On the Routh sphere problem

I A Bizyaev and A V Tsiganov

1 Institute of Computer Science, Udmurt State University, Izhevsk, Russia
2 St Petersburg State University, St Petersburg, Russia

E-mail: bizaev-90@mail.ru and andrey.tsiganov@gmail.com

Received 29 October 2012, in final form 13 January 2013
Published 6 February 2013
Online at stacks.iop.org/JPhysA/46/085202

Abstract

We discuss an embedding of a vector field for the nonholonomic Routh sphere into a subgroup of commuting Hamiltonian vector fields on six-dimensional phase space. The corresponding Poisson brackets are reduced to the canonical Poisson brackets on the Lie algebra $e^*(3)$. It allows us to relate the nonholonomic Routh system with the Hamiltonian system on a cotangent bundle to the sphere with a canonical Poisson structure.

PACS numbers: 02.30.Ik, 45.10.Na
Mathematics Subject Classification: 34D20, 70E40, 37J35

1. Introduction

Let us consider a smooth manifold $\mathcal{M}$ with coordinates $x_1, \ldots, x_m$ and a dynamical system defined by the following equations of motion:

$$\dot{x}_i = X_i, \quad i = 1, \ldots, m. \quad (1.1)$$

We can identify this system of ODEs with the vector field

$$X = \sum_{i=1}^{m} X_i \frac{\partial}{\partial x_i}, \quad (1.2)$$

which is a linear operator on a space of the smooth functions on $\mathcal{M}$ that encodes the infinitesimal evolution of any quantity

$$\dot{F} = X(F) = \sum X_i \frac{\partial F}{\partial x_i}$$

along the solutions of the system of equations (1.1).

In Hamiltonian mechanics, a Hamilton function $H$ on $\mathcal{M}$ generates a vector field $X$ describing a dynamical system

$$X = X_H \equiv P \, \partial H. \quad (1.3)$$
Here, $dH$ is a differential of $H$ and $P$ is some bivector on the phase space $\mathcal{M}$. By adding some other assumptions we can prove that $P$ is a Poisson bivector. In fact, it is enough to add energy conservation
\[
\dot{H} = X_H(H) = (P \, dH, \, dH) = 0
\]
and the compatibility of dynamical evolutions associated with two functions $H_{1,2}$
\[
X_{H_1}(X_{H_2}(F)) = X_{H_2}(X_{H_1}(F)) + X_{X_{H_1}(H_2)}(F),
\]
see [1, 11] and references therein.

For a considerable collection of dynamical systems, vector fields are created using a Hamilton function $H$ and a nowhere vanishing smooth function $g$. They are of the form
\[
X = g P \, dH.
\] (1.4)
This vector field $X$ (1.4) is the so-called conformally Hamiltonian vector field.

If the Hamiltonian vector field $X$ has the form (1.4) with respect to the second Poisson bivector, we have the so-called quasi-bi-Hamiltonian system. Among such systems are found the Kepler problem, the Euler problem of two fixed centres, the Jacobi system on the ellipsoid etc. In nonholonomic mechanics, conformally Hamiltonian vector fields appear for the so-called Chaplygin-type systems after a nonholonomic reduction by the symmetry group. These systems include the nonholonomic Veselova and Suslov problems, the Chaplygin ball and their generalizations [3–5, 2, 21, 22].

Below we discuss the nonholonomic Routh sphere problem with the vector field $X$, which is a sum of commuting vector fields $P \, dH_k$ determined by integrals of motion $H_1, \ldots, H_n$:
\[
X = g_1 P \, dH_1 + \cdots + g_n P \, dH_n.
\] (1.5)
Recall that the Hamiltonian vector fields for separable systems always admit such decompositions with respect to the second Poisson structure. For example, such decompositions for the Lagrange and Kowalevski top, for the Toda lattice and Henon–Heiles systems and other integrable systems can be found in [15–18].

In nonholonomic mechanics for separable systems, we have similar decompositions [20, 22]. Moreover, we suppose that such generalized conformally Hamiltonian vector fields naturally appear for the Chaplygin-type systems after a partially nonholonomic reduction by some part of the symmetry group. We try to prove this fact by starting with the nonholonomic Routh sphere problem. Similar partially reduced nonholonomic systems, such as motion of a body of revolution on a plane, motion of a homogeneous ball on a surface of revolution and on a cylindrical surface, will be discussed in forthcoming publications.

If we have decomposition (1.5), then we can see that the common levels of integrals of motion form a Lagrangian foliation associated with the Poisson bivector $P$ with all the ensuing consequences. We want to highlight that we discuss only the properties of foliations and do not discuss the linearization of the corresponding flows.

It is known that a proper momentum mapping for the non-Hamiltonian vector field $X$ associated with the Routh sphere has a ‘focus–focus’ singularity [8]. According to [9], the nontriviality of the corresponding monodromy is the coarsest obstruction to existence of the global action-angle variables. We will observe the marks of this singularity in the corresponding Poisson structure.

