Algebraically special perturbations of the Schwarzschild solution in higher dimensions

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Abstract
We study algebraically special perturbations of a generalized Schwarzschild solution in any number of dimensions. There are two motivations. First, to learn whether there exist interesting higher-dimensional algebraically special solutions beyond the known ones. Second, algebraically special perturbations present an obstruction to the unique reconstruction of general metric perturbations from gauge-invariant variables analogous to the Teukolsky scalars and it is desirable to know the extent of this non-uniqueness. In four dimensions, our results generalize those of Couch and Newman, who found infinite families of time-dependent algebraically special perturbations. In higher dimensions, we find that the only regular algebraically special perturbations are those corresponding to deformations within the Myers–Perry family. Our results are relevant for several inequivalent definitions of ‘algebraically special’.

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1. Introduction

The study of spacetimes with algebraically special Weyl tensor played an important role in the discovery of some important solutions of the four-dimensional Einstein equation, e.g. the Kerr solution [1] and the spinning C-metric [2]. It seems worthwhile investigating whether the algebraically special property is useful for finding new explicit solutions of the Einstein equation in higher dimensions. In this paper we will consider the vacuum Einstein equation, allowing for a cosmological constant.

Several inequivalent definitions of ‘algebraically special’ have been proposed in higher dimensions [3–6]. (See [7] for an introductory review or [8] for a more detailed review.) The Myers–Perry black hole solution [9] is algebraically special according to any of these definitions. However, it is not known whether the class of algebraically special spacetimes in higher dimensions is as rich as in four dimensions. To investigate this, we will study...
algebraically special perturbations of a known solution, namely the higher-dimensional Schwarzschild solution. From this we can learn about algebraically special solutions which admit the Schwarzschild solution as a limit.

A second motivation for this project comes from work on linearized gravitational perturbations of algebraically special solutions such as the Myers–Perry black hole. Reference [10] showed that there exists a gauge-invariant quantity $\delta \Omega_{ij}$ (defined below) which depends linearly and locally on the metric perturbation and has the same number of physical degrees of freedom as the metric perturbation. $\delta \Omega_{ij}$ is the higher-dimensional analogue of one of the Teukolsky scalars used in the study of gravitational perturbations of a Kerr black hole [11]. Knowing $\delta \Omega_{ij}$ determines the metric perturbation up to addition of a solution of $\delta \Omega_{ij} = 0$. As we will see, this equation is closely related to the condition for the perturbation to be algebraically special. Hence by determining such perturbations we can identify the extent to which $\delta \Omega_{ij}$ fails to characterize metric perturbations. (For perturbations of a Kerr black hole, this problem was studied in [12].)

In 4D, algebraically special perturbations of the Schwarzschild solution were studied by Couch and Newman [13]. They decomposed perturbations into harmonics on $S^2$, labelled by $l = 0, 1, \ldots$. For $l = 0, 1$ there are algebraically special perturbations corresponding to a change in the mass, and to the Kerr solution linearized for small angular momentum. For each $l > 1$ there are time-dependent algebraically special perturbations of two types. The first type decays exponentially as a function of the retarded time coordinate $u$, and vanishes on the future horizon. This corresponds to the linearization about the Schwarzschild solution of the Robinson–Trautman class of algebraically special solutions [14, 15], for which the repeated principal null direction has vanishing rotation. The second type grows exponentially with $u$ and hence diverges on the future horizon. This corresponds to the linearization about the Schwarzschild solution of the class of algebraically special solutions for which the repeated principal null direction has non-vanishing rotation.$^3$

We will consider the $d$-dimensional generalized Schwarzschild solution for which the $(d-2)$-sphere of the metric is replaced by an arbitrary compact Einstein manifold $K^{d-2}$ with curvature of any sign. We also include a cosmological constant. We will exploit the results of Kodama and Ishibashi [16, 19], who showed that linearized metric perturbations can be decomposed into scalar, vector and tensor types on $K^{d-2}$. For each type there is a ‘master equation’, a wave equation for a single scalar quantity. This can be reduced to a wave equation in 2D by expansion in harmonics on $K^{d-2}$. For each type of perturbation, we show that imposing the algebraically special condition gives an additional equation so the problem reduces to solving this equation simultaneously with the master equation. Our results are as follows.

We find that algebraically special perturbations arise in the tensor sector if, and only if, there exist infinitesimal traceless perturbations of the metric on $K^{d-2}$ that preserve the Einstein condition.

For $d = 4$, there are two infinite classes of algebraically special perturbations: one of vector type and one of scalar type. These correspond to the perturbations discovered by Couch and Newman, generalized to allow for planar or hyperbolic symmetry and a cosmological constant.

For $d > 4$, we find that algebraically special vector and scalar perturbations are much more restricted then for $d = 4$. In the vector sector, algebraically special perturbations arise only when $K^{d-2}$ has non-negative curvature and admits an isometry, in which case

$^3$ Two more types of algebraically special perturbation are related to these by the Schwarzschild time-reversal isometry. This gives solutions which vanish on the past horizon and grow exponentially with the advanced time coordinate $v$, and solutions which diverge on the past horizon and decay exponentially with $v$. 


the perturbation corresponds to adding angular, or linear, momentum in the direction of the isometry. For $K^{d-2} = S^{d-2}$ this corresponds to the Myers–Perry solution linearized around the Schwarzschild solution. Finally, the only algebraically special perturbation in the scalar sector corresponds to a variation of the mass of the Schwarzschild solution.

Our result shows that $d > 4$ algebraically special perturbations are stationary. We have obtained essentially the same class of perturbations as was found in [20], which determined all stationary perturbations of Schwarzschild that are regular at the horizon and vanish at infinity, for the case in which $K^{d-2}$ is a space of constant curvature.

In summary, the only algebraically special perturbations of the higher-dimensional Schwarzschild solution correspond to simple variations of parameters in known solutions. In contrast with $d = 4$, there are no time-dependent algebraically special perturbations. From the perspective of looking for new algebraically special solutions, this is disappointing. However, it is encouraging from the point of view of using the quantity $\delta \Omega_{ij}$ to study linearized metric perturbations since our result implies that this quantity determines the perturbation up to the addition of a ‘variation of parameters’ perturbation. This is better than in four dimensions, where boundary conditions at the horizon and infinity are required to eliminate the Couch–Newman algebraically special perturbations [12].

A loophole in our result (and that of Couch and Newman) is that we have assumed that the perturbation is smooth on $K^{d-2}$ to perform the scalar/vector/tensor decomposition and expansion in harmonics. So we miss algebraically special perturbations which are not smooth on $K^{d-2}$. This includes the perturbation corresponding to turning on NUT charge or, in 4D, acceleration (i.e. the C-metric) [12]. It turns out that the former can be studied using our approach by considering singular eigenfunctions of the Laplacian on $K^{d-2}$. However, the singular perturbation corresponding to the 4D C-metric does not arise from a singular eigenfunction of the Laplacian on $S^2$. See [21] for a discussion of higher dimensional analogues of this perturbation.

This paper is organized as follows. Section 2 starts with the definition of algebraically special perturbations. Appendix A describes the formalism used to derive the condition for an algebraically special perturbation in a type D Einstein background. In section 3 we restrict our analysis to the generalized Schwarzschild black hole solutions. We find the solutions of the algebraically special condition in these backgrounds that also solve the Kodama–Ishibashi master equation, for the sensor, vector and scalar sector of perturbations. A final discussion of our conclusions is given in section 4.

2. Algebraically special perturbations

As discussed in the introduction, there are various inequivalent definitions of ‘algebraically special’ in higher dimensions [3–6]. We will see in this section how each of these definitions leads to the same necessary condition for a perturbation to be algebraically special.

Introduce a null basis $\{e_a\} = \{\ell, n, m_{(i)}\}$, with $a = 0, 1, \ldots, d - 1$, $i = 2, \ldots, d - 1$, which obeys the orthogonality relations,

$$\ell^2 = n^2 = m_{(i)} \cdot m_{(i)} = 0, \quad \ell \cdot n = 1, \quad m_{(i)} \cdot m_{(j)} = \delta_{ij}. \quad (2.1)$$

We will use Latin indices to label components of a tensor in the null basis and Greek letters for abstract indices. The following notation is used for certain Weyl tensor components [6]

$$\Omega_{ij} = C_{0i0j} = \ell^a m_{(i)}^b \ell^a m_{(j)}^c c_{ab \nu \mu}, \quad (2.2)$$

$$\Psi_{ijk} = C_{0i0jk} = \ell^a m_{(i)}^b m_{(j)}^c m_{(k)}^d c_{ab \rho \sigma}, \quad \Psi = \Psi_{ij}. \quad (2.3)$$
Following [4], the null vector $\ell$ is called a Weyl aligned null direction (WAND) if

$$\Omega_{ij} = 0.$$  \hspace{1cm} (2.5)

It is called a multiple WAND if

$$\Omega_{ij} = \Psi_{ijk} = 0.$$ \hspace{1cm} (2.6)

These definitions do not depend on how the other basis vectors are chosen [4]. In 4D, a WAND is the same as a principal null direction and a multiple WAND the same as a repeated principal null direction. Hence, in 4D, a spacetime is algebraically special if, and only if, it admits a multiple WAND. In $d > 4$ dimensions, some references define a spacetime to be algebraically special if it admits a WAND [4] and others require a multiple WAND [6]. The definition of [5] is stronger still: a solution which is algebraically special according to this definition must admit a multiple WAND and satisfy some additional conditions.

