A Nadel-type vanishing theorem concerning the asymptotic multiplier ideal sheaf

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Abstract

In this paper we establish a Nadel-type vanishing theorem on a projective manifold $X$ concerning the asymptotic multiplier ideal sheaf.

1. Introduction

The celebrated Nadel vanishing theorem says that

**Theorem 1.1 (Nadel, [Nad90]).** Let $X$ be a projective manifold of dimension $n$, and let $(L, \varphi)$ be a big line bundle on $X$. Then

$$H^q(X, K_X \otimes L \otimes \mathcal{I}(\varphi)) = 0$$

for $q > 0$.

Here $\mathcal{I}(\varphi)$ refers to the multiplier ideal sheaf [Nad90] associated to $\varphi$. This theorem can be seen as the analytic counterpart of the Kawamata–Viehweg vanishing theorem [Kaw82, Vie82] in algebraic geometry, and has great applications. Then it is natural to generalise it to a pseudo-effective line bundle. One could refer to [Cao14, Eno93, Hua19, Mat14, Mat15a, Mat15b, Mat18, WaZ19] and the references therein for several generalisations. In practice, we find that the lower bound of the order $q$ such that $H^q(X, K_X \otimes L \otimes \mathcal{I}(\varphi)) = 0$ usually depends on the numerical dimension $\text{nd}(L)$ or Iitaka dimension $\kappa(L)$ [Laz04a] of $L$. For example, in Theorem 1.1, $\kappa(L) = \text{nd}(L) = n$, and the lower bound is just $0 = n - \kappa(L)$.

In this paper, we present the following Nadel-type vanishing theorem concerning the asymptotic multiplier ideal sheaf $\mathcal{I}(\|L\|)$ (see Sect.2.1).

**Theorem 1.2.** Let $X$ be a projective manifold of dimension $n$, and let $L$ be a pseudo-effective line bundle. Then we have

$$H^q(X, K_X \otimes L \otimes \mathcal{I}(\|L\|)) = 0$$

for $q > n - \kappa(L)$.

In particular, if $L$ is nef and abundant (see Sect.2.2), we have

**Corollary 1.1.** Let $X$ be a projective manifold of dimension $n$, and let $L$ be a nef and abundant line bundle. Then we have

$$H^q(X, K_X \otimes L) = 0$$

for $q > n - \text{nd}(L)$.

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Theorem 1.2 can be seen as a generalisation of the original Nadel vanishing theorem in the view of the following variant [Laz04b] of Theorem 1.1.

**Theorem 1.3 (=Theorem 11.2.12, [Laz04b]).** Let $X$ be a projective manifold of dimension $n$, and let $L$ be a big line bundle on $X$. Then

$$H^q(X, K_X \otimes L \otimes \mathcal{I}(\|L\|)) = 0$$

for $q > 0$.

On the other hand, when we deal with the asymptotic multiplier ideal sheaf instead of the multiplier ideal sheaf, the presentation of the vanishing result for a pseudo-effective line bundle is considerably simplified comparing with [Cao14, Mat14, Mat15a, Mat15b].

The proof of Theorem 1.2 uses the same strategy as [Mat14]. We first prove an injectivity theorem and an asymptotic estimate for the order of the cohomology group as follows:

**Theorem 1.4 (Injectivity theorem).** Let $X$ be a compact Kähler manifold of dimension $n$. Let $L$ and $H$ be line bundles on $X$ with $\kappa(L) \geq 0$ and $\kappa(H) \geq 0$. Let $\varphi_L$ and $\varphi_L \otimes H$ be the singular metrics on $L$ and $L \otimes H$, which is associated to $\mathcal{I}(\|L\|)$ and $\mathcal{I}(\|L \otimes H\|)$ respectively (see Sect. 2.1). Assume that $i\Theta_{L, \varphi_L} \geq \delta i\Theta_{L \otimes H, \varphi_L \otimes H}$ for some positive number $\delta$.

For a (non-zero) section $s$ of $H$, the multiplication map induced by the tensor product with $s$

$$\Phi : H^q(X, K_X \otimes L \otimes \mathcal{I}(\|L\|)) \to H^q(X, K_X \otimes L \otimes H \otimes \mathcal{I}(\|L \otimes H\|))$$

is well-defined and injective for any $q \geq 0$.

Notice that the assumptions $\kappa(L) \geq 0$ and $\kappa(H) \geq 0$ are required in order to define $\mathcal{I}(\|L\|)$ and $\mathcal{I}(\|H\|)$. To my best acknowledgement, Theorem 1.4 cannot be obtained by directly applying the available injectivity theorems, such as those in [Eno93, Fuj12, Ko86a, Mat14, Mat15a, Mat15b, Mat18]. The basic reason is that if $\varphi_H$ is the singular metric associated to $\mathcal{I}(\|H\|)$, in general we do not have $\varphi_L + \varphi_H = \varphi_L \otimes H$.

For any coherent sheaf $\mathcal{F}$, let $h^q(\mathcal{F})$ be the dimension of $H^q(X, \mathcal{F})$. Let $L^k$ be the $k$-th tensor product of a line bundle $L$. Then we have

**Theorem 1.5 (Asymptotic estimate).** Let $X$ be a projective manifold of dimension $n$, and let $L$ be a pseudo-effective line bundle on $X$. Then for any coherent sheaf $\mathcal{G}$ and $q \geq 0$, we have

$$h^q(L^k \otimes \mathcal{G} \otimes \mathcal{I}(\|L^k\|)) = O(k^{n-q}).$$

Combining with Theorems 1.4 and 1.5 we then finish the proof of Theorem 1.2. The details are provided in the text.

Eventually, we present a relative version of Theorem 1.2.

**Theorem 1.6.** Let $f : X \to Y$ be a surjective morphism between projective manifolds, and let $L$ be a pseudo-effective line bundle on $X$. Let $l$ be the dimension of a general fibre $F$ of $f$. Then

$$R^q f_* (K_X \otimes L \otimes \mathcal{I}(f, \|L\|)) = 0$$

for $q > l - \kappa(L, f)$. Here $\mathcal{I}(f, \|L\|)$ is the relative version of the asymptotic multiplier ideal sheaf (see Sect. 2.7) and $\kappa(L, f)$ is the relative Iitaka dimension (see Sect. 2.3).

We remark here that Theorem 1.6 cannot be easily obtained by applying Theorems 1.4 and 1.5 on the general fibre.
The plan of this paper is as follows. In Sect. 2 we give a brief introduction on all the background materials including the asymptotic multiplier ideal sheaf, abundant line bundle and so on. In Sect. 3 we develop the harmonic theory and prove Theorem 1.4. In Sect. 4 we prove the asymptotic estimate, i.e. Theorem 1.5. Then we prove Theorem 1.2 in next section. In the final section we extend everything to the relative setting and prove Theorem 1.6.

2. Preliminary

In this section we will introduce some basic materials. For clarity and for convenience of later reference, it will be done in a general setting, i.e. on a Kähler manifold.

Let \((X, \omega)\) be a compact Kähler manifold, and let \(L\) be a pseudo-effective line bundle on \(X\).

2.1 The asymptotic multiplier ideal sheaf

This part is mostly collected from [Laz04b].

Recall that for an arbitrary ideal sheaf \(\mathfrak{a} \subset O_X\), the associated multiplier ideal sheaf is defined as follows: let \(\mu : \tilde{X} \rightarrow X\) be a smooth modification such that \(\mu^*\mathfrak{a} = O_{\tilde{X}}(-E)\), where \(E\) has simple normal crossing support. Then given a positive real number \(c > 0\) the multiplier ideal sheaf is defined as

\[ I(c \cdot \mathfrak{a}) := \mu^* O_{\tilde{X}}(K_{\tilde{X}/X} - \lfloor cE \rfloor). \]

Here \(K_{\tilde{X}/X}\) is the relative canonical divisor and \(\lfloor cE \rfloor\) means the round-down.

Now assume that \(\kappa(L) \geq 0\). Fix a positive real number \(c > 0\). For \(k > 0\) consider the complete linear series \(|L_k|\), and form the multiplier ideal sheaf \(I(c\cdot|L_k|) \subseteq O_X\), where \(I(c\cdot|L_k|) := I(c \cdot \mathfrak{a}_k)\) with \(\mathfrak{a}_k\) being the base-ideal of \(|L_k|\). It is not hard to verify that for every integer \(p \geq 1\) one has the inclusion

\[ I\left(\frac{c}{k} |L_k|\right) \subseteq I\left(\frac{c}{pk} |L^{pk}|\right). \]

Therefore the family of ideals

\[ \{I(\frac{c}{k} |L_k|)\}_{(k \geq 0)} \]

has a unique maximal element from the ascending chain condition on ideals.

**Definition 2.1.** The asymptotic multiplier ideal sheaf associated to \(c\) and \(|L|\),

\[ I(c \|L\|) \]

is defined to be the unique maximal member among the family of ideals \(\{I(\frac{c}{k} |L_k|)\}\).

By definition, \(I(c \|L\|) = I(\frac{c}{k} |L_k|)\) for some \(k\). Let \(u_1, \ldots, u_m\) be a basis of \(H^0(X, L^k)\), then the base-ideal of \(|L_k|\) is just \(I(u_1, \ldots, u_m)\). Let \(\varphi = \log(|u_1|^2 + \cdots + |u_m|^2)\), which is a singular metric on \(L^k\). We verify that

\[ I\left(\frac{c}{k} |L_k|\right) = I\left(\frac{c}{k} \varphi\right). \]

Indeed, let \(\mu : \tilde{X} \rightarrow X\) be the smooth modification such that \(\mu^*I(u_1, \ldots, u_m) = O_{\tilde{X}}(-E)\), where \(E\) has simple normal crossing support. Then it is computed in [Dem12] that

\[ I\left(\frac{c}{k} \varphi\right) = \mu^* O_{\tilde{X}}(K_{\tilde{X}/X} - \lfloor \frac{c}{k} E \rfloor), \]
which coincides with the definition of $\mathcal{I}(\frac{c}{k}|L^k|)$. In summary, we have

$$\mathcal{I}(c\|L\|) = \mathcal{I}(\frac{c}{k}\varphi),$$

and $\frac{1}{k}\varphi$ is called the singular metric on $L$ associated to $\mathcal{I}(\|L\|)$.

