The Isotropy Representation of a Real Flag Manifold: Split Real Forms

Mauro Patrão *  Luiz A. B. San Martin †

Abstract

We study the isotropy representation of real flag manifolds associated to simple Lie algebras that are split real forms of complex simple Lie algebras. For each Dynkin diagram the invariant irreducible subspaces for the compact part of the isotropy subgroup are described. Contrary to the complex flag manifolds the decomposition into irreducible components is not in general unique, since there are cases with infinitely many invariant subspaces.

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1 Introduction

This paper studies the isotropy representation of (generalized) real flag manifolds associated to a noncompact real simple Lie algebra \( \mathfrak{g} \). Here we consider the case where \( \mathfrak{g} \) is a split real form of a complex simple Lie algebra.

A flag manifold of \( \mathfrak{g} \) is a coset space \( \mathbb{F}_\Theta = G/P_\Theta \) where \( G \) is any connected Lie group with Lie algebra \( \mathfrak{g} \) and \( P_\Theta \subset G \) is a parabolic subgroup. The Lie
algebra \( p_\Theta \) of \( P_\Theta \) is a parabolic subalgebra which is the sum of the eigenspaces of the nonnegative eigenvalues of \( \text{ad}(H_\Theta) \) with \( H_\Theta \in g \) a suitable chosen element. If \( K \subset G \) is a maximal compact subgroup and \( K_\Theta = K \cap P_\Theta \) then \( \mathbb{F}_\Theta = K/K_\Theta \) as well.

The two presentations \( \mathbb{F}_\Theta = G/P_\Theta \) and \( \mathbb{F}_\Theta = K/K_\Theta \) yield the isotropy representations of \( P_\Theta \) and \( K_\Theta \) on the tangent space \( T_{b_\Theta}\mathbb{F}_\Theta \) at the origin \( b_\Theta \). The \( K_\Theta \)-representation is obtained by restricting the \( P_\Theta \)-representation.

Our objective in this paper is to describe the isotropy representation of \( K_\Theta \). This means that the invariant and irreducible subspaces of \( T_{b_\Theta}\mathbb{F}_\Theta \) must be obtained as well as the possible decompositions

\[
T_{b_\Theta}\mathbb{F}_\Theta = V_1 \oplus \cdots \oplus V_k \tag{1}
\]

into \( K_\Theta \)-invariant irreducible components.

The description of the isotropy representation of \( K_\Theta \) is essential to get \( K \)-invariant geometries on \( \mathbb{F}_\Theta \). For example the \( K \)-invariant Riemannian metrics on \( \mathbb{F} \) are given by the \( K_\Theta \)-invariant inner products on \( T_{b_\Theta}\mathbb{F}_\Theta \), which in turn are direct sum of invariant inner products on the components of a decomposition (1).

Too look at the \( K_\Theta \)-representation we consider first the the isotropy representation of \( P_\Theta \). It is completely determined by the restriction to its Levi component \( Z_\Theta \), which is the centralizer in \( G \) of \( H_\Theta \). The group \( Z_\Theta \) is reductive, so that its representation decomposes as a sum of invariant irreducible subspaces. This decomposition is unique and coincide with the decomposition for the ensuing representation of the Lie algebra \( z_\Theta \) of \( Z_\Theta \). In fact, each \( z_\Theta \)-irreducible component is a sum of root spaces (for a Cartan subalgebra) associated to different roots for different components. This implies uniqueness of the decomposition. For the moment we write the \( z_\Theta \)-decomposition as

\[
T_{b_\Theta}\mathbb{F}_\Theta = W_1^3 \oplus \cdots \oplus W_n^3.
\]

The subspaces \( W_i^3 \) are invariant by \( K_\Theta \) since \( K_\Theta \subset Z_\Theta \). Hence we are faced to the following problems:

1. Find the \( K_\Theta \)-invariant irreducible subspaces inside each \( W_i^3 \). This includes the question of deciding whether \( W_i^3 \) is \( K_\Theta \)-irreducible.

2. Among the invariant subspaces of item (1), find those pairs \( U_1, U_2 \) such that the \( K_\Theta \)-representations on them are equivalent. Given such a pair
we get further invariant irreducible subspaces contained in $U_1 \oplus U_2$ as graphs of operators $T : U_1 \to U_2$, intertwining the representations on $U_1$ and $U_2$.

The answers to these two questions give the full picture of the $K_\Theta$-invariant subspaces.

At this point it is worthwhile to compare the real flag manifolds with the complex ones. In the complex case the above questions have trivial answers: The subspaces $W^1_z$ are $K_\Theta$-irreducible and no two of them are equivalent. This is due to the fact that in a complex Lie group $K_\Theta$ is a compact real form of the semi-simple component $G(\Theta)$ of $Z_\Theta$, which is also a complex group. So that the equivalence classes of $K_\Theta$-representations are in bijection to the $G(\Theta)$-representations.

On the contrary for real flag manifolds new phenomena occur: There are $\mathfrak{z}_\Theta$-irreducible subspaces that are not $K_\Theta$-irreducible and there are equivalent $K_\Theta$-invariant irreducible subspaces. Such equivalence gives rise to continuous sets of invariant subspaces and to the nonuniqueness of the decompositions (1).

The basic differences of the real case to the complex one is that $K$ is not in general simple and $K_\Theta$ is not connected (if $\mathfrak{g}$ is a split real form). When $K$ is not simple we get a supply of $K_\Theta$-invariant subspaces as tangent spaces to the orbits through the origin $b_\Theta$ of the simple components of $K$. In many cases these tangent spaces decompose the $\mathfrak{z}_\Theta$-irreducible subspaces. The fact that $K_\Theta$ is not connected requires a separate analysis of the representations of its group of connected components, the so called $M$-group.

Now we describe the contents of the paper. Section 2 contains generalities about isotropy representations.

The main technical part of the paper starts at Section 3 where we look at the representations of the discrete group $M$. This is the centralizer in $K$ of the Cartan subalgebra $\mathfrak{a}$ and contains information about the group of connected components of any $K_\Theta$. Also $M = K_\Theta$ if $F_\Theta$ is the maximal flag manifold. The one dimensional root spaces $\mathfrak{g}_\alpha$ are $M$-invariant thus defining representations of $M$. For the roots $\alpha$ and $\beta$ we put $\alpha \sim_M \beta$ if the representations of $M$ on $\mathfrak{g}_\alpha$ and $\mathfrak{g}_\beta$ are equivalent. The purpose of Section 3 is to find $M$-equivalence classes of roots. After some preparations we proceed to a case by case analysis of the diagrams. For each case the $M$-equivalence classes are described at the beginning of the corresponding subsection. For the classical diagrams there are exceptions since in low dimension the sizes of
the classes tend to increase. The determination of the $M$-equivalence classes furnishes the complete picture of the isotropy representation on the maximal flag manifolds. They will be also a basic tool to detect inequivalent subrepresentations in the other flag manifolds.

Section 4 is preparatory. There we prove several lemmas to be applied in the study of isotropy representations on the partial flag manifolds. Some of these lemmas have independent interest, like Lemma 4.3 which ensures transitivity of the Weyl group on the set of weights of a given representation. This fact is far from to be true for general representations.

In Section 5 we go through the isotropy representations of the partial flag manifolds, again in a case by case analysis. For the classical diagrams we use their standard realizations as algebras of matrices: $A_l = \mathfrak{sl}(l + 1, \mathbb{R})$, $B_l = \mathfrak{so}(l, l + 1)$, $C_l = \mathfrak{sp}(l, \mathbb{R})$ and $D_l = \mathfrak{so}(l, l)$. These realizations allow the use of nice expressions for the roots. The analysis of the classical diagrams have the following pattern: First we describe the $\mathfrak{g}$-irreducible components. Then we check their $K_\Theta$-irreducibility and finally we look at equivalence between irreducible subspaces. The results are summarized at the end of each corresponding subsection. Regarding to the exceptional diagrams, $G_2$ is clear by its low dimensionality. For $E_6$, $E_7$ and $E_8$, it follows easily by the general lemmas of Section 4 that the $K_\Theta$-invariant subspaces are the $\mathfrak{g}$-irreducible components. As to $F_4$ we refrain to make a detailed and annoying description of the fifteen flag manifolds. Besides the maximal flag manifold, where the picture is given by the $M$-equivalence classes, we just look at a minimal flag manifold.

In conclusion we say that our initial motivation to study the isotropy representation came from the attempt to understand the $K$-invariant Riemannian metrics on the real flag manifolds. There is an extensive literature on invariant Riemannian geometry on complex flag manifolds. See for example Burstall-Rawnsley [1], Burstall-Salamon [2], Negreiros [6], San Martin-Negreiros [8], San Martin-Silva [9], and Wang-Ziller [10], and references therein. In a complex flag manifold the isotropy representation has a unique decomposition into invariant irreducible components, which makes the set of invariant Riemannian metrics a finite dimensional manifold. Our results in this paper show the existence of infinitely many decompositions on a real flag manifold, pointing to a great richness of the invariant Riemannian geometry.
2 Isotropy representation

Let \( \mathfrak{g} \) be a split real form of a complex simple Lie algebra, \( \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s} \) be a Cartan decomposition and \( \mathfrak{a} \subset \mathfrak{s} \) be a maximal abelian subalgebra. Denote by \( \Pi \) the associated set of roots and by

\[
\mathfrak{g} = \mathfrak{g}_0 \oplus \sum_{\alpha \in \Pi} \mathfrak{g}_\alpha
\]

the associated root space decomposition. Denote by \( G \) the group of inner automorphisms of \( \mathfrak{g} \), which is the subgroup of \( \text{Gl}(\mathfrak{g}) \) generated by \( \exp \text{ad}(\mathfrak{g}) \). Let \( K \) be the subgroup of \( G \) generated by \( \text{ad}(\mathfrak{k}) \). Fixing a set \( \Pi^+ \) of positive roots let \( \Sigma \) be the corresponding set of simple roots. We denote by \( \mathfrak{a}^+ = \{ H \in \mathfrak{a} : \forall \alpha \in \Sigma, \alpha(H) > 0 \} \) the Weyl chamber associated to \( \Sigma \).

A subset \( \Theta \subset \Sigma \) defines the parabolic subalgebra of type \( \Theta \) given by

\[
\mathfrak{p}_\Theta = \mathfrak{g}_0 \oplus \sum_{\alpha \in \Pi^+} \mathfrak{g}_\alpha \oplus \sum_{\alpha \in \langle \Theta \rangle^-} \mathfrak{g}_\alpha,
\]

where \( \langle \Theta \rangle^- \) is the set of negative roots generated by \( \Theta \). The standard parabolic subgroup \( P_\Theta \) defined by \( \Theta \) is the normalizer of \( \mathfrak{p}_\Theta \) in \( G \). The associated flag manifold is defined by \( F_\Theta = G/P_\Theta \). Since \( K \) acts transitively on \( F_\Theta \), this flag manifold can be given by \( F_\Theta = K/K_\Theta \), where \( K_\Theta = P_\Theta \cap K \).

When \( \Theta = \emptyset \) we get the minimal parabolic subalgebra \( \mathfrak{p} = \mathfrak{p}_\emptyset \). In this case the subscript is omitted and the maximal flag manifold is written \( F = G/P \).

We have \( F = K/M \), where \( M = K_\emptyset \) is the centralizer of \( \mathfrak{a} \) in \( K \).

For an alternative description of the parabolic subalgebra write

\[
\mathfrak{a}_\Theta = \{ H \in \mathfrak{a} : \forall \alpha \in \Theta, \alpha(H) = 0 \}
\]

for the anihilator of \( \Theta \). Let \( H_\Theta \) be characteristic for \( \Theta \), that is \( H_\Theta \) is in the “partial chamber” \( \mathfrak{a}_\Theta \cap \mathfrak{a}^+ \) and satisfies

\[
\Theta = \{ \alpha \in \Sigma : \alpha(H_\Theta) = 0 \}.
\]

Then

\[
\mathfrak{p}_\Theta = \sum_{\lambda \geq 0} V_\lambda(H_\Theta)
\]

where \( V_\lambda(H_\Theta) = \sum_{\alpha(H_\Theta) = \lambda} \mathfrak{g}_\alpha \) is the \( \lambda \)-eigenspace of \( \text{ad}(H_\Theta) \). Clearly any \( H_\Theta \) satisfying \( \langle ?? \rangle \) yield the same \( \mathfrak{p}_\Theta \), although the eigenspaces \( V_\lambda(H_\Theta) \) may change.
The centralizer of \( H_\Theta \), \( \mathfrak{z}_\Theta = \text{cent}_g (H_\Theta) = \sum_{\alpha(H_\Theta) = 0} g_\alpha \) is the Levi component of \( p_\Theta \). It is a reductive Lie algebra that decomposes as

\[
\mathfrak{z}_\Theta = g(\Theta) \oplus a_\Theta
\]

where the semi-simple component \( g(\Theta) \) is the subalgebra generated by \( g_\alpha \), \( \alpha \in \pm \Theta \). Since \( g \) is a split real form, it follows that \( g(\Theta) \) is also a split real form, having Cartan subalgebra the subspace \( a(\Theta) \) spanned by \( H_\alpha \), \( \alpha \in \Theta \) (where \( \alpha (\cdot) = \langle H_\alpha, \cdot \rangle \)). Put \( G(\Theta) = \langle \exp g(\Theta) \rangle \) for the connected subgroup with Lie algebra \( g(\Theta) \).

With this notation we have that \( K_\Theta = \text{Cent}_K (H_\Theta) \) is the centralizer of \( H_\Theta \) in \( K \) and its Lie algebra \( \mathfrak{k}_\Theta = \text{Cent}_\mathfrak{k} (H_\Theta) = \mathfrak{z}_\Theta \cap \mathfrak{k} \). Also, \( (K_\Theta)_0 \subset G(\Theta) \).

The nilpotent subalgebra

\[
\mathfrak{n}_\Theta = \sum_{\alpha \in \Pi^{-} \setminus \langle \Theta \rangle^{-}} g_\alpha = \sum_{\lambda < 0} V_\lambda (H_\Theta)
\]

complements \( p_\Theta \) in \( g \). Hence we identify \( \mathfrak{n}_\Theta \) with the tangent space \( T_{b_\Theta} F_\Theta \) at the origin \( b_\Theta \). Under this identification the isotropy representations of \( K_\Theta \) and \( G(\Theta) \) are just the adjoint representation, since \( \mathfrak{n}_\Theta \) is normalized by these groups. The same statement holds for the representations of the Lie algebras \( \mathfrak{t}_\Theta \), \( g(\Theta) \) and \( \mathfrak{z}_\Theta \).

Since \( \mathfrak{z}_\Theta \) is reductive its representation on \( \mathfrak{n}_\Theta \) is a direct sum

\[
\mathfrak{n}_\Theta = \sum_{\sigma} \mathfrak{V}_\Theta^\sigma
\]

where the subspaces \( \mathfrak{V}_\Theta^\sigma \) are \( \mathfrak{z}_\Theta \)-invariant and irreducible. Here we use \( \sigma \) to distinguish the different invariant subspaces.

**Proposition 2.1** Each \( \mathfrak{z}_\Theta \)-invariant and irreducible subspace \( \mathfrak{V}_\Theta^\sigma \) is a direct sum of root spaces,

\[
\mathfrak{V}_\Theta^\sigma = \sum g_\alpha
\]

where the sum extended to a subset of roots \( \Pi_\Theta^\sigma \subset \Pi^{-} \setminus \langle \Theta \rangle^{-} \). Conversely if \( \alpha \in \Pi^{-} \setminus \langle \Theta \rangle^{-} \) then \( g_\alpha \) is contained in a unique \( \mathfrak{z}_\Theta \)-component denoted by \( V_\Theta (\alpha) \). We write \( \Pi_\Theta (\alpha) \) for the roots \( \beta \) with \( g_\beta \subset V_\Theta (\alpha) \).

**Proof:** This follows by a standard argument using the fact that \( a \subset \mathfrak{z}_\Theta \). In fact, if \( V \) is a \( \mathfrak{z}_\Theta \)-invariant subspace and \( X = \sum X_\alpha \in V \) then

\[
\text{ad} (H) X = \sum \alpha (H) X_\alpha \in V
\]
if $H \in \mathfrak{a}$. By taking suitable values of $H \in \mathfrak{a}$ one concludes that each component $X_\alpha \in V$, so that $\mathfrak{g}_\alpha \subset V$. The last statement follows directly from the fact that $\mathfrak{n}_\Theta$ is the direct sum of the roots spaces as well as the $\mathfrak{z}_\Theta$-components.

The restriction to $\mathfrak{g}(\Theta)$ of the $\mathfrak{z}_\Theta$-representation on $V_\Theta^g$ is also irreducible. This is because $\mathfrak{z}_\Theta = \mathfrak{g}(\Theta) \oplus \mathfrak{a}_\Theta$ with $\mathfrak{a}_\Theta$ the center of $\mathfrak{g}(\Theta)$, so that $\text{ad}(H)$ is a scalar $\lambda \cdot \text{id}$ in $V_\Theta^g$ for any $H \in \mathfrak{a}_\Theta$. Hence, any $\mathfrak{g}(\Theta)$-invariant subspace $U \subset V_\Theta^g$ is also $\mathfrak{z}_\Theta$-invariant, ensuring that $V_\Theta^g$ is $\mathfrak{g}(\Theta)$-irreducible.

The weight spaces of the representation of $\mathfrak{g}(\Theta)$, w.r.t. $\mathfrak{a}(\Theta)$, are root spaces of $\mathfrak{g}$, so that the weights of the representation are restrictions to $\mathfrak{a}(\Theta)$ of some roots $\alpha \in \Pi \setminus \langle \Theta \rangle^-$. There is just one highest weight, say $\mu$, and two representations of $\mathfrak{g}(\Theta)$ on $V_{\Theta}^{g_1}$ and $V_{\Theta}^{g_2}$ are equivalent if and only if $\mu_{g_1} = \mu_{g_2}$. (We note that different representations of $\mathfrak{z}_\Theta$ on $V_{\Theta}^{g_1}$ and $V_{\Theta}^{g_2}$ cannot be equivalent, even if the $\mathfrak{g}(\Theta)$-representations are equivalent.)

The subspaces $V_\Theta^g$ are also invariant and irreducible by $G(\Theta)$, since by definition this group is connected. Hence $V_\Theta^g$ is invariant by the identity component $(K_\Theta)_0$ of $K_\Theta$, because $(K_\Theta)_0 \subset G(\Theta)$. As to $K_\Theta$ we have $K_\Theta = M \cdot (K_\Theta)_0$ which ensures that $V_\Theta^g$ is $K_\Theta$-invariant, because $M$ leaves invariant each root space.

