Geodesic dual spacetime

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\textbf{ABSTRACT:} A duality between spacetime manifolds, the geodesic duality, is introduced. Two manifolds are geodesic dual, if the transformation between their metrics is also the transformation between their geodesics. That is, the transformation that transforms the metric to the metric of the dual manifold is also the transformation that transforms the geodesic to the geodesic of the dual manifold. On the contrary, for nondual spacetime manifolds, a geodesic is no longer a geodesic after the transformation between the metrics. We give a general result of the duality between spacetime manifolds with diagonal metrics. The geodesic duality of spherically symmetric spacetime are discussed for illustrating the concept. The geodesic dual spacetime of the Schwarzschild spacetime and the geodesic dual spacetime of the Reissner-Nordström spacetime are presented.

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1 Introduction

Geodesics, the trajectory of a free particle, reflects the geometry of spacetime. Generally speaking, a transformation between the metrics of two spacetimes cannot transform the geodesic of one spacetime to the geodesic of another spacetime. In this paper, we show that there exist spacetimes whose geodesics can be transformed to each other by the transformation between the metrics of spacetimes. This is a duality between spacetimes, the geodesic duality.

By spacetime with diagonal metrics, we illustrate the concept of the geodesic duality. The spherically symmetric spacetime is generally discussed. The dual spacetime of the Schwarzschild spacetime and dual spacetime of the Reissner-Nordström spacetime are presented.

Duality bridges superficially different things. An important duality is the AdS/CFT duality [1–5]. There exists dualities between spacetime manifolds and fluids, i.e., the gravity/Fluid dual [6, 7], such as the duality between $d+2$-dimensional Ricci-flat metrics and $d+1$-dimensional fluids [8, 9], the geometrical duality of both Brans-Dicke theory and general relativity [10], the duality between the nonlinear equations of boundary fluid and gravity theory [11, 12], the duality between the conformal Navier Stokes equation and the long wavelength solution of gravity [13–15], the duality between $d+2$-dimensional Ricci-flat metrics and $d+1$-dimensional relativistic fluids [16], a dual relation of $d+2$-dimensional metrics corresponding to $d+1$-dimensional fluids [17], and the fluid/gravity dual in spacetime with general non-rotating weakly isolated horizons [18]. The gravoelectric duality decomposes the Riemann curvature into electric and magnetic parts and introduces the gravoelectric
duality transformation by interchange of active and passive electric parts which amounts to interchange of the Ricci and Einstein tensors. The gravoelectric duality can be used to find the solution of the Einstein equation \cite{19-22}. Invariances under the gravoelectric duality are also considered \cite{23}. There are various dualities between physical systems, such as the duality between the two-dimensional Ising and planar Heisenberg models to gauge theories in four dimensions \cite{24}, the duality between the $SU(2)$ Higgs-Kibble model and the relativistic hydrodynamics of Freedman coupled to Higgs scalars \cite{25}, the quantum equivalence of dual field theories \cite{26}, the duality in many-component Ising models in two dimensions on a square lattice \cite{27}.

In section 2, we define the concept of the geodesic duality. In section 3, we give a general discussion on the spherically symmetric geodesic dual spacetime. In sections 4 and 5, we give the dual spacetime of the Schwarzschild spacetime and the Reissner-Nordström spacetime.

# 2 Geodesic duality

The geodesic duality is defined as follows.

*Two manifolds are geodesic dual, if the transformation between their metrics is also the transformation between their geodesics.*

We illustrate the concept of the geodesic duality through examples of spacetime manifolds with diagonal metrics.

A geodesic on an $n+1$-dimensional manifold is determined by $n+1$ geodesic equations. A geodesic on a manifold, after a transformation between the metrics of two manifolds, will be transformed to a curve on the other manifold. If these two manifolds are geodesic dual, the curve will still be a geodesic.

In the following, we consider the geodesic duality between two spacetime manifolds with diagonal metrics. The idea applies also to more general cases.

