Raise and fall of a bright soliton in an optical lattice

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We study an ultracold atomic gas with attractive interactions in a one-dimensional optical lattice. We find that its excitation spectrum displays a quantum soliton band, corresponding to \(N\)-particle bound states, and a continuum band of other, mostly extended, states. For a system of a finite size, the two branches are degenerate in energy for weak interactions, while a gap opens above a threshold value for the interaction strength. We find that the interplay between degenerate extended and bound states has important consequences for both static and dynamical properties of the system. In particular, the solitonic states turn out to be protected from spatial perturbations and random disorder. We discuss how such dynamics implies that our system effectively provides an example of a quantum many-body system that, with the variation of the bosonic lattice filling, crosses over from integrable non-ergodic to non-integrable ergodic dynamics, through non-integrable non-ergodic regimes.

I. INTRODUCTION AND SUMMARY OF THE RESULTS.

Solitary waves in classical fluids may arise when wave dispersion effects are compensated by non-linear interactions \[1\]. The study of their mathematical properties has defined important domains in mathematical research \[2\]-[4], with far reaching implications for pure and applied modern science \[5\]-[9]. Although quantum mechanics is an intrinsically linear theory, solitons may emerge also in quantum fluids: in this case, non-linearities do arise as a result of effective cooperative phenomena occurring in quantum-many-particle systems. Indeed, solitons were demonstrated to emerge in different quantum mechanical contexts, ranging from quantum material science to particle physics \[10\]-[11].

In this work, we are primarily motivated by the recent investigations in quantum fluids as provided by ultracold atoms trapped in one dimensional optical potentials. In these systems, bright solitons may emerge for attractive atom-atom interactions \[12\] (see also \[12\]) and they have been unequivocally observed in several experiments \[13\]-[15]. From a conceptual point of view, however, bosonic systems with attractive interactions must be treated with care: because of the Bose statistics, the lowest energy state can be macroscopically occupied with a density that is magnified by interactions. Important progress for the problem have been achieved describing the bosonic fluid through a famous integrable theory as provided by the Lieb-Liniger model, which is amenable to the exact analysis \[16\]. Relying on that, it was demonstrated that the ground state energy may display instabilities that can be nonetheless cured by a suitable choice of interactions and density \[17\]. Indeed, the limits of vanishing interaction with a finite density, unexpectedly leading to mean-field results for large number of bosons \[18\], or of vanishing density with finite interactions \[19\], have been thoroughly explored. In particular, by analysing the solutions of the mean-field Gross-Pitaevskii equation it was found that \[20\]-[21] a critical value of the attraction exists for which the ground state density undergoes to a transition from a uniform profile to a bright-soliton-type one (with the logic implied by the onset to modulational instabilities in the condensate). In the same limit, it was exactly proven that density-density and higher-order correlation functions display a qualitative change of behaviour in correspondence to the critical value predicted by the mean-field theory.

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In this work we focus on a bosonic system described by the Bose-Hubbard Model (BHM) Eq. \[ \hat{H} \], confined in a one dimensional lattice in which the density and interactions strength are both finite. We perform a numerical study of energy bands and quantum correlations using Density Matrix Renormalization Group (DMRG) methods \[ \[24, 26\]. Among different aspects implied by the lattice, here we exploit the energy-band structure of the system, featuring characteristic bendings, foldings and energy gaps. For our system of attractive bosons, such effects can indeed define new physical regimes of the system with peculiar bound states of solitonic type (see \[27-29\]).

We find that for finite attractive interactions \(N\)-body bound states are formed; for weak interactions, however, these bound states are degenerate with a second band of other states, mostly extended or involving higher-order bound states. Increasing the interaction strength, the bound states get more and more energetically favourable, until a value of the interaction strength for which the band of bound states completely separates in energy from the rest of the spectrum. Fig. \[1\] summarizes the low-energy region of the band structure as well as the critical interactions for which the systems displays a gap opening between the two bands. We characterize the nature of the excited states in the spectrum by monitoring the density-density correlations functions, which display a different spatial behaviour for the \(N\)-particle bound states and for extended states (see Fig. \[2\]). The calculation of the dynamical structure factor, i.e the Fourier transform of the density-density correlation \(\langle n_l(t) n_0(0) \rangle\) with respect to both time and space, provides the portrait of the band structure of the system, displaying a typical two-branch dispersion at low energies as showed in Fig. \[3\]. Such quantity is experimentally detectable by means of the cold-atoms quantum technology \[30, 31\].