Our objective is to get the invariant irreducible subspaces of $\mathfrak{n}_\Theta$ by the $K_\Theta$ representation, which is equivalent to the isotropy representation of the flag $\mathcal{F}_\Theta$.

In view of the above discussion we are reduced to the following questions:

1. Describe the irreducible components $V_\Theta^g$ of the $\mathfrak{z}_\Theta$ representation.
2. Find the $K_\Theta$-invariant subspaces of each $V_\Theta^g$.
3. Find the pairs of irreducible subspaces having equivalent $K_\Theta$-representations.

Finally we note that if $H_\Theta \in \mathfrak{a}_\Theta \cap \mathfrak{cl} \mathfrak{a}^+$ then an eigenspace $V_\lambda(\mathfrak{H}_\Theta) = \sum_{\alpha(H_\Theta) = \lambda} \mathfrak{g}_\alpha$ is contained in $\mathfrak{n}_\Theta$ if $\lambda < 0$ and is invariant by $\mathfrak{z}_\Theta$. Hence $V_\lambda(\mathfrak{H}_\Theta)$ is the direct sum of some irreducible components $V_\Theta^g$. This remark will be used later to determine the irreducible components $V_\Theta^g$. Actually, in some cases an eigenspace $V_\lambda(\mathfrak{H}_\Theta)$ is irreducible and hence is itself a component.
3 $M$-equivalence classes

Let $M = \text{Cent}_K(a)$ be the centralizer of $a$ in $K$. It is known that $M \subset K_\Theta = M(K_\Theta)_0$. Also, any root space $g_\alpha$ is $M$-invariant. In this section we determine the pairs of root spaces $g_\alpha$, $g_\beta$ having equivalent representations of $M$. This will be used later to check equivalence or nonequivalence of $K_\Theta$-representations on invariant subspaces.

**Definition 3.1** The roots $\alpha$ and $\beta$ are said to be $M$-equivalent (in symbols $\alpha \sim_M \beta$) if the representations of $M$ on $g_\alpha$ and $g_\beta$ are equivalent. We write $[\alpha]_M$ for the $M$-equivalence class of the root $\alpha$.

If $g$ is a split real form of a complex semi-simple Lie algebra then $M$ is a discrete abelian subgroup equals to

$$M = \{m_\gamma = \exp(\pi i H_\gamma^\vee) : \gamma \in \Pi}\,$$

where $H_\gamma^\vee = \frac{2H_\gamma}{\langle \gamma, \gamma \rangle}$ is the co-root associated to $\gamma$ and $H_\gamma$ is defined by $\gamma(H) = \langle H_\gamma, H \rangle$, $H \in \mathfrak{a}$. In the above formula the exponential $\exp(\pi i H_\gamma^\vee)$ is in the complex group $\text{Aut}(\mathfrak{g}_C)$, where $\mathfrak{g}_C$ is the complexification of $\mathfrak{g}$ (see [5, Theorems 7.53 and 7.55]).

The following statement gives a necessary and sufficient condition for the $M$-equivalence between the roots $\alpha$ and $\beta$.

**Proposition 3.2** The root $\alpha$ and $\beta$ are $M$-equivalent if and only if, for every $\gamma \in \Pi$ we have

$$\frac{2\langle \gamma, \alpha \rangle}{\langle \gamma, \gamma \rangle} \equiv \frac{2\langle \gamma, \beta \rangle}{\langle \gamma, \gamma \rangle} \mod 2.$$

**Proof:** Take a root $\gamma$ and write as above $m_\gamma = \exp(\pi i H_\gamma^\vee)$. If $X \in \mathfrak{g}_\alpha$ and $Y \in \mathfrak{g}_\beta$ then

$$\text{Ad}(m_\gamma)X = e^{\pi i \alpha(H_\gamma^\vee)}X \quad \text{and} \quad \text{Ad}(m_\gamma)Y = e^{\pi i \beta(H_\gamma^\vee)}Y,$$

by definition of $m_\gamma$. It follows that $\alpha \sim_M \beta$ if and only if $e^{\pi i \alpha(H_\gamma^\vee)} = e^{\pi i \beta(H_\gamma^\vee)}$, which is equivalent to

$$\frac{2\langle \gamma, \alpha \rangle}{\langle \gamma, \gamma \rangle} \equiv \frac{2\langle \gamma, \beta \rangle}{\langle \gamma, \gamma \rangle} \mod 2$$

as desired. \qed

As a corollary we get the following necessary condition.
Corollary 3.3 If $\alpha \sim_M \beta$ then $\langle \alpha, \beta \rangle = 0$.

Proof: Suppose that $\langle \alpha, \beta \rangle \neq 0$. Then we have the following possibilities for the Killing numbers:

1. $\alpha$ and $\beta$ have the same length and the angle between them is $60^\circ$ or $120^\circ$. Then
   \[
   \frac{2\langle \alpha, \alpha \rangle}{\langle \alpha, \alpha \rangle} = 2 \quad \text{and} \quad \frac{2\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle} = \pm 1
   \]
   showing that $\alpha$ and $\beta$ are not $M$-equivalent.

2. The angle between $\alpha$ and $\beta$ is $45^\circ$ or $135^\circ$. If $\alpha$ is the long root then
   \[
   \frac{2\langle \alpha, \alpha \rangle}{\langle \alpha, \alpha \rangle} = 2 \quad \text{and} \quad \frac{2\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle} = \pm 1
   \]
   and $\alpha$ and $\beta$ are not $M$-equivalent. If otherwise $\beta$ is the long root then we interchange the roles of $\alpha$ and $\beta$ to get the same result.

3. The angle between $\alpha$ and $\beta$ is $30^\circ$ or $150^\circ$. Then
   \[
   \frac{2\langle \alpha, \alpha \rangle}{\langle \alpha, \alpha \rangle} = 2 \quad \text{and} \quad \frac{2\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle} = \pm 1, \pm 3
   \]
   concluding the proof.

In the sequel we apply the above proposition and its corollary to find for each Dynkin diagram the classes of $M$-equivalence between roots. The following simple remarks are used throughout with no further reference.

1. $\alpha \sim_M (-\alpha)$. In fact the Cartan involution $\theta$ is an equivalence between the $M$-representations in $g_\alpha$ and $g_{-\alpha}$ because $\theta(m) = m$ if $m \in M$. This implies that $\text{Ad}(m) \circ \theta = \theta \circ \text{Ad}(m)$ if $m \in M$ and since $\theta(g_\alpha) = g_{-\alpha}$ the equivalence follows. Hence we are reduced to check $M$-equivalence between positive roots alone.

2. The criterion of Proposition 3.2 implies easily that if $w \in W$ then $\alpha \sim_M \beta$ if and only if $w\alpha \sim_M w\beta$. Hence it will will be enough to get the $M$-equivalence classes for just one element in each orbit of the Weyl group, that is, for one root in the simply laced diagrams and for one long root and a short root in diagrams with multiple edges.

We proceed now to look at the $M$-equivalences for each Dynkin diagram.
3.1 Diagram $A_l$, $l \geq 1$

We use the standard realization of $A_l$ where the positive roots are written as $\lambda_i - \lambda_j$, $1 \leq i < j \leq l + 1$. There are two cases:

3.1.1 $A_l$, $l \neq 3$

The classes of $M$-equivalence on the positive roots are singletons. (That is the $M$-representation on different root spaces are not equivalent.)

Since the Weyl group is transitive on the set of roots it is enough to fix a specific root $\alpha$ and check that any positive root $\beta \neq \alpha$ is not $M$-equivalent to $\alpha$.

Suppose that $l > 3$ and take $\alpha = \lambda_1 - \lambda_2$. The positive roots orthogonal to $\alpha$ are $\lambda_i - \lambda_j$ with $3 \leq i < j$. By Corollary 3.3 we are reduced to check that these roots are not $M$-equivalent to $\alpha = \lambda_1 - \lambda_2$. There are the following cases for $3 \leq i < j$:

1. If $j < l+1$ then $\langle \gamma, \lambda_i - \lambda_j \rangle \neq 0$ but $\langle \gamma, \lambda_1 - \lambda_2 \rangle = 0$ where $\gamma = \lambda_j - \lambda_{j+1}$. Hence

$$\frac{2\langle \gamma, \lambda_1 - \lambda_2 \rangle}{\langle \gamma, \gamma \rangle} = 0$$

so that $\lambda_1 - \lambda_2$ is not $M$-equivalent to $\lambda_i - \lambda_j$, $3 \leq i < j$.

2. If $i > 3$ then $\langle \gamma, \lambda_i \pm \lambda_j \rangle \neq 0$ but $\langle \gamma, \lambda_1 - \lambda_2 \rangle = 0$ where $\gamma = \lambda_i - \lambda_j$, and again $\lambda_1 - \lambda_2$ is not $M$-equivalent to $\lambda_3 - \lambda_j$, $3 < j$.

3. If $i = 3$ and $j = l+1$ then $\lambda_4 - \lambda_{l+1}$ is a root orthogonal to $\lambda_1 - \lambda_2$ but not orthogonal to $\lambda_3 - \lambda_{l+1}$.

Finally if $l = 1$ there is just one positive root. If $l = 2$ then the positive roots are not orthogonal to each other so by Corollary 3.3 they are not $M$-equivalent.

3.1.2 $A_3$

The $M$-equivalence classes on the positive roots are $\{\lambda_1 - \lambda_2, \lambda_3 - \lambda_4\}$, $\{\lambda_1 - \lambda_3, \lambda_2 - \lambda_4\}$ and $\{\lambda_1 - \lambda_4, \lambda_2 - \lambda_3\}$.

In this case the unique root orthogonal to $\alpha = \lambda_1 - \lambda_2$ is $\lambda_3 - \lambda_4$ and hence, by Corollary 3.3, $\lambda_3 - \lambda_4$ is the only candidate to be $M$-equivalent to
\(\lambda_1 - \lambda_2\). To see that indeed \(\lambda_3 - \lambda_4 \sim_M \lambda_1 - \lambda_2\) note that a root \(\gamma = \lambda_i - \lambda_j\) with \((i,j) \neq (1,2)\) or \((3,4)\) is not orthogonal to \(\lambda_1 - \lambda_2\) neither to \(\lambda_3 - \lambda_4\). So that the Killing numbers \(\frac{2(\gamma,\lambda_1-\lambda_2)}{(\gamma,\gamma)}\) and \(\frac{2(\gamma,\lambda_3-\lambda_4)}{(\gamma,\gamma)}\) are \(\pm 1\), that is, the condition of Proposition 3.2 is satisfied showing that \(\lambda_3 - \lambda_4 \sim_M \lambda_1 - \lambda_2\), By applying the Weyl group (permutation group) we see that the classes of \(\lambda_i\) with \((i,j)\) satisfying the condition of Proposition 3.2 is satisfied showing that \(\lambda_3 - \lambda_4 \sim_M \lambda_1 - \lambda_2\), By applying the Weyl group (permutation group) we see that the classes of \(\lambda\)-equivalences are as stated.

### 3.2 Diagram \(B_l, l \geq 2\)

In the standard realization of \(B_l = \mathfrak{so}(l, l+1)\) the positive roots are written as \(\lambda_i \pm \lambda_j, 1 \leq i < j \leq l\) and \(\lambda_i, 1 \leq i \leq l\). These are the long and short roots respectively.

The \(M\)-equivalence classes depend on the rank \(l\), according to the following cases:

#### 3.2.1 \(B_l, l \geq 5\)

The \(M\)-equivalence classes on the positive roots are \(\{\lambda_i - \lambda_j, \lambda_i + \lambda_j\}\) and \(\{\lambda_i\}, 1 \leq i < j \leq l\).

We find the equivalence classes of the long and short roots.

Take long root \(\lambda_1 - \lambda_2\). We must check \(M\)-equivalence only for the roots orthogonal to it, namely \(\lambda_1 + \lambda_2, \lambda_i \pm \lambda_j\) and \(\lambda_i\) with \(3 \leq i < j\). The roots \(\lambda_i, 3 \leq i\), are not \(M\)-equivalent to \(\lambda_1 - \lambda_2\). In fact, \(\lambda_i \pm \lambda_{i+1}\) is a root because \(l \geq 5\). Now, \(\langle \lambda_i, \lambda_i \pm \lambda_{i+1} \rangle \neq 0\) and the Killing number \((\langle \lambda_i \pm \lambda_{i+1} \rangle, \lambda_i) = \pm 1\) because \(\lambda_i\) is a short root. Since \(\langle \lambda_i, \lambda_1 - \lambda_2 \rangle = 0\) the condition of Proposition 3.2 is violated by \(\gamma = \lambda_i \pm \lambda_{i+1}\). The same argument used in the \(A_l\) case show that \(\lambda_i \pm \lambda_j, 3 \leq i < j\), is not \(M\)-equivalent to \(\lambda_1 - \lambda_2\) (when \(l \geq 5\)). On the other hand \(\lambda_1 - \lambda_2 \sim_M \lambda_1 + \lambda_2\) because for any root \(\gamma\) it holds \(\langle \gamma, \lambda_1 - \lambda_2 \rangle = \pm \langle \gamma, \lambda_1 + \lambda_2 \rangle\). It follows that \(\{\lambda_1 - \lambda_2, \lambda_1 + \lambda_2\}\) is an \(M\)-equivalence class. To conclude this case we note that \(w \in \mathcal{W}\) acts on \(\lambda_i\) by a permutation followed by a change of sign, that is, \(w\lambda_i = \pm \lambda_j\), for some index \(j\). Hence \(\lambda_i - \lambda_j \sim_M \lambda_i + \lambda_j, 1 \leq i < j \leq l\) and the sets \(\{\lambda_i - \lambda_j, \lambda_i + \lambda_j\}\) are the only \(M\)-equivalence classes containing a long root.

By the previous paragraph no long root is \(M\)-equivalent to the short root \(\lambda_i\). Finally two short roots \(\lambda_i\) and \(\lambda_j\), \(i \neq j\), are not \(M\)-equivalent. For example \(\gamma = \lambda_i + \lambda_k\), \(k \neq i, j\) satisfies \(\langle \gamma, \lambda_i \rangle = 1\) while \(\langle \gamma, \lambda_j \rangle = 0\).
3.2.2 \( B_4 \)

The \( M \)-equivalence classes on the positive roots are \( \{\lambda_1 - \lambda_2, \lambda_1 + \lambda_2, \lambda_3 - \lambda_4, \lambda_3 + \lambda_4\}, \{\lambda_1 - \lambda_3, \lambda_1 + \lambda_3, \lambda_2 - \lambda_4, \lambda_2 + \lambda_4\}, \{\lambda_1 - \lambda_4, \lambda_1 + \lambda_4, \lambda_2 - \lambda_3, \lambda_2 + \lambda_3\} \), and the short roots \( \{\lambda_i\}, 1 \leq i \leq 4 \).

The difference from the general case is that \( \lambda_3 - \lambda_4 \sim_M \lambda_1 - \lambda_2 \). In fact if \( \lambda_i \) is a short root then \( \langle \lambda_i^\vee, \lambda_1 - \lambda_2 \rangle \) and \( \langle \lambda_i^\vee, \lambda_3 - \lambda_4 \rangle \) equals 0 or 2. Also if \( \gamma \) is a long root different from \( \lambda_1 - \lambda_2 \) and \( \lambda_3 - \lambda_4 \) then \( \langle \gamma^\vee, \lambda_1 - \lambda_2 \rangle \) and \( \langle \gamma^\vee, \lambda_3 - \lambda_4 \rangle \) equals \( \pm 1 \).

Again the same arguments show that a long root and a short root as well as two short roots are not \( M \)-equivalent.

3.2.3 \( B_3 \)

The \( M \)-equivalence classes on the positive roots are \( \{\lambda_1 - \lambda_2, \lambda_1 + \lambda_2, \lambda_3\} \), \( \{\lambda_1 - \lambda_3, \lambda_1 + \lambda_3, \lambda_2\} \) and \( \{\lambda_2 - \lambda_3, \lambda_2 + \lambda_3, \lambda_1\} \).

Here \( \lambda_3 \sim_M \lambda_1 - \lambda_2 \). The point is that if \( \gamma \neq \lambda_1 - \lambda_2 \) is a long root then \( \langle \gamma^\vee, \lambda_1 - \lambda_2 \rangle = \pm 1 \) and since \( \gamma \) cannot be orthogonal to \( \lambda_3 \) we have \( \langle \gamma^\vee, \lambda_3 \rangle = \pm 1 \) as well. On the other hand if \( \gamma \) is short then \( \langle \gamma^\vee, \lambda_1 - \lambda_2 \rangle \) and \( \langle \gamma^\vee, \lambda_3 \rangle \) are even.

3.2.4 \( B_2 \)

The \( M \)-equivalence classes on the positive roots are the long roots \( \{\lambda_1 - \lambda_2, \lambda_1 + \lambda_2\} \) and the short roots \( \{\lambda_1, \lambda_2\} \).

3.3 Diagram \( C_l, l \geq 3 \)

In the standard realization of \( C_l = \mathfrak{sp}(l, \mathbb{R}) \) the positive roots are written as \( \lambda_i \pm \lambda_j, 1 \leq i < j \leq l \) and \( 2\lambda_i, 1 \leq i \leq l \). These are the short and long roots respectively.

Here any two long roots \( 2\lambda_i \) and \( 2\lambda_j \) are \( M \)-equivalent. In fact, for any root \( \gamma \) the Killing number \( \langle \gamma^\vee, 2\lambda_i \rangle \) is even (0 or \( \pm 2 \)). In fact, if \( \gamma \) is a short root then \( \langle \gamma^\vee, 2\lambda_i \rangle \) is either 0 (orthogonal roots) or \( \pm 2 \) (Killing number between a short root and a long root). On the other hand two long roots are either equal or orthogonal.

As in the previous diagrams the \( M \)-equivalence classes increase for small ranks. For \( C_l \) the exception is when \( l = 4 \).
3.3.1 \( C_l, l \neq 4 \)

The \( M \)-equivalence classes are \( \{ \lambda_i - \lambda_j, \lambda_i + \lambda_j \} \) and the set of long roots \( \{ 2\lambda_1, \ldots, 2\lambda_l \} \).

