GRAPH STRUCTURE OF COMMUTING FUNCTIONS

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Abstract. The problem of finding graph structure of functions commuting with a given function in terms of their functional graphs is considered. Functional graphs of commuting functions are described. Several cases involving bijective or arbitrary functions as well as finite or arbitrary sets are considered.

Key words. functional graph, commuting functions, graph homomorphism

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1. Introduction.

1.1. The subject of study. Composition of functions is an important binary operation in function sets. This operation is so omnipresent and important in mathematics, that its basic property - associativity has been abstracted and accepted as a basic feature of algebraic structures such as groups. Commutativity is the second most useful property of algebraic structures, its importance originates from commutativity of set-theoretic union and intersection. Functions commuting with respect to the composition operation have been studied for both purely theoretical and applied reasons. See [5] for an example of studies of commuting rational functions dating back to the early 20th century. Commutativity of linear algebraic objects such as matrices with respect to multiplication has been studied since Frobenius, see [1]. Generalizations of commuting functions, e.g. commuting matrices and operators, are important in applications such as quantum physics.

In this paper we study graph structure of commuting functions and the results involve graph models of functions - functional graphs. The answer is well known for both functions being bijective in finite sets. Permutations commuting with a given permutation form a subgroup of the total permutation group, its algebraic structure has been studied and generalized, see [4]. The general case does not seem to have been described in the literature, therefore some further study and description of commuting functions seems sufficiently motivated. These studies may provide additional links between algebra and discrete mathematics. Our motivation and goal of this paper is to fill this gap - to describe functions commuting with a given function in terms of

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their functional graphs with the functions being arbitrary. In graph-theoretic terms this amounts to descriptions of homomorphic images of functional graphs.

1.2. Structure and notations. Basic notations and facts are reviewed in subsections 1.3, 1.4. The subsection 2.1 contains the results for finite sets, four main subcases are considered - permutations commuting with a permutation (subsection 2.1.1), permutations commuting with an arbitrary function (subsection 2.1.2), arbitrary functions commuting with a permutation (subsection 2.1.3), arbitrary functions commuting with an arbitrary function (subsection 2.1.4). The section 2.2 contains results for infinite sets.

In this paper we denote the Cartesian product of sets \( A_1 \times A_2 \times \ldots \times A_n \) as \( \bigotimes_{i=1}^{n} A_i \) (not to be confused with tensor products). Sequences (including pairs) of elements or sets are denoted using square brackets. For example, the sequence having elements \( a_1, a_2, \ldots, a_n \), is denoted as \([a_1, a_2, \ldots, a_n]\). Cycles are denoted using brackets. The power set of the set \( A \) is denoted as \( 2^A \). We use normal letters to denote fixed objects and \( \mathcal{M} \) letters to denote objects as function values.

1.3. Endofunctions, functional graphs and their mappings. We denote the set of endofunctions of a set \( S \) by \( \text{Fun}(S) \) and the set of bijective \( S \)-endofunctions (\( S \)-permutations) by \( \text{Bij}(S) \). Given a set \( S \) and \( f \in \text{Fun}(S) \) we denote the set of all \( S \)-endofunctions commuting with \( f \) (\( f \)-centralizer) by \( \mathcal{C}(f) \):

\[
\mathcal{C}(f) = \{ g \in \text{Fun}(S) | fg = gf \}.
\]

We denote the set of all \( S \)-permutations commuting with \( f \) by \( \mathcal{C}_{\text{bij}}(f) \):

\[
\mathcal{C}_{\text{bij}}(f) = \{ g \in \text{Bij}(S) | fg = gf \}.
\]

The graph \( \Gamma \) with a vertex set \( V \) and an edge set \( E \) is denoted by \( \Gamma = (V, E) \), \( \mathcal{V}(\Gamma) = V \), \( \mathcal{E}(\Gamma) = E \). Notation \( \Delta \leq \Gamma \) means that \( \Delta \) is a subgraph of \( \Gamma \). We denote the directed cycle \( (V, E) \), where \( V = \{x_0, \ldots, x_{n-1}\} \) (in this case and often in this paper we denote indices of cycle elements as residues mod \( n \)), \( E = \bigcup_{i=0}^{n-1} [x_i, x_{i+1}] \) by \( (x_0, \ldots, x_{n-1}) \).

Given a set \( S \) and an endofunction \( f \in \text{Fun}(S) \) we define as usual the functional graph of \( f \) (we call it the \( f \)-graph): it is the directed graph \( \Gamma(f) = (S, E_f) \), where \( E_f = \bigcup_{s \in S} [s, f(s)] \).

Example 1.1. Let \( S = \{0, 1, \ldots, 8\} \) - residues mod 9. Let \( f : S \to S \), \( f(x) \equiv x^2 \text{mod} 9 \). The \( f \)-graph is shown below.
Fig. 1. - the $f$-graph for Example [1,2]

Given two endofunctions $f$ and $g$ we can construct the weighted $(f, g)$-graph $\Gamma_{f,g} = (S, E_f \cup E_g)$ where edges of the sets $E_f$ and $E_g$ are weighted by $f$ and $g$, respectively.

We remind the reader some basic graph theory definitions for notational purposes. Suppose we are given two directed graphs $\Gamma_1 = (V_1, E_1)$ and $\Gamma_2 = (V_2, E_2)$. We call a function $f : V_1 \to V_2$ a graph homomorphism, denoted also as $f : \Gamma_1 \to \Gamma_2$, provided $[v, w] \in E_1$ implies $[f(v), f(w)] \in E_2$. A graph homomorphism from $\Gamma$ to itself is called a $\Gamma$-endomorphism.

Given a directed graph $\Gamma$ we can forget orientations of its arrows and get a undirected graph $\hat{\Gamma}$. Strictly speaking, in general we get a multigraph since there may be pairs of vertices with two directed edges between them. In case of functional graphs this can happen only if there are cycles of length 2, the distinction between graphs and multigraphs does not seem important for purposes of our paper. Two directed graphs are called weakly isomorphic if the corresponding undirected graphs are isomorphic. A weakly connected component of (a directed graph) $\Gamma$ is an induced subgraph $\Delta \leq \Gamma$ such that $\hat{\Delta}$ is a connected component of $\hat{\Gamma}$. Clearly, $f : \Gamma_1 \to \Gamma_2$ is a graph homomorphism iff its restriction to every weakly connected component is a graph homomorphism. Thus any graph homomorphism is obtained by composing graph homomorphisms from weakly connected components of the domain. Homomorphisms from weakly connected components can be constructed independently.

A directed graph $T$ is called a directed tree with the root $x$ provided 1) $\hat{T}$ is a tree and 2) there is a unique directed path from any other vertex to $x$.

The set of homomorphisms (endomorphisms, isomorphisms, automorphisms) from $\Gamma$ to $\Delta$ is denoted by $\text{Hom}(\Gamma, \Delta)$ ($\text{End}(\Gamma)$, $\text{Iso}(\Gamma, \Delta)$, $\text{Aut}(\Gamma)$).