2. The Routh sphere

Following [4, 7, 8, 14], let us consider a rolling of a dynamically symmetric and nonbalanced spherical rigid body, the so-called Routh sphere, over a horizontal plane without slipping
under the influence of a constant vertical gravitational force. Dynamically nonbalanced means that the geometric centre differs on the centre of mass, whereas dynamically symmetric means that two momenta of inertia coincide with each other, for instance \( I_1 = I_2 \). The line joining the centre of mass and the geometric centre is an axis of inertial symmetry.

The corresponding symmetry group \( G = E(2) \times SO(2) \) consists of two subgroups. The outer \( E(2) \) symmetry of Routh’s sphere is generated by translations of the horizontal plane and its rotations about a vertical axis. The inner \( SO(2) \) symmetry is generated by rotations of the sphere about the axis of inertial symmetry.

The moving sphere is subject to two kinds of constraint: a holonomic constraint of moving over of a horizontal plane and no slip nonholonomic constraint associated with the zero velocity in the point of contact

\[
v + \omega \times r = 0. \tag{2.1}
\]

Here, \( \omega \) and \( v \) are the angular velocity and velocity of the centre of mass of the ball, respectively, \( r \) is the vector joining the centre of mass with the contact point and \( \times \) means the vector product in \( \mathbb{R}^3 \). All the vectors are expressed in the so-called body frame, which is firmly attached to the ball, its origin is located at the centre of mass of the body and its axes coincide with the principal inertia axes of the body.

In the body frame, the angular momentum \( M \) of the ball with respect to the contact point is equal to

\[
M = I_Q \omega, \quad I_Q = I + m r^2 E - m r \otimes r. \tag{2.2}
\]

Here \( E \) is a unit matrix, \( m \) is a mass and \( I = \text{diag}(I_1, I_2, I_3) \) is an inertia tensor of the rolling ball.

If \( \gamma = (\gamma_1, \gamma_2, \gamma_3) \) is the unit normal vector to the plane at the contact point, then

\[
r = (R \gamma_1, R \gamma_2, R \gamma_3 + a),
\]

where \( R \) is a radius of the ball and \( a \) is a distance from the geometric centre to the centre of mass.

The phase space, initial equations of motion, reduction of symmetries and deriving of the reduced equation of motion are discussed in [4–6, 7, 8]. We omit this step and begin directly with the reduced equations of motion on the six-dimensional phase space \( M \) with local coordinates \( x = (\gamma, M) \):

\[
\dot{M} = M \times \omega + m r \times (\omega \times r) + \gamma \times \frac{\partial U}{\partial \gamma}, \quad \dot{\gamma} = \gamma \times \omega. \tag{2.3}
\]

Here, \( U = -mg(r, \gamma) \) and \( g \) is a gravitational acceleration. These equations on the six-dimensional phase space were obtained using nonholonomic reduction by the \( E(2) \) symmetry subgroup [8]. The completely reduced equations and the corresponding rank-2 Poisson structures are discussed in [4, 12, 13].

A straightforward calculation shows that these equations (2.3) possess four integrals of motion:

\[
H_1 = \frac{1}{2}(M, \omega) + U, \quad H_2 = (M, M) - m r^2 (\omega, M) + 2I_1 U, \quad H_3 = (M, r), \quad H_4 = (\gamma, \gamma), \tag{2.4}
\]

and the following invariant measure:

\[
\mu = g^{-1}(\gamma) d\gamma dM, \quad g(\gamma) = \sqrt{I_1 I_3 + I_1 m R^2 (H_4 - \gamma_3^2) + I_3 m (R \gamma_3 + a)^2}. \tag{2.5}
\]

As for the symmetry Lagrange top, there are two linear in momenta integrals of motion. The first integral

\[
H_3 = (M, r)
\]
is a well-known Jellet integral \[10\], see also section 243, p 192 in \[14\]. The second integral
\[
\hat{H}_2 = g(\gamma)\omega_3
\]
was found by Routh in 1884 \[14\] and recovered later by Chaplygin in \[7\]. These linear integrals
are related to the quadratic in momenta integral of motion
\[
H_2 = 2I_1H_1 + \frac{I_3 - I_1}{I_1}H_1^2 - \frac{m}{I_1}H_3.
\]
According to \[4\] this quadratic integral survives at \(I_1 \neq I_2\) in contrast with the linear integral \(\hat{H}_2\).

The vector field \(X\) defined by equations (2.3) has homoclinic trajectories when the Routh
sphere is either spinning very slowly about a vertical axis which passes through the centre of
mass and the geometric centre, or the axis is in the same position but the value of the Jellet
integral is slightly less than the threshold value \[8\].

