Wild Knots as limit sets of Kleinian Groups*

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Dedicated to Alberto Verjovsky on the occasion of his 60th anniversary.

Abstract

In this paper we study kleinian groups of Schottky type whose limit set is a wild knot in the sense of Artin and Fox. We show that, if the "original knot" fibers over the circle then the wild knot Λ also fibers over the circle. As a consequence, the universal covering of $S^3 - Λ$ is $\mathbb{R}^3$. We prove that the complement of a dynamically-defined fibered wild knot can not be a complete hyperbolic 3-manifold.

1 Introduction

Kleinian and (quasi)-fucshian groups were originally defined by H. Poincaré in the 1880’s. In particular, quasi-fucshian groups acting on the Riemann sphere provide us with a most natural way to construct fractal circles with the property of being self-similar. Kleinian groups in higher dimensions may be defined as subgroups of Möebius groups acting conformally on the sphere $S^n$. B. Maskit (see [14]), M. Kapovich (see [11]), have described kleinian groups of schottky type whose limit set is a wild knot in the sense of Artin and Fox.

The world of wild topology born with the work of Antoine and Alexander in the 1920’s, who discovered that a Cantor set and 2-sphere (respectively)
can be knotted in $S^3$. Many works in this direction have been done since then. However, the classical knot theory has excluded the wild case. Recently, Montesinos ([17], [18]) has exhibited a relationship between open 3-manifolds and wild knots and arcs. This generalizes the relationship between closed manifolds and tame links. Montesinos also proved that there exists a universal wild knot, i.e. every closed orientable 3-manifold is a 3-fold branched covering of $S^3$ with branched set a wild knot. This shows how rich the wild knot theory can be.

The purpose of this paper is to study the topological properties of dynamically-defined wild knots. In the first sections we describe geometrically the action of these kleinian groups, their limit sets and fundamental domains. In section 4, we study the complement $\Omega$, of dynamically-defined wild knots. We prove that if the template (see definition 2.2) of the corresponding wild knot is a non-trivial tame fibered knot, then the complement of it also fibers over the circle with fiber an infinite-genus surface with one end. As a consequence, we obtain that the universal covering of $\Omega$ is $\mathbb{R}^3$. This is not a trivial fact, e.g. there are Whitehead manifolds which are infinite cyclic covering spaces of other non-compact 3-manifolds (see, for instance [19]). Although a generalization of a classical theorem of R. H. Bing characterizes $\mathbb{R}^3$ among all contractible open 3-manifolds ([4]), its application to prove that $\mathbb{R}^3$ is the universal covering of the complement of any knot is not entirely obvious, at least for us. As far as we know, this question remains open. P. H. Doyle ([5]) gave an example of a wild knot whose fundamental group is $\mathbb{Z} \ast G$, where $G \neq \mathbb{Z}$ is a group. We suspect that the universal covering of its complement is a contractible space not homeomorphic to $\mathbb{R}^3$. In section 5, we describe completely the monodromy of wild knots whose complements fiber over the circle. The monodromy allows us to recognize if two dynamically-defined wild knots are isotopic.

In section 6, we consider the question of whether the complement of a dynamically-defined, fibered wild knot can be given the structure of a complete hyperbolic 3-manifold. We recall that W. Thurston ([27]) proved that if $K$ is a tame knot, then $S^3 - K$ admits a complete hyperbolic structure if and only if $K$ is neither a torus knot nor a satellite knot. This motivates us to discover if there exists an equivalent result for dynamically-defined wild knots. In this direction, we answer the following question formulated by E. Ghys: Let $K$ be a tame knot such that its complement is a complete hyperbolic 3-manifold. Is the complement of the wild knot obtained from $K$ still a complete hyperbolic 3-manifold?

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In the last section, we give the definition of $q$-fold cyclic covering space over $S^3$ with branched set a dynamically-defined wild knot $\Lambda$, when the “original knot” fibers over the circle. Recently, Montesinos ([17], [18]) has proved amazing results about branched coverings of $S^3$ with branched sets that are wild knots. However, in our case the covering spaces are not manifolds.

This paper contains some results from my Ph.D. thesis which was directed by A. Verjovsky. I would like to thank him for many councils and pivotal suggestions. I would also like to thank M. Kapovich for his very helpful corrections and comments, F. González-Acuña and V. Núñez for the valuable discussions.

2 Preliminaries

Let $M\ddot{o}b(S^n)$ denote the group of Möbius transformations of the $n$-sphere $S^n = \mathbb{R}^n \cup \{\infty\}$. For a discrete group $G \subset M\ddot{o}b(S^n)$ the discontinuity set is defined

$$\Omega(G) = \{x \in S^n : x \text{ has a neighbourhood } U(x) \text{ such that } U(x) \cap g(U(x)) \text{ is empty for all but finite elements } g \in G\}$$

The complement $S^n - \Omega(G) = \Lambda(G)$ is the limit set (see [11]). A subgroup $G \subset M\ddot{o}b(S^n)$ is called a kleinian group if $\Omega(G)$ is not empty. We will be concerned with very specific kleinian groups of schottky type (see [14], page 82).

We recall that a conformal map $\psi$ on $S^n$ can be extended in a natural way to the hyperbolic space $\mathbb{H}^{n+1}$ as an orientation-preserving isometry with respect to the Poincaré metric. Hence we can identify the group $M\ddot{o}b(S^n)$ with the group of orientation-preserving isometries of the hyperbolic $(n+1)$-space $\mathbb{H}^{n+1}$. Thus we can also define the limit set of a kleinian group through sequences (see [14] section II.D).

