The Lamb shift contribution of very light millicharged particles

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Abstract

The leading order vacuum polarization contribution of very light millicharged fermions and scalar (spin–0) particles with charge $\varepsilon e$ and mass $\mu$ to the Lamb shift of the hydrogen atom is shown to imply universal, i.e. $\mu$–independent, upper bounds on $\varepsilon$ : $\varepsilon \lesssim 10^{-4}$ for $\mu \lesssim 1$ keV in the case of fermions, and for scalars this bound is increased by a factor of 2. This is in contrast to expectations based on the commonly used approximation to the Uehling potential relevant only for conventionally large fermion (and scalar) masses.
The recent observation [1] of an optical rotation of linearly polarized laser light generated in vacuum by a magnetic field may be due [2] to the photon initiated pair production of very light charged fermions with mass \( \mu \simeq 0.1 \text{ eV} \) and charge \( \varepsilon e \) where \( \varepsilon \simeq 10^{-6} \). There exist, however, very strong astrophysical, cosmological and laboratory constraints [3, 4, 5] which exclude the quoted values of \( \varepsilon \) and \( \mu \). Some of these constraints may nevertheless be relaxed in specific paraphoton scenarios [5]. Further constraints on the mass and charge of light charged fermions may be obtained from their leading order vacuum polarization contribution [3, 7, 8] to the Lamb shift of the hydrogen atom

\[
\delta E = E(2S_{1/2}) - E(2P_{1/2})
\]  

(1)

or from their higher order contribution [7] to the anomalous magnetic moment of the muon. The Lamb shift constraints for conventionally large fermion masses studied in [3, 7, 8] were, as usual, based on the commonly used *approximate* leading order vacuum polarization contribution

\[
\delta E_{VP} \simeq -\alpha \frac{m_e}{30\pi} \left( \frac{m_\varepsilon}{\mu} \right)^2 \varepsilon^2.
\]

(2)

However, the upper bound presented in [3, 9] is not appropriate for \( \mu \lesssim 1 \text{ keV} \). Even much lower values of \( \mu \), \( \mu < 1 \text{ eV} \), were considered in [2], for example, and it is therefore necessary to study the consequences of going beyond this standard approximation relevant only for \( \mu > \alpha m_e \).

The lowest order Coulomb interaction \( V = -\alpha/r \) for a point nucleus is modified at the 1-loop level according to \( V = -\alpha/r + \delta V \) where [10, 11]

\[
\delta V(r) = -\frac{\alpha}{r} \frac{2\alpha \varepsilon^2}{3\pi} \int_1^\infty du e^{-2\mu u} \left( 1 + \frac{1}{2u^2} \right) \frac{\sqrt{u^2 - 1}}{u^2}
\]

(3)

and \( \alpha = e^2/4\pi = 1/137.036 \). This leading order Uehling potential yields, instead of the approximate equation (2), the *exact* leading order expression

\[
\delta E_{VP} = \int_0^\infty dr r^2 \delta V(r) \left[ R_{20}^2 (r) - R_{21}^2 (r) \right]
\]

(4)
with the normalized radial hydrogen wave functions \( R_{n\ell} \) given by
\[
R_{20}(r) = \frac{1}{\sqrt{2}} \frac{1}{a^{3/2}} \left( 1 - \frac{\rho}{2} \right) e^{-\rho/2}, \quad R_{21}(r) = \frac{1}{2\sqrt{6}} \frac{1}{a^{3/2}} \rho e^{-\rho/2}
\]
where \( \rho = r/a \) and \( a^{-1} = \alpha m_e \). The integration over \( r \) in (4) yields
\[
\delta E_{VP} = -\frac{4\alpha^2 m_e}{3\pi} \varepsilon^2 \alpha^* I(\alpha^*)
\]
where
\[
I(\alpha^*) = \int_1^\infty du \left( 1 + \frac{1}{2u^2} \right) \frac{\sqrt{u^2 - 1}}{(\alpha^* + 2u)^4}
\]
with the effective coupling \( \alpha^* = \alpha m_e/\mu \). Requiring this leading order vacuum polarization contribution \( |\delta E_{VP}|/2\pi\hbar \) to the Lamb shift not to exceed 0.01 MHz, corresponding to a \( 1\sigma \) discrepancy between the measured and calculated shifts [12, 13], yields the constraints on \( \varepsilon \) shown in Fig. 1. (Imposing a 2\( \sigma \) error, i.e. increasing the experimental uncertainty to 0.02 MHz, increases the upper bound on \( \varepsilon \) by a factor of \( \sqrt{2} \).) It is interesting to note that, contrary to what one might naively expect, it turns out that the integral \( I(\alpha^*) \) in (6) entirely compensates the large enhancement factor \( \alpha^* \) already at \( \mu \lesssim 1 \) keV due \( \alpha^* I(\alpha^*) = 1/24 \) for \( \alpha^* \gg 1 \). (This asymptotic result is easily obtained by observing that the maximum of the integrand in (7) moves to \( \infty \) as \( \alpha^* \to \infty \) where the integrand reduces to \( u(\alpha^* + 2u)^{-4} \).) Therefore the upper bound in Fig. 1 saturates below \( \mu \simeq 1 \) keV which implies a rather moderate but universal, i.e. \( \mu \)-independent, upper bound
\[
\varepsilon \leq 1.085 \times 10^{-4} \quad \text{for} \quad \mu \lesssim 1 \text{ keV}.
\]
Although in the different context of electronic vacuum polarization corrections to the energy levels of exotic atoms, the exact result (7) and its asymptotic limit was previously presented [14] in a different form.\(^1\) Despite the fact that our eqs. (6) and (7) are particularly suited for directly understanding the reasons for the universal bound on \( \varepsilon \) discussed

