Distributed Robust Nash Equilibrium Seeking for Mixed-Order Games by a Neural-Network-Based Approach

Maojiao Ye®, Member, IEEE, Lei Ding®, Senior Member, IEEE, and Jizhao Yin®

Abstract—In practical applications, decision makers with heterogeneous dynamics may be engaged in the same decision-making process. This motivates us to study distributed Nash equilibrium seeking for games in which players are mixed-order (first- and second-order) integrators influenced by unknown dynamics and external disturbances in this article. To solve this problem, we employ an adaptive neural network to manage unknown dynamics and disturbances, based on which a distributed Nash equilibrium seeking algorithm is developed by further adapting concepts from gradient-based optimization and multiagent consensus. By constructing appropriate Lyapunov functions, we analytically prove the convergence of the reported method. Theoretical investigations suggest that players’ actions would be steered to an arbitrarily small neighborhood of the Nash equilibrium, which is also testified by simulations.

Index Terms—Distributed network, mixed-order integrators, Nash equilibrium, neural network.

I. INTRODUCTION

GAME theory acts as an effective technique for investigating interactive decision-making situations involving multiple participants. Typical examples that fall into the game theoretic framework include economic dispatch problems [1], charging coordination among electric vehicles [2], energy consumption coordination in the smart grid [3], global optimization [4], and formation control of multiagent systems [5]. The wide applications of games motivate many researchers to direct their energies to the development of Nash equilibrium seeking algorithms, leading to fruitful results in this field. For instance, games in which players are first-order integrators, second-order integrators, high-order integrators, and linear dynamic ones were, respectively, investigated in [6], [7], [8], [9], [10], [11], [12], and [13]. Hybrid games, in which both discrete-time players and continuous-time players are engaged, were addressed by Ye et al. [14]. It is worth mentioning that most of the existing works focused on games in which players have homogeneous dynamics and related results on games with heterogeneous players are relatively limited. However, with distinct computation abilities, working environment and dynamics, decision makers exhibit significant and remarkable heterogeneity in various perspectives. Steered by the incentive to accommodate heterogeneity among different entities, heterogeneous multiagent systems have attracted quite a few attention. For example, Jiang et al. [15], Hua et al. [16], Yaghaie et al. [17], and Li et al. [18] were concerned with formation control, output regulation, as well as optimal coordination in linear multiagent systems, respectively, in which the constant matrices associated with the agents’ dynamics are different from each other. Nonlinear systems with heterogeneous dimensions were considered in [19]. Moreover, second-order systems with time-varying gains and distinct inertia were explored in [20]. Among various kinds of heterogeneities, systems with both first- and second-order agents are of great interest since velocity-driven vehicles may work and collaborate with acceleration-driven ones [21].

Zheng and Wang [22] dealt with consensus for a category of multiagent systems where the engaged agents are described by first-order integrators as well as second-order integrators without using knowledge on the agents’ velocity information. In addition, average consensus problems were addressed in [21] under similar settings. Consensus protocols were considered to be subject to bounded delays for mixed-order systems in a discrete-time scenario in [23]. However, few results on games in which players’ dynamics are of different order have been reported, especially, when both nonlinear dynamics and disturbances are involved. Therefore, this article focuses on the establishment of distributed Nash equilibrium seeking algorithms for games with mixed-order participants. Moreover, as in many practical situations, e.g., physical hydraulic systems [24], air hybrid vehicles [25], and marine surface vessels [26], external disturbances and unmodeled dynamics are inevitable due to complex working conditions.

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environment of engineering actuators and limited knowledge about the explicit system model, this article further considers that players’ dynamics are influenced by unknown nonlinear dynamics and time-varying disturbances. Noticing that the radial basis function neural network (RBFNN) has been proven to be effective for approximating unknown continuous mappings over a compact domain (see, e.g., [27], [28], [29], [30]), this article takes the benefits of RBFNN to establish robust Nash equilibrium seeking strategies for the considered mixed-order games. With some preliminary findings presented in [31], we give the core contributions and novelties of this manuscript as follows.

1) This article accommodates games with mixed-order integrator-type players who are suffering from both unknown nonlinear dynamics and time-varying disturbances. In comparison with the state of art, the setting has rarely been explored. The presented exploration provides a unified viewpoint on how to simultaneously deal with first- and second-order players and offers convenience for the applications of games in mixed-order multi-agent systems.

2) Unmodeled but Lipschitz nonlinear dynamics and disturbances are addressed through adapting an adaptive neural network. By compensating players’ dynamics with the approximated value generated by the neural network, a distributed Nash equilibrium seeking strategy is developed for mixed-order games. This article significantly improves its conference version [31] by considering the disturbances and nonlinear dynamics that are unknown beforehand.

3) The convergence property of the reported algorithm is analytically investigated on the basis of Lyapunov stability analysis. The mathematical investigations show that the reported method is capable of steering players’ actions and velocities, respectively, to be arbitrarily close to the Nash equilibrium and zero.

We organize the remaining sections as follows. Related preliminary knowledge is offered in Sections II and III shows the problem in consideration. Method development and the corresponding analysis are provided in Section IV. Numerical illustrations are offered in Section V. Furthermore, concluding statements are illustrated in Section VI.

II. PRELIMINARIES

Algebraic Graph Theory: Graph $G = (N, E)$ is given by a vertex set $N = \{1, 2, \ldots, N\}$, together with the associated edge set $E \subseteq N \times N$. This article considers that $G$ is undirected in the sense that for any $(i, j) \in E$, we can derive that $(j, i) \in E$. Furthermore, the graph is connected provided that for every pair of distinct vertices, there exists a path. The adjacency matrix and Laplacian matrix of $G$ are defined as $A = [a_{ij}]$ and $L = D - A$, respectively, in which $a_{ij} = 1$ if $(i, j) \in E$, else, $a_{ij} = 0$. $a_{ii} = 0$ is a diagonal matrix with its $i$th diagonal entry being $\sum_{j=1}^{N} a_{ij}$ and the notation $B = [b_{ij}]$ defines a matrix whose $(i, j)$th entry is $b_{ij}$ [6].

RBFNNs: A continuous function $l(z) : \mathbb{R}^N \rightarrow \mathbb{R}^N$ can be estimated on a compact domain $z \in \Omega_z \subseteq \mathbb{R}^N$ by

$$l_{NN}(z) = W^T S(z) \tag{1}$$

in which $W \in \mathbb{R}^{q \times N}$ is an adjustable weight matrix and $q$ is the neuron number. Moreover, $S(z) = [s_1(z), s_2(z), \ldots, s_q(z)]^T$ is the activation function given by

$$s_i(z) = \exp \left[\frac{-(z - \mu_i)^T (z - \mu_i)}{\rho_i^2}\right], i = 1, 2, \ldots, q \tag{2}$$

in which $\mu_i = [\mu_{i1}, \mu_{i2}, \ldots, \mu_{iN}]^T$ denotes the center of the receptive field, and $\rho_i$ denotes the width of the Gaussian function [27].

Then, for $z \in \Omega_z$ and any arbitrary small positive constant $\tilde{\epsilon}$, there exist a weight matrix $W^* \in \mathbb{R}^{q \times N}$ and a neural number $q$ so that

$$l(z) = W^* S(z) + \epsilon \tag{3}$$

in which $\epsilon$ is the estimation error that satisfies $|\epsilon| \leq \tilde{\epsilon}$ [27].

**Lemma 1 [27]:** Assume that $V(t) \geq 0$ is a continuous function defined for $t \geq 0$ and $V(0)$ is bounded. Then, if

$$\dot{V}(t) \leq -aV(t) + b \tag{4}$$

where $a > 0$ and $b > 0$ are constants, we can obtain that

$$V(t) \leq V(0)e^{-at} + \frac{b}{a}(1 - e^{-at}) \tag{5}$$

**Lemma 2 [34]:** For any $\epsilon > 0$ and $\eta \in \mathbb{R}$

$$0 \leq |\eta| - \eta \tanh \left(\frac{\eta}{\epsilon}\right) \leq K \epsilon \tag{6}$$

where $K > 0$ is a constant that satisfies $K = e^{-(K+1)}$, indicating that $K = 0.2785$.

