Semigroups and Controllability of Invariant Control Systems on $\text{Sl}(n, \mathbb{H})$

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Abstract: Let $\text{Sl}(n, \mathbb{H})$ be the simple Lie group of $n \times n$ quaternionic matrices $g$ with $|\det g| = 1$. We prove that a subsemigroup $S \subseteq \text{Sl}(n, \mathbb{H})$ with nonempty interior is equal to $\text{Sl}(n, \mathbb{H})$ if $S$ contains a subgroup isomorphic to $\text{Sl}(2, \mathbb{H})$. As application we give sufficient conditions on $A, B \in \mathfrak{sl}(n, \mathbb{H})$ to ensuring that the invariant control system $\dot{g} = Ag + uBg$ is controllable on $\text{Sl}(n, \mathbb{H})$. We prove also that these conditions are generic in the sense that we obtain an open and dense set of controllable pairs $(A, B) \in \mathfrak{sl}(n, \mathbb{H})^2$.

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1 Introduction and main results

Let $\text{Sl}(n, \mathbb{H})$ be the simple Lie group of quaternionic $n \times n$ matrices $g$ with $|\det g| = 1$. In this paper we give conditions ensuring that a subsemigroup $S \subseteq \text{Sl}(n, \mathbb{H})$ is in fact the whole group. These conditions are applied to the controllability problem of an invariant control system

$$\dot{g} = Ag + uBg$$

(1)
on \( \text{Sl}(n, \mathbb{H}) \). In such a system, \( A \) and \( B \) are quaternionic \( n \times n \) matrices having trace with zero real part, i.e., \( A, B \) are elements of the Lie algebra \( \mathfrak{sl}(n, \mathbb{H}) \) of \( \text{Sl}(n, \mathbb{H}) \). This is a real simple Lie algebra that complexifies to a complex Lie algebra isomorphic to \( \mathfrak{sl}(2n, \mathbb{C}) \).

Our approach to prove that a subsemigroup \( S \) equals \( \text{Sl}(n, \mathbb{H}) \) follows the same lines of the results proved by Dos Santos and San Martin in [8] and [9]. These papers work in a general noncompact semi-simple Lie group \( G \) and develop a method based on the topology of flag manifolds of \( G \). That topological method permits to show that a subsemigroup \( S \) with \( \text{int} S \neq \emptyset \) must be the group \( G \) provided it contains certain classes of subgroups given by a root of the Lie algebra of \( G \).

For the group \( \text{Sl}(n, \mathbb{H}) \), our main result proves that \( S = \text{Sl}(n, \mathbb{H}) \) if \( \text{int} S \neq \emptyset \) and \( S \) contains the subgroup \( \langle \exp \mathfrak{g}_{\pm \alpha} \rangle \) generated by the root spaces \( \mathfrak{g}_{\pm \alpha} \) of a root \( \alpha \). Explicitly, for a pair \( (r, s) \), \( 1 \leq r < s \leq n \) let \( \text{Sl}(2, \mathbb{H})_{r,s} \) be the subgroup of \( \text{Sl}(n, \mathbb{H}) \), isomorphic to \( \text{Sl}(2, \mathbb{H}) \), of the matrices in the space \( \text{span}\{e_r, e_s\} \) where \( \{e_1, \ldots, e_n\} \) is the standard basis of \( \mathbb{H}^n \), plugged into the \( n \times n \) matrices. That is, \( \text{Sl}(2, \mathbb{H})_{r,s} = \langle \exp \mathfrak{sl}(2, \mathbb{H})_{r,s} \rangle \) where the Lie algebra \( \mathfrak{sl}(2, \mathbb{H})_{r,s} \) is given by the matrices in \( \mathfrak{sl}(n, \mathbb{H}) \) having nonzero entries only at the positions \((a, b)\) with \( a, b \in \{r, s\} \).

Then we apply this theorem to the controllability of (1). We follow an idea that goes back to the papers by Jurdjevic and Kupka [3] and [4]. More precisely, we find conditions on the matrices \( A \) and \( B \) ensuring that \( \text{Sl}(2, \mathbb{H})_{1,n} \) is contained in the control semigroup \( S_{A,B} \), that is, the subsemigroup of \( \text{Sl}(n, \mathbb{H}) \) generated by the 1-parameter semigroups \( e^{t(A+uB)} \), \( t \geq 0 \), \( u \in \mathbb{R} \). Hence if we assume also the Lie algebra rank condition (so that \( \text{int} S_{A,B} \neq \emptyset \)) then Theorem 3.1 applies to get \( S_{A,B} = \text{Sl}(n, \mathbb{H}) \), which means that (1) is controllable.

In this context we prove that controllability for invariant systems on \( \text{Sl}(n, \mathbb{H}) \) is a generic property in the sense that there is an open and dense set \( C \subset \mathfrak{sl}(n, \mathbb{H})^2 \) such that the control system \( \dot{g} = A(g) + uB(g) \) with unrestricted controls \( (u \in \mathbb{R}) \) is controllable for all pairs \( (A, B) \in C \). To get the set \( C \subset \mathfrak{sl}(n, \mathbb{H})^2 \) we first write a sufficient condition for controllability by taking \( B \) to be diagonal (see Theorem 4.1 below). Afterwards we check that in \( \mathfrak{sl}(n, \mathbb{H}) \) there is just one conjugacy class of Cartan subalgebras. This allows to build the dense set \( C \) as a set of conjugates of a controllable pair \( (A, B) \) with \( B \) diagonal (see Section 5 below).