The band structure of the system and the opening of a gap has important consequences for the dynamics of the system. To study the soliton dynamics, we devise a protocol in which we prepare a quantum soliton in a particular position, say at lattice site \(i_0\). Then we let it expand under the guidance of Hamiltonian Eq. \[1\] for different values of the interactions strength \(U\). Such a protocol leads to different effects, depending on the availability of extended states. Namely: for \(U = 0\) only extended states are available in the dynamics; for \(0 < U < U_c\) the extended and bound states are available; for \(U > U_c\), at low energies, solely bound states exist. As a striking feature, in this latter regime the density keeps more and more the shape of the initial state and only a small fraction of it spreads over the chain. We note that the expansion velocity carries over from being constant (in time) for \(U < U_c\) to displaying specific oscillations for \(U > U_c\). These features, that have been predicted for two particles \[32\], clearly emerge in Fig. \[4\].

We found that the occurrence of degenerate scattering and bound states in the system implies nontrivial results for the time asymptotics of the correlations. Indeed, we observe that the time-evolving density-density correlations are not function solely of the energy, but they display a dependence on the projection of the initial state over scattering or bound states. This lack of ergodicity, is supported by showing that a random disorder is not able to turn the time asymptotics of solitonic bound states into the asymptotics of scattering states (see Fig. \[5\]). Besides being interesting in the context of general non-equilibrium dynamics of many-body systems, we stress that such a result provides a particular stability condition for the bright soliton in the lattice.

The article is organized as follows. In Sec. \[II\] we introduce the Bose-Hubbard Hamiltonian that is the main model system we will be dealing with. In Sects. \[III\]-\[IV\] we obtain a portion of the spectrum, correlation functions and the dynamical structure factor. Sections \[V\] and \[VI\] are devoted to dynamics: in the former section we study the dynamics of the bright soliton initially pinned in a given lattice position, and in the latter section we study the dynamics of the density-density correlation functions and its features under random-noise perturbations. In Sec. \[VII\] we draw our conclusions and comment on the relevance of our results in the general context of the thermalization dynamics of quantum many-body systems. In the Appendices, we discuss more technical aspects of the work.

**II. THE MODEL**

We employ the Bose-Hubbard Model (BHM) describing \(N\) interacting bosons in a one-dimensional lattice. The Hamiltonian reads

\[
\hat{H} = -J \sum_{j=1}^{L} (\hat{a}_{j}^{\dagger} \hat{a}_{j+1} + \text{h.c.}) - \frac{|U|}{2} \sum_{j=1}^{L} \hat{n}_{j} (\hat{n}_{j} - 1) \tag{1}
\]

where the operators \(\hat{n}_{i} := \hat{a}_{i}^{\dagger} \hat{a}_{i}\) count the number of bosons at the site \(i\); operators \(\hat{a}_{i}, \hat{a}_{i}^{\dagger}\) obey the canonical commutation relations \([\hat{a}_{i}, \hat{a}_{j}^{\dagger}] = \delta_{ij}\), and \(L\) is the number of sites. The parameters \(J, U\) in Eq. \[1\] are the hopping amplitude and the strength of the on-site interaction, respectively. Throughout this article, only attractive interactions are considered, i.e. we take \(U > 0\); all energies will be expressed in units of the tunnel energy \(J\). The BHM Eq. \[1\] is not solvable by the coordinate Bethe ansatz, the failure arising because of finite probabilities that a given site is occupied by more the two particles \[33, 34\] and thus, \(n\)-body interactions cannot be factorized in 2-body ones. Nevertheless the plane-wave ansatz works well in the so called two-particle-sector, for which such probabilities are vanishing and where exact solutions are still possible \[32, 36\]. Despite the fact that the BHM is not integrable, its continuous limit is the Bose-gas integrable field theory \[37\] (see Appendices). We note that a similar logic works also for discretization for the classical non-linear- Schrödinger equation \[38\].
III. SPECTRAL PROPERTIES AND CORRELATION FUNCTIONS LANDSCAPE

