Nonlinear filtration of Ricci data as a tool for the phase measurements: aspects of a theory

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Nonlinear filtration of Rician data as a tool for the phase measurements: aspects of a theory

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Abstract. The paper presents a new theoretical approach to high precision measuring of the signals’ phase shift by statistical analysis and processing of the amplitude’s sampled data. The amplitude of a quasi-harmonic signal is shown to obey to the Rice statistical distribution. The approach is based upon the application of methods of the Rician data’s two-parameter analysis ensuring an efficient separation of the signal’s informative and noise components. A meaningful peculiarity of the proposed approach consists in the possibility of determination of the signals’ sought-for parameters only on the basis of the sampled measurements, without any a priori assumptions. Moreover, as applied to the specific task of high-precision measuring the phase shift between the signals’ propagation by various channels, the approach implies the implementation of the amplitude measurements only, thus providing an essential decrease in technical requirements to the equipment. The technique is meaningful for a wide spectrum of various scientific and applied tasks, in particular, in metrology, in ranging and communication systems.

1. Introduction

At solving the tasks of processing the signals propagating in inhomogeneous medium one faces the problem of the noised signals’ filtration as any initially determined signal’s characteristic is transformed into a random variable due to the inevitable noise influence.

The mathematical statistical methods are known to be widely used for the random signals’ processing, in particular, at handling the problem of noise suppression. Obviously, at applying such methods the peculiarities of the statistical distribution of the specific data being analyzed have a substantial significance for solving the task efficiently.

In recent years the Rice statistical model, [1], has become widely used for solving the problems connected with the random signals processing as it is known to adequately describe many physical processes of various nature, namely the Rice distribution describes a wide range of information processing problems when the output signal is composed as a sum of the required initial signal and a random noise generated by many independent normally-distributed summands of zero mean value. The amplitude, or the envelope of such a resulting signal is known to obey to the Rice statistical distribution, [1], and in many scientific and applied tasks just this variable is to be measured and analyzed what makes the applicability of the Rice model quite extended.

The special peculiarities of the Rice distribution have appeared to open new possibilities for random data filtering allowing an efficient separation of the informative and the noise components of the signals to be processed.
The two Rice distribution parameters which determine the probability density function are known to correspond to the initial un-noised signal value and the noise dispersion, correspondingly. Therefore the task of the efficient reconstruction of the informative signal value against the noise background comes down to the calculation of the both Rician parameters.

The so-called two-parameter approach to the Rician signals’ analysis which recently has been theoretically developed in [2, 3] consists in solving the task of joint determination of both parameters of the Rice distribution. In contrast to the traditional one-parameter approximation this approach is free of limitations that are inherent to the one-parametric approximation based upon the supposition that one of the task statistical parameters – the noise dispersion – is known a priori, [4-6]. That’s why the technique of the two-parametric task solution ensures much more correct estimation of the required signal and noise parameters’ values, [2, 3, 7-9].

The paper presents an original application of the elaborated techniques of Rician data two-parameter analysis for solving the tasks of the high-precision calculation of the phase shift between the quasi-harmonic signals from the sampled measurements of the signals’ envelope, or the amplitude values.

2. Relation to prior work
A significant interest to solving a task of joint estimation of the Rice distribution’s both parameters has appeared in 60-th years of the 20th century because of the understanding that in conditions of the Rice distribution only the knowledge of both Rician parameters allows efficient reconstructing the initial, undistorted signal against the noise background.

In paper [4] there was first formulated the significance of solving the two-parameter task applicable to radar signals’ analysis. However this task has appeared to be connected with finding the solution of a system of two essentially nonlinear equations what is conjugated with considerable difficulties of both the theoretical and the computational character. Partly due to this reason in [4] the mathematical consideration of the task is limited by the determination of the lower bounds for the standard deviation of these parameters’ estimations on the basis of the Cramer-Rao inequality.

Later the simplified methods of the Rician data analysis have been elaborated in the conditions of the so-called one-parameter approximation consisting in estimating of only one of the two unknown parameters – the signal value, in supposition that the second parameter – the noise dispersion – is known a priori.

The fundamental papers considering the problem within the one-parameter approximation are the papers [5] and [6], in which the required signal parameter is being estimated on the basis of the method of moments and the maximum likelihood techniques, respectively.

