Finite dimensional simple modules of \((q, Q)\)-current algebras

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Abstract. The \((q, Q)\)-current algebra associated with the general linear Lie algebra was introduced by the second author in the study of representation theory of cyclotomic \(q\)-Schur algebras. In this paper, we study the \((q, Q)\)-current algebra \(U_q(\mathfrak{sl}_n(Q)[x])\) associated with the special linear Lie algebra \(\mathfrak{sl}_n\). In particular, we classify finite dimensional simple \(U_q(\mathfrak{sl}_n(Q)[x])\)-modules.

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§ 0. Introduction

0.1. The \((q, Q)\)-current algebra associated with the general linear Lie algebra was introduced in \(W16\) to study the representation theory of cyclotomic \(q\)-Schur algebras. (In fact, the algebra introduced in \(W16\) is isomorphic to a \((q, Q)\)-current algebra, considered in this paper, with special parameters. See Appendix D for these
connections.) We expected that the \((q, Q)\)-current algebra has good properties like quantum groups.

0.2. In this paper, we study the \((q, Q)\)-current algebra \(U_q(\mathfrak{sl}_n^{(Q)}[x])\) associated with the special linear Lie algebra \(\mathfrak{sl}_n\). The \((q, Q)\)-current algebra \(U_q(\mathfrak{sl}_n^{(Q)}[x])\) has parameters \(q \in \mathbb{C}^\times\) and \(Q = (Q_1, Q_2, \ldots, Q_{n-1}) \in \mathbb{C}^{n-1}\).

In the case where \(q = 1\), the algebra \(U_1(\mathfrak{sl}_n^{(Q)}[x])\) is isomorphic to the universal enveloping algebra of the deformed current Lie algebra \(\mathfrak{sl}_n^{(Q)}[x]\) given in [W18] under avoiding some ambiguities of signs (see Remark 0.3 ii). We remark that the deformed current Lie algebra \(\mathfrak{sl}_n^{(Q)}[x]\) is isomorphic to the polynomial current Lie algebra \(\mathfrak{sl}_n[x]\) if \(Q = (0, \ldots, 0)\).

On the other hand, in the case where \(Q = 0 = (0, \ldots, 0)\), the algebra \(U_q(\mathfrak{sl}_n^{(0)}[x])\) is a subalgebra of the quantum loop algebra \(U_q(\mathfrak{Lsl}_n)\). This connection corresponds to the fact that the polynomial current Lie algebra \(\mathfrak{sl}_n[x] = \mathfrak{sl}_n \otimes \mathbb{C}[x]\) is a subalgebra of the loop Lie algebra \(\mathfrak{Lsl}_n = \mathfrak{sl}_n \otimes \mathbb{C}[x, x^{-1}]\) in the natural way (see Remark 2.5 (i)). By using the explicit description in [FT] for the coproduct of \(U_q(\mathfrak{Lsl}_n)\) under Drinfeld’s new generators, we see that the coproduct of \(U_q(\mathfrak{Lsl}_n)\) induces the coproduct of \(U_q(\mathfrak{sl}_n^{(0)}[x])\) by the restriction (see Proposition 3.2).

In general, we prove that the algebra \(U_q(\mathfrak{sl}_n^{(Q)}[x])\) is a subalgebra of a shifted quantum affine algebra \(U_{b,0}\) introduced in [FT]. The quotient is obtained by regarding some central elements in \(U_{b,0}\) as scalars depending on the parameters \(Q_1, \ldots, Q_{n-1}\). (see Proposition 2.3 for details). Then, by applying an analogy of the argument in [FT], we have the following theorem.

**Theorem 0.3** (Proposition 1.12 Theorem 3.9 and Proposition 3.10).

(i) There exist injective algebra homomorphisms \(\iota_+^{(Q)}\) and \(\iota_-^{(Q)}\) from \(U_q(\mathfrak{sl}_n^{(Q)}[x])\) to \(U_q(\mathfrak{sl}_n^{(0)}[x])\).

(ii) The algebra \(U_q(\mathfrak{sl}_n^{(Q)}[x])\) is a right (resp. left) coideal subalgebra of \(U_q(\mathfrak{sl}_n^{(0)}[x])\) through the injection \(\iota_-^{(Q)}\) (resp. \(\iota_+^{(Q)}\)).

0.4. The goal of this paper is to classify the finite dimensional simple modules of \(U_q(\mathfrak{sl}_n^{(Q)}[x])\) in the case where \(q\) is not a root of unity. The algebra \(U_q(\mathfrak{sl}_n^{(Q)}[x])\) has a triangular decomposition (Theorem 1.10), and we see that every finite dimensional simple \(U_q(\mathfrak{sl}_n^{(Q)}[x])\)-module is a highest weight module in the usual manner. Thus, it is enough to classify the highest weights such that the corresponding highest weight modules are finite dimensional. The highest weight for a highest weight \(U_q(\mathfrak{sl}_n^{(Q)}[x])\)-module is described by an element of \((\mathbb{C}^\times \times \prod_{t \geq 0} \mathbb{C})^{n-1}\). For \(u = (\lambda_i, (u_{i,t})_{t \geq 0})_{1 \leq i \leq n-1} \in (\mathbb{C}^\times \times \prod_{t \geq 0} \mathbb{C})^{n-1}\), we denote the highest weight \(U_q(\mathfrak{sl}_n^{(Q)}[x])\)-module of the highest weight \(u\) by \(L(u)\) (see [W19] for details).

In order to describe the highest weights for finite dimensional simple \(U_q(\mathfrak{sl}_n^{(Q)}[x])\)-modules, we prepare some combinatorics as follows.
For \( t, k \in \mathbb{Z}_{>0} \), we define the symmetric polynomial \( p_t(q)(x_1, x_2, \ldots, x_k) \) with variables \( x_1, x_2, \ldots, x_k \) by
\[
p_t(q)(x_1, x_2, \ldots, x_k) := \sum_{\lambda, \ell(\lambda) \leq t} q^{-\ell(\lambda)}(q - q^{-1})^{\ell(\lambda) - 1} m_\lambda(x_1, x_2, \ldots, x_k),
\]
where we denote by \( \lambda \vdash t \) if \( \lambda \) is a partition of \( t \), denote by \( \ell(\lambda) \) the length of \( \lambda \) and denote by \( m_\lambda(x_1, x_2, \ldots, x_k) \) the monomial symmetric polynomial associated with \( \lambda \). For \( t, k \in \mathbb{Z}_{>0} \) and \( Q, \beta \in \mathbb{C}^\times \), we also define the symmetric polynomial \( p_t^{(Q)}(q; \beta)(x_1, x_2, \ldots, x_k) \) by
\[
p_t^{(Q)}(q; \beta)(x_1, x_2, \ldots, x_k) := p_t(q)(x_1, x_2, \ldots, x_k) + \tilde{\beta}Q^{-t} + (q - q^{-1}) \sum_{z=1}^{t-1} \tilde{\beta}Q^{-t+z} p_z(q)(x_1, x_2, \ldots, x_k),
\]
where we put \( \tilde{\beta} = (q - q^{-1})^{-1}(1 - \beta^{-2}) \). We remark that, in the case where \( q = 1 \), the polynomial \( p_t(1)(x_1, x_2, \ldots, x_k) \) coincides with the power sum symmetric polynomial of degree \( t \). We also remark that, in the case where \( \beta = \pm 1 \), we have \( p_t^{(Q)}(q; \pm 1)(x_1, x_2, \ldots, x_k) = p_t(q)(x_1, x_2, \ldots, x_k) \).

Let \( \mathbb{C}[x] \) be the polynomial ring over \( \mathbb{C} \) with an indeterminate variable \( x \). For \( \varphi \in \mathbb{C}[x] \), we denote the leading coefficient of \( \varphi \) by \( \beta_\varphi \). Then we define a map \( u^{(Q)} : \mathbb{C}[x] \setminus \{0\} \to \mathbb{C}^\times \times \prod_{t>0} \mathbb{C} \) by
\[
u^{(Q)}(\varphi) = \begin{cases} (\beta_\varphi, (0)_{t>0}) & \text{if } Q = 0 \text{ and } \deg \varphi = 0, \\
(\beta_\varphi q^{\deg \varphi}, (p_t(q)(\gamma_1, \gamma_2, \ldots, \gamma_k))_{t>0}) & \text{if } Q = 0 \text{ and } \deg \varphi > 0, \\
(\beta_\varphi, (\beta_\varphi Q^{-t})_{t>0}) & \text{if } Q \neq 0 \text{ and } \deg \varphi = 0, \\
(\beta_\varphi q^{\deg \varphi}, (p_t^{(Q)}(q; \beta_\varphi)(\gamma_1, \gamma_2, \ldots, \gamma_k))_{t>0}) & \text{if } Q \neq 0 \text{ and } \deg \varphi > 0 \end{cases}
\]
for \( \varphi = \beta_\varphi(x - \gamma_1)(x - \gamma_2)\ldots(x - \gamma_k) \in \mathbb{C}[x] \setminus \{0\} \). For \( Q \in \mathbb{C} \), set
\[
\mathbb{C}[x]^{(Q)} = \begin{cases} \{ \varphi \in \mathbb{C}[x] \setminus \{0\} \mid \beta_\varphi = \pm 1 \} & \text{if } Q = 0, \\
\{ \varphi \in \mathbb{C}[x] \setminus \{0\} \mid \beta_\varphi^{-2}Q^{-1} \text{ is not a root of } \varphi \} & \text{if } Q \neq 0. 
\end{cases}
\]
Then we have the following classification of the isomorphism classes of finite dimensional simple \( U_q(\mathfrak{sl}^{(Q)}[x]) \)-modules.

**Theorem 0.5 (Theorem 10.5).** There exists the bijection between \( \prod_{1 \leq i \leq n-1} \mathbb{C}[x]^{(Q_i)} \) and the isomorphism classes of finite dimensional simple \( U_q(\mathfrak{sl}^{(Q)}[x]) \)-modules given by \( (\varphi_i)_{1 \leq i \leq n-1} \mapsto L((u^{(Q_i)}(\varphi_i))_{1 \leq i \leq n-1}) \).

We remark that the simple highest weight module \( L((u^{(Q_i)}(\varphi_i))_{1 \leq i \leq n-1}) \) is finite dimensional even if \( \varphi_i \not\in \mathbb{C}[x]^{(Q_i)} \) for some \( i \) such that \( Q_i \neq 0 \) although it is infinite dimensional if \( \varphi_i \not\in \mathbb{C}[x]^{(0)} \) for some \( i \) such that \( Q_i = 0 \). In the case where \( Q \neq 0 \),
the map \( u^{(Q)} : \mathbb{C}[x] \setminus \{0\} \to \mathbb{C}^\times \times \prod_{t>0} \mathbb{C} \) is not injective, and we have the following proposition.

**Proposition 0.6** (Proposition [9.5]). For \( \varphi, \varphi' \in \mathbb{C}[x] \setminus \{0\} \) such that \( \deg \varphi \geq \deg \varphi' \), we have that \( u^{(Q)}(\varphi) = u^{(Q)}(\varphi') \) if and only if

\[
\varphi = q^{-(\deg \varphi - \deg \varphi')} \varphi' \prod_{i=1}^{\deg \varphi - \deg \varphi'} (x - q^{-(z-1)} \beta^{-2} Q^{-1}).
\]

Thanks to this proposition, we can take the set \( \prod_{1 \leq i \leq n-1} \mathbb{C}[x]^{(Q_i)} \) as an index set for the isomorphism classes of finite dimensional simple \( U_q(s^i_n) \)-modules.

We also remark that, in the case where \( Q = 0 = (0, \ldots, 0) \), the algebra \( U_q(s^0_n) \) is a subalgebra of the quantum loop algebra \( U_q(Lsl_n) \), and the argument to classify finite dimensional simple \( U_q(s^0_n) \)-modules is essentially same as the argument for \( U_q(Lsl_n) \) given in [CP91] and [CP94a]. However, in the case where \( Q \neq (0, \ldots, 0) \), we need more careful treatments.

0.7. In the theory of quantum loop algebras and shifted quantum affine algebras, we usually use generating functions for generators. In order to describe the corresponding statements for \( U_q(s^i_n) \), we need other generators \( \Psi^+_i \in U_q(s^i_n) \) \((1 \leq i \leq n, t \geq -b_i)\) defined by [2.6.1] and [2.6.2]. We consider the generating function \( \Psi^+_i(\omega) = \prod_{t \geq -b_i} \Psi^+_i(w^t) \). We also define a map \( \varphi : \mathbb{C}[x] \to \mathbb{C}[w] \) \((\varphi \mapsto \varphi^+(\omega))\) by

\[
\varphi^+(\omega) = (1 - \gamma_1 \omega) (1 - \gamma_2 \omega) \ldots (1 - \gamma_k \omega)
\]

if \( \varphi = \beta \varphi(x - \gamma_1)(x - \gamma_2) \ldots (x - \gamma_k) \). Then we have the following corollary.

**Corollary 0.8** (Corollary [8.13] and Corollary [9.7]). For \( (\varphi_i)_{1 \leq i \leq n-1} \in \prod_{1 \leq i \leq n-1} \mathbb{C}[x]^{(Q_i)} \), let \( v_0 \) be a highest weight vector of \( L((u^{(Q_i)}(\varphi_i))_{1 \leq i \leq n-1}) \). Then we have

\[
\Psi^+_i(\omega) \cdot v_0 = \begin{cases} 
\beta \varphi_i q^{\deg \varphi_i} \frac{\varphi^+_i(q^{-2} \omega)}{\varphi^+_i(\omega)} v_0 & \text{if } Q_i = 0, \\
q^{\deg \varphi_i} \frac{\varphi^+_i(q^{-2} \omega)}{\varphi^+_i(\omega)} (\beta^{-1} Q_i \beta^{-1} \omega - 1) v_0 & \text{if } Q_i \neq 0
\end{cases}
\]

for \( i = 1, 2, \ldots, n-1 \).

We remark that Corollary [0.8] is an analogue of the statement for shifted Yangians given in [BK] Corollary 7.10 and [KTWWY] Theorem 3.5.

0.9. After writing the first version of this paper, Alexander Tsymbaliuk informed us that he and Michael Finkelberg obtained a classification of finite dimensional simple modules of shifted quantum affine algebras of type \( A \). Unfortunately, their work is unpublished.

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§ 1. THE \((q, Q)\)-CURRENT ALGEBRA \(U_q(\mathfrak{sl}_n^Q[x])\)

In this section, we give a definition of the \((q, Q)\)-current algebra \(U_q(\mathfrak{sl}_n^Q[x])\) associated with the special linear Lie algebra \(\mathfrak{sl}_n\). We also give some basic properties of \(U_q(\mathfrak{sl}_n^Q[x])\).

1.1. For \(v \in \mathbb{C}^\times\) and any elements \(x, y\) of an associative algebra over \(\mathbb{C}\), we put 
\[ [x, y]_v = xy - vyx. \]
In the case where \(v = 1\), we denote \([x, y]_1 = xy - yx\) by \([x, y]\) simply.

Put \(I = \{1, 2, \ldots, n - 1\}\). Let \(A = (a_{ij})_{i,j \in I}\) be the Cartan matrix of type \(A_{n-1}\), namely we have \(a_{ii} = 2, a_{i,i \pm 1} = -1\) and \(a_{ij} = 0\) if \(j \neq i, i \pm 1\).

Take \(q \in \mathbb{C}^\times\). Put \(|k| = (q-q^{-1})^{-1}(q^k-q^{-k})\) for \(k \in \mathbb{Z}\), and \([k]! = [k][k-1] \ldots [1]\) for \(k \in \mathbb{Z}_{>0}\) with \([0]! = 1\).

We define the \((q, Q)\)-current algebra \(U_q(\mathfrak{sl}_n^Q[x])\) associated with the special linear Lie algebra \(\mathfrak{sl}_n\), as follows.

**Definition 1.2.** For \(q \in \mathbb{C}^\times\) and \(Q = (Q_1, Q_2, \ldots, Q_{n-1}) \in \mathbb{C}^I\), we define an associative algebra \(U_q(\mathfrak{sl}_n^Q[x])\) over \(\mathbb{C}\) by the following generators and defining relations:

**Generators:** \(X^\pm_{i,t}, J^\pm_{i,t}, K^\pm_{i} \ (i \in I, t \in \mathbb{Z}_{\geq 0})\).

**Defining relations:**

\[
\begin{align*}
\text{(Q1-1)} & & [K^+_i, K^-_j] = [K^+_i, J^+_{j,t}] = [J^+_{i,s}, J^+_{j,t}] = 0, \\
\text{(Q1-2)} & & K^+_i K^-_i = 1 - (q - q^{-1})J^+_{i,0}, \\
\text{(Q2)} & & X^+_{i,t+1} X^-_{j,s} - q^{a_{ij}} X^+_{j,s} X^+_{i,t+1} = q^{a_{ij}} X^+_{i,t} X^+_{j,s+1} - X^+_{j,s+1} X^+_{i,t}, \\
\text{(Q3)} & & X^-_{i,t+1} X^-_{j,s} - q^{-a_{ij}} X^-_{j,s} X^-_{i,t+1} = q^{-a_{ij}} X^-_{i,t} X^-_{j,s+1} - X^-_{j,s+1} X^-_{i,t}, \\
\text{(Q4-1)} & & K^+_i X^+_{j,t} = q^{a_{ij}} X^+_{j,t}, \\
\text{(Q4-2)} & & q^{a_{ij}} J^+_{i,0} X^+_{j,t} - q^{-a_{ij}} X^-_{j,t} J^+_{i,0} = [a_{ij}] X^+_{j,t}, \\
\text{(Q4-3)} & & [J^+_{i,s+1}, X^+_{j,t}] = q^{a_{ij}} J^+_{i,s} X^+_{j,t+1} + q^{-a_{ij}} X^+_{j,t+1} J^+_{i,s}, \\
\text{(Q5-1)} & & K^+_i X^-_{j,t} = q^{-a_{ij}} X^-_{j,t}, \\
\text{(Q5-2)} & & q^{-a_{ij}} J^-_{i,0} X^-_{j,t} - q^{-a_{ij}} J^-_{j,t} X^-_{i,0} = [-a_{ij}] X^-_{j,t}, \\
\text{(Q5-3)} & & [J^-_{i,s+1}, X^-_{j,t}] = q^{a_{ij}} J^-_{i,s} X^-_{j,t+1} - q^{-a_{ij}} X^-_{j,t+1} J^-_{i,s}, \\
\text{(Q6)} & & [X^+_{i,t}, X^-_{j,t}] = \delta_{ij} K^+_i (J^+_{i,s+t} - Q_t J^-_{i,s+t+1}), \\
\text{(Q7)} & & [X^+_{i,t}, X^+_{j,s}] = 0 \text{ if } j \neq i, i \pm 1, \\
& & X^+_{i,t+1,u} (X^+_{i,s} X^+_{i,t} + X^+_{i,t} X^+_{i,s}) + (X^+_{j,s} X^+_{i,t} + X^+_{i,t} X^+_{j,s}) X^+_{i,t+1,u} \\
& & = (q + q^{-1})(X^+_{i,s} X^+_{i+1,u} X^+_{i,t} + X^+_{i,t} X^+_{i+1,u} X^+_{i,s}), \\
\text{(Q8)} & & [X^-_{i,t}, X^-_{j,s}] = 0 \text{ if } j \neq i, i \pm 1, \\
& & X^-_{i,t+1,u} (X^-_{i,s} X^-_{i,t} + X^-_{i,t} X^-_{i,s}) + (X^-_{i,s} X^-_{i,t} + X^-_{i,t} X^-_{i,s}) X^-_{i,t+1,u} \\
& & = (q + q^{-1})(X^-_{i,s} X^-_{i+1,u} X^-_{i,t} + X^-_{i,t} X^-_{i+1,u} X^-_{i,s}).
\end{align*}
\]
\[ = (q + q^{-1})(X_{i,s}^- X_{i,\pm 1, u}^- X_{i,t}^- + X_{i,t}^- X_{i,\pm 1, u}^- X_{i,s}^-) \]

We call \( U_q(\mathfrak{sl}^{(Q)}_n[x]) \) the \((q, Q)\)-current algebra associated with \( \mathfrak{sl}_n \). We denote \( U_q(\mathfrak{sl}^{(Q)}_n[x]) \) by \( U_q^{(Q)} \) simply unless there is any confusion.

**Remarks 1.3.**

(i) If \( q \neq 1 \), the relation (Q4-2) (resp. (Q5-2)) follows from the relations (Q1-2) and (Q4-1) (resp. (Q1-2) and (Q5-1)).

(ii) In the case where \( q = 1 \), we see easily that \( U_1(\mathfrak{sl}^{(Q)}_n[x])/\langle K_i^+ - 1 \mid i \in I \rangle \) is isomorphic to the universal enveloping algebra of the deformed current Lie algebra \( \mathfrak{sl}^{(Q)}_n[x] \) given in [W18, Definition 1.1], where \( \langle K_i^+ - 1 \mid i \in I \rangle \) is the two-sided ideal of \( U_1(\mathfrak{sl}^{(Q)}_n[x]) \) generated by \( \{ K_i^+ - 1 \mid i \in I \} \). Under this isomorphism, the generators \( X_{i,t}^\pm \) and \( J_{i,t} \) of \( U_1(\mathfrak{sl}^{(Q)}_n[x]) \) correspond to the generators of the enveloping algebra of \( \mathfrak{sl}^{(Q)}_n[x] \) denoted by the same symbols respectively. We note that \( \mathfrak{sl}^{(Q)}_n[x] \) is isomorphic to the polynomial current Lie algebra \( \mathfrak{sl}_n[x] \) if \( Q = (0, \ldots, 0) \).

\[ \hat{\gamma} \text{From the defining relations, we can easily check the following lemma.} \]

**Lemma 1.4.** There exists the algebra anti-involution \( \hat{\gamma} : U_q^{(Q)} \to U_q^{(Q)} \) such that

\[ \hat{\gamma}(X_{i,t}^\pm) = X_{i,t}^\mp, \quad \hat{\gamma}(J_{i,t}) = J_{i,t} \quad \text{and} \quad \hat{\gamma}(K_i^\pm) = K_i^\pm \quad \text{for} \quad i \in I \quad \text{and} \quad t \in \mathbb{Z}_{\geq 0}. \]

**1.5.** The relation (Q1-2) implies that

\[ J_{i,0} = \frac{1 - (K_i^-)^2}{q - q^{-1}} \]

if \( q^2 \neq 1 \). By the relations (Q4-2), (Q4-3), (Q5-2) and (Q5-3), we have

\[ [J_{i,1}, X_{i,t}^\pm] = \pm [2]X_{i,t+1}^\pm. \]

This implies that

\[ X_{i,t+1}^\pm = \pm \frac{1}{[2]}[J_{i,1}, X_{i,t}^\pm] \]

if \( q^2 \neq -1 \). The relations (Q1-2) and (Q6) imply that

\[ J_{i,t+1} = \begin{cases} K_i^- [X_{i,t+1}, X_{i,0}^-] & \text{if} \quad Q_i = 0, \\ Q_i^{-1} J_{i,t} - Q_i^{-1} K_i^- [X_{i,t}, X_{i,0}^-] & \text{if} \quad Q_i \neq 0. \end{cases} \]

Thanks to the relations (1.5.1), (1.5.3) and (1.5.4), we have the following lemma.

**Lemma 1.6.** Assume that \( q^2 \neq \pm 1 \). The algebra \( U_q(\mathfrak{sl}^{(Q)}_n[x]) \) is generated by \( X_{i,0}^\pm \), \( J_{i,1} \) and \( K_i^\pm \) for \( i \in I \).
1.7. Let $U_{q,Q}^\pm$ be a subalgebra of $U_q^{(Q)}$ generated by $X_{i,t}^\pm ((i,t) \in I \times \mathbb{Z}_{\geq 0})$, and $U_{q,Q}^0$ be a subalgebra of $U_q^{(Q)}$ generated by $J_{i,t} ((i,t) \in I \times \mathbb{Z}_{\geq 0})$ and $K_i^\pm (i \in I)$. From the defining relations, we see that

\begin{equation}
U_q^{(Q)} = U_{q,Q}^- \cdot U_{q,Q}^0 \cdot U_{q,Q}^+.
\end{equation}

1.8. Through the connection with the shifted quantum affine algebra given in the next section, and using the PBW theorem for the quantum loop algebra in $[T]$, we can obtain the PBW theorem for $U_q^{(Q)}$. In this section, we give only the statement of PBW theorem for $U_q^{(Q)}$, and a proof is given in Appendix $[A]$.

1.9. Let $\{\alpha_1, \alpha_2, \ldots, \alpha_{n-1}\}$ be the set of simple roots of $\mathfrak{sl}_n$, and

$$\Delta^+ = \{\alpha_{i,j} := \alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-1} | 1 \leq i < j \leq n\}$$

be the set of positive roots. We define a total order on $\Delta^+$ by

$$\alpha_{i,j} \leq \alpha_{i',j'} \text{ if } i < i' \text{ or } i = i', j \leq j'.$$

We also define a total order on $\Delta^+ \times \mathbb{Z}$ by

$$(\beta, t) \leq (\beta', t') \text{ if } \beta < \beta' \text{ or } \beta = \beta', t \leq t'.$$

Let $H_{\geq 0}$ denote the set of all functions $h : \Delta^+ \times \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ with finite support. For $(\alpha_{i,j}, t) \in \Delta^+ \times \mathbb{Z}_{\geq 0}$, put

$$X_{\alpha_{i,j}}^+ (t) := [[\cdots [[X_{j-1,0}^+, X_{j-2,0}^+]_q, X_{j-3,0}^+]_q, \ldots, X_{i+1,0}^+]_q, X_{i,t}^+]_q,$$

$$X_{\alpha_{i,j}}^- (t) := [X_{i,t}^-, [X_{i+1,0}^-, \ldots, [X_{j-3,0}^-, X_{j-2,0}^-]_q, \ldots]_q]_q.$$

For $h \in H_{\geq 0}$, put

$$X_h^+ := \prod_{(\beta, t) \in \Delta^+ \times \mathbb{Z}_{\geq 0}} X_{\beta}^+ (t)^{h(\beta, t)}, \quad X_h^- := \prod_{(\beta, t) \in \Delta^+ \times \mathbb{Z}_{\geq 0}} X_{\beta}^- (t)^{h(\beta, t)}.$$

We define a total order on $I \times \mathbb{Z}_{\geq 0}$ by $(i, t) \leq (i', t')$ if $i < i'$ or $i = i', t \leq t'$. Let $H_0$ denote the set of all functions $h_0 : I \times \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ with finite support, and put

$$J_{h_0} := \prod_{(i, t) \in I \times \mathbb{Z}_{\geq 0}} J_{i,t}^{h_0(i,t)}$$

for $h_0 \in H_0$. For $k = (k_1, k_2, \ldots, k_{n-1}) \in \mathbb{Z}^I$, put

$$K^k = K_1^{k_1} K_2^{k_2} \cdots K_{n-1}^{k_{n-1}}.$$
(Note that $J_{i,0} = (q - q^{-1})^{-1}(1 - (K_i)^2)$ by the relation (Q1-1).) Then we have the following theorem.

**Theorem 1.10.** Assume that $q \neq \pm 1$, then we have the following.

(i) The multiplication map

$$U_{q,\mathbb{Q}}^- U_{q,\mathbb{Q}}^0 U_{q,\mathbb{Q}}^+ \rightarrow U_{q,\mathbb{Q}}^{(Q)}$$

(gives an isomorphism of vector spaces.

(ii) (a) $\{X_h^+ | h \in H_{\geq 0}\}$ gives a $\mathbb{C}$-basis of $U_{q,\mathbb{Q}}^+$.

(b) $\{X_h^- | h \in H_{\geq 0}\}$ gives a $\mathbb{C}$-basis of $U_{q,\mathbb{Q}}^-$.

(c) $\{K_h^k | k \in \mathbb{Z}^I, h_0 \in H_0\}$ gives a $\mathbb{C}$-basis of $U_{q,\mathbb{Q}}^0$.

(d) $\{X_h^- K_h^k J_{h_0} | h, h' \in H_{\geq 0}, h_0 \in H_0, k \in \mathbb{Z}^I\}$ gives a $\mathbb{C}$-basis of $U_{q,\mathbb{Q}}^{(Q)}$.

(iii) (a) The algebra $U_{q,\mathbb{Q}}^{+}$ is generated by $\{X_{i,t}^+ | (i,t) \in I \times \mathbb{Z}_{\geq 0}\}$ subject to the defining relations (Q2) and (Q7).

(b) The algebra $U_{q,\mathbb{Q}}^-$ is generated by $\{X_{i,t}^- | (i,t) \in I \times \mathbb{Z}_{\geq 0}\}$ subject to the defining relations (Q3) and (Q8).

(c) The algebra $U_{q,\mathbb{Q}}^0$ is generated by $\{J_{i,t}, K_i^\pm | i \in I, t \in \mathbb{Z}_{\geq 0}\}$ subject to the defining relations (Q1-1) and (Q1-2).

