An HDG method using a hybridized numerical flux

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Abstract In this paper, we propose a new hybridizable discontinuous Galerkin (HDG) method for steady-state diffusion problems. In the proposed method, both the trace and flux of the exact solution are hybridized, whereas only the trace is hybridized and the flux is approximated by the numerical flux. We prove that our method is superconvergent if finite element spaces admit the $M$-decomposition. The so-called Lehrenfeld-Schöberl stabilization is implicitly included in our method, so that the orders of convergence in all variables are optimal without postprocessing and computation of any projection if finite element spaces are appropriately chosen. Numerical results are present to validate our theoretical results.

Keywords Discontinuous Galerkin Method · Hybridization · Superconvergence

Mathematics Subject Classification (2000) 65N12 · 65N30

1 Introduction

We consider the following steady-state diffusion problem with Dirichlet boundary condition as a model problem:

\begin{align}
q + \nabla u &= 0 \quad \text{in } \Omega, \\
\nabla \cdot q &= f \quad \text{in } \Omega, \\
\n\quad \quad u &= g_D \quad \text{on } \partial \Omega,
\end{align}

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where $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$) is a bounded convex polygonal or polyhedral domain and $f$ and $g_D$ are given functions. For simplicity, we deal only with the homogeneous case, i.e., $g_D \equiv 0$.

In this paper, we propose a new hybridizable discontinuous Galerkin (HDG) method in which both the trace of $u$ and flux of $q$ are hybridized. In the original method [7], the trace of $u$ is hybridized, denoted by $\hat{u}_h$, and the flux is approximated by the numerical flux defined as

$$\hat{q}_h \cdot n = q_h \cdot n + \tau(u_h - \hat{u}_h),$$

where $u_h$ and $q_h$ are unknown variables approximating to $u$ and $q$, respectively, $n$ is the outer unit normal vector to the boundary of an element, and $\tau$ is a stabilization parameter. As is well known, we can eliminate the variables $u_h$ and $q_h$ in an element-by-element fashion and obtain condensed equations only in terms of $\hat{u}_h$. In [3], another formulation was proposed, in which $\hat{q}_h$ is unknown and $\hat{u}_h$ is given by

$$\hat{u}_h = u_h + \tau^{-1}(q_h - \hat{q}_h) \cdot n.$$  (3)

Note that the above equality is equivalent to (2). Roughly speaking, the method proposed in [3] is derived by swapping $\hat{u}_h$ and $\hat{q}_h$ in the original method.

It is natural to consider a method using both $\hat{u}_h$ and $\hat{q}_h$ as unknown variables. However, it is not trivial to devise such a method because we do not know how to give an appropriate connection between the hybrid variables. Our idea is to impose (2) in variational form, not in strong form.

Our method as well as the original method has superconvergence properties in some cases, for example, when triangular meshes and polynomials of the same degree to approximate all unknowns are used. In the paper, we prove the superconvergence of our method by making use of the $M$-decomposition theory [6,4,5].

In [10,11], it was shown that the HDG method using the so-called Lehrenfeld-Schöberl (LS) stabilization [9] can achieve optimal convergence in all variables for any polygonal or polyhedral element if polynomials of degree $k$, $k + 1$, and $k$ are used to approximate $q$, $u$, and the trace of $u$ on inter-element boundaries. The LS stabilization is obtained by introducing the $L^2$-projection onto a finite element space for approximating the trace of $u$, denoted by $P_M$, in the numerical flux (2):

$$\hat{q}_h^{LS} \cdot n := q_h \cdot n + \tau(P_M u_h - \hat{u}_h).$$

Remarkably, it turns out that the LS stabilization is implicitly included in our method, which means that the method gives such optimal convergence without the use of any projection.

The rest of the paper is organized as follows. In Section 2, we introduce some notation and present our method. In addition, its well-posedness and local solvability are verified. In Section 3, we prove the superconvergence property of our method, assuming the $M$-decomposition. In Section 4, numerical results are presented to validate our theoretical results.
2 A New method

2.1 Notation

To begin with, we introduce some notation to define our method. Let \( \{T_h\}_h \) be a family of meshes satisfying the quasi-uniform condition, where \( h \) stands for the mesh size. Let \( \mathcal{E}_h \) denote the set of all facets of elements in \( T_h \). Let \( L^2(\mathcal{E}_h) \) denote the \( L^2 \)-space on \( \bigcup_{e \in \mathcal{E}_h} e \) and we define \( L^2_D(\mathcal{E}_h) = \{ \mu \in L^2(\Omega) : \mu|_{\partial\Omega} = 0 \} \). We use the usual symbols of Sobolev spaces \([1]\), such as \( H^m(D) \), \( H^m(D) := H^m(\Omega) \) for a domain \( D \) and an integer \( m \). When \( D = \Omega \), we simply write as \( \| \cdot \|_m,\Omega = \| \cdot \|_m \), \( \| \cdot \|_0,\Omega = \| \cdot \|_0 \), and \( | \cdot |_m = | \cdot | \). The piecewise Sobolev space of order \( m \) is denoted by \( H^m(\mathcal{T}_h) \). The inner products are denoted as

\[
(q,v)_K = \int_K q \cdot v \, dx, \quad (u,w)_K = \int_K u \cdot w \, dx, \quad (u,w)_{\partial K} = \int_{\partial K} u \cdot w \, ds,
\]

\[
(u,w)_{T_h} = \sum_{K \in T_h} (u,w)_K, \quad (u,w)_{\partial T_h} = \sum_{K \in T_h} (u,w)_{\partial K},
\]

\[
(q,v)_{T_h} = \sum_{K \in T_h} (q,v)_K, \quad (u,w) = \int_\Omega u \cdot w \, dx.
\]

Throughout the paper, we use the symbol \( C \) to denote a generic constant independent of the mesh size \( h \).

