A Study of Some Geometric Properties of Meromorphic Univalent Functions Associated with Ruscheweyh Derivative

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Abstract: We presented in this paper a new class S(λ, α, β) containing meromorphic univalent in functions in the punctured unit disk. We obtained many geometric properties, like coefficient inequality, distortion and growth theorems, radii of starlikeness and convexity, extreme points, convex linear combination, arithmetic mean, hadamard product, closure theorems, neighborhood and partial sums results.

Keywords: Univalent functions, meromorphic functions, starlike and convex functions.

1. Introduction

Let S denote the class of functions of the form

\[ f(z) = z^{-1} + \sum_{n=1}^{\infty} a_n z^n, a_n \geq 0, n \in \mathbb{N} = \{1, 2, \ldots\} \]  \hspace{1cm} (1.1)

which are meromorphic univalent functions in the punctured unit disk \( U^* = \{z \in \mathbb{D} : 0 < |z| < 1\} \). The hadamard product (or convolution) of two functions \( f \) and \( g \)

\[ (f * g)(z) = z^{-1} + \sum_{n=1}^{\infty} a_n b_n z^n, \]  \hspace{1cm} (1.2)

where \( a_n, b_n \geq 0 \).

Also, a function \( f \in S \) is meromorphic starlike function of order \( \delta \) (0 ≤ \( \delta < 1 \)) if

\[ -Re \left\{ \frac{f'(z)}{f(z)} \right\} > \delta \]  \hspace{1cm} (5)

and a function \( f \in S \) is meromorphic convex of order \( \delta \) (0 ≤ \( \delta < 1 \)) if

\[ -Re \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \delta \]  \hspace{1cm} (6)

The Ruscheweyh derivative of order \( \lambda \) is denoted by \( D^\lambda f \) and is defined as follows

\[ D^\lambda f(z) = \frac{1}{(1 - z)\lambda + z} \left[ \sum_{n=1}^{\infty} D_n(\lambda) a_n z^n \right], \lambda > -1, z \in U^*. \]

where

\[ D_n(\lambda) = \frac{(\lambda + 1)(\lambda + 2)\ldots(\lambda + n)}{(n + 1)!} \]

Our aim of this paper is to study the class \( S(\lambda, \alpha, \beta) \) containing function \( f \) and satisfying

\[ \left| \frac{z (D^\lambda f)(z)}{(D^\lambda f)(z)} \right| < \beta \]  \hspace{1cm} (1.3)

where

\[ D^\lambda f(z) = z^{-1} + \sum_{n=1}^{\infty} D_n(\lambda) a_n z^n. \]  \hspace{1cm} (1.4)

Many authors studied classes of meromorphic functions which are univalent and multivalent like W. G. Atshan in [1,2], B. A. Frasin and M. Darus in [3], A. R. S. Juma and H. Zirar in [4], L. Liu in [5], L. Liu and M. Srivastava in [6], J. E. Miller in [7], and M. L. Mogra in [8], R. K. Raina and H. M. Srivastava in [9] and N. Xu and D. Yang in [10].

2. Coefficient Inequality

Theorem (2.1) Let the function \( f \) defined by (1.1). Then \( f \in S(\lambda, \alpha, \beta) \) if and only if

\[ \sum_{n=1}^{\infty} [n + 1 + 2\beta(n + \alpha)] D_n(\lambda) a_n < 2\beta(1 - \alpha) \]  \hspace{1cm} (2.1)

where 0 ≤ \( \alpha < 1 \), 0 < \( \beta \leq 1 \).

Proof. Assume the condition (2.1) is satisfied, therefore

\[ |z (D^\lambda f(z))| + |(D^\lambda f(z))'| \leq \beta |2z (D^\lambda f(z))' + 2\alpha (D^\lambda f(z))| = |z^{-1} + \sum_{n=1}^{\infty} D_n(\lambda) a_n z^n + z^{-1} + \sum_{n=1}^{\infty} D_n(\lambda) a_n z^n| - \beta \]

\[ \leq 2z^{-1} + 2 \sum_{n=1}^{\infty} D_n(\lambda) a_n z^n + 2\alpha z^{-1} + 2 \sum_{n=1}^{\infty} D_n(\lambda) a_n z^n \leq \sum_{n=1}^{\infty} D_n(\lambda) (n + 1) a_n z^n \]

Then by maximum modulus theorem, we get the result.

