EVOLUTION OF STARSHAPED HYPERSURFACES BY GENERAL CURVATURE FUNCTIONS

ALI FARDOUN AND RACHID REGBAOUI

Abstract. We consider the evolution of starshaped hypersurfaces in the Euclidean space by general curvature functions. Under appropriate conditions on the curvature function, we prove the global existence and convergence of the flow to a hypersurface of prescribed curvature.

1. Introduction

Let $M_0$ be a smooth closed compact hypersurface in $\mathbb{R}^{n+1}$ ($n \geq 2$). We suppose that $M_0$ is starshaped with respect to a point, which we assume to be the origin of $\mathbb{R}^{n+1}$ for simplicity, and in the rest of the paper all starshaped hypersurfaces are with respect to the origin of $\mathbb{R}^{n+1}$. This means that for every point $P \in M_0$, we have $P \not\in T_PM_0$, where $T_PM_0$ is the tangent space of $M_0$ at $P$. If we let $\pi : M_0 \to S^n$ to be the projection on $S^n$ defined by $\pi(P) = P/|P|$, $P \in M_0$, then one can prove that $M_0$ is starshaped if and only if $\pi$ is a diffeomorphism. It follows that the inverse diffeomorphism $X_0 := \pi^{-1} : S^n \to M_0$ can be used as a parametrization of $M_0$. The function $\rho_0 : S^n \to \mathbb{R}^+$ defined by $\rho_0(x) = |X_0(x)|$ is called the radial function of $M_0$. Thus we have

$$X_0(x) = \rho_0(x)x, \quad x \in S^n. \quad (1.1)$$

From now on, we say that a smooth embedding $X : S^n \to \mathbb{R}^{n+1}$ is a starshaped embedding if $M := X(S^n)$ is a starshaped hypersurface in $\mathbb{R}^{n+1}$, so by composing by a smooth diffeomorphism of $S^n$ if necessary, we may suppose that $X$ is of the form (1.1).

We consider the evolution problem

$$\begin{cases}
\partial_t X(t, x) = \left( K \circ \kappa(X)(t, x) - f \circ X(t, x) \right) \nu(t, x) \\
X(0, x) = X_0(x)
\end{cases} \quad (1.2)$$

where $X(t, \cdot) : S^n \to \mathbb{R}^{n+1}$ is a smooth starshaped embedding, $\nu$ is the outer unit normal vector field of the hypersurface $M_t := X(t, S^n)$, $K$ is a suitable function of the principal curvatures vector $\kappa(X) = (\kappa_1(X), ..., \kappa_n(X))$ of $M_t$, referred as the curvature function, and $f : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}$ is a given smooth function referred as the prescribed function. We suppose that the function $K$ is expressed as an inverse function of the principal curvatures, that is

$$K \circ \kappa(X) = \frac{1}{F \circ \kappa(X)} = \frac{1}{F \circ (\kappa_1(X), ..., \kappa_n(X))};$$

2000 Mathematics Subject Classification. 35K55, 58J35, 53A07.

Key words and phrases. Geometric flow, Starshaped hypersurface, curvature function.
where \( F \in C^\infty(\Gamma) \cap C^0(\overline{\Gamma}) \) is a positive, symmetric function on an open, convex symmetric cone \( \Gamma \subset \mathbb{R}^n \) with vertex at the origin, which contains the positive cone

\[
\Gamma^+ = \{ (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n : \lambda_i > 0 \ \forall i \in [1, \ldots, n] \}.
\]

This implies in particular that

\[
\Gamma \subset \{ (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n : \lambda_1 + \cdots + \lambda_n > 0 \}.
\]

The function \( F(\lambda) = F(\lambda_1, \ldots, \lambda_n) \) is assumed to satisfy the following structure conditions

\[
\frac{\partial F}{\partial \lambda_i} > 0 \text{ on } \Gamma \quad \forall i \in [1, \ldots, n] \quad (1.3)
\]

\[
F \text{ is homogeneous of degree } k > 0 \text{ on } \Gamma \quad \text{and} \quad F \equiv 0 \text{ on } \partial \Gamma \quad (1.4)
\]

\[
\log F \text{ is concave on } \Gamma. \quad (1.5)
\]

By scaling, we may suppose

\[
F(1, \ldots, 1) = 1. \quad (1.6)
\]

The above conditions on \( F \) are usually assumed in the study of fully nonlinear partial differential equations. Condition (1.3) ensures that the system (1.2) is parabolic, which is of great importance in proving short time existence of solutions. The other conditions will be used to control the \( C^1 \) and \( C^2 \)-norms of solutions. Some examples of suitable curvature functions satisfying (1.3)-(1.6) are

\[
F(\lambda_1, \ldots, \lambda_n) = \binom{k}{n}^{-1} S_k(\lambda_1, \ldots, \lambda_n) = \binom{k}{n}^{-1} \sum_{1 \leq i_1 < \cdots < i_k \leq n} \lambda_{i_1} \cdots \lambda_{i_k}
\]

the \( k \)-th elementary symmetric functions normalised so that \( F(1, \ldots, 1) = 1 \). In this case we take \( \Gamma \) to be the component of the set where \( S_k \) is positive which contains the positive cone. Thus we obtain the mean curvature when \( k = 1 \) and the Gauss curvature when \( k = n \). Other examples of curvature functions are

\[
F(\lambda_1, \ldots, \lambda_n) = \binom{k}{n} (S_k(\lambda_1^{-1}, \ldots, \lambda_n^{-1}))^{-1}.
\]

In this case, we take \( \Gamma = \Gamma^+ \). A particular case of interest in the previous example is the harmonic mean curvature when \( k = 1 \).

Finally, we notice that if a function \( F \) satisfies conditions (1.3)-(1.6) above, then for any \( \alpha > 0 \), the function \( F^\alpha \) satisfies the same conditions where \( k \) is replaced by \( \alpha k \). This invariance property is due to the fact that the convexity condition (1.5) concerns \( \log F \) but not \( F \).

When the prescribed function \( f \equiv 0 \), problem (1.2) has been studied by J. Urbas [12] and independently by C. Gerhardt [5], assuming that the curvature function \( F \) satisfies (1.3)-(1.6) with \( k = 1 \) and that \( F \) is concave instead of \( \log F \) concave. They showed the existence of a global solution on \([0, +\infty)\), and for the convergence at infinity, they prove that if \( \hat{M}_t \) is the hypersurface parametrized by \( \hat{X}(t, .) = e^{-t}X(t, .) \), then \( \hat{M}_t \) converges to a sphere in the \( C^\infty \) topology as \( t \to +\infty \). See also the recent work of C. Gerhardt [6] where he considers curvature functions \( F \) with homogeneity degree \( 0 < k \neq 1 \) and \( f \equiv 0 \). There is an extensive literature on curvature
Moreover, that, for every $t \in \mathbb{R}$, equation:

$$\frac{\partial}{\partial t} \left( \rho^{-k} f(X) \right) > 0, \ X \in \mathbb{R}^{n+1} \setminus \{0\}$$  \hspace{1cm} (1.7)

where $\rho = |X|$. We will also assume that there exist two positive real numbers $r_1 \leq r_2$ such that

$$\begin{cases} 
    f(X) \leq r_1^k & \text{if } |X| = r_1 \\
    f(X) \geq r_2^k & \text{if } |X| = r_2. 
\end{cases} \hspace{1cm} (1.8)$$

These assumptions were made by L. Caffarelli, L. Nirenberg and J. Spruck [4] for the existence by elliptic methods of starshaped embedding $X$ whose $\frac{1}{F}$-curvature is equal to $f$, i.e. satysfying the equation:

$$\frac{1}{F(\kappa(X))} = f(X).$$  \hspace{1cm} (1.9)

Our main result in this paper is that conditions (1.7)-(1.8) on the prescribed function $f$ are also sufficient to study the evolution problem (1.2). Moreover the solution of such flow converges to a smooth starshaped embedding satisfying (1.9). Our first result concerns the case where the homogeneity degree $k$ of $F$ satisfies $0 < k \leq 1$. We have

**Theorem 1.1.** Let $F \in C^\infty(\Gamma) \cap C^0(\Gamma)$ be a positive symmetric function satisfying conditions (1.3)-(1.6) such that the homogeneity degree $k$ of $F$ satisfies $0 < k \leq 1$, and let $f \in C^\infty(\mathbb{R}^{n+1} \setminus \{0\})$ be a positive smooth function satisfying (1.7)-(1.8). Let $M_0$ a closed compact starshaped hypersurface in $\mathbb{R}^{n+1}$, parametrized by an embedding $X_0 : \mathbb{S}^n \to \mathbb{R}^{n+1}$ of the form (1.1) such that

$$\kappa(X_0) \in \Gamma \quad \text{and} \quad \frac{1}{F(\kappa(X_0))} - f(X_0) \geq 0.$$  \hspace{1cm} (1.10)

Then the evolution problem (1.2) admits a unique smooth solution $X(t, \cdot)$ defined on $[0, +\infty)$ such that, for every $t \in [0, +\infty)$, $X(t, \cdot) : \mathbb{S}^n \to \mathbb{R}^{n+1}$ is a starshaped embedding satisfying $\kappa(X(t, \cdot)) \in \Gamma$. Moreover, $X(t, \cdot)$ converges in $C^\infty(\mathbb{S}^n, \mathbb{R}^{n+1})$ to a starshaped embedding $X_\infty : \mathbb{S}^n \to \mathbb{R}^{n+1}$ as $t \to +\infty$, satisfying

$$\frac{1}{F(\kappa(X_\infty))} = f(X_\infty),$$

and for any $m \in \mathbb{N}$, $t \in [0, +\infty)$, we have

$$\|X(t, \cdot) - X_\infty\|_{C^m(\mathbb{S}^n, \mathbb{R}^{n+1})} \leq C_m e^{-\lambda_m t},$$  \hspace{1cm} (1.11)

where $C_m$ and $\lambda_m$ are positive constants depending only on $m, f, F, r_1, r_2$ and $X_0$.

**Remark 1.1.** There are many starshaped embeddings $X_0 : \mathbb{S}^n \to \mathbb{R}^{n+1}$ satisfying condition (1.10) in Theorem 1.1. Indeed, it suffices to take $X_0(x) = rx, x \in \mathbb{S}^n$, where $r$ is any positive constant such that $0 < r \leq r_1$, with $r_1$ as in (1.8). Using conditions (1.7)-(1.8), it is easy to see that (1.10) is satisfied.
As a consequence of Theorem 1.1, we recover the existence result for Weingarten hypersurfaces of Cafarelli-Nirenberg-Spruck [4] stated above. Moreover, we prove the uniqueness of starshaped solutions of (1.9). Namely we have:

**Corollary 1.1.** Let $F \in C^\infty(\Gamma) \cap C^0(\overline{\Gamma})$ be a positive symmetric function satisfying (1.3)-(1.6), and let $f$ be a smooth positive function satisfying (1.7)-(1.8). Then there exists a smooth starshaped embedding $X : \mathbb{S}^n \to \mathbb{R}^{n+1}$ such that $\kappa(X) \in \Gamma$, and satisfying

$$\frac{1}{F(\kappa(X))} = f(X).$$

(1.12)

Moreover, $X$ is the unique starshaped solution of (1.12) with $\kappa(X) \in \Gamma$.