We first present the DMRG results for the spectral properties of the BHM Eq. (1). Details for the numerical simulations can be found in the Appendices. In Fig. 1 (a), (b), we see that a critical value of $U$ exists for which a finite bandgap opens in the spectrum. Specifically, we show the energy of the first 40 exited states of the lattice of $L = 30$ sites with open boundary conditions (hard-wall box), for $N = 3$ and $N = 7$ particles.

The energy gap between the bands is usually defined as the difference in energy between the last bound state, i.e. the $L$-th exited state, and the next one: $\Delta E = E_{L+1} - E_L$. For open boundary conditions, special care has to be taken due to the presence of the two edge states appearing at the border of the chain [99] (doublets in Figs. 1(a) and (b)) – we therefore adopt an alternative definition, $\Delta E = E_{L+1} - E_{L-2}$. The inspection of the gap as a function of a rescaled interaction strength, $NU$, in Fig. 1(c), allows us to conclude that there exists a critical value of the interaction strength, $U_c$, for which an emerging bandgap is displayed in the spectrum. In particular, the behaviour of $U_c$ in Fig. 1(d) shows that the gap-opening mechanism is general and does not depend on the choice of boundary conditions.

We point out that the regime in which extended and bound states result degenerate arises because of the bending of the energy bands. In particular, such effect cannot be captured by neither mean-field nor continuous Lieb-Liniger since both theories describe the solitonic states with a single parameter (the interaction $U$), without reference to any specific feature of the energy bands.

In a translationally invariant system, the ground-state density is homogeneous and it hides any spatial structure associated with the presence of bound states. Even in the system with open boundary conditions, the density is not uniform, but it remains almost unchanged when varying the interaction strength. In order to characterize bound states, we study the density-density correlation function: $C(r) = \langle n_{L/2} n_{L/2+r} \rangle$.

From our numerical analysis, the results of which are displayed in Fig. 2(a), we clearly see that the increasing interaction strength makes $C(r)$ more peaked, showing an exponential-like decay as a function of the distance, $C(r) \sim \exp(-r/\xi)$, with a correlation length $\xi$, shown in the inset of Fig. 2(a), that decreases when interaction strength increases, reflecting the trend of particles in the bound state to cluster together. In Fig. 2(b) we show the correlation function $C(r)$ for several excited states.
in the regime of $U < U_c$. The difference in the correlations between states belonging to the two branches appears clearly: for all states belonging to the lowest excitation branch, we find that $C(r)$ decays exponentially, with the same length $\xi$, which is therefore a characteristic of the system, depending solely on $U$, thus confirming the hypothesis that the lowest branch is made of bound many-body states. On the other hand, for the states belonging to the second branch, at intermediate distances $C(r)$ approaches a plateau shape, with a value comparable to the squared density, $n_{as} = (N/L)^2$, before dropping down when approaching the walls of the box. We thus conclude that the higher branch contains extended states. This strong difference in the correlations in the two bands can be quantified by studying the density-density correlations at large distance, as defined by the operator

$$C^{LD} = \sum_{i=2}^{L/4} C(i)/N$$

where $N$ is the number of sites over which this operator has its support. While for scattering states $C^{LD} \approx n_{as}$, see Fig. 3 (a), for bound states the magnitude of the correlations is several orders of magnitude smaller because of the faster decay of the corresponding $C(r)$. This bimodal distribution, that we show in Fig. 3 (a), and, in particular, the co-existence, for $U < U_c$, of the two families of states at the same energy gives rise to fundamental consequences at the level of quantum dynamics, see Section IV.

