Classification of Higher Dimensional Spacetimes

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Abstract

We algebraically classify some higher dimensional spacetimes, including a number of vacuum solutions of the Einstein field equations which can represent higher dimensional black holes. We discuss some consequences of this work.
1 Introduction

Many higher dimensional spacetimes are now known, including a number of vacuum solutions of the Einstein field equations which can represent higher dimensional black holes. These N-dimensional (ND) black holes are of physical interest, particularly in view of the development of string theory. It is of importance to classify these higher dimensional spacetimes algebraically [1, 2].

Higher dimensional generalizations of the Schwarzschild solution, the Schwarzschild-Tangherlini (ST) solution [3], which are spherically symmetry on spacelike $(N-2)$-surfaces, are of algebraic Weyl type D [1, 2]. Higher dimensional generalizations of Reissner-Nordstrom black holes are also of type D [4].

A class of 5D Kaluza-Klein vacuum solutions [5] are also of physical interest. As we shall see, the non-black hole solutions (i.e., all solutions except the 5D generalized Schwarzschild solution) are not of type D (but of a more general algebraic Weyl type). A related class of non-static spherically symmetric solutions [6] is also of type D.

The Myers-Perry solution in five and higher dimensions [7], a direct generalization of the 4D asymptotically flat, rotating black hole Kerr solution, is also of type D [2]. A class of higher dimensional Kerr-(anti) de Sitter solutions, which are given in N-dimensions and have $(N-1)/2$ independent rotation parameters, have been given in Kerr-Schild form [8]. These rotating black hole solutions with a non-zero cosmological constant reduce to the 5D solution of [9] and the Kerr-de Sitter spacetime in 4D, and the Myers-Perry solution in the absence of a cosmological constant.

Non-rotating uncharged black string Randall-Sundrum braneworlds were first discussed in [10]. The rotating black ring solutions (“black rings” – BR) are vacuum, asymptotically flat, stationary black hole solutions of toplogy $S^1 \times S^2$ [11]. These solution have subsequently been generalized to the non-supersymmetric black ring solutions of minimal supergravity in [12]. There are also supersymmetric rotating black holes that exist in five dimensions. There is the extremal charged rotating BMPV black hole of [13] in minimal supergravity, with a horizon of spherical topology (see also [14]). The first supersymmetric black ring (solution of 5D minimal supergravity) was presented in [15] (and subsequently generalized in [16, 17, 18, 19]).

There are many other higher dimensional spacetimes of interest. A class of dimensional relativistic gyratons (RG) [20], which are vacuum solutions of the Einstein equations of the generalized Kundt class (representing a beam pulse of spinning radiation), are of type III.

1.0.1 Black Hole Uniqueness

Many of these higher-dimensional black hole solutions are of particular physical interest, especially regarding black hole uniqueness. In the static case, the unique asymptotically flat vacuum black hole is the N-dimensional Schwarzschild-Tangherlini solution [21] (when the assumption of asymptotic flatness is dropped, uniqueness fails even with same topology as in ST).

In more than four dimensions, even for pure gravity, stationary five-dimensional black holes are not uniquely characterized by their asymptotic conserved charges, such as mass and angular momenta. In particular, the vacuum solutions of asymptotically flat rotating black rings with event horizon homeomorphic to $S^1 \times S^2$ [11] have the same conserved charges as the stationary Myers-Perry rotating black hole; if electromagnetic fields are included, then the black rings can carry charge and, moreover, the number of parameters required to specify black ring solutions now exceeds the number of conserved quantities that they carry [22]. However, the $U(1) \times U(1)$ supersymmetric solutions are only specified by a finite number of parameters. Recently, Bena and Warner [17] have studied a family of supersymmetric solutions of five-dimensional supergravity that is specified by seven arbitrary functions of one variable. However, it has been argued that the only solutions which are smooth belong to the original $U(1) \times U(1)$ invariant family (or superpositions thereof [18]), and for this reason the Bena and Warner [17] solution cannot be extended through a $C^2$ horizon [23].

We begin with a brief review of the classification procedure. We then present the classification of several important classes of spacetimes (more details will be presented in [24]). These results are summarized in the table in the last section. We conclude with a brief discussion. A bibliography is appended.
2 Classification of the Weyl Tensor

The algebraic classification of the Weyl tensor in higher dimensional Lorentzian manifolds is achieved by characterizing algebraically special Weyl tensors by means of the existence of aligned null vectors of various orders of alignment \[2\]. We consider a null frame \(\ell = m_0\), \(n = m_1\), \(m_2 \ldots m_{l}\) (\(\ell\), \(n\) null with \(\ell^\alpha \ell_\alpha = n^\alpha n_\alpha = 0\), \(\ell^\alpha n_\alpha = 1\), \(m^\alpha\) real and spacelike with \(m_i^a m_j^a = \delta_{ij}\); all other products vanish) in an \(N\)-dimensional Lorentz-signature space(time), so that \(g_{ab} = 2(n a \nu_b) + \delta_{ij} m_i^a m_j^b\). Indices \(a, b, c\) range from 0 to \(N - 1\), and space-like indices \(i, j, k\) also indicate a null-frame, but vary from 2 to \(N - 1\) only. The frame is covariant relative to the group of linear Lorentz transformations. There are null rotations about \(n\) and \(\ell\) and spins of the spatial frame vectors \(m_i\). In particular, a boost is a transformation of the form

\[
\hat{n} = \lambda^{-1} n, \quad \hat{m}_i = m_i, \quad \hat{\ell} = \lambda \ell, \quad \lambda \neq 0.
\]

