Degrees of polarization for a quantum field

L L Sánchez-Soto\textsuperscript{1}, J Söderholm\textsuperscript{2}, E C Yustas\textsuperscript{1}, A B Klimov\textsuperscript{3} and G Björk\textsuperscript{4}
\textsuperscript{1} Departamento de Óptica, Facultad de Física, Universidad Complutense, 28040 Madrid, Spain
\textsuperscript{2} Institute of Quantum Science, Nihon University, 1-8 Kanda-Surugadai, Chiyoda-ku, Tokyo 101-8308, Japan
\textsuperscript{3} Departamento de Física, Universidad de Guadalajara, 44420 Guadalajara, Jalisco, Mexico
\textsuperscript{4} School of Information and Communication Technology, Royal Institute of Technology (KTH), Electrum 229, SE-164 40 Kista, Sweden

Abstract. Unpolarized light is invariant with respect to any SU(2) polarization transformation. Since this fully characterizes the set of density matrices representing unpolarized states, we introduce the degree of polarization of a quantum state as its distance to the set of unpolarized states. We discuss different candidates of distance, and show that they induce fundamentally different degrees of polarization.

1. Introduction
The polarization properties of light are essential both in the classical and quantum domain. The polarization state is a robust characteristic, which is relatively simple to manipulate without inducing more than marginal losses. It is thus hardly surprising that this variable has been crucial to experimentally demonstrate fundamental quantum properties and applications, such as Bell tests, complementarity, quantum cryptography and tomography, or teleportation [1].

In classical optics, the polarization of a light beam is determined by the four Stokes parameters [2], and can be elegantly visualized by resorting to the Poincaré sphere. The Stokes parameters present unique advantages: they are easily measured, they can be extended to the quantum domain (where they become the mean values of the Stokes operators) and they allow us to classify the states of light according to a degree of polarization [3].

This classical degree of polarization is simply the modulus of the Stokes vector. While this affords a very intuitive image, it has also serious drawbacks in the quantum domain, which can be traced back to the fact that the Stokes parameters are proportional to the second-order correlations of the field amplitudes. For this reason, the Stokes parameters do not distinguish between quantum states having remarkably different polarization properties [4]. These unwanted consequences call for alternative measures.

Recently, we have addressed this problem from a different point of view [5]. Our starting point is the fact that unpolarized light can be considered to be described by quantum states that are invariant with respect to any SU(2) polarization transformation [6]. This requirement fixes the set of density operators admissible to represent unpolarized fields. It is then suggestive to look at the degree of polarization as a distance from a given state to this set of unpolarized states. In this contribution, we explore a suitable definition that avoids at least some of the aforementioned difficulties that previous approaches based on Stokes parameters encounter.
2. Polarization and the group SU(2)

We consider a monochromatic plane wave propagating in the $z$ direction, whose electric field lies in the $xy$ plane. Denoting the annihilation operators of the horizontally and vertically polarized modes as $a_H$ and $a_V$, respectively, the Stokes operators can be expressed as

\[
\hat{S}_0 = \hat{a}_H^\dagger \hat{a}_H + \hat{a}_V^\dagger \hat{a}_V, \quad \hat{S}_1 = \hat{a}_H^\dagger \hat{a}_V + \hat{a}_V^\dagger \hat{a}_H, \quad \hat{S}_2 = i(\hat{a}_H^\dagger \hat{a}_H - \hat{a}_V^\dagger \hat{a}_V), \quad \hat{S}_3 = \hat{a}_H^\dagger \hat{a}_H - \hat{a}_V^\dagger \hat{a}_V, \quad (1)
\]

and the Stokes parameters are given by the corresponding average values $\langle \hat{S}_k \rangle$. Using the bosonic commutation relations

\[
[\hat{a}_j, \hat{a}_k^\dagger] = \delta_{jk}, \quad j, k \in \{H, V\}, (2)
\]

one finds that the operators $(\hat{S}_1, \hat{S}_2, \hat{S}_3)$ all commute with $\hat{S}_0$, and satisfy

\[
[\hat{S}_1, \hat{S}_2] = 2i\hat{S}_3 \quad (3)
\]

and its cyclic permutations. The Stokes operators $(\hat{S}_1, \hat{S}_2, \hat{S}_3)$ thus form an su(2) algebra, and it is clear from their noncommutability that simultaneous definite values of all the Stokes parameters can only be obtained for the two-mode vacuum state.

