Inequalities for two systems of subspaces with prescribed intersections

Gábor Hegedüs
Antal Bejczy Center For Intelligent Robotics
Kiscelli utca 82, Budapest, Hungary, H-1032
hegedus.gabor@nik.uni-obuda.hu

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Abstract

Let $W$ denote a linear space over a fixed field $\mathbb{F}$. We define the notions of weak ISP-system and weak $(u, v)$-system $\mathcal{S} = \{(U_i, V_i) : 1 \leq i \leq m\}$ of subspaces of $W$. We give upper bounds for the size of weak ISP-systems and weak $(u, v)$-systems.

Keywords. $q$-binomial coefficient, Bollobás Theorem, extremal set theory

1 Introduction

First we recall the notion of $q$-binomial coefficients.

The $q$-binomial coefficient $\left[\begin{array}{c} n \\ m \end{array}\right]_q$ is a $q$-analog for the binomial coefficient, also called a Gaussian coefficient or a Gaussian polynomial. The $q$-binomial coefficient is given by

$$\left[\begin{array}{c} n \\ m \end{array}\right]_q := \frac{[n]_q!}{(n - m)_q! \cdot [m]_q!}$$

for $n, m \in \mathbb{N}$, where $[n]_q!$ is the $q$-factorial (see [3], p. 26)

$$[n]_q! := (1 + q) \cdot (1 + q + q^2) \cdots (1 + q + q^2 + \ldots + q^{n-1}).$$
Clearly we have $\binom{n}{k}_q = \binom{n}{n-k}_q$. If we substitute $q = 1$ into (1), then this substitution reduces this definition to that of binomial coefficients.

Bollobás proved in [1] the following two remarkable results in extremal combinatorics.

**Theorem 1.1** Let $A_1, \ldots A_m$ and $B_1, \ldots B_m$ be finite sets satisfying the conditions

(i) $A_i \cap B_i = \emptyset$ for each $1 \leq i \leq m$;

(ii) $A_i \cap B_j \neq \emptyset$ for each $i \neq j$ ($1 \leq i, j \leq m$).

Then

$$\sum_{i=1}^{m} \frac{1}{\binom{|A_i| + |B_i|}{|A_i|}} \leq 1.$$  

**Theorem 1.2** Let $A_1, \ldots A_m$ be $r$-element sets and $B_1, \ldots B_m$ be $s$-element sets such that

(i) $A_i \cap B_i = \emptyset$ for each $1 \leq i \leq m$;

(ii) $A_i \cap B_j \neq \emptyset$ for each $i \neq j$ ($1 \leq i, j \leq m$).

Then

$$m \leq \binom{r + s}{s}.$$  

Tuza proved the following two versions of Bollobás Theorem.

**Theorem 1.3** Let $p$ be an arbitrary real number, $0 < p < 1$ and $t := 1 - p$.

Let $A_1, \ldots A_m$ and $B_1, \ldots B_m$ be finite sets satisfying the conditions

(i) $A_i \cap B_i = \emptyset$ for each $1 \leq i \leq m$;

(ii) $A_i \cap B_j \neq \emptyset$ or $A_j \cap B_i \neq \emptyset$ for $i \neq j$ ($1 \leq i, j \leq m$).

Then

$$\sum_{i=1}^{m} p^{|A_i|} t^{|B_i|} \leq 1.$$  

**Theorem 1.4** Let $A_1, \ldots A_m$ be $r$-element sets and $B_1, \ldots B_m$ be $s$-element sets satisfying the conditions
(i) \( A_i \cap B_i = \emptyset \) for each \( 1 \leq i \leq m \);

(ii) \( A_i \cap B_j \neq \emptyset \) or \( A_j \cap B_i \neq \emptyset \) for \( i \neq j \) (\( 1 \leq i, j \leq m \)).

Then

\[
m \leq \frac{(r+s)^{r+s}}{r^s s^r}.
\]

Z. Tuza raised in [6] the following question: Let \( a, b \) be fixed positive integers. Determine the largest integer \( m := m(a, b) \) such that there exists a system \( S = \{(A_i, B_i) : 1 \leq i \leq m\} \) of \( m(a, b) \) pairs of sets satisfying the conditions:

(i) \( A_1, \ldots, A_m \) are \( r \)-element sets and \( B_1, \ldots, B_m \) are \( s \)-element sets;

(ii) \( A_i \cap B_i = \emptyset \) for each \( 1 \leq i \leq m \);

(iii) \( A_i \cap B_j \neq \emptyset \) or \( A_j \cap B_i \neq \emptyset \) for \( i \neq j \) (\( 1 \leq i, j \leq m \)).

