Polylogarithm identities in a conformal field theory in three dimensions

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The $N = \infty$ vector $O(N)$ model is a solvable, interacting field theory in three dimensions ($D$). In a recent paper with A. Chubukov and J. Ye [1], we have computed a universal number, $\tilde{c}$, characterizing the size dependence of the free energy at the conformally-invariant critical point of this theory. The result [1] for $\tilde{c}$ can be expressed in terms of polylogarithms. Here, we use non-trivial polylogarithm identities to show that $\tilde{c}/N = 4/5$, a rational number; this result is curiously parallel to recent work on dilogarithm identities in $D = 2$ conformal theories. The amplitude of the stress-stress correlator of this theory, $c$ (which is the analog of the central charge), is determined to be $c/N = 3/4$, also rational. Unitary conformal theories in $D = 2$ always have $c = \tilde{c}$; thus such a result is clearly not valid in $D = 3$. 
Consider a conformally-invariant field theory in $D$ dimensions. Place it in a slab which is infinite in $D - 1$ dimensions, but of finite length $L$ in the remaining direction. Impose periodic boundary conditions along this finite direction. An old result of Fisher and de Gennes \cite{2} states that if hyperscaling is valid, the free energy density $F = -\log Z/V$ ($Z$ is the partition function and $V$ is the total volume of the slab) satisfies

$$F = F_\infty - \frac{\Gamma[D/2] \zeta(D)}{\pi^{D/2}} \frac{\bar{c}}{L^D}. \quad (1)$$

Here $F_\infty$ is the free energy density in the infinite system and $\bar{c}$ is a universal number. The coefficient of $1/L^D$ has been chosen such that $\bar{c} = 1$ for a single component, massless free scalar field theory. A similar parametrization has also been discussed recently by Castro Neto and Fradkin \cite{3}.

A second universal number characterizing the conformal theory is the amplitude of the two-point correlation function of the stress tensor $T_{\mu\nu}$ in infinite flat space. Cardy has proposed a normalization convention for $T_{\mu\nu}$ in arbitrary dimensions \cite{4}, and shown that its two-point correlation is of the following form

$$\langle T_{\mu\nu}(r)T_{\lambda\sigma}(0) \rangle = \frac{c}{r^{2D}} \left[ \left( \delta_{\mu\lambda} - \frac{2r_{\mu}r_{\lambda}}{r^2} \right) \left( \delta_{\nu\sigma} - \frac{2r_{\nu}r_{\sigma}}{r^2} \right) + \left( \delta_{\mu\sigma} - \frac{2r_{\mu}r_{\sigma}}{r^2} \right) \left( \delta_{\nu\lambda} - \frac{2r_{\nu}r_{\lambda}}{r^2} \right) - \frac{2}{D} \delta_{\mu\nu} \delta_{\lambda\sigma} \right]. \quad (2)$$

This defines a universal amplitude, $c$, which is the analog of the central charge in $D = 2$ conformal field theories. A key property of $D = 2$ unitary conformal field theories is $\bar{c} = c$ \cite{5}. The generalization of this result to arbitrary $D$, and in particular $D = 3$, remains an important open problem.

It would clearly be interesting to obtain results for $\bar{c}$ and $c$ for specific models in dimensions other than $D = 2$. In a recent paper by A. Chubukov, myself and J. Ye \cite{1} on the critical properties of two-dimensional quantum antiferromagnets, the value of $\bar{c}$ was computed for
the vector $O(N)$ model in $D = 3$ in a $1/N$ expansion. In this note we highlight some features of the computation of $\tilde{c}$ at $N = \infty$, as we believe the results may be of interest to a broader audience of conformal field theorists. The result for $\tilde{c}$ at $N = \infty$ can be expressed in terms of di- and trilogarithm functions. Below, we use some known polylogarithm identities to simplify the result for $\tilde{c}$. The appearance of these identities is surprisingly parallel to recent work establishing a connection between dilogarithmic identities and the rational central charge of $D = 2$ conformal theories. We will also compute the value of $c$ at $N = \infty$ in the $D = 3$ $O(N)$ model.

We consider the field theory with the action

$$S = \frac{N}{2g} \int d^3x (\partial n)^2$$

where $n$ is a $N$-component real vector of unit length, $n^2 = 1$. The fixed length constraint is actually not crucial and identical universal properties can be obtained in a soft-spin theory with a $n^4$ interaction term. The theory has to be suitably regulated in the ultraviolet by a momentum cutoff $\Lambda$. It becomes conformally invariant at a critical value $g = g_c = \alpha/\Lambda$ which separates the $g < g_c$ Goldstone phase with broken $O(N)$ invariance, from the $g > g_c$ massive phase. The location of the critical point, $\alpha$, is of course non-universal and will depend upon the cutoff scheme.