For the Converse, assume that

\[ \left| \frac{z (D^\lambda f)(z)}{(D^\lambda f)(z)} \right| < \beta \]

is satisfied.

Therefore,

\[ \left| \sum_{n=1}^{\infty} D_n(\lambda) (n+1) a_n z^n \right| < \beta \].

Since \( |Re(z)| \leq |z| \) for all \( z \), we have

\[ \sum_{n=1}^{\infty} D_n(\lambda) (n+1) a_n z^n < \beta \].

Then by choosing the value of \( z \) on the real axis and letting \( z \to 1 - \) through values, we get

\[ \sum_{n=1}^{\infty} [n + 1 + 2\beta(n + \alpha)] D_n(\lambda) a_n < 2\beta(1 - \alpha). \]

Volume 6 Issue 7, July 2017

www.ijsr.net

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Paper ID: ART20175494
DOI: 10.21275/ART20175494
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Corollary (2.1) If \( f \in S(\lambda, \alpha, \beta) \), then
\[
2\beta(1-\alpha)
\]
where, \( 0 \leq \alpha < 1 \), \( 0 < \beta \leq 1 \).

3. Distortion and Growth Theorem

Theorem (3.1) If the function \( f \) defined by (1.1) is in the class \( S(\lambda, \alpha, \beta) \), then for \( 0 < |z| = r < 1 \), we have
\[
|f(z)| \leq r \frac{r-\beta(1-\alpha)}{1+\beta(1+\alpha)} r \quad \text{and} \quad |f(z)| \geq r \frac{1-\beta(1-\alpha)}{1+\beta(1+\alpha)} r \tag{3.1}
\]

Proof

Since \( f(z) \) be a function in \( S(\lambda, \alpha, \beta) \), then we have from Theorem (2.1)
\[
\sum_{n=1}^{\infty} n^2 a_n < 2\beta(1-\alpha).
\]

Then
\[
|f(z)| = |z^{-1} + \sum_{n=1}^{\infty} a_n z^n| \leq |z|^{-1} + \sum_{n=1}^{\infty} a_n |z|^n
\leq r \sum_{n=1}^{\infty} a_n
\]

By Theorem (2.1),
\[
\text{we get } \sum_{n=1}^{\infty} a_n \leq \frac{2\beta(1-\alpha)}{1+\beta(1+\alpha)}(1+\lambda+\lambda^2)
\]

Then
\[
|f(z)| \leq r \frac{r-\beta(1-\alpha)}{1+\beta(1+\alpha)}(1+\lambda+\lambda^2)
\]

Also,
\[
|f(z)| \geq r \frac{1-\beta(1-\alpha)}{1+\beta(1+\alpha)}(1+\lambda+\lambda^2)
\]

Corollary (3.2) If the function \( f \) defined by (1.1) is in the class \( S(\lambda, \alpha, \beta) \), then for \( 0 < |z| = r < 1 \), we have
\[
\frac{1}{r^2} - \frac{2\beta(1-\alpha)}{1+\beta(1+\alpha)}(1+\lambda+\lambda^2) \leq |f'(z)| \leq \frac{1}{r^2} + \frac{2\beta(1-\alpha)}{1+\beta(1+\alpha)}(1+\lambda+\lambda^2)
\tag{3.2}
\]

4. Radii of Star Likeness and Convexity

The following results giving the radii of starlikeness and convexity of the function \( f(z) \in S(\lambda, \alpha, \beta) \).