The transformations generated by $(\hat{S}_1, \hat{S}_2, \hat{S}_3)$ form the group SU(2). As $\hat{S}_2$ is the infinitesimal generator of geometrical rotations around the direction of propagation, $\hat{S}_3$ is the infinitesimal generator of differential phase shifts between the two modes, and $\hat{S}_1$ is related to these operators according to (3), it follows that any SU(2) transformation can be realized with linear optics [7].

This can be accomplished by an appropriate combination of phase plates and rotators. The set of transformations realized by linear optics actually form the group U(2), which also includes transformations generated by $\hat{S}_0$, i.e., a common phase shift in the two modes. However, it is clear that these latter transformations do not change the polarization state, and the unpolarized states have therefore been defined as those invariant under any SU(2) transformation [6].

In analogy with the classical definition, one can define a semiclassical degree of polarization by [8]

\[
P_{sc} = \sqrt{\langle \hat{S}_1 \rangle^2 + \langle \hat{S}_2 \rangle^2 + \langle \hat{S}_3 \rangle^2} / \langle \hat{S}_0 \rangle. \quad (4)
\]

The set of unpolarized states is then different from the one above, and characterized by $\langle \hat{S}_1 \rangle = \langle \hat{S}_2 \rangle = \langle \hat{S}_3 \rangle = 0$, i.e., the origin of the Poincaré sphere [9]. However, this leads to several peculiarities, e.g., quantum states with “hidden” polarization [10]. That is, one arrives at the paradoxical conclusion that unpolarized light has a polarization structure, which is latent when the average intensities are measured and detectable when the noise intensities are measured [11].

If one instead uses the definition of unpolarized light as the states that remain invariant under any SU(2) transformation, there is no “hidden” polarization structure. Any state satisfying this invariance condition will also fulfill the classical definition of an unpolarized state, but the converse is not true. It has been shown [6] that density operators satisfying the invariance condition must be of the form

\[
\hat{\sigma} = \bigoplus_{N=0}^{\infty} \lambda_N \mathbb{1}_N, \quad (5)
\]

where $\mathbb{1}_N$ is the identity operator in the subspace with exactly $N$ photons. The coefficients $\lambda_N$ are real and nonnegative, and the unit-trace condition of density operators implies

\[
\sum_{N=0}^{\infty} (N+1)\lambda_N = 1. \quad (6)
\]
3. Quantum degree of polarization as a distance

Measures of nonclassicality have been defined as the minimum distance to an appropriate set representing classical states [12]. Similarly, the minimum distance to the (convex) set of separable states has been used to introduce measures of entanglement [13]. In the same vein, we propose to quantify the degree of polarization as

$$P(\hat{\rho}) \propto \inf_{\hat{\sigma} \in U} D(\hat{\rho}, \hat{\sigma}),$$

where $U$ denotes the set of unpolarized states of the form (5) and $D(\hat{\rho}, \hat{\sigma})$ is any measure of distance between the density matrices $\hat{\rho}$ and $\hat{\sigma}$, such that $P(\hat{\rho})$ satisfies some requirements motivated by both physical and mathematical concerns. The constant of proportionality in (7) must be chosen such that $P$ is normalized to unity, i.e., $\sup_{\hat{\rho}} P(\hat{\rho}) = 1$. In Ref. [14] a check list of six simple, physically-motivated criteria that should be satisfied by any good measure of distance between quantum processes can be found. A discussion in terms of polarization is presented in Ref. [5].

Among the many possible choices of $D(\hat{\rho}, \hat{\sigma})$, let us start with the Hilbert-Schmidt distance

$$D_{\text{HS}}(\hat{\rho}, \hat{\sigma}) = \text{Tr} \left[ (\hat{\rho} - \hat{\sigma})^2 \right],$$

which has been previously studied in the context of entanglement [15]. In order to determine the corresponding degree of polarization for a given state $\hat{\rho}$, we should thus look for the unpolarized state $\hat{\sigma}$ that minimizes the distance

$$D_{\text{HS}}(\hat{\rho}, \hat{\sigma}) = \text{Tr} \left( \hat{\rho}^2 \right) + \sum_{N=0}^{\infty} \left[ (N + 1)\lambda_N^2 - 2p_N \lambda_N \right].$$

