Statistical Optimality of Divide and Conquer Kernel-based Functional Linear Regression†

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Abstract

Previous analysis of regularized functional linear regression in a reproducing kernel Hilbert space (RKHS) typically requires the target function to be contained in this kernel space. This paper studies the convergence performance of divide-and-conquer estimators in the scenario that the target function does not necessarily reside in the underlying RKHS. As a decomposition-based scalable approach, the divide-and-conquer estimators of functional linear regression can substantially reduce the algorithmic complexities in time and memory. We develop an integral operator approach to establish sharp finite sample upper bounds for prediction with divide-and-conquer estimators under various regularity conditions of explanatory variables and target function. We also prove the asymptotic optimality of the derived rates by building the mini-max lower bounds. Finally, we consider the convergence of noiseless estimators and show that the rates can be arbitrarily fast under mild conditions.

Keywords and phrases: Functional linear regression, Reproducing kernel Hilbert space, Divide-and-conquer estimator, Hard learning scenario, Mini-max optimal rates

1 Introduction

Functional data analysis (FDA) has been an intense recent study, achieving remarkable success in a wide range of fields, including, among many others, chemometrics, linguistics, medicine, and economics [24, 43]. Under an FDA framework, the explanatory variable is usually a random function. We consider the following functional linear regression model to characterize the functional nature of explanatory variables. Let $Y$ be a scalar response, and $X$ be a random variable taking values in $L^2(T)$. Throughout the paper, we use $L^2(T)$ to denote the Hilbert space of square integrable functions defined over a domain $T \subseteq \mathbb{R}^D$ for some integer $D \geq 1$. In the functional linear regression model, the dependence of $Y$ and $X$ is expressed as

$$Y = \int_T \beta_0(t)X(t)dt + \epsilon,$$

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where $\beta_0 \in \mathcal{L}^2(\mathcal{T})$ is the slope function and $\epsilon$ is a random noise independent of $X$ with zero mean and bounded variance. The goal of functional linear regression is to construct an estimator $\hat{\beta}$ to approximate $\beta_0$ based on training samples of $(X, Y)$. The performance of an estimator can be measured by the prediction risk, given by

$$ R(\hat{\beta}) := \mathbb{E} \left[ \left( Y - \int_{\mathcal{T}} \hat{\beta}(t) X(t) dt \right)^2 \right], $$

(1.2)

or equivalently, the excess prediction risk $R(\hat{\beta}) - R(\beta_0)$.

The research on model (1.1) can be traced back to the 1990s [10] [22] [32]. Subsequently, a vast amount of literature has emerged to study the prediction and estimation problems under this model. A flourishing line of research is based on the functional principal component analysis (FPCA), leveraging spectral expansions of the covariance kernel of $X$ and its empirical counterpart to estimate the slope function (see, e.g., [3] [21] [24] [45]). A necessary condition for the success of the FPCA-based approaches is that the slope function $\beta_0$ can be efficiently represented by the leading functional principal components, which, however, fails to hold in many applications. To address this issue, another influential line of research utilizes kernel-based estimators to approximate the target $\beta_0$ in a suitable reproducing kernel Hilbert space (RKHS) [7] [17]. More concretely, given a training sample $S := \{(X_i, Y_i)\}_{i=1}^N$ consisting of $N$ independent copies of $(X, Y)$, one can employ an RKHS $(\mathcal{H}_K, \|\cdot\|_K)$ induced by a reproducing kernel $K: \mathcal{T} \times \mathcal{T} \to \mathbb{R}$ to estimate $\beta_0$ through the regularized least squares (RLS) estimators defined by

$$ \hat{\beta}_{S,\lambda} := \arg\min_{\beta \in \mathcal{H}_K} \left\{ \frac{1}{N} \sum_{i=1}^N \left( Y_i - \int_{\mathcal{T}} \beta(t) X_i(t) dt \right)^2 + \lambda \|\beta\|_K^2 \right\}. $$

(1.3)

Here we choose a tuning parameter $\lambda > 0$ to balance fidelity to the data and complexity of the estimators (measured by its squared $\mathcal{H}_K$ norm). According to the Representer Theorem proved in [17], $\hat{\beta}_{S,\lambda}$ can be uniquely expressed as $\hat{\beta}_{S,\lambda}(\cdot) = \sum_{i=1}^N c_i \int_{\mathcal{T}} K(\cdot, t) X_i(t) dt$ with $(c_1, \cdots, c_N)^T = (\lambda N I_N + \mathbb{K}_X)^{-1} \mathbf{Y}$, where $I_N$ is the identity matrix on $\mathbb{R}^N$, $\mathbb{K}_X \in \mathbb{R}^{N \times N}$ is the kernel matrix evaluated on $X := \{X_1, \cdots, X_N\}$ with the $(i, j)$-entrance $[\mathbb{K}_X]_{i,j} = \int_{\mathcal{T}} X_i(s) K(s, t) X_j(t) ds dt$, and $\mathbf{Y} := (Y_1, \cdots, Y_N)^T$. Under the assumption that the slope function $\beta_0$ belongs to the RKHS $\mathcal{H}_K$, it is shown in [7] that the excess prediction risk of $\hat{\beta}_{S,\lambda}$ can achieve the mini-max optimal convergence rates.

In this paper, we aim to further advance the line of research on the kernel-based approach designed for functional linear regression model (1.1). Specially, we will study the convergence behavior of divided-and-conquer RLS estimators without requiring the unknown slope function $\beta_0$ to be contained in the RKHS $\mathcal{H}_K$. As a generalization of classical kernel ridge regression (see, e.g., [36]), algorithm (1.3) suffers from the same complexity issue that seriously limits its performance when dealing with massive data. To make the computational problem more tractable for large-scale sample sets, we implement algorithm (1.3) via the divide-and-conquer approach. We randomly partition the entire sample set $S$ into $m$ disjoint equal-sized subsets $S_1, \cdots, S_m$. On each $S_j$, a local estimator $\hat{\beta}_{S_j,\lambda}$ is obtained according to algorithm (1.3), i.e.,

$$ \hat{\beta}_{S_j,\lambda}(\cdot) = \sum_{i: (X_i, Y_i) \in S_j} c_i \int_{\mathcal{T}} K(\cdot, t) X_i(t) dt \text{ where } (c_i)_{\{i: (X_i, Y_i) \in S_j\}} = (\lambda |S_j| I_{|S_j|} + \mathbb{K}_{S_j})^{-1} \mathbf{Y}_j. $$
Here $|S_j|$ denotes the cardinality of $S_j$, $X_j$ is the set of $X$’s sample in $S_j$, and $Y_j \in \mathbb{R}^{|S_j|}$ is a vector composed of $Y$’s sample in $S_j$. Divide-and-conquer RLS estimator is then computed by simply averaging $\{\hat{\beta}_{S_j,\lambda}\}_{j=1}^m$, which is given by
\[ \bar{\beta}_{S,\lambda} := \frac{1}{m} \sum_{j=1}^m \hat{\beta}_{S_j,\lambda}. \] (1.4)

This approach is appealing due to its easy exercisable partitions. Dividing randomly the sample set into $m$ equally-sized subsets and performing algorithm (1.3) on each subset in parallel roughly reduce the algorithmic complexities in time and memory to $\frac{1}{m}$ of the original. In the context of regression analysis for massive data, divide-and-conquer kernel ridge regression and its variants have been extensively studied in statistics and machine learning communities \[19, 30, 35, 40, 48\]. In the present paper, we evaluate the prediction performance of averaged estimator $\bar{\beta}_{S,\lambda}$ in (1.4) via its excess prediction risk:
\[ R(\bar{\beta}_{S,\lambda}) - R(\beta_0) \] (1.5)
in a more general setting which allows $\beta_0 \notin \mathcal{H}_K$. In kernel-based methods, if the target function does not reside in the underlying RKHS, this scenario is often referred to as a hard learning problem (see, e.g., [37]). More recently, convergence behaviors of kernel ridge regression in hard learning problems have been investigated in \[15, 29, 40\], which showed asymptotically mini-max optimal rates in many situations. In practice, canonical choices of $\mathcal{H}_K$ in (1.3) are the Sobolev spaces of smoothness $s$ (see [17] and the references therein). Though such an RKHS is dense in $L^2(\mathcal{T})$, the assumption that $\beta_0$ lies precisely in it is too restrictive in many real applications, as this assumption requires the derivatives of $\beta_0$ up to order $s-1$ are absolute continuous and its $s$-th derivative belongs to $L^2(\mathcal{T})$. This raises the question of whether the global RLS estimator (1.3) and its averaged version (1.4) can still maintain excellent prediction performances in the hard learning scenario $\beta_0 \notin \mathcal{H}_K$. We positively answer this question by establishing a tight convergence analysis with an integral operator technique. Furthermore, we also consider the noiseless circumstance when the model (1.1) has no additive noise. The noiseless condition means no ambiguity of the response $Y$ given the explanatory $X$; in other words, the response $Y$ is determined uniquely by the input $X$. The noiseless linear model has been widely adopted in many areas, including image classification and sound recognition (see, e.g., [25]). The convergence of estimators in a noiseless model is very important but has not been considered till the very recent papers \[4, 25, 40\].

The main contribution of this paper is to present new finite sample bounds on the prediction risk (1.5) concerning various regularity conditions. These conditions characterize the complexity of the prediction problem in functional linear regression model (1.1), which is measured through the regularities of explanatory variable $X$, optimum $\beta_0$, and their images under the kernel operators. See Section 2 and Section 3 for precise definitions and statements. Our analysis of convergence incorporates these regularity conditions into the integral operator techniques, substantially generalizing previously published bounds, which only considered the case $\beta_0 \in \mathcal{H}_K$, to the hard learning scenario $\beta_0 \notin \mathcal{H}_K$ and the divide-and-conquer estimators. For prediction using the noised model, the established convergence is tight as in most cases we prove upper and lower bounds on the performance of estimators that almost match. For prediction using the noiseless model, we prove that the estimator can converge with arbitrarily fast polynomial rates if the reproducing kernel or the covariance kernel is sufficiently smooth. Thus the estimators show some adaptivity to the complexity of the prediction problem.
The rest of this paper is organized as follows. We start in Section 2 with an introduction to notations, general assumptions, and some preliminary results. In Section 3, we describe the regularity conditions and present main theorems and their corollaries. In Section 4, we give further comments on these regularity conditions and main results and compare them with other related contributions. All proofs can be found in Section 5 and Appendix A.

2 Preliminaries

In this section we will provide basic notations and some preliminary results necessary for the further statement. We first recall some basic notations in operator theory (e.g., see [12]). Let \( A : \mathcal{H} \to \mathcal{H}' \) be a linear operator, where \((\mathcal{H}, \langle \cdot, \cdot \rangle_{\mathcal{H}})\) and \((\mathcal{H}', \langle \cdot, \cdot \rangle_{\mathcal{H}'})\) are Hilbert spaces with the corresponding norms \( \| \cdot \|_{\mathcal{H}} \) and \( \| \cdot \|_{\mathcal{H}'} \). The set of bounded linear operators from \( \mathcal{H} \) to \( \mathcal{H}' \) is a Banach space with respect to the operator norm \( \| A \|_{\mathcal{H}, \mathcal{H}'} = \sup_{\| f \|_{\mathcal{H}} = 1} \| Af \|_{\mathcal{H}'} \), which is denoted by \( \mathcal{B}(\mathcal{H}, \mathcal{H}') \) or \( \mathcal{B}(\mathcal{H}) \) if \( \mathcal{H} = \mathcal{H}' \). When \( \mathcal{H} \) and \( \mathcal{H}' \) are clear from the context, we will omit the subscript and simply denote the operator norm as \( \| \cdot \| \). Let \( A^* \) be the adjoint operator of \( A \) such that \( \langle Af, f' \rangle_{\mathcal{H}} = \langle f, A^* f' \rangle_{\mathcal{H}'} \), \( \forall f \in \mathcal{H}, f' \in \mathcal{H}' \). If \( A \in \mathcal{B}(\mathcal{H}, \mathcal{H}') \), then \( A^* \in \mathcal{B}(\mathcal{H}', \mathcal{H}) \) with \( \| A^* \| = \| A \| \). We say that \( A \in \mathcal{B}(\mathcal{H}) \) is self-adjoint if \( A^* = A \), and positive if \( A \) is self-adjoint and \( \langle Af, f \rangle_{\mathcal{H}} \geq 0 \) for all \( f \in \mathcal{H} \). The operator norm of a positive operator \( A \in \mathcal{B}(\mathcal{H}) \) has an equivalent expression:

\[
\| A \| = \sup_{x \in \mathcal{H}, \| x \|_{\mathcal{H}} = 1} \langle Ax, x \rangle_{\mathcal{H}}. \tag{2.1}
\]

For \( f \in \mathcal{H} \) and \( f' \in \mathcal{H}' \), define a rank-one operator \( f \otimes f' : \mathcal{H} \to \mathcal{H}' \) by \( f \otimes f'(h) = \langle f, h \rangle_{\mathcal{H}} f'(h), \forall h \in \mathcal{H} \). If \( A \in \mathcal{B}(\mathcal{H}) \) is compact and positive, Spectral Theorem ensures that, there exists an orthonormal basis \( \{ e_k \}_{k \geq 1} \) in \( \mathcal{H} \) consisting of eigenfunctions of \( A \) such that \( A = \sum_{k \geq 1} \lambda_k e_k \otimes e_k \), where the eigenvalues \( \{ \lambda_k \}_{k \geq 1} \) (with geometric multiplicities) are non-negative and arranged in decreasing order, and either the set \( \{ \lambda_k \}_{k \geq 1} \) is finite or \( \lambda_k \to 0 \) when \( k \to \infty \). Moreover, for any \( r > 0 \), we define the \( r \)-th power of \( A \) as \( A^r = \sum_{k \geq 1} \lambda_k^r e_k \otimes e_k \), which is itself a positive compact operator on \( \mathcal{H} \). An operator \( A \in \mathcal{B}(\mathcal{H}, \mathcal{H}') \) is Hilbert-Schmidt if \( \sum_{k \geq 1} \| A e_k \|_{\mathcal{H}'}^2 < \infty \) for some (any) orthonormal basis \( \{ e_k \}_{k \geq 1} \) of \( \mathcal{H} \). The space of Hilbert-Schmidt operators is also a Hilbert space endowed with the inner product \( \langle A, B \rangle_{HS} = \sum_{k \geq 1} \langle A e_k, B e_k \rangle_{\mathcal{H}'} \) and we denote by \( \| \cdot \|_{HS} \) the corresponding norm. In particular, a Hilbert-Schmidt operator \( A \) is compact and we have the following inequality to relate its two different norms:

\[
\| A \| \leq \| A \|_{HS}. \tag{2.2}
\]

An operator \( A \in \mathcal{B}(\mathcal{H}, \mathcal{H}') \) is trace class if \( \sum_{k \geq 1} \| (A \ast A)^{1/2} e_k \|_{\mathcal{H}} < \infty \) for some (any) orthonormal basis \( \{ e_k \}_{k \geq 1} \) of \( \mathcal{H} \). All trace class operators constitute a Banach space endowed with the norm \( trace(A) = \sum_{k \geq 1} \| (A \ast A)^{1/2} e_k \|_{\mathcal{H}} \). It is obviously for any positive operator \( A \in \mathcal{B}(\mathcal{H}) \),

\[
trace(A) = \sum_{k \geq 1} \langle Ae_k, e_k \rangle_{\mathcal{H}}. \tag{2.3}
\]

In the following, we fix a reproducing kernel Hilbert space \( \mathcal{H}_K \) of functions \( f : \mathcal{T} \to \mathbb{R} \) such that all the evaluation functionals are bounded. Then there is a unique symmetric positive definite kernel function \( K : \mathcal{T} \times \mathcal{T} \to \mathbb{R} \), called reproducing kernel, associated with \( \mathcal{H}_K \). Let
Recall that \( \beta \) since we are mainly interested in the hard learning scenario \( \int \) such that \( K \) kernel holds for all \( \| \cdot \| \). Then \( K_t \in \mathcal{H}_K \) and the reproducing property
\[
f(t) = \langle f, K_t \rangle_K
\]
holds for all \( t \in \mathcal{T} \) and \( f \in \mathcal{H}_K \). It is also well-known that any symmetric positive definite kernel \( K \) uniquely defines a reproducing kernel Hilbert space whose reproducing kernel is \( K \) (see, for instance, \([1]\)). Throughout the paper, we assume that \( K \) is measurable on \( \mathcal{T} \times \mathcal{T} \) such that
\[
\int_{\mathcal{T}} \int_{\mathcal{T}} K^2(t, t') dt dt' < \infty.
\]

Recall that \( \mathcal{L}^2(\mathcal{T}) \) is the Hilbert space of functions from \( \mathcal{T} \) to \( \mathbb{R} \) square-integrable with respect to Lebesgue measure. Denote by \( \| \cdot \|_{\mathcal{L}^2} \) the corresponding norm of \( \mathcal{L}^2(\mathcal{T}) \) induced by the inner product \( \langle f, g \rangle_{\mathcal{L}^2} = \int_{\mathcal{T}} f(t) g(t) dt \). For a reproducing kernel \( K \), the integral operator \( L_K : \mathcal{L}^2(\mathcal{T}) \to \mathcal{L}^2(\mathcal{T}) \), given by, for \( f \in \mathcal{L}^2(\mathcal{T}) \) and \( t \in \mathcal{T} \),
\[
L_K(f)(t) = \int_{\mathcal{T}} K(s, t) f(s) ds,
\]
is a positive, compact operator on \( \mathcal{L}^2(\mathcal{T}) \). Then \( L_K^{1/2} \) is well-defined and compact, and \( L_K^{1/2} \) is an isomorphism from \( \overline{\mathcal{H}_K} \), the closure of \( \mathcal{H}_K \) in \( \mathcal{L}^2(\mathcal{T}) \), to \( \mathcal{H}_K \), i.e., for each \( f \in \overline{\mathcal{H}_K} \), \( L_K^{1/2} f \in \mathcal{H}_K \) and
\[
\| f \|_{\mathcal{L}^2} = \| L_K^{1/2} f \|_K.
\]

Since we are mainly interested in the hard learning scenario \( \beta_0 \notin \mathcal{H}_K \), we will always assume \( \mathcal{H}_K \) is dense in \( \mathcal{L}^2(\mathcal{T}) \), i.e., \( \mathcal{L}^2(\mathcal{T}) = \overline{\mathcal{H}_K} \).

Besides the reproducing kernel \( K \), another important kernel in our setting is the covariance kernel. Without loss of generality, we let the explanatory variable \( X \) satisfy \( \mathbb{E}[X] = 0 \) and further assume \( \mathbb{E} \left[ \| X \|_{2, \mathcal{T}}^2 \right] < \infty \). Then the covariance kernel \( C : \mathcal{T} \times \mathcal{T} \to \mathbb{R} \), given by \( C(s, t) := \mathbb{E} [X(s) X(t)] \), \( \forall s, t \in \mathcal{T} \), can define a compact positive integral operator \( L_C : \mathcal{L}^2(\mathcal{T}) \to \mathcal{L}^2(\mathcal{T}) \) via
\[
L_C(f)(t) = \int_{\mathcal{T}} C(s, t) f(s) ds, \quad \forall f \in \mathcal{L}^2(\mathcal{T}) \text{ and } \forall t \in \mathcal{T}.
\]

We next use \( L_K \) and \( L_C \) to give an expression of estimator \( \hat{\beta}_{S, \lambda} \) in \([1, 3]\). Recall that \( \mathcal{L}^2(\mathcal{T}) = \overline{\mathcal{H}_K} \) and norms relation \((2.4)\). We can express \( \hat{\beta}_{S, \lambda} \) as \( \beta_{S, \lambda} = L_K^{1/2} \hat{f}_{S, \lambda} \) with
\[
\hat{f}_{S, \lambda} = \arg \min_{f \in \mathcal{L}^2(\mathcal{T})} \left\{ \frac{1}{N} \sum_{i=1}^{N} \left( Y_i - \langle L_K^{1/2} f, \beta \rangle_{\mathcal{L}^2} \right)^2 + \lambda \| f \|_{\mathcal{L}^2}^2 \right\}.
\]

Following the proof of Theorem 6.2.1 in \([23]\), we can solve \( \hat{f}_{S, \lambda} \) explicitly and obtain the following proposition.

**Proposition 1.** The estimator \( \hat{\beta}_{S, \lambda} \) in \([1, 3]\) can be expressed as \( \hat{\beta}_{S, \lambda} = L_K^{1/2} \hat{f}_{S, \lambda} \) with
\[
\hat{f}_{S, \lambda} = (\lambda I + T_X)^{-1} \left[ \sum_{(X_i, Y_i) \in \mathcal{S}} L_K^{1/2} X_i Y_i \right],
\]
\((2.5)\).
where $I$ denotes the identity operator on $L^2(\mathcal{T})$, $|S| = N$ is the cardinality of $S = \{(X_i, Y_i)\}_{i=1}^{N}$, and $T_X : L^2(\mathcal{T}) \rightarrow L^2(\mathcal{T})$ is an empirical operator with $X = \{X_1, \cdots, X_N\}$ defined by

$$T_X = \frac{1}{|S|} \sum_{X_i \in X} L_X^{1/2} X_i \otimes L_K^{1/2} X_i. \quad (2.6)$$

Recall that $S = \bigcup_{j=1}^{m} S_j$ with $S_j \cap S_k = \emptyset$ for $j \neq k$ and $|S_j| = \frac{N}{m}$. One can define the empirical operators $T_{X_j}$ with $X_j = \{X_i \mid (X_i, Y_i) \in S_j\}$ according to (2.6) and compute the local estimator $\hat{f}_{S_j, \lambda}$ as (2.5), i.e.,

$$T_{X_j} = \frac{1}{|S_j|} \sum_{X_i \in X_j} L_X^{1/2} X_i \otimes L_K^{1/2} X_i$$

and

$$\hat{f}_{S_j, \lambda} = (\lambda I + T_{X_j})^{-1} \frac{1}{|S_j|} \sum_{(X_i, Y_i) \in S_j} L_X^{1/2} X_i Y_i.$$

Then the averaged estimator $\overline{\beta}_{S, \lambda}$ in (1.4) is given by $\overline{\beta}_{S, \lambda} = L_K^{1/2} \overline{f}_{S, \lambda}$ with $\overline{f}_{S, \lambda} := \frac{1}{m} \sum_{j=1}^{m} \hat{f}_{S_j, \lambda}$.

To derive the upper bounds of excess prediction error, for any estimator $\hat{\beta} \in L^2(\mathcal{T})$, we rewrite $\mathcal{R}(\hat{\beta}) - \mathcal{R}(\beta_0)$ as

$$\mathcal{R}(\hat{\beta}) - \mathcal{R}(\beta_0) = \mathbb{E} \left[ \left( X^2 \hat{\beta} - \beta_0 \right)^2 \right] = \left\| L_C^{1/2} (\hat{\beta} - \beta_0) \right\|_{L^2}^2. \quad (2.7)$$

Notice $T_X$ and $\frac{1}{|S|} \sum_{(X_i, Y_i) \in S} L_X^{1/2} X_i Y_i$ are empirical versions of $L_X^{1/2} L_C L_X^{1/2}$ and $L_K^{1/2} L_C \beta_0$, respectively. We thus introduce intermediate function $f_\lambda := (\lambda I + L_K^{1/2} L_C L_K^{1/2})^{-1} L_K^{1/2} L_C \beta_0$ which can be expected to approximate $\hat{f}_{S, \lambda}$ and its averaged version $\overline{f}_{S, \lambda}$. According to (2.7), we then split $\mathcal{R}(\overline{\beta}_{S, \lambda}) - \mathcal{R}(\beta_0)$ into two parts:

$$\mathcal{R}(\overline{\beta}_{S, \lambda}) - \mathcal{R}(\beta_0) = \left\| L_C^{1/2} \left( L_K^{1/2} \overline{f}_{S, \lambda} - L_K^{1/2} f_\lambda + L_K^{1/2} f_\lambda - \beta_0 \right) \right\|_{L^2}^2 \leq 2 \mathcal{S}(S, \lambda) + 2 \mathcal{A}(\lambda), \quad (2.8)$$

where $\mathcal{S}(S, \lambda) := \left\| L_C^{1/2} L_K^{1/2} \overline{f}_{S, \lambda} - L_C^{1/2} L_K^{1/2} f_\lambda \right\|_{L^2}^2$ and $\mathcal{A}(\lambda) := \left\| L_C^{1/2} L_K^{1/2} f_\lambda - L_K^{1/2} \beta_0 \right\|_{L^2}^2$.

