"Local Realism", Bell’s Theorem and Quantum "Locally Realistic" Inequalities

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Based on the new general framework for the probabilistic description of experiments, introduced in Ref. 6, 7, we analyze in mathematical terms the link between the validity of Bell-type inequalities under joint experiments upon a system of any type and the physical concept of "local realism". We prove that the violation of Bell-type inequalities in the quantum case has no connection with the violation of "local realism". In a general setting, we formulate in mathematical terms a condition on "local realism" under a joint experiment and consider examples of quantum "locally realistic" joint experiments. We, in particular, show that quantum joint experiments of the Alice/Bob type are "locally realistic". For an arbitrary bipartite quantum state, we derive quantum analogs of the original Bell inequality. In view of our results, we argue that the violation of Bell-type inequalities in the quantum case cannot be a valid argument in the discussion on locality or non-locality of quantum interactions.

Keywords: information states, joint experiments, Bell-type inequalities, locality

1 INTRODUCTION

The Bell inequality (Refs. 1, 2) and the Clauser-Horne-Shimony-Holt (CHSH) inequality (Ref. 3) describe the relation between the statistical data observed under joint measurements. The original
derivations of these inequalities (and their further numerous generalizations and strengthenings) are based on the structure of probability theory\(^1\) associated with the formalism of random variables. The latter probabilistic formalism is often referred to as \textit{classical} probability.

The sufficient mathematical condition used for the derivation of the above inequalities in the \textit{classical} probabilistic frame is usually linked with the physical concept of "\textit{local realism}". The latter refers (see, for example, in Ref. 5, page 160) to those situations where, under a joint experiment, set-ups of marginal experiments are chosen independently.

In the quantum case, the Bell inequality is, in general, violated and Bell’s theorem\(^2\) states that a "locally realistic" model cannot describe statistics under joint quantum measurements.

In the present paper, we analyze this statement from the point of view of the general framework for the probabilistic description of experiments introduced in Refs. 6, 7. Based on the notions of an information state and a generalized observable, this new probabilistic formalism allows to describe both classical and quantum measurements in a unified way.

In Sec. 2, we discuss in a general setting the description of a joint experiment performed on a system of any type represented initially by an information state.

In Sec. 3, we formulate a general mathematical condition sufficient for the validity a CHSH-form inequality under joint measurements\(^3\) upon a system of any type. This sufficient condition concerns only a factorizable form of joint generalized observables describing the corresponding joint measurements and does not, in general, result in the existence of a local hidden variable model for a system information state. We underline that though joint generalized observables describing classical joint measurements are factorizable, the converse is not true and factorizable generalized observables may represent quantum joint measurements.

For factorizable generalized observables, we further specify the general condition sufficient for the validity of the original Bell inequality\(^4\). We prove that Bell’s correlation restriction on the observed outcomes (see in Refs. 1, 2) represents only a particular case of the general sufficient condition that we introduce in this paper. Under the latter sufficient condition, the Bell inequality holds even if the observed outcomes are not perfectly correlated or anticorrelated.

We discuss possible mathematical reasons for the violation of a CHSH-form inequality and point out that the sufficient condition for its validity does not, in general, represent mathematically the

\(^1\)See, for example, in Ref. 4.

\(^2\)As explained by A Shimony in private communication, the first published use of the term \textit{Bell’s theorem} appeared in Ref. 3 and referred to the acknowledgment of J. Bell’s results in Ref. 1.

\(^3\)An experiment with real-valued outcomes is usually referred to as a measurement.

\(^4\)Our derivation of the Bell inequality is valid for any type of outcomes and does not exploit generally accepted "measurement result" restrictions introduced in Ref. 1, 2.
physical concept of "local realism".

In Sec. 4, we formulate in mathematical terms a general condition on "local realism" under a joint experiment upon a system of any type and consider examples of quantum "locally realistic"5 joint experiments. We, in particular, show that quantum joint measurements of the Alice/Bob type are "locally realistic".

From our presentation it follows that, under "locally realistic" joint measurements, a CHSH-form inequality may be violated whenever joint generalized observables, describing these measurements, do not have a factorizable form. The latter is just a general situation under quantum joint measurements of Alice and Bob.

In Sec. 5, for an arbitrary bipartite quantum state, we derive quantum analogs of the original Bell inequality.

In Sec. 6, we argue that the violation of Bell-type inequalities under quantum Alice/Bob joint measurements does not point to non-locality of quantum interactions.

2 DESCRIPTION OF JOINT MEASUREMENTS

Consider the description of an experiment with outcomes in a set Λ and performed on a system of any type.

Let, before an experiment, a system be characterized in terms of some properties θ ∈ Θ of any nature and the uncertainty of possible θ be specified by a σ-algebra F_Θ of subsets of Θ and a probability distribution π on (Θ, F_Θ).

We refer to a measurable space (Θ, F_Θ) as a system information space and call a triple

\[ I := (Θ, F_Θ, π) \] (1)

an information state of a system (see Refs. 6, 7, for details). We say that an information state I has the support on a set F ∈ F_Θ if π(F) = 1.

The above mathematical setting on initial representation of a system is rather general and covers a broad class of probabilistic situations arising under the description of experiments, in particular, those of classical probability, of quantum measurement theory and, more generally, all those situations where each θ is interpreted as a "bit" of information available on a system and the uncertainty of possible "bits" is specified by a probability distribution π.

5If, under a joint experiment, the physical concept of "local realism" is not violated then, for short, we refer to this joint experiment as "locally realistic".
According to our consideration in Ref. 6, 7, any experiment, with an outcome set \( \Lambda \), performed on a system, described initially by an information space \((\Theta, \mathcal{F}_\Theta)\), is uniquely represented on this information space by a \textit{generalized observable} \( \Pi \). If \( I \) is a system initial information state then the probability \( \mu^{(\Pi)}(B; I) \) that an outcome \( \lambda \) belongs to a subset \( B \) of \( \Lambda \) is given by (see Ref. 6, 7):

\[
\mu^{(\Pi)}(B; I) = \int_\Theta (\Pi(B))(\theta)\pi(d\theta),
\]

where: (i) for any outcome subset \( B \subseteq \Lambda \), the real-valued function \((\Pi(B))(\cdot)\) on \((\Theta, \mathcal{F}_\Theta)\) is measurable, with values in \([0,1]\); (ii) for any \( \theta \in \Theta \), the mapping \((\Pi(\cdot))(\theta)\) represents a probability distribution of outcomes in \( \Lambda \). Thus, \( \Pi \) is a normalized measure with values \( \Pi(B) \), \( \forall B \subseteq \Lambda \), that are nonnegative real-valued measurable functions on \((\Theta, \mathcal{F}_\Theta)\). By its measure structure, \( \Pi \) is similar to the notion of a positive operator-valued (POV) measure in quantum measurement theory.\(^6\)