Let \(T_{A_1 \ldots A_p}\) be a rank \(p\) tensor. For a fixed list of indices \(A_1, \ldots, A_p\), we call the corresponding \(T_{A_1 \ldots A_p}\) a null-frame scalar. These scalars transform under a boost \(1\) according to

\[
\hat{T}_{A_1 \ldots A_p} = \lambda^b T_{A_1 \ldots A_p}, \quad b = b_{A_1} + \ldots + b_{A_p},
\]

where \(b_0 = 1\), \(b_i = 0\), \(b_1 = -1\). We call the above \(b\) the boost-weight of the scalar. We define the boost order of the tensor \(T\) to be the boost weight of its leading term. Introducing the notation

\[
T_{\{pqr\}} = \frac{1}{2} (T_{[ab][cd]} + T_{[cd][ab]}),
\]

the components of the Weyl tensor can be decomposed and sorted by boost weight \[2\]:

\[
C_{abcd} = 4C_{000j} n_\alpha m_i^\alpha n_\beta m_j^\beta m_k^\gamma m_l^\delta + 8C_{0101} n_\alpha n_\beta n_\gamma n_\delta m_i^\alpha m_j^\beta m_k^\gamma m_l^\delta + 4C_{0ijk} n_\alpha m_i^\alpha m_j^\beta m_k^\gamma m_l^\delta + 8C_{01ij} n_\alpha n_\beta m_i^\alpha m_j^\beta m_k^\gamma m_l^\delta + C_{ijkl} m_i^\alpha m_j^\beta m_k^\gamma m_l^\delta \]

The Weyl tensor is generically of boost order 2. If all \(C_{000j}\) vanish, but some \(C_{010c}\), or \(C_{0ijk}\) do not, then the boost order is 1, etc. A null rotation about \(\ell\) fixes the leading terms of a tensor, while boosts and spins subject the leading terms to an invertible transformation. It follows that the boost order of a tensor is a function of the null direction \(\ell\) (only). We shall therefore denote boost order by \(B(\ell)\) \[2\]. We will define a null vector \(\ell\) to be aligned with the Weyl tensor whenever \(B(\ell) \leq 1\) (and we shall refer to \(\ell\) as a Weyl aligned null direction (WAND)). We will call the integer \(1 - B(\ell) \in \{0, 1, 2, 3\}\) the order of alignment. We will say that the principal type of a Lorentzian manifold is \(I, II, III, N\) according to whether there exists an aligned \(\ell\) of alignment order \(0, 1, 2, 3\) (i.e. \(B(\ell) = 1, 0, -1, -2\)), respectively. If no aligned \(\ell\) exists we will say that the manifold is of (general) type \(G\). If the Weyl tensor vanishes, we will say that the manifold is of type \(O\). The algebraically special types are summarized as follows:

\[
\begin{align*}
\text{Type } I & : \quad C_{000j} = 0 \\
\text{Type } II & : \quad C_{000j} = C_{0ijk} = 0 \\
\text{Type } III & : \quad C_{000j} = C_{0ijk} = C_{ijkl} = C_{01ij} = 0 \\
\text{Type } N & : \quad C_{000j} = C_{0ijk} = C_{ijkl} = C_{01ij} = C_{1ijk} = 0
\end{align*}
\]

Further categorization can be obtained by specifying alignment type \[2\], whereby we try to normalize the form of the Weyl tensor by choosing both \(\ell\) and \(n\) in order to set the maximum number of leading and trailing null frame scalars to zero. Let \(\ell\) be a WAND whose order of alignment is as large as possible. We then define the principal (or primary) alignment type of the tensor to be \(b_{max} - b(\ell)\). Supposing such a WAND \(\ell\) exists, we
then let \( \mathbf{n} \) be a null vector of maximal alignment subject to \( \ell_n n^a = 1 \). We define the secondary alignment type of the tensor to be \( b_{\text{max}} - b(\mathbf{n}) \). The alignment type of the Weyl tensor is then the pair consisting of the principal and secondary alignment type \([2]\). In general, for types \( \text{I}, \text{II}, \text{III} \) there does not exist a secondary aligned \( \mathbf{n} \) (in contrast to the situation in 4D), whence the alignment type consists solely of the principal alignment type. Alignment types \( (1,1), (2,1) \) and \( (3,1) \) therefore form algebraically special subclasses of types \( \text{I}, \text{II}, \text{III} \) respectively (denoted types \( \text{I}_i, \text{II}_i, \text{III}_i \)). There is one final subclass possible, namely type \( (2,2) \), which is a further specialization of type \( (2,1) \); we shall denote this as type \( \text{II}_{ii} \), or simply as type \( \text{D} \). Therefore, a type \( \text{D} \) Weyl tensor in canonical form has no terms of boost weights \( 2, 1, -1, -2 \) (i.e., all terms are of boost weight zero for type \( \text{D} \)).

In the case in which the Weyl tensor is reducible, it is possible to obtain more information by decomposing the Weyl tensor and classifying its irreducible parts. In a full classification it is necessary to count aligned directions, the dimension of the alignment variety, and the multiplicity of principal directions \([2]\). In most applications \([5,26,25]\) the Weyl classification is relatively simple and the details of the more complete classification are not necessary. In \([2]\) it was shown that the present classification reduces to the classical 4D Petrov classification.

### 2.1 Necessary conditions for WANDs

It would be useful to be able to find a more practical way of determining the Weyl type, such as for example employing certain higher dimensional scalar invariants. A set of necessary conditions for various classes, which can significantly simplify the search for WANDs, can be given \([27]\):

\[
\ell^b \ell^c [e_{[c} C_{a]b|d} \ell_f] = 0 \iff \ell \text{ is WAND, at most primary type I}; \\
\ell^b \ell^c [e_{[c} C_{a]b|d} \ell_f] = 0 \iff \ell \text{ is WAND, at most primary type II}; \\
\ell^c C_{a[b} \ell_{d} \ell_f] = 0 \iff \ell \text{ is WAND, at most primary type III}; \\
\ell^c C_{abcd} = 0 \iff \ell \text{ is WAND, at most primary type N}. 
\]

(6)

For type \( \text{I} \), equivalence holds in arbitrary dimension. However, this is not the case for more special types.

### 2.2 Methods

Therefore, there are essentially three methods currently available to determine the Weyl type. In a straightforward approach the alignment equations are studied, which are \( \frac{1}{2} N(N-3) \) degree-4 polynomial equations in \( (N-2) \) variables (and are generally overdetermined and hence have no solutions for \( N > 4 \)), to determine if there exist non-trivial solutions. A second method, in which the necessary conditions are investigated, is more practical (and results in studying essentially the same equations but in a more organized form). This approach is followed in classifying the Black Ring solutions (see below \([27]\)). Finally, in many applications which are simple generalizations of 4D solutions in which the preferred 4D null frame is explicitly known, the 5D null frame can be guessed directly. We will begin with some examples of this latter approach.

