Ill-posedness for the Burgers equation in Sobolev spaces

Jinlu Li\textsuperscript{1}, Yanghai Yu\textsuperscript{2,*} and Weipeng Zhu\textsuperscript{3}

\textsuperscript{1} School of Mathematics and Computer Sciences, Gannan Normal University, Ganzhou 341000, China
\textsuperscript{2} School of Mathematics and Statistics, Anhui Normal University, Wuhu 241002, China
\textsuperscript{3} School of Mathematics and Big Data, Foshan University, Foshan, Guangdong 528000, China

March 23, 2021

\textbf{Abstract:} In this paper, we considered the Cauchy problem for the Burgers equation and proved that the problem is ill-posed in Sobolev spaces $H^s$ with $s \in [1, \frac{3}{2})$.

\textbf{Keywords:} Burgers equation, Ill-posedness.

\textbf{MSC (2010):} 35Q35, 35B30.

\section{Introduction}

\subsection{The Concept of Well-posedness}

A Cauchy problem

$$\partial_t f = F(f), \quad f(0, x) = f_0(x)$$

is said to be Hadamard well-posed in a Banach space $X$ if for any data $f_0 \in X$ there exists $T > 0$ and a unique solution in the space $C([0, T), X)$ which depends continuously on the data. In particular, solutions describe continuous curves in $X$ at least for a short time. The problem is said to be ill-posed in $X$ if it is not well-posed in the above sense. Based on the definition of well-posedness, there are at least three types of ill-posedness were studied in the literature: nonexistence, non-uniqueness, and discontinuous dependence on the data. In this paper we are interested in discontinuity with respect to the data.

*E-mail: lijinlu@gnnu.edu.cn; yuyanghai214@sina.com (Corresponding author); mathzwp2010@163.com
1.2 The Burgers equation

The Burgers equation with fractional dissipation is written as

\[
\begin{aligned}
\partial_t u + uu_x + \Lambda^\gamma u &= 0, \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}, \\
u(0, x) &= u_0(x), \quad x \in \mathbb{R},
\end{aligned}
\]  \tag{1.1}

where \( \gamma \in [0, 2] \) and the fractional power operator \( \Lambda^\gamma \) is defined by Fourier multiplier with the symbol \(|\xi|^\gamma\)

\[\Lambda^\gamma u(x) = \mathcal{F}^{-1}(|\xi|^\gamma \mathcal{F} u(\xi)).\]

The Burgers equation (1.1) with \( \gamma = 0 \) and \( \gamma = 2 \) has received an extensive amount of attention since the studies by Burgers in the 1940s. If \( \gamma = 0 \), the equation is perhaps the most basic example of a PDE evolution leading to shocks. If \( \gamma = 2 \), it provides an accessible model for studying the interaction between nonlinear and dissipative phenomena. Kiselev et al. [7] gave a complete study for general \( \gamma \in [0, 2] \) for the periodic case. In particular, for the case \( \gamma = 1 \), they proved the global well-posedness of the equation in the critical Hilbert space \( H^{\frac{3}{2}}(T) \) by using the method of modulus of continuity. Subsequently, Miao-Wu [11] proved the global well-posedness of the critical Burgers equation in critical Besov spaces \( B_{p,1}^{1/p}(\mathbb{R}) \) with \( p \in [1, \infty) \) with the help of Fourier localization technique and the method of modulus of continuity. For more results on the fractional Burgers equation and dispersive perturbations of Burgers equations, we refer the readers to see [1, 4, 8–10] and the references therein. We should mention that Molinet et al. [10] proved that the Cauchy problem for a class of dispersive perturbations of Burgers equations is locally well-posed in \( H^s(\mathbb{R}) \).

