NONLOCAL CAPILLARITY FOR ANISOTROPIC KERNELS

ALESSANDRA DE LUCA, SERENA DIPIERRO, AND ENRICO VALDINOCI

Abstract. We study a nonlocal capillarity problem with interaction kernels that are possibly anisotropic and not necessarily invariant under scaling. In particular, the lack of scale invariance will be modeled via two different fractional exponents $s_1, s_2 \in (0, 1)$ which take into account the possibility that the container and the environment present different features with respect to particle interactions.

We determine a nonlocal Young’s law for the contact angle and discuss the unique solvability of the corresponding equation in terms of the interaction kernels and of the relative adhesion coefficient.

CONTENTS

1. Introduction and main results .......................... 1
   1.1. Interaction kernels .................................. 2
   1.2. Preliminary results: existence theory and Euler-Lagrange equation .......................... 4
   1.3. Main results: nonlocal Young’s law .................. 5
   1.4. Organization of the paper ............................ 11
   2. The cancellation property in the anisotropic setting and proof of Proposition 1.3 ......... 11
   3. Nonlocal Young’s law and proofs of Theorems 1.4 and 1.6, of Corollary 1.5 and of Proposition 1.9 .......................... 17
   4. Proofs of Theorems 1.7 and 1.8 ......................... 29
   5. Unique determination of the contact angle and proof of Theorem 1.10 .................... 39
Appendix A. Existence of minimizers and proof of Proposition 1.1 .......................... 43
References ............................................ 44

1. INTRODUCTION AND MAIN RESULTS

In the classical capillarity theory (see e.g. [dG85, dGBWQ04]) the contact angle is defined as the angle $\vartheta$ at which a liquid interface meets a solid surface. At the equilibrium, this angle is expressed by the Young’s law in terms of the relative adhesion coefficient $\sigma$ via the classical formula

$$\cos(\pi - \vartheta) = \sigma.$$
The contact angle plays also an important role in the fluid spreading on a solid surface, determining also the velocity of the moving contact lines (see e.g. [dGHL90] and the references therein).

The contact angle is certainly the “macroscopic” outcome of several complex “microscopic” phenomena, involving physical chemistry, statistical physics and fluid dynamics, and ultimately relying on the effect of long-range and distance-dependent interactions between atoms or molecules (such as van der Waals forces). It is therefore of great interest to understand how the interplay between different microscopic effects generates an effective contact angle at a macroscopic scale, and to detect the proximal regions of the interfaces (likely, at a very small distance from the contact line) in which the effect of the singular long-range potentials may produce a significant effect, see e.g. [KDV79,DV83].

To further understand the role of long-range particle interactions in models related to capillarity theory, a modification of the classical Gauß free energy functional has been introduced in [MV17] that took into account surface tension energies of nonlocal type and modeled on the fractional perimeter presented in [CRS10]. These new variational principles lead to suitable equilibrium conditions that determine a specific contact angle depending on the relative adhesion coefficient and on the properties of the interaction kernel. The classical limit angle was then obtained from this long-range prescription via a limit procedure, and precise asymptotics have been provided in [DMV17]. Local minimizers in the fractional capillarity model have been studied in [DMV], where their blow-up limits at boundary points have been considered, showing, by means of a new monotonicity formula, that these blow-up limits are cones, and giving a complete characterization of such cones in the planar case.

The main goal of this paper is to present a capillarity theory of nonlocal type in which the long-range particle interactions are possibly anisotropic and not necessarily invariant under scaling. This setting is specifically motivated by the case in which the potential interactions of the droplet with the container and those with the environment are subject to different van der Waals forces. These two different interactions will be modeled here by two different fractional exponents. In this setting, we determine a nonlocal Young’s law for the contact angle, which extends the known one in the nonlocal isotropic setting and recovers the classical one as a limit case.

We now discuss in further detail the type of particle interactions that we take into account and the variational structure of the corresponding anisotropic nonlocal capillarity theory.

1.1. Interaction kernels. Owing to [CRS10], the most widely studied interaction kernel of singular type in problems related to nonlocal surface tension is

\[
K_s(\zeta) := \frac{1}{|\zeta|^{n+s}} \quad \text{for all } \zeta \in \mathbb{R}^n \setminus \{0\},
\]

with \( s \in (0,1) \). Here, we aim at considering more general kernels than the one in (1.1), with a twofold objective: on the one hand, we wish to initiate and consolidate a nonlocal capillarity theory in an anisotropic scenario; on the other hand, we want to also model the case in which the particle interaction of the container has a different structure with respect to the one of the external environment.

The first of these two goals will be pursued by considering interaction kernels that are not necessarily invariant under rotation, the second by taking into account interactions with different homogeneity inside the container and in the external environment.

More specifically, the mathematical setting in which we work is the following. Given \( n \geq 2 \), \( s \in (0,1) \), \( \lambda \geq 1 \) and \( \varrho \in (0,\infty] \), we consider the family of interaction kernels, denoted by \( \mathbf{K}(n,s,\lambda,\varrho) \), containing the even functions \( K : \mathbb{R}^n \setminus \{0\} \to [0,\infty) \) such that, for all \( \zeta \in \mathbb{R}^n \setminus \{0\} \),

\[
\frac{\chi_{B_\varrho}(\zeta)}{\lambda|\zeta|^{n+s}} \leq K(\zeta) \leq \frac{\lambda}{|\zeta|^{n+s}}.
\]
Here, we are using the notation $B_g = \mathbb{R}^n$ when $g = \infty$. Also, for every $h \in \mathbb{N}$, we consider the class $K^h(n, s, \lambda, g)$ of all the kernels $K \in K(n, s, \lambda, g) \cap C^h(\mathbb{R}^n \setminus \{0\})$ such that, for all $\zeta \in \mathbb{R}^n \setminus \{0\}$,

$$|D^jK(\zeta)| \leq \frac{\lambda}{|\zeta|^{n+s+j}} \quad \text{for all } 1 \leq j \leq h.$$  

We also say that the kernel $K$ admits a blow-up limit if for every $\zeta \in \mathbb{R}^n \setminus \{0\}$ the following limit exists:

$$K^*(\zeta) := \lim_{r \to 0^+} r^{n+s}K(r\zeta).$$

For each kernel $K$ we consider the interaction induced by $K$ between any two disjoint (measurable) subsets $E, F$ of $\mathbb{R}^n$ defined by

$$I_K(E, F) := \int_E \int_F K(x - y) \, dx \, dy.$$  

For instance, with this definition, the so-called $K$-nonlocal perimeter of a set $E$ associated to $K$ is given by the quantity $I_K(E, E)$, which is the interaction of the set $E$ with its complement with respect to $\mathbb{R}^n$ (here, as usual, we use the notation $E^c := \mathbb{R}^n \setminus E$). See [CSV19] for several results on the $K$-nonlocal perimeter. In particular, if $K$ is the fractional kernel in (1.1), then the notion of $K$-perimeter boils down to the one introduced by Caffarelli, Roquejoffre and Savin in [CRS10].

Given an open set $\Omega \subseteq \mathbb{R}^n$, $s_1, s_2 \in (0, 1)$ and $\sigma \in \mathbb{R}$, for every $K_1 \in K(n, s_1, \lambda, g)$ and $K_2 \in K(n, s_2, \lambda, g)$ and every set $E \subseteq \Omega$ we define the functional

$$\mathcal{E}(E) := I_1(E, E^c \cap \Omega) + \sigma I_2(E, \Omega^c).$$

Here above and in what follows, we use\(^1\) the short notation $I_1 := I_{K_1}$ and $I_2 := I_{K_2}$. Moreover, given a function $g \in L^\infty(\Omega)$, we let

$$\mathcal{C}(E) := \mathcal{E}(E) + \int_E g(x) \, dx.$$  

The setting that we take into account is general enough to include anisotropic nonlocal perimeter functionals as in [Lud14, CSV19], which, in turn, can be seen as nonlocal modifications of the classical anisotropic perimeter functional. In this spirit, the functional in (1.6) can be seen as a nonlocal generalization of classical anisotropic capillarity problems, such as the ones in [DPM15]. As customary in the analysis of nonlocal problems arising from geometric functionals, the long-range interactions involved in (1.6) produce significant energy contributions which will give rise to structural differences with respect to the classical case.

The goal of this article is to study the minimizers of the nonlocal capillarity functional $\mathcal{C}$ among all the sets $E$ with a given volume.

The case in which $K_1(\zeta) = K_2(\zeta) = K^*(\zeta)$ as in (1.1) has been studied in [MV17, DMV17, DMV]. Here instead we are specifically interested in the nonlocal capillarity energy in (1.6) with two different types of interactions between $E$ and $\Omega \cap E^c$ and between $E$ and $\Omega^c$, as modeled in (1.5).

\(^1\)We observe that when $\sigma > 0$, one could reabsorb it into the second interaction kernel up to redefining $K_2$ into $\sigma K_2$. In general, one can think that $\sigma$ “simply plays the role of a sign”, say it suffices to understand the matter for $\sigma \in \{-1, +1\}$, up to changing $K_2$ into $|\sigma|K_2$: indeed, if $K_2 := |\sigma|K_2$ we have that

$$\sigma I_{K_2}(E, \Omega^c) = \text{sign}(\sigma)|\sigma|\int_{E^c} \int_{\Omega^c} K_2(x - y) \, dx \, dy = \text{sign}(\sigma) I_{K_2}(E, \Omega^c).$$

However, we thought it was convenient to consider $\sigma$ as an “independent parameter”, since this makes it easier to compare with the classical case.
1.2. Preliminary results: existence theory and Euler-Lagrange equation. We now describe some basic features of the capillarity energy functional $C$ in (1.6). First of all, we have that the volume constrained minimization of this functional is well-posed, according to the following statement:

**Proposition 1.1** (Existence of minimizers). Let $K_1 \in K(n, s_1, \lambda, \varrho)$ and $K_2 \in K(n, s_2, \lambda, \varrho)$.

Let $\Omega$ be an open and bounded set with $I_1(\Omega, \Omega^c) + I_2(\Omega, \Omega^c) < +\infty$.

Let $m \in (0, |\Omega|)$ and $g \in L^\infty(\Omega)$.

Then, there exists a minimizer for the functional $C$ in (1.6) among all the sets $E$ with Lebesgue measure equal to $m$.

Moreover, $I_1(E, E^c \cap \Omega) < +\infty$ for every minimizer $E$.

In the formulation given here, Proposition 1.1 is new in the literature, though its proof relies on an appropriate variation of standard techniques, see e.g. [CRS10, MV17]. Nevertheless, we provide its proof in Appendix A, since here we would like to point out some modifications due to the facts that $\sigma \in \mathbb{R}$ and the kernels have different homogeneities, differently from [MV17].

The volume constrained minimizers (and, more generally, the volume constrained critical points) obtained in Proposition 1.1 satisfy (under reasonable regularity assumptions on the domain and on the interaction kernels) a suitable Euler-Lagrange equation, according to the following result. To state it precisely, it is convenient to denote by $\text{Reg}_E$ the collection of all those points $x_0 \in \Omega \cap \partial E$ for which there exists $\rho > 0$ and $\alpha \in (s_1, 1)$ such that $B_\rho(x_0) \cap \partial E$ is a manifold of class $C^{1,\alpha}$ possibly with boundary, and the boundary (if any) is contained in $\partial \Omega$, see Figure 1.

**Figure 1.** The geometry involved in the definition of $\text{Reg}_E$.

Given a kernel $K \in K(n, s_1, \lambda, \varrho)$, it is also convenient to recall the notion of $K$-mean curvature, that is defined, for all $x \in \Omega \cap \text{Reg}_E$, as

\begin{equation}
H^K_{\partial E}(x) := \text{p.v.} \int_{\mathbb{R}^n} K(x - y)(\chi_{E^c}(y) - \chi_E(y)) \, dy.
\end{equation}

Here p.v. stands for the principal value, that we omit from now on for the sake of simplicity of notation.

We also say that $E \subseteq \Omega$ is a critical point of $C$ among sets with prescribed Lebesgue measure if

\[
\frac{d}{dt} \bigg|_{t=0} C(f_t(E)) = 0,
\]

for every family of diffeomorphisms $\{f_t\}_{|t|<\delta}$ such that, for each $|t| < \delta$, one has that $f_0 = \text{Id}$, the support of $f_t - \text{Id}$ is a compact set, $f_t(\Omega) = \Omega$ and $|f_t(E)| = |E|$.

With this notation, we have the following result:
**Proposition 1.2** (Euler-Lagrange equation). Let $K_1 \in \mathbf{K}^1(n,s_1,\lambda,g)$ and $K_2 \in \mathbf{K}^1(n,s_2,\lambda,g)$. Let $\Omega$ be an open bounded set with $C^1$-boundary, $m \in (0,|\Omega|)$ and $g \in C^1(\mathbb{R}^n)$.

Let $E$ be a critical point of $\mathcal{C}$ in (1.6) among all the sets with Lebesgue measure equal to $m$. Then, there exists $c \in \mathbb{R}$ such that
\begin{equation}
\begin{aligned}
\int_{E \times (E^c \cap \Omega)} \text{div}_{(x,y)} \left( K_1(x-y)(T(x),T(y)) \right) \, dx \, dy \\
+ \sigma \int_{E \times \Omega} \text{div}_{(x,y)} \left( K_2(x-y)(T(x),T(y)) \right) \, dx \, dy + \int_{E} \text{div}(g \, T) = c \int_{E} \text{div} T
\end{aligned}
\end{equation}
for every $T \in C_c^\infty(\mathbb{R}^n,\mathbb{R}^n)$ with $T \cdot \nu_\Omega = 0$ on $\partial \Omega$.

Moreover, if $K_1 \in \mathbf{K}^2(n,s_1,\lambda,g)$ and $K_2 \in \mathbf{K}^2(n,s_2,\lambda,g)$, then
\begin{equation}
H_{\partial \Omega}^{K_1}(x) - \int_{\Omega^c} K_1(x-y) \, dy + \sigma \int_{\Omega^c} K_2(x-y) \, dy + g(x) = c
\end{equation}
for all $x \in \Omega \cap \text{Reg} \, E$.

The proof of Proposition 1.2 relies on a modification of techniques previously exploited in [CRS10, FFM+15, MV17]. We omit the proof here since one can follow precisely the proof of Theorem 1.3 in [MV17] with obvious modifications due to the presence of different kernels.

We now present the main results of this paper, which are focused on the determination of the contact angle.

### 1.3. Main results: nonlocal Young’s law
One of the pivotal steps of any capillarity theory is the determination of the contact angle between the droplet and the container (in jargon, the Young’s law). In our setting, this Young’s law is very sensitive to the relative homogeneity of the interacting kernels.

Loosely speaking, when $s_1 < s_2$, at small scales (which are the ones which we believe are more significant in the local determination of the contact angle), the interaction between the droplet and the exterior of the container prevails\(^2\) with respect to the one between the droplet and the interior of the container. Thus, in this situation, the sign of the relative adhesion coefficient $\sigma$ becomes determinant: in the hydrophilic regime $\sigma < 0$ the droplet is “absorbed” by the boundary of the container, thus producing a zero contact angle; instead, in the hydrorepellent regime $\sigma > 0$ the droplet is “held off” the boundary of the container, thus producing a contact angle equal to $\pi$; finally, in the neutral case $\sigma = 0$ the behavior of the second interaction kernel becomes irrelevant. When $\sigma = 0$ and additionally the problem is isotropic, the contact angle becomes $\pi/2$.

Conversely, when $s_1 > s_2$, the interaction between the droplet and the interior of the container is, at small scales, significantly stronger than that between the droplet and the exterior of the

\(^2\)For instance, when $s_1 < s_2$, $\Omega := \{x_n > 0\}$, $E := \{0 < x_n < b |x'|\}$ and $r \in (0,\varrho)$, one sees from (1.2) and the change of variables $(X,Y) := (\frac{x}{r}, \frac{y}{r})$ that
\begin{align*}
I_1(E \cap B_r, E^c \cap \Omega \cap B_r) & \leq \lambda^2 \frac{\int_{(E \cap B_r) \times (E^c \cap \Omega \cap B_r)} \frac{dx \, dy}{|x-y|^{n+s_1}}}{\int_{(E \cap B_r) \times (\Omega \cap B_r)} \frac{dx \, dy}{|x-y|^{n+s_2}}} \\
& = \lambda^2 \rho^{s_2-s_1} \frac{\int_{(E \cap B_{r\rho}) \times (E^c \cap \Omega \cap B_{r\rho})} \frac{dX \, dY}{|X-Y|^{n+s_1}}}{\int_{(E \cap B_{r\rho}) \times (\Omega \cap B_{r\rho})} \frac{dX \, dY}{|X-Y|^{n+s_2}}},
\end{align*}
which is infinitesimal when $r \searrow 0$. This suggests that in the small vicinity of contact points, when $s_1 < s_2$, the effect of the kernel $K_2$ in the determination of the energy minimizers and of their geometric properties plays a dominant role with respect to that played by $K_1$. 

5
In this situation, the relative adhesion coefficient $\sigma$ does not play any role and the contact angle is determined by an integral cancellation condition (that will be explicitly provided in (1.20)). When the first kernel is isotropic, this condition simplifies and the contact angle is proved to be $\pi/2$.

More precisely, the determination of the contact angle relies on a delicate cancellation of the singular kernel contributions, which requires the determination of an auxiliary angle which is “symmetric” (in a suitable sense of “measuring singular interactions”) with respect to the contact angle itself: this “dual contact angle” will be denoted by $\hat{\vartheta}$ and the cancellation property will be described in detail in the forthcoming formula (1.20).

The detailed analysis of the contact angle when $s_1 \neq s_2$ is given in the forthcoming Theorem 1.4. When instead $s_1 = s_2$, the internal and external interactions equally contribute at small scales. This situation will be analyzed in Theorem 1.6 and will lead to a contact angle described by an integral condition (given explicitly in (1.25) and reformulated in (1.28) below).

We now dive into the technicalities required for the determination of the contact angle. Namely, using the Euler-Lagrange equation in (1.9) and taking blow-ups along sequences of interior points converging to $\partial \Omega \cap \text{Reg}_{E}$, we derive two versions of the nonlocal Young’s law depending on whether $s_1 \neq s_2$ or $s_1 = s_2$. For this, we introduce the following notations that will be used throughout all this paper:

- given a set $F \subseteq \mathbb{R}^n$, $x_0 \in \mathbb{R}^n$ and $r > 0$, we let
  $$F^{x_0,r} := \frac{F - x_0}{r};$$
  (1.10)
- for any two angles $\vartheta_1$, $\vartheta_2 \in [0, 2\pi)$, with $\vartheta_1 < \vartheta_2$, we define
  $$J_{\vartheta_1,\vartheta_2} := \{ x \in \mathbb{R}^n : \exists \beta \in (\vartheta_1,\vartheta_2), \rho > 0 \text{ such that } (x_1, x_n) = \rho (\cos \beta, \sin \beta) \};$$
  (1.11)
- for any angle $\alpha$, we set
  $$e(\alpha) := \cos \alpha e_1 + \sin \alpha e_n.$$
  (1.12)

In order to establish the nonlocal Young’s law, we consider $K_1 \in K^2(n, s_1, \lambda, \varrho)$ and $K_2 \in K^2(n, s_2, \lambda, \varrho)$ such that the associated blow-up kernels defined as in (1.4) are well-defined and given by

$$K_1^* (\zeta) = \frac{a_1(\zeta)}{|\zeta|^{n+s_1}} \quad \text{and} \quad K_2^* (\zeta) = \frac{a_2(\zeta)}{|\zeta|^{n+s_2}},$$
(1.13)

where $\overrightarrow{\zeta} := \frac{\zeta}{|\zeta|}$ and $a_1, a_2$ are continuous functions on $\partial B_1$, bounded from above and below by two positive constants and satisfying

$$a_i(\omega) = a_i(-\omega)$$
(1.14)

for all $\omega \in \partial B_1$ and $i \in \{1, 2\}$.

Before exhibiting the main results of this paper, we premise the following result which has been thought in order to reproduce a cancellation of terms as in [MV17]. This result points out that in this context a new construction is needed since the function $a_1$ is anisotropic.

**Proposition 1.3.** Given $\vartheta \in (0, \pi)$, for every $\bar{\vartheta} \in (0, 2\pi)$ let

$$D_{\bar{\vartheta}} := \int_{J_{\vartheta,\vartheta+\bar{\vartheta}}} a_1(\frac{\overrightarrow{x - e(\bar{\vartheta})}}{|x - e(\vartheta)|^{n+s_1}}) dx - \int_{J_{\vartheta,\vartheta}} a_1(\frac{\overrightarrow{x - e(\vartheta)}}{|x - e(\vartheta)|^{n+s_1}}) dx.$$
(1.15)

Then,

$$D_{\bar{\vartheta}} \text{ is well-defined in the principal value sense; }$$
(1.16)
$$D_{\bar{\vartheta}} \text{ is continuous in } (0, 2\pi);$$
(1.17)
$$\lim_{\bar{\vartheta} \to \vartheta} D_{\bar{\vartheta}}(\bar{\vartheta}) = -\infty;$$
(1.18)
Moreover, for every $c \in \mathbb{R}$ and every angle $\vartheta \in (0, \pi)$, there exists a unique angle $\hat{\vartheta} \in (0, 2\pi)$ such that
\begin{equation}
\mathcal{D}_{\vartheta}(\hat{\vartheta}) = c.
\end{equation}

\textbf{Theorem 1.4.} Let $K_1 \in K^2(n, s_1, \lambda, \varrho)$ and $K_2 \in K^2(n, s_2, \lambda, \varrho)$. Suppose that $K_1$, $K_2$ admit blow-up limits $K_1^*$, $K_2^*$ (according to (1.4)) that satisfy assumption (1.13).

Let $g \in C^1(\mathbb{R}^n)$. Let $\Omega$ be an open bounded set with $C^1$-boundary and $E$ be a volume-constrained critical set of $\mathcal{E}$.

