NONTRIVIAL SOLUTIONS FOR THE CHOQUARD EQUATION
WITH INDEFINITE LINEAR PART AND UPPER CRITICAL
EXPONENT

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Abstract. This paper is dedicated to studying the Choquard equation
\[
\begin{align*}
-\Delta u + V(x)u &= (I_\alpha \ast |u|^p)|u|^{p-2}u + g(u), \quad x \in \mathbb{R}^N,
\end{align*}
\]
where $N \geq 4$, $\alpha \in (0, N)$, $V \in C(\mathbb{R}^N, \mathbb{R})$ is sign-changing and periodic, $I_\alpha$ is the Riesz potential, $p = \frac{N+\alpha}{N-2}$ and $g \in C(\mathbb{R}, \mathbb{R})$. The equation is strongly indefinite, i.e., the operator $-\Delta + V$ has infinite-dimensional negative and positive spaces. Moreover, the exponent $p = \frac{N+\alpha}{N-2}$ is the upper critical exponent with respect to the Hardy-Littlewood-Sobolev inequality. Under some mild assumptions on $g$, we obtain the existence of nontrivial solutions for this equation.

1. Introduction. Consider the Choquard equation
\[
\begin{align*}
-\Delta u + V(x)u &= (I_\alpha \ast |u|^p)|u|^{p-2}u + g(u), \quad x \in \mathbb{R}^N,
\end{align*}
\]
where $N \geq 4$, $\alpha \in (0, N)$, $p = \frac{N+\alpha}{N-2}$ and $I_\alpha : \mathbb{R}^N \setminus \{0\} \to \mathbb{R}$ is the Riesz potential defined by
\[
I_\alpha(x) = \frac{\Gamma\left(\frac{N-\alpha}{2}\right)}{2\Gamma\left(\frac{\alpha}{2}\right)\pi^{N/2}\sqrt{\alpha}} |x|^{N-\alpha}.
\]
Meanwhile, $V : \mathbb{R}^N \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ satisfy the following basic assumptions:
(V) $V \in C(\mathbb{R}^N, \mathbb{R})$, $V(x)$ is 1-periodic in each of $x_1$, $x_2$, \ldots, $x_N$ and
\[
\sup_{x \in \mathbb{R}^N} |V(x)| < \infty ,
\]
(G1) $g \in C(\mathbb{R}^N, \mathbb{R})$, $\lim_{s \to \infty} \frac{G(s)}{s^2} = +\infty$, where $G(s) := \int_0^s g(\tau)d\tau \geq 0$;
(G2) $\lim_{s \to 0} \frac{G(s)}{s^2} = 0$, $\lim_{s \to \infty} \frac{G(s)}{s^{2+\epsilon}} = 0$, where $2^* = \frac{2N}{N-2}$.

Let $A = -\Delta + V$. Then $A$ is self adjoint in $L^2(\mathbb{R}^N)$ with domain $D(A) = H^2(\mathbb{R}^N)$ (see [10, Theorem 4.26]). Let $\{E(\lambda) : -\infty < \lambda < +\infty\}$ and $|A|$ be the spectral family and the absolute value of $A$, respectively. And $|A|^{1/2}$ be the square root of $|A|$. Set

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\( \mathcal{U} = id - \mathcal{E}(0) - \mathcal{E}(0) . \) Then \( \mathcal{U} \) commutes with \( \mathcal{A} \), \( |\mathcal{A}| \) and \( |\mathcal{A}|^{1/2} \), while \( \mathcal{A} = \mathcal{U}|\mathcal{A}| \) is the polar decomposition of \( \mathcal{A} \) (see [9, Theorem IV 3.3]). Let

\[
E = \mathcal{D}(|\mathcal{A}|^{1/2}), \quad E^- = \mathcal{E}(0)E, \quad E^+ = [id - \mathcal{E}(0)]E. \tag{2}
\]

For any \( u \in E \), it is easy to see that \( u = u^- + u^+ \), where

\[
u^- := \mathcal{E}(0)u \in E^-, \quad u^+ := [id - \mathcal{E}(0)]u \in E^+. \tag{3}
\]

The case \( p = 2 \) goes back to the description of the quantum theory of a polaron at rest by Pekar [24] in 1954. The existence of solutions for (12) was proved via variational methods [4, 5, 6, 13, 14, 20, 29, 30] and ordinary differential equations techniques [7, 21], respectively. There are not a few approaches devoted to the existence and multiplicity of solutions of (12) and their qualitative properties, see, for example, the survey paper [23] and the references therein.

To study problem (12) variationally, the well-known Hardy-Littlewood-Sobolev inequality is the starting point. Usually, \( \frac{N+\alpha}{N-2} \) (or \( \frac{N+\alpha}{N-2} \)) is called the upper (or
lower) critical exponent with respect to the Hardy-Littlewood-Sobolev inequality (see [15]). For the subcritical autonomous case \( p \in (\frac{N+\alpha}{N}, \frac{N+\alpha}{N-2}) \) and \( V(x) \equiv 1 \), Moroz and Van Schaftingen [22], as well as established the regularity, positivity and the decay estimates of groundstates, they obtained some nonexistence results under the range \( p \geq \frac{N+\alpha}{N-2} \) or \( p \leq \frac{N+\alpha}{N} \). Thus, for the upper critical case, the subcritical perturbation is necessary to secure the existence of a nontrivial solution.

Recently, there are a few papers study the nontrivial solutions for problem (1). In [31], Van Schaftingen and Xia considered the lower critical problem

\[
-\Delta u + u = (I_\alpha * |u|^{\frac{N+\alpha}{N-2}})|u|^{\frac{N+\alpha}{N-2}} - 2u + f(u) \text{ in } \mathbb{R}^N. \tag{13}
\]

Under some assumptions on \( f(u) \), they obtained a groundstate of (13) by using the Mountain-Pass lemma, the Brezis-Lieb lemma and the concentration compactness principle. Li and Tang [17] considered the upper critical problem

\[
-\Delta u + u = (I_\alpha * |u|^p)|u|^{p-2}u + g(u) \text{ in } \mathbb{R}^N. \tag{14}
\]

with \( g \) subcritical and super-quadratic; they obtained a groundstate of (14) by using the compactness lemma of Struwe [25]. Recently, Li, Ma and Zhang [18] considered the subcritical autonomous Choquard equation

\[
-\Delta u + \lambda u = (I_\alpha * |u|^{\frac{N+\alpha}{N-2}})|u|^{\frac{N+\alpha}{N-2}} - 2u + g(u) \text{ in } \mathbb{R}^N. \tag{15}
\]

They studied the existence of the nontrivial solutions of (15) under \( \lambda > 0 \), \( \frac{N+\alpha}{N} < p < \frac{N+\alpha}{N-2} \) and \( q \in (2, 2^*) \).

In [1], Ackermann discussed the strongly indefinite Choquard-Pekar equation

\[
-\Delta u + V(x)u = (W * u^2)u \text{ in } \mathbb{R}^3. \tag{16}
\]

By using the generalized linking theorem of Kryszewski-Szulkin and Bartsch-Ding, the author obtained infinitely many geometrically distinct weak solutions of (16). Chabrowski and Szulkin [3] investigated the semilinear Schrödinger equation with indefinite linear part

\[
-\Delta u + V(x)u = K(x)|u|^{2^* - 2}u + g(x, u) \text{ in } \mathbb{R}^N. \tag{17}
\]

They estimated the function value in a suitable interval and obtained nontrivial solutions for (17) by means of the linear operator theory. For more results about Schrödinger equation, see [26] and references therein.

