Fourier Eigenspaces of Waldspurger’s Basis

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Abstract

In this paper we investigate invariant distributions on $p$-adic $\mathfrak{sp}_{2n}$ defined by Waldspurger in [20] and find the Fourier eigenspaces in their span. We prove that there is a single eigenvalue if $n$ can be represented as a sum of triangular numbers or is triangular itself and that none exist otherwise. We determine that the dimension of this lone eigenspace is equal to the number of noncommuting representations of $n$ as the sum of at most two triangular numbers. Each such representation corresponds to what we call a Lusztig distribution. These distributions belong to the generating set defined by Waldspurger and form a basis for the eigenspace. Finally, we show that the eigenspace contains a 1-dimensional subspace consisting of stable distributions when $n = 2\Delta$, $\Delta$ a triangular number, but otherwise consists of distributions that are not stable.

1 Introduction

In this paper we study the eigenspaces of the Fourier transform in the span of certain invariant distributions on symplectic Lie algebras over a $p$-adic field that appear, denoted by $\phi_{\theta}(X_T, -)$, in the book [20] by Waldspurger as precursors to the various bases he produces there for the purpose of studying stability and endoscopy. One of those bases is obtained from the Fourier transforms of what are called generalized Green functions, which motivates the question of finding the Fourier eigendistributions in the span of Waldspurger’s distributions.

Let $G$ be a connected reductive group, $\mathfrak{g}$ be its Lie algebra, both
defined over a $p$-adic field $F$, and let $G = G(F)$ and $\mathfrak{g} = \mathfrak{g}(F)$. In [20], Waldspurger finds a basis for the set $\mathcal{D}^{\text{st}} \cap \mathcal{D}^{\text{nil}}$ of stable invariant distributions supported on the nilpotent set $\mathfrak{g}^{\text{nil}}$ in $\mathfrak{g}$.

The set $\mathcal{D}$ of invariant distributions on $\mathfrak{g}$ consists of elements of the dual of the convolution algebra $C_c^\infty(\mathfrak{g})$ of locally constant, compactly supported, complex-valued functions on $\mathfrak{g}$ that are invariant under the action induced by the adjoint action of $G$ on $\mathfrak{g}$. The set of distributions supported on a particular subset $\omega$ is denoted $\mathcal{D}(\omega)$, except in certain cases: when $\omega$ is $\mathfrak{g}^{\text{nil}}$ or $\mathfrak{g}^{\text{ent}}$ we adopt Waldspurger’s notation and write $\mathcal{D}^{\text{nil}}$ for $\mathcal{D}(\mathfrak{g}^{\text{nil}})$ and $\mathcal{D}^{\text{ent}}$ for $\mathcal{D}(\mathfrak{g}^{\text{ent}})$. Harish-Chandra has proved (see [9]) that a basis for $\mathcal{D}^{\text{nil}}$ is given by the nilpotent orbital integrals. Stable distributions are harder to define explicitly, but $\mathcal{D}^{\text{st}}$ can be briefly described as the closure of the span of the stable regular semisimple orbital integrals, which have a straightforward, if lengthy, description$^2$. Aside from the regular semisimple orbital integrals, determining the stability of a given distribution (in particular those with nilpotent support) is a difficult task. This being the case Waldspurger made use of a bridge from distributions with nilpotent support and those with support on the “integral elements” of $\mathfrak{g}$, denoted $\mathfrak{g}^{\text{ent}}$, where more can be said about stability.

The bridge takes the form of a homogeneity result proved earlier by Waldspurger (in [19]), which states that

$$\text{res}_H \mathcal{D}^{\text{ent}} = \text{res}_H \mathcal{D}^{\text{nil}}.$$ 

Here, the restriction is to the set

$$H = \sum_C C_c(\mathfrak{g}/\mathfrak{g}_C),$$

where $C_c(\mathfrak{g}/\mathfrak{g}_C)$ is the subset of $C_c^\infty(\mathfrak{g})$ consisting of functions invariant under translation by elements of the parahoric subalgebra $\mathfrak{g}_C$, and the sum is over all alcoves of the Bruhat-Tits building $\mathcal{B}(G)$ of $G$. This allowed him to find his basis in terms of distributions that are supported on $\mathfrak{g}^{\text{ent}}$, which is composed of the elements of $\mathfrak{g}$ with integral eigenvalues; alternately,

$$\mathfrak{g}^{\text{ent}} = \bigcup_{x \in \mathcal{B}(G)} \mathfrak{g}_{x,0}.$$ 

$^1$In general, boldface letters denote algebro-geometric objects and their unbolded equivalents set-theoretic ones.

$^2$For which see, e.g., [7].
The distributions Waldspurger defines and then restricts to \( \mathcal{H} \) are associated to generalized Green functions, which are \( G \)-conjugation invariant functions on Lie groups defined over finite fields supported on the unipotent elements. Transported via the exponential map (whose existence, it should be noted, restricts the characteristic of the finite field to the set of good primes for \( G \)), they are also defined for Lie algebras and supported on the nilpotent elements. They can be divided into two basic types, the majority arising as Deligne-Lusztig characters, the rest as characteristic functions of cuspidal unipotent character sheaves as defined by Lusztig (in [11]). The latter kind are called “Lusztig functions” by Waldspurger, and were originally distinguished by their special properties vis à vis the Fourier transform: they are, up to scalar multiplication, the only nilpotently supported functions on the Lie algebra whose Fourier transform is also supported on \( g_{\text{nil}} \). We show that these functions are tied to the Fourier eigenfunctions in Waldspurger’s proto-basis, though the correspondence is not one-to-one.

In this paper, we deal solely with the case where \( G = \text{Sp}_{2n} \) and \( g = \text{sp}_{2n} \) over a \( p \)-adic field \( F \). We briefly recall the definition of Lusztig functions along with some facts about the parametrization of nilpotent orbits needed to support the definition. We go on to consider the inflation of generalized Green functions to \( g \) and show that among these functions the Fourier eigenfunctions are inflations of products of Lusztig functions, which leads us to define and enumerate the class of \( p \)-adic Lusztig functions. These functions do not correspond bijectively with the set of Lusztig functions—there are always more \( p \)-adic Lusztig functions than there are Lusztig functions. In fact, \( p \)-adic Lusztig functions exist on \( p \)-adic \( \text{sp}_{2n} \) over a \( p \)-adic field \( F \) when no such functions exist on \( \text{sp}_{2n} \) over the residue field of \( F \).

