ABSTRACT. We prove several Künneth formulas in motivic homotopy categories and deduce a Verdier pairing in these categories following SGA5, which leads to the characteristic class of a constructible motive, an invariant closely related to the Euler-Poincaré characteristic. We prove an additivity property of the Verdier pairing using the language of derivators, following the approach of May and Groth-Ponto-Shulman; using such a result we show that in the presence of a Chow weight structure, the characteristic class for all constructible motives is uniquely characterized by proper covariance, additivity along distinguished triangles, refined Gysin morphisms and Euler classes. In the relative setting, we prove the relative Künneth formulas under some transversality conditions, and define the relative characteristic class.

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1. Introduction

1.1. The Euler-Poincaré characteristic.

1.1.1. The Euler-Poincaré characteristic (or Euler characteristic) is an important invariant in topological spaces in algebraic topology which gives rise to various generalizations in geometry, homological algebra and category theory. In topology, the Euler characteristic of a finite CW-complex is the alternating sum of the dimensions of its singular homology groups. In algebraic geometry, this notion is generalized for étale sheaves as follows: let $X$ be a separated scheme of finite type over a perfect field $k$ of characteristic $p$; if $\ell$ is a prime different from $p$ and $\mathcal{F}$ is a constructible complex of $\ell$-adic étale sheaves over $X$, then the Euler characteristic (with compact support)

$$
\chi_c(X_k, \mathcal{F}) = \sum_{i \geq 0} (-1)^i \cdot \dim H^i_c(X_k, \mathcal{F})
$$

is a well-defined integer, by the finiteness theorems in [SGA4.5].

1.1.2. Morel and Voevodsky introduced motivic homotopy theory ([MV98]) where one study cohomology theories over algebraic varieties by means of the homotopy theory relative to the affine line $\mathbb{A}^1$, leading to several triangulated categories of motives: the stable motivic homotopy category $\text{SH}$ classifies cohomology theories which satisfy $\mathbb{A}^1$-homotopy invariance, and Voevodsky’s category of motivic complexes $\text{DM}$ ([VSF00]) computes motivic cohomology. These categories are built in a style very close to the derived category of $\ell$-adic étale sheaves: the work of Ayoub ([Ayo07]) and Cisinski-Déglise ([CD19]) establish a six functors formalism similar to the powerful machinery in [SGA4], and the étale realization functor ([Ayo14], [CD16]) gives a map from motives to the derived category étale sheaves which preserves the six functors, generalizing the cycle class map in étale cohomology [SGA4.5, Cycle].

1.1.3. A natural question arises to define the Euler characteristic of a motive. However, for constructible objects in the categories of motives mentioned above, the very definition with (1.1.1.1) apparently does not work, since motivic cohomology groups, or equivalently Bloch’s higher Chow groups ([Blo86]), are in general infinite-dimensional as vector spaces. Instead, there is a more categorical approach using the trace of a morphism: recall that if $\mathcal{C}$ is a symmetric monoidal category with unit $\mathbb{1}$, $M$ is a (strongly) dualizable object in $\mathcal{C}$ (which corresponds to locally constant or smooth sheaves in the étale setting) with dual $M^\vee$ and $u : M \to M$ is an endomorphism of $M$, then the trace of $u$ is the map

$$
\text{Tr}(u) : \mathbb{1} \xrightarrow{\eta} M^\vee \otimes M \xrightarrow{id \otimes u} M^\vee \otimes M \simeq M \otimes M^\vee \xrightarrow{\epsilon} \mathbb{1}
$$

considered as an endomorphism of the unit $\mathbb{1}$, where $\eta$ and $\epsilon$ are unit and counits of the duality. The Euler characteristic of $M$ is defined as the trace of the identity map of $M$. If $k$ is a field, in the stable motivic homotopy category $\text{SH}(k)$ the endomorphism ring of the unit $\mathbb{1}_k$ is identified as

$$
\text{End}_{\text{SH}(k)}(\mathbb{1}_k) \simeq GW(k)
$$

where $GW(k)$ is the Grothendieck-Witt ring of $k$, that is, the Grothendieck group of non-degenerate quadratic forms over $k$. Therefore the Euler characteristic of motives in this case is an invariant in terms of quadratic forms, refining the usual integer-valued Euler characteristic.

1.1.4. For example, if $f : X \to Y$ is a smooth and proper morphism, then the motive $M_f(X) = f_! \mathbb{1}_X$ is dualizable ([Hoy15], [Lev18a]); the motivic Gauss-Bonnet formula states that the Euler characteristic of $M_f(X)$ can be computed as the degree of the (motivic) Euler class of the tangent bundle of $f$ ([Lev18a, Theorem 1], [DJK18, Theorem 4.6.1]), and is a refinement of the classical Gauss-Bonnet formula ([SGA5, VII 4.9]). There are other examples of dualizable motives ([Lev18b]), and more generally if $k$ is a perfect field which has resolution of singularities, then every constructible object in $\text{SH}(k)$ is dualizable.
1.1.5. The Lefschetz trace formula ([SGA4.5, Cycle]) plays an important role in Grothendieck’s approach to the Weil conjectures via a cohomological interpretation of the $L$-functions ([SGA4.5, Rapport]); in [SGA5, III], this formula for constant coefficients is generalized to a more general form, called the Lefschetz-Verdier formula. In order to express this last formula, a very general cohomological pairing, called the Verdier pairing, is constructed from several Künneth type formulas for étale sheaves and a delicate analysis on the six functors. The idea behind this construction is the formalism of Grothendieck-Verdier local duality ([CD19, 4.4.23]) which, in the setting of the six functors, gives rise to a generalized trace map (see (1.3.2.1) below), in the way that the usual formalism of (strong) duality produces the trace map (1.1.3.1); the construction works not only for dualizable objects, but also for all constructible ones, which can be considered as weakly dualizable. If $X$ is a scheme, $\mathcal{F}$ is a constructible complex of $\ell$-adic étale sheaves over $X$ and $u$ is an endomorphism of $\mathcal{F}$, this generalized trace for $u$ is an element in the group of global sections of the dualizing complex over $X$, called the characteristic class of $u$, or the characteristic class of $\mathcal{F}$ when $u$ is the identity map ([AS07, Definition 2.1.1]).

1.1.6. The characteristic class is closely related to the Euler characteristic: when $X$ is the spectrum of the base field $k$, the characteristic class agrees with the Euler characteristic; more generally, if $f : X \to \text{Spec}(k)$ is a proper morphism, the Lefschetz-Verdier formula implies that the degree of the characteristic class of $\mathcal{F}$ agrees with the Euler characteristic of $Rf_*\mathcal{F}$.

1.1.7. The main goal of this paper is to define the characteristic class for constructible motives and study its properties. Given the six functors formalism in the motivic context, analogous to the classical one, we would like to use the very definition of [SGA5, III] to define the Verdier pairing. For this, a major input is the proof of some Künneth formulas for motives.

1.2. Künneth formulas for motives.

1.2.1. In Section 2, we prove several general Künneth formulas for motives that would lead to the Verdier pairing, summarized as follows:

**Theorem 1.2.2** (see Theorem 2.4.6). Let $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$ be two morphisms between separated schemes of finite type over a field $k$, with the following commutative diagram

$$
\begin{array}{ccc}
X_1 & \xrightarrow{p_1} & X_1 \times_k X_2 \xrightarrow{p_2} X_2 \\
\downarrow{f_1} & \downarrow{f} & \downarrow{f_2} \\
Y_1 & \xleftarrow{p'_1} & Y_1 \times_k Y_2 \xleftarrow{p'_2} Y_2.
\end{array}
$$

(1.2.2.1)

Let $\mathbf{T}_c$ be the category of constructible motivic spectra $\mathbf{SH}_c$ or the category of constructible cdh-motives $\mathbf{DM}_{cdh,c}$ (more generally, $\mathbf{T}_c$ can be the subcategory of constructible objects in a motivic triangulated category, see Definition 2.0.1).

We assume resolution of singularities (by blowups or by alterations, see the condition (RS) in 2.1.12 below). For $i = 1, 2$, consider objects $L_i \in \mathbf{T}_c(X_i)$ and $M_i, N_i \in \mathbf{T}_c(Y_i)$. Then there are canonical isomorphisms

(1.2.2.2) $p_1^* f_1^* L_1 \otimes p_2^* f_2^* L_2 \to f_*(p_1^* L_1 \otimes p_2^* L_2)$;

(1.2.2.3) $p_1^* f_1^! L_1 \otimes p_2^* f_2^! L_2 \to f_!(p_1^* L_1 \otimes p_2^* L_2)$;

(1.2.2.4) $p_1^! f_1^* M_1 \otimes p_2^* f_2^* M_2 \to f^!(p_1^* M_1 \otimes p_2^* M_2)$;

(1.2.2.5) $p_1^* \text{Hom}(M_1, N_1) \otimes p_2^* \text{Hom}(M_2, N_2) \to \text{Hom}(p_1^* M_1 \otimes p_2^* M_2, p_1^* N_1 \otimes p_2^* N_2)$. 

1.2.3. The proof of Theorem 2.4.6 is quite different from the classical case: the main ingredient of the proof is the strong devissage property (Definition 2.1.10), which says that under resolution of singularities, the category of constructible motives is generated by (relative) Chow motives as a thick subcategory; we therefore reduce to the case of Chow motives, in which case a careful manipulation of the functors gives the desired isomorphisms. The isomorphism (1.2.2.3) involving $f_i$ is quite formal, and holds more generally when we replace the base field by any base scheme, while the other ones fail to hold in general. We will see later in Section 6 that under some assumptions they also hold in the relative case.

1.3. **The Verdier pairing and the characteristic class.**

1.3.1. In Section 3, we use the Künneth formulas to define the Verdier pairing, following [SGA5, III]. Let $X_1$ and $X_2$ be two separated schemes of finite type over $k$. Let $c : C \to X_1 \times_k X_2$ and $d : D \to X_1 \times_k X_2$ be two morphisms. We denote by $E = C \times_{X_1 \times_k X_2} D$. For $i = 1, 2$, we denote by $p_i : X_1 \times_k X_2 \to X_i$ the projections, $c_i = p_i \circ c : C \to X_i$, $d_i = p_i \circ d : D \to X_i$ and let $L_i \in T_c(X_i)$. Then given two maps $u : c_1^* L_1 \to c_2^* L_2$ and $v : d_2^* L_2 \to d_1^* L_1$, the Verdier pairing $(u, v)$ is an element of the bivariant group (or Borel-Moore theory group) $H_0(E/k)$ (see Definition 5.1.3) seen as a map

$$(1.3.1.1) \quad (u, v) : \mathbb{1}_E \to \mathcal{K}_E$$

where $\mathcal{K}_E = \mathbb{D}(\mathbb{1}_E)$ is the dualizing object (Definition 3.1.8). The Lefschetz-Verdier formula (Proposition 3.1.6) states that this pairing is compatible with proper direct images. In Proposition 3.2.5 we show that this pairing can always be reduced to a generalized trace map, which we explicitly identify in Proposition 3.2.8.

1.3.2. Let $X$ be a scheme, $M$ be a motive over $X$ and $u : M \to M$ be an endomorphism of $M$. We define the characteristic class $C_X(M, u) := (u, 1_M)$ as a particular case of the Verdier pairing (Definition 5.1.3). Explicitly, the characteristic class is the composition

$$(1.3.2.1) \quad \mathbb{1}_X \xrightarrow{u} \text{Hom}(M, M) \to \mathbb{D}(M) \otimes M \cong M \otimes \mathbb{D}(M) \xrightarrow{\varepsilon_M} \mathcal{K}_X$$

where the second map is deduced from the Künneth formulas. We denote $C_X(M) = C_X(M, 1_M)$. The bivariant group $H_0(X/k)$ in which it lives can be computed in many cases:

- If $M$ is a constructible cdh-motive in $\text{DM}_{cdh,c}(X, \mathbb{Z}[1/p])$, the characteristic class $C_X(M)$ is a 0-cycle in the Chow group $CH_0(X)[1/p] = CH_0(X) \otimes_{\mathbb{Z}} \mathbb{Z}[1/p]$ of $X$ up to $p$-torsion.
- If $M$ is a constructible element in the homotopy category of $\text{KGL}$-modules over $X$, the characteristic class $C_X(M)$ is an element in the 0-th algebraic $G$-theory group $G_0(X)$.
- If $M$ is a constructible motivic spectrum in $\text{SH}_c(X)$ and when we apply the $\mathbb{A}^1$-regulator map with values in the Milnor-Witt spectrum ([DJK18, Example 4.4.6]), then the Milnor-Witt-valued characteristic class $C_X^{MW}(M)$ is an element in the Chow-Witt group $\text{CH}_0(X)$.

In other words, the characteristic class associates to every constructible motive a concrete object, which is realized as either a 0-cycle, a formal sum of coherent sheaves or a Milnor-Witt 0-cycle. It lifts the $\ell$-adic characteristic class to the cycle-theoretic level, therefore giving an illustration of the general philosophy of mixed motives.

1.3.3. The results in Section 3 are rather transcriptions of classical results in our context, but will be important for the next sections. A particular case of the construction is already given in [Ols16]. For $h$-motives in $\text{DM}_h$, the characteristic class is also defined independently in [Cis19]. Note that since the étale realization functor is compatible with the six functors, our constructions are compatible with the ones in [AS07] and [SGA5, III].

1.4. **Additivity of traces.**
1.4.1. Given a distinguished triangle \( L \to M \to N \to L[1] \) of motives, one can naturally ask about the relations between the characteristic classes of \( L, M \) and \( N \). It is well known that for monoidal triangulated categories, the trace map fails to be additive along distinguished triangles in general ([Fer05]). This failure may be explained by the defect of the axioms of a triangulated category, where the cone of a morphism exists uniquely only up to isomorphism but not up to unique isomorphism; more concretely, a commutative diagram between distinguished triangles in the derived category does not always reflect a commutative diagram in the category of complexes, which would mean to be “truly commutative”. Behind such a phenomena lies the idea of higher category theory as illustrated by the vast theory of \((\infty, 1)\)-categories ([Lur09]).

1.4.2. A landmarking breakthrough in this direction is made by [May01], where it is shown that the trace map is additive along distinguished triangles for triangulated categories satisfying some extra axioms that arise naturally from topology; with the same spirit, the additivity of traces is generalized to stable derivators in [GPS14], where it is shown that the additivity holds for an endomorphism of distinguished triangles if the diagram commutes in the strong sense of derivators. A derivator, in some sense, lies between a 1-category and an \((\infty, 1)\)-category, in which one can define left and right homotopy Kan extensions using only the 1-categorical language, and which carries enough information to characterize homotopy limits and colimits by 1-categorical universal properties. In particular the axioms of a stable derivator produce functorial cone objects, fixing the problem above for triangulated categories.

1.4.3. We use a very similar approach for the generalized trace map: in Section 4, we prove the additivity of the characteristic class using the language of derivators in the motivic setting ([Ayo07, Section 2.4.5]). Using the same notations as 1.3.1, the main result is the following additivity for the Verdier pairing:

**Theorem 1.4.4** (see Theorem 4.2.8). Let \( T_c \) be a constructible motivic derivator (see Definition 4.2.4) whose underlying motivic triangulated category \( T \) satisfies resolution of singularities (see condition \( (RS) \) in 2.1.12). For \( i \in \{1, 2\} \), let

\[
\begin{align*}
L_i & \to M_i \\
\gamma_i \downarrow & \quad \downarrow \\
\ast & \to N_i
\end{align*}
\]

be a coherent biCartesian square in \( T_c(X_i, \square) \). Let \( f : c_1^i \Gamma_1 \to c_2^i \Gamma_2 \) and \( g : d_2^i \Gamma_2 \to d_1^i \Gamma_1 \) be morphisms of coherent squares in \( T_c(C, \square) \) and \( T_c(D, \square) \). Then the Verdier pairing satisfies

\[
\langle f_M, g_M \rangle = \langle f_L, g_L \rangle + \langle f_N, g_N \rangle
\]

where \( f_M : c_1^i M_1 \to c_2^i M_2 \) is the restriction of \( f \), and similarly for the other maps.

1.4.5. The above result corresponds to [SGA5, III (4.13.1)] which claims the additivity of the Verdier pairing in the filtered derived category. The strategy of the proof is to first use Proposition 3.2.5 to reduce the pairing to a generalized trace map with one single entry. Then we follow closely the same steps of proof as in [GPS14], where we need to check the same axioms for the local duality functor instead of the usual duality functor. Note that all the usual examples such as \( \text{SH} \) or \( \text{DM}_{cdh,c} \) arise from constructible motivic derivators, so working with derivators is not a restriction in practice.

1.5. A characterization of the characteristic class of a motive.

1.5.1. There has been an extensive study in the literature around the Euler characteristic of étale sheaves via ramification theory, see for example [AS07], [KS08] and [Sai17]. In this paper, we use a different approach to give a description of the characteristic class for \( cdh \)-motives in \( \text{DM}_{cdh,c} \). In Section 5 we start with the study of some elementary properties of the characteristic class, using the (Fulton-style) intersection theory developed in [DJK18]. The main result is the following characterization of the characteristic class for \( cdh \)-motives over a perfect field:
Theorem 1.5.2 (see Theorem 5.2.6). Assume that the base field $k$ is perfect, and let $X$ be a scheme. Then the map

\begin{equation}
\text{DM}_{cdh,c}(X, \mathbb{Z}[1/p]) \to CH_0(X)[1/p] \\
M \mapsto C_X(M)
\end{equation}

is the unique map satisfying the following properties:

1. For any distinguished triangle $L \to M \to N \to L[1]$ in $\text{DM}_{cdh,c}(X)$, $C_X(M) = C_X(L) + C_X(N)$.
2. Let $f : Y \to X$ be a proper morphism with $Y$ smooth of dimension $d$ over $k$ and let $M$ be the direct summand of the Chow motive $f_* \mathbb{1}_Y(n)$ defined by an endomorphism $u$. Then $u$ is identified as a cycle $u' \in CH_d(Y \times_X Y)[1/p]$, and we have

\begin{equation}
C_X(M) = C_X(f_* \mathbb{1}_Y(n), u) = f_* \Delta^1 u' \in CH_0(X)[1/p]
\end{equation}

where $f_* : CH_0(Y)[1/p] \to CH_0(X)[1/p]$ is the proper push-forward and $\Delta^1 : CH_d(Y \times_X Y)[1/p] \to CH_0(Y)[1/p]$ is the refined Gysin morphism ([Ful98, 6.2]) associated to the Cartesian square

\begin{equation}
\begin{array}{ccc}
Y & \delta_{Y/X} & Y \times_X Y \\
\downarrow & \Delta & \downarrow \\
Y & \delta_{Y/k} & Y \times_k Y.
\end{array}
\end{equation}

There is an alternative description using the Euler class (i.e. top Chern class), see Proposition 5.1.15 below.

1.5.3. The idea is as follows: Bondarko’s theory of weight structures ([Bon10], [BI15]) implies that $\text{DM}_{cdh,c}$ is generated by Chow motives not only as a thick triangulated category but also as a triangulated category, and therefore by additivity of traces, it suffices to compute the characteristic class for Chow motives, which can be achieved using intersection theory. This description also holds when we replace $\text{DM}_{cdh,c}$ by homotopy category of $KGL$-modules, since the Chow weight structure also exists by the results of [BL16]. In general, the characterization holds over the sub-triangulated category generated by direct summands of Chow motives.

1.5.4. While our result gives an abstract characterization for the characteristic class, we expect it to be related with the Grothendieck-Ogg-Shafarevich type results in [AS07]. When the base field is not perfect, there is a similar characterization by perfection using the work of [EK18], see Remark 5.2.7 below.

1.5.5. In Section 5.3 we show the compatibility between the characteristic class and Riemann-Roch transformations. If $X$ is a scheme and $M \in \mathbb{S}H_c(X)$, then we can canonically associate to $M$ a constructible element in the homotopy category of $KGL$-modules over $X$, as well as an element in $\text{DM}_{cdh,c}(X)$ (see 5.3.1). Then the Riemann-Roch transformation

$\tau_X : G_0(X) \to \oplus_{i \in \mathbb{Z}} CH_i(X)_{\mathbb{Q}}$

constructed in [Ful98, Theorem 18.3] sends the characteristic class of the former to that of the latter. In Corollary 5.3.4 we prove a more general version of such a result.

1.6. The relative case.
1.6.1. In Section 6 we prove some relative Künneth formulas, following the approach in [YZ18]. We first introduce the transversality conditions (Definition 6.1.3), which are closely related to the notion of purity in [DJK18] (see 6.1.5); instead of making use of the geometric notion of singular support as in [YZ18], our definition is a more categorical one extracted from the spirit of [Sai17]. We show that under such conditions and some smoothness assumptions, the Künneth formulas (1.2.2.4) and (1.2.2.5) still holds over a general base scheme (see Theorem 6.2.7). The proof uses the Künneth formulas over a field in Section 2. As a special case, we obtain the following result:

**Corollary 1.6.2.** (see Corollary 6.2.4) Let $T_c$ be the subcategory of constructible objects in a motivic triangulated category which satisfies resolution of singularities (see condition (RS) in 2.1.12). Let $S$ be a smooth $k$-scheme, let $\pi: X \to S$ be a smooth morphism and let $F \in T_c(X)$. If $\pi$ is universally $F$-transversal (see Definition 6.1.3 below), then there is a canonical isomorphism

\[
p_1^* F \otimes p_2^* \text{Hom}(F, \pi^! 1_S) \stackrel{\sim}{\to} \text{Hom}(p_2^* F, p_1^! F)
\]

where $p_i: X \times_S X \to X$ is the projection for $i = 1, 2$.

1.6.3. These Künneth formulas are sufficient to define the relative Verdier pairing, as well as the relative characteristic class (Definition 1.6.3). These Künneth formulas are sufficient to define the relative Verdier pairing, as well as the relative characteristic class (Definition 1.6.3). These Künneth formulas are sufficient to define the relative Verdier pairing, as well as the relative characteristic class (Definition 1.6.3).

1.6.4. In Section 6.3 we establish an equivalence between several notions of local acyclicity and transversality conditions (Proposition 6.3.5 and Proposition 6.3.8). We also give an application using the Fulton style specialization map in [DJK18, 4.5.6] (Corollary 6.3.10).

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**Notation and Conventions.**

1. Throughout the paper, we denote by $k$ a field, and a scheme stands for a separated scheme of finite type over $k$. The category of schemes is denoted by $\text{Sch}$.

2. For any pair of adjoint functors $(F, G)$ between two categories, we denote by $ad_{(F,G)}: 1 \to GF$ and $ad'_{(F,G)}: GF \to 1$ the unit and counit maps of the adjunction.

3. We say that a morphism of schemes $f : X \to Y$ is local complete intersection (abbreviated as “lci”) if it factors as the composition of a regular closed immersion followed by a smooth morphism. ¹ We denote by $L_f$ or $L_X/Y$ its virtual tangent bundle in $K_0(X)$.

4. If $A, B$ are objects in a closed symmetric monoidal category, we denote by

\[
\eta_A : 1 \to \text{Hom}(A, A)
\]

\[
\epsilon_A : A \otimes \text{Hom}(A, B) \to B
\]

the unit and counit maps of the monoidal structure.

¹This notion is called “smoothable lci” in [DJK18].
2. KÜNNETH FORMULAS FOR MOTIVES

In this section we prove several Künneth formulas for motives whose analogues for \(\ell\)-adic étale sheaves are proven in [SGA4.5] and [SGA5]. We start by recalling the axioms of a motivic triangulated category in the sense of [CD19, Definition 2.4.45]. We denote by \(SMTR\) the 2-category of symmetric monoidal triangulated categories with (strong) monoidal functors.

**Definition 2.0.1.** A motivic triangulated category is a (non-strict) 2-functor \((T, \otimes) : Sch^{op} \to SMTR\) satisfying the following properties:

1. The value of \(T\) at the empty scheme \(T(\emptyset)\) is the zero category.
2. For every morphism of schemes \(f : Y \to X\), the functor \(f^+ : T(X) \to T(Y)\) is monoidal.
3. For every smooth morphism \(f : Y \to X\), the functor \(f^* : T(X) \to T(Y)\) has a left adjoint \(f_\#\), such that
   (a) For any commutative square of schemes
   \[
   \begin{array}{ccc}
   Y & \xrightarrow{g} & X \\
   \downarrow & \xrightarrow{\Delta} & \downarrow f \\
   T & \xrightarrow{p} & S
   \end{array}
   \]
   (2.0.1.1)
   The following natural transformation is an isomorphism:
   \[
   q_#g^* \xrightarrow{ad(p_#, p^*)} q_#g^*p_#^* = q_#q^*f^*p_#^* \xrightarrow{ad(q_#, q^*)} f^*p_#^*.
   \]
   (2.0.1.2)
   (b) For any smooth morphism \(f : Y \to X\) and any objects \(M \in T(Y), N \in T(X)\), the following transformation is an isomorphism:
   \[
   f^*_\#(M \otimes f^* N) \xrightarrow{ad(f^*_\#, f^*)} f^*_\#(f^* f^*_\# M \otimes f^* N) \cong f^*_\#(f^*_\# M \otimes N) \xrightarrow{ad'(f^*_\#, f^*)} f^*_\# M \otimes N.
   \]
   (2.0.1.3)
   (4) For every morphism of schemes \(f : Y \to X\), the functor \(f^* : T(X) \to T(Y)\) has a right adjoint \(f_*\).
   (5) For any scheme \(S\) with \(p : \mathbb{A}^1_S \to S\) the canonical projection, the unit map \(1 \xrightarrow{ad(p^*, p^*)} p_*p^*\) is an isomorphism.
   (6) For every proper morphism \(f : Y \to X\), the functor \(f_* : T(Y) \to T(X)\) has a right adjoint \(f^!\).
   (7) For every smooth morphism \(f : X \to S\) with a section \(s : S \to X\), the functor \(f^!s_* : T(S) \to T(X)\) is an equivalence of categories.
   (8) For any closed immersion \(i : Z \to X\) with open complement \(j : U \to X\), the pair of functors \((j^*, i^*)\) is conservative, and the counit map \(i^*i_* \xrightarrow{ad(i^*, i_*)} 1\) is an isomorphism.

In other words, for any scheme \(X\) we have a triangulated category \(T(X)\), and such a formation satisfies the six functors formalism by [CD19, Theorem 2.4.50]. Examples are given by the stable motivic homotopy category \(\mathbf{SH}\), the category of cdh-motivic complexes \(\mathbf{DM}_{cdh}\) or the category of modules over a motivic ring spectrum. We gradually recall the formal properties we need in this section.

For any scheme \(X\), we denote by \(1_X \in T(X)\) the unit object. We are mainly interested in constructible motives, see the discussion in 2.1.12 below.

2.1. Local acyclicity and Künneth formula for \(f_*\). In this section, we work with a motivic triangulated category \(T\).
2.1.1. For any commutative square of schemes

\[ Y \xrightarrow{q} X \]
\[ g \downarrow \Delta \downarrow f \]
\[ T \xrightarrow{p} S \]

there is a canonical natural transformation

\[ f^* p^* \xrightarrow{\text{ad}(q^*, g_*)} q_* q^* f^* p^* = q_* g^* p^* p^* \xrightarrow{\text{ad}'(p^*, q_*)} q_* g^* \]

which is compatible with horizontal and vertical compositions of squares.

2.1.2. The map (2.1.1.2) is an isomorphism if \( f \) is smooth, or if \( p \) is proper. More generally, we give the following definition:

**Definition 2.1.3.** Let \( f : X \to S \) be a morphism of schemes. We say that the category \( T \) satisfies \( f \)-**base change** if for any morphism \( p : T \to S \) with a Cartesian square \( \Delta \) as in (2.1.1.1), the map (2.1.1.2) is an isomorphism. If \( S \) is a class of morphisms, we say that \( T \) satisfies \( S \)-base change if it satisfies \( f \)-base change for any \( f \in S \).

