The rigid limit of $N = 2$ supergravity

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Abstract

In this paper we elucidate the rigid limit of $N = 2$ supergravity coupled to vector and hypermultiplets. In particular we show how the respective scalar field spaces reduce to their global counterparts. In the hypermultiplet sector we focus on the relation between the local and rigid c-map.

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1. Introduction

Supergravity theories are mainly discussed as effective low-energy theories of some ultraviolet complete fundamental theory such as string theory. In this low-energy limit the heavy string modes are integrated out and only massless and light modes are retained. However, gravitational interactions of the light states are kept and thus the Planck scale $M_{Pl}$ appears in the effective low-energy Lagrangian. For some applications it is of interest to decouple gravity in a second step by taking the rigid limit $M_{Pl} \rightarrow \infty$. In many cases this is straightforward but it can also be a confusing issue.

In this paper we focus on $N = 2$ supergravity coupled to vector and hypermultiplets and study its rigid limit in a Minkowski background. Both multiplets contain scalar fields and $N = 2$ supersymmetry dictates how the geometry of their respective field spaces as well as the other couplings in the low-energy effective theory have to behave in the rigid limit. For vector multiplets the scalar geometry reduces from a projective special Kähler manifold (which is sometimes called a 'local special Kähler manifold') to a special Kähler manifold (which is sometimes called a 'rigid special Kähler manifold'). For hypermultiplets the scalar geometry...
reduces from a quaternionic Kähler to a hyper-Kähler manifold. This limiting procedure has previously been discussed in [1–6] (for reviews see, for example, [7, 8]).

This paper is inspired by the situation where \( N = 2 \) supergravity appears as the low-energy limit of string theory, but our considerations hold for any UV theory with similar properties. In string theory one typically has two different classes of light scalar fields with different types of couplings in the effective theory. On the one hand, there are scalars (often denoted as moduli) which only couple gravitationally and whose background values can be as large as \( M_{Pl} \). These fields are essentially frozen to their background values with only harmonic fluctuations left. On the other hand, one can have charged scalars which can have gauge interactions as well as gravitational interactions. In an unbroken gauge theory their background values are zero and they contribute to non-trivial dynamics at low energies. These scalars also can have a non-zero background value which induces spontaneous symmetry breaking at a scale set by the background values. Here we do not specify the details of the gauge dynamics but we do distinguish scalar fields \( \Phi \) with background values \( \Phi_0 = O(M_{Pl}) \) and scalar fields \( \varphi \) with background values \( \varphi_0 \ll O(M_{Pl}) \). Furthermore, if the effective theory has no additional scale \( \Lambda < M_{Pl} \) (such as the QCD- or Seiberg–Witten scale [3]) any scalar field space reduces to flat space. Therefore we allow for a generic \( \Lambda \) and perform the rigid limit in the presence of non-zero \( \Lambda \). In addition, we also assume throughout the paper that both supersymmetries are unbroken.

The paper is organized as follows. In section 2 we recall that without any additional scale \( \Lambda < M_{Pl} \) in the theory the rigid limit of the scalar field space is flat. In section 3 we take the rigid limit in the vector multiplet sector and show how the rigid prepotential characterizing the rigid special Kähler geometry is related to the prepotential of the local special Kähler geometry. In section 4 we consider the rigid limit in the hypermultiplet sector. Here we focus on special quaternionic Kähler manifolds which arise at the tree-level of type II compactifications and which are characterized by the (local) c-map. We show that in the rigid limit these spaces reduce to hyper-Kähler manifolds characterized by the rigid c-map [9, 10].

2. Preliminaries

We shall first briefly recall the spectrum and couplings of four-dimensional \( N = 2 \) supergravity (for a review see e.g. [7, 8]). The theory consists of a gravitational multiplet, \( n_v \) vector multiplets and \( n_h \) hypermultiplets. The gravitational multiplet \( (g_{\mu\nu}, \Psi_{\mu A}, A_{\mu}^I) \) contains the spacetime metric \( g_{\mu\nu} \), \( \mu, \nu = 0, \ldots, 3 \), two gravitini \( \Psi_{\mu A} \), \( A = 1, 2 \), and the graviphoton \( A_\mu^0 \). A vector multiplet \( (A_\mu, \lambda^A, \tau) \) contains a vector \( A_\mu \), two gaugini \( \lambda^A \) and a complex scalar \( \tau \). Finally, a hypermultiplet \( (\zeta_\alpha, q^\alpha) \) contains two hyperini \( \zeta_\alpha \) and four real scalars \( q^\alpha \). The bosonic Lagrangian is given by

\[
\mathcal{L} = \frac{1}{2\kappa^2} R + \frac{1}{4} \text{Im} N_{\mu I}(t, \bar{t}) F_{\mu I}^I F_{\mu I}^J - \frac{1}{8} \text{Re} N_{\mu I}(t, \bar{t}) \epsilon^{\mu\nu\rho\sigma} F_{\mu I}^I F_{\nu J}^J - g_{\mu I}(t, \bar{t}) D_\mu t^J D^J t - h_{\mu I}(q) D_\mu q^J D^J q - V(t, \bar{t}, q),
\]

where \( \kappa^{-1} = 8\pi M_{Pl} \). We have chosen canonical mass dimension 1 for the vector and scalar fields and thus the sigma-model metrics \( g_{\mu I}(t, \bar{t}) \), \( i, \bar{i} = 1, \ldots, n_v \) and \( h_{\mu I}(q), u, v = 1, \ldots, 4n_h \), together with the kinetic matrix \( N_{\mu I}(t, \bar{t}) \), \( I, J = 0, \ldots, n_h \) are dimensionless couplings which we specify in more detail in the following sections. \( V \) is the dimension-4 scalar potential.