Finally, we recall the definition of Waldspurger’s distributions, \( \phi_\theta(X_T, -) \), and investigate the Fourier eigenspaces in their span. Our main result is the following:

**Theorem 4.1** Let \( g = \text{sp}_{2n}(F) \). Then

1. \( \hat{\phi}_\theta(X_T, -) \in \text{span}\{\phi_\theta(X_T, -)\}_{(\theta, X_T)} \) if and only if \( T \) is trivial (i.e., \( X_T = 0 \)).

2. If \( n \) be a triangular number or the sum of two triangular numbers, \( \text{span}\{\phi_\theta(X_T, -)\}_{(\theta, X_T)} \) contains a single
Fourier eigenspace $E$ with eigenvalue

$$\left( \text{sgn}(-1)q^{-\frac{1}{2}} \sum_{x \in \mathbb{F}_q} \text{sgn}(x) \psi(x) \right)^n.$$ 

Moreover, up to $G$-conjugacy,

$$\dim(E) = \begin{cases} 
2(d_1(8n + 2) - d_3(8n + 2)) & \text{n not triangular,} \\
2(d_1(8n + 2) - d_3(8n + 2) + 1) & \text{n triangular,}
\end{cases}$$

where $d_i(m)$ denotes the number of divisors of $m$ congruent to $i$ mod 4.

We additionally show that the eigenspace $E$ contains a 1-dimensional subspace consisting of stable distributions precisely when $n$ is twice a triangular number. To conclude, we describe a conjectural geometrization of the eigendistributions in Theorem 4.1.

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2 Preliminaries

2.1 Notation

Thoughout, $F$ is a non-archimedean local field with ring of integers $\mathcal{O}_F$, uniformizer $\varpi$, and finite residue field $\mathbb{F}_q$ of characteristic $p > 0$. We will take the valuation on $F$ to be normalized so that $v_F(\varpi) = 1$. Certain results, including the parametrization of nilpotent orbits, the existence of a Killing form and of a $G$-equivariant isomorphism between the unipotent subvariety of the Lie group and the nilpotent subvariety of the Lie algebra require that $p$ not be too small. In particular, the latter two require that $p$ be a good prime (for which see [17] and [18]).

Even though some results (particularly background ones) hold more generally, we will only consider the linear algebraic group $G = \text{Sp}_{2n}$ over $F$ and its Lie algebra $\mathfrak{g} = \text{sp}_{2n}$, unless declared otherwise. Let $G = G(F)$ and $\mathfrak{g} = \mathfrak{g}(F)$. If we wish to consider $G$, $G$, $\mathfrak{g}$, or $\mathfrak{g}$ over
alternate fields, like an algebraic closure $\overline{F}$ of $F$ or the residue field $\mathbb{F}_q$, that field will be included as a subscript: $G(\overline{F}) = G_{\overline{F}}, g(\overline{F}) = g_{\overline{F}},$ and so on.

Although for the most part it is unnecessary, there will be some instances where it will be useful to choose a representation for $G$ or $g$. When we do, $(V,q_V)$ will be a $2n$-dimensional vector space over $F$ with symplectic form $q_V$ given by $q_V(x,y) = x^tJy$, where $J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$. Therefore, after embedding in $\text{GL}_{2n}(F)$, $G(V) = \{g \in \text{GL}_{2n}(F) \mid g^tJg = J\}$ and $g(V) = \{h \in \text{gl}_{2n}(F) \mid h^tJ - Jh = 0\}$.

### 2.2 Nilpotent Orbits

This section contains a brief recollection of the parametrization of nilpotent orbits in semisimple Lie algebras, which plays a role in the definition of Lusztig functions. All of the material here is modelled on the paper by Nevins, [14].

First, recall that as long as $F$ has characteristic 0 or $p > 3(h - 1)$, $h$ the Coxeter number of $G$ ($h = 2n$ in the case considered here), as a consequence of Jacobson-Morozov theory (see, e.g., [4]), the nilpotent $G$-orbits of a finite semisimple Lie algebra $\mathfrak{g}$ over an algebraically closed field (which henceforward will be called geometric orbits) can be parametrized by appropriate partitions of $\dim V = 2n$. A partition of a number $c$ is a non-increasing sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_d)$ of positive integers such that $\lambda_1 + \cdots + \lambda_d = c$. For any $1 \leq i \leq n$, let $m_i = |\{\lambda_j \in \lambda \mid \lambda_j = i\}|$ be the multiplicity of $i$. In the case of $\mathfrak{g} = \mathfrak{sp}_{2n}(F)$, the admissible partitions are those where $m_i$ is even for every odd $i$. In the correspondence, the parts of a partition $\lambda$ give the sizes of the blocks in the Jordan normal form of the elements in the corresponding nilpotent orbit variety, which will be denoted $O_\lambda$.

Since every such orbit variety contains an $F$-rational point we can consider the set of $F$-rational $G_F$-orbits within $O_\lambda(F)$. For $\mathfrak{g} = \mathfrak{sp}_{2n}(F)$, these orbits are parametrized by isometry classes of quadratic forms, as in the following proposition.

**Proposition 2.1** ([14], Prop. 5.1). *Let $\lambda$ be a partition of $2n$ and let $m_j$ be the multiplicity of $j$ in $\lambda$. Assume that $m_j$ is even whenever $j$ is odd. The $G_F$-orbits in $O_\lambda(F)$ are parametrized by $n$-tuples $Q = (Q_2, Q_4, \ldots, Q_{2n})$.***
where $Q_j$ represents the isometry class of a nondegenerate quadratic form over $F$ of dimension $m_j$ ($Q_j = 0$ if $m_j = 0$).

