Deciding Monotone Duality and Identifying Frequent Itemsets in Quadratic Logspace

Abstract

The monotone duality problem is defined as follows: Given two monotone formulas $f$ and $g$ in irredundant DNF, decide whether $f$ and $g$ are dual. This problem is the same as duality testing for hypergraphs, that is, checking whether a hypergraph $H$ consists of precisely all minimal transversals of a hypergraph $G$. By exploiting a recent problem-decomposition method by Boros and Makino (ICALP 2009), we show that duality testing for hypergraphs, and thus for monotone DNFs, is feasible in $DSPACE[\log^2 n]$, i.e., in quadratic logspace. As the monotone duality problem is equivalent to a number of problems in the areas of databases, data mining, and knowledge discovery, the results presented here yield new complexity results for those problems, too. For example, it follows from our results that whenever, for a Boolean-valued relation (whose attributes represent items), a number of maximal frequent itemsets and a number of minimal infrequent itemsets are known, then it can be decided in quadratic logspace whether there exist additional frequent or infrequent itemsets.

1 Introduction

This paper derives new complexity bounds for the problem $\text{Dual}$ of deciding whether two irredundant monotone Boolean formulas in DNF are mutually dual, or, equivalently, of deciding whether two simple hypergraphs are dual, i.e., whether each of these hypergraphs consists precisely of the minimal transversals of the other. While the exact complexity remains open, there is progress: We prove in the present paper a $DSPACE[\log^2 n]$ upper bound for $\text{Dual}$, and another, presumably tighter bound for the same problem, that is expressed in terms of sophisticated machine-bounded complexity classes. The $\text{Dual}$ problem is actually one of the most mysterious problems in theoretical computer science. It has many applications, especially in the database, data mining, and knowledge discovery areas [7, 8, 26, 27], some of which will be mentioned below. Let us first describe the $\text{Dual}$ problem more formally.

$\text{Duality testing for monotone DNFs and hypergraphs}$. A pair of Boolean formulas $f$ and $g$ on propositional variables $x_1, x_2, \ldots, x_n$ are dual if

$$f(x_1, x_2, \ldots, x_n) \equiv \neg g(\neg x_1, \neg x_2, \ldots, \neg x_n).$$

A monotone DNF is irredundant if the set of variables in none of its disjuncts is covered by the variable set of any other disjunct. The duality testing problem $\text{Dual}$ is the problem of testing whether two irredundant monotone DNFs $f$ and $g$ are dual.

A hypergraph $\mathcal{H}$ is a finite family of finite sets (also called hyperedges) defined over some set of vertices $V(\mathcal{H})$. $\mathcal{H}$ is simple if no hyperedge is contained in another one. By default, if $V(\mathcal{H})$ contains fewer hyperedges than vertices, $\mathcal{H}$ is hyper dense, otherwise $\mathcal{H}$ is hyper sparse.

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is not explicitly specified, the set of vertices of $\mathcal{H}$ is $\bigcup_{E \in \mathcal{H}} E$. A transversal of $\mathcal{H}$ is a subset of $V(\mathcal{H})$ that meets all hyperedges of $\mathcal{H}$, and a minimal transversal of $\mathcal{H}$ is a transversal of $\mathcal{H}$ that does not contain any other transversal as subset. The set of all minimal transversals of a hypergraph $\mathcal{H}$ is denoted by $tr(\mathcal{H})$. The Hypergraph Duality Problem is the problem of deciding for two simple hypergraphs $\mathcal{G}$ and $\mathcal{H}$ whether $\mathcal{G} = tr(\mathcal{H})$. Assume $\mathcal{G} \subseteq tr(\mathcal{H})$, then, in case $\mathcal{G} \neq tr(\mathcal{H})$, to witness this, one may want to exhibit a new transversal of $\mathcal{H}$ with respect to $\mathcal{G}$. This is a transversal of $\mathcal{H}$ that has no hyperedge of $\mathcal{G}$ as subset. Obviously, every new transversal $H$ contains at least one new minimal transversal of $\mathcal{H}$ w.r.t. $\mathcal{G}$, but it need not be minimal itself.

It is well known that DNF duality and hypergraph duality are actually the same problem (see [7]). In fact, two irredundant monotone DNFs $f$ and $g$ are dual iff their hypergraphs are dual. The hypergraph associated to a monotone DNF has precisely one hyperedge for each disjunct, consisting of the set of all variables of this disjunct. Vice versa, one can trivially associate an irredundant DNF to each simple hypergraph and thus reduce hypergraph duality to DNF duality. Given that these problems essentially coincide (and can be reduced to each other via trivial reductions that are much easier than logspace reductions), we regard them as one and the same problem, which we refer to as Dual.

The duality problem in data mining, database theory, and knowledge discovery.

The Dual problem is at the core of a number of important data mining and database problems. It is central, for example, to the determination of the maximal frequent and minimal infrequent sets in data mining. More precisely, consider a Boolean-valued data relation $M$ over a set $S$ of attributes called items, and a threshold $z$ with $0 < z \leq |M|$. Each subset $U \subseteq S$ is called an itemset. For each tuple $t$ of $M$, let $items(t) = \{A \in S \mid t[A] = 1\}$. The frequency $f(U)$ for an itemset $U$ is the number of tuples $t$ of $M$, such that $U \subseteq items(t)$. $U$ is frequent if $f(U) > z$ and infrequent otherwise.

In data mining, one considers the maximal frequent itemsets and the minimal infrequent itemsets (under set inclusion) for $M$ and $z$. Let us refer to the former as $IS^+(M, z)$ and to the latter as $IS^-(M, z)$. Clearly, both $IS^+(M, z)$ and $IS^-(M, z)$ are simple hypergraphs over $S$, and we abbreviate them by $IS^+$ and $IS^-$, respectively, when $M$ and $z$ are understood.

The maximal frequent itemsets $IS^+$ are of great interest in the context of data mining, but they are hard to compute. In fact, as shown in [2] that for a given Boolean-valued relation $M$, a threshold $z$ and a set $S \subseteq IS^+(M, z)$, deciding whether there are additional maximal frequent itemsets, i.e., whether $S \neq IS^+(M, z)$, is NP complete. It follows that, assuming $NP \neq \text{P}$, there cannot be any algorithm for enumerating $IS^+(M, z)$ with polynomial delay, and that under the slightly weaker assumption $NP \not\subseteq \text{DTIME}[n^{\text{polylog } n}]$, there is no algorithm enumerating $IS^+(M, z)$ with quasipolynomial delay either. For this reason, rather than computing $IS^+$ only, one often computes $IS^+ \cup IS^-$, which may be exponentially larger in the worst case, but has the advantage of being computable with quasipolynomial delay [3]. As a fundamental result towards the aim of jointly computing $IS^+$ and $IS^-$, it was shown in [26] that the minimal infrequent itemsets are exactly the minimal transversals of the complements of the maximal frequent itemsets, i.e. $IS^- = tr(IS^c)$, and thus also $IS^+ = tr(IS^-)^c$, where for $A \subseteq 2^S$, $A^c = \{S - A \mid A \in S\}$. Let MaxFreq-MinInfreq-Identification be the following decision problem in data mining: Given $M, z$, a set $\mathcal{G} \subseteq IS^-(M, z)$, and a set $\mathcal{H} \subseteq IS^+(M, z)$, decide whether $\mathcal{H} = IS^+(M, z)$ and $\mathcal{G} = IS^-(M, z)$, that is, whether there exists no additional maximal frequent or minimal infrequent itemset for $M$ and $z$, that is not already in $\mathcal{G} \cup \mathcal{H}$. In [26] it was shown that that there exist no such additional itemset iff $\mathcal{G} = tr(\mathcal{H}^c)$. With regard to the computational complexity, we thus have:

**Proposition 1.1** ([26]). MaxFreq-MinInfreq-Identification is logspace-equivalent to Dual.
The results of [26] are at the base of a host of algorithms for maximal frequent itemset
generation, that compute both IS+ and IS− incrementally. These algorithms initialize G and
Hc with some easy to compute subsets of IS− and IS+c, respectively. Then, at each step they
check whether for the current sets G = tr(Hc) is true, and if not, compute one or more new
transversals from which new maximal frequent itemsets or minimal infrequent itemsets can be
computed easily, see, e.g. [39, 36, 25, 2, 43]. Thus, not only the decision problem DUAL is of
relevance to data mining, but also the problem of effectively computing a new transversal that
acts a witness that G ≠ tr(Hc). In the present paper, we will obtain results on the complexity
of this latter problem, too.

Another interesting related database problem is the ADDITIONAL KEY FOR INSTANCE
problem for explicitly given relational instances. Given a relational instance R over attribute set
S, and a set K of minimal keys for R, determine if there exists a minimal key for R that is not
already contained in K. This problem, which has been shown equivalent to DUAL in the early
nineties [7], may be of renewed interest in the age of Big Data, where massive data tables arise
and have to be analyzed, and where the automatic recognition of structural features such as
minimal keys may be useful.

Proposition 1.2 ([7]). The ADDITIONAL KEY FOR INSTANCE problem is logspace equivalent
to DUAL. Moreover, enumerating the minimal keys of a relational instance R is equivalent to
enumerating the set tr(H) for some hypergraph H that is logspace-computable from R.

Other related problems equivalent to DUAL or to DUAL deal with the construction of
Armstrong relations for sets of functional dependencies [7], see also [23, 6].

We also wish to briefly mention a problem from the area of distributed databases. For
quorum-based updates [35] in distributed databases, the concept of coterie, which is essentially
a hypergraph of intersecting quorums has been introduced, and one is specifically interested in
so called non-dominated coteries (for definitions and details, see [16, 30], and for more recent
results and applications, see [37, 38, 28]). The following was proven:

Proposition 1.3 ([30, 7]). A coterie H is non-dominated iff tr(H) = H.

There are a large number of applications of the DUAL problem and of hypergraph dual-
ization in the areas of knowledge discovery, machine learning, and more generally in AI and
knowledge representation. Just to mention a few: Learning monotone Boolean CNFs and
DNFs with membership queries [26], model-based diagnosis [41, 24], computing a Horn ap-
proximation to a non-Horn theory [33, 19], and computing minimal abductive explanations to
observations [10]. Surveys of these and other applications and further references can be found in
[8, 7, 27].

Known complexity results. The exact complexity of DUAL has remained an open prob-
lem. Fredman and Khachiyan [15] have shown that DUAL is in DTIME[n^o(\log n)], more
precisely, that it is contained in DTIME[n^4\chi(n) + O(1)], where \chi(n) is defined by \chi(n)\chi(n) = n.
Eiter, Gottlob, and Makino [9], and independently, Kavvadias and Stavropoulos [34] have
shown that DUAL is in the complexity class co-\beta_2 P, which means that showing that the com-
plement of DUAL can be solved in polynomial time with O(\log^2 n) nondeterministic bits. This
small amount of nondeterminism can actually be lowered to O(\chi(n) \log n) which is o(\log^2 n),
see [9].

Research question tackled The question about the space-efficiency of DUAL, namely,
whether DUAL can be solved using sub-polynomial or even polylogarithmic space was not
satisfactorily answered. It was posed (explicitly or implicitly) several times since 1995, for
example in [7, 44, 11]. This is the main problem we tackle. In addition, we aim at obtaining a
better understanding of the DUAL problem in terms of machine-based structural complexity.
Results. We show in this paper that the decision problem Dual is in the complexity class DSPACE[$\log^2 n$], which is a very low class in POLYLOGSPACE. Modulo the assumption that PTIME $\not\subseteq$ POLYLOGSPACE, which is widely believed, we thus obtain satisfactory evidence that Dual is not PTIME-hard, which answers another complexity question posed in [7]. Our results are based on a careful analysis of a recent problem decomposition method by Boros and Makino [4]. Their decomposition method actually yields a parallel algorithm that solves Dual on an EREW PRAM in $O(\log^2 n)$ time using $n \log^2 n$ processors. However, it is currently not known whether such EREW PRAMS can be simulated in DSPACE[$\log^2 n$], and this is actually considered to be rather unlikely. However, Boros’ and Makino’s algorithm does not seem to exploit the full potential of a PRAM, and by taking into account the restricted pattern of information flow imposed by the specific self-reductions used in their algorithm, we succeeded to show that Dual is in DSPACE[$\log^2 n$].

Complexity theorists have very good reasons to assume that the space class DSPACE[$\log^2 n$] is incomparable with respect to containment to the class co-$\beta_2$P. It is thus somewhat unsatisfactory to have two upper bounds for Dual that are incomparable, which suggests that, most likely, there exist better bounds. This encouraged us to look for a tighter upper bound for Dual in terms of machine-based complexity models, that would be contained in both DSPACE[$\log^2 n$] and co-$\beta_2$P, and we succeeded to find one. We can, in fact, show that Dual belongs to the “guess and check” class GC($\log^2 n$, [LOGSPACE$_{pol}$]$^{\log^3}$). This somewhat exotic new machine-based complexity class consists of all problems that can be solved by first guessing $O(\log^2 n)$ bits and then checking the correctness of this guess by a procedure in [LOGSPACE$_{pol}$]$^{\log^3}$, which is a complexity class contained in PTIME we will define in the present paper. We hope that this tighter new bound will provide a better insight into the very nature of the Dual problem, and possibly hint at the right direction for future research towards finding a matching upper bound.

