SOLVING FUZZY LINEAR FRACTIONAL SET COVERING PROBLEM BY A GOAL PROGRAMMING BASED SOLUTION APPROACH

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Abstract. In this paper, a fuzzy linear fractional set covering problem is solved. The non-linearity of the objective function of the problem as well as its fuzziness make it difficult and complex to be solved effectively. To overcome these difficulties, using the concepts of fuzzy theory and component-wise optimization, the problem is converted to a crisp multi-objective non-linear problem. In order to tackle the obtained multi-objective non-linear problem, a goal programming based solution approach is proposed for its Pareto-optimal solution. The non-linearity of the problem is linearized by applying some linearization techniques in the procedure of the goal programming approach. The obtained Pareto-optimal solution is also a solution of the initial fuzzy linear fractional set covering problem. As advantage, the proposed approach applies no ranking function of fuzzy numbers and its goal programming stage considers no preferences from decision maker. The computational experiments provided by some examples of the literature show the superiority of the proposed approach over the existing approaches of the literature.

1. Introduction. Most of the engineering and non-engineering optimization problems are done by mathematical programming. The procedure is to formulate a problem to one or more objective function to be optimized according to some constraints which are obtained according to real limitations of the problem environment. As an especial case among the mathematical programming problems, the set covering problem is one of the most applicable problems to the real world situations. It can be applied to the popular real world problems such as facility location problem,
airlines scheduling problem, etc. For example, in the case of facility location problem, the set covering problem can be applied to find the locations for a new facility to serve all of the existing facilities. To be more close to real situations, the engineering and non-engineering problems are modeled with fuzzy parameters in the literature (see [4, 5, 7, 16, 17, 24, 25, 33]). Furthermore, the studies of Niroomand et al. [22], Mahmoodirad et al. [19], Mahmoodirad and Niroomand [17] are some recent studies that applied fuzziness and uncertainty in optimization problems of supply chain network design. In the same topic, the studies of Niroomand et al. [23] and Sanei et al. [27] can be of interest. The studies of Niroomand et al. [21], Garg [10], and Akram et al. [1] mention good applications of fuzziness in decision making problems. The set covering problem becomes difficult to be solved when its parameters are of uncertain type. In the case of fuzzy parameters, the problem is very interesting as many researchers paid attention to tackle the fuzzy version of the problem. Zimmermann [36] generally introduced the set covering problem in its fuzzy form. Saxena and Arora [28] introduced a linearization technique for solving the quadratic set covering problem. Li and Kwan [15] developed a meta-heuristic with orthogonal experiment for the set covering problem. Shavandi and Mahlouji [32], applied fuzzy hierarchical queuing models for the location set covering problem in congested systems. Sahraeian and Kazemi [26] developed a fuzzy set covering-clustering algorithm for facility location problem. Saxena and Gupta [29] developed an enumeration technique for solving linear fuzzy set covering problem. Optimizing the ratio functions (fractional functions) may happen in several real-world social and engineering problems like transportation problems, production engineering problems, government related problems, economical problems, etc. A rich and complete review of generalized fractional mathematical models and its applied area has been prepared by Frenk and Schaible [9](see also [8, 18, 20, 30, 31, 35]).

A set covering problem with fractional objective function where the objective function includes either triangular or trapezoidal fuzzy parameters or mix of them is called fuzzy linear fractional set covering problem (FLFSCP). This problem has been solved in the literature very rarely. The only study of literature on this problem has been done by Gupta and Saxena [14]. Their method has some disadvantages such as

1) an initial feasible solution must be generated and the method starts with this solution by applying the gradient of the objective function,
2) the method is highly depended on the generated initial feasible solution,
3) for the case of problem instances with high number of variables, the method becomes inefficient.

This study focuses to solve the fuzzy linear fractional set covering problem (FLFSCP). The FLFSCP has two core difficulties to be solved such as its uncertain nature and its non-linear nature arisen from the fractional objective function. To overcome these difficulties, a solution approach based on the concepts of fuzzy theory and goal programming approach is proposed as a contribution. The advantages of this proposed approach are summarized as,

1) it combines the concepts of fuzzy numbers and goal programming approach to solve the FLFSCP as a linear programming model,
2) it converts the FLFSCP to a multi-objective problem and gives an efficient (Pareto-optimal) solution to the objective functions where there is no available importance weights for the objective functions,
3) no ranking function of fuzzy numbers is applied in the solution procedure. The performance of the proposed approach is evaluated by some numerical examples taken from the literature or generated randomly. The obtained results show the superiority of the proposed approach over those of the literature such as the method of Gupta and Saxena [14].

