KODaira-Saito Vanishing Via Higgs Bundles In Positive Characteristic

DONU ARAPURA

Abstract. The goal of this paper is to give a new proof of a special case of the Kodaira-Saito vanishing theorem for a variation of Hodge structure on the complement of a divisor with normal crossings. The proof does not use the theory of mixed Hodge modules, but instead reduces it to a more general vanishing theorem for semistable nilpotent Higgs bundles, which is then proved by using some facts about Higgs bundles in positive characteristic.

In 1990, Saito [SI, prop 2.33] gave a far reaching generalization of Kodaira’s vanishing theorem using his theory of mixed Hodge modules. A number of interesting applications have been found in recent years; we refer to Popa’s survey [P2] for a discussion of these, and to [P1, SI] for a discussion of the theorem itself. We would like to explain the easiest, but still important, case of the theorem where the mixed Hodge module is of the form \( R^j \ast V \) (in standard “non perverse” notation), where \( V \) is a polarized variation of pure Hodge structure on the complement \( j : U \to X \) of a divisor with simple normal crossings \( D \). Let us also assume that \( V \) has unipotent local monodromies around components of \( D \). By Deligne [D], the flat vector bundle \( O_U \otimes V \) has a canonical extension \( V \) such that the original connection extends to a logarithmic connection

\[
\nabla : V \to \Omega_X^1 (\log D) \otimes V
\]

with nilpotent residues. By a theorem of Schmid [Sc], \( V \) has a filtration \( F \) by subbundles extending the Hodge filtration. This induces a filtration on the de Rham complex

\[
DR(V, \nabla) = V \bigoplus \Omega_X^k (\log D) \otimes V \bigoplus \Omega_X^k (\log D) \otimes V \ldots
\]

which, for us, starts in degree 0. Saito’s theorem tell us that if \( L \) is an ample line bundle and \( i > \dim X \), then

\[
H^i(X, Gr^i_F DR(V, \nabla) \otimes L) = 0
\]

More generally, this holds when \( V \) is replaced by an admissible variation of mixed Hodge structure.

The first goal of this paper is to give a short proof of this special case by reduction to characteristic \( p > 0 \). Illusie [I] had previously given a proof by such a reduction, when \( V \) arises geometrically from a semistable map of varieties. Our proof, however, is different and it works even for nongeometric cases (and this seems important for certain applications, e.g. to Shimura varieties [SU]). We first replace the variation of Hodge structure \( V \) by the vector bundle \( E = Gr_F(V) \), together with the so called Higgs field \( \theta = Gr_F(\nabla) \). By work of Simpson, this pair is semistable. In addition, the rational Chern classes \( c_i(E) = 0 \), and \( \theta \) is nilpotent. In fact, our main result is

\[
\text{Partially supported by the NSF .}
\]
a vanishing theorem for Higgs bundles \((E, \theta)\) satisfying these conditions, regardless of whether or not they arise from variations of Hodge structure. We show that

\[ H^i(X, \text{DR}(E, \theta) \otimes L) = 0 \]

for \(i > \dim X\), where \(\text{DR}(E, \theta)\) is the complex \(\Omega^\bullet_X(\log D) \otimes E\) with \(\theta\) as the differential. In order to prove this, we reduce to where the ground field is the algebraic closure \(\mathbb{F}_p\) of a finite field with large characteristic. We can now avail ourselves of the theory of Higgs bundles in characteristic \(p\) initiated by Ogus and Vologodsky [OV], and extended to the log setting by Schepler [Sr]. When combined with work of Langer [L1], we get an operation \(B\) from the class of Higgs bundles, satisfying the previous conditions, to itself that satisfies a “bootstrapping” inequality

\[ \dim H^i(\text{DR}(E, \theta) \otimes L) \leq \dim H^i(\text{DR}(B(E, \theta)) \otimes L^p) \]

By iterating \(B\), we get a sequence of Higgs bundles \((E_j, \theta_j) = B^j(E, \theta)\). (Such sequences were first considered – for different reasons – by Lan, Shen and Zuo [LSZ1]; they call them Higgs-de Rham sequences.) Langer [L2], and also Lan, Shen, Yang, Zuo [LSYZ], show that this sequence is eventually periodic, and this is the place where we need to work over \(\mathbb{F}_p\). These facts together with Serre vanishing implies the theorem. Replacing Serre vanishing by methods developed by the author [A] yields a stronger result. Suppose \(M\) is a vector bundle such that for some integer \(m > 0\) and some effective divisor \(D'\) supported on \(D\) with coefficients less than \(m\), \(S^m(M)(-D')\) is ample; or more succinctly, suppose that \(M(-\Delta)\) is ample for some fractional \(\mathbb{Q}\)-divisor \(\Delta\) supported on \(D\) (e.g. \(\Delta = \frac{1}{m}D'\)). Then

\[ H^i(X, \text{Gr}_F \text{DR}(V, \nabla) \otimes M) = 0 \]

for \(i \geq \dim X + \text{rank } M\). When \(M\) is a line bundle, this due to Suh [Su], who deduced it from Saito’s theory; also see [W].

As an application of the first vanishing theorem, we prove a Fujita type result that given a complex variation of Hodge structure \(V\) with the same assumptions as above, \(\wedge^a \text{Gr}^b_F(V) \otimes \det(F^{b+1})\) is numerically semipositive for every \(a\) and \(b\) (we define \(\det 0 = \mathcal{O}_X\)). From the strengthened form of the vanishing theorem stated at the end of the previous paragraph, we deduce an extension of the Kollár-Saito vanishing theorem. Suppose that \(Y\) is an arbitrary complex projective variety and \(V\) is a variation of Hodge structure supported on a smooth Zariski open of \(Y\), with quasipotent monodromy at infinity. Then there is a natural extension \(S(V)\) of \(F^{\text{max}}V\) to a coherent sheaf on \(Y\). If \(M\) is an ample vector bundle on \(Y\) then

\[ H^i(Y, S(V) \otimes M) = 0 \]

for \(i \geq \dim Y + \text{rank } M\).

My thanks to Yohan Brunebarbe, Adrian Langer, Mihnea Popa, and Kang Zuo for various comments, and to Junecue Suh for sending me [Su].

1. Statement of the main results

In order to put the main results in context, let us quickly summarize the relevant ideas of Simpson [Si], and their extensions due to Mochizuki [M1][M2]. We wish to point out from the beginning that since we will be applying these to a variation of Hodge structure, we will not need any of the deeper existence theorems from these papers. The only hard result needed is Simpson’s semistability theorem explained
below. Let $X$ be a $d$ dimensional smooth projective variety defined over an algebraically closed field $k$. Fix a reduced effective divisor with simple normal crossings $D$ and an ample line bundle $L$ on $X$. These assumptions will hold throughout the paper. We also assume for the next few paragraphs that the ground field $k = \mathbb{C}$.

We may identify the symmetric space $\text{Gl}(r)/U(r)$ with the space of positive definite hermitian matrices by sending $M \in \text{GL}(r)$ to $M^* M$. Therefore given a local system $V$ on $(X - D)^{an}$ of rank say $r$, a hermitian metric $h$ on it can be viewed as a $\pi_1((X - D)^{an})$-equivariant $C^\infty$ map from the universal cover $(X - D)^{an}$ to $GL(r)/U(r)$. The pair $(V, h)$ is called a tame harmonic bundle if $h$, viewed as a map as above, is harmonic with mild singularities near $D$. Simpson (when $d = 1$) and Mochizuki ($d$ arbitrary) show how to associate to any tame harmonic bundle, a so called parabolic or filtered Higgs bundle. This consists of a holomorphic vector bundle $E$ on $X$, an $O_X$-linear map $\theta : E \to \Omega^1_X(\log D) \otimes E$ such that $\theta^2 = \theta \wedge \theta$ viewed as a section of $H^0(\Omega^2_X(\log D) \otimes E)$ is zero, and certain filtrations along $D$. We will not need to make the last part precise, since these filtrations will be trivial for the cases of interest to us that will be described in the next paragraph.