The roots orthogonal to the short root \( \lambda_1 - \lambda_2 \) are \( \lambda_1 + \lambda_2, \lambda_i \pm \lambda_j \) and \( 2\lambda_i \) with \( 3 \leq i < j \). As in the \( B_l \) case (with \( l \geq 5 \)) the roots \( \lambda_i \pm \lambda_j, 3 \leq i < j \), are not \( M \)-equivalent to \( \lambda_1 - \lambda_2 \). On the other hand if \( 3 \leq i \) then \( \gamma = \lambda_1 - \lambda_j \) with \( j \neq i \) violates the criterion of Proposition 3.2 for \( M \)-equivalence between \( \lambda_1 - \lambda_2 \) and \( 2\lambda_i \). In fact, \( \langle \gamma^\vee, \lambda_1 - \lambda_2 \rangle = \pm 1 \) (non orthogonal roots of same length) and \( \langle \gamma^\vee, 2\lambda_i \rangle = 0 \).

Since the long roots are equivalent to each other it follows that \( \lambda_1 - \lambda_2 \) is not \( M \)-equivalent to any long root. Hence we get the classes stated above.

These arguments remain true if \( l = 3 \). (Differently from \( B_3 \) in \( C_3 \) long roots are not \( M \)-equivalent to short roots.)

3.3.2 \( C_4 \)

The \( M \)-equivalence classes are \( \{ \lambda_1 - \lambda_2, \lambda_1 + \lambda_2, \lambda_3 - \lambda_4, \lambda_3 + \lambda_4 \} \), \( \{ \lambda_1 - \lambda_3, \lambda_1 + \lambda_3, \lambda_2 - \lambda_4, \lambda_2 + \lambda_4 \} \), \( \{ \lambda_1 - \lambda_4, \lambda_1 + \lambda_4, \lambda_2 - \lambda_3, \lambda_2 + \lambda_3 \} \) and the long roots \( \{ 2\lambda_1, 2\lambda_2, 2\lambda_3, 2\lambda_4 \} \).

This is seen as in \( B_4 \) where \( \lambda_3 - \lambda_4 \sim_M \lambda_1 - \lambda_2 \).

3.4 Diagram \( D_l, l \geq 4 \)

In the standard realization of \( D_l = \mathfrak{so} (l, l) \) the positive roots are written as \( \lambda_i \pm \lambda_j, 1 \leq i < j \leq l \).

3.4.1 \( D_l, l > 4 \)

The \( M \)-equivalence classes on the positive roots are \( \{ \lambda_i - \lambda_j, \lambda_i + \lambda_j \} \), \( 1 \leq i < j \leq l \).

This is verified by arguments similar to the \( B_l \) case, simplified by the fact that the roots have the same length.

First the only root \( M \)-equivalent to \( \lambda_1 - \lambda_2 \) is \( \lambda_1 + \lambda_2 \). In fact, the roots orthogonal to \( \lambda_1 - \lambda_2 \) are \( \lambda_1 + \lambda_2 \) and \( \lambda_i \pm \lambda_j, 3 \leq i < j \). A root \( \lambda_i \pm \lambda_j \) with \( 3 \leq i < j \) is not \( M \)-equivalent to \( \lambda_1 - \lambda_2 \) by the following reasons:

1. If \( j < l \) and \( \gamma = \lambda_j - \lambda_{j+1} \) then \( \langle \gamma, \lambda_1 - \lambda_2 \rangle = 0 \) and \( \langle \gamma, \lambda_i \pm \lambda_j \rangle \neq 0 \) which implies that \( \langle \gamma^\vee, \lambda_i \pm \lambda_j \rangle = \pm 1 \). Thus by Proposition 3.2 \( \lambda_1 - \lambda_2 \) is not \( M \)-equivalent to \( \lambda_i \pm \lambda_j \).
2. If $i > 3$ and $\gamma = \lambda_{i-1} - \lambda_i$ then $\langle \gamma, \lambda_1 - \lambda_2 \rangle = 0$ and $\langle \gamma^{\vee}, \lambda_i \pm \lambda_j \rangle = \pm 1$.

3. Since $l > 4$, $\lambda_4 - \lambda_l$ is a root satisfying $\langle \gamma, \lambda_1 - \lambda_2 \rangle = 0$ and $\langle \gamma^{\vee}, \lambda_3 \pm \lambda_l \rangle = \pm 1$.

Finally $\lambda_1 - \lambda_2 \sim_M \lambda_1 + \lambda_2$, because $\langle \gamma, \lambda_1 - \lambda_2 \rangle = 0$ if and only if $\langle \gamma, \lambda_1 + \lambda_2 \rangle = 0$ for any root $\gamma$. Also, if $\gamma$ is not orthogonal to both roots then the Killing numbers are $\pm 1$, since the roots have the same length.

Since the Weyl group is transitive on the set of roots we get the equivalence classes stated above.

3.4.2 $D_4$

The $M$-equivalence classes on the positive roots are $\{\lambda_1 - \lambda_2, \lambda_1 + \lambda_2, \lambda_3 - \lambda_4, \lambda_3 + \lambda_4\}$, $\{\lambda_1 - \lambda_3, \lambda_1 + \lambda_3, \lambda_2 - \lambda_4, \lambda_2 + \lambda_4\}$ and $\{\lambda_1 - \lambda_4, \lambda_1 + \lambda_4, \lambda_2 - \lambda_3, \lambda_2 + \lambda_3\}$.

In this case, apart from $\lambda_1 + \lambda_2$ the roots $\lambda_3 - \lambda_4$ and $\lambda_3 + \lambda_4$ are $M$-equivalent to $\lambda_1 - \lambda_2$ (see the discussion for $B_4$). Hence an application of the Weyl group yield the stated classes.

3.5 Diagram $G_2$

The $M$-equivalence classes on the positive roots are the pairs $\{\alpha_1, \alpha_1 + 2\alpha_2\}$, $\{\alpha_1 + \alpha_2, \alpha_1 + 3\alpha_2\}$ and $\{\alpha_2, 2\alpha_1 + 3\alpha_2\}$ where $\alpha_1$ and $\alpha_2$ are the simple roots with $\alpha_1$ the long one.

The reason is that these are the only pairs of positive roots orthogonal to each other. Moreover if two roots are not orthogonal then their Killing are odd ($\pm 1$ ou $\pm 3$).

3.6 Diagrams $E_6$, $E_7$ and $E_8$

For these diagrams the $M$-equivalence classes on the positive roots are singletons.

Since these diagrams are simply-laced it is enough to find a positive root which is not $M$-equivalent to any other positive root.

In any of the diagrams $E_6$, $E_7$ and $E_8$ we choose the highest root $\mu$. To check that $\{\mu\}$ is an $M$-equivalence class we prove the
• **Claim:** For every $\beta > 0$ with $\langle \mu, \beta \rangle = 0$ there exists $\gamma \neq \beta$ such that $\langle \mu, \gamma \rangle = 0$ and $\langle \beta, \gamma \rangle \neq 0$.

From the claim we get $\langle \gamma^\vee, \mu \rangle = 0$ and $\langle \gamma^\vee, \beta \rangle$ odd because the diagrams are simply laced. Hence, by Proposition 3.2, no $\beta$ orthogonal to $\mu$ is $M$-equivalent to $\mu$. By Corollary 3.3 we conclude that $\{\mu\}$ is an $M$-equivalence class.

Now the roots orthogonal to the highest root $\mu$ have the following simple description: Denote by $\Sigma = \{\alpha_1, \ldots, \alpha_l\}$ the simple system of roots, and let $\{\omega_1, \ldots, \omega_l\}$ be the fundamental weights, defined by

$$\langle \alpha_i^\vee, \omega_j \rangle = \delta_{ij}.$$  

It is known that in the diagrams $E_6$, $E_7$ and $E_8$ the highest root $\mu = \omega_i$ for some fundamental weight. (The formula for $\mu$ in terms of the fundamental weights can be read off from the affine Dynkin diagrams. The extra root is precisely $-\mu$, see [4], Chapter X, Table of Diagrams S(A).) Let $\alpha = b_1\alpha_1 + \cdots + b_l\alpha_l$, $b_i \geq 0$, be a positive root. Since $\mu = \omega_i$ we have by definition

$$\langle \alpha, \mu \rangle = \frac{\langle \alpha_i, \alpha_i \rangle}{2} a_i b_i.$$  

So that $\langle \alpha, \mu \rangle = 0$ if and only if $a_i b_i = 0$. Therefore the roots orthogonal to $\mu$ are those spanned by $\Sigma \setminus \{\alpha_i\}$. This set of roots is a root system whose Dynkin diagram is the subdiagram of $\Sigma$ given by $\Sigma \setminus \{\alpha_i\}$. A glance at the affine Dynkin diagrams provides the diagrams $\Sigma \setminus \{\alpha_i\}$, namely,

- $\Sigma \setminus \{\alpha_i\} = A_5$ if $\Sigma = E_6$.
- $\Sigma \setminus \{\alpha_i\} = D_6$ if $\Sigma = E_7$.
- $\Sigma \setminus \{\alpha_i\} = E_7$ if $\Sigma = E_8$.

Now it is clear that in any of the root systems spanned by $\Sigma \setminus \{\alpha_i\}$ (A5, D6 or E7), the conclusion of the claim holds, that is, given $\beta$ there exists $\gamma$ with $\langle \beta, \gamma \rangle \neq 0$. This concludes the proof that the $M$-equivalence classes on the positive roots are singletons.
3.7 $F_4$

The 24 positive roots of

$$F_4 \xrightarrow{\alpha_1} \xrightarrow{\alpha_2} \xrightarrow{\alpha_3} \xrightarrow{\alpha_4}$$

split into the following $M$-equivalence classes:

- 12 singletons $\{\alpha\}$ with $\alpha$ running through the set of short roots.
- 3 sets of long roots $\{2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4, \alpha_2, \alpha_2 + 2\alpha_3 + 2\alpha_4\}$,
  $\{\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4, \alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4\}$ and
  $\{\alpha_1 + 2\alpha_2 + 4\alpha_3 + 2\alpha_4, \alpha_1, \alpha_1 + 2\alpha_2 + 2\alpha_3, \alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4\}$.

We let $\{\omega_1, \omega_2, \omega_3, \omega_4\}$ be the fundamental weights. The fundamental weight $\omega_4$ is also the short positive root

$$\omega_4 = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4.$$ 

We look at its $M$-equivalence class by the same method of the $E_l$’s. The set of roots orthogonal to the fundamental weight $\omega_4$ is spanned by $\{\alpha_1, \alpha_2, \alpha_3\}$ which is a $B_3$ Dynkin diagram. Now if $\beta$ is a root of $B_3$ then there exists a root $\gamma$ (in $B_3$) such that $\langle \gamma^\vee, \beta \rangle$ is odd. It follows by Proposition 3.2 and its Corollary 3.3 that $\{\omega_4\}$ is an $M$-equivalence class. This gives the classes of the short roots.

As to the long roots we first recall that they form a $D_4$ root system (see ???). Now if $\gamma$ is a short root and $\alpha$ a long root then $\langle \gamma^\vee, \alpha \rangle$ is even. Hence to check if $\alpha \sim_M \beta$ for the two long roots $\alpha$ and $\beta$ it is enough to test the condition of Proposition 3.2 when $\gamma$ is also a long root. This means that two long roots are $M$-equivalent if and only if they are equivalent as roots of $D_4$. Since no short root is $M$-equivalent to a long root we conclude that the classes of $D_4$ are also $M$-equivalence classes in $F_4$. These are the three sets with four orthogonal roots each as stated. (To get these sets start with the highest root $\omega_1 = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4$. Then the first set is $\omega_1$ together with the long roots orthogonal to it. The next two sets are obtained by applying first the reflection $r_{\alpha_1}$ and then $r_{\alpha_2}$.)

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4 Auxiliary lemmas

In this section we prove some lemmas to be used later in the determination of the irreducible $K_{\Theta}$-invariant subspaces of $\mathfrak{n}_\Theta$. We choose once and for all a generator $E_\alpha \in \mathfrak{g}_\alpha$ for each root space.

Recall that in Section 2 we denoted the irreducible components for the adjoint representation of $\mathfrak{z}_\Theta$ on $\mathfrak{n}_\Theta$ by $V_\sigma^{\mathfrak{z}_\Theta}$. Write $\Pi_\sigma^{\mathfrak{z}_\Theta} \subset \Pi^- \backslash \langle \Theta \rangle^-$ for the set of roots such that

$$V_\sigma^{\mathfrak{z}_\Theta} = \sum_{\alpha \in \Pi_\sigma^{\mathfrak{z}_\Theta}} \mathfrak{g}_\alpha.$$

The subgroup $K_{\Theta}$ leave invariant $V_\sigma^{\mathfrak{z}_\Theta}$, hence $\Pi_\sigma^{\mathfrak{z}_\Theta}$ is $W_\Theta$-invariant.

Our first results give conditions ensuring that a $\mathfrak{z}_\Theta$-invariant subspace is $K_{\Theta}$-irreducible.

**Lemma 4.1** Let $V = \sum_{\alpha \in \Pi_V} \mathfrak{g}_\alpha$ be a $\mathfrak{z}_\Theta$-invariant subspace and suppose that $W \subset V$ is a $K_{\Theta}$-invariant subspace. Take $X = \sum_{\alpha \in \Pi_V} a_\alpha E_\alpha \in W$ and let $\alpha$ be a root such that $a_\alpha \neq 0$. Define

$$V_{[\alpha]} = \sum_{\beta \in [\alpha] \cap \Pi_V} \mathfrak{g}_\beta.$$

Then $W \cap V_{[\alpha]} \neq \{0\}$.

**Proof:** Let $c_{X,\alpha}$ be the cardinality of $\{\beta \notin [\alpha]_M : a_\beta \neq 0\}$. If $c_{X,\alpha} = 0$ we are done. Otherwise we find $0 \neq Y \in U$ with $c_{Y,\alpha} < c_{X,\alpha}$. In fact, if $c_{X,\alpha} > 0$ then there are $\beta \notin [\alpha]_M$ with $a_\alpha, a_\beta \neq 0$. So that there exists $m \in M$ with $mE_\alpha = E_\alpha$ and $mE_\beta = -E_\beta$.

Now, $M \subset K_{\Theta}$, hence $Y = X + mX \in W$. Clearly $Y \neq 0$ and since the $\beta$ component of $Y$ is zero we have $c_{Y,\alpha} < c_{X,\alpha}$. Repeating this argument successively we arrive at $Z \in W$ such that $c_{Z,\alpha} = 0$, concluding the proof. \(\square\)

**Lemma 4.2** Take a subset $\Pi_V \subset \Pi^- \backslash \langle \Theta \rangle^-$ such that the subspace $V = \sum_{\alpha \in \Pi_V} \mathfrak{g}_\alpha$ is $\mathfrak{z}_\Theta$-invariant. Suppose that

1. $W_\Theta$ acts transitively on $\Pi_V$, and
2. two different roots in $\Pi_V$ are not $M$-equivalent.
Then $V$ is $K_\Theta$-irreducible.

**Proof:** Let $U \subset V$ be a nontrivial $K_\Theta$-invariant subspace. By transitivity of $W_\Theta \subset K_\Theta$ it is enough to prove that $U$ contains a root space $g_\alpha$, $\alpha \in \Pi_V$. But this follows by the previous lemma and the assumption that different roots in $\Pi_V$ are not $M$-equivalent. \qed

As a complement of the above lemma we exhibit next general cases where $W_\Theta$ acts transitively on sets of roots.

**Lemma 4.3** Let $\Pi^\sigma_\Theta \subset \Pi^\sigma - \langle \Theta \rangle^-$ be the set of roots corresponding to an irreducible component $V^\sigma_\Theta = \sum_{\alpha \in \Pi^\sigma_\Theta} g_\alpha$. In each of the following cases $W_\Theta$ acts transitively on $\Pi^\sigma_\Theta$.

1. The Dynkin diagram of $\mathfrak{g}$ has only simple edges ($A_l$, $D_l$, $E_6$, $E_7$ and $E_8$).

2. For the diagrams $B_l$, $C_l$ and $F_4$ there are the cases:

   (a) The roots in $\Theta \subset \Sigma$ are long.

   (b) The roots in $\Pi^\sigma_\Theta$ are short.

**Proof:** Let $\mu$ be the highest root of $\Pi^\sigma_\Theta$. By representation theory we know that any other root $\beta \in \Pi^\sigma_\Theta$ (weight of the representation) is given by

$$
\beta = \mu - \alpha_1 - \cdots - \alpha_k
$$

with $\alpha_i \in \Theta$ and such that any partial difference $\mu - \alpha_1 - \cdots - \alpha_i$, $i \leq k$, is also a root. Fix $i \leq k$ put $\delta = \mu - \alpha_1 - \cdots - \alpha_{i-1}$ and let $r_{\alpha_i}$ be the reflection with respect to $\alpha_i$.

We claim that $\delta - \alpha_i = r_{\alpha_i}(\delta)$. This follows by the Killing formula applied to the string of roots $\delta + k\alpha_i$. There are the following cases:

1. In the simply laced diagrams of (1) the Killing number

$$
\langle \alpha_i^\vee, \delta \rangle = \frac{2\langle \alpha_i, \delta \rangle}{\langle \alpha_i, \alpha_i \rangle}
$$

is 0 or $\pm 1$. Since $\delta - \alpha_i$ is a root we have $\langle \alpha_i^\vee, \delta \rangle = 1$, and hence $r_{\alpha_i}(\delta) = \delta - \alpha_i$. \hfill 18
2. If the roots in $\Theta$ are long as in (2a) then $\alpha_i$ is a long root implying that 
$\langle \alpha_i^\vee, \delta \rangle$ is 0 or $\pm 1$. Again, the fact that $\delta - \alpha_i$ is a root implies that 
$\langle \alpha_i^\vee, \delta \rangle = 1$, so that $r_{\alpha_i}(\delta) = \delta - \alpha_i$.

3. If the roots in $\Pi^\sigma_\Theta$ are short in a double laced diagram as in (2b) then $\delta$ and $\delta - \alpha_i$ are short roots. If $\alpha_i$ is a long root then $\delta$ and $\delta - \alpha_i$ are the only roots of the form $\delta + k\alpha_i$, $k \in \mathbb{Z}$. Hence by the Killing formula 
$\langle \alpha_i^\vee, \delta \rangle = 1$, that is $r_{\alpha_i}(\delta) = \delta - \alpha_i$.

