INITIAL COEFFICIENT BOUNDS FOR A GENERAL CLASS OF BI-UNIVALENT FUNCTIONS

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ABSTRACT. Inspired by the recent works of Srivastava et al. [14], Frasin and Aouf [6] and others [1, 5, 7, 17, 18], we propose to investigate the coefficient estimates for a general class of analytic and bi-univalent functions. Also, we obtain estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in this new class. Some interesting remarks, corollaries and applications of the results presented here are also discussed.

1. Introduction

Let $A$ denote the class of functions of the form
\[ f(z) = z + \sum_{n=2}^{\infty} a_n z^n \] (1.1)
which are analytic in the open unit disk $U = \{ z : z \in \mathbb{C} \text{ and } |z| < 1 \}$. Further, by $S$ we shall denote the class of all functions in $A$ which are univalent in $U$.

For two functions $f$ and $g$, analytic in $U$, we say that the function $f(z)$ is subordinate to $g(z)$ in $U$, and write
\[ f(z) \prec g(z), \quad z \in U, \]
if there exists a Schwarz function $w(z)$, analytic in $U$, with
\[ w(0) = 0 \quad \text{and} \quad |w(z)| < 1, \quad z \in U, \]
such that
\[ f(z) = g(w(z)), \quad z \in U. \]
In particular, if the function $g$ is univalent in $U$, the above subordination is equivalent to
\[ f(0) = g(0) \quad \text{and} \quad f(U) \subset g(U). \]

Some of the important and well-investigated subclasses of the univalent function class $S$ include (for example) the class $S^*(\alpha)$ of starlike functions of order $\alpha$ in $U$ and the class $K(\alpha)$ of convex functions of order $\alpha$ in $U$. By definition, we have
\[ S^*(\alpha) := \left\{ f : f \in S \text{ and } \Re \left( \frac{z f'(z)}{f(z)} \right) > \alpha; \ z \in U; \ 0 \leq \alpha < 1 \right\} \] (1.2)
and
\[ K(\alpha) := \left\{ f : f \in S \text{ and } \Re \left( 1 + \frac{z f''(z)}{f'(z)} \right) > \alpha; \ z \in U; \ 0 \leq \alpha < 1 \right\}. \] (1.3)

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It readily follows from the definitions (1.2) and (1.3) that
\[ f \in K(\alpha) \iff zf' \in S^*(\alpha). \]
Also, let \( S_\beta^\alpha(\alpha) \) and \( C_\beta^\alpha(\alpha) \) denote the subclasses of \( S \) consisting functions \( f(z) \) which are defined, respectively by
\[
S_\beta^\alpha(\alpha) := \left\{ f : f \in S \text{ and } \Re \left( e^{i\beta} \frac{zf'(z)}{f(z)} \right) > \alpha \cos \beta ; \ z \in \mathbb{U} ; \ 0 \leq \alpha < 1, \beta \in \left( -\frac{\pi}{2}, \frac{\pi}{2} \right) \right\}
\]
and
\[
C_\beta^\alpha(\alpha) := \left\{ f : f \in S \text{ and } \Re \left( e^{i\beta} \frac{zf'(z)}{f'(z)} \right) > \alpha \cos \beta ; \ z \in \mathbb{U} ; \ 0 \leq \alpha < 1, \beta \in \left( -\frac{\pi}{2}, \frac{\pi}{2} \right) \right\}.
\]
It is easy to see that
\[ f \in C_\beta^\alpha(\alpha) \iff zf' \in S_\beta^\alpha(\alpha). \]

It is well known that every function \( f \in S \) has an inverse \( f^{-1} \), defined by
\[ f^{-1}(f(z)) = z, \ z \in \mathbb{U} \]
and
\[ f(f^{-1}(w)) = w, \ |w| < r_0(f); \ r_0(f) \geq \frac{1}{4}, \]
where
\[ f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \ldots. \]

A function \( f \in A \) is said to be bi-univalent in \( \mathbb{U} \) if both \( f(z) \) and \( f^{-1}(z) \) are univalent in \( \mathbb{U} \). Let \( \Sigma \) denote the class of bi-univalent functions in \( \mathbb{U} \) given by (1.1). Examples of functions in the class \( \Sigma \) are
\[
z, \ -\log(1-z), \ \frac{1}{2} \log \left( \frac{1+z}{1-z} \right)
\]
and so on. However, the familiar Koebe function is not a member of \( \Sigma \). Other common examples of functions in \( S \) such as
\[
z - \frac{z^2}{2} \text{ and } \frac{z}{1-z^2}
\]
are also not members of \( \Sigma \) (see [6, 14]).

