A Possibilistic Portfolio Model with Fuzzy Liquidity Constraint

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Investors are concerned about the reliability and safety of their capital, especially its liquidity, when investing. This paper sets up a possibilistic portfolio selection model with liquidity constraint. In this model, the asset return and liquidity are fuzzy variables which follow the normal possibility distributions. Liquidity is measured as the turnover rate of the asset. On the basis of possibility theory, we transform the model into a quadratic programming problem to obtain its solution. We illustrate that, in the process of investment, investors can make better use of capital by choosing their investment portfolios according to their expected return and asset liquidity.

1. Introduction

Recently, portfolio selection has received extensive attention [1–4]. The portfolio selection model studies how to allocate investment funds among different assets to guarantee profits and disperse investment risk. Markowitz [5] proposed a mean-variance model (MVM) for portfolio selection, which played an important role in the development of modern portfolio selection theory. The MVM uses mean and variance to describe, respectively, the expected return and risk of a portfolio. The basic rule is the investors’ trade-off between expected return and risk.

In the MVM, using the variance of return of a portfolio as a risk measure has some limitations and computational difficulties to construct large-scale portfolio. In order to overcome the difficulty, some researchers extended the MVM in various ways. Examples are the semivariance model [6], mean absolute deviations model [7], semialbsolute deviation model [8], Value-at-Risk (VaR) model [9], Conditional Value-at-Risk (CVaR) model [10], and so on. But whether there is a risk measure that is best for all portfolios is still an open problem [11]. The main reason is that each measure performs the best in its domain, but not when considered in another measure domain [12].

The MVM combines probability theory with optimization techniques to model investment behavior under uncertainty. However, Black and Litterman [13] showed that the solutions of the MVM are very sensitive to perturbations of input parameters. This means that a small uncertainty in the parameters can make the usual optimal solution practically meaningless. Therefore, it is necessary to develop models that are immune, as far as possible, to data uncertainty. Robust optimization is a good tool to solve this problem [14]. In order to systematically counter the sensitivity of the optimal portfolio to statistical and modeling errors in the estimation of relevant market parameters, Goldfarb and Iyengar [15] established a robust portfolio selection problem. They introduced “uncertainty structures” for the market parameters and transformed the robust portfolio selection problem corresponding to these uncertainty structures into a second-order cone program. Piri et al. [16] studied the robust models of the mean-Conditional Value-at-Risk (M-CVaR) portfolio selection problem, in which the mean return risk is estimated in both the interval and ellipsoidal uncertainty sets. The corresponding robust models are a linear programming problem and a quadratic programming problem, respectively. Kara et al. [17] established a robust optimization problem based on the data. In order to obtain the robust optimal solution of the
et al. [36] proposed two kinds of portfolio selection models based on upper and lower possibilistic means and variances, and the notions of upper and lower possibilistic efficient portfolios. In order to better integrate an uncertain decision environment with vagueness and ambiguity, in 2009, Zhang et al. [37] proposed a possibilistic mean-variance portfolio selection model based on the definitions of the possibilistic return and possibilistic risk. In this model, the returns of assets were fuzzy variables with LR-type possibility distributions. Based on the definitions of the upper and lower possibilistic mean, Zhang et al. [38] defined the interval-valued possibilistic mean and possibilistic variance. Based on these, a new portfolio selection model with the maximum utility was established. Li et al. [39] developed a possibilistic portfolio model under VaR and risk-free investment constraints. Liu and Zhang [40] defined the product of multiple fuzzy numbers’ possibilistic mean and variance, and, then, based on these definitions, they developed three multiperiod fuzzy portfolio optimization models. To solve these models, they applied a novel fuzzy programming approach-based self-adaptive differential evolution algorithm.

Several modifications to the basic MVM have been suggested in the literature to consider more realistic factors such as liquidity, budget, and lower/upper bound constraints, while deciding the allocation of money among assets. Liquidity refers to the ability to transact a large number of shares at prices that do not vary substantially from past prices unless new information enters the market. There are various ways of measuring liquidity, among which trading volume, number of trades, transaction amount, turnover rate, and velocity of circulation are commonly used. The turnover rate was introduced by Datar et al. [41]. The turnover rate is the total amount of traded shares divided by the total net asset value of the fund over a particular period. Marshalla and Young [42] argued that liquidity is the most important factor in a portfolio. In our model, turnover rate is controlled through fuzzy chance constraint. Fuzzy portfolio selection with chance constraint, different assumptions, and estimation methods has been discussed in the literature (see [43–46]). Barak et al. [47] developed a mean-variance-skewness fuzzy portfolio model with cardinality constraint and considered the fuzzy chance constraint to measure portfolio liquidity. Furthermore, they designed a genetic algorithm to solve the model.

