Hybrid control for a dynamical love model

Wei Deng$^{1,*}$, Yan Yang$^{1}$

$^{1}$ Information Technology Center, Southwest University, Chongqing, China

*Corresponding author e-mail: dengwei@swu.edu.cn

Abstract. In this paper, strategy of hybrid control for a dynamical love system is taken into account. Hybrid controller is used to control the behaviour of love systems for the first time. By choosing suitable bifurcation parameter, it can be found that the Hopf bifurcation occurs when bifurcation parameter passes through a critical value. Stability analysis shows that by setting proper values of hybrid controller, the appearance of bifurcation will be delayed or advanced, and some interesting behaviors can be achieved. The stability of the bifurcating cycle for love dynamics are also derived. Finally, numerical simulation illustrate this control strategy is useful in controlling the appearance of Hopf bifurcation which has taken place in love model.

1. Introduction

Since Strogatz’s pioneering one page paper [1] which presented an elementary linear ordinary model of Romeo and Juliet has been published in 1988, there are many attempts in describing or predicting love relationship dynamics between two individuals or among three individuals in detail, and these love models are based on ordinary differential equations [2,3,4], fractional-order differential equations [5-8] or delay differential equations [9-14], etc. Until now, all the studies about love dynamical systems have been focused on finding chaos or bifurcations [9,15], the influence of random noise [16,17], the effect of delayed time [9-14], the reaction to partner’s appeal [18], the prediction of love relationship [4], the effect of time fluctuation [19], etc.

Hybrid control strategy [20-26], has been proposed and used in a wide range of application domains, in which the bifurcation was controlled by state feedback and parameter perturbation. The aim of our work here is to find a control law which can influence Hopf bifurcation in love relationship, Hybrid control strategy is chosen here. In love relationship, we want to delay (advance) the onset of bifurcating oscillation when it is harmful (necessary). Time delays are unavoidable and have significant influence on the behaviors of nonlinear dynamical models such as love dynamical systems [9,11,13]. Hopf bifurcation for love dynamical models has attracted more and more interest in recent years. The influence of control strategy on the appearance of bifurcation for the love dynamical systems with time delays has not been taken into account. In fact, it is very meaningful and useful to discuss the influence of strategy for control on love dynamical relationship, because by our nature, we always hope to develop a relationship toward perfectly stable and lasting direction, and try to avoid or delay the appearance of any bifurcation. Furthermore, there are few results with respect to discuss the influence of hybrid control for dynamical love system with time delays until now.

Motivated by the above discussions, based on the model in [9], bifurcation control for time delayed love dynamical system is discussed in this paper. In this paper, the hybrid controller is designed to control the interactive behaviors in love model for the first time. The influence of feedback gain for the interactive relationship has been paid attention to. The following parts are as follows. In Section 2,
uncontrolled love model with Hopf bifurcation is briefly introduced. In Section 3, the Hybrid controlled model is constructed and the corresponding Hopf bifurcation is discussed by regarding \( r_1 + r_2 = r, \ r_1 = r_2 = r \). In Section 4, by using the normal form approach and center manifold theory [27], the stability of the controlled model is discussed. In Section 5, numerical simulations are applied to mimic the theoretic findings. At last, conclusion appears in Section 6.

2. Controlled model’s Hopf bifurcation based on Hybrid control strategy

To accomplish the goal that maintaining sweet relationship as well as terminating the relationship of pain, control strategy of hybrid is implemented to that model in [9] as follows and the controlled Romeo-Juliet love model becomes

\[
\begin{align*}
\dot{x}_t &= \alpha [\alpha - a_x x_t + b_x f(x_t, (t - \tau_2)) + r_1 A_t] + (1 - \alpha)(x_t - x_t^*) \\
\dot{y}_t &= \alpha [\alpha - a_y y_t + b_y g(y_t, (t - \tau_2)) + r_2 A_t] + (1 - \alpha)(y_t - y_t^*),
\end{align*}
\]

where \( x_t \) and \( y_t \) express the feeling at time \( t \) of Romeo and Juliet. \( a_x, b_x, r_1, A \) are constant control parameters. \( \tau_2 \) is positive time delay. The negative term \( -a_x x_t \) means the forgetting process in the absence of the sweetheart, \( g(y_t, (t - \tau_2)) \) describes ‘i’s reaction to ‘j’s love and \( r_1 A \) is ‘i’s reaction to ‘j’s appeal, \( \alpha \) is a control parameter and \( \alpha \neq 1 \), \((x_t^*, y_t^*)\) be the nonzero equilibrium point.

