On the Poincaré Index of Isolated Invariant Sets

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Abstract

In this paper, we use Conley index theory to examine the Poincaré index of an isolated invariant set. We obtain some limiting conditions on a critical point of a planar vector field to be an isolated invariant set. As a result we show the existence of infinitely many homoclinic orbits for a critical point with the Poincaré index greater than one.

Keywords: Conley index, Homoclinic orbit, Poincaré-Lefchetz duality, Poincaré index

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1 Introduction

The Conley index has proved to be a useful tool in the investigation of qualitative properties of dynamical systems. It has generalized Morse theory for an isolated invariant set of a continuous flow on a locally compact metric space[1,3]. For this reason, Conley index is known as a generalization of Morse theory. In[2] and[3], Conley and Zehender used this index to show the existence of periodic solutions for Hamiltonian systems. This was a landmark in the proof of the Arnold conjecture on the existence of periodic orbits of Hamiltonian systems on symplectic manifolds. Conley index theory has also some applications in the existence of solutions for a class of differential equations. (See[13,17,1] and references therein.)
In Conley index theory, we deal with pairs of closed sets called index pair for an isolated invariant set \( I \). The homotopy type of these index pairs is independent of the index pair chosen, which is called the Conley index of \( I \) and denoted by \( h(I) \). This paper concerns the relation between Conley index theory and the Poincaré-Hopf theorem \([5]\). We define the Poincaré index of an isolated invariant set \( I \) to be \( \chi(h(I)) \). This definition coincides with the classical Poincaré index when the invariant set is a single point. We use some topological properties of the Conley index to obtain restrictions on the Poincaré index of isolated invariant sets in dimension two. It is well-known that on a two-dimensional manifold \( M \), the Poincaré index of an isolated critical point of a gradient vector field is not greater than one. Here we show that a critical point \( x \) with \( \text{ind}(x) > 1 \) cannot be an isolated invariant set. This concludes the existence of infinitely many homoclinic orbits for such a critical point.

We first present some basic results from Conley index theory. Then we define the concept of continuation and provide a new proof for the results of \([5]\) based on continuation to gradient \([12, 13]\). Finally we apply these results to show the existence of infinitely many homoclinic orbits in dimension two.

## 2 Conley Index

Let \( \varphi^t \) be a \( C^1 \)-flow on a smooth manifold \( M \). A subset \( I \subset M \) is called an isolated invariant set if it is the maximal invariant set in some compact neighborhood of itself. Such neighborhood is called an isolating neighborhood.

**Definition.** A closed pair \((N, L)\) is called an index pair for \( I \) if

1. \( \overline{N-L} \) is an isolating neighborhood for \( I \).

2. \( L \) is positively invariant relative to \( N \), i.e., if \( x \in L, t \geq 0, \varphi^{[0,t]}(x) \subset N \), then \( \varphi^{[0,t]}(x) \subset L \).

3. \( L \) is the exit set of \( N \), i.e., if \( x \in N, t \in \mathbb{R}^+ \) and \( \varphi^t(x) \notin N \), then there is a \( t' \in [0, t] \) such that \( \varphi^{t'}(x) \in L \).
In \[1, 3, 16, 17\] it has been shown that every isolated invariant set \(I\) admits an index pair \((N, L)\) and the homotopy type of \((N/L, [L])\) is independent of the index pair chosen. We denote the homotopy type of \((N/L, [L])\) by \(h(I)\) and call it the Conley index of \(I\). The homology Conley index of \(I\) is defined by \(CH_\ast(I) = H_\ast(N/L, [L])\).

**Example 2.1.** Let \(x \in M\) be a nondegenerate critical point for \(f : M \xrightarrow{C^2} \mathbb{R}\). Then \(\{x\}\) is an isolated invariant set for \(-\nabla f\) and by Morse Lemma \([4]\), it is easy to show that \(h(\{p\})\) is a pointed k-sphere where \(k\) is the number of positive eigenvalues of Hessian matrix \(f\) at \(p\). Therefore the Conley index can be considered as a generalization of Morse index.

