ON THE METRIC GEOMETRY OF TORIC AND PLANAR LAMPLIGHTER GROUPS

F. BAUDIER, P. MOTAKIS, TH. SCHLUMPRECHT, AND A. ZSÁK

ABSTRACT. Let $\mathbb{Z} \wr \mathbb{Z}^2$ be the planar lamplighter group. We show that there is a Lipschitz map $f$ from $\mathbb{Z} \wr \mathbb{Z}^2$ into $L_1$ whose compression modulus $\rho_f$ satisfies $t/\rho_f(t) \leq \log(t) \log^{-\omega}(t)$. We use a divide-and-conquer recursive approach to show that the sequence $(\mathbb{Z} \wr \mathbb{Z}^2_n)_{n \in \mathbb{N}}$ of lamplighter groups over 2-dimensional discrete tori admits an equi-coarse embedding with the same compression function, from which we deduce the result. Our approach also provides an example of a bi-Lipschitz embedding of $\mathbb{Z} \wr \mathbb{Z}^2_n$ into $L_1$ with distortion $O((\log(n))^\beta)$ for some positive constant $\beta$.

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1. INTRODUCTION

In [NP11], Naor and Peres showed that if $\Gamma$ is a finitely generated group with polynomial growth at least quadratic then for every $\varepsilon \in (0, 1)$ there exist a constant $C > 0$ and a map $f : \mathbb{Z} \wr \Gamma \to L_1$ such that for all $u, v \in \mathbb{Z} \wr \Gamma$,

$$d_{\mathbb{Z} \wr \Gamma}(u, v)^{1-\varepsilon} \leq \|f(u) - f(v)\|_1 \leq Cd_{\mathbb{Z} \wr \Gamma}(u, v),$$

where $d_{\mathbb{Z} \wr \Gamma}$ denotes the word metric on the restricted wreath product $\mathbb{Z} \wr \Gamma$. In the language of geometric group theory, it follows from inequality (1) that $\alpha_1(\mathbb{Z} \wr \mathbb{Z}^2) = 1$, where the numerical parameter $\alpha_1(\Gamma)$, introduced by Guentner and Kaminker [GK04], denotes the...
$L_1$-compression of the finitely generated group $\Gamma$. In the same paper, Naor and Peres raised the following problem.

**Problem A** ([NP11 Question 10.1]). Does the planar lamplighter group $\mathbb{Z}_2 \wr \mathbb{Z}^2$ embed bi-Lipschitzly into $L_1$?

Thus, Problem A asks whether we can find a map that satisfies inequality (1) with $\varepsilon = 0$. This challenging problem is in part motivated by Jones’ travelling salesman theorem [Jon90]. While the combinatorial travelling salesman problem is concerned with finding the shortest path in a graph that visits a given set of vertices, the analyst travelling salesman problem is asking under which quantitative conditions a bounded set $K$ in a metric space is contained in a rectifiable curve (i.e., a curve with finite length) and how to estimate the length of the shortest curve containing $K$, should such a curve exist. Given a bounded set $K$ in the Euclidean plane $\mathbb{R}^2$, Jones introduced a sequence of numbers $(\beta_k(Q))_{Q}$, indexed by dyadic squares $Q$, such that the quantity $\beta_k^2 := \text{diam}(K) + \sum_Q \beta_k^2(Q) \text{diam}(Q)$ is essentially equivalent to the length of the shortest curve containing $K$ (assuming such a curve exists).

The Jones $\beta$-numbers are scale-invariant geometric quantities that measure how far a given set deviates from a best fitting line at each scale and location.

The metric geometry of lamplighter groups is intimately connected to a combinatorial travelling salesman problem. Given a finitely generated group $\Gamma$, the set of all functions $f : \Gamma \to \mathbb{Z}_2$ with finite support, denoted by $\mathbb{Z}_2^{\Gamma}$, is a group with pointwise multiplication. If we identify $\mathbb{Z}_2^{\Gamma}$ with $[\Gamma]^{<\omega}$ the set of finite subsets of $\Gamma$, the group operation becomes the operation of taking the symmetric difference. The (restricted) lamplighter group $\mathbb{Z}_2 \wr \Gamma$ can then be described as the group of all pairs $(A,x)$, where $A \in [\Gamma]^{<\omega}$ and $x \in \Gamma$, equipped with the group operation given by $(A,x) \cdot (B,y) = (A \triangle B, xy)$, where $xB = \{xb : b \in B\}$. It is then easy to verify that $\mathbb{Z}_2 \wr \Gamma$ is finitely generated and that $d_{\mathbb{Z}_2 \wr \Gamma}((A,x),(B,y))$ is (up to multiplicative factors depending on the generating set) equal to $\text{tp}_\Gamma(x,A \triangle B,y) + |A \triangle B|$, where $\text{tp}_\Gamma(x,C,y)$ denotes the length of the shortest path in $\Gamma$ that starts at $x$, visits all the vertices in $C$, and terminates at $y$.

Given two finite sets $A$ and $B$ in $\mathbb{Z}_2^2$, one can compute their Jones $\beta$-numbers, $\beta_A^2$ and $\beta_B^2$, and use them to estimate the length of the shortest path that covers $A$ and $B$ respectively. If we wanted to estimate the length of the shortest path that covers $A \triangle B$, we would not be able to do so using $\beta_A^2$ and $\beta_B^2$, and we would have to compute the Jones $\beta$-number $\beta_{A \triangle B}^2$. However, since the lamplighter group $\mathbb{Z}_2 \wr \mathbb{Z}^2$ is amenable, a potential bi-Lipschitz embedding $f : \mathbb{Z}_2 \wr \mathbb{Z}^2 \to L_1$ could be upgraded to an equivariant bi-Lipschitz embedding $\tilde{f} : \mathbb{Z}_2 \wr \mathbb{Z}^2 \to L_1$ (cf. [NP11 Theorem 9.1]). After an appropriate translation we can assume that $\tilde{f}(\theta,(0,0)) = 0$ and it would then follow from the equivariance property of $\tilde{f}$ that for all $(A,x),(B,y) \in \mathbb{Z}_2 \wr \mathbb{Z}^2$,

$$\|\tilde{f}(A,x) - \tilde{f}(B,y)\|_1 = \|\tilde{f}(A \triangle B)\|_1 = \|\tilde{f}(A \triangle ((x-y) + B))\|_1 = \|\tilde{f}(A \triangle (x-y) + B)\|_1.$$

If for a bounded subset $C$ of $\mathbb{Z}^2$ we define the vector $\gamma_C := \tilde{f}(C,(0,0))$, then the number $\|\gamma_C\|_1$ would be, up to some universal constants, the length of the shortest path that starts at the origin, covers $C$, and returns at the origin. Moreover, given bounded subsets $A$ and $B$, the length of the shortest path that starts at the origin, covers $A \triangle B$, and returns at the origin, could now be estimated using $\gamma_A$ and $\gamma_B$ simply by computing $\|\gamma_A - \gamma_B\|_1$ (and without computing $\gamma_{A \triangle B}$) since it follows from (2) that $\|\gamma_A - \gamma_B\|_1 = \|\gamma_{A \triangle B}\|_1$. Therefore a positive solution to Problem A would provide an “equivariant strengthening” of the Jones travelling salesman theorem.

Problem A admits an equivalent form in terms of embeddability of finite lamplighter groups. Recall that for two metric spaces $X$ and $Y$, the $Y$-distortion of $X$ is

$$c_Y(X) = \inf\{\text{dist}(f) : f : X \to Y \text{ is a bi-Lipschitz embedding}\},$$
where
\[ \text{dist}(f) = \sup_{x \neq y} \frac{d_Y(f(x), f(y))}{d_X(x, y)} \sup_{x \neq y} \frac{d_X(x, y)}{d_Y(f(x), f(y))}. \]
We will simply write \( c_1(Y) \) for \( c_L(Y) \). Because embeddability into \( L_1 \) can be checked on finite subsets, we will focus on the \( L_1 \)-embeddability of the sequence of toric lamplighter groups \( (Z_2 \wr Z_n)_{n \in \mathbb{N}} \), since every finite subset of \( Z_2 \wr Z^2 \) isometrically embeds into \( Z_2 \wr Z^2_n \) for some large enough \( n \) (for well-chosen sets of generators). These facts will be justified in Section 5. Therefore Problem A can be reformulated as follows.

**Problem B.** Is \( \sup_{n \in \mathbb{N}} c_1(Z_2 \wr Z_n^2) < \infty ? \)

In Section 4 we give a proof of the following upper estimate on the \( L_1 \)-distortion of the toric lamplighter groups.

**Theorem A.** There exists a constant \( \beta > 0 \) such that for all \( n \in \mathbb{N} \) we have,
\[ c_1(Z_2 \wr Z_n^2) \lesssim \log(n)^\beta. \]

Note that since \( Z_2 \wr Z_n^2 \) has \( 2^{n^2} \cdot n^2 \) vertices the upper bound above is an exponential improvement over Bourgain’s upper bound for arbitrary finite metric spaces. It can be derived from Naor and Peres coarse embedding of \( Z_2 \wr Z^2 \), that Theorem A holds with \( \beta = 1 \). This follows essentially from the fact that for their embedding with compression exponent \( 1 - \varepsilon \), the dependency in \( \varepsilon \) for the constant \( C \) in equation (1) above is in \( 1/\varepsilon \). Our proof gives \( \beta \lesssim \log_2(35) \). At the expense of clarity, it is possible to carry out a slightly more tedious analysis that would give a slightly better upper bound on \( \beta \). It is not possible in general to obtain compression estimates from distortion estimates or vice versa, and thus we cannot readily use the quantitative information about the distortion of the discrete toric lamplighter groups in order to construct a coarse embedding of the planar lamplighter group. Nevertheless, our recursive approach for the proof of Theorem A, which relies on the construction of an embedding into \( L_1 \) of \( Z_2 \wr Z_n^2 \) assuming one is provided with an embedding of \( Z_2 \wr Z_n^2 \) into \( L_1 \), admits a refinement that allows us to construct a coarse embedding of the planar lamplighter group with a super-polynomial compression function.

**Theorem B.** There exist a constant \( \beta > 0 \), a map \( f : Z_2 \wr Z^2 \to L_1 \) and a constant \( C \in (0, \infty) \) such that for all \( u, v \in Z_2 \wr Z^2 \)
\[ \rho(d(u, v)) \leq \| f(u) - f(v) \|_1 \leq C d(u, v), \]
where
\[ \rho(t) = \frac{t}{\log(t)^{\beta+2 \log \log t + 1} (\log \log(t^2))^2}. \]

It is straightforward to verify that our proofs of Theorem A and Theorem B work for \( Z_2 \wr Z^d \) for any \( d \geq 2 \) with the caveat that the multiplicative constants will depend on the dimension \( d \). The work of Naor and Peres applies to the larger class of all groups with polynomial growth at least quadratic but it is unclear if it can go beyond compression functions of polynomial order. It is worth noting that the techniques developed by Tessera to obtain super-polynomial compression functions does not apply to \( Z_2 \wr Z^d \) for \( d \geq 2 \).

The work in [Tes08] applies to groups of sub-exponential growth, while the machinery of controlled sequences of Følner pairs in [Tes11] only provides a polynomial compression function for \( Z_2 \wr Z^d \), when \( d \geq 2 \).

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2. Preliminaries

In this section we recall some basic notions and prove some technical results that will be needed later on. We use standard results and notation from graph theory as can be found in [Bol98]. Throughout this article we will use the notation \( \preceq \) or \( \succeq \) to denote the corresponding inequalities up to universal constant factors.

2.1. Travelling salesman problem and minimal cost spanning trees. In the general travelling salesman problem there is given a set \( V \) together with a cost function \( c(x, y) \), \( x, y \in V \), that represents the cost of travelling from \( x \) to \( y \). The problem, which we denote by \( \text{TSP}_V(x, A, y) \), of finding the minimal cost \( \text{tsp}_V(x, A, y) \) of a route from \( x \) to \( y \) that visits all points in the finite set \( A \) is known to be NP-hard in general. (We will often drop the subscript \( V \) when this is clear from the context.) In the special case when the cost function is a metric, however, there is a simple polynomial-time algorithm that gives the optimum solution within a constant factor. We now proceed to describe this algorithm.