Next, we introduce the relative variant. Let $f : X \to Y$ be a surjective morphism between projective manifolds, and $L$ a line bundle on $X$ whose restriction to a general fibre of $f$ has non-negative Iitaka dimension. Then there is a naturally defined homomorphism

$$\rho : f^*f_*L \to L.$$ 

Let $\mu : \tilde{X} \to X$ be a smooth modification of $|L|$ with respect to $f$, having the property that the image of the induced homomorphism

$$\mu^*\rho : \mu^*f^*\mu_*L \to \mu^*L$$

is the subsheaf $\mu^*L \otimes \mathcal{O}_{\tilde{X}}(-E)$ of $\mu^*L$, $E$ being an effective divisor on $\tilde{X}$ such that $E+\operatorname{exception}(\mu)$ has simple normal crossing support. Here $\operatorname{exception}(\mu)$ is the exceptional divisor of $\mu$. Given $c > 0$ we define

$$\mathcal{I}(f, c\|L\|) = \mu_*\mathcal{O}_{\tilde{X}}(K_{\tilde{X}/X} - [cE]).$$

Similarly, $\{\mathcal{I}(f, \frac{c}{k}|L^k|)\}_{(k \geq 0)}$ has a unique maximal element.

**Definition 2.2.** The relative asymptotic multiplier ideal sheaf associated to $f, c$ and $|L|$, 

$$\mathcal{I}(f, c\|L\|)$$

is defined to be the unique maximal member among the family of ideals $\{\mathcal{I}(f, \frac{c}{k}|L^k|)\}$.

By definition, $\mathcal{I}(f, c\|L\|) = \mathcal{I}(f, \frac{c}{k}|L^k|)$ for some $k$. Let $\rho$ be the naturally defined homomorphism

$$\rho : f^*f_*L^k \to L^k$$

by abusing the notation. Let $\mu : \tilde{X} \to X$ be the smooth modification of $|L^k|$ with respect to $f$ such that $\operatorname{Im}(\mu^*\rho) = \mu^*L^k \otimes \mathcal{O}_{\tilde{X}}(-E)$. Consider $\mu_*\mathcal{O}_{\tilde{X}}(-E)$ which is an ideal sheaf on $X$. Pick a local coordinate ball $U$ of $Y$, and let $u_1, \ldots, u_m$ be the generators of $\mu_*\mathcal{O}_{\tilde{X}}(-E)$ on $f^{-1}(U)$. The existence of these generators is obvious concerning the fact that $\operatorname{Im}(\mu^*\rho) = \mu^*L^k \otimes \mathcal{O}_{\tilde{X}}(-E)$. Moreover they can be seen as the sections of $\Gamma(f^{-1}(U), L^k)$.

Now let $\varphi_U = \log(|u_1|^2 + \cdots + |u_m|^2)$, which is a singular metric on $L^k|_{f^{-1}(U)}$. It is then easy to verify that

$$\mathcal{I}(\frac{c}{k}\varphi_U) = \mathcal{I}(f, \frac{c}{k}|L^k|) \text{ on } f^{-1}(U).$$

Furthermore, if $v_1, \ldots, v_m$ are alternative generators and $\psi_U = \log(|v_1|^2 + \cdots + |v_m|^2)$, obviously we have $\mathcal{I}(\frac{c}{k}\varphi_U) = \mathcal{I}(\frac{c}{k}\psi_U)$. Hence all the $\mathcal{I}(\frac{c}{k}\varphi_U)$ patch together to give a globally defined multiplier ideal sheaf $\mathcal{I}(\frac{c}{k}\varphi)$ such that

$$\mathcal{I}(\frac{c}{k}\varphi) = \mathcal{I}(f, \frac{c}{k}|L^k|) = \mathcal{I}(f, c\|L\|) \text{ on } X.$$ 

One should be careful that $\{\frac{1}{k}\varphi_U\}$ won’t give a globally defined metric on $L$ in general. Hence $\frac{1}{k}\varphi$ is interpreted as the collection of functions $\{\frac{1}{k}\varphi_U\}$ by abusing the notation, which is called the (local) singular metric on $L$ associated to $\mathcal{I}(f, c\|L\|)$.
Now we collect some elementary properties from [Laz04b]. Recall that for an ideal sheaf $\mathcal{I}$ on $X$, the corresponding ideal sheaf relative to $f$ is defined as

$$\mathcal{I}_f := \text{Im}(f^*f_*\mathcal{I} \to \mathcal{O}_X).$$

**Proposition 2.1.** Let $f : X \to Y$ be a surjective morphism between compact Kähler manifolds, and $H_1, H_2$ are line bundles on $X$ whose restriction to a general fibre of $f$ has non-negative Iitaka dimension. Let $L_1, L_2$ be line bundles on $X$ with non-negative Iitaka dimension. $m$ and $k$ are non-negative integers.

(i) The natural inclusion

$$H^0(X, L^k \otimes \mathcal{I}(\|L^1\|)) \to H^0(X, L^k)$$

is an isomorphism for every $k \geq 1$.

(ii) Let $a_m = a(|L^m_1|)$ be the base-ideal of $|L^m_1|$, where by convention we set $a_m = (0)$ if $|L^m_1| = \emptyset$. Then

$$a_m \cdot \mathcal{I}(\|L^k_1\|) \subseteq \mathcal{I}(\|L^m_1 \otimes L^k_1\|).$$

(iii) $\mathcal{I}(\|L^k_1\|) \supseteq \mathcal{I}(\|L^{k+1}_1\|)$ for every $k$.

(iv) Let $a_{k,f} = a(f, |H^k_1|)$ be the base-ideal of $|H^k_1|$, relative to $f$. There exits a integer $k_0$ such that for every $k \geq k_0$, the canonical map $\rho_k : f^*f_*H^k_1 \to H^k_1$ factors through the inclusion $H^k_1 \otimes \mathcal{I}(f, \|H^k_1\|)$, i.e.

$$a_{k,f} \subseteq \mathcal{I}(f, \|H^k_1\|).$$

Equivalently, the natural map

$$f_*(H^k_1 \otimes \mathcal{I}(f, \|H^k_1\|)) \to f_*(H^k_1)$$

is an isomorphism.

(v) $a_{m,f} \cdot \mathcal{I}(f, \|H^k_1\|) \subseteq \mathcal{I}(f, \|H^m_1 \otimes H^k_1\|)$.

(vi) $\mathcal{I}(f, \|H^k_1\|) \supseteq \mathcal{I}(f, \|H^{k+1}_1\|)$ for every $k$.

**Proof.** (i) is proved in [Laz04b], Proposition 11.2.10. 

(ii) Fix $p \gg 0$ and divisible enough that computes all of the multiplier ideals $\mathcal{I}(\|L^m_1\|)$, $\mathcal{I}(\|L^k_1\|)$ and $\mathcal{I}(\|L^m_1 \otimes L^k_1\|)$. Let $b_k$ be the base-ideal of $|L^k_2|$, and let $c_{m,k}$ be the base-ideal of $|L^m_1 \otimes L^k_2|$. Let $\mu : X \to X$ be the smooth modification of $a_m$, $a_{pm}$ $b_{pk}$ and $c_{pm,pk}$, such that

$$\mu^*a_m = O_X(-E), \mu^*a_{pm} = O_X(-F), \mu^*b_{pk} = O_X(-G) \text{ and } \mu^*c_{pm,pk} = O_X(-H),$$

where $E = \sum a_i E_i$, $F = \sum b_i E_i$, $G = \sum c_i E_i$ and $H = \sum d_i E_i$ have simple normal crossing support. Then for every $i$,

$$d_i \leq b_i + c_i \leq p a_i + c_i$$

and consequently

$$-a_i - \left\lfloor \frac{c_i}{p} \right\rfloor \leq -\left\lfloor \frac{d_i}{p} \right\rfloor.$$

Thus

$$a_m \cdot \mathcal{I}(\|L^k_1\|) \subseteq \mu_*(O_X(-E + K_{X/X} - \left\lfloor \frac{1}{p}G \right\rfloor))$$

$$\subseteq \mu_*(O_X(K_{X/X} - \left\lfloor \frac{1}{p}H \right\rfloor))$$

$$= \mathcal{I}(\|L^m_1 \otimes L^k_1\|).$$
(iii) is proved in [Laz04b], Proposition 11.1.8.
(iv) is proved in [Laz04b], Proposition 11.2.15.
(v) is similar with (ii), and we omit it here.

(vi) Fix $p \gg 0$ and divisible enough that computes both of the multiplier ideals $\mathcal{I}(f, k\|H_1\|)$ and $\mathcal{I}(f, \|H_1^k\|)$. Then

\[
\mathcal{I}(f, \|H_1^k\|) = \mathcal{I}(f, \frac{1}{p}|H_1^k|) \\
= \mathcal{I}(f, k|H_1^k|) \\
= \mathcal{I}(f, k\|H_1\|).
\]

Now we have

\[
\mathcal{I}(f, \|H_1^k\|) = \mathcal{I}(f, k\|H_1\|) \\
\supseteq \mathcal{I}(f, (k + 1)\|H_1\|) \\
= \mathcal{I}(f, \|H_1^{k+1}\|).
\]

\[\square\]

### 2.2 Abundant line bundle

**Definition 2.3.** A line bundle $L$ is said to be abundant if $\kappa(L) = \text{nd}(L)$.

This notion arises naturally. Moreover, A nef and abundant line bundle can be characterised by asymptotic multiplier ideal sheaf as follows.

**Theorem 2.1 (Russo, [Rus09]).** Assume that $\kappa(L) \geq 0$. Then

\[
\mathcal{I}(\|L^k\|) = \mathcal{O}_X
\]

for all $k$ if and only if $L$ is nef and abundant.

Corollary 1.1 now follows immediately from Theorems 1.2 and 2.1.

### 2.3 Relative Iitaka dimension

Let $f : X \to Y$ be a surjective morphism between projective manifolds, and $L$ a line bundle on $X$. Let $l$ be the dimension of a general fibre $F$ of $f$. We have

**Proposition 2.2.** For every coherent sheaf $\mathcal{G}$ on $X$, there is $C > 0$ (independent of $L$) such that

\[
\text{rank}(f_* (\mathcal{G} \otimes L^k)) \leq Ck^l \text{ for all } k \gg 0.
\]

**Proof.** Let us write $L = A \otimes B^{-1}$, with $A$ and $B$ are very ample line bundles. For every $k$, if we choose $E$ general in the complete linear system $|B^k|$, then a local defining function of $E$ is a non-zero divisor on $\mathcal{G}$, in which case we have an inclusion

\[
 f_* (\mathcal{G} \otimes L^k) \hookrightarrow f_* (\mathcal{G} \otimes A^k).
\]

Since $A$ is very ample, we know that there is a polynomial $P \in \mathbb{Q}[t]$ with $\deg(P) \leq l$ such that $h^0(F, \mathcal{G} \otimes A^k) = P(k)$ for $k \gg 0$. Therefore $h^0(F, \mathcal{G} \otimes L^k) \leq P(k) \leq Ck^l$ for a suitable $C > 0$ and all $k \gg 0$. \[\square\]
A Nadel-type vanishing theorem concerning the asymptotic multiplier ideal sheaf

**Definition 2.4.** The relative Iitaka dimension \( \kappa(L, f) \) of \( L \) is the biggest integer \( m \) such that there is \( C > 0 \) satisfying

\[
\text{rank}(f_*L^k) \geq Ck^m \quad \text{for all } k \gg 0
\]

with the convention that \( \kappa(L, f) = -\infty \) if \( \text{rank } f_*L^k = 0 \).