In real financial markets, short-time and institutional investors hope to not only reach the expected rate of return, but also ensure that the liquidity of the portfolio should not be lower than the expected value. Moreover, in addition to return, risk, and liquidity, the threshold constraint is also the major concern for researchers and practitioners because, in order to manage the portfolio more effectively, it is necessary to limit the upper and lower bounds (threshold constraints) of the capital invested in each asset. The core of this study was inspired by Li et al. [39] and Barak et al. [47] and has been subsequently further promoted and developed by other researchers. Numerous studies have been done about possibilistic portfolio selection but a few papers consider liquidity constraint and regard asset liquidity as fuzzy variables obey the possibility distribution.
Our goal is to analyze the return-risk trade-off with liquidity constraint and threshold constraints under an uncertain market environment. To increase the applicability of the model, the return rate of assets is expressed as a fuzzy variable which is associated with a normal possibility distribution. Liquidity is measured by turnover rate and is also represented by a fuzzy variable associated with a normal possibility distribution. Thus, we established a possibilistic portfolio selection model with fuzzy chance constraints. By using the possibility theory, we transformed the chance-constrained model into a deterministic mathematical model and obtained the solution for the model.

The rest of this paper is organized as follows. In Section 2, we present some basic concepts regarding the possibility theory and the notions of the possibilistic mean and variance of a fuzzy number. At the same time, in this section, we recall the notion of normal possibility distribution and introduce a theorem about it. In Section 3, we propose a possibilistic portfolio model with liquidity constraint and threshold constraints. We suppose the expected rate of return and the turnover rate of the assets by a fuzzy distribution and introduce a theorem about it. In Section 4, we present some basic concepts regarding the possibility theory and the notions of the possibilistic mean and variance of a fuzzy number. At the same time, in this section, we recall the notion of normal possibility distribution and introduce a theorem about it. In Section 5, we provide a numerical example to illustrate the proposed approach. Section 5 concludes and provides directions for future research.

2. Preliminaries

In this section, we review some definitions and properties. Let \( \xi \) be a fuzzy variable with membership function \( \mu \), and let \( r \) be a real number. Then, the possibility of \( \xi \) is defined as follows:

\[
\text{Pos}\{\xi \geq r\} = \sup_{x \in \mathbb{R}} \mu(x). \tag{1}
\]

**Definition 1.** The upper possibilistic mean value of \( \bar{A} \) with \( \alpha \)-level set \( A_\alpha = [a(\alpha), b(\alpha)] \) is defined as

\[
M^*(\bar{A}) = \frac{\int_{0}^{1} \text{Pos}[\bar{A} \geq b(a)]b(a)da}{\int_{0}^{1} \text{Pos}[\bar{A} \geq b(a)]da} = 2 \int_{0}^{1} ab(a)da, \tag{2}
\]

where \( \text{Pos} \) denotes the possibility measure.

**Definition 2.** The lower possibilistic mean value of \( \bar{A} \) is defined as

\[
M_*(\bar{A}) = \frac{\int_{0}^{1} \text{Pos}[\bar{A} \leq a(a)]a(a)da}{\int_{0}^{1} \text{Pos}[\bar{A} \leq a(a)]da} = 2 \int_{0}^{1} aa(a)da. \tag{3}
\]

**Definition 3.** The possibilistic mean value of \( \bar{A} \) is defined as

\[
M(\bar{A}) = \frac{M_*(\bar{A}) + M^*(\bar{A})}{2} = \int_{0}^{1} a(a(\alpha) + b(\alpha))da. \tag{4}
\]

**Definition 4.** The possibilistic variance of \( \bar{A} \) is defined as

\[
\text{Var}(\bar{A}) = \int_{0}^{1} \text{Pos}[\bar{A} \leq a(a)]\left(\frac{a(a) + b(a)}{2} - a(a)\right)^2da + \int_{0}^{1} \text{Pos}[\bar{A} \geq b(a)]\left(\frac{a(a) + b(a)}{2} - b(a)\right)^2da
\]

\[
= \int_{0}^{1} a\left(\frac{a(a) + b(a)}{2} - a(a)\right)^2da + \int_{0}^{1} a\left(\frac{a(a) + b(a)}{2} - b(a)\right)^2da
\]

\[
= \frac{1}{2} \int_{0}^{1} a[b(a) - a(a)]^2da. \tag{5}
\]