7 These manifolds have also been of interest recently in relation to wall-crossing phenomena. (For recent reviews see, for example, [11, 12] and references therein.)
Before we study the rigid limit of this theory let us make some general observations. The (dimensionless) spacetime metric is expanded around a Minkowski background $\eta_{\mu\nu}$ as

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu} + \cdots,$$

while the scalar fields are expanded around their background values $t_0$ and $q_0$

$$t^i = t^i_0 + \delta t^i; \quad q^i = q^i_0 + \delta q^i.$$

The couplings in (2.1) can be expanded for small fluctuations accordingly

$$N_{IJ}(t, \bar{t}) = N_{IJ}(t_0, \bar{t}_0) + \partial_t N_{IJ}(t_0, \bar{t}_0) \delta t^i + \partial_{\bar{t}} N_{IJ}(t_0, \bar{t}_0) \delta \bar{t}^i + \cdots$$

$$g_{ij}(t, \bar{t}) = g_{ij}(t_0, \bar{t}_0) + \partial_t g_{ij}(t_0, \bar{t}_0) \delta t^i + \partial_{\bar{t}} g_{ij}(t_0, \bar{t}_0) \delta \bar{t}^i + \cdots$$

$$h_{vw}(q) = h_{vw}(q_0) + \partial_w h_{vw}(q_0) \delta q^w + \cdots.$$  

Since we have chosen canonical mass dimension 1 for the scalar fields and the gauge bosons, the couplings $N_{IJ}$, $g_{ij}$, $h_{vw}$ are dimensionless while their respective derivatives have mass dimension $-1$. Therefore, if the theory under consideration has no scale other than $M_{Pl}$ then all higher order terms in the expansions (2.4), i.e. all terms including derivatives of the couplings, scale with $\kappa$ and thus vanish in the rigid limit $\kappa \to 0$. This implies that for each quantity only the first term survives, where the couplings are evaluated at the constant background values of the scalar fields. At that point in field space all three matrices $N_{IJ}$, $g_{ij}$, $h_{vw}$ are constant and hence can be diagonalized. Thus, in this situation the scalar field space of the rigid theory is flat and the gauge kinetic matrix can be chosen diagonal. Of course what we are recalling here is that without an intermediate scale any kinetic term reduces to its renormalizable form in the rigid limit.

The situation changes if the theory has a second, possibly dynamical scale $\Lambda$ (such as the QCD scale or the Seiberg–Witten scale) which is well below the Planck scale $M_{Pl}$.

In this case the above reasoning does not hold as the derivatives of the couplings $\partial_t g_{ij}(t_0, \bar{t}_0)$ etc do not necessarily scale with $\kappa$ but could instead have a $\Lambda^{-1}$ dependence and thus do not have to vanish in the $\kappa \to 0$ limit. As a consequence there can be a non-trivial field space and non-trivial gauge couplings in the rigid theory—Seiberg–Witten theory [3] and generalizations thereof being a prominent example. It is this generic situation which we are concerned with in this paper.

In order to proceed we need to make one further generic distinction. The theory under consideration can have two classes of scalar fields, which for now we denote by $\Phi$ and $\varphi$. The $\Phi$s have Planck-sized background values, i.e., $\Phi_0 = O(M_{Pl})$, and are typically flat directions (i.e., moduli) of the potential $V$:

The fields $\varphi$ have vanishing or small background values $\varphi_0 \ll M_{Pl}$. Since all couplings in (2.1) are dimensionless they depend on the ratios $\frac{\Phi_0}{M_{Pl}}$, $\frac{\Phi}{M_{Pl}}$, $\frac{\varphi}{M_{Pl}}$, $\frac{\varphi_0}{M_{Pl}}$. The dependence $\frac{\varphi}{M_{Pl}}$ cannot occur as in this case the couplings would diverge, while any term that depends on the ratio $\frac{\varphi_0}{M_{Pl}}$ disappears in the $M_{Pl} \to \infty$ limit. Thus, in the rigid limit only the dependence on $\frac{\Phi_0}{M_{Pl}}$ and $\frac{\varphi}{M_{Pl}}$ needs to be kept. Furthermore, in that limit any derivative with respect to $\Phi$ necessarily scales like $M_{Pl}^{-1}$ while derivatives with respect to $\varphi$ scale like $\Lambda^{-1}$. Hence, in the limit $M_{Pl} \to \infty$ the fluctuations of $\Phi$ are suppressed, or frozen, and only fluctuations in $\varphi$ survive. Schematically we thus have

$$g(\Phi, \varphi) = g(\Phi_0, \varphi_0) + \partial_\varphi g(\Phi_0, \varphi_0) \varphi + \cdots,$$

i.e., we can replace $\Phi$ by its background value $\Phi_0$ while we keep $\varphi$ as a dynamical field. Thus $\Phi$ can be viewed as part of a hidden sector while $\varphi$ denotes the observable sector.