Tuza proved the following properties of the numbers \( m(a, b) \) in [6].

**Proposition 1.5** \( m(a, 1) = 2a + 1 \) for each \( a \geq 1 \). For every \( a, b, \geq 1 \)

\[
m(a, b) \geq m(a, b - 1) + m(a - 1, b).
\]

Proposition [1.5] gives a lower bound for \( m(a, b) \) near to \( 2^{(a+b)} \) for every \( a \) and \( b \).

Lovász used in [4] tensor product methods to prove the following skew version of Bollobás’ Theorem for subspaces.

**Theorem 1.6** Let \( \mathbb{F} \) be an arbitrary field. Let \( U_1, \ldots, U_m \) be \( r \)-dimensional and \( V_1, \ldots, V_m \) be \( s \)-dimensional subspaces of a linear space \( W \) over the field \( \mathbb{F} \). Assume that

(i) \( U_i \cap V_i = \{0\} \) for each \( 1 \leq i \leq m \);

(ii) \( U_i \cap V_j \neq \{0\} \) whenever \( i < j \) (\( 1 \leq i, j \leq m \)).

Then

\[
m \leq \binom{r+s}{r}.
\]
In this paper our main aim is to give a subspace version of Theorem 1.3 and 1.4.

The following definitions were motivated by Theorem 1.4 and 1.6.

Definition 1.7 Let $F$ be a fixed field. We say that a system $S = \{(U_i, V_i) : 1 \leq i \leq m\}$ is a weak ISP-system of subspaces of an $n$-dimensional linear space $W$ over the field $F$, if $S$ satisfies the following conditions:

(i) $U_i \cap V_i = \{0\}$ for each $1 \leq i \leq m$;

(ii) $U_i \cap V_j \neq \{0\}$ or $U_j \cap V_i \neq \{0\}$ for $i \neq j$ $(1 \leq i, j \leq m)$.

Definition 1.8 Let $F$ be a fixed field. We say that a system $S = \{(U_i, V_i) : 1 \leq i \leq m\}$ of subspaces of a linear space $W$ over the field $F$ is a weak $(u, v)$-system, if $S$ satisfies the conditions

(i) $S$ is a weak ISP-system;

(ii) $\dim(U_i) = u$ and $\dim(V_i) = v$ for each $1 \leq i \leq m$.

Our main results are upper bounds for the size of weak ISP-systems and weak $(u, v)$-systems.

Theorem 1.9 Let $S = \{(U_i, V_i) : 1 \leq i \leq m\}$ be a weak ISP-system of subspaces of a linear space $W$ over the finite field $F_q$. Let $u_i := \dim(U_i)$ and $v_i := \dim(V_i)$ for each $1 \leq i \leq m$. Let $0 \leq j \leq n$ be an arbitrary, but fixed integer. Then we have

$$\sum_{i=1}^{m} \left[ \frac{n-v_i-u_i}{j-u_i} \right] q^{(j-u_i)v_i} \leq 1.$$  

Theorem 1.10 Let $S = \{(U_i, V_i) : 1 \leq i \leq m\}$ be a weak $(u, v)$-system of subspaces of an $n$-dimensional linear space $W$ over the finite field $F_q$. Then

$$m \leq \left( \frac{q}{q-1} \right)^n q^{uv}.$$
2 Proofs of our main results

In the proof of our main results we use the following bounds for the $q$-binomial coefficients.

**Lemma 2.1** Let $0 \leq j \leq n$ be natural numbers. Then

$$\binom{n}{j}_q \leq \left(\frac{q}{q-1}\right)^n q^{j(n-j)}.$$  

**Proof.** This follows immediately from the inequalities

$$q^{\binom{n}{2}} \leq [n]_q! \leq \left(\frac{q}{q-1}\right)^n q^{\binom{n}{2}}.$$

$\square$

We use also in the proof of Theorem 1.9 the following simple Lemma (see Lemma 2.2 in [8]).