Proof

It is sufficient to show that
\[
\left| \frac{zf''(z) + 2zf'(z)}{f(z)} \right| \leq 1 - \delta
\]

The last expression must bounded by \( 1 - \delta \). Therefore,

\[
\sum_{n=1}^{\infty} n(n+1) a_n |z|^n \leq 1 - \delta
\]

\[
\sum_{n=1}^{\infty} n(n+\delta+2) a_n |z|^n \leq 1 - \delta
\]

Then by theorem (2.1), we have
\[
\sum_{n=1}^{\infty} n(n+1+2\beta(n+\alpha)) a_n < 1
\]

Hence \( R_1 = \frac{1}{n} \left[ \frac{n+1+2\beta(n+\alpha)D_n(\lambda)(1-\delta)}{2\beta(1-\alpha)(n+\delta+2)} \right]^{\frac{1}{n+1}} \tag{4.1}

Theorem (4.1) If a function \( f \in S(\lambda, \alpha, \beta) \), then \( f \) is meromorphically starlike function of order \( \delta \), \( 0 \leq \delta < 1 \) in the disk \( |z| < R_1 \), where

\[
R_1 = \frac{1}{n} \left[ \frac{n+1+2\beta(n+\alpha)D_n(\lambda)(1-\delta)}{2\beta(1-\alpha)(n+\delta+2)} \right]^{\frac{1}{n+1}} \tag{4.1}

Proof

It is sufficient to show that
\[
\left| \frac{zf''(z) + 2zf'(z)}{f(z)} \right| \leq 1 - \delta
\]

The last expression must bounded by \( 1 - \delta \). Therefore,

\[
\sum_{n=1}^{\infty} n(n+1) a_n |z|^n \leq 1 - \delta
\]

\[
\sum_{n=1}^{\infty} n(n+\delta+2) a_n |z|^n \leq 1 - \delta
\]

Therefore, we get the result.

Theorem (4.2). If a function \( f \in S(\lambda, \alpha, \beta) \), then \( f \) is meromorphically convex function of order \( \delta \), \( 0 \leq \delta < 1 \) in the disk \( |z| < R_2 \), where

\[
R_2 = \frac{1}{n} \left[ \frac{n+1+2\beta(n+\alpha)D_n(\lambda)(1-\delta)}{2\beta(1-\alpha)(n+\delta+2)} \right]^{\frac{1}{n+1}} \tag{4.2}

5. Convex Linear Combination

In the following theorem, the class \( S(\lambda, \alpha, \beta) \) is closed under convex linear combination.
Theorem (5.1). The class $S(\lambda, \alpha, \beta)$ is closed under convex linear combination.

Proof: We want to show that function $k(z) = (1-\Delta)f_1(z) + \bigoplus f_2(z)$, $0 \leq \beta \leq 1$, is in the class $S(\lambda, \alpha, \beta)$ where $f_1(z), f_2(z) \in S(\lambda, \alpha, \beta)$ and

$$f_1(z) = z^{-1} + \sum_{n=1}^{\infty} a_n z^n, \quad f_2(z) = z^{-1} + \sum_{n=1}^{\infty} b_n z^n, \quad (5.1)$$

where $a_n, b_n \geq 0$.

By theorem (2.1) we have

$$\sum_{n=1}^{\infty} [n + 1 + 2\beta(n + \alpha)] D_n(\lambda) a_n \leq 2\beta(1-\alpha)$$

and

$$\sum_{n=1}^{\infty} [n + 1 + 2\beta(n + \alpha)] D_n(\lambda) b_n \leq 2\beta(1-\alpha).$$

Therefore, $k(z) = (1-\Delta)f_1(z) + \bigoplus f_2(z)$,

$$= (1-\Delta)(z^{-1} + \sum_{n=1}^{\infty} a_n z^n) + \sum_{n=1}^{\infty} b_n z^n$$

$$= z^{-1} + \sum_{n=1}^{\infty} [(1-\Delta) a_n + b_n] z^n.$$

Further,

$$\sum_{n=1}^{\infty} [(n + 1) + 2\beta(n + \alpha)] [(1-\Delta) a_n + b_n] D_n(\lambda)$$

$$\leq \sum_{n=1}^{\infty} [(n + 1) + 2\beta(n + \alpha)] (1-\Delta) a_n + \sum_{n=1}^{\infty} [(n + 1) + 2\beta(n + \alpha)] b_n$$

$$\leq 2(1-\beta) \beta(1-\alpha) + 2\mu \beta(1-\alpha) = 2\beta(1-\alpha).$$

Therefore the result is follows that $k(z)$ is in the class $S(\lambda, \alpha, \beta)$.