If $\Gamma_1$ and $\Gamma_2$ are graphs with vertex weight functions $w_i : \mathcal{V}(\Gamma_i) \to k$, where $k$ is some weight set, then for $f : \mathcal{V}(\Gamma_1) \to \mathcal{V}(\Gamma_2)$ to be a graph isomorphism it must satisfy $w_1 = w_2 \circ f$.

Given a set $S$ and two $S$-endofunctions $f$ and $g$ we can consider $g$ as a mapping
for the graph $\Gamma(f)$ and vice versa.

**Lemma 1.2.** Let $S$ be a set. Let $f$ and $g$ be commuting $S$-endofunctions: $fg = gf$.

Then

1. $g$ is a $\Gamma(f)$-endomorphism.
2. If $g \in \text{Bij}(S)$, then $g$ is an $\Gamma(f)$-automorphism.

**Proof.**

1. If $[v, w] \in E(\Gamma(f))$, then $w = f(v)$. We have that $f(g(v)) = (fg)(v) = (gf)(v) = g(w)$, therefore $[g(v), g(w)] \in E(\Gamma(f))$.

2. By 1. $g \in \text{End}(\Gamma(f))$. In the other direction, since $g \in C(f)$ and $g \in \text{Bij}(S)$, we have $g^{-1} \in C(f)$, thus $g^{-1} \in \text{End}(\Gamma(f))$ and hence $g \in \text{Aut}(\Gamma(f))$. 

1.4. **Weakly connected components of functional graphs.** Isomorphism types of weakly connected components of functional graphs seem to be well known, we remind them below.

**1.4.1. Bijectons.** Recall that if $f$ is a bijection, then weakly connected components of the $f$-graph are directed cycles (called $f$-cycles in this paper) or directed lines (called $f$-lines or infinite $f$-cycles in this paper). We denote the isomorphism type of a directed cycle on $n$ vertices by $Z_n$, we can assume that $Z_n \simeq (\mathbb{Z}_n, E_n)$, where $\mathbb{Z}_n$ is the set of residue classes $\mod n$ and $[i, j] \in E_n$ iff $i + 1 \equiv j \mod n$. We denote the isomorphism type of directed line by $L$, we assume that $L \simeq (\mathbb{Z}, E)$, where $\mathbb{Z}$ is the set of integers and $[i, j] \in E$ iff $i + 1 = j$. Thus for an arbitrary bijective function $f$ we may assume that a weakly connected component of the $f$-graph is isomorphic to $Z_n$ or $L$. This follows from the observation, that in the $f$-graph every vertex has exactly one incoming and outgoing edge.

**1.4.2. Endofunctions on a finite set.**

**Pseudocycles.** If $S$ is a finite set and $f \in \text{Fun}(S)$ arbitrary, then the $f$-graph is a directed pseudoforest - weakly connected components of the $f$-graph are traditionally called directed pseudotrees, see [3], but in this paper we call them $f$-pseudocycles. We call a directed graph a pseudocycle provided it contains exactly one directed cycle (which may be a loop) and there is a unique directed path from any other vertex to the closest cycle vertex, thus any vertex of the directed cycle is a root of a directed tree (which may consist of a single vertex). This description follows from the observation, that in the $f$-graph every vertex has exactly one outgoing edge. See [2] for another description. In terms of vertex weighted graphs a pseudocycle is a directed cycle with vertices weighted by rooted directed trees. Thus we think and denote a pseudocycle
as a tree cycle \((T_0, \ldots, T_{m-1})\), where \(T_i\) is isomorphic to a rooted directed tree, two pseudocycles \(P_1 = (T_0, \ldots, T_{m-1})\) and \(P_2 = (T'_0, \ldots, T'_{m-1})\) are isomorphic iff there is an cyclic permutation \(\zeta \in \Sigma_m\) of the sequence \(P_1\) such that \(\zeta(P_1) \simeq P_2\), see below.

Cyclic permutations of pseudocycles. Given a pseudocycle \(x = (x_0, \ldots, x_{m-1})\) with vertices from the multiset \(X = \{\{y_1, \ldots, y_n\}\}\) we call the vertex multiset permutation \(\sigma_x = (x_0 \ldots x_{m-1})\) (in cycle notation) the elementary shift of \(x\). We denote the set of all permutations of \(X\) by \(\Sigma_X\). We call a permutation \(\rho \in \Sigma_X\) cyclic permutation of \(x\) if \(\rho = \sigma_k^x\) for some \(k \in \mathbb{N}\). For any \(x \in X^m\) minimal \(k \in \mathbb{N}\) such that \(\sigma_k^x(x) = x\) is called the order of \(x\), denoted by \(\text{ord}(x)\). We call \(s_x = \frac{|x|}{\text{ord}(x)}\) the index of \(x\). We define two sequences (or pseudocycles) \(x = (x_0, \ldots, x_{m-1})\) and \(y = (y_0, \ldots, y_{m-1})\) cyclic isomorphic if there is a cyclic permutation \(\zeta\) such that \(\zeta(x) = y\).

Example 1.3. If \(x = (a, b, c, a, b, c)\), then \(\sigma_x(x) = (c, a, b, c, a, b)\), \(\text{ord}(x) = 3\), \(s_x = 2\).

Rooted directed trees of pseudocycles. We denote the directed cycle of the pseudocycle \(P\) by \(\mathcal{Z}(P)\). Given \(x \in \mathcal{V}(P)\) we denote the rooted directed tree with the root \(x\) as \(\mathcal{T}(x)\). We have that \(\mathcal{T}(x) \cap \mathcal{Z}(P) = \{x\}\) if \(x \in \mathcal{V}(\mathcal{Z}(P))\) or \(\emptyset\) otherwise. By a (full) directed tree of a pseudocycle \(P\) we call \(\mathcal{T}(z)\) with \(z \in \mathcal{V}(\mathcal{Z}(P))\): \(\mathcal{T}(z)\) an induced subgraph of \(P\) such that \(\mathcal{T}(z) \cap \mathcal{Z}(P) = \{z\}\) and \(\mathcal{V}(\mathcal{T}(z))\) contains all vertices of \(P\) having directed paths to \(z\). We denote the pseudocycle having the directed cycle \((z_0, \ldots, z_{m-1})\) and corresponding full directed trees \(T_0, \ldots, T_{m-1}\) by \((T_0, \ldots, T_{m-1})\).

In any directed tree we can introduce the corresponding tree order: given two vertices \(x, y\) of a directed tree we define \(x < y\) provided there is a directed path from \(y\) to \(x\). Given a directed tree \(T\) denote by \(\mathcal{D}_i(T)\) the set of vertices being in distance \(i\) to the root, thus \(\mathcal{V}(T) = \bigcup_i \mathcal{D}_i(T)\). For vertices of directed trees we define a height function \(\phi\): \(\phi(x) = i\) iff \(x \in \mathcal{D}_i(T)\). Given a directed tree \(T\) with root \(x\), we denote the root-truncated graph \(T \setminus \{x\}\) by \(\overline{T}\).

1.4.3. Endofunctions on an arbitrary set.

If \(S\) is infinite, then there are two additional types of weakly connected components of functional graphs, which we call pseudolines (infinite pseudocycles) and pseudorays.

