On nonextensive thermo-statistics: systematization, clarification of scope and interpretation, and illustrations

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Abstract

When dealing with certain kind of complex phenomena the theoretician may face some difficulties – typically a failure to have access to information for properly characterize the system – for applying the full power of the standard approach to the well established, physically and logically sound, Boltzmann-Gibbs statistics. To circumvent such difficulties, in order to make predictions on properties of the system and looking for an understanding of the physics involved (for example in analyzing the technological characteristics of fractal-structured devices) can be introduced large families of auxiliary statistics. We present here a systematization of these different styles in what can be termed as Unconventional Statistical Mechanics, accompanied of an analysis of the construction and a clarification of its scope and interpretation. As illustrations are derived heterotypical Bose-Einstein, Fermi-Dirac and Maxwell-Boltzmann distributions, and several applications including studies of experimental works are briefly described.

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I. INTRODUCTION

Over twenty years ago Montroll and Shlesinger wrote that in the world of investigation of complex phenomena that requires statistical modeling and interpretation, several competing styles have been emerging, each with its own champions [1]. Lately a large amount of efforts – with a flood of papers related to one particular case – has been given to the topic. What is at play consists in that in the study of certain physicochemical systems we may face difficulties when handling situations involving fractal-like structures, correlations (spatial and temporal) with some type of scaling, turbulent and chaotic motion, small size (nanometer scale) system with eventually a low number of degrees of freedom and complicate boundary conditions, generalized hydrodynamics, and so on. The interest on them has been recently enhanced as a consequence that such situations are present in electronic and opto-electronic devices of the nowadays point-first technologies and also in technological areas involving the use of disordered systems, polymeric solutions, conducting glasses, the case of microbatteries, and others. The question is that these difficulties consist, as a rule, in that the theoretician cannot properly satisfy Fisher’s criteria of efficiency and sufficiency [2] in the conventional, well established, physically and logically sound Boltzmann-Gibbs statistics, meaning an impairment to correctly include the presence of large fluctuations (and eventually higher-order variances) and to account for the relevant and proper characteristics of the system, respectively. Then, just out of necessity and convenience and to be able to make predictions – and in the way providing an understanding, even partial, of the physics of the system of interest, for example in analyzing technological characteristic of physico-chemical systems as illustrated in Sect. III – one may resort to the use of statistics other than the usual form of the Boltzmann-Gibbs (BG) one, which are not at all extensions of the latter but, as said, introduce an auxiliary detour for analyzing the problem in hands.

Among existing approaches we can mention what can be called Generalized Statistical Mechanics as used by P. T. Landsberg showing that functional properties of the (informational) entropies (see next section) give origin to different types of thermostatistics, and rise the question of how to select a “proper” one, that is, are some better than others? [3]; for decades has been in use Levy Statistics, introducing a modified non-Gaussian distribution which has been applied to a variety of problems (see for example Ref. [4]); the approach of W. Ebeling who has addressed the question of the treatment of a class of systems, in nature
and society, which are determined by their total history; it can be referred-to as *Ebeling Statistics* [3]; *Superstatistics* developed by C. Beck and E. G. D. Cohen for nonequilibrium systems with complex dynamics in stationary states with large fluctuations on long-time scales [5]; *Non-Extensive Statistics* based on Havrda-Charvat statistics [7] applied to a number of cases are described in the Conference Proceedings of Ref. [8]: It consists in the so-called Tsallis statistics [9]; *Renyi statistics* – devised by the renowned theoretical statistician A. Renyi [10] – has been introduced in the scientific literature, as noticed in Ref. [11], with, for example, P. Grassberger and I. Procaccia [12] using it as a valuable method for characterizing experimental chaotic signals, and recently P. Jizba and T. Arimitsu [13] have presented an extensive analysis of it in a paper called “The world according to Renyi”; *Sharma-Mittal Statistics* [14] or better to say a variation of it (called *Kappa or Deformational statistics*) was used by V. M. Vasyliunas in problems of astrophysical plasma [15] and by G. Kaniadakis in the case of relativistic systems [16]; and so on.

It has been stated that some of these statistics are *nonextensive* as compared with BG statistics (e.g. Ref. [8]) but this is not quite correct once in the strict equilibrium of adiabatically closed systems, described by the microcanonical ensemble, the BG entropy is nonextensive only becoming practically extensive in the *thermodynamic limit* (e.g. Ref. [17]). The latter is one case of idealization in science [18], useful for particular situations however not applicable in general: notice for example the case of small systems [19] present, e.g., in nano-science and technology which are of a quite large interest and in ample development nowadays.

We recall that in Statistical Mechanics the probability distribution (statistical operator), usually derived from heuristic arguments, can also be derived from a variational method, once is made connection with Information Theory [20, 21, 22]. It consists into making extremal – a maximum –, subject to certain constraints, a functional (superoperator) of the probability distribution (statistical operator). Such quantity, first introduced in Shannon’s Theory of Communication [23], can be referred to as *measure of uncertainty of information*. It has also been called statistical measure and entropy, with the understanding that it is *informational entropy*. It is worth to emphasize – in view of some confusion that has recently pervaded the scientific literature – that the different possible informational entropies are not to be interpreted as the thermodynamic entropy of the physical system. R. T. Cox has noticed that the meaning of such entropies is not the same in all respects as that of anything which
has a familiar name in common use, and it is therefore impossible to give a simple verbal description of it, which is, at the same time, an accurate definition [24]. E. T. Jaynes has also commented that it is an unfortunate terminology, and a major occupational disease in that there exists a persistent failure to distinguished between the informational entropy, which is a property of any probability distribution, and the experimental entropy of thermodynamics, which is instead a property of the thermodynamic state: Many research papers are flawed fatally by the authors’ failure to distinguish between these entirely different things, and in consequence providing nonsense results [25]. Along such erroneous line, recently it has been considered the utterly wrong proposition that a particular informational entropy – among the infinitely-many that can be defined – comes to supersede the Boltzmann-Gibbs one as the entropy of physical systems [8, 26]: Such “entropy” has a form of the so-called structural Havrda-Charvat one [7], which is, we restate, a generating functional for deriving so-called heterotypical probability distributions [27] to contour, as noticed, the difficulties of application of Boltzmann-Gibbs statistics when we face limitation (ours, not of BG statistics) in satisfying Fisher’s criteria in statistics [28].

As noticed in the Abstract we present a systematization of these alternative statistics, dubbed Unconventional Statistical Mechanics (USM), within the variational formalism MaxEnt-NESEF (for Maximization of Informational Entropy in the Non-Equilibrium Statistical Ensemble Formalism [22, 29]) with the use of non-conventional statistical measures (informational entropies) to derive heterotypical nonequilibrium probability distributions. This is discussed in the next Section, and after that we illustrate the matter with the derivation of heterotypical Fermi-Dirac, Bose-Einstein, and Maxwell-Boltzmann single-particle distribution functions, and with a brief description of the application of USM in several situations including analysis of experimental data (see also Refs. [30, 31, 32]).