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8 In principle they could also be fixed by $V$, but in that case they would generically have masses of order $O(M_{Pl})$ and thus would have been integrated out of the low-energy theory.

9 We also allow for the case $\varphi_0 > \Lambda$ but then insist on $\Lambda < \varphi_0 \ll M_{Pl}$. 
Let us stress that the distinction between $\Phi$ and $\varphi$ usually can only be maintained locally. It is often the case that at specific subspaces in the field space heavy modes, which have been integrated out, become light. This effect generically shows up as a singularity in some couplings. In this case the choice of low-energy excitations is no longer consistent and one has to go to a different effective theory with different field variables and couplings. In the following we will always assume that we work in a region of the moduli space where the distinction between $\Phi$ and $\varphi$ is unambiguous. This is also consistent with the fact that taking the rigid limit zooms-in on a specific region of the moduli space. Only the points in the neighbourhood of such a region are kept in the rigid scalar field space, i.e. a generic point is at distance $M_{pl}$ from this region and it will be sent to infinite distance in the rigid limit.

3. Vector multiplet sector

Let us start with the vector multiplet sector. In this case the metric $g_{ij}$ in (2.1) is defined on the $2n_v$-dimensional special Kähler manifold $M_v$ [13, 14]. This means that $g_{ij}$ obeys

$$g_{ij} = \partial_i \partial_j K', \quad \text{for} \quad K' = -\kappa^{-2} \ln iY, \quad Y = \kappa^2 (\bar{X}^I F_I - X^I \bar{F}_I), \quad (3.1)$$

where both $X^I(t)$ and $F_I(t)$, $I = 0, 1, \ldots, n_v$, are dimension-1 holomorphic functions of the scalars $t^i$. $F_I = \partial F / \partial X^I$ is the derivative of a holomorphic prepotential $F(X)$ which is homogeneous of degree 2 in $X$. Furthermore, it is possible to go to a system of 'special coordinates' where $X^I = (M_{pl}, t^i)$ or more generally $t^i = M_{pl} \bar{t}^i$. Due to the homogeneity property of $F(X)$ it is convenient to define a dimensionless function $\mathcal{F}$ by

$$F = (X^0)^2 \mathcal{F}(X^j / X^0) = \kappa^{-2} \mathcal{F}(t^i). \quad \text{In terms of } \mathcal{F}, \text{ the Kähler potential (3.1) reads} \quad K' = -\kappa^{-2} \ln iY, \quad Y = 2(\mathcal{F} - \bar{\mathcal{F}}) - (t - \bar{t})(\mathcal{F} + \bar{\mathcal{F}}), \quad (3.2)$$

The kinetic matrix $N_{IJ}$ in (2.1) is also a function of the $t^i$ given by

$$N_{IJ} = \bar{F}_{IJ} + 2i \frac{\text{Im} F_{IK} X^K \text{Im} F_{JL} X^L}{X^\alpha \text{Im} F_{\alpha K} X^K}, \quad (3.3)$$

where $F_{IJ} = \partial_i F_j$. (See e.g., [14] for further details.)

In terms of the set-up of the previous section we now need to distinguish between $\Phi$ and $\varphi$ for the vector multiplets. Let us denote all scalar fields in vector multiplets with Planck-sized background values by $\Phi$, where we suppress the index for notational simplicity. By a slight abuse of notation we will continue to label the subset of $\tilde{n}_v$ scalars in the observable sector by $t^i, i = 1, \ldots, \tilde{n}_v$.

Before any expansion we can choose the prepotential to be of the generic form

$$\mathcal{F} = \mathcal{F}^\Phi(\kappa \Phi) + \mathcal{F}^\varphi(\kappa \Phi, t^i, \kappa, \Lambda), \quad (3.4)$$

where $\mathcal{F}^\varphi(\kappa \Phi, t^i = 0, \kappa, \Lambda) = 0$ holds$^{10}$. With this choice the first term $\mathcal{F}^\Phi(\kappa \Phi)$ encodes the geometry of the hidden sector, and since $\mathcal{F}^\Phi$ is dimensionless it only depends on the product $\kappa \Phi$. The second term $\mathcal{F}^\varphi$ captures any non-trivial gauge dynamics of the observable sector (and thus depends on $\Lambda$ and $t$) and the interactions between hidden and observable sector, or in other words between $\Phi$ and $t$. The condition $\mathcal{F}^\varphi(\kappa \Phi, t^i = 0) = 0$ says that any field independent term is chosen to be part of $\mathcal{F}^\Phi$.

