A REMARK ON THE HARDY-LITTLEWOOD MAXIMAL OPERATORS

WU-YI PAN

Abstract. We investigate the magnitude relation of the non-centered Hardy-Littlewood maximal operators and centered one. By using a discretization technique, we prove two facts: the first one is that the space is ultrametric if and only if the two maximal operators are identical for all discrete measure; the second is, the uncentered maximal operator is strictly greater than the centered one if $(M, d)$ is a Riemannian manifold and $\mu$ is the Riemannian volume measure.

1. Introduction

Let $(X, d)$ denote a metric space and $\mu$ a (positive Borel) measure on $X$. We use the symbol $B$ to denote an closed ball. To facilitate research, we will assume that the measure of each ball is finite and the support of $\mu$ is nonempty. We consider the Hardy–Littlewood maximal function centered and non-centered on $X$ and, for $f$ locally $\mu$-integrable,

$$M^c_\mu f \overset{\text{def}}{=} \sup_r \frac{1}{\mu(B(x, r))} \int_{B(x, r)} f \, d\mu; \quad M_\mu f \overset{\text{def}}{=} \sup_{B: B \ni x} \frac{1}{\mu(B)} \int_B f \, d\mu.$$ 

We will write them simply $M^c f, M f$ when no confusion can arise. It is easy to check that maximal functions are defined everywhere in the support of $\mu$. In order to avoid trivialities, we are required to $\int_B f \, d\mu / \mu(B) = 0$ if $\mu(B) = 0$.

In an ultrametric space, every point in a ball is a center, and so, all maximal functions coincide. When is the converse true? To be more specific, the question is that the statement:

**Question 1.1.** Whether coincidence of the two operators (in the support of the measure) implies that the metric is ultra? If not, what is the structure of the metric inherited from the parent space on the support? Is it ultra?
Regretfully, a simple delta in a non-ultrametric space provides a counter-example to the first issue. Here, we solve Question I partly. Indeed, we show

**Theorem 1.2.** Let \((X, d)\) be a metric space. The metric \(d\) is ultra iff the two maximal operators are the same for every discrete measure on \(X\).

In addition, we also show a rigid result revealing how the size relation of maximal functions changes affect the geometric structure of the support of the measure.

**Theorem 1.3.** If there is a continuous measure \(\mu\) such that \(M^c_\mu f = M_\mu f\) for all \(f \in L^1_{\text{loc}}(\mu)\), then there does not exist a series of the point \(\{x_n\}_{n\geq 1} \subseteq \text{supp}(\mu)\) so that each point in the sequence is the midpoint of two other different points in the sequence and \(\{x_n\}_{n\geq 1}\) has a limit that is also in \(\text{supp}(\mu)\).

It provides a verification of intuition that the non-center maximal operator is strictly greater than centered one for a large class of common spaces. Clearly, Theorem 1.3 implies

**Corollary 1.4.** Let \((M, d_g)\) be a Riemannian manifold, where the metric \(d_g\) induced by the inner product \(g\). Then for each continuous measure \(\mu\) satisfying that the support of the measure \(\mu\) contains a geodesic curve, there exists a function \(f \in L^1(\mu)\) such that its uncentred maximal function and centered maximal function take different values at least at one point.

2. **Proofs**

The complete proof of the following can be found in [PD22, Lemma 3.1]. See also [Ko15].

**Lemma 2.1.** Define the action of \(M^c\) on a finite Borel measure by

\[
M^c_\mu \nu \overset{\text{def}}{=} \sup_r \frac{\nu(B(x, r))}{\mu(B(x, r))},
\]

for every \(x \in \text{supp}(\mu)\). \(M_\mu \nu\) is similarly considered.

Consider a sequence of finite Borel measures \(\nu_n\) on \(X\). If \(\nu_n\) converges weakly to a finite Borel measure \(\nu\), then \(\liminf_{n \to \infty} M\nu_n(x) \geq M\nu(x)\) for all \(x \in \text{supp}(\mu)\). The same fact holds for \(M^c\).

For our purpose, we next apply a discretization method to establish a crucial lemma.
Lemma 2.2. If there is a measure $\mu$ such that $M^c f = M f$ for $f \in L^1(\mu)$, then for all pairs of points $(x, y) \in \text{supp}(\mu)$, we have

$$ (2.1) \quad \mu B(y, d(x, y)) \leq \inf_{B \ni x, y} \mu B. $$

Furthermore, $\mu B(y, d(x, y)) = \mu B(x, d(x, y))$.

