Equivariant Cohomotopy implies orientifold tadpole cancellation

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Abstract

There are fundamental open problems in the precise global nature of RR-field tadpole cancellation conditions in string theory. Moreover, the non-perturbative lift as M5/MO5-anomaly cancellation in M-theory had been based on indirect plausibility arguments, lacking a microscopic underpinning in M-brane charge quantization. We provide a framework for answering these questions, crucial not only for mathematical consistency but also for phenomenological accuracy of string theory, by formulating the M-theory C-field on flat M-orientifolds in the generalized cohomology theory called \textit{Equivariant Cohomotopy}. This builds on our previous results for smooth but curved spacetimes, showing in that setting that charge quantization in twisted Cohomotopy rigorously implies a list of expected anomaly cancellation conditions. Here we further expand this list by proving that brane charge quantization in unstable equivariant Cohomotopy implies the anomaly cancellation conditions for M-branes and D-branes on flat orbi-orientifolds. For this we (a) use an unstable refinement of the equivariant Hopf-tom Dieck theorem to derive local/twisted tadpole cancellation, and (b) the lift to super-differential cohomology to establish global/untwisted tadpole cancellation. Throughout, we use (c) the unstable Pontrjagin-Thom theorem to identify the brane/O-plane configurations encoded in equivariant Cohomotopy and (d) the Boardman homomorphism to equivariant K-theory to identify Chan-Paton representations of D-brane charge. We find that unstable equivariant Cohomotopy, but not its image in K-theory, distinguishes D-brane charge from the finite set of types of O-plane charges.
1 Introduction

Organizing and formalizing results in the string theory literature, we start by noticing the following curious systematics, to be elaborated upon throughout the paper.

Toroidal orientifolds with ADE-singularities – A curious pattern. Consider type II superstring vacua compactified on fluxless toroidal orientifolds ([Sag88], [DLP99], [Mu97]; see also [IU12], 5.3.4, 10.1.3; [BLT13], 15.3) with ADE-type singularities ([AM97], [In97]; see [HSS18]), hence on orbifold quotients $\mathbb{T}^4 / G$ (e.g. [Ra06], 13) of 4-tori by crystallographic point groups (22). These are finite subgroups $G \subset SU(2)$ $\simeq$ Sp(1) of the group unit quaternions acting by left multiplication on the space $\mathbb{H} \cong \mathbb{R}^4$ of all quaternions (17).

The consistency condition on such compactifications known as (Ramond-Ramond) RR-field tadpole anomaly cancellation ([GP96], Sec. 3; [Wi12], Sec. 9.3; see [IU12], 4.4; [BLT13], 9.4), essentially says that the joint D-brane and O-plane charge in such compact orientifolds has to vanish, albeit with some subtle fine print. Explicitly, we observe that a case-by-case analysis of the string worldsheet superconformal field theory shows (Table 1) that, for single wrapping number, RR-field tadpole anomaly cancellation is the following condition on the $G$-representation of D-brane charge and the $G$-set of O-planes:

(i) Local/twisted tadpole cancellation: D-brane charge is a combination of a regular representation $k_{\text{reg}}$ and the trivial one $1_{\text{triv}}$, with coefficients the number of integral and fractional branes, respectively:

$$ c_{\text{Dbrane}} = N_{\text{brane}} \cdot k_{\text{reg}} + N_{\text{brane}} \cdot 1_{\text{triv}}. $$

(ii) Global/untwisted tadpole cancellation: The dimension of D-brane charge is the cardinality of the $G$-set of O-planes:

$$ \dim(c_{\text{Dbra}}) = \text{card}(c_{\text{Opla}}). $$

In particular, $c_{\text{Dbra}}$ comes from, and $c_{\text{Opla}}$ gives rise to, a permutation representation, in the image of $\beta$ ([BSS18]):

$$ c_{\text{Dbra}} \in RO(G) \cong K^0 \left( \frac{G}{G} \right) \cong A(G) \cong G \text{-set} \cong c_{\text{Opla}} \quad (1) $$

| Single D-brane species on toroidal orientifold | Local/twisted tadpole cancellation condition | Global/untwisted tadpole cancellation condition | Comments |
|---------------------------------------------|---------------------------------------------|---------------------------------------------|---------|
| Branes on $\mathbb{T}^4$ / $G\text{-ADE}$   | $c_{\text{Dbra}} = N_{\text{brane}} \cdot k_{\text{reg}}$ | $\dim(c_{\text{Dbra}}) = \text{card}(c_{\text{Opla}})$ | The general pattern of the following case-by-case results |
| D5/D9-branes on $\mathbb{T}^4$ / $Z_2$     | $c_{\text{Dbra}} = N \cdot k_{\text{reg}}$ ([BST99], 19) | $c_{\text{Dbra}} = 16 \cdot k_{\text{reg}}$ ([BST99], 18) | Following ([GP96], [GJ96]) |
| D5/D9-branes on $\mathbb{T}^4$ / $Z_4$     | $c_{\text{Dbra}} = N \cdot k_{\text{reg}}$ ([BST99], 19) | $c_{\text{Dbra}} = 8 \cdot k_{\text{reg}}$ ([BST99], 18) | Re-derived via M5-branes below in §4 |
| D4-branes on $\mathbb{T}^4$ / $Z_k$        | $c_{\text{Dbra}} = N \cdot k_{\text{reg}}$ ([AFIRU00a], 4.2.1) | $c_{\text{Dbra}} = 4 \cdot k_{\text{reg}} + 4 \cdot 1_{\text{triv}}$ ([HS02], 4.1)-(17), | The special case of $k = 3$ (review in [Ma03], 4) |
| D8-branes on $\mathbb{T}^4$ / $Z_3$        | $c_{\text{Dbra}} = N \cdot k_{\text{reg}}$ ([AFIRU00b], 7.2) | $c_{\text{Dbra}} = 4 \cdot k_{\text{reg}} + 4 \cdot 1_{\text{triv}}$ ([HS01], 4, [HS02], 29) | Equivalent by T-duality to previous case ([HM01], p.1, [HM02], 6) |
| D3-branes on $\mathbb{T}^4$ / $Z_3$        | $c_{\text{Dbra}} = N \cdot k_{\text{reg}}$ ([FHKU01], 25) | $c_{\text{Dbra}} = 8 \cdot k_{\text{reg}}$ ([IKS99], 25) | |
| D7-branes on $\mathbb{T}^4$ / $Z_3$        | $c_{\text{Dbra}} = N \cdot k_{\text{reg}}$ ([FHKU01], 5, 6) | $c_{\text{Dbra}} = 8 \cdot k_{\text{reg}}$ ([IKS99], 25) | |

Table 1 – Tadpole cancellation conditions between D-branes and O-planes on toroidal ADE-orientifolds as derived from case-by-case analysis in perturbative string theory. The geometric content of this situation is illustrated in Figure A.
The D-brane species in \(\text{Table 1}\) with the most direct lift to M-theory are the D4-branes, lifting to M5-branes under double dimensional reduction \([\text{APPS97a}, 6]\)[\text{APPS97a}, 6][\text{LPSS11}]\); see \(\text{Table 7}\). With an actual formulation of M-theory lacking, indirect plausibility arguments have been advanced \([\text{DM95}]\)[\text{Wi95b}, 3.3][\text{Ho98}, 2.1]\) that for M5-branes on M-theoretic orientifolds of the form \(T^5_{\text{sgn}} // Z_2\), anomaly cancellation implies \(\text{Table MO5}\).

| Single M-brane species on toroidal orientifold | Local/twisted tadpole cancellation condition | Global/untwisted tadpole cancellation condition | Comments |
|-----------------------------------------------|-----------------------------------------------|-----------------------------------------------|----------|
| M5-branes on \(T^5_{\text{sgn}} // Z_2\) | \(c_{\text{Mbra}} = N \cdot 2_{\text{reg}}\) \([\text{DM95}]\)[\text{Wi95b}, 3.3][\text{Ho98}, 2.1]\) | \(c_{\text{Mbra}} = 16 \cdot 2_{\text{reg}}\) | plausibility arguments |

\(\text{Table 2 – M5/MO5 anomaly cancellation in M-theory}\) according to Folklore \([4.1]\). While it has remained open in which cohomology theory the charge \(c_{\text{Mbra}}\) is quantized, the geometric content is again that illustrated in Figure A.

We highlight in Figure A the geometric interpretation of these tadpole cancellation conditions from \(\text{Table 1}\) and \(\text{Table 2}\). The left side of Figure A shows a 2-dimensional slice through the toroidal orbifold \(T^4_H // Z_4 = (\mathbb{R}^4 / \mathbb{Z}^2) // Z_4\) with transversal branes/O-plane charges appearing as points. The O-plane charges (shown as open circles) are stuck one-to-one to the fixed points of the point reflection subgroup \(Z_2 \hookrightarrow Z_4\) (see also \(\text{Table RT}\)) and, in the example shown, are permuted by the full orbifold group action of \(Z_4\) according to the permutation representation \(2 \cdot 1_{\text{triv}} + 1 \cdot 2_{\text{perm}}\). The local/twisted tadpole cancellation condition says that the branes (shown as filled circles) appear in the vicinity of the O-planes with all their distinct mirror images under the full group action, thus contributing Chan-Paton fields in the regular representation \(4_{\text{reg}}\). The global/untwisted tadpole cancellation condition says that the total charge of branes minus O-planes, hence the net charge if all branes/O-planes could freely move and pairwise annihilate, vanishes:

\[
\text{local/twisted tadpole cancellation:} \quad c_{\text{Mbra}} = 4 \cdot (1 \cdot 4_{\text{reg}} - (2 \cdot 1_{\text{triv}} + 1 \cdot 2_{\text{perm}})) \quad \text{dim} \quad 4 \cdot (1 - 4 - (2 + 1)) = 0
\]

\[x_2 = \frac{1}{2}\]

\[x_2 = 0\]

\[x_1 = 0\]

\[x_1 = \frac{1}{2}\]

\[x_1 = 0\]

\[x_1 = \frac{1}{2}\]

Figure A – Illustration of the geometric situation of tadpole cancellation on toroidal ADE-orientifolds according to \(\text{Table 1}\) shown for the case \(G^{\text{ADE}} = Z_4\). This is for single wrapping number of the branes along any further compact dimensions; but the general statement is just the tensor product of this situation with the cohomology of these further compact spaces.

In view of the evident pattern evidenced by \(\text{Table 1}\) and \(\text{Table 2}\), here we ask the following question:

Is there a generalized cohomological brane charge quantization which enforces tadpole anomaly cancellation?

We show in this paper that (see Figure U), for fluxless toroidal ADE-orientifolds, the answer to this question is unstable equivariant Cohomotopy theory; see (3) below. Before explaining this, we put the open problem in perspective:
The open problem – Systematic understanding of tadpole cancellation by charge quantization. While the RR-field tadpole cancellation conditions are thought to be crucial not just for mathematical consistency, but also for phenomenological accuracy of string model building [IU12 Sec. 4.4], a real understanding of the full set of rules has remained an open problem; see [BDS06, p. 2] [Mo14 4.6.1] [HMSV19, p. 2] for critical discussion. In particular, most of the existing literature on tadpole cancellation simply regards D-brane charge as being in ordinary cohomology, while widely accepted arguments say that D-brane charge instead must be regarded in (a twisted differential enhancement of) K-theory; in this context, see [BMSS18, Sec. 1] for review, and see [GS17] [GS19a] [GS19b] for detailed constructions and accounts of the twisted differential case. D-brane charge in K-cohomology may be understood as a generalized charge quantization rule, in analogy to how Dirac’s classical argument for charge quantization [Dir31] (see [Fra97, 16.4e]) expresses the electromagnetic field as a cocycle in (the differential refinement of) ordinary cohomology; see [Fre00]. Notice that cohomological charge quantization concerns the full non-perturbative structure of a physical theory, including its instanton/soliton charge content.

Accordingly, in [Ur00, 5] it was suggested that RR-tadpole cancellation must be a consistency condition expressed in K-theory. Specifically, for orientifolds this could be Atiyah’s Real K-theory [At66], i.e., KR-theory restricting on O-planes to KO-theory [GS19b], which has been argued to capture D-brane charges on orientifolds in [Wi98] [Gk99] [BGS01, Sec. 3]; explicit constructions are in [DMDR14] [DMDR15] [HMSV13] [HMSV19]. In more detail, D-brane charge on orbifolds is traditionally expected [Wi98 5.1] [dBD+02, 4.5.2] [Gk99] to be in equivariant K-theory KU"G, KO"G (see [Gr05]). Hence orientifolds are expected to have charge quantization in a combination of these aspects in some Real equivariant orbifold K-theory [Mou11] [Mou12] [FM12] [Go17]. However, before even formulating tadpole cancellation in Real equivariant K-theory, the full formulation of O-plane charge has remained open:

**Open issue 1: Single O-plane charge.** While O-plane charge is not supposed to vary over all integers, perturbative string theory predicts it to vary in the set \{0, \pm 1\} (e.g. [HIS00, p. 2]), illustrated in Figure B. 

![Figure B](image-url)  
Figure B – The charge carried by a single O-plane takes values in the set \{0, \pm 1\} (in units of corresponding integral D-brane charge), visualized here following the geometric illustration of Figure A. For O4-planes this situation lifts to MO5-planes in M-theory [Ho98] [Gi98] [AKY98, II.B] [HK00, 3.1]. (The notation for O^i originates with [Ho98, p. 29] [Gi98, p. 4]; see Figure T for more.)

But in plain KR-theory all O-planes are O^−-planes. To capture at least the presence of O^+ -planes requires adding to KR-theory an extra sign choice [DMDR14]. In some cases this may be regarded as part of a twisting of KR-theory [HMSV19], but the situation remains inconclusive [HMSV19, p. 2].

**Open issue 2: Total O-plane charge.** As highlighted in [BGS01, p. 4, p. 25], it remains open whether a putative formalization of tadpole cancellation via Real K-theory reflects the absolute total charge to be carried by O-planes. This is a glaring open problem, since the absolute total charge -32 of Op-planes in toroidal orientifolds (see Table 3) fixes the gauge algebra so(32) of type I string theory required for duality with heterotic string theory (see, e.g., [BLT13, p. 250] [APT97]) with Green-Schwarz anomaly cancellation. This core result of string theory, is the basis of the “first superstring revolution” [Sw11, p. 21], and a successful formalization of tadpole cancellation ought to reproduce it.

1 Note that [HMSV19 footnote 1] claims a problem with the sign choice in [DMDR14], and hence also in [Mou12]. These continuing issues with orbifold K-theory for D-brane charge may motivate but do not affect the discussion here, where instead we propose equivariant Cohomotopy-theory for M-brane charge as an alternative.
A proposal for capturing absolute background charge of O-planes by equipping K-theory with a quadratic pairing has been briefly sketched in [DFM11], but the implications remain somewhat inconclusive [Mo14, p. 22]. We notice that the implications on M-brane charge quantization of analogous quadratic functions in M-theory [HS05] are reproduced by charge quantization in twisted Cohomotopy cohomology theory [FSS19]. Here we further check this alternative proposal: That brane charge quantization is in Cohomotopy cohomology theory, which lifts K-theory through the Boardman homomorphism; see (4) below.