Let us first focus on $\mathcal{F}^\Phi$ and expand $\Phi = \Phi_0 + \delta \Phi$. For small $\delta \Phi$ this implies the Taylor expansion

$$\mathcal{F}^\Phi = \mathcal{F}^\Phi(\kappa \Phi_0) + \kappa \partial \mathcal{F}^\varphi(\kappa \Phi_0) \delta \Phi + \frac{1}{2} \kappa^2 \partial^2 \mathcal{F}^\Phi(\kappa \Phi_0) \delta \Phi^2 + \mathcal{O}(\kappa^3) \quad (3.5)$$

$^{10}$Prepotentials of this form do not allow for terms like $\psi^\beta M_{pl}$, for example. However, when such a term is expanded around $\kappa \sim 0$, it has actually a term constant in $t$ and one term vanishing at $t = 0$. Hence, since at the end we are interested in the limit $\kappa \to 0$, we can start for our generic considerations from prepotentials of the form (3.4).
with the first term being constant. Note that we are assuming \( \Phi_0 = O(M_{Pl}) \) and thus \( \kappa \Phi_0 \) is a dimensionless parameter. Inserting this into the definition of \( K^\nu \) (3.2), expanding the \( \ln \) and ignoring \( F^\nu \) for the moment yields [4]

\[
K^\nu = -\ln Y(\kappa \Phi_0) + Y^{-1}(\kappa \Phi_0) K^\nu_0 + O(\kappa),
\]

where

\[
K^\nu_0 = h(\delta \Phi) + \tilde{h}(\delta \Phi) + g_{\Phi \Phi}(\kappa \Phi_0) \delta \Phi \delta \bar{\Phi}.
\]

(3.6)

\( h(\Phi) \) is holomorphic in \( \delta \Phi \) and thus does not enter the Kähler metric. Note that the \( \ln Y(\kappa \Phi_0) \) term and the normalization factor \( Y^{-1}(\kappa \Phi_0) \) in (3.6) are constant and thus can be absorbed by redefining \( h \) and \( g_{\Phi \Phi} \) or \( \delta \Phi \) in (3.7). We have chosen to display the rigid limit in the form (3.6) for later convenience and to stress that a non-zero \( Y(\kappa \Phi_0) \) is essential in order to expand the \( \ln \). As anticipated, only the quadratic term of \( K^\nu_0 \) contributes in the \( \kappa \rightarrow 0 \) limit, leading to a constant (and thus flat) metric \( g_{\Phi \Phi} = g_{\Phi \Phi}(\kappa \Phi_0) \) which can be computed straightforwardly in terms of \( F^\Phi \). The precise expression is not very illuminating and, furthermore, by an appropriate redefinition of the \( \delta \Phi \) one can always choose \( Y^{-1} g_{\Phi \Phi} = \delta \Phi \).

Let us now turn to the second piece \( F^\nu(\kappa \Phi, t', \kappa, \Lambda) \) in (3.4). First of all this term can in principle induce (constant) corrections to the metric \( g_{\Phi \Phi} \) depending on \( \Phi \). As such corrections can always be absorbed into a redefinition of \( \delta \Phi \) we will not consider them in the following. As we argued in the previous section we can replace \( \Phi \) by its background value and only consider \( F^\nu(\kappa \Phi_0, t', \kappa, \Lambda) \)\textsuperscript{11}. It is convenient to make the \( \kappa \)-dependence explicit by defining

\[
F^\nu(\kappa \Phi_0, t', \kappa, \Lambda) = \sum_{n=0}^{\infty} \kappa^n F^{\nu(n)}(\kappa \Phi_0, t', \Lambda),
\]

(3.8)

where the \( F^{\nu(n)} \) have mass dimension \( n \). For dimensional reasons \( F^{\nu(0)} \) has to be constant and with our convention \( F^\nu(\Phi, t' = 0, \kappa, \Lambda) = 0 \) we have chosen this constant to be part of \( F^\Phi(\Phi_0) \), implying that \( F^{\nu(0)} = 0 \). Similarly, \( F^{\nu(1)} \) would be linear in \( t' \) and so violate gauge invariance. More importantly, this linear term would correspond to terms of the form \( F \sim X^0 X^i \) and/or \( F \sim X^\Phi X^i \) in the homogenous degree-2 prepotential \( F \) appearing in (3.1) and (3.3). This violates our assumptions that the \( t' \) decouple from the hidden sector and, as we will see shortly, would lead to mixing with the graviphoton. Thus, consistency constrains the expansion (3.8) and requires

\[
F^{\nu(0)} = F^{\nu(1)} = 0.
\]

(3.9)