2.1.4. For any morphism \( q : Y \to X \) and objects \( K \in T(X), K' \in T(Y) \), there is a canonical natural transformation

\[ (\text{ad}(q^*, g_*) : K \to q_* q^* (K \otimes q_* K') \xrightarrow{\text{ad}'(q^*, g_*)} q_* (q^* K \otimes q_* K')) \]

The map (2.1.4.1) is an isomorphism for any proper morphism \( q \).

2.1.5. For any commutative square \( \Delta \) as in (2.1.1.1), any \( K \in T(X) \) and any \( L \in T(T) \), there is a canonical natural transformation

\[ \text{Ex}(\Delta^*, \otimes : K \otimes f^* p_* L \to q_* (q^* K \otimes g^* L) \]

defined as the composition

\[ K \otimes f^* p_* L \xrightarrow{(2.1.1.2)} K \otimes q_* g^* L \xrightarrow{(2.1.4.1)} q_* (q^* K \otimes g^* L). \]

2.1.6. For any Cartesian square \( \Delta \) as in (2.1.1.1) with \( p \) proper, the map (2.1.5.1) is an isomorphism. Our aim is to study the cases where the map (2.1.5.1) is an isomorphism for arbitrary \( p \). The following definition is inspired by [SGA4.5, Th. finitude, Définition 2.12]:

**Definition 2.1.7.** Let \( f : X \to S \) be a morphism of schemes and \( K \in T(X) \). We say that \( f \) is **strongly locally acyclic** relatively to \( K \) if for any morphism \( p : T \to S \) with a Cartesian square \( \Delta \) as in (2.1.1.1) and any object \( L \in T(T) \), the map (2.1.5.1) is an isomorphism. We say that \( f \) is **universally strongly locally acyclic** relatively to \( K \) if for any Cartesian square

\[ X' \xrightarrow{\phi} X \]
\[ f' \downarrow \phi \downarrow f \]
\[ S' \xrightarrow{\psi} S \]

the base change \( f' \) of \( f \) is strongly locally acyclic relatively to \( K|_{X'} = \phi^* K \).

**Lemma 2.1.8.** Let \( S \) is a class of morphisms which is stable by base change and suppose that \( T \) satisfies \( S \)-base change. Let \( f : X \to S \) be a morphism, \( \phi : W \to X \) be a proper morphism such that the composition \( f \circ \phi : W \to X \) lies in \( S \). Then \( f \) is universally strongly locally acyclic relatively to the object \( \phi_* \mathbb{1}_W \).
Proposition 2.1.11. where we use the fact that the morphism $\xi$ (2.1.8.3) is an isomorphism. On the left hand side, by assumptions we have

\[ (2.1.8.2) \quad Ex(\Delta^*_+ \otimes) : \xi^* \phi^* \mathbb{1}_W \otimes f^* p_* L \to q_*(q^* \xi^* \phi^* \mathbb{1}_W \otimes g^* L) \]

is an isomorphism. On the left hand side, by assumptions we have

\[ (2.1.8.3) \quad \xi^* \phi^* \mathbb{1}_W \otimes f^* p_* L \cong \phi'^* \mathbb{1}_{W'} \otimes f'^* p_* L \cong \phi'^* f'^* p_* L \cong \phi'^* r_* \psi^* g^* L \]

where we use the fact that the morphism $f' \circ \phi'$ lies in $S$. For the right hand side of (2.1.8.2), we have

\[ (2.1.8.4) \quad q_*(q^* \xi^* \phi^* \mathbb{1}_W \otimes g^* L) \cong q_*(\psi_1^* \mathbb{1}_V \otimes g^* L) \cong \psi_1^* \psi^* g^* L \cong \phi'^* r_* \psi^* g^* L. \]

It is not hard to check that the composition of (2.1.8.3) and the inverse of (2.1.8.4) agrees with the map (2.1.8.2), and therefore (2.1.8.2) is an isomorphism.

2.1.9. The category $T$ has a family of Tate twists $\mathbb{1}(n)$, which are $\otimes$-invertible objects that form a Cartesian section. By [FHMO, 3.2], we have a canonical isomorphism $f_*(\mathbb{1} \otimes M) \cong f_* (\mathbb{1} \otimes M)$, and therefore Tate twists commute with both $f^*$ and $f_*$ in a canonical way. More generally, all the six functors commute with Tate twists ([CD19, Section 1.1.d]) via canonical isomorphisms, and therefore it is safe to ignore them in the proof of K"unneth formulas.

Definition 2.1.10. (1) For any scheme $X$, a projective motive in $T(X)$ is an object of the form $\varphi_! \mathbb{1}_W(n)$, where $\varphi : W \to X$ is a projective morphism, and a primitive Chow motive is a projective motive with $W$ over a finite purely inseparable extension of $k$. 2

(2) We say that $T$ satisfies weak devissage if for any scheme $X$, the category $T(X)$ agrees with the smallest thick triangulated subcategory which is stable by direct sums and contains all projective motives.

(3) We say that $T$ satisfies strong devissage if for any scheme $X$, the category $T(X)$ agrees with the smallest thick triangulated subcategory which is stable by direct sums and contains all primitive Chow motives.

(4) We denote by $S_k$ the family of morphisms $p_2 : X \times_k Y \to Y$ where $X, Y$ are schemes and $p_2$ is the projection onto the second factor. We say that $T$ satisfies $S_k$-strong local acyclicity if any morphism of the form $f : X \to k$ is universally strongly locally acyclic relatively to any object in $T(X)$.

We denote by $R$ the family of finite surjective radical morphisms, namely the family of universal homeomorphisms.

Proposition 2.1.11. We suppose that $T$ satisfies weak devissage. Then

(1) $T$ satisfies $S_k$-strong local acyclicity if and only if it satisfies $S_k$-base change.

(2) If $T$ satisfies strong devissage and one of the following conditions hold:

(a) $k$ is perfect;

(b) $T$ satisfies $R$-base change.

Then $T$ satisfies $S_k$-strong local acyclicity.

\[ \text{In [Jin16] it is shown that for } T = \mathbf{DM}_{cdh,c} \text{ and } X \text{ quasi-projective over a perfect field, the idempotent completion of the additive subcategory generated by primitive Chow motives is equivalent to the category of relative Chow motives over } X \text{ defined by Corti and Hanamura, whence the terminology.} \]
Proof.  (1) The $S_k$-base change property is a particular case of $S_k$-strong local acyclicity since strong local acyclicity relative to the unit object is equivalent to base change property. The other direction follows from weak devissage by applying Lemma 2.1.8.

(2) Since strong local acyclicity is stable under distinguished triangles, direct summands, direct sums and Tate twists, the result is straightforward from Lemma 2.1.8.

2.1.12. Following [BD17, 2.4.1], we consider the following conditions on resolution of singularities:

(RS 1) The field $k$ is perfect, over which the strong resolution of singularities holds, in the sense that
(a) For every separated integral scheme $X$ of finite type over $k$, there exists a proper birational surjective morphism $X' \to X$ with $X'$ regular;
(b) For every separated integral regular scheme $X$ of finite type over $k$ and every nowhere dense closed subscheme $Z$ of $X$, there exists a proper birational surjective morphism $b : X' \to X$ such that $X'$ is regular, $b$ induces an isomorphism $b^{-1}(X - Z) \cong X - Z$, and $b^{-1}(Z)$ is a strict normal crossing divisor in $X'$.

(RS 2) The category $T$ is $\mathbb{Z}[1/p]$ linear where $p$ is the characteristic exponent of $k$, and there exists a premotivic adjunction $SH \leftrightarrows T$.

(RS) We say that the category $T$ satisfies (RS) if it satisfies one of the above conditions (RS 1) and (RS 2).

2.1.13. We recall the following facts about devissage:

(1) ([Ayo07, Lemme 2.2.23]) Any motivic triangulated category satisfies weak devissage.
(2) ([BD17, Corollary 2.4.8], [EK18, Proposition 3.1.3]) If $T$ is a motivic triangulated category which satisfies the condition (RS), then it satisfies strong devissage.

By [EK18, Remark 2.1.13], if $T$ satisfies the condition (RS 2) above, then $T$ satisfies $R$-base change. As a consequence, Proposition 2.1.11 implies that

**Corollary 2.1.14.** If $T$ is a motivic triangulated category which satisfies (RS) in 2.1.12, then it satisfies $S_k$-strong local acyclicity (or equivalently $S_k$-base change).

2.1.15. For any scheme $X$, we denote by $T_c(X)$ the subcategory of constructible objects, which is the thick triangulated subcategory of $T(X)$ generated by elements of the form $f_{\#}1_Y(n)$, where $f : Y \to X$ is a smooth morphism ([CD19, Definition 4.2.1]). By [CD15, Theorem 6.4], if $T$ satisfies (RS), the six functors preserve constructible objects. The devissage condition can be translated as follows:

(1') If $T$ is motivic, then for any scheme $X$, the category $T_c(X)$ agrees with the smallest thick triangulated subcategory which contains all projective motives;
(2') If $T$ is motivic and satisfies the condition (RS) in 2.1.12, then for any scheme $X$, the category $T_c(X)$ agrees with the smallest thick triangulated subcategory which contains all primitive Chow motives.

**Remark 2.1.16.**

(1) $S_k$-strong local acyclicity can be generalized to quasi-compact quasi-separated schemes over a field by a passing to the limit argument ([SGA4.5, Th. finitude Corollaire 2.16], [Hoy15, Appendix C]).

(2) In the derived category of étale sheaves, $S_k$-base change is a particular case of Deligne’s generic base change theorem ([SGA4.5, Th. finitude, Théorème 2.13]). This theorem is proved for $h$-motives in [Cis19] using a similar method.

(3) Under the assumption (RS), it follows from strong devissage that every object in $T_c(k)$ is (strongly) dualizable, which is a well-known result (see for example [Hoy15, Section 3] or [EK18, Theorem 3.2.1]).

The following notation will be used repeatedly in the study of Künneth formulas:
Notation 2.1.17. Let $S$ be a scheme and let $f_1 : X_1 \to Y_1$, $f_2 : X_2 \to Y_2$ be two $S$-morphisms. Denote by $p_i : X_1 \times_S X_2 \to X_i$, $p'_i : Y_1 \times_S Y_2 \to Y_i$ the projections, and $f_1 \times_S f_2 : X_1 \times_S X_2 \to Y_1 \times_S Y_2$ the fiber product. We have the following commutative diagram

$$
\begin{array}{ccc}
X_1 & \xrightarrow{p_1} & X_1 \times_S X_2 & \xrightarrow{p_2} & X_2 \\
\downarrow{f_1} & & \downarrow{f} & & \downarrow{f_2} \\
Y_1 & \xrightarrow{p'_1} & Y_1 \times_S Y_2 & \xrightarrow{p'_2} & Y_2.
\end{array}
$$

For $K_1 \in \mathbf{T}(X_1)$ and $K_2 \in \mathbf{T}(X_2)$, we denote

$$K_1 \otimes_S K_2 := p_1^* K_1 \otimes p_2^* K_2$$

which is an object of $\mathbf{T}(X_1 \times_S X_2)$.

2.1.18. The first Künneth formula is related to the functor $f_*$. For any morphism $f : X \to S$ and any $A, B \in \mathbf{T}(X)$, there is a canonical map

$$f_* A \otimes f_* B \to f_*(A \otimes B)$$

defined as the composition

$$f_* A \otimes f_* B \xrightarrow{ad(f^*, f_*)} f_* f^*(f_* A \otimes f_* B)$$

$$= f_*(f^* f_* A \otimes f^* f_* B) \xrightarrow{ad'(f^*, f_*) \cdot ad(f^*, f_*)} f_*(A \otimes B).$$

2.1.19. For $i = 1, 2$, let $L_i$ be an element of $\mathbf{T}(X_i)$. We have a canonical map

$$K\text{unn}_* : p_1^* f_1* L_1 \otimes p_2^* f_2* L_2 \to f_*(p_1^* L_1 \otimes p_2^* L_2)$$

defined as the composition

$$p_1^* f_1* L_1 \otimes p_2^* f_2* L_2 \xrightarrow{\text{Ex}(\Delta_1^*) \otimes \text{Ex}(\Delta_2^*)} f_*(p_1^* L_1 \otimes f_* p_2^* L_2) \xrightarrow{(2.1.18.1)} f_*(p_1^* L_1 \otimes p_2^* L_2).$$

By a classical argument (see [SGA5, III 1.6.4]), the following is a consequence of $S_k$-strong local acyclicity by Proposition 2.1.11:

Proposition 2.1.20 (Künneth formula for $f_*$). Let $\mathbf{T}$ be a motivic triangulated category. If $X_i \to S$ is universally strongly locally acyclic relatively to $L_i$ for $i = 1, 2$, then the map (2.1.19.1) is an isomorphism.

In particular, if $S = \text{Spec}(k)$ and if $\mathbf{T}$ satisfies the condition (RS) in 2.1.12, the map (2.1.19.1) is an isomorphism for any $L_i \in \mathbf{T}(X_i)$, $i = 1, 2$.

2.2. Künneth formula for $f_!$.

2.2.1. The second Künneth formula is concerned with the exceptional direct image functor. As part of the six functors formalism, for any morphism of schemes $f : X \to S$, there is an exceptional direct image functor (or direct image with compact support)

$$f_! : \mathbf{T}(X) \to \mathbf{T}(S)$$

which is compatible with compositions, such that $f_* = f_!$ if $f$ is proper. We also have

1. For any morphism $f : X \to S$, any object $K \in \mathbf{T}(X)$ and any $L \in \mathbf{T}(S)$, there is an invertible natural transformation

$$\text{Ex}(f^*, \otimes) : (f_* K) \otimes L \to f_!(K \otimes f^* L)$$

which agrees with the map (2.1.4.1) if $f$ is proper.
(2.2.1.3) All the maps involved are isomorphisms, and the result follows.

Lemma 2.2.3. We now state a Künneth formula for the functor $f_1$. We use the assumptions and notation as in Notation 2.1.17, with the following diagram

(2.2.2.1)

Ex. We use the assumptions and notation as in Notation 2.1.17, with the following diagram

(2.2.2.1) We now state a Künneth formula for the functor $f_1$. We use the assumptions and notation as in Notation 2.1.17, with the following diagram

Lemma 2.2.3. For $i = 1, 2$, let $L_i$ be an element of $T(X_i)$. There is a canonical isomorphism

(2.2.3.1) We now state a Künneth formula for the functor $f_1$. We use the assumptions and notation as in Notation 2.1.17, with the following diagram

Proof. We have the following commutative diagram

(2.2.3.2) and the following composition:

(2.2.3.3) All the maps involved are isomorphisms, and the result follows.

Remark 2.2.4. (1) The Künneth formula for $f_1$ is very formal and is valid over a general base $S$, while all the other ones are valid only when $S$ is the spectrum of a field.
(2) By definition, the map (2.2.3.1) agrees with the composition

\[
p_1^! f_1! L_1 \otimes p_2^! f_2! L_2 \xrightarrow{\mu_{\alpha}} (id_{Y_1} \times_S f_2)! (p_1^! f_1! L_1 \otimes p_2^! L_2) \xrightarrow{\mu_{\beta}} f_! (p_1^! L_1 \otimes p_2^! L_2).
\]

(2.2.4.1)

In other words, the map (2.2.3.1) is the composition of two maps of the same type where one of the \(f_i\)'s equals identity.

2.3. Künneth formula for \(f^!\).

2.3.1. The third Künneth formula is concerned with the exceptional inverse image functor. We recall the following facts from the six functors formalism:

1. For any morphism \(f : X \to S\), the functor \(f_!\) has a right adjoint, with the following pair of adjoint functors

\[
(2.3.1.1) \quad f_! : \mathcal{T}(X) \xrightarrow{\cong} \mathcal{T}(S) : f^!,
\]

such that \(f^! = f^*\) if \(f\) is étale.

2. For any closed immersion \(i\) with complementary open immersion \(j\), the functor \(i_*\) is conservative, and there is a canonical distinguished triangle

\[
(2.3.1.2) \quad i_* f^! \xrightarrow{\alpha_{i_*f^!}} 1 \xrightarrow{\alpha_d(j^*, j^*)} j_* j^* \xrightarrow{\phi_1} i_* f^![1].
\]

3. For any Cartesian square

\[
(2.3.1.3) \quad \begin{array}{ccc}
Y & \xrightarrow{g} & X \\
\downarrow{q} & & \downarrow{f} \\
T & \xrightarrow{p} & S
\end{array}
\]

which will be denoted as \(Y \times_X S \times T\) (this notation will be used in the proof of Proposition 2.3.5), there is a canonical invertible natural transformation

\[
(2.3.1.4) \quad Ex(\Delta^!_f) : q_* f^! \simeq g^! p_*.
\]

When \(p\) and \(q\) are proper, the map (2.3.1.4) agrees with the composition

\[
(2.3.1.5) \quad q_* f^! = q_* f^! \xrightarrow{\alpha_{d(f_!, f^!)}} f_! f_! q_* g^* \simeq f_! p_* g^* \xrightarrow{\alpha_{d(q^*, g^*)}} f^! p_1 = f^! p_*.
\]

4. (2.3.1.6) For any Cartesian square as (2.3.1.3), we deduce from the map \(Ex(\Delta^!_f)\) in (2.2.1.4) a canonical natural transformation

\[
(2.3.1.6) \quad Ex(\Delta^s_f) : g^* p^! \xrightarrow{\alpha_{d(q^*, g^*)}} q^* q^* g^* \xrightarrow{(Ex(\Delta^s_f))^{-1}} q^* f^* p^! \xrightarrow{\alpha_{d(p^!, p^*)}} q^! f^* s,
\]

which is an isomorphism when \(f\) is smooth.

**Lemma 2.3.2.** Take a Cartesian square \(\Delta\) as in (2.3.1.3). Assume that \(\mathcal{T}\) satisfies \(f'\)-base change for any morphism \(f'\) obtained from \(f\) by pull-back. Then the map \(Ex(\Delta^s_f)\) in (2.3.1.6) is an isomorphism.

**Proof.** By localizing \(p\) we may suppose \(p\) quasi-projective, and therefore we only need to deal with the cases where \(p\) is either a closed immersion or a smooth morphism. The smooth case follows from purity, and the case of a closed immersion follows from the localization sequence 2.3.1.2 and the base change property.

In particular, if \(\mathcal{T}\) is a motivic triangulated category which satisfies the condition (RS) in 2.1.12, by Proposition 2.1.11, Lemma 2.3.2 applies whenever \(f \in \mathcal{S}_k\) (see Definition 2.1.10).
2.3.3. We now state a Künneth formula for the functor $f^!$. We use the assumptions and notation as in Notation 2.1.17, with the following diagram

\[
\begin{array}{c}
X_1 \xrightarrow{p_1} X_1 \times_S X_2 \xrightarrow{p_2} X_2 \\
\downarrow f_1 \quad \quad \quad \downarrow f_2 \\
Y_1 \xleftarrow{p'_1} Y_1 \times_S Y_2 \xleftarrow{p'_2} Y_2.
\end{array}
\]

(2.3.3.1)

For $i = 1, 2$, let $M_i$ be an object of $\mathbf{T}(Y_i)$. Taking $L_i = f^!_i M_i$ in Lemma 2.2.3, we obtain a canonical map

\[
K u n^!_{(f_1, f_2)}(M_1, M_2) : p^*_1 f^!_1 M_1 \otimes p^*_2 f^!_2 M_2 \to f^!(p^*_1 M_1 \otimes p^*_2 M_2)
\]

by the composition

\[
p^*_1 f^!_1 M_1 \otimes p^*_2 f^!_2 M_2 \xrightarrow{\text{ad}_{(f'_1, f_2)}} f^!(p^*_1 f^!_1 M_1 \otimes p^*_2 f^!_2 M_2)
\]

(2.3.3.2)

We now establish the same principle as in Remark 2.2.4. Consider the following diagram

\[
\begin{array}{c}
X_1 \xrightarrow{p_1} X_1 \times_S X_2 \\
\downarrow f_1 \quad \downarrow f_2 \\
Y_1 \xrightarrow{p'_1} Y_1 \times_S Y_2 \xrightarrow{p'_2} Y_2
\end{array}
\]

(2.3.3.4)

As particular cases of the (2.3.3.2), we have the following maps:

\[
K u n^!_{(f_1, f_2)}(M_1, M_2) : p^*_1 f^!_1 M_1 \otimes p^*_2 f^!_2 M_2 \to (f_1 \times_S f_2)^!(p^*_1 M_1 \otimes p^*_2 M_2)
\]

(2.3.3.5)

\[
K u n^!_{(f_1, id_{X_2})}(M_1, f^!_2 M_2) : p^*_1 f^!_1 M_1 \otimes p^*_2 f^!_2 M_2 \to (f_1 \times_S id_{X_2})^!(p^*_1 M_1 \otimes p^*_2 f^!_2 M_2)
\]

(2.3.3.6)

Lemma 2.3.4. The map

\[
p^*_1 f^!_1 M_1 \otimes p^*_2 f^!_2 M_2 \xrightarrow{(2.3.3.6)} (f_1 \times_S id_{X_2})^!(p^*_1 M_1 \otimes p^*_2 f^!_2 M_2)
\]

(2.3.4.1)

\[
\xrightarrow{(2.3.3.5)} f^!(p^*_1 M_1 \otimes p^*_2 f^!_2 M_2)
\]

obtained from (2.3.3.5) and (2.3.3.6) agrees with the map (2.3.3.2).

Proof. By definition, the map (2.3.4.1) is the composition

\[
p^*_1 f^!_1 M_1 \otimes p^*_2 f^!_2 M_2 \xrightarrow{\text{ad}_{(f'_1, f_2)}} (f_1 \times f_2)^!(p^*_1 f^!_1 M_1 \otimes p^*_2 f^!_2 M_2)
\]

(2.3.4.2)

\[
\xrightarrow{(2.2.3.1)} (f_1 \times f_2)^!(p^*_1 f^!_1 M_1 \otimes p^*_2 f^!_2 M_2)
\]

\[
\xrightarrow{\text{ad}_{(f'_1, f_2) \otimes ad_{(f_1, f_2)}}} (f_1 \times f_2)^!(p^*_1 M_1 \otimes p^*_2 f^!_2 M_2)
\]

(2.3.4.3)

\[
\xrightarrow{(2.3.3.1)} f^!(p^*_1 M_1 \otimes p^*_2 M_2)
\]

(2.3.4.4)
The results then follows from Remark 2.2.4 and the naturality of the functors \( f \mapsto f' \) and \( f \mapsto f_! \). \( \square \)

**Proposition 2.3.5.** Suppose that \( T \) is a motivic triangulated category which satisfies the condition (RS) in 2.1.12. Then for \( S = \text{Spec}(k) \), the map (2.3.3.2) is an isomorphism.

**Proof.** By Lemma 2.3.4, it suffices to show that the maps (2.3.3.5) and (2.3.3.6) are isomorphisms. By symmetry, it suffices to show that (2.3.3.5) is an isomorphism. In other words we can assume that \( X_1 = Y_1 \) and \( f_1 = \text{id}_{X_1} \). By weak devissage we may suppose that \( M_1 = p_* 1_W \), where \( p : W \to Y_1 = X_1 \) is a proper morphism. The map (2.3.3.2) then becomes the following

\[
(2.3.5.1) \quad \text{Kun}^! : p_1^* p_* 1_W \otimes p_2^* f_2^! M_2 \to (\text{id}_{X_1} \times_k f_2)^!(p_1^* p_* 1_W \otimes p_2^* M_2).
\]

We have the following commutative diagram

\[
(2.3.5.2)
\]

and with the convention in 2.3.1 (3) we denote by

- \( \Delta_1 \) the Cartesian square formed by \( W \times_k Y_2 \times_k Y_1 \times_k Y_2 \times_k X_1 \times_k W \);
- \( \Delta_2 \) the Cartesian square formed by \( W \times_k Y_2 \times_k X_2 \times_k Y_1 \times_k X_2 \times_k W \);
- \( \Delta_3 \) the Cartesian square formed by \( W \times_k X_2 \times_k Y_2 \times_k Y_1 \times_k X_1 \times_k X_2 \times_k Y_2 \times_k W \times_k X_2 \times_k Y_2 \times_k X_1 \times_k X_2 \times_k Y_2 \times_k W \);
- \( \Delta_4 \) the Cartesian square formed by \( W \times_k Y_2 \times_k Y_2 \times_k Y_2 \times_k Y_1 \times_k X_1 \times_k X_2 \times_k Y_2 \times_k X_2 \times_k Y_2 \times_k W \);
- \( \Delta_5 \) the Cartesian square formed by \( X_1 \times_k X_2 \times_k X_1 \times_k X_2 \times_k Y_2 \times_k Y_2 \times_k X_2 \times_k X_2 \times_k Y_2 \times_k W \).

To show that the map (2.3.5.1) is an isomorphism, we transform it into other maps which are known to be isomorphisms: we have the following two composition maps, where all arrows are isomorphisms by Lemma 2.2.2:

\[
(2.3.5.3)
\]

\[
(2.3.5.4)
\]
using the fact that $p$ is proper and $p_{2W} : W \times X_2 \to X_2$ belongs to $\mathcal{S}_k$. Therefore we are reduced to show that the following diagram is commutative:

$$(id_{Y_1} \times k \ f_2)^! (p^!_{1*} p_* \mathbb{1}_W \otimes p_{2*}^! f_2^* M_2)$$

This follows from a diagram chase of the following form:

$$(id_{X_1} \times k \ f_2)^! (p_{1*}^! p_* \mathbb{1}_W \otimes p_{2*}^! f_2^* M_2)$$

The commutativity of the hexagon follows from the following general fact: for any Cartesian diagram of the form

$$(2.3.5.6)$$

and any object $M \in T(X)$, we have a commutative diagram

$$(2.3.5.7)$$

where the square $\Delta'$ is the transpose of the square $\Delta$, oriented as $Y \cdot T \cdot S \cdot X$. The rest of the diagram follows from standard adjunctions of functors and the fact that the maps of the form $Ex(\Delta'_1)$ are compatible with horizontal and vertical compositions of Cartesian squares, given that

- The square $\Delta_2$ is the composition of $\Delta_1$ and $\Delta'_3$;
- The square $\Delta_4$ is the composition of $\Delta_3$ and $\Delta_5$.

$\square$

2.4. Küneth formula for $\text{Hom}$. 

$^3$Alternatively, we could also use strong devissage and suppose that $p_{2W}$ is smooth.
2.4.1. The last Künneth formula is concerned with the internal Hom functor, the remaining one of the six functors. If \( T \) is motivic, its monoidal structure is closed, namely the tensor product on \( T \) has a right adjoint given by the internal Hom functor \( \text{Hom}(\cdot, \cdot) \), such that we have natural bijections

\[
\text{Hom}(A \otimes B, C) = \text{Hom}(A, \text{Hom}(B, C)).
\]

For any \( A \), we denote by \( \eta_A : B \to \text{Hom}(A, A \otimes B) \) the unit (or coevaluation) map, and \( \epsilon_A : A \otimes \text{Hom}(A, B) \to B \) the counit (or evaluation) map. We have the following exchange structures: for any morphism \( f : X \to Y \) and objects \( K \in T(X), L, M \in T(Y) \) we have the following natural isomorphisms:

\[
\begin{align*}
\text{Ex}(f_1, \text{Hom}) & : \text{Hom}(f_1 \cdot K, L) \cong f_1 \cdot \text{Hom}(K, f^1 L); \\
\text{Ex}(f^1, \text{Hom}) & : f^1 \cdot \text{Hom}(L, M) \cong \text{Hom}(f^* L, f^1 M).
\end{align*}
\]

We deduce the following isomorphism:

\[
\text{Hom}(f_1 \cdot X, L) \cong f_1 \cdot \text{Hom}(\cdot X, f^1 L).
\]

When \( f \) is proper, the isomorphism (2.4.1.4) fits into the two following commutative diagrams:

\[
\begin{array}{ccc}
\text{Hom}(f_1 \cdot X, L) & \xrightarrow{(2.4.1.4)} & \text{Hom}(f_1 \cdot X, L) \otimes f_1 \cdot X \\
\downarrow \text{ad}'_{(f_1, f^1)} & & \downarrow \epsilon_{f_1 \cdot X} \\
f_1 f^1 L & \xrightarrow{\text{ad}_{(f_1, f^1)}} & L
\end{array}
\]

\[
\begin{array}{ccc}
f_1 M & \xrightarrow{\text{ad}_{(f_1, f^1)}} & \text{Hom}(f_1 \cdot X, f_1 M \otimes f_1 \cdot X) \\
\downarrow \nabla_{f_1 \cdot X} & & \downarrow \epsilon_{(2.4.1.4)} \\
f_1 f^1 f_1 M & \xrightarrow{f_1 f^1 f_1} & f_1 f^1 f_1 (f_1 M \otimes f_1 \cdot X)
\end{array}
\]

where the two unlabeled horizontal maps are induced by the unit \( 1_Y \to f_1 \cdot X = f_1 \cdot X \).