The proposal – Charge quantization on orientifolds in Equivariant Cohomotopy cohomology theory. When educated guesswork gets stuck, it is desirable to identify principles from which to systematically derive charge quantization in M-theory, if possible, and seek the proper generalized cohomology theory to describe the M-theory fields, as was advocated and initiated in [Sa05a][Sa05b][Sa06][Sa10]. A first-principles analysis of super p-brane sigma-models in rational homotopy theory shows [Sa13][FSS15][FSS16a][FSS16b] that rationalized M-brane charge is quantized in rational Cohomotopy cohomology theory; see [FSS19] for review. This naturally suggests the following hypothesis about charge quantization in M-theory [Sa13][FSS19][FSS19c]:

**Hypothesis H.** The M-theory C-field is charge-quantized in Cohomotopy theory.

Applied to toroidal orbifolds, the relevant flavor of unstable Cohomotopy theory is (see Table 4) unstable equivariant Cohomotopy ([D79] 8.4)[Cr03], denoted $\pi^*_G$. This is the cohomology theory whose degrees are labeled by orthogonal linear $G$-representations, called the RO-degree (see, e.g., [Bl17, 3])

\[
\text{orthogonal linear } G\text{-representation } G \subset \text{O}(\dim(V)) \quad \xymatrix{ G \ar@{~}[r] & \text{O}(\dim(V)) } \quad \xymatrix{ G & \text{RO}(G) & \ar[l] \text{representation ring} }
\]

and whose value on a global $G$-quotient orbifold $X \sslash G$ with specified point at infinity $\infty \in X$ – see diagram (11) – is the set of $G$-homotopy classes $[\Sigma^X]_G$ of pointed $G$-equivariant continuous functions $\Sigma^X$ from $X$ to the $V$-representation sphere $S^V$ (21) (see §3 for details and illustration):

\[
\pi^*_G(X) := \left\{ \left[ \begin{array}{c} G \\ X \end{array} \right] \xymatrix{ c \ar[r] & S^V } \right\} / \sim
\]

(3)

This is the evident enhancement to unstable $G$-equivariant homotopy theory (see [Bl17, 1]) of unstable plain Cohomotopy cohomology theory $\pi^*$ ([Bo36][Sp49][KMT12][FSS19b 3.1]).

Equivariant Cohomotopy is a non-abelian (i.e. “unstable”) Cohomology theory [SSS12][NSS12] that maps to equivariant K-theory via stabilization followed by the Boardman homomorphism, see §3.1.2 and [BSS18].

\[
\xymatrix{ \pi^*_G & \Sigma^* \ar[l] & S_G \ar[l] & \beta \ar[l] \ar[r] & \text{KO}_G }
\]

(4)

The solution – From Hypothesis H. In this article we explain how lifting brane charge quantization to ADE-equivariant Cohomotopy, regarded as the generalized Dirac charge quantization of the M-theory C-field (e.g. [Du99]) on toroidal M-orientifolds ([DM95][W95b][Ho93][HSS18]), gives the local O-plane charges in $\{0, \pm 1\}$ from Figure B and enforces on D-brane charge in the underlying equivariant K-theory (4) the RR-field tadpole cancellation constraints from Table 1 via their M-theory lift from Table 2.

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Table 3 – Absolute O-plane charge [LU12 (5.52)] [BLT13 10.212] –32 is not implied by K-theory [BGS01], but is implied by Cohomotopy.
Overall picture – M-Theory and Cohomotopy. As we further explain in [SS], unstable equivariant Cohomotopy theory is the incarnation on flat orbifolds of unstable twisted Cohomotopy cohomology theory, which we showed in [FSS19b] [FSS19c] implies a list of M-theory anomaly cancellation conditions on non-singular (i.e., “smooth”) but topologically non-trivial spacetimes; see Table 4.

| Spacetime | Flat                      | Curved                                  |
|-----------|---------------------------|-----------------------------------------|
| Smooth    | plain Cohomotopy          | twisted Cohomotopy                       | (FSS15) [BMSS18] | (FSS19b) [FSS19c] |
| Orbi-singular | equivariant Cohomotopy | orbifold Cohomotopy                     | (FSS18, §4)     | (SS)            |

Table 4 – M-theory anomaly cancellation by C-field charge quantization in Cohomotopy. On smooth but curved spacetimes, Cohomotopy theory is twisted via the J-homomorphism by the tangent bundle. On flat orbi-orientifolds the space-time curvature is all concentrated in the $G$-singularities, around which the tangent bundle becomes a $G$-representation and twisted Cohomotopy becomes equivariant Cohomotopy. In each case the respective charge quantization implies expected anomaly cancellation conditions. See also Table 8.

Each entry in Table 4 supports Hypothesis H in different corners of the expected phase space of M-theory. This suggests that Hypothesis H is a correct assumption about the elusive mathematical foundation of M-theory.

The necessity of unstable = non-abelian charge quantization for O-planes. We highlight that most authors who discuss equivariant Cohomotopy consider stable equivariant Cohomotopy cohomology theory (e.g. [Seg71] [Ca84] [Lu05]), represented by the equivariant sphere spectrum $S_G$ in equivariant stable homotopy theory ([LMS86] [HHR16, Appendix]); see §3.1.2 below. There are comparison homomorphisms (4) from equivariant unstable Cohomotopy to stable Cohomotopy and further to $K$-theory but each step forgets some information (has a non-trivial kernel) and produces spurious information (has a non-trivial cokernel); see [BSS18]. For the result presented here (just as for the previous discussion in [FSS19b] [FSS19c]), it is crucial that we use the richer unstable version of the Cohomotopy theory, hence the non-abelian Cohomology theory [SSS12] [NSS12], which is the one that follows from analysis of super $p$-brane cocycles [Sa13] [FSS19a]. We find that:

(a) the difference in the behavior between the O-plane charges and the D-brane charges (in Table 1, Table 2 and Figure P) and

(b) the unstable/non-abelian nature of O-plane charge itself (Figure OP)

are reflected in the passage from the unstable to the stable range in unstable ADE-equivariant Cohomotopy, where the O-plane charges are distinguished as being in the unstable range; see Figure C.

Characterizing brane/O-plane charges – Unstable (equivariant) differential topology. Since in Figure C the fixed locus in the classifying space is just a 0-sphere, and since the Hopf degree of maps $X^n \to S^n$ stabilizes only for $n \geq 1$ – see diagram (13) – the fixed points in the spacetime (= O-planes) carry “unstable” or “non-linear”
charge: not given by a group element, but by a subset, distinguishing $O^\pm$-planes from $O^0$-planes as in Figure OP. The further distinction between $O^-$-planes and $O^+$-planes is implied by normal framing that enters in the unstable Pontrjagin-Thom theorem (discussed in §2.2). Moreover, the local/twisted tadpole cancellation condition in the vicinity of O-planes is implied by the unstable equivariant Hopf degree theorem (discussed in §3.1). Last but not least, it is the unstable Pontrjagin-Thom theorem, discussed in §2, which identifies all these charges with submanifolds, hence with actual brane/O-plane worldvolumes as shown in Figure A (while the stable PT-theorem instead relates stable Cohomotopy to manifolds equipped with any maps to spacetime).

| Classical theorem | Reference | Interpretation for brane charge quantization in unstable Cohomotopy (Hypothesis H) | Discussed in |
|-------------------|-----------|---------------------------------------------------------------------------------|-------------|
| Unstable Pontrjagin-Thom theorem | [Kos93, IX (5.5)] | Cohomotopy charge is sourced by submanifolds hence by worldvolumes of branes and O-planes | §2.1 |
| Unstable Hopf degree theorem | [Kos93, IX (5.8)] | Charge of flat transversal branes is integer while charge of flat transversal O-planes is in \{0, 1\} | §2.2 |
| Unstable equivariant Hopf degree theorem | [tD79, 8.4] | Branes appear in regular reps around O-planes = local/twisted tadpole anomaly cancellation | §3 |

**Organization of the paper.** In §2 we discuss how the classical unstable Pontrjagin-Thom isomorphism says that plain Cohomotopy classifies charge carried by brane worldvolumes. In §3 we introduce the enhancement of this situation to equivariant Cohomotopy on toroidal orbifolds, where it encodes joint D-brane and O-plane charge. We explain in §3 that now the equivariant Hopf degree theorem encodes the form of local/twisted tadpole cancellation conditions, and explain in §3.2 that super-differential refinement at global Elmendorf stage encodes the form of global/untwisted tadpole cancellation conditions as in Table 1 and Table 2. The Pontrjagin-Thom theorem now serves to map these charges precisely to the geometric situations of the form shown in Figure A. Finally, in §4 we specify these general considerations to the physics of M5-branes at MO5-planes in toroidal ADE-orientifolds in M-theory, with the C-field charge-quantized in equivariant Cohomotopy theory, according to Hypothesis H. To set the scene, we first recall in §4.1 the situation of heterotic M-theory on ADE-orbifolds and highlight subtleties in the interpretation of MO5-planes. With this in hand, we apply in §4.2 the general discussion of equivariant Cohomotopy from §3 to ADE-singularities intersecting MO9-planes in M-theory, and find (Cor. 4.4, Cor. 4.6) that this correctly encodes the expected anomaly cancellation of M5-branes at MO5-planes, and this, upon double dimensional reduction (see Table 7 and Figure U), the RR-field tadpole anomaly cancellation for D-branes on ADE-orientifolds.

## 2 Cohomotopy and brane charge

Before turning to equivariant/orbifold structure in §3, we first discuss basics of plain unstable Cohomotopy on plain manifolds. The key point is that the unstable Pontrjagin-Thom theorem, reviewed in §2.1, identifies cocycles in unstable Cohomotopy theory with cobordism classes of submanifolds carrying certain extra structure (normal framing). These submanifolds are naturally identified with the worldvolumes of branes that source the corresponding Cohomotopy charge, and the normal structure they carry corresponds to the charge carried by the branes, distinguishing branes from anti-branes. In §2.2 we highlight that coboundaries in unstable Cohomotopy accordingly correspond to brane pair creation/annihilation processes. This way the Pontrjagin-Thom theorem establishes Cohomotopy as a natural home for brane charges, as proposed in Sa13.

### 2.1 Pontrjagin-Thom theorem and brane worldvolumes

The special case of unstable $G$-equivariant Cohomotopy (5) with $G = 1$ the trivial group is unstable plain Cohomotopy theory (Bo36, Sp49, KMT12, FSS19b, 3.1), denoted $\pi^* := \pi_1^*$. This is the unstable/non-abelian cohomology theory whose degrees are natural numbers $n \in \mathbb{N}$ and which assigns to a topological space $X$ the Cohomotopy set of homotopy classes of continuous maps into the $n$-sphere:
\[
\pi^n(X) \ := \ \{ X \xrightarrow{c} S^n \} / \sim
\] (5)

Thinking of \( X \) here as spacetime, we are interested in the case that \( X = X^D \) admits the structure of closed smooth manifold of dimension \( D \in \mathbb{N} \). In this case, the unstable Pontrjagin-Thom theorem (7) identifies (see e.g. [Kos93 IX.5]) the degree-\( n \) Cohomotopy set of \( X^D \) (5) with the set of cobordism classes of normally framed codimension-\( n \) submanifolds of \( X^D \) (see e.g. [Kos93 IX.2]), hence of submanifolds \( \Sigma^d \hookrightarrow X^D \) which are of dimension \( d = D - n \) and equipped with a choice of trivialization

\[
N/\Sigma \sim \xrightarrow{\text{normal framing}} \Sigma \times \mathbb{R}^n
\] (6)

of their normal vector bundle:

\[
\overset{\text{Unstable}}{\overset{\text{Pontrjagin-Thom}}{\overset{\text{theorem}}{\overset{\pi^n(X^D)}{\sim}}}} \xrightarrow{\text{"PT collapse"}} \left\{ \text{Submanifolds } \Sigma \xrightarrow{\text{dir. asympt. dist.}} X^D \text{ of dimension } d = D - n \text{ and equipped with normal framing} \right\} / \text{cobordism}
\] (7)

The construction which exhibits this bijection is traditionally called the Pontrjagin-Thom \textit{collapse}, but a more suggestive description, certainly for our application to brane charges, is this: \textit{The Cohomotopy class corresponding to a submanifold/brane is represented by the function which assigns directed asymptotic distance from the submanifold/brane, as measured with respect to the given normal framing (6) upon identifying the normal bundle with a tubular neighborhood and regarding all points outside the tubular neighborhood as being at infinite distance; see \textit{Figure D}}:

\[
\begin{array}{c}
\text{X} \\
\text{manifold} \\
\hline
\text{Cohomotopy cocycle} \\
\xrightarrow{c} \text{S}^n = (\mathbb{R}^n)^{\text{cpt}} \\
\text{n-sphere} \\
\hline
\text{Cohomotopy coefficient}
\end{array}
\]

\textit{Figure D – The Pontrjagin-Thom construction} which establishes the unstable Pontrjagin-Thom theorem (7). The cocycle \( c \) in Cohomotopy eq. (5) is the continuous function which sends each point to its directed asymptotic distance from the given submanifold.

\textbf{One-point compactifications by adjoining the point at infinity.} Here and in all of the following, we are making crucial use of the fact that the \( n \)-sphere is the one-point compactification \((-)^{\text{cpt}}\) of the Cartesian space \( \mathbb{R}^n \),

\[
S^n \overset{\text{homeo}}{\sim} (\mathbb{R}^n)^{\text{cpt}} := \{ x \in \mathbb{R}^n \text{ or } x = \infty \}, \tau_{\text{cpt}} \quad \text{for all } n \in \mathbb{N},
\] (8)

as indicated on the right of \textit{Figure D}. Here the one-point compactification \( X^{\text{cpt}} \) of a topological space \( X \) is defined (e.g. [Ke55 p. 150]) by adjoining one point to the underlying set of \( X \) – denoted “\( \infty \)” as it becomes literally the \textit{point at infinity} – and by declaring on the resulting set a topology \( \tau_{\text{cpt}} \) whose open subsets are those of \( X \), not
containing $\infty$, and those containing $\infty$ but whose complement in $X$ is compact. Notice that this construction also applies to topological spaces that already are compact, in which case the point at infinity appears disconnected

$$X \text{ already compact } \Rightarrow X^\text{cpt} = X_+ := X \sqcup \{\infty\}. \quad (9)$$

This means that (8) indeed holds also in the “unstable range” of $n = 0$:

$$\left(\mathbb{R}^0\right)^\text{cpt} = (\{0\})^\text{cpt} = \{0\} \sqcup \{\infty\} = S^0. \quad (10)$$

**Cohomotopy charge vanishing at infinity.** In view of the Pontrjagin–Thom theorem [7], it makes sense to say that a cocycle in Cohomotopy vanishes wherever it takes as value the point at infinity $\infty \in (\mathbb{R}^n)^\text{cpt} \simeq S^n$ in the coefficient sphere, identified under (8). This means to regard the coefficient sphere as a pointed topological space, with basepoint $\infty \in S^n$. Given then a non-compact (spacetime) manifold $X$ (such as $X = \mathbb{R}^n$), a Cohomotopy cocycle $X \to S^n$ *vanishes at infinity* if it extends to the one-point compactification $X^\text{cpt}$ (8) such as to send the actual point at infinity $\infty \in X^\text{cpt}$ to the point at infinity in the coefficient sphere.

