Abstract The concepts of space, time, and matter are of central importance in any theory of the gravitational field. Here I discuss the role that these concepts might play in quantum theories of gravity. To be concrete, I will focus on the most conservative approach, which is quantum geometrodynamics. It turns out that spacetime is absent at the most fundamental level and emerges only in an appropriate limit. It is expected that the dynamics of matter can only be understood from a fundamental quantum theory of all interactions.

1 From classical to quantum gravity

In his famous habilitation colloquium on June 10, 1854, Bernhard Riemann concluded

The question of the validity of the hypotheses of geometry in the infinitely small is bound up with the question of the ground of the metric relations of space. . . . Either therefore the reality which underlies space must form a discrete manifoldness, or we must seek the ground of its metric relations outside it, in binding forces which act upon it. . . . This leads us into the domain of another science, of physic, into which the object of this work does not allow us to go to-day. Riemann (1868); translated by William Kingdon Clifford 1873.1

1 The German original reads (Jost (2013), p. 43): “Die Frage über die Gültigkeit der Voraussetzungen der Geometrie im Unendlichkleinen hängt zusammen mit der Frage nach dem innern Grunde der Massverhältnisse des Raumes. . . . Es muss also entweder das dem Raume zu Grunde liegende Wirkliche eine discrete Mannigfaltigkeit bilden, oder der Grund der Massverhältnisse ausserhalb, in darauf wirkenden bindenden Kräften, gesucht werden. . . . Es führt dies hinüber in das Gebiet einer andern Wissenschaft, in das Gebiet der Physik, welches wohl die Natur der heutigen Veranlassung nicht zu betreten erlaubt.” The English translation can be found in Jost (2016). For the role of Clifford in the development of these ideas, see e.g. Giulini (2018).
Riemann’s pioneering ideas are important for at least two reasons. First, although Riemann did not take into account the time dimension, his ideas led to the mathematical formalism that enabled Albert Einstein to formulate his theory of general relativity (GR) in 1915. In GR, gravity is understood as the manifestation of a dynamical geometry of space and time, which are unified into a four-dimensional spacetime.

Second, as is clear from the sentences quoted above, matter and geometry are no longer imagined as independent from each other; the metric now depends on the “binding forces which act upon it”. The metrical field is no longer given rigidly once and for all, but stands in causal dependence on matter. This idea is at the core of in GR.

In his commentary on Riemann’s text from 1919, Hermann Weyl emphasized the unification of geometry and field theory in physics,

For geometry, here the same step happened that Faraday and Maxwell performed within physics, in particular electricity theory, which was done by the transition from an action-at-a-distance to a local-action theory: carrying out the principle to understand the world from its behaviour in the infinitely small. See Jost (2013), p. 45.

Riemann’s approach turned out to be much more powerful than alternative ideas on the foundation of geometry, for example those of Hermann von Helmholtz, see e.g. Jost (2016), p. 119. Helmholtz starts from experience and postulates the possibility of free motion of bodies. As he can prove mathematically, for this free motion a space with constant curvature is required. From the later perspective of GR, this turns out to be too narrow. Riemann’s idea, that bodies can carry geometry with them, is realized in GR, which allows spaces, in fact spacetimes, to have arbitrary curvature, as determined by the Einstein field equations. These equations read

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}.$$  \hfill (1)

Here, $g_{\mu\nu}$ denotes the spacetime metric, $R_{\mu\nu}$ the Ricci tensor, and $R$ the Ricci scalar. Non-gravitational degrees of freedom (for simplicity called ‘matter’) are described by a symmetric energy–momentum tensor $T_{\mu\nu}$; it obeys the covariant conservation law

$$T_{\nu\mu}^\gamma = 0.$$  \hfill (2)

It is important to emphasize that this is not a standard conservation law (with a partial instead of a covariant derivative) from which a conserved current and charge can be derived. If the energy–momentum tensor obeys the dominant energy condition

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2 “Für die Geometrie geschah hier der gleiche Schritt, den Faraday und Maxwell innerhalb der Physik, speziell der Elektrizitätslehre, vollzogen durch den Übergang von der Fernwirkungstheorie zur Nahwirkungstheorie: das Prinzip, die Welt aus ihrem Verhalten im Unendlichkleinen zu verstehen, gelangt zur Durchführung.”

3 The title of Helmholtz’s article, “Ueber die Thatsachen, die der Geometrie zu Grunde liegen” (“On the Facts which Lie at the Bases of Geometry”), makes a dig at the title of Riemann’s work.

4 This is only possible in the presence of a symmetry, as expressed by a Killing vector.
(energy densities dominate over pressures), causality is implemented in the sense that no influence from outside the lightcone can enter its inside.

There are two free parameters in the gravitational sector: \( \kappa \) and \( \Lambda \). From the Newtonian limit, one can identify

\[
\kappa = 8\pi G/c^4, \tag{3}
\]

with \( G \) the gravitational (Newton) constant and \( c \) the speed of light. In 1917, Einstein had recognized that another free parameter is allowed – the cosmological constant \( \Lambda \) which has the physical dimension of an inverse length squared. From observations we find the value \( \Lambda \approx 1.2 \times 10^{-52} \text{m}^{-2} \approx 0.12 \text{(Gpc)}^{-2} \). The relation of this value to naive estimates from quantum field theory is an open question.