When the homogeneity degree $k$ of the curvature function $F$ satisfies $k > 1$, we need additional conditions on the initial embedding $X_0$. More precisely, we have

**Theorem 1.2.** Let $F \in C^\infty(\Gamma) \cap C^0(\overline{\Gamma})$ be a positive symmetric function satisfying conditions (1.3)-(1.6) such that the homogeneity degree $k$ of $F$ satisfies $k > 1$, and let $f \in C^\infty(\mathbb{R}^{n+1} \setminus \{0\})$ be a positive smooth function satisfying (1.7)-(1.8). Let $M_0$ be a closed compact starshaped hypersurface in $\mathbb{R}^{n+1}$, paramatrized by an embedding $X_0 : \mathbb{S}^n \to \mathbb{R}^{n+1}$ of the form (1.1) such that

$$\kappa(X_0) \in \Gamma \text{ and } 0 \leq -\left(\frac{1}{F(\kappa(X_0))} - f(X_0)\right) \frac{\nabla X_0}{|X_0|} \leq \frac{kR_1}{(k+1)R_2} \min_{|Y| \leq R_2} f(Y),$$

(1.13)

where

$$R_1 = \min\left(r_1, \min_{x \in \mathbb{S}^n} |X_0(x)|\right), \quad R_2 = \max\left(r_2, \max_{x \in \mathbb{S}^n} |X_0(x)|\right)$$

and $r_1, r_2$ are as in (1.8). Then the evolution problem (1.2) admits a unique smooth solution $X(t, \cdot)$ defined on $[0, +\infty)$ such that, for every $t \in [0, +\infty)$, $X(t, \cdot) : \mathbb{S}^n \to \mathbb{R}^{n+1}$ is a starshaped embedding satisfying $\kappa(X(t, \cdot)) \in \Gamma$. Moreover, $X(t, \cdot)$ converges in $C^\infty(\mathbb{S}^n, \mathbb{R}^{n+1})$ to a starshaped embedding $X_\infty : \mathbb{S}^n \to \mathbb{R}^{n+1}$ as $t \to +\infty$, satisfying

$$\frac{1}{F(\kappa(X_\infty))} = f(X_\infty),$$

and for any $m \in \mathbb{N}$, $t \in [0, +\infty)$, we have

$$\|X(t, \cdot) - X_\infty\|_{C^m(\mathbb{S}^n, \mathbb{R}^{n+1})} \leq C_m e^{-\lambda_m t},$$

(1.14)

where $C_m$ and $\lambda_m$ are positive constants depending only on $m, f, r_1, r_2$ and $X_0$.

**Remark 1.2.** There are many smooth starshaped embeddings $X_0 : \mathbb{S}^n \to \mathbb{R}^{n+1}$ satisfying condition (1.13) in Theorem 1.2. Indeed, by applying Corollary 1.1 to the functions $F^{1/k}, f^{1/k}$ instead of $F, f$ (as it can easily be seen, conditions (1.3)-(1.6) and (1.7)-(1.8) are still satisfied with a new homogeneity degree $k = 1$ for $F^{1/k}$), then we get a smooth starshaped embedding $X : \mathbb{S}^n \to \mathbb{R}^{n+1}$ satisfying:

$$\frac{1}{F(\kappa(X))} = f(X).$$

If we take $X_0 = rX$, where $r$ is any positive constant such that $r \in [1, 1 + \varepsilon)$, with $\varepsilon > 0$ small enough, then it is not difficult to see, by using condition (1.7)-(1.8), that $X_0$ satisfies condition (1.13) in Theorem 1.2.
2. Preliminaries

In this section, we recall some expressions for the relevant geometric quantities of smooth compact starshaped hypersurfaces $M \subset \mathbb{R}^{n+1}$. As we saw in the previous section, there is a smooth embedding $X : S^n \to \mathbb{R}^{n+1}$ parametrizing $M$, which is of the form

$$X(x) = \rho(x)x, \ x \in S^n.$$  

For any local orthonormal frame $\{e_1, ..., e_n\}$ on $S^n$ (endowed with its standard metric), covariant differentiation with respect to $e_i$ will be denoted by $\nabla_i$, $\nabla_{ij}$, $\nabla_{ijk}$, ..., and we let $\nabla$ be the gradient on $S^n$. Then in terms of the radial function $\rho$, the metric $g = [g_{ij}]$ induced by $X$ and its inverse $g^{-1} = [g^{ij}]$ are given by

$$g_{ij} = \langle \nabla_i X, \nabla_j X \rangle = \rho^2 \delta_{ij} + \nabla_i \rho \nabla_j \rho, \quad g^{ij} = \rho^{-2} \left( \delta_{ij} - \frac{\nabla_i \rho \nabla_j \rho}{\rho^2 + |\nabla \rho|^2} \right),$$

(2.1)

where $\langle , \rangle$ is the standard metric on $\mathbb{R}^{n+1}$, and $\delta_{ij}$ are Kronecker symbols. The unit outer normal to $M$ is

$$\nu = \frac{\rho x - \nabla \rho}{\sqrt{\rho^2 + |\nabla \rho|^2}}$$

(2.2)

and the the second fundamental form of $M$ is given by

$$h_{ij} = -\langle \nabla_{ij} X, \nu \rangle = \left( \rho^2 + |\nabla \rho|^2 \right)^{-\frac{1}{2}} \left( \rho^2 \delta_{ij} + 2 \nabla_i \rho \nabla_j \rho - \rho \nabla_{ij} \rho \right),$$

(2.3)

The principal curvatures of $M$ are the eigenvalues of the second fundamental form with respect to the induced metric $g$. Thus, $\lambda$ is a principal curvature if

$$\det[h_{ij} - \lambda g_{ij}] = 0,$$

or equivalently

$$\det[a_{ij} - \lambda \delta_{ij}] = 0,$$

where the symmetric matrix $[a_{ij}]$ is given by

$$[a_{ij}] = [g^{ij}] \frac{1}{2} [h_{ij}] [g^{ij}] \frac{1}{2}$$

(2.4)

and where $[g^{ij}] \frac{1}{2}$ is the positive square root of $[g^{ij}]$ which is given by

$$[g^{ij}] \frac{1}{2} = \rho^{-1} \left[ \delta_{ij} - \frac{\nabla_i \rho \nabla_j \rho}{\sqrt{\rho^2 + |\nabla \rho|^2} (\rho + \sqrt{\rho^2 + |\nabla \rho|^2})} \right].$$

(2.5)

Let us now make some remarks about the curvature function $F$. Since $F$ is symmetric, it is well known that $F$ can be seen as a smooth function on the set of real symmetric $n \times n$ matrices $[a_{ij}]$. More precisely, we have

$$F \in C^\infty(M(\Gamma)) \cap C^0(\overline{M(\Gamma)})$$

where $M(\Gamma)$ is the convex cone of real symmetric $n \times n$ matrices with eigenvalue vector in the cone $\Gamma$. One can also prove that conditions (1.3)-(1.6) are also valid when $F$ is seen as function on $M(\Gamma)$. We have

$$[F_{ij}] \text{ is positive definite on } M(\Gamma),$$

(2.6)

where $F_{ij} = \frac{\partial F}{\partial a_{ij}}$.

$$F \text{ is homogeneous of degree } k > 0 \text{ on } M(\Gamma) \text{ and } F \equiv 0 \text{ on } \partial M(\Gamma)$$

(2.7)
\log F \text{ is concave on } M(\Gamma). \quad (2.8)

\[ F(\delta_{ij}) = 1. \quad (2.9) \]

We note here that a smooth function \( G \) on \( M(\Gamma) \) is concave if

\[ \sum_{i,j=1}^{n} \sum_{k,l=1}^{n} G_{ij,kl} \eta_{ij} \eta_{kl} \leq 0 \text{ on } M(\Gamma) \]

for all real symmetric \( n \times n \) matrices \((\eta_{ij})\), where

\[ G_{ij,kl} = \frac{\partial^2 G}{\partial a_{kl} \partial a_{ij}}. \]

Now, we will show that equation (1.2) is equivalent to an evolution equation depending on the radial function \( \rho \). We proceed as in [12], first suppose that \( X(t, .) \) is a solution of (1.2) such that for each \( t \in [0, +\infty) \), \( X(t, .) \) is an embedding of a smooth closed compact hypersurface \( M_t \) in \( \mathbb{R}^{n+1} \), which is starshaped with respect to the origin and such that the vector of its principal curvatures \( \kappa = (\kappa_1, ..., \kappa_n) \) lies in the cone \( \Gamma \). If we choose a family of suitable diffeomorphisms \( \phi(t, .) : S^n \to S^n \) then

\[ X(t, x) = \rho(t, \varphi(t, x)) \varphi(t, x), \]

where \( \rho(t, .) : S^n \to \mathbb{R}^+ \) is the radial function of \( M_t \). We have

\[ \partial_t X = (\nabla \rho, \partial_t \varphi + \partial_t \rho) \varphi + \rho \partial_t \varphi \]

and the unit outer normal is given by

\[ \nu = \frac{\rho \varphi - \nabla \rho}{\sqrt{\|\nabla \rho\|^2 + \rho^2}}. \]

Using the fact that \( \partial_t \varphi \) is tangential to \( S^n \) at \( \varphi \), it follows that

\[ \langle \partial_t X, \nu \rangle = (\rho^2 + \|\nabla \rho\|^2)^{-\frac{1}{2}} \rho \partial_t \rho \]

hence \( \rho \) satisfies the initial value problem

\[ \begin{cases} 
\partial_t \rho = F[\rho(t, .)] \\
\rho(0, x) = \rho_0(x), \quad x \in S^n 
\end{cases} \quad (2.10) \]

where the nonlinear operator \( F \) is defined on smooth functions \( \rho : S^n \to (0, +\infty) \), such that the matrix \([a_{ij}]\) given in (2.4) lies in \( M(\Gamma) \), by

\[ F[\rho](x) = \left( \frac{1}{F(a_{ij}(x))} - f(\rho(x) x) \right) \sqrt{\rho^2(x) + \|\nabla \rho(x)\|^2}. \quad (2.11) \]

From now on, what we mean by admissible function is a smooth function \( \rho : [0, T] \times S^n \to (0, +\infty) \) such that the matrix \([a_{ij}]\) defined by (2.4) lies in the cone \( M(\Gamma) \) defined above. Conversely, suppose that \( \rho : [0, T] \times S^n \to (0, +\infty) \) is an admissible solution of (2.10). If we set

\[ X(t, x) = \rho(t, \varphi(t, x)) \varphi(t, x), \quad (t, x) \in [0, T] \times S^n, \]
where $\varphi(t, .) : S^n \to S^n$ is a diffeomorphism satisfying the ODE

$$
\begin{align*}
\partial_t \varphi(t, x) &= Z(t, \varphi(t, x)) \\
\varphi(0, x) &= x, \quad x \in S^n
\end{align*}
$$

(2.12)

with

$$Z(t, y) = -\left( \frac{1}{F(a_{ij}(t, y))} - f(\rho(t, y)y) \right) \frac{\nabla \rho(t, y)}{\rho \sqrt{\langle \nabla \rho(t, y) \rangle^2 + \rho^2(t, y)}}, \quad (t, y) \in [0, T] \times S^n,
$$

(2.13)

then it is not difficult to see that $X$ is a starshaped embedding which is a solution of (1.2) with $X_0(x) = \rho_0(x)x$.

The condition (2.6) implies that (2.10) is parabolic on admissible functions $\rho$. The classical theory of parabolic equations yields the existence and uniqueness of a smooth admissible solution $\rho$ defined on a small intervall $[0, T]$. From the classical theory of ordinary differential equations, the exists a family of diffeomorphisms $\varphi(t, .)$ defined on a small interval $[0, T]$ and satisfying (2.12). Thus by taking $X(t, x) = \rho(t, \varphi(t, x)) \varphi(t, x)$ we obtain a solution of (1.2) defined on $[0, T]$.

Usually in order to get high order estimates it is useful to represent the hypersurface locally as graph over an open set $\Omega \subset \mathbb{R}^n$. Locally, after rotating the coordinates axes, we may suppose that $M$ is the graph of a smooth function $u : \Omega \to \mathbb{R}$. Hence the metric of $M$, the outer normal vector and the second fundamental form can be written respectively

$$
g_{ij} = \delta_{ij} + D_i u D_j u, \quad g^{ij} = \delta_{ij} - \frac{D_i u D_j u}{1 + |Du|^2}
$$

(2.14)

$$
\nu = \frac{1}{\sqrt{1 + |Du|^2}} (Du, -1),
$$

(2.15)

$$
h_{ij} = \frac{D_{ij} u}{\sqrt{1 + |Du|^2}}
$$

(2.16)

where $D_k, D_{ij}$ are the usual first and second order derivatives in $\mathbb{R}^n$, and $Du = (D_1 u, ..., D_n u)$. The principal curvatures of $M$ are the eigenvalues of the symmetric matrix $[a_{ij}]$ given by

$$
[a_{ij}] = [g^{ij}] \frac{1}{2} [h_{ij}] [g^{ij}] \frac{1}{2}
$$

(2.17)

where $[g^{ij}]^{\frac{1}{2}}$ is the positive square root of $[g^{ij}]$. On ca compute

$$
a_{ij} = \frac{1}{v} \left( D_{ij} u - \frac{D_{ij} u D_{kl} u}{v(1 + v)} - \frac{D_{ij} u D_{kl} u}{v(1 + v)} + \frac{D_{ij} u D_{kl} u D_{kl} u}{v^2(1 + v)^2} \right)
$$

(2.18)

with $v = \sqrt{1 + |Du|^2}$.