However, to our knowledge, there seem to be no results on nontrivial solutions for (1) with strongly indefinite linear part and upper critical exponent. Motivated by [3, 26] and aforementioned works, in the present paper, we deal with the case when 0 lies in a gap of the spectrum \( \sigma(A) \) and \( p = \frac{N+\alpha}{N-2} \).

In addition to \((G_1)-(G_2)\), we also assume that

\( (G_3) \ G(s) := \frac{1}{2}g(s)s - G(s) > 0 \text{ if } s \neq 0, \) and there exist \( c_0 > 0, \sigma \in (0, 1) \) and \( r_0 > 0 \) such that

\[
\left( \frac{|g(s)|^{\frac{4+(N-2)\sigma}{N+\alpha}}}{|s|^\sigma} \right)^\frac{N}{N-2} \leq c_0G(s) \text{ if } |s| \geq r_0.
\]

**Remark 1.** \((G_3)\) was first introduced in [26], which is similar to the following assumption (DL) given by Ding and Lee [8]:

\( (DL) \ F(x,t) > 0 \text{ if } t \neq 0, \) and there exist \( c_0 > 0, r_0 > 0 \) and \( \kappa' > \max\{1, N/2\} \) such that

\[
|f(x,t)|^{\kappa'} \leq c_0F(x,t)|t|^{\kappa'} \text{ for } |t| \geq r_0.
\]
However, they are different conditions (see the following example).

**Example 1.** Let \( N \geq 4 \) and \( G(s) = \frac{|s|^{2^*}}{\ln(e + |s|^{2^*})} \). Then
\[
g(s) = \frac{2^*|s|^{2^*-2}s}{\ln(e + |s|^{2^*})} - \frac{(2^* - 2)|s|^{2(2^*-2)}s}{(e + |s|^{2^*})[\ln(e + |s|^{2^*})]^2},
\]
\[
G(s) = \left( \frac{2^* - 2}{2} \right) \frac{|s|^{2^*}}{\ln(e + |s|^{2^*})} - \left( \frac{2^* - 2}{2} \right) \frac{|s|^{2(2^*-1)}}{(e + |s|^{2^*})[\ln(e + |s|^{2^*})]^2}.
\]
It is easy to see that \( g \) satisfies \((G_1)-(G_3)\) with \( \sigma \in (0, 1) \), but not \((DL)\).

Moreover, it is easy to see that the following functions satisfy \((G_1)-(G_3)\).

**Example 2.** Let \( N \geq 4 \) and \( g(s) = \lambda |s|^{q-2}s \), where \( \lambda > 0 \), \( q \in (2, 2^*) \). Then
\[
G(s) = \frac{\lambda}{q} |s|^q, \quad G(s) = \left( \frac{\lambda}{2} - \frac{\lambda}{q} \right) |s|^q.
\]

Now we can state our main result.

**Theorem 1.1.** Let \( N \geq 4 \), \( \alpha \in (0, N) \). If \((V)\), \((G_1)-(G_3)\) are satisfied. Then problem \((1)\) has a nontrivial solution.

To check our result, we must overcome two difficulties in verifying the boundedness and the non-vanishing of the Cerami sequence \( \{u_n\} \). Firstly, since the equation involves the strongly indefinite linear part, it is difficult to show that
\[
\frac{1}{\|u_n\|^2} \langle \Psi'_1(u_n), u_n^+ - u_n^- \rangle = o(1).
\]

We will overcome this difficulty by estimating the norm of \( \Psi'_1(u_n) \) (see (23)), similar to that of Ackermann [1]. Secondly, since the equation with critical growth brings many obstacles for recovering the compactness, it becomes very hard to prove that the Cerami sequence \( \{u_n\} \) is non-vanishing; we can’t apply the arguments in [3] directly because of the absorption of the convolution term. We shall introduce some new tricks to overcome this difficulty.

The paper is organized as follows. We present some preliminary results in section 2; give our variational framework in section 3; give the proof of Theorem 1.1 in section 4.

Throughout this paper, we denote by \( c_1, c_2, \ldots \) and \( C_1, C_2, \ldots \) different positive constants.

**2. Preliminary lemmas.** Let \( X \) be a real Hilbert space with \( X = X^- \oplus X^+ \) and \( X^- \perp X^+ \). For a functional \( \varphi \in C^1(X, \mathbb{R}) \), \( \varphi \) is said to be weakly sequentially lower semi-continuous if for any \( u_n \rightharpoonup u \) in \( X \) one has \( \varphi(u) = \lim \inf_{n \to \infty} \varphi(u_n) \), and \( \varphi' \) is said to be weakly sequentially continuous if \( \lim_{n \to \infty} \langle \varphi'(u_n), v \rangle = \langle \varphi'(u), v \rangle \) for each \( v \in X \).

**Lemma 2.1 ([12, 16]).** Let \((X, \| \cdot \|)\) be a real Hilbert space, with \( X = X^- \oplus X^+ \) and \( X^- \perp X^+ \), and let \( \varphi \in C^1(X, \mathbb{R}) \) with the form
\[
\varphi(u) = \frac{1}{2} \langle u^+ \rangle^2 - \langle u^- \rangle^2 - \psi(u), \quad u = u^+ + u^- \in X^- \oplus X^+.
\]
Suppose that the following assumptions are satisfied:
\((KS1)\) \( \psi \in C^1(X, \mathbb{R}) \) is bounded from below and weakly sequentially lower semi-continuous;
(KS2) $\psi'$ is weakly sequentially continuous;
(KS3) there exist $r > \rho > 0$ and $e \in X^+$ with $\|e\| = 1$ such that

$$\kappa := \inf \varphi(S^+_{\rho}) > \sup \varphi(\partial Q),$$

where

$$S^+_{\rho} = \{u \in X^+ : \|u\| = \rho\}, \quad Q = \{v + se : v \in X^-, \ s \geq 0, \ \|v + se\| \leq r\}.$$  

Then there exist a constant $c \in [\kappa, \sup \varphi(Q)]$ and a sequence $\{u_n\} \subset X$ satisfying

$$\varphi(u_n) \to c, \quad \|\varphi'(u_n)\|(1 + \|u_n\|) \to 0.$$  

The following well-known Hardy-Littlewood-Sobolev inequality which can be found in [15] will be repeatedly used in this paper.

**Lemma 2.2.** Let $0 < \alpha < N$, and $s, r > 1$ be constants such that

$$\frac{1}{s} + \frac{N - \alpha}{N} = 1 + \frac{1}{r}.$$  

Assume that $f \in L^s(\mathbb{R}^N)$. Then there exists a sharp constant $C(N, \alpha, s)$, independent of $f$, such that

$$I_\alpha * f \in L^r(\mathbb{R}^N) \text{ and } |I_\alpha * f|_r \leq C(N, \alpha, s)|f|_s.$$  

As in the proof of [31, Proposition 2.4], we have the following lemma.

**Lemma 2.3.** If (V), (G$_1$) and (G$_2$) are satisfied, then there exists $\rho > 0$ such that

$$\tilde{\kappa} := \inf \{\Phi(u) : u \in E^+, \|u\| = \rho\} > 0. \quad (18)$$  

The following quasi-Cauchy-Schwarz inequality is the key to estimating the norm of $\Psi'_{1}(u_n)$.

**Lemma 2.4 ([19]).** Let $N \in \mathbb{N}$, $\|\cdot\|_1$ and $\|\cdot\|_2$ be any two norms on $\mathbb{R}^N$. Then there is a constant

$$C = C(N, \|\cdot\|_1, \|\cdot\|_2) \in [1, \infty)$$

such that the inequality

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \psi(\|x - y\|_1) d\mu(x) d\nu(y) \leq C \cdot \left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \psi(\|x - y\|_2) d\mu(x) d\nu(y) \right)^{1/2} \left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \psi(\|x - y\|_2) d\nu(x) d\nu(y) \right)^{1/2}$$

holds for every decreasing function $\psi : [0, \infty) \to [0, \infty]$ and every pair $(\mu, \nu)$ of nonnegative Radon measures on $\mathbb{R}^N$.