Roadmap. The paper is organized as follows. In the next section we discuss decomposition methods for Dual and give a succinct description of the method of Boros and Makino, which we consider to be the currently most advanced method. In Section 3, we define complexity classes based on iterated self-compositions of functions and prove a useful complexity-theoretic lemma. In Section 4, we use this lemma to prove our main result, namely that Dual is in DSPACE[$\log^2 n$]. In section 5 we provide our tighter structural complexity bound for Dual. The paper is concluded in Section 6, where we also exhibit a diagram (Fig. 1) that puts all relevant complexity classes in relation, and highlights the new upper bounds.

2 Decomposition Method by Boros and Makino

Most algorithms for deciding Dual rely on decompositions that start with an original Dual instance and recursively transform it into a conjunction of smaller instances, until each instance is either seen to be a no-instance because it violates necessary conditions for duality, or until it is small and efficiently decidable. Such decompositions are also known as self-reductions, see, e.g., Section 5.3 of [9]. The decomposition process corresponds in the obvious way to a decomposition tree. Different decomposition methods give rise to decomposition trees of different shapes and depths. For example, the well-known algorithm A by Fredman and Khachiyan [15] produces a “skinny” binary decomposition tree of depth linear in the input volume $|\mathcal{G}| \times |\mathcal{H}|$, while their algorithm B produces a non-binary tree of similar depth, but with fewer nodes. Later, decomposition methods giving rise to trees of polylogarithmic depth were published. In particular, the methods of Kavvadias and Stavropoulos [34] as well as the two methods by Elbassioni in [12] give rise to decomposition trees of polylogarithmic depth. Finally, decomposition methods yielding trees of logarithmic depth were presented by Gaur [17]
A set

A set of vertices

(iii)

An instance of

(iv)

Here is how the tree is built. The input instance \( G \) in detail. At each stage of the algorithm, let us denote the set of current leaf-nodes by \( \Lambda \). \( \alpha \) consisting of the root \( t \). Thus \( T \) will be a leaf of the final tree will be marked by the following procedure, and will then not be further expanded and will

\[ t(\alpha) : = \text{DONE}; \ t(\alpha) : = \emptyset \]

\[ \text{CASE 3. If } \mathcal{H}_{S_{\alpha}} = \{ H \} \text{ and } \{ \{ i \} | i \in H \} \subseteq G^{S_{\alpha}}, \text{ then } \{ \text{mark}(\alpha) : = \text{DONE}; \ t(\alpha) : = \emptyset \}. \]

\[ \text{CASE 4. OTHERWISE, let } H \text{ denote the only hyperedge of } \mathcal{H}_{S_{\alpha}} \text{ and set mark}(\alpha) : = \text{FAIL}, \text{ and } t(\alpha) : = S_{\alpha} - \{ i \} \text{ for some arbitrarily chosen } i \in H \text{ with } \{ i \} \not\subseteq G^{S_{\alpha}}. \]

A unique label \( label(\alpha) \) consisting of a sequence in \( N_{\mathcal{H}} \). In particular, the root \( \alpha_{0} \) of \( T(\mathcal{G}, \mathcal{H}) \) is labeled by \( () \), and the \( i \)-th child of a node labeled \( (j_{1}, \ldots, j_{k}) \) is labeled \( (j_{1}, \ldots, j_{k}, i) \).

A set \( S_{\alpha} \subseteq V(\mathcal{G}) \).

An instance of DUAL \( \text{inst}(\alpha) = (G^{S_{\alpha}}, \mathcal{H}_{S_{\alpha}}), \) where \( G^{S_{\alpha}} = \{ E \cap S_{\alpha} | E \in G \} \) and \( \mathcal{H}_{S_{\alpha}} = \{ E \in \mathcal{H} | E \subseteq S_{\alpha} \} \).

A marking \( \text{mark}(\alpha) \in \{ \text{DONE}, \ \text{FAIL}, \ \text{NIL} \} \), where each leaf of the final decomposition tree will be marked with \( \text{DONE} \) or \( \text{FAIL} \), and each non-leaf will be marked with dummy value \( \text{NIL} \). Intuitively, each leaf marked \( \text{DONE} \) identifies a branch that does not contradict \( \mathcal{H} = tr(\mathcal{G}) \), whereas a leaf marked \( \text{FAIL} \) identifies a branch that proves that \( \mathcal{H} \neq tr(\mathcal{G}) \).

A set of vertices \( t(\alpha) \subseteq V(\mathcal{G}) \). This set will be the empty set for each node not marked \( \text{FAIL} \), and, in case \( \alpha \) is marked \( \text{FAIL} \), will contain a witness for \( \mathcal{H} \neq tr(\mathcal{G}) \) in form of a new transversal of \( \mathcal{G} \) with respect to \( \mathcal{H} \).

Let us now describe the method for building \( T(\mathcal{G}, \mathcal{H}) \) and deciding whether \( \mathcal{H} = tr(\mathcal{G}) \) in detail. At each stage of the algorithm, let us denote the set of current leaf-nodes by \( \Lambda \). Here is how the tree is built. The input instance \( (\mathcal{G}, \mathcal{H}) \) is first transformed into a initial tree consisting of the root \( \alpha_{0} \) with \( label(\alpha_{0}) = () \), \( S_{\alpha_{0}} = V \), \( inst(\alpha_{0}) = (G, \mathcal{H}) \), \( \text{mark}(\alpha_{0}) = \text{NIL} \), and \( t(\alpha_{0}) = \emptyset \). At each stage of the decomposition, first, each leaf \( \alpha \in \Lambda \) where \( |\mathcal{H}_{S_{\alpha}}| \leq 1 \), will be marked by the following procedure, and will then not be further expanded and will thus be a leaf of the final tree \( T(\mathcal{G}, \mathcal{H}) \):

**PROCEDURE MARKSMALL(\( \alpha \))**:

CASE 1. IF \( \mathcal{H}_{S_{\alpha}} = \emptyset \) and \( \emptyset \not\subseteq G^{S_{\alpha}} \), THEN

\[ \{ \text{mark}(\alpha) := \text{FAIL}; \ t(\alpha) := S_{\alpha} \}. \]

CASE 2. IF \( \mathcal{H}_{S_{\alpha}} = \emptyset \) and \( \emptyset \in G^{S_{\alpha}} \), THEN

\[ \{ \text{mark}(\alpha) := \text{DONE}; \ t(\alpha) := \emptyset \}. \]

CASE 3. IF \( \mathcal{H}_{S_{\alpha}} = \{ H \} \) and \( \{ \{ i \} | i \in H \} \subseteq G^{S_{\alpha}} \), THEN \( \{ \text{mark}(\alpha) := \text{DONE}; \ t(\alpha) := \emptyset \} \).