In Section 3, we will describe the assumptions that are used to estimate $\mathcal{S}(S, \lambda)$ and $\mathcal{A}(\lambda)$, and then state the main results of this paper. Before that, we give a further characterization of the operators which are crucial in our estimation. For simplicity, let $T := L_K^{1/2} L_C L_K^{1/2}$ and $T_* := L_C^{1/2} L_K L_C^{1/2}$. Note that

$$T = L_K^{1/2} L_C^{1/2} (L_K^{1/2} L_C^{1/2})^*$$

and

$$T_* = (L_K^{1/2} L_C^{1/2})^* L_K^{1/2} L_C^{1/2}.$$

Due to the compactness of $L_K^{1/2} L_C^{1/2}$, both $T$ and $T_*$ are compact and positive operators on $L^2(\mathcal{T})$. Singular value decomposition of $L_K^{1/2} L_C^{1/2}$ (see, e.g., Theorem 4.3.1 in [23]) leads to

$$L_K^{1/2} L_C^{1/2} = \sum_{i=1}^{d} \sigma_i \hat{v}_i \hat{u}_i,$$
the following expansions:

\[ L_{K}^{1/2} L_{C}^{1/2} = \sum_{k \geq 1} \sqrt{\mu_k} \varphi_k \otimes \phi_k, \]

\[ L_{C}^{1/2} L_{K}^{1/2} = \sum_{k \geq 1} \sqrt{\mu_k} \phi_k \otimes \varphi_k, \]

\[ T = \sum_{k \geq 1} \mu_k \phi_k \otimes \phi_k, \]

\[ T_{*} = \sum_{k \geq 1} \mu_k \varphi_k \otimes \varphi_k, \]

where \( \{ \mu_k \}_{k \geq 1} \) is a non-negative, non-increasing and summable sequence, \( \{ \phi_k \}_{k \geq 1} \) and \( \{ \varphi_k \}_{k \geq 1} \) are two orthonormal bases of \( L^2(\mathcal{T}) \). Actually, for any \( \mu_k > 0 \), \( \sqrt{\mu_k} \) is the singular values of \( L_{K}^{1/2} L_{C}^{1/2} \) and the corresponding left and right singular vectors are given by \( \varphi_k \) and \( \phi_k \), which are the eigenvectors (with the same eigenvalue \( \mu_k \)) of \( T \) and \( T_{*} \). In particular, the system \( \{ \mu_k, \phi_k, \varphi_k \}_{k \geq 1} \) plays an important role in describing the regularities of explanatory variable \( X \) and the slope function \( \beta_0 \) which we will explain in details in Section 3.

3 Main Results

In this section, we will present our main theoretical results on the convergence of divide-and-conquer estimator \([1, 4]\) for the functional linear regression model \([1, 1]\). These main results are based on several key assumptions, including the regularity conditions of the slope function and the functional explanatory. We begin with a regularity condition on the slope function \( \beta_0 \) expressed in terms of covariance operator \( L_{C} \) and operator \( T_{*} \) given in \( 2.9 \).

**Assumption 1.** The slope function \( \beta_0 \) in functional linear regression model \([1, 1]\) satisfies

\[ L_{C}^{1/2} \beta_0 = T_{*}^{\theta}(\gamma_0) \text{ with } 0 < \theta \leq 1/2 \text{ and } \gamma_0 \in L^2(\mathcal{T}). \]  

(3.1)

This assumption implies that \( L_{C}^{1/2} \beta_0 \) belongs to the range space of \( T_{*}^{\theta} \) expressed as

\[ \text{ran} T_{*}^{\theta} := \left\{ f \in L^2(\mathcal{T}) : \sum_{k \geq 1} \frac{\langle f, \varphi_k \rangle_{L^2}^2}{\mu_k^{2\theta}} < \infty \right\}, \]

where \( \{ \mu_k, \varphi_k \}_{k \geq 1} \) is the eigensystem of \( T_{*} \). Then \( \text{ran} T_{*}^{\theta_1} \subseteq \text{ran} T_{*}^{\theta_2} \) whenever \( \theta_1 \geq \theta_2 \). The regularity of functions in \( \text{ran} T_{*}^{\theta} \) is measured by the decay rate of its expansion coefficients in terms of \( \{ \phi_k \}_{k \geq 1} \). Condition \( 3.1 \) means that \( \langle L_{C}^{1/2} \beta_0, \varphi_k \rangle_{L^2}^2 \) decays faster than the \( 2\theta \)-th power of the eigenvalues of \( T_{*} \). Larger parameter \( \theta \) will result in faster decay rates, and thus indicate higher regularities of \( L_{C}^{1/2} \beta_0 \). In particular, for \( \theta = 0 \) we have \( \text{ran} T_{*}^{\theta} = L^2(\mathcal{T}) \) implying \( \beta_0 \in L^2(\mathcal{T}) \) and \( \beta_0 \in H_{K} \) ensures regularity condition \( 3.1 \) is satisfied with \( \theta = 1/2 \) as \( \text{ran} T_{*}^{1/2} = \text{ran} L_{C}^{1/2} L_{K}^{1/2} \). From this point of view, condition \( 3.1 \) allows \( \beta_0 \notin H_{K} \) which extends the previous regularity assumption in \([7, 47]\). This condition is also known as Hölder-type source condition involving the operator \( T_{*} \), which is a classical smoothness assumption in the theory of inverse problems. Similar conditions defined by the operator \( L_{K} \) are widely used in the literature of learning theory, see, for instance, \([3, 5, 9, 39]\). We will provide more discussions on Assumption \([1] \) in Section \([4]\).
Throughout of the paper, we assume the following noise condition.

**Assumption 2.** The random noise $\epsilon$ in functional linear regression model (1.1) is independent of $X$ satisfying $\mathbb{E}[\epsilon] = 0$ and $\mathbb{E}[\epsilon^2] \leq \sigma^2$.

We first establish a mini-max lower bound of the excess prediction risk under Assumption 1 and Assumption 2. To this end, we also need to assume that $\{\mu_k\}_{k \geq 1}$, i.e., the eigenvalues of $T_*$ (and $T$), satisfy a polynomially decaying condition. For two positive sequences $\{a_k\}_{k \geq 1}$ and $\{b_k\}_{k \geq 1}$, we say $a_k \preceq b_k$ holds if there exits a constant $c > 0$ independent of $k$ such that $a_k \leq cb_k, \forall k \geq 1$. In addition, $a_k \asymp b_k$ if and only if $a_k \preceq b_k$ and $b_k \preceq a_k$. For the sake of simplicity, we write $L_C^{1/2} \beta_0 \in \text{ran} T^\theta_*$ if $\beta_0$ satisfied the regularity condition (3.1).

**Theorem 1.** Suppose that Assumption 1 and Assumption 2 are satisfied with $0 < \theta \leq 1/2$ and $\sigma > 0$, $\{\mu_k\}_{k \geq 1}$ satisfy $\mu_k \asymp k^{-1/p}$ for some $0 < p \leq 1$. Then the excess prediction risk satisfies

$$\liminf_{\gamma \to 0} \liminf_{N \to \infty} \sup_{\beta_0, \beta_S \in L^2(T)} \mathbb{P} \left\{ \mathcal{R}(\hat{\beta}_S) - \mathcal{R}(\beta_0) \geq \gamma N^{-\frac{\theta}{2p+\theta}} \right\} = 1,$$

where the supremum is taken over all $\beta_0 \in L^2(T)$ satisfying $L_C^{1/2} \beta_0 \in \text{ran} T^\theta_*$ and the infimum is taken over all possible predictors $\hat{\beta}_S \in L^2(T)$ based on the training sample $S = \{(X_i, Y_i)\}_{i=1}^N$.

In Theorem 1 and subsequent statements, the case $p = 1$ corresponds to the case in which we only require $\{\mu_k\}_{k \geq 1}$ to be summable. The lower bound for $p = 1$ is also known as capacity-independent optimal in some literature (for instance, see [46]). That is, the bound is optimal in the mini-max sense without requiring capacity assumption, e.g., the decaying condition on the eigenvalues $\{\mu_k\}_{k \geq 1}$.

We next consider the upper bound of excess prediction risk and show that the lower bound established in Theorem 1 can be achieved by the estimator $\hat{\beta}_{S,\lambda}$ in (1.4). The following assumption on moment condition plays a crucial role in establishing upper bounds of convergence rates of (2.7).

**Assumption 3.** There exists a constant $c_1 > 0$, such that for any $f \in L^2(T)$,

$$\mathbb{E} \left[ \langle X, f \rangle_{L^2}^4 \right] \leq c_1 \left[ \mathbb{E} \langle X, f \rangle_{L^2}^2 \right]^2.$$  

(3.3)

Assumption 3 has been introduced in [7, 47]. Condition (3.3) states that linear functionals of $X$ have bounded kurtosis which is satisfied in particular with $c_1 = 3$ when $X$ follows a Gaussian process. For the convenience of further statements, define the effective dimension as

$$\mathcal{N}(\lambda) := \sum_{k \geq 1} \frac{\mu_k}{\lambda + \mu_k},$$

(3.4)

where $\lambda > 0$ and $\{\mu_k\}_{k \geq 1}$ are non-negative eigenvalues of $T$ (with geometric multiplicities) arranged in decreasing order. The effective dimension is widely used in the convergence analysis of kernel ridge regression [9, 15, 30, 48]. Now under a polynomially decaying condition of eigenvalues $\{\mu_k\}_{k \geq 1}$, we can give the following theorem on the upper bound for excess prediction risk of $\hat{\beta}_{S,\lambda}$, with $S = \cup_{j=1}^m S_j = \{(X_i, Y_i)\}_{i=1}^N$ and $|S_j| = \frac{N}{m}$. We employ $o(\alpha_N)$ to denote a little-o sequence of $\{a_N\}_{N \geq 1}$ if $\lim_{N \to \infty} o(\alpha_N)/a_N = 0$. We
Theorem 2. Under Assumption 7 with $0 < \theta \leq 1/2$, Assumption 2 with $\sigma > 0$ and Assumption 4 with $c_1 > 0$, suppose that $\{\mu_k\}_{k \geq 1}$ satisfy $\mu_k \lesssim k^{-1/p}$ for some $0 < p \leq 1$. Then

1. For $p/2 < \theta \leq 1/2$, there holds

$$\lim_{\Gamma \to \infty} \sup_{N \to \infty} \sup_{\beta_0} \mathbb{P} \left\{ \mathcal{R}(\beta_{S,\lambda}) - \mathcal{R}(\beta_0) \geq \Gamma N^{-\frac{2\theta}{p}} \right\} = 0 \quad (3.5)$$

provided that $m \leq o(N^{\frac{2\theta - p}{2p + \theta p}})$ and $\lambda = N^{-\frac{1}{2(1 + \theta)}}$.

2. For $0 < \theta \leq p/2$, there holds

$$\lim_{\Gamma \to \infty} \sup_{N \to \infty} \sup_{\beta_0} \mathbb{P} \left\{ \mathcal{R}(\beta_{S,\lambda}) - \mathcal{R}(\beta_0) \geq \Gamma N^{-\frac{1}{p} (\log N)^{\frac{r}{p}}} \right\} = 0 \quad (3.6)$$

provided that $m \leq (\log N)^r$ for some $r > 0$ and $\lambda = N^{-\frac{1}{2} (\log N)^{\frac{r}{p}}}$, and

$$\lim_{\Gamma \to \infty} \sup_{N \to \infty} \sup_{\beta_0} \mathbb{P} \left\{ \mathcal{R}(\beta_{S,\lambda}) - \mathcal{R}(\beta_0) \geq \Gamma N^{-\frac{(1-r)\theta}{p} \log N} \right\} = 0 \quad (3.7)$$

provided that $m \leq N^r$ for some $0 \leq r < 1$ and $\lambda = N^{-\frac{1}{2} (\log N)^{\frac{r}{p}}}$.

Here the supremum is taken over all $\beta_0 \in \mathcal{L}^2(T)$ satisfying $L^{1/2}_C \beta_0 \in \text{ran} T^\theta_*$ with $0 < \theta \leq 1/2$.

Actually, we show that if the eigenvalue decay satisfies a polynomial upper bound of order 1/p with $0 < p < 1$ and the regularity parameter $\theta$ satisfies $0 < \theta \leq p/2$, there holds

$$\lim_{\Gamma \to \infty} \sup_{N \to \infty} \sup_{\beta_0} \mathbb{P} \left\{ \mathcal{R}(\beta_{S,\lambda}) - \mathcal{R}(\beta_0) \geq \Gamma \lambda^{2\theta} \right\} = 0$$

provided that $m^2 \lambda^{-2p} \leq o(N)$. From Theorem 2, the bound (3.5) implies when $\theta \in (p/2, 1/2]$, the excess prediction risk of $\beta_{S,\lambda}$ attains the convergence rate of the lower bound given by Theorem 1 and is therefore rate-optimal. Additionally, if $\theta = p/2$, from (3.6) taking $m \leq (\log N)^r$ and $\lambda = N^{-\frac{1}{2} (\log N)^{\frac{r}{p}}}$ with some $r > 0$ yields

$$\lim_{\Gamma \to \infty} \sup_{N \to \infty} \sup_{\beta_0} \mathbb{P} \left\{ \mathcal{R}(\beta_{S,\lambda}) - \mathcal{R}(\beta_0) \geq \Gamma N^{-\frac{1}{p} (\log N)^{\frac{r}{p}}} \right\} = 0.$$ 

This convergence rate is also optimal up to a logarithmic factor. The bound (3.5) generalizes previous results of [7], which only considered the case $\beta_0 \in \mathcal{H}_K$, to the hard learning scenario $\beta_0 \notin \mathcal{H}_K$ and the divide-and-conquer estimators. Actually, when $\theta = 1/2$, taking $m = 1$ and $\lambda = N^{-\frac{1}{2p + \theta p}}$, we recover Theorem 2 of [7] which establishes minimax upper bound for the estimator $\hat{\beta}_{S,\lambda}$ in (1.3) when $\beta_0 \in \mathcal{H}_K$.

We next introduce a higher-order moment condition on $X$ such that one can establish the strong convergence in expectation. To this end, given a reproducing kernel $K$, we shall introduce various regularities of explanatory variable $X$ defined through its image under $L^{1/2}_K$. Recall that $X$ is a random variable taking values in $L^2(T)$ with $\mathbb{E}[X] = 0$ and $\mathbb{E}[\|X\|^2_L] < \infty,$
and \( \{\mu_k, \phi_k\}_{k \geq 1} \) is the eigensystem of \( T \) according to 2.29. Consider the principal component decomposition of \( L_K^{1/2} X \) with respect to \( T \) (see 2 for details), which is expressed as

\[
L_K^{1/2} X = \sum_{k \geq 1} \sqrt{\mu_k} x_k \phi_k
\]  

(3.8)

where the \( x_k \)’s are zero-mean, uncorrelated real-valued random variables with \( \mathbb{E}[x_k^2] = 1 \). We assume the following moment condition to characterize the regularity of \( L_K^{1/2} X \).

**Assumption 4.** For some integer \( \ell \geq 2 \), there exists a constant \( \rho < \infty \) such that \( \{x_k\}_{k \geq 1} \) in decomposition (3.8) satisfy \( \sup_{k \geq 1} \mathbb{E}[x_k^{4\ell}] \leq \rho^{4\ell} \). Moreover, there exists a constant \( c_2 > 0 \) such that

\[
\mathbb{E} \left[ \langle X, f \rangle \right] \leq c_2^2 \left[ \mathbb{E} \langle X, f \rangle^2 \right]^{\ell}, \quad \forall f \in \mathcal{L}^2(T).
\]  

(3.9)

Since \( \mathbb{E}[x_k^2] = 1 \), we always have \( \rho \geq 1 \). When \( X \) is a Gaussian random element in \( \mathcal{L}^2(T) \), Assumption 4 is satisfied for any integer \( \ell \geq 2 \). In fact, given an integer \( \ell \geq 2 \), the linear functional of a Gaussian random element \( X \) satisfies

\[
\mathbb{E} \left[ \langle X, f \rangle^{4\ell} \right] \leq (4\ell - 1)!! \left[ \mathbb{E} \langle X, f \rangle^2 \right]^{2\ell}, \quad \forall f \in \mathcal{L}^2(T).
\]

Then taking \( f = L_K^{1/2} x_k \) implies Assumption 4 with \( \rho = [(4\ell - 1)!!]^{1/2} \) and \( c_2 = 105 \) (by letting \( \ell = 2 \)). We need condition (3.9) to bound \( \mathbb{E} \left[ \langle X, \beta_0 - L_K^{1/2} f, \lambda \rangle \right] \) when \( \beta_0 \notin \mathcal{H}_K \), which is crucial in the estimation of

\[
\mathcal{S}(S, \lambda) = \left\| L_C^{1/2} L_K^{1/2} S, \lambda - L_C^{1/2} L_K^{1/2} f_0 \right\|_{L^2(T)}^2.
\]

Now we can establish the following upper bounds of (2.7) in expectation under Assumption 2 and 4.

**Theorem 3.** Suppose that Assumption 2 is satisfied with \( 0 < \theta \leq 1/2 \) and \( \gamma_0 \in \mathcal{L}^2(T) \). Under Assumption 4 with \( \sigma > 0 \) and Assumption 4 with some integer \( \ell \geq 2 \) and \( c_2 > 0 \), take \( \lambda \leq 1 \), then if \( 2 \leq \ell < 8 \), there holds

\[
\mathbb{E} \left[ \left( \mathcal{R}(S, \lambda) - \mathcal{R}(\beta_0) \right) \right]
\leq 2\lambda^{2\theta} \left\| \gamma_0 \right\|_{L^2(T)}^2 + 16 \frac{N(\lambda)}{N}(c_2 \lambda^{2\theta} \left\| \gamma_0 \right\|_{L^2(T)}^2 + \sigma^2) + 8c_2 \frac{m}{N} N(\lambda) \lambda^{2\theta} \left\| \gamma_0 \right\|_{L^2(T)}^2
\]

\[
+ b_1(\ell) \lambda^{1 - 4} \left[ 1 + \left( \frac{mN^2(\lambda)}{N} \right)^{\frac{\ell}{4}} + \lambda^{-\frac{\ell}{4}} \left( \frac{mN^2(\lambda)}{N} \right)^{\frac{\ell}{8}} \right] \left( \frac{mN^2(\lambda)}{N} \right)^{\frac{\ell}{4}} \frac{4 + 2m}{N} \left( 1 + \lambda^{2\theta} N(\lambda) \right)
\]

\[
+ b_2(\ell) \lambda^{1 - 4} \left[ 1 + \left( \frac{mN^2(\lambda)}{N} \right)^{\frac{\ell}{4}} + \lambda^{-\frac{\ell}{4}} \left( \frac{mN^2(\lambda)}{N} \right)^{\frac{\ell}{8}} \right] \left( \frac{mN^2(\lambda)}{N} \right)^{\frac{\ell}{4}} \frac{4\sigma^2}{N} N(\lambda).
\]  

(3.10)

If \( \ell \geq 8 \), there holds

\[
\mathbb{E} \left[ \left( \mathcal{R}(S, \lambda) - \mathcal{R}(\beta_0) \right) \right]
\leq 2\lambda^{2\theta} \left\| \gamma_0 \right\|_{L^2(T)}^2 + 16 \frac{N(\lambda)}{N}(c_2 \lambda^{2\theta} \left\| \gamma_0 \right\|_{L^2(T)}^2 + \sigma^2) + 8c_2 \frac{m}{N} N(\lambda) \lambda^{2\theta} \left\| \gamma_0 \right\|_{L^2(T)}^2
\]

\[
+ b_1(\ell) \left[ 1 + \frac{mN^2(\lambda)}{N} + \frac{1}{\lambda^2} \left( \frac{mN^2(\lambda)}{N} \right)^2 \right] \left( \frac{mN^2(\lambda)}{N} \right)^{\frac{\ell}{4}} \frac{4 + 2m}{N} \left( 1 + \lambda^{2\theta} N(\lambda) \right)
\]  

(3.11)
Corollary 1. Under the assumptions of Theorem 3, suppose that \( \{\mu_k\}_{k \geq 1} \) satisfy \( \mu_k \lesssim k^{-1/p} \) for some \( 0 < p \leq 1 \).