2.4.2. Let \( Y_1 \) and \( Y_2 \) be two \( S \)-schemes. For \( i = 1, 2 \), we denote by \( p^i_1 : Y_1 \times_S Y_2 \to Y_i \) the canonical projection, and let \( M_i \) and \( N_i \) be objects of \( T(Y_i) \). We have a canonical map

\[
p^{1*} M_1 \otimes p^{1*}_1 \text{Hom}(M_1, N_1) \otimes p^{2*}_2 M_2 \otimes p^{2*}_2 \text{Hom}(M_2, N_2) \to p^{1*}_1 N_1 \otimes p^{2*}_2 N_2
\]

which induces a map

\[
p^{1*} \text{Hom}(M_1, N_1) \otimes p^{2*}_2 \text{Hom}(M_2, N_2) \to \text{Hom}(p^{1*}_1 M_1 \otimes p^{2*}_2 M_2, p^{1*}_1 N_1 \otimes p^{2*}_2 N_2).
\]

**Proposition 2.4.3.** Suppose that \( T \) is a motivic triangulated category which satisfies the condition \((RS)\) in 2.1.12 and \( M_1, M_2 \) are constructible. Then for \( S = \text{Spec}(k) \), the map (2.4.2.2) is an isomorphism.

**Proof.** By weak devissage we may suppose that \( M_i = f_i \cdot X_i \) where \( f_i : X_i \to Y_i \) is a proper morphism. We use the following notations:

\[
\begin{array}{ccc}
X_1 & \xrightarrow{p_1} & X_1 \times_k X_2 & \xrightarrow{p_2} & X_2 \\
\downarrow f_1 & & \downarrow f & & \downarrow f_2 \\
Y_1 & \xrightarrow{p'_1} & Y_1 \times_k Y_2 & \xrightarrow{p'_2} & Y_2.
\end{array}
\]

Then the map (2.4.2.2) becomes

\[
p^{1*}_1 \text{Hom}(f_1 \cdot X_1, N_1) \otimes p^{2*}_2 \text{Hom}(f_2 \cdot X_2, N_2) \to \text{Hom}(p^{1*}_1 f_1 \cdot X_1, p^{2*}_2 f_2 \cdot X_2, p^{1*}_1 N_1 \otimes p^{2*}_2 N_2).
\]
As in the proof of Proposition 2.3.5, we transform this map into other isomorphisms. By Lemma 2.2.3 and Proposition 2.3.5, we have the following two composition maps, where all arrows are isomorphisms:

\[ p_1^* \text{Hom}(f_1 \triangledown X_1, N_1) \otimes p_2^* \text{Hom}(f_2 \triangledown X_2, N_2) \stackrel{(2.4.1.4)}{=} p_1^* f_1^* f_1^! N_1 \otimes p_2^* f_2^* f_2^! N_2 \]

\[ f_*(p_1^* f_1^! N_1 \otimes p_2^* f_2^! N_2) \stackrel{(2.3.3.2)}{=} f_* f_! (p_1^* N_1 \otimes p_2^* N_2); \]

\[ \text{Hom}(p_1^* f_1 \triangledown X_1 \otimes p_2^* f_2 \triangledown X_2, p_1^* N_1 \otimes p_2^* N_2) \stackrel{(2.2.3.1)}{=} \text{Hom}(f_1 \triangledown X_1, X_2, p_1^* N_1 \otimes p_2^* N_2) \]

\[ f_* f_! (p_1^* N_1 \otimes p_2^* N_2). \]

Therefore we are reduced to show that the following diagram is commutative:

\[ p_1^* \text{Hom}(f_1 \triangledown X_1, N_1) \otimes p_2^* \text{Hom}(f_2 \triangledown X_2, N_2) \]

\[ \text{Hom}(p_1^* f_1 \triangledown X_1 \otimes p_2^* f_2 \triangledown X_2, p_1^* N_1 \otimes p_2^* N_2) \]

\[ f_* f_! (p_1^* N_1 \otimes p_2^* N_2). \]

By the same argument as in the proof of Proposition 2.3.5, we can assume that \( X_1 = Y_1 \) and \( f_1 = id_{X_1} \). The result then follows from a diagram chase, using the following general fact: for any Cartesian diagram of the form

\[ Y \xrightarrow{g} X \]

\[ q \downarrow \quad \Delta \quad \downarrow p \]

\[ T \xrightarrow{f} S \]

with \( f \) and \( g \) proper, and any objects \( A \in T(X), B \in T(S) \), there is a canonical isomorphism

\[ A \otimes p^* f_1 f_1^! B \xrightarrow{Ex(\Delta^!)} A \otimes g_* q^* f_1 f_1^! B = g_!(g^* A \otimes q^* f_1 f_1^! B) \]

and a commutative diagram of the form

\[ A \otimes p^* \text{Hom}(f_1 \triangledown T, B) \xrightarrow{\eta p^* f_1^{\triangledown} T} \text{Hom}(p^* f_1 \triangledown T, A \otimes p^* \text{Hom}(f_1 \triangledown T, B) \otimes p^* f_1 \triangledown T) \xrightarrow{\epsilon f_1^{\triangledown} T} \text{Hom}(p^* f_1 \triangledown T, A \otimes p^* B) \]

\[ A \otimes p^* f_1 f_1^! B \xrightarrow{\epsilon f_1^{\triangledown} T} \text{Hom}(g_1 \triangledown Y_1, A \otimes p^* \text{Hom}(f_1 \triangledown T, B) \otimes p^* f_1 \triangledown T) \xrightarrow{Ex(\Delta^!)} \text{Hom}(g_1 \triangledown Y_1, A \otimes p^* B) \]

\[ g_!(g^* A \otimes q^* f_1 f_1^! B) \xrightarrow{\epsilon f_1^{\triangledown} T} g_* g_! (A \otimes p^* \text{Hom}(f_1 \triangledown T, B) \otimes p^* f_1 \triangledown T) \xrightarrow{\epsilon f_1^{\triangledown} T} g_* g_! (A \otimes p^* B) \]

\[ g_1 g_! (g^* A \otimes q^* f_1 f_1^! B) \xrightarrow{\epsilon f_1^{\triangledown} T} g_* g_! (A \otimes p^* f_1 f_1^! B \otimes p^* f_1 \triangledown T) \xrightarrow{\epsilon f_1^{\triangledown} T} g_* g_! (A \otimes p^* f_1 f_1^! B) \]

using the diagrams (2.4.1.5) and (2.4.1.6).

\[ \square \]

**Remark 2.4.4.** When \( T = DM_h \) is the category of \( h \)-motives, Proposition 2.4.3 also holds when \( M_1 \) and \( M_2 \) are *locally constructible* ([CD16, Definition 6.3.1]). This is because to show that the map (2.4.2.2) is an isomorphism it suffices to check it étale locally, and for any étale morphism \( f : Y \to X \) and objects \( M, N \in T(X) \), the canonical map \( f^* \text{Hom}(M, N) \to \text{Hom}(f^* M, f^* N) \) is an isomorphism.
2.4.5. We end this section by summarizing the Künneth formulas we have obtained:

**Theorem 2.4.6.** Let $S$ be a scheme and let $f_1: X_1 \to Y_1$, $f_2: X_2 \to Y_2$ be two $S$-morphisms. Denote by $f: X_1 \times_S X_2 \to Y_1 \times_S Y_2$ be their product. Let $\mathcal{T}$ be a motivic triangulated category. For $i = 1, 2$, consider objects $L_i \in \mathcal{T}(X_i)$ and $M_i, N_i \in \mathcal{T}(Y_i)$. Then with the notations of 2.1.17 there are canonical maps

\[(2.4.6.1) \quad f_1, L_1 \otimes_S f_2, L_2 \to f_*(L_1 \otimes_S L_2)\]

\[(2.4.6.2) \quad f_1!, L_1 \otimes_S f_2!, L_2 \to f_!(L_1 \otimes_S L_2)\]

\[(2.4.6.3) \quad f_1^! M_1 \otimes_S f_2^! M_2 \to f^!(M_1 \otimes_S M_2)\]

\[(2.4.6.4) \quad \text{Hom}(M_1, N_1) \otimes_S \text{Hom}(M_2, N_2) \to \text{Hom}(M_1 \otimes_S M_2, N_1 \otimes_S N_2)\]

such that

1. The map (2.4.6.2) is an isomorphism.
2. If $\mathcal{T}$ satisfies the condition (RS) in 2.1.12 and $S = \text{Spec}(k)$, then the maps (2.4.6.1), (2.4.6.3) are isomorphisms.
3. If $\mathcal{T}$ satisfies the condition (RS) in 2.1.12, $S = \text{Spec}(k)$ and $M_1, M_2$ are constructible, then the map (2.4.6.4) is an isomorphism.

3. **The Verdier Pairing**

In this section, we define the Verdier pairing using the Künneth formulas in Section 2, following [SGA5, III]. This pairing is compatible with proper direct images, which implies the Lefschetz-Verdier formula. We assume that $\mathcal{T}_c$ is a motivic triangulated category of constructible objects which satisfies the condition (RS) in 2.1.12.

3.1. **The pairing.**

3.1.1. Let $X_1$ and $X_2$ be two schemes. We denote by $X_12 = X_1 \times_k X_2$ and $p_i: X_{12} \to X_i$ the projections. Let $L_i \in \mathcal{T}_c(X_i)$ and let $q_i: X_i \to \text{Spec}(k)$ be the structure map for $i = 1, 2$. We denote by $\mathbb{D}(L_i) = \text{Hom}(L_i, \mathcal{K}_{X_i}) = \text{Hom}(L_i, q_i^! \mathbb{1}_k)$. See [CD19, Section 4.4] for the detailed duality formalism related to this functor. By Proposition 2.3.5 and Proposition 2.4.3, we have the following isomorphism

\[(3.1.1.1) \quad p_1^! \mathbb{D}(L_1) \otimes p_2^* L_2 \simeq \text{Hom}(p_1^* L_1, p_2^! L_2),\]

given by the composition

\[(3.1.1.2) \quad p_1^! \mathbb{D}(L_1) \otimes p_2^* L_2 \simeq p_1^! \text{Hom}(L_1, \mathcal{K}_{X_1}) \otimes p_2^* L_2 \]

By symmetry, we also get an isomorphism

\[(3.1.1.3) \quad p_2^! \mathbb{D}(L_2) \otimes p_1^* L_1 \simeq \text{Hom}(p_2^* L_2, p_1^! L_1).\]

The particular case of (3.1.1.1) for $L_1 = \mathbb{1}_{X_1}$ and $L_2 = \mathcal{K}_{X_2}$ gives an isomorphism

\[(3.1.1.4) \quad p_1^! \mathcal{K}_{X_1} \otimes p_2^* \mathcal{K}_{X_2} \simeq \mathcal{K}_{X_{12}}.\]
3.1.2. Now let $c : C \to X_{12}$ and $d : D \to X_{12}$ be two morphisms, and form the following Cartesian square

$$
\begin{array}{ccc}
E & \xrightarrow{e} & D \\
\downarrow{d'} & \Delta & \downarrow{d} \\
C & \xrightarrow{c} & X_{12}
\end{array}
$$

(3.1.2.1)

Denote by $e : E \to X_{12}$ the canonical morphism. For any two objects $P, Q \in \mathcal{T}(X_{12})$, we have a canonical map

$$
d'^{\ast} c^{\ast} P \otimes c'^{\ast} d^{\ast} Q \xrightarrow{ad(c, c')} e'^{\ast} e^{\ast} (d'^{\ast} c^{\ast} P \otimes c'^{\ast} d^{\ast} Q)
$$

(3.1.2.2)

where we use Lemma 2.2.3 relative to the base scheme $X_{12}$ and the morphisms $c$ and $d$. Therefore we deduce a canonical map

$$
c^{\ast} c^{\ast} P \otimes d^{\ast} d^{\ast} Q \to e^{\ast} e' (P \otimes Q)
$$

(3.1.2.3)

which is given by the composition

$$
c^{\ast} c^{\ast} P \otimes d^{\ast} d^{\ast} Q \xrightarrow{ad(c^{\ast}, c^{\ast})} e^{\ast} e^{\ast} (c^{\ast} c^{\ast} P \otimes d^{\ast} d^{\ast} Q) = e^{\ast} (e^{\ast} c^{\ast} P \otimes e^{\ast} d^{\ast} d^{\ast} Q)
$$

(3.1.2.4)

3.1.3. For $i = 1, 2$ denote by $c_i = p_i \circ c : C \to X_i$, $d_i = p_i \circ d : D \to X_i$, and let $L_i \in \mathcal{T}_C(X_i)$. By 3.1.1 and the projection formula for $\mathcal{H}om$ we have an isomorphism

$$
\mathcal{H}om(c_i^{\ast} L_1, c_i^{\ast} L_2) \xrightarrow{\text{isom}} \mathcal{H}om(c^{\ast} p_1^{\ast} L_1, c^{\ast} p_2^{\ast} L_2)
$$

(3.1.3.1)

Then by (3.1.3.1), (3.1.3.2) and (3.1.2.3), we define a map

$$
c^{\ast} \mathcal{H}om(c_i^{\ast} L_1, c_i^{\ast} L_2) \otimes d^{\ast} \mathcal{H}om(d_i^{\ast} L_2, d_i^{\ast} L_1) \to e^{\ast} \mathcal{K}_E
$$

(3.1.3.3)

as the composition

$$
c^{\ast} \mathcal{H}om(c_i^{\ast} L_1, c_i^{\ast} L_2) \otimes d^{\ast} \mathcal{H}om(d_i^{\ast} L_2, d_i^{\ast} L_1) \xrightarrow{(3.1.2.3)} e^{\ast} e' (p_1^{\ast} \mathcal{D}(L_1) \otimes p_2^{\ast} \mathcal{D}(L_2) \otimes p_1^{\ast} L_1)
$$

(3.1.3.4)

$$
\xrightarrow{(3.1.3.4)} e^{\ast} e' (p_1^{\ast} \mathcal{K}_X \otimes p_2^{\ast} \mathcal{K}_X) \xrightarrow{(3.1.1.4)} e^{\ast} \mathcal{K}_{X_{12}} = e^{\ast} \mathcal{K}_E.
$$
3.1.4. We now construct a proper functoriality of the map (3.1.3.3). Let \( X'_1 \) and \( X'_2 \) be two schemes, and denote by \( X'_{12} = X'_1 \times_k X'_2 \). Let \( f_1 : X_1 \to X'_1 \) and \( f_2 : X_2 \to X'_2 \) be two morphisms, and denote by \( f = f_1 \times_k f_2 : X_{12} \to X'_{12} \) their product. For \( i = 1, 2 \), we denote by \( p'_i : X'_{12} \to X_i \) the projection, and let \( L_i \in \mathcal{T}_c(X_i) \). There is a canonical isomorphism

\[
(3.1.4.1) \quad f_* \text{Hom}(p'_1 L_1, p'_2 L_1) \cong \text{Hom}(p'_1 f_1 L_1, p'_2 f_2 L_2)
\]

given by the composition

\[
(3.1.4.2) \quad f_* \text{Hom}(p'_1 L_1, p'_2 L_1) \cong f_* (p'_1 \mathbb{D}(L_1) \otimes p'_2 L_2) \cong p'_1 f_1 \mathbb{D}(L_1) \otimes p'_2 f_2 L_2 \cong \text{Hom}(p'_1 f_1 L_1, p'_2 f_2 L_2)
\]

where we use the canonical duality isomorphism \( \mathbb{D}(f_1 L_1) \cong f_1 \mathbb{D}(L_1) \) ([CD19, Corollary 4.4.24]).

3.1.5. Now we consider another Cartesian square of schemes

\[
(3.1.5.1)
\]

\[
E' \longrightarrow D' \quad \text{such that there is a commutative cube}
\]

\[
(3.1.5.2)
\]

\[
C' \quad \text{such that there is a commutative cube}
\]

\[
(3.1.5.3)
\]

Assume that \( f_1, f_2, c, c', d, d' \) are proper. Consider the following commutative diagram

\[
(3.1.5.4) \quad f_* c_* c' = c'_* f_{C' \times X'_{12}} c'_* c'_* c'_* f_* c'_* c'_* c'_* f_* \cong c'_* c'_* f_*
\]

For \( i = 1, 2 \), denote by \( c'_i = p'_i \circ c' : C' \to X'_i \), \( d'_i = p'_i \circ d' : D' \to X'_i \). Then there is a canonical map

\[
(3.1.5.5) \quad f_* c_* \text{Hom}(c'_1 L_1, c'_2 L_2) \to c'_* \text{Hom}(c'_1 f_1 L_1, c'_2 f_2 L_2)
\]

given by the composition

\[
(3.1.5.6) \quad f_* c_* \text{Hom}(c'_1 L_1, c'_2 L_2) \cong f_* c_* c'_* \text{Hom}(p'_1 L_1, p'_2 L_2) \cong c'_* c'_* f_* \text{Hom}(p'_1 L_1, p'_2 L_2) \cong c'_* \text{Hom}(c'_1 f_1 L_1, c'_2 f_2 L_2).
\]
By symmetry we have a map

\[(3.1.5.7) \quad f_*d_*\mathcal{H}om(d_2^*L_2, d_1^*L_1) \to d_*\mathcal{H}om(d_2^*f_*L_2, d_1^*f_*L_1).\]

**Proposition 3.1.6.** The square

\[
\begin{array}{ccc}
  f_*c_*\mathcal{H}om(c_1^*L_1, c_2^*L_2) \otimes f_*d_*\mathcal{H}om(d_2^*L_2, d_1^*L_1) & \to & f_*c_*\mathcal{K}_E \\
\downarrow & & \downarrow \text{(3.1.3.3)} \\
  c_*\mathcal{H}om(c_1^*f_*L_1, c_2^*f_*L_2) \otimes d_*\mathcal{H}om(d_2^*f_*L_2, d_1^*f_*L_1) & \to & c_*\mathcal{K}_E,
\end{array}
\]

is commutative, where the left vertical map is given by the maps \((3.1.5.5)\) and \((3.1.5.7)\), and the upper horizontal row is the composition

\[
\text{(3.1.6.2)} \quad f_*c_*\mathcal{H}om(c_1^*L_1, c_2^*L_2) \otimes f_*d_*\mathcal{H}om(d_2^*L_2, d_1^*L_1) \xrightarrow{(2.1.18.1)} f_*(c_*\mathcal{H}om(c_1^*L_1, c_2^*L_2) \otimes d_*\mathcal{H}om(d_2^*L_2, d_1^*L_1)) \xrightarrow{(3.1.3.3)} c_*\mathcal{K}_E.
\]

The proof is identical to the one given in [SGA5, III 4.4], see also [YZ18, Theorem 3.3.2].

**Remark 3.1.7.** Following [DJK18, 4.2.1] (see also 6.1.5 below), there is a natural transformation \(f^* \to f^!\), given a lci morphism \(f\) together with an isomorphism \(Th_X(L_f) \simeq \mathbb{1}_X\), where the left hand side is the Thom space of the trivial virtual tangent bundle. \(^4\) For example étale morphisms satisfy this property, but the class of such morphisms is bigger. The map \((3.1.3.3)\) is also compatible with pull-backs along these morphisms, which we do not develop here.

**Definition 3.1.8.** With the notations of \((3.1.2)\), for two maps \(u : c_1^*L_1 \to c_2^*L_2\) and \(v : d_2^*L_2 \to d_1^*L_1\) which are called \((\text{cohomological})\) \(\text{correspondences}\), we define the **Verdier pairing**

\[
\langle u, v \rangle : \mathbb{1}_E \to \mathcal{K}_E
\]

obtained by adjunction from the composition

\[
\text{(3.1.8.2)} \quad \mathbb{1}_{X_{k2}} \xrightarrow{c_*\mathcal{C} \otimes d_*\mathcal{D}} c_*\mathcal{H}om(c_1^*L_1, c_2^*L_2) \otimes d_*\mathcal{H}om(d_2^*L_2, d_1^*L_1) \xrightarrow{(3.1.3.3)} c_*\mathcal{K}_E.
\]

**Remark 3.1.9.**

1. The map \(\langle u, v \rangle\) can be seen as an element of the \(\text{bivariant group} H_0(X/k)\), and the map \((3.1.10.3)\) below corresponds to the natural proper functoriality (see [DJK18]). We will come back to this point of view later in Section 5.

2. In the case where every scheme is equal to \(\text{Spec}(k)\), the assumptions imply that every object \(L \in \mathcal{T}_c(k)\) is dualizable and \(\mathbb{D}(L) = L^\vee\) is the dual object. In this case, given two maps \(u : L_1 \to L_2\), \(v : L_2 \to L_1\), the map \(\langle u, v \rangle\) is the composition

\[
\text{(3.1.9.1)} \quad \mathbb{1}_k \xrightarrow{\eta_1 \otimes L_1} L_1^\vee \otimes L_1 \otimes L_2^\vee \otimes L_2 \xrightarrow{\langle u \otimes 1 \otimes v \rangle} L_1^\vee \otimes L_2 \otimes L_2^\vee \otimes L_1 \xrightarrow{\epsilon_L \otimes L_2} \mathbb{1}_k.
\]

It is not hard to check that \(\langle u, v \rangle\) agrees with the trace of the composition \(v \circ u : L_1 \to L_1\). We will see a more general result in Proposition 3.2.5 below.

3. For \(\mathcal{T}_c = \mathbf{DM}_{\text{cdh}, c}\), we recover the construction in [Ols16, 5.8] as a particular case of the Verdier pairing.

---

\(^4\)In particular this condition means that \(f\) has relative dimension 0. If \(\mathcal{T}_c\) is oriented (i.e. endowed with a trivialization of Thom spaces of all vector bundles), then any lci morphism of relative dimension 0 satisfy this property.
3.1.10. Consider the setting of 3.1.4. We can define the proper direct image of correspondences as follows: given two correspondences \( u : c_1^*L_1 \to c_2^*L_2 \) and \( v : d_2^*L_2 \to d_1^*L_1 \), using the maps (3.1.11.5) and (3.1.1.7), we obtain the following maps

\[
\begin{align*}
(3.1.10.1) & \quad f_u : c_1^*f_1^*L_1 \to c_2^*f_2^*L_2, \\
(3.1.10.2) & \quad f_v : d_2^*f_2^*L_2 \to d_1^*f_1^*L_1
\end{align*}
\]

regarded as correspondences between \( f_1^*L_1 \) and \( f_2^*L_2 \). On the other hand, since the canonical morphism \( f_E : E \to E' \) is proper, for any map \( w : \Xi_E \to \mathcal{K}_E \) we define the proper direct image as

\[
(3.1.10.3) \quad f_{E*}w : \Xi_{E'} \to f_{E*}\Xi_E \xrightarrow{w} f_{E*}\mathcal{K}_E = f_{E*}f_{1*}^!\mathcal{K}_{E'} \xrightarrow{ad'(f_{E*},f_{E*}')} \mathcal{K}_{E'}.
\]

With the conventions above, the following is a consequence of Proposition 3.1.8.1:

**Corollary 3.1.11.** The Verdier pairing is compatible with proper direct images, i.e. we have

\[
(3.1.11.1) \quad \langle f_{E*}u, f_{E*}v \rangle = f_{E*}\langle u, v \rangle : \Xi_{E'} \to \mathcal{K}_{E'}.
\]

When \( X'_1 = X'_2 = C' = D' = \text{Spec}(k) \), the maps \( f_{E*}u : f_1^*L_1 \to f_2^*L_2 \) and \( f_{E*}v : f_2^*L_2 \to f_1^*L_1 \) are maps between dualizable objects in \( T_c(\text{Spec}(k)) \), and the map \( f_{E*}w \) is known as the degree map and is traditionally written as \( \mathcal{E}_w \). As a particular case of Corollary 3.1.11, we obtain the following Lefschetz-Verdier formula ([SGA5, III 4.7]):

**Corollary 3.1.12 (Lefschetz-Verdier formula).** When \( X'_1 = X'_3 = C' = D' = \text{Spec}(k) \), we have the following equality

\[
(3.1.12.1) \quad Tr(f_{E*}v \circ f_{E*}u) = \int_E \langle u, v \rangle
\]

as an endomorphism of \( \Xi_k \).

**Remark 3.1.13.** The formula in [Hoy15, Theorem 1.3] has a similar appearance, but is indeed of different nature.

3.2. Composition of correspondences and generalized traces. In this section, we study compositions of correspondences and show that the general Verdier pairing (3.1.8.1) can be reduced to a generalized trace map, which will be a key ingredient for the general form of additivity of traces in Section 4.

3.2.1. Let \( X_1, X_2 \) and \( X_3 \) be three schemes. Denote by \( X_{ij} = X_i \times_k X_j \times_k X_i \), and \( p_{ij}^i : X_{ij} \to X_i \) and \( p_i^j : X_i \to k \) the canonical projections, and we use similar notations for other schemes and morphisms.