Note that \( \kappa(L, f) \) takes value in \( \{-\infty, 0, 1, \ldots, l\} \) by Proposition 2.2. In particular, if \( \kappa(L, f) = l \), we say that \( L \) is \( f \)-big.

### 3. The harmonic theory

Let \( (L, \varphi) \) be a pseudo-effective line bundle on a compact Kähler manifold \( (X, \omega) \). Assume that there exit integers \( k_0, m \) and sections \( s_1, \ldots, s_m \in L^{k_0} \) such that

\[
(|s_1|^2 + \cdots + |s_m|^2)e^{-k_0\varphi}
\]

is bounded on \( X \). In this section, we will develop the harmonic theory on such a line bundle.

#### 3.1 The harmonic forms

The Laplacian operator associated to a singular metric \( \varphi \) is not well-defined in canonical harmonic theory. Fortunately, by Demailly’s approximation technique \cite{DPS01}, we can find a family of metrics with properties (a’), (b), (c) and (d).

(a) \( \varphi_\varepsilon \) is smooth on \( X - Z_\varepsilon \) for a closed subvariety \( Z_\varepsilon \);

(b) \( \varphi \leq \varphi_{\varepsilon_1} \leq \varphi_{\varepsilon_2} \) holds for any \( 0 < \varepsilon_1 \leq \varepsilon_2 \);

(c) \( \mathcal{I}(\varphi) = \mathcal{I}(\varphi_\varepsilon) \); and

(d) \( i\Theta_{L,\varphi_\varepsilon} \geq -\varepsilon \omega \).

Since

\[
(|s_1|^2 + \cdots + |s_m|^2)e^{-k_0\varphi}
\]

is bounded on \( X \), the pole-set of \( \varphi_\varepsilon \) for every \( \varepsilon > 0 \) is contained in the subvariety

\[
Z := \{ x | s_1(x) = \cdots = s_m(x) = 0 \}
\]

by property (b). Hence, instead of (a), we can assume that

(a’) \( \varphi_\varepsilon \) is smooth on \( X - Z \), where \( Z \) is a closed subvariety of \( X \) independent of \( \varepsilon \).

Throughout this paper, when saying that \( \{ \psi_\varepsilon \} \) is a regularising sequence of a singular metric \( \psi \), we always refer to such a family of metrics with properties (a’), (b), (c) and (d).

Now let \( Y = X - Z \). We use the method in \cite{Dem82} to construct a complete Kähler metric on \( Y \) as follows. Since \( Y \) is weakly pseudo-convex, we can take a smooth plurisubharmonic exhaustion function \( \psi \) on \( X \). Define \( \tilde{\omega}_l = \omega + \frac{1}{l}i\partial\bar{\partial}\psi^2 \) for \( l \gg 0 \). It is easy to verify that \( \tilde{\omega}_l \) is a complete Kähler metric on \( Y \) and \( \tilde{\omega}_{l_1} \geq \tilde{\omega}_{l_2} \geq \omega \) for \( l_1 \leq l_2 \).

Let \( L_{(2)}^{n,q}(Y, L)_{\varphi_\varepsilon, \tilde{\omega}_l} \) be the \( L^2 \)-space of \( L \)-valued \((n, q)\)-forms on \( Y \) with respect to the inner product given by \( \varphi_\varepsilon, \tilde{\omega}_l \). Then we have the orthogonal decomposition

\[
L_{(2)}^{n,q}(Y, L)_{\varphi_\varepsilon, \tilde{\omega}_l} = \text{Im} \partial \bigoplus \mathcal{H}_{\varphi_\varepsilon, \tilde{\omega}_l}^{n,q}(L) \bigoplus \text{Im} \bar{\partial}^* \varphi_\varepsilon
\]

where

\[
\text{Im} \bar{\partial} = \text{Im}(\bar{\partial} : L_{(2)}^{n,q-1}(Y, L)_{\varphi_\varepsilon, \tilde{\omega}_l} \to L_{(2)}^{n,q}(Y, L)_{\varphi_\varepsilon, \tilde{\omega}_l}),
\]

\[
\mathcal{H}_{\varphi_\varepsilon, \tilde{\omega}_l}^{n,q}(L) = \{ \alpha \in L_{(2)}^{n,q}(Y, L)_{\varphi_\varepsilon, \tilde{\omega}_l} | \bar{\partial}alpha = 0, \bar{\partial}^* \varphi_\varepsilon alpha = 0 \}.
\]
and
\[ \text{Im} \bar{\partial}^*_{\varphi} = \text{Im}(\bar{\partial}^*_{\varphi} : L^{n,q+1}_{(2)}(Y,L)_{\varphi,\tilde{\omega}} \to L^{n,q}_{(2)}(Y,L)_{\varphi,\tilde{\omega}}). \]

We give a brief explanation for the decomposition (3.1). Usually \( \text{Im} \bar{\partial} \) is not closed in the \( L^2 \)-space of a noncompact manifold even if the metric is complete. However, in the situation we consider here, \( Y \) has the compactification \( X \), and the forms on \( Y \) are bounded in \( L^2 \)-norms. Such a form will have good extension properties. Therefore the set \( L^{n,q}_{(2)}(Y,L)_{\varphi,\tilde{\omega}} \cap \text{Im} \bar{\partial} \) behaves much like the space
\[ \text{Im}(\bar{\partial} : L^{n,q-1}_{(2)}(X,L)_{\varphi,\omega} \to L^{n,q}_{(2)}(X,L)_{\varphi,\omega}), \]
which is surely closed. The complete explanation can be found in [Fuj12, Wu17].

Now we have all the ingredients for the definition of \( \square_0 \)-harmonic forms. We denote the Laplacian operator on \( Y \) associated to \( \tilde{\omega} \) and \( \varphi \) by \( \square_{\varphi,\tilde{\omega}} \). Recall that for two \( L \)-valued \( (n,q) \)-forms \( \alpha, \beta \) (not necessary to be \( \bar{\partial} \)-closed), we say that they are cohomologically equivalent if there exists an \( L \)-valued \( (n,q-1) \)-form \( \gamma \) such that \( \alpha = \beta + \bar{\partial} \gamma \). We denote by \( \alpha \in [\beta] \) this equivalence relationship.

**Definition 3.1.** Let \( \alpha \) be an \( L \)-valued \( (n,q) \)-form on \( X \) with bounded \( L^2 \)-norm with respect to \( \omega, \varphi \). Assume that for every \( \varepsilon \ll 1 \) and \( l \gg 1 \), there exists a Dolbeault cohomological equivalent class \( \alpha_{\varepsilon,l} \in [\alpha|_{\gamma}] \) such that

(i) \( \square_{\varepsilon,l} \alpha_{\varepsilon,l} = 0 \) on \( Y \);
(ii) \( \alpha_{\varepsilon,l} \to \alpha|_{\gamma} \) in \( L^2 \)-norm.

Then we call \( \alpha \) a \( \square_0 \)-harmonic form. The space of all the \( \square_0 \)-harmonic forms is denoted by
\[ \mathcal{H}^{n,q}(X,L \otimes \mathcal{I}(\varphi)). \]

### 3.2 The Hodge-type isomorphism

Firstly, we prove a regularity result concerning the \( \square_0 \) operator.

**Proposition 3.1.** Let \( \alpha \) be an \( L \)-valued \( (n,q) \)-form (not necessary to be smooth) on \( X \) whose \( L^2 \)-norm against \( \varphi \) is bounded. Then

(i) if \( \alpha \) is \( \square_0 \)-harmonic, \( \bar{\partial}(\ast \alpha) = 0 \). Equivalently, \( \ast \alpha \) is holomorphic.
(ii) if \( \square \alpha = 0 \), \( \alpha \) must be smooth.

**Proof.** (i) We denote \( \alpha|_{\gamma} \) simply by \( \alpha_{\gamma} \). Since \( \alpha \) is \( \square_0 \)-harmonic, there exists an \( \alpha_{l,\varepsilon} \in [\alpha_{\gamma}] \) with \( \alpha_{l,\varepsilon} \in \mathcal{H}^{n,q}_{\varphi,\tilde{\omega}}(L) \) for every \( \varepsilon, l \) such that \( \lim \alpha_{l,\varepsilon} = \alpha_{\gamma} \). In particular, \( \bar{\partial} \alpha_{l,\varepsilon} = \bar{\partial}^*_{\varphi} \alpha_{l,\varepsilon} = 0 \).

Let’s recall the generalized Kodaira–Akizuki–Nakano formula in [Tak95]. Let \( \psi \) be a smooth real-valued function on \( X \), and let \( \chi \) be a smooth metric on \( L \). Then we have
\[
\| \sqrt{\eta}(\bar{\partial} + \partial \psi) \alpha \|_\chi^2 + \| \sqrt{\eta} \bar{\partial}^* \alpha \|_\chi^2 = \| \sqrt{\eta}(\partial - \partial \psi) \ast \alpha \|_\chi^2 + \| \sqrt{\eta} \partial \chi \alpha \|_\chi^2 + \eta |\Theta_{L,\chi} + \bar{\partial} \partial \psi, \Lambda| \alpha, \alpha >_\chi
\]
for any \( \alpha \in A^{p,q}(X,L) \) and \( \eta = e^{\psi} \). Here \( \chi \) is the \((1,0)\)-part of the Chern connection associated to \( \chi \). We remark here that formula (3.2) is also valid on \( Y \) since the real codimension of \( Y \) is at least 2.

