Capacity Preserving Operators

Jonathan Leake

Department of Mathematics
UC Berkeley

IML, 2018
Our Goal

Definition

For $p \in \mathbb{R}[x] \equiv \mathbb{R}[x_1, \ldots, x_n]$, we say $p$ is real stable whenever $p(x) \neq 0$ for $x \in \mathbb{H}^n_+$.

Main goal: obtain bounds on combinatorial info via real stable polynomials which encode that info.

- Matching polynomial — matchings of a graph
- Product of linear forms — permanent of a matrix

objects $\rightarrow$ multivariate polynomials $\rightarrow$ apply operators $\rightarrow$ information

Can we use and/or emulate the Borcea-Brändén characterization to transfer quantitative information about coefficients/evaluations?
Two Motivating Examples

(BB) Multivariate matching polynomial = MAP(\(\prod_{(i,j) \in E}(1 - x_i x_j)\))
- \((1 - x_i x_j)\) is real stable, products are real stable.
- MAP = “Multi-Affine Part” preserves real-stability.
- Plug in \(x\) for all variables \(\rightarrow\) univariate matching poly is real-rooted.
- What about bounds on coefficients?

(Gurvits) Doubly stochastic matrix \(M \rightarrow \prod_{r \in \text{rows}} r \cdot x\)
- \(p_M(x) := \prod_i \sum_j m_{ij} x_j\) is real stable.
- (coefficient of \(x_1 x_2 \cdots x_n\)) = \(\partial_{x_1} \cdots \partial_{x_n} p\) is the permanent of \(M\).
- We can obtain a bound on the permanent by analyzing \(\partial_{x_k}\).

Both cases: want to obtain bounds on how certain linear operators affect the coefficients of a real stable polynomial.
An Explicit Example: Schrijver’s Inequality

Let $G$ be a $d$-regular bipartite graph with $2n$ total vertices.

Bipartite adjacency matrix, $M$:

$$
\begin{bmatrix}
1 & 1 & 0 & 1 \\
1 & 0 & 1 & 1 \\
0 & 1 & 1 & 1 \\
1 & 1 & 1 & 0
\end{bmatrix}
$$

$\#$ perfect matchings $=$ permanent

$$p_M = (x_1 + x_2 + x_4)(x_1 + x_3 + x_4)(x_2 + x_3 + x_4)(x_1 + x_2 + x_3)$$

- $\text{pm}(G) = \text{per}(M) = \partial_{x_1} \cdots \partial_{x_n} p_M$
- Schrijver: $\text{pm}(G) \geq \left( \frac{(d-1)^{d-1}}{d^{d-2}} \right)^n$
- $\#$ $k$-edge matchings $\sim \sum_{S \in \binom{[n]}{k}} \partial_x^S p_M(1) \geq ?$
Gurvits’ Method

Throughout: $x$ is a vector, $x > 0$ is element-wise, $x^\alpha := \prod_{k=1}^{n} x_k^{\alpha_k}$, etc.

**Definition (Gurvits)**

For $p \in \mathbb{R}_+[x]$ and $\alpha \in \mathbb{R}_+^n$, we define $\text{Cap}_\alpha(p) := \inf_{x > 0} \frac{p(x)}{x^\alpha}$.

**Theorem (Gurvits)**

Let $p \in \mathbb{R}_+[x] \equiv \mathbb{R}_+[x_1, \ldots, x_n]$ be $n$-homogeneous and real stable. Then:

$$\text{Cap}_{(1^{n-1})}(\partial_{x_k} p |_{x_k=0}) \geq \left( \frac{n-1}{n} \right)^{n-1} \text{Cap}_{(1^n)}(p)$$

- Gives a simple proof of the van der Waerden lower bound for the permanent of a doubly stochastic matrix ($\text{per}(M) \geq \frac{n!}{n^n}$)
- Essentially implies Schrijver’s perfect matching inequality
- Can be interpreted as a capacity preservation result for $\partial_{x_k} |_{x_k=0}$

