Manin supertriples and Drinfel’d superdoubles in low dimensions

Ladislav Hlavatý and Jan Vysoký *

Faculty of Nuclear Sciences and Physical Engineering,
Czech Technical University in Prague,
Břehová 7, 115 19 Prague 1, Czech Republic

July 16, 2010

Abstract

Defining the real Lie superalgebra as real $\mathbb{Z}_2$–graded vector space we classify real Manin supertriples and Drinfel’d superdoubles of superdimensions $(2, 2)$, $(4, 2)$ and $(2, 4)$. They can be used for construction of $\sigma$–models on supergroups related by Poisson–Lie T–plurality.

1 Introduction

Manin triples and Drinfel’d doubles as well as their supersymmetric counterparts are used for several purposes, e.g., for construction of solution of the Yang–Baxter equations and their supersymmetric analogs [1][2]. Because of that, many cases of Manin triples, especially for simple superalgebras were constructed.

Further investigation and classification of Manin triples and Drinfel’d doubles was boosted by discovery of Poisson–Lie T–duality of $\sigma$–models by Klimčík and Ševera [3]. The low dimensional cases became a convenient laboratory for investigation of the Poisson–Lie T–duality and its extension to the concept of Poisson–Lie T–plurality [3] [4].

As $\sigma$–models, where the Poisson–Lie T–plurality is used, are related to the string theory it is natural to extend the Poisson–Lie T–plurality to the supersymmetric structures either on the source space or in the target space. Useful tool for understanding features of the Poisson–Lie T–plurality of $\sigma$–models with supersymmetric target spaces is construction of simple and potentially solvable cases. The first step for this is to find and classify the low dimensional Manin supertriples and Drinfel’d superdoubles.

Several separate cases of low dimensional Manin supertriples or, equivalently, superbialgebras were investigated in [5] and an attempt to classify the Manin

*E–mail: hlavaty@fjfi.cvut.cz, vysokjan@fjfi.cvut.cz
supertriples was done in [6]. Unfortunately, there are some omissions and redundancies in that paper, probably because the authors considered only the indecomposable superalgebras given in [7] and in some cases did not take into account all isomorphisms to reduce the list of Manin supertriples.

In this paper we give lists of nonisomorphic real Manin supertriples in dimensions four and six, and classify the corresponding Drinfel’d superdoubles. We consider only the cases with nontrivial odd and even parts as purely even Drinfel’d doubles were classified in [8],[9],[10] and classification of the purely odd ones is trivial as the superalgebras are (super)abelian.

2 Lie superalgebras, Manin supertriples and Drinfel’d superdoubles

The real Lie superalgebra $S$ is defined [11] as real $\mathbb{Z}_2$-graded vector space $V = V_0 + V_1$ provided with Lie superbracket $[\ , \ ]$ satisfying

$$[x, y] = -(-1)^{|x||y|}[y, x],$$

(1)

$$(-1)^{|x|+|z|}[x, [y, z]] + (-1)^{|y|+|z|}[y, [z, x]] + (-1)^{|z|}[z, [x, y]] = 0,$$

(2)

where $x, y, z \in V_0 \cup V_1$ are so called homogeneous elements of $V$ and

$$|x| := 0 \text{ if } x \in V_0, \ |x| := 1 \text{ if } x \in V_1.$$

(3)

We say that the superalgebra is of the superdimension $(m, n)$ iff $\dim V_0 = m$ and $\dim V_1 = n$. We can always choose so called homogeneous basis $\{X_I\}$ in $V$

$$\{X_I\}_{I=1}^{m+n} = \{b_i, f_\alpha\}_{i=1,\alpha=1}^{m,n}, \ |b_i| = 0, \ |f_\alpha| = 1.$$

The indecomposable superalgebras up to the dimension 4 were classified in [7]. In the Tables [11,12] we list all two and three-dimensional superalgebras with nontrivial odd part [1]. All of them are solvable. For later use we include also the decomposable ones as they can produce indecomposable Manin supertriples.

A bilinear form $\langle \ , \ \rangle$ on $S$ is called supersymmetric iff

$$\langle x, y \rangle = (-1)^{|x||y|}\langle y, x \rangle$$

and it is called super ad-invariant iff

$$\langle [x, y], z \rangle + (-1)^{|x||y|}\langle y, [x, z] \rangle = 0.$$

(4)

To evade the problems with definitions of supergroups we shall define the Drinfel’d superdoubles only on the algebraic level. The Lie superalgebra $D$ provided with bilinear nondegenerate supersymmetric and super ad-invariant form $\langle \ , \ \rangle$ will be called Drinfel’d superdouble iff it can be decomposed into a pair of maximally isotropic subalgebras $S, \bar{S}$ such that $D = S + \bar{S}$. The triple $(D, S, \bar{S})$

---

1By $A_{m,n}$ we denote the Abelian superalgebra of the superdimension $(m, n)$. 
is called Manin supertriple\[12\]. It follows immediately from the properties of \( \langle \ldots \rangle \) that \( \dim S = \dim \tilde{S} \) so that the dimension of \( D \) is always even.

Let \( D \) and \( D' \) are Drinfel’d superdoubles with bilinear forms \( \langle \ldots \rangle \) and \( \langle \ldots \rangle' \). They will be called isomorphic iff there is an isomorphism \( P : D \to D' \) of the superalgebras such that

\[
(\forall x, y \in D) \ (\langle x, y \rangle = \langle Px, Py \rangle').
\]

The Manin supertriples \((D, S, \tilde{S}), (D', S', \tilde{S}')\) will be called isomorphic iff there is an isomorphism \( P \) of their Drinfel’d superdoubles such that

\[
P(S) = S', \ P(\tilde{S}) = \tilde{S}'.
\]

Below we shall denote the Manin supertriples \((D, S, \tilde{S})\) as \((S|\tilde{S})\).

