Optimal transport by omni-potential flow and cosmological reconstruction

Uriel Frisch,1,∗ Olga Podvigina,2,† Barbara Villone,3,‡ and Vladislav Zheligovsky2,§

1UNS, CNRS, Lab. Lagrange, OCA, B.P. 4229, 06304 Nice Cedex 4, France
2Institute of Earthquake Prediction Theory and Mathematical Geophysics of the Russian Academy of Sciences,
84/32 Profsoyuznaya St., 117997 Moscow, Russian Federation
3INAF, Osservatorio Astrofisico di Torino, Via Osservatorio, 20, 10025 Pino Torinese, Torino, Italy

(Dated: February 28, 2012)

One of the simplest models used in studying the dynamics of large-scale structure in cosmology, known as the Zeldovich approximation, is equivalent to the three-dimensional inviscid Burgers equation for potential flow. For smooth initial data and sufficiently short times it has the property that the mapping of the positions of fluid particles at any time $t_1$ to their positions at any time $t_2 \geq t_1$ is the gradient of a convex potential, a property we call omni-potentiality. Are there other flows with this property, that are not straightforward generalizations of Zeldovich flows? This is answered in the affirmative in both two and three dimensions. How general are such flows? Using a WKB technique we show that in two dimensions, for sufficiently short times, there are omni-potential flows with arbitrary smooth initial velocity. Mappings with a convex potential are known to be associated with the quadratic-cost optimal transport problem. This has important implications for the problem of reconstructing the dynamical history of the Universe from the knowledge of the present mass distribution.

Dedicated to the memory of Roman Juszkiewicz

I. INTRODUCTION

Reconstruction in cosmology considers the following problem: one assumes that the present spatial distribution of masses (galaxies and clusters, including their dark-matter components) is known from observations, and one wants to reconstruct the dynamical history of the Universe all the way to the earliest epoch, when matter and radiation decoupled (nearly 14 billion years ago). Peebles [22] introduced the reconstruction problem, and proposed a variational formulation for solving it on a relatively small spatial scale, that of the Local Group (which includes our own galaxy and neighboring ones). On scales much larger than that of the Local Group, which have been mapped in recent years through various projects such as the Sloan Digital Sky Survey [27], reconstruction may be posed in the simplest cases as an instance of optimal mass transport. Indeed, Frisch et al. [13] showed that when the Zeldovich approximation [26] or a refinement thereof (cf. below) are applied to the relevant cosmological fluid equations, the correspondence between the positions of mass elements initially (at decoupling) and finally (at the present epoch) is the solution to an optimal mass transport problem with quadratic cost. This solution is uniquely prescribed by the marginals: the mass distribution at decoupling (essentially uniform) and its highly non-uniform present distribution. A striking feature is that the sole knowledge of the current positions of galaxies, without knowledge of their (proper) velocities, yields nevertheless a unique solution for this kind of large-scale reconstruction.

It was then shown by Brenier et al. and by Loeper [7, 17] that, with prescribed marginals, unique reconstruction, not only of the Lagrangian map, but of the full dynamical history of matter elements, carries over to the Euler–Poisson model, whose validity extends much beyond that of the Zeldovich approximation. Its unique solution is again obtained from an optimal transport problem with a convex cost function, expressible as a space-time integral of a suitable action, a problem whose numerical resolution remains a challenge.

As is well known, the mass transport problem was introduced by Monge [19] more than two hundred years ago, and the theory took its modern shape after the 1942 work of Kantorovich [16] (see, e.g., Villani [25] for review).

The Zeldovich approximation [26] was introduced in 1970 as a first formulation in terms of Lagrangian coordinates of the growth of density perturbations. It replaces the full Euler–Poisson equations by basically the three-dimensional inviscid Burgers equation (written here in standard fluid dynamical notation)

$$\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} = 0, \quad \mathbf{v} = \nabla \varphi. \quad (1)$$

The validity of the Zeldovich approximation is controlled by how close one is to decoupling, but in a scale-dependent way: at very large scales, the Zeldovich approximation remains valid up to the present epoch; at very small scales, the formation of multi-stream caustics quickly ruins not only the validity of the Zeldovich approximation, but even that of the Euler–Poisson model. An immediate consequence of (1) is that the velocity of any fluid particle remains constant in the course of time.

∗Electronic address: uriel@oca.eu
†Electronic address: olgap@mitp.ru
‡Electronic address: villone@to.infn.it
§Electronic address: vlad@mitp.ru
and that the trajectories are straight lines. We denote by $q$ the initial (Lagrangian) fluid particle positions and by $x$ their (Eulerian) positions at the current epoch $t = T$. The Lagrangian map associated with the Zeldovich approximation is

$$q \mapsto x = \nabla q \left( \frac{|q|^2}{2} + T \varphi_0(q) \right), \quad (2)$$

where $\varphi_0(q)$ is the initial velocity potential. For sufficiently small $T$ and a sufficiently smooth initial potential, the Lagrangian map is thus the gradient of a convex function, a property shared by the next-order approximation, which will be discussed in Sec. V. This is why reconstruction is linked to optimal transport; indeed, a theorem of Brenier [5] states that the solution to the Monge optimal transport problem with quadratic cost is a gradient of a convex function, which satisfies a Monge–Ampère equation. The method of cosmological reconstruction in which one assumes that the Lagrangian map has a convex potential and then numerically solves a quadratic-cost optimal transport problem (after suitable discretization) is called the Monge–Ampère–Kantorovich (MAK) method [13].

The Zeldovich approximation gives us some insight into the full temporal history of mass elements. An important consequence of (1) is that for any $0 \leq t_1 < t_2 \leq T$ the mapping of particle positions at time $t_1$ to their positions at time $t_2$ is also a gradient of a convex function. When the flow-induced mapping between any two times is potential, the flow is here called omni-potential. As we shall see, the velocity field associated with such flows has the property of being simultaneously potential in Eulerian coordinates (in cosmology, this constraint stems from the potential character of gravity and the expansion of the Universe), as well as in Lagrangian coordinates (which allows reconstruction by solving an optimal transport problem).

We are of course led to ask whether there exist omni-potential flows other than Zeldovich/Burgers ones or trivial variants thereof. Investigating this issue is the central topic of our paper.

In Section II we show that omni-potentiality can be reexpressed geometrically and algebraically in terms of Hessian matrices and recast as a set of one or several partial differential equations (depending on the space dimension). In Section III we use an algebraic method to construct explicit non-trivial examples of omni-potential flows in two and three dimensions. These are rather special and we are led to ask how general are omni-potential flows. In Section IV we construct a fairly general class of two-dimensional omni-potential flows, leaving the problem open in higher dimensions. In Section V we return to questions of cosmological relevance: To what extent are the full solutions to the cosmological equations omni-potential? Why is the validity of MAK reconstruction better than that of the Zeldovich approximation, as pointed out by Mohayee et al. [18]? In Section VI we list some open problems and make concluding remarks. Finally, in the Appendix we characterize sets of commuting symmetric matrices by constructing suitable invariants.

II. CRITERIA FOR OMNI-POTENTIALITY OF FLOWS

In the present paper we study the kinematics of omni-potential flows. We start by recalling the basic definitions of the Lagrangian and Eulerian description of a flow — a motion of fluid regarded as a continuum of infinitesimal fluid particles (whose mathematical abstraction is a point particle).

Denote by $v(x, t)$ the velocity of the fluid measured at point $x$ in space and at time $t$. It is usually called the Eulerian velocity (the velocity measured at a fixed position in the laboratory frame). The motion of a fluid particle satisfies the ordinary differential equation

$$\dot{x} = v(x, t),$$

which has to be supplemented by the initial condition

$$x|_{t=0} = q.$$

If the velocity field $v(x, t)$ is prescribed and sufficiently regular, one can solve this initial value problem, at least locally in time, and obtain the mapping $q \mapsto x(q, t)$ called the Lagrangian map. It takes a particle at the Lagrangian position $q$ and carries it to the Eulerian position $x$. For a fixed $q$, the curve $x(q, t)$, parameterized by varying time $t$, is the trajectory of the particle, whose Lagrangian position is $q$. When, in a field associated with the flow, we perform the substitution $x \mapsto x(q, t)$, we obtain its Lagrangian description, in which the field is now a function of the Lagrangian coordinates $q$. For instance, $v(x(q, t), t)$ is called the Lagrangian velocity.

In this paper, we consider flows $v(x, t)$ defined, for simplicity, in the entire space $\mathbb{R}^d$, but restricted to a finite time interval $[0, T]$. The flow induces a set of mappings of space: given two arbitrary times $t$ and $\tau$, such that $0 \leq t < \tau \leq T$, the mapping from fluid particle positions at time $t$ to their positions at time $\tau$ is here called the $(t, \tau)$-mapping. The $(0, \tau)$-mapping is just the standard Lagrangian map.

As stated in the Introduction, for any two times $t$ and $\tau$, such that $0 \leq t < \tau \leq T$, the $(t, \tau)$-mappings induced by omni-potential flows are required to be the gradients of a convex potential $\Phi(q, t; \tau)$:

$$q \mapsto x = \nabla q \Phi(q, t; \tau). \quad (3)$$

Such a mapping is here called potential and $\Phi(q, t; \tau)$ is called the $(t, \tau)$-potential. Given any three times $t_0$, $t$ and $\tau$ such that $0 \leq t_0 \leq t \leq \tau \leq T$, the $(t_0, t)$-mapping composed with the $(t, \tau)$-mapping obviously yields the $(t_0, \tau)$-mapping. This semigroup associativity, combined with omni-potentiality, implies

$$\Phi(q, t_0; \tau) = \Phi(\nabla q \Phi(q, t_0; t), t; \tau), \quad (4)$$

where
whose Hessian is the identity. As the two times separate, loss of convexity would require one or several eigenvalues of the Hessian to change sign and thus to go through zero; at such an instant, the Jacobian matrix (which for a potential mapping coincides with the Hessian of the potential) becomes degenerate; then, generically, the inverse mapping ceases to exist, i.e., the property of the \((t, \tau)\)-mapping to be bijective gets lost (in the cosmological context this amounts to shell crossing leading to multi-streaming).

Below, we establish criteria for omni-potentiality. In Sec. II A we prove that a flow is omni-potential, whenever the Hessians of the potentials \(\Phi(q, t; \tau)\), calculated at any two points of a trajectory, commute. In Sec. II B we present an equivalent condition for the commutativity of Hessians: along any trajectory, at any time \(t\), the Hessian \(H(t) = H(\Phi(q; 0, t))\) and its time derivative should commute; this is used to show that omni-potentiality is equivalent to potentiality of both the Lagrangian and the Eulerian flow velocities (plus the convexity constraints). In Sec. II C we discuss simple examples of omni-potentiality: Zeldovich and Zeldovich-type flows. Finally, in Sec. II D we derive a partial differential equation for the potential of a two-dimensional omni-potential flow. It states that a suitable expression constructed from second-order derivatives depends only on the Lagrangian coordinates, but not on time. In other words, the Hessian of the potential possesses an invariant, whose value depends only on the trajectory. We construct similar invariants in higher-dimensional spaces in the Appendix.

A. Commutation of Hessians of the potential

The semigroup associativity (4) involves the composition of two potential maps. We shall show that, in general, such a composition is potential if and only if the Hessians commute. Basically this stems from the well-known theorem that the product of two symmetric matrices is symmetric only when they commute. The problem we are now addressing is illustrated in Fig. 1, which sketches the action of three mappings along the same trajectory. We observe that the \((t, \tau)\)-mapping is the composition of the inverse of the \((t_0, t)\)-mapping with the \((t_0, \tau)\)-mapping. We shall now show that its potentiality is equivalent to the commutation of the Hessians of the \((t_0, t)\)-mapping and of the \((t_0, \tau)\)-mapping.