The Einstein field equations (1) describe a non-linear interaction between geometry and matter. In this sense, \( T_{\mu \nu} \) must not be interpreted as the source from which the metric is determined. For the description of matter, the metric is also needed, since it enters the field equations for matter as well as the equation of motion for test bodies given by

\[
\ddot{x}^\mu + \Gamma^\mu_{\alpha \beta} x^\alpha \dot{x}^\beta = 0. \tag{4}
\]

Here, \( \Gamma^\mu_{\alpha \beta} \) are the components of the Levi–Civita connection, which is determined by the metric, and the dots denote derivatives with respect to proper time (for timelike geodesics) or with respect to an affine parameter (null geodesics). Equation (4) is the geodesic equation which reflects the universal coupling of gravity to matter. In contrast to its Newtonian analogue, it corresponds to free motion in the geometry described by \( g_{\mu \nu} \) (‘equivalence principle’). So for the description of matter, the pair \( (T_{\mu \nu}, g_{\alpha \beta}) \) is needed, and one needs a rather involved initial value formulation to determine the spacetime metric (see next section).

A major feature of GR, and one that is particularly relevant for its quantization, is background independence. This must be carefully distinguished from mere general covariance, which means form invariance of equations under an arbitrary change of coordinates. In contrast, background independence means that there are no absolute (non-dynamical) fields in the theory – this applies to GR, where the metric is a dynamical quantity that acts on matter and is acted upon by it. As Jürgen Ehlers has remarked (Ehlers (2007), p. 91): “Conceptually, the background independence must be seen as the principal achievement of general relativity theory; it is, however, at the same time the main obstacle to overcome if general relativity theory and quantum theory are to be united.” In GR, the law of motion (4) cannot be formulated independently from the field equations (1) – in fact, it follows from them by employing (2). This would not be possible in a theory with an absolute background, that is, with an absolute non-dynamical spacetime.

In 1918, Weyl generalized the notion of the Levi-Civita connection that occurs in (4) to a symmetric linear connection, Weyl (1918), see also the extended discussion.

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5 Recent doubts on this \( \Lambda \)-observation are expressed e.g. in Di Valentino et al. (2020).

6 Test bodies in GR cannot be mass points. The mass-to-radius ratio of objects has an upper bound of \( c^2/2G \); the concept of a mass point is replaced by a black hole.
in *Raum, Zeit, Materie*, Weyl (1993). For this concept, a metrical structure on the manifold is not needed, only the notion of a parallel transport for vectors and tensors, which provides the means to connect different points on the manifold. In contrast to the Levi-Civita connection, his more general connection need not be derivable from a metric. Weyl distinguishes, in fact, three levels of geometry: the first level is the topological manifold (which he calls *situs manifold or empty world*); the second level the affinely connected manifold, and the third level the metric continuum (which he also calls “ether”); see also Schrödinger (1954) for a lucid presentation.

The notion of a symmetric linear connection allowed Weyl to construct a generalization of Einstein’s theory. In his theory, the magnitude of vectors is not fixed, but the connection allows the comparison of magnitudes in different points. This introduces a new freedom into the theory – the freedom to perform gauge transformations. The metric is here determined only up to a (spacetime-dependent) factor. The exponent of this factor can be connected with a function that behaves as the electromagnetic vector potential (here interpreted as a one-form). Weyl thought that he has constructed in this way a unified theory of gravity and electromagnetism; for details, see Weyl (1993), p. 121 ff. In the above hierarchy, Weyl’s theory can be located between the second and third level: in it, spacetime has a conformal structure, which provides a more general framework than the structure of Riemannian geometry.

Weyl was convinced that fundamental geometric relations should only refer to infinitesimally neighbouring points (*Nahgeometrie* instead of *Ferngeometrie*). This principle plays a key role in both the 1918 and the 1929 versions of gauge theories. In Weyl (2000), p. 115, he writes (emphasis by Weyl): “Only in the infinitely small can we expect to encounter everywhere the same elementary laws, thus the world must be understood from its behaviour in the infinitely small.”

In spite of its formal elegance, Weyl’s theory is empirically wrong, as was soon realized by Einstein. The reason is that a non-integrable connection leads to path-dependent frequencies for atomic spectra, in contrast to observations. But his theory can nevertheless be seen as the origin of our modern gauge theories. A non-integrable connection is manifested there, for example, in the Aharonov–Bohm effect. In its non-Abelian generalization, gauge invariance is a key ingredient to the Standard Model of particle physics, see e.g. Dosch (2007) for a review. The Standard Model (extended by massive neutrinos) is experimentally extremely well

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7 *Analysis situs* is an older name for topology.

8 Here, the words ‘gauge’ (*Eichung*), ‘to gauge’ (*eichen*), and ‘gauge invariance’ (*Eich-Invarianz*) enter. Their original meaning arises from providing standards for physical quantities (including distances), which is different from their later abstract use in the description of intrinsic symmetries in gauge theories.

9 The German original reads: “Nur im Unendlichkleinen dürfen wir erwarten auf die elementaren, überall gleichen Gesetze zu stoßen, darum muß die Welt aus ihrem Verhalten im Unendlichkleinen verstanden werden.”
tested, and no obvious deviation from it is seen so far in experiments at the Large Hadron Collider (LHC) and elsewhere.