Apply formula (3.2) on \( Y \) with \( \eta = 1 \), we have
\[
0 = \| \bar{\partial} \alpha_{l,\varepsilon} \|_{\varphi,\tilde{\omega}}^2 + \| \bar{\partial}^*_{\varphi} \alpha_{l,\varepsilon} \|_{\varphi,\tilde{\omega}}^2
= \| \partial_{\varphi} \alpha_{l,\varepsilon} \|_{\varphi,\tilde{\omega}}^2 + \| \bar{\partial} \alpha_{l,\varepsilon} \|_{\varphi,\tilde{\omega}}^2 + \eta |\Theta_{L,\varphi,\tilde{\omega}} + \bar{\partial} \partial \psi, \Lambda| \alpha_{l,\varepsilon}, \alpha_{l,\varepsilon} >_{\varphi,\tilde{\omega}}.
\]
A Nadel-type vanishing theorem concerning the asymptotic multiplier ideal sheaf

Remember that $i\Theta_{L,\varphi_\varepsilon} \geq -\varepsilon \omega$, thus

$$<i(\Theta_{L,\varphi_\varepsilon}, \Lambda)\alpha_l,\alpha_l>_{\varphi_\varepsilon,\tilde{\omega}_l} \geq -\varepsilon' \tilde{\omega}_l$$

by elementary computation. In particular, $\varepsilon' \to 0$ as $\varepsilon$ tends zero and $l$ tends infinity. Now take the limit on the both sides of formula (3.3) with respect to $l, \varepsilon$, we eventually obtain that

$$\lim \|\tilde{\partial}_{\varphi_\varepsilon}^*\alpha_{l,\varepsilon}\|_{\varphi_\varepsilon,\tilde{\omega}_l}^2 = \lim <i(\Theta_{L,\varphi_\varepsilon}, \Lambda)\alpha_l,\alpha_l>_{\varphi_\varepsilon,\tilde{\omega}_l} = 0.$$  

In particular,

$$0 = \lim \tilde{\partial}_{\varphi_\varepsilon}^*\alpha_{l,\varepsilon} = \ast \tilde{\partial} \ast \lim \alpha_{l,\varepsilon} = \ast \tilde{\partial} \ast \alpha$$

in $L^2$-topology. Equivalently, $\tilde{\partial} \ast \alpha = 0$ on $Y$ in analytic topology. Hence $\ast \alpha$ is a holomorphic $L$-valued $(n-q,0)$-form on $Y$. On the other hand, since $\ast \alpha$ has the bounded $L^2$-norm on $Y$, it extends to the whole space by classic $L^2$-extension theorem [Ohs02]. The extension is still denoted by $\ast \alpha$, which is an $L$-valued holomorphic $(n-q,0)$-form on $X$.

(ii) Since $\alpha = c_{n-q,\varphi_\varepsilon} \wedge \ast \alpha$, $\alpha$ must be smooth. \qed

Next, we generalise Hodge’s theorem to show that Definition 3.1 is meaningful.

**Proposition 3.2.** Let $(X, \omega)$ be a compact Kähler manifold. $(L, \varphi)$ is a pseudo-effective line bundle on $X$, and $E$ is an arbitrary vector bundle. Then the following isomorphism holds for all $k$:

$$\mathcal{H}^{n-q}(X, L^k \otimes E \otimes \mathcal{A}(k\varphi)) \simeq H^{n,q}(X, L^k \otimes E \otimes \mathcal{A}(k\varphi))$$  

(3.4)

In particular, when $\varphi$ is smooth, $\alpha \in \mathcal{H}^{n,q}(X, L)$ if and only if $\alpha$ is $\Box_0$-harmonic in the usual sense.

**Proof.** We only prove this isomorphism when $k = 1$ and $E = \mathcal{O}_X$. The general case follows the same way.

Let $\| \cdot \|_{\varphi, \omega}$ be the $L^2$-norm defined by $\varphi$ and $\omega$. We use the de Rham–Weil isomorphism

$$H^{n,q}(X, L \otimes \mathcal{A}(\varphi)) \simeq \frac{\text{Ker} \tilde{\partial} \cap L^{n,q}_{(2)}(X, L)_{\varphi, \omega}}{\text{Im} \tilde{\partial}}$$

to represent a given cohomology class $[\alpha] \in H^{n,q}(X, L \otimes \mathcal{A}(\varphi))$ by a $\tilde{\partial}$-closed $L$-valued $(n, q)$-form $\alpha$ with $\|\alpha\|_{\varphi, \omega} < \infty$. Since $\tilde{\omega}_l \geq \omega$, it is easy to verify that

$$|\alpha_Y|^2_{\omega_l} e^{-\varphi_\varepsilon} dV_{\tilde{\omega}_l} \leq |\alpha_Y|^2_{\omega} e^{-\varphi} dV_{\omega},$$

which leads to the inequality $\int_Y |\alpha_Y|^2_{\omega_l} e^{-\varphi_\varepsilon} \leq \int_Y |\alpha_Y|^2_{\omega} e^{-\varphi}$. Then we have $\|\alpha_Y\|_{\varphi_\varepsilon, \tilde{\omega}_l} \leq \|\alpha\|_{\varphi, \omega}$ which implies

$$\alpha_Y \in L^{n,q}_{(2)}(Y, L)_{\varphi_\varepsilon, \tilde{\omega}_l}.$$  

By decomposition (3.1), we have a harmonic representative $\alpha_{\varepsilon,l}$ in

$$\mathcal{H}^{n,q}_{\varphi_\varepsilon, \tilde{\omega}_l}(L),$$

which means that $\Box_{\varepsilon,l} \alpha_{\varepsilon,l} = 0$ on $Y$ for all $\varepsilon, l$. Moreover, since a harmonic representative minimizes the $L^2$-norm, we have

$$\|\alpha_{\varepsilon,l}\|_{\varphi_\varepsilon, \tilde{\omega}_l} \leq \|\alpha_Y\|_{\varphi_\varepsilon, \tilde{\omega}_l} \leq \|\alpha\|_{\varphi, \omega}.$$  

So we can take the limit $\tilde{\alpha}$ of (a subsequence of) $\{\alpha_{\varepsilon,l}\}$ such that

$$\tilde{\alpha} \in [\alpha_Y].$$
It is left to extend it to $X$.

Indeed, by the proof of Proposition 3.1 (i), $\tilde{\alpha}$ maps to a $\bar{\partial}$-closed $L$-valued $(n - q, 0)$-form $\alpha$ on $X$. We denote this morphism by $S^q(\hat{\alpha})$. Furthermore, it is shown by Proposition 2.2 in [Wu17] that $\hat{\alpha} := c_{n-q}^q \wedge S^q(\hat{\alpha})$ is an $L$-valued $(n, q)$-form with

$$\hat{\alpha}|_Y = \tilde{\alpha}.$$ 

Therefore we finally get an extension $\hat{\alpha}$ of $\tilde{\alpha}$. By definition,

$$\hat{\alpha} \in \mathcal{H}^{n,q}(X, L \otimes \mathcal{J}(\varphi)).$$

We denote this morphism by $i([\alpha]) = \hat{\alpha}$.

On the other hand, for a given $\alpha \in \mathcal{H}^{n,q}(X, L \otimes \mathcal{J}(\varphi))$, by definition there exists an $\alpha_{\varepsilon, l} \in [\alpha_Y]$ with $\alpha_{\varepsilon, l} \in \mathcal{H}^{n,q}_{\varphi_{\varepsilon, \omega}}(L)$ for every $\varepsilon, l$. In particular, $\bar{\partial}\alpha_{\varepsilon, l} = 0$. So all of the $\alpha_{\varepsilon, l}$ together with $\alpha_Y$ define a common cohomology class $[\alpha_Y]$ in $H^{n,q}(Y, L \otimes \mathcal{J}(\varphi))$. Here we use the property (c) that $\mathcal{J}(\varphi_\varepsilon) = \mathcal{J}(\varphi)$ for every $\varepsilon$. It is left to extend this class to $X$.

We use the $S^q$ again. It maps $[\alpha_Y]$ to

$$S^q(\alpha_Y) \in H^0(X, \Omega^{n-q}_X \otimes L \otimes \mathcal{J}(\varphi)).$$

Furthermore,

$$c_{n-q}^q \wedge S^q(\alpha_Y) \in H^{n,q}(X, L \otimes \mathcal{J}(\varphi))$$

with $[(c_{n-q}^q \wedge S^q(\alpha_Y))|_Y] = [\alpha_Y]$. Here we use the fact that $\omega$ is a Kähler metric. We denote this morphism by $j(\alpha) = [c_{n-q}^q \wedge S^q(\alpha_Y)]$. It is easy to verify that $i \circ j = id$ and $j \circ i = id$. The proof is finished. $\square$

### 3.3 A Kollár-type injectivity theorem

We prove Theorem 1.4 to finish this section.

**Proof of Theorem 1.4** Let $\varphi_1$ and $\varphi_2$ be the singular metrics on $L$ and $L \otimes H$ respectively mentioned in Sect 2.1 such that

$$\mathcal{J}(\|L\|) = \mathcal{J}(\varphi_1)$$

and

$$\mathcal{J}(\|L \otimes H\|) = \mathcal{J}(\varphi_2).$$

In particular, it is easy to verify that there exits sections $u_1, ..., u_m \in L^{k_1}$ and $v_1, ..., v_l \in L^{k_2} \otimes H^{k_2}$ such that $(|u_1|^2 + \cdots + |u_m|^2)e^{-k_1\varphi_1}$ and $(|v_1|^2 + \cdots + |v_l|^2)e^{-k_2\varphi_2}$ are both bounded on $X$. Moreover, $(L, \varphi_1)$ and $(L \otimes H, \varphi_2)$ are pseudo-effective. Now we apply Proposition 3.2 to obtain that

$$H^q(X, K_X \otimes L \otimes \mathcal{J}(\|L\|)) \simeq H^q(X, K_X \otimes L \otimes \mathcal{J}(\varphi_1)) \simeq \mathcal{H}^{n,q}(X, L \otimes \mathcal{J}(\varphi_1)).$$

and

$$H^q(X, K_X \otimes L \otimes H \otimes \mathcal{J}(\|L \otimes H\|)) \simeq H^q(X, K_X \otimes L \otimes H \otimes \mathcal{J}(\varphi_2)) \simeq \mathcal{H}^{n,q}(X, L \otimes H \otimes \mathcal{J}(\varphi_2)).$$

It remains to prove that

$$\otimes s : \mathcal{H}^{n,q}(X, L \otimes \mathcal{J}(\varphi_1)) \rightarrow \mathcal{H}^{n,q}(X, L \otimes H \otimes \mathcal{J}(\varphi_2))$$

is a well-defined morphism. If so, the injectivity is obvious. Let $\{\varphi_{\varepsilon, 1}\}$ be the regularising sequence of $\varphi_1$, and let $\{\varphi_{\varepsilon, 2}\}$ be the regularising sequence of $\varphi_2$. In particular, they are smooth on an
A Nadel-type vanishing theorem concerning the asymptotic multiplier ideal sheaf

open subvariety $Y$. Consider

$$
\alpha \in \mathcal{H}^{n,q}(X, L \otimes \mathcal{I}(\varphi_1)).
$$

By definition, there exists a sequence $\{\alpha_{\varepsilon,l}\}$ such that $\square_{\varepsilon,l} \alpha_{\varepsilon,l} = 0$ and $\lim \alpha_{\varepsilon,l} = \alpha$ in the sense of $L^2$-topology. In particular, $\lim \partial_{\varphi_{\varepsilon,1}, \alpha_{\varepsilon,l}} = 0$ and

$$
\lim < i[\Theta_{L,\varphi_{\varepsilon,1}, 1}, \Lambda] \alpha_{\varepsilon,l}, \alpha_{\varepsilon,l} >_{\varphi_{\varepsilon,1}, \omega_l} = 0
$$

as is shown in the proof of Proposition 3.1. Now for an $s \in H^0(X, H)$, we claim that $\square_0(s\alpha) = 0$. If so, we can verify that $s\alpha \in \mathcal{H}^{n,q}(X, L \otimes H \otimes \mathcal{I}(\|L \otimes H\|))$ as follows: since $s$ is a section of $H$, $s \in \mathfrak{a}(\|H\|)$. Thus, $s\alpha \in \mathcal{H}^{n,q}(X, L \otimes H \otimes \mathcal{I}(\|L \otimes H\|))$ by Proposition 2.1 (ii).

