001), one can show $H^{1}_{dx}(\mathbb{R})$ is continuously embedded in $C_{0}(\mathbb{R})$, so the condition for well-posedness means it is enough for the initial data to be continuous with sufficient decay at infinity. In summary, ‘the problem on the full line is well-posed for continuous initial data with sufficient decay at infinity’.

The problem on the half-line was considerably more complex. Again, we implemented the non-local approach and formally solved the problem, but the solution contained the functions $\eta_{xx}(0, t)$ and $\eta_{xxx}(0, t)$, which were unknown. The approach we used to eliminate these unknowns relied heavily on the analysis of the global relation (3.8), and was motivated by a new, unified approach to boundary value problems introduced by Fokas (2008). The problem was reduced to solving a complex-valued integral equation (4.1), the real part of which yielded a Volterra equation of the first kind. This equation is of the convolution type, so in heed of lemma 4.5, can be solved using the Laplace transform. However, strong conditions are needed on the initial profile so that the integral equation is well-posed. In proposition 4.9 and the discussion that followed, we show that the integral equation is well-posed if we restrict the initial data to the space $H^{5}_{0}(\mathbb{R}^{+})$.

Again using standard embedding results, one can show that this restriction means the initial data must belong to the space $C^{4}_{0}(\mathbb{R})$, with support contained in $\mathbb{R}^{+}$. To summarize, ‘the problem on the half-line is well-posed if the initial profile is four times continuously differentiable up to the hinge and has sufficient decay at infinity’.

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