If a system initial information space \((\Theta, \mathcal{F}_\Theta)\) provides "no knowledge"\(^7\) on an experiment upon this system then this experiment is represented on \((\Theta, \mathcal{F}_\Theta)\) by a \textit{trivial} generalized observable \((\Pi(B))(\cdot)\) \(\forall \theta \in \Theta\). Thus, \( \Pi \) is a normalized measure with values \( \Pi(B) \), \( \forall B \subseteq \Lambda \), that are nonnegative real-valued measurable functions on \((\Theta, \mathcal{F}_\Theta)\). By its measure structure, \( \Pi \) is similar to the notion of a positive operator-valued (POV) measure in quantum measurement theory.\(^6\)

Let \( \Pi \), with an outcome set \( \Lambda_1 \times \Lambda_2 \), be a generalized observable, representing on \((\Theta, \mathcal{F}_\Theta)\) a \textit{joint} experiment. For any \( B_1 \subseteq \Lambda_1 \), \( B_2 \subseteq \Lambda_2 \), the relations

\[
\Pi_1(B_1) := \Pi(B_1 \times \Lambda_2), \quad \Pi_2(B_2) := \Pi(\Lambda_1 \times B_2)
\]

(3)

define the generalized observables \( \Pi_1 \), with the outcome set \( \Lambda_1 \), and \( \Pi_2 \), with the outcome set \( \Lambda_2 \). Each of the latter generalized observables is called \textit{marginal} and represents on \((\Theta, \mathcal{F}_\Theta)\) the corresponding \textit{marginal} experiment.\(^9\) With respect to \( \Pi_1 \) and \( \Pi_2 \), the generalized observable \( \Pi \) is called \textit{joint}.

We further consider joint experiments with real-valued outcomes \( \lambda_i \in \Lambda_i \) (that is, joint measurements) and, for simplicity, suppose that outcomes are bounded \( |\lambda_i| \leq C_i \), \( i = 1, 2 \).

Under a joint measurement performed on a system in an initial information state \( \mathcal{I} \), consider the expectation values

\[
\langle \lambda_i \rangle_{\mathcal{I}}^{(\Pi)} := \int_{\Lambda_1 \times \Lambda_2} \lambda_i \mu^{(\Pi)}(d\lambda_1 \times d\lambda_2; \mathcal{I}), \quad i = 1, 2,
\]

(4)

\(^6\)For quantum measurement theory, see Refs. 8 - 10.

\(^7\)In this case, the probability distribution of outcomes does not depend on information on a system specified by \((\Theta, \mathcal{F}_\Theta)\).

\(^8\)An experiment with outcomes in a product set \( \Lambda_1 \times \Lambda_2 \) is called \textit{joint}.

\(^9\)That is, such an experimental situation where under a joint experiment outcomes either in \( \Lambda_2 \) or in \( \Lambda_1 \) are ignored completely.
of the observed outcomes $\lambda_i \in \Lambda_i$ and the expectation value
\[
\langle \lambda_1 \lambda_2 \rangle_{I}^{(\Pi)} := \int_{\Lambda_1 \times \Lambda_2} \lambda_1 \lambda_2 \mu^{(\Pi)}(d\lambda_1 \times d\lambda_2; I)
\] (5)
of the product $\lambda_1 \lambda_2$ of the observed outcomes. Due to Eqs. (2) and (3), we have:
\[
\langle \lambda_i \rangle_{I}^{(\Pi)} = \int_{\Lambda_i} \int_{\Theta} \lambda_i (\Pi_i(d\lambda_i))(\theta) \pi(d\theta), \quad i = 1, 2,
\] (6)
\[
\langle \lambda_1 \lambda_2 \rangle_{I}^{(\Pi)} = \int_{\Lambda_1 \times \Lambda_2} \int_{\Theta} \lambda_1 \lambda_2 (\Pi(d\lambda_1 \times d\lambda_2))(\theta) \pi(d\theta),
\] (7)
where
\[
f_i(\theta) := \int_{\Lambda_i} \lambda_i (\Pi_i(d\lambda_i))(\theta), \quad |f_i(\theta)| \leq C_i, \quad \forall \theta \in \Theta, \quad i = 1, 2,
\] (8)
\[
f_{\text{joint}}(\theta) := \int_{\Lambda_1 \times \Lambda_2} \lambda_1 \lambda_2 (\Pi(d\lambda_1 \times d\lambda_2))(\theta), \quad |f_{\text{joint}}(\theta)| \leq C_1 C_2, \quad \forall \theta \in \Theta,
\] (9)
are real-valued measurable functions on $(\Theta, \mathcal{F}_\Theta)$. (In probability theory, a measurable real-valued function is usually referred to as a random variable, see, for example, in Ref. 4.)

From Eqs. (6) - (9) it follows that, under a joint measurement upon a system of any type represented initially by an information state $I$, the expectation values are always expressed in terms of random variables on $(\Theta, \mathcal{F}_\Theta)$.

However, in contrast to the generally accepted formalism of probability theory (see, for example, in Ref. 4), the values of random variables $f_i$, $i = 1, 2$, in Eq. (8) do not, in general, represent outcomes observed under this joint measurement while the random variable $f_{\text{joint}}$ in Eq. (7) does not, in general, coincide with the product $f_1 f_2$.

**Remark 1 (On classical measurements)** The values of the random variables $f_1$ and $f_2$ in (8) do represent the observed real-valued outcomes iff a joint generalized observable has the "image" form (see in Ref. 6, 7), that is:
\[
(\Pi^{(\text{image})}(B_1 \times B_2))(\theta) = \chi_{f_1^{-1}(B_1) \cap f_2^{-1}(B_2)}(\theta)
\] (10)
for any $\theta \in \Theta$ and any outcome subsets $B_1 \subseteq \Lambda_1$, $B_2 \subseteq \Lambda_2$. Here, $\chi_F(\theta)$ is an indicator function\(^{10}\) of a set $F \in \mathcal{F}_\Theta$ and
\[
f_i^{-1}(B_i) := \{ \theta \in \Theta : f_i(\theta) \in B_i \}
\] (11)
\(^{10}\)That is: $\chi_F(\theta) = 1$, $\forall \theta \in F$ and $\chi_F(\theta) = 0$, $\forall \theta \notin F$. 

is the preimage in $\mathcal{F}_\Theta$ of a subset $B_i \subseteq \Lambda_i$. "Image" generalized observables describe ideal measurements on a classical system (usually referred to as classical measurements).

Due to Eqs. (2), (7) - (10), under a classical joint measurement, the probability distribution $\mu^{(cl)}$ of outcomes in $\Lambda_1 \times \Lambda_2$ has the "image" form:

$$\mu^{(cl)}(B_1 \times B_2; I) = \pi(f_1^{-1}(B_1) \cap f_2^{-1}(B_2)), \quad (12)$$

the random variable $f_{\text{joint}} = f_1f_2$ and the expectation value of the product of outcomes is given by:

$$\langle \lambda_1 \lambda_2 \rangle^{(\text{class})}_I = \int_{\Theta} f_1(\theta)f_2(\theta)\pi(d\theta). \quad (13)$$

As we discuss this in detail in Sec. 3.1, the relation $f_{\text{joint}} = f_1f_2$ holds not only for a classical joint measurement but for any joint measurement described on $(\Theta, \mathcal{F}_\Theta)$ by a generalized observable of the product form (see Eq. (19)). However, in the latter case, the values of $f_1$ and $f_2$ do not, in general, represent the observed outcomes.