\(^1\)The integral in (7) as well as the ones for the (physically irrelevant) individual 2S and 2P contributions can in general be calculated analytically. The rather lengthy analytic expressions of the latter ones, together with their asymptotic limits (\( \alpha^* \gg 1 \)), are given in Appendix A of [14] by identifying \( \kappa_2 = \alpha^*/2 \). Although these expressions are less transparent for understanding our universal (scaling) upper bound on \( \varepsilon \), it should be mentioned that the individual 2S and 2P contributions do not scale asymptotically but increase as \( \ln \alpha^* - \frac{7}{3} \) and \( \ln \alpha^* - \frac{8}{3} \), respectively, in units of \( -\alpha^3 \varepsilon^2 m_e/6\pi \). The advantage of our more direct approach is that it can be immediately applied also to the case of scalar particles studied below.
above, we also present for completeness the analytic result for the integral in (7):

\[
\alpha^2 I(\alpha^*) = \frac{1}{8\kappa^3} \left[ 2\pi + \frac{\kappa}{3(\kappa^2 - 1)^2} \left( -12 + 22\kappa^2 - \frac{13}{2}\kappa^4 + \kappa^6 \right) \right. \\
- \frac{1}{(\kappa^2 - 1)^2} \left( 4 - 10\kappa^2 + \frac{15}{2}\kappa^4 \right) L \right]
\]

(9)

with \( \kappa = \alpha^*/2 \) and \( L = \frac{\ln(\kappa + \sqrt{\kappa^2 - 1})}{\sqrt{\kappa^2 - 1}} \) relevant for \( \kappa > 1 \), and \( L = \frac{\arc \cos \kappa}{\sqrt{1 - \kappa^2}} \) for \( \kappa < 1 \). This analytic result has originally been given explicitly in [15] in the context of mesonic atoms.

For a comprehensive review on radiative QED corrections to hydrogenlike atoms we refer the reader to [16].

The dashed curve in Fig. 1 displays the upper bound as obtained from using [3, 8] the approximate expression (2) which, imposing \( |\delta E_{VP}|/2\pi \hbar \leq 0.01 \) MHz, implies \( \varepsilon \leq \mu/(26.62 \) MeV) as was assumed to hold for \( \mu > 1 \) keV [8]. It can be already seen from Fig. 1 that this approximate bound is not correct for \( \mu \) much below \( 10^5 \) eV. To illustrate this more clearly, we confront in Fig. 2 the approximate and exact upper bounds on \( \varepsilon \), as obtained from (2) and (6), respectively, using a linear scale for \( \varepsilon \). Their convergence at \( \mu \gtrsim 10^5 \) eV is due to \( I(\alpha^*) \to \frac{1}{40} \) for \( \alpha^* \to 0 \) in (7).

It should be mentioned that corrections to the anomalous magnetic moments of electrons and muons, or to the hyperfine splitting, for example, are genuinely subleading and thus the resulting bounds are much weaker [7]. Furthermore, considerations of the level shifts of exotic atoms [14] obviously can not improve the upper bounds on \( \varepsilon \) as obtained for the hydrogen atom, since the associated relative theoretical uncertainties are generally more significant [17].

Next we also calculate the exact upper bound for light millicharged scalar (spin–0) particles of mass \( \mu \). In this case the standard leading order fermionic QED contribution \( x(1 - x) \ln[1 - x(1 - x) q^2/\mu^2] \) appearing in the integrand of the Feynman–parameter integral in the vacuum polarization tensor [10, 18] has to be replaced by
\[ \frac{1}{8}(2x - 1)^2 \ln [1 - x(1 - x) q^2/\mu^2] \]. Performing now the exact integrations as outlined in [10], for example, one arrives at

\[
\delta V_s(r) = -\frac{\alpha}{r} \frac{\alpha \varepsilon^2}{6\pi} \int_1^\infty du e^{-2\mu u} \left(1 - \frac{1}{u^2}\right) \frac{\sqrt{u^2 - 1}}{u^2}
\]  