**Definition 2.1** A point $x$ is a limit point for the kleinian group $G$ if there exist a point $z \in S^n$ and a sequence $\{g_m\}$ of distinct elements of $G$, with $g_m(z) \rightarrow x$. The set of limit points is $\Lambda(G)$.  

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One way to illustrate the action of a kleinian group $G$ is to draw a picture of $\Omega/G$. For this purpose a fundamental domain is very helpful. Roughly speaking, it contains one point from each equivalence class in $\Omega$ (see [12] pages 78-79, [14] pages 29-30). Let $D$ be the fundamental domain of $G$ and consider the orbit space $D^* = \overline{D} \cap \Omega/\sim_G$ with the quotient topology. Then $D^*$ is homeomorphic to $\Omega/G$.

**Definition 2.2** A necklace $T$ of $n$-pearls ($n \geq 3$), is a collection of $n$ consecutive 2-spheres $\Sigma_1, \Sigma_2, \ldots, \Sigma_n$ in $S^3$, such that $\Sigma_i \cap \Sigma_j = \emptyset$ ($i \neq i+1, i-1 \mod n$), except that $\Sigma_i$ and $\Sigma_{i+1}$ are tangent $i = 1, 2, \ldots, n-1$ and $\Sigma_1$ and $\Sigma_n$ are tangent. Each 2-sphere is called a pearl.

**Remark 2.3** If the points of tangency are joined by spherical geodesic segments in $S^3$, we obtain a tame knot $K$ which is called the template of $T$. Observe that the ordering of pearls gives us an orientation for $K$.

Conversely, a pearl-necklace $T$ subordinate to a polygonal knot $K$, is a collection of round 2-spheres $\Sigma_1, \Sigma_2, \ldots, \Sigma_n$ in $S^3$, such that $\Sigma_i$ is tangent to $\Sigma_{i+1}$ (index $i \mod n$) and $K$ is totally covered by them. For simplicity, we also require that each segment of $K$ contained in the interior of each pearl is an unknotted tangle.

We define the filling of $T$ as $|T| = \bigcup_{i=1}^{n} B_i$, where $B_i$ is the round closed 3-ball whose boundary $\partial B_i$ is $\Sigma_i$.

**Example 2.4** $K = \text{Trefoil knot}$.

![Figure 1: A pearl-necklace whose template is the trefoil knot.](image)

Let $\Gamma$ be the group generated by reflections $I_j$, through $\Sigma_j$ ($j = 1, \ldots, n$). Then $\Gamma$ is a kleinian group of Schottky type. It is easy to verify that the fundamental domain for the kleinian group $\Gamma$ is $D = S^3 - |T|$. 

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3 Geometric Description of the Limit Set

To describe the limit set of $\Gamma$, we need to find all the accumulation points of its orbits. We shall thus consider all the possible sequences of elements of $\Gamma$. We will do this in stages:

1. First stage: Observe that each reflection map $I_j$ ($j = 1, 2, \ldots, n$), sends a copy of the exterior of $K$ into the ball $B_j$. In particular, a mirror image of $K$ is sent into $B_j$ (see Figure 2).

![Figure 2: Image of the reflection map $I_k$.](image)

After doing this for each $j$, we obtain a new necklace $T_1$ of $n(n - 1)$ pearls, subordinate to a new knot $K_1$; which is in turn isotopic to the connected sum of $n$ copies of the mirror image of $K$ and one copy of $K$.

2. Second stage: Now, we reflect with respect to each pearl of $T_1$. When we are finished, we obtain a new necklace $T_2$ of $n(n - 1)^2$ pearls, whose template is a tame knot $K_2$; which is in turn isotopic to the connected sum of $n^2 - n + 1$ copies of $K$ and $n$ copies of its mirror image (recall that composition of an even number of reflections is orientation-preserving). Observe that $|T_2| \subset |T_1|$ (see Figure 3).

3. $k^{th}$-stage: In this case, we reflect with respect to each pearl of $T_{k-1}$, and we obtain a necklace $T_k$ of $n(n-1)^k$ pearls, subordinate to a tame knot $K_k$. By construction, $|T_k| \subset |T_{k-1}|$.

Let $x \in \bigcap_{k=1}^{\infty} |T_k|$. We shall prove that $x$ is a limit point. Indeed, there exists a sequence of closed balls $\{B_m\}$ with $B_m \subset |T_m|$ such that $x \in B_m$ for each $m$. For any $z \in S^3 - |T|$ we can find a sequence $\{w_m\}$ of distinct elements of $\Gamma$, such that $w_m(z) \in B_m$. Since $\text{diam}(B_m) \to 0$, it follows that $w_m(z)$ converges to $x$ independently of $z$. Then, $x$ is a limit point of $\Gamma$. 

Figure 3: Image of the reflection map $I_k$ after reflecting with respect to $\Sigma_j$.

(see definition 2.1). The other inclusion clearly holds, since the fundamental domain of $\Gamma$ is $S^3 - |T|$. Therefore, the limit set is given by

$$\Lambda(\Gamma) = \lim_{k \to \infty} |T_k| = \bigcap_{k=1}^{\infty} |T_k|.$$  

**Lemma 3.1** ([14]). Let $T$ be a pearl-necklace subordinate to the non-trivial, tame knot $K$. Let $\Gamma$ be the group generated by reflections through each pearl. Then the limit set $\Lambda(\Gamma)$ is homeomorphic to $S^1$.

**Lemma 3.2** Let $T$ be a pearl-necklace subordinate to the non-trivial tame knot $K$. Then $\Lambda(\Gamma)$ is wildly embedded in $S^3$. Moreover, $\Lambda(\Gamma)$ is wild at every point.