It is not true that \(H_\ast(N, L) \cong H_\ast(N/L, [L])\) for every index pair \((N, L)\). In \([16]\), Salamon introduced a class of index pairs for which the above isomorphism holds.

**Definition.** An index pair \((N, L)\) is called regular if the exit time map

\[
\tau_+ : N \rightarrow [0, +\infty], \quad \tau_+(x) = \begin{cases} 
\sup\{t | \varphi^{t,0}(x) \subset N - L\} & \text{if } x \in N - L, \\
0 & \text{if } x \in L,
\end{cases}
\]

is continuous. (See \([16]\) for more details about regular index pairs.) For every regular index pair \((N, L)\), we define the induced semi-flow on \(N\) by

\[
\varphi_2^t : N \times \mathbb{R}^+ \rightarrow N, \quad \varphi_2^t(x) = \varphi_{\min\{t, \tau_+(x)\}}(x)
\]

**Proposition 2.2.** If \((N, L)\) be a regular index pair for a continuous flow \(\varphi^t\), then \(L\) is a neighborhood deformation retract in \(N\). In particular, the natural map \(\pi : N \rightarrow N/L\) induces an isomorphism \(H_\ast(N, L) \cong H_\ast(N/L, [L])\).

**Proof.** Consider the induced semi-flow \(\varphi_2\) on \(N\) and the neighborhood \(U := \tau_+^{-1}[0, 1]\) of \(L\). Now \(\varphi_2|_{U \times [0, 1]}\) gives the desired deformation retraction. \(\square\)

In \([14]\), Robbin and Salamon proved that every isolated invariant set admits a regular index pair which is stable under perturbation. They first showed the existence of a smooth Liapunov function on a neighborhood of the isolated invariant set.
**Theorem 2.3.** Let $N$ be an isolating neighborhood of $I$. Then there is a neighborhood $U$ of $N$ and a smooth function $f: U \to \mathbb{R}$ satisfying

(i) $f(x) = 0$ for all $x \in I$.

(ii) $\frac{d}{dt}|_{t=0} f(\varphi^t(x)) < 0$ for all $x \in N - I$. ($f$ decreases along orbits in $U - I$.)

Then they used this Liapunov function to construct a triple $(N, L^-, L^+)$, such that $(N, L^+)$ is a regular index pair for $I$ with respect to the forward flow and $(N, L^-)$ is a regular index pair for $I$ with respect to the reverse flow. Furthermore $L^+$ and $L^-$ can be chosen to be $(n-1)$-manifolds with boundary, so that $N$ is a manifold with corners with those corners contained in $L^- \cap L^+$ and $N = L^- \cup L^+$. We call such a triple $(N, L^-, L^+)$ as a regular index triple for $I$ in $M$. The Conley indices of $I$ related by the forward and reverse flow are represented by $h^+(I)$ and $h^-(I)$. If $M$ is orientable in a neighborhood of $I$, the indices for the forward and reverse flows are related by Poincaré-Lefschetz duality isomorphism $H_\ast(N, L^+) \cong H_{m-\ast}(N, L^-)$ where $m = \dim M$. (See [8, 18, 18].) If we consider the homology with coefficients in $\mathbb{Z}_2$, the Poincaré-Lefschetz duality is valid without the assumption of orientability.

**Definition.** $A \subset M$ is called an attractor set if it is the $\omega$-limit set of a compact neighborhood of itself. A repeller set is an attractor set for the reverse flow.

**Proposition 2.4.** $I$ is an attractor set for $\varphi^t$ if and only if there is an index pair $(N, L)$ for $I$ which $L = \emptyset$.

**Proof.** Let $I$ be an attractor and $V$ be a neighborhood of $I$ such that $\omega(V) = I$. Then there is a $T > 0$ such that $\varphi^{[T, \infty)}(V) \subset \text{int}(V)$. If we set $N := \bigcap_{0 \leq s \leq T} \varphi^s(V)$, then $(N, \emptyset)$ is an index pair for $I$. Now assume that $(N, \emptyset)$ is an index pair for $I$. According to the property (3) of the definition of index pair, we imply that $N$ is positively invariant, hence $\omega(N) \subset N$. Since $N$ is an isolating neighborhood for $I$, it follows that $\omega(N) \subset I$. Since $I$ is an invariant set, we conclude that $\omega(N) = I$. □

**Theorem 2.5.** Suppose that $I \subset M$ is a connected isolated invariant set.

(i) If $I$ is not an attractor, then $H_0(h^+(I)) = 0$.  