The Myers–Perry solution admits a multiple WAND, in fact it admits two distinct multiple WANDs [22, 23], which implies that it is type D in the classification of [4]. It is also algebraically special according to the definition of [5]. Of course the same remarks apply to the Schwarzschild solution.

To obtain a necessary condition for a perturbation to preserve the algebraically special property, consider a one-parameter family of algebraically special solutions specified by a parameter $\lambda$, such that for $\lambda = 0$, the solution is the Schwarzschild solution of mass $M$. A necessary condition for the family of solutions to be algebraically special according to any of the definitions of [4–6] is that there exists a WAND. We assume that this depends smoothly on $\lambda$. For $\lambda = 0$ this must reduce to one of the multiple WANDs of the Schwarzschild solution (since every WAND of Schwarzschild is a multiple WAND4). Hence, by choosing the basis vector $\ell^a$ to be this WAND, we have $\Omega_{ij} = 0$ for all $\lambda$.

Differentiating with respect to $\lambda$ and evaluating at $\lambda = 0$ gives

$$\delta \Omega_{ij} = 0.$$ \hspace{1cm} (2.7)

This is a necessary condition for a linearized perturbation of the Schwarzschild solution to be the linearization around Schwarzschild of a larger family of solutions admitting a WAND which depends smoothly on $\lambda$. We will define an algebraically special perturbation to be a solution of this equation. The LHS of this equation has the desirable property that it is gauge invariant under infinitesimal coordinate transformations and infinitesimal basis transformations [10].

Our assumption that the WAND depends smoothly on $\lambda$ is highly non-trivial. For example, in 4D, any metric admits a WAND. However, one can show that, in 4D, some components of a WAND (with respect to a basis depending smoothly on $\lambda$) generically behave as $\sqrt{\lambda}$ as $\lambda \to 0$ and hence are not differentiable with respect to $\lambda$ at $\lambda = 0$.

We will now show that the WAND does depend smoothly on $\lambda$ if it satisfies the additional (basis-independent) condition $\Psi_i = 0$. Introduce a null basis $\{\tilde{\ell}, \tilde{n}, \tilde{m}_i\}$ which depends smoothly on $\lambda$ with the property that $\tilde{\ell}(0)$ is a multiple WAND of Schwarzschild. Now expand the WAND $\ell(\lambda)$ in this basis: $\ell^a = x(\tilde{\ell}^i + z_i \tilde{m}^a - (1/2)z_i z_j \tilde{n}^a)$ where $x, z_i$ are functions of $\lambda$ with $x(0) = 1$, $z_i(0) = 0$ (and we have used the fact that $\tilde{\ell}^a$ is null). Without loss of generality we can set $x \equiv 1$. Then $\ell^a$ is related to $\tilde{\ell}^a$ by a null rotation about $\tilde{n}^a$ with parameters $z_i$. Using the transformation properties of $\Psi_i$ under a null rotation [6] we have

$$\Psi_i = \tilde{\Psi}_i - \tilde{\Phi} z_i + 3 \tilde{\Phi}_{j}^{A} z_{j} - \tilde{\Phi}_{ij}^{S} z_{j} + O(z^2),$$ \hspace{1cm} (2.8)

4 Any null vector can be related to one of the multiple WANDs by a ‘null rotation’. But one can show that the only such rotation that preserves the WAND condition is the trivial one.
where the overbar refers to the smooth basis. If we assume that our WAND obeys $\Psi_t = 0$ then the implicit function theorem implies that $z_i$ must depend smoothly on $\lambda$ in a neighbourhood of $\lambda = 0$. We just have to check that $\det \partial_x \Psi_t \neq 0$ in a neighbourhood of $\lambda = 0$, $z_i = 0$. This is indeed true because, at $\lambda = 0$, $z_i = 0$ we have

$$
\partial_x \Psi_t = -\Phi \delta_{ij} + 3 \bar{\Phi}_{ij}^A - \Phi_{ij}^S = -\frac{d-1}{d-2} \Phi \delta_{ij}, \tag{2.9}
$$

where on the RHS we made use of the relations $\bar{\Phi}_{ij}^S = (\Phi/(d-2)) \delta_{ij}$ and $\Phi_{ij}^S = 0$ which are satisfied in the Schwarzschild spacetime. The determinant of the RHS is non-zero (because $\Phi \neq 0$ in the Schwarzschild solution) and so the result follows.

We have shown that if our family of solutions admits a WAND satisfying the extra conditions $\Psi_t = 0$ then the WAND depends smoothly on $\lambda$ and hence the resulting linearized perturbation of Schwarzschild will obey (2.7). There is a partial converse to this. Assume that we have a linearized perturbation of Schwarzschild which satisfies (2.7). Then equation (2.8) linearized in $z_i$ shows that we can perform an infinitesimal null rotation to set $\delta \Psi_t = 0$, which is the linearized version of $\Psi_t = 0$.

In 4D, the condition $\Psi_t = 0$ is equivalent to $\Psi_{ijk} = 0$. So in 4D, existence of a WAND with $\Psi_t = 0$ is the standard condition for the family of solutions to be algebraically special. In $d \geq 4$, $\Psi_t = 0$ is weaker than the multiple WAND condition $\Psi_{ijk} = 0$. In the classification of [4], existence of a WAND with $\Psi_t = 0$ is the condition for a solution to be type I(a) or more special. Hence any one-parameter family of solutions that is type I(a) or more special will give a solution of (2.7) when linearized around the Schwarzschild solution.

So far we have discussed the notions of algebraically special arising in [4–6]. We will now discuss the definition of [3]. The latter definition is restricted to $d = 5$ and is based on a spinorial classification of the Weyl tensor. In this definition, the basic object is the Weyl spinor $P_{ABCD} = (\text{CT}^{ab})_{AB}(\text{CT}^{cd})_{CD}C_{abcd}$, where $C$ denotes the charge-conjugation matrix and spinor indices $A, B, C, D$ take values from 1 to 4. $P_{ABCD}$ is totally symmetric [3] so contracting with a Dirac spinor $\psi^A$ gives $P(\psi) \equiv P_{ABCD}\psi^A \psi^B \psi^C \psi^D$, a homogeneous polynomial of degree 4 depending on the four complex components of $\psi^A$. Reference [3] classified the Weyl tensor according to whether, and how, this polynomial factorizes into polynomials of lower degree. An algebraically special solution is one for which the polynomial factorizes.

The Myers–Perry and Schwarzschild solutions are algebraically special: $P(\psi)$ is the square of a quadratic polynomial [24]. This means that these solutions are type 22 in the classification of [3]. Now, as above, consider a family of solutions that are algebraically special according to this definition, and reduce to Schwarzschild when $\lambda = 0$. Reference [25] showed that the possible factorizations of $P(\psi)$ are restricted by a reality condition. The only allowed factorization that can reduce to the square of a quadratic polynomial when $\lambda = 0$ corresponds to $P(\psi)$ being the product of two quadratic factors. If these are distinct then this corresponds to type 22 in the classification of [3].