About the structure of this paper, we first present the fundamental concepts for this paper. In the third section, we develop and prove our main
result which gives necessary conditions for a subsemigroup of $\text{Sl} (n, \mathbb{H})$ to be equal to $\text{Sl} (n, \mathbb{H})$. In the fourth section we apply the previous results to prove a theorem that provides sufficient conditions for controllability of invariant systems on $\text{Sl} (n, \mathbb{H})$. Finally, in the last section we show that controllability, for the above system, is a generic property.

2 Background

In this section we establish some necessary notations, concepts and results.

In a matrix Lie algebra a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$ is given by skew symmetric and symmetric (or hermitian) matrices. Hence the natural Cartan decomposition of $\mathfrak{sl} (n, \mathbb{H})$ is

$$\mathfrak{k} = \{ X \in M_{n \times n} (\mathbb{H}) : X = -X^T \} \quad \mathfrak{s} = \{ X \in M_{n \times n} (\mathbb{H}) : X = X^T \}$$

where $\cdot^*$ is a quaternionic conjugation. The algebra $\mathfrak{k}$ of the quaternionic skew Hermitian matrices is denoted by $\mathfrak{k} = sp (n)$ and is the compact real form of $C_n = sp (n, \mathbb{C})$.

The maximal abelian subalgebra $\mathfrak{a} \subset \mathfrak{s}$ is given by the diagonal matrices $\Lambda = \text{diag} \{ a_1, \ldots, a_n \}$ with $a_i \in \mathbb{R}$ and $\text{tr} \Lambda = 0$. The roots of $\mathfrak{a}$ are the following linear functionals

$$\alpha_{rs} (\Lambda) = (\lambda_r - \lambda_s) (\Lambda) = a_r - a_s \quad r \neq s.$$ 

The vector space $\mathfrak{g}_{\alpha_{rs}}$ corresponding to the root $\alpha_{rs}$ is given by the quaternionic matrices with non zero entries only in the position $rs$. Then all roots have multiplicity 4. The set of simple system of roots is given by

$$\Sigma = \{ \lambda_1 - \lambda_2, \ldots, \lambda_{n-1} - \lambda_n \} = \{ \alpha_1, \ldots, \alpha_{n-1} \}.$$ 

With this choice, the set of positive roots is given by $\alpha_{rs}$ with $r < s$. Hence an Iwasawa decomposition is

$$\mathfrak{sl} (n, \mathbb{H}) = sp (1) \oplus \mathfrak{a} \oplus \mathfrak{n}^+$$

where $\mathfrak{n}^+$ is the Lie algebra of upper triangular quaternionic $n \times n$ matrices with zero entries in the diagonal.
Now we recall an important result for our goals, which is related to the flag type of a semigroup in the special case of $\text{Sl}(n, \mathbb{H})$ (for the general context see San Martin and Tonelli [6], San Martin [7], Dos Santos and San Martin [9] and references therein). For $d = 1, \ldots, n - 1$, we denote by $\text{Gr}_d(\mathbb{H})$ the Grassmannian of $d$-dimensional quaternionic subspaces of $\mathbb{H}^n$. The group $\text{Sl}(n, \mathbb{H})$ acts transitively on each $\text{Gr}_d(\mathbb{H})$. The compact subgroup $\text{Sp}(n) \subset \text{Sl}(n, \mathbb{H})$ also acts transitively on $\text{Gr}_d(\mathbb{H})$.

**Theorem 2.1.** Let $S \subset \text{Sl}(n, \mathbb{H})$ be a proper subsemigroup with $\text{int}S \neq \emptyset$. Then there are $d \in \{1, \ldots, n - 1\}$ and a subset $C_d \subset \text{Gr}_d(\mathbb{H})$ satisfying

1. $C_d$ is closed, has nonempty interior and is invariant by the action of $S$. ($C_d$ is the unique invariant control set of $S$ in $\text{Gr}_d(\mathbb{H})$).

2. $C_d$ is contractible in $\text{Gr}_d(\mathbb{H})$ in the sense that there exists $H \in \mathfrak{sl}(n, \mathbb{H})$ such that $e^{tH}C_d$ shrinks to a point as $t \to +\infty$.

In the context of the above theorem, the Grassmannian $\text{Gr}_d(\mathbb{H})$ is called the flag type of the semigroup $S$.

## 3 Transitivity of a subsemigroup of $\text{Sl}(n, \mathbb{H})$

In this section, following the same construction and notation in the introduction, we prove our main result that gives sufficient condition for a semigroup $S$ to be equal to $\text{Sl}(n, \mathbb{H})$. The proof of this result is based on the existence of a flag type of a proper semigroup $S$ with $\text{int}S \neq \emptyset$ and it follows the same pattern as the proof of the results in [8] and [9]. By Theorem 2.1, we get that $S = \text{Sl}(n, \mathbb{H})$ if we can prove that $S$ does not leave invariant contractible subsets in the Grassmannians $\text{Gr}_d(\mathbb{H})$, $d = 1, \ldots, n - 1$.

Now we state the main theorem of this paper.

**Theorem 3.1.** Let $S \subset \text{Sl}(n, \mathbb{H})$ be a subsemigroup with $\text{int}S \neq \emptyset$ and suppose that $\text{Sl}(2, \mathbb{H})_{r,s} \subset S$ for some pair of indices $(r, s)$, $1 \leq r < s \leq n$. Then $S = \text{Sl}(n, \mathbb{H})$.

