EXPLICIT CONSTRUCTIONS OF K3 SURFACES AND UNIRATIONAL NOETHER–LEFSCHETZ DIVISORS

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ABSTRACT. We provide methods to construct explicit examples of K3 surfaces. This leads to unirational constructions of Noether–Lefschetz divisors inside the moduli space of K3 surfaces of genus $g$. We implement Mukai’s unirationality construction of the moduli spaces of K3 surfaces of genus $g \in \{6, \ldots, 10, 12\}$, and we also present a new constructive proof of the unirationality of the moduli space of K3 surfaces of genus 11. Furthermore, we show the existence of three unirational hypersurfaces in any moduli space of K3 surfaces of genus $g$.

INTRODUCTION

The moduli space $\mathcal{F}_g$ of (quasi-)polarized K3 surfaces of genus $g$ is a fundamental object of interest in algebraic geometry. In order to understand its geometry one has to know/compute its Kodaira dimension. There are many striking results in this direction. On the one hand the unirationality of these moduli spaces is known for small genus (see e.g. [Muk02, FV18, FV21]) and on the other hand for large genus the space is of general type (see e.g. [GHS07]). We give a complete list of known results in Section 7. One says that the moduli space $\mathcal{F}_g$ is unirational if there exists a dominant rational map $\mathbb{P}^n \to \mathcal{F}_g$. In this paper, we are more interested in the explicit unirationality of $\mathcal{F}_g$, meaning that the proof of unirationality provides an explicit algorithm which can be implemented in a computer algebra program such as Macaulay2 [GS21]; so, in particular, we shall be able to determine the equations of the general member of $\mathcal{F}_g$.

The natural next step to go is to ask for the Kodaira dimension of lattice-polarized K3 surfaces. A main goal of this article is to provide explicit unirational constructions of K3 surfaces of Picard rank 2 (or equivalently, Noether–Lefschetz divisors inside $\mathcal{F}_g$). We provide a Macaulay2 package, named K3s [HS21], where we have implemented all our constructions (see Section 8).

We recall the content of the article and some applications. Due to the work of Mukai (see [Muk88, Muk89, Muk92a]), we know that the 19-dimensional moduli space $\mathcal{F}_g$ of (quasi-)polarized K3 surfaces of genus $g$ (and degree $2g-2$ given by a quasi-polarization $L$) is explicitly unirational for $g \leq 12$ and $g \neq 11$. Using Mukai’s descriptions, we will construct such K3 surfaces containing a further curve class $C$ of small degree and/or small genus, giving a K3 surface with Picard lattice of rank at least 2 (see also Section 2). Let $S$ be such a K3 surface whose Picard lattice contains a (very) ample class of high self intersection given as a linear combination of the quasi-polarization $L$ and $C$, e.g.

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$H = aL + bC$ for $a, b \in \mathbb{Z}$. In our Macaulay2 package $K3s$, we provide a function computing the projective model $S \to \mathbb{P}(H^0(S,H)^*)$. Examples of projective models of $K3$ surfaces of genus 14, 22, 44 are presented in our notes. As an application, we find an unirational Noether–Lefschetz divisor in $F_{44}$ which is itself of general type (see Subsection 5.3). Beyond that we show the following result in Section 4.

**Theorem.** There are at least three unirational Noether–Lefschetz divisors in the moduli space $F_g$ for any $g \geq 3$.

In a forthcoming work of the first named author with G. Mezzedini, we prove the existence of a fourth unirational divisors in the moduli space $F_g$ for any $g \geq 3$. It is natural to ask for the number $n(g)$ of unirational divisors in $F_g$ depending on the genus $g$, for example we construct 7 unirational divisors in $F_{44}$. We think that the number $n(g)$ is unbounded for $g \to \infty$. But even more could be true. Are there infinitely many unirational divisors in the moduli space of $K3$ surfaces of genus $g$?

In a work in progress of A. Auel, M. Bolognesi and the first named author, we apply the explicit construction presented in this article to study codimension 2 subvarieties inside the moduli space of cubic fourfolds and their relation to Noether–Lefschetz divisors of their associated $K3$ surfaces.

Beside the implementation of Mukai’s unirationality constructions, our second major result is an explicit construction of $K3$ surfaces of genus 11. Mukai showed in [Muk96] that $F_g$ is also unirational for $g = 11$, but his proof does not seem to lead in an obvious way to an algorithm. In Section 6 we fill this gap by showing how to determine equations for a general $K3$ surface of genus 11.

**Theorem.** The moduli space $F_{11}$ of $K3$ surfaces of genus 11 is explicitly unirational.

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1. **Moduli spaces and Mukai models**

1.1. **Moduli spaces of lattice polarized $K3$ surfaces.** Let $(S, L)$ be a quasi-polarized $K3$ surface of genus $g$ (that is, $L$ is pseudo-ample and $L^2 = 2g - 2$) and let $F_g$ be the 19-dimensional moduli space of quasi-polarized $K3$ surfaces of genus $g$ (in particular, the projective model of a quasi-polarized $K3$ surface can have at most isolated singularities). There are countably many Noether–Lefschetz divisors in $F_g$ describing $K3$ surfaces of Picard-rank $\geq 2$. Such a divisor is birational to the moduli space $F_{\Lambda_{d,n}^g}$ of lattice-polarized $K3$ surfaces for some rank 2 lattice $\Lambda_{d,n}^g$ with the intersection matrix

$$(2g - 2 \quad d \quad d \quad n)$$

with respect to a basis $\{h_1, h_2\}$. The moduli space $F_{\Lambda_{d,n}^g}$ of lattice polarized $K3$ surfaces parametrizes pairs $(S, \varphi)$ consisting of a $K3$ surface and a primitive lattice embedding $\varphi : \Lambda_{d,n}^g \to \text{Pic}(S)$ such that $\varphi(\Lambda)$ contains the pseudo-ample class. It is a quasi projective irreducible 18 ($= 20 - \text{rk}(\Lambda)$)-dimensional variety by [Dol96]. There exists a non-empty
open subset of $F_{Λ_{g}}^{d,n}$ such that $L := \varphi(h_{1})$ is pseudo-ample and hence, inducing a quasi-polarization of genus $g$. By abuse of notation, we write $F_{Λ_{g}}^{d,n} \subset F_{g}$.