$n+1$ geodesic equations, which determine an $n+1$-dimensional geodesic, have $n+1$ first integrals, say $Q_0, Q_1, \ldots, Q_n$. Of its $n+1$ first integrals, suppose that $n$ first integrals, $Q_0, \ldots, Q_{\eta-1}, Q_{\eta+1}, \ldots, Q_n$, are known and one first integral $Q_\eta$ is left unknown. For a diagonal metric

\[
ds^2 = g_{\mu\nu} (dx^\mu)^2,
\]

the geodesic is given by

\[
\frac{dx^\lambda}{dx^\eta} = \frac{1/g_{\lambda\lambda}}{\sqrt{\frac{1}{g_{\eta\eta} Q_\lambda^2} - \frac{1}{g_{\eta\eta}} \left( \frac{Q_0^2}{Q_\lambda^2} \frac{1}{g_{00}} + \cdots + \frac{Q_{\eta-1}^2}{Q_\lambda^2} \frac{1}{g_{\eta-1\eta-1}} + \frac{Q_{\eta+1}^2}{Q_\lambda^2} \frac{1}{g_{\eta+1\eta+1}} + \cdots + \frac{1}{g_{\lambda\lambda}} + \cdots + \frac{Q_n^2}{Q_\lambda^2} \frac{1}{g_{nn}} \right)}}
\]

where $\lambda \neq \eta$.

Geodesic duality means that after a duality transformation, Eq. (2.2), the geodesic equation of the manifold with the metric (2.1), must also be a geodesic equation of its geodesic dual manifold. The geodesic equation on the dual manifold is still of the form of
Eq. (2.2). To obtain the geodesic equation of the dual manifold, we exchange the term \( \frac{1}{g_{\lambda\lambda}} \) with another term in the square root of Eq. (2.2):

\[
\frac{Q_\lambda^2}{Q^2_{\lambda}} \frac{1}{g_{aa}} \leftrightarrow \frac{1}{g_{\lambda\lambda}}.
\]

(2.3)

We then arrive at a new equation. Requiring the new equation to be a geodesic equation described by the metric \( \tilde{g}_{\mu\nu} \) gives a transformation

\[
x^\mu = \chi^\mu (y^\nu),
\]

(2.4)

\[
Q_\lambda = \Omega_\lambda (q_\kappa)
\]

(2.5)

determined by

\[
\frac{\Omega_\mu^2 (q_\kappa)}{\Omega_\nu^2 (q_\kappa) g_{aa} (\chi^\mu (y^\nu))} = \frac{1}{g_{\lambda\lambda} (y^{\nu})}.
\]

(2.6)

where \( y^{\nu} \) is the coordinate of the manifold described by \( \tilde{g}_{\mu\nu} \), and \( q_\kappa \) is the first integral of the new geodesic equation. Rewriting the new geodesic equation in the form of the geodesic equation (2.2):

\[
dy^\nu = \frac{1}{\sqrt{g_{\lambda\lambda} (y^{\nu})}} d\tilde{g}_{\mu\nu} (dy^\mu)^2.
\]

(2.7)

where \( \lambda \neq \eta \), we can read out the metric of the dual manifold, \( \tilde{g}_{\mu\nu} \), directly. It can be found from Eq. (2.7) that the metric of the dual manifold, \( \tilde{g}_{\mu\nu} \), is also diagonal, i.e.,

\[
d\sigma^2 = \tilde{g}_{\mu\nu} (dy^\mu)^2.
\]

(2.8)

The manifold with the metric \( \tilde{g}_{\mu\nu} \) is the geodesic duality of the manifold with the metric \( g_{\mu\nu} \).

In the following, we will illustrate how to solve the duality transformations (2.4) and (2.5) through examples.

It should be emphasized that an \( n+1 \)-dimensional spacetime has \( n-1 \) geodesic dual spacetimes. This is because the number of the term \( \frac{Q_\lambda^2}{Q^2_{\lambda}} \frac{1}{g_{aa}} \) is \( n-1 \), and then there are \( n-1 \) ways to exchange the term \( \frac{1}{g_{\lambda\lambda}} \) with the terms \( \frac{Q_\lambda^2}{Q^2_{\lambda}} \frac{1}{g_{aa}} \).

3 Geodesic duality: spherically symmetric spacetime

In this section, we consider spherically symmetric spacetime.

3.1 Duality transformation

The metric of a spherically symmetric spacetime is of the form

\[
ds^2 = g_{00} (x^1) (dx^0)^2 + g_{11} (x^1) (dx^1)^2 + (x^1)^2 \left[ (dx^2)^2 + \sin^2 x^2 (dx^3)^2 \right].
\]

(3.1)
In the spherical coordinate $x^0 = t$, $x^1 = r$, $x^2 = \theta$, $x^3 = \varphi$ and then $g_{00} = -f(r)$, $g_{11} = g(r)$, $g_{22} = r^2$, and $g_{33} = r^2 \sin^2 \theta$ with $f(r)$ and $g(r)$ the functions of $r$.