**Proof.** See Appendix A

1.11. In the next section, we give a connection with the shifted quantum affine algebras introduced in [11]. In particular, we see that $U_q^{(0)}$, where $\mathbf{0} = (0, \ldots, 0)$, turns out to be a Hopf subalgebra of the quantum loop algebra $U_q(L\mathfrak{sl}_n)$ associated with $\mathfrak{sl}_n$. Then the injective algebra homomorphisms $\iota_{\pm}^{(Q)} : U_q^{(Q)} \rightarrow U_q^{(0)}$ given in the following proposition have an important role in this paper.

**Proposition 1.12.** Assume that $q^2 \neq \pm 1$. We have the followings.

(i) There exists an injective algebra homomorphism $\iota_+^{(Q)} : U_q^{(Q)} \rightarrow U_q^{(0)}$ such that $X_{i,t}^+ \mapsto X_{i,t}^+ - Q_i X_{i,t+1}^+, \quad X_{i,t}^- \mapsto X_{i,t}^-, \quad K_i^\pm \mapsto K_i^\pm, \quad J_{i,t} \mapsto J_{i,t}$.

(ii) There exists an injective algebra homomorphism $\iota_-^{(Q)} : U_q^{(Q)} \rightarrow U_q^{(0)}$ such that $X_{i,t}^+ \mapsto X_{i,t}^+, \quad X_{i,t}^- \mapsto X_{i,t}^- - Q_i X_{i,t+1}^+, \quad K_i^\pm \mapsto K_i^\pm, \quad J_{i,t} \mapsto J_{i,t}$.

**Proof.** We can prove the well-definedness of the homomorphisms $\iota_{\pm}^{(Q)}$ by checking the defining relations directly.

In order to show the injectivity, it is enough to show that the restrictions of $\iota_{\pm}^{(Q)}$ to each subalgebras $U_q^{+}, U_q^{-}$ and $U_q^0$ are injective thanks to Theorem 1.10. By Theorem 1.10(iii) and the definitions of $\iota_{\pm}^{(Q)}$, it is clear that the restrictions $\iota_+^{(Q)} |_{U_q^-, \mathbb{Q}}, \quad \iota_-^{(Q)} |_{U_q^-, \mathbb{Q}}$ and $\iota_+^{(Q)} |_{U_q^0, \mathbb{Q}}, \quad \iota_-^{(Q)} |_{U_q^0, \mathbb{Q}}$ are injective. We prove the restriction $\iota_+^{(Q)} |_{U_q^+, \mathbb{Q}}$ is injective.

The injectivity of $\iota_-^{(Q)} |_{U_q^-, \mathbb{Q}}$ is similar.

Let $U^+$ be the associative algebra generated by $X_{i,t}^+ ((i,t) \in I \times \mathbb{Z}_{\geq 0})$ subject to the defining relations (Q2) and (Q7). Then both $U_q^+ \mathbb{Q}$ and $U^+_q$ are isomorphic to the algebra $U^+$ in natural way by Theorem 1.10 (iii)-(a), and the homomorphism
We prove that the endomorphism \( \iota \) of Definition 2.1 whose shifts are at most 1 since we need only these shifts.

suitable shift. We recall the definition of the shifted quantum affine algebra in [FT]

the shifted quantum affine algebra introduced in [FT]. In fact, the (\( q, a \))

Finite dimensional simple modules of (\( q, Q \))-current algebras

Remark 1.13. The injections in Proposition 1.12 are certain modifications of ones in [FT] Lemma 10.18 (see Remarks 2.5 (ii)).

\[ \psi_{i,t}^+ = 0, \quad \text{and} \quad \psi_{i,t}^- = 1, \]

\[ e_{i,t} = 0, \quad \text{and} \quad f_{i,t} = 1 \]

\[ [\psi_{j,s}, \psi_{i,t}] = [\psi_{i,s}, \psi_{j,t}] = \psi_{i,t} = 0 \quad (s_i \geq -b_i, t_j \geq b_j, s, t \leq 0), \]

\[ \psi_{i,-b_i}^- = (\psi_{i,-b_i})^{-1} = (\psi_{i,0}^-) = 1, \]

\[ e_{i,t+1} e_{j,s} - q^{a_{ij}} e_{j,s} e_{i,t+1} = q^{a_{ij}} e_{i,t} e_{j,s+1} - e_{j,s+1} e_{i,t} \quad (s, t \in \mathbb{Z}), \]

\[ f_{i,t+1} f_{j,s} - q^{-a_{ij}} f_{j,s} f_{i,t+1} = q^{-a_{ij}} f_{i,t} f_{j,s+1} - f_{j,s+1} f_{i,t} \quad (s, t \in \mathbb{Z}), \]

\[ \psi_{i,-b_i}^- e_{j,s} = q^{a_{ij}} e_{j,s}, \quad \psi_{i,0}^- e_{j,s} = q^{a_{ij}} e_{j,s}, \]

\[ q^{a_{ij}} e_{j,s} = q^{a_{ij}} e_{j,s-1} \psi_{i,t}^- = q^{a_{ij}} e_{j,s-1} \psi_{i,t}^- = q^{a_{ij}} e_{j,s-1} \psi_{i,t}^- \quad (s \in \mathbb{Z}, t \geq -b_i), \]

\[ \psi_{i,-b_i}^- f_{j,s} = q^{-a_{ij}} f_{j,s}, \quad \psi_{i,0}^- f_{j,s} = q^{-a_{ij}} f_{j,s}, \]

\[ [e_{i,t}, f_{j,s}] = \delta_{i,j} \]

\[ \begin{cases} \psi_{i,t+s}^+ = q - q^{-1} & \text{if } s + t > 0, \\ \psi_{i,0}^+ - \psi_{i,0}^- = q - q^{-1} & \text{if } s + t = 0, \\ \psi_{i,-1}^- - \psi_{i,-1}^- = q - q^{-1} & \text{if } s + t = -1 \text{ and } b_i = 1, \\ \psi_{i,t+s}^- = q - q^{-1} & \text{if } s + t < -b_i, \end{cases} \]
\[(U7) \quad [e_{i,t}, e_{j,s}] = 0 \quad \text{if } j \neq i, i \pm 1 \quad (s, t \in \mathbb{Z}),
\]

\[e_{i+1,u} e_{i,t} + e_{i,t} e_{i+1,u} \quad (s, t, u \in \mathbb{Z}),
\]

\[(U8) \quad [f_{i,t}, f_{j,s}] = 0 \quad \text{if } j \neq i, i \pm 1 \quad (s, t \in \mathbb{Z}),
\]

\[f_{i+1,u} f_{i,t} + f_{i,t} f_{i+1,u} \quad (s, t, u \in \mathbb{Z}).
\]

We define the elements \(\{h_{i,t}\}_{i \in I, t > 0}\) by

\[(\psi_{i,0}^+ z^b_i)^{-1} \left( \sum_{t \geq b_i} \psi_{i,0}^+ z^{-t} \right) = \exp((q - q^{-1}) \sum_{t > 0} h_{i,t} z^{-t}).
\]

In particular, we have

\[h_{i,1} = (q - q^{-1})^{-1} (\psi_{i,0}^+)^{-1} \psi_{i,0}^+.
\]

**Remarks 2.2.**

(i) For each \(i \in I\), the element \(\psi_{i,0}^+ \psi_{i,0}^-\) is a central element of \(U_{0,0}\).

(ii) In the case where \(b = (0, \ldots, 0)\), the algebra\( U_{0,0}/(\psi_{i,0}^+ \psi_{i,0}^- - 1 \mid i \in I)\) is isomorphic to the quantum loop algebra \(U_q(\mathfrak{gl}_n)\) associated with \(\mathfrak{sl}_n\).

2.3. For \(Q = (Q_1, Q_2, \ldots, Q_{n-1}) \in \mathbb{C}^I\), put \(b_Q = (b_1, b_2, \ldots, b_{n-1}) \in \{0, 1\}^I\) with

\[b_i = \begin{cases} 0 & \text{if } Q_i = 0, \\ 1 & \text{if } Q_i \neq 0. \end{cases}
\]

Let \(\mathcal{I}(Q)\) be the two-sided ideal of \(U_{b_Q,0}\) generated by \(\{\psi_{i,0}^+ \psi_{i,0}^- + Q_i + b_i - 1 \mid i \in I\}\), and we denote the quotient algebra \(U_{b_Q,0}/\mathcal{I}(Q)\) by \(U_{b_Q,0}^Q\). Then we have

\[(\psi_{i,0}^+)^{-1} \text{ if } Q_i = 0, \quad \text{and } \psi_{i,-1}^+ = -Q_i (\psi_{i,0}^-)^{-1} \text{ if } Q_i \neq 0
\]

in \(U_{b_Q,0}^Q\). In particular, we have \(U_{0,0}^{(0)} \cong U_q(\mathfrak{sl}_n)\) if \(Q = (0, \ldots, 0)\).

**Proposition 2.4.** Assume that \(q \neq \pm 1\). There exists an injective algebra homomorphism

\[
\Theta^Q : U_q^Q \rightarrow U_{b_Q,0}^Q,
\]

\[
X_{i,t}^+ \mapsto e_{i,t}, \quad X_{i,t}^- \mapsto f_{i,t}, \quad K_i^+ \mapsto (\psi_{i,0}^-)^{-1}, \quad K_i^- \mapsto \psi_{i,0}^-,
\]

\[
J_{i,t} \mapsto \begin{cases} (q - q^{-1})^{-1} (1 - (\psi_{i,0}^-)^2) & \text{if } t = 0, \\ (q - q^{-1})^{-1} \psi_{i,t}^+ \psi_{i,0}^- & \text{if } t > 0 \text{ and } Q_i = 0, \\ (q - q^{-1})^{-1} (Q_i^{-t} - \sum_{k=1}^{t} Q_i^{-k} \psi_{i,t-k}^+ \psi_{i,0}^-) & \text{if } t > 0 \text{ and } Q_i \neq 0. \end{cases}
\]
Proof. In order to prove the well-definedness of $\Theta^{(Q)}$, we check the relations only (Q4-3), (Q5-3) and (Q6) since other defining relations of $U_{q}^{(Q)}$ are clear. For the relation (Q4-3), we have

\[
\Theta^{(Q)}([J_{i,s+1}, X_{j,t}^+]) = \begin{cases} (q-q^{-1})-\psi_{i,s+1}^+\psi_{j,t}^- - e_{j,t}(q-q^{-1})-\psi_{i,s+1}^+\psi_{j,t}^- & \text{if } Q_i = 0, \\
(q-q^{-1})-\{(Q^{-1}_{i})-(s+1)\} \sum_{k=1}^{s+1} Q_i^{-k}\psi_{i,s+1-k}^+\psi_{j,t}^- & \text{if } Q_i \neq 0 \\
\end{cases}
\]

where we note that (Q5-3) is similar.

We check the relation (Q6). If $s = t = 0$, we have

\[
\Theta^{(Q)}([X_{i,t}^+, X_{j,s}^-]) = [e_{i,t}, f_{j,s}] = \delta_{i,j} (q-q^{-1})^{-1}(\psi_{i,t}^+ - \psi_{j,s}^-) \\
= \delta_{i,j} \begin{cases} (\psi_{i,t}^-)^{-1}(q-q^{-1})^{-1}(1 - (\psi_{i,t}^-)^2) & \text{if } Q_i = 0, \\
(\psi_{j,s}^-)^{-1}(q-q^{-1})^{-1}(1 - (\psi_{j,s}^-)^2) - Q_i(q-q^{-1})^{-1}(Q^{-1}_{i} - Q^{-1}_{i}\psi_{i,t}^+\psi_{j,s}^-) & \text{if } Q_i \neq 0 \\
\end{cases}
\]

where we note that $\psi_{i,t}^+\psi_{j,s}^- = -Q_i \in \mathcal{U}_{bq,0}^{(Q)}$ if $Q_i \neq 0$ by (2.3.1). The relation (Q5-3) is similar.

We check the relation (Q6). If $s = t = 0$, we have

\[
\Theta^{(Q)}([X_{i,t}^+, X_{j,s}^-]) = [e_{i,t}, f_{j,s}] = \delta_{i,j} (q-q^{-1})^{-1}(\psi_{i,t}^+ - \psi_{j,s}^-) \\
= \delta_{i,j} \begin{cases} (\psi_{i,t}^-)^{-1}(q-q^{-1})^{-1}(1 - (\psi_{i,t}^-)^2) & \text{if } Q_i = 0, \\
(\psi_{j,s}^-)^{-1}(q-q^{-1})^{-1}(1 - (\psi_{j,s}^-)^2) - Q_i(q-q^{-1})^{-1}(Q^{-1}_{i} - Q^{-1}_{i}\psi_{i,t}^+\psi_{j,s}^-) & \text{if } Q_i \neq 0 \\
\end{cases}
\]

where we note that $\psi_{i,t}^+\psi_{j,s}^- = -Q_i \in \mathcal{U}_{bq,0}^{(Q)}$ if $Q_i \neq 0$ by (2.3.1). If $s + t > 0$, we have

\[
\Theta^{(Q)}([X_{i,t}^+, X_{j,s}^-]) = [e_{i,t}, f_{j,s}] = \delta_{i,j} (q-q^{-1})^{-1}\psi_{i,t+s}^+ \\
= \delta_{i,j} \begin{cases} (\psi_{i,t+s}^-)^{-1}(q-q^{-1})^{-1}\psi_{i,t+s}^- & \text{if } Q_i = 0, \\
(\psi_{j,s}^-)^{-1}((q-q^{-1})(Q^{-1}_{i} - Q^{-1}_{i}\psi_{i,t+s}^+\psi_{j,s}^-) & \text{if } Q_i \neq 0 \\
\end{cases}
\]

where we note that $\psi_{i,t+s}^+\psi_{j,s}^- = -Q_i \in \mathcal{U}_{bq,0}^{(Q)}$ if $Q_i \neq 0$ by (2.3.1). If $s + t > 0$, we have

\[
\Theta^{(Q)}([X_{i,t}^+, X_{j,s}^-]) = [e_{i,t}, f_{j,s}] = \delta_{i,j} (q-q^{-1})^{-1}\psi_{i,t+s}^+ \\
= \delta_{i,j} \begin{cases} (\psi_{i,t+s}^-)^{-1}(q-q^{-1})^{-1}\psi_{i,t+s}^- & \text{if } Q_i = 0, \\
(\psi_{j,s}^-)^{-1}((q-q^{-1})(Q^{-1}_{i} - Q^{-1}_{i}\psi_{i,t+s}^+\psi_{j,s}^-) & \text{if } Q_i \neq 0 \\
\end{cases}
\]
The injectivity of $\Theta^{(Q)}$ follows from Theorem 10.10 and [FT] Proposition 5.1. □

Remarks 2.5.

(i) In the case where $Q = 0 = (0, \ldots, 0)$, we see that $b_0 = 0$ and $U^{(0)}_{0,0} \cong U_q(\mathfrak{sl}_n)$. The quantum loop algebra is a $\mathbb{Z}$-graded algebra with \( \text{deg}(e_{i,t}) = \text{deg}(f_{i,t}) = t, \text{deg}(\psi^+_{i,s}) = s \) and \( \text{deg}(\psi^-_{i,-s}) = -s \) for \( i \in I, t \in \mathbb{Z} \) and \( s \in \mathbb{Z}_{\geq 0} \).

By the injection $\Theta^{(0)} : U_q^{(0)} \to U^{(0)}_{0,0}$, we can regard $U_q^{(0)}$ as the subalgebra of $U_q(L\mathfrak{sl}_n)$ generated by the elements with nonnegative degree. Namely, $U_q^{(0)}$ is the counter part of the polynomial current Lie algebra $\mathfrak{sl}_n[x]$ which is a Lie subalgebra of the loop Lie algebra $L\mathfrak{sl}_n = \mathfrak{sl}_n[x, x^{-1}]$.

(ii) There are injective algebra homomorphisms

\[
\begin{align*}
\iota'_+ : U_{bQ,0} & \to U_{0,0}, \quad e_{i,t} \mapsto e_{i,t} - Q_i e_{i,t+1}, \quad f_{i,t} \mapsto f_{i,t}, \quad \psi^+_{i,t} \mapsto \psi^+_{i,t} - Q_i \psi^+_{i,t+1}, \\
\iota'_{-} : U_{bQ,0} & \to U_{0,0}, \quad e_{i,t} \mapsto e_{i,t}, \quad f_{i,t} \mapsto f_{i,t} - Q_i f_{i,t+1}, \quad \psi^-_{i,t} \mapsto \psi^-_{i,t} - Q_i \psi^-_{i,t+1},
\end{align*}
\]

where we put $\psi^+_{i,-1} = \psi^-_{i,1} = 0$ in $U_{0,0}$. We easily see that the injections $\iota'_\pm$ induce the injections $\iota'_+ : U_{bQ}^{(Q)} \to U^{(0)}_{0,0} \cong U_q(L\mathfrak{sl}_n)$. The injection $\iota'_+$ (resp. $\iota'_-$) is a certain modification of the injection $\iota_{\mu,-\mu,0}$ (resp. $\iota_{\mu,0,-\mu}$) given in [FT] Lemma 10.18 for the suitable $\mu$ through the isomorphism $U_{\mu,0}^{sc} \cong U_{0,\mu}^{sc}$.

We need this modification to obtain the injections from $U_{bQ,0}^{(Q)}$ to $U_{0,0}^{(0)}$. Then, we can check the diagram

\[
\begin{array}{ccc}
U_{bQ,0}^{(Q)} & \xrightarrow{\iota'_+} & U_{0,0}^{(0)} \\
\Theta^{(Q)} \downarrow & & \downarrow \Theta^{(0)} \\
U_{bQ,0}^{(Q)} & \xrightarrow{\iota'_-} & U_{0,0}^{(0)} \cong U_q(L\mathfrak{sl}_n)
\end{array}
\]

commutes.

2.6. In arguments for quantum loop algebras and shifted quantum affine algebras, we usually use generating functions for generators. In order to compare with such arguments, we prepare generating functions for $U_q^{(Q)}$ as follows.

We define generators $\Psi^+_{i,t} \in U_q^{(Q)}$ ($i \in I, t \geq -b_i$) by

\[
\Psi^+_{i,0} = K_i^+, \quad \Psi^+_{i,t} = (q - q^{-1}) K_i^+J_{i,t} \quad (t > 0)
\]

if $Q_i = 0$, and by

\[
\begin{align*}
\Psi^+_{i,-1} & = -Q_i K_i^+, \quad \Psi^+_{i,0} = K_i^+ - (q - q^{-1}) Q_i K_i^+ J_{i,1}, \\
\Psi^+_{i,t} & = (q - q^{-1}) K_i^+ (J_{i,t} - Q_i J_{i,t+1}) \quad (t > 0)
\end{align*}
\]
if \(Q_i \neq 0\). Then, Proposition \[2.4\] implies that \(\Theta^Q(\Psi^+_i) = \psi^+_i\) for \(i \in I\) and \(t \geq -b_i\).

Set

\[
X^+_i(\omega) := \sum_{t \geq 0} X^+_{i,t} \omega^t, \quad \Psi^+_i(\omega) := \sum_{t \geq -b_i} \Psi^+_{i,t} \omega^t
\]

for \(i \in I\), then we have

\[
\Theta^Q(X^+_i(\omega)) = \sum_{t \geq 0} e_{i,t} \omega^t, \quad \Theta^Q(X^-_i(\omega)) = \sum_{t \geq 0} f_{i,t} \omega^t, \quad \Theta^Q(\Psi^+_i(\omega)) = \sum_{t \geq -b_i} \psi^+_i \omega^t.
\]

\section{Algebra homomorphisms \(\Delta^Q_r\) and \(\Delta^Q_l\)}

\subsection*{3.1. In the case where \(Q = 0 = (0, \ldots, 0)\), we recall the injective homomorphism \(\Theta^{(0)} : U_q^{(0)} \rightarrow U_{q,0} \cong U_q(L\mathfrak{sl}_n)\) in Proposition \[2.4\]. Let \(\Delta : U_q(L\mathfrak{sl}_n) \rightarrow U_q(L\mathfrak{sl}_n) \otimes U_q(L\mathfrak{sl}_n)\) be the Drinfeld-Jimbo coproduct on \(U_q(L\mathfrak{sl}_n)\) (see \[FT, \text{Theorem 10.13}\] for the coproduct \(\Delta\)). Then we denote the composition of \(\Theta^{(0)}\) and \(\Delta\) by

\[
\Delta^{(0)} = \Delta \circ \Theta^{(0)} : U_q^{(0)} \rightarrow U_q(L\mathfrak{sl}_n) \otimes U_q(L\mathfrak{sl}_n).
\]

We regard \(U_q^{(0)} \otimes U_q^{(0)}\) as a subalgebra of \(U_q(L\mathfrak{sl}_n) \otimes U_q(L\mathfrak{sl}_n)\) through the injection \(\Theta^{(0)} \otimes \Theta^{(0)}\). Then we have the following proposition.

\[\textbf{Proposition 3.2.}\ Assume that \(q \neq \pm 1\) and \(Q = 0 = (0, \ldots, 0)\), then we have \(\Delta^{(0)}(U_q^{(0)}) \subset U_q^{(0)} \otimes U_q^{(0)}\). In particular, the homomorphism \(\Delta^{(0)}\) induces the algebra homomorphism

\[
\Delta^{(0)} : U_q^{(0)} \rightarrow U_q^{(0)} \otimes U_q^{(0)}.
\]

Moreover, we have

\[
\Delta^{(0)}(X^+_{i,0}) = 1 \otimes X^+_{i,0} + X^+_{i,0} \otimes K^+_i, \quad \Delta^{(0)}(X^-_{i,0}) = X^-_{i,0} \otimes 1 + K^-_i \otimes X^-_{i,0},
\]

\[
\Delta^{(0)}(K^+_i) = K^+_i \otimes K^+_i.
\]
and

\[
\Delta^{(0)}(J_{i,1}) = J_{i,1} \otimes 1 + 1 \otimes J_{i,1} - (q^2 - q^{-2})X_{i,0}^+ \otimes X_{i,1}^- \\
+ (q - q^{-1}) \sum_{l > i + 1} \tilde{X}_{\alpha_{i+1,l}}^+(0) \otimes X_{\alpha_l}^-(1) \\
+ (q - q^{-1}) \sum_{k < i} q^{k+1-i}X_{\alpha_{k,i}}^+(0) \otimes X_{\alpha_k}^-(1) \\
+ q^{-2}(q - q^{-1}) \sum_{l > i + 1} [X_{i,0}^+, \tilde{X}_{\alpha_{i+1,l}}^+(0)]q^3 \otimes X_{\alpha_l}^- (1) \\
- (q - q^{-1}) \sum_{k < i} q^{k-i-1}[X_{i,0}^+, X_{\alpha_{k,i}}^+(0)]q^3 \otimes X_{\alpha_{k,i+1}}^- (1) \\
+ (q - q^{-1})^2 \sum_{l > i + 1} q^{k-i}(\tilde{X}_{\alpha_{i,l}}^+(0)X_{\alpha_{k,i}}^+(0) - \tilde{X}_{\alpha_{i+1,l}}^+(0)X_{\alpha_{k,i+1}}^+(0)) \otimes X_{\alpha_{k,l}}^- (1),
\]

where

\[
X_{\alpha_{i,j}}^+(0) = \left[\ldots [X_{i-1,0}^+, X_{j-2,0}^+]q, \ldots, X_{i+1,0}^+, X_{i,0}^+]q, \ldots \right], \\
\tilde{X}_{\alpha_{i,j}}^+(0) = \left[\ldots [X_{i-1,0}^+, X_{j-2,0}^+]q^{-1}, \ldots, X_{i+1,0}^-, X_{i,0}^+]q^{-1}, \ldots \right], \\
X_{\alpha_{i,j}}^-(1) = [X_{i+1,0}^-, [X_{i+1,0}^-, \ldots, [X_{j-1,0}^-, X_{j-1,0}^]q]q].
\]

Proof. By Lemma 1.6 it is enough to check that \(\Delta^{(0)}(X_{i,0}^\pm), \Delta^{(0)}(J_{i,1})\) and \(\Delta^{(0)}(K_{i}^\pm)\) belong to \(U_q^{(0)} \otimes U_q^{(0)}\) for \(i \in I\). Note that

\[
\Theta^{(0)}(K_{i}^+) = (\psi_{i,0}^-)^{-1} = \psi_{i,0}^+, \quad \Theta^{(0)}(K_{i}^-) = \psi_{i,0}^-, \\
\Theta^{(0)}(J_{i,1}) = (q - q^{-1})^{-1}q_{i,0}^+\psi_{i,0}^- = h_{i,1}, \\
\Theta^{(0)}(X_{i,0}^+) = e_{i,0}, \quad \Theta^{(0)}(X_{i,0}^-) = f_{i,0}.
\]

Moreover, we see that

\[
\Theta^{(0)}(X_{\alpha_{i,j}}^+(0)) = [[\ldots [e_{j-1,0}, e_{j-2,0}]q, \ldots, e_{i+1,0}]q, e_{i,0}]q, \\
\Theta^{(0)}(\tilde{X}_{\alpha_{i,j}}^+(0)) = [[\ldots [e_{j-1,0}, e_{j-2,0}]q^{-1}, \ldots, e_{i+1,0}]q^{-1}, e_{i,0}]q^{-1}, \\
\Theta^{(0)}(X_{\alpha_{i,j}}^-(1)) = [f_{i,1}, [f_{i+1,0}, \ldots, [f_{j-1,0}, f_{j-1,0}]q]\ldots]q].
\]

Then, the proposition follows from [FT Theorem 10.13]. (In [FT Theorem 10.13], the elements \(\Theta^{(0)}(X_{\alpha_{i,j}}^+(0)), \Theta^{(0)}(\tilde{X}_{\alpha_{i,j}}^+(0))\) and \(\Theta^{(0)}(X_{\alpha_{i,j}}^-(1))\) are denoted by \(\tilde{E}_{ij}^{(0)}, E_{ij}^{(0)}\) and \(F_{ji}^{(1)}\) respectively.)

\[\square\]

Remark 3.3. In fact, the statement \(\Delta^{(0)}(U_q^{(0)}) \subset U_q^{(0)} \otimes U_q^{(0)}\) immediately follows from the RTT presentation of the quantum loop algebra and the Drinfeld-Jimbo
coproduct. The RTT presentation is crucially used in [FT, Appendix G] to derive the formula for $\Delta(h_{i,1})$ which we recalled in the above proof. These were pointed out by Alexander Tsymbaliuk after writing the first version of this paper.

**Remark 3.4.** In the case where $q = \pm 1$ and $Q = 0$, we can define the algebra homomorphism $\Delta^{(0)} : U_q^{(0)} \rightarrow U_q^{(0)} \otimes U_q^{(0)}$ by

$$
\Delta^{(0)}(X_{i,1}^+) = 1 \otimes X_{i,1}^+ + X_{i,1}^+ \otimes K_i^+,
\Delta^{(0)}(X_{i,0}^-) = X_{i,0}^- \otimes 1 + K_i^- \otimes X_{i,0}^-,
\Delta^{(0)}(K_i^+) = K_i^+ \otimes K_i^+,
\Delta^{(0)}(J_{i,t}) = J_{i,t} \otimes 1 + 1 \otimes J_{i,t}.
$$

In this case, we can check the well-definedness by direct calculations.

**3.5.** By (1.5.3), we have

$$
\Delta^{(0)}(X_{i,1}^+) = \frac{1}{2} (\Delta^{(0)}(J_{i,1})\Delta^{(0)}(X_{i,0}^+) - \Delta^{(0)}(X_{i,0}^+)\Delta^{(0)}(J_{i,1})),
$$

$$
\Delta^{(0)}(X_{i,1}^-) = -\frac{1}{2} (\Delta^{(0)}(J_{i,1})\Delta^{(0)}(X_{i,0}^-) - \Delta^{(0)}(X_{i,0}^-)\Delta^{(0)}(J_{i,1})).
$$

Thus, Proposition 3.2 implies the following corollary.

**Corollary 3.6.** We have

$$
\Delta^{(0)}(X_{i,1}^+) = 1 \otimes X_{i,1}^+ + X_{i,1}^+ \otimes K_i^+ + (q - q^{-1})X_{i,0}^+ \otimes K_i^+ J_{i,1} - q^{-1}(q - q^{-1})^2 X_{i,0}^+ X_{i,0}^+ \otimes X_{i,1}^- K_i^+
$$

$$
+ q(q - q^{-1}) \sum_{l > i + 1} \tilde{X}_{\alpha, l}^+ (0) \otimes X_{\alpha_{i+1, l}}^-(1) K_i^+
$$

$$
- (q - q^{-1})^2 \sum_{l > i + 1} X_{i,0}^+ \tilde{X}_{\alpha, l}^+ (0) \otimes X_{\alpha_{i, l}}^-(1) K_i^+
$$

$$
- q(q - q^{-1}) \sum_{k < i} q^{k-i} X_{\alpha_{k,i+1}}^+ (0) \otimes X_{\alpha_{k,i}}^- (1) K_i^+
$$

$$
- (q - q^{-1})^2 \sum_{k < i} q^{k-i} X_{i,0}^+ X_{\alpha_{k,i+1}}^+ (0) \otimes X_{\alpha_{k,i}}^- (1) K_i^+
$$

$$
- (q - q^{-1})^2 \sum_{l > i + 1} q^{k-i} \tilde{X}_{\alpha, l}^+ (0) X_{\alpha_{k,i+1}}^+ (0) \otimes X_{\alpha_{k,l}}^- (1) K_i^+.
$$

$$
\Delta^{(0)}(X_{i,1}^-) = X_{i,1}^- \otimes 1 + K_i^+ \otimes X_{i,1}^- + q^{-1}(q - q^{-1}) \sum_{l > i + 1} \tilde{X}_{\alpha_{i+1, l}}^+ (0) K_i^+ \otimes X_{\alpha_{i,l}}^-(1)
$$

$$
- (q - q^{-1}) \sum_{k < i} q^{k-i} X_{\alpha_{k,i}}^+ (0) K_i^+ \otimes X_{\alpha_{k,i+1}}^- (1)
$$

$$
- (q - q^{-1})^2 \sum_{k < i} q^{k-i} \tilde{X}_{\alpha_{i+1, l}}^+ (0) X_{\alpha_{k,i}}^+ (0) K_i^+ \otimes X_{\alpha_{k,l}}^- (1).
$$
Remark 3.7. The explicit form of $\Delta^{(0)}(X_{i,1}^+)$ in Corollary 3.6 follows directly from one of $\Lambda(f_{i,1})$ given in [FT] Theorem 10.13 through the injection $\Theta^{(0)} \otimes \Theta^{(0)}$.