Before to prove this theorem, we need some remarks and lemmas. Taking into account the assumption, of the above theorem, that $\text{Sl}(2, \mathbb{H})_{r,s} \subset S$ we consider separately the case where $(r, s) = (1, n)$, that is, $\text{Sl}(2, \mathbb{H})_{1,n} \subset S$. Then the strategy is to prove that an $S$-invariant subset $C_d \subset \text{Gr}_d(\mathbb{H})$, which
is closed and has nonempty interior, contains an orbit of \( \text{Sl}(2, \mathbb{H})_{1,n} \) that is not contractible to a point in \( \text{Gr}_d(\mathbb{H}) \). In the next lemma we describe the noncontractible \( \text{Sl}(2, \mathbb{H})_{1,n} \)-orbits in the Grassmannians that are proved to be contained in the invariant control sets \( C_d \). They are 4-dimensional spheres (cf. Lemma 3.7 of [8]).

**Lemma 3.2.** For \( d = 1, n - 1 \) let \( V_d \) be the subspace of \( \mathbb{H}^n \) spanned by the first \( d \) basic vectors,

\[
V_d = \{(q_1, \ldots, q_d, 0, \ldots, 0) : q_r \in \mathbb{H}\}.
\]

Then the \( \text{Sl}(2, \mathbb{H})_{1,n} \)-orbit in \( \text{Gr}_d(\mathbb{H}) \) through \( V_d \) is diffeomorphic to \( S^4 \).

**Proof:** The orbit is diffeomorphic to the coset space \( \text{Sl}(2, \mathbb{H})_{1,n} / P \) where \( P = \{g : gV_d = V_d\} \) is the isotropy subgroup. By a direct check one sees that \( P \) is the subgroup of matrices in \( \text{Sl}(2, \mathbb{H})_{1,n} \) that are upper triangular. This is a parabolic subgroup of \( \text{Sl}(2, \mathbb{H})_{1,n} \) hence \( \text{Sl}(2, \mathbb{H})_{1,n} / P \) is a flag manifold of \( \text{Sl}(2, \mathbb{H})_{1,n} \). Now \( \text{Sl}(2, \mathbb{H}) \) is a real rank 1 group so that it has just one flag manifold which is diffeomorphic to a sphere. The dimension of the sphere equals the codimension of \( P \) which is 4. Therefore \( \text{Sl}(2, \mathbb{H})_{1,n} / P \) as well as the orbit through \( V_d \) is a sphere \( S^4 \). \( \square \)

The next step is to check that for any \( d = 1, \ldots, n - 1 \) the orbit

\[
\text{Sl}(2, \mathbb{H})_{1,n} V_d \approx S^4
\]

is not contractible in \( \text{Gr}_d(\mathbb{H}) \), that is, is not homotopic to a point. In other words we are required to prove that the 4-sphere \( \text{Sl}(2, \mathbb{H})_{1,n} V_d \) is not a representative of the identity of the homotopy group \( \pi_4 (\text{Gr}_d(\mathbb{H})) \). To this purpose we recall the cellular decomposition of \( \text{Gr}_d(\mathbb{H}) \) given in Rabelo and San Martin [5]. From that decomposition the homology \( H_* (\text{Gr}_d(\mathbb{H})) \) of a Grassmannian \( \text{Gr}_d(\mathbb{H}) \) is freely generated by the Schubert cells and \( H_r (\text{Gr}_d(\mathbb{H})) = \{0\} \) if \( r \) is not a multiple of 4. In \( \text{Gr}_d(\mathbb{H}) \) there is just one 4-dimensional cell which is the orbit \( \text{Sl}(2, \mathbb{H})_{d,d+1} V_d \). Here, \( \text{Sl}(2, \mathbb{H})_{d,d+1} = \langle \exp \mathfrak{sl}(2, \mathbb{H})_{d,d+1} \rangle \approx \text{Sl}(2, \mathbb{H}) \) and \( \mathfrak{sl}(2, \mathbb{H})_{d,d+1} \) is the algebra of matrices with nonzero entries only in the entries \((d, d), (d, d + 1), (d + 1, d)\) and \((d + 1, d + 1)\). Analogous to the Lemma 3.2 we have that \( \text{Sl}(2, \mathbb{H})_{d,d+1} V_d \) is diffeomorphic to \( S^4 \).

Now, by the Hurewicz homomorphism \( \pi_4 (\text{Gr}_d(\mathbb{H})) \approx H_4 (\text{Gr}_d(\mathbb{H})) \) because the homology is trivial in degrees less than 4. It follows that \( \pi_4 (\text{Gr}_d(\mathbb{H})) \)
\[ \approx \mathbb{Z} \] and the equivalence class of the orbit \( \text{Sl}(2, \mathbb{H})_{d,d+1} V_d \approx S^4 \) is a generator of \( \pi_4(\text{Gr}_d(\mathbb{H})) \). The next lemma shows that \( \text{Sl}(2, \mathbb{H})_{1,n} V_d \approx S^4 \) is a generator as well.

**Lemma 3.3.** The orbits \( \text{Sl}(2, \mathbb{H})_{d,d+1} V_d \approx S^4 \) and \( \text{Sl}(2, \mathbb{H})_{1,n} V_d \approx S^4 \) are homotopic to each other.