Specifically, we assume that, for any times \(t\) and \(\tau\), such that \(0 \leq t \leq \tau \leq T\), the \((t, \tau)\)-mapping (3) of \(\mathbb{R}^d\) into itself is a bijection that, together with its inverse, is smooth (i.e., has as many continuous derivatives, as we might need). We denote

\[\Phi_1(q) \equiv \Phi(q, t_0; t), \quad \Phi_2(q) \equiv \Phi(q, t_0; \tau),\]

where \(\Phi(q, t_0; t)\) and \(\Phi(q, t_0; \tau)\) are the potentials of the \((t_0, t)\)-mapping and the \((t_0, \tau)\)-mapping, respectively. The required potentiality of the \((t, \tau)\)-mapping implies
that the Jacobian matrix $\| \frac{\partial \xi}{\partial x} \|$ is symmetric:

$$\partial_{x_i} \xi_i = \partial_{x_j} \xi_j$$

for any pair of indices $1 \leq i, j \leq d$. The converse is also true, at least locally in space. By the chain rule,

$$\mathcal{H}_{mn}(\Phi_2) = \frac{\partial q_n}{\partial x_m} \xi_n = \sum_{k=1}^{d} \partial_{x_k} \xi_m \frac{\partial q_k}{\partial x_n} = \sum_{k=1}^{d} \partial_{x_k} \xi_m \mathcal{H}_{kn}(\Phi_1).$$

Therefore,

$$\| \frac{\partial \xi}{\partial x} \| = \mathcal{H}(\Phi_2)\mathcal{H}^{-1}(\Phi_1), \quad (5)$$

where $\mathcal{H}^{-1}(\Phi_1)$ denotes the matrix inverse to $\mathcal{H}(\Phi_1)$. For the r.h.s. of (5) to be a symmetric matrix, the matrices $\mathcal{H}(\Phi_2)$ and $\mathcal{H}^{-1}(\Phi_1)$ must commute, which is equivalent to the commutation of the two Hessians $\mathcal{H}(\Phi_2)$ and $\mathcal{H}(\Phi_1)$.

Reversing the arguments, we establish that commutation of Hessians $\mathcal{H}(\Phi(q, t_0; t))$ along a trajectory,

$$\mathcal{H}(\Phi(q, t_0; t))\mathcal{H}(\Phi(q, t_0; \tau)) = \mathcal{H}(\Phi(q, t_0; \tau))\mathcal{H}(\Phi(q, t_0; t)),$$

at any times $t$ and $\tau$ is necessary for the mapping $\xi(x)$ to be potential. In particular, commutation of Hessians of the potential $\Phi(q, 0; t)$ of the Lagrangian map along each trajectory (for each fixed $q$), together with invertibility or convexity, is equivalent to omni-potentiality.

By the theorem on codiagonalizability of commuting (real) symmetric matrices (see, e.g., Ref. [15], pp. 50–51) the equivalent condition is that the Hessians of the potential $\mathcal{H}(\Phi(q, t_0; t))$ calculated at different times $t$ for the same coordinate $q$ and the same $t_0$ are codiagonalizable, i.e., can be transformed into the diagonal form using the same unitary matrix. In other words, along any trajectory, only the eigenvalues of the Hessian of the potential but not the eigendirections are allowed to vary.

So far, we have shown the commutation — along a given trajectory — of the Hessians of the potentials of the $(t_0, t)$-mapping and the $(t_0, \tau)$-mapping for the same starting time $t_0$ (e.g., for the $(t_0, t_\tau)$-mapping and the $(t_0, t_1)$-mapping of Fig. 2). A similar argument proves the commutation of the Hessians of the potentials of two mappings, such that the ending time of one of them coincides with the starting time of the second one (e.g., for $(t_0, t_1)$-mapping and the $(t_1, t_\tau)$-mapping). Combining these two results and relying again on the theorem on codiagonalizability of symmetric commuting matrices, we find that the Hessians of the potentials of the $(t_0, t_\tau)$-mapping and of the $(t_1, t_\tau)$-mapping commute. Thus we have established that, along any given trajectory, the Hessians associated with arbitrary pairs of times commute.

B. Commutation of Hessians and their time derivatives: bi-potential velocities

Here we give an alternative formulation of omni-potentiality in terms of commutation of Hessians and their time derivatives. We need some preparatory material regarding $d \times d$ symmetric matrices with smooth time dependence. Suppose at any two times $t$ and $t'$ they commute:

$$H(t)H(t') = H(t')H(t). \quad (6)$$

Differentiating this equation in $t'$ and letting $t' = t$, we find that at any time $H(t)$ commutes with its time derivative $\dot{H}(t)$:

$$H(t)\dot{H}(t) = \dot{H}(t)H(t). \quad (7)$$

We shall show now that the converse is also true: if at any time $t$ (i) relation (7) is satisfied, and (ii) all eigenvalues $\lambda_i$ of the symmetric matrix $\dot{H}(t)$ are distinct, then (6) holds true for any times $t$ and $t_1$.

Since $H(t)$ is symmetric, it can be expressed as

$$H(t) = U^t(t)\Lambda(t)U(t), \quad (8)$$

where $U$ is a unitary $d \times d$ matrix, and $\Lambda$ is diagonal. Consider the identity $U(t)U^t(t) = I$, where $I$ is the identity matrix; differentiating it in time yields

$$U\dot{U}^t = -\dot{U}U^t = -\left(U\dot{U}^t\right)^t.$$

Thus $X = U\dot{U}^t$ is an antisymmetric matrix.

We substitute (8) into (7), and multiply it by $U$ on the left and by $U^t$ on the right, obtaining

$$\Lambda X\Lambda - \Lambda^2 X = X\Lambda^2 - \Lambda X\Lambda,$$

i.e., the matrix $X\Lambda - \Lambda X$ commutes with $\Lambda$. This matrix is symmetric due to the antisymmetry of $X$. By the
theorem on codiagonalizability of commuting symmetric matrices, the matrices $X \Lambda - \Lambda X$ and $\Lambda$ are thus simultaneously diagonalizable; but $\Lambda$ is already diagonal, and hence so is $X \Lambda - \Lambda X$. The entries of the latter matrix are $(\lambda_j - \lambda_i) X_{ij}$, and therefore $X_{ij} = 0$ for all $i \neq j$ (we use here the condition that all eigenvalues of $H$ are distinct). The antisymmetry of $X$ implies that all diagonal entries of $X$ also vanish, and thus $X = 0$. Therefore,

$$\dot{U} = -XU$$

vanishes. In other words, variation of $H$ in time consists solely of variation of its eigenvalues $\lambda_i$. This implies (6).

The restriction that all the eigenvalues be distinct does not significantly affect the generality of the statement: if they are not distinct at some isolated times, the relation $\dot{U} = 0$ remains satisfied at these times by continuity.

Returning to the problem of omni-potentiality, we now take

$$H(t) = H(\Phi(q,0; t)) = \equiv H.$$  

This Hessian $H$ is the Jacobian of the Lagrangian map $\nabla \Phi(q,0; t)$. By the above result, omni-potentiality is equivalent to the commutation, at any time, of the Jacobian and its time derivative.

We have seen earlier that omni-potential flow has a velocity which is bi-potential, i.e., potential in both Eulerian and Lagrangian coordinates. The statement just derived allows to prove the converse, namely that a flow with a bi-potential velocity (and some convexity requirements) is omni-potential. Let us denote by $v^L(q,t)$ and $v^E(x,t)$ the Lagrangian and Eulerian velocity, respectively. Since $x(q,t)$ is the Lagrangian map, we obviously have

$$v^E(x,t) = v^L(q(x,t),t),$$  

where $q(x,t)$ is the inverse Lagrangian map, whose Jacobian is $H^{-1}$. We now calculate the Eulerian velocity gradient, using (9). By the chain rule, for any $i$ and $j$, we have:

$$\partial_{x_i} v^E_j(x,t) = \sum_{m=1}^{d} (H^{-1})_{im} \partial_{q_m} v^L_j(q,t)$$
$$= \sum_{m=1}^{d} (H^{-1})_{im} \partial_{q_m} \Phi(q,0; t)$$
$$= \sum_{m=1}^{d} (H^{-1})_{im} \dot{H}_{mj}.$$  

Thus, the Eulerian velocity gradient is the product of the matrices $H^{-1}$ and $\dot{H}$. For the Eulerian velocity to be potential, it is necessary and (locally) sufficient that this product be a symmetric matrix. The commutation of the symmetric matrices $H^{-1}$ and $\dot{H}$ is equivalent to the commutation of $H$ and $\dot{H}$. Equivalence to omni-potentiality follows from the statement derived above.

C. Zeldovich and Zeldovich-type flows

In the Zeldovich approximation each particle keeps its initial velocity unaltered in the course of time, and hence particles move along straight lines (at least before multi-streaming occurs). The Lagrangian map at time $t$ is

$$q \mapsto x = \nabla_q \left( \frac{|q|^2}{2} + t\varphi_0(q) \right),$$

where $\varphi_0(q)$ is the velocity potential, prescribed at $t = 0$. The Hessian of this map is $I + t\dot{H}(\varphi_0)$, where $I$ is the identity matrix and the matrix $\dot{H}(\varphi_0)$ is the Hessian of the initial potential. For a given $q$, the eigendirections of the associated Hessian are those of the Hessian of the initial velocity potential. Clearly, all these Hessians commute and, by the results of Sec. II A, such a flow is omni-potential.

More general examples of omni-potential flows can be constructed by performing an arbitrary nonlinear transformation of the time and by time-dependent zooming of space. In space of any dimension $d \geq 2$, consider the flows defined by the potentials

$$\Phi(q,0; t) = \mu(t) \frac{|q|^2}{2} + \eta(t) \varphi_0(q),$$  

(10)

where $\mu(t)$ and $\eta(t)$ are arbitrary functions of time. Clearly, these are again omni-potential.

In general, the trajectories associated with (10) are not straight lines. However, if we look at them with a time-dependent magnifying glass which applies a zooming factor $1/\mu(t)$, they become straight. Furthermore, if we introduce a new time variable $t' = \eta(t)/\mu(t)$, particles move again with a constant velocity. Hence, the flows defined by (10) are trivial generalizations of Zeldovich flows, and will here be called Zeldovich-type flows.

Our goal is to find omni-potential flows that are not of this type.

D. A linear second-order PDE for two-dimensional omni-potential flow

We derive here a differential equation for the potential of a two-dimensional omni-potential flow. It turns out to be a linear second-order PDE.

Consider a symmetric $2 \times 2$ matrix $H$. Suppose its eigenvector associated with eigenvalue $\lambda$ makes angle $\theta$ with the cartesian axis $q_1$. Thus,

$$H_{11} \cos \theta + H_{12} \sin \theta = \lambda \cos \theta,$$
$$H_{12} \cos \theta + H_{22} \sin \theta = \lambda \sin \theta.$$  

In order to eliminate the eigenvalue $\lambda$, we multiply the first of these equations by $\sin \theta$ and the second one by
cos $\theta$. Subtracting afterwards the second equation from the first one, we obtain

$$\frac{H_{11} - H_{22}}{H_{12}} = \cot 2\theta. \quad (11)$$

Prescribing the r.h.s. of (11) uniquely defines the orthogonal frame of the two eigendirections. (The values of $\cot 2\theta$ define the angle $\theta$ modulo $\pi/2$; however, changing $\theta \to \theta + \pi/2$ swaps the eigendirections, but does not affect the set of eigendirections.)

