KNAPSACK AND THE POWER WORD PROBLEM IN
SOLVABLE BAUMSLAG-SOLITAR GROUPS

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Abstract. We prove that the power word problem for certain metabelian sub-
groups of GL(2, C) (including the solvable Baumslag-Solitar groups BS(1, q) =
⟨a, t | tat−1 = aqt⟩) belongs to the circuit complexity class TC0. In the power
word problem, the input consists of group elements g1, . . . , gd and binary en-
coded integers n1, . . . , nd and it is asked whether g1n1 · · · gdnd = 1 holds. More-
over, we prove that the knapsack problem for BS(1, q) is NP-complete. In the
knapsack problem, the input consists of group elements g1, . . . , gd, h and it is
asked whether the equation g1x1 · · · gdxd = h has a solution in Nd. For the more
general case of a system of so-called exponent equations, where the exponent
variables xi can occur multiple times, we show that solvability is undecidable
for BS(1, q).

1. Introduction

1.1. The power word problem. The study of multiplicative identities and equa-
tions has a long tradition in computational algebra, and has recently been ex-
tended to the non-abelian case. Here, the multiplicative identities we have in mind
have the form g1n1g2n2 · · · gdn = 1, where g1, . . . , gd are elements of a group G and
n1, n2, . . . , nd ∈ N are non-negative integers (we may also allow negative ni, but
this makes no difference, since we can replace a gi by its inverse gi−1). Typically,
the numbers ni are given in binary representation, whereas the representation of
the group elements gi depends on the underlying group G. Here, we consider the
case where G is a finitely generated (f.g. for short) group, and elements of G are
represented by finite words over a fixed generating set Σ (the concrete choice of Σ
is not relevant). In this setting, the question whether g1n1g2n2 · · · gdn = 1 is a true
identity has been recently introduced as the power word problem for G [31]. It ex-
tends the classical word problem for G (does a given word over the group generators
represent the group identity?) in the sense that the word problem trivially reduces
to the power word problem (take an identity w1 = 1). Recent results on the power
word problem in specific f.g. groups are:

• For every f.g. free group the power word problem belongs to deterministic
logspace [34]. This result has been recently generalized in [31], where it is
shown that the power word problem in a fixed graph product of groups is
logspace-reducible (even AC0-Turing-reducible) to the word problem of the
free group of rank two and the power word problem of the base groups of
the graph product.

• For the following groups the power word problem belongs to the circuit
complexity class TC0 f.g. nilpotent groups [34], iterated wreath products


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groups.

1 TC0 is a very small complexity class within polynomial time; see Section 2.1 for more details.
In this paper, TC0 always refers to the DLOGTIME-uniform version.
of f.g. free abelian groups and (as a consequence of the latter) free solvable groups \cite{15}.

- If \( G \) is a so-called uniformly efficiently non-solvable group (this is a large class of non-solvable groups that was recently introduced in \cite{4} and that includes all finite non-solvable groups and f.g. free non-abelian groups) then the power word problem for the wreath product \( G \wr \mathbb{Z} \) is \( \text{coNP} \)-hard \cite{15}.

As a consequence, the power word problem for Thompson’s group \( F \) is \( \text{coNP} \)-complete \cite{15}.

Historically, the power word problem appeared earlier in the area of computational (commutative) algebra. Ge \cite{20} proved that one can check in polynomial time whether an identity \( \alpha_1^{n_1} \alpha_2^{n_2} \cdots \alpha_d^{n_d} = 1 \), where the \( n_i \) are binary encoded integers and the \( \alpha_i \) are from an algebraic number field (and suitable encoded), holds.

In this paper we investigate the power word problem for certain 2-generated subgroups of \( \text{GL}(2, \mathbb{C}) \): for a fixed complex number \( \alpha \in \mathbb{C} \setminus \{0\} \) we consider the group \( T(\alpha) \) generated by the two matrices

\[
\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}.
\]

In a purely group theoretic context, these groups were studied in \cite{22,23}. Important special cases are the wreath product \( \mathbb{Z} \wr \mathbb{Z} \) (for \( \alpha \) transcendental) and the solvable Baumslag-Solitar groups \( BS(1, q) = \langle a, t \mid tat^{-1} = a^q \rangle \) (for \( \alpha = q \geq 2 \) an integer).

Our first main result states that the power word problem for every group \( T(\alpha) \) belongs to \( \text{TC}^0 \) (Theorem 3.1). For the word problem of \( T(\alpha) \), membership in \( \text{TC}^0 \) follows from \cite{28} since \( T(\alpha) \) is a linear solvable group. Theorem 3.1 is directly related to recent results on matrix powering problems \cite{1,18}. These problems can be quite difficult to analyze. For instance, it is not known whether a certain bit of the \((1,1)\)-entry of a matrix power \( A^n \) can be computed in polynomial time, when \( n \) is given in binary notation and \( A \) is a \((2 \times 2)\)-matrix over \( \mathbb{Z} \). The related problem of checking whether the \((1,1)\)-entry (or any other entry) of \( A^n \) is positive can be solved in polynomial time by \cite{18}.

1.2. The knapsack problem. If one replaces in the power word problem the exponents \( n_i \) by pairwise different variables \( x_i \) and the right-hand side 1 by an arbitrary group element \( h \in G \), one obtains a so-called knapsack equation \( g_1^{x_1} g_2^{x_2} \cdots g_d^{x_d} = h \).

The question, whether such an equation has a solution in \( \mathbb{N}^d \) is known as the knapsack problem for \( G \). In the general context of finitely generated groups the knapsack problem has been introduced by Myasnikov, Nikolaev, and Ushakov \cite{39}. As for the power word problem, this problem has been studied in the commutative setting before. For the case \( G = \mathbb{Z} \) one obtains a variant of the classical \( \text{NP} \)-complete knapsack problem; a proof of the \( \text{NP} \)-hardness of our variant of the knapsack problem for the integers can be found in \cite{24}. For this hardness result it is important that integers are represented in binary notation. For unary encoded integers the complexity of the knapsack problem goes down to \( \text{TC}^0 \). For the case that the \( g_i \) are commuting matrices over an algebraic number field, the knapsack problem has been studied in \cite{3,12}.

For the case of (in general) non-commutative groups, the knapsack problem has been studied in \cite{13,15,17,19,29,33,35,39}. In these papers, group elements are usually represented by finite words over the generators (although in \cite{39} a more succinct representation by so-called straight-line programs is studied as well). Note that for the group \( \mathbb{Z} \) this corresponds to a unary representation of integers. Hyperbolic groups, which are of fundamental importance in the area of geometric

\footnote{For the special case \( BS(1,q) \) membership of the word problem in \( \text{TC}^0 \) was shown in \cite{43}.}
group theory, are an important class of groups where knapsack can be decided in polynomial time (even in LogCFL, i.e., the closure of the context-free languages under logspace reductions). This result can be extended to the class of all groups that can be built from hyperbolic groups by the operations of (i) direct products with \(\mathbb{Z}\) and (ii) free products [35]. On the other hand, for many groups the knapsack problem is NP-complete. Examples are certain right-angled Artin groups (like the direct product of two free groups of rank two [35]), wreath products (e.g., the wreath product \(\mathbb{Z} \wr \mathbb{Z}\) [19]) and free solvable groups [15]. For wreath products \(G \wr \mathbb{Z}\), where \(G\) is finite non-solvable or free of rank at least two, the knapsack problem is complete for \(\Sigma^P_2\) (the second existential level of the polynomial time hierarchy) [15]. Finally, for finitely generated nilpotent groups, the knapsack problem is in general undecidable [19, 38].