**Proof:** The homotopy is performed by the product of two one-parameter subgroups. Let \( A, B \in \text{sl}(n, \mathbb{H}) \) be the matrices such that \( Ae_1 = e_d, Ae_d = -e_1, Be_{d+1} = e_n, Be_n = -e_{d+1} \) and \( Ae_r = Be_r = 0 \) elsewhere. Put \( P(t) = e^{tA}_t B \). Then for all \( t, P(t)V_d = V_d \) and \( P(\pi/2) \) permutes the subspaces spanned by \( \{e_d, e_{d+1}\} \) and \( \{e_1, e_n\} \) so that \( P(\pi/2) \text{Sl}(2, \mathbb{H})_{d,d+1} P(\pi/2)^{-1} = \text{Sl}(2, \mathbb{H})_{1,n} \). Hence

\[
P(\pi/2) \text{Sl}(2, \mathbb{H})_{d,d+1} V_d = P(\pi/2) \text{Sl}(2, \mathbb{H})_{d,d+1} P(\pi/2)^{-1} P(\pi/2) V_d = \text{Sl}(2, \mathbb{H})_{1,n} V_d,
\]

showing that the map \( t \mapsto P(t) \text{Sl}(2, \mathbb{H})_{d,d+1} V_d \) is a homotopy between the orbits \( \text{Sl}(2, \mathbb{H})_{d,d+1} V_d \) and \( \text{Sl}(2, \mathbb{H})_{1,n} V_d \).

Now we can start the proof of Theorem 3.1 in the case when \( \text{Sl}(2, \mathbb{H})_{1,n} \subset S \). Denote by \( N \) the nilpotent group of lower triangular matrices in \( \text{Sl}(n, \mathbb{H}) \) having 1’s at the diagonal. It is well known (and easy to prove) that \( NV_d \) is an open and dense set in \( \text{Gr}_d(\mathbb{H}) \). Hence \( NV_d \cap C_d \neq \emptyset \) because \( \text{int}C_d \neq \emptyset \) where \( C_d \) is the invariant control set of \( S \) in \( \text{Gr}_d(\mathbb{H}) \).

The assumption \( \text{Sl}(2, \mathbb{H})_{1,n} \subset S \) of Theorem 3.1 implies that \( gC_d \subset C_d \) for any \( g \in \text{Sl}(2, \mathbb{H})_{1,n} \). Since \( C_d \) is closed it follows that any limit \( \lim gtx \) with \( x \in C_d \) and \( g_t \in \text{Sl}(2, \mathbb{H})_{1,n} \) also belongs to \( C_d \).

Now, take \( x = gV_d \in NV_d \cap C_d \) with \( g \in N \) and

\[
h = \text{diag}\{\lambda, 1, \ldots, 1, \lambda^{-1}\} \in \text{Sl}(2, \mathbb{H})_{1,n}
\]

with \( \lambda > 1 \). As \( l \to +\infty \) the sequence of conjugations \( h^lgh^{-l} \) converges to the matrix \( g_1 \in N \) that has zeros at the first column and the last row outside the diagonal. We have \( h^{-l}V_d = V_d \) so that \( h^lx = h^lgV_d = h^lg^{-l}V_d \) implying that \( W = \lim h^lx = g_1V_d \in C_d \). Therefore the orbit \( \text{Sl}(2, \mathbb{H})_{1,n} W \) is entirely contained in \( C_d \).

The next step is to prove that the orbit \( \text{Sl}(2, \mathbb{H})_{1,n} W \subset C_d \) is a sphere \( S^4 \) homotopic to \( \text{Sl}(2, \mathbb{H})_{1,n} V_d \). By the zeros in the first column and the

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Proof: Let so that the semigroup $P T P$ be a matrix in $\text{Sl}(n, \mathbb{H})$ that permutes the subspaces $\langle e_1 \rangle$ and $\langle e_r \rangle$ and the subspaces $\langle e_n \rangle$ and $\langle e_s \rangle$. Then $\text{PSl}(2, \mathbb{H})_{r,s} P^{-1} = \text{Sl}(2, \mathbb{H})_{1,n}$ so that the semigroup $P T P^{-1}$ contains $\text{Sl}(2, \mathbb{H})_{1,n}$. Since $\text{int} P T P^{-1} \neq \emptyset$ we conclude, by Theorem 3.1, that $P T P^{-1} = \text{Sl}(n, \mathbb{H})$, hence $T = \text{Sl}(n, \mathbb{H})$. 

Corollary 3.4. Let $T \subset \text{Sl}(n, \mathbb{H})$ be a semigroup with nonempty interior. Suppose that $\text{Sl}(2, \mathbb{H})_{r,s} \subset T$ for some pair $(r,s)$, $r \neq s$. Then $T = \text{Sl}(n, \mathbb{H})$.

Proof: Let $P$ be a matrix in $\text{Sl}(n, \mathbb{H})$ that permutes the subspaces $\langle e_1 \rangle$ and $\langle e_r \rangle$ and the subspaces $\langle e_n \rangle$ and $\langle e_s \rangle$. Then $\text{PSl}(2, \mathbb{H})_{r,s} P^{-1} = \text{Sl}(2, \mathbb{H})_{1,n}$ so that the semigroup $P T P^{-1}$ contains $\text{Sl}(2, \mathbb{H})_{1,n}$. Since $\text{int} P T P^{-1} \neq \emptyset$ we conclude, by Theorem 3.1, that $P T P^{-1} = \text{Sl}(n, \mathbb{H})$, hence $T = \text{Sl}(n, \mathbb{H})$. 