Remainder of the paper is organized by five sections. Section 2, describes some basic definitions and concepts of fuzzy theory and optimization theory. Section 3, introduces the FLFSCP and analyzes its formulation. Section 4, organizes the goal programming based solution approach proposed for the FLFSCP. Section 5, contains some illustrative numerical examples from the literature to analyze the performance of the proposed solution approach provided by Section 4. Finally, the paper is ended by remarking some conclusions in Section 6.

2. Basic definitions. Some basic definitions from fuzzy theory which will be used in the paper, are given here.

Definition 2.1. Let \( X \) be a non-empty set of elements \( x \). A fuzzy set \( \tilde{A} \) in \( X \) is a set of pairs \( (x, \mu_{\tilde{A}}(x)) \) for \( x \in X \), that \( \mu_{\tilde{A}} : X \rightarrow [0,1] \) is its membership function.

Definition 2.2. A trapezoidal fuzzy number is shown by \( \tilde{A} = (a_1, a_2, a_3, a_4) \) with its membership function as follow,

\[
\mu_{\tilde{A}}(x) = \begin{cases} 
\frac{x-a_1}{a_2-a_1}, & a_1 \leq x \leq a_2, \\
1, & a_2 \leq x \leq a_3, \\
\frac{a_4-x}{a_4-a_3}, & a_3 \leq x \leq a_4, \\
0, & \text{o.w}
\end{cases} \tag{1}
\]

Definition 2.3. If \( \tilde{Z} = (Z^1, Z^2, Z^3, Z^4) \) is a trapezoidal fuzzy objective function being calculated from trapezoidal fuzzy parameters or variables, according to the concept of component-wise optimization (see also Gupta and Mehlawat [13]) the optimization problem,

\[
\min \tilde{Z} = (Z^1, Z^2, Z^3, Z^4) \tag{2}
\]

is equivalent to the following multi-objective optimization problem,

\[
\begin{cases} 
\min Z^1 \\
\min Z^2 \\
\min Z^3 \\
\min Z^4
\end{cases} \tag{3}
\]

3. Linear set covering problems. As mentioned by Gupta and Saxena [14], the set covering problem is one of the most applicable problems to the real world situations. It can be applied to the popular real world problems such as facility location problem, airline scheduling problem, etc. For example, in the case of facility location problem, the set covering problem can be applied to find the locations for a new facility to serve all of the existing facilities (suppose that a new facility can serve an existing facility, if their distance is less than a predetermined distance).

To generalize the problem, consider a set of \( I = \{1, 2, \ldots, m\} \) where each element of the set should be covered by at least one element of the set \( J = \{1, 2, \ldots, n\} \). The set of coverable elements of the set \( I \) by \( j \in J \) is shown by \( P_j \). If \( i \in P_j \), therefore \( a_{ij} = 1 \) is defined as a parameter, otherwise, \( a_{ij} = 0 \). If element \( j \) is selected to cover the elements of its associated set \( P_j \), the binary variable \( X_j = 1 \) is considered,
otherwise, \( X_j = 0 \). Of course if an item \( j \in J \) is selected to cover the elements of its associated set \( P_j \), the cost of \( c_j \) is charged. According to these definitions, the linear set covering problem (LSCP) is formulated as the following model to minimize the total covering cost by covering all of the elements of set \( I \).

\[
\text{Min } Z = \sum_{j=1}^{n} c_j X_j \\
\text{subject to} \\
\sum_{j=1}^{n} a_{ij} X_j \geq 1 \quad i \in I \\
X_j \in \{0,1\} \quad j \in J
\]

As an extension, the LSCP is converted to the linear fractional set covering problem (LFSCP) as follow,

\[
\text{Min } Z = \frac{\sum_{j=1}^{n} c_j X_j + P}{\sum_{j=1}^{n} d_j X_j + q} \\
\text{subject to} \\
\sum_{j=1}^{n} a_{ij} X_j \geq 1 \quad i \in I \\
X_j \in \{0,1\} \quad j \in J
\]

where, if an item \( j \in J \) is selected to cover the elements of its associated set \( P_j \), the benefit of \( d_j \) is obtained. The values \( p \) and \( q \) are positive values by cost and benefit natures, respectively. Of course, the condition \( \sum_{j=1}^{n} d_j X_j + q > 0 \) should be considered.