In order to prove that the Cerami sequence is non-vanishing, we need the following lemma.

**Lemma 2.5 ([3]).** If $V$ satisfies (V). Then $\|u\|_{W^{1,\infty}(\mathbb{R}^N)} \leq C_0 |u|_2$ for some constant $C_0 > 0$ and all $u \in E^-$. 
3. Variational setting. We set
\[ \Psi(u) = \Psi_1(u) + \Psi_2(u), \forall u \in E, \]
where \( \Psi_1, \Psi_2 \) are defined by (7).

**Lemma 3.1.** Suppose that \((G_1)\) and \((G_2)\) are satisfied. Then \( \Psi \) is nonnegative, weakly sequentially lower semi-continuous, and \( \Psi' \) is weakly sequentially continuous.

**Proof.** Let \( u_n \to u \) in \( E \), up to a subsequence, \( u_n \to u \), a.e. on \( \mathbb{R}^N \). Then, by Fatou's lemma, we have
\[ \Psi(u) = \Psi_1(u) + \Psi_2(u) \leq \liminf_{n \to \infty} \Psi_1(u_n) + \liminf_{n \to \infty} \Psi_2(u_n) \]
\[ \leq \liminf_{n \to \infty} [\Psi_1(u_n) + \Psi_2(u_n)] = \liminf_{n \to \infty} \Psi(u_n). \]

Next, we show that \( \psi' \) is weakly sequentially continuous. Assume that \( u_n \to u \) in \( E \). By the Sobolev embedding theorem, \( u_n \) is bounded in \( L^{2^*}(\mathbb{R}^N) \). Therefore, the sequence \( \{ |u_n|^{2^*} - |u|^{2^*} \} \) is bounded in \( L^{\frac{2N}{N-2}}(\mathbb{R}^N) \), and
\[ \left| |u_n|^{2^*} - |u|^{2^*} \right| \to 0 \text{ in } L^{\frac{2N}{N-2}}(\mathbb{R}^N). \]

Using Lemma 2.2, we have
\[ I_\alpha \left( |u_n|^{2^*} - |u|^{2^*} \right) \to 0 \text{ in } L^{\frac{2N}{N-2}}(\mathbb{R}^N). \] (19)

For any \( R > 0 \), by (19), there holds
\[ \int_{B_R(0)} I_\alpha \left( |u_n|^{2^*} - |u|^{2^*} \right) dx \to 0 \text{ as } n \to \infty. \]

Therefore, for every \( \varepsilon > 0 \) and every \( R > 0 \),
\[ \lim_{n \to \infty} \text{meas} \left( \left\{ x \in B_R(0) : I_\alpha \left( |u_n|^{2^*} - |u|^{2^*} \right) \geq \varepsilon \right\} \right) = 0. \]

Then, by Riesz Lemma, up to a subsequence, one gets \( I_\alpha \left( |u_n|^{2^*} - |u|^{2^*} \right) \to 0 \), a.e. on \( \mathbb{R}^N \). Hence,
\[ I_\alpha \left( |u_n|^{2^*} \right) \to I_\alpha \left( |u|^{2^*} \right), \text{ a.e. on } \mathbb{R}^N \]
and
\[ (I_\alpha \left( |u_n|^{2^*} \right)) u_n \left| \frac{2^*_\alpha - 2}{2^*_\alpha} u_n \to (I_\alpha \left( |u|^{2^*} \right)) |u| \left| \frac{2^*_\alpha - 2}{2^*_\alpha} u \right|, \text{ a.e. on } \mathbb{R}^N. \]

By Lemma 2.2, the sequence \( \left\{ (I_\alpha \left( |u_n|^{2^*} \right)) |u_n| \left| \frac{2^*_\alpha - 2}{2^*_\alpha} u_n \right\} \) is bounded in \( L^{\frac{2N}{N+2}}(\mathbb{R}^N) \), and so
\[ (I_\alpha \left( |u_n|^{2^*} \right)) |u_n| \left| \frac{2^*_\alpha - 2}{2^*_\alpha} u_n \to (I_\alpha \left( |u|^{2^*} \right)) |u| \left| \frac{2^*_\alpha - 2}{2^*_\alpha} u \right|, \text{ in } L^{\frac{2N}{N+2}}(\mathbb{R}^N). \]

Therefore, for any \( v \in E \),
\[ \int_{\mathbb{R}^N} (I_\alpha \left( |u_n|^{2^*} \right)) |u_n| \left| \frac{2^*_\alpha - 2}{2^*_\alpha} u_n \right| v dx \to \int_{\mathbb{R}^N} (I_\alpha \left( |u|^{2^*} \right)) |u| \left| \frac{2^*_\alpha - 2}{2^*_\alpha} u \right| v dx. \]
Lemma 3.2. Suppose that $R > 0$ and $v \in E$

$$\int_{\mathbb{R}^N \setminus B_R(0)} |g(u_n)v| \, dx \leq \int_{\mathbb{R}^N \setminus B_R(0)} C_1(\|u_n\| + |u_n|^{2^*-1})|v| \, dx$$

$$\leq C_2 \left( \int_{\mathbb{R}^N \setminus B_R(0)} |u_n|^2 \, dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^N \setminus B_R(0)} |v|^2 \, dx \right)^{\frac{1}{2}}$$

$$+ C_2 \left( \int_{\mathbb{R}^N \setminus B_R(0)} |u_n|^2 \, dx \right)^{\frac{2^*-1}{2}} \left( \int_{\mathbb{R}^N \setminus B_R(0)} |v|^{2^*} \, dx \right)^{\frac{1}{2^*}}.$$  

On the other hand, it follows from $(G_2)$ that

$$\int_{B_R(0)} |g(u_n)|^{\frac{2^*}{2}} \, dx \leq C_3,$$

and then $g(u_n) \rightharpoonup g(u)$, in $L^{\frac{2^*}{2}}(B_R(0))$. Therefore,

$$\int_{B_R(0)} g(u_n)v \, dx \to \int_{B_R(0)} g(u)v \, dx.$$  

Taking $R$ large enough, one has

$$\int_{\mathbb{R}^N} g(u_n)v \, dx \to \int_{\mathbb{R}^N} g(u)v \, dx.$$  

The proof is thus finished.

The following lemma is crucial to the verification of the link geometry of $\Phi$.