4 Application to Controllability

In this section we apply Theorem 3.1 to show the following sufficient conditions for controllability of an invariant system like $\text{(I)}$ in $\text{Sl}(n, \mathbb{H})$. The statement of this theorem is, in some sense, a $\text{sl}(n, \mathbb{H})$-version of a well-known approach to controllability (see e.g., El Alssoudi, Gauthier and Kupka $\text{(I)}$).

Theorem 4.1. Let $\dot{g} = A(g) + uB(g)$ be an invariant control system in $\text{Sl}(n, \mathbb{H})$ where $A, B \in \text{sl}(n, \mathbb{H})$. Such a system with unrestricted controls ($u \in \mathbb{R}$) is controllable if the following conditions are satisfied.
H1. The pair \((A, B)\) generates \(\mathfrak{sl}(n, \mathbb{H})\) as a Lie algebra (Lie algebra rank condition).

H2. \(B = \text{diag}\{a_1 + i b_1, \ldots, a_n + i b_n\}\) with \(a_1 > a_2 \geq \cdots \geq a_{n-1} > a_n\), \(b_n \neq 0 \neq b_1\) and \(b_1/b_n\) is irrational.

H3. Denote the 1, \(n\) and \(n, 1\) entries of the matrix \(A\) by \(p \in H\) and \(q \in H\), respectively. Let \(H_{1,i}\) and \(H_{j,k}\) be the (real) subspaces of \(H\) spanned by \(\{1, i\}\) and \(\{j, k\}\) respectively. Then \(p\) and \(q\) do not belong to \(H_{1,i} \cup H_{j,k}\).

The proof of the Theorem 4.1 will be made throughout this section and it is immediate from Theorem 3.1 combined with the following proposition ensuring that \(\text{Sl}(2, \mathbb{H})_{1,n}\) is contained in the control semigroup of the system. Although the Lie algebra rank condition will not be needed for this proposition, it allows us to conclude the proof of the Theorem 4.1 by ensuring that the control semigroup \(S\) has nonempty interior, leading us to the conditions required for Theorem 3.1.

Proposition 4.2. Under the conditions H2 and H3 of Theorem 4.1, the semigroup \(S\) of the system contains the group \(\text{Sl}(2, \mathbb{H})_{1,n}\).

To prove this proposition, let \(S\) be the control semigroup for the invariant system (11) and write

\[ c(S) = \{X \in \mathfrak{sl}(n, \mathbb{H}) : \forall t \geq 0, e^{tX} \in \text{cl}S\} \]

for the Lie wedge of \(S\) (see [3], [4] and Hilgert, Hofmann and Lawson [2]). The main properties of \(c(S)\) are:

1) \(c(S)\) is a closed convex cone in the Lie algebra \(\mathfrak{sl}(n, \mathbb{H})\);
2) \(c(S) \cap (-c(S))\) is a Lie subalgebra and
3) If \(X \in c(S) \cap (-c(S))\) then \(e^{\text{ad}(X)}c(S) = c(S)\).

By definition of \(S\) we have that \(A + uB \in c(S)\) for all \(u \in \mathbb{R}\) (since we consider unrestricted controls). Hence \(A \in c(S)\) and if \(u \neq 0\) then

\[ \frac{1}{|u|} A + \frac{u}{|u|} B = \frac{1}{|u|} (A + uB) \in c(S). \]

Taking limits as \(u \to \pm \infty\) we see that \(\pm B \in c(S)\), that is, \(B \in c(S) \cap (-c(S))\). It follows that \(e^{t\text{ad}(B)}A \in c(S)\) and hence \(e^{-t(a_1-a_n)}e^{t\text{ad}(B)}A \in c(S)\) for all \(t \in \mathbb{R}\) where \(a_1, \ldots, a_n\) are the real parts of the entries of \(B\).
Now by assumption we have $a_1 > a_2 > \cdots > a_n$ so that as $t \to +\infty$ the entries $e^{-t(a_1-a_n)}e^{\text{tad}(B)}A$ converge to 0 except for the $(1,n)$-entry. The $(1,n)$-entry of $e^{-t(a_1-a_n)}e^{\text{tad}(B)}A$ is $e^{it(b_1-b_n)}p$ where $p$ is as in the statement of the theorem and $b_1, \ldots, b_n$ are the imaginary parts of the entries of $B$. Choosing a sequence $t_k \to +\infty$ such that $e^{it(b_1-b_n)} \to 1$ we conclude that

$$X = \begin{pmatrix} 0 & \cdots & p \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \in \mathfrak{c}(S).$$

Using again the properties of $\mathfrak{c}(S)$ as a Lie wedge we have that for all $t, s \in \mathbb{R}$,

$$e^{-t(a_1-a_n)}e^{\text{tad}(B)}X = \begin{pmatrix} 0 & \cdots & e^{itb_1pe^{-itb_n}} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \in \mathfrak{c}(S). \quad (2)$$

The following lemma about conjugation of quaternions shows that $\mathfrak{g}_{a_1n} = \text{span}_\mathbb{H}\{X\}$ is contained in $\mathfrak{c}(S)$.