In this paper, we focus on the well-posedness problem of the following Burgers equation.

\[
\begin{aligned}
\partial_t u + uu_x &= 0, \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}, \\
u(0, x) &= u_0(x), \quad x \in \mathbb{R},
\end{aligned}
\]  \tag{1.2}

Roughly speaking, (1.2) can be viewed as the simplest in the family of partial differential equations modeling the Euler and Navier-Stokes equation nonlinearity. The local well-posedness of the Burgers equation (1.2) for data in \( H^s(\mathbb{R}) \) with any \( s > 3/2 \) can be proved by combining the Sobolev embedding \( H^{s-1}(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R}) \) and the classical energy estimate

\[\|u\|_{H^s} \leq \|u_0\|_{H^s} \exp \left(C \int_0^t \|u_x\|_{L^\infty} d\tau \right).\]

Also, in [12] it is obtained that the solution map is continuous dependence while not uniformly continuous dependence on initial data for the Burgers equation (1.2) in the same space \( H^s(\mathbb{R}) \) with \( s > 3/2 \). For the endpoint case, Linares et al. in [9] proved that the Cauchy problem for (1.2) is ill-posed in \( H^{3/2}(\mathbb{R}) \), where the key point is that the available local well-posedness theory in \( H^s(\mathbb{R}) \) with any \( s > 3/2 \) have been used, for more details see Remark 1.6. Using the idea developed in [9], Guo et al. in [5] to prove the ill-posedness for the Camassa-Holm equation in the critical Sobolev
space $H^{3/2}(\mathbb{R})$ and even in the Besov space $B^{1+1/p}_{p,r}(\mathbb{R})$ with $r > 1$. Precisely speaking, their main idea is to construct a blow-up smooth solution $u(t) \in \mathcal{C}([0, T^*), B^{1+1/p}_{p,r}) \cap \mathcal{C}([0, T^*), H^2)$ such that

$$
\|u_0\|_{B^{1+1/p}_{p,r}} \leq \varepsilon \quad \text{and} \quad \lim_{t \uparrow T^*} \|u(t)\|_{Lip} = \infty, \quad T^* < \varepsilon.
$$

then using the blow-up result and the following inequality

$$
\|u(t)\|_{Lip} \leq C\|u(t)\|_{H^2} \leq C(1 + \|u_0\|_{H^2}) \exp \exp \left( C \int_0^t \|u_x\|_{B^{1}_{\infty,\infty}} d\tau \right).
$$

they deduce that $\lim_{t \uparrow T^*} \|u(t)\|_{B^{1}_{\infty,\infty}} = \infty$ which in turn implies $\lim_{t \uparrow T^*} \|u(t)\|_{B^{1+1/p}_{p,r}} = \infty$.

It remains nevertheless an interesting issue to prove that the Cauchy problem for the Burgers equation (1.2) in the Sobolev space $H^s(\mathbb{R})$ with $s < \frac{3}{2}$ is ill-posed. However, due to the absence of the embedding $H^s(\mathbb{R}) \hookrightarrow B^{1}_{\infty,\infty}(\mathbb{R})$ for $s < 3/2$, the method in [5, 9] is invalid for $s < 3/2$. In this paper, we shall develop a new method to study this problem and give a partial answer.

1.3 Main Result

Now let us state our main ill-posedness result of this paper.

**Theorem 1.1** Let $1 \leq s < \frac{3}{2}$. For any $\delta > 0$, there exists initial data satisfying

$$
\|u_0\|_{H^s} \leq \delta,
$$

such that a solution $u(t) \in \mathcal{C}([0, T_0]; H^s)$ of the Cauchy problem (1.2) satisfies

$$
\|u(T_0)\|_{H^s} \geq \frac{1}{\delta} \quad \text{for some} \quad 0 < T_0 < \delta.
$$

**Remark 1.1** Theorem 1.1 indicates that the solutions of (1.2) with arbitrarily short time which initially have an arbitrarily small $H^s$-norm that grows arbitrarily large. This result shows the ill-posedness of (1.2) in $H^s(\mathbb{R})$ with $1 \leq s < \frac{3}{2}$ in the sense that the solution map $u_0 \in H^s \mapsto u \in H^s$ is discontinuous with respect to the initial data.

**Strategies to Proof.** We shall outline the main ideas in the proof of Theorem 1.1.

- Firstly, we construct an explicit example for initial data $u_0$, where the norm $\|u_0\|_{H^s}$ is sufficiently small while $\|u'_0\|_{L^\infty}$ can be large enough.