However in practice the condition when the Gaussian noise dispersion is known a priori never takes place and so this supposition is a severe restriction of the one-parameter approach.

An undertaken in paper [10] attempt of solving the two-parameter task theoretically had some restrictions, namely: the suppositions concerning the task solution’s properties are based mainly on the graphical illustrations instead of the strict mathematical analysis. That’s why some of these suppositions have appeared to be partially mistakable, for instance, concerning the number of the maximum likelihood equations’ solutions.

Therefore the theoretical problem consisting in joint estimation of both parameters of the Rice statistical distribution, not limited by any a priori conditions, has remained unsolved for a few decades, since the 60-th years of the 20th century.

In [2, 3, 7-9] an accurate theory of the Rician signals statistical processing has been developed: new mathematical methods have been elaborated and strictly substantiated for the two-parameter approach to Rician data analysis ensuring an efficient joint signal and noise estimation.
3. Theoretical basics and definitions

In the tasks of the Rician signal analysis the value to be measured is an amplitude $x = \sqrt{x_{\text{Re}}^2 + x_{\text{Im}}^2}$ of the complex variable with the real $x_{\text{Re}}$ and the imaginary $x_{\text{Im}}$ components characterized by their mean value $\nu$ and distorted by the normally distributed Gaussian noise with the dispersion $\sigma^2$. These conditions characterize many tasks of processing the signals of various physical nature. The amplitude $x = \sqrt{x_{\text{Re}}^2 + x_{\text{Im}}^2}$ obeys to the Rice distribution with the probability density function:

$$P(x|\nu, \sigma^2) = \frac{x}{\sigma^2} \cdot \exp\left(-\frac{x^2 + \nu^2}{2\sigma^2}\right) \cdot I_0\left(\frac{x\nu}{\sigma^2}\right),$$

where $I_{\alpha}(z)$ is the modified Bessel function of the first type of the order $\alpha$.

The task consists in determining the unknown parameters $\nu$ and $\sigma^2$ on the basis of data measured in the samples. In virtue of the specific peculiarities of Rice statistical distribution the Rician data analysis demands a development of the particular methods and the corresponding mathematical apparatus.

As it is known at processing the Gaussian data an efficient and traditional filtration tool is the data averaging. However, as it has been noticed above, in contrast to the case of the Gaussian distribution an average value of the Rician signal $\bar{x}$ does not coincide to the requires useful signal’s value $\nu$. This is illustrated in figure 1 where the average Rician signal’s value $\bar{x}$ that is depicted by a dotted curved while the useful signal’s value $\nu$ depends is depicted by a straight line.

The analytical expression for the average Rician signal’s value $\bar{x}$ as a function of the Rician parameters $\nu$ and $\sigma$ is as follows:

$$\bar{x} = \sigma \cdot \sqrt{\pi/2} \cdot L_{1/2}(-\nu^2/2\sigma^2)$$

In (2) $L_{1/2}(z)$ is a Laguerre polynomial.

![Figure 1](image.png)

**Figure 1.** The illustration of the divergency of the Rician signal’s averaged value $\bar{x}$ and Rician parameter $\nu$, as dependent on the signal-to-noise ratio $\text{SNR} = \nu / \sigma$.

The plots in figure 1 correspond to the fixed values of the parameter $\sigma$: $\sigma = 1$, so the values at the abscissa axis correspond to the signal-to-noise ratio $\text{SNR} = \nu / \sigma$.

Thereby, figure 1 illustrates the following: if one applies the filtration methods by averaging to the Rician data then in a range of small values of $\text{SNR}$ just the smoothing of the true values of the signal takes place.

So, the traditional averaging is not efficient at the Rician data noise filtering. The specific features of the Rice statistical model cause the necessity of the development of a special approach to the data
Let \( \tilde{r}(r,\psi) \) be a denotation of a noise component that is superimposed on initial signal \( \tilde{A} \). The components \( r_x, r_y \) of the noise vector \( \tilde{r} \) are independent and obey the normal distribution:

\[
\bar{r}_x = \bar{r}_y = 0, \quad r_x^2 = r_y^2 = \sigma^2,
\]

where \( \sigma^2 \) is a noise dispersion value. Amplitude \( r \) and phase \( \psi \) are statistically distributed as follows: amplitude \( r \) obeys the Rayleigh distribution, while the noise components’ phase \( \psi \) is distributed uniformly within interval \((0, 2\pi)\), [11].