On the other hand if $\alpha_i$ is short then there are two possibilities for the string of roots $\delta + k\alpha_i$: i) $\delta - \alpha_i$, $\delta$ and $\delta + \alpha_i$ are roots in which case 
$\langle \alpha_i, \delta \rangle = 0$ and $\delta - \alpha_i$ and $\delta + \alpha_i$ are long roots; ii) $\delta - \alpha_i$ and $\delta$ are roots and 
$\langle \alpha_i, \delta \rangle = 1$. The first case is ruled out because otherwise we would 
have the long roots $\delta - \alpha_i, \delta + \alpha_i \in \Pi^\sigma_\Theta$, contradicting the assumption. 
Therefore $\langle \alpha_i, \delta \rangle = 1$, that is, $r_{\alpha_i}(\delta) = \delta - \alpha_i$.

Since $r_{\alpha_i} \in \mathcal{W}_\Theta$, it follows by induction that $\beta$ belongs to the $\mathcal{W}_\Theta$-orbit 
of $\mu$, proving transitivity of $\mathcal{W}_\Theta$. \hfill $\square$

We turn now to the equivalence of irreducible representations.

**Lemma 4.4** Let $V^\sigma_\Theta$ and $V^\tau_\Theta$ be $\mathfrak{g}_\Theta$-irreducible components. Suppose that 
there exists $\alpha \in \Pi^\sigma_\Theta$ which is not $M$-equivalent to any $\beta \in \Pi^\tau_\Theta$. Then $V^\sigma_\Theta$ and 
$V^\tau_\Theta$ are not $K_\Theta$-equivalent.

**Proof:** Suppose to the contrary that there exists an isomorphism $T : V^\sigma_\Theta \rightarrow V^\tau_\Theta$ intertwining the $K_\Theta$-representations. In particular

$$TmX = mTX,$$

for all $m \in M \subset K_\Theta$ and $X \in V^\sigma_\Theta$.

Take $0 \neq E_\alpha \in \mathfrak{g}_\alpha$. Then for every $m \in M$ we have $mE_\alpha = \varepsilon_mE_\alpha$ with 
$\varepsilon_m = \pm 1$. Write

$$TE_\alpha = \sum_{\beta \in \Pi^\sigma_\Theta} a_\beta E_\beta.$$

Then for $m \in M$ we have

$$\varepsilon_m \sum_{\beta \in \Pi^\sigma_\Theta} a_\beta E_\beta = \varepsilon_mE_\alpha = mTE_\alpha = \sum_{\beta \in \Pi^\sigma_\Theta} a_\beta mE_\beta.$$
Since \( mE_\beta = \pm E_\beta \) and the set \( E_\beta \) is linearly independent, it follows that \( mE_\beta = \varepsilon mE_\alpha \) if \( a_\beta \neq 0 \). For any such \( \beta \) the representation of \( M \) on \( g_\beta \) is equivalent to the representation on \( g_\alpha \). This contradicts the assumption that \( \alpha \) is not \( M \)-equivalent to \( \beta \in \Pi_\Theta \).

The next statement gives a sufficient condition for equivalence.

**Proposition 4.5** Let \( V^\sigma_\Theta \) and \( V^\tau_\Theta \) be \( \mathfrak{g}_\Theta \)-irreducible components. Suppose that there is a bijection \( \iota : \Pi^\sigma_\Theta \to \Pi^\tau_\Theta \) such that \( g_\alpha \) and \( g_{\iota(\alpha)} \) are \( M \)-equivalent for every \( \alpha \in \Pi^\sigma_\Theta \). Assume also that the linear map \( T : V^\sigma_\Theta \to V^\tau_\Theta \), given by \( TE_\alpha = E_{\iota(\alpha)} \), commutes with \( \text{ad} (X) \), \( X \in \mathfrak{t}_\Theta \) for every \( \alpha \in \Pi^\sigma_\Theta \). Then \( T \) is an intertwining operator for the \( K_\Theta \)-representations on \( V^\sigma_\Theta \) and \( V^\tau_\Theta \). Moreover the subspaces

\[
V_{[(x,y)]} = \{ xX + yTX : X \in V^\sigma_\Theta \},
\]

where \( [(x,y)] \in \mathbb{R}P^2 \), are the only \( K_\Theta \)-invariant subspaces in \( V^\sigma_\Theta \oplus V^\tau_\Theta \).

**Proof:** The first assumption implies that \( T \) intertwines the \( M \)-representations, while the second assumption means that \( T \) intertwines the representations of \( (K_\Theta)_0 \). Since \( K_\Theta = M (K_\Theta)_0 \), we conclude that \( T \) is in fact an intertwining operator for the \( K_\Theta \)-representations.

This implies that \( V_{[(x,y)]} \) is a \( K_\Theta \)-invariant subspace in \( V^\sigma_\Theta \oplus V^\tau_\Theta \) for any \( [(x,y)] \in \mathbb{R}P^2 \).

Now, if \( V \) is a \( K_\Theta \)-invariant subspace in \( V^\lambda_\Theta \oplus V^\mu_\Theta \) different from \( V^\sigma_\Theta = V_{[(1,0)]} \) or \( V^\tau_\Theta = V_{[(0,1)]} \), then there exist a linear isomorphism \( L : V^\sigma_\Theta \to V^\tau_\Theta \) such that

\[
V = \{ X + LX : X \in V^\lambda_\Theta \}
\]

and

\[
LkX = kLX,
\]

for every \( k \in K_\Theta \). Since \( M \) is a subset of \( K_\Theta \) and since \( mE_\alpha = \varepsilon mE_\alpha \), we can argue as in the proof of the previous Lemma to show that

\[
LE_\alpha = y_i E_{\iota(i)}.
\]

Since \( W_\Theta \) acts transitively in the set of the directions \( \{ g^\lambda_1, \ldots, g^\lambda_{n_\lambda} \} \), we conclude that \( y_i = y \), is independent of the index \( i \in \{ 1, \ldots, n_\lambda \} \). Thus we have that \( V = V_{[(1,y)]} \), concluding the proof.

The previous results are complemented by the following standard basic fact in representation theory.
Proposition 4.6 Let $V$ be the space of a finite dimensional representation of a group $L$. Suppose that

$$V = V_1 \oplus \cdots \oplus V_s$$

with $V_i$ invariant and irreducible. If the representations of $L$ on different components $V_i, V_j, i \neq j$, are not equivalent then the only $L$-invariant subspaces are sums of the components.

Proof: (Sketch) If $\{0\} \neq W \subset V$ is an invariant subspace then the projection $W_i$ to $V_i$ is invariant and hence either $\{0\}$ or $V_i$. Suppose that there are $i \neq j$ such that $W_i = V_i$ and $W_j = V_j$ and write $W_{ij}$ for the projection on $V_i \oplus V_j$. Then $W_{ij} \cap V_i = \{0\}$ or $V_i$. If $W_{ij} \cap V_i = V_i$ then $W_{ij} = V_i \oplus V_j$, which implies that $V_i \oplus V_j \subset W$. Otherwise $W_{ij} \cap V_i = W_{ij} \cap V_j = \{0\}$. In this case $W_{ij}$ is the graph of an isomorphism $V_i \to V_j$, intertwining the representations on $V_i$ and $V_j$.

Finally for several split simple Lie algebras the compact subalgebra $\mathfrak{k}$ is not simple. Via the next lemma we exploit this fact to get $K_\Theta$-invariant subspaces in $n^-_\Theta$.

Lemma 4.7 Let $U \subset K$ be a normal subgroup and denote by $V \subset n^-_\Theta$ the tangent space to the $U$-orbit $U \cdot b_\Theta$ through the origin. Then $V$ is $K_\Theta$-invariant.

Proof: The orbit $U \cdot b_\Theta$ is invariant by $K_\Theta$. In fact, if $u \cdot b_\Theta \in U \cdot b_\Theta$ and $k \in K_\Theta$ then $ku^{-1} \in U$ so that $ku \cdot b_\Theta = kuk^{-1} \cdot b_\Theta = kuk^{-1} \cdot b_\Theta$ belongs to $U \cdot b_\Theta$. Hence its tangent space at $b_\Theta$ is invariant by the isotropy representation.

5 Irreducible $K_\Theta$-invariant subspaces

In this section we describe the previous results to each diagram.
5.1 Flags of $A_l = \mathfrak{sl}(l+1, \mathbb{R})$

As checked in Section 3, no two different negative roots of $A_l$ are $M$-equivalent if $l \neq 3$. On the other hand by Lemma 4.3, on any flag manifold of $A_l$, the subgroup $W_\Theta$ acts transitively on each set of roots $\Pi_\Theta$ corresponding to an irreducible representation of $\mathfrak{z}_\Theta$ on $V_\Theta^\sigma$. Therefore by Lemma 4.2 we conclude that $K_\Theta$ is irreducible on each $V_\Theta^\sigma$. Looking again the $M$-equivalence classes we see that two different irreducible subspaces are not $K_\Theta$-equivalent. Hence we get the following description of the $K_\Theta$-invariant irreducible subspaces in a flag manifold of $A_l$.

**Proposition 5.1** For any flag manifold $F_\Theta$ of $A_l$, $l \neq 3$, the $K_\Theta$-invariant irreducible subspaces are the irreducible components $V_\Theta^\sigma$ for the $\mathfrak{z}_\Theta$ representation. Two such representations are not $K_\Theta$-equivalent.

The irreducible components $V_\Theta^\sigma$ are easily described in terms of the matrices in $\mathfrak{sl}(n, \mathbb{R})$, $n = l + 1$. In fact, let

$$H_\Theta = \text{diag}\{a_1, \ldots, a_n\} \in \mathfrak{a}_\Theta \cap \text{cl} \mathfrak{a}^+ \quad a_1 \geq \cdots \geq a_n$$

be characteristic for $\Theta$. The multiplicities of the eigenvalues of $H_\Theta$ determine the sizes of a block decomposition of the $n \times n$ matrices. With respect to this decomposition the matrices in $\mathfrak{z}_\Theta$ are block diagonal while a block outside the diagonal determines a $\mathfrak{z}_\Theta$-irreducible component. These are also the $K_\Theta$-irreducible components.

Now we look at the case $l = 3$. The matrix

$$\begin{pmatrix}
\ast & a & b & c \\
a & \ast & c & b \\
b & c & \ast & a \\
c & b & a & \ast
\end{pmatrix}$$

summarizes the $M$-equivalence classes of $\mathfrak{sl}(4, \mathbb{R})$, where root spaces represented by the same letter are $M$-equivalent (see Section 3).

This shows that in the maximal flag manifold $F_\Theta$, $\Theta = \emptyset$, the $K_\Theta = M$ invariant irreducible subspaces are the one-dimensional subspaces of $\mathfrak{g}_{21} \oplus \mathfrak{g}_{43}$, $\mathfrak{g}_{31} \oplus \mathfrak{g}_{42}$ or $\mathfrak{g}_{32} \oplus \mathfrak{g}_{41}$. The irreducible subspaces in the other flag manifolds are easily obtained from this $M$-equivalence.
We discuss further the instructive case when $\mathbb{F}_\Theta = \text{Gr}_2(4)$, the Grassmanian of two dimensional subspaces of $\mathbb{R}^4$. In this case $\mathfrak{n}_\Theta$ is the subalgebra of matrices written in $2 \times 2$ blocks as

$$X = \begin{pmatrix} 0 & 0 \\ B & 0 \end{pmatrix}.$$ 

The representations of $\mathfrak{z}_\Theta$ and $\mathfrak{g}(\Theta)$ on $\mathfrak{n}_\Theta$ are irreducible. Here $K_\Theta = \text{SO}(2) \times \text{SO}(2)$ whose representation on $\mathfrak{n}_\Theta$ decomposes into two 2-dimensional irreducible subspaces. This is due to the fact that $\mathfrak{so}(4) = \mathfrak{so}(3)_1 \oplus \mathfrak{so}(3)_2$ is a sum of two copies of $\mathfrak{so}(3)$. The matrices in these components have the form

$$\mathfrak{so}(3)_1 : \begin{pmatrix} A & -B^T \\ B & A \end{pmatrix} \quad \mathfrak{so}(3)_2 : \begin{pmatrix} A & -B^T \\ B & -A \end{pmatrix}$$

with $A + A^T = 0$ where $B$ is symmetric with $\text{tr}B = 0$ for $\mathfrak{so}(3)_1$ while

$$B = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

for $\mathfrak{so}(3)_2$. Hence by Lemma 4.7, the tangent spaces $V_i$ to orbits of $\text{SO}(3)_i = \langle \exp \mathfrak{so}(3)_i \rangle$, $i = 1, 2$, are $K_\Theta$-invariant. The subspace $V_i$, $i = 1, 2$, is given by the matrices in $\mathfrak{n}_\Theta$ with $B$ as $\mathfrak{so}(3)_1$ or $\mathfrak{so}(3)_2$, respectively.

5.2 Flags of $B_l = \mathfrak{sl}(l + 1, l)$

In the standard realization $\mathfrak{sl}(l + 1, l)$ is the algebra of matrices

$$\begin{pmatrix} 0 & a & b \\ -b^T & A & B \\ -a^T & C & -A^T \end{pmatrix} \quad B + B^T = C + C^T = 0.$$ 

In this case $\mathfrak{a}$ is the subalgebra of matrices

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & \Lambda & 0 \\ 0 & 0 & -\Lambda \end{pmatrix}$$

with $\Lambda = \text{diag}\{a_1, \ldots, a_l\}$. The set of roots are i) the long ones $\pm(\lambda_i - \lambda_j)$ and $\pm(\lambda_i + \lambda_j)$, $1 \leq i < j \leq l$ and ii) the short ones $\pm \lambda_i$, $1 \leq i \leq l$. The
set of simple roots is \( \Sigma = \{ \lambda_1 - \lambda_2, \ldots, \lambda_{l-1} - \lambda_l, \lambda_l \} \), which we write also as 
\( \Sigma = \{ \alpha_1, \ldots, \alpha_l \} \), that is, \( \alpha_i = \lambda_i - \lambda_{i+1} \) if \( i < l \) and \( \alpha_l = \lambda_l \).

The Weyl chamber \( a^+ \subset a \) is defined by the inequalities  

\[ a_1 > a_2 > \cdots > a_{l-1} > a_l > 0, \]

and a partial chamber \( a_\Theta \cap \text{cl} a^+ \) is defined by a similar relations where some of the strict inequalities are changed by equalities (e.g. if \( \lambda_i - \lambda_j \in \Theta \) then \( a_i = a_j \)). In particular a characteristic element \( H_\Theta \) for the subset \( \Theta = \{ \alpha \in \Sigma : \alpha (H_\Theta) = 0 \} \subset \Sigma \) is defined by one of these relations.

The subalgebra \( k \) is composed of the skew-symmetric matrices in \( \mathfrak{sl}(l, l) \), that is,

\[
\begin{pmatrix}
0 & a & a \\
-a^T & A & B \\
-a^T & B & A
\end{pmatrix}
\]

\( A + A^T = B + B^T = 0 \).

It is isomorphic to \( \mathfrak{so}(l + 1) \oplus \mathfrak{so}(l) \). The isomorphism is provided by the decomposition

\[
\begin{pmatrix}
0 & a & a \\
-a^T & A & B \\
-a^T & B & A
\end{pmatrix} = \begin{pmatrix}
0 & a & a \\
-a^T (A + B) / 2 & (A + B) / 2 \\
-a^T (A + B) / 2 & (A + B) / 2
\end{pmatrix} + \begin{pmatrix}
0 & 0 & 0 \\
0 & (A - B) / 2 & -(A - B) / 2 \\
0 & -(A - B) / 2 & (A - B) / 2
\end{pmatrix},
\]

so that \( \mathfrak{k} = \mathfrak{k}_{l+1} \oplus \mathfrak{k}_l \approx \mathfrak{so}(l + 1) \oplus \mathfrak{so}(l) \) where the ideals are given by matrices as follows

\[
\mathfrak{k}_{l+1} : \begin{pmatrix}
0 & a & a \\
-a^T & A & A \\
-a^T & A & A
\end{pmatrix}, \quad \mathfrak{k}_l : \begin{pmatrix}
0 & 0 & 0 \\
0 & A & -A \\
0 & -A & A
\end{pmatrix}.
\]

In both cases \( A \) is skew-symmetric. We write \( K_{l+1} = \langle \exp \mathfrak{k}_{l+1} \rangle \) and \( K_{l+1} = \langle \exp \mathfrak{k}_l \rangle \).

We start our analysis by describing the irreducible components \( V_\Theta^* \) defined by the set of roots \( \Pi_\Theta^* \). For this we separate the cases where \( \lambda_l \) belongs or not to \( \Theta \).

**Lemma 5.2** Suppose that \( \lambda_l \notin \Theta \) and let

\[
V_\Theta^* = \sum_{\alpha \in \Pi_\Theta^*} g_\alpha
\]

be an irreducible component. Then \( \Pi_\Theta^* \) contains only short roots or long roots. These sets are described as follows:
1. **Short roots:** Take a simple root $\alpha_i \not\in \Theta$. Then there are two possibilities:

   (a) $\alpha_{i-1} \not\in \Theta$. Then $g_{-\lambda_i}$ is $z_{\Theta}$-invariant and hence is an irreducible component.

   (b) $\alpha_{i-1} \in \Theta$. Let $j(i) < i$ be the smallest index such that $\{\alpha_{j(i)}, \ldots, \alpha_{i-1}\} \subset \Theta$. Then $\Pi_{\Theta}^\sigma = \{-\lambda_{j(i)}, \ldots, -\lambda_{i-1}\}$ defines a $z_{\Theta}$-irreducible component.

   These sets form a disjoint union of the negative short roots $-\lambda_i$, $1 \leq i \leq l$. (Note that this disjoint union completely determines $\Theta$.)

2. **Long roots:** A subset $\Pi_{\Theta}$ contains only roots of the type $\lambda_i - \lambda_j$ or of the type $-\lambda_i - \lambda_j$.

   **Proof:** To see the components corresponding to the short roots take an index $i$ with $\alpha_i \not\in \Theta$. An easy check shows that the only simple roots $\alpha$ such that $-\lambda_i + \alpha$ is a root are $\alpha = \lambda_i - \lambda_{i+1}$ or $\alpha = \lambda_l$. By assumption these simple roots are not in $\Theta$. This implies that $-\lambda_i$ is the highest weight of an irreducible representation of $g(\Theta)$. The weights of this representation are restrictions of $\Delta(\Theta)$ of roots. They have the form $-\lambda_i - \beta_1 - \cdots - \beta_k$ with $\beta_i \in \Theta$. But these successive differences are roots only when $\beta_1 = \lambda_{i-1} - \lambda_i$, $\beta_2 = \lambda_{i-2} - \lambda_{i-1}$, and so on, obtaining the $-\lambda_i, -\lambda_{i-1}, \ldots, -\lambda_{j(i)}$ with $j(i)$ as in (b). This concludes the case of the short roots.