II. UNCONVENTIONAL STATISTICAL MECHANICS

In Statistical Mechanics the variational approach MaxEnt-NESEF – mentioned in the Introduction – [21, 22, 29, 33] provides a powerful, practical and soundly based procedure, of a quite broad scope, for building a nonequilibrium ensemble formalism. It is encompassed in what is sometimes referred-to as Informational-Entropy Optimization Principles (see for example Ref. [27]), or, to be more precise, we would say constrained optimization, that
is, restricted by constraints consisting in the available information. Such optimization is performed through calculus of variation with Lagrange’s method for finding the constrained extremum being the preferred one.

In the conventional approach, let it be in the cases of equilibrium [34, 35], or nonequilibrium (in the linear or Onsagerian regime [36], or for arbitrarily far-from-equilibrium systems [22, 29, 33]), one proceeds to maximize Shannon-Jaynes measure of information – or informational entropy which in equilibrium is Gibbs entropy giving rise to Boltzmann entropy in the microcanonical ensemble in equilibrium –, namely

$$S_{SJ}(t) = -Tr \{ \rho(t) \ln \rho(t) \} ,$$  \hspace{1cm} (1)

under the given constraints. The calculus leads to statistical operators of an exponential form, with the exponent depending on the set of basic dynamical variables present in the constraints ($\{\hat{P}_j(r)\}$, see below), and the set of Lagrange multipliers ($\{F_j(r, t)\}$, see below), the latter being the nonequilibrium (or equilibrium according to the case) thermodynamic variables. In fact we must proceed to the maximization of the SJ-informational entropy with the constraints

$$Tr \{ \rho(t) \} = 1 ,$$  \hspace{1cm} (2)

$$Q_j(r, t) = Tr \{ \hat{P}_j(r) \rho(t) \} ,$$  \hspace{1cm} (3)

with $j = 1, 2, \ldots$. Equation (2) is the normalization condition and Eq. (3) consists in the set of average values over the nonequilibrium ensemble (characterized by the sought after statistical operator $\rho(t)$) of the set of microdynamical variables $\{\hat{P}_j(r)\}$ chosen for providing the constraints in the variational method. The resulting statistical operator is given by

$$\rho_\epsilon(t) = \exp \left\{ -\hat{S}(t, 0) + \int_{-\infty}^{t} dt' e^{(t'-t)} \frac{d}{dt'} \hat{S}(t', t'-t) \right\} ,$$  \hspace{1cm} (4)

where

$$\hat{S}(t', t'-t) = \exp \left\{ -\frac{1}{i\hbar} (t' - t) \hat{H} \right\} \hat{S}(t', 0) \exp \left\{ \frac{1}{i\hbar} (t' - t) \hat{H} \right\} ,$$  \hspace{1cm} (5)
and

\[ \hat{S}(t, 0) = \phi(t) + \sum_j \int d^3 r F_j(r, t) \hat{P}_j(r) \equiv -\ln \bar{\rho}(t, 0) \quad (6) \]

is the so-called informational-statistical-entropy operator \[37\], and \(\bar{\rho}(t, 0)\) is an auxiliary (sometime called “instantaneously frozen quasiequilibrium”) statistical operator having an exponential form resembling a canonical-like distribution \[29\]. In these expressions, \(\hat{H}\) is the system Hamiltonian, \(\{F_j(r, t)\}\) the set of Lagrange multipliers associated to the set of basic microdynamical variables \(\{\hat{P}_j(r)\}\), and \(\phi(t)\) ensures the normalization condition of Eq. \(2\) and can be considered as being the logarithm of a nonequilibrium partition function, say, \(\phi(t) \equiv \ln \bar{Z}(t)\). Moreover, the term containing the positive infinitesimal \(\epsilon\) – which goes to zero after the calculation of averages has been performed – results from introducing Abel’s kernel (in the convergence of integral transforms) in the integral on time \[22, 29, 33\]. We recall that it introduces in the theory the concept of Bogoliubov’s quasiaverages \[38\] leading to irreversible evolution from an initial condition, what it does by selecting the retarded solutions of the Liouville equation that \(\rho(t)\) satisfies: the advanced solutions are discarded in a quite similar way as done by Gell-Mann and Goldberger in the case of Schrödinger equation in scattering theory \[39\]. The statistical operator of Eq. \(4\) satisfies, as it should, the question of historicity – in Kirkwood’s and Mori’s sense \[40, 41\] – and irreversibility is introduced here in the so-called interventionistic picture in logic, resorting to Krylov’s “jolting” approach \[42, 43\], using a Poissonian distribution (Abel’s kernel) to account for the nonisolation of any physical system in the real world \[22, 29, 33\].

In that way it is built a general ensemble formalism in Boltzmann-Gibbs Statistical Mechanics for any kind of thermodynamic state of the system, and fundamental for the study of the physics of condensed matter: it allows to describe successfully any possible physical situation in many-body systems. Of course, such success is possible if we are able to properly handle it, what means, as already mentioned, that in any problem we need to satisfy Fisher’s criteria \[2\] for it to give satisfactory predictions. As noticed, we may face difficulties for some kind of situations where the use of BG statistics is simply impaired because of either becomes not possible to handle the required information relevant to the problem in hands, or, more important and fundamental, is involved a failure on the part of the theoretician to have a correct access to such information. Typical cases are that of small systems
(few particles) when we may not be satisfying Fisher’s criterion of efficiency (fluctuations and higher-order variances cannot be ignored), and the case of complex systems with some type or other of fractal-like structures or with long-range space correlations or particular long-time correlations, when we have deficiencies in the proper knowledge of the characterization of the states of the system (failure to satisfy Fisher’s criterion of sufficiency). We reemphasize that what is present is a practical difficulty (a limitation on the part of the theoretician) in the otherwise complete, physically and logically sound BG-statistics, and this is illustrated in Section III.

When facing such difficulties, out of necessity (obstacles in accounting for the relevant information) or, mainly, the force of circumstances (lack of the proper characterization of the system) a way to circumvent them has consisted into resorting to auxiliary (or unconventional) statistics (see the first sentence in the Introduction).

This auxiliary Unconventional Statistical Mechanics consists of two steps:

1. The choice of a heterotypical probability distribution, obtained through application of MaxEnt to different measures of information, that is, other than the Shannon-Jaynes one that leads to the conventional BG-statistics;

2. The use of the escort probability in terms of the chosen heterotypical probability of item 1 above, in the calculation of averages.

