The Normal Dynamic Stiffness Model of Joint Interfaces Based on Fractal Theory

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Abstract. To solve the problem of the poor dynamic performance of the machine due to low stiffness of joint interfaces, a theoretical normal dynamic stiffness model of joint interfaces based on fractal theory was proposed. An improved three-dimensional WM fractal function was used to characterize the contact surface, the vibration differential equation of the sphere-plane dynamic contact was established, the linearized stiffness of asperities in elastic deformation and elastoplastic deformation regime are derived respectively, and the normal dynamic stiffness model of the joint interfaces was established. At the same time, the relationship between the dynamic stiffness and the static stiffness of the joint interfaces was established. The analysis results show that, the dynamic stiffness of the joint was approximately equal to the static stiffness when the excitation frequency was much smaller than the natural frequency, and the dynamic stiffness was much larger than the static stiffness when the excitation frequency is much larger than the natural frequency, there also existed a resonant frequency that minimizes the dynamic stiffness.

1. Introduction

The contact characteristic parameters of the joint interfaces directly affect the static performance and dynamic performance of the whole machine, and the contact stiffness research of the joint is the key point to study the static and dynamic performance of the whole machine. Huang et al.[1] proposed a classification method for the influencing factors of contact characteristics of joint interfaces, and established a model of normal contact characteristic parameters. Hijink and Wolf[2] studied the stiffness and damping characteristics of the horizontal milling machine, and its calculation model was also established. Liu et al.[3,4] established the normal and tangential contact stiffness models based on the fractal theory, and the contact stiffness model was verified by modal analysis. Yuan et al.[5] established the normal and tangential stiffness and damping models of the blade shroud and lacing based on fractal theory, and analyzed the effects of lacing and shroud on blade stiffness and vibration reduction performance. Based on the contact theory and the revised MB fractal theory, Zhang[6] established a tangential contact damping and damping loss factor model for the joint interfaces. Tian et al.[7] established the normal contact stiffness and damping model, Miao and Huang[8] studied the contact characteristics of the joint interfaces using a multi-scale analysis method.

The aim of this study is to reveal the dynamic performance of the joint interfaces. Based on the vibration differential equation of the sphere-plane dynamic contact, the linearized stiffness of asperities in elastic deformation and elastoplastic deformation regime are derived respectively, and the normal dynamic stiffness model of the joint interfaces is proposed, finally, the influencing parameters on the normal dynamic stiffness of joint interfaces are analyzed.
2. Contact model of normal static stiffness

A revised Weierstrass-Mandelbrot (W-M) function \[ W-M \] is widely used to characterize rough surface, and can be represented as

\[
z(x) = L(G/L)^{D-1} \left( \ln \gamma \right)^{1/2} \sum_{n=n_0}^{n_{\text{max}}} \gamma^{(D-2)n} \left[ \cos \phi_n - \cos \left( 2\pi \gamma x / L - \phi_n \right) \right]
\]  

(1)

Where, \( z(x) \) is the surface profile height, \( x \) is the lateral distance, \( L \) is the sample length, \( D \) is the fractal dimension, \( G \) is the fractal roughness parameter, \( \gamma \) is the scaling parameter, \( \phi_n \) is a random phase, and \( n \) is a frequency index ranging from \( n_{\text{min}} \) which represents the lowest level corresponding to the lowest cutoff frequency to \( n_{\text{max}} \) which represents the highest level corresponding to the highest cutoff frequency.

It can be seen from the literature[10] that in the elastic deformation stage, the microcontact deformation \( \delta \), the microcontact area \( a_\varepsilon \) and microcontact load \( F_\varepsilon \) of asperities when the sphere of radius \( R \) contact with a smooth rigid plane can be, respectively, expressed as,

\[
a_\varepsilon = \pi R \delta, \quad F_\varepsilon = \frac{4}{3} ER^{1/2} \delta^{3/2}
\]

(2)

In the plastic deformation stage, the microcontact area \( a_p \) and the plastic contact load \( F_p \) of the contact asperities can be, respectively, expressed as

\[
a_p = 2\pi R \delta, \quad F_p = Ha_p
\]

(3)

In the elastic and plastic deformation stage, the microcontact area \( a_{ep} \) and the plastic contact load \( F_{ep} \) of the contact asperities can be, respectively, expressed as

\[
a_{ep} = \pi R \delta \left( \frac{\delta}{\delta_{\text{av}}} \right)^b, \quad F_{ep} = 2\pi KHR \delta \left( \frac{\delta}{\delta_{\text{av}}} \right)^d
\]

(4)

From the literature[11], the specific expressions of the normal microcontact stiffness of the elastic deformation region and the elastoplastic deformation region can be obtained respectively as

\[
k_{we} = \frac{4(3-D)E}{3(2-D)(2\pi)^{1/2}} a^{n/2}
\]

(5)

\[
k_{wep} = \frac{4}{3} E \left( \frac{d-dD+D}{(2\pi)^{1/2}(2-D)} \right) \left( \frac{\pi KH}{2E} \right)^2 \left( \frac{1}{2^7-2D} \right) a^{(3-2d)/2}
\]

