Joint Distribution of the Wave Crest and Its Associated Period for Nonlinear Random Waves of Finite Bandwidth

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Abstract: The theoretical treatment of statistical properties relevant to nonlinear random waves of finite bandwidth, such as the joint distribution of wave crest and its associated wave period, is an overdue task hampered by the complicated form of the analytical model for sea surface elevation. In this study, we first derive the wave crest distribution based on the simplified version of the Longuet-Higgins’ wave model and proceed to derive the joint distribution of the wave crest and its associated period, and the conditional wave period distribution with a given wave crest, which are of great engineering value. It is shown that the bandwidth of the wave spectrum has a significant influence on the crest distribution, and the significant wave crest is getting larger in an increasing manner as nonlinearity is increased as expected. It also turns out that the positive correlation of wave crest height with its associated period is extended to more massive waves as nonlinearity is enhanced contrary to the general perception in the coastal engineering community that the wave crest is a statistically independent random process with wave period over large waves. The peak period decreases due to the destructive interference of second-order free harmonics.

Keywords: nonlinear random waves of finite bandwidth; joint distribution of wave crest and its associated period; Wallops spectrum; conditional wave period distribution; significant wave slope

1. Introduction

Offshore structures have to be designed robust enough to survive harsh environmental conditions and to secure sufficient safety against fatigue failures as well. In addition, offshore structures are often exposed to flow induced vibrations, which significantly worsen the durability of offshore structures against fatigue. As a result, an interval between two successive wave loads that comprise most of the environmental loads is a very critical design factor. Despite its great engineering value, the probability distribution of wave periods has received less attention than that of wave crests to the point that even the effect of nonlinearities on wave periods has not been fully explored [1]. This lack of attention is caused by the absence of an analytical model for the wave period, even under the assumption that waves are linear and narrow banded. To overcome these difficulties, the probability distribution of wave periods has been derived as a marginal distribution from the empirical joint distribution of wave crests and periods. The earlier efforts were made by Longuet-Higgins [2,3] based on the assumption of a narrow banded spectral density function. These Longuet-Higgins models [2,3] are still frequently referred to even though their application should be limited to narrow banded waves, and the widespread perception such that the wave crest is a statistically independent random process with wave period over large waves [4] can be attributed to these Longuet-Higgins models [2,3].

On the other hand, a variety of empirical, semi-empirical, and theoretical wave crest models have been proposed [5–15].
Among these, empirical models have been the most preferred ones in the design practice, in which random waves are assumed to follow a standard probability distribution such as the Gaussian, Weibull, and Rayleigh, and the unknown parameters are estimated empirically using sample data [16]. Even though the popularity of empirical models is understandable, considering its simple structural form, which makes its practical applications more amenable, it also has flaws such that the structural form of empirical models and the parameters hardly bear any physical meaning.

On the contrary, in the theoretical models, the structural form of probability distribution and the model parameters are derived based on an analytical model that approximates the physics underlying the ocean waves. For nonlinear random waves of arbitrary bandwidth, Longuet-Higgins [17] gave a second-order analytical model for sea surface elevation using the perturbation method, the complex form of which has impeded further development of the theoretical study of statistical properties relevant to nonlinear random waves. In order to avoid these difficulties, it has also been assumed in the past studies that the linear components of sea surface elevation are narrow-banded, and Rayleigh distributed, under which sea surface elevation is known to take the familiar form of an amplitude-modulated Stokes wave with a mean frequency and random phase to the second-order [18]. For example, Tayfun [15,18,19] utilized the second-order Stokes wave and the quadratic transformation of random variables to derive the probability distribution of wave crests and shows that nonlinearities have significant effects, and wave crests are non-Rayleigh distributed. However, despite its advantage over empirical models to the point that the probability distribution and model parameters reflect some physical insight, the theoretical Rayleigh-Stokes model by Tayfun [15,18,19] has not been widely accepted in the design practice since its functional form is more complex [20]. Moreover, the assumption made in its derivation, such as narrow banded, made its application difficult since narrow-banded waves are rare in the real ocean.

With high-quality data from full-scale measurement being increasingly available, there are many efforts to develop the semi-empirical models by correcting the flaw of empirical models such as its lack of physical insight into the waves. Similar to theoretical models, the structural form of the semi-empirical model is derived based on an analytical model for waves. The model parameters are estimated empirically from the sample data using moment-based parameter estimation methods [20]. Therefore, the underlying structural form and its parameters bear some physical insight [21]. In the hierarchy of semi-empirical models, the theoretical model by Tayfun [15,18,19] can be classified as a one-parameter Rayleigh-Stokes model. However, the structural form in most of the semi-empirical models is of the second-order Stokes wave first introduced by Tayfun [18], and hence, its application should be limited to narrow-banded waves. For example, Izadparast and Niedzwecki [21] in their study of a probability distribution of ocean wave power based on a three-parameter Rayleigh-Stokes model found some discrepancies between numerical results and the measured data over small waves and later attributed these differences to the fact that the narrow-banded assumption is not fully satisfied [21]. In light of all the facts mentioned above, it can be easily perceived that the narrowband assumption needs to be relaxed for the design of offshore structures robust enough to survive extremely energetic waves. Under these conditions, waves are steep, and their frequency spectrum is not necessarily narrow [22].

The search for a way more straightforward than that of Longuet-Higgins [17] to describe nonlinear waves of finite bandwidth was carried out by Tung et al. [23]. Based on the studies of Tayfun [18,24], Tung et al. [23] proposed a simple but accurate expression for second-order nonlinear sea surface elevation for waves of finite bandwidth. With this wave model being available, this study intends to theoretically derive the probability distribution of wave crests, the joint distribution of the wave crest and its associated period. In doing so, the joint distribution of sea surface elevation, its first and second derivative, is also derived, which is indispensable for studying the statistical properties relevant to nonlinear random waves of finite bandwidth. Moreover, following Cavanie et al. [25], the probability distribution of wave periods for waves of finite bandwidth is given as a marginal distribution from
the joint distribution of wave crests and periods. In this study, attention is centered on deep water waves only.

2. Nonlinear Wave Model

Longuet-Higgins [17] derived a second-order nonlinear wave model for random waves of arbitrary bandwidth, which can be written as:

\[ \zeta(x, t) = \sum_{i=1}^{\infty} a_i \cos \chi_i + \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_i a_j \omega_i^2 \cos(\chi_i + \chi_j) - \frac{1}{2g} \sum_{i=1}^{\infty} \sum_{j>i} a_i a_j (\omega_i^2 - \omega_j^2) \cos(\chi_i - \chi_j) \]  

(1)

where \( \zeta \) is the sea surface elevation, \( \chi_i = k_i x - \omega_i t + \phi_i \), \( k_i \) is the wave number, \( \omega_i = \left( g k_i \right)^{1/2} \) is the wave frequency, \( \phi_i \) is the random phase uniformly distributed over the interval \((0, 2\pi)\), and \( a_i \) is the amplitude of the component wave. It is evident that the straightforward use of Equation (1) for the analysis of the statistical properties relevant to nonlinear random waves of moderate bandwidth would be rather cumbersome.