III. PROBLEM DESCRIPTION

Consider a game containing $N$ players, in which the player set is given by $N = \{1, 2, \ldots, N\}$. Suppose that $n$ ($n \geq 1$ and $n < N$) of them are first-order integrators whose actions are steered by

$$\dot{x}_i = u_i + g_i(x) + d_i(t), i \in \mathcal{N}_f \tag{7}$$

in which $x_i \in \mathbb{R}, u_i \in \mathbb{R}, g_i(x) \in \mathbb{R}$, and $d_i(t) \in \mathbb{R}$, respectively, represent for the action, the control signal to be designed, the unknown dynamics and the external, time-varying disturbance of player $i$. Moreover, $x$ is a vector containing all players’ actions, i.e., $x = [x_1, x_2, \ldots, x_N]^T$ and $\mathcal{N}_f$ is the set of first-order players, i.e., $\mathcal{N}_f = \{1, 2, \ldots, n\}$. Furthermore, assume that the rest of players are second-order integrators whose actions evolve according to

$$\dot{x}_i = v_i \tag{8}$$

$$\dot{v}_i = u_i + g_i(x) + d_i(t), \quad i \in \mathcal{N}_s$$

in which $x_i \in \mathbb{R}, v_i \in \mathbb{R}, u_i \in \mathbb{R}, g_i(x) \in \mathbb{R}$ and $d_i(t) \in \mathbb{R}$, respectively, denote the action, velocity, control signal, unknown dynamics and disturbance of player $i$. In addition, $\mathcal{N}_s$ is the set of second-order integrators, i.e., $\mathcal{N}_s = \{n+1, n+2, \ldots, N\}$. Based on the above notations, it can be clear that $N = \mathcal{N}_f \cup \mathcal{N}_s$. Associate each player $i, i \in N$ with a cost function $f_i(x)$, which can be alternatively denoted as $f_i(x, x_{-i})$ by defining $x_{-i} = [x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_N]^T$. Authorize licensed use limited to the terms of the applicable license agreement with IEEE. Restrictions apply.
Then, each player $i$ in the considered game is involved in the following optimization problem

$$
\min_{x_i \in \mathbb{R}} \ f_i(x_i, x_{-i})
$$

s.t. (7) or (8)  \hspace{1cm} (9)

resulting in the Nash equilibrium. Therefore, the purpose of this manuscript is to construct control signals $u_i$, $i \in \mathcal{N}$ so that players’ actions $x$ can be steered to a small neighborhood of the Nash equilibrium $x^* = (x^*_i, x^*_i)_{i \in \mathcal{N}}$, that satisfies

$$
\dot{f_i}(x^*_i, x^*_i) \leq -\phi_i(x^*_i, x^*_i) \hspace{1cm} (10)
$$

for $x_i \in \mathbb{R}, i \in \mathcal{N}$.

For notational simplicity, let $\nabla f_i(x) = [\partial f_i(x)/\partial x_i]$ and $\nabla^2 f_i(x) = [\partial^2 f_i(x)/\partial x_i \partial x_j]$. The mathematical development of this article is based on the subsequent conditions.

Assumption 1: For each $i \in \mathcal{N}$, $f_i(x)$ is $C^2$ and $\nabla f_i(x)$ is globally Lipschitz with constant $l_i$ for $x \in \mathbb{R}^N$.

Assumption 2: The players can exchange information through an undirected and connected graph $\mathcal{G}$.

For notational convenience, let $A_0 = \text{diag}(a_{ij})$ for $ij \in \mathcal{N}$ denote a diagonal matrix with its diagonal elements successively being $a_{11}, a_{12}, \ldots, a_{1N}, a_{21}, \ldots, a_{2N}$. Moreover, let $\otimes$ denote the Kronecker product. Then, under Assumption 2, $- (L \otimes I_{N \times N} + A_0)$, in which $I_{N \times N}$ is an identity matrix of dimension $N \times N$, is Hurwitz. Hence, $P(L \otimes I_{N \times N} + A_0) + (L \otimes I_{N \times N} + A_0)P = Q$ for some positive diagonal matrix $P$ and symmetric positive definite matrix $Q$ of compatible dimensions [35, Th. 2.9].

Assumption 3: For $x, z \in \mathbb{R}^N$

$$
(x - z)^T (P(x) - P(z)) \geq m ||x - z||^2
$$

where $m > 0$ is a constant and $P(x) = [\nabla f_1(x), \nabla f_2(x), \ldots, \nabla f_N(x)]^T$.

Remark 1: This strong monotonicity assumption ensures that the Nash equilibrium $x^*$ exists and is unique. Moreover, it is achieved at $P(x) = 0_N$ [6].

Assumption 4: For $x \in \mathbb{R}^N$, $\nabla f_i(x)$ is bounded for $i \in \mathcal{N}_s, j \in \mathcal{N}$.

Assumption 5: For each $i \in \mathcal{N}$, $g_i(x)$ is globally Lipschitz with constant $l_i$ and $d_i(t)$ is bounded.

Remark 2: Note that in [33], it is required that the unmodeled dynamics $g_i(x)$ is sufficiently smooth with its first two partial derivatives being bounded provided that $x$ is bounded. Similarly, the disturbance $d_i(t)$ is supposed to be sufficiently smooth with $\dot{d}_i(t)$ and $\ddot{d}_i(t)$ being bounded in [7] and [33]. From Assumption 5, we see that these conditions are relaxed to some extent in this article. Moreover, compared with internal model-based approaches in [9] and [10], we do not assume disturbances to be of specific forms and different from [12] that considered quadratic games, this article considers games with general costs. Besides, this article considers mixed-order system dynamics while in the aforementioned works, players’ dynamics are of the same order. The heterogeneity would further introduce some difficulties in the establishment and analytical study of the seeking algorithms.

IV. MAIN RESULTS

In this section, a distributed Nash equilibrium seeking strategy will be developed on the basis of adaptive neural networks, consensus algorithms and gradient-based optimization algorithms. Moreover, the corresponding convergence analysis will be provided.

A. Method Establishment

To realize disturbance rejection in the considered game, the core idea of this article is to adapt RBFNN (see, e.g., [27] and many other references) to accommodate the unknown disturbances and dynamics. With RBFNN, the control input of player $i$ for $i \in \mathcal{N}_f$ is designed as

$$
u_i = -k_1(x_i - z_i) - \hat{W}_i^T S_i(y_i) - \phi_i
$$

in which $k_1$ is a positive constant, $S_i(\cdot) \in \mathbb{R}^{q_i \times 1}$ is the activation function defined in (1) and (2), $z_i \in \mathbb{R}$ and $\hat{W}_i \in \mathbb{R}^{q_i \times 1}$ ($q_i$ is the number of neurons for player $i$) are adaptively updated according to

$$
\dot{z}_i = -k_2 \nabla f_i(y_i) \hspace{1cm} (13)
$$

in which $k_2$ is a positive constant. $\nabla f_i(y_i) = \nabla f_i(x)|_{x=y_i}$ and

$$
\dot{\hat{W}}_i = \beta S_i(y_i)(x_i - z_i) \hspace{1cm} (14)
$$

if $\hat{W}_i^T \hat{W}_i < W_{\text{max}}$ or alternatively, $\hat{W}_i^T \hat{W}_i = W_{\text{max}}$ and $(x_i - z_i)\hat{W}_i^T S_i(y_i) < 0$, where $\beta$ and $W_{\text{max}}$ are positive constants. In addition

