A Fourth-Order Cumulants Orthonormal Propagator Rooting Method Based on Toeplitz Approximation

Heping Shi

Email: shiheping@tju.edu.cn

*Corresponding author

Ning Ma

Email: maning@tju.edu.cn

Zhiwei Guan

Email: zhiwguan@163.com

Lizhu Zhang

Email: zlzjcb@163.com

Shan Jiang

Email: jiangshan@catarc.ac.cn

1. School of Automobile and Transportation, Tianjin University of Technology and Education, Tianjin 300222, China

2. School of Electronic Engineering, Tianjin University of Technology and Education, Tianjin 300222, China

3. Tianjin Sino-German University of Applied Sciences, Tianjin 300350, China

4. Auto Testing Research Institute of China Automotive Technology and Research Center, Tianjin, 300300, China

Abstract: A novel Toeplitz fourth-order cumulants (FOC) orthonormal propagator rooting method (TFOC-OPRM) to direction-of-arrival (DOA) estimation for uniform linear array (ULA) is
addressed in this paper. Specifically, the modified (reduced-dimension) FOC (MFOC) matrix is achieved at first via removing the redundant information encompassed in the primary FOC matrix, and then the TFOC matrix which possesses Toeplitz structure can be recovered by utilizing the Toeplitz approximation method. To reduce computational complexity, we adopt an effective method which depends on the polynomial rooting technology. Finally, the DOAs of incident signals can be estimated by exploiting orthonormal propagator rooting method. The theoretical analysis coupled with simulation results show that the proposed resultant algorithm can reduce computational complexity significantly, as well as improve the estimation performance in both spatially-white noise and spatially-color noise environments.

**Keywords:** Direction-of-arrival (DOA); fourth-order cumulants (FOC); polynomial rooting; Toeplitz approximation; orthonormal propagator method (OPM)

1 Introduction

Direction-of-arrival estimation based on antenna array is one of the important directions of research hotspot in array signal processing, which has wide prospects of application in military and civil fields such as wireless communications, radar, passive sonar, biomedicine and seismic exploration [1-4]. Various high-resolution algorithms, such as multiple signal classification algorithm (MUSIC) [5] and estimating signal parameter via rotational invariance techniques (ESPRIT) [6] approaches, have been proposed to estimate direction-of-arrivals (DOAs) estimation of narrowband far-field signal sources. However, these subspace-based DOA estimation algorithms described above are not only very sensitive to the noise, but also require the noise’s characteristics of the sensors in advance. Furthermore, it is restricted that the total number of sources acting on the array must be less than or equal with that of
sensors [7]. When the constrained condition cannot be met in practical environments, the estimation performance of those aforementioned algorithms may run into a stone wall. Fortunately, much more attentions has been paid to this issue, and much more efforts has been made to overcome the above drawbacks. Motivated by the truth that high-order cumulants-based (HOC) has been recognized as a promising technique for direction finding by adopting sensor array [8-10]. Besides, another key motivation of using HOC is the ability to resolve more number of sources than or equal to that of the array elements [11]. However, the process of eigenvalue decomposition (EVD) or singular value decomposition (SVD) requires large amount of calculation and time taken, which greatly affects the development of rapid source location. Marcos and co-workers [12-13] firstly proposed so-named propagator method (PM) to obtain the signal and noise subspaces by executing a linear-partition operation, which can decrease the computational complexity effectively. Specifically, the performance of the PM algorithm under the conditions of medium and high signal-to-noise ratio (SNR) can achieve the same as that of the traditional high-resolution algorithms but with higher calculation efficiency. Base on [12-13], numerous modifications of PM methods have been proposed, such as [14-15], to achieve low-complexity DOA estimation. In [16], an efficient HOPM algorithm is proposed by making full use of intrinsic multi-dimensional characteristics and affordable computability. A FOC-based and OPM-like (FOC-OPM) algorithm [17] is proposed to gain good location performance. However, the computational complexity of this method is high due to a great number of redundant information is still existed in the FOC matrix. To mitigate this shortcoming, the improved FOC algorithm [18] is proposed to low the computational complexity. However, the performance of the algorithm cannot be asymptotically optimal due to the estimation error of the FOC matrix. Zhang et al. [19] derives a root-MUSIC method using a co-prime linear array to improve the estimation accuracy
with low complexity. In [20-22], a similar polynomial root-based method is chosen to realize low-complexity for DOAs estimation.