(6)

Therefore, the total normal static stiffness of the entire joint interfaces produced by the deformation of the microntacts is

\[
K_{nS} = \int_{a_\varepsilon}^{a_p} k_{we} n(a') da' + \int_{a_p}^{a_{ep}} k_{wep} n(a') da'
\]

(7)

3. Contact model of normal dynamic stiffness

3.1. Dynamic differential equation

The contact dynamic stiffness model of the joint interfaces is shown in Fig. 1, and \( K_{nDS} \) denotes the total normal dynamic stiffness.
In the elastic and elastoplastic deformation regime

In the plastic deformation regime

Substructure 2

Figure 1. The dynamic contact stiffness model of joint interfaces

For the steady state vibration problem, acting on a ball-plane contact on an exciting force \( F_i + \Delta F \, e^{i\omega t} \), the exciting force includes an average force and a period of resonance force, as shown in Fig. 2.

![Figure 2. Ball-plane contact vibration model](image)

The asperity under the action of the exciting force stores or consumes energy. For a single-degree-of-freedom nonlinear system of ball-plane contact, a dynamic model can be established by equivalent stiffness and equivalent resistance. Single-degree-of-freedom nonlinear harmonic differential equation[12].

\[
M \ddot{z} + C \dot{z} = F_i + \Delta F \, e^{i\omega t} - F_D
\]

where, \( F_D \) is the contact load.

Therefore, regardless of elastic deformation or elastoplastic deformation, \( F_D \) has the following form

\[
F_D = \alpha (z_s + z)^{\beta}
\]

where, the \( \alpha \) denotes spherical plane contact constant, \( \beta \) denotes the parameter related to the deformation phase of the sphere. Then, Eq. (8) can be written as

\[
M \ddot{z} + C \dot{z} + \alpha (z_s + z)^{\beta} = F_i + \Delta F \, e^{i\omega t}
\]

Theoretical static equilibrium position can be written as

\[
z_s = (F_i / \alpha)^{1/\beta}
\]

3.2. Linearization stiffness of elastic deformation

Firstly, the quasi-static process is considered, for the elastic contact, according to the Hertz elastic contact theory, the relationship between the normal load and the normal deformation for the elastic ball and the rigid plane contact model is obtained, the following equations can be inferred by Eqs.(2) and (9),

\[
\alpha = 4 \frac{E}{3} \left[ \frac{2^{3(\alpha - \xi)/2}}{G^{(\alpha-1)} \left( \ln \gamma \right)^{\xi/2}} \right]^{(\beta - D)/2}
\]
\[ \beta = \frac{3-D}{2-D} \quad (13) \]

Linearize the nonlinearity of the contact load using the Taylor expansion at the static equilibrium position,

\[ \alpha (z_s + z)^\beta = \alpha z_s^\beta + \alpha \beta z_s^{\beta-1}z = F_s + \frac{(3-D)}{(2-D)} \alpha z_s^{1(2-D)}z \quad (14) \]

Linear stiffness in the elastic deformation stage can be expressed as,

\[ k' = \frac{(3-D)}{(2-D)} \alpha z_s^{1(2-D)} \quad (15) \]

Substituting equations (2), (12), and (11) into equation (15), the following equation can be get,

\[ k' = \frac{4(3-D)}{3(2\pi)^{1/2} (2-D)} E a^{\gamma/2} \quad (16) \]

From the comparison of equations (5) and (16), the linearization stiffness of the elastic stage is equal to the static stiffness of the asperity, ie,

\[ k' = k_{se} \quad (17) \]

3.3. Linearization stiffness of elastoplastic deformation

Similarly, in the elastoplastic deformation stage, the relationship between the normal load \( F_{op} \) and the normal deformation \( \delta \) for the elastoplastic ball and the rigid plane contact model can be obtained from Eqs.(4) and (9), ie,

\[ \alpha = \frac{1}{3} KH \frac{a_y^{(d-1)(D-1)}}{2^{1-D} \pi^{(D-2)/2} G^{D-1} (\ln \gamma)^{1/2}} \left[ \frac{1}{2^{2-D} \pi^{(D-2)/2} G^{D-1} (\ln \gamma)^{1/2}} \right]^{(d-1)(1-D)+1} (2-D) \quad (18) \]

\[ \beta = \frac{2[(d-1)(1-D)+1]}{(2-D)} \quad (19) \]

Linearize the contact force using the Taylor expansion at the static equilibrium position

\[ \alpha (z_s + z)^\beta = \alpha z_s^\beta + \alpha \beta z_s^{\beta-1}z = F_s + \frac{2[(d-1)(1-D)+1]}{2-D} \alpha z_s^{1(2-D)z} \quad (20) \]

Linear stiffness of the elastoplastic deformation can be expressed as:

\[ k' = \alpha \beta z_s^{\beta-1} \quad (21) \]