For \( i \in \{1, 2, 3\} \), let \( L_i \) be an object of \( T_c(X_i) \). We have a canonical map

\[
(3.2.1.1) \quad p_{12}^{123*}\text{Hom}(p_1^{12*}L_1, p_2^{12l}L_2) \otimes p_{23}^{123*}\text{Hom}(p_2^{23*}L_2, p_3^{23l}L_3) \to p_{13}^{123*}\text{Hom}(p_1^{13*}L_1, p_3^{13l}L_3)
\]

given by the composition

\[
(3.2.1.2) \quad \begin{align*}
& \approx p_{12}^{123*}\text{Hom}(p_1^{12*}L_1, p_2^{12l}L_2) \otimes p_{23}^{123*}\text{Hom}(p_2^{23*}L_2, p_3^{23l}L_3) \\
& \approx p_1^{123*}\mathcal{D}(L_1) \otimes p_2^{123*}L_2 \otimes p_2^{123*}\mathcal{D}(L_2) \otimes p_3^{123*}L_3 \\
& \approx p_1^{123*}\mathcal{D}(L_1) \otimes p_2^{123*}K_{X_2} \otimes p_3^{123*}L_3 \approx p_2^{123*}K_{X_2} \otimes p_1^{123*}\mathcal{D}(L_1) \otimes p_3^{123*}L_3 \\
& \approx p_1^{123*}L_1 \otimes p_3^{13l}L_3 \approx p_1^{123*}\text{Hom}(p_1^{13*}L_1, p_3^{13l}L_3).
\end{align*}
\]
3.2.2. Now consider two morphisms \( c_{12} : C_{12} \to X_{12} \) and \( c_{23} : C_{23} \to X_{23} \). Let \( C_{13} = C_{12} \times_{X_{12}} C_{23} \) together with a canonical morphism \( c_{13}^{13} : C_{13} \to X_{123} \). We denote by \( c_{13} = p_{13}^{12} \circ c_{12}^{13} : C_{13} \to X_{13} \), and \( c_{ij}^{ij} = p_{ij}^{ij} \circ c_{ij} : C_{ij} \to X_{i} \), etc. Consider the following diagram

\[
\begin{array}{c}
\begin{array}{ccc}
C_{13} & \xrightarrow{q_3} & C_{23} \\
\downarrow c_{13}^{13} & & \uparrow c_{23} \\
C_{12} & \xrightarrow{q_1} & X_{12} \\
\end{array}
\end{array}
\begin{array}{ccc}
\begin{array}{ccc}
q_1 & \downarrow c_{12} & \rightarrow \ \\
\downarrow c_{12}^{12} & & \downarrow c_{23}^{12} \\
q_2 & \downarrow c_{12} & \rightarrow \ \\
\downarrow c_{23}^{12} & & \downarrow c_{23}^{23} \\
q_3 & \downarrow c_{23} & \rightarrow \ \\
\end{array}
\end{array}
\begin{array}{ccc}
\begin{array}{ccc}
C_{13} & \xrightarrow{c_{13}} & C_{23} \\
\downarrow p_{13}^{12} & & \uparrow p_{23}^{23} \\
X_{12} & \xrightarrow{p_{12}^{12}} & X_{23} \\
\end{array}
\end{array}
\]

where all the four squares are Cartesian. For any objects \( P \in \mathcal{C}(X_{12}) \) and \( Q \in \mathcal{T}_c(X_{23}) \), we have a canonical map

\[
q_1^* c_{12}^{1} P \otimes q_3^* c_{23}^{1} Q = q_1^* c_{12}^{1} P \otimes q_3^* c_{23}^{1} Q
\]

Therefore we deduce a map

\[
q_1^* \text{Hom}(c_{12}^{12} L_1, c_{22}^{12} L_2) \otimes q_3^* \text{Hom}(c_{23}^{23} L_2, c_{33}^{23} L_3) \to \text{Hom}(c_{13}^{13} L_1, c_{33}^{13} L_3)
\]

given by the composition

\[
q_1^* \text{Hom}(c_{12}^{12} L_1, c_{22}^{12} L_2) \otimes q_3^* \text{Hom}(c_{23}^{23} L_2, c_{33}^{23} L_3)
\]

\[
\overset{\text{2.4.1.3}}{\simeq} q_1^* c_{12}^{1} \text{Hom}(p_{12}^{12} L_1, p_{22}^{12} L_2) \otimes q_3^* c_{23}^{1} \text{Hom}(p_{23}^{23} L_2, p_{33}^{23} L_3)
\]

\[
\overset{\text{2.3.2.2}}{\to} c_{13}^{13} \text{Hom}(p_{12}^{12} L_1, p_{22}^{12} L_2) \otimes p_{23}^{23} \text{Hom}(p_{23}^{23} L_2, p_{33}^{23} L_3)
\]

\[
\overset{\text{2.3.1.1}}{\to} c_{13}^{13} \text{Hom}(p_{13}^{13} L_1, p_{33}^{13} L_3)
\]

\[
\overset{\text{2.4.1.3}}{\simeq} \text{Hom}(c_{13}^{13} L_1, c_{33}^{13} L_3).
\]

**Definition 3.2.3.** Given two correspondences \( u : c_{12}^{1} L_1 \to c_{22}^{12} L_2 \) and \( v : c_{23}^{23} L_2 \to c_{33}^{23} L_3 \), we deduce from the map (3.2.2.3) a correspondence from \( L_1 \) to \( L_3 \)

\[
u u : c_{13}^{13} L_1 \to c_{33}^{13} L_3
\]

which we call the composition of the correspondences \( u \) and \( v \).

3.2.4. Now assume that \( X_1 = X_3 = X_1 \) and \( L_1 = L_3 \). Consider two morphisms \( c : C \to X_{12} \), \( d : D \to X_{21} \). For \( i \in \{1, 2\} \), we denote by \( c_i = p_{i}^{21} \circ c : C \to X_i \) and \( d_i = p_{i}^{21} \circ d : D \to X_i \) the canonical maps. Let \( F = C \times X_2 \) and \( E = C \times X_{12} \), then there is a Cartesian diagram of the form

\[
\begin{array}{ccc}
E & \xrightarrow{f'} & X_{12} \\
\downarrow & & \downarrow p_{12}^{12} \\
F & \xrightarrow{f} & X_{121}
\end{array}
\]

and \( E \) is canonically identified with the fiber product \( F \times X_{11} \) via the diagonal map \( \delta : X_1 \to X_{11} \). As in 3.2.2 we denote by \( q_1 : F \to C \) and \( q_3 : F \to D \) the canonical maps, and \( f_1, f_3 \) the compositions

\[
f_1 : F \xrightarrow{q_1} C \xrightarrow{c} X_{12} \xrightarrow{p_{12}^{12}} X_1
\]

\[
f_3 : F \xrightarrow{q_3} D \xrightarrow{d} X_{21} \xrightarrow{p_{21}^{21}} X_1.
\]
Suppose that $u : c_1^* L_1 \rightarrow c_2^* L_2$ and $v : d_2^* L_2 \rightarrow d_1^* L_1$ are two correspondences. By Definition 3.2.3, there is a composition $vu : f_1^* L_1 \rightarrow f_2^* L_1$. The following is stated in [SGA5, III (5.2.10)] without proof:

**Proposition 3.2.5.** The Verdier pairing (3.1.8.1) satisfies the following equality

\[
\langle u, v \rangle = \langle vu, 1 \rangle : \mathbb{I}_E \rightarrow \mathcal{K}_E
\]

where 1 is the identity correspondence $id : \mathbb{I}_{X_1} \rightarrow \mathbb{I}_{X_1}$.

**Proof.** Denote by $p_{11}^1, p_{31}^1 : X_{11} \rightarrow X_1$ the projections to the first and the second summands. We have a canonical map

\[
\mathbb{I}_{X_1} \xrightarrow{\eta_{L_1}} \text{Hom}(L_1, L_1) \cong \delta^* \text{Hom}(p_{31}^{11} L, p_{11}^{11} L) \cong \delta^*(p_{31}^{11} \mathbb{D}(L) \otimes p_{11}^{11} L),
\]

from which we deduce a canonical map

\[
\delta^* f^* f_\ast \text{Hom}(f_1^* L_1, f_3^* L_1) \rightarrow f^\prime \delta^* \text{Hom}(L_1) \otimes L_1
\]

given by the composition

\[
\delta^* f^* f_\ast \text{Hom}(f_1^* L_1, f_3^* L_1) \cong \delta^* f^* f_\ast f^! \delta^! f \text{Hom}(p_{31}^{11} L, p_{11}^{11} L) \cong \delta^! \delta^* f^* f_\ast f^! \text{Hom}(p_{31}^{11} L, p_{11}^{11} L) \cong \delta^! \delta^* f^* f_\ast f^! \text{Hom}(L_1) \otimes p_{31}^{11} L)
\]

\[
\delta^* f^* f_\ast f^! \text{Hom}(L_1) \otimes p_{31}^{11} L \otimes p_{11}^{11} L \cong \delta^* f^* f_\ast f^! \text{Hom}(L_1) \otimes p_{31}^{11} L \otimes p_{11}^{11} L)
\]

\[
\delta^* f^* f_\ast f^! \text{Hom}(L_1) \otimes p_{31}^{11} L \otimes p_{11}^{11} L \cong \delta^* f^* f_\ast f^! \text{Hom}(L_1) \otimes p_{31}^{11} L \otimes p_{11}^{11} L)
\]

\[
\delta^* f^* f_\ast f^! \text{Hom}(L_1) \otimes p_{31}^{11} L \otimes p_{11}^{11} L \cong \delta^* f^* f_\ast f^! \text{Hom}(L_1) \otimes p_{31}^{11} L \otimes p_{11}^{11} L)
\]

Note that we have natural transformations $f^* c_\ast \rightarrow f^* f_\ast q_1^\ast$ and $f^* d_\ast \rightarrow f^* f_\ast q_3^\ast$. We want to show that both the maps $\langle u, v \rangle$ and $\langle vu, 1 \rangle$ are equal to the following composition

\[
\mathbb{I}_E \rightarrow \delta^* f^* f_\ast \mathbb{I}_F \xrightarrow{\mathbb{I}_F} \delta^* f^* f_\ast \text{Hom}(f_1^* L_1, f_3^* L_1) \rightarrow f^\prime \delta^* \text{Hom}(\mathbb{D}(L_1) \otimes L_1) \rightarrow f^\prime \delta^* \mathcal{K}_{X_1} = \mathcal{K}_E.
\]

This follows from the commutativity of the following diagram

\[
\begin{array}{ccc}
\mathbb{I}_F & \xrightarrow{f^*} & \mathbb{I}_F \\
\delta^* f^! p_{31}^{11} & \xrightarrow{f^*} & \delta^* f^! p_{31}^{11} \\
\mathbb{I}_F & \xrightarrow{f^*} & \mathbb{I}_F \\
\delta^* f^* f_\ast f^! & \xrightarrow{f^*} & \delta^* f^* f_\ast f^!
\end{array}
\]

where each subdiagram follows either from definition or from a straightforward check. \qed

**Remark 3.2.6.** Proposition 3.2.5 says that the Verdier pairing (3.1.8.1) can always be reduced to the case where $X_1 = X_2$, $L_1 = L_2$ and one of the correspondences is the identity. As such we reduce the pairing with two entries to a generalized trace map, therefore making it much easier to deal with additivity along distinguished triangles.
3.2.7. We now give a more explicit description of the map \( \langle u, 1 \rangle \) (see [AS07, Proposition 2.1.7])\(^5\). Let \( X \) be a scheme and \( c : C \to X \times_k X \) be a morphism. We use the notation in 3.1.2, with \( D = X \), \( d = \delta : X \to X \times_k X \) the diagonal morphism and the Cartesian diagram

\[
\begin{array}{ccc}
E & \xrightarrow{c'} & X \\
\delta' \downarrow & & \delta \downarrow \\
C & \xrightarrow{c} & X \times_k X.
\end{array}
\]

(3.2.7.1)

**Proposition 3.2.8.** Let \( L \in \mathbf{T}_c(X) \) and \( u : c^*_1L \to c^*_2L \) be a correspondence. Denote by \( 1 = id_L : L \to L \) the identity correspondence, and by \( u' \) the following map:

\[
\mathbb{1}_C \xrightarrow{\eta_{c^*_1L}} \text{Hom}(c^*_1L, c^*_1L) \xrightarrow{u_*} \text{Hom}(c^*_1L, c^*_2L) \xrightarrow{(2.4.1.3)} \text{Hom}(p^*_1L, p^*_2L) \xrightarrow{(3.1.1.1)} c'(p^*_1L \otimes p^*_1L).
\]

Then the map \( \langle u, 1 \rangle : \mathbb{1}_E \to \mathcal{K}_E \) is obtained by adjunction from the map

\[
c'_E \mathbb{1}_E \xrightarrow{(Ex(\Delta'))^{-1}} \delta^* c_1 \mathbb{1}_C \xrightarrow{u'} \delta^* c_1 c'(p^*_1L \otimes p^*_2L) \xrightarrow{ad'_{c_1,c'}} \delta^* (p^*_1L \otimes p^*_2L) = \mathbb{D}(L) \otimes L \simeq L \otimes \mathbb{D}(L) \xrightarrow{\epsilon_L} \mathcal{K}_X.
\]

**Proof.** Similarly to the map (3.2.5.2), we denote by \( \eta' \) the following map

\[
\mathbb{1}_X \xrightarrow{\eta_L} \text{Hom}(L, L) \xrightarrow{(2.4.1.3)} \delta^* \text{Hom}(p^*_1L, p^*_1L) \xrightarrow{(3.1.1.1)} \delta^* (p^*_1L \otimes p^*_1L).
\]

We are reduced to show the commutativity of the following diagram:

\[
\begin{array}{ccc}
\mathbb{1}_E & \xrightarrow{ad'_{c_1,c'}} & c'_E \mathbb{1}_E \\
\delta^* c'(p^*_1L \otimes p^*_2L) & \xrightarrow{(2.3.1.6)} & c^\delta c'_E \mathbb{1}_C \\
\delta^* c'(p^*_1L \otimes p^*_2L) & \xrightarrow{(3.1.2.2)} & c^\delta (p^*_1L \otimes p^*_2L) \otimes c^\delta (p^*_1L \otimes p^*_2L) \xrightarrow{(6.11.1)} c^\delta (\mathbb{D}(L) \otimes L) \xrightarrow{(2.3.1.6)} (\mathbb{D}(L) \otimes L) \otimes (\mathbb{D}(L) \otimes L) \\
\epsilon_L \otimes \epsilon_L & \xrightarrow{c^\delta K_X \otimes c^\delta K_X} & c^\delta (\mathbb{D}(L) \otimes L) \xrightarrow{\epsilon_L} \mathbb{D}(L) \otimes \mathbb{D}(L) \\
\end{array}
\]

\(^5\)The statement in loc. cit. holds for \( c \) a closed immersion, but our modified version holds in general.
We show the commutativity of the octagon, while the rest of the diagram follows from a straightforward check. We are reduced to the following diagram:

\[
\begin{align*}
\delta^*(p_1^* D(L) \otimes p_2^* L) 
& \xrightarrow{\eta'} \delta^*(p_1^* D(L) \otimes p_2^* L) \otimes \delta^*(p_2^* D(L) \otimes p_1^* L) \\
& \xrightarrow{\delta \delta^*} \delta^* (p_1^* D(L) \otimes p_2^* L) \otimes \delta^* (p_1^* D(L) \otimes p_2^* L) \\
& \xrightarrow{\sim} \delta^* (\delta (L \otimes D(L)))
\end{align*}
\]

where \( p : X \to \text{Spec}(k) \) and \( q : X \times_k X \to \text{Spec}(k) \) are structural morphisms. We are reduced to the commutativity of the pentagon, namely the following diagram:

\[
\begin{align*}
\delta^* (p_1^* D(L) \otimes p_2^* L) 
& \xrightarrow{\sim} \delta (L \otimes D(L)) \\
& \xrightarrow{a_{(\delta, \delta')}} \delta^* (p_1^* D(L) \otimes p_2^* L) \otimes \delta^* (p_2^* D(L) \otimes p_1^* L) \\
& \xrightarrow{\sim} \delta^* (\delta (L \otimes D(L)))
\end{align*}
\]

By considering the hexagon in the diagram above, we are reduced to the following diagram:

\[
\begin{align*}
q_! (p_1^* D(L) \otimes p_2^* L \otimes \delta_1 1_X) 
& \xrightarrow{\sim} q_! \delta (L \otimes D(L)) \\
& \xrightarrow{a_{(\delta, \delta')}} q_! (p_1^* D(L) \otimes p_2^* L \otimes \delta^* (p_2^* D(L) \otimes p_1^* L)) \\
& \xrightarrow{\sim} q_! (p_1^* D(L) \otimes p_2^* L \otimes \delta^* (p_1^* D(L) \otimes p_2^* L)) \\
& \xrightarrow{\sim} q_! (p_1^* (L \otimes D(L)) \otimes p_2^* (L \otimes D(L))) \\
& \xrightarrow{\sim} q_! (p_1^* (L \otimes D(L)) \otimes p_2^* K_L)
\end{align*}
\]

The right part is a straightforward check, and the left part is reduced to the following diagram:

\[
\begin{align*}
\delta_1^* \otimes \delta_1^* \otimes 1_X 
& \xrightarrow{\sim} \delta_1^* \otimes \delta_1^* \otimes 1_X \\
& \xrightarrow{\delta_1^* \otimes \delta_1^*} \delta_1^* \otimes \delta_1^* \otimes 1_X \\
& \xrightarrow{\sim} \delta_1^* \otimes \delta_1^* \otimes 1_X \\
& \xrightarrow{\sim} \delta_1^* \otimes \delta_1^* \otimes 1_X \\
& \xrightarrow{\sim} \delta_1^* \otimes \delta_1^* \otimes 1_X
\end{align*}
\]

which commutes since the composition

\[
(3.2.48) \quad \delta_1 L = \delta_1 \delta^* p_2^* L \to \delta_1 (\delta^* p_2^* L \otimes \Hom(\delta^* p_2^* L, \delta^* p_1^* L)) \to \delta_1 \delta^* p_1^* L = \delta_1 L
\]

is the identity map. \( \square \)

**Remark 3.2.9.** In the special case where \( c = \delta : X \to X \times_k X \) is the diagonal map, the K"unneth formulas indeed produce a map

\[
(3.2.9.1) \quad \id_X = \delta^* \delta_1 \id_X \to \delta^* (p_2^* D(L) \otimes p_1^* L) = D(L) \otimes L.
\]
Together with the map $\epsilon_L : L \otimes D(L) \to K_X$, they can be seen as the counit and unit maps of a duality formalism similar to the usual (strong) duality, where the usual dualizing functor is replaced by the local duality functor $D$, which gives rise to trace maps without requiring $L$ to be strongly dualizable; the general Verdier pairing is a more general form of the trace map in such a duality formalism. In Section 4 we will combine this point of view with the approach in [May01] in the study of additivity of traces. In particular, as mentioned in the introduction, the characteristic class of $L$ is the composition

\[(3.2.9.2) \quad 1_X \xrightarrow{(3.2.9.1)} D(L) \otimes L = L \otimes D(L) \xrightarrow{\epsilon_L} K_X\]

which will be studied in more details in Section 5.

4. Additivity of the Verdier Pairing

In this section we prove the additivity of the Verdier pairing following [May01] and [GPS14], using the language of derivators ([Ayo07], [GPS14]).

4.1. May’s axioms in stable derivators. In this section we recall the notion of closed symmetric monoidal stable derivators and obtain May’s axioms following [GPS14]; the statements we need are slightly different from loc.cit. and can be obtained with very minor changes from the original proof. Since we are mostly interested in constructible motives, we only consider finite diagrams for convenience.

**Notation 4.1.1.** We denote by $\text{FinCat}$ the 2-category of finite categories, $\text{CAT}$ the 2-category of categories and $\text{TR}$ the subcategory of $\text{CAT}$ of triangulated categories and triangulated functors.

We denote by $\emptyset$ the empty category, $\mathbf{0}$ the terminal category and $\mathbf{1} = (0 \to 1)$ the category with two objects and one non-identity morphism between them. We denote by $\square$ the category $\mathbf{1} \times \mathbf{1}$ written as

\[(4.1.1.1) \quad (0,0) \to (0,1) \quad \downarrow \quad \downarrow \quad (1,0) \to (1,1) .\]

We denote by $\sqcap$ and respectively $\sqcup$ the full subcategories $\square \setminus \{(1,1)\}$ and $\square \setminus \{(0,0)\}$, with $i_r : \sqcap \to \square$ and $i_\sqcup : \sqcup \to \square$ the inclusions.

**Definition 4.1.2.** A (strong) stable derivator is a (non-strict) 2-functor $\mathcal{T}_c : \text{FinCat}^{op} \to \text{CAT}$ satisfying the following properties:

1. $\mathcal{T}_c$ sends coproducts to products. In particular $\mathcal{T}_c(\emptyset) = \mathbf{0}$.
2. For any $I, J \in \text{FinCat}$, the canonical functor $\mathcal{T}_c(I \times J) \to \text{Fun}(I^{op}, \mathcal{T}_c(J))$ is conservative.
3. For any $A \in \text{FinCat}$, the canonical functor $\mathcal{T}_c(A \times \mathbf{1}) \to \text{Fun}(\mathbf{1}, \mathcal{T}_c(A))$ is full and essentially surjective.
4. For every functor $u : A \to B$ in $\text{FinCat}$, the functor $u^* : \mathcal{T}_c(B) \to \mathcal{T}_c(A)$ has a right adjoint $u_*$ and a left adjoint $u_\#$.
5. For any functor $u : A \to B$ and object $b$ of $B$, denote by $j_{A/b} : A/b \to A$, $j_{bnA} : bnA \to A$, $p_{A/b} : A/b \to \emptyset$ and $p_{bnA} : bnA \to \emptyset$ the canonical projections. Then the following canonical transformations are invertible:

\[b^* u_* \Rightarrow p_{bnA} p_{bnA}^* b^* u_* \Rightarrow p_{bnA} j_{A/b} j_{A/b}^* b^* u_* \Rightarrow p_{bnA} j_{A/b} j_{A/b}^* u_* \Rightarrow p_{bnA} j_{A/b} j_{A/b}^* u_* \Rightarrow b^* u_\# .\]

\[p_{bnA} j_{bnA} \Rightarrow p_{bnA} j_{bnA} j_{bnA}^* u_* u_\# \Rightarrow p_{bnA} p_{bnA}^* b^* u_\# \Rightarrow b^* u_\# .\]

6. For any $I \in \text{FinCat}$ the category $\mathcal{T}_c(I)$ has a zero object, i.e. an object which is both initial and terminal.
7. For any $I \in \text{FinCat}$ and $X$ an object of $\mathcal{T}_c(\square \times I)$, $X$ is Cartesian (i.e. the canonical map $X \to i_r i_\sqcup X$ is invertible) if and only if $X$ is coCartesian (i.e. the canonical map $i_r i_\sqcup^* X \to X$ is invertible). We also say that $X$ is biCartesian.
An object in $\mathcal{T}_c(A)$ is called an (A-shaped) coherent diagram, who has an underlying incoherent diagram in $\text{Func}(A, \mathcal{T}_c(\mathcal{U}))$.

The cofiber functor $\text{cof}: \mathcal{T}_c(\mathbf{1}) \to \mathcal{T}_c(\mathbf{1})$ is the composition

$$\mathcal{T}_c(\mathbf{1}) \xrightarrow{(0,1)_*} \mathcal{T}_c(\mathbf{1}) \xrightarrow{(1,1)_*} \mathcal{T}_c(\mathbf{1}).$$

If $\mathcal{T}_c$ is a stable derivator, for any $I \in \text{FinCat}$ we define a functor $\Sigma: \mathcal{T}_c(I) \to \mathcal{T}_c(I)$ by setting $\Sigma = (1,1)^*(i_r)_#(i_r)^*(0,0)_*$. For a biCartesian $X \in \mathcal{T}_c(\boxtimes I)$ depicted as

$$x \xrightarrow{f} y \xrightarrow{g} z$$

with $x, y, z \in \mathcal{T}_c(I)$, we define a canonical map $z \xrightarrow{h} \Sigma x$ as follows:

$$h: z = (1,1)^*X \xrightarrow{\sim} (1,1)^*(i_r)_#(i_r)^*X \to (1,1)(i_r)_#(i_r)^*(0,0)_*X = \Sigma X.$$

Then the category $\mathcal{T}_c(I)$ has the structure of a triangulated category by considering $\Sigma$ as the shift functor and letting distinguished triangles to be the ones isomorphic to a triangle of the form

$$x \xrightarrow{f} y \xrightarrow{g} z \xrightarrow{h} \Sigma x.$$

Therefore we can also see stable derivators as 2-functors $\mathcal{T}_c: \text{FinCat}^{op} \to TR$.

**Notation 4.1.3.** We denote by $\text{SMTR}$ the 2-category of symmetric monoidal triangulated categories with (strong) monoidal functors.

As in [GPS14], if $\otimes: \mathcal{T}_{c,1} \times \mathcal{T}_{c,2} \to \mathcal{T}_{c,3}$ is a two-variable morphism of stable derivators, it induces an internal product denoted as

$$\otimes_A: \mathcal{T}_{c,1}(A) \times \mathcal{T}_{c,2}(A) \to \mathcal{T}_{c,3}(A).$$

We define the external product $\otimes: \mathcal{T}_{c,1}(A) \times \mathcal{T}_{c,2}(B) \to \mathcal{T}_{c,3}(A \times B)$ as the composition

$$\mathcal{T}_{c,1}(A) \times \mathcal{T}_{c,2}(B) \xrightarrow{\pi_A \times \pi_B} \mathcal{T}_{c,1}(A \times B) \times \mathcal{T}_{c,2}(A \times B) \xrightarrow{\otimes_A \otimes_B} \mathcal{T}_{c,3}(A \times B).$$

where $\pi_A: A \times B \to A$ and $\pi_B: A \times B \to B$ are the canonical projections.

**Definition 4.1.4.** A symmetric monoidal stable derivator is a 2-functor $(\mathcal{T}_c, \otimes): \text{FinCat}^{op} \to \text{SMTR}$ such that

1. The composition $\text{FinCat}^{op} \xrightarrow{\mathcal{T}_c} \text{SMTR} \to TR$ is a stable derivator.
2. For any $A, B, C \in \text{FinCat}$, $u: A \to B$, $X \in \mathcal{T}_c(A)$, $Y \in \mathcal{T}_c(C)$, the following canonical map of external products is an isomorphism:

$$\begin{align*}
(u \times 1)_#(X \otimes Y) &\xrightarrow{\sim} (u \times 1)^*(u^*u_#X \otimes Y) \\
&\xrightarrow{\sim} (u \times 1)^*(u^*u_#X \otimes Y) \to u_#^*X \otimes Y.
\end{align*}$$

It is closed if the functor $\otimes$ has a right adjoint $\text{Hom}(\cdot, \cdot)$.

The following is [GPS14, Corollary 4.5]:

**Proposition 4.1.5** (TC1). Let $\mathcal{T}_c$ be a symmetric monoidal stable derivator and denote by $\mathbb{1}$ the unit in $\mathcal{T}_c(\mathcal{U})$. Then for any object $x$ in $\mathcal{T}_c(\mathcal{U})$ there is a natural equivalence $\alpha: \Sigma x \simeq x \otimes \Sigma \mathbb{1}$ such that the composition

$$\Sigma \Sigma \mathbb{1} \xrightarrow{\alpha} \Sigma \mathbb{1} \otimes \Sigma \mathbb{1} \xrightarrow{s} \Sigma \mathbb{1} \otimes \Sigma \mathbb{1} \xrightarrow{\alpha^{-1}} \Sigma \Sigma \mathbb{1}$$

is the multiplication by $-1$, where $s$ is the isomorphism that exchanges the two summands.
The following is [GPS14, Theorems 4.8 and 9.12]:

**Proposition 4.1.6 (TC2).** Let $\mathcal{T}_c$ be a closed symmetric monoidal stable derivator. Then for any distinguished triangle

\[(4.1.6.1)\]

\[x \overset{f}{\to} y \overset{g}{\to} z \overset{h}{\to} \Sigma x\]

in $\mathcal{T}_c(\mathcal{O})$ and any $t \in \mathcal{T}_c(\mathcal{O})$, the following triangles are distinguished:

\[(4.1.6.2)\]

\[x \otimes t \overset{f \otimes 1}{\to} y \otimes t \overset{g \otimes 1}{\to} z \otimes t \overset{h \otimes 1}{\to} \Sigma(x \otimes t).\]

\[(4.1.6.3)\]

\[t \otimes x \overset{1 \otimes f}{\to} t \otimes y \overset{1 \otimes g}{\to} t \otimes z \overset{1 \otimes h}{\to} \Sigma(t \otimes x).\]

\[(4.1.6.4)\]

\[\Sigma^{-1} \mathbb{H}om(x,t) \xrightarrow{\mathbb{H}om(h,t)} \mathbb{H}om(z,t) \xrightarrow{\mathbb{H}om(g,t)} \mathbb{H}om(y,t) \xrightarrow{\mathbb{H}om(f,t)} \mathbb{H}om(x,t).\]

\[(4.1.6.5)\]

\[\mathbb{H}om(t,x) \xrightarrow{\mathbb{H}om(t,f)} \mathbb{H}om(t,y) \xrightarrow{\mathbb{H}om(t,g)} \mathbb{H}om(t,z) \xrightarrow{\mathbb{H}om(t,h)} \Sigma \mathbb{H}om(t,z).\]

4.1.7. The following notions are specific to derivators and play a key role in the additivity of traces:

**Definition 4.1.8.** Let $A$ be a small category. The **twisted arrow category** $\text{tw}(A)$ is defined as follows: its objects are morphisms $a \overset{f}{\to} b$ in $A$, and morphisms from $a_1 \overset{f_1}{\to} b_1$ to $a_2 \overset{f_2}{\to} b_2$ are pairs of morphisms $b_1 \overset{h}{\to} b_2$ and $a_2 \overset{g}{\to} a_1$ such that $f_2 = h f_1 g$. There is a canonical map $(t^{op}, s^{op}) : \text{tw}(A)^{op} \to A^{op} \times A$ sending $a \overset{f}{\to} b$ to $(b, a)$.