3.8. We recall the injective homomorphisms $\iota^{(Q)}_\pm: U_q^{(Q)} \to U_q^{(0)}$ in Proposition 1.12 and we consider the algebra homomorphisms

$$
\Delta_q^{(Q)} := \Delta^{(0)} \circ \iota^{(Q)}_- : U_q^{(Q)} \to U_q^{(0)} \otimes U_q^{(0)}, \quad \Delta_q^{(Q)} := \Delta^{(0)} \circ \iota^{(Q)}_+ : U_q^{(Q)} \to U_q^{(0)} \otimes U_q^{(0)}.
$$

Then we have the following theorem by a similar argument as one in [FT] Theorem 10.20.

**Theorem 3.9.** (i) We have $\Delta_q^{(Q)}(U_q^{(Q)}) \subset \iota^{(Q)}_-(U_q^{(Q)}) \otimes U_q^{(0)}$. In particular, the homomorphism $\Delta_q^{(Q)}$ induces the algebra homomorphism

$$
\Delta_q^{(Q)} : U_q^{(Q)} \to U_q^{(0)} \otimes U_q^{(0)}.
$$

Moreover, we have

$$
\Delta^{(Q)}_q(X_{i,0}^+) = 1 \otimes X_{i,0}^+ + X_{i,0}^+ \otimes K_i^+,
$$

$$
\Delta^{(Q)}_q(X_{i,0}^-) = X_{i,0}^- \otimes 1 + K_i^- \otimes X_{i,0}^-.
$$

$$
- Q_1 \{ K_i^+ \otimes X_{i,1}^- + q^{-1}(q - q^{-1}) \sum_{l > i + 1} \tilde{X}_{\alpha_{i+1,l}}^+(0) K_i^+ \otimes X_{\alpha_{i,l}}^-(1) \\
- (q - q^{-1}) \sum_{k < i} q^{-k-i} X_{\alpha_{k,i}}^+(0) K_i^+ \otimes X_{\alpha_{k,i+1}}^-(1) \\
- (q - q^{-1})^2 \sum_{l > i + 1} q^{-k-i-1} \tilde{X}_{\alpha_{i+1,l}}^+(0) X_{\alpha_{k,l}}^+(0) K_i^+ \otimes X_{\alpha_{k,l}}^-(1) \}
$$

$$
\Delta^{(Q)}_q(K_i^+) = K_i^+ \otimes K_i^+,
$$

and $\Delta^{(Q)}_q(J_{1,1})$ is given by the right-hand side of (3.2.1).

(ii) We have $\Delta^{(Q)}_q(U_q^{(Q)}) \subset U_q^{(0)} \otimes \iota^{(Q)}_+(U_q^{(Q)})$. In particular, the homomorphism $\Delta^{(Q)}_q$ induces the algebra homomorphism

$$
\Delta^{(Q)}_q : U_q^{(Q)} \to U_q^{(0)} \otimes U_q^{(Q)}.
$$

Moreover, we have

$$
\Delta^{(Q)}_q(X_{i,0}^+) = 1 \otimes X_{i,0}^+ + X_{i,0}^+ \otimes K_i^+ \\
- Q_1 \{ X_{i,0}^+ \otimes K_i^+ + (q - q^{-1}) X_{i,0}^+ \otimes K_i^+ J_{1,1} - q^{-1}(q - q^{-1})^2 X_{i,0}^+ \otimes X_{i,0}^- K_i^+ \\
+ q(q - q^{-1}) \sum_{l > i + 1} \tilde{X}_{\alpha_{i+1,l}}^+(0) \otimes X_{\alpha_{i+1,l}}^-(1) K_i^+ \\
- (q - q^{-1})^2 \sum_{l > i + 1} X_{i,0}^+ \tilde{X}_{\alpha_{i,l}}^+(0) \otimes X_{\alpha_{i,l}}^-(1) K_i^+ \}
$$
\[ -q(q-q^{-1}) \sum_{k<i} q^{k-i} X^+_{\alpha_{k,i}+1}(0) \otimes X^-_{\alpha_{k,i}}(1) K^+_i \]
\[ -(q-q^{-1})^2 \sum_{k<i} q^{k-i} X^+_{i,0} X^+_{\alpha_{k,i}+1}(0) \otimes X^-_{\alpha_{k,i}}(1) K^+_i \]
\[ -(q-q^{-1})^2 \sum_{l>i+1} q^{k-i} \tilde{X}^+_{\alpha_{k,i},l}(0) X^+_{\alpha_{k,i}}(0) \otimes X^-_{\alpha_{k,i}}(1) K^+_i \}
\]
\[ \Delta_i^{(Q)}(X_{i,0}^-) = X_{i,0}^- \otimes 1 + K_i^- \otimes X_{i,0}^- \]
\[ \Delta_j^{(Q)}(K_i^\pm) = K_i^\pm \otimes K_j^\pm , \]
and \( \Delta_i^{(Q)}(J_{i,1}) \) is given by the right-hand side of (3.2.1).

**Proof.** We prove the statement (i). By Lemma [1.6], it is enough to check the relations for the generators \( X_{i,0}^\pm, J_{i,1} \) and \( K_i^\pm \) \((i \in I)\). By the definition of \( \iota^{(Q)} \), it is clear for the generators \( X_{i,0}^\pm, K_i^\pm \) and \( J_{i,1} \) \((i \in I)\). On the other hand, we have

\[ \Delta_i^{(Q)}(X_{i,0}^-) = \Delta_0^{(Q)}(X_{i,0}^- - Q_i X_{i,1}) \]
\[ = X_{i,0}^- \otimes 1 + K_i^- \otimes X_{i,0}^- \]
\[ - Q_i \{ X_{i,1}^- \otimes 1 + K_i^- \otimes X_{i,1}^- + q^{-1}(q-q^{-1}) \sum_{l>i+1} \tilde{X}^+_{\alpha_{i+1,l}}(0) K^+_i \otimes X^-_{\alpha_{i,l}}(1) \]
\[ -(q-q^{-1}) \sum_{k<i} q^{k-i} X^+_{\alpha_{k,i}}(0) K^+_i \otimes X^-_{\alpha_{k,i}+1}(1) \]
\[ -(q-q^{-1})^2 \sum_{l>i+1} q^{k-i} \tilde{X}^+_{\alpha_{k,i},l}(0) X^+_{\alpha_{k,i}}(0) K^+_i \otimes X^-_{\alpha_{k,i}}(1) \}
\]
\[ = (X_{i,0}^- - Q_i X_{i,1}) \otimes 1 + K_i^- \otimes X_{i,0}^- \]
\[ - Q_i \{ K_i^+ \otimes X_{i,1}^- + q^{-1}(q-q^{-1}) \sum_{l>i+1} \tilde{X}^+_{\alpha_{i+1,l}}(0) K^+_i \otimes X^-_{\alpha_{i,l}}(1) \]
\[ -(q-q^{-1}) \sum_{k<i} q^{k-i} X^+_{\alpha_{k,i}}(0) K^+_i \otimes X^-_{\alpha_{k,i}+1}(1) \]
\[ -(q-q^{-1})^2 \sum_{l>i+1} q^{k-i} \tilde{X}^+_{\alpha_{k,i},l}(0) X^+_{\alpha_{k,i}}(0) K^+_i \otimes X^-_{\alpha_{k,i}}(1) \} \]

Then we see that \( \Delta_i^{(Q)}(X_{i,0}^-) \subset \iota^{(Q)}(U_{q}^{(Q)}) \otimes U_{q}^{(0)} \) by the definition of \( \iota^{(Q)} \), and we have the statement (i). The statement (ii) is proven in a similar way.  

The homomorphisms \( \Delta_i^{(Q)} \) and \( \Delta_j^{(Q)} \) satisfy the following coassociativity.

**Proposition 3.10** (cf. [FKPRW, Proposition 4.14]). We have the following commutative diagrams.
Proof. We note that the coassociativity of the coproduct $\Delta^{(0)}$ on $U_q^{(0)}$ follows from the coassociativity of the Drinfeld-Jimbo coproduct $\Delta$ on $U_q(L\mathfrak{sl}_n)$. By Theorem 3.9 and the coassociativity of $\Delta^{(0)}$, we see that the diagram

\[
\begin{array}{cccccc}
U_q^{(Q)} & \xrightarrow{\Delta^{(Q)}} & U_q^{(Q)} \otimes U_q^{(0)} & \xrightarrow{Id \otimes \Delta^{(0)}} & U_q^{(0)} \otimes U_q \otimes U_q^{(0)} \\
\Delta^{(Q)} & & \Delta^{(Q)} \otimes Id & & U_q^{(0)} \otimes U_q^{(0)} \otimes U_q^{(0)} \\
U_q^{(0)} \otimes U_q^{(Q)} & \xrightarrow{Id \otimes \Delta^{(Q)}} & U_q^{(0)} \otimes U_q^{(0)} \otimes U_q^{(0)} \\
\end{array}
\]

commutes, and this diagram implies (i). The commutative diagram (ii) is proven in a similar way. \qed

§ 4. Evaluation homomorphisms

In this section, we recall the evaluation homomorphisms from $U_q(\mathfrak{sl}_n)$ to $U_q(\mathfrak{gl}_n)$ given in [3.6], and we prepare some results on evaluation modules along the calculation in [CP94b, 3.6]. In this section, we assume that $\mathbb{C}^\times \ni q \neq \pm 1$.

4.1. Put $\widehat{I} = I \cup \{0\}$, and let $\widehat{A} = (a_{ij})_{i,j \in \widehat{I}}$ be the Cartan matrix of type $A_{n-1}^{(1)}$. Namely, the submatrix $(a_{ij})_{i,j \in I}$ is the Cartan matrix of type $A_{n-1}$, and we have $a_{0,0} = 2$, $a_{0,1} = a_{1,0} = a_{0,n-1} = a_{n-1,0} = -1$ and $a_{0j} = a_{j0} = 0$ if $j \neq 0, n - 1$. Then the quantum affine algebra $U_q(\mathfrak{sl}_n)$ of type $A_{n-1}^{(1)}$ is an associative algebra over $\mathbb{C}$ generated by $e_i, f_i, k_i^\pm (i \in \widehat{I})$ subject to the following defining relations:

\[
k_i^+ k_i^- = k_i^- k_i^+ = 1, \quad [k_i^+, k_j^-] = 0, \quad k_i^+ e_j k_i^- = q^{a_{ij}} e_j, \quad k_i^+ f_j k_i^- = q^{-a_{ij}} f_j, \quad [e_i, f_j] = \delta_{ij} \frac{k_i^+ - k_i^-}{q - q^{-1}}, \quad \sum_{s=0}^{1-a_{ij}} (-1)^s e_i^{(1-a_{ij}^s)} e_j e_i^{(s)} = 0, \quad \sum_{s=0}^{1-a_{ij}} (-1)^s f_i^{(1-a_{ij}^s)} f_j f_i^{(s)} = 0,
\]

where we put $c^{(s)} = (e_i)^s/[s]!$ and $f^{(s)} = (f_i)^s/[s]!$ for $s \geq 0$. We note that $c := k_0^+ k_1^+ \ldots k_{n-1}^+$ is the canonical central element of $U_q(\mathfrak{sl}_n)$.

We also consider the quantum group $U_q(\mathfrak{gl}_n)$ associated with the general linear Lie algebra $\mathfrak{gl}_n$ which is an associative algebra over $\mathbb{C}$ generated by $E_i, F_i (i \in I)$.
Moreover, the homomorphism on the loop algebra \((D, B)\)

\[ T_i^+ T_i^- = T_i^- T_i^+, \quad [T_i^+, T_j^+] = 0, \]

\[ T_i^+ E_j T_i^- = q^{\delta_{ij} - \delta_{i,j+1}} E_j, \quad T_i^+ F_j T_i^- = q^{-(\delta_{ij} - \delta_{i,j+1})} F_j, \]

\[ [E_i, F_j] = \delta_{ij} \frac{T_i^+ T_{i-1} - T_i^- T_{i+1}}{q - q^{-1}}, \]

\[ \sum_{s=0}^{1-a_{ij}} (-1)^s E_i^{(1-a_{ij}-s)} E_j E_i^{(s)} = 0, \quad \sum_{s=0}^{1-a_{ij}} (-1)^s F_i^{(1-a_{ij}-s)} F_j F_i^{(s)} = 0, \]

and where we put \(E_i^{(s)} = (E_i)^{s}/[s]!\) and \(F_i^{(s)} = (F_i)^{s}/[s]!\) for \(s \geq 0\).

For \(\gamma \in \mathbb{C}^\times\), we have the following evaluation homomorphism \(e\nu_{\gamma} : U_q(\hat{s}l_n) \to U_q(gl_n)\).

**Proposition 4.2** ([I]). For \(\gamma \in \mathbb{C}^\times\), there exists an algebra homomorphism \(e\nu_{\gamma} : U_q(\hat{s}l_n) \to U_q(gl_n)\) such that

\[ e_i \mapsto E_i, \quad f_i \mapsto F_i, \quad h_i^+ \mapsto T_i^+ T_{i+1}^- (i \in I), \quad h_0^+ \mapsto T_1^- T_n^+, \]

\[ e_0 \mapsto \gamma q^{-1} (T_1^+ T_n^-)[F_{n-1}, [F_{n-2}, \ldots, [F_2, F_1]_q^{-1}, \ldots]_q^{-1}]_q^{-1}, \]

\[ f_0 \mapsto (-1)^{n-1} \gamma^{-1} q^{-n} (T_1^- T_n^+)[E_{n-1}, [E_{n-2}, \ldots, [E_2, E_1]_q^{-1}, \ldots]_q^{-1}]_q^{-1}. \]

Moreover, the homomorphism \(e\nu_{\gamma}\) factors through the quotient algebra \(U_q(\hat{s}l_n)/\langle c - 1 \rangle\), where \(\langle c - 1 \rangle\) is the two-sided ideal of \(U_q(\hat{s}l_n)\) generated by \(c - 1\).

It is known that the quotient algebra \(U_q(\hat{s}l_n)/\langle c - 1 \rangle\) is isomorphic to the quantum loop algebra \(U_q(Ls\hat{l}_n) \cong \mathcal{U}_{0,0}^{(0)}\) as follows.

**Proposition 4.3** ([II], [III]). There exists an algebra isomorphism \(\Psi : U_q(\hat{s}l_n)/\langle c - 1 \rangle \to \mathcal{U}_{0,0}^{(0)} \cong U_q(Ls\hat{l}_n)\) such that

\[ e_i \mapsto e_{i,0}, \quad f_i \mapsto f_{i,0}, \quad h_i^+ \mapsto \psi_{i,0}^+, \quad h_0^+ \mapsto \psi_{1,0}^- \psi_{2,0}^- \cdots \psi_{n-1,0}^-, \]

\[ e_0 \mapsto [f_{n-1,0}, [f_{n-2,0}, \ldots, [f_{2,0}, f_{1,1}]_q^{-1}, \ldots]_q^{-1}]_q^{-1} (\psi_{1,0}^+ \psi_{2,0}^- \cdots \psi_{n-1,0}^-), \]

\[ f_0 \mapsto \mu(\psi_{1,0}^+ \psi_{2,0}^- \cdots \psi_{n-1,0}^-) [e_{n-1,0}, [e_{n-2,0}, \ldots, [e_{2,0}, e_{1,-1}]_q^{-1}, \ldots]_q^{-1}]_q^{-1}, \]

where \(\mu \in \mathbb{C}^\times\) is determined by the formula \([\Psi(e_0), \Psi(f_0)] = (q - q^{-1})^{-1}(\Psi(h_0^+) - \Psi(h_0^-))\).

**4.4.** Thanks to Proposition 4.2 and Proposition 4.3, we have the algebra homomorphism \(e\nu_{\gamma} \circ \Psi^{-1} : U_q(Ls\hat{l}_n) \to U_q(gl_n)\), and we denote it by \(e\nu_{\gamma}\) again.

Let \(P = \bigoplus_{\epsilon_i \in \mathbb{Z}} \mathbb{Z}\epsilon_i\) be the weight lattice of \(gl_n\), and put \(\omega_i = \epsilon_i + \epsilon_{i+1} + \cdots + \epsilon_n\) for \(i \in I\). Let \(V(\omega_i)\) be the simple highest weight \(U_q(gl_n)\)-module of highest weight \(\omega_i\), and \(v_0^{(i)} \in V(\omega_i)\) be a highest weight vector. Then we have

\[ E_j \cdot v_0^{(i)} = 0 \text{ for all } j \in I, \quad F_j \cdot v_0^{(i)} = 0 \text{ if } j \neq i, \quad F_i \cdot v_0^{(i)} \neq 0, \quad F_i^2 \cdot v_0^{(i)} = 0, \]
\( T^\pm_j \cdot v_0^{(i)} = \begin{cases} q^\pm 1 v_0^{(i)} & \text{if } 1 \leq j \leq i, \\ v_0^{(i)} & \text{if } i < j \leq n. \end{cases} \)

For each \( \gamma \in \mathbb{C}\times \), we regard \( U_q(\mathfrak{g}_n) \)-module \( V(\omega_i) \) as a \( U_q(\mathfrak{sl}_n) \)-module through the homomorphism \( \text{ev}_\gamma : U_q(\mathfrak{sl}_n) \to U_q(\mathfrak{g}_n) \), and denote it by \( V(\omega_i)^{\text{ev}_\gamma} \). The following proposition is obtained by the same argument with one in \([\text{CP}94b, 3.6]\).

**Proposition 4.5** (cf. \([\text{CP}94b, 3.6]\)). For the \( U_q(\mathfrak{sl}_n) \)-module \( V(\omega_i)^{\text{ev}_\gamma} \) (\( i \in I, \gamma \in \mathbb{C}\times \)), we have

\[
e_{j,0} \cdot v_0^{(i)} = 0 \text{ for all } j \in I, \quad f_{j,0} \cdot v_0^{(i)} = 0 \text{ if } j \neq i, \quad f_{i,1} \cdot v_0^{(i)} = \gamma q^{-i+2} f_{i,0} \cdot v_0^{(i)}.
\]

4.6. Recall the injective homomorphism \( \Theta^{(0)} : U_q^{(0)} \to U_{0,0}^{(0)} \cong U_q(\mathfrak{sl}_n) \) in Proposition 2.4. Then we have the algebra homomorphism \( \text{ev}_\gamma \circ \Theta^{(0)} : U_q^{(0)} \to U_q(\mathfrak{g}_n) \), and we denote it by \( \text{ev}_\gamma^{(0)} \). We cannot define the evaluation homomorphism \( \text{ev}_0 : U_q(\mathfrak{sl}_n) \to U_q(\mathfrak{g}_n) \) at \( \gamma = 0 \). However, if we restrict \( U_q(\mathfrak{sl}_n) \) to \( U_q^{(0)} \), we can also define the evaluation homomorphism \( \text{ev}_0^{(0)} : U_q^{(0)} \to U_q(\mathfrak{g}_n) \) at \( \gamma = 0 \) by

\[
X_{i,t}^+ \mapsto \delta_{t,0} E_i, \quad X_{i,t}^- \mapsto \delta_{t,0} F_i, \quad J_{i,t} \mapsto \delta_{t,0} \frac{1 - (T_{i-T_{i+1}}^+)^2}{q - q^{-1}}, \quad K_i^+ \mapsto T_{i+n}^+ \bar{T}_{i+n}^-.
\]

For each \( \gamma \in \mathbb{C} \), we regard the \( U_q(\mathfrak{g}_n) \)-module \( V(\omega_i) \) as a \( U_q^{(0)} \)-module through the homomorphism \( \text{ev}_\gamma^{(0)} : U_q^{(0)} \to U_q(\mathfrak{g}_n) \), and denote it by \( V(\omega_i)^{\text{ev}_\gamma^{(0)}} \). Then we have the following proposition.

**Proposition 4.7.** For the \( U_q^{(0)} \)-module \( V(\omega_i)^{\text{ev}_\gamma^{(0)}} \) (\( i \in I, \gamma \in \mathbb{C} \)), we have

\[
X_{j,t}^+ \cdot v_0^{(i)} = 0, \quad J_{j,t} \cdot v_0^{(i)} = \begin{cases} q^{-i} (\gamma q^{-i+2})^t v_0^{(i)} & \text{if } j = i, \\ 0 & \text{if } j \neq i, \end{cases} \quad K_i^+ \cdot v_0^{(i)} = \begin{cases} q v_0^{(i)} & \text{if } j = i \\ v_0^{(i)} & \text{if } j \neq i \end{cases}
\]

for \( j \in I \) and \( t \geq 0 \).

**Proof.** From the definitions, we have

\[
(4.7.1) \quad K_i^+ \cdot v_0^{(i)} = \begin{cases} q v_0^{(i)} & \text{if } j = i \\ v_0^{(i)} & \text{if } j \neq i \end{cases}
\]

for \( j \in I \) immediately. By Proposition 4.5 in the case where \( \gamma \neq 0 \) and direct calculation in the case where \( \gamma = 0 \), we have

\[
(4.7.2) \quad X_{j,0}^+ \cdot v_0^{(i)} = 0 \text{ for all } j \in I, \quad X_{j,0}^- \cdot v_0^{(i)} = 0 \text{ if } j \neq i, \quad X_{i,1}^- \cdot v_0^{(i)} = \gamma q^{-i+2} X_{i,0}^- \cdot v_0^{(i)}.
\]
By the relation (Q1-1), we see that $J_{j,t}$ acts on $v_0^{(i)}$ as a scalar multiplication since the weight space of $V(\omega_i)$ with the weight $\omega_i$ is one-dimensional. Then, by the induction on $t$ using (4.7.2) and (1.5.3), we have

\begin{equation}
X_{j,t}^+ \cdot v_0^{(i)} = 0 \text{ for all } j \in I \text{ and } t \geq 0.
\end{equation}

The equations $X_{j,0}^- \cdot v_0^{(i)} = 0$ if $j \neq i$ in (4.7.2) and (4.7.3) together with the relation (Q6) imply that $J_{j,t} \cdot v_0^{(i)} = 0$ for all $j \in I \setminus \{i\}$ and $t \geq 0$.

For $t \geq 0$, applying $X_{i,t}^+$ to both sides of the equation $X_{i,1}^- \cdot v_0^{(i)} = q^{-i-2}X_{i,0}^- \cdot v_0^{(i)}$ in (4.7.2), we have $X_{i,t}^+ X_{i,1}^- \cdot v_0^{(i)} = q^{-i+2}X_{i,t}^+ X_{i,0}^- \cdot v_0^{(i)}$. By the relations (Q1-2), (Q6) and the equation (4.7.3), the above equation implies $J_{i,t+1} \cdot v_0^{(i)} = q^{-i+2}J_{i,t} \cdot v_0^{(i)}$.

Thus we have

\begin{equation}
J_{i,t} \cdot v_0^{(i)} = (\gamma q^{-i+2})^t J_{i,0} \cdot v_0^{(i)} = q^{-1}(\gamma q^{-i+2})^t \cdot v_0^{(i)}
\end{equation}

for $t \geq 0$, where the second equation follows from (4.7.1), (Q1-1) and (Q1-2). □

§ 5. Highest weight $U_q^{(Q)}$-modules

In the rest of the paper, we assume that the parameter $q$ is not a root of unity.

In this section, we give a notion of highest weight $U_q^{(Q)}$-modules with respect to the triangular decomposition (4.7.1). The argument is standard, so we give only notation and some statements.

5.1. Highest weight modules. For a $U_q^{(Q)}$-module $M$, we say that $M$ is a highest weight module if there exists $v_0 \in M$ satisfying the following conditions:

(i) $M$ is generated by $v_0$ as a $U_q^{(Q)}$-module.

(ii) $X_{i,t}^+ \cdot v_0 = 0$ for all $(i, t) \in I \times \mathbb{Z}_{\geq 0}$.

(iii) There exists $u = (\lambda_i, (u_{i,t})_{t > 0})_{i \in I} \in (\mathbb{C}^x \times \prod_{t > 0} \mathbb{C})^I$ such that $K_i^+ \cdot v_0 = \lambda_i v_0$ and $J_{i,t} \cdot v_0 = u_{i,t} v_0$ for each $i \in I$ and $t \in \mathbb{Z}_{>0}$.

In this case, we say that $u$ is the highest weight of $M$, and that $v_0$ is a highest weight vector of $M$. We remark that $J_{i,0} \cdot v_0 = (q - q^{-1})^{-1}(1 - \lambda_i^{-2})$ by the relation (Q1-2).

5.2. Verma modules. For $u = (\lambda_i, (u_{i,t})_{t > 0})_{i \in I} \in (\mathbb{C}^x \times \prod_{t > 0} \mathbb{C})^I$, let $\mathfrak{J}(u)$ be the left ideal of $U_q^{(Q)}$ generated by $X_{i,t}^+ ((i, t) \in I \times \mathbb{Z}_{\geq 0})$, $K_i^+ - \lambda_i$ ($i \in I$) and $J_{i,t} - u_{i,t} ((i, t) \in I \times \mathbb{Z}_{>0})$. Then, we define the Verma module as the quotient module $M(u) = U_q^{(Q)} / \mathfrak{J}(u)$. By the standard argument, the Verma module $M(u)$ has the unique maximal proper submodule $\text{rad} \, M(u)$, and we have the unique simple top $L(u) := M(u) / \text{rad} \, M(u)$. We have the following proposition whose proof is also standard.

Proposition 5.3. For $u \in (\mathbb{C}^x \times \prod_{t > 0} \mathbb{C})^I$, a highest weight simple $U_q^{(Q)}$-module of highest weight $u$ is isomorphic to $L(u)$. Moreover, any finite dimensional simple $U_q^{(Q)}$-module is isomorphic to $L(u)$ for some $u \in (\mathbb{C}^x \times \prod_{t > 0} \mathbb{C})^I$. 
§ 6. SOME SYMMETRIC POLYNOMIALS

In this section, we introduce some symmetric polynomials, and give some properties of them. These symmetric polynomials will be used to describe the highest weights of finite dimensional $U'_q\mathfrak{q}$-modules.

6.1. A partition is a non-increasing sequence $\lambda = (\lambda_1, \lambda_2, \ldots)$ of non-negative integers with only finitely many non-zero terms. The size of a partition $\lambda$, denoted by $|\lambda|$, is $|\lambda| = \sum_{i \geq 1} \lambda_i$. We denote by $\lambda \vdash t$ if $\lambda$ is a partition of size $t$. The length of a partition $\lambda$ is the number of non-zero terms, and we denote it by $\ell(\lambda)$.

Let $\mathbb{C}[x_1, x_2, \ldots, x_k]$ be the polynomial ring over $\mathbb{C}$ with indeterminate variables $x_1, \ldots, x_k$. For $t, k \in \mathbb{Z}_{>0}$, put

$$e_t(x_1, \ldots, x_k) = \sum_{1 \leq i_1 < i_2 < \cdots < i_t \leq k} x_{i_1}x_{i_2} \cdots x_{i_t} \in \mathbb{C}[x_1, \ldots, x_k],$$

$$p_t(x_1, \ldots, x_k) = x_1^t + x_2^t + \cdots + x_k^t \in \mathbb{C}[x_1, \ldots, x_k]$$

and $e_0(x_1, \ldots, x_k) = 1$. Namely, these polynomials are the elementary symmetric polynomial and the power sum symmetric polynomial respectively. For a partition $\lambda = (\lambda_1, \ldots, \lambda_k) \vdash t$ such that $\ell(\lambda) \leq k$, put

$$m_\lambda(x_1, \ldots, x_k) = \sum_{\mu \in \mathfrak{S}_k : \lambda} x_1^{\mu_1}x_2^{\mu_2} \cdots x_k^{\mu_k} \in \mathbb{C}[x_1, \ldots, x_k],$$

where $\mathfrak{S}_k \cdot \lambda = \{ \mu = (\mu_1, \ldots, \mu_k) \in \mathbb{Z}_{\geq 0}^k \mid \mu_i = \lambda_{\sigma(i)} (1 \leq i \leq k) \text{ for some } \sigma \in \mathfrak{S}_k \}$. Namely, the polynomial $m_\lambda(x_1, \ldots, x_k)$ is the monomial symmetric polynomial associated with $\lambda$.