The coefficients of $P(\psi)$ depend smoothly on $\lambda$. However, we cannot expect the coefficients of the quadratic factors to be differentiable with respect to $\lambda$ at $\lambda = 0$. This is because the quadratic factors become coincident at $\lambda = 0$. For example, write $\psi^A = (w, x, y, z)$ and consider $P(\psi) = (w^2 + \sqrt{\lambda}(x^2 + y^2 + z^2))(w^2 - \sqrt{\lambda}(x^2 + y^2 + z^2))$: the coefficients of $P$ are smooth at $\lambda = 0$ but the coefficients of the quadratic factors are not. However, if our family is of type 22 for all $\lambda$ then $P(\psi) = Q(\psi)^2$ for some quadratic polynomial $Q$ and the coefficients of $Q$ will be smooth functions of $\lambda$, even at $\lambda = 0$. To see this, pick $\psi_0$ such that $P(\psi_0) \neq 0$ at $\lambda = 0$. In a neighbourhood of $\psi = \psi_0$, $\lambda = 0$ we have $P(\psi) \neq 0$ and then $Q(\psi) = \sqrt{P(\psi)}$ implies that the coefficients in $Q$ depend smoothly on $\lambda$ at $\lambda = 0$. 
The Weyl tensor of a type 22 solution can be written as an expression quadratic in an antisymmetric tensor $A_{\mu\nu}$ [25] which is constructed linearly from $Q$ and hence depends smoothly on $\lambda$. If one introduces a null basis as above then one has [25]

$$\Omega_{ij} = A_{0k}A_{0\ell}\delta_{ij} - 3A_{0[\mu}A_{\nu]}_{|0],}.$$  

(2.10)

Linearizing around the Schwarzschild background gives

$$\delta\Omega_{ij} = 2A_{0k}\delta A_{0\ell}\delta_{ij} - 6A_{0[\mu}\delta A_{\nu]}_{|0],}.$$  

(2.11)

where $A_{0\mu}$ is evaluated in the Schwarzschild geometry. Let us now choose our null basis as before, i.e. so that $\ell^a$ is a (multiple) WAND of Schwarzschild when $\lambda = 0$. This implies that $A_{0\mu} = 0$ [25] so (2.11) reduces to (2.7). Hence (2.7) is a necessary condition for a perturbation to correspond to the linearization about Schwarzschild of a family of type 22 solutions.

In summary, we have shown that (2.7) is a necessary condition for a linearized perturbation of the Schwarzschild solution to arise from a one-parameter family of solutions which is type I(α) or more special in the classification of [4] (as will be the case if it is algebraically special according to the definitions of [5, 6]) or, in 5D, is of type 22 in the classification of [3].

As mentioned in the introduction, equation (2.7) arises also in the study of general perturbations of algebraically special solutions. For perturbations of a solution admitting a multiple WAND (e.g. Myers–Perry), $\delta\Omega_{ij}$ is gauge invariant under infinitesimal coordinate transformations and infinitesimal basis transformations [10]. This makes $\delta\Omega_{ij}$ a natural quantity to consider when studying perturbations of such a solution. Furthermore, since $\delta\Omega_{ij}$ is a $(d-2) \times (d-2)$ traceless symmetric matrix, it has the same number of degrees of freedom as the gravitational field and so it seems likely that the metric perturbation could be reconstructed from knowledge of $\delta\Omega_{ij}$. However, $\delta\Omega_{ij}$ determines the metric perturbation only up to addition of a solution of (2.7). Hence it is desirable to determine all solutions of (2.7) in order to determine the extent to which $\delta\Omega_{ij}$ uniquely characterizes the metric perturbation.

Our definition of an algebraically special perturbation as a solution of (2.7) requires choosing our basis vector $\ell^a$ to reduce to one of the multiple WANDs of Schwarzschild when $\lambda = 0$. But there are two such multiple WANDs. A perturbation which is algebraically special with respect to one choice generically will not be special with respect to the other choice. However, the two multiple WANDs of Schwarzschild are related by the time-reversal isometry. Hence by applying this isometry to our solutions we will ensure that we do not miss any algebraically special perturbations. For $d = 4$ the result of doing this is described in footnote 3.

The calculation of $\delta\Omega_{ij}$ for a general metric perturbation of a type D Einstein spacetime is explained in appendix A.

### 3. Kodama–Ishibashi decomposition

#### 3.1. Introduction

Our background geometry is the generalized Schwarzschild solution in $d$ dimensions. In ingoing Eddington–Finkelstein coordinates, this is

$$ds^2 = -f(r) \, dv^2 + 2 \, dv \, dr + r^2 \gamma_{ij} \, dx^i \, dx^j \quad \text{with} \quad f(r) = K - \lambda c r^2 - \frac{\rho d-3}{r d-3}. \quad (3.1)$$

This result generalizes to a family of type 22 if one assumes that the quadratic polynomials depend smoothly on $\lambda$. However, as discussed in the text, this assumption is not expected to be generally valid.

**Footnote 3:**

If the metric perturbation is of $\text{type~}I(a)$ or more special in the classification of [4], then a perturbation which is algebraically special with respect to one choice of WAND will not be special with respect to the other choice. However, the two multiple WANDs of Schwarzschild are related by the time-reversal isometry. Hence by applying this isometry to our solutions we will ensure that we do not miss any algebraically special perturbations. For $d = 4$ the result of doing this is described in footnote 3.

The calculation of $\delta\Omega_{ij}$ for a general metric perturbation of a type D Einstein spacetime is explained in appendix A.
Here $\gamma_{ij}$ is the metric on a $(d - 2)$-dimensional compact Einstein manifold $K^{d-2}$ with Ricci tensor $K(d - 3)\gamma_{ij}$ where $K \in [0, \pm 1]$. $r_\infty$ is a mass parameter while $\lambda_c$ is the cosmological constant parameter appearing in the Einstein equation $R_{\mu\nu} = (d - 1)\lambda_c g_{\mu\nu}$.

The standard Schwarzschild–(de Sitter) solution has $K = 1$ and $K^{d-2} = S^{d-2}$. If $\lambda_c < 0$, the solutions with $K = 0$ and $K = -1$ are also regular black holes. These include the planar and hyperbolic AdS–Schwarzschild black holes, with $T^{d-2}$ and (compactified) $H^{d-2}$ horizon topology, respectively.

We will sometimes write the metric in the form
\[
  ds^2 = g_{AB} dx^A dx^B + r^2 \gamma_{ij} dx^i dx^j, \tag{3.2}
\]
where $g_{AB}$ is the Lorentzian metric of the two-dimensional orbit spacetime.

In our computations we will use the null basis
\[
  \ell = dv, \quad n = dr - \frac{1}{f} dv, \quad m_i = r \hat{e}_i, \tag{3.3}
\]
where $\hat{e}_i$ are a vielbein for the metric $\gamma_{ij}$. Note that $\ell$ and $n$ are the multiple WANDs of Schwarzschild.

We can decompose perturbations according to how they transform under diffeomorphisms of $K^{d-2}$. This decomposition was worked out by Kodama and Ishibashi [16]. An arbitrary metric perturbation $h_{\mu\nu}$ can be decomposed into perturbations of scalar, transverse vector, and transverse traceless tensor types on $K^{d-2}$. Note that in performing this decomposition we exploit our assumption that $K^{d-2}$ is compact, and assume that $h_{\mu\nu}$ is regular on $K^{d-2}$.

Kodama and Ishibashi showed that each type of perturbation can be encoded in a gauge invariant master variable for a tensorial perturbation is
\[
  \delta \Omega_{ij} = 0 \quad \text{for a general metric perturbation of a type D Einstein spacetime is given in (A.15) of appendix A. In the Schwarzschild background it simplifies considerably. With the harmonic decomposition we can write it in terms of the gauge invariant master variable and it reduces to the vanishing of a product of two factors. One factor depends only on the harmonic and its derivatives. The other contribution depends only on the two-dimensional orbit spacetime coordinates $v, r$. Therefore, to have $\delta \Omega_{ij} = 0$, one of these two factors must vanish. Typically we will find that it is the orbit spacetime factor that constrains our search, although we will also encounter special cases where the algebraically special condition is automatically obeyed because the contribution from the harmonic vanishes (these cases describe the perturbations that just shift the mass and angular momentum).}

\section{Tensor perturbations}

Tensor-type perturbations have the form [17, 18]
\[
  h_{AB} = 0, \quad h_{Ai} = 0, \quad h_{ij} = 2i^2 H_T T_{ij}, \tag{3.4}
\]
where $H_T = H_T(v, r)$, and $T_{ij}(x^k)$ are eigentensors obeying
\[
  (\Delta_L - \lambda_L) T_{ij} = 0, \quad \text{with} \quad \Delta_L T_{ij} = -D^2 T_{ij} - 2R_{ikjl} T^{kl} + 2(d - 3)K T_{ij}, \tag{3.5}
\]
where $\Delta_L$ is the Lichnerowicz operator, $\lambda_L$ is the associated eigenvalue, $D$ is the derivative defined by the metric $\gamma_{ij}$ of the Einstein base space $K^{d-2}$, and $R_{ikjl}$ is the associated Riemann tensor [17, 18].