In this case equation (1.2) takes the forme

$$
\partial_t u = -\left( \frac{1}{F(a_{ij})} - f(x, u) \right) \sqrt{1 + |Du|^2}.
$$

(2.19)

In what follows, what we mean by an admissible solution of (2.19) is a smooth function $u : [0, T] \times \Omega \to \mathbb{R}$ such that the matrix $[a_{ij}]$ defined by (2.18) lies in the cone $M(\Gamma)$ defined above, and satisfying (2.19).
In this section we prove $C^1$-estimates on solutions $\rho$ of (2.10) and exponential decay of its derivatives $\partial_t \rho$. First we prove $C^0$-estimates.

**Proposition 3.1.** Suppose that $F$ satisfies conditions (1.3)-(1.6) and that $f$ satisfies conditions (1.7)-(1.8). Let $\rho : [0, T] \times \mathbb{S}^n \to (0, +\infty)$ be an admissible solution of (2.10). Then we have, for all $(t, x) \in [0, T] \times \mathbb{S}^n$,

$$R_1 \leq \rho(t, x) \leq R_2$$

(3.1)

where

$$R_1 = \min \left( r_1, \min_{x \in \mathbb{S}^n} \rho_0(x) \right) \quad \text{and} \quad R_2 = \max \left( r_2, \max_{x \in \mathbb{S}^n} \rho_0(x) \right)$$

and where $r_1, r_2$ are as in (1.8).

**Proof.** Let $\rho : [0, T] \times \mathbb{S}^n \to (0, +\infty)$ be an admissible solution of (2.10). Let $(t_0, x_0) \in [0, T] \times \mathbb{S}^n$ such that

$$\rho(t_0, x_0) = \max_{(t, x) \in [0, T] \times \mathbb{S}^n} \rho(t, x).$$

(3.2)

We want to prove

$$\rho(t_0, x_0) \leq R_2.$$

(3.3)

If $t_0 = 0$, then

$$\rho(t_0, x_0) = \rho_0(x_0) \leq R_2,$$

so (3.3) is proved in this case. Suppose now that $t_0 > 0$. Then we have

$$\partial_t \rho(t_0, x_0) \geq 0$$

(3.4)

$$\nabla \rho(t_0, x_0) = 0$$

(3.5)

and the matrix

$$[\nabla_{ij} \rho(t_0, x_0)]$$

is negative semi-definite.

(3.6)

It follows from (3.5) and (3.6) that the matrix $[a_{ij}]$ defined by (2.4) satisfies in the sense of operators

$$a_{ij}(t_0, x_0) \geq \rho^{-1}(t_0, x_0) \delta_{ij}.$$  

(3.7)

Since by (1.3) $F$ is monotone, then by using (3.7) we have at $(t_0, x_0)$

$$F(a_{ij}) \geq F(\rho^{-1} \delta_{ij}) = \rho^{-k} F(\delta_{ij}) = \rho^{-k},$$

(3.8)

where we have used the fact that $F$ is homogeneous of degree $k$ and $F(\delta_{ij}) = 1$. Using equation (2.10) and (3.8), we obtain

$$\partial_t \rho(t_0, x_0) \leq \rho^k(t_0, x_0) - f(\rho(t_0, x_0) x_0).$$

(3.9)

Combining (3.4) and (3.9) gives

$$f(\rho(t_0, x_0) x_0) \leq \rho^k(t_0, x_0).$$

(3.10)

But from (1.7) and (1.8) we have that if $X \in \mathbb{R}^{n+1}$ satisfies $|X| > r_2$, then $f(X) > |X|^k$. So it follows from (3.10) that $\rho(t_0, x_0) \leq r_2$. This proves (3.3) since $r_2 \leq R_2$.

It remains now to prove that $\rho(t, x) \geq R_1$. As before, if we let $(t_0, x_0) \in [0, T] \times \mathbb{S}^n$ such that

$$\rho(t_0, x_0) = \min_{(t, x) \in [0, T] \times \mathbb{S}^n} \rho(t, x),$$

(3.11)
then in the same way as before, we prove that \( \rho(t_0, x_0) \geq R_1 \). This achieves the proof of Proposition 3.1.

\[ \square \]

We prove now the exponential decay of \( \partial_t \rho \).

**Proposition 3.2.** Assume that \( F \) satisfies conditions (1.3)-(1.6) and that \( f \) satisfies conditions (1.7)-(1.8). Let \( \rho : [0, T] \times \mathbb{S}^n \to (0, +\infty) \) be an admissible solution of (2.10). We suppose that \( F[\rho_0] \geq 0 \) if \( k \leq 1 \) and \( F[\rho_0] \leq 0 \) if \( k > 1 \), where the operator \( F \) is given by (2.11), and \( k \) is the homogeneity degree of \( F \). Then we have, for any \( (t, x) \in [0, T] \times \mathbb{S}^n \),

\[ \partial_t \rho(t, x) \geq 0 \quad \text{if} \quad k \leq 1 \]

and

\[ \partial_t \rho(t, x) \leq 0 \quad \text{if} \quad k > 1. \]

Moreover, there exists a positive constant \( \lambda \) depending only on \( f, r_1, r_2 \) and \( \rho_0 \) such that, for any \( t \in [0, T] \), we have

\[ \max_{x \in \mathbb{S}^n} |\partial_t \rho(t, x)| \leq \frac{R_2}{R_1} \max_{x \in \mathbb{S}^n} |F[\rho_0](x)| e^{-\lambda t} , \]  

(3.11)

where

\[ R_1 = \min \left( r_1, \min_{x \in \mathbb{S}^n} \rho_0(x) \right) \quad \text{and} \quad R_2 = \max \left( r_2, \max_{x \in \mathbb{S}^n} \rho_0(x) \right) \]

and where \( r_1, r_2 \) are as in (1.8).

The proof of the above proposition is based on the following lemma which asserts that the function \( \rho^{-1} \partial_\rho \) satisfies a second order parabolic equation.

**Lemma 3.1.** Suppose that \( F \) satisfies conditions (1.3)-(1.6). Let \( \rho : [0, T] \times \mathbb{S}^n \to (0, +\infty) \) be an admissible solution of (2.10) and set \( G = \rho^{-1} \partial_\rho \). Then we have for some smooth functions \( A_l, l = 1, \ldots, n \) (depending on \( \rho \) and its derivatives),

\[ \partial_t G = \sum_{i,j=1}^{n} A_{ij} \nabla_{ij} G + \sum_{l=1}^{n} A_l \nabla_l G - \frac{\sqrt{\rho^2 + |\nabla \rho|^2}}{\rho^2} \left( \rho \partial_\rho f - f - \frac{k-1}{F} \right) G \]

where

\[ A_{ij} = \frac{1}{\rho^2 F^2} \sum_{i,m=1}^{n} \gamma_{il} F_{lm} \gamma_{mj} \]  

(3.12)

and

\[ \gamma_{ij} = \delta_{ij} - \frac{\nabla_i \rho \nabla_j \rho}{\sqrt{\rho^2 + |\nabla \rho|^2} \left( \rho + \sqrt{\rho^2 + |\nabla \rho|^2} \right) \} . \]  

(3.13)
Proof. We recall that by (2.10), $\rho$ satisfies
\[ \partial_t \rho = F[\rho] \] (3.14)
where
\[ F[\rho] = \left( \frac{1}{F(a_{ij})} - f(\rho x) \right) \sqrt{\rho^2 + |\nabla \rho|^2} \rho \] (3.15)
and where $a_{ij}$ is given by (2.4).

In view of the definition of $G$ and (3.15) it will be useful to work with the function $r = \log \rho$ instead of $\rho$. Equation (3.14) becomes then
\[ \partial_t r = \left( \frac{1}{F(a_{ij})} - f(e^r x) \right) e^{-r} \sqrt{1 + |\nabla r|^2} \] (3.16)
where $a_{ij}$ takes the form
\[ a_{ij} = \frac{e^{-r} b_{ij}}{\sqrt{1 + |\nabla r|^2}} \] (3.17)
with
\[
\begin{cases} 
  b_{ij} = \gamma_{il} (\delta_{lm} + \nabla_l r \nabla_m r - \nabla_{lm} r) \gamma_{mj} \\
  \gamma_{ij} = \delta_{ij} - \frac{\nabla_i r \nabla_j r}{\sqrt{1 + |\nabla r|^2} \left(1 + \sqrt{1 + |\nabla r|^2}\right)}.
\end{cases}
\] (3.18)

Now, we have
\[ G = \rho^{-1} \partial_t \rho = \partial_t r = \left( \frac{1}{F(a_{ij})} - f(e^r x) \right) e^{-r} \sqrt{1 + |\nabla r|^2}, \] (3.19)
so
\[ \partial_t G = -e^{-r} \sqrt{1 + |\nabla r|^2} \sum_{i,j=1}^{n} \frac{F_{ij}}{F^2} \partial_t a_{ij} - \sqrt{1 + |\nabla r|^2} \partial_t f(e^r x) \partial_t r \]
\[ + \left( \frac{1}{F(a_{ij})} - f(e^r x) \right) e^{-r} \left( -\sqrt{1 + |\nabla r|^2} \partial_t r + \frac{\langle \nabla \partial_t r, \nabla r \rangle}{\sqrt{1 + |\nabla r|^2}} \right). \] (3.20)

Using (3.17) and (3.18), one can check that for some smooth functions $B_{ij}^l(t, x)$ ($l = 1, ..., n$), we have
\[ \partial_t a_{ij} = -a_{ij} \partial_t r - \frac{e^{-r}}{\sqrt{1 + |\nabla r|^2}} \sum_{i,m=1}^{n} \gamma_{il} \gamma_{mj} \nabla_l m \partial_t r + \sum_{l=1}^{n} B_{ij}^l \nabla_l \partial_t r, \] (3.21)
and since $\partial_t r = G$, it follows from (3.20) and (3.21) that
\[ \partial_t G = \sum_{i,j=1}^{n} A_{ij} \nabla^2 G + \sum_{l=1}^{n} A_l \nabla_l G - \partial_p f(e^r x) \sqrt{1 + |\nabla r|^2} G - G^2 \]
\[ + e^{-r} \sqrt{1 + |\nabla r|^2} \sum_{i,j=1}^{n} \frac{F_{ij}}{F^2} a_{ij} G, \] (3.22)
EVOLUTION OF STARSHAPED HYPERSURFACES BY GENERAL CURVATURE FUNCTIONS

where

\[ A_{ij} = \frac{e^{-2r}}{F^2} \sum_{l,m=1}^{n} \gamma_{il} \gamma_{mj} F_{lm} \]

and \( A_l(t, x) \) (\( l = 1, ..., n \)) are smooth functions. Since \( F \) is homogeneous of degree \( k \), then

\[ \sum_{i,j=1}^{n} A_{ij} \nabla_{ij} G = \frac{k}{F}, \]

so it follows from (3.22) that

\[ \partial_t G = \sum_{i,j=1}^{n} A_{ij} \nabla_{ij} G + \sum_{l=1}^{n} A_l \nabla_l G - \sqrt{1 + |\nabla r|^2} e^{-r} \left( e^{\rho} \partial_{\rho} f - \frac{k}{F} \right) G - G^2 \]

This achieves the proof Lemma 3.1.

\[ \square \]

We need also the following lemma which is a well known version of the maximum principle for parabolic equations.

