Random vectorial fields representing the local structure of turbulence

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Abstract. We propose a method to build up a random homogeneous, isotropic and incompressible turbulent velocity field that mimics turbulence in the inertial range. The underlying Gaussian field is given by a modified Biot-Savart law. The long range correlated nature of turbulence is then incorporated heuristically using a non linear transformation inspired by the recent fluid deformation imposed by the Euler equations. The resulting velocity field shows a non vanishing mean energy transfer towards the small scales and realistic alignment properties of vorticity with the eigenframe of the deformation rate.

1. The underlying Gaussian vectorial field

This work starts with defining a Gaussian homogeneous, isotropic and incompressible vectorial field \( u(x) \), which correlation structure is consistent with K41 scalings. The first naive way to do this would be to start from a Gaussian homogeneous, isotropic and divergence-free vorticity field \( \omega(x) \) and then, reconstruct the associated velocity field from the Biot-Savart law. To do so, we have to build first this vorticity field using a white vectorial noise. In an equivalent manner, a more straightforward way to do this is to directly build the velocity field from a Gaussian vectorial white noise. It reads in 3-dimensions

\[
u_c(x) = \int_{\mathbb{R}^3} \varphi_L(x-y) \frac{x-y}{|x-y|^{\frac{3}{2} + \frac{q}{2}}} \wedge dW(y),
\]

where \( dW(y) = (dW_1(y), dW_2(y), dW_3(y)) \) is a Gaussian vectorial white noise in 3-dimensions, \( \varphi_L \) a large scale cut-off that introduces the integral length scale \( L \). The deterministic kernel entering in Eq. (1) is regularized over the small length scale \( \epsilon \), namely \( |.|_{\epsilon} = \theta_{\epsilon} * |.| \) (\( * \) stands for the convolution product) with a mollifier \( \theta_{\epsilon}(x) = \frac{1}{\epsilon^3} \theta \left( \frac{x}{\epsilon} \right) \) and \( \int \theta(x)dx = 1 \). It is shown in Ref. (1) that the velocity \( u_c(x) \) has a well-defined limit when \( \epsilon \to 0 \), denoted by \( u(x) \), and such that \( \langle |u(x + \ell e) - u(x)|^q \rangle \sim C_q (\ell/L)^{q/3} \) when \( \ell \to 0 \) with \( C_q \) a constant independent on the vector \( e \).

The Gaussian vectorial field (Eq. (1)) is poor representation of turbulence since it does not reproduce several important features such as a mean energy transfer towards small scales (i.e. the skewness phenomenon), the non gaussianity of velocity increments (i.e. the intermittency...
phenomenon) and the peculiar alignment of vorticity with the intermediate eigenvector of the deformation rate. We will see in the following how to modify this Gaussian vectorial field such that to make it more realistic of fluid turbulence.

2. Linear stretching of an initial Gaussian vorticity field

In this section, we are focussing on the stretching of an initial Gaussian vorticity field, given as \( \omega(x) = \nabla \wedge u(x) \) where \( u(x) \) is the Gaussian velocity field of Eq. 1. Let us now introduce a flavor of Euler dynamics in the picture. The Euler equation writes:

\[
\begin{cases}
\frac{\partial u}{\partial t} + (u \cdot \nabla) u = -\nabla p \\
\nabla \cdot u = 0
\end{cases}
\]  
(2)

It is classical to introduce the vorticity field \( \omega(t, x) = \nabla \wedge u(t, x) \), and take the curl of (2) to eliminate the pressure and get the Beltrami equation:

\[
\frac{\partial \omega}{\partial t} = (\nabla u) \omega - (u \cdot \nabla) \omega,
\]  
(3)

which together with the system \( \nabla \wedge u = \omega \) and \( \nabla \cdot u = 0 \) gives a closed equation in \( \omega(t, x) \). If vorticity vanishes at infinity, the solution of this system is given by the classical Biot-Savart formula:

\[
u(t, x) = -\frac{1}{4\pi} \int \frac{x-y}{|x-y|^3} \wedge \omega(t, y) dy.
\]  
(4)