**Lemma 3.2.** Suppose that $(V)$, $(G_1)$ and $(G_2)$ are satisfied. Then for any $c \in E \setminus E^-$, there exists $R_c > 0$ such that

$$\Phi(u) \leq 0, \ u \in E^- \oplus \mathbb{R}^+e^+, \ \|u\| \geq R_c.$$  

(20)

**Proof.** Arguing indirectly, suppose that, for some sequence $\{w_n + s_n e^+\} \subset E^- \oplus \mathbb{R}^+e^+$ with $\|w_n + s_n e^+\| \to \infty$ as $n \to \infty$, $\Phi(w_n + s_n e^+) > 0$ for all $n \in \mathbb{N}$. Set $s_n = \|w_n + s_n e^+\| = v_n^+ + t_n e^+$. Then $\|v_n^+ + t_n e^+\| = 1$, up to a subsequence, $v_n^+ \rightharpoonup v_0 \in E^-$, $v_n^+ \to v_0$ a.e. on $\mathbb{R}^N$, $t_n \to t_0 \geq 0$, $v_n^+ + t_n e^+ \to v_0 + t_0 e^+$ a.e. on $\mathbb{R}^N$. Hence, it follows from (10) and $(G_1)$ that

$$0 < \Phi(w_n + s_n e^+) \leq \frac{1}{2} \|t_n^2 e^+\|^2 - \|v_n^-\|^2 = \frac{1}{2} t_n^2 \|e^+\|^2 - \frac{1}{2} \to t_0^2 \|e^+\|^2 - \frac{1}{2}$$

which yields $t_0 > 0$, $v_0 + t_0 e^+ \neq 0$. Set $\Omega := \{x \in \mathbb{R}^N : (v_0 + t_0 e^+)(x) \neq 0\}$. Then $\text{meas}(\Omega) > 0$. We may assume that $v_n^+ + t_n e^+ \neq 0$ on $\Omega$. Now it follows from $(G_1)$ and Fatou’s Lemma that

$$\lim_{n \to \infty} \int_{\mathbb{R}^N} \frac{G(w_n + s_n e^+)}{\|w_n + s_n e^+\|^2} \, dx \geq \lim_{n \to \infty} \int_{\Omega} \frac{G(w_n + s_n e^+)}{(w_n + s_n e^+)^2} (v_n^+ + t_n e^+)^2 \, dx$$

$$\geq \lim_{n \to \infty} \int_{\Omega} \frac{G(w_n + s_n e^+)}{(w_n + s_n e^+)^2} (v_n^+ + t_n e^+)^2 \, dx = +\infty.$$  

Hence,

$$0 < \Phi(w_n + s_n e^+) \leq \frac{1}{2} \|t_n^2 e^+\|^2 - \int_{\mathbb{R}^N} \frac{G(w_n + s_n e^+)}{\|w_n + s_n e^+\|^2} \, dx \to -\infty,$$

this is a contradiction. 

\[\square\]
Lemma 3.3. Suppose that (V), (G₁) and (G₂) are satisfied. Then there exist a constant \( c > 0 \) and a sequence \( \{u_n\} \subset E \) satisfying
\[
\Phi(u_n) \to c, \quad \|\Phi'(u_n)\|(1 + \|u_n\|) \to 0. \tag{21}
\]

Proof. Lemma 3.3 is a direct corollary of Lemmas 2.1, 2.3, 3.1 and 3.2. □

Lemma 3.4. Suppose that (V), (G₁)-(G₃) are satisfied. Then any sequence \( \{u_n\} \subset E \) satisfying (21) is bounded in \( E \).

Proof. Suppose on the contrary that \( \|u_n\| \to \infty \). Let \( v_n = u_n/\|u_n\| \). Then \( \|v_n\| = 1 \). In view of (21), there exists a constant \( C_1 > 0 \) such that
\[
C_1 \geq \Phi(u_n) - \frac{1}{2} \langle \Phi'(u_n), u_n \rangle = \left( \frac{1}{2} - \frac{1}{2^*} \right) \int_{\mathbb{R}^N} (I_\alpha * |u_n|^{\frac{2^*}{2}})|u_n|^{\frac{2^*}{2}} \, dx + \int_{\mathbb{R}^N} G(u_n) \, dx.
\]
Since \( G(s) \geq 0 \), there exists a constant \( C_2 > 0 \) such that
\[
\int_{\mathbb{R}^N} (I_\alpha * |u_n|^{\frac{2^*}{2}})|u_n|^{\frac{2^*}{2}} \, dx \leq C_2, \quad \int_{\mathbb{R}^N} G(u_n) \, dx \leq C_2. \tag{22}
\]
Using Lemmas 2.2, 2.4 and the Hölder inequality, we can get for any \( v \in E \)
\[
\left|\langle \Phi'(u_n), v \rangle\right| \leq \int_{\mathbb{R}^N} (I_\alpha * |u_n|^{\frac{2^*}{2}})|u_n|^{\frac{2^*}{2}} |v| \, dx \leq \left( \int_{\mathbb{R}^N} (I_\alpha * |u_n|^{\frac{2^*}{2}})|u_n|^{\frac{2^*}{2}} \, dx \right)^{\frac{2^* - 2}{2^*}} \left( \int_{\mathbb{R}^N} (I_\alpha * |u_n|^{\frac{2^*}{2}})|v|^{\frac{2^*}{2}} \, dx \right)^{\frac{2}{2^*}} \leq C_3 \left( \int_{\mathbb{R}^N} (I_\alpha * |u_n|^{\frac{2^*}{2}})|u_n|^{\frac{2^*}{2}} \, dx \right)^{\frac{2^* - 2}{2^*}} \left( \int_{\mathbb{R}^N} (I_\alpha * |u_n|^{\frac{2^*}{2}})|u_n|^{\frac{2^*}{2}} \, dx \right)^{\frac{2}{2^*}} \times \left( \int_{\mathbb{R}^N} (I_\alpha * |v|^{\frac{2^*}{2}})|v|^{\frac{2^*}{2}} \, dx \right)^{\frac{2}{2^*}} \leq C_4 \left( \int_{\mathbb{R}^N} (I_\alpha * |u_n|^{\frac{2^*}{2}})|u_n|^{\frac{2^*}{2}} \, dx \right)^{\frac{2^* + 1}{2^*}} \|v\|. \tag{23}
\]
Hence, by virtue of (7), (22) and (23), one can get
\[
\|\Phi'(u_n)\| = O(1). \tag{24}
\]
From (G₂), for \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that
\[
\left|\frac{g(s)}{s}\right| < \varepsilon. \tag{25}
\]
And from (G₃), there exist \( c_0 > 0, \sigma \in (0, 1) \) and \( r_0 > 0 \) such that
\[
\left( \frac{|g(s)|^{\frac{N+2\sigma}{N+2}}}{|s|^\sigma} \right)^{\frac{N+2}{2}} \leq c_0 G(s) \text{ if } |s| \geq r_0.
\]
For the above \( \delta \) and \( r_0 \), we set
\[
A_{n,\delta} = \{ x \in \mathbb{R}^N : |u_n(x)| \leq \delta \}, \\
B_{n,\delta,r_0} = \{ x \in \mathbb{R}^N : \delta < |u_n(x)| < r_0 \}, \\
C_{n,r_0} = \{ x \in \mathbb{R}^N : |u_n(x)| \geq r_0 \}.
\]
From \((G_3)\) and \((22)\), we have

\[
meas(B_{n,\delta,r_0}) \leq \frac{1}{\inf_{\delta<s<r_0} \mathcal{G}(s)} \int_{\mathbb{R}^N} g(u_n) dx \leq \frac{C_2}{\inf_{\delta<s<r_0} \mathcal{G}(s)} := C_5. \tag{26}
\]

In view of \((26)\), one has

\[
\frac{1}{\|u_n\|^{1/2}} \int_{B_{n,\delta,r_0}} \frac{|g(u_n)|}{|u_n|^{1/2}} |v_n^+| dx \leq \frac{C_5^{1/4}}{\|u_n\|^{1/2}} \left( \sup_{\delta<s<r_0} \frac{|g(s)|}{|s|^{1/2}} \right) |v_n^+|_2 = o(1). \tag{27}
\]