Let \( y_h(t) = x_h(t) - x_t^* \), \( y_j(t) = y_t(t) - y_t^* \), we have

\[
\begin{align*}
\dot{y}_h(t) &= \alpha [-a_x y_h(t) + b_x g(y_h(t), (t - \tau_2)) + (1 - \alpha)y_h(t - \tau_2)] \\
\dot{y}_j(t) &= \alpha [-a_y y_j(t) + b_y g(y_j(t), (t - \tau_2)) + (1 - \alpha)y_j(t - \tau_2)],
\end{align*}
\]

where

\[
\begin{align*}
g(y_j(t - \tau_2)) &= f(y_j(t - \tau_2) + y_j^*) - f(y_j^*) \\
g(y_h(t - \tau_2)) &= f(y_h(t - \tau_2) + y_h^*) - f(y_h^*).
\end{align*}
\]

The right side of system (2) is expanded to Taylor series around the equilibrium point \((x_t^*, y_t^*)\) and linearizing the result, we have

\[
\begin{align*}
\dot{V}_h(t) &= \alpha [-a_x V_h(t) + b_x d V_h(t - \tau_2)] + (1 - \alpha)V_h(t - \tau_2) \\
\dot{V}_j(t) &= \alpha [-a_y V_j(t) + b_y d V_j(t - \tau_2)] + (1 - \alpha)V_j(t - \tau_2),
\end{align*}
\]

where \( d = g'(0) \).

The following assumption are written to simplify the discussion:

(H1) \( \tau_1 + \tau_2 = \tau, \ \tau_1 = \tau_2 = \tau \), where \( \tau \) is bounded.

Corresponding characteristic equation for Eq. (4) is

\[
\lambda^2 + d_1 \lambda + d_2 + (d_3 \lambda + d_4) e^{\omega \tau} + d_5 e^{-\omega \tau} = 0,
\]

where

\[
d_1 = \alpha (a_x + a_y), \ d_2 = \alpha^2 a_x a_y, \ d_3 = -2(1 - \alpha), \ d_4 = -\alpha (1 - \alpha) (a_x + a_y) - \alpha^2 b_x b_y d^2, \ d_5 = (1 - \alpha)^2.
\]

By multiplying \( \sin^2(\omega \tau) = 1 \) on every side of Eq. (5), the following equation can be obtained:

\[
(\lambda^2 + d_3 \lambda + d_4) e^{\omega \tau} + (d_5 \lambda + d_5) e^{-\omega \tau} = 0.
\]

When Eq. (6) has zero eigenvalue or pure imaginary pair, the stability can change. When \( d_3 + d_4 + d_5 = 0 \), the zero eigenvalue \( \lambda = 0 \) can be obtained in Eq. (6). Here, the condition \( d_3 + d_4 + d_5 \neq 0 \) is defined.

Assume Eq. (6) has a pair of roots \( \pm i \omega (\omega > 0) \) which is purely imaginary, then the following equation should be satisfied

\[
(-\omega^2 + d_3) \cos \omega \tau + i \sin \omega \tau + (d_1 \omega + d_5) \cos \omega \tau + i \sin \omega \tau = 0.
\]

When the real part and imaginary part of Eq. (7) are separated, there are

\[
\begin{align*}
(d_2 + d_3 - \omega^2) \cos \omega \tau - d_1 \omega \sin \omega \tau + d_5 = 0, \\
(d_2 - d_3 - \omega^2) \sin \omega \tau + d_1 \omega \cos \omega \tau + d_5 = 0.
\end{align*}
\]
From Eq. (8) and (9), we obtain the following equations:

\[
\cos \omega \tau = -\frac{d_1 e_2 + d_2 \omega e_3}{e_1 e_2 + e_1^2}, \quad \sin \omega \tau = \frac{d_1 e_1 - d_2 \omega e_3}{e_1 e_2 + e_1^2},
\]

where

\[e_1 = d_2 + d_4 - \omega^2, \quad e_2 = d_2 - d_4 - \omega^2, \quad e_3 = d_1 \omega.\]