**Definition 5.** The possibilistic covariance between fuzzy numbers \( \bar{A} \) and \( \bar{B} \) is defined as

\[
\text{Cov}(\bar{A}, \bar{B}) = 2 \int_{0}^{1} a(b_1(a) - a_1(a))(b_2(a) - a_2(a))da. \tag{6}
\]

**Lemma 1** (see[25]). Let \( \alpha, \beta \in \mathbb{R} \) and let \( \bar{A} \) and \( \bar{B} \) be fuzzy numbers. Then,

1. \( M(\alpha \bar{A} + \beta \bar{B}) = \alpha M(\bar{A}) + \beta M(\bar{B}) \),
2. \( \text{Var}(\alpha \bar{A} + \beta \bar{B}) = \alpha^2 \text{Var}(\bar{A}) + \beta^2 \text{Var}(\bar{B}) + 2\alpha\beta \text{Cov}(\bar{A}, \bar{B}) \).

**Definition 6.** The fuzzy variable \( \xi \) is obeying the normal distribution, if its membership function is

\[
\mu_{\xi}(x) = \exp\left\{-(x - \mu)^2 \sigma^2\right\}. \tag{7}
\]

Then, it can be written as \( \xi \sim FN(\mu, \sigma^2) \).

**Example 1.** If the fuzzy variable \( \xi \sim FN(\mu, \sigma^2) \), then its \( \lambda \)-level set is

\[
(\xi)_\lambda = [\mu - \sigma \sqrt{\lambda}^{-1}, \mu + \sigma \sqrt{\lambda}^{-1}], \quad \lambda \in (0, 1). \tag{8}
\]

In fact, from (4), we obtain

\[
M(\xi) = \frac{M_*(\xi) + M^*(\xi)}{2} = \int_{0}^{1} \lambda(a(\lambda) + b(\lambda))d\lambda
\]

\[
= \int_{0}^{1} \lambda\left(\mu + \sigma \sqrt{\lambda}^{-1}\right)d\lambda - \mu \sigma \int_{0}^{1} \sqrt{\lambda}^{-1}d\lambda
\]

\[
= \int_{0}^{1} 2\mu d\lambda = \mu. \tag{9}
\]

It follows from (5) that...
\[
\text{Var}(\xi) = \frac{1}{2} \int_0^1 \lambda \left( \mu + \sigma \sqrt{\ln \lambda^{-1}} - \mu + \sigma \sqrt{\ln \lambda^{-1}} \right)^2 d\lambda \\
= \frac{1}{2} \int_0^1 4\lambda \sigma^2 \ln \lambda^{-1} d\lambda \\
= 2\sigma^2 \int_0^1 \lambda \ln \lambda^{-1} d\lambda \\
= 2\sigma^2 \frac{1}{4} = \frac{1}{2} \sigma^2.
\]

Next, we show the distribution of \(\sum^n_{k=1} \lambda_k \xi_k\).

**Theorem 1.** If \(\xi_k, k = 1, 2, \ldots, n\), are \(n\) normally distributed fuzzy variables expressed as \(\xi_k \sim FN(\mu_k, \sigma_k^2)\) and \(\lambda_k, k = 1, 2, \ldots, n\), are \(n\) real numbers, then

\[
\sum^n_{k=1} \lambda_k \xi_k \sim FN \left( \sum^n_{k=1} \lambda_k \mu_k, \frac{1}{2} \left( \sum^n_{k=1} |\lambda_k| \sigma_k \right)^2 \right).
\]

**Proof.** According to Lemma 1 and (9), one has

\[
M \left( \sum^n_{k=1} \lambda_k \xi_k \right) = \sum^n_{k=1} \lambda_k M(\xi_k) = \sum^n_{k=1} \lambda_k \mu_k.
\]