The geodesic equation on spherically symmetric spacetime, by Eq. (2.2), is

$$
\frac{dx^3}{dx^1} = \frac{1}{g_{33} \sqrt{\frac{1}{g_{11} g_{33}} - \frac{1}{g_{11}} \left( \frac{Q_0^2}{Q_3^2} \frac{1}{g_{00}} + \frac{1}{g_{33}} \right)}}.
$$

(3.2)

In spherically symmetric cases the geodesic orbit is in a plane, so we can choose $x^2 = \pi/2$, then, $g_{33} = (x^1)^2$ and the first integral $Q_2 = 0$ which is the $x^2$-component of the angular momentum.

In this case, the only possible exchange in Eq. (2.3) is

$$
\frac{Q_0^2}{Q_3^2} \frac{1}{g_{00}} \Leftrightarrow \frac{1}{g_{33}}.
$$

(3.3)

In spherically symmetric cases, only the radial coordinate $x^1 = r$ needs to be considered, so there is only one coordinate transformation

$$
x^1 = \chi^1 (y^1).
$$

(3.4)

The duality relation is determined by

$$
\frac{Q_0^2}{Q_3^2} \frac{1}{g_{00} (\chi (y^1))} = \frac{1}{(y^1)^2},
$$

(3.5)

so the duality relation (2.6) is

$$
x^1 \rightarrow \chi^1 (y^1) = g_{00}^{-1} \left( \frac{Q_0^2}{Q_3^2} (y^1)^2 \right).
$$

(3.6)

Note that the duality relation of the angular coordinate is trivial: $x^3 \rightarrow \beta y^3$.

Now consider the transformation of the angular coordinate. Suppose the transformation is

$$
x^3 \rightarrow \chi^3 (y^1) y^3 (y^1).
$$

(3.7)

Substituting Eq. (3.7) into the geodesic equation (3.2) gives

$$
\frac{dy^3 (y^1)}{dx^1} = \frac{1}{\chi^3 (y^1) g_{33} \sqrt{\frac{1}{g_{11} g_{33}} - \frac{1}{g_{11}} \left( \frac{Q_0^2}{Q_3^2} \frac{1}{g_{00}} + \frac{1}{g_{33}} \right)}} - \frac{y^3 (y^1) d\chi^3 (y^1)}{\chi^3 (y^1) dx^1}.
$$

(3.8)

If Eq. (3.8) is still a geodesic equation, we must have

$$
\frac{y^3 (y^1)}{\chi^3 (y^1)} \frac{d\chi^3 (y^1)}{dx^1} = 0,
$$

(3.9)

so

$$
\chi^3 (y^1) = \beta
$$

(3.10)
with $\beta$ a constant.

The geodesic equation on the dual spacetime then can be obtained by submitting the duality relation (3.6) into the geodesic equation (3.2):

$$
\frac{dy^3}{dy^1} = \frac{1}{(y^1)^2} \left\{ \frac{1}{g_{11}} \left( g_{00}^{-1} \left( \frac{Q_0^2}{Q_3^2} (y^1)^2 \right) \right) \right\} \left\{ \frac{1}{(y^1)^2} \left[ g_{00}^{-1} \left( \frac{Q_0^2}{Q_3^2} (y^1)^2 \right) \right] \right\}^2 \frac{1}{Q_3^2} \\
- \frac{1}{g_{11}} \left( g_{00}^{-1} \left( \frac{Q_0^2}{Q_3^2} (y^1)^2 \right) \right) \left\{ \frac{1}{(y^1)^2} \left[ g_{00}^{-1} \left( \frac{Q_0^2}{Q_3^2} (y^1)^2 \right) \right] \right\}^2 \left\{ \frac{1}{\alpha g_{00}^{-1} \left( \frac{Q_0^2}{Q_3^2} (y^1)^2 \right)} \right\}^{2\alpha^2 + \frac{1}{(y^1)^2}} \right\}^{-1/2}.
$$

(3.11)

Rewriting the new equation to the standard form of the geodesic equation given by Eq. (2.2),

$$
\frac{dy^3}{dy^1} = \frac{1}{\sqrt{\frac{1}{g_{11}g_{33}} - \frac{1}{g_{11}} \left( \frac{q_0^2}{q_3^2 g_{00}} + \frac{1}{g_{33}} \right)}},
$$

(3.12)

we can read out the metric of the dual spacetime:

$$
\tilde{g}_{00} (y^1) = \left[ \alpha g_{00}^{-1} \left( \frac{Q_0^2}{Q_3^2} (y^1)^2 \right) \right]^2,
$$