Let us consider both concepts. The escort probability of order $\gamma$ of a given probability distribution $\rho$ is defined as

$$D_\gamma\{\rho\} = \frac{\rho^\gamma}{\text{Tr}\{\rho^\gamma}\},$$

where $\gamma$ is a positive real number, which seems to have been introduced by A. Renyi but in terms of Renyi’s heterotypical probability distribution (see Ch. 10 in the book of Ref. [10]), being generalized, and the name given, by C. Beck and F. Schlögl (see Ch. 5 in the book of Ref. [44]). Its role is to add to the normal definition of average value the presence of second and higher-order variances (in that way trying to improve upon the possible failure of distribution $\rho$ to satisfy Fisher’s criterion of efficiency): For the average value of an observable $\hat{A}$, if we write $\gamma = 1 + \epsilon$ and make an expansion in powers of $\epsilon$ we find that
\[ \langle \hat{A} \rangle_\gamma = Tr \{ \hat{A} \mathcal{D}_\gamma \{ \rho \} \} \]
\[ = Tr \{ \hat{A} \rho \} + \epsilon \left[ \langle \hat{A} \hat{S} \rangle - \langle \hat{A} \rangle \langle \hat{S} \rangle \right] \]
\[ + \frac{\epsilon^2}{2} \left[ \langle \hat{A} \hat{S} \hat{S} \rangle - \langle \hat{A} \rangle \langle \hat{S} \hat{S} \rangle + 2 \langle \hat{A} \rangle \langle \hat{S} \rangle^2 - 2 \langle \hat{A} \hat{S} \rangle \langle \hat{S} \rangle \right] + O(\epsilon^3), \] (8)

where
\[ \langle ... \rangle = Tr \{ ... \rho \}, \] (9)

stands for the normal average. For illustration let us take for \( \hat{A} \) the Hamiltonian \( \hat{H} \) and for \( \rho \) a canonical distribution \( \rho = Z^{-1} \exp \{ -\beta \hat{H} \} \), and then using Eq. (8) up to second order in \( \epsilon = \gamma - 1 \) it follows that the energy is given by
\[ E = \langle \hat{H} \rangle_\gamma = \langle \hat{H} \rangle + \epsilon \beta \Delta_2 E + \frac{\epsilon^2}{2} \beta^2 \Delta_3 E, \] (10)

where
\[ \Delta_2 E = \left\langle \left( \hat{H} - \langle \hat{H} \rangle \right)^2 \right\rangle = \langle \hat{H}^2 \rangle - \langle \hat{H} \rangle^2, \] (11)
\[ \Delta_3 E = \left\langle \left( \hat{H} - \langle \hat{H} \rangle \right)^3 \right\rangle = \langle \hat{H}^3 \rangle - 3 \langle \hat{H} \rangle \langle \hat{H}^2 \rangle + 2 \langle \hat{H} \rangle^3, \] (12)

with \( \Delta_2 E \) and \( \Delta_3 E \) being the second and the third order variances of the energy. The departure of \( \gamma \) from the value 1 gives an indication of the influence of the variances in the process of prediction of average values.

On the other hand, for obtaining the heterotypical distributions one needs to introduce alternative measures of information (i.e. informational entropies). A large family is the one provided by I. Csiszár ([45] and see also the book in Ref. [27]) and, among the infinitely-many possibilities, we list in Table I five of them (\( W^{-1} \) is the constant probability in the uniform distribution). Kullback-Leibler measure [46], which is parameter independent, corresponds to Shannon-Jaynes informational entropy and produces the usual results in BG-statistics. The others are dependent on what has been dubbed as infoentropic index(es): The one of the Havrda-Charvat [7] has been used in Physics and other disciplines with the name of Tsallis entropy [9] (calling \( q \) the infoentropic index); Sharma-Mittal informational entropy [14] depends on two infoentropic indexes (taking index \( \beta = 1 \) it goes over the one of
Havrda-Charvat, and for a particular relation between both indexes results Kappa-statistics of Refs. [15, 16], which is a kind of weighted superposition of two Havrda-Charvat measures. Renyi measure [10] has been used in a number of problems in several disciplines, including Physics [11, 12, 13] (it can be noticed that Renyi and Havrda-Charvat measures produce, via MaxEnt, the same statistical operator, differing only in the interpretation of the Lagrange multipliers). Kapur measure [47], depending on two infoentropic indexes is a superposition of two Renyi measures.

Two important points need be stressed: On the one hand, Shannon-Küllback-Leibler measure is derived from a set of fundamental axioms, and Havrda-Charvat and Renyi measures follow from the modification of one of such axioms [7, 10, 46]. On the other hand, in the process is introduced an open parameter (the infoentropic index α, recently indicated by q), to be determined by fitting of the index-dependent prediction with observation.

The role of these heterotypical probability distributions is to introduce a selective weighting of, for example, occupation functions of quantum states, amplitudes of movement in hydrodynamics, etc. This is illustrated in next Section.

| Table I: Informational-Statistical Entropies |
|---------------------------------------------|
| Conventional ISE                            |
| Boltzmann-Gibbs-Shannon-Jaynes ISE          |
| (from Küllback-Leibler measure)              |
| -Tr{ρln ρ}                                  |
| Unconventional (entropic-indexes-dependent) ISEs |
| From Havrda-Charvat measure                  |
| -1/α−1Tr{ρα − ρ}                            |
| α > 0 and α ≠ 1                              |
| From Sharma-Mittal measure                  |
| −Wα−1β−1Tr{[Wα−βρα−β+1 − ρ]ρβ−1}            |
| α > 1, β ≤ 1 or α < 1, β ≥ 1                |
| From Renyi measure                           |
| −1/α−1ln Tr{ρα}                             |
| α > 0 and α ≠ 1                              |
| From Kapur Measure                           |
| −1/α−β[ln Tr{ρα} − ln Tr{ρβ}]               |
| α > 0, β > 0 and α ≠ β                      |
III. ILLUSTRATIVE EXAMPLES

In this Section we illustrate the use of the Unconventional Statistical Mechanics described in the previous ones. We present comments on a few cases and is given the reference where details can be consulted. One example is the case of “anomalous” luminescence in semiconductor heterostructures, which requires the introduction of heterotypical Fermi-Dirac distributions. We begin with this case.

A. Photoluminescence in Semiconductor Heterostructures

There exists nowadays a large interest on the question of optical properties of quantum wells in semiconductor heterostructures, which have been extensively investigated in the last decades as they have large relevance for the high performance of electronic and optoelectronic devices (see for example Ref. [48]). To deal with these kind of systems, because of the constrained geometry that they present (where phenomena develop in nanometer scales) the researcher has to face difficulties with the theoretical analysis: A most relevant question to be dealt with is the one related to the interface roughness, usually having a kind of fractal-like structure which leads to energies and wavefunction depending on boundary conditions which need account for spatial correlations. As a consequence the different physical properties of these systems appear as, say, “anomalous” when the results are compared with those that are observed in bulk materials, particularly, the case of photoluminescence which we briefly describe here. In the study of photoluminescence in nanometric quantum wells the conventional treatment via the well established Boltzmann-Gibbs formalism has its application impaired because of the spatial correlations resulting from the non smooth confinement, relevant in the characterization of the system, on which one does not have access to (obviously the interface roughness varies from sample to sample and one does not have any easy possibility to determine the topography of the interface). This is then the reason why the criterion of sufficiency is not satisfied in this case.

Let us consider a system of carriers (electrons and holes) produced, in the quantum well of an heterostructure, by a ultrafast laser pulse. They are out of equilibrium and their nonequilibrium macroscopic state can be described in terms of a nonequilibrium statistical thermodynamics [22], with the nonequilibrium thermodynamic state characterized by the
time evolving quantities energy and density (see for example Ref. [49]). Electrons and holes do recombine producing a luminescence spectrum which, we recall, is theoretically expressed as

$$I(\omega|t) \propto \sum_{n,n',k_{\perp}} f_{n,k_{\perp}}^{e}(t) f_{n',k_{\perp}}^{h}(t) \delta(h\Omega - \epsilon_{n,k_{\perp}}^{e} - \epsilon_{n',k_{\perp}}^{h}),$$  \hspace{1cm} (13)

where $f^{e}$ and $f^{h}$ are the populations of electrons and holes, $h\Omega = h\omega - E_{G}$, with $\omega$ being the frequency of the emitted photon and $E_{G}$ the energy gap, and $\epsilon_{n,k_{\perp}}^{e(h)}$ are the electron (hole) individual energy levels in the quantum well (index $n$ for the discrete levels and $k_{\perp}$ for the free movement in the $x - y$ plane), and the delta function accounts for energy conservation.

