SECOND ORDER MCKEAN-VLASOV SDES AND KINETIC FOKKER-PLANCK-KOLMOGOROV EQUATIONS

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Abstract. In this paper we study second order stochastic differential equations with measurable and density-distribution dependent coefficients. Through establishing a maximum principle for kinetic Fokker-Planck-Kolmogorov equations with distribution-valued inhomogeneous term, we show the existence of weak solutions under mild assumptions. Moreover, by using the Hölder regularity estimate obtained recently in [16], we also show the well-posedness of generalized martingale problems when diffusion coefficients only depend on the position variable (not necessarily continuous). Even in the non density-distribution dependent case, it seems that this is the first result about the well-posedness of SDEs with measurable diffusion coefficients.

Keywords: Maximum principle, De-Giorgi’s iteration, Stochastic differential equation, Krylov’s estimate, kinetic Fokker-Planck-Kolmogorov equation.

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1. Introduction

In this paper we are concerned with the following second order stochastic differential equation with density-distribution dependent coefficients (also called McKean-Vlasov SDE or simply DDSDE):

\[
\begin{align*}
    d\ddot{X}_t &= b_Z(t, Z_t)dt + \sigma_Z(t, X_t)dW_t, \\
    \dot{X}_t &= b_Z(t, Z_t)dt + \sigma_Z(t, X_t)dW_t,
\end{align*}
\]

(1.1)

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where \( Z_t := (X_t, \dot{X}_t) \) and \( \dot{X}_t := V_t \) stands for the velocity, \((W_t)_{t \geq 0}\) is a \( d \)-dimensional standard Brownian motion defined on some stochastic basis \((\Omega, \mathcal{F}, P; (\mathcal{F}_t)_{t \geq 0})\), and
\[
 b_Z(t, z) := \int_{\mathbb{R}^{2d}} b(t, z, \rho Z_t(z), z') \mu_{Z_t}(dz'),
\]
and
\[
 \sigma_Z(t, x) := \sqrt{2} a_Z(t, x), \quad a_Z(t, x) := \int_{\mathbb{R}^d} a(t, x, \rho X_t(x), z') \mu_{Z_t}(dz').
\]
Here \( \mu_{Z_t}(dz) = P \circ Z_t^{-1}(dz) = \rho Z_t(z)dz, P \circ X_t^{-1}(dx) = \rho X_t(x)dx \), and
\[
 b(t, z, r, z') : \mathbb{R}_+ \times \mathbb{R}^{2d} \times \mathbb{R}_+ \times \mathbb{R}^{2d} \to \mathbb{R}^d
\]
are Borel measurable functions, \( \mathcal{M}_{sym}^d \) is the set of all symmetric \( d \times d \)-matrices. Throughout the paper, the density dependence means that \( b_Z, a_Z \) point-wisely depend on the density of \( Z \), while the distribution dependence means that \( b_Z, a_Z \) non-locally depend on the distribution of \( Z \) in the whole space. Below we suppose that \( \alpha \) satisfies that for some \( 0 < \kappa_0 < \kappa_1 \) and all \( t, x, r, z' \),
\[
 \kappa_0 |\xi|^2 \leq \xi \cdot \alpha(t, x, r, z') \xi \leq \kappa_1 |\xi|^2, \quad \xi \in \mathbb{R}^d,
\]
where the dot stands for the inner product of two vectors. Since the coefficients depend on the density and distribution of the solution itself, equation (1.1) is usually regarded as a strong nonlinear SDE. Suppose that DDSDE (1.1) has a solution, whose meaning is given in Definition 1.1 below. By Itō’s formula, it is easy to see that the density \( \rho(t, z) := \rho Z_t(z) \) solves the following nonlinear kinetic Fokker-Planck-Kolmogorov equation (abbreviated as FPKE) in the distributional sense:
\[
 \partial_t \rho = \partial_v \cdot \partial_{ij}(\bar{a}_{ij}(\rho)\rho) - v \cdot \nabla_x \rho + \text{div}_x(\bar{b}(\rho)\rho),
\]
where for \((t, x) \in \mathbb{R}_+ \times \mathbb{R}^d\),
\[
 \bar{a}(\rho; t, x) := \int_{\mathbb{R}^{2d}} a(t, x, \langle \rho \rangle(t, x), z') \rho(t, z')(dz'),
\]
and for \((t, z) \in \mathbb{R}_+ \times \mathbb{R}^{2d}\),
\[
 \bar{b}(\rho; t, z) := \int_{\mathbb{R}^{2d}} b(t, z, \rho(t, z), z') \rho(t, z')(dz').
\]
Here and below, \( \langle \rho \rangle(t, x) := \int_{\mathbb{R}^d} \rho(t, x, v)dv \) stands for the mass density, and we use the convention that repeated indices are summed automatically.

An important prototype of PDE (1.5) is the following Landau equation or its variant form (see [21, 22, 1, 30]):
\[
 \partial_t \rho + v \cdot \nabla \rho = \partial_v \left\{ \int_{\mathbb{R}^d} a_{ij}(v - v') \left[ \rho(v') \partial_{v_j} \rho(v) - \rho(v) \partial_{v_j} \rho(v') \right] dv' \right\},
\]
where \( \rho = \rho(t, x, v) \) and for \( v = (v_1, \cdots, v_d) \),
\[
 a_{ij}(v) := c_{d, \gamma} |\delta_{ij} - v_i v_j / |v|^2| |v|^\gamma + d, \quad \gamma \in [-d, +\infty), \quad c_{d, \gamma} > 0,
\]
which models the behavior of a dilute plasma interacting through binary collisions (see [28]). While nonlinear SDE (1.1) with distribution dependent coefficients was firstly proposed by McKean [24] to give a probabilistic explanation for nonlinear Vlasov equations. Nowadays, McKean-Vlasov SDEs have been widely used in the study of interacting particle systems and mean-field games (cf. [32, 6] and references therein). Recently, there are many works to study the well-posedness of first order nondegenerate DDSDEs with rough coefficients. When \( b \) and \( a \) does not depend on the density variable \( r \) and of at most linear growth, by using the classical Krylov estimates, Mishura and Veretennikov [25] showed the existence of weak solutions for first order DDSDEs.
The uniqueness is also proved when \( a \) does not depend on \( z' \) and is Lipschitz continuous in \( x \). Later, in [27] their results were extended to the case that drift \( b \) is in \( L^p \)-spaces. More results and references about McKean-Vlasov SDEs are summarized in the paper of [14] (see also the references in [27]). For nonlinear SDEs with density dependent coefficients (also called McKean-Vlasov SDEs of Nemytskii-type), it was firstly studied in [3] to give a probabilistic representation for the solution of nonlinear FPKEs. In [4], for a large class of time independent coefficients \( b, a \), Barbu and Röckner obtained the existence of weak solutions for first order density dependent SDEs. The strategy in [3] and [4] is to solve the associated nonlinear FPKE and then by the well-known superposition principle to establish the existence of a weak solution to the first order DDSDE. In [11], Hao, Röckner and the present author give a purely probabilistic proof for the existence of weak solutions for the first order density dependent SDE by Euler’s scheme. Our current degenerate kinetic settings are more general and is not studied in the literature yet.

As usual, to study DDSDE (1.1), we must have better understanding for the associated kinetic FPKEs. In particular, we need to develop the hypoelliptic regularity estimates for kinetic FPKEs with rough coefficients. In [26], Pascucci and Polidoro obtained the upper bound estimate of weak solutions for a class of ultraparabolic equations of divergence form via Moser’s method, especially, including the following kinetic equation

\[
\partial_t u = \text{div}_v(a \cdot \nabla_v u) + v \cdot \nabla_x u.
\]  

(1.7)

In [34, 35], Wang and Zhang showed the Hölder regularity of weak solutions to the above equation. In [16], Golse, Imbert, Mouhot and Vasseur showed the Harnack inequality for (1.7) with bounded first order and inhomogeneous terms. As an application, they also showed the conditional Hölder regularity for the Landau equation (1.6). More recently, weak Harnack inequality and some quantitative estimates for kinetic FPKEs were obtained in [17, 18]. More historical backgrounds about Harnack inequalities and kinetic equations are referred to [17, Section 1.2].

1.1. Main results. Let \( \mathcal{P}(\mathbb{R}^{2d}) \) be the space of all probability measures over \( \mathbb{R}^{2d} \). We first introduce the following notion of weak solutions to DDSDE (1.1).

**Definition 1.1.** Let \( \nu \in \mathcal{P}(\mathbb{R}^{2d}) \) and \( \mathfrak{F} := (\Omega, \mathcal{F}, \mathbb{P}; (\mathcal{F}_t)_{t \geq 0}) \) be a stochastic basis, \( \mathcal{Z} = (\mathcal{Z}_t, \mathcal{V}_t) \) and \( \mathcal{W}_t \) be \( \mathbb{R}^{2d} \) and \( \mathbb{R}^d \)-valued \( \mathcal{F}_t \)-adapted processes, respectively. We call \((\mathfrak{F}, \mathcal{Z}, \mathcal{W})\) a weak solution of DDSDE (1.1) with initial distribution \( \nu \) if

(i) \( \mathbb{P} \circ \mathcal{Z}_0^{-1} = \nu \), and for Lebesgue almost all \( t \geq 0 \),

\[
\mathbb{P} \circ \mathcal{Z}_t^{-1}(dz) = \rho_{\mathcal{Z}_t}(z)dz, \quad \mathbb{P} \circ \mathcal{X}_t^{-1}(dx) = \rho_{\mathcal{X}_t}(x)dx.
\]

(ii) \( \mathcal{W} \) is a standard \( d \)-dimensional \( \mathcal{F}_t \)-Brownian motion.

(iii) For all \( t \geq 0 \), it holds that \( \mathcal{X}_t = \mathcal{X}_0 + \int_0^t \mathcal{V}_s ds \) and

\[
\mathcal{V}_t = \mathcal{V}_0 + \int_0^t b_2(s, Z_s)ds + \int_0^t \sigma_Z(s, X_s) d\mathcal{W}_s, \quad \mathbb{P} - a.s.,
\]

where \( b_2 \) and \( \sigma_Z \) are defined by (1.2) and (1.3), respectively.

Before stating our main results, we make the following assumptions about \( a \) and \( b \):

**\((H_1)\)** For any \( m \in \mathbb{N} \) and bounded domain \( Q \subset \mathbb{R}_+ \times \mathbb{R}^{2d} \), it holds that

\[
\lim_{h \to 0} \left\| \sup_{r, r' \leq m, |r - r'| \leq h} |b(\cdot, \cdot, r, \cdot) - b(\cdot, \cdot, r', \cdot)| \right\|_{L^1(Q)} = 0,
\]

(1.8)

and for all \((t, z, r, z') \in \mathbb{R}_+ \times \mathbb{R}^{2d} \times \mathbb{R}_+ \times \mathbb{R}^{2d} \),

\[
|b(t, z, r, z')| \leq h(t, z - z') \quad \text{with} \quad \|h\|_{L^{q_1}(\mathbb{R}^d)} \leq \kappa_2,
\]

(1.9)

where \( q_1 \in (2, 4) \) and \( p_1 \in (2, \infty)^{2d} \) satisfy \( a \cdot \frac{1}{p_1} + \frac{2}{q_1} < 1 \), the localized norm \( \| \|_{L^{q_1}(\mathbb{R}^d)} \) is defined by (5.2) below. Here \( a = (3, \ldots, 3, 1, \ldots, 1) \in \mathbb{R}^{2d} \) (see (3.12) below).
(H₂) In addition to ellipticity assumption (1.4), we assume that for each \((t, x, z) \in \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^{2d}, [0, \infty) \ni r \mapsto a(t, x, r, z) \in \mathbb{M}^d_{\text{sym}}\) is continuous.

**Remark 1.2.** The continuity of \([0, \infty) \ni r \mapsto a(t, x, r, z) \in \mathbb{M}^d_{\text{sym}}\) together with the boundedness of \(a\) implies that for any bounded \(Q \subset \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^{2d},\)

\[
\lim_{h \to 0} \sup_{r, r' \in \Xi, \|r-r'| \leq h} |a(\cdot, \cdot, r, \cdot) - a(\cdot, \cdot, r', \cdot)|_{L^1(Q)} = 0. \tag{1.10}
\]

Indeed, it follows by the dominated convergence theorem and for each \((t, x, z'),\)

\[
\lim_{h \to 0} \sup_{r, r' \in \Xi, \|r-r'| \leq h} |a(t, x, r, z') - a(t, x, r', z')| = 0.
\]

First of all, we have the following existence result of weak solutions.

**Theorem 1.3.** Under (H₁) and (H₂), for any \(\nu \in \mathcal{P}(\mathbb{R}^{2d})\), there exists at least one weak solution to SDE (1.1) with initial distribution \(\nu\) in the sense of Definition 1.1. In particular, the probability distributional density \(\rho_\tau\) of \(Z_t\) solves (1.5) with initial value being the probability measure \(\nu\). Moreover, \(\rho\) enjoys the following regularity: for any \(\alpha \in (0, 1)\) and \((q, p) \in (1, \infty)^{1+2d}\) satisfying

\[
\frac{2}{q} < 1 + \alpha, \quad \frac{2}{q} + a \cdot \left(\frac{1}{p} - 1\right) > 2\alpha,
\]

it holds that

\[
\|\rho 1_{[0, T]}\|_{L^q(\mathbb{B}_{p,a}^\alpha)} < \infty, \quad T > 0, \tag{1.11}
\]

where \(\mathbb{B}_{p,a}^\alpha\) is the anisotropic Besov space in Definition 2.1.

**Remark 1.4.** (i) Suppose that \(b\) and \(a\) do not depend on the density variable \(r\), then we can drop the assumption \(q_1 \in (2, 4)\) in (H₁) but require \(\frac{1}{p_1'} + \frac{1}{q_1'} < \frac{1}{2}\) (see Remark 6.12 below).

(ii) We explain the use of multi-integrability index \(p \in (2, \infty)^d\) (see (2.1) below). Consider the following second order SDE in \(\mathbb{R}^2;\)

\[
\begin{cases}
\dot{X}_1^1 = b_1(X_1^1 - X_2^1)dt + dW_1^1, \\
\dot{X}_1^2 = b_2(X_1^1 - X_2^1)dt + dW_2^1,
\end{cases}
\]

where \(b_1, b_2 : \mathbb{R} \to \mathbb{R}\) are two measurable functions. If we write

\[
X_t := (X_t^1, X_t^2), \quad Z_t := (X_t, \dot{X}_t), \quad W_t := (W_t^1, W_t^2)
\]

and

\[
b(x_1, x_2, v_1, v_2) := (b_1, b_2)(x_1 - x_2) : \mathbb{R}^4 \to \mathbb{R}^2,
\]

then the above SDE can be written in the form of (1.1). Suppose now that \(b_1, b_2 \in \overline{L}^p\) for some \(p > 3\). Then by the definition (5.1) below, it is easy to see that (1.9) holds for \(p = (p, p', p', p')\), where \(p'\) is chosen large enough so that \(a \cdot \frac{1}{p} = \frac{2}{q} + \frac{2}{q'} + \frac{2}{q} < 1\). However, if we do not use the above multi-index \(p\), then we have to require \(p > 8\). To the author’s knowledge, Ling and Xie [23] firstly used the mixed-\(L^p\) norm for studying singular first order SDEs. We emphasize that such a feature naturally appears in the study of interacting particle systems with singular interaction force, especially, propagation of chaos (cf. [32]). We shall study this in a future work.

Although we have shown the existence of weak solutions under very mild assumptions in the above theorem, the uniqueness is a more subtle problem. In fact, in the non-distribution dependent case, that is, \(a\) and \(b\) does not depend on \((r, z')\), when \(a\) is uniformly continuous and \(b\) is Hölder continuous or in some \(L^p\)-spaces, the uniqueness of weak solutions or strong solutions for second order SDEs was obtained in [7, 36, 37, 8] via Zvonkin’s transformation. Recently, when \(a\) is the identity matrix and \(b = b_1 + b_2\), where \(b_1(t, z)\) is a distribution of \(z\) and \(b_2(t, z, z')\) is bounded measurable, the well-posedness of generalized martingale problem associated with SDE (1.1) was obtained in [13], where the generalized martingale problem is taken in the sense of Either and Kurtz.
It should be noted that the notion of generalized martingale solutions strongly depends on the solvability of the associated backward Kolmogorov equations (see Definition 7.1 below for a precise description). To show the uniqueness, we now suppose that (H₃) \( a = a(t, x) \) is independent of \((r, z')\) and satisfies (1.4), and \( b \) is bounded measurable and satisfies that for some \( C > 0 \) and all \((t, z, z') \in \mathbb{R}_+ \times \mathbb{R}^{2d} \times \mathbb{R}^{2d} \) and \( r, r' \geq 0 \),

\[
|b(t, r, z, z') - b(t, r', z, z')| \leq C|r - r'|.
\]

We have the following well-posedness result of generalized martingale problems.

**Theorem 1.5.** Under (H₃), for any \( s \geq 0 \) and \( \nu(dz) = \rho_0(z)dz \), where \( \rho_0 \in C^b_0(\mathbb{R}^{2d}) \), there is a unique generalized martingale solution \( \mathbb{P} \in \hat{\mathcal{M}}_{a,b}^{\nu} \) with initial distribution \( \nu \) at time \( s \) in the sense of Definition 7.1 below. Moreover, its density \( \rho_t(z) \) enjoys the regularity (1.11) and solves the following nonlinear kinetic FPK equation in the distributional sense:

\[
\partial_t \rho_t = \partial_{v_i} \partial_{x_j} (a_{ij} \rho_t) - v \cdot \nabla_x \rho_t + \text{div}_v (b \rho_t), \quad t \geq s,
\]

where \( b(t, z) := \int_{\mathbb{R}^{2d}} b(t, z, \rho_t(z), z') \rho_t(z')dz' \).

**Remark 1.6.** (i) If \( b \) does not depend on the density variable \( r \), then we can drop the regularity assumption on the initial distribution \( \nu \) (see Remark 7.3 below). In particular, suppose that \( a = a(x) \) satisfies (1.4) and \( b = b(z) \) is bounded measurable, then for each starting point \( z \in \mathbb{R}^{2d} \), the following second order SDE admits a unique generalized martingale solution \( \mathbb{P}_z \in \hat{\mathcal{M}}_{a,b}^{\nu} \) in the sense of Definition 7.1:

\[
dX_t = b(Z_t)dt + \sqrt{a}(X_t)dW_t, \quad Z_0 = z.
\]

Moreover, let \( b_z \) and \( a_z \) be the smooth approximation of \( b \) and \( a \). Consider the following SDE:

\[
dX^*_t = b_z(Z^*_t)dt + \sqrt{a_z}(X^*_t)dW_t, \quad Z^*_0 = z.
\]

By the uniqueness, for each \( z \in \mathbb{R}^{2d} \), the law of \( Z^*_t \) weakly converges to \( \mathbb{P}_z \), not necessarily subtracting a subsequence. In particular, the family of probability measures \( (\mathbb{P}_z)_{z \in \mathbb{R}^{2d}} \) on the space of continuous functions forms a family of strong Markov processes (see [9, p.184, Theorem 4.4.2]). In fact, from the proof below, one can see that for each \( z \in \mathbb{R}^{2d} \), \( \mathbb{P}_z \) is also a classical martingale solution in the sense of Definition 6.1. However, it is not known whether any classical martingale solution priorily is a generalized martingale solution. This seems to be an open problem.

(ii) Consider the following first order SDE in \( \mathbb{R}^d \),

\[
dX_t = b(X_t)dt + \sqrt{a}(X_t)dW_t, \quad X_0 = x.
\]

When \( b \) is bounded measurable and \( a \) satisfies the ellipticity condition (1.4), it is well known that there is at least one weak solution to the above SDE (see [19] or [31, Ex. 7.3.2]). Moreover, for \( d = 1, 2 \), the weak uniqueness also holds (see [31, Ex. 7.3.3, 7.3.4]). However, for \( d \geq 3 \), to get the weak uniqueness, one usually needs to assume \( a \) being continuous (see [31, Theorem 7.2.1]). For the strong uniqueness, the well-known best condition for \( a \) seems to be \( a \in W^{2,p} \), the second order Sobolev space, where \( p > d \) (see [38]). It is interesting that for second order SDEs with discontinuous diffusion coefficients, we have the well-posedness of generalized martingale problems.

The crucial point in our proof is to establish a maximum principle for the following kinetic FPK:

\[
\partial_t u = \text{tr}(a \cdot \nabla^2 u) + v \cdot \nabla_x u + b_1 \cdot \nabla_v u + \text{div}_v(b_2u) + f, \quad u(0) = \varphi,
\]

where an important observation is that since the diffusion-matrix \( a \) does not depend on the velocity variable \( v \), the principal second order term can be written in divergence form:

\[
\text{tr}(a \cdot \nabla^2 u) = \text{div}_v(a \cdot \nabla_v u).
\]

Thus one can invoke the classical De Giorgi method to show the apriori \( L^\infty \)-estimates. Compared with the recent works [16, 17, 18], the novelty of our boundedness estimates is that \( b \) can be in some \( L_{t,x}^{q_1}(L_{t,v}^{p_1}) \)-space, where \( (q_1, p_1) \in (2, \infty)^{1+2d} \) satisfies \( a \cdot \frac{1}{p_1} + \frac{2}{q_1} < 1 \), and \( f \) is in some
\( L^q(\mathbb{L}^p_z) \)-space, where \((q_0, p_0) \in (1, \infty)^{1+2d} \) satisfies \( a \cdot \frac{1}{p_0} + \frac{2}{q_0} < 2 \). More importantly, \( f \) is also allowed to be a distribution in some Besov spaces with negative differentiability index, which plays a crucial role to show the existence of weak solutions in Theorem 1.3. That is, the distribution valued \( f \) provides us the regularity of the density so that one can show the strong convergence of the densities of approximating equations (see Lemma 6.9 below).

1.2. Plan of the paper. This paper is organized as follows. In Section 2, we prepare some preliminary results for later use. More concretely, in Subsection 2.1, we introduce necessary anisotropic Besov spaces and some basic properties related to the Besov spaces. It is noted that our \( L^p \)-spaces have different integrability indices along different components as explained in Remark 1.4 above. In Subsection 2.2 we provide an abstract criterion for the local bound of a function in De-Giorgi’s class. Compared with the classical notion of De-Giorgi’s class [20], our definition does not depend on any structure of PDEs and is only in the scope of \( L^p \)-spaces.

In Section 3, we recall some well-known properties about weak (sub)-solutions. In Subsection 3.2, we show how to improve the regularity in Besov spaces via Duhamel’s formula for kinetic equations, which is essentially proved in [12, 39]. Moreover, we also show the existence and uniqueness of weak solutions for the Cauchy problem of a linear kinetic FPKE (see Theorem 3.9), which seems to be new.

Section 4 is devoted to showing the local boundedness of weak solutions via De-Giorgi’s method. Two cases are considered. In the first case that the nonhomogeneous \( f \) is an \( L^p \)-function, we use the comparison method as in [16] to improve the regularity of weak sub-solutions. In this case, we have best integrability conditions on drift \( b \) and inhomogeneous \( f \) as stated above. In the second case that the nonhomogeneous \( f \) is allowed to be a distribution, we use the simple fact that for a weak solution \( u \), \((u^+)^2\) is still a weak solution with \( 2fu^+ - 2\langle a \cdot \nabla u^+, \nabla u^+ \rangle \) in place of \( f \) (see Lemma 3.5 below). In this case, the \( L^\infty \)-estimate of weak solutions in Theorem 4.5 shall provide extra Besov regularity for the density of second order SDEs so that we can treat the density dependent SDEs (see Corollary 6.7).

In Section 5, we show the well-posedness and stability of weak solutions for PDE (3.1). In particular, we obtain the global bound estimate and stability of weak solutions, where the key point is to use the localization norm \( \| \cdot \|_{L^q(\mathbb{L}^p_z)} \) introduced in [39]. For the stability, we use the H"older regularity estimate established in [16] (see also [34, 17, 18]).

In Section 6, we prove the existence of weak solutions. As usual, in terms of the equivalence between weak solutions and martingale problems, we show the existence of classical martingale solutions associated with DDSDE (1.1) via mollifying the coefficients. Since our diffusion coefficient does not depend on velocity variable, the Kolmogorov equation associated SDEs (1.1) can be written in the divergence form as explained in (1.14). Thus one can utilize the maximum principle obtained in Section 5 to show the crucial Krylov estimate for approximating equation. Then by Aubin-Lions’ lemma, one can find a strong convergence subsequence of the densities.

In Section 7, we show the well-posedness of generalized martingale problems. As said above, such a notion strongly depends on the solvability of the associated Kolmogorov equations. When the solution of the associated Kolmogorov equations has enough regularity, saying \( C^2 \)-smooth, the classical martingale solution must be a generalized martingale solution. In general, these two notions are not comparable. The advantage of using the notion of generalized martingale solutions is that the uniqueness is an easy consequence of the unique solvability of the associated Kolmogorov equation. The disadvantage is that it has few flexibility and Itô’s formula is not applicable so that it is not direct to see that the density of the solution solves the Fokker-Planck-Kolmogorov equation.

We conclude this introduction by introducing the following convention: Throughout this paper, we use \( C \) with or without subscripts to denote an unimportant constant, whose value may change.
in different occasions. We also use := as a way of definition and \( a^+ := 0 \vee a \). By \( A \lesssim_C B \) and \( A \asymp_C B \) or simply \( A \lesssim B \) and \( A \asymp B \), we mean that for some unimportant constant \( C \geq 1 \),
\[
A \lesssim CB, \quad C^{-1}B \lesssim A \lesssim CB.
\]

2. Preliminary

2.1. Anisotropic Besov spaces. Fix \( N \in \mathbb{N} \). For multi-index \( p = (p_1, \cdots, p_N) \in (0, \infty)^N \), we define
\[
\| f \|_{L^p} := \left( \int_{\mathbb{R}} \left( \int_{\mathbb{R}} \cdots \left( \int_{\mathbb{R}} |f(z_1, \cdots, z_N)|^{p_N} dz_N \right)^{\frac{p_{N-1}}{p_N}} \cdots dz_2 \right)^{\frac{1}{p_1}} \right)^{\frac{1}{p}}.
\]
(2.1)

When \( p = (p, \cdots, p) \in (0, \infty]^N \), we shall simply write
\[
L^p = 1^p.
\]

Note that if \( p \in [1, \infty]^N \), then \( \| f \|_{L^p} \) satisfies the triangle inequality and is a norm, and for any permutation \( p' \) of \( p \),
\[
\| f \|_{L^p} \neq \| f \|_{L^{p'}}.
\]

For multi-indices \( p, q \in (0, \infty]^N \), we shall use the following notations:
\[
\frac{1}{p} := \left( \frac{1}{p_1}, \cdots, \frac{1}{p_N} \right), \quad p \cdot q := \sum_{i=1}^{N} p_i q_i,
\]
and
\[
p > q \ (p \geq q; \ p = q) \iff p_i > q_i \ (p_i \geq q_i; \ p_i = q_i), \quad i = 1, \cdots, N.
\]
Moreover, we use bold number to denote constant vector in \( \mathbb{R}^N \):
\[
1 = (1, \cdots, 1), \quad 2 = (2, \cdots, 2).
\]

With a little confusion, for a set \( Q \subset \mathbb{R}^N \), we also use \( 1_Q \) to denote the indicator function of \( Q \).

For any multi-indices \( p, q, r \in (0, \infty)^N \) with \( \frac{1}{p} + \frac{1}{r} = \frac{1}{q} \), the following H"older’s inequality holds
\[
\| fg \|_{L^q} \leq \| f \|_{L^p} \| g \|_{L^r}.
\]

For any multi-indices \( p, q, r \in [1, \infty)^N \) with \( \frac{1}{p} + \frac{1}{r} = 1 + \frac{1}{q} \), the following Young’s inequality holds
\[
\| f * g \|_{L^q} \leq \| f \|_{L^p} \| g \|_{L^r}.
\]

These two inequalities are easy consequences of the well-known H"older and Young’s inequalities for \( p = (p, \cdots, p) \).