Proof. If the support of the measure contains only one point, the theorem follows, so we can take two different points in $\text{supp}(\mu)$. For fixed different points $x, y \in \text{supp}(\mu)$, choose $\delta > 0$ and let

$$ f_\delta = \frac{\chi_{U(x, \delta) \setminus U(y, d(x, y))}}{\mu(U(x, \delta) \setminus U(y, d(x, y)))} $$

where $\chi_A$ is the characteristic function of the set $A$ and $U(x, r)$ denotes the open ball centered at $x$ with radius $r > 0$. Clearly, $f_\delta \in L^1$, and $f_\delta d\mu$ converges weakly to $\delta_x$ where $\delta_x$ denotes the Dirac delta concentrated at $x \in X$. Applying Lemma 2.1 with $f_\delta$, we know

$$ \liminf_{\delta \to 0} M f_\delta(z) \geq M \delta_x(z) $$

for all $z \in \text{supp}(\mu)$. Now observe that

$$ M^c f_\delta(y) = \sup_{r \in [d(x, y), d(x, y) + \delta]} \frac{\mu(B(y, r) \cap U(x, \delta) \cap U^c(y, d(x, y)))}{\mu(U(x, \delta) \cap U^c(y, d(x, y))) \mu(B(y, r))} \leq \frac{1}{\mu B(y, d(x, y))}. $$

Thus $\limsup_{\delta \to 0} M^c f_\delta(y) \leq \frac{1}{\mu B(y, d(x, y))}$. Comparing these inequalities, we conclude that

$$ (2.2) \quad M \delta_x(y) \leq 1/\mu B(y, d(x, y)). $$

It follows from the definition that $M \delta_x(y) = 1/\inf_{B \ni x, y} \mu B$. Substituted it into the above inequality (2.2), the proof is completed. \hfill \Box

The lemma has made the function of “facing a kick of the door”. Now we are in the position of the proofs of Theorem 1.2 and 1.3.

Proof of Theorem 1.2. The adequacy is clear. Conversely, supposing that for every discrete measure on $X$, $M^c f = M f$ for $f \in L^1(\mu)$, we then need to show the metric is ultra. We prove it by contradiction. Suppose, if possible, that the metric space $(X, d)$ is not ultrametric, then there exists three points $x, y, z$ such that $z \in B(x, d(x, y))$ and $d(z, y) > d(x, y)$. This means $z \notin B(y, d(x, y))$. Now pick a measure $\nu = \delta_x + \delta_y + \delta_z$. Applying
Lemma 2.2 with \( \nu \), we have \( \nu B(y, d(x, y)) = 2 = 3 = \nu B(x, d(x, y)) \). It is impossible so we have proved Theorem 1.2.

\[ \Box \]

**Proof of Theorem 1.3.** We also argue it by contradiction. Suppose, if possible, that there exists a sequence \( \{x_n\}_{n \geq 1} \) described in Theorem 1.3. By passing to a subsequence, we can suppose that the sequence satisfies that \( x_{i+2} \) is the midpoint of \( x_i \) and \( x_{i+1} \) for \( i \geq 1 \) and \( \lim_{n \to \infty} x_n = x_0 \) for some \( x_0 \in \text{supp}(\mu) \). First consider the closed balls \( B_i = B(x_{i+2}, d(x_{i+1}, x_{i+2})) \) for all \( i \geq 1 \). Next we use (2.1) to get \( \mu(B_i) \leq \mu(B_{i+1}) \). It follows from the sequence \( B_i \) is monotonically decreasing that \( \mu(B_i) = \mu(B_{i+1}) \), and hence \( \mu(B_1) = \lim_{i \to \infty} \mu(B_i) = \mu(\{x_0\}) = 0 \) by the continuity of the measure. It contradicts the fact that \( x_0 \in \text{supp}(\mu) \). This finishes the proof.

\[ \Box \]

**References**

[Ko15] D. Kosz, *On the discretization technique for the Hardy-Littlewood maximal operators*. Real Anal. Exchange, 41 (2016), no. 2, 287–292.

[PD22] W. Pan, X. Dong, *Lower bounds for uncentered maximal functions on metric measure space*. Available at the Mathematics ArXiv.

**Key Laboratory of Computing and Stochastic Mathematics (Ministry of Education), School of Mathematics and Statistics, Hunan Normal University, Changsha, Hunan 410081, P. R. China**

*Email address: pwyyys@163.com*