FIBER ORDERS AND COMPACT SPACES OF UNCOUNTABLE WEIGHT

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Abstract. We study an order relation on the fibers of a continuous map and its application to the study of the structure of compact spaces of uncountable weight.

1. Introduction and main results

This work is motivated by the following general problem: Given two compact convex sets $K$ and $L$ (sitting in some locally convex linear topological spaces), are $K$ and $L$ homeomorphic? When $K$ and $L$ are metrizable (that is, they have countable weight) the well known Keller’s theorem, cf. [7], implies that $K$ and $L$ are homeomorphic if and only if they have the same dimension. Thus, when restricting our attention to compact sets of countable weight, only one topological invariant has to be computed to answer our question: the dimension, ranging from 0 to $\omega$.

When we pass to the case when the weight is uncountable, the situation is not that simple. A number of usual topological invariants, like chain conditions, cardinal functions, functional-analytic properties, etc. can be used to identify many different types of compact convex sets. Just to recall an elementary example, we may compare an uncountable product of intervals $[0,1]^\kappa \subset \mathbb{R}^\kappa$ with the ball $B(\kappa)$ of the Hilbert space $\ell_2(\kappa)$ in the weak topology. In $B(\kappa)$ we may find an uncountable family of disjoint open sets but $[0,1]^\kappa$ has the countable chain condition. Another argument would be that $B(\kappa)$ cannot be homeomorphic to an uncountable product since it contains $G_\delta$ points (we will obtain in this paper a much subtler fact: $B(\kappa)$ is not homeomorphic even to a finite product of compact spaces of uncountable weight).

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Very often, however, the standard topological technology is not so helpful as in the mentioned case of $B(\kappa)$ and $[0, 1]^\kappa$. An example of this is when we restrict our attention to weakly compact sets of the Hilbert space $\ell_2(\kappa)$. The first example a nonmetrizable weakly compact convex set not homeomorphic to $B(\kappa)$ may be traced back to constructions of Corson and Lindenstrauss [8, 16], who provided such a set in which all points are $G_\delta$. Such sets, however, cannot be symmetric and to the best of our knowledge, only recently the first author [1] provided a first example of an absolutely convex weakly compact subset of $\ell_2(\kappa)$ of weight $\kappa$ which is not homeomorphic to $B(\kappa)$. This was done by proving that $B(\kappa)$ satisfies a certain chain condition of Ramsey type introduced by Bell [4] and constructing ad hoc a compact convex set failing such property.

Let us provide now some natural examples of compact convex sets, all of them indeed representable as weakly compact convex subsets of $\ell_2(\kappa)$, for which apparently the standard techniques from topology give us no clue about the problem whether they are homeomorphic to each other or not. Letter $\kappa$ always denotes an uncountable cardinal.

- The ball of the Hilbert space $B(\kappa) = \{x \in \ell_2(\kappa) : \|x\|_2 \leq 1\}$.
- The space $P(A(\kappa))$ of Radon probability measures on $A(\kappa) = \kappa \cup \{\infty\}$, the one-point compactification of the discrete set $\kappa$.
- The spaces $P(A(\kappa)^n)$, $2 \leq n < \omega$.
- The spaces $P(\sigma_n(\kappa))$ of probability measures on $\sigma_n(\kappa) = \{x \in \{0, 1\}^\kappa : |\text{supp}(x)| \leq n\}$, $2 \leq n < \omega$.
- The finite and countable powers of the previous spaces.

We shall develop some new tools which will allow us to conclude that all these spaces are not homeomorphic to each other, with perhaps the exception of $B(\kappa)$ and $P(A(\kappa))$ for which our techniques are unable to determine whether they are homeomorphic or not. We also studied other examples, not embeddable into a Hilbert space, namely the compact sets $P([0, \kappa]^n)^m$ for $\kappa$ uncountable regular cardinal and $n, m \in \mathbb{N}$. In addition, we will obtain other applications concerning the structure of these spaces, regarding the two following kind of questions:

- The classification of the points of a compact space $K$, that is, for which points $x, y \in K$ there exists a homeomorphism $f : K \to K$ such that $f(x) = y$. 
• When a compact space $K$ can be homeomorphic to some power compact of the form $L^n$, or when it can be homeomorphic to a product of the form $L_1 \times \cdots \times L_n$.

The way to address all these questions goes through the beautiful technique of Shchepin of inverse limits and the spectral theorem developed in [19] and [20]. We explain this in detail in Section 2, but roughly speaking, given a compact space $K$ of uncountable weight, this technique allows to study the topological structure of $K$ by studying the continuous surjections $p : X \to Y$ for $X$ and $Y$ quotients of $K$ of countable weight. And here comes the key idea of our work, to study a certain preorder relation induced on the fibers of a continuous map:

**Definition 1.1.** Let $f : K \to L$ be a continuous map and $x \in L$. We define a preorder relation $\leq$ on the fiber $f^{-1}(x)$ by letting $s \leq t$ if and only if for every neighborhood $U$ of $s$ there exists a neighborhood $V$ of $t$ such that $f(V) \subset f(U)$.

In other words, $s \leq t$ if and only if

$$\{f(U) : U \text{ is a neighborhood of } s\} \subset \{f(V) : V \text{ is a neighborhood of } t\},$$

if and only if $f^{-1}f(U)$ is a neighborhood of $t$ for every neighborhood $U$ of $s$. We shall call $\mathcal{F}_x(f) = f^{-1}(x)$ to the fiber of $x$ endowed with the preorder $\leq$ (and also with its inherent topology, though we shall not use the topological structure here). We denote by $\mathcal{O}_x(f) = \mathcal{F}_x(f)/\sim$ the ordered set obtained by making a quotient by the equivalence relation $t \sim s \iff t \leq s$ and $s \leq t$.

In his study of the spaces $exp_n(2^\kappa)$ [19], Shchepin considered what in our language would be the cardinality of $\mathcal{O}_x(f)$. This was already useful in that discrete context but not in spaces like convex sets, where one needs to consider the ordered structure of $\mathcal{O}_x(f)$ to get some information.

Let us indicate how fiber orders may be helpful in the problem of classification of points of a compact space, and in the homeomorphic classification of compact sets. Consider a compact space $K$ of uncountable weight and a point $x \in K$. We can consider then the family of all fiber orders of type $\mathcal{O}_{p_L(x)}(q)$ for every continuous surjection $q : L' \to L$ between metrizable quotients of $K$ with projections $p_{L'} : K \to L'$, $p_L : K \to L$, $qp_{L'} = p_L$.

This collection of ordered sets may be in principle rather complicated, but in the examples that we deal with it happens that almost all these sets are
order isomorphic to the same ordered set that we can call \( \mathbb{O}_x(K) \). For instance, for a finite power of the ball of the nonseparable Hilbert space \( B(\kappa) \) we get the following picture:

**Theorem 1.2.** Let \( K = B(\kappa)^n \) and \( x = (x_1, \ldots, x_n) \in K \). Let \( r = |\{i : \|x_i\| < 1\}| \). Then \( \mathbb{O}_x(K) \cong [0, 1]^r \).

We view \([0, 1]^r\) as an ordered set endowed with the pointwise order, i.e.

\[
(t_1, \ldots, t_r) \leq (s_1, \ldots, s_r) \text{ iff } t_i \leq s_i \text{ for every } i.
\]

Notice other consequences of this result other than the fact that the finite powers of the ball are nonhomeomorphic. It is a standard fact that the points of \( B(\kappa)^n \) whose all coordinates belong to the sphere are the \( G_\delta \) points of \( B(\kappa)^n \) and hence, topologically different from the rest. We obtained something much less evident: that points with different number of coordinates in the sphere are topologically different. This is a complete classification of the points of \( B(\kappa)^n \) because if two points have the same number of coordinates in the sphere, then there is an automorphism of \( B(\kappa)^n \) which moves one to the other.

Apart from the euclidean ball, the other spaces that we studied are spaces of probability measures on scattered spaces. We developed a general method for computing fiber orders in these cases, which constitutes the part of our work which is technically the most involved. One of the key steps in this task is our Lemma 5.1 which probably has an independent interest. Every Radon probability measure on a scattered compact space is discrete, thus a certain (finite or infinite) convex combination of Dirac measures \( \delta_x \). The following result (which follows immediately from Theorem 6.2 below) reduces the computation of fiber orders in spaces \( P(K) \) to Dirac measures:

**Theorem 1.3.** Let \( K \) be a scattered compact and \( \mu = \sum_{i \in I} r_i \delta_{x_i} \in P(K) \), where \( x_i \in K \) are pairwise distinct and \( r_i > 0 \) for \( i \in I \). Then

\[
\mathbb{O}_\mu(P(K)) \cong \prod_{i \in I} \mathbb{O}_{\delta_{x_i}}(P(K)).
\]

The picture of the fiber orders of Dirac measures in our examples of probability measures spaces is the following:

**Theorem 1.4.** Let \( K = \sigma_n(\kappa) \) and \( x \in K \). Set \( k = n - |x| \). Then \( \mathbb{O}_{\delta_x}(P(K)) \cong \{(t_1, \ldots, t_k) \in [0, 1]^k : t_1 \leq \ldots \leq t_k\} \) where the order is defined as \( t \leq s \) if and only if \( t_j \leq s_j \) for every \( j \).

In the next result, we denote by \( 2^k \) the power set of \( \{1, \ldots, k\} \).
Theorem 1.5. Let $K = A(\kappa)^n$ or $K = [0, \omega_1]^n$, and let $x = (x_1, \ldots, x_n) \in K^n$. Set $k$ to be the number of coordinates of $x$ which are not $G_\delta$-points of $K$. Then $O_{\delta x}(P(K)) \cong \{(t_A)_{A \subseteq 2^k} \in [0, 1]^{2^k} : \sum_{A \subseteq 2^k} t_A = 1\}$ endowed with the order $(t_A) \leq (s_A)$ if and only if $\sum_{A \subseteq A} t_A \leq \sum_{A \subseteq A} s_A$ for every upwards closed family $A$ of subsets of $\{1, \ldots, k\}$.

A similar statement as Theorem 1.5 holds for compact spaces $K = [0, \tau]$ with $\tau$ an uncountable regular cardinal, but for a modified version of the ordered sets $O_x(L)$ relative to the cardinal $\tau$. Finally we state the kind of results that we prove using these techniques that refer to decomposition of compact spaces as products:

Theorem 1.6. Let $K = B(\kappa), P(\sigma_n(\kappa)), P(A(\kappa)^n)$ or $P([0, \tau]^n)$ for $\tau$ an uncountable regular cardinal, and let $k, m \in \mathbb{N}$. Suppose that there exists a compact $L$ such that $K^k \approx L^m$. Then, $k$ is a multiple of $m$.

Theorem 1.7. Let $K = B(\kappa), P(A(\kappa))$ or $P([0, \omega_1])$ and let $n, m \in \mathbb{N}$. Suppose that $L_1, \ldots, L_m$ are compact spaces of uncountable weight such that $K^n \approx \prod_{i=1}^m L_i$. Then, $m \leq n$.

Theorem 1.8. Let $\tau$ be a regular cardinal, $n, m$ natural numbers, and $L_1, \ldots, L_m$ compact spaces of weight $\tau$. If $P([0, \tau])^n \approx L_1 \times \cdots \times L_m$, then $m \leq n$.

We make two remarks about these results. First, our methods do not allow to decide whether these compact spaces can be expressed as a non-trivial product with one factor metrizable. This appears not to be an easy question. Using a result of [21] and its variants, the second author [14] has obtained that $P(K)$ is homeomorphic to $P(K) \times [0, 1]$, for any compact scattered $K$. However it is unknown to us whether $B(\kappa)$ is homeomorphic to $B(\kappa) \times [0, 1]$. Second, the first author [3] has obtained with different techniques an improvement of Theorem 1.7: If $B(\kappa)^n$ maps continuously onto a product of nonmetrizable compacta of the form $\prod_{i=1}^m L_i$, then $m \leq n$. These techniques do not apply to the case of Theorem 1.6 for $K \not= B(\kappa)$, and actually $P(A(\kappa)^n)$ and $P(\sigma_n(\kappa))$ map continuously onto $B(\kappa)^n$.

2. Spectral theory

In this section, we summarize in a self-contained way what we need about spectral theory, which is essentially taken from [19] and [20]. We also introduce the invariants $F_x(K)$ and $O_x(K)$, which play a central role in the paper.
Let $K$ be a compact space. We denote by $Q(K)$ the set all Hausdorff quotient spaces of $K$, that is the set all Hausdorff compact spaces of the form $K/E$ endowed with the quotient topology, for $E$ an equivalence relation on $K$. An element of $Q(K)$ can be represented either by the equivalence relation $E$ or by the quotient space $L = K/E$ together with the canonical projection $p_L : K \rightarrow L$.

On the set $Q(K)$ there is a natural order relation. In terms of equivalence relations $E \leq E'$ if and only if $E' \subset E$. Equivalently, in terms of the quotient spaces, $L \leq L'$ if and only if there is a continuous surjection $q : L' \rightarrow L$ such that $qp_{L'} = p_L$. The set $Q(K)$ endowed with this order relation is a complete semilattice, that is, every subset has a least upper bound or supremum: if $\mathcal{E}$ is a family of equivalence relations of $Q(K)$, its least upper bound is the relation given by $xE_0y$ if and only if $xEy$ for all $E \in \mathcal{E}$, in other words $E_0 = \sup \mathcal{E} = \bigcap \mathcal{E}$. It is easy to check that $E_0$ gives a Hausdorff quotient if each element of $\mathcal{E}$ does.

Let $Q_\omega(K) \subset Q(K)$ be the family of all quotients of $K$ which have countable weight. Notice that $\sup A \in Q_\omega(K)$ for every countable subset $A \subset Q_\omega(K)$ and also that $\sup Q_\omega(K) = K$. A family $S \subset Q_\omega(K)$ is called cofinal if for every $L \in Q_\omega(K)$ there exists $L' \in S$ such that $L \leq L'$. The family $S$ is called a $\sigma$-semilattice if for every countable subset $A \subset S$, the least upper bound of $A$ belongs to $S$.