More details of the calculations will be presented in \([24]\). It is clear that the current method of finding WANDs is very cumbersome. It is consequently important to derive a more practical method for determining Weyl type; for example, by utilizing invariants as in the case of 4D \([28]\).
3 Weyl types of some 5D vacuum spacetimes

A number of solutions of the five dimensional vacuum Einstein field equations are known, some of which represent higher dimensional black hole solutions. A null rotation about $n$ and a null rotation about $\ell$, subgroups of the Lorentz group, yield primary and secondary classifications, since positive and negative boost weight components can only be made to vanish using null rotations about $n/\ell$. Explicitly, a boost is given by

$$\hat{\ell} = b_{1} \ell, \quad \hat{n} = b_{1}^{-1} n, \quad \hat{m}_{i} = m_{i}. \quad (7)$$

A null rotation about $n$ is given by

$$\hat{\ell} = \ell - \frac{1}{2} \delta^{ij} d_{i} n + d_{i} m^{i}, \quad \hat{n} = n, \quad \hat{m}_{i} = m_{i} - d_{i} n. \quad (8)$$

A null rotation about $l$ is given by

$$\hat{\ell} = \ell, \quad \hat{n} = n - \frac{1}{2} \delta^{ij} c_{i} c_{j} \ell + c_{i} m^{i}, \quad \hat{m}_{i} = m_{i} - c_{i} \ell. \quad (9)$$

In the following examples the null coframe has the form

$$\ell = A^{2} dt + AB dr, \quad n = \frac{1}{2} (-dt + \frac{A}{B} dr), \quad m_{1} = Cr \sin \theta d\phi, \quad m_{2} = Cr \sin \theta d\phi, \quad m_{3} = D dy \quad (10)$$

with the functions $A, B, C$ and $D$ being specified in each case. The corresponding metric is given by

$$ds^{2} = -A^{2} dt^{2} + B^{2} dr^{2} + C^{2} (r^{2} d\theta^{2} + r^{2} \sin^{2} \theta d\phi^{2}) + D^{2} dy^{2}. \quad (11)$$

3.1 5D Schwarzschild: ST

The 4D Schwarzschild solution is spherically symmetric on spacelike 2-surfaces; an obvious generalization to five dimensions is spherical symmetry on spacelike 3-surfaces. We let $y$ be a cyclic coordinate and set (in 10)

$$A = (1 - \frac{2M}{r})^{1/2}, \quad B = (1 - \frac{2M}{r})^{-1/2}, \quad C = 1, \quad D = r \sin \theta \sin \phi. \quad (12)$$

This solution is the unique asymptotically flat static black hole solution in 5D. It follows immediately, for this null frame, that the Weyl basis components all have boost weight zero and therefore this spacetime is of type D. Higher dimensional generalizations of the Schwarzschild solution are also of type D.

3.2 Sorkin-Gross-Perry-Davidson-Owen soliton: GP

Another generalization of the Schwarzschild solution is obtained by setting

$$A = \left(\frac{ar}{ar+1}\right)^{2\kappa}, \quad B = \left(\frac{ar}{ar+1}\right)^{-2\kappa} \left(\frac{ar+1}{ar-1}\right)^{(\kappa-1)}, \quad C = B, \quad D = \left(\frac{ar+1}{ar-1}\right)^{\epsilon}. \quad (13)$$

To ensure that this is a vacuum soliton solution the consistency relation, $\epsilon^{2}(\kappa^{2} - \kappa + 1) = 1$, must be satisfied. In the limit as $\epsilon \to 0, \kappa \to \infty$ while $\epsilon \kappa \to 1$, we obtain the special case $S^{*}$ in which the hypersurface $y = \text{const}$ gives the 4D Schwarzschild solution in isotropic coordinates; we find that $S^{*}$ is of type D.

In the null frame given by (10), the Weyl tensor has components with boost weight $+2,0,-2$. Performing a null rotation about $n$ shows that with arbitrary $\epsilon$ and $a$ no solution exists for $d_{i}$ such that the Weyl basis components with boost weight $+2$ and $+1$ vanish. However, we note that for $\epsilon^{2} = 1/3$ and $\kappa = -1$ (special case GP$_{a}$) a null rotation about $n$ can be found that will make these positive boost weight Weyl components vanish, namely

$$d_{1} = d_{2} = 0, \quad d_{3} = \pm 2 \left(\frac{ar+1}{ar-1}\right)^{\epsilon}, \quad (14)$$
resulting in a primary classification of type II. Using $\epsilon^2 = 1/3$ and $\kappa = -1$ we can then perform a null rotation about $\ell$. We then find that the boost weight -1 and -2 Weyl basis components can be made to vanish with the following parameters

$$c_1 = c_2 = 0, \quad c_3 = \pm \frac{1}{2} \left( \frac{ar-1}{ar+1} \right)^{\epsilon}.$$ \hspace{1cm} (15)

Therefore, the special case $GP_s$ of (13), where $\epsilon^2 = 1/3$ and $\kappa = -1$, is of type D.

Returning to the general case of arbitrary $\epsilon$ and $a$, we have shown that a null rotation about $n$ cannot give type II. However, by choosing $d_1 = d_2 = 0$ and $d_3$ as a solution of

$$d_3^4 + 8 \left( \frac{ar-1}{ar+1} \right)^{2\kappa} \left( 1 + 2\kappa - \frac{4ar\epsilon(1+\kappa)}{a^2r^2 + 2ar\epsilon + 1} \right) d_3^2 + 16 \left( \frac{ar-1}{ar+1} \right)^{4\epsilon} = 0,$$ \hspace{1cm} (16)

we see that the GP metric is of type I.