Let \( a = (a_1, \cdots, a_N) \in [1, \infty)^N \) be a multi-scaling parameter. For \( z, z' \in \mathbb{R}^N \), we introduce the following anisotropic distance in \( \mathbb{R}^N \)
\[
|z - z'|_a := \sum_{i=1}^{N} |z_i - z'_i|^{1/a_i}.
\]

For \( r > 0 \) and \( z \in \mathbb{R}^N \), we also introduce the ball with respect to the above distance
\[
B^a_r(z) := \{ z' \in \mathbb{R}^N : |z' - z|_a \leq r \}, \quad B^a_r := B^a_r(0).
\]

Let \( \phi_0^a \) be a symmetric \( C^\infty \)-function on \( \mathbb{R}^N \) with
\[
\phi_0^a(\xi) = 1 \text{ for } \xi \in B_1^a \text{ and } \phi_0^a(\xi) = 0 \text{ for } \xi \notin B_2^a.
\]

For \( j \in \mathbb{N} \), we define
\[
\phi_j^a(\xi) := \phi_0^a(2^{-ja}\xi) - \phi_0^a(2^{-(j-1)a}\xi),
\]
where for \( s \in \mathbb{R} \) and \( \xi = (\xi_1, \cdots, \xi_N) \),
\[
2^{sa}\xi = (2^{sa_1}\xi_1, \cdots, 2^{sa_N}\xi_N).
\]
For an $L^1$-integrable function $f$, let $\hat{f}$ be the Fourier transform of $f$ defined by
$$\hat{f}(\xi) := (2\pi)^{-N/2} \int_{\mathbb{R}^N} e^{-i\xi z} f(z) dz, \quad \xi \in \mathbb{R}^N,$$
and $\check{f}$ the Fourier inverse transform of $f$ defined by
$$\check{f}(z) := (2\pi)^{-N/2} \int_{\mathbb{R}^N} e^{i\xi z} f(\xi) d\xi, \quad z \in \mathbb{R}^N.$$

Let $\mathcal{S}$ be the space of all Schwartz functions on $\mathbb{R}^N$ and $\mathcal{S}'$ the dual space of $\mathcal{S}$, called the tempered distribution space. For given $j \in \mathbb{N}_0$, the block operator $\mathcal{R}^a_j$ is defined on $\mathcal{S}'$ by
$$\mathcal{R}^a_j f(z) := (\hat{\phi}^a_j \hat{f})(z) = \hat{\phi}^a_j \ast f(z), \quad (2.2)$$
where the convolution is understood in the distributional sense and by scaling,
$$\hat{\phi}^a_j(z) = 2^{(j-1)|a|} \hat{\phi}_1^a(2^{(j-1)|a|} z). \quad (2.3)$$

Here and after, we denote
$$|a| := a_1 + \cdots + a_N = a \cdot 1.$$

For $j \geq 0$, by definition it is easy to see that
$$\mathcal{R}^a_j = \mathcal{R}^a_{j-1} \mathcal{R}^a_j, \quad \text{where } \mathcal{R}^a_{j-1} := \mathcal{R}^a_{j-1} + \mathcal{R}^a_j + \mathcal{R}^a_{j+1} \text{ with } \mathcal{R}^a_{-1} = 0, \quad (2.4)$$
and by the symmetry of $\phi^a_j$,
$$\langle \mathcal{R}^a_j f, g \rangle = \langle f, \mathcal{R}^a_j g \rangle, \quad f \in \mathcal{S}', \ g \in \mathcal{S}.$$

The cut-off low frequency operator $S_k$ is defined by
$$S_k f := \sum_{j=0}^{k-1} \mathcal{R}^a_j f \rightarrow f, \quad k \rightarrow \infty. \quad (2.5)$$

For $f, g \in \mathcal{S}'$, define
$$f \prec g := \sum_{j=0}^{\infty} S_{k-1} f \mathcal{R}^a_k g, \quad f \circ g := \sum_{|i-j| = 1} \mathcal{R}^a_i f \mathcal{R}^a_j g. \quad (2.6)$$

The Bony decomposition of $fg$ is formally given by (cf. [2, 5])
$$fg = f \prec g + f \circ g + g \prec f. \quad (2.7)$$

Now we introduce the following anisotropic Besov spaces (cf. [33, Chapter 5]).

**Definition 2.1.** (Anisotropic Besov space) For $s \in \mathbb{R}$ and $p \in [1, \infty]^N$, we define
$$\mathbf{B}^s_{p,a} := \left\{ f \in \mathcal{S}': \|f\|_{\mathbf{B}^s_{p,a}} := \sup_{j \geq 0} \left( 2^{sj}\|\mathcal{R}^a_j f\|_{L^p} \right) < \infty \right\}.$$

**Remark 2.2.** For $s \in (0, 1)$ and $p \in [1, \infty]^N$, it is well known that $\mathbf{B}^s_{p,a}$ has the following characterization (see [33] or [13, Theorem 2.7]):
$$\|f\|_{\mathbf{B}^s_{p,a}} \asymp \|f\|_{L^p} + \sup_h \|f(\cdot + h) - f(\cdot)\|_{L^p}/|h|_a^s. \quad (2.8)$$

Although the proof in [13] is for $p = (p, \cdots, p)$, it also works for general $p \in [1, \infty]^N$.

The following inequality of Bernstein’s type is quite useful (cf. [2]).

**Lemma 2.3.** Let $p, q \in [1, \infty]^N$ with $p \leq q$. For any $k \in \mathbb{N}_0$ and $i = 1, \cdots, N$, there is a constant $C = C(p, q, a, k, i) > 0$ such that for all $j \geq 0$,
$$\|\partial^k_{z_i} \mathcal{R}^a_j f\|_{L^q} \lesssim_C 2^{j(a_i + k + \alpha \left(1 - \frac{1}{p} \right))} \|\mathcal{R}^a_j f\|_{L^p}, \quad (2.9)$$
where $\partial^k_{z_i}$ denotes the $k$-order partial derivative with respect to $z_i$. Moreover,
$$\|\mathcal{R}^a_j f\|_{L^p} \lesssim_C \|f\|_{L^p}, \quad j \geq 0. \quad (2.10)$$
Proof. We only prove (2.9) for \( \nu \geq 1 \). Let \( r \in [1, \infty]^N \) be defined by
\[
\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{\theta}.
\]
By (2.4), (2.2) and Young’s inequality, we have
\[
\|\partial_z^k R^a_{x,y} f\|_{L^q} = \|\partial_z^k \tilde{R}^a_{x,y} R^a_{x,y} f\|_{L^q} = \| (\partial_z^k \tilde{\phi}^a_{x,y}) \ast R^a_{x,y} f\|_{L^q} \lesssim \|\partial_z^k \tilde{\phi}^a_{x,y}\|_{L^r} \|R^a_{x,y} f\|_{L^p},
\]
where
\[
\tilde{\phi}^a_{x,y} := \phi^a_{x,y-1} + \phi^a_{x,y} + \phi^a_{x,y+1}.
\]
By (2.3) and the change of variables, we have
\[
\|\partial_z^k \tilde{\phi}^a_{x,y}\|_{L^r} = 2^{(j-1)a} \|R^a_{x,y} f\|_{L^p} \|\partial_z^k \tilde{\phi}^a_{x,y}\|_{L^r}.
\]
Thus we get (2.9). \( \square \)

We have the following interpolation inequality of Gagliardo-Nirenberg’s type.

**Lemma 2.4.** Let \( p, q, r \in [1, \infty]^N \) and \( s, s_0, s_1 \in \mathbb{R}, \) \( \theta \in [0, 1] \). Suppose that
\[
\frac{1}{p} \leq \frac{1-\theta}{q} + \frac{\theta}{r}, \quad s - a \cdot \frac{1}{p} = (1-\theta)(s_0 - a \cdot \frac{1}{q}) + \theta(s_1 - a \cdot \frac{1}{r}).
\]
Then there is a constant \( C = C(p, q, r, s, s_0, s_1, \theta) > 0 \) such that
\[
\|f\|_{B^s_{p,\alpha}} \lesssim C \|f\|_{B^s_{p,\alpha}}^{1-\theta} \|f\|_{B^s_{q,\alpha}}^\theta .
\]

**Proof.** For \( \theta = 0, 1 \), (2.12) is direct by (2.9). Below we assume \( \theta \in (0, 1) \). Let \( r \leq w \in [1, \infty]^N \)
and \( s_2 \in \mathbb{R} \) be defined by
\[
\frac{1}{p} = \frac{1-\theta}{q} + \frac{\theta}{w}, \quad s_0(1-\theta) + s_2\theta = s.
\]
By Hölder’s inequality, we have
\[
\|R_{x,y}^a f\|_{L^p} = \| (R_{x,y}^a f)^{1-\theta}(R_{x,y}^a f)^\theta \|_{L^p} \lesssim \|R_{x,y}^a f\|_{L^q}^{1-\theta} \|R_{x,y}^a f\|_{L^w}^\theta .
\]
Hence,
\[
2^{jx_0} \|R_{x,y}^a f\|_{L^p} \lesssim \left(2^{jx_0} \|R_{x,y}^a f\|_{L^q}\right)^{1-\theta} \left(2^{js_2} \|R_{x,y}^a f\|_{L^w}\right)^\theta .
\]
Note that by (2.11) and (2.13),
\[
s_1 = s_2 + a \cdot \left(\frac{1}{r} - \frac{1}{w}\right),
\]
and by (2.9),
\[
\|R_{x,y}^a f\|_{L^w} \lesssim 2^{j\alpha} \|f\|_{L^w}, \quad j \geq 0.
\]
Substituting it into (2.14), we obtain
\[
2^{jx} \|R_{x,y}^a f\|_{L^p} \lesssim \left(2^{jx_0} \|R_{x,y}^a f\|_{L^q}\right)^{1-\theta} \left(2^{js_1} \|R_{x,y}^a f\|_{L^q}\right)^\theta \lesssim \|f\|_{B^s_{p,\alpha}}^{1-\theta} \|f\|_{B^s_{q,\alpha}}^\theta .
\]
The proof is complete. \( \square \)

We also need the following two simple results. Since they are crucial for treating distribution-valued inhomogeneous \( f \), we provide detailed proofs here for readers’ convenience.

**Lemma 2.5.** For any \( s' > s > 0 \) and \( p, q \in [1, \infty]^N \) with \( \frac{1}{p} + \frac{1}{q} = 1 \), there is a constant \( C = C(p, q, s, s', N) > 0 \) such that for all \( f, g \in B^{s'}_{p,\alpha} \) and \( g \in B^s_{q,\alpha} \),
\[
\|f \ast g\|_{B^s_{p,\alpha}} \lesssim C \|f\|_{B^{s'}_{p,\alpha}} \|g\|_{B^s_{q,\alpha}},
\]
where \( \langle \cdot, \cdot \rangle \) stands for the dual pair between \( S' \) and \( S \).
Proof. By (2.5), (2.4) and Hölder’s inequality, we have
\[
\|fg\|_{B^{t-a}_{1,1}} \lesssim_C \|f\|_{B^{p-a}_{p,a}} \|g\|_{B^{q-a}_{q,a}},
\]
where the last series converges due to $s' > s$. Next we show
\[
\|fg\|_{B^{t-a}_{1,1}} \lesssim_C \|f\|_{B^{p-a}_{p,a}} \|g\|_{B^{q-a}_{q,a}}.
\]
By Bony’s decomposition (2.7), it suffices to show the following three estimates:
\[
\begin{align*}
\|f \cdot g\|_{B^{t-a}_{1,1}} &\lesssim_C \|f\|_{B^{p-a}_{p,a}} \|g\|_{B^{q-a}_{q,a}}, \quad (2.15) \\
\|g \cdot f\|_{B^{t-a}_{1,1}} &\lesssim_C \|g\|_{L^q} \|f\|_{B^{p-a}_{p,a}}, \quad (2.16) \\
\|f \circ g\|_{B^{t-a}_{1,1}} &\lesssim_C \|f\|_{B^{p-a}_{p,a}} \|g\|_{B^{q-a}_{q,a}}, \quad (2.17)
\end{align*}
\]
Noting that (see [13, (2.9)]),
\[
R^a_j(S_{k-1}fR^a_k g) = 0, \quad |k - j| > 2,
\]
by definition (2.6), (2.10) and Hölder’s inequality, we have
\[
\|R^a_j(f \cdot g)\|_{L^1} \lesssim \sum_{|k-j| \leq 2} \|S_{k-1}fR^a_k g\|_{L^1} \lesssim \sum_{|k-j| \leq 2} \|S_{k-1}f\|_{L^p} \|R^a_k g\|_{L^q}
\]
\[
\lesssim \sum_{|k-j| \leq 2} \sum_{i=0}^{k-2} 2^{si2^{-k}a} \|f\|_{B^{p-a}_{p,a}} \|g\|_{B^{q-a}_{q,a}} \lesssim 2^{j(s-a')} \|f\|_{B^{p-a}_{p,a}} \|g\|_{B^{q-a}_{q,a}},
\]
which gives (2.15) by definition. Similarly,
\[
\|R^a_j(g \cdot f)\|_{L^1} \lesssim \sum_{|k-j| \leq 2} \|S_{k-1}gR^a_k f\|_{L^1} \lesssim \sum_{|k-j| \leq 2} \|S_{k-1}g\|_{L^q} \|R^a_k f\|_{L^q}
\]
\[
\lesssim \sum_{|k-j| \leq 2} 2^{sk2^{-k}a} \|g\|_{B^{q-a}_{q,a}} \|f\|_{B^{p-a}_{p,a}} \lesssim 2^{j(s)} \|g\|_{L^q} \|f\|_{B^{p-a}_{p,a}},
\]
which yields (2.16). Moreover, we also have
\[
\|R^a_j(f \circ g)\|_{L^1} \lesssim \|f \circ g\|_{L^1} \lesssim \sum_{|i-k| \leq 2} \|R^a_i fR^a_k g\|_{L^1} \lesssim \sum_{|i-k| \leq 2} \|R^a_i f\|_{L^p} \|R^a_k g\|_{L^q}
\]
\[
\lesssim \sum_{|i-k| \leq 2} 2^{si2^{-k}a} \|f\|_{B^{p-a}_{p,a}} \|g\|_{B^{q-a}_{q,a}} \lesssim \|f\|_{B^{p-a}_{p,a}} \|g\|_{B^{q-a}_{q,a}},
\]
which gives (2.17). The proof is complete. \qed

Lemma 2.6. For any $p \in [1, \infty]^N$ and $s \in (0, 1)$, there is a constant $C = C(p, a, s) > 0$ such that for any nonnegative $u$,
\[
\|u\|_{B^{p/2}_{p,a}}^2 \lesssim \|u\|_{B^{1/2}_{1,a}}^2.
\]
Proof. Since $|a - b| \leq \sqrt{|a^2 - b^2|}$ for $a, b \geq 0$, by (2.8), we have
\[
\|u\|_{B^{1/2}_{1,a}} \approx \|u\|_{L^2} + \sup_h \|u(\cdot + h) - u(\cdot)\|_{L^2} \approx \|u\|_{L^2} + \sup_h \|u(\cdot + h) - u(\cdot)\|_{L^2} \approx \|u\|_{L^2}^2.
\]
The proof is complete. \qed
2.2. De-Giorgi’s class. In this subsection we present a general criterion for the local bound of a function in De-Giorgi’s class. Let \( \mathcal{I} \subset (1, \infty)^N \) be an open multi-index set and \( Q := (Q_\tau)_{\tau \in [1,2]} \) be a family of increasing bounded open sets in \( \mathbb{R}^N \) with
\[
Q_\tau \cap Q_\sigma = \emptyset \quad \text{for} \quad \tau < \sigma \quad \text{and} \quad \bigcap_{\sigma > \tau} Q_\sigma = \bar{Q}_\tau.
\]
(2.18)

We introduce the following De-Giorgi class associated with \( \mathcal{I} \) and \( Q \).

**Definition 2.7.** We call a function \( u \in L^1(Q_2) \) being in the De-Giorgi class \( DG^+_\mathcal{I}(Q) \) if there are \( p_i \in \mathcal{I}, \ i = 1, \cdots, m, \ 1 \leq j < m \) and \( \lambda, A \geq 0 \) such that for any \( p \in \mathcal{I} \), there is a constant \( C_p > 0 \) such that for any \( 1 \leq \tau < \sigma \leq 2 \) and \( \kappa \geq 0 \),
\[
(\sigma - \tau)\lambda \| \mathbf{1}_{Q_\sigma}(u - \kappa)^+ \|_{L^p} \lesssim C_p \sum_{i=1}^j \| \mathbf{1}_{Q_\sigma}(u - \kappa)^+ \|_{L^{p_i}} + A \sum_{i=j+1}^m \| \mathbf{1}_{\{u > \kappa\}\cap Q_\sigma} \|_{L^{p_i}}.
\]
(2.19)

The aim of this subsection is to give an upper bound estimate of \( u \in DG^+_\mathcal{I}(Q) \). First of all, we recall the following two iteration lemmas (see [10]).

**Lemma 2.8.** Let \( (a_n)_{n \in \mathbb{N}} \) be a sequence of nonnegative real numbers. Suppose that for some \( C_0, \lambda > 1 \) and \( \delta > 0 \),
\[
a_{n+1} \leq C_0 \lambda^{n+\delta} a_n.
\]
Then under \( a_1 \leq (C_0 \lambda^{(1+\delta)/\delta} - 1/\delta) \), we have
\[
\lim_{n \to \infty} a_n = 0.
\]

**Lemma 2.9.** Let \( h(\tau) \geq 0 \) be bounded in \([\tau_1, \tau_2]\) with \( \tau_1 \geq 0 \). Suppose that for some \( \alpha \geq 0 \), \( \theta \in (0, 1) \) and any \( \tau_1 \leq \tau < \tau' \leq \tau_2 \),
\[
h(\tau) \leq \theta h(\tau') + (\tau' - \tau)^{-\alpha} A + B.
\]
Then there is a constant \( C = C(\alpha, \theta) > 0 \) such that
\[
h(\tau_n) \leq C (\tau_2 - \tau_1)^{-\alpha}(A + B).
\]

Now we can show the following main result of this subsection (cf. [39]).

**Theorem 2.10.** Let \( u \in DG^+_\mathcal{I}(Q) \). For any \( p > 0 \), there are constants \( \gamma, C > 0 \) only depending on \( p \), \( \text{Vol}(Q_2) \) and the parameters \( p_i, \lambda, C_p \) in the definition of \( DG^+_\mathcal{I}(Q) \) such that for all \( 1 \leq \tau < \sigma \leq 2 \),
\[
\|u^+ \mathbf{1}_{Q_\sigma}\|_{L^\infty} \lesssim C (\sigma - \tau)^{-\gamma} \|u^+ \mathbf{1}_{Q_\sigma}\|_{L^p} + A.
\]
(2.20)

**Proof.** Fix \( 1 \leq \tau < \sigma \leq 2 \) and \( \kappa > 0 \), which will be determined below. For \( n \in \mathbb{N} \), define
\[
\tau_n = \tau + (\sigma - \tau)2^{1-n}, \quad \kappa_n := \kappa (1 - 2^{1-n}),
\]
and
\[
w_n := (u - \kappa_n)^+.
\]
Clearly, by (2.18),
\[
\kappa_n \uparrow \kappa, \quad \tau_n \downarrow \tau, \quad Q_{\tau_n+1} \subset Q_{\tau_n} \downarrow \bar{Q}_\tau.
\]
Since \( \mathcal{I} \) is an open set, for any \( p \in \mathcal{I} \), there is a \( p' \in \mathcal{I} \) such that \( p' > p \). Let \( q = (q_1, \cdots, q_N) \in (1, \infty)^N \) be defined by
\[
\frac{1}{p'} + \frac{1}{q} = \frac{1}{p}.
\]
By Hölder’s inequality, there is a constant \( C = C(Q_2, q) > 0 \) such that
\[
\| \mathbf{1}_{Q_{\tau_n+1}} w_{n+1} \|_{L^p} \lesssim \| \mathbf{1}_{Q_{\tau_n+1}} w_{n+1} \|_{L^{p'}} \| \mathbf{1}_{\{w_{n+1} \neq 0\}\cap Q_{\tau_n+1}} \|_{L^q} \lesssim C \| \mathbf{1}_{Q_{\tau_n+1}} w_{n+1} \|_{L^{p'}} \| \mathbf{1}_{\{w_{n+1} \neq 0\}\cap Q_{\tau_n}} \|_{L^q} \lesssim \| \mathbf{1}_{Q_{\tau_n+1}} w_{n+1} \|_{L^{p'}} \| \mathbf{1}_{\{w_{n+1} \neq 0\}\cap Q_{\tau_n}} \|_{L^q} \|_{L^1}^{\max, q_i},
\]
(2.21)
Now let us choose $a_n := \sum_{i=1}^{m} \|1_{Q_{\tau n}} w_n\|_{L^{p_i}}$, then by (2.21), there are $p'_i \in \mathcal{I}$ and $C_1, \delta > 0$ such that

$$a_{n+1} \lesssim C_1 \sum_{i=1}^{m} \|1_{Q_{\tau n+1}} w_{n+1}\|_{L^{p'_i}} \|1_{\{w_{n+1} \neq 0\} \cap Q_{\tau n}}\|_{L^{\delta i}}. \tag{2.22}$$

On the other hand, noting that $w_{n}|_{w_{n+1} \neq 0} = (u - \kappa_{n+1} + \kappa_{n+1} - \kappa_n)^+|_{w_{n+1} \neq 0} \geq \kappa_{n+1} - \kappa_n = \kappa 2^{-n}$, for any $p \in [1, \infty)^N$, we have

$$\|1_{Q_{\tau n}} w_n\|_{L^{p}} \geq \|1_{\{w_{n+1} \neq 0\} \cap Q_{\tau n}} w_n\|_{L^{p}} \geq \kappa 2^{-n} \|1_{\{w_{n+1} \neq 0\} \cap Q_{\tau n}}\|_{L^{p}}. \tag{2.23}$$

Suppose $\kappa \geq A$.

For each $i = 1, \ldots, m$, since $p'_i \in \mathcal{I}$, by (2.19), (2.23) and $w_{n+1} \leq w_n$, we have

$$n^n |w_{n+1}| \leq (\tau_n - \tau_{n+1})^\lambda \|1_{Q_{\tau n+1}} w_{n+1}\|_{L^{p'_i}} \leq \sum_{i=1}^{m} \|1_{Q_{\tau n}} w_{n+1}\|_{L^{p'_i}} + A \sum_{i=j+1}^{m} \|1_{\{w_{n+1} > 0\} \cap Q_{\tau n}}\|_{L^{p_i}} \leq 2^n a_n.$$ 

Substituting this into (2.22) and by $\tau_n - \tau_{n+1} = (\sigma - \tau) 2^{-n}$ and (2.23), we obtain that for $\kappa \geq A$,

$$(\sigma - \tau)^\lambda 2^{-\kappa} a_{n+1} \leq 2^n a_n \|1_{\{w_{n+1} \neq 0\} \cap Q_{\tau n}}\|_{L^{\delta}} \leq 2^n a_n \|1_{\{w_{n+1} \neq 0\} \cap Q_{\tau n}}\|_{L^{p_i}} \leq 2^n a_n (2^{n-1})^\delta \|1_{Q_{\tau n}} w_n\|_{L^{p_i}} \leq 2^n a_n (2^{n-1})^\delta a_n.$$ 

In particular, there is a $C_2 > 0$ such that for all $n \in \mathbb{N}$,

$$a_{n+1} \leq C_2 (\sigma - \tau)^{-\kappa} 2^{(\lambda + \delta + 1)n - \delta^\delta} a_n^{1 + \delta}.$$ 

Now let us choose

$$\kappa = \left(\left[ C_2 (\sigma - \tau)^{-\lambda} 2^{(\lambda + \delta + 1)(1 + \delta)/\delta}\right]^{1/\delta} a_1 \right) \cup A.$$ 

Then by Fatou’s lemma and Lemma 2.8,

$$\|(u - \kappa)^+ 1_{Q_{\tau}}\|_{L^{p_i}} \leq \liminf_{n \to \infty} \|w_n 1_{Q_{\tau n}}\|_{L^{p_i}} \leq \limsup_{n \to \infty} a_n = 0,$$ 

which in turn implies that $(u - \kappa)^+ = 0$ on $Q_{\tau}$ and by the choice of $\kappa$,

$$\|u^+ 1_{Q_{\tau}}\|_{\infty} \leq \left(\left[ C_2 (\sigma - \tau)^{-\lambda} 2^{(\lambda + \delta + 1)(1 + \delta)/\delta}\right]^{1/\delta} a_1 \right) \cup A \leq C_3 (\sigma - \tau)^{-\delta} \sum_{i=0}^{m} \|u^+ 1_{Q_{\tau}}\|_{L^{p_i}} + A. \tag{2.24}$$

To show (2.20), without loss of generality, we may assume $p \leq \gamma/2, \gamma := \max_{i,j} p_{ij}, p_i = (p_{i1}, \ldots, p_{iN}).$
By (2.24), Hölder’s inequality and Young’s inequality, we have
\[ \|u^+1_Q\|_\infty \leq C(\sigma - \tau)^{-\frac{2}{3}}\|u^+1_Q\|_{L^\gamma} + A \]
\[ \leq C(\sigma - \tau)^{-\frac{2}{3}}\|u^+1_Q\|^{\frac{1}{2}}_{L^p} + A \]
\[ \leq \frac{1}{2}\|u^+1_Q\|_{\infty} + C(\sigma - \tau)^{-\frac{2}{3}}\|u^+1_Q\|_{L^p} + A, \]
where \( C \) is independent of \( \sigma, \tau \). Thus by Lemma 2.9, we obtain (2.20).

\[ \square \]

3. Weak (sub-)solutions of kinetic equations

In this section we present some basic properties about weak (sub-)solutions of linear kinetic FPKEs and show the gain of regularity in \( x \) via Duhamel's formula. Consider the following kinetic equation of divergence form:
\[ \partial_t u = \text{div}_v(a \cdot \nabla_v u) + v \cdot \nabla_x u + b \cdot \nabla_v u + f, \]
where \( a : \mathbb{R}^{1+2d} \rightarrow M_{\text{sym}}^d, b : \mathbb{R}^{1+2d} \rightarrow \mathbb{R}^d, f : \mathbb{R}^{1+2d} \rightarrow \mathbb{R} \)
are Borel measurable functions. Suppose that for some \( 0 < \kappa_0 < \kappa_1 \),
\[ \kappa_0 I \leq a \leq \kappa_1 I, \]
and for any bounded \( Q \subset \mathbb{R}^{1+2d} \),
\[ b1_Q \in L^2, \quad f1_Q \in L^1, \]
where for \( p, q \in [1, \infty] \),
\[ L^p_q(L^q) := L^q(\mathbb{R}; L^p(\mathbb{R}^{2d})), \quad L^p := L^p_1(L^p). \]
We introduce the following space of solutions: for an open set \( Q \subset \mathbb{R}^{1+2d} \),
\[ \mathcal{V}_Q := \left\{ f \in L^{1}_{loc} : \|f\|_{\mathcal{V}_Q} := \|1_Qf\|_{L^\infty(L^2)} + \|1_Q\nabla_v f\|_{L^2} < \infty \right\}. \]
For simplicity, we write for any \( T > 0 \),
\[ \mathcal{Y}_T := \mathcal{Y}_{[0,T] \times \mathbb{R}^{2d}}, \quad \mathcal{V} := \mathcal{Y}_{\mathbb{R}^{1+2d}}, \quad \mathcal{V}_{loc} := \bigcap_{\text{bounded} Q} \mathcal{Y}_Q, \]
and for given \( t \in \mathbb{R} \) and \( r > 0 \),
\[ I_t := 1_{(-\infty,t]} \quad Q_r := \{(t,x,v) : |t| < r^2, |x| < r^3, |v| < r\}. \]

3.1. Weak (sub-)solutions. In this subsection we introduce the notion of weak (sub-)solutions and their basic properties for later use.