**Theorem 2.1** (A version of Shchepin’s spectral theorem). Let $K$ be a compact space of uncountable weight and let $S$ and $S'$ two cofinal $\sigma$-semilattices in $Q_\omega(K)$. Then $S \cap S'$ is also a cofinal $\sigma$-semilattice in $Q_\omega(K)$.

Proof: The point is in proving that $S \cap S'$ is cofinal. Let $L_0 \in Q_\omega(K)$ be arbitrary. Since $S$ is cofinal there exists $L_1 \in S$ with $L_0 \leq L_1$, similarly find $L_2 \in S'$ with $L_1 \leq L_2$, and continue by induction an increasing sequence with $L_{2n+1} \in S$, $L_{2n} \in S'$. Finally $L = \sup \{L_n : n < \omega\} \in S \cap S'$ since both sets are $\sigma$-semilattices. $\square$

It is not so obvious to check whether a given $\sigma$-semilattice is cofinal, so this theorem must be applied together with the following criterion:

**Lemma 2.2.** Let $K$ be a compact space of uncountable weight and $S$ a $\sigma$-semilattice in $Q_\omega(K)$. Then, $S$ is cofinal if and only if $\sup S = K$.

Proof: If $S$ is cofinal, then $\sup S = \sup Q_\omega(K) = K$. Conversely, suppose that $\sup S = K$. Consider the family $A$ of all continuous functions $f : K \rightarrow \mathbb{R}$ such that there exists $L \in S$ such that $f$ factors through $p_L : K \rightarrow L$. 
$K \to L$, that is, there exists $\hat{f} : L \to \mathbb{R}$ with $f = \hat{f}p_L$. As $\mathcal{S}$ is a $\sigma$-semilattice, $\mathcal{A}$ is a subalgebra of the algebra $C(K)$ of real-valued continuous functions on $K$. Clearly, constant functions belong to $\mathcal{A}$ and since $\text{sup}\mathcal{S} = K$, $\mathcal{A}$ separates the points of $K$. Hence, by the Stone-Weierstrass theorem every $f \in C(K)$ is the limit of a sequence of functions from $\mathcal{A}$. But indeed $\mathcal{A}$ is closed under limits of sequences, namely if $f_n$ factors through $L_n \in \mathcal{S}$, then $\lim f_n$ factors through $\text{sup}\{L_n : n < \omega\} \in \mathcal{S}$. We conclude that $\mathcal{A} = C(K)$. Now, if $p : K \to L$ is an arbitrary element of $\mathcal{Q}_\omega(K)$, then we can take an embedding $L \subset \mathbb{R}^\omega$ and consider the functions $e_n p : K \to \mathbb{R}$ obtained by composing with the coordinate functions $e_n : \mathbb{R}^\omega \to \mathbb{R}$. For every $n$ we know, since $\mathcal{A} = C(K)$, that there exists $L_n \in \mathcal{S}$ such that $e_n p$ factors through $L_n$. Finally, this implies that $p$ factors through $L_\infty = \text{sup}\{L_n : n < \omega\}$, so $L \leq L_\infty \in \mathcal{S}$. □

The importance of this machinery is that it allows to study a compact space of uncountable weight through the study of a cofinal $\sigma$-semilattice of metrizable quotients, and particularly through the natural projections between elements of the $\sigma$-semilattice. In this way, the study of compact spaces of uncountable weight is related to the study of continuous surjections between compact spaces of countable weight. The following language will be useful:

**Definition 2.3.** Let $K$ be a compact space of uncountable weight and let $\mathcal{P}$ be a property. We say that the $\sigma$-typical surjection of $K$ satisfies property $\mathcal{P}$ if there exists a cofinal $\sigma$-semilattice $\mathcal{S} \subset \mathcal{Q}_\omega(K)$ such that for every $L < L'$ elements of $\mathcal{S}$, the natural projection $p : L' \to L$ satisfies property $\mathcal{P}$.

The consequence of the spectral theorem is that the fact whether the $\sigma$-typical surjection of $K$ has a certain property can be checked on any given cofinal $\sigma$-semilattice, namely:

**Theorem 2.4.** Let $K$ be a compact space of uncountable weight, let $\mathcal{P}$ be a property, and let $\mathcal{S}$ be a fixed cofinal $\sigma$-semilattice in $\mathcal{Q}_\omega(K)$. Then, the $\sigma$-typical surjection of $K$ has property $\mathcal{P}$ if and only if there exists a cofinal $\sigma$-semilattice $\mathcal{S}' \subset \mathcal{S}$ such that for every $L < L'$ elements of $\mathcal{S}'$, the natural projection $p : L' \to L$ satisfies property $\mathcal{P}$.

The main kind of properties $\mathcal{P}$ that we shall be interested concern the fiber orders of the surjections and the order relation that we defined on them. Given a point $x \in K$, we can study properties of the point $x$ by looking to fiber order of $p_L(x)$ in the $\sigma$-typical $p : L' \to L$. It may be
a useful language to call $F_x(K)$ to this $\sigma$-typical fiber, which we certainly cannot define as a concrete set, but rather as an abstract object of which we can predicate some properties.

**Definition 2.5.** Let $K$ be a compact space of uncountable weight and $x \in K$ and let $\mathcal{P}$ be a property. We say that $F_x(K)$ has property $\mathcal{P}$ if $F_{pL}(x)(p)$ has property $\mathcal{P}$ for the $\sigma$-typical surjection $p : L' \to L$.

In a similar way we shall talk about $O_x(K)$. It is worth to notice that a point $x$ is a $G_\delta$-point of $K$ if and only if $|F_x(K)| = 1$. In other words, the information given by $F_x(K)$ is trivial only when $x$ is a $G_\delta$ point of $K$. Namely, if $x$ is a $G_\delta$-point of $K$ then there is a continuous function $f : K \to [0, 1]$ such that $x = f^{-1}(0)$. Then, $f$ can be viewed as an element $L_0 \in Q_\omega(K)$ and we find that $|F_{pL}(x)(p)| = 1$ for all $L' > L > L_0$, $p : L' \to L$. Conversely, if $|F_x(K)| = 1$ then we can find $L \in Q_\omega(K)$ such that $x = p^{-1}_L(pL(x))$. Another elementary example is the compact $K = L^\kappa$ where $L$ is a metrizable compact. In this case, one can see as an exercise that for every $x \in K$, $F_x(K)$ is homeomorphic to $L^\omega$ and $|O_x(K)| = 1$.

### 3. Decomposition into products

In this section, apart from providing some basic facts that will be needed in the sequel, we prove two results, Theorem 3.7 and Theorem 3.9, which establish some sufficient conditions in terms of fiber orders in order that a compact $K$ cannot be decomposed as product of other spaces in a certain way. In further sections, when computing the fiber orders of specific spaces, we will find that several compact spaces satisfy the assumptions of these results.

**Definition 3.1.** Let $P$ be a set and $\leq$ be a binary relation on $P$. We say that $(P, \leq)$ is a preordered set if

1. $t \leq t$ for every $t \in P$,
2. If $t \leq s$ and $s \leq u$, then $t \leq u$, for every $t, s, u \in P$.

If, moreover, we have that for every $t, s \in P$, if $t \leq s$ and $s \leq t$ then $t = s$, then we say that $(P, \leq)$ is an ordered set. An ordered set $(O, \leq)$ is said to be linearly ordered if for every $t, s \in O$, either $t \leq s$ or $s \leq t$.

There is a canonical way of constructing an ordered set from a given preordered set $(P, \leq)$, namely we consider the equivalence relation on $P$ given by $t \sim s$ iff $t \leq s$ and $s \leq t$, and then the quotient set $P/ \sim$ is an ordered set when endowed with the relation induced from $P$. We call this
the ordered set associated to \( P \). When we write \( p < q \) in a preordered set, it means that \( p \leq q \) but \( q \not\leq p \).

An isomorphism between the preordered sets \( P \) and \( Q \) is a bijection \( f : P \to Q \) such that \( f(t) \leq f(s) \) if and only if \( t \leq s \).

**Definition 3.2.** Let \( \{Q_i : i \in I\} \) be a family of preordered sets. The product of this family is the preordered set whose underlying set is the cartesian product \( \prod_{i \in I} Q_i \) endowed by the preorder relation given by: \( (t_i)_{i \in I} \leq (s_i)_{i \in I} \) if and only if \( t_i \leq s_i \) for every \( i \in I \).

The product of an empty family of preordered sets is considered to be a singleton, with its only possible preordered structure. The product operation of preordered sets arises naturally in the context of fiber orders at least in two different situations, related to probability measures (cf. Theorem 6.2) and to products of compact spaces:

**Proposition 3.3.** Let \( \{f_i : K_i \to L_i : i \in I\} \) be a family of continuous surjections, let \( f : \prod_{i \in I} K_i \to \prod_{i \in I} L_i \) be its product and let \( x = (x_i)_{i \in I} \) be a point of \( \prod_{i \in I} L_i \). Then, the natural map \( F_x(f) : \prod_{i \in I} F_{x_i}(f_i) \) is an order-isomorphism. In particular, \( F_x(f) \cong \prod_{i \in I} F_{x_i}(f_i) \) and \( O_x(f) \cong \prod_{i \in I} O_{x_i}(f_i) \).

The proof of this statement is straightforward. If we have \( K \) a finite or countable product of compact spaces, then a cofinal \( \sigma \)-semilattice in \( Q_\omega(K) \) is formed by all quotients of countable weight of \( K \) which can be expressed as the product of a quotient of every factor. In this way, we see that the fibers of the \( \sigma \)-typical surjection of the product are the product of the fibers of the \( \sigma \)-typical surjection of every factor. We are thus allowed to write expressions like for instance \( F_{(x,y)}(K \times L) \cong F_x(K) \times F_y(L) \) or \( F_{(x_1,x_2,\ldots)}(\prod_{n<\omega} K_n) \cong \prod_{n<\omega} F_{x_n}(K_n) \).

**Definition 3.4.** An ordered set \( O \) is called **irreducible** if whenever \( O \) is isomorphic to a product \( Q \times R \) we have that either \( Q \) or \( R \) is a singleton.

An elementary example of an irreducible ordered set is a linearly ordered set. An ordered set \( O \) is called **connected** if whenever it is expressed as the disjoint union of two nonempty subsets \( O = A \cup B \), there exists \( a \in A \) and \( b \in B \) such that either \( a \leq b \) or \( b \leq a \). All the ordered sets that appear in this note happen to be connected since indeed they have a minimum. The following Theorem 3.5 and its Corollary 3.6 are due to Hashimoto [12] and assert that any two decompositions of a connected ordered set as product
have a common refinement, and consequently, a decomposi-
tion of a connected ordered set as a product of irreducible ordered sets is unique. Among other applications, this is a useful criterion to decide immediately that two
given ordered sets are not isomorphic.

**Theorem 3.5.** Let $O$ be a connected ordered set, $\{O_i : i \in I\}$ and $\{Q_j : j \in J\}$ two families of ordered sets such that $O \cong \prod_{i \in I} O_i \cong \prod_{j \in J} Q_j$. Then, there is a further family $\{Z_{ij} : (i, j) \in I \times J\}$ such that $O_i \cong \prod_{j \in J} Z_{ij}$ for every $i \in I$, and $Q_j \cong \prod_{i \in I} Z_{ij}$ for every $j \in J$.

**Corollary 3.6.** Let $O$ be a connected ordered set, $\{O_i : i \in I\}$ a family of irreducible ordered sets and $\{Q_j : j \in J\}$ a family of arbitrary ordered sets. Assume that $O \cong \prod_{i \in I} O_i \cong \prod_{j \in J} Q_j$. Then, there is a partition $I = \bigcup_{j \in J} F_j$ of the set $I$ such that $Q_j \cong \prod_{i \in I} O_i$ for every $j \in J$.

In the sequel we shall make use of the following terminology: Two continuous maps $f : U \rightarrow V$ and $f' : U' \rightarrow V'$ are said to be homeomorphic if there exists homeomorphisms $u : U \rightarrow U'$ and $v : V \rightarrow V'$ such that $vf = f'u$.

**Theorem 3.7.** Let $K$ be a compact space of uncountable weight and let $O$ be a connected irreducible ordered set. Assume that there is $x \in K$ such that $O_x(K) \cong O$ and that $O_y(K) \not\cong O^k$ for each $y \in K$ and each $k > 1$. If $K^n \cong L^m$ for some natural numbers $n$, $m$ and some space $L$, then $n$ is a multiple of $m$.

**Remark 3.8.** Note that the assertion $O_y(K) \not\cong O^k$ is not the negation of $O_y(K) \cong O^k$. It rather means that for the $\sigma$-typical surjection $p$ we have $O_y(p) \not\cong O^k$ (another remark about notation: we write $O_y(p) = O_y(p)$, where $y'$ is the projection of $y$ on the range of $p$).