A Cohomotopy cocycle on a non-compact space $X$ which vanishes at infinity is a Cohomotopy cocycle on the one-point compactification $X^\text{cpt}$ that sends the point at infinity in the domain to that in the coefficient $n$-sphere.

$$X^\text{cpt} \xrightarrow{\text{c}} (\mathbb{R}^n) \simeq S^n \quad (11)$$

**Example 2.1** (Figure E). For $X = \mathbb{R}^n$, we have that Cohomotopy $n$-cocycles on $X$ vanishing at infinity are equivalently maps from an $n$-sphere to itself:

$$(\mathbb{R}^n)^\text{cpt} \quad \xrightarrow{c = 1 - 3 = -2} \quad S^n \equiv (\mathbb{R}^n)^\text{cpt}$$

![Figure E](image)

Figure E – Cohomotopy in degree $n$ of Euclidean $n$-space vanishing at infinity is given by Cohomotopy cocycles [5] on the one-point compactification $(\mathbb{R}^n) \simeq S^n$ [8] that send $\infty$ to $\infty$ [11].

Of course, this is just the cohomotopical version of *instantons* in ordinary gauge theory:

**Instantons and solitons.** If $G$ is a compact Lie group with classifying space $BG$ equipped with the canonical point $* \simeq B\{e\} \longrightarrow BG$, then a *G-instanton sector* on Euclidean space $X = \mathbb{R}^n$ is the homotopy class of a continuous function from the one-point compactification of $X$ to $BG$, which takes the base points to each other.

$$A \text{ G instanton sector is a cocycle in degree-1 G-cohomology which vanishes at infinity in that it is a cocycle on the one-point compactification } X^\text{cpt} \text{ which sends the point at infinity in the domain to the base point in the classifying space } BG.$$
Cohomotopy and SU($N$)-instanton sectors. Specifically for $n = 4$ and $G = SU(N)$ any map $S^4 \xrightarrow{\varepsilon} BSU(N)$ representing a generator $1 \in \mathbb{Z} \simeq \pi_4(\text{BSU}(N))$ of the 4th homotopy group of the classifying space exhibits a bijection between the 4-Cohomotopy of $\mathbb{R}^4$ vanishing at infinity (11), and the set of SU($N$)-instanton sectors

$$
\pi^4(\mathbb{R}^4^{\text{cpt}}) = \{ (\mathbb{R}^4)^{\text{cpt}} \xrightarrow{\varepsilon} S^4 \} / \sim \simeq \{ (\mathbb{R}^4)^{\text{cpt}} \xrightarrow{} BSU(N) \} / \sim \simeq \{ \text{SU($n$)-instanton sectors on } \mathbb{R}^4 \}.
$$

Under this identification of SU($N$)-instanton sectors with Cohomotopy vanishing at infinity, the Pontrjagin-Thom construction (7) produces precisely the distribution of instanton center points, again illustrated by the left hand side in Figure E.

2.2 Hopf degree theorem and brane-antibrane annihilation

The classical Hopf degree theorem describes the $n$-Cohomotopy (5) of orientable closed $D$-manifolds $X$ (7) in the special case where $n = D$. It says that, in the “stable range” $n \geq 1$, the Cohomotopy set is in bijection with the set of integers, where the bijection is induced by sending the continuous function representing a Cohomotopy cocycle to its mapping degree (see, e.g., [Kob16, 7.5]):

$$
\pi^n(X) \xrightarrow{S^n \xrightarrow{\varepsilon} K(\mathbb{Z}, n)} H^n(X, \mathbb{Z}) \xrightarrow{} \mathbb{Z} \quad \text{detected by } \deg(c).
$$

Under the Pontrjagin-Thom theorem (7) the Hopf degree theorem (13) translates into the following geometric situation for signed (charged) points in $X^n$ (see [Kos93, IX.4]): A codimension-$n$ submanifold in an $n$-manifold $X^n$ is a set of points in $X^n$, and a choice of normal framing (6) is, up to normally framed cobordism, the same as choice of sign (charge) in $\{\pm 1\}$ for each point, as shown in Figure F:

$$
X^n \xrightarrow{c} S^n \rightarrow D(\mathbb{R}^n)/S(\mathbb{R}^n)
$$

Figure F – Charge in Cohomotopy carried by submanifolds, under the PT-isomorphism (7) is encoded in their normal framing (6). In full codimension the normal framing is a normal orientation and hence a choice in $\{\pm 1\}$, which we indicate graphically by $\bullet \leftrightarrow 1/\pm 1$.

Under this geometric translation, we have the correspondence

$$
\text{Hopf degree of Cohomotopy cocycle on } X \xrightarrow{\text{PT}} \text{Net number of } \pm \text{-charges carried by points in } X
$$

The mechanism which implements this on the geometric right hand side is that points of opposite sign/normal framing are cobordant to the empty collection of points, hence mutually annihilate each other via coboundaries in Cohomotopy, as shown in Figure G.
\[ \pi^0(X^{\text{cpt}}) = \{ X \overset{c}{\longrightarrow} S^0 \} \]

are in bijection to the subsets \( S \subset X \) of \( X \), by the assignment that sends \( c \) to the pre-image \( c^{-1}(\{0\}) \) of \( 0 \in S^0 \) under \( c \). We may think of these subsets as elements of the power set \( \{0, 1\}^X \) and as such call them the sets \( \deg(c) \) of Hopf degrees in \( \{0, 1\} \) for \( n = 0 \):

\[ \begin{array}{c}
\text{Hopf degree theorem} \\
\text{in unstable range } n = 0
\end{array} \]

\[ \begin{array}{c}
\pi^0(X^{\text{cpt}}) \overset{\cong}{\longrightarrow} \text{Subsets}(X) \\
X^0 \overset{c}{\longrightarrow} S^0 \end{array} \]

\[ \approx \quad \{0, 1\}^X \quad \text{sets of unstable Hopf degrees} \]

\[ \deg(c) \quad (14) \]

**Example 2.2.** For \( X = \{0\} \) the single point so that, with \( \{0\}^{\text{cpt}} \), \( X^{\text{cpt}} \) is the 0-sphere, we have \( \pi^0(\{0\}^{\text{cpt}}) \approx \{0, 1\} \), as illustrated in the following figure:

\[ \begin{array}{c}
\mathbb{R}^0 \overset{c = 0}{\longrightarrow} S^0 \\
\text{vanishing at infinity} \quad \text{no charge}
\end{array} \]

\[ \begin{array}{c}
\mathbb{R}^0 \overset{c = 1}{\longrightarrow} S^0 \\
\text{vanishing at infinity} \quad \text{not a unit charge at the single point}
\end{array} \]
The point of unstable Hopf degree in \{0, 1\} is that it exhibits \textit{homogeneous behavior under suspension} \(\Sigma^1\) across the unstable and stable range of Hopf degrees, with the unstable Hopf degrees in \{0, 1\} injecting into the full set of integers in the stable range:

\[
\begin{array}{cccc}
\{0, 1\} & \text{injection} & \mathbb{Z} & \text{injection} & \mathbb{Z} & \text{injection} & \mathbb{Z} & \ldots \\
\end{array}
\]

\[\pi^0(S^0) \xrightarrow{\Sigma^1} \pi^1(S^1) \xrightarrow{\Sigma^1} \pi^2(S^2) \xrightarrow{\Sigma^1} \pi^3(S^3) \xrightarrow{\Sigma^1} \ldots\]  

As we next turn from plain to equivariant Cohomotopy in §3 we find that unstable and stable Hopf degrees unify in the equivariant Hopf degree theorems, and that the \textit{D-brane charge is what appears in the stable range}, while the \textit{O-plane charge is what appears in the unstable range} (in particular, via the proof of Theorem 3.13 below).

### 3 Equivariant Cohomotopy and tadpole cancellation

We now turn to the equivariant enhancement (5) of Cohomotopy theory. We discuss in §3.1 and in §3.2, respectively, how this captures the form of the local/twisted (see Diagram (58) in §3.2) and of the global/untwisted tadpole cancellation conditions (see §4.1) according to Table 1 and Table 2 by appeal to the equivariant enhancement of the Hopf degree theorem applied to representation spheres, which we state as Theorem 3.10 and Theorem 3.13.  

#### Basic concepts of unstable equivariant homotopy theory.

To set up notation, we start with reviewing a minimum of underlying concepts from unstable equivariant homotopy theory (see [BHT], [HSS] 3.1 for more).

*Topological G-spaces.* For \(G\) a finite group, a \textit{topological G-space} \(X\) (or just \textit{G-space}, for short) is a topological space \(X\) equipped with a continuous \(G\)-action, hence with a continuous function \(G \times X \to X\) such that for all \(g_i \in G\) and \(x \in X\) we have \(g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x\) and \(e \cdot x = x\) (where \(e \in G\) is the neutral element).

Here we are concerned with the \textit{classes of examples of G-spaces} shown in Table 5.

| \(G\)-representation | \(G\)-space | \(G\)-orbifolds | Terminology |
|----------------------|-------------|----------------|-------------|
| \(G\) finite group \(V \in RO(G)\) | \(\mathbb{R}^V\) Euclidean \(G\)-space [17] | \(\mathbb{R}^V \sslash G\) Euclidean orbifold | singularity |
| \(G\) crystallographic group | \(\mathbb{R}^V\) G-representation sphere [37] | \(\mathbb{R}^V \sslash G\) Euclidean orbifold including point at infinity [37] | vicinity of singularity |
| \(G \rtimes \mathbb{Z}^{\text{dim}(V)} \subset \text{Iso}(\mathbb{R}^V)\) | \(\mathbb{R}^V\) G-representation space [37] | \(\mathbb{T}^V \sslash G = \left(\mathbb{R}^V \sslash G\right)/\mathbb{Z}^{\text{dim}(V)}\) toroidal orbifold | flat, compact singular space |

Table 5 – Flat \(G\)-orbifolds and the \(G\)-spaces covering them. Examples arising in application to M-theory are discussed in §4.1.

#### Linear \(G\)-representations.

The \(G\)-spaces of interest for the discussion of toroidal orbifolds all come from \textit{orthogonal linear \(G\)-representations} \(V\): finite-dimensional Euclidean vector spaces equipped with a linear action by \(G\) factoring through the canonical action of the orthogonal group. We will denote concrete examples of such \(V\) of dimension \(n \in \mathbb{N}\) and characterized by some label “\(i\)” in the form \(V = n_i\), and also refer to them as an \(RO\)-degree (2).

\(\text{For our purposes here, the covering \(G\)-space \(X\) is all we need to speak about the corresponding orbifold } X \sslash G.\) For a dedicated discussion of geometric orbifolds we refer to [Ra06, 13][SS]. Note that [Ra06, 13] says “Euclidean orbifold” for any flat orbifold.
The key class of examples of interest here are finite subgroups (see, e.g., \[BSS18, A.1\])

\[ G^{\text{ADE}} \subset SU(2) \simeq Sp(1) \simeq U(1,\mathbb{H}) \simeq S(\mathbb{H}) \]  

of the multiplicative group of unit norm elements \( q \in S(\mathbb{H}) \) in the vector space \( \mathbb{H} \simeq \mathbb{R}^4 \) of quaternions, and their defining 4-dimensional linear representation on this space (by left quaternion multiplication), which we denote by

\[ 4_{\mathbb{H}} \in RO(G^{\text{ADE}}). \]  

All of these, except the cyclic groups of odd order, contain the subgroup

\[ \mathbb{Z}_2^{\text{refl}} := \langle -1 \in S(\mathbb{H}) \rangle \subset G^{\text{ADE}} \]

generated by the quaternion \(-1 \in \mathbb{H}\). This acts on the 4-dimensional quatemionic representation \( 17 \) by point reflection at the origin, hence as the 4-dimensional sign representation

\[ \mathbb{R}^4_{\text{sgn}} \]

as illustrated for 2 of 4 dimensions in \( \text{Figure I} \).

**Euclidean G-Spaces.** The underlying Euclidean space of a linear \( G \)-representation \( V \) is of course a \( G \)-space, hence a \( \text{Euclidean G-space} \), which we suggestively denote by \( \mathbb{R}^V \):

\[ \text{linear } G \text{-representation } V \in RO(G) \Rightarrow \mathbb{R}^V \text{ Euclidean G-space} \]  

**Example 3.1 (Figure I).** With \( G = \mathbb{Z}_2 \) and \( V = 2_{\text{sgn}} \) its 2-dimensional sign representation, the Euclidean \( G \)-spaces \( \mathbb{R}^2_{\text{sgn}} \) is the Cartesian plane equipped with the action of point reflection at the origin:

\[ \mathbb{R}^2_{\text{sgn}} = \mathbb{R}^2 \subset \mathbb{R}^V, \]

as illustrated for 2 of 4 dimensions in \( \text{Figure I} \).

Notice that for \( V, W \in RO(G) \) two orthogonal linear \( G \)-representations, with \( V \oplus W \in RO(G) \) their direct sum representation, the Cartesian product of their Euclidean \( G \)-spaces \( 19 \) is the Euclidean \( G \)-space of their direct sum:

\[ \mathbb{R}^V \times \mathbb{R}^W \simeq \mathbb{R}^{V \oplus W}. \]  

**G-Representation spheres.** The one-point compactification \( 8 \) of a Euclidean space \( \mathbb{R}^V \) \( 19 \) becomes itself a \( G \)-space, with the point at infinity declared to be fixed by all group elements; this is called the representation sphere of \( V \) (see, e.g., \[BL17, 1.1.5\]):

\[ S^V := (\mathbb{R}^V)^{\text{cpt}} \simeq D(\mathbb{R}^V)/S(\mathbb{R}^V) \simeq S(\mathbb{R}^{1\text{triv}} \oplus V). \]  

**Example 3.2 (Figure J).** With \( G := \mathbb{Z}_2 \) the group of order 2 and \( 1_{\text{sgn}} \) its 1-dimensional sign-representation, the corresponding representation sphere \( 21 \) is the circle equipped with the \( \mathbb{Z}_2 \)-action that reflects across an equator:
Figure J – The \( \mathbb{Z}_2 \)-representation sphere of the 1-dimensional sign representation \( 1_{\text{sgn}} \) is the \( \mathbb{Z}_2 \)-space whose underlying topological space is the circle, and equipped with the \( \mathbb{Z}_2 \)-action that reflects points across the equator through 0 and the point at infinity.

\[
\begin{align*}
\mathbb{Z}_2, & \\
& \bigotimes
\end{align*}
\]

**G-Representation tori.** Similarly, consider the linear \( G \)-representation \( V \) such that \( G \subset \text{Iso}(\mathbb{R}^{\dim(V)}) \) is the point group of a crystallographic group \( C \) (see, e.g., [Far81]) of the underlying Euclidean space \( \mathbb{R}^{\dim(V)} \) with corresponding translational sub-lattice \( \mathbb{Z}^n \subset \text{Iso}(n) \) inside the Euclidean group in \( n = \dim(V) \) dimensions. This means we have an exact sequence of this form:

\[
\begin{array}{ccccccccc}
1 & \to & \mathbb{Z}^n & \to & C & \to & G & \to & 1 \\
1 & \to & \mathbb{R}^n & \to & \text{Iso}(n) & \to & \text{O}(n) & \to & 1
\end{array}
\]  \quad (22)

Then the corresponding torus \( T^n := \mathbb{R}^n / \mathbb{Z}^n \) inherits a \( G \)-action from \( \mathbb{R}^V \). We may call the resulting \( G \)-space the representation torus of \( V \). This is the type of \( G \)-space whose global quotients are toroidal orbifolds:

\[
V \in \text{RO}(G) \quad \Rightarrow \quad \bigotimes^G_{\mathbb{R}^V} \quad \Rightarrow \quad \bigotimes^G_{\mathbb{R}^V / \mathbb{Z}^n} \quad \Rightarrow \quad \bigotimes^G_{\mathbb{R}^V / \mathbb{Z}^n} \quad \Rightarrow \quad T^n / G.
\]  \quad (23)