CASE 4. OTHERWISE, let \( H \) denote the only hyperedge of \( \mathcal{H}_{S_{\alpha}} \) and set \( \text{mark}(\alpha) := \text{FAIL} \), and \( t(\alpha) := S_{\alpha} - \{ i \} \) for some arbitrarily chosen \( i \in H \) with \( \{ i \} \not\subseteq G^{S_{\alpha}} \).
Then, each leaf $\alpha$ of $\Lambda$ not yet marked is subjected to the following procedure:

**Procedure Process($\alpha$):**

1. Let $I_\alpha$ consist of those vertices of $\mathcal{H}_{S_\alpha}$ that occur in more than $|\mathcal{H}_{S_\alpha}|/2$ hyperedges of $\mathcal{H}_{S_\alpha}$;

2. If $I_\alpha$ is a new transversal of $\mathcal{G}^{S_\alpha}$ with respect to $\mathcal{H}_{S_\alpha}$, then
   \[
   \{ \text{mark}(\alpha) := \text{FAIL}; t(\alpha) := I_\alpha; \text{EXIT PROCEDURE} \};
   \]

3. Otherwise if there is a $G \in \mathcal{G}^{S_\alpha}$ such that $G \cap I_\alpha = \emptyset$ then let
   \[
   C = \{ S_\alpha - (E - \{i\}) | E \in \mathcal{G}^{S_\alpha}_G \text{ and } i \in E \cap G \},
   \]
   where $\mathcal{G}^{S_\alpha}_G = \mathcal{G}^{S_\alpha} - \{ E' \in \mathcal{G}^{S_\alpha} | E' \subseteq S_\alpha - G \}$;

4. Otherwise if there exists a $H \in \mathcal{H}_{S_\alpha}$ such that $H \subseteq I_\alpha$ then let
   \[
   C = \{ S_\alpha - \{i\} | i \in H \} \cup \{ H \};
   \]

5. Let $\kappa(\alpha) = |C|$ and assume $C = \{ C_1, C_2, \ldots, C_{\kappa(\alpha)} \}$. For each $C_i$, $1 \leq i \leq \kappa(\alpha)$, create a new child $\alpha_i$ with $\text{label}(\alpha_i) = (\text{label}(\alpha), i)$, $S_{\alpha_i} = C_i$, $\text{inst}(\alpha_i) = (\mathcal{G}^{S_{\alpha_i}}, \mathcal{H}_{S_{\alpha_i}})$, $\text{mark}(\alpha_i) = \text{NIL}$, and $t(\alpha_i) = \emptyset$.

Exhaustively apply the procedures MARKSMALL (to unmarked leaves $\alpha$ having $|\mathcal{H}_{S_\alpha}| \leq 1$) and PROCESS (to all other unmarked leaves), until there are no unmarked leaves left in the tree. The resulting tree is then $T(\mathcal{G}, \mathcal{H})$.

Note that, due to the possible multiple choices of $i$ in case 4 of the MARKSMALL procedure, and of $G$ in Step 3 and of $H$ in Step 4 of the PROCESS procedure, the decomposition tree $T(\mathcal{G}, \mathcal{H})$ is actually not uniquely defined. However, this is not a problem. To obtain a well-defined decomposition tree $T(\mathcal{G}, \mathcal{H})$, we may resort to any pair of deterministic versions of MARKSMALL and of PROCESS, for example, we may use the version of MARKSMALL where in case 4 the smallest $i \in H$ fulfilling $\{i\} \notin \mathcal{G}^{S_\alpha}$ is chosen, and the version of PROCESS where in Step 3 the lexicographically first edge $G \in \mathcal{G}^{S_\alpha}$ with $G \cap I_\alpha = \emptyset$ is chosen, and similarly for $H$ in Step 4.

The following proposition summarizes important results by Boros and Makino [4].

**Proposition 2.1** (Boros and Makino [4]).

1. $\mathcal{H} = \text{tr}(\mathcal{G})$ iff all leaves of $T(\mathcal{G}, \mathcal{H})$ are marked DONE.

2. The depth of $T(\mathcal{G}, \mathcal{H})$ is bounded by $\log |\mathcal{H}|$.

3. Each node $\alpha$ of $T(\mathcal{G}, \mathcal{H})$ has at most $|V| \cdot |\mathcal{G}|$ children, i.e., $\kappa(\alpha) \leq |V| \cdot |\mathcal{G}|$, where $V$ is the set of vertices of $\mathcal{G}$ and $\mathcal{H}$.

4. If $\mathcal{H} \neq \text{tr}(\mathcal{G})$, then $T(\mathcal{G}, \mathcal{H})$ has at least one leaf labeled FAIL, and the set $t(\alpha)$ associated to each leaf $\alpha$ labeled FAIL is a new transversal of $\mathcal{G}$ w.r.t. $\mathcal{H}$. 

3 A Complexity-Theoretic Lemma

For a space-constructible numerical function \( z, \) DSPACE[\( z(n) \)] (resp. FDSPACE[\( z(n) \)]), denotes, as usual, the class of all decision problems (resp. computation problems) solvable deterministically in \( O(z(n)) \) space. For a function \( f, \) let \( f^1 = f \) and for \( i \geq 1, \) let \( f^{i+1} = f \circ f^i, \) where \( o \) is the usual function composition, i.e., where for each \( x \) in the domain of \( g, \) \( (f \circ g)(x) = f(g(x)). \) Let \( Q_{\log} \) denote the set of all functions \( \rho \) from strings over some input alphabet to the non-negative natural numbers, where for each input string \( I, \) \( \rho(I) \) is \( O(\log |I|) \) and such that \( \rho \) is logspace-computable, i.e., \( \rho(I) \) is computable in logarithmic space from \( I. \) For each function \( f, \) and for each function \( \rho \in Q_{\log}, \) let \( f^\rho \) denote the function that to each input \( I \) associates the output \( f^\rho(I) = f^{\rho(I)}(I). \) If FC denotes a functional complexity class, then \( [\text{FC}]^{\log} \) denotes the class of functions that can be built from some function \( f \) in FC via a logarithmic number \( \rho(I) = O(\log n) \) of self-compositions of \( f \) for each input \( I \) of size \( n: \)
\[
[\text{FC}]^{\log} = \bigcup_{f \in \text{FC}, \rho \in Q_{\log}} \{ f^\rho \}.
\]

For a functional complexity class FC, the subclass FC_{pol} is defined as the set of all functions \( f \) of FC for which there exists a polynomial \( \gamma \) such that for each input \( I, \) and for each \( i \geq 1, \) \( |f^i(I)| \leq \gamma(|I|). \) In general, FC_{pol} is a proper subclass of FC. This is, in particular so for FDSPACE[\( \log n \)], i.e., functional logspace, a.k.a. FLOGSPACE. To see this, let \( f \) be the function that associates to an input of size \( n \) an output consisting of \( n^2 \) zeros. Clearly, \( f \in \text{FDSPACE}[\log n], \) but the output sizes of the \( f^i \) are not bounded by any fixed polynomial when \( i \) grows. Thus, \( f \not\in [\text{FDSPACE}[\log n]_{\text{pol}}], \) hence \( [\text{FDSPACE}[\log n]_{\text{pol}}] \) is a proper subclass of the class FDSPACE[\( \log n \)].