The Kodama–Ishibashi gauge invariant master variable for a tensorial perturbation is
\[
  \Phi_T = \epsilon^{(d-2)/2} H_T. \tag{3.6}
\]
It obeys the master equation [17, 18]
\[
\left( \Box_2 - \frac{V_T}{f} \right) \Phi_T = 0, \quad V_T = \frac{f}{r^2} \left[ \frac{d(d-2)}{4} f + \frac{(d-1)(d-2)r^d_\text{m}}{2r^{d-3}} + \lambda_L - (3d - 8)K \right],
\] (3.7)
where \(\Box_2\) is the d’Alembertian operator in the two-dimensional orbit spacetime with metric \(g_{AB}\).

Written in terms of the master variable we find that the algebraically special condition is
\[
\delta \Omega_{ij} = 0 \quad \Leftrightarrow \quad \partial^2_r (r^{-\lambda} \Phi_T) + \frac{2}{r} \partial_r (r^{-\lambda} \Phi_T) = 0,
\] (3.8)
with solution
\[
\Phi_T = r^{-\lambda} \left( \frac{H_1(v)}{r} + H_2(v) \right)
\] (3.9)
for arbitrary functions \(H_1\) and \(H_2\) of the Eddington–Finkelstein coordinate \(v\). Inserting this expression for \(\Phi_T\) into the master equation (3.7) we find that the latter reduces to a polynomial in \(r\):
\[
(d - 2)[H'_2(v) + \lambda_L H_1(v)]r^{d-1} + [(d - 4)H'_1(v) - (\lambda_L - 2(d - 3)K)H_2(v)]r^{d-2} - [\lambda_L - (d - 2)K]H_1(v) r^{d-3} - r^d_\text{m} H_1(v) = 0.
\] (10.10)
For \(\lambda_L \neq 2(d - 3)K\), only the trivial solution \(H_1(v) = 0 = H_2(v)\) satisfies this equation (we assume \(r_\text{m} \neq 0\)). For \(\lambda_L = 2(d - 3)K\), the general solution is \(H_1 = 0, H_2 = \text{constant}\). In summary, the algebraically special tensorial perturbations are given by:
\[
\lambda_L = 2(d - 3)K, \quad \Phi_T = H_2 r^{-\lambda_2}.
\] (11.11)
Tensor harmonics with \(\lambda_L = 2(d - 3)K\) are linearized perturbations of the metric on \(\mathcal{K}^{d-2}\) which preserve the Einstein condition (and the volume of \(\mathcal{K}^{d-2}\)). So algebraically special tensor perturbations exist if, and only if, \(\mathcal{K}^{d-2}\) admits such perturbations. If \(\mathcal{K}^{d-2}\) is a space of constant curvature then such perturbations exist only if either (i) \(K = 0\) or (ii) \(d = 4\) and \(K = -1\) (see e.g. [20]).

3.3. Vector perturbations

Vector perturbations are constructed out of vector harmonics \(V_i(x^k)\) [16]
\[
h_{AB} = 0, \quad h_{\alpha i} = rf_A V_i, \quad h_{ij} = -\frac{2}{k^r} H_T D_i V_j,
\] (12.12)
where \(f_A\) and \(H_T\) are functions of \(\{x^k\} = \{v, r\}\), and \(V_i\) denotes a transverse vector harmonic on \(\mathcal{K}^{d-2}\):
\[
D_i V^i = 0, \quad (D^2 + k_i^2) V_i = 0.
\] (13.13)
If \(\mathcal{K}^{d-2} = S^{d-2}\) then the eigenvalues are
\[
k_i^2 = l(l + d - 3) - 1, \quad l = 1, 2, \ldots
\] (14.14)
Harmonics with \(k_i^2 = (d - 3)K\) \((l = 1\) above) are special: they are Killing vectors on (compact) \(\mathcal{K}^{d-2}\), occurring only for \(K = 0, 1\). The cases \(k_i^2 \neq (d - 3)K\) and \(k_i^2 = (d - 3)K\) are described by different gauge invariant quantities and thus we will discuss them separately in the following two subsections.

Vectorial gauge transformations, \(h_{\mu\nu} \rightarrow h_{\mu\nu} + \mathcal{L}_\xi g_{\mu\nu}\), are generated by an infinitesimal gauge vector \(\xi\) with harmonic decomposition [16],
\[
\xi = rL(x^k) V_i \, dx^i.
\] (15.15)
3.3.1. Modes with $k^2_l \neq (d - 3)K$. If $K^{d-2} = S^{d-2}$ then this corresponds to $l > 1$.

A master variable $\Phi_V$ can be constructed in terms of a variable $F_A$ which is a gauge invariant combination of $f_A$ and $H_T$,

$$F_A \equiv f_A + \frac{r}{k^2_V} D_s H_T = r^{-(d-3)} \epsilon_{AB} D^B (s^{(d-2)/2} \Phi_V), \quad (3.16)$$

where $\epsilon_{AB}$ denotes the volume form on the 2D orbit spacetime. The quantity $\Phi_V$ obeys the Kodama–Ishibashi master equation [16]

$$\left(\square - \frac{V_V}{f}\right) \Phi_V = 0, \quad V_V = \frac{f}{r^2} \left[ k^2_V - (d-3)K + \frac{d(d-2)}{4} f - \frac{d-2}{2} rf' \right]. \quad (3.17)$$

We find that the algebraically special condition is

$$\delta \Omega_{ij} = 0 \iff \partial_r^2 \Phi_V + \frac{2}{r} \partial_r \Phi_V - \frac{(d-2)(d-4)}{4} \frac{\Phi_V}{r^2} = 0, \quad (3.18)$$

with solution

$$\Phi_V = \frac{C_0(v)}{r^{(d-3)/2}} + C_1(v) r^{(d-4)/2} \quad (3.19)$$

for arbitrary functions $C_{0,1}(v)$ of the ingoing Eddington–Finkelstein coordinate $v$. Inserting this into the master equation (3.17) gives a polynomial in $r$:

$$-(d-4)C'_1(v) r^{d-2} + (k^2_v + (d-3)K) C_1(v) r^{d-3} + (d-2)C'_0(v) r + \frac{[(k^2_v - (d-3)K) C_0(v) - (d-1)(d-3)r^{d-3} C_1(v)] = 0. \quad (3.20)$$

For $d \geq 5$ this polynomial involves four distinct powers of $r$. However, $d = 4$ is special because then the polynomial has degree 1. For this reason, we analyse the $d = 4$ and $d \geq 5$ cases separately.

(1) Case $d = 4$. For $d = 4$, equation (3.20) reduces to

$$- \left[ (K - k^2_v) C_0(v) + 3r_m C_1(v) \right] + r \left[ (K + k^2_v) C_1(v) + 2C'_0(v) \right] = 0, \quad (3.21)$$

which can be satisfied only if the two coefficients of the polynomial vanish independently. This gives

$$C_0(v) = -\frac{3r_m C_1(v)}{K - k^2_v}, \quad C_1(v) = A_0 (K - k^2_v) e^{-(k^2_v - K^2)v/(6r_m)}, \quad (3.22)$$

for an arbitrary constant $A_0$ (recall that modes with $k^2_v = (d-3)K = K$ are excluded from the analysis of this subsection). The associated metric perturbations can be reconstructed using (3.16) once a gauge is chosen. This map simplifies if we take advantage of the gauge transformation described by (3.15) to pick the gauge $H_T = 0$. The perturbation just obtained is then

$$f_v = -f_r \left( f(r) + \frac{(K + k^2_v)}{6r_m} \left[ (K - k^2_v) r - 3r_m \right] \right), \quad f_r = -\frac{A_0 (K - k^2_v)}{r} \frac{e^{-(k^2_v - K^2)v/(6r_m)}}{r}. \quad (3.23)$$

Recall that $f(r)$ is the function appearing in the Schwarzschild metric.

We have found an infinite class of algebraically special perturbations labelled by the eigenvalue $k^2_v$. In general, these perturbations decay exponentially with $v$ and blow up on the past horizon\(^6\). For $K = 1$ and $\Lambda = 0$ these perturbations were first identified by Couch and Newman [13]. They correspond to the linearization about the Schwarzschild

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\({^6}\) As discussed at the end of section 2, one obtains another solution by applying the time reversal isometry. This solution grows exponentially with the retarded time coordinate $u$ and diverges on the future horizon.
solution of the class of algebraically special solutions for which the repeated principal null direction has non-vanishing rotating.

