Locally octahedral and locally almost square 
Köthe-Bochner spaces

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Abstract. It has been proved in [19] that a Köthe-Bochner space $E(X)$ is locally octahedral/locally almost square if $X$ has the respective property and the simple functions are dense in $E(X)$. Here we show that the result still holds true without the density assumption. The proof makes use of the Kuratowski-Ryll-Nardzewski Theorem on measurable selections.

1 Introduction

Let $X$ be a real Banach space. We denote by $X^*$ its topological dual, by $B_X$ its closed unit ball and by $S_X$ its unit sphere.

$X$ is locally octahedral (LOH) if the following holds: for every $x \in S_X$ and every $\varepsilon > 0$ there exists $y \in S_X$ such that $\|x \pm y\| \geq 2 - \varepsilon$. This notion was introduced in [15] in connection with the so called diameter-two-properties.

$X$ is locally almost square (LASQ) if the following holds: for every $x \in S_X$ and every $\varepsilon > 0$ there exists $y \in S_X$ such that $\|x \pm y\| \leq 1 + \varepsilon$. This notion was introduced in [2].

For more information on these and related properties the reader may consult [1–9,11,13–22,24], [10, Theorem 2.5, p.106] and references therein.

Now consider a complete, $\sigma$-finite measure space $(\mathcal{S}, \mathcal{A}, \mu)$. For a set $A \subseteq \mathcal{S}$ we denote by $\chi_A$ the characteristic function of $A$. Let $(E, \|\cdot\|_E)$ be a Banach space of real-valued measurable functions on $\mathcal{S}$ (modulo equality $\mu$-almost everywhere) such that the following holds:

(i) $\chi_A \in E$ for every set $A \in \mathcal{A}$ with $\mu(A) < \infty$.

(ii) If $f \in E$ and $A \in \mathcal{A}$ with $\mu(A) < \infty$, then $f$ is $\mu$-integrable over $A$.

(iii) If $g$ is measurable and $f \in E$ such that $|g(t)| \leq |f(t)|$ $\mu$-a.e. then $g \in E$ and $\|g\|_E \leq \|f\|_E$.

Then $(E, \|\cdot\|_E)$ is called a Köthe function space over $(\mathcal{S}, \mathcal{A}, \mu)$.
Standard examples are the spaces $L^p(\mu)$ for $1 \leq p \leq \infty$. More generally, Orlicz spaces with the Luxemburg norm (see [25]) are examples of Köthe function spaces.

A function $f : S \to X$ is called simple if there are finitely many pairwise disjoint sets $A_1, \ldots, A_n \in \mathcal{A}$ such that $\mu(A_i) < \infty$ for all $i = 1, \ldots, n$, $f$ is constant on each $A_i$ and $f(t) = 0$ for every $t \in S \setminus \bigcup_{i=1}^{n} A_i$.

$f$ is said to be Bochner-measurable if there exists a sequence $(f_n)_{n \in \mathbb{N}}$ of simple functions such that $\lim_{n \to \infty} \|f_n(t) - f(t)\| = 0 \mu$-a.e.

According to the well-known Pettis Measurability Theorem, the following assertions are equivalent:

(i) $f$ is Bochner-measurable.

(ii) $x^* \circ f$ is measurable for every $x^* \in X^*$ and there is a separable, closed subspace $Y$ of $X$ such that $f(s) \in Y$ for $\mu$-a.e. $s \in S$.

(iii) $f$ is measurable (with respect to $\mathcal{A}$ and the Borel-$\sigma$-algebra of $X$) and there is a separable, closed subspace $Y$ of $X$ such that $f(s) \in Y$ for $\mu$-a.e. $s \in S$.

We denote by $E(X)$ the space of all Bochner-measurable functions $f : S \to X$ (modulo equality a.e.) such that $\|f(\cdot)\| \in E$.

We define $\|f\|_{E(X)} = \|\|f(\cdot)\|_E\|_E$ for $f \in E(X)$. Then $E(X)$ becomes a Banach space, the so called Köthe-Bochner space induced by $E$ and $X$.

For $E = L^p(\mu)$ we obtain the usual Lebesgue-Bochner spaces $L^p(\mu, X)$ for $1 \leq p \leq \infty$. For more information on Köthe-Bochner spaces the reader is referred to the book [23].

In [19, Theorems 4.1 and 4.5] the author proved the following results:

(a) If $X$ is LOH/LASQ and the simple functions are dense in $E(X)$, then $E(X)$ is also LOH/LASQ.

(b) If $X$ is LOH/LASQ, then $L^\infty(\mu, X)$ is also LOH/LASQ.

The proofs in [19] are based on a general reduction theorem and corresponding results for absolute sums of LOH/LASQ spaces that were obtained in [2, Proposition 5.3].