**Proof.** (Compare [14]). Let $x \in \Lambda(\Gamma)$ be a limit point. Since the knot $K$ keeps reproducing either itself or its mirror image, we have that given an open, connected neighborhood $U$ of $x$, there are infinitely many copies $K$ and its mirror image in $U$.

We will denote the complement of the wild knot $\Lambda$ contained in $U$ by $U - \Lambda(\Gamma)$, and the complement of the pearl-necklace $T_k$ in $U$ by $U - |T_k|$. 

Hence, using Van-Kampen’s Theorem and the fact that $\Pi_1(U - \Lambda(\Gamma))$ is the direct limit of $\{\Pi_1(U - |T_k|), \ k = l, l+1, \ldots; \ f_k, \ k = l, l+1, \ldots\}$. Where $f_k : \Pi_1(S^3 - |T_k|) \to \Pi_1(S^3 - |T_{k+1}|)$ is the inclusion map. The index $l$ is the smallest integer such that some balls of $T_l$ are totally contained in $U$ (see Lemma 2.4.1 in [22]). We have that,

$$\Pi_1(U - \Lambda(\Gamma)) \cong (\cdots ((\Pi_1(K) *_{\{z\}} \Pi_1(K)) *_{\{z\}} \cdots *_{\{z\}} \Pi_1(K)) *_{\{z\}} \cdots,$$
is an infinite free product of the fundamental group of the knot with itself. This implies that \( \Pi_1(U - \Lambda(\Gamma)) \) is not isomorphic to a finitely generated group, i.e. it is infinitely generated. Therefore, the result follows. ■

4 Fibration of \( S^3 - \Lambda(\Gamma) \) over \( S^1 \)

We recall that a knot or link \( L \) in \( S^3 \) is fibered if there exists a locally trivial fibration \( f : (S^3 - L) \to S^1 \). We require that \( f \) be well-behaved near \( L \), that is, each component \( L_i \) is to have a neighbourhood framed as \( \mathbb{D}^2 \times S^1 \), with \( L_i \cong \{0\} \times S^1 \), in such a way that the restriction of \( f \) to \( (\mathbb{D}^2 - \{0\}) \times S^1 \) is the map into \( S^1 \) given by \( (x, y) \to \frac{y}{|y|} \). It follows that each \( f^{-1}(x) \cup L \), \( x \in S^1 \), is a 2-manifold with boundary \( L \): in fact a Seifert surface for \( L \) (see [21], page 323).

**Examples 4.1** The right-handed trefoil knot and the figure-eight knot are fibered knots with fiber the punctured torus.

**Lemma 4.2** Let \( T \) be an \( n \)-pearl necklace subordinate to the fibered tame knot \( K \), with fiber \( S \). Then \( \Omega(\Gamma)/\Gamma \) fibers over the circle with fiber \( S^* \), the closure of \( S \) in \( S^3 \) with \( n \) boundary points removed.

**Proof.** Let \( \widetilde{P} : (S^3 - K) \to S^1 \) be the given fibration with fiber the surface \( S \). Observe that \( \widetilde{P} \mid_{S^3 - T} \equiv P \) is a fibration and, after modifying \( \widetilde{P} \) by isotopy if necessary, we can consider that the fiber \( S \) cuts each pearl \( \Sigma_i \in T \) in arcs \( a_i \), whose end-points are \( \Sigma_{i-1} \cap \Sigma_i \) and \( \Sigma_i \cap \Sigma_{i+1} \). These two points belong to the limit set (see Figure 4).

![Figure 4: The fiber intersects each pearl in arcs.](image)

Hence the space \( \overline{D} \cap \Omega(\Gamma) \) fibers over the circle with fiber the 2-manifold \( S^* \), which is the closure of the surface \( S \) in \( S^3 \) with \( n \) boundary points removed (points of tangency of the pearls). Since \( \Omega(\Gamma)/\Gamma \cong D^* \) and in our case \( D^* = \overline{D} \cap \Omega(\Gamma) \), the result follows. ■
By the above Lemma, in order to describe completely the orbit space \((S^3 - \Lambda(\Gamma))/\Gamma\) in the case that the original knot is fibered, we only need to determine its monodromy: which is precisely the monodromy of the knot.

Consider the orientation-preserving index two subgroup \(\widetilde{\Gamma} \subset \Gamma\). Since \(\widetilde{\Gamma}\) is a normal subgroup of \(\Gamma\), it follows by Lemma 8.1.3 in [25] that \(\widetilde{\Gamma}\) has the same limit set that \(\Gamma\). Therefore \(S^3 - \Lambda(\Gamma) = S^3 - \Lambda(\widetilde{\Gamma})\).

**Lemma 4.3** Let \(T\) be an \(n\)-pearl necklace subordinate to the fibered tame knot \(K\) with fiber \(S\). Then \(\Omega(\widetilde{\Gamma})/\widetilde{\Gamma}\) fibers over the circle with fiber a surface \(S^{**}\), which is homeomorphic to the surface \(S^*\) joined along an arc to a copy of itself in \(S^3\), with \(2(n-1)\) boundary points removed (the points of tangency).

**Proof.** It is easy to see that the fundamental domain for \(\widetilde{\Gamma}\) is

\[
\widetilde{D} = (S^3 - |T|) \cup (B_j - I_j(|T - \Sigma_j|)),
\]

for some \(j\). The rest of the proof is very similar to the proof of the above Lemma. \(\blacksquare\)

**Theorem 4.4** Let \(\Sigma_1, \Sigma_2, \ldots, \Sigma_n\) be round 2-spheres in \(S^3\) which form a necklace whose template is a non-trivial tame fibered knot. Let \(\Gamma\) be the group generated by reflections \(I_j\) on \(\Sigma_j\) \((j = 1, 2, \ldots, n)\) and let \(\widetilde{\Gamma}\) be the orientation-preserving index two subgroup of \(\Gamma\). Let \(\Lambda(\Gamma) = \Lambda(\widetilde{\Gamma})\) be the corresponding limit set. Then:

1. There exists a locally trivial fibration \(\psi: (S^3 - \Lambda(\Gamma)) \to S^1\), where the fiber \(\Sigma_\theta = \psi^{-1}(\theta)\) is an orientable infinite-genus surface with one end.