In the following we shall assume that the Manin supertriples are so called boson – fermion orthogonal\[2\], i.e.

\[
\langle S_0, \tilde{S}_1 \rangle = \langle S_1, \tilde{S}_0 \rangle = 0.
\] (5)

It follows from the boson – fermion orthogonality that the superdimensions of the superalgebras \(S\) and \(\tilde{S}\) coincide, i.e. \((m, n) = (\tilde{m}, \tilde{n})\), and in such Manin supertriples we can choose dual homogeneous basis

\[
\{X_I, \tilde{X}^J\}_{I,J=1}^{m+n} = \{b_i, f_\alpha, \tilde{b}_j, \tilde{f}_\beta\}_{i,\alpha,j,\beta=1}^{m,n,m,n}
\] (6)

where

\[
|b_i| = |\tilde{b}_j| = 0, \ |f_\alpha| = |\tilde{f}_\beta| = 1,
\]

\[
\langle b_i, \tilde{b}_j \rangle = \langle \tilde{b}_j, b_i \rangle = \delta_i^j, \ \langle f_\alpha, \tilde{f}_\beta \rangle = \langle \tilde{f}_\beta, f_\alpha \rangle = \delta_\alpha^\beta = 0,
\] (7)

\[
\langle b_i, b_j \rangle = \langle b_i, f_\alpha \rangle = \langle b_i, \tilde{f}_\beta \rangle = \langle f_\alpha, b_j \rangle = \langle f_\alpha, f_\beta \rangle = \langle f_\alpha, \tilde{b}_j \rangle = \langle \tilde{f}_\beta, b_j \rangle = \langle \tilde{f}_\beta, \tilde{f}_\alpha \rangle = \langle \tilde{f}_\beta, \tilde{f}_\alpha \rangle = 0
\]

The block matrix of the bilinear form \(\langle \ldots \rangle\) in this basis is

\[
B = \begin{pmatrix}
0 & 0 & 1_m & 0 \\
0 & 0 & 0 & 1_n \\
1_m & 0 & 0 & 0 \\
0 & -1_n & 0 & 0
\end{pmatrix},
\] (8)

where \(1_k\) is the identity matrix of dimension \(k\). It is obvious that the superdimension of these Manin supertriples and Drinfel’d superdoubles is \((2m, 2n)\).

A special type of the Drinfel’d superdouble isomorphism is \(T\)-duality that is the linear transformation \(T : D \to D\)

\[
T : b_i \mapsto \tilde{b}^i, \ f_\alpha \mapsto \tilde{f}^\alpha, \ \tilde{b}^i \mapsto b_j, \ \tilde{f}^\alpha \mapsto -f_\beta.
\]

\[2\]The other, rather exotic, possibilities were discussed in the bachelor thesis of J. Vysoký
Its transformation matrix is equal to $B$. Clearly $T(\mathcal{S}|\tilde{\mathcal{S}}) = (\tilde{\mathcal{S}}|\mathcal{S})$ and we shall call the Manin supertriple $(\mathcal{S}|\tilde{\mathcal{S}})$ dual to $(\tilde{\mathcal{S}}|\mathcal{S})$. They are not isomorphic in general.

Due to the super ad-invariance of $(.,.)$ structure coefficients of the Manin supertriple $(\mathcal{S}|\tilde{\mathcal{S}})$ in the dual basis are given by structure coefficients of the subalgebras $\mathcal{S}$ and $\tilde{\mathcal{S}}$

$$[X_I, X_J] = F_{IJ}^K X_K, \quad [\tilde{X}^I, \tilde{X}^J] = \tilde{F}^{IJ} K \tilde{X}^K$$

(9)

as

$$[X_I, \tilde{X}^J] = \tilde{F}^{JK} I X_K + F_{KI}^J \tilde{X}^K.$$  

(10)

Jacobi identities $^2$ for the Drinfel’d superdouble then imply the compatibility conditions for the subalgebras of the Manin supertriple $(\mathcal{S}|\tilde{\mathcal{S}})$.

We are going to classify the Manin supertriples and Drinfel’d superdoubles with nontrivial odd and even parts. As stated above their dimension is always even and two dimensional cases are (super)abelian. It means that the simplest interesting cases have dimensions four and six.

3 Method of classification

To classify the Manin supertriples and Drinfel’d superdoubles of superdimension $(2m, 2n)$ we start with a superalgebra $\mathcal{S}$ of the superdimension $(m, n)$ and look for structure coefficients of dual superalgebra $\tilde{\mathcal{S}}$ by solving Jacobi identities

$$(-1)^{|\tilde{X}^I||\tilde{X}^K|} [\tilde{X}^I, [\tilde{X}^J, \tilde{X}^K]] + \text{cyclic\{I, J, K\}} = 0$$

(11)

and

$$(-1)^{|\tilde{X}^I||X_K|} [\tilde{X}^I, [\tilde{X}^J, X_K]] + \text{cyclic\{I, J, K\}} = 0.$$  

(12)

We assume that the super Lie brackets are in the form $^9$ that means that we work in the dual homogeneous bases $^6$ of Manin supertriples.

Next step is search for a list of different classes of isomorphic Manin supertriples and a choice of ”canonical” forms of structure coefficients $\tilde{F}^{IJ} K$ for each class. The isomorphic triples are related by the transformations of dual homogeneous bases that can be written in the block form as

$$\left( \begin{array}{c} X' \\ \tilde{X}' \end{array} \right) = \left( \begin{array}{cc} A & 0 \\ 0 & (A^{-1})^T \end{array} \right) \left( \begin{array}{c} X \\ \tilde{X} \end{array} \right),$$

(13)

where $(X, \tilde{X})^T$ and $(X', \tilde{X}')^T$ are dual homogeneous bases of the Manin supertriples and $A$’s are (block diagonal) matrices of the group of automorphisms of the superalgebra $\mathcal{S}$.

Finally we classify the Drinfel’d superdoubles by looking for the classes of Manin supertriples that are isomorphic as Drinfel’d super doubles. By definition the Drinfel’d superdoubles $\mathcal{D}$ and $\mathcal{D}'$ are isomorphic if there is a linear bijection that transforms grading, bilinear form and algebraic structure of $\mathcal{D}$ into those
of $D'$. If dual homogeneous bases are chosen in both of them then it means that there is $2(m + n) \times 2(m + n)$ matrix $C$ of the block form

$$C = \begin{pmatrix} P & Q \\ R & U \end{pmatrix}$$

(14)

where $P, Q, R, U$ are $(m + n) \times (m + n)$ block diagonal matrices and elements of $C$ satisfies equations

$$C_a^p C_b^q B_{pq} = B_{ab}, \quad C_a^p C_b^q F_{pq}^r = F_{ab}^r C_c^r,$$

(15)

where $B$ is the matrix $\mathfrak{S}$ and $F, F'$ are structure coefficients of the Drinfel’d superdoubles $D$ and $D'$.