**Definition 3.1.** Let \( Q \subset \mathbb{R}^{1+2d} \) be a bounded domain. A function \( u \in \mathcal{Y}_Q \cap L^\infty_\text{loc} \) is called a weak sub-solution of PDE (3.1) in \( Q \) if for any nonnegative \( \varphi \in C^\infty_c(Q) \),
\[ -\int_Q u \partial_t \varphi \leq -\int_Q \langle a \cdot \nabla_v u, \nabla_v \varphi \rangle - \int_Q u \langle v, \nabla_x \varphi \rangle + \int_Q \langle b \cdot \nabla_v u, \varphi \rangle + \int_Q f \varphi. \]

If both \( u \) and \( -u \) are weak sub-solutions, then we call \( u \) a weak solution of PDE (3.1) in \( Q \). If \( u \in \mathcal{V}_{loc} \cap L^\infty_\text{loc} \) and (3.5) holds for any bounded domain \( Q \), then it is called a global weak solution.

**Remark 3.2.** By (3.2) and (3.3), each term in (3.5) is well-defined. Let \( u \) be a global weak solution. By a standard approximation, one sees that (3.5) is equivalent that for any \( \varphi \in C^\infty_c(\mathbb{R}^{2d}) \) and Lebesgue almost all \( t_0 < t_1 \),
\[ \langle u, \varphi \rangle_{t_0}^{t_1} \leq -\int_{t_0}^{t_1} \langle a \cdot \nabla_v u, \nabla_v \varphi \rangle - \int_{t_0}^{t_1} \langle v \cdot \nabla_x u, \varphi \rangle + \int_{t_0}^{t_1} \langle b \cdot \nabla_v u, \varphi \rangle + \int_{t_0}^{t_1} f \varphi, \]
where \( \langle u, \varphi \rangle := \int_{\mathbb{R}^{2d}} u(z) \varphi(z) \mathrm{d}z \). In particular, let \( u \) be a weak solution of the Cauchy problem of (3.1) with initial value \( u(0) = 0 \). Then it can be extended to be a global weak solution by setting \( u(t) = f(t) \equiv 0 \) for \( t \leq 0 \).
The following two lemmas are well known to experts. For readers’ convenience, we provide detailed proofs here.

**Lemma 3.3.** Let $u \in \mathcal{V}_Q \cap L^\infty_Q$ be a nonnegative weak sub-solution of PDE (3.1) in $Q$. Under (3.2) and (3.3), for any nonnegative $\eta \in C^\infty_c(Q)$ and $t \in \mathbb{R}$, it holds that

\begin{equation} \label{eq:3.6}
\frac{1}{2} \int_{\mathbb{R}^{d+2}} |(u\eta)(t)|^2 \leq \int_{\mathbb{R}^{1+2d}} u^2 \eta (\partial_s \eta - v \cdot \nabla_x \eta) \mathcal{L}_t - \int_{\mathbb{R}^{1+2d}} \langle a \cdot \nabla \eta, \nabla_v (u\eta)^2 \rangle \mathcal{L}_t \\
+ \int_{\mathbb{R}^{1+2d}} (b \cdot \nabla_v u) u\eta^2 \mathcal{L}_t + \int_{\mathbb{R}^{1+2d}} f u\eta^2 \mathcal{L}_t.
\end{equation}

Moreover, if $u$ is a weak solution, then the above inequality becomes an equality.

**Proof.** Let $\Gamma$ be a nonnegative symmetric smooth function in $\mathbb{R}^{1+2d}$ with compact support. For $\varepsilon \in (0, 1)$, we introduce the following mollifier:

\begin{equation} \label{eq:3.7}
\Gamma_\varepsilon(t, x, v) := \varepsilon^{-(4d+2)} \Gamma\left(\frac{\varepsilon^{-1} t \cdot \varepsilon^{-1} x}{\varepsilon^2}, \frac{\varepsilon^{-1} v}{\varepsilon^2}\right), \quad u_\varepsilon := u \ast \Gamma_\varepsilon.
\end{equation}

Let $\phi \in C^\infty_c(Q)$ be nonnegative. By the integration by parts, we have

\begin{equation} \label{eq:3.5}
- \frac{1}{2} \int_{\mathbb{R}^{1+2d}} u_\varepsilon^2 \partial_t \phi = \int_{\mathbb{R}^{1+2d}} \partial_t u_\varepsilon \phi = - \int_{\mathbb{R}^{1+2d}} u_\varepsilon \partial_t (u_\varepsilon \phi) = - \int_{\mathbb{R}^{1+2d}} u_\varepsilon \partial_t (\Gamma_\varepsilon \ast (u_\varepsilon \phi)).
\end{equation}

Thus by (3.5) with $\varphi = \Gamma_\varepsilon \ast (u_\varepsilon \phi) \in C^\infty_c(Q)$ provided $\varepsilon$ small enough, we get

\begin{equation} \label{eq:3.8}
- \frac{1}{2} \int_{\mathbb{R}^{1+2d}} u_\varepsilon^2 \partial_t \phi \leq - \int_{\mathbb{R}^{1+2d}} \langle a \cdot \nabla_v u, \Gamma_\varepsilon \ast \nabla_v (u_\varepsilon \phi) \rangle - \int_{\mathbb{R}^{1+2d}} u \langle v, \Gamma_\varepsilon \ast \nabla_x (u_\varepsilon \phi) \rangle \\
+ \int_{\mathbb{R}^{1+2d}} (\Gamma_\varepsilon \ast (b \cdot \nabla_v u)) u_\varepsilon \phi + \int_{\mathbb{R}^{1+2d}} (\Gamma_\varepsilon \ast f) u_\varepsilon \phi.
\end{equation}

Fix $t \in \mathbb{R}$. Let $\chi_n$ be a family of smooth functions in $\mathbb{R}$ so that

$$
\chi_n \to 1_{(-\infty, t]}, \quad \chi'_n \to -\delta_{\{t\}} \quad \text{as } n \to \infty,
$$

where $\delta_{\{t\}}$ stands for the Dirac measure at point $t$. Using $\chi_n \eta^2$ in place of $\phi$ in (3.8) and then taking limits $n \to \infty$, we obtain

\begin{equation} \label{eq:3.9}
\frac{1}{2} \left( \int_{\mathbb{R}^{d+2}} |(u_\varepsilon \eta)(t)|^2 - \int_{\mathbb{R}^{1+2d}} u_\varepsilon^2 \partial_s \eta \mathcal{L}_t \right) \\
\leq - \int_{\mathbb{R}^{1+2d}} \langle a \cdot \nabla_v u, \Gamma_\varepsilon \ast \nabla_v (u_\varepsilon \eta^2) \rangle - \int_{\mathbb{R}^{1+2d}} u \langle v, \Gamma_\varepsilon \ast \nabla_x (u_\varepsilon \eta^2) \rangle \\
+ \int_{\mathbb{R}^{1+2d}} (\Gamma_\varepsilon \ast (b \cdot \nabla_v u)) u_\varepsilon \eta^2 \mathcal{L}_t + \int_{\mathbb{R}^{1+2d}} (\Gamma_\varepsilon \ast f) u_\varepsilon \eta^2 \mathcal{L}_t.
\end{equation}

By the definition of convolutions, we have

\begin{align*}
&\left| \int_{\mathbb{R}^{1+2d}} u \langle v, \nabla_x \Gamma_\varepsilon \ast (u_\varepsilon \mathcal{L}_t \eta^2) \rangle - v \cdot \nabla_x (u \ast \Gamma_\varepsilon)(u_\varepsilon \mathcal{L}_t \eta^2) \right| \\
&= \left| \int_{\mathbb{R}^{1+2d}} \left( \int_{\mathbb{R}^{1+2d}} u(\tilde{s}, \tilde{x}, \tilde{v})(\tilde{v} - v) \cdot \nabla_x \Gamma\left(\frac{\tilde{x} - x}{\varepsilon}, \frac{\tilde{v} - v}{\varepsilon}\right) \right) (u_\varepsilon \mathcal{L}_t \eta^2)(s, x; v) \right| \\
&\lesssim \varepsilon^2 \left( \int_{\mathbb{R}^{1+2d}} \int_{\mathbb{R}^{1+2d}} |u(\tilde{s}, \tilde{x}, \tilde{v})| \left| \nabla_x \Gamma\left(\frac{\tilde{x} - x}{\varepsilon}, \frac{\tilde{v} - v}{\varepsilon}\right) \right| \right) |(u_\varepsilon \mathcal{L}_t \eta^2)(s, x; v)|,
\end{align*}

which converges to zero as $\varepsilon \to 0$. Note that

\begin{equation*}
\int_{\mathbb{R}^{1+2d}} \langle v, \nabla_x u_\varepsilon \rangle (u_\varepsilon \mathcal{L}_t \eta^2) = \frac{1}{2} \int_{\mathbb{R}^{1+2d}} \langle v, \nabla_x u_\varepsilon \rangle \eta^2 \mathcal{L}_t = - \frac{1}{2} \int_{\mathbb{R}^{1+2d}} (v \cdot \nabla_x \eta^2) u_\varepsilon^2 \mathcal{L}_t.
\end{equation*}

Substituting these into (3.9) and letting $\varepsilon \to 0$, we get

\begin{equation*}
\frac{1}{2} \int_{\mathbb{R}^{d+2}} |(u \eta)(t)|^2 \leq \int_{\mathbb{R}^{1+2d}} u^2 \eta \partial_s \eta \mathcal{L}_t - \int_{\mathbb{R}^{1+2d}} \langle a \cdot \nabla_v u, \nabla_v (u \eta^2) \rangle \mathcal{L}_t
\end{equation*}
\[- \frac{1}{2} \int_{\mathbb{R}^{1+2d}} (v \cdot \nabla x \eta')^2 u^2 I_t + \int_{\mathbb{R}^{1+2d}} (b \cdot \nabla_v u) u \eta^2 I_t + \int_{\mathbb{R}^{1+2d}} f u \eta^2 I_t,\]

where \( u \in L^\infty_0 \) is used for taking limits for

\[
\lim_{\varepsilon \to 0} \int_{\mathbb{R}^{1+2d}} (\Gamma_\varepsilon \ast (b \cdot \nabla_v u)) u \eta^2 I_t = \int_{\mathbb{R}^{1+2d}} (b \cdot \nabla_v u) u \eta^2 I_t
\]

and

\[
\lim_{\varepsilon \to 0} \int_{\mathbb{R}^{1+2d}} (\Gamma_\varepsilon \ast f) u \eta^2 I_t = \int_{\mathbb{R}^{1+2d}} f u \eta^2 I_t.
\]

Thus we obtain (3.6). If \( u \) is a weak solution, then from the above proof, one sees that (3.6) takes equality. \( \square \)

**Remark 3.4.** If \( b = 0 \) and \( 1 \leq f \in L^1_0(L^2_0) \), it suffices to require \( u \in \mathcal{Y}_Q \) in Lemma 3.3.

The second part of the following lemma shall be used to deal with the distribution-valued inhomogeneous \( f \).

**Lemma 3.5.** (i) Suppose that \( u \in \mathcal{Y}_{loc} \cap L^\infty_{loc} \) is a weak sub-solution of (3.1). Then \( u^+ \) is still a weak sub-solution of (3.1) with \( f \mathbb{I}_{\{u > 0\}} \) in place of \( f \).

(ii) Suppose that \( u \in \mathcal{Y}_{loc} \cap L^\infty_{loc} \) is a weak solution of (3.1). Then \( (u^+)^2 \) is still a weak solution of (3.1) with \( 2f u^+ - 2(a \cdot \nabla u^+, \nabla v u^+) \) in place of \( f \).

**Proof.** Let \( \Gamma_\varepsilon \) be as in (3.7). By taking \( \varphi = \Gamma_\varepsilon(t - \cdot, x - \cdot, v - \cdot) \) in (3.5), we obtain

\[
\partial_t u_\varepsilon \leq \text{div}_v (a \cdot \nabla_v u) \ast \Gamma_\varepsilon + (v \cdot \nabla_x u + b \cdot \nabla_v u) \ast \Gamma_\varepsilon + f_\varepsilon.
\]

First of all, we assume that \( \beta : \mathbb{R} \to \mathbb{R} \) is a smooth non-decreasing function and \( u \) is a weak sub-solution. By the chain rule and \( \beta' \geq 0 \), we have

\[
\Gamma_\varepsilon \ast \beta(u_\varepsilon) \leq \text{div}_v (a \cdot \nabla_v u) \ast \Gamma_\varepsilon + (v \cdot \nabla_x u + b \cdot \nabla_v u) \ast \Gamma_\varepsilon + f_\varepsilon + \beta'(u_\varepsilon)
\]

\[
= \text{div}_v (a \cdot \nabla_v u_\varepsilon) + v \cdot \nabla_x u_\varepsilon + b \cdot \nabla_v u_\varepsilon + H_\varepsilon + f_\varepsilon + \beta'(u_\varepsilon),
\]

where

\[
H_\varepsilon := \text{div}_v (a \cdot \nabla_v u) \ast \Gamma_\varepsilon - \text{div}_v (a \cdot \nabla_v u_\varepsilon) + [\Gamma_\varepsilon \ast, v \cdot \nabla_x] u + [\Gamma_\varepsilon \ast, b \cdot \nabla_v] u,
\]

and

\[
[\Gamma_\varepsilon \ast, v \cdot \nabla_x] u := \Gamma_\varepsilon \ast (v \cdot \nabla_x u) - v \cdot \nabla_x (\Gamma_\varepsilon \ast u).
\]

For any \( \varphi \in C^\infty_c(\mathbb{R}^{1+2d}) \), by the chain rule, we have

\[
- \int \beta(u_\varepsilon) \partial_t \varphi \leq - \int \langle a \cdot \nabla u_\varepsilon, \nabla_v (\beta'(u_\varepsilon) \varphi) \rangle - \int (v \cdot \nabla_x \varphi) \beta(u_\varepsilon)
\]

\[
+ \int b \cdot \nabla_v \beta(u_\varepsilon) \varphi + \int (H_\varepsilon + f_\varepsilon) \beta'(u_\varepsilon) \varphi.
\]

Note that \( u \) is locally bounded and

\[
\langle a \cdot \nabla u_\varepsilon, \nabla_v (\beta'(u_\varepsilon) \varphi) \rangle = \langle a \cdot \nabla_v \beta(u_\varepsilon), \nabla_v \varphi \rangle + \langle a \cdot \nabla_v u_\varepsilon, \nabla_v \beta'(u_\varepsilon) \varphi \rangle.
\]

By easy calculations, we have

\[
\lim_{\varepsilon \to 0} \int H_\varepsilon \beta'(u_\varepsilon) \varphi = 0,
\]

and by taking limits \( \varepsilon \downarrow 0 \) for (3.10),

\[
- \int \beta(u) \partial_t \varphi \leq - \int \langle a \cdot \nabla u, \nabla_v \varphi \rangle - \int \langle a \cdot \nabla_v u, \nabla_v \varphi \rangle - \int (v \cdot \nabla x) \beta(u) 
\]

\[
- \int b \cdot \nabla_v \beta(u) \varphi + \int (H + f) \beta'(u) \varphi.
\]

(i) For \( \varepsilon > 0 \), define

\[
\beta_\varepsilon(r) := (\sqrt{(r - \varepsilon)^2 + \varepsilon^3} + r - \varepsilon)/2.
\]
It is easy to see that $\beta_\varepsilon \geq 0$ is a smooth non-decreasing function and
\[ \beta_\varepsilon''(r) \geq 0, \quad \lim_{\varepsilon \downarrow 0} \beta_\varepsilon(r) = r^+, \quad \lim_{\varepsilon \downarrow 0} \beta_\varepsilon'(r) = 1_{\{r > 0\}}, \quad r \in \mathbb{R}. \]

Using $\beta_\varepsilon$ in place of the $\beta$ in (3.11) and taking limits $\varepsilon \downarrow 0$, by the dominated convergence theorem, one sees that
\[ -\int u^+ \partial_t \varphi \leq -\int (a \cdot \nabla v u^+ \varphi) - \int (v \cdot \nabla_x \varphi) u^+ + \int b \cdot \nabla v u^+ \varphi + \int f 1_{\{u > 0\}} \varphi. \]

(ii) Note that
\[ \lim_{\varepsilon \downarrow 0} (\beta_\varepsilon^2(r))' = 2r^+, \quad \lim_{\varepsilon \downarrow 0} (\beta_\varepsilon^2(r))'' = 21_{r > 0}. \]

If $u$ is a weak solution, then (3.11) becomes an equality. Using $\beta_\varepsilon^2$ in place of the $\beta$ in (3.11) and taking limits $\varepsilon \downarrow 0$, one sees that
\[ -\int (u^+)^2 \partial_t \varphi = -\int (a \cdot \nabla v (u^+)^2) \varphi - 2 \int (a \cdot \nabla v u^+ , \nabla v u^+ ) \varphi - \int (v \cdot \nabla_x (u^+)^2) \varphi + 2 \int f u^+ \varphi. \]

The proof is complete. \qed

3.2. Gain of regularity via Duhamel’s formula. Below we take
\[ \mathbf{a} = \left( \underbrace{3, \ldots , 3}_{d}, \underbrace{1, \ldots , 1}_{d} \right) \in \mathbb{R}^{2d}, \quad |\mathbf{a}| = 4d. \tag{3.12} \]

Consider the following simple model equation:
\[ \partial_t u = \Delta_v u + v \cdot \nabla_x u + f, \quad u|_{t=0} = 0. \]

By Duhamel’s formula, it’s unique solution can be represented by
\[ u(t, z) = \int_0^t P_{t-s} f(s, z) ds, \tag{3.13} \]

where $P_t$ is the heat semigroup of operator $\Delta_v + v \cdot \nabla_x$ given by
\[ P_t f(z) = \mathcal{E} f \left( x + tv + \sqrt{2} \int_0^t W_z ds, v + \sqrt{2} W_z \right), \quad z = (x, v). \tag{3.14} \]

The following estimate is important for improving the regularity of $u$.

**Lemma 3.6.** Let $1 \leq q \leq \nu \leq \infty$, $\gamma \in \mathbb{R}$ and $p \in [1, \infty]^{2d}$. For any $T > 0$ and $\beta < \gamma + 2(1 + \nu - \frac{1}{q})$, there is a constant $C = C(T, d, p, q, \nu, \gamma, \beta) > 0$ such that
\[ \| u \|^2_{L^q_t(B^{\beta}_{p, \infty})} \leq C \| f \|^2_{L^q_t(B^{\beta}_{p, \infty})}, \quad t \in (0, T], \tag{3.15} \]

where $u$ is defined by (3.13). Here we use the convention $f(t)|_{t=0} = 0$.

**Proof.** Without loss of generality, we assume $\gamma \leq \beta$. Since $\beta < \gamma + 2(1 + \nu - \frac{1}{q})$, one can choose
\[ 0 < \kappa \in (1 - \frac{2}{\nu}, (1 - \frac{\beta - \gamma}{2}) q). \]

Noting that by [39, Lemma 3.3],
\[ \| P_t f \|^2_{B^{\beta}_{p, \infty}} \leq t^{-\frac{d-\beta}{2}} \| f \|^2_{B^{\beta}_{p, \infty}}, \quad t \in (0, T], \]

by definition (3.13) and Hölder’s inequality with respect to $(t-s)^{\kappa-1} ds$, we have
\[ \| u(t) \|^2_{B^{\beta}_{p, \infty}} \leq \int_0^t (t-s)^{-\frac{d-\beta}{2}} \| f(s) \|^2_{B^{\beta}_{p, \infty}} ds \]
\[ \leq \left( \int_0^t (t-s)^{\kappa-1} \| f(s) \|^q_{B^{\beta}_{p, \infty}} ds \right)^{1/q}. \]
Thus by Minkowskii’s inequality, for any $t \in [0, T]$,

$$
\left( \int_0^t \|u(s)\|_{B^q_{p,a}}^{\nu} \, ds \right)^{1/\nu} \leq \left( \int_0^t \left( \int_0^s (s - r)^{\kappa - 1} \|f(r)\|_{B^q_{p,a}}^{\nu/q} \, dr \right)^{q/\nu} \, ds \right)^{1/q} 
$$

$$
\leq C_{q,r,t} \left( \int_0^t \|f(r)\|_{B^q_{p,a}}^{q} \, dr \right)^{1/q},
$$

which gives (3.15). The proof is complete. \( \square \)

We also need the following simple interpolation lemma.

**Lemma 3.7.** For any $\beta > 0$, $1 \leq q \leq \nu \leq \infty$ and $r \in [2, \infty]^{2d}$ with

$$
\frac{2}{p} \geq \left( 1 - \frac{\beta}{2} \right) 1, \quad a \cdot \left( \frac{1}{p} - \frac{1}{2} \right) + \frac{\beta}{p} > 0,
$$

there is a constant $C = C(\beta, \nu, q, r, a) > 0$ such that

$$
\|u\|_{L^p_t(L^q_z)} \lesssim \|u\|_{L^{p,a}_t(L^{q,a}_z)} + \|u\|_{L^{p,a}_t(B^q_{p,a})}.
$$

**Proof.** Let $\theta := \frac{2}{p} \in [0, 1]$. Since $\frac{1}{r} \geq \frac{1+\frac{\beta}{2}}{2}$, one can choose $p \in [2, \infty]^{2d}$ so that

$$
\frac{\theta}{p} + \frac{1-\theta}{2} = \frac{1}{r}.
$$

By Hölder’s inequality and Sobolev’s embedding (2.12), we have

$$
\|u\|_{L^q_z} \leq \|u\|_{L^2_z}^{1-\theta} \|u\|_{L^p_z}^\theta \lesssim \|u\|_{L^2_z}^{1-\theta} \|u\|_{B^q_{p,a}}^\theta \lesssim \|u\|_{L^2_z}^{1-\theta} \|u\|_{B^q_{p,a}}^\theta,
$$

where the second inequality is due to $s = \beta - a \cdot \left( \frac{1}{2} - \frac{1}{p} \right) > 0$ by the assumption. Hence,

$$
\|u\|_{L^p_t(L^q_z)} \lesssim \|u\|_{L^p_t(L^{q,a}_z)} \|u\|_{L^p_t(B^q_{p,a})}.
$$

The proof is complete. \( \square \)

**Remark 3.8.** The first condition in (3.16) means that for fixed $q$, if $\nu$ goes to infinity, then $r$ has to be close to 2 whatever $\beta$ is how large and the second condition holds. However, for $\beta \in (0, \frac{1}{2}]$,

$$
\frac{1}{p} + \frac{1-\theta}{2} = \frac{1}{r}.
$$

Indeed, for $r = (r_1, \cdots, r_{2d}) \in [2, \infty]^{2d}$, and for each $i$, we have

$$
0 < a \cdot \left( \frac{1}{q} - \frac{1}{2} \right) + \frac{\beta}{q} \leq a_i \cdot \left( \frac{1}{r_i} - \frac{1}{2} \right) + \frac{\beta}{r_i} \leq \frac{1}{r_i} - \frac{1}{2} + \frac{\beta}{r_i}.
$$

For any $T > 0$, $q \in [1, \infty]$ and a Banach space $B$, we denote

$$
L^q_T(B) := L^q([0, T]; B).
$$

Now we consider the following Cauchy problem of the kinetic equation

$$
\partial_t u = \text{div}_v(a \cdot \nabla_v u) + v \cdot \nabla_x u + f, \quad u|_{t=0} \equiv 0,
$$

where the inhomogeneous $f$ satisfies that for any $T > 0$,

$$
(H_f) \quad f = \text{div}_v F + f_1 + \cdots + f_m, \quad F \in L^q_t(L^2_x) \quad \text{and for each } i = 1, \cdots, m, \quad f_i \in L^q_t(L^2_x)\text{ for some } (q_i, p_i) \in [1, 2]^{1+2d} \text{ with}
$$

$$
\frac{1}{p_i} \leq \left( \frac{2}{q_i} - \frac{1}{2} \right), \quad a \cdot \left( \frac{1}{p_i} - \frac{1}{2} \right) + \frac{\beta}{q_i} < 2.
$$

We have the following solvability result for equation (3.18).
Theorem 3.9. Under (3.2) and (H_1), there is a unique weak solution u to PDE (3.18) so that for any T > 0 and \( \beta \in (0, 1) \), there is a constant \( C = C(T, a, \kappa, q, p, \beta) > 0 \) such that
\[
\| u_I \| \leq C \sum_{i=1}^{m} \| f_i I_i \|_{L^q(\mathbb{R}^d_x)} + \| F I \|_{L^2}, \quad t \in [0, T]. \tag{3.20}
\]
Here we use the convention \( f_i(t) = F(t) = 0 \) for \( t \leq 0 \).

Proof. We only prove the apriori estimate (3.20). The existence follows by the standard continuity method. Let \( \chi \in C_c^\infty(\mathbb{R}^{1+2d}) \) with \( \chi = 1 \) on \( Q_1 \) and \( \chi = 0 \) on \( Q_2 \). For \( R \geq 1 \), let
\[
\chi_R(t, x, v) := \chi(t/R^2, x/R^3, v/R). \tag{3.21}
\]
By taking \( \eta = \chi_R \) in Lemma 3.3, we have for any \( t \in (0, T], \)
\[
\frac{1}{2} \int_{\mathbb{R}^d} \| (u \chi_R)(t) \|^2 = \int_{\mathbb{R}^{1+2d}} u^2 \chi_R(\partial_s \chi_R - v \cdot \nabla_x \chi_R) I_t - \int_{\mathbb{R}^{1+2d}} \langle a \cdot \nabla v, \nabla u(\chi_R^2) \rangle I_t + \int_{\mathbb{R}^{1+2d}} f u \chi_R^2 I_t.
\]
Noting that for some \( C > 0 \) independent of \( R > 0, \)
\[
\| \partial_s \chi_R - v \cdot \nabla_x \chi_R \| \leq C/R^2,
\]
by the dominated convergence theorem and letting \( R \to \infty \), we obtain
\[
\frac{1}{2} \int_{\mathbb{R}^d} |u(t)|^2 = - \int_{\mathbb{R}^{1+2d}} \langle a \cdot \nabla u, \nabla v u \rangle I_t - \int_{\mathbb{T} \times \mathbb{R}^d} F \cdot \nabla v u I_t + \sum_{i=1}^{m} \int_{\mathbb{R}^{1+2d}} f_i u I_t.
\]
By the uniform ellipticity (3.2) of \( a \), we have
\[
\int_{\mathbb{R}^{1+2d}} \langle a \cdot \nabla u, \nabla v u \rangle I_t \geq \kappa_0 \| \nabla v u I_t \|_{L^2}.
\]
By Hölder’s inequality and Young’s inequality, we have
\[
\left| \int_{\mathbb{R}^{1+2d}} F \cdot \nabla v u I_t \right| \leq \| F I \|_{L^2} \| \nabla v u I_t \|_{L^2} \leq \frac{\epsilon}{\kappa_0} \| \nabla v u I_t \|_{L^2}^2 + \frac{1}{\epsilon} \| F I \|_{L^2}^2,
\]
and for \( \frac{1}{r_1} + \frac{1}{r_2} = 1, \frac{1}{q_1} + \frac{1}{q_2} = 1 \) and any \( \epsilon > 0, \)
\[
\left| \int_{\mathbb{R}^{1+2d}} f_i u I_t \right| \leq \| u I_t \|_{L^q(\mathbb{R}^d_x)} \| f_i I_t \|_{L^p(\mathbb{R}^d_x)} \leq \| u I_t \|_{L^q(\mathbb{R}^d_x)}^\epsilon \| f_i I_t \|_{L^p(\mathbb{R}^d_x)} + \frac{1}{\epsilon} \| f I \|_{L^q(\mathbb{R}^d_x)}.
\]
Combining the above calculations, we obtain that for any \( \epsilon > 0, \)
\[
\| u I_t \| \leq C \| F I \|_{L^2} + \epsilon \sum_{i=1}^{m} \| u I_t \|_{L^q(\mathbb{R}^d_x)} + \frac{1}{\epsilon} \| f I \|_{L^q(\mathbb{R}^d_x)}.
\tag{3.22}
\]
On the other hand, we may write
\[
\partial_t u = \Delta_v u + v \cdot \nabla_x u + \text{div}_v((a - \mathbb{I}) \cdot \nabla_v u + F) + \sum_{i=1}^{m} f_i.
\]
By Duhamel’s formula, we have
\[
u(t) = \int_0^t P_{t-s} \text{div}_v((a - \mathbb{I}) \cdot \nabla_v u + F) ds + \sum_{i=1}^{m} \int_0^t P_{t-s} f_i ds =: I_0(t) + \sum_{i=1}^{m} I_i(t).
\]
Below we fix \( \beta \in (0, 1) \) and estimate each \( I_i(t) \) as following.
• For \( I_0(t) \), by (3.15) with \((β, \nu, \nu, p) = (-1, 2, 2, 2)\) and Lemma 2.3, we have
\[
\|I_0 I_t\|_{L^2_0(B_{2n})} \lesssim \|\text{div}_v ((a - I) \cdot \nabla v) + F I_t\|_{L^2_0(B_{2n})} \\
\lesssim \|((a - I) \cdot \nabla v + F I_t\|_{L^2} \\
\lesssim \kappa_1 \|\nabla v I_t\|_{L^2} + \|FI_t\|_{L^2}.
\]

• For \( I_i(t) \), let \( \beta_i > 0 \) be defined by
\[
\beta_i := \beta + a \cdot (\frac{1}{p_i} - \frac{1}{q_i}) < \beta + 2 - \frac{2}{p_i} < 3 - \frac{2}{q_i}.
\]

By (2.12) and (3.15) with \((β, \nu) = (0, 2)\), we have for \(i = 1, \cdots, m\),
\[
\|I_i I_t\|_{L^2(B_{2n})} \lesssim \|I_i I_t\|_{L^2(B_{2n})} \lesssim \|f_i I_t\|_{L^2(B_{p_i})} \lesssim \|f_i I_t\|_{L^2(B_{p_i})}.
\]

Combining the above calculations, we obtain
\[
\|u I_t\|_{L^2(B_{2n})} \lesssim \|\nabla v u I_t\|_{L^2} \|FI_t\|_{L^2} + \sum_{i=1}^m \|f_i I_t\|_{L^2(B_{p_i})},
\]
which, together with (3.22), yields that for any \( \varepsilon \in (0, 1) \) and \( \beta \in (0, 1) \),
\[
\|u I_t\|_{L^2} + \|u I_t\|_{L^2(B_{2n})} \leq C\|FI_t\|_{L^2} + \varepsilon \sum_{i=1}^m \|u I_t\|_{L^2(B_{p_i})} + C \varepsilon \sum_{i=1}^m \|f_i I_t\|_{L^2(B_{p_i})}.
\]  
(3.23)

Recalling \( \frac{1}{r_1} + \frac{1}{r_2} = 1 \) and \( \frac{1}{p_i} + \frac{1}{q_i} = 1 \), by (3.19), we have
\[
\left(\frac{2}{p_i} \right) \geq (1 - \frac{2}{p_i})1, \quad \beta_i := \frac{2}{p_i} \left(a \cdot (\frac{1}{p_i} - \frac{1}{q_i})\right) < 1.
\]

Thus, for \( \beta_i \in (\bar{\beta}, 1) \), by Lemma 3.7 and (2.3), we have for any \( \varepsilon \in (0, 1) \),
\[
\sum_{i=1}^m \|u I_t\|_{L^2(B_{p_i})} \leq \|FI_t\|_{L^2} + \varepsilon \sum_{i=1}^m \|u I_t\|_{L^2(B_{p_i})} + C \varepsilon \sum_{i=1}^m \|f_i I_t\|_{L^2(B_{p_i})}.
\]

Choosing \( \varepsilon \) small enough, we get
\[
\sum_{i=1}^m \|u I_t\|_{L^2(B_{p_i})} \lesssim C\|FI_t\|_{L^2} + \sum_{i=1}^m \|f_i I_t\|_{L^2(B_{p_i})}.
\]

Substituting it into (3.23), we conclude the proof. \( \Box \)

4. LOCAL BOUNDEDNESS OF WEAK SOLUTIONS

In this section we derive the local bounds of weak solutions for PDE (3.1) by considering two cases: \( L^p \)-integrable inhomogeneous \( f \) and distribution-valued inhomogeneous \( f \).

