Acyclicity for Groups and Vector Spaces ∗†

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Abstract

A matching in an Abelian group $G$ is a bijection $f$ from a subset $A$ to a subset $B$ in $G$ such that $a + f(a) \not\in A$, for all $a \in A$. This notion was introduced by Fan and Losonczy who used matchings in $\mathbb{Z}^n$ as a tool for studying an old problem of Wakeford concerning canonical forms for symmetric tensors. The notion of acyclic matching property was provided by Losonczy and it was proved that torsion-free groups admit this property. In this paper, we introduce a duality of acyclic matching as a tool for classification of some Abelian groups; moreover, we study matchings for vector spaces and give a connection between matchings in groups and vector spaces. Our tools mix additive number theory, combinatorics and algebra.

1 Introduction

Let $G$ be a group and $A$ and $B$ be two non-empty subsets of $G$. If $f : A \to B$ is a matching, we define $m_f : G \to \mathbb{Z} \cup \{\infty\}$ by $m_f(x) = \#\{a \in A : a + f(a) = x\}$. A matching $f$ is called acyclic if for any matching $g : A \to B$ with $m_f = m_g$, we have $f = g$. A group $G$ possesses the finite matching property if for every pair $A$ and $B$ of non-empty finite subsets satisfying $\#A = \#B$ and $0 \not\in B$, there exists at least one matching from $A$ to $B$. Furthermore, $G$ possesses the finite acyclic matching property, if for every pair $A$ and $B$ of non-empty finite subsets satisfying $\#A = \#B$ and $0 \not\in B$, there exists at least one acyclic matching from $A$ to $B$. We say $G$ fails to have the acyclic matching property at order $m \in \mathbb{N} \cup \{\infty\}$, if there exist subsets
Let $A$ be a subset of $\mathbb{Z}_p$ and $f : A \to A$ be a bijection, where $p$ is prime. Then $\text{ord}_f(a)$ denotes the minimum positive integer $n$ for which $f^n(a) = a$, where $a \in A$. Losonczy in [10] proved the following theorems:

**Theorem 1.1.** If $G$ is an Abelian group, then $G$ has the finite matching property if and only if $G$ is torsion-free or cyclic of prime order.

**Theorem 1.2.** If $G$ is an Abelian torsion-free group, then $G$ has the finite acyclic matching property.

For more results on matchings see [2, 4, 5, 6, 7, 8, 10, 11 and 13]. Also, the interested reader is referred to [12] to see more details on Wakeford’s problem. Here, we prove the following theorem as a connection between acyclic matching property and its duality.

**Theorem 1.3.** Let $G$ be an Abelian group and $G \neq \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_5$. If $G$ has the finite acyclic matching property, then it fails to have the acyclic matching property at order $m$, for some $m \in \mathbb{N} \cup \{\infty\}$.

2 Acyclic matching in a special case for some cyclic groups

In the following theorem, we show that $\mathbb{Z}_p$ fails to have the acyclic matching property at order $\frac{p-1}{2}$ for $p > 5$. 

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Theorem 2.1. Let $p > 5$ be a prime. Then $\mathbb{Z}_p$ has the cyclic matching property of order $\frac{p-1}{2}$.

Proof. Choose $a$ and $b \in \{1, \ldots, p-1\}$ such that $a \neq b$, \(\left(\frac{a}{p}\right) = \left(\frac{b}{p}\right) = 1\) and \(\left(\frac{a+1}{p}\right) = \left(\frac{b+1}{p}\right) = -1\), where \(\left(\cdot\right)\) denotes Legendre symbol. See [3] for more results on quadratic residue modulo $p$. Set $A := \{n^2 : n \in \mathbb{Z}_p \setminus \{0\}\} \subseteq \mathbb{Z}_p$ and define the bijections $f$ and $g : A \rightarrow A$ by $f(n^2) = an^2$ and $g(n^2) = bn^2$ for any $n \in \mathbb{Z}_p \setminus \{0\}$. Now it is clear that $f$ and $g$ are matchings with $m_f = m_g$. This follows $\mathbb{Z}_p$ fails to have the acyclic matching property at order $\frac{p-1}{2}$ for $p > 5$. $\Box$

In the next section, we generalize Theorem 2.1 without invoking the results on quadratic residue.