6. Arithmetic Mean

In the following, we shall prove that class $S(\lambda, \alpha, \beta)$ is closed under arithmetic mean.

Theorem (6.1). Let $f_1(z), f_2(z), \ldots, f_l(z)$ be functions defined by $f_i(z) = z^{-1} + \sum_{n=1}^{\infty} a_{n,i} z^n$, $a_{n,i} \geq 0$, $i = 1, 2, \ldots, l$ in the class $S(\lambda, \alpha, \beta)$. Then the arithmetic mean of $f_i(z)$, $i = 1, \ldots, l$, defined by

$$h(z) = \frac{1}{l} \sum_{i=1}^{l} f_i(z)$$

is also in the class $S(\lambda, \alpha, \beta)$.

Proof: By hypothesis we get

$$h(z) = \frac{1}{l} \sum_{i=1}^{l} z^{-1} + \frac{1}{l} \sum_{i=1}^{l} \sum_{n=1}^{\infty} a_{n,i} z^n$$

$$= z^{-1} + \sum_{n=1}^{\infty} \left( \frac{1}{l} \sum_{i=1}^{l} a_{n,i} \right) z^n$$

since $f_i(z) \in S(\lambda, \alpha, \beta)$ for every $i = 1, 2, \ldots, l$. So by using theorem (2.1), we get

that $\sum_{n=1}^{\infty} [(n + 1) + 2\beta(n + \alpha)] D_n(\lambda) \left( \frac{1}{l} \sum_{i=1}^{l} a_{n,i} \right)$$

$$\leq \frac{1}{l} \sum_{i=1}^{l} 2\beta(1-\alpha) = 2\beta(1-\alpha).$$

7. Hadamard Product

In the following theorem, we obtain the convolution result for function belongs to the class $S(\lambda, \alpha, \beta)$.

Theorem (7.1). Let the functions $f, g$ of the form (1.1) be in the class $S(\lambda, \alpha, \beta)$. Then $(f \cdot g) \in S(\lambda, \alpha, \beta)$.

Proof. Let $f, g \in S(\lambda, \alpha, \beta)$ and so

$$\sum_{n=1}^{\infty} \frac{[n + 1 + 2\beta(n + \alpha)] D_n(\lambda)}{2\beta(1-\alpha)} a_n \leq 1$$

and

$$\sum_{n=1}^{\infty} \frac{[n + 1 + 2\beta(n + \alpha)] D_n(\lambda)}{2\beta(1-\alpha)} b_n \leq 1$$

We have to find the smallest number $l$ such that

$$\sum_{n=1}^{\infty} \frac{[n + 1 + 2\beta(n + \alpha)] D_n(\lambda)}{2\beta(1-\alpha)} a_n b_n \leq 1$$

By the Cauchy-Schwarz inequality, we get

$$\sum_{n=1}^{\infty} \frac{[n + 1 + 2\beta(n + \alpha)] D_n(\lambda)}{2\beta(1-\alpha)} \alpha b_n \leq 1$$

It is sufficient to show that

$$\frac{[n + 1 + 2\beta(n + \alpha)] D_n(\lambda)}{2\beta(1-\alpha)} \alpha b_n \leq \frac{[n + 1 + 2\beta(n + \alpha)] D_n(\lambda)}{2\beta(1-\alpha)} \sqrt{a_n b_n} \leq 1$$

That is,

$$\sqrt{a_n b_n} \leq \frac{[n + 1 + 2\beta(n + \alpha)]}{\beta(1-\alpha) \sqrt{a_n b_n}}$$

But from (7.4)

$$\sqrt{a_n b_n} \leq \frac{[n + 1 + 2\beta(n + \alpha)]}{\beta(1-\alpha) \sqrt{a_n b_n}}$$

Thus it is enough to show that

$$\frac{2\beta(1-\alpha)}{[n + 1 + 2\beta(n + \alpha)] D_n(\lambda) - 4\beta^2(1-\alpha)(n + \alpha)}$$

which simplifies to

$$l \geq \frac{2\beta^2(1-\alpha)(n + 1)}{[n + 1 + 2\beta(n + \alpha)] D_n(\lambda) - 4\beta^2(1-\alpha)(n + \alpha)}$$

which proves theorem (7.1).