But what about gravity and spacetime? In its standard formulation, GR is not a gauge theory. The reason is that the connection $\Gamma^\alpha_{\mu\beta}$ is not independent there, but is derived from a metric. On thus has the chain

$$g_{\mu\nu} \rightarrow \Gamma^\mu_{\alpha\beta} \rightarrow R^\mu_{\alpha\beta\gamma},$$

where $R^\mu_{\alpha\beta\gamma}$ denotes the Riemann curvature tensor. For gauge theories, the first step in this chain is lacking. Gauge theories of gravity do, however, exist, and they are needed for the consistent implementation of fermions, see Blagojević and Hehl (2013). Weyl’s original theory is a special case of this general class, but it is important to emphasize that the coupling of Weyl’s vector potential is not to the electrodynamic current – as its creator believed – but to the dilaton current (because the one-parameter dilation group is gauged), see Hehl et al. (1988).

One of the striking properties of GR is that it exhibits its own incompleteness. This is expressed in the singularity theorems which state that, under general conditions, singularities in spacetime are unavoidable, see Hawking and Penrose (1996). Singularities are here understood in the sense of geodesic incompleteness – timelike or null geodesics as found from (4) terminate at finite proper time or finite affine parameter value. In most physically relevant cases, the occurrence of singularities is connected with regions of infinite curvature or energy density; notable examples are the singularities characterizing the beginning of the Universe (“big bang”) and the interior of black holes. One of the hopes connected with the construction of a quantum theory of gravity is that such a theory will avoid singularities. This hope may be extended to a different type of singularities in our present physical theories – the infinities that arise in almost every local quantum field theory. One has learnt to cope with the latter singularities by employing sophisticated methods of regularization and renormalization. Nevertheless, one would expect that a truly fundamental theory will be finite from the onset. The reason is that the occurrence of singularities is connected with an unsufficient understanding of the microstructure of spacetime. True infinities should not occur in any sensible description of Nature, cf. Ellis et al. (2018).

One possible solution to the singularity problem is to avoid a continuum for the spacetime structure and to assume instead that spacetime is built up from discrete entities. There are indications for such a discrete structure in some approaches to quantum gravity, but the last work has not yet been spoken. Interestingly, Riemann himself envisaged the possibility of a continuous as well as a discrete manifold; the smallest entities he calls *quanta* (Jost (2016), p. 32):

Definite portions of a manifoldness, distinguished by a mark or by a boundary, are called Quanta. Their comparison with regard to quantity is accomplished in the case of discrete

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10 See also the contributions by Hehl and Obukhov and by Scholz to this volume.
magnitudes by counting, in the case of continuous magnitudes by measuring. (Translated by William Kingdon Clifford 1873)

Weyl, in his commentary to Riemann’s text, speculates that the final answer to the problem of space may be found in its discrete nature.

What happens to this when the quantum of action $\hbar$ comes into play? One of the early pioneers of attempts to quantizing gravity, Matvei Bronstein, through the application of thoughts experiments, arrived at the necessity of introducing minimal distances in spacetime, thus abandoning the idea of a metric continuum. He writes:

The elimination of the logical inconsistencies connected with this [his thought experiments] requires a radical reconstruction of the theory, and in particular, the rejection of a Riemannian geometry dealing, as we see here, with values unobservable in principle, and perhaps also the rejection of our ordinary concepts of space and time, modifying them by some much deeper and nonevident concepts. Wer’s nicht glaubt, bezahlt einen Taler.

In Bronstein’s analysis, quantities appear that can be found by combining $G$, $c$, and $\hbar$ into units of length, time, and mass (or energy). They were first presented by Max Planck in 1899 (one year before the ‘official’ introduction of the quantum of action into physics!) and are called Planck units in his honour. They read:

\[
\begin{align*}
  l_P &= \sqrt{\frac{\hbar G}{c^3}} \approx 1.62 \times 10^{-35} \text{ m} \\
  t_P &= \frac{l_P}{c} \sqrt{\frac{\hbar G}{c^5}} \approx 5.40 \times 10^{-44} \text{ s} \\
  m_P &= \frac{\hbar}{l_P c} \sqrt{\frac{hc}{G}} \approx 2.17 \times 10^{-8} \text{ kg} \approx 1.22 \times 10^{19} \text{ GeV}/c^2.
\end{align*}
\]

At the end of his 1899 paper, Planck wrote the following prophetic sentences, see Kiefer (2012), p. 6:

These quantities retain their natural meaning as long as the laws of gravitation, of light propagation in vacuum, and the two laws of the theory of heat remain valid; they must therefore, if measured in various ways by all kinds of intelligent beings, always turn out to be the same.

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11 The German original reads (Jost (2013), p. 31): “Bestimmte, durch ein Merkmal oder eine Grenze unterschiedene Theile einer Mannigfaltigkeit heissen Quanta. Ihre Vergleichung der Quantität nach geschieht bei den discreten Grössen durch Zählung, bei den stetigen durch Messung.”