Lemma 3.1.

\([\text{FDSPACE}[\log n]_{\text{pol}}]^{\log} \subseteq \text{FDSPACE}[\log^2 n].\]

Proof. The proof is similar the well-known proof that for any two functions \( f, g \) that are in the functional class FDSPACE[\( \log n \)], their composition \( g \circ f \) is in FDSPACE[\( \log n \)], too. (See, e.g. [40]). However, here, the logarithmic (rather than constant) number of compositions is responsible for the blowup of the required space by a logarithmic factor. Let \( f \) be a function from strings to strings in FDSPACE[\( \log n \)] pol, realized by a logspace Turing Machine \( T \), and let \( \rho \in Q_{\log}. \) In order to prove the lemma, it is sufficient to show that one can construct a single functional Turing machine \( T^* \) with space bound \( O(\log^2 n) \) that simulates the pipelined application \( T^\rho(I) \) that outputs \( f^\rho(I)(I). \)

\( T^* \) first computes \( \rho(I) \) in logspace and then simulates an arrangement of \( \rho(I) \) copies of \( T, \) say, \( T_1, T_2, \ldots, T_{\rho(I)}, \) such that the input string \( v_1 \) to \( T_1 \) is \( I, \) and such that for \( i \geq 1, \) the input string \( v_{i+1} \) to \( T_{i+1} \) is equal to the output string \( w_1 \) of \( T_i \). Given that the size of \( w_1 = T^i(I) \) is bounded by some fixed polynomial \( \gamma, \) there are numbers \( a \) and \( b \) such that each \( T_i \) requires no more than space \( a + b \log n. \) When simulating the pipelined computation \( T_{\rho(I)}(T_{\rho(I)-1}(\cdots (T_2(T_1(I))))) \) on a single Turing machine \( T^*, \) we have to avoid the effective storage of any intermediate output \( w_i \) (or, equivalently, input \( v_{i+1} \)). To this aim, \( T^* \) simulates each \( T_i \) via a logspace procedure \( P_i \) that maintains its own space area on the worktape of \( T^*. \) Each \( P_i \) acts like \( T_i, \) except for the following modifications: For \( 1 < i < \rho(I) \), \( P_i \) has a single output bit which is stored on the worktape of \( T^*; \) moreover \( P_i \) takes as input a dedicated special index register \( d_i \) that specifies which output bit of \( T_i \) is to be computed, and computes only this output bit (suppressing all other output bits) and stores it in a single-bit register \( a_i. \) \( T_i \)’s access to its \( j \)-th input bit is then simulated by \( P_i \) writing ”\( j \)” (in binary) into the special
index register $d_{i-1}$, starting $P_{i-1}$, and then waiting until $P_{i-1}$ writes the desired output bit into $a_{i-1}$ which corresponds to the correct value of the $j$-th output of $T_{i-1}$, and thus the $j$-th input bit to $T_i$. $P_1$ and $P_{\rho(I)}$ work in a similar way, except that $P_1$ directly accesses the input string $I$ from the input tape of $T^*$, and $P_{\rho(I)}$, rather than suppressing some output bits, writes all output bits to the output tape of $T^*$.

The workspace required by each procedure $P_i$ is easily seen to be bounded by $a' + b' \log n$ for some fixed constants $a'$ and $b'$ independent of $n$. This reflects the $a + b \log n$ bits required to execute $T_i$, plus the little extra space $P_i$ may require for its index $d_i$, for the output bit $a_i$, and for a constant number of auxiliary counters and pointers (of size at most $a + b \log n$ bits each) for control and stack management for the $P_i$ procedures. Given that $\rho(I)$ is $O(\log n)$, $T^*$ requires $O(\log^2 n)$ space in total.

Note that the same space bound doesn’t hold for the complexity class $[\text{FDSPACE}[\log n]]^{\log}$. In fact, with functions $f$ in this class, intermediate outputs $j^t(I)$ may be of superpolynomial size, and in the worst case, even of exponential size $n^{\Theta(n)}$. Therefore, when omitting the “pol” restriction, $[\text{FDSPACE}[\log n]]^{\log} \subseteq \text{FPSPACE}$ is the best space upper bound we are able to show. Since an $\text{FDSPACE}[\log^2 n]$ Turing machine has an output of size at most $n^{O(\log n)}$, it it actually holds that $[\text{FDSPACE}[\log n]]^{\log} \not\subseteq \text{FDSPACE}[\log^2 n]$.

### 4 The New Space Bound

The main result of this section is that for a pair $(G, H)$, the entire decomposition tree $T(G, H)$ (with all markings and labels) produced by the decomposition method of Boros and Makino as outlined in Section 2 can be computed with quadratic logspace. The other space-complexity results follow from this as simple corollaries.

We start with a lemma that provides us with a logarithmic space bound for computing the $i$-th child of a node $\alpha$ of the decomposition tree from the fully labeled node $\alpha$ and from the set $V$ of vertices of the original input instance, or for discovering that such a child does not exist. If $\alpha$ is a node of the decomposition tree, let us denote by $\text{attr}(\alpha)$ the attributes of $\alpha$, i.e., the tuple $(\text{label}(\alpha), S_\alpha, \text{inst}(\alpha), \text{mark}(\alpha), t(\alpha))$.

**Lemma 4.1.** There exists a deterministic logspace procedure $\text{NEXT}(V, \text{attr}(\alpha), i)$, which for each dual instance $(G, H)$ over vertex set $V$, for each attribute set attr($\alpha$) of a node $\alpha$ of $T(G, H)$, and for each positive integer $i \leq |V| \cdot |G|$ outputs:

- $\text{attr}(\alpha_i)$ if $\alpha_i$ is the $i$-th child of $\alpha$ in $T(G, H)$;
- IMPOSSIBLE otherwise (i.e., if $\alpha$ has less than $i$ children).

**Proof.** First note that by simple inspection it is immediate that the procedures $\text{MARKSMALL}$ and $\text{PROCESS}$ given in Section 2 can be implemented by deterministic logspace transducers. In fact, these procedures only perform a fixed composition of simple cardinality checks, counting, assignments, and set theoretic operations that are all well-known to run in logspace.

A procedure next, as required, can be constructed as follows. If $\text{label}(\alpha) \in \{\text{DONE}, \text{FAIL}\}$ then output IMPOSSIBLE, else perform $\text{MARKSMALL}^*(\text{PROCESS}^*(\alpha))$, where:

- $\text{PROCESS}^*$ works like $\text{PROCESS}$ except that it outputs only the $i$-th child of $\alpha$, if such a child exists, rather than outputting all children, and output IMPOSSIBLE otherwise; and

- $\text{MARKSMALL}^*$ works like $\text{MARKSMALL}$, except that it also accepts the input IMPOSSIBLE, in which case it also outputs IMPOSSIBLE.