1. When \( 2 \leq \ell < 8 \), if \( \frac{p\ell + 8}{4\ell + 16} \leq \theta \leq \frac{1}{2} \), then

\[
\mathbb{E} \left[ \mathcal{R}(\overline{\beta}_{S,\lambda}) - \mathcal{R}(\beta_0) \right] \lesssim N^{-\frac{2\theta}{2\theta + p}}
\]

provided that

\[
m \leq \min \left\{ N^{\frac{8 + p\ell - 4\theta - 3\theta \ell}{8 + 4\theta + 3\theta \ell}}, N^{\frac{8 + p\ell + 8 - 4\theta \ell}{8 + 4\theta + 3\theta \ell}} \right\}
\]

and

\[
\lambda = N^{-\frac{1}{2\theta + p}};
\]

if \( 0 < \theta < \frac{p\ell + 8}{4\ell + 16} \), then

\[
\mathbb{E} \left[ \left( \mathcal{R}(\overline{\beta}_{S,\lambda}) - \mathcal{R}(\beta_0) \right) \right] 
\lesssim \max \left\{ N^{\frac{2\theta (4 + 2\ell)(r - 1)}{8 + 8\theta + 2\theta p - r}}, N^{\frac{2\theta (4 + 2\ell)(r - 1)}{8 + 8\theta + 3\theta p}}, N^{\frac{2\theta (r - 1)}{8 + 4p + 8\theta + 3\theta p}} \right\}
\]

provided that

\[
m \leq N^r \text{ for some } 0 \leq r \leq \frac{2\theta}{2\theta + p}
\]

and

\[
\lambda = \max \left\{ \frac{N^{\frac{8 + p\ell - 4\theta - 3\theta \ell}{8 + 4\theta + 3\theta \ell}}, N^{\frac{8 + p\ell + 8 - 4\theta \ell}{8 + 4\theta + 3\theta \ell}}, N^{\frac{2\ell (r - 1)}{8 + 4p + 8\theta + 3\theta p}} \right\}.
\]

2. When \( \ell \geq 8 \), if \( \frac{p\ell + 8}{2\ell + 16} \leq \theta \leq \frac{1}{2} \), then

\[
\mathbb{E} \left[ \mathcal{R}(\overline{\beta}_{S,\lambda}) - \mathcal{R}(\beta_0) \right] \lesssim N^{-\frac{2\theta}{2\theta + p}}
\]

provided that

\[
m \leq \min \left\{ N^{\frac{8 + p\ell - 4\theta + 16\theta - 2\theta \ell}{(12 + \ell)(2\theta + p)}}, N^{\frac{8 + p\ell - 24\theta - 2\theta \ell}{(12 + \ell)(2\theta + p)}} \right\}
\]

and

\[
\lambda = N^{-\frac{1}{2\theta + p}};
\]

if \( 0 < \theta < \frac{p\ell + 8}{2\ell + 16} \), then

\[
\mathbb{E} \left[ \left( \mathcal{R}(\overline{\beta}_{S,\lambda}) - \mathcal{R}(\beta_0) \right) \right] 
\lesssim \max \left\{ N^{\frac{\theta (4 + \ell)(r - 1)}{4\ell p + 4p + 8\theta + 3\theta p}}, N^{\frac{\theta (12 + \ell)(r - 1)}{4\ell p + 4p + 8\theta + 3\theta p}}, N^{\frac{\theta (8 + \ell)(r - 1)}{4\ell p + 4p + 8\theta + 3\theta p}} \right\}
\]
provided that

\[ m \leq N^r \text{ for some } 0 \leq r \leq \frac{2\theta}{2\theta + p} \]

and

\[ \lambda = \max \left\{ \frac{(1+c)(r-1)}{N^{2\theta + 2}}, \frac{(r-1)}{N^{2\theta + m}}, \frac{(12+c)(r-1)}{N^{2\theta + m}}, \frac{(8+c)(r-1)}{N^{2\theta + m}} \right\}. \]

According to Theorem 4, the expectation bounds (3.12) and (3.14) are minimax optimal. Due to the well-known Markov’s inequality, convergence in expectation given by Theorem 3 and Corollary 1 is stronger, leading to bounds in a similar form as that of Theorem 2. However, the possible ranges of $\theta$ that achieve the optimal rates in (3.12) and (3.14), given respectively by $[\frac{p+8}{2\theta}, 1/2]$ and $[\frac{p+8}{2\theta+16}]$, both of which are covered by $(p/2, 1/2)$, become smaller compared to the previous range of $\theta$ in the minimax bound (3.5). Moreover, we also observe from Corollary 1 that as the integer $\ell$ in Assumption 1 diverges to infinity, the possible ranges of $\theta$ that achieve the minimax expectation bounds will increase to $(p/2, 1/2)$ which is exactly the range of $\theta$ leading to the minimax bound (3.5). Motivated by this observation, we introduce another regularity condition on $X$ to establish optimal expectation error bounds for any $\theta \in (0, 1/2]$.

**Assumption 5.** There exists a constant $\rho < \infty$ such that $\{\xi_k\}_{k \geq 1}$ in decomposition (3.8) satisfy $\sup_{k \geq 1} |\xi_k| \leq \rho$ and the fourth-order moment condition (3.3) is satisfied with $c_1 > 0$.

One can verify that Assumption 5 holds true if the expansion of $L^{1/2}X$ in (3.8) is a summation of finite terms. Recall that the trace of operator $T$ is given by

\[ \text{trace}(T) := \sum_{k \geq 1} \mu_k. \] (3.16)

Then we have the following theorem.

**Theorem 4.** Suppose that Assumption 4 is satisfied with $0 < \theta \leq 1/2$ and $\gamma_0 \in L^2(\mathcal{T})$. Under Assumption 2 with $\sigma > 0$ and Assumption 5 with $\rho, c_1 > 0$, take $\lambda \leq 1$, then there holds

\[
\mathbb{E} \left[ (\mathcal{R}(\mathcal{S}_{\lambda}) - \mathcal{R}(\beta_0)) \right] \leq 2\lambda^{2\theta\gamma^2} + 16\frac{\mathcal{N}(\lambda)}{N} \left( c_1 \lambda^{2\theta} \gamma_0^2 + \sigma^2 \right) + 8c_1 \frac{m}{N} \mathcal{N}(\lambda) \lambda^{2\theta} \gamma_0^2 + c_3 c_4 \mu_1 \frac{4 + 2m}{N^{2\theta - 2\theta} (1 + \frac{m}{N})^{1/2} (\lambda) \exp \left( - \frac{c_5 N}{2m \mathcal{N}(\lambda)} \right)} \]

where $c_3, c_4$ and $c_5$ are universal constants which will be specified in the proof.

We further obtain a corollary of Theorem 4.

**Corollary 2.** Under the assumptions of Theorem 4, suppose that $\{\mu_k\}_{k \geq 1}$ satisfy $\mu_k \leq k^{-1/p}$ for some $0 < p \leq 1$. There holds

\[
\mathbb{E} \left[ \mathcal{R}(\mathcal{S}_{\lambda}) - \mathcal{R}(\beta_0) \right] \leq N^{-\frac{2\theta}{2\theta + p}} \] (3.18)

provided that $m \leq o\left( \frac{N^{2\theta + p}}{\log N} \right)$ and $\lambda = N^{-\frac{2\theta}{2\theta + p}}$. 

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As far as we know, the expectation bound (3.18) establishes the first mini-max optimal rates for all possible $0 < \theta \leq 1/2$. One can refer to Section 4 for more discussions.

At the end of this section, we establish fast convergence rates for noiseless functional linear model (i.e., $\epsilon = 0$ in (1.1)).

**Theorem 5.** Under Assumption $\mathcal{A}$ with $0 < \theta \leq 1/2$, Assumption $\mathcal{B}$ with $\sigma = 0$ and Assumption $\mathcal{C}$ with $c_1 > 0$, suppose that $\{\mu_k\}_{k \geq 1}$ satisfy $\mu_k \lesssim k^{-1/p}$ for some $0 < p \leq 1$. For any $0 < \eta \leq 1/2$, there holds

$$
\lim_{\Gamma \to \infty} \sup_{N \to \infty} \sup_{\beta_0} \mathbb{P} \left\{ \mathcal{R}(\overline{\beta}_S, \lambda) - \mathcal{R}(\beta_0) \geq \Gamma N^{-\frac{\eta(1-2\eta)}{p}} \right\} = 0 \quad (3.19)
$$

provided that $m \leq o(N^\eta)$ and $\lambda = N^{-\frac{1-2\eta}{2p}}$, where the supremum is taken over all $\beta_0 \in \mathcal{L}^2(T)$ satisfying $L_C^{1/2} \beta_0 \in \text{ran} T_*$ with $0 < \theta \leq 1/2$.

Follow from (3.19), given any $s > 0$ such that $0 < s < 2/s$ and $sp < \theta \leq 1/2$, taking $0 < \eta \leq \frac{1}{2} - \frac{s}{2p}$, $m \leq o(N^\eta)$ and $\lambda = N^{-\frac{1-2\eta}{2p}}$ yields

$$
\lim_{\Gamma \to \infty} \sup_{N \to \infty} \sup_{\beta_0} \mathbb{P} \left\{ \mathcal{R}(\overline{\beta}_S, \lambda) - \mathcal{R}(\beta_0) \geq \Gamma N^{-s} \right\}
$$

$$
\leq \lim_{\Gamma \to \infty} \sup_{N \to \infty} \sup_{\beta_0} \mathbb{P} \left\{ \mathcal{R}(\overline{\beta}_S, \lambda) - \mathcal{R}(\beta_0) \geq \Gamma N^{-\frac{\eta(1-2\eta)}{p}} \right\}
$$

$$
= 0,
$$

where the inequality follows from $\frac{\eta(1-2\eta)}{p} \geq s$. The difference between noised and noiseless models is significant: rates faster than $N^{-1}$ for model (1.1) are impossible with non-zero additive noise, while we prove that the divided-and-conquer estimators for the noiseless model can converge with arbitrarily fast polynomial rates when $p$ is small enough. To our best knowledge, Theorem 5 and the related convergence rates (3.19) are also new to the literature, constituting another contribution of this paper. We will prove all these results in Section 5.

### 4 Discussions and Comparisons

In this section, we first comment on Assumption $\mathcal{A}$ and then compare our convergence analysis with some related results. Regularity condition (3.1) in Assumption $\mathcal{A}$ was first introduced by [14] and then adopted in the subsequent work [11]. From the discussion in Section 3 we see that $\beta_0 \in \mathcal{H}_K$ implies condition (3.1) is satisfied with $\theta = 1/2$, while the former is equivalent to $\beta_0 = L_K^{1/2} \gamma_0$ for some $\gamma_0 \in \mathcal{L}^2(T)$. Actually, due to Theorem 3 in [11], if $L_K \succeq \delta L_C^\nu$ for some $\delta > 0$ and $\nu > 0$, then for any $\beta_0 \in \mathcal{L}^2(T)$, there exists some $\gamma_0 \in \mathcal{L}^2(T)$ such that condition (3.1) is satisfied with $\theta = 1/(2 + 2\nu)$. Here for any bounded self-adjoint operators $A_1$ and $A_2$ on $\mathcal{L}^2(T)$, we write $A_1 \succeq A_2$ if $A_1 - A_2$ is positive. As a special case when $L_K$ and $L_C$ can be simultaneously diagonalized, let $\{\rho_k\}_{k \geq 1}$ and $\{\lambda_k\}_{k \geq 1}$ be eigenvalues of $L_K$ and $L_C$ respectively (both are sorted in decreasing order with geometric multiplicities). When $\rho_k \asymp k^{-1/\omega}$ with $\omega > 1$ and $\lambda_k \asymp k^{-1/\tau}$ with $\tau > 1$, then $\beta_0 \in \text{ran} L_K^s$ for some $s \in [0, 1/2]$ implies condition (3.1) is satisfied with $\theta = (\omega + 2s\tau)/(2\omega + 2\tau)$, where $\text{ran} L_K^s$ denotes the range space of $L_K^s$. When $K$ is an analytic kernel on $T$, the eigenvalues of $L_K$
decay exponentially, and then condition (3.1) can be satisfied for $\theta$ arbitrarily closed to $1/2$ (but still strictly less than $1/2$). From the discussions above, Assumption 1 is mild and provides an intrinsic measurement for the complexity of the prediction problem through the regularity condition (3.1). Recently, under Assumptions 1, 2 and the boundedness condition on $K$ and $C$ (which implies (4.1) is satisfied with $t = 1$), the work [11, 18] applied stochastic gradient descent to solve functional linear regression model (1.1) and established convergence rates for prediction and estimation.

Convergence performance of kernel ridge regression and its variants for hard learning problems in which the optimal predictor is outside of the kernel space has been intensively studied recently by [15, 28, 29, 37, 38, 40]. Among all available literature, the work [15] obtained the best known convergence rates by applying the integral operator techniques combined with an embedding property (see condition (EMB) in [15]). As far as we know, our paper is the first work to consider functional linear regression in a hard learning scenario. To make a further comparison, we first introduce an equivalent embedding condition (see (4.1)) under the functional linear regression setting. Then we apply this condition to derive convergence rates and compare them with related results in Section 3.

**Assumption 6.** There exist constants $\kappa > 0$ and $0 < t \leq 1$ such that $\{\xi_k\}_{k \geq 1}$ in decomposition (3.8) satisfy

$$\sum_{k \geq 1} \mu_k^t \xi_k^2 \leq \kappa^2. \tag{4.1}$$

Moreover, the fourth-order moment condition (3.3) is satisfied with some $c_1 > 0$.

Condition (4.1) actually describes the $L^\infty$-embedding property of $\text{ran}T^{t/2}$ for $0 < t \leq 1$ which follows directly from Theorem 9 in [15]. Then we obtain the following result which also deserve attention in its own right.

**Theorem 6.** Under Assumption 1 with $0 \leq \theta < 1/2$, Assumption 2 with $\sigma > 0$ and Assumption 6 with $0 < t \leq 1$, suppose that $\{\mu_k\}_{k \geq 1}$ satisfy $\mu_k \lesssim k^{-1/p}$ for some $0 < p \leq 1$.

1. When $\max\{0, t/2 - p/2\} < \theta \leq 1/2$, then

$$\mathbb{E} \left[ \mathcal{R}(\overline{\beta}_{S,\lambda}) - \mathcal{R}(\beta_0) \right] \lesssim N^{-\frac{2\theta}{2\theta + p}} \tag{4.2}$$

provided that

$$m \leq o \left( \frac{N^{-\frac{2\theta + p - 1}{2\theta + p}}}{\log N} \right) \quad \text{and} \quad \lambda = N^{-\frac{1}{2\theta + p}}.$$  

2. When $0 \leq \theta \leq \max\{0, t/2 - p/2\}$, then

$$\mathbb{E} \left[ \mathcal{R}(\overline{\beta}_{S,\lambda}) - \mathcal{R}(\beta_0) \right] \lesssim N^{-\frac{2\theta}{\tau}} (\log N)^{-\frac{4\theta}{\tau}} \tag{4.3}$$

provided that

$$m \leq o(\log N) \quad \text{and} \quad \lambda = N^{-\frac{1}{\tau}}(\log N)^{-\frac{2}{\tau}}.$$  

The proof of Theorem 6 is also postponed to Section 5. Condition (4.1) characterizes the regularity of $L_1 K X$ through the parameter $t \in (0, 1]$, of which the most general case is taking $t = 1$, i.e., $\sum_{k=1}^{\infty} \mu_k \xi_k^2 \leq \kappa^2$, or equivalently,

$$\|L_1^{1/2} X\|_{\mathcal{L}^2} \leq \kappa. \tag{4.4}$$
One can check that condition (4.1) can be satisfied if the two kernels $K$ and $C$ are bounded on $\mathcal{T} \times \mathcal{T}$. When $t = 1$, we obtain the mini-max rates in expectation for $\theta \in (1/2 - p/2, 1/2]$ from bound (1.2). However, as $p \downarrow 0$, which implies the eigenvalues of $T^*$ decay even faster, the rate-optimal interval of $\theta$ is getting smaller. It seems unreasonable that higher regularity of $T^*$ could instead reduce the possible scale of $\theta$ that leads to the optimal convergence. This phenomenon is widely observed in the convergence analysis of regularized kernel regression for the hard learning scenario, e.g., see [15, 28, 29, 38]. Note that verifying the embedding condition (4.1) for $t < 1$ is highly non-trivial. This condition is automatically satisfied for all $0 < t \leq 1$ if the expansion of $L_{K}^{1/2}X$ in (3.8) is a summation of finite terms. However, it is a wide-open question whether this condition holds for more general cases. It is also pointed out by [15] that how to obtain the optimal rates for $t > p$ and $\theta \in (0, t/2 - p/2]$ is an outstanding problem that cannot be addressed by introducing the embedding condition. Comparing Assumption 4 to Assumption 1 in Theorem 3 and Corollary 1, it is difficult to tell which problem that cannot be addressed by introducing the embedding condition. Comparing regularities indicated by larger $\ell$ in Assumption 4 or smaller $p$ in the eigenvalue decaying of $T^*$ will result in larger scale of $\theta$ in which the estimators are rate-optimal. We believe that convergence analysis based on Assumption 4 is more insightful from this perspective. We then illustrate that in most cases, the index $p$ can be closed to zero arbitrarily if one of the kernels $K$ and $C$ is smooth enough. To this end, we need the following lemma.

**Lemma 1.** Consider two positive, compact operators $L_A$ and $L_B$ on a separable Hilbert space $H$. Assume $\text{ran}(L_A^{1/2}) = H$, then we have

$$\rho_k(L_A^{1/2}L_BL_A^{1/2}) \leq \rho_k(L_B)\|L_A\|,$$

where the $\rho_k(L_A^{1/2}L_BL_A^{1/2})$ and $\rho_k(L_B)$ denote the $k$-th eigenvalue (sorted in decreasing order) of operators $L_A^{1/2}L_BL_A^{1/2}$ and $L_B$, respectively.

We include its proof in the Appendix A for the sake of completeness. From Lemma 1 with $H = L^2(\mathcal{T})$ and the fact $\overline{\text{ran}(L_K^{1/2})} = L^2(\mathcal{T})$, or equivalently $\text{ran}(L_K^{1/2}) = L^2(\mathcal{T})$, we have $\mu_k = \rho_k(L_K^{1/2}L_CL_K^{1/2}) \leq \rho_k(L_C)\|L_K\| \leq \rho_k(L_C)$. Moreover, if $\text{ran}(L_C^{1/2}) = L^2(\mathcal{T})$, one can deduce $\mu_k = \rho_k(L_C^{1/2}L_KL_C^{1/2}) \leq \rho_k(L_C)\|L_C\| \leq \rho_k(L_K)$ by the same argument. For example, when $\mathcal{T} = \mathbb{R}$ and $K$ is the reproducing kernel of fractional Sobolev space $W^{\beta,2}(\mathbb{R})$ with $\beta > 1/2$, we have $\mu_k \leq \rho_k(L_K) \asymp k^{-2\beta}$ and then $p \leq \frac{1}{2\beta}$ can arbitrarily approach zero if $K$ is smooth enough, i.e., $\beta$ can be sufficiently large. Another notable example is that when $\mathcal{T} = [0,1]^D$ for some integer $D \geq 1$ and $X$ is a Gaussian random element in $L^2(\mathcal{T})$ with zero mean and covariance kernel $C_\gamma : \mathcal{T} \times \mathcal{T} \rightarrow \mathbb{R}$ (which is called a square-exponential kernel) defined by $C_\gamma(x,x') := \exp(-\frac{\|x-x'\|^2}{\gamma^2}), \forall x,x' \in \mathcal{T}$, i.e., $X \sim \mathcal{N}(0,L_{C_\gamma})$. Here $\gamma > 0$ is a constant and $L_{C_\gamma}$ denotes the covariance operator induced by $C_\gamma$. According to the existing literature about Gaussian process (see, for example, [26]), we know that $\{\rho_k(L_{C_\gamma})\}_{k \geq 1}$ enjoys an exponential decay. For this case, we can prove that the divided-and-conquer estimators are mini-max optimal for all possible $\theta \in (0,1/2]$ according to Corollary 1.

We now compare Theorem 6 with Corollary 2 of Theorem 4. Under the uniformly boundedness condition on $\{\xi_k\}_{k \geq 1}$ in Assumption 5, we simplify the embedding condition (4.1) by
only requiring the sequence \( \{\mu_k^t\}_{k \geq 1} \) to be summable, i.e., \( \sum_{k \geq 1} \mu_k^t < \infty \), which is satisfied for \( t = p + \epsilon \) if \( \mu_k \lesssim k^{-1/p} \). Here \( \epsilon > 0 \) can be arbitrarily small. Therefore, under the same assumptions of Corollary 2, the first claims in Theorem 6 ensures that for all sufficiently small \( \epsilon > 0 \) and \( \epsilon/2 < \theta \leq 1/2 \), there holds

\[
\mathbb{E} \left[ \mathcal{R}(\bar{\beta}_{S,\lambda}) - \mathcal{R}(\beta_0) \right] \lesssim N^{-\frac{2\theta}{2\theta + p}}
\]

with \( \lambda = N^{-\frac{1}{2\theta + p}} \) and \( m \leq o \left( \frac{N^{2\theta + p}}{\log N} \right) \). Since one can choose an arbitrarily small \( \epsilon > 0 \), the above result actually indicates the rate-optimal convergence for all \( 0 < \theta \leq 1/2 \). We see from Corollary 2 in Section 3 that, under Assumption 6, one can obtain the same convergence result with a slightly better estimate on \( m \) which only requires \( m \leq o \left( \frac{N^{2\theta + p}}{\log N} \right) \).

When we finished this paper, we found that the work [42] also studies the divide and conquer functional linear regression but under a regularity condition different from (3.1) which actually requires \( \beta_0 \in \mathcal{H}_K \), and a boundedness assumption equivalent to that (4.1) is satisfied with \( t = 1 \). And we also note that to achieve optimal convergence rate under condition \( \beta_0 \in \mathcal{H}_K \), Theorem 2.1 in [42] requires the number of partitions \( m = 1 \), while with an additional assumption that the fourth-moment condition (3.3) is satisfied, Theorem 6 in this paper allows the number of partitions \( m \leq o \left( \frac{N^{2\theta + p}}{\log N} \right) \).

There is an intense recent research on the performance of different estimators in the noiseless linear model. The authors in [25, 40] study kernel regularized least-squares and find that the rate of convergence improves on noiseless data compared to noisy data. The recent work [4] consider applying stochastic gradient descent to solve the noiseless linear model in a general Hilbert space but they only focus on the attainable case where the optimal predictor is in this space. As far as we know, the convergence of estimator in RKHS as well as its divide-and-conquer counterpart has not been considered in the context of noiseless functional linear model. We establish the first convergence result in this setting when the optimal predictor is outside of the underlying RKHS. The framework and estimations developed in this paper can be extended to study more complex models of nonparametric supervised learning, such as [17, 31, 33, 41], which we will leave as future work.

## 5 Convergence Analysis

In this section, we first derive the finite sample upper bounds presented in the main results of Section 2 and Theorem 6 of Section 4. Then we prove the mini-max lower bound in Theorem 4.

### 5.1 Upper Bounds

Recalling the decomposition (2.8), one can bound \( \mathcal{R}(\bar{\beta}_{S,\lambda}) - \mathcal{R}(\beta_0) \) through estimating \( \mathcal{S}(S, \lambda) = \left\| L_C^{1/2} L_K^{1/2} f_{S,\lambda} - L_C^{1/2} L_K^{1/2} f_\lambda \right\|_{L^2}^2 \) and \( \mathcal{A}(\lambda) = \left\| L_C^{1/2} L_K^{1/2} f_\lambda - L_C^{1/2} \beta_0 \right\|_{L^2}^2 \), respectively. We apply the following lemma to estimate \( \mathcal{A}(\lambda) \).
Lemma 2. Suppose Assumption \( \mathcal{L} \) is satisfied with \( 0 < \theta \leq 1/2 \) and \( \gamma_0 \in L^2(T) \), then for any \( \lambda > 0 \), there holds
\[
\mathcal{A}(\lambda) \leq \lambda^{2\theta} \|\gamma_0\|_{L^2}^2. \tag{5.1}
\]

Proof. Write \( \gamma_0 = \sum_{k \geq 1} a_k \varphi_k \), according to singular value decomposition of \( T_* \) in (2.9), we have \( L_{C}^{1/2} \beta_0 = T_*^\theta (\gamma_0) = \sum_{k \geq 1} \mu_k^\theta a_k \varphi_k \) and
\[
L_{C}^{1/2} L_{K}^{1/2} f_\lambda = L_{C}^{1/2} L_{K}^{1/2} (\lambda I + T)^{-1} L_{K}^{1/2} L_{C} \beta_0 = \sum_{k = 1}^{\infty} \frac{\mu_k^{1+\theta}}{\lambda + \mu_k} a_k \varphi_j.
\]
Therefore,
\[
\mathcal{A}(\lambda) = \left\| L_{C}^{1/2} (L_{K}^{1/2} f_\lambda - \beta_0) \right\|_{L^2}^2 = \sum_{k = 1}^{\infty} \left( \frac{\mu_k^{1+\theta}}{\lambda + \mu_k - \mu_j^\theta} \right)^2 a_j^2 = \sum_{k = 1}^{\infty} \frac{\lambda^2 \mu_k^{2\theta}}{\lambda + \mu_k} a_k^2.
\]

While we see that for \( 0 < \theta \leq 1/2 \),
\[
\frac{t^\theta}{\lambda + t} \leq \theta^\theta (1 - \theta)^{1-\theta} \lambda^{\theta - 1} \leq \lambda^{\theta - 1}, \quad \forall t > 0,
\]
which implies that
\[
\mathcal{A}(\lambda) \leq \sum_{k = 1}^{\infty} \frac{\lambda^2 \mu_k^{2\theta}}{\lambda + \mu_k} a_k^2 \leq \lambda^{2\theta} \sum_{k = 1}^{\infty} a_k^2 = \lambda^{2\theta} \|\gamma_0\|_{L^2}^2.
\]
The proof is then finished.