From (6), we deduce

\[
\text{Cov}(\xi_i, \xi_j) = \frac{1}{2} \int_0^1 \alpha 2\sigma_i \sqrt{\ln \alpha^{-1}} 2\sigma_j \sqrt{\ln \alpha^{-1}} d\alpha \\
= 2\sigma_i \sigma_j \int_0^1 \alpha \ln \alpha^{-1} d\alpha \\
= 2\sigma_i \sigma_j \frac{1}{2} = \frac{1}{2} \sigma_i \sigma_j, \quad i, j = 1, 2, \ldots, n.
\]

It follows from Lemma 1 and (10) that

\[
\text{Var} \left( \sum^n_{k=1} \lambda_k \xi_k \right) = \sum^n_{k=1} \lambda_k^2 \text{Var}(\xi_k) + 2 \sum^n_{i=k} |\lambda_i| \text{Cov}(\xi_i, \xi_j) \\
= \sum^n_{k=1} \lambda_k^2 \sigma_k^2 + 2 \sum^n_{i=j} |\lambda_i| \sigma_j |\lambda_j| \\
= \frac{1}{2} \left( \sum^n_{k=1} \lambda_k^2 \sigma_k^2 + 2 \sum^n_{i=j} |\lambda_i| \sigma_j |\lambda_j| \right) \\
= \frac{1}{2} \left( \sum^n_{k=1} |\lambda_k| \sigma_k \right)^2.
\]

Thus,

\[
\sum^n_{k=1} \lambda_k \xi_k \sim FN \left( \sum^n_{k=1} \lambda_k \mu_k, \frac{1}{2} \left( \sum^n_{k=1} |\lambda_k| \sigma_k \right)^2 \right).
\]

This completes the proof. \(\square\)

### 3. Model Foundation

#### 3.1. Possibilistic Portfolio Model with Fuzzy Liquidity Constraint and Risk-Free Investment

Suppose that there are \(n\) risky assets and one risk-free asset available for investment. Let \(\tilde{r}_k\) be the return rate of asset \(k, k = 1, 2, \ldots, n\), which is a fuzzy number. Let \(x_k\) represent the proportion invested in asset \(k, k = 1, 2, \ldots, n\), and let \(r_j\) be the return of the risk-free asset. Thus, the return \(\tilde{R}\) on the portfolio can be written as

\[
\tilde{R} = \sum^n_{k=1} x_k \tilde{r}_k + r_f \left( 1 - \sum^n_{k=1} x_k \right).
\]

Obviously \(\tilde{R}\) is a fuzzy number.

To establish a new model, we need the following values:

The possibilistic mean of the portfolio return \(\tilde{R}\) is given by

\[
M(\tilde{R}) = \sum^n_{k=1} x_k M(\tilde{r}_k) + r_f \left( 1 - \sum^n_{k=1} x_k \right).
\]

The possibilistic variance of \(\tilde{R}\) is written as

\[
\text{Var}(\tilde{R}) = \sum^n_{k=1} x_k^2 \text{Var}(\tilde{r}_k) + 2 \sum^n_{i,j=1 \atop i \neq j} x_i x_j \text{Cov}(\tilde{r}_i, \tilde{r}_j)
\]

Now, we can establish the following possibilistic portfolio selection model with fuzzy liquidity constraint:

\[
\begin{align*}
\min \text{Var}(\tilde{R}) = & \sum^n_{k=1} x_k^2 \text{Var}(\tilde{r}_k) + 2 \sum^n_{i,j=1 \atop i \neq j} x_i x_j \text{Cov}(\tilde{r}_i, \tilde{r}_j), \\
\text{s.t.} & \sum^n_{k=1} x_k M(\tilde{r}_k) + r_f \left( 1 - \sum^n_{k=1} x_k \right) \geq \mu, \\
& \text{Pos} \left[ \sum^n_{k=1} l_k x_k \geq l_0 \right] \geq 1 - \alpha, \\
& \sum^n_{k=1} x_k \leq 1, \\
& 0 \leq d_k \leq x_k \leq g_k, k = 1, 2, \ldots, n,
\end{align*}
\]

where \(\mu\) is the underestimated expected return rate and \(l_0\) is the predetermined value. \(d_k\) and \(g_k\) represent, respectively, the lower and the upper bounds on investment in asset \(k, k = 1, 2, \ldots, n\). \(l_k\) is the turnover rate of asset \(k, k = 1, 2, \ldots, n\), which reflects the liquidity of the asset. \(\alpha\) reflects the sensitivity of the investor. If the value of \(\alpha\) is close to 0, then the investor is sensitive to liquidity of the portfolio. Otherwise, the investor is not sensitive to the portfolio’s liquidity.