For us, the key example of a harmonic bundle arises as follows. A polarized complex variation of Hodge structure of weight $n$ on $X - D$ is a local system $V$ on $(X - D)^{an}$ with an indefinite form $Q$ (the polarization) and a bigrading of the associated $C^\infty$ vector bundle $C^\infty_X \otimes \mathbb{C} V = \bigoplus_{p+q=n} V^{pq}$ These are required to satisfy Griffiths transversality (appropriately formulated) and the Hodge-Riemann relations. However, unlike the usual notion, there is no requirement about the existence of a $\mathbb{Z}$ or $\mathbb{Q}$ lattice in $V$. Let us also suppose that the monodromies of the local system about components of $D$ are unipotent. This assumption simplifies the story, and in the geometric case, it is close to automatic; more precisely it can always be achieved by pulling back to a branched cover. After adjusting signs, the polarization determines a positive definite metric $h = \sum (-1)^p Q|_{V^{pq}}$ on $V$ such that the pair is tame harmonic. The associated Higgs bundle is given explicitly by $E \cong \text{Gr}_F(V)$ with Higgs field $\theta = \text{Gr}_F(\nabla) : E \to \Omega^1_X(\log D) \otimes E$ induced by the connection. The unipotency of local monodromies, forces the parabolic structure to be trivial. For our purposes, a Higgs bundle on $(X, D)$ will simply refer to a pair $(E, \theta)$ as above without any parabolic structure.

Suppose that $(E, \theta)$ is a Higgs bundle arising from a complex variation of Hodge structure with unipotent local monodromy. Then it has a number of special features that we need to explain.

1. The Chern classes of $E$, in rational cohomology, all vanish. This is because $c_i(V) = 0$ by [EV1 cor B3] and $c_i(V) = c_i(E)$ since $V$ and $E$ have the same class in the Grothendieck group.
The Higgs bundle is semistable in the sense that $\mu(E') \leq \mu(E) = 0$ for any proper coherent subsheaf $E' \subset E$ stable under $\theta$, where

$$\mu(E) = \frac{\deg E}{\text{rank } E} = \frac{c_1(E) \cdot L^{d-1}}{\text{rank } E}$$

is the slope. We can argue as follows. After replacing $L$ by a power, we can assume that it is very ample. Let $C \subset X$ be a curve given as a complete intersection of general divisors in $|L|$. Then it is enough to show that $\mu(E'|_C) \leq 0$. So it suffices to prove the semistability of $(E, \theta)$ after replacing $X$ by $C$, $V$ by $V|_C$ etc. For curves semistability is proved by Simpson [Si, thm 5]. Simpson proves this more generally for Higgs bundles arising from tame harmonic bundles. In this generality, the parabolic structure may be nontrivial so the definition of semistability is a bit more complicated; however, it reduces to what we gave when the parabolic structure is trivial, as is the case here.

Finally observe that our bundle carries a grading $E = \bigoplus E^p$ such that $\theta(E^p) \subset E^{p-1}$. This implies that $\theta$ is nilpotent in the sense that $E$ carries a filtration $E = N^0 \supset N^1 \ldots$ such that it stable under that action of $\theta$, and the induced action on the associated graded is zero. The length of the shortest such filtration is called the level of nilpotence of $\theta$.

Higgs bundles of this type also arise from certain variations of mixed Hodge structures called admissible variations. We will not recall the precise axioms, which are rather technical, but instead we use a weaker notion. Let us say that a weak complex variation of mixed Hodge structures with unipotent local monodromies around $D$, consists of local system $V$ on $(X-D)^{an}$ with unipotent local monodromy, a filtration $W \subset V$ by sublocal systems, a filtration of the Deligne extension $V$ of $V \otimes \mathcal{O}_{X-D}$ by subbundles $F^\bullet \subset V$ satisfying Griffiths transversality

$$\nabla(F^p) \subset \Omega^1_X(\log D) \otimes F^{p-1}$$

We require that the associated graded with respect to $W$ is a polarizable pure variation of Hodge structure. We can see that $\text{Gr}_F(V)$ is again a Higgs bundle, with a filtration by Higgs bundles induced by $W$.

**Lemma 1.1.** An extension of two nilpotent, semistable Higgs bundles with vanishing Chern classes has the same property.

**Proof.** The nilpotence and vanishing of Chern classes are immediate from the definition and the Whitney sum formula. Given an exact sequence

$$0 \to (E_1, \theta_1) \to (E, \theta) \to (E_2, \theta_2) \to 0$$

of Higgs bundles and a sub Higgs sheaf $(E', \theta') \subset (E, \theta)$

$$\deg E' = \deg E' \cap E_1 + \deg \text{im}(E' \to E_2) \leq 0$$

$$\Box$$

**Corollary 1.2.** The Higgs bundle $\text{Gr}_F(V)$ associated to a weak complex variation of mixed Hodge structures is nilpotent, and semistable with vanishing Chern classes.

Given a Higgs bundle $(E, \theta)$ on $(X, D)$, since $\theta^2 = 0$, we get a “de Rham” complex

$$\text{DR}(E, \theta) = E \xrightarrow{\theta} \Omega_X^1(\log D) \otimes E \xrightarrow{\theta} \Omega_X^2(\log D) \otimes E \ldots$$
If this arises from a weak complex variation of mixed complex of Hodge structure \( V \) as above, \( \text{DR}(E, \theta) \) can be identified with the associated graded \( \text{Gr}_F \text{DR}(V, \nabla) \) of the de Rham complex with respect to the filtration
\[
\text{Gr}_F \text{DR}(V, \nabla) = \text{Gr}_p \text{DR}(V, \nabla) = \Omega^1_X (\log D) \otimes F^{p-1} V \ldots
\]

Finally, observe that the notions of Higgs bundle, semistability, nilpotence and DR make sense over any field \( k \). Following [DI], we say that the pair \((X, D)\) is liftable modulo \( p^2 \) if it lifts to a smooth scheme over \( \text{Spec} W_2(k) \) with a relative normal crossing divisor.

**Theorem 1.** Let \((X, D, L)\), as above, be defined over an algebraically closed field \( k \). Let \((E, \theta)\) be a nilpotent semistable Higgs bundle on \((X, D)\) with vanishing Chern classes in \( H^*(X_{et}, \mathbb{Q}_p) \). Suppose that either
\begin{itemize}
  \item[(a)] \( \text{char } k = 0 \), or \( \text{char } k = p > 0 \), \( (X, D) \) is liftable modulo \( p^2 \), \( d + \text{rank } E < p \).
\end{itemize}

Then
\[
H^i(X, \text{DR}(E, \theta) \otimes L) = 0
\]
for \( i > d \).

We should remark that we only really need to assume that \( c_1(E) = 0 \) and \( c_2(E) \cdot L^{d-2} = 0 \).

**Corollary 1.3** (Saito). If \( V \) is a weak complex variation of Hodge mixed structure on \((X, D)\) with unipotent local monodromies around \( D \), then
\[
H^i(X, \text{Gr}_F \text{DR}(V, \nabla) \otimes L) = 0
\]
for \( i > d \).

The next result is a refinement of the semipositivity results of Fujita [F], Kawamata [Kw] and Peters [Pa]. It also overlaps with extensions due to Fujino, Fujisawa and Saito [FFS] and Brunebarbe [B]; see remark 3.2 for further discussion. Recall that a vector bundle on \( X \) is nef, or numerically semipositive, if any quotient line bundle of the pullback of \( E \) to a curve has nonnegative degree. Numerous other characterizations can be found in [L2]. As usual, we let \( \text{det}(E) = \wedge^{\text{rank } E} E \) if \( E \) is a nonzero vector bundle. For notational convenience, we define \( \text{det}(0) = \mathcal{O}_X \).

**Theorem 2.** Suppose that \( V \) is a weak complex variation of mixed Hodge structure on \((X, D)\) with unipotent local monodromies around \( D \). Then \( \wedge^a \text{Gr}^b_F(V) \otimes \text{det}(F^{b+1}) \) is nef for every \( a \) and \( b \).

**Corollary 1.4.** Let \( F^{\text{max}} \subset V \) be the smallest nonzero Hodge bundle, then \( F^{\text{max}} \) is nef. For every \( b \), \( \text{det } F^b \) is nef.

Suh [Su] considered a generalization of Saito’s theorem where the ampleness of \( L \) is relaxed. We will prove a version of this as well. Let \( \Delta \) be a \( \mathbb{Q} \)-divisor and let \( m \) be the least common multiple of the denominators of the coefficients of \( \Delta \). We will say that it is effective and fractional if the coefficients lie in \([0, 1)\). Given a vector bundle \( M \), let us say that \( M(-\Delta) \) is ample if \( S^m(M)(-m\Delta) \) is ample in the usual sense.
Theorem 3. Make the same assumptions as in theorem 1. Let $M$ be a vector bundle when $\text{char } k = 0$, or a line bundle when $\text{char } k = p$, such that $M(-\Delta)$ is ample for some fractional effective $\mathbb{Q}$-divisor supported on $D$. Then

$$H^i(X, \text{DR}(E, \theta) \otimes M(-D)) = 0$$

for $i \geq d + \text{rank } M$.