4 Classification in superdimensions $(2,2)$, $(4,2)$ and $(2,4)$

As mentioned in the Introduction it is relevant to consider only the superalgebras with nontrivial both even and odd part i.e. $dim V_0, \ dim V_1 \geq 1$, i.e. the Manin supertriples and Drinfel’d superdoubles of superdimensions $(2,2)$, $(4,2)$ and $(2,4)$. Applying the method from the previous Section we get the following results.

4.1 Manin supertriples and Drinfel’d superdoubles of the superdimension $(2,2)$

The maximal isotropic subalgebras $S$ and $\tilde{S}$ of Manin supertriples of superdimension $(2,2)$ must have superdimensions $(1,1)$. They must be isomorphic to those given in the Table 1 and one can choose the super Lie brackets of $S$ in the form given there.

Table 1: Nonisomorphic Lie superalgebras of the superdimension $(1,1)$

| Name | Nonzero super Lie brackets | 
|------|-----------------------------|
| $A_{11}$ | Abelian                      |
| $N_{11} = q(1)$ | $[f_1, f_1] = b_1$ | Nilpotent |
| $S_{11} = P(1)$ | $[b_1, f_1] = f_1$ | Solvable |

For each $S$ from the Table 1 it is then rather easy to calculate the structure coefficients of the dual algebras $\tilde{S}$ satisfying (11) and (12) and use the groups of automorphisms of $S$ for finding five classes of nonisomorphic Manin supertriples $(S|\tilde{S})$ of the superdimension $(2,2)$. They are listed in the Table 2. Note that even though the superalgebra $A_{11}$ is decomposable the semiabelian Manin
supertriples \((N_{11}|A_{11})\) and \((S_{11}|A_{11})\) are indecomposable. Traditional notation for \((C^2_1 + A)\) is \(gl(1|1)\) and the T–dual to the Manin supertriple 4 (i.e. \(\epsilon = +1\) appears in [13] as a bialgebra on \(q(1)\).

Table 2: List of nonisomorphic Manin supertriples of the superdimension \((2, 2)\) up to T–duality

| \((S|\tilde{S})\) | Nonzero super Lie brackets | Backhouse’s classification |
|-----------------|-----------------------------|---------------------------|
| 1 \((A_{11}|A_{11})\) | \([f_1, f_1] = b_1, [f_1, \tilde{b}_1] = \tilde{f}_1,\) | Abelian |
| 2 \((N_{11}|A_{11})\) | \([b_1, f_1] = f_1, [b_1, \tilde{f}_1] = -\tilde{f}_1, [f_1, \tilde{f}_1] = \tilde{b}_1\) | \((C^3 + A)\) |
| 3 \((S_{11}|A_{11})\) | \([b_1, f_1] = f_1, [b_1, \tilde{f}_1] = \epsilon f_1 - \tilde{f}_1, [f_1, \tilde{f}_1] = \tilde{b}_1\) | \((C^2_1 + A)\) |
| 4, 5 \((S_{11}|N_{11})\) | \([b_1, f_1] = f_1, [b_1, \tilde{f}_1] = \epsilon f_1 - \tilde{f}_1, [f_1, \tilde{f}_1] = \epsilon \tilde{b}_1\) | \(\cong (C^2_1 + A)\) |

As indicated in the Table 2 the Manin supertriples \((S_{11}|A_{11})\) and \((S_{11}|N_{11})\) are isomorphic as superalgebras and one can find an isomorphism that preserve also the bilinear form \(\langle \cdot, \cdot \rangle\) so that they belong (together with their T–duals) to the same Drinfel’d superdouble. The isomorphism of Manin supertriples between \((S_{11}|A_{11})\) and \((S_{11}|N_{11})\) is given by the matrix

\[
C = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & \frac{\epsilon}{2} & 0 & 1
\end{pmatrix},
\]

i.e. by the transformation

\[
(b'_1, f'_1, \tilde{b}'_1, \tilde{f}'_1) = (b_1, f_1, \tilde{b}_1, \tilde{f}_1 + \frac{1}{2} \epsilon f_1),
\]

where \((b_1, f_1, \tilde{b}_1, \tilde{f}_1)\) are generators of the Manin supertriple \((S_{11}|A_{11})\) and \((b'_1, f'_1, \tilde{b}'_1, \tilde{f}'_1)\) are generators of the Manin supertriple \((S_{11}|N_{11})\). We get the following theorem.

**Theorem 1** Any real Drinfeld superdouble of the superdimension \((2, 2)\) belongs just to one of the following 3 classes and allows decomposition into all Manin supertriples listed in the class and their duals \((S \leftrightarrow \tilde{S})\).

\[
DD_{(2,2)} I : (A_{11}|A_{11}),
\]

6
\[ DD_{(2,2)} \ II : \ (N_{11}|A_{11}), \]
\[ DD_{(2,2)} \ III : \ (S_{11}|A_{11}), \ (S_{11}|N_{11}), \ \epsilon = \pm 1. \]

### 4.2 Manin supertriples and Drinfel’d superdoubles of the superdimension \( (4,2) \)

The real Lie superalgebras of the superdimension \( (2,1) \) and their groups of automorphisms are listed\(^3\) in the Table 3.

#### Table 3: Three–dimensional superalgebras of the superdimension \( (2,1) \)

| Name | Nonzero super Lie brackets | Group of automorphisms | Comments |
|------|-----------------------------|-------------------------|----------|
| \( A_{21} \) | \([f_1, f_1] = b_1\) | \(\begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & k \end{pmatrix}\) | |
| \( N_{21} \) | \([b_1, f_1] = f_1\) | \(\begin{pmatrix} d^2 & 0 & 0 \\ b & a & 0 \\ 0 & d & 0 \end{pmatrix}\) | \(= N_{11} \oplus A_{10}\) |
| \( S_{21} \) | \([b_1, f_1] = f_1\) | \(\begin{pmatrix} 1 & b & 0 \\ 0 & c & 0 \\ 0 & d & 0 \end{pmatrix}\) | \(= S_{11} \oplus A_{10}\) |
| \( C_p^1 \) | \([b_1, b_2] = b_2, [b_1, f_1] = p f_1\) | \(\begin{pmatrix} 1 & b & 0 \\ 0 & c & 0 \\ 0 & d & 0 \end{pmatrix}\) | \(p \in \mathbb{R}\) |
| \( F \) | \([b_1, b_2] = b_2, [b_1, f_1] = \frac{1}{2} f_1, [f_1, f_1] = b_2\) | \(\begin{pmatrix} 1 & b & 0 \\ 0 & d^2 & 0 \\ 0 & 0 & d \end{pmatrix}\) | |

By checking the Jacobi identities and employing the groups of automorphisms of the superalgebra \( S \) we find that there are 14 classes of real Manin supertriples of the superdimension \( (4,2) \) up to the T–duality. They are listed in the Table 3.