In this paper, a novel TFOC-OPRM algorithm is introduced. The contributions of this paper is twofold: Firstly, the reduced dimension matrix is obtained to reduce computational complexity by removing a large number of redundant elements from the original FOC matrix while maintaining the effective aperture of the virtual array in unchanged state. Secondly, the Toeplitz structure is recovered by the Toeplitz operation of the reduced dimension FOC matrix, and the DOA estimation of the recovered Toeplitz structure matrix is performed based on the polynomial root method.

2 Data Model

Consider $M$ narrowband far-field sources $s_l(t), (l = 1, \ldots, M)$ impinging on a uniform linear array (ULA) with $N$ equispaced omnidirectional sensors, where the distance between adjacent sensors is equal to half the wavelength. Assume that the incoming sources are stationary and mutually independent. The noise is the additive white/color Gaussian one, and statistically independent of the sources. Let the first sensor be the reference, and then the observed data received in time $t$ at the $k$th sensor can be expressed as

$$x_k(t) = \sum_{i=1}^{M} a_k(\theta_i) s_i(t) + n_k(t), \quad k = 1, \ldots, N$$

(1)

where $s_i(t)$ is the $i$th source, $n_k(t)$ is the Gaussian noise at the $k$th sensor and $a_k(\theta_i)$ is the response of $k$th sensor corresponding to the $i$th source.

$$a_k(\theta_i) = \exp(j2\pi(d/\lambda)k \sin \theta_i)$$

(2)

where $\lambda$ is the central wavelength, $d$ is the spacing between two adjacent sensors. Therefore, the matrix form of (1) can be expressed as

$$X(t) = AS(t) + N(t)$$

(3)
where \( \mathbf{X}(t) = [x_1(t), \cdots, x_N(t)]^T \) is the \( N \times 1 \) received source vector, \( \mathbf{S}(t) = [s_1(t), \cdots, s_M(t)]^T \) is the \( M \times 1 \) radiating source vector, \( \mathbf{A} = [a(\theta_1), \cdots, a(\theta_M)] \) is the \( N \times M \) array manifold matrix and \( \mathbf{N}(t) = [n_1(t), \cdots, n_N(t)]^T \) denotes the \( N \times 1 \) complex Gaussian noise vector.

Assume that the source signals are zero-mean stationary random process, the FOC can be defined as

\[
\begin{align*}
\text{cum}(k_1, k_2, k_3, k_4) &= E(x_{k_1}(t)x_{k_2}(t)x_{k_3}(t)x_{k_4}(t)) - E(x_{k_1}(t)x_{k_2}(t))E(x_{k_3}(t)x_{k_4}(t)) - \\
& E(x_{k_1}(t)x_{k_2}(t)x_{k_3}(t))E(x_{k_4}(t)) - E(x_{k_1}(t)x_{k_2}(t)x_{k_3}(t)x_{k_4}(t)) \\
& k_1, k_2, k_3, k_4 \in [1, \cdots, N]
\end{align*}
\]

where \( x_{k_m} \) \((m = 1, 2, 3, 4)\) is the stochastic process. Apparently, \( \text{cum}(k_1, k_2, k_3, k_4) \) has \( N^4 \) values with the change of \( k_1, k_2, k_3, k_4 \). For simplicity, equation (4) can be written in matrix form, which is denoted by cumulants matrix \( \mathbf{C}_4 \), and \( \text{cum}(k_1, k_2, k_3, k_4) \) appears as the \([(k_1 - 1)N + k_2]^{th} \) row and \([(k_3 - 1)N + k_4]^{th} \) column of \( \mathbf{C}_4 \).

\[
\begin{align*}
\mathbf{C}_4[(k_1 - 1)N + k_2, (k_3 - 1)N + k_4] &= \text{cum}(k_1, k_2, k_3, k_4) \\
& = \mathbf{BC}_3\mathbf{B}^H
\end{align*}
\]

where \( \mathbf{B} \) and \( \mathbf{C}_3 \) represent the extended array manifold and the FOC matrix of incident source signals, respectively. \( \mathbf{B} = \mathbf{A} \otimes \mathbf{A} \), and each column of \( \mathbf{B} \) is \( \mathbf{b}(\theta) = a(\theta) \otimes a(\theta) \). It is obvious that \( \mathbf{b}(\theta) \) is a \( N^2 \times 1 \) vector, which means that the array aperture of ULA is extended. That is, the number of resolved source signals is no less than that of sensors.