In this section, we suppose that the return rate of asset \(k, k = 1, 2, \ldots, n\), is a normally distributed fuzzy variable denoted as \(\tilde{r}_k \sim FN(\mu_k, \sigma_k^2)\), and its membership function is

\[
A_{\tilde{r}_k}(x) = \exp \left\{ -\left( \frac{x - \mu_k}{\sigma_k} \right)^2 \right\}.
\]
The turnover rate of asset \( k, k = 1, 2, \ldots, n \), is also a normally distributed fuzzy variable denoted as \( \bar{I}_k \sim FN(a_k, b_k^2) \). So, its membership function is
\[
A_{\bar{I}_k}(x) = \exp \left\{ -\left( \frac{x - a_k}{b_k} \right)^2 \right\}.
\] (21)

From Theorem 1, we obtain
\[
\bar{R} \sim FN \left( \sum_{k=1}^{n} x_k \mu_k + r_f, \left( 1 - \sum_{k=1}^{n} x_k \right) \frac{1}{2} \left( \sum_{k=1}^{n} x_k \sigma_k \right)^2 \right),
\]
and
\[
\sum_{k=1}^{n} \bar{I}_k x_k \sim FN \left( \sum_{k=1}^{n} x_k a_k, \frac{1}{2} \left( \sum_{k=1}^{n} x_k b_k \right)^2 \right).
\] (22)

Then, the membership function of \( \sum_{k=1}^{n} \bar{I}_k x_k \) is defined by
\[
A(u) = \exp \left\{ -\frac{(u - \sum_{k=1}^{n} x_k a_k)^2}{(1/2) \left( \sum_{k=1}^{n} x_k b_k \right)^2} \right\}.
\] (23)

Furthermore,
\[
\text{Pos} \left[ \sum_{k=1}^{n} \bar{I}_k x_k \geq l_0 \right] = \sup_{u \geq l_0} \exp \left\{ -\frac{(u - \sum_{k=1}^{n} x_k a_k)^2}{(1/2) \left( \sum_{k=1}^{n} x_k b_k \right)^2} \right\}
\]
\[
= \begin{cases} 
\exp \left\{ \frac{(l_0 - \sum_{k=1}^{n} x_k a_k)^2}{(1/2) \left( \sum_{k=1}^{n} x_k b_k \right)^2} \right\}, & \text{for } \sum_{k=1}^{n} x_k a_k < l_0, \\
1, & \text{for } \sum_{k=1}^{n} x_k a_k \geq l_0.
\end{cases}
\] (24)

From (19) and (24), we have
\[
-\left( l_0 - \sum_{k=1}^{n} a_k x_k \right)^2 \geq \frac{1}{2} \ln (1 - \alpha) \left( \sum_{k=1}^{n} x_k b_k \right)^2.
\] (25)

From (22) and (25), (PL1) can be transformed into
\[
\begin{aligned}
\min \text{Var}(\bar{R}) &= \frac{1}{2} \left( \sum_{k=1}^{n} x_k \sigma_k \right)^2, \\
\text{s.t.} \sum_{k=1}^{n} x_k (\mu_k - r_f) + r_f \geq \mu, \\
-\left( l_0 - \sum_{k=1}^{n} a_k x_k \right)^2 &\geq \frac{1}{2} \ln (1 - \alpha) \left( \sum_{k=1}^{n} x_k b_k \right)^2, \\
\sum_{k=1}^{n} x_k &\leq 1, \\
0 &\leq d_k \leq x_k \leq g_k, \quad k = 1, 2, \ldots, n.
\end{aligned}
\] (PL1)

3.2. Possibilistic Portfolio Model without Risk-Free Investment.
Similar to the previous section, we propose a portfolio selection model without risk-free investment as follows:
\[
\begin{aligned}
\min \text{Var}(\bar{R}) &= \frac{1}{2} \left( \sum_{k=1}^{n} x_k \sigma_k \right)^2, \\
\text{s.t.} \sum_{k=1}^{n} x_k \mu_k \geq \mu, \\
-\left( l_0 - \sum_{k=1}^{n} a_k x_k \right)^2 &\geq \frac{1}{2} \ln (1 - \alpha) \left( \sum_{k=1}^{n} x_k b_k \right)^2, \\
\sum_{k=1}^{n} x_k &= 1, \\
0 &\leq d_k \leq x_k \leq g_k, \quad k = 1, 2, \ldots, n.
\end{aligned}
\] (NPL2)

(NPL1) and (NPL2) are quadratic programming problems and can be solved by MATLAB, Lingo, etc.