3.3 Non-static spherically symmetric solution: AC

The following solution, obtained by Abolghasem and Coley [6], is a spherically symmetric vacuum solution containing two arbitrary functions $\tilde{A}(t,y)$ and $\tilde{C}(t,y)$. By an appropriate specification of these functions it immediately follows that this solution contains the type D GP solution given above. In the null frame of (10), the solution is

$$A = \left( \frac{1-m}{1+m} \right)^{-\frac{1}{\sqrt{3}}} \tilde{A}(t,y), \quad B = \left( 1 + \frac{m}{2r} \right)^2 \left( \frac{1-m}{1+m} \right)^{1+\frac{1}{\sqrt{3}}}, \quad C = B, \quad D = \left( \frac{1-m}{1+m} \right)^{-\frac{1}{\sqrt{3}}} \tilde{C}(t,y),$$ \hspace{1cm} (17)

where $\tilde{A}$ and $\tilde{C}$ satisfy $(\tilde{C}^{-1}\tilde{A})_y = (\tilde{A}^{-1}\tilde{C})_t$ for vacuum. The non-vanishing Weyl tensor basis components have boost weights +2,0,-2. Interestingly, the vacuum condition implies that the Weyl basis components contain $\tilde{A}$, but are independent of $\tilde{C}$ or any derivatives of $\tilde{A}$ and $\tilde{C}$. Moreover, $\tilde{A}^2/\tilde{A}^{-2}$ appears only in the boost weight +2/-2 components; therefore, by a boost $b_1 = \tilde{A}^{-1}$, we can transform away any occurrence of the arbitrary function, $\tilde{A}$, in the Weyl tensor. Next, we perform a null rotation about $n$, using

$$d_1 = d_2 = 0, \quad d_3 = 2 \left( \frac{2r+m}{2r-m} \right)^{\frac{1}{\sqrt{3}}},$$ \hspace{1cm} (18)

to eliminate boost weight +2 and +1 Weyl components. This is followed by a null rotation about $\ell$, given by

$$c_1 = c_2 = 0, \quad c_3 = \frac{1}{2} \left( \frac{2r-m}{2r+m} \right)^{\frac{1}{\sqrt{3}}},$$ \hspace{1cm} (19)

to eliminate boost weight -2 and -1 Weyl components. In this new frame we are left with a Weyl tensor containing only boost weight 0 components; therefore, AC is of Weyl type D.

4 Higher dimensional Kerr-(anti) de Sitter solutions: K(A)S

A class of rotating black hole solutions with a non-zero cosmological constant, which have $(N-1)/2$ independent rotation parameters and reduce to the 5D solution of [9] and the Kerr-de Sitter solution in 4D and the Myers-Perry solution in the absence of a cosmological constant, have been given in Kerr-Schild form [8].
4.1 K(A)S is of type D

In Kerr-Schild form the 5D Kerr-de Sitter metric \[ \mathcal{S} \] is

\[
\begin{align*}
\text{ds}^2 &= -\frac{(1 - \lambda r^2)\Delta dt^2}{(1 + \lambda a^2)(1 + \lambda b^2)} + \frac{r^2 \rho^2 dr^2}{(1 - \lambda r^2)(r^2 + a^2)(r^2 + b^2)} + \frac{\rho^2 d\theta^2}{\Delta} + \frac{r^2 + a^2}{1 + \lambda a^2} \sin^2 \theta d\phi^2 + \frac{r^2 + b^2}{1 + \lambda b^2} \cos^2 \theta d\psi^2
\end{align*}
\]

where \( \Delta \) is the de Sitter metric and

\[
\begin{align*}
\rho^2 &\equiv r^2 + a^2 \cos^2 \theta + b^2 \sin^2 \theta, \\
\Delta &\equiv 1 + \lambda a^2 \cos^2 \theta + b^2 \sin^2 \theta.
\end{align*}
\]

The null vector \( k_\mu \) is

\[
\begin{align*}
k_\mu dx^\mu &= \frac{\Delta dt}{(1 + \lambda a^2)(1 + \lambda b^2)} + \frac{r^2 \rho^2 dr}{(1 - \lambda r^2)(r^2 + a^2)(r^2 + b^2)} - \frac{a \sin^2 \theta d\phi}{1 + \lambda a^2} - \frac{b \cos^2 \theta d\psi}{1 + \lambda b^2}.
\end{align*}
\]

We construct the following null coframe,

\[
\ell = k, \quad n = Adt + Bdr + Jd\phi + Kd\psi, \quad m_1 = \frac{\theta}{\sqrt{\Delta}} d\theta, \quad m_2 = Hdt + Fd\phi, \quad m_3 = Wdt + Zd\phi + Xd\psi
\]

where

\[
\begin{align*}
A &= \frac{\Delta(2Mr^2 - R)}{2r^2 \rho^2 (1 + \lambda a^2)(1 + \lambda b^2)}, \quad B = \frac{1}{2} + \frac{Mr^2}{R}, \quad J = -\frac{a \sin^2 \theta (2Mr^2 - R)}{2r^2 \rho^2 (1 + \lambda a^2)}, \quad K = -\frac{b \cos^2 \theta (2Mr^2 - R)}{2r^2 \rho^2 (1 + \lambda b^2)},
\end{align*}
\]

\[
\begin{align*}
H = -\frac{\sqrt{\Delta}(1 - \lambda r^2)a \sin \theta}{(1 + \lambda a^2)\sqrt{S}}, \quad F = \frac{\Delta(r^2 + a^2) \sin \theta}{(1 + \lambda a^2)\sqrt{S}}, \quad W = -\frac{\Delta(r^2 + a^2)(1 - \lambda r^2)b \cos \theta}{r \rho(1 + \lambda a^2)(1 + \lambda b^2)\sqrt{S}},
\end{align*}
\]

and we have set \( R = (r^2 + a^2)(r^2 + b^2)(1 - \lambda r^2), \quad S = \rho^2 - (1 - \lambda r^2)b^2 \sin^2 \theta \). It turns out that with respect to this frame the Weyl tensor has only boost weight 0 components and hence is of type D. In this case the null frame in \( \mathcal{S} \) is already aligned thus eliminating the need to consider Lorentz transformations. The method used to obtain \( \mathcal{S} \) was to first determine a null frame associated with the de Sitter metric \( \mathcal{S} \), under the requirement that \( \ell = k \) is one of the null directions. It then follows that in Kerr-Schild form the Kerr-de Sitter metric will partly contain a common factor \( \ell = k \); this results in a redefinition of the null vector \( n \) which will now have an \( M \) dependence.