3 The Proposed Method

3.1 The effective array aperture extended

As proven in [23], an array of \( N \) arbitrary identical omnidirectional sensors can be extended to at most \( N^2 - N + 1 \). Especially, the number of virtual elements is \( 2N - 1 \) for ULA according to [23]. In order to discuss the effective aperture of ULA, four real elements \((N = 4)\) are considered, and \( \mathbf{b}(\theta) \) can be expressed in detail as follows
\[ b(\theta) = a(\theta) \otimes a(\theta) \]
\[ = [1, z, z^2, z^3, z^4, z^5, z^6]^\top \]  \hspace{1cm} (6)

where \( z = \exp(j2\pi(d/\lambda)\sin \theta) \). Equation (6) shows that there is a lot of redundancy in expanded steering vector \( b(\theta) \). That is, only from \( 1 \)th to \( N \)th and all \( kN \)th \( (k = 2, \cdots, N) \) items of the \( b(\theta) \) are valid, while others are redundant ones. To eliminate these repetitive elements, a \((2N-1) \times (2N-1)\) matrix \( R_4 \) is defined firstly. Next, the \( 1 \)th to \( N \)th and all \( kN \)th \( (k = 2, \cdots, N) \) rows of \( C_4 \) are taken out in sequence, and then store these rows in the \( 1 \)th to \((2N-1)\)th row of the new matrix \( R_4 \). The same operation is performed on the \( 1 \)th to \( N \)th and all \( kN \)th \( (k = 2, \cdots, N) \) columns of \( C_4 \) to obtain the \( 1 \)th to \((2N-1)\)th columns of \( R_4 \). Similar to equation (5), \( R_4 \) can be expressed as

\[ R_4 = DC_4^H D \]  \hspace{1cm} (7)

where \( D \) denotes the extended array manifold without redundancy, and each column of \( D \) has the form of \( d(\theta) = [1, \cdots, z^{2N-2}]^\top \). Therefore, the reduced-dimension \( R_4 \) not only contains all of the information about original matrix \( C_4 \), but also keeps the extended array aperture unchanged.

### 3.2 The TFOC-OPRM Method

When the incident targets is a statistically independent signal sources, the ideal \( R_4 \) is with a Toeplitz structure. However, in practical applications, for example, due to finite sampling snapshots and the low SNR, the matrix \( R_4 \) obtained at this time does not meet the Toeplitz structure any more, instead, it becomes a diagonally dominant matrix. The happening of such condition will have a negative impact on the performance of the final DOA estimation. In order to improve the DOA estimation accuracy of the antenna array, the first task is to recover the Toeplitz structure of matrix \( \hat{R}_4 \), that is, to get Toeplitz matrix \( \hat{R}_{4T} \). Then, a \( R_{4T} \) of Toeplitz matrix can be approached to the real reduced dimension by solving the following optimization problem

\[ \min_{R_{4T} \in S_T} \| R_{4T} - R_4 \| \]  \hspace{1cm} (8)

where \( S_T \) represents Toeplitz matrices, and the entries of the Toeplitz matrix \( R_{4T} \) can be written as
\[ \gamma_h = (2N - 1 - h + 1)^{-1} \sum_{p=1}^{2N-1-h+1} r_{p(p+h-1)} \]  
(9)

where the element \( r_{p(p+h-1)} \) denotes the \( p \)th row and \( (p + h - 1) \)th column of \( \mathbf{R}_h \), \( h \in [1, \cdots, 2N - 1] \).

And then \( \mathbf{R}_{4T} \) can be obtained by the following Toeplitization operator

\[ \mathbf{R}_{4T} = \text{Toep}(\gamma_1, \cdots, \gamma_{2N-1}) \]  
(10)

where \( \text{Toep} \) stands for the Toeplitization operator.

Although conventional algorithms, such as MUSIC and ESPRIT, can be applied to estimate DOAs based on the \( \mathbf{R}_{4T} \), the computational burden is much heavier due to the EVD and SVD involved. Therefore, we apply OPM for estimating the DOAs to reduce the complex computations effectively.

