COMPLETE CLASSIFICATION OF THE TORSION STRUCTURES OF RATIONAL ELLIPTIC CURVES OVER QUINTIC NUMBER FIELDS

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ABSTRACT. We classify the possible torsion structures of rational elliptic curves over quintic number fields. In addition, let $E$ be an elliptic curve defined over $\mathbb{Q}$ and let $G = E(\mathbb{Q})_{\text{tors}}$ be the associated torsion subgroup. We study, for a given $G$, which possible groups $G \subseteq H$ could appear such that $H = E(K)_{\text{tors}}$, for $[K : \mathbb{Q}] = 5$. In particular, we prove that at most there is one quintic number field $K$ such that the torsion grows in the extension $K/\mathbb{Q}$, i.e., $E(\mathbb{Q})_{\text{tors}} \subsetneq E(K)_{\text{tors}}$.

1. INTRODUCTION

Let $E/K$ be an elliptic curve defined over a number field $K$. The Mordell-Weil Theorem states that the set of $K$-rational points, $E(K)$, is a finitely generated abelian group. Denote by $E(K)_{\text{tors}}$, the torsion subgroup of $E(K)$, which is isomorphic to $C_m \times C_n$ for two positive integers $m, n$, where $m$ divides $n$ and where $C_n$ is a cyclic group of order $n$.

One of the main goals in the theory of elliptic curves is to characterize the possible torsion structures over a given number field, or over all number fields of a given degree. In 1978 Mazur [25] published a proof of Ogg’s conjecture (previously established by Beppo Levi), a milestone in the theory of elliptic curves. In that paper, he proved that the possible torsion structures over $\mathbb{Q}$ belong to the set:

$$\Phi(1) = \{C_n \mid n = 1, \ldots, 10, 12\} \cup \{C_2 \times C_{2m} \mid m = 1, \ldots, 4\},$$

and that any of them occurs infinitely often. A natural generalization of this theorem is as follows. Let $\Phi(d)$ be the set of possible isomorphic torsion structures $E(K)_{\text{tors}}$, where $K$ runs through all number fields $K$ of degree $d$ and $E$ runs through all elliptic curves over $K$. Thanks to the uniform boundedness theorem [26], $\Phi(d)$ is a finite set. Then the problem is to determine $\Phi(d)$. Mazur obtained the rational case ($d = 1$). The generalization to quadratic fields ($d = 2$) was obtained by Kamienny, Kenku and Momose [17, 22]. For $d \geq 3$ a complete answer for this problem is still open, although there have been some advances in the last years.

However, more is known about the subset $\Phi^\infty(d) \subseteq \Phi(d)$ of torsion subgroups that arise for infinitely many $\overline{\mathbb{Q}}$-isomorphism classes of elliptic curves defined over number fields of degree $d$. For $d = 1$ and $d = 2$ we have $\Phi^\infty(d) = \Phi(d)$, the cases $d = 3$ and $d = 4$ have been determined by Jeon et al. [15, 16], and recently the cases $d = 5$ and $d = 6$ by Derickx and Sutherland [7].

Restricting our attention to the complex multiplication case, we denote $\Phi^{CM}(d)$ the analogue of the set $\Phi(d)$ but restricting to elliptic curves with complex multiplication (CM elliptic curves in the sequel). In 1974 Olson [30] determined the set of possible torsion structures over $\mathbb{Q}$ of CM elliptic curves:

$$\Phi^{CM}(1) = \{C_1, C_2, C_3, C_4, C_6, C_2 \times C_2\}.$$ 

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The quadratic and cubic cases were determined by Zimmer et al. [27, 8, 31] and recently, Clark et al. [5] have computed the sets $\Phi_{\text{CM}}(d)$, for $4 \leq d \leq 13$. In particular, they proved

$$\Phi_{\text{CM}}(5) = \Phi_{\text{CM}}(1) \cup \{ C_{11} \}.$$  

In addition to determining $\Phi(d)$, there are many authors interested in the question of how the torsion grows when the field of definition is enlarged. We focus our attention when the underlying field is $\mathbb{Q}$. In analogy to $\Phi(d)$, let $\Phi_{\text{Q}}(d)$ be the subset of $\Phi(d)$ such that $H \in \Phi_{\text{Q}}(d)$ if there is an elliptic curve $E/\mathbb{Q}$ and a number field $K$ of degree $d$ such that $E(K)_{\text{tors}} \simeq H$. One of the first general results is due to Najman [29], who determined $\Phi_{\text{Q}}(d)$ for $d = 2, 3$. Chou [4] has given a partial answer to the classification of $\Phi_{\text{Q}}(4)$. Recently, the author with Najman [11] have completed the classification of $\Phi_{\text{Q}}(4)$ and $\Phi_{\text{Q}}(p)$ for $p$ prime. Moreover, in [11] it has been proved that $E(K)_{\text{tors}} = E(\mathbb{Q})_{\text{tors}}$ for all elliptic curves $E$ defined over $\mathbb{Q}$ and all number fields $K$ of degree $d$, where $d$ is not divisible by a prime $\leq 7$. In particular, $\Phi_{\text{Q}}(d) = \Phi(1)$ if $d$ is not divisible by a prime $\leq 7$.

Our first result determines $\Phi_{\text{Q}}(5)$.

**Theorem 1.** The sets $\Phi_{\text{Q}}(5)$ and $\Phi_{\text{CM}}(5)$ are given by

$$\Phi_{\text{Q}}(5) = \{ C_n | n = 1, \ldots, 12, 25 \} \cup \{ C_2 \times C_{2m} | m = 1, \ldots, 4 \},$$

$$\Phi_{\text{CM}}(5) = \{ C_1, C_2, C_3, C_4, C_6, C_{11}, C_2 \times C_2 \}.$$  

**Remark.** $\Phi_{\text{Q}}(5) = \Phi_{\text{Q}}(1) \cup \{ C_{11}, C_{25} \}$ and $\Phi_{\text{CM}}(5) = \Phi_{\text{CM}}(5) = \Phi_{\text{CM}}(1) \cup \{ C_{11} \}$.

For a fixed $G \in \Phi(1)$, let $\Phi_{\text{Q}}(d, G)$ be the subset of $\Phi_{\text{Q}}(d)$ such that $E$ runs through all elliptic curves over $\mathbb{Q}$ with $E(\mathbb{Q})_{\text{tors}} \simeq G$. For each $G \in \Phi(1)$ the sets $\Phi_{\text{Q}}(d, G)$ have been determined for $d = 2$ in [23, 13], for $d = 3$ in [12] and partially for $d = 4$ in [10].

Our second result determines $\Phi_{\text{Q}}(5)$ for any $G \in \Phi(1)$.

**Theorem 2.** For $G \in \Phi(1)$, we have $\Phi_{\text{Q}}(5, G) = \{ G \}$, except in the following cases:

| $G$    | $\Phi_{\text{Q}}(5, G)$ |
|--------|-------------------------|
| $C_1$  | $\{ C_1, C_5, C_{11} \}$ |
| $C_2$  | $\{ C_2, C_{10} \}$     |
| $C_5$  | $\{ C_5, C_{25} \}$     |

Moreover, there are infinitely many $\mathbb{T}$-isomorphism classes of elliptic curves $E/\mathbb{Q}$ with $H \in \Phi_{\text{Q}}(5, G)$, except for the case $H = C_{11}$ where only the elliptic curves $121a2, 121c2, 121b1$ have eleven torsion over a quintic number field.

In fact, it is possible to give a more detailed description of how the torsion grows. For this purpose, for any $G \in \Phi(1)$ and any positive integer $d$, we define the set

$$H_{\text{Q}}(d, G) = \{ S_1, \ldots, S_n \}$$

where $S_i = [H_1, \ldots, H_m]$ is a list of groups $H_j \in \Phi_{\text{Q}}(d, G) \setminus \{ G \}$, such that, for each $i = 1, \ldots, n$, there exists an elliptic curve $E_i/\mathbb{Q}$ that satisfies the following properties:

- $E_i(\mathbb{Q})_{\text{tors}} \simeq G$, and
- there are number fields $K_1, \ldots, K_m$ (non-isomorphic pairwise) whose degrees divide $d$ with $E_i(K_j)_{\text{tors}} \simeq H_j$, for all $j = 1, \ldots, m$; and for each $j$ there does not exist $K'_j \subset K_j$ such that $E_i(K'_j)_{\text{tors}} \simeq H_j$. 

We are allowing the possibility of two (or more) of the $H_j$ being isomorphic. The above sets have been completely determined for the quadratic case ($d = 2$) in [14], for the cubic case ($d = 3$) in [12] and computationally conjectured for the quartic case ($d = 4$) in [10]. The quintic case ($d = 5$) is treated in this paper, and the next result determined $H_Q(5, G)$ for any $G \in \Phi(1)$:

**Theorem 3.** For $G \in \Phi(1)$, we have $H_Q(5, G) = \emptyset$, except in the following cases:

| $G$ | $H_Q(5, G)$ |
|-----|--------------|
| $C_1$ | $C_5$ |
| $C_5$ | $C_{11}$ |
| $C_2$ | $C_{10}$ |
| $C_{11}$ | $C_{25}$ |

In particular, for any elliptic curve $E/\mathbb{Q}$, there is at most one quintic number field $K$, up to isomorphism, such that $E(K)_{\text{tors}} \neq E(\mathbb{Q})_{\text{tors}}$.