For $t, k \in \mathbb{Z}_{>0}$, we define a polynomial $p_t(q)(x_1, \ldots, x_k) \in \mathbb{C}[x_1, \ldots, x_k]$ by

$$p_t(q)(x_1, \ldots, x_k) := \sum_{\lambda \vdash t, \ell(\lambda) \leq k} q^{-\ell(\lambda)}(q^{t-1} - q^{-1})^{\ell(\lambda)-1} m_\lambda(x_1, \ldots, x_k).$$

From the definition, we see that $p_t(q)(x_1, \ldots, x_k)$ is a symmetric polynomial.

**Remark 6.2.** In the case where $q = 1$, we have $p_t(1)(x_1, \ldots, x_k) = p_t(x_1, \ldots, x_k)$. Thus, the polynomial $p_t(q)(x_1, \ldots, x_k)$ is a $q$-analogue of the power sum symmetric polynomial.

**Lemma 6.3.** For $t, k \in \mathbb{Z}_{>0}$, the polynomial $p_t(q)(x_1, \ldots, x_k)$ satisfies the following equations:

(i) $p_t(q)(x_1, \ldots, x_k)$

$$= p_t(q)(x_1, \ldots, x_{k-1}) + q^{-1}x_k^t + q^{-1}(q - q^{-1}) \sum_{z=1}^{t-1} p_z(q)(x_1, \ldots, x_{k-1})x_k^{t-z}. $$

(ii) $p_t(q)(x_1, \ldots, x_k)$

$$= (-1)^{t-1}q^{-t} [t] e_t(x_1, \ldots, x_k) + \sum_{z=1}^{t-1} (-1)^{t+z-1} p_z(q)(x_1, \ldots, x_k)e_{t-z}(x_1, \ldots, x_k).$$
(iii) \( p_{k+t}(q)(x_1, \ldots, x_k) = \sum_{z=0}^{k-1} (-1)^{k+z-1} p_{t+z}(q)(x_1, \ldots, x_k) e_{k-z}(x_1, \ldots, x_k) \).

Proof. (i). By the definition of the monomial symmetric polynomials, for \( 1 \leq l \leq k \), we see that

\[
\sum_{\lambda \vdash \ell} \sum_{\ell(\lambda) = l} m_\lambda(x_1, \ldots, x_{k-1}) = \delta_{l \neq k} \sum_{\lambda \vdash \ell} m_\lambda(x_1, \ldots, x_{k-1}) + \delta_{l = 1} x_k^t + \delta_{l \neq 1} \sum_{z=1}^{t-1} \sum_{\lambda \vdash \ell} m_\lambda(x_1, \ldots, x_{k-1}) x_k^z,
\]

where \( \delta_{(*)} = 1 \) if the condition * is true, and \( \delta_{(*)} = 0 \) if the condition * is false. Thus, we have

\[
p_t(q)(x_1, \ldots, x_k) = \sum_{l=1}^{k-1} q^{-l}(q - q^{-1})^{l-1} \sum_{\lambda \vdash \ell} m_\lambda(x_1, \ldots, x_{k-1}) + q^{-1} x_k^t
\]

\[
+ \sum_{l=2}^k q^{-l}(q - q^{-1})^{l-1} \sum_{z=1}^{t-1} \sum_{\lambda \vdash \ell} m_\lambda(x_1, \ldots, x_{k-1}) x_k^z
\]

\[
= \sum_{\ell(\lambda) \leq k-1} q^{-\ell(\lambda)}(q - q^{-1})^{\ell(\lambda)-1} m_\lambda(x_1, \ldots, x_{k-1}) + q^{-1} x_k^t
\]

\[
+ \sum_{z=1}^{t-1} \left( \sum_{\lambda \vdash \ell} q^{-\ell(\lambda)}(q - q^{-1})^{\ell(\lambda)-1} m_\lambda(x_1, \ldots, x_{k-1}) \right) z_k^x
\]

\[
= p_t(q)(x_1, \ldots, x_{k-1}) + q^{-1} x_k^t + q^{-1}(q - q^{-1}) \sum_{z=1}^{t-1} p_{t-z}(q)(x_1, \ldots, x_{k-1}) x_k^z.
\]

(ii). Put

\[
\tilde{p}_t(q)(x_1, \ldots, x_k) = (-1)^{t-1} q^{-t} t e_t(x_1, \ldots, x_k) + \sum_{z=1}^{t-1} (-1)^{t+z-1} \tilde{p}_z(q)(x_1, \ldots, x_k) e_{t-z}(x_1, \ldots, x_k).
\]

We can prove the equation (i) replacing \( p_t(q)(x_1, \ldots, x_k) \) with \( \tilde{p}_t(q)(x_1, \ldots, x_k) \) by the induction on the degree \( t \). Then we can prove that \( p_t(q)(x_1, \ldots, x_k) = \tilde{p}_t(q)(x_1, \ldots, x_k) \) by the induction on the number \( k \) of variables using the equation (i) for both polynomials. As a consequence, we obtain (ii).
(iii). Note that \( e_t'(x_1, \ldots, x_k) = 0 \) if \( t' > k \), and the equation (iii) follows from the equation (ii). \( \square \)

**Corollary 6.4.** For \( k \in \mathbb{Z}_{>0} \), the set of polynomials

\[
\{ p_t(q)(x_1, \ldots, x_k) \mid 1 \leq t \leq k \}
\]

is algebraically independent over \( \mathbb{C} \).

**Proof.** Note that we assume that \( q \) is not a root of unity. Then we can prove the corollary in the same way with the corresponding statement for power sum symmetric polynomials using the equation (ii) in Lemma 6.3. \( \square \)

**Corollary 6.5.** For \( t = 1, 2, \ldots, k \), there exist the unique \( a_{\lambda} \in \mathbb{C} \) (\( \lambda \vdash t \)) such that

\[
e_t(x_1, x_2, \ldots, x_k) = \sum_{\lambda \vdash t} a_{\lambda} p_{\lambda}(q)(x_1, x_2, \ldots, x_k),
\]

where we put \( p_{\lambda}(q)(x_1, \ldots, x_k) = \prod_{i=1}^{\ell(\lambda)} p_{\lambda_i}(q)(x_1, \ldots, x_k) \).

**Proof.** We can prove the existence of the numbers \( a_{\lambda} \) (\( \lambda \vdash t \)) by the induction on \( t \) using Lemma 6.3 (ii). The uniqueness of \( a_{\lambda} \) (\( \lambda \vdash t \)) follows from Corollary 6.4. \( \square \)

**Proposition 6.6.** For \( k \in \mathbb{Z}_{>0} \), let \( P^{(k)}(\omega) = 1 + (q - q^{-1}) \sum_{t>0} p_t(q)(x_1, \ldots, x_k) \omega^t \) be the generating function. Then we have

\[
P^{(k)}(\omega) = \frac{(1 - q^{-2}x_1 \omega)(1 - q^{-2}x_2 \omega) \ldots (1 - q^{-2}x_k \omega)}{(1 - x_1 \omega)(1 - x_2 \omega) \ldots (1 - x_k \omega)}.
\]

**Proof.** In this proof, we denote \( p_t(q)(x_1, \ldots, x_k) \) (resp. \( e_t(x_1, \ldots, x_k) \)) by \( p_t(q) \) (resp. \( e_t \)) simply. We consider the generating function \( E^{(k)}(\omega) = \sum_{t \geq 0} (-1)^t e_t \omega^t \). Then, we have

\[
P^{(k)}(\omega) E^{(k)}(\omega) = \left( 1 + (q - q^{-1}) \sum_{t>0} p_t(q) \omega^t \right) \left( \sum_{t \geq 0} (-1)^t e_t \omega^t \right)
\]

\[
= \sum_{t \geq 0} (-1)^t e_t \omega^t + \sum_{t>0} \left( (q - q^{-1}) \sum_{z=1}^{t} (-1)^{t-z} p_z(q) e_{t-z} \right) \omega^t.
\]

Applying Lemma 6.3 (ii), we have

\[
P^{(k)}(\omega) E^{(k)}(\omega) = \sum_{t \geq 0} (-1)^t e_t \omega^t + \sum_{t>0} (q - q^{-1})(-1)^{t-1} q^{-[t]} e_t \omega^t
\]

\[
= \sum_{t \geq 0} (-1)^t q^{-2t} e_t \omega^t
\]

\[
= E^{(k)}(q^{-2} \omega).
\]
On the other hand, we have

\[ E^{(k)}(\omega) = \sum_{t=0}^{k} (-1)^t c_t(x_1, \ldots, x_k) \omega^t = (1 - x_1 \omega)(1 - x_2 \omega) \ldots (1 - x_k \omega) \]

since \( c_t(x_1, \ldots, x_k) = 0 \) if \( t > k \). As a consequence, we have

\[ P^{(k)}(\omega) = \frac{E^{(k)}(q^{-2}\omega)}{E^{(k)}(\omega)} = \frac{(1 - q^{-2}x_1 \omega)(1 - q^{-2}x_2 \omega) \ldots (1 - q^{-2}x_k \omega)}{(1 - x_1 \omega)(1 - x_2 \omega) \ldots (1 - x_k \omega)}. \]

**Remark 6.7.** The formula in Lemma 6.3 (ii) corresponds to the definition (8.1.1) which is identified with [CP91 Proposition 3.5 (ii)] under the injective algebra homomorphism \( \Theta^{(0)} \) (see the paragraph 8.1). Thus, the formula in Lemma 6.3 (ii) is a \( q \)-analogue of Newton’s formula relating the elementary symmetric polynomials and the power sums suggested in [CP91 Remark 3.5]. Under this correspondence, Proposition 6.6 corresponds to [CP91 Corollary 3.5] (see also Corollary 8.13).

**6.8.** For \( t, k \in \mathbb{Z}_{>0} \) and \( Q, \beta \in \mathbb{C}^\times \), we define a polynomial \( p_t^{(Q)}(q; \beta)(x_1, \ldots, x_k) \in \mathbb{C}[x_1, \ldots, x_k] \) by

\[
\begin{align*}
(6.8.1) \quad p_t^{(Q)}(q; \beta)(x_1, \ldots, x_k) & := p_t(q)(x_1, \ldots, x_k) + \tilde{\beta} Q^{-t} + (q - q^{-1}) \sum_{z=1}^{t-1} \tilde{\beta} Q^{-t+z} p_z(q)(x_1, \ldots, x_k),
\end{align*}
\]

where we put \( \tilde{\beta} = (q - q^{-1})^{-1}(1 - \beta^{-2}) \).

By definition, the polynomial \( p_t^{(Q)}(q; \beta)(x_1, \ldots, x_k) \) is a symmetric polynomial. In the case where \( \beta = \pm 1 \), we have \( p_t^{(Q)}(q; \pm 1)(x_1, \ldots, x_k) = p_t(q)(x_1, \ldots, x_k) \).

By the definition (6.8.1) together with Corollary 6.4, we have the following lemma.

**Lemma 6.9.** For \( k \in \mathbb{Z}_{>0} \) and \( Q, \beta \in \mathbb{C}^\times \), the set of polynomials

\[ \{ p_t^{(Q)}(q; \beta)(x_1, \ldots, x_k) \mid 1 \leq t \leq k \} \]

is algebraically independent over \( \mathbb{C} \).

**Lemma 6.10.** Fix \( k \in \mathbb{Z}_{>0} \) and we put \( x = (x_1, \ldots, x_k) \) for simplicity. For \( t \in \mathbb{Z}_{>0} \) and \( Q, \beta \in \mathbb{C}^\times \), the polynomial \( p_t^{(Q)}(q; \beta)(x) \) satisfies the following equations:

(i) \( p_t^{(Q)}(q; \beta)(x) \)

\[
\begin{align*}
&= (-1)^{t-1} q^{-t} [t] e_t(x) + \tilde{\beta} Q^{-t} + \sum_{z=1}^{t-1} (-1)^{t-z+1} (p_z^{(Q)}(q; \beta)(x) - q^{-2(t-z)} \tilde{\beta} Q^{-z}) e_{t-z}(x).
\end{align*}
\]
The equation (i) follows from the definition (6.8.1) and Lemma 6.3 (ii). We prove (ii). Note that \( e_{\ell}(x) = 0 \) if \( t' > k \), then the equation (i) implies

\[
p_k^{(Q)}(q; \beta)(x) - Q^{-1}p_{k+1}^{(Q)}(q; \beta)(x)
= \tilde{\beta}Q^{-(k+t)} + \sum_{z=t}^{k+t-1} (-1)^{k+z+1}(p_z^{(Q)}(q; \beta)(x) - q^{-2(k+z)}\tilde{\beta}Q^{-z})e_{k+z}(x)
\]

\[
- Q^{-1}\{\delta_{t,1}(1)^{-1}q^{-k}[k]e_k(x) + \tilde{\beta}Q^{-(k+t-1)}
+ \sum_{z=\max(1,t-1)}^{k+t-2} (-1)^{k+z-2}(p_z^{(Q)}(q; \beta)(x) - q^{-2(k+z-1)}\tilde{\beta}Q^{-z})e_{k+z-1}(x)\}
\]

\[
\begin{cases}
\sum_{z=t}^{k+t-1} (-1)^{k+z+1}(p_z^{(Q)}(q; \beta)(x) - Q^{-1}p_{z-1}^{(Q)}(q; \beta)(x))e_{k+z}(x) & \text{if } t > 1, \\
(1)^{k+1}p_1^{(Q)}(q; \beta)(x) - q^{-2k}\tilde{\beta}Q^{-1} - Q^{-1}q^{-k}[k]e_k(x) \\
+ \sum_{z=2}^{k} (-1)^{k+z+2}(p_z^{(Q)}(q; \beta)(x) - Q^{-1}p_{z-1}^{(Q)}(q; \beta)(x))e_{k+z}(x) & \text{if } t = 1.
\end{cases}
\]

Note that \( q^{-2k}\tilde{\beta} + q^{-k}[k] = (q - q^{-1})^{-1}(1 - (\beta q^{-k})^{-2}) = p_0^{(Q)}(q; \beta)(x) \), we have the equation (ii) by replacing \( z - t \) with \( z \).

\[\square\]

\section{7. One-dimensional \( U_q^{(Q)} \)-modules}

In this section, we classify one-dimensional \( U_q^{(Q)} \)-modules.

\subsection{7.1.} Let \( L = \mathbb{C}v \) be a one-dimensional \( U_q^{(Q)} \)-module with a basis \( v \). Then \( K_i^+(i \in I) \) acts on \( v \) as a scalar multiplication. We denote the eigenvalue of the action of \( K_i^+ \) by \( \beta_i \). By the relation (Q1-2), we have \( \beta_i \neq 0 \).

For \((j, t) \in I \times \mathbb{Z}_{\geq 0} \), the element \( X_{\pm}^{j,t} \cdot v \) is an eigenvector of the eigenvalue \( q^{\pm a_{ji}}\beta_i \) for the action of \( K_i^+ \) if \( X_{\pm}^{j,t} \cdot v \neq 0 \) by the relations (Q4-1) and (Q5-1). However, \( L \) is one-dimensional, thus we have \( X_{\pm}^{j,t} \cdot v = 0 \) for all \((j, t) \in I \times \mathbb{Z}_{\geq 0} \).

For \((i, t) \in I \times \mathbb{Z}_{\geq 0} \), we have \((K_i^+J_{i,t} - Q_iK_i^+J_{i,t+1}) \cdot v = [X_{i,t}^+, X_{i,0}^-] \cdot v = 0 \) by the relation (Q6). This equation implies that \( J_{i,t} \cdot v = 0 \) if \( Q_i = 0 \), and \( J_{i,t} \cdot v = Q_iJ_{i,t+1} \cdot v \) if \( Q_i \neq 0 \). Thus, we have \( J_{i,t} \cdot v = Q_i^{-1}J_{i,0} \cdot v \) if \( Q_i \neq 0 \). On the other hand, by the relation (Q1-2), we have \( J_{i,0} \cdot v = (q - q^{-1})^{-1}(1 - \beta_i^{-2})v \). Then we have \( \beta_i = \pm 1 \) if
$Q_i = 0$ since $J_{i,0} \cdot v = 0$ in this case. As a consequence, we have

$$(7.1.1) \quad X_{i,t}^+ \cdot v = 0, \quad K_i^+ \cdot v = \beta_i^{\pm 1} \cdot v, \quad J_{i,t} \cdot v = \begin{cases} 0 & \text{if } Q_i = 0, \\ \frac{1 - \beta_i^{-2}}{q - q^{-1}} Q_i^{-1} v & \text{if } Q_i \neq 0 \end{cases}$$

for $i \in I$ and $t \in \mathbb{Z}_{\geq 0}$, where $\beta_i = \pm 1$ if $Q_i = 0$.

**Proposition 7.3.** Any one-dimensional $U_q^{(Q)}$-module is isomorphic to $D_{\beta}^{(Q)}$ for some $\beta \in \mathbb{B}^{(Q)}$.

By [2.6.1], [2.6.2] and (7.1.1), we have the following corollary.

**Corollary 7.4.** For $D_{\beta}^{(Q)} = Cv (\beta \in \mathbb{B}^{(Q)})$, we have

$$\Psi_i^+(\omega) \cdot v = \begin{cases} \beta_i v & \text{if } Q_i = 0, \\ (\beta_i^{-1} - Q_i \beta_i^{-1} \omega) v & \text{if } Q_i \neq 0. \end{cases}$$

§ 8. Finite dimensional simple modules of $U_q(\mathfrak{sl}_2^{(0)}[x])$

In this section, we classify the isomorphism classes of finite dimensional simple modules of the algebra $U_q^{(0)} = U_q(\mathfrak{sl}_2^{(0)}[x])$ in the case of rank one and of $Q = 0$.

We recall that, in the case where $Q = 0$, the algebra $U_q^{(0)}$ is a subalgebra of the quantum loop algebra $U_q(\mathfrak{ls}l_2)$ through the injective homomorphism $\Theta^{(0)}$ in Proposition 2.2. In this case, the argument to classify the finite dimensional simple $U_q^{(0)}$-modules is essentially the same as the argument for $U_q(\mathfrak{ls}l_2)$ given in [CP91]. However, we discuss the case where $Q = 0$ in this section for completeness, and it is also useful in order to consider the case where $Q \neq 0$ in the next section.

In this and next sections, we consider only the case of rank one, namely $I = \{1\}$, so we omit the indices for $I$, e.g. we denote $X_{i,t}^\pm$ by $X_t^\pm$ simply, and so on.

**8.1.** For $t, k \in \mathbb{Z}_{\geq 0}$, put

$$X_t^{+(k)} = \frac{(X_t^+)^k}{[k]!}, \quad X_t^{-(k)} = \frac{(X_t^-)^k}{[k]!}.$$

For $t \in \mathbb{Z}_{\geq 0}$, we define the element $j_{[t]}^{(0)} \in U_q^{(0)}$ inductively by

$$(8.1.1) \quad j_{[0]}^{(0)} = 1 \quad \text{and} \quad j_{[t]}^{(0)} = q^t \frac{1}{[t]} \sum_{z=1}^{t} (-1)^{z-1} J_z j_{[t-z]}^{(0)} \quad \text{for } t > 0.$$
For examples, we have

\[ J_{[0]}^{(0)} = 1, \quad J_{[1]}^{(0)} = qJ_1, \quad J_{[2]}^{(0)} = \frac{1}{2}(q^3J_1^2 - q^2J_2), \]

\[ J_{[3]}^{(0)} = \frac{1}{3!}(q^6J_1^3 - (2q^5 + q^3)J_1J_2 + (q^4 + q^2)J_3). \]

Compare the definition [8.1.1] with [CP91, Proposition 3.5 (ii)], under the injective algebra homomorphism \( \Theta : U_q^{(0)} \rightarrow U_{q,0}^{(0)} \cong U_q(sl_2) \), then we see that \( \Theta(J_{[i]}^{(0)}) = (-1)^iP_i \) for \( t \in \mathbb{Z}_{\geq 0} \), where \( P_i \in U_q(sl_2) \) is an element given in [CP91, Proposition 3.5]. The following lemma is a slight variation of [CP91, Proposition 3.5 (iii)].

**Lemma 8.2.** For \( k \in \mathbb{Z}_{> 0} \), we have

\[ X_1^{+(k)}X_0^{-(k+1)} \equiv q^{-k(k+1)} \sum_{z=0}^{k} (-1)^z X_z^{-}(K^+)k J_{[k-z]}^{(0)} \mod \mathfrak{x}_+, \]

where \( \mathfrak{x}_+ \) is the left ideal of \( U_q^{(0)} \) generated by \( \{ X_t^+ \mid t \geq 0 \} \).

*Proof.* See Appendix B. \( \square \)

**8.3.** By Proposition 5.3 any finite dimensional simple \( U_q^{(0)} \)-module is isomorphic to the highest weight module \( L(u) \) for some \( u = (\lambda, (u_t)_{t>0}) \in \mathbb{C}^\times \times \prod_{t>0} \mathbb{C} \). We have the following necessary condition for \( L(u) \) to be finite dimensional.

**Proposition 8.4.** For \( u = (\lambda, (u_t)_{t>0}) \in \mathbb{C}^\times \times \prod_{t>0} \mathbb{C} \), if the highest weight simple \( U_q^{(0)} \)-module \( L(u) \) is finite dimensional, then there exist \( k \in \mathbb{Z}_{\geq 0} \) and \( \gamma_1, \gamma_2, \ldots, \gamma_k \in \mathbb{C} \) such that

\[ (8.4.1) \quad \lambda = \pm q^k, \quad u_t = \begin{cases} 0 & \text{if } k = 0, \\ p_t(q)(\gamma_1, \gamma_2, \ldots, \gamma_k) & \text{if } k > 0 \end{cases} \quad (t > 0). \]

*Proof.* Let \( v_0 \in L(u) \) be a highest weight vector. By the relation (Q5-1), we have \( K^+X_0^{-(k)} \cdot v_0 = q^{-2k}\lambda X_0^{-(k)} \cdot v_0 \). Namely \( X_0^{-(k)} \cdot v_0 \) is an eigenvector of the eigenvalue \( q^{-2k}\lambda \) for the action of \( K^+ \) if \( X_0^{-(k)} \cdot v_0 \neq 0 \). Thus, there exists a non-negative integer \( k \) such that \( X_0^{-(k)} \cdot v_0 \neq 0 \) and \( X_0^{-(k+1)} \cdot v_0 = 0 \) since \( L(u) \) is finite dimensional.

In the case where \( k = 0 \), we can easily check that \( L(u) \) is one-dimensional. Then we have (8.4.1) by Proposition 7.3.

Assume that \( k > 0 \). By the induction on \( c \) using the relation (Q1-2) and (Q6), we can show that \( [X_0^+, X_0^{-(c)}] = X_0^{-(c-1)}(q - q^{-1})^{-1}(q^{-c+1}K^+ - q^{c-1}K^-) \) for \( c > 0 \). Then we have

\[ 0 = X_0^+X_0^{-(k+1)} \cdot v_0 = \frac{q^{-k}\lambda - q^k\lambda^{-1}}{q - q^{-1}}X_0^{-(k)} \cdot v_0. \]
This implies that $\lambda = \pm q^k$ since $X_0^{-(k)} \cdot v_0 \neq 0$.

By Corollary 6.4, there exist $\gamma_1, \gamma_2, \ldots, \gamma_k \in \mathbb{C}$ such that $u_t = p_t(q)(\gamma_1, \gamma_2, \ldots, \gamma_k)$ for $t = 1, 2, \ldots, k$.

By the induction on $t$ using (8.4.1) and Lemma 6.3 (ii), we see that

(8.4.2) 
$$J_{[t]}^0 \cdot v_0 = e_t(\gamma_1, \gamma_2, \ldots, \gamma_k)v_0$$

for $t = 1, 2, \ldots, k$.

By Lemma 8.2 and the relation (Q6), for $t > 0$, we have

$$0 = X_1^+ X_1^{+(k)} X_0^{-(k+1)} \cdot v_0 = q^{-k(k+1)} \lambda^{k+1} \sum_{z=0}^{k} (-1)^z J_{t+z} J_{[k-z]}^0 \cdot v_0,$$

where we note that $X_1^+(K^+)J_{[k-z]}^0 \cdot v_0 = 0$ since $v_0$ is a highest weight vector. Note that $J_{[0]}^0 = 1$, this equation implies that

(8.4.3) 
$$J_{t+k} \cdot v_0 = \sum_{z=0}^{k-1} (-1)^{k-z+1} J_{t+z} J_{[k-z]}^0 \cdot v_0.$$ 

Then we can show that $u_{t+k} = p_{t+k}(q)(\gamma_1, \ldots, \gamma_k)$ for $t > 0$ by the induction on $t$ using (8.4.2), (8.4.3) and Lemma 6.3 (iii).

8.5. In order to prove that the highest weight simple module $L(\mathfrak{u})$ is finite dimensional if $\mathfrak{u}$ is given by (8.4.1), we use evaluation modules through the following evaluation homomorphisms from $U_q^{(0)}$ to the quantum group $U_q(s\mathfrak{l}_2)$. Let $e, f$ and $K^\pm$ be the usual Chevalley generators of $U_q(s\mathfrak{l}_2)$. For $\gamma \in \mathbb{C}$, we have the algebra homomorphism $\overline{ev}_{\gamma} : U_q^{(0)} \to U_q(s\mathfrak{l}_2)$ such that

$$X_1^+ \mapsto \gamma^t q^{-t} (K^+)^t e, \quad X_1^{-} \mapsto \gamma^t q^{-t} f(K^-)^t, \quad K^\pm \mapsto K^\pm,$$

$$J_t \mapsto \gamma^t q^{-t} (K^+) \frac{1 - (K^-)^2}{q - q^{-1}} - \gamma^t (q^t - q^{-t})(K^+)^t f e.$$

We remark that, if $\gamma \neq 0$, the homomorphism $\overline{ev}_{\gamma}$ is the restriction of the evaluation homomorphism $ev_{\gamma} : U_q(Ls\mathfrak{l}_2) \to U_q(s\mathfrak{l}_2)$ given in [CP91 Proposition 4.1] through the injection $\Theta^{(0)} : U_q^{(0)} \to U_q(Ls\mathfrak{l}_2)$. In the case where $\gamma = 0$, we can easily check the well-definedness of $\overline{ev}_{\gamma}$ by direct calculations.

Let $V_1 = \mathbb{C}v_0 \oplus \mathbb{C}v_1$ be the two-dimensional simple $U_q(s\mathfrak{l}_2)$-module of type 1, namely the action of $U_q(s\mathfrak{l}_2)$ is given by

$$K^+ \cdot v_0 = q v_0, \quad e \cdot v_0 = 0, \quad f \cdot v_0 = v_1, \quad K^+ \cdot v_1 = q^{-1} v_1, \quad e \cdot v_1 = v_0, \quad f \cdot v_1 = 0.$$
For $\gamma \in \mathbb{C}$, we regard $V_1$ as a $U_q^{(0)}$-module through the homomorphism $\tilde{ev}_\gamma^{(0)}$, and we denote it by $V_1^{\tilde{ev}_\gamma^{(0)}}$. By definition, we have

\begin{equation}
(8.5.1) \quad X_t^+ \cdot v_0 = 0, \quad K^+ \cdot v_0 = q v_0, \quad J_t \cdot v_0 = q^{-1} \gamma^t v_0 \quad (t \geq 0).
\end{equation}

**Remark 8.6.** We can also discuss by using the evaluation homomorphisms $ev_\gamma^{(0)} : U_q^{(0)} \to U_q(\mathfrak{gl}_2)$ given in [4]. Both arguments are essentially the same although the eigenvalues for the action of $J_t$ are different. In this section, we use $\tilde{ev}_\gamma^{(0)}$ instead of $ev_\gamma^{(0)}$ for a compatibility with the argument in [CP91] (see Remark 8.14).

**Proposition 8.7.** For $u = (\lambda,(u_t)_{t \geq 0}) \in \mathbb{C}^x \times \prod_{t \geq 0} \mathbb{C}$, if there exist $k \in \mathbb{Z}_{\geq 0}$ and $\gamma_1, \gamma_2, \ldots, \gamma_k \in \mathbb{C}$ such that

\begin{equation}
\lambda = \pm q^k, \quad u_t = \begin{cases} 0 & \text{if } k = 0, \\ p_t(q)(\gamma_1, \gamma_2, \ldots, \gamma_k) & \text{if } k > 0 \quad (t > 0), \end{cases}
\end{equation}

then $L(u)$ is finite dimensional.