The assumption that the simple functions are dense in $E(X)$ holds true whenever $E$ is order continuous, in particular for $E = L^p(\mu)$ with $1 \leq p < \infty$.

Here we will show that the result is still true without any additional assumptions on $E$ or $X$. The proof makes use of the Kuratowski-Ryll-Nardzewski Theorem on the existence of measurable selections, which we will recall in the next section.
2 Measurable selections

Let \((S, \mathcal{A})\) be a measurable space and \((Y, d)\) a metric space. Denote by \(\mathcal{P}(Y)\) the power-set of \(Y\) and by \(\mathcal{B}(Y)\) the Borel-\(\sigma\)-Algebra of \((Y, d)\). For a subset \(M \subseteq Y\) we denote by \(\overline{M}\) the closure of \(M\) in \((Y, d)\).

Let \(F : S \to \mathcal{P}(Y)\) be a set-valued map. For \(M \subseteq Y\) we put

\[
F_-(M) := \{s \in S : F(s) \cap M \neq \emptyset\}.
\]

\(F\) is said to be weakly \(\mathcal{A}\)-measurable if \(F_-(U) \in \mathcal{A}\) for every open set \(U \subseteq Y\).

If \(F_-(C) \in \mathcal{A}\) for every closed set \(C \subseteq Y\), then \(F\) is called \(\mathcal{A}\)-measurable.

\(\mathcal{A}\)-measurability of \(F\) implies weak \(\mathcal{A}\)-measurability (this follows from the fact that every open subset in a metric space is an \(F_\sigma\)-set).

The following standard lemma follows directly from the definition.

**Lemma 2.1.** Suppose that \(F, G : S \to \mathcal{P}(Y)\) are two set-valued maps such that \(F(s) \subseteq G(s) \subseteq F(s)\) for every \(s \in S\). Then \(F\) is weakly \(\mathcal{A}\)-measurable if and only if \(G\) is weakly \(\mathcal{A}\)-measurable.

The next lemma is also standard but we include a sketch of the proof here for the reader’s convenience.

**Lemma 2.2.** Suppose that \((Y, d)\) is separable and \(g : S \times Y \to \mathbb{R}\) is a Carathéodory function, i.e.

(i) \(g(s, \cdot)\) is continuous for every \(s \in S\),

(ii) \(g(\cdot, y)\) is \(\mathcal{A}\)-measurable for every \(y \in Y\).

Let \(\alpha \in \mathbb{R}\) and put

\[
F(s) := \{y \in Y : g(s, y) > \alpha\} \quad \text{for all } s \in S.
\]

Then \(F : S \to \mathcal{P}(Y)\) is \(\mathcal{A}\)-measurable.

**Proof.** Let \(C \subseteq Y\) be nonempty and closed. Choose a sequence \((y_n)_{n \in \mathbb{N}}\) such that \(C = \{y_n : n \in \mathbb{N}\}\) and put \(A_n := \{s \in S : g(s, y_n) > \alpha\}\) for every \(n \in \mathbb{N}\). Then we have \(A_n \in \mathcal{A}\) for each \(n\) and it is easy to see that \(F_-(C) = \bigcup_{n \in \mathbb{N}} A_n\). Thus \(F_-(C) \in \mathcal{A}\). \(\Box\)

A classical result on the existence of measurable selections is the Kuratowski-Ryll-Nardzewski Theorem (see for instance [12, Theorem 2.1]).

**Theorem 2.3** (Kuratowski-Ryll-Nardzewski Selection Theorem). Let \((S, \mathcal{A})\) be a measurable space and \((Y, d)\) a complete, separable metric space. Let \(F : S \to \mathcal{P}(Y)\) be a weakly \(\mathcal{A}\)-measurable set-valued map such that \(F(s)\) is non-empty and closed in \(Y\) for every \(s \in S\). Then there is an \(\mathcal{A}\)-\(\mathcal{B}(Y)\)-measurable map \(f : S \to Y\) such that \(f(s) \in F(s)\) for every \(s \in S\).
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Now we are ready to prove the general stability result.

**Theorem 3.1.** Let \((S, \mathcal{A}, \mu)\) be a complete, \(\sigma\)-finite measure space and let \(E\) be a K"{o}the function space over \((S, \mathcal{A}, \mu)\). If \(X\) is a real Banach space which is LOH/LASQ, then the K"{o}the-Bochner space \(E(X)\) is also LOH/LASQ.

**Proof.** 1) Assume that \(X\) is LOH. Fix \(\varepsilon > 0\) and \(f \in E(X)\) with \(\|f\|_{E(X)} = 1\). Since \(f : S \to X\) is Bochner-measurable there is a separable, closed subspace \(Y\) of \(X\) such that \(f(s) \in Y\) for \(\mu\)-a.e. \(s \in S\) (Pettis Measurability Theorem). Without loss of generality we may assume that this holds even for all \(s \in S\).