Interestingly, the above perturbation (3.22) becomes time independent when \( k_0^2 = -K \) (recall \( k_0^2 = K \) is excluded from this section). Indeed, (3.22) and (3.23) reduce to

\[
C_0(v) = -\frac{3\mu_0 C_N}{2K}, \quad C_1(v) = C_N;
\]

\[
f_v = \frac{C_N f}{r}, \quad f_r = -\frac{C_N}{r}, \quad H_T = 0, \tag{3.24}
\]

where we redefined the arbitrary constant of the problem as \( C_N = 2KA_0 \). The interpretation of this perturbation has been discussed in [20].

If \( K = 1 \) then \( k_0^2 = -K = -1 \) cannot correspond to a regular vector harmonic on \( S^2 \). For \( K = -1 \), we will show at the end of this section that if the harmonic is regular then the perturbation is locally pure gauge. However one can find non-trivial singular harmonics with this eigenvalue (which corresponds to setting \( l = 0 \) in (3.14)) [20]. To see this, let \((\theta, \phi)\) be spherical polars on \( S^2 \). An example of a vector harmonic on \( S^2 \) with \( k_0^2 = -1 \) is \( \mathbf{V} = \cos \theta \mathbf{d} \phi \). This is singular at \( \theta = 0, \pi \).

This perturbation adds NUT charge, proportional to the NUT parameter \( N \equiv C_N/2 \), to the Schwarzschild black hole [20]. To confirm this statement, recall that the (A)dS–Taub–NUT solution is described by the line element [15]

\[
ds^2 = -g(r)(dt + 2N \cos \theta \, d\phi)^2 + \frac{dr^2}{g(r)} + (r^2 + N^2) \, d\Omega_2^2,
\]

with \( g(r) = \frac{r^2 - 2Mr - N^2 - \lambda_\alpha(r^4 + 6N^2r^2 - 3N^2)}{r^2 + N^2} \).

Converting to Eddington–Finkelstein coordinates and linearizing in \( N \) reproduces the perturbation just discussed. A similar result holds for \( K = -1 \).

(2) Case \( d > 5 \): addition of NUT charge. For \( d \geq 5 \), there are no time-dependent solutions of equation (3.20). The only solution is

\[
k_0^2 = -(d - 3)K, \quad C_0(v) = -\frac{(d - 1)(d - 3)\kappa_{GN}}{2(d - 3)K}, \quad C_1(v) = C_N, \tag{3.25}
\]

for arbitrary constant \( C_N \) and \( K = \pm 1 \). This solution is the natural higher dimensional generalization of (3.24). In the gauge \( H_T = 0 \), the associated metric perturbation is \( f_v = C_N f/r, f_r = -C_N/r \). Converting to Schwarzschild coordinates this gives

\[
h_{\mu\nu} \, dx^\mu \, dx^\nu = 2C_N fV_t \, dt \, dx^t. \tag{3.26}
\]

For even \( d \), taking \( K^{d-2} \) to be a positive Kähler–Einstein space, this perturbation corresponds to the linearization of the higher-dimensional Taub–NUT solution of [26]. For general \( d > 5 \), taking \( K^{d-2} \) to be a product \( \mathcal{K}_1^{2n} \times \mathcal{K}_2^{d-2-2n} \) where \( \mathcal{K}_1^{2n} \) is Kähler–Einstein and \( \mathcal{K}_2^{d-2-2n} \) is Einstein, this perturbation corresponds to the linearization of the higher-dimensional Taub–NUT solutions of [27, 28]. For \( d = 6 \), the algebraic type of some of these solutions was discussed in [29], where they were shown to be type D in the classification of [4].

This perturbation does not correspond to a regular harmonic on \( K^{d-2} \). This is obvious for \( K = 1 \) because it has \( k_0^2 < 0 \) and hence must be singular on \( K^{d-2} \). For \( K = -1 \), note that \( k_0^2 = -(d - 3)K \) implies that the 1-form \( V_t \) is harmonic with respect to the Hodge–de Rham Laplacian. If \( V_t \) is assumed regular on compact \( K^{d-2} \), this implies that \( V_t \) is closed so locally on \( K^{d-2} \) we can write \( V_t = \partial_\alpha \) for some function \( \alpha \). But then from (3.26) one sees that the perturbation can be gauged away by a shift \( t \rightarrow t + C_N \alpha \). So if the perturbation is regular on \( K^{d-2} \) then it is locally trivial. This is not the case for the Taub–NUT perturbation just discussed so it does not correspond to a regular harmonic on \( K^{d-2} \).
3.3.2. Vector modes with $k_S^2 = (d - 3)K$. Addition of angular momentum. Considering only regular harmonics on $\mathcal{K}^{d-2}$, we must have $K = 0, 1$ and these harmonics are Killing vector fields so there is no $H_T$ contribution in (3.12). The unique gauge invariant variable in this case is $F_{AB}$. Its definition, the master equation it must obey, and the respective solution are [16]

$$F_{AB} = 2 e D_i (r^{-1} f_i B), \quad D^B (\rho^{d-1} F_{AB}) = 0 \iff F_{AB} = C_j \rho^{-d-1},$$

(3.27)

where $D$ is the covariant derivative in the two-dimensional orbit spacetime with metric $g_{AB}$.

$f_A = \{ f_i, f_i \}$ is defined in (3.12), $\epsilon_{AB}$ denotes the anti-symmetric tensor in the orbit spacetime, and $C_j$ is an arbitrary integration constant.

Recall that after the harmonic expansion, $\delta \Omega_{ij}$ is given by the product of an orbit space and a base space contributions. For the regular perturbations with $k_S^2 = (d - 3)K$, the algebraically special condition $\delta \Omega_{ij} = 0$ is trivially obeyed because its base space factor vanishes.

By a choice of gauge, the algebraically special perturbation (3.27) can be written in the form (3.12) with (no $H_T$ contribution)

$$f_i = \frac{C_j}{(d - 1)\rho^{d-2}}, \quad f_i = 0.$$  

(3.28)

For $K = 1$, this perturbation corresponds to adding angular momentum. For $\mathcal{K}^{d-2} = S^{l-2}$ it arises from the Kerr–Myers–Perry–(A)dS black hole [9, 30–32] linearized for small angular momentum parameter(s) [16]. For $K = 0$, the only regular Killing vectors on compact $\mathcal{K}^{d-2}$ are translations (i.e. covariantly constant) and this perturbation corresponds to a boost along a translationally invariant direction.

3.4. Scalar perturbations

Scalar perturbations are given by

$$h_{AB} = f_{AB} S_i, \quad h_{Ai} = r f_A S_i, \quad h_{ij} = 2 r^2 (H_L \gamma_{ij} S + H_T S_{ij}),$$

(3.29)

with $f_{AB}, f_A, H_T, H_L$ functions of $\{ x^4 \} = \{ v, r \}, \gamma_{ij}$ is the base space $\mathcal{K}^{d-2}$ metric. $S$ is the scalar harmonic which satisfies the eigenvalue equation

$$(D^2 + k_S^2) S = 0.$$  

(3.30)

Out of this scalar harmonic we can construct a scalar-type vector harmonic $S_i$ and a traceless scalar-type tensor harmonic $S_{ij}$ as (for $k_S^2 \neq 0$)

$$S_j = -\frac{1}{k_S^2} D_i S, \quad S_{ij} = \frac{1}{k_S^2} D_i D_j S + \frac{1}{d - 2} \gamma_{ij} S.$$  

(3.31)

For $\mathcal{K}^{d-2} = S^{l-2}$ the eigenvalues of (3.30) are

$$k_S^2 = l(l + d - 3), \quad l = 0, 1, \ldots.$$  

(3.32)

We must distinguish two special cases [16]. In the first special case $k_S^2 = 0$, i.e. constant $S$, we define $S_{ij} = S_i = 0$. These modes preserve the symmetry of the background solution. Birkhoff’s theorem implies that the only such solution arises from variation of the mass parameter.

The second special case $k_S^2 = (d - 2)K \neq 0$ is possible only for $K = 1$. In this case, $S_{ij} = 0$ so $S_i$ is a conformal Killing vector (but not a Killing vector) on $\mathcal{K}^{d-2}$. This happens only for $\mathcal{K}^{d-2} = S^{l-2} [33]$, for which it corresponds to $l = 1$ perturbations. For $d = 4$, such perturbations are known to be pure gauge. We will show below that the same is true for $d > 4$. First we treat the case $k_S^2 \neq 0, (d - 2)K$. 


3.4.1. Modes with \( k_3^2 \neq 0, (d - 2)K \). For \( K^{d-2} = S^{d-2} \) this case corresponds to assuming \( l \geq 2 \).