Let $x_0 \in \text{Reg}_E \cap \partial \Omega$ and suppose that $H$ and $V$ are open half-spaces such that
\begin{equation}
\Omega^{x_0, r} \to H \quad \text{and} \quad E^{x_0, r} \to H \cap V \quad \text{in } L^1_{\text{loc}}(\mathbb{R}^n) \text{ as } r \to 0^+.
\end{equation}

Let also $\vartheta \in [0, \pi]$ be the angle between the half-spaces $H$ and $V$, that is $H \cap V = J_{0, \vartheta}$ in the notation of (1.11).

Then, the following statements hold true.
1) If $s_1 < s_2$ and $\sigma < 0$ then $\vartheta = 0$.
2) If $s_1 < s_2$ and $\sigma > 0$ then $\vartheta = \pi$.
3) If $s_1 = s_2$, or $\sigma = 0$,

then $\vartheta \in (0, \pi)$. Also, letting $\widehat{\vartheta} \in (0, 2\pi)$ be as in (1.20) with $c = 0$, we have that $\widehat{\vartheta} = \pi - \vartheta$. Moreover, for all $v \in H \cap \partial V$,
\begin{equation}
H^{K_1^*}_{0(H \cap V)}(v) - \int_{H^c} K_1^*(v - y) dy = 0.
\end{equation}

The asymptotics in (1.21) are depicted in Figure 2. As a particular case of Theorem 1.4, we single out the special situation in which the kernel $K_1^*$ is isotropic. In this setting, the cancellation condition in (1.20) boils down to an explicit condition for the contact angle, and we have:

\textbf{Corollary 1.5.} Under the same assumptions of Theorem 1.4, we additionally suppose that $a_1 \equiv \text{const}$.

Then, the following statements hold true.
1) If $s_1 < s_2$ and $\sigma < 0$ then $\vartheta = 0$.
2) If $s_1 < s_2$ and $\sigma > 0$ then $\vartheta = \pi$. 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2.png}
\caption{The geometry involved in the asymptotics in (1.21).}
\end{figure}
3) If either \( s_1 < s_2 \) and \( \sigma = 0 \), or \( s_1 > s_2 \), then \( \vartheta = \frac{\pi}{2} \).

We exhibit below the nonlocal Young’s law in the case \( s_1 = s_2 \), which was left out of Theorem 1.4.

**Theorem 1.6.** Let \( s \in (0, 1) \) and \( K_1, K_2 \in K^2(n, s, \lambda, g) \). Suppose that \( K_1, K_2 \) admit blow-up limits \( K_1^*, K_2^* \) (according to (1.4)) that satisfy assumption (1.13). Assume that there exists \( \varepsilon_0 \in (0, 1) \) such that
\[
|\sigma| K_2(\zeta) \leq (1 - \varepsilon_0) K_1(\zeta) \quad \text{for all } \zeta \in B_{\varepsilon_0} \setminus \{0\}.
\]

Let \( g \in C^1(\mathbb{R}^n) \). Let \( \Omega \) be an open bounded set with \( C^1 \)-boundary and \( E \) be a volume-constrained critical set of \( g \).

Let \( x_0 \in \text{Reg}_E \cap \partial \Omega \) and suppose that \( H \) and \( V \) are open half-spaces such that
\[
\Omega^{x_0, r} \to H \quad \text{and} \quad E^{x_0, r} \to H \cap V \quad \text{in } L^1_{\text{loc}}(\mathbb{R}^n) \text{ as } r \to 0^+.
\]

Let also \( \vartheta \in [0, \pi] \) be the angle between the half-spaces \( H \) and \( V \), that is \( H \cap V = J_{0, \vartheta} \) in the notation of (1.11), and let \( \nu_E(x_0) := \nu_V(0) \).

Then, we have that \( \vartheta \in (0, \pi) \) and, for all \( v \in H \cap \partial V \),
\[
H^{K_1^*}_{\vartheta(H \cap V)}(v) - \int_{H^c} K_1^*(v - z) \, dz + \sigma \int_{H^c} K_2^*(v - z) \, dz = 0.
\]

Even in the very special situation in which \( K_1(\zeta) = K_2(\zeta) = \frac{1}{|\zeta|^{n+\tau}} \), Theorem 1.6 here can be seen as a strengthening of Theorem 1.4 in [MV17] (and, in particular, of formula (1.24) there): indeed, the result here establishes explicitly the nondegeneracy of the contact angle \( \vartheta \) by proving that \( \vartheta \in (0, \pi) \).

We point out that the case \( \sigma = 0 \) makes indistinguishable the setting \( s_1 = s_2 \) from that of \( s_1 \neq s_2 \): consistently with this, we observe that the contact angle prescription when \( s_1 = s_2 \), as given in (1.25), reduces to (1.23) when additionally \( \sigma = 0 \).

Also, we remark that when \( \sigma = 0 \) condition (1.24) is automatically satisfied. Furthermore, when \( K_1 = K_2 \), condition (1.24) reduces to \( \sigma \in (-1, 1) \), which is precisely the assumption taken in [MV17].

Besides, we think that the detection of a contact angle in a nonlocal capillarity setting is an interesting feature in itself, especially when we compare this situation with the stickiness phenomenon for the nonlocal minimal surfaces, as discovered in [DSV17]. More specifically, for nonlocal minimal surfaces, the long-range interactions make it possible for the surface to stick to a domain (even if the domain is smooth and convex), thus changing dramatically the boundary analysis (moreover, this phenomenon is essentially “generic”, see [DSV20]). The possible detection of the contact angle for the nonlocal capillarity theory instead highlights the fact that the boundary analysis of this theory is somewhat “sufficiently robust” with respect to the classical case. Roughly speaking, we believe that this important difference between nonlocal minimal surfaces and nonlocal capillarity theory is due to the fact that in the latter the mass is always placed in a bounded region, whence the energy contributions coming from infinity have a different nature than the ones occurring for nonlocal perimeter functionals.

We also stress that conditions (1.22) and (1.24) basically state that if the kernel \( K_2 \) is “too strong”, then one cannot expect nontrivial minimizers. Roughly speaking, while Proposition 1.1 always guarantees the existence of a minimizer, when conditions (1.22) and (1.24) are violated the minimizer can “detach from the boundary” (or “completely stick to the boundary”), hence the notion of contact angle becomes degenerate or void. That is, while for the existence of minimizers we do not need to require any bound on the relative adhesion coefficient \( \sigma \) in dependence of the interaction kernels, to speak about a contact angle some quantitative condition is in order (roughly speaking, otherwise the droplet does not meet the boundary of the container with a nontrivial angle, rather preferring to either detach from the container and float, or to completely stick at the boundary by surrounding it).
The configuration in which the droplet tends to stick to the container.

The configuration in which the droplet tends to be squashed on the container, thus producing a contact angle \( \vartheta \) close to zero, is sketched in Figure 3. The opposite situation in which the droplet tends to detach from the container, thus producing a contact angle \( \vartheta \) close to \( \pi \), is depicted in Figure 4.

These concepts are made explicit in the following exemplifying observations:

**Theorem 1.7.** Let \( \sigma > 0 \), \( \Omega := B_1 \), \( g := 0 \), \( K_1(\xi) := \frac{k_1}{|\xi|^{n+2}} \) and \( K_2(\xi) := \frac{k_2}{|\xi|^{n+2}} \), for some \( k_1, k_2 > 0 \).

Let \( E \) be a volume-constrained minimizer of \( C \). Assume that there exist \( p \in \partial B_1 \) and \( \varepsilon_0 > 0 \) such that \( B_{\varepsilon_0}(p) \cap B_1 \subseteq E \). Assume also that \( \text{Reg}_E \cap \Omega \neq \emptyset \).

Then, either \( s_1 > s_2 \), or \( s_1 = s_2 \) and \( k_1 > \sigma k_2 \).

**Theorem 1.8.** Let \( \sigma < 0 \), \( \Omega := B_1 \), \( g := 0 \), \( K_1(\xi) := \frac{k_1}{|\xi|^{n+2}} \) and \( K_2(\xi) := \frac{k_2}{|\xi|^{n+2}} \), for some \( k_1, k_2 > 0 \).

Let \( E \) be a volume-constrained minimizer of \( C \). Assume that there exist \( p \in \partial B_1 \) and \( \varepsilon_0 > 0 \) such that \( B_{\varepsilon_0}(p) \cap B_1 \subseteq (\Omega \setminus E) \). Assume also that \( \text{Reg}_E \cap \Omega \neq \emptyset \).

Then, either \( s_1 > s_2 \), or \( s_1 = s_2 \) and \( -k_1 < \sigma k_2 \).

We now reformulate the condition of contact angle according to the following result:

**Proposition 1.9.** Let \( K_1^* \) and \( K_2^* \) be as in (1.13). Let \( \sigma \in \mathbb{R} \). Assume that

(1.26)  
either s_1 = s_2, or \( \sigma = 0 \).

Let \( H \) and \( V \) be open half-spaces and let \( \vartheta \in (0, \pi) \) be the angle between \( H \) and \( V \), that is \( H \cap V = J_{0,\vartheta} \) in the notation of (1.11). Let also \( \hat{\vartheta} \in (0, 2\pi) \) be as in (1.20) with \( c := 0 \).

Suppose that there exists \( v \in H \cap \partial V \) such that

(1.27)  
\[ H^w_{\partial(H \cap V)}(v) - \int_{H^w} K_1^*(v - z) \, dz + \sigma \int_{H^w} K_2^*(v - z) \, dz = 0. \]
Then, we have that \( \vartheta \) and \( \sigma \) satisfy the relation

\[
\int_{J_{\vartheta, e}} \frac{a_1(e(\vartheta) - x)}{|e(\vartheta) - x|^{n+1}} \, dx - \int_{J_{\vartheta, e}} \frac{a_1(e(\vartheta) - x)}{|e(\vartheta) - x|^{n+1}} \, dx + \sigma \int_{H^e} \frac{a_2(e(\vartheta) - x)}{|e(\vartheta) - x|^{n+1}} \, dx = 0.
\]

A topical question in view of Proposition 1.9 is therefore to understand whether or not equation (1.28) identifies a unique contact angle \( \vartheta \). This is indeed the case, precisely under the natural condition in (1.26), according to the following result in Theorem 1.10. To state it in full generality, it is convenient to introduce some notation. Indeed, in the forthcoming computations, it comes in handy to reduce the problem to a two-dimensional situation. For this, one revisits the setting in (1.11) by defining its two-dimensional projection onto the variables \((x_1, x_n)\), namely one sets

\[
J_{\vartheta_1, \vartheta_2} := \{(x_1, x_n) \in \mathbb{R}^2 : \exists \beta \in (\vartheta_1, \vartheta_2), \rho > 0 \text{ such that } (x_1, x_n) = \rho(\cos \beta, \sin \beta)\}.
\]

Let also \( e^*(\vartheta) := (\cos \vartheta, \sin \vartheta) \) and, for every \( x = (x_1, x_2) \in \partial B_1 \subseteq \mathbb{R}^2 \) and \( j \in \{1, 2\} \),

\[
a_j^*(x) := \begin{cases} a_j(x) & \text{if } n = 2, \\ \int_{\mathbb{R}^{n-2}} a_j(x_1 e_1 + x_2 e_n + |x|(0, \bar{y}, 0)) (1 + |\bar{y}|^2)^{n+1} \, d\bar{y} & \text{if } n \geq 3. \end{cases}
\]

Let also

\[
\phi_j(\vartheta) := a_j^*(\cos \vartheta, \sin \vartheta).
\]

We remark that, as a byproduct of (1.14),

\[
a_j^*(x) = a_j^*(-x) \quad \text{and} \quad \phi_j(\vartheta) = \phi_j(\pi + \vartheta).
\]

With this framework, we can state the existence and uniqueness result for the contact angle equation as follows:

**Theorem 1.10.** Let \( K_1^* \) and \( K_2^* \) be as in (1.13). Let \( \sigma \in \mathbb{R} \) and assume that (1.26) holds true.

Then, there exists at most one \( \vartheta \in (0, \pi) \) satisfying the contact angle condition in (1.28).

Furthermore, if

\[
(1.33) \quad |\sigma| < \frac{\int_0^\pi \phi_1(\alpha) (\sin \alpha)^s \, d\alpha}{\int_0^\pi \phi_2(\alpha) (\sin \alpha)^s \, d\alpha},
\]

then there exists a unique solution \( \vartheta \in (0, \pi) \) of (1.28).

We stress once again that when \( a_1 = a_2 \) (and in particular for constant \( a_1 = a_2 \)), assumption (1.33) reduces to the structural assumption \( |\sigma| < 1 \) that was taken in [MV17].

Moreover, if \( K_1(\xi) := \frac{k_1}{|\xi|^s} \) and \( K_2(\xi) := \frac{k_2}{|\xi|^s} \) for some \( k_1, k_2 > 0 \), then assumption (1.33) boils down to \( |\sigma| < \frac{k_1}{k_2} \), which is precisely the condition for nontrivial minimizers obtained in Theorems 1.7 and 1.8.

For these reasons, Theorem 1.10 showcases the interesting fact that the equation prescribing the contact angle in (1.28) admits one and only one solution precisely in the natural range of kernels given by (1.26) and (1.33).

Additionally, as we will point out in Remark 5.3 at the end of Section 5, the uniqueness statement in Theorem 1.10 heavily depends on the strict positivity of the kernel and it fails for kernels that are merely nonnegative.
1.4. Organization of the paper. The rest of the paper is organized as follows. In Section 2 we provide the proof of the cancellation property stated in Proposition 1.3.

In Section 3 we prove the nonlocal Young’s law in Theorems 1.4 and 1.6 and Proposition 1.9, as well as Corollary 1.5. Section 4 deals with the possible complete stickiness or detachment of the nonlocal droplets and it presents the proofs of Theorems 1.7 and 1.8. Section 5 is devoted to the existence and uniqueness theory of the equation prescribing the contact angle and contains the proof of Theorem 1.10.

Finally, the proof of Proposition 1.1 is contained in Appendix A.

2. The cancellation property in the anisotropic setting and proof of Proposition 1.3

In this section we prove the desired cancellation property stated in Proposition 1.3. The argument relies on a delicate analysis of the geometric properties of the integrals involved in the definition of the function in (1.15).

Proof of Proposition 1.3. We focus on the proof of (1.16), (1.17), (1.18) and (1.19): once these statements are proved, we can conclude that there exists an angle \( \hat{\theta} \in (0, 2\pi) \) such that \( \mathcal{D}_\theta(\hat{\theta}) = 0 \), and this angle is unique since \( \mathcal{D}_\theta \) is strictly increasing, thus establishing (1.20).

We start by proving (1.16). For this, we observe that the definition in (1.15) has to be interpreted in the principal-value sense, namely

\[
\mathcal{D}_\theta(\hat{\theta}) = \lim_{\rho \searrow 0} \left( \int_{J_{\theta,\vartheta+\hat{\theta}} \setminus B_\rho(\epsilon(\hat{\theta}))} \frac{a_1(x - e(\hat{\theta}))}{|x - e(\hat{\theta})|^{n+s_1}} \, dx - \int_{J_{\theta,\vartheta} \setminus B_\rho(\epsilon(\hat{\theta}))} \frac{a_1(x - e(\hat{\theta}))}{|x - e(\hat{\theta})|^{n+s_1}} \, dx \right).
\]

Hence, to establish (1.16), we want to show that the limit in (2.1) does exist and is finite. To this end, given \( \hat{\theta} \in (0, 2\pi) \), we let \( \delta := \min\{\sin \hat{\theta}, \sin \vartheta\} \) and we note that \( B_\rho(\epsilon(\hat{\theta})) \) is contained in \( J_{\theta,\vartheta+\hat{\theta}} \). Then, for every \( \rho \in (0, \delta] \) we set

\[
f(\rho) := \int_{J_{\theta,\vartheta+\hat{\theta}} \setminus B_\rho(\epsilon(\hat{\theta}))} \frac{a_1(x - e(\hat{\theta}))}{|x - e(\hat{\theta})|^{n+s_1}} \, dx - \int_{J_{\theta,\vartheta} \setminus B_\rho(\epsilon(\hat{\theta}))} \frac{a_1(x - e(\hat{\theta}))}{|x - e(\hat{\theta})|^{n+s_1}} \, dx.
\]

We also define \( A_{\delta,\hat{\theta}}(\epsilon(\hat{\theta})) := B_\delta(\epsilon(\hat{\theta})) \setminus B_\rho(\epsilon(\hat{\theta})) \), see Figure 5. By the change of variable \( x \mapsto 2\epsilon(\hat{\theta}) - x \), we see that

\[
\int_{J_{\theta,\vartheta+\hat{\theta}} \cap A_{\delta,\hat{\theta}}(\epsilon(\hat{\theta}))} \frac{a_1(x - e(\hat{\theta}))}{|x - e(\hat{\theta})|^{n+s_1}} \, dx - \int_{J_{\theta,\vartheta} \cap A_{\delta,\hat{\theta}}(\epsilon(\hat{\theta}))} \frac{a_1(x - e(\hat{\theta}))}{|x - e(\hat{\theta})|^{n+s_1}} \, dx = 0,
\]

since \( a_1 \) is symmetric. From this, we deduce that for every \( \rho \in (0, \delta] \)

\[
f(\rho) - f(\delta) = \int_{J_{\theta,\vartheta+\hat{\theta}} \cap A_{\delta,\hat{\theta}}(\epsilon(\hat{\theta}))} \frac{a_1(x - e(\hat{\theta}))}{|x - e(\hat{\theta})|^{n+s_1}} \, dx - \int_{J_{\theta,\vartheta} \cap A_{\delta,\hat{\theta}}(\epsilon(\hat{\theta}))} \frac{a_1(x - e(\hat{\theta}))}{|x - e(\hat{\theta})|^{n+s_1}} \, dx = 0.
\]

Hence we conclude that

\[
\lim_{\rho \searrow 0} f(\rho) = f(\delta),
\]

thus proving the existence and finiteness of the limit in (2.1).

This completes the proof of (1.16) and we now focus on the proof of (1.17).

For this, we notice that, if \( \bar{\theta}, \hat{\theta} \in (0, 2\pi) \) with \( \bar{\theta} \geq \hat{\theta} \),

\[
\mathcal{D}_\theta(\hat{\theta}) - \mathcal{D}_\theta(\bar{\theta}) = \int_{J_{\theta,\vartheta+\hat{\theta}}} \frac{a_1(x - e(\hat{\theta}))}{|x - e(\hat{\theta})|^{n+s_1}} \, dx - \int_{J_{\theta,\vartheta+\bar{\theta}}} \frac{a_1(x - e(\bar{\theta}))}{|x - e(\bar{\theta})|^{n+s_1}} \, dx.
\]
\[ \Xi = \lim_{a \to 0} a \frac{(x - e(\vartheta))}{|x - e(\vartheta)|^{n+s_1}} \, dx. \]

Since the denominator in the latter integral is bounded from below by a positive constant (depending on $\vartheta$), the claim in (1.17) follows from the Dominated Convergence Theorem.

We now deal with the proof of (1.18) and (1.19). To this end, we first prove that

\[ \lim_{\varepsilon \to 0} \left( \int_{J_{\theta - \varepsilon, \theta + \varepsilon}} a_1 \frac{(x - e(\vartheta))}{|x - e(\vartheta)|^{n+s_1}} \, dx - \int_{J_{\theta, \theta + 2\varepsilon}} a_1 \frac{(x - e(\vartheta))}{|x - e(\vartheta)|^{n+s_1}} \, dx \right) = -\infty \]

(2.3)

and

\[ \lim_{\varepsilon \to 0} \left( \int_{J_{\theta - 2\varepsilon, \theta + \varepsilon}} a_1 \frac{(x - e(\vartheta))}{|x - e(\vartheta)|^{n+s_1}} \, dx - \int_{J_{\theta, \theta + \varepsilon}} a_1 \frac{(x - e(\vartheta))}{|x - e(\vartheta)|^{n+s_1}} \, dx \right) = +\infty. \]

We focus on the proof of the first claim in (2.3) since a similar argument would take care of the second one. For this, let $\Xi$ be the first limit in (2.3) and $R$ be the rotation by an angle $\vartheta$ in the $(x_1, x_n)$ plane that sends $e(\vartheta)$ in $e_1 = (1, 0, \ldots, 0)$. Let also $a_{1, \vartheta} := a_1 \circ R$ and notice that $a_{1, \vartheta}$ inherits the properties of $a_1$, that is $a_{1, \vartheta}$ is a continuous functions on $\partial B_1$, bounded from above and below by two positive constants and satisfying $a_{1, \vartheta}(\omega) = a_{1, \vartheta}(-\omega)$ for all $\omega \in \partial B_1$.

With this notation, we have

\[ \Xi = \lim_{\varepsilon \to 0} \left( \int_{J_{\theta - \varepsilon, \theta}} a_{1, \vartheta} \frac{(x - e_1)}{|x - e_1|^{n+s_1}} \, dx - \int_{J_{\theta, \theta + 2\varepsilon}} a_{1, \vartheta} \frac{(x - e_1)}{|x - e_1|^{n+s_1}} \, dx \right). \]

(2.4)

We also remark that, in view of the boundedness of $a_{1, \vartheta}$,

\[ \int_{J_{-2\varepsilon, \varepsilon}} a_{1, \vartheta} \frac{(x - e_1)}{|x - e_1|^{n+s_1}} \, dx \leq \int_{\mathbb{R}^n \setminus B_1} \frac{C}{|y|^{n+s_1}} \, dy \leq C, \]

for a suitable constant $C \geq 1$ possibly varying from step to step.