(3.13)

$$
\tilde{g}_{11} (y^1) = g_{11} \left( g_{00}^{-1} \left( \frac{Q_0^2}{Q_3^2} (y^1)^2 \right) \right) \left\{ \frac{1}{(y^1)^2} \left[ g_{00}^{-1} \left( \frac{Q_0^2}{Q_3^2} (y^1)^2 \right) \right] \right\}^2 \left\{ \beta \left[ g_{00}^{-1} \left( \frac{Q_0^2}{Q_3^2} (y^1)^2 \right) \right] \right\}^{2\alpha^2 + \frac{1}{(y^1)^2}}.
$$

(3.14)

$$
q_3 = Q_3, \quad \frac{q_0}{q_3} = \alpha,
$$

(3.15)

(3.16)

where we choose $y^2 = \pi/2$, then $\tilde{g}_{33} = (y^1)^2$, the first integral $q_2 = 0$ which is the $y^2$-component of the angular momentum, and $\alpha$ is a constant. Then the metric of the dual manifold is

$$
d\sigma^2 = \tilde{g}_{00} (y^1) (dy^0)^2 + \tilde{g}_{11} (y^1) (dy^1)^2 + (y^1)^2 \left[ (dy^2)^2 + \sin^2 y^2 (dy^3)^2 \right],
$$

(3.17)

which is also spherically symmetric.

We might as well choose the constant

$$
\alpha = \frac{Q_0}{Q_3},
$$

(3.18)

so that

$$
q_0 = Q_0, \quad q_3 = Q_3.
$$

(3.19)

(3.20)
Consequently, the duality relation between two geodesics on dual spacetimes can be summarized as

\[ Q_3 \frac{g_{00}(x^1)}{Q_0} = q_0 \tilde{g}_{33}(y^1) \]  
\[ y^3 = \beta x^3. \]  

The geodesic on the spacetime (3.17), Eq. (3.12), can be achieved directly from the geodesic on its geodesic dual spacetime (3.1), Eq. (3.2), through the duality transformation (3.21)-(3.22). This result can be verified directly by substituting the duality transforms (3.21)-(3.22) into the geodesic equation.

3.2 Inverse duality transformation

In addition, the inverse transformations of the duality transformations (3.21) and (3.22) are

\[ y^1 \rightarrow \tilde{g}_{00}^{-1} \left[ \frac{q_0^2}{q_3^2} (x^1)^2 \right], \]  
\[ y^3 \rightarrow \frac{1}{\beta} x^3. \]  

By the transformations (3.23) and (3.24), Eqs. (3.13) and (3.14) become

\[ g_{00}(x^1) = \left[ \alpha \tilde{g}_{00}^{-1} \left( \frac{q_0^2}{q_3^2} (x^1)^2 \right) \right]^2, \]  
\[ g_{11}(x^1) = \tilde{g}_{11} \left( \tilde{g}_{00}^{-1} \left( \frac{q_0^2}{q_3^2} (x^1)^2 \right) \right) \left( \frac{(x^1)^2}{\beta} \tilde{g}_{00}^{-1} \left( \frac{q_0^2}{q_3^2} (x^1)^2 \right) \right)^2 \left( \frac{1}{\tilde{g}_{00}^{-1} \left( \frac{q_0^2}{q_3^2} (x^1)^2 \right)} \right)^2, \]
\[ Q_3 = q_3, \]  
\[ Q_0 \frac{Q_3}{Q_0} = \alpha. \]

Then the geodesic equation (3.12) returns to the geodesic equation (3.2).

3.3 Duality

To sum up, the duality relation between two geodesic dual spacetimes is

\[ Q_3 \frac{g_{00}(x^1)}{Q_0} = q_0 \tilde{g}_{33}(y^1), \]  
\[ q_3 \tilde{g}_{00}(y^1) = Q_0 g_{33}(x^1), \]  
\[ \frac{g_{11}(x^1)}{(x^1)^4 \tilde{g}_{00}^{-1} \left( \frac{q_0^2}{q_3^2} (x^1)^2 \right)} = \beta^2 \frac{\tilde{g}_{11}(y^1)}{(y^1)^4 \tilde{g}_{00}^{-1} \left( \frac{q_0^2}{q_3^2} (y^1)^2 \right)}. \]

This duality transformation transforms both the metric and the geodesic to that of the dual spacetime.
3.4 Spherical coordinate form

As a complement, we rewrite the above result with explicit spherical coordinates.