In 1967, Lewin [9] investigated the bi-univalent function class \( \Sigma \) and showed that \( |a_2| < 1.51 \). On the other hand, Brannan and Clunie [2] (see also [3, 4, 15]) and Netanyahu [11] made an attempt to introduce various subclasses of the bi-univalent function class \( \Sigma \) and obtained non-sharp coefficient estimates on the first two coefficients \( |a_2| \) and \( |a_3| \) of (1.1). But the coefficient problem for each of the following Taylor-Maclaurin coefficients \( |a_n| \) \( n \in \mathbb{N} \setminus \{1, 2\} \) is still an open problem. Following Brannan and Taha [4], many researchers (see [11, 12, 13, 14, 15, 16]) have recently introduced and investigated several interesting subclasses of the bi-univalent function class \( \Sigma \) and they have found non-sharp estimates on the first two Taylor-Maclaurin coefficients \( |a_2| \) and \( |a_3| \).

Motivated by the above mentioned works, we define the following subclass of function class \( \Sigma \).
Definition 1.1. Let $h : \mathbb{U} \to \mathbb{C}$, be a convex univalent function such that $h(0) = 1$ and $h(z) = h(z)$, for $z \in \mathbb{U}$ and $\Re(h(z)) > 0$. A function $f(z)$ given by (1.1) is said to be in the class $NP_{\Sigma}^{\mu,\lambda}(\beta, h)$ if the following conditions are satisfied:

$$f \in \Sigma, \ e^{i\beta} \left(1 - \lambda \left(\frac{f(z)}{z}\right)^{\mu} + \lambda f'(z) \left(\frac{f(z)}{z}\right)^{\mu-1}\right) < h(z) \cos \beta + i \sin \beta, \ z \in \mathbb{U} \ (1.7)$$

and

$$e^{i\beta} \left(1 - \lambda \left(\frac{g(w)}{w}\right)^{\mu} + \lambda g'(w) \left(\frac{g(w)}{w}\right)^{\mu-1}\right) < h(w) \cos \beta + i \sin \beta, \ w \in \mathbb{U}, \ (1.8)$$

where $\beta \in (-\pi/2, \pi/2)$, $\lambda \geq 1$, $\mu \geq 0$ and the function $g$ is given by

$$g(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \ldots \ (1.9)$$

the extension of $f^{-1}$ to $\mathbb{U}$.

Remark 1.2. If we set $h(z) = \frac{1+Az}{1+Bz}$, $-1 \leq B < A \leq 1$, in the class $NP_{\Sigma}^{\mu,\lambda}(\beta, h)$, we have $NP_{\Sigma}^{\mu,\lambda}(\beta, \frac{1+Az}{1+Bz})$ and defined as

$$f \in \Sigma, \ e^{i\beta} \left(1 - \lambda \left(\frac{f(z)}{z}\right)^{\mu} + \lambda f'(z) \left(\frac{f(z)}{z}\right)^{\mu-1}\right) < \frac{1 + Az}{1 + Bz} \cos \beta + i \sin \beta, \ z \in \mathbb{U}$$

and

$$e^{i\beta} \left(1 - \lambda \left(\frac{g(w)}{w}\right)^{\mu} + \lambda g'(w) \left(\frac{g(w)}{w}\right)^{\mu-1}\right) < \frac{1 + Aw}{1 + Bw} \cos \beta + i \sin \beta, \ w \in \mathbb{U},$$

where $\beta \in (-\pi/2, \pi/2)$, $\lambda \geq 1$, $\mu \geq 0$ and the function $g$ is given by (1.9).