2. \(\Sigma_\theta - \Sigma_\theta = \Lambda(\Gamma)\), where \(\Sigma_\theta\) is the closure of \(\Sigma_\theta\) in \(S^3\).

**Proof.** We will first prove that \(S^3 - \Lambda(\Gamma)\) fibers over the circle. We know that \(\zeta: \Omega(\widetilde{\Gamma}) \to \Omega(\Gamma)/\widetilde{\Gamma}\) is an infinite-fold covering since \(\widetilde{\Gamma}\) acts freely on \(\Omega(\Gamma)\). By the previous lemma, there exists a locally trivial fibration \(\phi: \Omega(\Gamma)/\widetilde{\Gamma} \to S^1\) with fiber \(S^{**}\). Then \(\psi = \phi \circ \zeta: \Omega(\Gamma) \to S^1\) is a locally trivial fibration. The fiber is \(\Gamma(S^{**})\), i.e. the orbit of the fiber.

We now describe \(\Sigma_\theta = \widetilde{\Gamma}(S^{**})\) in detail. Let \(\widetilde{P}: (S^3 - K) \to S^1\) be the given fibration. The fibration \(\widetilde{P}|_{S^3 - |T|} \equiv P\) has been chosen as in the lemmas above. The fiber \(\widetilde{P}^{-1}(\theta) = P^{-1}(\theta)\) is a Seifert surface \(S^*\) of \(K\), for
each $\theta \in S^1$. We suppose $S^*$ is oriented. Recall that the boundary of $S^*$
cuts each pearl $\Sigma_j$ in an arc $a_j$ going from one point of tangency to another.

The reflection $I_j$ maps both a copy of $T - \Sigma_j$ (called $T^j$) and a copy of $S^*$ (called $S_i^j$) into the ball $B_j$, for $j = 1, 2, \ldots, n$. Observe that both $T^j$ and $S_i^j$ have opposite orientation and that $S^*$ and $S_i^j$ are joined by the arc $a_j$ (see Figure 5) which, in both surfaces, has the same orientation.

![Figure 5: Sum of two surfaces $S^*$ and $S_i^j$ along arc $a_j$.](image)

The necklaces $T^j$ and $T$ are joined by the points of tangency of the pearl $\Sigma_j$ and the orientation of these two points is preserved by the reflection $I_j$. Thus, we have obtained a new pearl-necklace isotopic to the connected sum $T \# T^j$, whose complement also fibers over the circle with fiber the sum of $S^*$ with $S_i^j$ along arc $a_j$, namely the fiber is $S^* \#_{a_j} S_i^j$.

Now do this for each $j = 1, \ldots, n$. At the end of the first stage, we have a new pearl-necklace $T_1$ whose template is the knot $K_1$ (see section 3). Its complement fibers over the circle with fiber the Seifert surface $S_1^*$, which is in turn homeomorphic to the sum of $n+1$ copies of $S^*$ along the respective arcs.

Continuing this process, we have from the second step onwards, that $n - 1$ copies of $S^*$ are added along arcs to each surface $S_i^*$, (the $i$th copy of $S^*$ corresponding to the $k$th stage). Notice that in each step, the points of tangency are removed since they belong to the limit set, and the length of the arcs $a_j$ tends to zero.

From the remarks above, we have that $\Sigma_\theta$ is homeomorphic to an orientable infinite-genus surface. In fact, it is the sum along arcs of an infinite number of copies of $S^*$. To determine what kind of surface it is, according to the classification theorem of non-compact surfaces (see [20]), we need only
describe its set of ends.

Consider the fuchsian model (see [14]). In this case, the necklace is formed by pearls of the same size, and each pearl is orthogonal to the unit circle (its template). Then its limit set is the circle and its complement fibers over $S^1$ with fiber the disk.

In each step we are adding handles to this disk in such a way that they accumulate on the boundary. If we intersect this disk with any compact set, we have just one connected component. Hence it has only one end. Therefore, our surface has one end.

![Figure 6: Disk with handles intersected with a compact set.](image)

The first part of the theorem has been proved. For the second part, observe that the closure of the fiber in $S^3$ is the fiber union its boundary. Therefore $\Sigma_{\theta} - \Sigma_{\theta} = \Lambda(\Gamma)$. ■

**Remarks 4.5**

1. Theorem 4.4 allows us to compute the monodromy of $\Lambda$ (see next section), and describe completely its complement. We would like to point out that this fact provides a way to determine if two dynamically-defined, fibered wild knots are isotopic.

2. This theorem can be generalized to fibered links.

3. This theorem gives an open book decomposition ([21] pages 340-341) of $S^3 - \Lambda(\Gamma)$, where the “binding” is the wild knot $\Lambda(\Gamma)$, and each “page” is an orientable, infinite-genus surface with one end (the fiber).