Can we generalize this result to other operators?
General Form of the Method

Fix \( p \in \mathbb{R}_+^{\lambda}[x] \) (degree at most \( \lambda_k \) in \( x_k \)) and linear \( T : \mathbb{R}_+^{\lambda}[x] \rightarrow \mathbb{R}_+^{\gamma}[x] \).

\[
\text{Cap}_\beta(T[p]) \geq c_{T,\alpha,\beta,\lambda} \cdot \text{Cap}_\alpha(p)
\]

What we need to happen:

- Series of linear operators which lead to a desired quantity.
- Capacity of starting polynomial is easy to compute.
- If \( T \) is a functional and \( \beta = \emptyset \), then \( T[p] = \text{Cap}_\beta(T[p]) \).

Bounds are achieved when \( p \) is real stable and \( T \) preserves real stability: can theoretically lower-bound any quantity which is derivable in this way.
First Idea: Inner Product Bounds

Certain differential operators can be interpreted via (real) inner products.

- E.g., \( \text{per}(M) = q(\partial_x) p_M(x) \big|_{x=0} \) for \( q = x_1 \cdots x_n \).
- Can we obtain/utilize bounds on inner products of polynomials?

### Definition

For \( p, q \in \mathbb{R}^{\lambda}[x] \), define \( \langle p, q \rangle^{\lambda} := \sum_{0 \leq \mu \leq \lambda} \binom{\lambda}{\mu}^{-1} p_\mu q_\mu \).

### Observation

\( \text{per}(M) = \partial_{x_1} \cdots \partial_{x_n} p_M = \langle x_1 \cdots x_n, p_M \rangle^{\lambda} \cdot \prod_k \lambda_k \)

Why this inner product?

- Practical — inductive structure leads to the bounds we want
- Useful — amenable to BB-style ideas (similar to apolarity form)
- Natural — unique \( SO_2^n \)-invariant bilinear form (up to degree)
First Idea: Inner Product Bounds

Theorem (Anari-Gharan, 2017)

For real stable multiaffine $p, q \in \mathbb{R}_+[x]$ and $\alpha \in \mathbb{R}^+_n$, we have:

$$\langle p, q \rangle^{(1^n)} \geq \alpha^\alpha (1 - \alpha)^{1-\alpha} \text{Cap}_\alpha(p) \text{Cap}_\alpha(q)$$

Proof: Strongly Rayleigh inequalities.

Theorem (Anari-Gharan, 2017)

For real stable $p, q \in \mathbb{R}_+[x]$ and $\alpha \in \mathbb{R}^+_n$, we have:

$$q(\partial_x)p(x)|_{x=0} \geq e^{-\alpha} \alpha^\alpha \text{Cap}_\alpha(p) \text{Cap}_\alpha(q)$$

Already: $\text{per}(M) \geq e^{-(1^n)(1^n)(1^n)} \text{Cap}_{(1^n)}(p_M) = e^{-n} \text{Cap}_{(1^n)}(p_M)$

Lemma (Gurvits)

If $M$ is doubly stochastic, then $\text{Cap}_{(1^n)}(p_M) = 1$. 
Can we do better if we know the degree of the polynomial?

**Theorem**

For real stable \( p, q \in \mathbb{R}_+^\lambda[x] \) and \( \alpha \in \mathbb{R}^n \), we have:

\[
\langle p, q \rangle^\lambda \geq \frac{\alpha^\lambda (\lambda - \alpha)^{\lambda - \alpha}}{\lambda^\lambda} \text{Cap}_\alpha(p) \text{Cap}_\alpha(q)
\]

Proof: Capacity and \( \langle \cdot, \cdot \rangle \) play nice with polarization; follows from the prior multiaffine result.