Remark 1.3. Taking $h(z) = \frac{1+(1-2\alpha)z}{1-z}, \ 0 \leq \alpha < 1$ in the class $NP_{\Sigma}^{\mu,\lambda}(\beta, h)$, we have $NP_{\Sigma}^{\mu,\lambda}(\beta, \alpha)$ and $f \in NP_{\Sigma}^{\mu,\lambda}(\beta, \alpha)$ if the following conditions are satisfied:

$$f \in \Sigma, \ \Re \left(e^{i\beta} \left(1 - \lambda \left(\frac{f(z)}{z}\right)^{\mu} + \lambda f'(z) \left(\frac{f(z)}{z}\right)^{\mu-1}\right)\right) > \alpha \cos \beta, \ 0 \leq \alpha < 1; \ z \in \mathbb{U}$$

and

$$\Re \left(e^{i\beta} \left(1 - \lambda \left(\frac{g(w)}{w}\right)^{\mu} + \lambda g'(w) \left(\frac{g(w)}{w}\right)^{\mu-1}\right)\right) > \alpha \cos \beta, \ 0 \leq \alpha < 1; \ w \in \mathbb{U},$$

where $\beta \in (-\pi/2, \pi/2)$, $\lambda \geq 1$, $\mu \geq 0$ and the function $g$ is given by (1.9).

Remark 1.4. Taking $\lambda = 1$ and $h(z) = \frac{1+(1-2\alpha)z}{1-z}, \ 0 \leq \alpha < 1$ in the class $NP_{\Sigma}^{\mu,\lambda}(\beta, h)$, we have $NP_{\Sigma}^{\mu,1}(\beta, \alpha)$ and $f \in NP_{\Sigma}^{\mu,1}(\beta, \alpha)$ if the following conditions are satisfied:

$$f \in \Sigma, \ \Re \left(e^{i\beta} f'(z) \left(\frac{f(z)}{z}\right)^{\mu-1}\right) > \alpha \cos \beta, \ 0 \leq \alpha < 1; \ \mu \geq 0; \ z \in \mathbb{U}$$

and

$$\Re \left(e^{i\beta} g'(w) \left(\frac{g(w)}{w}\right)^{\mu-1}\right) > \alpha \cos \beta, \ 0 \leq \alpha < 1; \ \mu \geq 0; \ w \in \mathbb{U},$$

where $\beta \in (-\pi/2, \pi/2)$ and the function $g$ is given by (1.9).
Remark 1.5. Taking $\mu + 1 = \lambda = 1$ and $h(z) = \frac{1+(1-2\alpha)z}{1-z}$, $0 \leq \alpha < 1$ in the class $N^\alpha \mathcal{P}_\Sigma^\mu,\lambda(\beta, h)$, we have $N^\alpha \mathcal{P}_\Sigma^0(\beta, \alpha)$ and $f \in N^\alpha \mathcal{P}_\Sigma^1(\beta, \alpha)$ if the following conditions are satisfied:
\[
f \in \Sigma, \quad \mathcal{R}\left(e^{i\beta} \frac{zf'(z)}{f(z)}\right) > \alpha \cos \beta, \quad 0 \leq \alpha < 1; \quad z \in \mathbb{U}
\]
and
\[
\mathcal{R}\left(e^{i\beta} \frac{wzg'(w)}{g(w)}\right) > \alpha \cos \beta, \quad 0 \leq \alpha < 1; \quad w \in \mathbb{U},
\]
where $\beta \in (-\pi/2, \pi/2)$ and the function $g$ is given by (1.9).

Remark 1.6. Taking $\mu = 1$ and $h(z) = \frac{1+(1-2\alpha)z}{1-z}$, $0 \leq \alpha < 1$ in the class $N^\alpha \mathcal{P}_\Sigma^\mu,\lambda(\beta, h)$, we have $N^\alpha \mathcal{P}_\Sigma^1(\beta, \alpha)$ and $f \in N^\alpha \mathcal{P}_\Sigma^1(\beta, \alpha)$ if the following conditions are satisfied:
\[
f \in \Sigma, \quad \mathcal{R}\left(e^{i\beta} \left((1-\lambda)\frac{zf(z)}{z} + \lambda f'(z)\right)\right) > \alpha \cos \beta, \quad 0 \leq \alpha < 1; \quad \lambda \geq 1; \quad z \in \mathbb{U}
\]
and
\[
\mathcal{R}\left(e^{i\beta} \left((1-\lambda)\frac{g(w)}{w} + \lambda g'(w)\right)\right) > \alpha \cos \beta, \quad 0 \leq \alpha < 1; \quad \lambda \geq 1; \quad w \in \mathbb{U},
\]
where $\beta \in (-\pi/2, \pi/2)$ and the function $g$ is given by (1.9).