Scalar perturbations can be expressed in terms of a single gauge invariant scalar \( \Phi_S \) whose definition can be found in [16]. This quantity obeys the Kodama–Ishibashi master equation [16],

\[
\left( \Box_2 - \frac{V_S}{f} \right) \Phi_S = 0, \quad V_S = \frac{f(r)Q(r)}{16r^2[\gamma + (d - 1)(d - 2)x/2]^2},
\]

where \( \Box_2 \) is the d’Alembertian defined by \( g_{AB} \) and

\[
x = \frac{r^d}{r^d - 3}, \quad \gamma = k_3^2 - (d - 2)K,
\]

\[
Q(r) = -[d - 2]^{3}d(d - 1)^2x^2 - 12(d - 2)^2(d - 1)(d - 4)\gamma x + 4(d - 4)(d - 6)\gamma^2)r_c r^2
\]
\[
+ (d - 2)(d - 1)[4((2(d - 2)^2 - 3(d - 2) + 4)\gamma
\]
\[
+ (d - 1)(d - 2)(d - 4)(d - 6)K]x^2 + (d - 2)^2(d - 1)^2x^3
\]
\[
- 12(d - 2)(d - 4)\gamma x + (d - 1)(d - 2)(d - 4)\gamma x + 16\gamma^2 + 4d(d - 2)\gamma^2.
\]

(3.34)

In terms of the master variable \( \Phi_S \), we find that the algebraically special condition \( \delta \Omega_{ij} = 0 \) reads

\[
2r^2 f \partial_r \partial_r \Phi_S - \frac{r(2P_{2r} - Q_{x} + Q_{x})}{4H} \partial_r \Phi_S + \frac{r f(Q_{x} - Q_{x})}{4H} \partial_r \Phi_S + \frac{f(P_{1r} - P_{1r})}{16H^2} \Phi_S = 0,
\]

(3.35)

where \( P_{1r}, P_{1r}, P_{2r}, Q_{x}, Q_{x}, \) and \( H \) are functions of \( r \) that can be found in equation (3.10) of [16].

An algebraically special perturbation is a solution of (3.33) and (3.35). A combination of these equations yields a necessary but not sufficient condition for an algebraically special perturbation,

\[
\partial_{r}^{2} \Phi_{S} = \frac{d - 2}{r} \frac{(d - 4)(d - 1)r_{m}^{d - 3} - 2r^{d - 3}[k_{3}^{2} - (d - 2)K]}{(d - 2)(d - 1)r_{m}^{d - 3} + 2r^{d - 3}[k_{3}^{2} - (d - 2)K]} \partial_{r} \Phi_{S} + \frac{(d - 4)(d - 2)}{4r^{2}} \Phi_{S} = 0.
\]

(3.36)

We will solve this and then substitute into (3.33) and (3.35). Note the presence of \( (d - 4) \) factors in (3.36), namely, in the linear term in \( \Phi_S \) and in one of the contributions to \( \partial_r \Phi_S \). We thus anticipate that the \( d = 4 \) and \( d \geq 5 \) cases have distinct properties and we analyse them separately.

(1) Case \( d = 4 \): Robinson–Trautman perturbations. In this case, (3.36) has the general solution

\[
\Phi_{S} = \frac{C_{1}(v)}{(2K - k_{3}^{2})(3r_{m} - r(2K - k_{3}^{2}))} + C_{2}(v)
\]

(3.37)

for arbitrary functions \( C_{1}(v), C_{2}(v) \) (recall that modes with \( k_{3}^{2} = (d - 2)K = 2K \) are excluded in the study of this subsection). The requirement that the original equations (3.33) and (3.35) are obeyed fixes

\[
C_{1}(v) = -3r_{m}(2K - k_{3}^{2})C_{2}(v), \quad C_{2}(v) = Ae^{-\frac{[x - k_{3}^{2}]}{r_{m}}}v
\]

(3.38)

for some constant \( A \). Putting this together gives the solution

\[
\Phi_{S} = \frac{Ar(k_{3}^{2} - 2K)}{r(k_{3}^{2} - 2K) + 3r_{m}}e^{-\frac{[x - k_{3}^{2}]}{r_{m}}}v.
\]

(3.39)
The metric perturbation can be reconstructed in a particular gauge using the linear differential map \( h_{\mu \nu} = h_{\mu \nu}(\Phi_3) \) given in [16]. Note that (3.37)–(3.39) are independent of the cosmological constant, but the corresponding \( h_{\mu \nu}(\Phi_3) \) is not.

We have found an infinite class of algebraically special perturbations labelled by the eigenvalue \( k_2^2 \). These solutions vanish on the past horizon and grow exponentially with \( v \).\(^7\) For \( K = 1 \) and \( \Lambda = 0 \) these perturbations were first identified by Couch and Newman [13]. They correspond to the linearization around the Schwarzschild solution of the Robinson–Trautman class of algebraically special solutions.

(2) \( \text{Case } d \geq 5. \) For \( d \geq 5 \), the most general solution of (3.36) is

\[
\Phi_3 = \frac{2r^{d-2} [A_1(v) + rA_2(v)]}{2[(d-2)K]^{d-3} + (d-2)(d-1)r_{m}^{-3}},
\]

where \( A_1(v), A_2(v) \) are arbitrary functions. Plugging (3.40) into the original equations (3.33) and (3.35) leads to the trivial solution

\[
A_1(v) = 0 = A_2(v) \quad \Leftrightarrow \quad \Phi_3 = 0.
\]

Hence there are no algebraically special perturbations of scalar type with \( k_2^2 \neq 0, (d-2)K \) and \( d \geq 5 \).

3.4.2. \( l = 1 \) perturbations on \( S^{d-2} \). We now take \( K^{d-2} = S^{d-2} \) and consider the \( l = 1 \) scalar perturbations. For \( \lambda_c = 0, d = 4 \), it is known that the only such perturbations are pure gauge [34]. Here we will show that the same is true for any \( \lambda_c, d \) by generalizing the \( d = 4 \) argument as presented in [35].

The equation of motion (3.30) and the conditions \( S_{ij} = 0 \) enable us to express all the second derivatives of the scalar harmonic \( S \) with \( k_2^2 = (d-2) \) as a function of \( S \) and its first derivatives. Consequently, we can derive the results below without ever introducing the explicit expression for the \( l = 1 \) scalar harmonics.

An infinitesimal scalar gauge vector \( \xi \) can be decomposed in terms of scalar harmonics as

\[
\xi = P_0(x^\theta) S dx^4 + r L(x^\theta) S_i dx^i.
\]

Choose the gauge \( f_{rr} = f_r = H_L = 0 \) which is preserved by the gauge parameter

\[
P_c = (f - 1)\alpha_1(v) - r\alpha_2(v),
\]

\[
P_r = -\alpha_1(v),
\]

\[
L = k_2\{[\alpha_1(v) + r\alpha_2(v)],
\]

for arbitrary functions \( \alpha_1(v) \) and \( \alpha_2(v) \) of the advanced time \( v \). Under this gauge transformation the other components of the metric perturbation transform as

\[
f_{uu} \rightarrow \tilde{f}_{uv} = f_{uv} + [\alpha_1(v) + r\alpha_2(v)]f' - 2(1-f)\alpha_1'(v) - 2r\alpha_2'(v)
\]

\[
f_{ur} \rightarrow \tilde{f}_{ur} = f_{ur} - [\alpha_2(v) + \alpha_1'(v)]
\]

\[
f_u \rightarrow \tilde{f}_u = f_u + \frac{k_2}{r}[(1-f)\alpha_1(v) + r(\alpha_2(v) + \alpha_1'(v) + r\alpha_2'(v))],
\]

where the ‘ denotes differentiation w.r.t. to the argument of the function. Consider now the linearized Einstein equation \( E_{AB} = 0 \) (explicitly written in (A.1) of [16]) for this perturbation. The equation \( E_{rr} = 0 \) implies that \( f_{uv} \) is a function of \( v \) only and we can set it to zero with the

\(^7\) Time reversal as discussed at the end of section 2 gives a solution decaying exponentially with \( u \) and vanishing on the future horizon. This is the usual form of the Robinson–Trautman solutions.
gauge parameter choice $\alpha_2(v) = -\alpha_1'(v)$; see (3.44). In these conditions, equation $E_{ii} = 0$ is solved by

$$f_v = r\gamma_1(v) + \gamma_2(v)/r^{d-2}$$  \hspace{1cm} (3.45)

for arbitrary functions $\gamma_{1,2}(v)$. Equation $E_{ii} = 0$ then implies that

$$f_{vv} = \frac{1}{rd-2} \gamma_2(v) - \frac{1}{k_S} \left( 2r\gamma_1(v) - \frac{(d-3)}{rd-2} \gamma_2(v) + \frac{(d-1)}{rd-3} \gamma_2'(v) \right)$$  \hspace{1cm} (3.46)

where $\gamma_2(v)$ is a new arbitrary function of $v$. We must set $\gamma_2(v) = 0$ to solve $E_{ii} = 0$. Finally, $E_{vv} = 0$ implies

$$\gamma_1 = -\frac{(d-1)r^2 \gamma_2'' + [(d-3)(f-1) + rf']\gamma_2}{rd-1[2(1-f) + rf']}. \hspace{1cm} (3.47)$$