3 The acyclicity in general case for some cyclic groups

Let $A \subseteq \mathbb{Z}_p \setminus \{0\}$ and $f : A \rightarrow A$ be a bijection. If $a \in A$, then $B = \{f^i(a) : i \in \mathbb{N}\}$ is invariant under $f$, i.e., $f(B) \subseteq B$. It is clear that there exist $a_1, \ldots, a_n \in A$ such that $A = \{f^i(a_j) : 1 \leq j \leq n, i \in \mathbb{N}\}$. Let $A \subseteq \mathbb{Z}_p \setminus \{0\}$ and $f : A \rightarrow A$ be a matching for which $f^2 \neq id_A$. There exists $a \in A$ with $\text{ord}_f(a) = m > 2$. Now, suppose there exists $b \in A$ such that $b \not\in \{f^i(a) : i \in \mathbb{N}\}$ and define $B = \{f^i(b) : i \in \mathbb{N}\}$. Then

$f|_{A \setminus B} : A \setminus B \rightarrow A \setminus B$ is a matching with $f \circ f|_{A \setminus B} \neq id_{A \setminus B}$.

In the following theorem, we show that the torsion groups $\mathbb{Z}_p$ fail to have the acyclic matching property at order $k$, where $2 < k < p - 2$. It is a remarkable fact
that \( m_f = m_f^{-1} \), where \( f \) is a matching from a non-empty subset \( A \) of a group \( G \) to \( A \) and it is applied in the proof of the next theorem. Also, we already have seen in elementary group theory that if the distinct cyclic representation of a permutation \( \sigma \in S_n \) has a cycle with a length greater than \( 2 \), then \( \sigma \neq \sigma^{-1} \).

**Theorem 3.1.** \( \mathbb{Z}_p \) fails to have the acyclic matching property at order \( k \) for \( 2 < k < p - 2 \), where \( p \) is a prime greater than \( 5 \).

**Proof.** First, we prove that \( \mathbb{Z}_p \) fails to have the acyclic matching property at order \( p - 3 \). Set \( A := \mathbb{Z}_p \setminus \{0, 1, p - 1\} \) and define \( f : A \to A \) by

\[
f = (4 \ p - 4)(5 \ p - 5) \cdots (\frac{p - 1}{2} \ \frac{p + 1}{2})(3 \ p - 3 \ 2 \ p - 2),
\]

where the notation \((a_1 a_2 \ldots a_n)\) denotes the permutation of the set \( \{a_1, a_2, \ldots, a_n\} \) with \( a_i \to a_{i+1} \) \( 1 \leq i \leq n - 1 \) and \( a_n \to a_1 \).

Obviously, \( f \) is a matching. If \( g = f^{-1} \), then \( f \neq g \) and \( m_f = m_g \). Now, we show that \( \mathbb{Z}_p \) fails to have the acyclic matching property at order \( p - 4 \). Let \( A = \mathbb{Z}_p \setminus \{0, 4, p - 4, p - 1\} \). Define \( f : A \to A \) by

\[
f = (5 \ p - 5)(6 \ p - 6) \cdots (\frac{p - 1}{2} \ \frac{p + 1}{2})(3 \ p - 3 \ 2 \ p - 2 \ 1).
\]

Thus \( f \) is a matching. Assume that \( g = f^{-1} \), then \( f \neq g \) and \( m_f = m_g \). This yields \( G \) fails to have the acyclic matching property at order \( p - 3 \) and \( p - 4 \). If we remove the transpositions of the distinct cyclic representation of \( f \), then \( f \) still remains a matching on the omitted subsets and if \( B_i \)'s are the omitted subsets, then \( f\big|_{A \setminus B_i} \neq id_{A \setminus B_i} \), for every \( i, 1 \leq i \leq n \). Suppose \( g_i = \left( f\big|_{A \setminus B_i} \right)^{-1} \), then \( f\big|_{A \setminus B_i} \neq g_i \) and \( m_f\big|_{A \setminus B_i} = m_{g_i} \). Hence \( \mathbb{Z}_p \) fails to have the acyclic matching property at orders \( p - 3 - 2k \) and \( p - 4 - 2k' \) for any \( 1 \leq k \leq \frac{p - 7}{2} \) and \( 1 \leq k' \leq \frac{p - 9}{2} \). Then \( \mathbb{Z}_p \) fails to
have the acyclic matching property at order \( k \), for \( 3 < k < p - 2 \). For \( k = 3 \), define \( A = \{1, 2, 4\} \) and \( f : A \to A \) by \( f = (1 \ 2 \ 4) \). Assuming \( g = f^{-1} \) we get the desired result.