So: \( \text{per}(M) = \langle x_1 \cdots x_n, p_M \rangle^\lambda \cdot \prod_k \lambda_k \geq \left( \frac{\lambda - 1}{\lambda} \right)^{\lambda - 1} \text{Cap}_{(1^n)}(p_M) \)

- Limits to the \( e^{-n} \) bound as \( \lambda \to \infty \).
- Looks similar to Gurvits’ theorem, but not quite as strong/general.
- Easy to achieve Schrijver’s inequality as a corollary.
Proof of Schrijver’s Inequality

$G$ is a $d$-regular bipartite graph on $2n$ vertices, with incidence matrix $M$.

$$
M = \begin{bmatrix}
1 & 1 & 0 & 1 \\
1 & 0 & 1 & 1 \\
0 & 1 & 1 & 1 \\
1 & 1 & 1 & 0
\end{bmatrix}
$$

$$
p_M = (x_1 + x_2 + x_4)(x_1 + x_3 + x_4)(x_2 + x_3 + x_4)(x_1 + x_2 + x_3)
$$

Recall: $pm(G) = \text{per}(M) \geq \left(\frac{\lambda - 1}{\lambda}\right)^{\lambda - 1} \text{Cap}_{(1^n)}(p_M)$

- $d$-regularity implies $\frac{1}{d}M$ is doubly stochastic
- Lemma implies $\text{Cap}_{(1^n)}(p_M) = d^n \cdot \text{Cap}_{(1^n)}(p_{\frac{1}{d}M}) = d^n$
- $d$-regularity implies $p_M$ is of degree $\lambda = (d, d, \ldots, d)$
- $\left(\frac{\lambda - 1}{\lambda}\right)^{\lambda - 1} = \prod_{k=1}^{n} \left(\frac{d-1}{d}\right)^{d-1} = \left(\frac{d-1}{d}\right)^{n(d-1)}$

Therefore: $pm(G) \geq \left(\frac{d-1}{d}\right)^{n(d-1)} \cdot d^n = \left(\frac{(d-1)^{d-1}}{d^{d-2}}\right)^n$
Other Bounds on Matchings

What about non-bipartite $G$? Via the matching polynomial?

Unfortunate problem: matching polynomial does not have non-negative coefficients, and this is essentially unavoidable for non-bipartite $G$.

What about counting $k$-matchings for bipartite $G$?

**Theorem (Csikvári, 2014)**

Let $G$ be a $d$-regular bipartite graph with $2n$ vertices. Then:

$$\mu_k(G) \geq \binom{n}{k} d^k \left( \frac{nd - k}{nd} \right)^{nd-k} \left( \frac{n}{n-k} \right)^{n-k}$$

- Reduces to Schrijver's inequality for $k = n$ (here $0^0 = 1$).
- Implies Friedland’s lower matching conjecture.
- Actually able to bound $k$-matchings for biregular bipartite graphs.
- Can prove these bounds using capacity-preservers.
The Symbol for Capacity

BB: stability properties shared between operator and its symbol.

Recall: $\langle p, q \rangle^\lambda = \sum_{\mu \leq \lambda} \binom{\lambda}{\mu}^{-1} p_\mu q_\mu$

**Definition**

Given linear $T : \mathbb{R}^\lambda[x] \to \mathbb{R}^\gamma[x]$, we define $\text{Symb}(T) \in \mathbb{R}^{(\lambda, \gamma)}[z, x]$ via:

$$T[p](x) = \langle \text{Symb}(T)(z, x), p(z) \rangle^\lambda$$

**Lemma**

For a given linear operator $T : \mathbb{R}^\lambda[x] \to \mathbb{R}^\gamma[x]$, we have:

$$\text{Symb}(T)(z, x) = T \left[ (1 + xz)^\lambda \right] = \sum_{\mu \leq \lambda} \binom{\lambda}{\mu} z^\mu T(x^\mu)$$

Is there a similar operator-symbol correspondence for capacity?
From Inner Products to Operators