The LFSCP is extended to the fuzzy linear fractional set covering problem (FLF-SCP) as follow,

\[
\text{Min } \tilde{Z} = \frac{\sum_{j=1}^{n} \tilde{c}_j X_j + \tilde{p}}{\sum_{j=1}^{n} \tilde{d}_j X_j + \tilde{q}} \\
\text{subject to} \\
\sum_{j=1}^{n} a_{ij} X_j \geq 1 \quad i \in I \\
X_j \in \{0,1\} \quad j \in J
\]

where, the parameters \( \tilde{c}_j, \tilde{d}_j, \tilde{p} \) and \( \tilde{q} \) are of either triangular or trapezoidal type fuzzy numbers. In this paper we use them as trapezoidal fuzzy numbers as \( \tilde{c}_j = (c^1_j, c^2_j, c^3_j, c^4_j) \), \( \tilde{d}_j = (d^1_j, d^2_j, d^3_j, d^4_j) \), \( \tilde{p} = (p^1, p^2, p^3, p^4) \) and \( \tilde{q} = (q^1, q^2, q^3, q^4) \).

As a solution approach, Gupta and Saxena [14] proposed a method to solve the FLFSCP. Their method has some disadvantages that are summarized by below points,

- An initial feasible solution must be generated. Then the method starts with this solution by applying the gradient of the objective function.
- Referring to the previous point, the method is highly depended on the generated initial feasible solution.
For the case of problem instances with high number of variables, the method becomes inefficient.

4. Proposed solution approach. The FLFSCP formulated by the model (6) is solved in this section. This problem has two core difficulties to be solved, (1) its uncertain nature, (2) its non-linear nature arisen from the fractional objective function. To overcome these difficulties, a solution approach based on the concepts of fuzzy theory and goal programming approach is proposed and detailed here. The advantages of this proposed approach are summarized below,

1) The proposed approach combines the concepts of fuzzy numbers and goal programming approach to solve the FLFSCP as a linear programming model. Goal programming is a useful solution approach to solve multi-objective problems. It is actually an extension of linear programming to tackle several objectives simultaneously. For each objective function a goal is considered and sum of deviations from the goals is minimized as a single objective function.

2) The proposed approach converts the FLFSCP to a multi-objective problem and gives an efficient (Pareto-optimal) solution to the objective functions where there is no available importance weights for the objective functions.

3) No ranking function of fuzzy numbers is applied in the solution procedure. The proposed approach is explained in the following steps. It is notable to say that the proposed approach is explained for the FLFSCP with trapezoidal fuzzy parameters, but it can be easily developed for the FLFSCP with triangular fuzzy parameters.

Step 1: As the objective function of the FLFSCP contains trapezoidal fuzzy parameters, it is expanded as below,

\[ \tilde{Z} = \left( \sum_{j=1}^{n} c^1_j X_j + p^1, \sum_{j=1}^{n} c^2_j X_j + p^2, \sum_{j=1}^{n} c^3_j X_j + p^3, \sum_{j=1}^{n} c^4_j X_j + p^4 \right) \]

that is more expanded as,

\[ \tilde{Z} = \left( \frac{\sum_{j=1}^{n} d^1_j X_j + p^1, \sum_{j=1}^{n} d^2_j X_j + p^2, \sum_{j=1}^{n} d^3_j X_j + p^3, \sum_{j=1}^{n} d^4_j X_j + p^4}{\sum_{j=1}^{n} d^1_j X_j + q^1, \sum_{j=1}^{n} d^2_j X_j + q^2, \sum_{j=1}^{n} d^3_j X_j + q^3, \sum_{j=1}^{n} d^4_j X_j + q^4} \right) \]

In summary, the objective function of model (6) is converted to the objective function (9).

Step 2: The concept of component-wise optimization was explained by Definition 3 in Section 2. According to this definition, the FLFFCP is reformulated as
the following multi-objective problem,

\[
\begin{align*}
\text{Min } Z_1 &= \sum_{j=1}^{n} c_j X_j + p^1 \\
&\quad + \frac{p^1}{\sum_{j=1}^{n} d_j X_j + q^1}
\text{Min } Z_2 &= \sum_{j=1}^{n} c_j X_j + p^2 \\
&\quad + \frac{p^2}{\sum_{j=1}^{n} d_j X_j + q^2}
\text{Min } Z_3 &= \sum_{j=1}^{n} c_j X_j + p^3 \\
&\quad + \frac{p^3}{\sum_{j=1}^{n} d_j X_j + q^3}
\text{Min } Z_4 &= \sum_{j=1}^{n} c_j X_j + p^4 \\
&\quad + \frac{p^4}{\sum_{j=1}^{n} d_j X_j + q^4}
\end{align*}
\]

subject to

\[
\sum_{j=1}^{n} a_{ij} X_j \geq 1 \quad i \in I
\]
\[
X_j \in \{0, 1\} \quad j \in J
\]

Step 3: A goal for each objective function of the formulation (10) is obtained by solving the following models separately. The goal for the objective function \(Z_1\) is shown by \(Z_1^*\) and so on.