**Example 3.3 (Figure K).** For \( G = \mathbb{Z}_4 \) the cyclic group of order 4 and \( 2_{\text{rot}} \) its 2-dimensional linear representation given by rotations around the origin by integer multiples of \( \pi/2 \), this action descends to the 2-torus quotient to give the representation torus \( T^2_{2_{\text{rot}}} \):

Figure K – The \( \mathbb{Z}_4 \)-representation torus (23) of the 2-dimensional rotational representation \( 2_{\text{rot}} \). The underlying topological space is the 2-torus \( T^2 = \mathbb{R}^2 / \mathbb{Z}^2 \), of which we show the canonical covering \( \mathbb{R}^2 \)-coordinate chart. Due to the coordinate identifications

\[
(\{x_1\}, [x_2]) = (\{x_1 + n\}, [x_2 + m]) \in T^2 = \mathbb{R}^2 / \mathbb{Z}^2
\]

the fixed point set (24) of the \( \mathbb{Z}_4 \)-subgroup has four points

\[
(\mathbb{R}^2)^{\mathbb{Z}_4} = \left\{(\{0\}, [0]), ([1/4], [1/4]), ([0], [1]), ([1/2], [0])\right\} \subset T^2.
\]

while that of the full group has two points

\[
(\mathbb{R}^2)^{\mathbb{Z}_4} = \left\{(\{0\}, [0]), ([1/2], [1/2])\right\} \subset T^2.
\]

**H-Fixed subspaces and isotropy groups.** For \( X \) a \( G \)-space and \( H \subset G \) any subgroup, the \( H \)-fixed subspace

\[
X^H := \left\{ x \in X \mid h \cdot x = x \text{ for all } h \in H \right\} \subset X
\]  \quad (24)

is the topological subspace of \( X \) on those points which are fixed by the action of \( H \). In particular, for \( 1 \subset G \) the trivial group we have \( X^1 = X \). We also write

\[
\text{Isotr}_X(G) := \left\{ \text{Stab}_G(x) \subset G \mid x \in X \right\}
\]  \quad (25)

for the set of isotropy subgroups of \( G \), hence those that appear as stabilizer groups of some point, namely as maximal subgroups fixing a point: \( \text{Stab}_G(x) := \left\{ g \in G \mid g \cdot x = x \right\} \subset G \). It is the isotropy subgroups (25), but not
necessarily the generic subgroups, which serve to filter a $G$-space in a non-degenerate way, since if one isotropy subgroup is strictly larger than another, then its fixed subspace is strictly smaller.

$$H_1 \subsetneq H_2 \in \text{Isotr}_X(G) \implies X^{H_2} \subsetneq X^{H_1}.$$  

**Example 3.4** (fixed subspaces of ADE-singularities). The non-trivial fixed subspaces of the Euclidean $G$-space of the quaternionic representation $4\mathbb{H}$ are all the singleton sets consisting of the origin:

$$(\mathbb{R}^4)^H = \begin{cases} \mathbb{R}^4 & \text{if } H = 1 \\ \mathbb{R}^0 & \text{otherwise.} \end{cases}$$

(26)

**Example 3.5** (Figure K). For $G = \mathbb{Z}_2$ and $n_{sgn}$ the $n$-dimensional sign representation, the corresponding representation torus has as $\mathbb{Z}_2$-fixed space the 0-dimensional space which is the set of points whose canonical coordinates are all either 0 mod $\mathbb{Z}$ or $\frac{1}{2}$ mod $\mathbb{Z}$:

$$T^{n_{sgn}} = (\mathbb{R}^n / \mathbb{Z}^n) = \{[0], [\frac{1}{2}]\}^n \subset \mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n.$$ (27)

**Example 3.6** (Kummer surface). The reflection ADE-action $\mathbb{R}^4_{\mathbb{H}}$ is clearly crystallographic. The orbifold presented by the corresponding representation torus is the Kummer surface (e.g. [BDP17, 5.5]). The cardinality of its fixed point set is (by Example 3.5)

$$|\left(\mathbb{T}^4_{\mathbb{H}}\right)_{\mathbb{Z}_2}| = |\{[0], [\frac{1}{2}]\}|^4 = 16.$$ (27)

**Residual action on fixed spaces.** There is a residual group action on any $H$-fixed subspace $X^H$ inherited from the $G$-action on all of $X$, with the residual group being the “Weyl group” $W_G(H)$:

$$W_G(H) := N_G(H)/H$$

(28)

which is the quotient group of the maximal subgroup $N_G(H) \subset G$ for which $H$ is a normal subgroup (the normalizer of $H$ in $G$) by $H$ itself. Thereby any $H$-fixed subspace becomes itself a $W_G(H)$-space:

$$X^H : \quad (H \subset G) \quad \mapsto \quad X^H.$$ (29)

Notice the two extreme cases of the Weyl group:

$$W_G(1) = G \quad \text{and} \quad W_G(G) = 1.$$ (30)

**Maps between $G$-spaces and their Elmendorf stages.** The relevant maps between $G$-spaces are continuous functions between the underlying spaces that are $G$-equivariant:

$$X^G \xrightarrow{f} Y^G \quad \iff \quad X \xrightarrow{f} Y \quad \text{such that } f(g \cdot x) = g \cdot f(x) \quad \text{for all } g \in G \text{ and all } x \in X.$$ (31)

This $G$-equivariance implies that $H$-fixed points are sent to $H$-fixed points, for every subgroup $H \subset G$, hence that every $G$-equivariant continuous function induces a system of plain continuous functions between $H$-fixed
point spaces (24), which are each equivariant with respect to the residual $W_G(H)$-action (28) and compatible with each other with respect to inclusions $H_i \subset H_j$ of subgroups:

$$\begin{array}{c}
G \xrightarrow{f} X \xrightarrow{f^H} Y \xrightarrow{W_G(H)} G
\end{array}$$

for all $H_i \subset H_j$ of subgroups.

We will refer to the component $f^H$ here as the Elmendorf stage labeled by $H$ [Bl17, 1.3] [HSS18, 3.1].

Finally, a $G$-homotopy between two $G$-equivariant functions $f_1, f_2$ (31)

$$\begin{array}{c}
G \xrightarrow{f_0} X \xrightarrow{\eta} Y \xrightarrow{f_1} G
\end{array}$$

is a homotopy $[0, 1] \times X \xrightarrow{\eta} X$ between the underlying continuous functions, hence such that $f_i = \eta(i, -)$, which is equivariant as a function on the product $G$-space $X \times [0, 1]$, where the $G$-action on the interval $[0, 1]$ is taken to be trivial.

3.1 Equivariant Hopf degree on spheres and Local tadpole cancellation

We discuss the unstable (Theorem 3.10) and the stabilized (Theorem 3.13) equivariant Hopf degree theorem for representation spheres, which characterizes equivariant Cohomotopy in compatible RO-degree (Def. 3.7 below), on Euclidean $G$-spaces and vanishing at infinity, hence of the vicinity of $G$-singularities inside flat Euclidean space (Def. 3.9 below). Using this we show (Prop. 3.14) that equivariant Cohomotopy implies the form of the local/twisted tadpole cancellation conditions from \textit{Table 1}, \textit{Table 2}.

3.1.1 Unstable equivariant Hopf degree

For stating the equivariant Hopf degree theorem, we need the following concept of compatible RO-degree for equivariant Cohomotopy. This condition is really a reflection of the structure of $J$-twisted Cohomotopy (as in [FSS19]) in its version on flat orbifolds, and as such is further developed in [SS].

\textbf{Definition 3.7 (Compatible RO-degree).} Given a $G$-space $X$ such that each $H$-fixed subspace $X^H$ (24) for isotropy groups $H \in \text{Isotr}_X(G)$ (25) admits the structure of an orientable manifold, we say that an orthogonal linear $G$-representation $V$ is a compatible RO-degree for equivariant Cohomotopy of $X$ if for each isotropy subgroup $H \in \text{Isotr}_X(G)$ (25) the following two conditions hold:

\begin{enumerate}
  \item \textbf{Compatible fixed space dimensions:} the dimension of the $H$-fixed subspace of $V$ equals that of the $H$-fixed subspace of $X$:
    \begin{equation}
    \dim(X^H) = \dim(V^H).
    \end{equation}
\end{enumerate}

\textsuperscript{4} These conditions are a specializations of the conditions stated in [D79] p. 212-213, streamlined here for our purpose.
(ii) **Compatible orientation behavior:** the action \((29)\) of an element \( [g] \in \Pi_G(H) \) \((28)\) on \(V^H \) is orientation preserving or reversing, respectively, precisely if it is so on \(X^H \)

\[
\text{orient} \left( \bigcup_{[g] \in \Pi_G(H)} X^H \right) = \text{orient} \left( \bigcup_{[g] \in \Pi_G(H)} (S^V)^H \right).
\]

(35)

**Example 3.8 (Compatible RO-degree for representation-spheres and -tori).** We observe that every real linear \(G\)-representation \(V\) is a compatible RO-degree (Def. 3.7)

(i) for the corresponding representation sphere \(S^V \) \((21)\);

(ii) and for the corresponding representation torus \(\mathbb{T}^V \) \((23)\).

If the latter exists, hence if \(G\) is the point group of a crystallographic group on \(\mathbb{R}^V \) \((22)\).

For brevity, we introduce the following terminology, following Table 5 for the situation in which we will now consider equivariant Cohomotopy in compatible RO-degree:

**Definition 3.9 (Cohomotopy of vicinity of the singularity).** Given a finite group \(G\) and an orthogonal linear \(G\)-representation \(V \in \text{RO}(G)\), we say that the **Cohomotopy of the vicinity of the singularity** is the unstable \(G\)-equivariant Cohomotopy \(\pi^V_G((\mathbb{R}^V)_{\text{cpt}}) = \pi^V_G(S^V)\)
in compatible RO-degree \(V \) (Def. 3.7 Example 3.8) of the Euclidean \(G\)-space \(\mathbb{R}^V \) \((19)\) and vanishing at infinity \((11)\), hence of the representation sphere \(S^V \) \((21)\) and preserving the point at infinity.

The key implication of the first clause \((34)\) on compatible RO-degrees is that each Elmendorf stage \(c^H \) \((32)\) of a \(G\)-equivariant Cohomotopy cocycle \(c\) is a cocycle in ordinary Cohomotopy \(\pi^{\dim(X)}(X)\) to which the ordinary Hopf degree theorem applies, either in its stable range \((13)\) or in the unstable range \((14)\):

\[
\begin{align*}
\text{deg}(f) &\in \mathbb{Z} \\
\text{deg}(c^H) &\in \mathbb{Z} \\
\text{deg}(c^K) &\in \mathbb{Z} \\
\text{deg}(c^I) &\in \{0, 1\}^{(X^I)}
\end{align*}
\]

(36)
Theorem 3.10 (Unstable equivariant Hopf degree theorem for representation spheres). The unstable Cohomotopy of the vicinity of a $G$-singularity $\mathbb{R}^V$ (Def. 3.9) is in bijection to the product set of one copy of the integers for each isotropy group $\{25\}$ with positive-dimensional fixed subspace $\text{Isotr}_{X}^{d_{\text{fix}} > 0}(G) \{24\}$, and one copy of $\{0,1\}$ if there is an isotropy group with 0-dimensional fixed subspace $\text{Isotr}_{X}^{d_{\text{fix}} = 0}(G)$ (which is then necessarily unique and, in fact, the group $G$ itself):

$$\pi^V_G\left( (\mathbb{R}^V)^{\text{cpt}} \right) \xrightarrow{(H \to N_H) \cong} \mathbb{Z}^{\text{Isotr}_{X}^{d_{\text{fix}} > 0}(G)} \times \{0,1\}^{\text{Isotr}_{X}^{d_{\text{fix}} = 0}(G)}, \quad (37)$$

where, for $H \in \text{Isotr}_{X}^{d_{\text{fix}} > 0}(G)$, the ordinary Hopf degree at Elmendorf stage $H \{36\}$ is of the form

$$\deg(c^H) = \phi_H\left( \{ \deg(K) \mid K \supseteq H \in \text{Isotr}_X(G) \}\right) - N_H \cdot |(W_G(H))| \in \mathbb{Z}. \quad (38)$$

The isomorphism $\{37\}$ is exhibited by sending an equivariant Cohomotopy cocycle to the sequence of the integers $N_H$ from $\{38\}$ in positive fixed subspace dimensions, together with possibly the choice of an element of $\{0,1\}$, which is the unstable Hopf degree in dimension $0 \{14\}$, at Elmendorf stage $G$ (if $\dim(V^G) = 0$).

Proof. In the special case that no subgroup $H \subset G$ has a fixed subspace of vanishing dimension, this is $\{D79\}$ Theorem 8.4.1] (the assumption of positive dimension is made “for simplicity” in $\{D79\}$ middle of p. 212). Hence we just need to convince ourselves that the proof given there generalizes: in the present case of representation spheres, the only possible 0-dimensional fixed subspace is the 0-sphere. Hence we need to consider the case that $(S^V)^G = S^0$.

To generalize the inductive argument in $\{D79\}$ p. 214 to this case, we just need to see that every function $(S^V)^G \to (S^V)^G$ extends to a $W_G(H)$-equivariant function $(S^V)^H \to (S^V)^H$ on a next higher Elmendorf stage $H$. This holds in the present case: every function from $S^0 = \{0,\infty\}$ to itself (as in $\{\text{Figure H}\}$) readily extends even to a $G$-equivariant function $S^V \to S^V$, and by assumption of vanishing at infinity $\{11\}$ one of exactly two extensions will work, namely either the identity function or the function constant on $\infty \in S^V$:

$$\begin{array}{ccc}
S^0 \xrightarrow{\text{id}_{S^0}} S^0 & \iff & S^V \xrightarrow{\text{id}^V} S^V \\
S^0 \xrightarrow{\text{const}_{S^0}} S^0 & \iff & S^V \xrightarrow{\text{const}^V} S^V
\end{array} \quad (39)$$

From this induction forward, the proof of $\{D79\}$ 8.4.1 applies verbatim and shows that on top of this initial Hopf degree number of -1 (a charge at $0 \in S^0$) or 0 (no charge at $0 \in S^0$) there may now be further $N_H \cdot |W_G(H)|$-worth of Hopf degree at the next higher Elmendorf stage $H$, and so on.