2.2 Finite element spaces

Let \( V(K) \), \( W(K) \) and \( M(e) \) be finite-dimensional spaces on an element \( K \in T_h \) or a facet \( e \in \mathcal{E}_h \) for approximating \( u|_K \), \( q|_K \) and the trace of \( u \) on \( e \), respectively. It is assumed that

\[
\nabla W(K) \subset V(K), \quad \nabla \cdot V(K) \subset W(K),
\]

namely, it holds that \( \nabla w \in V(K) \) for any \( w \in W(K) \) and \( \nabla \cdot v \in W(K) \) for any \( v \in V(K) \). We define an approximate space of \( \hat{q}_h \) by

\[
N(e) = \{ r \in L^2(e) := L^2(e)^d : (I - n \otimes n)r = 0 \}.
\]

The tangential part of \( \hat{q}_h \) is not used in the HDG method, so we let it be zero. We make the following assumptions:

\[
\mu n \in N(e) \quad \forall \mu \in M(e), \quad (A1)
\]
\[
(n \otimes n)v|_e \in N(e) \quad \forall v \in V(K), \quad (A2)
\]
\[
r \cdot n \in M(e) \quad \forall r \in N(e), \quad (A3)
\]
\[
v|_e \cdot n \in M(e) \quad \forall v \in V(K), \quad (A4)
\]
\[
w|_e \in M(e) \quad \forall w \in W(K). \quad (A5)
\]
where $K$ is any element in $\mathcal{T}_h$, $e$ is any edge of $K$, and $n$ is a unit normal vector to $e$. If $\mathbf{N}(e)$ is a subspace of $M(e)^d$, then $(A1)$ and $(A3)$ are automatically satisfied. We use $(A1) -(A4)$ when proving the well-posedness of our method. Assumption $(A5)$ is needed to make $\mathbf{V}(K) \times W(K)$ admit the $M$-decomposition. Hereinafter, we may write as $\text{tr} \mathbf{V} \subset M$ and $\text{tr} W \subset M$ and to indicate $(A4)$ and $(A5)$, respectively. Note that Assumptions (A1)–(A5) are in fact satisfied, for example, if all the spaces are polynomials of the same degree.

Finally, finite element spaces are constructed as:

$$\mathbf{V}_h := \{ v \in L^2(\Omega) : v|_K \in \mathbf{V}(K) \ \forall K \in \mathcal{T}_h \},$$

$$W_h := \{ w \in L^2(\Omega) : w|_K \in W(K) \ \forall K \in \mathcal{T}_h \},$$

$$M_h := \{ \mu \in L^2(\mathcal{E}_h) : \mu|_e \in M(e) \ \forall e \in \mathcal{E}_h \},$$

$$\mathbf{N}_h := \{ r \in L^2(\mathcal{E}_h) : r|_e \in \mathbf{N}(e) \ \forall e \in \mathcal{E}_h \}.$$

Let $P_V$, $P_W$, and $P_M$ denote the $L^2$-projections onto $\mathbf{V}_h$, $W_h$, and $M_h$, respectively. We simply write as $P_M(w|_{\partial K}) = P_M w$ for $w \in H^2(\mathcal{T}_h)$. Note that $P_M w$ does not belong to $M_h$ in general although $P_M w$ belongs to $M_h$ when $w$ is single-valued on inter-element boundaries.

Let $k$ be a non-negative integer. We assume that the following approximation properties of the spaces:

$$\| q - P_V q \| \leq Ch^s |q|_s,$$  \hspace{1cm} (6a)

$$\| q \cdot n - (P_V q) \cdot n \|_{\partial \mathcal{T}_h} \leq Ch^{s-1/2} |q|_s,$$  \hspace{1cm} (6b)

$$\| u - P_W u \| \leq Ch^s |u|_s,$$  \hspace{1cm} (6c)

$$\| u - P_W u \|_{\partial \mathcal{T}_h} \leq Ch^{s-1/2} |u|_s,$$  \hspace{1cm} (6d)

$$\| u - P_M u \|_{\partial \mathcal{T}_h} \leq Ch^{s-1/2} |u|_s,$$  \hspace{1cm} (6e)

for $1 \leq s \leq k + 1$.

2.3 The Method

Our method is defined as follows: Find $(q_h, u_h, \tilde{q}_h, \tilde{u}_h) \in \mathbf{V}_h \times W_h \times \mathbf{N}_h \times M_h$ such that

$$(q_h, v)_{\mathcal{T}_h} - (u_h, \nabla \cdot v)_{\mathcal{T}_h} + \langle \tilde{q}_h, v \cdot n \rangle_{\partial \mathcal{T}_h} = 0 \ \forall v \in \mathbf{V}_h, \hspace{1cm} (7a)$$

$$- (q_h, \nabla w)_{\mathcal{T}_h} + \langle \tilde{q}_h \cdot n, w \rangle_{\partial \mathcal{T}_h} = (f, w) \ \forall w \in W_h, \hspace{1cm} (7b)$$

$$\langle (q_h - \tilde{q}_h) \cdot n + \tau (u_h - \tilde{u}_h), \mu \rangle_{\partial \mathcal{T}_h} = 0 \ \forall \mu \in M_h, \hspace{1cm} (7c)$$

$$\langle (q_h - \tilde{q}_h) \cdot n + \tau (u_h - \tilde{u}_h), r \cdot n \rangle_{\partial \mathcal{T}_h} = 0 \ \forall r \in \mathbf{N}_h, \hspace{1cm} (7d)$$

where $\tau$ is a positive parameter. The equations $(7a)$ and $(7b)$ are the same as in the original HDG method. The difference from the original method is
that (2) is imposed in variational form, not in strong form. The transmission conditions for \( \hat{u}_h \) are automatically satisfied:

\[
\langle \hat{u}_h, r \cdot n \rangle_{\partial T_h} = 0 \quad \forall r \in N_h,
\]

\[
\langle \hat{q}_h \cdot n, \mu \rangle_{\partial T_h} = 0 \quad \forall \mu \in M_h.
\]