It is then straightforward to see that after expanding the \( \ln \) in (3.2) and taking the limit \( \kappa \rightarrow 0 \) only \( F^{\nu(2)}(\kappa \Phi_0, t', \Lambda) \) survives. Inserting this into (3.2) and expanding the \( \ln \), and including the \( \Phi \)-dependence given in (3.7), we obtain (3.6) with

\[
K^\nu_0 = h(\delta \Phi, t' \rightarrow 0) + \tilde{h}(\delta \Phi, t' \rightarrow 0) - i(\delta \Phi \partial_i F^\Phi(2) - \delta \Phi \bar{\partial}_i F^\Phi(2)) - i\tilde{F}^\nu_i \bar{F}^{\nu(2)}_i - t' F^{\nu(2)}_i
\]

(3.10)

where we have defined \( F^{\nu(2)} = \frac{1}{2} g_{\Phi \Phi}(\kappa \Phi_0) \delta \Phi^2 \) and \( h(\Phi, t') \) is a holomorphic function which does not enter the metric. Indeed (3.10) is the Kähler potential of a rigid special Kähler manifold with the metric [8]

\[
g = \begin{pmatrix} g_{\Phi \Phi} & 0 \\ 0 & g_{ij} \end{pmatrix} = 2 \begin{pmatrix} \text{Im} F^{\nu(2)}_{\Phi i} & 0 \\ 0 & \text{Im} F^{\nu(2)}_{ij} \end{pmatrix}.
\]

(3.11)

\textsuperscript{11} We argued in the previous section that a dependence on the ratio \( \frac{\Phi_0}{T} \) cannot occur as it would diverge in the \( M_{Pl} \rightarrow \infty \) limit and thus also \( F^\nu \) can only depend on \( \kappa \Phi \). Replacing \( \Phi \) by its background value then means that we are neglecting all derivatives of \( F^\nu \) with respect to \( \Phi \) as they disappear in the rigid limit.
so that $\mathcal{F}^{(2)} + \mathcal{F}^{(1)}$ can be identified as the rigid prepotential. Note that these standard relations only hold up to the constant normalization factor $Y^{-1}(\kappa \Phi)$ which, however, can be absorbed into the rigid prepotential or by a field redefinition of $\delta \Phi$ and $t$. Let us also reiterate that a non-zero $Y^{-1}(\kappa \Phi)$ is essential for a consistent expansion of the In.

Let us now turn to the gauge kinetic matrix defined in (3.3). It parameterizes the gauge couplings of the graviphoton together with the gauge bosons in the vector multiplets. Since we distinguish the purely gravitationally coupled scalars $\Phi$ and the observable scalars $t'$ we need to make the same distinction for the gauge bosons as they reside in the same multiplet. Thus $A_\mu^0$ denotes the graviphoton, $A_\mu^k$ the gauge bosons which form vector multiplets with the scalars $\Phi$ and $A_\mu^l$ the gauge bosons which form vector multiplets with the scalars $t'$. In the rigid limit the graviphoton $A_\mu^0$ can mix with the gravitationally coupled $A_\mu^k$ but there should be no couplings with the observable $A_\mu^l$. In other words, $N_{ij}$ should be block-diagonal with $N_{00} = N_{0k} = 0$ in the rigid limit. Let us now explicitly check the consistency of this requirement with the constraints on the prepotential that we discussed above. First, we observe that both $F_0 \sim O(\kappa)$ and $F_{0k} \sim O(\kappa)$, as the derivative with respect $t'$ (or rather $X'$) means that only the Planck suppressed $\mathcal{F}^{(2)}$ contributes. For the components $N_{00}$ and $N_{0k}$ the non-holomorphic second term in (3.3) is suppressed because in both case it contains the Planck suppressed $\mathcal{F}^{(2)}$ in the numerator. Hence we see that $N_{00} \sim O(\kappa)$ and $N_{0k} \sim O(\kappa)$, and so the gravitational and visible sectors decouple. The components $N_{00}, N_{0k}, N_{k\Phi}$, on the other hand, are constant at leading order as they depend on $\mathcal{F}^{(2)}$. Finally, in $N_{ij}$ the non-holomorphic second term is Planck suppressed, as the numerator only depends on $\mathcal{F}^{(2)}$, while the denominator can be $O(1)$. Thus we have

$$N_{ij} = F_{ij} = \mathcal{F}^{(2)}_{ij}, \quad (3.12)$$

implying that $\text{Im} N_{ij} = -\text{Im} \mathcal{F}^{(2)}_{ij} = -\frac{1}{2} g_{ij}$, which is indeed the correct relation in global $\mathcal{N}' = 2$ supersymmetry [8].