If $\mathcal{T}_c$ is a stable derivator and $X \in \mathcal{T}_c(A^{op} \times A)$, the **coend** of $X$ is defined as

\[(4.1.8.1)\]

\[
\int^A X := (\pi_{\text{tw}(A)^{op}})^\# (t^{op}, s^{op})^* X \in \mathcal{T}_c(\mathcal{O})
\]

where $\pi_{\text{tw}(A)^{op}} : \text{tw}(A)^{op} \to \mathcal{O}$ is the canonical map.

If $\otimes : \mathcal{T}_{c,1} \times \mathcal{T}_{c,2} \to \mathcal{T}_{c,3}$ is a two-variable morphism of stable derivators and $X \in \mathcal{T}_{c,1}(A^{op})$, $Y \in \mathcal{T}_{c,2}(A)$, the **cancelling tensor product** of $X$ and $Y$ is defined as

\[(4.1.8.2)\]

\[
X \otimes_{[A]} Y := \int^A (X \otimes Y) \in \mathcal{T}_{c,3}(\mathcal{O}).
\]

If $\mathcal{T}_c$ is a closed symmetric monoidal derivator, we have two natural two-variable morphisms $\otimes : \mathcal{T}_c \times \mathcal{T}_c \to \mathcal{T}_c$ and $\mathbb{H}om : \mathcal{T}_c^{op} \times \mathcal{T}_c \to \mathcal{T}_c$. We denote by $X \otimes_{[A]} Y$ and $\mathbb{H}om_{[A]}(X, Y)$ respectively the corresponding cancelling tensor products.

**Proposition 4.1.9 (TC3).** Let $\mathcal{T}_c$ be a closed symmetric monoidal derivator and let $x \overset{f}{\to} y$ and $x' \overset{f'}{\to} y'$ be two maps which give rise to distinguished triangles in $\mathcal{T}_c(\mathcal{O})$

\[(4.1.9.1)\]

\[x \overset{f}{\to} y \overset{g}{\to} z \overset{h}{\to} \Sigma x.
\]

\[(4.1.9.2)\]

\[x' \overset{f'}{\to} y' \overset{g'}{\to} z' \overset{h'}{\to} \Sigma x'.
\]

Then the following properties hold:

1. For $v := \mathbb{H}om_{A}(\text{cof}(f), f')$ there are distinguished triangles

\[(4.1.9.3)\]

\[
\mathbb{H}om(y,x') \xrightarrow{p_1} v \xrightarrow{j_1} \mathbb{H}om(z,x') \xrightarrow{\mathbb{H}om(h,f')} \Sigma \mathbb{H}om(y,x')
\]

\[(4.1.9.4)\]

\[
\Sigma^{-1} \mathbb{H}om(x,z') \xrightarrow{p_2} v \xrightarrow{j_2} \mathbb{H}om(y,y') \xrightarrow{-\mathbb{H}om(f,g') \mathbb{H}om(x,z')}
\]

\[(4.1.9.5)\]

\[
\mathbb{H}om(z,y') \xrightarrow{p_3} v \xrightarrow{j_3} \mathbb{H}om(x,x') \xrightarrow{\mathbb{H}om(h,f')} \Sigma \mathbb{H}om(z,y').
\]
with a coherent diagram of the form

\[ \begin{array}{cccccc}
\Sigma^{-1} \text{Hom}(y,x') & \xrightarrow{p_1} & \text{Hom}(y,x') & \xrightarrow{p_2} & \text{Hom}(z,x') & \xrightarrow{p_3} \text{Hom}(z,x') \\
\Sigma^{-1} \text{Hom}(f,1_{x'}) & \xrightarrow{\Sigma^{-1} \text{Hom}(f,1_{x'})} & \text{Hom}(y,x') & \xrightarrow{\Sigma^{-1} \text{Hom}(f,1_{x'})} & \text{Hom}(z,x') & \xrightarrow{\Sigma^{-1} \text{Hom}(f,1_{x'})} \text{Hom}(z,x') \\
\text{Hom}(f,1_{x'}) & \xrightarrow{\text{Hom}(f,1_{x'})} & \text{Hom}(y,x') & \xrightarrow{\text{Hom}(f,1_{x'})} & \text{Hom}(z,x') & \xrightarrow{\text{Hom}(f,1_{x'})} \text{Hom}(z,x') \\
\text{Hom}(1_{x'},f') & \xrightarrow{\text{Hom}(1_{x'},f')} & \text{Hom}(y,x') & \xrightarrow{\text{Hom}(1_{x'},f')} & \text{Hom}(z,x') & \xrightarrow{\text{Hom}(1_{x'},f')} \text{Hom}(z,x') \\
\text{Hom}(1_{x'},f') & \xrightarrow{\text{Hom}(1_{x'},f')} & \text{Hom}(y,x') & \xrightarrow{\text{Hom}(1_{x'},f')} & \text{Hom}(z,x') & \xrightarrow{\text{Hom}(1_{x'},f')} \text{Hom}(z,x') \\
\text{Hom}(1_{x'},f') & \xrightarrow{\text{Hom}(1_{x'},f')} & \text{Hom}(y,x') & \xrightarrow{\text{Hom}(1_{x'},f')} & \text{Hom}(z,x') & \xrightarrow{\text{Hom}(1_{x'},f')} \text{Hom}(z,x') \\
\text{Hom}(1_{x'},f') & \xrightarrow{\text{Hom}(1_{x'},f')} & \text{Hom}(y,x') & \xrightarrow{\text{Hom}(1_{x'},f')} & \text{Hom}(z,x') & \xrightarrow{\text{Hom}(1_{x'},f')} \text{Hom}(z,x') \\
\text{Hom}(1_{x'},f') & \xrightarrow{\text{Hom}(1_{x'},f')} & \text{Hom}(y,x') & \xrightarrow{\text{Hom}(1_{x'},f')} & \text{Hom}(z,x') & \xrightarrow{\text{Hom}(1_{x'},f')} \text{Hom}(z,x') \\
\end{array} \]

(2) For \( w := \text{Hom}(1_{x'}) \text{Hom}(f,h) \), there are distinguished triangles

\[ \begin{align*}
\text{Hom}(z,x') & \xrightarrow{k_1} w \xrightarrow{q_1} \text{Hom}(x,y') \xrightarrow{\text{Hom}(h,g')} \Sigma \text{Hom}(z,x') \\
\text{Hom}(y,y') & \xrightarrow{k_2} w \xrightarrow{q_2} \Sigma \text{Hom}(z,x') \xrightarrow{-\Sigma \text{Hom}(g,f')} \Sigma \text{Hom}(y,y') \\
\text{Hom}(x,x') & \xrightarrow{k_3} w \xrightarrow{q_3} \text{Hom}(y,z') \xrightarrow{\text{Hom}(f,h')} \Sigma \text{Hom}(x,x')
\end{align*} \]

with a similar coherent diagram.

(3) For \( u := f \otimes_{[1]} \text{Hom}(f') \), there are distinguished triangles

\[ \begin{align*}
x \otimes z' & \xrightarrow{l_1} u \xrightarrow{r_1} z \otimes y' \xrightarrow{h \otimes g'} \Sigma x \otimes z' \\
y \otimes y' & \xrightarrow{l_2} u \xrightarrow{r_2} \Sigma x \otimes x' \xrightarrow{-\Sigma(y \otimes f')} \Sigma y \otimes y' \\
z \otimes x' & \xrightarrow{l_3} u \xrightarrow{r_3} y \otimes z' \xrightarrow{f \otimes h'} \Sigma z \otimes x'
\end{align*} \]

with a similar coherent diagram.

We call these statements respectively \((TC3D)\), \((TC3D')\) and \((TC3')\).

**Proof.** The proof of \((TC3D)\) is the same as that of [GPS14, Theorem 6.2], where we replace everywhere \(\otimes\) by \(\text{Hom}\). The statement of \((TC3')\) is proved in [GPS14, Section 7], and \((TC3D')\) follows from a similar argument. \(\square\)

The following has the same proof as [GPS14, Theorem 7.3]:

**Proposition 4.1.10** (TC4). With the notations in Proposition 4.1.9, there is a biCartesian square

\[ \begin{array}{ccc}
\text{Hom}(y,y') & \xrightarrow{\text{Hom}(y,y')} & \text{Hom}(y,y') \\
\text{Hom}(z,z') & \xrightarrow{\text{Hom}(z,z')} & \text{Hom}(z,z') \\
\text{Hom}(x,x') & \xrightarrow{\text{Hom}(x,x')} & \text{Hom}(x,x') \\
\end{array} \]

\[ \begin{array}{ccc}
\text{Hom}(z,z') & \xrightarrow{\text{Hom}(z,z')} & \text{Hom}(z,z') \\
\text{Hom}(x,x') & \xrightarrow{\text{Hom}(x,x')} & \text{Hom}(x,x') \\
\text{Hom}(y,y') & \xrightarrow{\text{Hom}(y,y')} & \text{Hom}(y,y') \\
\end{array} \]

\[ \begin{array}{ccc}
\text{Hom}(z,z') & \xrightarrow{\text{Hom}(z,z')} & \text{Hom}(z,z') \\
\text{Hom}(x,x') & \xrightarrow{\text{Hom}(x,x')} & \text{Hom}(x,x') \\
\text{Hom}(y,y') & \xrightarrow{\text{Hom}(y,y')} & \text{Hom}(y,y') \\
\end{array} \]

\[ \begin{array}{ccc}
\text{Hom}(z,z') & \xrightarrow{\text{Hom}(z,z')} & \text{Hom}(z,z') \\
\text{Hom}(x,x') & \xrightarrow{\text{Hom}(x,x')} & \text{Hom}(x,x') \\
\text{Hom}(y,y') & \xrightarrow{\text{Hom}(y,y')} & \text{Hom}(y,y') \\
\end{array} \]

Note that when \( x, y, z \) are dualizable, up to taking their duals, Propositions 4.1.9 and 4.1.10 are precisely [GPS14, Theorems 6.2 and 7.3].
4.1.11. Let $\mathcal{T}_c$ be a closed symmetric monoidal derivator. Consider a distinguished triangle in $\mathcal{T}_c(\mathbf{Q})$

$$x \to y \to z \to \Sigma x$$

and let $t \in \mathcal{T}_c(\mathbf{Q})$. By Proposition 4.1.6, we have a distinguished triangle

$$\xymatrix{ \text{Hom}(z, t) \ar[r]^-{\text{Hom}(g, t)} & \text{Hom}(y, t) \ar[r]^-{\text{Hom}(f, t)} & \text{Hom}(x, t) \ar[r]^-{\Sigma \text{Hom}(\Sigma^{-1} h, t)} & \Sigma \text{Hom}(z, t).}$$

By [GPS14, Lemma 7.1], we have

$$\text{Hom}(f, t) \otimes [1] f \simeq \text{Hom}(g, t) \otimes [1] g \simeq \text{Hom}(h, t) \otimes [1] h$$

and we denote by $u$ this object. The following follows from the proof of [GPS14, Theorem 10.3]:

**Proposition 4.1.12** (TC5a). With the notations above, there is a map $\tilde{e} : u \to t$ in $\mathcal{T}_c(\mathbf{Q})$ such that the following incoherent diagrams commute:

$$\xymatrix{ \text{Hom}(x, t) \otimes x \ar[r]^-{l_1} \ar[d]_-{\epsilon_x} & u \ar[d]^-{\tilde{e}} & \text{Hom}(y, t) \otimes y \ar[r]^-{l_2} \ar[d]_-{\epsilon_y} & u \ar[d]^-{\tilde{e}} & \text{Hom}(z, t) \otimes z \ar[r]^-{l_3} \ar[d]_-{\epsilon_z} & u \ar[d]^-{\tilde{e}}.}$$

4.1.13. We now discuss the last one of May’s axioms, where we work with local duality instead of the usual duality. The following definition is standard ([CD19, Definition 4.4.4]):

**Definition 4.1.14.** We say that an object $t \in \mathcal{T}_c(\mathbf{Q})$ is **dualizing** if for any $x \in \mathcal{T}_c(\mathbf{Q})$, the following canonical map is an isomorphism:

$$x \to \text{Hom}(\text{Hom}(x, t), t).$$

We denote by $\mathbb{D}_t := \text{Hom}(-, t) : \mathcal{T}_c^{op} \to \mathcal{T}_c$ the $t$-**dual** functor. We have clearly $\mathbb{D}_t \circ \mathbb{D}_t = id$.

**Lemma 4.1.15.** If $t \in \mathcal{T}_c(\mathbf{Q})$ is a dualizing object, then for any $a \in \mathcal{T}_c(A)$ and $b \in \mathcal{T}_c(B)$, the following canonical map is an isomorphism in $\mathcal{T}_c(A^{op} \times B)$:

$$\text{Hom}(a, b) \to \mathbb{D}_t(a \otimes \mathbb{D}_t(b)).$$

**Proof.** The proof is the same as [CD19, Corollary 4.4.24]: we have a canonical isomorphism

$$\mathbb{D}_t(a \times c) \simeq \text{Hom}(a, \mathbb{D}_t(c))$$

and since $t$ is dualizing, the result follows by replacing $c$ by $\mathbb{D}_t(b)$ in the previous map. □

Thanks to Lemma 4.1.15, the proof of the following is similar to [GPS14, Theorem 11.12]:

**Proposition 4.1.16** (TC5b). Consider a distinguished triangle

$$x \to y \to z \to \Sigma x$$

in $\mathcal{T}_c(\mathbf{Q})$ and let $t \in \mathcal{T}_c(\mathbf{Q})$ be a dualizing object. Then the (TC3') diagram specified in (TC5a) for the triangles $(\mathbb{D}_t g, \mathbb{D}_t f, \mathbb{D}_t \Sigma^{-1} h)$ and $(f, g, h)$ is isomorphic to the $t$-dual of the (TC3D) diagram for the triangles $(f, g, h)$ and $(f, g, h)$.

**Remark 4.1.17.** The original (TC5b) statement also requires the (TC4) axiom for the dual diagram up to an involution, which is also true in our case if we write down the corresponding (TC3) and (TC3D’) diagrams; but such a fact is not used in [GPS14] for the proof of the additivity of traces.

The following is obtained by taking the $t$-dual of the (TC5a):
**Corollary 4.1.18.** Assume that $\mathcal{T}(\mathcal{D})$ has a dualizing object and consider a distinguished triangle in $\mathcal{T}(\mathcal{D})$

\[(4.1.18.1)\]

Let $v$ be element specified in the (TC3D) diagram for the triangles $(f,g,h)$ and $(f,g,h)$. Then there is a map $\bar{\eta} : \mathbb{1} \to v$ in $\mathcal{T}(\mathcal{D})$ such that the following incoherent diagrams commute:

\[(4.1.18.2)\]

**4.2. Motivic derivators and additivity of traces.**

**Definition 4.2.1.** The 2-category $\text{DiaSch}$ is defined as follows:

- An object of $\text{DiaSch}$ is a pair $(F,J)$ where $J \in \text{FinCat}$ and $F : J \to \text{Sch}$ is a covariant functor.
- An 1-morphism from $(G,J')$ to $(F,J)$ is the data of a functor $\alpha : J' \to J$ together with a natural transformation of functors $f : G \to F \circ \alpha$.
- A 2-morphism from $(f,\alpha)$ to $(f',\alpha')$ as above is a natural transformation $t : \alpha \to \alpha'$ such that $f' = t \circ f$.

We say that a 1-morphism $(f,\alpha) : (G,J') \to (F,J)$ is **Cartesian** if $\alpha$ is an equivalence of categories and for any morphism $i \to j$ in $J'$, the following square is Cartesian:

\[(4.2.1.1)\]

If $X \in \text{Sch}$ and $J \in \text{FinCat}$, we denote by $(X,J)$ the object in $\text{DiaSch}$ with constant value $X$.

The following definition is almost identical to [Ayo07, Definition 2.4.13]:

**Definition 4.2.2.** A **stable algebraic derivator** is a (non-strict) 2-functor $\mathcal{T}_c : (\text{DiaSch})^{op} \to \text{TR}$ satisfying the following properties:

1. $\mathcal{T}_c$ sends coproducts to products.
2. For any 1-morphism $(f,\alpha) : (F,J) \to (G,J')$ in $\text{DiaSch}$, the functor $(f,\alpha)^*$ has a right adjoint $(f,\alpha)_*$.
3. For any 1-morphism $(f,\alpha) : (F,J) \to (G,J')$ in $\text{DiaSch}$ which is termwise smooth, the functor $(f,\alpha)^*$ has a left adjoint $(f,\alpha)_{\#}$.
4. If $f : G \to F$ is a morphism of $J$-diagrams of schemes and $\alpha : J' \to J$ is a functor in $\text{FinCat}$, then the exchange 2-morphism

\[(4.2.2.1)\]

associated to the following Cartesian square in $\text{DiaSch}$ is invertible:

\[(4.2.2.2)\]

5. In the situation of (4), if $f$ is Cartesian and termwise smooth, then the following exchange 2-morphism associated the square (4.2.2.2) is invertible:

\[(f_{\#})_{\#} \alpha^* \to (f_\#)^* \alpha_\#.\]
(6) For any $X \in Sch$, the 2-functor

$$\mathcal{T}_c(X, \cdot) : FinCat^{op} \to TR$$

is a stable derivator (Definition 4.1.2).

(7) The 2-functor

$$\mathcal{T}_c(\cdot, \emptyset) : Sch^{op} \to TR$$

is the subcategory of constructible objects in a motivic triangulated category $\mathcal{T}$ ([CD19, Definition 4.2.1]). We call $\mathcal{T}$ the underlying motivic triangulated category.

Note that by [CD19, Corollary 4.4.24] and [CD15, Theorem 7.3], if the underlying motivic triangulated category satisfies (RS), then for any scheme $X$, $\mathcal{K}_X$ is a dualizing object of $\mathcal{T}_c(X, \emptyset)$. We will then write $\mathbb{D}$ for $\mathbb{D}_{\mathcal{K}_X}$ in coherence with the previous sections.

4.2.3. By [Ayo07, Section 2.4.4], given a stable algebraic derivator $\mathcal{T}_c$, one can extend the four functors $f^!, f_!, f^*, f_*$ to diagrams of schemes in the following form:

- For any 1-morphism $(f, \alpha) : (F, J) \to (G, J')$ in $DiaSch$, there is a pair of adjoint functors

$$f^* : \mathcal{T}_c(F, J) \rightleftarrows \mathcal{T}_c(G, J') : (f, \alpha)_*.$$ (4.2.3.1)

- For any $J \in FinCat$ and any Cartesian $J$-shaped 1-morphism $f : (F, J) \to (G, J)$ in $DiaSch$, there is a pair of adjoint functors

$$f_! : \mathcal{T}_c(F, J) \rightleftarrows \mathcal{T}_c(G, J) : f^!.$$ (4.2.3.2)

If $f : X \to Y$ is a morphism of schemes, then the four functors associated to morphisms of the form $(f, J) : (X, J) \to (Y, J)$ commute with finite limit and colimits along diagrams, and therefore commute with the coend construction (Definition 4.1.8): we have a commutative diagram

$$\begin{array}{ccc}
\mathcal{T}_c(Y, A^{op} \times A) & \xrightarrow{f^*} & \mathcal{T}_c(X, A^{op} \times A) \\
\downarrow{f^\wedge} & & \downarrow{f^\wedge} \\
\mathcal{T}_c(Y, \emptyset) & \xrightarrow{f^*} & \mathcal{T}_c(X, \emptyset)
\end{array}$$

and similarly for the other functors $f_!, f^!$ and $f_!$.

We now deal with the monoidal structure.

**Definition 4.2.4.** A constructible motivic derivator is a (non-strict) 2-functor $(\mathcal{T}_c, \otimes) : DiaSch^{op} \to SMTR$ satisfying the following properties:

1. The composition $DiaSch^{op} \xrightarrow{\mathcal{T}_c} SMTR \to TR$ is a stable algebraic derivator (Definition 4.2.2), and the monoidal structure agrees with the one on the underlying motivic triangulated category.

2. For any scheme $X$, the 2-functor

$$(\mathcal{T}_c(X, \cdot), \otimes) : FinCat^{op} \to SMTR$$

is a closed symmetric monoidal stable derivator (Definition 4.1.4).

3. For any $J \in FinCat$, any Cartesian $J$-shaped 1-morphism $f : (F, J) \to (G, J)$ in $DiaSch$ and any pair of objects $(A, B) \in \mathcal{T}_c(G, J) \times \mathcal{T}_c(F, J)$, the following canonical map is an isomorphism:

$$f^#(f^*A \otimes_{G, J} B) \to A \otimes_{F, J} f^#B.$$ (4.2.4.2)
4.2.5. We now apply the formalism to the generalized trace map (3.2.8.2). The following is similar to [GPS14, Theorem 12.1]:

**Proposition 4.2.6.** Let $\mathcal{T}_c$ be a constructible motivic derivator whose underlying motivic triangulated category $\mathcal{T}$ satisfies (RS). We use the notations in 3.2.7. Let

\[
\begin{array}{c}
L \xrightarrow{f} M \\
\gamma \downarrow \quad \downarrow g \\
* \xrightarrow{} N
\end{array}
\]

be a biCartesian square in $\mathcal{T}_c(X, \square)$, and let $\phi : c_1^\Gamma \to c_2^\Gamma$ be a morphism of squares in $\mathcal{T}_c(C, \square)$. Then the pairing (3.2.8.2) satisfies

\[
(\phi_M, 1) = (\phi_L, 1) + (\phi_N, 1)
\]

where $\phi_M : c_1^M \to c_2^M$ is the restriction of $\phi$ to a map in $\mathcal{T}_c(C)$, and similarly for the other maps.

**Proof.** It suffices to construct the following incoherent diagram in $\mathcal{T}_c(X, 0)$

\[
\begin{tikzcd}
\delta^* c_1 (\text{Hom}(c_1^* L, c_1^* L) \oplus \text{Hom}(c_1^* N, c_1^* N)) & \delta^* c_1 v & \delta^* c_1 \text{Hom}(c_1^* M, c_1^* M) \\
\delta^* c_1 (\text{Hom}(c_1^* L, c_1^* L) \oplus \text{Hom}(c_1^* N, c_1^* N)) & \delta^* c_1 w & \delta^* c_1 \text{Hom}(c_1^* M, c_1^* M) \\
\delta^* c_1 (\text{Hom}(p_1^* L, p_1^* L) \oplus \text{Hom}(p_1^* N, p_1^* N)) & \delta^* c_1 w' & \delta^* c_1 \text{Hom}(p_1^* M, p_1^* M) \\
\delta^* (\text{Hom}(p_1^* L, p_1^* L) \oplus \text{Hom}(p_1^* N, p_1^* N)) & \delta^* w'' & \delta^* \text{Hom}(p_1^* M, p_1^* M)
\end{tikzcd}
\]

where the objects $v, w, w', w'', u, u'$ are specified by May's axioms in Section 4.1:

- $v \approx \text{Hom}_{\mathcal{T}_c}(c_1^* g, c_1^* f)$ is specified in the (TC3D) diagram for the triangles $(c_1^* f, c_1^* g, c_1^* h)$ and $(c_1^* f, c_1^* g, c_1^* h)$.
- $w \approx \text{Hom}_{\mathcal{T}_c}(c_1^* g, c_1^* g)$ is specified in the (TC3D') diagram for the triangles $(c_1^* f, c_1^* g, c_1^* h)$ and $(c_1^* f, c_1^* g, c_1^* h)$. 
The commutativity of the diagram follows from the following facts:

- The two triangles on the top commute by (TC5b), and the two on the bottom commute by (TC5a); the quadrilateral involving \( \delta^* c_2 w \) and \( \delta^* c_1 w \) commute by (TC4).
- Since \( \phi \) is a map of biCartesian squares, the functoriality of the (TC3D') diagram gives rise to a dotted map \( w \to w' \) making the two adjacent trapezoids commute.
- By 4.2.3, the functor \( c^j \) commutes with the coend construction, and the functoriality of the (TC3D') diagram together with the exchange isomorphisms (2.4.1.3) give rise to an isomorphism \( c^i w'' \approx w' \) making the two adjacent squares commute.
- Similarly, the functor \( \delta^* \) commutes with the coend construction, and the functoriality of the (TC3D') diagram together with the K"unneth formulas (3.1.1.1) give rise to an isomorphism \( u \approx \delta^* w'' \) making the two adjacent squares commute.
- The map \( \delta^i c_1 c_2 w'' \to \delta^* w'' \) is simply the counit of the adjunction, which clearly makes the two squares in the middle commute.
- By [GPS14, Lemma 6.9], there exists an isomorphism \( u \approx u' \) making the two adjacent squares commute.

\[ \square \]

Lemma 4.2.7. Consider the setting in 3.2.2. For \( i \in \{1, 2, 3\} \), let

\[
\begin{array}{c}
L_i \\
\downarrow \gamma_i \\
\ast \\
\downarrow \\
M_i \\
\end{array}
\]

be a biCartesian square in \( T_c(X_i, \Box) \). Let \( \phi : c_1^{12} \Gamma_1 \to c_2^{12} \Gamma_2 \) and \( \psi : c_2^{23} \Gamma_2 \to c_3^{23} \Gamma_3 \) be morphisms of squares in \( T_c(C_{12}, \Box) \) and \( T_c(C_{23}, \Box) \). Then the composition of correspondences (Definition 3.2.3) can be lifted to a morphism of squares

\[
\psi \phi : c_1^{13} \Gamma_1 \to c_3^{13} \Gamma_3.
\]

Proof. This is because we can lift the six functors to the level of diagrams in \( T_c(\cdot, \Box) \), and consequently the same is true for the composition of correspondences. \( \square \)

From Proposition 3.2.5, Proposition 4.2.6 and Lemma 4.2.7 we deduce the following

Theorem 4.2.8. Let \( T_c \) be a constructible motivic derivator whose underlying motivic triangulated category \( T \) satisfies (RS). We use the notations in 3.1.2. For \( i \in \{1, 2\} \), let

\[
\begin{array}{c}
L_i \\
\downarrow \gamma_i \\
\ast \\
\downarrow \\
M_i \\
\end{array}
\]

be a biCartesian square in \( T_c(X_i, \Box) \). Let \( \phi : c_1^1 \Gamma_1 \to c_2^1 \Gamma_2 \) and \( \psi : d_2^1 \Gamma_2 \to d_1^1 \Gamma_1 \) be morphisms of squares in \( T_c(C, \Box) \) and \( T_c(D, \Box) \). Then the pairing (3.1.8.1) satisfies

\[
(\phi_M : \psi_M) = (\phi_L : \psi_L) + (\phi_N : \psi_N),
\]

where \( \phi_M : c_1^1 M_1 \to c_2^1 M_2 \) is the restriction of \( \phi \), and similarly for the other maps.
Remark 4.2.9. One could also formulate the additivity of traces using the language of motivic \((\infty, 1)\)-categories developed in [Kha16]. Given our result for derivators, it suffices to check that the homotopy derivator of a motivic \((\infty, 1)\)-category of coefficients ([Kha16, Chapter 2 3.5.2]) satisfies the axioms of a constructible motivic derivator. As remarked in [GPS14], it is expected but is yet to be verified that monoidal structures are carried through this construction.