12 “Sehen wir von der ersten Möglichkeit ab, es könnte ‘das dem Raum zugrunde liegende Wirkliche eine diskrete Mannigfaltigkeit bilden’ (obschon in ihr vielleicht einmal die endgültige Antwort auf das Raumproblem enthalten sein wird, my emphasis) . . .

13 The quotation is from Kiefer (2012), p. 20.

14 See e.g. Kiefer (2012), p. 5.

15 The German original reads: “Diese Grössen behalten ihre natürliche Bedeutung so lange bei, als die Gesetze der Gravitation, der Lichtfortpflanzung im Vacuum und die beiden Hauptsätze der Wärmetheorie in Gültigkeit bleiben, sie müssen also, von den verschiedensten Intelligenzen nach den verschiedensten Methoden gemessen, sich immer wieder als die nämlichen ergeben.”
One can form a dimensionless number out of these Planck units by bringing the cosmological constant $\Lambda$ into play. Inserting the present observational value for $\Lambda$ (see above), this gives

$$l_p^2 \Lambda \equiv \frac{G \hbar \Lambda}{c^3} \approx 3.3 \times 10^{-122}.$$  

(8)

The smallness of this number is one of the biggest open puzzles in fundamental physics. Only a fundamental unified theory of all interactions is expected to provide a satisfactory explanation.

What are the general arguments that speak in favour of a quantum theory of gravity? First, as mentioned above, there is the singularity problem of classical general relativity, which points to the incompleteness of Einstein’s theory. Second, the search for a unified theory of all interactions should include quantum gravity: gravity interacts universally to all fields of Nature, and all non-gravitational fields are successfully described by quantum (field) theory so far, so a quantum description should apply to gravity, too. Third, a very general argument was put forward by Richard Feynman in 1957, see Kiefer (2012), p. 18: if we generate a superposition of two masses at different locations, their gravitational fields should also be superposed, unless the superposition principle of quantum theory breaks down. A quantum theory of gravity is needed to describe such superpositions. It is clear that such a state can no longer correspond to a classical spacetime. There are at present interesting suggestions for the possibility to observing the gravitational field generated by a quantum superposition in laboratory experiments, see Carlesso et al. (2019) and references therein.

Several approaches to quantum gravity exist, but there is so far no consensus in the community, see Kiefer (2012). The ideal case would be to construct a finite quantum theory of all interactions from which present physical theories can be derived as approximations (or “effective field theories”) in appropriate limits. The only reasonable candidate is string theory. In this theory, the dimension of spacetime assumes the number ten or eleven. Unfortunately, it is so far not clear how to recover the Standard Model from string theory and how to test it by experiments. Connected with this is the difficulty to proceed in a more or less unique way from the ten or eleven spacetime dimensions to the four dimensions of the observed world.

The main alternatives to finding a unified theory are the more modest attempts to construct first a quantum theory of the gravitational field and to relegate unification to a later step. The usual starting point is GR, but quantization methods may be applied to any other gravitational theory. Standard methods are path integral quantization and canonical quantization. We shall focus below on the canonical quantization of GR using metric variables, because conceptual issues dealing with space and time are most transparent in this approach, see Kiefer (2009).

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16 See e.g. Kiefer (2012) for a comprehensive discussion.
Besides ordinary three-dimensional space (or four-dimensional spacetime), the concept of configuration space plays an eminent role in physics. In mechanics, this is the N-dimensional space generated by all configurations, described by coordinates \( \{ q^a \} \), \( a = 1, \ldots, N \), that the system can assume. In field theory, it is infinite-dimensional of possible field configurations. In quantum theory, it will enter the argument of the wave function (functional) and lead to the central property of entanglement.

What is the configuration space in general relativity? As John Wheeler writes (Wheeler (1968), p. 245): “A decade and more of work by Dirac, Bergmann, Schild, Pirani, Anderson, Higgs, Arnowitt, Deser, Misner, DeWitt, and others has taught us through many a hard knock that Einstein’s geometrodynamics deals with the dynamics of geometry: of 3-geometry, not 4-geometry.” Most of these developments happened after Weyl’s death in 1955. In fact, upon application of the canonical (or Hamiltonian) formalism, Einstein’s theory can be written as a dynamical system for the three-metric \( h_{ab} \) and its canonical momentum \( \pi^{ab} \) on a spacelike hypersurface \( \Sigma \).

The ten Einstein equations can be formulated as four constraints, that is, restrictions on initial data \( h_{ab} \) and \( \pi^{ab} \) on \( \Sigma \), and six evolution equations. The four constraints read (per spacepoint)

\[
\mathcal{H}_\perp \approx 0 \quad \text{and} \quad \mathcal{H}^a \approx 0,
\]

where \( \mathcal{H}_\perp \) is called “Hamiltonian constraint”, while \( \mathcal{H}^a \) are called “momentum (diffeomorphism) constraints”. The symbol \( \approx 0 \) is Dirac’s weak equality and means “vanishing as a constraint”. The canonical momentum \( \pi^{ab} \) is related to the extrinsic curvature \( K_{cd} \) of \( \Sigma \) by

\[
\pi^{ab} = \frac{G^{ab \, cd} K_{cd}}{2 \kappa},
\]

where \( G^{ab \, cd} \) denotes the DeWitt metric itself (the inverse of the expression in (11)). This quantity plays the role of a metric in the space of all Riemannian three-metrics \( h_{ab} \), a space called Riem \( \Sigma \).