   Indeed, this decomposition can be thought of in the following way. For the above theorem, $S^3 - \Lambda(\Gamma)$ is $\Sigma_{\theta} \times [0, 1]$ modulo the identification of the top with the bottom through a characteristic homeomorphism. Consider $\Sigma_{\theta} \times [0, 1]$ and identify the top with the bottom. This is equivalent to keeping $\partial \Sigma_{\theta}$ fixed and spinning $\Sigma_{\theta} \times \{0\}$ with respect to $\partial \Sigma_{\theta}$ until it is glued with $\Sigma_{\theta} \times \{1\}$. Removing $\partial \Sigma_{\theta}$ we obtain the open book decomposition.
4. The fibering map $\psi$ can be chosen to be smooth. In fact, by a theorem of Nielsen ([10]) we know that every surface bundle over $S^1$ depends only on the monodromy isotopy class and by another theorem of Nielsen we have that every homeomorphism of a surface is isotopic to a diffeomorphism.

COROLLARY 4.6 The universal covering of $\mathbb{S}^3 - \Lambda(\Gamma)$ is $\mathbb{R}^3$.

Proof. Let $P : (\mathbb{S}^3 - \Lambda(\Gamma)) \to S^1$ be a locally trivial fibration given by the above theorem. If $\exp : \mathbb{R} \to S^1$ denotes the covering map $r \mapsto e^{ir}$, then the following diagram commutes

$$
\begin{array}{ccc}
\bar{X} & \xrightarrow{\bar{P}} & \mathbb{R}^1 \\
\pi \downarrow & & \exp \downarrow \\
\mathbb{S}^3 - \Lambda(\Gamma) & \xrightarrow{P} & S^1 
\end{array}
$$

where $\bar{P} : \bar{X} \to \mathbb{R}^1$ is the pull-back of the fibration $P : (\mathbb{S}^3 - \Lambda(\Gamma)) \to S^1$.

Any fibration with contractible base space is trivial, i.e. the total space is homeomorphic to the product of the fiber with the base space, $\bar{X} \cong \mathbb{R}^1 \times \Sigma_\theta$, where $\Sigma_\theta = \text{fiber}$. The universal covering of $S^1$ is $\mathbb{R}$ and the universal covering of $\Sigma_\theta$ is $\mathbb{R}^2$. Therefore, the result follows. ■

Remark 4.7 We already know by the Sphere Theorem (see [21] page 102) that the universal covering of $\mathbb{S}^3 - \Lambda(\Gamma)$ must be contractible. We point out that there are uncountable contractible open 3-manifolds which are not homeomorphic to $\mathbb{R}^3$ (see [15]). P. H. Doyle ([5]) gave an example of a wild knot whose fundamental group is $\mathbb{Z} \ast G$, where $G \neq \mathbb{Z}$ is a group. We suspect that the universal covering of its complement is a contractible space not homeomorphic to $\mathbb{R}^3$.

Remarks 4.8 1. Through the reflecting process, we have constructed a “Seifert Surface” for the wild knot $\Lambda(\Gamma)$ with the property that its interior is a regular surface and its closure is a “crumpled surface” which is not, technically, a surface. This remains true for any template and not just for the fibered ones. Thus the wild knots obtained by our dynamic construction all admit Seifert surfaces in this generalized sense.

2. Let $E = \Sigma_\theta \cup \overline{\Sigma_{\theta+\pi}}$. Then $E$ is a wild equator of $\mathbb{S}^3$, i.e. there exists a homeomorphism $h : \mathbb{S}^3 \to \mathbb{S}^3$ such that $h(E) = E$ and $h(H^+) = H^-$, where $H^+ = \{x \in \mathbb{S}^3 - E : x \in \Sigma_\psi, \theta < \psi < \theta + \pi\}$ and $H^- = \mathbb{S}^3 - E - H^+$. 

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5 Monodromy

Let $K$ be a non-trivial fibered tame knot and let $S$ be the oriented fiber. Since $S^3 - K$ fibers over the circle, we know that $S^3 - K$ is a mapping torus equal to $S \times [0, 1]$ modulo a characteristic homeomorphism $\psi : S \to S$ that glues $S \times \{0\}$ with $S \times \{1\}$. This homeomorphism induces a homomorphism $\psi_# : \Pi_1(S) \to \Pi_1(S)$ called the monodromy of the fibration.

Another way to understand the monodromy is through the Poincaré first return map of a flow defined as follows. Let $M$ be a connected, compact manifold and let $f_t$ be a flow that possesses a transversal section $\eta$. It follows that if $x \in \eta$ then there exists a continuous function $t(x) > 0$ such that $f_t \in \eta$. We may define the first return Poincaré map $F : \eta \to \eta$ as $F(x) = f_{t(x)}(x)$. This map is a diffeomorphism and induces a homomorphism of $\Pi_1$, called the monodromy (see [28], chapter 5).

For the manifold $S^3 - K$, the flow that defines the first return Poincaré map $\Phi$, is the flow that cuts transversally each page of its open book decomposition. Notice that we can choose this flow, up to isotopy, in such a way that the first return Poincaré map can be extended to $K$ as the identity.

Consider a pearl-necklace $T$ subordinate to $K$. As we have observed during the reflecting process, $K$ and $S$ are copied in each reflection (preserving or reversing orientation). So the flow $\Phi$ is also copied, and its direction changes according to the number of reflections. Hence the Poincaré map can be extended at each stage, giving us in the limit a homeomorphism $\psi : \Sigma_\theta \to \Sigma_\theta$ that identifies $\Sigma_\theta \times \{0\}$ with $\Sigma_\theta \times \{1\}$ and which induces the monodromy of the wild knot.

From the above, if we know the monodromy of the knot $K$ then we know the monodromy of the wild knot $\Lambda(\Gamma)$.

By the long exact sequence associated to a fibration, we have

\[ 0 \to \Pi_1(\Sigma_\theta) \to \Pi_1(S^3 - \Lambda(\Gamma)) \xrightarrow{\psi} \mathbb{Z} \to 0, \]

which has a homomorphism section $\Psi : \mathbb{Z} \to S^3 - \Lambda(\Gamma)$. Therefore (1) splits. As a consequence $\Pi_1(S^3 - \Lambda(\Gamma))$ is the semi-direct product of $\mathbb{Z}$ with $\Pi_1(\Sigma_\theta)$. 