**Lemma 2.2** Let $V$ denote the $n$-dimensional vector space over the finite field $\mathbb{F}_q$ and fix an $(n-d)$-dimensional subspace $K$ of $V$, where $0 \leq d \leq n$. Let $U_1$ be a fixed $\ell_1$-subspace of $V$ such that $U_1 \cap K = \{0\}$. Let $u(n, d; \ell_1, \ell_2)$ denote the number of $\ell_2$-subspaces $U_2$ of $V$ satisfying $U_2 \cap K = \{0\}$ and $U_1 \subseteq U_2$. Then

$$u(n, d; \ell_1, \ell_2) = \binom{n}{\ell_2}_q \binom{\ell_2}{\ell_1}_q q^{(\ell_2-\ell_1)(n-d)}.$$

$\square$

**Proof of Theorem 1.9**

Let $1 \leq i \leq m$, $0 \leq j \leq n$ be fixed integers. Let $\mathcal{F}(i, j)$ denote the following subset of subspaces of $W$:

$$\mathcal{F}(i, j) := \{U \leq W : \dim(U) = j, U_i \subseteq U, V_i \cap U = \{0\}\}.$$

Then it follows immediately from Lemma 2.2 that

$$|\mathcal{F}(i, j)| = \frac{\binom{n-v_i}{j}_q \binom{j}{u_i}_q q^{(j-u_i)v_i}}{\binom{n-v_i}{u_i}_q}.$$

for each $0 \leq j \leq n$. 

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Lemma 2.3 Let $0 \leq j \leq n$ be fixed. Let $1 \leq i_1 < i_2 \leq m$ be two indices. Then
\[ \mathcal{F}(i_1, j) \cap \mathcal{F}(i_2, j) = \emptyset. \]

Proof. We can prove this statement by an indirect argument. Suppose that there exist two indices $1 \leq i_1 < i_2 \leq m$ such that $\mathcal{F}(i_1, j) \cap \mathcal{F}(i_2, j) \neq \emptyset$. Let $U \in \mathcal{F}(i_1, j) \cap \mathcal{F}(i_2, j)$ be an arbitrary, but fixed subspace. Then $U_{i_1} \subseteq U$ and $V_{i_1} \cap U = \{0\}$. Similarly $U_{i_2} \subseteq U$ and $V_{i_2} \cap U = \{0\}$. Hence we get that
\[ U_{i_1} \cap V_{i_2} = \{0\} \]
and
\[ U_{i_2} \cap V_{i_1} = \{0\}, \]
which gives a contradiction, because $S = \{(U_i, V_i) : 1 \leq i \leq m\}$ is a weak $(u, v)$-system of subspaces of the linear space $W$. \[ \square \]

In the following let $0 \leq j \leq n$ be a fixed integer. It follows from Lemma 2.3 that
\[ \sum_{i=1}^{m} |\mathcal{F}(i, j)| = |\bigcup_{i=1}^{m} \mathcal{F}(i, j)| \leq \left[ \begin{array}{c} n \\ j \end{array} \right]_q, \]
because $\mathcal{F}(i, j) \subseteq \{U \leq W : \text{dim}(U) = j\}$. Hence
\[ \sum_{i=1}^{m} \left[ \begin{array}{c} n-v_i \\ j \\ u_i \end{array} \right]_q \frac{\left[ \begin{array}{c} j \\ q \end{array} \right]_q \cdot \left[ \begin{array}{c} j-u_i \end{array} \right]_q}{\left[ \begin{array}{c} n-u_i \\ q \end{array} \right]_q} \leq \left[ \begin{array}{c} n \\ j \end{array} \right]_q \tag{2} \]
But it is easy to verify that
\[ \left[ \begin{array}{c} n-v_i \\ j \\ u_i \end{array} \right]_q \frac{\left[ \begin{array}{c} j \\ q \end{array} \right]_q \cdot \left[ \begin{array}{c} j-u_i \end{array} \right]_q}{\left[ \begin{array}{c} n-u_i \\ q \end{array} \right]_q} = \left[ \begin{array}{c} n-v_i-u_i \\ j-u_i \end{array} \right]_q, \]
hence it follows from inequality (2) that
\[ \sum_{i=1}^{m} \left[ \begin{array}{c} n-v_i-u_i \\ j-u_i \end{array} \right]_q \cdot j^{(j-u_i)} \leq \left[ \begin{array}{c} n \\ j \end{array} \right]_q, \]
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which was to be proved. □