Consider now two joint measurements, performed on a system of any type and represented on a system information space $(\Theta, \mathcal{F}_\Theta)$ by joint generalized observables $\Pi^{(1)}$ and $\Pi^{(2)}$. From Eq. (7) it follows:

$$\langle \lambda_1 \lambda_2 \rangle^{(\Pi^{(1)})}_I \pm \langle \lambda_1 \lambda_2 \rangle^{(\Pi^{(2)})}_I = \int_{\Theta} \{f_{\text{joint}}^{(1)}(\theta) \pm f_{\text{joint}}^{(2)}(\theta)\} \pi(d\theta). \quad (14)$$

Due to the relation

$$|x - y| \leq 1 - xy, \quad (15)$$

valid for any real numbers $|x| \leq 1, |y| \leq 1$, the inequality

$$\left| \langle \lambda_1 \lambda_2 \rangle^{(\Pi^{(1)})}_I \pm \langle \lambda_1 \lambda_2 \rangle^{(\Pi^{(2)})}_I \right| \leq C_1C_2 \pm \frac{1}{C_1C_2} \langle \lambda_1 \lambda_2 \rangle^{(\Pi^{(1)})}_I \langle \lambda_1 \lambda_2 \rangle^{(\Pi^{(2)})}_I \quad (16)$$

holds for any information state $I$ and any generalized observables $\Pi^{(1)}$ and $\Pi^{(2)}$.

In the following section, we derive an upper bound of the expression (14) for joint generalized observables of the special type.

### 3 FACTORIZABLE GENERALIZED OBSERVABLES

We say that a joint generalized observable $\Pi$ on an information space $(\Theta, \mathcal{F}_\Theta)$ is factorizable on a set $F \in \mathcal{F}_\Theta$ if $\Pi$ admits a representation

$$\langle \Pi(B_1 \times B_2) \rangle(\theta) = \int_{\Omega} (\Pi_{1,\omega}(B_1))(\theta)(\Pi_{2,\omega}(B_2))(\theta) \nu(d\omega), \quad (17)$$
for any $\theta \in F$ and any outcome subsets $B_i \subseteq \Lambda_i$, $i = 1, 2$. Here: (i) $(\Omega, \mathcal{F}_\Omega)$ is some measurable space; (ii) $\nu$ is a probability distribution on $(\Omega, \mathcal{F}_\Omega)$; (iii) $\Pi_{1,\omega}$ and $\Pi_{2,\omega}$ are generalized observables on $(\Theta, \mathcal{F}_\Theta)$ with outcome sets $\Lambda_1$ and $\Lambda_2$, respectively. To express Eq. (17) in short, we use the notation\textsuperscript{11}

$$
\Pi \overset{E}{=} \int_\Omega \Pi_{1,\omega} \times \Pi_{2,\omega} \nu(d\omega), 
$$

(18)

and we omit "$F$" whenever $\Pi$ is factorizable on all of a set $\Theta$.

If, in particular, $\nu = \delta_{\omega_0}, \forall \omega_0 \in \Omega$, is a Dirac measure, then, in Eq. (18), a joint generalized observable

$$
\Pi \overset{E}{=} \Pi_{1,\omega_0} \times \Pi_{2,\omega_0}
$$

(19)

has the product form on a set $F \in \mathcal{F}_\Theta$, with the generalized observables $\Pi_{1,\omega_0}$ and $\Pi_{2,\omega_0}$ representing the marginal experiments.

Notice that an "image" generalized observable (10), representing on $(\Theta, \mathcal{F}_\Theta)$ a classical joint measurement, is of the product form on all of $\Theta$.

### 3.1 Bell-type Inequalities

For simplicity, we first consider the case of product generalized observables.

For a joint measurement with outcomes $|\lambda_1| \leq C_1$, $|\lambda_2| \leq C_2$, let the corresponding generalized observable on $(\Theta, \mathcal{F}_\Theta)$ be product (see Eq. (19)) and have the form:

$$
\Pi^{(a,b)} \overset{E}{=} \Pi_1^{(a)} \times \Pi_2^{(b)}, \quad \forall F \in \mathcal{F}_\Theta.
$$

(20)

Here, in the left-hand side, a parameter standing in the first place of a pair specifies a set-up of the marginal measurement with outcomes in $\Lambda_1$ while a parameter standing in the second place of a pair - a set-up of the marginal measurement with outcomes in $\Lambda_2$. In the right-hand, the lower indices refer to outcome sets $\Lambda_1$ and $\Lambda_2$.

For a generalized observable of the form (20), the corresponding random variable $f_{\text{joint}}^{(a,b)}$ in Eq. (21) has the product form:

$$
f_{\text{joint}}^{(a,b)}(\theta) = f_1(\theta, a)f_2(\theta, b), \quad \forall \theta \in F,
$$

(21)

where

$$
f_1(\theta, a) = \int_{\Lambda_1} \lambda_1(\Pi_1^{(a)}(d\lambda_1))(\theta), \quad f_2(\theta, b) = \int_{\Lambda_2} \lambda_2(\Pi_2^{(b)}(d\lambda_2))(\theta)
$$

(22)

\textsuperscript{11}In measure theory, the notation $\mu \times \nu$ is generally accepted for the product measure with the marginal measures $\mu$ and $\nu.$
and \(|f_1(\theta, a)| \leq C_1, |f_2(\theta, b)| \leq C_2\), for any \(\theta \in F\).

From Eq. (21) it follows that, under a joint measurement, represented on a system information space \((\Theta, \mathcal{F}_\Theta)\) by a product generalized observable of the form (20), the expectation value (7) admits the representation:

\[
\langle \lambda_1 \lambda_2 \rangle^{(a,b)} := \langle \lambda_1 \lambda_2 \rangle^{(a,b)} = \int_F f_1(\theta, a) f_2(\theta, b) \pi(d\theta),
\]

for any initial information state \(I\) with the support on \(F \in \mathcal{F}_\Theta\).

**Lemma 1** For two product generalized observables of the form (20), the corresponding expectation values satisfy the relation

\[
|\gamma_1 \langle \lambda_1 \lambda_2 \rangle^{(a,b_1)} + \gamma_2 \langle \lambda_1 \lambda_2 \rangle^{(a,b_2)}| \leq C_1C_2 + \gamma_1 \gamma_2 \frac{C_1}{C_2} \langle \lambda'_2 \lambda_2 \rangle^{(\tilde{a})},
\]

for any information state \(I\) with the support on \(F \in \mathcal{F}_\Theta\) and any real-valued coefficients \(|\gamma_1| \leq 1, |\gamma_2| \leq 1\). Here,

\[
\langle \lambda'_2 \lambda_2 \rangle^{(\tilde{a})} := \int_F f_2(\theta, b_1) f_2(\theta, b_2) \pi(d\theta)
\]

\[
= \int_{\Lambda_2 \times \Lambda_2} \int_F \lambda'_2 \lambda_2 (\Pi(d\lambda'_2 \times d\lambda_2))(\theta) \pi(d\theta)
\]

and \(\tilde{\Pi} := \Pi_2^{(b_1)} \times \Pi_2^{(b_2)}\).