**Remark 1** As mentioned in the Introduction, the LS stabilization is hidden in the equations of (7). We rewrite the equations to explain it. Since \( \mu = P_M \mu \) and \( r \cdot n = P_M (r \cdot n) \), we see that (7c) and (7d) become

\[
\langle (q_h - \hat{q}_h) \cdot n + \tau (P_M u_h - \hat{u}_h), \mu \rangle_{\partial T_h} = 0 \quad \forall \mu \in M_h,
\]

\[
\langle (q_h - \hat{q}_h) \cdot n + \tau (P_M u_h - \hat{u}_h), r \cdot n \rangle_{\partial T_h} = 0 \quad \forall r \in N_h,
\]

respectively, which are the LS flux in variational form. Moreover, since \( \hat{q}_h \cdot n = P_M (\hat{q}_h \cdot n) \), (7c) is rewritten as

\[
- (q_h, \nabla w) + \langle \hat{q}_h \cdot n, P_M w \rangle_{\partial T_h} = (f, w) \quad \forall w \in W_h.
\]

Thus, all \( w \) and \( u_h \) appearing in the facet integrals of (7) can be replaced by \( P_M w \) and \( P_M u_h \), respectively. The rewritten equations are the same as those of (11), except that the numerical flux is given in variational form. As a result, error analysis can be done in the almost same manner as presented in (11).

### 2.4 Well-posedness

The goal of this section is to verify the well-posedness of our method by proving an a priori estimate under Assumptions (A1)–(A5). Although (A5) is not necessary to prove the well-posedness, we assume (A5) only for simplicity.

**Lemma 1** Let \( (q_h, u_h, \hat{q}_h, \hat{u}_h) \in V_h \times W_h \times N_h \times M_h \) be the solution of (7). Then we have

\[
\| q_h \|^2 - \langle (q_h - \hat{q}_h) \cdot n, u_h - \hat{u}_h \rangle_{\partial T_h} = (f, u_h).
\]

**Proof** Taking \( v = q_h \) in (7a) and \( w = u_h \) in (7b), we have

\[
\| q_h \|^2 - \langle u_h - \hat{u}_h, q_h \cdot n \rangle_{\partial T_h} + \langle \hat{q}_h \cdot n, u_h \rangle_{\partial T_h} = (f, u_h).
\]

From the transmission condition, it follows that \( \langle \hat{q}_h \cdot n, \hat{u}_h \rangle_{\partial T_h} = 0 \). Combining these equalities, we obtain the assertion. \( \square \)

An a priori estimate does not immediately follow from this lemma. We cannot take \( \mu = u_h \) in (7c) or \( r = (n \otimes n) q_h \) in (7c) since \( u_h \) and \( q_h \) may be double-valued on inter-element boundaries. Thus, we show the next lemma.
Lemma 2 Let \((q_h, u_h, \bar{q}_h, \bar{u}_h) \in V_h \times W_h \times N_h \times M_h\) be the solution of (7).

Then we have
\[
\langle (q_h - \bar{q}_h) \cdot n + \tau(u_h - \bar{u}_h), w \rangle_{\partial T_h} = 0 \quad \forall w \in W_h, \quad (9)
\]
\[
\langle (q_h - \bar{q}_h) \cdot n + \tau(u_h - \bar{u}_h), v \cdot n \rangle_{\partial T_h} = 0 \quad \forall v \in V_h. \quad (10)
\]

Proof We first prove (9). Let \(K \in T_h\) and \(e\) be an edge of \(K\). Let \(K'\) denote the adjacent element of \(K\) across \(e\), see Figure 1. We denote as \(z = w|_{K}\) and \(z' = w|_{K'}\) and let \(n\) and \(n'\) be the unit outer normal vector to \(\partial K\) and \(\partial K'\), respectively. Let \(\{\cdot\}\) and \([\cdot]\) be the usual average and jump operators (e.g. see [2]). Namely,
\[
\{w\}|_e = (z + z')/2, \quad [w]|_e = zn + z'n'. \quad (11)
\]
If \(K\) has no adjacent element across \(e\), we define as
\[
\{w\}|_e = 0, \quad [w]|_e = 2zn,
\]
which can be included in (11) as \(z' = -z\). In view of \(n' = -n\), we have
\[
\left( \{w\} + \frac{[w]}{2} \cdot n \right)|_{\partial K} = \frac{1}{2}(z + z') + \frac{1}{2}(zn + z'n') \cdot n = z.
\]
Therefore we have
\[
\left( \{w\} + \frac{[w]}{2} \cdot n \right)|_{\partial K} = w|_{\partial K}. \quad (12)
\]
Note that \(\{w\}\) and \([w]\) are single-valued, \(\{w\} = 0\) on \(\partial \Omega\), and \((I - n \otimes n)[w] = 0\) on inter-element boundaries. By (A5) and (A4), we can take \(\mu = \{z\}\) in (7c) and \(r = [z]/2\) in (7d). Then we get
\[
\langle (q_h - \bar{q}_h) \cdot n + \tau(u_h - \bar{u}_h), \{w\} + ([w]/2) \cdot n \rangle_{\partial T_h} = 0.
\]
By (12), we deduce that
\[
\langle (q_h - \bar{q}_h) \cdot n + \tau(u_h - \bar{u}_h), w \rangle_{\partial T_h} = 0. \quad (13)
\]
which implies \( \theta \). Next, we show \( \theta \). Let \( z = v|_K \) and \( z' = v|_{K'} \). The average and jump of \( v \) are given by