**Lemma 3.2.** Let \( G : [0, T] \times S^n \to \mathbb{R} \) be a smooth function satisfying

\[ \partial_t G \geq \sum_{i,j=1}^{n} A_{ij} \nabla_{ij} G + \sum_{l=1}^{n} A_l \nabla_l G + AG \quad (3.23) \]

for some smooth functions \( A, A_l, A_{ij}, (l, i, j = 1, ..., n) \), such that the matrix \( [A_{ij}] \) is positive semi-definite. Suppose

\[ \min_{x \in S^n} G(0, x) \geq 0, \]

then

\[ \min_{(t, x) \in [0, T] \times S^n} G(t, x) \geq 0. \]

**Proof.** Let \( \lambda \in \mathbb{R} \) such that

\[ \lambda < - \max_{(t, x) \in [0, T] \times S^n} |A(t, x)|, \quad (3.24) \]

and consider the function \( \tilde{G} \) defined by \( \tilde{G}(t, x) = e^{\lambda t} G(t, x) \). To prove the lemma it is equivalent to prove that

\[ \min_{(t, x) \in [0, T] \times S^n} \tilde{G}(t, x) \geq 0. \quad (3.25) \]

By using (3.23), \( \tilde{G} \) satisfies

\[ \partial_t \tilde{G} \geq \sum_{i,j=1}^{n} A_{ij} \nabla_{ij} \tilde{G} + \sum_{l=1}^{n} A_l \nabla_l \tilde{G} + (\lambda + A) \tilde{G}. \quad (3.26) \]
Let \((t_0, x_0) \in [0, T] \times \mathbb{S}^n\) such that
\[
\tilde{G}(t_0, x_0) = \min_{(t, x) \in [0, T] \times \mathbb{S}^n} \tilde{G}(t, x).
\]
We want to prove
\[
\tilde{G}(t_0, x_0) \geq 0. \tag{3.27}
\]
If \(t_0 = 0\), then
\[
\tilde{G}(t_0, x_0) = \tilde{G}(0, x_0) = G(0, x_0) \geq 0
\]
and (3.27) is proved in this case. If \(t_0 > 0\), then
\[
\partial_t \tilde{G}(t_0, x_0) \leq 0 \tag{3.28}
\]
\[
\nabla \tilde{G}(t_0, x_0) = 0 \tag{3.29}
\]
and the matrix
\[
\begin{bmatrix} \nabla_{ij} \tilde{G}(t_0, x_0) \end{bmatrix}
\]
is positive semi-definite. \tag{3.30}
It follows from (3.26), (3.28), (3.29) and (3.30) that
\[
(\lambda + A(t_0, x_0)) \tilde{G}(t_0, x_0) \leq 0
\]
which implies that \(\tilde{G}(t_0, x_0) \geq 0\) since \(\lambda + A(t_0, x_0) < 0\) by (3.24). Thus (3.27) is proved and the lemma follows.
\[
\square
\]
\textit{Proof of Proposition 3.2.} Let \(G = \rho^{-1} \partial_t \rho\). Then by Lemma 3.1 we have
\[
\partial_t G = \sum_{i,j} A_{ij} \nabla_{ij} G + \sum_{l=1}^{n} A_l \nabla_l G
\]
\[
- \frac{\sqrt{\rho^2 + |\nabla \rho|^2}}{\rho^2} \left( \rho \partial_\rho f - f - \frac{k - 1}{F} \right) G. \tag{3.31}
\]
By (1.3) (or equivalently (2.6)) the matrix \([F_{ij}]\) is positive definite. So it follows from (3.12) that \([A_{ij}]\) is positive semi-definite. We distinguish two cases:

First case: \(0 < k \leq 1\). Since \(G\) satisfies (3.31) and \(G(0, x) = \rho_0^{-1}(x) \partial_t \rho(0, x) = \rho_0^{-1}(x) F[\rho_0](x) \geq 0\) by hypothesis, then by Lemma 3.2 we have for any \(t \in [0, T]\),
\[
\min_{x \in \mathbb{S}^n} G(t, x) \geq 0. \tag{3.32}
\]
In particular, (3.32) implies that \(\partial_t \rho \geq 0\) since \(\partial_t \rho = \rho G\). Now we have, since \(\rho\) satisfies (2.10),
\[
G = \rho^{-1} \partial_\rho \rho = \left( \frac{1}{F(a_{ij})} - f(\rho x) \right) \frac{\sqrt{\rho^2 + |\nabla \rho|^2}}{\rho^2},
\]
so it follows from (3.32) that
\[
\frac{1}{F(a_{ij})} \geq f(\rho x)
\]
which implies that the last term in (3.31) is bounded from below as
\[
\frac{\sqrt{\rho^2 + |\nabla \rho|^2}}{\rho^2} \left( \rho \partial_\rho f - f - \frac{k - 1}{F} \right) \geq \frac{\sqrt{\rho^2 + |\nabla \rho|^2}}{\rho^2} (\rho \partial_\rho f - kf) \tag{3.33}
\]
Since \( f \) satisfies (1.7), then \( \rho \partial_\rho f - kf > 0 \), and since \( R_1 \leq \rho(t, x) \leq R_2 \) by Proposition 3.1, we deduce that

\[
\rho \partial_\rho f - kf \geq \delta_0
\]

for some constant \( \delta_0 > 0 \) depending only on \( f, R_1 \) and \( R_2 \). It follows from (3.33) and (3.34) by using Proposition 3.1 that

\[
\frac{\sqrt{\rho^2 + |\nabla \rho|^2}}{\rho^2} \left( \rho \partial_\rho f - f - \frac{k - 1}{F} \right) \geq \frac{\delta_0}{R_2}.
\]

By setting \( \lambda = \frac{\delta_0}{R_2} \) and \( \tilde{G}(t, x) = e^\lambda \rho \tilde{G}(t, x) \), it follows from (3.31) that \( \tilde{G} \) satisfies

\[
\partial_t \tilde{G} = \sum_{i,j=1}^n A_{ij} \nabla_{ij} \tilde{G} + \sum_{l=1}^n A_l \nabla_l \tilde{G}
- \frac{\sqrt{\rho^2 + |\nabla \rho|^2}}{\rho^2} \left( \rho \partial_\rho f - f - \frac{k - 1}{F} \right) \tilde{G} + \lambda \tilde{G}
\]

which gives by using (3.35) and the fact that \( \tilde{G} \geq 0 \),

\[
\partial_t \tilde{G} \leq \sum_{i,j=1}^n A_{ij} \nabla_{ij} \tilde{G} + \sum_{l=1}^n A_l \nabla_l \tilde{G}.
\]

It follows from (3.36) by applying Lemma 3.2 to the function \( -\tilde{G} + \max_{x \in \mathbb{S}^n} \tilde{G}(0, x) \) that

\[
-\tilde{G} + \max_{x \in \mathbb{S}^n} \tilde{G}(0, x) \geq 0
\]

which implies

\[
\max_{x \in \mathbb{S}^n} G(t, x) \leq e^{-\lambda t} \max_{x \in \mathbb{S}^n} G(0, x).
\]

But from the definition of \( G \) we have

\[
\partial_t \rho = \rho G,
\]

so it follows from (3.37) and (3.38) since \( \partial_t \rho \geq 0 \) and \( R_1 \leq \rho \leq R_2 \) by Proposition 3.1, that

\[
|\partial_t \rho| \leq R_2 e^{-\lambda t} \max_{x \in \mathbb{S}^n} G(0, x) = R_2 e^{-\lambda t} \max_{x \in \mathbb{S}^n} \left( \frac{\mathcal{F}[\rho_0](x)}{\rho_0(x)} \right) \leq \frac{R_2}{R_1} e^{-\lambda t} \max_{x \in \mathbb{S}^n} \mathcal{F}[\rho_0](x).
\]

This proves Proposition 3.2 in the case \( 0 < k \leq 1 \).

Second case : \( k > 1 \). Since \( G \) satisfies (3.31) and \( G(0, x) = \rho_0^{-1}(x) \partial_t \rho(0, x) = \rho_0^{-1}(x) \mathcal{F}[\rho_0](x) \leq 0 \) by hypothesis, then by Lemma 3.2 we have for any \( t \in [0, T] \),

\[
\max_{x \in \mathbb{S}^n} G(t, x) \leq 0.
\]

In particular, (3.39) implies that \( \partial_t \rho \leq 0 \) since \( \partial_t \rho = \rho \tilde{G} \). Now we have, since \( \rho \) satisfies (2.10),

\[
G = \rho^{-1} \partial_t \rho = \left( \frac{1}{\mathcal{F}(a_{ij})} - f(\rho x) \right) \frac{\sqrt{\rho^2 + |\nabla \rho|^2}}{\rho^2},
\]

so it follows from (3.39) that

\[
\frac{1}{\mathcal{F}(a_{ij})} \leq f(\rho x)
\]
which implies that the last term in (3.31) is bounded from below as
\[
\frac{\sqrt{\rho^2 + |\nabla \rho|^2}}{\rho^2} \left( \rho \partial_\rho f - f - \frac{k - 1}{F} \right) \geq \frac{\sqrt{\rho^2 + |\nabla \rho|^2}}{\rho^2} (\rho \partial_\rho f - kf).
\]
(3.40)

Since \( f \) satisfies (1.7), then \( \rho \partial_\rho f - kf > 0 \), and since \( R_1 \leq \rho(t, x) \leq R_2 \) by Proposition 3.1, we deduce that
\[
\rho \partial_\rho f - kf \geq \delta_0
\]
(3.41)

for some constant \( \delta_0 > 0 \) depending only on \( f, R_1 \) and \( R_2 \). It follows from (3.40) and (3.41) by using Proposition 3.1 that
\[
\frac{\sqrt{\rho^2 + |\nabla \rho|^2}}{\rho^2} \left( \rho \partial_\rho f - f - \frac{k - 1}{F} \right) \geq \frac{\delta_0}{R_1}.
\]
(3.42)

By setting \( \lambda = \frac{\delta_0}{R_1} \) and \( \tilde{G}(t, x) = e^{\lambda t} G(t, x) \), it follows from (3.31) that
\[
\partial_t \tilde{G} = \sum_{i,j=1}^n A_{ij} \nabla_{ij} \tilde{G} + \sum_{l=1}^n A_l \nabla_l \tilde{G}
\]
\[- \frac{\sqrt{\rho^2 + |\nabla \rho|^2}}{\rho^2} \left( \rho \partial_\rho f - f - \frac{k - 1}{F} \right) \tilde{G} + \lambda \tilde{G}
\]
which gives by using (3.42) and the fact that \( \tilde{G} \leq 0 \),
\[
\partial_t \tilde{G} \geq \sum_{i,j=1}^n A_{ij}(t, x) \nabla_{ij} \tilde{G} + \sum_{l=1}^n A_l(t, x) \nabla_l \tilde{G}.
\]
(3.43)

It follows from (3.43) by applying Lemma 3.2 to the function \( \tilde{G} - \min_{x \in S^n} \tilde{G}(0, x) \) that
\[
\tilde{G} - \min_{x \in S^n} \tilde{G}(0, x) \geq 0
\]
which implies
\[
\min_{x \in S^n} G(t, x) \geq e^{-\lambda t} \min_{x \in S^n} G(0, x).
\]
(3.44)

But from the definition of \( G \) we have
\[
\partial_t \rho = \rho G,
\]
(3.45)

so it follows from (3.44) and (3.45) since \( \partial_t \rho \leq 0 \) and \( \rho \leq R_2 \) by Proposition 3.1, that
\[
|\partial_\rho| \leq -R_2 e^{-\lambda t} \min_{x \in S^n} G(0, x) = R_2 e^{-\lambda t} \max_{x \in S^n} |G(0, x)| = R_2 e^{-\lambda t} \max_{x \in S^n} \left( \frac{|F[\rho_0](x)|}{\rho_0(x)} \right)
\]
\[
\leq \frac{R_2}{R_1} e^{-\lambda t} \max_{x \in S^n} |F[\rho_0](x)|.
\]

The proof of Proposition 3.2 is then complete.

\[ \square \]

Now we are in position to prove \( C^1 \)-estimates on the function \( \rho \).
Proposition 3.3. Suppose that $F$ satisfies conditions (1.3)-(1.6) and that $f$ satisfies conditions (1.7)-(1.8). Let $\rho : [0, T] \times S^n \to \mathbb{R}^+$ be an admissible solution of (2.10). We suppose that $F[\rho_0] \geq 0$ if $k \leq 1$ and $F[\rho_0] \leq 0$ if $k > 1$, where the operator $F$ is given by (2.11), and $k$ is the homogeneity degree of $F$. Then there exists a positive constant $C$ depending only on $f, r_1, r_2$ and $\rho_0$ such that

$$\max_{(t,x) \in [0,T] \times S^n} |\nabla \rho(t,x)| \leq C,$$

where $r_1$ and $r_2$ are as in (1.8).