4.1. \( L^p \)-integrable inhomogeneous \( f \). We first establish the following local energy estimate.

**Lemma 4.1.** Let \( Q \subset \mathbb{R}^{1+2d} \) be a bounded open set and \( u \in \mathcal{V}_Q \cap L^\infty_Q \) a nonnegative weak subsolution of PDE (3.1) in \( Q \). Let \((q_1, p_1), (\nu_1, r_1) \in (2, \infty)^{1+2d}\) satisfy
\[
a \cdot \frac{1}{p_i} + \frac{2}{q_i} < 1, \quad \frac{1}{p_i} + \frac{1}{r_i} = \frac{1}{2}, \quad \frac{1}{q_i} + \frac{1}{\nu_1} = \frac{1}{2}.
\]  
(4.1)

Under (3.2), there is a constant \( C = C(Q, \kappa_0, \nu_1, r_1) > 0 \) such that for any nonnegative \( \eta \in C_c^\infty(Q) \) and \( t \in \mathbb{R} \),
\[
\|u \eta I_t\|_{L^2} \lesssim C \left(1 + \|q I_{\nu_1}(1, L^r_\nu)\| + \Xi_\eta\right)\|1_Q u I_t\|_{L^2(\nu_1, L^r_\nu)} + \|\eta f q \eta I_t\|_{L^2_\nu}^{1/2},
\]
where \( I_t = 1_{(-\infty, t]} \) and
\[
\Xi_\eta := \|\eta\|_{L^\infty} + \|v \cdot \nabla_\nu \eta\|_{L^2} + \|\nabla \eta\|_{L^\infty}.
\]  
(4.2)
Proof. By (3.6), we have for any $t \in \mathbb{R}$,

\[
\frac{1}{2} \int_{\mathbb{R}^{2d}} |(u\eta)(t)|^2 \leq \int_{\mathbb{R}^{1+2d}} u^2 \eta(\partial_t \eta - v \cdot \nabla \eta) d\mathcal{L}_t - \int_{\mathbb{R}^{1+2d}} \langle a \cdot \nabla_v u, \nabla_v (u\eta^2) \rangle d\mathcal{L}_t + \int_{\mathbb{R}^{1+2d}} (b \cdot \nabla_v) u\eta^2 d\mathcal{L}_t + \int_{\mathbb{R}^{1+2d}} f u\eta^2 d\mathcal{L}_t.
\]

Noting that by Young’s inequality,

\[
\langle a \cdot \nabla_v u, \nabla_v (u\eta^2) \rangle = \langle a \cdot \nabla_v u, \nabla_v u \rangle \eta^2 + 2\langle a \cdot \nabla_v u, \eta \nabla_v \eta \rangle u \geq \kappa_0 |\nabla_v u|^2 - \kappa_1 |\nabla_v \eta| |u \nabla_v \eta| \geq \frac{\kappa_0}{2} |\eta \nabla_v u|^2 - \frac{\kappa_1}{2\kappa_0} |u \nabla_v \eta|^2,
\]

we have

\[
\int_{\mathcal{I}_t \times \mathbb{R}^d} \langle a \cdot \nabla_v u, \nabla_v (u\eta^2) \rangle \geq \frac{\kappa_0}{2} \|\eta \nabla_v u\|_{L^2}^2 - \frac{\kappa_1}{2\kappa_0} \|u \nabla_v \eta\|_{L^2}^2.
\]

By (4.1), Hölder’s inequality and Young’s inequality, we have

\[
\int_{\mathbb{R}^{1+2d}} (b \cdot \nabla_v) u\eta^2 d\mathcal{L}_t \leq \|\eta \nabla_v u\|_{L^2} \|1_Q b\|_1 \|1_Q u\|_{L^{q_1}(L^{p_1})} \|1_Q u\|_{L^{q_1}(L^{p_1})} \leq \frac{\kappa_0}{2} \|\eta \nabla_v u\|_{L^2}^2 + \frac{\kappa_1}{\kappa_0} \|1_Q b\|_1 \|1_Q u\|_{L^{q_1}(L^{p_1})}^2.
\]

Combining the above estimates, we obtain that for any $t \in \mathbb{R}$,

\[
\frac{1}{2} \|(u\eta)(t)|_{L^2}^2 + \frac{\kappa_0}{2} \|\nabla_v u\|_{L^2}^2 \leq \left(\|\partial_t \eta\|_{\infty} + \|v \cdot \nabla \eta\|_{\infty} + \frac{\kappa_1}{\kappa_0} \|\nabla_v \eta\|_{L^2}^2\right) \|1_Q u\|_{L^2}^2 + \frac{\kappa_1}{\kappa_0} \|1_Q b\|_1 \|1_Q u\|_{L^{q_1}(L^{p_1})}^2 + \|f \eta, u\eta\|_{L^1}^2,
\]

which in turn implies the desired estimate by noting that for some $C = C(Q, \nu_1, r_1) > 0$,

\[
\|1_Q u\|_{L^2}^2 \leq C \|1_Q u\|_{L^{q_1}(L^{p_1})}^2.
\]

The proof is complete.

Recalling $Q_r$ being defined by (3.4), we make the following assumption about the drift $b$:

\textbf{(H$_b$)} Suppose that $\|1_Q b\|_{L^{q_1}(L^{p_1})} \leq \kappa_2$ for some $(q_1, p_1) \in (2, \infty)^{1+2d}$ with

\[
a \cdot \frac{1}{p_1} + \frac{2}{q_1} < 1, \quad \frac{1}{p_1} < \left(\frac{1}{2} - \frac{1}{q_1}\right)1.
\]

\textbf{Remark 4.2.} The second condition comes from the first condition in (3.16). Note that if $p_1 = (p_1, \cdots, p_1) \in (2, \infty)^{2d}$, then

\[
a \cdot \frac{1}{p_1} + \frac{2}{q_1} < 1 \Rightarrow \frac{1}{p_1} < \left(\frac{1}{2} - \frac{1}{q_1}\right)1.
\]

Now we can show the following main result of this subsection.

\textbf{Theorem 4.3.} Under (3.2) and (H$_b$), for any $(q_0, p_0) \in (1, \infty)^{1+2d}$ with

\[
a \cdot \frac{1}{p_0} + \frac{2}{q_0} < 2, \quad \frac{1}{p_0} < (1 - \frac{1}{q_0})1,
\]

there is a constant $C = C(\kappa_i, q_i, p_i) > 0$ such that for any weak sub-solution $u \in \mathcal{V}_{Q_2} \cap L^\infty_{Q_2}$,

\[
\|1_Q u\|_{L^\infty} + \|1_Q \nabla_v u\|_{L^{q_0}(L^{p_0})} \leq C \|1_Q u\|_{L^{q_1}(L^{p_1})} + \|1_Q f\|_{L^{q_0}(L^{p_0})}, \quad \forall |t| \leq 1.
\]

\textbf{Proof.} Let $u \in \mathcal{V}_{Q_2} \cap L^\infty_{Q_2}$ be a weak sub-solution and for fixed $t \in [-1, 1]$,

\[
\tilde{Q}_t^+ := Q_r \cap (-\infty, t) \times \mathbb{R}^{2d}.
\]

To show the local upper bound estimate in (4.5), by Theorem 2.10, it suffices to show that

\[u \in DG_{\tilde{Q}_t, p_0}(\tilde{Q}_t^+) \text{ with } A \|1_Q f\|_{L^{q_0}(L^{p_0})},\]

where $A_0$ is an open index subset defined by (see (3.16))

\[
\mathcal{S}_0 := \left\{(\nu, r) \in (2, \infty)^{1+2d} : \frac{2}{p} > (1 - \frac{2}{p})1, \ a \cdot \left(\frac{1}{p} - \frac{1}{2}\right) + \frac{2}{p} > 0\right\}.
\]
More precisely, we want to show that for any \((\nu, r) \in \mathcal{F}_0\), there is a constant \(C > 0\) only depending on \(\kappa_0, \kappa_1, \nu, r\) and \(q_\nu, p_\nu, i = 0, 1\) such that for any \(1 \leq \tau < \sigma \leq 2\) and \(|t| \leq 1, \kappa \geq 0\),
\[
(\sigma - \tau)^2 \|1_{Q_\nu} (u - \kappa)^+ \mathcal{I}_t \|_{L_\nu^p (L_\nu^q)} \lesssim C \sum_{i=0,1} \|1_{Q_\nu} (u - \kappa)^+ \mathcal{I}_t \|_{L_{\nu_i}^{0} (L_{\nu_i}^\infty)} + \|1_{Q_\nu} f \mathcal{I}_t \|_{L_{\nu_i}^{0} (L_{\nu_i}^\infty)} \|1_{(u > \kappa) \cap Q_\sigma} \mathcal{I}_t \|_{L_{\nu_i}^{0} (L_{\nu_i}^\infty)},
\]
where for \(i = 0, 1\), \((\nu_i, r_i) \in \mathcal{F}_0\) is determined by \((q_i, p_i)\).

We follow the idea of [16] and divide the proof into three steps.

**Step 1** Let \(1 \leq \tau < \sigma \leq 2\) and \(\tau := \frac{t - \sigma}{2}\). Let \(\eta_0 \in C_0^\infty(Q_\nu; [0, 1])\) and \(\eta_1 \in C_0^\infty(Q_\sigma; [0, 1])\) with \(\eta_0|_{Q_\nu} \equiv 1\) and \(\eta_1|_{Q_\sigma} \equiv 1\) and satisfy that for some universal constant \(C > 0\),
\[
\Xi_\eta_i = \|\partial_\nu \eta_i \|_{L_\nu^\infty}^{1/2} + \|v \cdot \nabla \eta_i \|_{L_\nu^\infty}^{1/2} + \|\nabla \nu \eta_i \|_{L_\nu^\infty} \lesssim C(\sigma - \tau)^{-1}, \quad i = 0, 1.
\]
Since by Lemma 3.5, \((u - \kappa)^+\) is a nonnegative weak sub-solution of PDE (3.1) in \(Q_\nu\), without loss of generality, we may assume \(u\) itself being a nonnegative weak sub-solution of PDE (3.1), and prove (4.7) for \(u\) in place of \((u - \kappa)^+\). Note that
\[
\partial_t (u \eta_0) \leq \text{div} v \left( a \cdot \nabla v(u \eta_0) \right) + v \cdot \nabla_x (u \eta_0) - \text{div} v \left( a \cdot \nabla v \eta_0 \right) u - \left( a \cdot \nabla_x u, \nabla v \eta_0 \right) + \left( b \cdot \nabla_x u, \nabla v \eta_0 \right) u + f_1 \mathbb{1}_{\{u > 0\}} \eta_0
\]
where
\[
F := -(a \cdot \nabla v \eta_0) u, \quad f_1 := (\partial_t \eta_0 - v \cdot \nabla x \eta_0) u,
\]
and
\[
f_2 := f \mathbb{1}_{\{u > 0\}} \eta_0, \quad f_3 := (b \eta_0 - a \cdot \nabla v \eta_0) \cdot \nabla v u.
\]
Fix \(t \in [-1, 1]\). By supp(\(\eta_0\)) \(\subset Q_\sigma\), it is easy to see that
\[
\|F \mathcal{I}_t \|_{L_2} = \|\left( a \cdot \nabla v \eta_0 \right) u \mathcal{I}_t \|_{L_2} \leq \kappa_1 \Xi_\eta_0 \|1_{Q_\nu} u \mathcal{I}_t \|_{L_2},
\]
and
\[
\|f_1 \mathcal{I}_t \|_{L_2} = \|\left( \partial_t \eta_0 - v \cdot \nabla x \eta_0 \right) u \mathcal{I}_t \|_{L_2} \lesssim \Xi_\eta_0^2 \|1_{Q_\nu} u \mathcal{I}_t \|_{L_2}.
\]
For \(f_2\), let \((\nu_0, r_0) \in (2, \infty)^{1+2d}\) and \((\bar{q}_0, \bar{p}_0) \in (1, 2)^{1+2d}\) be defined by
\[
\frac{1}{\nu_0} + \frac{2}{r_0} = 1, \quad \frac{1}{\bar{q}_0} + \frac{2}{\bar{p}_0} = 1, \quad \frac{1}{\bar{q}_0} = \frac{1}{\nu_0} + \frac{1}{r_0}, \quad \frac{1}{\bar{p}_0} = \frac{1}{\nu_0} + \frac{1}{r_0}.
\]
By Hölder’s inequality, we have
\[
\|f_2 \mathcal{I}_t \|_{L_{\bar{q}_0}^{\bar{p}_0} (L_\nu^\infty)} = \|f \mathbb{1}_{\{u > 0\}} \eta_0 \mathcal{I}_t \|_{L_{\bar{q}_0}^{\bar{p}_0} (L_\nu^\infty)} \lesssim \|f \eta_0 \mathcal{I}_t \|_{L_{\bar{q}_0}^{\bar{p}_0} (L_\nu^\infty)} \|1_{(u \mathcal{I}_t > 0) \cap Q_\sigma} \|_{L_{\nu_i}^{0} (L_{\nu_i}^\infty)}.
\]
For \(f_3\), let \((\bar{q}_1, \bar{p}_1) \in (1, 2)^{1+2d}\) be defined by
\[
\frac{1}{\bar{q}_1} + \frac{1}{\bar{p}_1} = \frac{1}{\bar{q}_0}, \quad \frac{1}{\bar{q}_1} \frac{1}{\bar{p}_1} = \frac{1}{\bar{q}_0} \frac{1}{\bar{p}_0}.
\]
Since \(\eta_1 = 1\) on the support of \(\eta_0\), by Hölder’s inequality and Lemma 4.1, we have
\[
\|f_3 \mathcal{I}_t \|_{L_{\bar{q}_1}^{\bar{p}_1} (L_\nu^\infty)} = \|f \mathcal{I}_t \|_{L_{\bar{q}_1}^{\bar{p}_1} (L_\nu^\infty)} \lesssim \|f \mathcal{I}_t \|_{L_{\nu_i}^{0} (L_{\nu_i}^\infty)} \|1_{(u \mathcal{I}_t > 0) \cap Q_\sigma} \|_{L_{\nu_i}^{0} (L_{\nu_i}^\infty)^{1/2}},
\]
where \((\nu_1, r_1) \in (2, \infty)^{1+2d}\) is defined by (4.1). Moreover, by (4.11) and Hölder’s inequality,
\[
\|f \mathcal{I}_t \|_{L_{\nu_i}^{0} (L_{\nu_i}^\infty)} \lesssim \|f \mathcal{I}_t \|_{L_{\nu_i}^{0} (L_{\nu_i}^\infty)} \|1_{(u \mathcal{I}_t > 0) \cap Q_\sigma} \|_{L_{\nu_i}^{0} (L_{\nu_i}^\infty)} \|u \mathcal{I}_t \|_{L_{\nu_i}^{0} (L_{\nu_i}^\infty)}.
\]
Substituting this into (4.14) and by Young’s inequality, we get
\[
\|f_3 \mathcal{I}_t \|_{L_{\bar{q}_1}^{\bar{p}_1} (L_\nu^\infty)} \lesssim \left( \kappa_2 + \kappa_1 \Xi_\eta_0 \right) \|f \mathcal{I}_t \|_{L_{\nu_i}^{0} (L_{\nu_i}^\infty)} + \|f \mathcal{I}_t \|_{L_{\nu_i}^{0} (L_{\nu_i}^\infty)} \|1_{(u \mathcal{I}_t > 0) \cap Q_\sigma} \|_{L_{\nu_i}^{0} (L_{\nu_i}^\infty)}.
\]
Note that by (4.4) and (4.3), for \( i = 0, 1, (\nu_i, r_i) \in \mathcal{S}_0, \) i.e.,
\[
\frac{2}{r_i} > (1 - \frac{2}{p_i})1, \quad a \cdot (\frac{1}{p_i} - \frac{1}{2}) + \frac{2}{q_i} > 0.
\]

(Step 2) For \( \beta \in (0, 1) \) and \(|t| \leq 1\), we define
\[
\mathcal{A}_t^\beta := \|w_i^t\|_\nu + \|w_i^t\|_{L_t^p(B_{r_i}^\beta)}.
\]

Let \((\tilde{q}_i, \tilde{p}_i) \in (1, 2)^{1+2d}, i = 0, 1\) be defined as above. By (4.4) and (4.3), one sees that
\[
\frac{1}{p_i} < (\frac{3}{2} - \frac{1}{\tilde{q}_i})1, \quad a \cdot (\frac{1}{\tilde{p}_i} - \frac{1}{2}) + \frac{2}{\tilde{q}_i} < 2, \quad i = 0, 1.
\]
Thus, by Theorem 3.9, there is a unique weak solution \( u \) to the following PDE:
\[
\partial_t w = \text{div}_x(a \cdot \nabla_x w) + v \cdot \nabla_x w + \text{div}_z F + f_1 + f_2 + f_3, \quad w|_{t=0} = 0 \tag{4.16}
\]
so that for any \( \beta \in (0, 1) \) and \(|t| \leq 1\),
\[
\mathcal{A}_t^\beta \lesssim \|F_i^t\|_{L_x^2} + \|f_1 i^t\|_{L_x^2} + \|f_2 i^t\|_{L_x^1 r o} + \|f_3 i^t\|_{L_x^1 r o}.
\]

Furthermore, by (4.10), (4.11), (4.13) and (4.15), we get
\[
\mathcal{A}_t^\beta \lesssim (1 + \xi_\rho^2)\|\eta_{i^t}\|_{L_x^2} + \|f\eta_{i^t}\|_{L_x^1 r o} \|\mathbf{1}_{(u_i^t > 0) \cap Q_r^o} i^t_{1} \|_{L_x^1 r o} + \|\mathbf{1}_{Q_r^o} i^t_{1} \|_{L_x^1 r o} i^t_{1} \|_{L_x^1 r o} + \|\mathbf{1}_{(u_i^t > 0) \cap Q_r^o} i^t_{1} \|_{L_x^1 r o} i^t_{1} \|_{L_x^1 r o} (4.17)
\]
\[
\lesssim \sum_{i=0,1} \|\mathbf{1}_{Q_r^o} i^t_{1} \|_{L_x^1 r o} i^t_{1} \|_{L_x^1 r o} + \|f\eta_{i^t}\|_{L_x^1 r o} \|\mathbf{1}_{(u_i^t > 0) \cap Q_r^o} i^t_{1} \|_{L_x^1 r o} i^t_{1} \|_{L_x^1 r o} (4.18)
\]
where the implicit constant is independent of \( \tau, \sigma \) and \( t \), but may depend on \( \beta \).

(Step 3) By (4.9) and (4.16), one sees that \( \bar{w} := w_0 - w \in \mathcal{V} \) is a weak sub-solution of
\[
\partial_t \bar{w} \leq \text{div}_x(a \cdot \nabla_x \bar{w}) + v \cdot \nabla_x \bar{w}, \quad \bar{w}|_{t=0} = 0.
\]
By Lemma 3.5, \( \bar{w}^+ \) is still a weak sub-solution of the above equation. By Lemma 4.1 with \( b = f = 0, \eta = \chi_R, \) where \( \chi_R \) is the same as in (3.21), and letting \( R \to \infty \), it is easy to see that
\[
\bar{w}^+ \equiv 0 \Rightarrow 0 \leq w_0 \leq w.
\]
For given \((\nu, r) \in \mathcal{S}_0, \) let
\[
\bar{\beta} := \frac{\nu}{2}(a \cdot (\frac{1}{r} - \frac{1}{2})).
\]

By Lemma 3.7 with \( q = 2 \) and \( \beta \in (\bar{\beta}, 1) \), we have
\[
\|\mathbf{1}_{Q_r^o} i^t_{1} \|_{L_x^1 r o} \lesssim \|u_\nu_{i^t}\|_{L_x^1 r o} \lesssim \|w_i^t\|_{L_x^1 r o} \lesssim \mathcal{A}_t^\beta,
\]
which combining with (4.18) yields (4.7). Thus, by Theorem 2.10,
\[
\|\mathbf{1}_{Q_{2r}^o} u^+ i^t\|_{L_x^1} \lesssim \|\mathbf{1}_{Q_{r}^o} u^+ i^t\|_{L_x^1} + \|\mathbf{1}_{Q_{2r}^o} f i^t\|_{L_x^1 r o}.
\]
Finally, let \( \chi \in C_0^\infty(Q_{3/2}) \) be nonnegative and \( \chi = 1 \) on \( Q_1 \). By (4.5) with \( \eta = \chi \) and the above estimate, we also have
\[
\|u^+ \chi i^t\|_{L_x^1} \lesssim \|\mathbf{1}_{Q_{3r}^o} u^+ i^t\|_{L_x^1 r o} + \|\mathbf{1}_{Q_{3r}^o} u^+ \chi i^t\|_{L_x^1 r o}^{1/2}
\]
\[
\lesssim \|\mathbf{1}_{Q_{3r}^o} u^+ i^t\|_{L_x^1} + \|\mathbf{1}_{Q_{3r}^o} u^+ i^t\|_{L_x^1}^{1/2} + \|\mathbf{1}_{Q_{3r}^o} u^+ i^t\|_{L_x^1}^{1/2}
\]
\[
\lesssim \|\mathbf{1}_{Q_{2r}^o} u^+ i^t\|_{L_x^1} + \|\mathbf{1}_{Q_{2r}^o} f i^t\|_{L_x^1 r o}.
\]
where we have used that \( \|\mathbf{1}_{Q_{2r}^o} f i^t\|_{L_x^1 r o} \lesssim \|\mathbf{1}_{Q_{2r}^o} f i^t\|_{L_x^1 r o} \). The proof is complete. \( \square \)
4.2. Distribution-valued inhomogeneous $f$. In this subsection, we consider the case of $f$ being a distribution and suppose that

$$\langle H_0 \rangle \quad f = g + \text{div}_x G,$$

where $g \in L^{q_0}_t (B^{-\alpha_0}_{p_0, a}) \cap L^1$ for some $\alpha_0 \in (0, 1)$ and $(q_0, p_0) \in (1, \infty)^{1+2d}$ with

$$1 - \alpha_0 < \frac{a}{q_0}, \quad \frac{a}{q_0} + a \cdot \frac{1}{p_0} < 2 - 2\alpha_0,$$  \hspace{1cm} (4.19)

and $G \in L^{q_2}_t (L^{P_2}_x)$ for some $q_2 \in (2, 4)$ and $p_2 \in (2, \infty)^{2d}$ with

$$\frac{a}{q_2} + a \cdot \frac{1}{p_2} < 1.$$  \hspace{1cm} (4.20)

$$\langle H_0 \rangle \quad \| \mathbb{1}_{Q_2} b \|_{L^m_1 (L^b_1)} \leq \kappa_2 \text{ for some } q_1 \in (2, 4) \text{ and } p_1 \in (2, \infty)^2 \text{ with }$$

$$\frac{a}{q_1} + a \cdot \frac{1}{p_1} < 1.$$  \hspace{1cm} (4.21)

**Remark 4.4.** Condition $\langle H_0 \rangle$ does not imply $\langle H_0 \rangle$ due to the extra requirement $\frac{1}{p_1} + \frac{1}{q_1} < \frac{1}{2}$ in (4.3). In other words, when $q_1 \in (2, 4)$, we can drop the assumption $\frac{1}{p_1} + \frac{1}{q_1} < \frac{1}{2}$ in $\langle H_0 \rangle$ in the following theorem because we shall use Lemma 3.7 with $q = 4$ and $\beta \in (0, \frac{1}{2})$ (see (3.17)).

**Theorem 4.5.** Under (3.2), $\langle H_0 \rangle$ and $\langle H_0 \rangle$, there is a constant $C > 0$ only depending on $\alpha_0, \kappa_i, q_i, p_i$ such that for any weak solution $u \in \mathcal{V}_{Q_2} \cap L^\infty_{Q_2}$ of PDE (3.1) and $|t| \leq 1$,

$$\| \mathbb{1}_{Q_2} u \mathcal{I}_t \|_{L^\infty_t} + \| \mathbb{1}_{Q_2} \mathcal{V}_v u \mathcal{I}_t \|_{L^2_t} \leq C \left( \| \mathbb{1}_{Q_2} u \mathcal{I}_t \|_{L^2_t} + \| g \mathcal{V}_2 \mathcal{I}_t \|_{L^{q_0}_t (B^{-\alpha_0}_{p_0, a})} + \| G \mathcal{V}_2 \mathcal{I}_t \|_{L^{q_2}_t (L^{P_2}_x)} \right),$$  \hspace{1cm} (4.22)

for any $\beta \in (0, 1)$,

$$\| u \mathcal{I}_t \|_{L^2_t (B^{2 \beta/3}_{2, 2})} \leq C \left( \| \mathbb{1}_{Q_2} u \mathcal{I}_t \|_{L^2_t} + \| g \mathcal{V}_2 \mathcal{I}_t \|_{L^{q_0}_t (B^{-\alpha_0}_{p_0, a})} + \| G \mathcal{V}_2 \mathcal{I}_t \|_{L^{q_2}_t (L^{P_2}_x)} \right),$$  \hspace{1cm} (4.23)

where $\chi_1$ and $\chi_2$ are defined by (3.21).