- Secondly, we express the solution to the Burgers equation (1.2) by exploring fully the properties of the flow map and give the explicit blow-up time $T^*$.

- Lastly, we mainly observe that the transport term does cause growth of the $L^2$-norm of $u_x$ as $t$ tends to $T^*$. Precisely speaking, we estimate the $L^2$-norm of $u_x$ over $(-\psi(t, q_0), \psi(t, q_0))$ and obtain that its lower bound can be arbitrarily large as $t$ tends to $T^*$.
The structure of the paper. In Section 2 we provide several key Lemmas. In Section 3 we present the proof of Theorem 1.1.

Let us complete this section with some notations we shall use throughout this paper.

Notations. The notation $A \leq a \land b$ means that $A \leq a$ and $A \leq b$. $a \approx b$ means $C^{-1} b \leq a \leq C b$ for some positive harmless constants $C$. Given a Banach space $X$, we denote its norm by $\| \cdot \|_X$. For $I \subset \mathbb{R}$, we denote by $C(I; X)$ the set of continuous functions on $I$ with values in $X$. For all $f \in \mathcal{S}'$, the Fourier transform $\hat{f}$ is defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}} e^{-ix\xi} f(x) dx \quad \text{for any } \xi \in \mathbb{R}.$$ 

For $s \in \mathbb{R}$, the nonhomogeneous Sobolev space $H^s(\mathbb{R})$ is defined by its usual norm

$$\|f\|_{H^s}^2 = \int_{\mathbb{R}} (1 + \xi^2)^s |\hat{f}(\xi)|^2 d\xi.$$ 

2 Preliminary

In the section, we make some preparations for the proof of the main theorem.

2.1 Key Example for Initial Data

Firstly, we construct an explicit example as follows. Set

$$u_0(x) := p_0 \left( e^{-|x+q_0|} - e^{-|x-q_0|} \right),$$

where two positive numbers $p_0$ and $q_0 \in (0, 1)$ will be fixed later.

It is easy to check that $u_0(x)$ is an odd function. Furthermore, we can deduce that the following result holds:

**Lemma 2.1** For every $q_0 \in (0, 1)$ and $s \in \left( \frac{1}{2}, \frac{3}{2} \right)$, there exists $C = C_s > 0$ such that

$$C^{-1} p_0 q_0^{3/2-s} \leq \|u_0\|_{H^s} \leq C p_0 q_0^{3/2-s}. \quad (2.3)$$

**Proof.** The proof essentially follows that of Lemma 3.1 in [2] or Proposition 1 in [6]. For the sake of readability, we sketch the proof here. Defining the function

$$f(x) := e^{-|x+q_0|} - e^{-|x-q_0|},$$

then using the fact $\hat{e^{-|x|}}(\xi) = 2/(1 + \xi^2)$, we have

$$\hat{f}(\xi) = \frac{2(e^{i q_0 \xi} - e^{-i q_0 \xi})}{1 + \xi^2} = \frac{4i \sin(q_0 \xi)}{1 + \xi^2}.$$
Using the definition of the $H^s$-norm and the change of variable setup $y = q_0 \xi$, we have

$$
\|f\|^2_{H^s(\mathbb{R})} = 16 \int_{\mathbb{R}} (1 + \xi^2)^{s-2} \sin^2(q_0 \xi) d\xi \\
\geq 32 \left(1 + \frac{\pi^2}{q_0^2}\right)^{s-2} \int_{0}^{\infty} \sin^2(q_0 \xi) d\xi \\
= 32 \left(1 + \frac{\pi^2}{q_0^2}\right)^{s-2} \cdot \frac{1}{q_0} \int_{0}^{\infty} \sin^2 y dy \\
\geq 16\pi \left(1 + \frac{\pi^2}{q_0^2}\right)^{-3/2} q_0^{3-2s},$$

where we have used $s \in \left(\frac{1}{2}, \frac{3}{2}\right)$ and $q_0 \in (0, 1)$. This proves the lower bound.