These minor modifications of $\text{MARKSMALL}$ and $\text{PROCESS}$ clearly run in deterministic logspace, therefore their composition does, and hence so does the procedure $\text{NEXT}$. 


A path descriptor for a dual instance $I = (G, H)$ over a vertex set $V$ is a list of length $\leq \lceil \log |H| \rceil$, whose elements are integers bounded by $|V| \cdot |G|$. The set of all path descriptors for $I$ is denoted by $PD(I)$. Clearly, $PD(I) \subset \mathbb{N}_2$, and each label $\text{label}(\alpha)$ of a node $\alpha$ of $T(G, H)$ is contained in $PD(I)$. Intuitively, a path descriptor, exactly in the same way as a label, is intended to describe a sequence of child-indices, that, starting from the roof of $T(G, H)$ lead to a specific node $\alpha$ of $T(G, H)$. The root of $T(G, H)$ is identified by the empty path descriptor. If $\pi = (i_1, i_2, . . . , i_r)$ is a path descriptor, then $\text{head}(\pi) = i_1$ and $\text{tail}(\pi)$ is the path descriptor $(i_2, . . . , i_r)$. Two path descriptors of the form $(i_1, . . . , i_r)$ and $(i_1, . . . , i_r, i_{r+1})$ are said to be consecutive.

Lemma 4.2. There is a procedure $\text{PATHNODE}(I, \pi)$ that runs in deterministic space $O(\log^2(|I|))$, that for each dual input instance $I$ and path descriptor $\pi \in PD(I)$ outputs $\text{attr}(\alpha)$ if $\pi$ corresponds to the label $\text{label}(\alpha)$ of a node $\alpha$ in $T(G, H)$, and outputs $\text{WRONGPATH}$ otherwise.

Proof. Let $I = (G, H)$, $V = V(G)$, and $\pi \in PD(I)$, and let $\ell(\pi)$ denote the length of the sequence $\pi$ (recall that $\ell(\pi) \leq \log |I|$). The procedure $\text{PATHNODE}$ first computes in deterministic logspace $\text{attr}(\alpha_0)$ for the root $\alpha_0$ of $T(G, H)$. It then computes $f^{\ell(\pi)}(V, \text{attr}(\alpha_0), \pi)$, where $f$ is the function corresponding to the procedure $F$ described as follows. $F$ accepts as input either the string $\text{WRONGPATH}$, or a triple $(W, \text{attr}, \gamma)$ where $W$ is a set, $\text{attr}$ is a data structure of the same format as the attributes $\text{attr}(\beta)$ of some vertex $\beta$ in a decomposition tree, and $\gamma$ is a sequence of positive integers. On all other inputs, $F$ outputs the empty string. On input $\text{WRONGPATH}$, $F$ outputs $\text{WRONGPATH}$; otherwise $F$ computes $F'(\text{NEXT}(W, \text{attr}, \text{head}(\gamma)))$, where NEXT be as specified in Lemma 4.1 and where $F'$ is as follows. $F'$ outputs $\text{WRONGPATH}$ if $\text{NEXT}(W, \text{attr}, \text{head}(\gamma)) = \text{IMPOSSIBLE}$, and $F'$ outputs $(W, \text{Attr'}, \text{tail}(\gamma))$, whenever

$$\text{NEXT}(W, \text{attr}, \text{head}(\gamma)) = \text{Attr'}$$

for some attribute description $\text{Attr'}$. Since NEXT runs in deterministic logspace, also $F'$ and $F$ do, and therefore $f$ is a logspace computable function.

By construction and by Lemma 4.1, $\text{PATHNODE}$ precisely computes the attributes $\text{attr}(\alpha)$ if there is a node $\alpha$ with $\text{label}(\alpha) = \pi$ in $T(G, H)$, whereas otherwise $\text{PATHNODE}$ outputs $\text{WRONGPATH}$. Since the function $\ell$ (expressing the length $\ell(\pi)$) is clearly in $Q_{\log}$, and since $f^{i}(V, \text{attr}(\alpha_0), \pi)$, for each $i$, is of size polynomially bounded in the input size $(|V, \text{attr}(\alpha_0), \pi|)$, $f^{\ell(\pi)}(V, \text{attr}(\alpha_0), \pi)$ can be computed by a procedure in $[\text{FSPACE}]_{\log n}^{|\log n|_{\text{pol}}}^{|\log n|}$, and therefore, by Lemma 3.1, in deterministic space $O(\log^2 n)$. The same complexity bounds obviously hold for $\text{PATHNODE}$. 

By using a procedure $\text{PATHNODE}$ according to the above Lemma, we are now ready to formulate an algorithm $\text{DECOMPOSE}$ that computes the decomposition tree $T(G, H)$ to a dual instance $(G, H)$. In particular, the algorithms first lists the vertices and then the edges of the tree $T(G, H)$.

Algorithm $\text{DECOMPOSE}$:
Input: dual-instance $I = (G, H)$; Output: $T(G, H)$.
BEGIN
OUTPUT("Vertices:");
FOR each path descriptor $\pi \in PD(I)$ DO
IF $\text{PATHNODE}(I, \pi) \neq \text{WRONGPATH}$
THEN OUTPUT($\text{PATHNODE}(I, \pi)$);
OUTPUT("Edges:");
ENDFOR.
FOR each pair \((\pi, \pi')\) of consecutive path descriptors in \(PD(I)\) DO

BEGIN

\(\alpha := \text{PATHNODE}(I, \pi)\);
\(\alpha' := \text{PATHNODE}(I, \pi')\);
IF \(\alpha' \neq \text{WRONGPATH}\) THEN

OUTPUT( \(\langle \text{label}(\alpha), \text{label}(\alpha') \rangle \) );

END

END.

**Theorem 4.1.** The Algorithm \textsc{decompose} computes the decomposition tree \(T(G, H)\) to a \textsc{Dual} instance \((G, H)\) in space \(O(\log^2 n)\).

**Proof.** The correctness of the algorithm follows from the correctness of \textsc{pathnode} as shown in Lemma 4.2. For the space bound, note that each each path descriptor requires only \(O(\log^2 |I|) = O(\log^2 n)\) bits, and that we can thus can thus iterate (by re-using work-space) over all path descriptors and pairs of path descriptors in \(O(\log^2 n)\) space. Given that, by Lemma 4.2, \textsc{pathnode} also runs in \(O(\log^2 n)\) space, the entire \textsc{decompose} algorithm needs only \(O(\log^2 n)\) space. \(\Box\)

**Corollary 4.1.**

1. Deciding \textsc{Dual} is in \(\text{DSPACE}[\log^2 n]\).

2. If \(\text{tr}(G) \neq H\), then computing a new transversal of \(G\) w.r.t. \(H\) is in \(\text{FDSPACE}[\log^2 n]\).