Proof of Theorem 1.10: If $S = \{(U_i, V_i) : 1 \leq i \leq m\}$ is a weak $(u, v)$-system of subspaces of the linear space $W$, then $u_i = \text{dim}(U_i) = u$ and $v_i = \text{dim}(V_i) = v$ for each $1 \leq i \leq m$. It follows from Theorem 1.9 that

$$\sum_{i=1}^{m} \binom{n-u-v}{j-u} \frac{q^{(j-u)v}}{\binom{n}{j}} \leq 1$$

for each $1 \leq j \leq n$. Let $j := n - v$. This choice implies that

$$\sum_{i=1}^{m} \frac{q^{(n-v-u)v}}{\binom{n}{v}} \leq 1.$$ 

It follows from Lemma 2.1 that

$$\sum_{i=1}^{m} \frac{q^{(n-v-u)v}}{\left(\frac{q}{q-1}\right)^n q^{u(n-v)}} \leq 1.$$ 

But then

$$m \frac{q^{-uv}}{\left(\frac{q}{q-1}\right)^n} \leq 1,$$

which was to be proved. □

3 Concluding remarks

We can raise the following natural question: Let $u, v$ be fixed positive integers. Let $F$ be a fixed field. Determine the largest integer $t := t(u, v)$ such that there exists a weak $(u, v)$-system $S = \{(U_i, V_i) : 1 \leq i \leq t\}$ of $t(u, v)$ pairs of subspaces of an $n$-dimensional linear space $W$ over the field $F$.

If $F$ is the finite field $\mathbb{F}_q$, then we proved in Theorem 1.10 that

$$t(u, v) \leq \left(\frac{q}{q-1}\right)^n q^{uv}.$$
On the other hand, it is easy to verify the lower bound \( m(u, v) \leq t(u, v) \). Namely let \( \{e_1, \ldots, e_n\} \) denote a fixed basis of the \( n \)-dimensional linear space \( W \) over \( \mathbb{F} \). By the definition of the number \( m(u, v) \) there exists a system \( S = \{(A_i, B_i) : 1 \leq i \leq m(u, v)\} \) of \( m(u, v) \) pairs of sets satisfying the conditions:

(i) \( A_1, \ldots A_m \) are \( u \)-element sets and \( B_1, \ldots B_m \) are \( v \)-element sets;

(ii) \( A_i \cap B_i = \emptyset \) for each \( 1 \leq i \leq m \);

(iii) \( A_i \cap B_j \neq \emptyset \) or \( A_j \cap B_i \neq \emptyset \) for \( i \neq j \) \( (1 \leq i, j \leq m) \).

Define the generated subspaces \( U_i := \langle \{e_k : k \in A_i\} \rangle \) and \( V_i := \langle \{e_l : l \in B_i\} \rangle \) for each \( 1 \leq i \leq m(u, v) \).

Then it is easy to verify that the system \( S = \{(U_i, V_i) : 1 \leq i \leq m(u, v)\} \) of \( m(u, v) \) pairs of subspaces is a weak \((u, v)\)-system.

References

[1] B. Bollobás, On generalized graphs. *Acta Math. Hung.* **16**(3), (1965) 447-452.

[2] L. Babai and P. Frankl, *Linear algebra methods in combinatorics*, September 1992.

[3] W. Koepf, *Hypergeometric Summation: An Algorithmic Approach to Summation and Special Function Identities*, Vieweg, 1998.

[4] L. Lovász, Flats in matroids and geometric graphs, in: *Combinatorial surveys*, Proc. 6th British Comb. Conf., Egham 1977, Acad. Press, London 1977, 45–86.

[5] L. Lovász, Topological and algebraic methods in graph theory, in *Graph theory and related topics*, Proc. Conf., Univ. Waterloo, Waterloo, Ont., 1979, 1–14.

[6] Z. Tuza, Application of Set-Pair Method in Extremal Hypergraph Theory, in “Extremal problems for Finite Sets”, *Bolyai Soc. Math. Studies* **3**, János Bolyai Math. Soc., Budapest, 1994, 479–514.
[7] Z. Tuza, Inequalities for two set systems with prescribed intersections. 
*Graphs and Comb.*, **3**(1),(1987) 75-80.

[8] K. Wang and Z. Li, Lattices associated with vector spaces over a finite field, *Lin. Algebra and Its Applications* **429**(2), (2008), 439-446.