The presented propagator method is based on the following partition

\[ \mathbf{R}_{4T} = \begin{bmatrix} \mathbf{R}_{4T1} \\ \mathbf{R}_{4T2} \end{bmatrix} \]  
(11)

where the dimensions of \( \mathbf{R}_{4T1} \) and \( \mathbf{R}_{4T2} \) are respectively \( M \times (2N - 1) \) and \( (2N - 1 - M) \times (2N - 1) \).

The \( M \times (2N - 1 - M) \) propagator matrix \( \mathbf{P} \) is defined as a unique linear operator which satisfies the following condition

\[ \mathbf{P}^H \mathbf{R}_{4T1} = \mathbf{R}_{4T2} \]  
(12)

Defining \( \mathbf{Q}^H = [\mathbf{P}^H - \mathbf{I}_{2N-1-M}] \), and combine with equation (11)

\[ \mathbf{Q}^H \mathbf{R}_{4T} = \begin{bmatrix} \mathbf{P}^H & -\mathbf{I}_{2N-1-M} \end{bmatrix} \begin{bmatrix} \mathbf{R}_{4T1} \\ \mathbf{R}_{4T2} \end{bmatrix} = \mathbf{0}_{(2N-1-M) \times (2N-1)} \]  
(13)

The relation (13) shows that the \( \mathbf{R}_{4T} \) is orthogonal to the columns of \( \mathbf{Q}^H \), and the propagator matrix \( \mathbf{P} \) can be obtained by minimizing the cost function \( \xi(\mathbf{P}) \)

\[ \xi(\mathbf{P}) = \left\| \mathbf{R}_{4T2} - \mathbf{P}^H \mathbf{R}_{4T1} \right\|^2_F \]  
(14)

where \( \| \cdot \|_F \) indicates the Frobenius norm, and the optimal solution \( \mathbf{P} \) is given by

\[ \mathbf{P} = (\mathbf{R}_{4T1} \mathbf{R}_{4T1}^H)^{-1} \mathbf{R}_{4T1} \mathbf{R}_{4T2}^H \]  
(15)

In order to introduce the orthonormalization, the orthonormalized matrix \( \mathbf{Q}_0 \) is obtained as follows

\[ \mathbf{Q}_0 = \mathbf{Q}(\mathbf{Q}^H \mathbf{Q})^{-1/2} \]  
(16)

Therefore, the following spectral function \( p(\theta) \) can be formed to estimate the DOAs of source signals

\[ p(\theta) = \frac{1}{d^H(\theta) \mathbf{Q}_0 \mathbf{Q}_0^H d(\theta)} \]  
(17)
It can be seen from function (17) that the $M$ DOAs of the incoming signals can be obtained by means of one-dimensional (1-D) spectrum-peak search over $\theta$. However, to further reduce the computational burden, we can improve the function (17) in order to derive a computationally more efficient search-free modification estimator based on polynomial rooting [24]. In order to further reduce the computational complexity of the algorithm, the method based on polynomial roots is used to improve the spatial spectrum estimation function, so as to obtain more efficient estimators in the calculation, with the specific description of the algorithm given as follows.

Set $z = \exp(j2\pi(d/\lambda) \sin \theta)$, we have $d = d(z)$

$$d(z) = [1, \ldots, z^{2N-2}]^T$$

(18)

Then the denominator of the estimator (17) can be re-expressed with the following polynomial format

$$f(z) = d^H(z)Q_0Q_0^Hd(z)$$

(19)

In an ideal condition, there should be exactly $M$ numbers of roots, that is, $z_1, z_2, \cdots, z_M$ distributing over the unit circle, and these $M$ numbers of roots are exactly the roots of the polynomial $f(z)$. However, in practical application, due to the influence of various complex factors in the environment, leading to the failure of $M$ roots of the equation $f(z)$ in being strictly distributed over the unit circle. In this case, only $M$ roots close to the unit circle need to be selected, similarly to the Root-MUSIC approach [24-26]. When $M$ roots $\{z_1, \cdots, z_i, \cdots, z_M\}$ is obtained, DOA estimation of the incident target signal source can then be completed by the following formula

$$\theta_i = \arcsin\left(\frac{\lambda}{2\pi d} \angle(z_i)\right) \quad i = 1, \ldots, M$$

(20)

So far as it is concerned, the specific operational steps of the proposed Toeplitz fourth-order cumulant orthogonal propagation method based on polynomial roots under limited sampling snapshots can be summarized as follows:

Step 1 Estimate $C_4$ from the received data by (5).