**Proof.** Note that the coproduct $\Delta^{(0)}$ on $U_q^{(0)}$ is a restriction of the coproduct on $U_q(Ls\mathfrak{l}_2)$ through the injection $\Theta^{(0)}$. Then, by [CP91, Proposition 4.4], we have

\begin{equation}
\Delta^{(0)}(X_t^+) \equiv X_t^+ \otimes K^+ + 1 \otimes X_t^+ + (q - q^{-1}) \sum_{z=1}^{t'} X_{t'-z}^+ \otimes K^+ J_z \mod \mathfrak{x}_+^2 \otimes \mathfrak{x}_-,
\end{equation}

\begin{equation}
\Delta^{(0)}(J_t) \equiv J_t \otimes 1 + 1 \otimes J_t + (q - q^{-1}) \sum_{z=1}^{t-1} J_z \otimes J_{t-z} \mod \mathfrak{x}_+ \otimes \mathfrak{x}_-,
\end{equation}

for $t' \geq 0$ and $t > 0$, where $\mathfrak{x}_+^2$ (resp. $\mathfrak{x}_+, \mathfrak{x}_-$) is the left ideal of $U_q^{(0)}$ generated by $\{X_s^+ X_s^+ \mid s, s' \geq 0\}$ (resp. $\{X_s^+ \mid s \geq 0\}$, $\{X_s^- \mid s \geq 0\}$).

For each $\gamma_i (1 \leq i \leq k)$, we consider the evaluation module $V_1^{\tilde{ev}_\gamma^{(0)}}$ at $\gamma_i$, and let $v_0^{(i)} \in V_1^{\tilde{ev}_\gamma^{(0)}}$ be a highest weight vector. We also consider the one-dimensional $U_q^{(0)}$-module $\mathcal{D}_{\pm 1}^{(0)} = \mathbb{C} v$ given by (7.1.1). Through the coproduct $\Delta^{(0)}$, we consider the $U_q^{(0)}$-module $\mathcal{D}_{\pm 1}^{(0)} \otimes V_1^{\tilde{ev}_\gamma^{(0)}} \otimes V_1^{\tilde{ev}_{\gamma_2}^{(0)}} \otimes \cdots \otimes V_1^{\tilde{ev}_{\gamma_k}^{(0)}}$. Let $V(\pm 1; \gamma_1, \ldots, \gamma_k)$ be the $U_q^{(0)}$-submodule of $\mathcal{D}_{\pm 1}^{(0)} \otimes V_1^{\tilde{ev}_\gamma^{(0)}} \otimes V_1^{\tilde{ev}_{\gamma_2}^{(0)}} \otimes \cdots \otimes V_1^{\tilde{ev}_{\gamma_k}^{(0)}}$ generated by $v \otimes v_0^{(1)} \otimes v_0^{(2)} \otimes \cdots \otimes v_0^{(k)}$. Then, by definition together with (8.7.2), we see that

\begin{align*}
X_t^+ \cdot (v \otimes v_0^{(1)} \otimes \cdots \otimes v_0^{(k)}) &= 0 \quad (t \geq 0), \\
K^+ \cdot (v \otimes v_0^{(1)} \otimes \cdots \otimes v_0^{(k)}) &= \pm q^k v \otimes v_0^{(1)} \otimes \cdots \otimes v_0^{(k)}.
\end{align*}
For $t > 0$, we have $J_t \cdot (v \otimes v_0^{(1)}) = q^{-1} \gamma_1 v \otimes v_0^{(1)} = p_t(q)(\gamma_1) v \otimes v_0^{(1)}$ in $\mathcal{D}_{\pm 1}^{(0)} \otimes V^{(0)}_{1}$ by (7.1.1), (8.5.1) and (8.7.2). By the induction on $k$ using (7.1.1), (8.5.1), (8.7.2) and Lemma 6.3(i), we can show that

$$J_t \cdot (v \otimes v_0^{(1)} \otimes \cdots \otimes v_0^{(k)}) = p_t(q)(\gamma_1, \gamma_2, \ldots, \gamma_k) v \otimes v_0^{(1)} \otimes \cdots \otimes v_0^{(k)}.$$

As a consequence, the $U_q^{(0)}$-module $V(\pm 1; \gamma_1, \ldots, \gamma_k)$ is a highest weight module of the highest weight $u$ given by (8.7.1), and $L(u)$ is a quotient of $V(\pm 1; \gamma_1, \ldots, \gamma_n)$. Thus $L(u)$ is finite dimensional since $V(\pm 1; \gamma_1, \ldots, \gamma_k)$ is finite dimensional. □

8.8. Let $\mathbb{C}[x]$ be the polynomial ring over $\mathbb{C}$ with an indeterminate variable $x$. For $\varphi \in \mathbb{C}[x]$, we denote the leading coefficient of $\varphi$ by $\beta_{\varphi}$. Put

$$\mathbb{C}[x]^{(0)} = \{ \varphi \in \mathbb{C}[x] \setminus \{0\} \mid \beta_{\varphi} = \pm 1 \}.$$

We define a map $u^{(0)} : \mathbb{C}[x]^{(0)} \to \mathbb{C}^\times \times \prod_{t>0} \mathbb{C}$ by

$$u^{(0)}(\varphi) = \begin{cases} (\beta_{\varphi}, (0)_{t>0}) & \text{if } \deg \varphi = 0, \\ (\beta_{\varphi}^{\deg \varphi}, (p_t(q)(\gamma_1, \gamma_2, \ldots, \gamma_k))_{t>0}) & \text{if } \deg \varphi > 0 \end{cases}$$

for $\varphi = \beta_{\varphi}(x - \gamma_1)(x - \gamma_2)\ldots(x - \gamma_k) \in \mathbb{C}[x]^{(0)}$.

**Lemma 8.9.** The map $u^{(0)} : \mathbb{C}[x]^{(0)} \to \mathbb{C}^\times \times \prod_{t>0} \mathbb{C}$ is injective.

**Proof.** For $\varphi, \varphi' \in \mathbb{C}[x]^{(0)}$, write $\varphi = \beta_{\varphi}(x - \gamma_1)(x - \gamma_2)\ldots(x - \gamma_k)$ and $\varphi' = \beta_{\varphi'}(x - \gamma'_1)(x - \gamma'_2)\ldots(x - \gamma'_l)$. If $u^{(0)}(\varphi) = u^{(0)}(\varphi')$, then we have

$$\beta_{\varphi} q^k = \beta_{\varphi'} q^l, \quad p_t(q)(\gamma_1, \gamma_2, \ldots, \gamma_k) = p_t(q)(\gamma'_1, \gamma'_2, \ldots, \gamma'_l) \quad (t > 0).$$

The first equation implies that $\beta_{\varphi} = \beta_{\varphi'}$ and $k = l$ since $q$ is not a root of unity. Moreover, we have

$$\varphi = \beta_{\varphi} x^k + \beta_{\varphi} \sum_{z=1}^{k} (-1)^z e_z(\gamma_1, \gamma_2, \ldots, \gamma_k) x^{k-z}$$

$$= \beta_{\varphi} x^k + \beta_{\varphi} \sum_{z=1}^{k} (-1)^z \left( \sum_{\lambda^z} a_{\lambda}(q)(\gamma_1, \gamma_2, \ldots, \gamma_k) \right) x^{k-z}$$

$$= \beta_{\varphi'} x^k + \beta_{\varphi'} \sum_{z=1}^{k} (-1)^z e_z(\gamma'_1, \gamma'_2, \ldots, \gamma'_l) x^{k-z}$$

$$= \beta_{\varphi'} x^k + \beta_{\varphi'} \sum_{z=1}^{k} (-1)^z e_z(\gamma'_1, \gamma'_2, \ldots, \gamma'_l) x^{k-z}$$

$$= \varphi'$$
by Corollary 8.5 and (8.9.11).

Proposition 8.3, Proposition 8.7 and Lemma 8.9 imply the following theorem.

**Theorem 8.10.** There exists the bijection between \( \mathbb{C}[x]^{(0)} \) and the isomorphism classes of finite dimensional simple \( U_q(\mathfrak{sl}_2^{(0)}[x]) \)-modules given by \( \varphi \mapsto L(u^{(0)}(\varphi)) \).

**Corollary 8.11.** For \( \varphi, \psi \in \mathbb{C}[x]^{(0)} \), let \( v_0 \in L(u^{(0)}(\varphi)) \) (resp. \( w_0 \in L(u^{(0)}(\psi)) \)) be a highest weight vector. Let \( V(\varphi, \psi) \) be a \( U_q^{(0)} \)-submodule of \( L(u^{(0)}(\varphi)) \otimes L(u^{(0)}(\psi)) \) generated by \( v_0 \otimes w_0 \). Then \( V(\varphi, \psi) \) is a highest weight module of the highest weight \( u^{(0)}(\varphi, \psi) \). In particular, we have \( L(u^{(0)}(\varphi)) \otimes L(u^{(0)}(\psi)) \cong L(u^{(0)}(\varphi)) \) if \( L(u^{(0)}(\varphi)) \otimes L(u^{(0)}(\psi)) \) is simple.

**Proof.** For \( \varphi, \psi \in \mathbb{C}[x]^{(0)} \), write \( \varphi \) and \( \psi \) as \( \varphi = \varepsilon(x - \gamma_1)(x - \gamma_2) \ldots (x - \gamma_k) \) and \( \psi = \varepsilon'(x - \xi_1)(x - \xi_2) \ldots (x - \xi_l) \) respectively. Let \( v \) (resp. \( w \)) be a basis of one-dimensional \( U_q^{(0)} \)-module \( D^{(0)}_\varepsilon \) (resp. \( D^{(0)}_{\varepsilon'} \)), and let \( v_0^{(i)} \in V_{1}^{\varepsilon}(\varphi) \) (1 \( \leq i \leq k \)) (resp. \( w_0^{(i)} \in V_{1}^{\varepsilon'}(\psi) \) (1 \( \leq i \leq l \)) be a highest weight vector. As in a proof of Proposition 8.7, we have \( L(u^{(0)}(\varphi)) \cong \text{Top} V(\varepsilon; \gamma_1, \ldots, \gamma_k) \) and \( L(u^{(0)}(\psi)) \cong \text{Top} V(\varepsilon'; \xi_1, \ldots, \xi_l) \). By the definition of \( D^{(0)}_{\pm 1} \) given in (7.1.1), we can easily check that \( M \otimes D^{(0)}_{\pm 1} \cong D^{(0)}_{\pm 1} \otimes M \) as \( U_q^{(0)} \)-modules for any \( U_q^{(0)} \)-module \( M \). As a consequence, we see that the highest weight of \( V(\varphi, \psi) \) is same as the highest weight of \( V(\varepsilon \varepsilon'; \gamma_1, \ldots, \gamma_k, \xi_1, \ldots, \xi_l) \) given by \( u^{(0)}(\varphi, \psi) \).

**8.12.** We define a map \( b : \mathbb{C}[x] \to \mathbb{C}[\omega] \) (\( \varphi \mapsto \varphi^b(\omega) \)) by

\[
\varphi^b(\omega) = (1 - \gamma_1 \omega)(1 - \gamma_2 \omega) \ldots (1 - \gamma_k \omega)
\]

if \( \varphi = \beta \varphi(x - \gamma_1)(x - \gamma_2) \ldots (x - \gamma_k) \). Then, Theorem 8.10 together with Proposition 6.6 implies the following corollary.

**Corollary 8.13.** For \( \varphi \in \mathbb{C}[x]^{(0)} \), let \( v_0 \) be a highest weight vector of \( L(u^{(0)}(\varphi)) \). Then we have

\[
\Psi^+(\omega) \cdot v_0 = \beta q^{\deg \varphi} \frac{\varphi^b(q^{-2} \omega)}{\varphi^b(\omega)} v_0.
\]

(Note that \( p_t(q)(\gamma_1, \gamma_2, \ldots, \gamma_{k-1}, 0) = p_t(q)(\gamma_1, \gamma_2, \ldots, \gamma_{k-1}) \) by Lemma 6.3 (i).)

**Remark 8.14.** Let \( \mathbb{C}[x]^D \) be the set of polynomials over \( \mathbb{C} \) with an indeterminate variable \( x \) whose constant term is equal to 1. By [CP91 Theorem 3.4], there is a bijection between \( \mathbb{C}[x]^D \) and isomorphism classes of finite dimensional simple \( U_q(L\mathfrak{sl}_2) \)-modules of type 1. We call elements of \( \mathbb{C}[x]^D \) Drinfeld polynomials. For \( \varphi \in \mathbb{C}[x]^D \), let \( L^D(\varphi) \) be the corresponding finite dimensional simple \( U_q(L\mathfrak{sl}_2) \)-module. We regard \( L^D(\varphi) \) as a \( U_q^{(0)} \)-module through the injection \( \Theta^{(0)} \). Then, we see that \( L^D(\varphi) \) is still simple as a \( U_q^{(0)} \)-module (cf. [MTZ Remark 3.2]).

Let \( \sharp : \mathbb{C}[x]^D \to \mathbb{C}[x]^{(0)} \) (\( \varphi \mapsto \varphi^\sharp \)) be the injective map given by

\[
((1 - \gamma_1 x)(1 - \gamma_2 x) \ldots (1 - \gamma_k x))^\sharp = (x - \gamma_1)(x - \gamma_2) \ldots (x - \gamma_k),
\]
where $\gamma_i \neq 0$ (1 ≤ i ≤ k). Then we see that $L^D(\varphi) \cong L(u^{(0)}(\varphi^2))$ as $U_q^{(0)}$-modules by Corollary 8.13 and [CP91, Theorem 3.4].

§ 9. Finiteness of simple modules

In this section, we classify the isomorphism classes of finite dimensional simple modules of the algebra $U_q^{(Q)} = U_q(\mathfrak{sl}_2^{(Q)}[x])$ in the case of rank one and of $Q \neq 0$.

9.1. For $k \geq 0$, put $J^{(Q)}_{[k;0]} = q^{-k(k+1)}Q^k$, For $t = 1, 2, \ldots, k$, we define the element $J^{(Q)}_{[k;t]} \in U_q^{(Q)}$ inductively by

\[(9.1.1) \quad J^{(Q)}_{[k;t]} = q^{\frac{1}{1}} \sum_{z=1}^{t} (-1)^{z-1} (J_z - q^{2(k-t+z)}Q^{-z}J_0 + q^{k-2(t-z)[k]}Q^{-z}) J^{(Q)}_{[k;t-z]}.
\]

For examples, we have

\[
\begin{align*}
J^{(Q)}_{[1;0]} &= q^{-2}Q, \\
J^{(Q)}_{[2;0]} &= q^{-2}Q^2, \\
J^{(Q)}_{[2;2]} &= 1 - q^{2}QJ_0 + q^{-2}Q^2J_1,
\end{align*}
\]

We have the following relations in $U_q^{(Q)}$.

Lemma 9.2. For $k \in \mathbb{Z}_{>0}$, we have

\[X_0^{+(k)}X_0^{-(k+1)} \equiv \sum_{z=0}^{k} (-1)^{k-z}X_z^{-(k)}(K^+)^{k}J^{(Q)}_{[k;k-z]} \mod \mathfrak{X}_+,
\]

where $\mathfrak{X}_+$ is the left ideal of $U_q^{(Q)}$ generated by $\{X_t^+ \mid t \geq 0\}$.

Proof. See Appendix C.

By using the above lemma, we have the following condition for $L(u)$ to be finite dimensional.

Proposition 9.3. Assume that $Q \neq 0$. For $u = (\lambda, (u_t)_{t>0}) \in \mathbb{C}^\times \times \prod_{t>0} \mathbb{C}$, the simple $U_q^{(Q)}$-module $L(u)$ is finite dimensional if and only if there exist $k \in \mathbb{Z}_{>0}$, $\beta \in \mathbb{C}^\times$ and $\gamma_1, \gamma_2, \ldots, \gamma_k \in \mathbb{C}$ such that

\[(9.3.1) \quad \lambda = \beta q^k, \quad u_t = \begin{cases}
\tilde{\beta}Q^{-t} & \text{if } k = 0, \\
\frac{1}{k}(Q^t(\varphi; \beta)(\gamma_1, \gamma_2, \ldots, \gamma_k) & \text{if } k \neq 0 \quad (t > 0),
\end{cases}
\]

where we put $\tilde{\beta} = (q - q^{-1})^{-1}(1 - \beta^{-2})$.

Proof. We prove the only if part. Suppose that $L(u)$ is finite dimensional. Let $v_0 \in L(u)$ be a highest weight vector. By investigating the eigenvalues for the
action of $K^+$, there exists $k \in \mathbb{Z}_{\geq 0}$ such that $X_0^{-(k)} \cdot v_0 \neq 0$ and $X_0^{-(k+1)} \cdot v_0 = 0$ since $L(u)$ is finite dimensional.

In the case where $k = 0$, we can easily check that $L(u)$ is one-dimensional. Then the condition (9.3.1) follows from Proposition 7.3.

Assume that $k > 0$. Put $\beta = \lambda q^{-k}$. By Lemma 6.9, there exist $\gamma_1, \gamma_2, \ldots, \gamma_k \in \mathbb{C}$ such that $u_t = p_t^Q(q; \beta)(\gamma_1, \gamma_2, \ldots, \gamma_k)$ for $t = 1, 2, \ldots, k$. By the induction on $t$ using (9.1.1) and Lemma 6.10 (i), we can show that

(9.3.2) \[ J_{[k, h]}^{(Q)} \cdot v_0 = q^{-k(1+k)}Q^k e_t(\gamma_1, \gamma_2, \ldots, \gamma_k) v_0 \quad (1 \leq t \leq k) \]

where we note that $(q^{2k} J_0 - q^k [k]) \cdot v_0 = \overline{\beta} v_0$.

By Lemma 9.2 and the relation (Q6), for $t \geq 0$, we have

\[ 0 = X_t^+ X_0^{-(k)} X_0^{-(k+1)} \cdot v_0 = \sum_{z=0}^{k} (-1)^{k-z} (J_{t+z} - Q J_{t+z+1})(K^+) \cdot v_0. \]

Note that $(K^+) \cdot v_0 = \beta^k Q^k e_k = q^k(\gamma_1, \ldots, \gamma_k)$ by the choice of $\beta$ and (9.3.2), then the above equation implies that

(9.3.3) \[ J_{k+t+1} \cdot v_0 = Q^{-1} J_{k+t} \cdot v_0 + \sum_{z=0}^{k-1} (-1)^{k-z+1}(J_{t+1+z} - Q^{-1} J_{t+z}) e_k \cdot v_0. \]

Then we can show that $u_{k+t} = p_{k+t}^Q(q; \beta)(\gamma_1, \ldots, \gamma_k)$ for $t > 0$ by the induction on $t$ using (9.3.3) and Lemma 6.10 (ii).

Next we prove the if part. Recall the algebra homomorphism $\Delta_r^{(Q)} : U_q^{(Q)} \to U_q^{(Q)} \otimes U_q^{(0)}$ given in Theorem 3.9. In a similar way as in [CP91] Proposition 4.4, we can show that

(9.3.4) \[ \Delta_r^{(Q)}(X_{t}^+) \equiv X_{t}^+ \otimes K^+ + 1 \otimes X_{t}^+ + (q - q^{-1}) \sum_{z=1}^{t} X_{t-z}^+ \otimes K^+ \mod \mathfrak{x}_{+2}^{(Q)} \otimes \mathfrak{x}_{-}^{(0)}, \]

\[ \Delta_r^{(Q)}(J_t) \equiv J_t \otimes 1 + 1 \otimes J_t + (q - q^{-1}) \sum_{z=1}^{t-1} J_z \otimes J_{t-z} \mod \mathfrak{x}_{+}^{(Q)} \otimes \mathfrak{x}_{-}^{(Q)}, \]

for $t' \geq 0$ and $t > 0$, where $\mathfrak{x}_{+2}^{(Q)}$ (resp. $\mathfrak{x}_{+}^{(Q)}$, $\mathfrak{x}_{-}^{(Q)}$) is the left ideal of $U_q^{(Q)}$ generated by $\{X_{r}^{+} X_{s}^{+} | s, s' \geq 0\}$ (resp. $\{X_{r}^{+} \ | \ s \geq 0\}$, $\{X_{s}^{-} \ | \ s \geq 0\}$).

For $\gamma_1, \gamma_2, \ldots, \gamma_k \in \mathbb{C}$, put $\varphi = (x - \gamma_1)(x - \gamma_2) \ldots (x - \gamma_k) \in \mathbb{C}[x]^{(0)}$. Then $L(u^{(0)}(\varphi))$ is finite dimensional simple $U_q^{(0)}$-module by Theorem 8.10. Let $v_0 \in L(u^{(0)}(\varphi))$ be a highest weight vector. For $\beta \in \mathbb{C}^x$, we take the one-dimensional $U_q^{(Q)}$-module $D_\beta^{(Q)} = Cv$. We consider the $U_q^{(Q)}$-module $D_\beta^{(Q)} \otimes L(u^{(0)}(\varphi))$ through
the homomorphism $\Delta^{(Q)}$. Let $V(\beta; \gamma_1, \ldots, \gamma_k)$ be the $U_q^{(Q)}$-submodule of $D^{(Q)}_\beta \otimes L(u^{(Q)}(\varphi))$ generated by $v \otimes v_0$. Then we have $X_t^+ \cdot (v \otimes v_0) = 0$ for all $t \geq 0$ by \([9.3.4]\). On the other hand, we have

$$J_t \cdot (v \otimes v_0) = \langle \tilde{\beta} Q^{-t} + p_t(q)(\gamma_1, \ldots, \gamma_k) + (q - q^{-1}) \sum_{z=1}^{t-1} \tilde{\beta} Q^{-z} p_{t-z}(q)(\gamma_1, \ldots, \gamma_k) \rangle v \otimes v_0$$

for $t > 0$ by \([9.3.4]\), where $\tilde{\beta} = (q - q^{-1})^{-1}(1 - \beta^{-2})$. Thus, we have $J_t \cdot (v \otimes v_0) = p_t(q)(\beta)(\gamma_1, \ldots, \gamma_k)$ by \([6.8.1]\). We also see that $K^+ \cdot (v \otimes v_0) = \beta q^k v \otimes v_0$. As a consequence, the $U_q^{(Q)}$-module $V(\beta; \gamma_1, \ldots, \gamma_k)$ is a highest weight module of the highest weight $u$ given by \([9.3.1]\), and $L(u)$ is a quotient of $V(\beta; \gamma_1, \ldots, \gamma_k)$. Thus $L(u)$ is finite dimensional. \qed

**9.4.** In order to give a correspondence between the elements of $\mathbb{C}[x]$ and finite dimensional simple $U_q^{(Q)}$-modules, We define a map $u^{(Q)} : \mathbb{C}[x] \setminus \{0\} \to \mathbb{C}^\times \times \prod_{t>0} \mathbb{C}$ by

$$u^{(Q)}(\varphi) = \begin{cases} (\beta_\varphi, (\tilde{\beta}_\varphi Q^{-t})_{t>0}) & \text{if} \ \deg \varphi = 0, \\ (\beta_\varphi q^{\deg \varphi}, (p_t(q; \beta_\varphi)(\gamma_1, \gamma_2, \ldots, \gamma_k))_{t>0}) & \text{if} \ \deg \varphi > 0 \end{cases}$$

for $\varphi = \beta_\varphi(x - \gamma_1)(x - \gamma_2) \ldots (x - \gamma_k) \in \mathbb{C}[x] \setminus \{0\}$, where $\tilde{\beta}_\varphi = (q - q^{-1})^{-1}(1 - \beta_\varphi^{-2})$. Unfortunately, the map $u^{(Q)}$ is not injective. In order to obtain an index set of the isomorphism classes of finite dimensional simple $U_q^{(Q)}$-modules, we take a subset $\mathbb{C}[x]^{(Q)}$ of $\mathbb{C}[x]$ as

$$\mathbb{C}[x]^{(Q)} = \{ \varphi \in \mathbb{C}[x] \setminus \{0\} \mid \beta_\varphi^{-2}Q^{-1} \text{ is not a root of } \varphi \}.$$ 

Then we have the following proposition.

**Proposition 9.5.**

(i) For $\varphi, \varphi' \in \mathbb{C}[x] \setminus \{0\}$ such that $\deg \varphi \geq \deg \varphi'$, we have that $u^{(Q)}(\varphi) = u^{(Q)}(\varphi')$ if and only if

$$\varphi = q^{-(\deg \varphi - \deg \varphi')} \prod_{z=1}^{\deg \varphi - \deg \varphi'} (x - q^{-2(z-1)}\beta_\varphi^{-2}Q^{-1}).$$

(ii) The restriction of $u^{(Q)}$ to $\mathbb{C}[x]^{(Q)}$ is injective. Moreover, for any $\varphi \neq 0 \in \mathbb{C}[x]$, there exists the unique $\varphi' \in \mathbb{C}[x]^{(Q)}$ such that $u^{(Q)}(\varphi) = u^{(Q)}(\varphi')$.

**Proof.** We prove the statement (i) by the induction on $\deg \varphi - \deg \varphi'$. For $\varphi, \varphi' \in \mathbb{C}[x] \setminus \{0\}$, write $\varphi = \beta_\varphi(x - \gamma_1)(x - \gamma_2) \ldots (x - \gamma_k)$ and $\varphi' = \beta_{\varphi'}(x - \gamma'_1)(x - \gamma'_2) \ldots (x - \gamma'_l)$.

First, we consider the case where $k = l$. In this case, it is clear that $u^{(Q)}(\varphi) = u^{(Q)}(\varphi')$ if $\varphi = \varphi'$. Assume that $u^{(Q)}(\varphi) = u^{(Q)}(\varphi')$. If $k = l = 0$, we can easily
check that \( \varphi = \varphi' \) from definitions. If \( k = l > 0 \), the assumption \( u^{(Q)}(\varphi) = u^{(Q)}(\varphi') \) implies that \( \beta_{\varphi} = \beta_{\varphi'} \) and

\[
p_t(q)(\gamma_1, \ldots, \gamma_k) = p_t(q)(\gamma'_1, \ldots, \gamma'_k)
\]

for \( t > 0 \). Using this equation, we can prove that \( p_t(q)(\gamma_1, \ldots, \gamma_k) = p_t(q)(\gamma'_1, \ldots, \gamma'_k) \) for \( t > 0 \). Then we have \( \varphi = \varphi' \) in the same way as the proof of Lemma 8.9. As a consequence, we have \( \varphi = \varphi' \) if \( u^{(Q)}(\varphi) = u^{(Q)}(\varphi') \).

Next we consider the case where \( k = l + 1 \). If \( \varphi = q^{-1}\varphi'(x - \beta_{\varphi'}^{-2}Q^{-1}) \), we have \( \beta_{\varphi} = q^{-1}\beta_{\varphi'} \) and \( p_t^{(Q)}(q; \beta_{\varphi})(\gamma_1, \ldots, \gamma_k) = p_t^{(Q)}(q; q^{-1}\beta_{\varphi'})(\gamma'_1, \ldots, \gamma'_k, \beta_{\varphi'}^{-2}Q^{-1}) \) for \( t > 0 \). On the other hand, we have

\[
p_t^{(Q)}(q; q^{-1}\beta_{\varphi'})(\gamma'_1, \ldots, \gamma'_k, \beta_{\varphi'}^{-2}Q^{-1}) = p_t(q)(\gamma'_1, \ldots, \gamma'_k, \beta_{\varphi'}^{-2}Q^{-1}) + \frac{1 - q^2\beta_{\varphi'}^{-2}}{q - q^{-1}}Q^{-t} \]

\[
+ \sum_{z=1}^{t-1} (1 - q^2\beta_{\varphi'}^{-2})Q^{-t+z}p_z(q)(\gamma'_1, \ldots, \gamma'_k, \beta_{\varphi'}^{-2}Q^{-1})
\]

by the definition (6.8.1). Applying Lemma 6.3(i) to the right-hand side of the above equation, we have

\[
p_t^{(Q)}(q; \beta_{\varphi})(\gamma_1, \ldots, \gamma_k)
\]

\[
= p_t^{(Q)}(q; q^{-1}\beta_{\varphi'})(\gamma'_1, \ldots, \gamma'_k, \beta_{\varphi'}^{-2}Q^{-1})
\]

\[
= p_t(q)(\gamma'_1, \ldots, \gamma'_k) + \frac{1 - \beta_{\varphi'}^{-2}}{q - q^{-1}}Q^{-t} + \sum_{z=1}^{t-1} (1 - \beta_{\varphi'}^{-2})Q^{-t+z}p_z(q)(\gamma'_1, \ldots, \gamma'_k)
\]

\[
= p_t^{(Q)}(q; \beta_{\varphi'})(\gamma'_1, \ldots, \gamma'_k).
\]

Thus, we have \( u^{(Q)}(\varphi) = u^{(Q)}(\varphi') \) if \( \varphi = q^{-1}\varphi'(x - \beta_{\varphi'}^{-2}Q^{-1}) \).

Assume that \( u^{(Q)}(\varphi) = u^{(Q)}(\varphi') \). Then we have \( q\beta_{\varphi} = \beta_{\varphi'} \) and

\[
p_t^{(Q)}(q; \beta_{\varphi})(\gamma_1, \ldots, \gamma_k) = p_t^{(Q)}(q; \beta_{\varphi'})(\gamma'_1, \ldots, \gamma'_l)
\]

for \( t > 0 \). We note that \( \widetilde{\beta}_{\varphi'} = (q - q^{-1})^{-1}(1 - q^{-2}\beta_{\varphi'}^{-2}) = \widetilde{\beta}_{\varphi} + q^{-1}\beta_{\varphi'}^{-2} \) since \( \beta_{\varphi'} = q\beta_{\varphi} \).