In the rest part of this subsection, we focus on estimating \( \mathcal{A}(S, \lambda) \). Recall that \( S = \bigcup_{j=1}^{m} S_j \) with \( S_j \cap S_k = \emptyset \) for \( j \neq k \), the empirical operator \( T_{X_j} \) is defined with \( X_j = \{X_i : (X_i, Y_i) \in S_j\} \) according to (2.4). For any \( j = 1, 2, \ldots, m \), define the event
\[
\mathcal{U}_j = \left\{ X_j : \left\| (\lambda I + T)^{-1/2} (T_{X_j} - T)(\lambda I + T)^{-1/2} \right\| \geq 1/2 \right\},
\]
and denote its complement by \( \mathcal{U}_j^c \). Let \( \mathcal{U} = \bigcup_{j=1}^{m} \mathcal{U}_j \) be the union of above events. Then the complement of \( \mathcal{U} \) is given by \( \mathcal{U}^c = \cap_{j=1}^{m} \mathcal{U}_j^c \). Hereafter, let \( \mathbb{I}_{\mathcal{E}} \) denote the indicator function of the event \( \mathcal{E} \) and \( \mathbb{P}(\mathcal{E}) = \mathbb{E}[\mathbb{I}_{\mathcal{E}}] \). We first give the following estimation
\[
\left\| (\lambda I + T)^{-1/2} (\lambda I + T_{X_j})^{-1}(\lambda I + T)^{1/2} \right\| \mathbb{I}_{\mathcal{U}_j^c} \\
\leq 1 + \sum_{k = 1}^{\infty} \left\| (\lambda I + T)^{-1/2} (T - T_{X_j})(\lambda I + T)^{-1/2} \right\|^k \mathbb{I}_{\mathcal{U}_j^c} \\
\leq 1 + \sum_{k = 1}^{\infty} \frac{1}{2^k} = 2, \tag{5.2}
\]
where inequality (*) follows by expanding the inverse in Neumann series.

The following lemma plays a crucial role in bounding \( \mathcal{S}(S, \lambda) \).

**Lemma 3.** For any \( m \geq 1 \), there holds

\[
\mathbb{E} \left[ \mathcal{S}(S, \lambda) \right] \leq \frac{1}{m} \mathbb{E} \left[ \left\| L_{C}^{1/2} L_{K}^{1/2} \hat{f}_{S, \lambda} - L_{C}^{1/2} f_{\lambda} \right\|_{L_{2}}^{2} \right] + \left\| L_{C}^{1/2} L_{K}^{1/2} \mathbb{E} \left[ (\hat{f}_{S, \lambda} - f_{\lambda}) \right] \right\|_{L_{2}}^{2}.
\]

**Proof.** When \( m \geq 2 \), as

\[
\mathcal{S}(S, \lambda) = \left\| L_{C}^{1/2} L_{K}^{1/2} \hat{f}_{S, \lambda} - L_{C}^{1/2} L_{K}^{1/2} f_{\lambda} \right\|_{L_{2}}^{2} = \frac{1}{m} \sum_{i=1}^{m} \left( L_{C}^{1/2} L_{K}^{1/2} \hat{f}_{S, \lambda} - L_{C}^{1/2} L_{K}^{1/2} f_{\lambda} \right),
\]

then we have

\[
\mathbb{E} \left[ \mathcal{S}(S, \lambda) \right] \leq \frac{1}{m} \mathbb{E} \left[ \left\| L_{C}^{1/2} L_{K}^{1/2} \hat{f}_{S, \lambda} - L_{C}^{1/2} f_{\lambda} \right\|_{L_{2}}^{2} \right] \mathbb{E} \left[ \left\| (\hat{f}_{S, \lambda} - f_{\lambda}) \right\|_{L_{2}}^{2} \right].
\]

Here equality (i) follows from the binomial expansion. Equality (ii) uses the fact that \( \mathbb{I}_{U_{i}} = \mathbb{I}_{U_{i}} \mathbb{I}_{U_{j}} \mathbb{I}_{U_{j}} \mathbb{I}_{U_{n}} \) and for \( 1 \leq i \neq j \leq m \), \( (\hat{f}_{S, \lambda} - f_{\lambda}) \mathbb{I}_{U_{i}} \) is independent of \( (\hat{f}_{S, \lambda} - f_{\lambda}) \mathbb{I}_{U_{j}} \). Inequality (iii) is from

\[
\mathbb{E} \left[ \left\| L_{C}^{1/2} L_{K}^{1/2} \hat{f}_{S, \lambda} - L_{C}^{1/2} L_{K}^{1/2} f_{\lambda} \right\|_{L_{2}}^{2} \right].
\]

This completes the proof. \( \square \)

For simplicity of notation, in the rest of this paper, we always denote

\[
n := |S_{1}| = \frac{N}{m} \quad \text{and} \quad \{(X_{i,1}, Y_{i,1})\}_{i=1}^{n} := S_{1}.
\]

We establish the following bounds on the right hand side of (5.3) in Lemma 3.
Lemma 4. Suppose that Assumption 2 is satisfied with $\sigma > 0$ and Assumption 3 is satisfied with $c_1 > 0$. Then there hold

$$
\mathbb{E} \left[ \left\| L_C^{1/2}L_K^{1/2}(\hat{f}_{S,1,\lambda} - f_\lambda) \right\|^2_{L^2} \right] \leq 8 \frac{m}{N} \mathcal{N}(\lambda) \left( c_1 \lambda^{2\theta} \|\gamma_0\|^2_{L^2} + \sigma^2 \right)
$$

and

$$
\left\| L_C^{1/2}L_K^{1/2} \mathbb{E} \left[ (\hat{f}_{S,1,\lambda} - f_\lambda) \right] \right\|^2_{L^2} \leq 4c_1 \frac{m}{N} \mathcal{N}(\lambda) \lambda^{2\theta} \|\gamma_0\|^2_{L^2},
$$

where $\mathcal{N}(\lambda)$ is the effective dimension given by (3.4).

Proof. We first prove the second inequality (5.6). Recalling (5.4), we can write

$$
\hat{f}_{S,1,\lambda} = (\lambda I + T_{X_1})^{-1} \frac{1}{n} \sum_{i=1}^n L_K^{1/2} X_{1,i} Y_{1,i}.
$$

Then

$$
\left\| L_C^{1/2}L_K^{1/2} \mathbb{E} \left[ (\hat{f}_{S,1,\lambda} - f_\lambda) \right] \right\|^2_{L^2} 
= \left\| L_C^{1/2}L_K^{1/2} \mathbb{E} \left[ (\lambda I + T_{X_1})^{-1} \frac{1}{n} \sum_{i=1}^n L_K^{1/2} X_{1,i} Y_{1,i} - f_\lambda \right] \right\|^2_{L^2} 
\overset{(i)}{=} \left\| L_C^{1/2}L_K^{1/2} \mathbb{E} \left[ (\lambda I + T_{X_1})^{-1} \left( \frac{1}{n} \sum_{i=1}^n L_K^{1/2} X_{1,i} (X_{1,i}, \beta_0 - L_K^{1/2} f_\lambda)_2 - \lambda f_\lambda \right) \right] \right\|^2_{L^2} 
\overset{(ii)}{\leq} \mathbb{E} \left[ \left\| L_C^{1/2}L_K^{1/2} (\lambda I + T_{X_1})^{-1} \left( \frac{1}{n} \sum_{i=1}^n L_K^{1/2} X_{1,i} (X_{1,i}, \beta_0 - L_K^{1/2} f_\lambda)_2 - \lambda f_\lambda \right) \right\|^2_{L^2} \right] 
\overset{(iii)}{\leq} 4 \mathbb{E} \left[ \left( \lambda I + T^{-1/2} \left( \frac{1}{n} \sum_{i=1}^n L_K^{1/2} X_{1,i} (X_{1,i}, \beta_0 - L_K^{1/2} f_\lambda)_2 - \lambda f_\lambda \right) \right)^2 \right].
$$

Here equality (i) is from Assumption 2, inequality (ii) follows from the Jensen’s inequality, and inequality (iii) is due to (5.2) and the fact that

$$
\left\| L_C^{1/2}L_K^{1/2} (\lambda I + T)^{-1/2} \right\|^2 = \left\| (\lambda I + T)^{-1/2} L_K^{1/2} L_C^{1/2} (\lambda I + T)^{-1/2} \right\| = \left\| (\lambda I + T)^{-1/2} T (\lambda I + T)^{-1/2} \right\| \leq 1.
$$

Note that for $1 \leq i \leq n$, $L_K^{1/2} X_{1,i} (X_{1,i}, \beta_0 - L_K^{1/2} f_\lambda)_2 - \lambda f_\lambda$ is a zero-mean random element. Then we have

$$
\mathbb{E} \left[ \left( \lambda I + T^{-1/2} \left( \frac{1}{n} \sum_{i=1}^n L_K^{1/2} X_{1,i} (X_{1,i}, \beta_0 - L_K^{1/2} f_\lambda)_2 - \lambda f_\lambda \right) \right)^2 \right].
$$
estimations above, we have completed the proof of (5.6).

\[ \sum \infty \text{ follows from } A \text{ from Cauchy-Schwartz inequality. Inequality (ii) applies Assumption 3. Equality (iii) follows from } \lambda \text{ is given by the singular value decomposition of } T \text{ in } (2.9). \text{Inequality (i) follows from Cauchy-Schwartz inequality. Inequality (ii) applies Assumption 3. Equality (iii) follows from } \frac{1}{\lambda + \mu_j} \text{ and the calculation that } \sum_{j=1}^{\infty} \frac{1}{\lambda + \mu_j} = N(\lambda). \text{Inequality (iv) is due to Lemma } 2 \text{ and } n = N/m. \text{Combining the two estimations above, we have completed the proof of (5.6).}

Next we prove the first inequality (5.5). According to the expression of \( \hat{f}_{S_1, \lambda} \) and the triangular inequality, we have

\[
\mathbb{E} \left[ \left\| L_C^{1/2} L_K^{1/2} (\hat{f}_{S_1, \lambda} - f) \right\|_{L^2}^2 \right] 
\leq 2 \mathbb{E} \left[ \left\| L_C^{1/2} L_K^{1/2} (\lambda + T_{X_1})^{-1} \left( \frac{1}{n} \sum_{i=1}^{n} L_K^{1/2} X_{1,i} (X_{1,i}, \beta_0 - L_K^{1/2} f) \right) \right\|_{L^2}^2 \right] 
\leq 2 \mathbb{E} \left[ \left\| L_C^{1/2} L_K^{1/2} (\lambda + T_{X_1})^{-1} \left( \frac{1}{n} \sum_{i=1}^{n} L_K^{1/2} X_{1,i} \epsilon_{1,i} + \epsilon_{1,i} \right) \right\|_{L^2}^2 \right],
\]

where \( \epsilon_{1,i} := Y_{1,i} - (\beta_0, X_{1,i}) \). We have bounded the term (5.5a) in the proof of (5.6), which is given by

\[
\mathbb{E} \left[ \left\| L_C^{1/2} L_K^{1/2} (\lambda + T_{X_1})^{-1} \left( \frac{1}{n} \sum_{i=1}^{n} L_K^{1/2} X_{1,i} \epsilon_{1,i} \right) \right\|_{L^2}^2 \right] 
\leq 4 \frac{m}{N} N(\lambda) \lambda^{2\theta} \| \gamma_0 \|_{L^2}.
\]

Note that \( L^{1/2} X_{1,i} \epsilon_{1,i} \) is also a zero-mean random element. Analogously, one can bound (5.5b).
Proof. Recall (5.4). We first bound

\[ \text{Lemma 5.} \]

Then we obtain inequality (5.5) and the proof is finished. \( \square \)

We also need the following lemma to estimate the probability of event \( U_1 \). Recall that \( U_1 \) is defined as

\[ U_1 = \left\{ X_1 : \left\| (\lambda + T)^{-1/2}(TX_1 - T)(\lambda + T)^{-1/2} \right\| \geq 1/2 \right\}. \]

Lemma 5. Suppose Assumption 3 is satisfied with \( c_1 > 0 \), then

\[ \mathbb{P}(U_1) \leq 4c_1 \frac{m}{N} \mathcal{N}(\lambda), \]

where \( \mathcal{N}(\lambda) \) is the effective dimension given by (3.4).

Proof. Recall (5.24). We first bound \( \mathbb{E} \left[ \left\| (\lambda + T)^{-1/2}(TX_1 - T)(\lambda + T)^{-1/2} \right\| ^2 \right] \) as

\[
\begin{align*}
\mathbb{E} \left[ \left\| (\lambda + T)^{-1/2}(TX_1 - T)(\lambda + T)^{-1/2} \right\|^2 \right] \\
\quad \leq \mathbb{E} \left[ \left\| (\lambda + T)^{-1/2}(TX_1 - T)(\lambda + T)^{-1/2} \right\|^2 _{HS} \right] \\
\quad = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \mathbb{E} \left[ \left\| (\lambda + T)^{-1/2} \left( \frac{1}{n} \sum_{i=1}^{n} L_{K,i}^{1/2} X_{1,i} \otimes L_{K,i}^{1/2} X_{1,i} - T \right)(\lambda + T)^{-1/2} \phi_j, \phi_k \right\| _{L^2}^2 \right] \\
\quad \leq \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{\lambda + \mu_j} \frac{1}{\lambda + \mu_k} \mathbb{E} \left[ \left\| L_{K,i}^{1/2} X_{1,i}, \phi_j \right\| _{L^2}^2 \right] \mathbb{E} \left[ \left\| L_{K,i}^{1/2} X_{1,i}, \phi_k \right\| _{L^2}^2 \right] \\
\quad \leq \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{\lambda + \mu_j} \frac{1}{\lambda + \mu_k} \mathbb{E} \left[ \left\| L_{K,i}^{1/2} X_{1,i}, \phi_j \right\| _{L^2}^4 \right] \mathbb{E} \left[ \left\| L_{K,i}^{1/2} X_{1,i}, \phi_k \right\| _{L^2}^4 \right] \\
\quad \leq \frac{c_1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{\lambda + \mu_j} \frac{1}{\lambda + \mu_k} \mathbb{E} \left[ \left\| L_{K,i}^{1/2} X_{1,i}, \phi_j \right\| _{L^2}^2 \right] \mathbb{E} \left[ \left\| L_{K,i}^{1/2} X_{1,i}, \phi_k \right\| _{L^2}^2 \right]
\end{align*}
\]
\[ \frac{c_1}{n} \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{\lambda + \mu_j} \frac{1}{\lambda + \mu_k} \langle T \phi_j, \phi_j \rangle_{L^2} \langle T \phi_k, \phi_k \rangle_{L^2} \]
\[ = \frac{c_1}{n} \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \frac{\mu_j}{\lambda + \mu_j} \frac{\mu_k}{\lambda + \mu_k} = \frac{c_1 m}{N^2} N^2(\lambda). \]

Here \( \{\phi_j\}_{j=1}^{\infty} \) is given by the singular value decomposition of \( T \) in (2.9). Inequality (i) follows from (2.2). Inequality (ii) is from the fact that \( L^{1/2}_K X_i \otimes L^{1/2}_K X_i - T \) is a zero-mean random operator. Inequality (iii) uses Cauchy-Schwartz inequality. Inequality (iv) applies Assumption 3.

Combining the above estimation with Chebyshev inequality, we obtain
\[ P(U_1) = P\left( \left\{ X_1 : \left\| (\lambda I + T)^{-1/2} (TX_1 - T)(\lambda I + T)^{-1/2} \right\| \geq 1/2 \right\} \right) \]
\[ \leq 4E \left[ \left\| (\lambda I + T)^{-1/2} (TX_1 - T)(\lambda I + T)^{-1/2} \right\|^2 \right] \]
\[ \leq 4c_1 \frac{m}{N} N^2(\lambda). \]

Then we obtain the desired result and complete the proof.

The following lemma provides an estimation of \( N(\lambda) \) under the polynomial decaying condition of the eigenvalues.

**Lemma 6.** Suppose that \( \{\mu_k\}_{k=1}^{\infty} \) satisfy \( \mu_k \lesssim k^{-1/p} \) for some \( 0 < p \leq 1 \), then there holds
\[ N(\lambda) \lesssim \lambda^{-p}, \quad \forall 0 < \lambda \leq 1. \] (5.9)

The estimation in Lemma 6 can be found in [19, 30, 20].

We have established preliminary estimates for Theorem 2 and 5. We are in the position to prove these two theorems. To this end, we also need to introduce the notations \( o_P(1) \) and \( O_P(\cdot) \). For a sequence of random variables \( \{\xi_k\}_{k=1}^{\infty} \), we write \( \xi_k \leq o_P(1) \) if
\[ \lim_{k \to \infty} P(|\xi_k| \geq d) = 0, \forall d > 0. \]
And we write \( \xi_k \leq O_P(1) \) if
\[ \lim_{D \to \infty} \sup_{k \geq 1} P(|\xi_k| \geq D) = 0. \]

In addition, suppose there is a positive sequence \( \{a_k\}_{k=1}^{\infty} \). Then we write \( \xi_k \leq o_P(a_k) \) if \( \xi_k / a_k \leq o_P(1) \), and \( \xi_k \leq O_P(a_k) \) if \( \xi_k / a_k \leq O_P(1) \).

**Proof of Theorem 2** Combining the decomposition (2.8) and (5.1) in Lemma 2 yields
\[ \mathcal{R}(S) - \mathcal{R}(\beta_0) \leq 2\mathcal{S}(S, \lambda) + 2\mathcal{A}(\lambda) \]
\[ \leq 2\mathcal{S}(S, \lambda) + 2\lambda^{2p} \|\lambda_0\|^2_{L^2}. \] (5.10)

We first decompose \( \mathcal{S}(S, \lambda) \) as
\[ \mathcal{S}(S, \lambda) = \mathcal{S}(S, \lambda)\mathbb{I} + \mathcal{S}(S, \lambda)\mathbb{I}_{\mathcal{C}}. \]
For the term $\mathcal{I}(S, \lambda)\mathbb{I}_U$, following from (5.8) in Lemma 5, we have

$$E[\mathbb{I}_U] = \mathbb{P}(U) \leq \sum_{j=1}^{m} \mathbb{P}(U_j) = m\mathbb{P}(U_1) \leq 4c_1 \frac{m^2}{N} N^2(\lambda).$$

Then using Markov’s inequality, we can write

$$\mathcal{I}(S, \lambda)\mathbb{I}_U \leq O_P \left( \frac{m^2}{N} N^2(\lambda) \right).$$

For the term $\mathcal{I}(S, \lambda)\mathbb{I}_C$, combining (5.3) in Lemma 3 with (5.5) and (5.6) in Lemma 4 yields

$$E [\mathcal{I}(S, \lambda)\mathbb{I}_C] \leq 8 \frac{N(\lambda)}{N} (c_1 \lambda^{2\theta} \|\gamma_0\|_{L^2}^2 + \sigma^2) + 4c_1 \frac{m}{N} N(\lambda) \lambda^{2\theta} \|\gamma_0\|_{L^2}^2.$$

Then using Markov’s inequality, we can write

$$\mathcal{I}(S, \lambda)\mathbb{I}_C \leq O_P \left( \frac{N(\lambda)}{N} + \frac{m}{N} N(\lambda) \lambda^{2\theta} \right).$$

Therefore, we have

$$\left[ 1 - O_P \left( \frac{m^2}{N} N^2(\lambda) \right) \right] \mathcal{I}(S, \lambda) \leq O_P \left( \frac{N(\lambda)}{N} + \frac{m}{N} N(\lambda) \lambda^{2\theta} \right).$$

Then following from the estimation of $N(\lambda)$ (5.9) in Lemma 6, taking $\lambda \leq 1$, we can write

$$\left[ 1 - O_P \left( \frac{m^2}{N} \lambda^{-2p} \right) \right] \mathcal{I}(S, \lambda) \leq O_P \left( \frac{\lambda^{-p}}{N} + \frac{m}{N} \lambda^{2\theta - p} \right). \quad (5.11)$$

Take $m$ and $\lambda$ satisfying $m^2 \lambda^{-2p} \leq o(N)$ and $\lambda \leq 1$, then (5.11) implies that

$$[1 - o_p(1)] \mathcal{I}(S, \lambda) \leq O_P \left( \frac{\lambda^{-p}}{N} + \frac{m}{N} \lambda^{2\theta - p} \right), \text{ as } O_P \left( \frac{m^2}{N} \lambda^{-2p} \right) \leq o_p(1).$$

Thus, we can write

$$\mathcal{I}(S, \lambda) \leq O_P \left( \frac{\lambda^{-p}}{N} + \frac{m}{N} \lambda^{2\theta - p} \right).$$

Combining the above estimation with (5.10) yields

$$\mathcal{R}(\beta_{S,\lambda}) - \mathcal{R}(\beta_0) \leq O_P \left( \lambda^{2\theta} + \frac{\lambda^{-p}}{N} + \frac{m}{N} \lambda^{2\theta - p} \right) \quad (5.12)$$

provided that

$$m^2 \lambda^{-2p} \leq o(N) \text{ and } \lambda \leq 1.$$

When $p/2 < \theta \leq 1/2$, take $m \leq o \left( N^{2\theta - p/2} \right)$ and $\lambda = N^{-\frac{1}{2\theta + p} / N}$, then there hold $m^2 \lambda^{-2p} \leq o(N)$ and $\lambda \leq 1$. Therefore, following from (5.12), we have

$$\mathcal{R}(\beta_{S,\lambda}) - \mathcal{R}(\beta_0) \leq O_P \left( N^{-2\theta} \right),$$

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or equivalently,

$$\lim_{\Gamma \to \infty} \sup_{N \to \infty} \sup_{\beta_0} \mathbb{P} \left\{ R(\beta_{S,\lambda}) - R(\beta_0) \geq \Gamma N^{-\frac{2\theta}{2p}} \right\} = 0.$$  

This completes the proof of (3.5).

When $0 < \theta \leq p/2$, take $m$ and $\lambda$ satisfying $m^2 \lambda^{-2p} \leq o(N)$ and $\lambda \leq 1$, then following from (5.12), one can calculate

$$R(\beta_{S,\lambda}) - R(\beta_0) \leq O_P \left( \lambda^{2\theta} \right),$$

or equivalently,

$$\lim_{\Gamma \to \infty} \sup_{N \to \infty} \sup_{\beta_0} \mathbb{P} \left\{ R(\beta_{S,\lambda}) - R(\beta_0) \geq \Gamma \lambda^{2\theta} \right\} = 0,$$

which further implies (3.6) and (3.7). The proof of Theorem 2 is then completed.

Now we turn to prove Theorem 5.

Proof of Theorem 5. Recalling (5.4), under the noiseless condition, we can write

$$\hat{f}_{S,\lambda} = (\lambda I + T_{X_1})^{-1} \frac{1}{n} \sum_{i=1}^{n} L_{K}^{1/2} X_{1,i} \langle X_{1,i}, \beta_0 \rangle L_2.$$

Then an improved estimation of the left hand side of (5.5) is given by

$$\mathbb{E} \left[ \left\| L_{P}^{1/2} L_{K}^{1/2} (\hat{f}_{S,\lambda} - f_{\lambda}) \right\|_{L^2}^{2} \bar{U}_2 \right]$$

$$= \mathbb{E} \left[ \left\| L_{P}^{1/2} L_{K}^{1/2} (\lambda I + T_{X_1})^{-1} \left( \frac{1}{n} \sum_{i=1}^{n} L_{K}^{1/2} X_{1,i} \langle X_{1,i}, \beta_0 \rangle L_2 - \lambda f_{\lambda} \right) \right\|_{L^2}^{2} \bar{U}_2 \right] (5.13)$$

$$(*) \leq 4c_1 \frac{m}{N} \mathcal{N}(\lambda) \lambda^{2\theta} \| \gamma_0 \|_{L^2}^2,$$

where inequality (*) follows from (5.7).