In the last theorem, we showed that \( \mathbb{Z}_p \) fails to have the acyclic matching property at order \( k \), where \( 2 < k < p - 2 \). In the following theorem, we study its behavior at order \( p - 2 \).

**Theorem 3.2.** Let \( A \) be a subset of \( \mathbb{Z}_p \setminus \{0\} \) and \( \#A = p - 2 \). If \( f : A \to A \) is a matching, then \( f^2 = \text{id}_A \).

**Proof.** Let \( f^2 \neq \text{id}_A \) and choose \( a \in A \) and the positive integer \( m > 2 \), such that \( \text{ord}_f(a) = m \). Thus \( f^{i-1}(a) + f^i(a) \notin A \), for each \( i, 1 \leq i \leq m \). It is clear that \( f^{i-1}(a) + f^i(a) \neq f^i(a) + f^{i+1}(a) \) for any \( i, 1 \leq i \leq m \). Suppose \( m \) is even, since \( \#A = p - 2 \) and \( A \cap \{f^{i-1}(a) + f^i(a) : 1 \leq i \leq m\} = \emptyset \), then \( f^{i-1}(a) + f^i(a) = f^{i+1}(a) + f^{i+2}(a) \), for any \( i = 1, \ldots, m - 1 \). Let us \( a + f(a) = n \) and \( f(a) + f^2(a) = n' \), \( n = a + f(a) = f^2(a) + f^3(a) = \cdots = f^{m-1}(a) + f^m(a) = n \) and \( n' = f(a) + f^2(a) = f^3(a) + f^4(a) = \cdots = f^{m-2}(a) + f^{m-1}(a) = f^m(a) + a \). Therefore, \( \sum_{i=1}^{m} f^i(a) = (m + 1)n = (m + 1)n' \), so \( n = n' \) and it is a contradiction. If \( m \) is odd, there exists \( i, 1 \leq i \leq m \) for which \( f^{i-1}(a) + f^i(a) = f^i(a) + f^{i+1}(a) \). Since \( \# \{f^{i-1}(a) + f^i(a) : 1 \leq i \leq m\} \leq 2 \), therefore \( f^2(a) = a \), which is a contradiction.

**Remark 3.3.** There is only one matching \( f \) from \( \mathbb{Z}_p \setminus \{0\} \) to \( \mathbb{Z}_p \setminus \{0\} \). Then \( \mathbb{Z}_p \) does not fail to have the acyclic matching property at order \( p - 1 \).
4 Acyclicity for Abelian torsion-free groups

Theorem 4.1. Let $G$ be an Abelian group. If $G$ is non-divisible and torsion-free, then it fails to have the acyclic matching property at order $\infty$.

Proof. Assume that $n$ is the smallest positive integer such that $2nG \subsetneq G$. We break the proof into the following cases:

Case 1: If $n > 1$, then $G = 2G$. Choose $x \in G \setminus 2ng$ and let $2nG + x = \{2ng + x : g \in G\}$. Define $f, g : 2nG \to 2nG + x$ by $f(2nt) = 2nt + x$ and $g(2nt) = 2nt + (2n + 1)x$, for any $t \in G$. Since $G$ is torsion-free, $f$ and $g$ are matchings.

Choose $g_0 \in G$ such that $x = 2g_0$ and define $A_t = \{y \in G : 4ny + x = t\}$ and $B_t = \{y \in G : 4ny + (2n + 1)x = t\}$, for any $t \in G$. So, $\varphi : A_t \to B_t$ with $\varphi(y) = y - g_0$ is a bijection. Now, since $m_f(t) = \#A_t$ and $m_g(t) = \#B_t$, then $m_f = m_g$ and $G$ fails to have the acyclic matching property.