Remark 1.7. Taking $\mu = \lambda = 1$ and $h(z) = \frac{1+(1-2\alpha)z}{1-z}$, $0 \leq \alpha < 1$ in the class $N^\alpha \mathcal{P}_\Sigma^\mu,\lambda(\beta, h)$, we have $N^\alpha \mathcal{P}_\Sigma^1(\beta, \alpha)$ and $f \in N^\alpha \mathcal{P}_\Sigma^1(\beta, \alpha)$ if the following conditions are satisfied:
\[
f \in \Sigma, \quad \mathcal{R}\left(e^{i\beta} f'(z)\right) > \alpha \cos \beta, \quad 0 \leq \alpha < 1; \quad z \in \mathbb{U}
\]
and
\[
\mathcal{R}\left(e^{i\beta} g'(w)\right) > \alpha \cos \beta, \quad 0 \leq \alpha < 1; \quad w \in \mathbb{U},
\]
where $\beta \in (-\pi/2, \pi/2)$ and the function $g$ is given by (1.9).

We note that
(1) $N^\alpha \mathcal{P}_\Sigma^1(0, \alpha) = \mathcal{H}_\Sigma^\alpha$ (see [14])
(2) $N^\alpha \mathcal{P}_\Sigma^1(0, \alpha) = \mathcal{B}_\Sigma(\alpha, \lambda)$ (see [6])
(3) $N^\alpha \mathcal{P}_\Sigma^0(0, \alpha) = \mathcal{F}_\Sigma(\alpha)$ (see [10])
(4) $N^\alpha \mathcal{P}_\Sigma^1(0, \alpha) = \mathcal{N}_\Sigma^\mu(\alpha)$ (see [13])
(5) $N^\alpha \mathcal{P}_\Sigma^\mu,\lambda(0, \alpha) = \mathcal{N}_\Sigma^\mu,\lambda(\alpha)$ (see [5]).

In order to derive our main result, we have to recall here the following lemma.

Lemma 1.8. [12] [16] Let the function $\varphi(z)$ given by
\[
\varphi(z) = \sum_{n=1}^{\infty} B_n z^n, \quad z \in \mathbb{U}
\]
be convex in $\mathbb{U}$. Suppose also that the function $h(z)$ given by
\[
h(z) = \sum_{n=1}^{\infty} h_n z^n, \quad z \in \mathbb{U}
\]
is holomorphic in $\mathbb{U}$. If $h(z) \prec \varphi(z), \quad z \in \mathbb{U}$, then $|h_n| \leq |B_1|, \quad n \in \mathbb{N} = \{1, 2, 3, \ldots \}$. 
The object of the present paper is to introduce a general new subclass $\mathcal{N}P_{\Sigma}^{\mu,\lambda}(\beta, h)$ of the function class $\Sigma$ and obtain estimates of the coefficients $|a_2|$ and $|a_3|$ for functions in this new class $\mathcal{N}P_{\Sigma}^{\mu,\lambda}(\beta, h)$.

2. Coefficient bounds for the function class $\mathcal{N}P_{\Sigma}^{\mu,\lambda}(\beta, h)$

In this section we find the estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in the class $\mathcal{N}P_{\Sigma}^{\mu,\lambda}(\beta, h)$.

**Theorem 2.1.** Let $f(z)$ given by (1.1) be in the class $\mathcal{N}P_{\Sigma}^{\mu,\lambda}(\beta, h)$, $0 \leq \alpha < 1$, $\lambda \geq 1$ and $\mu \geq 0$, then

$$|a_2| \leq \sqrt{\frac{2|B_1| \cos \beta}{(1 + \mu)(2\lambda + \mu)}}$$

and

$$|a_3| \leq \frac{2|B_1| \cos \beta}{(2\lambda + \mu)(1 + \mu)},$$

where $\beta \in (-\pi/2, \pi/2)$.