Since \( \sin^2(\omega \tau) + \cos^2(\omega \tau) = 1 \), we can obtain

\[\omega^8 + s_1 \omega^6 + s_2 \omega^4 + s_3 \omega^2 + s_4 = 0,\]

where

\[s_1 = 2d_2^2 - 4d_2 + d_4^2, \quad s_2 = 6d_2^2 - 2d_2 - 4d_2 d_4 + 3d_4^2 - 4d_4 - 2d_4^2 + 2d_4 d_5 + 2d_5^2 d_4,
\]

\[s_3 = 2d_2^2 d_4^2 - 4d_2 d_4^2 - 4d_2^2 (d_4^2 - d_4^2) - (d_4^2 - 2d_4 + 4d_4) d_4^2 - d_4^2 (d_2 + d_4)^2, \quad s_4 = (d_2^2 - d_4^2)^2 - d_4^2 (d_2 + d_4)^2.\]

Letting \( z = \omega^2 \), then Eq. (11) can be rewritten as

\[z^4 + s_1 z^3 + s_2 z^2 + s_3 z + s_4 = 0,\]

Suppose \( F(z) = z^4 + \frac{3}{4} z^3 + \frac{1}{2} z^2 + \frac{1}{4} s_4 \), then the following lemma can be gotten.

**Lemma 1.** (27-32).

(i) When \( s_4 < 0 \), Eq. (12) has one positive root at least.

(ii) When \( s_4 \geq 0 \), Eq. (12) has one positive root at least if any of the following situation satisfies:

(1) If \( \Delta > 0, \quad z_1^* \geq 0, \quad h(z_1^*) < 0 \);

(2) If \( \Delta = 0, \quad z_1^* \geq 0, \quad h(z_1^*) < 0 \);

(3) If \( \Delta < 0, \quad z_1^* \geq 0, \quad h(z_1^*) < 0 \).

where \( \Delta = \frac{d_2^2 + d_4^2}{4} \) is called third-degree algebraic equation’s discriminant, \( g(y) = y^2 + py + q \), \( z^* \) is the maximum root, \( h(z) = z^4 + s_1 z^3 + s_3 z^2 + s_3 z + s_4 \).

Obviously, we can get the expression of \( \tau_4^k \) from Eq. (10).

\[
\tau_4^k = \frac{1}{\omega_k} \arccos \left( \frac{d_2 e_2 + d_4 \omega e_3}{e_1 e_2 + e_1^2} \right) + \frac{2j\pi}{\omega_k},
\]

and \( k = 1, 2, 3, 4; \quad j = 0, 1, \ldots.\)

Suppose \( \tau_n = \tau_0 = \lim_{s \to 0} \{ \tau_k \}, \quad \omega_0 = \omega_s.\)

When \( \tau = 0 \), Eq. (5) becomes

\[\lambda^2 + (d_1 + d_4 + d_4) \lambda + (d_2 + d_4 + d_4) = 0.\]

It is easy to find that the system will be stable and the real parts of the roots in Eq. (14) will be negative when the following situation holds.

**H2** \( D_1 = (d_1 + d_4) > 0, \quad D_2 = (d_2 + d_4 + d_4) > 0.\)

After that, the following assumptions can be made:

**H3** \( D_3 = (d_1 + d_4) > 0. \quad (H4) \quad \frac{d \text{Re}(\lambda(\tau))}{d\tau} \bigg|_{\tau = \tau_n} \neq 0.\)

**Lemma 2.** Let \( \lambda = \beta + i\omega \) be Eq. (7)’s root, the transversality condition can be held:

\[
\frac{d \text{Re}(\lambda(\tau))}{d\tau} \bigg|_{\tau = \tau_n} \neq 0.
\]

Then, the following equations can be obtained

\[
\begin{align*}
\frac{d \text{Re}(\lambda(\tau))}{d\tau} &= \gamma_1 \gamma_2 + \gamma_3 \gamma_4 \neq 0, \\
\frac{d \text{Im}(\lambda(\tau))}{d\tau} &= \gamma_3 \gamma_4 - \gamma_1 \gamma_4.
\end{align*}
\]

(15)
3. The Hopf Bifurcation’s stability and direction

In this part, we compute the Hopf bifurcation’s direction and the bifurcating periodic solution’s stability of Eq. (2) based on the center manifold and the normal form theories [27].