Now we prove the claim. Observe that $\bar{\partial}(s\alpha) = 0$,

$$
[s\alpha] \in H^q(X, K_X \otimes L \otimes H \otimes \mathcal{I}(\varphi_2)).
$$

By Proposition 3.2, there exists a sequence $\{\beta_{\varepsilon,l}\}$ on $Y$ such that $\square_{\varepsilon,l} \beta_{\varepsilon,l} = 0$ and $\beta_{\varepsilon,l} \in [s\alpha]$. It is left to prove that $\lim \beta_{\varepsilon,l} = (s\alpha)|_Y$ in the sense of $L^2$-topology. Indeed, since $\beta_{\varepsilon,l} \in [s\alpha]$, there exits an $(L \otimes H)$-valued $(n, q - 1)$-form $\gamma_{\varepsilon,l}$ such that $s\alpha = \beta_{\varepsilon,l} + \bar{\partial}\gamma_{\varepsilon,l}$ on $Y$. From $\square_{\varepsilon,l} \beta_{\varepsilon,l} = 0$, we obtain $\bar{\partial}_{\varphi_{\varepsilon,2}} \beta_{\varepsilon,l} = 0$. Now apply the formula (3.2) on $Y$, we get that

$$
\lim ||\bar{\partial}^*_{\varphi_{\varepsilon,2}, s\alpha}||^2_{\varphi_{\varepsilon,2} \omega_l} = \lim(||\bar{\partial}^*_{\varphi_{\varepsilon,2}, s\alpha}||^2_{\varphi_{\varepsilon,2} \omega_l} + < i[\Theta_{L,\varphi_{\varepsilon,2}, 1}, \Lambda](s\alpha), s\alpha >_{\varphi_{\varepsilon,2}, \omega_l}).
$$

Since $\bar{\partial}^*_{\varphi_{\varepsilon,2}, s\alpha} = s\bar{\partial}^* (s\alpha) = s s \bar{\partial}^* \alpha = s \bar{\partial}^*_{\varphi_{\varepsilon,1}} \alpha$,

$$
\lim ||\bar{\partial}^*_{\varphi_{\varepsilon,2}, s\alpha}||^2_{\varphi_{\varepsilon,2} \omega_l} = \lim ||s \bar{\partial}^*_{\varphi_{\varepsilon,1}} \alpha||^2_{\varphi_{\varepsilon,2} \omega_l} \leq C \sup_X |s|^2 e^{-\varphi_3} \lim ||\bar{\partial}^*_{\varphi_{\varepsilon,1}} \alpha||^2_{\varphi_{\varepsilon,1} \omega_l} = 0.
$$

Here $\varphi_3$ is the singular metric on $H$ defined by $\mathfrak{a}(\|H\|)$. Now we explain the inequality. Fix $p \gg 0$ and divisible enough that computes both of the multiplier ideals $\mathcal{I}(\|L\|)$ and $\mathcal{I}(\|L \otimes H\|)$. Let $\mu : \tilde{X} \rightarrow X$ be the smooth modification of $\mathfrak{a}(\|H\|)$, $\mathfrak{a}(\|L\|)$ and $\mathfrak{a}(\|L \otimes H\|)$, such that

$$
\mu^* \mathfrak{a}(\|H\|) = \mathcal{O}_X(-E), \mu^* \mathfrak{a}(\|L\|) = \mathcal{O}_X(-F) \text{ and } \mu^* \mathfrak{a}(\|L \otimes H\|) = \mathcal{O}_{\tilde{X}}(-G),
$$

where $E = \sum a_i E_i$, $F = \sum b_i E_i$ and $G = \sum c_i E_i$ have simple normal crossing support. Then for every $i$,

$$
c_i \leq p a_i + b_i
$$

and consequently

$$
\left| \frac{c_i}{p} \right| \leq a_i + \left| \frac{b_i}{p} \right|.
$$

Let $g_i$ be the local generator of $E_i$. Recall that the associated singular metrics are defined as follows:

$$
\varphi_1 = \mu_*(\Pi_i \log |g_i|^{2\frac{1}{p+1}}), \varphi_2 = \mu_*(\Pi_i \log |g_i|^{2\frac{1}{p}}) \text{ and } \varphi_3 = \mu_*(\Pi_i \log |g_i|^{2a_1})).
$$

Obviously, $\varphi_1 + \varphi_3 \leq \varphi_2 + C$ for some constant $C$, which leads to the desired inequality. Observe that $\sup_X |s|^2 e^{-\varphi_3}$ is bounded, the last equality follows.
Jingcao Wu

In summary, we obtain that $\lim \partial^*_{\varphi_{\varepsilon,2}}(s\alpha) = 0$. Similarly, 
\[
0 \leq \lim <i[\Theta_L \otimes H, \varphi_{\varepsilon,2}], \Lambda](s\alpha), s\alpha >_{\varphi_{\varepsilon,2}, \omega_l} \leq \sup_X |s|^2 e^{-\varphi_{\varepsilon,2}} \lim <i[\frac{1}{\delta}\Theta_L, \varphi_{\varepsilon,1}], \Lambda]\alpha, \alpha >_{\varphi_{\varepsilon,1}, \omega_l} = 0.
\]
We obtain that $\lim <i[\Theta_L \otimes H, \varphi_{\varepsilon,2}], \Lambda](s\alpha), s\alpha >_{\varphi_{\varepsilon,2}, \tilde{\omega}_l} = 0$. Therefore, 
\[
\lim \|\bar{\partial}^*_{\varphi_{\varepsilon,2}}(s\alpha)\|_{\varphi_{\varepsilon,2}, \omega_l} = 0.
\]

Then we have 
\[
\lim \|\bar{\partial}^*_{\varphi_{\varepsilon,2}}(s\alpha - \beta_{\varepsilon,l})\|_{\varphi_{\varepsilon,2}, \omega_l} = 0.
\]

In other words, $\lim \bar{\partial}^*_{\varphi_{\varepsilon,2}} \partial_{\gamma_{\varepsilon,l}} = 0$. Hence 
\[
\lim \|\partial_{\gamma_{\varepsilon,l}}\|_{\varphi_{\varepsilon,2}, \omega_l} = \lim <\bar{\partial}^*_{\varphi_{\varepsilon,2}}, \gamma_{\varepsilon,l} >_{\varphi_{\varepsilon,2}, \omega_l} = 0.
\]
We conclude that $\lim \bar{\partial}^*_{\varphi_{\varepsilon,2}} \partial_{\gamma_{\varepsilon,l}} = 0$. Equivalently, $\lim \beta_{\varepsilon,l} = s\alpha$ on $Y$. The proof is finished. \hfill $\square$

One refers to [Fuj12, Ko86a, Ko86b, Mat15a, Mat18] for a partial history of Kollár’s injectivity theorem.

4. An asymptotic estimate

In this section we should prove Theorem 1.5. The method is mainly borrowed from [Mat14]. Recall there is the following lemma given in [Mat14].

Lemma 4.1 (=Lemma 4.3, [Mat14]). Let $X$ be a projective manifold of dimension $n$. Let $L$ (resp. $G$) be a line bundle (resp. coherent sheaf) on $X$ and $\{I_k\}_{k=1}^\infty$ be ideal sheaves on $X$ with the following assumption:

There exists a very ample line bundle $A$ on $X$ such that $H^q(X, A^m \otimes G \otimes L^k \otimes I_k) = 0$ for any $q > 0$ and $k, m > 0$.

Then for any $q > 0$, we have 
\[
h^q(X, G \otimes L^k \otimes I_k) = O(k^{n-q}) \text{ as } k \to \infty.
\]

Apart from this, we also need the following generalisation of Nadel’s vanishing theorem.

Theorem 4.1. Let $X$ be a projective manifold of dimension $n$. Let $(E, H)$ be a Nakano (resp. semi-)positive [Dem12] vector bundle on $X$, and let $(L, \varphi)$ be a (resp. big) pseudo-effective line bundle on $X$. Then 
\[
H^q(X, K_X \otimes E \otimes L \otimes \mathcal{I}(\varphi)) = 0
\]
for $q > 0$.

Proof. Let $\{\varphi_{\varepsilon}\}$ be the regularising sequence mentioned at the beginning of Sect 3. Then 
\[
\{H \otimes e^{-\varphi_{\varepsilon}}\}
\]
A Nadel-type vanishing theorem concerning the asymptotic multiplier ideal sheaf

is a regularising approximation of $H \otimes e^{-\varphi}$. In particular, $H \otimes e^{-\varphi}$ is smooth outside a closed subvariety $Z_\varepsilon$. Let $D^{1,0}_\varepsilon$ be the $(1,0)$-part of the Chern connection on $(E \otimes L)|_{X-Z_\varepsilon}$ associated with $H \otimes e^{-\varphi_\varepsilon}$. Following the same argument in Sect.3.2, we can easily generalise Proposition 3.2 (with the same notations there) as follows:

$$H^{n,q}(X, E \otimes L \otimes \mathcal{J}(\varphi)) \simeq H^{n,q}(X, E \otimes L \otimes \mathcal{J}(\varphi)).$$

Now for any $\alpha \in H^{n,q}(X, E \otimes L \otimes \mathcal{J}(\varphi))$, we apply formula (3.2) with $\eta = 1$, $\chi = \varphi_\varepsilon$ on $X - Z_\varepsilon$ to obtain that

$$0 = \| (D^{1,0}_\varepsilon)^* \alpha \|^2_{\varphi_\varepsilon} + < i[\Theta_{E \otimes L, H \otimes e^{-\varphi_\varepsilon}}, \Lambda] \alpha, \alpha >_{\varphi_\varepsilon}.$$

Although formula (3.2) is formulated for a line bundle, we can arrange all the things for a higher rank vector bundle without obstacle. It is easy to verify that $< i[\Theta_{E \otimes L, H \otimes e^{-\varphi_\varepsilon}}, \Lambda] \alpha, \alpha >_{\varphi_\varepsilon} > 0$ if $\varepsilon \ll 1$ and $\alpha \neq 0$, which is a contradiction. Therefore we must have $\alpha = 0$. Equivalently,

$$H^q(X, K_X \otimes E \otimes L \otimes \mathcal{J}(\varphi)) = 0.$$

**Proof of Theorem 1.5.** Firstly, we prove it when $\mathcal{G}$ is locally free, i.e. it is isomorphic to a vector bundle $E$. It is sufficient to show that there exists a very ample line bundle $A$ on $X$ (independent of $k$) such that

$$H^q(X, A^m \otimes E \otimes L_k \otimes \mathcal{J}(\|L^k\|)) = 0$$

for $q > 0$ and $k, m > 0$. By taking a sufficiently ample line bundle $A$ on $X$, we may assume that $A$ is very ample and $K^{-1}_X \otimes E \otimes A$ is positive in the sense of Nakano. Since $A$ is ample, there is a smooth metric $\psi$ on $A$ with strictly positive curvature.