$$
\dot{\hat{W}}_i = \beta S_i(y_i)(x_i - z_i) - \beta (x_i - z_i)\hat{W}_i^T S_i(y_i) \hspace{1cm} (15)
$$

if $\hat{W}_i^T \hat{W}_i = W_{\text{max}}$ and $(x_i - z_i)\hat{W}_i^T S_i(y_i) \geq 0$. Note that it is required that $\hat{W}_i^T(0)\hat{W}_i(0) \leq W_{\text{max}}$, which can be achieved by choosing the initial value of $\hat{W}_i$ to be zero. Moreover, in (12)

$$
\phi_i = \delta \tanh \left(\frac{K\delta (x_i - z_i)}{\epsilon}\right)
$$

in which $\epsilon > 0$, $\delta > 0$ are constants. Furthermore, $y_i \in \mathbb{R}^N$ and is defined as $y_i = [y_{i1}, y_{i2}, \ldots, y_{iN}]^T$ where $y_{ij}$ is produced by

$$
\dot{y}_{ij} = -k_3 \sum_{k=1}^{N} a_{ik} \left(y_{ij} - y_{kj}\right) + a_{ij} \left(y_{ij} - \bar{x}_j\right), j \in \mathcal{N}_s \hspace{1cm} (17)
$$

where $k_3 > 0$ is a constant, $\bar{x}_j = z_j$ for $j \in \mathcal{N}_f$ and $\bar{x}_j = x_j$ for $j \in \mathcal{N}_s$.

Remark 3: The control input designed for first-order integrator-type players in (12) contains a regulation term $x_i - z_i$, which is employed to regulate $x_i$ to $z_i$. As the purpose of this article is to drive $x$ to $x^*$, such a regulation term actually transfers the problem to drive $x$ defined as $x = [z_1, z_2, \ldots, z_N]^T$, to $x^*$, which is achieved by (13) and (17). In addition, $\hat{W}_i^T S_i(y_i)$ and $\phi_i$ are designed based on RBFNN to address unknown dynamics and time-varying disturbances.

By similar ideas, for second-order players, the control input of player $i$ for $i \in \mathcal{N}_f$ is designed as

$$
u_i = -k_4 \nabla f_i(y_i) - k_4 \nu_i - \hat{W}_i^T S_i(y_i) - \phi_i
$$

(18)
where \( k_d > 0 \) is a constant and \( \hat{W}_i \) is updated according to
\[
\dot{\hat{W}}_i = \beta S_i(y_i)(k_d \nabla f_i(y_i) + v_i)
\]
(19)
if \( \hat{W}^T_i \dot{\hat{W}}_i < W_{\text{max}} \) or alternatively \( \hat{W}^T_i \dot{\hat{W}}_i = W_{\text{max}} \) and \((k_d \nabla f_i(y_i) + v_i)\hat{W}^T_i S_i(y_i) < 0 \). Moreover, if \( \hat{W}^T_i \dot{\hat{W}}_i = W_{\text{max}} \) and \((k_d \nabla f_i(y_i) + v_i)\hat{W}^T_i S_i(y_i) \geq 0 \)
\[
\dot{\hat{W}}_i = \beta S_i(y_i)(k_d \nabla f_i(y_i) + v_i) - \beta (k_d \nabla f_i(y_i) + v_i)\hat{W}^T_i S_i(y_i) \hat{W}_i
\]
where \( \hat{W}^T_i (0) \dot{\hat{W}}_i (0) \leq W_{\text{max}} \).

Furthermore
\[
\phi_i = \delta \tanh \left( \frac{K \delta (k_d \nabla f_i(y_i) + v_i)}{\epsilon} \right)
\]
(21)

and
\[
\dot{y}_i = -k_3 \left( \sum_{k=1}^N a_{ik} (y_i - y_k) + a_{ij} (y_i - \bar{x}_j) \right), j \in \mathcal{N}_i.
\]
(22)

Remark 4: The control input design for second-order players in (18) is similar to the control design in (12), where \( \hat{W}^T_i S_i(y_i) \) and \( \phi_i \) are included to accommodate unknown dynamics and disturbances. Different from (12), stabilization of players’ velocities \( v_i \) is needed and achieved by the negative feedback of velocity \( v_i \) in (18). Note that each player has only knowledge on its own information but not other players’ information. Therefore, for each player \( i \), it generates a local estimate on \( \bar{x}_j \), which is denoted by \( y_i \). By supposing that neighboring information exchange is allowed, players provide their own information \( \bar{x}_i \) as a leading reference to be followed by others through the communication graph [see (17) and (22)]. In this way, players achieve estimation of unknown information in a distributed fashion by utilizing (17) and (22), which are standard leader-following consensus protocols [6]. They follow the same format, but \( \bar{x}_i \) is for second-order players, while \( \bar{x}_j \) is for first-order integrators. In (17) and (22), \( y_i \) evolves to \( \bar{x}_i \) according to the leader-following consensus protocol. It is worth mentioning that as \( \bar{x}_i \) is known to player \( i \), \( y_i \) can be directly set to \( \bar{x}_i \), which does not affect the effectiveness of the proposed method.

Recalling the dynamics of first- and second-order integrator-type players in (7) and (8), we get that for the first-order players
\[
\dot{x}_i = -k_1 (x_i - z_i) - \left[ \hat{W}^T_i S_i(y_i) \right]_{\mathcal{N}_i} - \phi_i \lambda_i \left[ g_i(x) \right]_{\mathcal{N}_i} + [d_i(t)]_{\mathcal{N}_i} - \phi_i \lambda_i \left[ g_i(x) \right]_{\mathcal{N}_i} + [d_i(t)]_{\mathcal{N}_i}
\]
(23)
and for the second-order players
\[
\dot{x}_i = v_s - k_2 \left[ \nabla f_i(y_i) \right]_{\mathcal{N}_i} - \left[ \hat{W}^T_i S_i(y_i) \right]_{\mathcal{N}_i} - \phi_i \lambda_i \left[ g_i(x) \right]_{\mathcal{N}_i} + [d_i(t)]_{\mathcal{N}_i}
\]
(24)
and for \( y = [y_1, y_2, \ldots, y_N]^T \)
\[
\dot{y} = -k_3 (\mathcal{L} \otimes \mathbf{1}_{N \times N} + A_0) (y - \mathbf{1}_N \otimes \bar{x})
\]
(25)

where \( \bar{x} = [\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_N]^T \), \( y = [x_i]_{\mathcal{N}_i} \), \( z_i = [z_i]_{\mathcal{N}_i} \), \( x_i = [x]_{\mathcal{N}_i} \), and \( v_s = [v_i]_{\mathcal{N}_i} \) and the notation \([p_i]_{\mathcal{N}_i}([p_i]_{\mathcal{N}_i}) \) defines the concatenated vector of \( p_i \) for \( i \in \mathcal{N}_i \).

By similar analysis in [27], the subsequent result, which is needed in the convergence analysis of the proposed method, can be obtained.

Lemma 3: By the adaptive laws in (14), (15), (19), and (20)
\[
\hat{W}^T_i(t) \dot{\hat{W}}_i(t) \leq W_{\text{max}} \quad \forall i \in \mathcal{N}
\]
(26)

for all \( t \geq 0 \).

Proof: The analysis follows that of [27] and the details are provided in the Appendix A for the convenience of readers.

In the subsequent section, the analytical investigation on the proposed method will be presented.

B. Convergence Analysis

Since RBFNN is utilized to estimate the unknown dynamics over a compact domain (see preliminaries on RBFNN in Section II and references therein), we first show the boundedness of system trajectories before proceeding to present the convergence results.