**Remark.** Notice that for any CM elliptic curve $E/\mathbb{Q}$ and any quintic number field $K$ it has $E(K)_{\text{tors}} = E(\mathbb{Q})_{\text{tors}}$, except to the elliptic curve 121b1 and $K = \mathbb{Q}(\zeta_{11})^+ = \mathbb{Q}(\zeta_{11} + \zeta_{11}^{-1})$ where $E(\mathbb{Q})_{\text{tors}} \simeq C_1$ and $E(K)_{\text{tors}} \simeq C_{11}$.

Let us define

$$h_Q(d) = \max_{G \in \Phi(1)} \left\{ \#S \mid S \in H_Q(d, G) \right\}.$$ 

The values $h_Q(d)$ have been computed for $d = 2$ and $d = 3$ in [14] and [12] respectively. For $d = 4$ we computed a lower bound in [10]. For $d = 5$ we have:

**Corollary 4.** $h_Q(5) = 1$.

**Remark.** In particular, we have deduced the following:

| $d$ | 2 | 3 | 4 | 5 |
|-----|---|---|---|---|
| $h_Q(d)$ | 4 | 3 | $\geq 9$ | 1 |

**Notation.** We will use the Antwerp–Cremona tables and labels [1, 6] when referring to specific elliptic curves over $\mathbb{Q}$.

For conjugacy classes of subgroups of $\text{GL}_2(\mathbb{Z}/n\mathbb{Z})$ we will use the labels introduced by Sutherland in [34] §6.4.

We will write $G \simeq H$ (or $G \lesssim H$) for the fact that $G$ is isomorphic to $H$ (or to a subgroup of $H$ resp.) without further detail on the precise isomorphism.

For a positive integer $n$ we will write $\varphi(n)$ for the Euler-totient function of $n$.

We use $\mathcal{O}$ to denote the point at infinity of an elliptic curve (given in Weierstrass form).

2. **Mod $n$ Galois representations associated to elliptic curves**

Let $E/\mathbb{Q}$ be an elliptic curve and $n$ a positive integer. We denote by $E[n]$ the $n$-torsion subgroup of $E(\overline{\mathbb{Q}})$, where $\overline{\mathbb{Q}}$ is a fixed algebraic closure of $\mathbb{Q}$. That is, $E[n] = \{ P \in E(\overline{\mathbb{Q}}) \mid [n]P = \mathcal{O} \}$. The absolute Galois group $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on $E[n]$ by its action on the coordinates of the points, inducing a Galois representation

$$\rho_{E,n} : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{Aut}(E[n]).$$

Notice that since $E[n]$ is a free $\mathbb{Z}/n\mathbb{Z}$-module of rank 2, fixing a basis $\{ P, Q \}$ of $E[n]$, we identify $\text{Aut}(E[n])$ with $\text{GL}_2(\mathbb{Z}/n\mathbb{Z})$. Then we rewrite the above Galois representation as

$$\rho_{E,n} : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{GL}_2(\mathbb{Z}/n\mathbb{Z}).$$
Therefore we can view $\rho_{E,n}(\text{Gal}(\overline{Q}/Q))$ as a subgroup of $GL_2(\mathbb{Z}/n\mathbb{Z})$, determined uniquely up to conjugacy, and denoted by $G_E(n)$ in the sequel. Moreover, $Q(E[n]) = \{x, y \mid (x, y) \in E[n]\}$ is Galois and since $\ker \rho_{E,n} = \text{Gal}(\overline{Q}/Q(E[n]))$, we deduce that $G_E(n) \simeq \text{Gal}(Q(E[n])/Q)$.

Let $R = (x(R), y(R)) \in E[n]$ and $Q(R) = Q(x(R), y(R)) \subseteq Q(E[n])$, then by Galois theory there exists a subgroup $H_R$ of $\text{Gal}(Q(E[n])/Q)$ such that $Q(R) = Q(E[n])^{H_R}$. In particular, if we denote by $H_R$ the image of $H_R$ in $GL_2(\mathbb{Z}/n\mathbb{Z})$, we have:

- $[Q(R) : Q] = [G_E(n) : H_R]$.
- $\text{Gal}(Q(R)/Q) \simeq G_E(n)/N_{G_E(n)}(H_R)$, where $\overline{Q}(R)$ denotes the Galois closure of $Q(R)$ in $\overline{Q}$, and $N_{G_E(n)}(H_R)$ denotes the normal core of $H_R$ in $G_E(n)$.

We have deduced the following result.

**Lemma 5.** Let $E/Q$ be an elliptic curve, $n$ a positive integer and $R \in E[n]$. Then $[Q(R) : Q]$ divides $|G_E(n)|$. In particular $[Q(R) : Q]$ divides $|GL_2(\mathbb{Z}/n\mathbb{Z})|$.

In practice, given the conjugacy class of $G_E(n)$ we can deduce the relevant arithmetic-algebraic properties of the fields of definition of the $n$-torsion points: since $E[n]$ is a free $\mathbb{Z}/n\mathbb{Z}$-module of rank 2, we can identify the $n$-torsion points with $(a, b) \in (\mathbb{Z}/n\mathbb{Z})^2$ (i.e. if $R \in E[n]$ and $\{P, Q\}$ is a $\mathbb{Z}/n\mathbb{Z}$-basis of $E[n]$, then there exist $a, b \in \mathbb{Z}/n\mathbb{Z}$ such that $R = aP + bQ$). Therefore $H_R$ is the stabilizer of $(a, b)$ by the action of $G_E(n)$ on $(\mathbb{Z}/n\mathbb{Z})^2$. In order to compute all the possible degrees (jointly with the Galois group of its Galois closure in $\overline{Q}$) of the fields of definition of the $n$-torsion points we run over all the elements of $(\mathbb{Z}/n\mathbb{Z})^2$ of order $n$.

Now, observe that $(R) \subseteq E[n]$ is a subgroup of order $n$. Equivalently, $E/Q$ admits a cyclic $n$-isogeny (non-rational in general). The field of definition of this isogeny is denoted by $Q((R))$. A similar argument could be used to obtain a description of $Q((R))$ using Galois theory. In particular, if $(R)$ is Galois over $Q$, then the isogeny is defined over $Q$. To compute the relevant arithmetic-algebraic properties of the field $Q((R))$ is similar to the case $Q(R)$, replacing the pair $(a, b)$ by the $\mathbb{Z}/n\mathbb{Z}$-module of rank 1 generated by $(a, b)$ in $(\mathbb{Z}/n\mathbb{Z})^2$.

In the case $E/Q$ be a non-CM elliptic curve and $p \leq 11$ be a prime, Zywina [35] has described all the possible subgroups of $GL_2(\mathbb{Z}/p\mathbb{Z})$ that occur as $G_E(p)$.

For each possible subgroup $G_E(p) \subseteq GL_2(\mathbb{Z}/p\mathbb{Z})$ for $p \in \{2, 3, 5, 11\}$, Table I lists in the first and second column the corresponding labels in Sutherland and Zywina notations, and the following data:

- $d_0$: the index of the largest subgroup of $G_E(p)$ that fixes a $\mathbb{Z}/p\mathbb{Z}$-submodule of rank 1 of $E[p]$; equivalently, the degree of the minimal extension $L/Q$ over which $E$ admits a $L$-rational $p$-isogeny.
- $d_1$: is the index of the stabilizers of $v \in (\mathbb{Z}/p\mathbb{Z})^2$, $v \neq (0, 0)$, by the action of $G_E(p)$ on $(\mathbb{Z}/p\mathbb{Z})^2$; equivalently, the degrees of the extension $L/Q$ over which $E$ has a $L$-rational point of order $p$.
- $d$: is the order of $G_E(p)$; equivalently, the degree of the minimal extension $L/Q$ for which $E[p] \subseteq E(L)$.

Note that Table I is partially extracted from Table 3 of [34]. The difference is that [34] Table 3] only lists the minimum of $d_v$, which is denoted by $d_1$ therein.

For the CM case, Zywina [35] §1.9 gives a complete description of $G_E(p)$ for any prime $p$.

3. ISOGENIES.

In this paper a rational $n$-isogeny of an elliptic curve $E/Q$ is a (surjective) morphism $E \rightarrow E'$ defined over $Q$ where $E'/Q$ and the kernel is cyclic of order $n$. The rational $n$-isogenies of elliptic curves over $Q$, have been described completely in the literature, for all $n \geq 1$. The following result gives all the possible values of $n$. 