Our second main result is that for the Baumslag-Solitar groups \(BS(1, q)\) with \(q \geq 2\) the knapsack problem is NP-complete (Theorem 4.1). This extends a result from [13], where decidability (without any complexity bound) was shown for a restriction of the knapsack problem for \(BS(1, q)\). In this restriction, all group elements \(g_i\) must be represented by words where the exponent sum of all occurrences of \(t\) is zero (here we refer to the presentation \(\langle a, t \mid tat^{-1} = a^q \rangle\) of \(BS(1, q)\)). Showing NP-hardness of the knapsack problem for \(BS(1, q)\) is easy (based on the result that knapsack for \(\mathbb{Z}\) with binary encoded integers is NP-hard). For membership in NP we use a recent result of Guépin, Haase, and Worrell [21] according to which the existential fragment of Büchi arithmetic (an extension of Presburger arithmetic) belongs to NP. The NP-membership of the knapsack problem for \(BS(1, q)\) is a bit of a surprise, since one can show that minimal solutions of knapsack equations over \(BS(1, q)\) can be of size doubly exponential in the length of the equation, see Theorem 4.2. This rules out a simple guess-and-verify strategy.

1.3. Solvability of systems of exponent equations. In the final section of the paper we consider the following generalization of the knapsack problem for \(BS(1, q)\): the input is a conjunction

\[
\bigwedge_{i=1}^{n} g_{i1}^{x_{i1}} g_{i2}^{x_{i2}} \cdots g_{id_i}^{x_{id_i}} = h_i, 
\]

(1)

where the \(g_{ij}, h_i\) are elements of \(BS(1, q)\) and the \(x_{ij}\) are variables taking values in \(\mathbb{N}\). In contrast to the knapsack problem, we do not require these variables to be pairwise different (we also allow \(x_{ij} = x_{ik}\)). We call (1) a system of exponent equations. Aside from being a natural generalization of the knapsack problem, systems of exponent equations play a crucial role in a characterization of decidability of the knapsack problem for wreath products [6]: In order to understand for which wreath products \(G \wr H\) the knapsack problem is decidable, we need to clarify for which groups \(G\) one can decide solvability of systems of exponent equations. For example, the knapsack problem is decidable for \(\mathbb{Z} \wr G\) if and only if solvability of systems of exponent equations is decidable for \(G\) (this special case already follows from [19, Proposition 3.1, Theorem 5.3]).

For many groups, solvability of systems of exponent equations is decidable. This holds in fact for all so-called knapsack semilinear groups, i.e., groups where the set of solutions of a knapsack equation is an effectively computable semilinear set. Examples of knapsack semilinear groups are hyperbolic groups [33] and co-context-free groups [29]. Moreover, the class of knapsack semilinear groups is effectively closed under finite extensions [16], wreath products [19], graph products [16], and amalgamated products and HNN-extensions over finite groups [16]. On the other hand, solvability of systems of exponent equations is undecidable for the discrete Heisenberg group [29].
Our last main result states that solvability of systems of exponent equations is undecidable for every Baumslag-Solitar group $BS(1,q)$ with $q \geq 2$ (Theorem 5.1). We prove this result by a reduction from the existential theory of $(\mathbb{N}+, (x,y) \mapsto x \cdot 2^y)$, which was shown to be undecidable by Büchi and Senger [11, Corollary 5]. In contrast to this result, it has been shown recently that the Diophantine theory (or, equivalently, solvability of systems of word equations with variables ranging over $BS(1,q)$) is decidable for $BS(1,q)$ [27].

A preliminary version of this paper appeared in [36].

2. Preliminaries

For $a, b \in \mathbb{Z}$ we write $a \mid b$ ($a$ divided $b$) if $b = ka$ for some $k \in \mathbb{Z}$. We denote with $[a, b]$ the interval $\{z \in \mathbb{Z} \mid a \leq z \leq b\}$. For complex numbers $a_1, \ldots, a_k \in \mathbb{C}$ we denote with $\mathbb{Z}[a_1, \ldots, a_k]$ the subring of $\mathbb{C}$ obtained by adjoining to the ring of integers $\mathbb{Z}$ the complex numbers $a_1, \ldots, a_k$.

The set of polynomials with variable $x$ and coefficients from $\mathbb{Z}$ is denoted with $\mathbb{Z}[x]$. Let $p(x) = a_0 x^0 + a_1 x^1 + \cdots + a_n x^n \in \mathbb{Z}[x]$ with $a_n \neq 0$. Then we define $\deg(p) = n$ (the degree of $p(x)$) and $\height(p) = \max\{\{|a_0|, \ldots, |a_n|\}\}$. The dense representation of the above polynomial is the tuple $(a_0, a_1, \ldots, a_n)$, where every $a_i$ is given in binary encoding. We define the sparse representation of a polynomial $p(x) = a_0 x^{e_0} + a_1 x^{e_1} + \cdots + a_n x^{e_n}$ with $a_i \in \mathbb{Z} \setminus \{0\}$ for all $0 \in \{0, n\}$ and $0 \leq e_0 < e_1 < \cdots < e_n$ as the list $(a_0, e_0, a_1, e_1, \ldots, a_n, e_n)$ where all numbers in this list are written in binary representation.

A Laurent polynomial is a polynomial that may also contain powers $x^k$ with $k < 0$. Formally, a Laurent polynomial over $\mathbb{Z}$ is an expression $p(x) = \sum_{i \in \mathbb{Z}} a_i x^i$ with $a_i \in \mathbb{Z}$ such that only finitely many $a_i$ are non-zero. With $\mathbb{Z}[x, x^{-1}]$ we denote the set of all Laurent polynomials over $\mathbb{Z}$; it is a ring with the natural addition and multiplication operations. If $p(x) = \sum_{i=l}^{l+k} a_i x^i$ with $k, l \in \mathbb{Z}$, $k \leq l$ and $a_k \neq 0 \neq a_l$ then we define the dense unary (resp., dense binary) representation of the Laurent polynomial $P(x)$ as the list of unary (resp., binary) encoded integers $a_k, a_{k+1}, \ldots, a_l$ together with $k$ in unary encoding.

For a complex number $\alpha \in \mathbb{C} \setminus \{0\}$ we have a natural homomorphism from $\mathbb{Z}[x, x^{-1}]$ to $\mathbb{Z}[\alpha, \alpha^{-1}]$ obtained by evaluating a Laurent polynomial at $x = \alpha$. Clearly, for an integer $q \in \mathbb{Z} \setminus \{0\}$ we have $\mathbb{Z}[q, q^{-1}] = \mathbb{Z}[1/q]$. If $q \geq 2$, this is the set of all rational numbers that have finite expansion in base $q$, i.e., the set of all numbers $\sum_{a_i \leq q^i r_i q^i \neq 0} a_i r_i q^i$ with $r_i \in \{0, q-1\}$ and $a, b \in \mathbb{Z}$. If $u = \sum_{-k \leq \ell \leq \ell+k} r_i q^i \neq 0$ with $k, \ell \geq 0$ and $\ell + k$ minimal, we define $\|u\|_q = \ell + k + 1$. Under the assumption that $q$ is a constant (which will be always the case in this paper), $\|u\|_q$ is the number of digits in the $q$-ary representation of $u$.