The metric (3.1) and the geodesic equation (3.2) under the spherical coordinate, \( x^0 = t, \ x^1 = r, \ x^2 = \theta, \ x^3 = \varphi, \) are

\[
\begin{align*}
  ds^2 &= -f(r) \, dt^2 + g(r) \, dr^2 + r^2 \left( d\theta^2 + \sin^2 \theta \, d\varphi^2 \right) \quad (3.32) \\
  \frac{d\varphi}{dr} &= \frac{1}{r^2} \frac{1}{\sqrt{b^2 f(r)g(r)} - \frac{1}{g(r)} \left( \frac{1}{a^2} + \frac{1}{r^2} \right)}, \quad (3.33)
\end{align*}
\]

where \( b^2 = -Q_3^2/Q_0^2 = L^2/E^2 \) and \( a^2 = -Q_3^2 = L^2/m^2 \) with \( L \) the angular momentum and \( E \) the energy.

For the dual spacetime, the metric (3.17) and the geodesic equation (3.12) under the spherical coordinate \( y^0 = \tau, \ y^1 = \rho, \ y^2 = \vartheta, \ y^3 = \phi, \) become

\[
\begin{align*}
  d\sigma^2 &= -F(\rho) \, d\tau^2 + G(\rho) \, d\rho^2 + \rho^2 \left( d\vartheta^2 + \sin^2 \vartheta \, d\phi^2 \right) \quad (3.34) \\
  \frac{d\phi}{d\rho} &= \frac{1}{\rho^2} \frac{1}{\sqrt{B^2 F(\rho)G(\rho)} - \frac{1}{G(\rho)} \left( \frac{1}{A^2} + \frac{1}{\rho^2} \right)}, \quad (3.35)
\end{align*}
\]

where \( B^2 = -q_3^2/q_0^2 = \ell^2/E^2 \) and \( A^2 = -q_3^2 = \ell^2/m^2 \) with \( \ell \) the angular momentum and \( E \) the energy.

The duality relation here is

\[
\begin{align*}
  bf(r) &= -\frac{\rho^2}{B}, \quad (3.36) \\
  -\frac{\rho^2}{b} &= BF(\rho), \quad (3.37) \\
  \frac{g(r)}{r^4 \left[ F^{-1}\left( -\frac{\rho^2}{B^2} \right) \right]'} &= \beta^2 \frac{G(\rho)}{\rho^4 \left[ f^{-1}\left( -\frac{\rho^2}{b^2} \right) \right]'}, \quad (3.38)
\end{align*}
\]

4 Geodesic dual spacetime of Schwarzschild spacetime

For the Schwarzschild spacetime

\[
\begin{align*}
  ds^2 &= - \left(1 - \frac{2M}{r} \right) \, dt^2 + \frac{1}{1 - \frac{2M}{r}} \, dr^2 + r^2 \left( d\theta^2 + r^2 \sin^2 \theta \, d\varphi^2 \right) \quad (4.1)
\end{align*}
\]

the duality transformations (3.6) and (3.10) becomes

\[
\begin{align*}
  r &\rightarrow \frac{2b^2 M}{b^2 + \rho^2}, \quad (4.2) \\
  \varphi &\rightarrow \beta \varphi. \quad (4.3)
\end{align*}
\]
Then the metric of the geodesic dual spacetime of the Schwarzschild spacetime is
\[
\begin{align*}
\mathrm{d}\sigma^2 &= -\left(\frac{2bM}{b^2 + \rho^2}\right)^2 \mathrm{d}r^2 + \left(\frac{\rho^2}{\beta bM}\right)^2 \mathrm{d}\rho^2 - \rho^2 \left(\mathrm{d}\theta^2 + \sin^2 \theta \mathrm{d}\phi^2\right).
\end{align*}
\] (4.4)
The geodesic equations is
\[
\begin{align*}
\frac{d\varphi}{dr} &= \frac{1}{r^2} \sqrt{\frac{1}{b^2} - \left(1 - \frac{2M}{r}\right) \left(\frac{1}{b^2} + \frac{1}{r^2}\right)}; \\
\frac{d\phi}{d\rho} &= \frac{1}{\rho^2} \sqrt{\frac{1}{b^2} \left(\frac{2bM}{b^2 + \rho^2}\right)^2 - \left(\frac{1}{b^2 M + \rho^2}\right)^2 \left(\frac{1}{b^2} + \frac{1}{\rho^2}\right)}. 
\end{align*}
\] (4.5) (4.6)

It can be checked that Eqs. (4.1) and (4.4) satisfy the duality relation (3.36)-(3.38).