The formal structure of the $N \to \infty$ limit is quite standard. The fixed length constraint is imposed by an auxiliary field $\lambda$. After integrating out the $n$ field, the $N = \infty$ theory is given by the saddle point of the resulting functional integral. In this manner we find

$$\frac{\mathcal{F}}{N} = \frac{1}{2} \text{Tr} \log(-\partial^2 + m^2) - \frac{m^2}{2g}$$

where $m^2$ is the saddle-point value of $\lambda$. The critical point is at $g = g_c$, where

$$\frac{1}{g_c} = \int \frac{d^3p}{8\pi^2} \frac{1}{p^2}.$$
and \( m^2 = 0 \) in the infinite volume system. In the slab with thickness \( L \), however, we find at \( g = g_c \) that

\[
m = m_L = \frac{2 \log \tau}{L},
\]

(6)

where \( \tau = (\sqrt{5} + 1)/2 \) is the golden mean.

To compute \( \tilde{c} \), we now need to evaluate the \( \text{Tr log} \) in (4) in the slab geometry. The momentum along the finite direction is quantized in integer multiples of \( 2\pi/L \). The summation over these discrete modes can be accomplished with the identity

\[
\lim_{M \to \infty} \left[ \frac{1}{L} \sum_{n=-M}^{M} \log \left( \frac{4\pi^2 n^2}{L^2} + a^2 \right) - \int_{-2\pi M/L}^{2\pi M/L} \frac{d\omega}{2\pi} \log(\omega^2 + a^2) \right] = \frac{2}{L} \log \left( 1 - e^{-L|a|} \right),
\]

(7)

where \( a \) is any constant. The expression for \( F \) in the slab of width \( L \) at \( g = g_c \) is then easily shown to be

\[
\frac{F}{N} = \frac{1}{L} \int \frac{d^2 k}{4\pi^2} \log \left( 1 - e^{-L\sqrt{m_L^2 + k^2}} \right) + \frac{1}{2} \int \frac{d^3 p}{8\pi^3} \left[ \log \left( p^2 + m_L^2 \right) - \frac{m_L^2}{p^2} \right] - \frac{1}{2} \int_{-L}^{L} \frac{d \omega}{2\pi} \log(\omega^2 + a^2)
\]

(8)

The second integral is of course badly divergent in the ultraviolet. All divergences however disappear after the infinite volume result has been subtracted, in which case

\[
\frac{F - F_\infty}{N} = \frac{1}{L} \int \frac{d^2 k}{4\pi^2} \log \left( 1 - e^{-L\sqrt{m_L^2 + k^2}} \right) + \frac{1}{2} \int \frac{d^3 p}{8\pi^3} \left[ \log \left( p^2 + m_L^2 \right) - \frac{m_L^2}{p^2} \right] - \frac{1}{2} \int_{-L}^{L} \frac{d \omega}{2\pi} \log(\omega^2 + a^2)
\]

(9)

These integrals can be expressed in terms of polylogarithms. We will skip the straightforward intermediate steps and present our final result for \( \tilde{c} \) obtained from (9) and (1)

\[
(1/N) \text{Li}_3(1) \tilde{c} = \text{Li}_3(2 - \tau) - \log(2 - \tau) \text{Li}_2(2 - \tau) - \frac{1}{6} \log^3(2 - \tau)
\]

(10)

where \( 2 - \tau = 1/\tau^2 = (3 - \sqrt{5})/2 \), and the polylogarithm function is defined by analytic continuation of the series

\[
\text{Li}_p(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^p}.
\]

(11)

Note \( \text{Li}_p(1) = \zeta(p) \).
Remarkably, it turns out that $2 - \tau$ is one of only three real, positive, $z$ for which both $\text{Li}_2(z)$ and $\text{Li}_3(z)$ can be expressed in terms of elementary functions [8] (the other points are $z = 1$ and $z = 1/2$). As shown in the book by Lewin [8], the value of $\text{Li}_2(2 - \tau)$ follows from a combined analysis of the following identities

\begin{align*}
\text{Li}_2(z) + \text{Li}_2(1 - z) &= \frac{\pi^2}{6} - \log z \log(1 - z) \\
\text{Li}_2(z) + \text{Li}_2 \left( \frac{-z}{1 - z} \right) &= -\frac{1}{2} \log^2(1 - z) \\
\frac{1}{2} \text{Li}_2(z^2) + \text{Li}_2 \left( \frac{-z}{1 - z} \right) - \text{Li}_2(-z) &= -\frac{1}{2} \log^2(1 - z)
\end{align*}

(12)

To see the special role of the golden mean in these identities, note that two of the arguments $z^2$ and $-z/(1 - z)$ coincide when $z^2 - z - 1 = 0$. The solutions of this are $z = \tau, 1 - \tau$. It is not difficult to show that the above identities evaluated at $z = 2 - \tau, \tau - 1$, and $1 - \tau$, can be combined to uniquely determine $\text{Li}_2(2 - \tau)$ [8]:

\begin{align*}
\text{Li}_2(2 - \tau) &= \frac{\pi^2}{15} - \frac{1}{4} \log^2(2 - \tau) 
\end{align*}

(13)