Proof. As in the proof of Lemma 3.1, we introduce the function $r = \log \rho$. We have then

$$\partial_t r = \left( \frac{1}{F(a_{ij})} - f(e^r x) \right) e^{-r} \sqrt{1 + |\nabla r|^2}$$

where we recall that $a_{ij}$ takes the form

$$a_{ij} = \frac{e^{-r} b_{ij}}{\sqrt{1 + |\nabla r|^2}}$$

with

$$\begin{cases} 
  b_{ij} = \gamma_{il} (\delta_{lm} + \nabla_l r \nabla_m r - \nabla_{lm} r) \gamma_{mj} \\
  \gamma_{ij} = \delta_{ij} - \frac{\nabla_i r \nabla_j r}{\sqrt{1 + |\nabla r|^2} \left( 1 + \sqrt{1 + |\nabla r|^2} \right)}.
\end{cases}$$

Set $H = \frac{1}{2} |\nabla r|^2$, and let $(t_0, x_0) \in [0, T] \times S^n$ such that

$$H(t_0, x_0) = \max_{(t,x) \in [0,T] \times S^n} H(t,x).$$

Let $\{e_1, ..., e_n\}$ be an orthonormal frame in a neighborhood of $x_0$ such that $\nabla_i (e_j) = 0$ at $x_0$, for $i, j = 1, ..., n$.

If $t_0 = 0$, then

$$H(t_0, x_0) = H(0, x_0) = \max_{x \in S^n} H(0, x).$$

If $t_0 > 0$, then

$$\partial_t H(t_0, x_0) \geq 0$$

and the matrix

$$[\nabla_{ij} H(t_0, x_0)]$$

is negative semi-definite.

In what follows, to simplify the notation we shall write $F$ instead of $F(a_{ij})$, and $f$ instead of $f(e^r x)$. We have at $(t_0, x_0)$, by using (3.51),

$$\partial_t H = \langle \nabla \partial_t r, \nabla r \rangle = \left\langle \nabla \left( \frac{1}{F} - f \right) e^{-r} \sqrt{1 + |\nabla r|^2}, \nabla r \right\rangle$$

$$= -e^{-r} \sqrt{1 + |\nabla r|^2} \sum_{i,j=1}^n \frac{F_{ij}}{F^2} (\nabla a_{ij}, \nabla r) - 2 \sqrt{1 + |\nabla r|^2} \partial_r f H$$
\[ -\sqrt{1+|\nabla r|^2} \langle \nabla f, \nabla r \rangle - 2 \left( \frac{1}{F} - f \right) e^{-r} \sqrt{1+|\nabla r|^2} H. \]  

(3.53)

Using (3.47) and (3.48), one can check that for some smooth functions \( B^l_{ij}(t, x) \) (\( l = 1, ..., n \)), we have, for any \( \alpha = 1, ..., n \), at \( (t_0, x_0) \),

\[ \nabla_\alpha a_{ij} = -\frac{e^{-r}}{\sqrt{1+|\nabla r|^2}} \sum_{l,m=1}^{n} \gamma_i \gamma_m \nabla_{\alpha lm} r + \sum_{l=1}^{n} B^l_{ij} \nabla_{\alpha l} r - a_{ij} \nabla_\alpha r. \]

It follows that, at \( (t_0, x_0) \),

\[ \langle \nabla a_{ij}, \nabla r \rangle = \sum_{\alpha=1}^{n} \nabla_\alpha a_{ij} \nabla_\alpha r \]

\[ = -\frac{e^{-r}}{\sqrt{1+|\nabla r|^2}} \sum_{a,l,m=1}^{n} \gamma_i \gamma_m \nabla_{a lm} r \nabla_\alpha r - 2a_{ij} H. \]  

(3.54)

The formula for commuting the order of covariant differentiation gives at \( (t_0, x_0) \)

\[ \nabla_{a lm} r = \nabla_{lm a} r + \delta_{am} \nabla_{l} r - \delta_{lm} \nabla_{a} r. \]  

(3.55)

Combining (3.54) and (3.55) we get at \( (t_0, x_0) \)

\[ \langle \nabla a_{ij}, \nabla r \rangle = -\frac{e^{-r}}{\sqrt{1+|\nabla r|^2}} \sum_{a,l,m=1}^{n} \gamma_i \gamma_m \nabla_{a lm} r \nabla_\alpha r \\
- \frac{e^{-r}}{\sqrt{1+|\nabla r|^2}} \sum_{l,m=1}^{n} \gamma_i \gamma_m \nabla_{l} r \nabla_{m} r \\
+ 2 \frac{e^{-r}}{\sqrt{1+|\nabla r|^2}} \sum_{t=1}^{n} \gamma_i \gamma_{t} H - 2a_{ij} H. \]  

(3.56)

But we have at \( (t_0, x_0) \)

\[ \nabla_{lm} H = \frac{1}{2} \nabla_{lm} (|\nabla r|^2) = \sum_{\alpha=1}^{n} \nabla_{lma} r \nabla_\alpha r + \sum_{\alpha=1}^{n} \nabla_{la} r \nabla_{ma} r. \]  

(3.57)

Hence it follows from (3.50), (3.53), (3.56) and (3.57) that, at \( (t_0, x_0) \),

\[ 0 \leq e^{-2r} \sum_{i,j=1}^{n} A_{ij} \nabla_{ij} H - e^{-2r} \sum_{a,l,m=1}^{n} A_{lm} \nabla_{la} r \nabla_{ma} r \\
+ 2e^{-r} \sqrt{1+|\nabla r|^2} \sum_{i,j=1}^{n} \frac{F^i_j}{F^2} a_{ij} H + e^{-2r} \sum_{i,j=1}^{n} A_{ij} \nabla_{i} r \nabla_{j} r - 2e^{-2r} \text{Trace} [A_{ij}] H \\
- 2\sqrt{1+|\nabla r|^2} \partial_p f H - \sqrt{1+|\nabla r|^2} \langle \nabla f, \nabla r \rangle - 2 \left( \frac{1}{F} - f \right) e^{-r} \sqrt{1+|\nabla r|^2} H, \]  

(3.58)

where

\[ A_{ij} = \sum_{l,m=1}^{n} \frac{F^l_{lm}}{F^2} \gamma_i \gamma_m. \]
Since \([F_{ij}]\) is positive definite, then \([A_{ij}]\) is positive semi-definite. So we have at \((t_0, x_0)\), by using (3.52),
\[
\sum_{i,j=1}^{n} A_{ij} \nabla_{ij} H \leq 0, \tag{3.59}
\]
\[
\sum_{\alpha,l,m=1}^{n} A_{lm} \nabla_{l\alpha} r \nabla_{m\alpha} r \geq 0 \tag{3.60}
\]
and
\[
\sum_{i,j=1}^{n} A_{ij} \nabla_{i} r \nabla_{j} r - 2 \text{ Trace } [A_{ij}] H \leq 0. \tag{3.61}
\]

Since \(F\) is homogenous of degree \(k\), we have also
\[
\sum_{i,j=1}^{n} F_{ij} a_{ij} = \frac{k}{F}. \tag{3.62}
\]

Thus we get from (3.58), (3.59), (3.60), (3.61), (3.61) and (3.62), at \((t_0, x_0)\)
\[
0 \leq 2 e^{-r} \sqrt{1 + |\nabla r|^2} \frac{k}{F} H - 2 \sqrt{1 + |\nabla r|^2} \partial_{\rho} f H
\]
\[
- 2 \left( \frac{1}{F} - f \right) e^{-r} \sqrt{1 + |\nabla r|^2} H - \sqrt{1 + |\nabla r|^2} \langle \nabla f, \nabla r \rangle. \tag{3.63}
\]

But by Proposition 3.2 we have \(\partial_{\rho} \rho \geq 0\) if \(k \leq 1\), and \(\partial_{\rho} \rho \leq 0\) if \(k > 1\). This implies, since \(\rho\) satisfies (2.10), that \(\frac{1}{F(a_{ij})} - f(\rho x) \geq 0\) if \(k \leq 1\), and \(\frac{1}{F(a_{ij})} - f(\rho x) \leq 0\) if \(k > 1\). That is,
\[
\frac{k - 1}{F(a_{ij})} \leq (k - 1) f(\rho x)
\]

Hence it follows from (3.63) that at \((t_0, x_0)\)
\[
2 \left( e^{r} \partial_{\rho} f - kf \right) H \leq e^{r} \langle \nabla f, \nabla r \rangle. \tag{3.64}
\]

By (1.7) we have \(\rho \partial_{\rho} f(\rho x) - kf(\rho x) > 0\), which implies that
\[
\delta_0 = \min_{(\rho, x) \in [R_1, R_2] \times \mathbb{R}^n} (\rho \partial_{\rho} f(\rho x) - kf(\rho x)) > 0,
\]
where \(R_1\) and \(R_2\) are defined in Proposition 3.1. Since \(R_1 \leq \rho(t, x) \leq R_2\) by Proposition 3.1, then \(e^{r} \partial_{\rho} f - kf \geq \delta_0\). Thus it follows from (3.64) at \((t_0, x_0)\)
\[
2 \delta_0 H \leq e^{r} \langle \nabla f, \nabla r \rangle \leq R_2 |\nabla f| |\nabla r| = R_2 |\nabla f| \sqrt{2} \sqrt{H}
\]

that is
\[
H(t_0, x_0) \leq \frac{C_0^2 R_2^2}{2 \delta_0^2}, \tag{3.66}
\]
where
\[
C_0 = \sup_{R_1 \leq |y| \leq R_2} |\nabla f(y)|.
\]

It follows from (3.49) and (3.66) that
\[
H(t_0, x_0) \leq \max \left( \max_{x \in \mathbb{R}^n} H(0, x), \frac{C_0^2 R_2^2}{2 \delta_0^2} \right).
\]
This ends the proof of Proposition 3.3. \hfill \square

4. $C^2$-ESTIMATES AND PROOF OF THE MAIN RESULTS

To get $C^2$-estimates we need to control the principal curvatures.

**Proposition 4.1.** Suppose that $F$ satisfies conditions (1.3)-(1.6) and that $f$ satisfies conditions (1.7)-(1.8). Let $\rho : [0, T] \times S^n \rightarrow (0, +\infty)$ be an admissible solution of (2.10). We suppose that

\[
\begin{cases}
F[\rho_0] \geq 0 & \text{if } k \leq 1 \\
0 \leq -F[\rho_0] \leq \frac{kR_1}{(k + 1)R_2} \min_{r_1 \leq |Y| \leq r_2} f(Y) & \text{if } k > 1,
\end{cases}
\]

(4.1)

where the operator $F$ is given by (2.11), $k$ is the homogeneity degree of $F$, and

\[ R_1 = \min \left( r_1, \min_{x \in S^n} \rho_0(x) \right), \quad R_2 = \max \left( r_2, \max_{x \in S^n} \rho_0(x) \right) \]

with $r_1, r_2$ as in (1.8). Then there exists a positive constant $C$ depending only on $f, r_1, r_2$ and $\rho_0$ such that

\[ \max_{(t, x) \in [0, T] \times S^n} \max_{1 \leq i \leq n} |\kappa_i(t, x)| \leq C, \]

where $\kappa_1, ..., \kappa_n$ are the principal curvatures of the hypersurface $M_t$ parametrized by $X(t, x) = \rho(t, x)x$.