\[
\begin{align*}
\text{Min } Z_1^* &= \sum_{j=1}^{n} c_j X_j + p^1 \\
&\quad + \frac{p^1}{\sum_{j=1}^{n} d_j X_j + q^1}
\text{Min } Z_2^* &= \sum_{j=1}^{n} c_j X_j + p^2 \\
&\quad + \frac{p^2}{\sum_{j=1}^{n} d_j X_j + q^2}
\text{Min } Z_3^* &= \sum_{j=1}^{n} c_j X_j + p^3 \\
&\quad + \frac{p^3}{\sum_{j=1}^{n} d_j X_j + q^3}
\text{Min } Z_4^* &= \sum_{j=1}^{n} c_j X_j + p^4 \\
&\quad + \frac{p^4}{\sum_{j=1}^{n} d_j X_j + q^4}
\end{align*}
\]

subject to

\[
\sum_{j=1}^{n} a_{ij} X_j \geq 1 \quad i \in I
\]
\[
X_j \in \{0, 1\} \quad j \in J
\]
\[
\text{Min } Z^3 = \frac{\sum_{j=1}^{n} c^3_j X_j + p^3}{\sum_{j=1}^{m} d^3_j X_j + q^3}
\]

subject to

\[
\sum_{j=1}^{n} a_{ij} X_j \geq 1 \quad i \in I
\]

\[
X_j \in \{0, 1\} \quad j \in J
\]

The fractional objective functions of the models (11)-(14) are linearized using the method introduced by Charnes and Cooper [6]. To show this linearization procedure, as an example model (11) is linearized here. The models (12)-(14) will follow a similar linearization procedure.

To linearize model (11), first the conversion \( \sum_{j=1}^{n} d^4_j X_j + q^4 = \frac{T}{T} \) is applied where \( T \) is a continuous variable \((T > 0)\). Therefore, model (11) is converted to the following non-linear model (both sides of the constraints are multiplied by \( T \)),

\[
\text{Min } Z^1 = \sum_{j=1}^{n} c^1_j Y_j T + p^1 T
\]

subject to

\[
\sum_{j=1}^{n} d^4_j X_j T + q^4 T = 1
\]

\[
\sum_{j=1}^{n} a_{ij} X_j \geq 1 \quad i \in I
\]

\[
X_j \in \{0, 1\} , T \geq 0 \quad j \in J
\]

To linearize problem (15), another conversion is required. For this aim, the new continuous variable \( Y_j = X_j T \) is defined. Then problem (15) is converted to the following linearized form ( \( M \) is a large positive value),

\[
\text{Min } Z^1 = \sum_{j=1}^{n} c^1_j Y_j + p^1 T
\]

subject to

\[
\sum_{j=1}^{n} d^4_j Y_j + q^4 T = 1
\]

\[
\sum_{j=1}^{n} a_{ij} Y_j \geq T \quad i \in I
\]

\[
Y_j \leq M X_j \quad j \in J
\]

\[
Y_j \leq T \quad j \in J
\]

\[
Y_j \geq T - M(1 - X_j) \quad j \in J
\]

\[
X_j \in \{0, 1\} , Y_j , T \geq 0 \quad j \in J
\]

Model (16) is the linearized version of models (11) and (15), where, its third, fourth, and fifth constraints together are equivalent to the term \( Y_j = X_j T \). Non-linear problems (12)-(14) are linearized by the same way.
Step 4: After obtaining the goal of each objective function, we go back to solve model (6). As model (6) was reformulated to model (10), we introduce an effective goal programming approach for solving the reformulated model. So, considering the goals of each objective function obtained in Step 3, the model (10) is converted to the following non-linear formulation,

\[
\begin{align*}
\text{Min} \quad & D_1^+ + D_2^+ + D_3^+ + D_4^+ \\
\text{subject to} \quad & \sum_{j=1}^{n} c_j X_j + q^1 + D_1^1 - D_1^+ = Z_1^* \\
& \sum_{j=1}^{n} c_j X_j + q^2 + D_2^1 - D_2^+ = Z_2^* \\
& \sum_{j=1}^{n} c_j X_j + q^3 + D_3^1 - D_3^+ = Z_3^* \\
& \sum_{j=1}^{n} c_j X_j + q^4 + D_4^1 - D_4^+ = Z_4^* \\
& D_1^1 - D_1^+ = 0 \\
& D_2^1 - D_2^+ = 0 \\
& D_3^1 - D_3^+ = 0 \\
& D_4^1 - D_4^+ = 0 \\
& \sum_{j=1}^{n} a_{ij} X_j \geq 1 \quad i \in I \\
& X_j \in \{0, 1\} \quad j \in J \\
& D_1^-, D_2^-, D_3^-, D_4^- \geq 0
\end{align*}
\]

where \(D_1^-\) and \(D_1^+\) are negative and positive deviations of objective function \(Z_1^*\) from its goal \(Z_1^*\) respectively. The other variables are defined accordingly.