Substituting Eqs.(4) (11) and (20) into Eq. (21),

\[ k' = \frac{2[(d-1)(1-D)+1]}{3(2-D) \left[ 2^{1-D} \pi^{(D-2)/2} G^{D-1} (\ln \gamma)^{1/2} \right]^{(d-1)(1-D)+1} a^{(2-D)(2d-2d+3D-2)/2} \quad (22) \]

It can be seen from the comparison of Eqs.(6) and (22) that the linearization stiffness in the elastoplastic stage is also equal to the static stiffness of the asperity, ie,
\[ k' = k_{np} \]  

(23)

### 3.4. Normal dynamic stiffness model of the joint interfaces

The frequency response function of the linearized system:

\[ H(\omega) = \frac{X}{A} = \frac{1}{1 - (\omega/\omega_0)^2 + i2\xi\omega/\omega_0} \]  

(24)

where, \( X \) is the amplitude, and \( A = F/k' \).

Frequency characteristics of the system can be obtained as,

\[ G(\omega) = \frac{X}{F} = \frac{1/k'}{1 - (\omega/\omega_0)^2 + i2\xi\omega/\omega_0} \]  

(25)

Essentially, \( G(\omega) \) represents the dynamic flexibility of the mechanical structure \( \lambda(\omega) \), that is, the reciprocal of the dynamic stiffness of the joint interfaces, ie,

\[ k_n(\omega) = \frac{1}{\lambda(\omega)} = \frac{1}{G(\omega)} \]  

(26)

The total normal dynamic stiffness of the joint interfaces \( K_nD \) is equal to the total dynamic stiffness \( K_nDS \) generated by all asperities.

\[ K_{nD} = K_{nDS} = K_a \sqrt{\left(\frac{\omega}{\omega_n}\right)^2 \left(1 - \frac{\omega}{\omega_n}\right)^2 + \left(2\xi\frac{\omega}{\omega_0}\right)^2} \]  

(27)

### 4. Analysis of influence parameters of normal dynamic stiffness

From the derived total normal dynamic stiffness model of the joint interface, the relationship between dynamic stiffness and vibration frequency is illustrated in Figure 3.

![Figure 3](image.jpg)

Figure 3. the relationship between dynamic stiffness and vibration frequency

Figure 3 shows the variations of dynamic stiffness with the vibration frequency, when \( \omega = 0 \), the normal dynamic stiffness of the joint is

\[ k_{nD}(\omega)|_{\omega=0} = K_a \]  

(28)

when \( \omega \neq 0 \), the system has a minimum dynamic stiffness. Seeking the dynamic stiffness, the following equation,
\[ \frac{\partial K_{nd}}{\partial \omega} = K_n \left[ -2 \left( 1 - \frac{\omega^2}{\omega_n^2} \right) \frac{\omega}{\omega_n} + 4 \xi^2 \frac{\omega}{\omega_n} \right] \sqrt{\left( 1 - \frac{\omega^2}{\omega_n^2} \right) + \left( 2 \xi \frac{\omega}{\omega_n} \right)^2} \]  

(29)

When \( \frac{\partial K_{nd}}{\partial \omega} = 0 \), resonance frequency of second-order system can be gotten as,

\[ \omega_r = \omega_n \sqrt{1 - 2 \xi^2} \]  

(30)

Resonant peak can be obtained by substituting the resonant frequency into the amplitude-frequency characteristic. The dynamic flexibility has a maximum value, and the dynamic stiffness has a minimum value at this moment, that is,

\[ K_D(\omega)_{\text{min}} = 2 \xi K_n \sqrt{1 - \xi^2} \]  

(31)

It can be seen from Eq.(34) that the minimum dynamic stiffness amplitude of the system is approximately equal to

\[ K_D(\omega)_{\text{min}} = 2 \xi K_n \]  

(32)

Therefore, we can get the following conclusions,

When \( \omega / \omega_n \ll 1 \), the normal dynamic stiffness of the joint interfaces is approximately equal to the static stiffness, ie., \( K_{nd} = K_n \).

When \( \omega / \omega_n \gg 1 \), the relationship between the normal dynamic stiffness and the static stiffness of the joint interfaces is approximately expressed as \( K_{nd} = K_n \left( \omega^2 / \omega_n^2 - 1 \right) \).

When the excitation frequency is constant and small, the normal dynamic stiffness of the joint interfaces is proportional to the static stiffness; at this time, the greater the system damping, the greater the dynamic stiffness. It can be seen that increasing the damping ratio of the mechanical structure or increasing the static stiffness and natural frequency of the joint can effectively improve the dynamic stiffness of the system.

5. Conclusions

The vibration differential equations of sphere-plane dynamic contact are established, the linearized static stiffness of asperities in the elastic and elastoplastic deformation stage are derived respectively based on the fractal theory, and the relationship between the dynamic stiffness and the static stiffness of the joint interfaces is established. The results show that when the excitation frequency is much smaller than the natural frequency, the dynamic stiffness of the joint interfaces is approximately equal to the static stiffness, when the excitation frequency is much larger than the natural frequency, the dynamic stiffness is much larger than the static stiffness, there exists a resonant frequency that minimizes the dynamic stiffness, and the resonant frequency is less than the natural frequency.

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