   Now, take a long root, e.g. $\lambda_i - \lambda_j$. Then $\Pi_{\Theta} (\lambda_i - \lambda_j)$ does not contain short roots that were already exhausted. On the other hand, by assumption $\Theta$ is contained in the set of roots of the type $\lambda_r - \lambda_s$. Since this set is closed by sum we conclude that the roots in $\Pi_{\Theta} (\lambda_i - \lambda_j)$ have the type $\lambda_r - \lambda_s$. The same argument applies to $\Pi_{\Theta} (-\lambda_i - \lambda_j)$. $\square$

**Lemma 5.3** Suppose that $\lambda_l \in \Theta$ and let $i_0$ be the largest index such that $\lambda_{i_0} - \lambda_{i_0+1} \not\in \Theta$, that is, $\{\alpha_{i_0+1}, \ldots, \alpha_l = \lambda_l\}$ is the connected component of $\Theta$ containing $\lambda_l$. Then the sets of roots defining the $z_{\Theta}$-irreducible components are as follows:

1. **Components containing short roots:** If $i \leq i_0$ then $\Pi_{\Theta} (-\lambda_i)$ contains $-\lambda_i + \lambda_k$ and $-\lambda_i - \lambda_k$ for all $k \geq i_0 + 1$. (The short roots $-\lambda_i, i > i_0$, belong to $\langle \Theta \rangle^\ast$.)
Moreover the sets of short roots belonging to the same component are as in Lemma 5.2 (1), namely \(-\lambda_{j(i)}, \ldots, -\lambda_{i-1}\) where \(\alpha_{j(i)}, \ldots, \alpha_{i-1}\) is a connected component of \(\Theta\).

2. **Components containing only long roots**: If \(i < j \leq i_0\) then \(\Pi_\Theta (-\lambda_i + \lambda_j)\) has only roots \(\lambda_r - \lambda_s\) and \(\Pi_\Theta (-\lambda_i - \lambda_j)\) has only roots \(-\lambda_r - \lambda_s\).

(These sets exhaust the roots because \(-\lambda_i, -\lambda_i \pm \lambda_j \in \langle \Theta \rangle\) if \(i_0 + 1 \leq i < j\).)

**Proof:** By assumption \(\Theta\) contains the subdiagram simple roots \(B_{l-i_0} = \{\alpha_{i_0+1}, \ldots, \alpha_l\}\). This implies that the roots \(\pm \lambda_k \pm \lambda_j\) belong to \(\langle \Theta \rangle\) if \(i_0 + 1 \leq k < j\). Take a short root \(-\lambda_i\) with \(i \leq i_0\), which is not in \(\langle \Theta \rangle\). For any root \(\alpha \in \langle \Theta \rangle\) such that \(-\lambda_i + \alpha\) is a root we have \(-\lambda_i + \alpha \in \Pi_\Theta (-\lambda_i)\). If we take \(\alpha = \pm \lambda_k\), \(k \geq i_0 + 1\), we see that \(-\lambda_i \pm \lambda_k \in \Pi_\Theta (-\lambda_i)\), proving the first part of (1). By the same argument of the proof Lemma 5.2 we get the statement about the short roots.

Now, a long root \(-\lambda_i + \lambda_j\), \(i < j \leq i_0\), is orthogonal to every root in \(B_{l-i_0}\). Hence the only way to get new roots from \(-\lambda_i + \lambda_j\) is by adding or subtracting roots in \(\Theta \setminus B_{l-i_0}\). These roots have the type \(\lambda_r - \lambda_s\), so that as in the proof of Lemma 5.2 we see that \(\Pi_\Theta (-\lambda_i + \lambda_j)\) contains only roots of the type \(\lambda_r - \lambda_s\). The same argument works for \(-\lambda_i - \lambda_j\), \(i < j \leq i_0\), showing (2).

The next step is to look at the \(K_\Theta\)-irreducibility of the \(\mathfrak{z}_\Theta\)-irreducible components. For this we use the \(M\)-equivalence classes so we are led to consider separately different values of \(l\).

**Lemma 5.4** Take \(B_l\) with \(l \geq 5\).

1. **Suppose that** \(\lambda_l \notin \Theta\). **Then any component** \(V_\sigma^\Theta\) **is** \(K_\Theta\)-irreducible.

2. **If** \(\lambda_l \in \Theta\) **then** \(K_\Theta\) **is irreducible in the components** \(V_\sigma^\Theta\) **such that** \(\Pi_\Theta^\sigma\) **contains only long roots as in Lemma 5.3 (2).**

**Proof:** We just piece together different facts proved previously. First if \(\lambda_l \notin \Theta\) then \(\Theta\) contains only long roots. Hence by Lemma 4.3 the subgroup \(W_\Theta\) acts transitively on the sets \(\Pi_\Theta^\sigma\) for any irreducible component \(V_\sigma^\Theta\). Now
if \( l \geq 5 \) then the \( M \)-equivalence classes are the short roots \( \{ -\lambda_i \} \) and \( \{ -\lambda_i + \lambda_j, -\lambda_i - \lambda_j \} \), \( i < j \). Hence by Lemma 5.2 the intersection of a \( M \)-equivalence class with a set \( \Pi_\Theta^\sigma \) has just one root. Therefore, the assumptions of Lemma 4.2 are satisfied, and we get the conclusion that any \( V_\Theta^\sigma \) is \( K_\Theta \)-irreducible, proving (1).

The proof of (2) is similar. Take a subset \( \Pi_\Theta^\sigma \) as in the statement. Again no two roots in \( \Pi_\Theta^\sigma \) are \( M \)-equivalent. As to the transitive action of \( W_\Theta \) consider the subset \( B_l-i_0 = \{ \alpha_{i_0+1}, \ldots, \alpha_l \} \) defined in the proof of Lemma 5.3. Then \( W_\Theta \) is the direct product \( W_\Theta = W_{\Theta \setminus B_l-i_0} \times W_{B_l-i_0} \), and any \( w \in W_{B_l-i_0} \) is the identity in \( \Pi_\Theta^\sigma \), because the sets \( \Pi_\Theta^\sigma \) and \( B_l-i_0 \) are orthogonal (see the proof of Lemma 5.3). Now \( W_{\Theta \setminus B_l-i_0} \) acts transitively on \( \Pi_\Theta^\sigma \) since \( \Theta \setminus B_l-i_0 \) has only long roots (see the proof of Lemma 4.3).

It remains to analyze the components \( V_\Theta (-\lambda_i) \) containing short roots \(-\lambda_i \) in case \( \lambda_i \in \Theta \). Contrary to the others these are not \( K_\Theta \)-irreducible. Let us write them explicitly as follows: Let \( i_0 \) be, as in Lemma 5.3, the largest index such that \( \alpha_{i_0} = \lambda_{i_0} - \lambda_{i_0+1} \notin \Theta \), so that \(-\lambda_i \notin \langle \Theta \rangle^- \). For \( i \leq i_0 \) and \( k > i_0 \) put

\[
W_{ik}^i = g_{-\lambda_i + \lambda_k} \oplus g_{-\lambda_i} \oplus g_{-\lambda_i - \lambda_k} \quad \text{and} \quad W_i^i = \sum_{k \geq i_0 + 1} W_{ik}^i.
\]

(Note that the last sum is not direct.)

By Lemma 5.3 (2) we have one irreducible component \( V_\Theta (-\lambda_i) \) for each index \( i \leq i_0 \) such that \( \alpha_i = \lambda_i - \lambda_{i+1} \notin \Theta \). To write it in terms of the subspaces \( W_i^i \) let \( j (i) \leq i \) be defined by

1. \( j (i) = i \) if \( \alpha_{i-1} \notin \Theta \), and
2. \( j (i) \) is such that \( \{ \alpha_{j(i)}, \ldots, \alpha_{i-1} \} \) is a connected component of \( \Theta \) if \( \alpha_{i-1} \in \Theta \).

Then

\[
V_\Theta (-\lambda_i) = \sum_{k=j(i)}^i W_i^k.
\]

Now for \( i, j \) let

\[
E_{ij}^- = \begin{pmatrix} 0 & 0 & 0 \\ 0 & E_{ij} & 0 \\ 0 & 0 & -E_{ji} \end{pmatrix}, \quad E_{ij}^+ = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & E_{ij} - E_{ji} & 0 \end{pmatrix}, \quad E_i^0 = \begin{pmatrix} 0 & e_i & 0 \\ 0 & 0 & 0 \\ -e_i^T & 0 & 0 \end{pmatrix}
\]

(5)
where $E_{ij}$ and $e_i$ are basic $l \times l$ and $1 \times l$ matrices. These matrices are generators of $\mathfrak{g}_{-\lambda_i+\lambda_j}$, $\mathfrak{g}_{-\lambda_i-\lambda_j}$, and $\mathfrak{g}_{-\lambda_i}$, respectively. So that \{ $E_{ik}^{-}, E_{ik}^{+}, E_{ik}^{0}$ \} is a basis of $W_{\Theta}^{ik}$ and \{ $E_{ik}^{-}, E_{ik}^{+}, E_{ik}^{0} : k \geq i_0+1$ \} is a basis of $W_{\Theta}^{i}$.

Before proceeding we note that the subspace $W_{\Theta}^{ik}$ is invariant and irreducible by adjoint representation of the subalgebra $\mathfrak{g}(\lambda_k) \approx \mathfrak{sl}(2, \mathbb{R})$ generated by $\mathfrak{g}_{\pm \lambda_k}$, thus defining an irreducible representation of $\mathfrak{sl}(2, \mathbb{R})$. By dimensionality this representation is equivalent to the adjoint representation of $\mathfrak{sl}(2, \mathbb{R})$, which in turn is not $\mathfrak{so}(2)$-irreducible: It decomposes into the subspaces of skew-symmetric (1-dimensional) and symmetric (2-dimensional) matrices. The equivalence $\mathfrak{g}(\lambda_k) \approx \mathfrak{sl}(2, \mathbb{R})$ maps $\mathfrak{g}_{-\lambda_i+\lambda_k}$ and $\mathfrak{g}_{-\lambda_i-\lambda_k}$ onto the upper and lower triangular matrices, respectively and $\mathfrak{g}_{-\lambda_i}$ onto the diagonal matrices. So we get

**Lemma 5.5** The representation of $\mathfrak{g}(\lambda_k)$ decomposes $W_{\Theta}^{ik}$ into two $\mathfrak{t}_{\{\lambda_k\}}$-invariant subspaces, namely

\[
(W_{\Theta}^{ik})_1 = \text{span}\{E_{ik}^{-}, E_{ik}^{+}\} \quad \text{and} \quad (W_{\Theta}^{ik})_2 = \text{span}\{E_{ik}^{-}, E_{ik}^{+}, E_{ik}^{0}\}.
\]

Now we can decompose $V_{\Theta}(-\lambda_i), i \leq i_0$ when $\lambda_i \in \Theta$ into $K_{\Theta}$-irreducible subspaces.

**Lemma 5.6** In $B_l$, $l \geq 5$, suppose $\lambda_l \in \Theta$ and let $i_0$ be the largest index such that $\lambda_{i_0} - \lambda_{i_0+1} \notin \Theta$. If $i \leq i_0$ then $-\lambda_i \notin (\Theta)^-$ and the $\mathfrak{g}_{\Theta}$-irreducible component $V_{\Theta}(-\lambda_i)$ is the direct sum of the following $K_{\Theta}$-irreducible and invariant subspaces

\[
(W_{\Theta}^{i})_1 = \sum_{j=j(i)}^{i} \sum_{k \geq i_0+1} (W_{\Theta}^{jk})_1 \quad \text{and} \quad (W_{\Theta}^{i})_2 = \sum_{j=j(i)}^{i} \sum_{k \geq i_0+1} (W_{\Theta}^{jk})_2
\]

with $\dim (W_{\Theta}^{i})_1 = l-i_0+i-j(i)+1$ and $\dim (W_{\Theta}^{i})_2 = 2(l-i_0+i-j(i)+1)$.

Furthermore, $(W_{\Theta}^{i})_1 = V_{\Theta}(-\lambda_i) \cap T_{b_{\Theta}} K_l \cdot b_{\Theta}$ and $(W_{\Theta}^{i})_2 = V_{\Theta}(-\lambda_i) \cap T_{b_{\Theta}} K_{l+1} \cdot b_{\Theta}$.

**Proof:** The intersections in the last statement with the tangent space to the orbits $K_l \cdot b_{\Theta}$ and $K_{l+1} \cdot b_{\Theta}$ are readily obtained from the matrices in $\mathfrak{t}_l$ and $\mathfrak{t}_{l+1}$ given in (H) and the definition of the subspaces $(W_{\Theta}^{i})_1$ and $(W_{\Theta}^{i})_2$. It follows by Lemma 4.7 that these subspaces are $K_{\Theta}$-invariant.

To check irreducibility consider $(W_{\Theta}^{i})_2$ and take a nonzero $K_{\Theta}$-invariant subspace $Z \subset (W_{\Theta}^{i})_2$. 

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We claim that there are \( j \in [j(i), i] \) and \( k \geq i_0 + 1 \) such that \( (W_{\Theta}^{jk})_2 \subset Z \).

By Lemma 4.1 we have a nontrivial intersection of \( Z \) with a subspace \( \sum_{\alpha} g_{\alpha} \), with the sum extended to a \( M \)-equivalence class. Since we are assuming that \( l \geq 5 \), the \( M \)-equivalence classes are \( \{ -\lambda_s + \lambda_r, -\lambda_s - \lambda_r \} \) and \( \{ -\lambda_s \} \). Hence either there exists \( j \in [j(i), i] \) such that \( Z \cap g_{-\lambda_j} \neq \{0\} \) or there are \( j \in [j(i), i] \) and \( k \geq i_0 + 1 \) such that \( Z \cap (g_{-\lambda_i + \lambda_j} + g_{-\lambda_j - \lambda_k}) \neq \{0\} \). In both cases we have

\[
Z \cap (W_{\Theta}^{jk})_2 \neq \{0\}
\]

However, \( (W_{\Theta}^{jk})_2 \) is invariant and irreducible for \( \mathfrak{g}_{\{\lambda_s\}} \). Since \( k \geq i_0 + 1 \) we have \( \lambda_k \in \langle \Theta \rangle \) and \( \mathfrak{g}_{\{\lambda_k\}} \subset \mathfrak{g}_\Theta \). By \( K_{\Theta} \)-invariance of \( Z \) we conclude that \( (W_{\Theta}^{jk})_2 \subset Z \).

Now let \( B_{l-i_0+1} \) be the connected component of \( \Theta \) containing \( \lambda_i \). Then its Weyl group \( \mathcal{W}_{B_{l-i_0+1}} \subset \mathcal{W}_\Theta \) acts transitively on the set of its short root. This means that if \( k_1, k_2 \geq i_0 + 1 \) then there exists \( w \in \mathcal{W}_{B_{l-i_0+1}} \) such that \( w\lambda_{k_1} = \lambda_{k_2} \). Combining this transitivity with the claim it follows that \( (W_{\Theta}^{jk})_2 \subset Z \) for every \( k \geq i_0 + 1 \). Consequently, there exists \( j \in [j(i), i] \) such that \( \sum_{k \geq i_0 + 1} (W_{\Theta}^{jk})_2 \subset Z \).

To finish the proof we use the subgroup \( \mathcal{W}_{[j(i), i-1]} \) of \( \mathcal{W}_\Theta \) generated by the reflections with respect to the roots in the connected component \( \{\alpha_{j(i)}, \ldots, \alpha_{i-1}\} \subset \Theta \). This subgroup is the permutation group of \( \{j(i), \ldots, i-1, i\} \). Since \( \sum_{k \geq i_0 + 1} (W_{\Theta}^{jk})_2 \subset Z \) for some \( j \in [j(i), i] \) we conclude \( \sum_{k \geq i_0 + 1} (W_{\Theta}^{jk})_2 \subset Z \) for all \( s \in [j(i), i] \), so that \( (W_{\Theta}^{jk})_2 \subset Z \), showing irreducibility of \( (W_{\Theta}^{jk})_2 \).

The proof for \((W_{\Theta}^{jk})_1\) is similar. \( \square \)

Summarizing the above discussion we have the following \( K_{\Theta} \)-invariant irreducible subspaces for \( B_l, l \geq 5 \):

1. A \( \mathfrak{z}_\Theta \)-component \( V_\Theta (-\lambda_i) \) containing only short roots. These components occur only when \( \lambda_i \notin \Theta \).

2. \( \mathfrak{z}_\Theta \)-components \( V_\Theta (-\lambda_i + \lambda_j) \) and \( V_\Theta (-\lambda_i - \lambda_j), i < j \), containing only long roots. These subspaces occur in both cases when \( \lambda_i \) belongs or not to \( \Theta \). When \( \lambda_i \in \Theta \) the indexes \( i, j \) satisfy \( i < j \leq i_0 \) where \( \{\alpha_{i_0}, \ldots, \alpha_i = \lambda_i\} \) is the connected component of \( \Theta \) containing \( \lambda_i \).
3. The subspaces \((W^i_\Theta)_1\) and \((W^i_\Theta)_2\) contained in a \(j_\Theta\)-component \(V_\Theta(-\lambda_i)\) when \(\lambda_i \in \Theta\).

These are not the only invariant irreducible subspaces of \(K_\Theta\), since among them some pairs \(V_1 \neq V_2\) are \(K_\Theta\)-equivalent, enabling the existence of invariant subspaces inside \(V_1 \oplus V_2\). Among these pairs we can discard the following by \(M\)-equivalence we discard the following pairs: i) \(V_1\) is a subspace in item (1) and \(V_2\) in (1) or (2); ii) \(V_1\) is a subspace in (2) and \(V_2\) in (3); iii) \(V_1\) is a subspace \((W^i_\Theta)_{1,2}\) and \(V_2 = (W^j_\Theta)_{1,2}\) if \(i \neq j\). Since (1) and (3) are subspaces for different \(\Theta\) and \((W^i_\Theta)_{1,2}\) and \((W^j_\Theta)_{1,2}\) do not have the same dimension, it remains the subspaces \(V_\Theta(-\lambda_i + \lambda_j)\) and \(V_\Theta(-\lambda_i - \lambda_j)\) of (2). These are indeed equivalent.