4. Numerical Example
In this section, we give a real portfolio example to illustrate our approach. In this example, we selected eight stocks from the Shanghai Stock Exchange. We collected data on monthly returns and turnover rate for each of the eight stocks over the period of January 2006 to December 2006 from the RESSET Financial Research Database. We use SPSS to generate the frequency distributions of the monthly returns and turnover rates.

Table 1 shows the transaction codes and the frequency distributions of the monthly returns, while Table 2 shows the transaction codes and the frequency distributions of the monthly turnover rates.
Let the risk-free asset be a bank deposit. We use the three-month deposit interest rates as the return on risk-free assets. So, we get the return on risk-free assets $\text{rf} = 2.8\%$. If the possibility of the portfolio’s turnover rate must be more than 0.002 (in more than 90% times); that is, $l_0 \geq 0.2$ (\%), and $\alpha = 10\%$, then the lower bound of investment ratio $x_k$ must be $d = [0.02, 0, 0.1, 0, 0.02, 0.1, 0.1, 0.05]$ and the upper bound $g = [0.3, 0.2, 0.3, 0.2, 0.2, 0.4, 0.4, 0.3]$. By solving the model (NPL1) and (NPL2), the possibilistic efficient portfolios for the different $\mu$s are obtained as shown in Tables 3 and 4.

From these two tables we can see that the risk increases as $\mu$ increases. We also can see that when the value of $\mu$ is low (e.g., $\mu = 3\%$), the proportion that invests is low. For example, when the value of $\mu$ is equal to 3\%, the investment proportion with risk-free is the lower bound but the investment proportion without risk-free is 0.02, 0.0702, 0.1, 0.02, 0.1, 0.1, 0.05 and the upper bound $g = [0.3, 0.2, 0.3, 0.2, 0.2, 0.4, 0.4, 0.3]$. By solving the model (NPL1) and (NPL2), the possibilistic efficient portfolios for the different $\mu$s are obtained as shown in Tables 3 and 4.

From these two tables we can see that the risk increases as $\mu$ increases. We also can see that when the value of $\mu$ is low (e.g., $\mu = 3\%$), the proportion that invests is low. For example, when the value of $\mu$ is equal to 3\%, the investment proportion with risk-free is the lower bound but the investment proportion without risk-free is 0.02, 0.0702, 0.1, 0.02, 0.1, 0.1, 0.05 and the upper bound $g = [0.3, 0.2, 0.3, 0.2, 0.2, 0.4, 0.4, 0.3]$. By solving the model (NPL1) and (NPL2), the possibilistic efficient portfolios for the different $\mu$s are obtained as shown in Tables 3 and 4.

Analogous to the above, if the turnover rate of the portfolio must be more than 0.002, i.e., $l_0 = 0.2$ (\%), and the expected rate of return must reach 8\% or more, then the lower and upper bound of investment ratio $x_k$ should be $d = [0.02, 0, 0.1, 0, 0.02, 0.1, 0.1, 0.05]$ and $g = [0.3, 0.2, 0.3, 0.2, 0.2, 0.4, 0.4, 0.3]$, respectively. By solving the model (NPL1) with different investor sensitivity parameter values, we obtain the possibilistic efficient portfolios in Table 5. The relationship between investor sensitivity and the portfolio risk is shown in Figure 4.


equation

Table 1: The possibility distribution of returns.

| Stock code | $\mu_k$ | $\sigma_k$ |
|------------|---------|------------|
| 600058     | 0.06    | 0.167      |
| 600028     | 0.09    | 0.102      |
| 600089     | 0.10    | 0.207      |
| 600115     | 0.04    | 0.111      |
| 600170     | 0.03    | 0.049      |
| 600495     | 0.10    | 0.080      |
| 600526     | 0.01    | 0.086      |
| 600662     | 0.03    | 0.076      |

Table 2: The possibility distribution of turnover rates (%).

| Stock code | $a_k$ | $b_k$ |
|------------|-------|-------|
| 600058     | 40.70 | 31.762|
| 600028     | 1.77  | 0.802 |
| 600089     | 32.51 | 10.681|
| 600115     | 9.75  | 4.384 |
| 600170     | 14.86 | 8.796 |
| 600495     | 73.85 | 51.658|
| 600526     | 65.47 | 16.565|
| 600662     | 40.14 | 19.587|

Figure 1 shows these two efficient portfolios. From Figure 1, we can see that when the risk-free investment is included in the portfolio, the risk is lower at the same value of $\mu$. And from Figure 1, we also can draw the conclusion that the risk of the portfolios can be spread by diversification. The impact of liquidity constraint on the portfolio is shown in Figure 2. Figure 2 shows some possibilistic efficient portfolios with and without liquidity constraints.