**Proof.** Let $u \in \mathcal{V}_{Q_2} \cap L^\infty_{Q_2}$ be a weak solution of PDE (3.1). For proving (4.22), by Theorem 2.10, it suffices to show that for fixed $|t| \in [-1, 1]$,

$$u \in \mathcal{D}^{n_0} \left( \tilde{Q}^t \right)$$

with $\mathcal{A} = \| g \mathcal{V}_2 \mathcal{I}_t \|_{L^{q_0}_t (B^{-\alpha_0}_{p_0, a})} + \| G \mathcal{V}_2 \mathcal{I}_t \|_{L^{q_2}_t (L^{P_2}_x)}$, where $\tilde{Q}^t$ is defined by (4.6) and $\mathcal{A}$ is an open index subset defined by

$$\mathcal{A}_1 := \left\{ (\nu, \tau) \in (4, \infty) \times (2, \infty)^{2d} : a \cdot \left( \frac{1}{P} - \frac{1}{2} \right) + \frac{2}{P} > 0 \right\}. \hspace{1cm} (4.24)$$

More precisely, we want to show that for any $(\nu, \tau) \in \mathcal{A}_1$, there is a constant $C > 0$ only depending on $\nu, \tau$ and $\alpha_0, \kappa_i, q_i, p_i, i = 0, 1$, such that for any $1 \leq \tau < \sigma \leq 2$ and $|t| \leq 1, \kappa \geq 0$,

$$(\sigma - \tau) \| \mathbb{1}_{Q_2} w \mathcal{I}_t \|_{L^1_{t, \tau} (L^1_x)} \leq C \left( \| \mathbb{1}_{Q_2} w \mathcal{I}_t \|_{L^{2d}_{t, \tau} (L^2_x)} + A \left( \| \mathbb{1}_{Q_2} \mathcal{I}_t \|_{L^{q_0}_t (L^{q_0}_x)} + \| \mathbb{1}_{Q_2} \mathcal{I}_t \|_{L^{q_2}_t (L^{P_2}_x)} \right) \right),$$  \hspace{1cm} (4.25)

where $(\nu_i, \tau_i) \in \mathcal{A}_1$ are determined by $(q_i, p_i), i = 0, 1$, and

$$w := (u - \kappa)^+.$$

We divide the proof into four steps.

**(STEP 1)** Let $1 \leq \tau < \sigma \leq 2$ and $\eta \in C^\infty_c (Q_\sigma; [0, 1])$ with $\eta|_{Q_\sigma} \equiv 1$ and satisfy that for some universal constant $C > 0$

$$\Xi_\eta := 1 + \| \partial_t \eta \|_{L^2_t} + \| v \cdot \mathcal{V}_x \eta \|_{L^2_t} + \| \mathcal{V}_x \eta \|_{L^\infty_t} \leq C (\sigma - \tau)^{-1}. \hspace{1cm} (4.26)$$

Note that by (ii) of Lemma 3.5, $w^2$ is a weak solution of PDE

$$\partial_t w^2 = \text{div}_v (a \cdot \mathcal{V}_x w^2) + v \cdot \mathcal{V}_x w^2 + b \cdot \mathcal{V}_x w^2 - 2tr (a \cdot \mathcal{V}_v w \otimes \mathcal{V}_v w) + 2fw.$$

Let

$$\bar{w} := uw.$$
By definition, one sees that $\tilde{w}^2$ is a nonnegative weak solution of the following PDE
\begin{equation}
\partial_t \tilde{w}^2 = \text{div}_v(a \cdot \nabla_v \tilde{w}^2) + v \cdot \nabla_x \tilde{w}^2 + h, \tag{4.27}
\end{equation}
where, thanks to $f = g + \text{div}_v G$, $h$ is given by
\begin{align*}
h := & -\text{div}_v(a \cdot \nabla_v \eta^2 \tilde{w}^2) - \text{tr}(a \cdot \nabla_v \tilde{w}^2 \otimes \nabla_v \eta^2) - 2\text{tr}(a \cdot \nabla_v w \otimes \nabla_v \eta^2) \\
& + (\partial_t \eta^2 - v \cdot \nabla_x \eta^2) \tilde{w}^2 + (b \cdot \nabla_v \tilde{w}^2) \eta^2 + 2(g + \text{div}_v G) \eta^2.
\end{align*}
Integrating both sides of (4.27) over $\mathbb{R}^d$, by the integration by parts, we obtain
\begin{equation}
\partial_t \int \tilde{w}^2 = \int h = \int f_1 + \int f_2 + \int f_3 + 2\langle g\eta, \tilde{w} \rangle, \tag{4.28}
\end{equation}
where $\langle \cdot, \cdot \rangle$ stands for the duality between $L^1_2$ and $L^\infty_2$, and for $b := b\eta - 2a \cdot \nabla \eta$, \begin{align*}
f_1 := & -2\text{tr}(a \cdot \nabla_v w \otimes \nabla_v \eta)^2, \\
f_2 := & (\partial_t \eta^2 - v \cdot \nabla_x \eta^2) \tilde{w}^2 + (b \cdot \nabla_v \tilde{w}^2) \eta, \\
f_3 := & -2G \cdot (\nabla_v \eta^2 \tilde{w}^2 + \nabla \eta^2 \tilde{w}).
\end{align*}
For $f_1$, by the uniform ellipticity of $a$, we have
\begin{equation}
\int f_1 = -\int \text{tr}(a \cdot \nabla_v w \otimes \nabla_v \eta)^2 \eta \leq -\kappa_0 \int |\nabla_v \eta|^2 = -\kappa_0 \int |\nabla_v \eta|^2. \nonumber
\end{equation}
For $f_2$, since $\eta(t, z) \in [0, 1]$ has support in $Q_\sigma$, by Hölder and Young’s inequalities, we have
\begin{align*}
|f_2| & \leq 2|\langle \partial_t \eta, v \cdot \nabla_x \eta \rangle \rangle| + 2 |\langle b \cdot \nabla_v \eta \rangle \rangle| \\
& \leq \Xi^2_\eta \|1_{Q_\sigma} w\|_{L^2_2}^2 + \frac{\kappa_0}{\kappa_0} \|\nabla_v \eta\|_{L^2_2}^2 + \frac{2}{\kappa_0} |\langle b \cdot \nabla_v \eta \rangle \rangle|.
\end{align*}
For $f_3$, by $\nabla_v w = \nabla_v w 1_{w \neq \emptyset}$ (see [15, Lemma 7.6]) and Hölder’s inequality, we have
\begin{equation}
\|f_3\| \leq 2 \|G 1_{w \neq \emptyset} \eta\|_{L^1_2} \left( \|\nabla_v \eta\|_{L^2_2}^2 + 2 \|w \nabla_v \eta\|_{L^1_2}^2 + G 1_{w \neq \emptyset} \eta\|_{L^1_2}^2 + 2 \|\langle g\eta, \tilde{w} \rangle\| \right), \tag{4.29}
\end{equation}
Integrating both sides of (4.28) from $-\infty$ to $t$ and combining the above calculations, we obtain
\begin{align*}
\int \tilde{w}(t) \|_{L^2_2}^2 & \leq \int_{-\infty}^t \left(-\frac{\kappa_0}{\kappa_0} \|\nabla_v \eta\|_{L^2_2}^2 + C \Xi^2_\eta \|1_{Q_\sigma} w\|_{L^2_2}^2 + \frac{\kappa_0}{\kappa_0} \|\nabla_v \eta\|_{L^2_2}^2 + \frac{\kappa_0}{\kappa_0} \|G 1_{w \neq \emptyset} \eta\|_{L^2_2}^2 + 2 \|\langle g\eta, \tilde{w} \rangle\| \right) \text{d}s,
\end{align*}
which yields by taking supremum in $t$ that for fixed $t \in [-1, 1]$,
\begin{equation}
\int_{-\infty}^t \|\tilde{w}\|_{L^2_2}^2 + \|\nabla_v \eta \|_{L^2_2}^2 \|_{L^2_t}^2 + \|G 1_{w \neq \emptyset} \|_{L^2_t}^2 + 2 \|\langle g\eta, \tilde{w} \rangle\|_{L^1_t} \leq \Xi^2_\eta \|1_{Q_\sigma} w\|_{L^2_2}^2 + \|\nabla_v \eta\|_{L^2_2}^2 + \|G 1_{w \neq \emptyset} \eta\|_{L^2_2} + \|\langle g\eta, \tilde{w} \rangle\|_{L^1_2} \tag{4.30}
\end{equation}
(Step 2) Note that in the distributional sense,
\begin{align*}
\partial_t \tilde{w}^2 & = \Delta_v \tilde{w}^2 + v \cdot \nabla_x \tilde{w}^2 + \text{div}_v F + f_1 + f_2 + f_3 + 2 g \eta \tilde{w},
\end{align*}
where $f_1, f_2, f_3$ are defined as above and
\begin{align*}
F := (a - 1) \cdot \nabla_v \tilde{w}^2 - a \cdot \nabla_v \eta^2 \tilde{w}^2 + G \eta \tilde{w}^2.
\end{align*}
By Duhamel’s formula, we have
\begin{align*}
\tilde{w}(t)^2 &= \int_{-\infty}^t P_{t-s} \left(\text{div}_v F + f_1 + f_2 + f_3 + 2 g \eta \tilde{w} \right) \text{d}s \\
& =: J_0(t) + J_1(t) + J_2(t) + J_3(t) + J_4(t).
\end{align*}
We estimate each term by Lemma 3.6 as following.
For $I_0(t)$, by (3.15) with $\beta \in (0, 1)$ and $(\nu, q, \gamma, p) = (2, 2, -1, 1)$,
\[
\|I_0\|_{L_t^2(B_{1}^{2,q})} \lesssim \|\text{div}_u((\alpha - I) \cdot \nabla_v w^2 - a \cdot \nabla_v \eta^2 w^2 + G w \eta^2)I_t\|_{L_t^2(B_{1}^{2,q})} \\
\lesssim \|((\alpha - I) \cdot \nabla_v w^2 - a \cdot \nabla_v \eta^2 w^2 + G w \eta^2)I_t\|_{L_t^2(B_{1}^{2,q})} \\
\lesssim \|\nabla_v \eta \bar{w}I_t\|_{L_t^2(L_2^2)} + \|wu \nabla \eta \bar{w}I_t\|_{L_t^2(L_2^2)} + \|G \bar{w} \eta \bar{w}I_t\|_{L_t^2(L_2^2)} \\
\lesssim \|\nabla_v \eta \bar{w}I_t\|_{L_t^2(L_2^2)} + \|wu \nabla \eta \bar{w}I_t\|_{L_t^2(L_2^2)} + \|\bar{w}I_t\|_{L_t^2(L_2^2)}. 
\]

For $I_1(t)$, by (3.15) with $\beta \in (0, 1)$ and $(\nu, q, \gamma, p) = (2, 1, 0, 1)$,
\[
\|I_1I_t\|_{L_t^2(B_{1}^{2,q})} \lesssim \|tr(a \cdot \nabla_v w \otimes \nabla_v \eta)\|_{L_t^2(B_{1}^{2,q})} \\
\lesssim \|tr(a \cdot \nabla_v w \otimes \nabla_v \eta)\|_{L_t^2(B_{1}^{2,q})} \lesssim \|\nabla_v \eta \bar{w}I_t\|_{L_t^2(L_2^2)}. 
\]

For $I_2(t)$, by (3.15) with $\beta \in (0, 1)$ and $(\nu, q, \gamma, p) = (2, 1, 0, 1)$,
\[
\|I_2I_t\|_{L_t^2(B_{1}^{2,q})} \lesssim \|((\partial_t \eta^2 - v \cdot \nabla_x \eta^2)w^2 - (b \cdot \nabla_v w^2)\eta)I_t\|_{L_t^2} \\
\lesssim \|\partial_t \eta \bar{w}I_t\|_{L_t^2(L_2^2)} + \|bw \eta \bar{w}I_t\|_{L_t^2(L_2^2)}. 
\]

For $I_3(t)$, by (3.15) with $\beta \in (0, 1)$ and $(\nu, q, \gamma, p) = (2, 1, 0, 1)$, as in (4.29),
\[
\|I_3I_t\|_{L_t^2(B_{1}^{2,q})} \lesssim \|G \cdot (\nabla_v w \eta^2 + \nabla \eta^2 w)I_t\|_{L_t^2} \\
\lesssim \|G \cdot (\nabla_v w \eta^2 + \nabla \eta^2 w)I_t\|_{L_t^2(L_2^2)} + \|G I_{w \neq 0} \eta I_t\|_{L_t^0(L_2^2)} + \|\bar{w}I_t\|_{L_t^0(L_2^2)}. 
\]

For $I_4(t)$, by (3.15) with $\beta \in (0, 1)$ and $(\nu, q, \gamma, p) = (2, \frac{2}{2 - \alpha_0}, \alpha_0, 1)$,
\[
\|I_4I_t\|_{L_t^2(B_{1}^{2,q})} \lesssim \|\eta \bar{w}I_t\|_{L_t^2(B_{1}^{2,q})}. 
\]

Combining the above calculations, we obtain that for $q = \frac{2}{2 - \alpha_0}$ and any $\beta \in (0, 1)$,
\[
\|\bar{w}I_t\|_{L_t^2(B_{1}^{2,q})} \lesssim \|\bar{w}I_t\|_{L_t^2(L_2^2)} + \|\nabla_v \eta \bar{w}I_t\|_{L_t^2(L_2^2)} + \|G I_{w \neq 0} \eta I_t\|_{L_t^2(L_2^2)} + \|\bar{w}I_t\|_{L_t^2(L_2^2)} + \|\eta \bar{w}I_t\|_{L_t^2(B_{1}^{2,q})}. 
\]

Note that by Lemma 2.6,
\[
\|\bar{w}I_t\|_{L_t^2(B_{1}^{2,q})} \lesssim \|\bar{w}I_t\|_{L_t^2(B_{1}^{2,q})}. 
\]

If we define
\[
\mathcal{A}_t^\beta := \|\bar{w}I_t\|_{L_t^2(B_{1}^{2,q})} + \|\bar{w}I_t\|_{L_t^2(L_2^2)} + \|\nabla_v \eta \bar{w}I_t\|_{L_t^2(L_2^2)} + \|G I_{w \neq 0} \eta I_t\|_{L_t^2(L_2^2)} + \|\bar{w}I_t\|_{L_t^2(L_2^2)} + \|\eta \bar{w}I_t\|_{L_t^2(B_{1}^{2,q})}, 
\]

then by (4.32), (4.31) and (4.30), we get for $q = \frac{2}{2 - \alpha_0}$ and any $\beta \in (0, 1)$, $|t| < 1$,
\[
\mathcal{A}_t^\beta \lesssim \Xi_\eta^2 \|1_{\mathbb{Q}_w} \bar{w}I_t\|_{L_t^2(L_2^2)} + \|\bar{w}I_t\|_{L_t^2(L_2^2)} + \|G I_{w \neq 0} \eta I_t\|_{L_t^2(L_2^2)} + \|\eta \bar{w}I_t\|_{L_t^2(B_{1}^{2,q})}.
\]

(Step 3) In this step we estimate the last two terms by Lemma 2.5 and Hölder’s inequality. Let $q_2 \in (0, \infty)$ and $p_2 \in (1, 2]^{d}$ be defined by
\[
\frac{1}{q_2} + \frac{1}{q_2} = \frac{1}{q} = 1 - \frac{2\alpha}{p_0}, \quad \frac{1}{p_0} + \frac{1}{p_2} = 1. 
\]

By Lemma 2.5 and Hölder’s inequality, we have for any $\alpha_2 > \alpha_0$,
\[
\|\eta \bar{w}I_t\|_{L_t^2(L_2^2)} + \|\eta \bar{w}I_t\|_{L_t^2(B_{1}^{2,q})} \lesssim \|\eta \bar{w}I_t\|_{L_t^2(B_{1}^{2,q})}. 
\]

Now due to (4.19), one can choose $\theta \in (2\alpha_0, 1)$ being close to $2\alpha_0$ so that
\[
\frac{2}{q_0} + \frac{\alpha_2 - \frac{1}{\beta}}{p_0} < 2 - \alpha_0 - \frac{\theta}{2}. 
\]

Let $q_3 \in (0, \infty)$ and $p_3 \in (1, 2]^{d}$ be defined by
\[
\frac{1}{q_3} = \frac{1}{q_0} + \frac{\theta}{2}, \quad \frac{1}{p_0} = \frac{1}{p_3} + \frac{\theta}{2}. 
\]
In particular, since \( \frac{p}{2} > \alpha_0 \), one can choose \( \beta \in (0, 1) \) so that
\[
\alpha_2 := a \cdot \frac{1}{p_2^2} + (1 - \theta)(0 - a \cdot \frac{1}{p_2^1}) + \theta \left( \frac{\alpha}{2} - \frac{|\alpha|}{\infty} \right) = \frac{\theta \alpha}{2} > \alpha_0.
\]
Thus by Gagliardo-Nirenberge's inequality (2.12) and Hölder’s inequality, we have
\[
\| \tilde{w} t_i \|_{L^{\theta_2}(B_{p_2}^{-\alpha_2})} \lesssim \| \tilde{w} t_i \|_{L^{\theta_1}(B_{p_1}^{-\alpha_0})} \| \hat{w} t_i \|_{L^{\theta_2}(B_{p_2}^{-\alpha_2})} \lesssim \| \tilde{w} t_i \|_{L^{\theta_1}(L^{\infty})} \| \hat{w} t_i \|_{L^{\theta_2}(L^{\infty})}.
\]
On the other hand, for \( \nu_0 := \frac{2 - \theta \nu}{1 - \theta} \) and \( r_0 := \frac{2 - \theta \nu}{1 - \theta} \), by Hölder’s inequality,
\[
\| \tilde{w} t_i \|_{L^{\theta_2}(B_{p_2}^{-\alpha_2})} \lesssim \| \tilde{w} t_i \|_{L^{\theta_1}(L^{\infty})} \| \hat{w} t_i \|_{L^{\theta_2}(L^{\infty})} = \| \tilde{w} \|_{L^{\theta_1}(L^{\infty})} \| \hat{w} \|_{L^{\theta_2}(L^{\infty})}.
\]
Hence,
\[
\| \tilde{w} t_i \|_{L^{\theta_2}(B_{p_2}^{-\alpha_2})} \lesssim \| \tilde{w} \|_{L^{\theta_1}(L^{\infty})} \| \hat{w} \|_{L^{\theta_2}(L^{\infty})} \| \tilde{w} t_i \|_{L^{\theta_1}(L^{\infty})} \| \hat{w} t_i \|_{L^{\theta_2}(L^{\infty})}.
\]
(4.38)
Since \( 1 - \alpha_0 < \frac{2}{q_0} \), by (4.36), it is easy to see that \( (\nu_0, r_0) \in \mathcal{F}_1 \).

(Step 4) Combining (4.33), (4.35) and (4.38) and by Hölder’s inequality, we have
\[
\| \tilde{w} t_i \|_{L^{\theta_2}(L^{\infty})} \lesssim \| \tilde{w} \|_{L^{\theta_1}(L^{\infty})} \| \hat{w} \|_{L^{\theta_2}(L^{\infty})} \| \tilde{w} t_i \|_{L^{\theta_1}(L^{\infty})} \| \hat{w} t_i \|_{L^{\theta_2}(L^{\infty})} \]
where we have used \( \| \tilde{w} \|_{L^{\theta_1}(L^{\infty})} = \| \tilde{w} \|_{L^{\theta_1}(L^{\infty})} \| \hat{w} \|_{L^{\theta_2}(L^{\infty})} \| \tilde{w} t_i \|_{L^{\theta_1}(L^{\infty})} \| \hat{w} t_i \|_{L^{\theta_2}(L^{\infty})} \)
\[= \| \tilde{w} \|_{L^{\theta_1}(L^{\infty})} \| \hat{w} \|_{L^{\theta_2}(L^{\infty})} \| \tilde{w} t_i \|_{L^{\theta_1}(L^{\infty})} \| \hat{w} t_i \|_{L^{\theta_2}(L^{\infty})}.\]
Note that by (4.20) and (4.21),
\[\|
\]
Recalling \( \frac{1}{\nu} = \frac{2 - \theta \nu}{1 - \theta} \) for any \( \nu \in (0, 1) \), by Young’s inequality, we arrive at
\[
\| \tilde{w} t_i \|_{L^{\theta_2}(L^{\infty})} \lesssim \| \tilde{w} \|_{L^{\theta_1}(L^{\infty})} \| \hat{w} \|_{L^{\theta_2}(L^{\infty})} \| \tilde{w} t_i \|_{L^{\theta_1}(L^{\infty})} \| \hat{w} t_i \|_{L^{\theta_2}(L^{\infty})} \]
\[= \| \tilde{w} \|_{L^{\theta_1}(L^{\infty})} \| \hat{w} \|_{L^{\theta_2}(L^{\infty})} \| \tilde{w} t_i \|_{L^{\theta_1}(L^{\infty})} \| \hat{w} t_i \|_{L^{\theta_2}(L^{\infty})} \]
\[= \| \tilde{w} \|_{L^{\theta_1}(L^{\infty})} \| \hat{w} \|_{L^{\theta_2}(L^{\infty})} \| \tilde{w} t_i \|_{L^{\theta_1}(L^{\infty})} \| \hat{w} t_i \|_{L^{\theta_2}(L^{\infty})}. \]
(4.39)
For any \( \nu \in (0, 1) \), one can choose \( \beta \in (0, 1) \) so that \( a \cdot \left( \frac{1}{\nu} - \frac{1}{2} \right) + \frac{2 \beta}{\nu} > 0 \), and thus, by Lemma 3.7 with \( q = 4 \) and (3.17),
\[
\| \tilde{w} \|_{L^{\theta_1}(L^{\infty})} \lesssim \| \tilde{w} \|_{L^{\theta_1}(L^{\infty})} \| \hat{w} \|_{L^{\theta_2}(L^{\infty})} \]
\[= \| \tilde{w} \|_{L^{\theta_1}(L^{\infty})} \| \hat{w} \|_{L^{\theta_2}(L^{\infty})} \| \tilde{w} t_i \|_{L^{\theta_1}(L^{\infty})} \| \hat{w} t_i \|_{L^{\theta_2}(L^{\infty})} \]
\[= \| \tilde{w} \|_{L^{\theta_1}(L^{\infty})} \| \hat{w} \|_{L^{\theta_2}(L^{\infty})} \| \tilde{w} t_i \|_{L^{\theta_1}(L^{\infty})} \| \hat{w} t_i \|_{L^{\theta_2}(L^{\infty})}. \]
(4.40)
which, together with \( (\nu_0, r_0) \in \mathcal{F}_1 \), implies that the last term in (4.39) can be absorbed by the left
hand side if one lets \( \varepsilon \) being small enough. Moreover,
\[
\| G \tilde{q} \|_{L^{\theta_2}(L^{p})} = \| G^2 \tilde{q} \|_{L^{\theta_2}(L^{p})} \lesssim \| G \|_{L^{\theta_1}(L^{\infty})} \| \tilde{q} \|_{L^{\theta_2}(L^{p})} \]
and
\[
\| \tilde{q} \|_{L^{\theta_2}(L^{p})} \lesssim \| \tilde{q} \|_{L^{\theta_2}(L^{\infty})} = \| \tilde{q} \|_{L^{\theta_2}(L^{\infty})} \]
Substituting these into (4.39) and by (4.26), (4.40) and (4.28), we obtain (4.25) as well as (4.23).

Remark 4.6. Since \( \nu_0, \nu_1, \nu_2 \in (2, 4) \) are required to be greater than \( 4 \) (see (4.24)), we have to require \( 1 - \alpha_0 < \frac{2}{q_0} \) in (4.19) and \( q_1, q_2 \in (2, 4) \) in (4.21) and (4.20). Note that \( q_1 \in (2, 4) \) naturally leads to \( a \cdot \frac{1}{p_1} < \frac{1}{2} \), and when \( \alpha_0 = 0 \), the condition \( \frac{1}{\nu_0} \lesssim \frac{2 \beta}{\nu} < 1 \).

Remark 4.7. If \( \nu = \sum_{i=0}^{m} \nu_i \), where \( \nu_i \in \mathbb{R}^p_{\nu_i}(B_{P_0}^{-\alpha_0}) \) for some \( \alpha_i \in [0, 1) \) and \( (\nu_i, \nu) \in (1, \infty)^{1+2d} \) satisfying (4.19), then we still have (4.22) with \| G \tilde{q} \|_{L^{\theta_2}(L^{p})} \) replaced by \( \sum_{i=0}^{m} \| G \tilde{q} \|_{L^{\theta_2}(L^{p})} \)
5. Global boundedness and stability of weak solutions

Fix \( r > 0 \). For \( p \in [1, \infty)^{2d} \), we introduce the following localized \( LP \)-space:
\[
\|f\|_{L^p_{\tau}} := \sup_{z_0} \| f 1_{B^p_\tau(z_0)} \|_{L^p_{\tau}} = \sup_{z_0} \| f(\cdot - z_0) 1_{B^p_\tau(z_0)} \|_{L^p_{\tau}},
\]
and for \((q, p) \in [1, \infty]^{1+2d}\),
\[
\|f\|_{L^q_{t}p_{\tau}(\mathbb{R}^d, \mathbb{R}^d)} := \sup_{t_0, z_0} \| f(\cdot - t_0, \cdot - z_0) 1_{Q_{t}} \|_{L^q_{t}(L^p_{\tau})},
\]
and for \( \alpha \in \mathbb{R} \),
\[
\|f\|_{L^q_{t}(\mathbb{R}^d, \mathbb{R}^d)} := \sup_{t_0, z_0} \| f(\cdot - t_0, \cdot - z_0) \chi_2 \|_{L^q_{t}(L^p_{\tau})},
\]
where \( \chi_2 \) is a cutoff function as in (3.21). By a finitely covering technique, it is easy to see that for different \( r, r' > 0 \) (cf. [39]),
\[
\sup_{z_0} \| f 1_{B^p_\tau(z_0)} \|_{L^p_{\tau}} \asymp \sup_{z_0} \| f 1_{B^p_{\tau'}(z_0)} \|_{L^p_{\tau'}}
\]
and
\[
\sup_{t_0, z_0} \| f(\cdot - t_0, \cdot - z_0) 1_{Q_{t}} \|_{L^q_{t}(L^p_{\tau})} \asymp \sup_{t_0, z_0} \| f(\cdot - t_0, \cdot - z_0) 1_{Q_{t'}} \|_{L^q_{t}(L^p_{\tau})}.
\]
In particular, for any \( T > 0 \),
\[
\|1_{[0,T]}f\|_{L^q_{t}(\mathbb{R}^d, \mathbb{R}^d)} \asymp \sup_{z_0} \|1_{[0,T]}f 1_{B^p_\tau(z_0)} \|_{L^q_{t}(L^p_{\tau})} \leq (\int_{0}^{T} \| f(s) \|_{L^p_{\tau}}^{q} ds)^{1/q},
\]
and for any bounded \( Q \subset \mathbb{R}^{1+2d} \), there is a constant \( C = C(Q, d, q, p) > 0 \) such that
\[
\|1_{Q}f\|_{L^q_{t}(L^p_{\tau})} \lesssim C \|f\|_{L^q_{t}(\mathbb{R}^d, \mathbb{R}^d)}.
\]
We also introduce the following localized energy space for later use:
\[
\mathcal{V} := \left\{ f \in L^1_{t,loc} : \| f \|_{\mathcal{V}} := \| f \|_{L^q_{t}(\mathbb{R}^d, \mathbb{R}^d)} + \| \nabla_v f \|_{L^q_{t}(L^p_{\tau})} < \infty \right\}.
\]

Using the above notations, we make the following global assumption about drift \( b \):
(\( \mathbb{H}_b \)) Suppose that \( \|b\|_{L^q_{t}(\mathbb{R}^d, \mathbb{R}^d)} \leq \kappa_2 \) for some \((q_1, p_1) \in (2, \infty)^{1+2d}\) satisfying (4.3).