Lemma 4: Under Assumptions 1–5, there exists a positive constant \( k_2^* \) so that for each \( k_2 > k_2^* \), there exist positive constants \( k_1^*(k_2), k_3^*(k_2) \) so that for \( k_1 > k_1^*(k_2), k_3 > k_3^*(k_2) \), there exists a positive constant \( k_3^* (k_2, k_3) \) so that for \( k_4 > k_4^* \), \( x(t), z_i(t), y_i(t) \) and \( y(t) \) generated by the proposed method in (23)–(25) stay bounded given that their initial values are bounded.

Proof: Let \( \bar{v}_i = k_2 \nabla f_i(y_i) + v_i \) for \( i \in \mathcal{N}_i \) and \( \bar{v}_i = [\bar{v}_i]_{\mathcal{N}_i} \).

Then,
\[
\bar{v}_i = v_i + k_2 [\nabla f_i(y_i)]_{\mathcal{N}_i}.
\]
(27)
Therefore
\[
\dot{x}_i = \bar{v}_i - k_2 [\nabla f_i(y_i)]_{\mathcal{N}_i} - \phi_i \lambda_i \left[ g_i(x) \right]_{\mathcal{N}_i} + [d_i(t)]_{\mathcal{N}_i}
\]
(28)
and for \( y = [y_1, y_2, \ldots, y_N]^T \)
\[
\dot{y} = -k_3 (\mathcal{L} \otimes \mathbf{1}_{N \times N} + A_0) (y - \mathbf{1}_N \otimes \bar{x})
\]
(29)

Then, by Assumption 3
\[
\dot{\bar{v}}_i = (\bar{x} - x^*)^T \bar{v}_i
\]
\[
= (\bar{x} - x^*)^T [\bar{v}_i, \bar{v}_j]^T
\]
\[
= -k_2 (\bar{x} - x^*)^T [\nabla f_i(y_i)]_{\mathcal{N}_i} + (\bar{x} - x^*)^T [\bar{v}_i, \bar{v}_j]^T
\]
that where
\[
\begin{aligned}
&\nabla_2 = \nabla_3(T_k[\nabla_i(y_i)]_{i})_N \\
&+ \nabla_2(T_k[\nabla_i(y_i)]_{i})_N \\
&+ \nabla_2(T_k[\nabla_i(y_i)]_{i})_N \\
&\leq -k_3 \lambda_{\min}(Q) \|y - 1_N \otimes \hat{x}\|^2 \\
&+ 2k_2 \sqrt{N} \max_{i \in N} \{t_i\} \|P\| \|y - 1_N \otimes \hat{x}\|^2 \\
&+ 2k_2 N \max_{i \in N} \{t_i\} \|P\| \|y - 1_N \otimes \hat{x}\| \|\hat{x} - x^*\| \\
&+ 2\sqrt{N} \|P\| \|y - 1_N \otimes \hat{x}\| \|\hat{v}_s\| \\
& \tag{33}
\end{aligned}
\]
where the notation \(\lambda_{\min}(Q)\) denotes the minimum eigenvalue of \(Q\). Hence
\[
\hat{V} \leq -k_2 m \|\hat{x} - x^*\|^2 - k_1 \|\nabla_i(y_i)\|^2 - k_4 \|
\hat{v}_s\| \tag{34}
\]
Therefore
\[
\hat{V} \leq \hat{V}_1 \|\hat{x} - x^*\|^2 - \hat{V}_2 \|\nabla_i(y_i)\|^2 - \hat{V}_3 \|
\hat{v}_s\| \tag{35}
\]
where
\[
\hat{V}_1 = k_2 m - \frac{1}{2} \sqrt{N} \max_{i \in N} \{t_i\} \|y - 1_N \otimes \hat{x}\| \tag{35}
\]
and
\[
\hat{V}_4 = k_3 \lambda_{\min}(Q) - 2k_2 \sqrt{N} \max_{i \in N} \{t_i\} \|P\|^2 \\
- \frac{1}{2} \sqrt{N} \max_{i \in N} \{t_i\} \|y - 1_N \otimes \hat{x}\|^2 \\
- \frac{1}{2} \max_{i \in N} \{t_i\} \|\hat{x} - x^*\|^2 \\
\tag{36}
\]
Hence, by choosing \(k_2\) to be sufficiently large, \(\hat{V}_1 > 0\). Then, for fixed \(k_2\), we can choose \(k_1\) and \(k_3\) to be sufficiently large such that \(\hat{V}_2 > 0\) and \(\hat{V}_4 > 0\). Therefore, for fixed \(k_1, k_2\), and \(k_3\), we can choose \(k_4\) to be sufficiently large such that \(\hat{V}_3 > 0\). By such a tuning rule
\[
\hat{V} \leq -k_2 m \|\hat{x} - x^*\|^2 - k_1 \|\nabla_i(y_i)\|^2 - k_4 \|
\hat{v}_s\| \tag{36}
\]
i.e.,
\[
\dot{V} \leq -\frac{\min\{\hat{\Psi}_1, \hat{\Psi}_2, \hat{\Psi}_3, \hat{\Psi}_4\}}{2\max\{\lambda_{\text{max}}(P), \frac{1}{2}\}} V
\]
(37)
for \(\sqrt{V} \geq \left(2(\alpha + \beta)\max\{\lambda_{\text{max}}(P), (1/2)\}\right)/\left[\min\{\hat{\Psi}_1, \hat{\Psi}_2, \hat{\Psi}_3, \hat{\Psi}_4\}\right]\) from which the conclusion can be easily derived.

From Lemma 4, it can be concluded that \(\eta_i\) for \(i \in N\) would stay bounded given that the control gains are suitably chosen and the initial values of the variables are bounded. Hence, it is clear that for any positive constant \(\bar{\epsilon}\), there exist a weight matrix \(W_i^*\) and a neural number \(q_i\) that satisfy
\[
g_i(y_i) = W_i^* N \eta_i + \epsilon_i
\]
(38)
where \(\epsilon_i < \bar{\epsilon}\) by the results of RBFNN in Section II and references therein.

Therefore, by (23)–(25) and (38), it is derived that for first-order integrators
\[
\dot{x}_i = -k_1(x_i - z_i) - \left[\hat{W}_i^* N \eta_i\right]_{N_i} - [g_i(x) - g_i(y_i)]_{N_i} + [d_i(t) + \epsilon_i]_{N_i}
\]
and for second-order players
\[
\dot{x}_i = v_i
\]
\[
\dot{v}_i = -k_2v_i - k_2k_3\left[\hat{W}_i^* N \eta_i\right]_{N_i} - [g_i(x) - g_i(y_i)]_{N_i} + [d_i(t) + \epsilon_i]_{N_i}
\]
In addition
\[
\dot{y} = -k_3(\mathcal{L} \otimes I_{N \times N} + A_0)(y - I_N \otimes \bar{x}).
\]
(41)

The subsequent supportive lemmas are given before we provide the convergence results.

**Lemma 5:** Suppose that \(W_i^* W_i^* \leq W_{\text{max}}\) for \(i \in N\). Then, for \(i \in N_i\)
\[
\dot{\hat{W}}_i^* = \left(\frac{\dot{\hat{W}}_i}{\hat{\beta}} - S_i(y_i) (x_i - \bar{x}_i)\right) \leq 0
\]
(42)
and for \(i \in N_s\)
\[
\dot{\hat{W}}_i = \left(\frac{\dot{\hat{W}}_i}{\hat{\beta}} - S_i(y_i) \bar{v}_i\right) \leq 0
\]
(43)
in which \(\hat{W}_i = \hat{W}_i - W_i^*\) for \(i \in N\).

**Proof:** See the Appendix-B.

**Lemma 6:** Let \(\delta \geq |\epsilon_i| + |d_i(t)|\) for \(i \in N\) and \(t \geq 0\). Then, for each \(i \in N_i\)
\[
(x_i - \bar{z}_i)(d_i(t) + \epsilon_i - \phi_i) \leq \epsilon
\]
(44)
and for each \(i \in N_s\)
\[
\bar{v}_i(d_i(t) + \epsilon_i - \phi_i) \leq \epsilon
\]
(45)

**Proof:** See the Appendix-C.

We are now well prepared to provide the convergence analysis for the system in (39)–(41).