Pseudolines. We call a directed graph a pseudoline (infinite pseudocycle) provided it is isomorphic to a functional graph, which contains at least one subgraph isomorphic to the directed line. Note that the directed line \(L\) is a special case of
pseudoline.

Example 1.4. Let $S = \mathbb{Z}$ and $f(x) = x + a$ for a fixed $a \in \mathbb{Z}$. The $f$-graph has $a$ $f$-lines as weakly connected components.

We can note that in the pseudoline there is a unique directed path from any vertex to the closest vertex of $\Lambda$, thus any vertex of the directed line $\Lambda$ may be a root of a possibly infinite directed tree.

Pseudorays. We denote the isomorphism type of directed ray by $R$, we may assume that $R \simeq (\mathbb{N}, E_{\mathbb{N}})$, where $\mathbb{N}$ is the set of natural numbers and $[i, j] \in E_{\mathbb{N}}$ iff $i + 1 = j$. We call a functional graph a pseudoray provided it contains a subgraph isomorphic to a directed ray but no directed cycle or line.

Example 1.5. Let $S = \mathbb{Z}$ and $f(x) = x^2$. The $f$-graph has countably many pseudorays ($f$-rays) as weakly connected components. A fragment of an $f$-ray is shown below.

\[
\begin{align*}
2 & \rightarrow 4 \rightarrow 16 \rightarrow 256 \rightarrow \ldots \\
-2 & \rightarrow -4 \rightarrow -16 \rightarrow -256 \rightarrow \ldots
\end{align*}
\]

Fig. 2. - a pseudoray for Example 1.5

Similarly as in the case of pseudoline there is a unique finite directed path from any vertex to the closest vertex of $\Lambda$, thus any vertex of $\Lambda$ may be a root of a finite directed tree. Note that in this case directed trees must be finite since there are no directed line.

A pseudoline or a pseudoray may have more than one subgraph isomorphic to a directed line or ray.

For pseudolines and pseudorays we can generalize notions of directed trees with given roots etc. If $X$ is a directed line or a directed ray in a functional graph and $x \in \mathcal{V}(X)$, then $T(x)$ is the maximal directed tree such that $T(x) \cap X = \{x\}$.

2. Main results. In this section we describe endofunctions $g \in \mathcal{F}un(S)$ commuting with a given endofunction $f \in \mathcal{F}un(S)$. In terms of functional graphs we describe possible graph homomorphisms of $f$-graphs. Descriptions are given as correspondences $g \leftrightarrow \{A, B, C, \ldots\}$, where $A, B, \ldots$ are mappings or substructures related
2.1. Finite sets.

2.1.1. Permutations commuting with a fixed permutation. In this subsection S is a finite set, f is a permutation: \( f \in \text{Bij}(S) \). We want to describe \( C_{bij}(f) = C(f) \cap \text{Bij}(S) \) - f-commuting permutations. This description seems to be well known.

**Lemma 2.1.** Let S be a finite set, let f and g be commuting S-endofunctions: \( fg = gf \). Let \( Z = (x, f(x), ..., f^{k-1}(x)) \) be an f-cycle of length k.

Then \( g(Z) = (g(x), g(f(x)), ..., g(f^{k-1}(x))) \) is an f-cycle of length k.

**Proof.** \( fg = gf \) implies \( g(f^i(x)) = f^i(g(x)) \) for all \( x \in S \) and \( i \in \mathbb{Z} \). It follows that \( f(g(f^i(x))) = g(f^{i+1}(x)) \) for \( i \in \{0, ..., k - 2\} \). Since \( f^k(x) = x \) we have that \( f(g(f^{k-1}(x))) = g(f^k(x)) = g(x) \), therefore \( g(Z) = (g(x), f(g(x)), ..., f^{k-1}(g(x))) \) is an f-cycle of length k. \( g(Z) \) is independent of the choice of \( x \in Z \). \( \square \)

**Remark 2.2.** In terms of Lemma 2.1 the isomorphic f-cycles \( Z \) and \( g(Z) \) may be equal or different (vertex disjoint). An f-commuting permutation g is determined on any f-cycle \( Z \) by choosing \( g(x) \) in any f-cycle of length \( |Z| \) for any fixed \( x \in Z \).

In the next theorem we describe f-commuting permutations.

**Theorem 2.3.** Let S be a finite set, f and g - commuting S-permutations: \( f, g \in \text{Bij}(S) \), \( fg = gf \). Denote the set of f-cycles of length i as \( Z_i = \bigcup_{j=1}^{n_i} Z_{i,j} \), where \( Z_{i,j} \) denotes a cycle of length i. Define \( Z = \bigsqcup Z_i \). For any f-cycle \( Z_{i,j} \) choose an element \( x_{i,j} \in V(Z_{i,j}) \), define the set \( X = \bigsqcup_{i,j} x_{i,j} \).

Then g is bijectively defined by the pair \( [\overline{g}, g|_X] \), where

1. \( \overline{g} \) is a permutation \( \mathbf{Z} \rightarrow \mathbf{Z} \), such that \( \overline{g}(Z_i) = Z_i \) and
2. \( g|_X \) is the restriction of g on X, \( g|_X : X \rightarrow S \), where \( g(x_{i,j}) \in V(\overline{g}(Z_{i,j})) \);

**Proof.**

By Lemma 2.1 an f-commuting permutation g is bijectively determined on f-cycles of length i by the sequence \( g(x_{i,1}), ..., g(x_{i,n_i}) \), where \( g(x_{i,j}) \) belongs to some f-cycle of length i, denote this f-cycle by \( \overline{g}(Z_{i,j}) \). For each j \( g(x_{i,j}) \) determines \( g(Z_{i,j}) \), g is a permutation thus we get a permutation \( \overline{g}_i \) of \( Z_i \). Considering the set of all f-cycles \( \mathbf{Z} \) we can construct a permutation \( \overline{g}_i \) which fixes each \( Z_i \). For an arbitrary set \( X = \{x_{i,j}\}_{i,j} \) any function on X given by a set \( g(x_{i,j}) \), where \( x_{i,j} \) belongs to a
cycle of length $i$ and the corresponding function $Z \to Z$ is a permutation, the function $g$ can be extended to a permutation of $S$ commuting with $f$. □

The next theorem describes cycle structure of $f$-commuting permutations.

**Theorem 2.4.** Let $S$ be a finite set, $f$ and $g$ - commuting $S$-permutations: $f, g \in \text{Bij}(S), fg = gf$. Denote the set of $f$-cycles of length $i$ as $Z_i = \bigcup_{j=1}^{n_i} Z_{i,j}$. Let $\tilde{g}$ be defined as in Theorem 2.3.

Then

1. any cycle of $\tilde{g}|_{V(Z_i)}$ of length $k > 1$ decomposes into $i$ g-cycles of length $k$;
2. any cycle of $\tilde{g}|_{V(Z_i)}$ of length 1 corresponding a map $g_Z : Z \to Z$, where $Z = \{x, f(x), ..., f^{i-1}(x)\}$ is an $f$-cycle and $g_Z(f^k(x)) = f^{k+j}(x)$ ($0 \leq j \leq i - 1$) decomposes into $\frac{i}{\text{GCD}(i,j)}$ g-cycles of length $\frac{i}{\text{GCD}(i,j)}$;
3. $|C_{bij}(f)| = \prod_{i=1}^{n} n_i! i^{n_i}$.