Utilizing (5.13) and following the same arguments in the proof of Theorem 2 we have

$$R(\beta_{S,\lambda}) - R(\beta_0) \leq O_P \left( \lambda^{2\theta} + \frac{m}{N} \lambda^{2\theta - p} \right), (5.14)$$

provided that

$$m^2 \lambda^{-2p} \leq o(N) \text{ and } \lambda \leq 1.$$

For any $0 < \eta \leq 1/2$, take $m \leq o(N^{\eta})$ and $\lambda = N^{-\frac{1-2\eta}{2p}}$, then there hold $m^2 \lambda^{-2p} \leq o(N)$ and $\lambda \leq 1$. Therefore, following from (5.14), one can calculate

$$R(\beta_{S,\lambda}) - R(\beta_0) \leq O_P \left( \lambda^{2\theta} \right) \leq O_P \left( N^{-\frac{\eta(1-2\eta)}{p}} \right),$$

or equivalently,

$$\lim_{\Gamma \to \infty} \sup_{N \to \infty} \sup_{\beta_0} \mathbb{P} \left\{ R(\beta_{S,\lambda}) - R(\beta_0) \geq \Gamma N^{-\frac{\eta(1-2\eta)}{p}} \right\} = 0.$$
We obtain (3.19). The proof of Theorem 5 is then finished.

We next aim to prove Theorem 3 and Corollary 1. We also need several lemmas before proving them.

When Assumption 3 is enhanced to Assumption 4, we can estimate the probability of event $\mathcal{U}_1$ better than Lemma 5.

**Lemma 7.** Suppose that Assumption 4 is satisfied with some integer $\ell \geq 2$. Then there holds

$$\mathbb{P}(\mathcal{U}_1) \leq c(\ell)2^{2\ell} \rho^{4\ell} \left( \frac{mN^2(\lambda)}{N} \right)^\ell,$$

where $c(\ell)$ is a constant only depends on $\ell$ and $N(\lambda)$ is given by (3.4).

Lemma 7 can be proved by employing the Markov inequality combined with the following lemma.

**Lemma 8.** Suppose that Assumption 4 is satisfied with some integer $\ell \geq 2$. Then

$$\mathbb{E} \left[ \left\| (\lambda I + T)^{-1/2}(TX_1 - T)(\lambda I + T)^{-1/2} \right\|^2_{HS} \right] \leq c(\ell)2^{2\ell} \rho^{4\ell} \left( \frac{mN^2(\lambda)}{N} \right)^\ell$$

and

$$\mathbb{E} \left[ \left\| (\lambda I + T)^{-1/2}(TX_1 - T) \right\|^2_{HS} \right] \leq c(\ell)2^{2\ell} \rho^{4\ell} \text{trace}(T) \left( \frac{mN^2(\lambda)}{N} \right)^\ell,$$

where $\text{trace}(T) = \sum_{j=1}^{\infty} \mu_j$ denotes the trace of operator $T$, $N(\lambda)$ is defined by (3.4), and $c(\ell)$ is a constant only depends on $\ell$.

**Proof.** We first prove inequality (5.16). Recalling (5.4), for brevity of notations, we define

$$Q_i := (\lambda I + T)^{1/2} \left( L^1_K (X_{i_1} \otimes X_{i_1}) - T \right) (\lambda I + T)^{-1/2}, \quad i = 1, 2, \ldots, n.$$  

Then we can write

$$\mathbb{E} \left[ \left\| (\lambda I + T)^{-1/2}(TX_1 - T)(\lambda I + T)^{-1/2} \right\|^2_{HS} \right] = \mathbb{E} \left[ \left\| \frac{1}{n} \sum_{i=1}^{n} Q_i \right\|^2_{HS} \right] = \frac{1}{n^{2\ell}} \sum_{i_1=1}^{n} \cdots \sum_{i_{\ell}=1}^{n} \sum_{j_1=1}^{n} \cdots \sum_{j_{\ell}=1}^{n} \mathbb{E} \left[ (Q_{i_1}, Q_{j_1})_{HS} \cdots (Q_{i_{\ell}}, Q_{j_{\ell}})_{HS} \right].$$

When the indexes in group $\{i_1, \ldots, i_{\ell}, j_1, \ldots, j_{\ell}\}$ are all distinct, then following from the independence, there holds $\mathbb{E} \left[ (Q_{i_1}, Q_{j_1})_{HS} \cdots (Q_{i_{\ell}}, Q_{j_{\ell}})_{HS} \right] = 0$. We denote the set of all index-distinct groups by $\Omega(n, \ell)$. Let $\Theta(n, \ell) = \{1, \ldots, n\}^{2\ell} \setminus \Omega(n, \ell)$. Using these notations, we can write

$$\mathbb{E} \left[ \left\| (\lambda I + T)^{-1/2}(TX_1 - T)(\lambda I + T)^{-1/2} \right\|^2_{HS} \right] = \frac{1}{n^{2\ell}} \sum_{\{i_1, \ldots, i_{\ell}, j_1, \ldots, j_{\ell}\} \in \Theta(n, \ell)} \mathbb{E} \left[ (Q_{i_1}, Q_{j_1})_{HS} \cdots (Q_{i_{\ell}}, Q_{j_{\ell}})_{HS} \right].$$
We estimate the cardinality of $\Theta(n,k)$ as

$$|\Theta(n,\ell)| = |\Theta_i(n,\ell)| + \ldots + |\Theta_1(n,\ell)| \leq (2\ell)! \left( \begin{array}{c} n \\ \ell \end{array} \right) + \left( \begin{array}{c} n \\ \ell - 1 \end{array} \right) (\ell - 1)^2 + \ldots + \left( \begin{array}{c} n \\ 1 \end{array} \right) \leq (2\ell)! \ell^{2\ell+1} n^\ell := c(\ell)n^\ell,$$

(5.19)

where $c(\ell) := (2\ell)! \ell^{2\ell+1}$. Let $\Theta_i(n,\ell)$ denote a subset of $\Theta(n,\ell)$ consisting of all groups with exactly $i$ different indexes. Then $\Theta(n,\ell) = \cup_{i=1}^f \Theta_i(n,\ell)$ and $|\Theta_i(n,\ell)| \leq (2\ell)! \ell^{2\ell-i} \leq (2\ell)! \ell^{2\ell} n^\ell$.

For any $\{i_1, \ldots, i_\ell, j_1, \ldots, j_\ell\} \in \Theta(n,\ell)$, we have

$$E \left[ \sum_{j=1}^\ell \sum_{k=1}^\ell \frac{1}{\lambda + \mu_j} \langle (X_{l,j} \otimes X_{l,i} - T)\phi_j, \phi_k \rangle_{L^2}^2 \right]$$

\begin{equation}
= \sum_{j_1=1}^\ell \ldots \sum_{j_\ell=1}^\ell \sum_{k_1=1}^\ell \ldots \sum_{k_\ell=1}^\ell \left[ \frac{1}{\lambda + \mu_{j_1}} \frac{1}{\lambda + \mu_{k_1}} \langle (L_{l,j_1}^1 X_{l,i} \otimes L_{l,k_1}^1 X_{l,i} - T)\phi_{j_1}, \phi_{k_1} \rangle_{L^2}^2 \right] \leq \sum_{j_1=1}^\ell \sum_{j_\ell=1}^\ell \sum_{k_1=1}^\ell \ldots \sum_{k_\ell=1}^\ell \left[ \frac{1}{\lambda + \mu_{j_1}} \frac{1}{\lambda + \mu_{k_1}} \langle (L_{l,j_1}^1 X_{l,i} \otimes L_{l,k_1}^1 X_{l,i} - T)\phi_{j_1}, \phi_{k_1} \rangle_{L^2}^2 \right]^{\frac{1}{2}} \times \ldots \times \sum_{j_\ell=1}^\ell \sum_{j_1=1}^\ell \sum_{k_\ell=1}^\ell \ldots \sum_{k_1=1}^\ell \left[ \frac{1}{\lambda + \mu_{j_\ell}} \frac{1}{\lambda + \mu_{k_\ell}} \langle (L_{l,j_\ell}^1 X_{l,i} \otimes L_{l,k_\ell}^1 X_{l,i} - T)\phi_{j_\ell}, \phi_{k_\ell} \rangle_{L^2}^2 \right]^{\frac{1}{2}}.
\end{equation}

(5.21)

Here the last inequality also follows from the Hölder inequality. It remains to estimate

$$E \left[ \langle (L_{l,j_1}^1 X_{l,i} \otimes L_{l,k_1}^1 X_{l,i} - T)\phi_{j_1}, \phi_{k_1} \rangle_{L^2}^2 \right], \quad \forall 1 \leq i \leq n \text{ and } \forall 1 \leq j, k < \infty.$$

When $j \neq k$, we have

$$E \left[ \langle (L_{l,j_1}^1 X_{l,i} \otimes L_{l,k_1}^1 X_{l,i} - T)\phi_{j}, \phi_{k} \rangle_{L^2}^{2\ell} \right]$$

$$= E \left[ \langle L_{l,j_1}^1 X_{l,i} \phi_{j} \rangle_{L^2}^{2\ell} \langle L_{l,k_1}^1 X_{l,i} \phi_{k} \rangle_{L^2}^{2\ell} \right].$$
where inequality (i) is from Cauchy-Schwarz inequality and inequality (ii) utilizes Assumption 4.

When \( j = k \), we have

\[
\mathbb{E} \left[ \left( \langle L^{1/2} X_{1,i} \otimes L^{1/2} X_{1,i} - T \rangle \phi_j, \phi_j \rangle_{L^2} \right)^{2\ell} \right] \\
= \mathbb{E} \left[ \left( \langle L^{1/2} X_{1,i} \otimes L^{1/2} X_{1,i} \phi_j, \phi_j \rangle_{L^2} - \mu_j \right)^{2\ell} \right] \\
= 2^{2\ell} \mathbb{E} \left[ \left( \frac{1}{2} \langle L^{1/2} X_{1,i} \otimes L^{1/2} X_{1,i} \phi_j, \phi_j \rangle_{L^2} - \frac{1}{2} \mu_j \right)^{2\ell} \right] \\
\leq 2^{2\ell - 1} \left( \mathbb{E} \left[ \langle L^{1/2} X_{1,i} \otimes L^{1/2} X_{1,i} \phi_j, \phi_j \rangle_{L^2} \right]^2 + \mu_j^{2\ell} \right) \\
= 2^{2\ell - 1} \left( \mathbb{E} \left[ \langle L^{1/2} X_{1,i} \phi_j \rangle_{L^2} \right]^2 + \mu_j^{2\ell} \right)^{(ii)} \leq 2^{2\ell} \rho^\ell \mu_j^{2\ell},
\]

where inequality (i) is due to Jensen’s inequality and inequality (ii) follows from Assumption 4 and the fact that \( \rho \geq 1 \).

Combining the above estimations, for any \( 1 \leq i \leq n \) and \( 1 \leq j, k < \infty \), there holds

\[
\mathbb{E} \left[ \left( \langle L^{1/2} X_{1,i} \otimes L^{1/2} X_{1,i} - T \rangle \phi_j, \phi_k \rangle \right)^{2\ell} \right] \leq 2^{2\ell} \rho^\ell \mu_j \mu_k.
\]  

(5.22)

Recall that \( n = N/m \). Combining (5.18), (5.19), (5.20), (5.21) and (5.22) yields

\[
\mathbb{E} \left[ \left\| (\lambda + T)^{-1/2} (T \alpha_j - T)(\lambda + T)^{-1/2} \right\|_{HS}^{2\ell} \right] \leq C(\ell) 2^{2\ell} \rho^\ell \left( \frac{mN^2(\lambda)}{N} \right)^\ell.
\]

This completes the proof of (5.16).

Analogously, we can demonstrate the second inequality (5.17) through

\[
\mathbb{E} \left[ \left\| (\lambda + T)^{-1/2} (L^{1/2} X_{1,i} \otimes L^{1/2} X_{1,i} - T) \right\|_{HS}^{2\ell} \right] \leq 2^{2\ell} \rho^\ell \text{trace}^\ell (T) N^\ell (\lambda)
\]

and

\[
\mathbb{E} \left[ \left\| (\lambda + T)^{-1/2} (T \alpha_j - T) \right\|_{HS}^{2\ell} \right] \leq C(\ell) 2^{2\ell} \rho^\ell \text{trace}^\ell (T) \left( \frac{mN(\lambda)}{N} \right)^\ell.
\]

The proof of Lemma 9 is then finished. □

The following lemma plays a key role in estimating the upper bound of \( \mathcal{S}(S, \lambda) \) under Assumption 4.

**Lemma 9.** Suppose that Assumption 4 is satisfied with \( 0 < \theta \leq 1/2 \) and \( \gamma_0 \in L^2(T) \). Under Assumption 4 and Assumption 2, taking \( \lambda \leq 1 \) yields

\[
\mathbb{E} \left[ \left\| (\lambda + T)^{-1/2} \frac{1}{|S|} \sum_{X \in S} L^{1/2} X (X, \beta_0 - L^{1/2} f_\lambda)_{L^2} - \lambda f_\lambda \right\|_{L^2}^4 \right] \leq C_0 \frac{m^2}{N^2} (1 + \lambda^{4\theta} N^2(\lambda)),
\]  

(5.23)
where $c_6$ is a universal constant and $\mathcal{N}(\lambda)$ is given by (3.4).

**Proof.** Recalling (5.4), for simplicity of notations, we define

$$\alpha_i := L_K^{1/2} X_{1,i} \langle X_{1,i}, \beta_0 - L_K^{1/2} f_\lambda \rangle_{\mathcal{L}^2} - \lambda f_\lambda, \quad i = 1, 2, \ldots, n.$$  

We begin with the proof of the first inequality (5.23). Note that $\{ (\lambda I + T)^{-1/2} \alpha_i \}_{i=1}^n$ are independent operator-valued zero-mean random variables. Then

$$E \left\| (\lambda I + T)^{-1/2} \frac{1}{n} \sum_{i=1}^{n} \alpha_i \right\|_{\mathcal{L}^2}^4$$

$$= \frac{1}{n^4} \sum_{i_1=1}^{\infty} \sum_{i_2=1}^{\infty} \sum_{j_1=1}^{\infty} \sum_{j_2=1}^{\infty} E \left[ \left\langle (\lambda I + T)^{-1/2} \alpha_{i_1}, (\lambda I + T)^{-1/2} \alpha_{i_2} \right\rangle_{\mathcal{L}^2} \times \left\langle (\lambda I + T)^{-1/2} \alpha_{j_1}, (\lambda I + T)^{-1/2} \alpha_{j_2} \right\rangle_{\mathcal{L}^2} \right] \times \left( (\lambda I + T)^{-1/2} \alpha_{j_1}, (\lambda I + T)^{-1/2} \alpha_{j_2} \right)_{\mathcal{L}^2}$$

(5.24)

Here $\Theta(n, 2) = \{1, \ldots, n\} \setminus \Omega(n, 2)$ where $\Omega(n, 2)$ denotes the set of all index-distinct group $\{i_1, i_2, j_1, j_2\}$. Then

$$|\Theta(n, 2)| \leq 4! \left( \binom{n}{2} + \binom{n}{1} \right) \leq 24n^2, \quad \forall n \geq 1.$$  

(5.25)

And for any $\{i_1, i_2, j_1, j_2\} \in \Theta(n, 2)$, we have

$$E \left[ \left\langle (\lambda I + T)^{-1/2} \alpha_{i_1}, (\lambda I + T)^{-1/2} \alpha_{i_2} \right\rangle_{\mathcal{L}^2} \langle (\lambda I + T)^{-1/2} \alpha_{j_1}, (\lambda I + T)^{-1/2} \alpha_{j_2} \rangle_{\mathcal{L}^2} \right]$$

$$\leq E \left[ \left\| (\lambda I + T)^{-1/2} \alpha_{i_1} \right\|_{\mathcal{L}^2}^4 \times \left\| (\lambda I + T)^{-1/2} \alpha_{i_2} \right\|_{\mathcal{L}^2}^4 \times \left\| (\lambda I + T)^{-1/2} \alpha_{j_1} \right\|_{\mathcal{L}^2}^4 \times \left\| (\lambda I + T)^{-1/2} \alpha_{j_2} \right\|_{\mathcal{L}^2}^4 \right]$$

(5.26)

Here the last inequality follows from Hölder inequality.

It remains to estimate $E \left[ \left\langle (\lambda I + T)^{-1/2} \alpha_i \right\rangle_{\mathcal{L}^2}^4 \right]^\frac{1}{4}, \forall 1 \leq i \leq n$. For brevity of notations, we define

$$\tilde{\alpha}_i := L_K^{1/2} X_{1,i} \langle X_{1,i}, \beta_0 - L_K^{1/2} f_\lambda \rangle_{\mathcal{L}^2}, \quad i = 1, 2, \ldots, n.$$  

Then we see that $\alpha_i = \tilde{\alpha}_i - \lambda f_\lambda$ and for any $1 \leq i \leq n$,

$$E \left[ \left\| (\lambda I + T_0)^{-1/2} \tilde{\alpha}_i \right\|_{\mathcal{L}^2}^4 \right]$$

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\[
\mathbb{E} \left[ \left( \sum_{j=1}^{\infty} \left( \langle \lambda I + T \rangle^{-1/2} \tilde{\alpha}_i, \phi_j \rangle^{2} \right)^{2} \right)^{2} \right] \\
= \mathbb{E} \left[ \sum_{j_1=1}^{\infty} \sum_{j_2=1}^{\infty} \frac{1}{\lambda + \mu_{j_1}} \frac{1}{\lambda + \mu_{j_2}} \langle \tilde{\alpha}_i, \phi_{j_1} \rangle^{2} \langle \tilde{\alpha}_i, \phi_{j_2} \rangle^{2} \right] \\
\leq \sum_{j_1=1}^{\infty} \sum_{j_2=1}^{\infty} \frac{1}{\lambda + \mu_{j_1}} \frac{1}{\lambda + \mu_{j_2}} \left[ \mathbb{E} \langle \tilde{\alpha}_i, \phi_{j_1} \rangle^{4} \right]^{1/2} \left[ \mathbb{E} \langle \tilde{\alpha}_i, \phi_{j_2} \rangle^{4} \right]^{1/2}. 
\]

(5.27)

We further bound \( \mathbb{E} \left[ \langle \tilde{\alpha}_i, \phi_j \rangle^{4} \right] \) as

\[
\mathbb{E} \langle \tilde{\alpha}_i, \phi_j \rangle^{4} \\
= \mathbb{E} \left[ \left( L_{K}^{1/2} X_{1,i}, \phi_j \right)^{4} \left( X_{1,i}, \beta_0 - L_{K}^{1/2} f_{\lambda} \right)^{4} \right] \\
\leq \left( \mathbb{E} \left[ \left( L_{K}^{1/2} X_{1,i}, \phi_j \right)^{8} \right] \right)^{1/2} \left( \mathbb{E} \left[ \left( X_{1,i}, \beta_0 - L_{K}^{1/2} f_{\lambda} \right)^{8} \right] \right)^{1/2} \\
\leq c_2 \rho^4 \mu_j^2 \mathbb{E} \left[ \left( X_{1,i}, \beta_0 - L_{K}^{1/2} f_{\lambda} \right)^{2} \right] = c_2 \rho^4 \mu_j^2 \mathbb{E} \left[ \langle \tilde{\alpha}_i, \phi_j \rangle^{4} \right]. 
\]

(5.28)

Here inequality (i) again follows from Cauchy-Schwartz inequality. Inequality (ii) is due to Assumption [3]. Inequality (iii) is from Lemma 2.

Combining (5.27) and (5.28) yields

\[
\mathbb{E} \left[ \| (\lambda I + T)^{-1/2} \tilde{\alpha}_i \|^{4}_{L^2} \right] \leq c_2 \rho^4 \| \gamma_0 \|^4_{L^2} \lambda^{4\theta} N^2(\lambda), \quad \forall 1 \leq i \leq n.
\]

Then for any \( 1 \leq i \leq n \), we have

\[
\mathbb{E} \left[ \| (\lambda I + T)^{-1/2} \tilde{\alpha}_i \|^{4}_{L^2} \right] = \mathbb{E} \left[ \| (\lambda I + T)^{-1/2} (\tilde{\alpha}_i - \lambda f_{\lambda}) \|^{4}_{L^2} \right] \\
\leq 8 \mathbb{E} \left[ \| (\lambda I + T)^{-1/2} \tilde{\alpha}_i \|^{4}_{L^2} \right] + 8 \| (\lambda I + T)^{-1/2} \lambda f_{\lambda} \|^{4}_{L^2} \\
= 8 \mathbb{E} \left[ \| (\lambda I + T)^{-1/2} \tilde{\alpha}_i \|^{4}_{L^2} \right] + 8 \| (\lambda I + T)^{-1/2} \lambda (\lambda I + T)^{-1} L_{K}^{1/2} L_{C}^{1/2} T_{\theta} \gamma_0 \|^{4}_{L^2} \\
\leq 8 c_2 \rho^4 \| \gamma_0 \|^4_{L^2} \lambda^{4\theta} N^2(\lambda) + 8 \| (\lambda I + T)^{-1/2} \lambda \|^{4}_{L^2} \| (\lambda I + T)^{-1} L_{K}^{1/2} T_{\theta} \gamma_0 \|^{4}_{L^2} \\
\leq 8 c_2 \rho^4 \| \gamma_0 \|^4_{L^2} \lambda^{4\theta} N^2(\lambda) + 8 \| T_{\theta} \|^{4}_{L^2} \| \gamma_0 \|^{4}_{L^2} = 8 c_2 \rho^4 \| \gamma_0 \|^4_{L^2} \lambda^{4\theta} N^2(\lambda) + 8 \mu_1^{4\theta} \| \gamma_0 \|^{4}_{L^2}.
\]

(5.29)

Recall that \( n = N/m \) and take \( \lambda \leq 1 \). Combining with (5.24), (5.25), (5.26) and (5.29), we obtain

\[
\mathbb{E} \left[ (\lambda I + T_0)^{-1/2} \sum_{i=1}^{n} \alpha_i \right]^{4} \\
\leq \frac{192 m^2}{N^2} \left( c_2 \rho^4 \| \gamma_0 \|^4_{L^2} \lambda^{4\theta} N^2(\lambda) + \mu_1^{4\theta} \| \gamma_0 \|^{4}_{L^2} \right) \leq c_0 \frac{m^2}{N^2} (1 + \lambda \theta N^2(\lambda)),
\]

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where \( \epsilon_0^2 := 192(c_2\rho^4\|\gamma_0\|_{L^2}^4 + \max\{\mu_1^2, 1\}\|\gamma_0\|_{L^2}^4) \). We have completed the proof of Lemma 10.

We propose the following lemma to bound \( \mathbb{E}[ \mathcal{J}(S, \lambda) ] \).