**Theorem 1:** Assume that Assumptions 1-5 hold and \(\delta \geq |\epsilon_i| + |d_i(t)|\). \(W_i^* W_i^* \leq W_{\text{max}}\) for \(i \in N\), \(t \geq 0\). Then, for any pair of positive constants \(\Lambda\) and \(\Xi\), there exist positive constants \(\beta^*\) and \(k_2^*\) so that for \(\beta > \beta^*\) and \(k_2 > k_2^*\), there exist positive constants \(k_1^*\) and \(k_3^*\) so that for \(k_1 > k_1^*, k_2 > k_2^*\), there exists a positive constant \(k_4^*\) \((k_2, k_3)\) so that for \(k_4 > k_4^*\)
\[
\left\|x(t) - x^*\right\| \leq \Xi \forall t > T
\]
(46)
for some \(T \geq 0\) given that \(\|\left(\tilde{\mathbf{x}}(0) - x^*\right)^T, \tilde{v}(0)^T, (y(0) - I_N \otimes \bar{x}(0))^T, (y(0) - I_N \otimes \bar{x}(0))^T + \sum_{i=1}^N \tilde{W}_i(0)^T \tilde{W}_i(0)\) \(\leq\lambda\).

**Proof:** Let \(V = \sum_{i=1}^N V_i\), where
\[
V_1 = \frac{1}{2}(\bar{x} - x^*)^T (\bar{x} - x^*)
\]
\[
V_2 = \frac{1}{2} \tilde{v}_s^T \tilde{v}_s
\]
\[
V_3 = \frac{1}{2}(y - I_N \otimes \bar{x})^T (y - I_N \otimes \bar{x})
\]
\[
V_4 = \frac{1}{2} \sum_{i=1}^N \tilde{W}_i^T \tilde{W}_i.
\]
(47)

Then, following the analysis in Lemma 4, we get that
\[
\dot{V}_1 = -k_2m\left\|\tilde{x} - x^*\right\|^2 + \left\|\tilde{x} - x^*\right\| \left\|\tilde{v}_s\right\|
\]
\[
+ k_2\max_{i \in N} \left\|\bar{l}_i\right\| \left\|\tilde{x} - x^*\right\| \left\|y - I_N \otimes \bar{x}\right\|
\]
(48)

and
\[
\dot{V}_2 = \tilde{v}_s^T \tilde{v}_s
\]
\[
= \tilde{v}_s^T \left(-k_4 \tilde{v}_s + k_2 H_1 [\dot{\tilde{x}}(\bar{y})]_{N_i}\right)
\]
\[
+ \tilde{v}_s^T \left(-[\hat{W}^T N \eta_i]_{N_i} - [g_i(x) - g_i(y_i)]_{N_i} + [d_i(t) + \epsilon_i]_{N_i}\right)
\]
\[
+ \tilde{v}_s^T [g_i(x) - g_i(y_i)]_{N_i}
\]
\[
= -k_4 \left\|\tilde{v}_s\right\|^2 + k_2 k_3 b \left\|\tilde{v}_s\right\| \left\|y - I_N \otimes \bar{x}\right\| \left\|\tilde{v}_s\right\|
\]
\[
- \tilde{v}_s^T \tilde{W}_i^* N \eta_i + (N - n)\epsilon
\]
\[
+ \tilde{v}_s^T \left[g_i(x) - g_i(y_i)\right]_{N_i}
\]
(49)

where the result in Lemma 6 has been utilized.

Moreover
\[
\dot{V}_3 = (y - I_N \otimes \bar{x})^T (\dot{y} - z_f)
\]
\[
= (y - I_N \otimes \bar{x})^T \left(-k_1 (x_i - z_i) + k_2 \left[\hat{W} \eta_i\right]_{N_i}\right)
\]
\[
+ (x_i - z_i)^T \left(-[\hat{W}^T N \eta_i]_{N_i} - [g_i(x) - g_i(y_i)]_{N_i}\right)
\]
\[
+ (x_i - z_i)^T [d_i(t) + \epsilon_i]_{N_i}
\]
\[
+ (x_i - z_i)^T \left[g_i(x) - g_i(y_i)\right]_{N_i}
\]
\[
\leq -k_1 \left\|x_i - z_i\right\|^2 + ne
\]
\[
+ k_2 \sqrt{N} \max_{i \in N} \left\|\bar{l}_i\right\| \left\|x_i - z_i\right\| \left\|\bar{x} - x^*\right\|
\]
\[
+ k_2 \max_{i \in N} \left\|\bar{l}_i\right\| \left\|x_i - z_i\right\| \left\|y - I_N \otimes \bar{x}\right\|
\]
where Lemma 3 is utilized in the last inequality, we get that
\begin{align}
V_4 & \leq -k_3\lambda_{\min}(\mathbf{Q})\|\mathbf{y} - \mathbf{1}_N \otimes \bar{\mathbf{x}}\|^2 \\
& \quad + 2k_2\sqrt{N}\max_{i \in N}\|\tilde{\mathbf{l}}_i\|\|\mathbf{P}\|\|\mathbf{y} - \mathbf{1}_N \otimes \bar{\mathbf{x}}\|^2 \\
& \quad + 2k_2N\max_{i \in N}\|\tilde{\mathbf{l}}_i\|\|\mathbf{y} - \mathbf{1}_N \otimes \bar{\mathbf{x}}\|\big\|\bar{\mathbf{x}} - \mathbf{x}^*\big\| \\
& \quad + 2\sqrt{N}\|\mathbf{P}\|\|\mathbf{y} - \mathbf{1}_N \otimes \bar{\mathbf{x}}\|\big\|\bar{\mathbf{x}} - \mathbf{x}^*\big\|.
\end{align}

Furthermore
\begin{align}
\dot{V}_s &= \sum_{i=1}^{N}\frac{\tilde{W}_i^T\tilde{W}_i}{\beta} \\
&\leq \sum_{i=1}^{n}\frac{\tilde{W}_i^T\tilde{W}_i}{\beta} + \sum_{i=n+1}^{N}\frac{\tilde{W}_i^T\tilde{W}_i}{\beta}.
\end{align}

Hence
\begin{align}
\dot{V} & \leq -k_2m\big\|\bar{\mathbf{x}} - \mathbf{x}^*\big\|^2 - k_1\big\|\mathbf{x}_f - \mathbf{z}_f\big\|^2 - k_4\big\|\bar{\mathbf{v}}_s\big\|^2 \\
& \quad - \left(3\lambda_{\min}(\mathbf{Q}) - 2k_2\sqrt{N}\max_{i \in N}\|\tilde{\mathbf{l}}_i\|\|\mathbf{P}\|\big\|\mathbf{y} - \mathbf{1}_N \otimes \bar{\mathbf{x}}\big\|^2 \\
& \quad + 2k_2\sqrt{N}\max_{i \in N}\|\tilde{\mathbf{l}}_i\|\|\mathbf{y} - \mathbf{1}_N \otimes \bar{\mathbf{x}}\|\big\|\bar{\mathbf{x}} - \mathbf{x}^*\big\| \\
& \quad + 2\sqrt{N}\|\mathbf{P}\|\|\mathbf{y} - \mathbf{1}_N \otimes \bar{\mathbf{x}}\|\big\|\bar{\mathbf{x}} - \mathbf{x}^*\big\|
\right) \\
& \quad - \left(\sum_{i=1}^{N}\tilde{W}_i^T\tilde{W}_i + \mathbf{P}\right) + k_2\max_{i \in N}\|\tilde{\mathbf{l}}_i\|\|\mathbf{y} - \mathbf{1}_N \otimes \bar{\mathbf{x}}\|\big\|\bar{\mathbf{x}} - \mathbf{x}^*\big\| \\
& \quad + k_2\max_{i \in N}\|\tilde{\mathbf{l}}_i\|\|\mathbf{y} - \mathbf{1}_N \otimes \bar{\mathbf{x}}\|\big\|\bar{\mathbf{x}} - \mathbf{x}^*\big\| \\
& \quad + k_2\max_{i \in N}\|\tilde{\mathbf{l}}_i\|\mathbf{y} - \mathbf{1}_N \otimes \bar{\mathbf{x}}\|\big\|\bar{\mathbf{x}} - \mathbf{x}^*\big\| \\
& \quad + k_2\max_{i \in N}\|\tilde{\mathbf{l}}_i\|\|\mathbf{y} - \mathbf{1}_N \otimes \bar{\mathbf{x}}\|\big\|\bar{\mathbf{x}} - \mathbf{x}^*\big\| \\
& \quad + k_2\max_{i \in N}\|\tilde{\mathbf{l}}_i\|\|\mathbf{y} - \mathbf{1}_N \otimes \bar{\mathbf{x}}\|\big\|\bar{\mathbf{x}} - \mathbf{x}^*\big\|
\right)
\end{align}