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Observe that by applying Van-Kampen’s Theorem in each step coupled with lemma 2.4.1 in [22], \( \Pi_1(\Sigma_{\theta}) \) is isomorphic to the infinite free product of \( \Pi_1(S) \) with itself.

**Example 5.1** Let \( K \) be the trefoil knot. Then \( S \) is the punctured torus and its fundamental group is the free group in two generators, \( a \) and \( b \). The monodromy map \( \psi_# \) sends \( a \mapsto b^{-1} \) and \( b \mapsto ab \). Its order is six up to an outer automorphism (See [21] pages 330-333).

The monodromy in the limit \( \psi_# : \Pi_1(\Sigma_{\theta}) \to \Pi_1(\Sigma_{\theta}) \) is given by \( a_i \mapsto b_i^{-1} \) and \( b_i \mapsto a_i b_i \), where \( \Pi_1(\Sigma_{\theta}) = \{a_i, b_i\} \). So

\[
\Pi_1(S^3 - \Lambda(\Gamma)) \cong \Pi_1(S^1) \ltimes_{\psi_#} \Pi_1(\Sigma_{\theta})
\]

\[
= \{a_i, b_i, c : a_i \ast c = b_i^{-1}, \ b_i \ast c = a_i b_i \}
\]

\[
= \{a_i, c : c^{-1} a_i^{-1} c = a_i c^{-1} a_i^{-1} \}
\]

\[
= \{a_i, c : c = a_i c a_i^{-1} c^{-1} \}
\]

\[
= \{a_i, c : c = a_i c^{-1} a_i c^{-1} a_i^{-1} c^{-1} \}
\]

\[
= \{a_i, c : c = a_i c a_i^{-1} c^{-1} a_i^{-1} c^{-1} \}
\]

Let \( \alpha_i = c a_i c^{-1} ; \)

\[
= \{\alpha_i, c : c = \alpha_i c a_i^{-1} \alpha_i^{-1} \}
\]

\[
= \{\alpha_i, c : c a_i = \alpha_i c a_i \}
\]

This gives another method for computing the fundamental group of a wild knot whose complement fibers over the circle.

**Corollary 5.2** Let \( T \) be a pearl-necklace whose template is a non-trivial tame fibered knot, with fiber \( S \). Then \( \Pi_1(\Omega(\Gamma)/\Gamma) \cong \mathbb{Z} \ltimes \Pi_1(S) \) and \( \Pi_n(\Omega(\Gamma)/\Gamma) = 0 \) for \( n > 1 \).

6 **Hyperbolicity**

In this section, we consider the question of whether the complement of a dynamically-defined, fibered wild knot can be given the structure of a complete hyperbolic 3-manifold.
**Theorem 6.1** Let $K$ be a fibered tame knot, whose complement admits a complete hyperbolic structure. Then the complement of the wild knot $\Lambda(\Gamma)$, obtained from $K$ through a reflecting process is not hyperbolic.

**Proof.** Suppose that $\Omega = \mathbb{S}^3 - \Lambda$ is a complete hyperbolic 3-manifold. By virtue of the way $\Lambda$ was constructed (see section 3), we can decompose $\Omega$ into pieces $W_k = \mathbb{S}^3 - |T_k|$, such that $\Omega = \cup W_k$. Now, $i : W_2 \hookrightarrow \Omega$ induces an injective map $i_# : \pi_1(W_2) \hookrightarrow \pi_1(\Omega)$. If we denote the image of this group by $G = i_#(\pi_1(W_2))$, then $M = \mathbb{H}^3/G$ is a complete hyperbolic 3-manifold which covers $W_2$. Let $p : M \to W_2$ be this covering.

Since $G$ is finitely generated, by a theorem of Scott (see [3]) it follows that there is a compact submanifold $M_T$ of $M$, such that the inclusion map $i : M_T \to M$ induces an isomorphism of fundamental groups. In our case $M_T$ is homeomorphic to the closure of the copy of $W_2$ in $M$. Moreover, there is a bijective correspondence between the boundary components of $M_T$ and the topological ends of $M$. The $\mathbb{Z} \oplus \mathbb{Z}$-cusps correspond precisely to the toroidal components of $\partial M_T$.

By the above, $M$ has one end which is a $\mathbb{Z} \oplus \mathbb{Z}$-cusp and $M$ is diffeomorphic to $T^2 \times \mathbb{R}$. Any 2-torus $T^2_t := T^2 \times \{t\} \subset \mathbb{R}$, separates $M$ into two pieces $A_t$ with finite volume and $B_t$. One can assume, without loss of generality, that the hyperbolic diameter of $T^2_t$ tends to zero as $t \to \infty$. This implies that $p(A_t) \subset W_2$ has finite volume and $p(A_t)$ determines the only end of $W_2$. Therefore $W_2 - p(A_t)$ has finite volume and hence $W_2$ itself is of finite volume (compare [3] page 253).

Now, let $V$ be a closed tubular neighbourhood of $|T| \subset W_2$. The boundary of $V$ is an incompressible torus in $W_2$ and it is not isotopic to a component of its boundary. This contradicts Hyperbolization Theorem of Thurston. Therefore the result follows.

**Remark 6.2** We recall that a complete hyperbolic manifold $N$ is said to be tame if it is homeomorphic to the interior of a compact 3-manifold. Marden’s Tameness Conjecture says that a complete hyperbolic manifold with finitely generated fundamental group, is tame. Recently, Ian Agol has announced a proof of this conjecture (http://atlas-conferences.com/cgi-bin/abstract/camc-67). In our case this implies immediately that $W_2$ has finite volume.
On the other hand, we have the following positive result.