The following issues are of interest about the formulation (17).

- As the objective functions are of minimization type, therefore, never an objective function can be less than its goal. Thus, \(D_1^- = D_2^- = D_3^- = D_4^- = 0\). As a result, the fifth, sixth, seventh, and ninth constraints of the model can be removed from the model.
- As \(D_1^- = D_2^- = D_3^- = D_4^- = 0\), the variables \(D_1^-, D_2^-, D_3^-, D_4^-\) are removed from the first, second, and third constraints. As the non-linearity of these constraints are difficult to be linearized, a typical variable changing technique is proposed here. For this aim, the variables \(D_1^-, D_2^-, D_3^-, D_4^-\) are converted to the fractions \(R_1^+, R_2^+, R_3^+, R_4^+\), respectively, where, the non-negative variables \(R_1^+, R_2^+, R_3^+, R_4^+\) are used instead of the variables \(D_1^+, D_2^+, D_3^+, D_4^+\) in the objective function.
Therefore, the model (17) is linearized as below,

\[
\begin{align*}
\text{Min } & \ R^1 + R^2 + R^3 + R^4 \\
\text{subject to } & \ \\
\sum_{j=1}^{n} c_j^1 X_j + p^1 - R^1 &= Z^1 \\
\sum_{j=1}^{n} c_j^2 X_j + p^2 - R^2 &= Z^2 \\
\sum_{j=1}^{n} c_j^3 X_j + p^3 - R^3 &= Z^3 \\
\sum_{j=1}^{n} c_j^4 X_j + p^4 - R^4 &= Z^4 \\
\sum_{j=1}^{n} a_{ij} X_j &\geq 1, \quad i \in I \\
X_j &\in \{0,1\}, \quad j \in J \\
R^1, R^2, R^3, R^4 &\geq 0
\end{align*}
\]

Finally, solving model (22) will result in a solution for model (10). The obtained solution may be either a Pareto-optimal solution or only a satisfactory solution of model (10). For this aim a Pareto-optimality test is proposed in the next step. It is notable to say that the obtained solution should be either used in the objective functions of models (10) or objective function of model (6) for obtaining final fuzzy objective function value.

Step 5: To test whether or not the solution obtained by model (22) is a Pareto-optimal solution for model (10), the following model and the consequent theorem is considered. Notably, the solution obtained by model (22) is supplied in model (23). Therefore, except the notations \( R^1, R^2, R^3, R^4 \) and
$R^{4+}$, the other notations are constants here.

Max $R^{1+} + R^{2+} + R^{3+} + R^{4+}$

subject to

\[
\sum_{j=1}^{n} c_j^1 X_j + p^1 - R^{1+} = Z_1^1 \left( \sum_{j=1}^{n} d_j^1 X_j + q^1 \right)
\]

\[
\sum_{j=1}^{n} c_j^2 X_j + p^2 - R^{2+} = Z_2^2 \left( \sum_{j=1}^{n} d_j^2 X_j + q^2 \right)
\]

\[
\sum_{j=1}^{n} c_j^3 X_j + p^3 - R^{3+} = Z_3^3 \left( \sum_{j=1}^{n} d_j^3 X_j + q^3 \right)
\]

\[
\sum_{j=1}^{n} c_j^4 X_j + p^4 - R^{4+} = Z_4^4 \left( \sum_{j=1}^{n} d_j^4 X_j + q^4 \right)
\]

$R_1^+, R_2^+, R_3^+, R_4^+ \geq 0$

**Theorem 4.1.** If the optimal value of problem (24) for a solution obtained by model (22) is zero, then, the obtained solution is Pareto-optimal solution for model (10) and conversely.

**Proof.** In order to simplify the proof, problem (10) is represented by the below formulation,

Min $Z^r = \frac{f_r(x)}{g_r(x)}$ \quad $r = 1, 2, 3, 4$

subject to

$x \in S$

while, problem (21) is represented by the following formulation,

Max $\sum_{r=1}^{4} d_r$

subject to

$f_r(x) + d_r - Z_r^* g_r(x) = 0$ \quad $r = 1, 2, 3, 4$

d_r \geq 0 \quad r = 1, 2, 3, 4$

$x \in S$

Now, we assume that the optimal objective function value of problem (25) is zero and its associated optimal solution $x'$ (which are also obtained by the model (23)) is not Pareto-optimal for problem (10). So, there should be a solution say $x''$ which is better than $x'$ by below condition,