**Proof.** In both cases, it is possible to first compute the entire decomposition tree \(T(G, H)\) in \(\text{FDSPACE}[\log^2 n]\), and then (i) for problem 1 check by a \(\text{DLOGSPACE}\) procedure whether all leaves are marked \text{DONE}, and (ii) for problem 2, use a \(\text{FLOGSPACE}\) procedure to find a node \(\alpha\) marked \text{FAIL} in \(T(G, H)\) and output its component \(t(\alpha)\). Let \(\circ\) denote the composition operator for complexity classes in the obvious sense. Given that

\[\text{FDSPACE}[\log^2 n] \circ \text{DLOGSPACE} = \text{DSPACE}[\log^2 n],\]

and given that, moreover,

\[\text{FDSPACE}[\log^2 n] \circ \text{FLOGSPACE} = \text{FDSPACE}[\log^2 n],\]

the complexity bounds follow. Alternatively, we can solve the problems 1 and 2 directly by respective slight modifications of \textsc{decompose}. \(\Box\)

Note that if \(\text{tr}(G) \neq H\), the witness \(t(\alpha)\) produced is not necessarily a minimal transversal of \(G\), but is, in general, just a transversal of \(G\) that contains no edge of \(H\) and thus witnesses that \(\text{tr}(G) \neq H\), because \(t(\alpha)\) must contain a missing minimal transversal of \(G\). From \(t(\alpha)\), such a minimal transversal \(t\) can easily be computed in polynomial time by letting first \(t := t(\alpha)\) and by then successively eliminating vertices \(v\) from \(t\) for which \(t - \{v\}\) is still a transversal of \(G\). However this process requires linear space in the vertex set \(V\) to remember the eliminated vertices plus logarithmic space in the instance size \(|(G, H)|\) for checking. This is still better than polynomial space in the full instance size, but is not quite in quadratic logspace. It is currently not clear whether there exists a smarter algorithm that requires quadratic logspace only.
5 A Tighter Bound for DUAL

By the results of the previous section, DUAL and its complement $\overline{\text{DUAL}}$ are in quadratic logspace, i.e., in the class $\text{DSPACE}[\log^2 n]$. On the other hand, as already mentioned, the complement of DUAL is in $\beta_2$P, the class of problems solvable in polynomial time with $O(\log^2 n)$ nondeterministic guesses. $\beta_2$P is identical with the complexity class $\text{GC}(\log^2 n, \text{PTIME})$ of the so called Guess and Check model for limited nondeterminism [5, 20], where $O(\log^2 n)$ nondeterministic bits are guessed and are appended to the input before the proper PTIME computation starts. The Guess and Check classes are, more generally, defined as follows. Let $C$ be a complexity class and $s$ a numerical function. Then $\text{GC}(s(n), C)$ is the class of all languages $L$ for which there exists a language $A \in C$ such that an input string $I$ is in $L$ iff there is a string $J$ of $O(s(|I|))$ bits, such that $(I, J)$ is in $A$. In other words, $L$ is in $\text{GC}(s(n), C)$ iff the membership of a string $I$ in $L$ can be checked in $C$ after having guessed $O(s(n))$ nondeterministic bits that can be used as an additional input.

Given that PTIME is believed to be incomparable with $\text{DSPACE}[\log^2 n]$ (cf. [31]), and given that, obviously, $\text{PTIME} \subseteq \text{GC}(\log^2 n, \text{PTIME})$, it is very likely that also $\text{GC}(\log^2 n, \text{PTIME})$ and $\text{DSPACE}[\log^2 n]$ are incomparable. Since DUAL belongs to both classes, this suggests that neither well characterizes DUAL, and that DUAL is unlikely to be complete for either. This observation incited us to look out for a complexity class containing DUAL that would be contained in both classes $\text{GC}(\log^2(n), \text{PTIME})$ and $\text{DSPACE}[\log^2 n]$, that would thus constitute a tighter upper complexity bound for DUAL than all those we have seen so far. In this section, we present precisely such a complexity class. In order to describe this class, we state some definitions and prove a lemma.

The class $[\text{LOGSPACE}_{\text{pol}}]^{\log}$ is defined as the composition of $[\text{FDSPACE}[\log n]_{\text{pol}}]^{\log}$ with LOGSPACE. Formally, $[\text{LOGSPACE}_{\text{pol}}]^{\log}$ is equal to

$$[\text{FDSPACE}[\log n]_{\text{pol}}]^{\log} \circ \text{LOGSPACE}.$$ 

Here an input $I$ is first transformed to an output $O$ by a functional procedure that runs in deterministic space $[\text{FDSPACE}[\log n]_{\text{pol}}]^{\log}$, after which $O$ is submitted to a LOGSPACE decision procedure which will decide based on $O$ if the original input $I$ is accepted or rejected.

Note that $[\text{LOGSPACE}_{\text{pol}}]^{\log}$ is by all means a complexity class defined in terms of machines and resource bounds. In addition to the classical resources such as the amount of workspace, we here involve somewhat more unusual resources such as the allowed number of self-compositions, which is here bounded by $O(\log n)$, whence the superscript log, and the allowed size of intermediate outputs in compositions, which is here polynomially bounded, whence the subscript pol.

**Lemma 5.1.** Given a DUAL instance $I = (G, \mathcal{H})$ and a path descriptor $\pi \in PD(I)$, deciding if $\text{PATHNODE}(I, \pi)$ outputs a leaf of $T(G, \mathcal{H})$ whose mark-component is fail is feasible with complexity $[\text{LOGSPACE}_{\text{pol}}]^{\log}$.

**Proof.** The proof of Lemma 4.2 already shows that $\text{PATHNODE}$ lies in $[\text{FDSPACE}[\log n]_{\text{pol}}]^{\log}$. Deciding whether $\text{PATHNODE}(I, \pi)$ outputs a leaf of $T(G, \mathcal{H})$ whose mark-component is fail can thus be implemented by first executing $\text{PATHNODE}(I, \pi)$, and then checking whether the output is a node labeled fail. This is obviously in $[\text{FDSPACE}[\log n]_{\text{pol}}]^{\log} \circ \text{LOGSPACE} = [\text{LOGSPACE}_{\text{pol}}]^{\log}$.

We next consider $\text{GC}(\log^2 n, [\text{LOGSPACE}_{\text{pol}}]^{\log})$, the main complexity class studied in this section.