Is Riem \( \Sigma \) the configuration space of GR? Not yet. The constraints \( \mathcal{H}^a \approx 0 \) guarantee the invariance of the theory under infinitesimal three-dimensional coordinate transformations. The real configuration space is thus the space of all three-geometries, not the space of all three-metrics. This is what Wheeler called superspace, here
denoted by $S(\Sigma)$, see Wheeler (1968). It is the arena for classical and quantum geometrodynamics. One can formally write

$$S(\Sigma) := \text{Riem } \Sigma / \text{Diff } \Sigma,$$

where Diff $\Sigma$ denotes the group of three-dimensional diffeomorphisms (“coordinate transformations”). By going to superspace, the momentum constraints are automatically fulfilled. Whereas Riem $\Sigma$ has a simple topological structure, the topological structure of $S(\Sigma)$ is very complicated because it inherits (via Diff $\Sigma$) some of the topological information contained in $\Sigma$; see Giulini (2009) for details.

The DeWitt metric has pointwise a Lorentzian signature with one negative and five positive directions, that is, it has negative, null, and positive directions. Due to the minus sign, the kinetic term for the gravitational field is indefinite. It is important to note that this minus sign is unrelated to the signature of spacetime; starting with a four-dimensional Euclidean space instead of a four-dimensional spacetime, the same signature for the DeWitt metric is found. The presence of this minus sign is related to the attractive nature of gravity. It is also worth mentioning that the DeWitt metric reveals a surprising analogy with the elasticity tensor in three-dimensional elasticity theory and the local and linear constitutive tensor in four-dimensional electrodynamics, see Hehl and Kiefer (2018). This analogy could be of importance for theories of emergent gravity.

Constraints and evolution equations have an intricate relationship; see e.g. Giulini and Kiefer (2007). Let me summarize the main features as well as pointing out analogies with electrodynamics. First, there is an important connection with the (covariant) conservation law of energy–momentum. The constraints are preserved in time if and only if the energy–momentum tensor of matter has vanishing covariant divergence. In electrodynamics, the Gauss constraint is preserved in time if and only if electric charge is conserved.

Second, Einstein’s equations represent the unique propagation law consistent with the constraints. To be more concrete, if the constraints are valid on an “initial” hypersurface and if the dynamical evolution equations (the pure spatial components of the Einstein equations) hold, the constraints hold on every hypersurface. And if the constraints hold on every hypersurface, the dynamical evolution equations hold. Again, there is an analogy with electrodynamics: Maxwell’s equations are the unique propagation law consistent with the Gauss constraint.

It must be emphasized that the picture of a spacetime foliated by a one-parameter family of hypersurfaces only emerges after the dynamical equations are solved. Then, spacetime can be interpreted as a “trajectory of spaces”. Before this is done, one only has a three-dimensional manifold $\Sigma$ with given topology, equipped with the canonical variables satisfying the constraints (9) and (10).

This fact that spacetime is not given from the outset but must be constructed through an initial value formulation, is an expression of the background independence discussed in the previous section. In this sense, the analogy with electrodynamics on a given external spacetime breaks down. Background independence is related with the classical version of what is called the problem of time: if we restrict ourselves
to compact three-manifolds \( \Sigma \), the total Hamiltonian of GR is a combination of the constraints (9) and (10). Thus, no external time parameter exists; all physical time parameters are to be constructed from within our system, that is, as functional of the canonical variables. A priori, there is no preferred choice of such an intrinsic time parameter. It is this absence of an external time and the non-preference of an intrinsic one that is known as the problem of time in (classical) canonical gravity. Still, after the solution of the dynamical equations, spacetime as a trajectory of spaces exists. This is different in the quantum theory where it leads to the far-reaching quantum version of the problem of time (next section).

The possibility of constructing spacetime in the way just described, is also reflected in the closure of the Poisson algebra for the constraints (9) and (10):

\[
\{H_{\perp}(x), H_{\perp}(y)\} = -\sigma \delta_{ab}(x,y) \left( h^{ab}(x) H_{b}(x) + h^{ab}(y) H_{b}(y) \right)
\]

(13)

\[
\{H_{a}(x), H_{\perp}(y)\} = H_{b}(x) \delta_{ab}(x,y)
\]

(14)

\[
\{H_{a}(x), H_{b}(y)\} = H_{b}(x) \delta_{ab}(x,y) + H_{a}(y) \delta_{ab}(x,y)
\]

(15)

It is not a Lie algebra, though, because the Poisson bracket between two Hamiltonian constraints at different points also contains (the inverse of) the three-metric, \( h^{ab} \).

We also remark that the signature of the spacetime metric enters here in the form of the parameter \( \sigma \): in fact, \( \sigma = -1 \) corresponds to a four-dimensional spacetime, while \( \sigma = 1 \) corresponds to a four-dimensional space. It is a fundamental (and open) question whether the closure of this algebra also holds in quantum gravity.

The relation of the transformations generated by the constraints to the spacetime diffeomorphisms is a subtle one and will not be discussed here; see e.g. Sundermeyer (2014), Kiefer (2012).