In an omni-potential flow, the eigendirections of the Hessians of the $(0,t)$-potentials should depend only on the Lagrangian position $q$ and not on the time $t$. Let $\Phi(q,t)$ be a two-dimensional omni-potential flow and let us denote by $g(q)$ the common value of $\cot 2\theta$ along the particle trajectory emanating from $q$. It then follows from (11) that

$$(\partial_{q_1 q_1}^2 - \partial_{q_2 q_2}^2)\Phi = g(q) \partial_{q_1 q_2}^2 \Phi. \quad (12)$$

The search for two-dimensional omni-potential flow has thus been reduced to finding solutions to (12) for suitably prescribed functions $g(q)$.

### III. EXAMPLES OF OMNI-POTENTIAL FLOWS IN TWO AND THREE DIMENSIONS

The main question that we address in this paper is whether omni-potential flows exist that are not of Zeldovich type. In this section we give a positive reply to this question both in the two- and three-dimensional spaces by providing explicit examples of polynomial potentials for mappings induced by such flows.

An example of an omni-potential flow in a space of arbitrary dimension is provided by spherically-symmetric potentials of the form $\Phi(q(t),t)$: a simple calculation reveals the commutation of Hessians of such potentials, calculated at different times at different points of a trajectory. This example shares with Zeldovich flows the property that the trajectories are straight lines — in this case, in the radial direction. We would like to construct less symmetric examples.

#### A. Particular examples of two-dimensional omni-potential flow

In $\mathbb{R}^2$, the problem of finding omni-potential flows has been recast into the form of the partial differential equation (12) with the initial condition $|q|^2/2$ (which generates the identity map). We can therefore construct examples of two-dimensional omni-potential flows by finding different solutions to (12) for a given function $g(q)$ in the r.h.s. By linearity, any linear combination of such solutions with time-dependent coefficients is also a solution to (12). For example, if $\Phi_1(q)$ and $\Phi_2(q)$ are two sufficiently smooth independent solutions that are also independent of $|q|^2/2$ (which is always a solution to (12)), then the flow with the potential $|q|^2/2 + \alpha_1(t)\Phi_1(q) + \alpha_2(t)\Phi_2(q)$ is omni-potential, and is typically not of Zeldovich type; for this, the functions of time $\alpha_1$ and $\alpha_2$ must be linearly independent and sufficiently small, so as not to spoil the convexity stemming from the $|q|^2/2$ term.

When $g(q)$ is a ratio of homogeneous polynomials of $q$ (say, of the same degree $m$), solutions to (12) can be obtained by a purely algebraic method. A solution can be sought in the form of a homogeneous polynomial, $p_n^{(2)}(q)$, of degree $n \geq m + 2$; then (12) reduces to a system of $m+n-1$ equations for the coefficients of $p_n^{(2)}(q)$ and $g(q)$. (In what follows, $p_n^{(d)}$ denotes certain homogeneous polynomials of degree $n$ defined in $\mathbb{R}^d$.) The function $g(q)$ involves $2m + 1$ independent coefficients (since the numerator and denominator can be multiplied by any constant without changing $g(q)$), and the polynomial $p_n^{(2)}(q)$ involves $n$ independent coefficients (since a solution to (12) can be multiplied by any constant without yielding a new independent solution). Comparison of the number of equations, $m + n - 1$, with the total number of the unknown coefficients, $2m + n + 1$, suggests that we can construct a family of such solutions, parameterized by $m + 2$ coefficients of $g(q)$. However, the system of equations for the coefficients is, in general, nonlinear, and hence its solvability cannot be established just by counting the numbers of the unknowns and equations. When $g(q)$ is the ratio of linear functions, the equations for the coefficients of $p_n^{(2)}(q)$ are linear and can be solved for any prescribed coefficients of $g(q)$.

Since the potential $\Phi$ is required to be a convex function on the entire plane $\mathbb{R}^2$, we start by seeking homogeneous polynomials $p_n^{(2)}(q)$ involving only even powers of $q_1$ and $q_2$. An instance of a solvable linear system of equations yielding the coefficients of such polynomials is obtained for

$$g(q) = \frac{aq_1^2 - bq_2^2}{q_1 q_2}, \quad (13)$$

where the coefficients $a$ and $b$ may take arbitrary preset values. For such $g(q)$, a homogeneous polynomial of degree $2k$, $k \geq 2$, solving (12), is

$$p_{2k}^{(2)}(q_1, q_2) = \sum_{i=0}^{k} \left( i-1 \prod_{j=0}^{i-1} (2k - 1 + 2j(a-1)) \right) \times \prod_{j=0}^{k-1-i} 2(k-1) \frac{k! q_1^{2i} q_2^{2(k-i)}}{i!(k-i)!} \quad (14)$$

In particular, the first low-degree polynomial solutions are:

$$p_4^{(2)}(q_1, q_2) = (2a + 1)q_1^4 + 6q_1^2 q_2^2 + (2b + 1)q_2^4, \quad (15)$$

$$p_6^{(2)}(q_1, q_2) = (4a + 1)(2a + 3)q_1^6 + 15(2a + 3)q_1^4 q_2^2 + 15(2b + 3)q_1^2 q_2^4 + (4b + 1)(2b + 3)q_2^6. \quad (16)$$
As one can see, the polynomial (14) vanishes identically for integer \( j \geq 1 \) and \( \tilde{j} \geq 1 \) such that \( j + \tilde{j} \leq k - 1 \), and

\[
\tilde{a} = 1 - \frac{2k - 1}{2j}, \quad \tilde{b} = 1 - \frac{2k - 1}{2j}.
\]  
(17)

For these isolated values in the plane of parameters, there exist nevertheless two independent solutions, namely

\[
\frac{\partial}{\partial a} p_{2k}^{(2)} \bigg|_{a = \tilde{a}, \, b = \tilde{b}} \quad \text{and} \quad \frac{\partial}{\partial b} p_{2k}^{(2)} \bigg|_{a = \tilde{a}, \, b = \tilde{b}}.
\]

This can be easily seen by differentiating (12) in \( a \) and \( b \) and substituting the parameter values (17).

Clearly, \( p_{2k}^{(2)}(q) \) is convex, if all coefficients in (14) are non-negative (this condition is sufficient, but not necessary) and tend to zero fast enough to guarantee convergence of the series at any point \( q \) and termwise differentiability of (19) in the spatial variables. If the sum (19) is finite and the maximum degree of the polynomials involved is \( 2K \), then the potential is convex for 

\[
\min(a, b) \geq -1/(2K - 2).
\]  
(18)

Thus, the potentials

\[
\Phi(q, t) = \mu_2(t) \frac{|q|^2}{2} + \sum_{k \geq 2} \mu_{2k}(t) p_{2k}^{(2)}(q_1, q_2)
\]  
(19)

are convex for \( \min(a, b) \geq 0 \), if in addition all \( \mu_{2k}(t) \) are non-negative (this condition is sufficient, but not necessary) and tend to zero fast enough to guarantee convergence of the series in any point \( q \) and termwise differentiability of (19) in the spatial variables. If the sum (19) is finite and the maximum degree of the polynomials involved is \( 2K \), then the potential is convex for 

\[
\min(a, b) \geq -1/(2K - 2).
\]  
(18)

The initial condition is satisfied provided \( \mu_2(0) = 1 \) and \( \mu_{2k}(0) = 0 \) for all \( k > 1 \). The convex potentials (19) satisfy all requirements for omni-potential flows in the plane, and are not of Zeldovich type.

So far, we have considered only even-degree homogeneous polynomial solutions or linear combinations thereof. Is an admixture of odd-degree homogeneous polynomials permitted? If such an odd-degree addition has a degree higher than that of the highest even-degree homogeneous polynomial comprised in the solution, then convexity in the whole plane is ruled out. However, in a finite domain, convexity need not be lost, provided the odd polynomial is scaled by a sufficiently small factor. This is precisely what happens when \( g(q) \) is given by (13): a homogeneous polynomial of odd degree \( 2k + 1 \) can be a solution to (12) only for \( k \geq 2 \) and

\[
a = b = -1/(k - 1).
\]  
(20)

The solution for these parameter values is

\[
p_{2k+1}^{(2)}(q_1, q_2) = c_1 p_k(q_1, q_2) + c_2 q_2, \quad p_{2k}(q_1, q_2),
\]

where \( p_k(q_1, q_2) = \sum_{i=0}^{k-1} \prod_{j=0}^{i-1} \frac{(2k - j + 1)(k - 1 - j)}{(j + 1)(2j - 1)} \times q_1^{2i} q_2^{2(k-i)+1} \), and \( c_1 \) and \( c_2 \) are arbitrary constants. For instance, for \( a = b = -1 \) fifth-degree polynomial solutions are

\[
p_{5}^{(2)}(q_1, q_2) = c_1 q_1^5 - 5c_1 q_1^3 q_2^2 - 5c_2 q_1^2 q_2^3 + c_2 q_2^5.
\]

Comparison of conditions (18) and (20) shows that the degree of an odd-degree polynomial solution for \( g(q) \) defined by (13) is higher than the degree of any convex even-degree polynomials existing for the chosen \( g(q) \).

B. Examples of omni-potential flow in dimension \( d \geq 3 \)

The approach that has been applied in the previous subsection for construction of an example in dimension two cannot be immediately generalized to higher-dimensional spaces: while in \( \mathbb{R}^2 \) a single invariant determines whether two symmetric matrices are codiagonalizable and hence equation (11) fixes the set of eigendirections of a symmetric matrix \( H \), in \( \mathbb{R}^3 \) at least three such invariants must be considered simultaneously (see the Appendix). In dimension three, equations (A7)–(A9) applied for the entries of the Hessian of an unknown potential give rise to three partial differential equations in the potential,

\[
\frac{\partial^2 g_{2, q_1, q_2}}{\partial q_1^2} \Phi + \frac{\partial^2 g_{2, q_1, q_2}}{\partial q_2^2} \Phi + \frac{\partial^2 g_{2, q_1, q_2}}{\partial q_3^2} \Phi
\]

\[
= (g_1(q) + \frac{\partial^2 g_{2, q_1, q_2}}{\partial q_2^2} \Phi) - \frac{\partial^2 g_{2, q_1, q_2}}{\partial q_1^2} \Phi,
\]  
(21)

\[
\frac{\partial^2 g_{2, q_2, q_3}}{\partial q_1^2} \Phi + \frac{\partial^2 g_{2, q_1, q_2}}{\partial q_3^2} \Phi + \frac{\partial^2 g_{2, q_1, q_2}}{\partial q_1^2} \Phi
\]

\[
= (g_1(q) + \frac{\partial^2 g_{2, q_1, q_2}}{\partial q_1^2} \Phi) - \frac{\partial^2 g_{2, q_1, q_2}}{\partial q_3^2} \Phi,
\]  
(22)

\[
\frac{\partial^2 g_{2, q_2, q_3}}{\partial q_2^2} \Phi + \frac{\partial^2 g_{2, q_1, q_2}}{\partial q_3^2} \Phi + \frac{\partial^2 g_{2, q_1, q_2}}{\partial q_1^2} \Phi
\]

\[
= (g_1(q) + \frac{\partial^2 g_{2, q_1, q_2}}{\partial q_3^2} \Phi) - \frac{\partial^2 g_{2, q_1, q_2}}{\partial q_2^2} \Phi,
\]  
(23)

that must be satisfied simultaneously. Here the time-independent quantities \( g_k(q) \) are related to the invariants \( \gamma_{21}^{(3,k)} \) introduced in the Appendix:

\[
g_1(q) = \gamma_{21}^{(3,1)} + \gamma_{21}^{(3,3)},
\]

\[
g_2(q) = \gamma_{21}^{(3,2)} + 1,
\]

\[
g_3(q) = \gamma_{21}^{(3,3)}.
\]
take into account the properties of the Hessian, stemming from the specific structure of its entries (for instance, each row and a column of the Hessian is a gradient, which implies certain differential relations between the entries). It is unclear how to prescribe \( g_{\alpha}(q) \), taking into account these additional constraints, for the equations to have at least two distinct solutions.