**Lemma 4.3.** Consider the action of the circle group $S^1 = \{e^{it} : t \in [0,2\pi]\}$ in $\mathbb{H}$ given by conjugation $(t,q) \mapsto e^{it}qe^{-it}$. Write $q = a + b$ with $a = x_1 + ix_2 \in \mathbb{H}_{\{1,i\}}$ and $b = jx_3 + kx_4 \in \mathbb{H}_{\{j,k\}}$. Suppose that $a \neq 0 \neq b$, that is, $q \notin \mathbb{H}_{\{1,i\}} \cup \mathbb{H}_{\{j,k\}}$. Then the orbit $T^2q$ is a 2-dimensional torus and $\mathbb{H}$ is the convex cone generated by $T^2q$.

**Proof:** For the first statement it suffices to prove that the orbit is 2-dimensional because it is a quotient of $T^2$. For this purpose we note that the tangent space of $T^2q$ at $q$ is spanned by

$$\frac{\partial}{\partial t} (e^{it}qe^{-is})|_{(0,0)} = iq \quad \frac{\partial}{\partial s} (e^{it}qe^{-is})|_{(0,0)} = -qi.$$

Now,

$$iq = ix_1 - x_2 + kx_3 - jx_4 = v + w$$

$$-qi = -ix_1 + x_2 + kx_3 - jx_4 = -v + w$$

with $v = ix_1 - x_2$ and $w = kx_3 - jx_4$. The assumption about $q$ says that $v \neq 0 \neq w$ so that $\{v, w\}$ is linearly independent. Hence $\{iq, qi\}$ is linearly independent as well because $2v = iq + qi$ and $2w = iq - qi$. 

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To the convex cone $C$ generated by $T^2q$ take $r = e^{it}qe^{-is} \in T^2q$. Then $-r = e^{i\pi r} = e^{i(t+\pi)}qe^{-is}$ also belongs to $T^2q$. Hence $C$ is a subspace. The orbit contains $e^{i\pi/2}q = iq = ia + ib$ and $qe^{-i3\pi/2} = qi = ai + bi = ia - ib$. So that $C$ contains $ia$ and $ib$ and hence contains $H_{\{1,i\}}$ and $H_{\{j,k\}}$ because $C$ is invariant by left multiplication by $i$. Thus $H$ is the cone generated by $T^2q$.

Corollary 4.4. Let $c_1, c_2 \in \mathbb{R}$ with $c_1c_2 \neq 0$ and $c_1/c_2$ irrational. Take $q \in H$ with $q \notin H_{\{1,i\}} \cup H_{\{j,k\}}$. Then $H$ is the closed convex cone generated by the curve $e^{itc_1}qe^{-itec_2}$.

Proof: Since $c_1/c_2$ is irrational, the curve $(e^{itc_1}, e^{itec_2})$ is dense in the torus $T^2$. Hence $t \mapsto e^{itc_1}qe^{-itec_2}$ is dense in the orbit $T^2q$ which implies the corollary.

Applying this corollary to the curve (2) it follows, by the assumption on $B$ in Theorem 4.1, that the subspace $g_{\alpha_{1n}} = \text{span}_H \{X\}$ is contained in $\mathfrak{c}(S)$ and hence in $\mathfrak{c}(S) \cap (-\mathfrak{c}(S))$.

By similar arguments we get lower triangular matrices in $\mathfrak{c}(S)$: taking limits as $t \rightarrow -\infty$ of $e^{-t(a_1-a_n)}e^{tad(B)}A$ it follows that

$$Y = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ q & \cdots & 0 \end{pmatrix} \in \mathfrak{c}(S),$$

hence applying the same idea we conclude that $g_{\alpha_{1n}} = \text{span}_H \{Y\}$ is contained in $\mathfrak{c}(S)$ and hence in $\mathfrak{c}(S) \cap (-\mathfrak{c}(S))$.

Now, the Lie algebra generated by $g_{\alpha_{1n}}$ and $g_{\alpha_{1n}}$ is $\mathfrak{sl}(2,\mathbb{H})_{1n}$ so that this Lie algebra is contained in $\mathfrak{c}(S)$. It follows that $\mathfrak{sl}(2,\mathbb{H})_{1n}$ is contained in $S$, concluding the proof of Proposition 4.2.

5 Cartan subalgebras and genericity

In this section we prove that controllability for invariant control systems in $\mathfrak{sl}(n,\mathbb{H})$ is a generic property. To prove it we first show that the set of pairs conjugate to a pair $(A, B)$ satisfying the conditions of Theorem 4.1 is dense in $\mathfrak{sl}(n,\mathbb{H})^2$.

We start by observing that the algebra of diagonal matrices

$$\mathfrak{h} = \{\text{diag}\{a_1 + ib_1, \ldots, a_n + ib_n\} : a_r, b_r \in \mathbb{R}, \ a_1 + \cdots + a_n = 0\} \quad (3)$$
is a Cartan subalgebra of $\mathfrak{sl}(n, \mathbb{H})$ since it is maximal abelian and $\text{ad}(H)$ is semi-simple for any $H \in \mathfrak{h}$.

Next we prove that up to conjugation, $\mathfrak{h}$ is the only Cartan subalgebra of $\mathfrak{sl}(n, \mathbb{H})$. Then we recall the Cartan decomposition $\mathfrak{sl}(n, \mathbb{H}) = \mathfrak{sp}(n) \oplus \mathfrak{s}$ where $\mathfrak{s}$ is the subspace of Hermitian quaternionic matrices in $\mathfrak{sl}(n, \mathbb{H})$. The subspace $\mathfrak{a} \subset \mathfrak{s}$ of real diagonal matrices with zero trace is a maximal abelian subalgebra contained in $\mathfrak{s}$.