First of all we derive the following existence and uniqueness result of global weak solutions for FPKE (3.1) with initial value \( u(t)_{|_{t=0}} = 0 \).

**Theorem 5.1.** Under (3.2) and (\( \mathbb{H}_b \)), for any \( f \in L^q_{t}(\mathbb{R}^d, \mathbb{R}^d) \) with \((q_0, p_0) \in (1, \infty)^{1+2d}\), satisfying (4.4), and \( T > 0 \), there is a unique global weak solution \( u \) to PDE (3.1) in the sense of Definition 3.1 with initial value \( u(t)_{|_{t=0}} = 0 \) such that for some \( C = C(T, \kappa, q_1, p_1, \kappa_2) > 0 \) and any \( t \in [0, T] \),
\[
\| u \mathcal{I}_t \|_{L^\infty} + \| u \mathcal{I}_t \|_{\mathcal{V}} \lesssim C \| f \mathcal{I}_t \|_{L^q_{t}(\mathbb{R}^d, \mathbb{R}^d)}.
\]

**Proof.** We divide the proof into two steps.

(Step 1) We first show (5.6) for any weak solution \( u \). Without loss of generality, we assume \( T = 1 \) and
\[
u(t, x, v) := b(t, x, v) = f(t, x, v) \equiv 0, \quad \forall t \leq 0.
\]
Fix \( z_0 = (x_0, v_0) \in \mathbb{R}^{2d} \). Let \( \Gamma_t z_0 := (x_0 + t v_0, v_0) \) and define
\[
u_{z_0}(t, z) := u(t, z - \Gamma_t z_0), \quad a_{z_0}(t, z) := a(t, z - \Gamma_t z_0),
\]
and
\[
u_{z_0}(t, z) := b(t, z - \Gamma_t z_0), \quad f_{z_0}(t, z) := f(t, z - \Gamma_t z_0).
\]
By definition, it is easy to see that \( u_{z_0} \) is a weak solution of PDE (3.1) with coefficients \((a_{z_0}, b_{z_0}, f_{z_0})\).
By (\( \mathbb{H}_b \)) and (5.5), it is easy to see that
\[
\sup_{z_0} \|1_{Q_{z_0}} b_{z_0} \|_{L^q_{t}(\mathbb{R}^d, \mathbb{R}^d)} \leq C(\kappa_2).
\]
By applying Theorem 4.3 to nonnegative weak sub-solution $u^+_z$ and $u^-_z = (-u^+_z)^+$ separately, there is a constant $C = C(\kappa_i, Q_i, p_j)$ such that for each $t \in [0, 1]$ and $z \in \mathbb{R}^{2d}$,

$$
\|Q_1 u^+_z \|_{L^\infty} + \|Q_1 \nabla u^+_z \|_{L^2} \lesssim C \|Q_2 u^+_z \|_{L^2} + \|Q_2 f^+_z \|_{L^{p_0}(L^{\infty})}.
$$

In particular, for each $t \in [0, 1]$,

$$
\|Q_1 u^+_z (t) \|_{L^2}^2 \lesssim \text{Vol}(Q_1) \|Q_1 u^+_z (t) \|_{L^\infty}^2 \lesssim \int_0^t \|Q_2 u^+_z (s) \|_{L^2}^2 ds + \|Q_2 f^+_z \|_{L^{p_0}(L^{\infty})}.
$$

Taking supremum in $z \in \mathbb{R}^{2d}$ and by (5.3) and (5.4), we obtain

$$
\sup_{s \in [0, t]} \|u(s)\|_{L^2} \lesssim \|f\|_{L^{p_0}(L^{\infty})},
$$

which implies by Gronwall’s inequality that for any $t \in [0, 1]$,

$$
\|u(t)\|_{L^2}^2 \lesssim C \int_0^t \|u(s)\|_{L^2}^2 ds + \|f\|_{L^{p_0}(L^{\infty})}^2.
$$

By this estimate and taking supremum in $z_0$ for both sides of (5.7), we obtain (5.6). In particular, (5.6) implies the uniqueness of weak solutions.

**(STEP 2).** In this step we show the existence of weak solutions. Let $\Gamma_\varepsilon$ be as in (3.7). Define

$$
a_\varepsilon := a \ast \Gamma_\varepsilon, \; b_\varepsilon := b \ast \Gamma_\varepsilon, \; f_\varepsilon := f \ast \Gamma_\varepsilon.
$$

Under (3.2) and ($\mathcal{H}_b^\varepsilon$), we have

$$
\kappa_0 \| \lesssim a_\varepsilon \lesssim \kappa_1 \|, \; \sup_{\varepsilon \in (0, 1)} \|b_\varepsilon\|_{L^1_1} \lesssim C(\kappa_2),
$$

and for each $\varepsilon \in (0, 1)$,

$$
a_\varepsilon, b_\varepsilon, f_\varepsilon \in C^\infty_c (\mathbb{R}^{1+2d}).
$$

It is well known that there is a unique smooth solution $u_\varepsilon$ solving the following PDE:

$$
\partial_t u_\varepsilon = \text{div} (a_\varepsilon \nabla u_\varepsilon) + v \cdot \nabla u_\varepsilon + b_\varepsilon \cdot \nabla u_\varepsilon + f_\varepsilon, \; u_\varepsilon(0) = 0.
$$

In particular, by Remark 3.2, for any $\varphi \in C^\infty_c (\mathbb{R}^{1+2d})$,

$$
\int_{\mathbb{R}^{1+2d}} u_\varepsilon \partial_t \varphi = -\int_{\mathbb{R}^{1+2d}} \langle a_\varepsilon \nabla u_\varepsilon, \nabla \varphi \rangle - \int_{\mathbb{R}^{1+2d}} u_\varepsilon (v \cdot \nabla \varphi) + \int_{\mathbb{R}^{1+2d}} (b_\varepsilon \cdot \nabla u_\varepsilon) \varphi + \int_{\mathbb{R}^{1+2d}} f_\varepsilon \varphi.
$$

Moreover, by (5.8) and (5.6), we also have

$$
\sup_{\varepsilon \in (0, 1)} (\|u_\varepsilon I_1\|_{L^\infty} + \|u_\varepsilon I_1\|_{L^2}) < \infty.
$$

By the weak compactness of $\mathcal{Y}$, there are a sequence $\varepsilon_k \to 0$ and $u \in \mathcal{Y} \cap L^\infty$ such that for each $\varphi \in C^\infty_c (\mathbb{R}^{1+2d}$ and $\Phi \in L^2(\mathbb{R}^{1+2d}; \mathbb{R}^{4d})$,

$$
\int_{\mathbb{R}^{1+2d}} u_{\varepsilon_k} \varphi \to \int_{\mathbb{R}^{1+2d}} u \varphi, \; \int_{\mathbb{R}^{1+2d}} (\varphi \Phi) \cdot \nabla_v u_{\varepsilon_k} \to \int_{\mathbb{R}^{1+2d}} (\varphi \Phi) \cdot \nabla_v u.
$$

Taking limits for both sides of (5.10) along $\varepsilon_k$, one sees that $u$ is a weak solution of PDE (3.1). Let us only show the following limit since the others are completely the same: for $\varphi \in C^\infty_c (\mathbb{R}^{1+2d})$,

$$
\lim_{k \to \infty} \int_{\mathbb{R}^{1+2d}} (b_{\varepsilon_k} \cdot \nabla_v u_{\varepsilon_k}) \varphi = \int_{\mathbb{R}^{1+2d}} (b \cdot \nabla_v u) \varphi.
$$

Let $Q$ be the support of $\varphi$. By Hölder’s inequality and (5.11), we have

$$
\left| \int_{\mathbb{R}^{1+2d}} ((b_{\varepsilon_k} - b) \cdot \nabla_v u_{\varepsilon_k}) \varphi \right| \leq \|b_{\varepsilon_k} - b\|_{L^1} \|\nabla_v u_{\varepsilon_k} 1_Q \|_{L^2} \|\varphi\|_{L^\infty}
$$

$$
\leq C \|(b_{\varepsilon_k} - b) 1_Q\|_{L^2} \to 0, \; k \to \infty,
$$

where $1_Q$ is the characteristic function of $Q$. Therefore, we have shown that for each $\varphi \in C^\infty_c (\mathbb{R}^{1+2d})$,

$$
\int_{\mathbb{R}^{1+2d}} u \partial_t \varphi = \int_{\mathbb{R}^{1+2d}} (v \cdot \nabla \varphi) - \int_{\mathbb{R}^{1+2d}} \varphi (b \cdot \nabla_v u).
$$

This completes the proof.
and by the weak convergence of $\nabla_v u_{\varepsilon_k}$,
\[
\lim_{k \to \infty} \int_{\mathbb{R}^{1+2d}} b \cdot \nabla_v (u_{\varepsilon_k} - u) \varphi = 0.
\]
Thus we complete the proof. \qed

Now we present a stability result which shall be used to show the well-posedness of generalized martingale problem when the coefficients are bounded measurable.

**Theorem 5.2.** Let $(a_n, b_n, f_n)$ be a sequence of coefficients that satisfy the following assumptions:

(i) $a_n$ satisfies (3.2) uniformly in $n$ and $b_n, f_n$ are uniformly bounded.

(ii) $(a_n, b_n, f_n)$ converges to $(a, b, f)$ in Lebesgue measure as $n \to \infty$.

Let $u_n$ and $u$ be the respective weak solutions of PDE (3.1) corresponding to $(a_n, b_n, f_n)$ and $(a, b, f)$ with zero initial value. Then for any bounded domain $Q \subset \mathbb{R}^+ \times \mathbb{R}^{2d}$,
\[
\lim_{n \to \infty} \sup_{(t, z) \in Q} |u_n(t, z) - u(t, z)| = 0. \tag{5.12}
\]

**Proof.** First of all, by (i) and (5.6) we have for any $T > 0$,
\[
\sup_n (\|u_n I_T\|_{L^\infty} + \|u_n I_T\|_\varphi) < \infty. \tag{5.13}
\]
By Remark 3.2, one can extend $u_n$ to a global weak solution in $\mathbb{R}^{1+2d}$. Thus by (i) and [16, Theorem 3], there are $\alpha \in (0, 1)$ and constant $C > 0$ such that for any $n \in \mathbb{N}$ and all $(t, z), (t', z') \in Q_{1/2}$,
\[
|u_n(t, z) - u_n(t', z')| \leq C |(t, z) - (t', z')|^\alpha (\|1_{Q_1} u_n\|_{L^2} + \|1_{Q_1} f_n\|_{L^\infty}),
\]
which, together with (5.13) and a shifting argument as used in Step 1 of Theorem 5.1, yields that for any bounded domain $Q \subset \mathbb{R}^{1+2d}$, there is a constant $C > 0$ such that for all $n \in \mathbb{N}$ and $(t, z), (t', z') \in Q$,
\[
|u_n(t, z) - u_n(t', z')| \leq C |(t, z) - (t', z')|^\alpha.
\]

In particular, by (5.13), the weak compactness of $\overline{\mathcal{V}}$ and Ascoli-Arzelà’s theorem, there is a subsequence $n_k$ and $\bar{u} \in \overline{\mathcal{V}} \cap C(\mathbb{R}^{1+2d})$ such that for any $\Phi \in L^2(\mathbb{R}^{1+2d}; \mathbb{R}^d)$ with compact support,
\[
\int_{\mathbb{R}^{1+2d}} \Phi \cdot \nabla_v u_{n_k} \to \int_{\mathbb{R}^{1+2d}} \Phi \cdot \nabla_v \bar{u},
\]
and for any bounded domain $Q \subset \mathbb{R}^{1+2d}$,
\[
\lim_{k \to \infty} \sup_{(t, z) \in Q} |u_{n_k}(t, z) - \bar{u}(t, z)| = 0.
\]
As in Step 2 of Theorem 5.1, by (ii) one sees that $\bar{u} = u$ is the unique weak solution of PDE (3.1). By a contradiction method, one sees that (5.12) holds for the whole sequence. \qed

Below we assume that $a \in L_\infty^\infty(C_b^\infty(\mathbb{R}^{2d}))$ satisfies (3.2) and
\[
b_1, b_2 \in L_\infty^\infty(C_b^\infty(\mathbb{R}^{2d})), \ g \in L_\infty^\infty(C_c^\infty(\mathbb{R}^{2d})), \ \varphi \in C_b^\infty(\mathbb{R}^{2d}). \tag{5.14}\]
For applications in SDEs with density dependent coefficients, we need the following apriori $L^\infty$-estimate for the Cauchy problem of the following kinetic FPKE:
\[
\partial_t u = \div_v (a \cdot \nabla_v u) + v \cdot \nabla_x u + b_1 \cdot \nabla_v u + \div_v (b_2 u) + g, \ u(0) = \varphi.
\]
Note that under (5.14), it is well known that there is a unique smooth solution to the above PDE.

**Theorem 5.3.** Suppose that for some $q_i \in (2,4)$ and $p_i \in (2, \infty)^{2d}$ with $\frac{2}{q_i} + \frac{1}{p_i} < 1$, $i = 1, 2$,
\[
\|b_1\|_{L_1(T; L^p(\mathbb{R}^{2d}))} + \|b_2\|_{L_2(T; L^p(\mathbb{R}^{2d}))} \leq \kappa_2.
\]
If \((q_0, p_0) \in \mathbb{R}^{1+2d}\) satisfies (4.19) for some \(\alpha_0 \in [0,1]\), then for any \(T > 0\), there is a constant \(C > 0\) only depending on \(T, \alpha_0, \kappa, q_1, p_1\) such that for all \(t \in [0, T]\),

\[
\|u_t\|_{L^\infty} \lesssim C \|g_t\|_{L^\infty_{t,x}}(\tilde{\theta}_{p_0}^{-\alpha_0}) + \|\varphi\|_{C_1^t}.
\]  

(5.15)

**Proof.** First of all, let \(\bar{u} := P_t \varphi\), where \(P_t\) is defined by (3.14). Note that

\[
\partial_t \bar{u} = \Delta_v \bar{u} + v \cdot \nabla_x \bar{u}, \quad \bar{u}(0) = \varphi.
\]  

(5.16)

It is easy to see that \(\hat{u} := u - \bar{u}\) solves the following PDE:

\[
\partial_t \hat{u} = \text{div}_v(a \cdot \nabla_v \hat{u}) + v \cdot \nabla_x \hat{u} + b_1 \cdot \nabla_v \hat{u} + \text{div}_v G + \tilde{g} + \tilde{g}, \quad \hat{u}(0) = 0,
\]  

where

\[
G := (a - I) \cdot \nabla_v \bar{u} + b_2 u, \quad \tilde{g} := b_1 \cdot \nabla_v \bar{u}.
\]

Below we consider PDE (5.16) with zero initial value. Since \(q_1 \in (2, 4)\) and \(p_1 \in (2, \infty)^{2d}\) satisfy

\[
\frac{2}{q_1} + a \cdot \frac{1}{p_1} < 1,
\]

one can choose \(q_3 \in (1, \frac{q_1}{2})\) and \(p_3 \in (1, \infty)^{2d}\) so that

\[
p_3 \leq p_1, \quad \frac{2}{q_3} + a \cdot \frac{1}{p_3} < 2.
\]

Suppose \(T = 1\). As in the Step 1 of Theorem 5.1, by applying Theorem 4.5 to \(\hat{u}^+_{z_0}\) and \(\hat{u}^-_{z_0}\) (see Remark 4.7), there is a constant \(C = C(\kappa, q_1, p_1) > 0\) such that for each \(t \in [0, 1]\) and \(z_0 \in \mathbb{R}^{2d}\),

\[
\|1_{Q_1} \hat{u}_{z_0} I_t\|_{L^\infty} + \|1_{Q_1} \nabla_v \hat{u}_{z_0} I_t\|_{L^2} \lesssim_C \|1_{Q_2} \hat{u}_{z_0} I_t\|_{L^2} + G_{z_0}^t,
\]  

(5.17)

where

\[
G_{z_0}^t := \|g_{z_0} \chi_2 I_t\|_{L^q_0(B_{p_0}^{\alpha_0})} + \|\tilde{g}_{z_0} \chi_2 I_t\|_{L^{q_3}(L^{p_3})} + \|G_{z_0} \chi_2 I_t\|_{L^{q_2}(L^{p_2})}.
\]

In particular, for each \(t \in [0, 1]\),

\[
\|1_{Q_1} \hat{u}_{z_0}(t)\|_{L^2} \leq \text{Vol}(Q_1) \|1_{Q_1} \hat{u}_{z_0}(t)\|_{L^2} \lesssim \int_0^t \|1_{Q_1} \hat{u}_{z_0}(s)\|_{L^2}^2 ds + (G_{z_0}^t)^2.
\]

Taking supremum in \(z_0 \in \mathbb{R}^{2d}\) and by (5.3), we obtain

\[
\|\hat{u}(t)\|_{L^2}^2 \lesssim \int_0^t \|\hat{u}(s)\|_{L^2}^2 ds + \sup_{z_0} (G_{z_0}^t)^2,
\]

which implies by Gronwall’s inequality that for any \(t \in [0, 1]\),

\[
\sup_{s \in [0,t]} \|\hat{u}(s)\|_{L^2} \lesssim \sup_{z_0} G_{z_0}^t.
\]

Substituting it into (5.17) and taking supremum in \(z_0 \in \mathbb{R}^{2d}\), we get

\[
\|u_t\|_{L^\infty} \lesssim \sup_{s \in [0,t]} \|P_s \varphi\|_{L^\infty} + \|\tilde{u}_t I_t\|_{L^\infty} \lesssim_C \|\varphi\|_{L^\infty} + \sup_{z_0} G_{z_0}^t,
\]  

(5.18)

By \(\|\nabla_v \tilde{u}\|_{L^\infty} = \sup_t \|\nabla_v \tilde{P}_t \varphi\|_{L^\infty} \lesssim \|\varphi\|_{C_1^t}\), and \(p_3 \leq p_1\) and \(q_3 \leq q_1\),

\[
\sup_{z_0} \|\tilde{g}_{z_0} \chi_2 I_t\|_{L^{q_3}(L^{p_3})} \leq \|b_1\|_{L^{q_3}(L^{p_3})} \|\nabla_v \tilde{u}\|_{L^\infty} \lesssim \|b_1\|_{L^{q_3}(L^{p_3})} \|\varphi\|_{C_1^t},
\]

and

\[
\sup_{z_0} \|G_{z_0} \chi_2 I_t\|_{L^{q_2}(L^{p_2})} \lesssim \|\nabla_v \tilde{u}\|_{L^\infty} + \sup_{z_0} \|\tilde{b}_2 u\|_{L^{q_2}(L^{p_2})} \lesssim \|\varphi\|_{C_1^t} + \left(\int_0^t \|\tilde{b}_2 u\|_{L^{q_2}(L^{p_2})}^2 ds\right)^{1/2}.
\]

Substituting the above estimates into (5.18), we get for any \(t \in [0, 1]\),

\[
\|u(t)\|_{L^p} \lesssim C \|\varphi\|_{C_1^t} + \|g_t\|_{L^q_0(B_{p_0}^{\alpha_0})} + \left(\int_0^t \|\tilde{b}_2 u\|_{L^{q_2}(L^{p_2})}^2 ds\right)^{1/2},
\]

which implies the desired estimate (5.15) by Gronwall’s inequality. 

\[\square\]
6. Weak solutions of second order McKean-Vlasov SDEs

In this section we devote to showing the existence of weak solutions to the following second order McKean-Vlasov SDE:

\[ dX_t = V_t dt, \quad dV_t = b_Z(t, Z_t)dt + \sigma_Z(t, X_t)dW_t, \]  

(6.1)

where \( Z_t = (X_t, V_t) \) and \( b_Z, \sigma_Z \) are defined by (1.2) and (1.3), respectively.

Let \( \mathcal{C} \) be the space of all continuous functions from \( \mathbb{R}_+ \) to \( \mathbb{R}^{2d} \), which is endowed with the locally uniformly convergence topology. Let \( \omega_t \) be the canonical process on \( \mathcal{C} \), and \( \mathcal{B}_t \) the natural filtration associated with \( \omega_t \). Let \( \mathcal{P}(\mathcal{C}) \) be the space of all probability measures over \( \mathcal{C} \).

The notion of weak solutions has been introduced in Definition 1.1. Now we introduce the equivalent notion of classical martingale solutions (cf. [31]). The advantage of using martingale solutions is that it avoids the use of stochastic integrals.

**Definition 6.1.** Let \( \nu \in \mathcal{P}(\mathbb{R}^{2d}) \). We call a probability measure \( \mathbb{P} \in \mathcal{P}(\mathcal{C}) \) a classical martingale solution of DDSDE (6.1) with initial distribution \( \nu \) if

(i) \( \mathbb{P} \circ \omega_t^{-1} = \nu \), and for Lebesgue almost all \( t \geq 0 \), \( \mathbb{P} \circ \omega_t^{-1}(dz) = \rho_t(z)dz \).

(ii) For each \( f \in C_b^2(\mathbb{R}^{2d}) \), the process

\[ M_t(\omega) := f(\omega_t) - f(\omega_0) - \int_0^t (\text{tr}(a_\rho \cdot \nabla^2_v f) + v \cdot \nabla_x f + b_\rho \cdot \nabla_v f)(r, \omega_r)dr \]  

(6.2)

is a \( \mathcal{B}_t \)-martingale with respect to \( \mathbb{P} \), where for \( \langle \rho_t \rangle(x) := \int_{\mathbb{R}^d} \rho_t(x, v)dv \),

\[ a_\rho(t, x) := \int_{\mathbb{R}^d} a(t, x, (\rho_t)(x), z') \rho_t(z')dz', \]

(6.3)

\[ b_\rho(t, z) := \int_{\mathbb{R}^d} b(t, z, \rho_t(z), z') \rho_t(z')dz'. \]

(6.4)

The set of all the martingale solutions \( \mathbb{P} \in \mathcal{P}(\mathcal{C}) \) with initial distribution \( \nu \) is denoted by \( \mathcal{M}^{\mathcal{a}, \mathcal{b}}_\nu \).

**Remark 6.2.** Note that \( a_\rho \) and \( b_\rho \) are only well defined except for some Lebesgue null set. This does not affect the definition of (6.2) due to the point (i). Under (1.4), it is well known that weak solution in Definition 1.1 is equivalent to the above martingale solution (see [31]).

Our main result of this section is to show the following existence result.

**Theorem 6.3.** Under (H_1) and (H_2), for any initial probability measure \( \nu \in \mathcal{P}(\mathbb{R}^{2d}) \), there exists at least one classical martingale solution \( \mathbb{P} \in \mathcal{M}^{\mathcal{a}, \mathcal{b}}_\nu \) in the sense of Definition 6.1. Moreover, \( \rho \) enjoys the regularity (1.11).

To show the existence of a martingale solution, by mollifying the coefficients and the fact that \( a_\rho \) does not depend on \( v \), the key point is to use (5.6) and (5.15) to establish the uniform Krylov estimate for approximating DDSDEs. For \( N \in \mathbb{N} \), let \( \Gamma \in C_c^\infty(\mathbb{R}^N) \) be a probability density function and define

\[ \Gamma_n(\cdot) = n^N \Gamma(n\cdot), \quad n \in \mathbb{N}. \]

With a little confusion, we shall use the same notation \( \Gamma_n \) to denote different mollifiers for different dimension \( N \). For \( n \in \mathbb{N} \), define

\[ b_n(t, z, r, z') := (b(t, \cdot, \cdot, \cdot) * \Gamma_n)(z, r, z'), \]

\[ a_n(t, x, r, z') := (a(t, \cdot, \cdot, \cdot) * \Gamma_n)(x, r, z'), \]

and for a probability measure \( \mu \in \mathcal{P}(\mathbb{R}^{2d}) \),

\[ b^\mu_n(t, z) := \int_{\mathbb{R}^{2d}} b_n(t, z, \mu * \Gamma_n(z), z')\mu(dz'), \]

\[ a^\mu_n(t, x) := \int_{\mathbb{R}^{2d}} a_n(t, x, \mu * \Gamma_n(x), z')\mu(dz'). \]
For two probability measures $\mu_1, \mu_2 \in \mathcal{P}_2(\mathbb{R}^{2d})$ with finite second order moments, the Wasserstein metric is defined by

$$W_2(\mu_1, \mu_2) := \inf \left\{ \int_{\mathbb{R}^{2d}} |z - z'|^2 \pi(dz, dz') : \pi \in \mathcal{P}(\mathbb{R}^{4d}), \pi(\cdot, \mathbb{R}^{2d}) = \mu_1, \pi(\mathbb{R}^{2d}, \cdot) = \mu_2 \right\}^{\frac{1}{2}}.$$

The following lemma provides necessary estimates for $b^n_{\mu}$ and $a^n_{\mu}$.

**Lemma 6.4.** Under (H$_1$) and (H$_2$), for each $n \in \mathbb{N}$, $T > 0$ and $\mu \in \mathcal{P}(\mathbb{R}^{2d})$, we have

$$b^n_{\mu} \in L^\infty_T C^\infty(\mathbb{R}^{2d}), \quad a^n_{\mu} \in L^\infty_T C^\infty(\mathbb{R}^d),$$

and there is a constant $C = C(T, n) > 0$ such that for any $\mu_1, \mu_2 \in \mathcal{P}(\mathbb{R}^{2d})$,

$$\|b^n_{\mu_1} - b^n_{\mu_2}\|_\infty + \|\sqrt{a^n_{\mu_1}} - \sqrt{a^n_{\mu_2}}\|_\infty \leq C W_2(\mu_1, \mu_2).$$

Moreover, we also have

$$\sup_n \|b^n_{\mu}\|_{L^{\infty}_t(\mathcal{L}^p_{\mu}')} \leq \|h\|_{L^{\infty}_t(\mathcal{L}^p_{\mu}')},$$

and for all $\xi \in \mathbb{R}^d$ and $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^d$,

$$\kappa_0|\xi|^2 \leq \xi \cdot a^n_{\mu}(t, x) \xi \leq \kappa_1|\xi|^2.$$  \hfill (6.8)

**Proof.** Estimates (6.5) and (6.8) are obvious by definition. For (6.7), noting that by (1.9),

$$|b^n_{\mu}(t, z)| \leq \int_{\mathbb{R}^{2d}} \left( \int_{\mathbb{R}^{4d}} h(t, z - z_1 - z' + z'_1) \Gamma_n(z_1, z'_1) dz_1 dz'_1 \right) \mu(dz'),$$

by Minkowski’s inequality, we have

$$\|b^n_{\mu}\|_{L^{\infty}_t(\mathcal{L}^p_{\mu}')} \leq \int_{\mathbb{R}^{2d}} \left( \int_{\mathbb{R}^{4d}} \|h\|_{L^{\infty}_t(\mathcal{L}^p_{\mu}')} \Gamma_n(z_1, z'_1) dz_1 dz'_1 \right) \mu(dz') = \|h\|_{L^{\infty}_t(\mathcal{L}^p_{\mu}')}.$$  \hfill (6.7)

Below we prove the second estimate in (6.6), and divide the proofs into three steps.

**Step 1** In this step we show the following claim: Suppose that $A(r) : \mathbb{R} \rightarrow M_{sym}^d$ is a $C^1$-positive definite matrix-valued function so that $A(r) \geq \kappa_0 I$ for all $r$. Then (cf. [31, Theorem 5.2.2])

$$\|\partial_r A^\sharp(r)\|_{H^S} \leq \|\partial_r A(r)\|_{H^S}/(2\sqrt{\kappa_0}).$$  \hfill (6.9)

Indeed, by the chain rule, we have

$$\partial_r A(r) = \partial_r A^\sharp(r) \cdot A^\sharp(r) + A^\sharp(r) \cdot \partial_r A^\sharp(r).$$

Let $P$ be an orthogonal matrix so that $PA^\sharp(r)P^* = : D$ is an diagonal matrix, where the asterisk stands for the transpose of a matrix. Then

$$P\partial_r A(r)P^* = (P\partial_r A^\sharp(r)P^*)D + D(P\partial_r A^\sharp(r)P^*)$$

and

$$(P\partial_r A^\sharp(r)P^*)_{ij} = (P\partial_r A(r)P^*)_{ij}(D_{jj} + D_{ii}).$$

From this we derive (6.9).