Example 3.11 ($\mathbb{Z}_2$-equivariant Cohomotopy). Consider the $\mathbb{Z}_2$-equivariant Cohomotopy vanishing at infinity $\{11\}$ of the $n$-dimensional Euclidean orientifold $\mathbb{R}^n_{\text{or}} \{19\}$ underlying the $n$-dimensional sign representation $n_{\text{sgn}}$ (as in $\{\text{Figure J}\}$) hence the equivariant Cohomotopy of the representation sphere $S^{n_{\text{sgn}}} \{21\}$ (as in $\{\text{Figure J}\}$) in compatible RO-degree $n_{\text{sgn}}$ (by Example 3.8). In this case, the unstable equivariant Hopf degree theorem 3.10 says, when translated to a geometric situation via the unstable Pontrjagin-Thom theorem, that

(i) there either is, or is not, a single charge sitting at the finite fixed point $0 \in S^{n_{\text{sgn}}}$, corresponding, with $\{39\}$, to an offset of $-1$ or 0, respectively, in $\{38\}$;

(ii) in addition, there is any integer number (the $N_1 \in \mathbb{N}$ in $\{38\}$) of orientifold mirror pairs (since $|W_{\mathbb{Z}_2}(1)| = |\mathbb{Z}_2| = 2$, by $\{30\}$) of charges floating in the vicinity.
Figure L – The $\mathbb{Z}_2$-Equivariant Cohomotopy of Euclidean $n$-orientifolds vanishing at infinity according to the unstable equivariant Hopf degree theorem [3.10] applied to sign-representation spheres (Figure J) and visualized by the corresponding configurations of charged points via the unstable Pontrjagin-Thom construction (7), in equivariant enhancement of the situation show in Figure E The same situation, just crossed with an interval, appears in the application to M5/MO5 charge in Figure V.

It is possible and instructive to make this fully explicit in the simple special case of the 1-dimensional sign representation, where the statement of the equivariant Hopf degree theorem [3.10] may be found in elementary terms: It is readily checked that all the continuous functions $c^1: S^1 \to S^1$ which start at either of $0, \infty \in S^1$ and wind around at constant parameter speed are $\mathbb{Z}_2$-equivariant, hence are Elmendorf stages (32) of $\mathbb{Z}_2$-equivariant cocycles $c$:

$$\left(\mathbb{R}^{1,\text{sgn}}\right)^\text{cpt} \simeq S^1_{\text{sgn}} \xrightarrow{c} S^1_{\text{sgn}}.$$ (40)

If such a function vanishes at infinity (11), in that it takes $\infty \mapsto \infty$ as shown in Figure L, then we have one of two cases:

(i) either $c^1$ winds an odd number of times, so that (38) reads:

$$\deg(c^1) = \text{offset} \cdot 1 - N_1 \cdot 2,$$

in which case it satisfies $c^1(0) = 0$, so that under the PT-theorem (7) there is precisely one charge at the singular fixed point, together with the even integer number $2 \cdot N_1 \in \mathbb{Z}$ of net charges in its “vicinity” (namely: away from infinity) which are arranged in $\mathbb{Z}_2$-mirror pairs, due to the $\mathbb{Z}_2$-equivariance of $c$; this is what is shown on the left of Figure L.

(ii) or $c^1$ winds an even number of times so that (38) reads:

$$\deg(c^1) = \text{offset} \cdot 0 - N_1 \cdot 2,$$

in which case it satisfies $c^1(0) = \infty$, so that under the PT-theorem (7) there is no charge at the singular fixed point, but a net even integer number $2 \cdot N_1 \in \mathbb{Z}$ of charges in its vicinity, as before.

Remark 3.12 (Number of branes and offset). Notice that:

(i) For $N_1 = 0$ (no branes) this is the situation of (39): either there is a non-vanishing charge associated with the singular fixed point (O-plane charge), or not.
(ii) Furthermore, if there is, it is either +1 or -1, so that in general the charge associated with the singular fixed point is in \( \{0, \pm 1\} \), as befits O-plane charge according to

(iii) The offset is relevant only modulo 2, so that we could have chosen an offset of +1 instead of as −1 in the first case. This choice just fixes the sign convention for D-brane/O-plane charge.

**Characterizing the brane content around a singularity.** In the above example in RO-degree \( 1_{\text{sgn}} \), it is clear that the configurations of branes implied by the unstable equivariant Hopf degree theorem (Theorem 3.10) appear in multiples of the regular \( G \)-set around a fixed O-plane charge stuck in the singularity, as illustrated in

In order to prove that this is the case generally, we now turn to the stabilized equivariant Hopf degree theorem (Theorem 3.13 below), which concretely characterizes the (virtual) \( G \)-sets of branes that may appear classified by equivariant Cohomotopy.

### 3.1.2 Stable equivariant Hopf degree

In a homotopy-theoretic incarnation of perturbation theory, we may approximate unstable equivariant Cohomotopy (Theorem 3.13) by its homotopically linearized, namely stabilized (see (BMSS18)) version. We briefly recall the basics of stable equivariant Cohomotopy in RO-degree 0 ([Seg71], see [D79, 7.6 & 8.5][Lu05]) before applying this in Theorem 3.13 and Prop. 3.14 below.

**Equivariant suspension.** For \( V, W \in \text{RO}(G) \) two orthogonal linear \( G \)-representations, and for

\[
\left[ S^V \simeq (\mathbb{R}^V)^{\text{cpt}} \xrightarrow{c} (\mathbb{R}^V)^{\text{cpt}} \simeq S^V \right] \in \pi_G( (\mathbb{R}^V)^{\text{cpt}} )
\]

the class of a cocycle in the equivariant Cohomotopy (3) of the Euclidean \( G \)-space \( \mathbb{R}^V \) in compatible RO-degree \( V \) (Example 3.3) and vanishing at infinity (11), we obtain the class of a cocycle vanishing at infinity on the product \( G \)-space \( \mathbb{R}^V \oplus \mathbb{R}^W \) in compatible degree \( V \oplus W \), simply by forming the Cartesian product of \( c \) with the identity on \( \mathbb{R}^W \). This is the *equivariant suspension* of \( c \) by RO-degree \( W \):

\[
\sum^W c \quad := \quad \left[ \left( \mathbb{R}^V \times \mathbb{R}^W \right)^{\text{cpt}} \xrightarrow{c \times \text{id}} \left( \mathbb{R}^V \times \mathbb{R}^W \right)^{\text{cpt}} \right] \in \pi_G( (\mathbb{R}^V \oplus \mathbb{R}^W)^{\text{cpt}} ). \tag{41}
\]

Note that this reduces to the ordinary suspension operation eq. (15) for \( G = 1 \) the trivial group, hence for RO-degrees \( n_{\text{triv}} = n \). These equivariant suspension operations form a directed system on the collection of equivariant Cohomotopy sets (3), indexed by inclusions of orthogonal linear representations:

\[
(V \leftarrow V \oplus W) \quad \mapsto \quad \left( \pi^V_G( (\mathbb{R}^V)^{\text{cpt}} ) \xrightarrow{\sum^W} \pi^V_G( (\mathbb{R}^V \oplus \mathbb{R}^W)^{\text{cpt}} ) \right). \tag{42}
\]

**Stable equivariant Cohomotopy.** As a consequence of the above, one may consider the union of all unstable equivariant Cohomotopy sets of representation spheres in all compatible degrees, with respect to the identifications along the equivariant suspension maps (42) (the colimit of this system):

\[
\pi^V_G( (\mathbb{R}^V)^{\text{cpt}} ) \xrightarrow{\sum^W} \text{unstable equivariant Cohomotopy set in compatible RO-degree } V \quad \lim_{V \to V} \pi^V_G( (\mathbb{R}^V)^{\text{cpt}} ) = \pi^0_G \quad \text{union (colimit) of all unstable Cohomotopy sets in compatible RO-degrees}
\]

Since the resulting union/colimit is, by construction, stable under taking further such suspensions, this is called the *stable equivariant Cohomotopy in degree 0* ([Seg71, p. 1], see [Lu05, p. 9-10]) also called the 0th stable \( G \)-equivariant homotopy group of spheres or the *\( G \)-equivariant stable 0-stem* or similar (see [May96 IX.2][Schw3]). Notice that here the stable RO-degree is the formal difference of the unstable RO-degree by the RO-degree of
the singularity, so that vanishing stable RO-degree is another expression of compatibility of unstable degree, in the sense of Example 3.8

\[ \pi^V(S^W) = [S^V, S^W] = \pi^W(S^V) \sum W \rightarrow S^W - V. \]

Extensive computation of stable \( \mathbb{Z}_2 \)-equivariant Cohomotopy of representation spheres in non-vanishing RO-degrees, i.e., computation of the abelian groups \( \mathbb{S}^k_{\text{equiv}} + \mathbb{H}^{k-2}_{\text{equiv}} \), is due to \[\text{[Ada74, II.6]}\]; see also \[\text{[HIr82]}\] [\text{[Ir82]}]. Under \text{Hypothesis H} these groups are relevant for tadpole cancellation with branes wrapping orientifolds singularities non-transversally. This is of interest to us but goes beyond the scope of this article.

**Equivalence to the Burnside ring.** Due to the stabilization, the stable equivariant Cohomotopy set (43) has the structure of an abelian group, in fact the structure of a ring. As such, it is isomorphic to the **Burnside ring** \( A(G) \) of virtual \( G \)-sets (see \[\text{[BSS18, 1]}\]):

\[ \mathbb{S}^0_G \simeq A(G) = \{ \text{Virtual } G\text{-sets} \}. \tag{44} \]

This result is due to \[\text{[Seg71, p. 2]}\]; see \[\text{[D79]}\] 7.6.7 & 8.5.1 [\text{[Lu05]} 1.13], we highlight its geometric meaning below; see \[\text{[Figure M]}\]. This is a non-linear analog (more precisely, the analog over the absolute base “field” \( \mathbb{F} \)) of the fact that the equivariant K-theory in degree 0 is the representation ring of virtual linear \( G \)-representations over the field of real numbers (see, e.g., \[\text{[Gr05, 3]}\]):

\[ \text{KO}^0_G \simeq \text{RO}(G) = \{ \text{Virtual } G\text{-representations} \}. \tag{45} \]

In fact, the operation \( S \mapsto \mathbb{R}[S] \) that sends a (virtual) \( G \)-set \( S \in A(G) \) to its linearization, hence to its linear span \( \mathbb{R}[S] \), hence to the (virtual) permutation representation that it induces (see \[\text{[D79, 4]}\] [\text{[BSS18, 1]}]), is a ring homomorphism from the Burnside ring to the representation ring. Furthermore, it exhibits the value on the point of unique multiplicative morphism from equivariant stable Cohomotopy theory to equivariant K-theory, called the **Boardman homomorphism** \[\text{[Ada74]}\] II.6), which is the Hurewicz homomorphism generalized from ordinary cohomology to generalized cohomology theories:

\[ \text{S}^0_G \xrightarrow{\beta} \text{KO}^0_G \]

\[ \mathbb{S}^0_G \xrightarrow{\beta} \text{KO}^0_G \]

\[ \text{Burnside ring} \xrightarrow{\beta} \text{equivariant K-theory} \]

\[ \text{Boardman homomorphism} \]

\[ \text{Burnside ring} \xrightarrow{\beta} \text{equivariant K-theory} \]

In summary, the composite of the stabilization morphism (43) with the isomorphism (44) to the Burnside ring explicitly extracts from any cocycle \( c \) in unstable equivariant Cohomotopy a virtual \( G \)-set \{branes\}, hence a virtual \( G \)-permutation representation \( \mathbb{R}[\{\text{branes}\}] \). The following theorem explicitly identifies this \( G \)-set \{branes\} in terms of the Elmendorf stage-wise Hopf degrees of the cocycle \( c \); see \[\text{Figure M}\] below for illustration.

**Theorem 3.13 (Stabilized equivariant Hopf degree theorem for representation spheres).** Consider a cocycle \( c \) in unstable Cohomotopy of the vicinity of a \( G \)-singularity \( \mathbb{R}^V \) (Def. 3.9). Its image under stabilization in equivariant stable Cohomotopy (43) is, under the identification (44) with the Burnside ring, precisely that virtual \( G \)-set \{branes\} \( \in A(G) \) whose net number of \( H \)-fixed points (“Burnside marks”, see \[\text{[BSS18, 1]}\]), equals the Hopf degree of \( c \) at any Elmendorf stage \( H \in \text{Isotr}_X(G) \) (36). Hence if \( H = \langle g \rangle \) is a cyclic group generated by an element \( g \in G \), this number also equals the character value at \( g \) (i.e., the trace of the linear action of \( g \)) on the linear representation \( \mathbb{R}[\{\text{branes}\}] \):
Proposition 3.14 which consists of all the virtual representations of the form $G$ by, in particular, the trivial 1-dimensional representation, have the same image $\Sigma$

For ADE-singularities

Proof. By (26), the representation $4_{\mathbb{H}}$ is such that every non-trivial subgroup $H \subset G$ has a 0-dimensional fixed space:

$$\dim \left( \pi^G_{\text{4H}} \right)^H = \begin{cases} 4 & \text{if } H = 1 \\ 0 & \text{otherwise.} \end{cases}$$

This means that for $c \in \pi^G_{\text{4H}}$ an equivariant Cohomotopy cocycle in the vicinity of an ADE-singularity, its only Elmendorf stage-wise Hopf degree (36) in positive dimension is, by equation (38) in Theorem 3.10 of the form

$$\deg(c^H) = Q_{\text{Opla}} - N_1 \cdot |G|,$$

**Proof.** For the case that all fixed subspace dimensions are positive, this is essentially the statement of [D79, 8.5.1], after unwinding the definitions there (see [D79, p. 190]). We just need to see that the statement generalizes as claimed to the case where the full fixed subspace $(S^V)^G = S^0$ is the 0-sphere. But, under stabilization map $\Sigma^\infty$ (43), the image of a Cohomotopy cocycle $S^V \xrightarrow{c} S^V$ and its equivariant suspension $S^\Sigma^\infty_{\text{inv}} \xrightarrow{\Sigma^\infty_{\text{inv}}} S^\Sigma^\infty_{\text{inv}}$ by, in particular, the trivial 1-dimensional representation, have the same image $\Sigma^\infty(c) \simeq \Sigma^\infty(\Sigma^\infty_{\text{inv}}c)$. Now to the suspended cocycle $\Sigma^\infty_{\text{inv}}$ the theorem [D79, 8.5.1] applies, and hence the claim follows from the fact (15) that the unstable Hopf degree in $\{0, 1\}$ injects under suspension into the stable Hopf degrees:

$$\left( S^V \xrightarrow{G} S^V \right)^G = S^0 \simeq \left( S^V \right)^G \in \{0, 1\} \hookrightarrow \mathbb{Z}.$$
where we used the fact that $W_G(1) = G$. But, by Theorem 3.13, this implies that the virtual $G$-set $\{\text{branes}\}$ of branes corresponding to $c$ has the following Burnside marks

$$\{\text{branes}\}_H' = \begin{cases} Q_{\text{Opla}} - N_1 \cdot |G| & \text{if } H = 1 \\ Q_{\text{Opla}} & \text{otherwise,} \end{cases}$$

hence that the corresponding permutation representation of branes has the following characters:

$$\text{Tr}_g(\mathbb{R}\{\text{branes}\}) = \begin{cases} Q_{\text{Opla}} & \text{if } g = e - N_1 \cdot |G| \\ Q_{\text{Opla}} & \text{otherwise}. \end{cases}$$

The unique $G$-set/$G$-representation with these Burnside marks/characters is the sum of the $N_1$-fold multiple of the regular $G$-set/$G$-representation and the $Q_{\text{Opla}}$-fold multiple of the trivial representation (see Figure M):

$$\mathbb{R}\{\{\text{branes}\}\} = Q_{\text{Opla}} \cdot 1_{\text{triv}} - N_1 \cdot k_{\text{reg}},$$

The situation is illustrated by Figure M:

---

**Figure M** – Virtual $G$-representations of brane configurations classified by equivariant Cohomotopy in the vicinity of ADE-singularities (Def. 3.9), according to Prop. 3.14, following Theorem 3.10 and Theorem 3.13. The results reproduces the form of the local/twisted tadpole cancellation conditions in Table 1, Table 2. Shown is a situation for $G = \mathbb{Z}_4$ and $V = 2_{\text{rot}}$ as in Figure K.