2.1. Circuit complexity. We assume basic knowledge in complexity theory, in particular with the complexity class $\text{NP}$; see [2] for details. We deal with the circuit complexity class $\text{TC}^0$. It contains all languages $L \subseteq \{0, 1\}^\ast$ that can be solved by a family of threshold circuits of polynomial size and constant depth. More formally: we have a family $\mathcal{C} = \{C_n\}_{n \geq 0}$ of boolean circuits $C_n$ with the following properties:

- $C_n$ has exactly $n$ input gates $x_1, \ldots, x_n$ with fan-in zero (the fan-in of a gate is the number of incoming wires).
- All other gates are either not-gates (with fan-in one), and-gates (with arbitrary fan-in), or majority-gates (with arbitrary fan-in). A majority gate of fan-in $k$ evaluated to 1 if and only if at least $k/2$ many input wires carry the truth value 1.
- Every $C_n$ has a distinguished output gate.
• There is a constant $d$ such that the depth of very circuit $C_n$ is bounded by $d$, where the depth of a circuit is the length of a longest path from an input gate to the output gate.
• There is a polynomial $p(n)$ such that $C_n$ has at most $p(n)$ many gates.
• For every word $w = a_1a_2 \cdots a_n$ with $a_i \in \{0, 1\}$, we have $w \in L$ if and only if the output gate of the circuit $C_n$ evaluates to 1 when every input gate $x_i$ is set to $a_i$.

We can lift this definition to languages over an arbitrary alphabet $\Sigma$ by fixing a binary encoding of the symbols from $\Sigma$. We always assume such encodings implicitly. To compute a function $f : \{0, 1\}^* \to \{0, 1\}^*$ by a circuit family, we encode $f$ by the language $L_f = \{1^i0w \, | \, w \in \{0, 1\}^*, \text{the } i\text{-th bit of } f(w) \text{ is 1}\}$.

In this paper, we only deal with the $\text{DLOGTIME}$-uniform version of $\text{TC}^0$. In this variant, $\text{TC}^0$ is contained in deterministic logspace and hence in polynomial time. We do not give the quite technical definition of $\text{DLOGTIME}$-uniformity; see see [12] for details. In fact, all we need about $\text{TC}^0$ is the fact that the following problems belong $\text{DLOGTIME}$-uniform $\text{TC}^0$:

1. iterated addition/multiplication (i.e., addition/multiplication of an arbitrary number) of binary encoded numbers/polynomials that are given in dense representation [13] [25],
2. division with remainder of two binary encoded numbers/polynomials that are given in dense representation [11] [25],
3. computing the number $|w|_a$ of occurrences of a letter $a$ in a word $w$,
4. computing an image $h(w)$ where $h : \Sigma^* \to \Gamma^*$ is a homomorphism of free monoids [30].

The results on binary numbers hold for any basis, since one can transform between binary representation and $q$-ary representation; this is a consequence of the first two points.

In the rest of the paper, when we speak about $\text{TC}^0$, we always refer to $\text{DLOGTIME}$-uniform $\text{TC}^0$.

2.2. Algebraic numbers. An algebraic number is a complex number which is the root of a polynomial from $\mathbb{Z}[x]$. For every algebraic number $\alpha$ there is a unique polynomial $p(x) \in \mathbb{Z}[x]$ with $p(\alpha) = 0$ and such that $p(x)$ has minimal degree among all such polynomials and the coefficients of $p(x)$ have no common divisor $> 1$. This polynomial is called the minimal polynomial of $\alpha$. If $p(x)$ is the minimal polynomial of $\alpha$, then we define $\deg(\alpha) = \deg(p)$ and $\text{height}(\alpha) = \text{height}(p)$.

Sparse polynomial root testing is the following decision problem:

**Input:** A polynomial $P(x) \in \mathbb{Z}[x]$ given in sparse representation and an algebraic number $\alpha \in \mathbb{C}$ given by its minimal polynomial in dense representation.

**Question:** Is $P(\alpha) = 0$?

Note that we do not specify $\alpha$ uniquely: if $p(x)$ is the minimal polynomial then by writing down only $p(x)$, we cannot distinguish $\alpha$ from its conjugates. On the other hand, for sparse polynomial root testing there is no reason to make this distinction, because $P(\alpha) = 0$ if and only if $P(x)$ is a multiple of $p(x)$.

**Theorem 2.1.** Sparse polynomial root testing is in $\text{TC}^0$.

$^3$We are not aware of a $\text{TC}^0$-algorithm for testing whether a given polynomial is irreducible. Hence, in the statement of this theorem we consider sparse polynomial root testing as a promise problem. On the other hand, in our later application we will only deal with a fixed algebraic number $\alpha$ in case the minimal polynomial of $\alpha$ can be hard-wired in the algorithm.
Proof. Let \( p_\alpha(x) \) be the minimal polynomial of \( \alpha \). In [31] it was shown that
the problem belongs to polynomial time using the following gap theorem: Let
\( P(x) = P_0(x) + x^k P_1(x) \in \mathbb{Z}[x] \) be a polynomial with \( k + 1 \) monomials and \( u = \deg(P_0) \)
and let \( d \geq 1 \) be an integer such that
\[
s - u > \frac{\ln k + \ln \text{height}(P)}{c(d)}
\]
where
\[
c(1) = \ln 2 \quad \text{and} \quad c(d) = \frac{2}{d \cdot (\ln(3d))^{1/2}} \quad \text{for} \quad d \geq 2.
\]
If \( \alpha \) is an algebraic number of degree at most \( d \) which is not a root of unity then
\( P(\alpha) = 0 \) if and only if \( P_0(\alpha) = 0 \) and \( P_1(\alpha) = 0 \). Note that the number on
the right-hand side of (2) is polynomial in the input length if \( P \) is given in spare
representation and the minimal polynomial of \( \alpha \) is given in dense representation.
This allows to split in \( TC^0 \) the input polynomial \( P(x) \) into several polynomials
\( p_0(x), \ldots, p_k(x) \) such that \( P(\alpha) = 0 \) if and only if \( p_i(\alpha) = 0 \) for all \( 1 \leq i \leq k \).
Moreover, all \( p_i \) are computed in dense representation. Finally, we check for every
\( i \) whether \( p_\alpha(x) \) divides \( p_i(x) \).

It remains to consider the case where \( \alpha \) is a root of unity. The case \( \alpha = \pm 1 \)
is clear since iterated addition is in \( TC^0 \). Otherwise \( \alpha \) is an \( m^{th} \) primitive root of
unity for some \( m > 2 \) and the degree of \( p_\alpha(x) \) is \( d = \varphi(m) \), where \( \varphi \) is Euler’s
phi-function. We have \( m \leq 3 \varphi(m)^{3/2} = 3d^{3/2} \) [7]. Hence, given \( p_\alpha(x) \) of degree \( d \)
we simply test in parallel for every \( d + 1 \leq e \leq 3d^{3/2} \) whether \( p_\alpha(x) \) divides \( x^e - 1 \).
Once we found such an \( e \) we can replace in the polynomial \( P(x) \) every binary
encoded monomial \( x^n \) by \( x^n \mod e \). In this way we can compute a polynomial \( \tilde{P}(x) \)
in dense representation such that \( \tilde{P}(\alpha) = 0 \) if and only if \( P(\alpha) = 0 \). Finally, we
check whether \( p_\alpha(x) \) divides \( \tilde{P}(x) \). \( \square \)

2.3. Groups. We assume that the reader is familiar with the basics of group theory.
Let \( G \) be a group. We always write 1 for the group identity element. We say
that \( G \) is finitely generated (f.g.) if there is a finite subset \( \Sigma \subseteq G \) such that
every element of \( G \) can be written as a product of elements from \( \Sigma \); such a \( \Sigma \)
is called a (finite) generating set for \( G \). We always assume that \( a \in \Sigma \) implies
\( a^{-1} \in \Sigma \); such a generating set is also called symmetric. We write \( G = \langle \Sigma \rangle \) if \( \Sigma \)
is a symmetric generating set for \( G \). In this case, we have a canonical surjective
morphism \( h : \Sigma^* \to G \) that maps a word over \( \Sigma \) to its product in \( G \) (the so
called evaluation morphism). If \( h(w) = 1 \) we also say that \( w = 1 \) in \( G \). On
\( \Sigma^* \) we can define a natural involution \( \cdot^{-1} \) by \( (a_1a_2 \cdots a_n)^{-1} = a_n^{-1} \cdots a_2^{-1}a_1^{-1} \) for
\( a_1, a_2, \ldots, a_n \in \Sigma \).