Fix a metric space \( (M, d) \) and consider the complete graph \( K_M \) on \( M \) which has vertex set \( M \) and edge set \( |M|^2 \), the set of all subsets of \( M \) of size 2. For each edge \( e = xy \) of \( K_M \) we define the \textit{cost} \( c(e) \) of \( e \) to be the distance between the endvertices of \( e \), i.e., \( c(e) = d(x, y) \). For a finite set \( A \subset M \) and a subgraph \( T \) of \( K_M \), we say \( T \) is a \textit{minimum cost spanning tree} for \( A \) if \( T \) is a spanning tree for \( A \) (which means that \( T \) is a connected, acyclic graph with vertex set \( A \)) and \( c(T) \leq c(T') \) for all spanning trees \( T' \) for \( A \). Finally, we define the \textit{reduced minimum cost} \( \text{mc}(A) \) of \( A \) by \( \text{mc}(A) = c(T) \) where \( T \) is any minimum cost spanning tree for \( A \). The following relationship between minimum cost spanning trees and the travelling salesman problem is well known. We give a sketch proof for completeness.

Proposition 1. Let \( A \) be a finite subset of a metric space \( M \) and let \( x, y \in M \). Then
\[
\text{mc}(A \cup \{x, y\}) \leq \text{tsp}(x, A, y) \leq 2 \cdot \text{mc}(A \cup \{x, y\}).
\]

\textit{Proof.} The first inequality is clear. Indeed, for \( x \neq y \) a solution for \( \text{TSP}(x, A, y) \) is a minimum cost Hamiltonian path in \( A \cup \{x, y\} \) from \( x \) to \( y \). Since a Hamiltonian path is in particular a spanning tree, it follows that \( \text{mc}(A \cup \{x, y\}) \leq \text{tsp}(x, A, y) \). If \( x = y \), then a solution for \( \text{TSP}(x, A, y) \) is a minimum cost Hamiltonian cycle in \( A \cup \{x, y\} \). Removing an edge reduces the cost and creates a Hamiltonian path. So as before, we obtain \( \text{mc}(A \cup \{x, y\}) \leq \text{tsp}(x, A, y) \). For the second inequality, fix a minimum cost spanning tree \( T \) for \( A \cup \{x, y\} \). It is not too hard to check (see for example [BMSZ] Theorem 3) for a detailed proof) that there is a walk in \( T \) from \( x \) to \( y \) that visits every vertex and passes through each edge of \( T \) at most twice. It follows easily that \( \text{tsp}(x, A, y) \leq 2 \cdot c(T) = 2 \cdot \text{mc}(A \cup \{x, y\}) \). \( \square \)

We next recall a simple algorithm for producing minimum cost spanning trees for a finite subset \( A \) of \( M \). Given \( i \in \mathbb{N} \) and distinct edges \( e_1, \ldots, e_{i-1} \) of \( K_A \), we define \( E_A(e_1, \ldots, e_{i-1}) \) to be the set of edges \( e \) of \( K_A \) distinct from \( e_1, \ldots, e_{i-1} \) so that the subgraph of \( K_A \) with edge set \( \{e_1, \ldots, e_{i-1}, e\} \) is acyclic. We then let
\[
\text{mc}_A(e_1, \ldots, e_{i-1}) = \min \{ c(e) : e \in E_A(e_1, \ldots, e_{i-1}) \}
\]
\[
E_A^0(e_1, \ldots, e_{i-1}) = \{ e \in E_A(e_1, \ldots, e_{i-1}) : c(e) = \text{mc}_A(e_1, \ldots, e_{i-1}) \}.
\]

Proposition 2. Let \( A \) be a finite subset of a metric space \( M \) and let \( n = |A| \). If \( e_1, \ldots, e_{n-1} \) are edges of \( K_A \) such that \( e_i \in E_A^0(e_1, \ldots, e_{i-1}) \) for \( 1 \leq i \leq n-1 \), then the subgraph \( T \) of \( K_A \) with edge set \( \{e_1, \ldots, e_{n-1}\} \) is a minimum cost spanning tree for \( A \).

Conversely, if \( e_1, \ldots, e_{n-1} \) are edges of a minimum cost spanning tree for \( A \) with \( c(e_1) \leq \ldots \leq c(e_{n-1}) \), then \( e_i \in E_A^0(e_1, \ldots, e_{i-1}) \) for \( 1 \leq i \leq n-1 \).

\textit{Proof.} For the first part, let \( k \in \mathbb{N} \) be maximal so that there is a minimum cost spanning tree \( T' \) for \( A \) containing the edges \( e_i, 1 \leq i < k \). If \( k < n \), then \( T' + e_k \) contains a cycle that in turn contains at least one edge \( e \) that does not belong to \( \{e_1, \ldots, e_k\} \) since \( T \) is
acyclic. On the other hand, we have $e \in E_\delta(e_1, \ldots, e_{k-1})$ since $T'$ is acyclic. It follows that $c(e_k) \leq c(e)$. Hence $T'' = T' + e - e$ is connected and has $n - 1$ edges, and thus it is a spanning tree for $A$ with $c(T'') \leq c(T')$. We conclude that $T''$ is a minimum cost spanning tree for $A$ containing the edges $e_i$, $1 \leq i \leq k$. This contradiction with the choice of $k$ shows that $k = n$ and $T' = T$, as required.

The proof of the converse statement is similar. Choose $k \in \mathbb{N}$ maximal so that $e_i \in E^\delta_\omega(e_1, \ldots, e_{k-1})$ for $1 \leq i < k$. Assume that $k < n$, and pick $e \in E^\delta_\omega(e_1, \ldots, e_{k-1})$. Since $T$ is a tree, we certainly have $e_k \in E_\delta(e_1, \ldots, e_{k-1})$, but $c(e) > c(e_k)$ by the choice of $k$. In particular, $e \notin \{e_1, \ldots, e_{k-1}\}$, and thus $T + e$ contains a cycle. This cycle must contain an edge $e_i$ for some $i \geq k$ since $e \in E^\delta_\omega(e_1, \ldots, e_{k-1})$. Arguing as before, $T' = T + e - e_i$ is a spanning tree, and $c(T') < c(T)$. This contradiction shows that $k = n$, which completes the proof.

Note that the function $\overline{\mathfrak{mc}}$ is not monotone with respect to inclusion. Indeed, for $n \geq 2$ consider $A = \{x_1, \ldots, x_n\}$ and $B = \{x_0, x_1, \ldots, x_n\}$ with $d(x_0, x_i) = 1$ for $1 \leq i \leq n$ and $d(x_i, x_j) = 2$ for $1 \leq i < j \leq n$. Then $A \subseteq B$ and $\overline{\mathfrak{mc}}(B) = n$ whereas $\overline{\mathfrak{mc}}(A) = 2(n - 1)$. The factor 2 in this example is optimal since we have:

**Lemma 3.** Let $A \subset B$ be finite subsets of a metric space $M$. Then

$$\overline{\mathfrak{mc}}(A) \leq 2 \cdot \overline{\mathfrak{mc}}(B).$$

**Proof.** Let $x_1, \ldots, x_n$ be an ordering of $B$ so that $\sum_{i=2}^n d(x_{i-1}, x_i)$ is minimal. Then by Proposition 1 we have

$$\sum_{i=2}^n d(x_{i-1}, x_i) = \text{tsp}(x_1, B, x_n) \leq 2 \cdot \overline{\mathfrak{mc}}(B).$$

Let $1 \leq i_1 < \cdots < i_m \leq n$ be such that $A = \{x_{i_j} : 1 \leq j \leq m\}$. The edges $\{x_{i_{j-1}}x_{i_j} : 1 < j \leq m\}$ form a spanning tree for $A$. Hence, using the triangle inequality, we have

$$\overline{\mathfrak{mc}}(A) \leq \sum_{j=2}^m d(x_{i_{j-1}}, x_{i_j}) \leq \sum_{i=2}^n d(x_{i-1}, x_i) \leq 2 \cdot \overline{\mathfrak{mc}}(B).$$

Recall that $[M]^{<\omega}$ denotes the set of all finite subsets of $M$. Define for all $(A, x), (B, y) \in [M]^{<\omega} \times M$,

$$\tau_M((A, x), (B, y)) = \text{tsp}(x, A\triangle B, y).$$

It is clear that $\tau_M$ is a semimetric on $[M]^{<\omega} \times M$. We will drop the subscript $M$ when appropriate. Indeed, a walk from $x$ to $y$ visiting all vertices of $A\triangle B$ followed by a walk from $y$ to $z$ visiting all vertices in $B\triangle C$ constitutes a walk from $x$ to $z$ visiting all vertices in $A\triangle C$.

Our next result shows that if $M$ is a metric space with metric $d$ being the graph distance of a connected graph, then the semimetric $\tau_M$ behaves like a graph distance.

**Lemma 4.** Let $H = (V, E)$ be a connected graph equipped with its graph distance. Then for any $(A, x), (B, y) \in [V]^{<\omega} \times V$ with $\ell = \tau((A, x), (B, y)) \geq 1$, there is a sequence

$$(A, x) = (A_0, x_0), (A_1, x_1), \ldots, (A_\ell, x_\ell) = (B, y)$$

in $[V]^{<\omega} \times V$ such that $\tau((A_{i-1}, x_{i-1}), (A_i, x_i)) = 1$ for $1 \leq i \leq \ell$.

**Proof.** There is a walk $x_0, x_1, \ldots, x_\ell$ in $H$ from $x = x_0$ to $y = x_\ell$ visiting all vertices of $A\triangle B$. Let $A_0 = A$ and

$$A_i = A_{i-1} \triangle \{(x_{i-1}, x_i) \cap (A\triangle B)\} \quad \text{for} \quad 1 \leq i \leq \ell.$$

Then $(A_0, x_0) = (A, x)$, $(A_\ell, x_\ell) = (B, y)$ and for $1 \leq i \leq \ell$ we have

$$\tau((A_{i-1}, x_{i-1}), (A_i, x_i)) = \text{tsp}(x_{i-1}, A_{i-1} \triangle A_i, x_i) = 1.$$
since \(A_{i-1} \triangle A_i \subseteq \{x_{i-1}, x_i\}\) and \(x_{i-1} x_i\) is an edge in \(H\).

It will be convenient in the sequel to also work with another semimetric that can be defined based on a monotone variant of the reduced minimum cost.

**Definition.** Let \(M\) be a finite metric space. For a finite subset \(A\) of \(M\) we define the minimum cost \(\text{mc}_M(A)\) of \(A\) in \(M\) by

\[
\text{mc}_M(A) = \min c(G)
\]

where the minimum is taken over all finite connected subgraphs of \(K_M\) with vertex set containing \(A\).

We should drop the subscript \(M\) when appropriate. The advantage of \(\text{mc}\) over \(\text{mc}^\tau\) lies in its monotonicity. It is straightforward from the definition that \(\text{mc}(A) \leq \text{mc}(B)\) for finite subsets \(A \subset B\) of \(M\).

**Remarks.** 1. If \(\text{mc}(A) = c(G)\) for a finite connected graph \(G\) containing \(A\), then we call \(G\) a minimum cost graph for \(A\). Since every connected graph contains a spanning tree, by minimality of \(c(G)\) it follows that \(G\) is in fact a tree and we will also refer to it as a minimum cost tree for \(A\). A consequence of this and of Lemma\(^3\) is that \(\text{mc}(A) \leq \text{mc}^\tau(A) \leq 2\text{mc}(A)\) for all finite subsets \(A\) of \(M\).

2. The minimum cost of a finite subset \(A\) of \(M\) is at least the diameter of \(A\). Indeed, let \(G\) be a minimum cost graph for \(A\), and fix \(x, y \in A\) with \(d(x, y) = \text{diam}(A)\). Let \(x = x_0, x_1, \ldots, x_k = y\) be a path in \(G\) from \(x\) to \(y\). Then by the triangle-inequality we have

\[
\text{diam}(A) = d(x, y) \leq \sum_{i=1}^k d(x_{i-1}, x_i) = \sum_{i=1}^k c(x_{i-1}, x_i) \leq c(G) = \text{mc}(A).
\]

3. Suppose \(M\) is the vertex set of a connected graph \(H\) and \(d\) is the corresponding graph distance on \(M\). Let \(A\) be a finite subset of \(M\) and \(G\) be a finite connected subgraph of \(K_M\) with \(A \subseteq V(G)\). Replacing each edge \(e = xy\) of \(G\) with a path in \(H\) from \(x\) to \(y\) of length \(d(x, y) = c(e)\), we obtain a finite connected subgraph \(G'\) of \(H\) with \(A \subseteq V(G')\) and with cost \(c(G') \leq c(G)\). Note that \(c(G') = e(G')\) is the number of edges of \(G'\). It follows that \(\text{mc}(A) = \min\{e(G)\}\) where the minimum is over all finite connected subgraphs \(G\) of \(H\) with \(A \subseteq V(G)\). As before, we can further refine this noting that if the minimum is achieved at some graph \(G\), then \(G\) is in fact a tree.