**Proof.** Define the function $h : [0, T] \times S^n \rightarrow \mathbb{R}$ by

\[ h(t, x) = \log \max_{1 \leq i \leq n} \frac{\kappa_i(t, x)}{\langle X(t, x), \nu(t, x) \rangle} \]

(4.2)

where $\kappa_1, ..., \kappa_n$ are the principal curvatures of the hypersurface $M_t$ parametrized by $X(t, x) = \rho(t, x)x$, and $\nu(t, .)$ is its outer normal vector. First we shall give an upper bound on the function $h$. Let $(t_0, x_0) \in [0, T] \times S^n$ the point where $h$ achieves its maximum on $[0, T] \times S^n$, that is,

\[ h(t_0, x_0) = \max_{(t, x) \in [0, T] \times S^n} h(t, x) = \max_{(t, x) \in [0, T] \times S^n} \log \max_{1 \leq i \leq n} \frac{\kappa_i(t, x)}{\langle X(t, x), \nu(t, x) \rangle}. \]

We want to prove that

\[ h(t_0, x_0) \leq C_0, \]

(4.3)

where the constant $C_0$ depends only on $f, r_1, r_2$ and $\rho_0$. If $t_0 = 0$, then $h(t_0, x_0) = h(0, x_0)$, and (4.3) is trivially satisfied in this case. From now on, we suppose that $t_0 > 0$. Without loss of generality, we may suppose that $x_0$ is the south pole of $S^n$. Let $\Sigma$ the tangent hyperplane to $M_{t_0}$ at the point $Z_0 = X(t_0, x_0)$. Then near $(t_0, Z_0)$, the family of hypersurfaces $M_t$ can be represented as the graph of a smooth function $u$ defined on a neighbourhood of $(t_0, Z_0)$ in $[0, T] \times \Sigma$. Thus the function $u$ is an admissible solution of (2.18) (see section 2).
By choosing a new coordinate system in the hyperplane $\Sigma$, with origin at the point $Z_0$, then in the coordinate parallel to the new ones with centre at the original origin, denoted by $x_1, ..., x_n$, we have

$$Z_0 = (a_1, ..., a_n, -a),$$
for some constants $a_1, ..., a_n, a$, with $a > 0$,

and

$$X(t, x) = (a_1, ..., a_n, -a) + (x, u(t, x))$$
with $u(t_0, 0) = 0$.

By formula (2.16) of section 2, we have

$$\nu = \frac{1}{v(Du, -1)}$$
(4.4)

and

$$\langle X, \nu \rangle = \frac{1}{v} \left( a - u + \sum_{k=1}^{n} (x_k + a_k) D_k u \right),$$
(4.5)

where

$$v = (1 + |Du|^2)^{1/2}.$$  
(4.6)

By our choice of coordinates we have

$$u(t_0, 0) = 0$$
(4.7)

and

$$Du(t_0, 0) = (0, ..., 0).$$
(4.8)

By rotating the new $x_1, ..., x_n$ coordinates, we may suppose that $\max_{1 \leq i \leq n} \kappa_i(t_0, x_0)$ occurs in the $x_1$-direction. We have then by using formula (2.17) and (4.8)

$$\max_{1 \leq i \leq n} \kappa_i(t_0, x_0) = \kappa_1(t_0, x_0) = \frac{D_{11} u(t_0, 0)}{v(t_0, 0) (1 + (D_1 u(t_0, 0))^2)}$$

$$= D_{11} u(t_0, 0).$$

On a neighborhood of $(t_0, 0)$ define the function $H$ by

$$H = \log \left( \frac{D_{11} u}{\varphi v(1 + (D_1 u)^2)^{1/2}} \right),$$

where

$$\varphi = \langle X, \nu \rangle = \frac{1}{v} \left( a - u + \sum_{k=1}^{n} (x_k + a_k) D_k u \right).$$

Thus we have

$$H(t_0, 0) = h(t_0, x_0) = \max_{(t, x) \in [0, T] \times S^n} h(t, x).$$
(4.9)

We will give an upper bound on $H(t_0, 0)$. By our choice of coordinates we have

$$D_{1_\alpha} u(t_0, 0) = 0$$
for $\alpha > 1$,
(4.10)

so by rotating the $x_2, ..., x_n$ coordinates, we may suppose that the matrix $D^2 u(t_0, 0)$ is diagonal and that $D_{11} u(t_0, 0) > 0$.

We have, since $H$ attains a local maximum at $(t_0, 0)$, that

$$DH(t_0, 0) = 0$$
(4.11)
and
\[ \partial_t H(t_0, 0) \geq 0 \] (4.12)
since \( t_0 > 0 \). On the other hand, we have
\[ D_\alpha H = \frac{D_{11\alpha} u}{D_{11} u} - \frac{D_\alpha v}{v} - \frac{2D_1 u D_{1\alpha} u}{1 + (D_1 u)^2} - \frac{D_\alpha \varphi}{\varphi} \]
and
\[ D_\alpha \varphi = \sum_{k=1}^n \frac{(a_k + x_k) D_{\alpha k} u}{v} - \frac{\varphi D_\alpha v}{v}. \]
But by using (4.8) and (4.10) we have at \((t_0, 0)\)
\[ D_\alpha v = \sum_{k=1}^n \frac{D_k u D_{\alpha k} u}{v} = 0, \]
so
\[ D_\alpha \varphi = a_\alpha D_{\alpha\alpha} u \]
and
\[ D_\alpha H = \frac{D_{11\alpha} u}{D_{11} u} - \frac{a_\alpha D_{\alpha\alpha} u}{\varphi} \]
which together with (4.11) give at \((t_0, 0)\),
\[ \frac{D_{11\alpha} u}{D_{11} u} - \frac{a_\alpha D_{\alpha\alpha} u}{\varphi} = 0. \] (4.13)
Differentiating once again, we get at \((t_0, 0)\)
\[ D_{\alpha\alpha} v = (D_{\alpha\alpha} u)^2 \]
and
\[ D_\alpha \left( \frac{D_{\alpha\alpha} \varphi}{\varphi} \right) = \frac{1}{\varphi} \left( D_{\alpha\alpha} u + \sum_{k=1}^n a_k D_{\alpha k} D_{\alpha k} u \right) - \frac{(a_\alpha D_{\alpha\alpha} u)^2}{\varphi^2} - (D_{\alpha\alpha} u)^2. \]
So
\[ D_{\alpha\alpha} H = \frac{D_{11\alpha\alpha} u}{D_{11} u} - \left( \frac{D_{11\alpha} u}{D_{11} u} \right)^2 - 2(D_{1\alpha} u)^2 + \frac{(a_\alpha D_{\alpha\alpha} u)^2}{\varphi^2} \]
\[ - \frac{1}{\varphi} \left( D_{\alpha\alpha} u + \sum_{k=1}^n a_k D_{\alpha k} D_{\alpha k} u \right) \]
at \((t_0, 0)\). And using (4.13) we obtain then
\[ D_{\alpha\alpha} H = \frac{D_{11\alpha\alpha} u}{D_{11} u} - 2(D_{1\alpha} u)^2 - \frac{1}{a} \left( D_{\alpha\alpha} u + \sum_{k=1}^n a_k D_{\alpha k} D_{\alpha k} u \right) \] (4.14)
at \((t_0, 0)\) for \( \alpha = 1, \ldots, n \), where de have used the fact that \( a = \varphi(t_0, 0) \).

Now if we differentiate equation (2.19) in the \( x_1 \) direction, we get
\[
D_1 \partial_{k} u = -\frac{1}{\sqrt{1 + |Du|^2}} \left( \frac{1}{F} - f \right) \sum_{k=1}^n D_k u D_{k1} u
\]
\[ + \frac{\sqrt{1 + |Du|^2}}{F^2} \sum_{i,j=1}^n F_{ij} D_1 a_{ij} + \sqrt{1 + |Du|^2} (D_1 f + D_{n+1} f D_1 u). \]
Differentiating once again in the $x_1$ direction and using (4.7), (4.8) and (4.10) we get at $(t_0, 0)$

$$D_{11}\partial_t u = -\left(\frac{1}{F} - f\right) (D_{11}u)^2 + \frac{1}{F^2} \sum_{i,j=1}^n F_{ij} D_{11}a_{ij} - \frac{2}{F^3} \left(\sum_{i,j=1}^n F_{ij} D_{11}a_{ij}\right)^2$$

$$+ \frac{1}{F^2} \sum_{j,j',r,s=1}^n F_{ij,r} D_{11}a_{ij} D_{11}a_{rs} + D_{11}f + D_{n+1}f D_{11}u. \quad (4.15)$$

But since $\log F$ is concave, we have

$$-\frac{2}{F^3} \left(\sum_{i,j=1}^n F_{ij} D_{11}a_{ij}\right)^2 + \frac{1}{F^2} \sum_{j,j',r,s=1}^n F_{ij,r} D_{11}a_{ij} D_{11}a_{rs} \leq 0,$$

so it follows from (4.15) that

$$D_{11}\partial_t u \leq -\left(\frac{1}{F} - f\right) (D_{11}u)^2 + \frac{1}{F^2} \sum_{i,j=1}^n F_{ij} D_{11}a_{ij} + D_{11}f + D_{n+1}f D_{11}u \quad (4.16)$$

at $(t_0, 0)$. Now from the definition of the matrix $[a_{ij}]$ in (2.17), we have at $(t_0, 0)$ by using (4.7) and (4.8),

$$D_{11}a_{ij} = D_{11}i u - (D_{11}u)^2 D_{ij}u - 2D_{11}u D_{11}u D_{11}u,$$

and since $D^2 u$ is diagonal at $(t_0, 0)$, then we have at this point

$$D_{11}a_{11} = D_{1111}u - 3(D_{11}u)^3 \quad (4.17)$$

and

$$D_{11}a_{\alpha \alpha} = D_{11\alpha \alpha}u - D_{\alpha \alpha}u(D_{11}u)^2 \quad (4.18)$$

for $\alpha = 2, \ldots, n$. Combining (4.16), (4.17) and (4.18) we obtain, since $[F_{ij}]$ is diagonal at $(t_0, 0)$,

$$D_{11}\partial_t u \leq -\left(\frac{1}{F} - f\right) (D_{11}u)^2 + \frac{1}{F^2} \left(\sum_{\alpha=1}^n F_{\alpha \alpha} D_{11\alpha \alpha}u - (D_{11}u)^2 \sum_{\alpha=2}^n F_{\alpha \alpha} D_{\alpha \alpha}u\right)$$

$$- 3\frac{F_{11}}{F^2} (D_{11}u)^3 + D_{11}f + D_{n+1}f D_{11}u. \quad (4.19)$$

But from (4.14) we have

$$D_{11\alpha \alpha}u = D_{11}u D_{\alpha \alpha}H + 2D_{11}u (D_{1\alpha}u)^2 + \frac{D_{11}u}{a} \left(D_{\alpha \alpha}u + \sum_{k=1}^n a_k D_{\alpha \alpha k}u\right),$$

which gives by replacing in (4.19)

$$D_{11}\partial_t u \leq -\left(\frac{1}{F} - f\right) (D_{11}u)^2 + \frac{D_{11}u}{F^2} \sum_{\alpha=1}^n F_{\alpha \alpha} D_{\alpha \alpha}H - \frac{(D_{11}u)^2}{F^2} \sum_{\alpha=1}^n F_{\alpha \alpha} D_{\alpha \alpha}u$$

$$+ \frac{D_{11}u}{aF^2} \sum_{\alpha=1}^n F_{\alpha \alpha} D_{\alpha \alpha}u - \frac{D_{11}u}{aF^2} \sum_{\alpha,k=1}^n F_{\alpha \alpha k} D_{\alpha \alpha k}u + D_{11}f + D_{n+1}f D_{11}u, \quad (4.20)$$

and since $F$ is homogenous of degree $k$ we have at $(t_0, 0)$

$$\sum_{\alpha=1}^n F_{\alpha \alpha} D_{\alpha \alpha}u = kF.$$
So it follows from (4.20) that
\[
D_{11} \partial_t u \leq - \left( \frac{k+1}{F} - f \right) (D_{11} u)^2 + k \frac{D_{11} u}{aF} + \frac{D_{11} u}{F^2} \sum_{\alpha=1}^{n} F_{\alpha \alpha} D_{\alpha \alpha} H \\
+ \frac{D_{11} u}{aF^2} \sum_{\alpha,k=1}^{n} F_{\alpha \alpha} a_k D_{\alpha \alpha k} u + D_{11} f + D_{n+1} f D_{11} u.
\] (4.21)

Since \( H \) achieves a local maximum at \((t_0, 0)\), then the matrix \([D_{ij} H]\) is negative semi-definite at \((t_0, 0)\), and since \([F_{ij}]\) is positive semi-definite and diagonal at \((t_0, 0)\), then we have at \((t_0, 0)\)
\[
\sum_{\alpha=1}^{n} F_{\alpha \alpha} D_{\alpha \alpha} H \leq 0.
\]

Then using the fact that \(D_{11} u(t_0, 0) > 0\), we get from (4.21) at \((t_0, 0)\),
\[
D_{11} \partial_t u \leq - \left( \frac{k+1}{F} - f \right) (D_{11} u)^2 + k \frac{D_{11} u}{aF} + \frac{D_{11} u}{aF^2} \sum_{\alpha,k=1}^{n} F_{\alpha \alpha} a_k D_{\alpha \alpha k} u + D_{11} f + D_{n+1} f D_{11} u.
\] (4.22)