Table 1. Image groups $G_E(p)$, for $p \in \{2, 3, 5, 11\}$, for non-CM elliptic curves $E/\mathbb{Q}$.

| Sutherland | Zywina | $d_0$ | $d_\nu$ | $d$ |
|-------------|--------|-------|---------|-----|
| 2Cs         | $G_1$  | 1     | 1       | 1   |
| 2B          | $G_2$  | 1     | 1,2     | 2   |
| 2Cn         | $G_3$  | 3     | 3       | 3   |
| $\text{GL}(2, \mathbb{Z}/2\mathbb{Z})$ | | 3 | 3 | 6 |
| 3Cs.1.1     | $H_{1,1}$ | 1 | 1,2 | 2 |
| 3Cs         | $G_1$  | 1     | 2,4     | 4   |
| 3B.1.1      | $H_{3,1}$ | 1 | 1,6 | 6 |
| 3B.1.2      | $H_{3,2}$ | 1 | 2,3 | 6 |
| 3Ns         | $G_2$  | 2     | 4       | 8   |
| 3B          | $G_3$  | 1     | 2,6     | 12  |
| 3Nn         | $G_4$  | 4     | 8       | 16  |
| $\text{GL}(2, \mathbb{Z}/3\mathbb{Z})$ | | 4 | 8 | 48 |
| 11B.1.4     | $H_{1,1}$ | 1 | 5,110 | 110 |
| 11B.1.5     | $H_{2,1}$ | 1 | 5,110 | 110 |
| 11B.1.6     | $H_{2,2}$ | 1 | 10,55 | 110 |
| 11B.1.7     | $H_{1,2}$ | 1 | 10,55 | 110 |
| 11B.10.4    | $G_1$  | 1     | 10,110  | 220 |
| 11B.10.5    | $G_2$  | 1     | 10,110  | 220 |
| 11Nn        | $G_3$  | 12    | 120     | 240 |
| $\text{GL}(2, \mathbb{Z}/11\mathbb{Z})$ | | 12 | 120 | 13200 |
| 5Cs.1.1     | $H_{1,1}$ | 1 | 1,4 | 4 |
| 5Cs.1.3     | $H_{1,2}$ | 1 | 2,4 | 4 |
| 5Cs.4.1     | $G_1$  | 1     | 2,4,8   | 8   |
| 5Ns.2.1     | $G_3$  | 2     | 8,16    | 16  |
| 5Cs         | $G_2$  | 1     | 4     | 16  |
| 5B.1.1      | $H_{6,1}$ | 1 | 1,20 | 20 |
| 5B.1.2      | $H_{5,1}$ | 1 | 4,5 | 20 |
| 5B.1.4      | $H_{6,2}$ | 1 | 2,20 | 20 |
| 5B.1.3      | $H_{5,2}$ | 1 | 4,10 | 20 |
| 5Ns         | $G_4$  | 2     | 8,16    | 32  |
| 5B.4.1      | $G_6$  | 1     | 2,20    | 40  |
| 5B.4.2      | $G_5$  | 1     | 4,10    | 40  |
| 5Ns         | $G_7$  | 6     | 24      | 48  |
| 5B          | $G_8$  | 1     | 4,20    | 80  |
| 5S4         | $G_9$  | 6     | 24      | 96  |
| $\text{GL}(2, \mathbb{Z}/5\mathbb{Z})$ | | 6 | 24 | 480 |

Theorem 6 ([25 18 19 20 21]). Let $E/\mathbb{Q}$ be an elliptic curve with a rational $n$-isogeny. Then $n \leq 19$ or $n \in \{21, 25, 27, 37, 43, 67, 163\}$.

A direct consequence of the Galois theory applied to the theory of cyclic isogenies is the following (cf. Lemma 3.10 [4]).

Lemma 7. Let $E/\mathbb{Q}$ be an elliptic curve such that $E(K)[n] \cong C_n$ over a Galois extension $K/\mathbb{Q}$. Then $E$ has a rational $n$-isogeny.

4. $\mathcal{P}$-PRIMARY TORSION SUBGROUP

Let $E/K$ be an elliptic curve defined over a number field $K$. For a given set of primes $\mathcal{P} \subset \mathbb{Z}$, let $E(K)[\mathcal{P}^\infty]$ denote the $\mathcal{P}$-primary torsion subgroup of $E(K)_{\text{tors}}$, that is, the direct product of the $p$-Sylow subgroups of $E(K)$ for $p \in \mathcal{P}$. If $\mathcal{P} = \{p\}$, let us denote by $E(K)[p^\infty]$.

Proposition 8. Let $E/\mathbb{Q}$ be an elliptic curve and $K/\mathbb{Q}$ be a quintic number field.

(1) If $P$ is a point of prime order $p$ in $E(K)$, then $p \in \{2, 3, 5, 7, 11\}$. 

(2) If \( E(K)[n] = E[n] \), then \( n = 2 \).

Proof. (1) Lozano-Robledo \[24\] has determined that the set of primes \( p \) for which there exists a number field \( K \) of degree \( \leq 5 \) and an elliptic curve \( E/\mathbb{Q} \) such that the \( p \) divides the order of \( E(K)_{\text{tors}} \) is given by \( S_5(5) = \{2, 3, 5, 7, 11, 13\} \). Then to finish the proof we must remove the prime \( p = 13 \). This follows from Lemma 5 since 5 does not divide the order of \( GL_2(\mathbb{F}_{13}) \), that is \( 2^5 \cdot 3^2 \cdot 7 \cdot 13 \).

(2) Let \( E/K \) be the base change of \( E \) over the number field \( K \). If \( E[n] \subseteq E(K) \) then \( \mathbb{Q}(\zeta_n) \subseteq K \). In particular \( \varphi(n) | |K : \mathbb{Q}| \). The only possibility if \( |K : \mathbb{Q}| = 5 \) is \( n = 2 \). \( \square \)

4.1. \( p \)-primary torsion subgroup \((p \neq 5, 11)\).

Lemma 9. Let \( E/\mathbb{Q} \) be an elliptic curve and \( K/\mathbb{Q} \) a quintic number field. Then, for any prime \( p \neq 5, 11 \):

\[
E(K)[p^n] = E(\mathbb{Q})[p^n].
\]

In particular, if \( P \in E(K)[p^n] \) and \( p^n \) is its order, then \( n \leq 3, 2, 1 \), if \( p = 2, 3, 7 \), respectively, and \( n = 0 \) otherwise.

Proof. Let \( P \in E(K)[p^n] \). By Lemma 5, \( |Q(P) : \mathbb{Q}| \) divides \( |GL_2(\mathbb{Z}/p^n\mathbb{Z})| = p^{4n-3}(p^2 - 1)(p - 1) \). If \( p \in \{2, 3, 7\} \) then \( Q(P) = \mathbb{Q} \). Together with Proposition 8 (2), we deduce \( E(K)[p^n] = E(\mathbb{Q})[p^n] \). If \( p \geq 13 \) and \( n > 0 \), then \( [p^{n-1}]P \in E(K) \) is a point or order \( p \), a contradiction with Proposition 8 (1). That is, \( E(K)[p^n] = E(\mathbb{Q})[p^n] = \{O\} \) if \( p \geq 13 \). \( \square \)

4.2. 5-primary torsion subgroup.

Lemma 10. Let \( E/\mathbb{Q} \) be an elliptic curve and \( K/\mathbb{Q} \) a quintic number field. Then

\[
E(K)[5^n] \lesssim C_{25}.
\]

In particular if \( E(K)[5^n] \neq \{O\} \) then \( E \) has non-CM. Moreover:

(1) if \( E(\mathbb{Q})[5^n] \simeq C_5 \), then \( G_E(5) \) is labeled 5B.1.1 or 5Cs.1.1;
(2) if \( E(\mathbb{Q})[5^n] \simeq C_5 \) and \( E(K)[5^n] \neq \{O\} \), then \( G_E(5) \) is labeled 5B.1.2;
(3) if \( E(K)[5^n] \simeq C_{25} \), then \( E(\mathbb{Q})[5^n] \simeq C_5 \). Moreover, \( K \) is Galois if \( G_E(5) \) is labeled 5B.1.1.

Proof. First suppose that \( E \) has CM. Then by the classification \( \Phi_{5,\mathbb{Q}}(5) \) we deduce that \( E(\mathbb{Q})[5^n] = \{O\} \). From now on we assume that \( E \) is non-CM. First, it is not possible \( E[5] \subseteq E(K) \) by Proposition 8 (2). Now, the characterization of \( \Phi(1) \) tells us that \( E(\mathbb{Q})[5^n] \lesssim C_5 \). We observe in Table 1 that \( d_v = 1 \) (resp. \( d_v = 5 \)) for some \( v \in (\mathbb{Z}/5\mathbb{Z})^2 \) of order 5 if and only if \( G_E(5) \) is labeled by 5Cs.1.1 or 5B.1.1 (resp. 5B.1.2), which proves (1) (resp. (2)). We are going to prove that \( E(K)[5^n] \lesssim C_{25} \). First, we prove (3). Assume that there exists a quintic number field \( K \) such that \( E(K)[25] = \langle P \rangle \simeq C_{25} \). Then \( G_E(25) \) satisfies:

\[
G_E(25) \equiv G_E(5) \pmod{5} \quad \text{and} \quad [G_E(25) : H_P] = 5.
\]

Note that in general we do not have an explicit description of \( G_E(25) \), but using Magma \[2\] we do a simulation with subgroups of \( GL_2(\mathbb{Z}/25\mathbb{Z}) \).

First assume that \( G_E(5) \) is labeled by 5B.1.2, then \( G_E(5) \) is conjugate in \( GL_2(\mathbb{Z}/5\mathbb{Z}) \) to the subgroup (cf. \[35\] Theorem 1.4 (iii))

\[
H_{5,1} = \left\langle \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\rangle \subseteq GL_2(\mathbb{Z}/5\mathbb{Z}).
\]

Since we do not have a characterization of \( G_E(25) \), we check using Magma that for any subgroup \( G \) of \( GL_2(\mathbb{Z}/25\mathbb{Z}) \) satisfying \( G \equiv H \pmod{5} \) for some conjugate \( H \) of \( H_{5,1} \) in \( GL_2(\mathbb{Z}/5\mathbb{Z}) \), and for any
v ∈ (\mathbb{Z}/25\mathbb{Z})^2 of order 25, we have \([G : G_v] \neq 5\) (where \(G_v\) be the stabilizer of \(v\) by the action of \(G\) on \((\mathbb{Z}/25\mathbb{Z})^2\)). Therefore for any point \(P ∈ E[25]\) it has \([G_E(25) : H_P] \neq 5\). In particular this proves that if \(G_E(5)\) is labeled by 5B.1.2, then there is not \(5^n\)-torsion over a quintic number field, for \(n > 1\). This finishes the first part of (3).