### 1.2. Birational descriptions of Mukai models

We denote by $Σ_{g}$ the Mukai model of genus $g \in \{6,7,8,9,10,12\}$, these are (homogeneous) varieties with a canonical curve section. In the following table we recall their definitions and their rational parametrizations. We will write $G(r,n)$ for the Grassmannian parametrizing $r$-planes in $P_{n}$.

**Remark 1.1.** Note that the varieties $Σ_{6}$ and $Σ_{12}$ are not homogeneous varieties. The Fano threefold $Σ_{12}$ is a complete intersection with respect to the rank 9 vector bundle $Λ^{2}S \oplus 3$ on $G(2,6)$, where $S$ is the dual of the universal subbundle on $G(2,6)$.

**Remark 1.2.** The described parametrizations in Table 1 are explained in [Zak93, Thm. 3.8] for $g \in \{6,7,8\}$ and are obtained as the inverse of a tangential projection. The rational parametrization of the Lagrangian Grassmannian $LG(3,6)$ is described in [LM02, Section 2.1 and 3.1]. The special rational parametrization of $G_{2}/B \subset P_{13}$ is explained in [LM02, Section 2.2].

### 1.3. K3 surfaces of small genus as sections of Mukai models

For our explicit constructions of K3 surfaces, we use the fundamental work of Mukai (see [Muk02] for a summary). Mukai classified Brill–Noether general K3 surfaces of genus $g \in \{6,\ldots,10,12\}$. We recall his classification.

**Definition 1.3.** A quasi-polarized K3 surface $(S,L)$ is **Brill–Noether general** if the inequality $h^{0}(S,M) \cdot h^{0}(S,N) < h^{0}(S,L)$ holds for any pair $(M,N)$ of non-trivial line bundles such that $L \cong M \otimes N$.

**Remark 1.4.** The condition of being Brill-Noether general is open in the moduli space of K3 surfaces of fixed genus. The locus of Brill–Noether general K3 surfaces of genus

### Table 1. Parametrizations of Mukai models

| $g$ | $Σ_{g}$                              | dim($Σ_{g}$) | Parametrization                                                                 |
|-----|--------------------------------------|--------------|--------------------------------------------------------------------------------|
| 6   | $G(1,4)$                             | 7            | image of $P_{7}$ given by quadrics through the cone $P^{1} \times P^{2} \subset P^{6} \subset P^{7}$ |
| 7   | $OG(5,10)$                           | 10           | image of $P^{10}$ given by quadrics through $G(1,4) \subset P^{9} \subset P^{10}$   |
| 8   | $G(1,5)$                             | 8            | image of $P^{8}$ given by quadrics through $P^{1} \times P^{3} \subset P^{7} \subset P^{8}$ |
| 9   | $LG(3,6)$                            | 6            | image of $P^{6}$ given by cubics singular along the Veronese surface in $P^{5} \subset P^{6}$ |
| 10  | $G_{2}/B$                            | 5            | image of $P^{5}$ given by special quartics singular along a twisted cubic in $P^{3} \subset P^{5}$ |
| 12  | $X_{22}$                             | 3            | image of a quadric $Q \subset P^{4}$ given by quintics singular along a rational normal sextic $C \subset Q$ (see also Section 5.2) |

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\( g \in \{6, \ldots, 10, 12\} \) is described in [GLT15, Lemma 1.7]. In these low genera this condition is equivalent to all the smooth curves in the linear system \( |L| \) being Brill–Noether general.

**Theorem 1.5 (Mukai).** A primitively quasi-polarized K3 surface \((S, L)\) of genus \(6 \leq g \leq 10\) or \(12\) is Brill–Noether general if and only if it is birational to a complete intersection with respect to a vector bundle on a space \(\Sigma_g\), listed below:

- \(g = 6\): a quadratic and a codimension 4 linear section of the cone \(\Sigma_6 \subset \mathbb{P}^{10}\) over the Grassmannian \(G(1, 4)\),
- \(g = 7\): a codimension 8 linear section of the orthogonal Grassmannian \(\Sigma_7 = \mathcal{O}G(5, 10) \subset \mathbb{P}^{15}\),
- \(g = 8\): a codimension 6 linear section of the Grassmannian \(\Sigma_8 = G(1, 5) \subset \mathbb{P}^{14}\),
- \(g = 9\): a codimension 4 linear section of the Langrangian Grassmannian \(\Sigma_9 = LG(3, 6) \subset \mathbb{P}^{13}\),
- \(g = 10\): a codimension 3 linear section of the flag variety \(\Sigma_{10} \subset \mathbb{P}^{13}\) of dimension five associated with the adjoint representation of \(G_2\),
- \(g = 12\): a hyperplane section of a prime Fano-threefold \(\Sigma_{12} \subset \mathbb{P}^{13}\) of genus 12 and degree 22.

**Remark 1.6.** For polarized K3 surfaces of genus \(g\) (whence with a smooth model in \(\mathbb{P}^g\)), the strategy of the above theorem is presented in [Muk02, §4]. The extension of this result for quasi-polarized K3 surfaces is explained in [Muk02, §6 and §7].

**Remark 1.7.** For \(g = 6\), we emphasize using the cone of the Grassmannian \(G(1, 4)\) in the statement of the theorem since there are several articles which state Mukai’s result only for the Grassmannian. If the codimension 4 linear section contains the vertex of the cone, the K3 surface is a double cover of a del Pezzo quintic surface branched along a curve of genus 6. Such K3 surfaces were studied in [AK11, HK20] and lie in codimension 4 inside the moduli space of K3 surfaces of genus 6.

## 2. Overview of our results

We present explicit unirational construction of the moduli spaces \(\mathcal{F}^{\Lambda_{d,n}}\) for the lattices in Table 2.

**Remark 2.1.** (a) From K3 surfaces with Picard lattices as in Table 2 one can construct infinitely many other K3 surfaces with different geometric features. Indeed, for a K3 surface \(S \in \mathcal{F}^{\Lambda_{d,n}}\) from Table 2, let \(H\) and \(C\) be the two curves on \(S\) inducing the intersection lattice \(\Lambda_{d,n}\). Our Macaulay2 package also provides a function that embeds \(S\) by the linear system \(|aH + bC|\) for \((a, b) \in \mathbb{Z}^2\) if possible. Therefore, one can construct a K3 surface in \(\mathcal{F}^{\Lambda'_{d',n'}}\) whenever \(\Lambda_{d,n}\) and \(\Lambda'_{d',n'}\) are congruent to each other. For example, considering a K3 surface of genus 10 containing a conic (case (iii) in Table 2) is equivalent to a nodal K3 surface of genus 11.