As it was the case with the function \(\text{tsp}_M\), the function \(\text{mc}_M\) induces a semimetric on \([M]^{\ldots, 0} \times M\).

**Lemma 5.** Let \(M\) be a finite metric space and define \(\delta_M : ([M]^{\ldots, 0} \times M) \times ([M]^{\ldots, 0} \times M) \to \mathbb{R}\) by setting for \((A, x), (B, y) \in [M]^{\ldots, 0} \times M\),

\[
(6) \quad \delta_M((A, x), (B, y)) = \text{mc}_M((A \triangle B) \cup \{x, y\}).
\]

Then \(\delta_M\) is a semimetric on \([M]^{\ldots, 0} \times M\).

**Proof.** Given \((A_i, x_i) \in [M]^{\ldots, 0} \times M\) for \(i = 1, 2, 3\), set \(A'_i = (A_i \triangle A_k) \cup \{x_j, x_k\}\) for each \(i = 1, 2, 3\) with \(\{j, k\} = \{1, 2, 3\} \setminus \{i\}\). Then the required inequality

\[
\delta_M((A_1, x_1), (A_3, x_3)) \leq \delta_M((A_1, x_1), (A_2, x_2)) + \delta_M((A_2, x_2), (A_3, x_3))
\]

becomes \(\text{mc}(A'_2) \leq \text{mc}(A'_1) + \text{mc}(A'_1)\). To see this, choose a minimum cost graph \(G_i\) for \(A'_i\) for \(i = 1, 3\). Let \(G_2\) be the union of \(G_1\) and \(G_3\). Since \(A'_1 \cap A'_3 \neq \emptyset\) (it contains \(x_2\)), it follows that \(G_2\) is connected. Moreover, \(A'_2 \subseteq A'_1 \cup A'_3 \subseteq V(G_2)\), and thus

\[
\text{mc}(A'_2) \leq c(G_2) \leq c(G_1) + c(G_3) = \text{mc}(A'_1) + \text{mc}(A'_3),
\]

as required.

The relationship\(^4\) between \(\text{tsp}_M\) and \(\text{mc}^\tau_M\) can be extended to the two semimetrics \(\tau_M\) and \(\delta_M\). \(\square\)
Lemma 6. Let $M$ be a finite metric space. Then the two semimetrics $\tau_M$ and $\delta_M$ on $[M]^{c\omega} \times M$ satisfy:

$$\delta_M((A,x),(B,y)) \leq \tau_M((A,x),(B,y)) \leq 2\delta_M((A,x),(B,y))$$

for all $(A,x),(B,y) \in [M]^{c\omega} \times M$.

Proof. To verify (7) we use Proposition 1. On the one hand, we have

$$\delta_M((A,x),(B,y)) = \text{mc}_M((A\triangle B) \cup \{x,y\}) \leq 2\text{mc}_M((A\triangle B) \cup \{x,y\})$$

$$\leq \text{tsp}_M(x,A\triangle B,y) = \delta_M((A,x),(B,y)).$$

On the other hand, given a finite connected subgraph $G$ of $K_M$ with vertex set $V$ containing $(A\triangle B) \cup \{x,y\}$, we have

$$\text{tsp}_M(x,A\triangle B,y) \leq \text{tsp}_M(x,V,y) \leq 2\text{mc}_M(V) \leq 2c(G).$$

Taking minimum over all $G$, we obtain

$$\text{tsp}_M(x,A\triangle B,y) \leq 2\delta_M((A,x),(B,y)),$$

which completes the proof of (7). \qed

2.2. Lamplighter spaces. Even though our goal is to prove some embedding results about lamplighter groups, we will mostly work in the framework of lamplighter graphs which are generalizations of the former. Working with lamplighter graphs essentially amounts to fixing, once and for all, a particular finite generating set for which the graph metric has a convenient form. It also allows use to manipulate the lamplighter construction over graphs that are not necessarily Cayley graphs of groups, and indeed are not even necessarily regular.

Let $G = (V,E)$ be a graph, which may be finite or infinite, and in the latter case we do not necessarily assume $G$ to be locally finite. The lamplighter graph $\La(G)$ of $G$ is the graph with vertex set consisting of all pairs $(A,x)$ where $A$ is a finite subset of $V$ and $x \in V$. Two vertices $(A,x)$ and $(B,y)$ are connected by an edge if and only if either $A = B$ and $xy$ is an edge in $G$, or $x = y$ and $A\triangle B = \{x\}$. The usual interpretation of $\La(G)$ is as follows. Each vertex of $G$ has a light attached to it. A vertex $(A,x)$ of $\La(G)$ is the configuration in which $A$ is the set of lights that are switched on, and a lamplighter is standing at vertex $x$. The lamplighter can make one of two possible moves: he can either move to another vertex $y$ along an edge in $G$, or stay at $x$ and change the status of the light at vertex $x$. Lamplighter graphs are generalizations of lamplighter groups as follows. Given a group $\Gamma$ with generating set $S$, the lamplighter graph of the Cayley graph $\text{Cay}(\Gamma,S)$ of $\Gamma$ with respect to $S$ is the Cayley graph of the lamplighter group of $\Gamma$ with respect to an appropriate generating set.

When $G$ is connected, it becomes a metric space with the graph distance $d = d_G$ given by the length of a shortest path between vertices. In this case $\La(G)$ is also connected and its graph distance is given by the following formula (see for instance \cite[Proposition 1]{BMSZ}).

$$d_{\La(G)}((A,x),(B,y)) = \text{tsp}_G(x,A\triangle B,y) + |A\triangle B|$$

where $\text{tsp}_G(x,C,y)$ denotes the solution of the combinatorial travelling salesman problem in $G$ for $x,y \in G$ and $C$ a finite subset of $G$. Thus, $\text{tsp}_G(x,C,y)$ is the least $n \geq 0$ such that there is a sequence $x_0,x_1,\ldots,x_n$ of vertices in $G$ satisfying $x_0 = x$, $x_n = y$, $x_i-x_{i-1} \in E(G)$ for $1 \leq i \leq n$ and $C \subset \{x_0,\ldots,x_n\}$.

We are mainly interested in lamplighter graphs, however it turns out, that some of our embedding results are also valid for more general lamplighter spaces. By analogy with the lamplighter graph construction, we make the following definition. Recall that $[M]^{c\omega}$ denotes the set of all finite subsets of $M$.
Definition. Let $M$ be a metric space with metric $d_M$. We define the **lamplighter metric space** $La(M)$ of $M$ to be the set $[M]^{<\omega} \times M$ with metric

$$d_{La(M)}((x,y),(B,y)) = \text{tsp}_M(x,A\triangle B,y) + |A\triangle B|$$

(9)

We will drop the subscripts $M$ and $La(M)$ when appropriate.

The fact that $d_{La(M)}$ is a metric follows from the fact that it is the sum of the semimetric $\tau_M$ on $[M]^{<\omega} \times M$ and the classical metric $d_\triangle$ on $[M]^{<\omega}$ where $d_\triangle(A,B) = |A\triangle B|$. The fact that this is indeed an extension of the notion of lamplighter graph amounts to proving that $\text{tsp}_M(x,A,y) = \text{tsp}_G(x,A,y)$ for $x,y \in G$ in the case where the metric on $M$ coincides with the graph metric on associated on the vertex set of a connected graph $G$, i.e., $d = d_M = d_G$.

In this situation, the graph distance $d_{La(G)}$ on the vertex set of the lamplighter graph $La(G)$ is precisely the metric $d_{La(M)}$ of the lamplighter space $La(M)$ as defined above.

**Lemma 7.** Let $G = (V,E)$ be a connected graph and denote by $(M,d)$ the set $V$ equipped with the canonical graph metric. Then the combinatorial travelling salesman problem on $G$ coincides with the metric travelling salesman problem on $(M,d)$, i.e., $\text{tsp}_M(x,A,y) = \text{tsp}_G(x,A,y)$.

**Proof.** Given a walk $x_0,x_1,\ldots, x_n$ in $K_M$ from $x = x_0$ to $y = x_n$ visiting all vertices in $A$, for each $i = 1,\ldots,n$ let $w_i$ be a path in $G$ from $x_{i-1}$ to $x_i$ of length $d(x_{i-1},x_i)$. Concatenating $w_1,\ldots,w_n$ yields a walk in $G$ from $x$ to $y$ of length $\sum_{i=1}^n d(x_{i-1},x_i)$. This shows the inequality $\text{tsp}_M(x,A,y) \geq \text{tsp}_G(x,A,y)$. For the reverse inequality, consider a walk $x_0,x_1,\ldots, x_n$ in $G$ from $x = x_0$ to $y = x_n$ visiting all vertices in $A$. Then $d(x_{i-1},x_i) = 1$ for all $i$, and hence the total cost $\sum_{i=1}^n d(x_{i-1},x_i)$ is the length of the walk. It follows that $\text{tsp}_G(x,A,y) \geq \text{tsp}_M(x,A,y)$.

Since our main concern is the embeddability of lamplighter spaces into $L_1$ and $d_\triangle$ (the symmetric difference metric on $[M]^{<\omega}$) is an $L_1$-metric, we shall focus our attention to the semimetric $\tau_M$ and the equivalent semimetric $\delta_M$ (see Lemma 6). We now prove a few useful results on the semimetrics $\tau$ and $\delta$. The first one is easy and states that to embed a lamplighter space, it is sufficient to prove a pair of inequalities involving these two semimetrics. This fact will be used repeatedly in the ensuing arguments.

**Lemma 8.** Let $M$ be a finite metric space and $f: [M]^{<\omega} \times M \to L_1(\mu)$ be a function such that there exist constants $0 < \Gamma < \Lambda$ with

$$\Gamma \delta((A,x),(B,y)) \leq \|f(A,x) - f(B,y)\|_{L_1} \leq \Lambda \tau((A,x),(B,y))$$

for all $(A,x),(B,y) \in [M]^{<\omega} \times M$. Then there is a measure $\nu$ and there is a function $\tilde{f}: La(M) \to L_1(\nu)$ satisfying

$$\Gamma d((A,x),(B,y)) \leq \|\tilde{f}(A,x) - \tilde{f}(B,y)\|_{L_1} \leq 2 \Lambda \cdot d((A,x),(B,y))$$

for all $(A,x),(B,y) \in La(M)$.

**Proof.** Define $\tilde{f}: La(M) \to L_1(\mu) \otimes_1 L_1(\mu)$ by letting

$$\tilde{f}(A,x) = (2 \cdot f(A,x), \Gamma \cdot 1_A) .$$

Then

$$\|\tilde{f}(A,x) - \tilde{f}(B,y)\| = 2 \cdot \|f(A,x) - f(B,y)\| + \Gamma |A\triangle B| .$$

The result now follows from the assumption on $f$, from (7) and from definition (9) of the metric on $La(M)$.

The next result concerns a multi-coloured version of the lamplighter construction on a metric space. We will assume here that $M$ is a finite metric space. A remark on the general case is given after the proof. Let $S$ be an arbitrary set and denote by $S^M$ the set of all
functions $\alpha: M \to S$. For $\alpha, \beta \in S^M$ we let $\{\alpha \neq \beta\}$ stand for $\{u \in M: \alpha(u) \neq \beta(u)\}$. We define the multi-coloured lamplighter metric space $L_\alpha(M)$ to be the set $S^M \times M$ with distance $d = d_{L_\alpha(M)}$ defined by

$$d((\alpha,x),(\beta,y)) = tsp(x,\{\alpha \neq \beta\},y) + |\{\alpha \neq \beta\}|.$$ 

From now on we will identify $L_\alpha(M)$ and $La(M)$ with $S^M \times M$ and $[M]^0 \times M$ respectively. The observation that the restriction of a minimum cost spanning tree to a subset can be extended to a minimum cost spanning tree for the subset will be needed for the next proof. Recall that $[A]^2$ is the set of all subsets of $A$ of size 2, i.e., the set of all edges of $K_A$.