Here, $p_N$ is the probability distribution of the total number of photons given by

$$p_N = \sum_{k=0}^{N} \langle N, k | \hat{\rho} | N, k \rangle,$$

where $|N, k\rangle = |N\rangle_H \otimes |k\rangle_V$ and the subscripts $H$ and $V$ correspond to the horizontally and vertically polarized modes, respectively. In this case, it is easy to obtain the optimal coefficients $\lambda_N$. A straightforward calculation gives $\lambda_N = p_N / (N + 1)$. Since these coefficients satisfy the constraint (6), the corresponding density operator (5) minimizes the distance (9). The Hilbert-Schmidt degree of polarization can thus be written as

$$P_{\text{HS}}(\hat{\rho}) = \text{Tr} \left( \hat{\rho}^2 \right) - \sum_{N=0}^{\infty} \frac{p_N^2}{N + 1}.$$
our second candidate of distance we therefore consider the fidelity (or Uhlmann transition probability) \cite{17}

\[ F(\hat{\rho}, \hat{\sigma}) = \left( \text{Tr} \sqrt{\hat{\sigma}^{1/2} \hat{\rho} \hat{\sigma}^{1/2}} \right)^2. \] (12)

The fidelity has many attractive properties. Even though it is not obvious from (12), the fidelity is symmetric in its arguments \( F(\hat{\rho}, \hat{\sigma}) = F(\hat{\sigma}, \hat{\rho}) \). It can also be shown that \( 0 \leq F(\hat{\rho}, \hat{\sigma}) \leq 1 \), with equality in the second inequality iff \( \hat{\rho} = \hat{\sigma} \). Hence, the fidelity is not a metric as such, but serves rather as a generalized measure of the overlap between two quantum states. A common way of turning it into a metric is through the Bures metric

\[ D_B(\hat{\rho}, \hat{\sigma}) = 2 \left[ 1 - \sqrt{F(\hat{\rho}, \hat{\sigma})} \right]. \] (13)

The origin of this distance can be seen intuitively by considering the case when \( \hat{\rho} \) and \( \hat{\sigma} \) are both pure states. The Bures metric is just the Euclidean distance between the two pure states, with respect to the usual norm on the state space. Since the Bures distance \( D_B(\hat{\rho}, \hat{\sigma}) \) decreases with increasing fidelity \( F(\hat{\rho}, \hat{\sigma}) \), we can define the Bures degree of polarization as

\[ \mathbb{P}_B(\hat{\rho}) = 1 - \sup_{\hat{\sigma} \in \mathcal{U}} \sqrt{F(\hat{\rho}, \hat{\sigma})}. \] (14)

Finally, our third candidate is the so-called trace distance, which is defined as \cite{18}

\[ D_T(\hat{\rho}, \hat{\sigma}) = \text{Tr} |\hat{\rho} - \hat{\sigma}|, \] (15)

where \( |\hat{X}| = \sqrt{\hat{X}^\dagger \hat{X}} \). As we shall see in a moment, the appropriate normalization factor in this case is \( 1/2 \), so the corresponding degree of polarization is

\[ \mathbb{P}_T(\hat{\rho}) = \frac{1}{2} \inf_{\hat{\sigma} \in \mathcal{U}} D_T(\hat{\rho}, \hat{\sigma}). \] (16)

Unfortunately, we have not found a general expression of the unpolarized state \( \hat{\sigma} \) that gives the maximum fidelity or the minimum trace distance for a general \( \hat{\rho} \). Such a task must be performed case by case and will be illustrated below.

4. Comparison between different degrees of polarization

In this Section, we shall focus on the behavior of the different degrees of polarization for states with a fixed excitation number \( N \). The corresponding density operator can be written as

\[ \hat{\rho}^{(N)} = \sum_{k=0}^{N} \pi_k |\Psi_k^{(N)} \rangle \langle \Psi_k^{(N)}|, \] (17)

where \( \{|\Psi_k^{(N)} \rangle\}_{k=0}^{N} \) is a suitable set of orthonormal \( N \)-photon states and the probabilities satisfy \( \pi_0 \geq \pi_1 \geq \ldots \geq \pi_N \). One can easily work out that the Hilbert-Schmidt and Bures degrees of polarization for these states are \cite{5}

\[ \mathbb{P}_{HS}(\hat{\rho}^{(N)}) = \sum_{k=0}^{N} \pi_k^2 - \frac{1}{N+1}, \quad \mathbb{P}_B(\hat{\rho}^{(N)}) = 1 - \frac{1}{\sqrt{N+1}} \sum_{k=0}^{N} \sqrt{\pi_k}. \] (18)