**Step 2** Let $\mathbb{H}$ be a separable Hilbert space and $A : \mathbb{H} \rightarrow M_{sym}^d$ a $C^1$-positive definite matrix-valued function. Suppose that for all $h, h' \in \mathbb{H}$,

$$A(h) \geq \kappa_0 I, \quad \|A(h) - A(h')\|_{H^S} \leq \kappa_1 \|h - h'\|_{\mathbb{H}}.$$  \hfill (6.10)

Then it holds that

$$\|A^\sharp(h) - A^\sharp(h')\|_{H^S} \leq \kappa_1 \|h - h'\|_{\mathbb{H}}/(2\sqrt{\kappa_0}).$$
Indeed, without loss of generality, we may assume $H = \ell^2$, where $\ell^2$ is the usual sequence Hilbert space. For each $h = (h_1, h_2, \cdots) \in \ell^2$, the assumption implies that

$$\sum_{j \in \mathbb{N}} \| \partial_{h_j} A(h) \|^2_{HS} \leq \kappa_1^2.$$ 

Thus, by Hölder’s inequality and (6.9), we have for some $h^*_j \in \ell^2$,

$$\| A^i(h) - A^i(h^*) \|_{HS} \leq \sum_{j \in \mathbb{N}} |h_j - h^*_j| \|(\partial_{h_j} A^i)(h^*_j)\|_{HS}$$

$$\leq \|h - h^*\|_H \left( \sum_{j \in \mathbb{N}} \|(\partial_{h_j} A^i)(h^*_j)\|^2_{HS} \right)^{1/2} \leq \frac{\kappa_1 \|h - h^*\|_H}{2 \sqrt{\kappa_0}}.$$

(Step 3) For simplicity, we drop the variable $(t, x)$. Suppose that $X \in L^2(\Omega, \mathcal{F})$ has distribution $\mu$. Then we can write

$$a^m_\mu = \int_{\mathbb{R}^2} a_n(\mu \ast \Gamma_n, z') \mu(dz') = E a_n(\mu \ast \Gamma_n, X) := A(\mu \ast \Gamma_n, X).$$

Thus for any $X, Y \in L^2(\Omega, \mathcal{F})$ with distributions $\mu_1$ and $\mu_2$, we have

$$\sqrt{a_{\mu_1}^m} - \sqrt{a_{\mu_2}^m} = \sqrt{A(\mu_1 \ast \Gamma_n, X)} - \sqrt{A(\mu_2 \ast \Gamma_n, Y)},$$

and by (6.9) and (6.10),

$$\| \sqrt{a_{\mu_1}^m} - \sqrt{a_{\mu_2}^m} \|_{HS} \leq C_n |(\mu_1 \ast \Gamma_n - \mu_2 \ast \Gamma_n)| + \|X - Y\|_{L^2}$$

$$\leq C_n |E| \Gamma_n(\cdot - X) - \Gamma_n(\cdot - Y)| + \|X - Y\|_{L^2}$$

which in turn implies that

$$\| \sqrt{a_{\mu_1}^m} - \sqrt{a_{\mu_2}^m} \|_{HS} \leq C_n \mathbb{W}_2(\mu_1, \mu_2).$$

The proof is complete.

Now let $Z_0$ be an $\mathcal{F}_0$-measurable random variable with distribution $\nu$. Let us consider the following approximating DDSDE:

$$dX^n_t = V^n_t dt, \quad dV^n_t = b^n_{\mu \ast \Gamma_n}(t, Z^n_t) dt + \sqrt{a^n_{\mu \ast \Gamma_n}}(t, X^n_t) dW_t,$$

subject to the initial value

$$Z^n_0 := (-n) \vee Z_0 \wedge n,$$

where for a vector $z \in \mathbb{R}^{2d}$, $((-n) \vee z \wedge n)_i := (-n) \vee z_i \wedge n$.

By Lemma 6.4 and standard Picard’s iteration, it is well known that the above SDE admits a unique strong solution $Z^n_t$. Now we use Theorems 5.1 and 5.3 to derive the following crucial Krylov estimates.

**Lemma 6.5** (Krylov’s estimates). Let $\alpha_0 \in [0, 1)$ and $(q_0, p_0) \in (1, \infty)^{1+2d}$ satisfy (4.19). For any $T > 0$, there are $\theta \in (0, 1)$ and constant $C > 0$ depending only on $\kappa_i$ and $q_i, p_i$ such that for any $\delta \in (0, 1)$, stopping time $\tau \leq T$ and $f \in C_c^\infty(\mathbb{R}^{1+2d})$,

$$\sup_{n \in \mathbb{N}} E \left( \int_0^{\tau + \delta} f(r, Z^n_r) dr \right)^{q_0} \leq C \delta^\theta \| 1_{[0, T]} f \|_{L^{\infty}(\mathbb{B}_{p_0}^{-\alpha_0})} \leq C \delta^\theta \| 1_{[0, T]} f \|_{L^{\infty}(\mathbb{B}_{p_0}^{-\alpha_0})}. \quad (6.12)$$
Proof. Fix $T > 0$. By Lemma 6.4, we have
\[ b^n_{\mu_Z} \in L^\infty_c(\mathbb{R}^{2d}), \quad a^n_{\mu_Z} \in L^\infty_c(\mathbb{R}^{2d}). \]
Fix $0 \leq t_0 < t_1 \leq T$ and $f \in C_c^\infty(\mathbb{R}^{1+2d})$. Let $u_n \in L^\infty_{t_1}(C^\infty_0(\mathbb{R}^{2d}))$ solve the following backward Kolmogorov equation:
\[ \partial_t u_n + \text{tr}(a^n_{\mu_Z} \cdot \nabla u_n) + v \cdot \nabla_x u_n + b^n_{\mu_Z} \cdot \nabla_v u_n = f, \quad u_n(t_1) = 0. \]

(6.13)
Since $a \cdot \frac{1}{p_0} + \frac{2}{q_0} < 2$, one can choose $\tilde{q}_0 < q_0$ so that
\[ a \cdot \frac{1}{p_0} + \frac{2}{\tilde{q}_0} < 2. \]
Thus by (6.7), (6.8), (1.14) and (5.15), there is a constant $C = C(\kappa, q_i, p_i) > 0$ such that
\[ \sup_{n \in \mathbb{N}} \| 1_{[t_0, t_1]} u_n \| \lesssim_C \| 1_{[t_0, t_1]} f \|_{L^\infty_1(\mathbb{B}^{-\alpha_0})} \lesssim_C (t_1 - t_0)^{\frac{1}{q_0} - \frac{1}{\tilde{q}_0}} \| 1_{[t_0, t_1]} f \|_{L^\infty_1(\mathbb{B}^{-\alpha_0})}. \]
where the second step is due to Hölder’s inequality. Now by (6.13) and Itô’s formula, we have
\[ 0 = u_n(t_1, Z^n_{t_1}) = u_n(t_0, Z^n_{t_0}) + \int_{t_0}^{t_1} f(r, Z^n_r) dr + \int_{t_0}^{t_1} \left\langle \nabla_v u_n(r, Z^n_r), \sqrt{a^n_{\mu_Z}}(r, X^n_r) dW_r \right\rangle. \]
Hence, by taking conditional expectation with respect to $\mathcal{F}_{t_0}$ and (6.14),
\[ \mathbb{E} \left( \int_{t_0}^{t_1} f(r, Z^n_r) dr \mid \mathcal{F}_{t_0} \right) = -u_n(t_0, Z^n_{t_0}) \lesssim (t_1 - t_0)^{\frac{1}{q_0} - \frac{1}{\tilde{q}_0}} \| 1_{[t_0, t_1]} f \|_{L^\infty_1(\mathbb{B}^{-\alpha_0})}. \]

By discretization stopping time approximation (see [40, Remark 1.2]), we obtain the first estimate in (6.12). Note that by Lemma 2.5,
\[ \| 1_{[0,T]} f \|_{L^\infty_1(\mathbb{B}^{-\alpha_0})} \lesssim \sup_{z_0} \| 1_{[0,T]} f \|_{L^\infty_1(\mathbb{B}^{-\alpha_0})} \| \chi_{2} \|_{L^\infty_1(\mathbb{B}^{-\alpha_0})} \lesssim \| 1_{[0,T]} f \|_{L^\infty_1(\mathbb{B}^{-\alpha_0})}. \]

The second estimate in (6.12) then follows. \qed

Remark 6.6. Under (H3), one can use Theorem 5.1 to obtain the following Krylov estimate: for any $(q_0, p_0) \in (1, \infty)^{1+2d}$ satisfying (4.4),
\[ \sup_{n \in \mathbb{N}} \mathbb{E} \left( \int_{0}^{T} f(r, Z^n_r) dr \mid \mathcal{F}_{t} \right) \leq C \delta^{\theta} \| 1_{[0,T]} f \|_{L^1_1(\mathbb{L}^{q_0})}. \]

(6.15)
The following corollary is direct by (6.12).

Corollary 6.7. For Lebesgue almost all $t \geq 0$, $Z^n_t$ admits a density $\rho^n_t(z)$ with the following regularity: for any $\alpha \in [0, 1)$ and $(q, p) \in (1, \infty)^{1+2d}$ satisfying
\[ \frac{2}{q} < 1 + \alpha, \quad \alpha + a \cdot (\frac{1}{p} - 1) > 2\alpha, \]
it holds that
\[ \sup_{n \in \mathbb{N}} \| \rho^n 1_{[0,T]} \|_{L^1_1(\mathbb{B}^{-\alpha_0})} < \infty, \quad T > 0. \]

(6.16)
Proof. Let $\mu^n_t(dz)$ be the probability distribution of $Z^n_t$. By Krylov’s estimate (6.12), for any $a_0 \in [0, 1)$ and $(q_0, p_0) \in (1, \infty)^{1+2d}$ satisfying (4.19), there is a constant $C_T > 0$ such that for any $f \in C_c^\infty(\mathbb{R}^{2d})$,
\[ \sup_{n \in \mathbb{N}} \int_{0}^{T} \int_{\mathbb{R}^{2d}} f(r, z) \mu^n_t(dz) dr \leq C_T \| 1_{[0,T]} f \|_{L^1_1(\mathbb{B}^{-\alpha_0})}. \]
This implies that $\mu^n_t$ has a density $\rho^n_t(z)$, which belongs to the dual space of $L^\infty([0,T]; \mathbb{B}^{-\alpha_0})$. In particular, we have (6.16). \qed

By Krylov’s estimate (6.12), we can also show the following tightness result.
Lemma 6.8. The law $P_n$ of $Z^n$ in $\mathbb{C}$ is tight. Moreover, for any accumulation point $P$, it holds that $P \circ \omega_t^{-1}(dz) = \rho(t) dz$, where $\rho$ satisfies the same estimate (6.16).

Proof. Fix $T > 0$. Let $\tau$ be any stopping time bounded by $T$. By SDE (6.11), we have

$$V^n_{\tau + t} - V^n_\tau = \int_\tau^{\tau + t} b_{\mu,Z}(r, Z^n_r) dr + \int_\tau^{\tau + t} \sqrt{a_{\mu,Z}(r, Z^n_r)} dW_r,$$

and

$$X^n_{\tau + t} - X^n_\tau = \int_\tau^{\tau + t} V^n_r dr. \quad (6.17)$$

By Krylov’s estimate (6.12) and (6.7), there is a $\theta \in (0, 1)$ such that for all $\delta \in (0, 1)$ and $n \in \mathbb{N}$,

$$E \left( \int_\tau^{\tau + \delta} |b_{\mu,Z}^n(r, Z^n_r)| dr \right) \leq \delta^\theta \|b_{\mu,Z}^n\|_{L^q(\mathbb{P}^n_{\mathbb{Z}^n})} \lesssim \delta^\theta \|b\|_{L^q(\mathbb{P}^n_{\mathbb{Z}^n})},$$

and by BD’s inequality and (6.8),

$$E \left( \sup_{t \in [0, \delta]} \int_\tau^{\tau + \delta} \sqrt{a_{\mu,Z}^n(r, Z^n_r)} dW_r \right) \leq E \left( \int_\tau^{\tau + \delta} \|a_{\mu,Z}^n(r, X^n_r)\|_{HS} dr \right)^{1/2} \lesssim \delta^{1/2},$$

where the implicit constant is independent of $n$, $\tau$ and $\delta \in (0, 1)$. Hence,

$$\sup_n E \left( \sup_{t \in [0, \delta]} |V^n_{\tau + t} - V^n_\tau| \right) \lesssim \delta^{\theta \wedge (1/2)}. \quad (6.18)$$

Noting that for any $R > 0$,

$$P \left( \sup_{t \in [0, T + \delta]} |V^n_t| \geq R \right) \leq P \left( \sup_{t \in [0, T + \delta]} |V^n_t - V^n_0| \geq \frac{R}{2} \right) + P \left( |V^n_0| \geq \frac{R}{2} \right),$$

and by $|V^n_0| \leq |V_0|$, (6.18) and Chebyshev’s inequality, we get

$$\lim_{R \to \infty} \sup_n P \left( \sup_{t \in [0, T + \delta]} |V^n_t| \geq R \right) = 0. \quad (6.19)$$

Thus, by (6.17) and (6.19), we derive that for any $\varepsilon > 0$,

$$\lim_{\delta \to 0} \sup_n P \left( \sup_{t \in [0, \delta]} |X^n_{\tau + t} - X^n_\tau| > \varepsilon \right) = 0,$$

which, together with (6.18) and (6.19), and by [31, Theorem 1.3.2], yields the tightness of $P_n$.

Let $P$ be any accumulation point. By Prohorov’s theorem, there is a subsequence $n_m$ such that $P_{n_m}$ weakly converges to $P$ as $m \to \infty$. By taking weak limits, one sees that for any $T > 0$, $f \in C_c(\mathbb{R}^{1+2d})$ and any $a_0 \in [0, 1)$ and $(q_0, p_0) \in (1, \infty)^{1+2d}$ satisfying (4.19),

$$E^P \left( \int_0^T f(r, \omega_r) dr \right) \lesssim \|f\|_{L^{q_0}(B_{p_0}^{-a_0})}.$$

As in Corollary 6.7, $P \circ \omega_t^{-1}(dz) = \rho(t) dz$ and $\rho$ satisfies (6.16). \hfill \Box

The following lemma provides the strong convergence of the density.

Lemma 6.9. For fixed accumulation point $P$ of $(P_n)_{n \in \mathbb{N}}$, there is a subsequence $n_k$ such that $P_{n_k}$ weakly converges to $P$ as $k \to \infty$, and for any bounded domain $Q \subset \mathbb{R}_+ \times \mathbb{R}^{2d}$,

$$\lim_{k \to \infty} \| \rho^{n_k} - \rho \|_{L^1(Q)} = 0, \quad (6.20)$$

where $P^{n_k} \circ \omega_t^{-1}(dz) = \rho^{n_k}(t) dz$ and $P \circ \omega_t^{-1}(dz) = \rho(t) dz$. \hfill \Box
where the implicit constant may depend on $m$ by Bernstein’s inequality (2.9), it is easy to see that
\[
\|\partial_i (\rho^n \chi_R)\|_{L^2(B_{1/2}^n)} \leq \|\partial_i \partial_j (a_{\mu \alpha}^n \rho^n \chi_R \chi_R)\|_{L^2(B_{1/4}^n)} + \|v \cdot \nabla_x (\rho^n \chi_R)\|_{L^2(B_{1/4}^n)} + \|\div_x (b_{\mu \alpha}^n \rho^n \chi_R)\|_{L^2(B_{1/4}^n)}
\]
\[
\leq \|a_{\mu \alpha}^n \rho^n \chi_R\|_{L^2(L^1)} + \|\rho^n \chi_R\|_{L^2(L^1)} + \|b_{\mu \alpha}^n \rho^n \chi_R\|_{L^2(L^1)},
\]
\[
(6.21)
\]
where the implicit constant may depend on $R$, but is independent of $n$. For the last term, by Hölder’s inequality, we have
\[
\|b_{\mu \alpha}^n \rho^n \chi_R\|_{L^2(L^1)} \leq \|b_{\mu \alpha}^n\|_{L^{1}Q_{4k}} \|\rho^n\|_{L^{q_{1}}(L^{p_{1}})}
\]
\[
(6.22)
\]
where $\frac{1}{q} + \frac{1}{q_{1}} = \frac{1}{2}$ and $\frac{1}{p_{1}} + \frac{1}{p_{1}} = 1$. Since $\frac{2}{q} + a \cdot \frac{1}{p_{1}} < 1$, we have $\frac{2}{q} + a \cdot \frac{1}{p_{1}} - 1 > 0$. Thus by (6.16), (6.7), (6.21) and (6.22), we obtain
\[
\sup_n \|\partial_i (\rho^n \chi_R)\|_{L^2(B_{1/2}^n)} < \infty.
\]
On the other hand, by (6.16) we also have for some $\alpha > 0$,
\[
\sup_n \|\rho^n \chi_R\|_{L^2(L^1)} < \infty.
\]
Since $B_{1/2}^n$ is locally and compactly embedded in $L^1$, by Aubin-Lions’ lemma (see [29]) and a further diagonalization method, there is a subsequence $n_k$ of $n_m$ so that $\mathbb{P}_{n_k}$ weakly converges to $\mathbb{P}$ as $k \to \infty$ and (6.20) holds. The proof is complete. \(\square\)

**Convention:** For simplicity of notations, we shall assume that $(\mathbb{P}_n)_{n \in \mathbb{N}}$ weakly converges to $\mathbb{P}$ and (6.20) holds for the whole sequence below.

We need the following technical lemma for taking limits below.

**Lemma 6.10.** Fix $N \in \mathbb{N}$. Let $f \in L^1_{\text{loc}}(\mathbb{R}^{N+1})$ and $\rho_n, \rho \in L^1_{\text{loc}}(\mathbb{R}^N)$. Suppose that for any $m \in \mathbb{N}$ and bounded $Q \subset \mathbb{R}^N$,
\[
\lim_{h \to 0} \sup_{|z|,|z'| \leq m,|z-z'| \leq h} |f(z, r) - f(z', r')| dz = 0, \quad \lim_{n \to \infty} \|\rho_n - \rho\|_{L^1(Q)} = 0.
\]
Let $f_n(z, r) := (f * \Gamma_n)(z, r)$. Then for any $\varepsilon > 0$ and bounded $Q \subset \mathbb{R}^N$,
\[
\lim_{n \to \infty} \mathcal{L}\{z \in Q : |f_n(z, \rho_n(z)) - f(z, \rho(z))| \geq \varepsilon\} = 0,
\]
where $\mathcal{L}$ stands for the Lebesgue measure in $\mathbb{R}^N$.

**Proof.** Fix bounded $Q \subset \mathbb{R}^N$. For any $m, n \in \mathbb{N}$ and $h > 0$, we define
\[
A_m^n := \{z \in Q : |\rho_n(z)| \vee |\rho(z)| \leq m\}, \quad D_h^n := \{z \in Q : |\rho_n(z) - \rho(z)| \leq h\},
\]
and for $\varepsilon > 0$,
\[
Q_\varepsilon^n := \{z \in Q : |f_n(z, \rho_n(z)) - f(z, \rho(z))| \geq \varepsilon\}.
\]
By the assumption and Chebyshev’s inequality, we clearly have
\[
\lim_{h \to 0} \sup_n \mathcal{L}(A_m^n) = 0, \quad \lim_{n \to \infty} \mathcal{L}(D_h^n) = 0, \quad \forall h > 0.
\]
Note that
\[
Q_\varepsilon^n \subset \{z \in Q : |f(z, \rho_n(z)) - f(z, \rho(z))| \geq \varepsilon/2\}
\]
\[
\cup \{z \in Q : |f_n(z, \rho_n(z)) - f(z, \rho_n(z))| \geq \varepsilon/2\} = Q_\varepsilon^n \cup Q_\varepsilon^{n+1}.
\]
Thus, to show \(\lim_{n \to \infty} \mathcal{L}(Q_n^\varepsilon) = 0\), by (6.23), it suffices to show that for each \(m \in \mathbb{N}\),
\[
\lim_{h \to 0} \sup_{n} \mathcal{L}(Q_{n}^{\varepsilon,1} \cap A_{n}^{m} \cap D_{n}^{h}) = 0,
\]
and
\[
\lim_{n \to \infty} \mathcal{L}(Q_{n}^{\varepsilon,2} \cap A_{n}^{m}) = 0.
\]
For limit (6.24), it follows by the following observation and the assumption
\[
Q_{n}^{\varepsilon,1} \cap A_{n}^{m} \cap D_{n}^{h} \subset \left\{ z \in Q : \sup_{|r| \leq m, |r - r'| \leq h} |f(z, r) - f(z, r')| \geq \varepsilon/2 \right\}.
\]
For limit (6.25), note that by the definition of \(f_n\) and the assumption,
\[
\lim_{h \to 0} \sup_{n \in \mathbb{N}, |r| \leq m} \int_{Q} |f_n(z, r + h) - f_n(z, r)| dz \leq \lim_{h \to 0} \int_{Q'} |f(z, r + h) - f(z, r)| dz = 0,
\]
where \(Q \subset Q'\). Since for each \(r\),
\[
\lim_{n \to \infty} \int_{Q} |f_n(z, r) - f(z, r)| dz = 0,
\]
by (6.26) and a finitely covering technique, we have for each \(m \in \mathbb{N}\),
\[
\lim_{n \to \infty} \sup_{|r| \leq m} |f_n(z, r) - f(z, r)| dz = 0,
\]
which in turn implies (6.25) by noting that
\[
Q_{n}^{\varepsilon,2} \cap A_{n}^{m} \subset \left\{ z \in Q : \sup_{|r| \leq m} |f_n(z, r) - f(z, r)| \geq \varepsilon/2 \right\}.
\]
The proof is complete. \(\square\)

Now define
\[
a_n(t, x) := \int_{\mathbb{R}^{2d}} a_n \left( t, x, (\rho_{t}^{n} \ast \Gamma_{n})(x), z' \right) \rho_{t}^{n}(z') dz',
\]
where \(\langle \rho_{t}^{n}(x) \rangle := \int_{\mathbb{R}^{d}} \rho_{t}^{n}(x, v) dv\) and
\[
b_n(t, z) := \int_{\mathbb{R}^{2d}} b_n \left( t, z, (\rho_{t}^{n} \ast \Gamma_{n})(z), z' \right) \rho_{t}^{n}(z') dz'.
\]
We have

**Lemma 6.11.** For any bounded domain \(Q \subset \mathbb{R}_{+} \times \mathbb{R}^{2d}\), it holds that for any \((q_0, p_0) \in [1, \infty)^{1+2d}\),
\[
\lim_{n \to \infty} \| (a_n - a)_{Q} \|_{L_{q}^{q_0}(L_{p}^{p_0})} = 0,
\]
and for any \(q_0 < q_1\) and \(p_0 < p_1\), where \((q_1, p_1)\) is from (1.9),
\[
\lim_{n \to \infty} \| (b_n - b)_{Q} \|_{L_{q}^{q_0}(L_{p}^{p_0})} = 0,
\]
where \(a_{\rho}\) and \(b_{\rho}\) are defined by (6.3) and (6.4) in terms of \(\rho\) in Lemma 6.9.

**Proof.** We only prove (6.30) since (6.29) is completely the same. Note that by (1.9) and (5.5),
\[
\sup_{n} \| (b_n - b)_{Q} \|_{L_{q}^{q_{1}}(L_{p}^{p_{1}})} \lesssim \| h \|_{L_{q}^{q_{1}}(L_{p}^{p_{1}})}.
\]
It suffices to prove that
\[
\lim_{n \to \infty} \| (b_n - b)_{Q} \|_{L_{1}^{1}} = 0.
\]
For $R \geq 1$, let $B_R := \{ z \in \mathbb{R}^d : |z| \leq R \}$. By (6.28) and (6.4), we make the following decomposition:

$$\begin{align*}
(\bar{b}_n - b_\nu)(t, z) &= \int_{\mathbb{R}^d} b_n(t, z, (\rho_\nu^n \ast \Gamma_n)(z), z') \rho_\nu^n(z') dz' - \int_{\mathbb{R}^d} b(t, z, \rho_t(z), z') \rho_t(z') dz' \\
&= \left\{ \int_{B_R^c} b_n(t, z, (\rho_\nu^n \ast \Gamma_n)(z), z') \rho_\nu^n(z') dz' - \int_{B_R^c} b(t, z, \rho_t(z), z') \rho_t(z') dz' \right\} \\
&+ \left\{ \int_{B_R} b_n(t, z, (\rho_\nu^n \ast \Gamma_n)(z), z') \rho_\nu^n(z') dz' - \int_{B_R} b(t, z, \rho_t(z), z') \rho_t(z') dz' \right\} \\
&=: I^n_R(t, z) + J^n_R(t, z).
\end{align*}$$

For $I^n_R$, noting that by (1.9) and Minkowski’s inequality,

$$\|I^n_R\|_{\mathcal{E}^1} \leq \|h\|_{\mathcal{E}^1} \sup_{t \in [0,T]} \left( \int_{B_R^c} \rho_\nu^n(z') dz' + \int_{B_R^c} \rho_t(z') dz' \right),$$

by the tightness of $(\mathbb{P}_n)_{n \in \mathbb{N}}$, we get

$$\lim_{R \to \infty} \sup_{n \in \mathbb{N}} \|I^n_R\|_{\mathcal{E}^1} = 0.$$

Thus it remains to show that for each $R \geq 1$,

$$\lim_{n \to \infty} \|J^n_R 1_Q\|_{\mathcal{E}^1} = 0. \tag{6.31}$$

We make the following decomposition for $J^n_R$,

$$J^n_R(t, z) = \int_{B_R} \left( b_n(t, z, (\rho_\nu^n \ast \Gamma_n)(z), z') - b(t, z, \rho_t(z), z') \right) \rho_t(z') dz'$$

$$+ \int_{B_R} b_n(t, z, (\rho_\nu^n \ast \Gamma_n)(z), z') \left( \rho_\nu^n(z') - \rho_t(z') \right) dz' =: J^{1,n}_R(t, z) + J^{2,n}_R(t, z).$$

For $J^{1,n}_R$, noting that by (6.20),

$$\lim_{n \to \infty} \|(\rho_\nu^n \ast \Gamma_n - \rho) 1_Q\|_{\mathcal{E}^1} = 0,$$

by (1.8) and Lemma 6.10, we have for each $\varepsilon > 0$,

$$\lim_{n \to \infty} \mathcal{L} \left\{ (t, z, z') \in Q : |b_n(t, z, (\rho_\nu^n \ast \Gamma_n)(z), z') - b(t, z, \rho_t(z), z')| \geq \varepsilon \right\} = 0.$$

Thus, by the uniform integrability of $J^{1,n}_R 1_Q$,

$$\lim_{n \to \infty} \|J^{1,n}_R 1_Q\|_{\mathcal{E}^1} = 0. \tag{6.32}$$

For $J^{2,n}_R$, by (1.9), (6.20) and Hölder’s inequality, we have

$$\lim_{n \to \infty} \|J^{2,n}_R\|_{\mathcal{E}^1} \leq \|h\|_{\mathcal{E}^2(\mathcal{L}_1)} \lim_{n \to \infty} \|\rho^n - \rho\|_{\mathcal{L}^2(\mathcal{L}_1)} = 0,$$

which together with (6.32) yields (6.31). The proof is complete. \hfill \Box

Now we can give the proof of Theorem 6.3.