Step 2 Obtain the dimension reduction matrix $R_4$ by removing the redundant items from the expanded matrix $C_4$. 

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Step 3 Reconstruct the Toeplitz matrix $R_{i\tau}$ by performing Toeplitz approximation on $R_i$ as formulas (9) and (10).

Step 4 Estimate the linear operator $P$ according to (14) and (15), then calculate the standard orthonormalized matrix $Q_i$ based on (16).

Step 5 Obtain the polynomial Function (19), and further to it, obtain the $M$ roots closest to the unit circle, that is, the roots of the $f(z)$.

Step 6 Obtain the direction estimation of the incoming wave of the incident target signal source from (20).

### 3.3 Complexity analysis

As for the analysis of computational complexity, the main parts of computation are considered, that is, the construction of the cumulant matrix, the linear operation, the spectral peak search operation, the Toeplitz operation and the polynomial rooting operation. To further prove the superiority of the TFOC-OPRM algorithm in terms of computational complexity, FOC-OPM and MFOC-OPM are used as the comparative algorithms.

For the FOC-OPM technique, the main operation amount includes three major parts, that is, to calculate the $N^2 \times N^2$ cumulant matrix, to perform the linear operator of cumulant matrix and to execute once spectral search. Therefore, the computational complexity of the FOC-OPM technique is $O((9N^4 L) + (MN^4) + (180/\Delta \theta)N^4)$, in which $L$ and $\Delta \theta$ denotes respectively the number of snapshots and the interval of the angular scanning. For the MFOC-OPM algorithm, the mainly calculation amount is to construct one $(2N-1)(2N-1)$ cumulant matrix, perform the linear operator of cumulant matrix and execute once spectral search. Therefore, the computational load of the MFOC-OPM is $O(9(2N-1)^2 L + M(2N-1)^2 + (180/\Delta \theta)(2N-1)^2)$.

For the proposed TFOC-OPRM algorithm, the major computational complexity is to form one $(2N-1)(2N-1)$ cumulant matrix, to perform Toeplitz operation, to perform the linear operator of
cumulant matrix and to execute once polynomial rooting operation. Therefore, the computational complexity is \( O(9(2N-1)^2L + 2(2N-1)^2 - 1 + 2(2N-1)^2 - (2N-1)) + 2(2N-1)^2 - 1 + M(2N-1)^2 + MN) \).

From the above analysis, it can be obviously seen that the computational complexity of TFOC-OPRM algorithm proposed is significantly lower than that of both FOC-OPM algorithm and MFOC-OPM algorithm. The main reason is that the polynomial roots method has been involved to reduce the computational complexity further.

4 Results and Discussion

In this section, the proposed TFOC-OPRM algorithm, as well as FOC-OPM [17] and MFOC-OPM [18] algorithms that are used for the purpose of comparison are simulated in the environment of spatial white noise and spatial color noise respectively to verify the superiority of the proposed algorithm. In the simulation experiment, the ULA composed of three antenna elements \((N=3)\) is used, in which the interval between adjacent antenna elements is \(d = \lambda/2\). It is assumed that there are three far-field narrow-band statistically independent target signal sources \((M=3)\), whose incident angles are \([-45°, 15°, 40°]\) respectively, with the noise being considered as Gaussian white / color noise. Both the proposed TFOC-OPRM algorithm and the two comparative algorithms take 500 Monte-Carlo simulations each time as their estimated performance value. Two respectively performance indexes, namely, normalized probability of success (NPS) and estimated root-mean-square-errors (RMSEs) are defined to evaluate the performance of these three algorithms.

\[
\text{RMSEs} = \sqrt{\frac{1}{500M} \sum_{i=1}^{500} \sum_{a=1}^{M} (\hat{\theta}_a(i) - \theta_a)^2} \quad (21)
\]

\[
\text{NPS} = \frac{\gamma_{\text{succ}}}{\gamma_{\text{total}}} \quad (22)
\]
where $\hat{\theta}_n(i)$ refers to the estimated value of the real value $\theta_n$ in the $i$th time Monte Carlo trial. The $\Upsilon_{\text{succ}}$ and $T_{\text{total}}$ denote the times of success and Monte Carlo trial, respectively. Furthermore, it should be noted that the defined success of a simulation experiment satisfies $\max(|\hat{\theta}_n - \theta_n|) < \varepsilon$, and $\varepsilon$ in the formula equals 0.8 and 1.5 for experiment two and three, respectively.