The textbook expression for the energy levels corresponding to the use of perfectly smooth bidimensional boundaries is given by

$$\epsilon_{n,k_{\perp}}^{e(h)} = n^2 \frac{\pi^2 \hbar^2}{2m_{e(h)}^* L_{QW}^2} + \frac{\hbar^2 k_{\perp}^2}{2m_{e(h)}^*},$$  \hspace{1cm} (14)

where $L_{QW}$ is the quantum-well width and $m_{e(h)}^*$ is the effective mass and the populations $f$ take a form that resembles instantaneous in time Fermi-Dirac distributions: see Ch. 6 in the book of Ref. [22] and Ref. [49]. Using this expression in the calculation of $I$ of Eq. (13) and comparing it with the experimental results [50, 51] one finds a disagreement, and then it is used the name of “anomalous” luminescence for these experimental results. This is a consequence that we are using an improper description of the carriers’ energy levels – we are not satisfying the criterion of sufficiency (as discussed in previous Sections) –, resulting of ignoring the roughness of the boundaries (with self-affine fractal structure [52]) which needs be taken into account in these nanometric-scale geometries, and then the boundary conditions to be placed on the wavefunctions are space dependent. Hence complicated space correlations are to be introduced, but to which we do not have access (information), as already noticed. Hence, this limitation on the part of the researcher breaks the sufficiency criterion, and application of the Boltzmann-Gibbs-Shannon-Jaynes construction is impaired, and one can try to circumvent the difficulty introducing unconventional statistics based on parameter-dependent structural informational entropies in order to analyze the properties of the system.
1. Distribution Functions of Fermions and Boson in USM

According to Eq. (13) what we need is to express the carrier populations in a heterotypical statistics, and we resort to the Renyi one (see for example Ref. [13] and [30, 31, 32]). Using MaxEnt-NESEF in terms of Renyi statistical entropy it can be derived the corresponding auxiliary (“instantaneously frozen”) statistical operator $\bar{\rho}(t, 0)$ [cf. Eq. (6) – the case of the conventional one –, and we recall that the statistical operator is given in term of this auxiliary one in Eq. (4)]. After some straightforward mathematical manipulations it follows that it can be written in a convenient form for performing calculations, namely, (31)

$$\bar{\rho}_\alpha(t, 0) = \frac{1}{\tilde{\eta}_\alpha(t)} \left[ 1 + (\alpha - 1) \sum_j \tilde{F}_\alpha(t) \hat{P}_j \right]^{-\frac{1}{\alpha-1}}, \quad (15)$$

where

$$\tilde{\eta}_\alpha(t) = Tr \left\{ \left[ 1 + (\alpha - 1) \sum_j \tilde{F}_\alpha(t) \hat{P}_j \right]^{-\frac{1}{\alpha-1}} \right\}, \quad (16)$$

$$\tilde{F}_\alpha(t) = F_\alpha(t) \left[ 1 - (\alpha - 1) \sum_m F_\alpha(t) Q_m(t) \right]^{-1}. \quad (17)$$

Equation (16) stands for a modified form of the quantity that ensures the normalization condition, and Eq. (17) for modified Lagrange multipliers, where $F_\alpha$ are the original Lagrange multipliers, and $Q_m$ the average values of the microdynamical variables $\hat{P}_m$. Moreover, we recall that in Renyi statistics the associated escort probability is given by

$$\mathcal{D}_\alpha\{\rho\} = \rho_\alpha^\alpha / Tr\{\rho_\alpha^\alpha\}, \quad (18)$$

that is, the order of the escort probability is the same as the index in Renyi heterotypical distribution (see Ch. 10 in the book of Ref. [10]). In particular to derive the distribution functions for fermions and for bosons using USM in terms of Renyi statistical approach one chooses as basic dynamical variables, i.e. the $\hat{P}_j$ in Eq. (15), the set of occupation number operators

$$\{\hat{n}_k\} = \{c^*_k c_k\}, \quad (19)$$
where $c (c^\dagger)$ are the usual annihilation (creation) operators in states $|k\rangle$, satisfying the corresponding commutation and anticommutation rules of, respectively, bosons and fermions (the spin index is ignored). Their average values are the infoentropic-index $\alpha$-dependent distribution functions

$$f_k(t) = Tr \left\{ c_k^\dagger c_k D_{\alpha \epsilon} \{ \rho_{\alpha \epsilon} (t) \} \right\} = Tr \left\{ c_k^\dagger c_k D_{\alpha} \{ \bar{\rho}_\alpha (t, 0) \} \right\}, \quad (20)$$

which is a consequence that for the basic variables, and only for the basic variables, the average value coincides with the one calculated with the auxiliary distribution $^{[22, 33]}$. The auxiliary statistical operator is then [cf. Eq. (15)]

$$\bar{\rho}_\alpha (t, 0) = \frac{1}{\bar{n}_\alpha (t)} \left[ 1 - (\alpha - 1) \sum_k \bar{F}_{k\alpha} (t) c_k^\dagger c_k \right]^{-\frac{1}{\alpha - 1}}, \quad (21)$$

with [cf. Eq. (17)]

$$\bar{F}_{k\alpha} (t) = F_{k\alpha} (t) \left[ 1 + (\alpha - 1) \sum_{k'} F_{k'\alpha} (t) f_{k'} (t) \right]^{-1}. \quad (22)$$

The populations of Eq. (20), according to the calculation described in Ref. $^{[30]}$ take the form

$$f_k(t) = \bar{f}_k(t) + C_k(t), \quad (23)$$

where

$$\bar{f}_k = \frac{1}{\left[ 1 + (\alpha - 1) \bar{F}_{k\alpha} (t) \right]^{\frac{\alpha}{\alpha - 1}} \pm 1}, \quad (24)$$

where upper plus sign stands for fermions, and the lower minus sign for bosons, and

$$C_k(t) = \alpha(1 - \alpha)(1 - \bar{f}_k(t)) \sum_k \bar{F}_{k\alpha} (t) F_{k'\alpha} (t) Tr \left\{ c_k^\dagger c_k^\dagger c_{k'} c_{k'} D_{\alpha} \{ \bar{\rho} (t, 0) \} \right\} + \ldots, \quad (25)$$

involving two, three, etc. particle correlations, which in general are minor corrections to the first, and main, contribution, the one given by Eq. (24).
In the limit of $\alpha$ going to 1, which applies when the criteria of efficiency and sufficiency are satisfied, Renyi statistical entropy acquires the form of the Boltzmann-Gibbs-Shannon-Jaynes one, $C$ becomes null, $\tilde{F}_{k\alpha}(t)$ becomes $F_k(t)$, and then

$$f_k(t) = \frac{1}{e^{F_k(t)} \pm 1}. \quad (26)$$

(In equilibrium $F_k(t) \to (\epsilon_k - \mu)/k_B T$ and there follows the traditional Fermi-Dirac and Bose-Einstein distributions).

