The method of investigation of deformations of solids from fibre-reinforced compressible materials

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Abstract. In the paper a numerical method for studying the deformation of transversally isotropic hyperelastic materials is presented. The algorithm of an investigation of compressible Neo-Hookean solids with reinforcement is considered. A computational algorithm is based on the finite element method.

1. Introduction
In a civil engineering and in the aircraft industry the constrictions made of composite materials are widely used. The investigation of such materials is an actual problem of a solid mechanic. Recently, various composite materials have been studied as nonlinear anisotropic elastic solids. The nonlinear behavior of isotropic materials is widely studied in [1–11, 14–18, 24, 25, 33–41]. Such materials as composites, biological tissues are modeled as solid reinforced nonlinear materials. This work is devoted to the construction of methods for calculating such materials. There are many works devoted to the study of composite reinforced solids [19–23]. The behavior of biological tissues was considered by the authors [12, 13, 30, 31]. Also, various numerical methods for the study of composite materials are given by the authors [26–29, 42].

2. Transversally isotropic model.
Composite materials reinforced with one fiber family can be modeled as a transversally isotropic nonlinear solid. In the present paper, the compressible Neo-Hookean material reinforced with one family of fibers was considered:

\[ \Psi((B),\bar{a})=\frac{\mu}{2} (J_{1B} - 3) - \mu \ln(J) + \frac{\lambda}{2} (\ln(J))^2 + \frac{\mu \gamma}{2} (J_5 - 1)^2 \]  

where \( \mu \), \( \lambda \ (>0) \) – mechanical constants of the Neo-Hookean material, \( \gamma \ (>0) \) is a material constant that provides a measure of the strength of reinforcement in the fiber direction [32]. \( J_{1B} = \text{tr}(B), J = \sqrt{\text{tr}(B)} \) invariants of the measure of the left Cauchy-Green deformation tensor \( (B) \). In fiber-reinforced materials work with the function of strain energy containing two components. The first is an isotropic term. The second term, which brings anisotropy. In particular, for transversally isotropic materials the second term is invariants \( J_4 = \bar{a} \cdot \bar{a} \) and \( J_5 = \bar{a} \cdot (B)\bar{a} \). In the reference configuration, the unit vector of the reinforcement direction will be \( \bar{a} = \frac{(F) \cdot \bar{a}_0}{\| (F) \cdot \bar{a}_0 \|} \), where \( \bar{a}_0 \) is a
vector of the reinforcement direction in the deformation configuration, \((F)\) deformation gradient tensor.

The invariant \(J_s\) is characterized as the square of stretching of the material in the direction of reinforcement. \(J_s\) it is also associated with stretch, but at the same time takes into account the reaction of the reinforcement to shear deformations.

The isotropic part and the transversal part are isolated in the potential energy of deformation:

\[
\Psi = \Psi_{iso} + \Psi_{trans}
\]

\[
\Psi_{iso} = \frac{\mu}{2} (J^4_{iso} - 3) - \mu \ln(J) + \frac{\lambda}{2} (\ln(J))^2
\]

\[
\Psi_{trans} = \frac{\mu \gamma}{2} (J_s - 1)^2
\]

3. Constitutive relations

For formulation physical relations we will use a relation of:

\[
\partial \Psi = \frac{J}{2} (\Sigma) \cdot (\delta B)
\]

From this relation follows the formula for finding the Cauchy stress:

\[
(\Sigma) = \frac{2}{J} \left\{ \frac{\partial \Psi}{\partial (B)} \cdot (B) \right\}
\]

Just as the potential strain energy, the stress tensor will consist of two terms:

\[
(\Sigma) = (\Sigma_{iso}) + (\Sigma_{trans})
\]

The isotropic components of the stress tensor:

\[
(\Sigma_{iso}) = \frac{\mu}{2} (B) \cdot (I) + \lambda (J - 1)(I)
\]

Transversal component:

\[
(\Sigma_{trans}) = \mu \gamma \cdot J_s - 1 \cdot \dot{\alpha} \otimes (B) \dot{\alpha} + (B) \dot{\alpha} \otimes \dot{\alpha}
\]

The second derivative of the stress tensor is the fourth-order elasticity tensor:

\[
(\Delta) = \frac{4}{J} (B) \cdot \frac{\partial^2 (\Psi)}{\partial (B) \partial (B)} \cdot (B)
\]

\[
(\Sigma) = (\Delta) \cdot (d) + (h) \cdot (\Sigma) + (\Sigma) \cdot (h)^T - (\Sigma) I_{id}
\]

4. The variational principles

To construct the calculation method using the method incremental loading, it is necessary to derive the linearized relations. The principle of virtual work in terms of the virtual velocity is used:

\[
\int \frac{1}{2} (\Sigma) \cdot (\delta(d)) dV = \int (\vec{f} - \rho \dot{\vec{v}} \cdot \delta \vec{v}) dV + \int_{S^o} t_n \delta \vec{t} dS,
\]

where \(\vec{v}\) – velocity vector of a material point; \(S^o\) – part of the surface on which are defined forces; \(\vec{f}, t_n\) – vectors of body and surface forces respectively.