It can be seen from Figure 2 that, under the same expected return, those investors who have requirements for financial liquidity need to take on greater risks. In other words, if investors want to keep the risks they take constant, the demand for financial liquidity will reduce the expected return of the portfolio. In addition, the impact of liquidity floor $l_0$ on the portfolio is shown in Figure 3. Figure 3 shows some possibilistic efficient portfolios with different $l_0$. The impact of $l_0$ on the portfolio depends on the slope and the position of the $l_0$ line and the position of the possibilistic efficient frontier.

As shown in Figure 3, when the position of the possibilistic efficient frontier and the slope of the $l_0$ line remain unchanged, the intercept changes of the $l_0$ line will affect the optimal solution of the model. As $l_0$ moves from “a” point to “b” point, the optimal solution of the model would not change. But when $l_0$ moves from “a” point to “c” point, this change would affect the optimal solution of the model. From Figure 4, we can see that an increase in investor sensitivity is associated with a decrease in the portfolio risk, and when investor sensitivity reaches a certain threshold, the portfolio risk no longer reduces. That is to say, in this case, if investors seek at least 8\% of the profits, regardless of the investors’ preference for liquidity, the minimum risk of the portfolio is 0.5126 (or 51.26\%).

The impact of liquidity sensitivity $\alpha$ on the portfolio is shown in Figure 5. Figure 5 shows some possibilistic efficient portfolios with different $\alpha$. In Figure 5, when the position of the possibilistic efficient frontier and the intercept of the $\alpha$ line remain unchanged, the slope changes of the $\alpha$ line will affect the optimal solution of the model. As $\alpha$ moves from “a” point to “b” point, the optimal solution of the model

Complexity
Table 3: Possibilistic efficient portfolios with risk-free investment for the different μs with $l_0 = 0.2\%$ and $\alpha = 10\%$.

| μ (%) | 600058 | 600028 | 600089 | 600115 | 600170 | 600495 | 600526 | 600662 | Risk | $\sum_{k=1}^{8} x_k$ |
|-------|--------|--------|--------|--------|--------|--------|--------|--------|------|----------------|
| 3     | 0.1    | 0.02   | 0.1    | 0      | 0.02   | 0.1    | 0.1    | 0.05   | 0.0901| 0.39 |
| 5     | 0.1    | 0.02   | 0.1    | 0      | 0.02   | 0.2330 | 0.1    | 0.05   | 0.1441| 0.523|
| 6.5   | 0.1    | 0.0717 | 0.1    | 0      | 0.02   | 0.3737 | 0.1    | 0.05   | 0.2654| 0.7354|
| 7     | 0.1    | 0.1521 | 0.1    | 0      | 0.02   | 0.3717 | 0.1    | 0.05   | 0.3287| 0.8138|
| 7.5   | 0.1    | 0.2    | 0.1497 | 0      | 0.02   | 0.482  | 0.1    | 0.05   | 0.4408| 0.8879|
| 8     | 0.1    | 0.2    | 0.2725 | 0      | 0.02   | 0.2929 | 0.1    | 0.05   | 0.6485| 0.9554|
| 8.12  | 0.1    | 0.2    | 0.3    | 0.0197 | 0.02   | 0.2779 | 0.1    | 0.05   | 0.7252| 0.9876|

Table 4: Possibilistic efficient portfolios without risk-free investment for the different μs with $l_0 = 0.2\%$ and $\alpha = 10\%$.