**Theorem 5.1.** $\overline{\text{DUAL}} \in \text{GC}(\log^2 n, [\text{LOGSPACE}_{\text{pol}}]^{\log})$. 

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Proof. In order to find a new transversal \( t \) of \( \mathcal{G} \) with respect to \( \mathcal{H} \), for a Dual instance \( I = (\mathcal{G}, \mathcal{H}) \), rather than computing the entire decomposition tree \( T(\mathcal{G}, \mathcal{H}) \), it is sufficient to guess a branch of this tree that terminates in a leaf \( \alpha \) labeled \( \text{fail} \), and then compute \( t(\alpha) \). Guessing such a branch amounts to guess a path descriptor \( \pi \) and then checking that \( \text{pathnode}(I, \pi) \) outputs a node marked \( \text{fail} \). Guessing \( \pi \) amounts to guess \( \log^2 n \) bits, and this is all our guess-and-check algorithm guesses. Checking that \( \pi \) is a \( \text{fail} \) node is, by Lemma 5.1 in \([ \text{LOGSPACE}_{\text{pol}} ]^{\log} \), hence the overall computation is in \( \text{GC}(\log^2 n, [ \text{LOGSPACE}_{\text{pol}} ]^{\log}) \).

The last theorem of this section shows, as promised, that \( \text{GC}(\log^2 n, [ \text{LOGSPACE}_{\text{pol}} ]^{\log}) \) is effectively a subclass of the tightest known other upper bounds that are most likely incomparable to each other: the classes \( \text{DSPACE}[\log^2 n] \) and \( \text{GC}(\log^2 n, \text{PTIME}) = \beta_2 \text{P} \).

Theorem 5.2. \( \text{GC}(\log^2 n, [ \text{LOGSPACE}_{\text{pol}} ]^{\log}) \subseteq \text{DSPACE}[\log^2 n] \cap \text{GC}(\log^2 n, \text{PTIME}) \).

Proof. For the inclusion

\[
\text{GC}(\log^2 n, [ \text{LOGSPACE}_{\text{pol}} ]^{\log}) \subseteq \text{DSPACE}[\log^2 n],
\]

note that a decision procedure in the complexity class \( \text{GC}(\log^2 n, [ \text{LOGSPACE}_{\text{pol}} ]^{\log}) \) amounts to (i) guessing \( O(\log^2 n) \) bits, which can be simulated by an exhaustive enumeration of all possible guesses (under re-use of space), which is feasible in \( \text{DSPACE}[\log^2 n] \), and (ii) for each such simulated guess, performing a check in \([ \text{FDSPACE}[\log n]_{\text{pol}} ]^{\log} \cap \text{LOGSPACE} \). Since, by Lemma 1, \([ \text{FDSPACE}[\log n]_{\text{pol}} ]^{\log} \subseteq \text{DSPACE}[\log^2 n] \), and given that the composition of a function from the class \( \text{FDSPACE}[\log^2 n] \) with a \( \text{LOGSPACE} \) computation yields a \( \text{DSPACE}[\log^2 n] \) decision procedure, the overall computation is in \( \text{DSPACE}[\log^2 n] \).

To establish the reverse inclusion

\[
\text{GC}(\log^2 n, [ \text{LOGSPACE}_{\text{pol}} ]^{\log}) \subseteq \text{GC}(\log^2 n, \text{PTIME}),
\]

it is obviously sufficient to see that \([ \text{LOGSPACE}_{\text{pol}} ]^{\log} \) is contained in \( \text{PTIME} \). In fact, a decision procedure in \([ \text{LOGSPACE}_{\text{pol}} ]^{\log} \) amounts to a pipelined execution of \( O(\log n) \) instantiations of a logspace function \( f \), where the intermediate results are guaranteed to be of polynomial size in the original input, followed by the application of a logspace Boolean procedure \( g \). This can be replaced by the pipelined execution of \( O(\log n) \) instances of a \( \text{PTIME} \) procedure equivalent to \( f \), followed by the application of a Boolean \( \text{PTIME} \) procedure equivalent to \( g \). In total, this latter process is in \( \text{PTIME} \) because it amounts to a logarithmic number of invocations of a \( \text{PTIME} \) procedure, where each time the input size is bounded by a polynomial in the size \( n \) of the overall input. Therefore, \([ \text{LOGSPACE}_{\text{pol}} ]^{\log} \subseteq \text{PTIME} \).

6 Discussion and Conclusion

In this paper we have derived new complexity bounds for the Dual problem and its complement Dual, that show that these problems can, in principle, be implemented by space-efficient algorithms. These bounds are depicted in Figure 1 in relation to the other relevant complexity classes. Here, set-inclusion is visualized by ascending lines or paths. We believe that our results represent some progress in the long and rather tortuous battle towards a better understanding of the mysterious Dual problem. Our results are —for the time being— mainly of theoretical interest, and we do not claim they have immediate practical consequences. In fact, it is currently not clear whether our space-efficient version of the algorithm by Boros and Makino has any practical
advantage over its original version. Future research may look at the applicability of such space-efficient techniques in presence of extremely large hypergraphs or data relations. Our bounds do prove useful for other purposes, however.

For instance, the $O(\log^2 n)$ space bound helps telling the Dual problem apart from other problems that are candidates for completeness for intermediate classed between P and NP. For example, model-checking modal $\mu$-calculus formulas [11], and the equivalent problem of whether a given player has a winning strategy in a parity games on graphs [13, 45], are such problems. They are not known to be tractable but lie in $\text{UP} \cap \text{coUP}$ [32], and are thus most likely not NP complete either. Given that these problems bear a certain superficial similarity to Dual, the question arises, whether they are actually disguised versions of sub-problems of Dual, and can thus be reduced to Dual via simple low-level reductions (logspace or lower). By our results, this turns out not to be the case unless PTIME is in $\text{DSPACE}[\log^2 n]$, which is highly unlikely. In fact, the model-checking problem for the modal $\mu$-calculus and the equivalent parity game problem are known to be PTIME hard (see [29]).

We hope, moreover, that our results may help steering future research towards a matching bound for the Dual problem. We have reasons not to believe that Dual is hard for the class $\text{GC}(\log^2 n, [\text{LOGSPACE}\_{\text{pol}}])^{\log}$. This upper bound, however, gives us some intuition of where to dig further. For example, we conjecture that Dual lies in $\text{GC}(\log^2 n, \text{LOGSPACE})$, and hope to be able to prove this in the near future.

Other future work will include the further analysis of hypergraph decomposition techniques to deal with the Dual problem. It is known that Dual is tractable for hypergraphs of bounded degeneracy, and, in particular, for acyclic hypergraphs [9]. The latter coincide with all hypergraphs of hypertree width 1 (see [22, 21, 11]). However, it was shown in [8] that for hypergraphs whose hypertree width is bounded by some constant $k \geq 2$, the Dual problem remains as hard as in the general case. It would thus be interesting to find hypergraph decomposition methods that are more general than bounded degeneracy and still lead to tractable Dual instances. Other research directions are to look for new parameters that lead to tractable (or even fixed-parameter tractable) instances in case they are bounded. See, for instance, [27] for some fixed-parameter tractability results, and [42] for a new parameter...
leading to tractability.

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