Theorem

For real stable \( p, q \in \mathbb{R}_+^\lambda [x] \) and \( \alpha \in \mathbb{R}_+^n \), we have:

\[
\langle p, q \rangle^\lambda \geq \frac{\alpha^\lambda (\lambda - \alpha)^{\lambda - \alpha}}{\lambda^\lambda} \text{Cap}_\alpha (p) \text{Cap}_\alpha (q)
\]

For \( T \) and \( p \) with desired properties, and fixed \( x > 0 \):

\[
T[p](x) = \langle \text{Symb} (T)(z, x), p(z) \rangle^\lambda
\geq \frac{\alpha^\lambda (\lambda - \alpha)^{\lambda - \alpha}}{\lambda^\lambda} \text{Cap}_\alpha (p) \text{Cap}_\alpha (\text{Symb} (T)(\cdot, x))
\]

Divide by \( x^\beta \) and take \( \inf_{x > 0} \) on both sides (recall \( \text{Cap}_\beta (p) := \inf_{x > 0} \frac{p(x)}{x^\beta} \)):

\[
\text{Cap}_\beta (T[p]) \geq \frac{\alpha^\lambda (\lambda - \alpha)^{\lambda - \alpha}}{\lambda^\lambda} \text{Cap}_\alpha (p) \text{Cap}_{(\alpha, \beta)} (\text{Symb} (T))
\]
**Theorem**

Let $T : \mathbb{R}_+^\lambda [x] \rightarrow \mathbb{R}_+^\gamma [x]$ be such that $\text{Symb}(T)(z, x) \in \mathbb{R}_+^{(\lambda, \gamma)}[z, x]$ is real stable in $z$ for every $x > 0$. For any real stable $p \in \mathbb{R}_+^\lambda [x]$:

$$\text{Cap}_\beta(T[p]) \geq \left[ \frac{\alpha^\alpha (\lambda - \alpha)^{\lambda-\alpha}}{\lambda^\lambda} \text{Cap}_{(\alpha, \beta)}(\text{Symb}(T)) \right] \text{Cap}_\alpha(p)$$

Moreover, this bound is tight for any fixed $\alpha, \beta, \text{and } T$.

Tightness is demonstrated by considering $p(x) = (xy + 1)^\lambda$ for fixed $y > 0$.

**Corollary**

The above theorem holds for any operator preserving real stability and non-negative coefficients, which has image of dimension greater than 2.
Application: Gurvits’ Theorem

**Theorem (Gurvits)**

Let \( p \in \mathbb{R}_+[x] \equiv \mathbb{R}_+[x_1, \ldots, x_n] \) be \( n \)-homogeneous and real stable. Then:

\[
\text{Cap}_{(1^{n-1})}(\partial_{x_k} p|_{x_k=0}) \geq \left( \frac{n-1}{n} \right)^{n-1} \text{Cap}_{(1^n)}(p)
\]

Recall: \( \text{Cap}_\beta(T(p)) \geq \left[ \frac{\alpha^\alpha (\lambda - \alpha)^{\lambda - \alpha}}{\lambda^\lambda} \text{Cap}_{(\alpha, \beta)}(\text{Symb}(T)) \right] \text{Cap}_\alpha(p) \)

- \( \lambda = (n, \ldots, n), \alpha = (1^n), \beta = (1^{n-1}) \rightarrow \frac{\alpha^\alpha (\lambda - \alpha)^{\lambda - \alpha}}{\lambda^\lambda} = \left( \frac{(n-1)^{n-1}}{n^n} \right)^n \)
- \( \text{Symb}(\partial_{x_k}|_{x_k=0}) = \partial_{x_k}(xz + 1)^\lambda|_{x_k=0} = \lambda_k z_k (xz + 1)^\lambda' \)
- \( \text{Cap}_{(1^n, 1^{n-1})}(\lambda_k z_k (xz + 1)^\lambda') = n \left( \frac{n^n}{(n-1)^{n-1}} \right)^{n-1} \)