\[
\frac{f_r(x'')}{g_r(x'')} \leq \frac{f_r(x')}{g_r(x')} \quad r = 1, 2, 3, 4
\]

and $\exists j \in \{1, 2, 3, 4\}$ such that,

\[
\frac{f_j(x'')}{g_j(x'')} \leq \frac{f_j(x')}{g_j(x')} \quad (27)
\]
Considering a non-negative slack variable of \( \frac{d_r}{g_r(x''')} \), \( r = 1, 2, 3, 4 \), for the inequalities (26)-(27), the following equality is obtained,
\[
\frac{f_r(x''')}{g_r(x''')} + \frac{d_r}{g_r(x''')} = \frac{f_r(x')}{g_r(x')}
\]
where \( \bar{d} = (d_1, d_2, ..., d_p) \geq 0, \bar{d} \neq 0 \). If we multiply both sides of equation (28) by \( g_r(x''') \), the following equation is obtained,
\[
f_r(x''') + d_r = f_r(x') g_r(x') g_r(x''') = z_r g_r(x''') \quad r = 1, 2, 3, 4 (29)
\]
According to equation (29), it is concluded that \( (x''', \bar{d}) \) is a feasible solution for problem (25), where \( \bar{d} \neq 0 \). The value \( \bar{d} \neq 0 \) implies that there is a contradiction with the initial assumption (the optimal objective function value of problem (25) is zero) and also the related optimum objective function value must be positive. Therefore, \( x' \) is a Pareto-optimal solution for model (25).

In order to prove the theorem conversely, first we assume that \( x' \) is a Pareto-optimal solution for problem (10). Therefore, it should be shown that the optimal objective function value of problem (25) is zero. To do so, a contradiction is considered as assuming a positive value for the objective function of problem (25). Therefore \( (\bar{x}, \bar{d}) \) is a feasible solution to problem (25) with \( \bar{d} \neq 0 \). As \( g_r(\bar{x}) > 0, r = 1, 2, 3, 4 \), the constraints of the problem is rewritten by the following equality,
\[
\bar{Z}^r + \frac{d_r}{g_r(\bar{x})} = Z^r \quad r = 1, 2, 3, 4 (30)
\]
where \( \bar{Z}^r = \frac{L_r(x)}{g_r(x)} \). And the following relation can be obtained,
\[
\bar{Z}^r = Z^r - \frac{d_r}{g_r(\bar{x})} < Z^r \quad r = 1, 2, 3, 4 (31)
\]
As a result of relation (31), \( x' \) is not a Pareto-optimal solution for problem (25). This claim is in contradiction with the initial assumption \( (x' \) is a Pareto-optimal solution for problem (10)). Therefore, the optimal value of problem (25) must be zero.

5. **Illustrative examples.** In this section the proposed solution approach for the FLFSCP is evaluated over two numerical examples. The first example is taken from Gupta and Saxena [14] where the second one is generated randomly. For comparison purposes, both of the examples are solved by the proposed approach and the approach of Gupta and Saxena [14].

**Example 1.** This example of the FLFSCP which is taken from Gupta and Saxena [14] is represented by the following formulation,
\[
\begin{align*}
\text{Min} & \quad (5, 6, 7, 20)X_1 + (16, 17, 18, 40)X_2 + (24, 25, 26, 50)X_3 + (2, 5, 5, 9)X_4 \\
& \quad (4, 6, 6, 8)X_1 + (5, 7, 7, 12)X_2 + (6, 11, 11, 13)X_3 + (15, 16, 16, 17)X_4 \\
\text{subject to} & \quad X_1 + X_2 \geq 1 \\
& \quad X_2 + X_3 \geq 1 \\
& \quad X_1 + X_3 \geq 1 \\
& \quad X_1, X_2, X_3 \in \{0, 1\}
\end{align*}
\]
Now, the steps of the proposed approach of previous section is implemented here. According to Step 1 and Step 2 of the proposed approach, the following multi-objective approach is obtained.

\[
\begin{align*}
\text{Min } Z_1 &= \frac{5X_1 + 16X_2 + 24X_3 + 2}{8X_1 + 12X_2 + 13X_3 + 17} \\
\text{Min } Z_2 &= \frac{6X_1 + 17X_2 + 25X_3 + 5}{6X_1 + 7X_2 + 11X_3 + 16} \\
\text{Min } Z_3 &= \frac{7X_1 + 18X_2 + 26X_3 + 5}{6X_1 + 7X_2 + 11X_3 + 16} \\
\text{Min } Z_4 &= \frac{8X_1 + 5X_2 + 6X_3 + 15}{20X_1 + 40X_2 + 50X_3 + 9}
\end{align*}
\]

subject to

\[
\begin{align*}
X_1 + X_2 &\geq 1 \\
X_2 + X_3 &\geq 1 \\
X_1 + X_3 &\geq 1 \\
X_1, X_2, X_3 &\in \{0, 1\}
\end{align*}
\]