(b) Case (v) Table 2 implies that every moduli space \(\mathcal{F}_g\) of K3 surfaces of genus \(g \geq 3\) contains at least three unirational hypersurfaces (see Section 4).

For K3 surfaces of genus 3, 4 and 5 (hence, complete intersections), the unirational constructions are standard (see [FHM20] for details and an application to elliptic K3 surfaces). We briefly recall the general strategy in Section 3.
We use Mukai models to construct nodal K3 surfaces of genus \( g \in \{6, \ldots, 10, 12\} \). The projection from the node yields a K3 surface of genus one less and containing a conic. We explain the unirational construction in Section 5.3 for K3 surfaces of genus 11 containing a conic and note that the other cases for \( g \leq 9 \) are similar. All constructions are implemented in our Macaulay2 package. Section 5.2 gives a construction of K3 of genus 12 containing a conic.

### 3. K3 surfaces as complete intersections

#### 3.1. Hilbert scheme of curves and parameter count: general strategy.

For the rest of this subsection, let \( g \in \{3, 4, 5\} \), \( d \geq 1 \) and \( n = 2k - 2 \geq -2 \) be integers. It is well-known that polarized K3 surfaces of genus \( g \) are complete intersections in \( \mathbb{P}^g \) of type (4), (2, 3) and (2, 2, 2), respectively. Let \( S \in \mathcal{F}_{\Lambda_g^{d,n}} \) be a lattice polarized K3 surface with a projective model \( S \subset \mathbb{P}^g \). Hence, there exists a curve \( C \subset S \subset \mathbb{P}^g \) of degree \( d \) and genus \( k \). Together with the hyperplane section \( h \) of \( S \), the intersection matrix with respect to the basis \( \{h, [C]\} \) is the lattice \( \Lambda_g^{d,n} \). We explain the classical strategy to prove the unirationality of \( \mathcal{F}_{\Lambda_g^{d,n}} \) which can be reduced to a parameter count and the computation of a single example with the desired properties.

Let \( \mathcal{H} \) be the Hilbert scheme of curves of degree \( d \) and genus \( k \) in \( \mathbb{P}^g \). We assume that \( \mathcal{H} \) is unirational (e.g. the Hilbert scheme of rational/elliptic curves in \( \mathbb{P}^g \)). We consider the following incidence variety

\[
I = \{(S, C) : S \in \mathcal{F}_{\Lambda_g^{d,n}}, C \in \mathcal{H} \text{ a smooth curve on } S \} \subset \mathcal{F}_{\Lambda_g^{d,n}} \times \mathcal{H}/\text{PGL}(g+1),
\]

together with the two projections

\[
p_1 : I \to \mathcal{F}_{\Lambda_g^{d,n}}, \quad p_2 : I \to \mathcal{H}/\text{PGL}(g+1).
\]
Note that the fiber of $p_1$ over a general surface is a $\mathbb{P}^k(\cong |C|)$ and the fiber of $p_2$ over a curve $C$ is isomorphic to an iterated Grassmannian, and hence rational if non-empty. More precisely, in the case $g = 3$, the fiber is $|\mathcal{I}_C(4)|$, in the case $g = 4$, the fiber is isomorphic to a projective bundle $\mathbb{P}\mathcal{E}$ over $|\mathcal{I}_C(2)|$, whose fiber over $q \in |\mathcal{I}_C(2)|$ is $\mathbb{P}\mathcal{E}_q$ defined in the exact sequence

$$0 \to H^0(\mathcal{O}_{\mathbb{P}^4}(1)) \xrightarrow{\cdot q} H^0(\mathcal{I}_C(3)) \to E_q \to 0.$$ 

In the remaining case $g = 5$, the fiber of $p_2$ is $\mathcal{G}(2,|\mathcal{I}_C(2)|)$. It follows the unirationality of the incidence variety $\mathcal{I}$. Now if we check in a single example of a smooth curve $C$ that $\dim(p_1^{-1}(C))$ is minimal and that

$$h^0(C, \mathcal{N}_{C/\mathbb{P}^g}) - h^1(C, \mathcal{N}_{C/\mathbb{P}^g}) + \dim(p_2^{-1}(C)) = \dim \mathcal{F}^{d,n}_{\Lambda g} + k,$$

then we can conclude by semi-continuity that $\dim I = \dim \mathcal{F}^{d,n}_{\Lambda g} + k$ is as expected, the first projection $p_1$ is dominant, and hence, the unirationality of $\mathcal{F}^{d,n}_{\Lambda g}$ follows.

Remark 3.1. The Hilbert scheme of rational/elliptic curves in $\mathbb{P}^g$ for $g \in \{3, 4, 5\}$ is irreducible and unirational. For the cases in Table 2 (iv) and (vi), the fiber of $p_2$ over a general element is non-empty and therefore, we can apply the above strategy to conclude unirationality.

3.2. Application: K3 surfaces of genus 14. By [Nue15] and [FV18] the moduli space $\mathcal{F}_{14}$ of polarized K3 surfaces of genus 14 (and degree 26) is unirational. The starting point of their unirationality proofs is the following isomorphism. For a general cubic fourfold $X \subset \mathbb{P}^5$ of discriminant 26 the Fano scheme of lines on $X$ is isomorphic to the Hilbert square $S^{[2]}$ of a uniquely determined K3 surface $S$ of genus 14. By sending $X$ to $S$, we get a birational isomorphism between the moduli space $\mathcal{C}_{26}$ of cubic fourfolds of discriminant 26 and the moduli space $\mathcal{F}_{14}$ of polarized K3 surfaces of genus 14 (see [Has00]). We will give an explicitly unirational construction of a Noether–Lefschetz divisor in $\mathcal{F}_{14}$.