Now assume that \(G_E(5)\) is labeled by 5B.1.1. That is, \(G_E(5)\) is conjugate in \(GL_2(\mathbb{Z}/5\mathbb{Z})\) to the subgroup (cf. 35 Theorem 1.4 (iii))

\[
H_{6,1} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\} \subset GL_2(\mathbb{Z}/5\mathbb{Z}).
\]

A similar argument as the one used before, we check that for any subgroup \(G\) of \(GL_2(\mathbb{Z}/25\mathbb{Z})\) satisfying \(G \equiv H\) (mod 5) for some conjugate \(H\) of \(H_{6,1}\) in \(GL_2(\mathbb{Z}/5\mathbb{Z})\), and for any \(v ∈ (\mathbb{Z}/25\mathbb{Z})^2\) of order 25 such that \([G : G_v] = 5\) we have that \(G/N_G(G_v) \cong C_5\). Therefore we have deduced that if \(E/\mathbb{Q}\) is an elliptic curve such that \(G_E(5)\) is labeled by 5B.1.1 and there exists a quintic number field \(K\) with a \(K\)-rational point of order 25, then \(K\) is Galois. Note that in this case there does not exist a point of order \(5^n\) for \(n > 2\) over any quintic number field: suppose that \(K'\) is a quintic number field such that there exists \(P ∈ E(K'[5^n])\). Then \([5^{n-2}]P ∈ E(K'[25])\). Therefore \(K'\) is Galois and, by Lemma 7, \(E\) has a rational \(5^n\)-isogeny. In contradition with Theorem 6. This completes the proof of (3).

Finally we assume that \(G_E(5)\) is labeled by 5Cs.1.1. That is, \(G_E(5)\) is conjugate in \(GL_2(\mathbb{Z}/5\mathbb{Z})\) to the subgroup (cf. 35 Theorem 1.4 (iii))

\[
H_{1,1} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \right\} \subset GL_2(\mathbb{Z}/5\mathbb{Z}).
\]

In this case using a similar algorithm as above we check that if there exists a quintic number field \(K\) such that \(E(K)[25] \cong C_{25}\) then \(K\) is Galois or the Galois closure of \(K\) in \(\overline{\mathbb{Q}}\) is isomorphic to \(F_5\), where \(F_5\) denotes the Fröbenius group of order 20. In the former case, this proves that there does not exist a point of order \(5^n\) for \(n > 2\) over any Galois quintic number field. Now, assume that \(K\) is not Galois, then \(G_E(125)\) satisfies:

\[
G_E(125) \equiv G_E(5) \pmod{5}, \quad [G_E(125) : H_P] = 5,
\]

\[
G_E(125) \equiv G_E(25) \pmod{25}, \quad [G_E(25) : H_{5P}] = 5.
\]

We check that for any subgroup \(G\) of \(GL_2(\mathbb{Z}/125\mathbb{Z})\) satisfying \(G \equiv H\) (mod 5) for some conjugate \(H\) of \(H_{1,1}\) in \(GL_2(\mathbb{Z}/52\mathbb{Z})\), and for any \(v ∈ (\mathbb{Z}/125\mathbb{Z})^2\) of order 125 such that \([G : G_v] = 5\) and \(G/N_G(G_v) \cong F_5\) we obtain that \([G' : G'_w] \neq 5\) for any \(w ∈ (\mathbb{Z}/25\mathbb{Z})^2\) of order 25; where \(G' \equiv G\) (mod 25). We deduce that there do not exist points of order 125 over quintic number fields. So, \(E(K)[5^{\infty}] \lesssim C_{25}\).

This finishes the proof.

\[\square\]

4.3. 11-primary torsion subgroup.

**Lemma 11.** Let \(E/\mathbb{Q}\) be an elliptic curve and \(K/\mathbb{Q}\) a quintic number field. Then \(E(K)[11^{\infty}] \lesssim C_{11}\).

In particular, if \(E(K)[11^{\infty}] \neq \{O\}\) then \(E\) is labeled 121a2, 121c2, or 121b1, \(K = \mathbb{Q}(\zeta_{11})^+\) and \(E(K)_{\text{tors}} \simeq C_{11}\).
Proof. First, suppose that \( E/\mathbb{Q} \) is non-CM. Then Table 1 shows that there exists a point of order 11 over a quintic number field if and only if \( G_E(11) \) is labeled 11B.1.4 or 11B.1.5. Or in Zywina notation, \( G_E(11) \) is conjugate in \( \text{GL}_2(\mathbb{Z}/11\mathbb{Z}) \) to the subgroups \( H_{1,1} \) or \( H_{2,1} \). Then Zywina [35, Theorem 1.6(v)] proved that \( E \) is isomorphic (over \( \mathbb{Q} \)) to 121a2 or 121c2 respectively.

Now, let us suppose that \( E/\mathbb{Q} \) has CM. Recall that there are thirteen \( \mathbb{Q} \)-isomorphic classes of elliptic curve with CM (cf. [33, A §3]), each of them has CM by an order in the imaginary quadratic field with discriminant \( -D \), where \( D \in \{3, 4, 7, 8, 11, 19, 43, 67, 163\} \). In this context, Zywina [35, §1.9] gives a complete characterization of the conjugacy class of \( G_E(p) \) in \( \text{GL}_2(\mathbb{Z}/p\mathbb{Z}) \), for any prime \( p \). Let us apply these results for the case \( p = 11 \). The proof splits on whether \( j(E) \neq 0 \) (Proposition 1.14 [35]) or \( j(E) = 0 \) (Proposition 1.16 (iv) [35]):

\begin{itemize}
  \item \( j(E) \neq 0 \). Depending whether \( -D \) is a quadratic residue modulo 11:
    \begin{itemize}
      \item if \( D \in \{7, 8, 19, 43\} \) then \( G_E(11) \) is conjugate to 11Ns.
      \item if \( D \in \{3, 4, 6, 7, 163\} \) then \( G_E(11) \) is conjugate to 11Nn.
      \item if \( D = 11 \):
        \begin{itemize}
          \item if \( E \) is 121b1 then \( G_E(11) \) is conjugate to 11B.1.3.
          \item if \( E \) is 121b2 then \( G_E(11) \) is conjugate to 11B.1.8.
          \item otherwise \( G_E(11) \) is conjugate to 11B.10.3.
        \end{itemize}
    \end{itemize}
  \item \( j(E) = 0 \). Then \( G_E(11) \) is conjugate to 11Nn.1.4 or 11Nn.
\end{itemize}

The following table lists for each possible \( G_E(11) \) as above, the value \( d_1 \), the minimum of the indexes of the stabilizers of \( v \in (\mathbb{Z}/11\mathbb{Z})^2 \), \( v \neq (0,0) \), by the action of \( G_E(11) \) on \( (\mathbb{Z}/11\mathbb{Z})^2 \); equivalently, the minimum degree of the extension \( L/\mathbb{Q} \) over which \( E \) has a \( L \)-rational point of order 11.

| 11Ns | 11Nn | 11B.1.3 | 11B.1.8 | 11B.10.3 | 11Nn.1.4 |
|------|------|---------|---------|----------|----------|
| 20   | 120  | 5       | 10      | 10       | 40       |

The above table proves that \( E/\mathbb{Q} \) has a point of order 11 over a quintic number fields if and only if \( E \) is the curve 121b1.

Finally, Table 3 shows that the torsion of the elliptic curves 121a2, 121c2 and 121b1 grows in a quintic number field to \( \mathcal{C}_{11} \) only over the field \( \mathbb{Q}(\zeta_{11})^+ \), and over that field the torsion is \( \mathcal{C}_1 \).

\[ \square \]

Remark. If in the above statement the quintic number field is replaced by a number field \( K \) of degree \( d \) such that \( d \neq 5 \) and \( d \leq 9 \), then there does not exist any elliptic curve \( E/\mathbb{Q} \) with a point of order 11 over \( K \).

4.4. \( \{p,q\} \)-primary torsion subgroup.

Lemma 12. Let \( E/\mathbb{Q} \) be an elliptic curve and \( K/\mathbb{Q} \) a quintic number field. Let \( p,q \in \{2, 3, 5, 7, 11\} \), \( p \neq q \), such that \( pq \) divides the order of \( E(K)_{\text{tors}} \). Then

\[ E(\mathbb{Q})\{\{\{p,q\}^\infty\} \} = E(K)\{\{p,q\}^\infty\} \quad \text{or} \quad E(K)\{\{p,q\}^\infty\} \cong \mathcal{C}_{10}, \]

In the former case, \( E(\mathbb{Q})_{\text{tors}} = E(\mathbb{Q})\{\{p,q\}^\infty\} \cong G \), where \( G \in \{\mathcal{C}_5, \mathcal{C}_{10}, \mathcal{C}_5 \times \mathcal{C}_5\} \).

Proof. First we may suppose \( p \neq 11 \) by Lemma 11. Assume that \( p,q \in \{2, 3, 7\} \), then by Lemma 9 we have that the \( \{p,q\} \)-primary torsion is defined over \( \mathbb{Q} \). That is, \( E(K)\{\{p,q\}^\infty\} = E(\mathbb{Q})\{\{p,q\}^\infty\} \). Let \( G \in \Phi(1) \) such that \( E(\mathbb{Q})_{\text{tors}} \cong G \). Then \( G \in \{\mathcal{C}_5, \mathcal{C}_2 \times \mathcal{C}_5\} \).