While clearly the last theorem implies the first, we prefer to state and prove them separately. The proof of the first is somewhat easier and it serves as a model for the last.

From the previous theorem, we can deduce a Kollár-Saito type vanishing theorem for arbitrary varieties. Let $Y$ be possibly singular complex projective variety, and let $V$ be a complex variation of Hodge structures on smooth Zariski open $U \subset Y$ with quasipotent monodromies at infinity. The last condition means that every restriction of the pull back of $V$ to a punctured disk has quasipotent monodromy. For example, quasipotence holds when $V$ is of geometric origin. Choose a desingularization $\pi : X \rightarrow Y$ which is an isomorphism over $U$ and such that the preimage $D = \pi^{-1}(Y - U)$ has simple normal crossings. Let $j : U \rightarrow X$ denote the inclusion. Let

$$\nabla : V \rightarrow \Omega^1_X(\log D) \otimes V$$

be the Deligne extension of $\mathcal{O}_U \otimes V$ such that eigenvalues of the residues lie in $(-1, 0]$. We define $S'(V) = \omega_X \otimes (V \cap j_* F^\text{max}(\mathcal{O}_U \otimes V))$ and $S(V) = \pi_* S'(V)$. This coincides with the similarly named object defined by Saito in [S2] because of [loc. cit., (3.1.1), thm 3.2]. Consequently, $S(V)$ is independent of the choice of $X$, although this is easy to check directly as well.

Theorem 4. Let us keep the notation and assumptions of the previous paragraph. Let $M$ be an ample vector bundle on $Y$, then

(a) $H^i(X, S'(V) \otimes \pi^* M) = 0$ for $i \geq \dim X + \text{rank } M$.

(b) $R^i \pi_*(S'(V)) = 0$ for $i > 0$.

(c) $H^i(Y, S(V) \otimes M) = 0$ for $i \geq \dim X + \text{rank } M$.

When $M$ is a line bundle, this is due to Kollár [Ko] when $V$ is of geometric origin and Saito [S2] in general.

2. Proof of theorem

Before giving the proof, we need to explain the basic tools. Let $(X, D, L)$ be as in the previous section, but now defined over an arbitrary algebraically closed field $k$.

Lemma 2.1. If $L$ is an ample line bundle, then

$$H^i(X, \text{DR}(E, \theta) \otimes L^N) = 0$$

for $i > d$ and $N \gg 0$.

Proof. This follows from Serre vanishing and the spectral sequence

$$E_1^{ab} = H^b(\Omega^a_X(\log D) \otimes E \otimes L^N) \Rightarrow H^{a+b}(\text{DR}(E, \theta) \otimes L^N).$$
A vector bundle with an integrable logarithmic connection
\[ \nabla : V \rightarrow \Omega^1_X(\log D) \otimes V \]
defined in the usual way, will simply be referred to as a flat bundle below. If \((V, \nabla)\) is a flat bundle and \(F^*\) a filtration on \(V\) by subbundles satisfying Griffiths transversality \(\nabla(F^p) \subset F^{p-1}\), then \((\text{Gr}_F(V), \text{Gr}_F(\nabla))\) is a Higgs bundle exactly as above. We say that that the connection \((V, \nabla)\) is semistable if \(\mu(V') \leq \mu(V)\) for every \(\nabla\)-stable saturated subbundle \(V' \subseteq V\). (Saturation means that \(V/V'\) is torsion free.) We have the following fact:

**Theorem 5** (Langer [L1, thm 5.5]). If \((V, \nabla)\) is a semistable flat bundle, there exists a canonical filtration \(V = F^0 \subset V \supseteq F^1 \subset V \ldots\) satisfying Griffiths transversality such that \(\Lambda(V, \nabla) = (\text{Gr}_{F, \text{can}}(V), \text{Gr}_{F, \text{can}}(\nabla))\) is a semistable Higgs bundle.

The second ingredient is the positive characteristic version of Higgs bundle theory due to Ogus-Vologodsky [OV] and Schepler [Sr]. Assume now that the characteristic is \(p > 0\). Let \(F \Phi : X \rightarrow X\) be the absolute Frobenius given by the identity on the space and \(p\)th power on the structure sheaf. This factors as

\[ X \xrightarrow{F \Phi} X' \xrightarrow{\Phi} X \]

where the map \(\Phi\) is base change along the absolute Frobenius of \(k\). The map \(\Phi\) is an isomorphism of schemes because \(k\) is perfect. The first map \(F \Phi\), called the relative Frobenius, is \(k\)-linear and is what is used is [OV]. It is more convenient for us to work with \(X\) alone and transport objects from \(X'\) to \(X\) via \(\Phi\) when necessary.

We recall some basic facts about flat bundles in positive characteristic, and refer to [K] for more details. We have a new invariant for a flat bundle \((V, \nabla)\) called the \(p\)-curvature, which gives the obstruction for \(\nabla\) to commute with \(p\)th powers. We denote the pullback \(F \Phi^* M\) of a vector bundle by \(M^{(p)}\). This carries a canonical integrable connection \(\nabla_{\text{cart}} : M^{(p)} \rightarrow \Omega^1_X \otimes M^{(p)}\) with trivial \(p\)-curvature, that we call the Cartier connection. Under the natural embedding \(M \hookrightarrow M^{(p)} = \mathcal{O}_X \otimes \mathcal{O}_X\), \(\Phi^* M\) given by \(m \mapsto 1 \otimes m\), we have \(\ker \nabla_{\text{cart}} \cong M\).

**Lemma 2.2.** Let \(M\) be a vector bundle and let \((V, \nabla)\) be a flat bundle on \((X, D)\). Equip \(V \otimes M^{(p)}\) with the tensor product connection \(\nabla_T = \nabla \otimes \nabla_{\text{cart}}\). Then

\[ (F \Phi^* \text{DR}(V, \nabla)) \otimes M \cong F \Phi^* \text{DR}(V \otimes M^{(p)}, \nabla_T) \]
as complexes of \(\mathcal{O}_X\)-modules.

**Proof.** The projection formula yields isomorphisms

\[ \pi : F_{\Phi^*}(\Omega^1_X(\log D) \otimes V) \otimes M \cong F_{\Phi^*}(\Omega^1_X(\log D) \otimes V \otimes M^{(p)}) \]

If \(\alpha\) and \(m\) are local sections of \(F_{\Phi^*}(\Omega^1_X(\log D) \otimes V)\) and \(M\), then \(\nabla_T(\pi(\alpha \otimes m)) = \pi(\nabla(\alpha) \otimes m)\) because \(\nabla_{\text{cart}} m = 0\).

A flat bundle is called nilpotent (of level at most \(n\)) if the there exists a filtration (of length at most \(n\)) by flat bundles such that the \(p\)-curvature of the associated graded vanishes. To avoid confusion, we should point out that the word “nilpotent” will be used in three different senses in the next theorem. Two have already been explained. We say that the residue \(\text{Res}_{D, \nabla} \in \text{End}(V|_{D, \nabla})\) along a component \(D_1 \subset D\) is nilpotent of level \(\leq \ell\) if \((\text{Res}_{D, \nabla})^\ell = 0\).
Theorem 6 (Ogus-Vologodsky, Schepler). If $(X, D)$ lifts modulo $p^2$, then there exists an equivalence between the category of flat bundles which are nilpotent of level less than $p$ and with residues nilpotent of level $\leq p$, and the category of Higgs bundles which are nilpotent of level at most $p$. The functor giving the equivalence, which depends on the choice of lifting, is called the Cartier transform $C$.

Proof. The general result follows from [Sr, cor 4.11]. When $D = \emptyset$, this was first proved in [OV, thm 2.8]. A simplified construction of the correspondence in the last case can be found in [LSZ2]. □

Theorem 7 (Ogus-Vologodsky, Schepler). Suppose that $(X, D)$ lifts modulo $p^2$, $(V, \nabla)$ is a flat bundle which is nilpotent of level $\ell$ and with residues nilpotent of level $\leq p$. Let $(E, \theta) = C(V, \nabla)$. If $\ell + d < p$, there is an isomorphism

$$Fr_* DR(V, \nabla) \cong DR(E, \theta)$$

in the derived category.