2.3.1. Matrix groups. For a complex number \( \alpha \in \mathbb{C} \setminus \{0\} \) let \( T(\alpha) \) be the subgroup
of \( GL(2, \mathbb{C}) \) consisting of the upper triangular matrices
\[
\begin{pmatrix}
\alpha^k & u \\
0 & 1
\end{pmatrix}
\]
with \( k \in \mathbb{Z} \) and \( u \in \mathbb{Z}[\alpha, \alpha^{-1}] \). This means we have the multiplication
\[
\begin{pmatrix}
\alpha^k & u \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
\alpha^\ell & v \\
0 & 1
\end{pmatrix} =
\begin{pmatrix}
\alpha^{k+\ell} & u + \alpha^k \cdot v \\
0 & 1
\end{pmatrix}.
\]
This group can be also written as the semi-direct product \( \mathbb{Z}[\alpha, \alpha^{-1}] \rtimes \mathbb{Z} \), where \( \mathbb{Z} \)
acts on \( \mathbb{Z}[\alpha, \alpha^{-1}] \) by multiplication with \( \alpha \). The groups \( T(\alpha) \) are also studied in
[22] [23].
We encode the matrix \( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \) by the pair \((k, p)\), where \(k\) is given in unary encoding and \(p\) is a Laurent polynomial with \(u = p(\alpha)\) that is given in dense unary representation. The group \( T(\alpha) \) is generated by the two matrices

\[
\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}
\]

and their inverses. We denote with \(h: \{a, a^{-1}, t, t^{-1}\}^* \rightarrow T(\alpha)\) the canonical evaluation morphism. Hence, \(h(w)\) is the identity matrix if and only if \(w = 1\) in \(T(\alpha)\).

We now have two encodings of elements from \(T(\alpha)\): as pairs \((k, p)\) describing a matrix \(A\) and as words over the alphabet \(\{a, a^{-1}, t, t^{-1}\}\). By the following lemma, we can switch in \(TC^0\) between these encodings.

**Lemma 2.2.** Given a word \(w \in \{a, a^{-1}, t, t^{-1}\}^*\) we can compute in \(TC^0\) the matrix \(h(w)\) encoded as a pair \((k, p)\) as above. Vice versa, given a matrix \(A \in T(\alpha)\) in the above encoding, we can compute in \(TC^0\) a word \(w \in h^{-1}(A)\).

**Proof.** First consider a word \(w \in \{a, a^{-1}, t, t^{-1}\}^*\) and let \(h(w)\) be the matrix in \(A\). Then \(k = |w|_t - |w|_{t^{-1}}\), which can be computed in \(TC^0\). It remains to compute a Laurent polynomial \(p(x)\) in dense unary representation such that \(u = p(\alpha)\). Let \(w_1a^n, \ldots, w_la^{n'}\) be all prefixes of \(w\) that end with \(a\) or \(a^{-1}\) \((\epsilon_1, \ldots, \epsilon_l \in \{-1, 1\})\). Let \(k_i = |w_i|_t - |w_i|_{t^{-1}}\), which can be computed in \(TC^0\) in unary notation. Then, \(u = p(\alpha)\) with \(p(x) = \sum_{i=1}^l \epsilon_i x^{k_i}\). The dense unary representation of this polynomial can be easily computed in \(TC^0\).

The inverse transformation is straightforward: take the matrix \(A\), where \(k\) is given in unary encoding and \(u = p(x)\) for a Laurent polynomial \(p(x)\) in dense unary representation. A matrix of the form \(\begin{pmatrix} 1 & a^x \\ 0 & 1 \end{pmatrix}\) (for a unary encoded \(z\)) can be produced by the word \(t^0a^z\). By concatenating such words (which is possible in \(TC^0\) by point 4 from page 5), one can produce from a given Laurent polynomial \(p(x)\) in dense unary representation a word for the matrix \(\begin{pmatrix} 1 & \rho(\alpha) \\ 0 & 1 \end{pmatrix}\). Finally, one has to concatenate \(t^k\) on the right in order to produce the matrix \(A\). \(\square\)

### 2.3.2. Baumslag-Solitar groups.

For \(p, q \in \mathbb{Z} \setminus \{0\}\), the Baumslag-Solitar group \(BS(p, q)\) is defined as the finitely presented group \(BS(p, q) = \langle a, t \mid ta^pt^{-1} = a^q \rangle\). We can w.l.o.g. assume that \(q \geq 1\). Of particular interest are the Baumslag-Solitar groups \(BS(1, q)\) for \(q \geq 2\). They are solvable and linear. It is well-known (see e.g. [15, III.15.C]) that \(BS(1, q)\) is isomorphic to \(T(q)\). Moreover, the generator \(a\) (resp., \(t\)) of \(BS(1, q)\) corresponds to the matrix \(a\) (resp., \(t\)) from \(A\). From Lemma 2.2 we immediately get:

**Lemma 2.3.** Given a word \(w \in \{a, a^{-1}, t, t^{-1}\}^*\) we can compute in \(TC^0\) the matrix \(h(w)\) with matrix entries given in \(q\)-ary encoding. Vice versa, given a matrix \(A \in T(q)\) with \(q\)-ary encoded entries, we can compute in \(TC^0\) a word \(w \in h^{-1}(A)\).

By the previous lemma, we can represent elements of \(BS(1, q)\) either as words over the alphabet \(\{a, a^{-1}, t, t^{-1}\}\) or by matrices from \(T(q)\) with \(q\)-ary encoded entries. For the matrix \(A \in T(q)\) (with \(a = q\)) we define \(|A| = |k| + |u|_q\). Hence \(|A|\) is the length of the encoding of \(A\).

Another well known special case of the group \(T(\alpha)\) is obtained when \(\alpha\) is transcendental. In this case \(T(\alpha)\) is isomorphic to the wreath product \(\mathbb{Z} \wr \mathbb{Z}^2\). It is isomorphic to the group of all matrices

\[
\begin{pmatrix} x^k & P(x) \\ 0 & 1 \end{pmatrix}
\]

(6)
where \( k \in \mathbb{Z} \) and \( P(x) = x^k + x + 1 \) (see e.g. [37] Section 2.2). In contrast to \( \text{BS}(1, q) \) the group \( \mathbb{Z} \wr \mathbb{Z} \) is not finitely presented [5]. A well-known infinite presentation of \( \mathbb{Z} \wr \mathbb{Z} \) is \( \langle a, t \mid [a^t, a^t] = 1 (i, j \in \mathbb{Z}) \rangle \).