5 Black Ring: BR

The necessary conditions \( \text{II} \) can be used to classify the black ring solution \( \text{II} \). The Kerr solution and the Myers-Perry solution in five dimensions are of type D with geodesic principal null congruences. The rotating black ring solution, which is a vacuum, asymptotically flat, stationary black hole solution with a horizon of topology \( S^1 \times S^2 \), is given in \( \{t, x, y, \phi, \psi\} \) coordinates by \( \text{II} \)

\[
\begin{align*}
\text{ds}^2 &= -\frac{F(x)}{F(y)} \left( dt + R \sqrt{\lambda \nu(1 + y)d\psi} \right)^2 \\
&\quad + \frac{R^2}{(x - y)^2} \left[ -F(x) \left( G(y)d\psi^2 + \frac{F(y)}{G(y)} dy^2 \right) + F(y)^2 \left( \frac{dx^2}{G(x)} + \frac{G(x)}{F(x)} d\phi^2 \right) \right],
\end{align*}
\]

where

\[
F(\xi) = 1 - \lambda \xi, \quad G(\xi) = (1 - \xi^2)(1 - \nu \xi).
\]
5.1 Black ring is of type I

The black ring solution and its various special cases can be classified. The method is to solve the necessity conditions and then check that these solutions do indeed represent WANDs by calculating the components of the Weyl tensor in an appropriate frame [27]. In order to solve the first equation in (6), \( \ell^a \) is denoted by \((\alpha, \beta, \gamma, \delta, \epsilon)\). A set of fourth order polynomial equations in \(\alpha \ldots \epsilon\) is then obtained. An additional second order equation follows from \(\ell_a \ell^a = 0\). From an analysis of these equations, it can be shown [27] that the black ring solution is algebraically special and of type I.

5.1.1 Black ring is of type II on the horizon

A transformation leads to a metric regular on the horizon \(y = 1/\nu\). The second equation in (6) admits a solution \(L\). It can be checked that the boost order of the Weyl tensor in the frame with \(\ell = L\) is 0 and thus the black ring is of type II on the horizon.

5.1.2 Myers-Perry metric is of type D

By setting \(\lambda = 1\) in (25) we obtain the Myers-Perry metric [7] with a single rotation parameter. It turns out that the second equation in (6) admits two independent solutions \(L^\pm\). When we choose a frame with \(\ell \sim L^+\) and \(n \sim L^-\) all components of the Weyl tensor with boost weights 2,1,-1,-2 vanish and the spacetime is thus of type D [4].

5.2 Special Cases

Two other special cases were considered in [30].

5.2.1 Wrapped black string

In terms of the metric (10), this 5D vacuum solution is defined by

\[
A = \left(1 - \frac{2M}{r}\right)^{1/2}, \quad B = \left(1 - \frac{2M}{r}\right)^{-1/2}, \quad C = 1, \quad D = R,
\]

representing a so-called black string, wrapped around a circle of radius \(R\). It immediately follows from the chosen coframe of (10) that the non-vanishing Weyl tensor components have boost weight 0. Therefore, this spacetime is of type D. We mention in passing that \(C_{abcd}C^{abcd} = 48M^2/r^6\), indicating the presence of a singularity at \(r = 0\).

5.2.2 Homogeneous wrapped object

Setting

\[
A = \left(1 - \frac{2M}{r}\right)^{1/2}, \quad B = 1, \quad C = A, \quad D = R \left(1 - \frac{2M}{r}\right)^{-1/2},
\]

in (10), with \(R\) a constant, gives the so-called homogeneous wrapped object [11, 30]. This 5D vacuum solution contains singular points at \(r = 0\) and \(r = 2M\) as indicated by

\[
C_{abcd}C^{abcd} = \frac{24M^2(2r^2 - 4Mr + 3M^2)}{r^4(r - 2M)^4}.
\]

Initially the Weyl tensor has components with boost weight +2,0,-2. There does not exist a null rotation about \(n\) (for \(r \neq M\)) that will eliminate components with boost weight +2 and +1, but transformations exist that make only boost weight +2 components vanish, namely
\begin{align}
d_1 &= 0, \quad d_2 = 0, \quad d_3 = \pm \frac{2 \left[ (r - 2 M) \left( 4 M - 3 r \pm 2 \sqrt{2(r - M)(r - 2 M)} \right) \right]^{1/2}}{r}, \\
d_1 &= 0, \quad d_2 = \pm \frac{2 \sqrt{2(r - M)(r - 2 M)}}{r}, \quad d_3 = \pm \frac{2 i (r - 2 M)}{r}, \\
d_1 &= \pm \frac{\sqrt{8(r - M)(r - 2 M) - d_2^2 r^2}}{r}, \quad d_2 = d_2, \quad d_3 = \pm \frac{2 i (r - 2 M)}{r}.
\end{align}

Note that equations (31) are a special case of the 1-parameter family of solutions in (32). Following (30), we perform a null rotation about \( \ell \) and find that only boost weight -2 components can be made to vanish; therefore, we obtain Weyl type I (since boost weight +2 and -2 components are now zero). If instead we follow (31) by a null rotation about \( \ell \), we now find that both boost weight -2 and -1 components can be made to vanish with

\begin{align}
c_1 &= \pm \frac{\sqrt{\frac{i}{2(2(r - M)(r - 2 M))}}} {r}, \quad c_2 = \pm \frac{\sqrt{\frac{r}{2(2(r - M)(r - 2 M))}}} {r}, \quad c_3 = 0.
\end{align}

Since the non-vanishing Weyl basis components have boost weights +1 and 0, we obtain Weyl type II, which is more specialized than that found above. We briefly mention a second specialization of (32) with \( d_2 = 0 \). In this case the solution is similar to (31), but with \( d_1 \) and \( d_2 \) interchanged. A null rotation about \( \ell \) can then be used to set boost weight -2 and -1 components to zero (transformation parameters are similar to (33) but with \( c_1 \) and \( c_2 \) interchanged). Hence this second specialization of (32) also gives Weyl type II. The solution given by (28) is of type II.

6 Supersymmetric Black Ring: SBR

Next, we consider supersymmetric rotating black holes that exist in five dimensions. There is the BMPV black hole of \([13]\), with a horizon of spherical topology. The more general supersymmetric black ring (solution of 5D minimal supergravity) was presented in \([15, 16]\).