Case 2: If $n = 1$, choose $x \in G \setminus 2G$. The bijections $f, g : 2G \to 2G + x$ defined by $f(2t) = 2t + x$ and $g(2t) = 2t - 3x$ are matchings. Define $A_t = \{y \in G : 4y + x = t\}$ and $B_t = \{y \in G : 4y - 3x = t\}$, for any $t \in G$. Hence $\varphi : A_t \to B_t$ by $y \mapsto y + x$ is a bijection. This yields that $m_f = m_g$ and $G$ fails to have the acyclic matching property at order $\infty$. \qed

Example 4.2. For any integer $n$, $n\mathbb{Z}$ fails to have the acyclic matching property at order $\infty$.

In the proof of the Theorem 4.4, the following result on divisible torsion-free groups will be used. See [9] for more details.

Theorem 4.3. Let $G$ be an Abelian group. If $G$ is divisible and torsion-free, then it
is a direct-sum of isomorphic copies of $\mathbb{Q}$.

By the aforementioned theorem, we can consider $\mathbb{Q}$ as a subset of a group $G$ under the suitable hypotheses on $G$ and we get the following theorem:

**Theorem 4.4.** Let $G$ be an Abelian group. If $G$ is divisible and torsion-free, then it fails to have the acyclic matching property at order $\infty$.

**Proof.** By Theorem 4.3, $\mathbb{Q}$ is embedded in $G$. Set $A := \{2k : k \in \mathbb{Z}\}$ and $B := \{2k+1 : k \in \mathbb{Z}\}$ as subsets of $\mathbb{Q}$. Define the bijections $f, g : A \to B$ by $f(2n) = 2n + 1$ and $g(2n) = 2n + 5$. It is clear that $f$ and $g$ are matchings. Now, if $x \in G \setminus \{4k + 1 : k \in \mathbb{Z}\}$, then $m_f(x) = m_g(x) = 0$. On the other hand, if $x \in \{4k + 1 : k \in \mathbb{Z}\}$, then $m_f(x) = m_g(x) = 1$ and then, in all cases $m_f = m_g$. □

**Corollary 4.5.** By Theorem 4.1 and Theorem 4.4, if $G$ is an Abelian, torsion-free group then $G$ fails to have the acyclic matching property at order $\infty$.

**Example 4.6.** Two additive groups $\mathbb{R}$ and $\mathbb{Q}$ fails to have the acyclic matching property at order $\infty$.

Now, our result regarding the connection of matching properties for Abelian groups.

**Theorem 4.7.** Suppose $G$ is an Abelian group and $G \neq \mathbb{Z}_2, \mathbb{Z}_3$ and $\mathbb{Z}_5$. If $G$ has the finite acyclic matching property, then it fails to have the acyclic matching property at order $m$ for some $m \in \mathbb{N} \cup \{\infty\}$.

We will see the proof of this theorem in section 7.
5 Linear version of acyclicity for subspaces in a field extension

Let $G$ be an Abelian group and $f$ and $g : A \to B$ be two matchings where $A$ and $B$ are non-empty finite subsets of $G$ and $m_f = m_g$. For any $x \in G$, define $A_f^x = \{a \in A : a + f(a) = x\}$, $A_g^x = \{a \in A : a + g(a) = x\}$, $\mathcal{A}_f = \{A_f^x : m_f(x) \neq 0\}$ and $\mathcal{A}_g = \{A_g^x : m_g(x) \neq 0\}$. It is clear that $\mathcal{A}_f$ and $\mathcal{A}_g$ are distinct decompositions for $A$ and $\#\mathcal{A}_f < \infty$, $\#\mathcal{A}_g < \infty$. Define the function $\varphi : A \to A$ by the following method:

Define $\mathcal{A}_f = \{A_f^x_1, A_f^x_2, \ldots, A_f^x_m\}$. Since $m_f = m_g$, then $\mathcal{A}_g = \{A_g^x_1, A_g^x_2, \ldots, A_g^x_m\}$.

Assume that $a_1 \in A_f^x_1$, choose an arbitrary element $b_1$ of $A_g^x_1$ and put $\varphi(a_1) = b_1$.

If $a_2$ is another element of $A_f^x_1$, choose another arbitrary element $b_2$ of $A_g^x_1 \setminus \{b_1\}$ and put $\varphi(a_2) = b_2$. We can continue this procedure to define $\varphi$ on $A_f^x_1$, and by the similar way, we can define the function $f$ on whole $A$ which is bijective and satisfies $a + f(a) = \varphi(a) + g(\varphi(a))$ for any $a \in A$.