Noticing that
\begin{align}
\|\bar{\mathbf{x}} - \mathbf{x}^*\| & \leq \frac{1}{2}\|\bar{\mathbf{x}} - \mathbf{x}^*\|^2 + \frac{1}{2}\|\bar{\mathbf{v}}_s\|^2 \\
& \leq \left(2\sqrt{N}\|\mathbf{P}\| + \frac{k_2b}{2}\right)\|\mathbf{y} - \mathbf{1}_N \otimes \bar{\mathbf{x}}\|\big\|\bar{\mathbf{x}} - \mathbf{x}^*\big\|.
\end{align}

In addition
\begin{align}
\left(2\sqrt{N}\|\mathbf{P}\| + k_2b\right)\|\mathbf{y} - \mathbf{1}_N \otimes \bar{\mathbf{x}}\| \\
& \leq \left(\sqrt{N}\|\mathbf{P}\| + \frac{k_2b}{2}\right)\|\bar{\mathbf{v}}_s\|^2.
\end{align}

Furthermore
\begin{align}
k_2\sqrt{N}\max_{i \in N}\|\tilde{\mathbf{l}}_i\|\|\mathbf{x}_f - \mathbf{z}_f\|\big\|\bar{\mathbf{x}} - \mathbf{x}^*\big\| \\
& \leq \sqrt{N}\max_{i \in N}\|\tilde{\mathbf{l}}_i\|\|\mathbf{y} - \mathbf{1}_N \otimes \bar{\mathbf{x}}\|^2 \\
& \quad + \frac{k_2\sqrt{N}\max_{i \in N}\|\tilde{\mathbf{l}}_i\|k_2^2}{2}\|\mathbf{y} - \mathbf{1}_N \otimes \bar{\mathbf{x}}\|^2.
\end{align}

As \(\|\tilde{\mathbf{W}}_i\| = \|\tilde{\mathbf{W}}_i - W_i^*\| \leq \|W_i^*\| + \|\tilde{\mathbf{W}}_i\| \leq 2\sqrt{W_{\max}}\)
where Lemma 3 is utilized in the last inequality, we get that \(4W_{\max} - \sum_{i=1}^{N}\tilde{W}_i^T\tilde{W}_i \geq 0\). Hence, further utilizing the results in Lemma 5
\begin{align}
\dot{V} & \leq -k_2m\big\|\bar{\mathbf{x}} - \mathbf{x}^*\big\|^2 - k_1\big\|\mathbf{x}_f - \mathbf{z}_f\big\|^2 - k_4\big\|\bar{\mathbf{v}}_s\big\|^2 \\
& \quad - \left(3\lambda_{\min}(\mathbf{Q}) - 2k_2\sqrt{N}\max_{i \in N}\|\tilde{\mathbf{l}}_i\|\|\mathbf{P}\|\big\|\mathbf{y} - \mathbf{1}_N \otimes \bar{\mathbf{x}}\big\|^2 \\
& \quad + 2k_2\sqrt{N}\max_{i \in N}\|\tilde{\mathbf{l}}_i\|\|\mathbf{y} - \mathbf{1}_N \otimes \bar{\mathbf{x}}\|\big\|\bar{\mathbf{x}} - \mathbf{x}^*\big\| \\
& \quad + 2\sqrt{N}\|\mathbf{P}\|\|\mathbf{y} - \mathbf{1}_N \otimes \bar{\mathbf{x}}\|\big\|\bar{\mathbf{x}} - \mathbf{x}^*\big\|
\right) \\
& \quad - \left(\sum_{i=1}^{N}\tilde{W}_i^T\tilde{W}_i + \mathbf{P}\right) + k_2\max_{i \in N}\|\tilde{\mathbf{l}}_i\|\|\mathbf{y} - \mathbf{1}_N \otimes \bar{\mathbf{x}}\|\big\|\bar{\mathbf{x}} - \mathbf{x}^*\big\| \\
& \quad + k_2\max_{i \in N}\|\tilde{\mathbf{l}}_i\|\|\mathbf{y} - \mathbf{1}_N \otimes \bar{\mathbf{x}}\|\big\|\bar{\mathbf{x}} - \mathbf{x}^*\big\| \\
& \quad + k_2\max_{i \in N}\|\tilde{\mathbf{l}}_i\|\|\mathbf{y} - \mathbf{1}_N \otimes \bar{\mathbf{x}}\|\big\|\bar{\mathbf{x}} - \mathbf{x}^*\big\| \\
& \quad + k_2\max_{i \in N}\|\tilde{\mathbf{l}}_i\|\|\mathbf{y} - \mathbf{1}_N \otimes \bar{\mathbf{x}}\|\big\|\bar{\mathbf{x}} - \mathbf{x}^*\big\|
\right)
\end{align}
\[-\frac{k_2^2}{2} - \frac{k_2^2 b}{2} - \sqrt{N} \max_{\eta_i} \eta_i (\bar{l}_i) \right) (y - 1_N \otimes \bar{x})^2 \]

\[-\sum_{i=1}^{N} \tilde{W}_i^T \tilde{W}_i + (x_j - z_j)^T [g_i(x) - g_i(y)] \eta_i \bar{l}_i \]

\[+ \tilde{v}_s^T [g_i(x) - g_i(y)] \eta_i \bar{l}_i + N \epsilon + 4N \text{W}_{\max} \]

(60)

where \( \Phi_1 = k_2 m - (1/2) - \left( [\sqrt{N} \max_{\eta_i} \eta_i (\bar{l}_i)] |P| + \max_{\eta_i} \eta_i (\bar{l}_i)]^2 / 2 \right) - (1/\sqrt{N} \max_{\eta_i} \eta_i (\bar{l}_i) / 2) \). Furthermore

\[(x_j - z_j)^T [g_i(x) - g_i(y)] \eta_i \bar{l}_i \leq \sqrt{N} \max_{\eta_i} \eta_i \|x_j - z_j\|^2 + \max_{\eta_i} \eta_i \|x_j - z_j\| \|y - 1_N \otimes \bar{x}\| \]

(61)

and similarly

\[\tilde{v}_s^T [g_i(x) - g_i(y)] \eta_i \bar{l}_i \leq \sqrt{N} \max_{\eta_i} \eta_i \|x_j - z_j\| \|y - 1_N \otimes \bar{x}\| \]

(62)