By the Witt Decomposition Theorem, every nondegenerate quadratic form can be decomposed as a direct sum

$$Q = q_0^m \oplus Q_{\text{aniso}},$$

where $m \leq \frac{1}{2} \dim Q$, $q_0^m$ is the $m$-fold direct sum of the hyperbolic plane, which has matrix representative

$$q_0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

and $Q_{\text{aniso}}$ is $Q$’s anisotropic part (recall that a quadratic form is isotropic if there is an $x \neq 0$ such that $Q(x) = 0$ and anisotropic otherwise). Below, we reproduce a table (adapted from [14]), which lists diagonal matrix representatives for all isometry classes of quadratic forms by dimension. The quadratic forms $Q_{2i}$ in the proposition above can be chosen from among those in the table (replacing $\varpi^{-1}$ with 1 when working in $g_{\mathfrak{F}_q}$).

| Dimension | Representatives of Isometry Classes |
|-----------|-----------------------------------|
| 1         | $[t]$, $t \in \{1, \epsilon, \varpi^{-1}, \epsilon \varpi^{-1}\}$ |
| 2         | $\text{diag}(t, \epsilon t)$, $t \in \{1, \varpi^{-1}\}$ |
|           | $\text{diag}(t \varpi^{-1}, -t')$, $t, t' \in \{1, \epsilon\}$ |
| 3         | $\text{diag}(1, -\epsilon, t \varpi^{-1})$, $t \in \{1, \epsilon\}$ |
|           | $\text{diag}(\varpi^{-1}, -\epsilon \varpi^{-1}, t)$, $t \in \{1, \epsilon\}$ |
| 4         | $\text{diag}(1, -\epsilon, -\varpi^{-1}, \epsilon \varpi^{-1})$ |

Figure 1: Representatives of isometry classes of quadratic forms listed by dimension. The element $\epsilon$ is a nonsquare unit in $\mathcal{O}_F$.

Henceforward, we will denote by $\text{Nil}(g)$ the set of rational nilpotent orbits in $g$, and identify those elements with pairs $(\lambda, \mathcal{Q})$ consisting of a partition of $2n$ and an $n$-tuple of quadratic form representatives. Geometric orbits will simply be identified with partitions of $2n$.

**Example 2.2.** For reasons that will be made clear in the next section, we will be interested in rational orbits contained in the geometric orbit matching a partition of the form $\lambda = (2i, 2i - 2, \ldots, 4, 2)$ for some $i$.
or the union of two such partitions, meaning \( \lambda = (2i, 2i - 2, \ldots, 2j + 2, 2j, 2j, 2j - 2, 2j - 2, \ldots, 2, 2) \) for \( i \geq j \). Representatives in the first case follow a simple pattern that the following low-dimensional examples should make clear.

\[
\begin{align*}
sp_2(F) : & \quad \begin{bmatrix} 0 & Q_2 \\ Q_2 & 0 \end{bmatrix}, \\
sp_6(F) : & \quad \begin{bmatrix} 0 & Q_2 & 0 \\ Q_2 & 0 & 0 \\ 0 & 0 & -Q_4 \end{bmatrix},
\end{align*}
\]

where \( Q_2, Q_4, Q_6 \in \{1, \epsilon, \epsilon^{-1}, \epsilon \omega^{-1}\} \). In these cases, where each part in \( \lambda \) has multiplicity 1, only 1-dimensional (in particular, only anisotropic) quadratic forms appear.

For partitions of the second type we can look at \( sp_4(F) \), which contains a geometric orbit parametrized by \( \lambda = (2, 2) \). Rational orbit representatives there take the form

\[
\begin{bmatrix}
0 & Q_2 & 0 & 0 & 0 \\
Q_2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0
\end{bmatrix}
\]

or

\[
\begin{bmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

where \( Q_2, Q'_2 \in \{1, \epsilon, \epsilon^{-1}, \epsilon \omega^{-1}\} \). In \( sp_8(F) \), there is a geometric orbit parametrized by \( \lambda = (4, 2, 2) \). There, most rational orbit representatives take the form

\[
\begin{align*}
\begin{bmatrix}
0 & Q_2 & 0 & 0 & 0 & 0 & 0 & 0 \\
Q_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\end{align*}
\]

where, again, all entries are drawn from \( \{1, \epsilon, \epsilon^{-1}, \epsilon \omega^{-1}\} \).
2.3 Lusztig Functions

Throughout this section all Lie groups and algebras are assumed to be defined over a finite field $\mathbb{F}_q$.

In his Utrecht paper, [12], Lusztig investigated the existence of nilpotently supported, $\mathbb{Q}_\ell$-valued functions on semisimple finite Lie algebras defined over $\mathbb{F}_q$ whose Fourier transform is also nilpotently supported. Employing his theory of character sheaves [11, et seq.], he determined that such functions are actually quite rare ([12, Cor. 10]). Up to scalar multiplication there is at most one such function on $\mathfrak{sp}_{2n}(\mathbb{F}_q)$ and it exists exactly when $n$ is a triangular number (i.e., $n = i(i+1)/2$ for some $i$).

So, let us fix one of these scalar multiples and denote it by $^o f$, in imitation of Waldspurger in [20], where these functions are officially called “Lusztig functions.” If more than one Lusztig function appears, we will distinguish them with subscripts: $^o f_{2n}$, and so on. These functions are supported on the geometric orbit parametrized by the partition of $^o \lambda := (2i, 2i-2, \ldots, 4, 2)$ of $2n$. These will be called the Lusztig orbit and Lusztig partition of $\mathfrak{sp}_{2n}$, respectively. Finally, since they will be relevant after shifting attention to the $p$-adic field $F$, we call any partition that is a union of two Lusztig partitions an extended Lusztig partition.

As in section 2.2 rational orbits contained in a Lusztig orbit are represented by tuples of quadratic form representatives, either $[\lambda]$ or $[\epsilon]$, where $\epsilon$ is a nonsquare in $\mathbb{F}_q^\times$. Let $e([\lambda, \mathcal{Q}])$ be the characteristic function of the orbit corresponding to the pair $(\lambda, \mathcal{Q}) \in \text{Nil}(\mathfrak{g}_{\mathbb{F}_q})$, and define

$$^o \text{sgn}(\mathcal{Q}) = \prod_{i \text{ even}} \text{sgn}(-1)^{\lfloor \dim(Q_i)/2 \rfloor} \det(Q_i),$$

where $\det$ means the class of the determinant of $Q_i$ in $\mathbb{F}_q^\times/(\mathbb{F}_q^\times)^2$. Then,

$$^o f_{2n} = \sum_{(\epsilon, \mathcal{Q}) \in \text{Nil}(\mathfrak{g}_{\mathbb{F}_q})} ^o \text{sgn}(\mathcal{Q}) e([\epsilon, \mathcal{Q}]).$$

Lusztig shows that—even more than having Fourier transform supported on $\mathfrak{g}_{\text{null}}$—these functions are actually eigenfunctions of the Fourier transform. Their eigenvalues have been determined by Wald-
spurger [20, Prop. V.8]: For any $x \in \mathbb{F}_q$, let

$$\text{sgn}(x) = \begin{cases} 
0 & \text{if } x = 0, \\
1 & \text{if } x \in (\mathbb{F}_q^\times)^2, \\
-1 & \text{if } x \in \mathbb{F}_q^\times - (\mathbb{F}_q^\times)^2.
\end{cases}$$