2.1. Narrow Band Approximation

For waves of narrow bandwidth, Tayfun [18,26] showed that sea surface elevation could be written in terms of an amplitude-modulated carrier wave with spectral mean wave number and frequency. Following Tayfun [18,26], the wave model in Equation (1) can be reduced to second-order Stokes waves such as:

\[ \zeta(x, t) = a \cos(k_m x - \omega_m t - \psi) + \frac{1}{2} a'^2 k \cos 2(k_m x - \omega_m t - \psi) \]  

(2)

where \( a \) is a slowly varying wave amplitude and follows the Rayleigh distribution, \( \psi \) is a random phase, \( k_m \) is the spectral mean wave number, and \( \omega_m \) is the spectral mean frequency. In Equation (2), the first order and second-order terms are phase-locked, and in phase.

2.2. Finite Band Approximation

Tayfun [26] scrutinized Equation (1) and showed that the third term on the right-hand side of Equation (1) contains little energy compared with the first two terms. Based on this observation, Tung et al. [23] elected to neglect the third term altogether and obtained a nonlinear wave model, which is more convenient to use. Upon introducing the following random processes:

\[ \eta_1 = \frac{1}{M_0^{1/2}} \sum_{i=1}^{\infty} a_i \cos \chi_i \]  

(3)

\[ \eta_2 = \frac{1}{M_2^{1/2}} \sum_{i=1}^{\infty} a_i \omega_i \sin \chi_i \]  

(4)

\[ \eta_3 = -\frac{1}{M_4^{1/2}} \sum_{i=1}^{\infty} a_i \omega_i^2 \cos \chi_i \]  

(5)

\[ \eta_4 = \frac{1}{M_4^{1/2}} \sum_{i=1}^{\infty} a_i \omega_i^2 \sin \chi_i \]  

(6)

\[ \eta_5 = \frac{1}{M_0^{1/2}} \sum_{i=1}^{\infty} a_i \sin \chi_i \]  

(7)
the normalized sea surface elevation $\zeta_1$ can be written as:

$$\zeta_1 = \frac{c}{m_0}^{1/2} \equiv (M_0/m_0)^{1/2} \left( \eta_1 - \frac{1}{2} \epsilon \eta_1 \eta_3 + \frac{1}{3} \epsilon \eta_4 \eta_5 \right)$$  \hfill (8)

where $M_i$ and $m_i$ are the $i$th spectral moments of the linear and nonlinear sea surface elevation, respectively and $\epsilon = M_4^{1/2}/g$. For a monochromatic wave of amplitude $a$ and frequency $\omega$, $M_4 = a^2 \omega^4/2$ so that $\epsilon = ak/2$ is a small quantity. For the problem under consideration, $\epsilon$ will be used as a perturbation parameter.

3. Joint Distribution of a Nonlinear Sea Surface Elevation, and Its First and Second Derivative

As will be shown in Sections 4 and 5, the joint distribution of sea surface elevation, its first and second derivative, is indispensable for the study of the statistical properties of nonlinear random waves of moderate bandwidth. In order to derive this joint distribution, we carry out the differentiation of nonlinear sea surface elevation with respect to time twice. The resulting expressions of the first and second derivative involve many random variables. Although the joint distribution of $\zeta$, $\dot{\zeta}$ and $\ddot{\zeta}$, $f_{\zeta\dot{\zeta}\ddot{\zeta}}(\cdot, \cdot, \cdot)$, may be obtained, the task is tedious (over dot denotes time derivative).

To facilitate subsequent computation of $f_{\zeta\dot{\zeta}\ddot{\zeta}}(\cdot, \cdot, \cdot)$, we further introduce some assumptions based on the analysis given by Tayfun [26]. We first note that $\eta_1$ in Equation (3) and $\eta_5$ in Equation (7) may be written as:

$$\eta_1 = \frac{1}{M_0^{1/2}} C \cos(k_m x - \omega_m t - \psi)$$  \hfill (9)

and

$$\eta_5 = \frac{1}{M_0^{1/2}} C \sin(k_m x - \omega_m t - \psi)$$  \hfill (10)

where $C(x, t)$ is an amplitude process.

Furthermore, we note that:

$$\eta_2 = \bar{\eta}_1 (M_0/M_2)^{1/2}$$  \hfill (11)

$$\eta_3 = \bar{\eta}_1 (M_0/M_4)^{1/2}$$  \hfill (12)

$$\eta_4 = \bar{\eta}_5 (M_0/M_4)^{1/2}$$  \hfill (13)

It was shown by Tayfun [26] that if $C$ and $\psi$ are $O(1)$, $\dot{C}$ and $\dot{\psi}$ are $O(\nu)$, and all higher-order derivatives of $C$ and $\psi$ are of corresponding higher-order smallness of $\nu$, as:

where

$$\nu = (M_0 M_2/M_1^2 - 1)^{1/2}$$  \hfill (14)

is a measure of the bandwidth of the frequency spectrum which, for all practical purposes, is a small quantity. Based on this assumption, the normalized sea surface elevation $\zeta_1$ can be rewritten, to the order of $\nu$, as:

$$\zeta_1 \equiv (M_0/m_0)^{1/2} \left[ \eta_1 - \frac{1}{2} \epsilon \eta_1 \eta_3 + \frac{1}{3} \epsilon \eta_4 \eta_5 \right]$$

$$\equiv (M_0/m_0)^{1/2} \left[ C \cos \chi + \frac{1}{2} \epsilon C (\omega_m + \dot{\psi})^2 \cos 2 \chi \right]$$  \hfill (15)

where

$$\chi = k_m x - \omega_m t - \psi.$$  \hfill (16)

Then, it follows that, to the order of $\nu$:

$$\zeta_2 \equiv (m_0/m_2)^{1/2} \zeta_1$$

$$\equiv (M_2/m_2)^{1/2} \left[ \dot{C} \cos \chi + C (\omega_m + \dot{\psi}) \sin \chi + 2 \epsilon C (\omega_m + \dot{\psi}) \sin \chi \cdot C (\omega_m + \dot{\psi})^2 \cos \chi \right]$$  \hfill (17)
\[
\begin{align*}
\zeta_3 &\equiv (m_0/m_4)^{1/2} \zeta_1 \\
&\equiv (M_4/m_4)^{1/2} \left[-C(\omega_m + \psi)^2 \cos x - 2\epsilon C^2(\omega_m + \psi)^4 \cos^2 x + 2\epsilon C^2(\omega_m + \psi)^4 \sin^2 x \right]
\end{align*}
\]