IV. DYNAMICAL STRUCTURE FACTOR

Information on the available excitations in the system as a function of the momentum $k$ and energy $\omega$ transferred to the condensate is provided by the dynamical structure factor $S(k, \omega)$, which features characteristic peaks at points representing the allowed many-body excitations:

$$S(k, \omega) = \sum_{\alpha \neq 0} |\langle \alpha | n_k | 0 \rangle|^2 \delta(\omega - \omega_\alpha).$$

In particular, following the peaks in $S(k, \omega)$ we can reconstruct the energy bands of the system [40, 41]. In the many-body state, we observe that for any infinitesimally small $U$ a low-energy band separates from the rest of the spectrum. Corroborated by the exact results obtained for the BHM in the two-particle [32, 33] and three-particle [40, 41] sectors (see the summary in Appendix B), we conclude that the lower band is made of $L$ bound states, while the upper band is made of scattering states. Such conclusion is further analysed and supported by the study of the correlation functions presented above. Remarkably, because of the lattice bending of the energy band, for $U < U_c$ we observe that some of the bound states belonging to the lowest band are degenerate with respect to some states in the band of extended states (see Fig. 3 (a)). Only for $U > U_c$, the two bands are fully separated by an energy gap which linearly grows with the interaction strength (see Fig. 3 (b)).

FIG. 3: Dynamical structure factor $S(k, \omega)$. Upper row: analytical results for $N = 2$ particles in the two regimes $U = 2 < U_c$ (a) and $U = 5 > U_c$ (b) in a lattice with $L = 30$ sites and open boundary conditions. The black lines are a guide to the eye, outlining two bands of bound (lower) and scattering (upper) states. $S(k, \omega)$ unambiguously reveals the presence of a gap in the spectrum, providing information concerning available excitations in the system. Lower row: $S(k, \omega)$, as produced by the DMRG analysis of excited states in the system. Simulations have been performed, respectively, for $N = 5$ particles in the lattice with $L = 30$ sites and open boundary conditions, in two different regimes, for $U = 0.75 < U_c$ (c) and $U = 1.2 > U_c$ (d). For the many-body problem with $N > 2$, where no exact solution are available, $S(k, \omega)$ reveals the presence of band structure.

V. DYNAMICAL EXPANSION OF PINNED SOLITONS

Here, we devise a specific dynamical protocol to trace effects which may be observed in the band structure and dynamical structure factor $S(k, \omega)$ discussed above. The intention is to capture a soliton, pinning it at a given position in the lattice, and then to let it spread. The pinning procedure consists in breaking the discrete translational symmetry of the underlying lattice by adding an attractive local potential to the Hamiltonian:

$$\mathcal{H}_i(\mu, U) = \mathcal{H}(U) + \mu(U)n_{i0},$$

whose ground state $\psi_0$ will be the initial state of the dynamics. The dynamics in the system, after the pinning term has been lifted, will be governed by $\mathcal{H}(U)$. The strength of the pinning $\mu(U)$ introduced in this context,
must be chosen satisfying the requirement that the energy injected in the system by the perturbation is equal to the width of the bound-state band (see Appendix B for details): as the soliton that we want to prepare is fully localized in space, and the bound states are, at fixed momentum, configurations with the lowest energy, the minimum requirement is to populate all of them. From these considerations we directly conclude that, while for small $U$ the pinning procedure will populate several scattering and bound states coexistent in the same energy window, for $U > U_c$, when the gap separates the two bands, almost all the populated modes are bound states.