Let us close this section with two explicit examples. As a first, simple example we will consider the prepotential

$$F = \frac{i}{4} ((X^0)^2 - \eta_{ij} X^i X^j), \quad i = 1, \ldots, n_v, \quad (3.13)$$

with $\eta_{ij}$ real. Inserted into (3.1) or (3.2) and using $t' = \kappa^{-1} \frac{X^i}{X^0}$ yields

$$F = \frac{i}{4} (1 - \kappa^2 \eta_{ij} t' t''), \quad K = -\ln(1 - \kappa^2 \eta_{ij} t' t'') + \text{const.}, \quad (3.14)$$

which is the Kähler potential of the space $SU(1, n_v) / U(1, n_v)$. In terms of the notation (3.8) we infer $\mathcal{F}^{(2)} = -\frac{1}{2} \eta_{ij} t' t''$ and inserting this into (3.11) we obtain the flat metric $g_{ij} = \eta_{ij}$. As a second example we consider the prepotential

$$F = \frac{i}{4} X^1 (X^2 X^3 - \eta_{ij} X^i X^j), \quad i, j = 4, \ldots, n_v, \quad (3.15)$$

with $\eta_{ij}$ again real. Inserting this into (3.1) or (3.2) and using

$$S = \kappa^{-1} \frac{X^1}{X^0}, \quad T = \kappa^{-1} \frac{X^2}{X^0}, \quad U = \kappa^{-1} \frac{X^3}{X^0}, \quad t' = \kappa^{-1} \frac{X^i}{X^0} \quad (3.16)$$

yields

$$F = \frac{i}{4} \kappa^3 S(U - \eta_{ij} t' t''), \quad (3.17)$$

12 Note that we could add terms of the form $X^0 X^i$ in $F$ which for a purely quadratic $F$ can always be rotated away. Furthermore, any imaginary part of $\eta_{ij}$ does not contribute to $K$ (and thus the metric) but does contribute to the $\theta$-angle as can be seen from (3.12).
and
\[ K = - \ln(S - \bar{S}) - \ln((T - \bar{T})(U - \bar{U})) - \eta_{ij}(t - \bar{t})^i(t - \bar{t})^j + \text{const.}, \tag{3.18} \]
which is the \( \text{Kähler} \) potential of the space
\[ \frac{SU(1, 1)}{U(1)} \times \frac{SO(2, n_v - 1)}{SO(n_v - 1) \times SO(2)}. \tag{3.19} \]
Here \( S, T \) and \( U \) are scalars of gravitationally coupled vector multiplets, while the \( t' \) can be part of the observable sector. In the spirit of this paper \( S, T \) and \( U \) are assumed to have Planck-sized background values \( S_0, T_0, U_0 \) and thus \( F^{(2)}_{\tau} = -\frac{i}{4} \kappa S_0 \eta_{ij} t' \). Inserted into (3.11) we obtain the flat metric \( g_{ij} = \kappa S_0 \eta_{ij} \). If the \( t' \) parameterize the Coulomb-branch of a non-Abelian gauge theory then \( F^{(2)}_{\tau} \) is corrected at one-loop and non-perturbatively. For \( SU(2) \) with one modulus \( t \) one finds [3]
\[ F^{(2)}_{\tau} = -\frac{i}{4} \kappa S_0 t'^2 + t^2 \ln \frac{t^2}{\Lambda^2} + t^2 \sum_{k=1}^{\infty} F_k \left( \frac{t}{\Lambda} \right)^{4k}. \tag{3.20} \]
The generalization to arbitrary gauge groups and the derivation of \( F^{(2)}_{\tau} \) from string theory is reviewed in [15].

Before turning to the hypermultiplet sector let us also note that at special points in the \( T - U \) plane a non-Abelian gauge enhancement can occur. For example, near \( T \approx U \) one observes the enhancement \( U(1)^2 \rightarrow SU(2) \times U(1) \). This is precisely the situation that we mentioned above, in that on a subspace of the moduli space additional states become light which change the effective description. In this case it is convenient to introduce the variables \( T_\pm = \frac{1}{2}(T \pm U) \) such that near \( T \approx U \) the background value of \( T_\pm \) is small, as is assumed for the \( t' \). In other words, near \( T \approx U \) we should treat \( T_\pm \) as an observable scalar field. Inserted into (3.17) this yields
\[ F = \frac{i}{4} \kappa^3 (T_+^2 - T_-^2 - \eta_{ij} t'^i t'^j), \tag{3.21} \]
which displays the similarity of \( T_\pm \) and the \( t' \). The full perturbative and non-perturbative corrections of the \( STU \)-model in string theory were derived in [16–20], while the rigid limit yielding the rigid prepotential (3.20) was explicitly performed in [21].

4. Hypermultiplet sector

4.1. Generic case

Let us now consider the rigid limit of the hypermultiplet geometry. In this sector \( h_m(q) \) appearing in the Lagrangian (2.1) is constrained to be the metric of a \((4n_v\text{-dimensional})\) quaternionic \( \text{Kähler} \) manifold \( M_h \) [1, 8, 22]. Such manifolds have holonomy group \( Sp(1) \times Sp(n_h) \) and they admit a triplet of complex structures \( J^x, x = 1, 2, 3 \), which satisfy the quaternionic algebra \( J^x J^y = -\delta^{xy} 1 + \epsilon^{xyz} J^z \). The metric \( h_m \) is Hermitian with respect to all three of complex structures. The associated hyper-\( \text{Kähler} \) 2-forms given by \( K_{\alpha\beta}^x = h_m(J^x)^{\alpha\beta} \) are covariantly closed with respect to the \( Sp(1) \) connection \( \omega^x \), i.e., \( \nabla K^x = dK^x + \epsilon^{xyz} \omega^y \wedge K^z = 0 \). This implies that \( K^x \) can be viewed as \( Sp(1) \) field strength of \( \omega^x \) given by
\[ K^x = d\omega^x + \frac{i}{2} \epsilon^{xyz} \omega^y \wedge \omega^z. \tag{4.1} \]

