ON THE GENUS OF BIRATIONAL MAPS BETWEEN 3-FOLDS

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In this note we present two equivalent definitions for the genus of a birational map \( \varphi : X \rightarrow Y \) between smooth complex projective 3-folds. The first one is the definition introduced by M. A. Frumkin in [Fru73], the second one was recently suggested to me by S. Cantat. By focusing first on proving that these two definitions are equivalent, one can obtain all the results in [Fru73] in a much shorter way. In particular, the genus of an automorphism of \( \mathbb{C}^3 \), view as a birational self-map of \( \mathbb{P}^3 \), will easily be proved to be 0.

1. Preliminaries

By a \( n \)-fold we always mean a smooth projective variety of dimension \( n \) over \( \mathbb{C} \).

Let \( \varphi : X \rightarrow Y \) be a birational map between \( n \)-folds.

We call base locus of \( \varphi \) the base locus of the linear system associated with \( \varphi \): this is a subvariety of codimension at least 2 of \( X \), which corresponds to the indeterminacy set of the map.

Another subvariety of \( X \) associated with \( \varphi \) is the exceptional set, which is defined as the complement of the maximal open subset where \( \varphi \) is a local isomorphism. If \( X = Y = \mathbb{P}^3 \) the exceptional set has pure codimension 1 (given by the single equation Jacobian = 0), but this not the case in general: consider for instance the case of a flop between smooth 3-folds, where the exceptional set coincides with the base locus.

A regular resolution of \( \varphi \) is a morphism \( \sigma : Z \rightarrow X \) which is a sequence of blow-ups \( \sigma = \sigma_1 \circ \cdots \circ \sigma_r \) along smooth irreducible centers, such that \( Z \rightarrow Y \) is a birational morphism, and such that each center \( B_i \) of the blow-up \( \sigma_i : Z_i \rightarrow Z_{i-1} \) is contained in the base locus of the induced map \( Z_{i-1} \rightarrow Y \).

We shall use the following basic observations about the exceptional set and the base locus of a birational map.

Lemma 1. (1) Let \( \tau : X \rightarrow Y \) be a birational morphism between 3-folds. Then through a general point of any component of the exceptional set of \( \tau \), there exists a rational curve contracted by \( \tau \).
(2) Let $\varphi: X \dashrightarrow Y$ be a birational map between 3-folds, and let $E \subset X$ be an irreducible divisor contracted by $\varphi$. Then $E$ is birational to $\mathbb{P}^1 \times C$ for a smooth curve $C$.

Proof. For the first assertion (which is in fact true in any dimension), see for instance [Deb01, Proposition 1.43]. When $\varphi$ is a morphism, the second assertion is in fact what is first proved in [Deb01]. Finally, when $\varphi$ is not a morphism, we reduce to the previous case by considering a resolution of $\varphi$. □

Lemma 2. Let $\varphi: X \dashrightarrow Y$ be a birational map between $n$-folds, and consider

\[
\begin{array}{c}
\sigma \\
\downarrow \varphi \\
X \\
\downarrow \\
\tau \\
Z \\
\downarrow Y
\end{array}
\]

a regular resolution of $\varphi$. Then a point $p \in X$ is in the base locus of $\varphi$ if and only if the set $\tau(\sigma^{-1}(p))$ has dimension at least 1.

Proof. If $p$ is not in the base locus of $\varphi$ than by regularity of the resolution $\sigma^{-1}(p)$ is a single point, and thus $\tau(\sigma^{-1}(p))$ as well.

Now suppose that $p$ is in the base locus of $\varphi$, and consider $H_Y$ a general hyperplane section of $Y$. Denote by $H_X, H_Z$ the strict transform of $H_Y$ on $X$ and $Z$ respectively. By definition of the base locus, we have $p \in H_X$, hence

$\sigma^{-1}(p) \cap H_Z \neq \emptyset$ and $\tau(\sigma^{-1}(p)) \cap H_Y \neq \emptyset$.

This implies that $\tau(\sigma^{-1}(p))$ has positive dimension. □

We will consider blow-ups of smooth irreducible centers in 3-folds. If $B$ is such a center, $B$ is either a point or a smooth curve. We define the genus $g(B)$ to be 0 if $B$ is a point, and the usual genus if $B$ is a curve. Similarly, if $E$ is an irreducible divisor contracted by a birational map between 3-folds, then by Lemma 1 $E$ is birational to a product $\mathbb{P}^1 \times C$ where $C$ is a smooth curve, and we define the genus $g(E)$ of the contracted divisor to be the genus of $C$.

2. THE TWO DEFINITIONS

Consider now a birational map $\varphi: X \dashrightarrow Y$ between 3-folds, and let $\sigma: Z \to Y$ be a regular resolution of $\varphi^{-1}$.

Frumkin [Fru73] defines the genus $g(\varphi)$ of $\varphi$ to be the maximum of the genus among the centers of the blow-ups in the resolution $\sigma$. Remark that this definition depends a priori from a choice of regular resolution, and Frumkin spends a few pages in order to show that in fact it does not.