\[
\{v\}_e = (z + z')/2, \quad [v]_e = z \cdot n + z' \cdot n',
\]
respectively. When \( K \) has no adjacent element across the edge, we set \( z' = z \) so that

\[
\{v\}_e = z, \quad [v]_e = 0.
\]

We have

\[
\left( \frac{[v]}{2} + \{v\} \cdot n \right)_{|\partial K} = \frac{1}{2} (z \cdot n + z' \cdot n') + \frac{1}{2} (z + z') \cdot n = z \cdot n. \tag{14}
\]

Substituting \( \mu = [v]/2 \) in \( \partial \) and \( r = n \otimes n \{v\} \) in \( \partial \) and noting that \( r \cdot n = \{v\} \cdot n \) in this case, we get

\[
\langle (q_h - \hat{q}_h) \cdot n + \tau (u_h - \hat{u}_h), [v]/2 + \{v\} \cdot n \rangle_{\partial \Omega} = 0.
\]

By \( \langle 14 \rangle \), \( ([v]/2 + \{v\} \cdot n)_{|\partial K} = (v \cdot n)_{|\partial K} \). Therefore we obtain \( \theta \). \( \square \)

**Theorem 1** Let the setting be the same as in Lemma 2. Then we have

\[
\|q_h\|^2 + \|v^{-1/2} (q_h - \hat{q}_h) \cdot n \|^2_{\partial \Omega} + \|v^{1/2} (u_h - \hat{u}_h)\|^2_{\partial \Omega} \leq C \|f\|^2.
\]

**Proof** Taking \( w = u_h \) in \( \theta \) and \( \mu = \hat{u}_h \) in \( \partial \), we have

\[
\|v^{1/2} (u_h - \hat{u}_h)\|^2_{\partial \Omega} = -\langle (q_h - \hat{q}_h) \cdot n, u_h - \hat{u}_h \rangle_{\partial \Omega}. \tag{15}
\]

Combining this with \( \theta \), we have

\[
\|q_h\|^2 + \|v^{1/2} (u_h - \hat{u}_h)\|^2_{\partial \Omega} = (f, u_h).
\]

It is known that

\[
\|u_h\| \leq C \left( \|q_h\|^2 + \|v^{1/2} (u_h - \hat{u}_h)\|^2_{\partial \Omega} \right)^{1/2}, \tag{16}
\]

see Theorem 3 in the Appendix for the detail proof. By the above inequality and the Schwarz inequality, we obtain

\[
\|q_h\|^2 + \|v^{1/2} (u_h - \hat{u}_h)\|^2_{\partial \Omega} \leq C \|f\|^2. \tag{17}
\]

In a similar way to \( \theta \), we can get

\[
\left\langle (q_h - \hat{q}_h) \cdot n \right\rangle_{\partial \Omega} = -\langle (q_h - \hat{q}_h) \cdot n, \tau (P_M u_h - \hat{u}_h) \rangle_{\partial \Omega}
\]
and

\[
\|q_h\|^2 + \|v^{-1/2} (q_h - \hat{q}_h) \cdot n \|^2_{\partial \Omega} \leq C \|f\|^2, \tag{18}
\]
which completes the proof. \( \square \)
Remark 2 We do not need Assumption \((A5)\) to prove the well-posedness. Even if \(\text{tr} \, W \not\subset M\), we can show the following instead of Lemma \(^2\) and Theorem \(^3\)
\[
\langle (q_h - \tilde{q}_h) \cdot n + \tau (P_M u_h - \tilde{u}_h), w \rangle_\partial T_h = 0 \quad \forall w \in W_h, \tag{19}
\]
\[
\langle (q_h - \tilde{q}_h) \cdot n + \tau (P_M u_h - \tilde{u}_h), v \cdot n \rangle_\partial T_h = 0 \quad \forall v \in V_h, \tag{20}
\]
and
\[
\|q_h\|^2 + \|\tau^{-1/2} (q_h - \tilde{q}_h) \cdot n\|^2_\partial T_h + \|\tau^{1/2} (P_M u_h - \tilde{u}_h)\|^2_\partial T_h \leq C \|f\|^2.
\]

We present a further result on the jumps of the numerical trace and flux.