2.1. Poisson brackets
For the Routh sphere, six equations of motion (2.3) possess four integrals of motion and an
invariant measure and, therefore, they are integrable by quadratures according to the Euler–
Jacobi theorem. It allows us to suppose that common level surfaces of integrals form a direct
sum of symplectic and Lagrangian foliations of a dual dynamical system which is Hamiltonian
with respect to the Poisson bivector \(P\), so that
\[
[P, P] = 0, \quad P dC_{1,2} \equiv P dH_{i,j} = 0, \quad (P dH_i, dH_k) \equiv \{H_i, H_k\} = 0.
\]
Here \([\ldots]\) is the Schouten bracket and \((i, j, l, m)\) is the arbitrary permutation of \((1, 2, 3, 4)\). In
fact, here we suppose that the Euler–Jacobi integrability of the non-Hamiltonian system (2.3)
is equivalent to the Liouville integrability of the dual Hamiltonian dynamical system with the
same integrals of motion, see \[17\].

The first equation in (2.7) guarantees that \(P\) is a Poisson bivector. In the second equation,
we define two Casimir elements \(C_1 = H_i\) and \(C_2 = H_j\) of bivector \(P\). It is a necessary condition
because by fixing its values one obtains the four-dimensional symplectic phase space of our
dynamical system if we assume that \(\text{rank } P = 4\). The third equation provides that the two
remaining integrals \(H_l\) and \(H_k\) are in involution with respect to the Poisson bracket associated
with \(P\).

In this note, we discuss the solutions of equations (2.7) in the space of the linear in
momenta \(M_i\) and bivectors \(P\) at the different choice of the Casimir functions:

**Case 1.** \(C_1 = (\gamma, \gamma), \quad C_2 = (M, r), \quad H_l = H_1, \quad H_k = H_2;\)

**Case 2.** \(C_1 = (\gamma, \gamma), \quad C_2 = g(\gamma)\omega_3, \quad H_l = H_1, \quad H_k = H_3;\)

**Case 3.** \(C_1 = (M, r), \quad C_2 = g(\gamma)\omega_3, \quad H_l = H_1, \quad H_k = H_4.\)

In the generic case, linear in momenta Casimir functions look like
\[
C_{1,2} = a_{1,2}(\gamma, \gamma) + b_{1,2}(M, r) + c_{1,2}g(\gamma)\omega_3, \quad a_{1,2}, b_{1,2}, c_{1,2} \in \mathbb{C}.
\]
However, the corresponding complete solutions of (2.7) have the same properties as particular
solutions obtained in the above listed three special cases.

In \[16\], we have solved the same system of equations (2.7) for the symmetric Lagrange
top and proved that solutions may be useful for the construction of the variables of separation
and the recursion Lenard–Magri relations for this Hamiltonian system.
If we have some solution \( P \) of (2.7), then we can obtain the decomposition of the initial vector field \( X \) by commuting Hamiltonian vector fields \( P \, dH_I \) and \( P \, dH_k \). The existence of such decomposition by the basis of the Hamiltonian vector fields requires one to impose one more condition rank \( P = 4 \).

### 3. First Poisson bracket

Substituting linear in momenta \( M_i \) ansatz for the entries of the Poisson bivector

\[
P_{ij} = \sum_{k=1}^{3} a_{jk}(\gamma) M_k + b_{ij}(\gamma)  
\]

into (2.7) in case 1 in (2.8) and solving the resulting system of algebro-differential equations, one obtains the following proposition.

**Proposition 1.** In this case, the generic solution of (2.7) is parameterized by two functions \( \alpha(\gamma_1/\gamma_2) \) and \( \beta(\gamma_3) \):

\[
P = \alpha g \begin{pmatrix} 0 & \Gamma_\alpha \\ -\Gamma_\alpha^T & M_\alpha \end{pmatrix} + \beta \begin{pmatrix} 0 & \Gamma_\beta \\ -\Gamma_\beta^T & M_\beta \end{pmatrix}.
\]

(3.2)

Here, matrices \( \Gamma_{\alpha,\beta} \) are equal to

\[
\Gamma_\alpha = \begin{pmatrix} \gamma_1 \gamma_2 (R \gamma_3 + a) & \gamma_1 \gamma_2 (R \gamma_3 + a) & \gamma_1 \gamma_2 (R \gamma_3 + a) \\ \gamma_1 \gamma_2 (R \gamma_3 + a) & \gamma_1 \gamma_2 (R \gamma_3 + a) & \gamma_1 \gamma_2 (R \gamma_3 + a) \\ \gamma_1 \gamma_2 (R \gamma_3 + a) & \gamma_1 \gamma_2 (R \gamma_3 + a) & \gamma_1 \gamma_2 (R \gamma_3 + a) \\ \end{pmatrix},
\]

\[
\Gamma_\beta = \begin{pmatrix} 0 & \frac{\gamma_1 \gamma_2}{\gamma_1^2 + \gamma_2^2} & 0 \\ \frac{\gamma_1 \gamma_2}{\gamma_1^2 + \gamma_2^2} & \gamma_2^2 & \gamma_1 \gamma_2 \\ 0 & \gamma_1 \gamma_2 & 0 \\ \end{pmatrix},
\]