We then let $v_0^t$, the ground state of Eq. (4), evolve by removing the pinning and allowing the configuration to expand. In Fig. 4(a-b-c) we show the expansion dynamics of the density for three cases: $U < U_c$, $U \approx U_c$ and $U > U_c$. Increasing the interaction strength, we see that the density profile stays closer and closer to the shape of the initial state, only its small fraction spreading into the chain. This can be seen more quantitatively by studying the expansion velocity \( \nu_\infty \), calculated in the following way:

\[
\nu(t) = \frac{d}{dt} \sqrt{R^2(t) - R^2(0)} \quad R^2(t) = \frac{1}{N} \sum_{i=1}^{L} n_i(t) (i - i_0)^2
\]

\( i_0 \) being the pinning position. In Fig. 4(d-e) we show, respectively, $\nu(t)$ and its asymptotic value, $\nu_\infty$, at large times. For $U \approx U_c$, a crossover appears clearly between the regions of $U < U_c$ and $U > U_c$. In fact, while in the first regime we have the light-cone-like expansion at a constant speed, in the latter one almost no expansion is present, while strong oscillations of $\nu(t)$ start to appear. The asymptotic expansion velocity $\nu_\infty$ is identified by fitting it to a phenomenological expression,

\[
\nu(t) \approx \nu_\infty + \cos(At)/t^B,
\]

where $A$ and $B$ are fitting parameters. The inspection of $\nu_\infty$ in Fig. 4(e) makes the difference between the two regimes even clearer: with the increase of the strength of the attractive interaction above $U_c$, the asymptotic expansion velocity falls from a finite value to an almost vanishing one.

VI. QUANTUM DYNAMICS

Thermalization in quantum many-body systems is usually expressed in terms of the well-known Eigenstates Thermalization Hypothesis (ETH) [13, 14]: if the expectation values of local observables for individual eigenstates are a smooth function of energy, then the system behaves ergodically and one can exchange, for such observable, the long-time average by the Gibbs ensemble average. What we described in Section II indicates a violation of ETH, as, in the same energy window, one can find different eigenstates characterized by values of $C^{LD}$ that differ by several orders of magnitude. Addressing rigorously the lack of ergodicity in this system is an extremely hard task: it requires either an analysis of all the eigenvalue of $C^{LD}$ while enlarging the system’s size, or the study of the out-of-equilibrium dynamics at very large times (or, possibly, developing both approaches). Here, instead, we ask the following question: in the case of $U < U_c$, would a small random perturbation, comparable, in its magnitude, to the spacing of the energy spectrum, be able to make the expectation values of $C^{LD}$ for (quasi) degenerate bound and scattering states converging to the same asymptotic limit, as a result of the long-time evolution? If the answer is affirmative, then the system behaves ergodically; otherwise, some of the information from the initial state, encoded in $C^{LD}$, will persist, leading to asymptotic stability of the solitonic state. To probe such a stability, we address the evolution of an excited state, belonging to the bound-state branch, $|B⟩$, and of a scattering state $|S⟩$ with adjacent energy eigenvalues in the spectrum for $U < U_c$, after having perturbed the system by adding a random-noise source in it. The dynamics is then governed by a Hamiltonian which depends both on the interaction strength $U$.
FIG. 5: Panel (a): Eigenstate expectation values \( \langle \phi_n | C_{LD} | \phi_n \rangle \) of \( C_{LD} \) for eigenstates \( \phi_n \) of Hamiltonian (ref) as a function of the eigenstate energy \( E \) for \( N = 5 \) particles in the lattice with \( L = 21 \) sites and open boundary conditions in the case of \( U = 0.5 < U_c \). The expectation values for the bound and scattering states differ by two orders of magnitude, leading to breaking of ETH. The black line at \( C_{LD} = (N/L)^2 \approx 0.056 \) is a guide to the eye. Panel (b): the evolution of \( C_{LD} \), starting from both a scattering and a bound state with adjacent energy eigenvalues in the spectrum in the same case as in panel (a). On the present time scale, \( t \sim 5 \times 10^5 \), the two states exhibit completely different behaviour.

and on the intensity of the perturbation, \( W \):

\[
\mathcal{H}(W, U) = \mathcal{H}(U) + \sum_i \epsilon_i n_i,
\]

where \( \epsilon_i \) is a random variable chosen uniformly from interval \([-W, +W]\). In Fig. 5 (b) we observe that, on the present time scale, correlations for the bound state are not strongly affected by the random perturbations (within the time scale available in our numerical simulations). In fact, \( C^{LD}(t) \) remains almost constant in the course of the evolution, without reaching the value observed for the scattering states. This is a strong evidence, on the reasonable timescale, of the soliton stability.