2.3.3. Knapsack, exponent equations and the power word problem. Let \( G = (\Sigma) \) be a f.g. group. Moreover, let \( x_1, x_2, \ldots, x_d \) be pairwise distinct variables. A knapsack expression over \( G \) is an expression of the form

\[
E = v_0 u_1^i v_1 u_2^j v_2 \cdots u_d^k v_d
\]

with \( d \geq 1 \), words \( v_0, \ldots, v_d \in \Sigma^* \) and non-empty words \( u_1, \ldots, u_d \in \Sigma^* \). A tuple \((n_1, \ldots, n_d) \in \mathbb{N}^d\) is a \( G \)-solution of \( E \) if \( v_0 u_1^{n_1} v_1 u_2^{n_2} v_2 \cdots u_d^{n_d} v_d = 1 \) in \( G \). With \( \text{sol}(G, E) \) we denote the set of all \( G \)-solutions of \( E \). The size of \( E \) is defined as \( |E| = |v_0| + \sum_{d=1}^d |u_i| + |v_i| \). The knapsack problem for \( G \), \( \text{Knapsack}(G) \) for short, is the following decision problem:

**Input:** A knapsack expression \( E \) over \( G \).

**Question:** Is \( \text{sol}(G, E) \) non-empty?

It is easy to observe that the concrete choice of the generating set \( \Sigma \) has no influence on the decidability/complexity status of \( \text{Knapsack}(G) \). W.l.o.g. we can restrict to knapsack expressions of the form \( u_1^i u_2^j \cdots u_d^k v \): for \( E = v_0 u_1^{n_1} v_1 u_2^{n_2} v_2 \cdots u_d^{n_d} v_d \) and

\[
E' = (v_0 u_1 v_0^{-1})^{x_1} (v_0 u_2 v_2^{-1} v_0^{-1})^{x_2} \cdots (v_0 \cdots v_{d-1} u_d v_d^{-1} \cdots v_0^{-1})^{x_d} v_0 \cdots v_{d-1} v_d
\]

we have \( \text{sol}(G, E) = \text{sol}(G, E') \).

An exponent expression over \( G = (\Sigma) \) is a formal expression \( E \) as in [7], but in contrast to knapsack expressions, we allow \( x_i = x_j \) for \( i \neq j \). The set of solutions \( \text{sol}(G, E) \) for the exponent expression \( E \) can be defined analogously to knapsack expressions. We define solvability of systems of exponent equations over \( G \), \( \text{ExpEq}(G) \) for short, as the following decision problem:

**Input:** A finite list of exponent expressions \( E_1, \ldots, E_n \) over \( G \).

**Question:** Is \( \bigcap_{i=1}^n \text{sol}(G, E_i) \) non-empty?

This problem has been studied for various groups in [15, 19, 33, 35].

A power word (over \( \Sigma \)) is a tuple \((u_1, k_1, u_2, k_2, \ldots, u_d, k_d)\) where \( u_1, \ldots, u_d \in \Sigma^* \) are words over the group generators and \( k_1, \ldots, k_d \in \mathbb{Z} \) are integers that are given in binary notation. Such a power word represents the word \( u_1^{k_1} u_2^{k_2} \cdots u_d^{k_d} \). Quite often, we will identify the power word \((u_1, k_1, u_2, k_2, \ldots, u_d, k_d)\) with the word \( u_1^{k_1} u_2^{k_2} \cdots u_d^{k_d} \). The power word problem for the f.g. group \( G \), \( \text{PowerWP}(G) \) for short, is defined as follows:

**Input:** A power word \((u_1, k_1, u_2, k_2, \ldots, u_d, k_d)\).

**Question:** Does \( u_1^{k_1} u_2^{k_2} \cdots u_d^{k_d} = 1 \) hold in \( G \)?

Due to the binary encoded exponents, a power word can be seen as a succinct description of an ordinary word. The size of the above power word \( w \) is \( \sum_{i=1}^d |u_i| + \lceil \log_2 k_i \rceil \) which is the length of the binary encoding of \( w \).

3. Power word problem for \( \text{BS}(1, q) \)

In this section we prove our first main result:

**Theorem 3.1.** For every \( \alpha \in \mathbb{C} \setminus \{0\} \), \( \text{PowerWP}(T(\alpha)) \) belongs to \( \text{TC}^0 \).

**Proof.** If \( \alpha \) is transcendental, then \( T(\alpha) \) is isomorphic to \( \mathbb{Z} \wr \mathbb{Z} \) for which the power word problem belongs to \( \text{TC}^0 \) [53]. For the rest of the proof we assume that \( \alpha \) is algebraic. We show that in this case, \( \text{PowerWP}(T(\alpha)) \) is \( \text{TC}^0 - \) reducible to sparse polynomial root testing, which belongs to \( \text{TC}^0 \) by Theorem 2.4.
Let us fix an algebraic number $\alpha \in \mathbb{C} \setminus \{0\}$. Consider a power word of the form
\[
\left(\alpha^{k_1} \cdot u_1 \right)^{n_1} \left(\alpha^{k_2} \cdot u_2 \right)^{n_2} \cdots \left(\alpha^{k_l} \cdot u_l \right)^{n_l}
\]
Here, the $n_i$ are binary encoded integers and every $u_i$ is of the form $u_i = p_i(\alpha)$ for a Laurent polynomial $p_i$ over $\mathbb{Z}$ that is given in dense unary representation. Note that
\[
\left(\begin{array}{c}
\alpha^m \\
0
\end{array} \right) \left(\begin{array}{c}
\alpha^n \\
0
\end{array} \right) \left(\begin{array}{c}
\alpha^m \\
0
\end{array} \right) = \left(\begin{array}{c}
\alpha^n \cdot \alpha^m u \\
0
\end{array} \right).
\]
By this we can assume that all $p_i$ are ordinary polynomials over $\mathbb{Z}$. We have
\[
\left(\begin{array}{c}
\alpha^{k_1} \\
0
\end{array} \right)^n = \left(\begin{array}{c}
\alpha^{k_1} \cdot (1 + \alpha^{k_1} + \cdots + \alpha^{(n-1)k_1}) \cdot u_1 \\
0
\end{array} \right)
\]
\[
= \left\{ \begin{array}{ll}
\alpha^{k_1} \cdot \frac{\alpha^{n-1} - 1}{\alpha - 1} \cdot u_1 & \text{if } \alpha^{k_1} \neq 1 \\
\frac{n \cdot u_1}{\alpha} & \text{if } \alpha^{k_1} = 1
\end{array} \right.
\]
Hence, our power word can be written as
\[
\left(\begin{array}{c}
\alpha^{k_1} \cdot v_1 \\
0
\end{array} \right) \left(\begin{array}{c}
\alpha^{k_2} \cdot u_2 \\
0
\end{array} \right) \cdots \left(\begin{array}{c}
\alpha^{k_l} \cdot v_l \\
0
\end{array} \right) = \left(\begin{array}{c}
\alpha^{k_1} \cdot v_1 + \alpha^{k_2} \cdot v_2 + \cdots + \alpha^{k_l} \cdot v_l \\
0
\end{array} \right)
\]
Here, $v_i = n_i \cdot p_i(\alpha)$ (if $\alpha^{k_i} = 1$) or $\frac{\alpha^{n_i} - 1}{\alpha - 1} \cdot p_i(\alpha)$ (if $\alpha^{k_i} \neq 1$).

We have to check whether
\[
\alpha^{k_1} \cdot v_1 + \alpha^{k_2} \cdot v_2 + \cdots + \alpha^{k_l} \cdot v_l = 1 \quad \text{(8)}
\]
Equality (8) can be easily checked in $\mathbb{TC}^0$: we compute in $\mathbb{TC}^0$ the binary encoding of $s = k_1 n_1 + \cdots + k_l n_l$. If $\alpha$ is not a root unity then we check whether $s = 0$. On the other hand, iff $\alpha$ is a primitive, say $d$th, root of unity, then we check whether $d$ divides $s$.