We consider the lattice $\Lambda_{5,0}$ defined by the intersection matrix

$$\begin{pmatrix} 8 & 9 \\ 9 & 0 \end{pmatrix}$$

with respect to an ordered basis $\{h_1, h_2\}$. We may assume that the class $h_1 + h_2$ is big and nef or even ample in general (see [BHPdV04, VIII, Prop. 3.10] and note that there are no $(-2)$-classes in $\Lambda_{5,0}$). Since $(h_1 + h_2)^2 = 26$ and there is a primitive lattice embedding $\langle 26 \rangle \hookrightarrow \Lambda_{5,0}$, the moduli space $\mathcal{F}^{5,0}_{\Lambda}$ is birational to a Noether–Lefschetz divisor in $\mathcal{F}_{14}$. Therefore we write $\mathcal{F}^{5,0}_{\Lambda} \subset \mathcal{F}_{14}$. Applying the above strategy, we get the following proposition.

**Proposition 3.2.** The Noether–Lefschetz divisor $\mathcal{F}^{5,0}_{\Lambda} \subset \mathcal{F}_{14}$ is unirational.

**Lemma 3.3.** A polarized K3 surface $(S, L) \in \mathcal{F}^{5,0}_{\Lambda} \subset \mathcal{F}_{14}$ with Picard lattice $\Lambda_{5,0}$ is Brill–Noether general.
Proof. Let $L, E \in \text{Pic}(S)$ be classes on $S$ such that $L^2 = 26$, $L.E = 9$ and $E^2 = 0$, hence $L$ is a polarization of genus 14. Let $aL + bE \in \text{Pic}(S)$ be an effective class on $S$. Then $a = \frac{(aL + bE).E}{b} \geq 0$.

Let $M = aL + bE, N = a'L + b'E$ be non-trivial line bundles such that $L = M + N$. Since $a, a' \geq 0$ and $E$ is effective, we may assume that $M = bE$ and $N = L - bE$ with $b > 0$. But $(L - bE)^2 = 26 - 18b \geq 0$ and therefore, $b = 1$. For the only non-trivial decomposition of $L$, the inequality

$$h^0(S, L - E) \cdot h^0(S, E) = 6 \cdot 2 < 15 = h^0(S, L)$$

is satisfied. □

3.3. Application: K3 surfaces of genus 22. In [FV21] Farkas and Verra prove the unirationality of the moduli space $\mathcal{F}_{22}$ of K3 surfaces of genus 22. As in the previous section we will give an unirational construction of a Noether–Lefschetz divisor in $\mathcal{F}_{22}$ where our construction is implemented in Macaulay2.

Let $\Lambda_5^{3,-2}$ be the rank 2 lattice defined by the intersection matrix

$$
\begin{pmatrix}
8 & 3 \\
3 & -2
\end{pmatrix}
$$

with respect to an ordered basis $\{h_1, h_2\}$. We may assume that the class $2h_1 + h_2$ is big and nef or even ample in general (see [BHPdV04, VIII, Prop. 3.10]). Note that $h_2$ is the only $(-2)$-class in $\Lambda_5^{3,-2}$. Since $(2h_1 + h_2)^2 = 42$ and there is a primitive lattice embedding $(42) \to \Lambda_5^{3,-2}$, the moduli space $\mathcal{F}_{22}$ is birational to a Noether–Lefschetz divisor in $\mathcal{F}_{22}$. Therefore we write $\mathcal{F}_{22} \subset \mathcal{F}_{22}$ and get the following proposition.

Proposition 3.4. The Noether–Lefschetz divisor $\mathcal{F}_{22} \subset \mathcal{F}_{22}$ is unirational.

Remark 3.5. Let $S \in \mathcal{F}_{22}$ be a K3 surface. Let $H$ and $C$ be the classes in $\text{Pic}(S)$ with $H^2 = 8, H.C = 3$ and $C^2 = -2$. The polarization of genus 22 is given by $L = 2H + C$. Then $S \subset \mathbb{P}^{22}$ is not Brill–Noether general, since $h^0(L) < h^0(H) \cdot h^0(H + C)$.

4. Tri-, tetra- and pentagonal K3 surfaces

We briefly recall general facts about K3 surfaces contained in a rational normal scroll following [TJK04] (see also [Sch86]). All results in this section should be already known, but not in the context of unirational hypersurfaces inside moduli spaces of polarized K3 surfaces. Similar results for curves can be found in [Sch86, Bop15, DP15].

Let $S \subset \mathbb{P}^{g}$ be a polarized K3 surface of genus $g$ with polarization $L$ such that $L$ can be decomposed into $L \sim E + F$ with $E$ a base point free elliptic pencil (that is, $h^0(S, E) = 2$ and $D \subset |E|$ a smooth irreducible elliptic curve) and $h^0(S, F) \geq 2$. Then

$$X = \bigcup_{D \subset |E|, \deg D \geq 2} \overline{D} \subset \mathbb{P}^{g}$$

is a rational normal scroll of dimension $d = h^0(S, L) - h^0(S, L - E)$ and degree $f = h^0(S, L - D)$, where $\overline{D}$ denotes the span of the elliptic curve $D$. Conversely, if $S$ is contained in a rational normal scroll $X$ of degree $f \geq 2$, the ruling on $X$ sweeps out a pencil of divisors on $S$ which induces a decomposition of the polarization as above.
We restrict our considerations to $K3$ surfaces carrying an elliptic pencil of degree 3 (4 and 5), so-called trigonal (tetragonal and pentagonal, resp.) $K3$ surfaces. Therefore, they are lying on a scroll of dimension 3 (4 and 5, respectively) and using structure theorems of codimension up to 3, we have the following proposition.

**Proposition 4.1.** [TJK04, Section 9] For $3 \leq d \leq 5$, the moduli space $\mathcal{F}^{\Lambda_d,0}_{K3}$ of lattice polarized $K3$ surfaces is unirational. In other words, the moduli space of tri-, tetra- and pentagonal $K3$ surfaces is unirational.

We get immediately the following result for any moduli space of polarized $K3$ surfaces.

**Corollary 4.2.** The moduli space $\mathcal{F}_g$ of $K3$ surfaces of genus $g$ contains at least three unirational Noether–Lefschetz divisors for any genus $g \geq 3$.

In [TJK04, Section 9], the authors describe all possible families of smooth $K3$ surfaces as in the proposition on rational normal scrolls and compute their number of moduli (see also [SD74] for the trigonal case).