To classify the Drinfel’d superdoubles it is useful to determine the dimensions of multiple commutants \(C_1 = [D, D], C_2 = [C_1, C_1], \ldots \) of the Manin supertriples. This sorts them out into eight classes that finally turn out to be the Drinfel’d superdoubles. The only exception are the Manin supertriples with \(dim C_1 = 3, \ dim C_2 = 1\) but the superdimensions of their subalgebras \( C_1 \) are \((1,2)\) and \((3,0)\) so that they belong to nonisomorphic Drinfel’d superdoubles. The classes

---

\(^3\)Differently from \[7\] we have denoted the superalgebra \(C_p^1\) as \(F\) because it differs from \(C_{p=\frac{1}{2}}^1\).
are displayed in the Table. The Drinfel’d superdouble isomorhisms among the corresponding Manin supertriples are given in the Appendix A.

**Theorem 2** Any real Drinfel’d superdouble of the superdimension (4,2) belongs just to one of the following classes and allows decomposition into nonisomorphic Manin supertriples listed in the class and their duals ($\mathcal{S} \leftrightarrow \tilde{\mathcal{S}}$).

- $DD_{(4,2)} I : (A_{21}|A_{21})$,
- $DD_{(4,2)} II : (N_{21}|A_{21})$,
- $DD_{(4,2)} III : (S_{21}|A_{21}), (S_{21}|N_{21}), (S_{21}|S_{21})$,
- $DD_{(4,2)} IV_\alpha : (C_0^1|A_{21}) = (C_0^1|C_0^0|\kappa), (C_p^1|C_1^-|\kappa|\alpha), p = \kappa = 0$,
- $DD_{(4,2)} IV_\kappa : (C_0^1|C_0^0|\kappa), \kappa \neq 0$,
- $DD_{(4,2)} V : (C_0^1|N_{21})$,
- $DD_{(4,2)} VI_\rho : (C_{1\rho}|A_{21}), (C_{1\rho}|N_{21}), (C_{1\rho}|C_1^-|\kappa|\alpha), p > 0$,
- $DD_{(4,2)} VII : (F|A_{21}), (C_1^1|F|N_{21}), (F|F|\epsilon, (F|C_1^1|\epsilon|\rho|1/2), (F|F|\epsilon|\rho|1/2))$,
- $DD_{(4,2)} VIII_\kappa : (F|F|\epsilon, \kappa)$.

### 4.3 Manin supertriples and Drinfel’d superdoubles of the superdimension (2,4)

The real Lie superalgebras of the superdimension (1,2) together with their groups of automorphisms are listed in the Table.

The Jacobi identities imply that the Manin supertriples of the superdimension (2,4) are only of the forms $(C|N_{12}^{\alpha,\beta,\gamma})$ or their $T$–duals, where $C = A_{12}, C_0^0, C_0^1, C_1^1, C_p^1$, and $N_{12}^{\alpha,\beta,\gamma}$ are superalgebras with super Lie products

\[
[f_1, f_2] = \alpha \vec{b}^1, \quad [\vec{f}_1, f_2] = \beta \vec{b}^1, \quad [\vec{f}_1, \vec{f}_2] = \gamma \vec{b}^1.
\]

They are isomorphic to $N_{12}^{1,0,\epsilon} = N_{12}^{1,0,\epsilon}$ or $N_{12}^{0,1,0} = N_{12}^{0,1,0}$ or $A_{12} = N_{12}^{1,0,0}$.

Using the automorphisms of $A_{12}, C_0^0, C_0^1, C_1^1, C_p^1$ displayed in the Table we find that there are 31 classes of Manin supertriples of the superdimension (2,4) up to the $T$–duality. They are listed in the Table.

We shall show that in many cases the Manin supertriples $(C|N_{12}^{\alpha,\beta,\gamma})$ belong to the same Drinfel’d superdouble as the semiabelian $(C|A_{12})$, in other words, there are linear transformations between the basis $(b_1, f_1, f_2, \vec{b}^1, f_3, \vec{f}^2)$ of $(C|A_{12})$ and $(b_1', f_1', f_2', \vec{b}^1', \vec{f}^2')$ of $(C|N_{12}^{\alpha,\beta,\gamma})$ that preserve the bilinear form $(\cdot, \cdot)$ and transform the super Lie brackets of $(C|A_{12})$ into those of $(C|N_{12}^{\alpha,\beta,\gamma})$.

Let the Manin supertriple $(C|N_{12}^{\alpha,\beta,\gamma})$ is given by super Lie brackets of $\mathcal{S}$ and $\tilde{\mathcal{S}}$

\[
[b_1', f_2'] = H_{j\beta}^k f_k', \quad [\vec{f}^j', \vec{f}^k'] = G^{j\beta} \vec{b}^1', \quad j, k \in \{1, 2\},
\]
\[ G^{jk} = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix}, \]

The only nonzero super Lie brackets of \((C|A_{12})\) are

\[ [b_1, f_j] = H_j^k f_k, \quad [f_j, \tilde{f}^k] = H_j^k \tilde{b}^1, \quad [b_1, \tilde{f}^k] = -H_j^k \tilde{f}^j. \quad (19) \]

Let the transformation of bases is given by the symmetric matrix \(R^{jk}\)

\[ (b'_1, f'_1, f'_2, \tilde{b}'_1, \tilde{f}'_1, \tilde{f}'_2) = (b_1, f_1, f_2, \tilde{b}^1, \tilde{f}_1 + R^{1k} f_k, \tilde{f}_2 + R^{2k} f_k). \quad (20) \]

It is easy to see that this transformation preserves the bilinear form \(\langle \, , \rangle\) and

\[ [\tilde{f}'_j, \tilde{f}'^k] = (R^{j1} H_1^k + R^{j2} H_2^k) \tilde{b}'^1. \quad (21) \]