Let us prove that the first term in the right side of (4.22) is negative, that is
\[
\frac{k+1}{F} - f \geq 0.
\] (4.23)

If \(0 < k \leq 1\), then by Proposition 3.2, we have
\[
\frac{1}{F} - f \geq 0
\] (4.24)
since \(\frac{1}{F} - f = \frac{\rho}{\sqrt{\rho^2 + |\nabla \rho|^2}} \partial_t \rho \geq 0\). It is clear that (4.24) implies (4.23) since \(F > 0\). Now if \(k > 1\), then by Proposition 3.2 we have
\[
0 \leq - \left( \frac{1}{F} - f \right) = -\frac{\rho}{\sqrt{\rho^2 + |\nabla \rho|^2}} \partial_t \rho \leq |\partial_t \rho| \leq \frac{R_2}{R_1} \max_{x \in S^n} |F[\rho_0](x)|,
\]
that is,
\[
\frac{1}{F} \geq f - \frac{R_2}{R_1} \max_{x \in S^n} |F[\rho_0](x)|.
\] (4.25)

Now it is easy to see that (4.23) is a consequence of (4.25) and the second part of condition 4.1 in Proposition 4.1. Thus it follows from (4.22) and (4.23) that at \((t_0, 0)\),
\[
\partial_t D_{11} u \leq k \frac{D_{11} u}{aF} + \frac{D_{11} u}{aF^2} \sum_{\alpha,k=1}^{n} F_{\alpha \alpha} a_k D_{\alpha \alpha k} u + D_{11} f + D_{n+1} f D_{11} u.
\] (4.26)

On the other hand, since at \((t_0, 0)\) we have
\[
D_k a_{ij} = D_{ijk} u,
\]
then by differentiating equation (2.19) we get at \((t_0, 0)\)
\[
D_k \partial_t u = \frac{1}{F^2} \sum_{i,j=1}^{n} F_{ij} D_{ijk} u + D_k f = \frac{1}{F^2} \sum_{\alpha=1}^{n} F_{\alpha \alpha} D_{\alpha \alpha k} u + D_k f
\] (4.27)
since \([F_{ij}]\) is diagonal at \((t_0, 0)\).
Now differentiating $H$ with respect to $t$, we see that at $(t_0, 0)$
\[
\partial_t H = \frac{\partial_t D_{11} u}{D_{11} u} - \frac{\partial_t \varphi}{\varphi} = \frac{\partial_t D_{11} u}{D_{11} u} + \frac{1}{a} \partial_t u - \frac{1}{a} \sum_{k=1}^{n} a_k D_k \partial_t u
\]
and using equation (2.18) and (4.27) we obtain then
\[
\partial_t H = \frac{\partial_t D_{11} u}{D_{11} u} - \frac{1}{a} \left( \frac{1}{F} - f \right) - \frac{1}{a F^2} \sum_{\alpha, k=1}^{n} F_{\alpha \alpha} a_k D_{\alpha \alpha k} u - \frac{1}{a} \sum_{k=1}^{n} a_k D_k f. \tag{4.28}
\]
Thus we obtain from (4.26) and (4.28)
\[
\partial_t H \leq \frac{k - 1}{a F} + \frac{D_{11} f}{D_{11} u} + D_{n+1} f - \frac{1}{a} \sum_{k=1}^{n} a_k D_k f + \frac{1}{a} f \tag{4.29}
\]
at $(t_0, 0)$. Since by (4.12) we have $\partial_t H(t_0, 0) \geq 0$, then it follows from (4.29) that
\[
0 \leq \frac{k - 1}{a F} + \frac{D_{11} f}{D_{11} u} + D_{n+1} f - \frac{1}{a} \sum_{k=1}^{n} a_k D_k f + \frac{1}{a} f. \tag{4.30}
\]
And since
\[
D_{n+1} f(a_1, ..., a_n, -a) - \frac{1}{a} \sum_{k=1}^{n} a_k D_k f(a_1, ..., a_n, -a) = -\frac{1}{a} \rho \partial_\rho f(a_1, ..., a_n, -a),
\]
then (4.30) becomes
\[
0 \leq \frac{k - 1}{a F} + \frac{D_{11} f}{D_{11} u} + \frac{1}{a} \rho \partial_\rho f + \frac{1}{a} f \tag{4.31}
\]
at $(t_0, 0)$. But by Proposition 3.2 we have $\partial_\rho \rho \geq 0$ if $k \leq 1$, and $\partial_\rho \rho \leq 0$ if $k > 1$. This implies that
\[
\frac{k - 1}{F(a_{ij})} - f(\rho x) \geq 0 \text{ if } k \leq 1, \text{ and } \frac{1}{F(a_{ij})} - f(\rho x) \leq 0 \text{ if } k > 1.
\]
That is,
\[
\frac{k - 1}{F(a_{ij})} \leq (k - 1) f(\rho x). \tag{4.32}
\]
It follows from (4.31) and (4.32) that
\[
\frac{1}{a} (\rho \partial_\rho f - k f) \leq \frac{D_{11} f}{D_{11} u} \tag{4.33}
\]
at $(t_0, 0)$. Since $f$ satisfies (1.7), then $\rho \partial_\rho f - k f > 0$, which implies
\[
\delta_0 = \min_{(\rho, x) \in [R_1, R_2] \times \mathbb{S}^n} (\rho \partial_\rho f(\rho x) - k f(\rho x)) > 0,
\]
and since $R_1 \leq \rho(t, x) \leq R_2$ by Proposition 3.1, then $\rho \partial_\rho f - k f \geq \delta_0$. Thus we get from (4.33) at $(t_0, 0)$
\[
\frac{\delta_0}{a} \leq \frac{D_{11} f}{D_{11} u} \leq C \frac{D_{11} f}{D_{11} u}, \tag{4.34}
\]
where
\[
C = \|f\|_{C^2(A_{R_1, R_2})}, \quad A_{R_1, R_2} = \{X \in \mathbb{R}^{n+1} : R_1 \leq |X| \leq R_2\}.
\]
We recall that by definition of $H$, we have $D_{11} u = aeH$ at $(t_0, 0)$. It follows from (4.34) that
\[ e^{H(t_0,0)} \leq \delta_0^{-1}C \]

or equivalently
\[ H(t_0,0) \leq \log \frac{C}{\delta_0}. \]  

(4.35)

Thus the estimate (4.3) is proved by taking
\[ C_0 = \max \left( \log \frac{C}{\delta_0}, \max_{x \in \mathbb{S}^n} h(0,x) \right). \]

(4.3) implies then, for any \((t,x) \in [0,T] \times \mathbb{S}^n,\)
\[ h(t,x) \leq C_0. \]  

(4.36)

We have by (4.2)
\[ \max_{1 \leq i \leq n} \kappa_i = \langle X, \nu \rangle e^h \]
and since by Proposition 3.1 we have
\[ \langle X, \nu \rangle = \frac{\rho^2}{\sqrt{\rho^2 + |\nabla \rho|^2}} \leq \rho \leq R_2, \]
then we get from (4.36) the upper bound
\[ \max_{1 \leq i \leq n} \kappa_i \leq R_2 e^{C_0}. \]  

(4.37)

Now, to get a lower bound on the principal curvatures, it suffices to observe that \(\kappa_1 + \cdots + \kappa_n > 0\) since \(\kappa = (\kappa_1, \ldots, \kappa_n) \in \Gamma,\) and then use the upper bound (4.37). Indeed, we have for all \(i = 1, \ldots, n,\)
\[ 0 < \kappa_1 + \cdots + \kappa_n \leq \kappa_i + (n-1)R_2 e^{C_0} \]
so
\[ \kappa_i \geq -(n-1)R_2 e^{C_0}. \]

The proof of Proposition 4.1 is complete.

\[ \square \]

The previous proposition allows us to get higher order estimates on our solutions.

**Proposition 4.2.** Let \(\rho : [0,T] \times \mathbb{S}^n \to (0, +\infty)\) be an admissible solution of (2.10) as in Proposition 4.1. Then for any \(m \in \mathbb{N},\) there exist two positive constants \(C_m\) and \(\lambda_m\) depending only on \(m, f, F, r_1, r_2\) and \(\rho_0\) such that
\[ \|\rho\|_{C^m([0,T] \times \mathbb{S}^n)} \leq C_m \]  

(4.38)

and for all \(t \in [0,T],\)
\[ \|\partial_t \rho(t,\cdot)\|_{C^m(\mathbb{S}^n)} \leq C_m e^{-\lambda_m t}. \]  

(4.39)

Moreover, there exists a compact set \(K \subset M(\Gamma)\) depending only on \(f, F, r_1, r_2\) and \(\rho_0,\) such that for any \((t,x) \in [0,T] \times \mathbb{S}^n,\)
\[ [a_{ij}(t,x)] \in K, \]  

(4.40)

where the cone \(M(\Gamma)\) is defined in section 2, and the matrix \([a_{ij}]\) is given by (2.4) in section 2.
Proof. The principal curvatures $\kappa_i$ of the hypersurface $M_t$ parametrized by $X(t, x) = \rho(t, x)x$, are the eigenvalues of the matrix $[a_{ij}]$ (see section 2) defined by
\[ [a_{ij}] = [g^{ij}]^{\frac{1}{2}} [h_{ij}] [g^{ij}]^{\frac{1}{2}} \]  \tag{4.41}
where $[g^{ij}]^{\frac{1}{2}}$ is the positive square root of $[g^{ij}]$ which is given by
\[ [g^{ij}]^{\frac{1}{2}} = \rho^{-1} \left[ \delta_{ij} - \frac{\nabla_i \rho \nabla_j \rho}{\sqrt{\rho^2 + |\nabla \rho|^2 (\rho + \sqrt{\rho^2 + |\nabla \rho|^2})}} \right] \]  \tag{4.42}
and $[h_{ij}]$ is the matrix representing the second fundamental form of $M_t$, given by
\[ h_{ij} = (\rho^2 + |\nabla \rho|^2)^{-\frac{1}{2}} (\rho^2 \delta_{ij} + 2 \nabla_i \rho \nabla_j \rho - \rho \nabla_{ij} \rho). \]  \tag{4.43}
It is clear from Proposition 4.1, Proposition 3.1 and Proposition 3.3 by using (4.41), (4.42) and (4.43) that
\[ \sup_{t \in [0, T]} \| \rho(t, .) \|_{C^2(S^\alpha)} \leq C, \]  \tag{4.44}
where $C$ depends only on $f, r_1, r_2$ and $\rho_0$. In order to get higher order estimates, let us first prove (4.40). By Proposition 3.2 we have
\[ |\partial_\tau \rho| \leq C e^{-\lambda t} \leq C, \]  \tag{4.45}
where the constant $C$ depends only on $f, r_1, r_2$ and $\rho_0$. Since $\rho$ satisfies (2.10), then it follows from (4.45)
\[ \frac{1}{F(a_{ij})} \leq \frac{1}{F(a_{ij})} - f(\rho x) \leq \frac{1}{F(a_{ij})} \leq f(\rho x) + C \leq C_0 \]
that is,
\[ F(a_{ij}) \geq \frac{1}{C_0}, \]  \tag{4.46}
where
\[ C_0 = C + \max_{R_1 \leq |X| \leq R_2} |f(X)|. \]
Since $F \equiv 0$ on $\partial M(\Gamma)$, it follows from (4.46) that there exists a constant $\delta_0 > 0$ depending only on $f, F, R_1, R_2$ and $\rho_0$ such that
\[ \text{dist}([a_{ij}], \partial M(\Gamma)) \geq \delta_0, \]  \tag{4.47}
where $\partial M(\Gamma)$ is the boundary of the cone $M(\Gamma)$ and $\text{dist}([a_{ij}], \partial M(\Gamma))$ is the distance of $[a_{ij}]$ to $\partial M(\Gamma)$. It is clear from (4.47) that there exists a compact set $K \subset M(\Gamma)$ depending only on $f, F, R_1, R_2$ and $\rho_0$ such that $[a_{ij}] \in K$. Thus (4.40) is proved.