Now let $\varphi_k$ be the singular metric on $L^k$ (see Sect.2.1) such that

$$\mathcal{J}(\|L^k\|) = \mathcal{J}(\varphi_k).$$

Then $(A^m \otimes L^k, m\psi + \varphi_k)$ is a big line bundle for all $m$ and $k$. By Theorem 4.1 we obtain

$$H^q(X, A^m \otimes E \otimes L^k \otimes \mathcal{J}(\|L^k\|)) = H^q(X, K_X \otimes L^{-1}_X \otimes A^m \otimes E \otimes L^k \otimes \mathcal{J}(\varphi_k)) = 0.$$

Here we use the fact that $\mathcal{J}(m\psi + \varphi_k) = \mathcal{J}(\varphi_k)$.

In the end, for a general $\mathcal{G}$, there exits a free resolution [Har77] of $\mathcal{G}$, there exits a free resolution

$$0 \to E_m \to \cdots \to E_1 \to \mathcal{G} \to 0.$$

We briefly explain the existence of such a resolution. Indeed, we can assume without loss of generality that $\mathcal{G}$ is globally generated by tensoring with $A$. Then the existence is elementary (see [Kob87] for a suggestive argument). From this resolution, we eventually obtain the desired vanishing result for $\mathcal{G}$. The proof is finished.

5. Vanishing theorem

Firstly, we prove Theorem 1.2.

**Proof of Theorem 1.2.** All this really makes sense only if $\kappa(L) \geq 0$, so we shall assume this.

Next, we prove the vanishing result by contradiction. Firstly, we claim that if

$$H^{n,q}(X, L \otimes \mathcal{J}(\|L\|))$$
is non-zero,

\[ h^0(L^{k-1}) \leq \text{dim } H^{n,q}(X, L^k \otimes \mathcal{I}([L^k])) . \]

In fact, let \( \{s_j\} \) be a basis of \( H^0(X, L^{k-1}) \). Then for any \( \alpha \in H^{n,q}(X, L \otimes \mathcal{I}([L])) \), \( \{s_j \alpha\} \) is linearly independent in \( H^{n,q}(X, L^k \otimes \mathcal{I}([L^k])) \) by Theorem 1.4. Indeed, let \( \varphi_1 \) and \( \varphi_k \) be the metrics on \( L \) and \( L^k \) associated to \( \mathcal{I}([L]) \) and \( \mathcal{I}([L^k]) \) respectively. As a by-product of Proposition 2.1 (iii), we have \( \varphi_k = k \varphi_1 \). So \( i \Theta_{L^k, \varphi_k} = k i \Theta_{L, \varphi_1} \). Thus, Theorem 1.4 applies here, and it leads to the inequality.

Now suppose that \( H^{n,q}(X, L \otimes \mathcal{I}([L])) \) is non-zero for \( q > n - \kappa(L) \). We have

\[ h^0(L^{k-1}) = h^0(L^{k-1} \otimes \mathcal{I}([L^{k-1}])) \leq \text{dim } H^{n,q}(X, L^k \otimes \mathcal{I}([L^k])) . \]

The first equality comes from the Proposition 2.1 (i), and the second inequality is due to the claim. By the definition of Iitaka dimension \([\text{Laz04a}]\), we have

\[ \limsup_{k \to \infty} \frac{h^0(L^{k-1})}{(k-1)^{\kappa(L)}} > 0. \]

It means that

\[ \limsup_{k \to \infty} \frac{\text{dim } H^{n,q}(X, L^k \otimes \mathcal{I}([L^k]))}{(k-1)^{\kappa(L)}} > 0. \]

On the other hand, we have

\[ \text{dim } H^{n,q}(X, L^k \otimes \mathcal{I}([L^k])) = O(k^{n-q}) \]

by Theorem 1.5 so \( n - q \geq \kappa(L) \). It contradicts to the fact that \( q > n - \kappa(L) \). Hence

\[ H^{n,q}(X, L^k \otimes \mathcal{I}([L^k])) = 0 \]

for \( q > n - \kappa(L) \).

6. Further discussion

In order to prove Theorem 1.6 we need to extend Theorems 1.4 and 1.5 to the relative setting.

6.1 The harmonic forms: local theory

Let \((X, \omega)\) be a compact Kähler manifold of dimension \( n \), and let \( L \) be a line bundle on \( X \).

Firstly, we recall the harmonic theory in a local setting \([\text{Tak95}]\). Let \( V \) be a bounded domain with smooth boundary \( \partial V \) on \( X \). Moreover, there is a smooth plurisubharmonic exhaustion function \( r \) of \( V \) on \( X \), with \( \sup_X (|r| + |dr|) < \infty \). In particular, \( V = \{r < 0\} \) and \( dr \neq 0 \) on \( \partial V \). The volume form \( dS \) of the real hypersurface \( \partial V \) is defined by \( dS := * (dr)/|dr|_\omega \). Let \( \varphi \) be a smooth Hermitian metric on \( L \). Let \( L_{p,q}^{(2)}(V, L)_{\varphi, \omega} \) be the space of \( L \)-valued \((p,q)\)-forms on \( V \) which are \( L^2 \)-bounded with respect to \( \varphi, \omega \). Setting \( \tau := dS/|dr|_\omega \) we define the inner product on \( \partial V \) by

\[ [\alpha, \beta]_{\varphi} := \int_{\partial V} <\alpha, \beta >_\varphi \tau \]

for \( \alpha, \beta \in L_{p,q}^{(2)}(V, L)_{\varphi, \omega} \). For a smooth \((p,q)\)-form \( \gamma \), let \( e(\gamma) \) be the morphism \( \gamma \wedge \cdot \). Then by
A Nadel-type vanishing theorem concerning the asymptotic multiplier ideal sheaf

Stokes' theorem we have the following:
\[
<\bar{\partial}\alpha, \beta >_\varphi = <\alpha, \bar{\partial}^* \beta >_\varphi + \alpha, e(\bar{\partial}r)^* \beta >_\varphi \\
<\partial \alpha, \beta >_\varphi = <\alpha, \partial^* \beta >_\varphi + \alpha, e(\partial r)^* \beta >_\varphi
\]  
(6.1)

where \(\bar{\partial}^*, \partial^*\) are the adjoint operators defined on \(X\).

Now we furthermore assume that \(i\Theta_{L,\varphi} \geq -C\omega\). Based on (6.1), if \(e(\bar{\partial}r)^* \alpha = 0\), it is proved in [Tak95] that the Bochner formula on \(V\) can be formulated as
\[
\lim_{\varphi \rightharpoonup 0} \| \sqrt{\eta} (\bar{\partial} + e(\bar{\partial}r)\chi) \alpha \|^2_{\varphi, \omega} + ||\sqrt{\eta} \partial^* \alpha ||^2_{\varphi, \omega} = ||\sqrt{\eta} (\partial^* - e(\partial r)^*) \alpha ||^2_{\varphi, \omega} \\
+ ||\sqrt{\eta} \partial \alpha ||^2_{\varphi, \omega} + <\eta i [\Theta_{L,\varphi} + \partial \bar{\partial} \chi, \Lambda] \alpha, \alpha >_{\varphi, \omega}.
\]  
(6.2)

where \(\eta\) is a positive smooth function on \(X\) with \(\chi := \log \eta\).

We then define the space of harmonic forms on \(V\) by
\[
\mathcal{H}^{n,q}_{\varphi}(V, L, r, \omega) := \{ \alpha \in L^{n,q}_{(2)}(V, L)_{\varphi, \omega}; \bar{\partial} \alpha = \partial^* \alpha = e(\bar{\partial}r)^* \alpha = 0 \}.
\]

Now \((L, \varphi)\) is a pseudo-effective line bundle. Assume that there exists integers \(k_0, m\) and sections \(s_1, \ldots, s_m \in L^{k_0}\) such that
\[
(|s_1|^2 + \cdots + |s_m|^2) e^{-k_0 \varphi}
\]
is bounded on \(X\). Let \(\{\varphi_\varepsilon\}\) be the regularising sequence given at the beginning of Sect 3. Using the same notations there, the harmonic space with respect to \(\varphi\) is defined as
\[
\mathcal{H}^{n,q}(V, L \otimes I(\varphi), r) := \{ \alpha \in L^{n,q}_{(2)}(V, L)_{\varphi, \omega}; \text{there exits } \alpha_{t, \varepsilon} \in [\alpha] \text{ such that } \alpha_{t, \varepsilon} \in \mathcal{H}^{n,q}_{\varphi_\varepsilon}(V - Z, L, r, \omega_1) \text{ and } \alpha_{t, \varepsilon} \to \alpha \text{ in } L^2\text{-limit} \}.
\]

We then generalise the work in [Tak95] here.

**Proposition 6.1.** We have the following conclusions:

(i) Assume \(\alpha \in L^{n,q}_{(2)}(X, L)_{\varphi, \omega}\) satisfied \(e(\bar{\partial}r)^* \alpha = 0\) on \(V\). Then \(\alpha\) satisfies \(\bar{\partial} \alpha = \lim \partial^*_r \alpha = 0\) on \(V\) if and only if \(\bar{\partial}^* \alpha = 0\) and \(\lim <i e(\Theta_{L,\varphi_\varepsilon} + \partial \bar{\partial} r) \Lambda \alpha, \alpha >_{\varphi_\varepsilon} = 0\) on \(V\).

(ii) \(\mathcal{H}^{n,q}(V, L \otimes I(\varphi), r)\) is independent of the choice of exhaustion function \(r\).

(iii) \(\mathcal{H}^{n,q}(V, L \otimes I(\varphi), r) \simeq H^q(V, K_V \otimes L \otimes I(\varphi))\).