### 3 Quantum geometrodynamics

In the last section, we have reviewed the canonical (Hamiltonian) formulation of GR. Here, we discuss the quantum version of this, see e.g. Kiefer (2012) for a comprehensive treatment. We follow Dirac’s heuristic approach and transform the classical constraints (9) and (10) into conditions on physically allowed wave functionals. These wave functionals are defined on the space of all three-metrics (the above space Riem \( \Sigma \)) and matter fields on \( \Sigma \). The quantum version of (9) reads

\[
\hat{H}_{\perp} \Psi \equiv \left( -16\pi G \hbar^2 G_{abcd} \frac{\delta^2}{\delta h_{ab} \delta h_{cd}} \right. - (16\pi G)^{-1} \sqrt{\hbar} \left. (3 \mathcal{R} - 2\Lambda) + \sqrt{\hbar} \hat{\rho} \right) \Psi = 0
\]

(16)

\[\text{In the asymptotically flat case, additional boundary terms are present.}\]
and is called the **Wheeler–DeWitt equation**. We note that the kinetic term in this equation only has formal meaning before the issues of factor ordering and regularization are successfully addressed.\(^{18}\) The quantum implementation of \((10)\) reads

\[
\hat{\mathcal{H}}^a \Psi \equiv -2\nabla_b \left( \frac{\hbar}{i} \frac{\delta \Psi}{\delta h_{ab}} \right) + \sqrt{h} \hat{j}^a \Psi = 0 \tag{17}
\]

and is called the momentum (or diffeomorphism) constraints. These latter equations have a simple interpretation. Under a coordinate transformation

\[
x^a \mapsto \bar{x}^a = x^a + \delta N^a(x),
\]

the three-metric transforms as

\[
h_{ab}(x) \mapsto \bar{h}_{ab}(x) = h_{ab}(x) - D_a \delta N_b(x) - D_b \delta N_a(x).
\]

The wave functional then transforms according to

\[
\Psi[h_{ab}] \mapsto \Psi[\bar{h}_{ab}] = 2 \int d^3x \frac{\delta \Psi[A]}{\delta h_{ab}(x)} D_a \delta N_b(x).
\]

Assuming the invariance of the wave functional under this transformation, one is led to

\[
D_a \frac{\delta \Psi}{\delta h_{ab}} = 0.
\]

This is exactly \((17)\) (restricted here to the vacuum case).

A simple analogy to \((17)\) is Gauss’s law in quantum electrodynamics (or its generalization to the non-Abelian case). The quantized version of the constraint \(\nabla E \approx 0\) reads

\[
\frac{\hbar}{i} \nabla \frac{\delta \Psi[A]}{\delta A} = 0,
\]

where \(A\) is the vector potential. This equation reflects the invariance of \(\Psi\) under spatial gauge transformations of the form \(A \rightarrow A + \nabla \lambda\).

The constraints can only be implemented in the form \((16)\) and \((17)\) if the quantum version of the constraint algebra \((13) - (15)\) holds without extra c-number terms on the right-hand side. Otherwise, only a part of the quantum constraints (or even none) holds in this form. The situation is reminiscent of string theory where the Virasoro algebra displays such extra (central or Schwinger) terms. More general quantum constraints hold there provided the number of spacetime dimensions is restricted to a specific number (ten in the case of superstrings). It is imaginable that a restriction in the number of spacetime dimensions arises also here from a consistent treatment of the quantum constraint algebra. But so far, this is not clear at all.\(^{19}\)

\(^{18}\) For a recent attempt into this direction, see Feng (2018).

\(^{19}\) This problem was already known to Dirac and was the reason why he abandoned working on quantum gravity. In his last contribution to this field, he remarked, Dirac (1968), p. 543: “The problem of the quantization of the gravitational field is thus left in a rather uncertain state. If one accepts Schwinger’s plausible methods, the problem is solved. [Dirac refers to a heuristic
In the last section, we have seen that we can interpret spacetime as a generalized trajectory of spaces. In its construction, the four constraint equations and the six dynamical equations are inextricably interwoven. What happens in the quantum theory? There, the trajectory of spaces has disappeared, in the same way as the ordinary mechanical trajectory of a particle has disappeared in quantum mechanics. The three-metric $h_{ab}$ and its momentum $\pi^{cd}$ play the role of the $q^i$ and $p_j$ in mechanics, so it is clear that in quantum gravity $h_{ab}$ and $\pi^{cd}$ cannot be “determined simultaneously”, which means that spacetime is absent at the most fundamental level, and only the configuration space of all three-metrics respective three-geometries remains. This is clearly displayed in Table 1 on p. 248 in Wheeler (1968).

From this point of view it is clear that in the quantum theory only the constraints survive. The evolution equations lose their meaning in the absence of a spacetime. In a certain sense, this is anticipated in the classical theory by the strong connection between constraints and evolution equations as discussed in the previous section.

The absence of spacetime, and in particular of time, is usually understood as the quantum version of the problem of time. It means that the quantum world at the fundamental level is timeless – it just is. Weyl has attributed such a static picture already to the classical spacetime of GR. In Weyl (2000) p. 150, he writes:

> The objective world just is, it does not happen. Only from the view of the consciousness crawling upwards in the worldline of my life a sector of this world “lives up” and passes by at him as a spatial picture in temporal transformation.