Proof of Theorem 3.7. Along this proof, it is important to have in mind that if $S$ is a cofinal $\sigma$-semilattice in $Q_\omega(X)$, then $S^k = \{p_Z^k : X^k \rightarrow Z^k : Z \in S\}$ is a cofinal $\sigma$-semilattice in $Q_\omega(X^k)$, $k \leq \omega$. Assume that $n$ is not a multiple of $m$ and that $K^n \cong L^m$. Choose $x \in K$ with $O_x(K) \cong O$. By Proposition 3.3 we get $O_{(x, \ldots, x)}(K^n) \cong O^n$. Let $w = (w_1, \ldots, w_m) \in L^m$ be the point corresponding to $(x, \ldots, x)$ by the homeomorphism. Then, of course, $O_w(L^m) \cong O^n$. Further, by Proposition 3.3 we have that for the $\sigma$-typical surjection $q$ of $L$,

$$O^n \cong O_w^m(q) \cong \prod_{i=1}^m O_{w_i}(q).$$
Using Corollary 3.6 and the fact that \( n \) is not a multiple of \( m \), we get that

for the \( \sigma \)-typical surjection \( q \) of \( L \) there is \( k \in \{1, \ldots, m\} \) and \( n/m < s \leq n \)

such that \( \mathcal{O}_{w_k}(q) \cong O^s \). It follows that there are \( k \in \{1, \ldots, m\} \) and \( n/m < s \leq n \)

such that in each cofinal \( \sigma \)-semilattice in \( L \) there is some surjection \( q \) with \( \mathcal{O}_{w_k}(q) \cong O^s \) (this follows from Theorem 2.1) if not, for each \( k, s \)

there would be the corresponding cofinal \( \sigma \)-semilattice \( S_{k,s} \), and then \( \cap S_{k,s} \)

gives a contradiction). Set \( \tilde{w} = (w_k, \ldots, w_k) \) and let \( y = (y_1, \ldots, y_n) \in \)

\( K^n \) correspond by the homeomorphism to \( \tilde{w} \). By our assumptions there

is a cofinal \( \sigma \)-semilattice \( T \subset Q_\omega(K) \) such that \( \mathcal{O}_{p(y_i)}(p) \not\cong O^j \) for any

surjection \( p \) inside \( T \), \( i = 1, \ldots, n \) and \( j > 1 \). Consider the cofinal \( \sigma \)-lattice

\( U = \{ Z \in Q_\omega(L) : Z^m \in T^n \} \). By the previous argument, there exists a surjection \( q \) inside \( U \) such that \( \mathcal{O}_{w_k}(q) \cong O^s \). The surjection \( q^m \) corresponds to a surjection \( p^n \) inside \( T^n \) for which we have that:

\[
\prod_{i=1}^{n} \mathcal{O}_{y}(p) \cong \mathcal{O}_{y}(p^n) \cong O^{sm}
\]

As \( sm > n \), by Corollary 3.6 we get that \( \mathcal{O}_{p(y_i)}(p) \cong O^j \) for some \( i \in \{1, \ldots, n\} \) and some \( j > 1 \), a contradiction. \( \square \)

**Theorem 3.9.** Let \( K \) be a compact space of uncountable weight, \( n \) a natural number, and \( O \) a connected irreducible ordered set with \( |O| > 1 \). Assume that the \( \sigma \)-typical surjection of \( K \), \( p : X \rightarrow Y \) has the following properties:

1. For every \( y \in Y \), \( \mathcal{O}_{y}(p) \) is a connected ordered set.
2. There is no point \( y \in Y \) with \( \mathcal{O}_{y}(p) \cong O \times P \) with \( |P| > 1 \).
3. There exists a point \( x \in Y \) such that \( \mathcal{O}_{x}(p) \cong O \).
4. For any point \( x \in Y \) with \( \mathcal{O}_{x}(p) \cong O \) the preordered set \( F_x(p) \) has an equivalence class which is a singleton.

Then, if \( L_1, \ldots, L_m \) are compact spaces of uncountable weight such that \( K^n \cong L_1 \times \cdots \times L_m \), then \( m \leq n \).

Proof: Let \( S \) be a cofinal \( \sigma \)-semilattice in \( Q_\omega(K) \) in which all the natural projections satisfy properties (1) to (4). Let \( S^n \) be the cofinal \( \sigma \)-semilattice in \( Q_\omega(K^n) \), like defined in the proof of Theorem 3.7. Consider \( T \) the cofinal \( \sigma \)-semilattice in \( Q_\omega(L_1 \times \cdots \times L_m) \) whose elements are the quotients of \( L_1 \times \cdots \times L_m \) which are products of quotients of each coordinate, that is, of the form

\[ q_1 \times \cdots \times q_m : L_1 \times \cdots \times L_m \rightarrow Z_1 \times \cdots \times Z_m, \]

for \( q_i : L_i \rightarrow Z_i \) element of \( Q_\omega(L_i) \). Since \( K \cong L_1 \times \cdots \times L_m \), the \( \sigma \)-semilattices \( S^n \) and \( T \) can be viewed as cofinal \( \sigma \)-semilattices of metrizable quotients over the same compact, so by Theorem 2.1 they intersect
in a further cofinal \(\sigma\)-semilattice, and in particular, we can find a natural projection inside \(S^n\), \(p^n : X^n \rightarrow Y^n\) and a natural projection inside \(T\), \(q = q_1 \times \cdots \times q_m : Z_1 \times \cdots \times Z_m \rightarrow W_1 \times \cdots \times W_m\) which are homeomorphic. Of course, we have enough freedom to choose it in such a way that \(W_i \neq Z_i\) for every \(i\). Consider a point \(w = (w_1, \ldots, w_m)\) in \(W_1 \times \cdots \times W_m\) which corresponds by the homeomorphism to a point \((x, x, \ldots) \in Y^n\) with \(O_x(p) \cong O\). Thus, \(\prod_{r=1}^n O_{w_r}(q_r) \cong O^n\). After reordering if necessary, by Corollary 3.6 we know that \(|O_{w_r}(q_r)| = 1\) for \(r > n\).

Claim A: \(|O_v(q_r)| = 1\) for every \(v \in W_r\) and every \(r > n\).

Proof of the claim: Suppose for instance that there exists \(v \in W_{n+1}\) with \(|O_v(q_{n+1})| > 1\). Let \(w' = (w_1, \ldots, w_n, v, w_{n+2}, \ldots)\) and \(y = (y_1, \ldots, y_n) \in Y^n\) which corresponds to \(w'\) by the homeomorphism. Then \(\prod_{r=1}^n O_{w_r}(p) \cong O_{w'}(q) \cong O^n \times O_{v}(q_{n+1})\). Using Theorem 3.5 and the fact that \(O\) is irreducible, we conclude that there must exist \(i\) such that \(O_{y_i}(p)\) is isomorphic to something of the form \(O \times P\) with \(|P| > 1\), which is a contradiction.

Claim B: \(|F_v(q_r)| = 1\) for every \(v \in W_r\) and every \(r > n\).

Proof of the claim: Suppose for instance that there exists \(v \in W_{n+1}\) with \(|F_v(q_{n+1})| > 1\), let \(w' = (w_1, \ldots, w_n, v, w_{n+2}, \ldots)\) and \(y = (y_1, \ldots, y_n) \in Y^n\) which corresponds to \(w'\) by the homeomorphism. We know by Claim A that \(|O_v(q_{n+1})| = 1\), which means that \(F_v(q_{n+1})\) consists of one equivalence class which is not a singleton. By Proposition 3.3 this translates into the fact that \(F_{w'}(q) \cong \prod_{i=1}^n F_{y_i}(p)\) has no equivalence class which is a singleton, and this further implies that for some \(i\), \(F_{y_i}(p)\) has no equivalence class which is a singleton. Moreover, \(\prod_{i=1}^n O_{y_i}(p) \cong O_{w'}(q) \cong O^n\), so by Corollary 3.6 and our hypothesis (2), \(O_{y_i}(p) \cong O\) for every \(i\). In this way, we found a contradiction with our hypothesis (4).

Finally, notice that \(|F_v(q_r)| = 1\) simply means that \(q_r\) is one-to-one for \(r > n\), that is \(Z_r = W_r\). Since we supposed that \(Z_r \neq W_r\) for all \(r\), we conclude that \(m \leq n\). \(\Box\)

Remark 3.10. Note that the previous theorem cannot be formulated just using \(O_x(K)\) and \(F_x(K)\) (while Theorem 3.7 is formulated in this way). Indeed, if \(L\) is first countable, then \(F_x(L)\) is singleton for each \(x \in L\). Therefore \(K\) and \(K \times L\) cannot be distinguished using just the objects \(O_x(K)\) and \(F_x(K)\) and there are first countable compact spaces of uncountable weight.
4. The ball of the Hilbert space, $P(A(\kappa))$ and $M(A(\kappa))$

In this section, we shall compute $O_\kappa x$ for the ball of the Hilbert space and its finite powers. In particular, we shall prove Theorem 1.2 and Theorems 1.6 and 1.7 for the case $B(\kappa)$. Recall that

$$B(\kappa) = \left\{ (x_i)_{i<\kappa} \in \mathbb{R}^\kappa : \sum_{i<\kappa} |x_i|^2 \leq 1 \right\}$$

endowed with the weak topology of the Hilbert space $\ell_2(\kappa)$. The weak topology clearly coincides with the pointwise one. We can identify this space by the obvious homeomorphism with

$$B(\kappa) \approx \left\{ (x_i)_{i<\kappa} \in \mathbb{R}^\kappa : \sum_{i<\kappa} |x_i| \leq 1 \right\} \subset \mathbb{R}^\kappa$$

with the pointwise topology. This compact is also homeomorphic to the ball of $c_0(\kappa)$ for $1 < p < \infty$ in the weak topology and to the dual ball of $c_0(\kappa)$ in the weak* topology. It is to be noticed that all the results proved in this section hold true (with essentially identical proof) if we substitute the space $B(\kappa)$ by $P(A(\kappa)) \approx \left\{ (x_i)_{i<\kappa} \in [0,1]^\kappa : \sum_{i<\kappa} x_i \leq 1 \right\}$. The fiber orders of $P(A(\kappa))$ will be computed again as one particular case of our methods in spaces of probability measures. We shall also notice that $P(A(\kappa))$ is not homeomorphic to the dual unit ball of the Banach space of continuous functions $C(A(\kappa))$ in its weak* topology.

For a subset $M$ of $\kappa$, we consider $B(M) = \left\{ (x_i)_{i \in M} \in \mathbb{R}^M : \sum_{i \in M} |x_i| \leq 1 \right\}$, and for $M \subset N$ we have the natural projection $p_{MN} : B(M) \longrightarrow B(N)$ given by $p((x_i)_{i \in M}) = (x_i)_{i \in N}$. Thus every $B(M)$ can be seen as a quotient of $B(\kappa)$ through the projection $p_{KM} : B(\kappa) \longrightarrow B(M)$, and all quotients of this type for $M$ countably infinite subset of $\kappa$ constitute a cofinal $\sigma$-semilattice of $Q_\kappa(B(\kappa))$, as it easily follows from Lemma 2.2. Hence, the $\sigma$-typical surjection of $B(\kappa)$ is of the form $p_{MN} : B(M) \longrightarrow B(N)$ and its fiber orders are computed in the following way:

**Lemma 4.1.** Let $p_{MN} : B(M) \longrightarrow B(N)$ be as above, $x \in B(N)$ and $y^1, y^2 \in p_{MN}^{-1}(x)$. Then $y^1 \leq y^2$ if and only if $\sum_{i \in M \setminus N} |y^1_i| \leq \sum_{i \in M \setminus N} |y^2_i|$.\[\text{Proof:}\] Set $M^* = M \setminus N$. Let $y$ be any point of $p_{MN}^{-1}(x)$. A basic neighborhood of $y$ is of the form

$$U = \left\{ z \in B(M) : z_i \in W_i \text{ for } i \in F \right\}$$

where $F$ is a finite subset of $M$ and $W_i$ is an open real interval containing $y_i$, for every $i \in F$. Let $a_i = \inf \{|t| : t \in W_i\}$ be the distance of the interval
$W_i$ to 0. Then, the image of the above typical basic neighborhood $U$ under $p_{NM}$ is the following:

$$p_{MN}(U) = \{ z \in B(N) : z_i \in W_i \text{ for } i \in F \cap N \},$$

if $0 \in W_i$ for all $i \in F \cap M^*$;

$$p_{MN}(U) = \begin{cases} z \in B(N) : z_i \in W_i \text{ for } i \in F \cap N \\ \text{and } \sum_{i \in N} |z_i| < 1 - \sum_{i \in F \cap M^*} a_i \end{cases} \text{ otherwise.}$$

This means that the images of the basic neighborhoods of $y$ are the sets of the following form:

- If $y_i = 0$ for all $i \in M^*$, then the images of the basic neighborhoods of $y$ are the basic neighborhoods of $x$.
- Otherwise, the images of basic neighborhoods of $y$ are the sets of the form $V \cap \{ z : \sum_{i \in N} |z_i| < 1 - r \}$ where $V$ is a basic neighborhood of $x$ and $r$ is any real number such that $0 \leq r < \sum_{i \in M^*} |y_i|$.

From this description, it is already clear that if $\sum_{i \in M^*} |y_1^2| \leq \sum_{i \in M^*} |y_2^2|$ then $y_1^2 \leq y_2^2$. For the converse implication, it is enough to check that if $r < s < 1 - \sum_{i \in N} |x_i|$ there is no neighborhood $V$ of $x$ such that $V \cap \{ z : \sum_{i \in N} |z_i| < 1 - r \} \subseteq \{ z : \sum_{i \in N} |z_i| < 1 - s \}$. This follows from the fact that $N$ is infinite: Suppose $V = \{ z \in B(N) : z_i \in W_i \text{ for } i \in F \}$ where $F$ is some finite subset of $N$ and $W_i$ are intervals; take a number $1 - s < t < 1 - r$ and $n \in N \setminus F$; consider the element $y$ which agrees with $x$ on $F$, $y_n = t - \sum_{i \in F} |x_i|$, and $y_i$ is 0 in all other coordinates. Then $1 - s < t = \sum_{i \in N} |y_i| < 1 - r$ and $y \in V$, so $y \in V \cap \{ z : \sum_{i \in N} |z_i| < 1 - r \} \setminus \{ z : \sum_{i \in N} |z_i| < 1 - s \}$. □

It follows from Lemma 1.1 that $\mathcal{O}_x(p_{MN})$ is order isomorphic to an interval $[a, b]$ if $\sum_{i \in N} |x_i| < 1$, and $|\mathcal{O}_x(p_{MN})| = 1$ if $\sum_{i \in N} |x_i| = 1$. From this, it is also clear that for $x \in B(\kappa)$, $\mathcal{O}_x(B(\kappa)) \cong [0, 1]$ if $\sum_{i < \kappa} |x_i| < 1$, and $|\mathcal{O}_x(B(\kappa))| = 1$ if $\sum_{i < \kappa} |x_i| = 1$. Theorem 1.2 follows immediately now.