VII. CONCLUSIONS

In this work, we studied the spatial correlations and dynamical properties of condensates composed of attractive bosons in one-dimensional lattices. The presence of the lattice induces a characteristic energy band structure, for which the soliton gives rise to peculiar features, with distinctive correlation functions. Such features, in turn, substantially affect the dynamics of the system. Overall, we have demonstrated how a bright solitonic bound state can be created in the system and, by studying the expansion dynamics, we have provided a way to test its stability against external perturbations. As an outlook, we stress the relevance of our work to studies on the ergodicity of quantum systems.

Discrete lattices play an important role in many areas of fundamental physics. In particular, we note that the finite lattice spacing adds an ultraviolet cut-off in continuous field theories for which it is often possible to eliminate divergences from physical observables by means of the so-called lattice regularization. In the framework of such a procedure, however, some features of the continuous field theory may be lost in the regularized theory. For our system, the continuous theory is the Bose-gas field theory (encompassing the Lieb-Liniger Hamiltonian model), which is well known to be integrable [46]. Indeed, the BHM, which is considered in the present work, accounts for the non-integrable regularization of the Bose-gas field theory [37] where, at finite densities, the lattice spacing \( \Delta \) vanishes proportionally to the filling factor \( N/L \).

Our approach, then, implies that the integrability of the Bose-gas is destroyed by increasing the filling fraction. Therefore, we can interpret the results stemming from our dynamical protocols in the following way.

I) At sufficiently low filling, our bosonic system is integrable (described by the Bose-gas field theory). According to the general theory, in this limit the system is expected to thermalize to a Generalized Gibbs Ensemble.

II) Increasing the filling, the system does not remain integrable, being described by the Bose-Hubbard model. For such system, when bound states and scattering states coexist with equal energies (i.e. for \( U < U_c \)), the long-times asymptotic states strongly depend on the initial states.

III) At larger filling, the system is far from integrability, as the Bose-Hubbard corrections to the Bose-gas grow stronger. In this case, the solitonic band is nearly flat, making the coexistence between bound and scattering states impossible. In this limit, therefore, the system is ergodic.

Such a scenario, summarized in Fig. 5 indicates that going from I) to III), the integrability, controlled by the filling (instead of the perturbation added to the Hamiltonian, in the framework of more standard approaches), is destroyed by entering an intermediate regime, in which the system keeps some trace of integrability in that the
dynamics is not ergodic.

We believe that our analysis is within the current activity in atomic physics quantum technology. Specifically, ultracold atoms with tunable interactions have been already loaded in 1D optical lattices in several experiments [47]. Atomtronic circuits can also provide appropriate tools to explore the dynamics of the system [48–51].

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Appendix A: Derivation of the continuum model

In this appendix, we sketch the derivation of the Bose Gas and Lieb-Liniger models as continuous limit of the Bose-Hubbard quantum dynamics.

To see that, let’s define the density of bosons in the lattice as \( D = N/(L\Delta) \), \( \Delta \) being the lattice spacing. In the continuous limit \( \Delta \to 0 \), which implies that the filling factor \( \nu = N/L = D\Delta \) must be accordingly small. In the continuous limit the bosonic operators must be rescaled: \( a_i = \sqrt{\Delta} \hat{\Psi}(x), n_i = \Delta \hat{\Psi}^\dagger(x) \hat{\Psi}(x), x = \Delta i \).

Then, the Bose-Hubbard model reduces to the Bose gas quantum field theory \[ H_{BH} = \frac{1}{2} \Delta^2 \mathcal{H}_{BG}, \mathcal{H}_{BG} = \int dx \left[ (\partial_z \hat{\Psi}^\dagger)(\partial_z \hat{\Psi}) + c \hat{\Psi}^\dagger \mathcal{U} \hat{\Psi}^\dagger \hat{\Psi} \right], \text{ with } c = U/(t\Phi) \].