In what follows, we shall suppose that we have a smooth solution of the system (3), (4) with initial data \( \omega_0 \). Then, it will be convenient to introduce the associated Lagrangian flow \( X(t, x) \) defined by the ordinary differential equation \( \frac{dX(t, x)}{dt} = u(t, X(t, x)) \) and \( X(0, x) = x \). Using \( X(t, x) \), it is easy to see that (3) then writes \( \frac{d\omega(t, X(t, x))}{dt} = (\nabla u) \omega(t, X(t, x)) \) or equivalently:

\[
\frac{d\omega(t, X(t, x))}{dt} = S\omega(t, X(t, x))
\]  
(5)

where \( S \) is the deformation rate tensor defined by the splitting of the tensor \( \nabla u \) into antisymmetric and symmetric parts \( \nabla u = \frac{1}{2} \omega \wedge . + S \). Let us now focus on the short time evolution of the system (5). Since we suppose that the solution is regular, we can linearize (5) in the neighborhood of zero, replacing thus \( S \) by \( S_0 \) (the strain associated to the initial vorticity \( \omega_0 \)), which gives:

\[
\omega(t, x) \approx e^{tS_0} \omega_0(x - tu_0(x))
\]  
(6)

using the fact that \( X(t, x) \approx x + tu_0(x) \). In a first step, we will neglect the advection of the vorticity by the velocity field and only consider the stretching of the vorticity by the initial strain tensor \( S_0 \), which gives at time \( t \):

\[
u(t, x) = -\frac{1}{4\pi} \int \frac{x-y}{|x-y|^3} \wedge e^{tS_0(y)} \omega_0(y) dy.
\]  
(7)

Starting with the Biot-Savart formula, classical calculations (5; 6) give:

\[
S_0(y) = \frac{3}{8\pi} \text{P.V.} \int \left[ \frac{(y - \sigma) \otimes [(y - \sigma) \wedge \omega_0(\sigma)]}{|y - \sigma|^5} + \frac{[(y - \sigma) \wedge \omega_0(\sigma)] \otimes (y - \sigma)}{|y - \sigma|^5} \right] d\sigma,
\]  
(8)

where the integral is understood as a Cauchy Principal Value (P.V.) and \( \otimes \) the tensor product, i.e. \( x \otimes y = x_i y_j \). Now, it is tempting to introduce in formula (7) a random field \( \omega_0 \) which is
Figure 1. Numerical simulations of the process given in (9). (a) PDF of longitudinal (solid line) and transverse (dashed line) velocity gradients for the $N = 512$ case. (b) Skewness ($S$) and flatness ($F$) of longitudinal (open symbols) and transverse (filled symbols) for the three resolutions: $N = 128$ (.), $N = 256$ (□) and $N = 512$ (○). (c) Contour plots of the logarithm of the joint probability of the two invariants of $A$ ($N = 512$ case) non-dimensionalized by the average strain $Q^* = Q/\langle S_{ij}S_{ij} \rangle$ and $R^* = R/\langle S_{ij}S_{ij} \rangle^{3/2}$. The thick line corresponds to the zero discriminant (Vieillefosse) line. Contour lines correspond to probabilities $10^{-4}, 10^{-3}, 10^{-2}, 10^{-1}, 1$. (d) PDF of the cosine of the angle $\theta$ between vorticity and the eigenvectors of the strain (see text) associated to three eigenvalues $\lambda_1$ (dashed-dot), $\lambda_2$ (solid) and $\lambda_3$ (dashed).

divergence-free, homogeneous, isotropic, Gaussian and with K41 scaling, as we did for a Gaussian velocity field (Eq. 1), that is formally (1):

$$\omega_0(x) = \int \frac{x - y}{|x - y|^{3/2 + 1}} \wedge dW(y).$$

A more straightforward way to do this is to take for $\omega_0(y)$ the white noise $dW(y)$ and only change to appropriate values the exponents of the denominators in the kernels giving $u(x)$ ($|x - y|^{-3}$ is replaced by $|x - y|^{-\left(\frac{3}{2} + \frac{1}{3}\right)}$) and $S_0(x)$ ($|x - y|^{-5}$ by $|x - y|^{-\beta}$, with $\beta = 2 + \frac{2}{3} + \frac{2}{3}$). Notice that now, the integral in the modified (8) is no more a principal value. These considerations lead finally to define the random field:

$$u(t, x) = -\frac{1}{4\pi} \int \frac{x - y}{|x - y|^{\beta + 2}} \frac{1}{e^{(t - y)^{\frac{3}{2} + 2}}} \wedge e^{i S_0(y)} dW(y),$$