and skew symmetric matrices \( M_{\alpha,\beta} \) have the form

\[
M_\alpha = \begin{pmatrix} 0 & \gamma_1 \gamma_2 (R \gamma_3 + a) & \gamma_1 \gamma_2 (R \gamma_3 + a) \\ \gamma_1 \gamma_2 (R \gamma_3 + a) & 0 & \gamma_1 \gamma_2 (R \gamma_3 + a) \\ \gamma_1 \gamma_2 (R \gamma_3 + a) & \gamma_1 \gamma_2 (R \gamma_3 + a) & 0 \\ \end{pmatrix},
\]

\[
M_\beta = \begin{pmatrix} 0 & \gamma_1 \gamma_2 & \gamma_1 \gamma_2 \\ \gamma_1 \gamma_2 & 0 & \gamma_1 \gamma_2 \\ \gamma_1 \gamma_2 & \gamma_1 \gamma_2 & 0 \\ \end{pmatrix},
\]

where \( g \equiv g(\gamma) \) and

\[
\sigma = mR(m, \gamma)C_2 + I_5(\gamma M_1 + \gamma_2 M_2) + I_1(M_3, \gamma_3).
\]

The proof is a straightforward solution of (2.7) using linear in momenta ansatzs.

The corresponding Poisson brackets read as

\[
[M_1, \gamma_1] = -\alpha g \frac{\gamma_1 \gamma_2 R (\gamma_3 + a)}{\gamma_1^2 + \gamma_2^2} + \frac{\beta \gamma_1 \gamma_2 \gamma_3}{\gamma_1^2 + \gamma_2^2},
\]

\[
[M_1, \gamma_3] = -\beta \gamma_2,
\]

\[
[M_2, \gamma_1] = -\frac{\alpha g R \gamma_2 (R \gamma_3 + a)}{\gamma_1^2 + \gamma_2^2} - \frac{\beta \gamma_1 \gamma_2 \gamma_3}{\gamma_1^2 + \gamma_2^2},
\]

\[
[M_2, \gamma_3] = \beta \gamma_1,
\]

\[
[M_3, \gamma_1] = \alpha g \gamma_2, [M_3, \gamma_2] = -\alpha g \gamma_1, [M_1, \gamma_3] = 0, \quad [\gamma_1, \gamma_3] = 0.
\]

(3.3)
and
\[
[M_1, M_2] = \frac{\alpha g(\gamma_1 M_1 + \gamma_2 M_2)(R\gamma_3 + a)}{(\gamma_1^2 + \gamma_2^2)g^2} + \beta \left( M_3 = \frac{\gamma_3(\gamma_1 M_1 + \gamma_2 M_2)}{\gamma_1^2 + \gamma_2^2} - \frac{\sigma(R\gamma_3 + a)}{g^2} \right),
\]
\[
[M_1, M_3] = -\alpha g M_2 + \frac{\beta \sigma R}{g^2} \gamma_2, \quad \{M_2, M_3\} = \alpha g M_1 - \frac{\beta \sigma R}{g^2} \gamma_1.
\]
In the generic case, rank \( P = 4 \); however, if \( \alpha = 0 \) or \( \beta = 0 \) there are additional Casimir function \( \gamma_1 \) and \( \gamma_1/\gamma_2 \), respectively. A particular form of these brackets was obtained in [13].

Using this Poisson bivector, we can obtain the basis of the commuting Hamiltonian vector fields
\[
X_1 = P \, dH_1 \quad \text{and} \quad X_2 = P \, dH_2,
\]
and try to expand the initial non-Hamiltonian vector field \( X (2.3) \) by these vector fields.

**Proposition 2.** Using Poisson brackets (3.3), we can rewrite the reduced equations of motion for the Routh sphere (2.3) in the following form:
\[
\dot{x}_k = g_1 \{x_k, H_1\} + g_2 \{x_k, H_2\}, \quad k = 1, \ldots, 6,
\]
if and only if
\[
\alpha(\gamma_1/\gamma_2) = \text{const}, \quad \beta(\gamma_3) = \alpha g \left( 1 + \frac{a}{R\gamma_3} \right).
\]
In this case, coefficients are equal to
\[
g_1 = -\frac{(R\gamma_3 + a)I_1 - R\gamma_3 I_3}{\alpha g(I_1 - I_3)(R\gamma_3 + a)}, \quad g_2 = \frac{a}{2\alpha g(I_1 - I_3)(R\gamma_3 + a)}.
\]
The proof is a straightforward verification of equations (3.4).

There are other special values of the functions \( \alpha \) and \( \beta \) according to the following:

**Proposition 3.** If
\[
\alpha(\gamma_1/\gamma_2) = \text{const} \quad \text{and} \quad \beta(\gamma_3) = \frac{\alpha g(I_1 R\gamma_3 - I_3 (R\gamma_3 + a))}{R((I_1 - I_3)\gamma_3 + am(r, \gamma))},
\]
then the Poisson bivector \( P (3.2) \) is compatible with the canonical Poisson bivector \( P_0 \) on the Lie algebra \( e^*(3) \)
\[
P_0 = \begin{pmatrix} 0 & \Gamma & \text{M} \end{pmatrix},
\]
where
\[
\Gamma = \begin{pmatrix} 0 & \gamma_3 & -\gamma_2 \\ -\gamma_3 & 0 & \gamma_1 \\ \gamma_2 & -\gamma_1 & 0 \end{pmatrix}, \quad \text{M} = \begin{pmatrix} 0 & M_3 & -M_2 \\ -M_3 & 0 & M_1 \\ M_2 & -M_1 & 0 \end{pmatrix}.
\]

The proof is a calculation of the Schouten bracket \([P_0, P] = 0\).