Then, by applying the definition (6.8.1) to both sides of (9.5.1), we have

\[
p_t(q)(\gamma_1, \ldots, \gamma_k)
\]

\[
= p_t(q)(\gamma'_1, \ldots, \gamma'_l) + q^{-1}\beta_{\varphi'}^{-2}Q^{-t} + (q - q^{-1}) \sum_{z=1}^{t-1} (\widetilde{\beta}_{\varphi} + q^{-1}\beta_{\varphi'}^{-2})Q^{-t+z}p_z(q)(\gamma'_1, \ldots, \gamma'_l)
\]
Applying Lemma \[6.3\] (i), we have

\[\left(9.5.2\right)\]

and the induction hypothesis. Thus, we have the statement (i).

Finally, we consider the case where \(k > l + 1\) by the induction on \(k - l\). Put \(\varphi'' = q^{-1}\varphi'\left(x - q^2\varphi^{-2}Q^{-1}\right)\), and we have \(\beta_{\varphi''} = q^{-1}\beta_{\varphi'}\). Then, we have \(u(Q)(\varphi'') = u(Q)(\varphi')\) by the above argument. On the other hand, we have

\[u(Q)(\varphi) = u(Q)(\varphi'') \iff \varphi = q^{-\deg \varphi - \deg \varphi''} \prod_{z=1}^{\deg \varphi - \deg \varphi''} (x - q^{-2(2z-1)}\beta_{\varphi}^{-2}Q^{-1})\]

by the induction hypothesis. Thus, we have the statement (i).

The statement (ii) follows from the statement (i) and the definition of \(\mathbb{C}[x]^{(Q)}\).

Proposition \[9.3\] and Proposition \[9.5\] imply the following theorem.
**Theorem 9.6.** Assume that \( Q \neq 0 \). There exists the bijection between \( \mathbb{C}[x]^{(Q)} \) and the isomorphism classes of finite dimensional simple \( \mathfrak{u}(\mathfrak{sl}_2) \)-modules given by \( \varphi \mapsto L(\mathfrak{u}(\mathfrak{sl}_2)(\varphi)) \).

**Corollary 9.7.** For \( \varphi \in \mathbb{C}[x]^{(Q)} \), let \( v_0 \) be a highest weight vector of \( L(\mathfrak{u}(\mathfrak{sl}_2)(\varphi)) \). Then we have

\[
\Psi^{+}(\omega) \cdot v_0 = q^{\deg \varphi} \frac{\varphi^b(q^{-2} \omega)}{\varphi^b(\omega)} (\beta_\varphi^{-1} - Q \beta_\varphi \omega^{-1}) v_0.
\]

**Proof.** For \( \varphi \in \mathbb{C}[x]^{(Q)} \), write \( \varphi = \beta_\varphi(x - \gamma_1)(x - \gamma_2) \cdots (x - \gamma_k) \). By [2.6.2], for \( t > 0 \), we have

\[
\Psi^+_t \cdot v_0 = (q - q^{-1})K^+ (J_t - QJ_{t+1}) \cdot v_0 = (q - q^{-1})\beta_\varphi q^{\deg \varphi} \left( p_t^{(Q)}(q; \beta_\varphi)(\gamma_1, \ldots, \gamma_k) - Qp_{t+1}^{(Q)}(q; \beta_\varphi)(\gamma_1, \ldots, \gamma_k) \right) v_0.
\]

Applying (6.8.1), we have

\[
\Psi^+_t \cdot v_0 = (q - q^{-1})\beta_\varphi q^{\deg \varphi} (\beta_\varphi^{-2} p_t(q)(\gamma_1, \ldots, \gamma_k) - Qp_{t+1}(q)(\gamma_1, \ldots, \gamma_k)) v_0
\]

for \( t > 0 \), where we note that \( 1 - (q - q^{-1})\beta_\varphi = \beta_\varphi^{-2} \). We also have

\[
\Psi^-_{-1} \cdot v_0 = -Q\beta_\varphi q^{\deg \varphi} v_0, \quad \Psi^+_0 \cdot v_0 = \beta_\varphi q^{\deg \varphi} (\beta_\varphi^{-2} - (q - q^{-1})Qp_1(q)(\gamma_1, \ldots, \gamma_k)) v_0.
\]

Thus, we have

\[
\Psi^+(\omega) \cdot v_0 = \left\{ -Q\beta_\varphi q^{\deg \varphi} \omega^{-1} + \beta_\varphi q^{\deg \varphi} (\beta_\varphi^{-2} - (q - q^{-1})Qp_1(q)(\gamma_1, \ldots, \gamma_k)) + \sum_{t>0} (q - q^{-1})\beta_\varphi q^{\deg \varphi} (\beta_\varphi^{-2} p_t(q)(\gamma_1, \ldots, \gamma_k) - Qp_{t+1}(q)(\gamma_1, \ldots, \gamma_k)) \omega^t \right\} v_0
\]

\[
= q^{\deg \varphi} \left( 1 + (q - q^{-1}) \sum_{t>0} p_t(q)(\gamma_1, \ldots, \gamma_k) \omega^t \right) (\beta_\varphi^{-1} - Q \beta_\varphi \omega^{-1}) v_0
\]

\[
= q^{\deg \varphi} \frac{\varphi^b(q^{-2} \omega)}{\varphi^b(\omega)} (\beta_\varphi^{-1} - Q \beta_\varphi \omega^{-1}) v_0
\]

by Proposition 6.6. \( \square \)
§ 10. Finite dimensional simple modules of $U_q(\mathfrak{sl}^\mathbb{C}_n[Q][x])$

In this section, we classify the isomorphism classes of finite dimensional simple $U_q(\mathfrak{sl}^\mathbb{C}_n[Q][x])$-modules. In this section, we denote by $U_q^Q$ the $(q, Q)$-current algebra $U_q(\mathfrak{sl}_n^\mathbb{C})[x]$ of rank $n - 1$ with a parameter $Q = (Q_1, \ldots, Q_{n-1}) \in \mathbb{C}^{n-1}$.

10.1. For each $i \in I$, we can easily check that there exists the algebra homomorphism $\iota_i : U_q(\mathfrak{sl}_2^Q[x]) \to U_q^Q$ such that $\iota_i(X_i^\pm) = X_i^\pm$, $\iota_i(J_i) = (J_i)$, and $\iota_i(K_i) = K_i$. For $u = ((\lambda_i, (u_{i,t})_{t>0}))_{i \in I} \in (\mathbb{C}^\times \times \prod_{t>0} \mathbb{C})^I$, we regard the highest weight simple $U_q^Q$-module $L(u)$ as a $U_q(\mathfrak{sl}_2^Q[x])$-module through the homomorphism $\iota_i$. Let $v_0 \in L(u)$ be a highest weight vector. Then we can easily check that the $U_q(\mathfrak{sl}_2^Q[x])$-submodule of $L(u)$ generated by $v_0$ is a highest weight simple $U_q(\mathfrak{sl}_2^Q[x])$-module of the highest weight $(\lambda_i, (u_{i,t})_{t>0})$. Thus, Theorem 8.10 and Theorem 5.6 imply the following proposition.

Proposition 10.2. For $u = ((\lambda_i, (u_{i,t})_{t>0}))_{i \in I} \in (\mathbb{C}^\times \times \prod_{t>0} \mathbb{C})^I$, if the highest weight simple $U_q^Q$-module $L(u)$ is finite dimensional, then there exists $\varphi_i \in \prod_{i \in I} \mathbb{C}[x](Q_i)$ such that $((\lambda_i, (u_{i,t})_{t>0}))_{i \in I} = (u(Q_i)(\varphi_i))_{i \in I}$.

10.3. Recall the algebra homomorphism $\Delta_r^Q : U_q^Q \to U_q^Q \otimes U_q^0$ given in Theorem 3.9. In a similar way as in [CP91 Proposition 4.4], we can show that

(10.3.1)
$$
\Delta_r^Q(X_{i,t}^+) \equiv X_{i,t'}^+ \otimes K_i^+ + 1 \otimes X_{i,t'}^+ + (q - q^{-1}) \sum_{z=1}^{t'} X_{i,t' - z}^+ \otimes K_i^+ J_{i,z} \mod \mathfrak{X}_2^Q \otimes \mathfrak{X}_2^-^Q,
$$
$$
\Delta_r^Q(J_{i,t}) \equiv J_{i,t} \otimes 1 + 1 \otimes J_{i,t} + (q - q^{-1}) \sum_{z=1}^{t-1} J_{i,z} \otimes J_{i,t - z} \mod \mathfrak{X}_2^Q \otimes \mathfrak{X}_2^-^Q,
$$

for $i \in I$, $t' \geq 0$, and $t > 0$, where $\mathfrak{X}_2^Q$ (resp. $\mathfrak{X}_2^Q$, $\mathfrak{X}_2^-^Q$) is the left ideal of $U_q^Q$ generated by $\{X_{j,s}^+, X_{j,s'}^- \mid (j, s), (j', s') \in I \times \mathbb{Z}_{\geq 0}\}$ (resp. $\{X_{j,s}^+ \mid (j, s) \in I \times \mathbb{Z}_{\geq 0}\}$, $\{X_{j,s}^- \mid (j, s) \in I \times \mathbb{Z}_{\geq 0}\}$).

For $(\varphi_i)_{i \in I} \in \prod_{i \in I} \mathbb{C}[x](Q_i)$, write $\varphi_i$ as $\varphi_i = \beta_{\varphi_i}(x - \gamma_1(i)) \cdots (x - \gamma_{k_i}(i))$ for each $i \in I$. Put $\gamma_{i,p} = q^{i-2} - q^{p(i)}$ for $i \in I$ and $1 \leq p \leq k_i$. We consider the evaluation module $V(\omega_i)^{ev}_{i,p}$ at $\gamma_{i,p}$ for $i \in I$ and $1 \leq p \leq k_i$. Let $v_0^{(i)} \in V(\omega_i)^{ev}_{i,p}$ be a highest weight vector. We also consider the one-dimensional $U_q^Q$-module $D_\beta^Q = \mathbb{C}v$, where $\beta = (\beta_{\varphi_i})_{i \in I} \in \prod_{i \in I} \mathbb{C}[Q_i]$. Then we have the $U_q^Q$-module $D_\beta^Q \otimes \bigotimes_{i \in I} \bigotimes_{p=1}^{k_i} V(\omega_i)^{ev}_{i,p}$ through the algebra homomorphisms $\Delta_r^Q$ and $\Delta^Q$.

Let $V(((\varphi_i)_{i \in I})$ be the $U_q^Q$-submodule of $D_\beta^Q \otimes \bigotimes_{i \in I} \bigotimes_{p=1}^{k_i} V(\omega_i)^{ev}_{i,p}$ generated by $v \otimes (v_1^{(1)} \otimes \cdots \otimes v_{k_i}^{(1)}) \otimes \cdots \otimes (v_1^{(n-1)} \otimes \cdots \otimes v_{k_n}^{(n-1)})$. By Proposition 4.7 and (10.3.1), we can show that $V(((\varphi_i)_{i \in I})$ is a highest weight $U_q^Q$-module of the highest weight $(u(Q_i)(\varphi_i))_{i \in I}$ in a similar way as in the proofs of Proposition 8.7 and of Proposition
Thus, the highest weight simple $U_q^{(Q)}$-module $L((u^{(Q)}(\varphi_i))_{i \in I})$ is a quotient of $V((\varphi_i)_{i \in I})$. In particular, it is finite dimensional. As a consequence, we have the following proposition.

**Proposition 10.4.** For $(\varphi_i)_{i \in I} \in \prod_{i \in I} C[x]^{(Q)}$, the highest weight simple $U_q^{(Q)}$-module $L((u^{(Q)}(\varphi_i))_{i \in I})$ is finite dimensional.

We have the following theorem by Proposition 10.2 and Proposition 10.4.

**Theorem 10.5.** There exists the bijection between $\prod_{i \in I} C[x]^{(Q)}$ and the isomorphism classes of finite dimensional simple $U_q(\mathfrak{sl}_n^{(Q)})$-modules given by $(\varphi_i)_{i \in I} \mapsto L((u^{(Q)}(\varphi_i))_{i \in I})$.

**Appendix A. A proof of Theorem 1.10**

**A.1.** Let $A = C[v, v^{-1}]$ be the Laurent polynomial ring over $C$ with an indeterminate element $v$, and let $K = C(v)$ be the quotient field of $A$. We also consider the localization $C[v]_{(v = 1)}$ of the polynomial ring $C[v]$ at $v = 1$.

For $X \in \{A, K, C[v]_{(v = 1)}\}$, we define an associative algebra $A_X$ over $X$ by generators $x_{i,t} \ (i, t) \in I \times \mathbb{Z}_{\geq 0}$ with defining relations

\[(A.1.1)\]
\[x_{i,t+1}x_{j,s} - v^{a_{i,j}}x_{j,s}x_{i,t+1} = v^{a_{i,j}}x_{i,t}x_{j,s+1} - x_{j,s+1}x_{i,t},\]
\[[x_{i,t}, x_{j,s}] = 0 \quad \text{if} \quad j \neq i, i \pm 1,
\]
\[x_{i+1,u}(x_{i,s}x_{i,t} + x_{i,t}x_{i,s}) + (x_{i,s}x_{i,t} + x_{i,t}x_{i,s})x_{i+1,u} = (v + v^{-1})(x_{i,s}x_{i+1,u}x_{i,t} + x_{i,t}x_{i+1,u}x_{i,s}).\]

For $q \in C^\times$, let $A$ be the scalar extension $C \otimes_A A^\times$ of $A^\times$ through the ring homomorphism $A \rightarrow C \ (v \mapsto q)$. Clearly, the algebra $A$ is isomorphic to an associative algebra over $C$ generated by $x_{i,t} \ (i, t) \in I \times \mathbb{Z}_{\geq 0}$ with defining relations (A.1.1), where we replace $v$ with $q$. Then we have the surjective algebra homomorphisms

\[\pi^+: A \rightarrow U_{q,Q}^+ (x_{i,t} \mapsto X_{i,t}^+), \quad \pi^- : A^{opp} \rightarrow U_{q,Q}^- (x_{i,t} \mapsto X_{i,t}^-),\]

where $A^{opp}$ is the opposite algebra of $A$.

**A.2.** Let $Q = \bigoplus_{i \in I} \mathbb{Z}_{\geq 0}$ be the root lattice of $\mathfrak{sl}_n$, and we put $Q^+ = \sum_{i \in I} \mathbb{Z}_{\geq 0} \alpha_i$. From the definition, we see that the algebra $A_X$ is a $Q$-graded algebra with $\deg_Q(x_{i,t}) = \alpha_i$, and $A_X^\times$ is also a $\mathbb{Z}$-graded algebra with $\deg(x_{i,t}) = t$. Then the algebra $A_X^\times$ decomposes into

\[A_X^\times = \bigoplus_{\gamma \in Q^+, s \geq 0} A^\times_{\gamma,s}, \quad A^\times_{\gamma,s} := \{x \in A_X^\times \mid \deg_Q(x) = \gamma, \ \deg(x) = s\}\]
as $\mathbb{X}$-modules. It is clear that, for each $(\gamma, s) \in Q^+ \times \mathbb{Z}_{\geq 0}$, the $\mathbb{X}$-module $A_{\gamma,s}^X$ is generated by $\{x_{i_1,t_1}, x_{i_2,t_2}, \ldots x_{i_k,t_k} \mid \alpha_{i_1} + \cdots + \alpha_{i_k} = \gamma, \ t_1 + \cdots + t_k = s\}$, and $A_{\gamma,s}^X$ is finitely generated over $\mathbb{X}$.

**A.3.** For $(\alpha_{i,j}, t) \in \Delta^+ \times \mathbb{Z}_{\geq 0}$, put

$$x_{\alpha_{i,j},t}(t) := [[[\cdots [x_{j-1,0}, x_{j-2,0}]_v, x_{j-3,0}]_v, \ldots, x_{i+1,0}]_v, x_{i,0}]_v$$

as an element of $A^X$. For $h \in H_{\geq 0}$, put

$$x_h := \prod_{(\beta,t) \in \Delta^+ \times \mathbb{Z}_{\geq 0}} x_\beta(t)^{h(\beta,t)}.$$

We also set

$$B^X := \{x_h \mid h \in H_{\geq 0}\},$$

$$B_{\gamma,s}^X := \{x_h \in B^X \mid \sum_{(\beta,t) \in \Delta^+ \times \mathbb{Z}_{\geq 0}} h(\beta,t) \cdot \beta = \gamma, \sum_{(\beta,t) \in \Delta^+ \times \mathbb{Z}_{\geq 0}} h(\beta,t) \cdot t = s\}.$$

**A.4.** Let $U_v(L\mathfrak{sl}_n)$ be the quantum loop algebra over $\mathbb{K}$ associated with $\mathfrak{sl}_n$. Then we have an algebra homomorphism $\theta: A^K \to U_v(L\mathfrak{sl}_n)$ by $\theta(x_{i,t}) = e_{i,t}$. By [1] Theorem 2.17, we see that the set $\{\theta(x_h) \mid h \in H_{\geq 0}\}$ is linearly independent, and we have

(A.4.1) \[ \dim_{\mathbb{K}} A_{\gamma,s}^K \geq \# B_{\gamma,s}^K \]

for $(\gamma, s) \in Q^+ \times \mathbb{Z}_{\geq 0}$.

**A.5.** We note that the scalar extension $\mathbb{C} \otimes_{\mathbb{C}[\![v]\!]_{(\nu=1)}} A_{\mathbb{C}[\![v]\!]_{(\nu=1)}}$ through the ring homomorphism $\mathbb{C}[\![v]\!]_{(\nu=1)} \to \mathbb{C} (v \mapsto 1)$ is isomorphic to the universal enveloping algebra of the positive part of the polynomial current Lie algebra $\mathfrak{sl}_n[x]$. Then, by the same argument using (A.4.1) as one of [1] the proof of Proposition 1.13], we see that $B_{\gamma,s}^K$ gives a $\mathbb{K}$-basis of $A_{\gamma,s}^K$ for each $(\gamma, s) \in Q^+ \times \mathbb{Z}_{\geq 0}$. As a consequence, we have the following lemma.

**Lemma A.6.** The set $B^K$ gives a $\mathbb{K}$-basis of $A^K$, and the algebra homomorphism $\theta: A^K \to U_v(L\mathfrak{sl}_n)$ is injective.

**A.7.** For $(\beta,t) \in \Delta^+ \times \mathbb{Z}_{\geq 0}$, put $\tilde{x}_\beta(t) := (v - v^{-1})x_\beta(t) \in A^K$, and we set $\tilde{x}_h := \prod_{(\beta,t) \in \Delta^+ \times \mathbb{Z}_{\geq 0}} \tilde{x}_\beta(t)^{h(\beta,t)}$ for $h \in H_{\geq 0}$. Let $\tilde{A}^K$ be the $\mathbb{A}$-subalgebra of $A^K$ generated by $\{\tilde{x}_\beta(t) \mid (\beta,t) \in \Delta^+ \times \mathbb{Z}_{\geq 0}\}$. By definitions, we have $\{\tilde{x}_h \mid h \in H_{\geq 0}\} \subset \tilde{A}^K$, and we see that the set $\{\tilde{x}_h \mid h \in H_{\geq 0}\} \subset \tilde{A}^K$ is linearly independent over $\mathbb{A}$ thanks to Lemma A.6. On the other hand, for any $X \in \tilde{A}^K$, we can write $X = \sum_{h \in H_{\geq 0}} c_h \tilde{x}_h$ $(c_h \in \mathbb{K})$ uniquely since $\tilde{A}^K \subset A^K$ and $\{\tilde{x}_h \mid h \in H_{\geq 0}\}$ is a $\mathbb{K}$-basis of $A^K$ by Lemma A.6. Then we have $\theta(X) = \sum_{h \in H_{\geq 0}} c_h \theta(\tilde{x}_h)$, and we see that $c_h \in \mathbb{A}$ by [1] Theorem
2.19 (b)], where we note that the element \( \theta(\tilde{e}_h) \) coincides with the element \( \tilde{e}_h \) in [Theorem 2.19 (b)] by definitions. As a consequence, we have the following lemma.

**Lemma A.8.** The set \( \{ \tilde{e}_h \mid h \in H_{\geq 0} \} \) gives a free \( \mathbb{A} \)-basis of \( \tilde{A}^\mathbb{A} \).

**A.9.** Recall that \( \mathcal{A} = \mathbb{C} \otimes_\mathbb{A} \tilde{A}^\mathbb{A} \) through the ring homomorphism \( \mathcal{A} \to \mathbb{C} (v \mapsto q) \). Note that \( \tilde{A}^\mathbb{A} \subset \mathcal{A}^\mathbb{A} \), this embedding induces the algebra homomorphism \( \Phi : \mathbb{C} \otimes_\mathbb{A} \tilde{A}^\mathbb{A} \to \mathbb{C} \otimes_\mathbb{A} \mathcal{A}^\mathbb{A} = \mathcal{A} \). On the other hand, we can check that there exists an algebra homomorphism \( \Psi : \mathcal{A} \to \mathbb{C} \otimes_\mathbb{A} \tilde{A}^\mathbb{A} \) such that \( \Psi(x_{i,t}) = (q - q^{-1})^{-1} \otimes \tilde{x}_{t,i} \) if \( q \neq \pm 1 \). Without the homomorphism \( \Psi \) is not surjective. From definitions, we have \( \Phi \circ \Psi(x_{i,t}) = x_{i,t} \) for \( (i, t) \in I \times \mathbb{Z}_{\geq 0} \). Thus, the homomorphism \( \Phi \circ \Psi \) is the identity. In particular, \( \Psi \) is an isomorphism. As a consequence, we have the following proposition.

**Proposition A.10.** Assume that \( q \neq \pm 1 \), the set \( \{ x_h \mid h \in H_{\geq 0} \} \) gives a \( \mathbb{C} \)-basis of \( \mathcal{A} \).

**A.11.** Let \( \mathcal{A}^0 \) be an associative algebra over \( \mathbb{C} \) generated by \( \{ J_{i,t}, K_i^\pm \mid i \in I, t \in \mathbb{Z}_{\geq 0} \} \) subject to the defining relations (Q1-1) and (Q1-2). Then we have the surjective algebra homomorphism

\[
\pi^0 : \mathcal{A}^0 \to U_{q,\mathbb{Q}}^0 (J_{i,t} \mapsto J_{i,t}, K_i^\pm \mapsto K_i^\pm).
\]

By definition, we see easily that \( \{ K^k J_{h_0} \mid k \in \mathbb{Z}^{n-1}, h_0 \in H_0 \} \) gives a \( \mathbb{C} \)-basis of \( \mathcal{A}^0 \), where we use the same notation with one in [1].

By (1.11), we have the surjective linear map

\[
\pi : \mathcal{A}^{opp} \otimes_\mathbb{C} \mathcal{A}^0 \otimes_\mathbb{C} \mathcal{A} \xrightarrow{\pi \otimes \pi^0 \otimes \pi^+} U_{q,\mathbb{Q}}^- \otimes_\mathbb{C} U_{q,\mathbb{Q}}^0 \otimes_\mathbb{C} U_{q,\mathbb{Q}}^+ \xrightarrow{\text{multiplication}} U_q^{(\mathbb{Q})}.
\]

Moreover, we see that the set

\[
\{ \Theta^{(\mathbb{Q})} \circ \pi(x_h \otimes K^i J_{h_0} \otimes x_{h'}) \mid h, h' \in H_{\geq 0}, k \in \mathbb{Z}^{n-1}, h_0 \in H_0 \}
\]

is linearly independent by [1] Proposition 5.1 and [1] Theorem 2.15. Thus, the set \( \{ x_h \otimes K^k J_{h_0} \otimes x_{h'} \mid h, h' \in H_{\geq 0}, k \in \mathbb{Z}^{n-1}, h_0 \in H_0 \} \) is linearly independent.

Combining with Proposition [A.10] we see that \( \pi \) (resp. \( \pi^\pm, \pi^0 \)) is an isomorphism, and we obtain Theorem [1.10]

**APPENDIX B. A PROOF OF LEMMA [B.2]**

In this appendix, we give a proof of Lemma [B.2], so we consider some relations of \( U_q(\mathfrak{sl}_2^{(0)} \langle x \rangle) \) in the case where rank one and \( Q = 0 \).

**B.1.** By the induction on \( k \geq 0 \), we can show that

\[
(B.1.1) \quad X_{t+1}^+ X_t^{+(k)} = q^k \frac{1}{[2]} (J_1 X_t^{+(k+1)} - X_t^{+(k+1)} J_1),
\]
(B.1.2) \[ X_t^{- (k)} X_{t+1}^- = -q^k \frac{1}{2} (J_1 X_t^{- (k+1)} - X_t^{- (k+1)} J_1) \]

for \( t \geq 0 \), where we note that \( X_t^{+ (k)} X_{t+1}^+ = q^{-2k} X_{t+1}^+ X_t^{+ (k)} \) by the relation (Q2). We also remark that (B.1.2) follows from (B.1.1) by applying the algebra anti-involution \( \dagger \) given in Lemma 1.1. The relations (B.1.1) and (B.1.2) also hold in the case of \( k = 1 \) if we put \( X_1^{+ (1)} = 0 \).

**B.2.** By the induction on \( k > 0 \), we can show that

(B.2.1) \[ X_1^+ X_0^{-(k)} = X_0^{-(k)} X_1^+ + q^{-k+1} X_0^{-(k-1)} K^+ J_1 - q^{-2(k-1)} X_0^{-(k-2)} X_1^- K^+ \]

(B.2.2) \[ X_1^{+ (k)} X_0^- = X_0^- X_1^{+ (k)} + q^{-k+1} K^+ J_1 X_1^{+ (k-1)} - q^{-2(k-1)} K^+ X_2^+ X_1^{+ (k-2)} \]

where we put \( X_0^{-(1)} = X_1^{+ (1)} = 0 \).

For \( k > 0 \), applying (B.2.1) and (B.2.2) to the right-hand side of the equation

\[ X_1^{+ (k)} X_0^{-(k+1)} = q^{-k} (\{k+1\} - q^{-1}[k]) X_1^{+ (k)} X_0^{-(k+1)} \]

we have

\[ X_1^{+ (k)} X_0^{-(k+1)} = q^{-k} X_1^{+ (k)} X_0^- X_0^{-(k)} - q^{-k-1} X_1^{+ (k-1)} X_1^+ X_0^{-(k+1)} \]

Applying (B.1.1) and (B.1.2) to the right-hand side of this equation, we have

(B.2.3) \[ X_1^{+ (k)} X_0^{-(k+1)} = q^{-k} \frac{1}{[k]} X_0^- X_1^{+ (k)} X_0^{-(k)} + q^{-2k} \frac{1}{[2]} (J_1 X_1^{+ (k-1)} X_0^{-(k)} - X_1^{+ (k-1)} X_0^{-(k)} J_1) K^+ \]

- \[ q^{-k-1} X_1^{+ (k-1)} X_0^{-(k+1)} J_1 \].

**B.3.** We prove Lemma 8.2 by the induction on \( k \). If \( k = 1 \), the statement follows from (B.2.1). If \( k > 1 \), we have

\[ X_1^{+ (k)} X_0^{-(k+1)} = q^{-k} \frac{1}{[k]} X_0^- X_1^{+} \{q^{-1}[k-1] \sum_{z=0}^{k-1} (-1)^z X_z^- (K^+)^{k-1} J_{[k-1-z]}^{(0)} \} \]

+ \[ q^{-2k} \frac{1}{[2]} J_1 \{q^{-1}[k-1] \sum_{z=0}^{k-1} (-1)^z X_z^- (K^+)^{k-1} J_{[k-1-z]}^{(0)} \} K^+ \]
\[-q^{-2k} \frac{1}{[2]} \{ q^{-(k-1)k} \sum_{z=0}^{k-1} (-1)^z X_z^- (K^+) (k^{-1} J_{[k-1-z]}^{(0)}) \} J_1 K^+ \mod X_+ \]

by applying the induction hypothesis to the right-hand side of the equation (B.2.3). This equation together with the relations (Q6) and (1.5.2) implies that

\[
X_1^+(k) X_0^{-(k+1)} = q^{-k^2} \frac{1}{[k]} \sum_{z=0}^{k-1} (-1)^z (K^+) (k) J_{z+1}^{(0)} \]

\[+ q^{-k^2-k} \frac{1}{[2]} \sum_{z=0}^{k-1} (-1)^z (X_z^- J_1 - [2] X_{z+1}^-) (K^+) (k) J_{[k-1-z]}^{(0)} \]

\[- q^{-k^2-k} \frac{1}{[2]} \sum_{z=0}^{k-1} (-1)^z X_z^- (K^+) (k) J_{z+1}^{(0)} \mod X_+ \]

\[= q^{-k(k+1)} X_0^{-(k+1)} (K^+) (k) q^{-k} \frac{1}{[k]} \sum_{z=0}^{k-1} (-1)^{(z+1)-1} J_{z+1}^{(0)} \]

\[+ q^{-k(k+1)} \sum_{z=0}^{k-1} (-1)^{z+1} X_{z+1}^- (K^+) (k) J_{[k-(z+1)]}^{(0)} \]

Note the definition (8.1.1), and this equation implies the statement of Lemma 8.2.