(9)
with

\[ S_0(y) = \frac{3}{8\pi} \int \left[ \frac{(y - \sigma) \otimes (y - \sigma) \wedge dW(\sigma)}{|y - \sigma|_e^3} + \frac{(y - \sigma) \wedge dW(\sigma) \otimes (y - \sigma)}{|y - \sigma|_e^2} \right] \varphi_L(y - \sigma). \]

We would like now to study the statistical properties of the velocity field defined by Eq. (9). Analytical formulas are difficult to obtain, thus we will focus on numerical simulations. To do so, one has to choose the short time scale \( t = \tau \). It is easy to check that the variance of the matrix \( S_0 \) (defined as \( \langle \text{tr} \ S_0 \rangle \)) goes to infinity as the small scale parameter \( \epsilon \) goes to zero. So we take for \( \tau \) the local normalizing value \( \tau = (\text{tr} \ S_0)^{-1/2} \), in the spirit of the RFD closures provided in Ref. (2).

The simulation is performed in a 1-periodic box with \( N^3 \) collocation points. The infinitesimal volume is given by \( dV = dx^3 \), with \( dx = 1/N \). We choose as a regularizing function and large-scale cut-off the isotropic normalized Gaussian function \( \varphi_L(x) = \theta_L(x) = \left( \frac{2}{\pi L^2} \right)^{3/2} \exp(-6|x|^2/L^2) \). This allows to compute analytically the regularized norm of a vector \( x \), useful for numerical purposes: \( |x|_e = \frac{\sqrt[6]{2}}{\sqrt{\pi}} \epsilon^{-6} |x|^2 + \left( |x| + \frac{\epsilon^2}{12|x|} \right) \operatorname{erf}\left( \frac{\sqrt{6}|x|}{\epsilon} \right) , \) with \( \operatorname{erf} \) the error function. The small-scale cut-off is chosen as \( \epsilon = 2dx \) and the large one as \( L = 1/2 \).

The kernels of the form \( x/|x|^a \) entering in Eq. (9), with \( a = \frac{3}{2} + \frac{3}{2} \) or \( a = \beta \), are estimated in the physical space in a periodic fashion. White noise components \( dW_i \), of zero mean and of variance \( dV \), are generated in the physical space using a standard random Gaussian generator. Convolution products are then performed in the Fourier space. The matrix exponential is evaluated at each point of space using a Padé approximant with scaling and squaring. We choose \( N = 128, 256, 512 \). Results are displayed in Fig. 1.

In Fig. 1(a), we represent the longitudinal and transverse velocity gradient PDFs for the \( N = 512 \) case. We see indeed that the longitudinal PDF is skewed, but not the transverse one (for symmetry reasons). To further characterize the structure in scale of this velocity field, we represent in Fig. 1(b), the dependence on the scale \( \ell \) of the Skewness \( S = \langle (\delta_\ell u)^3 \rangle / \langle (\delta_\ell u)^2 \rangle^{3/2} \) and Flatness \( F = \langle (\delta_\ell u)^4 \rangle / \langle (\delta_\ell u)^2 \rangle^2 \) of the velocity increments \( \delta_\ell u = u(x + \ell) - u(x) \), in both the longitudinal (open symbols) and transverse (filled symbols) cases. We see that \( S \) vanishes and \( F \) is consistent with a Gaussian process (i.e. \( F = 3 \)) in the inertial range. This means that the weak non-Gaussianity observed on the velocity gradients does not survive in the inertial range for the velocity increments. To further characterize the local structure of this field, we represent the joint probability of two important invariants of the velocity gradient tensor, namely \( Q = -\frac{1}{2} \text{tr}(A^2) \) and \( R = -\frac{1}{3} \text{tr}(A^3) \). This so-called RQ-plane has been extensively studied experimentally and numerically. As in empirical data, the RQ-plane is elongated along the right-tale of the Vieillefosse line, showing predominance of both enstrophy-enstrophy production (upper-left quadrant) and dissipation-dissipation production (lower-right) regions. Finally, an important nontrivial property of 3D turbulence is the preferential alignments of vorticity with the intermediate eigenvector of the deformation. We represent in 1(d), the probability density of the cosine of the angle between vorticity and the eigenvectors \( e_{\lambda_i} \) of the deformation \( \theta = (\omega, e_{\lambda_i}) \) with \( \lambda_1 < \lambda_2 < \lambda_3 \). We first see that the vorticity is preferentially orthogonal to \( e_{\lambda_1} \). It has been observed that the vorticity gets preferentially aligned with \( e_{\lambda_2} \). We see in Fig. 1(d) that the opposite is observed in our synthetic field, namely, vorticity gets preferentially aligned with \( e_{\lambda_3} \). In the following, we will see that including multifractality will allow us to predict, among other features, correct alignments.