Recall that compatibility means that a linear combination of these bivectors
\[
P_0 = P_0 + \lambda P, \quad \lambda \in \mathbb{C},
\]
is a Poisson bivector at any value of \( \lambda \). It also means that \( P (3.2) \) is a trivial deformation of \( P_0 \), see details in [19].

It is interesting that this condition of compatibility allows us to expand the initial vector field (2.3) by a basis of Hamiltonian vector fields
\[
\dot{x}_k = \hat{g}_1 \{x_k, H_1\} + \hat{g}_2 \{x_k, H_2\}, \quad k = 1, \ldots, 6,
\]
associated with the linear in momenta Routh integral \( \hat{H}_2 \) (2.6). Here, coefficients
\[
\hat{g}_1 = -\frac{1}{\beta^3}, \quad \hat{g}_2 = \frac{a}{\alpha I_1 (I_1 R\gamma_3 - I_3 (R\gamma_3 + a))},
\]
deck on coordinates and momenta in contrast with the previous decomposition.
3.1. Properties of the first Poisson brackets

Similar to the Chaplygin sphere problem [21] and nonholonomic Veselova problem [22], we can reduce this Poisson bracket to the canonical Poisson brackets on the Lie algebra $e^*(3)$ and identify the Routh sphere model with the Hamiltonian system on the two-dimensional sphere.

One of the possible reductions is given by the following proposition.

**Proposition 4.** After a change of momenta

\[
\begin{align*}
L_1 &= \frac{1}{\gamma_1^2 + \gamma_2^2} \left( \gamma_1 \gamma_3 \left( R \gamma_1 M_1 + \gamma_2 M_2 - b \gamma_1 I_1 \right) \frac{1}{\alpha g (R \gamma_3 + a)} + \frac{\gamma_2 (\gamma_2 M_1 - \gamma_1 M_2)}{\beta} + c \gamma_1 \right), \\
L_2 &= \frac{1}{\gamma_1^2 + \gamma_2^2} \left( \gamma_2 \gamma_3 \left( R \gamma_1 M_1 + \gamma_2 M_2 - b \gamma_1 I_1 \right) \frac{1}{\alpha g (R \gamma_3 + a)} - \frac{\gamma_1 (\gamma_2 M_1 - \gamma_1 M_2)}{\beta} + c \gamma_2 \right), \\
L_3 &= \frac{M_3}{\alpha g} + \frac{bm(R \gamma_3 + a)}{\alpha g I_1},
\end{align*}
\]

the Poisson brackets \( \{L_i, L_j\} \) coincide with the canonical Poisson brackets on the Lie algebra $e^*(3)$

\[
\{L_i, L_j\} = \epsilon_{ijk} L_k, \quad \{L_i, \gamma_j\} = \epsilon_{ijk} \gamma_k, \quad \{\gamma_i, \gamma_j\} = 0,
\]

where $\epsilon_{ijk}$ is a completely antisymmetric tensor.

It is easy to see that the Poisson map (3.8) is locally defined in the region

\[
\gamma_1^2 + \gamma_2^2 \equiv 1 - \gamma_3^2 \neq 0.
\]

In this region of the phase space the vector field for the Routh sphere $X (2.3)$ does not have homoclinic orbits [8].

At $\alpha = \text{const}$ and $c = 0$ the images of the initial Hamiltonian are the nonhomogeneous second-order polynomial in momenta

\[
H = \frac{1}{2R^2 (1 - \gamma_3^2)} \left( \alpha^2 L_2^2 I_1^2 (R^2 (\gamma_3^2 - 1) - (R \gamma_3 + a)^2 I_3) + \frac{\beta^2 (\gamma_2 L_1 - \gamma_1 L_2)^2}{(I_1 + m(r, r))} \right)
\]

\[
- \frac{2agb (R \gamma_3 + a) L_3}{I_1} + \frac{I_1 + m(R \gamma_3 + a)^2 b^2}{I_1^2} + U(\gamma_3)
\]

which defines a Hamiltonian system on cotangent bundle $T^* S^2$ to the sphere $S^2$.