Combining this information with (2.4) we find that

\[ \Xi \leq \lim_{\varepsilon \to 0} \left( \int_{J_{-\varepsilon, \varepsilon}} a_{1, \vartheta} \frac{(x - e_1)}{|x - e_1|^{n+s_1}} \, dx - \int_{J_{0, 2\varepsilon}} a_{1, \vartheta} \frac{(x - e_1)}{|x - e_1|^{n+s_1}} \, dx + C \right). \]

(2.5)

Now we claim that, if $\varepsilon$ is sufficiently small,

\[ B_{\varepsilon/10} \left( e_1 + \frac{3\varepsilon}{2} e_n \right) \subseteq J_{\varepsilon, 2\varepsilon} \cap B_2. \]

(2.6)
To check this, let $y \in B_{\varepsilon/10} \left( e_1 + \frac{3\varepsilon}{2} e_n \right)$. Then,

$$
\frac{\varepsilon^2}{100} \geq |y_1 - 1|^2 + \left| y_n - \frac{3\varepsilon}{2} \right|^2
$$

and accordingly $y_1 \in \left[ 1 - \frac{\varepsilon}{10}, 1 + \frac{\varepsilon}{10} \right]$ and $y_n \in \left[ \frac{7\varepsilon}{5}, \frac{8\varepsilon}{5} \right]$. As a consequence, if $\varepsilon$ is conveniently small,

$$
y_n \over y_1 \in \left[ \frac{7\varepsilon}{5}, \frac{8\varepsilon}{5} \right] \subseteq \left[ \frac{6\varepsilon}{5}, \frac{9\varepsilon}{5} \right] \subseteq \left[ \tan \varepsilon, \tan(2\varepsilon) \right],
$$

which, recalling (1.11), establishes (2.6).

Using (2.6) and the assumption that $a_{1,\vartheta}$ is bounded from below away from zero, we obtain that

$$
\int_{J_{-\varepsilon,\varepsilon}} \frac{a_{1,\vartheta}(x-e_1)}{|x-e_1|^{n+1}} \, dx \geq \frac{1}{C} \int_{B_{\varepsilon/10} \left( e_1 + \frac{3\varepsilon}{2} e_n \right)} \frac{dx}{|x-e_1|^{n+1}} \geq \frac{1}{C\varepsilon^{s_1}}.
$$

This and (2.5) entail that

$$
(2.7) \quad \Xi \leq \lim_{\varepsilon \searrow 0} \left( \int_{J_{-\varepsilon,\varepsilon}} \frac{a_{1,\vartheta}(x-e_1)}{|x-e_1|^{n+1}} \, dx - \int_{J_{0,\varepsilon}} \frac{a_{1,\vartheta}(x-e_1)}{|x-e_1|^{n+1}} \, dx - \frac{1}{C\varepsilon^{s_1}} + C \right).
$$

Now we observe that

$$
J_{-\varepsilon,\varepsilon} = \left\{ x \in \mathbb{R}^n : |x_n| < \tan \varepsilon x_1 \right\}
$$

and we define

$$
J_{\varepsilon} := 2e_1 - J_{-\varepsilon,\varepsilon}, \quad R_{\varepsilon} := J_{-\varepsilon,\varepsilon} \cap J_{\varepsilon}^2, \quad J_{\varepsilon}^* := J_{0,\varepsilon} \setminus R_{\varepsilon}
$$

and

$$
B_{\varepsilon} := \left\{ x \in J_{\varepsilon}^* : x_n > \frac{x_1 - 1}{|\log \varepsilon|} \right\},
$$

see Figure 6.

![Figure 6](image_url)

**Figure 6.** The set decomposition involved in the proof of (2.3).

The intuition behind this set decomposition is that, on the one hand, the set $R_{\varepsilon}$ accounts for the cancellations due to the symmetry of $a_{1,\vartheta}$ (corresponding to the reflection through $e_1$, namely $x \mapsto 2e_1 - x$); on the other hand, the remaining integral contributions in $J_{\varepsilon}^*$ cancel.
exactly when $a_{1,\vartheta}$ is constant, thanks to the reflection through the horizontal hyperplane $x \mapsto (x',-x_n)$, but they may provide additional terms when $a_{1,\vartheta}$ is not constant. To overcome this difficulty, our idea is to exploit the continuity of $a_{1,\vartheta}$ together with the reflection through the horizontal hyperplane in order to “approximately cancel” as many contributions as possible.

This idea by itself however does not exhaust the complexity of the problem, because two adjacent points can end up being projected far away from each other on the sphere (for instance, if a point is close to $e_1 + \tan \varepsilon e_n$ and the other to $e_1 - \tan \varepsilon e_n$). To overcome this additional complication, we exploit the set $B_\varepsilon$: roughly speaking, points outside $B_\varepsilon$ remain sufficiently close after they get projected on the sphere (and here we can take advantage of the continuity of $a_{1,\vartheta}$), while the points in $B_\varepsilon$ provide an additional, but small, correction, in view of the location of $B_\varepsilon$ and of its measure.

The details of the quantitative computation needed to implement this combination of ideas go as follows.

We stress that
\begin{equation} \tag{2.8} \end{equation}
if $x$ belongs to $R_\varepsilon$, then so does $2e_1 - x$.

Indeed, if $x \in R_\varepsilon$ then $x \in J_{-\varepsilon,\varepsilon}$ and $x \in 2e_1 - J_{-\varepsilon,\varepsilon}$ and consequently $2e_1 - x \in 2e_1 - J_{-\varepsilon,\varepsilon}$ and $2e_1 - x \in J_{-\varepsilon,\varepsilon}$, which gives (2.8).

Also, we see that
\[ J_{-\varepsilon,0} \cap R_\varepsilon = R_\varepsilon \setminus \{x_n < 0\} \quad \text{and} \quad J_{0,\varepsilon} \cap R_\varepsilon = R_\varepsilon \setminus \{x_n > 0\}. \]
Thus, using (2.8), the change of variable $x \mapsto 2e_1 - x$ and the symmetry of $a_{1,\vartheta}$, taking into account that under this transformation some vectors end up outside the ball $B_2$,
\[
\int_{J_{-\varepsilon,0} \cap B_2 \cap R_\varepsilon} \frac{a_{1,\vartheta}(x - e_1)}{|x - e_1|^{n+1}} \, dx = \int_{R_\varepsilon \setminus \{x_n < 0\} \cap B_2} \frac{a_{1,\vartheta}(x - e_1)}{|x - e_1|^{n+1}} \, dx
\leq \int_{R_\varepsilon \setminus \{x_n < 0\} \cap B_2} \frac{a_{1,\vartheta}(e_1 - x)}{|e_1 - x|^{n+1}} \, dx + C \int_{\mathbb{R}^n \setminus B_2} \frac{dx}{|x - e_1|^{n+1}}
\leq \int_{J_{0,\varepsilon} \cap B_2 \cap R_\varepsilon} \frac{a_{1,\vartheta}(x - e_1)}{|x - e_1|^{n+1}} \, dx + C.
\]
Plugging this cancellation into (2.7), we conclude that
\[
\Xi \leq \lim_{\varepsilon \searrow 0} \left( \int_{(J_{-\varepsilon,0} \cap B_2) \setminus R_\varepsilon} \frac{a_{1,\vartheta}(x - e_1)}{|x - e_1|^{n+1}} \, dx - \int_{(J_{0,\varepsilon} \cap B_2) \setminus R_\varepsilon} \frac{a_{1,\vartheta}(x - e_1)}{|x - e_1|^{n+1}} \, dx - \frac{1}{C_{\varepsilon^{s_1}}} + C \right).
\]
Using the change of variable $x \mapsto (x',-x_n)$ and noticing that $|(x',-x_n) - e_1| = |x - e_1|$, we thus find that
\[
\Xi \leq \lim_{\varepsilon \searrow 0} \left( \int_{(J_{0,\varepsilon} \cap B_2) \setminus R_\varepsilon} \frac{a_{1,\vartheta}(x',-x_n) - e_1)}{|x - e_1|^{n+1}} \, dx - \frac{1}{C_{\varepsilon^{s_1}}} + C \right)
\]
\begin{equation} \tag{2.9} \end{equation}
= \lim_{\varepsilon \searrow 0} \left( \int_{J_{1,\varepsilon} \cap B_2} \frac{a_{1,\vartheta}(x',-x_n) - e_1)}{|x - e_1|^{n+1}} \, dx - \frac{1}{C_{\varepsilon^{s_1}}} + C \right).
\]
We point out that
\begin{equation} \tag{2.10} J_{1,\varepsilon} \subseteq \{x_1 \geq 1\}. \end{equation}

Indeed, if $x \in J_{1,\varepsilon}^*$, then $x \in J_{0,\varepsilon}$, whence
\begin{equation} \tag{2.11} x_n \in (0, \tan \varepsilon x_1). \end{equation}

Also, we have that $x \notin R_\varepsilon$ and therefore either $x \notin J_{-\varepsilon,\varepsilon}$ or $x \notin J_{1,\varepsilon}$. In fact, since $J_{0,\varepsilon} \subseteq J_{-\varepsilon,\varepsilon}$, we have that necessarily $x \notin J_{1,\varepsilon}$ and, as a result, $2e_1 - x \notin J_{-\varepsilon,\varepsilon}$. This gives that $|x_n| \geq \tan \varepsilon (2 - x_1)$. Therefore, by (2.11),
\begin{equation} \tag{2.12} 2 - x_1 \leq \frac{|x_n|}{\tan \varepsilon} = \frac{x_n}{\tan \varepsilon} \leq x_1, \end{equation}
and this entails (2.10).

Now we claim that
\begin{equation}
B_ε \subseteq \{ x \in \mathbb{R}^n : |x_1 - 1| \leq 2ε |\log ε|, |x_n - \tan ε| \leq 2ε^2 |\log ε| \}.
\end{equation}

To check this, let \( x \in B_ε \). Then,
\begin{equation}
\frac{x_1 - 1}{|\log ε|} \leq x_n \leq \tan ε \leq (x_1 - 1) + \tan ε.
\end{equation}

Recalling (2.10), we thus find that
\begin{equation}
\left( \frac{1}{|\log ε|} - \tan ε \right) |x_1 - 1| = \left( \frac{1}{|\log ε|} - \tan ε \right) (x_1 - 1) \leq \tan ε.
\end{equation}

Consequently, if \( ε \) is conveniently small,
\begin{equation}
\frac{9}{10} |x_1 - 1| \leq (1 - \tan ε |\log ε|) |x_1 - 1| \leq \tan ε |\log ε| \leq \frac{11}{10} ε |\log ε|.
\end{equation}

Furthermore, by the second inequality in (2.14) and (2.15),
\begin{equation}
(x_n - \tan ε \leq \tan ε (x_1 - 1) \leq \tan ε |x_1 - 1| \leq \frac{11}{9} ε \tan ε |\log ε| \leq 2ε^2 |\log ε|.
\end{equation}

Moreover, from (2.12),
\begin{equation}
x_n \geq \tan ε (2 - x_1),
\end{equation}

whence, utilizing again (2.15),
\begin{equation}
\tan ε - x_n \leq \tan ε + \tan ε (x_1 - 2) \leq \tan ε |x_1 - 1| \leq \frac{11}{9} ε \tan ε |\log ε| \leq 2ε^2 |\log ε|.
\end{equation}

From this, (2.15) and (2.16) we obtain (2.13), as desired.

Now, using (2.13) and the changes of variable \( y := \frac{x-e_1}{\tan ε} \) and \( z := (y', y_n) \), we see that
\begin{equation}
\int_{B_ε} \frac{dx}{|x-e_1|^{n+1}} \leq \int \frac{dx}{|x-e_1|^{n+1}} = \frac{1}{(\tan ε)^{n+1}} \int \frac{dy}{|y|^{n+1}}
\end{equation}
\begin{equation}
\leq \frac{2}{ε^{n+1}} \int \frac{dy}{|y|^{n+1}}
\end{equation}
\begin{equation}
= \frac{2}{ε^{n+1}} \int \frac{dz_n}{z_n} \left| \frac{dz_n}{z_n} \right|^{1+4ε |\log ε|}
\end{equation}
\begin{equation}
\leq C \frac{1+4ε |\log ε|}{z_n^{n+1}}
\end{equation}
\begin{equation}
\leq C ε^{1-s_1} |\log ε|,
\end{equation}

up to renaming the positive constant \( C \) line after line.

We also recall that \( |(x', -x_n) - e_1| = |x - e_1| \) and accordingly
\begin{equation}
\left( \frac{(x', -x_n) - e_1}{x - e_1} \right) = \left( \frac{|(x', -x_n) - e_1 - (x - e_1)|}{|x - e_1|} \right) = \frac{2|x_n|}{|x - e_1|}.
\end{equation}

As a result, recalling (2.10) and (2.11), if \( x \in J_ε^* \setminus B_ε \) then
\begin{equation}
|x_n| = x_n \leq \frac{x_1 - 1}{|\log ε|} = \frac{|x_1 - 1|}{|\log ε|}.
\end{equation}

This and (2.18) give that
\begin{equation}
\left( \frac{(x', -x_n) - e_1 - x - e_1}{x - e_1} \right) \leq \frac{2}{|log ε|}.
\end{equation}
Consequently, if we consider the modulus of continuity of \(a_{1,\vartheta}\), namely
\[
\sigma(t) := \sup_{v, w \in B_1} |a_{1,\vartheta}(v) - a_{1,\vartheta}(w)|,
\]
we deduce that if \(x \in J^*_\varepsilon \setminus B_\varepsilon\) then
\[
|a_{1,\vartheta}(x',-x_n) - e_1| - a_{1,\vartheta}(x - e_1)| \leq \sigma \left( \frac{2}{|\log \varepsilon|} \right)
\]
and thus
\[
\int_{J^*_\varepsilon \setminus B_\varepsilon} a_{1,\vartheta}(x',-x_n) - e_1 - a_{1,\vartheta}(x - e_1) \, dx \leq \sigma \left( \frac{2}{|\log \varepsilon|} \right) \int_{J^*_\varepsilon \setminus B_\varepsilon} dx < \sigma \left( \frac{2}{|\log \varepsilon|} \right) \int_{J^*_\varepsilon \setminus B_\varepsilon} \frac{dx}{|x - e_1|^{n+s_1}}.
\]
Notice also that
\[
\int_{J^*_\varepsilon \setminus B_\varepsilon} dx < C \varepsilon \cdot \left( \int_{|\log \varepsilon|} dx < C \varepsilon \cdot \left( \int_{|\log \varepsilon|} dx < C \varepsilon \cdot \left( \int_{|\log \varepsilon|} dx < C \varepsilon \cdot \left( \int_{|\log \varepsilon|} dx < C \varepsilon \cdot \left( \int_{|\log \varepsilon|} dx < \right) \right) \right) \right)
\]
which together with (2.17) leads to
\[
\int_{J^*_\varepsilon \setminus B_\varepsilon} dx \leq C \varepsilon^{1-s_1} |\log \varepsilon|.
\]
Joining this information with (2.9) we find that
\[
\Xi \leq \lim_{\varepsilon \to 0} \left[ \frac{C}{\varepsilon^{s_1}} \sigma \left( \frac{2}{|\log \varepsilon|} \right) + C \varepsilon^{1-s_1} |\log \varepsilon| - \frac{1}{C \varepsilon^{s_1}} + C \right]
\]
\[
= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^{s_1}} \left[ C \sigma \left( \frac{2}{|\log \varepsilon|} \right) + C \varepsilon \cdot |\log \varepsilon| - 1 + C \varepsilon^{s_1} \right]
\]
\[
\leq \lim_{\varepsilon \to 0} \left( - \frac{1}{2C \varepsilon^{s_1}} \right)
\]
\[
= -\infty.
\]
This completes the proof of (2.3).
Now, using (2.3),
\[
\lim_{\vartheta \to 0} \mathcal{D}_\vartheta(\tilde{\vartheta})
= \lim_{\vartheta \to 0} \left( \int_{J_{0,\vartheta+\vartheta}} a_{1}(x - e(\tilde{\vartheta})) \, dx - \int_{J_{0,\vartheta-3\vartheta}} a_{1}(x - e(\tilde{\vartheta})) \, dx - \int_{J_{0,\vartheta-2\vartheta}} a_{1}(x - e(\tilde{\vartheta})) \, dx \right)
\]
\[
\leq \lim_{\vartheta \to 0} \left( \int_{J_{0,\vartheta+\vartheta}} a_{1}(x - e(\tilde{\vartheta})) \, dx - \int_{J_{0,\vartheta-3\vartheta}} a_{1}(x - e(\tilde{\vartheta})) \, dx - \int_{J_{0,\vartheta-2\vartheta}} a_{1}(x - e(\tilde{\vartheta})) \, dx \right) = -\infty,
\]
which proves (1.18), and
\[
\lim_{\vartheta \to 2\pi} D_{\vartheta}(\vartheta) = \lim_{\alpha \to 0} D_{\vartheta}(2\pi - \alpha) = \lim_{\alpha \to 0} \left( \int_{J_{\vartheta,\vartheta+2\pi-\alpha}} \frac{a_1(x - e(\vartheta))}{|x - e(\vartheta)|^{n+s_1}} \, dx - \int_{J_{0,\vartheta}} \frac{a_1(x - e(\vartheta))}{|x - e(\vartheta)|^{n+s_1}} \, dx \right)
\]
\[
\geq \lim_{\alpha \to 0} \left( \int_{J_{\vartheta,\vartheta+2\alpha}} \frac{a_1(x - e(\vartheta))}{|x - e(\vartheta)|^{n+s_1}} \, dx - \int_{J_{0,\vartheta-\alpha}} \frac{a_1(x - e(\vartheta))}{|x - e(\vartheta)|^{n+s_1}} \, dx \right) = +\infty,
\]
which proves (1.19).

\[\square\]

3. Nonlocal Young's Law and Proofs of Theorems 1.4 and 1.6, of Corollary 1.5 and of Proposition 1.9,

In order to prove Theorems 1.4 and 1.6, Corollary 1.5 and Proposition 1.9, we first recall an ancillary result on the continuity of the nonlocal $K$-mean curvature defined in (1.7) (for the usual fractional mean curvature, that is when the kernel $K$ is as in (1.1), similar continuity results were presented in [FFM*15, Coz15]).

From now on, we denote points $x \in \mathbb{R}^n$ as $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$ and we set
\[
C := \{x = (x', x_n) \in \mathbb{R}^n : |x'| < 1, |x_n| < 1\}
\]
and
\[
D := \{z \in \mathbb{R}^{n-1} : |z| < 1\}.
\]

**Lemma 3.1.** Let $\lambda \geq 1$, $s \in (0, 1)$ and $\alpha \in (s, 1)$. Let $\{F_k\}_{k \in \mathbb{N}}$ be a sequence of Borel sets in $\mathbb{R}^n$ such that $0 \in \partial F_k$ and
\[
F_k \to F \text{ in } L^1_{\text{loc}}(\mathbb{R}^n) \text{ for some } F \subseteq \mathbb{R}^n.
\]
and $u_k, u \in C^{1,\alpha}(\mathbb{R}^{n-1})$ be such that
\[
C \cap F_k = \{x \in C : x_n \leq u_k(x')\}
\]
and
\[
\lim_{k \to +\infty} \|u_k - u\|_{C^{1,\alpha}(D)} = 0.
\]
Let $K_k, K \in K(n, s, \lambda, 0)$ be such that $K_k \to K$ pointwise in $\mathbb{R}^n \setminus \{0\}$ as $k \to +\infty$.

Then
\[
\lim_{k \to +\infty} H^K_{\partial F_k}(0) = H^K_{\partial F}(0).
\]

For the proof of Lemma 3.1 here, see Lemma 4.1 in [MV17].

We will also need a technical lemma to distinguish between the nondegenerate case $\vartheta \in (0, \pi)$ and the particular cases in which $\vartheta \in \{0, \pi\}$.

**Lemma 3.2.** Let $K_1 \in K^2(n, s_1, \lambda, \vartheta)$ be such that it admits a blow-up limit $K_1^*$ (according to (1.4)). Let $\Omega$ be an open bounded set with $C^1$-boundary and $E$ be a volume-constrained critical set of $\mathcal{E}$.

Let $x_0 \in \text{Reg}_E \cap \partial \Omega, x_k \in \text{Reg}_E \cap \Omega$ such that $x_k \to x_0$ as $k \to +\infty$ and $r_k > 0$ such that $r_k \to 0$ as $k \to +\infty$.

Suppose that $H$ and $V$ are open half-spaces such that
\[
(3.1) \quad \Omega^{x_0, r_k} \to H \quad \text{and} \quad E^{x_0, r_k} \to H \cap V \text{ in } L^1_{\text{loc}}(\mathbb{R}^n) \text{ as } k \to +\infty.
\]

Set $v_k := \frac{x_0 - x_k}{r_k}$ and suppose that there exists $v \in H \cap \partial V$ such that $v_k \to v$ as $k \to +\infty$.

Let $\vartheta \in [0, \pi]$ be the angle between the half-spaces $H$ and $V$, that is $H \cap V = J_{0, \vartheta}$ in the notation of (1.11). Then,
\[\text{i) if } \vartheta = 0 \text{ then} \lim_{k \to +\infty} r_k^{s_1} \left[ H_{\partial E}^{K_1}(x_k) - \int_{\Omega^c} K_1(x_k - y) \, dy \right] = +\infty; \]

\[\text{ii) if } \vartheta = \pi \text{ then} \lim_{k \to +\infty} r_k^{s_1} \left[ H_{\partial E}^{K_1}(x_k) - \int_{\Omega^c} K_1(x_k - y) \, dy \right] = -\infty; \]

\[\text{iii) if } 0 < \vartheta < \pi \text{ then} \lim_{k \to +\infty} r_k^{s_1} \left[ H_{\partial E}^{K_1}(x_k) - \int_{\Omega^c} K_1(x_k - y) \, dy \right] = H_{\partial(H \cap V)}^{K_1}(v) - \int_{\Omega^c} K_1(v - y) \, dy \in \mathbb{R}. \]

**Proof.** We start by proving i). For this, we notice that

\[\Xi_k := r_k^{s_1} \left[ H_{\partial E}^{K_1}(x_k) - \int_{\Omega^c} K_1(x_k - y) \, dy \right] = r_k^{s_1} \left[ \int_{\Omega^c} K_1(x_k - y) \, dy - \int E K_1(x_k - y) \, dy \right] = r_k^{n+s_1} \left[ \int_{(E^{x_0, r_k}) \cap \Omega^{x_0, r_k}} K_1(x_k - x_0 - r_k z) \, dz \right] - \int_{E^{x_0, r_k}} K_1(x_k - x_0 - r_k z) \, dz \]

where the change of variable \( z = \frac{y - x_0}{r_k} \) has been used.