(18)

In terms of \(\eta_1, \zeta_2, \) and \(\zeta_3\) can be written as:

\[
\begin{align*}
\zeta_2 &\equiv (M_2/m_2)^{1/2} (\eta_2 - 2\epsilon \eta_2 \eta_3) \\
\zeta_3 &\equiv (M_4/m_4)^{1/2} (\eta_3 - 2\epsilon \eta_2^2 + 2\epsilon \eta_4^2)
\end{align*}
\]

(19) (20)

In Equations (15), (19), and (20), to the first order of \(\epsilon, m_o = M_o, m_2 = M_2, \) and \(m_4 = M_4.\) The random variables \(\zeta_1, \zeta_2, \) and \(\zeta_3\) are functions of \(\eta_1, \eta_2, \eta_3, \eta_4, \) and \(\eta_5.\) Therefore, the joint distribution of \(\eta_1, \eta_2, \eta_3, \eta_4, \) and \(\eta_5\) can be written as:

\[
f_{\eta_1\eta_2\eta_3\eta_4\eta_5}(\cdot; \cdot; \cdot; \cdot) = f_{\eta_1\eta_3}(\cdot; \cdot) f_{\eta_2\eta_4\eta_5}(\cdot; \cdot; \cdot; \cdot)
\]

(21)

where

\[
f_{\eta_1\eta_3}(\cdot; \cdot) = \frac{1}{2\pi \sqrt{1 - \rho_1^2}} \exp\left[-\frac{1}{2(1 - \rho_1^2)} (\eta_1^2 + \eta_3^2 - 2\rho_1 \eta_1 \eta_3) \right]
\]

(22)

and

\[
f_{\eta_2\eta_4\eta_5}(\cdot; \cdot; \cdot; \cdot) = \frac{1}{(2\pi)^{3/2}|S|^{1/2}} \exp\left[-\frac{1}{2|S|} \sum_{j=1}^{3} \left| \sum_{k=1}^{3} S_{i,j} \eta_j \eta_k \right| \right]
\]

(23)

In Equation (22), the correlation coefficient \(\rho_1\) is \(E[\eta_1 \eta_3]\) and \(E[\cdot; \cdot]\) denotes the expected value of quantity enclosed in the bracket. In Equation (23), the matrix \(S\) of co-variances of \(\eta_2, \eta_4, \) and \(\eta_5\) is given by:

\[
S = \begin{bmatrix}
E[\eta_2^2] & E[\eta_2 \eta_4] & E[\eta_2 \eta_5] \\
E[\eta_4 \eta_2] & E[\eta_4^2] & E[\eta_4 \eta_5] \\
E[\eta_5 \eta_2] & E[\eta_5 \eta_4] & E[\eta_5^2]
\end{bmatrix}
\]

(24)

and \(|S|_{i,j}\) is the cofactor of the element in the \(i\)th row and \(j\)th column of \(S.\) After denoting the correlation coefficients \(E[\eta_2 \eta_4] = E[\eta_4 \eta_2] = \rho_2\) and \(E[\eta_2 \eta_5] = E[\eta_5 \eta_2] = \rho_3,\) \(S\) can be written as:

\[
S = \begin{bmatrix}
1 & \rho_2 & \rho_3 \\
\rho_2 & 1 & \rho_1 \\
\rho_3 & \rho_1 & 1
\end{bmatrix}
\]

(25)

By introducing the following auxiliary random variables:

\[
\zeta_4 = \eta_4
\]

(26)

\[
\zeta_5 = \eta_5
\]

(27)

the joint distribution \(f_{\zeta_1\zeta_2\zeta_3\zeta_4\zeta_5}(\cdot)\) can be obtained by utilizing the standard method of transformation of random variables [28].

That is:

\[
f_{\zeta_1\zeta_2\zeta_3\zeta_4\zeta_5}(\cdot; \cdot; \cdot; \cdot; \cdot) = f_{\eta_1\eta_3}(\eta_1, \eta_3) f_{\eta_2\eta_4\eta_5}(\eta_2, \eta_4, \eta_5) \left| \frac{\zeta_1\zeta_2\zeta_3\zeta_4\zeta_5}{\eta_1\eta_2\eta_3\eta_4\eta_5} \right|^{-1}
\]

(28)

where \(J\) is the Jacobian of the random variable transformation. From Equations (8), (19), (20), (26), and (27), and following the perturbation technique [28], it may be shown that, to the order of \(\epsilon:\)

\[
|J| = 1 - \frac{13}{2} \epsilon \eta_3
\]

(29)
Here, \( \eta_1, \eta_2, \eta_3, \eta_4, \) and \( \eta_5 \) on the right hand side of Equation (28) are to be replaced by the inverse of the functions in Equations (8), (19), (20), (26), and (27). These inverses may also be obtained approximately by the perturbation method.

To the order of \( \varepsilon \):

\[
\begin{align*}
\eta_1 &\equiv c_1 + \frac{1}{2} \varepsilon c_2 c_3 - \frac{1}{2} \varepsilon c_4 c_5 & \quad & (30) \\
\eta_2 &\equiv c_2 + 2 \varepsilon c_2 c_3 & \quad & (31) \\
\eta_3 &\equiv c_3 + 2 \varepsilon c_2^2 c_3 - 2 \varepsilon c_4^2 & \quad & (32) \\
\eta_4 &\equiv c_4 & \quad & (33) \\
\eta_5 &\equiv c_5 & \quad & (34)
\end{align*}
\]

Substituting Equations (30)–(34) into Equation (28) and carrying out the integration with respect to \( \varepsilon_4 \) and \( \varepsilon_5 \) such as:

\[
f_{\varepsilon_4 \varepsilon_5}(\varepsilon, \varepsilon') = \int \int f_{\varepsilon_4 \varepsilon_5}(\varepsilon, \varepsilon', \varepsilon_4, \varepsilon_5) d\varepsilon_4 d\varepsilon_5
\]

the joint distribution of normalized sea surface elevation, its first and second derivative can be obtained. The integration in Equation (35) can be easily carried out although the task is lengthy and tedious.