It is clear that these expressions are maximized by pure states \( (\pi_0 = 1) \), and that they for any pure state go to unity in the limit \( N \to \infty \).
Next, we consider the trace distance. Denoting the largest integer \( k \) satisfying \( \pi_k \geq \lambda_N \) as \( m \), we have

\[
D_T(\rho^{(N)}, \sigma) = \sum_{M=0}^{\infty} \sum_{M \neq N} (M + 1)\lambda_M + \sum_{k=0}^{N} |\pi_k - \lambda_N| = 1 - (N + 1)\lambda_N + \sum_{j=0}^{m} (\pi_j - \lambda_N) + \sum_{k=m+1}^{N} (\lambda_N - \pi_k),
\]

where the last sum vanishes for \( m = N \). Since \( m \) is non-negative and cannot increase with increasing \( \lambda_N \), the minimum distance is obtained for the \( N \)-photon unpolarized state characterized by \( \lambda_N = 1/(N + 1) \), as could have been expected. Hence, we have

\[
\mathbb{P}_T(\rho^{(N)}) = \frac{\sum_{j=0}^{n} \pi_j - \frac{n + 1}{N + 1}}{\frac{n + 1}{N + 1}},
\]

where \( n \) is the largest integer satisfying \( \pi_n \geq 1/(N+1) \). Introducing the parameter \( P = \sum_{j=0}^{n} \pi_j \), we thus have \( 1/(N + 1) < P \leq 1 \) and \( 0 \leq n \leq (N + 1)P - 1 \). The \( N \)-photon states with the highest polarization degree of \( N/(N + 1) \) are again pure states. However, since the trace-norm degree of polarization depends on two parameters, and \( n \) is an integer, \( \mathbb{P}_T \) is very different from the Hilbert-Schmidt and Bures degrees of polarization. This makes it easy to show that there exist pairs of states that are ordered differently by \( \mathbb{P}_T \) and \( \mathbb{P}_{HS} \) or \( \mathbb{P}_B \). To this end, let us consider two-photon states with \( \pi_0 > 2/3 \): the trace-norm degree of polarization for any such state is independent of \( \pi_1 \) and is given by

\[
\mathbb{P}_T(\pi_0) = \pi_0 - \frac{1}{3}.
\]

In contrast, the Hilbert-Schmidt and Bures degrees of polarization, \( \mathbb{P}_{HS}(\pi_0, \pi_1) \) and \( \mathbb{P}_B(\pi_0, \pi_1) \), vary with \( \pi_1 \). For any given \( \pi_0 > 2/3 \), their maximum values are obtained for \( \pi_1 = 1/2 - \pi_0/2 \), and their minimum values for \( \pi_1 = 1 - \pi_0 \). It is therefore clear that there exist pairs of states characterized by the probabilities \( (\pi_0, 1/2 - \pi_0/2) \) and \( (\pi'_0, 1 - \pi'_0) \) with \( \pi_0 \) slightly smaller than \( \pi'_0 \), implying \( \mathbb{P}_T(\pi_0) < \mathbb{P}_T(\pi'_0) \), that satisfy \( \mathbb{P}_{HS}(\pi_0, 1/2 - \pi_0/2) > \mathbb{P}_{HS}(\pi'_0, 1 - \pi'_0) \) and \( \mathbb{P}_B(\pi_0, 1/2 - \pi_0/2) > \mathbb{P}_B(\pi'_0, 1 - \pi'_0) \).

As we have previously shown [5] that there exist pairs of states that are ordered differently by \( \mathbb{P}_{HS} \) and \( \mathbb{P}_B \), it thus follows that all three degrees of polarization considered here (\( \mathbb{P}_{HS}, \mathbb{P}_B, \mathbb{P}_T \)) are fundamentally different.

5. Conclusion

The problem with a “hidden” polarization structure for quantum states resulting from the definition of polarization degree based on the Stokes operators can be avoided by taking the unpolarized states as a starting point. We have proposed a generic definition of the degree of polarization as the minimum distance to the set of unpolarized states. All three measures of distance considered here give rise to fundamentally different degrees of polarization. It is noteworthy that the Hilbert-Schmidt degree of polarization for an arbitrary state can be expressed analytically.