We notice that $B(\kappa)$ satisfies the hypotheses of both Theorem 3.7 and Theorem 3.9, taking $[0, 1]$ as irreducible ordered set. Every fiber of $p_{NM}$ has an equivalence class which is a singleton, namely the class of the minimum element, the one with $y_i = 0$ for all $i \in M^*$. This yields the proof of the case $B(\kappa)$ of Theorems 1.6 and 1.7.
We stated in the introduction that whenever two points \( x, y \in B(\kappa)^n \) have the same number of coordinates in the sphere then there is a homeomorphism \( f : B(\kappa)^n \rightarrow B(\kappa)^n \) with \( f(x) = y \). Let us indicate why. It is enough to consider the case \( n = 1 \). If \( \|x\| = \|y\| \) then we can find a linear isometry of the Hilbert space \( \ell_2(\kappa) \) onto itself sending \( x \) to \( y \). After this, it remains to find some automorphism of \( B(\kappa) \) sending some element of norm \( \lambda \) to some element of norm \( \mu \) for every \( \lambda, \mu \in [0, 1) \). View now again \( B(\kappa) = \{(x_i)_{i<\kappa} : \sum |x_i| \leq 1\} \). By the standard homeomorphism, the norm function is transformed into \( \|x\|^2 = \sum |x_i| \). Consider an increasing homeomorphism \( \phi : [-1, 1] \rightarrow [-1, 1] \) such that \( \phi(\lambda) = \mu \) and there exist the side derivatives \( \phi'_a(-1) = 1 = \phi'_i(1) \). Consider then \( f : B(\kappa) \rightarrow B(\kappa) \) given by \( f((x_i)_{i<\kappa}) = (y_i)_{i<\kappa} \) where \( y_0 = \phi(x_0) \) and \( y_i = \frac{1-|\phi(x_0)|}{1-|x_0|} x_i \) for \( i > 0 \).

Notice that \( f \) is a homeomorphism and \( f(\lambda, 0, 0, \ldots) = (\mu, 0, 0, \ldots) \).

Let \( M(K) \) denote the set of all Radon measures of variation at most one (in other words, the dual ball of the Banach space \( C(K) \)) endowed with the weak* topology. We know that \( Q_x(P(A(\kappa))) \) is either a singleton or order-isomorphic to \([0, 1]\) for \( x \in P(A(\kappa)) \). A cofinal \( \sigma \)-semilattice for \( Q_\omega(M(K)) \) is formed by the quotients of the form \( M(p) : M(K) \rightarrow M(L) \) where \( p : K \rightarrow L \) is an element of \( Q_\omega(K) \), and hence the \( \sigma \)-typical surjection of \( M(K) \) is of the form \( M(g) : M(X) \rightarrow M(Y) \), where \( g : X \rightarrow Y \) is the \( \sigma \)-typical surjection of \( K \). Hence, in order to prove that \( P(A(\kappa)) \not\cong M(A(\kappa)) \), it is enough to check the following:

**Proposition 4.2.** Let \( g : K \rightarrow L \) be continuous surjection and let \( f = M(g) : M(K) \rightarrow M(L) \) be the induced map between the spaces of Radon measures of variation at most 1. If there exists \( x \in L \) such that \( |Q_x(g)| > 1 \), then \( Q_0(f) \) is not linearly ordered.

Proof: Take \( y, z \in g^{-1}(x) \) such that \( y \not\leq z \), so that there is a neighborhood \( U \) of \( y \) such that \( g(U) \) does not contain the image of any neighborhood of \( z \), and moreover \( z \not\in \overline{U} \). There is a net \( (z_\alpha) \) in \( K \) that converges to \( z \) and with \( g(z_\alpha) \not\in g(U) \) for every \( \alpha \). Consider the measures \( \nu = \frac{1}{2}\delta_y - \frac{1}{2}\delta_z \) and \( \mu = -\frac{1}{2}\delta_y + \frac{1}{2}\delta_z \). We claim that these are two incomparable elements of \( f^{-1}(0) \). We prove that \( \nu \not\leq \mu \) (that \( \mu \not\leq \nu \) is done by analogy). We consider \( W = \{ \lambda \in M(K) : \lambda(U) > \frac{3}{8} \} \), which is a neighborhood of \( \nu \). We claim that \( f(W) \) does not contain the \( f \)-image of any neighborhood of \( \mu \). Notice that \( f(W) \) does not contain the \( f \)-image of any neighborhood of \( \mu \). In particular \( f(\mu) \not\in f(W) \) for each \( \alpha \). This witnesses that \( \nu \not\leq \mu \). \( \square \)
5. Computing images of neighborhoods in spaces of probability measures

In order to compute the order of the fiber of a certain point \( y \in K_2 \) in a surjection \( f : K_1 \to K_2 \), we have to know how to compute the images \( f(U) \) of basic neighborhoods \( U \) of points \( x \in f^{-1}(y) \). The surjections which appear in the cases that we are going to study now are of the form \( f = P(g) : P(K) \to P(L) \) were \( g : K \to L \) is a surjection between scattered compacta and \( P \) is the functor of probability measure spaces. In this case, a neighborhood basis of a measure \( \mu \in P(K) \) is formed by the sets of the following form:

\[
U = \{ \nu \in P(K) : \nu(U_i) > c_i, \ i = 1, \ldots, n \}
\]

where \( U_i \) are disjoint clopen subsets of \( K \) from a given basis of clopen sets, and \( c_i \) are any numbers with \( \mu(U_i) > c_i \). The following lemma provides a computation of the image \( f(U) \) of such a neighborhood and will be applied repeatedly in the future.

**Lemma 5.1.** Let \( g : K \to L \) be a surjection between compact spaces, let \( f = P(g) : P(K) \to P(L) \), \( U_1, \ldots, U_n \) be disjoint closed subsets of \( K \), \( c_1, \ldots, c_n \geq 0 \), and

\[
U = \{ \nu \in P(K) : \nu(U_i) > c_i, \ i = 1, \ldots, n \}.
\]

Then

\[
f(U) = \left\{ \lambda \in P(L) : \lambda \left( g \left( \bigcup_{i \in A} U_i \right) \right) > \sum_{i \in A} c_i \text{ for } A \subset \{1, \ldots, n\}, A \neq \emptyset \right\}.
\]

The fact that \( f(U) \) is included in the righthand side expression is trivial. The other inclusion is included in the righthand side expression is trivial.

The proof of Lemma 5.2. We consider numbers \( c_1, \ldots, c_n \geq 0 \) and \( \alpha_A \geq 0 \) for \( A \subset \{1, \ldots, n\}, A \neq \emptyset \) such that for every nonempty \( A \subset \{1, \ldots, n\} \) we have that

\[
\sum_{B \cap A \neq \emptyset} \alpha_B > \sum_{i \in A} c_i.
\]

Then for every nonempty \( A \subset \{1, \ldots, n\} \) there exist numbers \( \beta_{A,i} \), \( i \in A \) such that

\[
\sum_{i \in A} \beta_{A,i} = \alpha_A \text{ and moreover, for every } i = 1, \ldots, n, \sum_{A \ni i} \beta_{A,i} > c_i.
\]

Let us make some comment about the history of the lemmas. We first had a long proof by induction of Lemma 5.2. After speaking about it with Richard Haydon, he indicated to us a more elegant and shorter proof using combinatorial optimization that we reproduce below. Later, after David Fremlin heard about it in the Marczewski Centennial Conference in Bedlewo, he wrote a note where he shows that actually Lemma 5.1...
holds under more general assumptions in \( K \)-analytic spaces (our original statement was only for scattered or metrizable compact sets).

We first notice how Lemma 5.1 follows from Lemma 5.2 in the cases when \( K \) is either scattered or metrizable, which is enough for the applications that we present (for the general case we refer to [9]). Given a measure \( \lambda \) in the righthand side of the conclusion of Lemma 5.1, we consider \( X_\Lambda = \bigcap_{i \in \Lambda} g(U_i) \setminus \bigcup_{i \notin \Lambda} g(U_i) \) and the numbers \( \alpha_\Lambda = \lambda(X_\Lambda) \) (note that each \( X_\Lambda \) is Borel as it is the difference of two closed sets), to which we can apply Lemma 5.2 and obtain the numbers \( \beta_{\Lambda,i} \). We define a measure \( \nu \in \mathcal{P}(K) \) with \( f(\nu) = \lambda \) in the following way.

Suppose first that \( K \) and \( L \) are scattered, so that all Radon measures on them are discrete, that is, determined by the measures of singletons (in this case, we do not even need that the sets \( U_i \) are closed). If \( y \in L \setminus \bigcup_{i=1}^n g(U_i) \) then we pick a point \( x_y \in g^{-1}(y) \) and declare \( \nu(\{x_y\}) = \lambda(\{y\}) \). If \( y \in X_\Lambda \) for some nonempty \( A \subset \{1, \ldots, n\} \) with \( \alpha_\Lambda > 0 \), then we can choose elements \( x_{y,i} \in f^{-1}(y) \cap U_i \) for every \( i \in A \) and then we declare \( \nu(\{x_{y,i}\}) = \frac{\beta_{\Lambda,i}}{\alpha_\Lambda} \lambda(\{y\}) \). In any other points \( \nu(\{x\}) = 0 \). This \( \nu \) is a probability measure on \( K \) with \( f(\nu) = \lambda \) and moreover \( \nu \in \mathcal{U} \) since \( \nu(U_i) = \sum_{A \ni i} \beta_{\Lambda,i} \).

In the other case, suppose that \( K \) and \( L \) are metrizable. In this case, Radon and Borel measures coincide. For each \( A \subset \{1, \ldots, n\} \) and each \( i \in A \), we set \( Y_{A,i} = g^{-1}(X_A) \cap U_i \) which is a nonempty Borel set. We also define \( X_\emptyset = L \setminus \bigcup_{i=1}^n g(U_i) \) and \( Y_\emptyset = g^{-1}(X_\emptyset) \). By the Jankov-Von Neumann Uniformization Theorem [15, Theorem 18.1], there exists a measurable selection \( s_{A,i} : X_A \rightarrow Y_{A,i} \) for the the inverse of \( g|_{Y_{A,i}} \) and also a measurable selection \( s_\emptyset : X_\emptyset \rightarrow Y_\emptyset \) of the inverse of \( g|_{Y_\emptyset} \). Consider

\[
\nu = s_\emptyset(\lambda|_{X_\emptyset}) + \sum_{i \in A} \frac{\beta_{A,i}}{\alpha_\Lambda} \cdot s_{A,i}(\lambda|_{X_{A,i}}).
\]

Then \( f(\nu) = \lambda \), and \( \nu(U_i) = \sum_{A \ni i} \beta_{A,i} \), so \( \nu \in \mathcal{U} \).

For the proof of Lemma 5.2 we shall use the so called max-flow min-cut theorem, Theorem 5.3 below, from combinatorial optimization. This result is originally due to Ford and Fulkerson [11] and Dantzig and Fulkerson [10], and can be found in the book [18, Theorem 10.3]. We have to recall some concepts from this area. A directed graph (digraph for short) is a couple \( G = (V, A) \) where \( V \) is a finite set whose elements are called vertices, and \( A \subset V \times V \) is a set whose elements are called arcs. An \( s - t \)-flow is a function \( f : A \rightarrow (0, \infty) \) which satifies the flow conservation law at all points except \( s \) and \( t \):
\[
\sum_{(x,u) \in A} f(x,u) = \sum_{(u,x) \in A} f(u,x) \quad \text{for every } u \in V \setminus \{s,t\}.
\]

In words, the flow entering \(u\) equals the flow leaving \(u\). The value of the flow \(f\) is the net amount of flow leaving \(s\), which happens to be equal to the net amount of flow entering \(t\),

\[
\text{value}(f) = \sum_{(s,x) \in A} f(s,x) - \sum_{(x,s) \in A} f(x,s) = \sum_{(x,t) \in A} f(x,t) - \sum_{(t,x) \in A} f(t,x).
\]

Let us consider a function \(c : A \rightarrow (0, +\infty)\) that we call a capacity function. A flow \(f\) is said to be under \(c\) if \(f(u,v) \leq c(u,v)\) for every \((u,v) \in A\). Given a set \(B \subset A\), the capacity of \(B\) is \(c(B) = \sum_{(u,v) \in B} c(u,v)\).

For a subset \(U \subset V\), we denote by \(\delta(U)\) the set of all arcs which leave \(U\) and enter \(V \setminus U\), that is,

\[
\delta(U) = \{(u,v) \in A : u \in U, v \not\in U\}.
\]

For \(s, t \in V\), an \(s - t\) cut is a set of arcs of the form \(\delta(U)\), where \(U \subset V\) with \(s \in U\) and \(t \not\in U\).

**Theorem 5.3** (max-flow min-cut theorem). Let \(G = (V, A)\) be a digraph, \(t, s \in V\) and \(c : A \rightarrow \mathbb{R}_+\) a capacity function. Then the maximum value of an \(s - t\) flow under \(c\) equals the minimum capacity of an \(s - t\)-cut,

\[
\max \{\text{value}(f) : f \leq c \text{ is an } s-t \text{ flow}\} = \min \{c(\delta(U)) : U \subset V, s \in U, t \not\in U\}.
\]

Proof of Lemma 5.2. We shall denote by \(P_n\) the family of all nonempty subsets of \(\{1, \ldots, n\}\). First we consider numbers \(c'_i > c_i\) for every \(i \leq n\) such that the inequalities

\[
\sum_{B \cap A \neq \emptyset} \alpha_B > \sum_{i \in A} c'_i
\]

still hold. We consider a digraph \(G = (V, A)\) where the set of vertices is

\[
V = \{s\} \cup \{p_i : 1 \leq i \leq n\} \cup \{q_A : A \in P_n\} \cup \{t\},
\]

and the set of arcs is

\[
A = \{(s, p_i) : 1 \leq i \leq n\} \cup \{(p_i, q_A) : i \in A\} \cup \{(q_A, t) : A \in P_n\}.
\]

Let \(M \in (0, +\infty)\) be such that \(M > \sum_{i=1}^n c'_i\). We define a capacity function \(c : A \rightarrow \mathbb{R}_+\) as

- \(c(s, p_i) = c'_i\).
- \(c(p_i, q_A) = M\).
- \(c(p_i, q_A) = \alpha_A\).