The proposition follows from a finite number of cases. Note that after substracting a multiple of the elliptic pencil from the polarization $L$, we may reduce the proof of the proposition to the following cases. Let $(S, L)$ be a polarized $K3$ surface with an elliptic pencil $|E|$ of degree $d \in \{3, 4, 5\}$ such that

$$L^2 = 2g - 2 \in \begin{cases} 
\{4, 6, 8\} & \text{for } d = 3, \\
\{4, 6, 8, 10\} & \text{for } d = 4, \\
\{4, 6, 8, 10, 12\} & \text{for } d = 5.
\end{cases}$$

The only non-trivial cases, not covered in Section 3, are the following examples. All constructions are unirational since generators/matrices of some bidegrees in the coordinate ring of the rational normal scroll are parametrized linearly. One can compute that the number of moduli is 18 in all cases.

**Example 4.3.** (Trigonal $K3$ surface of genus 5) Let $S \subset \mathbb{P}^5$ be a smooth $K3$ surface of genus 5 carrying an elliptic pencil of degree 3. Then, the surface $S$ lies on a cubic scroll $X$ isomorphic to $\mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$ and is cut out by a form of bidegree $(2, 3)$ on $X$.

**Example 4.4.** (Tetragonal $K3$ surface of genus 6) Let $S \subset \mathbb{P}^6$ be a smooth $K3$ surface of genus 6 carrying an elliptic pencil of degree 4. Then, the surface $S$ lies on a cubic scroll $X$ isomorphic to the cone of $\mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^6$ and is cut out by two forms of bidegree $(1, 2)$ and $(2, 2)$ on $X$.

**Example 4.5.** (Pentagonal $K3$ surface of genus 6) Let $S \subset \mathbb{P}^6$ be a smooth $K3$ surface of genus 6 carrying an elliptic pencil of degree 5. Then, the surface $S$ lies on a singular quadric $X$ of rank 4 and is generated by the $4 \times 4$ Pfaffians of $5 \times 5$ skew-symmetric matrix on $X$ with homogeneous entries of bidegree $(1, 1)$.

**Example 4.6.** (Pentagonal $K3$ surface of genus 7) Let $S \subset \mathbb{P}^7$ be a smooth $K3$ surface of genus 7 carrying an elliptic pencil of degree 5. Then, the surface $S$ lies on a cubic scroll $X$ isomorphic to the cone of $\mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^7$ and is generated by the $4 \times 4$ Pfaffians of $5 \times 5$ skew-symmetric matrix on $X$ with homogeneous entries given in the following
matrix
\[
\begin{pmatrix}
0 & 0 & (1,1) & (1,1) & (1,1) \\
0 & 0 & (1,1) & (1,1) & (1,1) \\
(1,1) & (1,1) & 0 & (1,0) & (1,0) \\
(1,1) & (1,1) & (1,0) & 0 & (1,0) \\
(1,1) & (1,1) & (1,0) & (1,0) & 0
\end{pmatrix}.
\]

**Remark 4.7.** It turns out that the moduli space \( \mathcal{F}^{A_g,0} \) of hexagonal K3 surfaces is unirational as well. These are K3 surfaces of genus \( g \) which contain an elliptic pencil of degree 6. This will be part of a forthcoming paper of the first named author and G. Mezzedimi.

5. Nodal K3 surfaces and K3 surfaces containing a rational curve

5.1. Rational parametrizations of Fano threefolds. A prime Fano threefold \( X \) is a smooth projective variety of dimension 3 whose Picard group is generated by the anti-ample canonical class \( K_X \). The genus \( g(X) \) is defined by the equality \((−K_X)^3 = 2g(X)−2\).

We use several methods to parametrize prime Fano threefolds of genus \( g \in \{6, \ldots, 10, 12\} \) embedded into \( \mathbb{P}^{g+1} \) together with either a distinguished point or a rational curve on it. There are parametrizations given by the inverse of a double projections from a line for \( g \leq 10 \) (as treated in [Isk77, Š79]) or by a Sarkisov link for \( g = 12 \) (see Section 5.2).

Note that all such prime Fano threefolds are sections of the corresponding Mukai models (see e.g. [Muk92a]). In Table 3, we list our chosen parametrizations (see also Table 1).

| \( g \) | Parametrization |
|-------|----------------|
| 6     | section of the Grassmannian \( \mathcal{G}(1,4) \) containing special Schubert cycles |
| 7     | parametrization of the Mukai model \( \Sigma_7 \) containing a distinguished point/a rational curve |
| 8     | section of the Grassmannian \( \mathcal{G}(1,5) \) containing special Schubert cycles |
| 9     | parametrization of the Mukai model \( \Sigma_9 \) containing a distinguished point/a rational curve |
| 10    | image of a quadric \( Q \subset \mathbb{P}^4 \) given by quintics singular along a curve \( C \subset Q \) of degree 7 and genus 2 |
| 12    | image of a quadric \( Q \subset \mathbb{P}^4 \) given by quintics singular along a rational normal sextic \( C \subset Q \) |

**Table 3.** Parametrizations of prime Fano threefolds

We remark that we get a unirational variety parametrizing pairs of Fano threefolds together with either a point or a rational curve of small degree. Indeed, let \( \mathcal{H}_X(\mathbb{P}^{g+1}) \) be the Hilbert scheme of Fano threefolds \( X \subset \mathbb{P}^{g+1} \) of genus \( g \) which is unirational by construction. If \( X \) is rigid, the universal Fano threefold \( \mathcal{X} \) parametrizing pairs of
Fano threefolds with a distinguished point is unirational too, since $X$ is rational. In the remaining cases (if $X$ is not rigid), we construct $X$ as a general section of $\Sigma_g$ containing a general point, and hence, the unirationality of the universal Fano threefold $X$ follows. Let $\mathcal{H}_C(X)$ be the Hilbert scheme of rational curves of low degree on $X$. By our constructions and a similar parameter count as in Section 3, the incidence variety
\[ \{(X, C) | C \subset X\} \subset \mathcal{H}_X(\mathbb{P}^{g+1}) \times \mathcal{H}_C(X) \]
is unirational.

**Nodal K3 surfaces.** Having a prime Fano threefold $X$ together with a distinguished point $p$, we can compute the tangent space $T_p X$ to $X$ at $p$. Choosing a general hyperplane section of $X$ containing $T_p X$ yields a K3 surface with one simple double point. Such hyperplane sections are parametrized by a projective space. Hence, the moduli space parametrizing such K3 surfaces is unirational.