To get the upper bound, we split the domain of integration as

$$
\|f\|^2_{H^s(\mathbb{R})} = 32 \int_{0}^{\infty} (1 + \xi^2)^{s-2} \sin^2(q_0 \xi) d\xi \\
= 32 \left(\int_{0}^{1/q_0} + \int_{1/q_0}^{\infty}\right) (1 + \xi^2)^{s-2} \sin^2(q_0 \xi) d\xi \\
=: 32(I_1 + I_2).
$$

Due to the simple fact $\sin^2(q_0 \xi) \leq |q_0 \xi|^2 \wedge 1$, we have

$$
I_1 \leq 32 q_0^2 \int_{0}^{1/q_0} \xi^{2s-2} d\xi \leq \left(\frac{32}{2s-1}\right) q_0^{3-2s}, \\
I_2 \leq 32 \int_{1/q_0}^{\infty} \xi^{2s-4} d\xi \leq \left(\frac{32}{3-2s}\right) q_0^{3-2s},
$$

which completes the proof of Lemma 2.1.

### 2.2 Existence and Blow-up criterion

**Lemma 2.2** For every $s \in [1, \frac{3}{2})$, there exists a solution $u \in C([0, T^*); H^s) \cap L^\infty([0, T^*); \text{Lip})$ for the Burgers equation (1.2), where $T^* < \infty$ is the maximal time for initial data $u_0$.

Furthermore, we have

$$
\lim_{t \uparrow T^*} \left(\|u(t)\|_{H^s} + \|u(t)\|_{\text{Lip}}\right) = +\infty \quad \Leftrightarrow \quad \lim_{t \uparrow T^*} \|u(t)\|_{\text{Lip}} = +\infty.
$$

**Proof.** Easy computations give that

$$
u_0'(x) = \begin{cases} 
-p_0(e^{-q_0} - e^{q_0})e^x, & \text{if } x \in (-\infty, -q_0), \\
-p_0(e^{-q_0}(e^x + e^{-x}), & \text{if } x \in (-q_0, q_0), \\
-p_0(e^{-q_0} - e^{q_0})e^{-x}, & \text{if } x \in (q_0, +\infty),
\end{cases}
$$

from which and Lemma 2.1, we get that $u_0 \in H^s \cap \text{Lip}$. 

5
Following the proof of Lemma 2.4 in [3], we can get

\[ \|u(t)\|_{H^s} \leq \|u_0\|_{H^s} \exp(C \int_0^t \|u(\tau)\|_{\text{Lip}} d\tau) \]

and

\[ \|u(t)\|_{\text{Lip}} \leq \|u_0\|_{\text{Lip}} \exp(C \int_0^t \|u(\tau)\|_{\text{Lip}} d\tau). \]

This is enough to complete the proof of Lemma 2.2.

### 2.3 The Equation Along the Flow

Given a Lipschitz velocity field \( u \), we may solve the following ODE to find the flow induced by \( u \):

\[
\begin{cases}
\frac{d}{dt} \psi(t, x) = u(t, \psi(t, x)), \\
\psi(0, x) = x,
\end{cases}
\tag{2.5}
\]

which is equivalent to the integral form

\[ \psi(t, x) = x + \int_0^t u(\tau, \psi(\tau, x)) d\tau. \]

Furthermore, we get from (1.2) that

\[ \frac{d}{dt} u(t, \psi(t, x)) = u_t(t, \psi(t, x)) + u_x(t, \psi(t, x)) \frac{d}{dt} \psi(t, x) = 0, \]

which means that

\[ u(t, \psi(t, x)) = u_0(x), \quad \text{namely,} \quad u(t, x) = u_0(\psi^{-1}(t, x)). \tag{2.6} \]

Thus we can give the explicit expression of the flow as:

\[ \psi(t, x) = x + tu_0(x). \tag{2.7} \]

Let \( y = \psi(t, x) \), then we have

\[ \psi^{-1}(t, y) = y - tu_0(\psi^{-1}(t, y)) = y - tu(t, y). \tag{2.8} \]

Differentiating (1.2) with respect to space variable \( x \), we find

\[ u_{tx} + uu_{xx} + (u_x)^2 = 0. \]