**Proof.** For a state \(I\) with the support on \(F \in \mathcal{F}_\Theta\), the proof is based on the representation (23), the inequality (15), the relation \(\pi(F) = 1\), Eq. (22) and the notation (25). Specifically:

\[
|\gamma_1 \langle \lambda_1 \lambda_2 \rangle^{(a,b_1)} + \gamma_2 \langle \lambda_1 \lambda_2 \rangle^{(a,b_2)}| \leq \int_F |f_1(\theta, a)\{\gamma_1 f_2(\theta, b_1) + \gamma_2 f_2(\theta, b_2)\}| \pi(d\theta)
\]

\[
\leq C_1 \int_F |\gamma_1 f_2(\theta, b_1) + \gamma_2 f_2(\theta, b_2)| \pi(d\theta)
\]

\[
\leq C_1C_2 \int_F \{1 + \frac{\gamma_1 \gamma_2}{C_2} f_2(\theta, b_1) f_2(\theta, b_2)\} \pi(d\theta)
\]

\[
= C_1C_2 + \gamma_1 \gamma_2 \frac{C_1}{C_2} \langle \lambda'_2 \lambda_2 \rangle^{(\tilde{a})}.
\]

From Lemma 1 it follows the following general statement. (For simplicity, we further consider the case \(C_1 = C_2 = 1\)).

**Proposition 1 (The Bell inequality)** Let a system be represented initially by an information state \(I = (\Theta, \mathcal{F}_\Theta, \pi)\) with the support on \(F \in \mathcal{F}_\Theta\) and three joint measurements, with outcomes
\(|\lambda_1| \leq 1 \text{ and } |\lambda_2| \leq 1\), performed on this system, be described on \((\Theta, F_\Theta)\) by the product generalized observables of the form \((21)\), specified by pairs \((a, b_1), (a, b_2)\) and \((b_1, b_2)\) of measurement parameters. If
\[
\int_{\Lambda_2} \lambda_2 (\Pi_2^{(b_1)}) (d\lambda_2)(\theta) = \pm \int_{\Lambda_1} \lambda_1 (\Pi_1^{(b_1)}) (d\lambda_1)(\theta) \quad \iff \quad f_2(\theta, b_1) = \pm f_1(\theta, b_1),
\]
\(\pi\)-almost everywhere\(^\ddagger\) on \(F\), then
\[
\left| \langle \lambda_1 \lambda_2 \rangle^{(a, b)}_I - \langle \lambda_1 \lambda_2 \rangle^{(a, b_2)}_I \right| \leq 1 \mp \langle \lambda_1 \lambda_2 \rangle^{(b_1, b_2)}_I. \quad (28)
\]

Notice that, in Proposition 1, we derive the original Bell inequality \((28)\) without the so-called "measurement result" restrictions on the observed outcomes, introduced in Refs. 1, 2 and generally accepted in the literature on the Bell inequality.

**Proposition 2** Let four joint measurements, with outcomes \(|\lambda_1| \leq 1 \text{ and } |\lambda_2| \leq 1\), be represented by the product generalized observables of the form \((20)\), specified by pairs \((a_k, b_m), \forall k, m = 1, 2\), of measurement parameters. Then the extended CHSH inequality\(^\ddagger\)
\[
| \sum_{k,m=1,2} \gamma_{km} \langle \lambda_1 \lambda_2 \rangle^{(a_k, b_m)}_I \rangle \leq 2 \quad (29)
\]
holds for any initial information state \(I\) with the support on \(F \in F_\Theta\) and any real-valued coefficients \(|\gamma_{km}| \leq 1, \forall k, m = 1, 2\), such that \(\gamma_{11} \gamma_{12} = -\gamma_{21} \gamma_{22}\) or \(\gamma_{11} \gamma_{21} = -\gamma_{12} \gamma_{22}\).

**Proof.** In view of Lemma 1, we have:
\[
| \sum_{k,m=1,2} \gamma_{km} \langle \lambda_1 \lambda_2 \rangle^{(a_k, b_m)}_I \rangle \leq | \sum_{m=1,2} \gamma_{1m} \langle \lambda_1 \lambda_2 \rangle^{(a_1, b_m)}_I | + | \sum_{m=1,2} \gamma_{2m} \langle \lambda_1 \lambda_2 \rangle^{(a_2, b_m)}_I | \quad (30)
\]
\[
\leq 2 + (\gamma_{11} \gamma_{12} + \gamma_{21} \gamma_{22}) \langle \lambda_1^2 \lambda_2^2 \rangle^{(i)}_I .
\]
The latter relation proves the statement for the case \(\gamma_{11} \gamma_{12} = -\gamma_{21} \gamma_{22}\). Combining in the left hand side of the inequality \((29)\) the first term with the third and the second term with the fourth, we prove, quite similarly, the statement for the case \(\gamma_{11} \gamma_{21} = -\gamma_{12} \gamma_{22}\). \(\blacksquare\)

Clearly, the extended CHSH inequality is always true under classical joint measurements\(^\ddagger\) and the original derivation of the CHSH inequality in Ref. 3 just corresponds to the classical case.

\(^\ddagger\)The term "\(\pi\)-almost everywhere on \(F^\pi\)" (a.e., for short) means that some relation holds on \(F\) excluding the null sets of a probability distribution \(\pi\).

\(^\ddagger\)Introduced in Ref. 12.

\(^\ddagger\)"Image" generalized observable describing classical joint measurements are product, see Eq. \((10)\).
Consider now a more general situation where four joint measurements, specified by pairs \((a_k, b_m), \forall k, m = 1, 2,\) of measurement parameters, are represented on \((\Theta, F_\Theta)\) by factorizable generalized observables (see Eq. 18) of the form:

\[
\Pi^{(a_k, b_m)} = \int_{\Omega} \Pi_{1,\omega}^{(a_k)} \times \Pi_{2,\omega}^{(b_m)} \nu_{a_1, a_2}^{(b_1, b_2)}(d\omega), \quad \forall k, m = 1, 2, \quad \forall F \in F_\Theta,
\]

where, in general, a probability distribution \(\nu_{a_1, a_2}^{(b_1, b_2)}\) depends on set-ups of marginal measurements. Under these joint measurements, the expectation values admit the representations:

\[
\langle \lambda_1 \lambda_2 \rangle^{(a_k, b_m)}_I = \int_{F} \int_{\Omega} f_1(\theta, \omega, a_k) f_2(\theta, \omega, b_m) \nu_{a_1, a_2}^{(b_1, b_2)}(d\omega) \pi(d\theta), \quad \forall k, m = 1, 2
\]

\[
f_1(\theta, \omega, a_k) = \int_{\Lambda_1} \lambda_1(\Pi_{1,\omega}^{(a_k)}(d\lambda_1))(\theta),
\]

\[
f_2(\theta, \omega, b_m) = \int_{\Lambda_2} \lambda_2(\Pi_{2,\omega}^{(b_m)}(d\lambda_2))(\theta),
\]

for any information state \(I\) with the support on \(F \in F_\Theta\).