**Lemma 9.** Let $A \subset B$ be finite subsets of a metric space $M$. Let $T$ be a minimum cost spanning tree for $B$. Then there exists a minimum cost spanning tree $T'$ for $A$ such that $E(T) \cap [A]^2 \subset E(T')$. In particular, we have

$$\sum_{e \in E(T) \cap [A]^2} c(e) \leq \text{mcc}(A).$$

**Proof.** Order the edges $e_1,e_2,\ldots, e_n$ of $T$ so that $c(e_1) \leq c(e_2) \leq \cdots \leq c(e_n)$. By Proposition 2 we have

$$e_i \in E_B(e_i,\ldots,e_{i-1}) \quad \text{for } 1 \leq i \leq n.$$ 

Next, choose $1 \leq r_1 < r_2 < \cdots < r_m \leq n$ so that $E(T) \cap [A]^2 = \{e_{r_j}: 1 \leq \ell \leq m\}$. We may clearly assume that $m \geq 1$. We are going to choose edges $f_1,f_2,\ldots$ of $K_A$ and integers $1 \leq s_1 < s_2 < \cdots < s_m$ such that

$$f_i \in E_A(f_i,\ldots,f_{i-1}) \quad \text{for all } i,$$

$$f_{s_i} = e_{r_i} \quad \text{for all } \ell.$$ 

By Proposition 2 these edges will then form a minimum cost spanning tree $T'$ for $A$ that contains $E(T) \cap [A]^2$, as required.

Assume that for some $1 \leq k \leq m$ we have already chosen $1 \leq s_1 < \cdots < s_k-1$ and edges $f_1,f_2,\ldots,f_{s_k-1}$ so that (11) holds together with one further assumption. Let $C_1,\ldots,C_{m-k+1}$ be the connected components of $T - \{e_{r_i}: k \leq \ell \leq m\}$. For every edge $e$ of $K_B$ the endvertices of $e$ lie in components $C_{u(e)}$ and $C_{v(e)}$ for some $1 \leq u(e) \leq v(e) \leq m-k+2$.

We let $F = F_k = \{e \in [B]^2: u(e) \neq v(e)\}$. The further assumption is that $f_i \notin F$ for all $1 \leq i \leq s_{k-1}$. Note that these assumptions hold for $k = 1$ (we put $s_0 = 0$).

We now continue picking edges $f_{s_k+1},f_{s_k+2},\ldots,f_{j-1}$ so that

$$f_i \in E_A(f_i,\ldots,f_{i-1})$$

remains true for all $i$ and so that $j$ is minimal with $E_A(f_1,f_2,\ldots,f_{j-1}) \cap F \neq \emptyset$. The choice of $j$ implies that $f_i \notin F$ for all $1 \leq i \leq j-1$. It follows that

$$F \cap [A]^2 \subset E_A(f_1,\ldots,f_{j-1}).$$

By the definition of $F$, we have

$$F \cap E(T) = \{e_{r_\ell}: 1 \leq \ell \leq m\}$$

and

$$F \subset E_B(e_1,e_2,\ldots,e_{r_{k-1}}).$$

It follows from (12) and (13) that $e_{r_k} \in E_A(f_1,\ldots,f_{j-1})$. By (10) we have $e_{r_k} \in E_A^0(e_1,e_2,\ldots,e_{r_k-1})$, which together with (14) implies $c(e_{r_k}) \leq \min \{c(e): e \in F\}$. Finally, as $E_A^0(f_1,f_2,\ldots,f_{j-1}) \cap F \neq \emptyset$, we have $e_{r_k} \in E_A^0(f_1,f_2,\ldots,f_{j-1})$. So we can choose $f_j = e_{r_k}$ and $s_k = j$. If $k = m$ we now complete the construction of $T'$ by picking edges $f_i \in E_A^0(f_1,f_2,\ldots,f_{i-1})$ for $i > s_m$ until $E_A^0(f_1,f_2,\ldots,f_{j-1})$ becomes empty. If $k < m$, we need to verify our induction hypothesis for $k+1$. Observe that the components of $T - \{e_{r_\ell}: 1 \leq \ell \leq m\}$ are $C_\ell$ with $\ell \in \{1,\ldots,m-k+2\} \setminus \{u(e_{r_k}),v(e_{r_k})\}$ together with $C_{u(e_{r_k})} \cup C_{v(e_{r_k})}$. Since $f_i \notin F = F_k$
for all $1 \leq i \leq j - 1$, and $f_j = e_{k_j}$, it follows at once that $f_i \notin F_{k+1}$ for all $1 \leq i \leq s_k$. This completes the induction step. \hfill \Box

The next lemma and its proof are inspired by \cite{NP1} Lemma 2.1, but our argument differs from theirs for the lower bound of the norm-inequality for $G$ as it uses Lemma \cite{P}

**Lemma 10.** Let $M$ be a finite metric space and $S$ an arbitrary set. Assume there is a function $g : \La(M) \to L_1(\mu)$ and constants $0 < \Gamma \leq \Lambda$ such that

$$
\Gamma \delta((A,x),(B,y)) \leq \|g(A,x) - g(B,y)\|_{L_1} \leq \Lambda \tau((A,x),(B,y))
$$

for all $(A,x),(B,y) \in \La(M)$. Then there is a probability measure $\pi$ and a function $G : \La_2(M) \to L_1(\pi, L_1(\mu))$ such that

$$
\frac{\Gamma}{8} \mc(\{\alpha \neq \beta\} \cup \{x,y\}) \leq \|G(\alpha,x) - G(\beta,y)\|_{L_1} \leq \Lambda \cdot \tsp(x, \{\alpha \neq \beta\}, y)
$$

for all $(\alpha,x), (\beta,y) \in \La_2(M)$.

**Proof.** Fix iid $\{0,1\}$-valued Bernoulli random variables $(\varepsilon_i)_{i \in S}$ on some probability space $(\Omega, \pi)$. For $\alpha \in S^M$ set $C_\alpha = \{u \in M : \varepsilon_{\alpha(u)} = 1\}$ which is a random subset of $M$. We define $G : \La_2(M) \to L_1(\pi, L_1(\mu))$ by letting $G(\alpha,x) = g(C_\alpha, x)$.

Let us now fix $(\alpha,x), (\beta,y) \in \La_2(M)$. It is clear that $\mathcal{C}_{\alpha} \triangle \mathcal{C}_{\beta} \subset \{\alpha \neq \beta\}$. Hence, using the assumption on $g$, we obtain

$$
\|G(\alpha,x) - G(\beta,y)\| = \mathbb{E}_\pi\|g(C_\alpha, x) - g(C_\beta, y)\|_{L_1} \\
\leq \Lambda \cdot \tsp(x, C_\alpha \triangle C_\beta, y) \\
\leq \Lambda \cdot \tsp(x, \{\alpha \neq \beta\}, y).
$$

Turning to the lower bound, set $D = (C_\alpha \triangle C_\beta) \cup \{x,y\}$ and let $T$ be a minimum cost spanning tree for $(\alpha \neq \beta) \cup \{x,y\}$. Using the assumption on $g$, Lemma \cite{P} and the linearity of expectation, we obtain

$$
2\|G(\alpha,x) - G(\beta,y)\| = 2\mathbb{E}_\pi\|g(C_\alpha, x) - g(C_\beta, y)\|_{L_1} \\
\geq 2\mathbb{E}_\pi \mc((C_\alpha \triangle C_\beta) \cup \{x,y\}) = 2\mathbb{E}_\pi \mc(D) \\
\geq \Gamma \mathbb{E}_\pi \mc(D) = \Gamma \sum_{e \in E(T)} c(e) \\
= \Gamma \sum_{e \in E(T)} c(e) \mathbb{P}(e \in |D|^2).
$$

Fix an edge $e = uv \in E(T)$. Then $\mathbb{P}(e \in |D|^2) = \mathbb{P}(u,v \in (C_\alpha \triangle C_\beta) \cup \{x,y\})$. We now claim that this probability $\rho$ is at least $\frac{1}{3}$. It will then follow from above that

$$
\|G(\alpha,x) - G(\beta,y)\| \geq \frac{\Gamma}{6} \sum_{e \in E(T)} c(e) = \frac{\Gamma}{6} c(T) \geq \frac{\Gamma}{8} \mc(\{\alpha \neq \beta\} \cup \{x,y\})
$$

completing the proof.

To prove our claim, we consider three cases. If $u,v \in \{x,y\}$, then clearly $\rho = 1$. If $u \in \{x,y\}$ and $v \notin \{x,y\}$, then $\rho = \mathbb{P}(v \in C_\alpha \triangle C_\beta) = \mathbb{P}(\varepsilon_{\alpha(v)} \neq \varepsilon_{\beta(v)}) = \frac{1}{2}$ In the third case, $u,v \notin \{x,y\}$, and so

$$
\rho = \mathbb{P}(u,v \in C_\alpha \triangle C_\beta) = \mathbb{P}(\varepsilon_{\alpha(u)} \neq \varepsilon_{\beta(u)} \text{ and } \varepsilon_{\alpha(v)} \neq \varepsilon_{\beta(v)}).
$$

We now have three further cases depending on whether $\{\alpha(u), \beta(u)\} \cup \{\alpha(v), \beta(v)\}$ has size 2, 3 or 4. A straightforward computation gives the values $1/2$, $1/4$ and $1/4$, respectively in each case, for the value of $\rho$. \hfill \Box

**Remark 1.** For general $M$ we consider a set $S$ with a distinguished point 0 and replace $S^M$ with the set $S^{(M)}$ of all functions $\alpha : M \to S$ whose support $\{u \in M : \alpha(u) \neq 0\}$ is finite. The above result then still holds. The proof is modified by letting $C_\alpha = \{u \in M : \alpha(u) \neq 0\}$.
0 and $e_{2u(i)} = 1$. The last part of the proof will then involve checking more cases but otherwise it will be similar to the proof above.

3. A minimum cost estimate in the discrete torus

For $n \geq 2$ we denote by $\mathbb{Z}_n$ the one-dimensional discrete torus, i.e., the quotient group $\mathbb{Z}/n\mathbb{Z}$ operated with the quotient metric. We let $\mathbb{Z}_2^n = \mathbb{Z}_n \times \mathbb{Z}_n$ denote the two-dimensional discrete torus. For technical reasons we equip this with the $\ell_2$-distance

$$d((x_1, y_1), (x_2, y_2)) = \max\{d(x_1, x_2), d(y_1, y_2)\}$$

which is 2-equivalent to the $\ell_1$-distance. The latter is the graph distance in the Cayley graph of $\mathbb{Z}_2^n$ with respect to the standard generating set $\{(\pm 1, 0), (0, \pm 1)\}$, whereas the former uses the generating set $\{\pm 1, 0, (0, \pm 1)\}$. Define $[a, b] = \{k \in \mathbb{N} : a \leq k \leq b\}$.

We will simply write $[n]$ for $[0, n - 1] = \{0, 1, \ldots, n - 1\}$ and consider the finite grid $[n]^2$ which will also be equipped with the $\ell_\infty$-distance. We will always identify $[n]^2$ with a subset of $\mathbb{Z}_{2n}^2$ via the isometric embedding given by

$$(i, j) \mapsto (i + 2nZ, j + 2nZ).$$

For the rest of this section we fix $n \in \mathbb{N}$. In later sections we will consider various inductive considerations going from the torus of size $2n$ to the torus of size $2n^2$. Here we lay the ground for this by first introducing a fair amount of notation. We begin with $V \subset \mathbb{Z}_{2n}^2$ defined as

$$V = \{(i + 2nZ, j + 2nZ) : i, j \in [n]\}.$$ 

For every $s = (i, j) \in \mathbb{Z}_{2n}^2$ we let

$$C_s = (in, jn) + V$$

which is a subset of $\mathbb{Z}_{2n}^2$ (the map $a \mapsto an$ is a homomorphism $\mathbb{Z}_{2n} \to \mathbb{Z}_{2n^2}$) and is naturally isometric to $[n]^2$; and hence $C_s$ will be identified as a subset of $\mathbb{Z}_{2n}^2$. Note that $V = C_0$, for $s = (0, 0)$, however we keep the different notations as we will always think of $V$ as a subset of $\mathbb{Z}_{2n}^2$, and not as a subset of $\mathbb{Z}_{2n}^2$.

We next set $C'_v = v + C_s$ for $v \in V$ and $s \in \mathbb{Z}_{2n}^2$. Note that each of the sets $C'_v$ is isometric to $[n]^2$ and so they will be identified as subsets of $\mathbb{Z}_{2n}^2$. As it will follow from Lemmas 11 and 12 below, for any $v \in V$, $s \in \mathbb{Z}_{2n}^2$ and $A \subset C'_v$, the value of $mc(A)$ is the same whether it is calculated in $\mathbb{Z}_{2n}^2$, in $C'_v$ or in $\mathbb{Z}_{2n}^2$. This observation, which will be used frequently and implicitly in the sequel, is not completely immediate. Indeed, the functions $mc$ and $\delta$ depend on the ambient space $M$. If $N$ is a subspace of $M$, then in general for a finite subset $A$ of $N$ we have $mc_N(A) \leq mc_M(A)$, but this inequality may be strict. Thus the restriction of $\delta$ to $La(N)$ is dominated by but possibly different from $\delta$. However, in some situations as shown in the first of the next two lemmas, these two semi-metrics on $La(N)$ are the same. The second lemma then gives an example of such a situation in $\mathbb{Z}_{2n}^2$.