Combining the above and (2.5), we obtain

\[
\frac{d}{dt} u_x(t, \psi(t, x)) = u_{tx}(t, \psi(t, x)) + u_{xx}(t, \psi(t, x)) \frac{d}{dt} \psi(t, x),
\]

\[ = u_{tx}(t, \psi(t, x)) + u_{xx}(t, \psi(t, x)) u(t, \psi(t, x))
\]

\[ = -(u_x)^2(t, \psi(t, x)), \]
which reduces to

$$u_x(t, \psi(t, x)) = \frac{1}{t + u_0(x)}.$$ \hspace{1cm} (2.9)

We should mention that the above can also be deduced from (2.6) and (2.7).

According to the definition of $u_0$, we can deduce that

$$T^* = -\frac{1}{u'_0(q_0)} \in \left(\frac{1}{2p_0}, \frac{1}{p_0}\right).$$

Because the velocity field is Lipschitz, then we get that for $t \in [0, T^*)$

$$\psi_x(t, x) = \exp \left(\int_0^t u_x(\tau, \psi(\tau, x))d\tau\right) > 0.$$ This shows that $\psi(t, \cdot)$ is an increasing diffeomorphism over $\mathbb{R}$, that is, for all $x, y \in \mathbb{R}$, there holds that $\psi(t, x) < \psi(t, y)$ if $x < y$.

3 Proof of Main Theorem

In this section, we prove Theorem 1.1.

Proof of Theorem 1.1 By the definition of $u'_0(x)$ and (2.9), we know that $u_x(t, \psi(t, x))$ is continuous in $[0, T^*) \times (-q_0, q_0)$. We should emphasize that $u_x(t, x)$ is discontinuous in $[0, T^*) \times \mathbb{R}$, but we can claim that $u_x(t, x)$ is continuous in $[0, T^*) \times (-\psi(t, q_0), \psi(t, q_0))$. In fact, for any $x, y \in (-\psi(t, q_0), \psi(t, q_0))$, we have $\psi^{-1}(t, x), \psi^{-1}(t, y) \in (-q_0, q_0)$. Moreover, we deduce from (2.8) that for $t \in [0, T^*)$

$$|\psi^{-1}(t, x) - \psi^{-1}(t, y)| \leq |\partial_x \psi^{-1}(t, x)||x - y|$$

$$\leq |x - y|(1 + T^*\|u_x\|_{L^\infty(L^\infty)}).$$ \hspace{1cm} (3.10)

Also, it follows from (2.8) and (1.2) that

$$|\psi^{-1}(t, x) - \psi^{-1}(s, x)| = |tu(t, x) - su(s, x)|$$

$$\leq \|u_0\|_{L^\infty}|t - s| + T^*\int_s^t \|\partial_t u(\tau, \cdot)\|_{L^\infty}d\tau$$

$$\leq C(1 + T^*\|u_x\|_{L^\infty(L^\infty)})|t - s|.$$ \hspace{1cm} (3.11)

Thus, we obtain from (2.4) and (3.10)-(3.11) for $s, t \in [0, T^*)$ and $x, y \in (-\psi(t, q_0), \psi(t, q_0))$

$$|u_x(t, x) - u_x(s, y)| \leq |u_x(t, x) - u_x(t, y)| + |u_x(t, y) - u_x(s, y)|$$

$$\leq |u_x(t, \psi(t, \psi^{-1}(t, x))) - u_x(t, \psi(t, \psi^{-1}(t, y))))|$$

$$+ |u_x(t, \psi(t, \psi^{-1}(t, y))) - u_x(s, \psi(s, \psi^{-1}(s, y))))|$$

$$\rightarrow 0 \text{ as } (t, x) \rightarrow (s, y).$$
By the Burgers equation $\partial_t u = -uu_x$, we can deduce that $\partial_t u(t, x)$ is continuous in $[0, T^*) \times (-\psi(t, q_0), \psi(t, q_0))$. That is $u(t, x) \in C^1([0, T^*) \times (-\psi(t, q_0), \psi(t, q_0)))$. Furthermore, one has

$$u_{xx}(t, \psi(t, x)) = \frac{u''_0(x)}{(1 + tu'_0(x))^3}.$$ 

The similar argument shows that $u_x(t, x) \in C^1([0, T^*) \times (-\psi(t, q_0), \psi(t, q_0)))$.