It remains to prove the case \( p = 5 \) and \( q \in \{2, 3, 7\} \). Without loss of generality we can assume that the 5-primary torsion is not defined over \( \mathbb{Q} \), otherwise \( E(K)\{\{5,q\}^\infty\} = E(\mathbb{Q})\{\{5,q\}^\infty\} \) and the unique possibility is \( \mathcal{C}_{10} \). In particular, by Lemma 10 we have that \( E \) has non-CM and the 5-primary torsion of \( E \) over \( K \) is cyclic of order 5 or 25, and \( E(\mathbb{Q})[5^\infty] = \{O\} \) or \( E(\mathbb{Q})[5^\infty] \cong \mathcal{C}_5 \) respectively. Depending on \( q \in \{2, 3, 7\} \) we have:

\begin{itemize}
  \item \( q = 2 \):
\end{itemize}
\* \(E(K)[5^\infty] \simeq C_5\). If \(E(K)[2^\infty] \simeq C_2\) then there are infinitely many elliptic curves such that \(E(K)[\{2, 5\}^\infty] \simeq C_{10}\) (see Proposition 15). In fact, the above 2-primary torsion is the unique possibility since if \(C_4 \subseteq E(Q)\) then \(C_{20} \not\subseteq E(K)\) and if \(E[2] \subseteq E(Q)\) then \(C_2 \times C_{10} \not\subseteq E(K)\) (see Remark below Theorem 7 of 10).

\* \(E(K)[5^\infty] \simeq C_{25}\). Assume that \(E(K)[2] \neq \{O\}\). If \(G_E(5)\) is labeled 5B.1.1 then \(K\) is Galois and therefore, by Lemma 7, \(E\) has a rational 50-isogeny, that is not possible by Theorem 6. Now suppose that \(G_E(5)\) is labeled 5Cs.1.1. Since \(E(K)[2^\infty] = E(Q)[2^\infty]\) and \(E(Q(\zeta_5)) = E[5]\) (by Table 1) we deduce \(C_5 \times C_{10} \subseteq E(Q(\zeta_5))\). But this is not possible since Bruin and Najman 3. Theorem 6] have proved that any elliptic curve defined over \(Q(\zeta_5)\) have torsion subgroup isomorphic to a group in the following set

\[\Phi(\zeta_5) = \{C_n \mid n = 1, \ldots, 10, 12, 15, 16\} \cup \{C_2 \times C_{2m} \mid m = 1, \ldots, 4\} \cup \{C_5 \times C_5\}\]

\(\star\) A remarkable fact is that this genus 2 curve is new modular of level 45 (see 9).

- \(q = 3\): A necessary condition if 15 divides \(E(K)_{\text{tors}}\) is that the 5-torsion is not defined over \(Q\) and the 3-torsion is defined over \(Q\). By Lemma 10, \(G_E(5)\) is labeled 5B.1.2. Zywina [35, Theorem 1.4] has showed that its \(j\)-invariant is of the form

\[J_5(t) = \frac{(t^4 + 228t^3 + 494t^2 - 228t + 1)^3}{t(t^2 - 11t - 1)^5}, \quad \text{for some } t \in Q.\]

On the other hand, we have proved that the 3-torsion is defined over \(Q\). Then, by Table 1 \(G_E(3)\) is labeled 3Cs.1.1 or 3B.1.1. Again Zywina [35, Theorem 1.2] characterizes the \(j\)-invariant of \(E/Q\) depending on the conjugacy class of \(G_E(3)\):

- \(3Cs.1.1\): \(J_1(s) = 27(s + 1)^3(s + 3)^3(s^2 + 3)^3 \quad \text{for some } s \in Q\). We must have an equality of \(j\)-invariants: \(J_1(s) = J_5(t)\). In particular, grouping cubes we deduce:

\[t(t^2 - 11t - 1)^2 = r^3, \quad \text{for some } t, r \in Q.\]

This equation defines a curve \(C\) of genus 2, which in fact transforms (according to Magma) to \(C'\): \(y^2 = x^6 + 22x^3 + 125\). The Jacobian of \(C'\) has rank 0, so we can use the Chabauty method, and determine that the points on \(C'\) are

\[C'(Q) = \{(1 : \pm 1 : 0)\}\]

Therefore \(C'\) has no affine points and we obtain

\[C(Q) = \{(0, 0)\} \cup \{(1 : 0 : 0)\}\]

Then \(t = 0\), and since \(t\) divides the denominator of \(J_5(t)\) we have reached a contradiction to the existence of such curve \(E\).

- \(3B.1.1\): \(J_3(s) = 27(s + 1)(s + 9)^3 \quad \text{for some } s \in Q\). A similar argument with the equality \(J_3(s) = J_5(t)\) gives us the equation:

\[C : 27(s + 1)(s + 9)^3t(t^2 - 11t - 1)^5 = s^3(t^4 + 228t^3 + 494t^2 - 228t + 1)^3.\]

In this case the above equation defines a genus 1 curve which has the following points:

\[\{(-2/27, -1/8), (-27/2, -2), (-27/2, 1/2), (0, 0), (-2/27, 8)\} \cup \{(0 : 1 : 0), (1/27 : 1 : 0), (1 : 0 : 0)\}\]

The curve \(C\) is \(Q\)-isomorphic to the elliptic curve 15a3, which Mordell-Weil group (over \(Q\)) is of order 8. Therefore we deduce that \(s = -2/27, -27/2, \) and in particular

\[j(E) \in \{-5^2/2, -5^2 \cdot 241^3/2^3\}\]
Therefore there are two $\mathbb{Q}$-isomorphic classes of elliptic curves. Each pair of elliptic curves in the same $\mathbb{Q}$-isomorphic class is related by a quadratic twist. Najman [28] has made an exhaustive study of how the torsion subgroup changes upon quadratic twists. In particular Proposition 1 (c) [28] asserts that if $E/\mathbb{Q}$ is neither 50a3 nor 450b4, and it satisfies $E(\mathbb{Q})_{\text{tors}} \simeq C_3$ and the $(-3)$-quadratic twist $E^{-3}$, satisfies $E^{-3}(\mathbb{Q})_{\text{tors}} \not\simeq C_3$, then for any quadratic twist we must have $E^d(\mathbb{Q}) \simeq C_1$ for all $d \in \mathbb{Q}^*/(\mathbb{Q}^*)^2$. We apply this result to the elliptic curves 50a1 and 450b2 that have $j$-invariant $-5^2/2$ and $-5^2 \cdot 2413/9^3$ respectively. Both curves have cyclic torsion subgroup (over $\mathbb{Q}$) of order 3 and the corresponding torsion subgroup of the $(-3)$-quadratic twist is trivial. Thus we are left with two elliptic curves (50a1 and 450b2) to finish the proof. Applying the algorithm described in Section 7 we compute that the 5-torsion does not grow over any quintic number field for both curves.

$q = 7$. Similar to the the case $q = 3$, we deduce that $E/\mathbb{Q}$ has the 7-torsion defined over $\mathbb{Q}$ and $G_E(5)$ is labeled 5B.1.2. Looking at Table 1 we deduce that $E/\mathbb{Q}$ has a rational 5-isogeny, since $d_0 = 1$ for 5B.1.2. Then, since $E/\mathbb{Q}$ has a point of order 7 defined over $\mathbb{Q}$, there exists a rational 35-isogeny, which contradicts Theorem 6.

4.5. \{p,q,r\}-primary torsion subgroup.

Lemma 13. Let $E/\mathbb{Q}$ be an elliptic curve and $K/\mathbb{Q}$ a quintic number field. Let $p, q, r \in \{2, 3, 5, 7, 11\}$, $p \neq q \neq r$, such that $pqr$ divides the order of $E(K)_{\text{tors}}$. Then $E(K)[\{p,q,r\}^\infty] = \{O\}$.

Proof. Lemma [12] shows that there do not exist three different primes $p, q, r$ such that $pqr$ divides the order of $E(K)_{\text{tors}}$.

5. Proof of Theorems 1, 2 and 3

We are ready to prove Theorems 1, 2 and 3.

Proof of Theorem 1. Since we have $\Phi_{\mathbb{Q}}(1) \subseteq \Phi_{\mathbb{Q}}(5)$, let us prove that the unique torsion structures that remain to add to $\Phi_{\mathbb{Q}}(1)$ to obtain $\Phi_{\mathbb{Q}}(5)$ are $C_{11}$ and $C_{25}$. Let $H \in \Phi_{\mathbb{Q}}(5)$ be such that $H \not\in \Phi_{\mathbb{Q}}(1)$. Lemma [12] shows that $|H| = p^n$, for some prime $p$ and a positive integer $n$. Now, Lemma 9 shows that $p \in \{5, 11\}$. If $p = 11$ then $n = 1$ by Lemma 11. If $p = 5$ then $n = 2$ by Lemma 10 and an example with torsion subgroup isomorphic to $C_{25}$ is given in Table 3. This finish the proof for the set $\Phi_{\mathbb{Q}}(5)$.