Set \(\beta = \frac{4(1-\sigma)}{4+\sigma(N-2)}. \) By virtue of \((G_3), \ (22)\) and the Hölder inequality,

\[
\frac{1}{\|u_n\|^\beta} \int_{C_{n,r_0}} \frac{|g(u_n)|}{|u_n|^{1-\beta}} |v_n|^{1-\beta} |v_n^+| dx \\
\leq \frac{1}{\|u_n\|^\beta} \left[ \int_{C_{n,r_0}} \left( \frac{|g(u_n)|}{|u_n|^{1-\beta}} \right)^{\frac{2^{*}-2+\beta}{2}} dx \right]^{\frac{2}{2^{*}-2+\beta}} |v_n|_{2^{*}} |v_n^+|_{2^{*}} \\
= \frac{1}{\|u_n\|^\beta} \left[ \int_{C_{n,r_0}} \left( \frac{|g(u_n)|^{\frac{4+\sigma(N-2)}{\sigma}}}{{|u_n|}^{\sigma}} \right)^{\frac{2}{N+2}} dx \right]^{\frac{2}{2^{*}-2+\beta}} |v_n|_{2^{*}} |v_n^+|_{2^{*}} \\
\leq C_3 \left( \int_{\mathbb{R}^N} (I_\alpha |u_n|^{2^{*}}) |u_n|^{2^{*}} dx \right)^{\frac{2}{2^{*}}} \left( \int_{\mathbb{R}^N} (I_\alpha |u_n|^{2^{*}}) |u_n|^{2^{*}} dx \right)^{\frac{1}{2^{*}}} \\
\leq \frac{1}{\|u_n\|^\beta} \left( c_0 \int_{C_{n,r_0}} g(u_n) dx \right)^{\frac{2}{2^{*}}} |v_n|_{2^{*}} |v_n^+|_{2^{*}} = o(1). \tag{28}
\]

Hence, from \((7), (25), (27)\) and \((28)\), we have

\[
\frac{1}{\|u_n\|^2} |\langle \Psi'_2(u_n), v_n^+ \rangle| \\
\leq \int_{\mathbb{R}^N} \frac{|g(u_n)|}{\|u_n\|} |v_n^+| dx \\
= \int_{A_{n,\delta}} \frac{|g(u_n)|}{|u_n|} |v_n| |v_n^+| dx + \frac{1}{\|u_n\|^{1/2}} \int_{B_{n,\delta,r_0}} \frac{|g(u_n)|}{|u_n|^{1/2}} |v_n|^{1/2} |v_n^+| dx \\
+ \frac{1}{\|u_n\|^\beta} \int_{C_{n,r_0}} \frac{|g(u_n)|}{|u_n|^{1-\beta}} |v_n|^{1-\beta} |v_n^+| dx \\
\leq \epsilon |v_n|_2 |v_n^+|_2 + o(1), \tag{29}
\]

which implies

\[
\frac{1}{\|u_n\|^2} |\langle \Psi'_2(u_n), v_n^+ \rangle| = o(1). \tag{30}
\]

Similarly,

\[
\frac{1}{\|u_n\|^2} |\langle \Psi'_2(u_n), u_n^- \rangle| = o(1). \tag{31}
\]
Hence, from (9), (21), (24), (30) and (31), we have
\[ 1 = \frac{1}{\|u_n\|^2} \langle \Phi'(u_n), u_n^+ - u_n^- \rangle + \frac{1}{\|u_n\|^2} \langle \Psi_1'(u_n), u_n^+ - u_n^- \rangle + \frac{1}{\|u_n\|^2} \langle \Psi_2'(u_n), u_n^+ - u_n^- \rangle = o(1). \]
This contradiction shows that \( \{u_n\} \) is bounded.

Use \( S_\alpha \) to denote the best constant defined by
\[ S_\alpha := \inf \left\{ \int_{\mathbb{R}^N} |\nabla u|^2 \, dx : u \in D^{1,2}(\mathbb{R}^N), \int_{\mathbb{R}^N} (I_\alpha * |u|^{2^*}) |u|^{\frac{2^*_\alpha}{2}} \, dx = 1 \right\}. \]
The constant \( S_\alpha \) is achieved by
\[ U_\varepsilon := C \left( \frac{\varepsilon^{\frac{N}{2}}}{\varepsilon + \|x\|^2} \right)^{\frac{N-2}{2}}, \]
where \( C > 0 \) is a fixed constant and \( \varepsilon > 0 \) is a parameter (see [11, 15]). Let \( \psi \in C_0^\infty(\mathbb{R}^N, [0, 1]) \) be a cut-off function satisfying \( \psi = 1 \) for \( x \in B_\rho \) and \( \psi = 0 \) for \( x \in \mathbb{R}^N \setminus B_{2\rho} \), where \( \rho \) is some positive constant. Let
\[ v_\varepsilon := \frac{\psi U_\varepsilon}{\left( \int_{\mathbb{R}^N} (I_\alpha * |\psi U_\varepsilon|^{\frac{2^*_\alpha}{2}}) |\psi U_\varepsilon|^{\frac{2^*_\alpha}{2}} \, dx \right)^{\frac{1}{2^*}}}. \]
We shall need the following asymptotic estimates as \( \varepsilon \to 0^+ \) (see [2, 11, 32]):
\[ \int_{\mathbb{R}^N} (I_\alpha * |v_\varepsilon|^{\frac{2^*_\alpha}{2}}) |v_\varepsilon|^{\frac{2^*_\alpha}{2}} \, dx = 1, \quad \int_{\mathbb{R}^N} |\nabla v_\varepsilon|^2 \, dx = S_\alpha + O(\varepsilon^{\frac{N-2}{4}}), \]
\[ \int_{\mathbb{R}^N} |v_\varepsilon| \, dx = O(\varepsilon^{\frac{N}{4}}), \quad \int_{\mathbb{R}^N} |\nabla v_\varepsilon| \, dx = O(\varepsilon^{\frac{N-2}{4}}), \]
\[ \int_{\mathbb{R}^N} |v_\varepsilon|^2 \, dx = \begin{cases} O(\varepsilon^{\frac{N-2}{4}}), & N \geq 5; \\ O(\varepsilon^{\frac{N}{4}} \ln \varepsilon), & N = 4, \end{cases} \]
and
\[ \int_{\mathbb{R}^N} |v_\varepsilon|^{2^* - \delta} \, dx = \begin{cases} O(\varepsilon^{\frac{(N-2)\delta}{4}}), & 0 < \delta < 2^*/2; \\ O(\varepsilon^{\frac{N}{4}} \ln \varepsilon), & \delta = 2^*/2; \\ O(\varepsilon^{\frac{N}{4} - \frac{(N-2)\delta}{4}}), & 2^*/2 < \delta < 2^*. \end{cases} \]

Now we are ready to estimate the “Cerami level” \( c \) given in Lemma 2.1.

**Lemma 3.5.** Let \( N \geq 4, \alpha \in (0, N) \). If (V), (G1)-(G3) are satisfied, then there exist \( \varepsilon > 0 \) such that
\[ \sup_{E^{\varepsilon} \ominus \mathbb{R}^+ v_\varepsilon} \Phi < \left( \frac{\alpha + 2}{2(N + \alpha)} \right)^{\frac{N+\alpha}{N+2}} S_\alpha^{\frac{N+\alpha}{N+2}}. \]