The verification of (8) can be reduced to sparse polynomial root testing as follows. First, we compute all binary encoded numbers $s_i = k_1 n_1 + \cdots + k_i n_i$ for $i \in [1, l + 1]$. By multiplying (9) with a power $\alpha^m$ for $m \geq 0$ sufficiently large, we can assume that all $s_i$ are non-negative. We have to check whether
\[
\sum_{i=1}^{l} \alpha^{s_i} v_i = 0. \tag{10}
\]
Let $J = \{i \in [1, l] \mid \alpha^{k_i} \neq 1\}$ and define the polynomial
\[
q(x) = \prod_{i \in J} (x^{k_i} - 1).
\]
Note that $q(\alpha) \neq 0$. We can compute in $\mathbb{TC}^0$ the dense representation of $q(x)$ (recall from Section 3.1 that iterated multiplication of densely represented polynomials is in $\mathbb{TC}^0$). Then, we compute for all $i \in [1, l]$ the sparse representation of the polynomial
\[
q_i(x) := \left\{ \begin{array}{ll}
\alpha^{k_i} \cdot q(x) \cdot p_i(x) \cdot x^{s_i} & \text{if } \alpha^{k_i} = 1 \\
(x^{k_i} - 1) \cdot \prod_{j \in J \setminus \{i\}} (x^{k_j} - 1) \cdot p_i(x) \cdot x^{s_i} & \text{if } \alpha^{k_i} \neq 1.
\end{array} \right.
\]
This is possible in $\mathbb{TC}^0$. For instance, in the second case ($\alpha^{k_i} \neq 1$), we first compute in $\mathbb{TC}^0$ the the dense representation of $\prod_{j \in J \setminus \{i\}} (x^{k_j} - 1) \cdot p_i(x)$ (this is iterated...
Proof.

**Theorem 4.1.** Our second main result gives a positive answer and also settles the computational complexity:

Every solution of \( x^{k,n_i} - x^{n_i} = x^{\phi(x) - 1} \) yields the sparse representation of \( q_i(x) \).

Finally, we compute in \( \text{TC}^0 \) the sparse representation of the polynomial.

\[
Q(x) = \sum_{i=1}^{l} q_i(x).
\]

We obtain

\[
Q(\alpha) = q(\alpha) \cdot \sum_{i=1}^{l} \alpha^{s_i} v_i.
\]

Since \( q(\alpha) \neq 0 \), (10) is equivalent to \( Q(\alpha) = 0 \). This concludes our reduction to sparse polynomial root testing. \( \square \)

4. Knapsack for \( BS(1,q) \)

Whether the knapsack problem is decidable for \( BS(1,q) \) was left open in [13].

Our second main result gives a positive answer and also settles the computational complexity:

**Theorem 4.2.** For every \( q \geq 2 \), Knapsack(\( BS(1,q) \)) is \( \text{NP-complete} \).

Let us first remark that \( BS(1,q) \) is unusual in terms of its knapsack solution sets. In almost all groups where knapsack is known to be decidable, knapsack equations have semilinear solution sets [15, 16, 19, 29, 33, 55]. After the discrete Heisenberg group [29], the groups \( BS(1,q) \) are the only second known example where this is not the case: the knapsack equation \( t^{-z} a^{x_2 t^{x_3}} = a \) has the non-semilinear solution set \( \{(k, q^k, k) \mid k \in \mathbb{N}\} \).

Another unusual aspect is that knapsack is in \( \text{NP} \) although there are knapsack equations over \( BS(1,2) \) whose solutions are all at least doubly exponential in the size of the equation:

**Theorem 4.3.** There is a family \( E_k = E_k(x,y,z) \), \( k \geq 1 \), of solvable knapsack expressions over \( BS(1,2) \) such that \( |E_k| = \Theta(k) \) and \( z \geq (2^k 3^{k-1} - 1)/3^k - 1 \) for every solution of \( E_k = 1 \).

**Proof.** It is a well-known fact in elementary number theory that for every \( k \geq 1 \), 2 is a primitive root modulo \( 3^k \), i.e., 2 generates the group \( (\mathbb{Z}/3^k\mathbb{Z})^* \) (the group of units of \( \mathbb{Z}/3^k\mathbb{Z} \)). See, for example, Theorem 3.6 and the remarks before Theorem 3.8 in [10]. Consider the knapsack equation

\[
\begin{pmatrix}
2 & 0 \\
0 & 1
\end{pmatrix}^x
\begin{pmatrix}
1 & 1 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
2^{-1} & 0 \\
0 & 1
\end{pmatrix}^y
\begin{pmatrix}
1 & -3^k \\
0 & 1
\end{pmatrix}^z
= \begin{pmatrix}
1 & 3^k + 1 \\
0 & 1
\end{pmatrix}
\]

in \( BS(1,2) \). In the top-left entry, it implies \( 2^x 2^{-y} = 1 \). Therefore, we must have \( x = y \) in every solution. In this case, the left-hand side of eq. (11) is

\[
\begin{pmatrix}
2^x & 0 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 1 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
2^{-x} & 0 \\
0 & 1
\end{pmatrix}^y
\begin{pmatrix}
1 & -3^k \\
0 & 1
\end{pmatrix}^z
= \begin{pmatrix}
1 & 2^x - z \cdot 3^k \\
0 & 1
\end{pmatrix}.
\]

Therefore, eq. (11) is equivalent to \( x = y \) and \( 2^x - z \cdot 3^k = 3^k + 1 \). Since some non-zero power of 2 is congruent to 1 modulo \( 3^k \), eq. (11) has a solution. Moreover, any solution must satisfy \( 2^x \equiv 1 \mod 3^k \). Since 2 is a primitive root modulo \( 3^k \), \( x \) must be a multiple of \( |(\mathbb{Z}/3^k\mathbb{Z})^*| = \varphi(3^k) = 2 \cdot 3^{k-1} \) (here, \( \varphi \) is Euler’s phi-function). Moreover, \( x \) must be non-zero, because \( 1 - z \cdot 3^k = 3^k + 1 \) is not possible for \( z \in \mathbb{N} \). We obtain \( x \geq 2 \cdot 3^{k-1} \). Since \( 2^x - z \cdot 3^k = 3^k + 1 \), this yields \( z = (2^x - 3^k - 1)/3^k \geq (2^{2 \cdot 3^{k-1} - 1})/3^k - 1 \). \( \square \)
Remark 4.3. Subject to Artin’s conjecture on primitive roots [26], a similar doubly-
exponential lower bound results for every BS(1, q) where q ≥ 2 is not a perfect
square. Moreover, Theorem 4.2 holds even if the variables x, y, z range over Z. For
this, one replaces 3^k + 1 with the inverse of 2 in (Z/3^kZ)^* in eq. (11).

Theorem 4.2 rules out a simple guess-and-verify strategy to show Theorem 4.1. If one has an exponential upper bound (in terms of input length) on the size of a
smallest solution of a knapsack equation, then one can guess the binary representa-
tion of a solution and verify, using the power word problem, whether the guess is
indeed a solution. The second step (verification of a solution using the power word
problem) would work for BS(1, q) in polynomial time due to Theorem 3.1, but the
first step (guessing a binary encoded candidate for a solution) does not work for
BS(1, 2) due to Theorem 4.2.

Our main tool for the proof of Theorem 4.1 is a recent result from [21] concerning
the existential fragment of B"uchi arithmetic.

4.1. B"uchi arithmetic. B"uchi arithmetic [10] is the first-order theory of the struc-
ture (Z, +, ≥, 0, V_q). Here, V_q is the function that maps n ∈ Z to the largest power
of q that divides n. It is well-known that B"uchi arithmetic is decidable (this was
first claimed in [10]; a correct proof was given in [8]). We will rely on the follow-

Theorem 4.4 (c.f. [21]). The existential fragment of B"uchi arithmetic belongs to
NP. 4

We will also make use of the following simple lemma:

Lemma 4.5. Given the q-ary representation of a number r ∈ Z[1/q], we can con-
struct in polynomial time an existential Presburger formula over (Z, +) of size
O(∥r∥_q) which expresses y = r · x for x, y ∈ Z.