Now we point out that

\[r_k^{n+s_1} \int_{\mathbb{R}^n \setminus B_{1/2}(v_k)} K_1(r_k(v_k - z)) \, dz \leq \lambda \int_{\mathbb{R}^n \setminus B_{1/2}(v_k)} \frac{dz}{|u_k - z|^{n+s_1}} \leq C, \]

thanks to (1.2), for some positive constant \( C \), depending only on \( n, s_1 \) and \( \lambda \).

From these observations we conclude that

\[\Xi_k \geq r_k^{n+s_1} \left[ \int_{(E^{x_0, r_k}) \cap \Omega^{x_0, r_k} \cap B_{1/2}(v_k)} K_1(r_k(v_k - z)) \, dz \right. \]

\[\left. - \int_{E^{x_0, r_k} \cap B_{1/2}(v_k)} K_1(r_k(v_k - z)) \, dz \right] - C. \]

Now we notice that \( E^{x_0, r_k} \cap B_{1/2}(v_k) \) can be written as a portion of space included between the graphs of the functions describing \( \partial \Omega^{x_0, r_k} \) and \( \partial E^{x_0, r_k} \), that we denote respectively by \( \psi_k \) and \( u_k \).

More precisely, recalling that \( x_0 \in \text{Reg}_E \cap \partial \Omega \), in the vicinity of \( x_0 \) we can describe \( \partial \Omega \) and \( \partial E \) by the graphs of two functions \( \psi \) and \( u \), respectively, with \( \psi \) of class \( C^1 \) and \( u \) of class \( C^{1, \alpha} \) with \( \alpha \in (s_1, 1) \), and \( \psi(x_0') = u(x_0') = x_{0,n} \). Up to a rotation, we also assume that \( \nabla \psi(x_0') = 0 \).

In this way,

\[\psi_k(x') = \frac{\psi(x_0' + r_k x') - x_{0,n}}{r_k} \quad \text{and} \quad u_k(x') = \frac{u(x_0' + r_k x') - x_{0,n}}{r_k}. \]

Moreover,

\[E^{x_0, r_k} \cap B_{1/2}(v_k) = \left\{ x \in B_{1/2}(v_k) : x_n \in (\psi_k(x'), u_k(x')) \right\} \]

and notice that, since \( E \subseteq \Omega \), it follows that \( \psi \leq u \) and so \( \psi_k \leq u_k \). As a result,

\[\{ x \in B_{1/2}(v_k) : x_n > u_k(x') \} \subseteq (E^{x_0, r_k})^c \cap \Omega^{x_0, r_k} \cap B_{1/2}(v_k). \]
Hence, from (3.2) we obtain that

\[
\Xi_k \geq r_k^{n+s_1} \left[ \int_{B_{1/2}(v_k) \cap \{x_n > u_k(x')\}} K_1(r_k(v_k - z)) \, dz 
- \int_{B_{1/2}(v_k) \cap \{x_n \in (\psi_k(x'), u_k(x'))\}} K_1(r_k(v_k - z)) \, dz \right] - C.
\]

(3.4)

We now define

\[
\tilde{u}_k(x') := u_k(v'_k) + \nabla u_k(v'_k) \cdot (x' - v'_k)
\]

and we point out that, if \(|x' - v'_k| \leq 3,

\[
|u_k(x') - \tilde{u}_k(x')| = \left| \frac{u(x'_0 + r_k x') - u(x'_0 + r_k v'_k)}{r_k} - \nabla u(x'_0 + r_k v'_k) \cdot (x' - v'_k) \right|
= \left| \frac{u(x'_k + r_k (x' - v'_k)) - u(x'_k)}{r_k} - \nabla u(x'_k) \cdot (x' - v'_k) \right|
= \left| \int_0^1 \nabla u(x'_k + tr_k(x' - v'_k)) \cdot (x' - v'_k) \, dt - \nabla u(x'_k) \cdot (x' - v'_k) \right|
\leq \|u\|_{C^{1,\alpha}(B'_r(x'_0))} r_k^{\alpha} |x' - v'_k|^{1+\alpha},
\]

for a suitable \(\rho > 0\). As a consequence,

\[
r_k^{n+s_1} \int_{\{x_n > u_k(x')\} \Delta \{x_n > \tilde{u}_k(x')\} \cap B_{1/2}(v_k)} K_1(r_k(v_k - z)) \, dz
\leq \lambda \int_{\{x_n > u_k(x')\} \Delta \{x_n > \tilde{u}_k(x')\} \cap B_{1/2}(v_k)} \frac{dz}{|v_k - z'|^{n+s_1}}
\leq \lambda \|u\|_{C^{1,\alpha}(B'_r(x'_0))} r_k^{\alpha} \int_{B'_{1/2}(v'_k)} \frac{|v'_k - z'|^{1+\alpha}}{|v'_k - z'|^{n+s_1}} \, dz'
\leq C r_k^{\alpha},
\]

up to renaming \(C\), possibly in dependence of \(u\) as well.

Plugging this information into (3.4), and possibly renaming \(C\) again, we obtain that

\[
\Xi_k \geq r_k^{n+s_1} \left[ \int_{B_{1/2}(v_k) \cap \{x_n > u_k(x')\}} K_1(r_k(v_k - z)) \, dz 
- \int_{B_{1/2}(v_k) \cap \{x_n \in (\psi_k(x'), u_k(x'))\}} K_1(r_k(v_k - z)) \, dz \right] - C.
\]

(3.5)
Now, from (3.3) we see that \( \psi_k(x') \to \nabla \psi(x'_0) \cdot x' \) and \( u_k(x') \to \nabla u(x'_0) \cdot x' \) as \( k \to +\infty \). Hence, if \( \vartheta = 0 \) it follows that \( \nabla \psi(x'_0) = \nabla u(x'_0) \). Consequently, if \( x' \in B'_{1/2}(v'_k) \) then

\[
\left| \bar{u}_k(x') - \psi_k(x') \right| = \left| u_k(v'_k) + \nabla u_k(v'_k) \cdot (x' - v'_k) - \frac{\psi(x'_0 + r_k x') - \psi(x'_0)}{r_k} \right|
\]

\[
= \frac{u(x'_0 + r_k v'_k) - u(x'_0)}{r_k} + \nabla u(x'_0 + r_k v'_k) \cdot (x' - v'_k) - \int_0^1 \nabla \psi(x'_0 + tr_k x') \cdot x' \, dt
\]

\[
= \left| \int_0^1 \nabla u(x'_0 + tr_k v'_k) \cdot v'_k \, dt + \nabla u(x'_0 + r_k v'_k) \cdot (x' - v'_k) - \int_0^1 \nabla \psi(x'_0 + tr_k x') \cdot x' \, dt \right| + \delta_k
\]

\[
= \delta_k,
\]

for a suitable \( \delta_k \) such that \( \delta_k \to 0 \) as \( k \to +\infty \).

This and (3.5) give that

\[
\Xi_k \geq r^n_k + \int_{B_{1/2}(v_k) \cap \{ x_n > \bar{u}_k(x') \}} K_1(r_k(v_k - z)) \, dz - C.
\]

Now we define the map \( Y(z) := 2v_k - z \) and we show that

\[
Y \left( B_{1/2}(v_k) \cap \{ x_n \in (\bar{u}_k(x') - \delta_k, \bar{u}_k(x')) \} \right) \subseteq B_{1/2}(v_k) \cap \{ x_n \in (\bar{u}_k(x'), \bar{u}_k(x') + \delta_k) \}.
\]

Indeed, let \( z \in B_{1/2}(v_k) \cap \{ x_n \in (\bar{u}_k(x') - \delta_k, \bar{u}_k(x')) \} \) and call \( y := Y(z) \). We have that \( |y - v_k| = |v_k - z| < 1/2 \). Moreover,

\[
y_n - \bar{u}_k(y') = 2v_k,n - z_n - \bar{u}_k(2v'_k - z')
= 2u_k(v'_k) - z_n - \bar{u}_k(2v'_k - z')
\in \begin{pmatrix} 2u_k(v'_k) - \bar{u}_k(z') - \bar{u}_k(2v'_k - z') \end{pmatrix}, 2u_k(v'_k) - \bar{u}_k(z') - \bar{u}_k(2v'_k - z') + \delta_k
= \begin{pmatrix} 2u_k(v'_k) - \bar{u}_k(v'_k), 2u_k(v'_k) - \bar{u}_k(v'_k) + \delta_k \end{pmatrix}
= (0, \delta_k)
\]

and the proof of (3.8) is thus complete.

Using (3.8) and changing variable \( y = Y(z) \) we see that

\[
\int_{B_{1/2}(v_k) \cap \{ x_n \in (\bar{u}_k(x') - \delta_k, \bar{u}_k(x')) \}} K_1(r_k(v_k - z)) \, dz
\]

\[
\leq \int_{B_{1/2}(v_k) \cap \{ x_n \in (\bar{u}_k(x'), \bar{u}_k(x') + \delta_k) \}} K_1(r_k(y - v_k)) \, dy
\]

\[
= \int_{B_{1/2}(v_k) \cap \{ x_n \in (\bar{u}_k(x'), \bar{u}_k(x') + \delta_k) \}} K_1(r_k(v_k - y)) \, dy.
\]

Combining this and (3.7), and recalling (1.2), we arrive at

\[
\Xi_k \geq r^n_k + \int_{B_{1/2}(v_k) \cap \{ x_n > \bar{u}_k(x') + \delta_k \}} K_1(r_k(v_k - z)) \, dz - C
\]

\[
\geq \frac{1}{\lambda} \int_{B_{1/2}(v_k) \cap \{ x_n > \bar{u}_k(x') + \delta_k \}} \frac{dz}{v_k - z} |z|^{n+1} - C.
\]
Now we define
\[(3.10)\quad \nu_k := \frac{(-\nabla u_k(v'_k), 1)}{\sqrt{1 + |\nabla u_k(v'_k)|^2}} \quad \text{and} \quad \zeta_k := v_k + 3\delta_k \nu_k\]
and we claim that, if \(k\) is sufficiently large,
\[(3.11)\quad B_{\delta_k}(\zeta_k) \subseteq B_{1/2}(v_k) \cap \{x_n > \tilde{u}_k(x') + \delta_k\}.
\]
To check this, we observe that
\[
\lim_{k \to +\infty} |\nabla u_k(v'_k)| = \lim_{k \to +\infty} |\nabla u(x'_k)| = |\nabla u(x'_0)| = |\nabla \psi(x'_0)| = 0
\]
and consequently
\[(3.12)\quad \lim_{k \to +\infty} \frac{3}{1 + |\nabla u_k(v'_k)|^2} - \frac{4|\nabla u_k(v'_k)| - 2}{1} = 1.
\]
Now, pick \(w \in B_{\delta_k}(\zeta_k)\). We have that
\[
|w - v_k| \leq |w - \zeta_k| + |\zeta_k - v_k| < \delta_k + 3\delta_k = 4\delta_k
\]
and thus \(w \in B_{1/2}(v_k)\) as long as \(k\) is large enough.
Moreover,
\[
w_n - \tilde{u}_k(w') - \delta_k \geq (\zeta_k,n - \delta_k) - u_k(v'_k) - \nabla u_k(v'_k)(w' - v'_k) - \delta_k
\]
\[
= \left(v_{k,n} + \frac{3\delta_k}{\sqrt{1 + |\nabla u_k(v'_k)|^2}} - \delta_k\right) - v_{k,n} - \nabla u_k(v'_k)(w' - v'_k) - \delta_k
\]
\[
= \frac{3\delta_k}{\sqrt{1 + |\nabla u_k(v'_k)|^2}} - \nabla u_k(v'_k)(w' - v'_k) - 2\delta_k
\]
\[
\geq \frac{3\delta_k}{\sqrt{1 + |\nabla u_k(v'_k)|^2}} - |\nabla u_k(v'_k)| |w' - v'_k| - 2\delta_k
\]
\[
\geq \left(\frac{3}{\sqrt{1 + |\nabla u_k(v'_k)|^2}} - 4|\nabla u_k(v'_k)| - 2\right) \delta_k
\]
\[
> 0,
\]
thanks to (3.12).

The proof of (3.11) is thereby complete.

Thus, exploiting (3.9) and (3.11), we find that
\[
\Xi_k \geq \int_{B_{\delta_k}(\zeta_k)} \frac{dz}{|v_k - z|^{n+1}} dz - C.
\]
Notice also that if \(z \in B_{\delta_k}(\zeta_k)\) then \(|v_k - z| \leq |v_k - \zeta_k| + |\zeta_k - z| \leq 3\delta_k + \delta_k = 4\delta_k\) and accordingly
\[
\Xi_k \geq \int_{B_{\delta_k}(\zeta_k)} \frac{dz}{(4\delta_k)^{n+1}} dz - C = \frac{c}{\delta_k^{n+1}} - C,
\]
for some \(c > 0\). This establishes the claim in i), as desired.

The claim in ii) can be proved similarly.

As for the claim in iii), we suppose that \(\vartheta \in (0, \pi)\) and, for every \(k \in \mathbb{N}\), we denote by \(F_k\) the set obtained by a suitable rigid motion of the set \(E^{x_0, r_k} - v_k\) so as to have that \(0 \in \partial F_k\) and
\[(3.13)\quad C \cap F_k = \{x \in C : x_n \leq u_k(x')\},
\]
for some \(u_k \in C^{1,\alpha}(\mathbb{R}^{n-1})\). Let also \(u\) be the linear function such that \(u_k \to u\) in \(C^{1,\alpha}(D)\) as \(k \to +\infty\). We notice that, by (3.1), up to a rigid motion,
\[(3.14)\quad F_k \to F := H \cap V - v \in L^1_{\text{loc}}(\mathbb{R}^n)\text{ as } k \to +\infty.
\]
Furthermore, recalling the definition of mean curvature in (1.7) and exploiting the change of variable \( y = x_0 + r_k z \), we see that
\[
\mathbf{H}^{K_i}_E(x_k) = \int_{\mathbb{R}^n} K_1(x_k - y) \left( \chi_{E^c}(y) - \chi_E(y) \right) dy
\]
(3.15)
\[
= r_k^{-s_1} \int_{\mathbb{R}^n} r_k^{s_2} K_1(x_k - x_0 - r_k z) \left( \chi_{(E^{\geq 0} \cap r_k)}(z) - \chi_{E^{\geq 0} \cap r_k}(z) \right) dz.
\]
We also introduce, for every \( \zeta \in \mathbb{R}^n \setminus \{0\} \), the kernel
\[
K_{1,k}(\zeta) := r_k^{s_1} K_1(r_k \zeta),
\]
and we observe that, in light of (3.15),
\[
\mathbf{H}^{K_i}_E(x_k) = r_k^{-s_1} \mathbf{H}^{K_{1,k}}_{\partial E}(0).
\]
Furthermore, we recall that \( K_{1,k} \to K_1^* \) pointwise in \( \mathbb{R}^n \setminus \{0\} \), hence one can infer from (3.13), (3.14), (3.16) and Lemma 3.1 that
\[
\lim_{k \to +\infty} r_k^{s_1} \mathbf{H}^{K_i}_E(x_k) = \mathbf{H}^{K_1^*}_{\partial (H \cap V)}(v).
\]
Moreover, since \( \vartheta \in (0, \pi) \), one can use the Lebesgue’s Dominated Convergence Theorem and find that
\[
\lim_{k \to +\infty} r_k^{s_1} \int_{\Omega^c} K_1(x_k - y) dy = \lim_{k \to +\infty} \int_{(\Omega^{\geq 0} \cap r_k)^c} r_k^{s_1} K_1(r_k(v_k - y)) dy
\]
\[
= \int_{H^c} K_1^*(v - y) dy.
\]
From this and (3.17) we obtain the desired result in iii). \( \square \)

Now we showcase a refinement of Lemma 3.2 which will be needed to exclude the degenerate blow-up limits \( \vartheta \in \{0, \pi\} \) in the case \( s_1 > s_2 \).

**Lemma 3.3.** Let \( s_1 > s_2 \), \( K_1 \in \mathbf{K}^2(n, s_1, \lambda, \varrho) \) and \( K_2 \in \mathbf{K}^2(n, s_2, \lambda, \varrho) \). Let \( \Omega \) be an open bounded set with \( C^1 \)-boundary and \( E \) be a volume-constrained critical set of \( \mathcal{C} \).

Let \( x_0 \in \text{Reg}_E \cap \partial \Omega \), \( x_k \in \text{Reg}_E \cap \Omega \) such that \( x_k \to x_0 \) as \( k \to +\infty \) and \( r_k > 0 \) such that \( r_k \to 0 \) as \( k \to +\infty \).

Suppose that \( H \) and \( V \) are open half-spaces such that
\[
\Omega^{x_0,r_k} \to H \quad \text{and} \quad E^{x_0,r_k} \to H \cap V \quad \text{in } L^1_\text{loc}(\mathbb{R}^n) \quad \text{as } k \to +\infty.
\]
Let \( \vartheta \in [0, \pi] \) be the angle between the half-spaces \( H \) and \( V \), that is \( H \cap V = J_{0,\vartheta} \) in the notation of (1.11).

Then,
\begin{enumerate}
  \item[i)] if \( \vartheta = 0 \) then
  \[
  \lim_{k \to +\infty} r_k^{s_1} \left[ \mathbf{H}^{K_i}_E(x_k) - \int_{\Omega^c} K_1(x_k - y) dy \right] + \sigma r_k^{s_1-s_2} r_k^{s_2} \int_{\Omega^c} K_2(x_k - y) dy = +\infty;
  \]
  \item[ii)] if \( \vartheta = \pi \) then
  \[
  \lim_{k \to +\infty} r_k^{s_1} \left[ \mathbf{H}^{K_i}_E(x_k) - \int_{\Omega^c} K_1(x_k - y) dy \right] + \sigma r_k^{s_1-s_2} r_k^{s_2} \int_{\Omega^c} K_2(x_k - y) dy = -\infty.
  \]
\end{enumerate}

**Proof.** We focus on the proof of i), since the proof of ii) is similar, up to sign changes. To this end, we exploit the notation introduced in Lemma 3.2, and specifically (3.5), and we
set \( v_k := \frac{x_k - x_0}{r_k} \), to see that

\[
\mathcal{Y}_k := r_k^{n_1} \left[ \frac{H_{\Omega k}^K(x_k)}{\Omega^e} - \int_{\Omega^e} K_1(x_k - y) \, dy \right] + \sigma r_k^{n_1-s_2} r_k^{s_2} \int_{\Omega^e} K_2(x_k - y) \, dy
\]

\[
\geq \Xi_k - |\sigma| r_k^{n_1-s_2} r_k^{s_2} \int_{\Omega^e} K_2(x_k - y) \, dy
\]

\[
\geq r_k^{n_1} \int_{B_{1/2}(v_k) \cap \{x_n > \bar{u}_k(x')\}} K_1(r_k(v_k - z)) \, dz
\]

\[
- \int_{B_{1/2}(v_k) \cap \{x_n \in (\psi_k(x'), \bar{u}_k(x'))\}} K_1(r_k(v_k - z)) \, dz
\]

\[
- |\sigma| r_k^{n_1-s_2} r_k^{n_1} \int_{\mathbb{R}^n \setminus \Omega^e} K_2(r_k(v_k - z)) \, dz - C
\]

(3.18)

up to changing \( C > 0 \) from line to line.

Also, by (3.6),

\[
\int_{B_{1/2}(v_k) \cap \{x_n \leq \psi_k(x')\}} K_2(r_k(v_k - z)) \, dz
\]

\[
= \int_{B_{1/2}(v_k) \cap \{x_n \leq (\bar{u}_k(x') - \delta_k, \psi_k(x'))\}} K_2(r_k(v_k - z)) \, dz + \int_{B_{1/2}(v_k) \cap \{x_n > (\bar{u}_k(x') - \delta_k)\}} K_2(r_k(v_k - z)) \, dz.
\]

Therefore, we can write (3.18) as

\[
\mathcal{Y}_k \geq r_k^{n_1} \int_{B_{1/2}(v_k) \cap \{x_n > \bar{u}_k(x')\}} K_1(r_k(v_k - z)) \, dz
\]

\[
- \int_{B_{1/2}(v_k) \cap \{x_n \in (\psi_k(x'), \bar{u}_k(x'))\}} K_1(r_k(v_k - z)) \, dz
\]

(3.19)

\[
- |\sigma| r_k^{n_1-s_2} r_k^{n_1} \int_{B_{1/2}(v_k) \cap \{x_n \in (\bar{u}_k(x') - \delta_k, \psi_k(x'))\}} K_2(r_k(v_k - z)) \, dz
\]

\[
- |\sigma| r_k^{n_1-s_2} r_k^{n_1} \int_{B_{1/2}(v_k) \cap \{x_n < \bar{u}_k(x') - \delta_k\}} K_2(r_k(v_k - z)) \, dz - C.
\]

Now we set

\[
\mathcal{Z}_k(x) := \max \left\{ r_k^{n_1} K_1(x), |\sigma| r_k^{n_1-s_2} r_k^{n_1} K_2(x) \right\}.
\]

(3.20)
In this way, we deduce from (3.19) that

\[
\begin{align*}
\mathcal{Y}_k &\geq r_k^{n+s_1} \int_{\partial B_{1/2}(v_k) \cap \{ x_n \geq \bar{u}_k(x') \}} K_1(r_k(v_k - z)) \, dz \\
&\quad - \int_{\partial B_{1/2}(v_k) \cap \{ x_n \leq \bar{u}_k(x') - \delta_k, \bar{u}_k(x') \}} Z_k(r_k(v_k - z)) \, dz \\
&\quad - |\sigma| r_k^{s_1-s_2} r_k^{n+s_2} \int_{\partial B_{1/2}(v_k) \cap \{ x_n < \bar{u}_k(x') - \delta_k \}} K_2(r_k(v_k - z)) \, dz - C.
\end{align*}
\]

(3.21)

Let \( Y(z) := 2v_k - z \). We also use the short notation

\[
\begin{align*}
\mathcal{P}_k &:= B_{1/2}(v_k) \cap \{ x_n > \bar{u}_k(x') \}, \\
\mathcal{Q}_k &:= B_{1/2}(v_k) \cap \{ x_n \in (\bar{u}_k(x') - \delta_k, \bar{u}_k(x')) \} \\
\text{and} \quad \mathcal{R}_k &:= B_{1/2}(v_k) \cap \{ x_n < \bar{u}_k(x') - \delta_k \}.
\end{align*}
\]

We know from (3.8) that

\[
Y(\mathcal{Q}_k) \subseteq B_{1/2}(v_k) \cap \{ x_n \in (\bar{u}_k(x'), \bar{u}_k(x') + \delta_k) \} \subseteq \mathcal{P}_k.
\]

We also claim that

\[
Y(\mathcal{R}_k) \subseteq \mathcal{P}_k \setminus Y(\mathcal{Q}_k).
\]

Indeed, if there were a point \( y \in Y(\mathcal{Q}_k) \cap Y(\mathcal{R}_k) \) we would have that \( y = 2v_k - Q = 2v_k - R \) for some \( Q \in \mathcal{Q}_k \) and \( R \in \mathcal{R}_k \), but this would entail that \( Q = R \in \mathcal{Q}_k \cap \mathcal{R}_k = \emptyset \), which is a contradiction. This shows that \( Y(\mathcal{R}_k) \) lies in the complement of \( Y(\mathcal{Q}_k) \), thus, to complete the proof of (3.23), it only remains to show that \( Y(\mathcal{R}_k) \subseteq \mathcal{P}_k \). To this end, we observe that if \( z_n < \bar{u}_k(z') - \delta_k \) and \( y = Y(z) \), then

\[
\begin{align*}
y_n - \bar{u}_k(y') &= 2v_{k,n} - z_n - \bar{u}_k(y') = 2\bar{u}_k(v'_k) - z_n - \bar{u}_k(2v'_k - z') \\
&> 2\bar{u}_k(v'_k) - \bar{u}_k(z') + \delta_k - \bar{u}_k(2v'_k - z') = \delta_k > 0.
\end{align*}
\]

This completes the proof of (3.23).