The result is:

\[
f_{\varepsilon_4 \varepsilon_5}(\varepsilon, \varepsilon') = \frac{Q}{(2\pi)^{3/2}} \sqrt{1 - \rho_1^2} \exp \left[ -\frac{1}{2(1 - \rho_1^2)} \left( \varepsilon^2 + \varepsilon'^2 - 2 \rho_1 \varepsilon \varepsilon' \right) \right] \exp (-\frac{1}{2} \varepsilon_2^2)
\]

where

\[
Q = Q_1 + Q_2 + Q_3
\]

\[
Q_1 = 1 + \frac{\varepsilon \rho_2}{2(1 - \rho_1^2)} (\rho_3 - 4 \rho_1 \rho_2) \varepsilon^2 c_3^2 - \frac{\varepsilon}{2(1 - \rho_1^2)} c_1^2 c_3^2 - \frac{\varepsilon}{2(1 - \rho_1^2)} (3 \rho_1 + \rho_1 \rho_2 - 4 \rho_1 \rho_2^2) c_1
\]

\[
Q_2 = \left[ \frac{\varepsilon \rho_2}{2(1 - \rho_1^2)} (4 \rho_2 - \rho_1 \rho_3) - 2 \varepsilon \rho_1 \rho_3 \right] c_2 c_3^2 + \frac{\varepsilon}{2(1 - \rho_1^2)} (17 - 14 \rho_1^2 + \rho_1 \rho_2 \rho_3 - 4 \rho_2^2) c_3
\]

\[
Q_3 = \frac{\varepsilon}{2(1 - \rho_1^2)} 5 \rho_1 c_1 c_3^2 - \frac{\varepsilon}{2(1 - \rho_1^2)} 4 c_3^3
\]

4. Wave Crest Distribution

For the stationary random process \( \varepsilon(t) \) of arbitrary bandwidth, the probability density function of crests \( f_{\varepsilon_o}(\varepsilon_0) \) can be written as [29,30]:

\[
f_{\varepsilon_o}(\varepsilon_0) = F_p(\varepsilon_0)/N_p
\]

where

\[
F_p(\varepsilon_0) \quad \text{is the mean number of occurrences per unit time for which a crest of } \varepsilon \text{ takes value between } \varepsilon_0 \text{ and } \varepsilon_0 + d\varepsilon_0, \text{ and } N_p \text{ is the mean number of occurrences of a crest of } \varepsilon \text{ per unit time regardless of}
\]

\[
F_p(\varepsilon_0) = \int_{-\infty}^{\infty} \int_{-\infty}^{0} [\varepsilon] f_{\varepsilon_0 \varepsilon_0}(\varepsilon_0, 0, \varepsilon) d\varepsilon d\varepsilon_0
\]

\[
N_p = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [\varepsilon] f_{\varepsilon_0 \varepsilon_0}(\varepsilon_0, 0, \varepsilon) d\varepsilon_0 d\varepsilon
\]
magnitude. Substituting Equation (36) into Equation (41) and after some elaboration, we can obtain the probability density function of wave crests \( f_\omega(\zeta_0) \) such as:

\[
f_\omega(\zeta_0) = \frac{1}{\sqrt{2\pi}[1 - \epsilon(1 - \rho_1^2)]} \left[ M \int_{-\infty}^{\rho_1\zeta_0/\sqrt{1 - \rho_1^2}} \exp\left(-\frac{z^2}{2}\right) dz + \bar{N} \right]
\] (44)

where

\[
M = \frac{1}{\sqrt{2\pi}} \left[ -\frac{\epsilon}{2}(5 - 2\rho_1^2 + \rho_1\rho_2\rho_3 - 4\rho_2^2) - \rho_1\zeta_0 + \frac{\epsilon}{2}(1 - 5\rho_1^2 + \rho_1\rho_2\rho_3)\zeta_0^2 + \frac{\epsilon}{2}\rho_1^2\zeta_0^3 \right] \exp\left(-\frac{1}{2}\zeta_0^2\right)
\] (45)

and

\[
\bar{N} = \frac{1}{\sqrt{2\pi}} \left[ 1 + \frac{\epsilon}{2}(4\rho_1 - \rho_2\rho_3)\zeta_0 - \frac{\epsilon}{2}\rho_1\zeta_0^3 \right] \exp\left(-\frac{1}{2}\zeta_0^2\right)
\] (46)

To quantify the above results, we must specify the wave spectrum from which the quantities \( \rho_1, \rho_2, \rho_3, \) and \( \epsilon \) can be estimated. In this study, we shall use the Wallops spectrum [31] which can be written as:

\[
\Phi(z) = \frac{aG^2}{z^m\omega_0^{3-m}} \exp\left[-\frac{m}{4}\left(\frac{\omega_0}{a}\right)^4\right]
\] (47)

where

\[
m = \left| \frac{\log(2\pi^2\zeta^2)}{\log 2} \right|
\] (48)

is the absolute value of the slope of the spectrum (on the log-log scale) in the high frequency range and:

\[
\xi = M_0^{1/2}/L_o = \sigma k/2\pi = \epsilon/2\pi
\] (49)

is the significant slope where \( \sigma = M_0^{1/2} \) is the standard deviation of the sea surface elevation and \( L_o \) is the wave length whose frequency \( \omega_0 \) corresponds to the peak of the Wallops spectrum.

In Equation (47), the coefficient \( a \) is given by:

\[
a = \frac{(2\pi^2)^{m(m-1)/4}}{4^{(m-5)/4}} \frac{1}{\Gamma[(m-1)/4]}
\] (50)

where \( \Gamma[\cdot] \) is the gamma function [32]. From Equation (47), it may be shown that:

\[
\rho_1 = -\frac{\Gamma[(m - 3)/4]}{\Gamma^{1/2}[(m - 1)/4]\Gamma^{1/2}[(m - 5)/4]}
\] (51)

\[
\rho_2 = \frac{\Gamma[(m - 4)/4]}{\Gamma^{1/2}[(m - 3)/4]\Gamma^{1/2}[(m - 5)/4]}
\] (52)

\[
\rho_3 = -\frac{\Gamma[(m - 2)/4]}{\Gamma^{1/2}[(m - 1)/4]\Gamma^{1/2}[(m - 3)/4]}
\] (53)

and:

\[
\epsilon = 2\pi \xi \left[ m\Gamma[(m - 5)/4]/4\Gamma[(m - 1)/4] \right]^{1/2}
\] (54)

so that \( \epsilon, \rho_1, \rho_2, \) and \( \rho_3 \) are dependent solely on the value of \( \xi \) which was shown to rarely exceed 0.03 in the ocean [33]. At \( \epsilon = 0 \), the wave crest distribution in Equation (44) is reduced to:

\[
f_\omega(\zeta_0) = \frac{1}{\sqrt{2\pi}} \left[ -\rho_1\zeta_0 \exp\left(-\frac{1}{2}\zeta_0^2\right) \right] \left[ \int_{-\infty}^{\rho_1\zeta_0/\sqrt{1 - \rho_1^2}} \exp\left(-\frac{1}{2}\zeta_0^2\right) dz \right] + \sqrt{1 - \rho_1^2} \exp\left(-\frac{1}{2}\zeta_0^2\right)
\] (55)
which is precisely the same as the probability density function of the crest of the Gaussian process of finite bandwidth derived by Cartwright and Longuet-Higgins [34].