At this point, all components of the Einstein equation are obeyed. Define $\gamma(v)$ by

$$\gamma_2(v) = \frac{r^{d-3}}{d-1} [2(1-f) + rf'] \gamma(v), \hspace{1cm} (3.48)$$

We conclude that the only $l = 1$ scalar modes are

$$f_{vv} = \frac{1}{k_S} [2r\gamma''(v) - 2(1-f) + rf'] \gamma'(v) + f' \gamma(v),$$

$$f_{vr} = 0,$$

$$f_v = -r \gamma''(v) + \frac{1-f}{r} \gamma(v)$$  \hspace{1cm} (3.49)

in the gauge $f_{rr} = f_r = H_L = 0$. There is a remaining gauge freedom described by (3.43) and (3.44) with $\alpha_2(v) = -\alpha_1'(v)$. Setting the gauge parameter $\alpha_1(v) = \gamma(v)/k_S$ we find that this gauge transformation yields

$$\tilde{f}_{AB} = 0, \hspace{1cm} \tilde{f}_A = 0, \hspace{1cm} \tilde{H}_L = 0.$$  \hspace{1cm} (3.50)

Therefore, the $l = 1$ scalar modes can indeed be gauged away.

4. Discussion

To summarize: if there exists a family of vacuum solutions that is type I(a) (or more special) in the classification of [4], or type 22 in the 5D classification of [3], and contains the generalized Schwarzschild solution (3.1), then the linearization of this family about Schwarzschild yields a solution of the linearized Einstein equation which also satisfies $\delta/\Omega_{ij} = 0$. We have determined all such perturbations which are regular on (compact) $\mathcal{K}^{d-2}$. For $d = 4$ we find infinite classes of time-dependent perturbations corresponding to those discovered by Couch and Newman. However, for $d > 4$, the only perturbations that we find are those corresponding to variation of parameters in the Schwarzschild solution (i.e. the mass, and moduli of $\mathcal{K}^{d-2}$), or to turning on angular or linear momentum.

One of our motivations was to learn whether there exist new families of algebraically special solutions that contain the Schwarzschild solution. The answer is no, at least if one insists that the family be a smooth deformation of Schwarzschild. Algebraically special solutions appear to be much scarcer in higher dimensions than in $d = 4$, even if one adopts the definition of an algebraically special as one of type I(a), which is weaker than that of most recent work.

Special cases of our result follow from previous work on certain classes of solutions admitting multiple WANDs. Reference [36] determined all axisymmetric vacuum spacetimes admitting a multiple WAND. No non-trivial family of solutions containing Schwarzschild
was discovered. Recent work of [29] implies there exists no non-trivial family of vacuum solutions which admits a multiple WAND which is geodesic and non-twisting, and contains the Schwarzschild solution. To prove this, note that theorem 1.1 of [29] implies that any such family must be Robinson–Trautman. For $d = 4$, the Robinson–Trautman family contains a large class of time-dependent spacetimes. But the only $d > 4$ Robinson–Trautman solution with non-vanishing ‘mass function’ is the generalized Schwarzschild solution (3.1) [38].

It would be desirable to classify algebraically special perturbations without assuming that the perturbation is regular on $K^{d-2}$. However, this seems difficult even for $d = 4$. Reference [12] determined all perturbations of the Kerr solution that satisfy the linearization of the type D condition but without assuming regularity on $S^2$. However, the method relied heavily on the work of [2], in which all type D solution of the vacuum Einstein equation were determined. In higher dimensions we have no analogue of the analysis of [2].

Our result has significance for the study of general perturbations of higher-dimensional black holes. Reference [10] showed that $\delta \Omega_{ij}$ is gauge invariant for perturbations of any algebraically special vacuum solution. Since $\delta \Omega_{ij}$ is gauge-invariant and contains the same number of degrees of freedom as a generic metric perturbation, it was suggested that it should be possible to reconstruct a metric perturbation from the corresponding $\delta \Omega_{ij}$ up to the freedom to add modes corresponding to variation of parameters in the background solution. Our results show that this is indeed the case for perturbations of Schwarzschild: if two metric perturbations lead to the same $\delta \Omega_{ij}$ then their difference has $\delta \Omega_{ij} = 0$ and therefore, by our results, corresponds to a variation of parameters (allowing for a change in angular momenta or a boost).

For $d = 4$, $\Lambda = 0$ it is known that the Couch–Newman algebraically special perturbations are closely related to quasinormal (QN) modes with purely imaginary frequency (see [39] for a discussion). Our results show that this relation does not extend to $\Lambda < 0$ (allowing for black holes with $K = 0, -1$). Our $\Lambda < 0$ generalizations of the Couch–Newman modes have purely imaginary frequencies, but they are not QN modes because they do not satisfy the appropriate (normalizable) boundary conditions at infinity. Conversely, there exist QN modes with purely imaginary frequency [40] but these frequencies differ from those of our algebraically special modes. Hence for $\Lambda < 0$ it appears that there is no relation between algebraically special perturbations and QN modes with purely imaginary frequency. For $d > 4$, $\Lambda = 0$ we have found no time-dependent algebraically special perturbations and numerical work indicates that purely imaginary QN modes also do not exist [41]. However, we know of no reason why these observations should be related.

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Appendix A. Perturbations of a type D Einstein spacetime

We use the higher-dimensional generalization of the GHP formalism [42], which was developed in [6]. We ask the reader to see section 2 of [6] for the GHP notation and properties required to follow the derivation of this appendix.

We are interested in linearized gravitational perturbations of a type D Einstein spacetime. Such a geometry is defined by the conditions

\[ \Omega_{ij}^{(0)} = 0, \quad \Omega_{ij}^{(0)} = 0, \quad \Psi_{ijk}^{(0)} = 0, \quad \Psi_{ijk}^{(0)} = 0, \quad \Psi_{ij}^{(0)} = 0, \quad \Psi_{ij}^{(0)} = 0, \]  

(A.1)

\[ R_{\alpha\beta} = \frac{2\lambda_c}{d-2} g_{\alpha\beta}. \]  

(A.2)

For a quantity \( X \), we shall write \( X = X^{(0)} + \delta X \) where \( X^{(0)} \) is the value in the background spacetime and \( \delta X \) is the perturbation. As described in section 2, we want to find the expression for \( \delta \Omega_{ij} \) (the perturbation in \( \Omega_{ij} \)) which is gauge invariant under infinitesimal coordinate and basis transformations. An algebraically special perturbation obeys (2.7), i.e. \( \delta \Omega_{ij} = 0 \).

The variation \( \delta \Omega_{ij} \) includes two main contributions, one that comes from the variation of the basis under a perturbation, and the other that is due to the variation of the Weyl tensor itself,

\[ \delta \Omega_{ij} = \delta C_{00ij} = \delta (\ell^a m_{ij})^\beta \ell^\mu m_{ij}^\nu C_{\alpha\beta\mu\nu} = 2 C_{\alpha(0)ij} \delta \ell^a + (\delta C)_{00ij} \]  

(A.3)

where in the last equality we used the symmetries of the Weyl tensor and we assumed that the background is Petrov type D. We use the notation \( (\delta C)_{00ij} \equiv \ell^a m_{ij})^\beta \ell^\mu m_{ij}^\nu \delta (C_{\alpha\beta\mu\nu}) \). Also, we use \( \delta \ell^a \equiv \delta (\ell^a) \) to represent the variation of the vector \( \ell^a \). Then, \( \delta \ell_a \equiv \eta_{ab} \delta \ell^b \) and recall that \( \delta \ell^a = e^a_{\beta} \delta \ell^\beta \). 10 To find the variation of a vector of the null basis we vary the expression for the background metric in terms of the null basis vectors,

\[ -h^{\mu\nu} \equiv \delta g^{\mu\nu} = \delta (2 \ell^a n^{\nu} + \delta j_{ij} m_{ij}^\nu + m_{ij}^\nu), \]  

(A.4)

which, for example, allows to find that the null basis components of the variation of \( \ell \) are

\[ \delta \ell^a = \ell^\mu \delta \ell^\nu = -h^{\mu\nu} \delta n_{\nu} - \ell^\mu \delta s_{\nu} n_0 - \delta j_{ij} m_{ij}^\nu \delta m_{ij}^\mu, \]  

\[ \rightarrow 2 \delta \ell^0 = h_{00} - \delta n_0, \quad \delta \ell^1 = h_{01} - \delta s_{1}, \quad \delta \ell^2 = h_{02} - \delta s_{2}, \quad (A.5) \]

where we used the orthogonality conditions (2.1) for the null basis.