Proof. This is [Sr, cor 5.7], and [OV, cor 2.27] in the non-log case. □

Corollary 2.3. If $(E, \theta)$ is a nilpotent Higgs bundle of level less than $p - d$, then for any vector bundle $M$, we have

$$H^i(X, DR(E, \theta) \otimes M) = H^i(X, DR(C^{-1}(E, \theta) \otimes M^{(p)}))$$

In the above, $M^{(p)}$ is equipped with $\nabla_{\text{cart}}$ and $C^{-1}(E, \theta) \otimes M^{(p)}$ should be understood as the tensor product of bundles with connection.

Proof. The theorem together with lemma [2.2] and finiteness of $Fr$ shows that

$$H^i(X, DR(E, \theta) \otimes M) = H^i(X, Fr_*(DR(C^{-1}(E, \theta) \otimes M^{(p)})))$$

$$= H^i(X, DR(C^{-1}(E, \theta) \otimes M^{(p)}))$$

□

Let $h^i(-)$ denote the dimension $\dim H^i(-)$ below.

Corollary 2.4. If $(E, \theta)$ is a nilpotent Higgs bundle of level less than $p - d$, then for any vector bundle $M$, we have

1) $$h^i(X, DR(E, \theta) \otimes M) \leq h^i(X, DR(\Lambda C^{-1}(E, \theta)) \otimes M^{(p)})$$

In particular, if $L$ is a line bundle then

2) $$h^i(X, DR(E, \theta) \otimes L) \leq h^i(X, DR(\Lambda C^{-1}(E, \theta)) \otimes L^p)$$

Proof. By corollary [2.3] for (1) it is enough to prove that

$$h^i(X, DR(C^{-1}(E, \theta) \otimes M^{(p)})) \leq h^i(X, DR(\Lambda C^{-1}(E, \theta)) \otimes M^{(p)})$$

This follows from the next lemma [2.4]. For (2), we use the isomorphism $L^{(p)} \cong L^p$. □

Lemma 2.5. If $(V, \nabla)$ is a semistable flat bundle on $(X, D)$ and $N$ another vector bundle, we have

$$h^i(X, DR(V, \nabla) \otimes N) \leq h^i(X, DR(\Lambda(V, \nabla) \otimes N))$$
Proof. The spectral sequence abutting to $H^\ast(DR(V, \nabla) \otimes N)$ associated to the filtration $F_{can}^\ast$ of $\text{DR}(V, \nabla) \otimes N$ has
\[
\bigoplus_{a + b = i} E_{1}^{ab} = H^i(X, \text{DR}(\Lambda(V, \nabla)) \otimes N)
\]
The lemma is now a consequence of the standard inequality $\dim E_{1}^{ab} \geq \dim E_{\infty}^{ab}$. □

Remark 2.6. Since the level is bounded by the rank, the previous corollary applies when $d + \text{rank } E < p$.

Set $B(E, \theta) = \Lambda_{C^1}(E, \theta)$. This operator, which will give a map from the set of Higgs bundles of the appropriate type to itself, was first introduced implicitly in [LSZ1].

Theorem 8 (Langer). Assume that $(X, D)$ lifts modulo $p^2$.

1. Suppose that $(E, \theta)$ is a Higgs bundle nilpotent of level at most $p$. Then $(E, \theta)$ is semistable if and only if $C^{-1}(E, \theta)$ is semistable.

2. Suppose that $k = \overline{\mathbb{F}}_p$ is the algebraic closure of a finite field. If $(E, \theta)$ is semistable with vanishing $\ell$-adic Chern classes and rank $\leq p$, then the sequence $(E_i, \theta_i) = B^i(E, \theta)$ is eventually periodic in the sense that $(E_n, \theta_n) = (E_m, \theta_m)$ for some $m > n$.

Proof. The first item is [L1, cor 5.10]. The second is [L2, prop 1] together with the remarks at the beginning of section 3.1 of [L3]. When $D = \emptyset$, a different proof of (2) can be found in [LSYZ]. □

For the reader's convenience, we restate theorem 1.

Theorem 1. Let $(X, D, L)$, as above, be defined over an algebraically closed field $k$. Let $(E, \theta)$ be a nilpotent semistable Higgs bundle on $(X, D)$ with vanishing Chern classes in $H^\ast(X_{et}, \mathbb{Q}_\ell)$. Suppose that either

(a) $\text{char } k = 0$, or

(b) $\text{char } k = p > 0$, $(X, D)$ is liftable modulo $p^2$, $d + \text{rank } E < p$.

Then $H^i(X, \text{DR}(E, \theta) \otimes L) = 0$ for $i > d$.

Proof. Suppose that we are in the case (a) where $\text{char } k = 0$. We choose a finitely generated subring $A \subset k$, such that $X, D, E, \theta, L$ are all defined over it. In other words, we have a projective scheme $\mathcal{X} \to \text{Spec } A = S$ such that the geometric generic fibre of $\mathcal{X}$ is $X$, and objects with $\mathcal{D}, \mathcal{E}, \Theta, \mathcal{L}$ over $\mathcal{X}$ which restrict to $D$ etc. After shrinking $S$ if necessary, we can assume that

1. The characteristics of the residue fields of closed points of $S$ are bigger than $d + \text{rank } E$,

2. $\mathcal{X} \to S$ is smooth, and $\mathcal{D}$ is a divisor with relative normal crossings,

3. $\mathcal{E}$ is a relatively ample line bundle,

4. $\mathcal{E}$ is flat over $S$, and $\Theta$ has constant rank

5. The restrictions of $(\mathcal{E}, \Theta)$ to the geometric fibres are semistable (by the openness of semistability, c.f. [Ma, thm 1.14]),
In case (b), char \( k = p > d + \text{rank} E \) and we have a lifting of \((X,D)\) over \( W_2(k) \). We choose a finitely generated subring \( A \subset W_2(k) \), so that \((X,D)\) is defined over it, and such that the remaining objects \( L, \ldots \) are defined over \( A/(p) \).

Let \( S = \text{Spec} \, A/(p) \), and \( X \rightarrow S, D \rightarrow S \) be the correspond families. The pair lifts to a pair \((\tilde{X}, \tilde{D})\) over \( \tilde{S} = \text{Spec} \, A \). We may shrink \( \tilde{S} \) so that \((\tilde{X}, \tilde{D})/\tilde{S} \) is smooth with relative normal crossings and such that the above assumptions (3)-(5) hold.

From this point on, we will treat both cases in parallel. Let \( \mathcal{L}_s, \mathcal{L}_{\tilde{s}}, \ldots \) denote the restrictions of these object to the fibre \( \mathcal{X}_s \) and geometric fibre \( \mathcal{X}_{\tilde{s}} = \mathcal{X} \times_S \text{Spec} \, k(s) \). Then by semicontinuity of cohomology, it is enough to prove that

\[
H^i(\mathcal{X}_s, \text{DR}(\mathcal{E}_s, \Theta_s) \otimes L_s) = H^i(\mathcal{X}_{\tilde{s}}, \text{DR}(\mathcal{E}_{\tilde{s}}, \Theta_{\tilde{s}}) \otimes L_{\tilde{s}}) \otimes k(s) = 0
\]

for all closed points \( s \in S \). Now fix such an \( s \). The residue field \( k(s) \) is finite of characteristic \( p > d + \text{rank} E \). By theorem [8] corollary [2.4] and remark [2.6] there is an eventually periodic sequence of Higgs bundle \((E_j, \theta_j)\) such that \((E_0, \theta_0) = (\mathcal{E}_s, \Theta_s)\) and

\[
h^i(\text{DR}(E_0, \theta_0) \otimes L_s) \leq h^i(\text{DR}(E_1, \theta_1) \otimes L_s^p) \leq h^i(\text{DR}(E_2, \theta_2) \otimes L_s^{p^2}) \ldots
\]

Since the sequence of bundles is eventually periodic, we can bound the initial term by

\[
h^i(\text{DR}(E_n, \theta_n) \otimes L_s^{p^n})
\]

for some \( n \) and \( j \) arbitrarily large. This is zero by lemma [2.1] and this completes the proof. \( \square \)

**Remark 2.7.** Langer pointed out to the author that the proof of case (b) can be simplified slightly. It is not necessary to reduce to finite fields. Instead, arguing as in the proof of [L2] prop 1, one sees that the set of Higgs bundle \((E, \theta)\), satisfying the above conditions, forms a bounded family. Therefore there is a uniform constant \( N_0 \) such that \( H^i(\text{DR}(E, \theta) \otimes L^N) = 0 \) for \( N \geq N_0 \) and all \((E, \theta)\). Now the result follows from corollary [2.4].