Let \( \tilde{R}(R,\phi) \) be a vector denoting the resulting signal that is formed by summing initial signal \( \tilde{A} \) and noise \( \tilde{r} : \tilde{R} = \tilde{A} + \tilde{r} \). The real and imaginary parts of \( \tilde{R} \) can be written as follows:

\[
R \cos \phi = A \cos \phi_0 + r \cos \psi; \quad R \sin \phi = A \sin \phi_0 + r \sin \psi
\]  

(3)

The statistical distribution of amplitude \( R \) and phase \( \phi \) of resulting signal \( \tilde{R} \) is determined by their joint distribution function, [11], which can be obtained with taking into account the statistical distributions of amplitude \( r \) and phase \( \psi \). The joint distribution function for amplitude \( R \) and phase \( \phi \) looks as follows:

\[
W(R,\phi)dRd\phi = \frac{1}{2\pi\sigma^2} \cdot \exp\left\{ -\frac{1}{2\sigma^2}[A^2 + R^2 - 2AR\cos(\phi - \phi_0)] \right\} R dR d\phi
\]  

(4)

As one can see from (4), the distributions of the resulting signal’s amplitude \( R \) and its phase \( \phi \) are not independent, and phase \( \phi \) as distinct from phase \( \psi \) is not a uniformly distributed random value.

Having integrated (4) by \( \phi \) within the limits from 0 to \( 2\pi \) one can obtain an expressions for the distribution function for amplitude \( R \) of resulting signal \( \tilde{R} = \tilde{A} + \tilde{r} : \)

\[
W_R(R)dR = \frac{RdR}{\sigma^2} \cdot I_0\left(\frac{RA}{\sigma^2}\right)e^{-\left(r_x^2 + r_y^2\right)/2\sigma^2}
\]  

(5)

At obtaining (5) an integral representation for the modified Bessel function has been taken into account, [12]:

\[
I_0(z) = \frac{1}{2\pi} \int_0^{2\pi} e^{z\cos t} \, dt.
\]

From (4) and (1) it follows that amplitude \( R \) obeys the Rice distribution with parameters \( A, \sigma^2 \) (\( \sigma^2 \) is the Gaussian noise dispersion value). So, the influence of noise can be mathematically described as “blurring” the initial signal’s vector \( \tilde{A} \) of amplitude \( A \) so that its amplitude becomes a random value \( R = |\tilde{R}| \) that obeys the Rice distribution.

4. The essence of the proposed phase shift measuring technique by means of the Rician data nonlinear filtration

The statement of the problem being considered here as a specific application of the Rician data two-parameter analysis is as follows: two initially sine-shaped optical signals of the same frequency propagate through the different channels thus accumulating the phase shift to be measured.

In practice the propagation of the harmonic signal in a medium is inevitably accompanied by the noise influence what results in the random variations of the signal’s amplitude. Therefore, instead of a sine-shaped signal one has to consider just the quasi-harmonic, or quasi-sinusoidal signal. According to the above the amplitude of such a signal is a random value that satisfies to the Rice statistical distribution.
The time dependence of any quasi-harmonic signal $x(t)$ can be presented as follows:

$$x(t) = R(t) \cdot \sin(\omega t + \varphi(t))$$ (6)

where $\omega$ is the frequency, $R(t)$ is the signal’s amplitude, or envelope that randomly varies in time $t$ due to the inevitable Gaussian noise influence, and $\varphi(t)$ is the phase shift that also changes randomly in time due to the so-called amplitude-phase modulation.

For convenience of the mathematical analysis we’ll consider signal (1) as a complex value having denoted it as $S(t)$:

$$S(t) = R(t) \cdot \exp\left[i(\omega t + \varphi(t))\right] = s(t) \cdot \exp(i\omega t)$$ (7)

For measuring the signals’ phase values we’ll analyze the “slow” signal’s component $s(t) = R(t) \cdot \exp\left[i\varphi(t)\right]$.

Let the initial, undistorted complex signal be denoted as vector $\vec{A}(A, \varphi_0)$. It is characterized by an initially determined amplitude $A$ and phase $\varphi_0$. The signal’s propagation through a medium is inevitably accompanied by its noising, namely – the initial signal’s real $A \cos \varphi_0$ and imaginary $A \sin \varphi_0$ parts are independently varied by a lot of random noise components.