**Lemma 5.7** In \(B_l\), \(l \geq 5\), the subspaces \(V_\Theta(-\lambda_i + \lambda_j)\) and \(V_\Theta(-\lambda_i - \lambda_j)\) as in (2) above are \(K_\Theta\)-equivalent if both roots \(-\lambda_i + \lambda_j\) and \(-\lambda_i - \lambda_j\) do not belong to \((\Theta)^-\).

**Proof:** To prove equivalence we shall exhibit a proper \(K_\Theta\)-invariant subspace \(\{0\} \neq V \subset V_\Theta(-\lambda_i + \lambda_j) \oplus V_\Theta(-\lambda_i - \lambda_j)\) different from the irreducible components \(V_\Theta(-\lambda_i + \lambda_j)\) and \(V_\Theta(-\lambda_i - \lambda_j)\). This implies equivalence by Proposition 4.6.

The required subspace \(V\) will be obtained from the tangent space at the origin of the orbit of the normal subgroup \(K_l\). By (4) the matrices in the Lie algebra \(\mathfrak{t}_l\) of \(K_l\) are

\[
\begin{pmatrix}
0 & 0 & 0 \\
0 & A & -A \\
0 & -A & A
\end{pmatrix}
\]

Looking at these matrices we see that after identifying \(T_{b_\Theta} \mathbb{F}_\Theta\) with \(\mathfrak{n}_\Theta^-\), the tangent space \(T_{b_\Theta} (K_l \cdot b_\Theta)\) is identified to the subspace \(T_l \subset \mathfrak{n}_\Theta^-\) spanned by \(\text{pr} (E_{rs}^- - E_{rs}^+)\), \(r > s\), where \(E_{rs}^\pm\) were defined in [5] and \(\text{pr} : \mathfrak{n}^- \rightarrow \mathfrak{n}_\Theta^-\) is the projection w.r.t. the root spaces decomposition.

The tangent space \(T_{b_\Theta} (K_l \cdot b_\Theta)\) is invariant by the isotropy representation of \(K_\Theta\), by Lemma 4.7. Hence \(T_l\) is invariant by the adjoint action of \(K_\Theta\).

Now if \(-\lambda_r + \lambda_s\), \(-\lambda_r - \lambda_s\) \(\in \Pi^- \setminus (\Theta)^-\) then \(E_{rs}^- - E_{rs}^+ = \text{pr} (E_{rs}^- - E_{rs}^+)\). Hence the following vectors form a basis of \(T_l\):

1. \(E_{rs}^-\) such that \(-\lambda_r + \lambda_s \in \Pi_\Theta (-\lambda_i + \lambda_j)\) and \(-\lambda_r - \lambda_s \notin \Pi_\Theta (-\lambda_i - \lambda_j)\).
2. $E_{rs}^+ \text{ such that } -\lambda_r - \lambda_s \in \Pi_\Theta (-\lambda_i - \lambda_j)$ and $-\lambda_r + \lambda_s \notin \Pi_\Theta (-\lambda_i + \lambda_j)$.

3. $E_{rs}^- - E_{rs}^+ \text{ such that } -\lambda_r + \lambda_s \in \Pi_\Theta (-\lambda_i + \lambda_j)$ and $-\lambda_r - \lambda_s \in \Pi_\Theta (-\lambda_i - \lambda_j)$.

The third case is not empty (e.g. $(r, s) = (i, j)$ fall in this case), which means that $E_{rs}^- - E_{rs}^+ \in T_i$ for some pair $(r, s)$. For this pair $T_i \cap V_\Theta (\lambda_r - \lambda_s) = T_i \cap V_\Theta (-\lambda_r - \lambda_s) = \{0\}$, which shows that $T_i$ is proper and different from $V_\Theta (-\lambda_i + \lambda_j)$ and $V_\Theta (-\lambda_i - \lambda_j)$. By Proposition 4.6 it follows that these irreducible subspaces are $K_\Theta$-equivalent. 

In conclusion we have:

**Theorem 5.8** Let $F_\Theta$ be a flag manifold of $B_l = \mathfrak{so}(l + 1, l), l \geq 5$. Then the $K_\Theta$-invariant irreducible subspaces of $n_\Theta$ are in the following classes:

1. **Isolated subspaces:**

   (a) $V_\Theta (-\lambda_i)$ when $\lambda_i \notin \Theta$. These subspaces contain root spaces of short roots only.

   (b) $V_\Theta (-\lambda_i + \lambda_j), i < j$, when $-\lambda_i - \lambda_j \notin \langle \Theta \rangle^-$. Any such pair occur if $-\lambda_i \in \Theta$. Otherwise we have $i < j \leq i_0$, where $\{\alpha_{i_0+1}, \ldots, \alpha_l = \lambda_i\}$ is the connected component of $\Theta$ containing $\lambda_i$.

   (c) The same as (b) interchanging the roles of $-\lambda_i + \lambda_j$ and $-\lambda_i - \lambda_j$.

   (d) The subspaces

   $$(W_\Theta^i)_1 = \sum_{j=j(i)}^{i} \sum_{k \geq i_0+1} (W_\Theta^{jk})_1 \quad \text{and} \quad (W_\Theta^i)_2 = \sum_{j=j(i)}^{i} \sum_{k \geq i_0+1} (W_\Theta^{jk})_2$$

   defined in Lemma 5.6. These subspaces decompose $V (-\lambda_i)$ when $\lambda_i \in \Theta$.

2. **A continuum of invariant subspaces** parametrized by $[(x, y)] \in \mathbb{R}P^2$ given by

   $$V_{[x,y]}^{ij} = \{xX + yTX : X \in V_\Theta (-\lambda_i + \lambda_j)\} \quad i < j.$$  

   The indexes $ij$ are as in (1.b), and here both $-\lambda_i + \lambda_j$ and $-\lambda_i - \lambda_j$ are not in $\langle \Theta \rangle^-$.  

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The low dimensional cases \( l = 2, 3, 4 \) must be treated separately because of the difference in the \( M \)-equivalence classes.

For instance \( B_2 \) has three flag manifolds. The maximal one whose irreducible components are detected by the \( M \)-equivalence classes \( \{ \lambda_1 - \lambda_2, \lambda_1 + \lambda_2 \} \) and \( \{ \lambda_1, \lambda_2 \} \). Hence, there are two continuous families of 1-dimensional irreducible subspaces. In the flag \( F_{\{\lambda_1-\lambda_2\}} \) there are two \( \mathfrak{z}_{\Theta} \)-irreducible components defined by the sets \( \{ -\lambda_2, -\lambda_1 \} \) and \( \{ -\lambda_1 - \lambda_2 \} \). Both are \( K_{\Theta} \)-irreducible and clearly they are not equivalent. On the other hand the flag \( F_{\{\lambda_2\}} \) has just one 3-dimensional \( \mathfrak{z}_{\Theta} \)-irreducible component which decomposes into a 1-dimensional plus a 2-dimensional irreducible subspaces of \( K_{\Theta} \) (as happens to the adjoint representation of \( \mathfrak{sl}(2, \mathbb{R}) \)).

For \( B_3 \) and \( B_4 \) the compact subalgebra \( \mathfrak{k} = \mathfrak{so}(3) \oplus \mathfrak{so}(4) \) and \( \mathfrak{so}(4) \oplus \mathfrak{so}(5) \), respectively) splits once more because \( \mathfrak{so}(4) = \mathfrak{so}(3) \oplus \mathfrak{so}(3) \). By Lemma \( \ref{lem:split} \) these simple components of \( \mathfrak{k} \) can yield new \( K_{\Theta} \)-invariant subspaces. The example with \( D_4 \) below, which has a similar splitting, illustrates this occurrence of new invariant subspaces. Another aspect that differs \( B_3 \) and \( B_4 \) from the general case are the \( M \)-equivalence classes that have more elements. This can introduce more \( K_{\Theta} \)-equivalence than the general case. The example with \( C_4 \) below illustrates this fact.

### 5.3 Flags of \( C_l = \mathfrak{sp}(l, \mathbb{R}) \)

The symplectic Lie algebra \( \mathfrak{sp}(l, \mathbb{R}) \) is composed of the real \( 2l \times 2l \) matrices

\[
\begin{pmatrix}
A & B \\
C & -A^T
\end{pmatrix}
\]

\( B - B^T = C - C^T = 0 \)

written in the basis \( \{ e_1, \ldots, e_l, f_1, \ldots, f_l \} \). In this case \( \mathfrak{a} \) is the subalgebra of matrices

\[
\begin{pmatrix}
\Lambda & 0 \\
0 & -\Lambda
\end{pmatrix}
\]

with \( \Lambda = \text{diag}\{a_1, \ldots, a_l\} \). The set of roots are i) the long ones \( \pm 2\lambda_i, 1 \leq i \leq l \) and ii) the short ones \( \pm (\lambda_i - \lambda_j) \) and \( \pm (\lambda_i + \lambda_j), 1 \leq i < j \leq l \). The set of simple roots is \( \Sigma = \{ \lambda_1 - \lambda_2, \ldots, \lambda_{l-1} - \lambda_l, 2\lambda_l \} \), which we write also as \( \Sigma = \{ \alpha_1, \ldots, \alpha_l \} \), that is, \( \alpha_i = \lambda_i - \lambda_{i+1} \) if \( i < l \) and \( \alpha_l = 2\lambda_l \).

The Weyl chamber \( \mathfrak{a}^+ \subset \mathfrak{a} \) is defined by the inequalities

\[
a_1 > a_2 > \cdots > a_{l-1} > a_l > 0,
\]
and a partial chamber \( a_\Theta \cap \text{cl} \alpha^+ \) is defined by a similar relations where some of the strict inequalities are changed by equalities (e.g. if \( \lambda_i - \lambda_j \in \Theta \) then \( a_i = a_j \)). In particular a characteristic element \( H_\Theta \) for the subset \( \Theta = \{ \alpha \in \Sigma : \alpha (H_\Theta) = 0 \} \subset \Sigma \) is defined by one of these relations.

The subalgebra \( \mathfrak{t} \) is composed of the skew-symmetric matrices in \( \mathfrak{sp} (l, \mathbb{R}) \), that is,

\[
\begin{pmatrix}
  A & -B \\
  B & A
\end{pmatrix} \quad A + A^T = B - B^T = 0.
\]

It is isomorphic to \( \mathfrak{u} (l) = \mathfrak{su} (l) + \mathbb{R} \), where the isomorphism associates the above matrix the complex matrix \( A + iB \).

To describe the \( \mathfrak{g}_\Theta \)-irreducible components \( V^g_\Theta \) defined by the set of roots \( \Pi^g_\Theta \) we consider first the components \( V_\Theta (-2\lambda_i) \) containing the long roots.

**Lemma 5.9** Let \( i = 1, \ldots, l \) be an index such that \( \alpha_i \notin \Theta \).

1. If \( i = 1 \) or \( \alpha_{i-1} \notin \Theta \) then \( V_\Theta (-2\lambda_i) = \mathfrak{g}_{-2\lambda_i} \).

2. Otherwise let \( j(i) < i \) be such that \( \{ \alpha_{j(i)}, \ldots, \alpha_{i-1} \} \) is the connected component of \( \Theta \) containing \( \alpha_{i-1} \). Then

\[
V_\Theta (-2\lambda_i) = \sum_{k,r=j(i)}^i \mathfrak{g}_{-\lambda_k - \lambda_r}.
\]

If \( \alpha_i \in \Theta \) then either \(-2\lambda_i \in \langle \Theta \rangle\) if \( 2\lambda_i \in \Theta \) and \( \alpha_i \) and \( 2\lambda_i \) are in the same connected component of \( \Theta \) or \(-2\lambda_i \in \Pi_\Theta (-2\lambda_j) \) where \( j > i \) is the smallest index such that \( \alpha_j \notin \Theta \).

**Proof:** Since \( \alpha_i = \lambda_i - \lambda_{i+1} \notin \Theta \) the only way that \(-2\lambda_i \pm \alpha \) is a root with \( \alpha \in \Theta \) is in the string \(-2\lambda_i - (\lambda_{i-1} - \lambda_i) = -\lambda_{i-1} - \lambda_i \) and \(-2\lambda_i - 2 (\lambda_{i-1} - \lambda_i) = -2\lambda_{i-1} \). Hence if \( \alpha_{i-1} \notin \Theta \) (or \( i = 1 \)) no such sum occurs and \( V_\Theta (-2\lambda_i) = \mathfrak{g}_{-2\lambda_i} \).

On the other hand if \( \alpha_{i-1} = \lambda_{i-1} - \lambda_i \in \Theta \) then the roots \(-\lambda_{i-1} - \lambda_i = -2\lambda_i - (\lambda_{i-1} - \lambda_i) \) and \(-2\lambda_i = -2\lambda_i - 2 (\lambda_{i-1} - \lambda_i) \) belong to \( \Pi_\Theta (-2\lambda_i) \). Proceeding successively it follows that \(-\lambda_k - \lambda_{k+1}, -2\lambda_k \in \Pi_\Theta (-2\lambda_i) \) if \( k = j(i), \ldots, i-1 \). The roots \(-\lambda_k - \lambda_r, j(i) \leq k < r-1 \leq i-1 \), also belong to \( \Pi_\Theta (-2\lambda_i) \), since \(-\lambda_k - \lambda_r = (\lambda_k - \lambda_{k+1}) + (\lambda_{k+1} - \lambda_r) \) and \( \lambda_{k+1} - \lambda_r \in \langle \Theta \rangle \). Hence \( V_\Theta (-2\lambda_i) \) contains the subspace \( \sum_{k,r=j(i)}^{i-1} \mathfrak{g}_{-\lambda_k - \lambda_r} \). This subspace is
plectic matrices in the subspace spanned by

\{ \text{Sp} (j(i), \ldots, i_{i-1}) \} \text{ is orthogonal to the roots } -\lambda_k - \lambda_r, \ k, r = j(i), \ldots, i.

The last statement is a consequence of the expression for \( V_\Theta (-2\lambda_i) \) in (2).

\[ \square \]

\textbf{Remark:} The first case of the above lemma is included in the second case by taking \( j(i) = i \).

To look at the representation of \( K_\Theta \) on the subspace \( V(-2\lambda_i) \) of \( \mathfrak{g} \) we make use of the following geometric meaning: Let \( \text{sp} (j(i), i) \) be the subalgebra generated by the root spaces \( \mathfrak{g}_{\pm(\lambda_k \pm \lambda_r)} \), \( k, r = j(i), \ldots, i \). Its elements are symplectic matrices

\[
\begin{pmatrix}
A & -B \\
B & A
\end{pmatrix}
\]

with \( A, B \) and \( C \) having non-zero entries only at the positions \( k, r = j(i), \ldots, i \), which shows that it is isomorphic to the Lie algebra of symplectic matrices in the subspace spanned by \{ \( e_{j(i)}, \ldots, e_i, f_{j(i)}, \ldots, f_i \} \). Let \( \text{Sp} (j(i), i) = \langle \exp \text{sp} (j(i), i) \rangle \) be the connected subgroup with Lie algebra \( \text{sp} (j(i), i) \) and put \( U (j(i), i) = \text{Sp} (j(i), i) \cap K \) for its maximal compact subgroup, which is isomorphic to the unitarian group \( U (i - j(i) + 1) \).

The inclusion \( \{ \alpha_{j(i)}, \ldots, \alpha_{i-1} \} \subset \Theta \) shows that the root spaces \( \mathfrak{g}_{\lambda_k - \lambda_r} \), \( k, r = j(i), \ldots, i \), are contained in the isotropy subalgebra at the origin \( b_\Theta \in \mathbb{F}_\Theta \). From this it is easily seen that the orbit \( \text{Sp} (j(i), i) \cdot b_\Theta = U (j(i), i) \cdot b_\Theta \) is a flag manifold of \( \text{Sp} (j(i), i) \) and identifies to the coset \( U (j(i), i) / \text{SO} (j(i), i) \) where \( \text{SO} (j(i), i) \) is the subgroup isomorphic to \( \text{SO} (i - j(i) + 1) \), whose Lie algebra is contained in \( \sum_{k=r=j(i)}^i \mathfrak{g}_{\lambda_k - \lambda_r} \). Further \( V_\Theta (-2\lambda_i) = \sum_{k,r=j(i)}^i \mathfrak{g}_{-\lambda_k - \lambda_r} \) is the tangent space at the origin \( b_\Theta \in \mathbb{F}_\Theta \) of the orbit \( \text{Sp} (j(i), i) \cdot b_\Theta = U (j(i), i) \cdot b_\Theta \).

Now we can get the \( K_\Theta \)-irreducible components of \( V_\Theta (-2\lambda_i) \).

\textbf{Lemma 5.10} The \( \mathfrak{g}_\Theta \)-irreducible subspace \( V_\Theta (-2\lambda_i) = \sum_{k,r=j(i)}^i \mathfrak{g}_{-\lambda_k - \lambda_r} \) has two \( K_\Theta \)-irreducible components if \( j(i) < i \). They are given as follows:

1. The one-dimensional subspace \( V_\Theta (-2\lambda_i) \) cent \( \subset \sum_{k=j(i)}^i \mathfrak{g}_{-2\lambda_k} \) spanned by the matrix

\[
\begin{pmatrix}
0 & 0 \\
0 & I_{j(i), i}
\end{pmatrix} \in \text{sp} (j(i), i) \subset \text{sp} (l, \mathbb{R})
\]

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where $I_{(j(i), i)}$ is the diagonal matrix with 1 in the positions $j(i), \ldots, i$ and 0 otherwise.

2. The subspace $V_{\Theta}(-2\lambda_i)_{su(j(i), i)}$ given by the matrices

$$
\begin{pmatrix}
  A & 0 \\
  B & -A^T
\end{pmatrix} \in \mathfrak{sp}(j(i), i)
$$

with $A$ lower triangular and $tr B = 0$.

**Proof:** The compact group $U(j(i), i)$ being isomorphic to $U(i - j(i) + 1)$ is the product of its center $Z_{(j(i), i)}$ by $SU(j(i), i)$. The Lie algebra of the center is given by matrices

$$
\begin{pmatrix}
  0 & -B \\
  B & 0
\end{pmatrix}
$$

with $B \in \mathbb{R} \cdot I_{(j(i), i)}$ (corresponding to the scalar matrices in $u(i - j(i) + 1)$).