**Proof.**

1. We have to find the cycle decomposition of the union of several $f$-cycles of length $i$, which are cyclically permuted by $g$. If $g$ cyclically permutes $k$ $f$-cycles $Z_1, ..., Z_k$ from $Z_i, k > 1$, then due to the bijectivity of $g$ the restriction of $g$ to the union $\bigcup_{j=1}^{k} V(Z_j)$ decomposes into $i$ cycles of length $k$.

2. We have to find the cycle decomposition of the $f$-cycle $Z = (x, f(x), ..., f^{i-1}(x))$ under $g_Z$, where $g_Z(f^k(x)) = f^{k+j}(x)$. $g_Z$ is the restriction of $g$ to $Z$ and $Z$ is fixed by $g$, the definition of $g_Z$ follows from the commutativity condition. We have that every element of form $f^c(x)$ lies in a $g$-cycle $(f^k(x), f^{k+c}(x), ..., f^{k+c})$, where $c$ is the minimal natural solution of the equation $jc \equiv 0(\mod i)$ or the equivalent equation $jc \equiv 0(\mod \text{GCD}(i,j))$. It follows that $c = \frac{i}{\text{GCD}(i,j)}$. Thus the cycle length of every $g$-cycle of $Z$ is equal to $c$ and the number of $g$-cycles is equal to $\frac{i}{c} = \frac{i}{\text{GCD}(i,j)}$.

3. For any $i \in \{1, ..., n\}$ any $g$ is bijectively determined by a permutation of $f$-cycles of length $i$ and sequence of elements belonging to each such $f$-cycle thus the number of restrictions of $f$-commuting permutations on $Z_i$ is $n_i! i^{n_i}$. For each $i$ the action of $g$ can be chosen independently, therefore the statement follows by the product rule. □

**Remark 2.5.** Thus an $f$-commuting permutation permutes $f$-cycles of the same length. It can be shown that, as a group, $C_{bij}(f)$ can be expressed as a direct product of wreath products of certain subgroups.
2.1.2. Permutations commuting with a fixed endofunction. In this subsection we describe permutations commuting with an arbitrary fixed endofunction \( f \) given on a finite set \( S \) - \( C_{bij}(f) = C(f) \cap Bij(S) \).

**Lemma 2.6.** Let \( S \) be a finite set, \( f \in \text{Fun}(S) \), \( g \) - an \( f \)-commuting permutation: \( g \in Bij(S) \), \( fg = gf \). Let \( P = (T_0, ..., T_{m-1}) \) be an \( f \)-pseudocycle.

Then \( g(P) \) is an \( f \)-pseudocycle, which is cyclic isomorphic to \( P \).

**Proof.** \( g \) is an \( \Gamma(f) \)-automorphism, therefore \( P \) isomorphic to \( Z(P) \) by a cyclic permutation. Each directed tree \( T(z) \) of \( P \) is mapped isomorphically to the directed tree \( T(g(z)) \). \( \square \)

**Lemma 2.7.** Let \( S \) be a finite set, \( f \in \text{Fun}(S) \), \( g \) - an \( f \)-commuting permutation: \( g \in Bij(S) \), \( fg = gf \). \( P = (T_0, ..., T_{m-1}) \) is an \( f \)-pseudocycle.

Then \( \zeta : V(P) \to V(P) \) is a \( P \)-automorphism if and only if \( \zeta \) is a cyclic permutation on \( Z(P) \) and \( \zeta(T_i) \cong T_i \) for any \( i \in \{0, ..., m-1\} \).

**Proof.** \( \zeta \) must cyclically permute vertices of \( Z(P) \) since it is the only oriented cycle of \( P \). \( \zeta \) must send each directed tree \( T_i \) to an isomorphic directed tree. \( \square \)

Now we describe \( f \)-commuting permutations with an arbitrary \( f \).

**Theorem 2.8.** Let \( S \) be a finite set, \( f \in \text{Fun}(S) \), \( g \) - an \( f \)-commuting permutation: \( g \in Bij(S) \), \( fg = gf \). Denote the set of \( f \)-pseudocycles with a tree cycle \( T \) as \( P_T = \bigsqcup_{j=1}^{n_T} P_{T,j} \), \( P_{T,j} \) denotes an \( f \)-pseudocycle with a tree cycle \( T \). Denote \( P = \bigsqcup_T P_T \).

For any \( f \)-pseudocycle \( P_{T,j} \) choose an element \( x_{T,j} \in V(Z(P_{T,j})) \), let \( X_T = \bigsqcup_j x_{T,j} \), \( X = \bigsqcup_T X_T \).

Then for every \( T \) restriction \( g|_{V(P_T)} \) is bijectively determined by the triple \( \tau = [\tilde{g}T, g|_{X_T}, A_T] \), where

1. \( \tilde{g}T \) is a permutation of the set of \( f \)-pseudocycles \( P_T \);
2. \( g|_{X_T} \) is the restriction of \( g \) on \( X_T \), \( g|_{X_T} : X_T \to S \), where \( T(f^k(x_{T,j})) \cong T(g(f^k(x_{T,j}))) \) for all \( j \in \{1, ..., n_T\} \), \( k \in \{0, ..., |T| - 1\} \) (\( |T| \) denotes the length of \( Z(T) \));
3. \( A_T = [\alpha_{jk}]_{j=1}^{n_T} |T|^{-1} \), where \( \alpha_{jk} : T(f^k(x_{T,j})) \to T(g(f^k(x_{T,j}))) \) is an isomorphism of directed trees (\( A_T \) is a two dimensional array of graph isomorphisms).

**Proof.** By Lemma 2.6 and Lemma 2.7 any \( f \)-pseudocycle \( P_{T,j} \) is necessarily mapped by the isomorphism \( g \) to an \( f \)-pseudocycle with cyclic isomorphic sequence...
of directed trees, i.e.

$$T(f^k(x_{T,j})) \simeq T(g(f^k(x_{T,j}))), \ k \in [0,\ldots,|T| - 1].$$

Every directed tree is mapped isomorphically to the corresponding directed tree, which gives the array of directed tree isomorphisms $\alpha_{j,k}$. For each $T$ and $j$, $g(x_{T,j})$ determines $Z(g(P_{T,j}))$, by adding the action of $g$ on directed trees by isomorphisms $\alpha_{j,k}$ we get the restriction of $g$ on $P_T$. \qed

The next theorem describes cycle structure of $f$-commuting permutations for an arbitrary $f$.

**Theorem 2.9.** Let $S$ be a finite set, $f \in \text{Fun}(S)$, $g$ - an $f$-commuting permutation: $g \in \text{Bij}(S)$, $fg = gf$. Denote the set of $f$-pseudocycles with a tree cycle $T$ as $P_T = \bigcup_{j=1}^{n_T} P_{T,j}$, $P_{T,j}$ denotes an $f$-pseudocycle with a tree cycle $T$. Let $\tilde{g}_T$ be defined as in Theorem 2.8.