**Proof of Theorem 6.3.** Let $\bar{P}$ be the accumulation point in Lemma 6.9. We aim to show $\bar{P} \in \mathcal{M}^{b,a}_{\nu}$. More precisely, we need to show that for each $t_1 > t_0 \geq 0$ and bounded continuous $\mathcal{B}_{t_0}$-measurable functional $G_{t_0}$,

$$\mathbb{E}^P(M_{t_1} G_{t_0}) = \mathbb{E}^\bar{P}(M_{t_0} G_{t_0}), \tag{6.33}$$

where $M_t$ is defined by (6.2). Below we simply write $\mathbb{E} := \mathbb{E}^P$ and $\mathbb{E}_n := \mathbb{E}^{\bar{P}_n}$. By SDE (6.11) and Itô’s formula, it is easy to see that

$$\mathbb{E}_n(M_{t_1}^n G_{t_0}) = \mathbb{E}(M_{t_1}^n(Z^n) G_{t_0}(Z^n)) = \mathbb{E}(M_{t_0}^n(Z^n) G_{t_0}(Z^n)) = \mathbb{E}_n(M_{t_0}^n G_{t_0}), \tag{6.34}$$

Therefore, for each $t_1 > t_0 \geq 0$ and bounded continuous $\mathcal{B}_{t_0}$-measurable functional $G_{t_0}$,

$$\mathbb{E}(M_{t_1} G_{t_0}) = \mathbb{E}(M_{t_0} G_{t_0}).$$

This completes the proof of Theorem 6.3.
where
\[ M_t^n(\omega) := f(\omega_t) - f(\omega_0) - \int_0^t \left( \text{tr}(\tilde{a}_n \cdot \nabla_v f) + v \cdot \nabla_v f + \tilde{b}_n \cdot \nabla_v f \right) (r, \omega_r) \, dr, \]
where \( \tilde{a}_n \) and \( \tilde{b}_n \) are defined by (6.27) and (6.28). Note that (6.34) is equivalent to
\[ E_n \left( G_{t_0} \int_{t_0}^{t_1} \left( \text{tr}(\tilde{a}_n \cdot \nabla_v f) + v \cdot \nabla_v f + \tilde{b}_n \cdot \nabla_v f \right) (r, \omega_r) \, dr \right) = 0. \tag{6.35} \]

We use Lemma 6.5 and Lemma 6.11 to show the following limits
\[ \lim_{n \to \infty} E_n \left( G_{t_0} \int_{t_0}^{t_1} (\tilde{a}_n \cdot \nabla_v f)(r, \omega_r) \, dr \right) = E \left( G_{t_0} \int_{t_0}^{t_1} (a_p \cdot \nabla_v f)(r, \omega_r) \, dr \right) \tag{6.36} \]
and
\[ \lim_{n \to \infty} E_n \left( G_{t_0} \int_{t_0}^{t_1} (\tilde{b}_n \cdot \nabla_v f)(r, \omega_r) \, dr \right) = E \left( G_{t_0} \int_{t_0}^{t_1} (b_p \cdot \nabla_v f)(r, \omega_r) \, dr \right). \tag{6.37} \]

We only show (6.37) since (6.36) is completely the same. For (6.37), it follows from the following two limits:
\[ \lim_{n \to \infty} E_n \left( G_{t_0} \int_{t_0}^{t_1} ((\tilde{b}_n - b_p) \cdot \nabla_v f)(r, \omega_r) \, dr \right) = 0, \tag{6.38} \]
and
\[ \lim_{n \to \infty} E_n \left( G_{t_0} \int_{t_0}^{t_1} (b_p \cdot \nabla_v f)(r, \omega_r) \, dr \right) = E \left( G_{t_0} \int_{t_0}^{t_1} (b_p \cdot \nabla_v f)(r, \omega_r) \, dr \right). \tag{6.39} \]

We first show (6.38). Let \( q_0 := q_1/2, \quad p_0 := p_1/2. \)

For any \( R \geq 1, \) since \( \frac{q_0}{q_0} + a \cdot \frac{1}{p_0} < 2, \) by Krylov’s estimate (6.12) and (6.30), we have
\[ E_n \left( \int_{t_0}^{t_1} \left| 1_{B_R}(\tilde{b}_n - b_p) \right| (r, \omega_r) \, dr \right) \lesssim \left\| 1_{[t_0, t_1] \times B_R} (\tilde{b}_n - b_p) \right\|_{L_{t_0}^{q_0}(L_{p_0})} \to 0, \]
and by Hölder’s inequality,
\[ E_n \left( \int_{t_0}^{t_1} \left| 1_{B_R}(\tilde{b}_n - b_p) \right| (r, \omega_r) \, dr \right) \lesssim \left( \int_{t_0}^{t_1} P_n(|\omega_r| \geq R) \, dr \right)^{\frac{1}{2}} \left( E_n \int_{t_0}^{t_1} |\tilde{b}_n - b_p|^2 (r, \omega_r) \, dr \right)^{\frac{1}{2}} \]
\[ \lesssim \sup_{r \in [t_0, t_1]} \left( P_n(|\omega_r| \geq R) \right)^{\frac{1}{2}} \left\| \tilde{b}_n - b_p \right\|_{L_{t_0}^{q_0}(L_{p_0})} \]
\[ \lesssim \sup_{r \in [t_0, t_1]} \left( P(|Z_{r}^m| \geq R) \right)^{\frac{1}{2}}, \]
which converges to zero uniformly in \( n \) as \( R \to \infty \) by the proof of Lemma 6.8. Thus we obtain (6.38). Next we look at (6.39). Let
\[ \tilde{b}_p^m(t, z) := b_p(t, \cdot) * \Gamma_{m}(z). \]
As above, by Krylov’s estimate (6.12), we have
\[ \lim_{m \to \infty} \sup_n E_n \left( \int_{t_0}^{t_1} \left| (\tilde{b}_p^m - b_p) \cdot \nabla_v f \right| (r, \omega_r) \, dr \right) = 0. \tag{6.40} \]
Moreover, for fixed \( m \in \mathbb{N}, \) since \( \omega \to G_{t_0}(\omega) \int_{t_0}^{t_1} (b_p^m \cdot \nabla_v f)(r, \omega_r) \, dr \) is a bounded continuous functional, we have
\[ \lim_{n \to \infty} E_n \left( G_{t_0} \int_{t_0}^{t_1} (b_p^m \cdot \nabla_v f)(r, \omega_r) \, dr \right) = E \left( G_{t_0} \int_{t_0}^{t_1} (b_p^m \cdot \nabla_v f)(r, \omega_r) \, dr \right), \]
which together with (6.40) yields (6.39). Thus, by taking limits \( n \to \infty \) for (6.35), we obtain
\[
\mathbb{E} \left( G_{t_0}^1 \int_{t_0}^{t_1} \left( \text{tr}(a_{\rho} \cdot \nabla_v f) + v \cdot \nabla_x f + b_{\rho} \cdot \nabla_v f \right) (r, \omega_r) dr \right) = 0,
\]
which implies (6.33). As for the regularity (1.11), it follows by (6.16) and (6.20). The proof is complete.

**Remark 6.12.** Suppose that the coefficients do not depend on the density, that is, \( b(t, z, r, z') \) and \( a(t, x, r, z') \) are independent of \( r \), then we can drop the assumption \( q_1 \in (2, 4) \) in (H₁) but require \( \frac{1}{p_1} + \frac{1}{q_1} < \frac{1}{2} \) (see Remark 4.4). In this case, we can use the Kyrlov estimate (6.15) and follow the same argument as in [27] to take the limits.

### 7. Well-Posedness of Generalized Martingale Problems

In this section we show the well-posedness for a class of generalized martingale problems of SDE (6.1) when diffusion matrix does not depend on the distribution, but may be discontinuous in position variable \( x \). Instead of (H₁) and (H₂), we suppose that (H₃) holds. Fix \( T > 0 \) and \( f \in C^\infty_c(\mathbb{R}^{2d}) \). Consider the following backward Kolmogorov equation:
\[
\partial_t u + \text{tr}(a \cdot \nabla_v^2 u) + v \cdot \nabla_x u + b_{\rho} \cdot \nabla_v u = f, \quad u(T) = 0,
\](7.1)
where for a family of density functions \( \rho_t(z) \) in \( \mathbb{R}^{2d} \),
\[
b_{\rho}(t, z) := \int_{\mathbb{R}^{2d}} b(t, z, \rho_t(z), z') \rho_t(z') dz'.
\]
Since \( b_{\rho} \) is bounded measurable and \( a \) does not depend on \( v \), by reversing the time variable and Theorem 5.2, there is a unique weak solution \( u^f_{\rho} \) to (7.1) with
\[
u \in \tilde{\mathcal{T}}_T \cap C_b([0, T] \times \mathbb{R}^{2d}).
\]
Here the continuity of \( u^f_{\rho} \) is crux for the following notion of generalized martingale solutions.

**Definition 7.1** (Generalized martingale problem). Let \( s \geq 0 \) and \( \nu \in \mathcal{P}(\mathbb{R}^{2d}) \). A probability measure \( \mathbb{P} \in \mathcal{P}(\mathcal{C}) \) is called a generalized martingale solution of DDSDE (6.1) starting from \( \nu \) at time \( s \) if
(i) \( \mathbb{P} \circ \omega^{-1}_s = \nu \), and for Lebesgue almost all \( t \geq s \), \( \mathbb{P} \circ \omega^{-1}_t (dz) = \rho_t(z) dz \).
(ii) For any \( T > s \) and \( f \in C^\infty_c(\mathbb{R}^{2d}) \), the process
\[
M_t := u^f_{\rho}(t, \omega_t) - u^f_{\rho}(s, \omega_s) - \int_s^t f(\omega_r) dr, \quad t \in [s, T],
\]
is a \( \mathcal{B}_t \)-martingale with respect to \( \mathbb{P} \), where \( u^f_{\rho} \) is the unique solution of (7.1).

The set of all the generalized martingale solutions \( \mathbb{P} \in \mathcal{P}(\mathcal{C}) \) with initial distribution \( \nu \) at time \( s \) is denoted by \( \mathcal{M}^{\omega,s}_{\rho,\nu} \).

**Remark 7.2.** The above notion of martingale solutions was introduced in its most general form in [9, Chapter 4]. It should be noticed that if (7.1) has a \( C^2 \)-solution \( u \), then by Itô’s formula, any weak solution of (6.1) must be a generalized martingale solution. In general, these two notions are not equivalent.

Now we can give

**Proof of Theorem 1.5. (Existence)** Without loss of generality we assume \( s = 0 \). As in Section 6, we consider the approximating SDE (6.11). Let \( \mathbb{P} \in \mathcal{P}(\mathcal{C}) \) be an accumulation point of \( (\mathbb{P}_n)_{n \in \mathbb{N}} \). In particular, by Lemma 6.8,
\[
\mathbb{P}_n \circ \omega^{-1}_t(z) = \rho^n_t(z) dz, \quad \mathbb{P} \circ \omega^{-1}_t(z) = \rho_t(z) dz.
\]
In the following we use the convention before Lemma 6.10. Recall $a_n(t,x) = a(t,\cdot) * G_n(x)$ and $\bar{b}_n$ being defined by (6.28). Since for each $\varphi \in C_c^\infty(\mathbb{R}^{2d})$, by Itô's formula,

$$\partial_t \varphi_{\rho t} = \int \text{tr}(a_n \cdot \nabla_{\varphi}^2 \varphi) \rho_t + \int (v \cdot \nabla_x \varphi) \rho_t + \int (\bar{b}_n \cdot \nabla_v \varphi) \rho_t,$$

by Lemmas 6.9 and 6.11 and taking limits, it is easy to see that

$$\partial_t \varphi_t = \int \text{tr}(a \cdot \nabla_{\varphi}^2 \varphi) \rho_t + \int (v \cdot \nabla_x \varphi) \rho_t + \int (\bar{b} \cdot \nabla_v \varphi) \rho_t.$$

In other words, $\rho_t$ solves (1.12) in the distributional sense. Fix $T > 0$ and $f \in C_c^\infty(\mathbb{R}^{2d})$. Let $u_n \in L^\infty_T(C_c^\infty(\mathbb{R}^{2d}))$ be the unique smooth solution of the following backward PDE:

$$\partial_t u_n + \text{tr}(a_n \cdot \nabla_{\rho t}^2 u_n) + v \cdot \nabla_x u_n + \bar{b}_n \cdot \nabla_v u_n = f, \quad u_n(T) = 0. \quad (7.3)$$

By Lemma 6.11 and Theorem 5.2, we have for any bounded $Q \subset (0,T) \times \mathbb{R}^{2d},$

$$\lim_{n \to \infty} \sup_{(t,z) \in Q} |u_n(t,z) - u^f(T,t,z)| = 0, \quad (7.4)$$

where $u^f$ is the unique weak solution of PDE (7.1). Now we show that for each $T \geq t_1 > t_0 \geq 0$ and bounded continuous $\mathcal{B}_{t_0}$-measurable functional $G_{t_0}$,

$$\mathbb{E}^\rho_1(M_{t_1}G_{t_0}) = \mathbb{E}^\rho_2(M_{t_0}G_{t_0}), \quad (7.5)$$

where $M_t$ is defined by (7.2). By SDE (6.11), Itô's formula and (7.3), it is easy to see that

$$\mathbb{E}^\rho_n(M_{t_1}^nG_{t_0}) = \mathbb{E}^\rho_{n+1}(M_{t_0}^nG_{t_0}), \quad (7.6)$$

where

$$M_t^n := u_n(t,\omega_t) - u_n(0,\omega_0) - \int_0^t f(\omega_r)dr, \quad (7.7)$$

By taking limits for both sides of (7.6) and using the pointwise convergence (7.4), we obtain (7.5).

**Uniqueness** We divide the proof into three steps.

(Step 1) First of all, we show the uniqueness for linear SDE, i.e., $b(t, z, r, z') = b(t, z)$ does not depend on variables $r, z'$. Let $\mathbb{P}_1, \mathbb{P}_2 \in \mathcal{F}_{s,t}$ be two solutions of the generalized martingale problem. Fix $T > s$ and $f \in C_c^\infty(\mathbb{R}^{2d})$. Let $u^f$ be the unique weak solution of (see Theorem 5.2),

$$\partial_t u^f + \text{tr}(a \cdot \nabla_{\rho t}^2 u^f) + v \cdot \nabla_x u^f + b \cdot \nabla_v u^f = f, \quad u^f(T) = 0. \quad (7.8)$$

By Definition 7.1 and $u^f(T) = 0$, we have

$$\int_{\mathbb{R}^{2d}} u^f(s, z)\nu(dz) = -\mathbb{E}^\mathbb{P}_1 \int_s^T f(\omega_r)dr, \quad i = 1, 2,$$

which means that for each $T > s$,

$$\int_s^T \mathbb{E}^\mathbb{P}_1 f(\omega_r)dr = \int_s^T \mathbb{E}^\mathbb{P}_2 f(\omega_r)dr.$$

Hence, for any $f \in C_c^\infty(\mathbb{R}^{2d})$

$$\mathbb{E}^\mathbb{P}_1 f(\omega_T) = \mathbb{E}^\mathbb{P}_2 f(\omega_T), \quad \forall T > s. \quad (7.9)$$

From this, by a standard way (see Theorem 4.4.2 in [9]), we derive that

$$\mathbb{P}_1 = \mathbb{P}_2.$$

Indeed, it suffices to prove the following claim by induction:

(Cn) for given $n \in \mathbb{N}$, and for any $s \leq t_1 < t_2 < \cdots < t_n < T$ and strictly positive and bounded measurable functions $g_1, \cdots, g_n$ on $\mathbb{R}^{2d}$,

$$\mathbb{E}^\mathbb{P}_1(g_1(\omega_{t_1}) \cdots g_n(\omega_{t_n})) = \mathbb{E}^\mathbb{P}_2(g_1(\omega_{t_1}) \cdots g_n(\omega_{t_n})). \quad (7.10)$$
By (7.7), (C_1) holds. Suppose now that (C_n) holds for some n ≥ 2. For simplicity we write
\[ \eta := g_1(\omega_t) \cdots g_n(\omega_{t_n}) > 0, \]
and for i = 1, 2, we define new probability measures
\[ d\bar{P}_i := \eta dP_i/(E^\eta) \in \mathcal{P}(\mathcal{D}), \quad \bar{\nu}_i := \bar{P}_i \circ \omega_{t_n}^{-1} \in \mathcal{P}(R^{2d}). \]
Now we show
\[ \bar{P}_i \in \mathcal{M}_{t_n; \bar{\nu}_i}, \quad i = 1, 2. \]
For any \( f \in C^\infty_c(R^{2d}), \) let
\[ M_t := u^f(t, \omega_t) - u^f(t_n, \omega_{t_n}) - \int_{t_n}^t f(\omega_r)dr, \quad t \in [t_n, T]. \]
We only need to prove that for any \( T \geq T' > t \geq t_n \) and bounded \( B_t \)-measurable \( \xi, \)
\[ E^\tilde{P}_i(M_t, \xi) = E^\tilde{P}_i(M_t \xi) \Leftrightarrow E^{\bar{P}_i}(M_t, \xi) = E^{\bar{P}_i}(M_t \xi), \]
which follows from \( P_i \in \mathcal{M}^{a,b}_{t_n}, i = 1, 2. \) Thus, by induction hypothesis and (7.7),
\[ \bar{\nu}_1 = \bar{\nu}_2 \Rightarrow \bar{P}_1 \circ \omega_{t_{n+1}}^{-1} = \bar{P}_2 \circ \omega_{t_{n+1}}^{-1}, \quad \forall \ T \geq t_{n+1} > t_n. \]
which in turn implies that (C_{n+1}) holds.

(Step 2) For general nonlinear SDE, we use Girsanov’s transformation method (see [27, 13]). Without loss of generality, we assume \( s = 0. \) Let \( \mathcal{P}_1, \mathcal{P}_2 \in \mathcal{M}^{a,b}_{0,v} \) be two solutions of the generalized martingale problem. Let
\[ a_n := a \ast \Gamma_n, \quad b_n := b \ast \Gamma_n, \quad \rho_0^\ast := \rho_0 \ast \Gamma_n, \]
and for \( \bar{P}_i \circ \omega_{t_n}^{-1}(dz) = \rho_t^i(z)dz, \)
\[ \bar{b}_n^i(t, z) := \int_{R^{2d}} b_n(t, z, (\rho_t^i \ast \Gamma_n)(z), z')\rho_t^i(z')dz'. \]
Note that for any \( T > 0, \)
\[ a_n, \bar{b}_n^i \in L^\infty(T(C^\infty_b(\mathcal{D})), \quad \| a_n \| \infty \leq \| a \| \infty, \quad \| \bar{b}_n^i \| \infty \leq \| b \| \infty. \]
We consider the following approximation of linearized SDEs: for \( i = 1, 2, \)
\[ dX_t^{i,n} = V_t^{i,n}dt, \quad dV_t^{i,n} = \bar{b}_n^i(t, Z_t^{i,n})dt + \sqrt{2a_n(t, X_t^{i,n})}dW_t, \]
(7.10)
where \( Z_0^{i,n} \) is a \( \mathcal{F}_0 \)-measurable random variable and has smooth density \( \rho_0^\ast. \) Since by Lemma
6.10,
\[ \bar{b}_n^i(t, z) \xrightarrow{n \to \infty} b_{t}^i(t, z) = \int_{R^{2d}} b(t, z, \rho_t^i(z), z')\rho_t^i(z')dz' \] 
Lebesgue measure dtdz, as in the proof of the existence part, and due to the uniqueness of linear equations, for \( i = 1, 2, \)
the law of \( Z_t^{i,n} \) weakly converges to \( P_i, \) as \( n \to \infty. \) In particular, for any \( \varphi \in C_b(R^{2d}), \)
\[ E_{\varphi}(Z_t^{i,n}) \xrightarrow{n \to \infty} E_{\bar{P}_i}(\varphi(\omega_t)), \quad i = 1, 2. \]
(7.11)
Since \( a_n, \bar{b}_n^i \in L^\infty(T(C^\infty_b(\mathcal{D}))) \) and \( \rho_0^\ast \in C^\infty_b, \) it is well known that the density \( \rho_t^i \) of \( Z_t^{i,n} \) is smooth and uniquely solves the following FPKE:
\[ \partial_t \rho_t^i = \text{div}_v(a \cdot \nabla_v \rho_t^i) - v \cdot \nabla_x \rho_t^i + \text{div}_v(\bar{b}_n^i \rho_t^i), \]
where we have used that \( a_n \) does not depend on the velocity variable \( v. \) By Theorem 5.3 with \( b_1 = 0, b_2 = \bar{b}_n^i \) and \( g = 0, \) for any \( T > 0, \)
there is a constant \( C > 0 \) such that for all \( n \in \mathbb{N}, \)
\[ \| 1_{[0,T]} \rho_t^i \| \infty \leq C \| \rho_0^\ast \| \mathcal{L}_1 \leq C \| \rho_0 \| \mathcal{C}_1, \quad i = 1, 2. \]
(7.12)
(Step 3) To perform the Girsanov transformation, for \( i = 1, 2 \) and \( n \in \mathbb{N}, \) we define
\[ H_n^i(s, z) := (\sqrt{2a_n})^{-1}(s, x) \cdot \bar{b}_n^i(s, z), \quad z = (x, v), \]
(7.13)
In particular, for any \( \phi \)

\[
\frac{1}{2} \left| H_n^{(i)}(s, Z_s^{i,n}) \right|^2 \text{d}s
\]

Fix \( T > 0 \). Since for some \( C > 0 \) independent of \( n \),

\[
\| H_n^{(i)} \|_\infty \leq C, \quad i = 1, 2,
\]

by Girsanov’s theorem, for \( i = 1, 2 \), under the new probability measure \( Q_n^{i,n} := A_T^i \mathbf{P} \),

\[
\mathcal{W}^i_n := \int_0^t H_n^{(i)}(s, Z_s^{i,n}) \text{d}s + W_t, \quad t \in [0, T],
\]

is still a Brownian motion, and

\[
dX_t^{i,n} = V_t^{i,n} \text{d}t, \quad dV_t^{i,n} = \sqrt{2} \alpha_n(t, X_t^{i,n}) \text{d}\mathcal{W}_t^{i,n}.
\]

Since \( \sqrt{\alpha_n} \) is Lipschitz, the above SDE admits a unique weak solution. Thus

\[
Q_n^{1,n} \circ (Z^{1,n})^{-1} = Q_n^{2,n} \circ (Z^{2,n})^{-1},
\]

equivalently, for any nonnegative functional \( G \) on \( C_T := C([0, T]; \mathbb{R}^d) \),

\[
\mathbf{E}(G(Z^{1,n})A_T^i) = \mathbf{E}(G(Z^{2,n})A_T^i).
\]

In particular, for any \( \varphi \in C_b(\mathbb{R}^d) \),

\[
\mathbf{E}\varphi(Z^{1,n}) = \mathbf{E}(\varphi(Z^{2,n})Y^n_T),
\]

where

\[
Y^n_T := \exp \left\{ \int_0^T (H_n^{(1)} - H_n^{(2)})(s, Z_s^{2,n}) \text{d}W_s + \frac{1}{2} \int_0^T \left| H_n^{(1)} - H_n^{(2)} \right|^2(s, Z_s^{2,n}) \text{d}s \right\}.
\]

Thus, by Hölder’s inequality,

\[
|\mathbf{E}\varphi(Z^{1,n}_T) - \mathbf{E}\varphi(Z^{2,n}_T)| = |\mathbf{E}(\varphi(Z^{2,n}_T)(Y^n_T - 1))| \leq \|\varphi(Z^{2,n}_T)\|_{L^2(\Omega)} \|Y^n_T - 1\|_{L^2(\Omega)}
\]

\[
\lesssim \left( \|\varphi\|_\infty \wedge \|\varphi\|_{L^2(\Omega)} \right) \|Y^n_T - 1\|_{L^2(\Omega)},
\]

where the implicit constant is independent of \( n \), and we have used (7.12) to derive that

\[
\|\varphi(Z^{2,n}_T)\|_{L^2(\Omega)} = \left( \int_{\mathbb{R}^{2d}} |\varphi(z)|^2 \alpha^{2,n}(z) \text{d}z \right)^{1/2} \lesssim \|\varphi\|_{L^2(\Omega)}.
\]

On the other hand, by Itô’s formula, we have

\[
Y^n_T - 1 = \int_0^T Y^n_s (H_n^{(1)} - H_n^{(2)})(s, Z_s^{2,n}) \text{d}W_s
\]

\[
+ \frac{1}{2} \int_0^T Y^n_s \left| H_n^{(1)} - H_n^{(2)} \right|^2(s, Z_s^{2,n}) \text{d}s
\]

\[
+ \frac{1}{2} \int_0^T Y^n_s \left| H_n^{(1)} - H_n^{(2)} \right|^2(s, Z_s^{2,n}) \text{d}s.
\]

By BDG’s inequality and (7.14), we have

\[
\mathbf{E}|Y^n_T - 1|^2 \lesssim \int_0^T \mathbf{E}|Y^n_s - 1|^2 \text{d}s + \int_0^T \mathbf{E}(\left| H_n^{(1)} - H_n^{(2)} \right|^2(s, Z_s^{2,n})) \text{d}s,
\]

which implies by Gronwall’s inequality that

\[
\mathbf{E}|Y^n_T - 1|^2 \lesssim \int_0^T \mathbf{E}(\left| H_n^{(1)} - H_n^{(2)} \right|^2(s, Z_s^{2,n})) \text{d}s.
\]

Note that by (7.9), (7.13) and (H3),

\[
\left| H_n^{(1)} - H_n^{(2)} \right|(t, z) \leq \| (\sqrt{2}a_n)^{-1} \|_{\infty} |\tilde{b}^{(1)} - \tilde{b}^{(2)}|(t, z) \lesssim |\rho_1^2 - \rho_2^2| \ast \Gamma_n(z) + \|\rho_1^2 - \rho_2^2\|_{L^1(\Omega)}.
\]
Therefore, by (7.12) again,
\[
E|Y^n_T - 1|^2 \lesssim \int_0^T E\left(\rho_s^1 - \rho_s^2\right) \Gamma_s \left|Z^n_s\right|^2 \, ds + \int_0^T \|\rho_s^1 - \rho_s^2\|^2_{L^2_x} \, ds
\]
\[
\lesssim \int_0^T \|\rho_s^1 - \rho_s^2\|^2_{L^2_x} \, ds + \int_0^T \|\rho_s^1 - \rho_s^2\|^2_{L^2_x} \, ds
\]
\[
\lesssim \int_0^T \left(\|\rho_s^1 - \rho_s^2\|^2_{L^2_x} \vee \|\rho_s^1 - \rho_s^2\|^2_{L^2_x}\right) \, ds.
\]
Combining (7.15) and (7.16) and by (7.11), we obtain that for all \( \varphi \in C_b(\mathbb{R}^{2d}) \),
\[
|E^{P_1}\varphi(\omega_T) - E^{P_2}\varphi(\omega_T)| = \lim_{n \to \infty} |E\varphi(Z^n_T) - E\varphi(Z^{2,n}_T)|
\]
\[
\lesssim \left(\|\varphi\|_{\infty} \wedge \|\varphi\|_{L^2_x}\right) \left(\int_0^T \left(\|\rho_s^1 - \rho_s^2\|^2_{L^2_x} \vee \|\rho_s^1 - \rho_s^2\|^2_{L^2_x}\right) \, ds\right)^{1/2}.
\]
Since for Lebesgue almost all \( T > 0, \mathbb{P}_0 \circ \omega_T^{-1}(dz) = \rho^T_0(z) \, dz \), the above inequality means that for Lebesgue almost all \( T > 0 \),
\[
\|\rho_T^1 - \rho_T^2\|^2_{L^2_x} \vee \|\rho_T^1 - \rho_T^2\|^2_{L^2_x} \lesssim \left(\int_0^T \left(\|\rho_s^1 - \rho_s^2\|^2_{L^2_x} \vee \|\rho_s^1 - \rho_s^2\|^2_{L^2_x}\right) \, ds\right)^{1/2},
\]
which in turn implies that \( \rho_T^1 = \rho_T^2 \) by Gronwall’s inequality. Finally, as usual we use the uniqueness for linear equations in Step 1 to derive \( \mathbb{P}_1 = \mathbb{P}_2 \). The proof is complete.

Remark 7.3. (i) From the proof of the existence part, it is easy to see that the unique generalized martingale solution \( P \in \mathcal{P}^{a,b}_{\nu_0,\nu} \) also belongs to \( \mathcal{P}^{a,b}_{\nu} \), i.e., it is also a classical martingale solution.

(ii) If \( b \) does not depend on the density variable \( r \), then we can drop the assumption on the initial distribution \( \nu \) since in this case the boundedness (7.12) is not needed in the proof and only the total variational norm \( \|\rho^1_s - \rho^2_s\|^2_{L^2_x} \) is used in (7.17).

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