**Appendix C. A proof of Lemma 9.2**

**Lemma C.1.** For \( k \in \mathbb{Z}_{>0} \) and \( t = 1, 2, \ldots, k \), we have

(C.1.1) \[ J_{[k:t]}^{(Q)} = (K^-)^2 J_{[k-1:t-1]}^{(Q)} + q^{-2k} Q J_{[k-1:t]}^{(Q)} \quad \text{if } t < k, \]

(C.1.2) \[ J_{[k:k]}^{(Q)} = q^{-k} \frac{1}{[k]} \{(Q J_1 - q^2 k J_0 + q^k [k]) J_{[k-1:k-1]}^{(Q)} + \sum_{z=1}^{k-1} (-1)^{z-1} (J_z - Q J_{z+1}) J_{[k-1;k-z-1]}^{(Q)} \}. \]

**Proof.** We prove (C.1.1) by the induction on \( t \). In the case where \( t = 1 \), we can check (C.1.1) by direct calculations using definitions. Suppose that \( t > 1 \). Applying
the induction hypothesis to the right-hand side of the definition (9.1.1), we have

\[(C.1.3)\]

\[J_{[k;t]}^{(Q)} = q^t \frac{1}{[t]} \sum_{z=1}^{t-1} (-1)^{z-1} \left( J_z - q^{2(k-t+z)} Q^{-z} J_0 + q^{k-2(t-z)} [k] Q^{-z} \right) \]

\[\times \left\{ (K^-)^2 J_{[k-1;t-z-1]}^{(Q)} + q^{-2k} Q J_{[k-1;t-z]}^{(Q)} \right\} + q^t \frac{1}{[t]} (-1)^{t-1} \left( J_t - q^{2k} Q^{-t} J_0 + q^k [k] Q^{-1} \right) J_{[k;0]}^{(Q)} \]

\[= (K^-)^2 q^t \frac{1}{[t]} \sum_{z=1}^{t-1} (-1)^{z-1} \left( J_z - q^{2(k-t+z)} Q^{-z} J_0 + q^{k-2(t-z)} [k] Q^{-z} \right) J_{[k-1;t-z-1]}^{(Q)} + q^{-2k} Q q^t \frac{1}{[t]} \sum_{z=1}^{t-1} (-1)^{z-1} \left( J_z - q^{2(k-t+z)} Q^{-z} J_0 + q^{k-2(t-z)} [k] Q^{-z} \right) J_{[k-1;t-z]}^{(Q)} \]

where we note that \(J_{[k;0]}^{(Q)} = q^{-2k} Q J_{[k-1;0]}^{(Q)}\) by definition. Note that

\[-q^{2(k-t+z)} Q^{-z} J_0 + q^{k-2(t-z)} [k] Q^{-z} \]

\[= - \left\{ q^{2(k-1-t+z)} + (q - q^{-1}) q^{2(k-1-t+z)-1} \right\} Q^{-z} J_0 + \left\{ q^{k-1-2(t-z)} [k - 1] + q^{2(k-t+z)-1} \right\} Q^{-z} \]

\[= -q^{2(k-1-t+z)} Q^{-z} J_0 + q^{k-1-2(t-z)} [k - 1] Q^{-z} + q^{2(k-t+z)-1} (K^-)^2 Q^{-z} \]

by the relation (Q1-2), and the equation (C.1.3) implies that

\[(C.1.4)\]

\[J_{[k;t]}^{(Q)} = (K^-)^2 q^t \frac{1}{[t]} \sum_{z=1}^{t-1} (-1)^{z-1} \left( J_z - q^{2(k-t+z)} Q^{-z} J_0 + q^{k-2(t-z)} [k] Q^{-z} \right) J_{[k-1;t-z-1]}^{(Q)} + q^{-2k} Q q^t \frac{1}{[t]} \sum_{z=1}^{t-1} (-1)^{z-1} \left( J_z - q^{2(k-1-t+z)} Q^{-z} J_0 + q^{k-1-2(t-z)} [k - 1] Q^{-z} \right) J_{[k-1;t-z]}^{(Q)} + q^{-2k} Q q^t \frac{1}{[t]} \sum_{z=1}^{t-1} (-1)^{z-1} q^{2(k-t+z)-1} (K^-)^2 Q^{-z} J_{[k-1;t-z]}^{(Q)} \]

Note that

\[q^{-2k} Q q^t \frac{1}{[t]} \sum_{z=1}^{t} (-1)^{z-1} q^{2(k-t+z)-1} (K^-)^2 Q^{-z} J_{[k-1;t-z]}^{(Q)} \]

\[= (K^-)^2 q^t \frac{1}{[t]} \sum_{z=1}^{t} (-1)^{(z-1)-1} (q^{-2(t-(z-1))+1} Q^{-z-1}) J_{[k-1;t-(z-1)-1]}^{(Q)} \]
and $q^{k-2(t-z)}[k] - q^{-2(t-z)+1} = q^{(k-1)-2(t-1-z)}[k-1]$, then the equation (C.1.4) implies that

$$J^{(Q)}_{[k:t]} = (K^-)^2 q^{t} \frac{1}{[t]} \sum_{z=1}^{t-1} (-1)^{z-1} \left( J_z - q^{2(k-1)-(t-1-z)}Q^{-z}J_0 + q^{(k-1)-2(t-1-z)}[k-1]Q^{-z} \right) J^{(Q)}_{[k-1;(t-z-1)]}$$

$$+ q^{-2k}Qq^{t} \frac{1}{[t]} \sum_{z=1}^{t} (-1)^{z-1} \left( J_z - q^{2(k-1-2(t-z))}Q^{-z}J_0 + q^{k-1-2(t-z)}[k-1]Q^{-z} \right) J^{(Q)}_{[k-1;1]}$$

$$+ (K^-)^2 q^{t} \frac{1}{[t]} q^{-2t+1} J^{(Q)}_{[k-1;1]};$$

Applying the definition (9.1.1), we have

$$J^{(Q)}_{[k:t]} = (K^-)^2 q^{t} \frac{1}{[t]} q^{t+1} [t-1] J^{(Q)}_{[k-1;1]} + q^{-2k}Q J^{(Q)}_{[k-1;t]} + (K^-)^2 q^{t} \frac{1}{[t]} q^{-2t+1} J^{(Q)}_{[k-1;1]}$$

$$= (K^-)^2 J^{(Q)}_{[k-1;1]} + q^{-2k}Q J^{(Q)}_{[k-1;1]};$$

Next we prove (C.1.2). Applying (C.1.1) to the right-hand side of (9.1.1), we have

$$J^{(Q)}_{[k;k]} = q^{k} \frac{1}{[k]} \sum_{z=1}^{k-1} (-1)^{z-1} \left( J_z - q^{2z}Q^{-z}J_0 + q^{-k+2z}[k]Q^{-z} \right)$$

$$\times \left\{ (K^-)^2 J^{(Q)}_{[k-1;k-z-1]} + q^{-2k}Q J^{(Q)}_{[k-1;k-z]} \right\}$$

$$+ q^{k} \frac{1}{[k]} (-1)^{k-1} \left( J_k - q^{2k}Q^{-k}J_0 + q^{k}[k]Q^{-k} \right) J^{(Q)}_{[k;0]};$$

This implies that

$$J^{(Q)}_{[k;k]} = q^{-k} \frac{1}{[k]} \sum_{z=1}^{k-2} (-1)^{z-1} \left\{ (q^{2k}J_z - q^{2k+2z}Q^{-z}J_0 + q^{k+2z}[k]Q^{-z}) (K^-)^2 \right.$$

$$\left. - (QJ_{z+1} - q^{2z+2}Q^{-z}J_0 + q^{-k+2z+2}[k]Q^{-z}) \right\} J^{(Q)}_{[k-1;k-z-1]}$$

$$+ q^{-k} \frac{1}{[k]} (-1)^{k-1} \left\{ (QJ_k - q^{2k}Q^{-k+1}J_0 + q^k[k]Q^{-k+1}) \right.$$

$$\left. - (q^{2k}J_{k-1} - q^{4k-2}Q^{-k+1}J_0 + q^{3k-2}[k]Q^{-k+1}) (K^-)^2 \right\} J^{(Q)}_{[k-1;0]}$$

$$+ q^{-k} \frac{1}{[k]} (QJ_1 - q^{2}J_0 + q^{-k+2}[k]) J^{(Q)}_{[k-1;k-1]}$$

$$= q^{-k} \frac{1}{[k]} \left\{ (QJ_1 - q^{2k}J_0 + q^k[k])J^{(Q)}_{[k-1;1]} + \sum_{z=1}^{k-1} (-1)^{z-1} (J_z - QJ_{z+1}) J^{(Q)}_{[k-1;k-z-1]} \right\}$$

$$+ q^{-k} \frac{1}{[k]} \sum_{z=1}^{k-2} (-1)^{z-1} \left\{ (q^{2k}J_z - q^{2k+2z}Q^{-z}J_0 + q^{k+2z}[k]Q^{-z}) (K^-)^2 \right.$$
\[- (Q J_{z+1} - q^{2z+2} Q^{-z} J_0 + q^{-k+2z+2}[k] Q^{-z}) - (J_z - Q J_{z+1}) \} J_{[k-1; k-z+1]}^{(Q)}
\]
\[+ q^{-k} \frac{1}{[k]} \left( - (1)^{-k-1} \{ (Q J_k - q^{-k+2} Q^{-k+1} J_0 + q^k [k] Q^{-k+1})
\]
\[- (Q J_{k-1} - q^{-k+2} Q^{-k+1} J_0 + q^k [k] Q^{-k+1}) (K^-)^2 + (J_{k-1} - Q J_k) \} J_{[k-1; 0]}^{(Q)}
\]
\[+ q^{-k} \frac{1}{[k]} \left( (Q J_1 - q^2 J_0 + q^{-k+2} [k]) - (Q J_1 - q^2 J_0 + q^k [k]) \} J_{[k-1; k-1]}^{(Q)}
\]
\[= q^{-k} \frac{1}{[k]} \left( (Q J_1 - q^2 J_0 + q^k [k]) J_{[k-1; k-1]}^{(Q)} + \sum_{z=1}^{k-1} (-1)^{z-1} (J_z - Q J_{z+1}) J_{[k-1; k-z+1]}^{(Q)} \right)
\]
\[+ q^{-k} \frac{1}{[k]} \sum_{z=1}^{k-1} (-1)^{z-1} \{ q^{2k} (J_z - q^{2z} Q^{-z} J_0 + q^{-k+2z} [k] Q^{-z}) (K^-)^2
\]
\[- (J_z - q^{2z+1} Q^{-z} J_0 + q^{-k+2z+2} [k] Q^{-z}) \} J_{[k-1; k-z+1]}^{(Q)}
\]
\[- q^{-k} \frac{1}{[k]} \left( q^{k+1} (K^-)^2 - q^{-k+1} \right) [k - 1] J_{[k-1; k-1]}^{(Q)},
\]
where we use the relation (Q1-2) in the last term. Applying the definition (9.1.1) to the last term of the above equation, we have
\[
J_{[k; k]}^{(Q)} = q^{-k} \frac{1}{[k]} \left( (Q J_1 - q^{2k} J_0 + q^k [k]) J_{[k-1; k-1]}^{(Q)} + \sum_{z=1}^{k-1} (-1)^{z-1} (J_z - Q J_{z+1}) J_{[k-1; k-z+1]}^{(Q)} \right)
\]
\[+ q^{-k} \frac{1}{[k]} \sum_{z=1}^{k-1} (-1)^{z-1} \{ q^{k+2z} ([k] - q[k-1]) (K^-)^2
\]
\[- (q^{k+2z+1} (q - q^{-1}) J_0 + q^{-k+2z+2} ([k] - q^{-1}[k-1])) \} Q^{-z} J_{[k-1; k-z+1]}^{(Q)}
\]
\[= q^{-k} \frac{1}{[k]} \left( (Q J_1 - q^{2k} J_0 + q^k [k]) J_{[k-1; k-1]}^{(Q)} + \sum_{z=1}^{k-1} (-1)^{z-1} (J_z - Q J_{z+1}) J_{[k-1; k-z+1]}^{(Q)} \right),
\]
where we also use the relation (Q1-2).

\[\square\]

**C.2.** By the induction on \( k > 0 \), we can show that
\[
(C.2.1) \quad X_0^{+(k)} X_0^{-(k)} = X_0^{-(k)} X_0^{+(k)} + X_0^{-(k-1)} K^+ \left( q^{-k+1} J_0 - q^{-k+1} Q J_1 \right) - (X_0^+ - q^{-2} Q X_1^-) X_0^{-(k-2)} K^+,
\]
\[
(C.2.2) \quad X_0^{+(k)} X_0^- = X_0^- X_0^{+(k)} + K^+ \left( q^{-k+1} J_0 - q^{-k+1} Q J_1 \right) X_0^{+(k-1)} - K^+ X_0^{+(k-2)} (X_0^+ - q^{-2} Q X_1^-),
\]
where we put $X^{(+1)}_0 - X^{(-1)}_0 = 0$. We remark that the relation (C.2.2) follows from (C.2.1) by applying the algebra anti-involution $\dagger$ given in Lemma 1.4.

C.3. For $k > 0$, applying (C.2.1) and (C.2.2) to the right-hand side of the equation

\[ X^{(+k)}_0 X^{(-k+1)} = q^{-k}([k + 1] - q^{-1}[k])X^{(+k)}_0 X^{(-k+1)}_0 \]

we have

\[ X^{(+k)}_0 X^{(-k+1)} = q^{-k}\{X^{(-k+1)}_0 - q^{-k-1}X^{(+k-1)}_0 X^{(+k)}_0\} \]

Applying the relations (Q2), (Q3), (B.1.1) and (B.1.2) to the right-hand side of the above equation, we have

\[ \begin{align*}
X^{(+k)}_0 X^{(-k+1)} & = q^{-k}\{X^{(-k+1)}_0 - q^{-k-1}X^{(+k-1)}_0 X^{(+k)}_0\} \\
& = q^{-k}\{X^{(-k+1)}_0 - q^{-k-1}X^{(+k-1)}_0 X^{(+k)}_0\} K^+ \\
& - X^{(+k-1)}_0 X^{(-k)}_0 (q^{-1}J_0 - q^{-2}Q_1) K^+ - q^{-k-1}X^{(+k-1)}_0 X^{(-k+1)}_0 K^+.
\end{align*} \]

We note that the relations (B.1.1) and (B.1.2) also hold in the case where $Q \neq 0$. Applying the relations (Q2), (Q3), (B.1.1) and (B.1.2) to the right-hand side of the above equation, we have

\[ \begin{align*}
(C.3.1) \\
X^{(+k)}_0 X^{(-k+1)} & = q^{-k} \frac{1}{[k]} X^{(-k+1)}_0 - q^{-2k} \frac{1}{[2]} Q_1 \} X^{(+k-1)}_0 X^{(+k)}_0 K^+ \\
& - X^{(+k-1)}_0 X^{(-k)}_0 (q^{-1}J_0 - q^{-2}Q_1) K^+ - q^{-k-1}X^{(+k-1)}_0 X^{(-k+1)}_0 K^+.
\end{align*} \]

C.4. We prove Lemma 9.2 by the induction on $k$. In the case where $k = 1$, the statement follows from (C.2.1). Suppose that $k > 1$. By (C.3.1), we have

\[ \begin{align*}
X^{(+k)}_0 X^{(-k+1)} & \equiv q^{-k} \frac{1}{[k]} X^{(-k+1)}_0 - q^{-2k} \frac{1}{[2]} Q_1 \} X^{(+k-1)}_0 X^{(+k)}_0 K^+ \\
& - X^{(+k-1)}_0 X^{(-k)}_0 (q^{-1}J_0 - q^{-2}Q_1) K^+ \quad \text{mod } \mathfrak{X}_+.
\end{align*} \]

Applying the induction hypothesis, we have

\[ \begin{align*}
X^{(+k)}_0 X^{(-k+1)} & \equiv q^{-k} \frac{1}{[k]} X^{(-k+1)}_0 \{\sum_{z=0}^{k-1} (-1)^{k-z-1} X^{(k+1)}_z (K^+)^{k-1} j^{(Q)}_{[k-1,k-z-1]}\} \\
& + (q^{-3}J_0 - q^{-2k} \frac{1}{[2]} Q_1) \{\sum_{z=0}^{k-1} (-1)^{k-z-1} X^{(k+1)}_z (K^+)^{k-1} j^{(Q)}_{[k-1,k-z-1]}\} K^+.
\end{align*} \]
\[- \{ \sum_{z=0}^{k-1} (-1)^{k-z-1} X_z^- (K^+)^{k-1} J_{[k-1; k-z-1]}^{(Q)} \} (q^{-1} J_0 - q^{-2k} \frac{1}{[2]} Q J_1 - q^{-2}) K^+ \mod \mathfrak{X}^+ \]

\[
\equiv q^{-k} \frac{1}{[k]} \sum_{z=0}^{k-1} (-1)^{k-z-1} (J_z - Q J_{z+1}) (K^+)^{k} J_{[k-1; k-z-1]}^{(Q)} \\
+ \sum_{z=0}^{k-1} (-1)^{k-z-1} \{ q^{-3} (q^4 X_z^- J_0 - q^2 [2] X_z^-) - q^{-2k} \frac{1}{[2]} Q (X_z^- J_1 - [2] X_{z+1}^-) \} (K^+)^{k} J_{[k-1; k-z-1]}^{(Q)} \\
- \sum_{z=0}^{k-1} (-1)^{k-z-1} X_z^- (K^+)^{k} J_{[k-1; k-z-1]}^{(Q)} (q^{-1} J_0 - q^{-2k} \frac{1}{[2]} Q J_1 - q^{-2}) \mod \mathfrak{X}^+ 
\]

where we use the relations (Q1-1), (Q1-2), (Q6), (1.3.2) and the fact $X_s^+ U_{q,Q}^0 \subset \mathfrak{X}^+$ for all $s \geq 0$ which follows from defining relations immediately. This equation implies

\[
X_0^{+(k)} X_0^{-(k+1)} \\
\equiv (-1)^k X_0^- (K^+)^k \left\{ q^{-k} \frac{1}{[k]} \sum_{z=0}^{k-1} (-1)^{k-z-1} (J_z - Q J_{z+1}) J_{[k-1; k-z-1]}^{(Q)} + (1 - (q - q^{-1}) J_0) J_{[k-1; k-1]}^{(Q)} \right\} \\
+ \sum_{z=1}^{k-1} (-1)^{k-z} X_z^- (K^+)^k \left\{ 1 - (q - q^{-1}) J_0 \right\} J_{[k-1; k-z-1]}^{(Q)} \\
+ \sum_{z=0}^{k-1} (-1)^{k-(z+1)} X_{z+1}^- (K^+)^k \left\{ q^{-2k} Q \right\} J_{[k-1; k-(z+1)]}^{(Q)} \mod \mathfrak{X}^+ \\
= (-1)^k X_0^- (K^+)^k \\
\times q^{-k} \frac{1}{[k]} \left\{ (Q J_1 - q^{2k} J_0 + q^k [k]) J_{[k-1; k-1]}^{(Q)} + \sum_{z=1}^{k-1} (-1)^{k-z-1} (J_z - Q J_{z+1}) J_{[k-1; k-z-1]}^{(Q)} \right\} \\
+ \sum_{z=1}^{k-1} (-1)^{k-z} X_z^- (K^+)^k \left\{ (K^-)^2 J_{[k-1; k-z-1]}^{(Q)} + q^{-2k} Q J_{[k-1; k-z-1]}^{(Q)} \right\} \\
+ X_{k}^- (K^+)^k (q^{-2k} Q J_{[k-1; 0]}^{(Q)}),
\]

where we note that $1 - (q - q^{-1}) J_0 = (K^-)^2$ by (Q1-2). Applying Lemma [C.1] we have

\[
X_0^{+(k)} X_0^{-(k+1)} \equiv \sum_{z=0}^{k} (-1)^{k-z} X_z^- (K^+)^k J_{[k; k-z]}^{(Q)} \mod \mathfrak{X}^+,
\]

where we note that $J_{[k,0]}^{(Q)} = q^{-2k} Q J_{[k-1,0]}^{(Q)}$ by definition.
APPENDIX D. THE \((q, Q)\)-CURRENT ALGEBRA \(U_q(\mathfrak{gl}_n^Q[x])\) AND CYCLOMATIC \(q\)-SCHUR ALGEBRAS

In this appendix, we consider the \((q, Q)\)-current algebra \(U_q(\mathfrak{gl}_n^Q[x])\) associated with the general linear Lie algebra \(\mathfrak{gl}_n\). We show that the algebra \(U_q(\mathfrak{gl}_n^Q[x])\) with special parameters is isomorphic to the algebra \(U_qQ(\mathfrak{n})\) introduced in [W16] (see Proposition D.10). We also give some connection with cyclotomic \(q\)-Schur algebras according to [W16].

**D.1.** Recall that \(A = (a_{ij})_{1 \leq i, j \leq n-1}\) is the Cartan matrix of type \(A_{n-1}\). We also put \(\widetilde{a}_{ii} = 1, \widetilde{a}_{i+1,i} = -1\) and \(\widetilde{a}_{ij} = 0\) if \(i \neq j, j + 1\) for \(1 \leq i, j \leq n\). We define the \((q, Q)\)-current algebra \(U_q(\mathfrak{gl}_n^Q[x])\) associated with the general linear Lie algebra \(\mathfrak{gl}_n\) as follows.

**Definition D.2.** For \(q \in \mathbb{C}^\times\) and \(Q = (Q_1, Q_2, \ldots, Q_{n-1}) \in \mathbb{C}^{n-1}\), we define the associative algebra \(U_q(\mathfrak{gl}_n^Q[x])\) over \(\mathbb{C}\) by the following generators and defining relations:

**Generators:** \(X^\pm_{i,t}, I^\pm_{j,t}, \widetilde{K}^\pm_{i,j} (1 \leq i \leq n - 1, 1 \leq j \leq n, t \in \mathbb{Z}_{\geq 0})\),

**Defining relations:**

\[(Q'1-1) \quad [\widetilde{K}^+, \widetilde{K}^+] = [\widetilde{K}^+, I^\sigma_{j,t}] = [I^\sigma_{i,s}, I^\sigma_{j,t}] = 0 \quad (\sigma, \sigma' \in \{+, -, \}),\]
\[(Q'1-2) \quad \widetilde{K}^+_j \widetilde{K}^-_j = 1 = \widetilde{K}^-_j \widetilde{K}^+_j, \quad (\widetilde{K}^\pm_j)^2 = 1 \pm (q - q^{-1})I^\pm_j,\]
\[(Q'2) \quad X^+_{i,t+1}X^-_{i,s} - q^2X^+_{i,t}X^+_{i,t+1} = q^2X^+_{i,s}X^-_{i+1,t+1} - X^-_{i+1,s+1}X^+_{i,t},\]
\[(Q'3) \quad X^-_{i,t+1}X^+_{i,s} - q^2X^-_{i,s}X^-_{i,t+1} = q^2X^-_{i,s}X^-_{i+1,t+1} - X^-_{i+1,s+1}X^-_{i,t},\]
\[(Q'4-1) \quad \widetilde{K}^+_{i,j}X^+_{j,t} = q^{\tilde{a}_{ij}}X^+_{j,t},\]
\[(Q'4-2) \quad q^{\tilde{a}_{ij}}I^+_{i,0}X^+_{j,t} - q^{\tilde{a}_{ij}}X^+_{j,t}I^+_{i,0} = \tilde{a}_{ij}X^+_{j,t},\]
\[(Q'4-3) \quad [I^\pm_{i,s+1}, X^\pm_{j,t}] = q^{\tilde{a}_{ij}}I^\pm_{i,s}X^\pm_{j,t+1} - q^{\tilde{a}_{ij}}X^\pm_{j,t+1}I^\pm_{i,s},\]
\[(Q'5-1) \quad \widetilde{K}^-_{i,j}X^-_{j,t} = q^{\tilde{a}_{ij}}X^-_{j,t},\]
\[(Q'5-2) \quad q^{\tilde{a}_{ij}}I^-_{i,0}X^-_{j,t} - q^{\tilde{a}_{ij}}X^-_{j,t}I^-_{i,0} = -\widetilde{a}_{ij}X^-_{j,t},\]
\[(Q'5-3) \quad [I^\pm_{i,s+1}, X^-_{j,t}] = q^{\tilde{a}_{ij}}I^\pm_{i,s}X^-_{j,t+1} - q^{\tilde{a}_{ij}}X^-_{j,t+1}I^\pm_{i,s},\]
\[(Q'6) \quad [X^+_{i,t}, X^-_{j,s}] = \delta_{i,j}K^+_i(J_{i,s+t} - qJ_{i,s+t+1}),\]
\[(Q'7) \quad [X^+_{i,t}, X^-_{j,s}] = 0 \quad \text{if} \quad j \neq i, i \pm 1,\]
\[(Q'8) \quad [X^+_{i,t}, X^-_{j,s}] = 0 \quad \text{if} \quad j \neq i, i \pm 1,\]
where we put \( K_i^+ = \tilde{K}_i^+ \tilde{K}_{i+1}^-, \ K_i^- = \tilde{K}_i^- \tilde{K}_{i+1}^+ \).

\[
J_{i,t} = \begin{cases} 
I_{i,0}^+ - I_{i+1,0}^- + (q - q^{-1})I_{i,0}^- I_{i+1,0}^- & \text{if } t = 0, \\
q^{-t}I_{t,0}^+ - q^tI_{i+1,t}^- - (q - q^{-1}) \sum_{z=1}^{t-1} q^{-t+2z} I_{t,z}^+ I_{i+1,z}^- & \text{if } t > 0.
\end{cases}
\]

**Remark D.3.** In the case where \( q = 1 \), let \( \mathfrak{J} \) be the two-sided ideal of \( U_1(\mathfrak{g}_n \mathcal{Q}[x]) \) generated by \( \{ \tilde{K}_j^+ - 1, \ I_{j,t}^+ - I_{j,t}^- \mid 1 \leq j \leq n, \ t \in \mathbb{Z}_{\geq 0} \} \). Then we see easily that \( U_1(\mathfrak{g}_n \mathcal{Q}[x]) / \mathfrak{J} \) is isomorphic to the universal envelope algebra of the deformed current Lie algebra \( \mathfrak{g}_n \mathcal{Q}[x] \) given in [WIS] Definition 1.1.

From the defining relations, we can easily check the following lemma.

**Lemma D.4.** There exists the algebra anti-involution \( \dagger : U_q(\mathfrak{g}_n \mathcal{Q}[x]) \to U_q(\mathfrak{g}_n \mathcal{Q}[x]) \) such that \( \dagger(X_{i,t}^\pm) = X_{i,t}^\mp, \ \dagger(I_{i,t}^\pm) = I_{j,t}^\pm \) and \( \dagger(\tilde{K}_j^\pm) = \tilde{K}_j^\mp \) for \( 1 \leq i \leq n-1, \ 1 \leq j \leq n \) and \( t \in \mathbb{Z}_{\geq 0} \).

**Lemma D.5.** We have the following relations in \( U_q(\mathfrak{g}_n \mathcal{Q}[x]) \).

(i) \( K_i^+ X_{j,t}^\pm K_i^- = q^{+a_{ij}} X_{j,t}^\pm \).

(ii) \( (K_i^-)^2 = 1 - (q - q^{-1})J_{i,0} \).

(iii) \( q^{+a_{ij}} J_{i,0} X_{j,t}^\pm - q^{-a_{ij}} X_{j,t}^\pm J_{j,t} = [\pm a_{ij}] X_{j,t}^\pm \).

(iv) \( [J_{i,s+1}, X_{j,t}^\pm] = q^{a_{i,j}} J_{i,s} X_{j,t+1}^\pm - q^{a_{1,i,j}} J_{i,s} X_{j,t+1}^\pm J_{i,s} \).

(v) \( [J_{i,s+1}, X_{j,t}^\pm] = q^{a_{i,j}} J_{i,s} X_{j,t+1}^\pm - q^{a_{1,i,j}} X_{j,t+1}^\pm J_{i,s} \).