**Lemma 10.** For any \( m \geq 1 \), there holds

\[
\mathbb{E}[ \mathcal{J}(S, \lambda) ] \leq \frac{1}{m} \mathbb{E} \left[ \left\| L_C^{1/2} L_K^{1/2} (\hat{f}_{S, \lambda} - f_\lambda) \right\|_{L^2}^2 \right] + \frac{1}{m} \mathbb{E} \left[ \left\| L_C^{1/2} L_K^{1/2} \mathbb{E} \left[ (\hat{f}_{S, \lambda} - f_\lambda) \right] \right\|_{L^2}^2 \right].
\]

(5.30)

**Proof.** When \( m \geq 2 \), as

\[
\mathcal{J}(S, \lambda) = \left\| L_C^{1/2} L_K^{1/2} \hat{f}_{S, \lambda} - L_C^{1/2} L_K^{1/2} f_\lambda \right\|_{L^2}^2 = \frac{1}{m} \sum_{i=1}^{m} L_C^{1/2} L_K^{1/2} \hat{f}_{S, \lambda} - L_C^{1/2} L_K^{1/2} f_\lambda \right\|_{L^2}^2,
\]

then we can write

\[
\mathbb{E}[ \mathcal{J}(S, \lambda) ] = \mathbb{E} \left[ \left\| L_C^{1/2} L_K^{1/2} \left( \frac{1}{m} \sum_{i=1}^{m} \hat{f}_{S, \lambda} - f_\lambda \right) \right\|_{L^2}^2 \right] \leq \frac{1}{m} \sum_{i=1}^{m} L_C^{1/2} L_K^{1/2} \left( \hat{f}_{S, \lambda} - f_\lambda \right) \right\|_{L^2}^2 \]

\[+ \frac{1}{m^2} \sum_{i \neq j} \mathbb{E} \left[ \left\| L_C^{1/2} L_K^{1/2} \left( \hat{f}_{S, \lambda} - f_\lambda \right) , L_C^{1/2} L_K^{1/2} \left( \hat{f}_{S, \lambda} - f_\lambda \right) \right\|_{L^2} \right] \]

\[\leq \frac{1}{m} \mathbb{E} \left[ \left\| L_C^{1/2} L_K^{1/2} (\hat{f}_{S, \lambda} - f_\lambda) \right\|_{L^2}^2 \right] + \frac{1}{m} \mathbb{E} \left[ \left\| L_C^{1/2} L_K^{1/2} \mathbb{E} \left[ (\hat{f}_{S, \lambda} - f_\lambda) \right] \right\|_{L^2}^2 \right].
\]

Here equality (i) follows from the binomial expansion. Inequality (ii) is from

\[
\mathbb{E} \left[ \left( L_C^{1/2} L_K^{1/2} (\hat{f}_{S, \lambda} - f_\lambda) , L_C^{1/2} L_K^{1/2} (\hat{f}_{S, \lambda} - f_\lambda) \right) \right] = \left\| L_C^{1/2} L_K^{1/2} \mathbb{E} \left[ (\hat{f}_{S, \lambda} - f_\lambda) \right] \right\|_{L^2}^2.
\]

When \( m = 1 \), (5.30) is obvious.

Thus, we have completed the proof of Lemma 10.

Now we are in the position to prove Theorem 3.

**Proof of Theorem 3** Combining (2.8), (5.1) and (5.30) yields

\[
\mathbb{E}[ (\mathcal{R}(\beta_{S, \lambda}) - \mathcal{R}(\beta_0)) ] \leq 2 \mathbb{E}[ \mathcal{J}(S, \lambda) ] + 2\varphi(\lambda) \leq \frac{2}{m} \mathbb{E} \left[ \left\| L_C^{1/2} L_K^{1/2} (\hat{f}_{S, \lambda} - f_\lambda) \right\|_{L^2}^2 \right] + 2 \left\| L_C^{1/2} L_K^{1/2} \mathbb{E} \left[ (\hat{f}_{S, \lambda} - f_\lambda) \right] \right\|_{L^2}^2 + 2\varphi(\lambda)
\]

(5.31)
In the followings, we aim to bound $\mathbb{E} \left[ \left\| L_C^{1/2} L_K^{1/2} (\hat{f}_{S_1, \lambda} - f_{\lambda}) \right\|_{L^2}^2 \right]$ and $\mathbb{E} \left[ L_C^{1/2} L_K^{1/2} \mathbb{E} \left[ (\hat{f}_{S_1, \lambda} - f_{\lambda}) \right] \right]_{L^2}$, respectively. Recalling (5.24) and $Y_{1,i} = (X_{1,i}, \beta_0 + f_{\lambda}) L^2 + \epsilon_{1,i}$, for simplicity of notations, let

$$a_i := L_K^{1/2} X_{1,i}(X_{1,i}, \beta_0 - L_K^{1/2} f_{\lambda}) L^2 - \lambda f_{\lambda}, \quad i = 1, 2, \ldots, n.$$

Then

$$\hat{f}_{S_1, \lambda} - f_{\lambda} = (\lambda + T_{X_i})^{-1} \frac{1}{n} \sum_{i=1}^{n} (a_i + L_K^{1/2} X_{1,i}) \epsilon_{1,i}).$$

Using this expression, we can bound $\mathbb{E} \left[ \left\| L_C^{1/2} L_K^{1/2} (\hat{f}_{S_1, \lambda} - f_{\lambda}) \right\|_{L^2}^2 \right]$ as

$$\mathbb{E} \left[ \left\| L_C^{1/2} L_K^{1/2} (\hat{f}_{S_1, \lambda} - f_{\lambda}) \right\|_{L^2}^2 \right] = \mathbb{E} \left[ \left\| L_C^{1/2} L_K^{1/2} (\hat{f}_{S_1, \lambda} - f_{\lambda}) \right\|_{L^2}^2 \right] + \mathbb{E} \left[ \left\| L_C^{1/2} L_K^{1/2} (\hat{f}_{S_1, \lambda} - f_{\lambda}) \right\|_{L^2}^2 \right]$$

$$\leq 8 \frac{m}{N} \mathcal{N}(\lambda) \left( c_2 \lambda^2 \| \gamma_0 \|_{L^2}^2 + \sigma^2 \right) + 2 \mathbb{E} \left[ \left\| L_C^{1/2} L_K^{1/2} (\hat{f}_{S_1, \lambda} - f_{\lambda}) \right\|_{L^2}^2 \right]$$

$$\leq 8 \frac{m}{N} \mathcal{N}(\lambda) \left( c_2 \lambda^2 \| \gamma_0 \|_{L^2}^2 + \sigma^2 \right) + 2 \mathbb{E} \left[ \left\| L_C^{1/2} L_K^{1/2} (\hat{f}_{S_1, \lambda} - f_{\lambda}) \right\|_{L^2}^2 \right]$$

Here inequality (i) follows from (5.5) in Lemma 4 and the triangular inequality. Inequality (ii) is due to Assumption 2.

We next bound $\mathbb{E} \left[ \left\| L_C^{1/2} L_K^{1/2} \mathbb{E} \left[ (\hat{f}_{S_1, \lambda} - f_{\lambda}) \right] \right\|_{L^2}^2 \right]$ as

$$\left\| L_C^{1/2} L_K^{1/2} \mathbb{E} \left[ (\hat{f}_{S_1, \lambda} - f_{\lambda}) \right] \right\|_{L^2}^2$$

$$\leq \mathbb{E} \left[ \left\| L_C^{1/2} L_K^{1/2} (\hat{f}_{S_1, \lambda} - f_{\lambda}) \right\|_{L^2}^2 \right]$$

$$\leq 4 c_2 \frac{m}{N} \mathcal{N}(\lambda) \lambda^{2} \| \gamma_0 \|_{L^2}^2 + \mathbb{E} \left[ \left\| L_C^{1/2} L_K^{1/2} \mathbb{E} \left[ (\hat{f}_{S_1, \lambda} - f_{\lambda}) \right] \right\|_{L^2}^2 \right].$$
Here equality (i) is from Assumption \textit{2}. Inequality (ii) follows from Jensen’s inequality and (5.35) in Lemma \textit{4}.

The key point in the rest of the proof is to estimate 

$$
E \left[ \left\| L_{C}^{1/2} L_{K}^{1/2} (\lambda I + T_{X_{i}})^{-1} \frac{1}{n} \sum_{i=1}^{n} \alpha_{i} \right\|_{L^{2}}^{2} \right] \tag{5.34}
$$

and 

$$
E \left[ \left\| L_{C}^{1/2} L_{K}^{1/2} (\lambda I + T_{X_{i}})^{-1} L_{K}^{1/2} X_{1,i} \right\|_{L^{2}}^{2} \right].
$$

For the first term, we have 

$$
E \left[ \left\| L_{C}^{1/2} L_{K}^{1/2} (\lambda I + T_{X_{i}})^{-1} \frac{1}{n} \sum_{i=1}^{n} \alpha_{i} \right\|_{L^{2}}^{2} \right] \leq E \left[ \left\| L_{C}^{1/2} L_{K}^{1/2} (\lambda I + T)^{-1/2} \right\|^{2} \left\| (\lambda I + T)^{1/2} (\lambda I + T_{X_{i}})^{-1} \frac{1}{n} \sum_{i=1}^{n} \alpha_{i} \right\|_{L^{2}}^{2} \right] \leq (i) \leq E \left[ \left\| (\lambda I + T)^{1/2} (\lambda I + T_{X_{i}})^{-1} (\lambda I + T)^{1/2} \right\|^{4} \left\| (\lambda I + T)^{-1/2} \frac{1}{n} \sum_{i=1}^{n} \alpha_{i} \right\|_{L^{2}}^{4} \right]^{1/2}
$$

Here inequality (i) follows from the fact that 

$$
\left\| L_{C}^{1/2} L_{K}^{1/2} (\lambda I + T)^{-1/2} \right\|^{2} = \left\| (\lambda I + T)^{-1/2} L_{C}^{1/2} L_{K}^{1/2} (\lambda I + T)^{-1/2} \right\| = \left\| (\lambda I + T)^{-1/2} T (\lambda I + T)^{-1/2} \right\| \leq 1.
$$

Inequalities (ii) and (iii) are from Cauchy-Schwartz inequality.

Analogously, for the second term, we have 

$$
E \left[ \left\| L_{C}^{1/2} L_{K}^{1/2} (\lambda I + T_{X_{i}})^{-1} L_{K}^{1/2} X_{1,i} \right\|_{L^{2}}^{2} \right] \leq E \left[ \left\| (\lambda I + T)^{1/2} (\lambda I + T_{X_{i}})^{-1} (\lambda I + T)^{1/2} \right\|^{4} \left\| (\lambda I + T)^{-1/2} L_{K}^{1/2} X_{1,i} \right\|_{L^{2}}^{4} \right]^{1/2}
$$

While we can write 

$$
E \left[ \left\| (\lambda I + T)^{-1/2} L_{K}^{1/2} X_{1,i} \right\|_{L^{2}}^{4} \right] = E \left[ \left( \sum_{j=1}^{\infty} \frac{1}{\lambda + \mu_{j}} \left\langle L_{K}^{1/2} X_{1,i}, \phi_{j} \right\rangle_{L^{2}}^{2} \right)^{2} \right]
$$

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\[
\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{\lambda + \mu_j + \mu_k} \frac{1}{\lambda + \mu_j + \mu_k} \mathbb{E} \left[ \left( L_K^{1/2} X_{1,i} \phi_j \right)^2 \right]^{2} \mathbb{E} \left[ \left( L_K^{1/2} X_{1,i} \phi_k \right)^2 \right]^{2} \] (5.36)

\[
\leq \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{\lambda + \mu_j + \mu_k} \frac{1}{\lambda + \mu_j + \mu_k} \left[ \mathbb{E} \left( L_K^{1/2} X_{1,i} \phi_j \right)^4 \right]^{1/2} \left[ \mathbb{E} \left( L_K^{1/2} X_{1,i} \phi_k \right)^4 \right]^{1/2} (5.37)
\]

\[
\leq \rho^4 \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \frac{\mu_j}{\lambda + \mu_j + \mu_k} + \frac{\mu_k}{\lambda + \mu_j + \mu_k} = \rho^4 N^2(\lambda).
\]

Here \( \{\phi_k\}_{k=1}^{\infty} \) is given by the singular value decomposition of \( T \) in (2.9). Inequality (i) is from Cauchy-Schwartz inequality. Inequality (ii) is due to the decomposition of \( L_K^{1/2} X \) (3.8) and Assumption 4.

For the term \( \mathbb{E} \left[ \left\| (\lambda I + T)^{1/2} (\lambda I + T_{X_i})^{-1} (\lambda I + T)^{1/2} \right\|^8 \right] \), first applying the second-order decomposition, which was introduced in (19) (30) (20), to \( (\lambda I + T_{X_i})^{-1} \) yields that

\[
(\lambda I + T_{X_i})^{-1} = (\lambda I + T)^{-1} + (\lambda I + T_{X_i})^{-1}(T - T_{X_i})(\lambda I + T)^{-1}
\]

\[
= (\lambda I + T)^{-1} + (\lambda I + T)^{-1}(T - T_{X_i})(\lambda I + T)^{-1}
\]

\[
+ (\lambda I + T)^{-1}(T - T_{X_i})(\lambda I + T_{X_i})^{-1}(T - T_{X_i})(\lambda I + T)^{-1}. \] (5.37)

If \( 2 \leq \ell < 8 \), applying the above second-order decomposition of \( (\lambda I + T_{X_i})^{-1} \) and taking \( \lambda \leq 1 \), we have

\[
\mathbb{E} \left[ \left\| (\lambda I + T)^{1/2} (\lambda I + T_{X_i})^{-1} (\lambda I + T)^{1/2} \right\|^8 \right]
\]

\[
\leq (1 + \mu_1)^{8-\ell} \frac{1}{\lambda^{8-\ell}} \mathbb{E} \left[ \left\| (\lambda I + T)^{1/2} (\lambda I + T_{X_i})^{-1} (\lambda I + T)^{1/2} \right\|^8 \right]
\]

\[
\leq (1 + \mu_1)^{8-\ell} \frac{3^\ell}{\lambda^{8-\ell}} \mathbb{E} \left[ \left\| (\lambda I + T)^{-1/2}(T - T_{X_i})(\lambda I + T)^{-1/2} \right\|^\ell \right]
\]

\[
\leq (1 + \mu_1)^{8-\ell} \frac{3^\ell}{\lambda^{8-\ell}} \left\{ \left\| I \right\| + \mathbb{E} \left[ \left\| (\lambda I + T)^{-1/2}(T - T_{X_i})(\lambda I + T)^{-1/2} \right\|^\ell \right] \right\}
\]

\[
\leq (1 + \mu_1)^{8-\ell} \frac{3^\ell}{\lambda^{8-\ell}} \left\{ \left\| I \right\| + \mathbb{E} \left[ \left\| (\lambda I + T)^{-1/2}(T - T_{X_i})(\lambda I + T)^{-1/2} \right\|^2 \right]^{\ell/2} \right\}
\]

\[
\leq (1 + \mu_1)^{8-\ell} \frac{3^\ell}{\lambda^{8-\ell}} \left\{ \left\| I \right\| + \mathbb{E} \left[ \left\| (\lambda I + T)^{-1/2}(T - T_{X_i})(\lambda I + T)^{-1/2} \right\|^{2\ell} \right]^{\ell/2} \right\}
\]

\[
\leq (1 + \mu_1)^{8-\ell} \frac{3^\ell}{\lambda^{8-\ell}} \left\{ \left\| I \right\| + \mathbb{E} \left[ \left\| (\lambda I + T)^{-1/2}(T - T_{X_i})(\lambda I + T)^{-1/2} \right\|^{2\ell} \right] \right\}
\]

\[
\leq (1 + \mu_1)^{8-\ell} \frac{3^\ell}{\lambda^{8-\ell}} \left\{ 1 + c^2(\ell) 2^\ell \rho^{2\ell} \left( \frac{mN^2(\lambda)}{N} \right) \frac{\ell}{2} + c(\ell) 2^\ell \rho^{2\ell} \text{trace}^\ell(T) \frac{1}{\lambda^\ell} \left( \frac{mN^2(\lambda)}{N} \right) \right\}
\]

\[
\leq (1 + \mu_1)^{8-\ell} \frac{3^\ell}{\lambda^{8-\ell}} \left\{ 1 + \text{trace}^\ell(T) \frac{1}{\lambda^\ell} \left( \frac{mN^2(\lambda)}{N} \right) \right\}
\]

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where \( c_7^4 := (1 + \mu_1)^3 \sigma_c^2 \sigma^2 \max \{ 1, \text{trace}^8(T) \} \). Here equality (i) is from the second-order decomposition of \((\lambda I + T_{X_1})^{-1}(5.37)\). Inequality (ii) follows from Jensen’s inequality. Inequality (iii) is due to Cauchy-Schwartz inequality and (2.2). Inequality (iv) is from estimations (5.16) and (5.17) in Lemma 8.

Analogously, if \( \ell \geq 8 \), applying the second-order decomposition of \((\lambda I + T_{X_1})^{-1}(5.37)\) and taking \( \lambda \leq 1 \), we have

\[
\mathbb{E} \left[ \left\| (\lambda I + T)^{1/2} (\lambda I + T_{X_1})^{-1} (\lambda I + T)^{1/2} \right\|^8 \right] \\
\leq 3^7 \left[ 1 + c_7^4 (8)^2 \rho^2 \left( \frac{mN^2(\lambda)}{N} \right)^4 + c(8)^2 \rho^2 \text{trace}^8(T) \frac{1}{8} \left( \frac{mN(\lambda)}{N} \right)^8 \right] \\
\leq c_7^4 \left[ 1 + \left( \frac{mN^2(\lambda)}{N} \right)^4 + \frac{1}{8} \left( \frac{mN(\lambda)}{N} \right)^8 \right], \tag{5.39}
\]

where \( c_7^4 = (1 + \mu_1)^3 \sigma_c^2 \sigma^2 \max \{ 1, \text{trace}^8(T) \} \).

We can now prove (3.10).

If \( 2 \leq \ell < 8 \), taking \( \lambda \leq 1 \), for the term \( \mathbb{E} \left[ \left\| L_{1/2}^{1/2} L_{K}^{1/2} (\lambda I + T_{X_1})^{-1} \frac{1}{n} \sum_{i=1}^{n} \alpha_i \right\|^2 \mathbb{I}_{\ell_2} \right] \), combining (5.34) and (5.38) with (5.15) in Lemma 7 and (5.23) in Lemma 9 we have

\[
\mathbb{E} \left[ \left\| L_{1/2}^{1/2} L_{K}^{1/2} (\lambda I + T_{X_1})^{-1} \frac{1}{n} \sum_{i=1}^{n} \alpha_i \right\|^2 \mathbb{I}_{\ell_2} \right] \\
\leq c_6 c_7^4 \ell^2 \rho^\ell \lambda^{\ell/4} \left[ 1 + \left( \frac{mN^2(\lambda)}{N} \right)^{\frac{4}{\ell}} + \lambda^{-\ell} \left( \frac{mN(\lambda)}{N} \right)^{\ell} \right] \left( \frac{mN^2(\lambda)}{N} \right)^{\frac{4}{\ell}} m \left( 1 + \lambda^{4g} \mathcal{N}^2(\lambda) \right)^{\frac{1}{4}} \\
\leq c_6 c_7^4 \ell^2 \rho^\ell \lambda^{\ell/4} \left[ 1 + \left( \frac{mN^2(\lambda)}{N} \right)^{\frac{4}{\ell}} + \lambda^{-\ell} \left( \frac{mN(\lambda)}{N} \right)^{\ell} \right] \left( \frac{mN^2(\lambda)}{N} \right)^{\frac{4}{\ell}} m \left( 1 + \lambda^{2g} \mathcal{N}(\lambda) \right),
\]

where \( b_1(\ell) := c_6 c_7^4 \ell^2 \rho^\ell \).

For the term \( \mathbb{E} \left[ \left\| L_{1/2}^{1/2} L_{K}^{1/2} (\lambda I + T_{X_1})^{-1} L_{K}^{1/2} X_{1,i} \right\|^2 \mathbb{I}_{\ell_2} \right] \), combining (5.35), (5.36) and (5.38) with (5.15) in Lemma 7 we have

\[
\mathbb{E} \left[ \left\| L_{1/2}^{1/2} L_{K}^{1/2} (\lambda I + T_{X_1})^{-1} L_{K}^{1/2} X_{1,i} \right\|^2 \mathbb{I}_{\ell_2} \right] \\
\leq c_7^4 \ell^2 \rho^{\ell+2} \lambda^{\ell/4} \left[ 1 + \left( \frac{mN^2(\lambda)}{N} \right)^{\frac{4}{\ell}} + \lambda^{-\ell} \left( \frac{mN(\lambda)}{N} \right)^{\ell} \right] \left( \frac{mN^2(\lambda)}{N} \right)^{\frac{4}{\ell}} \mathcal{N}(\lambda)
\]

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\[ b_2(\ell) \lambda^{\frac{\ell - s}{4}} \left[ 1 + \left( \frac{m N^2(\lambda)}{N} \right)^{\frac{s}{2}} + \lambda^{-\frac{\ell}{4}} \left( \frac{m N(\lambda)}{N} \right)^{\frac{s}{2}} \right] \left( \frac{m N^2(\lambda)}{N} \right)^{\frac{\ell}{4}} N(\lambda), \]

where \( b_2(\ell) := c_7 c^\frac{1}{4}(\ell) 2^\ell \rho^{\ell+2}. \)

Then recall \( n = N/m, \) combining the above two estimations with (5.31), (5.32) and (5.33) yields

\[
\mathbb{E} \left[ (R(\beta_{S,\lambda}) - R(\beta_0)) \right] \\
\leq 2 \lambda^{2\theta} \| \gamma_0 \|^2_{L^2} + 16 \lambda N(\lambda) \lambda^{2\theta} \| \gamma_0 \|^2_{L^2} + 8 c_2 m \lambda N(\lambda) \lambda^{2\theta} \| \gamma_0 \|^2_{L^2} \\
+ b_1(\ell) \lambda^{\frac{\ell - s}{4}} \left[ 1 + \left( \frac{m N^2(\lambda)}{N} \right)^{\frac{s}{2}} + \lambda^{-\frac{\ell}{4}} \left( \frac{m N(\lambda)}{N} \right)^{\frac{s}{2}} \right] \left( \frac{m N^2(\lambda)}{N} \right)^{\frac{\ell}{4}} N(\lambda) \\
+ b_2(\ell) \lambda^{\frac{\ell - s}{4}} \left[ 1 + \left( \frac{m N^2(\lambda)}{N} \right)^{\frac{s}{2}} + \lambda^{-\frac{\ell}{4}} \left( \frac{m N(\lambda)}{N} \right)^{\frac{s}{2}} \right] \left( \frac{m N^2(\lambda)}{N} \right)^{\frac{\ell}{4}} 4 \lambda^{2\theta} N(\lambda). \]

This completes the proof of (5.10).

We next give the proof of (5.11).