Then, the eigenvalue of the Lusztig function on $\mathfrak{sp}_{2n}(\mathbb{F}_q)$ under the Fourier transform (defined with respect to a character $\psi: \mathbb{F}_q \to \mathbb{C}^\times$) is

$$\gamma_{2n}^\circ = \left( \text{sgn}(-1)q^{-\frac{1}{2}} \sum_{x \in \mathbb{F}_q} \text{sgn}(x)\psi(x) \right)^{\frac{k(k+1)}{2}}.$$

**Example 2.3.**
1. In the case of $\mathfrak{sp}_2(\mathbb{F}_q)$, the Lusztig function is

$$\gamma = e\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - e\begin{bmatrix} 0 & \epsilon \\ 0 & 0 \end{bmatrix},$$

the two terms being the characteristic functions of the rational orbit of the given representative.

2. For $\mathfrak{sp}_6(\mathbb{F}_q)$, the Lusztig function is

$$\gamma = e[(1, 1)] + e[(1, \epsilon)] - e[(\epsilon, 1)] - e[(\epsilon, \epsilon)].$$

### 3 $p$-adic Lusztig Functions

Our approach to finding the eigendistributions among Waldspurger’s distributions involves relating the known properties of generalized Green functions on finite Lie algebras to their inflations to $\mathfrak{g}$ over $F$, $F$ our $p$-adic field. The obvious way of mediating between these settings is through the parahoric subalgebras of $\mathfrak{g}$ and their associated reductive quotients.

For any point $x \in \mathcal{B}(\mathbb{G})$ of the Bruhat-Tits building of $\mathbb{G}$, we denote by $G_x$ and $\mathfrak{g}_x$ respectively the parahoric subgroup of $G$ and parahoric subalgebra of $\mathfrak{g}$ associated to $x$ (see [2] or [15]). Both depend only on the facet $\mathcal{F}$ of $\mathcal{B}(\mathbb{G})$ containing $x$, so we will also substitute $\mathcal{F}$ for $x$ in the notation, as in $G_{\mathcal{F}}$ and $\mathfrak{g}_{\mathcal{F}}$. If $r \in \mathbb{R}$, we denote the corresponding Moy-Prasad filtration subgroups and subalgebras (see [13]) by $G_{x,r}$ and $\mathfrak{g}_{x,r}$. Finally, we write $\mathcal{G}_x = G_{x,0}/G_{x,0+}$ and $\mathcal{G}_x = \mathfrak{g}_{x,0}/\mathfrak{g}_{x,0+}$ for the reductive quotients of $G$ and $\mathfrak{g}$ attached to $x$ (or a facet containing $x$).
3.1 Inflations

Generalized Green functions are defined in [20, Ch. II], where they are denoted \( Q_T \). Aside from mentioning that they come in two broad types, Deligne-Lusztig characters on one hand and Lusztig functions on the other, we will not reprise their definition here since finding the Fourier eigendistributions among the distributions Waldspurger defines only requires knowing the eigenfunctions among them. It is known, as a byproduct of the proof of [20, Prop. II.8], that these are precisely the Lusztig functions.

Definition 3.1. Let \( F \) be a facet of \( B(G) \) such that

\[
\mathfrak{g}_F \cong \mathfrak{sp}_{2\Delta_1}(\mathbb{F}_q) \times \mathfrak{sp}_{2\Delta_2}(\mathbb{F}_q) \times \cdots \times \mathfrak{sp}_{2\Delta_m}(\mathbb{F}_q),
\]

where \( \Delta_i \) is triangular for all \( i \). Let

\[
^0 f_F(Y) = \begin{cases} 
(^{0}f_{2\Delta_1} \times ^{0}f_{2\Delta_2} \times \cdots \times ^{0}f_{2\Delta_m}) \circ \rho_{F,0}(Y), & Y \in \mathfrak{g}_{F,0} \\
0, & Y \notin \mathfrak{g}_{F,0},
\end{cases}
\]

where \( \rho_{F,0} : \mathfrak{g}_{F,0} \to \mathfrak{g}_F \) is the reduction map. We call all such functions \( p \)-adic Lusztig functions.

The following proposition further justifies the name. In preparation, we fix a Fourier transform on \( \mathfrak{g} \). Fix a Killing form \( \langle \, , \rangle : \mathfrak{g} \times \mathfrak{g} \to \mathbb{F} \) for \( \mathfrak{g} \) and a nontrivial additive character \( \psi : \mathbb{F} \to \mathbb{C}^\times \) that is trivial on the maximal ideal \( \mathfrak{p}_F \) of \( F \). Then, for any \( f \in C_c^\infty(\mathfrak{g}) \),

\[
(\mathcal{F} f)(X) = \hat{f}(X) = \int_\mathfrak{g} \psi(\langle X, Y \rangle) f(Y) dY,
\]

with respect to the Haar measure normalized so that \( \text{meas}(\mathfrak{g}_{x,0+}) = \frac{1}{|\mathfrak{g}_x|^{\frac{1}{2}}} = \frac{1}{|\mathbb{F}_q|^{-\frac{\dim(\mathfrak{g}_x)}}} \) for all \( x \in B(G) \).

Proposition 3.2. In the setting of Definition 3.1, \( ^0 f_F \) is an eigenfunction of the Fourier transform on \( \mathfrak{g} \).

Proof. We begin by observing that in fact no more than two factors are required in Definition 3.1 to capture all \( p \)-adic Lusztig functions. As noted by J.-K. Yu in [10, §7.6], if \( \Phi \) is the set of roots of \( G \) with respect to a maximal \( F \)-torus \( T \), the root data of the reductive quotient

\footnote{The existence of such requires that \( p \neq 2 \).}
$\mathfrak{g}_F$ is $(X^*(T), \Phi_F, X_+(T), \Phi_F^>)$, where the subset $\Phi_F$ of $\Phi$ associated to the facet $F$ is the set of roots $\alpha \in \Phi$ such that there is an affine root $\alpha + n$ whose reflecting hyperplane contains $F$ (or, more precisely, the smallest affine space containing $F$ in the apartment determined by $T$). As Yu says, “this allows us to see the isomorphism class of $[\mathfrak{g}_F]$ by pure thought when $G$ is split adjoint or split simply connected and $F$ is a vertex.” We find ourselves in exactly this case, since we are indeed dealing with a split simply connected group, and if the reductive quotient is to have the shape required by Definition 3.1, its root system $(\mathcal{A}, \Phi_F, \mathcal{A}^\vee, \Phi_F^\vee)$ must be semisimple (i.e., $|\Phi_F| = \dim \mathcal{A}$). This cannot be the case if $F$ has positive dimension, since then the affine space containing $F$ must be spanned by some nonempty subset of $\Phi$. Thus, $F$ must be a vertex.