Hence, by (3.21), (3.22) and (3.23),

\[
\begin{align*}
\mathcal{Y}_k \geq r_k^{n+s_1} \int_{\mathcal{P}_k} K_1(r_k(v_k - z)) \, dz - \int_{\mathcal{P}_k} Z_k(r_k(v_k - z)) \, dz \\
&\quad - |\sigma| r_k^{s_1-s_2} r_k^{n+s_2} \int_{\mathcal{R}_k} K_2(r_k(v_k - z)) \, dz - C \\
(3.24) \quad &\quad = r_k^{n+s_1} \int_{\mathcal{P}_k} K_1(r_k(v_k - z)) \, dz - \int_{Y(\mathcal{Q}_k)} Z_k(r_k(v_k - y)) \, dy \\
&\quad - |\sigma| r_k^{s_1-s_2} r_k^{n+s_2} \int_{Y(\mathcal{R}_k)} K_2(r_k(v_k - y)) \, dy - C \\
&\quad = r_k^{n+s_1} \int_{\mathcal{P}_k \setminus (Y(\mathcal{Q}_k) \cup Y(\mathcal{R}_k))} K_1(r_k(v_k - z)) \, dz + \int_{Y(\mathcal{Q}_k)} \alpha_k(z) \, dz + \int_{Y(\mathcal{R}_k)} \beta_k(z) \, dz - C,
\end{align*}
\]

where

\[
\begin{align*}
\alpha_k(z) := r_k^{n+s_1} K_1(r_k(v_k - z)) - Z_k(r_k(v_k - z)) \\
\text{and} \quad \beta_k(z) := r_k^{n+s_1} K_1(r_k(v_k - z)) - |\sigma| r_k^{s_1-s_2} r_k^{n+s_2} K_2(r_k(v_k - z)).
\end{align*}
\]

We stress that up to now the condition \( s_1 > s_2 \) has not been used. We are going to exploit it now to bound \( \alpha_k \) and \( \beta_k \). For this, we note that, if \( z \in B_{1/2}(v_k) \) and \( k \) is large enough, then

\[
|\sigma| r_k^{s_1-s_2} r_k^{n+s_2} K_2(r_k(v_k - z)) \leq \frac{\lambda |\sigma| r_k^{s_1-s_2}}{|v_k - z|^{n+s_2}} \leq \frac{\lambda |\sigma| r_k^{s_1-s_2}}{|v_k - z|^{n+s_1}} = \frac{\lambda |\sigma| r_k^{s_1-s_2} r_k^{n+s_1}}{|r_k(v_k - z)|^{n+s_1}}.
\]
\[
\leq \lambda^2 |\sigma|^2 r_k^{s_1-s_2} r_k^{n+s_1} K_1(r_k(v_k - z)) \leq \frac{1}{2} r_k^{n+s_1} K_1(r_k(v_k - z)).
\]

This and (3.20) entail that if \( z \in B_{1/2}(v_k) \) and \( k \) is large enough, then \( Z_k(r_k(v_k - z)) = r_k^{n+s_1} K_1(r_k(v_k - z)) \), and therefore \( \alpha_k(z) = 0 \). In addition,

\[
\beta_k(z) \geq \frac{1}{2} r_k^{n+s_1} K_1(r_k(v_k - z)).
\]

From these observations and (3.24) we arrive at

\[
\mathcal{Y}_k \geq r_k^{n+s_1} \int_{\mathcal{P}_k \setminus (Y(Q_k) \cup Y(\mathcal{R}_k))} K_1(r_k(v_k - z)) \, dz + \frac{1}{2} r_k^{n+s_1} \int_{Y(\mathcal{R}_k)} K_1(r_k(v_k - z)) \, dz - C
\]

(3.25)

\[
\geq \frac{1}{2} r_k^{n+s_1} \int_{\mathcal{P}_k \setminus Y(Q_k)} K_1(r_k(v_k - z)) \, dz - C.
\]

Now we utilize the notation in (3.10), the inclusion in (3.11) and the first inclusion in (3.22) to see that

\[
\mathcal{P}_k \setminus Y(Q_k) \supset \mathcal{P}_k \setminus \left( B_{1/2}(v_k) \cap \{ x_n \in (\bar{u}_k(x'), \bar{u}_k(x') + \delta_k) \} \right)
\]

(3.26)

\[
= B_{1/2}(v_k) \cap \{ x_n \geq \bar{u}_k(x') + \delta_k \}
\]

\[
\supset B_{\delta_k}(\zeta_k).
\]

By plugging this information into (3.25), we thereby conclude that

\[
\mathcal{Y}_k \geq \frac{1}{2} r_k^{n+s_1} \int_{B_{\delta_k}(\zeta_k)} K_1(r_k(v_k - z)) \, dz - C
\]

(3.27)

\[
\geq \frac{1}{2} \int_{B_{\delta_k}(\zeta_k)} \frac{dz}{|v_k - z|^{n+s_1}} - C
\]

\[
= \frac{c}{\delta_k^s} - C,
\]

for some \( c > 0 \). From this, the desired result in i) plainly follows. \( \square \)

With this, we are in the position of providing the proof of Theorem 1.4, where we suppose that \( a_1 \) and \( a_2 \) are anisotropic functions and then, as a special case, we exhibit the proof of Corollary 1.5 where we take \( a_1 \equiv \text{const.} \).

**Proof of Theorem 1.4.** We fix a point \( x_0 \in \partial \Omega \cap \text{Reg}_E \) and a sequence of points \( x_k \in \Omega \cap \text{Reg}_E \) such that \( x_k \to x_0 \) as \( k \to +\infty \). We also set \( r_k := |x_k - x_0| \) and we observe that \( r_k \to 0 \) as \( k \to +\infty \). From (1.9) evaluated at \( x_k \), we get

\[
H^{K_1}_{\partial E}(x_k) - \int_{\Omega^c} K_1(x_k - y) \, dy + \sigma \int_{\Omega^c} K_2(x_k - y) \, dy + g(x_k) = c,
\]

where \( c \) does not depend on \( k \). Multiplying both sides by \( r_k^{s_1} \), we thereby obtain that

\[
r_k^{s_1} H^{K_1}_{\partial E}(x_k) - r_k^{s_1} \int_{\Omega^c} K_1(x_k - y) \, dy + \sigma r_k^{s_1-s_2} r_k^{s_2} \int_{\Omega^c} K_2(x_k - y) \, dy + r_k^{s_1} g(x_k) = c r_k^{s_1}.
\]

Notice that, since \( g \) is locally bounded, we have that \( r_k^{s_1} g(x_k) \to 0 \) as \( k \to +\infty \). As a consequence,

\[
\lim_{k \to +\infty} r_k^{s_1} \left[ H^{K_1}_{\partial E}(x_k) - \int_{\Omega^c} K_1(x_k - y) \, dy \right] + \sigma r_k^{s_1-s_2} r_k^{s_2} \int_{\Omega^c} K_2(x_k - y) \, dy = 0.
\]

(3.28)

Now, we prove the statement in 1) of Theorem 1.4. For this, we suppose that \( s_1 < s_2 \) and \( \sigma < 0 \). In this case,

\[
\sigma r_k^{s_1-s_2} r_k^{s_2} \int_{\Omega^c} K_2(x_k - y) \, dy \leq 0,
\]

25
Lemma 3.2 and conclude that \( \vartheta \neq \pi \). Hence, to prove 1) it remains to check that
\[
(3.29) \quad \vartheta \not\in (0, \pi).
\]
To this end, we suppose by contradiction that \( \vartheta \in (0, \pi) \). Then, by the Lebesgue’s Dominated Convergence Theorem,
\[
\lim_{k \to +\infty} \int_{\Omega^c} K_2(x_k - y) \, dy = \lim_{k \to +\infty} \int_{(\Omega^c \cap r_k)} r_k^{n+s_2} K_2(r_k(v_k - y)) \, dy
\]
\[
= \int_{H^c} K_2^*(v - y) \, dy
\]
and this limit is finite. Consequently,
\[
\lim_{k \to +\infty} \sigma r_k^{s_1-s_2} \int_{\Omega^c} K_2(x_k - y) \, dy = -\infty.
\]
This and iii) in Lemma 3.2 contradict (3.28), and thus (3.29) is proved.

Accordingly, if \( s_1 < s_2 \) and \( \sigma < 0 \), then necessarily \( \vartheta = 0 \), which establishes 1).

We now prove the statement in 2). Namely we consider the case in which \( s_1 < s_2 \) and \( \sigma > 0 \), and thus
\[
\sigma r_k^{s_1-s_2} \int_{\Omega^c} K_2(x_k - y) \, dy \geq 0.
\]
From this, i) in Lemma 3.2 and (3.28) we infer that \( \vartheta \neq 0 \). Hence, to establish 2) we show that
\[
(3.31) \quad \vartheta \not\in (0, \pi).
\]
We argue as before and we suppose by contradiction that \( \vartheta \in (0, \pi) \). Then, exploiting (3.30) we see that
\[
\lim_{k \to +\infty} \sigma r_k^{s_1-s_2} \int_{\Omega^c} K_2(x_k - y) \, dy = +\infty.
\]
This and iii) in Lemma 3.2 contradict (3.28), and thus (3.31) is proved.

As a consequence, if \( s_1 < s_2 \) and \( \sigma > 0 \), then \( \vartheta = \pi \), hence we have established 2) as well. Hence, we now focus on the statement in 3).

For this, we first suppose that \( s_1 < s_2 \) and \( \sigma = 0 \). Then, (3.28) becomes
\[
(3.32) \quad \lim_{k \to +\infty} r_k^{s_1} \left[ H_{\partial E}^K(x_k) - \int_{\Omega^c} K_1(x_k - y) \, dy \right] = 0.
\]
This and Lemma 3.2 give that \( \vartheta \in (0, \pi) \) in this case.

In the case in which \( s_1 > s_2 \), if \( \vartheta \in \{0, \pi\} \) then we would use Lemma 3.3 to find a contradiction with (3.28), hence we conclude that necessarily \( \vartheta \in (0, \pi) \) in this case as well.

Now, in order to prove (1.23), we take \( v \in H \cap \partial V \), then by (1.21) we have that, for every \( k \), there exists \( v_k \in \Omega^{x_0,r_k} \cap \partial E^{x_0,r_k} \) such that \( v_k \to v \) as \( k \to +\infty \), where \( r_k \) is an infinitesimal sequence as \( k \to +\infty \). As a consequence, for every \( k \), there exists \( x_k \in \text{Reg}_E \cap \Omega \) such that \( v_k = \frac{x_k - x_0}{r_k} \) and \( x_k \to x_0 \) as \( k \to +\infty \). Then, we are in the position to apply iii) in Lemma 3.2 and conclude that
\[
(3.33) \quad \lim_{k \to +\infty} r_k^{s_1} \left[ H_{\partial E}^K(x_k) - \int_{\Omega^c} K_1(x_k - y) \, dy \right] = H_{\partial (H \cap \Omega)}^K(v) - \int_{H^c} K_1^*(v - y) \, dy.
\]

Also, if \( s_1 > s_2 \), we recall that the limit in (3.30) is finite (since \( \vartheta \in (0, \pi) \)) and that \( r_k \) is infinitesimal to infer that
\[
\lim_{k \to +\infty} r_k^{s_1-s_2} \int_{\Omega^c} K_2(x_k - y) \, dy = 0.
\]
This, together with (3.28), gives that (3.32) holds true in this case as well.
Accordingly, from (3.32) and (3.33) we deduce that
\[ H^K_\partial(H \cap \partial V)(v) - \int_{H^c} K^*_1(v - y) \, dy = 0, \]
which establishes (1.23).

Hence, to complete the proof of the statement in 3), it remains to check that \( \hat{\vartheta} = \pi - \vartheta \), being \( \hat{\vartheta} \in (0, 2\pi) \) the angle given in (1.20) with \( c = 0 \).

For this, we exploit the notation in (1.12), the assumption in (1.13) and the change of variable \( z = y/\vert y \vert \), to see that, for all \( v \in H \cap \partial V \), the left hand side of (1.23) can be written as
\[
H^K_\partial(H \cap \partial V)(v) - \int_{H^c} K^*_1(v - y) \, dy = \int_{\mathbb{R}^n} \frac{\varrho_1(\vartheta - z)(\chi_{H \cap \partial V}(z) - \chi_{D_0}(z))}{\varrho(\vartheta) - z} \, dz.
\]

Therefore, by (1.23),
\[
(3.34) \qquad \int_{J_{\vartheta,\pi}} a_1(\varrho(\vartheta) - z) \, dz - \int_{J_{\vartheta,0}} a_1(\varrho(\vartheta) - z) \, dz = 0.
\]

Consequently, recalling the notation in (1.15) and exploiting (1.20) with \( c = 0 \), we have that
\[
\mathcal{D}_\varphi(\pi - \vartheta) = \int_{J_{\vartheta,\pi}} \frac{a_1(\varrho(\vartheta) - z)}{\varrho(\vartheta) - z} \, dz - \int_{J_{\vartheta,0}} \frac{a_1(\varrho(\vartheta) - z)}{\varrho(\vartheta) - z} \, dz = 0 = \mathcal{D}_\varphi(\hat{\vartheta}).
\]

By the uniqueness claim in Proposition 1.3, we conclude that \( \pi - \vartheta = \hat{\vartheta} \), as desired.

This completes the proof of 3), and in turn of Theorem 1.4.

As a consequence of Theorem 1.4 we now obtain the particular case in which \( a_1 \equiv \text{const} \) dealt with in Corollary 1.5.

**Proof of Corollary 1.5.** We point out that 1) and 2) in Corollary 1.5 follow from 1) and 2) in Theorem 1.4, respectively.

To prove 3) of Corollary 1.5, we first notice that \( \vartheta \in (0, \pi) \) in these cases. Also, if \( a_1 \equiv \text{const} \), then the cancellation property in (1.20) boils down to \( \mathcal{D}_\varphi(\vartheta) = 0 \), and therefore, by the uniqueness claim in Proposition 1.3 we obtain that \( \hat{\vartheta} = \vartheta \).

Furthermore, we recall that (1.23) holds true in this case, thanks to 3) of Theorem 1.4, and therefore, using the equivalent formulation of (1.23) given in (3.34) (with \( a_1 \equiv \text{const} \) in this case), we find that
\[
\mathcal{D}_\varphi(\pi - \vartheta) = \int_{J_{\vartheta,\pi}} \frac{a_1}{\vert \varrho(\vartheta) - z \vert^{n+1}} \, dz - \int_{J_{\vartheta,0}} \frac{a_1}{\vert \varrho(\vartheta) - z \vert^{n+1}} \, dz = 0 = \mathcal{D}_\varphi(\vartheta).
\]
Hence, using again the uniqueness claim in Proposition 1.3 we conclude that \( \pi - \vartheta = \vartheta \), which gives that \( \hat{\vartheta} = \frac{\pi}{2} \), as desired.

We now deal with the case \( s_1 = s_2 \), as given by Theorem 1.6. For this, we need a variation of Lemma 3.3 that takes into account the situation in which \( s_1 = s_2 \).
Lemma 3.4. Let $s \in (0, 1)$ and $K_1, K_2 \in K^2(n, s, \lambda, \varrho)$. Assume that there exists $\varepsilon_0 \in (0, 1)$ such that

$$\tag{3.35} |\sigma| K_2(\zeta) \leq (1 - \varepsilon_0) K_1(\zeta) \quad \text{for all } \zeta \in B_{\varepsilon_0} \setminus \{0\}. $$

Let $\Omega$ be an open bounded set with $C^1$-boundary and $E$ be a volume-constrained critical set of $\mathcal{C}$.

Let $x_0 \in \text{Reg}_E \cap \partial \Omega$, $x_k \in \text{Reg}_E \cap \Omega$ such that $x_k \to x_0$ as $k \to +\infty$ and $r_k > 0$ such that $r_k \to 0$ as $k \to +\infty$.

Suppose that $H$ and $V$ are open half-spaces such that

$$ \Omega^{x_0-r_k} \to H \quad \text{and} \quad E^{x_0-r_k} \to H \cap V \quad \text{in } L^1_{\text{loc}}(\mathbb{R}^n) \text{ as } k \to +\infty. $$

Let $\vartheta \in [0, \pi]$ be the angle between the half-spaces $H$ and $V$, that is $H \cap V = J_{0,\vartheta}$ in the notation of (1.11).

Then,

i) if $\vartheta = 0$ then

$$ \lim_{k \to +\infty} r_k^s \left[ H_{\partial E}^K(x_k) - \int_{\Omega^c} K_1(x_k - y) \, dy + \sigma \int_{\Omega^c} K_2(x_k - y) \, dy \right] = +\infty; $$

ii) if $\vartheta = \pi$ then

$$ \lim_{k \to +\infty} r_k^s \left[ H_{\partial E}^K(x_k) - \int_{\Omega^c} K_1(x_k - y) \, dy + \sigma \int_{\Omega^c} K_2(x_k - y) \, dy \right] = -\infty. $$

Proof. We establish i), being the proof of ii) analogous. For this, we use the notation introduced in the proof of Lemma 3.3, and specifically we recall formula (3.24), to be used here with $s_1 = s_2 = s$. In this case, we use (3.35) to see that, if $k$ is large enough, for all $z \in B_{1/2}(v_k)$ we have that

$$\tag{3.36} |\sigma| K_2(r_k(v_k - z)) \leq (1 - \varepsilon_0) K_1(r_k(v_k - z)).$$

This and (3.20) give that

$$Z_k(r_k(v_k - z)) = r_k^{n+s} \max \left\{ K_1(r_k(v_k - z)) , \ |\sigma| K_2(r_k(v_k - z)) \right\} \leq r_k^{n+s} K_1(r_k(v_k - z)),$$

which entails that $\alpha_k(z) = 0$.

Also, using again (3.36), it follows that

$$\beta_k(z) = r_k^{n+s} \left( K_1(r_k(v_k - z)) - |\sigma| K_2(r_k(v_k - z)) \right) \geq \varepsilon_0 r_k^{n+s} K_1(r_k(v_k - z)).$$

In light of these observations, (3.24) in this framework reduces to

$$\Upsilon_k \geq \varepsilon_0 r_k^{n+s} \int_{p_k \setminus \gamma_k} K_1(r_k(v_k - z)) \, dz - C.$$ 

We have thus recovered the last inequality in (3.25), with $1/2$ replaced by the constant $\varepsilon_0$. Then it suffices to proceed as in (3.26) and (3.27) to complete the proof.

Proof of Theorem 1.6. We fix a point $x_0 \in \partial \Omega \cap \text{Reg}_E$ and a sequence of points $x_k \in \Omega \cap \text{Reg}_E$ such that $x_k \to x_0$ as $k \to +\infty$. We also set $r_k := |x_k - x_0|$ and we observe that $r_k \to 0$ as $k \to +\infty$.

From (1.9) evaluated at $x_k$, we get

$$\tag{3.28} H_{\partial E}^{K_1}(x_k) - \int_{\Omega^c} K_1(x_k - y) \, dy + \sigma \int_{\Omega^c} K_2(x_k - y) \, dy + g(x_k) = c,$$
where $c$ does not depend on $k$. Thus, multiplying both sides by $r^*_k$, we find that
\[ r^*_k H^{K_1}_{\partial\Omega}(x_k) - r^*_k \int_{\Omega^c} K_1(x_k - y) dy + \sigma r^*_k \int_{\Omega^c} K_2(x_k - y) dy + r^*_k g(x_k) = c r^*_k. \]
Since $g$ is locally bounded, we have that $r^*_k g(x_k) \to 0$ as $k \to +\infty$, and therefore
\begin{equation}
(3.37) \quad \lim_{k \to +\infty} r^*_k \left[ H^{K_1}_{\partial\Omega}(x_k) - \int_{\Omega^c} K_1(x_k - y) dy + \sigma \int_{\Omega^c} K_2(x_k - y) dy \right] = 0.
\end{equation}
In light of Lemma 3.4 (which can be exploited here thanks to assumption (1.24)), this gives that the angle $\vartheta$ between $H$ and $V$ lies in $(0, \pi)$.