Put \(S' := \{s \in S : f(s) \neq 0\}\).

Since \(Y\) is separable the unit sphere \(S_Y\) is also separable. We fix a sequence \((y_n)_{n \in \mathbb{N}}\) which is dense in \(S_Y\).

Because \(X\) is LOH we can find a sequence \((z_n)_{n \in \mathbb{N}}\) in \(S_X\) such that \(\|y_n \pm z_n\| \geq 2 - \varepsilon/2\) for every \(n \in \mathbb{N}\). We put \(Z := \text{span}\{z_n : n \in \mathbb{N}\}\). This is again a separable, closed subspace of \(X\).

Next we define \(F : S' \to \mathcal{P}(S_Z)\) by

\[
F(s) := \left\{ z \in S_Z : \frac{f(s)}{\|f(s)\|} \pm z \geq 2 - \varepsilon \right\}
\]

for all \(s \in S'\).

We have \(F(s) \neq \emptyset\) for every \(s \in S'\). To see this note that \(f(s)/\|f(s)\| \in S_Y\) and hence there is some index \(N \in \mathbb{N}\) such that \(\|f(s)/\|f(s)\| - y_N\| < \varepsilon/2\). It follows that

\[
\left\| \frac{f(s)}{\|f(s)\|} \pm z_N \right\| \geq \|y_N \pm z_N\| - \left\| \frac{f(s)}{\|f(s)\|} - y_N \right\| \geq 2 - \frac{\varepsilon}{2} = 2 - \varepsilon.
\]

If we define

\[
g(s, z) := \min \left\{ \left\| \frac{f(s)}{\|f(s)\|} + z \right\|, \left\| \frac{f(s)}{\|f(s)\|} - z \right\| \right\}
\]

for \(s \in S', z \in S_Z\), then \(F(s) = \{ z \in S_Z : g(s, z) \geq 2 - \varepsilon \}\) and since \(g\) is a Carathéodory function, it follows from Lemma 2.2 and Lemma 2.1 that \(F\) is weakly measurable.

Thus by the Kuratowski-Ryll-Nardzewski Selection Theorem there exists a measurable function \(\tilde{f} : S' \to S_Z\) such that \(\tilde{f}(s) \in F(s)\) for every \(s \in S'\). Note that by Pettis Measurability Theorem \(\tilde{f}\) is also Bochner-measurable. We put \(h(s) := \|f(s)/\|f(s)\| \tilde{f}(s)\) for \(s \in S'\) and \(h(s) := 0\) for \(s \in S \setminus S'\). Then \(h\) is Bochner-measurable and \(\|h(s)\| = \|f(s)\|\) for every \(s \in S\) and thus \(\|h\|_{E(X)} = \|f\|_{E(X)} = 1\).

We further have \(\|f(s) \pm h(s)\| \geq (2 - \varepsilon)\|f(s)\|\) for every \(s \in S\). This implies \(\|f \pm h\|_{E(X)} \geq (2 - \varepsilon)\|f\|_{E(X)} = 2 - \varepsilon\).

This proves that \(E(X)\) is LOH.
2) Now assume that $X$ is LASQ. We take $\varepsilon, f, Y, S'$ and $(y_n)_{n \in \mathbb{N}}$ as in part 1). Since $X$ is LASQ there is a sequence $(z_n)_{n \in \mathbb{N}}$ in $S_X$ such that $\|y_n \pm z_n\| \leq 1 + \varepsilon/2$ for every $n \in \mathbb{N}$ and we put $Z := \text{span}\{z_n : n \in \mathbb{N}\}$ and

$$F(s) := \left\{ z \in S_Z : \frac{\|f(s)\|}{\|f(s)\|} \pm z < 1 + \varepsilon \right\} \quad \text{for all } s \in S'.$$

Analogously to the proof in part 1) we can see that each set $F(s)$ is non-empty and also that $F$ is weakly measurable. For the latter, put

$$g(s, z) := \max \left\{ \frac{\|f(s)\|}{\|f(s)\|} + z, \frac{\|f(s)\|}{\|f(s)\|} - z \right\} \quad \text{for } s \in S', \ z \in S_Z.$$

Then $g$ is a Carathéodory function and $F(s) = \{ z \in S_Z : g(s, z) < 1 + \varepsilon \}$.

By the Kuratowski-Ryll-Nardzewski Selection Theorem we find a measurable function $\tilde{f} : S' \to S_Z$ such that $\tilde{f}(s) \in F(s)$ for every $s \in S'$. Then if we define $h$ as in 1), we find that $h \in S_{E(X)}$ and $\|f(s) \pm h(s)\| \leq \|f(s)\|(1 + \varepsilon)$ for every $s \in S$. This implies $\|f \pm h\|_{E(X)} \leq 1 + \varepsilon$ and the proof is finished. \qed

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