Claim A: The minimal capacity of an $s-t$ cut in $G$ equals $\sum_{i=1}^{n} c'_i$.

Proof of Claim A: If $U = \{s\}$, then $c(\delta(U)) = \sum_{i=1}^{n} c'_i$. We suppose that $\delta(U)$ is an arbitrary $s-t$ cut and we show that its capacity is larger than $\sum_{i=1}^{n} c'_i$. Let $A = \{i \leq n : p_i \in U\}$. If there exists $B \in P_n$ such that $q_B \notin U$ and $A \cap B \neq \emptyset$, then there exists $(p_i, q_B) \in \delta(U)$ and in particular $c(\delta(U)) \geq c(p_i, q_B) = M > \sum_{i=1}^{n} c'_i$. Hence, we can suppose that $q_B \in U$ whenever $A \cap B \neq \emptyset$, therefore $(q_B, t) \in \delta(U)$ whenever $A \cap B \neq \emptyset$, and

$$c(\delta(U)) \geq \sum_{i \notin A} c(s, p_i) + \sum_{B \cap A \neq \emptyset} c(q_B, t) = \sum_{i \notin A} c'_i + \sum_{B \cap A \neq \emptyset} \alpha_B \geq \sum_{i \notin A} c'_i + \sum_{i \in A} c'_i = \sum_{i=1}^{n} c'_i.$$

By Claim A and Theorem 5.3 there exists an $r-s$ flow $f \leq c$ and value equal to $\sum_{i=1}^{n} c'_i$. Notice that $f(s, p_i) \leq c'_i$ but $\sum_{i=1}^{n} f(s, p_i) = \text{value}(f) = \sum_{i=1}^{n} c'_i$, hence $f(s, p_i) = c'_i$.

By the flow conservation law at the vertex $q_A$, for every $A \in P_n$ we have that

$$\sum_{i \in A} f(p_i, q_A) = f(q_A, t) \leq c(q_A, t) = \alpha_A,$$

therefore we can choose numbers $\beta_{A,i}$ for $i \in A$ such that $\beta_{A,i} \geq f(p_i, q_A)$ and $\sum_{i \in A} \beta_{A,i} = \alpha_A$. We claim that these numbers have the desired property. To check this, we use again the flow conservation law now at a vertex $p_i$,

$$c_i < c'_i = f(s, p_i) = \sum_{A \ni i} f(p_i, q_A) \leq \sum_{A \ni i} \beta_{A,i}. \quad \Box$$

6. Fiber orders of the probability measures on a scattered compact

As we already mentioned, it is a standard fact that if $K$ is a totally disconnected compact and $B$ is a basis for the topology of $K$ consisting of clopen sets, then a basis for the topology of $P(K)$ consists of the sets of the form $\{\mu \in P(K) : \mu(U_i) > c_i : i = 1, \ldots, n\}$, where the $c_i$’s are positive numbers and the $U_i$’s are disjoint basic clopen sets. When $K$ is scattered, all measures from $P(K)$ are discrete, and this allows to find a finer neighborhood basis which will be quite useful for us. To avoid heavy notation, we write $\mu(x)$ instead of $\mu(\{x\})$ to denote the measure of a singleton.

Lemma 6.1. Let $K$ be a scattered compact space, $\mu \in P(K)$ and let $B$ be a basis of the topology of $K$ consisting of clopen sets. A neighborhood basis of $\mu$ consists of the sets of the form $\{\nu : \nu(U_i) > c_i \text{ for } i = 1, \ldots, n\}$, where $U_1, \ldots, U_n$ are pairwise disjoint basic clopen neighborhoods of points $x_1, \ldots, x_n$ of $K$ and $c_1, \ldots, c_n$ are positive numbers with $\mu(x_i) > c_i$. 
Proof: Consider a neighborhood of $\mu$ of the form $V = \{\nu \in P(K) : \nu(V_j) > d_j, j = 1, \ldots, n\}$ for some disjoint basic clopen neighborhoods $V_j$ with $\mu(V_j) > d_j$. Since $\mu$ is discrete, for every $j$ we can find a finite family of points $\{x^i_j : i \in F_j\}$ such that $\sum \{\mu(x^i_j) : i \in F_j\} > d_j$. Consider numbers $d^j_i < \mu(x^i_j)$ such that $\sum_{i \in F_j} d^j_i > d_j$, and disjoint basic clopen sets $x^i_j \in U^j_i \subset V_j$, then

$$\mu \in \{\nu \in P(K) : \nu(U^j_i) > d^j_i, j = 1, \ldots, n, i \in F_j\} \subset V \quad \square$$

For the rest of the section, we fix $g : K \to L$ to be a surjection between scattered compact spaces and $f = P(g) : P(K) \to P(L)$ the induced map between the spaces of probability measures. Note that the norm we use below is the $\ell_1$-norm, i.e.

$$\|\mu - \nu\| = \sum_{k \in K} |\mu(k) - \nu(k)| \text{ for } \mu, \nu \in P(K).$$

**Theorem 6.2.** Let $\mu = \sum_{i \in I} r_i \delta_{x_i}$ be a probability measure on $L$, where $I = \mathbb{N}$ or $I = \{1, \ldots, N\}$ for some $N \in \mathbb{N}$, $x_i, i \in I$ are pairwise distinct points in $L$ and $r_i > 0$ for all $i \in I$. Then, the natural bijection $\prod_{i \in I} f^{-1}(\delta_{x_i}) \to f^{-1}(\mu)$ given by $(\nu_i)_{i \in I} \mapsto \sum_{i \in I} r_i \nu_i$ is an order-isomorphism. In particular $\mathbb{F}_{\mu}(f) \cong \prod_{i \in I} \mathbb{F}_{\delta_{x_i}}(f)$ and $\mathcal{O}_{\mu}(f) \cong \prod_{i \in I} \mathcal{O}_{\delta_{x_i}}(f)$.

Proof: Consider the mapping $\Phi : P(K)^I \to P(K)$ defined by

$$\Phi((\nu_i)_{i \in I}) = \sum_{i \in I} r_i \nu_i.$$  

It is easy to check that $\Phi$ is a continuous surjection. Moreover, as it is affine, it maps $\prod_{i \in I} f^{-1}(\delta_{x_i})$ bijectively onto $f^{-1}(\mu)$. We will show that the restriction of $\Phi$ to $\prod_{i \in I} f^{-1}(\delta_{x_i})$ is an order-isomorphism.

First, suppose that $\sum_{i \in I} r_i \nu_i \leq \sum_{i \in I} r_i \nu_i'$ with $\nu_i, \nu_i' \in f^{-1}(\delta_{x_i})$ for $i \in I$ and we shall prove that $\nu_1 \leq \nu'_1$. We consider a typical neighborhood of $\nu_1$ of the form

$$U = \{\nu \in P(K) : \nu(U_j) > c_j \text{ for } j = 1, \ldots, n\}$$

where each $U_j$ is a clopen neighborhood of some $a_j$ satisfying $\nu_1(a_j) > c_j$ and $U_j$‘s are pairwise disjoint. We want to find a neighborhood $V$ of $\nu'_1$ such that $f(V) \subset f(U)$.

We consider a very small number $\varepsilon > 0$, namely such that

$$\varepsilon < r_1 \text{ and } (r_1 + \varepsilon)(c_j + 2\varepsilon) < r_1 \nu_1(a_j) \text{ for } j = 1, \ldots, n.$$
Choose \( k \in I \) such that
\[
\sum_{i > k} r_i < \frac{1}{4} r_1 \varepsilon (r_1 - \varepsilon)
\]
and \( H \) a clopen subset of \( L \) such that \( x_1 \in H \) but \( x_i \notin H \) for \( 2 \leq i \leq k \).

The following is a neighborhood of \( \sum_{i \in I} r_i \nu_i : \)
\[
U^0 = \{ \nu \in P(K) : \nu(U_j \cap g^{-1}(H)) > (r_1 + \varepsilon)(c_j + 2\varepsilon) \text{ for } j = 1, \ldots, n \}
\]

By our assumption, there exists \( V^0 \) a neighborhood of \( \sum_{i \in I} r_i \nu'_j \) such that \( f(V^0) \subset f(U^0) \). We take \( V^0 \) to be of the typical form
\[
V^0 = \{ \nu \in P(K) : \nu(V_j) > d_j \text{ for } j \in J \}
\]
where \( J \) is a finite set, \( V_j \) is a clopen neighborhood of some \( b_j \) satisfying
\[
\sum_{j \in I} r_j \nu'_j(b_j) > d_j
\]
and \( V_j \)'s are pairwise disjoint. We let
\[
J_i = \{ j \in J : g(b_j) = x_i \}.
\]

Without loss of generality we suppose that \( V_j \subset g^{-1}(H) \) for \( j \in J_1 \), and \( V_j \cap g^{-1}(H) = \emptyset \) for \( j \in J_2 \cup \cdots \cup J_k \). Notice that
\[
(6.1) \quad \sum_{j \in J_i} d_j \left\{ J_i : j \in \bigcup_{i > k} J_i \right\} < \frac{1}{4} r_1 \varepsilon (r_1 - \varepsilon).
\]

Consider now
\[
V = \{ \nu \in P(K) : \nu(V_j) > d_j/r_1 \text{ for } j \in J_1 \} \cap \{ \nu \in P(K) : \nu(K \setminus g^{-1}(H)) < \varepsilon/2 \}.
\]
This is a neighborhood of \( \nu'_j \) (notice that \( \nu'_j(K \setminus g^{-1}(H)) = 0 \) and \( \nu'_j(V_j) \geq \nu'_j(b_j) > d_j/r_1 \text{ for } j \in J_1 \)). We claim that \( f(V) \subset f(U) \).

So take \( \xi_1 \in V \). We can easily find \( \xi_2 \in V \) with \( \| \xi_2 - \xi_1 \| < \varepsilon \) such that \( \xi_2(K \setminus g^{-1}(H)) = 0 \). We pick a measure \( \lambda \in P(K) \) with \( \lambda(g^{-1}(H)) = 0 \) and \( \lambda(V_j) > d_j/(1-r_1) \text{ for } j \in J_2 \cup \cdots \cup J_k \). Then the measure \( \xi_3 = r_1 \xi_2 + (1-r_1)\lambda \) satisfies \( \xi_3(V_j) > d_j \text{ for } j \in J_1 \cup \cdots \cup J_k \). By \( (6.1) \) we may find \( \xi_4 \in V^0 \) such that
\[
\| \xi_4 - \xi_3 \| < \frac{1}{2} r_1 \varepsilon (r_1 - \varepsilon).
\]

Set
\[
r = \xi_4(g^{-1}(H)) \text{ and } \xi_5 = \frac{1}{r} \xi_4|_{g^{-1}(H)}.
\]

We have
\[
(6.2) \quad |r_1 - r| = |\xi_3(g^{-1}(H)) - \xi_4(g^{-1}(H))| < \frac{1}{2} r_1 \varepsilon (r_1 - \varepsilon) < \varepsilon
\]
and
\[
\|\xi_2 - \xi_5\| = \sum_{t \in g^{-1}(H)} |\xi_2(t) - \xi_5(t)| = \sum_{t \in g^{-1}(H)} \left| \frac{\xi_3(t)}{r_1} - \frac{\xi_4(t)}{r} \right| \\
= \sum_{t \in g^{-1}(H)} \frac{1}{r_1 r} |r\xi_3(t) - r_1\xi_4(t)| \leq \frac{1}{r_1 r} \|r\xi_3 - r_1\xi_4\| \\
\leq \frac{1}{r_1 r} \|(r_1 - r)\xi_4\| + \frac{1}{r_1 r} \|r\xi_3 - r\xi_4\| \\
\leq \frac{1}{r_1 r} |r_1 - r| + \frac{1}{r_1} \|\xi_3 - \xi_4\| \\
< \frac{\varepsilon(r_1 - \varepsilon)}{2r} + \frac{1}{2} \varepsilon(r_1 - \varepsilon) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\]
The first inequality on the last line follows from the first inequality of (6.2), for the second one we use the fact that \( r_1 - \varepsilon < r \). It follows that \( \|\xi_1 - \xi_5\| < 2\varepsilon \), and hence
\[(6.3) \quad \|f(\xi_1) - f(\xi_5)\| < 2\varepsilon \]
as well.

Now, \( \xi_4 \in V_0 \), hence \( f(\xi_4) \in f(V_0) \subset f(U_0) \). By the description of \( f(U_0) \) given by Lemma 5.1, the fact that all clopen subsets of \( K \) appearing in the definition of \( U_0 \) are contained in \( g^{-1}(H) \) implies that
\[(6.4) \quad f(\xi_5) \in f(\{\nu \in P(K) : \nu(U_j \cap g^{-1}(H)) > (r_1 + \varepsilon)(c_j + 2\varepsilon)/r \text{ for } j = 1, \ldots, n\}) \\
\subset f(\{\nu \in P(K) : \nu(U_j) > c_j + 2\varepsilon \text{ for } j = 1, \ldots, n\}).
\]

The inclusion above follows from (6.2) – note that \( r_1 + \varepsilon > r \). Finally, using (6.4) and (6.3) it easily follows from Lemma 5.1 that \( f(\xi_1) \in f(U) \) which completes the proof of the first implication.