The super Lie brackets of \(\text{MANIN}\) then can be obtained by solving \(R^{jk}\) from the equation

\[ R^{j1} H_1^k + R^{j2} H_2^k = G^{jk}. \quad (22) \]

It can be solved for general \(G^{jk}\) iff \(\text{Trace } H \neq 0\) and \(\det H \neq 0\). These conditions are satisfied for \(C_4^1, C_p^2, p \neq -1, 0\) and \(C_p^0, p \neq 0\). In the exceptional cases the equation \(22\) can be solved for particular values of \(G^{jk}\), namely, for \(C_p^0\) and \(C_4^3\) only if \(G^{3j} = \gamma = 0\), for \(C_{2-1}\) only if \(G^{21} = \beta = 0\), and for \(C_6^0\) only if \(G^{33} = \gamma = -G^{22} = -\alpha\). These cases are not isomorphic to the semiabelian Manin supertriples \((C|A_{12})\), but e.g. to \((C|N_{12}^{0,1,1})\). The isomorphisms are given in the Appendix B.

We get the following theorem.

**Theorem 3** Any real Drinfel’d superdouble of the superdimension \((2,4)\) belongs just to one of the following classes and allows decomposition into all Manin supertriples listed in the class and their duals \((S \leftrightarrow \bar{S})\).

- \(DD_{(2,4)} I : (A_{12}|A_{12})\)
- \(DD_{(2,4)} II_p : (C_p^2|N_{12}^{\alpha,\beta,\gamma}), 0 < p < 1,\)
- \(DD_{(2,4)} II_1 : (C_1^2|N_{12}^{\alpha,\beta,\gamma}), (C_{-1}^2|N_{12}^{\lambda,\kappa,\rho}),\)
- \(DD_{(2,4)} II_0 : (C_0^2|N_{12}^{\alpha,\beta,0}),\)
- \(DD_{(2,4)} III : (C_3^3|N_{12}^{\alpha,\beta,0}), (A_{12}|N_{12}^{\lambda,\kappa,\gamma}), \lambda \gamma < \kappa^2,\)
- \(DD_{(2,4)} IV : (C_4^4|N_{12}^{\alpha,\beta,\gamma}), (C_{-2}^2|N_{12}^{\lambda,\kappa,\rho}), \kappa \neq 0,\)
- \(DD_{(2,4)} V_p : (C_p^0|N_{12}^{\alpha,\beta,\gamma}), p > 0,\)
- \(DD_{(2,4)} V_0 : (C_0^0|N_{12}^{\alpha,\beta,-\alpha}),\)
- \(DD_{(2,4)} VI : (C_0^2|N_{12}^{\alpha,\beta,\gamma}), \gamma \neq 0,\)
\[ DD_{(2,4)} \text{ VII} : (C^3|N_{12}^{\alpha,\beta,\gamma}), \gamma \neq 0, \]
\[ DD_{(2,4)} \text{ VIII} : (C^5|N_{12}^{\alpha,\beta,\gamma}), \alpha \neq -\gamma, \]
\[ DD_{(2,4)} \text{ IX} : (A_{12}|N_{12}^{\alpha,\beta,\gamma}), \alpha \gamma > \beta^2, \]
\[ DD_{(2,4)} \text{ X} : (A_{12}|N_{12}^{\alpha,\beta,\gamma}), (\alpha, \beta, \gamma) \neq (0,0,0), \alpha \gamma = \beta^2. \]

Note that the Drinfel’d superdoubles do not depend on the parameters of Manin supertriples \( \alpha, \beta, \gamma, \kappa \in \mathbb{R} \). The Drinfel’d superdoubles \( H_p - V_0 \) and \( IX, X \) can be called semiabelian as they are isomorphic to \( (S|A_{12}) \). Dimensions of commutants of some Drinfel’d superdoubles given above, are equal and we are able to prove their nonisomorphicity only by the computer calculations.

5 Conclusion

We have classified real Manin supertriples and Drinfel’d superdoubles of the superdimensions \( (2,2) \), \( (4,2) \) and \( (2,4) \), i.e real Lie superalgebras provided with with bilinear nondegenerate supersymmetric and super ad-invariant form such that they can be decomposed into a pair of maximally isotropic subalgebras.

It turned out that nearly all investigated Drinfel’d superdoubles contain more than one pair of nonisomorphic dual Manin supertriples. This gives us the possibility to apply the Poisson–Lie T–plurality to \( \sigma \)–models whose targets are supergroups.

Acknowledgement

This work was supported by the research plan LC527 of the Ministry of Education of the Czech Republic.

Appendix A. Isomorfism of the Drinfel’d superdoubles of the superdimension (4,2)

\[ DD_{(4,2)} \text{ III} : \]
\[ (S_{21}|A_{21}) \to (S_{21}|N_{21}^1) : C = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & \frac{\gamma}{2} & 0 & 0 & 1 \\
\end{pmatrix} \]
\[
(S_{21}|A_{21}) \rightarrow (S_{21}|S_{21}) \quad C = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

**DD\(_{(4,2)}\)IV\(_0\):**

\[
(C_{0}^1|A_{21}) \rightarrow (C_{p}^1|C_{-p}^1), \ p = 0 \quad C = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

**DD\(_{(4,2)}\)V:**

\[
(C_{0}^1|N_{21}^+) \rightarrow (C_{0}^1|N_{21}^-) \quad C = \begin{pmatrix}
-1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

**DD\(_{(4,2)}\)VI\(_p\):**

\[
(C_{p}^1|A_{21}) \rightarrow (C_{-p}^1|A_{21}) \quad C = \begin{pmatrix}
-1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

\[
(C_{p}^1|A_{21}) \rightarrow (C_{p}^1|N_{21}^+), \ C_{p} = 0 \\
(C_{p}^1|A_{21}) \rightarrow (C_{p}^1|N_{21}^-), \ C_{p} = 0 \\
(C_{p}^1|A_{21}) \rightarrow (C_{p}^1|N_{21}^\epsilon), \ C_{p} = 0
\]

\[
(C_{p}^1|N_{21}^+), \ C_{p}^1 \rightarrow (C_{p}^1|N_{21}^-), \ C_{p}^1 \rightarrow (C_{p}^1|N_{21}^\epsilon)
\]
Appendix B. Isomorphisms of the Drinfel’d superdoubles of the superdimension (2,4)

\[ DD_{(2,4)}^{\Pi, p} : \]