The Lie algebra $su((j(i), i))$ of $SU(j(i), i)$ is given by matrices

$$
\begin{pmatrix}
  A & -B \\
  B & A
\end{pmatrix} \in \mathfrak{sp}(j(i), i)
$$

with $A$ skew symmetric and $B$ symmetric with $tr B = 0$. It follows that the tangent spaces to the orbits $Z_{(j(i), i)} \cdot b_{\Theta}$ and $SU(j(i), i)$ are $V_{\Theta}(-2\lambda_i)_{cont}$ and $V_{\Theta}(-2\lambda_i)_{su(j(i), i)}$, respectively.

Hence, by Lemma 4.7 these subspaces are invariant by the isotropy representation of $SO(j(i), i) = U(j(i), i) \cap K_{\Theta}$. They are $K_{\Theta}$-invariant as well because the connected components of $\Theta$ besides $\{\alpha_{j(i)}, \ldots, \alpha_{i-1}\}$ are orthogonal to $\Pi_{\Theta}(-2\lambda_i)$. Hence the simple components of $K_{\Theta}$ different from $SO(j(i), i)$ act trivially on $V_{\Theta}(-2\lambda_i)$.

Finally, both subspaces $V_{\Theta}(-2\lambda_i)_{cont}$ and $V_{\Theta}(-2\lambda_i)_{su(j(i), i)}$ are irreducible. This is obvious for $V_{\Theta}(-2\lambda_i)_{cont}$ which is one dimensional. On the other hand the representation of $SO(j(i), i)$ on $V_{\Theta}(-2\lambda_i)$ is equivalent to the isotropy representation of the symmetric space $U(j(i), i)/SO(j(i), i)$, which is known to be irreducible.

The $z_{\Theta}$-irreducible components described in Lemma 5.9 contain all the root spaces of the long roots not in $\langle \Theta \rangle$. They include also the short roots $-\lambda_i - \lambda_j$ such that $\lambda_i - \lambda_j \in \langle \Theta \rangle$. The other $z_{\Theta}$-components are given as follows.
Lemma 5.11 Suppose the root $\lambda_i - \lambda_j$, $i < j$, does not belong to $\langle \Theta \rangle$.

1. If $2\lambda_i \notin \Theta$ then the set $\Pi_\Theta (-\lambda_i + \lambda_j)$ corresponding to the $3_\Theta$-irreducible component $V_\Theta (-\lambda_i + \lambda_j)$ contains only short roots of the type $-\lambda_r + \lambda_s$.

2. In case $2\lambda_i \in \Theta$ let $i_0$ be such that $C_{l-i_0+1} = \{ \alpha_{i_0+1}, \ldots, \alpha_i = 2\lambda_i \}$ is the connected component of $\Theta$ containing $2\lambda_i$. Then $V_\Theta (-\lambda_i + \lambda_j) = V_{\Theta \setminus \{2\lambda_i\}} (-\lambda_i + \lambda_j)$ and $V_\Theta (-\lambda_i - \lambda_j) = V_{\Theta \setminus \{2\lambda_i\}} (-\lambda_i - \lambda_j)$ if $j \leq i_0$.

3. On the other hand if $j \geq i_0 + 1$ then

\[
V_\Theta (-\lambda_i + \lambda_j) = V_\Theta (-\lambda_i - \lambda_j) = V_{\Theta \setminus C_{l-i_0+1}} (-\lambda_i + \lambda_j) \oplus V_{\Theta \setminus C_{l-i_0+1}} (-\lambda_i - \lambda_j).
\]

Moreover these $3_\Theta$-irreducible components are $K_\Theta$-irreducible.

Proof: The first statement is proved as Lemma 5.2 for $B_l$. The proof of Lemma 5.3 also works for the components in (2) with $j \leq i_0$. The direct sum in (7) comes from the pair of roots $-\lambda_i + \lambda_j$, $(-\lambda_i + \lambda_j) - 2\lambda_j$ and the fact that $2\lambda_j \in \langle \Theta \rangle^- \text{ if } j \geq i_0 + 1$.

The $K_\Theta$-irreducibility of the subspaces in (1) is a consequence of Lemma 4.2. In fact no two roots in $\Pi_\Theta (-\lambda_i + \lambda_j)$ or in $\Pi_\Theta (-\lambda_i - \lambda_j)$ are $M$-equivalent and by Lemma 4.3 2b, $W_\Theta$ acts transitively on these sets of short roots.

The same argument hold for the subspaces $V_\Theta (-\lambda_i + \lambda_j) = V_{\Theta \setminus \{2\lambda_i\}} (-\lambda_i + \lambda_j)$ and $V_\Theta (-\lambda_i - \lambda_j) = V_{\Theta \setminus \{2\lambda_i\}} (-\lambda_i - \lambda_j)$ when $j \leq i_0$, which are indeed $K_{\Theta \setminus \{2\lambda_i\}}$-irreducible.

Finally, in (2) if $j \geq i_0 + 1$ then $2\lambda_j \in \langle \Theta \rangle$. From the equality $-\lambda_i - \lambda_j = -\lambda_i + \lambda_j - 2\lambda_j$ we see that $g_{-\lambda_i + \lambda_j} \oplus g_{-\lambda_i - \lambda_j}$ is an irreducible subspace for the three dimensional subalgebra $g (2\lambda_j)$, isomorphic to $sl(2, \mathbb{R})$, generated by $g_{\pm 2\lambda_j}$. In this two dimensional subspace the compact part $t (2\lambda_j)$ of $g (2\lambda_j)$ is also irreducible. Now let $\{0\} \neq U \subset V_\Theta (-\lambda_i + \lambda_j)$ be a $K_\Theta$-invariant subspace. Since the roots $-\lambda_i \pm \lambda_j \in \Pi_\Theta (-\lambda_i + \lambda_j) = \Pi_\Theta (-\lambda_i - \lambda_j)$ are $M$-equivalent, it follows by Lemma 4.1 that $U \cap (g_{-\lambda_i + \lambda_j} \oplus g_{-\lambda_i - \lambda_j}) \neq \{0\}$. This subspace is $t (2\lambda_j)$-invariant, hence $g_{-\lambda_i + \lambda_j} \oplus g_{-\lambda_i - \lambda_j} \subset U$. Hence irreducibility of $V_\Theta (-\lambda_i + \lambda_j)$ is a consequence of Lemma 4.2 combined with the fact that $W_{\Theta \setminus \{2\lambda_i\}} \subset W_\Theta$ acts transitively on the sets $\Pi_{\Theta \setminus \{2\lambda_i\}} (-\lambda_i + \lambda_j)$ and $\Pi_{\Theta \setminus \{2\lambda_i\}} (-\lambda_i - \lambda_j)$. \qed
With this lemma we finish the description of the irreducible $K_{\Theta}$-components. Among them the only $K_{\Theta}$-equivalents are the following:

1. The one dimensional subspaces $V_\Theta (-2\lambda_i)_{\text{cont}}$ of Lemma 5.10 (1). The representation of $K_{\Theta}$ on each one of them is trivial.

2. $V_\Theta (-\lambda_i + \lambda_j) \approx V_\Theta (-\lambda_i - \lambda_j)$ when $2\lambda_l / \in \Theta$ as in Lemma 5.11 (1). This equivalence follows by Proposition 4.5, since there is a bijection between $\Pi_\Theta (-\lambda_i + \lambda_j)$ and $\Pi_\Theta (-\lambda_i - \lambda_j)$ mapping a root $-\lambda_r + \lambda_s \in \Pi_\Theta (-\lambda_i + \lambda_j)$ to the $M$-equivalent root $-\lambda_r - \lambda_s \in \Pi_\Theta (-\lambda_i - \lambda_j)$.

3. The subspaces $V_\Theta (-\lambda_i + \lambda_j) = V_{\Theta \setminus \{2\lambda_l\}} (-\lambda_i + \lambda_j)$ and $V_\Theta (-\lambda_i - \lambda_j) = V_{\Theta \setminus \{2\lambda_l\}} (-\lambda_i - \lambda_j)$ with $j \leq i_0$ as in Lemma 5.11 (2).

Any other pair of subspaces are not $K_{\Theta}$-equivalent because the lack of $M$-equivalence in the corresponding sets of roots (cf. Lemma 4.4).

Summarizing we get the $K_{\Theta}$-invariant subspaces for the flags of $C_l$, $l > 4$.

**Theorem 5.12** Let $F_\Theta$ be a flag manifold of $C_l = \mathfrak{sp}(l, \mathbb{R})$, $l \geq 5$. Then the $K_{\Theta}$-invariant irreducible subspaces of $n_\Theta$ are the following:

1. Continuous families:

   (a) One dimensional subspaces spanned by matrices

   $\begin{pmatrix} 0 & 0 \\ \Lambda & 0 \end{pmatrix}$

   where $\Lambda$ is a diagonal matrix $a_1 I_{[j(i_1),i_1]} + \cdots + a_k I_{[j(i_k),i_k]}$ where $[j(i_1),i_1], \ldots, [j(i_k),i_k]$ are the connected components of $\Theta$ not containing $2\lambda_l$, and $I_{[j(i_k),i_k]}$ is the identity matrix corresponding to these indexes.

   (b) The subspaces parametrized by $[(x,y)] \in \mathbb{R}P^2$ given by

   $V_{[x,y]}^{ij} = \{ xX + yTX : X \in V_\Theta (-\lambda_i + \lambda_j) \}$

   where $T : V_\Theta (-\lambda_i + \lambda_j) \to V_\Theta (-\lambda_i - \lambda_j)$ is an intertwining operator for the $K_{\Theta}$-representations. Here the indexes $ij$ are arbitrary if $2\lambda_l / \in \Theta$. Otherwise $j \leq i_0$, where $\{ \alpha_{i_0+1}, \ldots, \alpha_l = 2\lambda_l \}$ is the component of $\Theta$ containing $2\lambda_l$.  

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2. Isolated subspaces:

(a) The subspaces $V_\Theta (-2\lambda_i)_{su(j\iota,i)}$ of codimension 1 contained in $V_\Theta (-2\lambda_i)$ as defined in Lemma 5.10.

(b) The subspaces

$$V_\Theta (-\lambda_i + \lambda_j) = V_\Theta (-\lambda_i - \lambda_j)$$

$$= V_{\Theta\setminus C_l-i_{o+1}} (-\lambda_i + \lambda_j) \oplus V_{\Theta\setminus C_l-i_{o+1}} (-\lambda_i - \lambda_j),$$

when $2\lambda_i \in \Theta$ and $i < i_o + 1 \leq j$ where $\{\alpha_{i_o+1}, \ldots, \alpha_l = 2\lambda_i\}$ is the component of $\Theta$ containing $2\lambda_i$.

When $l = 4$ the $M$-equivalence classes of the short roots increase to \{\{\lambda_1 - \lambda_2, \lambda_1 + \lambda_2, \lambda_3 - \lambda_4, \lambda_3 + \lambda_4\}, \{\lambda_1 - \lambda_3, \lambda_1 + \lambda_3, \lambda_2 - \lambda_4, \lambda_2 + \lambda_4\}, \{\lambda_1 - \lambda_4, \lambda_1 + \lambda_4, \lambda_2 - \lambda_3, \lambda_2 + \lambda_3\}\} while the long roots are kept the same \{2\lambda_1, 2\lambda_2, 2\lambda_3, 2\lambda_4\}. Since there are more $M$-equivalent pair of roots we can have more $K_\Theta$-equivalent subspaces than in the general case.

For example consider flag $\mathbb{F}_{\{\lambda_2 - \lambda_3\}}$. By the general result the subspaces $V_{\{\lambda_2 - \lambda_3\}} (-\lambda_1 + \lambda_2)$ and $V_{\{\lambda_2 - \lambda_3\}} (-\lambda_1 - \lambda_2)$ are $K_\Theta$-equivalent. Their corresponding roots are $\Pi_{\{\lambda_2 - \lambda_3\}} (-\lambda_1 + \lambda_2) = \{-\lambda_1 + \lambda_2, -\lambda_1 + \lambda_3\}$ and $\Pi_{\{\lambda_2 - \lambda_3\}} (-\lambda_1 - \lambda_2) = \{-\lambda_1 - \lambda_2, -\lambda_1 - \lambda_3\}$. For $l = 4$ we have $(-\lambda_1 + \lambda_2) \sim_M (-\lambda_3 + \lambda_4)$ and $(-\lambda_1 + \lambda_3) \sim_M (-\lambda_2 + \lambda_4)$. Since the set of root for $V_{\{\lambda_2 - \lambda_3\}} (-\lambda_3 + \lambda_4)$ is $\Pi_{\{\lambda_2 - \lambda_3\}} (-\lambda_3 + \lambda_4) = \{-\lambda_3 + \lambda_4, -\lambda_2 + \lambda_4\}$ we conclude that $V_{\{\lambda_2 - \lambda_3\}} (-\lambda_1 + \lambda_2)$ is also $K_\Theta$-equivalent to $V_{\{\lambda_2 - \lambda_3\}} (-\lambda_3 + \lambda_4)$. The same way $V_{\{\lambda_2 - \lambda_3\}} (-\lambda_1 - \lambda_2)$ and $V_{\{\lambda_2 - \lambda_3\}} (-\lambda_3 - \lambda_4)$ are $K_\Theta$-equivalent. Thus we get new continuous families of invariant subspaces that are not present in the general case.

5.4 Flags of $D_l = so (l,l)$

The Dynkin diagram $D_l$ has no multiple edges. Hence on any flag manifold $\mathbb{F}_\Theta$ we have, by Lemma 4.3, that $W_\Theta$ acts transitively on each set of roots $\Pi_\Theta$ corresponding to an irreducible representation of $\mathfrak{g}_\Theta$ on $V_\Theta^\mathfrak{g} \subset \mathfrak{n}_\Theta^\mathfrak{g}$. This transitivity is one of the conditions of Lemma 4.2, ensuring that the subspaces $V_\Theta^\mathfrak{g}$ are $K_\Theta$-irreducible. To look at the condition involving the $M$-equivalence classes we work, as in Section 3, with the standard realization of $D_l = so (l,l)$.

In this realization $\mathfrak{a}$ is the subalgebra of matrices

$$\begin{pmatrix} \Lambda & 0 \\ 0 & -\Lambda \end{pmatrix}$$
with $\Lambda = \text{diag}\{a_1, \ldots, a_l\}$ and the set of simple roots is $\Sigma = \{\lambda_l - \lambda_2, \ldots, \lambda_l - \lambda_{l-1}, \lambda_{l-1} + \lambda_l\}$.

The Weyl chamber $a^+ \subset a^*$ is defined by the inequalities

$$a_1 > a_2 > \cdots > a_{l-1} > a_l > -a_{l-1}.$$  \hspace{1cm} (8)

A partial chamber $a_\Theta \cap \text{cla}^+$ is defined by a similar relation where some of the inequalities are changed by equalities. In particular a characteristic element $H_\Theta$ for the subset $\Theta = \{\alpha \in \Sigma : \alpha(H_\Theta) = 0\} \subset \Sigma$ is defined by one of these relations.

The following statement is specific for $D_l$ and will be used soon to check that $M$-equivalent root spaces are not contained in an irreducible component.

**Lemma 5.13** Given a subset $\Theta \subset \Sigma$ there exists characteristic element

$$H_\Theta = \begin{pmatrix} \Lambda_\Theta & 0 \\ 0 & -\Lambda_\Theta \end{pmatrix} \in a_\Theta \cap \text{cla}^+$$

with $\Lambda_\Theta = \text{diag}\{a_1, \ldots, a_l\}$ such that $a_i \neq 0$, $i = 1, \ldots, l$.

**Proof:** By the last two inequalities in (8) we have $a_{l-1} \geq -a_{l-1}$, that is, $a_{l-1} \geq 0$. Also, $a_{l-1} = 0$ if and only if $a_{l-1} = a_l = -a_{l-1}$, that is, $a_{l-1} - a_l = a_{l-1} + a_l = 0$, which means that both roots $\lambda_{l-1} - \lambda_l$ and $\lambda_l - \lambda_{l-1}$ belong to $\Theta$. This being so we consider the possibilities:

1. $\{\lambda_{l-1} - \lambda_l, \lambda_{l-1} + \lambda_l\} \subset \Theta$. Let $i < l-1$ be the maximum such that $\lambda_i - \lambda_{i+1} \notin \Theta$ (it is tacitly assumed that $\Theta \neq \Sigma$). Then the conditions to define a characteristic element for $\Theta$ have the form

$$a_1 \geq \cdots \geq a_i > a_{i+1} = a_{i+2} = \cdots = a_l.$$  

Thus we can choose a characteristic element having $a_{i+1} = a_{i+2} = \cdots = a_l > 0$, so that all the entries of $\Lambda_\Theta$ will be $> 0$.

2. One of the roots $\lambda_{l-1} - \lambda_l$ or $\lambda_{l-1} + \lambda_l$ does not belong to $\Theta$. In this case $a_{l-1} > 0 > -a_{l-1}$ for any $H_\Theta$ so that $a_i > 0$ for any $i \leq l-1$. Also the relations defining $a_\Theta \cap \text{cla}^+$ end with

$$a_{l-1} > a_l > -a_{l-1} \quad \text{or} \quad a_{l-1} \geq a_l > -a_{l-1} \quad \text{or} \quad a_{l-1} > a_l \geq -a_{l-1}.$$  

In each case we can choose $a_l \neq 0$ without violating the conditions.
From now on we distinguish the cases where \( l > 4 \) and \( l = 4 \).

If \( l > 4 \) then \( M \)-equivalence classes in the positive roots are \( \{ \lambda_i - \lambda_j, \lambda_i + \lambda_j \}, \ i < j \), and \( \{ \lambda_i - \lambda_j, -\lambda_i - \lambda_j \}, \ i > j \), in the negative roots. By the previous lemma we get easily that the corresponding \( M \)-equivalent root spaces are not contained in a single irreducible component.

**Lemma 5.14** Let \( V_{\sigma}^g \) be an irreducible component containing the root spaces \( g_{\alpha} \) and \( g_{\beta}, \alpha \neq \beta \). If \( l > 4 \) then \( \alpha \) and \( \beta \) are not \( M \)-equivalent.

**Proof:** Take a characteristic element \( H_{\Theta} \) with \( a_i \neq 0 \) as in the previous lemma. Then \((\lambda_i - \lambda_j)(H_{\Theta}) \neq -(\lambda_i + \lambda_j)(H_{\Theta})\) for otherwise \( a_i - a_j = -a_i - a_j \), that is, \( a_i = 0 \). The result follows, since \( V_{\sigma}^g \) is contained in an eigenspace of \( \text{ad}(H_{\Theta}) \).