Note that the Cartan subalgebra $\mathfrak{h}$ decomposes as $\mathfrak{h} = (\mathfrak{h} \cap \mathfrak{sp}(n)) \oplus \mathfrak{a}$. More generally $\mathfrak{j}$ is said to be a standard Cartan subalgebra if it decomposes as $\mathfrak{j} = \mathfrak{j}_\mathfrak{k} \oplus \mathfrak{j}_\mathfrak{a}$ with $\mathfrak{j}_\mathfrak{k} = \mathfrak{j} \cap \mathfrak{k}$ and $\mathfrak{j}_\mathfrak{a} = \mathfrak{j} \cap \mathfrak{a}$. The following statement is a basic fact for the classification of Cartan subalgebras in real semi-simple Lie algebras (Theorem of Kostant-Sugiura).

**Proposition 5.1.** Any Cartan subalgebra of $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{H})$ is conjugate (by an inner automorphism) to a standard Cartan subalgebra $\mathfrak{j}$.

**Proof:** See Warner [10], Section 1.3.1. \hfill \Box

In the next proposition, we prove that in $\mathfrak{sl}(n, \mathbb{H})$ there is a unique conjugacy class of Cartan subalgebras. We give a direct proof without relying in the general classification theorem (Theorem of Kostant-Sugiura).

**Proposition 5.2.** Every Cartan subalgebra of $\mathfrak{sl}(n, \mathbb{H})$ is conjugate (by an inner automorphism) to the subalgebra $\mathfrak{h}$ defined in (3).

**Proof:** Let $\mathfrak{j} = \mathfrak{j}_\mathfrak{k} \oplus \mathfrak{j}_\mathfrak{a}$ be a standard Cartan subalgebra. The following simple arguments show that $\mathfrak{j}_\mathfrak{k}$ is a Cartan subalgebra of $\mathfrak{sp}(n)$ and $\mathfrak{j}_\mathfrak{a} = \mathfrak{a}$. We have $\dim \mathfrak{j}_\mathfrak{a} \leq \dim \mathfrak{a} = n - 1$. Also, $\dim \mathfrak{j}_\mathfrak{k} \leq \text{rank} \mathfrak{sp}(n) = n$ because $\mathfrak{j}_\mathfrak{k}$ is an abelian subalgebra of $\mathfrak{sp}(n)$ and hence is contained in a Cartan subalgebra of $\mathfrak{sp}(n)$ whose dimension is $\text{rank} \mathfrak{sp}(n)$. On the other hand $\dim \mathfrak{j} = \text{rank} \mathfrak{sl}(n, \mathbb{H}) = \dim \mathfrak{h} = 2n - 1$. Hence we must have $\dim \mathfrak{j}_\mathfrak{k} = n$ and $\dim \mathfrak{j}_\mathfrak{a} = n - 1$. By the first equality $\mathfrak{j}_\mathfrak{k}$ is a Cartan subalgebra of $\mathfrak{sp}(n)$ while the second equality shows that $\mathfrak{j}_\mathfrak{a} = \mathfrak{a}$.

Now $\mathfrak{j}_\mathfrak{k}$ commutes with $\mathfrak{a}$ and hence is contained in the algebra $\mathfrak{m} \approx \mathfrak{sp}(1)^n$ of diagonal matrices with entries in the imaginary quaternions $\text{Im} \mathbb{H}$. Since $\dim \mathfrak{j}_\mathfrak{k} = n = \text{rank} \mathfrak{sp}(1)^n$ it follows that there is an inner automorphism $g = e^{\text{ad}(X)}$, $X \in \mathfrak{m}$, such that $g(\mathfrak{j}_\mathfrak{k}) = \mathfrak{h}_\mathfrak{k}$ and $g$ fixes $\mathfrak{a}$. Therefore $g(\mathfrak{j}) = \mathfrak{h}$ showing that any standard Cartan subalgebra is conjugate to $\mathfrak{h}$. By the above proposition, $\mathfrak{h}$ is a representative of the unique conjugacy class of Cartan subalgebras of $\mathfrak{sl}(n, \mathbb{H})$. \hfill \Box
Now denote by $\mathfrak{a}^+$ the Weyl chamber of real diagonal matrices

$$\text{diag}\{a_1, \ldots, a_n\} \quad a_1 > \cdots > a_n.$$ 

A matrix $B$ satisfying the second condition of Theorem 4.1 belongs to $\mathfrak{a}^+ + \mathfrak{h}_t$ where $\mathfrak{h}_t$ is as above the space of diagonal matrices with entries in $i\mathbb{R}$. Denote by $D_0 \subset \mathfrak{a}^+ + \mathfrak{h}_t$ the set of the matrices $B$ satisfying that condition. By definition if $H \in \mathfrak{a}^+$ and $X = \text{diag}\{ib_1, \ldots, ib_n\} \in \mathfrak{h}_t$ then $H + X \in D_0$ if and only if $b_1b_n \neq 0$ and $b_1/b_n$ is irrational. Hence $D_0$ is a dense subset of $\mathfrak{a}^+ + \mathfrak{h}_t$.