| μ (%) | 600058 | 600028 | 600089 | 600115 | 600170 | 600495 | 600526 | 600662 | Risk | $\sum_{k=1}^{8} x_k$ |
|-------|--------|--------|--------|--------|--------|--------|--------|--------|------|----------------|
| 3     | 0.2    | 0.0702 | 0.1    | 0      | 0.2    | 0.1    | 0.2098 | 0.3    | 0.3607| 1    |
| 5     | 0.2    | 0.0760 | 0.1    | 0      | 0.2    | 0.1547 | 0.1493 | 0.3    | 0.3640| 1    |
| 6.5   | 0.2    | 0.1554 | 0.1    | 0      | 0.2    | 0.2868 | 0.1    | 0.1378 | 0.4004| 1    |
| 7     | 0.2    | 0.1906 | 0.1    | 0      | 0.2    | 0.3281 | 0.1    | 0.0613 | 0.4165| 1    |
| 7.5   | 0.2    | 0.2    | 0.1817 | 0      | 0.1386 | 0.3097 | 0.1    | 0.05   | 0.5309| 1    |
| 8     | 0.2    | 0.2    | 0.2853 | 0      | 0.0671 | 0.2776 | 0.1    | 0.05   | 0.6910| 1    |
| 8.12  | 0.2    | 0.2    | 0.3    | 0.03   | 0.024  | 0.276  | 0.1    | 0.05   | 0.7252| 1    |

Figure 1: Variation of portfolio risks with $\mu$.

Figure 2: Some possibilistic efficient portfolios with and without liquidity constraint.
would not change. But when \( \alpha \) moves from “a” point to “c” point, this change would affect the optimal solution of the model.

5. Conclusions and Directions for Future Research

In this paper, we proposed a possibilistic portfolio model, which is different from the MVM proposed by Markowitz. Unlike the MVM, we measured the liquidity of asset as the turnover rate. Besides, we assumed that the expected rate of returns and turnover rates of assets are fuzzy variables, which follow the normal possibility distribution. We, then, applied the possibility theory and fuzzy chance constraint to consider the turnover rate and obtained the solution for the model. Furthermore, we illustrated our proposed effective approaches for the portfolio construction using numerical examples. In these examples, we analyzed the risk between the portfolios with and without risk-free constraints. We also analyzed changes in the portfolio’s risk with respect to

**Table 5: Possibilistic efficient portfolios with risk-free investment for the different \( \alpha \) values with \( l_0 = 0.2 \% \) and \( \mu = 8 \% \).**

| \( \alpha \) (%) | 600058 | 600028 | 600089 | 600115 | 600170 | 600495 | 600526 | 600662 | Risk | \( \sum_{k=1}^{8} x_k \) |
|------------------|--------|--------|--------|--------|--------|--------|--------|--------|------|------------------|
| 0.1              | 0.02   | 0.2    | 0.2725 | 0      | 0.02   | 0.2929 | 0.1    | 0.05   | 0.6485 | 0.9554         |
| 0.2              | 0.02   | 0.2    | 0.2544 | 0      | 0.02   | 0.3110 | 0.1    | 0.05   | 0.6241 | 0.9564         |
| 0.3              | 0.02   | 0.2    | 0.2315 | 0      | 0.02   | 0.3339 | 0.1    | 0.05   | 0.5942 | 0.9605         |
| 0.4              | 0.02   | 0.2    | 0.2012 | 0      | 0.02   | 0.3642 | 0.1    | 0.05   | 0.5560 | 0.9554         |
| 0.5              | 0.02   | 0.2    | 0.1654 | 0      | 0.02   | 0.4    | 0.1    | 0.05   | 0.5126 | 0.9554         |
| 0.6              | 0.02   | 0.2    | 0.1654 | 0      | 0.02   | 0.4    | 0.1    | 0.05   | 0.5126 | 0.9554         |

**Figure 3: Some possibilistic efficient portfolios with different \( l_0 \).**

**Figure 4: Variation of portfolio risk with investor sensitivity \( \alpha \).**

**Figure 5: Some possibilistic efficient portfolios with different \( \alpha \).**
changes in investor sensitivity. Finally, we drew a conclusion that our method can play a leading role in financial markets.

Future research can extend our model in the following ways. Firstly, researchers can use other methods to solve the problem and compare the results. Secondly, our fuzzy portfolio model can be extended to a multiperiod case. Thirdly, our model can be considered in a fuzzy random environment. Finally, with the development of behavioral finance, investor behavior has received more attention. We believe that investor behavior can be considered in fuzzy portfolio theory.

**Data Availability**

The data used to support the findings of this study are available from the corresponding author upon request.

**Disclosure**

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**Conflicts of Interest**

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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