6.1 BMPV is of type I

The BMPV metric is \([13]\)

\begin{equation}
\frac{\text{d} s^2}{f} = -f^2 [\text{d} t - J \text{d} \phi - K \text{d} \psi]^2 + f^{-2} \text{d} r^2 + r^2 (\text{d} \theta^2 + \sin^2 \theta \text{d} \phi^2 + \cos^2 \theta \text{d} \psi^2),
\end{equation}

where we have set

\begin{equation}
f = 1 - \frac{\mu}{r^2}, \quad J = \frac{\mu \omega \sin^2 \theta}{r^2 - \mu}, \quad K = \frac{\mu \omega \cos^2 \theta}{r^2 - \mu}.
\end{equation}

We choose the following null coframe:

\begin{align}
\ell &= f^2 \text{d} t + d r - f^2 J \text{d} \phi - f^2 K \text{d} \psi, \quad n = \frac{1}{2} (-\text{d} t + f^{-2} \text{d} r + J \text{d} \phi + K \text{d} \psi), \\
m_1 &= r \text{d} \theta, \quad m_2 = r \sin \theta \text{d} \phi, \quad m_3 = r \cos \theta \text{d} \psi.
\end{align}

The occurrence of off-diagonal metric components would suggest that the aligned \( m_2 \) and \( m_3 \) are more complicated than that given in \([57]\). Indeed, this frame yields Weyl tensor components with boost weights of all orders. We begin by carrying out a boost with \( b_1 = (r^2 - \mu)^{-1} \) to simplify the Weyl tensor. We then perform a null rotation about \( n \) and find that the boost weight +2 components can be transformed \(^3\) to zero using

\(^3\)At \( \theta = 0 \) no transformation exists such that the boost weight +1 components can be set to zero.
\[ d_1 = 0, \quad d_3 = -d_2 \cot \theta \]  
and \( d_2 \) is a root of one of the following equations

\[ \omega r^4 d_2^2 - 4r^3 \sin \theta d_2 + 4\omega \sin^2 \theta = 0 \]  
(39)

\[ \mu \omega r^4 d_2^2 + 4r^3(4r^2 - 5\mu) \sin \theta d_2 + 4\mu \omega \sin^2 \theta = 0. \]  
(40)

Equations (39) and (40) each yield different canonical forms for the Weyl tensor, thus two cases need to be considered. Notice that at the horizon \( r = \sqrt{\mu} \) there is only one possibility since equations (39) and (40) coincide. Let us first consider solutions obtained from (39). Performing a null rotation about \( \ell \), we find that boost weight -2 components can be transformed\(^4\) to zero using

\[ c_1 = 0, \quad c_3 = -c_2 \cot \theta \]  
(41)

and \( c_2 \) is a root of the equation

\[ 16(r^2 - \mu)(Rr - \omega)c_2^3 - 4r \sin \theta[2r(3\omega R - 2r)(r^2 - \mu) + \mu(r^2 - \omega^2)]c_2^2 
+ 4r^3 \sin^2 \theta [2r(r^2 - \mu) - \mu(r - \omega R)]c_2 - \mu r^5 \omega^2 \sin^3 \theta = 0, \]  
(42)

where \( R \) is determined from the root \( d_3 \) of (39) using the relation \( R = r^2 d_2/(2 \sin \theta) \). This consequently shows that the BMPV metric is of Weyl type \( \text{I}_1 \) (13).

In the second case, we consider solutions obtained from (40). Again, a null rotation about \( \ell \) can be performed to transform \(^4\) the boost weight -2 components to zero (the transformation parameters \( c_1 \) and \( c_3 \) are identical to the ones given in (11)). However, \( c_2 \) is now a root of the following equation

\[ 8(r^2 - \mu)[4R(r^2 - \mu) - \mu(Rr - 2\omega)]c_2^3 + 4\mu r \sin \theta[\mu(3\omega R + 5r)(r^2 - \mu) - (r^4 - \omega^2 \mu)]c_2^2 
+ 2\mu r^3 \sin^2 \theta [4r(r^2 - \mu) - \mu(2r - \omega R)]c_2 - \mu r^2 \omega^2 r^3 \sin^3 \theta = 0, \]  
(43)

and \( R \) is given by \( R = r^2 d_2/\sin \theta \), where \( d_2 \) is a root of (40). In this case, as before, the transformed frame yields Weyl type \( \text{I}_1 \).

6.2 SBR

Given a class of spacetimes, such as the two parameter family of BMPV metrics, we can define the Weyl type of the class to be the Weyl type of the most algebraically general member contained in the class. Since the Weyl type of any particular metric (for example, a fixed \( \mu \) and \( \omega \) in BMPV) is a geometric property determined at every point of the manifold, we generally expect the Weyl type to vary over the manifold. Similarly, the Weyl type of any particular metric is the Weyl type at the point having the most algebraically general Weyl tensor.

In [16] it was shown that the BMPV metric is a particular case of the supersymmetric black ring (having a horizon topology \( S^1 \times S^2 \)). It follows that the supersymmetric black ring is at most of Weyl type \( \text{I}_1 \), and possibly of type \( \text{I}_G \) or \( G \). The line element of the supersymmetric black ring is [16]

\[ ds^2 = -f^2(dt + \omega)^2 + f^{-1}ds^2(R^4), \]  
(44)

where

\[ f^{-1} = 1 + \frac{Q - q^2}{2R^2}(x - y) - \frac{q^2}{4R^2}(x^2 - y^2), \quad \omega = \omega_\phi d\phi + \omega_\psi d\psi \]  
(45)

\[ \omega_\phi = -\frac{q}{8R^2}(1 - x^2)[3Q - q^2(3 + x + y)], \quad \omega_\psi = \frac{3}{2}q(1 + y) + \frac{q}{8R^2}(1 - y^2)[3Q - q^2(3 + x + y)], \]  
(46)

and the four dimensional flat space is

\[ ds^2(R^4) = \frac{R^2}{(x - y)^2} \left[ \frac{dy^2}{y^2 - 1} + (y^2 - 1)d\psi^2 + \frac{dx^2}{1 - x^2} + (1 - x^2)d\phi^2 \right]. \]  
(47)

Admissible coordinates values are \(-1 \leq x \leq 1, -\infty < y \leq -1 \) and \( \phi, \psi \) are \( 2\pi \)-periodic; it is assumed that \( q > 0 \) and \( Q \geq q^2 \).