We pass now to the converse implication. So we assume that \( \nu_i \leq \nu'_i \) for every \( i \), and we shall see that \( \sum_{i \in I} r_i \nu_i \leq \sum_{i \in I} r_i \nu'_i \).

Let \( U \) be a neighborhood of \( \sum r_i \nu_i \) in \( P(K) \). By the continuity of \( \Phi \) and the definition of the product topology there is some \( k \in I \) and neighborhoods \( U_i \) of \( \nu_i \) for \( i \leq k \) such that
\[(6.5) \quad \left\{ \sum_{i \in I} r_i \lambda_i : \lambda_i \in P(K) \text{ for } i \in I, \lambda_i \in U_i \text{ for } i \leq k \right\} \subset U.
\]
As \( \nu_i \leq \nu'_i \) for all \( i \in I \), there is, for each \( i \leq k \), a neighborhood \( V_i \) of \( \nu'_i \) such that \( f(V_i) \subset f(U_i) \).

Now we are going to specify the form of \( V_i \)'s. Let \( H_1, \ldots, H_k \) be pairwise disjoint clopen subsets of \( L \) containing \( x_1, \ldots, x_k \), respectively. Then we
can without loss of generality suppose that for each $i \leq k$ we have
\[ V_i = \{ \lambda \in P(K) : \lambda(V_{i}^{j}) > d_{i}^{j} \text{ for } j \in J_i \} \]
where $J_i$ is a finite set, $d_{i}^{j} > 0$ for $j \in J_i$ and $V_{i}^{j}$, $j \in J_i$, are pairwise disjoint clopen subsets of $g^{-1}(H_i)$. Set
\[ V = \{ \lambda \in P(K) : \lambda(V_{i}^{j}) > r_i d_{i}^{j} \text{ for } j \in J_i, i \leq k \}. \]
Then $V$ is clearly a neighborhood of $\sum_{i=1}^{k} r_i V_{i}^{j}$. We claim that $f(V) \subset f(U)$.

Let $\lambda \in V$ be arbitrary. Choose $\delta > 0$ such that
\[ (1 + \delta) \sum_{j \in J_i} d_{i}^{j} < 1, \quad i = 1, \ldots, k; \tag{6.6} \]
\[ \lambda(V_{i}^{j}) > (1 + \delta)r_i d_{i}^{j}, \quad j \in J_i, i = 1, \ldots, k. \tag{6.7} \]
Further, define the following measures:
\[ \sigma_i = \sum_{j \in J_i} (1 + \delta)d_{i}^{j} \frac{\lambda|_{V_{i}^{j}}}{\lambda(V_{i}^{j})}, \quad i = 1, \ldots, k, \]
\[ \tau = \lambda - \sum_{i=1}^{k} r_i \sigma_i. \]
All $\sigma_i$’s are clearly positive measures. Moreover, $\tau$ is positive, too, by (6.7) as
\[ \tau|_{V_{i}^{j}} = \lambda|_{V_{i}^{j}} \left( 1 - \frac{(1 + \delta)r_i d_{i}^{j}}{\lambda(V_{i}^{j})} \right) \]
for $j \in J_i, i = 1, \ldots, k$.

It follows from (6.6) that $\sigma_i(K) < 1$ for each $i = 1, \ldots, k$, and so $\tau(K) > 0$. For $i = 1, \ldots, k$ set
\[ \theta_i = \sigma_i + \frac{1 - \sigma_i(K)}{\tau(K)} \tau. \]
Then $\theta_i \in P(K)$. Moreover,
\[ \theta_i(V_{i}^{j}) \geq \sigma_i(V_{i}^{j}) = (1 + \delta)d_{i}^{j} > d_{i}^{j} \]
for $j \in J_i$, hence $\theta_i \in V_i$ for $i = 1, \ldots, k$. We claim that
\[ \lambda \in \left\{ \sum_{i=1}^{k} r_i \lambda_i : \lambda_i \in V_i \text{ for } i = 1, \ldots, k \right\}. \tag{6.8} \]
Indeed, we can take $\lambda_i = \theta_i$ for $i = 1, \ldots, k$. To see this, we have to check that
\[ \vartheta = \lambda - \sum_{i=1}^{k} r_i \theta_i \]
is a nonnegative measure. Namely,
\[
\vartheta = \lambda - \sum_{i=1}^{k} r_i \vartheta_i = \lambda - \sum_{i=1}^{k} r_i \left( \sigma_i + \frac{1 - \sigma_i(K)}{\tau(K)} \right) \\
= \lambda - \sum_{i=1}^{k} r_i \sigma_i - \sum_{i=1}^{k} r_i \frac{1 - \sigma_i(K)}{\tau(K)} \tau = \tau - \sum_{i=1}^{k} r_i \frac{1 - \sigma_i(K)}{\tau(K)} \tau \\
= \frac{\tau}{\tau(K)} \left( \tau(K) - \sum_{i=1}^{k} r_i + \sum_{i=1}^{k} r_i \sigma_i(K) \right),
\]
which is positive because \( \sum_{i=1}^{k} r_i \leq 1 = \lambda(K) = \tau(K) + \sum_{i=1}^{k} r_i \sigma_i(K) \). Thus, (6.8) is proved. Using (6.8) and (6.5) we get by Lemma 5.1 that \( f(\lambda) \in f(U) \) which completes the proof. □

Let \( g : K \rightarrow L \) be a continuous surjection, \( x \in L \) and \( y_1, \ldots, y_n \) elements of the fiber \( g^{-1}(x) \), we define \( \langle y_1, \ldots, y_n \rangle \) to be the set of all elements \( z \in g^{-1}(x) \) such that for every neighborhoods \( U_1, \ldots, U_n \) of \( y_1, \ldots, y_n \) respectively there exists a neighborhood \( V \) of \( z \) such that \( g(V) \subset g(U_1) \cup \cdots \cup g(U_n) \).

Notice some elementary properties, for instance \( \langle y \rangle = \{ z \in g^{-1}(x) : z \geq y \} \) and \( \langle Y \rangle \subset \langle Y' \rangle \) whenever \( Y \subset Y' \). The \( \langle \cdot \rangle \)-operation provides in general a finer structure on the fiber \( g^{-1}(x) \) than the one given by the order, and it is needed to determine the fiber order on spaces \( P(K) \) in terms of the fibers of \( K \). To avoid heavy notation, for a measure \( \nu \), we often write \( \nu(\langle \cdot \rangle) \) and \( \nu(\{\cdot\}) \) instead of \( \nu(\langle \cdot \rangle) \) and \( \nu(\{\cdot\}) \).

**Theorem 6.3.** Let \( \nu, \nu' \) be elements of \( f^{-1}(\delta_x) \). Then \( \nu \leq \nu' \) if and only if for every elements \( y_1, \ldots, y_n \) of \( g^{-1}(x) \), \( \nu(y_1, \ldots, y_n) \leq \nu'(y_1, \ldots, y_n) \).

**Proof:** Suppose first that \( \nu \leq \nu' \), and let \( y_1, \ldots, y_n \) be elements of \( g^{-1}(x) \). If \( \nu(y_1, \ldots, y_n) > \nu'(y_1, \ldots, y_n) \), this would mean that we can find elements \( u_1, \ldots, u_r \) in \( \langle y_1, \ldots, y_n \rangle \), and elements \( v_1, \ldots, v_s \) in \( g^{-1}(x) \setminus \langle y_1, \ldots, y_n \rangle \), and a number \( \xi > 0 \) such that \( \sum_{i=1}^{r} \nu(u_i) > \xi \) and \( \sum_{i=1}^{s} \nu'(v_i) > 1 - \xi \). Since \( v_j \not\in \langle u_1, \ldots, u_r \rangle \) for any \( j \) we can find neighborhoods \( U_{ij} \) of \( u_i \) such that \( \bigcup_{i=1}^{r} g(U_{ij}) \) does not contain the \( g \)-image of any neighborhood of \( v_j \). Call \( U_i = \bigcap_{j=1}^{s} U_{ij} \). Consider a neighborhood of \( \nu \) of the form
\[
U = \{ \lambda \in P(K) : \lambda(U_i) > d_i, i = 1, \ldots, r \}
\]
where \( d_i < \nu(u_i) \) and \( \sum_{i=1}^{r} d_i > \xi \). We claim that \( f(U) \) does not contain the image of any neighborhood of \( \nu' \), contradicting the fact that \( \nu \leq \nu' \). So take any neighborhood of \( \nu' \), that we can suppose of the normal form
\[
V = \{ \lambda \in P(K) : \lambda(V_j) > e_j, j = 1, \ldots, k \}
\]
where the \(V_j\)'s are disjoint neighborhoods of points \(w_j\) with \(\nu'(w_j) > e_j\) and moreover we can assume that \(w_j = v_j\) for \(j = 1, \ldots, s\). For every \(j = 1, \ldots, s\) we can find a point \(x_j \in g(V_j) \setminus \bigcup_{i=1}^r g(U_i)\). Consider the measure \(\lambda \in P(L)\) such that

\[
\lambda(x_j) = \sum \{\nu'(v_{j'}) : x_{j'} = x_j\} \text{ for } j = 1, \ldots, s
\]

and \(\lambda(x) = 1 - \sum_{i=1}^s \nu'(v_i)\). Using Lemma 5.1 it easily follows that \(\lambda \in f(V)\). Further notice that

\[
\lambda \left( \bigcup_{i=1}^r g(U_i) \right) \leq 1 - \sum_{i=1}^s \lambda(x_i) = 1 - \sum_{i=1}^s \nu'(v_i) < \xi.
\]

This implies that \(\lambda \not\in f(U)\) because otherwise we should have that

\[
\lambda \left( \bigcup_{i=1}^r g(U_i) \right) > \sum_{i=1}^r d_i > \xi.
\]

Now we suppose that \(\nu(Y) \leq \nu'(Y)\) for every finite set \(Y \subset g^{-1}(x)\). We want to see that \(\nu \leq \nu'\) so we take a typical neighborhood of \(\nu\) of the form

\[
U = \{\lambda \in P(K) : \lambda(U_i) > c_i, i = 1, \ldots, n\}
\]

where the \(U_i\)'s are disjoint clopen neighborhoods of points \(y_i\) such that \(\nu(y_i) > c_i\). For every nonempty \(A \subset \{1, \ldots, n\}\) we have

\[
\nu' \langle y_i : i \in A \rangle \geq \nu \langle y_i : i \in A \rangle \geq \nu \{y_i : i \in A\} > \sum_{i \in A} c_i,
\]

and so there exists a finite set of points \(\{z_j : j \in F_A\} \subset \langle y_i : i \in A \rangle\) such that

\[
\sum_{j \in F_A} \nu'(z_j) > \sum_{i \in A} c_i.
\]

Pick numbers \(\xi_j < \nu'(z_j)\) such that \(\sum_{j \in F_A} \xi_j > \sum_{i \in A} c_i\). For every \(j \in F_A\), since \(z_j \in \langle y_i : i \in A \rangle\) we can find a clopen neighborhood \(V_j\) of \(z_j\) such that \(g(V_j) \subset \bigcup_{i \in A} g(U_i)\). We can suppose that \(V_j \cap V_{j'} = \emptyset\) for different \(j, j' \in F_A\). Now, for every \(A\) the following is a neighborhood of \(\nu'\):

\[
V^A = \{\lambda \in P(K) : \lambda(V_j) > \xi_j, j \in F_A\}
\]

Let \(V = \bigcap \{V^A : \emptyset \neq A \subset \{1, \ldots, n\}\}\). We claim that \(f(V) \subset f(U)\). Take \(\lambda \in f(V)\). According to Lemma 5.1 we have to check that for every nonempty \(A \subset \{1, \ldots, n\}\), \(\lambda(\bigcup_{i \in A} g(U_i)) > \sum_{i \in A} c_i\). Since \(\lambda \in f(V)\) \(\subset f(V^A)\), by the same lemma we know that \(\lambda(\bigcup_{j \in F_A} g(V_j)) > \sum_{j \in F_A} \xi_j > \sum_{i \in A} c_i\), and on the other hand \(\bigcup_{j \in F_A} g(V_j) \subset \bigcup_{i \in A} g(U_i)\). \(\square\)

We finish this section by the following proposition which will enable us to verify the assumptions of Theorems 3.7 and 3.9 for spaces of the form \(P(K)\).
Proposition 6.4. Let $K$ be a scattered compact space and $O$ a connected irreducible ordered set with $|O| > 1$.

(i) Suppose that there is a point $x \in K$ such that $\mathcal{O}_{\delta_x}(P(K)) \cong O$ and for each $y \in K \setminus \{x\}$ we have $\mathcal{O}_{\delta_y}(P(K)) \not\cong O^k$ for any $k \geq 1$. Then $P(K)$ satisfies the assumption of Theorem 3.7.

(ii) Suppose that for the $\sigma$-typical surjection $f : L \to M$ (where $L$ and $M$ are metrizable quotients of $K$) there is $x \in M$ such that $\mathcal{O}_{\delta_x}(P(f)) \cong O$, $\mathbb{F}_{\delta_x}(P(f))$ has one equivalence class which is a singleton and $\mathbb{F}_{\delta_y}(P(f))$ is a singleton for each $y \in M \setminus \{x\}$. Then $P(K)$ satisfies the assumptions of Theorem 3.7.

Proof: (i) We have $\mathcal{O}_{\delta_x}(P(K)) \cong O$. Further, suppose that $\mathcal{O}_{\mu}(P(K)) \cong O^k$ for some $k > 1$ for some $\mu \in P(K)$. Let $C$ be a countable set supporting $\mu$. Then it follows from Theorem 6.2 and Corollary 3.6 that for the $\sigma$-typical surjection $f$ of $K$ there is some $y \in C \setminus \{x\}$ such that $\mathcal{O}_{\delta_y}(P(f)) \cong O^j$ for some $j \geq 1$. Now, as $C$ is countable, it implies that there is $y \in C^* = C \setminus \{x\}$ such that in each cofinal $\sigma$-semilattice in $K$ there is a surjection $f$ such that $\mathcal{O}_{\delta_y}(P(f)) \cong O^j$ for some $j \geq 1$, which contradicts our assumptions. (Otherwise, for every $y \in C^*$ there would be a cofinal $\sigma$-lattice $S_y \subset Q_o(K)$ with $\mathcal{O}_{\delta_y}(P(f)) \not\cong O^j$, for every $j$; an obvious improvement of Theorem 2.1 shows that $\bigcap_{y \in C^*} S_k$ is a cofinal $\sigma$-semilattice which leads to a contradiction.) Thus we have verified the assumptions of Theorem 3.7.