In the rigid limit, on the other hand, global \( N = 2 \) supersymmetry constrains \( K^x \) to be closed and the metric to be hyper-\( \text{Kähler} \) with holonomy \( Sp(n) \) [23]. Hyper-\( \text{Kähler} \) manifolds are Ricci-flat while quaternionic \( \text{Kähler} \) manifolds are Einstein. It was shown in [1] that the
Sp(1) part of the curvature scales with $\kappa^2$ and the Riemann curvature tensor of a quaternionic Kähler manifold decomposes as
\[
R_{\text{av}} = \kappa^2 \hat{R}_{\text{av}} + W_{\text{av}},
\]
where $\hat{R}_{\text{av}}$ is the (dimensionless) $Sp(1)$ curvature while $W_{\text{av}}$ is the Ricci-flat Weyl-curvature of a hyper-Kähler manifold\(^{13}\). In terms of the metric $h_{\text{av}}$ this again implies that $h_{\Phi \Phi}$ is flat for the purely gravitationally coupled scalars, while one can have a non-trivial hyper-Kähler metric in an observable sector if there is an additional scale $\Lambda$. It is difficult to make further, general statements about the hyper-Kähler limit of quaternionic Kähler geometry, and in order to progress further one must proceed example by example (see, e.g., [6] for a recent discussion of a specific example). Rather than taking this approach, we shall consider the rigid limit of special quaternionic Kähler manifolds [9]. This large class of manifolds are constructed from special Kähler base manifolds, and will allow us to make use of our discussion of the previous section.

4.2. The rigid limit of special quaternionic Kähler manifolds

At the tree-level of type II compactifications $M_h$ takes a special form in that its metric is entirely determined in terms of the holomorphic prepotential of a $(2n_h - 2)$-dimensional special Kähler submanifold $M_h$. In this case $M_h$ is called ‘special quaternionic Kähler’ and its construction is known as the c-map [9].

Let us denote the complex coordinates of $M_h$ by $z^a, \bar{z}^\alpha, a = 1, \ldots, n_h - 1$, its Kähler potential by $K^h(z, \bar{z})$ and the holomorphic prepotential by $G$. The remaining scalars in the hypermultiplets are the dilaton $\phi$, the axion $\phi$ and $2n_h$ real Ramond–Ramond scalars $\xi^A, \bar{\xi}^A, A = 0, \ldots, n_h - 1$. An explicit form of the metric on $M_h$ is known as the Ferrara–Sabharwal metric which reads [10]
\[
\mathcal{L} = -(\partial \phi)^2 - e^{4\phi} (\partial \phi + \kappa \bar{\xi}^A \partial \xi^A - \kappa \bar{\xi}^A \partial \bar{\xi}^A)^2 + g_{\alpha \bar{\alpha}} \partial z^a \partial \bar{z}^{\bar{\alpha}}
\]
\[
- e^{2\phi} \text{Im} \mathcal{M}^{-1} (\partial \bar{\xi}^A + \mathcal{M} \partial \xi^A, \partial \bar{\xi}_B + \mathcal{M} \partial \xi_B).
\]

The metric $g_{\alpha \bar{\alpha}}$ denotes the special Kähler metric on $M_h$ which is determined in terms of a holomorphic prepotential $G(Z)$ by the relation (3.1) with $F(X)$ replaced by $G(Z)$. The couplings $\mathcal{M}_{AB}$ are also determined in terms of $G$ via the analogue of (3.3)
\[
\mathcal{M}_{AB} = \hat{G}_{AB} + 2i \frac{\text{Im}(G_{AC}) Z^C \text{Im}(G_{BD}) Z^D}{\text{Im}(G_{AC}) Z^A Z^B},
\]
where $Z^A = (M_{PI}, \xi^A)$ are the homogeneous coordinates on $M_h$.

In order to take the rigid limit we will make the same distinction as in the vector multiplet sector, in that we split the scalars on $M_h$ into only gravitationally coupled scalars with Planck-sized background values (denoted again by $\Phi$) and scalars in the observable sector which we continue to denote by $z^a$ and which have small background values. As a consequence, the rigid limit for the special Kähler metric $g_{\alpha \bar{\alpha}}$ is exactly as in the previous section and it reduces to a metric on a rigid special Kähler manifold determined by a Kähler potential analogous to the one given in (3.10) with $F^{(2)}$ defined in (3.8) replaced by $G^{(2)}(\kappa \Phi_0, z, \Lambda)$
\[
K^h = h(\Phi, z) + \bar{h}(\Phi, \bar{z}) - i(\delta \Phi \bar{g}_{\Phi}^{(2)} - \delta \Phi \bar{g}_{\Phi}^{(2)}) - i(\bar{z}^\alpha \bar{g}_{\bar{a}}^{(2)} - z^\alpha \bar{g}_{\bar{a}}^{(2)}),
\]
where $\bar{g}_{\Phi}^{(2)} = \frac{i}{2} \bar{g}_{\Phi}^{(2)}(\kappa \Phi_0) \delta \Phi^2$. As in the observable sector of the vector multiplets we have allowed for a non-trivial, possibly different scale $\Lambda < M_{Pl}$ also in the hypermultiplet sector.