When Taylor expansion is applied to the system (1) at the equilibrium point and based on assumption (H1), we have

\[
\begin{align*}
\dot{y}_i(t) = & c_i y_i(t) + c_2 y_2(t-\tau) + c_3 y_3(t-2\tau) + c_4 y_4(t-\tau_2) + O(y^4_i), \\
\dot{y}_j(t) = & c_6 y_j(t) + c_7 y_2(t-\tau) + c_8 y_3(t-\tau_1) + c_9 y_4(t-\tau_1) + O(y^4_j),
\end{align*}
\]

where

\[
\begin{align*}
c_i &= -aa_i, \
c_2 &= (1-\alpha), \
c_3 &= \alpha b_i g'(x_j^*), \
c_4 &= \frac{1}{2}ab_i g''(x_j^*), \
c_6 &= -aa_j, \
c_7 &= \alpha b_i g'(x_j^*), \
c_8 &= \frac{1}{2}ab_i g''(x_j^*), \
c_9 &= \frac{1}{6}ab_i g''(x_j^*).
\end{align*}
\]

Let \( \tau = \tau_0 + \mu(\mu \in \mathbb{R}) \), \( \mu(t) = (y_i(t), y_j(t)) \) and \( x_i(t) = (x_i(t), x_j(t)) \), then \( \mu = 0 \) is Hopf bifurcation value for Eq. (1). For initial condition \( \theta \in [-2\tau, 0] \), Eq. (1) can be written as

\[
\dot{x}_i(t) = L_{\mu,\theta} x_i(t) + f(\mu, x_i),
\]

where \( x_i(\theta) = x_i(t+\theta) \) and \( L_{\mu,\theta} : C \rightarrow \mathbb{R}^2 \), \( f : \mathbb{R} \times C \rightarrow \mathbb{R}^2 \) can be obtained by

\[
L_{\mu,\theta} \varphi = B_1 \varphi(0) + B_2 \varphi(-\tau) + B_3 \varphi(-\tau_1) + B_4 \varphi(-\tau_2),
\]

where

\[
B_1 = \begin{bmatrix} c_1 & 0 \\ 0 & c_0 \end{bmatrix}, \
B_2 = \begin{bmatrix} c_2 & 0 \\ 0 & c_1 \end{bmatrix}, \
B_3 = \begin{bmatrix} 0 & 0 \\ c_3 & 0 \end{bmatrix}, \
B_4 = \begin{bmatrix} 0 & c_3 \\ 0 & 0 \end{bmatrix}.
\]

For \( \varphi \in [-2\tau, 0] \), we define

\[
A_{\mu}(\varphi) = \begin{bmatrix} d\varphi \\ d\varphi \end{bmatrix}, \quad \theta \in [-2\tau, 0),
\]

\[
\int_{-2\tau}^{0} d\eta_\tau(\theta, \mu) \varphi(\theta), \quad \theta = 0,
\]

and

\[
R_{\mu}(\varphi) = \begin{cases} 0, & \theta = 0, \\ f(\mu, \varphi), & \theta = 0. \end{cases}
\]

\( A_\mu \)'s adjoint operator is \( A_\mu^* \), and can be defined by

\[
A_{\mu}^*(\varphi) = \begin{bmatrix} d\varphi(s) \\ ds \end{bmatrix}, \quad s \in (0, 2\tau),
\]

\[
\int_{-2\tau}^{0} d\eta_\tau^*(t, 0) \varphi_1(-t), \quad s = 0.
\]

When \( \varphi_1 \in [-2\tau, 0] \) and \( \psi_1 \in [0, 2\tau] \), the bilinear form can be defined as

\[
\langle \psi_1, \varphi_1 \rangle := \mathcal{V}(\varphi_1, \psi_1, 0) - \int_{-2\tau}^{0} \int_{t=0}^{\tau} d\eta_{\varphi_1}(\xi - \theta) d\eta_{\psi_1}(\xi) d\xi,
\]

where \( \eta_{\varphi_1}(\theta) = \eta_{\psi_1}(\theta, 0) \).