Therefore:

\[
\frac{\alpha^\alpha (\lambda - \alpha)^{\lambda - \alpha}}{\lambda^\lambda} \text{Cap}_{(\alpha, \beta)}(\text{Symb}(T)) = \left( \frac{n-1}{n} \right)^{n-1}
\]
Theorem (Csikvári, 2014)

Let $G$ be a $d$-regular bipartite graph with $2n$ vertices. Then:

$$\mu_k(G) \geq \left(\begin{array}{c} n \\ k \end{array}\right) d^k \left(\frac{nd - k}{nd}\right)^{nd-k} \left(\frac{n}{n - k}\right)^{n-k}$$

Recall: $\text{Cap}_\beta(T(p)) \geq \left[\frac{\alpha^\alpha(\lambda - \alpha)^{\lambda - \alpha}}{\lambda^\lambda} \text{Cap}_{(\alpha,\beta)}(\text{Symb}(T))\right] \text{Cap}_\alpha(p)$

- $M$ is bipartite adjacency matrix, $p_M$ is associated product of linears.
- $d$-regularity implies $\mu_k(G) = d^{k-n} \sum_{S \in \binom{[n]}{k}} \partial_x^S p_M(1) =: d^{k-n} T(p_M)$
- $d$-regularity implies $\text{Cap}_{1^n}(p_M) = d^n$
- $d$-regularity implies $\lambda = (d, ..., d)$
- $\alpha = (1^n)$ implies $\frac{\alpha^\alpha(\lambda - \alpha)^{\lambda - \alpha}}{\lambda^\lambda} = \frac{(d-1)^{nd-n}}{d^{nd}}$
- $\beta = \emptyset$ implies $\text{Cap}_\beta(T(p_M)) = T(p_M)$
Application: Csikvári’s Theorem (continued)

- \( \text{Symb}(T) = \sum_{S \in \binom{[n]}{k}} \partial_x^S (xz + 1)^\lambda \bigg|_{x=1} = \sum_{S \in \binom{[n]}{k}} d^k z^S (z + 1)^{\lambda - S} \)

Lemma

If \( p \in \mathbb{R}_+[x] \equiv \mathbb{R}_+[x_1, \ldots, x_n] \) is symmetric, then:

\[
\text{Cap}_{(t,\ldots,t)}(p) = \text{Cap}_{nt}(p(x_0, \ldots, x_0))
\]

- \( \text{Symb}(T) \) is symmetric:

\[
\text{Cap}_{(1^n)} \left[ \sum_{S \in \binom{[n]}{k}} d^k z^S (z + 1)^{\lambda - S} \right] = \text{Cap}_n \left[ \binom{n}{k} d^k z_0^k (z_0 + 1)^{dn-k} \right]
\]

- Easier: \( \text{Cap}_n \left[ \binom{n}{k} d^k z_0^k (z_0 + 1)^{dn-k} \right] = \binom{n}{k} d^k \frac{(nd-k)^{nd-k}}{(n-k)^{n-k} (nd-n)^{nd-n}} \)
Further Questions

Applications of capacity-preservers, beyond differential operators?

Can we get similar bounds based only on the total degree of a given homogeneous polynomial?

$SO_n$-invariant inner product: $\langle p, q \rangle_{SO_n}^d := \sum_{\mu} \binom{d}{\mu}^{-1} p_{\mu} q_{\mu}$

Conjecture (Gurvits, 2009)

For real stable $d$-homogeneous polynomials $p, q \in \mathbb{R}_+[x]$, we have:

$$\langle p, q \rangle_{SO_n}^d \geq n^{-d} \text{Cap}_\alpha(p) \text{Cap}_\alpha(q)$$

What about similar results for polynomials which take matrices as input?

- Some bound on Frobenius inner product? Some other inner product?
- Possibly related to $SO_n$ inner product above.