If \( \ell \geq 8, \) taking \( \lambda \leq 1, \) utilizing (5.39) and following the same arguments in the proof of (5.10), we obtain

\[
\mathbb{E} \left[ \left\| L_{C}^{1/2} L_{K}^{1/2}(\lambda I + T x_i)^{-1} \sum_{i=1}^{n} \alpha_i \right\|_{L^2}^2 \right] \\
\leq c_6 c_7 c^\frac{1}{4}(\ell) 2^\ell \rho^{\ell} \left[ 1 + \frac{m N^2(\lambda)}{N} + \frac{1}{\lambda^2} \left( \frac{m N(\lambda)}{N} \right)^2 \right] \left( \frac{m N^2(\lambda)}{N} \right)^{\frac{\ell}{4}} m N(\lambda) \\
= b_1(\ell) \left[ 1 + \frac{m N^2(\lambda)}{N} + \frac{1}{\lambda^2} \left( \frac{m N(\lambda)}{N} \right)^2 \right] \left( \frac{m N^2(\lambda)}{N} \right)^{\frac{\ell}{4}} m N(\lambda), \]

where \( b_1(\ell) = c_6 c_7 c^\frac{1}{4}(\ell) 2^\ell \rho^{\ell}. \)

And

\[
\mathbb{E} \left[ \left\| L_{C}^{1/2} L_{K}^{1/2}(\lambda I + T x_i)^{-1} L_{K}^{1/2} x_i \right\|_{L^2}^2 \right] \\
\leq c_3 c^\frac{1}{4}(\ell) 2^\ell \rho^{\ell+2} \left[ 1 + \frac{m N^2(\lambda)}{N} + \frac{1}{\lambda^2} \left( \frac{m N(\lambda)}{N} \right)^2 \right] \left( \frac{m N^2(\lambda)}{N} \right)^{\frac{\ell}{4}} N(\lambda) \\
= b_2(\ell) \left[ 1 + \frac{m N^2(\lambda)}{N} + \frac{1}{\lambda^2} \left( \frac{m N(\lambda)}{N} \right)^2 \right] \left( \frac{m N^2(\lambda)}{N} \right)^{\frac{\ell}{4}} N(\lambda), \]

where \( b_2(\ell) = c_7 c^\frac{1}{4}(\ell) 2^\ell \rho^{\ell+2}. \)

And then

\[
\mathbb{E} \left[ (R(\beta_{S,\lambda}) - R(\beta_0)) \right] \]
\[ \leq 2\lambda^{2\theta}\|\gamma_0\|_{L^2}^2 + 16\frac{N(\lambda)}{N}(c_2\lambda^{2\theta}\|\gamma_0\|_{L^2}^2 + \sigma^2) + 8c_2\frac{m}{N}N(\lambda)\lambda^{2\theta}\|\gamma_0\|_{L^2}^2 \\
+ b_1(\ell) \left[ 1 + \frac{mN^2(\lambda)}{N} + \frac{1}{\lambda^2} \left( \frac{mN(\lambda)}{N} \right)^2 \right] \left( \frac{mN^2(\lambda)}{N} \right)^{\frac{4}{7}} \frac{4 + 2m}{N} \left( 1 + \lambda^{2\theta}N(\lambda) \right) \\
+ b_2(\ell) \left[ 1 + \frac{mN^2(\lambda)}{N} + \frac{1}{\lambda^2} \left( \frac{mN(\lambda)}{N} \right)^2 \right] \left( \frac{mN^2(\lambda)}{N} \right)^{\frac{4}{7}} \frac{4\sigma^2}{N}N(\lambda). \]

We have completed the proof of inequality (3.11). The proof of Theorem 5 is then finished. \( \square \)

We next prove Corollary 1.

**Proof of Corollary 1** We prove the desired bounds in two cases, respectively.

When \( 2 \leq \ell < 8 \), taking \( \lambda \leq 1 \), (3.10) and (5.9) implies

\[ \mathbb{E} \left[ (\mathcal{R}(\mathcal{S}_S, \lambda) - \mathcal{R}(\beta_0)) \right] \leq \lambda^{2\theta} + \frac{\lambda^{-p}}{N} + \frac{\lambda^{2\theta-p}}{N} \]

\[ + \lambda^{\ell-8} \left[ 1 + \left( \frac{m\lambda^{-2p}}{N} \right)^{\frac{4}{7}} + \left( \frac{m\lambda^{-2p}}{N} \right)^{\frac{4}{7}} \right] \left( \frac{m\lambda^{-2p}}{N} \right)^{\frac{4}{7}} \left( \frac{m + m\lambda^{2\theta-p}}{N} + \frac{\lambda^{-p}}{N} \right) \]

Taking \( \lambda = N^{-\frac{1}{2\theta + p}} \) yields

\[ \mathbb{E} \left[ (\mathcal{R}(\mathcal{S}_S, \lambda) - \mathcal{R}(\beta_0)) \right] \leq N^{\frac{2\theta}{2\theta + p}} \]

provided that \( \frac{\ell+8}{4\ell} \leq \theta \leq 1 \) and \( m \leq \min \left\{ N^{\frac{\ell+8}{4\ell}} \right\} \).

If \( \theta < \frac{\ell+8}{4\ell} \), take \( m \leq N^r \) for some \( 0 \leq r \leq \frac{2\theta}{2\theta + p} \). Then we have

\[ \mathbb{E} \left[ (\mathcal{R}(\mathcal{S}_S, \lambda) - \mathcal{R}(\beta_0)) \right] \leq \max \left\{ N^{\frac{2\theta \ell}{8 + \theta + 2\theta \ell}}, \frac{N}{8 + \theta + 2\theta \ell}^{\frac{\ell-1-4}{\ell-1}}, N^{\frac{2\theta \ell}{8 + \theta + 2\theta \ell}} \right\} \]

providing that

\[ \lambda = \max \left\{ N^{\frac{\ell-1-4}{\ell-1}}, N^{\frac{\ell-1-4}{\ell-1}} \right\} \]

This completes the proof of case \( 2 \leq \ell < 8 \).

When \( \ell \geq 8 \), taking \( \lambda \leq 1 \), (3.11) and (5.9) implies

\[ \mathbb{E} \left[ (\mathcal{R}(\mathcal{S}_S, \lambda) - \mathcal{R}(\beta_0)) \right] \leq \lambda^{2\theta} + \frac{\lambda^{-p}}{N} + \frac{\lambda^{2\theta-p}}{N} \]

\[ + \left( 1 + \frac{m\lambda^{-2p}}{N} + \frac{m^2\lambda^{-2p-2}}{N^2} \right) \left( \frac{m\lambda^{-2p}}{N} \right)^{\frac{4}{7}} \left( \frac{m + m\lambda^{2\theta-p}}{N} + \frac{\lambda^{-p}}{N} \right). \]

Taking \( \lambda = N^{-\frac{1}{2\theta + p}} \) yields

\[ \mathbb{E} \left[ (\mathcal{R}(\mathcal{S}_S, \lambda) - \mathcal{R}(\beta_0)) \right] \leq N^{\frac{2\theta}{2\theta + p}} \]
provided that \( \frac{p + 8}{2c + 16} \leq \theta \leq 1/2 \) and \( m \leq \min \left\{ N \frac{8 + p - 4\theta - 2\theta}{(12 + r)(2p + r)}, N \frac{8 + p - 4\theta - 2\theta}{(12 + r)(2p + r)} \right\} \).

If \( \theta < \frac{p + 8}{2c + 16} \), take \( m \leq N^r \) for some \( 0 \leq r \leq \frac{2\theta}{2p + r} \). Then we have

\[
\mathbb{E} \left[ (\mathcal{R}(\beta_{S,\lambda}) - \mathcal{R}(\beta_0)) \right] \leq \max \left\{ N \frac{\theta(4 + \ell)(r - 1)}{4+4p+4\theta+p^2}, N \frac{\theta(12 + \ell)(r - 1)}{4+4p+4\theta+p^2}, N \frac{\theta(8 + \ell)(r - 1) - 4\theta}{4+4p+4\theta+p^2} \right\}
\]

provided that

\[
\lambda = \max \left\{ N \frac{(4 + \ell)(r - 1)}{4+4p+4\theta+p^2}, N \frac{\ell(r - 1) - 4\theta}{4+4p+4\theta+p^2}, N \frac{8 + \ell(r - 1) - 4\theta}{4+4p+4\theta+p^2} \right\}.
\]

We have completed the proof of case \( \ell \geq 8 \). Then proof is then finished. \( \square \)

We next turn to prove Theorem 4 and Corollary 2. If Assumption 5 is satisfied, we can estimate the probability of event \( \mathcal{U}_1 \) better than Lemma 5 and Lemma 7.

**Lemma 11.** Suppose Assumption 5 is satisfied, then there holds

\[
\mathbb{P}(\mathcal{U}_1) \leq c_4 \left( 1 + \frac{m^2 \mathcal{N}^2(\lambda)}{\mathcal{N}^2} \right) \mathcal{N}(\lambda) \exp \left( -c_5 \frac{N}{m \mathcal{N}(\lambda)} \right). \tag{5.40}
\]

Where \( c_4 \) and \( c_5 \) are universal constants and \( \mathcal{N}(\lambda) \) is given by (5.4).

**Proof.** Our proof relies on the Bernstein’s inequality for the sum of self-adjoint random operators (see, Lemma 16). Recalling (5.4), define

\[
\zeta_i := (\lambda I + T)^{-1/2} L_{\lambda K}^{1/2} X_{1,i} \otimes L_{\lambda K}^{1/2} X_{1,j} (\lambda I + T)^{-1/2} \quad \text{and} \quad \eta_i := \frac{1}{n} (\zeta_i - \mathbb{E}[\zeta_i]), \quad 1 \leq i \leq n.
\]

Then

\[
(\lambda I + T)^{-1/2} (T_{X_1} - T) (\lambda I + T)^{-1/2} = \sum_{i=1}^{n} \eta_i.
\]

Using expression (5.8), we have

\[
\left\| (\lambda I + T)^{-1/2} L_{\lambda K}^{1/2} X \otimes L_{\lambda K}^{1/2} X (\lambda I + T)^{-1/2} \right\| \\
= \sup_{\|f\|_{L^2} = 1, \|g\|_{L^2} = 1} \left\langle (\lambda I + T)^{-1/2} L_{\lambda K}^{1/2} X \otimes L_{\lambda K}^{1/2} X (\lambda I + T)^{-1/2} f, g \right\rangle_{L^2} \\
\leq \left\| (\lambda I + T)^{-1/2} L_{\lambda K}^{1/2} X \right\|_{L^2}^2 = \sum_{k=1}^{\infty} \frac{\mu_k}{\lambda + \mu_k} \xi_k^2 \\
\leq \rho^2 \sum_{k=1}^{\infty} \frac{\mu_k}{\lambda + \mu_k} = \rho^2 \mathcal{N}(\lambda). \tag{5.41}
\]

Here inequality (*) is from Assumption 5.

Then for any \( 1 \leq i \leq n \), one can calculate

\[
\|\eta_i\| = \left\| \frac{1}{n} (\zeta_i - \mathbb{E}[\zeta_i]) \right\| \overset{(i)}{\leq} \frac{1}{n} \|\zeta_i\| + \frac{1}{n} \mathbb{E} \left[ \|\zeta_i\| \right] \overset{(ii)}{\leq} 2 \rho^2 \frac{\mathcal{N}(\lambda)}{n}. \tag{5.42}
\]

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Here inequality (i) uses the triangle inequality and Jensen’s inequality. Inequality (ii) follows from (5.41) and the definition that $\zeta_i = (\lambda I + T)^{-1/2} L_K^{1/2} X_{1,i} \otimes L_K^{1/2} X_{1,i} (\lambda I + T)^{-1/2}$. And then we have

$$
\| \mathbb{E} [z_i^2] \| = \| \mathbb{E} \left( \sum_{i=1}^{n} \eta_i \right)^2 \|
$$

$$
\overset{(i)}{=} \sup_{f \in \ell^2(T), \|f\| = 1} \sum_{i=1}^{n} \langle f, \mathbb{E} [\eta_i^2] f \rangle \ell^2
$$

$$
= \frac{1}{n^2} \sup_{f \in \ell^2(T), \|f\| = 1} \sum_{i=1}^{n} \left( \langle f, \mathbb{E} [\zeta_i^2] f \rangle - \langle f, \mathbb{E} [\zeta_i^2] f \rangle \mathbb{E} \right) \ell^2
$$

$$
\leq \frac{1}{n^2} \sup_{f \in \ell^2(T), \|f\| = 1} \sum_{i=1}^{n} \sum_{k=1}^{\infty} \left( \frac{1}{\lambda + \mu_k} \right)^2 \langle f, \phi_k \rangle \ell^2 \leq \frac{1}{n^2} \mathcal{N}(\lambda)
$$

Here equality (i) is due to the equivalent expression of the operator norm of a positive operator (5.41) and the fact that $\mathbb{E} [\eta_i] = 0$. Inequality (ii) follows from the fact that

$$
\| (\lambda I + T)^{-1/2} L_K^{1/2} X_{1,i} \|^2 \ell^2 \leq \rho^2 \mathcal{N}(\lambda)
$$

which is given by (5.41). Equality (iii) is from the fact that

$$
\mathbb{E} \left( (L_K^{1/2} X, \phi_j) \ell^2 (L_K^{1/2} X, \phi_k) \ell^2 \right) = (T \phi_j, \phi_k) \ell^2 = \mu \delta_{jk}
$$

We also need the following estimates given by

$$
\text{trace} (\mathbb{E} [z_i^2]) \overset{(i)}{=} \sum_{k=1}^{n} \mathbb{E} \left[ \left( \sum_{i=1}^{n} \eta_i \right)^2 \right] \phi_k, \phi_k \ell^2
$$

$$
\overset{(i)}{=} \sum_{k=1}^{n} \sum_{i=1}^{n} \left( \mathbb{E} [\eta_i^2] \phi_k, \phi_k \right) \ell^2 \leq \frac{1}{n^2} \sum_{i=1}^{n} \sum_{k=1}^{\infty} \left( \mathbb{E} [\zeta_i^2] \phi_k, \phi_k \right) \ell^2
$$

$$
= \frac{1}{n^2} \sum_{i=1}^{n} \sum_{k=1}^{\infty} \frac{1}{\lambda + \mu_k} \mathbb{E} \left[ \| (\lambda I + T)^{-1/2} L_K^{1/2} X_{1,i} \|^2 \ell^2 (L_K^{1/2} X_{1,i}, \phi_k) \ell^2 \right]
$$

$$
\overset{(ii)}{=} \rho^2 \frac{\overline{N}(\lambda)}{n^2} \sum_{i=1}^{n} \sum_{k=1}^{\infty} \frac{1}{\lambda + \mu_k} \mathbb{E} \left[ (L_K^{1/2} X_{1,i}, \phi_k) \ell^2 \right] \overset{(iii)}{=} \rho^2 \frac{\overline{N}^2(\lambda)}{n}
$$

Here equality (i) is from the formulation of the trace norm of an operator (2.9). Inequality (ii) is due to the fact that $\| (\lambda I + T)^{-1/2} L_K^{1/2} X_{1,i} \|^2 \ell^2 \leq \rho^2 \mathcal{N}(\lambda)$. Inequality (iii) follows from the calculation that $\sum_{k=1}^{\infty} \frac{1}{\lambda + \mu_k} \mathbb{E} \left[ (L_K^{1/2} X_{1,i}, \phi_k) \ell^2 \right] = \frac{\mu_k}{\lambda + \mu_k} = \mathcal{N}(\lambda)$. 

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Recall that \( n = N/m \). Based on (5.42), (5.43) and (5.44), one can apply Lemma 16 with \( L = 2\rho^2 mN(\lambda) \), \( v = \rho^2 mN(\lambda) \), \( d = N(\lambda) \) and \( s = 1/2 \) to obtain

\[
P(\mathcal{U}_1) = P \left( \left\| \sum_{i=1}^{n} \eta_i \right\| \geq 1/2 \right) \leq \left[ 1 + 6 \left( \rho^2 \frac{4mN(\lambda)}{N} + \rho^4 \frac{4mN(\lambda)}{3N} \right)^2 \right] \mathcal{N}(\lambda) \exp \left(-\frac{3N}{32\rho^2 mN(\lambda)} \right) \leq c_4 \left( 1 + \frac{m^2N^2(\lambda)}{N^2} \right) \mathcal{N}(\lambda) \exp \left(-c_5 \frac{N}{mN(\lambda)} \right),
\]

where \( c_4 := \left[ 1 + 6 \left( 4\rho^2 + \frac{4\rho^4}{3} \right)^2 \right] \) and \( c_5 := \frac{3}{32\rho^2} \). The proof is then completed. \( \square \)

Now we can prove Theorem 4.

Proof of Theorem 4 Under Assumption 5 there holds

\[
\left\| L_{K}^{1/2} X \right\|_{\mathcal{L}^2} = \left( \sum_{k=1}^{\infty} \mu_k \xi_k^2 \right)^{\frac{1}{2}} \leq \rho \left( \sum_{k=1}^{\infty} \mu_k \right)^{\frac{1}{2}} = \rho \cdot \text{trace}^{\frac{1}{2}}(T).
\]

Therefore, recalling (5.41), for any \( 1 \leq i \leq n \), we can write

\[
\begin{align*}
\mathbb{E} \left[ \left\| L_{K}^{1/2} L_{K}^{1/2} (\lambda I + T_{X_1})^{-1} L_{K}^{1/2} X_{1,i} \right\|_{\mathcal{L}^2}^2 \right] &\leq \frac{\mu_1}{\lambda^2} \mathbb{E} \left[ \left\| L_{K}^{1/2} X_{1,i} \right\|_{\mathcal{L}^2}^2 \right] \leq \frac{\mu_1}{\lambda^2} \left[ \mathbb{E} \left[ \left\| L_{K}^{1/2} X_{1,i} \right\|_{\mathcal{L}^2}^4 \right] \right]^{\frac{1}{2}} \mathbb{P}^{\frac{1}{2}}(\mathcal{U}_1) \quad \text{(5.46)}
\end{align*}
\]

Here inequality (i) is from Cauchy-Schwartz inequality. Inequality (ii) follows from (5.40) in Lemma 11 and (5.46).

Then under Assumption 1 and 5 one can calculate

\[
\begin{align*}
\mathbb{E} \left[ \left\| L_{K}^{1/2} X (\beta_0 - L_{K}^{1/2} f_\lambda) \right\|_{\mathcal{L}^2}^4 \right] &\leq \rho^4 \text{trace}^2(T) \mathbb{E} \left[ \left\| X (\beta_0 - L_{K}^{1/2} f_\lambda) \right\|_{\mathcal{L}^2}^2 \right] \\
&\leq c_1 \rho^4 \text{trace}^2(T) \left( \mathbb{E} \left[ X (\beta_0 - L_{K}^{1/2} f_\lambda) \right]^2 \right)^2 \\
&= c_1 \rho^4 \text{trace}^2(T) \left\| L_{K}^{1/2} (\beta_0 - L_{K}^{1/2} f_\lambda) \right\|_{\mathcal{L}^2}^4 \\
&\leq c_1 \rho^4 \text{trace}^2(T) \left\| \gamma_0 \right\|_{\mathcal{L}^2}^4 \lambda^{4\theta}.
\end{align*}
\]

Here inequality (i) follows from (5.46), inequality (ii) uses the fourth-moment condition (3.3), and inequality (iii) is due to Lemma 2. Then utilize the above bound and follow the same
estimates in the proof of (5.23). By taking $\lambda \leq 1$, we obtain

$$
E \left[ \left\| \frac{1}{n} \sum_{i=1}^{n} \left( L_K^{1/2} X_{1,i} (X_{1,i}, \beta_0 - L_K^{1/2} f_\lambda) \right) \right\|^4_{L^2} \right] 
\leq 192m^2 \left( c_1 \rho^4 \text{trace}^2(T) \| \gamma_0 \|_{L^2}^4 \lambda^{4\theta} + \mu_1^{4\theta} \| \gamma_0 \|_{L^2}^4 \lambda^2 \right) \leq c_3 \frac{m^2}{N^2} \lambda^{4\theta},
$$

where $c_3 := 192 \left( c_1 \rho^4 \text{trace}^2(T) \| \gamma_0 \|_{L^2}^4 + \max\{\mu_1^2, 1\} \| \gamma_0 \|_{L^2}^4 \right)$.

For simplicity of notations, define

$$
\alpha_i := L_K^{1/2} X_{1,i} (X_{1,i}, \beta_0 - L_K^{1/2} f_\lambda) \lambda - \lambda f_\lambda, \quad i = 1, 2, \ldots, n.
$$

Then we can write

$$
E \left[ \left\| L_C^{1/2} L_K^{1/2} (\lambda I + T_{X_1})^{-1} \frac{1}{n} \sum_{i=1}^{n} \alpha_i \right\|^2_{L^2} \| \mathbb{I}_{U_1} \right] 
\leq \frac{\mu_1}{\lambda^2} E \left[ \left\| \frac{1}{n} \sum_{i=1}^{n} \alpha_i \right\|^2_{L^2} \right] \left( i \right) \leq \frac{\mu_1}{\lambda^2} \left( E \left[ \left\| \frac{1}{n} \sum_{i=1}^{n} \alpha_i \right\|^4_{L^2} \right] \right)^{1/2} \mathbb{P}^{1/2}(U_1) \leq \frac{c_3 c_4 \mu_1 m}{N^2 \lambda^{2-2\theta}} (1 + \frac{m N(\lambda)}{N}) N^2(\lambda) \exp \left( -\frac{c_5 N}{2m N(\lambda)} \right).
$$

Here inequality (i) is due to Cauchy-Schwartz inequality. Inequality (ii) is from (5.40) in Lemma [11] and (5.37).

Finally, utilizing estimates (5.46) and (5.48), we follow the same arguments in the proof of (5.10) and then obtain

$$
E \left[ (\frac{F(\bar{S}_{\lambda}) - F(\beta_0))}{N} \right] 
\leq 2 \lambda^{2\theta} \| \gamma_0 \|_{L^2}^2 + 16 \frac{N(\lambda)}{N} \left( c_1 \lambda^{2\theta} \| \gamma_0 \|_{L^2}^2 + \sigma^2 \right) + 8c_1 \frac{m N(\lambda)}{N} \lambda^{2\theta} \| \gamma_0 \|_{L^2}^2 
+ c_3 c_4 \mu_1 \frac{4 + 2m}{N \lambda^{2-2\theta}} (1 + \frac{m N(\lambda)}{N}) N^2(\lambda) \exp \left( -\frac{c_5 N}{2m N(\lambda)} \right) 
+ c_4 \mu_1 \rho^2 \text{trace}(T) \frac{4 \sigma^2}{N \lambda^2} (1 + \frac{m N(\lambda)}{N}) N^2(\lambda) \exp \left( -\frac{c_5 N}{2m N(\lambda)} \right) 
$$

The proof of Theorem [4] is finished.

We next turn to prove Corollary [2].

**Proof of Corollary [4]** Taking $m \leq o \left( \frac{N^{2\theta} \log N}{\log N} \right)$ and $\lambda = N^{-\frac{1}{2p}}$, (5.39) implies

$$
m \frac{N(\lambda)}{N} \leq m \frac{\lambda^{-p}}{N} \leq o \left( \frac{1}{\log N} \right).
$$

Therefore, for any $r > 0$, there holds

$$
\liminf_{N \to \infty} N^r \exp \left( -\frac{c_5 N}{2m N(\lambda)} \right) = 0.
$$
Then using Theorem 4, we obtain

\[ \mathbb{E} \left[ \mathcal{R}(\beta_{S_{N,\lambda}}) - \mathcal{R}(\beta_0) \right] \lesssim \lambda^{2\theta} + \frac{N(\lambda)}{N} + m \frac{N(\lambda)}{N} \lambda^{2\theta} \lesssim N^{-2\theta + p}. \]

The proof is then finished. \(\square\)

Finally, we will provide the proof of Theorem 6. Before that, we establish the following lemma to estimate \(\mathbb{P}(\mathcal{U}_1)\) based on Assumption 6.