**Theorem 2** Let the setting be the same as in Lemma \(^3\) We have
\[
\|\tau^{-1/2} (q_h - \tilde{q}_h) \cdot n\|^2_\partial T_h = \|\tau^{-1/2} (u_h - \tilde{u}_h)\|^2_\partial T_h.
\]

**Proof** Taking \(\mu = \tilde{u}_h\) in \((17c)\) and \(w = u_h\) in \((19)\), we get
\[
\|\tau^{1/2} (u_h - \tilde{u}_h)\|_\partial T_h = \langle (q_h - \tilde{q}_h) \cdot n, u_h - \tilde{u}_h \rangle_\partial T_h.
\]

Similarly, taking \(r = \tilde{q}_h\) in \((17d)\) and \(v = q_h\) yields
\[
\|\tau^{1/2} (q_h - \tilde{q}_h) \cdot n\|_\partial T_h = \langle u_h - \tilde{u}_h, (q_h - \tilde{q}_h) \cdot n \rangle_\partial T_h.
\]

Putting the two equalities together, we obtain the assertion. \(\square\)

**Remark 3** When \(\text{tr} \, W \not\subset M\), the following holds:
\[
\|\tau^{-1/2} (q_h - \tilde{q}_h) \cdot n\|^2_\partial T_h = \|\tau^{1/2} (P_M u_h - \tilde{u}_h)\|^2_\partial T_h.
\]

2.5 Local solvability

We verify that the local solvability of our method, i.e., \(q_h\) and \(u_h\) can be locally eliminated by the hybrid unknowns. To this end, it suffices to show that the equations \(^7\) for each element \(K \in T_h\) have the only zero solution if \(\hat{q}_h, \tilde{u}_h\) and \(f\) are all set to zeros. Let \((q_K, u_K) \in V(K) \times W(K)\) be a solution of the following:
\[
\begin{align*}
(q_K, v)_K - (u_K, \nabla \cdot v)_K &= 0 \quad \forall v \in V(K), \tag{21a} \\
(q_K, \nabla w)_K &= 0 \quad \forall w \in W(K), \tag{21b} \\
(q_K \cdot n + \tau u_K, \mu)_{\partial K} &= 0 \quad \forall \mu \in M(\partial K), \tag{21c} \\
(q_K \cdot n + \tau u_K, r \cdot n)_{\partial K} &= 0 \quad \forall r \in N(\partial K). \tag{21d}
\end{align*}
\]

We show that \(\hat{q}_K = 0\) and \(\hat{u}_K = 0\). Taking \(v = q_K\) in \(21a\) and \(w = u_K\) in \(21b\) yields
\[
\|q_K\|^2_K - (u_K, q_K \cdot n)_{\partial K} = 0.
\]

By \(21d\) with the choice of \(r = (n \otimes n) q_K\), we see that the second term in the above equals \(\|\tau^{-1/2} q_K \cdot n\|^2_{\partial K}\), from which it follows that \(q_K = 0\). Then choosing \(\mu = u_K\) in \(21c\), we get \(\|\tau^{1/2} u_K\|^2_{\partial K} = 0\). Taking \(v = \nabla u_K\) in \(21a\) and integrating it by parts, we have \(\|\nabla u_K\|^2_K = 0\), \(u_K\) is constant on \(K\), and it must be zero since \(u_K = 0\) on \(\partial K\). Therefore, we conclude that the only solution of the equations \(^2\) is zero.
3 Superconvergence by the \( M \)-decomposition

In this section, we prove the superconvergence property of our method, assuming the \( M \)-decomposition. We start by introducing the operator \( Q_h : H^1(T_h) \times L^2(\mathcal{E}_h) \to V_h \), defined by requiring

\[
(Q_h(w, \mu), v)_{T_h} - (w, \nabla \cdot v)_{T_h} + \langle \mu, v \cdot n \rangle_{\partial T_h} = 0 \quad \forall v \in V_h. \quad (22)
\]

It immediately follows that

\[ Q_h(u_h, \hat{u}_h) = q_h. \]

By using the operator, we rewrite (7a) and (7b) into an easier form to handle.

**Lemma 3** Let \((q_h, u_h, \hat{u}_h, \hat{q}_h)\) be the solution of our method (7). Then we have

\[
(q_h, Q_h(w, \mu))_{T_h} - \langle (q_h - \hat{q}_h) \cdot n, w - \mu \rangle_{\partial T_h} = (f, w) \quad \forall w \in W_h, \mu \in M_h. \quad (23)
\]

**Proof** Taking \( v = q_h \) in (22), we have

\[
(Q_h(w, \mu), q_h)_{T_h} - (w, \nabla \cdot q_h)_{T_h} + \langle \mu, q_h \cdot n \rangle_{\partial T_h} = 0. \quad (23)
\]

By integration by parts, the second term above becomes

\[
(w, \nabla \cdot q_h)_{T_h} = (q_h, \nabla w)_{T_h} - \langle q_h \cdot n, w \rangle_{\partial T_h}
\]

\[
= -\langle (q_h - \hat{q}_h) \cdot n, w \rangle_{\partial T_h} - (f, w).
\]

Recalling the transmission condition \( \langle \hat{q}_h \cdot n, \mu \rangle_{\partial T_h} = 0 \), we deduce the third term in (23) equals \( \langle (q_h - \hat{q}_h) \cdot n, \mu \rangle_{\partial T_h} \), which completes the proof. \( \square \)

3.1 The HDG-projection

If we assume that \( V_h \times W_h \) admits the \( M \)-decomposition, then the HDG-projection is well defined. We present the summary of results shown in [6] in the next theorem.