Because of this difficulty, instead of considering the invariants and solving equations \((21)–(23)\), we exploit the fact that omni-potentiality of flows amounts to commutation of the Hessians of the various \((t_1, t_2)\)-mappings along any trajectory (see Sec. II A). We shall construct our examples in \( \mathbb{R}^d \) using the following strategy. The potential is sought in the form of a linear combination of “building blocks” with time-dependent coefficients. One of the building blocks is prescribed; we take it to be a homogeneous polynomial, \( p_m^{(d)}(q) \), of degree \( m \). All the other building blocks are then required to have their Hessians commuting with that of the prescribed building block. The function \( |q|^2 \), whose Hessian is the identity matrix, constitutes a trivial solution. We can try finding other building blocks in the form of homogeneous polynomials \( \tilde{p}_n^{(d)}(q) \) of some degree \( n \). If we succeed, it is easy to show that any linear combination of such building blocks (with the convexity restriction) will define an omni-potential flow. We seek such polynomials by requiring the vanishing of the non-diagonal entries of the commutator of the two Hessians, viz.

\[
\begin{align*}
C(p_m^{(d)}, p_n^{(d)}) &\equiv \mathcal{H}(p_m^{(d)})\mathcal{H}(p_n^{(d)}) - \mathcal{H}(p_n^{(d)})\mathcal{H}(p_m^{(d)}).
\end{align*}
\]  

(24)

Unfortunately, in general this strategy does not work, as now explained. A homogeneous polynomial of degree \( n \) has

\[
\frac{(n + d - 1)!}{n!(d - 1)!}
\]

coefficients. Non-diagonal entries of the commutator \( C \) are homogeneous polynomials of degree \( m + n - 4 \). The commutator is antisymmetric (recall that the Hessians are symmetric matrices), hence we have to consider the \( d(d - 1)/2 \) non-diagonal entries of \( C \). Thus, in general, we have to solve

\[
\frac{d(m + n + d - 5)!}{2(m + n - 4)!(d - 2)!}
\]

equations, a number which is easily seen to exceed the number of coefficients,

\[
\frac{(m + d - 1)!}{m!(d - 1)!} + \frac{(n + d - 1)!}{n!(d - 1)!}.
\]

So, the problem is overdetermined.

Nevertheless, potentials having all the required properties can be constructed in \( \mathbb{R}^d \) \((d \geq 2)\), if all the building blocks are restricted to be homogeneous polynomials symmetric in all their arguments (i.e., invariant under any permutation of the spatial variables \( q_i \leftrightarrow q_j \)). Such building blocks have the following significant advantage: it suffices to consider any of the polynomial equations arising from non-diagonal entries of the commutator \((24)\) (referred to as “commutator equations”) – all these equations are equivalent by virtue of the symmetry. In what follows, we implement this “symmetric building block strategy” in two cases, in \( \mathbb{R}^d \) for \( d \geq 3 \) with just one unknown building block, and in \( \mathbb{R}^3 \) with infinitely many ones.

Now, we focus on the symmetric polynomials

\[
\begin{align*}
p_1^{(d)}(q) &= \sum_{i=1}^{d} q_i^4 + \tilde{c} \sum_{i=1}^{d-1} q_i^2 q_j^2, \quad (25) \\
p_0^{(d)}(q) &= \sum_{i=1}^{d} q_i^6 + \tilde{a} \sum_{i=1}^{d-1} q_i^4 q_j^2 + \tilde{b} \sum_{1 \leq i < j < k \leq d} q_i^2 q_j^2 q_k^2. \quad (26)
\end{align*}
\]

(When \( d = 3 \), the last sum in \( p_0^{(d)}(q) \) reduces to a single term \( \tilde{b} q_1^2 q_2^2 q_3^2 \)). We consider polynomials involving only even powers of the spatial variables \( q_j \), because we seek solutions that are convex functions.

To be specific, we consider the commutator equation \( C_{12}(p_0^{(d)}(q), p_1^{(d)}(q)) = 0 \) in \( \mathbb{R}^d, d \geq 3 \). The l.h.s. is a polynomial of degree 6. Since in \( p_1^{(d)}(q) \) and \( p_0^{(d)}(q) \) any power of \( q_1 \) and \( q_2 \) is even, \( C_{12} \) is proportional to \( q_1 q_2 \), and every variable enters into the polynomial \( C_{12}/(q_1 q_2 q_3^2 - q_1 q_2^3) \) only in an even power. Since both potentials are symmetric in \( q_1 \) and \( q_2 \), \( C_{12} = 0 \) for \( q_1 = q_2 \), and hence \( C_{12} \) is divisible by \( q_1^2 - q_2^2 \). The polynomial \( C_{12}/(q_1 q_2(q_1^2 - q_2^2)) \) is of the second degree; it is thus just a sum of \( q_j^2 \) with certain coefficients. Because the potentials are symmetric in \( q_j \), it has the form

\[
\alpha_1 q_1^2 + \alpha_2 q_2^2 + \alpha_2 \sum_{j=3}^{d} q_j^2.
\]

Hence, we have three independent parameters, \( \tilde{a} \), \( \tilde{b} \) and \( \tilde{c} \), and two equations to satisfy. Calculating the coefficients \( \alpha_1 \) and \( \alpha_2 \) and letting them vanish, we find that the Hessians of \( p_1^{(d)}(q) \) and \( p_0^{(d)}(q) \) commute for

\[
\begin{align*}
\tilde{a} &= 15 \tilde{c}^2/(12 - \tilde{c}), \\
\tilde{b} &= 75 \tilde{c}^2/(((12 - \tilde{c})(3 + \tilde{c})).
\end{align*}
\]

(27)

(28)

Thus, the potential

\[
\Phi(q, t) = \mu_2(t) q_1^2/2 + \mu_4(t)p_1^{(d)}(q) + \mu_6(t)p_0^{(d)}(q) \quad (29)
\]

defines a non-Zeldovich-type omni-potential flow in \( \mathbb{R}^d \) \((d \geq 3)\). Polynomials \( p_1^{(d)}(q) \) and \( p_0^{(d)}(q) \) are convex provided \( 0 \leq \tilde{c} < 12 \); hence, if all \( \mu_i(t) \geq 0 \), potential \( (29) \) is convex for \( \tilde{c} \) from this interval. For \( \tilde{c} \neq 2 \), \( p_1^{(d)}(q) \) and
The polynomials $p_n^d(q)$ do not possess spherical symmetry, and hence the potential (29) is not spherically symmetric. Restrictions of $p_4^d(q)$ and $p_6^d(q)$ onto the plane $q_3 = \ldots = q_d = 0$ coincide with the polynomials $p_4^2(q_1, q_2)$ (15) and $p_6^2(q_1, q_2)$ (16) for $a = b = (6 - c)/2(5)$.

Henceforth, for the sake of simplicity, we assume that the problem is three-dimensional. In the remainder of the subsection we shall implement the building block strategy with unknown blocks that are arbitrary even-degree polynomials $p_n^{(3)}(q)$ for $n > 2$. The polynomial $p_4^d(q)$ (25) for $d = 3$ remains our prescribed building block. By the theorem on codiagonalizability of symmetric matrices, commutation with the Hessian of $p_4^{(3)}(q)$ implies, that the Hessians of any two polynomials from this family commute. (This is taking place generically, i.e., at those points in $\mathbb{R}^3$, where the Hessian of $p_4^{(3)}(q)$ does not possess equal eigenvalues; at non-generic points the commutation follows from continuity of the Hessians and the commutation at generic points, which are present in any neighborhood of a non-generic point.)

The polynomial

$$p_n^{(3)}(q) = \sum_{i,j,k>0} a_{i,j,k} q_1^{i} q_2^{j} q_3^{k}$$

is symmetric, whenever $a_{i,j,k}$ does not depend on the order of subscripts $i, j, k$.

Straightforward algebra yields

$$C_{12}(p_n^{(3)}, p_4^{(3)}) = 8q_1 q_2 \sum_{i,j,k>0} a_{i,j,k} q_1^{2i} q_2^{2j} q_3^{2k} \times \left[i j (c - 6)(q_1^2 - q_2^2) + c (-j(2j - 1 + 2k)q_1^2 + i(2i - 1 + 2k)q_2^2)\right].$$

Collecting similar terms in this sum, we find that it vanishes as long as

$$\tilde{a}_{i,j,k} = \tilde{a}_{i+1,j-1,k} \chi_j/\chi_{i+1}$$

for any $i, j$ and $k$ such that $i + j + k = n$, where we have denoted

$$\chi_m = (c(2n + 2 - 3m) + 6(m - 1))/m.$$

Relation (32) can be regarded as a recurrence for coefficients $\tilde{a}_{i,j,k}$ for a fixed $k$. For $k = 0$, we start the recurrence assuming $a_{n,0,0} = 1$. This yields

$$\tilde{a}_{i,n-1,0} = \frac{\prod_{m=1}^{n-i} \chi_m \prod_{m=1}^{i} \chi_m}{\prod_{m=1}^{n} \chi_m}.$$

We obtain now the starting values for the recurrence (32) for $k > 0$ setting, in view of (31),

$$\tilde{a}_{n-k,0,k} = \tilde{a}_{n-k,k,k} = \frac{\prod_{m=1}^{n-k} \chi_m \prod_{m=1}^{k} \chi_m}{\prod_{m=1}^{n} \chi_m}.$$

and find

$$\tilde{a}_{i,j,k} = \frac{\prod_{m=1}^{i} \chi_m \prod_{m=1}^{j} \chi_m}{\prod_{m=1}^{k} \chi_m}.$$ (33)

Evidently, coefficients (33) satisfy the symmetry condition (31). Hence, (30) with the coefficients (33) is a symmetric homogeneous polynomial of degree $2n$, whose Hessian commutes with the Hessian of $p_4^{(3)}(q)$. The potential

$$\tilde{\Phi}(q, t) = \mu_2(t) \frac{\chi_3^2}{2} + \sum_{n \geq 2} \mu_2(t) p_n^{(3)}(q)$$

(34)

defines an omni-potential flow of a non-Zeldovich type in $\mathbb{R}^3$ (provided the coefficients $\mu_2(t)$ tend to zero sufficiently fast to guarantee convergence of the series (34) and to allow its termwise differentiation). Since the polynomial $p_n^{(3)}(q)$ is convex for

$$0 \leq \tilde{c} < 6(n - 1)/(n - 2),$$

the potential (34) is convex if all $\mu_2(t)$ are non-negative and $0 \leq \tilde{c} \leq 6$ (or for $0 \leq \tilde{c} < 6(N - 1)/(N - 2)$, if all $\mu_2(t)$ vanish for $n > N$).

Although we have constructed our example without prescribing the invariants, it might be of interest to calculate them for the solutions that have been obtained. Straightforward calculations yield the values of the invariants for the potential $\tilde{\Phi}(q, t)$, for instance,

$$\gamma_{21}^{(3,1)} = \frac{(6 + 3\tilde{c})q_2^2 - (6 + \tilde{c})q_1^2 + 2q_2(q_2^2 - q_3^2)}{2q_1 q_2} \quad \frac{2 \tilde{c} q_1 q_2}{q_1(q_2^2 - q_3^2)},$$

which is consistent with a non-trivial dependence of the eigendirections on the trajectories (labeled by the Lagrangian coordinates). We note that, although the solution is symmetric in the spatial coordinates, the symmetry is lost in the invariant. This stems from the invariant under consideration being a nonlinear function of the projections of the eigendirections on the plane $(q_1, q_2)$, and also from the components of the eigendirections not being invariant under all permutations of coordinates (an eigenvector is invariant under a permutation of the spatial variables $q_i \leftrightarrow q_j$ provided its $i$th and $j$th components are swapped).