From this, it follows that every reductive quotient of $\mathfrak{g}$ that carries a Lusztig function has the form $\mathfrak{sp}_{2\Delta_1}(\mathbb{F}_q) \times \mathfrak{sp}_{2\Delta_2}(\mathbb{F}_q)$, where $n = \Delta_1 + \Delta_2$ and $\Delta_1$ and $\Delta_2$ are triangular numbers (taking $\mathfrak{sp}_0(\mathbb{F}_q)$ to be the trivial algebra). To see this, consider the extended Dynkin diagram of type $C_n$:

```
/------\(\alpha_0\)
|      |
\alpha_1------\alpha_2------\alpha_3------\alpha_{n-1}------\alpha_n
```

The roots corresponding to the nodes are $\alpha_0 = -2e_1$, $\alpha_n = 2e_n$, and $\alpha_i = e_i - e_{i+1}$, $1 \leq i \leq n - 1$. The Dynkin diagrams resulting from deleting a node from the extended diagram tell us the isomorphism classes of reductive quotients at vertices. Removing the 0th or $n^{th}$ node gives the Dynkin diagram of type $C_n$, corresponding to a reductive quotient isomorphic to $\mathfrak{sp}_{2\Delta_1}(\mathbb{F}_q)$. Removing the $i^{th}$ node for $1 \leq i \leq n - 1$ gives a disconnected diagram corresponding to $\mathfrak{sp}_{2\Delta_1}(\mathbb{F}_q) \times \mathfrak{sp}_{2(n-i)}(\mathbb{F}_q)$.

Therefore, we can assume $^0f_F$ is the inflation of a Lusztig function or the product of two Lusztig functions. With the Haar measure on $\mathfrak{g}$ normalized as above, we have that $\mathcal{F}(^0f_F) = (\mathcal{F}(^0f_{\Delta_1} \times ^0f_{\Delta_2}))_F = \mathcal{F}(^0f_{\Delta_1} \times ^0f_{\Delta_2}) \circ \rho_{F,0}$, where $\rho_{F,0}$ is the restriction map; i.e., the Fourier transform commutes exactly with inflation. This follows from [6] Prop. 1.13. It suffices to show, then, that $^0f_{\Delta_1} \times ^0f_{\Delta_2}$ is an eigenfunction of the Fourier transform on $\mathfrak{sp}_{2\Delta_1}(\mathbb{F}_q) \times \mathfrak{sp}_{2\Delta_2}(\mathbb{F}_q)$.

By the convolution theorem, $\mathcal{F}(^0f_{\Delta_1} \times ^0f_{\Delta_2}) = c(\mathcal{F}(^0f_{\Delta_1} \times 1)) * (\mathcal{F}(1 \times ^0f_{\Delta_2}))$ for some constant $c$. With normalizations as above,
c = 1, so we have
\[
\mathcal{F}(\circ f_{2\Delta_1} \times \circ f_{2\Delta_2}) = (\mathcal{F}(\circ f_{2\Delta_1} \times 1)) \ast (\mathcal{F}(1 \times \circ f_{2\Delta_2}))(s, t)
\]
\[
= \int \mathcal{F}(\circ f_{2\Delta_1} \times 1)(\sigma, \tau) \mathcal{F}(1 \times \circ f_{2\Delta_2})(s - \sigma, t - \tau) d\sigma d\tau
\]
\[
= \int \delta_0(\tau) \mathcal{F}(\circ f_{2\Delta_1})(\sigma) \delta_0(s - \sigma) \mathcal{F}(\circ f_{2\Delta_2})(t - \tau) d\sigma d\tau
\]
\[
= \mathcal{F}(\circ f_{2\Delta_1})(s) \cdot \mathcal{F}(\circ f_{2\Delta_2})(t)
\]
\[
= \circ \gamma_{2\Delta_1} \circ \gamma_{2\Delta_2} \circ f_{2\Delta_1} \circ f_{2\Delta_2},
\]
where \(\circ \gamma_{2\Delta_1}\) and \(\circ \gamma_{2\Delta_2}\) are the eigenvalues of \(\circ f_{2\Delta_1}\) and \(\circ f_{2\Delta_2}\) respectively, which exist by [12, Cor. 10]. With that, we are done. \(\square\)

Remark 3.3. Note that, from the formula for the eigenvalues proved by Waldspurger (recalled in §2.3), \(\circ \gamma_{2\Delta_1}\) and \(\circ \gamma_{2\Delta_2}\) differ only in their exponents, which means that the eigenvalue of \(\circ f_{2\Delta_1} \times \circ f_{2\Delta_2}\) is

\[
\circ \gamma_{2\Delta_1} \circ \gamma_{2\Delta_2} = \left( \text{sgn}(-1)q^{\frac{1}{2}} \sum_{x \in \mathbb{F}_q} \text{sgn}(x) \psi(x) \right)^{\Delta_1 + \Delta_2}.
\]

With our normalizations, this is the eigenvalue of the inflation of \(\circ f_{2\Delta_1} \times \circ f_{2\Delta_2}\) to \(g\) as well.

Proposition 3.4. The distinct nontrivial \(p\)-adic Lusztig functions on \(g = sp_{2n}(F)\) can be classified up to conjugacy by the following conditions on \(n\):

1. If \(n\) is triangular, there exist two \(p\)-adic Lusztig functions on \(g\) up to conjugacy, corresponding to \(G\)-conjugacy classes of hyperspecial vertices in \(\mathcal{B}(Sp_{2n})\).

2. For each representation of \(n\) as a sum of distinct triangular numbers, \(\Delta_1\) and \(\Delta_2\), there exist two \(p\)-adic Lusztig functions on \(g\).

3. If \(n = 2\Delta, \Delta\) a triangular number, there exists one \(p\)-adic Lusztig function on \(g\) obtained by inflation from the reductive quotient \(sp_n \times sp_n\).

Therefore, the dimension of the \(p\)-adic Lusztig functions up to conjugacy is equal to the number of noncommuting representations of \(n\) as a sum of less than three triangular numbers.