After linearization, we received the resolving system of the linear algebraic equations.
\[
\left\{ \frac{1}{2}(\Sigma \cdot (\delta(h)^T \cdot k(h) + |h|^2 \cdot \delta(h)) + k(d) \cdot (\Lambda) \cdot \delta(d) - \left[ k \Delta_y \cdot \frac{\Delta \nu}{\Delta t}\right] f^T \cdot \delta \nu \right\} dV +
\]
\[
\int_{\Gamma} \left[ k \Delta_y \cdot \frac{\Delta \nu}{\Delta t}\right] f^T \cdot \delta \nu dS - \frac{1}{\Delta t}
\]
\[
\times \left\{ \int_{\Gamma} k(d) \cdot \delta(d) dV - \int_{\Gamma} k \frac{\Delta \nu}{\Delta t} \cdot \delta \nu dS - \int_{\Gamma} \frac{\Delta \nu}{\Delta t} \cdot \delta \nu dS \right\}
\]

The deformation process is represented as a sequence of equilibrium states. Each subsequent equilibrium state is calculated by increasing the load. At each step, the necessary stress tensor and strain measures are calculated.

5. The finite element discretization.

To solve the equations obtained, the finite element method is used, which is most widely used in modern solid mechanics. An 8-node isoparametric three-dimensional finite element is used. Approximate geometry and velocity:

\[
k y^i(\xi^i) = \sum_{i=1}^{8} k y_i N_i(\xi^i),
\]

\[
k \nu^i(\xi^i) = \sum_{i=1}^{8} k \nu_i N_i(\xi^i),
\]

there \( k y^i \) are the coordinates of the nodes in the global coordinate system.

\( N_i(\xi^i) = \frac{1}{8} (1 + \xi_1^i)(1 + \xi_2^i)(1 + \xi_3^i) \) functions forms with the local coordinate \(-1 \leq \xi_1, \xi_2, \xi_3 \leq 1\), \( \xi_i = \pm 1 \).

In the above approximation, we calculate all the necessary tensors:

\[
B^i = \sum_{i,j,m}^{8} y_i^j N_j m_s m_i s m,
\]

\[
F^i = \sum_{i,j}^{8} y_i^j N_i j,
\]

\[
d_{ij} = \sum_{i=1}^{8} \frac{1}{2} (C_{j}^i N_{i,m} + C_{i}^m N_{i,m})
\]

\( \nu = \frac{\Delta \nu}{\Delta t} \) velocity vector are unknown. We take \( \Delta t = 1 \), and find the displacement vector \( \Delta u \) from the solution obtained after all the transformations of the system of linear algebraic equations \( [K] \Delta u = \Delta P - \Delta H \), which allows us to calculate the stresses and geometry for the \( I + 1 \) state.

6. Conclusion

As a result of the work performed, a numerical algorithm for calculating the transversally isotropic hyperelastic model of the material was compiled. Compressible reinforced Neo-Hookean material was considered. The resolving equations are derived. The results do not contradict the previously published papers.

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References

[1] Golovanov A I, Konoplev Yu G, Sultanov L U 2008 Uchenye Zapiski Kazanskogo Universiteta. Seriya Fiziko-Matematicheskie Nauki 150 (3) 122–132

[2] Golovanov A I, Konoplev Yu G, Sultanov L U 2010 Uchenye Zapiski Kazanskogo Universiteta. Seriya Fiziko-Matematicheskie Nauki 151 (3) 108–120

[3] Bonet J and Wood R D 1997 Nonlinear continuum mechanics for finite element analysis (Cambridge University Press) 283

[4] Davydov R L and Sultanov L U 2013 PNRPU Mechanics Bulletin 1 81–93

[5] Golovanov A I and Sultanov L U 2005 International Applied Mechanics 41 (6) 614–620

[6] Shamim M R. and Berezhnoi D V 2016 IOP Conference Series: Materials Science and Engineering 158 (1) 012083