for the 1–loop correction of the leading order Coulomb interaction \( V = -\alpha/r + \delta V_s \). The calculation of this vacuum polarization contribution to the Lamb shift is now analogous to the above case of fermions, and the exact leading order contribution of scalars reads

\[
\delta E_{VP}^s = -\frac{\alpha^3 m_e}{3\pi} \varepsilon^2 \alpha^* I_s(\alpha^*)
\]  

where

\[
I_s(\alpha^*) = \int_1^\infty du \left(1 - \frac{1}{u^2}\right) \frac{\sqrt{u^2 - 1}}{\left(\alpha^* + 2u\right)^4},
\]  

to be compared with the fermionic result in (6) and (7). Imposing the same experimental upper bound on \(|\delta E_{VP}^s|/2\pi \hbar\) of 0.01 MHz as above for fermions yields the constraints on \( \varepsilon \) shown in Fig. 1 for scalars. The scalar upper bound in Fig. 1 again saturates below \( \mu \approx 1 \text{ keV} \) due to \( \alpha^* I_s(\alpha^*) = \frac{1}{24} \) for \( \alpha^* \gg 1 \), which is the same asymptotic result as in the fermionic case for \( \alpha^* I(\alpha^*) \). Therefore, since the scalar contribution in (11) is 4 times smaller than the fermionic one in (6), the universal, i.e. \( \mu \)-independent, upper bound for the charge of scalar particles is a factor of \( \sqrt{4} \) larger than the one given in (8), i.e.

\[
\varepsilon < 2.17 \times 10^{-4} \quad \text{for} \quad \mu \lesssim 1 \text{ keV}.
\]  

The dashed curve in Fig. 1 for scalars displays the upper bound as obtained from utilizing the approximate expression [7]

\[
\delta E_{VP}^s \simeq -\frac{\alpha^5 m_e}{24\pi\mu} \left(\frac{m_e}{\mu}\right)^2 \varepsilon^2
\]  

which is obtained from (11) using \( I_s(\alpha^*) \to \frac{1}{80} \) for \( \alpha^* \to 0 \). (Notice that this result is a factor of 8 smaller than the fermionic approximate result in (2).) It can be seen from Fig. 1 that the approximate scalar bound is not correct for \( \mu \) much below \( 10^5 \text{ eV} \) as in the fermionic case. Due to the different integrand in (12), as compared to (7), the difference
between the approximate and exact upper bounds is more pronounced than for fermions which is illustrated more clearly in Fig. 3 (to be compared with Fig. 2). Finally it should be mentioned that the scalar scenario is experimentally somewhat favored (cf. Table V of [2]) over the fermionic one.

Again, as in the fermionic case, our eqs. (11) and (12) are particularly suited for directly understanding the reasons for our universal bound (13) on \( \varepsilon \). Nevertheless we also present for completeness the analytic result for \( I_s \) in (12) as obtained from a straightforward integration:

\[
\alpha^*^2 I_s(\alpha^*) = \frac{1}{8\kappa^3} \left[-4\pi + \frac{\kappa}{\kappa^2 - 1} \left(-8 + \frac{20}{3}\kappa^2 + \frac{1}{3}\kappa^4\right) \right.
- \frac{1}{2(\kappa^2 - 1)} \left(16 - 24\kappa^2 + 6\kappa^4\right)L \right]
\]

with \( \kappa \) and \( L \) as in (9).

To conclude, we have shown that the contribution of very light fermions with charge \( \varepsilon e \) and mass \( \mu \) to the Lamb shift of the hydrogen atom implies a universal, i.e. \( \mu \)-independent, upper bound \( \varepsilon \lesssim 10^{-4} \) for \( \mu \lesssim 1 \) keV. This result is only obtainable by utilizing the exact expression for the Uehling potential rather than its standard approximation [11, 18] commonly used in QED. Since the scalar spin–0 scenario is experimentally somewhat favored [2], we calculated the exact upper bound for millicharged light scalar particles as well, which turns out to be a factor of 2 larger for \( \mu \lesssim 1 \) keV than for fermions.

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Figure 1: Allowed upper bounds of $\varepsilon$ according to the exact leading order contribution (6) and (11) of fermions and scalars, respectively, to the Lamb shift of the hydrogen atom corresponding to a 1$\sigma$ discrepancy of 0.01 MHz between theory and experiment. The dashed ‘fermion’ curve corresponds to the bound suggested in [3, 8] being based on the approximate expression (2) as discussed in the text. The dashed ‘scalar’ curve is obtained from the approximate expression (14). (In order to avoid any confusion it should be stressed that the area above the respective curves is excluded.)
Figure 2: As in Fig. 1 for fermions but using a linear scale for $\varepsilon$. 

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure2}
\caption{Figure 2: As in Fig. 1 for fermions but using a linear scale for $\varepsilon$.}
\end{figure}
Figure 3: As in Fig. 1 for scalars but using a linear scale for $\varepsilon$. 

$\varepsilon \times 10^4$