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Proof. 1. \( n \) is triangular: Given the foregoing, it is clear that the functions here arise as inflations of the Lusztig functions that exist on \( \mathfrak{sp}_{2n}(\mathbb{F}_q) \), and thus correspond to vertices that give this reductive quotient, \( i.e., \) the hyperspecial vertices. The only point that might require justification is the implicit assertion that there are only two conjugacy classes of hyperspecial vertices in \( B(\mathbb{S}p_n) \). This is proved in \[16\], see pg. 3], and correspond to the first and last nodes of the extended Dynkin diagram in the proof of Proposition 3.2.

2. \( n = \Delta_1 + \Delta_2 \): In light of 3.2, the claim here is just that whenever \( n \) has this form there are two vertices such that

\[
\overline{B}_x = \mathfrak{sp}_{2\Delta_1}(\mathbb{F}_q) \times \mathfrak{sp}_{2\Delta_2}(\mathbb{F}_q).
\]

From \[10\] §7.6, and as can be seen in the proof of Proposition 3.2 these two vertices exist: they correspond to the \( \Delta_1 + 1 \)th and \( \Delta_2 + 1 \)th nodes in the extended Dynkin diagram.

3. \( n = 2\Delta \): Same argument as above, yielding in this case only one vertex.

Example 3.5. 1. The Lusztig function on \( \mathfrak{sp}_2(\mathbb{F}_q) \) is

\[
\circ f_2 = e \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - e \begin{bmatrix} 0 & \epsilon \\ 0 & 0 \end{bmatrix}.
\]

In this case there are two conjugacy classes of parahorics available to inflate this function through. Working in the standard apartment, we take \( x_0 \) and \( x_1 \) to be hyperspecial vertices corresponding to these two classes of parahorics,

\[
\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \ \begin{bmatrix} 0 & \epsilon \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & \varpi^{-1} \\ 0 & 0 \end{bmatrix}, \ \begin{bmatrix} 0 & \epsilon \varpi^{-1} \\ 0 & 0 \end{bmatrix}
\]

as representatives for the four distinguished rational nilpotent orbits in the Lusztig orbit of \( \mathfrak{sp}_2(F) \), and write \( e[(1)] \), \( e[(\epsilon)] \), \( e[(\varpi)] \), and \( e[(\epsilon \varpi)] \) for the characteristic functions of those orbits. Then the \( p \)-adic Lusztig functions on \( \mathfrak{sp}_2 \) over \( F \) coming from this apartment are

\[
\circ f_{x_0} = \begin{cases} e[(1)] - e[(\epsilon)] & \text{on } \mathfrak{sp}_2(F)_{x_0} \\ 0 & \text{on } \mathfrak{sp}_2(F) \setminus \mathfrak{sp}_2(F)_{x_0} \end{cases}
\]

\[
\circ f_{x_1} = \begin{cases} e[(\varpi^{-1})] - e[(\epsilon \varpi^{-1})] & \text{on } \mathfrak{sp}_2(F)_{x_1} \\ 0 & \text{on } \mathfrak{sp}_2(F) \setminus \mathfrak{sp}_2(F)_{x_1} \end{cases}
\]
2. The Lusztig function on $\mathfrak{sp}_6(\mathbb{F}_q)$ is

$$\circ f = e[(1,1)] + e[(1,\epsilon)] - e[(\epsilon,1)] - e[(\epsilon,\epsilon)].$$

Again there are two relevant conjugacy classes of parahoric associated to hyperspecial vertices, denoted $\mathfrak{sp}_{6,x_0}$ and $\mathfrak{sp}_{6,x_1}$. The distinguished orbits for each parahoric are:

- $\mathfrak{sp}_{6,x_0}$: $(1,1)$, $(1,\epsilon)$, $(\epsilon,1)$, $(\epsilon,\epsilon)$
- $\mathfrak{sp}_{6,x_1}$: $((\omega^{-1},\omega^{-1}), (\omega^{-1},\epsilon\omega^{-1}), (\epsilon\omega^{-1},\omega^{-1}), (\epsilon\omega^{-1},\epsilon\omega^{-1})$

so the two $p$-adic Lusztig functions are

$$\circ f_{x_0} = e[(1,1)] + e[(1,\epsilon)] - e[(\epsilon,1)] - e[(\epsilon,\epsilon)]$$

$$\circ f_{x_1} = e[(\omega^{-1},\omega^{-1})] + e[(\omega^{-1},\epsilon\omega^{-1})] - e[(\epsilon\omega^{-1},\omega^{-1})] - e[(\epsilon\omega^{-1},\epsilon\omega^{-1})]$$

on their respective parahorics, and trivial outside them.

4 The Main Result

In this section, $G$ and $g$ are defined over the $p$-adic field $F$.

Recall the following definitions from the introduction: $\mathcal{C}_\infty^c(g)$ is the $\mathbb{C}$-algebra of locally constant, compactly supported complex-valued functions on $g$, $\mathcal{D}$ is the space of invariant distributions on $g$ (those elements of $(\mathcal{C}_\infty^c(g))^\vee$ that are invariant under the action of $G$), $\mathcal{D}(\omega)$ is the subset of elements of $\mathcal{D}$ supported on the subset $\omega$. Finally, the subset $\mathcal{H}$ is

$$\mathcal{H} = \sum_C \mathcal{C}_c(g/C),$$

the sum over all alcoves $C$ of $B(G)$. Each summand is the subset of $\mathcal{C}_\infty^c(g)$ consisting of elements invariant under translation by elements of the parahoric subalgebra $g_C$.

4.1 Distributions From Inflations of Generalized Green Functions

We begin by describing the distributions Waldspurger associates to generalized Green functions. In passing from $\mathbb{F}_q$ to the $p$-adic field
he makes use of a parameter set \( \Theta(V) \) consisting of 5-tuples of combinatorial data. The eigenfunctions are those indexed by \( \theta = (k', k'', \emptyset, \emptyset, \emptyset) \) where \( k'(k' + 1) + k''(k'' + 1) = 2n \), together with an element \( X_T \) of the torus associated to the corresponding generalized Green function, \( Q_T \).

To describe the distributions without wading into the intricacies of \( \Theta(V) \), we will say that \( \theta \in \Theta(V) \) consists of the following data:

- An orthogonal decomposition \( V = V_0 \oplus V_1 \);
- a maximal unramified torus \( T \leq G(V_1) \) defined over \( F \);
- a facet \( \mathcal{F} \) of \( \mathcal{B}(G(V_0)) \) such that there exists a \( p \)-adic Lusztig function \( ^o f_\mathcal{F} \), i.e., such that there is a Lusztig function or product of Lusztig functions defined on \( g(V_0) \).