Since \( \pm i\omega_0 \) are \( A_0 \) and \( A_0^* \)'s eigenvalues, here we suppose \( q_i(\theta) \) and \( q_i^*(\theta) \) are \( A \) and \( A^* \)'s eigenvectors corresponding to the eigenvalues \( \pm i\omega_0 \), and we can have

\[
\begin{bmatrix} A_0 q_i(\theta) = i\omega_0 q_i(\theta) \\ A_0^* q_i(\theta) = -i\omega_0 q_i^*(\theta) \end{bmatrix},
\]

and suppose
By Eq. (22), we have
\[
<q_i^* , q_i> = \frac{\delta^*}{\delta t}(0)\delta(t,0) - \int_{0}^{\pi} \int_{0}^{\pi} \tilde{\delta}^* (\xi - \theta) \bar{\delta}(\theta)\bar{q}_i(\xi) d\xi d\theta = \frac{1}{\rho} \left( 1 + k\tilde{k}^* \right) - \int_{0}^{\pi} \int_{0}^{\pi} \frac{1}{\rho} (k^*) e^{-ik\rho(\xi - \theta)} \\
= \frac{1}{\rho} \left( 1 + k\tilde{k}^* \right) \left( 1 + \tau^0 e^{-ik\rho B_2} + \tau^1 e^{-ik\rho B_3} + \tau^2 e^{-ik\rho B_4} \right).
\]

Hence we obtain
\[
\rho = \left( 1 + k\tilde{k}^* \right) \left( 1 + \tau^0 e^{-ik\rho B_2} + \tau^1 e^{-ik\rho B_3} + \tau^2 e^{-ik\rho B_4} \right).
\]

By Eq. (17)'s solution with \( \mu = 0 \), \( z(t) \) and \( W(t,\theta) \) can be defined as
\[
z(t) = <q_i(\theta), x_i(t) > , \tag{27}
\]
and
\[
W_i(t,\theta) = x_i(t, \theta) - z(t, \theta) \bar{q}_i(\theta) - \bar{Z}(t, \bar{q}_i(t, \theta)) = x_i(t, \theta) - 2 \text{Re}(z(t, \bar{q}_i(\theta))). \tag{28}
\]

where \( x_i(\theta) \) is Eq. (28)'s solution, then we can get
\[
W_i(t, \theta) = W_i(\bar{z}, t) = W_20(\theta) \frac{z^2}{2} + W_11(\theta) \bar{z} \bar{\bar{z}} + W_00 \frac{\bar{z}^2}{2} + \ldots. \tag{29}
\]

Since \( \mu = 0 \), we know that
\[
\dot{z}(t) = <q_i^* , x_i(t) > = \langle q_i(\theta), x_i(t) \rangle \tag{30}
\]

After that, we have
\[
\dot{W}_i(t, \theta) = A \dot{W}_i - 2 \text{Re}[^{q_i^*}(0) f(\bar{z}, t, 0) + 2 \text{Re}(z(t, \bar{q}_i(0))) = i\omega \bar{r}_a z + g(z, \bar{z}). \tag{31}
\]

where
\[
L(z, \bar{z}, t) = L_{20}(\theta) \frac{z^2}{2} + L_{11}(\theta) \bar{z} \bar{\bar{z}} + L_{00}(\theta) \frac{\bar{z}^2}{2} + \ldots. \tag{32}
\]

Using Eq. (29) and Eq. (30), we can get
\[
\begin{align*}
\langle A - 2i\tau_a \omega \rangle W_{20}(\theta) &= -L_{20}(\theta) \\
A W_{11}(\theta) &= -L_{11}(\theta) \\
\langle A + 2i\tau_a \omega \rangle W_{00}(\theta) &= -L_{00}(\theta)
\end{align*} \tag{33}
\]

So we have
\[
x_i(t, \theta) = \begin{cases} 
 y_i(t + \theta) = \frac{W_{11}(\bar{z}, t, \theta)}{W_{11}(\bar{z}, t, \theta)} + \frac{1}{2} \left( k \right)^2 e^{i\omega \theta} + \frac{1}{2} \left( k \right)^2 e^{-i\omega \theta} 
\end{cases} \tag{34}
\]

Notice that
\[
y_i(t, \bar{z}^2) = e^{i\omega \theta} z + e^{-i\omega \theta} \bar{z} + W_{11}(t, -\bar{z}^2), \tag{35}
\]
and
\[
y_i(t, \bar{z}^0) = e^{i\omega \theta} k z + e^{-i\omega \theta} \bar{k} + W_{11}(t, -\bar{z}^0). \tag{36}
\]

where
\begin{equation}
W_{i}^{(2)}(t,-t_{i}^{2})=W_{20}^{(2)}(-t_{i}^{2})\frac{z^2}{2}+W_{11}^{(2)}(-t_{i}^{2})z\tau+W_{02}^{(2)}(-t_{i}^{2})\frac{\tau^2}{2}+\ldots,
\end{equation}
\begin{equation}
W_{i}^{(1)}(t,-t_{i}^{2})=W_{20}^{(1)}(-t_{i}^{2})\frac{z^2}{2}+W_{11}^{(1)}(-t_{i}^{2})z\tau+W_{02}^{(1)}(-t_{i}^{2})\frac{\tau^2}{2}+\ldots.
\end{equation}