Let \( \Phi_2 = k_4 - (1/2) - \sqrt{N} |P| - [k_2^2 b / 2] - \left( [\sqrt{N} \max_{\eta_i} \eta_i (\bar{l}_i)] / 2 \right) - \left( \max_{\eta_i} \eta_i (\bar{l}_i) / 2 \right) - \sqrt{N} \max_{\eta_i} \eta_i (\bar{l}_i) / 2) - \left( [\sqrt{N} \max_{\eta_i} \eta_i (\bar{l}_i)] / 2 \right) \), and then

\[\tilde{v} \leq -KN \epsilon + 4N \text{W}_{\max} \]

(63)

where \( K = \min [2\Phi_1, 2\Phi_2, 2\Phi_3, (\Phi_4 / [\lambda_{\max}(P)])], 2\beta \). Hence, by Lemma 1

\[V(t) \leq V(0)e^{-Kt} + \frac{N \epsilon + 4N \text{W}_{\max}}{K} \]

(64)

where \( K \) can be arbitrarily large by the following tuning rule: choose \( k_3 \) to be large enough so that \( \Phi_1 \) is sufficiently large. Then, for fixed \( k_2 \), choose \( k_1 \) and \( k_3 \) such that \( \Phi_3 \) and \( \Phi_4 \) are sufficiently large. Then, for fixed \( k_3 \), choose \( k_2 \) to be large enough so that \( \Phi_2 \) is sufficiently large. If this is the case, \( K \) is sufficiently large with sufficiently large \( \beta \), indicating that \( V(t) \) is decaying to be arbitrarily close to zero. Recalling the definitions of the Lyapunov candidate function and \( \tilde{v}_s \), the conclusion can be obtained.

Remark 5: As \( A \) and \( \Xi \) can be any positive constants, Theorem 1 indicates that for any bounded initialization, the reported method (39)–(41) can drive \( x(t) \) to an arbitrary small neighborhood of the Nash equilibrium \( x^* \). The main content of this article focuses on distributed Nash equilibrium seeking for games involving mixed-order players. Note that when all players are first-order integrators, i.e., \( n = N \) (second-order integrators, i.e., \( n = 0 \)), Theorem 1 illustrates that the method in (12)–(17) [(18)–(22)] steers players’ actions to an arbitrarily small neighborhood of the Nash equilibrium \( x^* \) as well. Therefore, the presented analysis is a unified analysis of both first- and second-order players.

Remark 6: Theorem 1 indicates that one should first choose \( \beta \) and \( k_2 \) to be sufficiently large. Then, with fixed \( k_2 \), one should choose \( k_1 \) and \( k_3 \) to be sufficiently large, following which \( k_4 \) should be chosen to be sufficiently large to derive the result. It is worth mentioning that this tuning rule is in line with that of Lemma 4, indicating the existence of the control gains.

If unknown nonlinear and disturbance modulations (i.e., \( g_i(x) + d_i(t) \)) are not contained in the players’ dynamics, the corresponding estimation module can be removed from the proposed algorithm. If this is the case, we get that for the first-order players

\[\dot{x}_j = k_4 (x_j - z_j) \]

\[z_j = -k_2 [\nabla f_i (y)] \eta_i \bar{l}_i \]

(65)

and for the second-order players

\[\dot{x}_s = v_s \]

\[v_s = -k_4 v_s - k_2 k_4 [\nabla f_i (y)] \eta_i \bar{l}_i \]

(66)

with

\[y = -k_3 (L \otimes [I_{n \times N} + A_0])(y - 1_N \otimes \bar{x}) \]

(67)

where the definitions for the variables and gains follow those in (12)–(22). In this case, the subsequent result can be derived.

Theorem 2: Under Assumptions 1–5, there exists a positive constant \( k_2^* \) such that for \( k_2 > k_2^* \), there exist positive constants \( k_1^* \) and \( k_3^* \) so that for \( k_1 > k_1^*(k_2) \), \( k_3 > k_3^*(k_2) \), there exists a positive constant \( k_4^*(k_2, k_3) \) so that for \( k_4 > k_4^* \), the Nash equilibrium \( x^* \) is globally exponentially stable with the strategy in (65)–(67).

Proof: Define

\[V = \frac{1}{2} (x - x^*)^T (x - x^*) + \frac{1}{2} \tilde{v}_s^T \tilde{v}_s \]

\[+ \frac{1}{2} (x_j - z_j)^T (x_j - z_j) \]

\[+ (y - 1_N \otimes \bar{x})^T P (y - 1_N \otimes \bar{x}) \]

(68)

Then, following the proof of Theorem 1:

\[\dot{V} \leq -k_2 m \|x - x^*\|^2 + \|x - x^*\| \|\tilde{v}_s\| \]

\[+ k_4 \max_{i \in N} \|\tilde{t}_i\| \|\tilde{x} - x^*\| + k_3 \lambda_{\max} \|y - 1_N \otimes \tilde{x}\| \]

\[- k_4 \|\tilde{v}_s\|^2 + k_2 k_3 \|y - 1_N \otimes \tilde{x}\| \]

\[- k_1 \|x_j - z_j\|^2 \]

\[+ k_2 \sqrt{N} \max_{i \in N} \|\tilde{t}_i\| \|\tilde{x} - z_j\| \|\tilde{x} - x^*\| \]

\[+ k_2 \max_{i \in N} \|\tilde{t}_i\| \|x_j - z_j\| \|y - 1_N \otimes \tilde{x}\| \]

\[- k_3 \lambda_{\max} \|\tilde{t}_i\| \|\tilde{x} - z_j\| \|y - 1_N \otimes \tilde{x}\| \]

\[+ 2 k_2 \sqrt{N} \max_{i \in N} \|\tilde{t}_i\| \|P\| \|y - 1_N \otimes \tilde{x}\| \]

\[+ 2 k_3 \max_{i \in N} \|\tilde{t}_i\| \|P\| \|y - 1_N \otimes \tilde{x}\| \|\tilde{x} - x^*\| \]

\[+ 2 \sqrt{N} \|P\| \|y - 1_N \otimes \tilde{x}\| \|\tilde{v}_s\| \]

(69)

Let \( \Phi_1 = k_2 m - (5/2) \), \( \tilde{\Phi}_2 = k_4 - (1/2) - \left( [k_2 k_3 b^2 / 2] - \sqrt{N} \|P\| \right) \), \( \Phi_3 = k_3 \lambda_{\max} \|\tilde{t}_i\| \left( [k_2 \max_{i \in N} \|\tilde{t}_i\| / 2] \right) - (1/2) - \left( \max_{i \in N} \|\tilde{t}_i\| k_2 / 2 \right) - \left( \max_{i \in N} \|\tilde{t}_i\| \right) \).
$2k_2\sqrt{N}\max_{i\in\mathcal{N}}|\dot{u}_i||P| - (k_2N \max_{i\in\mathcal{N}}|\dot{u}_i||P|)^2 - \sqrt{N}||P||$, and $\Phi_4 = k_4 - (k_2N \max_{i\in\mathcal{N}}|\dot{u}_i|)N/2 - (k_2 \max_{i\in\mathcal{N}}|\dot{u}_i|)2/2$.

Then, by choosing $k_2 > (5/2m)$, we get that $\Phi_1 > 0$ and then, for fixed $k_1$ and $k_3$ to be sufficiently large such that $\Phi_3 > 0$ and $\Phi_4 > 0$. Moreover, for fixed $k_2$, $k_3$, we can choose $k_4$ such that $\Phi_2 > 0$. By such a tuning rule, we get that

$$\dot{V} \leq -\min\{\Phi_1, \Phi_2, \Phi_3, \Phi_4\}||E||^2$$

in which $E = ([\ddot{x} - \dot{x}^*]^T, \ddot{y}_1^T, (x_j - \bar{z}_j)^T, (y - 1_N \otimes \bar{x})^T)^T$.

Recalling the definition of $V$, the conclusion is drawn by [32, Th. 4.10].