By its structure, these representations are quite similar to the representation. That is why, the above propositions can be easily generalized.

**Proposition 3 (The extended CHSH inequality)** Under four joint measurements, described by factorizable generalized observables, the corresponding expectation values satisfy the extended CHSH inequality for any initial information state \(I\) with the support on \(F \in F_\Theta\).

Furthermore, let three joint measurements be described by factorizable generalized observables of the form:

\[
\Pi^{(a, b_m)} = \int_{\Omega} \Pi_{1,\omega}^{(a)} \times \Pi_{2,\omega}^{(b_m)} \nu_{a}^{(b_1, b_2)}(d\omega), \quad \forall F \in F_\Theta, \quad \forall m = 1, 2
\]

\[
\Pi^{(b_1, b_2)} = \int_{\Omega} \Pi_{1,\omega}^{(b_1)} \times \Pi_{2,\omega}^{(b_2)} \nu_{a}^{(b_1, b_2)}(d\omega), \quad \forall F \in F_\Theta
\]

It is easy to prove\(^{15}\) that, for these three joint measurements, the corresponding expectation values

\[
\langle \lambda_1 \lambda_2 \rangle^{(a, b_m)}_I = \int_{F} \int_{\Omega} f_1(\theta, \omega, a) f_2(\theta, \omega, b_m) \nu_{a}^{(b_1, b_2)}(d\omega) \pi(d\theta), \quad \forall m = 1, 2
\]

\[
\langle \lambda_1 \lambda_2 \rangle^{(b_1, b_2)}_I = \int_{F} \int_{\Omega} f_1(\theta, \omega, b_1) f_2(\theta, \omega, b_2) \nu_{a}^{(b_1, b_2)}(d\omega) \pi(d\theta)
\]

in a state \(I = (\Theta, F_\Theta, \pi)\) with the support on \(F \in F_\Theta\) satisfy the original Bell inequality whenever

\[
\int_{\Lambda_2} \lambda_2(\Pi_{2,\omega}^{(b_1)}(d\lambda_2))(\theta) = \pm \int_{\Lambda_1} \lambda_1(\Pi_{1,\omega}^{(b_1)}(d\lambda_1))(\theta) \iff
\]

\[
f_2(\theta, \omega, b_1) = \pm f_1(\theta, \omega, b_1),
\]

\(\pi \times \nu_{a}^{(b_1, b_2)}\)-almost everywhere on \(F \times \Omega\).

\(^{15}\) Quite similarly to our proof of Proposition 1.
Remark 2 (On perfect correlations/anticorrelations) It has been generally accepted to consider that the Bell inequality holds whenever (see Refs. 1, 2)

\[(\lambda_1 \lambda_2)_{(b_1, b_2)} = \pm 1.\]  

(36)

This Bell’s correlation restriction implies that outcomes \(\lambda_1, \lambda_2\) admit only two values \(\pm 1\) and are either perfectly correlated (plus sign) or anticorrelated (minus sign).

In contrast to Bell’s sufficient condition, the sufficient condition does not impose any restriction on a type of observed outcomes. Moreover, even in case of \((\pm 1)\)-valued outcomes, our ”average” condition on marginal generalized observables is more general than the Bell restriction on the observed outcomes.

Namely, since \(|f_i(\theta, \omega, b_1)| \leq 1, \forall i = 1, 2, Eqs. imply

\[f_1(\theta, \omega, b_1)f_2(\theta, \omega, b_1) = \pm 1 \Rightarrow f_1(\theta, \omega, b_1) = \pm f_2(\theta, \omega, b_1),\]  

(37)

\[\pi \times \nu_{(b_1, b_2)}\] -almost everywhere on \(F \times \Omega\). Thus, the validity of Bell’s correlation restriction implies the validity of the condition.

However, the converse of this statement is not true and, for factorizable generalized observables, satisfying the condition, the correlation function

\[\langle \lambda_1 \lambda_2 \rangle_{(b_1, b_2)} = \pm \int_F \int_{\Omega} f_1^2(\theta, \omega, b_1)\nu_{(b_1, b_2)}(d\omega)\pi(d\theta)\]  

(38)

may, in general, take any value in \([0, 1]\) - in case of plus sign and any value in \([-1, 0]\) - in case of minus sign.

3.2 Sufficient Condition

In view of our results in Sec. 3.1, let us now specify a general condition sufficient for the validity of CHSH-form inequalities under joint measurements upon a system of any type.

Condition 1 (Sufficient) If four joint measurements are represented on a system information space by the factorizable generalized observables, then the corresponding expectation values satisfy a CHSH-form inequality for any system initial information state with the support on \(F \in F_{\Theta}\).

Remark 3 (On the existence of formal LHV models) It is necessary to underline that since, in general, a probability distribution \(\nu_{(b_1, b_2)}\) in Eq. depends on settings of joint measurements,
the representations for the expectation values in a state $\mathcal{I}$ hold only under these joint measurements.

Therefore, the validity of the representations for an information state $\mathcal{I}$ does not, in general, mean the existence for this state of a local hidden variable (LHV) model.

If, however, a probability distribution $\nu$ does not depend on measurement settings and the representations hold for any measurement parameters then a state $\mathcal{I}$ admits an LHV model (in general, formal\textsuperscript{17}). The existence of formal LHV models for some bipartite quantum states was first specified in Ref. 11.

Notice that the existence of a formal LHV model under joint measurements does not imply "classicality" of the observed system.

**Remark 4 (On possible reasons for the violation )** Under joint measurements on an information state $\mathcal{I}$, the violation of a CHSH-form inequality may happen if: (i) generalized observables, describing these joint measurements, do not have the factorizable form; (ii) generalized observables, describing these joint measurements, are factorizable on a set $F \in \mathcal{F}_\Theta$ while the support of a state $\mathcal{I}$ is out of $F$.

The latter is just a general situation under Alice/Bob joint measurements on a bipartite quantum system and we discuss this in Sec. 4.1.

**Remark 5 (On "local realism")** As we show in Sec. 4, Condition 1 does not, in general, represent mathematically the physical concept of "local realism" under joint measurements. Therefore, the violation of a CHSH-form inequality cannot be linked with the violation of "local realism".

4 GENERAL "LOCAL REALISM" CONDITION

In a general setting, consider now a joint experiment, with outcomes in $\Lambda_1 \times \Lambda_2$, performed on a system of any type represented initially by an information space $(\Theta, \mathcal{F}_\Theta)$.