**Proof.** Note that \( K_i^\pm = \tilde{K}_i^\pm \tilde{K}_{i+1}^\mp \), then the relation (i) follows from the relations (Q’1-1), (Q’4-1) and (Q’5-1) immediately. We also have the relation (ii) by direct calculation using the relations (Q’1-1) and (Q’1-2). By the relations (i) and (ii), we have

\[
J_{i,0} X_{j,t}^\pm = \frac{1 - (K_i^-)^2}{q - q^{-1}} X_{j,t}^\pm = X_{j,t}^\pm \frac{1 - q^{+2a_{ij}} (K_i^-)^2}{q - q^{-1}} X_{j,t}^\pm + \frac{q^{-2a_{ij}}}{q - q^{-1}} X_{j,t}^\pm J_{i,0}.
\]

This implies the relation (iii). We prove (iv). By the definition of \( J_{i,s+1} \), we have

\[
J_{i,s+1} X_{j,t}^\pm = (q^{-(s+1)} I_{i,s+1}^+ - q^{s+1} I_{i+1,s+1}^- - (q - q^{-1}) \sum_{z=1}^{s} q^{-(s+1)+2z} I_{i,s-z+1}^+ I_{i+1,z}^-) X_{j,t}^+.
\]

Applying the relation (Q’4-3), we have

\[
J_{i,s+1} X_{j,t}^+ = X_{j,t}^+(q^{-(s+1)} I_{i,s+1}^+ - q^{s+1} I_{i+1,s+1}^- - (q - q^{-1}) \sum_{z=1}^{s} q^{-(s+1)+2z} I_{i,s-z+1}^+ I_{i+1,z}^-)
\]

\(= q^{-(s+1)}(q^{a_{ij}} I_{i,s} X_{j,t+1}^+ - q^{-a_{ij}} X_{j,t+1}^+ J_{i,s})
\)
\[-q^{s+1}(q^{-\tilde{a}_{i+1,j}}I_{i+1,s}X_{j,t}^+ - q^{-\tilde{a}_{i+1,j}}X_{j,t+1}^+ I_{i+1,s}^-)\]
\[-(q - q^{-1}) \sum_{z=1}^{s} q^{-(s+1) + 2z} (q^{\tilde{a}_{ij}}I_{i,s-z}^+ X_{j,t+1}^+ I_{i+1,z}^- - q^{-\tilde{a}_{i+1,j}} X_{j,t+1}^- I_{i+1,z}^+)\]
\[+ q^{-\tilde{a}_{i+1,j}}I_{i,s-z+1}^+ I_{i+1,z-1}^- X_{j,t+1}^+ - q^{-\tilde{a}_{i+1,j}} I_{i+1,z+1}^- X_{j,t+1}^+ I_{i+1,z-1}^-\]
\[= X_{j,t}^+ J_{i,s+1} \left\{ q^{\tilde{a}_{ij}} I_{i,s-z}^- - q^{-\tilde{a}_{i+1,j} + s + 1} I_{i,s}^- \right\} X_{j,t+1}^+ \]
\[-(q - q^{-1}) \sum_{z=1}^{s} q^{-(s+1) + 2z} (q^{\tilde{a}_{ij}} I_{i,s-z}^- X_{j,t+1}^+ I_{i+1,z}^- - q^{-\tilde{a}_{i+1,j}} I_{i,s-z+1}^- X_{j,t+1}^- I_{i+1,z-1}^-)\]

where we note that

\[\sum_{z=1}^{s} q^{-(s+1) + 2z} (q^{\tilde{a}_{ij}} I_{i,s-z}^- X_{j,t+1}^+ I_{i+1,z}^- - q^{-\tilde{a}_{i+1,j}} I_{i,s}^- I_{i+1,z}^+) = \sum_{z=1}^{s-1} q^{-(s+1) + 2z} (q^{\tilde{a}_{ij}} - q^{-\tilde{a}_{i+1,j} + 2}) I_{i,s-z}^+ X_{j,t+1}^+ I_{i+1,z}^- \]
\[+ q^{\tilde{a}_{ij} + s - 1} I_{i,0}^- X_{j,t+1}^+ I_{i+1,z}^- - q^{\tilde{a}_{i+1,j} - s + 1} I_{i,s}^- X_{j,t+1}^+ I_{i+1,0}^-\]

Thus, we have

\[[J_{i,s+1}, X_{j,t}^+]\]
\[= \left\{ q^{\tilde{a}_{ij}} I_{i,s}^- - q^{-\tilde{a}_{i+1,j} + s + 1} I_{i+1,s}^- \right\} X_{j,t+1}^+ \]
\[-X_{j,t+1}^+ \left\{ q^{\tilde{a}_{ij}} I_{i,s}^- - q^{-\tilde{a}_{i+1,j} + s + 1} I_{i+1,s}^- \right\} X_{j,t+1}^+ \]
\[-(q - q^{-1}) \sum_{z=1}^{s-1} q^{-(s+1) + 2z} (q^{\tilde{a}_{ij}} - q^{-\tilde{a}_{i+1,j} + 2}) I_{i,s-z}^+ X_{j,t+1}^+ I_{i+1,z}^- \]
\[-q^{s-1}(q - q^{-1}) (q^{\tilde{a}_{ij}} I_{i,0}^- X_{j,t+1}^+ - q^{-\tilde{a}_{ij}} X_{j,t+1}^+ I_{i+1,s}^-) I_{i+1,s}^- \]
\[+ q^{-s+1}(q - q^{-1}) I_{i,s}^- (q^{\tilde{a}_{i+1,j}} X_{j,t+1}^+ - q^{-\tilde{a}_{i+1,j}} I_{i+1,0}^- X_{j,t+1}^+).\]
Applying $J_{i,s} = q^{-s}I_{i,s}^+ - q^s I_{i+1,s}^+ - (q - q^{-1}) \sum_{z=1}^{s-1} q^{s-z}+2z I_{i,s-z}^+ I_{i+1,z}^-$ and the relation (Q'4-2), we have

\[(D.5.1)\]

\[
\begin{align*}
[J_{i,s+1}, X_{j,t}^+] &= q^{-a_{i+1,j+1}}J_{i,s}X_{j,t+1}^+ + q^{-s}(q^{-a_{ij}-1} - q^{-a_{i+1,j+1}})I_{i,s}^+ X_{j,t+1}^+ \\
&\quad - q^{-a_{ij}}X_{j,t+1}^+ J_{i,s} + q^s(q^{-a_{i+1,j+1}} - q^{-a_{ij}})X_{j,t+1}^+ I_{i+1,s}^- \\
&\quad - (q - q^{-1}) \sum_{z=1}^{s-1} q^{-s+2z}(q^{-a_{ij}-1} - q^{-a_{i+1,j+1}})I_{i,s-z}^+ X_{j,t+1}^+ I_{i+1,z}^-
\end{align*}
\]

On the other hand, by (Q'4-2) and (Q'4-3), we have

\[(D.5.2)\]

\[
\begin{align*}
I_{i,u}^+ X_{j,t+1}^+ &= X_{j,t+1}^+ I_{i,u}^+ \text{ if } i \neq j, j+1, \\
X_{j,t+1}^+ I_{i+1,u}^- &= I_{i+1,u}^- X_{j,t+1}^+ \text{ if } i \neq j - 1, j
\end{align*}
\]

for $u \geq 0$. Then, the equations (D.5.1) and (D.5.2) imply the relation (iv). The relation (v) follows from (iv) by applying the algebra anti-involution \(\dagger\).

\[\square\]

**Proposition D.6.** Put $Q_{[q]} = (q^{-1}Q_1, q^{-2}Q_2, \ldots, q^{-(n-1)}Q_{n-1})$. Then, there exists the algebra homomorphism $\Upsilon^{(Q)} : U_q(W_n(Q_{[q]})[x]) \to U_q(\mathfrak{g}_n^{(Q)}[x])$ such that

\[
\Upsilon^{(Q)}(X_{i,t}^\pm) = q^{t_i}X_{i,t}^\pm, \quad \Upsilon^{(Q)}(J_{i,t}) = q^{t_i}J_{i,t}, \quad \Upsilon^{(Q)}(K_{i}^\pm) = K_i^\pm.
\]

**Proof.** We can check the well-definedness of the homomorphism $\Upsilon^{(Q)}$ by direct calculations using the defining relations of $U_q(\mathfrak{g}_n^{(Q)}[x])$ and Lemma [D.5].

\[\square\]

**D.7.** For $q \in \mathbb{C}^\times$ and $\widehat{Q} = (\widehat{Q}_0, \widehat{Q}_1, \ldots, \widehat{Q}_{r-1}) \in \mathbb{C}^r$, let $\mathcal{H}_{m,r}$ be the Ariki-Koike algebra associated to the complex reflection group $\mathfrak{S}_m \ltimes (\mathbb{Z}/r\mathbb{Z})^m$ of type $G(r, 1, m)$ with parameters $q$ and $\widehat{Q}$. Namely, $\mathcal{H}_{m,r}$ is the associative algebra over $\mathbb{C}$ generated by $T_0, T_1, \ldots, T_{m-1}$ subject to the defining relations

\[
\begin{align*}
(T_0 - \widehat{Q}_0)(T_0 - \widehat{Q}_1)\ldots(T_0 - \widehat{Q}_{r-1}) &= 0, \quad (T_i - q)(T_i + q^{-1}) = 0 \quad (1 \leq i \leq m - 1), \\
T_0T_iT_0 &= T_iT_0T_i, \quad T_iT_{i+1}T_i = T_{i+1}T iT_{i+1} \quad (1 \leq i \leq m - 2), \\
T_iT_j &= T_jT_i \quad (|i-j| > 1).
\end{align*}
\]

For $n = (n_1, n_2, \ldots, n_r) \in \mathbb{Z}_{\geq 0}^r$, let $\mathcal{H}_{m,r}(n)$ be the cyclotomic $q$-Schur algebra associated to the Ariki-Koike algebra $\mathcal{H}_{m,r}$ with respect to $n$ defined in [DJKM] (see also [W16], §6 for definitions).

An $r$-tuple of partitions $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(r)})$ is called an $r$-partition. For an $r$-partition $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(r)})$, we denote $\sum_{k=1}^r |\lambda^{(k)}|$ by $|\lambda|$, and we call it the size of $\lambda$. Set $A_{m,r}^+ = \{\lambda = (\lambda^{(1)}, \ldots, \lambda^{(r)}) : r$-partition $|\lambda| = m, \ell(\lambda^{(r)}) \leq n_r\}$. For $\lambda \in A_{m,r}^+$, let $\Delta(\lambda)$ be the Weyl (cell) module corresponding to $\lambda$ constructed in
It is known that $\mathcal{S}_{m,r}(n)$ is a quasi-hereditary cellular algebra with the set of standard modules $\{\Delta(\lambda) \mid \lambda \in \Lambda_{m,r}^+\}$ if $n_k \geq m$ for all $k$ by [DJM].

**D.8.** For $n = (n_1, \ldots, n_r) \in \mathbb{Z}_{>0}^r$, set $n = n_1 + \cdots + n_r$, $\Gamma(n) = \{(i,k) \mid 1 \leq i \leq n_k, \ 1 \leq k \leq r\}$ and $\Gamma'(n) = \Gamma(n) \setminus \{(n,r)\}$. We identify $\Gamma(n)$ with $\{1, 2, \ldots, n\}$ by the bijection

$$\xi : \Gamma(n) \to \{1, 2, \ldots, n\} \text{ such that } \xi(i,k) = \sum_{j=1}^{k-1} n_j + i.$$  

Namely, $\Gamma(n)$ gives the separation of the set $\{1, 2, \ldots, n\}$ to $r$-parts with respect to $n$. Under this identification, we regard $(n_k + 1, k)$ (resp. $(0, k)$) as $(1, k + 1)$ (resp. $(n_k - 1, k - 1)$). For $(i,k), (j,l) \in \Gamma(n)$, set $\bar{a}_{(i,k)(j,l)} = \bar{a}_{\xi(i,k),\xi(j,l)}$. By [W16], the cyclotomic $q$-Schur algebra $\mathcal{S}_{m,r}(n)$ is realized as a quotient of the algebra $\mathcal{U}_q,\hat{Q}(n)$ defined as follows.

**Definition D.9 ([W16 Definition 4.2]).** We define the associative algebra $\mathcal{U}_q,\hat{Q}(n)$ over $\mathbb{C}$ by the following generators and defining relations:

**Generators:** $\mathcal{X}^{\pm}_{(i,k),t}$, $\mathcal{T}^\pm_{(j,l),t}$, $\mathcal{K}^\pm_{(j,l)}$ ($i, k) \in \Gamma'(n)$, $(j, l) \in \Gamma(n)$, $t \in \mathbb{Z}_{\geq 0}$.

**Defining relations:**

(R1) \[ \mathcal{K}^+_{(j,l)} \mathcal{K}^-_{(j,l)} = 1 = \mathcal{K}^-_{(j,l)} \mathcal{K}^+_{(j,l)}, \quad (\mathcal{K}^\pm_{(j,l)})^2 = 1 \pm (q - q^{-1}) \mathcal{T}^\pm_{(j,l)}, \]

(R2) \[ [\mathcal{K}^+_{(i,k)}, \mathcal{K}^+_{(j,l)}] = [\mathcal{K}^+_{(i,k)}, \mathcal{T}^\sigma_{(j,l),t}] = [\mathcal{T}^\sigma_{(i,k),s}, \mathcal{T}^{\sigma'}_{(j,l),t}] = 0 \quad (\sigma, \sigma' \in \{+, -\}), \]

(R3) \[ \mathcal{K}^+_{(i,k)} \mathcal{X}^+_{(j,l),t} \mathcal{K}^-_{(i,k)} = q^{\pm a_{(i,k)(j,l)}} \mathcal{X}^+_{(j,l),t}, \quad q^{\mp a_{(i,k)(j,l)}} \mathcal{X}^-_{(j,l),t} = q^{-\mp a_{(i,k)(j,l)}} \mathcal{X}^-_{(j,l),t}; \]

(R4) \[ q^{\mp a_{(i,k)(j,l)}} \mathcal{T}^+_{(i,k),0} \mathcal{X}^+_{(j,l),t} = q^{\pm a_{(i,k)(j,l)}} \mathcal{T}^-_{(i,k),0} \mathcal{X}^-_{(j,l),t}, \quad q^{\mp a_{(i,k)(j,l)}} \mathcal{T}^-_{(i,k),0} \mathcal{X}^-_{(j,l),t} = q^{\pm a_{(i,k)(j,l)}} \mathcal{T}^+_{(i,k),0} \mathcal{X}^+_{(j,l),t}; \]

(R5) \[ [\mathcal{I}^+_{(i,k),s+1}, \mathcal{X}^+_{(j,l),t}] = q^{a_{(i,k)(j,l)}} \mathcal{I}^+_{(i,k),s} \mathcal{X}^+_{(j,l),t+1} - q^{-a_{(i,k)(j,l)}} \mathcal{I}^+_{(i,k),s} \mathcal{X}^+_{(j,l),t+1}, \quad [\mathcal{I}^+_{(i,k),s+1}, \mathcal{X}^-_{(j,l),t}] = q^{-a_{(i,k)(j,l)}} \mathcal{I}^+_{(i,k),s} \mathcal{X}^-_{(j,l),t+1} - q^{a_{(i,k)(j,l)}} \mathcal{I}^+_{(i,k),s} \mathcal{X}^-_{(j,l),t+1}; \]

(R6) \[ [\mathcal{X}^+_{(i,k),t}, \mathcal{X}^-_{(j,l),s}, s] = \delta_{(i,k)(j,l)} \begin{cases} \mathcal{K}^+_{(i,k)} \mathcal{I}^+_{(i,k),s+1} & \text{if } i \neq n_k, \\ -q_{k} \mathcal{K}^+_{(n_k,k)} \mathcal{I}^+_{(n_k,k),s+1} + \mathcal{K}^+_{(n_k,k)} \mathcal{I}^+_{(n_k,k),s+1} & \text{if } i = n_k, \end{cases} \]

(R7) \[ [\mathcal{X}^+_{(i,k),t}, \mathcal{X}^-_{(j,l),s}, s] = 0 \quad \text{if } (j, l) \neq (i, k), (i \pm 1, k), \]

\[ \mathcal{X}^+_{(i,k),t+1} \mathcal{X}^+_{(i,k),s} = q^{-2} \mathcal{X}^+_{(i,k),s+1} \mathcal{X}^+_{(i,k),t} = q^{+2} \mathcal{X}^+_{(i,k),t} \mathcal{X}^+_{(i,k),s+1} - \mathcal{X}^+_{(i,k),s+1} \mathcal{X}^+_{(i,k),t}, \]

\[ \mathcal{X}^+_{(i,k),t+1} \mathcal{X}^+_{(i+1,k),s} = q^{-1} \mathcal{X}^+_{(i+1,k),s+1} \mathcal{X}^+_{(i,k),t+1} = \mathcal{X}^+_{(i,k),t+1} \mathcal{X}^+_{(i+1,k),s} - \mathcal{X}^+_{(i+1,k),s+1} \mathcal{X}^+_{(i,k),t+1}, \]

\[ \mathcal{X}^-_{(i+1,k),s} \mathcal{X}^-_{(i,k),t+1} = q^{-1} \mathcal{X}^-_{(i+1,k),s} \mathcal{X}^-_{(i,k),t+1} = \mathcal{X}^-_{(i+1,k),s} \mathcal{X}^-_{(i,k),t} - \mathcal{X}^-_{(i+1,k),s+1} \mathcal{X}^-_{(i,k),t+1}. \]
(R8)\[
\mathcal{X}^+_{(i\pm1,k),u} (\mathcal{X}^+_{(i,k),s} \mathcal{X}^+_{(i,k),t} + \mathcal{X}^+_{(i,k),t} \mathcal{X}^+_{(i,k),s}) + (\mathcal{X}^+_{(i,k),s} \mathcal{X}^+_{(i,k),t} + \mathcal{X}^+_{(i,k),t} \mathcal{X}^+_{(i,k),s}) \mathcal{Y}^+_{(i\pm1,k),u} \\
= (q + q^{-1}) (\mathcal{X}^+_{(i,k),s} \mathcal{X}^+_{(i\pm1,k),u} \mathcal{X}^+_{(i,k),t} + \mathcal{X}^+_{(i,k),t} \mathcal{X}^+_{(i\pm1,k),u} \mathcal{X}^+_{(i,k),s}) \\
\mathcal{X}^-_{(i\pm1,k),u} (\mathcal{X}^-_{(i,k),s} \mathcal{X}^-_{(i,k),t} + \mathcal{X}^-_{(i,k),t} \mathcal{X}^-_{(i,k),s}) + (\mathcal{X}^-_{(i,k),s} \mathcal{X}^-_{(i,k),t} + \mathcal{X}^-_{(i,k),t} \mathcal{X}^-_{(i,k),s}) \mathcal{Y}^-_{(i\pm1,k),u} \\
= (q + q^{-1}) (\mathcal{X}^-_{(i,k),s} \mathcal{X}^-_{(i\pm1,k),u} \mathcal{X}^-_{(i,k),t} + \mathcal{X}^-_{(i,k),t} \mathcal{X}^-_{(i\pm1,k),u} \mathcal{X}^-_{(i,k),s}),
\]

where we put
\[
\mathcal{K}^+_{(i,k)} = \tilde{\mathcal{K}}_{(i,k)} \mathcal{K}_{(i\pm1,k)}\] and
\[
\mathcal{K}^-_{(i,k)} = \tilde{\mathcal{K}}_{(i,k)} \mathcal{K}_{(i\pm1,k)}
\]

\[
\mathcal{J}_{(i,k),t} = \begin{cases} 
\mathcal{I}_{(i,k),0}^+ - \mathcal{I}_{(i+1,k),0}^- + (q - q^{-1}) \mathcal{I}_{(i,k),0}^- \mathcal{I}_{(i+1,k),0}^+ & \text{if } t = 0, \\
q^{-t} \mathcal{I}_{(i,k),t}^+ - q^t \mathcal{I}_{(i+1,k),t}^- - (q - q^{-1}) \sum_{z=1}^{t} q^{-t+2z} \mathcal{I}_{(i,k),t-z}^+ \mathcal{I}_{(i+1,k),z}^- & \text{if } t > 0.
\end{cases}
\]

We remark that the parameter \(\tilde{Q}_0\) does not appear, and we do not need it, in the definition of the algebra \(U_{q,Q}(\mathfrak{n})\). The parameter \(\tilde{Q}_0\) appears in the algebra homomorphism from \(U_{q,Q}(\mathfrak{n})\) to the cyclotomic \(q\)-Schur algebra \(\mathcal{S}_{m,r}(\mathfrak{n})\) given in [W16] Theorem 8.1.

We can easily prove the following proposition by checking defining relations.

**Proposition D.10.** Assume that \(\tilde{Q}_i \neq 0\) for all \(1 \leq i \leq r - 1\). Set \(Q' = (Q'_1, Q'_2, \ldots, Q'_{n-1}) \in \mathbb{C}^{n-1}\) as

\[
Q'_i = \begin{cases} 
\tilde{Q}_k^{-1} & \text{if } \xi^{-1}(i) = (n_k, k) \text{ for some } k, \\
0 & \text{otherwise.}
\end{cases}
\]

Then, there exists the algebra isomorphism \(\Omega_n^{(Q)} : U_q(\mathfrak{gl}^{(Q')}[x]) \to U_{q,\tilde{Q}}(\mathfrak{n})\) such that

\[
\Omega_n^{(Q)}(X^+_{i,j,t}) = \begin{cases} 
X^+_{\xi^{-1}(i),t} & \text{if } \xi^{-1}(i) \neq (n_k, k) \text{ for all } k, \\
Q_k^{-1} X^+_{\xi^{-1}(i),t} & \text{if } \xi^{-1}(i) = (n_k, k) \text{ for some } k,
\end{cases}
\]

\[
\Omega_n^{(Q)}(X^-_{i,j,t}) = X^-_{\xi^{-1}(i),t}, \quad \Omega_n^{(Q)}(I^+_{i,j,t}) = I^+_{\xi^{-1}(j),t}, \quad \Omega_n^{(Q)}(K^+_j) = K^+_\xi^{-1}(j).
\]

**D.11.** Let \(\Psi_n^{(Q)} : U_{q,\tilde{Q}}(\mathfrak{n}) \to \mathcal{S}_{m,r}(\mathfrak{n})\) be the algebra homomorphism given in [W16] Theorem 8.1. Assume that \(\tilde{Q}_i \neq 0\) for all \(1 \leq i \leq r - 1\). Set \(Q' = (Q'_1, \ldots, Q'_{n-1}) \in \mathbb{C}^{n-1}\) as \([D.10.1]\), and put \(Q = (Q_1, Q_2, \ldots, Q_{n-1}) = Q'_{[q]}\). Namely, we have

\[
Q_i = \begin{cases} 
q^{-(n_1 + \cdots + n_{k-1})} \tilde{Q}_k^{-1} & \text{if } \xi^{-1}(i) = (n_k, k) \text{ for some } k, \\
0 & \text{otherwise.}
\end{cases}
\]

Then, we have the algebra homomorphism

\[
\Phi_n^{(Q)} := \Psi_n^{(Q)} \circ \Omega_n^{(Q)} \circ \gamma^{(Q)} : U_q(\mathfrak{sl}^{(Q')}[x]) \to \mathcal{S}_{m,r}(\mathfrak{n}).
\]
Through the algebra homomorphism $\Phi_n^{(Q)}$, we regard $J_{m,r}(n)$-modules as $U_q(\mathfrak{sl}_n^{(Q)}[x])$-modules.

**Proposition D.12.** Assume that $q$ is not a root of unity, and $n_k \geq m$ for all $k$. For $\lambda \in A_{m,r}^+$, the Weyl module $\Delta(\lambda)$ is a highest weight $U_q(\mathfrak{sl}_n^{(Q)}[x])$-module, and the highest weight of $\Delta(\lambda)$ is given by $(u^{(Q)}(\phi_i))_{i \in I}$, where

$$
\varphi_i = \left\{ \begin{array}{ll}
\prod_{p=1}^{k-1} (x - q^{-2j+2\lambda_p^{(k)}} - 2(p-1)\widehat{Q}_{k-1}) & \text{if } i = \sum_{l=1}^{k-1} n_l + j \text{ for some } k \text{ and } 1 \leq j < n_k, \\
q^{-\lambda(k)} \prod_{p=1}^{k} (x - q^{-2n_k+2\lambda_p^{(k)}} - 2(p-1)\widehat{Q}_{k-1}) & \text{if } i = \sum_{l=1}^{k} n_l \text{ for some } k.
\end{array} \right.
$$

**Proof.** By the definition of $\Phi_n^{(Q)}$ together with the argument in [W16], we see that the Weyl module $\Delta(\lambda)$ ($\lambda \in A_{m,r}^+$) is a highest weight $U_q(\mathfrak{sl}_n^{(Q)}[x])$-module. Let $v_0 \in \Delta(\lambda)$ be a highest weight vector. For $i \in I$, put $(j,k) = \xi^{-1}(i)$. Then, by [W16] Theorem 8.3 together with the definition of $\Phi_n^{(Q)}$, we have

$$K_i \cdot v_0 = \tilde{K}^+(j,k) \tilde{K}^-(j+1,k) \cdot v_0 = \begin{cases} q^{\lambda_j^{(k)} - \lambda_{j+1}^{(k)}} v_0 & \text{if } j \neq n_k, \\
q^{\lambda_k^{(k)} - \lambda_{k+1}^{(k)}} v_0 & \text{if } j = n_k,\end{cases}$$

$$J_{i,t} \cdot v_0 = q^{t}\left(q^{-t}J_{j+1,k}^+ - q^{-t}J_{j+1,k}^- - (q - q^{-1}) \sum_{z=1}^{t-1} q^{-t+2z}J_{j+1,k}^+ \cdot v_0 \right) = \begin{cases} q^{t^{(i-2j)}}(\widehat{Q}_{k-1})^t q^{t(2t-1)}\lambda_j^{(k)} [\lambda_j^{(k)}] - q^{\lambda_{j+1}^{(k)}} [\lambda_{j+1}^{(k)}] \\
- (q - q^{-1}) \sum_{z=1}^{t-1} q^{2(t-z)\lambda_j^{(k)} + \lambda_{j+1}^{(k)}} [\lambda_j^{(k)}][\lambda_{j+1}^{(k)}] & \text{if } j \neq n_k, \end{cases}$$

$$= \begin{cases} q^{t^{(i-2j)}}(\widehat{Q}_{k-1})^t q^{t(2t-1)}\lambda_j^{(k)} [\lambda_j^{(k)}] - q^{\lambda_{j+1}^{(k)}} [\lambda_{j+1}^{(k)}] \\
- (q - q^{-1}) \sum_{z=1}^{t-1} q^{2t+z+(2t-z-1)\lambda_j^{(k)} + \lambda_{j+1}^{(k)}} [\lambda_j^{(k)}][\lambda_{j+1}^{(k)}] & \text{if } j = n_k, \end{cases}$$

$$= \begin{cases} q^{t^{(i-2j+2\lambda_j^{(k)})}}(\widehat{Q}_{k-1})^t q^{-\lambda_j^{(k)} - \lambda_{j+1}^{(k)}} [\lambda_j^{(k)}] - q^{\lambda_{j+1}^{(k)}} [\lambda_{j+1}^{(k)}] & \text{if } j \neq n_k, \\
q^{t^{(i-2j+2\lambda_j^{(k)})}}(\widehat{Q}_{k-1})^t q^{-\lambda_j^{(k)}} [\lambda_j^{(k)}] + (q - q^{-1}) [\lambda_{j+1}^{(k)}] (q^{-i}\widehat{Q}_{k-1})^{-t} \\
+ (q - q^{-1}) \sum_{z=1}^{t-1} (q^{t^{(i-2j+2\lambda_j^{(k)})}}(\widehat{Q}_{k-1})^t q^{-\lambda_j^{(k)}} [\lambda_j^{(k)}]) (q^{-i}\widehat{Q}_{k-1})^{-t} \\
x (q^{(i-2j+2\lambda_j^{(k)})}(\widehat{Q}_{k-1})^t q^{-\lambda_j^{(k)}} [\lambda_j^{(k)}]) & \text{if } j = n_k.
\end{cases}$$
On the other hand, for \( b \in \mathbb{C} \) and \( c, z \in \mathbb{Z}_{>0} \), we can show that
\[
p_z(q)(b, bq^{-2}, bq^{-2}, \ldots, bq^{-2(c-1)}) = b^zq^{-c}[c]
\]
by the induction on \( c \) using Lemma 6.3 (i). Thus, we have
\[
J_{i,t} \cdot v_0 = \begin{cases} 
  p_t(b, bq^{-2}, \ldots, bq^{-2(\lambda_j^k-\lambda_{j+1}^k)-1}) & \text{if } j \neq n_k, \\
  p_t(b, bq^{-2}, \ldots, bq^{-2(\lambda_j^k-1)}) + \tilde{\beta}Q_i^{-t} & \text{if } j = n_k, \\
  +(q-q^{-1}) \sum_{z=1}^{i-1} \tilde{\beta}Q_i^{-z}p_{t-z}(b, bq^{-2}, \ldots, bq^{-2(\lambda_j^k-1)}) & \text{if } j = n_k,
\end{cases}
\]
where we put \( b = q^{(i-2j+2\lambda_j^k)}(\tilde{Q}_k^{-1}) \) and \( \tilde{\beta} = -q^{\lambda_j^k}[\lambda_{j+1}^k] \), and we note that \( Q_i = q^{-i}\tilde{Q}_k^{-1} \) if \( j = n_k \) by (D.11.1). Moreover, applying the definition (6.8.1) to the above equation in the case where \( j = n_k \), we have
\[
J_{i,t} \cdot v_0 = p_t(Q_i^j)(q; \beta)(b, bq^{-2}, \ldots, bq^{-2(\lambda_j^k-1)}) \text{ if } j = n_k,
\]
where \( \beta = q^{-\lambda_j^k} \) since \( \tilde{\beta} = -q^{\lambda_j^k}[\lambda_{j+1}^k] = (1 - q^2\lambda_j^k)(q^{-1})^{-1} \). Note that \( i = \sum_{l=1}^{k-1} n_l + j \) since \((j, k) = \xi^{-1}(i)\), then we obtain the proposition. \( \square \)

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