\(^4\)At \( \theta = 0 \) no transformation exists such that the boost weight -1 components can be set to zero.
7 Discussion

We have algebraically classified a number of higher dimensional spacetimes. The results are summarized in the Table. In future, it would be useful to classify other higher dimensional solutions, such as other rotating black holes \[22\], higher-dimensional C-metrics and higher-dimensional Godel spacetimes. It is also clear that we need a more efficient way of classifying spacetimes, perhaps in terms of scalar invariants.

The Weyl types of the BR metrics might give a hint on the Weyl types of BR metrics that are missing (such as the doubly spinning neutral BR) \[31\]. The Bena and Warner \[17\] family of supersymmetric solutions of 5D supergravity are specified by seven arbitrary functions of one variable. These solutions are specified implicitly, although an exact solution with 3 arbitrary functions has been presented \[31, 32, 33\]. We have studied various subcases of these solutions, but we have made no substantial progress in their classification. We had hoped that it would be possible to identify which solutions are smooth and which are not \[23\] via their algebraic classification.
| Name | Ref | Comments | Type | ND | Special cases | Type |
|------|-----|----------|------|----|---------------|------|
| ST   | 3   | vacuum BH $R^2 \times S^3$ | $D$  | √  |               |      |
| GP   | 5   | vacuum soliton $R \times R^2 \times S^2$ | $I$  |    | $S^* GP_s AC$ | $D$  |
|      | 6   |                        |      |    |               |      |
| K(A)S| 8   | Rotating BH $\Lambda \neq 0$ | $D$  | √  | MP            |      |
|      | 7   |                        |      |    |               |      |
| BR   | 11  | Rotating BR $R \times R^2 \times S^2$ | $I_i$ |    | $BR_H MP$     | $II$ |
|      | 27  |                        |      |    |               |      |
|      | 12  |                        |      |    |               |      |
| BMPV | 13  | Supersymmetric         | $I_i$ |    |               |      |
| SBR  | 15  |                        | $I_i[*]$ |    |               |      |
| VSI  | 11  | Non BH                 | $N/III$ | √  |               |      |
| RG   | 20  | Rel. Gyration          | $III$ | √  |               |      |

Table 1: The solution (name) is identified by the acronym given in the text. In the comments, features of the solution are presented; i.e., whether it is a black hole (BH), and whether or not it is rotating, whether there is a non-zero cosmological constant $\Lambda$, its topology etc. In the "ND" (higher dimension) column, it is indicated whether there are higher (than 5) dimensional generalizations of these solutions. [*] indicates that the type is at most that specified.
Let us summarize the 4D static and stationary black hole solutions (with topology $S^2$): there is the Schwarzschild solution, the more general Kerr-Newman solutions, the non-vacuum Reissner-Nordström spacetimes, and the non-asymptotically flat vacuum solutions such as Schwarzschild-de Sitter spacetime. All of these solutions have a number of symmetries, which is reflected in their algebraic properties: namely, they are all of Weyl (Petrov) type D \cite{28}. All of the known higher dimensional black holes also have a great deal of symmetry \cite{32}. It is anticipated that this will again be reflected in their having special algebraic properties. Indeed, as can be seen from the Table, all of the higher dimensional black holes classified here are of algebraically special (Weyl) type. In addition, in 4D it is known that spherically symmetric spacetimes are of type D (or O) \cite{28}, and in arbitrary dimensions it has been shown that the Weyl tensor of a spherically symmetric and static spacetime is "boost invariant" \cite{35} (that is, of type D). This led to a conjecture that asserts that stationary higher dimensional black holes, perhaps with the additional conditions of vacuum and/or asymptotic flatness, are necessarily of Weyl type D \cite{2, 27}.

This conjecture has received support recently in a study of local (so that the results may be applied to surfaces of arbitrary topology) non-expanding null surfaces \cite{36}. Assuming the usual energy inequalities, it was found that the vanishing of the expansion of a null surface implies the vanishing of the shear so that a covariant derivative is induced on each non-expanding null surface. The induced degenerate metric tensor, locally identified with a metric tensor defined on the $N-2$ dimensional tangent space, and the induced covariant derivative, locally characterized by the rotation 2-form in the vacuum case, constitute the geometry of a non-expanding null surface. The remaining components of the surface covariant derivative lead to constraints on the induced metric and the rotation 2-form in the vacuum extremal isolated null surface case. This leads to the condition that at the non-expanding horizon (i.e., the isolated null horizon) the boost order of the null direction tangent to the surface is at most 0, so that the Weyl tensor is at least of type II \cite{36} (where the aligned null vectors tangent to the surface correspond to a double principal null direction (PND) of the Weyl tensor in the 4D case).

7.1 Future Work

As in 4D, there is clearly a relationship between algebraic type and the properties of the covariant derivative of the null (geodesic) $\ell$ (the $L$-tensor, defined below, which is the higher dimensional analogue for some of the Newman-Penrose (NP) spin coefficients). Indeed, such a relationship was exploited in the case of type II (and D) spacetimes in the work discussed above \cite{36}, and would likely be a necessary first step in a rigorous proof of the type-D conjecture. Ultimately, we seek a higher-dimensional version of the Goldberg-Sachs theorem. A first step was taken in \cite{29}, in which the Bianchi identities in higher dimensions were studied. Here we simply make some comments on the properties of the $L$-tensor for the spacetimes that have been classified.