Thus, in order to prove (1.25), we can take $v \in H \cap \partial V$ and we see that, for every $k$, there exists $v_k \in \Omega^{x_0, r_k} \cap \partial E^{x_0, r_k}$ such that $v_k \to v$ as $k \to +\infty$, where $r_k$ is an infinitesimal sequence as $k \to +\infty$. As a consequence, for every $k$, there exists $x_k \in \text{Reg}_E \cap \Omega$ such that $v_k = \frac{x_k - x_0}{r_k}$ and $x_k \to x_0$ as $k \to +\infty$. Thus, we are in the position to apply iii) in Lemma 3.2 and conclude that
\[ \lim_{k \to +\infty} r^*_k \left[ H^{K_1}_{\partial\Omega}(x_k) - \int_{\Omega^c} K_1(x_k - y) dy \right] = H^{K_1}_{\partial(H \cap V)}(v) - \int_{H^c} K_1^*(v - y) dy. \]

Also, by Lebesgue’s Dominated Convergence Theorem,
\[ \lim_{k \to +\infty} r^*_k \int_{\Omega^c} K_2(x_k - y) dy = \lim_{k \to +\infty} \int_{(\Omega^{x_0, r_k})^c} r^{n+s} K_2(r_k(v_k - y)) dy \]
\[ = \int_{H^c} K_2^*(v - y) dy \]
and this limit is finite.

These considerations and (3.37) give the desired result in (1.25).}

We are now in the position of establishing Proposition 1.9.

**Proof of Proposition 1.9.** We exploit the notation in (1.12), the assumption in (1.13) and the change of variable $z = y/v$, to see that (1.27) can be written as
\[ 0 = H^{K_1}_{\partial (H \cap V)}(v) - \int_{H^c} K_1^*(v - y) dy + \sigma \int_{H^c} K_2^*(v - y) dy \]
\[ = \int_{\mathbb{R}^n} K_1^*(v - y) (\chi_{H \cap V \cap H}(y) - \chi_{H \cap V}(y)) dy + \sigma \int_{H^c} K_2^*(v - y) dy \]
\[ = \int_{\mathbb{R}^n} \frac{a_1(v - \bar{y})}{|v - y|^{n+s+1}} (\chi_{H \cap V \cap H}(y) - \chi_{H \cap V}(y)) dy + \sigma \int_{H^c} \frac{a_2(v - \bar{y})}{|v - y|^{n+s+2}} dy \]
\[ = |v|^{-s_1} \int_{\mathbb{R}^n} \frac{a_1(e(\vartheta) - z)}{|e(\vartheta) - z|^{n+s_1}} \chi_{J_0, \vartheta}(z) \left( \chi_{J_0, \vartheta}(z) - \chi_{J_0, \vartheta}(z) \right) dz + \sigma |v|^{-s_2} \int_{H^c} \frac{a_2(e(\vartheta) - z)}{|e(\vartheta) - z|^{n+s_2}} dz \]
\[ = |v|^{-s_1} \int_{J_0, \vartheta} \frac{a_1(e(\vartheta) - z)}{|e(\vartheta) - z|^{n+s_1}} dz - |v|^{-s_1} \int_{J_0, \vartheta} \frac{a_1(e(\vartheta) - z)}{|e(\vartheta) - z|^{n+s_1}} dz \]
\[ + \sigma |v|^{-s_2} \int_{H^c} \frac{a_2(e(\vartheta) - z)}{|e(\vartheta) - z|^{n+s_2}} dz. \]

Hence, recalling the assumption in (1.26), this gives the desired result in (1.28).}

4. **Proofs of Theorems 1.7 and 1.8**

We now deal with the possibly degenerate cases in which the nonlocal droplets either detach from the container or adhere completely to its surfaces. These cases depend on the strong attraction or repulsion of the second kernel and are described in the examples provided in Theorems 1.7 and 1.8, which we are now going to prove. For this, we need some auxiliary integral
estimates to detect the interaction between “thin sets”. This is formalized in Lemmata 4.1 and 4.2 here below:

**Lemma 4.1.** Let \( r, \ t > 0, \ s \in (0, 1) \) and

\[
D := \{ x = (x', x_n) \in \mathbb{R}^n : |x'| < r \text{ and } x_n \in (0, t) \}.
\]

Then,

\[
\int\int_{D \times \{ y_n < 0 \}} \frac{dx \; dy}{|x - y|^{n+s}} = c_* r^{n-1} \ell^{1-s},
\]

for a suitable \( c_* > 0 \), depending only on \( n \) and \( s \).

**Proof.** We recall that the surface area of the \((n-1)\)-dimensional unit sphere is equal to \( \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)} \), where \( \Gamma \) is the Gamma Function. Furthermore,

\[
\int_0^{+\infty} \frac{\ell^{n-2} \; d\ell}{(\ell^2 + 1)^{\frac{n+s}{2}}} = \frac{\Gamma\left(\frac{n-1}{2}\right) \Gamma\left(\frac{1+s}{2}\right)}{2\Gamma\left(\frac{n+s}{2}\right)}.
\]

Hence, we use the substitution \( \xi := \frac{y' - x'}{x_n - y_n} \) to see that

\[
\int\int_{D \times \{ y_n < 0 \}} \frac{dx \; dy}{|x - y|^{n+s}}
\]

\[
= \int_0^t \left[ \int_{|x'|<r} \left[ \int_{-\infty}^0 \int_{-\infty}^{\infty} \frac{d\xi}{(x_n - y_n)^{1+s}} \right] \; dy' \right] \; dx_n
\]

\[
= \frac{2\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{n-1}{2}\right)} \int_0^t \left[ \int_{|x'|<r} \left[ \int_{-\infty}^0 \int_{0}^{+\infty} \frac{\ell^{n-2} \; d\ell}{(x_n - y_n)^{1+s}(\ell^2 + 1)^{\frac{n+s}{2}}} \right] \; dy' \right] \; dx_n
\]

\[
= \frac{\pi^{\frac{n-1}{2}} \Gamma\left(\frac{1+s}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)} \int_0^t \left[ \int_{|x'|<r} \left[ \int_{-\infty}^0 \frac{dy_n}{(x_n - y_n)^{1+s}} \right] \; dx_n \right]
\]

\[
= \frac{2\pi^{\frac{n-1}{2}} \Gamma\left(\frac{1+s}{2}\right)}{s \Gamma\left(\frac{n+2}{2}\right)} \int_0^t \left[ \int_{x_n}^{dx_n} \right] \; dx_n
\]

\[
= \frac{2\pi^{\frac{n-1}{2}} \Gamma\left(\frac{1+s}{2}\right)}{s (1-s) \Gamma\left(\frac{n+2}{2}\right)} \int_0^t \; dx_n
\]

\[
= \frac{2\pi^{\frac{n-1}{2}} \Gamma\left(\frac{1+s}{2}\right)}{s (1-s) \Gamma\left(\frac{n+2}{2}\right)} r^{n-1} \ell^{1-s},
\]

as desired. \( \square \)

**Lemma 4.2.** Let \( r, \ t > 0, \ s \in (0, 1) \),

\[
D := \{ x = (x', x_n) \in \mathbb{R}^n : |x'| < r \text{ and } x_n \in (0, t) \}
\]

and \( F := \{ x = (x', x_n) \in \mathbb{R}^n : |x'| > r \text{ and } x_n \in (0, t) \} \). Then,

\[
\int\int_{D \times F} \frac{dx \; dy}{|x - y|^{n+s}} \leq Ct r^{n-1-s},
\]

for some \( C > 0 \) depending only on \( n \) and \( s \).

**Proof.** Differently from the proof of Lemma 4.1, here it is convenient to exploit the substitutions \( \alpha := \frac{x_n}{|x' - y'|} \) and \( \beta := \frac{y_n}{|x' - y'|} \). In this way we see that

\[
\int\int_{D \times F} \frac{dx \; dy}{|x - y|^{n+s}}
\]

\[
= \int\int_{D \times \{ y_n < 0 \}} \frac{dx \; dy}{|x - y|^{n+s}}
\]

\[
= \frac{2\pi^{\frac{n-1}{2}} \Gamma\left(\frac{1+s}{2}\right)}{s (1-s) \Gamma\left(\frac{n+2}{2}\right)} r^{n-1} \ell^{1-s},
\]

as desired.
contributions. For this, we consider the map \( \eta : \mathbb{R}^n \to \mathbb{R}^n \) and \( \delta \). Then, there exist \( \delta \) and \( \eta \) such that if \( \delta < \delta_0 \) and \( \eta < \delta_0 \) then

\[
\int_{\{x'<r\}} \int_{\{y'>r\}} \int_0^{t/|x'-y'|} \int_0^{t/|x'-y'|} \frac{d\beta}{x'-y'|^{n+s-2}(1 + (\alpha - \beta)^2)^{\frac{n-s}{2}}} d\alpha \ dy' dx' \\
\leq \int_{\{x'<r\}} \int_{\{y'>r\}} \int_0^{t/|x'-y'|} \int_0^{+\infty} \frac{d\gamma}{x'-y'|^{n+s-2}(1 + \gamma^2)^{\frac{n-s}{2}}} d\alpha \ dy' dx' \\
= C \int_{\{x'<r\}} \int_{\{y'>r\}} \frac{dy'}{|x'-y'|^{n+s-1}} dx' \\
= Ct \int_{\{x'<r\}} \int_{\{y'>r\}} \frac{dy'}{|x'-y'|^{n+s-1}} dx' \\
= Ct \int_{\{X'<1\}} \int_{\{Y'>1\}} \frac{dY'}{|X'-Y'|^{n+s-1}} dX' \\
= Ct \int_{\{X'<1\}} \int_{\{Y'>1\}} \frac{dY'}{|X'-Y'|^{n+s-1}} dX',
\]

where, as customary, we took the freedom of renaming \( C \) line after line. \( \square \)

Now, in the forthcoming Lemma 4.3 we present a further technical result that detects suitable cancellations involving “thin sets”. This is a pivotal result to account for the nonlocal scenario. Indeed, in the classical capillarity theory, to look for a competitor for a given set, one can dig out a (small deformation of a) cylinder with base radius equal to \( \varepsilon \) and height \( \delta \varepsilon \) and then add a ball with the same volume. A very convenient fact in this scenario is that the surface error produced by the cylinder is of order \( \varepsilon^{n-1} \), while the one produced by the balls are of order \( \delta \varepsilon^n \). That is, for \( \delta \) suitably small, the surface tension produced by the new ball is negligible with respect to the surface tension of the cylinder, thus allowing us to construct competitors in a nice and simple way.

Instead, in the nonlocal setting, for a given value of the fractional parameter, the corresponding nonlocal surface tension produced by cylinders and balls of the same volume are comparable. This makes the idea of “adding a ball to compensate the loss of volume caused by removing a cylinder” not suitable for the nonlocal framework. Instead, as we will see in the proof of Theorem 1.7, the volume compensation should occur through the addition of a suitably thin set placed at a regular point of the droplet. The fact that the corresponding nonlocal surface energy produces a negligible contribution will rely on the following result:

**Lemma 4.3.** Let \( s \in (0, 1) \), \( 0 < \varepsilon < \delta < 1 \) and \( \eta \in (0, 1) \). Let \( f \in C_{0, \infty}^0 \left( \mathbb{R}^n, (-\frac{s}{2}, \frac{s}{2}) \right) \) for some \( \alpha \in (0, 1) \) and assume that \( f(0) = 0 \) and \( \partial_i f(0) = 0 \) for all \( i \in \{1, \ldots, n-1\} \).

Let \( \varphi \in C^\infty(\mathbb{R}^n, [0, +\infty)) \) be such that \( \varphi(x') = 0 \) whenever \( |x'| > 1 \) and \( \int_{\mathbb{R}^{n-1}} \varphi(x') \ dx' = 1 \). Let

\[
\psi(x') := \frac{\eta}{\varepsilon^{n-1}} \varphi \left( \frac{x'}{\varepsilon} \right),
\]

\[
\mathcal{P} := \{ x = (x', x_n) \in \mathbb{R}^n : |x'| < \delta \text{ and } x_n > f(x') + \psi(x') \},
\]

\[
\mathcal{Q} := \{ x = (x', x_n) \in \mathbb{R}^n : |x'| < \delta \text{ and } x_n \in (f(x'), f(x') + \psi(x')) \}
\]

and \( \mathcal{R} := \{ x = (x', x_n) \in \mathbb{R}^n : |x'| < \delta \text{ and } x_n < f(x') \} \).

Then, there exist \( \delta_0 \in (0, 1) \) and \( C > 0 \), depending only on \( n, s, \alpha, f \) and \( \varphi \), such that if \( \delta < \delta_0 \) and \( \eta < \delta_0 \varepsilon^n \) then

\[
\left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{dx \ dy}{|x-y|^{n+s}} - \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{dx \ dy}{|x-y|^{n+s}} \right| \leq C \left( \delta^n + \frac{\eta}{\varepsilon^n} \right) \varepsilon^{(n-1)s} \eta^{1-s}.
\]

**Proof.** The gist of this proof is to use a suitable reflection to simplify most of the integral contributions. For this, we consider the map

\[
T(x) := (-x', 2f(x') + \psi(x') - x_n).
\]
We observe that when $|x'| < \delta$ the distance between the Jacobian of $T$ and minus the identity matrix is bounded from above by

$$C \sup_{|x'| < \delta} \left( |\nabla f(x')| + |\nabla \psi(x')| \right) \leq C \sup_{|x'| < \delta} \left( |\nabla f(x') - \nabla f(0)| + \frac{\eta}{\varepsilon^n} \right) \leq C \left( \delta^\alpha + \frac{\eta}{\varepsilon^n} \right),$$

and the latter is a small quantity, as long as $\delta_0$ is chosen sufficiently small.

Moreover, the condition $T(x) \in \Omega$ is equivalent to $|x'| < \delta$ and $2f(x') + \psi(x') - x_n \in (f(x'), f(x') + \psi(x'))$, which is in turn equivalent to $x \in \Omega$.

Similarly, the condition $T(x) \in \mathcal{P}$ is equivalent to $x \in \mathcal{R}$, as well as the condition $T(x) \in \mathcal{R}$ is equivalent to $x \in \mathcal{P}$.

From these observations and the change of variable $(X, Y) := (T(x), T(y))$ we arrive at

$$\int \int_{\mathcal{P} \times \mathcal{Q}} \frac{dx \, dy}{|x - y|^{n+s}} = \left( 1 + O \left( \delta^\alpha + \frac{\eta}{\varepsilon^n} \right) \right) \int \int_{\mathcal{R} \times \mathcal{Q}} \frac{dX \, dY}{|X - Y|^{n+s}}.$$

As a result,

$$\left( 1 \right) \left| \int \int_{\mathcal{P} \times \mathcal{Q}} \frac{dx \, dy}{|x - y|^{n+s}} - \int \int_{\mathcal{R} \times \mathcal{Q}} \frac{dx \, dy}{|x - y|^{n+s}} \right| \leq C \left( \frac{\delta^\alpha + \eta}{\varepsilon^n} \right) \int \int_{\mathcal{R} \times \mathcal{Q}} \frac{dX \, dY}{|X - Y|^{n+s}}.$$

Now we consider the transformation $S(x) := (x', x_n - f(x'))$. When $|x'| < \delta$ the distance between the Jacobian of $S$ and the identity matrix is bounded from above by

$$C \sup_{|x'| < \delta} |\nabla f(x')| \leq C \delta^\alpha.$$

Besides, if $x \in \mathcal{R}$ then $S(x) \in \{ x \in \mathbb{R}^n : |x'| < \delta$ and $x_n < 0 \}$. Also, if $x \in \Omega$ then

$$S(x) \in \{ x \in \mathbb{R}^n : |x'| < \delta \text{ and } x_n \in (0, \psi(x')) \} \subseteq \left\{ x \in \mathbb{R}^n : |x'| < \varepsilon \text{ and } x_n \in \left( 0, \frac{C\eta}{\varepsilon^{n-1}} \right) \right\}.$$

We stress that we are using here the fact that $\psi(x') = 0$ when $|x'| \geq \varepsilon$.

From these remarks and (1), using now the change of variable $(X, Y) := (S(x), S(y))$, it follows that

$$\left| \int \int_{\mathcal{P} \times \mathcal{Q}} \frac{dx \, dy}{|x - y|^{n+s}} - \int \int_{\mathcal{R} \times \mathcal{Q}} \frac{dx \, dy}{|x - y|^{n+s}} \right| \leq C \left( \frac{\delta^\alpha + \eta}{\varepsilon^n} \right) \int \int_{\mathcal{R} \times \mathcal{Q}} \frac{dX \, dY}{|X - Y|^{n+s}}.$$

We can thus employ Lemma 4.1 with $r := \varepsilon$ and $t := \frac{C\eta}{\varepsilon^{n-1}}$ and conclude that

$$\left| \int \int_{\mathcal{P} \times \mathcal{Q}} \frac{dx \, dy}{|x - y|^{n+s}} - \int \int_{\mathcal{R} \times \mathcal{Q}} \frac{dx \, dy}{|x - y|^{n+s}} \right| \leq C \left( \frac{\delta^\alpha + \eta}{\varepsilon^n} \right) \varepsilon^{n-1} \left( \frac{\eta}{\varepsilon^{n-1}} \right)^{1-s},$$

from which the desired result follows.

With this preliminary work, we can now complete the proofs of Theorems 1.7 and 1.8.

**Proof of Theorem 1.7.** Up to a rigid motion we can suppose that $p = e_n$. We let $\varepsilon > 0$ and $\delta > 0$, to be taken as small as we wish in what follows. We also define

$$\mathcal{B} := \left\{ x = (x', x_n) \in B_1 \setminus B_{1-\delta} : x_n > 0 \text{ and } |x'| < \varepsilon \right\}.$$

We stress that $\mathcal{B} \subseteq B_{\varepsilon_0}(p) \cap B_1$ as long as $\varepsilon$ is small enough. Also, we pick a point $q \in \text{Reg}_\varepsilon \cap \Omega$ and we modify the surface of $\partial E$ in the normal direction in an $\varepsilon$-neighborhood of $q$ by a set $\mathcal{B'}$ with $|\mathcal{B}'| = |\mathcal{B}|$, see Figure 7 and notice that the geometry of Lemma 4.3 can be reproduced, up to a rigid motion. We stress that $\eta$ in Lemma 4.3 corresponds to the volume of the perturbation induced by $\psi$, therefore in this setting we will apply Lemma 4.3 with $\eta := |\mathcal{B}'| = |\mathcal{B}| \leq C \delta^\alpha$.

We also denote by $\Theta$ a cylinder centered at $q$ (oriented by the normal of $\mathcal{B'}$ at $q$) of height equal to $2\delta$ and radius of the basis equal to $\delta$. In this way, we have that if $x \in \mathcal{B'}$ and $y \in \mathbb{R}^n \setminus \Theta$
Figure 7. Removing the thin set $\mathcal{B}$ to $E$ near $p$ and adding the thin set $\mathcal{B}'$ with the same volume.

then $|x - y| \geq |y - q| - |q - x| \geq \frac{\delta}{2} - C\varepsilon \geq \frac{\delta}{4}$, as long as $\varepsilon$ is small enough, possibly in dependence of $\delta$, see Figure 8, whence

$$I_1(\mathcal{B}', B_1 \setminus \Theta) \leq C \int_{\mathcal{B}' \times B_1} \frac{dx \, dy}{\delta^{n+s_1}} \leq \frac{C|\mathcal{B}'|}{\delta^{n+s_1}}.$$

Consequently,

$$I_1(\mathcal{B}', B_1 \setminus E \setminus \mathcal{B}') - I_1(\mathcal{B}', E) \leq I_1(\mathcal{B}', (B_1 \setminus E \setminus \mathcal{B}') \cap \Theta) - I_1(\mathcal{B}', E \cap \Theta) + \frac{C|\mathcal{B}'|}{\delta^{n+s_1}}$$

$$\leq I_1(\mathcal{B}', (B_1 \setminus E \setminus \mathcal{B}') \cap \Theta) - I_1(\mathcal{B}', E \cap \Theta) + \frac{C\varepsilon^n}{\delta^{n-1+s_1}},$$

for some $C > 0$ that, as usual, gets renamed line after line.

Figure 8. Surrounding $\mathcal{B}'$ with a small cylinder $\Theta$.

In view of Lemma 4.3, we also know that

$$I_1(\mathcal{B}', (B_1 \setminus E \setminus \mathcal{B}') \cap \Theta) - I_1(\mathcal{B}', E \cap \Theta) \leq C\delta^\alpha \varepsilon^{(n-1)s_1} (\delta \varepsilon^n)^{1-s_1} = C\delta^{1-s_1+\alpha} \varepsilon^{n-s_1}.$$

This and (4.2) lead to

$$I_1(\mathcal{B}', B_1 \setminus E \setminus \mathcal{B}') - I_1(\mathcal{B}', E) \leq C\delta^{1-s_1+\alpha} \varepsilon^{n-s_1} + \frac{C\varepsilon^n}{\delta^{n-1+s_1}}.$$
Now we claim that
\begin{align*}
I_1(B, B_1 \setminus E) + I_1(B', E) + \sigma I_2(B, B_1^c) \\
\leq I_1(B, E \cup B' \setminus B) + I_1(B', B_1 \setminus E \setminus B') + I_1(B, B') + \sigma I_2(B', B_1^c). \tag{4.4}
\end{align*}
To prove this, we construct a competitor for the minimal set $E$ and compare their energies. Indeed, the set $\tilde{E} := (E \setminus B) \cup B'$ is a competitor for $E$, with the same volume of $E$, and accordingly, using the notation $X := E \setminus B$ and $Y := B_1 \setminus E \setminus B'$,
\begin{align*}
0 &> \varepsilon(E) - \varepsilon(\tilde{E}) \\
&= I_1(E, B_1 \setminus E) - I_1(\tilde{E}, B_1 \setminus \tilde{E}) + \sigma I_2(E, B_1^c) - \sigma I_2(\tilde{E}, B_1^c) \\
&= I_1(X \cup B, Y \cup B') - I_1(X \cup B', Y \cup B) + \sigma I_2(X \cup B, B_1^c) - \sigma I_2(X \cup B', B_1^c) \\
&= I_1(X, B') + I_1(B, Y) - I_1(X, B) - I_1(B', Y) + \sigma I_2(B, B_1^c) - \sigma I_2(B', B_1^c) \\
&= \left(I_1(E, B') - I_1(B, B')\right) + \left(I_1(B, B_1 \setminus E) - I_1(B, B')\right) \\
&= I_1(E, B') + I_1(B, B_1 \setminus E) - I_1(B, B') - I_1(E \cup B' \setminus B, B) - I_1(B', B_1 \setminus E \setminus B') \\
&\quad + \sigma I_2(B, B_1^c) - \sigma I_2(B', B_1^c).
\end{align*}
This proves (4.4).