The experimental data used to verify the numerical results was described in detail by Giovannozzi and Kobayashi [35]. The experiment was conducted in a wave tank that was 30 m long, 2.4 m wide, and 1.5 m high. The water depth in the tank was 0.9 m. Random waves were generated with a piston-type wave paddle. Waves were measured using capacitance wave gauges at ten stations along with the tank. The first three wave gauges were used to separate the incident and reflected waves using the linear wave theory [36]. The measured time series of water surface displacement were sampled at the rate of 20 HZ for 900 s. The peak spectral period $T_p$ of the incident random waves at wave gauge 1 was 4.8 s [RUN 1] and 1.6 s [RUN 2]. Each time series was band-pass filtered between 0.05 and 0.36 HZ for $T_p = 4.8$ s, and 0.15 and 1.09 Hz for $T_p = 1.6$ s to eliminate the very short fluctuations and much lower harmonic components than the dominant waves [37]. The measured water surface displacement for $T_p = 4.8$ s is plotted in Figure 1, where the envelope process interpolated by the cubic spline method is also included. Figure 2 shows the measured wave spectra, the significant slope, and bandwidth of which are $[\xi = 0.01, \nu = 0.429]$ for $T_p = 4.8$ s, and $[\xi = 0.004, \nu = 0.178]$ for $T_p = 1.6$ s, respectively.

Figure 1. Time series of measured water surface displacement with its envelope process (thick solid line).

In order to demonstrate the effect of nonlinearities on the wave crest distribution $f_{\omega}(\zeta_\omega)$, Equation (44) for $\varepsilon = 0.04$ is shown in Figure 3 where wave crest distributions by Cartwright
and Longuet-Higgins [34] are also included for the comparison. It is shown that the considerable probability density mass is shifted toward more massive waves, which complies with our general perception that the wave profile is peaky as nonlinearity is considered.

![Figure 3. Comparison of wave crest distribution of nonlinear random waves of finite bandwidth with linear counterpart by Cartwright and Longuet-Higgins [33].](image)

Figure 3. Comparison of wave crest distribution of nonlinear random waves of finite bandwidth with linear counterpart by Cartwright and Longuet-Higgins [33].

Figure 4 shows the evolution of wave crest distribution as nonlinearity is enhanced. It is shown that the probability density mass spreads toward the larger and smaller waves as nonlinearity is increased [38].

![Figure 4. Wave crest distribution for $\varepsilon = 0.04, 0.09, 0.15, 0.23, \text{ and } 0.34$ that corresponds to $\zeta = 0.005, 0.01, 0.015, 0.02, \text{ and } 0.025$.](image)

Figure 4. Wave crest distribution for $\varepsilon = 0.04, 0.09, 0.15, 0.23, \text{ and } 0.34$ that corresponds to $\zeta = 0.005, 0.01, 0.015, 0.02, \text{ and } 0.025$.

The comparison of $f_{\zeta_0}(\zeta_0)$ in Equation (44) with the Rayleigh distribution, and the measured data is shown in Figure 5 in order to demonstrate the effect of bandwidth on the wave crest distribution. The Rayleigh distribution underpredicts the wave crest heights over large waves and small waves as well, but $f_{\zeta_0}(\zeta_0)$ in Equation (44) represents the data well except for some discrepancies over small waves, which is concerned with the intrinsic limit of wave data analysis method rather than the theoretical model’s deficiencies in this study.
Figure 4. Wave crest distribution for $\varepsilon = 0.04, 0.09, 0.15, 0.23, \text{ and } 0.34$ that corresponds to $\xi = 0.005, 0.01, 0.015, 0.02, \text{ and } 0.025$. The comparison of ($\omega$) of $\varsigma$ in Equation (44) with the Rayleigh distribution, and the measured data is shown in Figure 5 in order to demonstrate the effect of bandwidth on the wave crest distribution. The Rayleigh distribution underpredicts the wave crest heights over large waves and small waves as well, but ($\omega$) of $\varsigma$ in Equation (44) represents the data well except for some discrepancies over small waves, which is concerned with the intrinsic limit of wave data analysis method rather than the theoretical model’s deficiencies in this study.

(a) Run 1.

(b) Run 2.

Figure 5. Comparison of wave crests distribution with Rayleigh distribution and measured data.

In this comparison, measured wave crest heights are of zero up-crossing waves, which work well for narrow banded waves. However, this zero-crossing wave analysis method can give erratic results for finite banded waves since more than one wave crest can occur between two successive zero up-crossing, and wave crests occurring below the mean water level cannot be detected as well. Moreover, these intrinsic deficiencies of the zero-crossing method cause some discrepancies with $f_\varsigma(\varsigma_o)$ in Equation (44) over small waves and below the mean water level. Here, it should also be noted that some discrepancies over small waves are prevalent in most theoretical models based on the narrow banded assumption, but this issue has never been addressed before in the literature.

In Figure 6, we plot the exceedance probability of wave crest height for varying nonlinearity. It is shown that as nonlinearity is increasing, the exceedance probability of a wave crest height increases implying that the wave crest height is more significant.

The comparison of the exceedance probability of wave crest height with measured data is shown in Figure 7. The observed exceedance probability is well described by $f_\varsigma(\varsigma_o)$ in Equation (44), but the general agreement deteriorates as the spectral bandwidth increases. These discrepancies are more discernable as the spectral bandwidth is broadened, and can be attributed to the abnormally sizeable spectral bandwidth in Run 2. Here, it should be recalled that the spectral bandwidth is used as a perturbation parameter in this study.
Figure 5. Comparison of wave crests distribution with Rayleigh distribution and measured data. In this comparison, measured wave crest heights are of zero up-crossing waves, which work well for narrow banded waves. However, this zero-crossing wave analysis method can give erratic results for finite banded waves since more than one wave crest can occur between two successive zero crossings, and wave crests occurring below the mean water level cannot be detected as well. Moreover, these intrinsic deficiencies of the zero-crossing method cause some discrepancies with Equation (44) over small waves and below the mean water level. Here, it should also be noted that some discrepancies over small waves are prevalent in most theoretical models based on the narrow banded assumption, but this issue has never been addressed before in the literature.

Figure 6. Evolution of exceedance probability $P[\zeta > \zeta_0]$ of wave crest height $\zeta$ for varying nonlinearity.