The quantum fields obey \( \mathcal{H}_{BG} = (\hat{\Psi}(x), \hat{\Psi}^\dagger(y) = \delta(x - y) \) and \( [\hat{\Psi}(x), \hat{\Psi}^\dagger(y)] = 0 \). The Bose gas field theory is the quantum field theory for the Lieb-Liniger model. Such statement can be demonstrated by writing the eigenstates of \( \mathcal{H}_{BG} \) as \( |\phi(\lambda)\rangle = \int dzX(z|\lambda)\hat{\Psi}^\dagger(z_1)\ldots\hat{\Psi}^\dagger(z_N)|0\rangle \). Then, it can be proved that \( \chi(z|\lambda) \) must be eigenfunctions of \( H_{LL} \):

\[
H_{LL} = - \sum_{j=1}^{L} \frac{\partial^2}{\partial z_j^2} - 2c \sum_{L>j>k} \delta(z_j - z_k) . \tag{A1}
\]

In the mean field limit, the Lieb-Liniger model reduces to the Gross-Pitaevskii equation (see i.e. [52])

\[
\left[ - \frac{\partial^2}{\partial x^2} - 2cN \left( 1 - \frac{1}{N} \right) |\phi(x)|^2 \right] \phi(x) = \mu \phi(x) . \tag{A2}
\]

The Gross-Pitaevskii equation above results to be the stationary solution of the functional \( E[\phi] = \int dz \chi_{GS}(z|\lambda)H_{LL}\chi_{GS}(z|\lambda) \) in which the ground state ansatz of \( H_{LL} \): \( \chi_{GS}(z|\lambda) = (1/N)^{\frac{1}{2}} \prod_{j} \phi(x_j) \) (\( \mu \) is the normalizing constant for \( \phi(x) \)).

We note that the integrability of the continuous theory can indeed be preserved through regularization, but the resulting lattice Hamiltonians do contain higher order non-linearities [57–59].

Appendix B: Few particle solution

In this appendix, we will discuss the exact results available for the few particle Bose-Hubbard model. Since the Hamiltonian Eq. \( (A1) \) commutes with the total number of particles, we can study the model separately in every sector with well defined number of particles.

The two-particle problem is integrable since only 2-particle interaction are possible. In this sector one can separate the center-of-mass and the relative coordinates and use the standard Bethe ansatz technique to provide an exact solution for the Schrödinger equation. As it was found by [42, 60], for a finite attractive interaction the spectrum decomposes in two bands: one band of scattering states and a second one composed by bound states which are characterized, respectively, by real and complex relative momenta. The band of bound states is composed by exactly \( L \) states, one for each center-of-mass momenta. It has also been pointed out that there exists a critical value of the interaction \( U_c = 4 \) for which both bands are completely separated in energy. Note that for any \( U \neq 0 \) the ground state is always a bound state however, for \( U < U_c \) some of the excited bound states occur to be degenerate in energy with some states in the scattering band. Only for \( U > U_c \) the two bands are fully separated by an energy gap that grows linearly with the interaction.

In Fig. \( (A1) \) (a), (b) we plot \( S(k, \omega) \) in the 2-particle case for \( U = 3 \) and \( U = 5 \), which are respectively below and above the critical value \( U_c \). In this image we also include the envelope of the scattering band and the energy of the bound state found with the Bethe-Ansatz solution [32].

The three-particle problem is already too complex to be completely solved but still the full energy spectrum can be obtained [40, 41]. Whereas the characterisation of all the states of the system is really complex, a band of 3-particle bound states at low energy can still be separated from the rest of the spectrum when interactions are larger than a critical value, that is different from the one of 2-particle case.