Compared with Theorem 1, it can be seen that Theorem 2 improves the semiglobal results in Theorem 1 to global versions without unknown dynamics and disturbances. In addition, Assumption 4 can be further relaxed in this case and the corresponding result is stated below.

**Corollary 1:** Assume that Assumptions 1-3 and 5 hold and $\nabla_{ij}(x)$ for $i \in \mathcal{N}, j \in \mathcal{N}_i$ are bounded if $x$ is bounded. Then, for any bounded initial condition, there exists a positive constant $k_0$ so that for $k_2 > k_0$, there exist positive constants $k_1^*$ and $k_2^*$ so that for $k_1 > k_1^*(k_2)$, $k_3 > k_2^*(k_2)$, there exists a positive constant $k_3^*(k_2, k_3)$ so that for $k_3 > k_3^*$, $x(t)$ exponentially converges to the Nash equilibrium $x^*$ under (65)–(67).

Compared with Theorem 2, Corollary 1 illustrates that if Assumption 4 is not satisfied, the corresponding result is degraded to a semiglobal counterpart by supposing that the initial values of the variables are bounded.

**Remark 7:** Though in this article, we suppose that the communication graph is undirected for presentation simplicity, the given results are applicable to deal with strongly connected digraphs by further utilizing Lemma 1 in [36].

V. NUMERICAL VERIFICATION

This section offers numerical verification of the reported methods by a connectivity control game involving five vehicles concerned in [33]. In the game, the cost function of vehicle $i$ is

$$f_i(x) = h_i(x_i) + l_i(x)$$

where $x_i = [x_{i1}, x_{i2}]^T \in \mathbb{R}^2$ and

$$h_i(x_i) = x_i^T m_{ii} x_i + x_i^T m_i + i$$

in which $m_{ii} = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$, $m_i = [i, i]^T$. Moreover, $l_1(x) = \|x_1 - x_2\|^2$, $l_2(x) = \|x_2 - x_3\|^2$, $l_3(x) = \|x_3 - x_2\|^2$, $l_4(x) = \|x_4 - x_3\|^2 + \|x_4 - x_5\|^2$, and $l_5(x) = \|x_5 - x_1\|^2$. In the presented example, $x_i^* = [-(1/2), -(1/2)]^T$ for $i \in \{1, 2, 3, 4, 5\}$ [33]. In the upcoming simulations, it is assumed that vehicles 1–3 are first-order integrators and vehicles 4 and 5 are second-order integrators.

To be more specific, for $i \in \{1, 2, 3\}$

$$\dot{x}_i = u_i + g_i(x) + d_i(t)$$

in which $g_1(x) + d_1(t) = [x_{21} + \sin(t), x_{22} + \sin(t)]^T$, $g_2(x) + d_2(t) = [x_{21}^2 + x_{31} + 2\sin(2t), x_{22} + 2\sin(2t)]^T$, and $g_3(x) + d_3(t) = [3x_{31} + 3\sin(3t), 3x_{32} + 3\sin(3t)]^T$. In addition, for $i \in \{4, 5\}$

$$\dot{x}_i = v_i$$

where $v_4(x) + d_4(t) = [4x_{41} + 4\sin(4t), 4x_{42} + 4\sin(4t)]^T$ and $g_5(x) + d_5(t) = [5x_{51} + 5\sin(5t), 5x_{52} + 5\sin(5t)]^T$.

In the simulation, the numbers of the neurons of the RBFNN are chosen as 11 and the centers of RBFNN activation functions are $-2.5$, $-2$, $-1.5$, $-1$, $-0.5$, $0$, $0.5$, $1$, $1.5$, $2$, $2.5$ for all vehicles. Furthermore, the variances are all set as $5\sqrt{2}$. In addition $W_{\text{max}} = 500$, $\beta = 100$, $\delta = 10$, $\epsilon = 0.01$, and $\dot{W}_i(0)$ is set as a zero matrix.

With $x(0) = [-5, 8, -4, -6, 1, 8, 0, -8, -1, 10]^T$, the numerical results produced by (23)–(25) are plotted in Figs. 2 and 3 by utilizing the communication graph in Fig. 1. Fig. 2 plots players’ actions from which it is clear that they would evolve to a small neighborhood of the Nash equilibrium. In addition, Fig. 3 illustrates the evolution of $v_i(t)$, from which it can be seen that velocities of the second-order players would be driven to be sufficiently small. Hence, the result in Theorem 1 is numerically testified.
corresponding numerical results plotted in Figs. 4 and 5.

Figs. 4 and 5 illustrate vehicles’ positions and velocities of
arbitrarily close to Nash equilibrium and zero, respectively.

By similar arguments, it can be derived that for each
i ∈ Nf, ̇Wi(t) ≤ Wmax holds as well, thus drawing the
conclusion.

B. Proof of Lemma 5

For each i ∈ Nf, ̇Wi = βSi(yi) ̇vi

Moreover, if ̇Wi = βSi(yi) ̇vi − β(( ̇yi ̇WiT Si(yi))/[ ĤWiT ĤWi]) ̇Wi, we
know that ̇WiT ̇Wi = Wmax and ̇vi ̇WiT Si(yi) ≥ 0. If this is the case,

in which

Hence, it can be obtained that

where

in which we have utilized the conclusions that ̇WiT ̇Wi = Wmax ≥ W̃TjWj and ̇WiT ĤWi ≥ 0.

Therefore,

for i ∈ Nf.

Summarizing the above two cases, it can be obtained that for
each i ∈ Nf,

By similar arguments, it can be easily obtained that

for i ∈ Nf.

VI. CONCLUSION

This article accommodates distributed Nash equilibrium
seeking for mixed-order games with both first-order integrator-type
participants and second-order integrator-type participants.
In particular, players’ dynamics are considered to be influ-
enced by unknown but Lipschitz nonlinear dynamics and
time-varying disturbances. To address unknown dynamics and
achieve disturbance rejection, an adaptive neural network-
based approach, i.e., RBFNN, is adapted. Through suitably
designing control inputs and choosing control parameters, it
is proven that the reported methods are able to steer play-
ers’ actions and velocities of second-order integrators to be
arbitrarily close to Nash equilibrium and zero, respectively.

APPENDIX

A. Proof of Lemma 3

For each i = Nf, ̇Wi is generated by (14), (15). For
notational clarity, define Yi = ̇WiT ̇Wi. Then, if Yi < Wmax,

indicating that Yi deceases and hence ̇WiT ̇Wi ≤ Wmax ∀i ∈ Nf holds. In addition, if Yi = Wmax and
(xj − zi) ̇WiT Sj(yj) < 0,

indicating that ̇WiT ̇Wi ≤ Wmax ∀i ∈ Nf holds.

Summarizing the above cases, we get that for each
i ∈ Nf, ̇WiT ̇Wi(t) ≤ Wmax holds.

Moreover, when there are no nonlinear dynamics and dis-
turbances, the strategy in (65)–(67) is testified with the
corresponding numerical results plotted in Figs. 4 and 5.
Figs. 4 and 5 illustrate vehicles’ positions and velocities of
the force-actuated vehicles, respectively. From these figures,
it is seen that vehicles’ positions evolve to be close to x^* and
velocities of the second-order ones evolve to be close to zero.
To this end, Theorem 2 is testified.
C. Proof of Lemma 6
For each $i \in N_T$, we have
\[
\begin{align*}
(x_i - z_i)(d_i(t) + \varepsilon_i - \phi_i) & \leq |x_i - z_i||d_i(t) + \varepsilon_i - (x_i - z_i)^T \phi_i | \\
& \leq \delta |x_i - z_i| - \delta(x_i - z_i)\tanh \left( \frac{K\delta(x_i - z_i)}{\varepsilon} \right) \\
& \leq \varepsilon
\end{align*}
\]
by Lemma 2. By similar arguments, (45) can be obtained.

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