For notational convenience we now set

$$\tilde{m}(t) := u_x(t, \psi(t, 0)) = \frac{1}{t + u'_0(0)}.$$

Therefore, we have

$$u_x(t, \psi(t, x)) \leq \tilde{m}(t) \quad \text{for all} \quad x \in (-q_0, q_0). \quad (3.12)$$

Set $w(t, x) := u_x(t, x)$, then we obtain from (1.2)

$$\partial_t w + \partial_x(uw) = 0,$$

which implies that for $(t, x) \in [0, T^*) \times (-\psi(t, q_0), \psi(t, q_0))$

$$\partial_t (w^2) + \partial_x(uw^2) + \partial_x uw^2 = 0. \quad (3.13)$$

Integrating (3.13) with respect to space variable $x$ over $[-\psi(t, q_0), \psi(t, q_0)]$, we have

$$\int_{|x| \leq \psi(t, q_0)} \partial_t (w^2) dx + \int_{|x| \leq \psi(t, q_0)} \partial_x (uw^2) dx + \int_{|x| \leq \psi(t, q_0)} \partial_x uw^2 dx = 0. \quad (3.14)$$

As $u_0(x)$ is odd, the solution of Burgers equation satisfies $u(t, x) = -u(t, -x)$, which tells us that $w(t, x) = w(t, -x)$. Thus we have

$$\int_{|x| \leq \psi(t, q_0)} \partial_t (w^2) dx = \frac{d}{dt} \int_{|x| \leq \psi(t, q_0)} w^2 dx - 2u(t, \psi(t, q_0)) w^2(t, \psi^-(t, q_0)), \quad (3.15)$$

and

$$\int_{|x| \leq \psi(t, q_0)} \partial_x (uw^2) dx = 2u(t, \psi(t, q_0)) w^2(t, \psi^-(t, q_0)). \quad (3.16)$$

Inserting (3.15) and (3.16) into (3.14) yields

$$\frac{d}{dt} \int_{|x| \leq \psi(t, q_0)} w^2 dx + \int_{|x| \leq \psi(t, q_0)} \partial_x u(t, x) w^2 dx = 0. \quad (3.17)$$

To simplify notation let

$$A(t) := \int_{|x| \leq \psi(t, q_0)} w^2(t, x) dx \quad \text{for} \quad t \in [0, T^*),$$

\[8\]
combining (3.12), then (3.17) reduces to
\[ A'(t) = \int_{|x| \leq \psi(t,q_0)} -u_x(t,x)w^2 \, dx \geq -\tilde{m}(t)A(t). \]
Solving the above differential inequality gives us that
\[ A(t) \geq A_0 \exp \left( \int_0^t -\tilde{m}(\tau) \, d\tau \right) = A_0 \cdot \frac{\tilde{m}(t)}{\tilde{m}(0)}. \]
which implies
\[ \|w\|_{L^2} \geq A_0^{\frac{1}{2}} \cdot \sqrt{\frac{\tilde{m}(t)}{u_0'(0)}}. \] (3.18)
Notice that
\[ A_0 = \int_{|x| \leq q_0} \left( u'_0(x) \right)^2 \, dx \approx p_0^2 q_0, \]
and
\[ \lim_{t \uparrow T^*} \tilde{m}(t) = \frac{1}{\frac{1}{u_0'(q_0^-)} + \frac{1}{u_0'(0)}} = \frac{u_0'(q_0^-)u_0'(0)}{u_0'(q_0^-) - u_0'(0)}, \]
combining the above and (3.18) yields
\[ \lim_{t \uparrow T^*} \|u(t)\|_{H^1} \geq \lim_{t \uparrow T^*} \|w\|_{L^2} \]
\[ \geq C p_0 \sqrt{q_0} \sqrt{\frac{u_0'(q_0^-)}{u_0'(q_0^-) - u_0'(0)}} \]
\[ \geq C \frac{p_0 \sqrt{q_0}}{1 - e^{-q_0}} \]
\[ \approx C p_0 q_0^{-\frac{1}{2}}, \]
where we have used that
\[ u_0'(0) = -2p_0 e^{-q_0} \quad \text{and} \quad u_0'(q_0^-) = -p_0 (e^{-2q_0} + 1) \]
and in the last step used
\[ \frac{q_0}{2} \leq 1 - e^{-q_0} \leq q_0 \quad \text{for} \quad q_0 \in (0,1). \]
By Lemma 2.1, one has
\[ \|u_0\|_{H^s} \leq c_1 p_0 q_0^{\frac{s}{2} - s} \leq \delta \quad \text{and} \quad T^* \leq \frac{1}{p_0} \leq \delta, \]
but
\[ \lim_{t \uparrow T^*} \|u(t)\|_{H^1} \geq c_2 p_0 q_0^{-\frac{1}{2}} \geq \frac{1}{\delta^2}. \]
if some large $p_0$ and small $q_0$ is chosen. In fact, we can take $p_0$ sufficiently large to make $p_0 \geq \frac{1}{8}$ and $q_0$ sufficiently small such that $q_0 \leq \left( \frac{\delta}{(c_1 p_0)} \right)^{2/(3-2s)} \land c_2 p_0^2 \delta^4$.