3. A realistic representation of the local structure of turbulence

Based on the recent fluid deformation imposed by the Euler flow (2), and further heuristic introduction of the multifractal structure as observed on empirical data (3), we proposed in Ref.
(4) the following 3d random vectorial field:

\[ \mathbf{u}_c(x) = \int_{\mathbb{R}^3} \varphi_L(x-y) \frac{x-y}{|x-y|^\frac{3}{2}+\frac{3}{4}} \wedge e^{S_c(y)} dW(y), \]  

where \( S \) is a tensorial Gaussian log-correlated noise of the form

\[ S_c(y) = \sqrt{\frac{5}{4\pi}} \lambda \int_{|y-\sigma| \leq L} (y-\sigma) \otimes [(y-\sigma) \wedge dW(\sigma)] + [(y-\sigma) \wedge dW(\sigma)] \otimes (y-\sigma). \]  

The form of the symmetric matrix \( S \) is inspired by the recent fluid deformation closure experienced by the fluid at short time (4) and the exponent \( \frac{3}{2} \) is such that the components of \( S \) are correlated logarithmically in space, following previous works (7; 8) proposing a stochastic representation of an intermittent and long range correlated field. A free parameter \( \lambda \) enters this construction and governs the level of intermittency of the field. We will take in the sequel \( \lambda^2 = 0.025 \) such as to be consistent with empirical findings (3).
Simulations of the vectorial field $u_i(x)$ can be done accurately and efficiently in periodic boxes using up to $1024^3$ collocations points. The computation of the matrix exponential is the limiting numerical step. It is estimated at each point of space using a Padé approximant with scaling and squaring (see (4) for details). We show in Fig. 2 several key statistical quantities obtained in our numerical simulations.

We start with considering the probability density functions (PDFs) of the longitudinal velocity increments at various scales. The continuous shape deformation of the PDFs from Gaussian statistics at the integral length scale $L$ down to highly stretched exponential shapes at the smallest scale $\epsilon$ as observed in experimental data (3) and characteristic of the intermittency phenomenon is shown in Fig. 2(a). Note also the slight asymmetry of the PDFs characteristic of mean energy transfer (skewness). We display in Fig. 2(b) the scale dependence of the Skewness $S$ and Flatness $F$ for three resolutions $N$. We can observe that the Flatness behaves as a power-law of exponent related to the parameter $\lambda$. A more detailed discussion on the intermittent properties of the velocity field is provided in Ref. (4). The Skewness is non zero in the longitudinal case and vanishes in the transverse case.

The so-called RQ-plane shows the characteristic teardrop shape as observed in experiments and underlines the predominance of enstrophy-enstrophy production and dissipation-dissipation production regions. Finally, we represent in Fig. 2(d) the probability density of the cosine of the angle between vorticity and the eigenvectors of the deformation rate. We can see the preferential alignment of vorticity with the eigenvector associated to the intermediate eigenvalue.

The proposed random velocity field (Eq. (10)) can be thus viewed as a stochastic representation of the local structure of turbulence, as depicted in the seminal works of Kolmogorov and Oboukhov (9; 10).

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