Figure 7. Comparison of exceedance probability $P[\zeta > \zeta_0]$ with measured data.
5. Joint Distribution of Wave Crest and Its Associated Period

Due to the intrinsic difficulties in obtaining an analytical model for wave period, the distribution of wave periods has been derived as a marginal distribution from the empirical joint distribution of wave crests and periods as in Longuet-Higgins [2] and Cavanie et al. [25]. The earlier efforts were made by Longuet-Higgins [2] based on the assumption of a narrow banded spectral density function, and later Longuet-Higgins [3] revised his first model to cure some inconsistencies with the measured data since wave crests and wave periods are not correlated even over lower waves in the first model. The revised distribution of wave period $T$ by Longuet-Higgins [3] can be written as:

$$f_T(\tau) = \left(1 + \frac{\nu^2}{4}\right) \frac{1}{2\nu^2 \tau^2} \left[1 + \left(1 - \frac{1}{\tau^2}\right)^2 \frac{1}{\sqrt{\nu^2}}\right]^{-3/2}$$  \hspace{1cm} (56)

where $\tau = m_1 T/m_o$ is a normalized period, and $\nu$ is a spectral bandwidth parameter defined in Equation (14).

Another model for the wave period distribution frequently referred to in the literature was derived by Cavanie et al. [25], which can be written as:

$$f_T(\tau) = \frac{A^3 B^2 \tau}{\left[(\tau^2 - A^2)^2 + A^4 B^2\right]^{3/2}}$$  \hspace{1cm} (57)

where $\tau$ is a normalized period as in Longuet-Higgins [3], and:

$$A = \frac{1}{2}\left[1 + \left(1 - v_L^2\right)^{1/2}\right]$$  \hspace{1cm} (58)

$$B = \frac{v_L}{\left(1 - v_L^2\right)^{1/2}}$$  \hspace{1cm} (59)

In Equations (58) and (59), $v_L$ is another spectral bandwidth parameter introduced by Cavanie et al. [25] that can be written as:

$$v_L = \left(1 - \frac{M_2^2}{M_4 M_6}\right)^{1/2}$$  \hspace{1cm} (60)

In this study, to derive the geometrical model for the wave period, we first expand the sea surface elevation $\zeta(t)$ by Taylor’s series with respect to $\zeta(t_o)$ in which $t_o$ denotes the time of occurrence of a maximum (see Figure 8).

To the second order, we have:

$$\zeta(t) = \zeta(t_o) + \frac{1}{2} \ddot{\zeta}(t_o)(t - t_o)^2 + \cdots$$  \hspace{1cm} (61)

Imposing the condition $\zeta(t) = 0$ whenever the sea surface elevation $\zeta(t)$ crosses the mean water level, we could have the functional relationship between the period of a wave crest $T$, the sea surface elevation $\zeta(t)$, and its second derivative $\ddot{\zeta}(t)$ at the maxima as:

$$0 = \zeta(t_o) + \frac{1}{2} \ddot{\zeta}(t_o)(t_c - t_o)^2 + \cdots$$  \hspace{1cm} (62)

In Equation (62), $(t_c - t_o)$ can be replaced by $T/4$ with an analogy to the corresponding sinusoidal wave profile [39,40] and in terms of $T = 4(t - t_o)$, Equation (62) can be rewritten as:

$$0 \equiv \zeta(t_o) + \frac{1}{32} \ddot{\zeta}(t_o)T^2$$  \hspace{1cm} (63)
Finally, from Equation (63), we could have the model for the period of wave crest as:

$$\tau^2 \equiv -\frac{32\zeta(t_o)}{\zeta(t_o)}$$  \hspace{1cm} (64)

Upon introducing the normalized quantities $\zeta_o = \zeta(t_o)/m_0^{1/2}$, $\zeta_o = \zeta(t_o)/m_4^{1/2}$, and $\tau = m_1 T/m_o$ as in Longuet-Higgins [3], the normalized wave period can be written as:

$$\tau \equiv \sqrt{\mu\zeta_o/\zeta_o}$$  \hspace{1cm} (65)

where

$$\mu = -\frac{32m_1^2/m_0^{3/2}m_4^{1/2}}{1/2\Gamma(1/2)[(m-2)/4]}$$  \hspace{1cm} (66)

and from Equation (47), it can be shown that:

$$\mu = -\frac{32\Gamma^2[(m-2)/4]}{1/2\Gamma(1/2)[(m-1)/4]\Gamma(1/2)[(m-5)/4]}$$  \hspace{1cm} (67)

Equation (65) shows that the wave period distribution can be found from the joint distribution of sea surface elevation and its second derivative at the maxima $f_{\zeta_o\zeta_o}(\cdot, \cdot)$, which can be written as:

$$f_{\zeta_o\zeta_o}(\cdot, \cdot) = \frac{1}{E_N} f_{\zeta_1\zeta_2\zeta_3}(\zeta_o, \zeta_2 = 0, \zeta_0)\left|\zeta_o\right|$$  \hspace{1cm} (68)

where $f_{\zeta_1\zeta_2\zeta_3}(\cdot, \cdot, \cdot)$ is given in Equation (36) and $E_N$ is a constant of normalization given by:

$$E_N = \int_R \int f_{\zeta_1\zeta_2\zeta_3}(\zeta_o, \zeta_2 = 0, \zeta_0)\left|\zeta_o\right|d\zeta_o d\zeta_0$$  \hspace{1cm} (69)

In Equation (69):

$$R = \left[\zeta_0(0, \infty), \zeta_0(-\infty, 0)\right].$$  \hspace{1cm} (70)

After noting that two random variables are involved in the right-hand side of Equation (65), but only one relationship is available as of now, the following auxiliary random variable is introduced in order to solve this ill-posed problem:

$$\zeta_o = \zeta_o$$  \hspace{1cm} (71)

By applying the transformation technique of random variables to Equations (65) and (71) [28], the joint distribution of wave crests $\zeta_o$ and its associated period $\tau, f_{\zeta_o\tau}(\zeta_o, \tau)$, can be written as:

---

**Figure 8.** Schematic sketch of geometrical model for the wave period.
where \( J \) is the Jacobian of the random variable transformation, and in the right-hand side of Equation (72), \( \hat{\psi}_o \) is replaced by \( \mu \frac{\hat{\psi}_o}{\tau^2} \) based on Equation (65).