**K3 surfaces containing a rational curve.** Having a prime Fano threefold $X$ together with a rational curve $C$, let $\overline{C}$ be the linear space of $C$. Choosing a general hyperplane section of $X$ containing $\overline{C}$ yields a K3 surface containing $C$. As above, such hyperplane sections are parametrized by a projective space. Hence, the moduli space parametrizing these K3 surfaces is unirational.

We will explain in the following section our procedure for prime Fano threefolds of genus 12.

### 5.2. Construction of prime Fano threefolds of genus 12.

Since the construction of a prime Fano threefold $V_{22}$ of genus 12 is more subtle, we recall a construction based on a Sarkisov link to a quadric threefold $Q \subset \mathbb{P}^4$. We follow [KP18] (see also [Tak89],[IP99]).

The moduli space $M_{\text{Fano}, 12}$ of prime Fano threefolds of genus 12 has three incarnations (see [Muk89], [Muk92a] and [Sch01]). It is birational to the moduli space $M_3$ of curves of genus 3, to the moduli space of nets of quadrics in $\mathbb{P}^3$ and to the moduli space of curves of genus 3 together with a non-vanishing theta characteristic. The birationality relies on geometric realizations of a Fano threefold $V_{22} \in M_{\text{Fano}, 12}$ as a Grassmannian $G(3, V_7, \eta)$ of isotropic 3-spaces in a 7-dimensional vector space $V_7$, as the variety of sums of powers presenting the equation of a plane quartic, or as the variety of twisted cubics in $\mathbb{P}^3$ whose quadric equations are annihilated by a net of quadrics. In particular, the moduli space $M_{\text{Fano}, 12}$ is unirational. But none of these realizations allows one to write down explicit equations of a general Fano threefold $V_{22}$. Using birational geometry we are able to do this.

Let $V_{22}$ be a general Fano threefolds of genus 12 such that the anti-canonical divisor $-K_{V_{22}}$ is very ample.

**Proposition 5.1.** [Tak89, Thm 0.2], [KPS18, Thm 1.1.1] There exists a smooth conic on $V_{22}$. Furthermore, the Hilbert scheme of conics on $V_{22}$ is isomorphic to $\mathbb{P}^2$.

**Theorem 5.2.** [KP18, Theorem 2.2, 2.6] Let $C \subset V_{22}$ be a smooth conic on $V_{22}$. There exists the following commutative diagram of birational maps
\begin{equation}
V_22 \subset P^{13} \xrightarrow{\varphi_C} \subset Q' \subset P^4
\end{equation}

where

- $Q$ is a smooth quadric in $P^4$,
- $\varphi_C$ is the blow up of $C$,
- $\varphi_\Gamma$ is the blow up of a smooth rational (quadratically normal) sextic curve $\Gamma \subset Q$, and $\chi$ is a flop.

The birational map $\varphi$ is given by the hyperplanes on $V_{22}$ double along $C$, that is, the linear system $|I_{C/V_{22}}^2(1)|$. The inverse map $\varphi^{-1}$ is given by quintic hypersurfaces double along $\Gamma$, that is, the linear system $|I_{\Gamma/Q}^5(5)|$. Furthermore, the compositions $\varphi_\Gamma \circ \chi$ and $\varphi_C \circ \chi^{-1}$ contract the strict transforms of the unique divisors in $|I_{C/V_{22}}^5(2)|$ and $|I_{\Gamma/Q}(2)|$, respectively.

To a smooth quadric $Q \subset P^4$ containing a rational quadratically normal sextic curve $\Gamma \subset Q$ there exists a smooth Fano threefold $V_{22}$ of genus 12 and a smooth conic $C \subset V_{22}$ related to $(Q, \Gamma)$ by (5.1).

Let $H$ be the quotient of the Hilbert scheme of plane conics in $P^{13}$ under the action of $PGL(14)$ and let $H'$ be the quotient of the Hilbert scheme of rational sextic curves $\Gamma \subset P^4$ under the action of $PGL(5)$. Note that the general element in $H'$ is quadratically normal. Let us consider the following incidence correspondences

$I_\Gamma = \{(\Gamma, Q) : \Gamma \subset Q \subset P^4 \} \subset H' \times \mathbb{P}(H^0(\mathcal{O}_{P^4}(2)))/PGL(5)$

and

$I_C = \{(C, V_{22}) : C \text{ smooth and } C \subset V_{22} \subset P^{13} \} \subset H \times \mathcal{M}_{Fano, 12}$.

Then $I_\Gamma$ is birational to a $\mathbb{P}^1$-bundle over $H'$ and hence, a 8-dimensional unirational variety. Similarly, by Proposition 5.1 $I_C$ is birational to a $\mathbb{P}^2$-bundle over $H$ and thus, a 8-dimensional unirational variety. Using a relative MMP one can extend Theorem 5.2 to smooth families. We get the following proposition.

**Proposition 5.3.** The incidence correspondences $I_\Gamma$ and $I_C$ are birational. In particular, there is an explicit unirational method to construct Fano threefolds of genus 12.

**Corollary 5.4.** The moduli space of $\Lambda_{12}^{2,-2}$-polarized $K3$ surfaces (that is, a polarized $K3$ surface of genus 12 containing a conic) is unirational.

**5.3. Application: K3 surfaces of genus 44.** Let $\Lambda_{12}^{0,-2}$ be the rank 2 lattice defined by the intersection matrix

$$
\begin{pmatrix}
22 & 0 \\
0 & -2
\end{pmatrix}
$$

with respect to an ordered basis $\{h_1, h_2\}$. Let $\mathcal{F}_{\Lambda_{12}^{0,-2}}$ be the moduli space of $\Lambda_{12}^{0,-2}$-polarized $K3$ surfaces.
Remark 5.5. As a Noether–Lefschetz divisors $F_{\Lambda_{12}^{0,-2}} \subset F_{12}$ it describes exactly those $K3$ surfaces of genus 12 with rational double points.

We may assume that the class $2h_1 - h_2$ is big and nef or even ample in general (see [BHPdV04, VIII, Prop. 3.10] and note that $h_2$ is the only $(-2)$-class in $\Lambda_{12}^{0,-2}$). Since $(2h_1 - h_2)^2 = 86$ and there is a primitive lattice embedding $\langle 86 \rangle \hookrightarrow \Lambda_{12}^{0,-2}$, the moduli space $F_{\Lambda_{12}^{0,-2}}$ is birational to a Noether–Lefschetz divisor in $F_{44}$. Therefore we write $F_{\Lambda_{12}^{0,-2}} \subset F_{44}$.