**Theorem 3** ([6]) If \( V_h \times W_h \) admits the \( M \)-decomposition, there exists an HDG-projection \((\Pi_V q, \Pi_W u) \in V_h \times W_h \) such that, for all \( K \in T_h \),

\[
(\Pi_W u, \nabla \cdot v)_K = (u, \nabla \cdot v)_K \quad \forall v \in V(K), \quad (24a)
\]

\[
(\Pi_V q, \nabla w)_K = (q, \nabla w)_K \quad \forall w \in W(K), \quad (24b)
\]

\[
\langle \Pi_V q \cdot n + \tau \Pi_W u, \mu \rangle_{\partial K} = \langle q \cdot n + \tau P_M u, \mu \rangle_{\partial K} \quad \forall \mu \in M(\partial K). \quad (24c)
\]
By the approximation properties, the errors of the HDG-projections can be estimated as
\[ \|u - \Pi_W u\| \leq Ch^{k+1}|u|_{k+1}, \]  
\[ \|q - \Pi_V q\| \leq Ch^{k+1}|u|_{k+1}. \]

In the following, we show several properties of the HDG-projection concerning the operator \( Q_h \).

**Lemma 4** We have, for all \( w \in W_h \) and \( \mu \in M_h \),
\[ (\Pi_V q, Q_h(w, \mu))_{T_h} + \langle \tau(\Pi_W u - P_M u), w - \mu \rangle_{\partial T_h} = (f, w). \]

**Proof** Taking \( v = \Pi_V q \) in (22),
\[ (Q_h(w, \mu), \Pi_V q)_{T_h} - (w, \nabla \cdot \Pi_V q)_{T_h} + \langle \mu, \Pi_V q \cdot n \rangle_{\partial T_h} = 0. \]

By (24b), the second term becomes
\[ (w, \nabla \cdot \Pi_V q)_{T_h} = -(\nabla w, \Pi_V q)_{T_h} + \langle w, \Pi_V q \cdot n \rangle_{\partial T_h} + \langle \tau(P_M u - \Pi_W u), P_M w \rangle_{\partial T_h} = (f, w) + \langle \tau(P_M u - \Pi_W u), P_M w \rangle_{\partial T_h}. \]

By substituting this into (29), it follows that
\[ (Q_h(\Pi_W u, P_M u), v)_{T_h} = (q, v)_{T_h}. \]

The proof is complete. \( \Box \)

**Lemma 5** We have
\[ Q_h(\Pi_W u, P_M u) = P_V q. \]

**Proof** Substituting \( w = \Pi_W u \) and \( \mu = P_M u \) in (22), we get
\[ (Q_h(\Pi_W u, P_M u), v)_{T_h} - (\Pi_W u, \nabla \cdot v)_{T_h} + \langle P_M u, v \cdot n \rangle_{\partial T_h} = 0 \quad \forall v \in V_h. \]

By (24a), we deduce
\[ -(\Pi_W u, \nabla \cdot v)_{T_h} = -(u, \nabla \cdot v)_{T_h} \]
\[ = (\nabla u, v)_{T_h} - \langle u, v \cdot n \rangle_{\partial T_h} \]
\[ = -(q, v)_{T_h} - \langle P_M u, v \cdot n \rangle_{\partial T_h}. \]

By substituting this into (29), it follows that
\[ (Q_h(\Pi_W u, P_M u), v)_{T_h} = (q, v)_{T_h}. \]

The proof is complete. \( \Box \)
3.2 Optimal convergence of \( q_h \)

We denote the projections of errors as
\[
e_q = \Pi_V q - q_h, \quad e_u = \Pi_W u - u_h, \quad e_{\tilde{u}} = P_{M}u - \tilde{u}_h
\]
and the approximate errors as
\[
\delta_q = q - \Pi_V q, \quad \delta_u = u - P_W u, \quad \delta_{M}u = u - P_{M}u.
\]

**Lemma 6** The following error equations hold:
\[
(e_q, Q_h(w, \mu))_{T_h} + (\tau(e_u - e_{\tilde{u}}), w - \mu)_{\partial T_h} = 0 \quad \forall w \in W_h, \mu \in M_h. \tag{30}
\]

*Proof* From Lemmas 3 and 4, it immediately follows.

**Theorem 4** We have
\[
\|e_q\|^2 + 2\|\tau^{1/2}(e_u - e_{\tilde{u}})\|^2_{\partial T_h} \leq \|\delta_q\|^2. \tag{31}
\]

*Proof* By Lemma 5 we have
\[
Q_h(e_u, e_{\tilde{u}}) = Q_h(\Pi_W u, P_{M}u) - Q_h(u_h, \tilde{u}_h) = P_V q - q_h.
\]

In (30), taking \( w = e_u \) and \( \mu = e_{\tilde{u}} \) yields
\[
(e_q, P_V q - q_h)_{T_h} + \|\tau^{1/2}(e_u - e_{\tilde{u}})\|^2_{\partial T_h} = 0. \tag{32}
\]

The \( L^2 \)-norm of \( e_q \) is computed as
\[
\|e_q\|^2 = (\Pi_V q - q_h, \Pi_V q - q_h)_{T_h} = (\Pi_V q - q_h, \Pi_V q - q_h)_{T_h} + (\Pi_V q - q_h, q - q_h)_{T_h} = -(e_q, \delta_q)_{T_h} + (e_q, P_V q - q_h)_{T_h}.
\]

In view of (32), we have
\[
\|e_q\|^2 + \|\tau^{1/2}(e_u - e_{\tilde{u}})\|^2_{\partial T_h} = -(e_q, \delta_q)_{T_h}.
\]

By applying the Young inequality to the right-hand side, we obtain the assertion.