Acknowledgments

We would like to acknowledge useful discussions with G. M. D’Ariano and A. Wünsche. We also acknowledge financial support from the Swedish Foundation for International Cooperation in Research and Higher Education (STINT), the Swedish Research Council (VR) and the Spanish Research Council (DGI) under the project FIS2005-06714.
References

[1] Ou Z Y and Mandel L 1988 Phys. Rev. Lett. 61 50
   Aspect A, Grangier P and Roger G 1981 Phys. Rev. Lett. 47 460
   Bennett C H, Brassard G, Salvail L and Smolin J 1992 J. Cryptology 5 3
   Kwiat P G, Mattle K, Weinfurter H, Zeilinger A, Sergienko A V and Shih Y 1995 Phys. Rev. Lett. 75 4337
   Muller A, Hertzog T, Huttner B, Tittel W, Zbinden H and Gisin N 1997 Appl. Phys. Lett. 70, 793
   Bouwmeester D, Pan J W, Mattle K, Eibl M, Weinfurter H and Zeilinger A 1997 Nature 390 575
   White A G, James D F V, Eberhard P H and Kwiat P 1999 Phys. Rev. Lett. 83 3103
   Trifonov A, Björk G and Söderholm J 2001 Phys. Rev. Lett. 86 4423
   Jennewein T, Weihs G, Pan J W and Zeilinger A 2002 Phys. Rev. Lett. 88 017903
   Barbieri M, De Martini F, Di Nepi G, Mataloni P, D’Ariano G M and Macchiavello C 2003 Phys. Rev. Lett. 91 227901

[2] Stokes G G 1852 Trans. Cambridge Philos. Soc. 9 399
   Born M and Wolf E 1980 Principles of Optics (Oxford: Pergamon)

[3] Jauch J M and Rohrlich F 1976 The Theory of Photons and Electrons (Berlin: Springer)
   Collett E 1970 Am. J. Phys. 38 563

[4] Tsegaye T, Söderholm J, Attäure M, Trifonov A, Björk G, Sergienko A V, Saleh B E A and Teich M C 2000 Phys. Rev. Lett. 85 5013
   Usachev P, Söderholm J, Björk G and Trifonov A 2001 Opt. Commun. 193 161

[5] Klimov A B, Sánchez-Soto L L, Yustas E C, Söderholm J and Björk G 2005 Phys. Rev. A 72 033813

[6] Prakash H and Chandra N 1971 Phys. Rev. A 4 796
   Agarwal G S 1971 Lett. Nuovo Cimento 1 53
   Lehner J, Leonhardt U and Paul H 1996 Phys. Rev. A 53 2727
   Björk G, Söderholm J, Trifonov A, Usachev P, Sánchez-Soto L L and Klimov A B 2002 Proc. SPIE 4750 1
   Wünsche A 2003 Fortschr. Phys. 51 262

[7] Yurke B, McCall S L and Klauder J R 1986 Phys. Rev. A 33 4033

[8] Chirkin A S, Alodjants A P and Arakelian S M 1997 Opt. Spectrosc. 82 919
   Alodjants A P, Arakelian S M and Chirkin A S 1998 App. Phys. B 66 53
   Alodjants A P and Arakelian S M 1999 J. Mod. Opt. 46 475

[9] Karassiov V P 1993 J. Phys. A 26 4345

[10] Klyshko D M 1992 Phys. Lett. A 163 349

[11] Karassiov V P 1995 Polarization structure of quantum light fields: A new insight. 2: Generalized coherent states, squeezing and geometric phases Preprint quant-ph/9503011

[12] Hillery M 1987 Phys. Rev. A 35 725
   Dodonov V V, Man’ko O V, Man’ko V I and Wünsche A 2000 J. Mod. Opt. 47 633
   Marian P, Marian T A and Scutaru H 2002 Phys. Rev. Lett. 88 153601

[13] Vedral V, Plenio M B , Rippin M A and Knight P L 1997 Phys. Rev. Lett. 78 2275
   Vedral V and Plenio M B 1998 Phys. Rev. A 57 1619

[14] Gilchrist A, Langford N K and Nielsen M A 2005 Phys. Rev. A 71 062310

[15] Witte C and Trucks M 1999 Phys. Lett. A 257 14
   Ozawa M 2000 Phys. Lett. A 268 158
   Bertlmann R A, Narnhofer H and Thirring W 2002 Phys. Rev. A 66 032319

[16] Horodecki P 2003 Phys. Rev. Lett. 90 167901
   Legrè M, Wegmuller M and Gisin N 2003 Phys. Rev. Lett. 91 167902

[17] Uhlmann A 1976 Rep. Math. Phys. 9 273

[18] Belavkin V P, D’Ariano G M and Raginsky M 2005 J. Math. Phys. 46 062106