From Equations (65) and (71), it can be shown that:

\[
\begin{align*}
\left| J \left( \frac{\hat{\psi}_o \, \hat{\tau}}{\tau} \right) \right| &= \left| \begin{vmatrix}
\frac{\partial \hat{\psi}_o}{\partial \hat{\psi}} & \frac{\partial \hat{\psi}_o}{\partial \hat{\tau}} \\
\frac{\partial \hat{\tau}}{\partial \hat{\psi}} & \frac{\partial \hat{\tau}}{\partial \hat{\tau}} 
\end{vmatrix} \right| \\
&= 2 \mu \frac{\hat{\psi}_o}{\tau^3} 
\end{align*}
\] (73)

After some elaboration, we have:

\[
f_{\hat{\psi}_o \hat{\tau}}(\hat{\psi}_o, \hat{\tau}) = \frac{2 \mu^2 D}{(2\pi)^{3/2} E_1} \frac{\hat{\psi}_o^2}{\tau^3} \exp \left[ -\frac{1}{2} \left( \frac{(\tau^2 - \rho_1 \mu)^2}{1 - \rho_1^2} + \mu^2 \right) \frac{\hat{\psi}_o^2}{\tau^4} \right] 
\] (74)

where

\[
D = D_1 + D_2 
\]

\[
D_1 = 1 - \frac{\epsilon}{2(1 - \rho_1^2)} \left[ (3\rho_1 + \rho_1 \rho_2 - 4\rho_1 \rho_2^2) \tau^2 - (17 - 14\rho_1^2 + \rho_1 \rho_2 \rho_3 - 4\rho_2^2) \mu \right] \frac{\hat{\psi}_o}{\tau^2} 
\]

\[
D_2 = -\frac{\epsilon \mu}{2(1 - \rho_1^2)} \left[ \tau^4 - 5\rho_1 \mu \tau^2 + 4\mu^2 \right] \frac{\hat{\psi}_o^3}{\tau^6} 
\] (76)

In Equation (74), \( E_1 \) denotes a normalization constant given by:

\[
E_1 = \int_{-\infty}^{\infty} \int_{0}^{\infty} f_{\hat{\psi}_o \hat{\tau}}(\hat{\psi}_o, \hat{\tau}) d\hat{\psi}_o d\hat{\tau} 
\] (78)

It is worth mentioning that contrary to the previous studies [1–3, 25], the structural form of Equation (74) implies that the waves crest and its associated period are significantly correlated for nonlinear random waves of finite bandwidth, whereas the wave crest and period have been perceived to be mutually independent random processes for more massive waves [40].

We will pursue this topic more in the analysis of numerical results, but it can be readily perceived from Equation (74) that the bandwidth of the spectrum substantially modifies the joint distribution of wave crest and its associated period since what distinguishes this study from the others is the relaxation of a narrow banded assumption made in the previous studies.

From Equation (74), the distribution of wave crest period \( f_{\hat{\tau}}(\hat{\tau}) \) can be defined as:

\[
f_{\hat{\tau}}(\hat{\tau}) = \int_{0}^{\infty} f_{\hat{\psi}_o \hat{\tau}}(\hat{\psi}_o, \hat{\tau}) d\hat{\psi}_o 
\] (79)

However, \( f_{\hat{\tau}}(\hat{\tau}) \) cannot be given in a closed-form since the complicated structural form of \( f_{\hat{\psi}_o \hat{\tau}}(\hat{\psi}_o, \hat{\tau}) \) does not allow any further integration, which also makes the direct comparison with the previous results such as Cavanie et al. [25] and Longuet-Higgins [3] difficult.

However, if the period of wave crest depends on the magnitude of wave crest as the structural form of \( f_{\hat{\psi}_o \hat{\tau}}(\hat{\psi}_o, \hat{\tau}) \) in Equation (74) implies, it is the period of wave crest concerned with the given wave crest rather than \( f_{\hat{\tau}}(\hat{\tau}) \) that is of great engineering value. It is worthy to note that the distribution of wave periods \( f_{\hat{\tau}}(\hat{\tau}) \) in the previous studies has no bearings with the magnitude of wave crest [3, 25]. In light of this fact, we are elected to pursue the distribution of wave crest period \( f_{\hat{\psi}_o \hat{\tau}}(\hat{\psi}_o, \hat{\tau}) \) associated with a given wave crest \( \hat{\psi}_o \), which can be obtained utilizing the definition of conditional probability.
From Equation (74), we can write the conditional period distribution \( f_{\tau \mid \zeta_o}(\tau) \) with a given wave crest \( \zeta_o \) as:

\[
f_{\tau \mid \zeta_o}(\tau) = \frac{f_{\zeta_o \tau}(\zeta_o, \tau)}{f_{\zeta_o}(\zeta_o)} \quad (80)
\]

where \( f_{\zeta_o}(\zeta_o) \) is given in Equation (44).

For a linear narrow-band process \((\nu \to 0, \rho_1 \to -1, \varepsilon \to 0)\), \( f_{\zeta_o}(\zeta_o) \) in Equation (44) reduces to Rayleigh distribution as expected:

\[
f_{\zeta_o}(\zeta_o) \to \zeta_o e^{-\frac{1}{2} \zeta_o^2} \quad (81)
\]

and \( f_{\tau \mid \zeta_o}(\tau) \) is reduced to:

\[
f_{\tau \mid \zeta_o}(\tau) = \frac{2\mu^2}{(2\pi)^{3/2} E} \frac{\zeta_o}{\tau^5} \exp \left[ -\frac{1}{2} \left( \frac{\left( \frac{\tau^2}{\rho_1^2} - \rho_1 \mu \right)^2}{1 - \rho_1^2} + \mu^2 \right) \frac{\zeta_o^2}{\tau^4} + \frac{1}{2} \frac{\zeta_o^2}{\tau^2} \right] \quad (82)
\]

In order to demonstrate the effect of nonlinearity on the joint distribution of wave crest and its associated period, \( f_{\zeta_o \tau}(\zeta_o, \tau) \) in Equation (74) with varying nonlinearity are shown in Figure 9. It is shown that as nonlinearity enhances, a considerable amount of probability mass shifts toward large waves and short waves as well, the area in \([\zeta_o, \tau]\) space where \( f_{\zeta_o \tau}(\zeta_o, \tau) \) has a meaningful value shrink and more clusters around the peak period \( \mu_\tau \) which is defined as the period where \( f_{\tau \mid \zeta_o}(\tau) \) reaches its maximum with a given wave crest.

![Figure 9](image-url)
Figure 9. Evolution of joint distribution of wave crest height and its associated period as nonlinearity is enhanced.