According to Step 3 of the proposed approach, the following models should be solved separately to obtain the goals of the objective functions as \(Z_1^*, Z_2^*, Z_3^*\) and \(Z_4^*\).

\[
\begin{align*}
\text{Min } Z_1 &= \frac{5X_1 + 16X_2 + 24X_3 + 2}{8X_1 + 12X_2 + 13X_3 + 17} \\
\text{subject to} \\
X_1 + X_2 &\geq 1 \\
X_2 + X_3 &\geq 1 \\
X_1 + X_3 &\geq 1 \\
X_1, X_2, X_3 &\in \{0, 1\}
\end{align*}
\]

\[
\begin{align*}
\text{Min } Z_2 &= \frac{6X_1 + 17X_2 + 25X_3 + 5}{6X_1 + 7X_2 + 11X_3 + 16} \\
\text{subject to} \\
X_1 + X_2 &\geq 1 \\
X_2 + X_3 &\geq 1 \\
X_1 + X_3 &\geq 1 \\
X_1, X_2, X_3 &\in \{0, 1\}
\end{align*}
\]

\[
\begin{align*}
\text{Min } Z_3 &= \frac{7X_1 + 18X_2 + 26X_3 + 5}{6X_1 + 7X_2 + 11X_3 + 16} \\
\text{subject to} \\
X_1 + X_2 &\geq 1 \\
X_2 + X_3 &\geq 1 \\
X_1 + X_3 &\geq 1 \\
X_1, X_2, X_3 &\in \{0, 1\}
\end{align*}
\]
Min \( Z^4 = \frac{20X_1 + 40X_2 + 50X_3 + 9}{4X_1 + 5X_2 + 6X_3 + 15} \)

subject to

\[
\begin{align*}
X_1 + X_2 & \geq 1 \\
X_2 + X_3 & \geq 1 \\
X_1 + X_3 & \geq 1 \\
X_1, X_2, X_3 & \in \{0, 1\}
\end{align*}
\] (37)

As models (34)-(37) are non-linear, those should be linearized by the method explained in Section 4 of the proposed approach. For this aim just the linearization procedure of model (34) is detailed here.

To linearize model (31), the term \( 8X_1 + 12X_2 + 13X_3 + 17 = \frac{1}{T} \) is assumed and the model is rewritten as,

Min \( Z^1 = 5X_1T + 16X_2T + 24X_3T + 2T \)

subject to

\[
\begin{align*}
8X_1T + 12X_2T + 13X_3T + 17T &= 1 \\
X_1T + X_2T & \geq T \\
X_2T + X_3T & \geq T \\
X_1T + X_3T & \geq T \\
X_1, X_2, X_3 & \in \{0, 1\}, T \geq 0
\end{align*}
\] (38)

Now, assuming the changing variable \( XT = Y \), model (38) is converted to the following linearized form,

Min \( Z^1 = 5Y_1 + 16Y_2 + 24Y_3T + 2T \)

subject to

\[
\begin{align*}
8Y_1 + 12Y_2 + 13Y_3 + 17T &= 1 \\
Y_1 + Y_2 & \geq T \\
Y_2 + Y_3 & \geq T \\
Y_1 + Y_3 & \geq T \\
Y_1 & \leq MX_1 \\
Y_1 & \leq T \\
Y_1 & \geq T - M(1 - X_1) \\
Y_2 & \leq MX_2 \\
Y_2 & \leq T \\
Y_2 & \geq T - M(1 - X_2) \\
Y_3 & \leq MX_3 \\
Y_3 & \leq T \\
Y_3 & \geq T - M(1 - X_3) \\
X_1, X_2, X_3 & \in \{0, 1\} \\
Y_1, Y_2, Y_3, T & \geq 0
\end{align*}
\] (39)
After obtaining the goals of the objective functions, the following single model is obtained according to the Step 4 of the proposed approach,

\[
\begin{align*}
\text{Min } & R_1^+ + R_2^+ + R_3^+ + R_4^+ \\
\text{subject to } & \\
5X_1 + 16X_2 + 24X_3 + 2 & - R_1^+ = Z_1^* (8X_1 + 12X_2 + 13X_3 + 17) \\
6X_1 + 17X_2 + 25X_3 + 5 & - R_2^+ = Z_2^* (6X_1 + 7X_2 + 11X_3 + 16) \\
7X_1 + 18X_2 + 26X_3 + 5 & - R_3^+ = Z_3^* (6X_1 + 7X_2 + 11X_3 + 16) \\
20X_1 + 40X_2 + 50X_3 + 9 & - R_4^+ = Z_4^* (4X_1 + 5X_2 + 6X_3 + 15) \\
X_1 + X_2 & \geq 1 \\
X_2 + X_3 & \geq 1 \\
X_1 + X_3 & \geq 1 \\
X_1, X_2, X_3 & \in \{0, 1\} \\
R_1^+, R_2^+, R_3^+, R_4^+ & \geq 0
\end{align*}
\]