Let $\mathcal{W}$ be the permutation group in $n$ letters (Weyl group) acting on the diagonal matrices by permutation of indices. The set of translates $\mathcal{W}\mathfrak{a}^+ = \{w\mathfrak{a}^+ : w \in \mathcal{W}\}$ is open and dense in $\mathfrak{a}$. Since $D_0$ is dense in $\mathfrak{a}^+ + \mathfrak{h}_t$ it follows that

$$\mathcal{W}D_0 = \{wD_0 : w \in \mathcal{W}\} \subset \mathfrak{a} + \mathfrak{h}_t = \mathfrak{h}$$

is dense in $\mathfrak{h}$.

We apply now Proposition 5.2 ensuring that every Cartan subalgebra is conjugate to $\mathfrak{h}$. This implies that the set $\{\text{Ad} (g) \mathfrak{h} : g \in \text{Sl} (n, \mathbb{H})\}$ is dense in $\mathfrak{sl} (n, \mathbb{H})$ because the set of regular elements is dense and each regular element is contained in a Cartan subalgebra. With these facts in mind we get the following density result.

**Proposition 5.3.** Let $D$ be the set of conjugates of the matrices $B$ satisfying the second condition of Theorem 4.1. Then $D$ is dense in $\mathfrak{sl} (n, \mathbb{H})$.

**Proof:** Take an open set $U \subset \mathfrak{sl} (n, \mathbb{H})$. Then there exists a regular element $X$ of $\mathfrak{sl} (n, \mathbb{H})$ with $X \in U$. Let $\mathfrak{h}_X$ be the unique Cartan subalgebra containing $X$. By Proposition 5.2 there exists $g \in \text{Sl} (n, \mathbb{H})$ such that $\text{Ad} (g) \mathfrak{h}_X = \mathfrak{h}$. So that $\text{Ad} (g) U \cap \mathfrak{h}$ is a nonempty open set of $\mathfrak{h}$ and hence $\text{Ad} (g) U \cap \mathcal{W}D_0 \neq \emptyset$. This means that $U$ meets $\text{Ad} (g^{-1}) \mathcal{W}D_0 \subset D$. Since $U$ is arbitrary this proves that $D$ is dense. \hfill \Box

Now we can show the main result of this section.

**Theorem 5.4.** There is an open and dense set $C \subset \mathfrak{sl} (n, \mathbb{H})^2$ such that the control system $\dot{g} = A (g) + uB (g)$ with unrestricted controls ($u \in \mathbb{R}$) is controllable for all pairs $(A, B) \in C$. 

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To prove this theorem, first note that the union $H_{1,i} \cup H_{j,k}$ is a nowhere dense subset of $H$, which implies that its complement is an open and dense subset of $H$. Consequently, the set of matrices $A$ satisfying the third condition of Theorem 4.1 is open and dense in $\mathfrak{sl}(n, \mathbb{H})$.

**Remark 5.5.** For $\Omega \subset M \times N$ open, the set $\pi_1 \left( \Omega \cap \pi_2^{-1}(b) \right) \subset M$ is open in $M$ for any $b \in N$. Here, $M$ and $N$ are arbitrary metric spaces and $\pi_1 : M \times N \to N$ and $\pi_2 : M \times N \to N$ are the canonical projections in the first and second coordinates, respectively. To see this just let the continuous map $i_b : M \to M \times N$, $i_b(x) = (x, b)$, and observe that

$$
\pi_1 \left( \Omega \cap \pi_2^{-1}(b) \right) = \{ \pi_1(x, b) \mid (x, b) \in \Omega \} = \{ x \in M \mid i_b(x) \in \Omega \} = (i_b)^{-1}(\Omega).
$$

We can now prove that the set $C \subset \mathfrak{sl}(n, \mathbb{H})^2$ of the conjugates of pairs satisfying the three conditions of the Theorem 4.1 is dense in $\mathfrak{sl}(n, \mathbb{H})^2$.

So, let $O$ be an open subset of $\mathfrak{sl}(n, \mathbb{H})^2$. Since the set of pairs $(A, B)$ satisfying $H_1$ is open and dense in $\mathfrak{sl}(n, \mathbb{H})^2$, there is $(A, B) \in O$ satisfying $H_1$. Further, there exists $O' \ni (A, B)$ for which every pair belonging to $O'$ satisfies $H_1$. Without loss of generality we can assume $O' \subset O$. Now, $\pi_2(O')$ is open in $\mathfrak{sl}(n, \mathbb{H})$ and by the Proposition 5.3 we can choose $\tilde{B} \in \pi_2(O') \cap D$. As the set $\pi_1 \left( O' \cap \pi_2^{-1}(\tilde{B}) \right)$ is open in $\mathfrak{sl}(n, \mathbb{H})^2$, by the above considerations we can take $\tilde{A} \in \pi_1 \left( \pi_2^{-1}(\tilde{B}) \cap O' \right)$ satisfying $H_3$. Thus the pair $(\tilde{A}, \tilde{B})$ has the following properties:

i) $(\tilde{A}, \tilde{B}) \in O' \subset O$.

ii) $(\tilde{A}, \tilde{B})$ is conjugate to a pair satisfying $H_1$, $H_2$ and $H_3$.

That is, $(\tilde{A}, \tilde{B}) \in O \cap C$ proving that $C$ is dense in $\mathfrak{sl}(n, \mathbb{H})^2$. Finally, as invariant systems remain controllable under small perturbations we can slightly enlarge the dense set $C$ to get the open and dense set as claimed in Theorem 5.4.

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