### 3. Proof of theorem 2

**Lemma 3.1.** Let \( E \) be a vector bundle and \( K, L \) line bundles on a projective variety \( Y \) with \( L \) very ample. Suppose that

\[
H^i(S^n(E) \otimes K \otimes L^m) = 0
\]

for all \( i > 0, n \gg 0 \) and \( m \gg 0 \). Then \( E \) is nef.

**Proof.** Choose \( m > \dim Y + 1 \). The hypothesis implies that for all \( n \), \( S^n(E) \otimes K \otimes L^m \) is 0-regular in the sense of Castelnuovo-Mumford, and therefore globally generated [Lz] §1.8. Choosing \( m \) large enough so that \( K \otimes L^m \) is ample allows us to apply [Lz] 6.2.13 to conclude that \( E \) is nef. \( \square \)

Suppose \( V \) and \( W \) are weak complex variations of mixed Hodge structures on \( X - D \) with unipotent monodromy around \( D \). Then \( V \otimes W \) also carries a variation of Hodge structure with unipotent monodromy around \( D \). Let \( F^a V \) and \( F^a W \) denote the Hodge filtrations of \( V \) and \( W \) respectively. Then

\[
F^c(V \otimes W) = \bigoplus_{a+b=c} F^a V \otimes F^b W
\]
is the Hodge filtration on the tensor product. It follows easily that
\[ Gr_F^a(V \otimes W) = \bigoplus_{a+b=c} Gr_F^a V \otimes Gr_F^b W \]

One can deduce the corresponding formulas for the symmetric powers
\[ Gr_F^b(S^n V) = \sum_{\lambda_1 + \ldots + \lambda_k = n} \text{im} \left[ S^{\lambda_1}(Gr_F^{\lambda_1} V) \otimes \ldots \otimes S^{\lambda_k}(Gr_F^{\lambda_k} V) \right] \]
and similarly for exterior powers.

Now we can prove:

**Theorem 2.** Suppose that \( V \) is a weak complex variation of mixed Hodge structure on \((X, D)\) with unipotent local monodromies around \( D \). Then \( \land^a Gr_F^a(V) \otimes \det(F^{b+1}) \) is nef for every \( a \) and \( b \).

**Proof.** Letting \( Gr_F^{max}V = F^{max}V \) denote the smallest nonzero Hodge bundle of \( V \) etc., we deduce from (3) that
\[ Gr_F^{max}(S^n V) = S^n(Gr_F^{max}V) \]
Let \( \omega_X = \Omega_X^{\dim X} \). Applying theorem 1 to \( S^n V \) shows that
\[ H^i(S^n(Gr_F^{max}V) \otimes \omega_X(D) \otimes L^m) = H^i(Gr_F^{max}S^n V \otimes \omega_X(D) \otimes L^m) = 0 \]
for any \( i > 0, n > 0, m > 0 \) and ample \( L \). Therefore, by lemma 3.1 \( Gr_F^{max}V \) is nef. This is a special case of the theorem corresponding to \( b = \text{max} \).

Fix \( b \), let \( r = \text{rank} F^{b+1} \) and \( 0 \leq a \leq \text{rank} F^{b+1} \). It is easy to see that
\[ Gr_F^a(\land^{r+a}V) \cong \land^a Gr_F^b(V) \otimes \det(Gr_F^{b+1}(V)) \otimes \det(Gr_F^{b+2}(V)) \ldots \cong \land^a Gr_F^b(V) \otimes \det(F^{b+1}(V)) \]
Therefore this is nef by what was proved in the previous paragraph.

**Remark 3.2.** Brunebarbe has pointed out to the author that the proof of [13] thm 3.4] carries over to complex variations without difficulty. In this form, it implies theorem 2 taking exterior powers as above.

### 4. Proof of theorem [13]

We recall some notions from [A] needed below. Let \( X \) be a smooth projective variety defined over an algebraically closed field \( k \). Suppose that \( M \) is a vector bundle on \( X \). The Frobenius amplitude is a number \( \phi(M) \) measuring the cohomological positivity in the following way. If \( \text{char } k = p > 0 \), define the Frobenius power by \( M^{(p^n)} = (Fr^n)^*M \). Then \( \phi(M) \) is the smallest natural number such that for any coherent sheaf \( F \)
\[ H^i(X, F \otimes M^{(p^n)}) = 0 \]
for \( i > \phi(M) \) and all \( n \gg 0 \). If \( \text{char } k = 0 \), then we “spread out” \((X, M)\) over the spectrum \( S \) of a finitely generated algebra, and take the supremum of \( \phi(M_s) \) over almost all closed fibres. More precisely \( \phi(M) = \sup_{U \subset \text{U}} \phi(M_s) \), as \( U \) runs over all nonempty opens of \( S \). So the smaller \( \phi(M) \) is, the more positive \( M \) is. This can be related to more familiar notions of positivity.

**Theorem 9 ([A thm 6.1]. Suppose \( \text{char } k = 0 \). If \( M \) is an ample vector bundle, then \( \phi(M) < \text{rank } M \).**
There is also a relative version of this. Suppose that $D \subset X$ is a reduced effective divisor with normal crossings. In characteristic $p$
\[ \phi(M, D) = \min \{ \phi(M^{\langle p^n \rangle}(-D')) \mid n \in \mathbb{N}, 0 \leq D' \leq (p^n - 1)D \} \]
It is useful to observe that if the minimum is achieved for $n = n_0$, then it is achieved for any $n \geq n_0$, because $\phi(M^{\langle p^{n_0} \rangle}(-D')) = \phi(M^{\langle p^{n_0 + 1} \rangle}(-pD'))$. The definition is extended to characteristic $0$ as above.

**Theorem 10** ([A] thm 6.7). Suppose $\text{char } k = 0$. If $M$ is a vector bundle, such that $M(-\Delta)$ is ample for some fractional effective $\mathbb{Q}$-divisor supported on $D$, then $\phi(M, D) = 0$.

The proof given there does not work in positive characteristic. However, we do get the same result for line bundles by an easy argument.

**Lemma 4.1.** Suppose that $\text{char } k$ is arbitrary. If $M$ is a line bundle, such that $M(-\Delta)$ is ample for some fractional effective $\mathbb{Q}$-divisor supported on $D$, then $\phi(M, D) = 0$.

**Proof.** As is well known, the ample divisors form an open cone in $\text{NS}(X) \otimes \mathbb{R}$. Thus for any small $\epsilon > 0$, $M(-\Delta - \epsilon D)$ is still ample. The set $S = \bigcup \frac{1}{\epsilon} \mathbb{Z}$ is clearly dense in $\mathbb{R}$. Therefore we can choose $\epsilon > 0$ so that $M(-\Delta - \epsilon D)$ is ample, and the coefficients of $\Delta + \epsilon D$ are in $S \cap [0, 1)$. Therefore $M^{\langle p^n \rangle}(-D')$ is ample for some integral divisor $0 \leq D' \leq (p^n - 1)D$. The equality $\phi(M^{\langle p^n \rangle}(-D')) = 0$ is an immediate consequence of Serre’s vanishing theorem.

Let $D' = \sum n_i D_i$ be a divisor supported on $D = \sum D_i$. If $f_i$ are the local equations of $D_i$, the formula
\[ d(\prod_i f_i^{-n_i}) = \sum_j \prod_i f_i^{-n_i} \cdot (-n_j \frac{df_j}{f_j}) \]
shows that the fractional ideal $\mathcal{O}_X(D') \subset k(X)$ is stable under differentiation. This rule endows $\mathcal{O}_X(D')$ with an integrable connection that we denote $dD'$. The above calculation shows that the residue of $dD'$ on $D_i$ is $-n_i$. If $(V, \nabla)$ is a flat bundle on $(X, D)$, then $\nabla$ can be extended to an operator on rational sections $V \otimes k(X)$. This operation restricts to the product connection $\nabla \otimes dD'$ on the subshaf $V \otimes \mathcal{O}_X(D') \subset V \otimes k(X)$. As above, we have
\[ \text{Res}_{D_i} \nabla \otimes dD' = \text{Res}_{D_i} \nabla - n_i I \]
([EV2] lemma 2.7). The next result is a generalization of [H] lemma 3.3.