Theorem (7.2). Let the functions $f, g$ of the form $f(z) = \sum_{n=1}^{\infty} a_n z^n, g(z) = \sum_{n=1}^{\infty} b_n z^n$ in the class $S(\lambda, \alpha, \beta)$. Then the function $h(z) = z^{-1} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) z^n$ is in the class $S(\lambda, \alpha, \gamma)$, where

$$\gamma = \frac{2\beta^4(1-\alpha)^2}{\frac{1}{l} \sum_{i=1}^{l} \sum_{n=1}^{\infty} (n + 1 + 2\beta(n + \alpha) D_n(\lambda) - 4\beta^2(1-\alpha)(n + \alpha)}$$

Proof. Since $f, g \in S(\lambda, \alpha, \beta)$, therefore, by theorem (2.1) yields

$$\sum_{n=1}^{\infty} \frac{[n + 1 + 2\beta(n + \alpha)] D_n(\lambda)}{2\beta(1-\alpha)} a_n \leq 1$$

and

$$\sum_{n=1}^{\infty} \frac{[n + 1 + 2\beta(n + \alpha)] D_n(\lambda)}{2\beta(1-\alpha)} b_n \leq 1$$

We obtain from the last two inequalities

$$\sum_{n=1}^{\infty} \frac{[n + 1 + 2\beta(n + \alpha)] D_n(\lambda)}{2\beta(1-\alpha)} (a_n^2 + b_n^2) \leq 1$$

But $h(z) \in S(\lambda, \alpha, \gamma)$ if and only if
\[
\sum_{n=1}^{\infty} \frac{[n+1+2\gamma(n+\alpha)]D_n(\lambda)}{2\gamma(1-\alpha)} a_n^2 \leq 1
\]  
(7.9)

Therefore, the inequality (7.9) satisfied if
\[
\frac{[n+1+2\gamma(n+\alpha)]D_n(\lambda)}{2\gamma(1-\alpha)} \leq 1 \left[ \frac{[n+1+2\beta(n+\alpha)]D_n(\lambda)}{2\beta(1-\alpha)} \right]^2
\]
which is imply
\[
\gamma \geq \frac{4\beta^2(n+1)(1-\alpha)^2}{(1-\alpha)[n+1+2\beta(n+\alpha)]D_n(\lambda) - 8\beta^2(n+\alpha)(1-\alpha)^2}
\]
which complete the proof.

8. Closure Theorem

We shall prove the following closure theorems for the class \( S(\lambda,\alpha,\beta) \).

**Theorem (8.1).** Let \( f \in S(\lambda,\alpha,\beta) \), \( j = 1, 2 \ldots \ s \), then
\[
g(z) = \sum_{j=1}^{s} c_j f_j(z) \in S(\lambda,\alpha,\beta)
\]  
(8.1)

for \( f_j(z) = z^{-1} + \sum_{n=1}^{\infty} a_{n,j} z^n \), where \( \sum_{j=1}^{s} c_j = 1 \).

**Proof.** Suppose that \( g(z) = \sum_{j=1}^{s} c_j f_j(z) \). Since \( f_j(z) = z^{-1} + \sum_{n=1}^{\infty} a_{n,j} z^n \),

Therefore, \( g(z) = z^{-1} + \sum_{n=1}^{\infty} a_n z^n \), where \( a_n = \sum_{j=1}^{s} c_j a_{n,j} \).