\( (F|A_{21}) \to (C_{p=\frac{1}{2}}^1|\overline{N}_{21}) : C = \begin{pmatrix}
-1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & -1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0
\end{pmatrix} \)

\( (F|A_{21}) \to (F|F.i, \epsilon) : C = \begin{pmatrix}
1 & 0 & 0 & 0 & \epsilon & 0 \\
0 & 1 & 0 & -\epsilon & 0 & 0 \\
0 & 0 & \epsilon & 0 & 0 & -1 \\
0 & \epsilon & 0 & 0 & 0 & 0 \\
-\epsilon & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0
\end{pmatrix} \)

\( (F|A_{21}) \to (F|C^{1,\epsilon}_{p=\frac{1}{2}}) : C = \begin{pmatrix}
1 & 0 & 0 & 0 & \epsilon & 0 \\
0 & 1 & 0 & -\epsilon & 0 & 0 \\
0 & 0 & \epsilon & 0 & 0 & -1 \\
0 & \epsilon & 0 & 0 & 0 & 0 \\
-\epsilon & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \epsilon
\end{pmatrix} \)
$\text{DD}_{(2,4)\Pi_1}$:

$\begin{align*}
(C^2_1|A_{12}) \rightarrow (C^2_1|N^\alpha_{12,\gamma}) : & \quad C = \\
& = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & \frac{\beta}{2} & 0 & 1 & 0 \\
0 & 0 & \frac{\beta}{2} & 0 & 0 & 1 \\
\end{pmatrix}, \\
(C^2_{-1}|A_{12}) \rightarrow (C^2_1|A_{12}) : & \quad C = \\
& = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & \frac{\alpha}{2} & \beta & 0 & 1 & 0 \\
0 & \beta & 0 & 0 & 0 & 1 \\
\end{pmatrix}
\end{align*}$

$\text{DD}_{(2,4)\Pi_0}$:

$\begin{align*}
(C^2_0|A_{12}) \rightarrow (C^2_0|N^\alpha_{12,0}) : & \quad C = \\
& = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & \frac{\alpha}{2} & \beta & 0 & 1 & 0 \\
0 & \beta & 0 & 0 & 0 & 1 \\
\end{pmatrix}
\end{align*}$

$\text{DD}_{(2,4)\Pi_3}$:

$\begin{align*}
(C^3|A_{12}) \rightarrow (C^3|N^\alpha_{12,\beta,0}) : & \quad C = \\
& = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & \frac{\alpha}{2} & \beta & 0 & 1 & 0 \\
0 & \beta & 0 & 0 & 0 & 1 \\
\end{pmatrix}
\end{align*}$

$\begin{align*}
(C^3|A_{12}) \rightarrow (A_{12}|N^\lambda_{12,\kappa,\gamma}) : & \quad C = \\
& = \begin{pmatrix}
\rho^2 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{2} \rho & 0 & 0 & 0 & 0 \\
0 & \frac{2 \rho}{2 \gamma} & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{\rho^2 - \rho}{2 \gamma} & 0 & \frac{2 \rho - \rho}{2 \gamma} & 0 \\
0 & 0 & \frac{2 \rho}{2 \gamma} & 0 & \frac{\rho^2 + \rho}{\rho} & 0 \\
\end{pmatrix}, \quad \rho = \sqrt{\kappa^2 - \lambda \gamma}
\end{align*}$

$\text{DD}_{(2,4)\Pi_4}$:

$\begin{align*}
(C^4|A_{12}) \rightarrow (C^4|N^\alpha_{12,\beta,\gamma}) : & \quad C = \\
& = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & \frac{\beta}{2} & \frac{\beta}{2} & 0 & 1 & 0 \\
0 & \frac{\beta}{2} & \frac{\beta}{2} & 0 & 0 & 1 \\
\end{pmatrix}
\end{align*}$

13
\((C^4|_{A_{12}}) \to (C^2_{-1}|_{N^\alpha,\kappa,\gamma}_{12}) : \quad C = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{1}{\kappa} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & -\alpha & 0 & 0 & -\kappa & 0 \\
0 & 0 & -1 & 0 & 0 & -\frac{\gamma}{2}
\end{pmatrix}, \kappa \neq 0\)

DD\((2,4)\)\(V_p\), \(p \neq 0\):

\((C^5_p|_{A_{12}}) \to (C^5_p|_{N^\alpha,\beta,\gamma}_{12}) : \quad C = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}\)

DD\((2,4)\)\(V_0\):

\((C^5_0|_{A_{12}}) \to (C^5_0|_{N^\alpha,\beta,\gamma}_{12}) : \quad C = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}\)

DD\((2,4)\)\(VI\):

\((C^2_0|_{N^0,0,0}_{12}) \to (C^2_0|_{N^\alpha,\beta,\gamma}_{12}) : \quad C = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{1}{\sqrt{\gamma}} & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & \frac{\alpha}{\sqrt{\gamma}} & -\frac{\beta}{\sqrt{\gamma}} & 0 & 1 & 0 \\
0 & \beta & 0 & 0 & 0 & -\sqrt{\gamma}
\end{pmatrix}, \gamma > 0\)

\((C^2_0|_{N^{0,0,1}}_{12}) \to (C^2_0|_{N^\alpha,\beta,\gamma}_{12}) : \quad C = \begin{pmatrix}
-1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & \frac{1}{\sqrt{\gamma}} & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 \\
0 & -1 & \frac{\beta}{\sqrt{\gamma}} & 0 & \frac{\alpha}{\sqrt{\gamma}} & 0 \\
0 & 0 & 0 & 0 & \beta & -\sqrt{\gamma}
\end{pmatrix}, \gamma < 0\)

14
DD\(_{(2,4)}\) VII:

\((C^3|N^{0,0,1}_{12}) \rightarrow (C^3|N^{\alpha,\beta,\gamma}_{12}) : C = \begin{pmatrix}
\gamma & 0 & 0 & 0 & 0 & 0 \\
0 & \gamma & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{\gamma} & 0 & 0 \\
0 & 0 & \frac{\alpha}{\gamma} & 0 & \frac{1}{\gamma} & 0 \\
0 & \frac{\alpha\gamma}{2} & \beta & 0 & 0 & 1
\end{pmatrix}, \gamma \neq 0\)