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(ii) If \( I \) is not a repeller, then \( H_m(h^+(I); \mathbb{Z}_2) = 0 \). Moreover if \( M \) is orientable in a neighborhood of \( I \), then \( H_m(h^+(I)) = 0 \). (\( m = \dim M \))

**Proof.** Consider a regular index triple \((N, L^+, L^-)\) for \( I \). We may assume that \( N \) is connected (otherwise replace \( N \) by the connected component of \( N \) that contains \( I \)). Since \( I \) is not an attractor set, \( L^+ \neq \emptyset \) by Proposition 2.4. Thus

\[
H_0(h^+(I)) = H_0(N, L^+) = 0
\]

Similarly we have \( L^- \neq \emptyset \) and \( H^0(h^-(I)) = 0 \). Now by Poincaré-Lefschetz duality \( H_m(h^+(I)) \cong H^0(h^-(I)) = 0 \). □

**Definition.** A **Morse decomposition** of \( I \) is a finite collection \( \{M_i\}_{i=1}^n \) of disjoint, nonempty isolated invariant subsets of \( I \) such that for each \( x \in I - (\bigcup_{i=1}^n M_i) \), there are \( 1 \leq i < j \leq n \) such that \( \alpha(x) \in M_j \) and \( \omega(x) \in M_i \).

**Example 2.6.** Consider the gradient flow of a smooth function \( f \) on a compact manifold \( M \). Suppose that \( \{x_1, \ldots, x_n\} \) are critical points of \( f \) with \( f(x_i) \leq f(x_j) \) for \( i < j \). Then \( \{x_1, \ldots, x_n\} \) is a Morse decomposition for \( M \).

**Theorem 2.7.** Let \( I \) be an isolated invariant set with a Morse decomposition \( \{M_i\}_{i=1}^n \). Define the Poincaré polynomial \( p(t, X) := \sum_{k=0}^\infty b_k(X)t^k \) that \( b_k(X) = \dim(H_k(X)) \) is \( k \)-th Betti number of \( X \). Then there is a polynomial \( Q \) with nonnegative coefficients such that

\[
\sum_{i=1}^n p(t, h(M_i)) = p(t, h(I)) + (1 + t)Q(t)
\]

The above theorem is known as the generalized Morse inequalities [3, 17]. If we apply this result to the gradient flow of a Morse function with the Morse decomposition described in Example 2.6., we obtain the classical Morse inequalities [4].
3 Continuation

A parametrized flow on $M$ is a collection of flows $\{\varphi^\lambda_t | \lambda \in I\}$ indexed by $I = [0,1]$ such that $\Phi_t(x, \lambda) = (\varphi^\lambda_t(x), \lambda)$ is a $C^1$-flow on $M \times I$. We say $S^0$, an invariant set for $\varphi^0_t$, and $S^1$, an invariant set for $\varphi^1_t$, are related by continuation if there is an isolated invariant set $\Sigma \subset M \times I$ for $\Phi_t$ such that $S^0 = \Sigma \cap M \times \{0\}$ and $S^1 = \Sigma \cap M \times \{1\}$.

The reason that continuations are interesting in Conley index theory is the following theorem \cite{1,17}.

**Theorem 3.1.** If $S^0$ and $S^1$ are isolated invariant sets related by continuation, then $h(S^0) = h(S^1)$.

If we have an isolated invariant set $S$ and make a small perturbation of the flow, then the new flow will have an isolated invariant set $S'$ near $S$, and by the above theorem $h(S) = h(S')$. Therefore Conley index is invariant under perturbation. In \cite{12,13} Reineck has shown that every isolated invariant set can be continued to an isolated invariant set in a Morse-Smale gradient flow. Since Morse-Smale flows are easy to deal with, we first prove our result for a Morse-Smale flow and then extend it to the general case by using the invariance of Conley index under continuation.