In the quantum theory, there is not even a spacetime and a worldline with a conscious observer, at least not at the most fundamental level. So how can we relate this picture of timelessness, forced upon us by a straightforward extrapolation of established physical theories, with the standard concept of time in physics? There are two points to be discussed here.

First, as already mentioned, the DeWitt metric (12) has an indefinite signature: one minus and five plus. This means that the Wheeler–DeWitt equation has a local hyperbolic structure through which part of the three-metric is distinguished as an intrinsic timelike variable. One can show that this role is played by the “local scale” $\sqrt{h}$. In simple cosmological models of homogeneous and isotropic (Friedmann–Lemaître) universes, this is directly related to the scale factor $a$. Using units with $2G/3\pi = 1$, the Wheeler–DeWitt equation for a closed Friedmann–Lemaître universe with a massive scalar field reads

$$ \frac{1}{2} \left( \frac{\hbar^2}{a^2} \frac{\partial}{\partial a} \left( a \frac{\partial}{\partial a} \right) - \frac{\hbar^2}{a^3} \frac{\partial^2}{\partial \phi^2} - a + \frac{\Lambda a^3}{3} + m^2 a^3 \phi^2 \right) \psi(a, \phi) = 0. $$ (18)

20 The German original reads: “Die objektive Welt ist schlechthin, sie geschieht nicht. Nur vor dem Blick des in der Weltlinie meines Lebens emporkriechenden Bewußtseins “lebt” ein Ausschnitt dieser Welt “auf” und zieht an ihm vorüber als räumliches, in zeitlicher Wandlung begriffenes Bild.” [emphasis by Weyl]
Additional gravitational and matter degrees of freedom come with kinetic terms that differ in sign from the kinetic term with respect to \(a\). For equations such as (18), one can thus formulate an initial value problem with respect to intrinsic time \(a\). The configuration space is here two-dimensional and spanned by the variables \(a\) and \(\phi\).

Standard quantum theory employs the mathematical structure of a Hilbert space in order to implement the probability interpretation for the quantum state. An important property is the unitary evolution of this state; it guarantees the conservation of the total probability with respect to the external time \(t\). But what happens when there is no external time, as we have seen is the case in quantum gravity? There is no common opinion on this, but it is at least far from clear whether a Hilbert-space structure is needed at all, and if yes, which one. This is also known as the Hilbert-space problem and is evidently related to the problem of time.

The second point concerns the recovery of the standard (general relativistic) notion of time from the fundamentally timeless theory of gravity. The standard way proceeds via a Born–Oppenheimer type of approximation scheme, similarly to molecular physics. For this to work, the quantum state, which is a solution of (16) and (17), must be of a special form. For such a state one can recover an approximate notion of semiclassical (WKB) time. One can show that this WKB time (which, in fact, is a “many-fingered time”) corresponds to the notion of time in Einstein’s theory. Equations (16) and (17) then lead to a functional Schrödinger equation describing the limit of quantum field theory in curved spacetime, the latter given by Einstein’s equations. It is in this limit that one can apply the standard Hilbert-space structure and the associated probability interpretation. Higher orders of this approximation allow the derivation of quantum-gravitational corrections terms, which, for example, give corrections to the Cosmic Microwave Background (CMB) anisotropy spectrum proportional to the inverse Planck-mass squared. Such terms follow from a straightforward expansion of (16) and (17) and could in principle give a first observational test of quantum geometrodynamics.

Quantum geometrodynamics, like practically all approaches to quantum gravity, is a linear theory in the quantum states and thus obeys the superposition principle. This means that most states do not correspond to any classical three-geometry. The situation resembles, of course, Schrödinger’s cat. Like there, one can employ the process of decoherence to understand why such weird superpositions are not observed, see Joos et al. (2003). Decoherence is the irreversible and unavoidable interaction of a quantum system with the irrelevant degrees of freedom of its “environment.” In quantum cosmology, one can consider, for example, the variables \(a\) and \(\phi\) in (18) as describing the (relevant) quantum system, while small density perturbations and tiny gravitational waves can play the role of the environment. The entanglement between

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21 Except phantom fields, which play a role in connection with discussions about dark energy, cf. Bouhmadi-López et al. (2019), Di Valentino et al. (2020).

22 See e.g. Kiefer (2012, 2013) for a detailed discussion of this and the other conceptual issues discussed below.

23 “Environment” is a metaphor here. It stands for other degrees of freedom in configuration space which become entangled with the quantum system, but which cannot be observed themselves.
system and environment leads to the suppression of interferences between different \( a \) and different \( \phi \) (within some limits); in this sense, classical geometry and classical universe emerge. The same holds for the emergence of structure in the universe from primordial quantum fluctuations, see Kiefer and Polarski (2009).

It is evident from the above that the question about the correct interpretation of quantum theory enters here with its full power. Since by definition the Universe as a whole is a strictly closed quantum system, one cannot invoke any classical measurement agent as acting from the outside. Following DeWitt (1967), the standard interpretation used, at least implicitly, is the Everett interpretation, which states that all components in the linear superposition are real.