Given such a parameter, choose an element \( X_T \in \text{Lie}(T)(F) \) "integral of regular reduction,” i.e., with \( Z_G(X_T) = T \) and whose image in \( g(F) \) has centralizer equal to the image of \( t \) in \( g(G) \).

Then, with Haar measure normalized as described in Section 3.1 define, for every \( f \in C_c^\infty(g) \),

\[
\phi_\theta(X_T, f) = \left| G_F \right|^{-\frac{1}{2}} \left| \mathfrak{g}_F \right|^{-\frac{1}{2}} \int_{T \backslash G} \int_{g(V_0)} f(x^{-1}(X_T + Y)x)^{\circ f_\mathcal{F}(Y)} dY dX,
\]

where \( ^o f_\mathcal{F} \) is the \( p \)-adic Lusztig function that comes packaged with \( \theta \). Let \( \Phi = \{ \phi_\theta(X_T, -) \mid \theta \in \Theta(V), X_T \in t_{1,r} \} \). By definition, \( \Phi \subseteq \mathcal{D}_{\text{ent}} \). Waldspurger goes on to prove that they no longer depend on the choice of \( X_T \) by the time they are restricted to \( \mathcal{H} \) (see [20, Cor. III.5]). Moreover, those restrictions, denoted \( \phi_\theta^\mathcal{H} \), form a basis for \( \text{res}_\mathcal{H} \mathcal{D}(\mathfrak{g}_{\text{ent}}) \) ([20, Cor. III.10(ii)]).

### 4.2 Fourier Eigendistributions in \( \text{span} \Phi \)

The Fourier transform of a distribution \( D \in C_c^\infty(g)^\vee \) is defined to be

\[
\hat{D}(f) = D(\hat{f}).
\]

**Theorem 4.1.** Let \( \mathfrak{g} = \mathfrak{sp}_{2n}(F) \). Then

1. \( \hat{\phi}_\theta(X_T, -) \in \text{span} \Phi \) if and only if \( T \) is trivial (i.e., \( X_T = 0 \)).

2. If \( n \) be a triangular number or the sum of two triangular numbers, \( \text{span} \Phi \) contains a single Fourier eigenspace \( E \) with eigenvalue

\[
\left( \frac{\text{sgn}(-1)q^{-\frac{1}{2}} \sum_{x \in \mathbb{F}_q} \text{sgn}(x)\psi(x)}{q^{\frac{1}{2}}} \right)^n.
\]
Moreover, up to $G$-conjugacy,

$$\dim(E) = \begin{cases} 
2(d_1(8n + 2) - d_3(8n + 2)) & n \text{ not triangular}, \\
2(d_1(8n + 2) - d_3(8n + 2) + 1) & n \text{ triangular},
\end{cases}$$

where $d_i(m)$ denotes the number of divisors of $m$ congruent to $i$ mod 4.

**Proof.**

1. Let $f \in C_c^\infty(g)$. Then $\hat{\phi_0}(X_T, f) = \phi_0(X_T, \hat{f})$. By [20, Prop. I.9], $\hat{f}$ is supported on the set of topologically nilpotent elements of $g$. Since $X_T$ is semisimple with regular reduction, the region of integration for $\phi_0$ does not intersect this set unless $X_T = 0$; hence $\text{res}(\hat{\phi_0}(X_T, -)) = 0$ when $X_T \neq 0$. Since the $\phi_0^H$ form a basis for $\text{res}_H(D_{\text{ent}})$, if $\hat{\phi_0}(X_T, -)$ were to lie in span $\Phi$, it would be trivial, which is not the case.

On the other hand, suppose $X_T = 0$, which can only be true if $T$ itself is trivial. Then, for any $f \in C_c^\infty(g)$,

$$\hat{\phi_0}(0, f) = |G_F|^{-\frac{1}{2}} |G_F|^{-\frac{1}{2}} \int_G \int_g \hat{f}(x^{-1}Yx)^\gamma f_Y(Y)dY dx$$

$$= |G_F|^{-\frac{1}{2}} |G_F|^{-\frac{1}{2}} \int_G \int_g f(x^{-1}Yx)^\gamma f_Y(Y)dY dx$$

$$= |G_F|^{-\frac{1}{2}} |G_F|^{-\frac{1}{2}} \int_G \int_g f(x^{-1}Yx)^\gamma f_Y(Y)dY dx$$

$$= |G_F|^{-\frac{1}{2}} |G_F|^{-\frac{1}{2}} \int_G \int_g f(x^{-1}Yx)^\gamma f_Y(Y)dY dx$$

The third equality follows from Proposition [3.2].

2. The first statement follows from the proof above together with the observations in Remark [3.3]. The rest follows from Proposition [3.4] together with the fact that the number of representations of $n$ as a sum of two triangular numbers is equal to the number of representations of $8n + 2$ as a sum of two odd squares, which is proved in [8], as is the fact that this number is equal to $d_1(8n + 2) - d_3(8n + 2)$ when it is not 0.

In light of the relation the eigendistributions have to Lusztig functions and the fact that they bear similar properties, we make the following definition.
Definition 4.2. The distributions in span Φ with $X_T = 0$ are called Lusztig distributions.

Corollary 4.3. A basis for the Fourier eigenspace $E$ in span Φ is given by the Lusztig distributions.

Hence, the original generating set defined by Waldspurger contains a basis for its Fourier eigenspace.

5 Stable Eigendistributions

Since much of [20] is concerned with stability of distributions, it’s worth asking whether any of the Fourier eigendistributions described in Theorem 4.1 are stable. Indeed there are stable eigendistributions, though they are relatively rare, and correspond to the $p$-adic Lusztig functions described in case 3 of Proposition 3.4.

Proposition 5.1. The Fourier eigenspace $E$ of span Φ on $\mathfrak{sp}_{2n}(F)$ contains a 1-dimensional subspace consisting of stable distributions exactly when $n = 2\Delta$, $\Delta$ a triangular number.

Proof. Establishing this without referring to Waldspurger’s parameter sets is impossible, but we will avoid explaining them in full here. The reader is referred to [20, III.1, IV.1] to verify our claims about them.

