ALL THE INFORMATION IN THE INTEGERS IS IN THE PRIMES

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ABSTRACT

Building upon the work of Chebyshev, Shannon and Kontoyiannis, it may be demonstrated that Chebyshev’s asymptotic result:

$$\ln N \sim \sum_{p \leq N} \frac{1}{p} \cdot \ln p$$

has a natural information-theoretic interpretation as the information-content of a typical integer $Z \sim U([1, N])$, sampled under the maximum entropy distribution, is asymptotically equivalent to the information content of all primes $p \leq N$ weighted by their typical frequency.

1 Theorem: All the Statistical Information in the integers is contained in the primes

The strategy in this section is to derive Chebyshev’s theorem (1852) via Maximum Entropy inference, using an adaptation of Kontoyiannis’ original arguments [4]. This theorem is an important precursor to the Prime Number Theorem.

As the Unique Prime Factorisation theorem implies that the prime numbers define a uniquely-decodable code for the integers, $Z \sim U([1, N])$ corresponds to a random prime factorisation in terms of the random variables $X_p$:

$$\forall Z \sim U([1, N]), Z = \prod_{p \leq N} p^{X_p} \implies H(Z) = \sum_{p \leq N} H(X_p) = \ln N$$

Now, under Maximum Entropy assumptions the integers have a stochastic encoding in terms of the prime numbers given by $P(Z \mod p = 0) \sim \frac{1}{p}$ since for a given mean $\mathbb{E}[X_p]$, the geometric distribution maximises the entropy of the integers:

$$\mathbb{E}[X_p] = \sum_{k \geq 1} P(X_p \geq k) = \sum_{k \geq 1} \frac{1}{N \cdot \left\lfloor \frac{N}{p^k} \right\rfloor} \sim \frac{1}{p}$$

so we have:

$$H(Z) = -\ln \prod_{p \leq N} P(Z \mod p = 0)^{P(Z \mod p = 0)} \sim \sum_{p \leq N} \frac{1}{p} \cdot \ln p$$

By combining (1) and (3) we may deduce Chebyshev’s theorem:

$$\sum_{p \leq N} \frac{1}{p} \cdot \ln p \sim \ln N$$

which implies that almost all integers have a random prime factorisation and that the average information gained from observing a prime number in the interval $[1, N]$, weighted by their typical frequency, is on the order of $\sim \ln N$. 

It is worth noting that the information-theoretic formulation of Chebyshev’s theorem that has been derived leads to a natural definition of the Statistical Information of a prime number. This definition is consistent with the hypothesis that the prime numbers have a uniform source distribution [1]. In fact, the information-theoretic derivation of Chebyshev’s theorem and the Prime Number theorem are consequences of this hypothesis.

2 Definition: The Statistical Information of a Prime Number

Given the information-theoretic derivation of the Prime Number Theorem, presented in [1], the average Statistical Information in a prime number in the interval \([1, N]\) is asymptotically on the order of \(\sim \ln N\) or to be precise,

\[
\frac{\log_2 2^N}{\pi(N)} \sim \ln N
\]  

(5)
as each integer in the interval \([1, N]\) is either prime or not prime.

3 Chebyshev’s theorem via Occam’s razor

Curiously, Chebyshev’s theorem may also be derived using Solomonoff’s theory of Algorithmic Probability [10,11] which is conventionally known as Occam’s razor.

As almost all integers are algorithmically random, \(Z \sim U([1, N])\) might as well be encoded using \(\sim \log_2 Z\) coin flips. It follows that the algorithmic probability of observing a prime \(p \in \mathbb{P}\) may be deduced from Levin’s coding theorem:

\[
K_U(x) \sim -\log_2 m(x) \implies m(p) \sim 2^{-K_U(p)} \sim 2^{-\log_2 p} = \frac{1}{p}
\]  

(6)

This has a natural frequentist interpretation, as we have a Bernoulli distribution:

\[
\forall Z \sim U([1, N]), P(Z \text{ mod } p = 0) + P(Z \text{ mod } p \neq 0) = \frac{1}{p} + (1 - \frac{1}{p}) = 1
\]  

(7)

Now, using the fact that the Expected Kolmogorov Complexity of a random variable equals its Shannon Entropy [5]:

\[
\mathbb{E}[K_U(Z)] = H(Z) + O(1)
\]  

(8)
we may deduce that:

\[
\forall Z \sim U([1, N]), \mathbb{E}[K_U(Z)] \sim -\log_2 \prod_{p \leq N} P(Z \text{ mod } p = 0)^{P(Z \text{ mod } p = 0)} \sim \sum_{p \leq N} \frac{1}{p} \cdot \log_2 p \sim \log_2 N
\]  

(9)
which yields Chebyshev’s theorem in base-2.

4 Discussion

In this article, we demonstrated that both Occam’s razor and Maximum Entropy inference may be used to derive Chebyshev’s theorem for the distribution of primes. The effective principle appears to be that the Expected Kolmogorov Complexity of a random variable equals its Shannon Entropy.

There are good reasons to believe that this common principle may be developed into a theory which may be applied to other non-trivial problems in probabilistic number theory.
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