5. The characteristic class of a motive

In this section we come back to 1-categorical concerns. Let \( \mathcal{T} \) be the underlying motivic triangulated category of a constructible motivic derivator which satisfies the condition (RS) in 2.1.12.

5.1. The characteristic class and first properties.

5.1.1. We recall briefly the formalism we need from [DJK18]:

Recall 5.1.2. For any scheme \( X \), the Thom space construction is a well-defined group homomorphism \( \text{Th}_X : K_0(X) \rightarrow Pic(\mathcal{T}(X)) \) ([DJK18, 2.1.4]).

Let \( f : X \rightarrow S \) be a morphism of schemes and let \( V \) be a virtual vector bundle over \( X \). The (twisted) bivariant group is defined as

\[
(5.1.2.1) \quad H_0(X/S, V) = \text{Hom}_{\mathcal{T}_c(X)}(\text{Th}_X(V), f^! \mathbb{1}_S).
\]

For any proper morphism \( p : Y \rightarrow X \), there is a proper covariant functoriality

\[
(5.1.2.2) \quad p_* : H_0(Y/k, p^*V) \rightarrow H_0(X/k, V).
\]

If \( V \) is a vector bundle over \( X \), the Euler class \( e(V) : \mathbb{1}_X \rightarrow \text{Th}_X(V) \) is an analogue of the top Chern class in the classical setting ([DJK18, Definition 3.1.2]). When \( V \) is the trivial virtual bundle, we use the notation \( H_0(X/k) = H_0(X/k, 0) \). If \( X \) is a smooth scheme, the class \( e(L_{X/k}) : \mathbb{1}_X \rightarrow \text{Th}_X(L_{X/k}) \cong \mathcal{K}_X \) is an element of \( H_0(X/k) \).

Definition 5.1.3. Let \( X \) be a scheme and \( M \in \mathcal{T}_c(X) \). The Verdier pairing in Definition 3.1.8 in the particular case where \( C = D = X_1 = X_2 = X \) and \( L_1 = L_2 = M \) is a pairing

\[
(5.1.3.1) \quad \langle , \rangle : \text{Hom}(M, M) \otimes \text{Hom}(M, M) \rightarrow H_0(X/k).
\]

For any endomorphism \( u \in \text{Hom}(M, M) \), the characteristic class of \( u \) is defined as the element \( C_X(M, u) := \langle u, 1_M \rangle \in H_0(X/k) \). The characteristic class of a motive \( M \) is the characteristic class of the identity \( C_X(M) = C_X(M, 1_M) \).

We now list some elementary properties of the characteristic class.

5.1.4. Since identity maps are particularly good choices of morphisms of distinguished triangles, Theorem 4.2.8 implies the additivity of the characteristic class:

Corollary 5.1.5. Let \( X \) be a scheme and let \( L \rightarrow M \rightarrow N \rightarrow L[1] \) be a distinguished triangle in \( \mathcal{T}_c(X) \). Then \( C_X(M) = C_X(L) + C_X(N) \).

Remark 5.1.6. (1) For every scheme \( X \), the additivity of traces yields a well-defined homomorphism of abelian groups

\[
(5.1.6.1) \quad K_0(\mathcal{T}_c(X)) \rightarrow H_0(X/k) \quad \quad [M] \mapsto C_X(M).
\]

---

\(^6\)The formalism in loc. cit. is constructed for the stable motivic homotopy category \( \text{SH} \), but since the construction is quite formal, it can also be done in any motivic triangulated category.

\(^7\)This is a particular case of the general formalism, see [DJK18, Definition 2.12].

\(^8\)Note that for \( \mathcal{T}_c = \text{DM}_{cd, c} \), the group \( H_0(X/k, V) \) is the Chow group of algebraic cycles of dimension the virtual rank of \( V \), and we recover the formalism in [Ful98]. The Euler class in Chow groups is the top Chern class.
where the left hand side is the Grothendieck group of the triangulated category $\mathbf{T}_c(X)$. For $T_c = \mathbf{DM}_{cdh,c}$, we have $H_0(X/k) \cong CH_0(X)$ is the Chow group of zero-cycles over $X$ (up to $p$-torsion). It is conjectured that the map (5.1.6.1) is related to the 0-dimensional part of the closure of the characteristic cycle ([Sai17, Conjecture 6.8]).

(2) One can also relate the characteristic class with the Grothendieck group of varieties over a base. Composing the map (5.1.6.1) with the obvious ring homomorphism

$$K_0(Var/X) \rightarrow K_0(T_c(X))$$

(5.1.6.2)

where $K_0(Var/X)$ is the Grothendieck group of varieties over $X$, we obtain a well-defined homomorphism of abelian groups

$$K_0(Var/X) \rightarrow H_0(X/k)$$

(5.1.6.3)

and gives an additive invariant on $K_0(Var/X)$. In the case $X = \text{Spec}(k)$, the group $H_0(X/k)$ is equal to the endomorphism ring $\text{End}(\mathbb{1}_k)$, and the map (5.1.6.3) is a ring homomorphism, which defines a motivic measure on $K_0(Var/k)$. When $T = \mathbf{SH}$ and $k$ has characteristic 0, this agrees with the construction in [Rön16].

5.1.7. It follows from Proposition 3.1.6 that the characteristic class is compatible with the proper functoriality:

**Corollary 5.1.8.** Let $f : X \rightarrow Y$ be a proper morphism. Then we have $f_* C_X(M, u) = C_Y(f_* M, f_* u)$.

5.1.9. The following lemma shows a relation between the characteristic class and the trace map:

**Lemma 5.1.10.** Let $X$ be a scheme and let $M, N$ be two objects in $\mathbf{T}_c(X)$ such that $M$ is dualizable. Let $u$ be an endomorphism of $M$ and let $v$ be an endomorphism of $N$. Then $C_X(M \otimes N, u \otimes v) = Tr(u) \cdot C_X(N, v)$.

**Proof.** This follows from Proposition 3.2.8, using the fact that the canonical map $M^\vee \otimes \mathbb{D}(N) \rightarrow \mathbb{D}(M \otimes N)$ is an isomorphism. □

5.1.11. By Corollary 5.1.8 and Lemma 5.1.10, if $f : X \rightarrow \text{Spec}(k)$ is a proper morphism, $M \in \mathbf{T}_c(X)$ and $u$ is an endomorphism of $M$, then the degree of the class $C_X(M, u)$ is the trace of the map $f_* u : f_* M \rightarrow f_* M$. In particular, the degree of the class $C_X(M)$ is the Euler characteristic of $M$.

5.1.12. We denote by $(-1)^k = \chi(\mathbb{1}_k(1)) \in \text{End}(\mathbb{1}_k)$ the Euler characteristic of the Tate twist, defined as the trace of the identity map. Then for any morphism $f : X \rightarrow \text{Spec}(k)$, we have $\chi(\mathbb{1}_X(n)) = (-1)_X^n$, where $(-1)_X = f^*(-1)^k \in \text{End}(\mathbb{1}_X)$. As a consequence we obtain

**Corollary 5.1.13.** Let $X$ be a scheme and let $M \in \mathbf{T}_c(X)$. Let $u \in \text{End}(M)$ be an endomorphism and denote by $u(n)$ be the corresponding endomorphism of $M(n)$. Then $C_X(M(n), u(n)) = (-1)^n_X \cdot C_X(M, u)$.

5.1.14. We can compute characteristic classes for endomorphisms of primitive Chow motives as follows:

**Proposition 5.1.15.** Let $X$ be a smooth $k$-scheme with tangent bundle $T_X/k$ and let $p : X \rightarrow S$ be a proper morphism. Then for any endomorphism $u$ of $p_* \mathbb{1}_X$, the characteristic class $C_X(p_* \mathbb{1}_X, u)$ is given by the composition

$$\mathbb{1}_S \rightarrow p_* \mathbb{1}_X \xrightarrow{u} p_* \mathbb{1}_X \xrightarrow{p_* e(T_X/k)} p_* \mathcal{K}_X \xrightarrow{\text{ad}'(p_* p)} \mathcal{K}_S.$$  

(5.1.15.1)

Note that formula (5.1.15.1) is quite similar to the motivic Gauss–Bonnet formula ([DJK18, Theorem 4.4.1]); when $S = \text{Spec}(k)$ and $u = id$ the two formulas are the same.

---

9For $T_c = \mathbf{SH}$, it is well-known that $(-1)_k \in \text{End}(\mathbb{1}_k) = GW(k)$ corresponds to the quadratic form $x \mapsto -x^2$ ([Hoy15, Example 1.7]). This element is reduced to identity in $\mathbf{DM}_{cdh,c}$ or $\ell$-adic étale cohomology.
Proof. We need to show the commutativity of the following diagram:

\[
\begin{array}{c}
\llbracket_S \xrightarrow{u} \text{Hom}(p_*\llbracket_S, p_*\llbracket_S) \xrightarrow{} \mathcal{D}(p_*\llbracket_S) \otimes p_*\llbracket_S \xrightarrow{} p_*\llbracket_S \otimes \mathcal{D}(p_*\llbracket_S) \\
p_*\llbracket_S \xrightarrow{u} p_*\llbracket_S \xrightarrow{p_*(L/X)} p_*\mathcal{K}_X \xrightarrow{\text{ad}_{\delta_*U'}} \mathcal{K}_S.
\end{array}
\]

(5.1.15.2)

While the two squares on the left and on the right are straightforward, it remains to show the commutativity of the square in the middle. Denote by $\delta : X \to X \times_k X$ the diagonal morphism and $p_1, p_2 : X \times_k X \to X$ the projections. By the self-intersection formula ([DJK18, Example 3.2.9]) as explained in the proof of [DJK18, Theorem 4.6.1], the Euler class $e(L_{X/k}) : \llbracket_S \to \mathcal{K}_X$ agrees with the composition

\[
\llbracket_S \to \delta^! p_*^! \mathcal{K}_X \to \delta^* p_*^! \mathcal{K}_X \simeq \mathcal{K}_X
\]

(5.1.15.3)

where the first map is induced by the fundamental class of the morphism $\delta$, and the second map is induced by the natural transformation $\delta^! \to \delta^*$ given by $\delta^! \simeq \delta^* \delta^! \xrightarrow{\text{ad}_{\delta_*U'}} \delta^*$. The commutativity then follows from a standard diagram chase.

Example 5.1.16. As a particular case of Proposition 5.1.15, for any smooth $k$-scheme $X$ we have $C_X(\llbracket_S) = e(L_{X/k})$. The class $e(L_{X/k})$ can be understood as the class of the “self-intersection of the diagonal”, and by 5.1.11 we recover the slogan “The degree of the self-intersection of the diagonal is the Euler characteristic” for smooth and proper schemes.

5.1.17. Alternatively, there is a more geometric description of the characteristic class $C_X(p_*\llbracket_S, u)$ using the refined Gysin map: denote by $p_1 : X \times_S X \to X$ the projection onto the first summand. Then base change and purity isomorphisms induce a canonical isomorphism

\[
\text{End}(p_*\llbracket_S) \simeq H_0(X \times_S X/k, p_1^{-1}L_{X/k}).
\]

(5.1.17.1)

Also consider the Cartesian diagram

\[
\begin{array}{c}
X \xrightarrow{\delta_X/S} X \times_S X \\
\xrightarrow{\Delta} X \times_k X
\end{array}
\]

(5.1.17.2)

since $\delta_X/k : X \to X \times_k X$ is a regular closed immersion, we have a refined Gysin map ([DJK18, Definition 4.3.1])

\[
\Delta^! : H_0(X \times_S X/k, p_1^{-1}L_{X/k}) \to H_0(X/k).
\]

(5.1.17.3)

Then for any endomorphism $u$ of $p_*\llbracket_S$ we have

\[
C_X(p_*\llbracket_S, u) = p_*\Delta^! u'
\]

(5.1.17.4)

where $u' \in H_0(X \times_S X/k, p_1^{-1}L_{X/k})$ is the image of $u$ by the map (5.1.17.1).

5.2. A characterization. In this section we give a characterization of the characteristic class for constructible motives.

5.2.1. Let $T$ be a motivic triangulated category and let $X$ be a scheme. We denote by $T_{\text{Chow}}(X)$ the idempotent completion of the additive subcategory generated by all primitive Chow motives over $X$, and $\langle T_{\text{Chow}}(X) \rangle$ the triangulated subcategory of $T(X)$ generated by $T_{\text{Chow}}(X)$.
5.2.2. If the condition (RS 1) in 2.1.12 holds, then \( (T_{	ext{Chow}}(X)) \) contains all strictly constructible motives in the sense of [Ayo07, 2.2.3] by [Ayo07, Proposition 2.2.27]. We now show that under some cancellation assumptions, it indeed contains all constructible motives:

**Definition 5.2.3.** If \( C \) is a triangulated category, we say that a collection of objects \( \mathcal{H} \) of \( C \) is **negative** if for any \( A, B \in \mathcal{H} \) any integer \( i > 0 \) we have

\[
\text{Hom}_C(A, B[i]) = 0.
\]

**Lemma 5.2.4.** Let \( T \) be a motivic triangulated category which satisfies the condition (RS) in 2.1.12. If \( X \) is a scheme such that the collection of primitive Chow motives over \( X \) is negative, then we have \( (T_{	ext{Chow}}(X)) = T_c(X) \).

**Proof.** By strong devissage and [Bon10, Theorem 4.3.2 and Proposition 5.2.2], if \( X \) is a scheme such that the collection of primitive Chow motives over \( X \) is negative, there exists a unique bounded weight structure on \( T_c(X) \) (called the Chow weight structure) whose heart is \( T_{	ext{Chow}}(X) \), and we conclude using [Bon10, Corollary 1.5.7]. \( \Box \)

5.2.5. The condition in Lemma 5.2.4 is satisfied for \( T = \text{DM}_{cdh} \) ([BL15]) or the homotopy category of KGL-modules ([BL16]), for every scheme \( X \).

**Theorem 5.2.6.** Assume that the base field \( k \) is perfect. Let \( T \) be the underlying motivic triangulated category of a constructible motivic derivator which satisfies the condition (RS) in 2.1.12. Let \( X \) be a scheme.

(1) The map

\[
(T_{	ext{Chow}}(X)) \rightarrow H_0(X/k)
\]

\[
M \mapsto C_X(M)
\]

is the unique map satisfying the following properties:

(a) For any distinguished triangle \( L \rightarrow M \rightarrow N \rightarrow L[1] \) in \( (T_{	ext{Chow}}(X)) \), \( C_X(M) = C_X(L) + C_X(N) \).

(b) If \( p : Y \rightarrow X \) is a proper morphism with \( Y \) smooth over \( k \) and \( M \) is the direct summand of a primitive Chow motive \( p_*
\]

(c) 1.5.7](Bon10, Definition 4.3.2). Let \( k' \) be the perfect closure of \( k \). Then for any scheme \( X \), the canonical morphism \( \phi_X : X_{k'} = X \times_k k' \rightarrow X \) is a universal homeomorphism, and by [EK18, Theorem 2.1.1] the functor \( \phi_X^* : T_c(X) \rightarrow T_c(X_{k'}) \) is an equivalence of categories. By [EK18, Remark 2.1.13] for any finite surjective radicial morphism \( f : Y \rightarrow X \) we have a canonical identification \( f^* = f^1 \), and therefore the Verdier pairing is contravariant for such morphisms (see Remark 3.1.7). If the collection of primitive Chow motives over \( X \) is negative, we conclude that the characteristic class of elements in \( T_c(X) \) is uniquely determined by the functor \( \phi_X^* \) and the description in Theorem 5.2.6 for the perfect field \( k^s \).
5.3. **The characteristic class and Riemann-Roch-transformations.** In this section we study the compatibility between the characteristic class and Riemann-Roch-transformations.

5.3.1. Assume that the pair \((\text{SH}, k)\) satisfies condition \((\text{RS})\) in 2.1.12. Let \(E \in \text{SH}(k)\) be a ring spectrum endowed with a unital associative commutative multiplication. Let \(f : S \to k\) be a morphism. Following [CD19, 7.2.2], the homotopy category of modules over \(E_S = f^*E\), \(Ho(E_S - \text{Mod})\) is a motivic triangulated category, and the functor

\[
\text{SH}(S) \to Ho(E_S - \text{Mod})
\]

\[
M \mapsto M \otimes E_S
\]

is a left adjoint of the forgetful functor \(Ho(E_S - \text{Mod}) \to \text{SH}(S)\), which preserves constructible objects. The unit map \(\phi : 1_S \to E_S\) induces the \(A^1\)-regulator map ([DJK18, Definition 4.1.2])

\[
\phi_* : H_0(S/k) \to E_0(S/k)
\]

where \(E_0(S/k) = \text{Hom}_{\text{SH}(S)}(1_S, f^!E)\).

5.3.2. Let \(M \in \text{SH}^c(S)\) be a constructible motivic spectrum, and let \(u : M \to M\) be an endomorphism of \(M\) in \(\text{SH}^c(S)\). Then \(u\) induces an endomorphism \(u^E : M \otimes E_S \to M \otimes E_S\) in \(Ho(E_S - \text{Mod})\).

By Definition 5.1.3, we have the characteristic class \(C^S_{\text{SH}}(M, u) \in H_0(S/k)\) in \(\text{SH}\), as well as the characteristic class \(C^E_S(M \otimes E_S, u^E) \in E_0(S/k)\) in \(Ho(E_S - \text{Mod})\).

**Proposition 5.3.3.** Via the map (5.3.1.2), the two classes above satisfy the identity

\[
\phi_* C^S_{\text{SH}}(M, u) = C^E_S(M \otimes E_S, u^E)
\]

in \(E_0(S/k)\).

**Proof.** We denote by \(\text{Hom}_E\) the internal \(\text{Hom}\) functor in \(Ho(E_S - \text{Mod})\) and \(\text{Hom}\) the internal \(\text{Hom}\) functor in \(\text{SH}\). We have a canonical identification \(\text{Hom}_E(A \otimes E_S, B) \simeq \text{Hom}(A, B)\). The result then follows from the following commutative diagram:

\[
\begin{array}{ccccccccc}
1_S & \xrightarrow{u} & \text{Hom}(M, M) & \xrightarrow{\phi} & M \otimes \mathbb{D}(M) & \xrightarrow{\epsilon_M} & K_S \\
\downarrow{\phi} & & \downarrow{\phi} & & \downarrow{\phi} \\
E_S & \xrightarrow{u^E} & \text{Hom}(M, M \otimes E_S) & \xrightarrow{\phi} & M \otimes \mathbb{D}(M) \otimes E_S & \xrightarrow{\epsilon_M} & K_S \otimes E_S \\
\downarrow{u^E} & & \downarrow{p^E\phi} & & \downarrow{p^E\phi} \\
\text{Hom}_E(M \otimes E_S, M \otimes E_S) & \xrightarrow{\phi} & \text{Hom}_E(M \otimes E_S, M \otimes E_S) & \xrightarrow{\epsilon_M} & p^E \otimes E_S \\
\downarrow{u^E} & & \downarrow{p^E} & & \downarrow{(6.1.1.1)} \\
M \otimes E_S \otimes \text{Hom}_E(M \otimes E_S, p^E) & \xrightarrow{\epsilon_{M \otimes E_S}} & E_S
\end{array}
\]

**Corollary 5.3.4.** Let \(E, F \in \text{SH}(k)\) be two ring spectra endowed with unital associative commutative multiplication. Let \(\phi : E \to F\) be a morphism of ring spectra. With the notations in 5.3.2, we have

\[
\phi_* C^S_{\text{SH}}(M \otimes E_S, u^E) = C^E_S(M \otimes F_S, u^E)
\]

where \(\phi_* : E_0(S/k) \to F_0(S/k)\) is the map induced by \(\phi\).

**Example 5.3.5.** Let \(E = \text{KGL}\) be the algebraic \(K\)-theory spectrum, \(F = \oplus_{i \in \mathbb{Z}} \text{H}_Q(i)[2i]\) be the periodized rational motivic Eilenberg-Mac Lane spectrum, and \(\phi = \text{ch} : \text{KGL} \to \oplus_{i \in \mathbb{Z}} \text{H}_Q(i)[2i]\) be the Chern character. Then for any scheme \(X\), the map \(\text{ch}_S = \tau_X : G_0(X) \to \oplus_{i \in \mathbb{Z}} CH_i(X)/Q\) is the Riemann-Roch transformation in [Ful98, Theorem 18.3] ([Dég18, Example 3.3.12]).
If $X$ is a smooth $k$-scheme of dimension $d$, then the KGL-valued characteristic class $C_X^{\mathrm{KGL}}(\mathbb{1}_X) \in G_0(X) = K_0(X)$ can be written as
\begin{equation}
C_X^{\mathrm{KGL}}(\mathbb{1}_X) = \sum_{i=0}^{d} (-1)^i [\Lambda^i L_{X/k}^\vee].
\end{equation}
On the other hand we have the $H^\mathbb{Q}$-valued characteristic class $C_X^{H^\mathbb{Q}}(\mathbb{1}_X) \in CH_0(X)$ given by the top Chern class $c_d(L_{X/k})$. By [Ful98, Example 3.2.5] we have
\begin{equation}
ch(C_X^{\mathrm{KGL}}(\mathbb{1}_X)) = \sum_{i=0}^{d} (-1)^i ch(\Lambda^i L_{X/k}^\vee) = c_d(L_{X/k}) \cdot Td(L_{X/k})^{-1} = C_X^{H^\mathbb{Q}}(\mathbb{1}_X) \cdot Td(L_{X/k})^{-1}.
\end{equation}
In other words we have $ch(C_X^{\mathrm{KGL}}(\mathbb{1}_X)) \cdot Td(L_{X/k}) = C_X^{H^\mathbb{Q}}(\mathbb{1}_X)$, where the left hand side is nothing but $\tau_X(C_X^{\mathrm{KGL}}(\mathbb{1}_X))$, and we recover a particular case of Corollary 5.3.4.

6. KÜNNETH FORMULAS OVER GENERAL BASES

In this section, we study transversality conditions following the method of [YZ18], and generalize the Küneth formulas (2.4.6.3) and (2.4.6.4) to a general base scheme $S$ under these conditions, which allows us to define the relative characteristic class. We consider $T$ a motivic triangulated category which satisfies the condition (RS) in 2.1.12.

6.1. The transversality conditions. In this Section 6.1, we introduce the transversality conditions and prove some elementary properties, which will be used in the formulation of Küneth formulas over a general base scheme in Section 6.2.

6.1.1. Let $f : X \to S$ be a morphism of schemes. For two objects $A$ and $B$ of $T(S)$, there is a canonical natural transformation
\begin{equation}
f^* B \otimes f^! A \to f^!(B \otimes A)
\end{equation}
given by the composition
\begin{equation}
f^* B \otimes f^! A \xrightarrow{ad(f^*, f^!)_{\mathbb{1}_S}} f^! f_*(f^* B \otimes f^! A) \xrightarrow{(2.2.1.2)} f^!(B \otimes f_! f^! A) \xrightarrow{\text{ad}(f_!, f^!)} f^!(B \otimes A).
\end{equation}
In particular when $A = \mathbb{1}_S$, the map (6.1.1.1) becomes
\begin{equation}
f^* B \otimes f^! \mathbb{1}_S \to f^! B.
\end{equation}

6.1.2. If $f : X \to S$ is a smooth morphism, or if $B \in T(S)$ is dualizable, then the map (6.1.1.1) is an isomorphism: the first case follows from purity, and the second case is [FHM03, 5.4].

**Definition 6.1.3.** Let $f : X \to S$ be a morphism of schemes.

1. ([Sai17, Definition 8.5]) Let $B$ be an object of $T(S)$. We say that the morphism $f : X \to S$ is $B$-transversal if the map (6.1.1.3) is an isomorphism.

2. Let $C$ be an object of $T(X)$. We say that the morphism $f : X \to S$ is $C$-transversal if the graph morphism $\Gamma_f : X \to X \times_k S$ of $f$ is $C \boxtimes_k D$-transversal for any object $D \in T_c(S)$. We say that $f$ is universally $C$-transversal if this property holds after any base change (cf. Definition 2.1.7).

**Remark 6.1.4.**

1. It is easy to see that in Definition 6.1.3 (2), $f$ is $C$-transversal if and only if $\Gamma_f$ is $C \boxtimes_k D$-transversal for any object $D \in T(S)$.

2. Let $T_c$ be a constructible motivic derivator and let $f : X \to S$ be a morphism of schemes. Then there is a stable derivator $Fun^{ex}(T_c(S), T_c(X))$ given by triangulated functors from $T_c(S)$ to $T_c(X)$. We lift the natural transformation $f^* B \otimes f^! \mathbb{1}_S \xrightarrow{(6.1.1.3)} f^! B$ to a coherent morphism in $Fun^{ex}(T_c(S), T_c(X))(\mathbb{1})$; the target of its cofiber (Definition 4.1.2) is a functor $f^\Delta$ in
Let \( f: X \to Y \) and \( g: Y \to Z \) be morphisms of schemes such that \( g \) is lci and \( \mathbb{1}_Z \) is \( g \)-pure. Let \( F \in \mathcal{T}(Z) \) be such that \( g \) is \( F \)-transversal. Then the following two conditions are equivalent:

1. \( f \) is \( g^* F \)-transversal.
2. \( g \circ f \) is \( F \)-transversal.

Proof. We show that the first condition implies the second, the converse being similar. Since \( g \) is \( F \)-transversal and \( f \) is \( g^* F \)-transversal, the following maps are isomorphisms:

\[
\begin{align*}
(f^* g^* F \otimes f^! \mathbb{1}_Y) & \xrightarrow{(6.1.1.3)} f^! g^* F, \\
(g^* F \otimes g^! \mathbb{1}_Z) & \xrightarrow{(6.1.1.3)} g^! F.
\end{align*}
\]

Since \( g \) is lci and \( \mathbb{1}_Z \) is \( g \)-pure, it follows that \( g^! \mathbb{1}_Z \) is dualizable, and by 6.1.2 the following canonical maps are isomorphisms:

\[
\begin{align*}
(f^* g^! \mathbb{1}_Z \otimes f^! \mathbb{1}_Y) & \xrightarrow{(6.1.1.3)} f^! g^! \mathbb{1}_Z, \\
(f^* g^! \mathbb{1}_Z \otimes f^! g^* F) & \xrightarrow{(6.1.1.1)} f^!(g^! \mathbb{1}_Z \otimes g^* F).
\end{align*}
\]

Therefore we have the following isomorphism

\[
\begin{align*}
f^* g^* F \otimes f^! g^! \mathbb{1}_Z & \xrightarrow{(6.1.6.3)} f^* g^* F \otimes f^! \mathbb{1}_Y \otimes f^* g^! \mathbb{1}_Z \xrightarrow{(6.1.6.1)} f^! g^* F \otimes f^* g^! \mathbb{1}_Z \\
& \xrightarrow{(6.1.6.4)} f^!(g^* F \otimes g^! \mathbb{1}_Z) \xrightarrow{(6.1.6.2)} g^! F.
\end{align*}
\]

It is straightforward to check that the map (6.1.6.5) is induced by the map (6.1.1.3), and therefore \( g \circ f \) is \( F \)-transversal. \( \square \)

Lemma 6.1.7 ([Sai17, Proposition 8.7]). Let \( f: X \to Y \) be a \( k \)-morphism of schemes. Then the following statements hold:

1. If \( f \) is lci and \( \mathbb{1}_Y \) is \( f \)-pure, then for any \( G \in \mathcal{T}_c(Y) \), \( f \) is \( G \)-transversal if and only if \( f \) is \( \mathbb{D}(G) \)-transversal.