**Lemma 12.** Suppose Assumption 6 is satisfied, then there holds

\[ \mathbb{P}(\mathcal{U}_1) \leq \left[ 1 + 6 \left( \frac{4m\kappa^2}{N\lambda^2} + \frac{4m\kappa^2}{3N\lambda^2} \right)^2 \right] N(\lambda) \exp \left( -\frac{3N\lambda^2}{32m\kappa^2} \right). \]  

(5.49)

**Proof.** Our proof relies on Lemma 16. Recalling (5.4) and the definition of \(\mathcal{U}_1\) given by

\[ \mathcal{U}_1 = \left\{ X_1 : \left\| (\lambda I + T)^{-1/2}(T_{X_1} - T)(\lambda I + T)^{-1/2} \right\| \geq 1/2 \right\}, \]

let

\[ \zeta_i := (\lambda I + T)^{-1/2} L_{1/2}^1 X_{1,i} \otimes L_{1/2}^1 X_{1,i} (\lambda I + T)^{-1/2} \quad \text{and} \quad \eta_i := \frac{1}{n} \left( \zeta_i - \mathbb{E}[\zeta_i] \right), \quad 1 \leq i \leq n. \]

Then

\[ (\lambda I + T)^{-1/2}(T_{X_1} - T)(\lambda I + T)^{-1/2} = \sum_{i=1}^n \eta_i. \]

Due to the decomposition of \(L_{1/2}^1 X\) in (3.8), there holds

\[ \left\| (\lambda I + T)^{-1/2} L_{1/2}^1 X \otimes L_{1/2}^1 X (\lambda I + T)^{-1/2} \right\| \]

\[ = \sum_{\|f\|_{L^2} = 1, \|g\|_{L^2} = 1} \left\langle (\lambda I + T)^{-1/2} L_{1/2}^1 X \otimes L_{1/2}^1 X (\lambda I + T)^{-1/2} f, g \right\rangle_{L^2} \]

\[ \leq \left\| (\lambda I + T)^{-1/2} L_{1/2}^1 X \right\|_{L^2}^2 = \sum_{k=1}^{\infty} \frac{\mu_k^{1/2} \mu_k^{1/2}}{\lambda + \mu_k} \leq \frac{1}{\lambda} \sum_{k=1}^{\infty} \mu_k^{1/2} \zeta_k^2 \leq \frac{\kappa^2}{\lambda}. \]

(5.50)

Here inequality (*) follows from Assumption 6. Then for any \(1 \leq i \leq n,\)

\[ \|\eta_i\| = \left\| \frac{1}{n} (\zeta_i - \mathbb{E}[\zeta_i]) \right\| \leq \frac{1}{n} \|\zeta_i\| + \frac{1}{n} \mathbb{E}[\|\zeta_i\|] \leq \frac{2\kappa^2}{n}. \]

(5.51)

Here inequality (†) follows from (3.50) and the definition that \(\zeta_i = (\lambda I + T)^{-1/2} L_{1/2}^1 X_{1,i} \otimes L_{1/2}^1 X_{1,i} (\lambda I + T)^{-1/2}.\)

Following the same arguments of (5.45), we have

\[ \left\| \mathbb{E} \left[ (\eta_i)^2 \right] \right\| \]

\[ \leq \frac{1}{n^2} \sup_{f \in L^2(\mathcal{T}), \|f\|_{L^2} = 1} \mathbb{E} \left[ \left\langle (\lambda I + T)^{-1/2} L_{1/2}^1 X_{1,i}, f \right\rangle_{L^2}^2 \left\| (\lambda I + T)^{-1/2} L_{1/2}^1 X_{1,i} \right\|_{L^2}^2 \right]. \]

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Here inequality (i) is from (5.50) and inequality (ii) is due to the decomposition of $L^{1/2}_K X$ in (3.8) and the fact that $\mathbb{E}[\xi_j \xi_k] = \delta^k_j$. And following the same arguments of (5.44), we have

\[
\text{trace}(\mathbb{E}[(\eta^T)^2]) \\
\leq \frac{1}{n^2} \sum_{i=1}^{n} \sum_{k=1}^{\infty} \frac{1}{\lambda + \mu_k} \mathbb{E} \left[ \left\| (\lambda I + T)^{-1/2} L^{1/2}_K X_{i,1} \right\|_{L^2}^2 \right] \\
= \frac{1}{n} \sum_{k=1}^{\infty} \frac{1}{\lambda + \mu_k} \mathbb{E} \left[ \left\| (\lambda I + T)^{-1/2} L^{1/2}_K X \right\|_{L^2}^2 \right] \\
\leq \frac{\kappa^2}{n \lambda^*} \sum_{k=1}^{\infty} \frac{\mu_k}{\lambda + \mu_k} \mathbb{E} \left[ \xi_k^2 \right] \leq \frac{\kappa^2 N(\lambda)}{n \lambda^*}. \tag{5.53}
\]

Here inequality (*) is from (5.50) and decomposition of $L^{1/2}_K X$ in (3.8).

Recall that $n = N/m$. Following from (5.51), (5.52) and (5.53), we employ Lemma 16 with $L = \frac{2m \kappa^2}{N \lambda^*}$, $v = \frac{m \kappa}{N \lambda^*}$, $d = N(\lambda)$ and $s = 1/2$ to obtain

\[
\mathbb{P}(U_1) = \mathbb{P} \left( \left\| \sum_{i=1}^{n} \eta_i \right\| \geq 1/2 \right) \\
\leq \left[ 1 + 6 \left( \frac{4m \kappa^2}{N \lambda^*} + \frac{4m \kappa^2}{3N \lambda^*} \right)^2 \right] N(\lambda) \exp \left( -\frac{3N \lambda^*}{32m \kappa^2} \right).
\]

This completes the proof. \qed

Now we are ready to prove Theorem 6.

**Proof of Theorem 6** Recalling (5.4) and $Y_{1,i} = (X_{1,i}, \beta_0)_{L^2} + \epsilon_{1,i}$, we define

\[
\alpha_i = L^{1/2}_K X_{1,i} (X_{1,i}, \beta_0 - L^{1/2}_K f_{\lambda})_{L^2} - \lambda f_{\lambda}, \quad i = 1, 2, \ldots, n.
\]

Under Assumption 4, we have

\[
\left\| L^{1/2}_K X \right\|_{L^2} = \left( \sum_{k=1}^{\infty} \mu_k \xi_k^2 \right)^{1/2} \leq \mu_{1/2} \left( \sum_{k=1}^{\infty} \mu_k \xi_k^2 \right)^{1/2} \leq \mu_{1/2} \kappa.
\]

Utilizing the above estimation and (5.49), following the same arguments in the proof of Theorem 4 we have

\[
\mathbb{E} \left[ \left\| L^{1/2}_C L^{1/2}_K (\lambda I + T_{X_1})^{-1} L^{1/2}_K X_{1,i} \right\|_{L^2}^2 \right]
\]
\[ \lesssim \frac{1}{\lambda^2} \left( 1 + \frac{m}{N\lambda^t} \right) \mathcal{N}^{1/2}(\lambda) \exp \left( -\frac{3N\lambda^t}{64m\kappa^2} \right) \]

and

\[
\mathbb{E} \left[ \left\| L_C^{1/2} L_K^{1/2} (\lambda I + T_{x_i})^{-1} \sum_{i=1}^n \alpha_i \right\|_{L^2}^2 \right] 
\lesssim \frac{m}{N\lambda^{2-2\theta}} \left( 1 + \frac{m}{N\lambda^t} \right) \mathcal{N}^{1/2}(\lambda) \exp \left( -\frac{3N\lambda^t}{64m\kappa^2} \right)
\]

and then

\[
\mathbb{E} \left[ (\mathcal{R}(\beta_{S,\lambda}) - \mathcal{R}(\beta_0)) \right] \lesssim \lambda^{2\theta} + \frac{\mathcal{N}(\lambda)}{N} + \frac{m}{N} \mathcal{N}(\lambda) \lambda^{2\theta}
+ \frac{m}{N\lambda^{2-2\theta}} \left( 1 + \frac{m}{N\lambda^t} \right) \mathcal{N}^{1/2}(\lambda) \exp \left( -\frac{3N\lambda^t}{64m\kappa^2} \right)
+ \frac{1}{N\lambda^2} \left( 1 + \frac{m}{N\lambda^t} \right) \mathcal{N}^{1/2}(\lambda) \exp \left( -\frac{3N\lambda^t}{64m\kappa^2} \right).
\] (5.54)

Recall that \( \{\mu_k\}_{k \geq 1} \) satisfy \( \mu_k \lesssim k^{-1/p} \) for some \( 0 < p \leq 1 \).

When \( \max\{0, t/2 - p/2\} \leq \theta \leq 1/2 \), taking \( m \leq o \left( \frac{N^{2\theta + p - 1}}{\log N} \right) \) and \( \lambda = N^{-\frac{1}{2\theta + p}} \) yields that for any \( r > 0 \), there holds

\[
\limsup_{N \to \infty} N^r \exp \left( -\frac{3N\lambda^t}{64m\kappa^2} \right) = 0, \quad \text{as} \quad \frac{m}{N\lambda^t} \leq o \left( \frac{1}{\log N} \right).
\]

Then combining with (5.9) and (5.51), we have

\[
\mathbb{E} \left[ (\mathcal{R}(\beta_{S,\lambda}) - \mathcal{R}(\beta_0)) \right] \lesssim \lambda^{2\theta} + \frac{\lambda^{-p}}{N} + \frac{m\lambda^{2\theta - p}}{N} \lesssim N^{-\frac{2\theta}{2\theta + p}}
\]

When \( \theta < \max\{0, t/2 - p/2\} \) which implies \( t > p > 0 \), taking \( m \leq o (\log N) \) and \( \lambda = N^{-\frac{1}{2\theta}} (\log N)^{-\frac{1}{2}} \) yields that for any \( r > 0 \), there holds

\[
\limsup_{N \to \infty} N^r \exp \left( -\frac{3N\lambda^t}{64m\kappa^2} \right) = 0, \quad \text{as} \quad \frac{m}{N\lambda^t} \leq o \left( \frac{1}{\log N} \right).
\]

Then combining with (5.9) and (5.51), we have

\[
\mathbb{E} \left[ (\mathcal{R}(\beta_{S,\lambda}) - \mathcal{R}(\beta_0)) \right] \lesssim \lambda^{2\theta} + \frac{\lambda^{-p}}{N} + \frac{m\lambda^{2\theta - p}}{N} \lesssim N^{-\frac{2\theta}{2\theta + p}} (\log N)^{-\frac{4\theta}{1}}.
\]

We have completed the proof of Theorem \ref{thm:6}.

\[ \square \]

### 5.2 Lower Rates

In this subsection, we establish the lower bound in Theorem \ref{thm:1}. Before that, we present some crucial results used in our proof. Our analysis of lower bound bases on Fano’s method, which
provides lower bound in nonparametric estimation problem and was proposed by \cite{27}. Fano’s method has been a crucial method in minimax lower bound estimation problem since it was proposed, and has inspired many following studies. (e.g., \cite{44}, \cite{10}, \cite{8}). The following lemma is a direct application of Fano’s method (see, \cite{44}). To this end, recall that the Kullback-Leibler divergence (KL-divergence) of two probability measures $P, Q$ on a general space $(\Omega, \mathcal{F})$ is defined as
\[
D_{kl}(P \| Q) := \int_{\Omega} \log \left( \frac{dP}{dQ} \right) dP,
\]
if $P$ is absolutely continuous with respect to $Q$, and otherwise $D_{kl}(P \| Q) := \infty$. Recall that for $\beta \in L^2(\mathcal{T})$, $L_C^{1/2} \beta \in \text{ran}T_*$ if $\beta$ satisfied the regularity condition (3.1), i.e.,
\[
L_C^{1/2} \beta = T_*^\theta(\gamma) \text{ with } 0 < \theta \leq 1/2 \text{ and some } \gamma \in L^2(\mathcal{T}).
\]

**Lemma 13.** Suppose that there exist constants $r, R > 0$ and $\beta_1, \beta_2, \cdots, \beta_L \in L^2(\mathcal{T})$ for some integer $L \geq 2$, such that
\[
L_C^{1/2} \beta_i \in \text{ran}T_*^\theta, \quad \left\| L_C^{1/2}(\beta_i - \beta_j) \right\|_{L^2} \geq 2r \quad \text{and} \quad D_{kl}(P_i \| P_j) \leq R, \quad \forall 1 \leq i \neq j \leq L,
\]
where $P_i$ denotes the joint probability distribution of $(X, Y)$ with
\[
Y = \int_{\mathcal{T}} \beta_i(t)X(t)dt + \epsilon.
\]
Here $\epsilon$ is independent of $X$ satisfying $\mathbb{E}[\epsilon] = 0$ and $\mathbb{E}[\epsilon^2] \leq \sigma^2$. Then we have
\[
\inf_{\beta_S} \sup_{\beta_0} \mathbb{P} \left\{ R(\hat{\beta}_S) - R(\beta_0) \geq r^2 \right\} \geq 1 - \frac{NR + \log 2}{\log L},
\]
where the supremum is taken over all $\beta_0 \in L^2(\mathcal{T})$ satisfying $L_C^{1/2} \beta_0 \in \text{ran}T_*^\theta$ and the infimum is taken over all possible predictors $\hat{\beta}_S \in L^2(\mathcal{T})$ based on the training sample $S = \{(X_i, Y_i)\}_{i=1}^N$ of $(X, Y)$ with
\[
Y = \int_{\mathcal{T}} \beta_0(t)X(t)dt + \epsilon.
\]

In the followings, we first construct a family of $\{\beta_i\}_{i=1}^L$ satisfying (5.55) with suitable $r, R$ and $L$, and then apply Lemma 13 to establish the lower bound. Note that any lower bound for a specific case yields immediately a lower bound for the general case. It therefore suffices to consider the case when $\epsilon$ is a zero-mean Gaussian random variable with $\mathbb{E}[\epsilon^2] = \sigma^2$. The following lemma is from the formulation of KL-divergence of two Gaussian distribution (see, e.g., example 2.7 of \cite{13}), which can further facilitate the calculation.

**Lemma 14.** Suppose that $\epsilon$ is a zero-mean Gaussian random variable independent of $X$ satisfying $\mathbb{E}[\epsilon^2] = \sigma^2$. For $\beta_i \in L^2(\mathcal{T}), i = 1, 2$, let $P_i$ denote the joint probability distribution of $(X, Y)$ with
\[
Y = \int_{\mathcal{T}} \beta_i(t)X(t)dt + \epsilon.
\]
Then
\[
D_{kl}(P_1 \| P_2) = \frac{1}{2\sigma^2} \left\| L_C^{1/2}(\beta_1 - \beta_2) \right\|_{L^2}^2.
\]
Our construction of $\{\beta_i\}_{i=1}^L$ relies on the following lemma which is known as Gilbert-Varshamov bound (see Lemma 7.5 in [13]).

**Lemma 15.** Let $M \geq 8$. There exists a subset $\Lambda \subset \mathcal{H}_M = \{-1,1\}^M$ of size $|\Lambda| \geq \exp(M/8)$ such that

$$\|i - i'\|_1 = 2 \sum_{i=1}^M \mathbb{I}_{\{i \neq i'\}} \geq M/2$$

for any $i \neq i'$ with $i, i' \in \Lambda$.

Now we are in the position to prove Theorem 1.

**Proof of Theorem 1.** Recall that the eigenvalues of $T$ denoted by $\{\mu_k\}_{k \geq 1}$ are sorted in decreasing order with geometric multiplicities and satisfy $\mu_k \asymp k^{-1/p}$ for some $0 < p \leq 1$, which implies there exists $c > 0$ independent of $j$ such that

$$\mu_{k+1} \leq \mu_k \text{ and } c k^{-1/p} \leq \mu_k \leq \frac{1}{c} k^{-1/p}, \quad \forall k \geq 1.$$  \hspace{1cm} (5.58)

We only consider the case that $\epsilon$ is from the Gaussian distribution $N(0, \sigma^2)$ and independent of $X$, then the Assumption 2 is satisfied with $\sigma > 0$.

For $L \geq 2$, we construct $\{\beta_i\}_{i=1}^L$ according to Lemma 15. Take $M = \lceil aN \frac{8}{\pi + 2\theta} \rceil$, which denotes the smallest integer larger than $aN \frac{8}{\pi + 2\theta}$ for some constant $a > 8$ to be specified later. Let $i^{(1)}, \ldots, i^{(L)} \in \{-1,1\}^M$ be given by Lemma 15 with $L \geq \exp(M/8)$. Given $0 < \theta \leq 1/2$, define

$$L_{C^2}^{1/2} \beta_i = \sum_{k=M+1}^{2M} \frac{1}{\sqrt{M}} \alpha_k^{(i)} e_k \varphi_k = T^\theta_1(\gamma_i), \quad i = 1, \ldots, L,$$  \hspace{1cm} (5.59)

where $\{\varphi_k\}_{k \geq 1}$ are the eigenvectors (corresponding to eigenvalue $\mu_k$) of $T$, which constitutes the orthonormal bases of $L^2(T)$, and $\gamma_i = \sum_{k=M+1}^{2M} \frac{1}{\sqrt{M}} \alpha_k^{(i)} e_k \varphi_k$ satisfies $\|\gamma_i\|_{L^2}^2 = 1$. Then $L_{C^2}^{1/2} \beta_i \in \text{ran} T^\theta_1$ with $0 < \theta \leq 1/2$ for $i = 1, \ldots, L$.

We next determine the positive constants $r$ and $R$ in (5.55) for $\{\beta_i\}_{i=1}^L$ defined above. For $1 \leq i, j \leq L$, we apply Lemma 5.55 and (5.58) to obtain

$$\|L_{C^2}^{1/2} (\beta_i - \beta_j)\|_{L^2}^2 = \sum_{k=M+1}^{2M} \frac{1}{M} \mu_k^{2\theta} \|\alpha_k^{(i)} - \alpha_k^{(j)}\|_{L^2}^2 \geq \mu_2^{2\theta} \frac{4}{M} \sum_{k=M+1}^{2M} \mathbb{I}_{\{\|\alpha_k^{(i)} - \alpha_k^{(j)}\|_{L^2}\}} \geq \mu_2^{2\theta} \frac{4}{M} \geq c^{2\theta} 2^{-2\theta} M^{-2\theta},$$

where the last two inequalities are from (5.58). Therefore, we can take $r = \frac{1}{2} \sqrt{c^{2\theta} 2^{-2\theta} M^{-2\theta}}$. To determine $R$, we turn to bound $D_{kl}(P_i||P_j)$ where $\{P_i\}_{i=1}^L$ are the joint probability distributions of $(X, Y)$ with $Y = (X, \beta_i)_{L^2} + \epsilon$ and $\epsilon \sim N(0, \sigma^2)$. Then, using lemma 14 and (5.59) yields

$$D_{kl}(P_i||P_j) = \frac{1}{2\sigma^2} \|L_{C^2}^{1/2} (\beta_i - \beta_j)\|_{L^2}^2.$$
where the last two inequalities are also due to (5.58). Thus, we can take \( R = \frac{2}{\sigma^2 c^{2 \theta}} M^{-\frac{2 \theta}{p}} \).

Finally, let \( r = \frac{1}{2} \sqrt{c^{2 \theta} 2^{-\frac{2 \theta}{p}} M^{-\frac{2 \theta}{p}}} \), \( R = \frac{2}{\sigma^2 c^{2 \theta}} M^{-\frac{2 \theta}{p}} \) in Lemma \( \text{13} \) with \( L \geq \exp (M/8) \) and \( M = \lceil aN^{\frac{p}{p+2\theta}} \rceil \). Then there holds

\[
\inf \lim \inf \sup_{N \to \infty} \mathbb{P} \left\{ \mathcal{R}(\hat{\beta}_S) - \mathcal{R}(\beta_0) \geq \frac{c^{2 \theta}}{4} 2^{-\frac{2 \theta}{p}} a^{-\frac{2 \theta}{p}} N^{-\frac{2 \theta}{p+2 \theta}} \right\} = 1 - a^{-\frac{2 \theta}{p}} \frac{16}{\sigma^2 c^{2 \theta}}
\]

and then

\[
\lim_{a \to \infty} \inf \lim \inf \sup_{N \to \infty} \mathbb{P} \left\{ \mathcal{R}(\hat{\beta}_S) - \mathcal{R}(\beta_0) \geq \frac{c^{2 \theta}}{4} 2^{-\frac{2 \theta}{p}} a^{-\frac{2 \theta}{p}} N^{-\frac{2 \theta}{p+2 \theta}} \right\} = 1.
\]

Taking \( \gamma = \frac{c^{2 \theta}}{4} 2^{-\frac{2 \theta}{p}} a^{-\frac{2 \theta}{p}} \), we have

\[
\lim \inf \lim \inf \sup_{\gamma \to 0} \mathbb{P} \left\{ \mathcal{R}(\hat{\beta}_S) - \mathcal{R}(\beta_0) \geq \gamma N^{-\frac{2 \theta}{p+2 \theta}} \right\} = 1.
\]

This completes the proof of Theorem \( \text{1} \). \( \square \)

**Appendix A.**

The lemma below provides Bernstein’s inequality for the sum of self-adjoint random operators on a Hilbert space. The proof of this lemma is given in \( \text{34} \).

**Lemma 16.** Consider a finite sequence \( \{\eta_i\}_{i \geq 1} \) of independent random self-adjoint operators on a separable Hilbert space \( H \). Assume that

\[
E[\eta_i] = 0 \quad \text{and} \quad ||\eta_i|| \leq L \quad \text{for each} \quad i
\]
Define the random operator \( \eta := \sum_{i \geq 1} \eta_i \). Suppose there are constant \( v, d > 0 \) such that 
\[ \|E[\eta^2]\| \leq v \text{ and } \text{trace}(E[\eta^2]) \leq vd. \]
Then for all \( s \geq 0 \),
\[ P(|\eta| \geq s) \leq 1 + 6 \left( \frac{v}{s^2} + \frac{L}{3s} \right)^2 d \exp \left( -\frac{s^2}{2(v + Ls/3)} \right) \]

We next give the proof of Lemma 1.

Proof of Lemma 1 Using the Courant-Fischer mini-max principle Theorem (see, for example, Theorem 4.2.7 in [23]), there holds
\[
\rho_k(L_A^{1/2}L_BL_A^{1/2}) = \max_{v_1, \ldots, v_k \in H} \min_{v \in \text{span}\{v_1, \ldots, v_k\}} \frac{\langle L_A^{1/2}L_BL_A^{1/2}v, v \rangle_H}{\|v\|_H^2} \\
= \max_{v_1, \ldots, v_k \in H} \min_{v \in \text{span}\{v_1, \ldots, v_k\}} \frac{\langle L_BL_A^{1/2}v, L_A^{1/2}v \rangle_H}{\|L_A^{1/2}v\|_H^2} \|L_A\| \\
\leq \max_{L_A^{1/2}v_1, \ldots, L_A^{1/2}v_k \in H} \min_{v \in \text{span}\{L_A^{1/2}v_1, \ldots, L_A^{1/2}v_k\}} \frac{\langle L_BL_A^{1/2}v, L_A^{1/2}v \rangle_H}{\|L_A^{1/2}v\|_H^2} \|L_A\| \\
\overset{(\dagger)}{\leq} \max_{e_1, \ldots, e_k \in H} \min_{e \in \text{span}\{e_1, \ldots, e_k\}} \frac{\langle L_BE, e \rangle_H}{\|e\|_H^2} \|L_A\| = \rho_k(L_B)\|L_A\|,
\]
where \( \{v_1, \ldots, v_k\} \) and \( \{e_1, \ldots, e_k\} \) are two groups of \( k \) linearly independent elements in \( H \).
Inequality (\dagger) uses the fact that \( L_A^{1/2}v_1, \ldots, L_A^{1/2}v_k \) are linearly independent which is deduced from the assumption \( \text{ran}(L_A^{1/2}) = H \). The proof is then finished. \( \square \)

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