(ii) Consider $f : L \to M$ and $x \in M$ as in the assumptions. Clearly $\mathbb{F}_{\delta_y}(P(f))$ is a singleton for each $y \in M \setminus \{x\}$. Therefore we get, by Theorem 6.2, that $\mathcal{O}_{\mu}(P(f)) \cong O$ if $\mu(x) > 0$ and $\mathbb{F}_{\mu}(P(f))$ is a singleton if $\mu(x) = 0$. In this way we have verified conditions (1)–(3) of Theorem 3.7. The remaining condition (4) follows immediately from Theorem 6.2. \qed

7. Examples of spaces of probability measures

7.1. The space $\sigma_n(\kappa)$. In this section we are going to prove Theorem 1.4 the case of $P(\sigma_n(\kappa))$ of Theorem 1.6 and the case of $P(\mathbb{A}(\kappa))$ of Theorem 1.7.

For $N \subseteq M$, let $g_{MN} : \sigma_n(M) \to \sigma_n(N)$ be the continuous surjection given by $g = g_{NM}(x) = x \cap N$. The $\sigma$-typical surjection of $P(\sigma_n(\kappa))$ is of the form $f = P(g) : P(\sigma_n(M)) \to P(\sigma_n(N))$ for $M \subseteq N$ infinite countable subsets of $\kappa$ such that $M^* = M \setminus N$ is infinite. The computation of the fiber order and the $(\cdot)^*$-operation is done as follows:

For $x \in \sigma_n(N)$, $g^{-1}(x) = \{x \cup y : y \subseteq M^*, |y| \leq n - |x|\}$. A basic neighborhood of such $x \cup y \in g^{-1}(x)$ is of the form

$$U = \{z \in \sigma_n(M) : x \cup y \subseteq z, z \cap u = \emptyset, z \cap v = \emptyset\},$$
where \( u \subset M^* \) and \( v \subset N \) are finite sets. The image of such a neighborhood equals

\[
g(U) = \{ z \in \sigma_n(N) : x \subset z, |z| \leq n - |x \cup y|, z \cap v = \emptyset \}.
\]

From this it is clear that for \( w, w' \in g^{-1}(x) \) we have that \( w \leq w' \) if and only if \( |w| \leq |w'| \) and also that

\[
\langle w_1, \ldots, w_k \rangle = \{ w \in g^{-1}(x) : |w| \geq \min(|w_1|, \ldots, |w_k|) \}.
\]

Thus, if we go now to the spaces of probabilities, for \( f = P(g) : P(\sigma_n(M)) \rightarrow P(\sigma_n(N)) \), for two measures \( \nu, \nu' \in f^{-1}(\delta_x) \) we have that \( \nu \leq \nu' \) if and only if for every \( k = 1, \ldots, n - |x| \) we have that

\[
\nu\{ w \in g^{-1}(x) : |w| \geq |x| + k \} \leq \nu'\{ w \in g^{-1}(x) : |w| \geq |x| + k \}.
\]

Notice that \( \langle x \rangle = g^{-1}(x) \), thus \( \nu(\langle x \rangle) = \nu'(\langle x \rangle) \) for all \( \nu, \nu' \in f^{-1}(\delta_x) \). The ordered set \( O_{\delta_x}(f) \) is thus, isomorphic to the following

\[
O_{\delta_x}(f) \cong \{ t \in [0, 1]^{n-|x|} : t_1 \leq \ldots \leq t_{n-|x|} \}.
\]

**Proposition 7.1.** The ordered set \( O_k = \{ t \in [0, 1]^k : t_1 \leq \ldots \leq t_k \} \) is an irreducible ordered set.

Proof: We proceed by induction on \( k \). For \( k = 0 \), \( O_0 \) is a singleton (by convention, if desired) and for \( k = 1 \) we have that \( O_1 = [0, 1] \) is linearly ordered, so we suppose that \( k \geq 2 \) and that we have an order-isomorphism \( \phi : O_k \rightarrow P \times Q \). We denote by the symbols 0 and 1 the minimum and the maximum respectively of any of the ordered sets \( O_k, P \) and \( Q \) (all must exist so that \( \phi(0) = (0, 0) \) and \( \phi(1) = (0, 1) \)). Let

\[
\Lambda = \{ t \in O_k : t_2 = t_3 = \ldots = t_k = 1 \} = \{ t \in O_k : \{ s : s \geq t \} \text{ is linearly ordered} \}
\]

Every element of \( \phi(\Lambda) \) must be either of the form \( (x, 1) \) or \( (1, x) \), since otherwise \( \{ s : s \geq \phi(\lambda) \} \) cannot be linearly ordered. Moreover, since \( \Lambda \) is linearly ordered, it follows that either \( \phi(\Lambda) \subset P \times \{ 1 \} \) or \( \phi(\Lambda) \subset \{ 1 \} \times Q \). We suppose that \( \phi(\Lambda) \subset P \times \{ 1 \} \). Now call \( \lambda = (0, 1, \ldots, 1) \in \Lambda \) and \( \phi(\lambda) = (u, 1) \). We have that

\[
O_{k-1} \cong \phi\{ t \in O_k : t \leq \lambda \} = \{ s \in P : s \leq u \} \times Q,
\]

so by the inductive hypothesis, either \( |Q| = 1 \) (which would finish the proof) or \( u = 0 \). So we suppose that \( u = 0 \), which implies that \( Q \cong O_{k-1} \) and also that \( \phi(\Lambda) = P \times \{ 1 \} \) (because we found that \( \phi(\lambda) = (0, 1) \in \phi(\Lambda) \) and this is an upwards closed set). Thus \( Q \cong O_{k-1} \) and \( P \cong \Lambda \cong [0, 1] \), and it remains to show that \( O_k \not\cong O_{k-1} \times [0, 1] \). The reason is that the elements \( p = ((0, 1, \ldots, 1), 1) \) and \( q = ((1, \ldots, 1), 0) \) are two incomparable elements
of \(O_{k-1} \times [0,1]\) with the property that \(\{t : t \geq p\}\) and \(\{t : t \geq q\}\) are linearly ordered. However we noticed that the set \(A\) of points with this property in \(O_k\) is linearly ordered. \(\square\)

Since one of our announced objectives was to show that \(P(\sigma_n(\kappa))\) is not homeomorphic to \(P(\sigma_m(\kappa))\) for \(n \neq m\) let us make explicit now why this is true. It is enough to notice that the irreducible ordered sets \(O_k = \{t \in [0,1]^k : t_1 \leq \cdots \leq t_k\}\) which appear in the fiber orders of these spaces are not order-isomorphic for different values of \(k\), since for \(n < m\), \(O_m\) does not appear as the fiber order of any point of \(P(\sigma_n(\kappa))\). This can be realized in many different ways. We propose to the reader one of them. Consider

\[e = (0,0,\ldots,1) = \max\{t \in O_k : \{s : s \leq t\} \text{ is linearly ordered}\}.\]

Then \(O_{k-1} \cong \{t \in O_k : t_k = 1\} = \{t : t \geq e\}\). This argument shows that \(O_{k-1}\) can be obtained in an intrinsic way from \(O_k\), and thus if \(O_k \cong O_j\), then \(O_{k-1} \cong O_{j-1}\). The inductive repetition of such argument leads to contradiction if \(k \neq j\).

Finally, let us show the appropriate parts of Theorems 1.6 and 1.7. We get easily that \(\sigma_n(\kappa)\) satisfies the assumptions of Proposition 6.4(i) with \(O = O_n\) and \(x = \emptyset\). Further, \(A(\kappa) = \sigma_1(\kappa)\) satisfies the assumptions of Proposition 6.4(ii) with \(O = O_1 = [0,1]\) and \(x = \emptyset\).

7.2. The spaces \(P([0,\omega_1]^n)\) and \(P(A(\kappa)^n)\). In this section we shall prove Theorem 1.5 and the appropriate part of Theorems 1.6 and 1.7.

The fiber orders of the two spaces of probability measures from the title can be computed in the same way. For \(M \supseteq N\), let \(p_{MN} : A(M) \rightarrow A(N)\) be the continuous surjection given by \(p_{MN}(x) = x\) if \(x \in A(N)\) and \(p_{MN}(x) = \infty\) otherwise. The \(\sigma\)-typical surjection of \(P(A(\kappa)^n)\) is of the form \(P(p_{MN}^\kappa) : P(A(M)^n) \rightarrow P(A(N)^n)\) for \(M \supseteq N\) infinite countable subsets of \(\kappa\) such that \(M \setminus N\) is infinite.

On the other hand, for countable ordinals \(\alpha < \beta\) let \(q_{\beta\alpha} : [0,\beta] \rightarrow [0,\alpha]\) be the continuous surjection given by \(q_{\beta\alpha}(\gamma) = \gamma\) for \(\gamma \leq \alpha\), and \(q_{\beta\alpha}(\gamma) = \alpha\) for \(\gamma > \alpha\). The \(\sigma\)-typical surjection of \(P([0,\omega_1]^n)\) is of the form \(P(q_{\beta\alpha}^\omega) : P([0,\beta]^n) \rightarrow P([0,\alpha]^n)\) where \(\alpha < \beta\) are countable limit ordinals.

From the point of view of fiber orders both surjections \(p_{MN}\) and \(q_{\beta\alpha}\) can be treated simultaneously since both can be viewed as a surjection \(g : K \rightarrow L\) satisfying the following properties:

\(\star\) There exist a point \(\varpi \in L\) and a point \(m \in g^{-1}(\varpi)\) such that \(|g^{-1}(x)| = 1\) for every \(x \in L \setminus \{\varpi\}\), and with respect to the fiber order of \(g^{-1}(\varpi)\), we have that \(m < t\) and \(t \sim s\) for every \(t, s \in g^{-1}(\varpi) \setminus \{m\}\).
In the case of $p_{MN}$ we should take $\varpi = \infty$ and $m = \infty$, while for $q_{3\alpha}$, $\varpi = \alpha$ and $m = \alpha$.

From now on, we shall concentrate in computing the fiber orders of $P(g^n)$ where $g : K \to L$ is a continuous surjection satisfying ($\star$), and with this information the computation of the fiber orders of $P(A(\kappa)^n)$ and $P([0, \omega_1]^n)$ will follow immediately.

We fix $x = (x_1, \ldots, x_n) \in L^n$, and we call $R(x) = \{ i \in \{1, \ldots, n\} : x_i = \varpi \}$. First step is to understand which are the sets of the form $(y^{(1)}, \ldots, y^{(k)})$ in $(g^n)^{-1}(x)$. For every $y \in (g^n)^{-1}(x)$ we call $S(y) = \{ i \in R(x) : y_i > m \}$.

Claim A: $(y^{(1)}, \ldots, y^{(k)}) = \{ z \in (g^n)^{-1}(x) : \exists j \in \{1, \ldots, k\} : (S(z) \supset S(y^{(j)})) \}$.

Proof of Claim A: Suppose first that $S(z) \supset S(y^{(j)})$ for some $j$. Then it follows immediately that $y^{(j)} \leq z$ since the inequality holds coordinatewise. Thus $z \in (y^{(1)}, \ldots, y^{(k)})$.

Now, for the converse inclusion suppose that for every $j$ there exists a coordinate $i(j) \in S(y^{(j)}) \setminus S(z)$. So $z_{i(j)} = m$ and $y^{(j)}_{i(j)} > m$. Since all the elements of $\mathbb{F}_x(g)$ which are greater than $m$ are equivalent, for every $y^{(j)} > m$ we can easily find a neighborhood $W^j_i$ such that $W = \bigcup_{i,j} g(W^j_i)$ does not contain the image of any neighborhood of $m$. For every $j \in \{1, \ldots, k\}$ set $U_j = \{ y \in K^n : y_{i(j)} \in W^j_{i(j)} \}$ which is a neighborhood of $y^{(j)}$. We claim that $g^n(U_1) \cup \cdots \cup g^n(U_k)$ contains no image of a neighborhood of $z$, which will finish the proof of Claim A. Namely, if $V$ is a neighborhood of $z$ of the form $V_1 \times \cdots \times V_n$ with $V_i$ neighborhood of $z_i$, then for every $i \in R(x) \setminus S(z)$ we can find a point $t_i \in g(V_i) \setminus W$. If we take $u \in V$ with $g(u_i) = t_i$ for $i \in R(x) \setminus S(z)$, then $g^n(u) \notin g^n(U_1) \cup \cdots \cup g^n(U_k)$.

Claim B: For $\nu, \nu' \in P((g^n)^{-1}(\delta_x))$, we have that $\nu \leq \nu'$ if and only if for every upwards closed subset of the power set of $R(x)$, $A \subset 2^{R(x)}$, we have that $\nu\{ z : S(z) \in A \} \leq \nu'\{ z : S(z) \in A \}$.

Proof of Claim B: It follows from Claim A that the subsets of $(g^n)^{-1}(x)$ of the form $(y^{(1)}, \ldots, y^{(k)})$ are exactly the sets of the form $\{ z : S(z) \in A \}$ for some upwards closed family $A$ of subsets of $R(x)$.

As a consequence, for $x \in L^n$ with $|R(x)| = k$, we have that $\mathcal{O}_{\delta_x}(P(g^n)) \cong \{ t \in [0, 1]^{2k} : \sum_{i \in 2^k} t_i = 1 \}$ endowed with the order $t \leq s$ if and only if $\sum_{i \in A} t_i \leq \sum_{i \in A} s_i$ for every upwards closed subset of $2^k$.