We have used two approaches for constructing examples of omni-potential flow. In three or more dimensions, we used the building-block strategy in which the field of eigendirections of the commuting Hessians is characterized by prescribing one of the blocks (in our construction the polynomial (25)). The problem of commutation of Hessians then reduces to three linear equations in the unknown block. These equations must be satisfied simultaneously, and we found that in general no solution exists for an arbitrary prescribed block. In two dimensions we have followed another approach, whereby the field of eigendirections of the commuting Hessians is characterized by prescribing the set of invariants, from which the field of eigendirections of the commuting Hessians...
can be uniquely determined. In $\mathbb{R}^2$ just one such invariant, $g(q)$ (see (12)), should be considered. In $\mathbb{R}^3$ one must consider three invariants, for instance, (A7)–(A9) of the Appendix, giving rise to three nonlinear equations (21)–(23). As in the former approach, these equations must be satisfied simultaneously, and hence a solution does not exist for an arbitrary set of prescribed invariants. Thus, whichever approach is used for construction of omni-potential flows in $\mathbb{R}^3$, the prescribed data must be tuned for a solution to exist. In the former approach the equations are linear and thus simpler, but one is left with just one function which can be tuned to achieve consistency of the three equations under consideration; in the latter approach the equations are nonlinear and thus more involved, but one has the freedom of tuning three a priori independent scalar functions to gain consistency of the equations. Of course, in three dimensions, the three invariants of the Hessian of the potential cannot be prescribed as arbitrary functions. In other words, some conditions on the invariants must hold for the three equations (21)–(23) to be compatible. Can such conditions on the invariants be expressed in more explicit form remains an open mathematical problem.

IV. A WKB APPROACH TO TWO-DIMENSIONAL OMNI-POTENTIALITY

So far we have obtained special cases of non-Zeldovich-type omni-potential flows. How general are they? Can we, for example, in the two-dimensional case prescribe an arbitrary smooth initial velocity potential $\varphi_0(q)$ or, more precisely, the invariant of its Hessian $H(\varphi_0(q))$:

$$g(q) = \left[\partial_{11} - \partial_{22}\right]^{2} \varphi_0(q) = \frac{H_{11} - H_{22}}{H_{12}}$$

that appears in the general equation (12)? In this section, we use only Lagrangian coordinates, $\partial_1$ and $\partial_2$ are short for $\partial_{q_1}$ and $\partial_{q_2}$; similarly, $\partial_{11}$, $\partial_{12}$ and $\partial_{22}$ denote the second Lagrangian derivatives.

We shall now show how the construction of non-Zeldovich-type omni-potential flow with arbitrary invariant function $g(q)$ can be done, using an idea of Arnold for solving the linear equation which controls the stability of solution to the Euler equation [1]. Rather than trying to find the most general solution to (12), we construct a special short-wavelength solution through the WKB ansatz

$$\Phi(q) = e^{iS(q)} \left[ A_0(q) + \frac{1}{\kappa} A_1(q) + \frac{1}{\kappa^2} A_2(q) + \ldots \right] + cc,$$

where the wavenumber $\kappa$ is taken very large and where $cc$ stands for complex conjugate (needed because we want real solutions). In WKB parlance, $S(q)$ is called the eikonal function and the functions $A_0(q), A_1(q), \ldots$ are the amplitudes.

To the leading order, $O(\kappa^2)$, the WKB ansatz turns the linear second-order PDE (12) into the following nonlinear first-order PDE:

$$\frac{(\partial_1 S)^2 - (\partial_2 S)^2}{(\partial_1 S)(\partial_2 S)} = g(q).$$

It is easily checked that (37) is equivalent to the statement that, in the leading order, $\nabla_q S(q)$ is an eigenvector of the Hessian $H(\varphi_0(q))$. Actually, this can be seen directly, by an argument which also applies in space dimensions $d$ higher than two. Assume that the leading WKB term for the potential has a fast spatial dependence involving the phase factor $e^{iS(q)}$, then the Hessian will involve in the leading order a matrix factor $-\kappa^2(\partial_q S)/(\partial_j S)$. This is a degenerate matrix with one eigenvector of non-vanishing eigenvalue in the direction of $\nabla_q S(q)$; all perpendicular vectors are associated with the eigenvalue zero, which has multiplicity $d - 1$. Omnipotentiality requires that this degenerate matrix commute with the Hessian of the initial potential or, equivalently, that $\nabla_q S(q)$ be an eigenvector of $H(\varphi_0(q))$.

Returning to the two-dimensional case, we now construct the eikonal function $S(q)$. This construction will be done only locally in a neighborhood $\Omega$, in which the Hessian $H(\varphi_0(q))$ is sufficiently smooth and its eigenvalues are everywhere distinct. (We recall that a double eigenvalue is an event of codimension two, which typically takes place at isolated locations.) Let $n^{(1)}(q)$ and $n^{(2)}(q)$ be two unit eigenvectors of $H(\varphi_0(q))$, chosen to depend smoothly on $q$ in $\Omega$. The condition that the gradient of the eikonal function be parallel to an eigendirection can now be expressed as

$$n^{(1)}(q) \cdot \nabla_q S(q) = 0 \quad \text{or} \quad n^{(2)}(q) \cdot \nabla_q S(q) = 0. \quad (38)$$

In words, these equations state that either $n^{(1)}(q)$ or $n^{(2)}(q)$ is normal to the level lines of the eikonal function. Equivalently, the level lines of $S$ are the integral curves defined by either $n^{(1)}(q)$ or $n^{(2)}(q)$. These form a set of orthogonal curves. We thus have two classes of solutions. We can prescribe $S$ arbitrarily on one of these curves $\mathcal{C}$ and extend it locally by demanding that it remains constant on all the curves orthogonal to $\mathcal{C}$. Note that these orthogonal curves play here the role of rays in geometrical optics and are thus conveniently called “rays”.

Next we write the equations for subleading corrections obtained by substituting (36) in (12) and identifying the coefficients of the various positive and negative powers of the large parameter $\kappa$. We shall only write the equations appearing at orders $\kappa^1$ and $\kappa^0$ (the higher-order equations have a similar structure). For what follows it is convenient to use the compact notation introduced by Monge in his theory of surfaces: $\hat{\rho}, \hat{q}, \hat{r}, \hat{s}$ and $\hat{t}$ stand respectively for $\hat{\partial}_1 S, \hat{\partial}_2 S, \hat{\partial}_{11} S, \hat{\partial}_{12} S$ and $\hat{\partial}_{22} S$. (We added hats to avoid possible confusions.) Furthermore, we write $g$ for $g(q)$. The leading-order equation, (37), repeated for
convenience, and the two first subleading equations are:
\[ \ddot{p}^2 - \dot{q}^2 - g \dot{p} \dot{q} = 0, \]  
\[ (\ddot{\hat{r}} - g \ddot{s} - \hat{t}) A_0 + 2(\dot{\hat{p}} \dot{\hat{r}} A_0 - \dot{\hat{q}} \dot{\hat{2}} A_0 - \ddot{\hat{q}} A_0 = 0, \]  
\[ (\ddot{\hat{r}} - g \ddot{s} - \hat{t}) A_1 + 2(\dot{\hat{p}} \dot{\hat{r}} A_1 - \dot{\hat{q}} \dot{\hat{2}} A_1 - \ddot{\hat{q}} A_1 = 0. \]

Using (39) to eliminate the function \( g \), we can rewrite (40) and (41) as
\[ \dot{\hat{q}} \partial \dot{\hat{1}} A_0 - \dot{\hat{q}} \partial \dot{\hat{2}} A_0 + \frac{\dot{\hat{p}} (\ddot{\hat{r}} - \dot{\hat{s}} - \ddot{\hat{q}} A_0 = 0, \]  
\[ \dot{\hat{q}} \partial \dot{\hat{1}} A_1 - \dot{\hat{q}} \partial \dot{\hat{2}} A_1 + \frac{\dot{\hat{p}} (\ddot{\hat{r}} - \dot{\hat{s}} - \ddot{\hat{q}} A_1 = 0. \]  

Equation (42) is a first-order linear homogeneous transport equation for the amplitude \( A_0 \) along the rays. It can be integrated starting from arbitrary non-zero data on any curve orthogonal to the rays. Equation (43) for the amplitude \( A_1 \) is of the same sort, except that it has an inhomogeneous term involving \( A_0 \). We may thus take vanishing data for \( A_1 \) on an arbitrary curve orthogonal to the rays. Higher-order amplitudes satisfy similar inhomogeneous transport equations.

Now we construct locally in space and time an omni-potential flow having a given invariant function \( q(q) \). We take the initial potential \( \varphi_0(q) \) arbitrary, but sufficiently smooth. Hence, by the WKB method described above we can construct a smooth eikonal function \( S(q) \) and smooth amplitude functions \( A_0(q), A_1(q), \ldots \). This, in principle, yields a smooth solution, \( \Phi(q) \), to (12). (We shall not address here the issue of the convergence of the WKB series (36).) Because of the imaginary exponential dependence on \( \kappa \), the potential \( \Phi(q) \) has very large second spatial derivatives \( O(\kappa^2) \) and has no reason to be convex. However, the following time-dependent potential defines an omni-potential Lagrangian map:
\[ \Phi(q, t) = \frac{|q|^2}{2} + t \varphi_0(q) + f(t) \frac{1}{\sqrt{\kappa^2}} \Phi(q). \]  

Here \( f(t) \) is an arbitrary smooth function of time that vanishes, together with \( \dot{f} \), at \( t = 0 \). For example, we can take \( f(t) = t^2 \). For sufficiently small \( t \) and sufficiently small \( \epsilon \), the last two terms in the r.h.s. of (44) will not spoil the convexity of the first term. We have thus constructed (locally) omni-potential flows for a quite arbitrary initial velocity. For large \( \kappa \), the trajectories resulting from (44) differ only minutely from the straight Zeldovich trajectories associated with the two first terms. However, these flows are not of Zeldovich type because of the third term in the r.h.s.

If we try to extend the above WKB procedure from two to three dimensions, we encounter an obstacle already in constructing the eikonal function \( S \). As we have seen, its gradient with respect to the Lagrangian position \( q \) should be everywhere parallel to an eigendirection of the Hessian \( H(\varphi_0) \). Denoting by \( n(q) \) a unit eigenvector, taken with a smooth \( q \)-dependence, we should then have
\[ \nabla_q S(q) = \mu(q) n(q), \]
where \( \mu(q) \) is a scalar function. In other words, the 1-form \( n(q) \cdot dq \) should have an integrating factor (a factor which makes it an exact 1-form). This is in general possible (locally) in two, but not in three dimensions.

V. COSMOLOGICAL IMPLICATIONS

So far our point of view has been kinematical: we constructed omni-potential flows without any underlying dynamical equations. In cosmology the dynamical setting is rather well known and discussed for example in Ref. [2]. Let us just recall a few salient points. The most widely accepted explanation of the large-scale structure seen in galaxy surveys is that it results from small primordial fluctuations that grew under gravitational self-interaction of collisionless cold dark matter particles in an expanding universe. The evolution of the mostly collisionless matter in the Universe is described by the Vlasov–Poisson system in the position-velocity phase space. At early times, i.e., close to the epoch of matter-radiation decoupling, the expansion of the Universe selects a single velocity solution at each position rather than a distribution of velocities. This feature persists until particle crossing (“shell-crossing” in the cosmological language), where multi-stream solutions are developed. As long as multi-streaming is ruled out or is confined to scales sufficiently small to be neglected, the Vlasov–Poisson system may be replaced by the Euler-Poisson system. Following the notation of Ref. [7] and using Eulerian comoving coordinates, \( x \), together with a time variable \( \tau \) based on the amplitude factor of the growing mode of linear theory, we can write the Euler–Poisson system as
\[ \partial_x v + (v \cdot \nabla x) v = -\frac{3}{2\tau} (v + \nabla x \varphi_g), \]  
\[ \partial_x \rho + \nabla x \cdot (\rho v) = 0, \]  
\[ \nabla^2 \varphi_g = \frac{\rho - 1}{\tau}. \]  

Here, \( v \) is the peculiar velocity, \( \rho \) the density (suitably normalized) and \( \varphi_g \) the gravitational potential. As \( \tau \to 0 \), to avoid singularities, the density must approach unity everywhere; thus the distribution of matter is in the leading order uniform as \( \tau \to 0 \). Similarly, \( v \to -\nabla x \varphi_g \) as \( \tau \to 0 \); thus the initial velocity is potential, but otherwise arbitrary. It then follows from (46) that the velocity stays potential at any later time.