The mathematical problem to be solved consists in measuring the phase shift between two quasi-harmonic signals that are propagating in different channels. Obviously, these signals’ phase difference is a characteristic of the object or the process to be studied. We can present these signals as the following vectors: $\vec{R}_1(R_1, \varphi_1), \vec{R}_2(R_2, \varphi_2)$ as illustrated in figure.2.

The quasi-harmonic signals’ amplitudes $R_1$ and $R_2$ obey to the Rice distribution with parameters $(A_1, \sigma^2)$, and $(A_2, \sigma^2)$, correspondingly, where $A_1$ and $A_2$ are the initial, undistorted signals’ amplitudes, $\sigma^2$ is the Gaussian noise dispersion. It is natural to suppose that such a dispersion value is the same for the both channels by which the two signals are propagating, although the mathematical analysis provided below can be easily generalized for a case of different dispersion values. In the further calculations we’ll use a-priori knowledge that the phase difference $\Delta \varphi = \varphi_2 - \varphi_1$ between the considered signals is unambiguously determined by the physical properties of the object or the process being studied.

The noised signals to be measured can be put down as follows: $\vec{R}_1 = \vec{A}_1 + \vec{r}_1$, $\vec{R}_2 = \vec{A}_2 + \vec{r}_2$, where vectors $\vec{A}_1$ and $\vec{A}_2$ denote the two initial, undistorted signals, $\vec{r}_1, \vec{r}_2$ - the noise vectors, each of them being characteristic for a corresponding channel of the signal propagation.

In figure 2 the noised signals $\vec{R}_1$, $\vec{R}_2$ and $\vec{R}_3$ are shown by the dashed lines while the initial, undistorted signals $\vec{A}_1$, $\vec{A}_2$ and their sum $\vec{A}_3$ are shown by solid lines.

The phase difference $\Delta \varphi$ between the two signals is equal to an angle between the corresponding vectors. Let us introduce the third vector that is equal to the sum of the two signals being analyzed. We denote it as vector $\vec{R}_3 = \vec{A}_3 + \vec{r}_3$, where $\vec{A}_3 = \vec{A}_1 + \vec{A}_2$ - the sum of the first two undistorted signals.

Vectors $\vec{R}_1$, $\vec{R}_2$ and $\vec{R}_3$ form a triangle and the phase difference between the two signals can be determined based on the geometrical consideration of this triangle, namely - by calculating the triangle sides’ values, i.e. the signals amplitudes’ values. However, the inevitable noise distorts each vector independently and the amplitudes measured in each moment of time would provide a false, distorted
value for the sought for phase shift. Obviously, the sought for phase difference between $\vec{A}_1$ and $\vec{A}_2$ could be correctly found only from the triangle formed by the initial, undistorted amplitudes: $A_1, A_2, A_3$.

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{figure2}
\caption{The vector representation of signals being measured and analyze dat calculating the phaseshift value $\Delta \phi$.}
\end{figure}

So, for the accurate phase shift calculation we have to “freeze” the triangle at the undistorted, noise-free state. Solving this problem has appeared to be possible by means of the amplitudes’ values processing with taking into account their statistical properties. As it has been shown above the signals’ amplitudes obey to the Rice distribution with the Rician parameters $(A_i, \sigma^2)$, $i = 1, 2$.

As for the third signal $\vec{R}_3 = \vec{A}_3 + \vec{n}$, its amplitude can be shown to obey the Rice distribution as well. The parameters of this distribution are: $(A_3, 2\sigma^2)$, where $A_3 = |\vec{A}_3|$. As the amplitudes measured in samples provide the distorted data for the lengths of the triangle sides, they need to be processed in such a way that would allow getting the undistorted values $A_1, A_2, A_3$. In other words we have to determine the corresponding Rician parameters’ values. The above mentioned two-parameter methods elaborated in [2,3,7-9] allow an accurate estimating of both the signal $(A_i, i = 1, 2, 3)$ and the noise $(\sigma^2)$ parameters based upon the sampled measurements.