\(^{13}\) See [8] for a review.
By exactly the same reasoning as in the previous section the matrix $M_{A\Phi}$ defined in (4.4) becomes block diagonal in the rigid limit

$$M_{A\Phi} = \begin{pmatrix} M_{\Phi\Phi} & 0 \\ 0 & G_{(2)}^{(2)} \end{pmatrix},$$

(4.6)

where $M_{\Phi\Phi}$ is constant and includes the $A = 0$ direction while $G_{(2)}^{(2)}$ is the second derivative of the prepotential $G_{(2)}^{(2)}$.

Finally we need to take the $\kappa \to 0$ limit in the first two terms in (4.3) which merely leaves $(\partial \phi)^2 + \text{Re}(\Phi^2)$. Thus the metric (4.3) splits into a flat part for the field directions $\phi, \bar{\phi}$ and the gravitationally coupled $\Phi$ of the special Kähler manifold $M_{\Phi}$ together with the corresponding RR-scalars, which we denote by $\xi^\Phi, \bar{\xi}_\Phi$ and which include the $A = 0$ direction. Being flat this component is also trivially hyper-Kähler. In the observable sector the hypermultiplets contain the scalars $(z^a, \bar{z}^b, \xi^\Phi, \bar{\xi}_\Phi)$. Taking the rigid limit of (4.3) in these directions one determines a Kähler metric characterized by the Kähler potential

$$K_{\kappa\gamma} = i\partial_\kappa \partial_{\gamma} G_{(2)}^{(2)} - \partial_\kappa \partial_{\gamma} \bar{G}_{(2)}^{(2)} - \frac{1}{2} \text{Im}(G_{(2)}^{(2)})^{-1} \sigma_{ab}(C + \bar{C})_{(2)}(C + \bar{C})_{(2)}.$$

(4.7)

where we defined [10]

$$C_{ab} = i(\xi^\Phi + G_{(2)}^{(2)} \xi^\Phi).$$

(4.8)

$K_{\kappa\gamma}$ is known as the Kähler potential of the rigid c-map and in [9] it is shown that the corresponding metric is hyper-Kähler. Flat directions can be added to $K_{\kappa\gamma}$ by replacing $G_{(2)}^{(2)}$ in (4.7) by $G_{(2)}^{(2)} = \frac{1}{2} \tau^2 + G_{(2)}^{(2)} + \bar{G}_{(2)}^{(2)}$, for $\tau = \phi + i\bar{\phi}$, and defining $C_{ab}$ in terms of $\xi^\Phi, \bar{\xi}_\Phi$ as in (4.8). Note that the rigid limit that we have just described associates to every special quaternionic Kähler manifold characterized by a prepotential $G$ a hyper-Kähler manifold characterized by a different prepotential $G_{(2)}^{(2)}$. This is in contrast to the recently discussed quaternionic–hyper-Kähler correspondence where the hyper-Kähler manifold is characterized by the same prepotential $G$ (see e.g. [12] and references therein).

## 5. Outlook

We have given a prescription for the rigid limit of the special Kähler and quaternionic Kähler geometry appearing in the sigma-model metrics of $N = 2$ supergravity. Using this, one could then study the rigid limit of the scalar potential generated by gauging the isometries of these two metrics. Indeed, it is straightforward to check that in the rigid limit the supergravity scalar potential reduces to that of a global $N = 2$ supersymmetric theory. We will return to this issue, as well as the rigid limit of spontaneously broken $N = 2$ theories, in future work.

To conclude, let us briefly mention the rigid limit of $N = 2$ supergravity on an anti-de Sitter (AdS) background. Recently, there has been much progress in the study of rigid supersymmetry in AdS space and, in particular, it has been realized that the structures of the sigma-models are different to their flat space counterparts. For instance, in rigid $N = 1$ AdS supersymmetry it has been shown that the usual Kähler manifold must have an exact Kähler form, and that this condition follows from the rigid limit of $N = 1$ supergravity [24]. In rigid $N = 2$ AdS supersymmetry it has been shown that one of the triplet of Kähler forms on the hyperKähler manifold must be exact, implying that the manifold is non-compact, and that the manifold must possess an additional $SO(2)$ isometry relative to its flat space counterpart [25]. It would be interesting to consider how these constraints arise in the rigid limit of $N = 2$ supergravity on an AdS background.
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