Reconstruction handles the Euler–Poisson system as a two-point boundary-value problem in which the initial density is prescribed as uniform and the final (current) density is given by astronomical observations. This
is a mass transport problem whose cost function is the action associated with the Euler–Poisson equations (see Refs. [7] and [17]). Unfortunately, because this action is a rather complicated functional, we do not yet possess efficient numerical algorithms allowing us to solve this mass transport problem.

The situation simplifies with the Zeldovich approximation, which amounts to setting to zero the r.h.s. of (46). The remaining equations are then (i) the three-dimensional inviscid Burgers equation, which implies that particles are moving with constant velocity (in the coordinates here chosen), and (ii) the continuity equation expressing mass conservation. The action to be minimized for reconstruction reduces then to its kinetic energy term. As a consequence, the cost function is just the mass-weighted integral of the squared displacement of fluid particles from their initial (Lagrangian) positions $q$ to their current (Eulerian) positions $x$, as required by a theorem of Brenier [6]. After discretization, this mass transport problem becomes an assignment problem, which can be solved by efficient algorithms (see Sec. 4 of Ref. [7] and Ref. [3]). This is the essence of the Monge–Ampère–Kantorovich (MAK) reconstruction method.

Mohayaee et al. [18] tested the quality of MAK reconstruction by applying it to final states of standard N-body simulations, that were performed for various random initial conditions of cosmological relevance. The authors of [18] noted the unprecedented accuracy of the reconstructions down to a few megaparsecs. In particular, they performed comparisons between three different Lagrangian maps: (i) the map based on the N-body integration, (ii) the map obtained by applying the MAK procedure to the current density field, calculated by the N-body integration, (iii) the map obtained by applying the Zeldovich approximation, starting from the same initial condition as for the N-body simulation. The conclusion of their comparisons is that the N-body map is approximated much better by the MAK-generated map than by the Zeldovich map. This is particularly striking in their Fig. 7, which shows the negative Lagrangian divergence of the displacement $x - q$, obtained by the three methods mentioned above.

Can we understand this good performance of MAK reconstruction on sufficiently large spatial scales? First, let us make the rather obvious observation that any Lagrangian map that is the (Lagrangian) gradient of a convex function (here called the “Brenier property”) will be reconstructed exactly (in a discretized version), if we solve the associated quadratic-cost optimal transport problem, for example, by using the MAK procedure. Here is a trivial example of this: if we let an initially quasi-uniform mass distribution evolve by pure Zeldovich dynamics to a final distribution and there is no shell crossing, then the MAK reconstruction is exact. If the solutions to the Euler–Poisson equations had the Brenier property, MAK would perform an exact reconstruction, but they don’t: flows which solve the Euler–Poisson equations have no Eulerian vorticity but do generate Lagrangian vorticity [10].

However there exists a refinement of the Zeldovich approximation which possesses the Brenier property. This is given by the second-order of the Lagrangian perturbation theory [4, 8, 9, 12, 20, 24]. Here we shall not describe the Lagrangian perturbation theory in any technical details since the reader can find them in the above publications. Nevertheless, in order to discuss some of its conceptual problems, we describe briefly a few key steps. One rewrites the Euler–Poisson equations in Lagrangian coordinates, to obtain a set of nonlinear equations for the displacement $x - q$ and its space and time partial derivatives. Assuming then that in a suitable sense (see below) the displacement is small, $O(\epsilon)$, one expands the equations in powers of the small parameter $\epsilon$. Here, the only perturbed quantities are the deviations of the particle trajectory from the homogeneous Hubble flow, i.e., from a purely expanding Universe. At the first order, $O(\epsilon)$, one has the Zeldovich approximation, which, as discussed in Sec. II C, is omni-potential. In particular, the Lagrangian map is potential (by the Brenier property) and the velocity is potential in both Eulerian and Lagrangian coordinates. As we have said, with the Lagrangian perturbation theory, one may refine the Zeldovich approximation. The second-order Lagrangian perturbation theory (usually denoted by L2) captures significant gravitational physics, for example some tidal effects [11], whose importance in the large scale structure formation has been widely recognized (see, e.g., [23]). L2 has the remarkable property that, for standard cosmological initial conditions, the Lagrangian map is still potential. As a consequence, the velocity is also potential in Lagrangian coordinates. With L2, in Eulerian coordinates the velocity is potential only up to second order. One would have to sum the whole series to arbitrarily high orders of the Lagrangian perturbation theory (i.e., to arbitrary orders of $\epsilon$) to recover the Eulerian potentiality of the velocity (but note that convergence of the asymptotic series is not guaranteed [21, 24]). L2 having the Brenier property, the Lagrangian and inverse Lagrangian maps can be reconstructed exactly as an optimal transport problem, for example, by the MAK technique. This is probably the main reason why MAK performs well (at sufficiently large scales). Beyond the second order of the Lagrangian perturbation theory, scales below the non-linearity scale are expected to play a decisive role and Lagrangian vorticity is unavoidable [9]. Such scales cannot be handled accurately by standard MAK reconstruction.

Finally, let us discuss briefly the thorny issue of the validity of the Lagrangian perturbation theory. As mentioned before, the Euler–Poisson equations are a consequence of the Vlasov–Poisson equations only as long as multi-streaming is absent. The problem is that, with any cosmologically realistic initial condition at decoupling, multi-streaming appears immediately or, anyway, well before the present epoch. The situation is somewhat similar to what we would have with a one-dimensional Burgers flow in which the initial velocity would be spatially...
VI. CONCLUDING REMARKS

The main question that we have addressed in this paper concerns omni-potentiality, the (convex) potential character of the mapping from any time \( t_1 \) to any time \( t_2 \geq t_1 \) with \( 0 \leq t_1 \leq t_2 \leq T \). First, we have considered a class of flows of “Zeldovich type”, comprised of pure Zeldovich/Burgers flows and those obtained from them by application of arbitrary nonlinear transformations of the time variable and arbitrary time-dependent scale factors. Such flows are trivially omni-potential. So are spherically symmetric flows. We then have investigated (i) the existence of non-trivial omni-potential flows, (ii) their genericity: can we prescribe the initial velocity potential in an arbitrary way?

The flows have been characterized by their Lagrangian maps \( q \mapsto x(q,t) = \nabla \Phi(q,t) \) in terms of the scalar potential \( \Phi \). As shown in Sec. II A, omni-potentiality implies that along any particle trajectory the Hessians \( \nabla^2 \Phi(q,t) \) commute and thus have common eigendirections. The field of such eigendirections is prescribed as a function of the initial (Lagrangian) position \( q \), for example, by the eigendirections of the Hessian of the initial velocity potential \( \Phi \). The set of eigendirections of a real symmetric \( d \times d \) matrix depends on \( d(d - 1)/2 \) parameters and can be characterized, for example, by \( d(d - 1)/2 \) of the invariants discussed in the Appendix. As we try to determine a single scalar function \( \Phi \), the situation is rather different in two and higher dimensions.

When \( d = 2 \), we have a single invariant expressible as a ratio of suitable combinations of spatial second derivatives of \( \Phi \). Thus, prescribing the field of the invariant values, \( g(q) \), we obtain a linear second-order PDE, (12), for \( \Phi \). For a suitable family of fields \( g(q) \), we have found in Sec. III A non-trivial omni-potential flows that are linear combinations of homogeneous polynomials, thus ensuring the existence of such non-trivial flows. Using a WKB method, in Sec. IV we have then been able to construct omni-potential flows, at least locally in space-time, for arbitrary smooth \( g(q) \). These flows are actually close to Zeldovich flows with straight trajectories (but are not of “Zeldovich type”). Extending this construction globally in space and avoiding the rapid spatial oscillations inherent to a WKB method constitute interesting open problems.

When \( d = 3 \), omni-potentiality can be expressed in either of two equivalent ways. One is to demand the commutation of the Hessians of \( \Phi \) with those of a prescribed \( \phi \); this gives \( d(d - 1)/2 \) linear homogeneous second-order PDEs. The other involves working with the invariants, introduced in the Appendix, which are rational functions of the entries of the Hessians of \( \Phi \); this gives \( d(d - 1)/2 \) nonlinear second-order PDEs. In both approaches we have one unknown scalar function, \( \Phi \), which has to satisfy more than one equation. Hence, there is an issue of compatibility of these equations. However, by restricting the potential \( \Phi \) to possess a suitable finite symmetry group, we have obtained a fairly large class of non-trivial solutions that are even-degree homogeneous polynomials. Whether non-Zeldovich-type omni-potential flows exist for an arbitrary smooth \( \phi \) remains an open problem. In dimensions \( d > 3 \) the situation is basically the same.

We have shown in Sec. II B that omni-potentiality of a flow is equivalent to having at each time a velocity field that is potential in both Eulerian and Lagrangian coordinates. Such double potentiality was frequently considered in cosmology. It is of particular relevance when performing reconstruction by convex optimization techniques such as the Monge–Ampère–Kantorovich (MAK) procedure. Note that the Euler–Poisson flow is potential in Eulerian coordinates, but not in general in Lagrangian coordinates. As discussed in Sec. V, the approximate Euler–Poisson flow obtained by the second-order Lagrangian perturbation theory (L2) is exactly potential in Lagrangian coordinates — and thus its inverse Lagrangian map can be obtained exactly by the Monge–Ampère–Kantorovich (MAK) procedure — however, it is only approximately potential in Eulerian coordinates and thus must be qualified as an approximately omni-potential flow. In particular, it does not represent an example of an exactly omni-potential three-dimensional flow for an arbitrary smooth initial velocity.

We finally wish to mention a concrete open problem of cosmological interest: as mentioned, MAK gives the exact inverse Lagrangian map for L2 (and this contributes to explaining why MAK works so well when tested with N-body simulations). However, in L2, between the Eulerian position \( x \) and its Lagrangian antecedent \( q \), the trajectory is not given exactly by the Zeldovich approxima-
tion that has a constant velocity \((x - q)/\tau\) (in the coordinates we used in Sec. V). The actual L2 trajectory is slightly curved and its current (peculiar) velocity differs slightly from \((x - q)/\tau\). It would be of interest to find how to perturbatively handle such discrepancies. Given that the full Euler–Poisson reconstruction problem and the Zeldovich approximation to it both have convex optimization formulations, the question arises, whether L2 and higher-order approximations possess such formulations.

Acknowledgments

We are particularly grateful to J. Bec, Y. Brenier, T. Buchert, S. Colombi and A. Sobolevskii for extensive fruitful discussions. Thanks are also due to F. Bouchet, K. Khanin, R. Mohayaee, A. Nusser and E.B. Vinberg. UF, OP and VZ were supported by the grant ANR-07-BLAN-0235 OTARIE from Agence Nationale de la Recherche, France. OP and VZ were supported by the grant 11-05-00167-a from the Russian foundation for basic research. Several visits of BV, OP and VZ to Observatoire de la Côte d’Azur (France) were supported by the French Ministry of Higher Education and Research.