At this stage, using (A.5) together with the symmetries of the Weyl tensor, table 2 of [6] and the relations (A.1) valid for a type D background, we can rewrite (A.3) as

\[ \delta \Omega_{ij} = 2 C_{1(i00j)} \delta \ell^1 + (\delta C)_{00ij} = -h_{00} \Phi_{ij}^0 + (\delta C)_{00ij} \]  

(A.6)

To compute \( (\delta C)_{00ij} \), we vary the expression that decomposes the Weyl tensor in terms of the Riemann and Ricci tensors and Ricci scalar, namely,

\[ C_{\alpha\beta\mu\nu} = R_{\alpha\beta\mu\nu} - \frac{1}{d-2} (g_{\alpha\mu} R_{\beta\nu} - g_{\beta\nu} R_{\alpha\mu} - g_{\alpha\nu} R_{\beta\mu} + g_{\beta\mu} R_{\alpha\nu}) + \frac{(g_{\alpha\nu} R_{\beta\mu} - g_{\beta\mu} R_{\alpha\nu}) R}{(d-1)(d-2)}. \]  

(A.7)

10 The variation of the covector \( \ell_a \) is then related to the variation of the dual vector through \( \delta (\ell_a) = \delta (\eta_{ab} \ell^b) = \delta \eta_{ab} \ell^b + \eta_{ab} \delta \ell^b = h_{ab} \ell^b + \delta \ell_a \).
This variation is accomplished if we use the expression for the variation of the affine connection, \( \delta \Gamma^\alpha_{\mu \nu} \), Palatini’s identity for the variation of the Riemann tensor \( \delta R^\alpha_{\rho \mu \nu} \), and the definitions of the variation of the Ricci tensor
\[
\delta R^\alpha_{\rho \mu \nu} = \nabla_\mu (\delta \Gamma^\alpha_{\rho \nu}) - \nabla_\nu (\delta \Gamma^\alpha_{\rho \mu}) - \delta \Gamma^\alpha_{\rho \mu} \Gamma^\beta_{\nu \beta} - \delta \Gamma^\alpha_{\rho \nu} \Gamma^\beta_{\mu \beta} + \delta \Gamma^\beta_{\mu \beta} \Gamma^\alpha_{\rho \nu} + \delta \Gamma^\beta_{\nu \beta} \Gamma^\alpha_{\rho \mu}.
\]
(A.8)
In these expressions, \( g \) is the background metric, \( \nabla \) is the associated background Levi-Civita connection, and we will henceforth take the background to be an Einstein spacetime (A.2).

In these conditions, at this point we can write
\[
2(\delta C)_{\alpha \beta \gamma \delta} = \epsilon^\mu m_{\gamma (i)} \epsilon^\nu m_{\delta (j)} D(2 \nabla_b h_{\alpha \gamma} - \nabla_a h_{\beta \delta}) - \epsilon^\mu m_{\gamma (i)} \epsilon^\nu \delta_{ji} \nabla_b h_{ac} + 2 \h_{00} \Phi_S^i \]
\[
+ \frac{2 \Lambda}{d-2} \h_{00} \delta_{ij} - \frac{1}{d-2} \delta_{ij} \epsilon^\alpha \epsilon^\beta (2 \nabla_a \nabla_b h_{c} - \nabla^2 h_{ac} - \nabla_c \nabla_b h),
\]
(A.9)
where we use the notation \( D \equiv \ell \cdot \nabla \), \( \Delta \equiv n \cdot \nabla \) and \( \delta_i \equiv m_{\gamma (i)} \cdot \nabla \) for the components of the covariant derivative operator in the null frame, and recall that \( \nabla_a = e^{a \mu}_{\nu} \nabla_\mu \). To proceed we make use of several definitions/notation/properties of the GHP formalism listed in section 2 of [6]. More concretely, we need to use the projection of a tensor into the null basis \( T_{ab} \equiv e^{a \mu}_{\nu} e^{\nu}_{\mu} \cdots e^{b \mu}_{\nu} T_{\mu \nu \cdots \alpha} \); the components of the covariant derivative in the null frame; the covariant derivative of the basis vectors,
\[
L_{ab} = \nabla_b n_a, \quad N_{ab} = \nabla_b m_{(i)a}, \quad M_{ab} = \nabla_b m_{(i)a},
\]
(A.10)
and associated notation listed in table A1; and the identities (which follow from the orthogonality properties of the basis vectors)
\[
N_{ab} + L_{ab} = 0, \quad M_{ab} + L_{ab} = 0, \quad M_{ab} + N_{ab} = 0, \quad M_{ja} + L_{ja} = 0, \quad N_{ja} + M_{ja} = 0, \quad L_{ia} = N_{ia} = M_{ia} = 0, \quad L_{10} = -N_{00}, \quad L_{11} = -N_{01} \quad \text{and} \quad L_{1i} = -N_{i0}.
\]
(A.11)
We also need to use the GHP derivative operators \( \hat{\cdot} \), \( \hat{\cdot}' \), \( \hat{\cdot}' \) that map GHP scalars to GHP scalars. They act on a GHP scalar \( T_{i_{1}i_{2}...i_{s}} \) of spin \( s \) and boost weight \( b \) as:
\[
\hat{\cdot} T_{i_{1}i_{2}...i_{s}} = (\ell \cdot \partial) T_{i_{1}i_{2}...i_{s}} - b L_{i_{0}} T_{i_{1}i_{2}...i_{s}} + \sum_{r=1}^{s} M_{i_{0}r} T_{i_{r}...i_{s}},
\]
(A.12)
\[
\hat{\cdot}' T_{i_{1}i_{2}...i_{s}} = (n \cdot \partial) T_{i_{1}i_{2}...i_{s}} - b L_{i_{1}} T_{i_{1}i_{2}...i_{s}} + \sum_{r=1}^{s} M_{i_{1}r} T_{i_{r}...i_{s}},
\]
(A.13)
\[
\hat{\cdot}' T_{j_{1}j_{2}...j_{s}} = (m_{(j)} \cdot \partial) T_{j_{1}j_{2}...j_{s}} - b L_{j_{1}} T_{i_{1}i_{2}...i_{s}} + \sum_{r=1}^{s} M_{j_{1}r} T_{j_{r}...j_{s}},
\]
(A.14)
These GHP derivative operators have properties that we use through our computation, namely they are GHP covariant, obey the Leibniz rule and are metric for $\delta_{ij}$ and satisfy the commutator relations listed in [6]. Our guideline and final target is to write $\delta \Omega_{ij}$ uniquely as a function of GHP scalar quantities, listed in tables A1 and A2, and their GHP derivatives. In the end of this process we find that an algebraically special perturbation must obey

$$
\delta \Omega_{ij} = \left[ -\frac{1}{2} \delta_{ij} \delta_{\bar{j}} + \tau_{i} \delta_{\bar{j}} + \frac{1}{2} \rho_{ij} \rho_{\bar{j}} - \frac{1}{2} \rho_{i(\bar{j})} \rho_{\bar{j})(j) + \kappa_{i} \kappa_{\bar{j}} \right] \left[ \frac{d}{2} \left( \frac{1}{2} \delta_{ij} \delta_{\bar{j}} + \frac{1}{2} \rho_{ij} \rho_{\bar{j}} - \frac{1}{2} \rho_{i(\bar{j})} \rho_{\bar{j})(j) + \kappa_{i} \kappa_{\bar{j}} \right) \right] h_{00}
$$

As required, $\delta \Omega_{ij}$ has the symmetries of $\Omega_{ij}$—it is symmetric and traceless—and it is a spin 2, boost weight 2 GHP scalar. Recall, the conditions for the validity of (A.15): the background must be type D and an Einstein spacetime and no gauge choice was made in the derivation of this expression.

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**Table A2.** Boost weight $b$ and spin $s$ of the GHP scalars built out of the metric perturbation.

| Quantity | Boost weight $b$ | Spin $s$ |
|----------|-----------------|----------|
| $h_{00}$ | 2               | 0        |
| $h_{01}$ | 0               | 0        |
| $h_{11}$ | -2              | 0        |
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