Thus
\[
g(z) \in S(\lambda,\alpha,\beta) \text{ if } \sum_{n=1}^{\infty} \frac{[n+1+2\beta(n+\alpha)]D_n(\lambda)}{2\beta(1-\alpha)} a_n \leq 1.\]

That is, if
\[
\sum_{n=1}^{\infty} \left[ \frac{[n+1+2\beta(n+\alpha)]D_n(\lambda)}{2\beta(1-\alpha)} \right] c_{n,j} \leq 1
\]
\[
= \sum_{j=1}^{s} \sum_{n=1}^{\infty} \frac{[n+1+2\beta(n+\alpha)]D_n(\lambda)}{2\beta(1-\alpha)} a_{n,j}
\]
\[
\leq \sum_{j=1}^{s} c_j = 1.
\]
Therefore the proof is completed.

9. Neighborhoods

Now, we define \((n,\gamma)\)-neighborhood of a function \( f \in S \) by
\[
N_{n,\gamma}(g) = \{ g \in S : g(z) = z^{-1} + \sum_{n=1}^{\infty} a_n z^n \}, \quad \sum_{n=1}^{\infty} n|a_n - b_n| \leq \gamma, \quad 0 \leq \gamma < 1 \}
\]  
(9.1)

For the identity function \( g(z) = z^{-1} \), we have
\[
N_{n,\gamma}(g) = \{ g \in S : g(z) = z^{-1} + \sum_{n=1}^{\infty} b_n z^n \}, \quad \sum_{n=1}^{\infty} n|b_n| \leq \gamma
\]
and \( \sum_{n=1}^{\infty} n|b_n| \leq \gamma \}

**Definition (9.1).** A function \( f \in S \) is said to be in the class \( S^\mu(\lambda,\alpha,\beta) \) if there exists a function \( g \in S(\lambda,\alpha,\beta) \) such that
\[
\left| \frac{g(z)}{f(z)} - 1 \right| < 1 - \mu, \quad (z \in U, 0 \leq \mu < 1)
\]

**Theorem (9.2).** If \( g \in S(\lambda,\alpha,\beta) \) and
\[
\mu = 1 - \frac{\gamma[1 + \beta(1 + \alpha)](\lambda + 1)(\lambda + 2)}{[1 + \beta(1 + \alpha)](\lambda + 1)(\lambda + 2) - 2\beta(1-\alpha)}
\]  
(9.2)

Then \( N_{n,\gamma}(g) \subset S^\mu(\lambda,\alpha,\beta) \).

**Proof.** Let \( f \in N_{\mu,\gamma}(g) \), we want to find from (9.1) that
\[
\sum_{n=1}^{\infty} n|a_n - b_n| \leq \gamma
\]
Which readily implies the following coefficient in equality
\[
\sum_{n=1}^{\infty} b_n \leq [1 + \beta(1+\alpha)](\lambda + 1)(\lambda + 2) - 2\beta(1-\alpha)
\]
So that
\[
\frac{f(z)}{g(z)} - 1 = \frac{f(z) - g(z)}{g(z)} = \frac{\sum_{n=1}^{\infty} a_n - b_n} {1 + \sum_{n=1}^{\infty} b_n z^{n+1}} \gamma
\]
\[
\sum_{n=1}^{\infty} a_n - b_n \leq \left[ 1 + \beta(1+\alpha)](\lambda + 1)(\lambda + 2) - 2\beta(1-\alpha) \right]
\]
Thus by definition (9.1), \( f \in S^\mu(\lambda,\alpha,\beta) \) for \( \mu \) given by (9.2)

10. Partial Sums

**Theorem (10.1).** Let \( f(z) \in S(\lambda,\alpha,\beta) \) is given by (1.1). Also \( f_1(z) \) and \( f_2(z) \) defined by
\[
f_1(z) = z^{-1} \quad \text{and} \quad f_2(z) = z^{-1} + \sum_{n=1}^{\infty} a_n z^n , \quad n \in N \setminus \{ 1 \} .
\]
Suppose that
\[
\sum_{n=1}^{\infty} d_n |a_n| \leq 1.
\]
Furthermore, \( \sum_{n=1}^{\infty} d_n \leq 1 \),

\[
\left| \frac{f(z)}{f_k(z)} - 1 \right| < 1 - \frac{d_{k+1}}{d_{k+1}}
\]
(10.4)  
**Proof.** For the coefficients \( d_n \) given by (10.2)
\[
d_n + d_{n+1} \leq 1
\]
(10.5)