**Corollary 1** Assume the approximation properties \( \| \) hold. Then we have
\[
\|q - q_h\| \leq 2\|\delta_q\| \leq C h^{k+1}|u|_{k+1}.
\]

*Proof* Apply the simple triangle inequality to Theorem 4. \(\square\)
3.3 Superconvergence of $u_h$

We consider the following adjoint problem: Find $(\theta, \xi) \in H^1(\Omega) \times (H^2(\Omega) \cap H^1_0(\Omega))$ such that

\begin{align}
\theta + \nabla \xi &= 0 \quad \text{in } \Omega, \\
\nabla \cdot \theta &= e_u \quad \text{in } \Omega, \\
\xi &= 0 \quad \text{on } \partial \Omega.
\end{align}

(33a) \quad (33b) \quad (33c)

It is well known that the elliptic regularity holds:

$$\|\theta\|_1 \leq C\|e_u\|.$$  

We prove the superconvergence of $\Pi W u - u_h$ by the Aubin-Nitsche technique.

**Theorem 5** We have

$$\|\Pi W u - u_h\| \leq C\|q - q_h\|.$$  

(34)

**Proof** Since Lemma [4] holds for the adjoint problem, it follows that

$$(\Pi V \theta, Q_h(w, \mu))_{T_h} + (\tau(\Pi W \xi - P_M \xi), w - \mu)_{\partial T_h} = (e_u, w)$$

for any $w \in W_h$ and $\mu \in M_h$. Choosing $w = e_u$ and $\mu = e_{\tilde{u}}$ above, in view of $Q_h(e_u, e_{\tilde{u}}) = P_V q - q_h$, leads to

$$(\Pi V \theta, P_V q - q_h)_{T_h} + (\tau(\Pi W \xi - P_M \xi), e_u - e_{\tilde{u}})_{\partial T_h} = \|e_u\|^2.$$  

(35)

By Lemma [4] for $w = \Pi W \xi$ and $\mu = P_M \xi$, noting that

$$Q_h(\Pi W \xi, P_M \xi) = P_V \theta,$$

we have

$$(e_q, P_V \theta) + (\tau(e_u - e_{\tilde{u}}), \Pi W \xi - P_M \xi)_{\partial T_h} = 0.$$  

(36)

From (35) and (36), it follows that

$$(\theta - \Pi V \theta, e_q)_{T_h} = \|e_u\|^2.$$  

By the Schwarz inequality, we have

$$\|e_u\|^2 \leq \|\theta - \Pi V \theta\|\|e_q\|.$$  

By (26) for $k = 0$ and the elliptic regularity, we have

$$\|\theta - \Pi V \theta\| \leq Ch|\theta|_1 \leq Ch\|e_u\|,$$

from which and Theorem [4] we obtain (34). The proof is complete. \qed
4 Numerical results

In this section, we examine the orders of convergence of our method by numerical experiments to validate our theoretical results. The test problem is as follows:

\[- \Delta u = 2\pi^2 \sin(\pi x) \sin(\pi y) \quad \text{in } \Omega := (0, 1)^2, \]
\[ u = 0 \quad \text{on } \partial \Omega, \]

where the exact solution is \( \sin(\pi x) \sin(\pi y) \). We use unstructured triangulations as a mesh. We carried out all numerical computations with FreeFEM++. \[\text{(8)}\]

4.1 Case 1: \( \text{tr} W \subset M \)

We use polynomials of the same degree \( k \) for all variables, which satisfies Assumption (A1)–(A5) and admit the \( M \)-decomposition. The stabilization parameter is set as \( \tau \equiv 1 \). The numerical results are displayed in Table 1. We observe that the orders of convergence in \( q \) are optimal, which fully agrees with Theorem 1. In addition, as expected by Theorem 2, the jump quantities, \( \| (q_h - \hat{q}_h) \cdot n \|_{\partial T_h} \) and \( \| u_h - \hat{u}_h \|_{\partial T_h} \), are equal each other.

Table 1: Convergence history in Case 1.

| \( k \) | \( 1/h \) | \( \| q - q_h \| \) error order | \( \| u - u_h \| \) error order | \( \| (q_h - \hat{q}_h) \cdot n \|_{\partial T_h} \) error order | \( \| u_h - \hat{u}_h \|_{\partial T_h} \) error order |
|-------|------|----------------|----------------|----------------|----------------|
| 1     | 4    | 8.78E-02       | 4.67E-02       | 3.19E-01       | 3.19E-01       |
|       | 8    | 1.96E-02       | 2.16           | 2.21           | 1.07E-01       |
|       | 16   | 5.6E-03        | 1.80           | 7.56E-03       | 1.07E-01       |
|       | 32   | 1.35E-03       | 2.07           | 1.15E-03       | 1.47E-02       |
| 2     | 4    | 7.58E-03       | 3.50           | 2.35E-02       | 3.23E-02       |
|       | 8    | 6.64E-04       | 5.59E-03       | 1.07E-01       | 1.07E-01       |
|       | 16   | 8.23E-05       | 3.02           | 6.35E-03       | 1.18E-03       |
|       | 32   | 9.39E-06       | 7.76E-06       | 1.05E-03       | 2.05E-04       |
| 3     | 4    | 5.86E-04       | 4.97           | 3.64E-03       | 3.64E-03       |
|       | 8    | 2.64E-05       | 4.74           | 2.36E-04       | 2.36E-04       |
|       | 16   | 2.25E-06       | 3.58           | 2.69E-05       | 2.69E-05       |
|       | 32   | 1.18E-07       | 4.22           | 4.15E-06       | 4.15E-06       |

4.2 Case 2: \( \text{tr} W \not\subset M \)

As mentioned in Remark [1], our method is optimally convergent in all variables if the degrees of polynomials of \( V_h, N_h, \) and \( M_h \) are equal to \( k \), that of \( W_h \) is \( k + 1 \), and we set \( \tau = 1/h \). We check the orders of convergence of the method by numerical experiments. The convergence history for \( k = 1, 2, 3 \) is shown in Table 2. Let us emphasize that any projection is not used or computed when solving the resulting equations. From the results, we see that the orders...
of convergence are optimal for both \( u \) and \( q \) in all cases, which supports our claim stated in Remark 1. Similarly to Case 1, the jump of \( q_h - \hat{q}_h \) coincides with the projected jump of \( u_h - \hat{u}_h \). The order of convergence of the projected jump is greater by one than the jump of \( u - P_M u \), which is a superconvergence property since \( \| h^{-1/2} (u - P_M u) \| = O(h^k) \).