According to [20] Thm. IV.13, a basis for $\text{res}_H(\mathcal{D}^{st} \cap \mathcal{D}_{ent})$ is given by a set of distributions denoted $\phi^H_{\xi,T}$ where the parameter $\xi$ comes from $\Xi^{st}(V)$. This parameter corresponds to possibly more than one 5-tuple $(k', k'', \mu^0, \mu', \mu'') = \theta \in \Theta(V)$ such that $k' = k''$, and $\phi^H_{\xi,T}$ is a linear combination of $\phi^H_{\theta}$ for those $\theta$. If any of these $\phi^H_{\theta}$ is restricted from a Fourier eigendistribution in span Φ, then $\theta = (k', k', \emptyset, \emptyset, \emptyset)$ where $k'$ is triangular and $k'(k' + 1) = n$. For any $\xi \in \Xi^{st}(V)$, there is at most one such $\theta$, which, characterized as in $\S$ 4.1, corresponds to

- the trivial decomposition $V = V$,
- $\mathbb{T} = \{1\}$,
- a facet $\mathcal{F}$ of $\mathcal{B}(G)$ such that $\mathfrak{T}_\mathcal{F} \cong \mathfrak{sp}_{2k'}(F) \times \mathfrak{sp}_{2k'}(F)$.

For $\phi_\theta(X_T, -)$ with such a $\theta$ we must have $X_T = 0$. Therefore, $\phi^H_{\xi,T}$ is the restriction of exactly one distribution from span Φ: $\phi_\theta(0, -)$ associated to a $p$-adic Lusztig function of the kind appearing in case 3 of Proposition 3.4 which exists if and only if $n = 2\Delta$ where $\Delta$ is triangular. In this case, $\Delta = k'$.

□
6 Geometrization

We conclude with some comments concerning a question that naturally emerges considering the origins of Lusztig functions noted in §2.3—namely, that they are of geometric origin, arising as characteristic functions of cuspidal unipotent character sheaves. Is the same is true of p-adic Lusztig functions? Do there exist character sheaves defined on some variety over $F$ from which p-adic Lusztig functions can be recovered? Or, since we’re speaking of character sheaves, it might be better to ask the same questions of Lusztig distributions.

In either case, the answer appears to be yes. Here is a sketch of how such a picture might be realized\footnote{Seeing as this last section is informal and conjectural, we elect to skip any definitions not already given, all of which can be found in, e.g., the author’s thesis \cite{3}.} Let $O_\lambda$ be a nilpotent orbit variety over $F$ matching an extended Lusztig partition $^\circ\lambda$ (an open subvariety of $\mathfrak{sp}_{2n}$). Every rank 1 $\mathbb{Q}_\ell$-local system $L$ (for a prime different from $p$) on the smooth part\footnote{which is the orbit minus a point, so we will simply refer to it as if it were the entire orbit here} of $O_\lambda$ is equivalent to an $\ell$-adic character $\chi_L: \pi_1(O_\lambda,F_q,x) \rightarrow \mathbb{Q}_\ell^\times$ of the étale fundamental group of $O_\lambda$. If $^\circ\lambda = ^\circ\lambda_1 \cup ^\circ\lambda_2$ is a decomposition of $^\circ\lambda$ as a union of two Lusztig partitions, this fundamental group is isomorphic to the product

$$\left(\pi_1(O_{^\circ\lambda_1,F_q},x) \times \pi_1(O_{^\circ\lambda_2,F_q},x)\right) \times \text{Gal}(\overline{F}/F),$$

where $O_{^\circ\lambda_i,F_q}$ is the nilpotent orbit variety for the Lusztig partition $^\circ\lambda_i$ over the algebraic closure of the residue field $F_q$ of $F$ and $\overline{F}$ is a separable closure of $F$.

The character $\chi_{^\circ f_1}$ of $\pi_1(O_{^\circ\lambda,\overline{F}_q},x)$ equivalent to the local system whose characteristic function is the Lusztig function $^\circ f_1$ supported on $O_{^\circ\lambda,\overline{F}_q}(\overline{F}_q)$ is known (and given by Waldspurger in \cite{20} II.IV, II.V). There are at most two conjugacy classes of $p$-adic Lusztig functions obtained by inflating $^\circ f_1 \times ^\circ f_2$. These two functions should be recoverable from the local systems on $O_\lambda$ equivalent to the characters $(\chi_{^\circ f_2} \otimes \chi_{^\circ f_3}) \otimes \text{id}$ and $(\chi_{^\circ f_2} \otimes \chi_{^\circ f_3}) \otimes \text{sgn}(\varpi)$ of the fundamental group of $O_\lambda$, where id is the trivial character of $\text{Gal}(\overline{F}/F)$ and $\text{sgn}(\varpi)$ is the character of $\text{Gal}(\overline{F}/F)$ that factors through the nontrivial character of the quotient group $\text{Gal}(F(\sqrt{\varpi})/F)$.
Recovering a $p$-adic Lusztig function $\circ f_{\mathcal{F}}$ from such a local system requires the existence of an appropriate integral model, $\mathcal{O}_{\lambda,\mathcal{F}}$, which will be matched to one or the other of the local systems above, whose $\mathcal{O}_F$-points are $\mathcal{O}_\lambda(F) \cap g_{\mathcal{F},0}$. When such a model exists the characteristic function of the nearby cycles of the local system will be $\circ f_1 \times \circ f_2$, from which $\circ f_{\mathcal{F}}$ can be obtained by inflation. A key feature of this machinery is the Galois characters distinguishing the two local systems, which give an action of $\text{Gal}(\overline{F}/F)$ on the nearby cycles, a sheaf on a variety over $\mathbb{F}_q$. This action makes it impossible to obtain a characteristic function from the nearby cycles when they are taken with respect to an integral model that is not matched to the local system on $\mathcal{O}_\lambda$.

A detailed recounting of this process, as well as how it can be used to produce distributions, can be found in the author’s thesis [3], where its validity is proved for $\mathfrak{sp}_2$. The only obstruction to establishing this picture in general for the symplectic case (and the odd orthogonal case, which is quite similar) is proving the existence of the appropriate integral models and carrying out the nearby cycles calculations for them. It seems this could be accomplished by proving the existence of a weak Neron model (see [1], §3.5 Def.1) for $\mathcal{O}_\lambda$. The author presumes this can be done but is ignorant of such a proof at present. Beyond idle curiosity, an answer to this question would be a meaningful step in extending recent work of Cunningham and Roe in [5] that expands Lusztig’s geometric approach to character theory beyond the realm of finite fields.

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