Then

1. each cycle of $\tilde{g}_T$ of length $k > 1$ decomposes into $g$-cycles of length $k$;
2. each cycle of $\tilde{g}_T$ of length 1 corresponding to a map $g_P : \mathcal{V}(P) \to \mathcal{V}(P)$, where $P$ is an $f$-pseudocycle, $Z(P) = \{x, f(x),\ldots, f^{i-1}(x)\}$ is an $f$-cycle and $g_P(f^j(x)) = f^{k+j}(x) \ (0 \leq j \leq i - 1)$ decomposes into $g$-cycles of length $\frac{GCD(i,j)}{GCD(i,j)}$;
3. $|C_{kij}(f)| = \left( \prod_{T} n_T! s_T^{n_T} \right) \cdot \left( \prod_{z \in \mathcal{V}(f)} |\text{Aut}(T(z))| \right)$, where $s_T$ is the index of $T$, $\mathcal{V}(f)$ is the union of all $f$-cycles of $\mathcal{V}(f))$.

**Proof.**

1. We have to find the cycle decomposition of the union of several $f$-pseudocycles each having vertex sets of size $|\mathcal{V}(P_{T,j})|$, which are cyclically permuted by $g$. If $g$ cyclically permutes $k$ $f$-pseudocycles $P_1,\ldots,P_k$ from $P_T$, then due to the bijectivity of $g$ the restriction of $g$ to the union $\bigcup_{j=1}^{k} P_j$ decomposes into $|\mathcal{V}(P_{T,j})|$ cycles of length $k$.

2. Proved similarly to 3. of Theorem 2.3. The cycle decomposition of $Z(P)$ has $GCD(i,j)$ $g$-cycles of length $\frac{i}{GCD(i,j)}$. It induces cycle decomposition of $P_T$ into $g$-cycles of the same length. The exact number of these $g$-cycle is not given here.

3. $P_T$ are permuted independently, for each tree cycle $T$ the number of permutations of $P_T$ is $n_T!$, the number of automorphisms of $Z(T)$ is $s_T$. Hence the number of different restrictions of commuting permutations on cycles of $P_T$ is $n_T! \cdot s_T^{n_T}$, each directed tree of the tree cycle $T$ can be twisted by an automorphism, formula follows
Remark 2.10. Thus an $f$-commuting permutation $g$ independently permutes $f$-pseudocycles having isomorphic cycles of directed trees. From Lemma 2.7 it follows that if the tree cycle $T$ is such that it is fixed up to isomorphism by cyclic permutations of order, which is a divisor of the total cycle order, then corresponding $f$-pseudocycles may allow more than one restriction of $f$-commuting permutation on their cycles (for each pair of pseudocycles).

Additionally directed trees can be independently twisted by automorphisms.

Example 2.11. Let $S = \{0, ..., 7\}$ and the $f$-graph is given in Figure 3 below.

![Fig.3. - the f-graph for Example 2.11](image)

In this case there is a single $T$, $n_T = 1$, $s_T = 2$. There are 2 directed trees each having 2 automorphism. Thus $|C_{bij}(f)| = 2 \cdot 2^2 = 8$.

2.1.3. Endofunctions commuting with a fixed permutation. In this subsection we describe arbitrary endofunctions commuting with a permutation $f$ of a finite set $S$.

Lemma 2.12. Let $S$ be a finite set, $f$ - a permutation on $S$, $g$ - an arbitrary $f$-commuting endofunction: $g \in \text{Fun}(S)$, $fg = gf$. Let $Z = (x, f(x), ..., f^{k-1}(x))$ be an $f$-cycle of length $k \in \mathbb{N}$.

Then there is $l \in \mathbb{N}$ such that $l|k$ and $g(Z) = (g(x), g(f(x)), ..., g(f^{l-1}(x)))$ is an $f$-cycle of length $l$. The $f$-cycle $g(Z)$ is determined by the $g$-image of any element of $Z$.

Proof. $fg = gf$ implies $g(f^i(x)) = f^i(g(x))$ for all $x \in S$ and $i \in \mathbb{N}$. Suppose that $g(x)$ belongs to an $f$-cycle of length $l$. Since $f^k(x) = x$ we must have $g(x) = f^k(g(x))$. It follows, that $l|k$ and for any $n \in \mathbb{N}$ we have $(gf^n)(x) = (f^{n(\text{mod } l)}g)(x)$, where $0 \leq n(\text{mod } l) < l$.

If $x \in Z$ and $y = f^i(x)$, then $g(y) = f^i(g(x))$. □

Remark 2.13. Lemma 2.12 amounts to the fact that a homomorphic image of a directed cycle of length $k$ is a directed cycle of length $l$ with $l|k$. For example, if
\[ f(x) = x \text{ and } fg = gf, \text{ then } f(g(x)) = g(x). \]

**Example 2.14.** A cycle \((0, 1, 2, 3)\) can be homomorphically mapped by \(g\) to the cycle \((4, 5)\) as shown below.

\[
\begin{array}{c}
\text{Fig.4. - the } f\text{-graph for Example 2.14}
\end{array}
\]

**Theorem 2.15.** Let \(S\) be a finite set, \(f\) - a permutation, \(g\) - an arbitrary \(f\)-commuting endofunction: \(fg = gf\). Denote the set of \(f\)-cycles of length \(i\) as \(Z_i = \bigcup_{j=1}^{n_i} Z_{i,j}\), denote \(Z = \bigcup_i Z_i\). For any \(f\)-cycle \(Z_{i,j}\) choose an element \(x_{i,j} \in V(Z_{i,j})\), denote \(X = \bigcup_{i,j} x_{i,j}\).

Then \(g\) is bijectively determined by the pair \([\tilde{g}, g|_X]\), where

1. \(\tilde{g}\) is a function \(\tilde{g} : Z \rightarrow Z\) such that \(|V(\tilde{g}(Z_{i,j}))|\) divides \(i\) for any \(i, j\).
2. \(g|_X\) is the restriction of \(g\) on \(X\), \(g|_X : X \rightarrow S\), where \(g(x_{i,j}) \in V(\tilde{g}(Z_{i,j}))\).

**Proof.** By Lemma 2.12 an \(f\)-cycle \(Z_{i,j}\) of length \(i\) can be homomorphically mapped only to an \(f\)-cycle of length \(l\), where \(l|i\), this defines \(\tilde{g}\). For any \(f\)-cycle \(\tilde{g}\) is determined by the \(g\) image of one element, say, \(x_{i,j} \in Z_{i,j}\). Images of \(f\)-cycles can be chosen independently. \(\Box\)

**Theorem 2.16.** Let \(S\) be a finite set, \(f\) - a permutation, \(g\) - an arbitrary \(f\)-commuting endofunction: \(fg = gf\). Denote the set of \(f\)-cycles of length \(i\) as \(Z_i = \bigcup_{j=1}^{n_i} Z_{i,j}\), denote \(Z = \bigcup_i Z_i\). Let \(\tilde{g}\) be defined as in Theorem 2.15.