Proof. Let r = ∑_{k≤ i ≤ ℓ} a_i q^i with k, ℓ ≥ 0 and 0 ≤ a_i < q for −k ≤ i ≤ ℓ. We
have y = rx if and only if q^k y = r' x for r' = ∑_{i=0}^{k+ℓ} a_i q^i ∈ Z. Using iterated
multiplication with the constant q (which can be replaced by addition) we can
easily define from x and y the integers q^k y and r' x by Presburger formulas of size
O(k + ℓ) = O(∥r∥_q).

4The paper [21] shows an NP upper bound for the structure (N, +, 0, V_q), but an existential
sentence over the structure (Z, +, ≥, 0, V_q) easily translates into one over (N, +, 0, V_q).

4.2. Proof of Theorem 4.1. We start with the lower bound. The multisubset
sum problem asks for integers a_1, . . . , a_d, b ∈ Z given in binary, whether there exist
natural numbers x_1, . . . , x_d ≥ 0 with x_1 a_1 + · · · + x_d a_d = b. It is known to be
NP-complete [24]. Since the knapsack equation
\[
\begin{pmatrix}
1 & a_1 \\
0 & 1
\end{pmatrix}^{x_1} \cdots \begin{pmatrix}
1 & a_d \\
0 & 1
\end{pmatrix}^{x_d} = \begin{pmatrix}
1 & b \\
0 & 1
\end{pmatrix}
\]
is equivalent to x_1 a_1 + · · · + x_d a_d = b, we obtain NP-hardness of knapsack over
BS(1, q). Note that computing the q-ary representation of a_i from the binary repre-
sentation is possible in logspace (even in TC^0).

For the upper bound we reduce Knapsack(BS(1, q)) to the existential fragment
of B"uchi arithmetic, which belongs to NP by Theorem 4.4. We proceed in three
steps.
Step 1: Expressing $M_q$ and $M_q^*$ using $S_\ell$. We first express a particular set of binary relations using existential first-order formulas over $(\mathbb{Z}, +, \geq, 0, V_q, (S_\ell)_{\ell \in \mathbb{Z}})$. Here, for $\ell \in \mathbb{Z}$, $S_\ell$ is the binary predicate with

$$x S_\ell y \iff \exists r \in \mathbb{N} \exists s \in \mathbb{N}: x = q^r \land y = q^{r+\ell s}.$$ 

Let $T_Z(q)$ denote the subset of matrices in $T(q)$ that have entries in $\mathbb{Z}$. We represent the matrix $\left( \begin{array}{c} m \\ n \end{array} \right) \in T_Z(q)$ by the pair $(m, n) \in \mathbb{Z} \times \mathbb{Z}$ (note that we must have $m \in \mathbb{N}$). Observe that we can define in the structure $(\mathbb{Z}, +, \geq, 0, V_q, (S_\ell)_{\ell \in \mathbb{Z}})$ the set of pairs $(m, n) \in \mathbb{Z}$ such that $\left( \begin{array}{c} m \\ n \end{array} \right) \in T_Z(q)$, because this is equivalent to $m$ being a power of $q$, which is expressed by $1.S_1 m$.

A key trick is to express solvability of a knapsack equation $g_1^{x_1} \cdots g_d^{x_d} = g$ without introducing variables in the logic for $x_1, \ldots, x_d$. Instead, we employ the following binary relations $M_q$ and $M_q^*$ on $T_Z(q)$, which allow us to express existence of powers implicitly. For $g \in T(q)$ and $x, y \in T_Z(q)$, we have:

- $x M_q y \iff y = x g$,
- $x M_q^* y \iff \exists s \in \mathbb{N}: y = x g^s$.

We construct existential formulas of size polynomial in $||g||$ over the structure $(\mathbb{Z}, +, \geq, 0, V_q, (S_\ell)_{\ell \in \mathbb{Z}})$, which define the relations $M_q$ and $M_q^*$. For the further consideration let

$$g = \left( \begin{array}{c} q^x \\ 0 \\ 1 \end{array} \right) .$$

Note that the relation $M_q$ is easily expressible because we can express multiplication with $q^x$ and $v$ by existential Presburger formulas of length $||g||$, see Lemma 4.5.

We now focus on the relations $M_q^*$ and express

$$\left( \begin{array}{c} q^k \\ 0 \\ 1 \end{array} \right) M_q^* \left( \begin{array}{c} q^m \\ 0 \\ 1 \end{array} \right) $$

Observe that for $\ell \neq 0$, we have

$$\left( \begin{array}{c} q^k \\ 0 \\ 1 \end{array} \right) \left( \begin{array}{c} q^\ell \\ 0 \\ 1 \end{array} \right)^s = \left( \begin{array}{c} q^k \\ 0 \\ 1 \end{array} \right) \left( \begin{array}{c} q^{s\ell} \\ 0 \\ 1 \end{array} \right) = \left( \begin{array}{c} q^{k+s\ell} \\ 0 \\ 1 \end{array} \right) .$$

Therefore, eq. (12) is equivalent to

$$\exists x \in \mathbb{Z} \exists s \in \mathbb{N}: q^m = q^{k+s\ell} \land w = u + vx \land (q^\ell - 1)x = q^m - q^k.$$ 

Here, we can quantify $x$ over $\mathbb{Z}$, because

$$\frac{q^{k+s\ell} - q^k}{q^\ell - 1} = q^k + q^{k+\ell} + \cdots + q^{k+(s-1)\ell}$$

must be an integer ($k$ and $k + s\ell = m$ are non-negative). Note that since we can express multiplication with $v$ and $q^\ell$ by existential Presburger formulas of size $O(||g||)$ (Lemma 4.5), we can also express $w = u + vx$ and $(q^\ell - 1)x = q^m - q^k$ by existential Presburger formulas of size $O(||g||)$. Finally, we can express $\exists s \in \mathbb{N}: q^m = q^{k+s\ell}$ using $q^\ell S_\ell q^m$.

It remains to express eq. (12) in the case $\ell = 0$. Note that

$$g^s = \left( \begin{array}{c} 1 \\ sv \\ 0 \\ 1 \end{array} \right)$$

in this case. Therefore, eq. (12) is equivalent to

(i) there exists $s \in \mathbb{N}$ with $w = u + q^k \cdot s \cdot v$ and
(ii) $q^m = q^k$. 

Note that condition (i) is equivalent to \( \exists t \in \mathbb{N}; V_q(t) \geq q^k \land w = u + v \cdot t \). This is because choosing \( t = q^k \cdot s \) yields (i). By Lemma \[13.5\] \( w = u + v \cdot t \) can be expressed by an existential Presburger formula of size \( O(||g||) \).

**Step 2:** Expressing \( S_\ell \) using \( V_q \). In our second step, we show that the binary relations \( M_q \) and \( M_q^* \) can be expressed using existential formulas over \((\mathbb{Z}, +, \geq, 0, V_q)\) of size \( \text{poly}(||g||) \). As shown above, for this it suffices to define \( S_\ell \) by an existential formula over \((\mathbb{Z}, +, \geq, 0, V_q)\) of size \( \text{poly}(\ell) \) (note that the relations \( S_\ell \) occur only positively in the formulas from Step 1). For \( m \in \mathbb{N} \), let \( P_m \) be the predicate where \( P_m(x) \) states that \( x \) is a power of \( m \). We first claim that for each \( \ell \geq 0 \), we can express \( P_q \) using an existential formula of size polynomial in \( \ell \) over \((\mathbb{Z}, +, \geq, 0, V_q)\). The case \( \ell = 0 \) is clear. For the case \( \ell \geq 1 \) we use the following observation from the proof of Proposition 7.1 in [9]. Note that \( P_q(x) \) is just \( V_q(x) = x \).