Appendix C: Critical pinning and soliton band width

In this appendix we will show how to pin the initial state (in space) while keeping a large projection over the bound states of the system. In section \( \text{V} \) we already discussed the pinning procedure, which consisted in breaking the translational symmetry of the system by adding
an attractive local potential to the Hamiltonian \[ H_i(\mu, U) = H(U) + \mu(U)n_{i_0} \] (C1)

whose ground state \( \psi_0^i \) will be the initial state of our dynamics. Since the soliton we want to prepare is fully localized in space and bound states are, at fixed momentum, the states with lowest energy, the minimum requirement is to populate all of them. We will therefore require that the energy we are introducing in the system (described by \( H(U) \)) with the pinning must be equal to the bound state band width \( \Delta E_B(U) \):

\[
\epsilon = \langle \psi_0^i | H(U) | \psi_0^i \rangle - \langle \psi_0 | H(U) | \psi_0 \rangle = \mu(U)N = \Delta E_B(U) \quad \text{(C3)}
\]

where \( \psi_0 \) is the ground state of \( H(U) \). \( \Delta E_B(U) \) can be estimated from a general expression [1] as:

\[
\Delta E_B = \sqrt{U^2 + K_{\text{min}}^2} - \sqrt{U^2 + K_{\text{max}}^2} \quad \text{(C4)}
\]

\[
= U - \sqrt{U^2 + K_{\text{max}}^2} \quad \text{(C5)}
\]

where \( U \) is the potential energy and \( K_{\text{max}} \), \( K_{\text{min}} \) are respectively the maximum and the minimum of the kinetic energy over the band. The second step in (C5) follows from \( K_{\text{min}} = 0 \). The contribution to the potential energy can be estimated as: \( \Delta = \frac{U}{2} N(N-1) \) and \( K = 2N \). With these assumptions, Eq. (C3) yields

\[
\mu(U) = \frac{|U|(1-N)}{2} \left( 1 - \sqrt{1 + \frac{16}{U^2(N-1)^2}} \right) \quad \text{(C6)}
\]

In Fig. 7 we plot \( \mu(U) \) for different number of particles \( N \) and interaction \( U \).

We observe that for a large value of \( N, \) \( \mu(U) \) tends to zero. This is compatible with the idea that, for large \( N, \) the kinetic energy of the solitonic state is negligible compared to the potential one. In this case, the band becomes effectively flat and all the states are degenerate.

By introducing a large attractive pinning, all the particle will populate only one site. This state will have an energy \( U N(N-1)/2 \), which corresponds to the upper limit of the soliton band and it has been proven for \( N = 2 \) particles to have [2] the largest projection over the band of bound state.

### Appendix D: Details of numerical simulations

In this section we provide more details about the numerical simulations, performed using Density matrix renormalization group (DMRG) techniques, that we presented in our paper. The DMRG is a numerical technique that hallow to study both statics and dynamical properties of general quantum systems in one dimension [23] [25].

Since in our paper we focused on system of particles with attractive interactions we never cut the local Hilbert space in our simulations, hallowing all the \( N \) particles to occupy a single site of our lattice. This requirement makes the Hilbert grow very fast like \( \approx N^L \), where \( L \) is the number of sites. Moreover increasing the number of particles \( N \) or the strength of the interaction massive degeneracies occurs in the spectrum (see Appendix C). These two difficulties make the simulations extremely challenging and clearly reduces up to \( N \approx 5 \sim 10 \) particles in systems with \( L \approx 50 \sim 100 \) lattice sites.

#### 1. Statics

In our paper we presented numerical results from DMRG simulations for energy, correlation functions and dynamical structure factor both for open and periodic boundary conditions. All the data have been obtained by targeting several excited states (up to 400 in Fig. 5). We kept up to 2000 states in order to limit the truncation error smaller than \( 10^{-8} \) at each step.

#### 2. Dynamics

In Section V and VI we present results obtained from time-dependent simulations based on the density-matrix renormalization group (DMRG). The time-evolution is based on a Runge-Kutta 4th order scheme, with a time step of \( \Delta t = 0.01 \). In all the results we presented, we kept a variable number of states (up to 1500 for the simulations at larger times) allowing us to keep the truncation error smaller than \( 10^{-6} \) at each time step. All time-dependent simulations have been performed imposing open boundary conditions in the lattice.

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