Proposition 7.2. Consider the ordered set $P_k = \{ t \in [0, 1]^{2^k} : \sum_{i \in 2^k} t_i = 1 \}$ endowed with the order $t \leq s$ if and only if $\sum_{i \in A} t_i \leq \sum_{i \in A} s_i$ for every upwards closed subset of $2^k$. Then $P_k$ is an irreducible ordered set.
Proof: We proceed by induction on \( k \). If \( k = 1 \), then \( P_k \cong [0, 1] \). Suppose that we had an isomorphism \( \phi : P_k \to Q \times R \). We shall use the symbols 0 and 1 to denote the minimum and maximum of any of these ordered sets (notice that that the minimum of \( P_k \) is the characteristic function of the empty set \( 0 = \chi_\emptyset \), and its maximum is \( 1 = \chi_{\{1, \ldots, n\}} \)). For every \( i \in \{1, \ldots, n\} \) we consider \( e_i \in P_k \) the characteristic function of the singleton \( \{i\} \). Notice that \( \{t \in P_k : t \leq e_i\} \) is linearly ordered since any such \( t \) satisfies \( \sum_{a \not\in \{i\}} t_a = 0 \). Thus \( \phi(e_i) \) must be of the form either \( \phi(e_i) = (u_i, 0) \) or \( \phi(e_i) = (0, u_i) \). Notice now that

\[
\{t \in P_k : t \geq e_i\} = \{t \in P_k : \sum_{i \in a} t_a = 1\} \cong P_{k-1}
\]

and \( P_{k-1} \) is irreducible by the inductive hypothesis, so \( u_i = 1 \) since \( \{(r, s) : (r, s) \geq (r_0, s_0)\} = \{r : r \geq r_0\} \times \{s : s \geq s_0\} \). Hence \( \phi(e_i) \in \{(1, 0), (0, 1)\} \) for every \( i \in \{1, \ldots, k\} \). If \( k > 2 \) this is already a contradiction, so we suppose that \( k = 2 \) and \( \phi(e_1) = (1, 0) \) and \( \phi(e_2) = (0, 1) \). We denote the elements of \( P_2 \) as \( t = (t_0, t_1, t_2) \). For every \( \lambda \in [0, 1] \), we call \( x^\lambda = (1 - \lambda, \lambda, 0, 0) \) and \( y^\lambda = (1 - \lambda, 0, \lambda, 0) \) in \( P_2 \). We have \( x^\lambda \leq e_1 \) and \( y^\lambda \leq e_2 \) and so \( \phi(x^\lambda) = (r^\lambda, 0) \) and \( \phi(y^\lambda) = (0, s^\lambda) \) for suitable \( r^\lambda \) and \( s^\lambda \). We consider the specific elements \( u = (0.7, 0.1, 0.1, 0.1) \) and \( u' = (0.8, 0.0, 0.2) \) of \( P_2 \). Say that \( \phi(u) = (r, s) \) and \( \phi(u') = (r', s') \). On the one hand \( x^{0.2} \) and \( y^{0.2} \) are lower than \( u \) and \( u' \) so \( r^{0.2} \leq r, r'^{0.2} \leq r', s^{0.2} \leq s \) and \( s^{0.2} \leq s' \). On the other hand, if \( \lambda > 0.2 \) then \( x^\lambda \not\leq u, x^\lambda \not\leq u' \), \( y^\lambda \not\leq u \), neither \( y^\lambda \not\leq u' \). Hence indeed \( r = r' = r^{0.2} \) and \( s = s' = s^{0.2} \). Thus \( \phi(u) = \phi(u') \), a contradiction. \( \square \)

Notice that \( P_k \) is not order-isomorphic to \( P_{k'} \) for \( k \neq k' \), since the set

\[
H = \{t \in P_k : \{s : s \leq t\} \text{ is linearly ordered}\}
\]

contains exactly \( k \) many maximal elements: \( \{e^1, \ldots, e^k\} \), where again \( e^i \in P_k \) denotes the characteristic function of the singleton \( \{i\} \). This also shows that these irreducible ordered sets are not isomorphic to the irreducible ordered sets \( O_k \) which appeared in the fiber orders of the spaces \( P(\sigma_n(\kappa)) \) (for \( n > 1 \)), because in those cases the set of all elements \( t \) such that \( \{s : s \leq t\} \) is linearly ordered was a linearly ordered set with precisely one maximal element.

The above calculation proves Theorem [15]. Further, both \( A(\kappa)^n \) and \([0, \omega_1]^n \) satisfy the assumptions of Proposition [6.3][1] with \( O = P_n \) and \( x = (\infty, \ldots, \infty) \) resp. \( x = (\omega_1, \ldots, \omega_1) \). This proves the appropriate part of
Finally, \([0, \omega_1]\) satisfies the assumptions of Proposition 6.3(ii) with \(O = [0, 1]\) and \(x = \alpha\) (using the above notation). This proves the appropriate part of Theorem 1.7.

We have not mentioned it so far but, despite the fact that the picture of fiber orders is similar, the spaces \(P(A(\kappa)^n)\) and \(P([0, \omega_1]^n)\) are very different, by other well known reasons. Namely, \(P(A(\kappa)^n)\) is an Eberlein compact, and hence Fréchet-Urysohn space, so it cannot contain any copy of \([0, \omega_1]\).

8. **Higher weights**

So far we used the version of spectral theorem that we stated as Theorem 2.1 but there is the possibility to use other versions. For example, for a regular cardinal \(\tau\), we consider \(Q_\tau(K)\) the family of quotients of weight strictly less than \(\tau\), and we call a \(\tau\)-semilattice to a subset \(S \subset Q(K)\) such that the supremum of every subset of \(S\) of cardinality less than \(\tau\) belongs to \(S\). The set \(S\) is cofinal in \(Q_\tau(K)\) if for every \(L \in Q_\tau(K)\) there exists \(L' \in S\) with \(L \leq L'\). We assume that \(\tau\) is a regular cardinal because otherwise there exists no cofinal \(\tau\)-semilattice in \(Q_\tau(K)\).

**Theorem 8.1.** Let \(K\) be a compact space with weight at least \(\tau\). The intersection of two cofinal \(\tau\)-semilattices in \(Q_\tau(K)\) is a further cofinal \(\tau\)-semilattice in \(Q_\tau(K)\).

**Proof:** It is completely analogous to the proof of Theorem 2.1. \(\square\)

**Lemma 8.2.** Let \(K\) be a compact space of weight at least \(\tau\) and \(S\) a \(\tau\)-semilattice in \(Q_\tau(K)\). Then, \(S\) is cofinal in \(Q_\tau(K)\) if and only if \(\sup S = K\).

**Proof:** Suppose \(\sup S = K\). By the same argument as in the proof of Lemma 2.2 every real-valued continuous function \(f \in C(K)\) factors through an element of \(S\). Now, if \(p : K \to L\) is an arbitrary element of \(Q_\tau(K)\), then we can take an embedding \(L \subset \mathbb{R}^\gamma\) for a cardinal \(\gamma < \tau\) and consider the functions \(e_i p : K \to \mathbb{R}\) obtained by composing with the coordinate functions \(e_i : \mathbb{R}^\gamma \to \mathbb{R}, i < \gamma\). For every \(i < \gamma\) we know that there exists \(L_i \in S\) such that \(e_i p\) factors through \(L_i\). Finally, this implies that \(p\) factors through \(L_\infty = \sup\{L_i : i < \gamma\}\), so \(L \leq L_\infty \in S\). \(\square\)

In a similar way as we did with \(\sigma\)-semilattices spectra, we can say that the \(\tau\)-typical surjection of \(K\) has a property \(P\) if there is cofinal \(\tau\)-semilattice in which all the natural surjections have property \(P\), and when this happens such a \(\tau\)-semilattice can be found as a subsemilattice of any given one. Also similarly, we can talk in this context of \(P_\tau(K)\) and \(O_\tau(K)\).
An application can be found in the study of of the space \( P([0, \tau]^n) \) and their finite powers, for \( \tau > \omega_1 \) a regular cardinal. The fiber orders of \( K = P([0, \alpha]^n) \) for any ordinal \( \alpha \geq \omega_1 \) can be computed using very similar arguments as in Section 7.2 and indeed \( \mathcal{O}_{\delta_x}(K) \cong P_k \) where \( k \) is the number of coordinates of \( x \in [0, \alpha]^n \) with uncountable cofinality. Therefore, for \( \alpha \geq \omega_1 + \omega_1 \), \( P([0, \alpha]^n) \) does not satisfy the assumptions of Proposition 6.4.

Indeed, \( [0, \alpha] \) has at least two non-\( G_\delta \)-points and hence \( [0, \alpha]^n \) contains several points \( x \) with \( \mathcal{O}_{\delta_x}(P(K)) \cong P_n \). Moreover, \( P([0, \alpha]^n) \) does not satisfy even the assumptions of Theorems 3.9 and 3.7 - for \( n = 1 \) it is witnessed by the fact that \( \mathcal{O}_{\frac{1}{2}(\delta_x + \delta_y)}(P([0, \alpha])) = [0, 1]^2 \) whenever \( x \) and \( y \) are two distinct points with uncountable cofinality.

However, still it is possible to get decomposition results about spaces \( P([0, \tau]^n) \) using \( \tau \)-semilattices, since analogues of Theorems 3.9 and 3.7 for the \( \tau \)-typical surjection hold, with identical proof. There is a natural cofinal \( \tau \)-semilattice for \( [0, \tau] \): For \( \alpha < \beta \) consider the continuous surjection \( p_{\beta \alpha} : [0, \beta] \to [0, \alpha] \) given by \( p_{\beta \alpha}(x) = x \) for \( x \leq \alpha \), and \( p_{\beta \alpha}(x) = \alpha \) for \( x > \alpha \). The \( \tau \)-semilattice consists of all quotients given by \( p_{\tau \alpha} \), \( \alpha < \tau \), and the \( \tau \)-typical surjection is of the form \( p_{\beta \alpha} \), \( \alpha < \beta < \tau \). Thus, the situation is completely analogous to that of \( P([0, \omega_1]^n) \), and we have the following result:

**Theorem 8.3.** Let \( \tau \) be a regular cardinal, \( K = P([0, \tau]^n), x = (x_1, \ldots, x_r) \in [0, \tau]^n \) and \( k = |\{i : x_i = \tau\}| \). Then \( \mathcal{O}_{\delta_x}(K) \cong P_k \) if \( k \in [0, 1]^2 : \sum_{i \in A} t_i = 1 \) endowed with the order \( t \leq s \) if and only if \( \sum_{i \in A} t_i \leq \sum_{i \in A} s_i \) for every upwards closed subset of \( 2^k \).

Now we get easily the remaining part of Theorem 1.6 and Theorem 1.8.

We mention that, answering a question posed to us by R. Deville and G. Godefroy, the ideas of this section are used in [2] to show that there exist \( 2^\kappa \) many nonhomeomorphic weakly compact convex subsets in \( \ell_2(\kappa) \).

### 9. Final Remarks and Open Problems

**Question 1.** Let \( M(K) \) denote the space of regular Borel measures of variation at most 1 (that is, the dual ball of the space of continuous functions \( C(K) \)) in its weak* topology. We show in this paper that \( M(A(\kappa)) \) is not homeomorphic to \( P(A(\kappa)) \) using fiber orders. We did not make a systematic study of the fiber orders of \( M(K) \) and this may be interesting. Analysing the relatively easy case of \( A(\kappa) \) it seems that the fiber orders of \( M(A(\kappa)) \) look similar to those of \( P(A(\kappa))^2 \), so we may ask: Is \( M(K) \approx P(K)^2 \) for each compact space? However, this question has negative answer. Let \( K \)
be the well-known “double arrow space”. Then $P(K)$ is first countable (for example by [13, Proposition 7]) while $M(K)$ is not first countable as $K$ is not metrizable. Therefore $M(K) \not\approx P(K)^2$. But we still can ask: Is $M(K) \approx P(K)^2$ for compact spaces considered in this paper ($A(\kappa)$, $\sigma_n(\kappa)$ etc.)?

**Question 2.** The analysis of the generic fibers of $B(\kappa)$ yields the same result as for $P(A(\kappa))$, namely all the non $G_\delta$ points have generic fibers order-isomorphic to an interval. Are the spaces $B(\kappa)$ and $P(A(\kappa))$ homeomorphic? In relation with this, it follows from [14] that $P(A(\kappa))$ is homeomorphic to $P(A(\kappa)) \times [0,1]$. Is $B(\kappa)$ homeomorphic to $B(\kappa) \times [0,1]$ or even to any product of two nontrivial spaces?

**Question 3.** In the various spaces of probability measures that we studied, fiber orders allow us to identify different types of points. Is this a complete classification? That is, we ask to determine exactly for which points $x, y \in P(K)$ there exists a homeomorphism $f : P(K) \rightarrow P(K)$ such that $f(x) = y$.

**Question 4.** Fiber orders are a good tool to determine whether two spaces are homeomorphic but they do not seem to help in determining whether a given space is the continuous image of another. In [3] the case of the spaces $B(\kappa)^n$ is studied, but the situation is not clear for the other spaces studied here. For instance we do not know whether $P(\sigma_n(\kappa))$ maps onto $P(\sigma_m(\kappa))$ for $n < m$, and so on. This is related also to the problem of the $A(\kappa)^\omega$-images, initiated by Benyamini, Rudin and Wage [6] and studied specially by Bell in [4] and [5]. It is proven in [1] that $P(A(\kappa))$ and $B(\kappa)$ are continuous images of $A(\kappa)^\omega$, but it is unclear to us whether $P(\sigma_n(\kappa))$ or $P(A(\kappa)^n)$ are continuous images of $A(\kappa)^\omega$ for $n > 1$.

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