Proposition 5.6. The Noether–Lefschetz divisor $F_{\Lambda_{12}^{0,-2}} \subset F_{44}$ is unirational.

Proof. Let $H$ be the Hilbert scheme of nodal $K3$ surface of genus 12 in $P_{12}$. Let us consider the incidence correspondence:

$I = \{(S, V_{22}) : S \text{ nodal and } S \subset V_{22}\} \subset \mathcal{H}/\text{PGL}(13) \times \mathcal{M}_{\text{Fano,12}}$

By [Muk02] the first projection $p_1$ is birational (there exists an unique stable rank 3 bundle with 7 global sections which induces the embedding $S \subset V_{22}$). For a general Fano threefold $V_{22}$, the fiber of $p_2$ is unirational. Indeed,

$(p_2)^{-1}(V_{22}) = \{(p, H) : p \in V_{22}, H \supset T_p(V_{22})\} = V_{22} \times \mathbb{P}^0 \cong \mathbb{P}^{12},$

since $V_{22}$ is rational. Hence, $I$ and $\mathcal{H}/\text{PGL}(13)$ is rational. Finally, the quotient $\mathcal{H}/\text{PGL}(13)$ is birational to $F_{\Lambda_{12}^{0,-2}}$ by sending a nodal $K3$ surface to its desingularization. \qed

Remark 5.7. Note that $F_{44}$ is not unirational since its Kodaira dimension is non-negative by [GHS07].

Remark 5.8. Let $S \in F_{\Lambda_{12}^{0,-2}}$ be a $K3$ surface. Let $H$ and $C$ be the classes in $\text{Pic}(S)$ with $H^2 = 22$, $H.C = 0$ and $C^2 = -2$. The polarization of genus 44 is given by $L = 2H - C$. Then $S \subset \mathbb{P}^{44}$ is not Brill–Noether general, since $h^0(L) < h^0(H) \cdot h^0(H - C)$.
6. Explicit construction of general K3 surfaces of genus 11

In [HS20], the explicit equations of a K3 surface of genus 11 containing a conic was the starting point to construct a new family of rational Gushel–Mukai fourfolds. A detailed study of the geometric connection of this family of Gushel–Mukai fourfolds and their associated K3 surfaces allows us to present an algorithm to compute the general K3 surface of genus 11. In [RS19] it is presented a construction of associated K3 surfaces to cubic fourfolds. We apply a similar method to our family of Gushel–Mukai fourfolds.

**Theorem 6.1.** The moduli space $F_{11}$ of K3 surfaces of genus 11 is explicitly unirational.

In particular, we have an algorithm to write down the equations of the general K3 surface of genus 11. Already in [Muk96], Mukai showed the unirationality of $F_{11}$. The unirationality of the moduli space of $n$-pointed K3 surfaces of genus 11 is shown by [FV18] for $n = 1$ and by [Bar18] for $n \leq 6$. Mukai’s indirect proof of the unirationality is based on a non explicit birational embedding of the moduli space $M_{11}$ of curves of genus 11 in the Mukai correspondence $F_{11}$ over $F_{11}$.

We explain our procedure and therefore, recall some results from [HS20]. Let $Y^5 = G(1, 4) \cap P^9 \subset P^8$ be a smooth hyperplane section of the Grassmannian $G(1, 4) \subset P^9$ parameterizing lines in $P^4$. In [RS19, HS20], it has been shown that there exists an irreducible, unirational, 25-dimensional family $S \subset \text{Hilb}_{Y^5}$ of rational surfaces of degree 9 and sectional genus 2 whose class in $G(1, 4)$ is given by $6\sigma_{3,1} + 3\sigma_{2,2}$. The family $S$ is explicitly described by an algorithm which is implemented in the Macaulay2 package SpecialFanoFourfolds [Sta21]. In particular, we are able to get equations for a surface $S \subset Y^5$ corresponding to a general point $[S] \in S$.

The Zariski-closure of the family of quadratic sections of $Y^5$, also known as (ordinary) Gushel-Mukai fourfolds, that contain some surface of the family $S$ describes a hypersurface in the 39-dimensional projective space $P(H^0(O_{Y^5}(2)))$ of all quadratic sections of $Y^5$. By passing to the quotient modulo the action of Aut$Y^5$, this hypersurface gives rise to a Noether–Lefschetz divisor $M_{20}$ in the 24-dimensional moduli space $M$ of Gushel-Mukai fourfolds; see [HS20, Theorem 3.3], and see [DIM15] for generalities on Gushel-Mukai fourfolds.

Let $S \subset Y^5$ be a general surface of the family $S$. Then $S$ admits inside $Y^5$ a congruence of 3-secants conics, that is, through the general point of $Y^5$ there passes a unique conic contained in $Y^5$ and that cuts $S$ at three points. Furthermore, the linear system of cubic hypersurfaces in $Y^5$ with double points along $S$ gives a dominant rational map $\mu : Y^5 \dasharrow P^4$ such that its general fibers are the conic curves of the congruence to $S \subset Y^5$; see [HS20].

Let $X \subset P^8$ be a general quadratic section of $Y^5$ through $S$, hence giving rise to a general Gushel-Mukai fourfold $[X] \in M_{20}$. The restriction of $\mu$ to $X$ gives a birational map $\mu|_X : X \dasharrow P^4$, and from this we deduced in [HS20, Theorem 3.4] that the Gushel-Mukai fourfolds in $M_{20}$ are rational.

The inverse map $(\mu|_X)^{-1} : P^4 \dasharrow X$ is defined by the linear system $|H^0(I_{U,P^4}(9))|$ of hypersurfaces of degree 9 with double points along a certain irreducible surface $U \subset P^4$. It turns out that $U$ can be realized as
a triple projection followed by a simple projection of a minimal K3 surface $\tilde{U} \subset \mathbb{P}^{11}$ of degree 20 and genus 11. In particular, $U$ has four apparent double points and contains an exceptional line $L$ and an exceptional twisted cubic curve $C$. Since $\tilde{U}$ is the associated K3 surface to a general Gushel-Mukai fourfold in $\mathcal{M}_{20}$ (see [DIM15] and see also [BP21]), we have that $\tilde{U}$ gives rise to a general point in the moduli space $\mathcal{F}_{11}$ of polarized K3 surfaces of genus 11 (and degree 20). Our goal is to determine (explicit) equations for $\tilde{U} \subset \mathbb{P}^{11}$.