Let a set-up of the marginal experiment with outcomes in $\Lambda_1$ be characterized by a parameter "a" while a set-up of the marginal experiment with outcomes in $\Lambda_2$ - by a parameter "b". In this setting, the set-up of a joint experiment is specified by a pair $(a, b)$ and we further denote by \textsuperscript{17}Recall that, in a Bell LHV model (see Refs. 1 - 3), the values of measurable functions represent the observed outcomes and, therefore, this model is classical.
Π\(^{(a,b)}\) a generalized observable, with an outcome set \(\Lambda_1 \times \Lambda_2\), representing this joint experiment on \((\Theta, \mathcal{F}_\Theta)\), and by \(\mu^{(a,b)}(\cdot; \mathcal{I})\) - the probability distribution of outcomes if a system is initially in an information state \(\mathcal{I}\). The marginal probability distributions

\[
\mu_1^{(a,b)}(B_1; \mathcal{I}) : = \mu^{(a,b)}(B_1 \times \Lambda_2; \mathcal{I}), \quad \forall B_1 \subseteq \Lambda_1, \tag{39}
\]

\[
\mu_2^{(a,b)}(B_2; \mathcal{I}) : = \mu^{(a,b)}(\Lambda_1 \times B_2; \mathcal{I}), \quad \forall B_2 \subseteq \Lambda_2,
\]

describe the statistics of outcomes under the marginal experiments with outcomes in \(\Lambda_1\) and \(\Lambda_2\), respectively. Recall that the marginal experiments are represented on \((\Theta, \mathcal{F}_\Theta)\) by the marginal generalized observables \(\Pi_1^{(a,b)}\) and \(\Pi_2^{(a,b)}\) (see Eq. (3)).

If, under the specified joint experiment, the physical concept of ”local realism” (see Ref. 5, page 160) is not violated then, for any information state \(\mathcal{I}\), the marginal probability distribution \(\mu_1^{(a,b)}(\cdot; \mathcal{I})\) must not depend on a parameter \(b\) while the marginal probability distribution \(\mu_2^{(a,b)}(\cdot; \mathcal{I})\) must not depend on a parameter \(a\), that is:

\[
\mu_1^{(a,b)}(\cdot; \mathcal{I}) = \mu_1^{(a)}(\cdot; \mathcal{I}), \quad \mu_2^{(a,b)}(\cdot; \mathcal{I}) = \mu_2^{(b)}(\cdot; \mathcal{I}), \tag{40}
\]

for any state \(\mathcal{I}\).

For short, we further refer to such joint experiments as ”locally realistic”.

Due to Eqs. (2) and (40), we have the following necessary and sufficient condition for a joint generalized observable \(\Pi^{(a,b)}\) to represent a ”locally realistic” joint experiment.

**Condition 2 (On ”local realism”)** A joint generalized observable \(\Pi^{(a,b)}\), with an outcome set \(\Lambda_1 \times \Lambda_2\), represents a ”locally realistic” joint experiment iff each of its marginal generalized observables depends only on a set-up of the corresponding marginal experiment, that is:

\[
\Pi^{(a,b)}(B_1 \times \Lambda_2) = \Pi_1^{(a)}(B_1), \quad \forall B_1 \subseteq \Lambda_1, \tag{41}
\]

\[
\Pi^{(a,b)}(\Lambda_1 \times B_2) = \Pi_2^{(b)}(B_2), \quad \forall B_2 \subseteq \Lambda_2.
\]

For short, we call a joint generalized observable satisfying Condition 2 as ”locally realistic”.

**Example 1** Consider a generalized observable \(\Pi_1^{(a)} \times \Pi_2^{(b)}\) which is product on all of \(\Theta\) and a generalized observable \(\int_\Omega \Pi_1^{(a)}(\omega) \times \Pi_2^{(b)}(\omega) \nu(d\omega)\) which is factorizable on all of \(\Theta\). Due to Condition 2, these joint generalized observables are ”locally realistic”. In particular, an ”image” joint generalized observable \(\Pi^{\text{im}}\), describing a classical joint measurement, has a product form and, hence, is ”locally realistic”.

13
However, in a general case, a "locally realistic" joint generalized observable is not necessarily product or factorizable.

Non-factorizable "locally realistic" generalized observables do not satisfy Condition 1. That is why, under "locally realistic" joint measurements described by these generalized observables, a CHSH-form inequality does not need to hold.

The latter is just a general situation under "locally realistic" joint measurements on a bipartite quantum system.

4.1 Quantum joint Measurements

In the quantum case, a system is described in terms of a separable complex Hilbert space $\mathcal{K}$. Denote by $\mathcal{R}_\mathcal{K}$ the set of all density operators $\rho$ on a Hilbert space $\mathcal{K}$.

For a quantum system, we take an information space to be represented by $(\mathcal{R}_\mathcal{K}, \mathcal{B}_{\mathcal{R}_\mathcal{K}})$ where $\mathcal{B}_{\mathcal{R}_\mathcal{K}}$ is the Borel $\sigma$-algebra.

Any quantum generalized observable on $(\mathcal{R}_\mathcal{K}, \mathcal{B}_{\mathcal{R}_\mathcal{K}})$, with an outcome set $\Lambda$, is convex linear in $\rho$ and is given by (see Ref. 7, section 5.2):

$$(\Pi_q(B))(\rho) = \text{tr}[\rho M(B)], \quad \forall \rho \in \mathcal{R}_\mathcal{K}, \quad \forall B \subseteq \Lambda,$$

(42)

where $M$ is a normalized measure with values $M(B)$, $\forall B$, that are positive bounded linear operators on $\mathcal{K}$, that is, a positive operator-valued (POV) measure.

Since any quantum generalized observable is convex linear in $\rho \in \mathcal{R}_\mathcal{K}$, under a quantum measurement, any two initial quantum information states $(\mathcal{R}_\mathcal{K}, \mathcal{B}_{\mathcal{R}_\mathcal{K}}, \pi_1)$ and $(\mathcal{R}_\mathcal{K}, \mathcal{B}_{\mathcal{R}_\mathcal{K}}, \pi_2)$, satisfying the relation $\int_{\mathcal{R}_\mathcal{K}} \rho \pi_1(d\rho) = \int_{\mathcal{R}_\mathcal{K}} \rho \pi_2(d\rho)$, give the same information on the statistics of the observed outcomes.

Consider the description of a joint quantum measurement with outcomes in $\Lambda_1 \times \Lambda_2$.

Suppose that, under this joint measurement, a set-up of the marginal measurement with outcomes in $\Lambda_1$ is specified by a parameter "$a$" while a set-up of a marginal measurement with outcomes in $\Lambda_2$ - by a parameter "$b$". Let $\Pi_{q(a,b)}$ be a generalized observable, representing this joint quantum measurement on the quantum information space $(\mathcal{R}_\mathcal{K}, \mathcal{B}_{\mathcal{R}_\mathcal{K}})$ and $M_{q(a,b)}$ be the POV measure uniquely corresponding to $\Pi_{q(a,b)}$ due to (42). The marginal POV measures $M_{q(a,b)}(B_1 \times \Lambda_2)$, $\forall B_1 \subseteq \Lambda_1$, and $M_{q(a,b)}(\Lambda_1 \times B_2)$, $\forall B_2 \subseteq \Lambda_2$, describe the corresponding marginal quantum measurements.

---

18 Representing the trace on $\mathcal{R}_\mathcal{K}$ of the Borel $\sigma$-algebra on the Banach space of trace class operators on $\mathcal{K}$.