DD\(_{(2,4)}\) VIII:

\((C^5|N^{0,0,1}_{12}) \rightarrow (C^5|N^{\alpha,\beta,\gamma}_{12}) : C = \begin{pmatrix}
\epsilon & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{1}{\sqrt{|\alpha + \gamma|}} & 0 & 0 & 0 & 0 \\
0 & 0 & -\epsilon \frac{1}{\sqrt{|\alpha + \gamma|}} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{\alpha}{2\sqrt{|\alpha + \gamma|}} & \epsilon & 0 \\
0 & -\frac{\beta}{2\sqrt{|\alpha + \gamma|}} & -\epsilon \frac{\alpha}{2\sqrt{|\alpha + \gamma|}} & 0 & -\sqrt{|\alpha + \gamma|} & 0 \\
0 & 0 & 0 & 0 & -\epsilon \sqrt{|\alpha + \gamma|} & 0
\end{pmatrix}\), \\
\alpha + \gamma \neq 0, \epsilon = \text{sign}(\alpha + \gamma)

References

[1] D. Leites and V. Serganova, Solutions of the classical Yang–Baxter equation for simple Lie superalgebras. *Theoret. Mat. Fiz* 58, 26 (1984).

[2] G. Karaali, Constructing \(r\)–matrices on simple Lie superalgebras. *J. Algebra* 282, 83 (2004).

[3] C. Klimčík and P. Ševera, Dual non–Abelian duality and the Drinfeld double. *Phys. Lett. B* 351, 455 (1995).

[4] R. von Unge, Poisson–Lie T–plurality, *J. High Energy Phys.* 02:07(2002)014.

[5] C. Juszczak and J.T. Sobczyk, Classification of low dimensional Lie superbialgebras. *J. Math. Phys.* 19, 2400 (1998).

[6] A. Eghbali, F. Heidapour and A. Rezaei-Aghdam, Classification of two and three dimensional Lie super-bialgebras. arXiv:0901.4471 [math-ph].

[7] N. Backhouse, A classification of four–dimensional Lie superalgebras. *J. Math. Phys.* 19, 2400 (1978).

[8] X. Gomez, Classification of three–dimensional Lie bialgebras. *J. Math. Phys.* 41, 4939 (2000).

[9] L. Hlavatý and L. Šnobl, Poisson–Lie T–dual models with two–dimensional targets. *Mod. Phys. Lett. A* 17, 429 (2002).
[10] L. Šnobl and L. Hlavatý, Classification of 6-dimensional real Drinfel’d doubles, *Int. J. Mod. Phys. A* **17**, 4043 (2002).

[11] A. Rogers, Supermanifolds, Theory and Applications, World Scientific, Singapore, 2007.

[12] N. Andruskiewitsch, Lie Superbialgebras and Poisson–Lie Supergroups. *Abh. Math. Sem. Univ. Hamburg* **63**, 147 (1993).

[13] G.I. Olshanski, Quantized universal enveloping superalgebra of type $Q$, *Lett. Math. Phys.* **24**, 93 (1992).
Table 4: List of nonisomorphic Manin supertriples of the superdimension $(4, 2)$ up to T–duality.

| $MT_{(4,2)}$ | $\mathcal{S}$ | $\tilde{\mathcal{S}}$ | Nonzero super Lie brackets | Comments |
|--------------|--------------|-----------------|---------------------------|----------|
| $1$          | $A_{21}$     | $A_{21}$        | $[f_1, f_1] = b_1$        |          |
| $2$          | $A_{21}$     |                 |                           |          |
| $3$          | $A_{21}$     |                 |                           |          |
| $4$          | $N_{21}$     |                 | $[\tilde{f}^1, \tilde{f}^1] = \epsilon \tilde{b}^1$ | $\epsilon = \pm 1$ |
| $5$          | $S_{21}$     |                 | $[\tilde{b}^2, \tilde{f}^1] = \tilde{f}^1$ |          |
| $C_p^1$      | $A_{21}$     |                 | $[b_1, b_2] = b_2, [b_1, f_1] = p f_1$ | $p \in \mathbb{R}$ |
| $6$          | $A_{21}$     |                 |                           |          |
| $7$          | $N_{21}$     |                 | $[\tilde{f}^1, \tilde{f}^1] = \epsilon \tilde{b}^1$ | $\epsilon = \pm 1$ |
| $8$          | $\tilde{C}_{p}^1$ |                 | $[\tilde{b}^1, \tilde{b}^2] = \tilde{b}^1, [\tilde{b}^2, \tilde{f}^1] = p \tilde{f}^1$ |          |
| $9$          | $\tilde{N}_{21}$ |                 | $[\tilde{f}^1, \tilde{f}^1] = \tilde{b}^2$ | $p = \frac{1}{2}$ |
| $10$         | $C_{0, \kappa}$ |                 | $[\tilde{b}^1, \tilde{b}^2] = \kappa \tilde{b}^2$ | $p = 0, \kappa \neq 0$ |
| $F$          | $A_{21}$     |                 | $[b_1, b_2] = b_2, [b_1, f_1] = \frac{1}{2} f_1, [f_1, f_1] = b_2$ |          |
| $11$         | $A_{21}$     |                 |                           |          |
| $12$         | $C_{p=\frac{1}{2}}$ |                 | $[\tilde{b}^1, \tilde{b}^2] = \epsilon \tilde{b}^1, [\tilde{b}^2, \tilde{f}^1] = \frac{1}{2} \epsilon \tilde{f}^1$ | $\epsilon = \pm 1$ |
| $13$         | $F. i, \epsilon$ |                 | $[\tilde{b}^1, \tilde{b}^2] = \epsilon \tilde{b}^1, [\tilde{b}^2, \tilde{f}^1] = -\frac{1}{2} \epsilon \tilde{f}^1, [\tilde{f}^1, \tilde{f}^1] = \epsilon \tilde{b}^1$ | $\epsilon = \pm 1$ |
| $14$         | $F. ii, \kappa$ |                 | $[\tilde{b}^1, \tilde{b}^2] = \kappa \tilde{b}^2, [\tilde{b}^2, \tilde{f}^1] = \frac{1}{2} \kappa \tilde{f}^1, [\tilde{f}^1, \tilde{f}^1] = \kappa \tilde{b}^2$ | $\kappa \neq 0$ |
### Table 5: Invariants of the Manin supertriples of the superdimension (4,2)