It is obvious that at the level of (18) there is no intrinsic difference between big bang and big crunch; both correspond to the region \( a \) approaching zero in configuration space. This has important consequences for cosmological models in which classically the universe expands and recollapses, see Zeh (2007). In the quantum version, there is no trajectory describing the expansion and the recollapse. The only structure available is an equation of the form (18) in which only the scale factor \( a \) (and other variables) enter. The natural way to solve such an equation is to specify initial values on constant-\( a \) hypersurfaces in configuration space and to evolve them from smaller \( a \) to larger \( a \). In more complicated models, one can evolve also the entanglement entropy between degrees of freedom in this way. If the entropy is low at small \( a \) (as is suggested by observations), it will increase all along from small \( a \) to large \( a \). There is then a formal reversal of the arrow of time at the classical turning point, although this cannot be noticed by any observer, because the classical evolution comes to an end before the region of the classical turning point is reached.

We have limited the discussion here to quantum geometrodynamics. The main conclusions also hold for the path-integral approach and to loop quantum gravity. In loop quantum gravity, there are analogies with gauge theories, for example with Faraday’s lines of forces, see Frittelli et al. (1994). Still, it is not a gauge theory by itself, and many conceptual issues such as the semiclassical limit are much less clear than in quantum geometrodynamics.

4 The role of matter

Very early on, Einstein was concerned with a fundamental duality observed in the physical description of Nature: the duality between fields and matter. This duality is the prime motivation for introducing the concept of light quanta in his important paper on the photoelectric effect from 1905. At that time, the only known dynamical

 Alternatives are the de Broglie–Bohm approach and collapse models, which both are more new theories than new interpretations.

 The situation in string theory so far is less clear; there are indications that not only the concept of spacetime, but also the concept of space is modified, as is discussed in the context of the AdS/CFT conjecture.
field was the electromagnetic field; ten years later, with GR, the gravitational field joined in.

In *Raum, Zeit, Materie*, Weyl writes at the end of the main text, see Weyl (1993), p. 317:

> In the darkness, which still wraps up the problem of matter, perhaps quantum theory is the first dawning light.

Here, the hope is expressed that quantum theory, which in 1918 was still in its infancy, may provide a solution for this duality. This is certainly along the lines of Einstein’s 1905 light quanta hypothesis. But, ten years later, the final quantum theory gave a totally different picture: central notions of the theory are wave functions and the probability interpretation. Einstein was repelled by this, especially by the feature of entanglement, which seems to provide a “spooky” action at a distance. This is why he focused on a unified theory of gravity and electrodynamics. He hoped to understand “particles” as solitonic solutions of field equations. His project did not succeed.

A somewhat different direction to understand ‘matter from space’ was pursued by John Wheeler in the 1950s, see Wheeler (1962). The idea is that mass, charge, and other particle properties originate from a non-trivial topological structure of space, the most famous example being Wheeler’s wormhole. This is most interesting, but has not led to anything close to a fundamental theory.

Weyl’s 1929 idea of understanding the interaction of electrons with the electromagnetic field by the gauge principle turned out to be more promising. The Standard Model of particle physics is an extremely successful gauge theory, and virtually all of its extensions make use of this principle, too. Gauge fields can also be described in a geometric way by adopting the mathematical structure of fibre bundles. Still, this is relatively far from the geometric concepts of GR, which deal with spacetime and not with the internal degrees of freedom of gauge theories. Perhaps gauge theories of gravity may help in finding a unified field theory, see Blagojević and Hehl (2013).

Our physical theories all employ a metric to represent matter fields and their interactions, so GR is always relevant, even in situations where its effects are small. As Jürgen Ehlers writes in Ehlers (2007), p. 91: “Since inertial mass is separable from active, gravity-producing mass, an ultimate understanding of mass can be expected only from a theory comprising inertia and gravity.” This should also apply for the origin of the masses in the Standard Model. The Higgs mechanism provides only a partial answer; the masses of elementary (non-composite) particles are given by the coupling to the Higgs, but the masses of composite particles such as proton and neutron cannot be explained. In fact, it seems that the mass of the proton mostly arises from the binding energy of its constituents – quarks and gluons – and not from their masses, which to first order are negligible. Invoking the inverse of Einstein’s famous formula, \( m = E/c^2 \), one can speculate that mass ultimately originates from energy, see Wilczek (1999/2000). It is hard to imagine that this origin can be understood.

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26 The German original reads: “In dem Dunkel, welches das Problem der Materie annonch umhüllt, ist vielleicht die Quantentheorie das erste anbrechende Licht.”

27 For a recent account of matter from (the topology of) space, see e.g. Giulini (2018).
without gravity. Perhaps a unified theory at the fundamental level is conformally invariant, similar to Weyl’s 1918 theory, expressing the irrelevance of masses at high energies (small scales); masses would then only emerge as an effective, low-energy concept.

Unfortunately, despite many attempts, the duality of matter and fields remains unresolved, even in present approaches to quantum gravity. An exception may be string theory, but this approach has its own problems and it is far from clear whether it can be tested empirically. Perhaps the solution to the problem of matter may arrive from a completely unexpected direction. Space, time, and matter continue to be central concepts for research in the 21st century. The question posed in the title of Einstein (1919), “Do gravitational fields play an essential role in the constitution of material elementary particles?” will most likely have to be answered by a definite yes.

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