Now the CM case. Notice that $\Phi_{\mathbb{Q}}^{\text{CM}}(1) \subseteq \Phi_{\mathbb{Q}}^{\text{CM}}(5) \subseteq \Phi_{\mathbb{Q}}^{\text{CM}}(5)$. We have that the unique torsion structure that belongs to $\Phi_{\mathbb{Q}}^{\text{CM}}(5)$ and not to $\Phi_{\mathbb{Q}}^{\text{CM}}(1)$ is $C_{11}$. But in Lemma 11 we have proved that the elliptic curve 121b1 has torsion subgroup isomorphic to $C_{11}$ over $\mathbb{Q}(\zeta_{11})^+$. Therefore $\Phi_{\mathbb{Q}}^{\text{CM}}(5) = \Phi_{\mathbb{Q}}^{\text{CM}}(5)$. This finishes the proof.

The determination of $\Phi_{\mathbb{Q}}(5, G)$ will rest on the following result:

Proposition 14. Let $E/\mathbb{Q}$ be an elliptic curve and $K/\mathbb{Q}$ a quintic number field such that $E(\mathbb{Q})_{\text{tors}} \simeq G$ and $E(K)_{\text{tors}} \simeq H$.

1. Let $p \in \{2, 3, 7\}$ and $G$ of order a power of $p$, then $H = G$.
2. If $H = C_{25}$, then $G = C_5$.

Proof. The item 1 follows from Lemma 9 and 2 from Lemma 10 (3).

Proof of Theorem 3. Let $E/\mathbb{Q}$ be an elliptic curve and $K/\mathbb{Q}$ a quintic number field such that

$E(\mathbb{Q})_{\text{tors}} \simeq G$ and $E(K)_{\text{tors}} \simeq H$.

The group $H \in \Phi_{\mathbb{Q}}(5)$ (row in Table 2) that does not appear in some $\Phi_{\mathbb{Q}}(5, G)$ for any $G \in \Phi(1)$ (column in Table 2), with $G \subseteq H$ can be ruled out using Proposition 14. In Table 2 we use:
• (1) and (2) to indicate which part of Proposition 14 is used,
• the symbol − to mean the case is ruled out because $G \nsubseteq H$,
• with a ✓, if the case is possible and, in fact, it occurs. There are two types of check marks in Table 2
  - ✓ (without a subindex) means that $G = H$.
  - ✓ means that $H \neq G$ can be achieved over a quintic number field $K$, and we have collected examples of curves and quintic number fields in Table 2.

| $H$ | $G$ | $C_1$ | $C_2$ | $C_3$ | $C_4$ | $C_5$ | $C_6$ | $C_7$ | $C_8$ | $C_{10}$ | $C_2 \times C_2$ | $C_2 \times C_4$ | $C_2 \times C_6$ | $C_2 \times C_8$
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| $C_1$ | ✓ | - | - | - | - | - | - | - | - | - | - | - | - |
| $C_2$ | ✓ | - | - | - | - | - | - | - | - | - | - | - | - |
| $C_3$ | - | ✓ | - | - | - | - | - | - | - | - | - | - | - |
| $C_4$ | ✓ | ✓ | - | - | - | - | - | - | - | - | - | - | - |
| $C_5$ | ✓ | - | - | - | - | - | - | - | - | - | - | - | - |
| $C_6$ | ✓ | ✓ | - | - | - | - | - | - | - | - | - | - | - |
| $C_7$ | - | ✓ | - | - | - | - | - | - | - | - | - | - | - |
| $C_8$ | ✓ | ✓ | - | - | - | - | - | - | - | - | - | - | - |
| $C_9$ | - | ✓ | - | - | - | - | - | - | - | - | - | - | - |
| $C_{10}$ | ✓ | ✓ | - | - | - | - | - | - | - | - | - | - | - |
| $C_{11}$ | ✓ | ✓ | - | - | - | - | - | - | - | - | - | - | - |
| $C_{25}$ | ✓ | ✓ | - | - | - | - | - | - | - | - | - | - | - |
| $C_2 \times C_2$ | ✓ | ✓ | - | - | - | - | - | - | - | - | - | - | - |
| $C_2 \times C_4$ | ✓ | ✓ | - | - | - | - | - | - | - | - | - | - | - |
| $C_2 \times C_6$ | ✓ | ✓ | - | - | - | - | - | - | - | - | - | - | - |
| $C_2 \times C_8$ | ✓ | ✓ | - | - | - | - | - | - | - | - | - | - | - |

Table 2. The table displays either if the case happens for $G = H$ (✓), if it occurs over a quintic (✓), if it is impossible because $G \nsubseteq H$ (−) or if it is ruled out by Proposition 14 (1) and (2).

It remains to prove that there are infinitely many $\mathbb{Q}$-isomorphism classes of elliptic curves $E/\mathbb{Q}$ with $H \in \Phi_\mathbb{Q}(5, G)$, except for the case $H = C_{11}$. Note that for any elliptic curve $E/\mathbb{Q}$ with $E(\mathbb{Q})_{\text{tors}}$, there is always an extension $K/\mathbb{Q}$ of degree 5 such that $E(K)_{\text{tors}} = E(\mathbb{Q})_{\text{tors}}$. Then for any $G \in \Phi(1) \cap \Phi_\mathbb{Q}(5)$ the statement is proved. Now, since $\Phi_\mathbb{Q}(5) \setminus \Phi(1) = \{C_{11}, C_{25}\}$, the only case that remains to prove is $H = C_{25}$. This case will be proved in Proposition 16.

Proof of Theorem 3. Let $E/\mathbb{Q}$ be an elliptic curve such that the torsion grows to $C_{11}$ over a quintic number field $K$. Then by Lemma 11 we know that $K = \mathbb{Q}(\zeta_{11})^+$ and the torsion does not grow for any other quintic number field. Therefore to finish the proof it remains to prove that there does not exist an elliptic curve $E/\mathbb{Q}$ and two non-isomorphic quintic number fields $K_1, K_2$ such that $E(K_i)_{\text{tors}} \simeq H \in \Phi_\mathbb{Q}(5)$, $i = 1, 2$, and $E(\mathbb{Q})_{\text{tors}} \nsubseteq H$. Note that the compositum $K_1K_2$ satisfies $[K_1K_2 : \mathbb{Q}] \leq [K_1 : \mathbb{Q}][K_2 : \mathbb{Q}] = 25$. Now, by Theorem 2 we deduce $H \in \{C_5, C_{10}, C_{25}\}$:

• First suppose that $H \in \{C_5, C_{10}\}$. Then by Lemma 10 $G_E(5)$ is labeled 5B.1.2. Now, since $K_1 \nsubseteq K_2$ we deduce $K_1K_2 = \mathbb{Q}(E[5])$ and, in particular, $\text{Gal}(K_1K_2/\mathbb{Q}) \simeq G_E(5)$. In this case we have that $G_E(5) \simeq F_5$, where $F_5$ denotes the Fröhrenius group of order 20. Diagram 1 shows the
lattice subgroup of $\mathcal{F}_5$, where $\mathcal{H}_{k,i}$ denotes the $k$-th subgroup of index $i$ in $\mathcal{F}_5$. Note that all the index 5 subgroups $\mathcal{H}_{k,5}$ are conjugates in $\mathcal{F}_5$. That is, their associated fixed quintic number fields are isomorphic. This proves that $K_1 \simeq K_2$.

\begin{center}
\textbf{Diagram 1. Lattice subgroup of $\mathcal{F}_5$}
\end{center}

- Finally suppose that $H = C_{25}$. In this case we use a similar argument as above but replacing $G_E(5)$ by $G_E(25)$. We know by Lemma 10 that $G_E(5)$ is labeled 5B.1.1 or 5Cs.1.1, but we do not have an explicit description of $G_E(25)$. For that reason we apply an analogous algorithm as the one used in the proof of Lemma 10. By [35, Theorem 1.4 (iii)] we have that $G_E(5)$ is conjugate in $\text{GL}_2(\mathbb{Z}/5\mathbb{Z})$ to

\[ H_{6,1} = \langle \left( \begin{array}{cc} 1 & 0 \\ 0 & 2 \end{array} \right), \left( \begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array} \right) \rangle \text{ or } H_{1,1} = \langle \left( \begin{array}{cc} 1 & 0 \\ 0 & 2 \end{array} \right) \rangle, \]

depending if $G_E(5)$ is labeled 5B.1.1 or 5Cs.1.1 respectively.

Suppose that $K_1 \not\simeq K_2$, then $K_1 K_2 = \mathbb{Q}(E[25])$. Therefore $\text{Gal}(K_1 K_2/\mathbb{Q}) \simeq G_E(25)$ and $|G_E(25)| \leq 25$. Now, we fix $\mathcal{H}$ to be $H_{6,1}$ or $H_{1,1}$ and since we do not have an explicit description of $G_E(25)$ we run a Magma program where the input is a subgroup $G$ of $\text{GL}_2(\mathbb{Z}/25\mathbb{Z})$ satisfying

- $|G| \leq 25$,
- $G \equiv H \pmod{5}$ for some conjugate $H$ of $\mathcal{H}$ in $\text{GL}_2(\mathbb{Z}/5\mathbb{Z})$,
- there exists $v \in (\mathbb{Z}/25\mathbb{Z})^2$ of order 25 such that $[G : G_v] = 5$.