Conversely, assume that $f$ and $g$ are two matchings from $A$ to $B$ and there exists a bijection $\varphi : A \to A$ for which $a + f(a) = \varphi(a) + g(\varphi(a))$, for any $a \in A$. We claim that $m_f = m_g$. Let us $x$ be an arbitrary element of $G$. We have the following cases:

Case 1: If $x \in A$, according to the definition of matching, $m_f(x) = m_g(x) = 0$.

Case 2: If $x \not\in A$, then $m_f(x) = \#\{a \in A : a + f(a) = x\} = \#\{a \in A : \varphi(a) + g(\varphi(a)) = x\} = \#\{a \in A : a + g(a) = x\} = m_g(x)$. So, $m_f = m_g$, as claimed.

So we get the following theorem:

*Theorem 5.1.* Let $A$, $B$, $f$ and $g$ be as above. Then, $m_f = m_g$ if and only if there
exists a bijection $\varphi : A \to A$ such that $a + f(a) = \varphi(a) + g(\varphi(a))$, for any $a \in A$.

By the aforementioned theorem, a natural generalization for the acyclic matching in vector spaces is inspired. To see this concept, we need to present some definitions from [5].

Definition 5.2. Let $K \subseteq L$ be a field extension and $A$, $B$ be $n$-dimensional $K$-subspaces of the field extension $L$. Let $\mathcal{A} = \{a_1, \ldots, a_n\}$ and $\mathcal{B} = \{b_1, \ldots, b_n\}$ be bases of $A$ and $B$, respectively. It is said $\mathcal{A}$ is matched to $\mathcal{B}$ if

$$a_i b \in A \Rightarrow b \in \langle b_1, \ldots, \hat{b}_i, \ldots, b_n \rangle$$

for all $b \in B$ and $i = 1, \ldots, n$, where $\langle b_1, \ldots, \hat{b}_i, \ldots, b_n \rangle$ is the hyperplane of $B$ spanned by the set $\mathcal{B} \setminus \{b_i\}$; moreover, it is stated that $A$ is matched with $B$ if every basis $\mathcal{A}$ of $A$ can be matched to a basis $\mathcal{B}$ of $B$. It is said that $L$ has the linear matching property if, for every $n \geq 1$ and every $n$-dimensional subspaces $A$, $B$ of $L$ with $1 \notin B$, the subspace $A$ is matched with $B$. A strong matching from $A$ to $B$ is a linear isomorphism $\varphi : A \to B$ such that any basis $\mathcal{A}$ of $A$ is matched to the basis $\varphi(\mathcal{A})$ of $B$.

Now, we are in the situation to give the linear version of acyclicity.

Definition 5.3. Let $K \subseteq L$ be a field extension and $A$ and $B$ be two $n$-dimensional $K$-subspaces in $L$ such that $n > 1$. Let $f$, $g : A \to B$ be two strong matchings. We say $f$ is equivalent to $g$ and denote it by $f \sim g$ if there exists a linear isomorphism $\varphi : A \to A$ such that $af(a) = \varphi(a)g(\varphi(a))$, for any $a \in A$; moreover, we state that the strong matching $f : A \to B$ is linear acyclic matching if for any strong matching $g : A \to B$, if $f \sim g$, then $f = cg$, for some $c \in K$. We say $K \subseteq L$ fails to have the linear acyclic matching property at order $m \in \mathbb{N}$, if there exist $K$-subspaces $A$
and $B$ in $L$ and strong matchings $f$ and $g : A \to B$ such that $f \neq g$, $f \sim g$ and $\dim_K A = \dim_K B = m$.

Eliahou and Lecouvey in [5] proved the following theorems. The interested reader is also referred to [1].

**Theorem 5.4.** Let $K \subset L$ be a field extension. Then $K$ has the linear matching property if and only if $L$ contains no proper finite-dimensional extension over $K$.

**Theorem 5.5.** Let $K \subset L$ be a field extension and $A$ and $B$ be $n$-dimensional $K$-subspaces distinct from $\{0\}$. There is a strong matching from $A$ to $B$ if and only if $AB \cap A = \{0\}$. In this case, any isomorphism $\varphi : A \to B$ is a strong matching.