2. If \( X \) is smooth over \( k \) and \( f \) factors through an open subscheme \( Y_0 \) of \( Y \) which is smooth over \( k \), then for any \( F \in \mathcal{T}_c(X) \), \( f \) is \( F \)-transversal if and only if \( f \) is \( \mathbb{D}(F) \)-transversal.
Proof. (1) By 6.1.5, we need to show that $G$ is $f$-pure if and only if $\mathbb{D}(G)$ is $f$-pure. By duality, the map

$$f^*G \otimes \text{Th}_X(L_f) \xrightarrow{(6.1.5.1)} f^!G$$

is an isomorphism if and only if its dual $\mathbb{D}(f^!G) \to \mathbb{D}(f^*G \otimes \text{Th}_X(L_f))$ is an isomorphism. This is equivalent to say that the canonical map

$$f^*\mathbb{D}(G) \xrightarrow{(6.1.5.1)} \mathbb{D}(G) \otimes \text{Th}_X(-L_f)$$

is an isomorphism, i.e. $\mathbb{D}(G)$ is $f$-pure.

(2) We know that the graph $\Gamma_f: X \to X \times_k Y$ is lci and $\mathbb{1}_{X \times_k Y}$ is $\Gamma_f$-pure. By (1) and duality, the following statements are equivalent:

(a) For any $H \in \mathcal{T}_c(Y)$, $\Gamma_f$ is $F \boxtimes_k H$-transversal.
(b) For any $H \in \mathcal{T}_c(Y)$, $\Gamma_f$ is $\mathbb{D}(F \boxtimes_k H)$-transversal.
(c) For any $H \in \mathcal{T}_c(Y)$, $\Gamma_f$ is $\mathbb{D}(F \boxtimes_k \mathbb{D}(H))$-transversal.

Denote by $p_1 : X \times_k Y \to X$ and $p_2 : X \times_k Y \to Y$ the projections. Then $p_2$ is a smooth morphism, and we have the following isomorphism:

$$(\mathbb{D}(F \boxtimes_k \mathbb{D}(H)))^{\mathbb{1}_{X \times_k Y}} \overset{(4.1.15.1)}{=} \text{Hom}(p_1^*F,p_2^*H) \cong \text{Hom}(p_1^*F,p_2^1H) \otimes \text{Th}_{X \times_k Y}(-L_{p_2})$$

(3.1.1.1) $\cong (\mathbb{D}(F) \boxtimes H) \otimes \text{Th}_{X \times_k Y}(-L_{p_2})$.

It follows that $\Gamma_f$ is $F \boxtimes_k H$-transversal for any $H \in \mathcal{T}_c(Y)$ if and only if $\Gamma_f: X \to X \times_k Y$ is $\mathbb{D}(F) \boxtimes_k H$-transversal for any $H \in \mathcal{T}_c(Y)$. This proves (2).

\begin{lemma}[(\text{YZ18, Lemma 2.3.4})] Let $X$ be a smooth $k$-scheme and let $F_1$ and $F_2$ be two objects of $\mathcal{T}_c(X)$. If the diagonal morphism $\delta : X \to X \times_k X$ is $\mathbb{D}(F_1) \boxtimes_k F_2$-transversal, then the following canonical map is an isomorphism:

$$\text{Hom}(F_1, \mathbb{1}) \otimes F_2 \to \text{Hom}(F_1,F_2).$$

\end{lemma}

Proof. Since $X$ is smooth over $k$, by purity we have $K_X \cong \text{Th}_X(L_{X/k})$ and $\delta^* \mathbb{1}_{X \times_k X} \cong \text{Th}(-L_{X/k})$. For $i = 1, 2$, denote by $p_i : X \times_k Y \to X$ the $i$-th projection. Since $\delta$ is $\mathbb{D}(F_1) \boxtimes_k F_2$-transversal, we have the following isomorphism:

$$\text{Hom}(F_1, \mathbb{1}) \otimes F_2 \cong \text{Hom}(F_1,K_X) \otimes F_2 \otimes \delta^* \mathbb{1}_{X \times_k X}$$

$$\overset{(6.1.1.3)}{=} \delta^* \left( \text{Hom}(F_1,K_X) \otimes_k F_2 \right) \otimes \delta^* \mathbb{1}_{X \times_k X}$$

$$\overset{(3.1.1.1)}{=} \delta^* \text{Hom}(p_1^*F_1,p_2^1F_2) \cong \text{Hom}(p_1^*F_1,p_2^1F_2) \otimes \text{Hom}(F_1,F_2).$$

One can check that the map (6.1.8.2) agrees with the canonical map, and the result follows.

\begin{lemma} Let $f : X \to Y$ be a lci morphism and let $F$ and $G$ be two objects of $\mathcal{T}(Y)$. If both $G$ and $\text{Hom}(F,G)$ are $f$-pure, then the following canonical map is an isomorphism:

$$f^* \text{Hom}(F,G) \to \text{Hom}(f^*F,f^*G).$$

\end{lemma}

Proof. By hypothesis, we have the following isomorphism:

$$f^* \text{Hom}(F,G) \otimes \text{Th}_X(L_f) \xrightarrow{(6.1.5.1)} f^! \text{Hom}(F,G) \overset{(2.4.1.3)}{=} \text{Hom}(f^*F,f^!G)$$

$$\xrightarrow{(6.1.5.1)} \text{Hom}(f^*F,f^*G \otimes \text{Th}_X(L_f)) \cong \text{Hom}(f^*F,f^*G) \otimes \text{Th}_X(L_f).$$

It is straightforward to check that (6.1.9.2) is induced by the canonical map, and the result follows.
Corollary 6.1.10. Let \( f : X \to Y \) be a morphism between smooth \( k \)-schemes. Let \( F \) be an object of \( \mathcal{T}_c(Y) \) such that \( f \) is \( F \)-transversal. Then the following canonical map is an isomorphism:

\[
(f^* \text{Hom}(F, \mathbb{1}_Y) \to \text{Hom}(f^* F, \mathbb{1}_X)).
\]

Proof. By Lemma 6.1.7, \( f \) is \( \mathcal{D}(F) \)-transversal. Since \( k_Y \simeq T_h Y(L_Y/k) \) is \( \otimes \)-invertible, we know that \( f \) is also \( \text{Hom}(F, \mathbb{1}_Y) \)-transversal. By hypothesis and 6.1.5, both \( F \) and \( \text{Hom}(F, \mathbb{1}_Y) \) are \( f \)-pure. We conclude by applying Lemma 6.1.9. \( \square \)

Lemma 6.1.11. Let \( p : X \to Y \) and \( g : Y \to S \) be two morphisms of schemes and let \( f = g \circ p : X \to S \) be their composition. Denote by \( \Gamma_p, \Gamma_g \) and \( \Gamma_f \) the graph morphisms of \( p, g \) and \( f \) respectively. Let \( F \) be an object of \( \mathcal{T}(X) \). Then the following statements hold:

1. If \( g \) is smooth and \( p \) is \( F \)-transversal, then \( f \) is \( F \)-transversal.
2. Assume that \( p \) is proper, \( f \) is \( F \)-transversal and the canonical map

\[
p^* \Gamma_1^1 \mathbb{1}_{Y \times_k S} \xrightarrow{(2.3.1.6)} \Gamma_f^1 \mathbb{1}_{X \times_k S}
\]

associated to the Cartesian square of schemes

\[
\begin{array}{ccc}
P \& Y \\
\Gamma_f \downarrow & & \downarrow \Gamma_g \\
X \times_k S & \xrightarrow{p \times \text{id}_S} & Y \times_k S.
\end{array}
\]

is an isomorphism. Then \( g \) is \( p \times (F) \)-transversal.

Proof. (1) We have a commutative diagram of schemes

\[
\begin{array}{ccc}
X & \xrightarrow{\Gamma_p} & X \times_k Y \\
\downarrow \Gamma_f & & \downarrow (\text{id}_X \times_k g) \\
X \times_k S & \to & X \times_k S.
\end{array}
\]

Let \( H \) be an object of \( \mathcal{T}_c(S) \). Since \( g \) is smooth, we have canonical isomorphisms

\[
\Gamma_f^*(F \boxtimes_k H) \otimes \Gamma_f^1 \mathbb{1}_{X \times_k S} = \Gamma_p^*(id_X \times_k g)^*(F \boxtimes_k H) \otimes \Gamma_p^1 (id_X \times_k g)^1 \mathbb{1}_{X \times_k S} \\
\simeq \Gamma_p^*(F \boxtimes_k g^1 H) \otimes \Gamma_p^1 \mathbb{1}_{X \times_k Y}.
\]

(6.1.11.5)

Since \( \Gamma_p \) is \( F \boxtimes_k g^1 H \)-transversal, the following canonical map is an isomorphism:

\[
\Gamma_p^*(F \boxtimes_k g^1 H) \otimes \Gamma_p^1 \mathbb{1}_{X \times_k Y} \xrightarrow{(6.1.1.3)} \Gamma_p^1 (F \boxtimes_k g^1 H).
\]

By (6.1.11.4), (6.1.11.5) and (6.1.11.6), the following canonical map is an isomorphism:

\[
\Gamma_f^*(F \boxtimes_k H) \otimes \Gamma_f^1 \mathbb{1}_{X \times_k S} \xrightarrow{(6.1.1.3)} \Gamma_f^1 (F \boxtimes_k H).
\]

In other words \( \Gamma_f \) is \( F \boxtimes_k H \)-transversal. Since this is true for any \( H \), by definition \( f \) is \( F \)-transversal.

(2) Let \( H \) be an object of \( \mathcal{T}_c(S) \). Then \( \Gamma_f \) is \( F \boxtimes_k H \)-transversal, and the following canonical map is an isomorphism:

\[
\Gamma_f^*(F \boxtimes_k H) \otimes \Gamma_f^1 \mathbb{1}_{X \times_k S} \xrightarrow{(6.1.1.3)} \Gamma_f^1 (F \boxtimes_k H).
\]
Since \( p \) is proper, we have canonical isomorphisms:

\[
p_*(\Gamma_f^* (\mathcal{F} \boxtimes_k H) \otimes \Gamma_f^1 \mathbb{I}_{X \times_k S}) \cong p_*(\Gamma_f^* (\mathcal{F} \boxtimes_k H) \otimes \Gamma_g^1 \mathbb{I}_{Y \times_k S}) \tag{6.1.11.1}
\]

(6.1.11.9)

\[
p_*(\Gamma_f^1 (\mathcal{F} \boxtimes_k H)) \cong p_*(\Gamma_g^1 (\mathcal{F} \boxtimes_k H)) \tag{6.1.13.1}
\]

By (6.1.11.8), (6.1.11.9) and (6.1.11.10), the following canonical map is an isomorphism:

\[
\Gamma_g^* (p_* \mathcal{F} \boxtimes_k H) \otimes \Gamma_g^1 \mathbb{I}_{Y \times_k S} \rightarrow \Gamma_f^1 (p_* \mathcal{F} \boxtimes_k H).
\]

In other words \( \Gamma_g \) is \( p_* \mathcal{F} \boxtimes_k H \)-transversal. Since this is true for any \( H \), by definition \( g \) is \( p_* \mathcal{F} \)-transversal.

**Remark 6.1.12.** Note that the map (6.1.11.1) is an isomorphism if \( p \) is smooth, or if \( g \) factors through an open subscheme of \( S \) which is smooth over \( k \).

**Proposition 6.1.13.** Let \( f_1 : X_1 \rightarrow Y_1 \) and \( f_2 : X_2 \rightarrow Y_2 \) be two morphisms of schemes, and let \( f : X_1 \times_k X_2 \rightarrow Y_1 \times_k Y_2 \) be their product. For \( i = 1, 2 \), let \( G_i \in \mathcal{T}(Y_i) \). Then we have the following K"unneth type result for the transversality condition: if \( f_i \) is \( G_i \)-transversal for \( i = 1, 2 \), then \( f \) is \( G_1 \boxtimes G_2 \)-transversal.

**Proof.** By assumption and Proposition 2.3.5, we have the following isomorphism:

\[
f^* (G_1 \boxtimes_k G_2) \otimes f_1^1 \mathbb{I}_{Y_1 \times_k Y_2} \cong (f_1^* (G_1) \boxtimes_k f_2^* (G_2)) \otimes (f_1^1 \mathbb{I}_{Y_1} \boxtimes_k f_2^1 \mathbb{I}_{Y_2}) \tag{6.1.13.1}
\]

It is straightforward to check that the map (6.1.13.1) agrees with (6.1.13.1), and therefore \( f \) is \( G_1 \boxtimes G_2 \)-transversal.

**Conjecture 6.1.14.** Let \( f_1 : X_1 \rightarrow Y_1 \) and \( f_2 : X_2 \rightarrow Y_2 \) be two morphisms of schemes, and let \( f : X_1 \times_k X_2 \rightarrow Y_1 \times_k Y_2 \) be their product. For \( i = 1, 2 \), let \( F_i \in \mathcal{T}(X_i) \). If \( f_i \) is \( F_i \)-transversal for \( i = 1, 2 \), then \( f \) is \( F_1 \boxtimes_k F_2 \)-transversal.

**Proposition 6.1.15.** Conjecture 6.1.14 is true if \( Y_2 = \text{Spec}(k) \): let \( f_1 : X_1 \rightarrow Y_1 \) be a morphism of schemes. Let \( F_1 \in \mathcal{T}(X_1) \) such that \( f_1 \) is \( F_1 \)-transversal. Then for any \( k \)-scheme \( X_2 \) and any object \( F_2 \in \mathcal{T}(X_2) \), the composition morphism \( f : X_1 \times_k X_2 \rightarrow Y_1 \times_k Y_2 \) is \( p_1^* F_1 \boxtimes_k F_2 \)-transversal.

**Proof.** Denote by \( p_1, p_2, p_3 \) the projections of \( X_1 \times_k Y_1 \times_k X_2 \) to its components. By definition, we need to show that for any \( G \in \mathcal{T}(Y_1) \), the graph of \( f \)

\[
\Gamma_f = \Gamma_{f_1} \times_k id_{X_2} : X_1 \times_k X_2 \rightarrow X_1 \times_k Y_1 \times_k X_2
\]

is \( p_1^* F_1 \boxtimes p_2^* G \boxtimes p_3^* F_2 \)-transversal, namely the following canonical map is an isomorphism:

\[
\Gamma_f^* (p_1^* F_1 \boxtimes p_2^* G \boxtimes p_3^* F_2) \otimes \Gamma_f^1 \mathbb{I}_{X_1 \times_k Y_1 \times_k X_2} \rightarrow \Gamma_f^* (p_1^* F_1 \boxtimes p_2^* G \boxtimes p_3^* F_2). \tag{6.1.15.1}
\]

Since \( \Gamma_{f_1} \) is \( F_1 \boxtimes_k G \)-transversal, the following canonical map is an isomorphism:

\[
\Gamma_{f_1}^* (F_1 \boxtimes_k G) \otimes \Gamma_{f_1}^1 \mathbb{I}_{X_1 \times_k Y_1} \rightarrow \Gamma_{f_1}^* (F_1 \boxtimes_k G). \tag{6.1.15.3}
\]
By Proposition 2.3.5, we have the following isomorphism:

\[
\Gamma_f^*(p_1^*F_1 \otimes p_2^*G \otimes p_3^*F_2) \otimes \Gamma_f^\dagger \mathbb{H}_{X_1 \times_k Y_1 \times_k X_2}
\]

\[
\cong (\Gamma_{f_1}^!(F_1 \boxtimes_k G) \boxtimes_k F_2) \otimes p_1^{12*}\Gamma_{f_1}^\dagger \mathbb{H}_{X_1 \times_k Y_1}
\]

\[
\cong \Gamma_f^!(p_1^*F_1 \otimes p_2^*G \otimes p_3^*F_2).
\]

(6.1.15.4)

(6.1.15.3)

(2.3.1.6)

(6.2.3.1)

(6.1.15.4)

It is straightforward to check that the map (6.1.15.4) agrees with (6.1.15.2), and therefore the latter is an isomorphism, which finishes the proof.

\[\square\]

6.2. Relative Künneth formulas and the relative Verdier pairing. In this section we use the results in Section 6.1 to extend the Künneth formulas to the relative setting, under some transversality assumptions. Using such results we define the relative Verdier pairing as in [YZ18].

6.2.1. Let \(S\) be a scheme, and let \(\pi_1: X_1 \rightarrow S\) and \(\pi_2: X_2 \rightarrow S\) be two morphisms. For \(i = 1, 2\), we denote by \(p_i: X_1 \times_S X_2 \rightarrow X_i\) the projections.

**Lemma 6.2.2.** We use the notations of 6.2.1, and assume that the morphism \(\pi_2\) is smooth. Let \(F_1\) be an object of \(T(X_1)\) such that the morphism \(\pi_1: X_1 \rightarrow S\) is universally \(F_1\)-transversal. Then the canonical closed immersion \(\iota: X_1 \times_S X_2 \rightarrow X_1 \times_k X_2\) is \(F_1 \boxtimes_k F_2\)-transversal for any \(F_2 \in T(X_2)\).

**Proof.** Denote by \(p_{13}: X_1 \times_S X_2 \times_k X_2 \rightarrow X_1 \times_k X_2\) the projection, and the graph of \(p_2\)

\[
(6.2.2.1)
\]

\[
\Gamma_{p_2}: X_1 \times_S X_2 \rightarrow X_1 \times_S X_2 \times_k X_2,
\]

with the commutative diagram

\[
(6.2.2.2)
\]

Let \(F_2\) be an object of \(T(X_2)\). Since \(\pi_1: X_1 \rightarrow S\) is universally \(F_1\)-transversal, the morphism \(p_2: X_1 \times_S X_2 \rightarrow X_2\) is also universally \(p_1^*F_1\)-transversal. Consequently \(\Gamma_{p_2}\) is \(p_1^*F_1 \boxtimes_k F_2 = p_{13}^*(F_1 \boxtimes_k F_2)\)-transversal. By assumption \(p_{13}\) is smooth, and by 6.1.2, \(p_{13}\) is universally \(F_1 \boxtimes_k F_2\)-transversal. By Lemma 6.1.6, the composition \(\iota = p_{13} \circ \Gamma\) is \(F_1 \boxtimes_k F_2\)-transversal, which finishes the proof.

\[\square\]

**Proposition 6.2.3 ([YZ18, Proposition 3.1.3]).** We use the notations of 6.2.1, and let \(E_i\) and \(F_i\) be objects of \(T_c(X_i)\) for \(i = 1, 2\). Assume that the following conditions are satisfied:

1. The morphisms \(\pi_1\) and \(\pi_2\) are smooth, and both \(X_1\) and \(X_2\) are smooth \(k\)-schemes.
2. For \(i = 1, 2\), the diagonal morphism \(X_i \rightarrow X_i \times_k X_i\) is \(\mathbb{D}(E_i) \boxtimes_k F_i\)-transversal.
3. For \(i = 1, 2\), \(\pi_i\) is universally \(E_i\)-transversal and universally \(F_i\)-transversal.

Then the following canonical map is an isomorphism:

\[
(6.2.3.1)
\]

\[
\text{Hom}(E_1, F_1) \boxtimes_S \text{Hom}(E_2, F_2) \xrightarrow{(2.4.2.2)} \text{Hom}(E_1 \boxtimes_S E_2, F_1 \boxtimes_S F_2).
\]

**Proof.** We use the following notation: if \(X\) is a scheme and \(F \in T(X)\), we denote \(F^\vee := \text{Hom}(F, \mathbb{1}_X)\).

By assumption the diagonal morphism \(X_i \rightarrow X_i \times_k X_i\) is \(\mathbb{D}(E_i) \boxtimes_k F_i\)-transversal, and by Lemma 6.1.8 the following canonical map is an isomorphism:

\[
(6.2.3.2)
\]

\[
F_i \otimes E_i^\vee = F_i \otimes \text{Hom}(E_i, \mathbb{1}_{X_i}) \xrightarrow{\sim} \text{Hom}(E_i, F_i).
\]

Hence we have the following isomorphism:

\[
(6.2.3.3)
\]

\[
\text{Hom}(E_1, F_1) \boxtimes_S \text{Hom}(E_2, F_2) \cong (F_1 \boxtimes E_1^\vee) \boxtimes_S (F_2 \boxtimes E_2^\vee) \cong (F_1 \boxtimes_S F_2) \otimes (E_1^\vee \boxtimes_S E_2^\vee).
\]
Denote by \( \iota \) the canonical closed immersion \( \iota: X_1 \times_S X_2 \to X_1 \times_k X_2 \). By Proposition 2.4.3, Corollary 6.1.10 and Lemma 6.2.2, we have the following isomorphism:

\[ (6.2.3.4) \quad E_1^\vee \otimes_S E_2^\vee = \iota^*(E_1^\vee \otimes_k E_2^\vee) = \iota^*(E_1 \otimes_S E_2)^\vee = (E_1 \otimes_S E_2)^\vee. \]

By assumption (2), Lemma 6.1.6, Lemma 6.1.7 and Lemma 6.2.2, the diagonal morphism \( X_i \to X_1 \times_S X_i \) is \( \mathcal{O}(E_i) \otimes_S F_i \)-transversal for \( i = 1, 2 \). By Proposition 6.1.13, the diagonal morphism \( X_1 \times_S X_2 \to (X_1 \times_S X_2) \times_k (X_1 \times_S X_2) = \mathcal{O}(E_1 \otimes_S E_2) \otimes_k (F_1 \otimes_S F_2) \)-transversal. Thus by Lemma 6.1.8, the following canonical map is an isomorphism:

\[ (6.2.3.5) \quad (F_1 \otimes_S F_2) \otimes (E_1 \otimes_S E_2)^\vee \to \mathcal{O}(E_1 \otimes_S E_2, F_1 \otimes_S F_2). \]

We deduce from (6.2.3.3), (6.2.3.4) and (6.2.3.5) that the map (6.2.3.1) is an isomorphism, which finishes the proof.

**Corollary 6.2.4.** We use the notations of 6.2.1, and let \( F_i \) be an object of \( \mathcal{T}_c(X_i) \) for \( i = 1, 2 \). Assume that the following conditions are satisfied:

1. The morphisms \( \pi_1 \) and \( \pi_2 \) are smooth, and both \( X_1 \) and \( X_2 \) are smooth \( k \)-schemes.
2. For \( i = 1, 2 \), \( \pi_i: X_i \to S \) is universally \( F_i \)-transversal.

Then the map

\[ (6.2.4.1) \quad F_1 \otimes_S \mathcal{O}(F_2, \pi_2^1 X_S) \to \mathcal{O}(p^*_2 F_2, p^*_1 F_1) \]

given by the composition

\[ (6.2.4.2) \quad F_1 \otimes_S \mathcal{O}(F_2, \pi_2^1 X_S) \to \mathcal{O}(p^*_2 F_2, p^*_1 F_1 \otimes p^*_2 \pi_2^1 X_S) \to \mathcal{O}(p^*_2 F_2, p^*_1 F_1). \]

is an isomorphism.

**Proof.** Since \( p_1 \) is smooth, by Proposition 6.2.3, we have the following isomorphism

\[ (6.2.4.3) \quad F_1 \otimes_S \mathcal{O}(F_2, \pi_2^1 X_S) \to \mathcal{O}(p^*_2 F_2, p^*_1 F_1 \otimes p^*_2 \pi_2^1 X_S) \]  
\[ \cong \mathcal{O}(p^*_2 F_2, p^*_1 F_1), \]

which shows that the map (6.2.4.1) is an isomorphism.

**Proposition 6.2.5 ([YZ18, Proposition 3.1.9]).** For \( i = 1, 2 \), consider a commutative diagram of \( S \)-morphisms of the form

\[ (6.2.5.1) \quad \begin{array}{ccc} X_i & \xrightarrow{f_i} & Y_i \\ \downarrow \pi_i & & \downarrow q_i \\ S, & & \end{array} \]

We denote \( X := X_1 \times_S X_2, Y := Y_1 \times_S Y_2 \) and \( f := f_1 \times_S f_2: X \to Y \). Let \( M_i \) be objects of \( \mathcal{T}_c(Y_i) \) for \( i = 1, 2 \). Assume that the following conditions are satisfied:

1. The morphisms \( \pi_i \) and \( q_i \) are smooth, and both \( X_i, Y_i \) are smooth \( k \)-schemes.
2. For \( i = 1, 2 \), \( q_i: Y_i \to S \) is universally \( M_i \)-transversal.

Then the following canonical map is an isomorphism:

\[ (6.2.5.2) \quad f^!_1 M_1 \otimes_S f^!_2 M_2 \to f^!(M_1 \otimes_S M_2). \]

**Proof.** By Lemma 2.3.4, we may assume that \( X_2 = Y_2 \) and \( f_2 = id_{X_2} \), i.e. it suffices to show that the following canonical map is an isomorphism:

\[ (6.2.5.3) \quad f^!_1 M_1 \otimes_S M_2 \to (f_1 \times id_{X_2})^!(M_1 \otimes_S M_2). \]
We assume that \( q_2: Y_2 \to S \) is universally \( M_2 \)-transversal, the other case is very similar. By assumption and duality, we have the following isomorphism:

\[
M_2 \xrightarrow{\text{\text{(4.1.14.1)}}} \text{Hom}(\mathbb{D}(M_2), K_{Y_2}) \simeq \text{Hom}(\mathbb{D}(M_2) \otimes \text{Th}(L_{q_2} - T_{Y_2}), q_2^1 \mathbb{1}_S).
\]

In other words \( M_2 \simeq \text{Hom}(L_2, q_2^1 \mathbb{1}_S) \) for some \( L_2 \in \text{T}(Y_2) \). By assumption, \( q_2^1 \mathbb{1}_S \) is a dualizing object in \( \text{T}(Y_2) \), and by Lemma 6.1.7, the morphism \( q_2: Y_2 \to S \) is universally \( L_2 \)-transversal. By Corollary 6.2.4, we have the following isomorphisms:

\[
(6.2.5.5) \quad M_1 \otimes S \text{Hom}(L_2, q_2^1 \mathbb{1}_S) \overset{\text{(6.2.4.1)}}{\simeq} \text{Hom}(p_2^* L_2, p_1^1 M_1),
\]

\[
(6.2.5.6) \quad f_1^! M_1 \otimes S \text{Hom}(L_2, q_2^1 \mathbb{1}_S) \overset{\text{(6.2.4.1)}}{\simeq} \text{Hom}((f_1 \times id_{X_2})^* p_2^* L_2, p_1^1 f_1^! M_1).
\]

We deduce from (6.2.5.5) and (6.2.5.6) the following isomorphism:

\[
(6.2.5.7) \quad (f_1 \times id)^!(M_1 \otimes S M_2) = (f_1 \times id_{X_2})^!(M_1 \otimes S \text{Hom}(L_2, q_2^1 \mathbb{1}_S))
\]

\[
\overset{\text{(6.2.5.5)}}{\simeq} (f_1 \times id_{X_2})^!(\text{Hom}(p_2^* L_2, p_1^1 M_1)) \overset{\text{(2.4.1.3)}}{\simeq} \text{Hom}((f_1 \times id_{X_2})^* p_2^* L_2, (f_1 \times id_{X_2})^! p_1^1 M_1)
\]

\[
= \text{Hom}((f_1 \times id_{X_2})^* p_2^* L_2, p_1^1 f_1^! M_1) \overset{\text{(6.2.5.6)}}{\simeq} f_1^! M_1 \otimes S \text{Hom}(L_2, q_2^1 \mathbb{1}_S) = f_1^! M_1 \otimes S M_2.
\]

One can check that the map (6.2.5.7) agrees with the map (6.2.5.3), and the result follows.