Then, we can obtain
\begin{equation}
f(\mu,\phi) = \left\{ \begin{array}{l}
c_{f}\phi_{0}^{2}(-t_{i}^{2})+c_{f}\phi_{0}^{3}(-t_{i}^{2}) \\
c_{g}\phi_{0}^{2}(-t_{i}^{2})+c_{g}\phi_{0}^{3}(-t_{i}^{2}) \\end{array} \right\} = \left\{ \begin{array}{l}
M_{11}z^{2}+M_{12}z\tau+M_{13}\tau^{2}+M_{14}z^{3}\tau \\
M_{21}z^{2}+M_{22}z\tau+M_{23}\tau^{2}+M_{24}z^{3}\tau \\end{array} \right\},
\end{equation}

where
\begin{align*}
M_{11} &= c_{j}e^{i\alpha x_{1}}t_{1}^{2},
M_{12} &= c_{j}e^{-i\alpha x_{1}}t_{2}^{2},
M_{13} &= c_{j}(2e^{i\alpha x_{1}}W_{i}^{(2)}(-t_{i}^{2})+e^{i\alpha x_{1}}W_{20}^{(2)}(-t_{i}^{2}))+3c_{j}e^{i\alpha x_{1}}t_{2}^{2}, \\
M_{21} &= c_{k}e^{i\alpha x_{1}}t_{1}^{2},
M_{22} &= 2c_{k}|k|^{2},
M_{23} &= c_{k}e^{i\alpha x_{1}}t_{2}^{2}, \\
M_{24} &= c_{k}(2ke^{i\alpha x_{1}}W_{i}^{(1)}(-t_{i}^{2})+ke^{i\alpha x_{1}}W_{20}^{(1)}(-t_{i}^{2}))+3c_{k}|k|^{2}e^{i\alpha x_{1}}t_{2}^{2}.
\end{align*}

Then we have
\begin{equation}
g(z,\tau) = \overline{q}_{i}(0)f(z,\tau)
= \frac{1}{\rho}[k(M_{11}z^{2}+M_{12}z\tau+M_{13}\tau^{2}+M_{14}z^{3}\tau)+(M_{21}z^{2}+M_{22}z\tau+M_{23}\tau^{2}+M_{24}z^{3}\tau)].
\end{equation}

We get the following equation when compare the coefficients of Eq. (40)
\begin{equation}
g_{20} = 2\frac{1}{\rho}\overline{k}(M_{11}+M_{21}),
g_{11} = \frac{1}{\rho}\overline{k}(M_{12}+M_{22}),
g_{02} = 2\frac{1}{\rho}\overline{k}(M_{13}+M_{23}),
g_{21} = \frac{2}{\rho}\overline{k}(M_{14}+M_{24}).
\end{equation}

Then, we will compute the values of \(W_{0}^{2}\) and \(W_{1}^{1}\). It is easy to get
\begin{equation}
L(z,\tau,\theta) = -2\Re[\overline{q}_{i}(0)f(z,\tau)q_{i}(\theta)] = -gq_{i}(\theta)-\overline{gq}_{i}(\theta)
= -(g_{20}\frac{z^2}{2}+g_{11}z\tau+g_{02}\frac{\tau^2}{2}+\ldots)q_{i}(\theta) - (\overline{g}_{20}\frac{z^2}{2}+\overline{g}_{11}z\tau+\overline{g}_{02}\frac{\tau^2}{2}+\ldots)\overline{q}_{i}(\theta).
\end{equation}

Comparing Eq.(42) with Eq. (32), we have
\begin{align*}
L_{20}(\theta) &= -g_{20}q_{i}(\theta)-\overline{g}_{20}\overline{q}_{i}(\theta) \\
L_{11}(\theta) &= -g_{11}q_{i}(\theta)-\overline{g}_{11}\overline{q}_{i}(\theta).
\end{align*}