We say that a map $\varphi : M \to N$ is a 1-Lipschitz retraction if it is 1-Lipschitz and is the identity on $N$; if such a map exists, then we say $N$ is a 1-Lipschitz retract of $M$.

Lemma 11. Let $N$ be a 1-Lipschitz retract of a metric space $M$. Then for every finite subset $A$ of $N$ we have $mc_N(A) = mc_M(A)$. It follows that $(La(N), \delta_N)$ is isometrically a subspace of $(La(M), \delta_M)$.

Proof. Let $\varphi : M \to N$ be a 1-Lipschitz retraction. Let $A$ be a finite subset of $N$, and let $G$ be a connected subgraph of $K_N$ such that $A \subset V = V(G)$ and $mc_M(A) = c(G)$. Let $G'$ be the graph with vertex set $V' = \{\varphi(x) : x \in V\}$ and edge set $E'$ defined as follows. For distinct $u, v \in V'$ we have $uv \in E'$ if and only if there is an edge $xy \in E$ with $\varphi(x) = u$ and $\varphi(y) = v$. Then $G'$ is a finite subgraph of $K_N$ and since $\varphi$ is the identity on $N$, we have $A \subset V'$. Given $u, v \in V'$, choose $x, y \in V$ with $\varphi(x) = u$ and $\varphi(y) = v$ and a path $x = x_0, x_1, \ldots, x_n = y$ in $G$. Then after deleting consecutive repetitions in the sequence $\varphi(x_0), \varphi(x_1), \ldots, \varphi(x_n)$, we
obtain a walk in $G'$ from $u$ to $v$. Thus, $G'$ is connected. Finally, since $\varphi$ is 1-Lipschitz, it follows that $c(G') \leq c(G)$, and hence $mc_v(A) \leq mc(A)$. The reverse inequality is clear, and thus the proof is complete. □

**Lemma 12.** Let $n \in \mathbb{N}$ and $a, b \in \llbracket n \rrbracket$. The set $V_a = \{i + 2n\mathbb{Z} : i \in [a]\}$ is a 1-Lipschitz retract of $\mathbb{Z}^2_{2n}$. Hence the set $V_a \times V_b$ is a 1-Lipschitz retract of $\mathbb{Z}^2_{2n}$.

**Proof.** We first ‘fold’ $\mathbb{Z}^2_{2n} \setminus V_n$ onto $V_n$ via the map $\varphi : \mathbb{Z}^2_{2n} \to V_n$ defined by

$$\varphi(i + 2n\mathbb{Z}) = \begin{cases} i + 2n\mathbb{Z} & \text{if } 0 \leq i < n \\ -i + 2n\mathbb{Z} & \text{if } n \leq i < 2n \end{cases}.$$

It is straightforward to check that this is a 1-Lipschitz retraction. Next we ‘contract’ $V_n$ to $V_0$ via the map $\kappa_n : V_n \to V_0$ defined by

$$\kappa_n(i + 2n\mathbb{Z}) = \begin{cases} i + 2n\mathbb{Z} & \text{if } 0 \leq i < a \\ a + 2n\mathbb{Z} & \text{if } a \leq i < n \end{cases}.$$

It is again easy to check that this is a 1-Lipschitz retraction, and hence so is the composite $\psi_n = \kappa_n \circ \varphi : \mathbb{Z}^2_{2n} \to V_a$. Finally, the product $\psi_n \times \psi_n$ is a 1-Lipschitz retraction of $\mathbb{Z}^2_{2n}$ onto $V_0 \times V_0$. □

Given an element $A$ of the power set $\mathcal{P}(\mathbb{Z}^2_{2n})$, we let $A_v = A \cap C_v$ and $A_v^\times = A \cap C_v^\times$ for $s \in \mathbb{Z}^2_{2n}$ and $v \in V$. We also define

$$\tilde{A}^\times = \{s \in \mathbb{Z}^2_{2n} : A_v^\times \neq \emptyset\}$$

which is simplified to $\tilde{A}$ when $v = (0,0)$. Our recursive construction is based on the following divide-and-conquer strategy to estimate a travelling salesman path $tsp(x, C, y)$ in $\mathbb{Z}^2_{2n}$:

1. partition the discrete torus of size $2n^2$ into $2n$ squared cells of size $n$;
2. solve travelling salesman subproblems in $\llbracket n \rrbracket^2$, or equivalently in $\mathbb{Z}^2_{2n}$, inside the cells that intersect the set $C \cup \{x, y\}$ (at most $4n^2$ subproblems);
3. solve a travelling salesman subproblem in $\mathbb{Z}^2_{2n}$ to connect in an optimal way the cells that intersect the set $C \cup \{x, y\}$;
4. build a travelling salesman path, that bounds $tsp(x, C, y)$ from above, combining the solutions of the previous travelling salesman subproblems.

In step (2) and (3) we would have to define precisely the subproblems we are considering. Also regarding step (4), it is easier to combine solutions of minimum cost problems than solutions of travelling salesman problems. This is one of the reason why we will work with the minimum cost semimetric, and use the upper bound formula for $mc(A)$ in Lemma [13] below. The above approach does not necessarily work if $C \cup \{x, y\}$ has small diameter and if we use only one partition. We will thus need to average over the solutions obtained for a partition and certain of its shifts. It is worth noting that in the lemma below, the first two minimum costs are taken in $\mathbb{Z}^2_{2n}$, while the last one is taken with respect to $\mathbb{Z}^2_{2n}$ which is not consider as a subset of $\mathbb{Z}^2_{2n}$ in this situation; this explains the scaling factor $2n$.

**Lemma 13.** For every $A \subset \mathbb{Z}^2_{2n}$ and for every $v \in V$ we have

$$mc(A) \leq \sum_{s \in \mathbb{Z}^2_{2n}} mc(A_v^\times) + 2n \cdot mc(\tilde{A})$$

**Proof.** Since $A_v^\times = v + (A - v)_v$ and $\tilde{A}^\times = (A - v)$ for $v \in V$ and $s \in \mathbb{Z}^2_{2n}$, and since translation on the torus is an isometry, and hence preserves minimum cost, without loss of generality we can assume $v \equiv 0$.

For each $s \in A$, let $T_s$ be a subtree of $K_v$ with vertex set containing $A_v$ such that $mc(A_v) = c(T_s)$. We also let $\tilde{T}$ be a subtree of $K_{\mathbb{Z}^2_{2n}}$ with vertex set containing $\tilde{A}$ such
Lemma 14. In other words we spread out $\phi$ into $s$ sufficient to restrict ourselves to $n$. Since we will not need this lower bound we will not include its rather delicate proof.

It clearly that $T$ is a subtree of $K_{2n^2}$ with vertex set containing $A$, and thus $mc(A) \leq c(T)$. Observe that for $s, t \in \mathbb{Z}_{2n}^2$ and for $x \in C_s$ and $y \in C_t$ we have

$$d(x, y) \leq (d(s, t) + 1) \cdot n \leq 2n \cdot d(s, t).$$

It follows that

$$mc(A) \leq c(T) = \sum_{s \in \mathbb{Z}_{2n}^2} c(T_s) + \sum_{s \in E(T)} d(x(s), x(t)) \leq \sum_{s \in \mathbb{Z}_{2n}^2} mc(A_s) + 2n \sum_{e \in E(T)} c(e) = \sum_{s \in \mathbb{Z}_{2n}^2} mc(A_s) + 2n \cdot mc(A).$$

This completes the proof.

Remark 2. It is possible to show that for every $A \subset \mathbb{Z}_{2n}^2$ and for every $v \in V$ we have

$$mc(A) \geq \frac{1}{n^2} \sum_{v \in \mathbb{Z}_{2n}^2} \left( \sum_{s \in \mathbb{Z}_{2n}^2} mc(A_s^v) + 2n \cdot mc(A^v) \right).$$

Since we will not need this lower bound we will not include its rather delicate proof.

4. Upper bound on the $L_1$-distortion of toric lamplighter groups

In this section we prove Theorem A. Recall that this is equivalent to proving the result for the lamplighter graph $La(\mathbb{Z}_n^2)$ instead of the lamplighter group $\mathbb{Z}_2 \wr \mathbb{Z}_2$. It will also be sufficient to restrict ourselves to $n \in \mathbb{N} = \{2 \cdot 2^k : k \geq 0\}$ since we have:

**Lemma 14.** For all $n, N \in \mathbb{N}$ with $n \leq N$, there is a bi-Lipschitz embedding from $La(\mathbb{Z}_n^2)$ into $La(\mathbb{Z}_N^2)$ with distortion at most 6.

**Proof.** Write $N = qn + r$ where $q \in \mathbb{N}$ and $0 \leq r < n$. Define $\varphi : \mathbb{Z}_n \to \mathbb{Z}_N$ by

$$\varphi(i) = \begin{cases} qi & \text{for } 0 \leq i \leq n - r \\ (q+1)i + r - n & \text{for } n - r + 1 \leq i \leq n - 1. \end{cases}$$

In other words we spread out $\mathbb{Z}_n$ inside $\mathbb{Z}_N$ as evenly as possible. It is straightforward to check that $\varphi$ has distortion at most $\frac{q+1}{q} \leq 2$. This induces the obvious way a 2-Lipschitz map $\varphi : \mathbb{Z}_n^2 \to \mathbb{Z}_N^2$. In turn, by the remark following [BMSZ, Lemma 18], this induces a Lipschitz map $\varphi : La(\mathbb{Z}_n^2) \to La(\mathbb{Z}_N^2)$ with distortion at most 6.

The proof will proceed by a recursive construction, and follows quite easily from the following recursive step whose proof is the core of the matter.

**Lemma 15 (Recursive Step Lemma).** Let $n \in \mathbb{N}$. Assume there is a function $g : La(\mathbb{Z}_n^2) \to L_1$ and constants $0 < \Gamma \leq \Lambda$ such that for all $(A, x), (B, y) \in La(\mathbb{Z}_n^2)$,

$$\Gamma \delta((A, x), (B, y)) \leq \|g(A, x) - g(B, y)\|_{L_1} \leq \Lambda \tau((A, x), (B, y)).$$

Then there exists a map $f : La(\mathbb{Z}_{2n^2}) \to L_1$ such that for all $(A, x), (B, y) \in La(\mathbb{Z}_{2n^2})$

$$\Gamma \delta((A, x), (B, y)) \leq \|f(A, x) - f(B, y)\|_{L_1} \leq 35 \Lambda \tau((A, x), (B, y)).$$

We postpone momentarily the technical proof of the recursive step and show how it is used to complete the proof of:
Theorem 16. For each $n \geq 2$, $c_1(\text{La}(\mathbb{Z}_n^2)) = O((\log n)^{\beta/2})$.

Proof. For each $k \geq 0$ let $n = 2 \cdot 2^{2^k}$ and let $f_k: \text{La}(\mathbb{Z}_n^2) \to L_1$ be a map such that

$$\delta(A,x),(B,y) \leq \| f_k(A,x) - f_k(B,y) \|_{L_1} \leq \Lambda_k \cdot \tau(A,x),(B,y)$$

holds for all $(A,x),(B,y) \in \text{La}(\mathbb{Z}_n^2)$, and such that $\Lambda_k$ is best possible. By Lemma 15 we have $\Lambda_{k+1} \leq 35 \cdot \Lambda_k$, and hence

$$\Lambda_k \leq 35^k \cdot \Lambda_0.$$

It follows from Lemma 8 that

$$c_1(\text{La}(\mathbb{Z}_n^2)) = O((\log n)^{\beta/2}) \quad \text{for } n \in \mathcal{N} = \{2 \cdot 2^{2^k} : k \geq 0\}$$

where $\beta = \log_2(35)$. For $n \notin \mathcal{N}$ choose $k \geq 1$ minimal such that $n < 2 \cdot 2^{2^k}$ and set $N = N_n = 2 \cdot 2^{2^k}$. By Lemma 14 there is a bi-Lipschitz embedding of $\text{La}(\mathbb{Z}_n^2)$ into $\text{La}(\mathbb{Z}_N^2)$ with distortion at most 6, and so we have $c_1(\text{La}(\mathbb{Z}_n^2)) \leq 6 \cdot c_1(\text{La}(\mathbb{Z}_N^2))$. The result now follows since $\log N = O(\log n)$.