19 For the notion of a POV measure, see Refs. 8 - 10.
From Condition 2 and Eq. [32] it follows that if a quantum joint measurement is described by a POV measure $M^{(a,b)}$ satisfying the relations:

$$M^{(a,b)}(B_1 \times \Lambda_2) = M_1^{(a)}(B_1), \quad M^{(a,b)}(\Lambda_1 \times B_2) = M_2^{(b)}(B_2),$$

(43)

for any outcome subsets $B_1 \subseteq \Lambda_1, B_2 \subseteq \Lambda_2$, then this joint quantum measurement is "locally realistic".

Consider an example of a quantum "locally realistic" joint measurement.

**Example 2 (Alice/Bob joint quantum measurement)** Consider a bipartite quantum system described in terms of a separable complex Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$ and let a joint quantum measurement on this system be represented by the POV measure

$$M_1^{(a)}(B_1) \otimes M_2^{(b)}(B_2),$$

(44)

for any $B_1 \subseteq \Lambda_1, B_2 \subseteq \Lambda_2$. This POV measure satisfies the condition (43) and, hence, represents a "locally realistic" joint quantum measurement. For convenience, $\Lambda_1$ and $\Lambda_2$ are referred to as sets of outcomes on the "sides" of Alice and Bob, respectively.

Thus, any Alice/Bob joint quantum measurement is "locally realistic".

Due to Eqs. (42), (43), under a quantum Alice/Bob joint measurement, the joint generalized observable has the form:

$$(\Pi_q^{(a,b)}(B_1 \times B_2))(\rho) = \text{tr}[\rho(M_1^{(a)}(B_1) \otimes M_2^{(b)}(B_2))], \quad \forall \rho \in \mathcal{R}_{\mathcal{H}_1 \otimes \mathcal{H}_2}. \quad (45)$$

On any separable density operator $\rho_S = \sum_j \gamma_j \rho_1^{(j)} \otimes \rho_2^{(j)}, \; \gamma_j > 0, \; \sum_j \gamma_j = 1$, this generalized observable admits a representation

$$(\Pi_q^{(a,b)}(B_1 \times B_2))(\rho_S) = \sum_j \gamma_j \text{tr}[\rho_1^{(j)} M_1^{(a)}(B_1)] \text{tr}[\rho_2^{(j)} M_2^{(b)}(B_2)]$$

(46)

and, hence, is factorizable. Due to Condition 1, under four quantum Alice/Bob joint measurements

$$M^{(a_k,b_m)}(B_1 \times B_2) = M_1^{(a_k)}(B_1) \otimes M_2^{(b_m)}(B_2), \quad \forall k, m = 1, 2,$$

(47)

performed on a separable quantum state, a CHSH-form inequality is satisfied.

On any density operator $\rho \in \mathcal{R}_{\mathcal{H}_1 \otimes \mathcal{H}_2}$, four joint generalized observables (45) do not, in general, admit the factorizable representations (31). That is why, under Alice/Bob joint measurements on an arbitrary bipartite quantum state, Condition 1 is not, in general, fulfilled and a CHSH-form inequality may be violated - though quantum Alice/Bob joint measurements are "locally realistic".
5 QUANTUM ANALOGS OF THE BELL INEQUALITY

In this section, for an arbitrary quantum state $\rho$ of two identical sub-systems, we introduce quantum Bell-form inequalities under Alice/Bob joint measurements.

In case of identical quantum sub-systems, $H_1 = H_2 = H$ and a bipartite state $\rho$ on $H \otimes H$ must be symmetric, that is: $S_2 \rho = \rho$, where $S_2$ is the symmetrization operator on the space of bounded linear operators on $H \otimes H$.

Moreover, each of marginal POV measures must have a symmetrized tensor product form and be specified by a set of outcomes on the "side" of Alice or Bob but not by the "side" of the tensor product. The latter means that, for an Alice/Bob joint quantum measurement on identical sub-systems, the POV measure has the form:

$$M^{(a,b)}(B_1 \times B_2) = \{M_1^{(a)}(B_1) \otimes M_2^{(b)}(B_2)\}_{\text{sym}}$$  \hspace{1cm} (48)$$

for any outcome subsets $B_1 \subseteq \Lambda_1$, $B_2 \subseteq \Lambda_2$.

For simplicity, we further suppose that outcomes $|\lambda_1| \leq 1$ and $|\lambda_2| \leq 1$.

Under an Alice/Bob joint quantum measurement $\rho$ on a symmetric state $\rho$, the expectation values $\mathcal{E}$ are given by:

$$\langle \lambda_1 \lambda_2 \rangle_{\rho}^{(a,b)} = \int_{\Lambda_1 \times \Lambda_2} \lambda_1 \lambda_2 \text{tr}[\rho \{M_1^{(a)}(d\lambda_1) \otimes M_2^{(b)}(d\lambda_2)\}_{\text{sym}}]$$  \hspace{1cm} (49)$$

where

$$A_1^{(a)} = \int_{\Lambda_1} \lambda_1 M_1^{(a)}(d\lambda_1), \quad A_2^{(b)} = \int_{\Lambda_2} \lambda_2 M_2^{(b)}(d\lambda_2)$$  \hspace{1cm} (50)$$

are self-adjoint bounded linear operators on $H$, with the operator norms $||A_1^{(a)}|| \leq 1, ||A_2^{(b)}|| \leq 1$.

For a state $\rho$, introduce a representation

$$\rho = \eta(\tau, \bar{\tau}) + \sigma^{(n)}_{\rho}$$  \hspace{1cm} (51)$$

via a separable density operator

$$\eta(\tau, \bar{\tau}) := \frac{1}{2} \sum_j \gamma_j (\tau_j \otimes \bar{\tau}_j + \bar{\tau}_j \otimes \tau_j), \quad \gamma_j > 0, \quad \sum_j \gamma_j = 1$$  \hspace{1cm} (52)$$

\footnote{Notice that $\text{tr}[\sigma W_1 \otimes W_2]_{\text{sym}} = \text{tr}[\sigma (W_1 \otimes W_2)]$, for any symmetric trace class operator $\sigma$.}
where \( \tau_j \) and \( \bar{\tau}_j \), \( N \leq \infty \), are any families of density operators on \( \mathcal{H} \). In Eq. \( 51 \), the operator \( \sigma^{(n)}_\rho \) is symmetric, self-adjoint, trace class and its trace norm \( \| \sigma^{(n)}_\rho \|_1 \) characterizes a "distance" between \( \rho \) and a separable state \( \eta(\tau, \bar{\tau}) \).

For concreteness, we further refer to Eq. \( 51 \) as a \((\tau, \bar{\tau})\)-representation of \( \rho \).