Now, our result regarding the connection of the linear matching properties for field extensions.

**Theorem 5.6.** Let $K \subsetneq L$ be a field extension admit the linear matching property and $\#K \geq 5$. Then it fails to have the linear acyclic matching property at order $m$, for some $m \in \mathbb{N}$.

We will see the proof of this theorem in section 7.

## 6 The linear acyclicity of a given order

In this section, we study the linear acyclicity for finite field extensions.

**Theorem 6.1.** Let $K \subsetneq L$ be a field extension with $[L : K] = n$, $\#K \geq 5$ and no
proper intermediate subfield. Then $K \subset L$ fails to have the linear acyclic matching property at order $m$, for any $1 \leq m \leq (n + 1)/4$.

**Proof.** Choose $m \in \mathbb{N}$ and $a \in L \setminus K$ for which $m \leq (n + 1)/4$. Set $A_m := \langle a, a^3, \ldots, a^{2m-1} \rangle$. Then $A_m \cap A_m^2 = \{0\}$, because $K(a) = L$, for any $A \in L \setminus K$. Using Theorem 5.5, there exists a strong matching $f_m : A_m \to A_m$. Next, set $g_m := f_m^{-1}$. One more time using Theorem 5.5, follows that $f_m^{-1}$ is a strong matching. Now, if $f_m \circ f_m \neq id_{A_m}$, then $f_m \neq g_m$ and $f_m \sim g_m$. On the other hand, if $f_m \circ f_m = id_{A_m}$, choose $c \in K$ such that $c^2 \notin \{0, 1\}$. Set $h_m := c^{-2}g_m$, then $h_m$ is a strong matching. We claim $f_m \sim h_m$. In order to prove, define $\varphi_m := cf_m$. We get $af_m(a) = \varphi_m(a)h_m(\varphi_m(a))$, for any $a \in A_m$. This tells us $f_m \sim h_m$, as claimed. \[ \square \]

**Theorem 6.2.** Let $K \subset L$ be a purely transcendental extension. Then, it fails to have the linear acyclic matching property at order $m$, for any $m \in \mathbb{N}$.

**Proof.** Let $a$ be an element of $L \setminus \{0, 1\}$ and set $A_m := \langle a, a^3, \ldots, a^{2m-1} \rangle$. Then $A_m \cap A_m^2 = \{0\}$ and by Theorem 5.5, there exists a strong matching $f_m$ from $A_m$ to $A_m$. By the same method in the previous theorem we can conclude that $K \subset L$ fails to have the acyclic linear matching property at order $m$, for any $m \in \mathbb{N}$. \[ \square \]

**Remark 6.3.** If a field extension $K \subseteq L$ has no finite-dimensional proper intermediate field extension and $\#K \geq 5$. Then, it fails to have the acyclic matching property at order $m$, for some $m \in \mathbb{N}$.

**Proof.** This is a direct consequence of Theorems 6.1 and 6.2. \[ \square \]
7 Main results

Theorem 7.1. Suppose $G$ is an Abelian group and $G \neq \mathbb{Z}_2, \mathbb{Z}_3$ and $\mathbb{Z}_5$. If $G$ has the finite acyclic matching property, then it fails to have the acyclic matching property at order $m$ for some $m \in \mathbb{N} \cup \{\infty\}$.

Proof. Assume $G$ has the finite acyclic matching property. Then $G$ has the finite matching property. Using Theorem 1.1, $G$ is cyclic of prime order or torsion-free. Invoking Corollary 4.5 and Theorem 2.1, $G$ fails to have the acyclic matching property at order $m$ for some $m \in \mathbb{N} \cup \{\infty\}$.

Theorem 7.2. Let $K \subset \mathbb{L}$ be a field extension admit the linear matching property and $\#K \geq 5$. Then it fails to have the linear acyclic matching property at order $m$, for some $m \in \mathbb{N}$.

Proof. If $K \subset L$ has the linear matching property, so Theorem 5.4 yields it has no proper finite-dimensional $K$-subspaces and by Remark 6.3, $K \subset L$ fails to have the acyclic matching property at order $m$, for some $m \in \mathbb{N}$.

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