### 3.2 Equivariant Hopf degree on tori and Global tadpole cancellation

We now globalize the characterization of equivariant Cohomotopy from the vicinity of singular fixed points to compact toroidal orbifolds, in Theorem 3.17 below. Prop. 3.18 below shows that the two are closely related, implying that the local/twisted tadpole cancellation carries over to toroidal orbifolds. Then we informally discuss the enhancement of unstable equivariant Cohomotopy to a super-differential cohomology theory (55) and show that its implications (58) on the underlying equivariant Cohomotopy enforce the form of the global/untwisted tadpole cancellation conditions.

**Globalizing from Euclidean orientifolds to toroidal orientifolds.** In §3.2 we discussed the characterization of equivariant Cohomotopy in the vicinity of singularities (according to Table 5). We may globalize this to compact toroidal orientifolds by applying this local construction in the vicinity of each singularity, using that the condition of “vanishing at infinity” (11) with respect to any one singularity means that the local constructions may be glued together. This local-to-global construction is indicated in Figure N.
Definition 3.15. We say that a $G$-space $X$ has well-isolated singularities if all the minimal subgroups with 0-dimensional fixed subspaces (24) are in the center of $G$, i.e. if the following condition holds:

$$H \subset G \text{ minimal such that } \dim(X^H) = 0 \Rightarrow H \subset \text{Center}(G).$$  

(49)

Example 3.16. The ADE-singularities (Table 5) with well-isolated fixed points in the sense of (49) are all those in the $A$-series, as well as the generalized quaternionic ones in the $D$-series – see Table 6. This is because, for ADE-singularities, all non-trivial subgroups have 0-dimensional fixed space (26), so that here the condition of well-isolated singularities (49) requires that all non-trivial minimal elements in the subgroup lattice be in the center. This is trivially true for the cyclic groups in the $A$-series, since they are abelian. For the generalized quaternionic groups in the $D$-series there is in fact a unique minimal non-trivial subgroup, and it in fact it is always the orientifold action $H_{\text{min}} = \mathbb{Z}_2$ which coincides with the center, as shown for the first few cases in Table 6.

The point of the notion of well-isolated fixed points (49) is that it is sufficient to guarantee that the action of the full group restricts to the union of the 0-dimensional fixed subspaces, since then

$$H \cdot x_{\text{fixed}} = x_{\text{fixed}} \Rightarrow H \cdot (g \cdot x_{\text{fixed}}) = (H \cdot g) \cdot x_{\text{fixed}} = (g \cdot H) \cdot x_{\text{fixed}} = g \cdot (H \cdot x_{\text{fixed}}) = g \cdot x_{\text{fixed}},$$

(50)

for all $g \in G$. Hence, with (49), the quotient set

$$\text{IsolSingPts}_G(X) := \left( \bigcup_{H \subset G, \dim(X^H) = 0} X^H \right) / G$$

(51)

exists and is the set of isolated singular points in the orbifold $X \sslash G$.  

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where, for $H \in \text{Isotr}^{\text{dis} > 0}_X(G)$, the ordinary Hopf degree at Elmendorf stage $H$ (56) is of the form

\[
\deg(\ell^H) = \phi_H\left(\{\deg(K) \mid K \supseteq H \in \text{Isotr}_X(G)\}\right) - N_H \cdot |W_G(H)| \in \mathbb{Z}. \quad (53)
\]

The isomorphism (52) is exhibited by sending an equivariant Cohomotopy cocycle to the sequence of the integers $N_H$ from (53) in positive fixed subspace dimensions, together with the collection of elements in $\{0,1\}$, which are the unstable Hopf degrees in dimension 0 (14), at Elmendorf stage $G$ at each one of the well-isolated singularities.
Proof. In the special case when no subgroup $H \subset G$ has a fixed subspace of vanishing dimension, this is \cite[Theorem 8.4.1]{D79}, where the assumption of positive dimension is made “for simplicity” in \cite[middle of p. 212]{D79}. Hence we just need to convince ourselves that the proof given there generalizes.

To that end, assume that $\dim(V^G) = 0$. To generalize the inductive argument in \cite[p. 214]{D79} to this case, we just need to see that every $G$-invariant function on the isolated fixed \cite[(51)\]

$$\text{IsolSingPts}_G(X) \xrightarrow{\eta} S^0$$

extends to a $W_G(K)$-equivariant function $(S^V)^K \to (S^V)^K$ on the next higher Elmendorf stage $K \in \text{Isotr}_{\text{fix}}^{d_\infty > 0}(G)$. For this, consider a $G$-equivariant tubular neighborhood around the well-isolated fixed points. This is guaranteed to exist on general grounds by the equivariant tubular neighborhood theorem, since, by assumption \cite[(49)\], the set of points (in the bottom left of \cite[(54)\]) is an equivariant (and of course closed) subspace, by \cite[(50)\]. In fact, in the present specific situation of global orthogonal linear actions on a Euclidean space we obtain a concrete such equivariant tubular neighborhood by forming the union of Euclidean open balls of radius $\epsilon$ around each of the points, for any small enough positive radius $\epsilon$. This kind of tubular neighborhood is indicated by the collection of dashed circles in \textit{Figure A} and \textit{Figure N}. Given this or any choice of equivariant tubular neighborhood, the extensions \cite[(39)\] in the proof of Theorem 3.10 apply to the vicinity of any one fixed points. This is a choice in \{0, 1\} for each element in $\text{IsolSingPts}_G(X)$ \cite[(51)\], hence in total is the choice of an element in $\{0, 1\}^{\text{IsolSingPts}_G(X)}$, as it appears in \cite[(37)\]. Since all these local extensions to the vicinity of any of the singularities “vanish at infinity” \cite[(11)\], i.e., at some distance $> \epsilon$ from any and all of the well-isolated fixed points, they may jointly be further extended to a global cocycle $\mathbb{T}^V \xrightarrow{c} S^V$ by declaring that $c$ sends every other point in $\mathbb{T}^V$ outside the given tubular neighborhood to $\infty \in S^V$ (shown in \textit{Figure N}). From this induction onwards, the proof of \cite[8.4.1]{D79} applies verbatim and shows that on top of this initial Hopf degree choice in \{0, 1\} $\text{IsolSingPts}_G(X)$ there may now be further $N_H \cdot |W_G(H)|$-worth of Hopf degree at the next higher Elmendorf stage $H$, and so on.

\section*{Stable equivariant Hopf degree of representation tori.} Note that the unstable equivariant Hopf degrees of representation spheres (Theorem 3.10) and of representation tori (Theorem 3.17) have the same form, \cite[(37)\] and \cite[(52)\], respectively, away from the unstable Hopf degrees in vanishing fixed space dimensions. It follows immediately that, up to equivariant homotopy, all brane charge may be thought of as concentrated in the vicinity of the “central” singularity (see \textit{Figure 1}):

\begin{align*}
\begin{pmatrix}
\pi_G^V \left( \left( \mathbb{R}^V \right)^{\mathbb{C}^P} \right) \xrightarrow{\pi_G^V \left( \left( \mathbb{C}^P \right)^{\mathbb{R}^V} \right) c} \\
\mathbb{R}^V \xrightarrow{x} S^V
\end{pmatrix} & \xrightarrow{i_*} \\
\begin{pmatrix}
\pi_G^V \left( \left( \mathbb{R}^V \right)^{\mathbb{R}^V} \right) \xrightarrow{\pi_G^V \left( \left( \mathbb{R}^V \right)^{\mathbb{R}^V} \right) c} \\
\mathbb{R}^V \xrightarrow{x} S^V
\end{pmatrix} & \xrightarrow{i_*} \\
\begin{pmatrix}
\mathbb{T}^V \xrightarrow{\mathbb{T}^V} S^V \\
\mathbb{T}^V \xrightarrow{x} S^V
\end{pmatrix} & \xrightarrow{i_*}
\end{align*}

is an isomorphism on Hopf degrees at Elmendorf stages $H > 0$ of non-vanishing fixed space dimension and an injection on the unstable Hopf degree set at Elmendorf stages with vanishing fixed subspace dimension (if any):

$$i_* : \begin{cases}
N_{H > 0} \hookrightarrow N_{H = 0} \\
N_{H > 0} \hookrightarrow N_{H > 0}
\end{cases}$$

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This is illustrated by Figure I:

\[ \pi_4^{eq}(\mathbb{R}^4_{cpt}) \xrightarrow{i_*} \pi_4^{eq}(\mathbb{R}^4_{+}) \]

Figure O – Pushforward in equivariant Cohomotopy from the vicinity of a singularity to the full toroidal orientifold is an isomorphism on brane charges and an injection on O-plane charges, by Prop. 3.18. Shown is a case with \( G = \mathbb{Z}_4 \), as in Figure M.

Local tadpole cancellation in toroidal ADE-orientifolds. Under the identification from Prop. 3.18 the stabilized equivariant Hopf degree theorem for representation spheres (Theorem 3.13) applies also to representation tori, and hence so does Prop. 3.14, showing now for the case of toroidal orbifolds with ADE-singularities that the brane charges classified by equivariant Cohomotopy are necessarily multiples of the regular representation. This result is visualized in Figure P.

Figure P – Local/twisted tadpole cancellation in a toroidal ADE-orientifold is enforced by equivariant Cohomotopy according to Prop. 3.18 which reduces to the situation in the vicinity of a single singularity, as in Figure O. Shown is a case with \( G = \mathbb{Z}_4 \) as in Figure M. This is the local/twisted tadpole cancellation in toroidal ADE-orientifolds according to Table 1 and Table 2.

Global/untwisted tadpole cancellation from super-differential Cohomotopy. This concludes our discussion of local tadpole cancellation in global (i.e. toroidal) ADE-orientifolds implied by C-field charge quantization in equivariant Cohomotopy. Finally, we turn to discuss how the global/untwisted tadpole cancellation condition on toroidal orbifolds follows from charge quantization in super-differential equivariant Cohomotopy. We state the concrete condition below in (58), but first we explain how this condition arises from super-differential refinement:
Super-differential enhancement of unstable equivariant Cohomotopy theory. Given any generalized cohomology theory for charge quantization, it is its corresponding enhancement to a differential cohomology theory which classifies not just the topological soliton/instanton sectors, but the actual geometric higher gauge field content, hence including the flux densities. For stable/abelian cohomology theories this is discussed for instance in [Fre00] [Bu12], while in the broader generality of unstable/non-abelian cohomology theories this is discussed in [FSS10] [SSS12] [FSS12] [FSS15]. For example, ordinary integral cohomology theory $BU(1)$ classifies magnetic charge sectors, but it is its differential cohomology enhancement $BU(1)_{\text{conn}}$ (Deligne cohomology) which is the universal moduli for actual electromagnetic field configurations. Similarly, plain (twisted) K-theory $KU$ and $KO$ classifies topological RR-charge sectors, but it is differential K-theory which classifies the actual RR-fields; see [GS17] [GS19a] [GS19b] [FSS10] [SSS12] [FSS12] [FSS15].

Hence with Hypothesis we are ultimately to consider the refinement of ADE-equivariant Cohomotopy theory $\pi^G_{ADE}$, discussed so far, to some differential equivariant Cohomotopy theory, denoted $(\pi^G_{ADE})_{\text{conn}}$ and characterized as completing a homotopy pullback diagram of geometric unstable cohomotopy theories of the following form:

![Diagram](https://example.com/diagram.png)

Discussing this construction in detail requires invoking concepts from $\infty$-stacks and $\mathcal{L}_{\infty}$-algebroids [FSS10] [SSS12], as well as their application to super-geometric orbifolds [SS], which is beyond the scope of this article. However, for the present purpose of seeing the global tadpole cancellation condition arise, all that matters are the following implications of super-differential refinement, which we make explicit by themselves:

**Rational flux constraints from equivariant enhancement of M2/M5-cocycle.** The homotopy pullback construction (55) amounts to equipping the rationalization of cocycles in plain unstable equivariant Cohomotopy (3) with equivalences (connection data) to prescribed flux super-forms in super-rational equivariant Cohomotopy $H$-equivariant enhancements (HSS18 5) of M2/M5-brane super WZW-terms jointly regarded as a cocycle in super-rational 4-Cohomotopy $\Omega_G((-), S^4)$ (57):

$$
\begin{array}{c}
\mathbb{R}^{10,1|32} \\
\mathcal{L}^4
\end{array}
\xrightarrow{\mu_{M2/M5}}
\begin{array}{c}
\mathcal{L}^4
\end{array}
\xrightarrow{\text{rationalized 4-sphere}}
\begin{array}{c}
\mathcal{L}^4
\end{array}
$$

(56)

Specifically, for $G_{ADE}$-equivariance (16) at ADE-singularities $4_H$ (17), a choice of equivariant extension of this cocycle is a choice of extension to an Elmendorf-stage diagram as in (56) – see [HSS18] (5).

$$
\begin{array}{c}
\mathbb{R}^{10,1|32} \\
\mathcal{L}^4
\end{array}
\xrightarrow{\mu_{M2/M5}}
\begin{array}{c}
\mathcal{L}^4
\end{array}
\xrightarrow{\text{rationalized 4-sphere}}
\begin{array}{c}
\mathcal{L}^4
\end{array}
$$

(57)

For more general actions this involves extension to a functor on the orbit category; see [HSS18] Lemma 5.4].
This involves a binary choice at lowest (and hence any other, by Example 3.4) Elmendorf stage. The homotopy in the diagram (55) enforces this local choice of rationalized flux globally onto the rationalized fluxes of the equivariant Cohomotopy cocycles. This has two effects:

1. Super-differential enhancement at global Elmendorf stage implies vanishing total flux. Note the M2/M5-brane super-cocycle $\mu_{M2/M5}$ appearing at global Elmendorf stage in (57) is purely supergeometric, with vanishing bosonic flux ($\mu_{M2/M5}|_{\psi=0} = 0$ by (56)). Also, the infinitesimal fermionic component $\psi$ does not contribute to the topology seen by plain equivariant Cohomotopy (see [SS] for details). Hence the homotopy in (55) forces the underlying classes in plain equivariant Cohomotopy to be pure torsion at global Elmendorf stage. But, since in compatible RO-degree (as in Example 3.8) the Hopf degree theorem (13) implies non-torsion Cohomotopy groups at all positive Elmendorf stages (36), this means that super-differential refinement (55) of equivariant Cohomotopy in compatible RO-degree enforces vanishing Hopf degrees at global Elmendorf stage $H = 1$ (36).