Then

1. Any \(\tilde{g}\)-cycle of length \(k > 1\) permuting \(f\)-cycles of length \(i\) decomposes into \(i\) \(g\)-cycles of length \(k\), any \(\tilde{g}\)-cycle of length \(1\) corresponding to a \(f\)-cycle of length \(i\) fixed by \(g\) decomposes into \(\text{GCD}(i, j)\) \(g\)-cycles of length \(\frac{\text{GCD}(i,j)}{\text{GCD}(i,j)}\) for some \(j\).
2. If \(g(Z_{i,j}) = Z_{k,l}\) with \(k|i\), then the restriction of \(g\) on \(Z_{i,j} \cup Z_{k,l}\) decomposes into a forest of \(k\) directed trees of size \(\frac{k}{i} + 1\).
3. \(|C(f)| = \prod_{d|i}(\sum_{n|d} n^a)^{n_a}\), where \(n_a\) is the number of \(f\)-cycles of length \(a\).
Proof. 1. Proved similarly to 2. and 3. of Theorem 2.8.

2. If \( g(Z_{i,j}) = Z_{k,l} \) with \( k \mid i \), then the inverse image of any element of \( \mathcal{V}(Z_{k,l}) \) contains \( \frac{i}{k} \) elements.

3. Any element of \( \mathcal{V}(Z_i) \) can be mapped to an admissible \( f \)-cycle of length \( d \) in \( d \) ways. Therefore by the sum rule the total number of admissible mappings of one vertex is \( \sum d \mid i \). The final formula follows by the product rule since images of \( f \)-cycles can be constructed independently. \( \Box \)

Remark 2.17. The function \( \tilde{g} \) described in Theorem 2.12 defines a pseudoforest on \( Z \). Thus to define a function commuting with a permutation \( f \) we need to choose an appropriate endofunction of \( Z \) and a set of representatives for \( f \)-cycles.

Remark 2.18. Combining proposals 2. and 3. of Theorem 2.15 we can deduce the pseudotree decomposition of \( g \): any \( \tilde{g} \)-pseudocycle \( P \) decomposes into \( g \)-pseudocycles, which can be recovered starting from the decomposition of \( Z(P) \).

Example 2.19. Let \( f \) be a permutation having cycle type \( 4^x3^y2^z1^t \) (where \( i^j \) means, that there are \( j \) cycles of length \( i \)). We can check, that

\[
|C(f)| = t^t(t + z)^z(t + 3y)^y(t + 2z + 4x)^x.
\]

2.1.4. Arbitrary endofunctions commuting with a fixed arbitrary endofunction. Finally in this subsection we consider the general case for a finite set \( S - C(f) \) with \( f \in \mathcal{F}_\text{un}(S) \). In this case we describe restrictions of commuting functions on individual weakly connected components.

Theorem 2.20. Let \( S \) be a finite set, \( f \) and \( g \) - commuting \( S \)-endofunctions: \( fg = gf \). Denote the set of directed cycles of length \( i \) of the \( f \)-pseudoforest as \( Z_i = \bigsqcup_{j=1}^i Z_{i,j} \). Denote \( Z = \bigcup_i Z_i \).

Then the restriction of \( g \) on \( \mathcal{V}(Z) \) determines a function \( \tilde{g} : Z \to Z \), such that \( |\tilde{g}(Z_{i,j})| \) divides \( i \) for all \( i,j \).

Proof. Similarly to Theorem 2.15 \( \Box \)

The next two lemmas and Theorem 2.25 deal with images of directed trees under graph homomorphisms.

Lemma 2.21. Let \( S \) be a finite set, \( f \) and \( g \) - commuting endofunctions: \( fg = gf \). Let \( P \) be an \( f \)-pseudocycle. Let \( Z \) be the union of \( f \)-cycles. Let \( x \in \mathcal{V}(P) \), \( y \in f^{-1}(x) \).

Then
1. if \( g(x) \in V(Z) \), then either \( g(y) \) and \( g(x) \) belong to the same \( f \)-cycle, or 
\[ \phi(g(y)) = 1; \]
2. if \( \phi(g(x)) = i > 0 \), then \( \phi(g(y)) = i + 1 \).

Proof. Follows from commutativity of \( f \) and \( g \). □

Lemma 2.22. Let \( S \) be a finite set, \( f \) and \( g \) - commuting endofunctions: \( fg = gf \).

Let \( P \) be an \( f \)-pseudocycle, \( T \) is the (full) directed tree of \( P \) with the root \( z \).

Then there is \( A \subseteq V(T\setminus\{z\}) \), such that:

1) for any \( x \in V(T) \) we have that \( \phi(g(x)) = 1 \) iff \( x \in A \);
2) there are no distinct elements \( x \in A, y \in A \), such that \( x < y \) in the tree order;
3) for any \( x \in A \), if \( x \leq y \), then \( g(x) \) and \( g(y) \) are in the same directed tree of \( g(P) \) and \( g(x) \leq g(y) \).

Proof. We construct \( A \) using 1) as its definition. Alternatively we construct it iteratively considering images under \( g \) of the sequence of subsets \( [D_1(T), D_2(T), \ldots] \).

Consider \( D_1(T) \). For any \( x \in D_1(T) \) either \( g(x) \in V(Z(g(P))) \) or \( \phi(g(x)) = 1 \). Define \( A(1) = \{ x \in D_1(T) | \phi(g(x)) = 1 \} \). If \( y \in V(T) \) is such that \( y \geq x \) for some \( x \in A_1 \), then \( \phi(g(y)) > 1 \) and \( g(y) \geq g(x) \).

Consider \( D_2(T) \). For any \( x \in D_2(T) \) either \( g(x) \in V(Z(g(P))) \), \( \phi(g(x)) = 1 \) or \( \phi(g(x)) = 2 \) (this happens if \( x > x_1 \) for some \( x_1 \in A(1) \)). Define \( A(2) = \{ x \in D_2(T) | \phi(g(x)) = 1 \} \). If \( y \in V(T) \) is such that \( y \geq x \) for some \( x \in A(2) \), then \( \phi(g(y)) > 1 \) and \( g(y) \geq g(x) \).

We continue this process until we reach the maximal \( k \) such that \( D_k(T) \neq \emptyset \). Define \( A = \bigcup_{h \geq 1} A(h) \). Statement 1) follows by construction, statements 2),3) follow by Lemma 2.21 □

Let \( T \) be a directed tree with the root \( z \). We call a \( T \)-vertex subset \( A \subseteq V(T\setminus\{z\}) \) incomparable vertex subset provided there are no two distinct \( x, y \in A \) such that \( x < y \) in the tree order. Denote by \( Inc(T) \) the set of all incomparable vertex subsets of \( T \).

Example 2.23. Let \( T \) be the directed tree given below:
Then $\text{Inc}(T)$ contains $\emptyset$, five 1-element subsets $\{1, 2\}, \{1, 5\}, \{2, 3\}, \{2, 4\}, \{3, 4\}, \{3, 5\}, \{4, 5\}$ and one 3-element subset $\{3, 4, 5\}$.

**Remark 2.24.** For any directed tree the minimal number of elements of an incomparable vertex subset is 0, but the maximal number of elements is the number of vertices of indegree 0 (leaves).