We can see that the distribution of Eq. (23) is composed of a term $\bar{f}$ corresponding to the individual particle in state $|k\rangle$, plus the contribution $C$ containing correlations (of order two, three, etc.) among the individual particles. This type of calculation but for systems in equilibrium and not using the average value defined in Eq. (8), in terms of the escort probability, was reported in Ref. [53], and then are not satisfactory.

Let us now give some attention to the Lagrange multipliers $F_{k\alpha}(t)$. The most general statistical operator for nonequilibrium systems can be expressed in the form of a generalized nonequilibrium grand-canonical statistical operator for a system of individual quasiparticles, where the basic variables are independent linear combinations of the single-quasiparticle occupation number operators [cf. Eq. (19)], consisting of the energy and particle densities and their fluxes of all order [22, 29]. This description follows from the choice [22]

$$\tilde{F}_{k\alpha}(t) = \tilde{\beta}_\alpha(t)[\epsilon_k - \tilde{\mu}_\alpha(t)] - \tilde{F}_{ha}(t) \cdot \epsilon_k u(k) - \tilde{F}_{na}(t) \cdot u(k)$$

$$- \sum_{r \geq 2} \left[ \tilde{F}_{ha}^{[r]}(t) \otimes \epsilon_k u^{[r]}(k) + \tilde{F}_{na}^{[r]}(t) \otimes u^{[r]}(k) \right], \quad (27)$$

where has been introduced the scalar quantities $\tilde{\beta}(t)$ and $\tilde{\mu}(t)$, $\tilde{F}_{ha}(t)$ and $\tilde{F}_{na}(t)$ are vectors, and $\tilde{F}_{ha}^{[r]}$ and $\tilde{F}_{na}^{[r]}$ $r$-th rank tensors. Moreover,

$$u^{[r]}(k) = [u(k) \ldots (r - \text{times}) \ldots u(k)], \quad (28)$$

is the tensorial product of $r$-times the characteristic velocity $u(k) = \hbar^{-1} \nabla_k \epsilon_k$, where $\epsilon_k$ is the energy dispersion relation of a single-particle, and then $u(k)$ is the group velocity in state $|k\rangle$. Dot stands as usual for scalar product of vectors, and $\otimes$ for fully contracted product of tensors.
To better illustrate the matter, we introduce a simplified description, or better to say a quite truncated description, proceeding to neglect in Eq. (27) all the contributions arising out of the fluxes, i.e. we put $F = 0$ and $F^{[r]} = 0$, retaining only the first term on the right-hand side. Therefore, we do have that

$$\bar{f}_k(t) = \frac{1}{1 + (\alpha - 1)\tilde{\beta}_\alpha(t) [\epsilon_k - \tilde{\mu}_\alpha(t)]} \pm 1,$$

where [cf. Eq. (22)]

$$\tilde{\beta}_\alpha(t) = \beta_\alpha(t)/\{1 - (\alpha - 1)\tilde{\beta}_\alpha(t)[E(t) - \mu(t)N(t)]\},$$

and $\tilde{\beta}\tilde{\mu} = \tilde{\beta}\mu$, then $\tilde{\mu} = \mu$ having the role of a quasichemical potential.

In this Eq. (30) $E(t)$ is the energy

$$E(t) \simeq \sum_k \epsilon_k \tilde{f}_k(t),$$

and $N$ the number of particles

$$N(t) \simeq \sum_k \tilde{f}_k(t),$$

where the correlations present in $C$ in Eq. (23) have been neglected. Moreover, in many cases we can use an approximate expression for the populations, that is, in the one of Eq. (24) we admit that $\pm 1$ can be neglected in comparison with the other term. This is considered as taking a statistical nondegenerate limit, once, if we put $\alpha$ going to 1 (what, we again stress, strictly corresponds to the situation when the principle of sufficiency is satisfied), the population takes the form of a Maxwell-Boltzmann distribution. In this condition the expression for the population can be written as

$$\bar{f}_k(t) = A_\alpha(t)[1 + (\alpha - 1)B_\alpha(t)\epsilon_k]^{-\frac{\alpha}{\alpha - 1}},$$

where

$$A_\alpha(t) = [1 - (\alpha - 1)\tilde{\beta}_\alpha(t)\tilde{\mu}_\alpha(t)]^{-\frac{\alpha}{\alpha - 1}},$$

$$B_\alpha(t) = \tilde{\beta}_\alpha(t)/[1 - (\alpha - 1)\tilde{\beta}_\alpha(t)\tilde{\mu}_\alpha(t)].$$
Consider a parabolic dispersion relation, that is, \( \epsilon_k = \frac{\hbar^2 k^2}{2m^*} \). Using Eq. (33) in Eqs. (31) and (32), we arrive at the result that

\[
n(t) = \frac{N(t)}{V} = A_{\alpha}^{3/2}(t) \frac{\lambda_{\alpha}^{-3}(t)}{4\pi^2} I_{1/2}(\alpha),
\]

\[
e(t) = \frac{E(t)}{V} = n(t) \frac{I_{3/2}(\alpha)}{I_{1/2}(\alpha)} k_B T_{\alpha}(t),
\]

with the integrals \( I_\nu(\alpha) \) related to Beta functions [30], and we have introduced the definition

\[
B_{\alpha}^{-1}(t) = k_B T_{\alpha}(t),
\]

where \( T \) plays the role of a pseudotemperature and where \( \lambda_{\alpha} \) in Eq. (36) is a characteristic length given by \( \lambda_{\alpha}^2(t) = \frac{\hbar^2}{m^* k_B T_{\alpha}(t)} \) (that is, de Broglie wave length for a particle of mass \( m^* \) and energy \( k_B T_{\alpha}(t) \)).

We can see that the above Eqs. (36) and (37) define the Lagrange multipliers \( \tilde{\beta}_{\alpha}(t) \) and \( \tilde{\mu}_{\alpha}(t) \) – present in \( A_{\alpha}(t), \lambda_{\alpha}(t), \) and \( B_{\alpha}(t) \) – in terms of the basic variables energy and number of particles. Moreover, using Eq. (36) we can obtain an expression for the quasi-chemical potential in terms of quasitemperature and density, namely

\[
1 - (\alpha - 1) \tilde{\beta}_{\alpha}(t) \tilde{\mu}_{\alpha}(t) = \left[ 4\pi^2 \lambda_{\alpha}^3(t)/I_{1/2}(\alpha) \right]^{\frac{2(\alpha-1)}{\alpha-3}} \left[ n(t) \right]^{\frac{2(\alpha-1)}{\alpha-3}}.
\]

Also, it can be noticed that for \( \alpha = 1 \) (the case when the condition of sufficiency is satisfied) one recovers the equivalent of the results of conventional nonequilibrium statistical mechanics [31, 36], which are

\[
e(t) = \frac{3}{2} n(t) k_B T^*(t),
\]

where we have introduced the so-called quasitemperature \( T^*(t) \), defined by \( k_B T^* = B_{\alpha=1}^{-1}(t) \), this equation standing for a kind of equipartition of energy at time \( t \), and

\[
\mu(t) = -k_B T^*(t) \ln[T^*(t)/\theta_{tr}(t)],
\]

where \( \theta_{tr}(t) = \frac{\hbar^2 n^{2/3}(t)}{2m^*} \) is the characteristic temperature (here in nonequilibrium conditions and at time \( t \)) for translational motion. This suggests to define a so-called “kinetic
temperature” $\Theta_K(t)$ by equating $e(t)$ to $(3/2)n(t)k_B\Theta_K(t)$, given, after Eq. (37) is used, by

$$\Theta_K(t) = \mathcal{T}_\alpha(t)/(5 - 3\alpha),$$

(42)

where we can see that $\alpha$ must be smaller than $5/3$, as shown in Ref. [31] where connection of theory with experiment is presented, together with other illustrations and discussions.