**Theorem 3.2.** Let $X$ be a smooth vector field on a Riemannian manifold $M$ and let $I$ be an isolated invariant set in $\varphi_t$, the flow generated by $X$, with isolating neighborhood $N$. Then $I$ can be continued to an isolated invariant set in a Morse-Smale gradient flow without changing the vector field on $M - N$.

**Definition.** We define the Poincaré index of an isolated invariant set $I$ to be the Euler characteristic of the Conley index of $I$, i.e. $\text{ind}_p(I) := \chi(h(I))$.

Suppose that the flow $\varphi^t$ is associated with a vector field $X$ on $M$. If $\{x\}$ is a critical point of $X$ and an isolated invariant set for $\varphi^t$, then $\text{ind}_p(x)$ coincides with the classical definition of Poincaré index of $x$ (up to a sign). This is a special case of the results of \cite{5} in which McCord developed the Poincaré-Hopf theorem and showed that $\text{ind}_p(I) = (-1)^m \sum \text{ind}(x)$, where the sum is taken over all critical points in $I$, $\text{ind}(x)$ is the Poincaré index of $x$ relative to vector field $X$ and $m = \text{dim}M$. We
provide another proof for this result based on Reneik’s continuation to gradient.

**Theorem 3.3.** Let $I$ be an isolated invariant set. Then

$$\text{ind}_p(I) = (-1)^m \sum_{x \in I} \text{ind}(x).$$

In particular, if $\text{ind}_p(I) \neq 0$, then there exists a critical point in $I$.

**Proof.** By Theorem 3.2., $I$ can be continued to an isolated invariant set $J$ in a Morse-Smale gradient flow $-\nabla f$ without changing the vector field on $M - N$ where $N$ is an isolating neighborhood for $I$. Thus $h(I) = h(J)$ by Theorem 3.1. If $\{y_i\}_{i=1}^n$ is the set of critical points of $f$ in $N$, then $\sum_{x \in I} \text{ind}(x) = \sum_{i=1}^n \text{ind}(y_i)$. If the Morse index of $y_i$ is $k$, then $\text{ind}(y_i) = (-1)^{m+k}$ and by Example 2.1, $h(\{y_i\}) = \Sigma_k$. Let $\mu_k$ denote the number of critical points of Morse index $k$. Thus $\sum_{x \in I} \text{ind}(x) = \sum_k (-1)^{m+k} \mu_k$. By Example 2.6., $\{y_i\}_{i=1}^n$ is a Morse decomposition for $J$ with respect to $-\nabla f$. According to the generalized Morse inequalities (Theorem 2.11), we have

$$\sum_{i=1}^n p(-1, h(y_i)) = p(-1, h(J)) = p(-1, h(I)).$$

Since $h(y_i) = \Sigma_k$, we get $p(t, h(y_i)) = t^k$ and $\sum_k (-1)^k \mu_k = p(-1, h(I))$. Therefore

$$(-1)^m \sum_{x \in I} \text{ind}(x) = p(-1, h(I)) = \sum_{k=0}^{\infty} (-1)^k b_k(h(I)) = \chi(h(I)) = \text{ind}_p(I). \quad \Box$$

**Proposition 3.4.** Suppose that $I \subset M$ is an NDR (Neighborhood Deformation Retract) isolated invariant set.

(i) If $I$ is an attractor, then $\text{ind}_p(I) = \chi(I)$.

(ii) If $I$ is a repeller, then $\text{ind}_p(I) = (-1)^m \chi(I)$. ($m = \dim M$)