By combining (4.3) and (4.4) we find that
\begin{align*}
I_1(B, B_1 \setminus E) + \sigma I_2(B, B_1^c) \\
\leq I_1(B, E \cup B' \setminus B) + I_1(B, B') + \sigma I_2(B', B_1^c) + C\delta^{1-s_1+\alpha} \varepsilon^{-s_1} + \frac{C\varepsilon^n}{\delta^{n-1+s_1}}. \tag{4.5}
\end{align*}
Besides, since the distance between $B'$ and $B_1^c$ is bounded from below by a uniform quantity, only depending on $q$ and $\varepsilon_0$ (and, in particular, independent of $\varepsilon$), we have that
\begin{align*}
I_2(B', B_1^c) \leq C|B'| = C|B| \leq C\varepsilon^n,
\end{align*}
for some $C > 0$ depending only on $n$, $s_2$, $k_2$, $\varepsilon_0$, $q$ and the regularity of $\partial E$ in the vicinity of $q$. This and (4.5) yield that
\begin{align*}
\sigma I_2(B, B_1^c) \leq I_1(B, E \cup B' \setminus B) + I_1(B, B') + C\varepsilon^n + C\delta^{1-s_1+\alpha} \varepsilon^{-s_1} + \frac{C\varepsilon^n}{\delta^{n-1+s_1}} \\
\leq I_1(B, B_1 \setminus B) + I_1(B, B') + C\varepsilon^n + C\delta^{1-s_1+\alpha} \varepsilon^{-s_1} + \frac{C\varepsilon^n}{\delta^{n-1+s_1}} \\
\leq I_1(B, B_1 \setminus B) + C\delta^{1-s_1+\alpha} \varepsilon^{-s_1} + \frac{C\varepsilon^n}{\delta^{n-1+s_1}}, \tag{4.6}
\end{align*}
up to renaming $C$ line after line.

Now, we use the change of variables $X := \frac{x - e_n}{\varepsilon}$ and $Y := \frac{y - e_n}{\varepsilon}$ to see that
\begin{align*}
\varepsilon^{s_1-n} I_1(B, B_1 \setminus B) = k_1 \varepsilon^{s_1-n} \int \int_{B \times (B_1 \setminus B)} \frac{dx \, dy}{|x - y|^{n+s_1}} = k_1 \int \int_{\mathcal{L}_\varepsilon \cup A_\varepsilon} \frac{dX \, dY}{|X - Y|^{n+s_1}}, \tag{4.7}
\end{align*}
where
\begin{align*}
\mathcal{Z}_\varepsilon &:= \frac{B - e_n}{\varepsilon} = \left\{ X \in \mathbb{R}^n : |X'| < 1, \ X_n > -\frac{1}{\varepsilon} \text{ and } \left| X + \frac{e_n}{\varepsilon} \right| \in \left[ \frac{1}{\varepsilon} - \delta, \frac{1}{\varepsilon} \right] \right\} \\
\mathcal{A}_\varepsilon &:= \frac{(B_1 \setminus B) - e_n}{\varepsilon} = \mathcal{L}_\varepsilon \cup \mathcal{M}_\varepsilon \cup \mathcal{N}_\varepsilon,
\end{align*}
with
\begin{align*}
\mathcal{L}_\varepsilon &:= \left\{ X \in \mathbb{R}^n : \left| X + \frac{e_n}{\varepsilon} \right| < \frac{1}{\varepsilon} - \delta \right\},
\end{align*}
\[ \mathcal{M}_\varepsilon := \left\{ X \in \mathbb{R}^n : |X'| \geq 1 \text{ and } \left| X + \frac{e_n}{\varepsilon} \right| \in \left[ \frac{1}{\varepsilon} - \delta, \frac{1}{\varepsilon} \right) \right\} \]

and \[ \mathcal{N}_\varepsilon := \left\{ X \in \mathbb{R}^n : |X'| < 1, \ X_n \leq -\frac{1}{\varepsilon} \text{ and } \left| X + \frac{e_n}{\varepsilon} \right| \in \left[ \frac{1}{\varepsilon} - \delta, \frac{1}{\varepsilon} \right) \right\}. \]

Similarly, \( \varepsilon^{s_2-n} I_2(B, B'_1) = k_2 \varepsilon^{s_2-n} \int_{B \times B'_1} \frac{dx \, dy}{|x-y|^{n+s_2}} = k_2 \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{dX \, dY}{|X-Y|^{n+s_2}}, \)

where \[ \mathcal{O}_\varepsilon := \left\{ X \in \mathbb{R}^n : \left| X + \frac{e_n}{\varepsilon} \right| \geq \frac{1}{\varepsilon} \right\}. \]

Plugging (4.7) and (4.8) into (4.6), we arrive at

\[ \sigma \varepsilon^{s_1-s_2} k_2 \left\langle \frac{dX \, dY}{|X-Y|^{n+s_2}} \right\rangle \leq k_1 \left\langle \frac{dX \, dY}{|X-Y|^{n+s_1}} + C\delta^{1-s_1+a} + \frac{C\varepsilon^{s_1}}{\delta^{n-1+s_1}} \right\rangle. \]

Now we claim that, if \( \varepsilon > 0 \) is suitably small, possibly in dependence of \( \delta \), then

\[ \mathcal{B} \subseteq \left\{ x = (x', x_n) \in \mathbb{R}^n : |x'| < \varepsilon \text{ and } x_n \in [1 - (1 + \delta)\delta \varepsilon, 1) \right\}. \]

Indeed, if \( x \in \mathbb{B} \) then

\[ x_n = \sqrt{|x|^2 - |x'|^2} \geq \sqrt{(1 - \delta \varepsilon)^2 - \varepsilon^2} = \sqrt{1 - 2\delta \varepsilon + \delta^2 \varepsilon^2 - \varepsilon^2} \]

\[ \geq \sqrt{1 - 2(1 + \delta)\delta \varepsilon + (1 + \delta)^2 \delta^2 \varepsilon^2} = \sqrt{(1 - (1 + \delta)\delta \varepsilon)^2} = 1 - (1 + \delta)\delta \varepsilon \]

and this establishes (4.10).

It follows from (4.10) that

\[ \mathcal{Z}_\varepsilon \subseteq \left\{ X = (X', X_n) \in \mathbb{R}^n : |X'| < 1 \text{ and } X_n \in [- (1 + \delta)\delta \varepsilon, 0) \right\} =: \mathcal{Z}_\varepsilon^*. \]

Note also that

\[ \mathcal{O}_\varepsilon \supseteq \{ Y_n > 0 \}. \]

We now claim that

\[ \mathcal{Z}_\varepsilon \supseteq \left\{ X \in \mathbb{R}^n : |X'| < 1, \ X_n \in (-\delta, 0) \text{ and } \left| X + \frac{e_n}{\varepsilon} \right| < \frac{1}{\varepsilon} \right\} =: \mathcal{W}_\varepsilon. \]

To check this, suppose by contradiction that there exists \( X \in \mathcal{W}_\varepsilon \) with \( \left| X + \frac{e_n}{\varepsilon} \right| < \frac{1}{\varepsilon} - \delta \). Then, we have that

\[ 0 < \left( \frac{1}{\varepsilon} - \delta \right)^2 - \left| X + \frac{e_n}{\varepsilon} \right|^2 = \frac{1}{\varepsilon^2} + \delta^2 - \frac{2\delta}{\varepsilon} - |X'|^2 - \left( X_n + \frac{1}{\varepsilon} \right)^2 \]

\[ = \delta^2 - \frac{2\delta}{\varepsilon} - |X'|^2 - X_n^2 - \frac{2X_n}{\varepsilon} \leq \delta^2 - |X'|^2 - X_n^2, \]

that is \( |X| < \delta \), and thus

\[ \frac{1}{\varepsilon} - \delta \geq \left| X + \frac{e_n}{\varepsilon} \right| \geq \left| \frac{e_n}{\varepsilon} \right| = \frac{1}{\varepsilon} - |X| > \frac{1}{\varepsilon} - \delta. \]

This is a contradiction which establishes (4.13).

Hence, by (4.12) and (4.13), we see that

\[ \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{dX \, dY}{|X-Y|^{n+s_2}} \geq \int_{\mathcal{W}_\varepsilon \times \{ Y_n > 0 \}} \frac{dX \, dY}{|X-Y|^{n+s_2}} \]

\[ \geq \int_{\mathcal{W}_\varepsilon^* \times \{ Y_n > 0 \}} \frac{dX \, dY}{|X-Y|^{n+s_2}} - \int_{(\mathcal{W}_\varepsilon^* \setminus \mathcal{W}_\varepsilon) \times \{ Y_n > 0 \}} \frac{dX \, dY}{|X-Y|^{n+s_2}}, \]

where

\[ \mathcal{W}_\varepsilon^* := \left\{ X \in \mathbb{R}^n : |X'| < 1 \text{ and } X_n \in (-\delta, 0) \right\}. \]
Since
\[\int \int_{(W^*_\delta \setminus W_{\varepsilon}) \times \{Y_n > 0\}} \frac{dX \, dY}{|X - Y|^{n+s_2}} \leq \int \int_{W^*_\delta \times \{Y_n > 0\}} \frac{dX \, dY}{|X - Y|^{n+s_2}} < +\infty\]
and
\[\lim_{\varepsilon \to 0} |W^*_\delta \setminus W_{\varepsilon}| = 0,
we have that
\[\lim_{\varepsilon \to 0} \int \int_{(W^*_\delta \setminus W_{\varepsilon}) \times \{Y_n > 0\}} \frac{dX \, dY}{|X - Y|^{n+s_2}} = 0\]
and, as a consequence, we infer from (4.14) that
\[\lim \inf_{\varepsilon \to 0} \int \int_{Z_{\varepsilon} \times \{Y_n > 0\}} \frac{dX \, dY}{|X - Y|^{n+s_2}} \geq \int \int_{W^*_\delta \times \{Y_n > 0\}} \frac{dX \, dY}{|X - Y|^{n+s_2}}.\]

We also note that if \(a \gg b > 0\) then
\[\sqrt{a^2 - b^2} = b^2 \leq \frac{b^2}{2a} = \frac{1}{2a} \leq \frac{a}{2a} \leq \frac{1}{a},\]
whence if \(X \in N_{\varepsilon}\) then
\[-X_n - \frac{2}{\varepsilon} = \left| X_n + \frac{1}{\varepsilon} \right| = \sqrt{\left| X_n + \frac{1}{\varepsilon} \right|^2 - \left| X' \right|^2 - \frac{1}{\varepsilon}}\]
\[\in \left[ \left| X + \frac{e_n}{\varepsilon} \right| - \frac{\left| X' \right|^2}{2 \left| X + \frac{e_n}{\varepsilon} \right|^2} - \frac{\left| X' \right|^4}{2 \left| X + \frac{e_n}{\varepsilon} \right|^3} - \frac{1}{\varepsilon} \right],\]
\[\subseteq [-\delta - 2\varepsilon, 2\varepsilon] \subseteq [-2\delta, 2\delta],\]
as long as \(\varepsilon\) is sufficiently small, leading to
\[|N_{\varepsilon}| \leq \left\{ X \in \mathbb{R}^n : |X'| < 1, \text{ and } X_n \in \left[ \left[ -\frac{2}{\varepsilon} - 2\delta, -\frac{2}{\varepsilon} + 2\delta \right] \right] \right\} \leq C\delta.\]

Furthermore, if \(X \in Z_{\varepsilon}\) then \(X_n \geq -(1 + \delta)\delta\), thanks to (4.10), and therefore if \(Y \in N_{\varepsilon}\) we have that
\[|X - Y| \geq X_n - Y_n \geq -(1 + \delta)\delta + \frac{1}{\varepsilon} \geq \frac{1}{\varepsilon}.\]
This and (4.16) yield that
\[\int \int_{Z_{\varepsilon} \times N_{\varepsilon}} \frac{dX \, dY}{|X - Y|^{n+s_1}} \leq C\varepsilon^{n+s_1} |Z_{\varepsilon}| |N_{\varepsilon}| \leq C\varepsilon^{n+s_1}.\]

Now we set
\[M'_{\varepsilon} := M_{\varepsilon} \cap B_2 \quad \text{and} \quad M''_{\varepsilon} := M_{\varepsilon} \setminus B_2.\]

We remark that, if \(\varepsilon > 0\) is suitably small, possibly in dependency of \(\delta\), then
\[M'_{\varepsilon} \subseteq \{ X \in \mathbb{R}^n : |X'| \in [1, 2] \text{ and } X_n \in [-1/2, 1/2 + \delta, 0] \} =: M^*_\varepsilon.\]

Indeed, if \(X \in M'_{\varepsilon}\) then \(|X'| \geq 1\) and \(|X'| \leq |X| < 2\). Furthermore,
\[1 + \left| X_n + \frac{1}{\varepsilon} \right|^2 \leq \left| X' \right|^2 + \left| X_n + \frac{1}{\varepsilon} \right|^2 = \left| X + \frac{e_n}{\varepsilon} \right|^2 \leq \frac{1}{\varepsilon^2},\]
which gives that \(X_n < 0\).
Moreover,
\[4 + \left| X_n + \frac{1}{\varepsilon} \right|^2 \geq \left| X' \right|^2 + \left| X_n + \frac{1}{\varepsilon} \right|^2 = \left| X + \frac{e_n}{\varepsilon} \right|^2 \geq \left( \frac{1}{\varepsilon} - \delta \right)^2.\]
Since $X_n \geq -|X| \geq -2$, this gives that
\[
X_n + \frac{1}{\varepsilon} = \sqrt{\left|X_n + \frac{1}{\varepsilon}\right|^2} \geq \sqrt{\left(\frac{1}{\varepsilon} - \delta\right)^2 - 4} = \sqrt{\frac{1}{\varepsilon^2} - \frac{2\delta}{\varepsilon} + \delta^2 - 4} = \frac{1}{\varepsilon}\sqrt{1 - 2\delta \varepsilon + \delta^2 \varepsilon^2 - 4 \varepsilon^2} \geq \frac{1}{\varepsilon}(1 - (1 + \delta)\delta)\varepsilon)
\]
and accordingly $X_n \geq -(1 + \delta)\delta$. These observations complete the proof of (4.18).

We now use (4.18) in combination with (4.11). In this way, we see that
\[
\text{(4.19)} \quad \int_{Z_x \times \mathcal{M}_x} dX dY \quad \leq \quad \int_{Z_x \times \mathcal{M}_x} dX dY \quad \leq \quad C |Z_x| \int_{\mathbb{R}^n \setminus B_{1/2}} \frac{dZ}{|Z|^{n+1}} \quad \leq \quad C\delta.
\]
Combining this and (4.19) we conclude that
\[
\text{(4.20)} \quad \lim_{\varepsilon \searrow 0} \sup \int_{Z_x \times \mathcal{A}_x} dX dY \quad \leq \quad \int_{Z_x \times \mathcal{M}_x} dX dY + C\delta.
\]
Using the latter inequality and (4.17) we obtain that
\[
\lim_{\varepsilon \searrow 0} \sup \int_{Z_x \times \mathcal{A}_x} dX dY \quad \leq \quad \int_{Z_x \times \mathcal{M}_x} dX dY + C\delta + \lim_{\varepsilon \searrow 0} \sup \int_{Z_x \times \mathcal{L}_x} dX dY.
\]
Now we consider the map
\[
\{ \mathbb{R}^n : |X'| < 2 \} \ni X = (X', X_n) \mapsto T(X) := \left( X', X_n - \sqrt{\left(\frac{1}{\varepsilon} - \delta\right)^2 - |X'|^2} + \frac{1}{\varepsilon} \right)
\]
and we observe that if $X \in Z_x$ then $X := T(X)$ satisfies $|X'| < 1$ and
\[
X_n = \left| X_n + \frac{1}{\varepsilon} - \sqrt{\left(\frac{1}{\varepsilon} - \delta\right)^2 - |X'|^2} + \sqrt{\left(\frac{1}{\varepsilon} - \delta\right)^2 - |X'|^2} - \sqrt{\left(\frac{1}{\varepsilon} - \delta\right)^2 - |X'|^2} \right| \in \left[ 0, \sqrt{\frac{1}{\varepsilon^2} - |X'|^2} - \sqrt{\left(\frac{1}{\varepsilon} - \delta\right)^2 - |X'|^2} \right] \subseteq [0, (1 + \delta)\delta].
\]
In addition, if $Y \in \mathcal{L}_x := \mathcal{L}_x \cap B_2$ and $Y := T(Y)$, we have that $|Y'| < 2$ and
\[
Y_n \leq \left| Y_n + \frac{1}{\varepsilon} - \sqrt{\left(\frac{1}{\varepsilon} - \delta\right)^2 - |Y'|^2} \right| \leq \left| Y_n + \frac{1}{\varepsilon} - \sqrt{\left(\frac{1}{\varepsilon} - \delta\right)^2 - |Y'|^2} \right| \leq 0.
\]
We also observe that the distance of the Jacobian matrix of $T$ from the identity is bounded from above by
\[
C \left| D_X \sqrt{\left(\frac{1}{\varepsilon} - \delta\right)^2 - |X'|^2} \right| \leq \frac{C|X'|}{{\sqrt{\left(\frac{1}{\varepsilon} - \delta\right)^2 - |X'|^2}}} \leq C\varepsilon,
\]
yielding that, in the above notation, $|X - Y| \leq (1 + C\varepsilon)|X - Y|$, with the freedom, as usual, of renaming $C$. 37


These observations allow us to conclude that
\[(4.21) \int\int_{Z_\delta \times L_\varepsilon'} \frac{dX \, dY}{|X - Y|^{n+s_1}} \leq (1 + \varepsilon) \int\int_{X^*_\delta \times y^*} \frac{dX \, dY}{|X - Y|^{n+s_1}}\]
where
\[X^*_\delta := \{ X \in \mathbb{R}^n : |X'| < 1 \text{ and } X_n \in (0, (1 + \delta)\delta) \}\]
and \[y^* := \{ X \in \mathbb{R}^n : |X'| < 2 \text{ and } X_n < 0 \}\].

Also, setting \(L_\varepsilon'' := L_\varepsilon \setminus B_2\), we have that
\[\int\int_{Z_\delta \times L_\varepsilon''} \frac{dX \, dY}{|X - Y|^{n+s_1}} \leq C |Z_\varepsilon| \int\int_{\mathbb{R}^n \setminus B_{1/2}} \frac{dZ}{Z} \leq C\delta.\]

Combining this inequality and (4.21) we find that
\[\int\int_{Z_\delta \times L_\varepsilon'} \frac{dX \, dY}{|X - Y|^{n+s_1}} \leq (1 + \varepsilon) \int\int_{X^*_\delta \times y^*} \frac{dX \, dY}{|X - Y|^{n+s_1}} + C\delta.\]

From this and (4.20) we arrive at
\[\limsup_{\varepsilon \searrow 0} \int\int_{Z_\delta \times A_\varepsilon} \frac{dX \, dY}{|X - Y|^{n+s_1}} \leq \int\int_{Z^*_\delta \times M^*_\delta} \frac{dX \, dY}{|X - Y|^{n+s_1}} + \limsup_{\varepsilon \searrow 0} (1 + \varepsilon) \int\int_{X^*_\delta \times y^*} \frac{dX \, dY}{|X - Y|^{n+s_1}} + C\delta.\]

Thus, given \(\delta > 0\), to be taken conveniently small, we consider the limit \(\varepsilon \searrow 0\) and we deduce from the latter inequality, (4.9) and (4.15) that, as \(\varepsilon \searrow 0\),
\[(4.22) \sigma \varepsilon^{s_1 - s_2} k_2 \left( \int\int_{W_{\delta} \times \{Y_n > 0\}} \frac{dX \, dY}{|X - Y|^{n+s_2}} + o(1) \right) \leq k_1 \left( \int\int_{Z^*_\delta \times M^*_\delta} \frac{dX \, dY}{|X - Y|^{n+s_1}} + (1 + \varepsilon) \int\int_{X^*_\delta \times y^*} \frac{dX \, dY}{|X - Y|^{n+s_1}} \right) + C\delta + C\delta^{1-s_1+\alpha} + \frac{C\varepsilon^{s_1}}{\delta^{n-1+s_1}}.\]

This yields that necessarily
\[(4.23) \quad s_1 \geq s_2.\]

Furthermore, if \(s_1 = s_2\) then we obtain, passing to the limit (4.22) as \(\varepsilon \searrow 0\), that
\[(4.24) \sigma k_2 \int\int_{W_{\delta} \times \{Y_n > 0\}} \frac{dX \, dY}{|X - Y|^{n+s_1}} \leq k_1 \left( \int\int_{Z^*_\delta \times M^*_\delta} \frac{dX \, dY}{|X - Y|^{n+s_1}} + \int\int_{X^*_\delta \times y^*} \frac{dX \, dY}{|X - Y|^{n+s_1}} \right) + C\delta + C\delta^{1-s_1+\alpha}.\]

We are now ready to send \(\delta \searrow 0\). To this end, we multiply (4.24) by \(\delta^{s_1-1}\) and we make use of Lemmata 4.1 and 4.2 to find that
\[\sigma \sigma k_2 = \lim_{\delta \searrow 0} \sigma k_2 \delta^{s_1-1} \int\int_{W_{\delta} \times \{Y_n > 0\}} \frac{dX \, dY}{|X - Y|^{n+s_1}} \leq \lim_{\delta \searrow 0} \left[ k_1 \delta^{s_1-1} \left( \int\int_{Z^*_\delta \times M^*_\delta} \frac{dX \, dY}{|X - Y|^{n+s_1}} + \int\int_{X^*_\delta \times y^*} \frac{dX \, dY}{|X - Y|^{n+s_1}} \right) + C\delta^{s_1} + C\delta^{\alpha} \right] \leq \lim_{\delta \searrow 0} [C\delta^{s_1}(1 + \delta) + c_* k_1 (1 + \delta)^{1-s_1} + C\delta^{s_1} + C\delta^{\alpha}] = c_* k_1.\]
and therefore $\sigma k_2 \leq k_1$. Thanks to this, we have that, to complete the proof of Theorem 1.7, it only remains to rule out the case $s_1 = s_2$ and $k_1 = \sigma k_2$. In this situation,

$$\mathcal{C}(F) = \mathcal{E}(F) = k_1 \int_{F \times F} \frac{dx \, dy}{|x - y|^{n+s_1}},$$

hence all the minimizers with prescribed volume correspond to balls, thanks to [FS08]. But this violates the assumptions about the point $p$ in Theorem 1.7. \hfill \Box

**Proof of Theorem 1.8.** This can be seen as a counterpart of Theorem 1.7 based on complementary sets. For this argument, we denote by $C_\sigma$, instead of $C$, the functional in (1.6), in order to showcase explicitly its dependence on the relative adhesion coefficient $\sigma$. Thus, in the setting of Theorem 1.8, if $F \subseteq \Omega$ and $\tilde{F} := \Omega \setminus F$,

$$C_\sigma(\tilde{F}) = I_1(\Omega \setminus F, (\Omega \setminus F)^c \cap \Omega) + \sigma I_2(\Omega \setminus F, \Omega^c) = I_1(\Omega \setminus F, F) + \sigma I_2(\Omega \setminus F, \Omega^c) = C_{-\sigma}(F) + \sigma I_2(F, \Omega^c) + \sigma I_2(\Omega \setminus F, \Omega^c).$$

Since the latter term does not depend on $F$, we see that if $E$, as in the statement of Theorem 1.8, is a volume-constrained minimizer of $C_\sigma$, then $\tilde{E} := \Omega \setminus E$ is a volume-constrained minimizer of $C_{-\sigma}$. Now, the set $\tilde{E}$ fulfills the assumptions of Theorem 1.7 with $\sigma$ replaced by $-\sigma$. It follows that either $s_1 > s_2$, or $s_1 = s_2$ and $k_1 > -\sigma k_2$, as desired. \hfill \Box

5. Unique determination of the contact angle and proof of Theorem 1.10

Here we discuss the existence and uniqueness theory for the equation that prescribes the non-local angle of contact between the droplet and the container. This analysis will ultimately lead to the proof of Theorem 1.10: for this, it is convenient to perform some integral computations in order to appropriately rewrite integral interactions involving cones, detecting cancellations, using a dimensional reduction argument and a well designed notation of polar angle with respect to the kernel singularity. The details go as follows.