Let us consider how does the $\alpha$-dependent distribution of Eq. (29) compares with the usual Fermi-Dirac and Bose-Einstein distributions. For illustration we consider the nondegenerate limit of Eq. (33), common to both, where parameter $B$ is related to the kinetic temperature $\Theta_K$ by Eqs. (38) and (42). Taking for $\mathcal{T}_\alpha$ of Eq. (38) the unique value of $300K$, Figs. 1 and 2 show the population of Eq. (33) corresponding to several values of the infoentropic index $\alpha$. It can be noticed that the formalism introduces the characteristic of a different weighting of the values of the standard distribution ($\alpha \simeq 1$), such that: (1) for $\alpha < 1$ the population of the modes at low energies are increased at the expense of those of higher energies ($\epsilon > 7 \times 10^{-3}eV$), while (2) for $\alpha > 1$ we can see the opposite behavior.

2. The "Anomalous" Photoluminescence Spectrum

Returning to the question of “anomalous” luminescence in nanometric quantum wells in semiconductor heterostructures, the carrier’s populations, to be used in Eq. (13), are of the form of Eq. (29), namely

$$\bar{f}_{\epsilon_n^{(h)}}(t) = \left\{ 1 + (\alpha - 1)\beta_{\epsilon_n^{(h)}}(t) \left[ \epsilon_{\epsilon_n^{(h)}} - \mu_{\epsilon_n^{(h)}}(t) \right] \right\}^{\frac{\alpha - 1}{\alpha}} \pm 1,$$

(43)

which depend on $\epsilon_{\epsilon_n^{(h)}}$, that is, the ideal single-carrier energy level of Eq. (14). Using these populations in the nondegenerate limit [cf. Eq. (33)], the luminescence spectrum of Eq. (13) is given by

$$I(\omega|t) \propto \left[ 1 + (\alpha - 1)\frac{m_e e^2 \mathcal{B}^{(c)}_{\alpha}(t) h\Omega}{m_e^2 m_h^2} \right]^{\frac{\alpha}{1 - \alpha}} \left[ 1 + (\alpha - 1)\frac{m_e e^2 \mathcal{B}^{(h)}_{\alpha}(t) h\Omega}{m_e^2 m_h^2} \right]^{\frac{\alpha}{1 - \alpha}}$$

$$= \left[ 1 + (\alpha - 1)\beta_{\epsilon_{eff}}(t) h\Omega + (\alpha - 1)^2 m_e e^2 \mathcal{B}^{(c)}_{\alpha}(t) B^{(h)}_{\alpha}(t) \frac{m_e^2}{m_e^2 m_h^2} (h\Omega)^2 \right]^{\frac{\alpha}{1 - \alpha}},$$

(44)
where $\beta_{\text{eff}} = \frac{m^{*}_e B^e(t)}{M} + \frac{m^{*}_h B^h(t)}{M}$, $m^{-1} = [m^{*}_e]^{-1} + [m^{*}_h]^{-1}$, and $M = m^{*}_e + m^{*}_h$. The second contribution in the last term in Eq. (44) is much smaller than the first, as verified a posteriori, and then the spectrum is approximately described by

$$I(\omega|t) \propto \left[ 1 + (\alpha - 1) \beta_{\text{eff}}(t)(\hbar \omega - E_G) \right]^{\frac{\alpha}{1-\alpha}}.$$  

(45)

The experiments reported in Ref. [50] are time integrated, that is, the spectrum is given by

$$I(\omega) = \frac{1}{\Delta t} \int_{t}^{t+\Delta t} dt' I(\omega|t'),$$  

(46)

where $\Delta t$ is the resolution time of the spectrometer. Using Eq. (45) in Eq. (46), and in the spirit of the mean-value theorem of calculus we write

$$I(\omega) \propto \left[ 1 + (\alpha - 1) \bar{\beta}_{\text{eff}}(\hbar \omega - E_G) \right]^{\frac{\alpha}{1-\alpha}},$$  

(47)

introducing the mean value $\bar{\beta}_{\text{eff}}$ (as an open parameter), which we rewrite as $[\bar{\beta}_{\text{eff}}]^{-1} = k_B \Theta$, defining an average, over the resolution time $\Delta t$, effective temperature of the nonequilibrium carriers, that is, a measure of their average kinetic energy (see Ref. [57]).

In Fig. 3 is shown the fitting of the experimental data with the theoretical curve as obtained from Eq. (47). It contains the results referring to four samples having different values of the quantum well width. The information-entropic index $\alpha$ depends, as expected, on the dimensions of the system: as the width of the quantum well increases the values of $\alpha$ keep increasing and tending to 1 (see Fig. 4). This is a clear consequence that the fractal-like granulation of the boundary surface becomes less and less relevant for influencing the outcome of the phenomenon, as the width of the quantum well falls outside the nanometer scale, and is approached the situation of a normal bulk sample. On the other hand the kinetic temperature of the carriers is smaller with increasing quantum well width, as also expected once the relaxation processes, mainly as a result of the interaction with the phonon system, become more effective and the cooling down of the hot carriers proceeds more rapidly.

Moreover, it can be empirically derive what we term as a law of “path to sufficiency”, namely,

$$\alpha(L) \simeq \frac{L + L_1}{L + L_2},$$  

(48)
where, by best fitting, $L_1 \simeq 139 \pm 17$, $L_2 \simeq 204 \pm 24$, all values given in Ångstrom. We do have here that as $L$ largely increases, the entropic index tends to 1, when one recovers the expressions for the populations in the conventional situation, but as $L$ decreases $\alpha$ tends to a finite value $L_1/L_2$, in this case $\simeq 0.7 \pm 0.06$. This indicates that the insufficiency of description when using Eq. (14) in the calculations (the ideal energy levels) becomes less and less relevant as the size of the system increases as already commented.

This illustration of the theory clearly evidences the already stated fact that the infoentropic index $\alpha$ is not a universal one for a given system, but it depends on the knowledge of the correct dynamics (the region of energy-momentum space that is involved in the experiment being analyzed), the geometry and size including the characteristics and influence of the boundary conditions (e.g. the fractality in the present case), the macroscopic (thermodynamic) state of the system in equilibrium or nonequilibrium conditions, and the experimental protocol.