**Lemma 4.2.** Suppose that $\text{char } k = p$ and that $0 \leq D' \leq (p - 1)D$ is a divisor. If the residues of $\nabla$ are nilpotent, the natural inclusion
\[ \text{DR}(V, \nabla) \subset \text{DR}(V \otimes \mathcal{O}_X(D'), \nabla \otimes dD') \]
is a quasiisomorphism.

**Proof.** Let $0 = D_0' \subset D_1' \subset \ldots D_N' = D'$ be a sequence of effective divisors such that the such $D'_{i+1} - D_i'$ consists of a single component, say $D_{j(i)}$. Then ([H] shows that $\text{Res}_{D_{j(i)}} \nabla \otimes dD'_{j(i)}$ is invertible. Therefore ([EV2] lemma 2.10) shows that
\[ \text{DR}(V \otimes \mathcal{O}(D_i'), \nabla \otimes dD') \subset \text{DR}(V \otimes \mathcal{O}_X(D'_{i+1}), \nabla \otimes dD'_{i+1}) \]
Lemma 4.3. Suppose that \( E, \theta \) is a quasiisomorphism. Then \( (E, \theta) \) is nilpotent of level at most \( \ell \), or semistable with vanishing Chern classes. Then \( c_1(E) \) is a nilpotent subsheaf, then \( \mu(E) \geq 0 \). If \( F \subset E \) is a saturated \( \theta^\vee \)-stable subbundle, then \( -\mu(F) = \mu(F^\vee) \geq \mu(E) = 0 \).

Proof. For the first statement, observe that if \( E \) carries a \( \theta \)-stable filtration \( E = N^0 \supset \ldots \supset N^\ell = 0 \) such that \( \text{Gr}(\theta) = 0 \), then \( \ker[E^\vee \to (N^i)^\vee] \) gives a filtration on the dual with the same property. Suppose that \( (E, \theta) \) is semistable with vanishing Chern classes. Then \( c_1(E^\vee) = \pm c_1(E) = 0 \). If \( F \subset E \) is a saturated \( \theta^\vee \)-stable subsheaf, then \( -\mu(F) = \mu(F^\vee) \geq \mu(E) = 0 \).

Observe that
\[
R\text{Hom}(DR(E, \theta), \omega_X[d]) \cong H\text{om}(DR(E, \theta), \omega_X[d]) \cong DR(E^\vee, \theta^\vee)(-D)[2d]
\]
in the derived category of quasicoherent sheaves. Thus Grothendieck-Serre duality \( [\text{H}] \) yields an isomorphism
\[
H^i(X, DR(E, \theta) \otimes M) = H^{2d-i}(X, DR(E^\vee, \theta^\vee)(-D) \otimes M^\vee)^\vee
\]

Lemma 4.4. Suppose that \( \text{char } k = p \), that \( M \) is a vector bundle and that \( (E, \theta) \) is a nilpotent semistable Higgs bundle of level less than \( p - d \).

1. If \( 0 \leq D' \leq (p - 1)D \), then
   \[
   h^i(X, DR(E, \theta) \otimes M) \leq h^i(X, DR(B(E, \theta)) \otimes M^{(p)}(D'))
   \]

2. If \( 0 \leq D' \leq (p - 1)D \), then
   \[
   h^i(X, DR(E, \theta) \otimes M(-D)) \leq h^i(X, DR(B(E, \theta)) \otimes M^{(p)}(-D - D'))
   \]

Proof. By theorem 7, lemma 22 and the projection formula
\[
H^i(X, DR(E, \theta) \otimes M) \cong H^i(X, (Fr_* DR(C^{-1}(E, \theta))) \otimes M)
\]
\[
\cong H^i(X, Fr_* DR(C^{-1}(E, \theta) \otimes (M^{(p)}), \nabla_{\text{cart}}))
\]
\[
\cong H^i(X, DR(C^{-1}(E, \theta) \otimes (M^{(p)}, \nabla_{\text{cart}})))(D'))
\]
Since the residues of \( C^{-1}(E, \theta) \otimes (M^{(p)}, \nabla_{\text{cart}}) \) are nilpotent, lemma 22 applies to show that this is isomorphic to
\[
H^i(X, DR(C^{-1}(E, \theta) \otimes (M^{(p)}, \nabla_{\text{cart}})))(D')
\]
Furthermore, we obtain
\[
h^i(X, DR(C^{-1}(E, \theta) \otimes M^{(p)})(D')) \leq h^i(X, DR(AC^{-1}(E, \theta)) \otimes M^{(p)}(D'))
\]
by lemma 2.3. This proves the first inequality.

The second inequality follows from the first using lemma 4.3.
Lemma 4.5. Suppose that $0 \leq D' \leq (p^n - 1)D$, and with the remaining assumptions of the previous lemma. Then

$$h^i(X, \text{DR}(E, \theta) \otimes M) \leq h^i(X, \text{DR}(B^n(E, \theta)) \otimes M^{(p^n)}(D'))$$

and

$$h^i(X, \text{DR}(E, \theta) \otimes M(-D)) \leq h^i(X, \text{DR}(B^n(E, \theta)) \otimes M^{(p^n)}(-D - D'))$$

Proof. We may write $D' = p^{n-1}D'_1 + p^{n-2}D'_2 + \ldots$, where $0 \leq D'_1 \leq (p - 1)D$. Then repeatedly applying lemma 4.4 gives

$$h^i(X, \text{DR}(E, \theta) \otimes M) \leq h^i(X, \text{DR}(B(E, \theta)) \otimes M^{(p)}(D'_1))$$

$$\leq h^i(X, \text{DR}(B^2(E, \theta)) \otimes M^{(p^2)}(pD'_1 + D'_2))$$

$$\ldots$$

This proves the first inequality. The second follows by duality. \qed

We are now ready to prove:

Theorem 3. With same assumptions as theorem 1, let $M$ be a vector bundle when char $k = 0$, or a line bundle when char $k = p$, such that $M(-\Delta)$ is ample for some fractional effective $\mathbb{Q}$-divisor supported on $D$. Then

$$H^i(X, \text{DR}(E, \theta) \otimes M(-D)) = 0$$

for $i \geq d + \text{rank } M$.

Proof. This is a modification of the proof of theorem 1, so we just outline the main steps. As above, we first reduce to the case where $k = \mathbb{F}_p$ with $p > d + \text{rank } E$. The sequence $(E_j, \theta_j) = B^1(E, \theta)$ is eventually periodic. This fact together with lemma 4.5 implies that

$$h^i(X, \text{DR}(E, \theta) \otimes M) \leq h^i(X, \text{DR}(E_n, \theta_n)(-D) \otimes (M^{(p^j)}(-D'))^{(p^j)})$$

for some $n, j$ and $0 \leq D' \leq (p^j - 1)D$ and $\ell$ arbitrarily large. Theorem 10 or lemma 4.1 shows that this is zero for $\ell$ large enough. \qed

5. Proof of theorem 4

Recall that we are given a complex projective variety $Y$, a desingularization $\pi : X \to Y$ such that the exceptional divisor $D$ has simple normal crossings, and a complex variation of Hodge structures $V$ on $U = X - D$ with quasiunipotent monodromies around $D$. By Kawamata ([Kv] thm 17) or [EV2, lemma 3.19), we can find a finite Galois cover $p : Z \to X$ such that $Z$ is smooth, the branch divisor $D' \supseteq D$ has simple normal crossings, and $W = p|_{p^{-1}U}V$ has unipotent monodromies around $E = p^*D'_\text{red}$. Since $V$ viewed as the filtered bundle with logarithmic connection is determined by $V|_{X - D'}$, we can and will replace $U$ by $X - D'$ and $D$ by $D'$ with causing any harm. Let $G = \text{Gal}(Z/X)$, and let $W$ denote the Deligne extension associated to $W$. By construction, $Z \to X$ is given as a tower of cyclic covers. It follows easily that $p^*\Omega^1_C(\log D) = \Omega^1_C(\log E)$ cf. [EV2 §3]. Let $\nabla'$ denote the dual connection on $W'$. Then the composite

$$p_*W^{\nabla'} \overset{p_*\nabla'}{\longrightarrow} p_*\Omega^1_C(\log E) \otimes W' \cong \Omega^1_C(\log D) \otimes p_*W^{\nabla'}$$

gives an induced logarithmic connection that we denote by $\nabla$.