In a 4D vacuum space-time the Goldberg-Sachs theorem \cite{28, 37} asserts that the Weyl tensor is of type II with repeated PND $\ell$ such that $\Psi_0 = \Psi_1 = 0$ if and only if (the spin coefficients) $\kappa = \sigma = 0$. Given an NP tetrad, and assuming that this result holds for both $\ell$ and $n$, implies that the Weyl tensor is type D with repeated PND's $\ell$ and $n$ such that $\Psi_0 = \Psi_1 = \Psi_3 = \Psi_4 = 0$ if and only if $\kappa = \sigma = \nu = \lambda = 0$ (the vanishing of the spin coefficients $\nu$ and $\lambda$ indicate that $n$ is a geodesic, shear-free congruence, respectively). \footnote{There are generalizations of the Goldberg-Sachs theorem to non-vacuum spacetimes \cite{28}.} For all black hole solutions in 4D of Petrov type D, this implies that an NP tetrad can always be chosen such that $\kappa = \sigma = \nu = \lambda = 0$ and $\Psi_2 \neq 0$. We note that an NP tetrad for the Schwarzschild metric can be chosen resulting in the spin coefficients $\kappa = \sigma = \nu = \lambda = \epsilon = \pi = \tau = 0$ and $\rho \neq 0$ \cite{37} (which immediately implies that the Schwarzschild metric is of Petrov type D in this NP tetrad and $\Psi_2$ is the only non-vanishing Weyl scalar). The condition $\epsilon = 0$ implies that the null geodesic defined by $\ell$ is affinely parametrized, and since $\rho = -\theta + i \omega$ is real, we also have that $\ell$ is twist-free with non-vanishing expansion. In addition, an NP tetrad for the Kerr metric giving $\kappa = \sigma = \nu = \lambda = \epsilon = 0$ can be chosen \cite{37}. Therefore, $\ell$ and $n$ are null geodesic and shear-free, and $\ell$ is affinely parametrized. Also, $\rho$ is non zero, showing that $\ell$ is twisting with non-vanishing expansion. Again, the Goldberg-Sachs theorem implies that Kerr is Petrov type D in this NP tetrad and $\Psi_2$ is the only non-vanishing Weyl scalar.

Let us now consider higher dimensions. Using the null frame $e_a = \{\ell, n, m_i\}$, where $i = 2, \ldots, N-1$ and assuming the usual inner product, we have by definition
\[
\ell_{\alpha;\beta} = L_{ab} e_{\alpha}^a e_{\beta}^b = L_{1a} \ell_a n_\beta + L_{11} \ell_\alpha \ell_\beta + L_{1i} \ell_\alpha m_i^\beta + L_{i0} m_i^\alpha n_\beta + L_{i1} m_i^1 \ell_\beta + L_{ij} m_i^1 m_j^\beta,
\]
where we have set \( L_{0a} = 0 \) as a consequence of \( \ell_\alpha \ell^\alpha = 0 \) \[1\]. Contracting \( 48 \) with \( \ell^\beta \) gives
\[
\ell_{\alpha;\beta} \ell^\beta = L_{10} \ell_\alpha + L_{i0} m_i^\alpha,
\]
from which we see that \( \ell \) is geodesic if \( L_{10} = 0 \) and affinely parametrized if, in addition, \( L_{10} = 0 \) (\( L_{00} \) and \( L_{10} \) are the analogues of \( \kappa \) and \( \epsilon + \theta \), respectively). We can decompose the purely spatial part of \( L \) as \( L_{ij} = S_{ij} + A_{ij} \) where \( S_{ij} = L_{(ij)} \) and \( A_{ij} = L_{[ij]} \) \[1\]. Further decomposing \( S_{ij} = \sigma_{ij} + \frac{2}{\theta^2} \delta_{ij} \) into its trace-free and trace parts identifies the shear and expansion of \( \ell \), respectively. From \( 48 \), the expansion of \( \ell \) is given by
\[
\theta := \frac{1}{N - 2} \ell^a \ell_a = \frac{1}{N - 2} (L_{10} + L_{ij} \delta^{ij}) = \frac{1}{N - 2} (L_{10} + S).
\]
Therefore, only when the null geodesic \( \ell \) is also affinely parametrized (\( L_{10} = 0 \)) can we identify \( Tr(S_{ij}) = S \) with the expansion of \( \ell \). \[4\] If \( \ell \) is geodesic we can always choose an affine parametrization by applying an appropriate boost; consequently we shall assume that \( L_{10} = 0 \).

Considering the 5D Schwarzschild (ST) metric and calculating \( L_{ab} \) shows that \( \ell \) is geodesic but is not affinely parametrized in the frame presented above. Performing a boost with \( b_1 = 1/A^2 \) affinely parametrizes \( \ell \), resulting in \( \theta = 1/r \) and \( \sigma_{ij} = A_{ij} = 0 \). Next we consider the non-static spherically symmetric AC metric and find that \( f \) satisfies \( (fC)_{tt} = (fA)_{yy} \), results in an affine parametrization; it then follows from \( L_{ab} \) that \( S_{ij} = A_{ij} = 0 \), and hence the null geodesic is expansion-free, shear-free and twist-free. Last, we calculate \( L_{ab} \) for the 5D Kerr-de Sitter (KS) metric in the frame considered. We find that \( \ell \) is geodesic and affinely parametrized. The expansion of \( \ell \) is
\[
\theta = \frac{3r^2 + a^2 \cos^2 \theta + b^2 \sin^2 \theta}{3r^2 + a^2 \cos^2 \theta + b^2 \sin^2 \theta}.
\]
However, unlike the 4D Kerr metric, in higher dimensions we find that \( \ell \) has non-zero shear; the shear invariants
\[
\sigma_{ij} \sigma_{ij} = \frac{2}{3} \frac{(a^2 \cos^2 \theta + b^2 \sin^2 \theta)^2}{3r^2 + a^2 \cos^2 \theta + b^2 \sin^2 \theta}, \quad \sigma_{ij} \sigma_{jk} \sigma_{ki} = \frac{2}{9} \frac{(a^2 \cos^2 \theta + b^2 \sin^2 \theta)^3}{3r^2 + a^2 \cos^2 \theta + b^2 \sin^2 \theta},
\]
are non-vanishing (unless \( a = b = 0 \)), even though this space-time is of Weyl type D (also see \[29\]). This implies that any higher dimensional version of the Goldberg-Sachs theorem will necessarily be more complicated.

### 7.2 Bibliography and Updates

We intend to keep an active version of this article, with periodic updates, on the arXiv. \[5\] In particular, in the arXiv version we will include a bibliography of known exact higher-dimensional solutions (and especially black hole solutions)

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\[4\] We note that in the study of isolated horizons, the space-time derivative operator induces a derivative operator on the null hypersurface \[39\]; it is with respect to this derivative operator that the (projected) expansion of the null normal \( \ell \) is zero.

\[5\] Authors wishing to send us references for exact higher dimensional solutions for us to add to the bibliography or exact solutions that have been classified for us to add to the table(s), please send e-mail to aac@mathstat.dal.ca with the subject header WClass.
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