During the social dinner of the conference Groups of Automorphisms in Birational and Affine Geometry\textsuperscript{1}, S. Cantat suggested to me another definition, which is certainly easier to handle in practice: define the genus of $\varphi$ to be the maximum of the genus among the irreducible divisors in $X$ contracted by $\varphi$.

Denote by $F_1, \ldots, F_r$ the exceptional divisors of the sequence of blow-ups $\sigma = \sigma_1 \circ \cdots \circ \sigma_r$, or more precisely their strict transforms on $Z$. On the other hand, denote by $E_1, \ldots, E_s$ the strict transforms on $Z$ of the irreducible divisors contracted by $\varphi$.

\textsuperscript{1}Levico Terme, October 29th- November 3rd, 2012.
Let $\varphi^{-1}$ be a morphism, then both collections $\{F_i\}$ and $\{E_i\}$ are empty. In this case, by convention we say that the genus of $\varphi$ is 0. In this section we prove:

**Proposition 3.** Assume $\varphi^{-1}$ is not a morphism. Then

$$\max_{i=1,\ldots,r} g(E_i) = \max_{i=1,\ldots,r} g(F_i).$$

In other words the definition of the genus by Frumkin coincides with the one suggested by Cantat, and in particular it does not depend on a choice of a regular resolution.

Denote by $B_i$ the center of the blow-up $\sigma_i$ producing $F_i$. We define a **partial order** on the divisors $F_i$ by saying that $F_j \succ F_k$ if one of the following conditions is verified:

1. $j = k$;
2. $j > k$, $B_k$ is a point, and $B_j$ is contained in the strict transform of $F_k$;
3. $j > k$, $B_k$ is a curve, and $B_j$ intersects the general fiber of the strict transform of the ruled surface $F_k$.

We say that $F_i$ is **essential** if $F_i$ is a maximal element for the order $\succ$.

**Lemma 4.** The maximum $\max_i g(F_i)$ is realized by an essential divisor.

**Proof.** We can assume that the maximum is not 0, otherwise there is nothing to prove. Consider $F_k$ realizing the maximum, and $F_j \succ F_k$ with $j > k$. Then the centers $B_j, B_k$ of $\sigma_j$ and $\sigma_k$ are curves, and $B_j$ dominates $B_k$ by a morphism. By Riemann-Hurwitz formula, we get $g(F_j) \geq g(F_k)$, and the claim follows.

**Lemma 5.** The subset of the essential divisors $F_i$ with $g(F_i) \geq 1$ is contained in the set of the contracted divisors $E_j$.

**Proof.** Let $B_i \subset Z_{i-1}$ be the center of a blow-up producing a non-rational essential divisor $F_i$, and consider the diagram:

\[
\begin{array}{c}
\tau \\
\downarrow \\
Z \\
\sigma = \sigma_{i-1} \circ \cdots \circ \sigma_1
\end{array}
\begin{array}{c}
\downarrow \\
X \\
\psi_{i-1} \\
\downarrow \\
Z_{i-1}
\end{array}
\]

By applying Lemma 2 to $\psi_{i-1} : Z_{i-1} \to X$, we get $\dim \tau((\sigma^{-1}(p)) \geq 1$ for any point $p \in B_i$. Since $F_i$ is essential, $l_p := \sigma^{-1}(p)$ is a smooth rational curve contained in $F_i$ for all except finitely many $p \in B_i$. So $\tau(l_p)$ is also a curve. If $\tau(l_p)$ varies with $p$, then $\tau(F_i)$ is a divisor, which is one of the $E_i$. Now suppose $\tau(l_p)$ is a curve independent of $p$, that means that $F_i$ is contracted to this curve by $\tau$. Consider $q$ a general point of $F_i$. By Lemma 1 there is a rational curve $C \subset F_i$ passing through $q$ and contracted by $\tau$, but this curve should dominate the curve $B_i$ of genus $\geq 1$: contradiction.

*Proof of Proposition 3.* Remark that the strict transform of a divisor contracted by $\varphi$ must be contracted by $\sigma$, hence we have the inclusion $\{E_i\} \subset \{F_i\}$. This implies $\max_i g(E_i) \leq \max_i g(F_i)$. If all $F_i$ are rational, then the equality is obvious.

Suppose at least one of the $F_i$ is non-rational. By Lemma 5 we have the inclusions

$$\{F_i; F_i \text{ is non-rational and essential}\} \subset \{E_i\} \subset \{F_i\}.$$

Taking maximums, this yields the inequalities

$$\max \{g(F_i); F_i \text{ is non-rational and essential}\} \leq \max_i g(E_i) \leq \max_i g(F_i).$$

By Lemma 4 we conclude that these three maximums are equal. □
3. SOME CONSEQUENCES

The initial motivation for a reworking of the paper of Frumkin was to get a simple proof of the fact that a birational self-map of $\mathbb{P}^3$ coming from an automorphisms of $\mathbb{C}^3$ admits a resolution by blowing-up points and rational curves:

**Corollary 6.** The genus of $\phi$ is zero in the following two situations:

1. $\phi \in \text{Bir}(\mathbb{P}^3)$ is the completion of an automorphism of $\mathbb{C}^3$;
2. $\phi: X \to Y$ is a pseudo-isomorphism (i.e. an isomorphism in codimension 1).