**Proof.** Let \( Z_\varepsilon = \{ u \in E^{\varepsilon} \ominus \mathbb{R}^+ v_\varepsilon : \|u^+\| \geq \|u^-\| \} \). If \( \|u^+\| < \|u^-\| \), then
\[ \Phi(u) = \frac{1}{2} (\|u^+\|^2 - \|u^-\|^2) - \frac{1}{2\alpha} \int_{\mathbb{R}^N} (I_\alpha * |u|^{\frac{2^*_\alpha}{2}}) |u|^{\frac{2^*_\alpha}{2}} \, dx - \int_{\mathbb{R}^N} G(u) \, dx < 0. \]
It suffices to show that
\[ \sup_{Z_\varepsilon} \Phi < \left( \frac{\alpha + 2}{2(N + \alpha)} \right)^{\frac{N+\alpha}{N+2}} S_\alpha^{\frac{N+\alpha}{N+2}}. \]
Since \(G(s) \geq 0\), we have
\[
\Phi(u) \leq \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2)dx - \frac{1}{2\alpha} \int_{\mathbb{R}^N} (I_{\alpha} * |u|_{N+\alpha}^{2\alpha})|u|_{N+\alpha}^{2\alpha} dx =: \Phi_0(u).
\]
For \(u \neq 0\), \(t \geq 0\), let \(h(t) := \Phi_0(tu)\). Then
\[
\max_{t \geq 0} h(t) = \frac{\alpha + 2}{2(\alpha + N)} \left( \frac{\int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2)dx}{\int_{\mathbb{R}^N} (I_{\alpha} * |u|_{N+\alpha}^{2\alpha})|u|_{N+\alpha}^{2\alpha} dx} \right)^{\frac{2\alpha}{N+\alpha}}.
\]
So it remains to show
\[
\sup_{w \in \mathbb{Z}_\epsilon} \left\{ \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2)dx : \int_{\mathbb{R}^N} (I_{\alpha} * |u|_{N+\alpha}^{2\alpha})|u|_{N+\alpha}^{2\alpha} dx = 1 \right\} < S_\alpha \tag{36}
\]
for small \(\epsilon > 0\). Suppose \(u = w + tv_\epsilon \in E^- \oplus \mathbb{R}^+v_\epsilon\), \(\|u^+\| \geq ||u^-||\) and \(\int_{\mathbb{R}^N} (I_{\alpha} * |u|_{N+\alpha}^{2\alpha})|u|_{N+\alpha}^{2\alpha} dx = 1\). We obtain, by convexity,
\[
|tv_\epsilon|^\frac{2\alpha}{\alpha} - \frac{2\alpha}{2} |tv_\epsilon|^\frac{2\alpha}{\alpha} - 1 |w| \leq |u|^\frac{2\alpha}{\alpha} \leq 2^\frac{2\alpha}{2} - 1 (|w|^\frac{2\alpha}{\alpha} + |tv_\epsilon|^\frac{2\alpha}{\alpha})
\]
and
\[
1 = \int_{\mathbb{R}^N} (I_{\alpha} * |u|_{N+\alpha}^{2\alpha})|u|_{N+\alpha}^{2\alpha} dx
\]
\[
\geq \int_{\mathbb{R}^N} (I_{\alpha} * |u|_{N+\alpha}^{2\alpha}) \left( |tv_\epsilon|^\frac{2\alpha}{\alpha} - \frac{2\alpha}{2} |tv_\epsilon|^\frac{2\alpha}{\alpha} - 1 |w| \right) dx
\]
\[
= t^\frac{2\alpha}{\alpha} \int_{\mathbb{R}^N} (I_{\alpha} * |v_\epsilon|_{N+\alpha}^{2\alpha})|v_\epsilon|_{N+\alpha}^{2\alpha} dx - \frac{2\alpha}{2} t^\frac{2\alpha}{\alpha} - 1 \int_{\mathbb{R}^N} (I_{\alpha} * |u|_{N+\alpha}^{2\alpha})|v_\epsilon|_{N+\alpha}^{2\alpha} - 1 |w| dx
\]
\[
\geq t^\frac{2\alpha}{\alpha} \int_{\mathbb{R}^N} (I_{\alpha} * |v_\epsilon|_{N+\alpha}^{2\alpha}) \left( |tv_\epsilon|^\frac{2\alpha}{\alpha} - \frac{2\alpha}{2} |tv_\epsilon|^\frac{2\alpha}{\alpha} - 1 |w| \right) dx
\]
\[
- \frac{2\alpha}{2} t^\frac{2\alpha}{\alpha} - 1 \int_{\mathbb{R}^N} (I_{\alpha} * |v_\epsilon|_{N+\alpha}^{2\alpha}) \left[ 2^\frac{2\alpha}{2} - 1 (|w|^\frac{2\alpha}{\alpha} + |tv_\epsilon|^\frac{2\alpha}{\alpha} \right] dx
\]
\[
= t^\frac{2\alpha}{\alpha} - \frac{2\alpha}{2} t^\frac{2\alpha}{\alpha} - 1 (1 + 2^\frac{2\alpha}{\alpha} - 1) \int_{\mathbb{R}^N} (I_{\alpha} * |v_\epsilon|_{N+\alpha}^{2\alpha})|v_\epsilon|_{N+\alpha}^{2\alpha} - 1 |w| dx
\]
\[
- \frac{2\alpha}{2} (2t)^\frac{2\alpha}{\alpha} - 1 \int_{\mathbb{R}^N} (I_{\alpha} * |w|_{N+\alpha}^{2\alpha})|v_\epsilon|_{N+\alpha}^{2\alpha} - 1 |w| dx
\]
\[
= t^\frac{2\alpha}{\alpha} - \frac{2\alpha}{2} t^\frac{2\alpha}{\alpha} - 1 (1 + 2^\frac{2\alpha}{\alpha} - 1)A(\epsilon) - \frac{2\alpha}{2} \left( 2t \right)^\frac{2\alpha}{\alpha} - 1 B(\epsilon), \tag{37}
\]
where
\[
A(\epsilon) = \int_{\mathbb{R}^N} (I_{\alpha} * |v_\epsilon|_{N+\alpha}^{2\alpha})|v_\epsilon|_{N+\alpha}^{2\alpha} - 1 |w| dx,
\]
\[
B(\epsilon) = \int_{\mathbb{R}^N} (I_{\alpha} * |w|_{N+\alpha}^{2\alpha})|v_\epsilon|_{N+\alpha}^{2\alpha} - 1 |w| dx.
\]
It follows from Lemma 2.5 that \(\|w\|_{W^{1,\infty}(\mathbb{R}^N)} \leq C_0 \|w\|\) for some constant \(C_0 > 0\). By (33) we obtain
\[
\left| \int_{\mathbb{R}^N} (\nabla w \cdot \nabla v_\epsilon^c + V(x)wv_\epsilon^c) dx \right| = \left| \int_{\mathbb{R}^N} (\nabla w \cdot \nabla v_\epsilon + V(x)wv_\epsilon) dx \right|
\]
\[
\leq |\nabla w|_\infty |\nabla v_\epsilon|_1 + |V|_\infty |w|_\infty |v_\epsilon|_1
\]
\[
\leq \|w\|O(\epsilon^{\frac{N-\alpha}{2}}).
\]
Set \(\delta = \frac{2N}{\alpha+\alpha}\). To show (36), we consider four possible cases.
Case 1: \( N \geq 5, N < \alpha + 4 \). Then \( \delta < \frac{2}{N} \). From Lemmas 2.2, 2.5 and expression (35), we have

\[
A(\varepsilon) \leq \|w\|_\infty \int_{\mathbb{R}^N} (J_\alpha * |v_c|^{\frac{2N}{N-2}})|v_c|^{\frac{2N}{N-2}-1} \, dx \\
\leq C(N, \alpha)\|w\|_\infty |v_c|^{\frac{2N}{N-2}} \left( \int_{\mathbb{R}^N} |v_c|^{2^*-\delta} \, dx \right)^{\frac{1}{2}} \\
= \|w\|_\infty O(\varepsilon^{\frac{N-2}{N}}) \\
\leq \|w\| O(\varepsilon^{\frac{N-2}{N}}) \tag{39}
\]

and

\[
B(\varepsilon) \leq C(N, \alpha)\|w\|_\infty |v_c|^{\frac{2N}{N-2}} \left( \int_{\mathbb{R}^N} |v_c|^{2^*-\delta} \, dx \right)^{\frac{1}{2}} \\
\leq \|w\|^{\frac{2N}{N-2}+1}O(\varepsilon^{\frac{N-2}{N}}). \tag{40}
\]