Substituting Eq. \( 51 \) into Eq. \( 49 \) and using the inequality \( 16 \), we derive that, for a state \( \rho \), the relation

\[
\left| \langle \lambda_1 \lambda_2 \rangle^{(a,b_1)}_\rho - \langle \lambda_1 \lambda_2 \rangle^{(a,b_2)}_\rho \right| - \langle z \rangle^{(n)}_\rho 
\leq 1 - \frac{1}{2} \sum_j \gamma_j \{ \text{tr}[\tau_j A_2^{(b_1)}] \text{tr}[\bar{\tau}_j A_2^{(b_2)}] + \text{tr}[\bar{\tau}_j A_2^{(b_1)}] \text{tr}[\tau_j A_2^{(b_2)}] \}
\]

holds for any \((\tau, \bar{\tau})\)-representation of \( \rho \). Here \(^21 \),

\[
\langle z \rangle^{(n)}_\rho := \text{tr}[\sigma^{(n)}_\rho (A_1^{(a)} \otimes (A_2^{(b_1)} - A_2^{(b_2)})]), \quad | \langle z \rangle^{(n)}_\rho | \leq \| \sigma^{(n)}_\rho \|_1 |A_2^{(b_1)} - A_2^{(b_2)}|.
\]

**Proposition 4 (Quantum analogs)** Let, under Alice/Bob joint measurements \( 48 \) with outcomes \( |\lambda_1| \leq 1 \) and \( |\lambda_2| \leq 1 \), the marginal POV measures satisfy the condition

\[
\int \lambda_2 M_2^{(b_1)}(d\lambda_2) = \int \lambda_1 M_1^{(b_1)}(d\lambda_1) \quad \iff \quad A_2^{(b_1)} = A_2^{(b_1)}.
\]

Then a quantum state \( \rho \) on \( \mathcal{H} \otimes \mathcal{H} \) satisfies the inequality

\[
\left| \langle \lambda_1 \lambda_2 \rangle^{(a,b_1)}_\rho - \langle \lambda_1 \lambda_2 \rangle^{(a,b_2)}_\rho \right| \leq \gamma^{(n)}_\rho - \langle \lambda_1 \lambda_2 \rangle^{(b_1,b_2)}_{\eta(\tau, \bar{\tau})},
\]

for every \((\tau, \bar{\tau})\)-representation \( 51 \) of \( \rho \). Here,

\[
\gamma^{(n)}_\rho = 1 + ||\rho - \eta(\tau, \bar{\tau})||_1 |A_2^{(b_1)} - A_2^{(b_2)}|,
\]

\[
\gamma^{(n)}_\rho \leq 1 + \frac{2}{\gamma_j} \sum_j (\tau_j \otimes \bar{\tau}_j + \bar{\tau}_j \otimes \tau_j).
\]

**Proof.** We use the inequality \( 53 \), Eq. \( 51 \), the condition \( 55 \) and then the notation \( 51 \).

The inequality \( 56 \) describes the relation between the expectation values under three Alice/Bob joint quantum measurements and we refer to it as a quantum analog of the Bell inequality.

Let us now specify the inequality \( 56 \) in case of a separable bipartite state. For a (symmetric) separable quantum state \( \rho_S \), there always exists a representation

\[
\rho_S = \frac{1}{2} \sum_j \gamma_j \left\{ \tau_j \otimes \bar{\tau}_j + \bar{\tau}_j \otimes \tau_j \right\} + \sigma^{(a)}_{\rho_S}, \quad \gamma_j > 0, \quad \sum_j \gamma_j = 1,
\]

\(^21\)We use the bound \( \| \text{tr}[\sigma W] \| \leq ||\sigma||_1 ||W|| \), valid for any trace class operator \( \sigma \) and any bounded linear operator \( W \).
where $||\sigma_{\rho_S}^{(s)}||_1 = 0$. Hence, for a separable state $\rho_S$, the inequality (55), corresponding to the representation (58), takes the form:

$$\left|\langle \lambda_1 \lambda_2 \rangle_{\rho_S}^{(a,b_1)} - \langle \lambda_1 \lambda_2 \rangle_{\rho_S}^{(a,b_2)}\right| \leq 1 - \langle \lambda_1 \lambda_2 \rangle_{\rho_S}^{(b_1,b_2)},$$

and coincides with the inequality (40) introduced in Ref. 12.

Suppose further that a separable quantum state $\rho_S$ is of the special form:

$$\rho_S = \sum_j \gamma_j \tau_j \otimes \tau_j + \sigma_{\rho_S}^{(s)}, \quad ||\sigma_{\rho_S}^{(s)}||_1 = 0. \quad (60)$$

For this state $\bar{\rho}_S = \rho_S - \sigma_{\rho_S}^{(s)}$ and $\langle \lambda_1 \lambda_2 \rangle_{\bar{\rho}_S}^{(b_1,b_2)} = \langle \lambda_1 \lambda_2 \rangle_{\rho_S}^{(b_1,b_2)}$.

**Corollary 1** Under Alice/Bob joint measurements (48), satisfying the condition (55), the perfect correlation form of the original Bell inequality

$$\left|\langle \lambda_1 \lambda_2 \rangle_{\rho_S}^{(a,b_1)} - \langle \lambda_1 \lambda_2 \rangle_{\rho_S}^{(a,b_2)}\right| \leq 1 - \langle \lambda_1 \lambda_2 \rangle_{\rho_S}^{(b_1,b_2)} \quad (61)$$

holds for any separable quantum state of the special form (60).

It is necessary to underline that the operator condition (55) on marginal POV measures is always true under Alice and Bob projective measurements of the same quantum observable on both sides. That is why, a separable state of the special form (60) satisfies the perfect correlation form (61) of the original Bell inequality under any three projective quantum measurements of Alice and Bob, specified on Alice and Bob sides by pairs of bounded quantum observables: $(A^{(a)}, A^{(b_1)})$, $(A^{(a)}, A^{(b_2)})$ and $(A^{(b_1)}, A^{(b_2)})$. Notice$^{22}$ that, satisfying the perfect correlation form of the Bell inequality for any bounded quantum observables $A^{(a)}$, $A^{(b_1)}$, $A^{(b_2)}$, a bipartite quantum state (60) does not necessarily exhibit perfect correlations.

### 6 ON LOCALITY OF QUANTUM INTERACTIONS

In the present paper, we discuss in a very general setting the description of joint experiments performed on a system of any type.

$^{22}$See also Remark 2.
Mathematically, any joint experiment is described by the notion of a joint generalized observable and this notion does not include any specifications on whether or not marginal experiments are separated in space and in time.

The main results of our paper indicate:

- The physical concept of "local realism" can be expressed in mathematical terms for a joint experiment upon a system of any type. The generally accepted mathematical specification of this concept in the frame of a hidden variable model corresponds only to a particular case of joint experiments represented by factorizable generalized observables;

- The general sufficient condition for a CHSH-form inequality to hold is not equivalent to the condition on "local realism" under joint experiments. Therefore, the violation of a CHSH-form inequality in the quantum case does not point to the violation of the physical concept of "local realism";

- Quantum joint experiments of the Alice/Bob type are "locally realistic". However, under these "locally realistic" joint experiments, the sufficient condition for a CHSH-form inequality to hold is not satisfied for any bipartite quantum state;

- Quantum analogs of the original Bell inequality, derived in this paper, specify the relation between the statistical data observed under quantum "locally realistic" joint experiments on an arbitrary bipartite quantum state.

*In the light of these results, we argue that the violation of Bell-type inequalities in the quantum case cannot be a valid argument in the discussion on locality or non-locality of quantum interactions.*

**Acknowledgments.** I am grateful to Klaus Molmer and Asher Peres for useful discussions. This work is partially supported by MaPhySto - A Network in Mathematical Physics and Stochastics, funded by The Danish National Research Foundation.

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