Substituting Eq. (43) into Eq. (33), the results in Eq. (44) can be obtained
\begin{align*}
W_{20}(\theta) &= 2i\tau_{e}\omega_{0}W_{20}(\theta)+g_{20}q_{i}(\theta)+\overline{g}_{20}\overline{q}_{i}(\theta) \\
W_{11}(\theta) &= g_{11}q_{i}(\theta)+\overline{g}_{11}\overline{q}_{i}(\theta).
\end{align*}

So
\begin{align*}
W_{20}(\theta) &= \frac{i\overline{g}_{20}q_{i}(\theta)}{\tau_{e}\omega_{0}}e^{i\tau_{e}\omega_{0}\theta}-\frac{\overline{g}_{20}}{3i\tau_{e}\omega_{0}}\overline{q}_{i}(\theta)e^{-i\tau_{e}\omega_{0}\theta}+E_{0}e^{i\tau_{e}\omega_{0}\theta} \\
W_{11}(\theta) &= \frac{g_{11}}{i\tau_{e}\omega_{0}}q_{i}(\theta)e^{i\tau_{e}\omega_{0}\theta}-\frac{\overline{g}_{11}}{i\tau_{e}\omega_{0}}\overline{q}_{i}(\theta)e^{-i\tau_{e}\omega_{0}\theta}+E_{2}.
\end{align*}

We have Eq.(46) from A's definition.
\begin{align*}
\int_{-\tau}^{\tau}d\eta(\theta)W_{20}(\theta) = AW_{20}(0) = 2i\tau_{e}\omega_{0}W_{20}(0)-L_{20}(0) \\
\int_{-\tau}^{\tau}d\eta(\theta)W_{11}(\theta) = AW_{11}(0) = -L_{11}(0).
\end{align*}

From \(f(\mu,\tau)\)'s definition and Eq. (43), we know
\begin{align*}
L_{20}(0) &= -g_{20}q_{i}(0)-\overline{g}_{20}\overline{q}_{i}(0)+\left[ \begin{array}{c}
M_{11} \\
M_{21}
\end{array} \right] \\
L_{11}(0) &= -g_{11}q_{i}(0)-\overline{g}_{11}\overline{q}_{i}(0)+\left[ \begin{array}{c}
M_{12} \\
M_{22}
\end{array} \right].
\end{align*}

Substituting Eq. (44), Eq. (34) into Eq. (41), and noticing that
From the above equations, the following solutions can be obtained
\begin{align}
\int_{-\tau}^{0} e^{i\tau_0^0\theta} d\eta(\theta)q_{i}(0) &= 0, \\
\int_{-\tau}^{0} e^{-i\tau_0^0\theta} d\eta(\theta)\overline{q}_{i}(0) &= 0.
\end{align}

(48)

From the above equations, the following solutions can be obtained
\begin{align}
E^{(1)}_1 &= \frac{M_{21}e^{i\tau_0^0\theta} - M_{11}(c_i + c_2e^{i\omega_0\theta} - 2i\omega_0)}{-c_1c_2e^{i\omega_0\theta}(c_i + c_2e^{i\omega_0\theta}) + (c_i + c_2e^{i\omega_0\theta}) - 2i\omega_0(c_i + c_2e^{i\omega_0\theta}) - 2i\omega_0}, \\
E^{(2)}_1 &= \frac{M_{21}(c_i + c_2e^{i\omega_0\theta}) - 2i\omega_0 - M_{11}ce^{i\omega_0\theta}}{c_1c_2e^{-i\omega_0\theta}(c_i + c_2e^{-i\omega_0\theta}) - (c_i + c_2e^{-i\omega_0\theta}) - 2i\omega_0(c_i + c_2e^{-i\omega_0\theta}) - 2i\omega_0}.
\end{align}

(49)

(50)

Similarly, we can get
\begin{align}
E^{(1)}_2 &= \frac{M_{12}(c_i + c_2) - M_{22}c_i}{(c_i + c_2)(c_i + c_2) - c_i c_i}, \\
E^{(2)}_2 &= \frac{M_{12}c_i - M_{22}(c_i + c_2)}{-(c_i + c_2)(c_i + c_2) + c_i c_i}.
\end{align}

(51)

(52)

From the above calculation, it is easy to find every $g_{ij}$ in Eq. (41) can be determined by the parameter values in Eq. (2). So based on the above analysis, the following equations are easy to compute:
\begin{align}
\mu_2 &= \frac{Re C_i(0)}{Re \lambda'(0)}, \\
\beta_2 &= 2 Re C_i(0), \\
T_2 &= \frac{Im C_i(0) + \mu_2 Im \lambda'(0)}{\omega_0}, \\
C_i(0) &= \frac{i}{2\omega_0} \left( g_{20}g_{11} - 2|g_{11}|^2 - |g_{20}|^2 \right) + \frac{g_{21}}{2},
\end{align}

where $C_i(0)$ is the Lyapunov coefficient.