If $\mathcal{H} = H_{6,1}$ the above algorithm does not return any subgroup $G$. In the case $\mathcal{H} = H_{1,1}$ all the subgroups returned are isomorphic either to $\mathcal{F}_5$ or to $C_{20}$. If $G \simeq \mathcal{F}_5$ then we have proved that it has five index 5 subgroups, all of them at the same conjugation class. If $G \simeq C_{20}$ there is only one subgroup of index 5. We have reached a contradiction with $K_1 \not\simeq K_2$. This finishes the proof. $\square$

6. INFINITE FAMILIES OF RATIONAL ELLIPTIC CURVES WHERE THE TORSION GROWS OVER A QUINTIC NUMBER FIELD.

Let $E/\mathbb{Q}$ be an elliptic curve and $K$ a quintic number field such that $E(\mathbb{Q})_{\text{tors}} \simeq G \in \Phi(1)$ and $E(K)_{\text{tors}} \simeq H \in \Phi_5(5)$. Theorem 3 shows that $G \not\simeq H$ in the following cases:

\[ (G, H) \in \{ (C_1, C_5), (C_1, C_{11}), (C_2, C_{10}), (C_5, C_{25}) \}. \]
By Lemma [11] we have that the pair \((C_1, C_{11})\) only occurs in three elliptic curves. For the rest of the above pairs we are going to prove that there are infinitely many non-isomorphic classes of elliptic curves and quintic number fields satisfying each pair.

6.1. \((C_1, C_5)\) and \((C_2, C_{10})\). Let \(E/\mathbb{Q}\) be an elliptic curve and \(K\) a quintic number field such that \(E(\mathbb{Q})[5] = \{0\}\) and \(E(K)[5] \cong C_5\). Then Theorem [2] tells us that:

\[ E(\mathbb{Q})_{\text{tors}} \cong C_1 \text{ and } E(K)_{\text{tors}} \cong C_5, \quad \text{or} \quad E(\mathbb{Q})_{\text{tors}} \cong C_2 \text{ and } E(K)_{\text{tors}} \cong C_{10}. \]

First notice that \(E\) has non-CM, since \(C_5\) is not a subgroup of any group in \(\Phi_{\text{CM}}(5)\). Then Lemma [10] shows that \(G_E(5)\) is labeled 5B.1.2 (\(H_{5,1}\) in Zywina’s notation). Then Zywina [35] Theorem 1.4(iii) proved that there exists \(t \in \mathbb{Q}\) such that \(E\) is isomorphic (over \(\mathbb{Q}\)) to \(E_{5,t}\):

\[ E_{5,t} : y^2 = x^3 - 27(5t^4 + 228t^3 + 494t^2 - 228t + 1)x + 54(t^6 - 522t^5 - 10005t^4 - 10005t^2 + 522t + 1). \]

Table [1] shows that the degree of the field of definition of a point of order 5 in \(E\) is 4 or 5. Moreover, we can compute explicitly the number fields factorizing the 5-division polynomial \(\psi_5(x)\) attached to \(E\). We define the following polynomial of degree 5:

\[ p_5(x) = x^5 + (15t^2 - 450t - 15)x^4 + (90t^4 - 65880t^3 + 22860t^2 + 11880t + 90)x^3 + (-270t^6 - 1015740t^5 - 7086690t^4 + 5725080t^3 - 4520610t^2 - 82620t - 270)x^2 + (405t^8 - 8874360t^7 - 58872420t^6 - 253721160t^5 - 1423822050t^4 + 637175160t^3 + 18109980t^2 + 223560t + 405)x - 243t^{10} - 2886626t^9 - 485812647t^8 + 3223702152t^7 - 3427289350t^6 - 21920572605t^5 + 53316735462t^4 - 2958964344t^3 - 74726631t^2 - 211410t - 243. \]

Then \(p_5(x)\) divides \(\psi_5(x)\) and we have \(E(\mathbb{Q}(\alpha))[5] = \langle R \rangle \cong C_5\), where \(p_5(\alpha) = 0\) and \(\alpha\) is the \(x\)-coordinate of \(R\).

Now suppose that \(E(\mathbb{Q})_{\text{tors}} \cong C_2\), then \(G_E(2)\) is labeled 2B. Then Zywina [35] Theorem 1.1 proved that its \(j\)-invariant is of the form

\[ J_2(s) = 256 \frac{(s + 1)^3}{s}, \quad \text{for some } s \in \mathbb{Q}. \]

Therefore we have \(J_2(s) = j(E_{5,t})\) for some \(s, t \in \mathbb{Q}\). In other words we have a solution of the next equation

\[ 256 \frac{(s + 1)^3}{s} = \frac{(t^4 + 228t^3 + 494t^2 - 228t + 1)^3}{t(t^2 - 11t - 1)^5}. \]

This equation defines a curve \(C\) of genus 0 with \((0, 0) \in C(\mathbb{Q})\), which can be parametrize according to Magma and making a linear change of the projective coordinate in order to simplify the parametrization by:

\[ (s, t) = \left( \frac{-512(5r + 1)(5r^2 - 1)^5}{(5r - 1)(5r + 3)(5r^2 + 10r + 1)^2}, \frac{2(5r + 3)^2}{(5r - 1)^2(5r + 1)} \right), \quad \text{where } r \in \mathbb{Q}. \]

Finally, replacing the above value for \(t\) in \(E_{5,t}\) and simplifying the Weierstrass equation we obtain:

\[ E_r : y^2 = x^3 - 2(5r^2 + 2r + 1)(5r^4 - 40r^3 - 30r^2 + 1)x^2 + 84375(5r - 1)(5r + 3)(5r^2 + 10r + 1)^5x. \]

Thus we have proved the following result:

**Proposition 15.** There exist infinitely many \(\overline{\mathbb{Q}}\)-isomorphic classes of elliptic curves \(E/\mathbb{Q}\) such that \(E(\mathbb{Q})_{\text{tors}} \cong C_1\) (resp. \(C_2\)) and infinitely many quintic number fields \(K\) such that \(E(K)_{\text{tors}} \cong C_5\) (resp. \(C_{10}\)).
6.2. $(C_5, \mathcal{C}_{25})$. Let $E/\mathbb{Q}$ be an elliptic curve such that $G_E(5)$ is labeled by $5B.1.1$ and there exists a quintic number field $K$ with the property $E(K)_{\text{tors}} \simeq \mathcal{C}_{25}$. Then, by Lemma \cite{1} (3), $K$ is Galois. In particular $E/\mathbb{Q}$ has a rational 25-isogeny. Then, we observe in \cite{22} Table 3 that its $j$-invariant must be of the form:

$$j_{25}(h) = \frac{(h^{10} + 10h^8 + 35h^6 - 12h^5 + 50h^4 - 60h^3 + 25h^2 - 60h + 16)^3}{(h^5 + 5h^3 + 5h - 11)^3}$$

for some $h \in \mathbb{Q}$.

On the other hand, Zywina \cite{35} Theorem 1.4(iii)] proved that there exists $s \in \mathbb{Q}$ such that $E$ is isomorphic (over $\mathbb{Q}$) to $\mathcal{E}_{6,s}$:

$$\mathcal{E}_{6,s} : y^2 = x^3 - 27(s^4 - 12s^3 + 14s^2 + 12s + 1)x + 54(s^6 - 18s^5 + 75s^4 + 75s^2 + 18s + 1).$$

The above $j$-invariants should be equal, so $j(\mathcal{E}_{6,s}) = j_{25}(h)$ for some $s, h \in \mathbb{Q}$. This equality defines a non-irreducible curve over $\mathbb{Q}$ whose irreducible components are a genus 1 curve and a genus 0 curve. It is possible to give a parametrization of the above genus 0 curve such that $s = t^5$, where $t \in \mathbb{Q}$. That is, there exists $t \in \mathbb{Q}$ such that $E$ is $\mathbb{Q}$-isomorphic to $\mathcal{E}_{6,t^5}$.

Now, let us define the quintic polynomial $p_{25}(x)$:

\begin{align*}
p_{25}(x) &= x^5 - (5t^{10} - 12t^8 - 12t^7 - 24t^6 + 30t^5 - 60t^4 + 36t^3 - 24t^2 + 12t - 5)x^4 \\
&\quad + (10t^{20} + 48t^{18} + 48t^{17} + 96t^{16} + 24t^{15} + 240t^{14} - 144t^{13} + 96t^{12} - 48t^{11} + 236t^{10} + 48t^9 + 48t^7 + 96t^6 \\
&\quad - 264t^5 + 240t^4 - 144t^3 + 96t^2 - 48t + 10)x^3 + (-10t^{30} - 72t^{28} - 72t^{27} - 144t^{26} - 252t^{25} - 360t^{24} \\
&\quad + 216t^{23} - 144t^{22} + 72t^{21} + 191t^{20} + 720t^{18} + 720t^{17} + 1440t^{16} - 1800t^{15} + 3600t^{14} - 2160t^{13} + 1440t^{12} \\
&\quad - 720t^{11} + 1914t^{10} - 72t^9 - 144t^8 + 612t^7 - 360t^6 + 216t^3 - 144t^2 + 72 - 10)x^2 \\
&\quad + (5t^{40} + 48t^{38} + 48t^{37} + 96t^{36} + 312t^{35} + 240t^{34} - 144t^{33} + 96t^{32} - 48t^{31} - 4516t^{30} - 1584t^{29} - 1584t^{28} \\
&\quad - 3168t^{26} + 19944t^{25} - 7920t^{24} + 4752t^{23} - 3168t^{22} + 1584t^{21} - 18114t^{20} - 1584t^{19} - 1584t^{18} - 3168t^{16} - 12024t^{15} - 7920t^{14} + 4752t^{13} - 3168t^{12} \\
&\quad + 1584t^{11} - 4516t^{10} - 48t^8 + 48t^7 + 96t^6 - 552t^5 + 240t^4 - 44t^3 \\
&\quad + 96t^2 - 48t + 5)x - (50 - 12t^{48} - 12t^{47} - 24t^{46} - 114t^{45} - 60t^{44} + 36t^{43} - 24t^{42} + 12t^{41} + 237t^{40} \\
&\quad + 816t^{38} + 816t^{37} - 1632t^{36} - 17880t^{35} + 4080t^{34} - 2448t^{33} + 1632t^{32} - 816t^{31} + 47294t^{30} - 13896t^{28} \\
&\quad - 13896t^{27} - 27792t^{26} + 34740t^{25} - 69480t^{24} + 41688t^{23} - 27792t^{22} + 13896t^{21} + 47294t^{20} + 816t^{18} + 816t^{17} + 1632t^{16} + 13800t^{15} + 4080t^{14} - 2448t^{13} + 1632t^{12} - 816t^{11} + 237t^{10} - 12t^8 - 24t^6 \\
&\quad + 174t^5 - 60t^4 + 36t^3 - 24t^2 + 12t - 1.
\end{align*}