| Dim. of $C_1, C_2, C_3$ | Manin supertriples  | Drinfel’d superdouble $DD_{(4,2)}$ |
|--------------------------|----------------------|-----------------------------------|
| 0, 0, 0                  | $(A_{21}|A_{21})$    | $I$                               |
| 2, 0, 0                  | $(N_{21}|A_{21})$    | $II$                              |
| 3, 1, 0                  | $(S_{21}|A_{21}), (S_{21}|N_{21}), (S_{21}|S_{21})$ | $III$                             |
| 3, 1, 0                  | $(C_0^1|A_{21}) = (C_0^1|C_{01}^1,k), (C_p^1|C_{-p}^1), p = k = 0$ | $IV_0$                            |
| 3, 3, 3                  | $(C_0^1|C_{01}^1,k)$ | $IV_k, k \neq 0$                  |
| 4, 1, 0                  | $(C_0^1|N_{21}^2)$   | $V$                               |
| 5, 1, 0                  | $(C_p^1|A_{21}), (C_p^1|N_{21}), (C_p^1|C_{-p}^1)$ | $VI_p, p > 0$                     |
| 5, 3, 0                  | $(F|A_{21}), (C_{p=\frac{1}{2}}^1|N_{21}), (F|F,i,\epsilon), (F|C_{p=\frac{1}{2}}^{1,\epsilon})$ | $VII$                             |
| 5, 5, 5                  | $(F|F.ii,k)$         | $VIII_k$                          |
Table 6: Superalgebras of the superdimension (1,2)

| Name | Nonzero super Lie brackets | Group of automorphisms |
|------|----------------------------|------------------------|
| $A_{12}$ | $[f_1, f_1] = b_1$ | $\begin{pmatrix} k & 0 & 0 \\ 0 & a & b \\ 0 & c & d \end{pmatrix}$ |
| $N_{12}^{0}$ | $[f_1, f_1] = b_1$ | $\begin{pmatrix} a^2 & 0 & 0 \\ 0 & a & b \\ 0 & 0 & d \end{pmatrix}$ |
| $N_{12}^{\epsilon}$ | $[f_1, f_1] = b_1, [f_2, f_2] = \epsilon b_1$ | $\begin{pmatrix} d^2 + \epsilon c^2 & 0 & 0 \\ 0 & \mp \epsilon d & \pm c \\ 0 & c & d \end{pmatrix}$ |
| $C_1^{-1}$ | $[b_1, f_1] = f_1, [b_1, f_2] = -f_2$ | $\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & b \\ 0 & c & 0 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & c \\ 0 & 0 & d \end{pmatrix}$ |
| $C_1^2$ | $[b_1, f_1] = f_1, [b_1, f_2] = p f_2$ | $\begin{pmatrix} 1 & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & d \end{pmatrix}$ |
| $C_1$ | $[b_1, f_1] = f_1, [b_1, f_2] = f_2$ | $\begin{pmatrix} a & 0 & 0 \\ 0 & ad & 0 \\ 0 & c & d \end{pmatrix}$ |
| $C^3$ | $[b_1, f_2] = f_1$ | $\begin{pmatrix} 1 & 0 & 0 \\ 0 & a & 0 \\ 0 & c & a \end{pmatrix}$ |
| $C^4$ | $[b_1, f_1] = f_1, [b_1, f_2] = f_1 + f_2$ | $\begin{pmatrix} 1 & 0 & 0 \\ 0 & a & -c \\ 0 & c & a \end{pmatrix}$ |
| $C_1^p$ | $[b_1, f_1] = p f_1 - f_2, [b_1, f_2] = f_1 + p f_2$ | $\begin{pmatrix} \pm 1 & 0 & 0 \\ 0 & \pm a & \mp c \\ 0 & c & a \end{pmatrix}$ |
| $C_0^5$ | $[b_1, f_1] = -f_2, [b_1, f_2] = f_1$ | $\begin{pmatrix} 0 & 0 & 0 \\ 0 & \pm a & \mp c \\ 0 & c & a \end{pmatrix}$ |
Table 7: List of nonisomorphic Manin supertriples of the superdimension (2, 4) up to T–duality, $\epsilon, \epsilon_1, \epsilon_2 = \pm 1$, $\delta = 0, 1$.

| $MT_{(2,4)}$ | $S$ | $\tilde{S}$ | Comments |
|--------------|-----|-------------|----------|
| 1–3          | $A_{12}$ | $A_{12}$, $N_{12}^{1,0,0}$, $N_{12}^{1,0,\epsilon}$ |          |
| 4 – 8        | $C_2^p$ | $A_{12}$, $N_{12}^{0,1,0}$, $N_{12}^{0,\delta,\epsilon}$, $N_{12}^{\epsilon,\delta,0}$, $N_{12}^{\epsilon_1,\epsilon_2,0}$ | $|p| < 1$, $\kappa \geq 0$ |
| 9 – 12       | $C_1^2$ | $A_{12}$, $N_{12}^{\epsilon,0,\epsilon}$, $N_{12}^{1,0,-1}$, $N_{12}^{0,0,\epsilon}$ |          |
| 13 – 17      | $C_{-1}^2$ | $A_{12}$, $N_{12}^{0,1,0}$, $N_{12}^{0,\delta,\epsilon}$, $N_{12}^{1,\epsilon,1}$, $N_{12}^{\epsilon,\epsilon,\epsilon}$ | $\kappa \geq 0$ |
| 18 – 22      | $C^3$ | $A_{12}$, $N_{12}^{\epsilon,0,1}$, $N_{12}^{1,0,0}$, $N_{12}^{0,\kappa,0}$, $N_{12}^{0,0,1}$ |          |
| 23 – 26      | $C^4$ | $A_{12}$, $N_{12}^{\epsilon,0,0}$, $N_{12}^{0,\epsilon,0}$, $N_{12}^{\kappa,0,0}$ | $\kappa \in \mathbb{R}$ |
| 27 – 29      | $C_p^5$ | $A_{12}$, $N_{12}^{\epsilon,0,\epsilon}$, $N_{12}^{-1,0,-1}$ | $p > 0$, $-1 < \kappa \leq 1$ |
| 30, 31       | $C_{p=0}^5$ | $A_{12}$, $N_{12}^{\epsilon,0,1}$ | $-1 \leq \kappa \leq 1$ |