By \( \|u^+\| \geq \|u^-\| \), (32) and (34), we have

\[
\|w\| \leq \sqrt{2t} \|v_c\| \leq C_1 t. \tag{41}
\]

Hence,

\[
A(\varepsilon) \leq tO(\varepsilon^{\frac{N-2}{N}}), \quad B(\varepsilon) \leq t^{\frac{2N}{N-2}+1}O(\varepsilon^{\frac{N-2}{N}}). \tag{42}
\]

According to (37) and (42),

\[
t^{2^*_\alpha} - t^{2^*_\alpha}O(\varepsilon^{\frac{N-2}{N}}) \leq 1. \tag{43}
\]

By virtue of (41) and (43), there exists a constant \( C_2 > 0 \), independent of \( \varepsilon \), such that

\[
t \leq C_2, \quad \|w\| \leq C_2. \tag{44}
\]

Using (40) and (44), we obtain \( B(\varepsilon) \leq \|w\|O(\varepsilon^{\frac{N-2}{N}}) \), which, together with (37), (39) and (44), yields \( t^{2^*_\alpha} \leq 1 + \|w\|O(\varepsilon^{\frac{N-2}{N}}) \). Thus,

\[
2^*_\alpha \leq 1 + \|w\|O(\varepsilon^{\frac{N-2}{N}}). \tag{45}
\]

By (V), we may assume that \( V(0) < 0 \), and there exists \( d > 0 \) such that

\[
\int_{\mathbb{R}^N} V(x)v_c^2 \, dx < -d\varepsilon. \tag{46}
\]

Note that

\[
-\|w\|^2 + \|w\|O(\varepsilon^{\frac{N-2}{N}}) \leq O(\varepsilon^{\frac{N-2}{N}}). \tag{47}
\]

Hence, it follows from (32), (38), (45), (46) and (47) that

\[
\int_{\mathbb{R}^N} (|\nabla w|^2 + V(x)u^2) \, dx \\
= -\|w\|^2 + 2t \int_{\mathbb{R}^N} (\nabla w \cdot \nabla v_c + V(x)wv_c) \, dx + t^2 \int_{\mathbb{R}^N} (|\nabla v_c|^2 + V(x)u^2) \, dx \\
\leq -\|w\|^2 + \|w\|O(\varepsilon^{\frac{N-2}{N}}) + (1 + \|w\|O(\varepsilon^{\frac{N-2}{N}}))(S_\alpha + O(\varepsilon^{\frac{N-2}{N}}) - d\varepsilon) \\
\leq -\|w\|^2 + \|w\|O(\varepsilon^{\frac{N-2}{N}}) + S_\alpha + O(\varepsilon^{\frac{N-2}{N}}) - d\varepsilon \\
\leq S_\alpha + O(\varepsilon^{\frac{N-2}{N}}) - d\varepsilon \\
< S_\alpha \tag{48}
\]
for \( \varepsilon > 0 \) sufficiently small.

Case 2: \( N \geq 5, N > \alpha + 4 \). Then \( \frac{2\varepsilon}{2} < \delta < 2, \frac{N}{2} - \frac{N-2}{2} > 1 \). Similar to (45), we have

\[
t^2 \leq 1 + \|w\|O(\varepsilon^{\frac{N}{2} - \frac{N-2}{2}}).
\]

Observe that

\[
-\|w\|^2 + \|w\|O(\varepsilon^{\frac{N}{2} - \frac{N-2}{2}}) \leq O(\varepsilon^{\frac{N}{2} - \frac{N-2}{2}}).
\]

So, (32), (38), (49) and (50) give

\[
\int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2)dx \\
\leq S_\alpha + O(\varepsilon^{\frac{N}{2} - \frac{N-2}{2}}) - d\varepsilon - \|w\|^2 + \|w\|O(\varepsilon^{\frac{N}{2} - \frac{N-2}{2}}) + \|w\|O(\varepsilon^{\frac{N}{2} - \frac{N-2}{2}}) \\
\leq S_\alpha + O(\varepsilon^{\frac{N}{2} - \frac{N-2}{2}}) - d\varepsilon \\
< S_\alpha
\]

for \( \varepsilon > 0 \) sufficiently small.

Case 3: \( N \geq 5, N = \alpha + 4 \). Then \( \delta = \frac{2\varepsilon}{2} \). Similar to (45), we have

\[
t^2 \leq 1 + \|w\|O(\varepsilon^{\frac{N}{2} - \frac{N-2}{2}} |\ln \varepsilon|^\frac{1}{2}).
\]

Note that

\[
-\|w\|^2 + \|w\|O(\varepsilon^{\frac{N}{2} - \frac{N-2}{2}} |\ln \varepsilon|^\frac{1}{2}) \leq O(\varepsilon^{\frac{N}{2} - \frac{N-2}{2}} |\ln \varepsilon|^\frac{1}{2}).
\]

Consequently, (32), (38), (52) and (53) imply

\[
\int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2)dx \\
\leq S_\alpha + O(\varepsilon^{\frac{N}{2} - \frac{N-2}{2}}) - d\varepsilon - \|w\|^2 + \|w\|O(\varepsilon^{\frac{N}{2} - \frac{N-2}{2}} |\ln \varepsilon|^\frac{1}{2}) + \|w\|O(\varepsilon^{\frac{N}{2} - \frac{N-2}{2}} |\ln \varepsilon|^\frac{1}{2}) \\
\leq S_\alpha + O(\varepsilon^{\frac{N}{2} - \frac{N-2}{2}}) - d\varepsilon + O(\varepsilon^{\frac{N}{2} - \frac{N-2}{2}} |\ln \varepsilon|^\frac{1}{2}) \\
< S_\alpha
\]

for \( \varepsilon > 0 \) sufficiently small.

Case 4: \( N = 4 \). Then \( N < \alpha + 4, \delta < \frac{2\varepsilon}{2} \). Note that

\[
-\|w\|^2 + \|w\|O(\varepsilon^{\frac{1}{2}}) \leq O(\varepsilon).
\]

We see using (32), (34), (38) and (55) that

\[
\int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2)dx \\
\leq S_\alpha + O(\varepsilon) - d\varepsilon |\ln \varepsilon| - \|w\|^2 + \|w\|O(\varepsilon^{\frac{1}{2}}) \\
\leq S_\alpha + O(\varepsilon) - d\varepsilon |\ln \varepsilon| \\
< S_\alpha
\]

for \( \varepsilon > 0 \) sufficiently small.

The above four cases confirm that (36) holds for \( N \geq 4 \) and \( \alpha \in (0, N) \).
4. **Proof of Theorem 1.1.** In this section, we give the proof of Theorem 1.1.