The case we presented consisted of experiments in time-integrated optical spectroscopy. The phenomenon of “anomalous” luminescence in nanometric quantum wells in semiconductor heterostructures is also present in the case of time-resolved experiments (nanosecond time resolution where the infoentropic index and the (nonequilibrium) kinetic temperature change in time accompanying the irreversible evolution of the system [58]). The use of USM in a completely analogous way as done above (note that Eq. (44) is valid for a time-resolved situation) allows to determine the evolution in time of the kinetic temperature $\Theta_\alpha(t)$, and the infoentropic index $\alpha(t)$, which is then changing in time as it accompanies the evolution of the irreversible processes in the nonequilibrium thermodynamic state of the carriers.

This case involves, as noticed, the difficulty of insufficiency in the characterization of the energy levels of the carriers in the quantum well, i.e. it is of a microscopic mechanical character. As shown before, use of the escort probability introduces ad hoc information trying to account for missing relevant correlations in the problem, and Renyi (heterotypical) statistical operator produces modifications in the carriers’ distributions trying to account for insufficiency of knowledge about the proper quantum mechanical levels.
B. "Anomalous" Diffusion

On the other side, insufficiency of description at a macroscopic level is present in questions involving hydrodynamics. So called "anomalous" situations are also presented here, e.g. "anomalous" diffusion (non-Fickian diffusion). The question now is which is the source of the lack of sufficiency in the well established Boltzmann-Gibbs formalism, which forces us to resort to the unconventional approaches. The answer resides in that is being used a quite incomplete hydrodynamic approach in situations which require a more detailed treatment. In its more general approach, the description of hydrodynamical (including rheological) motion should consist of an extended description, which can be referred-to as Non-Linear Higher-Order Hydrodynamics with fluctuations. Ignoring fluctuations (relevant for example in turbulent motion) one needs to introduce the densities of energy, \( h(\mathbf{r},t) \), and particles, \( n(\mathbf{r},t) \), and their fluxes of all order, namely \( I_p(\mathbf{r},t) \), for the first (vectorial) fluxes, \( I_p^{[r]}(\mathbf{r},t) \) for the higher-order (\( r \)-rank tensor) fluxes, \( r \geq 2 \), and \( p = h \) or \( n \) for energy and particle (material) motion respectively. The motion is then determined by a complicated set of equations of evolutions of the type

\[
\frac{\partial}{\partial t} I_p^{[r]}(\mathbf{r},t) + \nabla \cdot I_p^{[r+1]}(\mathbf{r},t) = J_p^{[r]}(\mathbf{r},t),
\]

where \( r = 0 \) for the density, \( r = 1 \) for the first (vectorial) flux, or current, and \( r \geq 2 \) for the all higher-order fluxes, \( J_p^{[r]} \) are collision integrals, and \( \nabla \cdot \) is the operator indicating to take the divergence of the tensor. Solving this set of coupled equations of evolution is a formidable, almost unmanageable, task. As a rule one uses, depending on each experimental situation, a truncation on the set of equations (i.e. they are considered from \( r = 0 \) up to a certain value, say \( n \), of the order \( r \)). Hence, if one is restricted to introduce a low order truncation it is faced a failure of the criterion of sufficiency when using the conventional, and universal, approach. Consequently, a way to circumvent the difficulty is, as shown before, to make calculations in term of unconventional statistical mechanics, e.g. resorting to Renyi’s approach.

The necessity to go over higher orders comes, for example, because of the presence of a fractal-like structure in the system. We have dealt in detail the question of polymeric solutions, where the question is presented and discussed in detail being shown how there follows in USM "anomalous" Maxwell-Cattaneo and diffusion equations. In this case,
macromolecules under flow, one faces the difficulty in the description resulting from the self-similarity in a type of average fractal structure that this shows (what we have called “Jackson Pollock Effect”, in view of the analogy with his paintings with the dripping method, showing to fractal structure [65]). Without entering into details, given elsewhere [60], the “anomalous” diffusion equation, derived in Renyi statistics, has the form

$$\frac{\partial}{\partial t} n(r, t) - D_\alpha(r, t) \nabla^2 n^{\gamma_\alpha}(r, t) = -\tau_\alpha(r, t) \nabla \cdot \nabla \cdot (n(r, t) [v(r, t) v(r, t)]) + ..., \quad (50)$$

where $D_\alpha(r, t)$ is a transport coefficient whose dependence on position and time is a consequence of its dependence on the nonequilibrium thermodynamic state of the system, and so is the case of the momentum relaxation time $\tau_\alpha(r, t)$; moreover, the first term on the right – depending on the velocity field – is the double divergence of the so-called convective pressure tensor ($[vv]$ stands for tensorial product of twice the velocity vector rendering a rank-2 tensor), and we have omitted to write down additional terms involving gradients of the transport coefficient and the momentum relaxation time. Finally, the power $\gamma_\alpha$ is

$$\gamma_\alpha = (5 - 3\alpha)/(3 - \alpha), \quad (51)$$

with $\alpha$ being, we recall, the infoentropic index in Renyi statistics, which is limited to the interval $1 \leq \alpha < 5/3$.

“Anomalous” diffusion is also called for the explanations of the results of measurements in the case of experiments of cyclic voltammetry in microbatteries with fractal-like structured electrodes [66]: As a consequence of the nowadays large interest associated to the development of microbatteries, the study of growth, annealing, and surface morphology of thin-films depositions used in cathodes, has acquired particular relevance [67]. These kind of systems are characterized by microroughnessed surfaces in a geometrically constrained region (nanometer scale), and then fractal characteristics can be expected to greatly influence the physical properties [52]. In cyclic voltammetry the resulting current between electrodes is determined by the charges arriving at the fractal electrode in a process of hydrodynamic motion. Characterizing such motion as a Fickian diffusion fails to explain the experimental data, what can be done using a postulated law of “anomalous” diffusion [60]. The use of the latter is, as noticed before, a fitting procedure in a situation that requires
a higher-order thermo-hydrodynamics (motion of the fluid of charges in the nano- and sub-nanometric spaces around the surface of the fractal-like structured electrode). The use of a zeroth-order thermo-hydrodynamics – the one leading to Fickian diffusion equations in a BG thermo-statistics – fails to satisfy Fisher’s criterion of sufficiency, and if one persists in using a zeroth-order hydrodynamics, it needs be handled in an unconventional approach. Such study has been performed resorting to Renyi statistics with details given in Refs. [31, 68], leading to Eq. (50). Here we only noticed that, keeping fixed all physical characteristics in the experiment, if the morphology of the fractal electrode surface is modified, it can be established a relation between Renyi infoentropic index and the average fractal dimension \( d_f \) in the form

\[
\alpha = \frac{(4d_f - 7)}{(2d_f - 3)}.
\]

We recall that values of \( \alpha \) are permitted in the interval \( 1 \leq \alpha < \frac{5}{3} \) (otherwise the theory gives rise to singularities), now a clear physical restriction once then \( 2 \leq d_f < 3 \) as it should.