After running the linearized form of models (34)-(37), the objective function values (goals) and their related solutions were obtained. Those are reported by Table 1.

| Objective | \(X_1\) | \(X_2\) | \(X_3\) | \text{function value} |
|-----------|--------|--------|--------|------------------------|
| Model (30) | 1      | 1      | 0      | \(Z_1^* = 0.622\)  |
| Model (31) | 1      | 1      | 0      | \(Z_2^* = 0.966\)  |
| Model (32) | 1      | 1      | 0      | \(Z_3^* = 1.034\)  |
| Model (33) | 1      | 1      | 0      | \(Z_4^* = 2.875\)  |

Applying the obtained goals of Table 1 into Step 4 of the proposed approach, model (40) was solved. The obtained solution contains the variable values \(X_1^* = 1\), \(X_2^* = 1\) and \(X_3^* = 0\). Using this solution in the objective functions of models (34)-(37) or the objective function of model (32), the fuzzy objective function value \(\tilde{Z} = (0.622, 0.966, 1.034, 2.875)\) was obtained for the example. The obtained solution and objective function value is the same as the values obtained by Gupta and Saxena [14] where they obtained the values \(X_1^* = 1\), \(X_2^* = 1\) and \(X_3^* = 0\), and the fuzzy objective function value \(\tilde{Z} = (0.622, 0.965, 1.034, 2.875)\). As can be seen, the performances of the proposed approach and the approach of Gupta and Saxena [14] for this example are similar. These solutions are represented by Figure 1.

\textbf{Example 2.} In order to further evaluate the proposed approach and the approach of Gupta and Saxena [14], the following formulation with triangular fuzzy coefficients which is larger than the formulation of Example 1, is solved by both
Figure 1. The fuzzy objective function obtained by the proposed approach and the approach of Gupta and Saxena [14] for Example 1.

After applying Steps 1-3 of the proposed approach on model (40), the goal values \( Z^* = 0.327, Z^2* = 1.111, \) and \( Z^3* = 2.667 \) were obtained. Following Step 4 of the proposed approach with the obtained goals, finally the solution \( X^1* = 0, X^2* = 0, X^3* = 1, X^4* = 1, X^5* = 1, X^6* = 1, X^7* = 0, \) and \( X^8* = 0, \) with the fuzzy objective function value of \( \tilde{Z} = (0.347, 1.148, 2.8) \) were obtained for the problem of Example 2.

For the case of this example, the obtained solution and objective function value is the same as the values obtained by implementation of the method of Gupta and Saxena [14] where the obtained values are \( X^1* = 1, X^2* = 0, X^3* = 1, X^4* = 1, X^5* = 1, X^6* = 1, X^7* = 0, \) and \( X^8* = 1, \) with the fuzzy objective function value of \( \tilde{Z} = (0.461, 1.25, 3.25). \) As can be seen, for this example, the fuzzy objective function value obtained by the proposed approach is strictly better than the value obtained by the method of Gupta and Saxena [14]. This comparison shows the superiority of the proposed approach over the approach of Gupta and Saxena [14] in the case of this example (see Figure 2).
6. **Concluding remarks.** In this study a fuzzy linear fractional set covering problem was considered. The non-linearity of the objective function of the problem as well as its fuzziness make it difficult and complex to be solved effectively. To overcome these difficulties, using the concepts of fuzzy theory and component-wise optimization, the problem was converted to a multi-objective crisp problem. In order to solve the obtained multi-objective problem, a goal programming based solution approach was proposed to obtain a Pareto-optimal solution. This Pareto-optimal solution is also a solution of the initial fuzzy linear fractional set covering problem.

Some of the advantages of the proposed approach are summarized below,

- It applies no ranking function of fuzzy numbers and
- In the goal programming stage, it considers no preferences from decision maker.
- Pareto-optimality is guaranteed.

The computational experiments provided by some examples of the literature showed the superiority of the proposed approach over those of the literature.

As future study, the proposed problem and solution approach can be considered in bipolar and m-polar fuzzy environments in order to apply different uncertainties. Further, the present method can be extended by considering the belief degree based uncertainty and some other kind of uncertainties under the diverse fuzzy environment [2, 3, 11, 12, 34].

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