**Theorem 2.25.** Let $S$ be a finite set, $f$ and $g$ - commuting endofunctions: $fg = gf$.

Let $P$ be a $f$-pseudocycle of cycle length $i$ with the $f$-cycle $Z = (z_0, \ldots, z_{i-1})$.

Then the restriction $g|_P$ is bijectively determined by the triple $\tau = [g(z_0), [A], [\Phi]]$, where

1. $[A]$ is a sequence of incomparable vertex subsets: $[A] = [A_0, \ldots, A_{i-1}]$, $A_k \subseteq \mathcal{V}(T(z_k), z_k)$ for $k \in \{0, \ldots, i-1\}$ and any $A_k$ is an incomparable vertex subset.
2. $[\Phi]$ is a sequence of ordered sets of directed tree homomorphisms - $[\Phi] = [\Phi_0, \ldots, \Phi_{i-1}]$ with $\Phi_k = [\varphi_x]_{x \in A_k}$, where $\varphi_x \in \text{Hom}(T(x), T(g(x)))$.

**Proof.** Considering the action of $g$ on $f$-cycles we get that the image of $Z$ is an $f$-cycle of length $l$, where $l | i$. This cycle is bijectively determined by $g(z_0)$. See Theorem 2.20. Considering images of directed trees consecutively increasing vertex heights we get $[A]$. Use Lemma 2.22. For any $k$ and $x \in A_k$, the subtree $T(x)$ is independently homomorphically mapped to $T(g(x))$ and taking sequences of homomorphisms over all $k$ and $x$ we get $[\Phi]$. □

**Remark 2.26.** Thus to define an $f$-commuting function on an $f$-pseudocycle we need

1. to map the $f$-cycle to an $f$-cycle of appropriate length,
2. to define vertices whose inverse images with respect ro $f$ leave the directed cycle (vertices forming the sets $A_k$) by travelling backwards the edges of directed
trees,
3. to define graph homomorphisms for remaining subtrees.

Note that an \( f \)-pseudocycle of cycle length \( i \) can be mapped by an \( f \)-commuting endofunction \( g' \) to any \( f \)-pseudocycle of cycle length \( l \), \( l \mid i \). To prove that the set of commuting functions is nonempty we can take \( g' \), which sends all vertices of positive height to the \( f \)-cycle.

**Remark 2.27.** Given an \( f \)-commuting function \( g \) we can determine \( g \)-pseudocycles by first considering the \( g \)-image of the set of \( f \)-cycles and then considering \( g \)-images of the directed trees of \( f \).

**Example 2.28.** Let \( S = \{0, ..., 9\} \) and the \( f \)-graph be given in Figure 4 below

![Fig.4. - the f-graph for Example 2.28](image)

Let us find \( |C(f)| \). There is one \( f \)-cycle \((0,1,2,3)\), therefore \( |C(f)| = \sum_{i=0}^{3} N_{0i}, \) where \( N_{0i} \) is the number of \( f \)-commuting functions sending \( 0 \) to \( i \). Furthermore, \( N_{0i} = \prod_{j=0}^{3} M_{ij} \), where \( M_{ij} \) is the number of possible ways to map the directed tree \( T_j \) (the full directed tree with root \( j \)) if \( 0 \) is mapped to \( i \). Below we give the table for \( M_{ij} \).

![Fig.4. - table of \( M_{ij} \) values for Example 2.28](image)

We have that \( |C(f)| = 2 \cdot 3^2 \cdot 3^2 + 2 \cdot 3^2 + 3 \cdot 2^2 + 3 \cdot 2^2 \cdot 3^2 = 300. \)

We end this section with an enumerative combinatorics result - a formula for
counting graph homomorphisms between two pseudocycles.

**Theorem 2.29.** Let \( P = (T_0, ..., T_{m-1}) \) be a pseudocycle of cycle length \( m \) with the directed cycle \( Z = (z_0, ..., z_{m-1}) \). Let \( P' = (T'_0, ..., T'_{l-1}) \) be a pseudocycle of cycle length \( l \) and the directed cycle \( (z'_0, ..., z'_{l-1}) \), where \( l \mid m \). Let a graph homomorphism \( g : P \to P' \) be defined by the triple \([g(z_0), [A], [\Phi]]\) as in Theorem 2.27.

Then

1. \( \text{Hom}(P, P') = \bigcup_{k=0}^{l-1} \text{Hom}(z_0, z'_k) \), where \( \text{Hom}(z_0, z'_k) \) is the set of graph homomorphisms \( P \xrightarrow{\Phi} P' \) sending \( z_0 \) to \( z'_k \);
2. \( \text{Hom}(z_0, z'_k) = \bigcup_{[A] \in \text{Inc}(P)} \text{Hom}(z_0, z'_k, [A]) \), where \( \text{Hom}(z_0, z'_k, [A]) \) is the set of graph homomorphisms \( P \xrightarrow{\Phi} P' \) sending \( z_0 \) to \( z'_k \) with the sequence of incomparable vertex subsets \([A] \), the disjoint union is taken over \( \text{Inc}(P) \) - all possible choices of sequences of incomparable subsets \([A] \);
3. \( \text{Hom}(z_0, z'_k, [A]) = \bigotimes_{i=0}^{m-1} \bigotimes_{h \geq 1 \ x \in A_i(h)} \text{Hom}(\mathcal{T}(x), \mathcal{T}'_{i+k-h+1(\text{mod} \ l)}) \)
4. \( |\text{Hom}(P, P')| = \sum_{k=0}^{l-1} |\text{Hom}(z_0, z'_k)| = \sum_{k=0}^{l-1} \sum_{[A] \in \text{Inc}(P)} |\text{Hom}(z_0, z'_k, [A])| = \)

\[
= \sum_{k=0}^{l-1} \prod_{i=0}^{m-1} \prod_{h \geq 1 \ x \in A_i(h)} |\text{Hom}(\mathcal{T}(x), \mathcal{T}'_{i+k-h+1(\text{mod} \ l)})|
\]

**Proof.** 1. A homomorphism \( P \xrightarrow{\Phi} P' \) sends \( z_0 \) to \( z' \).

2. A homomorphism \( P \xrightarrow{\Phi} P' \) determines the set \([A] = [A_0, ..., A_{m-1}]\) uniquely.

3. A homomorphism \( g : P \xrightarrow{\Phi} P' \) is uniquely determined by the sequence \([g|_{T_0}, ..., g|_{T_{m-1}}]\). Restrictions of homomorphisms to directed trees of \( P \) can be chosen independently. Furthermore, for any \( i \) this restriction \( g|_{T_i} \) is determined by \( A_i = [A_i(1), A_i(2), ...] \). For any \( x \in A_i(h) \) the subtree \( \mathcal{T}(x) \) is independently homomorphically mapped to the root-truncated directed tree \( \mathcal{T}'_{i+k-h+1(\text{mod} \ l)} \). The \(-h + 1\) term in the index corresponds to travelling along the cycle backwards \( h - 1 \) steps.