(iv) For Stein open subsets \(V_1, V_2\) in \(V\) such that \(V_2 \subset V_1\), the restriction map
\[
\mathcal{H}^{n,q}(V_1, I(\varphi), r) \to \mathcal{H}^{n,q}(V_2, I(\varphi), r)
\]
is well-defined, and further it satisfies the following commutative diagram:
\[
\begin{CD}
\mathcal{H}^{n,q}(V_1, I(\varphi), r) @>S^{n,q}_1>> H^0(V_1, \Omega_Y^{n-q} \otimes I(\varphi)) \\
@V{i_1}VV @VV{i_2}V \\
\mathcal{H}^{n,q}(V_2, I(\varphi), r) @>S^{n,q}_2>> H^0(V_2, \Omega_Y^{n-q} \otimes I(\varphi)).
\end{CD}
\]

**Proof.** The proof uses the same argument as Theorems 4.3 and 5.2 in [Tak95] with minor adjustment. So we only provide the necessary details.

(i) Let \(\psi = \varphi + r\) and \(\psi_\varepsilon = \varphi_\varepsilon + r\). If \(\bar{\partial} \alpha = \lim \partial^*_r \alpha = 0\), then \(\lim \partial^*_{\psi_\varepsilon} \alpha = 0\) and so \(\lim \partial \psi_\varepsilon \alpha = 0\). By formula (6.2) we obtain
\[
\lim(||\partial^*_{\psi_\varepsilon} \alpha ||^2_{\psi_\varepsilon} + <i e(\Theta_{L,\varphi_\varepsilon} + \partial \bar{\partial} r) \Lambda \alpha, \alpha >_{\psi_\varepsilon}) = 0
\]
on $V$. Since $\langle ie(\Theta_{L,\varphi_z})\Lambda \alpha, \alpha \rangle_{\psi_z} \geq -\varepsilon \omega$ and

$$\langle ie(\delta\partial r)\Lambda \alpha, \alpha \rangle_{\psi_z} \geq 0,$$

the equality above implies that

$$\ast \bar{\partial} \ast \alpha = 0 \text{ and } \lim \langle ie(\Theta_{L,\varphi_z})\Lambda \alpha, \alpha \rangle_{\psi_z} = \lim [ie(\delta\partial r)\Lambda \alpha, \alpha]_{\psi_z} = 0.$$  

Equivalently,

$$\bar{\partial} \ast \alpha = 0 \text{ and } \lim \langle ie(\Theta_{L,\varphi_z} + \delta\partial r)\Lambda \alpha, \alpha \rangle_{\psi_z} = 0.$$  

The necessity is proved.

Now assume that $\bar{\partial} \ast \alpha = 0$ and $\lim \langle imp(\Theta_{L,\varphi_z} + \delta\partial r)\Lambda \alpha, \alpha \rangle_{\psi_z} = 0$. Since $r$ is plurisubharmonic and $\lim \langle ie(\delta\partial r)\Lambda \alpha, \alpha \rangle_{\psi_z} = 0$, we have

$$\lim \langle ie(\Theta_{L,\varphi_z})\Lambda \alpha, \alpha \rangle_{\varphi_z} = \lim \langle ie(\Theta_{L,\varphi_z})\Lambda \alpha, \alpha \rangle_{\varphi_z} = 0.$$  

By formula (6.2) we have $\bar{\partial}\alpha = \lim \bar{\partial} \ast \alpha = 0$.

(ii) Let $\tau$ be an arbitrary smooth plurisubharmonic function on $V$. Donnelly and Xavier’s formula \cite{DoX84} implies that $\bar{\partial}\ast \alpha = ie(\delta\partial \tau)^\ast \alpha = ie(\delta\partial \tau)\Lambda \alpha$ if $\alpha \in H^{n,q}(V, L \otimes \mathcal{I}(\varphi), r)$.

Therefore

$$\langle ie(\delta\partial \tau)\Lambda \alpha, \alpha \rangle_{\varphi_z - \tau} = \langle \bar{\partial}\ast \alpha, \alpha \rangle_{\varphi_z - \tau} = \langle e(\partial \tau)^\ast \alpha, \bar{\partial} \ast \alpha \rangle_{\varphi_z - \tau} = \langle e(\partial \tau)^\ast \alpha, \bar{\partial} \ast \alpha \rangle_{\varphi_z - \tau} - \|e(\partial \tau)^\ast \alpha\|^2_{\varphi_z - \tau}.$$  

Here we use $\bar{\partial} \ast \alpha$, $\bar{\partial} \ast \alpha$ to denote the adjoint operators with respect to $\varphi_z - \tau$ and $\varphi_z$. Take the limit with respect to $\varepsilon$, we then obtain that

$$\langle ie(\delta\partial \tau)\Lambda \alpha, \alpha \rangle_{\varphi_z - \tau} = -\|e(\delta\partial \tau)^\ast \alpha\|^2_{\varphi_z - \tau}.$$  

Notice that $\tau$ is plurisubharmonic, we actually have

$$\langle ie(\delta\partial \tau)\Lambda \alpha, \alpha \rangle_{\varphi_z - \tau} = \|e(\delta\partial \tau)^\ast \alpha\|^2_{\varphi_z - \tau} = 0.$$  

Combine with (i), we eventually obtain that

$$H^{n,q}(V, L \otimes \mathcal{I}(\varphi), r) = H^{n,q}(V, L \otimes \mathcal{I}(\varphi), r + \tau)$$  

for any smooth plurisubharmonic $\tau$, hence the desired conclusion.

(iii) When $\varphi$ is smooth, it is proved in \cite{Tak95}, Theorem 4.5, (b). When $\varphi$ is singular, we could apply Theorem 4.5, (b) in \cite{Tak95} to its regularising sequence to obtain the desired conclusion. This approximation argument is similar with Proposition 3.2, and we omit the details here.

(iv) is intuitive due to the discussions in the global setting. In particular, $S^q_{V_i}$ with $i = 1, 2$ are similarly defined as in the proof of Proposition 3.2.

In the rest part of this paper, we are always working on the setting in Theorem 1.6. Namely, let $f : X \to Y$ be a surjective morphism between projective manifolds, and let $L$ be a pseudo-effective line bundle on $X$. Let $l$ be the dimension of a general fibre $F$ of $f$. 


Let \( \{U, r_U\} \) be a finite Stein covering of \( Y \) with smooth strictly plurisubharmonic exhaustion function \( r_U \). Let
\[
\mathcal{H}^{n,q}(f^{-1}(U), L \otimes \mathcal{I}(\varphi), f^*r_U)
\]
be the harmonic space defined above. Then the data
\[
\{\mathcal{H}^{n,q}(f^{-1}(U), L \otimes \mathcal{I}(\varphi), f^*r_U), i_U^1, i_U^2\}
\]
with the restriction morphisms
\[
i_U^1 : \mathcal{H}^{n,q}(f^{-1}(U_1), L \otimes \mathcal{I}(\varphi), f^*r_{U_1}) \to \mathcal{H}^{n,q}(f^{-1}(U_2), L \otimes \mathcal{I}(\varphi), f^*r_{U_2}),
\]
\((U_2, r_{U_2}) \subset \langle U_1, r_{U_1} \rangle \), yields a presheaf on \( Y \) by Proposition 6.1 (iv). We denote the associated sheaf by \( f_*\mathcal{H}^{n,q}(L \otimes \mathcal{I}(\varphi)). \) Since
\[
R^q f_* (K_X \otimes L \otimes \mathcal{I}(\varphi))
\]
is defined as the sheaf associated with the presheaf
\[
U \to H^q(f^{-1}(U), K_X \otimes L \otimes \mathcal{I}(\varphi)),
\]
the sheaf \( f_*\mathcal{H}^{n,q}(L \otimes \mathcal{I}(\varphi)) \) is isomorphic to \( R^q f_* (K_X \otimes L \otimes \mathcal{I}(\varphi)) \) by combing with Proposition 6.1 (iii) and (the proof of) Theorem 5.2, (i) in [Taylor95]. Moreover, the whole argument is even valid for a collection of local singular metrics \( \{f^{-1}(U), \varphi_U\} \) on \( L \) associated to \( \mathcal{I}(f, ||L||) \) (see Sect 2.1). Let \( \varphi \) denote the collection of \( \{\varphi_U\} \) by abusing the notation. Remember that \( \mathcal{I}(\varphi) \) is globally defined.

Then Proposition 6.2 is generalised as follows.

**Proposition 6.2.** Let \( \varphi \) be the associated metric (see Sect 2.1) of \( \mathcal{I}(f, ||L||) \). Then
\[
R^q f_* (K_X \otimes L \otimes \mathcal{I}(\varphi)) \simeq f_*\mathcal{H}^{n,q}(L \otimes \mathcal{I}(\varphi)).
\]

### 6.2 Injectivity theorem

In this subsection, we should extend Theorem 1.3.

**Theorem 6.1.** Let \( L \) be a line bundle on \( X \) with \( \kappa(L, f) \geq 0 \). For a (non-zero) section \( s \) of \( L \), the multiplication map induced by the tensor product with \( s \)
\[
\Phi : R^q f_* (K_X \otimes L \otimes \mathcal{I}(f, ||L||)) \to R^q f_* (K_X \otimes L^2 \otimes \mathcal{I}(f, ||L^2||))
\]
is well-defined and injective for any \( q \geq 0 \). In particular, \( R^q f_* (K_X \otimes L \otimes \mathcal{I}(f, ||L||)) \) is torsion-free for every \( q \).