Lemma 5.1. The eigenvalues of the residues of $\nabla$ are in $[0,1)$. 
Proof. The problem is local analytic, so we can reduce to the case where $Z \to X$ is given by $z_i^{m_i} = x_i$, and $D$ by $x_1 \ldots x_t = 0$. Since the residues of $\nabla'$ are nilpotent, we can find a local basis $v_1, \ldots, v_r$ of multivalued sections of $W^\vee$ such that
$$
\nabla'(v_j) \equiv 0 \mod v_{j+1}, v_{j+2}, \ldots
$$
We can see that
$$
\{z_1^{m_1} \ldots z_d^{m_d} \otimes v_j \mid 0 \leq m_i < n_i, 1 \leq j \leq r\}
$$
gives a basis of $p_*W^\vee$. We find that
$$
\nabla(z_1^{m_1} \ldots z_d^{m_d} \otimes v_j) \equiv \sum_i \frac{m_i}{n_i} z_i^{m_1} \ldots z_i^{m_d} \otimes v_j \otimes \frac{dx_i}{x_i} \mod v_{j+1}, \ldots
$$

\[\square\]

The group $G$ will act on the pair $(p_*W^\vee, \nabla)$.

Lemma 5.2.

(a) $(p_*W^\vee)^G = V^\vee$
(b) $p_*(\omega_Z \otimes W)^G = \omega_X \otimes V$
(c) $p_*(\omega_Z \otimes F^{\max}W)^G = S'(V)$

Proof. We can see easily that $(\pi_*\mathcal{O}_Z)^G \cong \mathcal{O}_X$, and that over $U$, this is isomorphism is compatible with the direct image or Gauss-Manin connection on the left and on the right. Therefore by the projection formula, it follows that $(\pi_*M)^G \cong M$ for any locally free module. Furthermore, if $M|_U$ is equipped with a connection, then the isomorphism is compatible with it. In particular, $(p_*W^\vee)^G|_U = V^\vee|_U$ as flat vector bundles. By the previous lemma, $(p_*W^\vee)^G$ gives the extension with eigenvalues of the residues in $[0,1]$. Since there is only one such extension $[\mathbb{D}]$, we must have $(p_*W^\vee)^G = V^\vee$.

By Grothendieck duality $[\mathbb{H}]$, we have an isomorphism
\[(5) \quad (p_*W^\vee)^\vee \cong p_*(\omega_{Z/X} \otimes W)\]
This isomorphism is canonical and therefore compatible with the $G$-action. The invariant part of a $G$-module $M$ is the image of the idempotent $\frac{1}{|G|} \sum_g g$, and therefore it is the same as the co-invariants. Thus
\[(6) \quad (M^\vee)^G = (M^G)^\vee\]
Taking $G$-invariants of both sides of $[\mathbb{A}]$, and combining this with (a) and (b) yields $p_*(\omega_{Z/X} \otimes W)^G = V$. This implies (b).

Let $\tilde{U} = p^{-1}U$ and let $\tilde{j} : U \to Z$ denote the inclusion. Then $F^{\max}W = \tilde{j}_*(F^{\max}\mathcal{O}_U \otimes W) \cap W$ as subsheaves of $\tilde{j}_*(\mathcal{O}_U \otimes W)$. This can also be expressed as the kernel of the difference of the inclusions
$$
F^{\max}W = \ker[\tilde{j}_*(F^{\max}\mathcal{O}_U \otimes W) \oplus W \to \tilde{j}_*(\mathcal{O}_U \otimes W)]
$$
Therefore
$$
(\omega_Z \otimes F^{\max}W)^G = \ker[(\omega_Z \otimes \tilde{j}_*(F^{\max}\mathcal{O}_U \otimes W))^G \otimes (\omega_Z \otimes W)^G \to (\omega_Z \otimes \tilde{j}_*(\mathcal{O}_U \otimes W))^G] \\
= \ker[(\omega_X \otimes \tilde{j}_*(F^{\max}\mathcal{O}_U \otimes V)) \otimes (\omega_X \otimes V) \to (\omega_X \otimes \tilde{j}_*(\mathcal{O}_U \otimes V))] \\
= \omega_X \otimes F^{\max}V
$$

\[\square\]

Theorem 4. Let $M$ be an ample vector bundle on $Y$, then
(a) \( H^i(X, S'(V) \otimes \pi^* M) = 0 \) for \( i \geq \dim X + \text{rank} M \).
(b) \( R^i\pi_*(S'(V)) = 0 \) for \( i > 0 \).
(c) \( H^i(Y, S(V) \otimes M) = 0 \) for \( i \geq \dim X + \text{rank} M \).

Proof. We can choose a fractional effective \( \mathbb{Q} \)-divisor \( \Delta \) supported on \( D \) such that \(-\Delta\) is relatively ample. Let \( H \) be an ample Cartier divisor on \( Y \). Then \( \epsilon \pi^*H - \Delta \) is ample for every \( \epsilon > 0 \). We can choose \( \epsilon > 0 \) so that \( M(-\epsilon H) \) remains ample. Therefore \( \pi^*M(-\epsilon H) \) is nef. Then [Lz] prop 6.2.11 implies that \( \pi^*M(-\Delta) = \pi^*M(-\epsilon H + \epsilon H - \Delta) \) is ample. Therefore \( q^*M(-\Delta) \) is ample where \( q = \pi \circ p \). Theorem 5 implies that for \( i \geq \dim X + \text{rank} M \),
\[
H^i(Z, \omega_Z \otimes F^{\text{max}}W \otimes q^*M) = 0
\]
which proves (a).

Suppose that \( M \) is an ample line bundle. By Serre, we can choose \( N \gg 0 \), so that, \( R^i\pi_*(\omega_X \otimes F^{\text{max}}V) \otimes M^N \) is globally generated for all \( i \), and
\[
H^j(Y, R^i\pi_*(\omega_X \otimes F^{\text{max}}V) \otimes M^N) = 0, \quad \forall i, \forall j > 0
\]
Therefore the Leray spectral sequence reduces to an isomorphism
\[
H^i(X, \omega_X \otimes F^{\text{max}}V \otimes \pi^* M^N) \cong H^0(Y, R^i\pi_*(\omega_X \otimes F^{\text{max}}V) \otimes M^N)
\]
By part (a), this must vanish for \( i > 0 \). Therefore (b) holds.

Part (b) implies that \( S(V) \otimes M = \mathbb{R}\pi_*(\omega_X \otimes F^{\text{max}}V \otimes \pi^* M) \). Therefore
\[
H^i(Y, S(V) \otimes M) \cong H^i(X, \omega_X \otimes F^{\text{max}}V \otimes \pi^* M)
\]
By part (a) this vanishes for \( i \geq \dim X + \text{rank} M \).

6. Stationary Higgs bundles

Let \((E, \theta)\) be a Higgs bundle on \((X, D)\), defined over \( k \), which is semistable, nilpotent with vanishing Chern classes. Let us say that \((E, \theta)\) is stationary if it is fixed under the bootstrapping operator \( B = \Lambda C^{-1} \) (for almost all mod \( p \) reductions). To be a bit more precise, let us first consider the case where the ground field \( k \) has characteristic \( p > 0 \), and \((X, D)\) is liftable mod \( p^2 \). Call \((E, \theta)\) stationary if it is nilpotent of level at most \( p \) and \( B(E, \theta) \cong (E, \theta) \). If \text{char} \( k = 0 \), call \((E, \theta)\) stationary if everything can be spread out, as in the proof of theorem 1 such that the restriction of \((E, \theta)\) to all the closed fibres are stationary. Note that we say “all the closed fibres”, because the base \( S \) may be shrunk.

Given a Higgs bundle \((E, \theta)\), define the associated \( i \)th cohomological jump locus by
\[
\Sigma^i(X; E, \theta) = \{ L \in \text{Pic}^0(X) \mid H^i(X, DR(E, \theta) \otimes L) \neq 0 \}
\]
When \((E, \theta) = (O_X, 0)\) is trivial, an important structure theorem was proved by Green and Lazarsfeld [GL] and Simpson [S2]. Pink and Roessler [PR] showed how to deduce these results from Deligne-Illusie. Their argument generalizes to any stationary Higgs bundle.