Remark 3. In [JV14] it was shown that

$$c_2(\mathbb{Z}_2 \times \mathbb{Z}_n^d) \geq \begin{cases} \frac{n}{\sqrt{\log n}} & \text{for } d = 2, \\ 1 \cdot n^2 & \text{for } d \geq 3. \end{cases}$$

By [NP11], $\mathbb{Z}_2 \times \mathbb{Z}_n^d$ coarsely embeds for $d \geq 2$ into $L_2$ with compression $\frac{1}{d}$, and the compression exponent is attained in this case (cf [NP11] Remark 3.2). It thus follows that $c_2(\mathbb{Z}_2 \times \mathbb{Z}_n^d) \approx n^2$. Therefore,

$$\frac{n}{\sqrt{\log n}} \lesssim c_2(\mathbb{Z}_2 \times \mathbb{Z}_n^2) \lesssim n,$$

and for $d \geq 3$

$$c_2(\mathbb{Z}_2 \times \mathbb{Z}_n^d) \approx n^d.$$

It would be interesting to understand the right order of magnitude for $c_2(\mathbb{Z}_2 \times \mathbb{Z}_n^d)$. 

4.1. Proof of the recursive step. For this entire section we fix $n \in \mathbb{N}$, a function $g: \text{La}(\mathbb{Z}_n^2) \to L_1(\mu)$ for some measure $\mu$, and constants $0 < \Gamma \leq \Lambda$ such that

$$\Gamma \delta((A,x),(B,y)) \leq \| g(A,x) - g(B,y) \|_{L_1} \leq \Lambda \tau((A,x),(B,y))$$

for all $(A,x),(B,y) \in \text{La}(\mathbb{Z}_n^2)$, and we will construct a map from $\text{La}(\mathbb{Z}_n^2)$ into some other $L_1(\mu')$-space. Recall that the inequality

$$\text{mc}(A) \leq \sum_{x \in \mathbb{Z}_n^2} \text{mc}(A^x) + 2n \cdot \text{mc}(\tilde{A}^v),$$

is valid for all $v \in V$. We are going to define two maps $p = p_v$ and $q = q_v$ associated to $g$. The first map $p$ will be used to estimate the term $\sum_{x \in \mathbb{Z}_n^2} \text{mc}(A^x)$ in (18), and the second map $q$ to estimate the term $2n \cdot \text{mc}(\tilde{A}^v)$ in (18).

Throughout this section and in the rest of this article, we are going to use the notation introduced in Section 3 following the proof Lemma 14. We also introduce a few additional pieces of notation. For each $s \in \mathbb{Z}_n^2$ and $v \in V$ we fix $c_s^v \in C_s^v$ such that $d(x,c_s^v) \leq n/2$ for all $x \in C_s^v$, and for $x \in \mathbb{Z}_n^2$ we let

$$x^v = \begin{cases} x & \text{if } x \in C_s^v, \\ c_s^v & \text{if } x \notin C_s^v. \end{cases}$$

Finally, we fix $v$ to be the uniform probability measure on the subset $V$ of $\mathbb{Z}_n^2$. 


4.1.1. Summing the local solutions: the map \( p \). For each \( v \in V \) we define

\[
p^v: \text{La}(\mathbb{Z}_2^{2n}) \to \bigoplus_{s \in \mathbb{Z}_2^{2n}} L_1(\mu)
\]

\[
(A, x) \mapsto (g(A^v_s, x^v_s) : s \in \mathbb{Z}_2^{2n})
\]

and then define

\[
p^v: \text{La}(\mathbb{Z}_2^{2n}) \to L_1\left( v, \bigoplus_{s \in \mathbb{Z}_2^{2n}} L_1(\mu) \right)
\]

\[
(A, x) \mapsto (p^v_p(A, x) : v \in V).
\]

The reason for the averaging resides in the fact that some vertices can be at distance 1 from each other but could end up being in two different cells of the partition, and this will incur too much distortion.

**Proposition 17.** Let \( g \) be as in (17) and \( p \) as above. Let \((A, x), (B, y) \in \text{La}(\mathbb{Z}_2^{2n}) \) and \( D = (A \triangle B) \cup \{x, y\} \). Then

\[
\Gamma \sum_{s \in \mathbb{Z}_2^{2n}} \text{mc}(D^v_s) \leq \|p^v_p(A, x) - p^v_p(B, y)\|_{L_1} \leq 3\Lambda \tau((A, x), (B, y)).
\]

**Proof.** For any \( v \in V \) and \( s \in \mathbb{Z}_2^{2n} \) we have

\[
D^v_s \subset (A \triangle B)^v_s \cup \{x^v_s, y^v_s\} = (A^v_s \triangle B^v_s) \cup \{x^v_s, y^v_s\},
\]

and hence

\[
\|p^v_p(A, x) - p^v_p(B, y)\|_{L_1} = \sum_{s \in \mathbb{Z}_2^{2n}} \|g(A^v_s, x^v_s) - g(B^v_s, y^v_s)\|_{L_1}
\]

\[
\geq \Gamma \sum_{s \in \mathbb{Z}_2^{2n}} \text{mc}((A^v_s \triangle B^v_s) \cup \{x^v_s, y^v_s\})
\]

\[
\geq \Gamma \sum_{s \in \mathbb{Z}_2^{2n}} \text{mc}(D^v_s).
\]

Taking expectation with respect to \( v \) yields the first inequality. To prove the second inequality, we may assume by Lemma[4] that \( \tau((A, x), (B, y)) \leq 1 \). If \( \tau((A, x), (B, y)) = 0 \), then \( x = y \) and \( A \triangle B \subset \{x\} \). It follows that \( x^v = y^v \) and \( A^v \triangle B^v \subset \{x^v\} \), and thus

\[
\|g(A^v_s, x^v_s) - g(B^v_s, y^v_s)\|_{L_1} \leq \Lambda \text{sp}(x^v, A^v \triangle B^v, y^v) = 0
\]

for all \( v \in V \) and \( s \in \mathbb{Z}_2^{2n} \). This immediately gives \( \|p^v_p(A, x) - p^v_p(B, y)\|_{L_1} = 0 \).

Next assume \( \tau((A, x), (B, y)) = 1 \). In this case \( d(x, y) = 1 \) and \( A \triangle B \subset \{x, y\} \). Hence \( D = \{x, y\} \) has diameter 1 and the set \( W = \{ v \in V : \exists t \in \mathbb{Z}_2^{2n}, D \subset C^v_t \} \) has size at least \((n - 1)^2\). Note that if \( D \subset C^v_t \), then \( x^v = y^v = c^v_t \) and \( A^v \triangle B^v = \emptyset \) for all \( s \neq t \), whereas \( x^v = x, y^v = y \) and \( A^v \triangle B^v = A \triangle B \). It follows that for any \( v \in W \) we have

\[
\|p^v_p(A, x) - p^v_p(B, y)\|_{L_1} = \sum_{s \in \mathbb{Z}_2^{2n}} \|g(A^v_s, x^v_s) - g(B^v_s, y^v_s)\|_{L_1}
\]

\[
\leq \Lambda \sum_{s \in \mathbb{Z}_2^{2n}} \text{sp}(x^v, A^v \triangle B^v, y^v)
\]

\[
= \Lambda \text{sp}(x, A \triangle B, y) = \Lambda.
\]

If \( v \notin W \), then for some \( t_1, t_2 \in \mathbb{Z}_2^{2n} \) with \( d(t_1, t_2) = 1 \) we have \( x \in \mathbb{C}^v\{t_1}\) and \( y \in \mathbb{C}^v\{t_2\} \). Then for \( s \notin \{t_1, t_2\} \) we have \( x^v = y^v = c^v_s \) and \( A^v \triangle B^v = \emptyset \), whereas \( x^v = x, y^v = c^v_s \) and \( A^v \triangle B^v \subseteq \{x\} \) and \( y^v = c^v_s \). It follows that \( \text{sp}(x^v, A^v \triangle B^v, y^v) = 0 \) for \( s \notin \{t_1, t_2\} \).
and \( \text{tsp}(x'_s, A'_x \triangle B'_s, y'_s) \leq n/2 \) for \( s \in \{t_1, t_2\} \). This yields

\[
\| p'_x(A, x) - p'_x(B, y) \|_{L_1} = \sum_{s \in \mathbb{Z}_{2n}^2} \| g(A'_x, x'_s) - g(B'_x, y'_s) \|_{L_1} \\
\leq \Lambda \sum_{s \in \mathbb{Z}_{2n}^2} \text{tsp}(x'_s, A'_x \triangle B'_s, y'_s) \\
= \Lambda \cdot \frac{n}{2} = \Lambda n.
\]

It follows that

\[
\| p_\beta(A, x) - p_\beta(B, y) \|_{L_1} = \mathbb{E}_v(\| p'_x(A, x) - p'_x(B, y) \|_{L_1} \mid v \in W) \cdot \mathbb{P}(v \in W) \\
+ \mathbb{E}_v(\| p'_x(A, x) - p'_x(B, y) \|_{L_1} \mid v \notin W) \cdot \mathbb{P}(v \notin W) \\
\leq \frac{(n-1)^2}{n^2} \Lambda + \frac{2n-1}{n^2} \Lambda n \leq 3 \Lambda.
\]

4.1.2. Connecting the local solutions: the map \( q \). Let \( S \) be the power set \( \mathcal{P}(\mathbb{Z}_{2n}^2) \). By Lemma 10 there is a probability measure \( \pi \) and a map \( G: \Lambda_\pi(\mathbb{Z}_{2n}^2) \to L_1(\pi, \pi_1(\mu)) \) such that

\[
\frac{1}{\Gamma} \text{mc}(\{ \alpha \neq \beta \} \cup \{x, y\}) \leq \| G(\alpha, x) - G(\beta, y) \|_{L_1} \leq \Lambda \cdot \text{tsp}(x, \{ \alpha \neq \beta \}, y)
\]

for all \( (\alpha, x), (\beta, y) \in \Lambda_\pi(\mathbb{Z}_{2n}^2) \). For \( A \subset \mathbb{Z}_{2n}^2 \) and \( v \in V \) define \( \alpha_{A, v}: \mathbb{Z}_{2n}^2 \to S \) by \( \alpha_{A, v}(x) = A'_x \). For \( x \in \mathbb{Z}_{2n}^2 \) and \( v \in V \), let \( \tilde{x}^v \) be the unique \( s \in \mathbb{Z}_{2n}^2 \) such that \( x \in C'_x \). Note that \( \tilde{\Lambda}' = \{ \tilde{x}^v : x \in A \} \) for \( A \subset \mathbb{Z}_{2n}^2 \). We are ready to define for each \( v \in V \) the map

\[
q'_v: \Lambda(\mathbb{Z}_{2n}^2) \to L_1(\pi, \pi_1(\mu)) \\
(\alpha, x) \mapsto 16nG(\alpha_{A, v}, \tilde{x}^v)
\]

and then define

\[
q_v: \Lambda(\mathbb{Z}_{2n}^2) \to L_1(\nu, \pi_1(\mu)) \\
(\alpha, x) \mapsto (q'_v(\alpha, x) : v \in V).
\]

**Proposition 18.** Let \( g \) be as in (17) and \( q_\beta \) as above. Let \((A, x), (B, y) \in \Lambda_\pi(\mathbb{Z}_{2n}^2) \) and \( D = (A \Delta B) \cup \{x, y\} \). Then

\[
2\Gamma n \mathbb{E}_v \text{mc}(\tilde{D}^v) \leq \| q'_v(A, x) - q'_v(B, y) \|_{L_1} \leq 32 \Lambda \tau((A, x), (B, y)).
\]

**Proof.** Set \( C = A \Delta B \). We begin with the observation that \( \{ \alpha_{A, v} \neq \alpha_{B, v} \} = \tilde{C}^v \) and

\[
\{ \alpha_{A, v} \neq \alpha_{B, v} \} \cup \{ \tilde{x}^v, \tilde{y}^v \} = \tilde{C}^v \cup \{ \tilde{x}^v, \tilde{y}^v \} = D^v,
\]

and thus by (19) we have

\[
2\Gamma n \mathbb{E}_v \text{mc}(\tilde{D}^v) \leq \| q'_v(A, x) - q'_v(B, y) \|_{L_1} \leq 16n \Lambda \text{tsp}(\tilde{x}^v, \tilde{C}^v, \tilde{y}^v).
\]