**Proof.** Let \( \{U, r_U\} \) be a finite Stein covering of \( Y \) with smooth strictly plurisubharmonic exhaustion function \( r_U \). From the discussion in Sect 2.1 there is a collection of (local) singular metrics \( \varphi_1 = \{f^{-1}(U), \varphi_{U,1}\} \) on \( L \) and \( \varphi_2 = \{f^{-1}(U), \varphi_{U,2}\} \) on \( L^2 \) such that
\[
\mathcal{I}(f, ||L||) = \mathcal{I}(\varphi_1)
\]
and
\[
\mathcal{I}(f, ||L^2||) = \mathcal{I}(\varphi_2)
\]
respectively. In particular, it is a by-product of Proposition 2.1 (vi), that \( \varphi_2 = 2\varphi_1 \), namely \( \varphi_{U,2} = 2\varphi_{U,1} \) for every \( U \). Then in the view of Proposition 6.2 it is left to prove that
\[
f_*\mathcal{H}^{n,q}(L \otimes \mathcal{I}(\varphi_1)) \to f_*\mathcal{H}^{n,q}(L^2 \otimes \mathcal{I}(\varphi_2))
\]
is well-defined and injective.
Let $\alpha \in H^{n,q}(f^{-1}(U), L \otimes \mathcal{I}(\varphi_1), f^*r_U)$, and let $\{\varphi_{\varepsilon,1}\}$ be a regularising sequence of $\varphi_1$. Certainly this regularising sequence is interpreted that $\varphi_{\varepsilon,1} = \{f^{-1}(U), \varphi_{U,\varepsilon,1}\}$ and $\{\varphi_{U,\varepsilon,1}\}$ is a regularising sequence of $\varphi_{U,\varepsilon,1}$ as in Sect 3.1. Obviously $\{2\varphi_{\varepsilon,1}\}$ is a regularising sequence of $\varphi_2$. By definition, there exists a sequence $\{\alpha_{\varepsilon,l}\}$ such that $\alpha_{\varepsilon,l} \in H^{n,q}_a(f^{-1}(U) - Z, L)$ and $\lim \alpha_{\varepsilon,l} = \alpha$ in the sense of $L^2$-topology. Apply formula (6.2) to $\alpha_{\varepsilon,l}$ on $f^{-1}(U) - Z$ and remember that $e(\partial r_U)^* \alpha_{\varepsilon,l} = 0$, we obtain

$$0 = \|\bar{\partial} \alpha_{\varepsilon,l}\|^2_{\varphi_{\varepsilon,1}, \omega_1} + \|\bar{\partial} \varphi_{\varepsilon,1} \alpha_{\varepsilon,l}\|^2_{\varphi_{\varepsilon,1}, \omega_1}$$

$$= \|\partial \varphi_{\varepsilon,1} \alpha_{\varepsilon,l}\|^2_{\varphi_{\varepsilon,1}, \omega_1} + i[\Theta L, \varphi_{\varepsilon,1}, \Lambda] \alpha_{\varepsilon,l}, \alpha_{\varepsilon,l} > \varphi_{\varepsilon,1}, \omega_1.$$

Remember that $i[\Theta L, \varphi_{U,\varepsilon,1}] \geq 0$. Thus, $\lim \partial \varphi_{\varepsilon,1} \alpha_{\varepsilon,l} = 0$ and

$$\lim < i[\Theta L, \varphi_{\varepsilon,1}, \Lambda] \alpha_{\varepsilon,l}, \alpha_{\varepsilon,l} > \varphi_{\varepsilon,1}, \omega_1 = 0.$$

Now for an

$$s \in H^0(X, L),$$

we have $\bar{\partial}(s \alpha) = 0$. Let $a$ be the base-ideal of $|L|$ relative to $f$, so $s \in a$. Then

$$[s \alpha] \in H^0(f^{-1}(U), K_X \otimes L^2 \otimes \mathcal{I}(\varphi_2))$$

by Proposition 2.1 (v). By Proposition 6.2 there exists a sequence $\{\beta_{\varepsilon,l}\}$ on $f^{-1}(U) - Z$ such that $\sqcap_{\varepsilon,l} \beta_{\varepsilon,l} = 0$ and $\beta_{\varepsilon,l} \in [s \alpha]$. It is left to prove that $\lim \beta_{\varepsilon,l} = (s \alpha)|_{f^{-1}(U) - Z}$ in the sense of $L^2$-topology. Indeed, since $\beta_{\varepsilon,l} \in [s \alpha]$, there exits an $L^2$-valued $(n, q-1)$-form $\gamma_{\varepsilon,l}$ such that $s\alpha = \beta_{\varepsilon,l} + \bar{\partial} \gamma_{\varepsilon,l}$ on $f^{-1}(U) - Z$. Since $\sqcap_{\varepsilon,l} \beta_{\varepsilon,l} = 0$, $\bar{\partial} \gamma_{\varepsilon,l} = 0$. Now apply the formula (6.2) on $f^{-1}(U) - Z$, we obtain that

$$\lim \|\bar{\partial} \varphi_{\varepsilon,1} (s \alpha)\|^2_{\varphi_{\varepsilon,1}, \omega_1}$$

$$= \lim \|\partial \varphi_{\varepsilon,1} (s \alpha)\|^2_{\varphi_{\varepsilon,1}, \omega_1} + i[\Theta L, 2\varphi_{\varepsilon,1}, \Lambda] (s \alpha) > 2\varphi_{\varepsilon,1}, \omega_1 \rangle.$$

Since $\partial \varphi_{\varepsilon,1} (s \alpha) = s \bar{\partial} * (s \alpha) = s \bar{\partial} * \alpha = s \partial \varphi_{\varepsilon,1} \alpha$, we have

$$\lim \|\partial \varphi_{\varepsilon,1} (s \alpha)\|^2_{\varphi_{\varepsilon,1}, \omega_1} = \lim \|s \partial \varphi_{\varepsilon,1} \alpha\|^2_{\varphi_{\varepsilon,1}, \omega_1}$$

$$\leq \sup_X |s|^2 e^{-\varphi_3} \lim \|\partial \varphi_{\varepsilon,1} \alpha\|^2_{\varphi_{\varepsilon,1}, \omega_1}$$

$$= 0.$$

Here $\varphi_3$ is the singular metric on $L|_{f^{-1}(U)}$ defined by $a(f, |L|)$, and the inequality has been explained in the proof of Theorem 1.4. It essentially follows from the fact that

$$a(f, |L|) \cdot \mathcal{I}(f, \|L\|) \subseteq \mathcal{I}(f, \|L^2\|).$$

Since $\sup_X |s|^2 e^{-\varphi_3}$ is obviously bounded, we obtain that $\lim \partial \varphi_{\varepsilon,1} (s \alpha) = 0$. Moreover,

$$0 \leq \lim < i[\Theta L, 2\varphi_{\varepsilon,1}, \Lambda] (s \alpha), s \varphi_{\varepsilon,1}, \omega_1 \rangle$$

$$\leq \sup_X |s|^2 e^{-\varphi_3} \lim < i[\Theta L, \varphi_{\varepsilon,1}, \Lambda] \alpha, \alpha > \varphi_{\varepsilon,1}, \omega_1 \rangle$$

$$= 0.$$

We obtain that $\lim < i[\Theta L, 2\varphi_{\varepsilon,1}, \Lambda] (s \alpha), s \varphi_{\varepsilon,1}, \omega_1 \rangle = 0$. Therefore,

$$\lim \|\partial \varphi_{\varepsilon,1} (s \alpha)\|^2_{\varphi_{\varepsilon,1}, \omega_1} = 0.$$
Then we have
\[
\lim \| \bar{\partial}^*_{2\varphi,1} \bar{\partial}^*_{\gamma,1,l} \|_{2\varphi,1,\omega_l}^2 = \lim \| \bar{\partial}^*_{2\varphi,1} (s\alpha - \beta_{\varepsilon,l}) \|_{2\varphi,1,\omega_l}^2 = 0.
\]

In other words, \( \lim \bar{\partial}^*_{\varphi,2} \bar{\partial}^*_{\gamma,l} = 0 \). Hence
\[
\lim \| \bar{\partial}^*_{\gamma,l} \|_{2\varphi,1,\tilde{\omega}_l}^2 = \lim < \bar{\partial}^*_{2\varphi,1} \bar{\partial}^*_{\gamma,l} >_{\tilde{\omega}_l} = 0.
\]

We conclude that \( \lim \bar{\partial}^*_{\gamma,l} = 0 \). Equivalently, \( \lim \beta_{\varepsilon,l} = s\alpha \) on \( f^{-1}(U) - Z \). In summary,
\[
s\alpha \in H^{0,q}(f^{-1}(U), L^2 \otimes \mathcal{I}(\varphi_2)).
\]

Then we have successfully proved that
\[
f_* H^{n,q}(L \otimes \mathcal{I}(\varphi_1)) \to f_* H^{n,q}(L^2 \otimes \mathcal{I}(\varphi_2))
\]
is well-defined. The injectivity is obvious. \( \square \)

### 6.3 Asymptotic estimate and vanishing theorem

In this subsection, we should extend Theorem 1.5.

**Theorem 6.2.** Let \( L \) be a pseudo-effective line bundle on \( X \). Then for any coherent sheaf \( \mathcal{G} \) and \( q \geq 0 \), we have
\[
\text{rank} R^q f_*(L \otimes \mathcal{G} \otimes \mathcal{I}(\|L\|)) = O(k^{l-q}).
\]

**Proof.** Apply Theorem 1.5 on the general fibre, we then obtain the desired result. \( \square \)

In the end, we prove Theorem 1.6.

**Proof of Theorem 1.6.** It is trivial when \( \kappa(L, f) = -\infty \).

When \( \kappa(L, f) \geq 0 \), we use the same argument as before. Firstly, we claim that if
\[
R^q f_*(X, K_X \otimes L \otimes \mathcal{I}(\|L\|))
\]
is non-zero,
\[
\text{rank} f_* L^{k-1} \leq \text{rank} R^q f_*(X, K_X \otimes L^k \otimes \mathcal{I}(\|L^k\|)).
\]
In fact, let \( \{s_j\} \) be a local basis of \( f_* L^{k-1} \). Then for any local section
\[
\alpha \in R^q f_*(X, K_X \otimes L \otimes \mathcal{I}(\|L\|)),
\]
\( \{s_j \alpha\} \) is linearly independent in \( R^q f_*(X, K_X \otimes L^k \otimes \mathcal{I}(\|L^k\|)) \) by Theorem 6.1. It leads to the inequality.

Now suppose that \( R^q f_*(X, K_X \otimes L \otimes \mathcal{I}(\|L\|)) \) is non-zero for \( q > l - \kappa(L, f) \). We have
\[
\text{rank} f_* L^{k-1} = \text{rank} f_*(L^{k-1} \otimes \mathcal{I}(\|L^{k-1}\|)) \leq \text{rank} R^q f_*(X, K_X \otimes L^k \otimes \mathcal{I}(\|L^k\|)).
\]
The first equality comes from the Proposition 2.1 (iv), and the second inequality is due to the claim. By the definition of relative Iitaka dimension (see Sect 2.3), we have
\[
\limsup_{k \to \infty} \frac{\text{rank} f_* L^{k-1}}{(k-1)^{\kappa(L, f)}} > 0.
\]
It means that
\[
\limsup_{k \to \infty} \frac{\text{rank } R^q f_*(X, K_X \otimes L^k \otimes \mathcal{F}(\|L^k\|))}{(k-1)^{\kappa(L,f)}} > 0.
\]
On the other hand, we have
\[
\text{rank } R^q f_*(L^k \otimes G \otimes \mathcal{F}(\|L^k\|)) = O(k^{l-q})
\]
by Theorem 6.2 so \(l - q \geq \kappa(L, f)\). It contradicts to the fact that \(q > l - \kappa(L, f)\). Hence
\[
R^q f_*(X, K_X \otimes L \otimes \mathcal{F}(\|L\|)) = 0
\]
for \(q > l - \kappa(L, f)\).

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