As for the Lagrange top, the existence of the linear in momenta integral of motion (2.6)

\[
\hat{H}_2 = \alpha I_1 L_3,
\]

allows us to explicitly integrate the corresponding Hamiltonian equations of motion by quadratures. Let us introduce spherical coordinates on the sphere

\[
\begin{align*}
\gamma_1 &= \sin \phi \sin \theta, \quad L_1 = \frac{\sin \phi \cos \theta - p_\phi - \cos \phi p_\theta}{\sin \theta}, \\
\gamma_2 &= \cos \phi \sin \theta, \quad L_2 = \frac{\cos \phi \cos \theta - p_\phi + \sin \phi p_\theta}{\sin \theta}, \\
\gamma_3 &= \cos \theta, \quad L_3 = -p_\phi
\end{align*}
\]

where $\phi, \theta$ are the Euler angles, $p_\phi$ and $p_\theta$ are the canonically conjugated momenta

\[
\{\phi, p_\phi\} = \{\theta, p_\theta\} = 1, \quad \{\phi, \theta\} = \{\phi, p_\theta\} = \{\theta, p_\phi\} = 0.
\]

For these variables, the initial integrals of motion are equal to

\[
H = \frac{A(\theta) p_\phi^2 + B(\theta) p_\theta^2 + bC(\theta) p_\phi + b^2 D(\theta)}{2} + U(\theta), \quad J_2 = -\alpha I_1 p_\phi.
\]
easily find variables of separation Proposition 5. Here, \( b \) is a value of the Jellet integral \( J_1, \alpha = \text{const} \)

\[
A(\theta) = \alpha^2 \left( I_1 + \frac{I_3(a^2 + 2aR \cos \theta + R^2 \cos^2 \theta)}{R^2 \sin^2 \theta} \right), \quad B(\theta) = \frac{\beta^2}{I_1 + m(a^2 + 2aR \cos \theta + R^2)}.
\]

\[
C(\theta) = \frac{2ag(R \cos \theta + a)}{I_1R^2 \sin^2 \theta}, \quad D(\theta) = \frac{I_1 + m(a^2 + 2aR \cos \theta + R^2)}{I_1R^2 \sin^2 \theta}.
\]

Using the expansion of the initial vector field (3.7), one obtains

\[
\dot{\theta} = \{H, \theta\}.
\]

Thus, similar to the Lagrange top, we have a standard equation on the nutation angle

\[
\dot{\theta} = B(\theta)p_\theta = \sqrt{B(\theta)(2E_1 - A(\theta)E_2^2 - bE_2C(\theta) - b^2D(\theta) - 2U(\theta))},
\]

where \( E_1 = H \) and \( E_2 = -J_2/aI_1 \) are constants of motion. Solving this equation by quadrature, one obtains the equation for the second Euler angle

\[
\dot{\phi} = \frac{(I_1 \sin^2 \theta + I_3 \cos^2 \theta) + aI_3 \cos \theta}{g(\theta)I_1R \sin^2 \theta} d\theta - \frac{\cos \theta}{I_1R \sin^2 \theta} b.
\]

### 3.2. Conformally Hamiltonian equations of motion

If the Jellet integral is equal to zero \( C_2 = (M, r) = 0 \), i.e., if \( b = 0 \), then integrals of motion \( H_{1,2} (3.10) \) become homogeneous quadratic polynomials in momenta. In this case, we can easily find variables of separation \( q_{1,2} \) in the corresponding Hamilton–Jacobi equation, if we diagonalize simultaneously two quadratic forms \( H_{1,2} (3.10) \). Then, using these variables of separation, we can rewrite the initial vector field \( X (1.5) \) in the conformally Hamiltonian form (1.4).

At \( C_2 = b = 0 \) integrals of motion \( H_{1,2} \) satisfy the following separated relations:

\[
\Phi_k(q_1, p_1, H_1, H_2) = 0, \quad k = 1, 2.
\]

Here, \( q_{1,2} \) and \( p_{1,2} \) are canonically conjugated variables of separation. In this case, according to [15], these integrals \( H_{1,2} \) are in involution

\[
\{H_1, H_2\}_f = 0
\]

with respect to the Poisson brackets

\[
\{q_1, p_1\}_f = f_1(q_1, p_1), \quad \{q_2, p_2\}_f = f_2(q_2, p_2), \quad \{q_1, q_2\}_f = \{p_1, p_2\}_f = 0,
\]

labelled by two arbitrary functions \( f_{1,2} \). The corresponding Poisson bivector \( P_f \) and integrals of motion \( H_{1,2} \) satisfy the equations

\[
P_f \, dH_i = F_{i1}P \, dH_1 + F_{i2}P \, dH_2, \quad i = 1, 2.
\]

Here, functions \( F_{ij} \) depend on \( f_{1,2} \) and form the so-called control matrix [15, 17].