First of all, we remark that if we have equations for $U$, $L$, and $C$, then using standard methods we are able to get a birational map $f : U \rightarrow \tilde{U}$, and hence equations for $\tilde{U}$ with two marked points, that is, the contractions of $L$ and $C$. Indeed, the map $f$ can be roughly defined by the linear system $|H + 3C + L|$, where $H$ denote the hyperplane section class (for this step, we also need to desingularize $U$ by blowing up its singular locus). Moreover, if we have equations for $S$ and $X$ we can also determine equations for $U$ (by computing the base locus of the inverse $\mu|_{X}$ induced by $S$). Thus we only needs to determine the curves $L$ and $C$ in $U$.

A key remark done in [RS19] in the contest of cubic fourfolds applies also in our case of Gushel-Mukai fourfolds: the surface $U$ depends on the pair $(S, X)$, but its exceptional curves $L$ and $C$ only depend on $S$ but not on $X$. In fact, the locus $L \cup C \subset \mathbb{P}^{4}$ can be also described as the closure of the locus of points $p \in \mathbb{P}^{4}$ such that the fiber $\mu^{-1}(p)$ has dimension at least 2. In practice, we can obtain the curves $L$ and $C$ as the top components of the intersection $U \cap U'$, where $U' \subset \mathbb{P}^{4}$ is the corresponding surface to another general quadratic section $X'$ of $\mathbb{P}^{5}$ through $S$, that is, $U'$ is the surface that determines the inverse map of the restriction of $\mu$ to $X'$.

In summary, let

$$I = \{(S, X) \mid S \subset X \} \subset S \times \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^5}(2)))/\text{PGL}(9)$$

be the 23-dimensional unirational incidence variety of pairs consisting of surfaces of degree 9 and sectional genus 2 and a Gushel–Mukai fourfold $X$ (note that PGL(9) acts diagonally). We get a rational map $\varphi : I \rightarrow \mathcal{F}_{11}$ which maps a pair $(S, X)$ to a K3 surface of genus 11.

We have implemented this construction in our Macaulay2 package $K3s$; see also the function associatedK3surface in the package SpecialFanoFourfolds.

\section{The unirationality of the moduli space of K3 surfaces - an overview}

We collect unirationality results concerning the moduli space $\mathcal{F}_g$ of (quasi)-polarized K3 surfaces of genus $g$ in the following table.

The Kodaira dimension $\kappa(\mathcal{F}_g)$ is non-negative for $g \in \{41, 43, 44, 47\}$ or $g \geq 49$, and $\mathcal{F}_g$ is of general type for $g \in \{47, 51, 55, 58, 59, 61\}$ or $g \geq 63$ by [GHS07]. Our goal is to find an implementation of the remaining unirationality constructions mentioned in the above table (that is, $g \in \{13, 16, 18, 20\}$).

The rationality of the universal K3 surface over $\mathcal{F}_8$ is shown in [Tul20].
In this section, we introduce the Macaulay2 package K3s [HS21] by referring to its documentation for more details.

The main function of the package is named K3. It can be used in two different ways. First way of using K3. The function accepts as input a sequence \((d, g, n)\) of three integers (as the ones in Table 2) and a coefficient ring \(K\) (this last input is optional and a particular large finite field is used by default). Then it constructs a random K3 surface \(S \subset \mathbb{P}^9_K\) of genus \(g\) that contains a curve \(C\) of degree \(d\) and self-intersection \(n\), as described previously in this paper. Hence, the curve \(C\) together with the hyperplane section class \(L\) of \(S\) span a rank 2 lattice \(\Lambda_{g}^{d,n}\) with the intersection matrix

\[
\begin{pmatrix}
2g - 2 & d \\
 d & n
\end{pmatrix}.
\]

In this case, the output object looks like a function, which takes as input a pair of integers \((a, b)\) and returns the image of the embedding of \(S\) defined by the complete linear system \([aL + bC]\) (an error is through if \(aL + bC\) is not very ample). As one specific example, the command \((\text{K3}(5,9,0))(1,1)\) returns a K3 surface of genus 14 and degree 26 in \(\mathbb{P}^{14}\) cut out by 66 quadric hypersurfaces and which contains an elliptic curve of degree 9; see also Subsection 3.2.
\$ M2 --no-preload

\begin{verbatim}
K3
S = K3(5,9,0)
o2 : K3 surface with rank 2 lattice defined by the intersection matrix: | 8 9 |
      | 9 0 |
o2 : Lattice-polarized K3 surface
S(1,1)
o3 = K3 surface of genus 14 and degree 26 in PP^{14}
o3 : Embedded K3 surface
i4 : degrees oo -- degrees of the generators of the ideal
o4 = {{(2), 66}}
\end{verbatim}

As another example, the command \((K3(11,2,-2))(2,1)\) returns a K3 surface of genus 44 and degree 86 in \(P^{44}\) cut out by 861 quadric hypersurfaces and which contains an irreducible conic; see also Subsection 5.3.

\begin{verbatim}
T = K3(11,2,-2)
o5 = K3 surface with rank 2 lattice defined by the intersection matrix: | 20 2 |
      | 2 -2 |
o5 : Lattice-polarized K3 surface
T(2,1)
o6 = K3 surface of genus 44 and degree 86 in PP^{44}
o6 : Embedded K3 surface
i7 : degrees oo -- degrees of the generators of the ideal
o7 = {{(2), 861}}
\end{verbatim}

On a standard laptop computer, the whole computation for the first example takes less than 2 seconds, while for the second example takes about 3 minutes and half.

Second way of using K3. If the input is just an integer \(g\) (with \(3 \leq g \leq 12\)) and a coefficient ring \(K\) (optional as before), then a random K3 surface of genus \(g\) and degree \(2g - 2\) in \(P^g\) is returned. The most relevant case is \(g = 11\) where the procedure described in Section 6 is performed, taking about 7 minutes. One can enable an option \texttt{Verbose} to display more details on the steps of the computation.

\begin{verbatim}
K3 11
o8 = K3 surface of genus 11 and degree 20 in PP^{11}
o8 : Embedded K3 surface
\end{verbatim}

Remark 8.1. All of our constructions also work over \(\mathbb{Q}\).

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