**Experiment 1: The spatial spectrum estimation**

In the first experiment, the input SNR and the number of snapshots are set to be 10dB and 500, respectively. Shown in the Fig. 1 is the spatial spectrum of the proposed TFOC-OPRM, FOC-OPM and MFOC-OPM algorithms in both spatially-white noise and spatially-color noise situations. It can be observed from the curves in the chart that all of the three algorithms have successfully located the peak corresponding to the incident angle. Further analysis indicates that the angular resolution of the proposed TFOC-OPRM algorithm is much higher than that of both MFOC-OPM and FOC-OPM algorithms. The reason is that the proposed TFOC-OPRM algorithm recovers the Toeplitz structure of $R_2$, making the Toeplitz matrix $R_{2T}$ closer to the real situation.

**Experiment 2: RMSEs and NPS versus SNR**

The main objective of this experiment is to evaluate the performance of TFOC-OPRM algorithm, FOC-OPM algorithm and MFOC-OPM algorithm in terms of RMSEs and NPS with the change of input SNR. The number of sampling snapshots is $L = 2000$, the input SNR changes from 8dB to 24dB, with the step being 2dB. In both Fig. 2 and Fig. 3 are the performance curves of RMSEs and NPS of the proposed algorithm and the comparison algorithms as the input SNR changes, respectively. It can be seen from Fig. 2 that the RMSEs of the three algorithms decrease monotonically with the increase of the input SNR. Further analysis shows that in the environment of spatial-white noise, with the increase of input SNR, the RMSEs performance curve of TFOC-OPRM algorithm is better than that
of FOC-OPM algorithm and that of MFOC-OPM algorithm; in the environment of spatial-color noise, the RMSEs performance curve of TFOC-OPRM algorithm is better than that of MFOC-OPM algorithm. In addition, when the input SNR changes between 8dB and 14dB, the proposed TFOC-OPRM algorithm manages to achieve almost the same RMSEs performance as the FOC-OPM algorithm. But when the input SNR is higher than 14dB, the performance of TFOC-OPRM becomes better than that of FOC-OPM. Moreover, the performance of the improved TFOC-OPRM algorithm is identical to that of the MFOC-OPM algorithm no matter whether in spatially-white noise situation or spatially-color noise situation. From Fig. 3, it can be concluded that the NPS performance of the proposed TFOC-OPRM algorithm is better than that of the FOC-OPM algorithm and MFOC-OPM algorithm in the case of low input SNR. With the increase of input SNR, the NPS of all of the three algorithms ultimately is 1. In addition to that, the proposed algorithm not only removes a lot of redundant data in the original FOC, but also restores the Toeplitz structure of the reduced dimensional FOC. Moreover, it adopts the method of finding roots of polynomials. Therefore, the proposed TFOC-OPRM algorithm not only reduces the computational complexity, but also improves the accuracy of DOA estimation.

**Experiment 3: RMSEs and NPS versus snapshots**

The main objective of this experiment is to verify the performance of the RMSEs and the NPS of TFOC-OPRM algorithm, FOC-OPM algorithm and MFOC-OPM algorithm when the number of sampling snapshots changes under the environment of Gaussian white noise and color noise. The input SNR is set to 10dB, the number of sampling snapshots changes from 400 to 2000, with the step of 200. Shown in Figure 4 and Figure 5 are the performance curves of RMSEs and NPS of the proposed algorithm and the comparative algorithm as the number of sampling snapshots changes. It can be seen from the performance curves demonstrated in Fig. 4 and Fig. 5 that when the number of sampling
snapshots varies from 400 to 800, both RMSEs and NPS show the presence of a large degree of jitter. The reason is that the number of sampling snapshots is relatively small, resulting in too less data acquired. In other words, the estimated matrix \( \hat{\mathbf{R}}_4 \) deviates greatly from the ideal matrix \( \mathbf{R}_4 \). With the increasing number of sampling snapshots, we can see that the performance curve tends to be stable gradually. At the same time, it can be observed that the TFOC-OPRM algorithm proposed in this paper achieves more satisfactory estimation performance than RMSEs algorithm and NPS algorithm, either in the condition of spatial-white noise or in the condition of spatial-color noise. Note that the computational complexity of proposed algorithm is significantly lower than that of the FOC-OPM algorithm due to the fact that the redundant information of the original cumulants matrix is removed. Moreover, the Toeplitz approximate method is performed on the reduced-rank \( \mathbf{R}_4 \) to improve estimation performance. Meanwhile, compared to MFOC-OPM method, the TFOC-OPRM algorithm has lower computational burden, which exploits polynomial rooting instead of spectral search.