### Table 2: Convergence history in Case 2.

| \( k \) | \( 1/h \) | Error of \( q - q_h \) | Error of \( u - u_h \) | Error of \( h^{-1/2} (q_h - \hat{q}_h) \cdot n \) | Error of \( h^{-1/2} (P_M u_h - \hat{u}_h) \) |
|--------|--------|------------------|------------------|------------------|------------------|
| 1      | 4      | 7.77E-02         | 1.29E-02         | 1.64E-01         | 1.64E-01         |
| 4      | 8      | 1.76E-02         | 2.14             | 1.19E-03         | 2.15             |
| 16     | 5.15E-03| 1.92E-04         | 2.63             | 1.11E-02         | 1.73             |
| 32     | 1.22E-03| 3.69E-02         | 3.69E-02         | 1.11E-02         | 1.73             |
| 2      | 4      | 5.24E-03         | 2.61E-03         | 3.09             | 2.62E-03         |
| 8      | 4.79E-04| 2.31E-03         | 2.31E-03         | 2.08             | 2.62E-03         |
| 16     | 5.62E-05| 4.38E-04         | 3.69             | 3.69E-02         | 2.31E-03         |
| 32     | 6.98E-06| 4.13E-04         | 3.69             | 3.69E-02         | 2.31E-03         |

A A proof of (16)

We prove (16) for the completeness of the paper. It is also worth presenting a proof using the operator \( Q_h \).

**Lemma 7** Let \( (q, u) \in H^1(\Omega) \times (H^2(\Omega) \cap H_0^1(\Omega)) \) be the exact solution of (1). Then we have

\[
(q, Q_h(w, \mu))_{\Gamma_h} + ((q - P_V q) \cdot n, w - \mu)_{\partial \Gamma_h} = (f, w)
\]

(37)

for any \( w \in H^2(\Gamma_h) \) and \( \mu \in L^2_{\Omega}(\mathcal{E}_h) \).

**Proof** Taking \( v = P_V q \) in (22) and integrating by parts, we have

\[
(q, Q_h(w, \mu))_{\Gamma_h} - (w, \nabla \cdot P_V q)_{\Gamma_h} + (\mu, P_V q \cdot n)_{\partial \Gamma_h} = (q, Q_h(w, \mu))_{\Gamma_h} + (\nabla w, q)_{\Gamma_h} - (w - \mu, P_V q \cdot n)_{\partial \Gamma_h} = 0.
\]

Since \( q \) satisfies

\[
-(q, \nabla w)_{\Gamma_h} + (q \cdot n, w)_{\partial \Gamma_h} = (f, w)
\]

and

\[
(q \cdot n, \mu)_{\partial \Gamma_h} = 0,
\]

we deduce

\[
(q, Q_h(w, \mu))_{\Gamma_h} + (w - \mu, (q - P_V q) \cdot n)_{\partial \Gamma_h} = (f, w),
\]

which completes the proof. \( \square \)
There exists a positive constant $C$ independent of $h$ such that

$$
\|w\| \leq C \left( \|Q_h(w, \mu)\|^2 + \|w - \mu\|_{\partial \Omega_h}^2 \right)^{1/2}
$$

for any $w \in H^1(\Omega_h)$ and $\mu \in L_2^2(\xi_h)$.

**Proof** We prove this by the duality argument. To this end, we consider the following problem:

Find $(\theta, \xi) \in H^1(\Omega) \times (H^2(\Omega) \cap H^1_0(\Omega))$ such that

$$
\begin{aligned}
&\theta + \nabla \xi = 0 \quad \text{in } \Omega, \\
&\nabla \cdot \theta = w \quad \text{in } \Omega, \\
&\xi = 0 \quad \text{on } \partial \Omega.
\end{aligned}
$$

Applying Lemma 7 for $\theta$ and $\xi$, we have

$$
(\theta, Q_h(w, \mu))_{\Omega_h} + \langle (\theta - PV \theta) \cdot n, w - \mu \rangle_{\partial \Omega_h} = \|w\|^2.
$$

By the Cauchy-Schwarz and Young’s inequalities, we have

$$
\|w\|^2 \leq \|\theta\| \|Q_h(w, \mu)\| + \|\theta - PV \theta\| \|n\|_{\partial \Omega_h} \|w - \mu\|_{\partial \Omega_h} \\
\leq \left( \|\theta\|^2 + \|\theta - PV \theta\| \|n\|_{\partial \Omega_h}^2 \right) \left( \|Q_h(w, \mu)\|^2 + \|w - \mu\|_{\partial \Omega_h}^2 \right) =: I_1.
$$

The term $I_1$ is bounded as

$$
|I_1| \leq \|\theta\|^2 + C h^{1/2} \|\theta\|^2 \\
\leq C \|w\|, \quad \text{(by (6b))}
$$

from which, the assertion immediately follows. \(\square\)

From this theorem for $w = u_h$ and $\mu = \hat{u}_h$, (16) follows.

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