Taking expectation with respect to \( v \) gives

\[
2\Gamma n \mathbb{E}_v \text{mc}(\tilde{D}^v) \leq \| q_v(A, x) - q_v(B, y) \|_{L_1} \leq 16n \Lambda \text{tsp}(\tilde{x}^v, \tilde{C}^v, \tilde{y}^v).
\]

It remains to show that

\[
\| q_v(A, x) - q_v(B, y) \|_{L_1} \leq 32 \Lambda \tau((A, x), (B, y)).
\]

By Lemma 4 we may assume that \( \tau((A, x), (B, y)) \leq 1 \). If \( \tau((A, x), (B, y)) = 0 \), then \( x = y \) and \( C \subset \{x\} \). It follows that \( \tilde{x}^v = \tilde{y}^v \) and \( \tilde{C}^v \subset \{ \tilde{x}^v \} \) for all \( v \in V \), and hence by (21) we have \( \| q_v(A, x) - q_v(B, y) \|_{L_1} = 0 \).
Assume now that \( \tau((A,x),(B,y)) = 1 \). Then \( d(x,y) = 1 \) and \( C \subset \{x,y\} \). It follows that \( D = \{x,y\} \) has diameter 1, and so \( W = \{v \in V : 3t \in \mathbb{Z}_{2^n}^2 D \subset C_t^v \} \) has at least \((n-1)^2\) elements. For all \( v \in W \), we have \( \bar{L} = y^v \) and \( \bar{C} \subset \{x^v\} \). Hence by (20) we have \( \|q^v_g((A,x) - q^v_g(B,y))\|_{L_1} = 0 \). Whereas if \( v \notin W \), then \( d(x^v,y^v) = 1 \) and \( \bar{C}^v \subset \{x^v, y^v\} \), which implies that \( \|q^v_g((A,x) - q^v_g(B,y))\|_{L_1} \leq 16nA \) by (20). Hence by (21) it follows that

\[
\begin{align*}
\|q^v_g(A,x) - q^v_g(B,y)\|_{L_1} &= \mathbb{E}_v(\|q^v_g(A,x) - q^v_g(B,y)\|_{L_1} | v \in W) \\
&\quad + \mathbb{E}_v(\|q^v_g(A,x) - q^v_g(B,y)\|_{L_1} | v \notin W) \cdot \mathbb{P}(v \notin W) \\
&\leq \frac{2n-1}{n^2} - 16nA \leq 32A,
\end{align*}
\]

as required. \( \square \)

We are now in position to prove the recursive step and thus complete the proof of Theorem 13.

**Proof of Lemma 15.** Define

\[
f : \text{La}(\mathbb{Z}^2_{2^n}) \to L_1(v, \bigoplus_{v \in \mathbb{Z}^2_{2^n}} L_1(\mu)) \oplus L_1(v, L_1(\pi, L_1(\mu))) \quad (A,x) \mapsto p_g(A,x) \oplus q_g(A,x).
\]

Fix \((A,x), (B,y) \in \text{La}(\mathbb{Z}^2_{2^n})\) and set \( D = (A \triangle B) \cup \{x,y\} \). Propositions 17 and 18 immediately provide the upper bound

\[
\|f(A,x) - f(B,y)\|_{L_1} \leq 35A \tau((A,x), (B,y)) .
\]

Combining Propositions 17 and 18 together with Theorem 13 we obtain

\[
\|f(A,x) - f(B,y)\|_{L_1} \geq \Gamma \mathbb{E}_v \left( \sum_{v \in \mathbb{Z}^2_{2^n}} mc(D^v) + 2nmc(D^v) \right) \geq \Gamma mc(D) = \Gamma \delta((A,x), (B,y)).
\]

\( \square \)

5. **A Coarse Embedding of the Planar Lamplighter Group**

The following localization result was proved by Ostrovskii in [Ost12].

**Theorem 19 (Ost12).** Let \( X \) be a locally finite metric space and \( Y \) be an infinite-dimensional Banach space. Then, there exists a universal constant \( C \in [1,\infty) \) such that if the collection of finite subsets of \( X \) \( (\rho, \alpha) \)-embeds into \( Y \) then \( X \) \( (\rho, C\alpha) \)-embeds into \( Y \).

Consider a finite subset \( F \) of \( \text{La}(\mathbb{Z}^2) \), then \( F \subset \text{La}([-n,n]^2) \) for some large enough \( n \) and it follows from Lemma 11 that \( F \) is isometrically a subset of \( \text{La}([-n,n]^2) \). Since \( \text{La}([-n,n]^2) \) is clearly isometric to \( \text{La}(\mathbb{I}^2) \), which in turn isometrically embeds into \( \text{La}(\mathbb{Z}^2_{2^n}) \), in order to prove Theorem 19 it is sufficient to show that the sequence of toric lamplighter groups embed into \( L_1 \) with the same compression function as in Theorem 13.

**Remark 4.** Since our target space is \( L_1 \), there are alternative ways to obtain the conclusion of Ostrovskii’s result. Ostrovskii’s argument is based on the barycentric gluing technique (see [Bau72] or [BL08]), basic ultraproduct techniques (cf. [DK72], [Hei80]), and a procedure to select basic subsequences from [KP65]. Since the \( \ell_1 \)-sum of \( L_1 \) is isometric to \( L_1 \) we could avoid the basic sequence selection argument and still obtain Theorem 19 for \( Y = L_1 \). In order to get the constant \( C = 1 + \varepsilon \) with \( \varepsilon \) arbitrarily small, we would need to invoke the fact that every finite subset of \( L_1 \) embeds isometrically into \( \ell_1 \) and trade the general barycentric gluing technique with the finer logarithmic spiral gluing from [OOT19]. Also, for \( Y = L_1 \) we could alternatively obtain Theorem 19 with \( C = 1 \) for all separable...
metric spaces using basic ultraproduct techniques and the classical ultraproduct theory of Banach lattices.

It is possible in some situations to exploit distortion estimates to obtain compression estimates (cf. for instance [Ant11]). Using a barycentric gluing argument, we could get some partial, but far from satisfactory, information. The principle in [ANT13] which relates compression estimates to distortion estimates for the Heisenberg group, uses crucially the fact that the Heisenberg group has polynomial growth. Since there exist groups (necessarily of exponential growth) not coarsely embeddable into $L_1$, there cannot be a similar principle in the context of groups with exponential growth, such as the planar lamplighter group. Nevertheless, in this section we show that for the planar lamplighter group we can enhance the recursive approach we used to obtain distortion bounds in order to get a satisfactory compression estimate.

Theorem 15 follows from Theorem 20 below, from Lemma 14 and from a straightforward modification of Lemma 8.

Theorem 20. There exists $C > 0$ such that for each $n \in \{2 \cdot 2^k : k \in \mathbb{N}\}$ there exists $h: \text{La}(\mathbb{Z}_n^2) \rightarrow L_1$ satisfying
\[
\rho(\delta((A,x),(B,y))) \leq \|h(A,x) - h(B,y)\|_{L_1} \leq C\tau((A,x),(B,y))
\]
for all $(A,x), (B,y) \in \text{La}(\mathbb{Z}_n^2)$, where
\[
\rho(t) = \frac{t}{\log(t)\log(35) + 2\log\log(t) + (\log\log(t))^2}.
\]

Remember from the previous section that if one is given a function $g$ from $\text{La}(\mathbb{Z}_n^2)$ into $L_1$, then we can construct two maps $p_g$ and $q_g$ defined on $\text{La}(\mathbb{Z}_n^2)$ with values in $L_1$. In this section we need finer assumptions on $g$. We assume that $n = 2^k$ for some $k \in \mathbb{N}$ and that there are positive constants $\Gamma_\ell$ for $0 \leq \ell < k$ such that
\[
\Gamma_\ell \delta((A,x),(B,y)) \leq \|g(A,x) - g(B,y)\|_{L_1} \leq \tau((A,x),(B,y))
\]
holds
- for all $(A,x),(B,y) \in \text{La}(\mathbb{Z}_n^2)$ if $\ell = k - 1$,
- and for all $(A,x),(B,y) \in \text{La}(\mathbb{Z}_n^2)$ with $\delta((A,x),(B,y)) \leq 2^\ell$ if $0 \leq \ell \leq k - 2$.

5.1. The folding map $\varphi$. Next, we consider the ‘folding map’ $\varphi: \mathbb{Z} \rightarrow \mathbb{Z}_{2^k}$ defined by
\[
\varphi(i + j) = \begin{cases} 
  j + 2nZ & \text{if } i \text{ is even} \\
  n - j - 1 + 2nZ & \text{if } i \text{ is odd}
\end{cases}
\]
for all $i \in \mathbb{Z}_n$ and $j \in [n]$. This is 1-Lipschitz and is isometric on $in + [n]$ for all $i \in \mathbb{Z}$. Hence it induces a 1-Lipschitz map $\Phi: \mathbb{Z}_{2^{2^k}} \rightarrow \mathbb{Z}_{2^{2^{k+1}}}$ which is isometric on $in + V_0$ for all $i \in \mathbb{Z}_{2^{2^k}}$ where $V_0 = \{ j + 2nZ : j \in [n] \}$. It follows that the product map $\Psi = \Phi \otimes \Phi: \mathbb{Z}_{2^{2^k}} \rightarrow \mathbb{Z}_{2^{2^{k+1}}}$ is 1-Lipschitz, and for any $s = (i,j) \in \mathbb{Z}_{2^{2^k}}$, it is isometric on $(in + V_0) \times (jn + V_0) = sn + V = C_s$.

For each $v \in V$ define $\psi^v: \mathbb{Z}_{2^{2^k}} \rightarrow \mathbb{Z}_{2^{2^k}}$ by $\psi^v(x) = \varphi(x - v)$. Then $\psi^v$ is 1-Lipschitz and isometric on $C_s^v$ for all $s \in \mathbb{Z}_{2^{2^k}}$. A Lipschitz map between metric spaces induces in a natural way a map between the corresponding lamplighter spaces (the special case of graphs is discussed in [BMSZ, Section 5]). However, this process does not sufficiently reflect the property of $\psi^v$ being isometric on the sets $C_s^v$. For this reason we proceed differently. Let us write $A + B$ for $A \triangle B$. Then for sets $A_1, \ldots, A_k$ we have $\sum_{i=1}^k A_i = A_1 \triangle A_2 \triangle \ldots \triangle A_k$. For each $v \in V$ we define
\[
\sigma^v(A) = \sum_{s \in \mathbb{Z}_{2^{2^k}}} \psi^v(A_s^v) \subset \mathbb{Z}_{2^{2^k}}^2 \quad \text{for } A \subset \mathbb{Z}_{2^{2^k}}^2.
\]
When \( v = 0 \) we simplify the notation to \( \sigma \) and \( \psi \).

**Lemma 21.** For each \( v \in V \), the induced map \( \psi^v \) defined above is 1-Lipschitz with respect to the semimetric \( \tau \). Moreover, given \( (A, x), (B, y) \in \text{La}(Z_{2m}^2) \), if \( (A \Delta B) \cup \{x, y\} \subset C^v_i \) for some \( s \in Z_{2m}^2 \), then

\[
\delta((\psi^v(A, x), \psi^v(B, y))) = \delta((A, x), (B, y)).
\]

**Proof.** Fix \( (A, x), (B, y) \in \text{La}(Z_{2m}^2) \) and set \( D = (A \Delta B) \cup \{x, y\} \). We can assume without loss of generality that \( v = 0 \). Observe that

\[
\sigma(A) \triangle \sigma(B) = \sum_{s \in Z_{2m}^2} \psi(A_s) + \sum_{s \in Z_{2m}^2} \psi(B_s) = \sum_{s \in Z_{2m}^2} \psi(A_s \triangle B_s) = \sigma(A \Delta B),
\]

and hence \( \sigma(A) \triangle \sigma(B) \subset \psi(A \Delta B) \). It follows that

\[
\tau(\psi(A, x), \psi(B, y)) = \text{tsp}(\psi(x), \sigma(A) \triangle \sigma(B), \psi(y)) \leq \text{tsp}(\psi(x), \psi(A \Delta B), \psi(y)).
\]

So in order to show that \( \psi^v \) is 1-Lipschitz, it remains to verify that

\[
(23) \quad \text{tsp}(\psi(x), \psi(C), \psi(y)) \leq \text{tsp}(x, C, y)
\]

for any \( x, y \in Z_{2m}^2 \) and any \( C \subset Z_{2m}^2 \). Let \( \ell = \text{tsp}(x, C, y) \) and \( x_0, x_1, \ldots, x_\ell \) be a walk in the graph \( Z_{2m}^2 \) from \( x = x_0 \) to \( y = x_\ell \) visiting all vertices in \( C \). Since \( \psi \) is 1-Lipschitz, it follows that for every \( i = 1, \ldots, \ell \), the vertices \( \psi(x_{i-1}) \) and \( \psi(x_i) \) in \( Z_{2m}^2 \) are either equal or adjacent. Deleting successive repetitions of vertices yields a walk of length at most \( \ell \) in \( Z_{2m}^2 \) from \( \psi(x) = \psi(x_0) \) to \( \psi(y) = \psi(x_\ell) \) visiting all vertices of \( \psi(C) \). This shows \( (23) \).