**Proof.** It follows from (1.7) and (1.8) that there exists $p, q \in \mathcal{P}$ such that

$$e^{i\beta} \left( (1 - \lambda) \left( \frac{f(z)}{z} \right)^\mu + \lambda f'(z) \left( \frac{f(z)}{z} \right)^{\mu-1} \right) = p(z) \cos \beta + i \sin \beta$$

and

$$e^{i\beta} \left( (1 - \lambda) \left( \frac{g(w)}{w} \right)^\mu + \lambda g'(w) \left( \frac{g(w)}{w} \right)^{\mu-1} \right) = p(w) \cos \beta + i \sin \beta,$$

where $p(z) \prec h(z)$ and $q(w) \prec h(w)$ have the forms

$$p(z) = 1 + p_1 z + p_2 z^2 + \ldots, \quad z \in \mathbb{U}$$

and

$$p(z) = 1 + q_1 w + q_2 w^2 + \ldots, \quad w \in \mathbb{U}.$$ 

Equating coefficients in (2.3) and (2.4), we get

$$e^{i\beta} (\lambda + \mu) a_2 = p_1 \cos \beta$$

and

$$e^{i\beta} \left[ \frac{a_2^2}{2} (\mu - 1) + a_3 \right] (2\lambda + \mu) = p_2 \cos \beta$$

and

$$e^{i\beta} (\lambda + \mu) a_2 = q_1 \cos \beta$$

and

$$e^{i\beta} \left[ (\mu + 3) \frac{a_2^2}{2} - a_3 \right] (2\lambda + \mu) = q_2 \cos \beta.$$

From (2.7) and (2.9), we get

$$p_1 = -q_1$$

and

$$2 e^{i\beta} (\lambda + \mu)^2 a_2^2 = (p_1^2 + q_1^2) \cos^2 \beta.$$

Also, from (2.8) and (2.10), we obtain

$$a_2^2 = \frac{e^{-i\beta} (p_2 + q_2) \cos \beta}{(1 + \mu)(2\lambda + \mu)}.$$
Since $p, q \in h(U)$, applying Lemma 1.8 we immediately have
\[ |p_m| = \left| \frac{p^{(m)}(0)}{m!} \right| \leq |B_1|, \ m \in \mathbb{N}, \] (2.14)
and
\[ |q_m| = \left| \frac{q^{(m)}(0)}{m!} \right| \leq |B_1|, \ m \in \mathbb{N}. \] (2.15)

Applying (2.14), (2.15) and Lemma 1.8 for the coefficients $p_1, p_2, q_1$ and $q_2$, we readily get
\[ |a_2| \leq \sqrt{\frac{2|B_1| \cos \beta}{(1 + \mu)(2\lambda + \mu)}}. \]
This gives the bound on $|a_2|$ as asserted in (2.1).

Next, in order to find the bound on $|a_3|$, by subtracting (2.10) from (2.8), we get
\[ 2(a_3 - a_2^2)(2\lambda + \mu) = e^{-i\beta}(p_2 - q_2) \cos \beta. \] (2.16)

It follows from (2.13) and (2.16) that
\[ a_3 = \frac{e^{-i\beta} \cos \beta (p_2 + q_2)}{(1 + \mu)(2\lambda + \mu)} + \frac{e^{-i\beta}(p_2 - q_2) \cos \beta}{2(2\lambda + \mu)}. \] (2.17)

Applying (2.14), (2.15) and Lemma 1.8 once again for the coefficients $p_1, p_2, q_1$ and $q_2$, we readily get
\[ |a_3| \leq \frac{2|B_1| \cos \beta}{(2\lambda + \mu)(1 + \mu)}. \]
This completes the proof of Theorem 2.1. \qed

3. Corollaries and Consequences

In view of Remark 1.2 if we set
\[ h(z) = \frac{1 + Az}{1 + Bz}, \ -1 \leq B < A \leq 1, \ z \in U \]
and
\[ h(z) = \frac{1 + (1 - 2\alpha)z}{1 - z}, \ 0 \leq \alpha < 1, \ z \in U, \]
in Theorem 2.1 we can readily deduce Corollaries 3.1 and 3.2 respectively, which we merely state here without proof.