Hence, we can choose $T_0 \in [0, T^*)$ such that

\[
\|u(T_0)\|_{H^s} \geq C \|u(T_0)\|_{H^1} \geq \frac{1}{\delta}.
\]

This completes the proof of Theorem 1.1.

**Acknowledgments**

J. Li is supported by the National Natural Science Foundation of China (Grant No.11801090). Y. Yu is supported by the Natural Science Foundation of Anhui Province (No.1908085QA05). W. Zhu is partially supported by the National Natural Science Foundation of China (Grant No.11901092) and Natural Science Foundation of Guangdong Province (No.2017A030310634).

**References**

[1] N. Alibaud, J. Droniou, Occurrence and non-apperance of shocks in fractal Burgers equations, J. Hyperbolic Differ. Equ. 4(3), 479-499 (2007).

[2] P. Byers, Existence time for the Camassa-Holm equation and the critical Sobolev index, Indiana Univ. Math. J. 55, 941-954 (2006).

[3] R. Danchin, A few remarks on the Camassa-Holm equation. Differential Integral Equations, 14, 953-988 (2001).

[4] H. Dong, D. Du, D. Li, Finite time singularities and global well-posedness for fractal Burgers equation, Indiana Univ. Math. J. 58(2009).

[5] Z. Guo, X. Liu, L. Molinet, Z. Yin, Ill-posedness of the Camassa-Holm and related equations in the critical space, J. Differential Equations, 266, 1698–1707 (2019).

[6] A. Himonas, C. Holliman, K. Grayshan, Norm inflation and Ill-Posedness for the Degasperis-Procesi Equation, Comm. Partial Differ. Equ. 39, 2198-2215 (2014).

[7] A. Kiselev, F. Nazarov, R. Shterenberg, Blow up and regularity for fractal Burgers equation, Dyn. Partial Differ. Equ. 5, 211-240 (2008).

[8] G. Karch, C. Miao, X. Xu, On convergence of solutions of fractal Burgers equation to ward rarefaction waves, SIAM J. Math. Anal. 39, 1536-1549 (2007).

[9] F. Linares, D. Pilod, J.-C. Saut, Dispersive perturbations of Burgers and hyperbolic equations I: local theory, 46(2), 1505-1537 (2014).

[10] L. Molinet, D. Pilod, S. Vento, On well-posedness for some dispersive perturbations of Burgers’equation, Ann. I. H. Poincaré-AN, 35, 1719-1756 (2018).
[11] C. Miao, G. Wu, Global well-posedness of the critical Burgers equation in critical Besov spaces, J. Differential Equations, 247, 1673-1693 (2009).

[12] N. Tzvetkov, Ill-posedness issues for nonlinear dispersive equations, arXiv, 2007.