**Theorem 5.** Suppose that \( k \) is an algebraically closed field of characteristic 0. If \((E, \theta)\) is stationary, then \( \Sigma^i(X; E, \theta) \) is a finite union of translates of subabelian varieties of \( \text{Pic}^0(X) \) by torsion elements.
Proof. Arguing as in the proof of [PR, thm 3.6], we can reduce to the case $k = \mathbb{Q}$. Then $X, D, L, E, \theta$ will be defined over a number field $k_0$. We can spread these out over $S \subset \text{Spec } O_{k_0}$, to obtain objects $X', D', L', E', \theta$ as in the proof of theorem 1. Standard semicontinuity arguments show that $\Sigma'^i(X'; E, \theta)$ is Zariski closed, and that it extends to a closed subscheme of $X$ with fibres $\Sigma'^i(X'_s; E_s, \theta_s)$. Choose a closed point $s \in S$ with residue characteristic $p \gg 0$. By corollary 2.4 and the hypothesis $h^i(X'_s, \text{DR}(E_s, \Theta_s) \otimes L_p) \leq h^i(X'_s, \text{DR}(E_s, \Theta_s)) \otimes L_p^p$)

Therefore $\Sigma'^i(X'_s; E_s, \Theta_s)$ is stable under multiplication by $p$. Since this property holds for almost all $s \in S$, it follows from [PR, thm 2.1] that this set is a finite union of torsion translates of subabelian varieties. □

A natural question asked by a referee is whether $(E, \theta)$ is stationary when it arises from a polarizable variation of Hodge structure with unipotent monodromy along $D$, as in section 1. A positive answer would have interesting consequences, but we are not very optimistic in general. However, for variations of Hodge structure of geometric origin we do expect this question to have a positive answer. Using [OV, thm 3.8], it is not difficult to prove:

**Proposition 6.1.** Let $f : Y \to X$ be a smooth projective map, and let $E = \bigoplus R^i_f \Omega^i_{Y/X}$ be equipped with the Higgs field $\theta$ induced by the Kodaira-Spencer class. Then $(E, \theta)$ is stationary.

We hope to pursue this question, and some consequences, in more detail elsewhere.

**References**

[A] D. Arapura, *Frobenius amplitude and strong vanishing theorems for vector bundles*, Duke Math. J. 121 (2004), no. 2, 231-267.

[B] Y. Brunebarbe, *Symmetric differentials and variations of Hodge structures*, Crelle’s Journal (to appear)

[D] P Deligne, *Équations différentielles à points singuliers réguliers*, Lecture Notes in Mathematics, Vol. 163. Springer-Verlag, Berlin-New York, (1970)

[DI] P. Deligne, L. Illusie, *Relèvements modulo p² et décomposition du complexe de de Rham*. Invent. Math. 89 (1987), no. 2, 247-270.

[GL] M. Green, R. Lazarsfeld, *Higher obstructions to deforming cohomology groups of line bundles*. J. Amer. Math. Soc. 4 (1991), no. 1, 87-103

[H] N. Hara, *A characterization of rational singularities in terms of injectivity of Frobenius maps*. Amer. J. Math. 120 (1998), no. 5, 981-996.

[EV1] H. Esnault, E. Viehweg, *Logarithmic de Rham complexes and vanishing theorems*. Invent. Math. 86 (1986), no. 1, 161-194.

[EV2] H. Esnault, E. Viehweg, *Lectures on vanishing theorems*. DMV Seminar, 20. Birkhäuser Verlag, Basel, 1992.

[F] T. Fujita, *On Kähler fiber spaces over curves*. J. Math. Soc. Japan 30 (1978), no. 4, 779-794.

[FFS] O. Fujino, T. Fujisawa, M. Saito, *Some remarks on the semipositivity theorems*. Publ. Res. Inst. Math. Sci. 50 (2014), no. 1, 85-112.

[H] R. Hartshorne, *Residues and duality*, Lecture Notes in Mathematics, No. 20 Springer-Verlag, Berlin-New York (1966)

[I] L. Illusie *Réduction semi-stable et décomposition de complexes de de Rham à coefficients*. Duke Math. J. 60 (1990), no. 1, 139-185.

[LSZ1] Guanlan, Mao Sheng, Kang Zuo, *Semistable Higgs bundles and representations of algebraic fundamental groups: Positive characteristic case*. [arXiv:1210.8320]
[LSZ2] Guitang Lan, Mao Sheng, Kang Zuo, Nonabelian Hodge theory in positive characteristic via exponential twisting. Math. Res. Lett. 22 (2015), no. 3, 859-879

[LSYZ] Guitang, Lan, Mao Sheng, Yanhong Yang, Kang Zuo, Semistable Higgs bundles of small ranks are strongly Higgs semistable, arXiv:1311.2405

[L1] A. Langer, Semistable modules over Lie algebroids in positive characteristic. Doc. Math. 19 (2014), 509-540.

[L2] A. Langer, Bogomolov’s inequality for Higgs sheaves in positive characteristic. Invent. Math. 199 (2015), no. 3, 889-920.

[L3] A. Langer, The Bogomolov-Miyaoka-Yau inequality for logarithmic surfaces in positive characteristic, Duke Math J. (2016)

[Lz] R. Lazarsfeld, Positivity in algebraic geometry. I, II. Springer-Verlag, Berlin, 2004.

[K] N. Katz, Nilpotent connections and the monodromy theorem: Applications of a result of Turrittin. Inst. Hautes Études Sci. Publ. Math. No. 39 (1970), 175-232.

[Kw] Y. Kawamata, Characterization of abelian varieties. Compositio Math. 43 (1981), no. 2, 253-276.

[Ko] J. Kollár, Higher direct images of dualizing sheaves. I. Ann. of Math. (2) 123 (1986), no. 1, 1142.

[Ma] Maruyama, Construction of moduli spaces of stable sheaves via Simpson’s idea. Moduli of vector bundles (Sanda, 1994; Kyoto, 1994), 147-187, Lecture Notes in Pure and Appl. Math., 179, (1996)

[M1] T. Mochizuki, Asymptotic behaviour of tame nilpotent harmonic bundles with trivial parabolic structure. J. Differential Geom. 62 (2002), no. 3, 351-559.

[M2] T. Mochizuki, Kobayashi-Hitchin correspondence for tame harmonic bundles and an application. Astérisque No. 309 (2006)

[Pe] C. Peters, A criterion for flatness of Hodge bundles over curves and geometric applications. Math. Ann. 268 (1984), no. 1, 119.

[PR] R. Pink, D. Roessler, A conjecture of Beauville and Catanese revisited. Math. Ann. 330 (2004), no. 2, 293-308.

[P1] M. Popa, Kodaira-Saito vanishing and applications, arXiv:1407.3294

[P2] M. Popa, Positivity for Hodge modules and geometric applications, arXiv 1605.0809

[OV] A. Ogus, V. Vologodsky, Nonabelian Hodge theory in characteristic p. Publ. Math. Inst. Hautes Études Sci. No. 106 (2007), 1138.

[S1] M. Saito, Mixed Hodge modules, Publ. Res. Inst. Math. Sci. 26 (1990), no. 2, 221-333

[S2] M. Saito, On Kollár’s conjecture, Several complex variables and complex geometry, Part 2, 509-517, Proc. Sympos. Pure Math., 52, Part 2, AMS (1991)

[Sr] D. Schepler, Logarithmic nonabelian Hodge theory in characteristic p, arXiv:0802.1977

[Sc] W. Schmid, Variation of Hodge structure: the singularities of the period mapping. Invent. Math. 22 (1973), 211-319.

[Si] C. Schnell, On Saito’s vanishing theorem. Math. Res. Lett. 23 (2016), no. 2, 499-527.

[Si1] C. Simpson, Harmonic bundles on noncompact curves. J. Amer. Math. Soc. 3 (1990), no. 3, 733-770.

[Si2] C. Simpson, Subspaces of moduli spaces of rank one local systems. Ann. Sci. École Norm. Sup. (4) 26 (1993), no. 3, 361-401.

[Su] J. Suh, Vanishing theorems for mixed Hodge modules and applications, J. Eur. Math. Soc. (to appear)

[W] L. Wu, Vanishing and injectivity theorems for Hodge modules, arXiv:1505.00881

Department of Mathematics, Purdue University, West Lafayette, IN 47907, U.S.A.
E-mail address: arapura@math.purdue.edu