Explicitly, this means that the super-differential enhancement (55) forces the underlying plain equivariant Cohomotopy cocycles of ADE-orientifolds in compatible RO-degree to be in the kernel of the forgetful map $(-)^1$ (36) from equivariant to ordinary Cohomotopy, which projects out the global Elmendorf stage at $H = 1$:

$$\pi^n_{Gade} \left( \left( T^4_{\text{H}} \right) + \right) \quad \text{unstable equivariant Cohomotopy } \left( \text{admitting super-differential refinement} \right)$$

$$\rightarrow \{0\} \quad \text{kernel}$$

$$\pi^n_{Gade} \left( \left( T^4_{\text{H}} \right) + \right) \quad \text{equivariant Cohomotopy of toroidal orbifold with ADE-singularities}$$

$$\rightarrow |Q_{\text{tot}}| = (-1)^1 \quad \text{project out global charge = Hopf degree at global Elmendorf stage}$$

$$\rightarrow \pi^1 \left( \left( T^4_{\text{H}} \right) + \right) \quad \text{plain Cohomotopy of plain 4-torus} \cong \mathbb{Z} \quad \text{global Hopf degree = net brane/O-plane charge}$$

It is now immediate, from Theorem 3.10 and Theorem 3.17 that this enforces the condition of vanishing net brane/O-plane charge, precisely in the form of the global/untwisted tadpole cancellation condition from Table 1 and Table 2. This is illustrated by the following figure:

**Figure Q – Global/untwisted tadpole cancellation as vanishing Cohomotopy charge at global Elmendorf stage (55), implied by enhancing unstable equivariant Cohomotopy theory to a super-differential cohomology theory (55). This yields the form of the global/untwisted tadpole conditions in Table 1 Table 2.**
The analogous situation for the group $G = \mathbb{Z}_4$ gives exactly what was shown in the motivating Figure A.

2. Super-differential enhancement at lower Elmendorf stage implies choice of O-plane charge. The globalization via (58) of the lower $S^0$-valued Elmendorf stage in the equivariantized M2/M5-brane cocycle (57) means to impose the chosen charge $c \in \{0, 1\}$ to all O-planes, via Prop. 3.14 as illustrated in Figure H. We will denote the ADE-equivariant Cohomotopy sets which admit super-differential refinement with the choice $-1 \in \{0, 1\}$ in (57) by a subscript $(-)_-$. 

Example 3.19 (Super-differentiable equivariant Cohomotopy of ADE-orbifolds). Locally, the super-differentiable equivariant Cohomotopy of the vicinity of an ADE-singularity (Table 5) with respect to the choice $-1 \in \{-0, -1\}$ in the equivariant enhancement (57) of the super-flux form (56) is

$$
\pi^{4\text{G}_{\text{ADE}}}_{\text{diff}}((\mathbb{R}^4_{\text{cpt}})^{-}) = \left\{ 1 \cdot 1_{\text{triv}} - N_{\text{brane}} \cdot 2_{\text{reg}} \mid N_{\text{brane}} \in \mathbb{Z} \right\}.
$$

Globally, the super-differentiable equivariant Cohomotopy specifically of the Kummer surface ADE-orbifold $T^4_{\text{Kummer}}$ (Example 3.6) is

$$
\pi^{4\text{G}_{\text{ADE}}}_{\text{diff}}((T^4)^{\text{refl}}) = \left\{ 16 \cdot 1_{\text{triv}} - N_{\text{brane}} \cdot 2_{\text{reg}} \mid 2N_{\text{brane}} - 16 = 0 \right\}.
$$

4 M5/MO5 anomaly cancellation

We now apply the general discussion of equivariant Cohomotopy in §3 to cohomotopical charge quantization of the M-theory C-field, according to Hypothesis H, for compactifications of heterotic M-theory on toroidal orbifolds with ADE-singularities. The resulting M5/MO5-anomaly cancellation is discussed in §4.2 below. In order to set the scene and to sort out some fine print, we first discuss in §4.1 relevant folklore regarding heterotic M-theory on ADE-orbifolds.

4.1 Heterotic M-theory on ADE-orbifolds

We now explain how the singularity structure (as in Table 5), which must really be meant when speaking of MO5-planes (61) coinciding with black M5-branes (62), is that of “1/2-M5-branes” (65) $SU(2) \cong Sp(1)$ (16); see Figure 5 below. This singularity structure goes back to [Sen97, 3] with further discussion and development in [FLO99, KSTY99, FLO00a, FLO00b, FLO00c, CHS19]; the type IIA perspective is considered in [GKST01] and also briefly in [KS02, p. 4]. We highlight the systematic picture behind the resulting heterotic M-theory on ADE-orbifolds and its string theory duals, further below in Table 7.

Critique of pure MO5-planes. We highlight the following:

(i) Seminal literature on M-theoretic orientifolds speaks of M5-branes parallel and/or coincident to MO5-singularities [DM95, Wi95b, Ho98, 2.1], namely to Euclidean $\mathbb{Z}_2$-orientifolds (19) of the form (see [HSS18, 2.2.2]):

$$
\begin{array}{cccc}
\mathbb{R}^5_{\text{MO5}} & \longrightarrow & \mathbb{R}^5_{\text{MO5}} \times \mathbb{R}^5_{\text{sign}},
\end{array}
$$

where $\mathbb{R}^5_{\text{sign}}$ is the Euclidean singularity (19) of the 5-dimensional sign representation of the group $\mathbb{Z}_2$.

(ii) But $1/2$BPS M5-brane solutions of $D = 11$ supergravity themselves have been classified [dMFO10, 8.3] and found to be given, in their singular far horizon limit [AFC99, 3], by singularities for finite subgroups $G_{\text{ADE}} \subset SU(2) \cong Sp(1)$ (16) of the type

$$
\begin{array}{cccc}
\mathbb{R}^5_{\text{M5}} & \longrightarrow & \mathbb{R}^5_{\text{M5}} \times \mathbb{R}^1 \times \mathbb{R}^4_{\text{ADE}},
\end{array}
$$

where the last factor is an ADE-singularity (17).
(iii) As orbifold singularities, this coincides with the far horizon geometry of coincident KK-monopole solutions to 11d supergravity (e.g. [MSY98 (47)] [As00 (18)]; see [HSS18 2.2.5])

\[
\text{MK6} \quad \mathbb{R}^{6,1} \hookrightarrow \mathbb{R}^{6,1} \times \mathbb{R}^{4_\text{H}},
\]

which, from the perspective of type IIA theory, reflects the fact that NS5-branes are domain walls inside D6-branes (e.g. [EGKRS00 p. 5], see [Fa17 3.3.1. 3.3.2]). This is illustrated by the central dot on the vertical axis in Figure S. Hence for the special case that \(G_{\text{ADE}} = \mathbb{Z}_{18}\), this yields the product \(\mathbb{R}^{1} \times \mathbb{R}^{4_{\text{sgn}}}\) of the 4-dimensional sign representation with the trivial 1-dimensional representation, instead of the 5-dimensional sign representation in (61).

(iv) In order to allow M5-singularities (62) to coincide with MO5-singularities (61) we have to consider intersecting a \(1/2\) BPS 5-brane solution with an MO9 locus ([HW95], see [HSS18 2.2.1]):

\[
\text{MO9} \quad \mathbb{R}^{9,1} \hookrightarrow \mathbb{R}^{9,1} \times \mathbb{R}^{4_{\text{sgn}}}.
\]

(v) This intersection is called the \(1/2\) M5 in [HSST18 2.2.7] [FSS19d 4]

\[
\frac{1}{2}\text{M5} = \text{MK6} \cap \text{MO9} \quad \mathbb{R}^{5,1} \hookrightarrow \mathbb{R}^{5,1} \times \mathbb{T}^{4_{\text{sgn}}} \times \mathbb{T}^{4_{\text{sgn}}}
\]

since its type IIA incarnation is known as the \(1/2\) NS5 [GST01 6][AF17 p. 18]. This is the brane configuration thought to geometrically engineer \(D = 6, \mathcal{N} = (1,0)\) field theories [HZ97][HKLY15][DHTV14 6].

Since the fixed point set of the toroidal orbifolds (23) for both the \(1/2\) M5 (65) and the MO5 (61) is the same set (27) of 32 points, all arguments about MO5 (61) which depend only on the set of orientifold fixed points, such as in [DM95][Wi95b 3.3][Ho98 2.1], apply to \(1/2\) M5 (65) as well. But the \(1/2\) M5 orientifold has in addition fixed lines, namely the MK6 loci, and fixed 4-planes, namely the MO9, as shown on the right of Figure S. This reflects the fact that, by the classification of [dMFO10 8.3], the black M5 not only may, but must appear as a domain wall inside an MK6 singular locus.

We **conclude** from this that: **The \(1/2\) M5 (65) orientifold is the correct model of orientifolded M5/MO5 geometry, while the pure MO5 (61) is just its restriction along the diagonal subgroup inclusion (66), as shown in Figure R.**

\[
\begin{align*}
\mathbb{Z}_{2}^{\text{refl} + \text{HW}} \hookrightarrow & \quad \mathbb{Z}_{2}^{\text{refl}} \times \mathbb{Z}_{2}^{\text{HW}} \\
\mathbb{Z}_{2}^{\text{refl} + \text{HW}} \quad & \quad \text{diag} \quad \mathbb{Z}_{2}^{\text{refl}} \times \mathbb{Z}_{2}^{\text{HW}}
\end{align*}
\]

In summary, this situation is illustrated by the following figure:
Orientifolds | MO5 | \( \frac{1}{2} \text{M5} 
\hline
Global quotient group | \( G = \mathbb{Z}_2 \) | \( \mathbb{Z}^{\text{HW}} \times G_{\text{ADE}} \) \\
Global quotient group action | \( (T^V)^G = \{ 0, \frac{1}{2} \}^5 = \mathbb{F}_2 \) | \( \mathbb{T}^5_{\text{sgn}} \times \mathbb{T}^4_{\text{sgn}} \times \mathbb{T}^4_{\text{sgn}} \) \\
Fixed/singular points | no | yes

Figure S – Singularity structure of heterotic M-theory on ADE-singularities, as in Figure R [HSS18 2.2.2, 2.2.7]. The corresponding toroidal orbifolds (as per Table 5) are illustrated in Figure V and Table 8.

O\(^0\)-planes and M2-brane CS level. There is one more ingredient to the G-space structure of heterotic M-theory on ADE-orbifolds (see Table 7 below for the full picture): While the MO5-planes (61) are supposed to be the M-theory lifts of the charged O\(^4\)^±-planes [Ho98, 3][Gi98, III.A][HK00, 3.1.1], the M-theory lift of the un-charged O\(^4\)^0-planes (see Figure OP) involves one more group action on spacetime [Gi98, III.B], which we indicate as follows:

\[
\begin{array}{c}
\text{IIA}^0 & \mathbb{R}^{9,1} \times \emptyset \xrightarrow{\text{KK}_{\text{rot}}} \mathbb{R}^{9,1} \times \mathbb{S}^1 . \\
\end{array}
\] (67)

Here on the right we have the circle regarded as a \( \mathbb{Z}_k \)-space (§3) via rigid rotation by multiples of \( 2\pi/k \). This is of course a free action (in particular, not a representation sphere (21)) hence with empty fixed subspace (24), whence the superscript \( (-)^0 \) in (69). But passing along the unique \( \text{G-equivariant function} \) (31)

\[
\begin{array}{c}
\mathbb{S}^1 \xrightarrow{\text{KK}_{\text{rot}}} \mathbb{S}^1 \xrightarrow{\text{KK-reduction}} \ast \\
\end{array}
\] (68)

from the circle to the point \( \ast \) with its necessarily trivial \( \mathbb{Z}_k^\text{rot} \)-action, as befits KK-reduction from M-theory to type IIA string theory (see [BMSS13] for discussion in the context of Hypothesis H). We obtain a non-empty fixed subspace:

\[
\begin{array}{c}
\text{IIA} & \mathbb{R}^{9,1} \xrightarrow{\text{KK}_{\text{rot}}} \mathbb{R}^{9,1} \times \ast . \\
\end{array}
\] (69)

In these terms, we may phrase the core of M/IIA duality as saying that

The lift of IIA (69) through KK\(\mathbb{S}^1_{\text{rot}} \) (68) is IIA\(^0 \) (67).

Notice in the case that the global 11d-spacetime is AdS\(_3 \) times \( S^7 \) regarded as an \( S^1_{\text{rot}} \)-fibration

\[
\begin{array}{c}
\mathbb{S}^1 \xrightarrow{\mathbb{Z}^\text{rot}_k} \mathbb{S}^7 \xrightarrow{\mathbb{S}^3_{\text{rot}}} \mathbb{C}P^3 \\
\end{array}
\]

the order \( k \) of \( \mathbb{Z}^\text{rot}_k \) in (67) is the level of the dual 3d Chern-Simons-matter theory [ABJM08].

The argument in [Gi98, III.B], together with our discussion above, suggests that the analogous statement for O\(^4\)^0-planes is this:

The lift of O\(^4\)^0 through KK\(\mathbb{S}^1_{\text{rot}} \) (68) is MO5\(^0 \) (70).

Here we define MO5\(^0 \) to be the following G-space/orbifold, combining MO5 (61) with IIA\(^0 \) (67):

\[
\begin{array}{c}
\text{MO5}^0 & \mathbb{R}^{5,1} \times \emptyset \xrightarrow{\text{KK}_{\text{rot}}} \mathbb{R}^{5,1} \times \mathbb{T}^4_{\text{sgn}} \times \mathbb{S}^1 . \\
\end{array}
\] (70)
As before in (67), the fixed subspace of the diagonal group action
\[ \mathbb{Z}_2^{\text{refl}+\text{rot}+\text{HW}} \subseteq \mathbb{Z}_2^{\text{refl}+\text{HW}} \times \mathbb{Z}_2^{\text{rot}} \]
in (70) is actually empty, since the action of \( \mathbb{Z}_2^{\text{rot}} \) and hence that of \( \mathbb{Z}_2^{\text{refl}+\text{rot}+\text{HW}} \) is free, whence the superscript \((-)^0\). But, as before in (69), under M/IIA KK-reduction (68) we have an equivariant projection map to the orbifold
\[ \mathbb{O}^0_4 \quad \mathbb{R}^{4,1} \xrightarrow{\mathbb{Z}_2^{\text{refl}+\text{HW}}} \mathbb{R}^{4,1} \times \mathbb{T}^{4\text{sign}} \times \ast, \]
with non-empty fixed/singular subspace being the O4-worldvolume – which is thereby exhibited as being uncharged, as its lift to M-theory in fact non-singular.

In the same manner, there is then the analogous \( \mathbb{Z}_k^{\text{rot}} \)-resolution of the MK6-singularity (63)
\[ \text{MK}_6^0 \quad \mathbb{R}^{6,1} \times \mathbb{O} \xrightarrow{\mathbb{Z}_k^{\text{rot}}} \mathbb{R}^{6,1} \times \mathbb{T}^{4\text{H}} \times \mathbb{S}^1, \]
as well as of the MO9-singularity (64):
\[ \text{MO}_9^0 \quad \mathbb{R}^{9,1} \times \mathbb{O} \xrightarrow{\mathbb{Z}_k^{\text{rot}}} \mathbb{R}^{9,1} \times \mathbb{T}^{4\text{H}} \times \mathbb{S}^1. \]
The reduction of the latter along KK\(_{\text{rot}}^\text{S}_1\) (68) is
\[ \text{O}_8^0 \quad \mathbb{R}^{8,1} \xrightarrow{\mathbb{Z}_k^{\text{rot}}} \mathbb{R}^{8,1} \times \mathbb{T}^{4\text{sign}} \times \ast. \]

In summary, the full singularity structure of heterotic M-theory on ADE-orbifolds, such as to admit
(i) black M5-branes coinciding with MO5-planes and
(ii) the MO5\(^0\)-lift of O4\(^0\)-planes
is as shown in Table 7.