If we assume that \( D \subset C \), for some \( s \in Z_{2m}^2 \), then

\[
\sigma(A) \triangle \sigma(B) \cup \{\psi(x), \psi(y)\} = \sigma(A \Delta B) \cup \{\psi(x), \psi(y)\} = \psi(D),
\]

and so we need to show that \( mc(\psi(D)) = mc(D) \). This follows since \( \psi \) is an isometry between \( C \) and \( \psi(C) \), and since \( \psi(C) \) is a 1-Lipschitz retract of \( Z_{2m}^2 \) by Lemma 21.

We now come to the definition of the map \( r_g \). For \( v \in V \) and any map \( g \) from \( \text{La}(Z_{2m}^2) \) into \( L_1(\mu) \) define

\[
r_g^v = g \circ \psi^v : \text{La}(Z_{2m}^2) \to L_1(\mu)
\]

and then define

\[
r_g : \text{La}(Z_{2m}^2) \to L_1(\nu, L_1(\mu))
\]

\[
(A, x) \mapsto (r_g^v(A, x) : v \in V).
\]

**Proposition 22.** Let \( g \) be as in \( (22) \) and \( r_g \) be as above, and let \( \Gamma_\ell = (1 - 2 \cdot 2^{\ell - k^2}) \cdot \Gamma_\ell \)

for \( 0 \leq \ell < k \). Then \( r_g \) is 1-Lipschitz with respect to \( \tau \), and

\[
(24) \quad \Gamma_\ell \delta((A, x), (B, y)) \leq \|r_g^v(A, x) - r_g^v(B, y)\|_{L_1}
\]

for all \( 0 \leq \ell \leq k - 1 \) and all \( (A, x), (B, y) \in \text{La}(Z_{2m}^2) \) with \( \delta((A, x), (B, y)) \leq 2^2 \).

**Proof.** Since \( \psi \) and \( g \) are 1-Lipschitz maps, so is \( r_g^v \) for every \( v \in V \). It follows that \( r_g \) is also 1-Lipschitz, and so it remains to show the lower bound \( (24) \).

Fix \( (A, x), (B, y) \in \text{La}(Z_{2m}^2) \), and set \( D = (A \Delta B) \cup \{x, y\} \). Assume that \( mc(D) = \delta((A, x), (B, y)) \leq 2^2 \) for some \( \ell \) with \( 0 \leq \ell \leq k - 1 \). Then \( m = \text{diam}(D) \leq \text{mc}(D) < n \), and thus the set \( W = \{v \in V : \exists y \in Z_{2m}^2, D \subset C^v_i\} \) has size at least \( (n - m)^2 \). By the assumptions \( (22) \) on \( g \) and by Lemma 21, we have that if \( D \subset C^v_i \), then

\[
\|r_g^v(A, x) - r_g^v(B, y)\|_{L_1} \geq \Gamma_\ell \delta((\psi(A, x), \psi(B, y)) = \Gamma_\ell \delta((A, x), (B, y)).
\]
exists a positive constant $\Gamma$.

Thus, letting $h(\delta) \equiv (\delta) $ for all $\delta \in W$. It follows that

$$
\|h(\delta)\| = \|\delta\| = \Gamma \, \|\delta\|
$$

and hence $h(\delta) \equiv (\delta) \, \Gamma \, \|\delta\|$. Now for each $k \in \mathbb{N}$, the recursive step is complete.

Now for each $j \in \mathbb{N}$ set $\gamma_j = \frac{1}{j}$ and let

$$
\gamma = \prod_{j=2}^{\infty} (1 - \gamma_j) \cdot \prod_{j=1}^{\infty} \left(1 - 2 \cdot 2^{-2^j} \cdot \gamma_j \right)
$$

and assume that for some $k \in \mathbb{N}$ we have a 1-Lipschitz map $h_k : \text{La}(\mathbb{Z}_2\mathbb{Z}) \to L_1$ with $n = 2^{2^k}$ and constants $\Gamma_k \leq 1$. Then, for all $(A, x), (B, y) \in \text{La}(\mathbb{Z}_2\mathbb{Z})$, the recursive step is complete.

Proof of Theorem 20. We begin with a recursive construction. Since $\text{La}(\mathbb{Z}_2\mathbb{Z})$ is finite there exists a positive constant $\Gamma_0^{(1)}$ and a map $h_1 : \text{La}(\mathbb{Z}_2\mathbb{Z}) \to L_1$ such that

$$
\Gamma_0^{(1)} \delta((A, x), (B, y)) = \|h_1(A, x) - h_1(B, y)\| \leq \tau((A, x), (B, y))
$$

for all $(A, x), (B, y) \in \text{La}(\mathbb{Z}_2\mathbb{Z})$. To see this define $h : \text{La}(\mathbb{Z}_2\mathbb{Z}) \to \ell_1(\mathbb{Z}_2\mathbb{Z}) \oplus \ell_1(\mathbb{Z}_2\mathbb{Z})$ by $h_1(A, x) = (1_A, 1_{\{1\}})$. Then,

$$
\|h(A, x) - h(B, y)\| = \|A \setminus \{x\} \triangle B \setminus \{y\} + |\{x\} \triangle \{y\}|
$$

and hence $\|h(A, x) - h(B, y)\| = 0$ if and only if $\text{mc}(A \triangle B) \cup \{x, y\} = 0$. It follows that $h_1 = h/C$ will do, where $C$ is the Lipschitz constant of $h$ with respect to the semimetric $\tau$.

Now for each $j \in \mathbb{N}$ set $\gamma_j = \frac{1}{j}$ and let

$$
\gamma = \prod_{j=2}^{\infty} (1 - \gamma_j) \cdot \prod_{j=1}^{\infty} \left(1 - 2 \cdot 2^{-2^j} \cdot \gamma_j \right)
$$

and assume that for some $k \in \mathbb{N}$ we have a 1-Lipschitz map $h_k : \text{La}(\mathbb{Z}_2\mathbb{Z}) \to L_1$ with $n = 2^{2^k}$ and constants $\Gamma_k \leq 1$. Then, for all $(A, x), (B, y) \in \text{La}(\mathbb{Z}_2\mathbb{Z})$, the recursive step is complete.

Define $h_{k+1} : \text{La}(\mathbb{Z}_2\mathbb{Z}) \to L_1$ by letting

$$
h_{k+1}(A, x) = \frac{\gamma_{k+1}}{35}p_{h_k}(A, x) \oplus \frac{\gamma_{k+1}}{35}h_k(A, x) \oplus (1 - \gamma_{k+1})r_{h_k}(A, x)
$$

Fix $(A, x), (B, y) \in \text{La}(\mathbb{Z}_2\mathbb{Z})$. It follows from Lemma 15 that

$$
\|h_{k+1}(A, x) - h_{k+1}(B, y)\| \leq \frac{\gamma_{k+1}}{35} \Gamma_k \delta((A, x), (B, y))
$$

If $\delta((A, x), (B, y)) \leq 2^{2^k}$ for some $0 \leq \ell \leq k - 1$, then by Proposition 22 we have

$$
\|h_{k+1}(A, x) - h_{k+1}(B, y)\| \leq (1 - \gamma_{k+1}) \cdot (1 - 2 \cdot 2^{2^\ell - 2^k}) \Gamma_k \delta((A, x), (B, y))
$$

Finally, from the upper bound in Lemma 15 and in Proposition 22 we obtain

$$
\|h_{k+1}(A, x) - h_{k+1}(B, y)\| \leq \left(\frac{\gamma_{k+1}}{35} \cdot 35 + (1 - \gamma_{k+1})\right) \tau((A, x), (B, y)) = \tau((A, x), (B, y))
$$

Thus, letting $\Gamma_{k+1} = (1 - \gamma_{k+1})(1 - 2 \cdot 2^{2^\ell - 2^k}) \Gamma_k$ for $0 \leq \ell < k - 1$, and letting $\Gamma_k = \frac{\gamma_{k+1}}{35} \Gamma_k$, the recursive step is complete.

Now fix $k \in \mathbb{N}$ and note that for each $0 \leq \ell < k$ we have

$$
\Gamma_k = \prod_{j=\ell+1}^{k-1} (1 - \gamma_j) \prod_{j=1}^{\ell} \gamma_j \prod_{j=1}^{2^\ell - 2^j} \Gamma_0^{(1)} \cdot 35 - 2^\ell \ell \gamma_j
$$
Let \((A, x), (B, y) \in \text{La}(\mathbb{Z}_2^n)\) and \(m = \delta((A, x), (B, y))\). Choose the smallest integer \(\ell \geq 0\) so that \(m \leq 2^\ell\). If \(0 \leq \ell \leq k - 2\), then
\[
\|h_k(A, x) - h_k(B, y)\|_{L_1} \geq \ell^{(k)} \delta((A, x), (B, y)) \geq m \cdot \gamma \cdot 35^{-\ell} \prod_{j=1}^{\ell} \gamma_j.
\]
Whereas if \(\ell \geq k - 1\), then
\[
\|h_k(A, x) - h_k(B, y)\|_{L_1} \geq \ell^{(k-1)} \delta((A, x), (B, y)) \geq m \cdot \gamma \cdot 35^{-(k-1)} \prod_{j=1}^{\ell-1} \gamma_j \geq m \cdot \gamma \cdot 35^{-\ell} \prod_{j=1}^{\ell} \gamma_j.
\]

It remains to estimate \(d := m \cdot 35^{-\ell} \prod_{j=1}^{\ell} \gamma_j\) from below and in terms of \(m\) only. If \(\ell \leq 1\) then \(d \geq \frac{m}{35}\). Now assume that \(\ell \geq 2\) and thus \(2^{\ell-1} < m \leq 2^\ell\). In particular \(m \geq 5\) and \(m^2 > 2^{2\ell}\). First observe that
\[
\prod_{j=1}^{\ell} \gamma_j = \prod_{j=1}^{\ell} 1 - \frac{1}{2^\ell} = \frac{1}{2^\ell},
\]
and thus
\[
d \geq \frac{m}{35^\ell 2^\ell}.
\]
The log functions below are all in base 2. Recall that \(\ell\) was chosen such that \(2^\ell < 2\log(m)\) and \(\ell < \log \log(m^2)\) and thus
\[
35^\ell \leq (2\log(m))^{\log_2(35)} = 35 \log(m)^{\log_2(35)},
\]
and
\[
\log(2^\ell) = 2\ell \log(\ell) \leq 2\log(\log(m^2)) \log \log \log(m^2)
\]
\[
= \log \left( \log(m^2)^{2\log \log \log(m^2)} \right)
\]
\[
= \log \left( (2\log(m))^{\log(\log(m^2))} \right)
\]
\[
= \log \left( \log^2(\log(m^2)) \log(m)^{\log^2(\log(m^2))} \right)
\]
\[
= \log \left( \log^2(\log(m^2)) \log(m)^{2\log \log(m^2)} \right)
\]

Combining (26), (27), and (28) we get
\[
d \geq \frac{m}{35\log(m)^{\log_2(35)}} \frac{2\log \log(m^2)}{(\log \log(m^2))^2}
\]

\[\square\]

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DEPARTMENT OF MATHEMATICS, TEXAS A&M UNIVERSITY, COLLEGE STATION, TX 77843, USA
E-mail address: florent@tamu.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN, URBANA, IL 61801, USA
E-mail address: pmotakis@illinois.edu

DEPARTMENT OF MATHEMATICS, TEXAS A&M UNIVERSITY, COLLEGE STATION, TX 77843, USA
AND FACULTY OF ELECTRICAL ENGINEERING, CZECH TECHNICAL UNIVERSITY IN PRAGUE, ZIKOVA 4, 166 27, PRAGUE
E-mail address: schlump@math.tamu.edu

PETERHOUSE, CAMBRIDGE, CB2 1RD, UK
E-mail address: a.zsak@dpmms.cam.ac.uk