**Proposition 5.** If \( X (1.5) \) is a linear combination of the commuting Hamiltonian vector fields

\[
X = g_1P \, dH_1 + g_2P \, dH_2,
\]

and coefficients \( g_{1,2} \) are special combinations of \( F_{ij} \)

\[
g_i = g(a_1F_{i1} + a_2F_{i2}), \quad i = 1, 2,
\]

then there is a Poisson bivector \( P_f \), which allows us to rewrite \( X \) in the conformally Hamiltonian form

\[
X = g_1P \, dH_1 + g_2P \, dH_2 = gP \, dH, \quad H = a_1H_1 + a_2H_2.
\]

In this case, \( H \) is a sum of initial physical integrals of motion \( H_{1,2} \).
For the Routh sphere at $C_2 = 0$, variables of separation $q_{1,2}$ are functions only on coordinates $\gamma_i$. Thus, the desired bivector $P_f$ may be directly obtained from $P$ (3.2) at

$$\alpha = -R, \quad \beta = -\frac{g(r, 1r)}{(\gamma', 1r)},$$

so we have

$$X = g_1P \, dH_1 + g_2P \, dH_2 = -\frac{1}{2\beta}P_f \, dH_1.$$

At $C_2 = 0$, variables of separation $q_{1,2}$ have to be functions on coordinates $\gamma_i$ and momenta $M_i$ and, therefore, entries of $P_f$ have to be more complicated functions on $M_i$. Unfortunately, we do not know how to obtain variables of separation for the nonhomogeneous polynomial integrals of motion (3.10) on the sphere.

4. Second and third Poisson brackets

Let us substitute linear in momenta $M_i$ ansatzs (3.1) into equations (2.7) in case 2 in (2.8).

**Proposition 6.** In this case, integrals of motion are in involution $\{H_1, H_2\} = 0$ if and only if the bivector

$$P' = \begin{pmatrix} 0 & \Gamma' \end{pmatrix}$$

is labelled by three arbitrary functions $\alpha_i(\gamma')$ entering into the matrix

$$\Gamma' = \begin{pmatrix} -\gamma_2(\alpha_1 + \gamma_2\alpha_3) & -\gamma_2\alpha_2 + \gamma_3\gamma_1\alpha_3 & \frac{mR\gamma_2(\gamma_1\alpha_1 + \gamma_2\alpha_2)(\gamma_3 + a)}{I_1 + m(\gamma_3 + a)^2} \\ \gamma_1\alpha_1 & \gamma_1\alpha_2 & \frac{mR\gamma_1(\gamma_1\alpha_1 + \gamma_2\alpha_2)(\gamma_3 + a)}{I_1 + m(\gamma_3 + a)^2} \\ -\gamma_1\gamma_2\alpha_3 & -\gamma_1^2\alpha_3 & 0 \end{pmatrix},$$

and into the skew symmetric matrix

$$M_{1,2}' = -\alpha_1M_1 - \alpha_2M_2 + \alpha_3(\gamma_1M_3 - \gamma_3M_1)$$

$$= \frac{\alpha_3\gamma_1R(\gamma_1M_3 + a)(m^2(\gamma', r)H_3 + m(I_1(\gamma_1M_1 + \gamma_2M_2) + I_1\gamma_3M_3))}{g^2},$$

$$M_{1,3}' = \frac{mR(\gamma_3 + a)(\gamma_1\alpha_1 + \gamma_2\alpha_2)}{I_1 + m(\gamma_3 + a)^2}M_2$$

$$+ \frac{\alpha_3\gamma_1\gamma_2R^2(m^2(\gamma', r)H_3 + m(I_3(\gamma_1M_1 + \gamma_2M_2) + I_1\gamma_3M_3))}{g^2}.$$
The proof is a straightforward solution of the differential equations using substitution (3.1).

Using this Poisson bivector, we can obtain a basis of Hamiltonian vector fields and an expansion of the initial vector field

$$X = g_1' P' dH_1 + g_2' P' dH_2$$

by these vector fields. The corresponding coefficients are equal to

$$g_1' = -\frac{1}{2\gamma_2\alpha_3 I_1}, \quad g_2' = \frac{I_1 (I_1 + ma(R\gamma_3 + a)) - mR(\gamma, \mathbf{L})}{2d^2R^2_1 (R\gamma_3 + a)}.$$

Similar to the first Poisson bracket (3.3), there is a transformation of momenta $M_i \to L_i$, which reduces this Poisson brackets to canonical Poisson brackets on the Lie algebra $\mathfrak{e}^* (3)$.

Now let us consider the third possible choice of the linear in momenta Casimir functions, i.e. case 3 in (2.8). In this case, the generic solution of (2.7) coincides with the previous solution $P'$ (4.1)

$$P'' = P'|_{\gamma_1\alpha_1 + \gamma_2\alpha_2 = 0},$$

at $\gamma_1\alpha_1 + \gamma_2\alpha_2 = 0$. It is easy to see that rank $P'' = 3$, and there is a third Casimir function $H_4 = (\gamma, \gamma')$:

$$P'' dC_3 = 0, \quad C_3 = H_4.$$

Consequently, in this case we have only one nontrivial Hamiltonian vector field $P'' dH_1$, which does not form a basis.

In the similar manner, we can consider generic linear in momenta Casimir functions (2.9).