**Fact 4.6.** For all \( \ell \geq 1 \), \( P_q(x) \) if and only if \( P_q(x) \) and \( q^\ell - 1 \) divides \( x - 1 \).

Proof. If \( x \) is a power of \( q^\ell \), then \( x = q^{\ell \cdot s} \) for some \( s \geq 0 \). So, \( x \) is a power of \( q \). Moreover,

\[
\frac{x - 1}{q^\ell - 1} = \frac{q^{\ell \cdot s} - 1}{q^\ell - 1} = \sum_{i=0}^{s-1} q^{\ell \cdot i}
\]

is an integer.

Conversely, suppose \( x \) is a power of \( q \) and \( q^\ell - 1 \) divides \( x - 1 \). Write \( x = q^{\ell \cdot s + r} \) with \( 0 \leq r < \ell \). Observe that

\[
x - 1 = q^{\ell \cdot s + r} - 1 = q^r (q^\ell - 1) + (q^r - 1).
\]

Since \( q^\ell - 1 \) divides \( x - 1 \) as well as \( q^r - 1 \), we conclude that \( q^\ell - 1 \) divides \( q^r - 1 \). As \( 0 \leq r < \ell \), this is only possible with \( r = 0 \). This shows the above fact. \( \square \)

Using the predicates \( P_q \), we can now express \( S_\ell \). Note that for \( \ell \geq 0 \), we have \( x S_\ell y \) if and only if

\[
y \geq x \land \bigvee_{i=0}^{\ell-1} P_q(q^i x) \land P_q(q^i y).
\]

Furthermore, for \( \ell < 0 \), we have \( x S_\ell y \) if and only if \( y S_{|\ell|} x \). Therefore, we can express each \( S_\ell \) using an existential formula of size polynomial in \( \ell \) over \((\mathbb{Z}, +, \geq, 0, V_q)\).

Hence, we can express \( M_q \) and \( M_q^* \) using existential formulas of size \( \text{poly}(||g||) \) over \((\mathbb{Z}, +, \geq, 0, V_q)\).

**Step 3:** Expressing solvability of knapsack. In the last step, we express solvability of a knapsack equation by an existential first-order sentence over \((\mathbb{Z}, +, \geq, 0, V_q)\), using the predicates \( M_q \) and \( M_q^* \). We claim that \( g_{q^1} \cdots g_{q^d} = g \) has a solution \((x_1, \ldots, x_d) \in \mathbb{N}^d \) if and only if there exist \( h_0, \ldots, h_d \in T_2(q) \) with

\[
h_0 M_q^* h_1 \land h_1 M_q^* h_2 \land \cdots \land h_{d-1} M_q^* h_d \land h_0 M_q h_d.
\]

This can be stated by an existential sentence over \((\mathbb{Z}, +, \geq, 0, V_q)\) of size polynomial in \( ||g|| + \sum_{i=1}^d ||g_i|| \).

If such \( h_0, \ldots, h_d \) exist, then for some \( x_1, \ldots, x_d \in \mathbb{N} \), we have \( h_i = h_{i-1} g_{q^i}^{-1} \) for all \( i \in [1, d] \) and \( h_d = h_0 g \), which implies \( g_{q^1}^{-1} \cdots g_{q^d}^{-1} = g \). For the converse, we observe that for each matrix \( A \in T(q) \), there is some large enough \( k \in \mathbb{N} \) such that \((q^k 0) k A \in T_2(q) \). Therefore, if \( g_{q^1}^{-1} \cdots g_{q^d}^{-1} = g \), then there is some large enough \( k \in \mathbb{N} \) so that for every \( i \in [1, d] \), the matrix \((q^k 0) g_{q^i}^{-1} \cdots g_{q^d}^{-1} \) has integer entries.

With this, we set \( h_0 = \left( q^k 0 \right) \) and \( h_i = h_{i-1} g_{q^i}^{-1} \) for \( i \in [1, d] \). Then we have \( h_0, \ldots, h_d \in T_2(q) \) and eq. \[13\] is satisfied. \( \square \)
5. Systems of exponent equations over BS(1,q)

Our algorithm for the knapsack problem in BS(1,q) cannot be extended to solvability of systems of exponent equations (not even to solvability of a single exponent equation). If we allow systems of exponent equations, we can show undecidability:

**Theorem 5.1.** For every $q \in \mathbb{N}$ with $q \geq 2$, $\text{ExpEq}(\text{BS}(1,q))$ is undecidable.

**Proof.** Consider the function $(x, y) \mapsto x \cdot 2^y$ on the natural numbers. Büchi and Senger [11, Corollary 5] have shown that the existential fragment of the first-order theory of $(\mathbb{N}, +, x \cdot 2^y)$ is undecidable. The proof generalizes to every function $(x, y) \mapsto x \cdot q^y$ for $q \in \mathbb{N}, q \geq 2$. We reduce this fragment to $\text{ExpEq}(\text{BS}(1,q))$. For this it suffices to consider an existentially quantified conjunction of formulas of the following form: $x \cdot q^y = z$, $x + y = z$, and $x < y$ (the latter allow to express inequalities). We replace each of these formulas by an equivalent exponent equation over BS(1,q).

For this it suffices to consider an existentially quantified conjunction of formulas of the following form:

- $x \cdot q^y = z$,
- $x + y = z$,
- $x < y$ (the latter allow to express inequalities).

We replace each of these formulas by an equivalent exponent equation over BS(1,q). For this we use the two generators $a$ and $t$ from (5) (for $\alpha = q$).

The formula $x + y = z$ is clearly equivalent to $a^x a^y = a^z$, i.e., $a^x a^y a^{-z} = 1$. The formula $x < y$ is equivalent $\exists z \in \mathbb{N}: a^x a^z a a^{-y} = 1$. Finally, $x \cdot q^y = z$ is equivalent to $t^y a^x t^{-y} a^{-z} = 1$. □

### 6. Open problems

Several open problems arise from our work:

- What is the complexity/decidability status of the power word/knapsack problem for Baumslag-Solitar groups $\text{BS}(p,q) = \langle a,t \mid ta^p t^{-1} = a^q \rangle$ for $p, q \geq 2$? Decidability of knapsack in case $\gcd(p,q) = 1$ was shown in [13], but the complexity as well as the decidability in case $\gcd(p,q) > 1$ are open. Since the word problem for BS(p,q) can be solved in logspace [44], one can easily show that the power word problem for BS(p,q) belongs to PSPACE. By using techniques from [34] one might try to find a logspace reduction from the power word problem for BS(p,q) to the word problem for BS(p,q) (the same was done for a free group in [37]); this would show that the power word problem for BS(p,q) can be solved in logspace.

- Baumslag-Solitar groups BS(1,q) are examples of f.g. solvable linear groups. In [28] it was shown that for every f.g. solvable linear group the word problem can be solved in $\text{TC}^0$. This leads to the question whether for every f.g. solvable linear group the word problem belongs to $\text{TC}^0$.

- The power word problem is a restriction of the compressed word problem, where it is asked whether the word produced by a so-called straight-line program (a context-free grammar that produces a single word) represents the group identity; see [32]. The compressed word problem for BS(1,q) belongs to $\text{coRP}$ (the complement of randomized polynomial time); this holds in fact for every f.g. linear group [32]. No better complexity bound is known for the compressed word problem for BS(1,q).

- Is the knapsack problem decidable for every matrix group $T(\alpha)$ with $\alpha \in \mathbb{C} \setminus \{0\}$?

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