**THEOREM 6.3** Let $K$ be a fibered tame knot whose complement is a complete hyperbolic 3-manifold. Then, the complement of the limit set $\Lambda(\Gamma)$ obtained from $K$, admits a canonical decomposition into a countable union of submanifolds $M_j$ with boundary. Their interiors are pairwise disjoint, and each interior of $M_j$ is hyperbolic. The boundary of $M_j$ is composed by a finite number of pairwise disjoint properly embedded incompressible cylinders.

**Proof.** Let $T$ be a necklace of $n$ pearls subordinate to $K$. Since $S^3 - |T|$ is the mapping torus of a homeomorphism $\psi : S \to S$, where $S$ is the fiber. Then by the Hyperbolization Theorem (see [16] chapter 3), it follows that $\psi$ is pseudo-Anosov.

As we discussed in the section 5, when we reflect with respect to a pearl $\Sigma_i$, both the monodromy $\psi$ and $T - \Sigma_i$ are copied into $\text{Int}(B_i)$, where $B_i$ is the ball whose boundary is $\Sigma_i$. Since $\text{Int}(B_i) - I(|T - \Sigma_i|)$ also fibers over the circle and $\psi$ is pseudo-Anosov, we have that it is hyperbolic (see [16] chapter 3). Notice that the hyperbolic pieces are equivalent to the complement of $K$. They are separated by the corresponding pearls without the two points of tangency, and clearly they are incompressible cylinders. Hence the result follows. ■

Observe that in the first step of the above decomposition, $n$ copies of $M_1$ are added to it along cylinder components of its boundary. From the second step onwards, $n - 1$ copies are added along cylinder components to each $M_j$ of the previous step.

In [27] W. Thurston stated some open questions concerning 3-manifolds and kleinian groups. The first one is: Do all 3-manifolds have decompositions into geometric pieces?. In this context, as a consequence of the above Theorem, we have the following

**COROLLARY 6.4** Let $K$ be a tame, fibered knot, whose complement is a hyperbolic 3-manifold. Then the complement of the limit set obtained from $K$, has a canonical decompositions into geometric pieces. In other words, it satisfies a recasting of the Thurston’s Geometrization Conjecture for non-compact manifolds.
Let $K$ be a non-trivial tame fibered knot with fiber the surface $S$. Then $S^3 - \Lambda(\Gamma)$ is the mapping torus of a homeomorphism $\psi : \Sigma_{\theta} \to \Sigma_{\theta}$, which was described in section 5. Notice that this map can be extended to $\Lambda(\Gamma)$ as the identity.

We construct the $q$-fold cyclic covering, $C^q(\Lambda)$, branched over the wild knot $\Lambda(\Gamma)$, as follows (see [21]). We consider $\Sigma_{\theta} \times [0, 1]$ and we identify $\partial \Sigma_{\theta} \times \{t\}$, $t \in (0, 1]$ with $\partial \Sigma_{\theta} \times \{0\}$ via the identity map and $\Sigma_{\theta} \times \{0\}$ with $\Sigma_{\theta} \times \{1\}$ via the $q^\text{th}$-iterate $\psi^q$, of the characteristic map. The resulting space is $C^q(\Lambda)$.

The covering map $p : C^q(\Lambda) \to S^3$ is defined in the usual way (see [21]).

Remarks 7.1
1. The branched covering $p : C^q(\Lambda) \to S^3$ is regular.
2. The space $C^q(\Lambda)$ is compact and path-connected.

**Proposition 7.2** The fundamental group of $C^q(\Lambda)$ is infinitely generated.

**Proof.** Let $K$ be a non-trivial tame fibered knot with fiber $S$. Suppose that the fundamental group of the surface $S$ is

$$\Pi_1(S) = \{a_1, a_2, \ldots, a_n : r_1, \ldots, r_l\}$$

Let $T$ be the pearl necklace subordinate to $K$ and let $\psi$ be the characteristic map of $\Lambda$. Then

$$\Pi_1(S^3 - |T|) \cong \Pi_1(K) \cong \Pi_1(S^1) \ltimes_{\psi'|_T} \Pi_1(S)$$

$$= \{c, a_1, a_2, \ldots, a_n : r_1, \ldots, r_l, a_k \ast c = \psi|_T(a_k)\}$$

and the fundamental group of the $q$-fold cyclic branched covering $C^q(K)$ of $S^3$ with branched set $K$ is

$$\Pi_1(C^q(K)) = \{c, a_1, a_2, \ldots, a_n : r_1, \ldots, r_l, a_k \ast c = \psi|_T(a_k), \psi^q|_T = 1\}.$$ 

By Van-Kampen’s theorem, we have that the fundamental group of the fiber $\Sigma_{\theta}$ is

$$\Pi_1(\Sigma_{\theta}) = \{a_{i1}, a_{i2}, \ldots, a_{in} : r_{i1}, \ldots, r_{il}\}$$

where $i \in \mathbb{N}$. Thus, the fundamental group of $S^3 - \Lambda$ is

$$\Pi_1(S^3 - \Lambda) \cong \Pi_1(\Lambda) \cong \Pi_1(S^1) \ltimes_{\psi} \Pi_1(\Sigma_{\theta})$$
\[ \{ c, a_{i1}, a_{i2}, \ldots, a_{in} : r_{i1}, \ldots, r_{id}, a_{ik} \cdot c = \psi_#(a_{ik}) \}, \]

where \( \psi_#(a_{ik}) \) is a product of \( a_{i1}, a_{i2}, \ldots, a_{in} \). This implies that the fundamental group of \( C^q(\Lambda) \) is

\[
\Pi_1(C^q(\Lambda)) = \{ c, a_{i1}, a_{i2}, \ldots, a_{in} : r_{i1}, \ldots, r_{id}, a_{ik} \cdot c = \psi_#(a_{ik}), \psi_#^q = 1 \}
\]

\[
\cong ( \cdots ((\Pi_1(C^q(K)) * \{c\} \Pi_1(C^q(K))) * \{c\} \cdots * \{c\} \Pi_1(C^q(K))) * \{c\} \cdots
\]

which is an infinite free product of the fundamental group of \( C^q(K) \). Hence \( \Pi_1(C^q(\Lambda)) \) is infinitely generated. \( \blacksquare \)