Then $p_{25}(x)$ divides the 25-division polynomial of $\mathcal{E}_{6,t^5}$. Fixing $t \in \mathbb{Q}$, we have that $Q(\alpha)/Q$ is a Galois extension of degree 5 and $E(Q(\alpha)) = \langle R \rangle \simeq \mathcal{C}_{25}$, where $p_{25}(\alpha) = 0$ and the $x$-coordinate of $R$ is $3\alpha$. Note that $[5]R = (3s^{10} - 18t^5 + 3, 108t^5) \in E(\mathbb{Q})$.

We have proved the following result:

**Proposition 16.** There exist infinitely many $\mathbb{Q}$-isomorphic classes of elliptic curves $E/\mathbb{Q}$ and infinitely many quintic number fields $K$ such that $E(K)_{\text{tors}} \simeq \mathcal{C}_{25}$. All of them satisfy $E(\mathbb{Q})_{\text{tors}} \simeq \mathcal{C}_5$. 

6.2.1. A 5-triangle tale. Let $E/\mathbb{Q}$ be an elliptic curve such that $G_E(5)$ is labeled by $5CS\cdot 1.1$ (H$_{1,1}$ in Zywina’s notation). Zywina \cite{35} Theorem 1.4(iii)] proved that there exists $t \in \mathbb{Q}$ such that $E$ is isomorphic (over $\mathbb{Q}$) to $\mathcal{E}_{6,t} = \mathcal{E}_{5,t^5}$. We observe in Table \cite{1} that there exists a $\mathbb{Z}/5\mathbb{Z}$-basis $\{P_1, P_2\}$ of $E[5]$ such that $E(\mathbb{Q})_{\text{tors}} = \langle P_2 \rangle \simeq \mathcal{C}_5$, $E(\mathbb{Q}(\zeta_5))_{\text{tors}} = E[5] = \langle P_1, P_2 \rangle$. Now, since $\langle P_1 \rangle$ and $\langle P_2 \rangle$ are distinct $\text{Gal}((\mathbb{Q}/\mathbb{Q})$-stable cyclic subgroups of $E(\mathbb{Q})$ of order 5, there exist two rational 5-isogenies:

\[
\begin{array}{ccc}
\phi_1 & E & \phi_2 \\
E_1 & \begin{array}{c} \rightarrow \\ \rightarrow \end{array} & E_2,
\end{array}
\]
where the elliptic curves \( E_1 = E/(P_1) \) and \( E_2 = E/(P_2) \) are defined over \( \mathbb{Q} \). Using Vellu’s formulæ we can compute explicit equations of these elliptic curves:

\[
E_1 = \mathcal{E}_{6,s(t)}, \quad E_2 = \mathcal{E}_{5,s(t)}, \quad \text{where } s(t) = \frac{t(t^4 + 3t^3 + 4t^2 + 2t + 1)}{t^4 - 2t^3 + 4t^2 - 3t + 1}.
\]

Then we have \( G_{E_1}(5) \) is labeled by 5B.1.1 and \( G_{E_2}(5) \) is labeled by 5B.1.2. We observe that the elliptic curve \( E_1 \) is the one obtained in the previous section, that is, \( E_1(\mathbb{Q}(\alpha)) = (R) \simeq \mathbb{C}_{25} \), where \( p_{25}(\alpha) = 0 \) and the \( x \)-coordinate of \( R \) is \( 3\alpha \). In particular, \( E_1 \) has a rational 25-isogeny. Note that \([5]R = Q_2 = (3t^{10} - 18t^5 + 3, 108t^5)\) is such that \( E_1(\mathbb{Q})[5] = \langle Q_2 \rangle \simeq \mathbb{C}_5 \) and \( E_1(L)[5] = E_1[5] = \langle Q_1, Q_2 \rangle \) with \( [L : \mathbb{Q}] = 20 \). If \( \phi_1 : E_1 \rightarrow E \) denotes the dual isogeny of \( \phi_1 \), then we have \( \phi_2 \circ \phi_1((R)) = 0 \in E_2 \). That is, \( \phi_2 \circ \phi_1, : E_2 \rightarrow E_1 \) is a rational 25-isogeny.

**Remark.** [There are] only seven elliptic curves (11a1, 550k2, 1342c2, 33825be2, 165066d2, 185163a2 and 192698c2) with conductor less than 350,000 such that the corresponding mod 5 Galois representation is labeled 5Cs.1.1. All of them give the corresponding 5-triangle with the associated elliptic curve (11a3, 550k3, 1342c1, 33825be3, 165066d1, 185163a1 and 192698c1 resp.) with \( \mathbb{C}_{25} \) torsion over the corresponding quintic number field. Notice that there are no more elliptic curves with conductor less than 350,000 and torsion isomorphic to \( \mathbb{C}_{25} \) over a quintic number field.

### 7. Examples

Given an elliptic curve \( E/\mathbb{Q} \), we describe a method to compute the quintic number field where the torsion could grow. If \( E \) is 121a2, 121c2 or 121b1 we have proved in Lemma 11 that the torsion grows to \( \mathbb{C}_{11} \) over the quintic number field \( \mathbb{Q}(\zeta_{11})^+ \). For the rest of the elliptic curves, we first compute \( E(\mathbb{Q})_{\text{tors}} \simeq G \in \Phi(1) \). If \( G \neq C_1, C_2, C_5 \), then by Theorem 2 the torsion remains stable under any quintic extension. If \( G = C_1 \) or \( C_2 \) then, by Theorem 2 the torsion could grow to \( C_5 \) or \( C_{10} \) respectively. Now compute the 5-division polynomial \( \psi_5(x) \). It follows that the quintic number fields where the torsion could grow are contained in the number fields attached to the degree 5 factors of \( \psi_5(x) \). In the case \( G = C_5 \) the torsion could grow to \( \mathbb{C}_{25} \), and the method is similar, replacing the 5-division polynomial by the 25-division polynomial. We explain this method with an example.

**Example.** Let \( E \) be the elliptic curve 11a2. We compute \( E(\mathbb{Q})_{\text{tors}} \simeq C_1 \). Now, the 5-division polynomial has two degree 5 irreducible factors: \( p_1(x) \) and \( p_2(x) \). Let \( \alpha_i \in \mathbb{Q} \) such that \( p_1(\alpha_i) = 0 \), \( i = 1, 2 \). We deduce \( \mathbb{Q}(\sqrt[5]{T_1}) = \mathbb{Q}(\alpha_1) = \mathbb{Q}(\alpha_2) \) and \( E(\mathbb{Q}(\sqrt[5]{T_1}))_{\text{tors}} \simeq C_5 \).

Table 3 shows examples where the torsion grows over a quintic number field. Each row shows the label of an elliptic curve \( E/\mathbb{Q} \) such that \( E(\mathbb{Q})_{\text{tors}} \simeq G \), in the first column, and \( E(\mathbb{K})_{\text{tors}} \simeq H \), in the second column, and the quintic number field \( K \) in the third column.

| \( G \)  | \( H \)  | Quintic  | Label  |
|---------|---------|----------|--------|
| \( C_1 \) | \( C_5 \) | \( \mathbb{Q}(\sqrt[5]{T_1}) \) | 11a2   |
| \( C_{11} \) | \( \mathbb{Q}(\zeta_{11})^+ \) | 121a2 , 121c2 , 121b1 |
| \( C_2 \) | \( C_{10} \) | \( \mathbb{Q}(\sqrt[5]{T_2}) \) | 66c3   |
| \( C_5 \) | \( \mathbb{C}_{25} \) | \( \mathbb{Q}(\zeta_{11})^+ \) | 11a3   |

**Table 3.** Examples of elliptic curves such that \( G \in \Phi(1), H \in \Phi(5, G) \) and \( G \neq H \).
Remark. Note that, although we have proved in Propositions 15 and 16 that there are infinitely many elliptic curves over $\mathbb{Q}$ such that the torsion grows over a quintic number field, these elliptic curves seem to appear not very often. We have computed for all elliptic curves over $\mathbb{Q}$ with conductor less than $350,000$ from [6] (a total of $2,188,263$ elliptic curves) and we have found only $1256$ cases where the torsion grows. Moreover, only $40$ cases when it grows to $C_{10}$ and $7$ to $C_{25}$ (the elliptic curves $11a3$, $550k3$, $1342c1$, $33825be3$ $165066d1$, $185163a1$ and $192698c1$).

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