Appendix: Invariants under variation of eigenvalues of symmetric matrices

Here we show how the set of commuting symmetric real \(d\)-dimensional matrices, having prescribed eigendirections, can be characterized by a certain number of invariants, which are rational functions of the matrix entries. The findings here may be of interest beyond the study of omni-potential flow. To the best of our knowledge, these results are not available in the published literature. If the reader is aware of any relevant reference, kindly inform the authors.

By the theorem on codiagonalizability of commuting symmetric matrices, in an omni-potential flow, Hessians of the potential of the Lagrangian map must have the form of all real symmetric matrices and the subspace of matrices having prescribed eigendirections, which is spanned by the set of all the powers, from zero to \(d-1\), of this matrix. The general problem can be, in principle, handled using Plücker coordinates [14]. For our problem, a more direct approach is available, as now explained. For \(d > 3\), our characterization involves fewer invariants than the corresponding number of Plücker coefficients.

We denote by \(h(\lambda_i)\) an eigenvector associated with the eigenvalue \(\lambda_i\), and assume without any loss of generality that no component of any eigenvector vanishes; one can always achieve this by suitably rotating the orthonormal basis in \(\mathbb{R}^d\), in which the eigenvectors are decomposed.

We construct the invariants as follows: We set, for some \(1 \leq m \neq n \leq d\) and \(k \leq d\),

\[
\gamma_{mn}^{(d,k)}(\beta_{mn,1}, \ldots, \beta_{mn,d}) = \gamma_{mn}^{(d,k)}
\]

where

\[
\beta_{mn,i} = h_m(\lambda_i)/h_n(\lambda_i)
\]

and \(P^{(d,k)}\) denote symmetric homogeneous polynomials of degree \(k \leq d\),

\[
P^{(d,k)}(y) = \sum_{1 \leq j_1 < \ldots < j_l < j_k \leq d} y_{j_1} \ldots y_{j_l} \ldots y_{j_k}
\]

for \(y \in \mathbb{R}^d\). By construction, the quantities \(\gamma_{mn}^{(d,k)}\) depend only on the eigendirections (through the ratios of components) and are invariant under permutations of the eigendirections; thus they depend only on the set of eigendirections. Then one substitutes into (A1) the respective components of the eigenvectors \(h(\lambda_i)\), expressed in terms of the associated eigenvalues \(\lambda_i\) and of the entries of the matrix \(H\). It is easily seen that this will produce rational functions of the matrix entries and of the eigenvalues. Furthermore, it may be shown that the eigenvalues enter only through symmetric polynomial combinations, which — by Viète’s theorem applied to the characteristic polynomial — have a polynomial dependence on the matrix entries. The actual derivation of the invariants can.

1. Construction of invariants in dimension \(d\)

Let us assume that all eigenvalues \(\lambda_i\) of a symmetric \(d \times d\) matrix \(H\) are distinct, and thus all eigendirections are uniquely defined. Description of an arbitrary set of \(d\) orthogonal eigendirections in \(\mathbb{R}^d\) requires \(d(d-1)/2\) parameters: \(d\) arbitrary directions require \(d(d-1)\) parameters, from which one must subtract the number of orthogonality conditions, \(d(d-1)/2\). We expect therefore that a set of \(d(d-1)/2\) suitably chosen invariants uniquely defines the eigendirections.

The problem of finding such invariants is an instance of a much more general problem of characterizing linear subspaces of a vector space; here the vector space is that of all real symmetric matrices and the subspace of matrices having prescribed eigendirections, which is spanned by the set of all the powers, from zero to \(d-1\), of this matrix. The general problem can be, in principle, handled using Plücker coordinates [14]. For our problem, a more direct approach is available, as now explained. For \(d > 3\), our characterization involves fewer invariants than the corresponding number of Plücker coordinates.

We denote by \(h(\lambda_i)\) an eigenvector associated with the eigenvalue \(\lambda_i\), and assume without any loss of generality that no component of any eigenvector vanishes; one can always achieve this by suitably rotating the orthonormal basis in \(\mathbb{R}^d\), in which the eigenvectors are decomposed.

We construct the invariants as follows: We set, for some \(1 \leq m \neq n \leq d\) and \(k \leq d\),

\[
\gamma_{mn}^{(d,k)}(\beta_{mn,1}, \ldots, \beta_{mn,d}) = \gamma_{mn}^{(d,k)}
\]

where

\[
\beta_{mn,i} = h_m(\lambda_i)/h_n(\lambda_i)
\]

and \(P^{(d,k)}\) denote symmetric homogeneous polynomials of degree \(k \leq d\),

\[
P^{(d,k)}(y) = \sum_{1 \leq j_1 < \ldots < j_l < j_k \leq d} y_{j_1} \ldots y_{j_l} \ldots y_{j_k}
\]

for \(y \in \mathbb{R}^d\). By construction, the quantities \(\gamma_{mn}^{(d,k)}\) depend only on the eigendirections (through the ratios of components) and are invariant under permutations of the eigendirections; thus they depend only on the set of eigendirections. Then one substitutes into (A1) the respective components of the eigenvectors \(h(\lambda_i)\), expressed in terms of the associated eigenvalues \(\lambda_i\) and of the entries of the matrix \(H\). It is easily seen that this will produce rational functions of the matrix entries and of the eigenvalues. Furthermore, it may be shown that the eigenvalues enter only through symmetric polynomial combinations, which — by Viète’s theorem applied to the characteristic polynomial — have a polynomial dependence on the matrix entries. The actual derivation of the invariants can.
be partially simplified by making use of the identity
\[ \prod_{k=1}^{d} (\lambda_k + c) = \det \| H + cI \|. \tag{A2} \]

2. Relations between invariants

We obtain thus \( d^2(d-1) \) invariants \( \gamma_{mn}^{(d,k)} \) in the form of rational functions of the entries of the symmetric matrix \( H \). Evidently, these invariants are too numerous to be independent. For instance, for any \( 1 \leq m \neq n \leq l \leq d \) and \( 0 < k < d \) they clearly satisfy the relations (no summation on repeated indices)
\[ \gamma_{mn}^{(d,d)} \gamma_{nm}^{(d,d)} = 1, \tag{A3} \]
\[ \gamma_{ml}^{(d,d)} \gamma_{ln}^{(d,d)} = \gamma_{mn}^{(d,d)} \tag{A4} \]
and
\[ \gamma_{mn}^{(d,k)} = \gamma_{mn}^{(d,d)} \gamma_{mn}^{(d,d-k)}. \tag{A5} \]

Identities (A3) and (A5) link invariants \( \gamma_{mn}^{(d,k)} \) for, say, \( m < n \) with those for \( m > n \); there are \( d^2(d-1)/2 \) of such independent relations between invariants. Equations (A4) imply
\[ \gamma_{mn}^{(d,d)} = \prod_{i=n}^{m-1} \gamma_{i+1,i}^{(d,d)} \]
for any \( m > n + 1 \); conversely, any relation (A4) follows from these relations together with (A3). Thus, the identities (A4) contribute further \( (d-1)(d-2)/2 \) independent relations.

For any \( n \) such that \( 1 \leq n \leq d \), the relation
\[ \sum_{m=1}^{d} \sum_{m \neq n} \gamma_{mn}^{(d,2)} = -\frac{d(d-1)}{2} \]
stems from orthogonality of the eigendirections (there are \( d \) such relations).

Another family of relations involves the invariants \( \gamma_{mn}^{(1,1)} \). For \( p > 0 \), let \( H^{(p)} \) denote the \( p \)th power of the matrix \( H \) and let \( H_{mn}^{(p)} \) denote its entries, which are of course readily expressed in terms of the entries of the matrix \( H \); we also set \( H^{(0)} \equiv I \). The relations
\[ \sum_{m=1}^{d} H_{mn}^{(p)} h_m(\lambda_i) = \lambda_i^p h_n(\lambda_i), \]
that hold true, for any \( i \) and \( n \), by definition of eigenvectors of \( H, h(\lambda_i) \), and the identity
\[ \sum_{i=1}^{d} \lambda_i^p = \tr H^{(p)} \]

imply, for each \( p > 0 \), the relation
\[ \sum_{m=1}^{d} \sum_{m \neq n} H_{mn}^{(p)} \gamma_{mn}^{(d,1)} \]
\[ = \sum_{i=1}^{d} \sum_{i \neq n} H_{mn}^{(p)} \frac{h_n(\lambda_i)}{h_n(\lambda_i)} \]
\[ = \sum_{i=1}^{d} \sum_{i = 1}^{d} (\lambda_i^p - H^{(p)}) = 0. \tag{A6} \]

Since, by the Cayley–Hamilton theorem, any matrix is a root of its characteristic polynomial, the entries \( H_{mn}^{(p)} \) for \( p \geq d \) are linear combinations of \( H_{mn}^{(p')} \) for \( p - d \leq p' \leq p - 1 \), the coefficients in these linear combinations being independent of indices \( p, m, \) and \( n \). Therefore, relation (A6) for \( p \geq d \) is a consequence of \( d \) such relations for \( p = p' \) such that \( p - d \leq p < p - 1 \). The relation (A6) for \( p = 0 \) is trivial, and hence there are \( d-1 \) independent relations (A6) for \( 1 \leq p \leq d - 1 \).

In principle, the total number \( d^2(d-1) \) of invariants \( \gamma_{mn}^{(d,k)} \) should exceed the number \( d(d-1)/2 \) of independent invariants by the number of relations between the invariants. For arbitrary \( d \), we have obtained above
\[ \frac{d^2(d-1)}{2} + \frac{(d-1)(d-2)}{2} + d = \frac{d(d+1)}{2} \]
\[ \frac{d^2(d-1)}{2} \]
independent relations. For \( d = 3 \), they constitute 15 relations constraining the 18 invariants that we have introduced; this fits our expectations that at most 3 invariants are independent, because the orthogonal frame of eigendirections of a \( 3 \times 3 \) symmetric matrix is described by 3 Euler angles. For \( d > 3 \) the number of the obtained independent relations is still insufficient to fill the gap, and more relations are to be identified.

3. Invariants for \( d = 2 \)

As an example, we present a detailed derivation of the invariant \( \gamma_{12}^{(2,1)} \) for \( d = 2 \). The \( i \)th eigenvector of a \( 2 \times 2 \) symmetric matrix \( H \) is \((H_{12}, \lambda_i - H_{11})\), and hence
\[ \gamma_{12}^{(2,1)} = \frac{H_{12}}{\lambda_1 - H_{11}} + \frac{H_{12}}{\lambda_2 - H_{11}} = \frac{H_{12}(\lambda_1 + \lambda_2 - 2H_{11})}{(\lambda_1 - H_{11})(\lambda_2 - H_{11})}. \]

The characteristic equation for the eigenvalues is
\[ \lambda^2 - (H_{11} + H_{22})\lambda + H_{11}H_{22} - H_{12}^2 = 0, \]
and hence \( \lambda_1 + \lambda_2 = H_{11} + H_{22} \). By virtue of (A2),
\[ (\lambda_1 - H_{11})(\lambda_2 - H_{11}) = \det \begin{vmatrix} 0 & H_{12} \\ H_{12} & H_{22} - H_{11} \end{vmatrix} = -H_{12}^2. \]

Consequently,
\[ \gamma_{12}^{(2,1)} = \frac{H_{11} - H_{22}}{H_{12}}. \]
The invariant $\gamma_{21}^{(2,1)}$ can be found by a similar calculation, or just by swapping subscripts in the expression for $\gamma_{12}^{(2,1)}$: clearly, the two invariants are interrelated: $\gamma_{21}^{(2,1)} = -\gamma_{12}^{(2,1)}$. The invariant $\gamma_{21}^{(2,2)}$ turns out to be degenerate:

$$\gamma_{21}^{(2,2)} = \frac{H_{12}^2}{(\lambda_1 - H_{11})(\lambda_2 - H_{11})} = -1.$$ 

Thus, we have obtained the same invariant as that found in Sec. II.D.