In particular for such a map $\phi$ any regular resolution only involves blow-ups of points and of smooth rational curves.

**Proof.** Both results are obvious using the definition via contracted divisors! \qed

I mention the following result for the sake of completeness, even if I essentially follow the proof of Frumkin (with some slight simplifications).

**Proposition 7** (compare with [Fru73, Proposition 2.2]). Let $\phi: X \to Y$ be a birational map between 3-folds, and let $\sigma: Z \to X$, $\sigma': Z' \to Y$ be resolutions of $\phi$, $\phi^{-1}$ respectively. Assume that $\sigma$ is a regular resolution, and denote by $h: Z \to Z'$ the induced birational map. Then $g(h) = 0$.

**Proof.** We write $\sigma = \sigma_1 \circ \cdots \circ \sigma_r$, where $\sigma_i: Z_i \to Z_{i-1}$ is the blow-up of a smooth center $B_i$. Note that $Z_0 = X$, and $Z_r = Z$. Assume that the induced map $h_i: Z_i \to Z'$ has genus 0 (this is clearly the case for $h_0 = \tau'^{-1}$), and let us prove the same for $h_{i+1} = h_i \circ \sigma_{i+1}$. We can assume that $\sigma_{i+1}$ is the blow-up of a non-rational smooth curve $B_{i+1}$, otherwise there is nothing to prove. Consider

$$
\begin{array}{cccc}
X & \xrightarrow{\sigma} & Z & \xrightarrow{h} & Z' \\
\downarrow \sigma & & \downarrow h & & \downarrow \sigma' \\
W_i & & Z_i & & Z' \\
\downarrow p_i & & \downarrow \sigma_i & & \downarrow q_i \\
\end{array}
$$

a regular resolution of $h_i^{-1}$. Since $W_i$ dominates $Y$ via the morphism $\sigma' \circ q_i$, the curve $B_{i+1}$ is in the base locus of $p_i^{-1}$; otherwise $B_{i+1}$ would not be in the base locus of $Z_i \to Y$, contradicting the regularity of the resolution $\sigma$. Thus for any point $x \in B_{i+1}$, $p_i^{-1}(x)$ is a curve, and there exists an open set $U \subset Z_i$ such that $p_i^{-1}(U \cap B_{i+1})$ is a non empty divisor. By applying over $U$ the universal property of blow-up (see [Har77, Proposition II.7.14]), we get that there exists an irreducible divisor on $W_i$ whose strict transform on $Z_{i+1}$ is the exceptional divisor $E_{i+1}$ of $\sigma_{i+1}$. Hence the birational map $p_i^{-1} \circ \sigma_{i+1}: Z_{i+1} \to W_i$ does not contract any divisor and so has genus 0. Composing by $q_i$ which also has genus 0 by hypothesis we obtain $g(h_{i+1}) = 0$. By induction we obtain $g(h_r) = 0$, hence the result since $h_r = h$. \qed

**Question 8.** In the setting of Proposition 7, if $\sigma'$ is also a regular resolution, is it always true that $h: Z \to Z'$ is a pseudo-isomorphism? I expect “no”, but I don’t have a concrete example.
The next result is less elementary.

**Proposition 9.** Let $X$ be a 3-fold with Hodge numbers $h^{0,1} = h^{0,3} = 0$, and let $\varphi : X \rightarrow X$ be a birational self-map. Then $g(\varphi) = g(\varphi^{-1})$.

For the proof, which relies on the use of intermediate Jacobians, I refer to the original paper of Frumkin [Fru73, Proposition 2.6], or to [LS12] where it is proved that the exceptional loci of $\varphi$ and $\varphi^{-1}$ are birational (and even more piecewise isomorphic). Note that Frumkin does not mention any restriction on the Hodge numbers of $X$, but it seems implicit since the proof uses the fact that the complex torus $\mathcal{J}(X)$ is a polarized abelian variety.

**Corollary 10.** Let $n \geq 0$ be an integer. The set of birational self-maps of $\mathbb{P}^3$ of genus less than $n$ is a subgroup of $\text{Bir}(\mathbb{P}^3)$.

**Proof.** Stability under taking inverse is Proposition 9, and stability under composition comes from the fact that any divisor contracted by $\varphi \circ \varphi'$ is contracted either by $\varphi$ or by $\varphi'$.

**Question 11.** The last corollary could be stated for any 3-fold satisfying the assumptions of Proposition 9, but I am not aware of any relevant example. For instance, if $X \subset \mathbb{P}^4$ is a smooth cubic 3-fold, is it true that any birational self-map of $X$ has genus 0?

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