Combining this lemma with Lemma 4.3 (about the transitivity of \( W_{\Theta} \)) we get at once \( K_{\Theta} \)-irreducibility of \( V_{\sigma}^g \), by Lemma 4.2.

**Proposition 5.15** In any flag manifold \( F_{\Theta} \) of \( D_l \), \( l > 4 \), the \( z_{\Theta} \)-irreducible components \( V_{\sigma}^g \) are also \( K_{\Theta} \)-irreducible.

To get the full picture of the invariant subspaces we must find the pairs of \( z_{\Theta} \)-irreducible components that are mutually \( K_{\Theta} \)-equivalent. Our method to check \( K_{\Theta} \)-equivalence is via the orbits on \( F_{\Theta} \) of the simple components of the maximal compact subgroup \( K \) of \( G \).

To this purpose we need some further notation concerning the standard realization of \( D_l \). The Lie algebra \( \mathfrak{so}(l, l) \) is the algebra of \( 2l \times 2l \) matrices of the form

\[
\begin{pmatrix}
A & B \\
C & -A^T
\end{pmatrix}
\]

\( B + B^T = C + C^T = 0 \).

We have that \( \mathfrak{f} \) is the subalgebra of skew-symmetric matrices in \( \mathfrak{so}(l, l) \), that is,

\[
\begin{pmatrix}
A & B \\
B & A
\end{pmatrix}
\]

\( A + A^T = B + B^T = 0 \).

\( \mathfrak{f} \) is the direct sum of two copies of \( \mathfrak{so}(l) \). In fact, via the decomposition

\[
\begin{pmatrix}
A & B \\
B & A
\end{pmatrix}
= \begin{pmatrix}
(A + B)/2 & (A + B)/2 \\
(A + B)/2 & (A + B)/2
\end{pmatrix} + \begin{pmatrix}
(A - B)/2 & -(A - B)/2 \\
-(A - B)/2 & (A - B)/2
\end{pmatrix}
\]
we get $\mathfrak{g} = \mathfrak{so}(l)_1 \oplus \mathfrak{so}(l)_2$ with

$$\mathfrak{so}(l)_1 : \begin{pmatrix} A & A \\ A & A \end{pmatrix} \quad \mathfrak{so}(l)_2 : \begin{pmatrix} A & -A \\ -A & A \end{pmatrix}$$

where in both cases $A$ is skew-symmetric. We write $\text{SO}(l)_i = \langle \exp \mathfrak{so}(l)_i \rangle$, $i = 1, 2$.

As to the root spaces we write

$$E_{ij}^- = \begin{pmatrix} E_{ij} & 0 \\ 0 & -E_{ij}^t \end{pmatrix} \quad \text{and} \quad E_{ij}^+ = \begin{pmatrix} 0 & 0 \\ E_{ij} - E_{ij}^t & 0 \end{pmatrix}$$

(9)

where $E_{ij}$ is a basic $l \times l$ matrix. Then $E_{ij}^-$ spans the root space $\mathfrak{g}_{\lambda_i - \lambda_j}$ and $E_{ij}^+$ spans $\mathfrak{g}_{-\lambda_i - \lambda_j}$.

We can return now to the question of $K_\Theta$-equivalence of the $\mathfrak{g}_\Theta$-components $V^\sigma_\Theta$.

For a root $\alpha \in \Pi^- \setminus \langle \Theta \rangle^-$ write $V_\Theta(\alpha)$ for the irreducible component containing $g_\alpha$ (cf. Proposition 2.1). By Lemma 5.14 we have $V_\Theta(\alpha) \neq V_\Theta(\beta)$ if $\alpha \sim_M \beta$ and $\alpha \neq \beta$. Moreover, by Lemma 4.4 a component $V^\sigma_\Theta$ is not $K_\Theta$-equivalent to $V_\Theta(\alpha)$ unless there exists $\beta \sim_M \alpha$ such that $V^\sigma_\Theta = V_\Theta(\beta)$.

Now by Section 3 we have that if $l \neq 4$ then the $M$-equivalent classes of $D_l$ (on the negative roots) has exactly two elements. If $\{\alpha, \beta\}$ is a $M$-equivalence class with say $\alpha \in \Pi^- \setminus \langle \Theta \rangle^-$ and $\beta \in \langle \Theta \rangle^-$ then $V_\Theta(\alpha)$ is not $K_\Theta$-equivalent to any other irreducible component $V^\sigma_\Theta$. On the other hand if both $\alpha, \beta \in \Pi^- \setminus \langle \Theta \rangle^-$ we have $K_\Theta$-equivalence between $V_\Theta(\alpha)$ and $V_\Theta(\beta)$.

Lemma 5.16 In $D_l$, $l > 4$, let $\{\alpha, \beta\}$ be a $M$-equivalence class contained in $\Pi^- \setminus \langle \Theta \rangle^-$. Then the $K_\Theta$ representations on $V_\Theta(\alpha)$ and $V_\Theta(\beta)$ are equivalent.

Proof: To prove equivalence we shall exhibit a $K_\Theta$-invariant subspace $\{0\} \neq V \subset V_\Theta(\alpha) \oplus V_\Theta(\beta)$ which is different from the irreducible components $V_\Theta(\alpha)$ and $V_\Theta(\beta)$. This will imply that the components are indeed $K_\Theta$-equivalent (see Proposition 4.6).

The required subspace $V$ will be obtained from the tangent space at the origin of the orbit of one of the normal subgroups $\text{SO}(l)_j$, $j = 1, 2$.

Take for instance $\text{SO}(l)_1$ whose Lie algebra $\mathfrak{so}(l)_1$ constitutes of the matrices

$$\begin{pmatrix} A & A \\ A & A \end{pmatrix} \quad A + AT = 0.$$
Looking at these matrices we see that after identifying $T_{b_{\Theta}}F_{\Theta}$ with $n_{\Theta}^-$ the tangent space $T_{b_{\Theta}}(SO(l) \cdot b_{\Theta})$ to the orbit $SO(l) \cdot b_{\Theta}$ is identified to the subspace $W_1 \subset n_{\Theta}^-$ spanned by $\text{pr}(E_{rs}^- + E_{rs}^+)$, $r > s$, where $E_{rs}^\pm$ were defined in (9) and $\text{pr}: n^- \rightarrow n_{\Theta}^-$ is the projection w.r.t. the root spaces decomposition.

The tangent space $T_{b_{\Theta}}(SO(l) \cdot b_{\Theta})$ is invariant by the isotropy representation of $K_{\Theta}$, by Lemma 4.7. Hence $W_1$ is invariant by the adjoint action of $K_{\Theta}$.

Now a $M$-equivalence class is given by $\{\lambda_i - \lambda_j, -\lambda_i - \lambda_j\}, i > j$, whose root spaces are spanned by $E_{ij}^-$ and $E_{ij}^+$, so that $E_{ij}^- \in V_{\Theta} (\lambda_i - \lambda_j)$ and $E_{ij}^+ \in V_{\Theta} (-\lambda_i - \lambda_j)$. If both roots are in $\Pi^- \setminus \langle \Theta \rangle^-$ we have

$$E_{rs}^- + E_{rs}^+ = \text{pr}(E_{rs}^- + E_{rs}^+) \in W_1 \cap (V_{\Theta} (\lambda_i - \lambda_j) \oplus V_{\Theta} (-\lambda_i - \lambda_j)).$$

Hence $W_1 \cap (V_{\Theta} (\lambda_i - \lambda_j) \oplus V_{\Theta} (-\lambda_i - \lambda_j)) \neq \{0\}$. This is a $K_{\Theta}$-invariant subspace different from $V_{\Theta} (\lambda_i - \lambda_j)$ and $V_{\Theta} (-\lambda_i - \lambda_j)$. It follows that the representation of $K_{\Theta}$ on the irreducible subspaces $V_{\Theta} (\lambda_i - \lambda_j)$ and $V_{\Theta} (-\lambda_i - \lambda_j)$ are equivalent by Proposition 4.6.

Summarizing we get the $K_{\Theta}$-invariant subspaces for the flags of $D_l$, $l > 4$.

**Theorem 5.17** In a flag $F_{\Theta}$ of $D_l$, $l > 4$, there are the following two classes of $K_{\Theta}$-invariant subspaces in $n_{\Theta}$.

1. The $z_{\Theta}$-irreducible component $V_{\Theta} (\alpha)$, containing the root space $g_{\alpha}$ in case $\alpha \in \Pi^- \setminus \langle \Theta \rangle^-$ is not $M$-equivalent to $\beta \in \Pi^- \setminus \langle \Theta \rangle^-$. (These are isolated invariant subspaces.)

2. Let $\{\alpha, \beta\}$ be a $M$-equivalence class contained in $\Pi^- \setminus \langle \Theta \rangle^-$. Then there is a continuum of invariant subspaces parametrized by $[(x, y)] \in \mathbb{R}P^1$ given by

$$V_{[(x,y)]} = \{xX + yTX : X \in V_{\Theta} (\alpha)\}$$

where $T: V_{\Theta} (\alpha) \rightarrow V_{\Theta} (\beta)$ is an isomorphism intertwining the $K_{\Theta}$-representations.

The case $l = 4$ differs from the general one in two aspects, namely each $M$-equivalence class has now 4 elements and the compact subalgebra $\mathfrak{k} = \mathfrak{so}(4) \oplus \mathfrak{so}(4)$ decomposes further into four copies of $\mathfrak{so}(3)$. These simple components of $\mathfrak{k}$ yield new invariant subspaces.

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To see what can happen let us consider the example with $\Theta = \{\lambda_1 - \lambda_2, \lambda_2 - \lambda_3, \lambda_3 - \lambda_4\}$. Then $\mathfrak{n}_\Theta$ is formed by matrices

$$Y = \begin{pmatrix} 0 & 0 \\ X & 0 \end{pmatrix}$$

with $X$ a $4 \times 4$ skew-symmetric matrix. Here $\mathfrak{t}$ is the Lie algebra of matrices

$$\begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix}$$

which is isomorphic to $\mathfrak{so}(4) \oplus \mathfrak{so}(4)$ by

$$\begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix} \mapsto \begin{pmatrix} \alpha + \beta & 0 \\ 0 & \alpha - \beta \end{pmatrix}.$$

Now $\mathfrak{so}(4)$ is the direct sum of two copies of $\mathfrak{so}(3)$ as in \[2\] (see the case $A_3$, above). Thus we can see that if we take $X$ in each one of the sets of matrices in \[2\] we get subspaces $V_1$ and $V_2$ of $\mathfrak{n}_\Theta$ that are the tangent spaces to the orbits of the simple components of $K$. Hence $\mathfrak{n}_\Theta = V_1 \oplus V_2$ is a decomposition into two 3-dimensional $K_\Theta$-invariant subspaces. These two representations are equivalent, since they are just the adjoint representation of $\mathfrak{so}(3)$ on each component.

### 5.5 Flags of $E_6$, $E_7$ and $E_8$

For a flag manifold $\mathbb{F}_\Theta$ of one of these exceptional Lie algebras the $K_\Theta$-invariant irreducible subspaces finite and coincide with the invariant irreducible components $V^\sigma_{\Theta} \subset \mathfrak{n}_\Theta$ for the representation of $\mathfrak{b}_\Theta$.

This is because the Dynkin diagrams are simply laced. Hence, by Lemma \[1.3\] it follows that $\mathcal{W}_\Theta$ acts transitively on each set of roots $\Pi^\sigma_{\Theta}$ corresponding to $V^\sigma_{\Theta}$. Also, as checked in Section \[3\] the classes of $M$-equivalence for these Lie algebras are singletons. Hence by Lemma \[4.2\] we have $K_\Theta$-irreducibility of each $V^\sigma_{\Theta}$. Furthermore the representations of $K_\Theta$ on different subspaces $V^\sigma_{\Theta1}$ and $V^\sigma_{\Theta2}$ are not $M$-equivalent, as follows by combining Lemma \[4.4\] and the fact that the $M$-equivalence classes are singletons.

### 5.6 Flags of $G_2$

Let $\alpha_1$ and $\alpha_2$ be the simple roots of $G_2$ with $\alpha_1$ the long one. There are three flag manifolds, $\mathbb{F}_\emptyset$, $\mathbb{F}_{\{\alpha_1\}}$ and $\mathbb{F}_{\{\alpha_2\}}$. The irreducible components on them are
easily obtained by direct inspection of the positive roots. Recall that the $M$-equivalence classes on the positive roots are \{\alpha_1, \alpha_1 + 2\alpha_2\}, \{\alpha_1 + \alpha_2, \alpha_1 + 3\alpha_2\} and \{\alpha_2, 2\alpha_1 + 3\alpha_2\}. They are listed below:

1. In $F = \mathbb{F}_g$ there are three families of $\mathfrak{g}_\Theta$ and $K_\Theta$-irreducible subspaces, parametrized by $\mathbb{R}P^1$, corresponding to the three $M$-equivalence classes on the negative roots.

2. For $F_{\{\alpha_1\}}$ there are three $\mathfrak{g}_\Theta$-irreducible components corresponding to the sets of roots \{\alpha_2, \alpha_1 + \alpha_2\}, \{\alpha_1 + 2\alpha_2\} and \{\alpha_1 + 3\alpha_2, 2\alpha_1 + 3\alpha_2\}. They are $K_\Theta$-irreducible because the 2-dimensional irreducible representation of $\mathfrak{sl}(2, \mathbb{R})$ is $\mathfrak{so}(2)$-irreducible. By checking the $M$-equivalence classes we see that the 2-dimensional subspaces are equivalent. Hence, we have the irreducible subspace $g_{-\alpha_1 - 2\alpha_2}$ and a family of 2-dimensional irreducible subspaces parametrized by $\mathbb{R}P^1$, contained in $g_{-\alpha_2} \oplus g_{-\alpha_1 - \alpha_2} g_{-\alpha_1 - 3\alpha_2} \oplus g_{-2\alpha_1 - 3\alpha_2}$.

3. For $F_{\{\alpha_2\}}$ the $\mathfrak{g}_\Theta$-irreducible components correspond to the sets of roots \{\alpha_1, \alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2, \alpha_1 + 3\alpha_2\} and \{2\alpha_1 + 3\alpha_2\}. The 4-dimensional irreducible representation of $\mathfrak{g}_\Theta \approx \mathfrak{sl}(2, \mathbb{R})$ decomposes into two $K_\Theta \approx \text{SO}(2)$-invariant irreducible 2-dimensional inequivalent representations. Hence there are three $K_\Theta$-invariant irreducible subspaces.

5.7 **Flags of $F_4$**

Recall that the $M$-equivalence classes on the positive roots of $F_4$ are given by

- 12 singletons \{\alpha\} with $\alpha$ running through the set of short roots.
- 3 sets of long roots \{2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4, \alpha_2, \alpha_2 + 2\alpha_3, \alpha_2 + 2\alpha_3 + 2\alpha_4\}, \{\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4, \alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + 2\alpha_3, \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4\} and \{\alpha_1 + 2\alpha_2 + 4\alpha_3 + 2\alpha_4, \alpha_1, \alpha_1 + 2\alpha_2 + 2\alpha_3, \alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4\}.

Hence in the maximal flag manifold the invariant subspaces are $g_{-\alpha}$, $\alpha$ short root, and the one dimensional subspaces contained in $g_{-\alpha} \oplus g_{-\beta} \oplus g_{-\gamma} \oplus g_{-\delta}$ with \{\alpha, \beta, \gamma, \delta\} a $M$-equivalence class of long roots.

We will not make an extensive analysis of the other 14 flag manifolds but look only at the specific flag manifold $F_\Theta$ where $\Theta = \{\alpha_2, \alpha_3, \alpha_4\}$ which
$C_3$ subdiagram. In this case the $\mathfrak{z}_\Theta$-representation decomposes into two irreducible components $V^1_\Theta$ and $V^2_\Theta$ with $\dim V^1_\Theta = 14$ and $\dim V^2_\Theta = 1$. The sets of roots $\Pi^i_\Theta$ corresponding to $V^i_\Theta$ are those having coefficient $-i$ ($i = 1, 2$) with respect to $\alpha_1$. Namely, $-\Pi^2_\Theta = \{2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4\}$ (the highest root) and $-\Pi^1_\Theta$ contains the remaining positive roots outside $\langle \Theta \rangle$.

Clearly the $K_\Theta$-representation on $V^2_\Theta$ is irreducible. The representation on $V^1_\Theta$ is irreducible as well. In fact, $-\alpha_1$ is the highest weight for the $\mathfrak{z}_\Theta$-representation. Since $\langle -\alpha_1, \alpha_2^\vee \rangle = 1$ and $\langle -\alpha_1, \alpha_3^\vee \rangle = \langle -\alpha_1, \alpha_4^\vee \rangle = 0$, it follows that this is a fundamental weight of $\mathfrak{sp}(3, \mathbb{R})$, namely the weight $\lambda_1 + \lambda_2 + \lambda_3$, where $\lambda_i$ has the same meaning as in Section 5. Hence $V^1_\Theta$ is the space of a basic representation of $\mathfrak{sp}(3, \mathbb{R})$. It is known that this basic representation is irreducible by the compact subalgebra $\mathfrak{u}(3)$. Hence $V^1_\Theta$ is $K_\Theta$-irreducible.

References

[1] Burstall, F. E. and J. H. Rawnsley: Twistor Theory for Riemannian Symmetric Spaces, Springer Lect. Notes in Math. 1424 (1990).

[2] Burstall, F.E. and S. Salamon: Tournaments, flags and harmonics maps, Math. Ann. 277 (1987), 249-265.

[3] Fulton, W. and J. Harris: Representation Theory. A first course. Springer-Verlag.

[4] Helgason, S.: Differential Geometry, Lie groups and Symmetric spaces. Ac. Press (1978).

[5] Knapp, A.W.: Lie Groups. Beyond an Introduction. Progress in Mathematics 140, Birkhäuser, 2004.

[6] Negreiros, C. J. C.: Some remarks about harmonic maps into flag manifolds, Indiana Univ. Math. J. 37 (1988), 617-636.

[7] San Martin, L.A.B.: Álgebras de Lie. Editora Unicamp (2010).

[8] San Martin, L. A. B. and C. J. C. Negreiros, Invariant almost Hermitian structures on flag manifolds, Advances in Math., 178 (2003), 277-310.
[9] San Martin, L. A. B. and R. C. J. Silva: Invariant nearly-Kahler structures, Geom. Dedicata 121 (2006), 143-154.

[10] Wang, M. and W. Ziller: On normal homogeneous Einstein metrics, Ann. Sci. Ecole norm. Sup.18(1985), 563-633.