**Proof.** When $I$ is an attractor, there is an index pair $(N, \emptyset)$ for $I$ by Proposition 2.4. Since $I$ is an NDR, there exists a neighborhood $U \subset N$ such that $I$ is deformation retract of $U$. By the definition of index pair, $N$ is positively invariant and $\omega(N) = I$. So there is a $T > 0$ such that $\varphi^T(N) \subset U$. Therefore $N$ can be deformed to $I$ and $H_i(N, \emptyset) = H_i(I)$ for every $i$. Thus $\chi(h(I)) = \chi(I)$ which proves (i).
Notice that for a finite CW-complex, the Euler characteristic does not depend on the coefficients field. Since $I$ is assume to be an NDR, it has the homotopy type of a finite CW-complex. If we consider the homology with coefficients in $\mathbb{Z}_2$, we obtain
\[
\chi(h^+(I)) = (-1)^m \chi(h^-(I))
\]
by the duality theorem. So if $I$ is an NDR repeller, then
\[
\text{ind}_p(I) = (-1)^m \chi(I).
\]
\[\square\]

4 Applications

In this section, we consider a smooth vector field on a surface $M$ with an isolated critical point $x$. We desire to show that if $\text{ind}(x) > 1$, then $x$ is accumulated by infinitely many homoclinic orbits.

**Lemma 4.1.** Let $I \subset M$ be a connected NDR isolated invariant set such that $\text{ind}_p(I) > 0$. Then $I$ is either an attractor or a repeller and $\text{ind}_p(I) = \chi(I)$.

**Proof.** Suppose that $I$ is neither an attractor nor a repeller. By Theorem 2.5.,
\[
H_2(h(I); \mathbb{Z}_2) \simeq H_0(h(I); \mathbb{Z}_2) = 0.
\]
Now we conclude that
\[
\text{ind}_p(I) = \chi(h(I))
\]
\[
= \text{rank}(H_2(h(I); \mathbb{Z}_2)) - \text{rank}(H_1(h(I); \mathbb{Z}_2)) + \text{rank}(H_0(h(I); \mathbb{Z}_2))
\]
\[
= -\text{rank}(H_1(h(I); \mathbb{Z}_2)) \leq 0.
\]
Since $m = 2$, the proof is complete by Proposition 3.4. \(\square\)

**Theorem 4.2.** Let $x$ be a critical point for a vector field on a surface $M$. If $\text{ind}(x) > 1$, then there exists a homoclinic orbit in any neighborhood of $x$.

**Proof.** We first show that $\{x\}$ cannot be an isolated invariant set. Suppose the contrary, then according to Theorem 3.3. and the above lemma, $\text{ind}(x) = \chi(\{x\}) = 1$ which is a contradiction. Consider a closed neighborhood $V$ of $x$ with no critical points rather than $x$. Let $I(V)$ be the maximal invariant set in $V$. The above argument says that there is a point $y \neq x$ in $I(V)$, hence $\omega(y) \subset I(V)$. Notice that there cannot be any cycle in $V$. To see this, suppose that is $\gamma$ is a cycle in $V$. Then the Poincaré index of $\gamma$ must be one, thus there exists a critical point inside $\gamma$. 

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Since the only critical point in \( V \) is \( x \), we get \( \text{ind}(x) = 1 \) which is a contradiction.

Now according to the Poincaré-Bendixon theorem \([9, 10]\), \( \omega(y) \) and \( \alpha(y) \) are critical points or homoclinic orbits. If neither of \( \omega(y) \) and \( \alpha(y) \) are homoclinic orbits, then \( \omega(y) = \alpha(y) = x \). So there exists a homoclinic orbit in \( V \). □

**Proposition 4.3** Let \( \gamma \) be a homoclinic orbit with no critical point inside of it. Then all the orbits inside \( \gamma \) are homoclinic.

**Proof.** Let \( x := \omega(\gamma) = \alpha(x) \) and \( \Omega \) be the region surrounded by \( \gamma \). Similar to the above argument, there is no cycles in \( \Omega \) and the limit sets of any orbit in \( \Omega \) are either \( \{x\} \) or homoclinic orbits. Since \( x \) is the only critical point in \( \Omega \), it belongs to all limit sets. On the other hand, it is known that if one of the limit sets is not a critical point, then the limit sets are disjoint \([3, 10]\). Therefore the limit sets of any orbit in \( \Omega \) must be \( \{x\} \). □

**Remark 4.4.** It is well-known that if \( x \) is a critical point of a gradient vector field, then \( \text{ind}(x) \leq 1 \). (See \([11]\) for another proof.) The above theorem clearly shows why this result is true.

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