6.2.6. We summarize the relative Künneth formulas we have obtained in Propositions 2.1.20, 6.2.3 and 6.2.5, extending Theorem 2.4.6:

**Theorem 6.2.7.** Let \( S \) be a scheme and let \( f_1: X_1 \to Y_1, f_2: X_2 \to Y_2 \) be two \( S \)-morphisms. Denote by \( f: X_1 \times S X_2 \to Y_1 \times S Y_2 \) be their product. Let \( \text{T} \) be a motivic triangulated category. For \( i = 1, 2 \), consider objects \( L_i \in \text{T}(X_i) \) and \( M_i, N_i \in \text{T}_c(Y_i) \). Then the following maps in Theorem 2.4.6

\[
(6.2.7.1) \quad f_1^* L_1 \otimes S f_2^* L_2 \overset{\text{(2.4.6.1)}}{\to} f^* (L_1 \otimes S L_2)
\]

\[
(6.2.7.2) \quad f_1^! M_1 \otimes S f_2^! M_2 \overset{\text{(2.4.6.3)}}{\to} f^!(M_1 \otimes S M_2)
\]

\[
(6.2.7.3) \quad \text{Hom}(M_1, N_1) \otimes S \text{Hom}(M_2, N_2) \overset{\text{(2.4.6.4)}}{\to} \text{Hom}(M_1 \otimes S M_2, N_1 \otimes S N_2)
\]

are such that

1. If for \( i = 1, 2 \), \( f_i \) is universally strongly locally acyclic relatively to \( L_i \), then then map (6.2.7.1) is an isomorphism.
2. If for \( i = 1, 2 \), both \( X_i \) and \( Y_i \) are smooth over \( S \) and smooth over \( k \), and the structure morphism \( Y_i \to S \) is universally \( M_i \)-transversal, then the map (6.2.7.2) is an isomorphism.
3. For \( i = 1, 2 \), denote by \( q_i: Y_i \to S \) the structure morphism. Then the map (6.2.7.3) is an isomorphism if the following conditions hold:
   a. The morphisms \( q_1 \) and \( q_2 \) are smooth, and \( Y_1 \) and \( Y_2 \) are smooth \( k \)-schemes.
   b. For \( i = 1, 2 \), the diagonal morphism \( Y_i \to Y_i \times_k Y_i \) is \( \mathbb{D}(M_i) \otimes_k N_i \)-transversal.
   c. For \( i = 1, 2 \), \( q_i \) is universally \( M_i \)-transversal and universally \( N_i \)-transversal.

**Remark 6.2.8.** By Proposition 6.3.5 below, the universal transversality conditions in Theorem 6.2.7 can be replaced by strong universal local acyclicity conditions.
6.2.9. Given Corollary 6.2.4, we are now ready to define the relative Verdier pairing in the same way as we have done in Section 3.1. We fix a base scheme $S$, and for any morphism $h : X \to S$ we denote $\mathcal{K}_{X/S} = h^! \mathcal{K}_S$.

Let $X_1$ and $X_2$ be two smooth $S$-schemes which are also smooth over $k$. We denote by $X_{12} = X_1 \times_S X_2$ and $p_i : X_{12} \to X_i$ the projections. Let $L_i \in \mathcal{T}_c(X_i)$ and let $q_i : X_i \to S$ be the structure map for $i = 1, 2$. Let $c : C \to X_{12}$ and $d : D \to X_{12}$ be two morphisms, and let $E = C \times_{X_{12}} D$ with $e : E \to X_{12}$ the canonical morphism. For $i = 1, 2$ denote by $c_i = p_i \circ c : X_i \to X_{12}$ and $d_i = p_i \circ d : D \to X_{12}$. Assume that for $i = 1, 2$, $q_i$ is universally $L_i$-transversal. By Corollary 6.2.4, we produce the following map in the same way as the map (3.1.3.3):

$$c_1 \mathcal{H}om(c_1^* L_1, c_1^* L_2) \otimes d_2 \mathcal{H}om(d_2^* L_2, d_2^* L_2) \to e_* \mathcal{K}_{E/S}$$

(6.2.9.1)

**Definition 6.2.10.** In the situation above, for two maps $u : c_1^* L_1 \to c_2^* L_2$ and $v : d_2^* L_2 \to d_1^* L_1$, we define the relative Verdier pairing

$$\langle u, v \rangle : 1_E \to \mathcal{K}_{E/S}$$

obtained by adjunction from the composition

$$1_{X_{12}} \to c_* 1_C \otimes d_* 1_D \to c_2 \mathcal{H}om(c_1^* L_1, c_1^* L_1) \otimes d_2 \mathcal{H}om(d_2^* L_2, d_2^* L_2)$$

(6.2.10.2)

$$u_* \otimes v_* \to c_2 \mathcal{H}om(c_1^* L_1, c_2^* L_2) \otimes d_2 \mathcal{H}om(d_2^* L_2, d_1^* L_1) \to e_* \mathcal{K}_{E/S}.$$ 

(6.2.9.1)

6.2.11. The relative Verdier pairing satisfies a proper covariance similar to Proposition 3.1.6 (see [YZ18, Theorem 3.3.2]). It satisfies an additivity property along distinguished triangles similar to Theorem 4.2.8.

6.2.12. We can define the relative characteristic class as in Definition 5.1.3:

**Definition 6.2.13.** Let $X$ be a smooth $S$-scheme which is also smooth over $k$. Let $M \in \mathcal{T}_c(X)$ be such that the structure morphism $X \to S$ is universally $M$-transversal. The Verdier pairing in Definition 6.2.10 in the particular case where $C = D = X_1 = X_2 = X$ and $L_1 = L_2 = M$ is a pairing

$$\{ \cdot, \cdot \} : \mathcal{H}om(M, M) \otimes \mathcal{H}om(M, M) \to H_0(X/S),$$

(6.2.13.1)

where $H_0(X/S) = \mathcal{H}om_{\mathcal{T}_c(X)}(1_X, \mathcal{K}_{X/S})$. For any endomorphism $u \in \mathcal{H}om(M, M)$, the **relative characteristic class** of $u$ is defined as the element $C_{X/S}(M, u) = \langle u, 1_M \rangle \in H_0(X/S)$. The relative characteristic class of $M$ is the characteristic class of the identity $C_{X/S}(M) = C_{X/S}(M, 1_M)$.

The following result is similar to Proposition 5.1.15:

**Proposition 6.2.14.** Let $X$ be a smooth $S$-scheme which is also smooth over $k$, with tangent bundle $L_{X/S}$. Then we have $C_{X/S}(1_X(n)) = (-1)^n e(L_{X/S})$.

6.2.15. We now establish a link between the relative characteristic class and the (absolute) characteristic class via specialization of cycles ([DJK18, 4.5.1]). Let $S$ be a smooth $k$-scheme and let $s : \text{Spec}(k) \to S$ be a $k$-rational point. Let $f : X \to S$ be a smooth morphism, and form the Cartesian square

$$\begin{array}{ccc}
X & \xrightarrow{s_X} & X \\
\downarrow f & & \downarrow f \\
S & \xrightarrow{s} & S
\end{array}$$

(6.2.15.1)

Then by [DJK18, 2.2.7(1)], there is a canonical specialization map induced by the base change

$$\Delta_* : H_0(X/S) \to H_0(X_s/k).$$

(6.2.15.2)

**Proposition 6.2.16.** Let $M \in \mathcal{T}_c(X)$ be such that $f : X \to S$ is universally $M$-transversal, and denote by $M_s := M_{|X_s} = s_X^* M \in \mathcal{T}_c(X_s)$. Let $u \in \mathcal{H}om(M, M)$ be an endomorphism of $M$, and denote by $u_s \in \mathcal{H}om(M_s, M_s)$ the induced endomorphism of $M_s$. Then via the specialization map (6.2.15.2), the
relative characteristic class $C_{X/S}(M,u) \in H_0(X/S)$ in Definition 6.2.13 and the characteristic class $C_{X_s}(M_s,u_s) \in H_0(X_s/k)$ in Definition 5.1.3 satisfy

\[(6.2.16.1) \quad \Delta^* C_{X/S}(M,u) = C_{X_s}(M_s,u_s).\]

**Proof.** By Corollary 6.1.10 and Lemma 6.3.4 below, the following canonical map is an isomorphism:

\[(6.2.16.2) \quad s_X^* \mathbb{D}_{X/S}(M) \xrightarrow{\sim} \mathbb{D}_{X_s}(M_s).\]

The result then follows from the following commutative diagram:

\[
\begin{array}{ccccccc}
\mathbb{1}_{X_s} & \xrightarrow{u_s} & \text{Hom}(M_s,M_s) & \xrightarrow{\mathbb{D}_{X_s}(M_s) \otimes M_s} & \sim & M_s \otimes \mathbb{D}_{X_s}(M_s) & \xrightarrow{\epsilon_{M_s}} & K_{X_s} \\
| & | & | & | & | & | & | \\
\downarrow s_X^{-1} & & \downarrow s_X^{-1} & & \downarrow s_X^{-1} & & \downarrow s_X^{-1} & & \downarrow s_X^{-1} \\
\mathbb{1}_X & \xrightarrow{s_X} & \text{Hom}(M,M) & \xrightarrow{s_X^* \mathbb{D}_{X/S}(M) \otimes s_X^* M} & \sim & s_X^* M \otimes s_X^* \mathbb{D}_{X/S}(M) & \xrightarrow{\epsilon_{s_X^* M}} & s_X^* K_{X/S}.
\end{array}
\]

\square

**Example 6.2.17.**

1. Assume that $S$ is a smooth $k$-scheme of dimension $n$. For $T_c = \text{DM}_{cdh,c}$, we have $H_0(X/S) \simeq CH_n(X)$ is the Chow group of $n$-cycles over $X$ (up to $p$-torsion), and the specialization map (6.2.15.2) is Fulton’s specialization map of algebraic cycles ([Ful98, Section 10.1]). By Proposition 6.2.16, the relative characteristic class of a motive can be seen as an $n$-cycle spanned by a family of $0$-cycles given by the characteristic classes of its fibers. It is conjectured that the relative characteristic class is related to the the relative characteristic cycle (see [YZ18, Conjecture 3.2.6]).

2. If we work in $T_c = \text{SH}_c$ and apply the $\mathbb{A}^1$ regulator map with values in the Milnor-Witt spectrum (see 5.3.1), then we get a quadratic refinement of the previous case, namely a family of Chow-Witt 0-cycles ([DJK18, Example 4.5.5]).

### 6.3. Purity, local acyclicity and transversality

In this section we clarify the link between the notions of purity, local acyclicity and transversality conditions. We also study the relation with the Fulton-style specialization map in [DJK18].

**6.3.1.** Let $T$ be a motivic triangulated category which satisfies the condition (RS) in 2.1.12. Let $f : X \to S$ be a morphism of schemes with $S$ smooth over $k$. Recall from 6.1.5 that if $\mathbb{1}_S$ is $f$-pure, then for any $B \in T(X)$, $f$ is $B$-transversal if and only if for any $C \in T_c(S)$, $B \otimes_k C$ is $\Gamma_f$-pure, where $\Gamma_f : X \to X \times_k S$ is the graph of $f$. This amounts to say that the following canonical map is an isomorphism:

\[(6.3.1.1) \quad B \otimes f^* C \otimes f^* \text{Th}_S(-L_{S/k}) \to \mathbb{D}(\mathbb{D}(B) \otimes \mathbb{D}(f^* C)).\]

**6.3.2.** The map (6.3.1.1) is always an isomorphism when $C$ is dualizable.

**Recall 6.3.3.** Consider a Cartesian square of schemes

\[\begin{array}{ccc}
Y & \xrightarrow{g} & X \\
\downarrow g & & \downarrow \Delta \\
T & \xrightarrow{p} & S
\end{array}\]

with $p$ a lci morphism. For $K \in T(X)$, there is a canonical map called **refined purity transformation**

\[(6.3.2.2) \quad q^* K \otimes g^* \text{Th}_T(L_p) \to q^! K\]

([DJK18, Definition 4.2.5]). We say that $K$ is $\Delta$-**pure** if the map (6.3.2.2) is an isomorphism.
Lemma 6.3.4 ([BG02, Lemma B.3]). Consider a Cartesian square of schemes

\[
\begin{array}{ccc}
Y & \xrightarrow{i'} & X \\
g & \downarrow & \downarrow f \\
T & \xrightarrow{i} & S
\end{array}
\]

with \(i\) a regular immersion. Let \(K \in \mathcal{T}(X)\).

(1) If both \(S\) and \(T\) are smooth over \(k\) and \(f\) is \(K\)-transversal, then \(K\) is \(\Delta\)-pure.

(2) If \(\mathbb{I}_S\) is \(i\)-pure and \(f\) is strongly locally acyclic relatively to \(K\), then \(K\) is \(\Delta\)-pure.

Proof. (1) We have an isomorphism

\[
K \otimes f^*i_*\mathbb{I}_T \cong f^*\text{Th}_S(-L_{S/k}) \rightarrow i'_*(i'^*K \otimes g^*\text{Th}_T(-L_{T/k}))
\]

given by the composition

\[
K \otimes f^*i_*\mathbb{I}_T \otimes f^*\text{Th}_S(-L_{S/k}) \cong \mathbb{D}(\mathbb{D}(K) \otimes \mathbb{D}(f^*i_*\mathbb{I}_T)) \\
\cong \mathbb{D}(\mathbb{D}(K) \otimes \mathbb{D}(i'_*g^*\mathbb{I}_T)) \cong i'_*(\mathbb{D}(i'^*K \otimes g^*\text{Th}_T(-L_{T/k})))
\]

where the first isomorphism follows from the transversality condition, and the last isomorphism follows from 6.3.2. The result follows by applying the functor \(i'^*\) to the map (6.3.4.2).

(2) Without loss of generality we can assume that \(i\) is a regular closed immersion. We have a canonical map

\[
K \otimes f^*i_*\mathbb{I}_S \rightarrow i'_*i'^*K
\]

given by the composition

\[
K \otimes f^*i_*\mathbb{I}_S \cong K \otimes i'_*g^*i'_*\mathbb{I}_S \cong K \otimes i'_*g^*\text{Th}_T(L_i) \\
\cong i'_*(i'^*K \otimes g^*\text{Th}_T(L_i)) \rightarrow i'_*i'^*K
\]

where the second isomorphism comes from purity, and the last map is the functor \(i'_*\) applied to the map (6.3.3.2). It follows from the local acyclicity and the localization sequence that the map (6.3.4.3) is an isomorphism, which implies that \(K\) is \(\Delta\)-pure.

\[\Box\]

Proposition 6.3.5 ([BG02, Theorem B.2]). Assume that \(k\) is a perfect field. Let \(f : X \rightarrow S\) be a morphism of schemes which factors through an open subscheme \(S_0\) of \(S\) which is smooth over \(k\). Let \(K \in \mathcal{T}(X)\). We consider Cartesian squares of the form

\[
\begin{array}{ccc}
Y & \xrightarrow{g} & X \\
g & \downarrow & \downarrow f \\
T & \xrightarrow{p} & S
\end{array}
\]

Then the following statements hold:

(1) If for any Cartesian square (6.3.5.1) with \(p\) smooth, \(g\) is strongly locally acyclic relatively to \(q^*K\), then \(f\) is \(K\)-transversal.

(2) If for any Cartesian square (6.3.5.1) with \(p\) smooth, \(g\) is \(q^*K\)-transversal, then \(f\) is strongly locally acyclic relatively to \(K\).

Proof. By hypothesis, \(f\) factors through a morphism \(f_0 : X \rightarrow S_0\). It is easy to see that \(f\) is strongly locally acyclic relatively to \(K\) (resp. \(K\)-transversal) if and only if \(f_0\) is strongly locally acyclic relatively to \(K\) (resp. \(K\)-transversal). Therefore by working with \(f_0\) we can assume that \(S = S_0\) is smooth over \(k\).
We consider a Cartesian square of schemes

\[
\begin{array}{c}
X' \xrightarrow{r} X \\
\downarrow f' \quad \Delta \quad \downarrow f \\
S' \xrightarrow{s} S
\end{array}
\]

where \( S' \) is smooth over \( k \), and a fortiori \( s \) is a lci morphism. Then we have the following diagram

\[
\begin{array}{c}
K \otimes f^* s_* 1_{S'} \otimes f^* Th_S (-L_{S'/k}) \xrightarrow{(a)} D(\mathbb{D}(K) \otimes \mathbb{D}(f^! s_* 1_{S'})) \\
\downarrow (b) \\
r_* r^* K \otimes f^* Th_S (-L_{S'/k}) \xrightarrow{(c)} r_* D(r^! K) \otimes D(f^! 1_{S'}) \\
\downarrow (d) \\
r_* D(r^* K \otimes f'^* Th_{S'} (-L_{S'}/k)) \otimes D(f^! 1_{S'}) \xrightarrow{(e)}
\end{array}
\]

where

- The map (a) is (6.1.5.1).
- The map (b) is (2.1.5.1).
- The map (c) is an isomorphism induced by base change and projection formula.
- The map (d) is an isomorphism deduced from (6.1.1.1) by 6.3.2.
- The map (e) is (6.3.3.2).

One can check that the diagram is commutative.

1. If \( f \) is strongly locally acyclic relatively to \( K \), then the map (b) above is an isomorphism. If the local acyclicity condition holds after any smooth base change, then by Lemma 6.3.4 \( K \) is \( \Delta \)-pure and the map (e) above is an isomorphism, which implies that the map (a) above is an isomorphism. It follows from strong devissage that \( f \) is \( K \)-transversal.
2. If \( f \) is \( K \)-transversal, then the map (a) above is an isomorphism. If the transversality condition holds after any smooth base change, then by Lemma 6.3.4 \( K \) is \( \Delta \)-pure and the map (e) above is an isomorphism, which implies that the map (b) above is an isomorphism. It follows from strong devissage that \( f \) is strongly locally acyclic relatively to \( K \).

\[ \square \]

6.3.6. We now show that the strong local acyclicity is equivalent to the following property, similar to the one in [Sai17, Proposition 8.11]:

**Definition 6.3.7.** Let \( f : X \to S \) be a morphism of schemes and let \( K \in \mathcal{T}(X) \). We say that the morphism \( f \) is **strongly fibrewise locally acyclic** relatively to \( K \) if the following condition holds:

For any schemes \( S' \) and \( S'' \) smooth over a finite extension \( k' \) of \( k \), and for any cartesian diagram of schemes

\[
\begin{array}{c}
X'' \xrightarrow{h} X' \xrightarrow{p'} X \\
\downarrow g \quad \downarrow \quad \downarrow f' \\
S'' \xrightarrow{i} S' \xrightarrow{p} S
\end{array}
\]

where \( p \) is proper and generically finite and \( i \) is a closed immersion, and for any \( L \in \mathcal{T}(S') \) the following composition map is an isomorphism:

\[
(6.3.7.2) \quad h^* (p'^* K) \otimes g^* i^! L \xrightarrow{(2.3.1.6)} h^* (p'^* K) \otimes h^! f'^* L \xrightarrow{(6.1.1.1)} h^! (p'^* K \otimes f'^* L)
\]

We say that \( f \) is **universally strongly fibrewise locally acyclic** relatively to \( K \) if the condition above holds after any base change.
The proof of the following statement is inspired by [CD15, Proposition 7.2]:

**Proposition 6.3.8.** Let \( f : X \to S \) be a morphism of schemes and let \( K \in \mathbb{T}(X) \). Then \( f \) is universally strongly locally acyclic relatively to \( K \) if and only if \( f \) is universally strongly fibrewise locally acyclic relatively to \( K \).

**Proof.** If \( f \) is universally strongly locally acyclic relative to \( F \), then by an argument similar to Lemma 6.3.4 we know that \( f \) is universally strongly fibrewise locally acyclic relatively to \( F \).

Now assume that \( f \) is universally strongly fibrewise locally acyclic relatively to \( F \). Then it follows that for any Cartesian diagram

\[
\begin{array}{ccc}
Y_U & \xrightarrow{j_Y} & Y' \\
\downarrow{f_U} & & \downarrow{f_{T'}} \\
U & \xrightarrow{j} & T'
\end{array}
\]

where \( f_{T'} : Y' \to T' \) is a base change of \( f \), \( q \) is proper and generically finite, \( j \) is an open immersion with complement a strict normal crossing divisor and both \( U \) and \( T' \) are smooth over a finite extension \( k' \) of \( k \), and any \( M \in \mathbb{T}(U) \), the following canonical map is an isomorphism:

\[
K_{\bar{V}} \otimes f_{T'q_*j_*M} \xrightarrow{(2.1.5.1)} q_{Y',Y'}(j_Y q_Y^* K_{\bar{V}} \otimes f_{U} M).
\]

We need to prove that for any Cartesian square

\[
\begin{array}{ccc}
Y_V & \xrightarrow{r_Y} & Y \\
\downarrow{f_V} & & \downarrow{f_T} \\
V & \xrightarrow{r} & T
\end{array}
\]

where \( f_T : Y \to T \) is a base change of \( f \), and any \( N \in \mathbb{T}(V) \), the following canonical map is an isomorphism:

\[
K_{\bar{V}} \otimes f_{T'r_*N} \xrightarrow{(2.1.5.1)} r_{Y',Y'}(r_Y^* K_{\bar{V}} \otimes f_{V} N).
\]

We prove this claim by noetherian induction on \( V \). By the existence of a compactification, we can factor the morphism \( r : V \to T \) above as an open immersion with dense image \( j_1 : V \to \bar{V} \) followed by a proper morphism \( p : \bar{V} \to T \).

\[
\begin{array}{ccc}
Y_V & \xrightarrow{j_{1Y}} & \bar{V} \\
\downarrow{f_V} & & \downarrow{f_{\bar{V}}} \\
V & \xrightarrow{j_1} & \bar{V}
\end{array}
\]

Since \( p \) is proper, it suffices to prove that under the assumption (RS 1) in 2.1.12 (respectively under the assumption (RS 2), for every prime number \( l \) different from the characteristic of \( k \), there exists a non-empty open immersion \( j_2 : V' \to V \) such that the following canonical map is an isomorphism (resp. is an isomorphism with coefficients in \( \mathbb{Z}_{(l)} \)):

\[
K_{\bar{V}} \otimes f_{V'}j_{1*}j_{2*}j_{2,1}^*N \xrightarrow{(2.1.5.1)} j_{1Y,*}(j_{1Y}^* K_{\bar{V}} \otimes f_{V'}j_{2*}j_{2,1}^*N).
\]

We can assume that \( \bar{V} \) is integral. By the assumption (RS 1) (resp. by de Jong-Gabber alteration ([ILO14, X. Theorem 2.1])), there exists a proper surjective morphism \( h : \bar{V} \to \bar{V} \) which is birational (respectively generically flat, generically finite with degree prime to \( l \)) such that \( \bar{V} \) is smooth over \( k \) (resp. smooth over a finite extension \( k' \) of \( k \) of degree prime to \( l \)), and such that the inverse image of \( \bar{V} \setminus V \) in \( \bar{V} \) is a strict normal crossing divisor. Let \( j_2 : V_0 \to \bar{V} \) be an open immersion such that the induced morphism
$h_{V_0} : V'_0 := h^{-1}(V_0) \to V_0$ is an isomorphism (resp. is finite lci with trivial virtual tangent bundle), and form the following Cartesian squares:

\[
\begin{array}{c}
V'_0 \\ h_{V_0} \downarrow \\
V_0 \\
\end{array}
\begin{array}{c}
\overset{j'_2}{\rightarrow} & \overset{i'_2}{\rightarrow} & \overset{\tilde{V}}{\rightarrow} \\
\downarrow h_{V'} & \downarrow h & \downarrow h \\
\overset{j_2}{\rightarrow} & \overset{i_2}{\rightarrow} & \overset{\tilde{V}}{\rightarrow} \\
\end{array}
\]

(6.3.8.7)

It follows that $j_{2*}j'_2^*N$ is equal to $j_{2*}h_{V_0}^{-1}h_{V'}^{-1}j_2^*N = h_{V*}j_{2*}j'_2^* h_{V'}^*N$ (resp. is a direct summand of $h_{V*}j_{2*}j'_2^* h_{V'}^*N$ by [EK18, Proposition 2.2.2]). By (6.3.8.2) the canonical map

\[
K_{V'} \otimes f_{V'}^*j_{1*}h_{V*}h_{V'}^*N \overset{\text{(2.1.5.1)}}{\longrightarrow} j_{1Y*}(j_{1Y}^* K_{[V']} \otimes f_{V'}^* h_{V*}h_{V'}^*N)
\]

is an isomorphism. Given the localization distinguished triangle

\[
h_{V*}i'_2 j'_2^* h_{V'}^*N \to h_{V*}h_{V'}^*N \to h_{V*}j_{2*}j'_2^* h_{V'}^*N \to h_{V*}i'_2 j'_2^* h_{V'}^*N[1]
\]

where $i'_2 : Z'_0 \to V'$ is the reduced closed complement of $j'_1$, and since $h_{V*}i'_2$ factors through $h(Z'_0)$ which is a proper closed subscheme of $V$, we conclude using the induction hypothesis. \hfill \Box

6.3.9. We now establish a link between strong local acyclicity and the Fulton-style specialization map defined in [DJK18]. Consider Cartesian squares of schemes

\[
\begin{array}{c}
X \overset{i_X}{\rightarrow} X \overset{j_X}{\rightarrow} X_U \\
\downarrow f \downarrow \downarrow f_U \\
S \overset{j}{\rightarrow} U
\end{array}
\]

(6.3.9.1)

where $i$ is a regular closed immersion and $j$ the complementary open immersion. Assume that the Euler class $\epsilon(-L_i)$, namely the Euler class of the normal bundle of $i$, is zero. Recall from [DJK18, 4.5.6] that for $A \in \mathfrak{T}(X)$ there is a natural transformation of functors

\[
i_{X*}(i_X^*A \otimes f_{Z}^* Th_Z(L_i)) \to j_{X!}j_X^! A.
\]

(6.3.9.2)

By Lemma 6.3.4, if $\mathcal{S}$ is $i$-pure and if $f$ is strongly locally acyclic relative to $A$, the following refined purity transformation (6.3.3.2) is an isomorphism:

\[
i_X^*A \otimes f_{Z}^* Th_Z(L_i) \to i_X^! A.
\]

(6.3.9.3)

In particular from the construction of (6.3.9.2) we deduce the following:

**Corollary 6.3.10.** We use the notations in (6.3.9.1) and assume that

(1) The Euler class $\epsilon(-L_i)$ is zero.
(2) $\mathcal{S}$ is $i$-pure.
(3) $f$ is strongly locally acyclic relative to $A \in \mathfrak{T}(X)$.

Then there exists a canonical map

\[
i_X^* i_X^! A \to j_{X!} j_X^! A
\]

(6.3.10.1)

such that the canonical map $i_X^* i_X^! A \overset{\text{ad}_{(i_X^*, i_X^!)}}{\longrightarrow} A$ agrees with the following composition

\[
i_X^* i_X^! A \overset{\text{(6.3.10.1)}}{\longrightarrow} j_{X!} j_X^! A \overset{\text{ad}_{(j_{X!}, j_X^!)}}{\longrightarrow} A.
\]

(6.3.10.2)

Equivalently, there exists a canonical map

\[
j_X^* j_X^! A \to i_X^* i_X^! A
\]

(6.3.10.3)
such that the canonical map $A \xrightarrow{\text{ad}(i_X^*j_X^*)} i_X^*i_X^* A$ agrees with the following composition

$A \xrightarrow{\text{ad}(j_X^*i_X^*)} j_X^*j_X^* A \xrightarrow{(6.3.10.3)} i_X^*i_X^* A$.

Note that by Lemma 6.3.4, a similar result holds for the transversality condition when $Z$ and $S$ are smooth over a field.

Remark 6.3.11. Corollary 6.3.10 is a consequence of the strong local acyclicity and therefore gives a criterion to detect it. In the usual derived category of étale sheaves, the vanishing cycle formalism ([SGA7 II, XIII]) gives an insightful interpretation of this phenomena: the failure of local acyclicity is precisely described by the stalks of the vanishing cycle complex. We do not know how to realize such a picture in the motivic world.

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