Let us now prove the estimates (4.38) and (4.39). Since $F$ satisfies (1.3)(or equivalently (2.6)), it follows from (4.40) and the estimate (4.44) that equation (2.10) is uniformly parabolic. Since by hypothesis the function $\log F$ is concave, then we can apply a result of B. Andrews [2] (Theorem 6, p.3), which is a generalisation of the result of N. Krylov [8] on fully nonlinear parabolic equations, to obtain the estimate
\[ \| \partial_\tau \rho \|_{C^\alpha([0, T] \times S^n)} + \| \nabla_{ij} \rho \|_{C^\alpha([0, T] \times S^n)} \leq C, \]  \tag{4.48}
where $C^\alpha([0, T] \times S^n)$ is the parabolic Hölder’s space, and where the constants $C > 0, \alpha \in (0, 1)$ depend only on $f, F, r_1, r_2$ and $\rho_0$. The higher order estimates (4.38) follows from (4.48) and the
standard theory of linear parabolic equations (see [3]). In order to prove (4.39) we use the following well known interpolation inequality, which is valid on any compact Riemannian manifold $M$,

$$\|\nabla u\|_{L^\infty(M)}^2 \leq 4\|u\|_{L^\infty(M)}\|\nabla^2 u\|_{L^\infty(M)}, \quad u \in C^\infty(M), \tag{4.49}$$

where $\nabla u$ and $\nabla^2 u$ denote respectively the gradient and the hessian of $u$. It suffices to apply (4.49) first to $u = \partial_t \rho$ and iterate it on the spatial higher order derivatives of $\partial_t \rho$ and using (4.38) and (3.11) to get (4.39). This achieves the proof of Proposition 4.2.

□

Now we are in position to prove our main result.

**Proof of Theorem 1.1 and Theorem 1.2.** Let $X_0(x) = \rho_0(x)x$ satisfies conditions (1.10) in Theorem 1.1 or conditions (1.13) in Theorem 1.2. Let $X : [0, T] \times S^n \rightarrow \mathbb{R}^{n+1}$ a local solution of (1.2). As we saw in section 2, $X$ is given by

$$X(t, x) = \rho(t, \varphi(t, x))\varphi(t, x), \quad (t, x) \in [0, T] \times S^n \tag{4.50}$$

where $\rho$ satisfies (2.10) and $\varphi(t, .) : S^n \rightarrow S^n$ is a diffeomorphism satisfying the ODE

$$\begin{cases}
\partial_t \varphi(t, x) = Z(t, \varphi(t, x)) \\
\varphi(0, x) = x,
\end{cases} \tag{4.51}$$

with

$$Z(t, y) = -\left( \frac{1}{F(a_{ij}(t, y))} - f(\rho(t, y)) \right) \frac{\nabla \rho(t, y)}{\rho \sqrt{\nabla \rho(t, y)^2 + \rho^2(t, y)}} = -\frac{\partial_t \rho(t, y) \nabla \rho(t, y)}{[\nabla \rho(t, y)]^2 + \rho^2(t, y)}, \quad (t, y) \in [0, T] \times S^n. \tag{4.52}$$

Since $X_0$ satisfies condition (1.10) in Theorem 1.1 or condition (1.13) in Theorem 1.2, then it is easy to check that the hypothesis of Proposition 4.1 (and then Proposition 4.2) concerning $\rho_0$ are satisfied. We can then apply Proposition 4.2 to the function $\rho$ given above. If we differentiate equation (4.51) and use the estimates (4.38)-(4.39) in Proposition 4.2, then it is not difficult to see that for any $m \in \mathbb{N}$, we have

$$\|\varphi\|_{C^m([0, T] \times S^n, S^n)} \leq C_m \tag{4.53}$$

and for any $t \in [0, T]$,

$$\|\partial_t \varphi(t, .)\|_{C^m(S^n, \mathbb{R}^{n+1})} \leq C_m e^{-\lambda_m t}, \tag{4.54}$$

where $C_m$ and $\lambda_m$ are positive constants depending only on $m, f, F, r_1, r_2$ and $X_0$. It follows from (4.50) by using the estimates (4.38)-(4.39) in Proposition 4.2 and (4.53)-(4.54) that, for any $m \in \mathbb{N}$,

$$\|X\|_{C^m([0, T] \times S^n, \mathbb{R}^{n+1})} \leq C_m \tag{4.55}$$

and for all $t \in [0, T]$,

$$\|\partial_t X(t, .)\|_{C^m(S^n, \mathbb{R}^{n+1})} \leq C_m e^{-\lambda_m t}, \tag{4.56}$$

with new constants $C_m$ and $\lambda_m$ depending only on $m, f, F, r_1, r_2$ and $X_0$. Also by Proposition 4.2 there exists a compact set $K \subset M(\Gamma)$ depending only on $m, f, F, r_1, r_2$ and $X_0$ such that for any $(t, x) \in [0, T] \times S^n$, we have

$$[a_{ij}(t, x)] \in K \subset M(\Gamma), \tag{4.57}$$
where the matrix \([a_{ij}]\) is given by (2.4). Since the constant \(C_m\) in (4.55) and the compact set \(K\) in (4.57) are independent of \(T\), then \(X\) can be extended to \([0, +\infty)\) as a solution of (1.2). The estimates (4.55), (4.56) and (4.57) become then

\[
\|X\|_{C^m([0, +\infty) \times \mathbb{S}^n, \mathbb{R}^{n+1})} \leq C_m
\]

(4.58)

\[
\|\partial_t X(t, .)\|_{C^m(\mathbb{S}^n, \mathbb{R}^{n+1})} \leq C_m e^{-\lambda_m t} \quad \text{for all } t \in [0, +\infty)
\]

(4.59)

and

\[
[a_{ij}(t, x)] \in K \subset M(\Gamma) \quad \text{for all } t \in [0, +\infty).
\]

(4.60)

Now it is clear from (4.58) and (4.59) that there exists a map \(X_{\infty} \in C^\infty(\mathbb{S}^n, \mathbb{R}^{n+1})\) such that \(X(t, .) \to X_{\infty}\) as \(t \to +\infty\) in \(C^m(\mathbb{S}^n, \mathbb{R}^{n+1})\) for all \(m \in \mathbb{N}\), and satisfying

\[
\|X(t, .) - X_{\infty}\|_{C^m(\mathbb{S}^n, \mathbb{R}^{n+1})} \leq C_m e^{-\lambda_m t} \quad \text{for all } t \in [0, +\infty).
\]

Since \(X(t, .)\) is starshaped, then it is easy to see that \(X_{\infty}\) is also starshaped, and from (4.60) we deduce that the principal curvatures of \(X_{\infty}\) lie in \(\Gamma\). By passing to the limit in equation (1.2) and using (4.59), we see that \(X_{\infty}\) satisfies

\[
\frac{1}{F(\kappa(X_{\infty}))} - f(X_{\infty}) = 0.
\]

This achieves the proofs of Theorem 1.1 and Theorem 1.2.

Proof of Corollary 1.1. As in Remark 1.1, if we take \(X_0(x) = rx\), where \(0 < r \leq r_1\) with \(r_1\) as in (1.8), then by using (1.7) and (1.8) one easily checks that condition (1.10) in Theorem 1.1 is satisfied by \(X_0\). Thus the evolution problem (1.2) admits a global solution \(X(t, .)\) which converges as \(t \to +\infty\), to a solution \(X_{\infty}\) of

\[
\frac{1}{F(\kappa(X_{\infty}))} = f(X_{\infty})
\]

(4.61)

which is starshaped and satisfying \(\kappa(X_{\infty}) \in \Gamma\). It remains then to prove that \(X_{\infty}\) is the unique starshaped solution of (4.61) such that \(\kappa(X_{\infty}) \in \Gamma\). Let \(X_1\) and \(X_2\) two starshaped solutions of (4.61) such that \(\kappa(X_l) \in \Gamma\), \(l = 1, 2\). We have then

\[
\frac{1}{F(\kappa(X_l))} = f(X_l) \quad , \quad l = 1, 2.
\]

(4.62)

Let \(\rho_l (l = 1, 2)\) be the radial function of \(X_l\), and set \(u_l(x) = \log \rho_l(x)\). Then we have by using formula (2.4) of section 2,

\[
\frac{1}{F(a_{ij}(u_l))} = f(e^{u_l} x) \quad , \quad l = 1, 2,
\]

(4.63)

where the matrix \([a_{ij}(u_l)]\) is given by

\[
[a_{ij}(u_l)] = \frac{e^{-u_l}}{\sqrt{1 + |\nabla u_l|^2}} \gamma_i \beta_{ij} \gamma_j
\]

(4.64)
with
\[
\begin{align*}
    b_{ij} &= \delta_{ij} + \nabla_i u_l \nabla_j u_l - \nabla_{ij} u_l \\
    \gamma_{ij} &= \delta_{ij} - \frac{\nabla_i u_l \nabla_j u_l}{\sqrt{1 + \|\nabla u_l\|^2} \left(1 + \sqrt{1 + \|\nabla u_l\|^2}\right)}, \quad l = 1, 2.
\end{align*}
\]

(4.65)

We shall prove that for any \(x \in S^n\), we have
\[
u_1(x) \geq \nu_2(x).
\]

(4.66)

It is clear that (4.66) would imply that \(u_1 = u_2\), and then \(\rho_1 = \rho_2\). To prove (4.66) define a function \(u : S^n \to \mathbb{R}\) by
\[
u_1(x) = u_1(x) - u_2(x), \quad x_0 \in S^n \text{ a point where } u \text{ achieves its minimum. Then we have at } x_0 \text{ that } \nabla u = 0 \text{ and the matrix } \nabla^2 u \text{ is positive semi-definite, that is, } \nabla u_1 = \nabla u_2 \text{ and } \nabla^2 u_1 \geq \nabla^2 u_2 \text{ (in the sense of operators) at } x_0. \text{ This implies by using (4.64) and (4.65) that at } x_0, \]
\[
e^{u_1[a_{ij}(u_1)]} \leq e^{u_2[a_{ij}(u_2)]}
\]

(4.67)

in the sense of operators. Since the function \(F\) is monotone (by (1.3)) and homogenous of degree \(k\), it follows from (4.63) and (4.67) that
\[
e^{-ku_1(x_0)} f(e^{u_1(x_0)} x_0) \geq e^{-ku_2(x_0)} f(e^{u_2(x_0)} x_0)
\]

which implies by using (1.7) that \(u_1(x_0) \geq u_2(x_0)\) or equivalently \(u(x_0) \geq 0\). This proves (4.66) and the proof of Corollary 1.1 is complete.

\[\square\]

References

[1] B. Andrews, Contraction of convex hypersurfaces in Euclidean space, Calc. Var. Partial Differential Equations 2 (1994), 151-171.

[2] B. Andrews, Fully Nonlinear Parabolic Equations in Two Space Variables, arXiv:math/0402235v1.

[3] B. Andrews, Pinching estimates and motion of hypersurfaces by curvature functions, J. Reine Angew. Math. 608 (2007), 17-33.

[4] L. Caffarelli, L. Nirenberg and J. Spruck, Nonlinear second order elliptic equations. IV. Starshaped compact Weingarten hypersurfaces, Current topics in partial differential equations, 1-26, Kinokuniya, Tokyo, 1986

[5] C. Gerhardt, Flow of nonconvex hypersurfaces into spheres, J. Differential Geom. 32 (1990), 299-314

[6] C. Gerhardt, Non-scale-invariant inverse curvature flows in Euclidean space, arXiv:1112.5626v2

[7] G. Huisken, Evolution of hypersurfaces by their curvature in Riemannian manifolds, Proceedings of the International Congress of Mathematicians, Vol. II, Berlin, 1998.

[8] N. Krylov, Nonlinear elliptic and parabolic equations of second order, Dordercht : Reidel, 1987.

[9] O. Layzhenskaya, V. Solonnikov, N. Uralt’seva Linear and quasilinear equations of parabolic type, Am. Math. Soc. Providence, 1968.

[10] K. Smoczyk, Starshaped hypersurfaces and the mean curvature flow, Manuscripta Math. 95 (1998), 225-236.

[11] K. Smoczyk, Evolution of hypersurfaces in central force field, J. Reine Angew. Math. 550 (2002), 77-95.

[12] J. Urbas, On the expansion of starshaped hypersurfaces by symmetric functions of their principal curvatures, Math. Z. 205 (1990), 355-372.

[13] J. Urbas, An expansion of convex hypersurfaces, J. Differential Geom. 33 (1991), 91-125.