**Lemma 5.1.** In the notation of (1.11), (1.29), (1.30) and (1.31), if $\vartheta \in (0, \pi)$, then

\[
\int_{J_{\vartheta,\pi}} \frac{a_1(x - e(\vartheta))}{|x - e(\vartheta)|^{n+s_1}} \, dx - \int_{J_{0,\vartheta}} \frac{a_1(x - e(\vartheta))}{|x - e(\vartheta)|^{n+s_1}} \, dx = \frac{1}{s_1(\sin \vartheta)^{s_1}} \left( \int_0^\vartheta \phi_1(\alpha) (\sin \alpha)^{s_1} \, d\alpha - \int_\vartheta^\pi \phi_1(\alpha) (\sin \alpha)^{s_1} \, d\alpha \right).
\]

\[
(5.1)
\]

![Figure 9. A geometric argument involved in the proof of Lemma 5.1.](image-url)
Proof. We stress that each of the integrals on the left hand side of (5.1) is divergent, hence the two terms have to be considered together, in the principal value sense. However, for typographical convenience, we will formally act on the integrals by omitting the principal value notation and perform the cancellations necessary to have only finite contributions to obtain the desired result.

To this end, we recall (1.11) and observe that \( x \in J_{0,\vartheta} \cap \{ x_n < 2 \sin \vartheta \} \) if and only if \( z := 2e(\vartheta) - x \in J_{0,\pi} \cap \{ x_n < 2 \sin \vartheta \} \), see Figure 9. Hence, by the symmetry of \( a_1 \),

\[
\int_{J_{0,\vartheta} \cap \{ x_n < 2 \sin \vartheta \}} \frac{a_1(x-e(\vartheta))}{|x-e(\vartheta)|^{n+1}} \, dx = \int_{J_{0,\pi} \cap \{ x_n < 2 \sin \vartheta \}} \frac{a_1(z-e(\vartheta))}{|z-e(\vartheta)|^{n+1}} \, dz.
\]

Consequently, if we denote by \( \Upsilon \) the left hand side of (5.1), we see after a cancellation that

\[
(5.2) \quad \Upsilon = \int_{J_{0,\vartheta} \cap \{ x_n < 2 \sin \vartheta \}} \frac{a_1(x-e(\vartheta))}{|x-e(\vartheta)|^{n+1}} \, dx - \int_{J_{0,\vartheta} \cap \{ x_n < 2 \sin \vartheta \}} \frac{a_1(x-e(\vartheta))}{|x-e(\vartheta)|^{n+1}} \, dx.
\]

It is useful now to reduce the problem to that in dimension 2. To this end, we adopt the notation in (1.29) and (1.30) and note that

\[
(5.3) \quad \int_{J_{0,\vartheta} \cap \{ x_n < 2 \sin \vartheta \}} \frac{a_1(x-e(\vartheta))}{|x-e(\vartheta)|^{n+1}} \, dx = \int_{\{ (x_1, x_n) \in J_{0,\vartheta} \cap x_n \in \mathbb{R}^{n-2}, \ x_n < 2 \sin \vartheta \}} \frac{a_1(x_1 - \cos \vartheta) e_1 + (x_n - \sin \vartheta) e_n + (0, \bar{x}, 0)}{(x_1 - \cos \vartheta)^2 + (x_n - \sin \vartheta)^2 + |\bar{x}|^2}^{\frac{n+1}{2}} \, dx_1 \, dx_n
\]

\[
= \int_{\{ y = (y_1, y_2) \in J_{0,\vartheta} \cap y_2 \in \mathbb{R}^{n-2}, y_2 < 2 \sin \vartheta \}} \frac{a_1((y_1 - \cos \vartheta) e_1 + (y_2 - \sin \vartheta) e_n + |y - e(\vartheta)|(0, \bar{y}, 0))}{|y - e(\vartheta)|^{2+n+1}} \, dy \, \bar{y}
\]

\[
= \int_{J_{0,\vartheta} \cap \{ y_2 > 2 \sin \vartheta \}} \frac{a_1(y - e(\vartheta))}{|y - e(\vartheta)|^{2+n+1}} \, dy.
\]

Similarly,

\[
\int_{J_{0,\vartheta} \cap \{ x_n < 2 \sin \vartheta \}} \frac{a_1(x-e(\vartheta))}{|x-e(\vartheta)|^{n+1}} \, dx = \int_{J_{0,\vartheta} \cap \{ y_2 > 2 \sin \vartheta \}} \frac{a_1(y - e(\vartheta))}{|y - e(\vartheta)|^{2+n+1}} \, dy.
\]

Thanks to these observations, we rewrite (5.2) in the form

\[
(5.4) \quad \Upsilon = \int_{J_{0,\vartheta} \cap \{ x_n > 2 \sin \vartheta \}} \frac{a_1^*(x-e^*(\vartheta))}{|x-e^*(\vartheta)|^{2+n+1}} \, dx - \int_{J_{0,\vartheta} \cap \{ x_n > 2 \sin \vartheta \}} \frac{a_1^*(x-e^*(\vartheta))}{|x-e^*(\vartheta)|^{2+n+1}} \, dx.
\]

Now we use polar coordinates centered at \( e^*(\vartheta) \). For this, if \( x \in J_{0,\vartheta} \cap \{ x_n > 2 \sin \vartheta \} \), we write \( x = (\cos \vartheta, \sin \vartheta) + \rho(\cos \alpha, \sin \alpha) \) with \( \alpha \in (0, \vartheta) \) and \( \rho > \frac{\sin \vartheta}{\sin \alpha} \). Similarly, if \( x \in J_{0,\pi} \cap \{ x_n > 2 \sin \vartheta \} \), we write \( x = (\cos \vartheta, \sin \vartheta) + \rho(\cos \beta, \sin \beta) \) with \( \beta \in (0, \pi) \) and \( \rho > \frac{\sin \vartheta}{\sin \beta} \), see Figure 10.

As a result, using the notation in (1.31), we deduce from (5.4) that

\[
\Upsilon = \int_{(0,\vartheta) \times (\sin \alpha, +\infty)} \frac{\phi_1(\alpha)}{\rho^{1+s_1}} \, d\alpha \, d\rho - \int_{(\vartheta, \pi) \times (\sin \beta, +\infty)} \frac{\phi_1(\beta)}{\rho^{1+s_1}} \, d\beta \, d\rho
\]

\[
= \frac{1}{s_1(\sin \vartheta)^{s_1}} \left( \int_0^{\vartheta} \phi_1(\alpha) (\sin \alpha)^{s_1} \, d\alpha - \int_{\vartheta}^\pi \phi_1(\beta) (\sin \beta)^{s_1} \, d\beta \right),
\]

which establishes (5.1). □
Lemma 5.2. Let the notation in (1.11), (1.29), (1.30) and (1.31) hold true. Then,

\[(5.5) \quad \int_{H^n} a_2(\vec{e}(\vartheta) - \vec{x}) |\vec{e}(\vartheta) - \vec{x}|^{n+s_1} d\vec{x} = \frac{1}{s_1 (\sin \vartheta)^{s_1}} \int_{-\pi}^{0} \phi_2(\alpha) |\sin \alpha|^{s_1} d\alpha.\]

Proof. As in (5.3), we have that the left hand side of (5.5) equals to

\[\Lambda := \int_{\mathbb{R} \times (-\infty,0)} \frac{a_2^* (y - e^*(\vartheta))}{|y - e^*(\vartheta)|^{2+s_1}} dy.\]

Now we use polar coordinates centered at \(e^*(\vartheta)\) by considering \(y = (\cos \vartheta, \sin \vartheta) + \rho (\cos \alpha, \sin \alpha)\) with \(\alpha \in (-\pi,0)\) and \(\rho > \frac{|\sin \vartheta|}{|\sin \alpha|}\), see Figure 11. In this way, and recalling (1.31), it follows that

\[\Lambda = \int_{(-\pi,0) \times \left(\frac{\sin \vartheta}{|\sin \alpha|}, +\infty\right)} \frac{\phi_2(\alpha)}{\rho^{1+s_1}} d\alpha d\rho = \frac{1}{s_1 (\sin \vartheta)^{s_1}} \int_{-\pi}^{0} \phi_2(\alpha) |\sin \alpha|^{s_1} d\alpha,\]

as desired. \(\square\)

With this, we can uniquely determine the contact angle, as presented in Theorem 1.10:

Proof of Theorem 1.10. We let

\[W(\vartheta) := s_1 (\sin \vartheta)^{s_1} \left( \int_{J_{\vartheta,\pi}} \frac{a_1(\vec{e}(\vartheta) - \vec{x})}{e(\vartheta) - \vec{x}|^{n+s_1}} d\vec{x} - \int_{J_{0,\vartheta}} \frac{a_1(\vec{e}(\vartheta) - \vec{x})}{e(\vartheta) - \vec{x}|^{n+s_1}} d\vec{x} - \sigma \int_{H^n} \frac{a_2(\vec{e}(\vartheta) - \vec{x})}{e(\vartheta) - \vec{x}|^{n+s_1}} d\vec{x}\right)\]

and we observe that solutions of (1.28) correspond to zeros of \(W\) in \([0, \pi]\).
Also, by Lemmata 5.1 and 5.2, and recalling (1.32),

\[(5.6)\]

\[
W(\vartheta) = \int_{0}^{\vartheta} \phi_1(\alpha)(\sin \alpha)^{s_1} d\alpha - \int_{\vartheta}^{\pi} \phi_1(\alpha)(\sin \alpha)^{s_1} d\alpha - \int_{-\vartheta}^{0} \phi_2(\alpha) |\sin \alpha|^{s_1} d\alpha
\]

\[
= \int_{0}^{\vartheta} \phi_1(\alpha)(\sin \alpha)^{s_1} d\alpha - \int_{\vartheta}^{\pi} \phi_1(\alpha)(\sin \alpha)^{s_1} d\alpha - \int_{0}^{\vartheta} \phi_2(\alpha + \sin(\alpha + \alpha))^{s_1} d\alpha
\]

\[
= \int_{0}^{\vartheta} \phi_1(\alpha)(\sin \alpha)^{s_1} d\alpha - \int_{\vartheta}^{\pi} \phi_1(\alpha)(\sin \alpha)^{s_1} d\alpha - \int_{0}^{\vartheta} \phi_2(\alpha)(\sin \alpha)^{s_1} d\alpha.
\]

In particular, \(W\) is continuous in \([0, \pi]\), differentiable in \((0, \pi)\) and, for each \(\vartheta \in (0, \pi)\),

\[
W'(\vartheta) = 2 \phi_1(\vartheta)(\sin \vartheta)^{s_1} > 0,
\]

which shows that \(W\) admits at most one zero in \((0, \pi)\). This establishes the uniqueness result stated in Theorem 1.10.

Now we show the existence result claimed in Theorem 1.10 under assumption (1.33). To this end, it suffices to notice that, by (1.33) and (5.6), we have that

\[
W(0) = - \int_{0}^{\pi} \phi_1(\alpha)(\sin \alpha)^{s_1} d\alpha - \sigma \int_{0}^{\pi} \phi_2(\alpha)(\sin \alpha)^{s_1} d\alpha < 0
\]

and

\[
W(\pi) = \int_{0}^{\pi} \phi_1(\alpha)(\sin \alpha)^{s_1} d\alpha - \sigma \int_{0}^{\pi} \phi_2(\alpha)(\sin \alpha)^{s_1} d\alpha > 0.
\]

From this and the continuity of \(W\), we obtain the existence of a zero of \(W\) in \((0, \pi)\). \(\square\)

**Remark 5.3.** We stress that the strict positivity of the kernel is essential for the uniqueness result in Theorem 1.10: indeed, if one allows degenerate kernels in which \(a_1\) is only nonnegative, such a uniqueness claim can be violated. As an example, consider \(\sigma := 0\) and pick \(\vartheta_0 \in \left(0, \frac{\pi}{2}\right)\).

Let \(\phi \in C^{\infty}(\mathbb{R})\) be such that \(\phi_1(\alpha) := 0\) for all \(\alpha \in [\vartheta_0, \pi - \vartheta_0]\). Assume also that \(\phi_1(\frac{\pi}{2} - \alpha) = \phi_1(\frac{\pi}{2} - \alpha)\) for all \(\alpha \in (0, \frac{\pi}{2})\) and that \(\phi_1(\alpha + \pi) = \phi_1(\alpha)\) for all \(\alpha \in (0, \pi)\). See e.g. Figure 12 for a sketch of this function.

**Figure 12.** A degenerate example of \(\phi_1\) leading to a multiplicity of the contact angle in (1.26).

Then, by (5.6), for every \(\vartheta \in [\vartheta_0, \frac{\pi}{2}]\),

\[
W(\vartheta) = \int_{0}^{\vartheta} \phi_1(\alpha)(\sin \alpha)^{s_1} d\alpha - \int_{\vartheta}^{\pi} \phi_1(\alpha)(\sin \alpha)^{s_1} d\alpha
\]

\[
= \int_{0}^{\vartheta_0} \phi_1(\alpha)(\sin \alpha)^{s_1} d\alpha - \int_{\vartheta_0}^{\vartheta} \phi_1(\alpha)(\sin \alpha)^{s_1} d\alpha
\]

\[
= \int_{0}^{\vartheta_0} \phi_1(\alpha)(\sin \alpha)^{s_1} d\alpha - \int_{\vartheta_0}^{\vartheta} \phi_1(\pi - \beta)(\sin(\pi - \beta))^{s_1} d\beta
\]
Lemma A.1

Lemma. Of the kernel cannot be dropped in Theorem 1.10. Accordingly, the assumption of strict positivity which shows that in this degenerate case every angle $\bar{\vartheta} > \frac{n}{2}$ for some $C > 0$ would be a zero of $W$, hence a solution of the contact angle equation in (1.28). Accordingly, the assumption of strict positivity of the kernel cannot be dropped in Theorem 1.10.

**Appendix A. Existence of minimizers and proof of Proposition 1.1**

The proof of the existence result in Proposition 1.1 is based on a semicontinuity argument and on a direct minimization procedure. We first provide the following lower semicontinuity lemma.

**Lemma A.1 (Semicontinuity of the energy).** If $I_2(\Omega, \Omega^c) < +\infty$, $E_j \subseteq \Omega$ and $E_j \to E$ in $L^1(\Omega)$, then

$$\liminf_{j \to +\infty} \mathcal{E}(E_j) \geq \mathcal{E}(E).$$

**Proof.** If $\sigma \geq 0$, the proof follows by Fatou’s Lemma. If instead $\sigma < 0$, then we observe that

$$I_2(\Omega, \Omega^c) = I_2(E, \Omega^c) + I_2(E^c \cap \Omega, \Omega^c),$$

and therefore, using that $\sigma = -|\sigma|$, we can write

$$\mathcal{E}(E) = I_1(E, E^c \cap \Omega) - |\sigma| I_2(E, \Omega^c) + (|\sigma| + 1) I_2(\Omega, \Omega^c) - (|\sigma| + 1) I_2(\Omega, \Omega^c)$$

$$= I_1(E, E^c \cap \Omega) + I_2(E, \Omega^c) + (|\sigma| + 1) I_2(E^c \cap \Omega, \Omega^c) - (|\sigma| + 1) I_2(\Omega, \Omega^c).$$

As a consequence, we can exploit Fatou’s Lemma and obtain the desired result. $\square$

With this we are able to prove Proposition 1.1:

**Proof of Proposition 1.1.** We observe that, if $K_1 \in \mathbf{K}(n, s_1, \lambda, \varrho)$, then, for any $p \in \mathbb{R}^n$,

$$A \mathbf{1}I_1(F, F^c) \geq \frac{1}{\lambda} I_{s_1}(F \cap B_{\varrho/2}(p), F^c \cap B_{\varrho/2}(p)), \quad \text{for every } F \subseteq \mathbb{R}^n. \tag{A.1}$$

To prove it, we notice that if $x, y \in B_{\varrho/2}(p)$, then $|x - y| \leq |x - p| + |p - y| < \varrho$, and therefore, recalling (1.2),

$$I_1(F, F^c) \geq \int_{F \cap B_{\varrho/2}(p)} \int_{F^c \cap B_{\varrho/2}(p)} K_1(x - y) \, dx \, dy \geq \frac{1}{\lambda} \int_{F \cap B_{\varrho/2}(p)} \int_{F^c \cap B_{\varrho/2}(p)} \frac{dx \, dy}{|x - y|^{n + s_1}},$$

which establishes (A.1).

Now, if $H$ is a half-space such that $|H \cap \Omega| = m$ and $R > 0$ is such that $\Omega \subseteq B_R$, then, using (1.2), we see that

$$I_1(H \cap B_R, H^c \cap B_R) = CR^{n - s_1},$$

for some $C > 0$ depending only on $n$ and $s_1$, and therefore

$$\mathcal{E}(H \cap \Omega) = I_1(H \cap \Omega, (H \cap \Omega)^c \cap \Omega) + |\sigma| I_2(H \cap \Omega, \Omega^c)$$

$$= I_1(H \cap \Omega, H^c \cap \Omega) + |\sigma| I_2(\Omega, \Omega^c)$$

$$\leq I_1(H \cap B_R, H^c \cap B_R) + |\sigma| I_2(\Omega, \Omega^c)$$

$$< +\infty.$$

As a consequence, we find that

$$\gamma := \inf \{ \mathcal{E}(E) : E \subseteq \Omega, |E| = m \} < +\infty.$$
Let now $E_j \subseteq \Omega$ be such that $|E_j| = m$ and $\mathcal{E}(E_j) = \mathcal{E}(E_j) + \int_{E_j} g \to \gamma$ as $j \to +\infty$. Then, if $j$ is large enough, we have that

$$\gamma + 1 + \int_{\Omega} |g| \geq \mathcal{E}(E_j) = I_1(E_j, E_j^c \cap \Omega) + \sigma I_2(E_j, \Omega^c) \geq I_1(E_j, E_j^c \cap \Omega) - |\sigma| I_2(\Omega, \Omega^c).$$

Consequently

$$I_1(E_j, E_j^c) = I_1(E_j, E_j^c \cap \Omega) + I_1(E_j, E_j^c \cap \Omega^c) \leq \gamma + 1 + \int_{\Omega} |g| + I_1(\Omega, \Omega^c) + |\sigma| I_2(\Omega, \Omega^c).$$

Since $E_j \subseteq B_R$, using (A.1) and the fact that the space $W^{1,2}(B_R)$ is compactly embedded in $L^1(B_R)$, we find that, up to a subsequence, $E_j \to E$ in $L^1_{\text{loc}}(\mathbb{R}^n)$ for some $E \subseteq \Omega$ with $|E| = m$. Hence, using the semicontinuity property in Lemma A.1, we conclude that $E$ is a minimizer.

We also remark that

$$I_2(E, \Omega^c) \leq I_2(\Omega, \Omega^c) < +\infty,$$

and therefore, since $\mathcal{E}(E) < +\infty$, we conclude that

$$I_1(E, E^c \cap \Omega) < +\infty,$$

as desired. \hfill \square

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