Theorem (10.6), we get that
\[
\left| \frac{g_1(z)}{g_1(z)+1} \right| \leq \frac{d_{n+1}+\sum_{n=k+1}^{\infty} a_n z^n}{2-2\sum_{n=1}^{\infty} a_n - d_{n+1}+\sum_{n=k+1}^{\infty} a_n} \leq 1
\]
(10.7)

Which readily yields the assertion (10.3) of Theorem (10.1). If the function

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**Volume 6 Issue 7, July 2017**

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Paper ID: ART20175494  
DOI: 10.21275/ART20175494  
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The result is sharp for the function
\( f(z) = z + \frac{d_{k+1}}{d_k} \).

(10.3)

Similarly, if we take
\[
g(z) = (1 + d_{k+1}) \left( \frac{f(z)}{f_k(z)} - \left( \frac{k+1}{d_{k+1}} \right) \right)
\]
\[
= 1 + \frac{\sum_{n=0}^{\infty} n a_n z^{n+1}}{\sum_{n=k+1}^{\infty} n a_n z^{n+1}} + \left( \frac{k+1}{d_{k+1}} \right)
\]
and making use of (10.6), we can deduce that
\[
\left| \frac{g(z)-1}{g(z)+1} \right| \leq \left| \frac{\sum_{n=k+1}^{\infty} n a_n z^{n+1}}{\sum_{n=0}^{\infty} n a_n z^{n+1}} \right| \leq 1 , \quad z \in U. \tag{10.9}
\]
Which leads to assertion (10.4) of Theorem (10.1). Therefore, the proof of Theorem (4.1) is complete.

**Theorem (10.2).** If \( f(z) \) of the form (1.1) satisfies the Theorem (2.1). Then
\[
\text{Re} \left( \frac{f(z)}{f_k(z)} \right) \geq \frac{k+1}{d_{k+1}}. \tag{10.10}
\]

\[\text{Proof:} \text{ By setting}
\]
\[
g(z) = \frac{d_{k+1}}{d_k} \left( \frac{f(z)}{f_k(z)} - \left( \frac{k+1}{d_{k+1}} \right) \right)
\]
\[
= \frac{\sum_{n=0}^{\infty} n a_n z^{n+1}}{\sum_{n=k+1}^{\infty} n a_n z^{n+1}} + \left( \frac{k+1}{d_{k+1}} \right)
\]
\[
= \left| \frac{g(z)-1}{g(z)+1} \right| \leq 1.
\]
Now
\[
\left| \frac{g(z)-1}{g(z)+1} \right| \leq 1 \text{ if } \sum_{n=1}^{k} n |a_n| + \frac{d_{k+1}}{d_k} \sum_{n=k+1}^{\infty} n |a_n| \leq 1. \tag{10.11}
\]
This is sharp for the function
\[f(z) = z + \frac{d_{k+1}}{d_k} \cdot\]

**Theorem (10.3).** If \( f(z) \) of the form (1.1) satisfies the Theorem (2.1) then
\[
\text{Re} \left( \frac{f(z)}{f_k(z)} \right) \geq \frac{d_{k+1}}{k+1 + d_{k+1}}. \tag{10.12}
\]

\[\text{Proof:} \text{ By setting}
\]
\[
g(z) = \frac{\left( (k+1) + d_{k+1} \right) \left( f(z) - \left( \frac{k+1}{d_{k+1}} \right) \right)}{k+1}
\]
\[
= \left| \frac{g(z)-1}{g(z)+1} \right| \leq 1.
\]
We can deduce that
\[
\left| \frac{g(z)-1}{g(z)+1} \right| \leq \left| \frac{\sum_{n=k+1}^{\infty} n a_n}{\sum_{n=0}^{\infty} n a_n} \right| \leq 1. \tag{10.11}
\]
Therefore, the result of Theorem (10.3) holds.

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