So, by means of calculating the initial, undistorted values of the three signals’ amplitudes $A_1, A_2, A_3$ we are able to “freeze” the picture as a noise-free one and thus calculate the needed phase difference value just on the basis of geometrical considerations by the formula:

$$\Delta \phi = \arccos \left( \frac{A_1^2 - A_2^2 - A_3^2}{2A_1A_2} \right)$$

From the above it follows that the proposed technique of the signals’ phase shift measuring differs in principle from other methods as it is based entirely upon measuring and processing the amplitude values only.

5. Mathematical methods of the phase shift measuring technique by means of the Rician data nonlinear filtration

From the above it follows that the proposed approach to solving the task of high-precision phase shift measuring is based upon the accurate joint signal and noise estimation of the Rician data. It means that for the undistorted amplitude values reconstruction the methods of the two-parameter Rician data analysis are to be applied.
This section provides a brief description of the principle mathematical techniques of the two-
parameter Rician signals' analysis which ensure an efficient reconstruction of the initial, un-noised
signal’s value.

The particular theoretical methods having been developed within the two-parameter analysis of the
Rician signal in \([2, 3, 7-9]\) differ in underlying statistical principles they are based upon. These
methods include the variants of the method of moments: based on the data for the 1-st and 2-nd
moments, designated as MM12; based on measurements of the 2-nd and 4-th moments, designated as
MM12; the two-parametric maximum likelihood method, designated as ML. Each method consists in
solving the corresponding equations’ system for the sought for parameters \(\nu\) and \(\sigma\), \([7-9]\).

The system of equation for method MM12 looks as follows:

\[
\begin{align*}
\sigma \cdot \sqrt{\pi / 2} & \cdot e^{\frac{\nu}{4\sigma^2}} \left[ \left( 1 + \frac{\nu^2}{2\sigma^2} \right) I_0 \left( \frac{\nu^2}{4\sigma^2} \right) + \frac{\nu^2}{2\sigma^2} I_1 \left( \frac{\nu^2}{4\sigma^2} \right) \right] = \overline{X}, \\
2\sigma^2 + \nu^2 &= \overline{X^2}.
\end{align*}
\]

(9)

Method MM24 is rather an original and simple in its realization with the following non-
complicated equations’ system:

\[
\begin{align*}
\overline{X^2} &= 2 \cdot \sigma^2 + \nu^2, \\
\overline{X^4} &= 8 \cdot \sigma^4 + 8 \cdot \sigma^2 \cdot \nu^2 + \nu^4.
\end{align*}
\]

(10)

As for the ML method, its system of equations looks as follows:

\[
\begin{align*}
\nu &= \frac{1}{n} \sum_{i=1}^{n} x_i \cdot I_1 \left( \frac{x_i \cdot \nu}{\sigma^2} \right) / I_0 \left( \frac{x_i \cdot \nu}{\sigma^2} \right), \\
\sigma^2 &= \left( \overline{X^2} - \nu^2 \right) / 2.
\end{align*}
\]

(11)

The existence and the uniqueness of the solutions of systems (9)-(11) have been strictly proved in
papers \([2,3,7,8]\).

As a results of the detailed mathematical analysis of the equations’ systems (9)-(11) the following
important mathematical results have been obtained: for each of the above mentioned methods the
corresponding system of two nonlinear equations for two variables \(\nu\) and \(\sigma\) can be reduced to one
equation for just one variable. This allows an essential decreasing of the computational resources
needed for the task solving.

6. Conclusion

The paper presents the theoretical basics of a new original approach to solving the task of high-
precision phase measuring based upon the statistical processing of the amplitudes values of
Rician signals to be analyzed.

The theoretical substantiation of the applicability of the Rice statistical model for solving the task
has been provided. The amplitude of a quasi-harmonic signal is shown to obey the Rice statistical
distribution.

The application of the proposed approach implies the analyzing and processing of the following
three signals: the two signals being compared and the third signal which is their sum. The sought for
phase difference is then calculated as an angle of a triangle formed by the reconstructed undistorted
signals’ amplitude values.

In order to reconstruct the undistorted amplitudes’ values against the noise background the methods
of the two-parameter Rician data analysis are to be applied. An important peculiarity of the proposed
technique consists in the fact that the required phase shift is obtained as a result of the amplitude
measurements only what significantly decreases the demands to the measuring equipment.
The proposed technique can be efficiently applied in a wide circle of scientific and technical tasks in numerous ranging and communication systems.

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