5 Geodesic dual spacetime of Reissner-Nordström spacetime

For the Reissner-Nordström spacetime
\[
\begin{align*}
\mathrm{d}\tau^2 &= -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) \mathrm{d}t^2 + \frac{1}{1 - \frac{2M}{r} + \frac{Q^2}{r^2}} \mathrm{d}r^2 + r^2 \mathrm{d}\theta^2 + r^2 \sin^2 \theta \mathrm{d}\phi^2, 
\end{align*}
\] (5.1)
the duality transformation (3.6) becomes
\[
\begin{align*}
&\quad r \rightarrow b^2 M + b^2 \sqrt{(M^2 - Q^2) - Q^2 \frac{\rho^2}{b^2}} \\
&\quad \varphi \rightarrow \beta \varphi. 
\end{align*}
\] (5.2) (5.3)
The metric of the geodesic dual spacetime is then
\[
\begin{align*}
\mathrm{d}\tau^2 &= -F(\rho) \mathrm{d}r^2 + G(\rho) \mathrm{d}\rho^2 + \rho^2 \mathrm{d}\theta^2 + \rho^2 \sin^2 \theta \mathrm{d}\phi^2 
\end{align*}
\] (5.4)
with
\[
\begin{align*}
F(\rho) &= \frac{-\left[\frac{b^2 M + b^2 \sqrt{(M^2 - Q^2) - Q^2 \frac{\rho^2}{b^2}}}{b^2 (b^2 + \rho^2)}\right]^2}{\rho^4} \\
G(\rho) &= \frac{-\beta^2 Q^2 \left[b^2 \left(1 - \frac{M^2}{Q^2}\right) - \rho^2\right]}{\rho^4},
\end{align*}
\] (5.5) (5.6)

Their geodesic equations are
\[
\begin{align*}
\frac{d\varphi}{dr} &= \frac{1}{r^2} \frac{1}{\sqrt{\frac{1}{b^2} - \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) \left(\frac{1}{b^2} + \frac{1}{r^2}\right)}}; \\
\frac{d\phi}{d\rho} &= \frac{1}{\rho^2} \frac{1}{\sqrt{\frac{2bM}{b^2 + \rho^2} \left(\frac{1}{b^2 M + \rho^2}\right)^2 - \left(\frac{1}{b^2 M + \rho^2}\right)^2 \left(\frac{1}{b^2} + \frac{1}{\rho^2}\right)}}. 
\end{align*}
\] (5.7) (5.8)

It can be checked that Eqs. (5.1) and (5.4) satisfy the duality relation (3.36)-(3.38).
6 Conclusions and outlook

In this paper, we introduce a duality between spacetimes. The duality relation transforms both the metrics and the geodesics of two dual spacetimes to each other. When two spacetimes are geodesically dual, their geodesics are related by the duality transformation which also relates the metrics of the spacetimes.

The geodesic dual transformation transforms the geodesic equation to the geodesic equation of the dual spacetime. Furthermore, the transformation that transforms another kinds of dynamical equation to a dynamical equation on another spacetime may lead to a new kind of duality of spacetimes. For example, the geodesic is the trajectory of a free particle on spacetime manifolds, while in quantum theory, various kinds of free particles are described by various kinds of wave equations, scalar particles described by scalar equations, spinor particles described by spinor equations, and so on. This inspires us to seek dualities between spacetimes based on various kinds of wave equations.

The geodesic corresponds only to free particles or free fields. It is worthy to consider the duality relation based on various dynamical equations with interactions. The geodesic is the solution of the geodesic equation. Furthermore, various physical quantities are also solutions of various dynamical equations. It is worthy to consider the duality relies on various physical quantities. For example, for bound states, we can consider the duality relation between bound-state eigenvalues. For scattering, the information of the dynamics is embedded in the scattering phase shift [28–35], and we can consider the duality relies on phase shifts.

A further problem is the relation between various physical quantities on dual spacetimes, e.g., the relation between eigenvalue spectra on dual manifolds. In quantum field theory, the physical qualities we are interested in are all spectral functions, such as the partition function (the global heat kernel) and the effective action [36–38]. The spectra and, then, the spectral function is determined by the field equation on spacetimes, so it is worthy to reveal the relation between the spectra and spectral functions on dual spacetimes.

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