In order to more clearly demonstrate how nonlinearity influences the joint distribution of wave crest and its associated period, the trajectories of peak period with a given wave crest height for varying nonlinearity are shown in Figure 10. It is shown that a positive correlation of wave crest height with its
associated period is extended to more massive waves, and the peak period decreases as nonlinearity is increasing.

\[ \nu = 0.34 \]

![Figure 10. Trajectory of peak period with given wave crest for varying nonlinearity over the sample space \((\nu, \tau)\).](image)

In order to verify the theoretical model in this study, the numerical results of \( f_{\nu,\tau}(\zeta_0, \tau) \) in Equation (74) are compared with the observed 256 and 361 wave crest heights and its associated period in RUN 1 and RUN 2, and comparison results are shown in Figure 11. In Run 2 [\( \nu = 0.429 \)], the region in \([\zeta_0, \tau]\) space which the observed data occupied becomes smaller and more densely packed, the extent of positive correlation of wave crest with the wave period over large waves is more visible, and the wave period becomes shorter by about 10% than in Run 1 [\( \nu = 0.178 \)]. These trends can also be found in Figures 9 and 10 and caused by the destructive interference of second-order harmonics of free mode which start to be involved as the spectral bandwidth is broadened, and as a result, nonlinearity is enhanced. Since these trends can precisely be reproduced with \( f_{\nu,\tau}(\zeta_0, \tau) \) in Equation (74) (see Figures 9–11), the validity of the theoretical model in this study is partially demonstrated. However, \( f_{\nu,\tau}(\zeta_0, \tau) \) in Equation (74) under-predicts the wave crest period, whereas the wave crest heights are well described.

\[ \nu = 0.04 \]

![Figure 11. Comparison of joint distribution of wave crests and its associated period with measured data.](image)
As mentioned in Section 4, these underpredictions are due to the intrinsic limit of the zero-crossing wave analysis method and have nothing to do with the deficiencies of $f_{\kappa,\tau}(\zeta_0, \tau)$ in Equation (74). Wave crest heights and periods in Figure 11 are of zero up-crossing waves, implying that only one wave crest and trough are counted between two successive up-crossings even though there can be few wave crests and troughs as the spectral bandwidth is not narrow (see Figure 1). As a result, the zero up-crossing wave periods in Figure 11 have a more significant value than the actual period due to the zero up-crossing wave analysis method’s intrinsic limit. These features have never been reported in the literature, and are what distinguishes this study from the previous ones [1–3,25,40], and needs more discussion: With only a few harmonic involved as in narrow banded waves, wave amplitude and phase are slowly varying, and second-order harmonics are bound to first-order harmonics. On the other hand, as spectral bandwidth increases, many harmonics appear in the random wave field, and among these, some harmonics are bound to each other while others are moving freely. Due to the presence of second-order free harmonics, the intervals of two successive wave crests are shorter due to the destructive interference of second-order free harmonics.

However, in the previous studies, only bound mode waves are considered, which eventually constitutes the widespread perception that the wave crest is a statistically independent random process with a wave period over large waves. In light of all these facts, it is easily conceived that our results are more physically plausible.

In order to demonstrate how the spectral bandwidth modifies the distribution of wave crest period, $f_{\kappa,\tau}(\tau)$ in Equation (80) with a given wave crest height is shown in Figure 12 where numerical results from Cavanie et al. [25] and Longuet-Higgins [3] are included for the comparison. Overall, the predicted wave crest period by the theoretical model in this study dramatically differs from that of Cavanie et al. [25] and Longuet-Higgins [3]. However, it should be noted that models of Cavanie et al. [25] and Longuet-Higgins [3] are based on the assumption of a narrow banded spectral density function. Hence, it is easily conceived that these discrepancies are occurring since the intended wave conditions of the theoretical model in this study are different from that of Cavanie et al. [25] and Longuet-Higgins [3].

![Figure 12](image-url)

**Figure 12.** Cont.
It is shown that the considerable probability density shifts toward the short waves due to amplified waves of free mode as nonlinearity is increased, leading to the spectral bandwidth to increase. It is also shown that the conditional wave crest period distribution $f_{\tau|\omega_0}(\tau)$ is asymmetrically distributed with respect to its peak period $\mu_\tau$ due to a positive correlation between wave crest height and its associated period, and the probability density mass of $f_{\tau|\omega_0}(\tau)$ are more clustered around its peak period $\mu_\tau$ as the wave crest height is more substantial.

6. Conclusions

Although a great deal of progress has recently been made on nonlinear waves’ theory, the complicated form of nonlinear random waves has made its application difficult. Theoretical treatment of the statistical properties relevant to nonlinear wave fields such as the peak of sea surface elevation and its period is an overdue task. If the underlying frequency spectrum is narrow, the stochastic representation of a nonlinear sea surface is reduced to a familiar form in which each realization is an amplitude modulated second-order Stokes wave. In contrast with the intricate complexity of the expression of nonlinear waves of finite bandwidth, such an approximation constitutes a more straightforward formulation to study numerically or analytically the nonlinear effects on the statistical properties of sea surfaces.
description of wave properties. However, considering the well-known wave-wave interaction in the
development of the wind-wave spectrum and its destructive nature, the narrow-band assumption is
hardly fulfilled in the ocean field [40]. For waves of finite bandwidth, an approximate wave model
proposed by Tung et al. [23] is a promising alternative from which the joint distribution of nonlinear
sea surface elevation, its first and second derivatives can be obtained and the structure of which is
simple enough so that statistical properties of nonlinear random waves can be obtained. Based on
this wave model, first, the joint distribution of wave elevation, its first and second derivative, which
were indispensable for the study of the statistical properties relevant to nonlinear random waves of
moderate bandwidth, was derived. Then, we proceeded to derive the wave crest distribution, the joint
distribution of wave crest and its associated period, and the conditional wave period distribution with
a given wave crest.

It was shown that as nonlinearity increases, the crest distribution deviates from its linear
counterpart increasingly. The general character of this deviation is in the form of a spreading of the
density mass toward the larger and smaller crests, which is consistent with the vertically asymmetric
properties of nonlinear waves, which are known to have shallower troughs, and sharper and larger
crests than the linear counterpart. In the case of the joint distribution of wave crest and its associated
period, the wave crest period is somewhat under-predicted, whereas the wave crest heights are well
described when compared with the measured data by Giovannozzi and Kobayashi [35]. However,
considering that the zero up-crossing wave analysis method can give erratic results for periods when
applied to finite banded random waves, some discrepancies are inevitable. It is also shown that as
the wave crest is substantial, its associated wave period is more prolonged, contrary to the general
perception that the wave crest is a statistically independent random process with the wave period.
Moreover, these positive correlations rapidly weaken as the wave crest surpasses a certain threshold.
However, these positive correlations over the larger values of wave crest are recovered as nonlinearity
becomes large, and the peak period decreases due to the destructive interference of second-order free
harmonics. These features cannot be reproduced with only second-order harmonics of the bound mode
being considered, as in the previous studies. Hence, the general perception in the coastal community
that the wave crest is a statistically independent random process with the wave period over the large
waves needs to be modified.

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