4. Numerical examples

The parameters are the same as in [9], that are $f(x) = \tanh(x)$, $a_1 = 1$, $a_2 = 1$, $b_1 = 1.5$, $b_2 = -2$, $r, A = 1.5$, $r, A = 0.5$. This case represents Romeo $(a_1 = 0.5, b_1 = 2)$ and Juliet $(a_2 = 1, b_2 = -3)$.

Here we control the appearance of Hopf bifurcation by hybrid controller. In this case, we describe hybrid control parameter as $\alpha = 0.7$, the system becomes to be controlled, and the corresponding critical value $\tau_0$ becomes 1.05 by direct calculation. By adding the appropriate role of hybrid control, Hopf bifurcation disappears and the limit cycle becomes an asymptotically stable equilibrium point (see Figure1). In real life, we sometimes may terminate the argument and have an impermanent or prolonged stable relationship with our spouse by the hybrid control of our parents, our family or some other internal or external pressure. Apparently, the appearance of Hopf bifurcation can be delayed and the stability interval for love relationship can be enlarged by hybrid controller.

Figure 1. Phase portrait of controlled system (3) when $\alpha = 0.7, \tau_1 = 0.5, \tau_2 = 0.4, \tau_1 + \tau_2 = 0.9 < \tau_0, \tau_3 = \tau_4 = 0.9$
On the other hand, when we choose $\alpha = 5$, the corresponding $r_0$ decreases and becomes 0.1266. Apparently, the Hopf bifurcation’s appearance can also be accelerated as well as be delayed by the hybrid controller.

\[ \frac{dx}{dt} = r_1 x - r_2 x^2 - r_3 x^3 + \alpha r_4, \quad \frac{dx}{dt} = r_5 x - r_6 x^2 - r_7 x^3 + \alpha r_8, \quad \frac{dx}{dt} = r_9 x - r_{10} x^2 - r_{11} x^3 + \alpha r_{12}, \quad \frac{dx}{dt} = r_{13} x - r_{14} x^2 - r_{15} x^3 + \alpha r_{16}. \]

**Figure 2.** Phase portrait of controlled system (3)

when $\alpha = 5, r_1 = 0.3, r_2 = 0.4, r_1 + r_2 = 0.7 > r_0, r_3 = r_4 = 0.7$

The relationship between $r_0$ and $\alpha$ is described in Figure 3. From Figure 3, we can easily know that $\alpha$ should be larger than 0.5. When $\alpha \leq 0.5$, there will be no suitable real solution of $r_0$ that meets the criteria value. Also from Figure 5, we can see that when $\alpha$ is increased, $r_0$ increases first and decreases then.

**Figure 3.** Illustration of bifurcating point $r_0$ versus hybrid control parameter $\alpha$

Finally, we can reach a decision that the controlled system (3) is more changeable, flexible and broader than the uncontrolled one. In other words, uncontrolled system is just a particular case of the controlled system when the value of control parameter $\alpha$ is 1. It is easy to change the limit cycle state into asymptotically stable state in the uncontrolled system or on the contrary, change asymptotically stable equilibrium point into limit cycle.

5. **Conclusions**

In the paper, a hybrid controller is considered with mimicking the love dynamics between two persons named Romeo and Juliet. We have formulated a novel model based on Liao’s model by taking the hybrid control strategy into consideration. By theoretical analysis and numerical simulations, we find that hybrid controller can successfully influence the appearance of Hopf bifurcation by choosing an appropriate control parameter when such a bifurcation is harmful or necessary. This foundation is very useful, effective and meaningful in dealing with love affairs. In addition, the relationship between $r_0$ and $\alpha$ is also described in this paper. It has been shown that the control parameter $\alpha$ greatly affects the value of $r_0$ and the corresponding dynamical behaviors are also effectively influenced. Furthermore, the stability and periodic solutions’ direction in the controlled system are also been discussed.
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