**Proposition 7.** We cannot rewrite equations of motion (2.3) on the six-dimensional phase space in the conformally Hamiltonian form

$$X = gP dF (H_1, H_2, H_3, H_4)$$

using linear in momenta Poisson bivector $P$ satisfying equations (2.7). Here, $F (H_1, H_2, H_3, H_4)$ is an arbitrary function on integrals of motion for the Routh sphere.

The proof is a straightforward verification of the fact that the common system of equations (4.4) and (2.7) is inconsistent if the Poisson bivector $P$ has linear in momenta entries.

5. Conclusion

It is well known that equations of motion for the nonholonomic Routh sphere are integrable by quadratures according to the Euler–Jacobi theorem. We identify the corresponding level sets of integrals of motion with the Lagrangian foliations associated with two different Poisson bivectors $P$ and $P'$. The corresponding expansions of the initial vector field $X$ (3.4) and (4.3)

$$X = g_1 P dH_1 + g_2 P dH_2 = g_1' P' dH_1 + g_2' P' dH_2$$

may be considered as a counterpart of the standard Lenard–Magri recurrence relations

$$X = P dH_1 = f_1 P' dH_2 + f_2 P' dH_3$$

for two-dimensional bi-Hamiltonian systems ($f_1 = 1, \quad f_2 = 0$), quasi bi-Hamiltonian systems ($f_2 = 0$) or bi-integrable systems ($\forall f_{1,2}$), which appear in Hamiltonian mechanics [15–17].
Acknowledgment

We would like to thank AV Bolsinov, AV Borisov and IS Mamaev for useful discussion of applications of the Poisson geometry to the different nonholonomic systems.

References

[1] Abraham R and Marsden J E 1978 Foundations of Mechanics 2nd edn (Reading, MA: Addison-Wesley)
[2] Bolsinov A V, Borisov A V and Mamaev I S 2011 Hamiltonization of nonholonomic systems in the neighborhood of invariant manifolds Regular Chaotic Dyn. 16 443–64
[3] Borisov A V and Mamaev I S 2001 The Chaplygin problem of the rolling motion of a ball is Hamiltonian Math. Notes 70 720–3
[4] Borisov A V and Mamaev I S 2002 The rolling motion of a rigid body on a plane and a sphere. Hierarchy of dynamics Regular Chaotic Dyn. 7 177–200
[5] Borisov A V, Mamaev I S and Kilin A A 2002 The rolling motion of a ball on a surface. New integrals and hierarchy of dynamics Regular Chaotic Dyn. 7 201–19
[6] Borisov A V and Mamaev I S 2008 Conservation laws, hierarchy of dynamics and explicit integration of nonholonomic systems Regular Chaotic Dyn. 13 443–90
[7] Chaplygin S A 1948 On motion of heavy rigid body of revolution on horizontal plane Collected Works vol 1 (Moscow: GITTL) pp 57–75 (in Russian) Chaplygin S A 2002 On a motion of a heavy body of revolution on a horizontal plane Regular Chaotic Dyn. 7 119–30 (Engl. transl.)
[8] Cushman R 1998 Routh’s sphere Rep. Math. Phys. 42 47–70
[9] Duistermaat J J 1986 On global action-angle variables Commun. Pure Appl. Math. 33 687–706
[10] Jellet J H 1872 A Treatise on the Theory of Friction (London: MacMillan)
[11] Jost R 1964 Poisson brackets (an unpedagogical lecture) Rev. Mod. Phys. 36 572–9
[12] Moschuk N K 1987 Reducing the equations of motion of certain nonholonomic Chaplygin systems to Lagrangian and Hamiltonian form J. Appl. Math. Mech. 51 172–7
[13] Ramos A 2004 Poisson structures for reduced non-holonomic systems J. Phys. A: Math. Gen. 37 4821–42
[14] Routh E J 1884 Advanced Rigid Bodies Dynamics (London: MacMillan) Routh E J 1960 Advanced Dynamics of a System of Rigid Bodies (New York: Dover) (reprint)
[15] Tsiganov A V 2007 On the two different bi-Hamiltonian structures for the Toda lattice J. Phys. A: Math. Theor. 40 6395–406
[16] Tsiganov A V 2008 On bi-Hamiltonian geometry of the Lagrange top J. Phys. A: Math. Theor. 41 315212
[17] Tsiganov A V 2011 On bi-integrable natural hamiltonian systems on Riemannian manifolds J. Nonlinear Math. Phys. 18 245–68
[18] Tsiganov A V 2011 On natural Poisson bivectors on the sphere J. Phys. A: Math. Theor. 44 105203
[19] Tsiganov A V 2011 Integrable Euler top and nonholonomic Chaplygin ball J. Geom. Mech. 3 337–62
[20] Tsiganov A V 2012 One invariant measure and different Poisson brackets for two non-holonomic systems Regular Chaotic Dyn. 17 72–96
[21] Tsiganov A V 2012 On the Poisson structures for the nonholonomic Chaplygin and Veselova problems Regular Chaotic Dyn. 17 439–50
[22] Tsiganov A V 2012 One family of conformally Hamiltonian systems Theor. Math. Phys. 173 1481–97