**Experiment 4: The calculation complexity versus snapshots**

In this simulation experiment, we further verify the advantages of TFOC-OPRM algorithm in terms of computational complexity, also by comparing the algorithms with FOC-OPM and MFOC-OPM. The number of incident target signal sources and the number of array elements of ULA are set as \( M = 3 \) and \( N = 3 \) respectively, with the interval of angular scanning being defined as \( \Delta \theta = 0.01 \). Fig. 6 shows the calculation complexity of the proposed TFOC-OPRM algorithm and the comparison algorithms as the number of sampling snapshots changes (the number of sampling snapshots changes from \( L = 400 \) to \( L = 2000 \)). Viewing from the simulation results in Fig. 6, with the increasing number of sampling snapshots, the computational complexity of the proposed TFOC-OPRM algorithm is far lower than that of the FOC-OPM algorithm and the MFOC-OPM algorithm, and this advantage will
be more obvious with the further increase of the number of sampling snapshots. The reason is that the proposed TFOC-OPRM algorithm not only eliminates a large number of redundant data in the original FOC, but also adopts the polynomial root method. This is consistent with the theoretical analysis given in section 3.3.

5 Conclusions

In this paper, a novel computationally efficient TFOC-OPRM localization algorithm have been proposed. Specifically, the extended effective array aperture can resolve the number of sources more than or equal to that of the array elements. Moreover, resorting to Toeplitz approximate method, the Toeplitz structure of the reduced-dimension $R_4$ matrix is recovered to provide a more satisfactory estimation performance than the compared algorithms. In addition, compared to MFOC-OPM algorithm, the proposed TFOC-OPRM algorithm can obtain good estimation performance, as well as has lower computational burden because of the advantage of the polynomial root method in computational complexity. Simulation results validate the effectiveness of the proposed algorithm both in spatially-white noise and spatially-color noise situations.

Abbreviations

FOC : fourth-order cumulants; TFOC-OPRM : Toeplitz fourth-order cumulants orthonormal propagator rooting method; DOA : direction-of-arrival; ULA : uniform linear array; MFOC : modified fourth-order cumulants; TFOC : Toeplitz fourth-order cumulants; OPM : orthonormal propagator method; MUSIC : multiple signal classification algorithm; ESPRIT : estimating signal parameter via rotational invariance techniques; DOAs : direction-of-arrivals; HOC : high-order cumulants; EVD : eigenvalue decomposition; SVD : singular value decomposition; SNR : signal-to-noise ratio; FOC-OPM : fourth-
order cumulants orthonormal propagator method; MFOC-OPM: modified fourth-order cumulants
fourth-order cumulants; NPS: normalized probability of success; RMSEs: root-mean-square-errors.

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Authors’ contributions
H.P.S proposed the main idea, designed the experiments, and discussed the results. N.M and Z.W.G
wrote the paper. L.Z.Z and S.J gave some important suggestions and revised the paper. All authors read
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Author details

1. School of Automobile and Transportation, Tianjin University of Technology and Education, Tianjin 300222, China
2. School of Electronic Engineering, Tianjin University of Technology and Education, Tianjin 300222, China
3. Tianjin Sino-German University of Applied Sciences, Tianjin 300350, China
4. Auto Testing Research Institute of China Automotive Technology and Research Center, Tianjin, 300300, China

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**Figure legends**

Figure 1 Spatial spectrum of different algorithms

Figure 2 RMSEs of the DOAs versus SNR

Figure 3 NPS of the DOAs versus SNR

Figure 4 RMSEs of the DOAs versus snapshots

Figure 5 NPS of the DOAs versus snapshots

Figure 6 Computational complexity comparison versus the number of snapshots