**Corollary 3.1.** Let $f(z)$ given by (1.1) be in the class $\mathcal{NP}_{\Sigma}(\mu, \lambda, \beta, \frac{1 + Az}{1 + Bz})$, then
\[ |a_2| \leq \sqrt{\frac{2(A - B) \cos \beta}{(1 + \mu)(2\lambda + \mu)}} \] (3.1)
and
\[ |a_3| \leq \frac{2(A - B) \cos \beta}{(2\lambda + \mu)(1 + \mu)}. \] (3.2)
where $\beta \in (-\pi/2, \pi/2)$, $\mu \geq 0$ and $\lambda \geq 1.$
Corollary 3.2. Let \( f(z) \) given by (1.1) be in the class \( \mathcal{NP}^{\mu,\lambda}_{\Sigma}(\beta, \alpha) \), \( 0 \leq \alpha < 1 \), \( \mu \geq 0 \) and \( \lambda \geq 1 \), then
\[
|a_2| \leq \sqrt{\frac{4(1-\alpha)\cos \beta}{(1+\mu)(2\lambda+\mu)}}
\] (3.3)
and
\[
|a_3| \leq \frac{4(1-\alpha)\cos \beta}{(2\lambda+\mu)(1+\mu)},
\] (3.4)
where \( \beta \in (-\pi/2, \pi/2) \).

Remark 3.3. When \( \beta = 0 \) the estimates of the coefficients \( |a_2| \) and \( |a_3| \) of the Corollary 3.2 are improvement of the estimates obtained in [5, Theorem 3.1].

Corollary 3.4. Let \( f(z) \) given by (1.1) be in the class \( \mathcal{NP}^{0,1}_{\Sigma}(\beta, \alpha) \), \( 0 \leq \alpha < 1 \) and \( \mu \geq 0 \), then
\[
|a_2| \leq \sqrt{\frac{4(1-\alpha)\cos \beta}{(1+\mu)(2+\mu)}}
\] (3.5)
and
\[
|a_3| \leq \frac{4(1-\alpha)\cos \beta}{(2+\mu)(1+\mu)},
\] (3.6)
where \( \beta \in (-\pi/2, \pi/2) \).

Corollary 3.5. Let \( f(z) \) given by (1.1) be in the class \( \mathcal{NP}^{0,1}_{\Sigma}(\beta, \alpha) \), \( 0 \leq \alpha < 1 \), then
\[
|a_2| \leq \sqrt{2(1-\alpha)\cos \beta}
\] (3.7)
and
\[
|a_3| \leq 2(1-\alpha)\cos \beta,
\] (3.8)
where \( \beta \in (-\pi/2, \pi/2) \).

Remark 3.6. Taking \( \beta = 0 \) in Corollary 3.5 the estimate (3.7) reduces to \( |a_2| \) of [10, Corollary 3.3] and (3.8) is improvement of \( |a_3| \) obtained in [10, Corollary 3.3].

Corollary 3.7. Let \( f(z) \) given by (1.1) be in the class \( \mathcal{NP}^{1,\lambda}_{\Sigma}(\beta, \alpha) \), \( 0 \leq \alpha < 1 \) and \( \lambda \geq 1 \), then
\[
|a_2| \leq \sqrt{\frac{2(1-\alpha)\cos \beta}{2\lambda+1}}
\] (3.9)
and
\[
|a_3| \leq \frac{2(1-\alpha)\cos \beta}{2\lambda+1},
\] (3.10)
where \( \beta \in (-\pi/2, \pi/2) \).

Remark 3.8. Taking \( \beta = 0 \) in Corollary 3.7 the inequality (3.10) improves the estimate of \( |a_3| \) in [6, Theorem 3.2].

Corollary 3.9. Let \( f(z) \) given by (1.1) be in the class \( \mathcal{NP}^{1,1}_{\Sigma}(\beta, \alpha) \), \( 0 \leq \alpha < 1 \), then
\[
|a_2| \leq \sqrt{\frac{2(1-\alpha)\cos \beta}{3}}
\] (3.11)
and
\[
|a_3| \leq \frac{2(1-\alpha)\cos \beta}{3},
\] (3.12)
where $\beta \in (-\pi/2, \pi/2)$.

Remark 3.10. For $\beta = 0$ the inequality (3.12) improves the estimate $|a_3|$ of [14, Theorem 2].

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