[7] Berezhnoi D V, Balafendieva I S, Sachenkov A A and Sekaeva L R 2016 IOP Conference Series: Materials Science and Engineering 158 (1) 012018

[8] Berezhnoi D V and Shamim R 2017 Procedia Engineering 206 1056–1062

[9] Sultanov L U 2014 Applied Mathematical Sciences 8 (143) 7117–7124

[10] Sultanov L U, Davydov R L and 2014 Applied Mathematical Sciences 8 (60) 2991–2996

[11] Sultanov L U 2015 Procedia Earth and Planetary Science 15 119–124

[12] Abdrakhmanova A I and Sultanov L U 2016 Materials Physics and Mechanics 26(1) 30–32

[13] Davydov R L and Sultanov L U 2013 Sixth International Conference on Nonlinear Mechanics (ICNM-VI) 64–67

[14] Davydov R L and Sultanov L U 2015 Journal of Engineering Physics and Thermophysics 88(5) 1280–1288

[15] Sultanov L U and Fakhruddinov L R 2013 Magazine of Civil Engineering 44 (9) 69–74

[16] Sultanov L U and Davydov R L 2013 Magazine of Civil Engineering 44 (9) 64–68

[17] Spencer A J M 1984 Continuum theory of the mechanics of fibre-reinforced composites 284

[18] Behrens B A, Rolffes R, Vucetic M, Reinoso J, Vogler M and Grbic N 2014 Procedia CIRP 18 250–255

[19] Advani S G and Talreja R 1993 Mechanics of Composite Materials 29 (2) 171–183

[20] Mortazavian S and Fatemi A 2015 Composites: Part B 72 116–129

[21] Vogler M, Rolffes R and Camanho P P 2013 Mechanics of Materials 59 50–64

[22] Golovanov A I and Sultanov L U 2008 Russian Aeronautics 51 (4) 2008 362–368

[23] Sultanov L U 2016 Lobachevskii Journal of Mathematics 37 (6) 787–793

[24] Badriev I B, Makarov M V and Paimushin V N 2015 Russian Mathematics 59 (10) 57–60

[25] Badriev I B, Garipova G Z, Makarov M V and Paimushin V N 2015 Research Journal of Applied Sciences 10 (8) 428–435

[26] Badriev I B, Garipova G Z, Makarov M V, Paimushin V N and Khabibullin R F 2015 Lobachevskii Journal of Mathematics 36 (4) 474–481

[27] Holzapfel G A, Gasser T C and Ogden R W 2000. J. Elasticity 61 1–48.

[28] Holzapfel G A, Gasser T C and Ogden R W 2004 J. Biomech. Engrg. 126 264–275

[29] Merodio J and Ogden R W 2005 Int. J Non-Linear Mech. 40 (2–3) 213–227

[30] Sultanov L U 2016 Lobachevskii Journal of Mathematics 37(6) 787–793

[31] Garifullin I R and Sultanov L U 2016 IOP Conference Series: Materials Science and Engineering 158 (1) 012035

[32] Davydov R L and Sultanov L U 2016 IOP Conference Series: Materials Science and Engineering 158 (1) 012030

[33] Sultanov L U 2016 IOP Conference Series: Materials Science and Engineering 158 (1) 012088

[34] Sultanov L U 2016 IOP Conference Series: Materials Science and Engineering 158 (1) 012090

[35] Fakhruddinov L R, Sultanov L U 2016 IOP Conference Series: Materials Science and Engineering 158 (1) 012031

[36] Abdrakhmanova A I, Garifullin I R and Sultanov L U 2016 IOP Conference Series: Materials Science and Engineering 158 (1) 012001
[37] Sultanov L U 2017 Uchenye Zapiski Kazanskogo Universiteta. Seriya Fiziko-Matematicheskie Nauki 159 (4) 4509–517
[38] Badriev I B and Banderov V V 2014 2015 Lobachevskii Journal of Mathematics 35 (4) 371–383
[39] Badriev I B, Makarov M V and Paimushin V N 2017 Russian Mathematics 51 (1) 69–75
Berezhnoi D V and Paymushin V N 2005 Naukoyemkiye tehnologii 6 (8-9) 59–64
[40] Berezhnoi D V and Sagdatullin M K 2015 Contemporary Engineering Sciences 8 (21-24)
1091–1098
[41] Sultanov L U 2018 Lobachevskii Journal of Mathematics 39 (9) 1478–1483
[42] Paimushin V N, Kayumov R A, Kholmogorov S A, Shishkin V M 2018 Russian Mathematics 62 (6) 75–79