4. Invariants for $d = 3$

We consider now invariants in the three-dimensional space. The $i$th eigenvector of a $3 \times 3$ symmetric matrix $H$ has components

$$(H_{12}H_{23} + H_{13}(\lambda_i - H_{22}), \ H_{12}H_{13} + H_{23}(\lambda_i - H_{11}),$$

$$(\lambda_i - H_{11})(\lambda_i - H_{22}) - H_{12}^2).$$

The procedure outlined above yields

$$\gamma_{21}^{(3,1)} = \frac{H_{22} - H_{11}}{H_{12}}$$

$$+ \frac{H_{13}(H_{11} - H_{22})H_{13}H_{23} + (H_{23} - H_{12})^2H_{12}}{H_{12}(H_{22} - H_{12}H_{13}H_{23} + (H_{13}^2 - H_{12})H_{23})}$$

$$+ \frac{(H_{11} - H_{13})H_{12}H_{23} + (H_{23} - H_{12})^2H_{13}}{(H_{22} - H_{12}H_{13} + (H_{13} - H_{12})H_{23})}.$$

(A7)

$\gamma_{21}^{(3,3)}$ is the ratio of two polynomials, which we calculate using (A2):

$$\gamma_{21}^{(3,3)} = \frac{H_{11} - H_{13}H_{12}H_{23} + (H_{23} - H_{12})^2H_{13}}{(H_{22} - H_{12}H_{13} + (H_{13} - H_{12})H_{23})}.$$  

(A8)

$\gamma_{21}^{(3,2)}$ can be found by applying identity (A5):

$$\gamma_{21}^{(3,2)} = \gamma_{21}^{(3,3)} - \gamma_{21}^{(3,1)}$$

(A9)

(here $\gamma_{21}^{(3,1)}$ can be obtained by permuting the subscripts 1 and 2 in (A7)).

The three invariants $\gamma_{21}^{(3,k)}$ for $1 \leq k \leq 3$ uniquely define the three ratios $\beta_{21i}$: by Viète’s theorem, they are roots of the cubic equation

$$\beta^3 - (\gamma_{21}^{(3,1)})\beta^2 + (\gamma_{21}^{(3,2)})\beta - \gamma_{21}^{(3,3)} = 0.$$ 

The eigendirections can be recovered in the form of three eigenvectors $(1, \beta_{211}, c_i)$. One can try to obtain the third components $c_i$ ($i = 1, 2, 3$) from the relations expressing orthogonality of the eigendirections. However, this produces two solutions: if $\{c_i\}$ is obtained in this way, then $\{-c_i\}$ is also a solution. Hence, the invariants $\gamma_{21}^{(3,k)}$, $1 \leq k \leq 3$, define two distinct sets of eigendirections. The non-uniqueness is eliminated, if in addition we know any of the invariants $\gamma_{21}^{(2,1)}$ or $\gamma_{21}^{(2,2)}$ for $i = 1, 3$ and $j = 1, 2$. It is unclear, whether one can choose a set of three invariants uniquely defining three eigendirections.

When all invariants $\gamma_{3m}^{(3,k)}$ are known for $k = 1$ and 3, the equations for the entries of symmetric matrix $H$ can be considerably simplified. In view of (A8) and the same equation with permuted subscripts 2 and 3, relation (A7) can be expressed as

$$\gamma_{21}^{(3,1)} + \gamma_{21}^{(3,3)} = \frac{H_{22} - H_{11}}{H_{12}} + \frac{H_{13}H_{31}^{(3,3)} - \gamma_{12}^{(3,3)}}{H_{12}}.$$  

(A10)

Adding (A10) to its analogue, where subscripts 1 and 2 are permuted, we obtain

$$-H_{12}^{(3,1)} + \gamma_{21}^{(3,1)} + \gamma_{12}^{(3,1)} + \gamma_{12}^{(3,3)}$$

$$+ H_{13}^{(3,3)} + H_{23}^{(3,3)} = 0.$$  

(A11)

Permuting subscripts in this equation, we obtain a linear system of equations for the non-diagonal entries of $H$. (Here, we note that the sum of (A11) and its counterparts with permuted subscripts reduces to (A6) for $p = 1$.) Upon solving this linear system, we find the differences $H_{mn} - H_{nm}$ from (A10) and the analogues of this equation with permuted subscripts, i.e., we determine all diagonal entries up to an additive constant. Further determination of the matrix $H$ would require, of course, the knowledge of its three eigenvalues. This implies that the non-diagonal entries, as determined from the above-mentioned linear system, involve two free parameters. This, in turn, requires that (A11) be equivalent to any equation, obtained from it by permutation of subscripts.

Consequently, we obtain further relations between the invariants: Permuting subscripts, say, 1 and 3 in (A11), we get

$$-H_{23}^{(3,1)} + \gamma_{21}^{(3,1)} + \gamma_{12}^{(3,1)} + \gamma_{12}^{(3,3)}$$

$$+ H_{13}^{(3,3)} + H_{12}\gamma_{12}^{(3,3)} = 0.$$  

This equation is equivalent to (A11) if and only if

$$\left(\gamma_{21}^{(3,1)} + \gamma_{21}^{(3,3)} + \gamma_{12}^{(3,1)} + \gamma_{12}^{(3,3)}\right)\gamma_{13}^{(3,3)} + \gamma_{31}^{(3,3)}\gamma_{12}^{(3,3)} = 0$$

and

$$\left(\gamma_{21}^{(3,1)} + \gamma_{21}^{(3,3)} + \gamma_{12}^{(3,1)} + \gamma_{12}^{(3,3)}\right)$$

$$\times \left(\gamma_{23}^{(3,3)} + \gamma_{23}^{(3,3)} + \gamma_{32}^{(3,3)} + \gamma_{32}^{(3,3)}\right) = \gamma_{32}^{(3,3)}\gamma_{12}^{(3,3)}.$$  

In view of (A4), the first of these relations is equivalent to

$$\gamma_{21}^{(3,1)} + \gamma_{21}^{(3,3)} + \gamma_{12}^{(3,1)} + \gamma_{12}^{(3,3)} + \gamma_{31}^{(3,1)}\gamma_{32}^{(3,3)} = 0,$$  

(A12)

and the second one follows from (A12) and the relation obtained from (A12) by permuting subscripts 1 and 3.
Relation (A12) and its analogues with permuted subscripts can, of course, be established directly. Such relations can be used in three dimensions for verifying the consistency of the invariants, instead of using (A6), one of our basic relations between invariants.

[1] Arnold, V.I., “Notes on the behavior of flows of the three-dimensional ideal fluid under a small perturbation of the initial velocity field,” Appl. Math. Mech. 36 2, 255–262 (1972).
[2] Bernardau, F., Colombi, S., Gaztañaga, E., and Scoccimarro, R., “Large-Scale Structure of the Universe and Cosmological Perturbation Theory,” Phys. Rep. 367, 1–248 (2002).
[3] Bertsekas, D.P., “Auction algorithms for network flow problems: A tutorial introduction,” Comput. Optim. Appl. 1, 7–66 (1992).
[4] Bouchet, F.R., Colombi, S., Hivon, E., and Juszkiewicz, R., “Perturbative Lagrangian approach to gravitational instability,” Astron. Astrophys. 296, 575–608 (1995).
[5] Brenier, Y., “Décomposition polaire et réarrangement monotone des champs de vecteurs”, C. R. Acad. Sci. Paris Série I Math. 305, 805–808 (1987).
[6] Brenier, Y., “Polar factorization and monotone rearrangement of vector-valued functions,” Comm. Pure Appl. Math. 44, 375–417 (1991).
[7] Brenier, Y., Frisch, U., Hénon, M., Loeper, G., Matarrese, S., Mohayaee, R., and Sobolevski, A., “A reconstruction of the early Universe as a convex optimization problem,” Mon. Not. R. Astron. Soc. 346, 501–524 (2003).
[8] Buchert, T., “Lagrangian theory of gravitational instability of Friedman–Lemaitre cosmologies and the ’Zel’dovich approximation”,’ Mon. Not. R. Astron. Soc. 254, 729–737 (1992).
[9] Buchert, T., “Lagrangian theory of gravitational instability of Friedman–Lemaitre cosmologies – a generic third-order model for non-linear clustering,” Mon. Not. R. Astron. Soc. 267, 811–820 (1994).
[10] Buchert, T., “Lagrangian perturbation approach to the formation of large-scale structure,” in Proc. IOP Enrico Fermi, Course CXXXII, Dark Matter in the Universe, Varenna 1995, eds.: S. Bonometto, J. Primack, A. Provenzale, IOS Press Amsterdam, pp. 543–564 (1996).
[11] Buchert, T. and Ehlers, J., “Lagrangian theory of gravitational instability of Friedman–Lemaître cosmologies-second order approach: an improved model for non-linear clustering,” Mon. Not. R. Astron. Soc. 264, 375–387 (1993).
[12] Catelan, P., “Lagrangian dynamics in non-flat universes and non-linear gravitational evolution,” Mon. Not. R. Astron. Soc. 276, 115–124 (1995).
[13] Frisch, U., Matarrese, S., Mohayaee, R., and Sobolevski, A., “A reconstruction of the initial conditions of the Universe by optimal mass transportation,” Nature 417, 260–262 (2002).
[14] Hodge, W.V.D. and Pedoe, D., Methods of Algebraic Geometry, Volume I (Book II). Cambridge University Press (1947).
[15] Horn, R.A. and Johnson, C.R., Matrix Analysis, Cambridge University Press, Cambridge (1990).
[16] Kantorovich, L., “On the translocation of masses,” C. R. Acad. Sci. URSS 37, 199–201 (1942).
[17] Loeper, G., “The reconstruction problem for the Euler–Poisson system in cosmology,” Arch. Rational Mech. Anal., 179, 153–216 (2006).
[18] Mohayaee, R., Mathis, H., Colombi, S., and Silk, J., “Reconstruction of primordial density fields,” Mon. Not. R. Astron. Soc. 365, 939–959 (2006).
[19] Monge, G., “Mémoire sur la théorie des déblais et des remblais,” Hist. Acad. R. Sci. Paris, 666–704 (1781).
[20] Moutarde, F., Alimi, J.M., Bouchet, F.R., Pellat, R., and Ramani, A., “Precollapse scale invariance in gravitational instability,” Astrophys. J. 382, 377–381 (1991).
[21] Nadkarni-Ghosh, S. and Chernoff, D.F., “Extending the domain of validity of the Lagrangian approximation,” Mon. Not. R. Astron. Soc. 410, 1454–1488 (2011).
[22] Peebles, P.J.E., “Tracing Galaxy Orbits Back in Time,” Astrophys. J. Lett. 344, 53–56 (1989).
[23] Peebles, P.J.E. and Groth E.J., “An integral constraint for the evolution of the galaxy two-point correlation function,” Astron. Astrophys. 53, 131–140 (1976).
[24] Sahni, V. and Shandarin, S., “Accuracy of Lagrangian approximations in voids,” Mon. Not. R. Astron. Soc. 282, 641–645 (1996).
[25] Villani, C., Optimal Transport, Old and New, Grundlehrer der mathematischen Wissenschaften, Springer Verlag, Berlin (2009).
[26] Zeldovich (Zel’dovich), Ya.B., “Gravitational instability: an approximate theory for large density perturbations,” Astron. Astrophys. 5, 84–89 (1970).
[27] www.sdss.org