**Proof of Theorem 1.1.** In view of Lemmas 3.3 and 3.4, there exists a bounded sequence \( \{u_n\} \subset E \) satisfying
\[
\Phi(u_n) \to c, \quad \|\Phi'(u_n)\|(1 + \|u_n\|) \to 0.
\]  
(57)

By Lemmas 2.1 and 3.5, we may assume that
\[
c < \left( \frac{\alpha + 2}{2(N + \alpha)} \right) S_{\alpha}^{\frac{N+\alpha}{2}}.
\]  
(58)

Next, we prove that \( \{u_n\} \) is non-vanishing, arguing by contradiction. If \( \{u_n\} \) is vanishing, then by Lions’ concentration compactness principle [32], \( u_n \to 0 \) in \( L^s(\mathbb{R}^N) \) for \( 2 < s < 2^* \). By Lemma 2.5, we see
\[
\|u_n\|_{W^{1,\infty}(\mathbb{R}^N)} \leq C_0 |u_n|^2.
\]  
(59)

And from \( (G_2) \), we see as well that for \( s \in (2,2^*) \),
\[
G(u_n) \leq \varepsilon (|u_n|^2 + |u_n|^2) + C_\varepsilon |u_n|^s.
\]  
(60)

Since \( \{u_n\} \) is bounded,
\[
\int_{\mathbb{R}^N} G(u_n) dx \to 0.
\]  
(61)

Similarly,
\[
\int_{\mathbb{R}^N} g(u_n) u_n dx \to 0, \quad \int_{\mathbb{R}^N} g(u_n) u_n^- dx \to 0.
\]  
(62)

Then from (57), we have
\[
\frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u_n|^2 + V(x)u_n^2) dx - \frac{1}{2\alpha} \int_{\mathbb{R}^N} (I_\alpha * |u_n|^\frac{2^*}{2}) |u_n|^\frac{2^*}{2} dx = c + o(1),
\]  
(63)

and
\[
\int_{\mathbb{R}^N} (|\nabla u_n|^2 + V(x)u_n^2) dx - \int_{\mathbb{R}^N} (I_\alpha * |u_n|^\frac{2^*}{2}) |u_n|^\frac{2^*}{2} dx = o(1).
\]  
(64)

Owing to (63) and (64), we may assume that
\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} (|\nabla u_n|^2 + V(x)u_n^2) dx = \lim_{n \to \infty} \int_{\mathbb{R}^N} (I_\alpha * |u_n|^\frac{2^*}{2}) |u_n|^\frac{2^*}{2} dx = b,
\]  
(65)

and
\[
\frac{\alpha + 2}{2(N + \alpha)} b = c.
\]  
(66)

From (57) and (62),
\[
o(1) = (\Phi'(u_n), u_n^-) = -\|u_n^-\|^2 - \int_{\mathbb{R}^N} (I_\alpha * |u_n|^\frac{2^*}{2}) |u_n|^\frac{2^*}{2-2} u_n^- u_n^- dx + o(1).
\]  
(67)

It follows from (59), Lemma 2.2 and the H"older inequality that
\[
\left| \int_{\mathbb{R}^N} (I_\alpha * |u_n|^\frac{2^*}{2}) |u_n|^\frac{2^*}{2-2} u_n^- u_n^- dx \right| \leq C |u_n|^\frac{2^*}{2} |u_n^-|^\frac{1}{2} |u_n^-|^\frac{1-\sigma}{2} |u_n^-|^\frac{2^*}{2^*} = C |u_n|^\frac{2^*}{2} |u_n^-|^\frac{1-\sigma}{2} |u_n^-|^\frac{2}{2^*} - \frac{2^*}{2^*} \|u_n^-\|_2^2.
\]  
(68)

Choose \( \sigma > 0 \) such that \( 2^*(\frac{2^* -2}{2^* -2 + 2\sigma}) \in (2,2^*) \). Since \( u_n \to 0 \) in \( L^s(\mathbb{R}^N) \), it follows from (68) that
\[
\int_{\mathbb{R}^N} (I_\alpha * |u_n|^\frac{2^*}{2}) |u_n|^\frac{2^*}{2-2} u_n^- u_n^- dx = o(1).
\]  
(69)

Thus, (67) leads to
\[
\|u_n^-\| = o(1).
\]  
(70)
Recall that \( \mathcal{E}(\lambda) = -\infty < \lambda < +\infty \) is the spectral family of \( -\Delta + V \) in \( L^2(\mathbb{R}^N) \). Let \( w_n = \mathcal{E}(\lambda)u_n^+ \), \( z_n = [id - \mathcal{E}(\lambda)]u_n^+ \). Then \( u_n^+ = w_n + z_n \). By [3, Proposition 2.4], there exist constants \( C_1 \) and \( C_2 \) such that

\[
|w_n|_{2N} \leq C_1|w_n|_2 \leq C_2\|u_n\|, \quad \text{if } N > 4,
\]

and

\[
|w_n|_q \leq C_1|w_n|_2 \leq C_2\|u_n\|, \quad \text{if } N = 4,
\]

where \( q \) may be taken arbitrarily large. Similar to (69), one has

\[
\int_{\mathbb{R}^N} (I_\alpha * |u_n|^{2^*_\alpha})|u_n|^{2^*_\alpha - 2} u_n w_n dx = o(1).
\]

Moreover,

\[
\|w_n\|^2 = \int_{\mathbb{R}^N} (I_\alpha * |u_n|^{2^*_\alpha})|u_n|^{2^*_\alpha - 2} u_n w_n dx + o(1) = o(1).
\]

Since \( z_n \in [id - \mathcal{E}(\lambda)]L^2 \cap E \),

\[
\int_{\mathbb{R}^N} (|\nabla z_n|^2 + V(x)z_n^2)dx \geq \lambda|z_n|_2^2.
\]

Then for each \( \delta > 0 \), we may find \( \lambda > 0 \) sufficiently large, such that

\[
\delta \int_{\mathbb{R}^N} |\nabla z_n|^2 dx + \int_{\mathbb{R}^N} V(x)z_n^2 dx \geq 0.
\]

By the definition of \( S_\alpha \), one has

\[
\int_{\mathbb{R}^N} |\nabla u_n|^2 dx \geq \left( \int_{\mathbb{R}^N} (I_\alpha * |u_n|^{2^*_\alpha})|u_n|^{2^*_\alpha} dx \right)^{\frac{2^*_\alpha}{2^*_\alpha - 2}} S_\alpha.
\]

Hence, it follows from (70), (74) and (76) that

\[
b = \int_{\mathbb{R}^N} (I_\alpha * |u_n|^{2^*_\alpha})|u_n|^{2^*_\alpha} dx + o(1) = \int_{\mathbb{R}^N} (|\nabla u_n|^2 + V(x)u_n^2)dx + o(1)
\]

\[
= \|u_n^+\|^2 + o(1) = \|z_n\|^2 + o(1) = \int_{\mathbb{R}^N} (|\nabla z_n|^2 + V(x)z_n^2)dx + o(1)
\]

\[
\geq (1 - \delta) \int_{\mathbb{R}^N} |\nabla z_n|^2 dx + o(1) = (1 - \delta) \int_{\mathbb{R}^N} |\nabla u_n|^2 dx + o(1)
\]

\[
\geq (1 - \delta) \left( \int_{\mathbb{R}^N} (I_\alpha * |u_n|^{2^*_\alpha})|u_n|^{2^*_\alpha} dx \right)^{\frac{2^*_\alpha}{2^*_\alpha - 2}} S_\alpha + o(1)
\]

\[
= (1 - \delta)\delta^{\frac{2^*_\alpha}{2^*_\alpha - 2}} S_\alpha + o(1).
\]

Clearly, (66) implies that \( b > 0 \). And (77) implies further that \( b \geq ((1 - \delta)S_\alpha)^{\frac{N + \alpha}{N - \alpha}} \).

Let \( \delta \to 0 \), we obtain

\[
b \geq S_\alpha^{\frac{N + \alpha}{N - \alpha}}.
\]

Employing (66) and (78), one has

\[
c \geq \left( \frac{\alpha + 2}{2(N + \alpha)} \right)^{\frac{N + \alpha}{N - \alpha}} S_\alpha^{\frac{N + \alpha}{N - \alpha}},
\]

which contradicts (58). Thus, \( \{u_n\} \) is a non-vanishing sequence. Then the result of Theorem 1.1 follows from (57) and a standard argument (e.g., [27]).
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