The following figure shows the corresponding subgroup lattice with its associated fixed/singular spaces:
4.2 Equivariant Cohomotopy charge of M5 at MO5

Applying the general mathematical results of §3 to the M_{HET}/ADE-singularities from §4.1, we finally show here (see Figure V) that Hypothesis H formalizes and validates the following widely accepted but informal Folklore 4.1 concerning the nature of M-theory:

**Folklore 4.1** (M5/MO5 anomaly cancellation [DM95][Wi95b 3.3][Ho98 2.1]). For M-theory on the toroidal orientifold $\mathbb{R}^{5,1} \times \mathbb{T}^5 \parallel \mathbb{Z}_2$ (Table 5) with MO5-singularities (61), consistency requires the situation shown in Table 2:

1. a charge of $q_{MO5}/q_{MS} = -1/2$ is carried by each of the fixed/singular MO5-planes (61);
2. the M5-brane charge is integral in natural units, hence on the covering $\mathbb{Z}_2$-space $\mathbb{T}_{5,W}^5$ the M5-branes appear in $\mathbb{Z}_2$-mirror pairs around the MO5-planes, as in Figure EC and Figure N;
3. the total charge of the $N_{MS}$ M5-branes has to cancel that of the 32 $O$-planes (27), $N_{MS}q_{MS} + 32q_{MO5} = 0$, as indicated in Figure Q.

Via the similarly widely accepted Folklore 4.2, the statement of Folklore 4.1 implies tadpole anomaly cancellation in string theory. Notice that this is not so much a claim than part of the defining criterion for M-theory:

**Folklore 4.2** (Double dimensional reduction of M5/MO5 to D4/O4 [Ho98 3][G98 III.A][HK00 3.1.1]). Under M/IIA duality, the situation of Folklore 4.1 become the string-theoretic tadpole cancellation condition from Table TC for D4-branes and $O^\perp$-planes.

**Folklore 4.3** (T-duality relating O-planes, e.g. [BLT13, p.317-318]). By iterative T-duality, the situation of Folklore 4.2 implies general tadpole cancellation for $D_p$-branes and $O_p^\perp$-planes (Table 3).
semi-complement (79) of MO5 \subset 1₂M5-singularities (Figure S) in heterotic M-theory on ADE-orbifolds (Table 7).

C-field charge quantization in ADE-equivariant Cohomotopy (3).

M-theory C-field charge-quantized by Hypothesis H as a cocycle in equivariant Cohomotopy.

\[ \left( \mathbb{R}^5 \times \mathbb{Z}^{\text{vert}+\text{HW}} / \mathbb{Z}^{\text{vert}+\text{HW}} \right) \times \mathbb{T}^4_{\text{eq}} \]

Figure V: Equivariant Cohomotopy of ADE-orbifolds in heterotic M-theory with singularity structure as in Figure S. The resulting charge classification (Cor. 4.4) implies, via the unstable PT isomorphism (§2.1), the 1₂M5 = MO9 \cap MK6-brane configurations (65) similarly shown in [FLO99, Fig. 1][KSTY99, p. 7][FLO00a, Fig. 1][FLO00b, Fig. 2][FLO00c, Fig. 1][GKST01, p. 4, 68, 71]. This is as in Figure 4 but with points (M5s) extended to half-line (MK6s), see Remark 4.7 and Table 8.

C-Field flux quantization at pure MO5-Singularities. To put the discussion below in perspective, it is instructive to first recall the success and the shortcoming of the existing argument [Ho98, 2] for M5/MO5-brane charge quantization around a pure MO5-singularity (61) (see the left column of Table 8): Following the classical argument of [Dir31], we consider removing the locus of the would-be M5-brane from spacetime and then computing the appropriate cohomology of the remaining complement. For the pure MO5-singularity (61) the complement spacetime is, up to homotopy equivalence, the 4-dimensional real projective space:

\[ X_{\text{MO5}}^{11} = \left( \mathbb{R}^{5,1} \times \mathbb{R}^{\text{eq}} / \mathbb{Z}^{\text{vert}+\text{HW}} \right) \setminus \{ \mathbb{R}^{5,1} \times \{0\} \} \cong_{\text{homotopy}} S(\mathbb{R}^{5,1}_{\text{eq}}) / \mathbb{Z}^{\text{vert}+\text{HW}} \cong \mathbb{R}P^4. \] (75)

Since this ambient spacetime (75) is smooth but curved (i.e. non-parallelizable) manifold, the flavor of Cohomotopy theory that measures its M-brane charge, according to Hypothesis H is, according to Table 4 the J-twisted Cohomotopy theory of [FSS19b, 3]. This implies, by [FSS19b, Prop. 4.12], that rationalized brane charge (bottom of (55)) is measured by the integral of a differential 4-form \( G_4 \in \Omega^4(X^{11}) \) (the C-field 4-flux density) which satisfies the half-integral shifted flux quantization condition

\[ [G_4] + [\frac{1}{2}p_1] \in H^4(X^{11}, \mathbb{Z}) \to H^4(X^{11}, \mathbb{R}) \] (76)

as is expected from the M-theory folklore (recalled in [FSS19b, 2.2]). Applying this to the complement \( X_{\text{MO5}}^{11} \) around a pure MO5-plane implies, as pointed out in [Ho98, 2.1], that there must be an odd integer of brane charge in the pure MO5-spacetime

\[ \int_{\mathbb{R}P^4} G_4 = 1 - 2N \mid N \in \mathbb{N} \] (77)

C-Field flux quantization at pure MO5-Singularities.
The need to resolve further microscopic details. If one could identify in (77) the offset of 1 mod 2 in (77) with the charge carried by the pure MO5-plane (61), and the remaining even charge $2N$ with that of $N$ M5-branes in its vicinity

$$1 - N \cdot 2 \equiv Q_{\text{MO5}} - N_{\text{brane}} \cdot Q_{\text{M5}}$$

(78)

this would be the local/twisted M5/MO5-anomaly cancellation condition of Table 2. Without such further information, the charge quantization (77) around pure MO5-planes (61) is only consistent with the local/twisted M5/MO5-anomaly cancellation from Table 2, as noticed in [Ho98, bottom of p. 5].

But with the results of §3 and in view of §4.1, we may now complete this old argument (see the right column of Table 8):

Equivariant Cohomotopy implies local/twisted M5/MO5-anomaly cancellation at $\frac{1}{2}$M5-singularities. We know from §3.1 that the identification (78) missing from the result (77) for twisted Cohomotopy on smooth but curved spacetimes is implied by the result of Prop. 3.14 for equivariant Cohomotopy of singular but flat spacetimes. Moreover, we have argued in §4.1 that having black M5-branes actually coinciding with MO5-planes are but the diagonally fixed sub-loci (shown in Table FMO) inside the richer $\frac{1}{2}M5 = \text{MK6} \cap \text{MO9-singularities}$ (65) of heterotic M-theory on ADE-orbifolds (Figure S). Hence for a rigorous M5/MO5-anomaly cancellation result not just consistent with (as in (77)), but actually implying Folklore 4.1, we need to compute the M-brane charge at MO5-singularities inside $\frac{1}{2}M5$-singularities (65). Concretely, this means with Hypothesis H that M5/MO5-charge at a single MO5-singularity is measured by the equivariant Cohomotopy of the following $\frac{1}{2}$M5-refinement of the naive MO5-complement spacetime (75):

$$X^\text{11}_{\frac{1}{2}M5} := \left( \mathbb{R}^{5,1} \times \mathbb{R}^{1,\text{sgn}} \times \mathbb{R}^{4}_{\text{HW}} \times \mathbb{Z}_{2}^{\text{refl}} \right) \backslash \left( \mathbb{R}^{5,1} \times \{0\} \times \mathbb{R}^{4}_{\text{HW}} \times \mathbb{Z}_{2}^{\text{refl}} \right)$$

As shown in the second line, this is homotopy-equivalent to a residual ADE-singularity (Table 5). Therefore, the discussion from §3.1 applies:

Corollary 4.4 (Equivariant Cohomotopy implies local/twisted M5/MO5-anomaly cancellation). The super-differentiable (55) equivariant Cohomotopy charge (3) of the vicinity (Def. 3.9) of the semi-complement spacetime of of a single charged MO5-singularity (79)

$$\pi^4_{\text{HW}} \left( \left( X^\text{11}_{\frac{1}{2}\text{MO5}} \right)^{\text{cpt}} \right)_{-} = \left\{ 1 \cdot 1_{\text{HW}} - N_{\text{M5}} \cdot 2_{\text{reg}} \right\} \left\{ N_{\text{M5}} \in \mathbb{Z} \right\}$$

(79)

as in Folklore 4.1 Table 2 regarding the local/twisted form of M5/MO5-anomaly cancellation.

Proof. By $G$-homotopy invariance of $G$-equivariant homotopy theory, this follows as the special case of Prop. 3.14 with (59) in Example 3.19 for $G = \mathbb{Z}_2$, hence with $k = |W_G(1)| = 2$. 

Remark 4.5 (Super-exceptional geometry of MO5 semi-complement). While here we consider only topological orientifold structure, the full super-exceptional geometry corresponding to (79) is introduced in [FSS19d, 4]; shown there to induce the M5-brane Lagrangian on any super-exceptional embedding of the $\frac{1}{2}$M5-locus.
Equivariant Cohomotopy implies local/untwisted M5/MO5-anomaly cancellation at $\frac{1}{2}$M5-singularities. It is immediate to consider the globalization of this situation to the semi-complement around one MO9 in heterotic M-theory compactified on the toroidal $\mathbb{Z}_2$-orbifold $\mathbb{T}^{5,8} / \mathbb{Z}_2^{\text{refl}}$ with MO5-singularities:

$$X_{\text{MHEF}/\mathbb{Z}_2}^{11} := \mathbb{R}^{5,1} \times \left( \mathbb{R}^{1,18} / \mathbb{Z}_2^{\text{HW}} \right) \times \mathbb{T}^{4} / \mathbb{Z}_2^{\text{refl}} .$$

(80)

To this toroidal ADE-orbifold the discussion in §3.2 applies as follows.

**Corollary 4.6 (Equivariant Cohomotopy implies global/untwisted M5/MO5-anomaly cancellation).** The super-differentiable ([55]) equivariant Cohomotopy charge (3) of the semi-complement spacetime (80) of heterotic M-theory on a toroidal MO5-orientifold (§4.1) with charged MO5-planes in compatible RO-degree (Example 3.8) and admitting equivariant super-differential refinement (58) is

$$\pi^4_{\text{HEF}} \left( \left( X_{\text{MHEF}/\mathbb{Z}_2}^{11} \right) + \right)_{\text{Subtle...}} = \left\{ 16 \cdot 1_{\text{triv}} - 8 \cdot 2_{\text{reg}} \right\}$$

as expected from Folklore 4.7 [Table 2] regarding the global/untwisted form of M5/MO5-anomaly cancellation (recalling that the semi-complement (80) is that around one of the two MO9-planes).

**Proof.** By $G$-homotopy invariance of equivariant Cohomotopy, this follows from the statement (60) in Example 3.19.

More generally we have the following:

**M5/MO5-anomaly cancellation in heterotic M-theory on general ADE-orbifolds.** The statements and proofs of Corollary 4.4 and Cor. 4.6 directly generalize to heterotic M-theory on general $G^{\text{ADE}}$-singularities $\mathbb{R}^{4} / \mathbb{Z}_2^{\text{ADE}}$ §4.1 because the underlying results in §3 apply in this generality. Hence **Hypothesis H** implies that on the semi-complement spacetime of an MO9 intersecting a toroidal ADE-orbifold

$$X_{\text{MHEF}/\mathbb{Z}_2}^{11} := \mathbb{R}^{5,1} \times \left( \mathbb{R}^{1,18} / \mathbb{Z}_2^{\text{HW}} \right) \times \mathbb{T}^{4} / \mathbb{Z}_2^{\text{refl}} .$$

(81)

the M5/MO5 charge, measured in equivariant Cohomotopy, is

$$Q_{\text{tot}} = 16 \cdot 1_{\text{triv}} - N_{\text{M5}} \cdot k_{\text{reg}} \quad |Q_{\text{tot}}| = 0 ,$$

for $k = |G^{\text{ADE}}|$ the order of the global quotient group. Under double dimensional reduction to type IIA string theory according to **Table 7** this implies the tadpole cancellation conditions for D4-branes in ADE-orientifolds, from **Table 7**.
Spacetimes on which to measure flux sourced by M5/MO5-charge

| Definition | $X_{\text{MO5}} \simeq_{\text{htpy}} S(\mathbb{R}^{1_{\text{gen}} + 4_{\text{gen}}}) / \mathbb{Z}^2_{\text{het} + \text{HW}}$ | $X_{\text{M5}} \simeq_{\text{htpy}} S(\mathbb{R}^{1_{\text{gen}}}) / \mathbb{Z}^2_{\text{HW}} \times \mathbb{T}^4_{\text{refl}} \parallel \mathbb{Z}^2_{\text{refl}}$ |
|-----------|---------------------------------------------------------------|----------------------------------------------------------------------------------|

Illustration

Illustration

Geometry | smooth but curved | singular but flat |

Cohomological charge quantization
by Hypothesis H

Cohomology theory
(by Table 4) $J$-twisted Cohomotopy $\pi^{TX}(X)$ |
| by [FSS19b] | [FSS19c] [FSS19e] | equivariant Cohomotopy $\pi^V(\mathbb{T}^V)$ |
| $\S^4$ |

Illustration
(Remark 4.7)

Charge classification

Table TP – Two ways of measuring M5/MO5-charge. On the left is the traditional approach not resolving the singularities. On the right (which shows the same situation as in Figure V but with the periodic identification indicated more explicitly) the fine-grained microscopic picture seen by C-field charge quantization in equivariant Cohomotopy.

With these result in hand, we highlight that not only did equivariant Cohomotopy inform us about M-theory, but M-theory also shed light on a subtle point regarding the interpretation of equivariant Cohomotopy:

**Remark 4.7 (Equivariant Cohomotopy and MK6 ending on M5).** (i) The heuristic way to see that ordinary Cohomotopy $\pi^4$ from (5) canonically measures charges of 5-branes inside 11-dimensional spacetime is that the ‘classifying space’ $\S^4$ of $\pi^4$ gets essentially identified with the (any) spacetime 4-sphere around a 5-brane in an 11-dimensional ambient space (see [HSS18] (6) for the heuristic picture, and [FSS19b] 4.5 for the full mathematical detail).

(ii) But as we pass from plain to equivariant Cohomotopy, this picture
Brane charge sourced in the center of $S^4$ may superficially appear to be in tension with the picture provided by the Pontrjagin-Thom theorem as in Figure D and Figure L where instead

Brane charge is sourced at the 0-pole of $S^4$.

However, in the orb-geometry of heterotic M-theory on ADE-singularities §4.1 indeed both pictures apply simultaneously, witnessing different but closely related brane species (see Table 8):

(iii) The black $\frac{1}{2}M5$-brane locus (Figure S) is the terminal point of an MK6-singularity which extends radially away from the M5. Hence, given any radial 4-sphere with the $\frac{1}{2}M5$ at its center, the MK6 will pierce this 4-sphere at one point. Since the $\frac{1}{2}M5$ and the MK6 are necessarily related this way, the 5-brane charge inside the $S^4$ may equivalently be measured by 6-brane charge piercing through $S^4$. This is exactly what the Pontrjagin-Thom theorem says happens in Corollary 4.4, as shown in Figure V and on the right of Table 8.

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