Similarly, the identities [8]

\begin{align*}
\frac{1}{4} \text{Li}_3(z^2) &= \text{Li}_3(z) + \text{Li}_3(-z) \\
\text{Li}_3(z) + \text{Li}_3 \left( \frac{-z}{1 - z} \right) + \text{Li}_3(1 - z) &= \text{Li}_3(1) + \frac{\pi^2}{6} \log(1 - z) \\
&\quad - \frac{1}{2} \log z \log^2(1 - z) + \frac{1}{6} \log^3(1 - z)
\end{align*}

(14)

evaluated at $z = 2 - \tau$ and $\tau - 1$ yield [8]

\begin{align*}
\text{Li}_3(2 - \tau) &= \frac{4}{5} \text{Li}_3(1) + \frac{\pi^2}{15} \log(2 - \tau) - \frac{1}{12} \log^3(2 - \tau)
\end{align*}

(15)

Inserting (13) and (15) into (10), we get one of our main results

\begin{align*}
\frac{\tilde{c}}{N} &= \frac{4}{5}
\end{align*}

(16)
Surprisingly, $\tilde{c}/N$ has turned out to be a rational number, although none of the intermediate steps suggested that this might be the case. Interestingly, this phenomenon is similar to that in recent determinations of $\tilde{c}$ from the size dependence of $\mathcal{F}$ in $D = 2$ conformal theories [3, 4]. There, the free energy was determined from integrable lattice models, or by evaluating the characters of a representation of the Virasoro algebra; in both cases the result was obtained in terms of dilogarithm sums, which thus must equal the rational central charge.

We turn next to the determination of $c$ for the $D = 3$, $N = \infty$ vector $O(N)$ model. The stress tensor $T_{\mu\nu}$ for (3) is

$$T_{\mu\nu} = \frac{4\pi}{g} \left( \partial_\mu n \partial_\nu n - \frac{\delta_{\mu\nu}}{2} (\partial n)^2 \right) - \delta_{\mu\nu} t$$

where $t$ is a cutoff-dependent subtraction needed to make $T_{\mu\nu}$ a proper scaling variable at the critical point. The general structure of these subtractions in the $1/N$ expansion for arbitrary operators in the $O(N)$ model with a hard momentum cutoff has been discussed by Ma [9]. Here, we simply note that dimensional regularization of the loop integrals in the vicinity of $D = 3$ leads to $t = 0$ at $N = \infty$. We evaluated $\langle T_{\mu\nu}(r)T_{\lambda\sigma}(0) \rangle$ at $N = \infty$ in the infinite system using dimensional regularization. There are two Feynman graphs which contribute at this order [3], including one involving fluctuation of the auxiliary field, $\lambda$, which imposed the fixed length constraint. The loop integrals are quite tedious, but straightforward. We found that our final result was indeed consistent with (2) with

$$\frac{c}{N} = \frac{3}{4}$$

Note that $c \neq \tilde{c}$, unlike $D = 2$. Instead, we have $c/\tilde{c} = 15/16$ in this theory.

We emphasize that all of the results of this paper are special to $D = 3$; $\tilde{c}$ can also be computed for general $D$, but the results simplify only in $D = 3$. The major question raised by this work is, of course, whether $\tilde{c}$ and $c$ have any of these special properties at finite $N$ in
$D = 3$. It would also be interesting to obtain the simple $D = 3$ results for $c$ and $\tilde{c}$ at $N = \infty$ by algebraic methods.

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References

[1] A.V. Chubukov, S. Sachdev and J. Ye, paper 9304046 on cond-mat@babbage.sissa.it.

[2] M.E. Fisher and P.-G. de Gennes, C.R. Acad. Sci. Ser. B 287, 207 (1978); V. Privman and M.E. Fisher, Phys. Rev. B 30, 322 (1984).

[3] A.H. Castro Neto and E. Fradkin, paper 9301009 on cond-mat@babbage.sissa.it

[4] J.L. Cardy, Nucl. Phys. B290, 355 (1987).

[5] H.W.J. Blote, J.L. Cardy and M.P. Nightingale, Phys. Rev. Lett. 56, 742 (1986); I. Affleck, Phys. Rev. Lett. 56, 746 (1986).

[6] V.V. Bazhanov and N. Yu. Reshetikhin, Int. J. Mod. Phys. A 4, 115 (1989); A.N. Kirillov, J. Sov. Math. 47 2450 (1989); T.R. Klassen and E. Melzer, Nucl. Phys. B 338, 485 (1990) and 370, 511 (1992); A. Klumper and P.A. Pearce, J. Stat. Phys. 64, 13 (1991); F. Ravanini, Phys. Lett. B 282, 73 (1992); W. Nahm, A. Recknagel, and M. Terhoeven, paper 9211034 on hep-th@xxx.lanl.gov.

[7] E. Brezin and J. Zinn-Justin, Phys. Rev. B 14, 3110 (1976).

[8] Polylogarithms and Associated Functions, by L. Lewin, North Holland, New York (1981).

[9] S.-k. Ma, Phys. Rev. 10, 1818 (1974).