**Example 7.3** Let \( K = T_{2,3} \) be the right-handed trefoil knot. Then its monodromy is order six, up to isotopy. Consider the 5\(^{th}\)-fold cyclic branched covering \( C^5(\Lambda) \) of \( S^3 \) with branched set the wild knot \( \Lambda \), obtained from \( K \).

Then

\[
\Pi_1(C^5(\Lambda)) \cong ( \cdots ((\Pi_1(C^5(K)) * \{c\} \Pi_1(C^5(K))) * \{c\} \cdots * \{c\} \Pi_1(C^5(K))) * \{c\} \cdots
\]

is the infinite free product of the fundamental group of \( C^5(K) \). In other words, it is the infinite free product of the binary icosahedral group.

On the other hand, by the Hurewicz homomorphism, we have than the first homology group of \( C^5(\Lambda) \) is

\[
H_1(C^5(\Lambda), \mathbb{Z}) \cong \Pi_1(C^5(\Lambda))/[\Pi_1(C^5(\Lambda)), \Pi_1(C^5(\Lambda))].
\]

We recall that \( C^5(K) \) is the Poincaré Sphere \( \Sigma_{2,3,5} \), hence \( H_1(C^5(K), \mathbb{Z}) = 0 \).

This implies that \( H_1(C^5(\Lambda), \mathbb{Z}) = 0 \).

**THEOREM 7.4** The space \( C^q(\Lambda) \) is not semilocally simply connected. In particular, it does not admit a universal covering.

**Proof.** Let \( K \) be a non-trivial tame fibered knot with fiber \( S \), and let \( \Lambda \) be the limit set obtained from \( K \) through the reflecting process.

Let \( x \in \Lambda \subset C^q(\Lambda) \). Consider \( U \) an open, connected neighbourhood of \( x \). Then we can think \( U \) as a \([0, 1]\)-family of connected neighbourhoods \( U_t \subset \Sigma_\theta \times \{t\} \) glued together along their boundaries, with \( U_0 \) identified with \( U_1 \) via the \( q^{th} \)-iterate, \( \psi^q \) of the characteristic map of \( \Lambda \).

Let \( U_x \) be the largest neighbourhood of \( x \) in \( \Sigma_\theta \) that satisfies \( U_x \subset U_t \), \( t \in \mathbb{S}^1 \). Then we have that a infinite number of copies of the surface \( S \) is
contained in $\text{Int}(U_x)$. Consider $\tilde{U} = U_x \times [0, 1]/(y, 1) \sim (\psi^q(y), 0) \subset U$. Then $\Pi_1(\tilde{U})$ is a subgroup of $\Pi_1(U)$ and $\tilde{U}$ contains an infinite number of copies of $\mathcal{C}^q(K)$. This implies that (see the proof of the above theorem)

$$\Pi_1(U) \cong \{c, a_{i1}, a_{i2}, \ldots, a_{im} : r_{i1}, \ldots, r_{il}, a_{ik} * c = \psi_#(a_{ik}), \psi^q_# = 1\}$$

$$\cong (\cdots (\Pi_1(\mathcal{C}^q(K)) *_{\{c}\{c\}} \Pi_1(\mathcal{C}^q(K)) *_{\{c\}} \cdots *_{\{c\}} \Pi_1(\mathcal{C}^q(K)) *_{\{c\}} \cdots$$

where $i \in \mathbb{N}$. Therefore, $\mathcal{C}^q(\Lambda)$ is not semilocally simply connected. ■

**COROLLARY 7.5** The space $\mathcal{C}^q(\Lambda)$ is not a manifold.

**Remarks 7.6**

1. The space $\mathcal{C}^q(\Lambda) - \Lambda$ is a 3-manifold.

2. The group of covering transformations of $\mathcal{C}^q(\Lambda)$ is $\mathbb{Z}/q\mathbb{Z}$. Hence the quotient space $\mathcal{C}^q(\Lambda)/\mathbb{Z}/q\mathbb{Z} \cong S^3$ is a manifold.

**THEOREM 7.7** Every closed, connected, orientable 3-manifold contains either a fibered wild knot or a fibered wild link.

**Proof.** Let $K$ be the figure-eight knot. Let $T$ be a pearl-necklace subordinate to $K$. Then the wild knot $\Lambda(\Gamma)$ obtained from $K$ through the reflecting process is contained in $|T|$.

By [9], we have that every closed, connected, orientable 3-manifold $M$ is a n-fold branched covering of $S^3$ with branched set $K$. Let $p$ be the covering map. Then $p^{-1}(|T|) \subset M$ is a finite number of copies of $|T|$ and in the interior of each copy of $|T|$, we have a copy of $\Lambda(\Gamma)$.

Therefore, $M$ contains a fibered wild knot or link. ■

**Remark 7.8** Even though any closed, connected, orientable 3-manifold contains a wild knot it is not so clear that this wild knot can be chosen to be fibered.

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