C. Ideal Gas in Finite Box

Let us now consider a simple but quite clarifying case, namely an ideal gas in a finite box (details in Ref. [31]). According to the exact result in R. K. Pathria textbook, [17] and see also [69], in the case of a finite but large box there follows for the energy per particle in the semiclassical limit the expression

\[
\frac{E}{N} \simeq \frac{3}{2} k_B T \left[ 1 + \frac{1}{12} \frac{A \lambda_T}{V} \right],
\]

where \( V \) is the volume and \( A \) the area of the box, and \( \lambda_T = h/\sqrt{mk_B T} \) is the mean thermal de Broglie wavelength of the particles in the grand-canonical ensemble at temperature \( T \). Evidently, in the thermodynamic limit it is recovered the result of energy equipartition. On the other hand, calculating in the thermodynamic limit but introducing the equivalent of the grand-canonical distribution in Renyi approach, in order to account for the insufficiency of description, we obtain that

\[
\frac{E}{N} \simeq \frac{3}{2} F_{\alpha}^{-1} \left[ 1 + \frac{1}{4} (\alpha - 1) \right],
\]
where $F_{\text{ha}}$ is the Lagrange multiplier associated to the energy and then, equating Eqs. (53) and (54) one finds that

$$\alpha \simeq 1 + 4 \left[ k_B T F_{\text{ha}} \left( 1 + \frac{\lambda T}{12 \frac{A}{V}} \right) - 1 \right].$$

(55)

In the thermodynamic limit $A/V$ goes to zero and $F_{\text{ha}}^{-1}$ goes to $k_B T$ and then $\alpha$ goes to 1 as it should. Moreover, from Eq. (55) we can clearly see that the infoentropic index $\alpha$ depends on the system dynamics, its geometry and size, and the thermodynamic state.

D. Black Body Radiation in Insufficient Description

Finally, we briefly mention the case of black-body radiation in insufficient description. We consider the gas of photons in the presence of an uniform flux of energy (generated, for example, as a result of the presence of different temperatures at both ends of the container). We look for the calculation of the energy what is done, on the one side, using the conventional approach in a description that includes the energy and the energy flux as basic variables (that is, in a first-order thermo-hydrodynamics; see discussion on higher-order thermo-hydrodynamics earlier in this Section) and, on the other side, using an incomplete description including only the energy (zeroth-order thermo-hydrodynamics) but dealt with using Renyi statistics trying to compensate for the insufficiency of description. Details are given in Ref. [31], and here we only notice that equating the values of the energy in both descriptions, it follows that the infoentropic index is given by

$$\alpha = 1 - \left( \frac{I}{I_0} \right)^2,$$

(56)

where $I$ is the energy flux and $I_0^2 = (8/15V^2)a^2c^2T^8$ with $a$ being Stefan-Boltzmann constant. It can be noticed that $I_0$ is a kind of a flux of energy composed of the energy density of the radiation, $aT^4/V$, traversing at the speed of light, while $I$ is roughly the density of energy traversing at a speed determined by the gradient of temperature, and then $I/I_0$ is very small, and then the infoentropic index is practically 1 meaning that the insufficiency in the characterization can be ignored.
IV. CONCLUDING REMARKS

However the enormous success and large application of Shannon-Jaynes approach to
derive the probability distributions of the ensemble formalism in the Laplace-Maxwell-
Boltzmann-Gibbs statistical foundations of physics, as it has been noticed, some cases are
difficult to be properly handled within such formulation, as a result of existing some kind
of fuzziness in data or information, that is, the presence of conditions of insufficiency in the
characterization of the (microscopic, macroscopic or mesoscopic) state of the system. Such
difficulty with the proper characterization of the system in the problem in hands can be
somehow compensated, as shown, with the introduction of peculiar parameter-dependent
alternative informational-entropies (see Table I), allowing for the construction of a vast
number of auxiliary heterotypical statistics. In that way one recovers the ability to make
improved predictions on the properties of the system, and be capable to get a picture of the
physical processes involved allowing, as for example illustrated in Section III, to evaluate
the influence of peculiar characteristics of the sample (like, e.g., its fractal structure) on its
physico-chemical behavior.

Returning to the question of the possible failure to satisfy Fisher’s criteria of efficiency
and sufficiency, in the case of particularly dealing with systems with some kind of fractal-like
structure the use of Shannon-Jaynes infoentropy would require to introduce as information
the highly correlated conditions that are in that case present. Two examples in condensed
matter physics have been described in the previous Section in which fractality enters via
the non-smooth topography of the boundary surfaces which have large influence on phe-
nomena occurring in constrained geometries (nanometer scales in the active region of the
sample). Hence the most general and complete Boltzmann-Gibbs formalism in Shannon-
Jaynes approach becomes hampered out and is difficult to handle, and then, as shown, use
of other types of informational-entropies leads to the derivation of heterotypical probability
distributions on the basis of the constrained maximization of unconventional informational-
statistical entropies (quantity of uncertainty of information), to be accompanied, as noticed
in the main text, with the use of the so-called escort probabilities, allowing for an analysis
and understanding of the observed physical behavior of the system.

Summarizing, Unconventional Statistical Mechanics consists of two steps: 1. The choice
of a deemed appropriate structural informational-entropy for generating the heterotypical
statistical operator, and 2. The use of a escort probability in terms of the heterotypical distribution of item 1.

As shown in the main text, and illustrated in Section III, the escort probability introduces corrections to an inefficient description by including correlations and higher-order variances of the observables involved. On the other hand, the heterotypical distribution introduces corrections to the insufficient description by modifying the statistical weight of the dynamical states of the conventional approach involved in the situation under consideration. Moreover, we have considered a particular case, namely the statistics as derived from the use of Renyi informational entropy, centering the attention on the derivation of an Unconventional Statistical Mechanics appropriate for dealing with far-removed-from-equilibrium systems. Moreover, we have reported the calculation, in such conditions, of the distribution functions of single fermions and bosons, the counterpart in these unconventional statistics of the usual Fermi-Dirac and Bose-Einstein distributions: These distributions are illustrated in Figs. 1 and 2.

In conclusion, we may say that USM appears as a valuable approach, in which the introduction of informational-entropic-indexes-dependent informational-entropies leads to a particularly convenient and sophisticated method for studying certain classes of physical systems for which the criteria of efficiency and sufficiency in its characterization cannot be properly satisfied.

Finally, it is relevant to emphasize again the fact that the infoentropic index(es) is(are) dependent on the dynamics involved, the system’s geometry and dimensions, boundary conditions, its macroscopic-thermodynamic state (in equilibrium, or out of it when becomes a function of time), and the experimental protocol.

Also, we recall the fundamental point, stressed in the Introduction, that these informational entropies (informational measures) are not at all to be interpreted as the physico-thermal entropy of the system.

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FIG. 1: The distribution of Eq. (33) for a kinetic temperature [cf. Eq. (42)] of 300 K and values of Renyi’s infoentropic-index \( \alpha \) smaller than 1.
FIG. 2: The distribution of Eq. (33) for a kinetic temperature [cf. Eq. (42)] of 300 K and values of Renyi’s infoentropic-index $\alpha$ larger than 1.
FIG. 3: The high-energy side of the photoluminescence spectra for the quantum-well widths indicated in the upper right inset. The continuous lines are the best fittings to experimental curves using Eq. (45).
FIG. 4: Dependence of the infoentropic index $\alpha$ with the quantum-well width following the approximate empirical law of Eq. (48).