4. The formula follows counting elements of \( \text{Hom}(P, P') \) using the previous statements of this theorem, applying the sum rule and the product rule. \( \square \)

**Remark 2.30.** Summation variables in statement 4 of Theorem 2.27 can be swapped.

**Example 2.31.**

Consider again the graph of Example 2.28. The vertex sets of root-truncated full directed trees are: \( V_0 = \mathcal{V}(\mathcal{T}_0) = \{4, 5\} \), \( V_1 = \{6, 7\} \), \( V_2 = \{8, 9\} \), \( V_3 = \emptyset \). The
incomparable vertex subsets are: \( \text{Inc}(T_0) = 2^{V_0} \setminus V_0, \text{Inc}(T_1) = 2^{V_1}, \text{Inc}(T_2) = 2^{V_2}, \text{Inc}(T_3) = \emptyset \). \( \text{Inc}(P) = \text{Inc}(T_0) \times \text{Inc}(T_1) \times \text{Inc}(T_2) \times \text{Inc}(T_3) \).

In terms of Example 2.28 and Theorem 2.29 we have that

\[
N_{\text{fg}} = \sum_{|A| \in \text{Inc}(P)} |\text{Hom}(0, i, [A])|.
\]

Let us check using Theorem 2.29 that \( N_{\text{fg}} = 12 \) which coincides with the computation in Example 2.28. Nonzero contributions can be given by incomparable subsets \( \emptyset, \{5\} \) of \( \text{Inc}(T_0) \), subset \( \emptyset \) of \( \text{Inc}(T_1) \), subsets \( \emptyset, \{8\}, \{9\} \), \( \{8, 9\} \) of \( \text{Inc}(T_2) \) and subset \( \emptyset \) of \( \text{Inc}(T_3) \). We have eight sequences of incomparable vertex subsets: four sequences each contributing 1 and four sequences each contributing 2 (for example, \( \{\{5\}, \emptyset, \{8\}\} \)), thus \( N_{\text{fg}} = 12 \).

2.2. Generalizations to functions on infinite sets. Results for finite sets can be transferred to the case of infinite sets, where there are additional types of weakly connected components of functional graphs.

Lemma 2.32. Let \( S \) be a set, \( f \) and \( g \) - commuting \( S \)-endofunctions: \( fg = gf \). Let \( L \leq \Gamma(f) \) be an \( f \)-line, let \( R \leq \Gamma(f) \) be an \( f \)-ray.

Then

1. \( g(L) \) is an \( f \)-line, an \( f \)-cycle or an \( f \)-cycle with an infinite directed tree - infinite directed path.
2. \( g(R) \) is an \( f \)-ray, an \( f \)-cycle or an \( f \)-cycle with a finite directed path.

Proof. Assume that \( L = (V_L, E_L) \), where \( V_L = \bigcup_{a \in \mathbb{Z}} x_a, E_V = \bigcup_{a \in \mathbb{Z}} [x_a, x_{a+1}] \), \( R = (V_R, E_R) \), where \( V_R = \bigcup_{a \in \mathbb{Z}} x_a, E_R = \bigcup_{a \in \mathbb{Z}} [x_a, x_{a+1}] \).

If \( g \) is injective on \( L \) (or \( R \)), then \( g(L) \) (or \( g(R) \)) is an \( f \)-line (or \( f \)-ray).

If \( g \) is not injective on \( L \) (or \( R \)), then there are two \( L \) (or \( R \)) vertices \( x_m \) and \( x_{m+a} \), \( a > 0 \), such that \( g(x_{m+a}) = g(x_m) \). It follows, that \( g(x_m) = g(f^a(x_m)) = f^a(g(x_m)) \) and \( g(x_p) \) for all \( p \geq m \) belong to a finite \( f \)-cycle \( Z \). If for all \( x \in V(L) \) (or \( x \in V(R) \)) we have that \( g(x) \in V(Z) \), then \( g(L) = Z \) (or \( g(R) = Z \)).

If there is \( n < m \) such that \( g(x_n) \) does not belong to \( V(Z) \), then \( g(L) \) (or \( g(R) \)) is \( Z \) with an infinite (or finite) directed path having its root in \( V(Z) \).

Proposition 2.33. Let \( S \) be a set, \( f \) and \( g \) - commuting \( S \)-endofunctions: \( fg = gf \). Let \( X \leq \Gamma(f) \) be an \( f \)-line or \( f \)-ray.

Then \( g(X) \) contains a directed \( f \)-cycle iff \( g \) is not injective on \( V(X) \).
Proof. If there is \( k \in \mathbb{N} \) and \( x, y \in S \) such that \( f^k(x) = y \) and \( g(x) = g(y) \), then \( g(f^{k+\alpha}(x)) = f^\alpha(g(f^k(x))) = f^\alpha(g(y)) \) for all \( \alpha \in \mathbb{N} \). It follows that the induced \( f \)-subgraph having vertex set \( \bigcup_{k \geq 0} f^k(x) \) is a directed \( f \)-cycle.

If \( g(X) \) contains an \( f \)-cycle \( Z \) as a subgraph, then there is \( v \in V(X) \) such that \( f^k(g(v)) = g(v) \) for some \( k \in \mathbb{N} \). It follows that \( g(v) = g(f^k(v)) \), hence \( g \) is not injective on \( X \). \( \blacksquare \)

**Lemma 2.34.** Let \( S \) be a set, \( f \) and \( g \) - commuting \( S \)-endofunctions: \( fg = gf \).

Let \( P \) be an \( f \)-pseudoline (or \( f \)-pseudoray) with a directed line (ray) \( L \) (or \( R \)). \( T \) is a directed tree of \( P \) with the root \( z \in V(L) \) (or \( z \in V(R) \)).

Then there is \( A \subseteq V(T \setminus \{z\}) \) such that

1) there are no distinct elements \( x, y \in A \), such that \( x < y \) in the tree order;
2) \( \phi(g(x)) = 1 \) iff \( x \in A \);
3) for any \( x \in A \) if \( x \leq y \) then \( g(x) \) and \( g(y) \) are in the same directed tree.

**Proof.** Similar to Lemma 2.22. \( \blacksquare \)

**Theorem 2.35.** Let \( S \) be a set, \( f \) and \( g \) - commuting \( S \)-endofunctions: \( fg = gf \).

Then

1. the \( g \)-image of an \( f \)-pseudocycle of cycle length \( m \) is an \( f \)-pseudocycle of cycle length \( l \), where \( l \mid m \);
2. if \( P \) is an \( f \)-pseudocycle with directed cycle \( Z = (z_0, ..., z_{m-1}) \) and directed tree cycle \( T = (T_0, ..., T_{m-1}) \), then the restriction \( g_{|V(P)} \) is bijectively defined by the triple \( \tau = [g_Z, [A], [\Phi]] \), where
   a) \( [A] = [A_0, ..., A_{m-1}] \), where \( A_i \subseteq V(T_i \setminus z_i) \) such that for any \( a \in V(P) \) we have that \( \phi(g(a)) = 1 \) iff \( a \in A_i \) for some \( i \) (\( \phi \) is meant with respect to \( g(Z) \));
   b) \( [\Phi] = [\Phi_0, ..., \Phi_{m-1}] \), where \( \Phi_i = [\varphi_{i,x}]_{x \in A_i} \), where \( \varphi_{i,x} \in \text{Hom}(T(x), T(g(x))) \).

**Proof.** Similar to Theorem 2.26. \( \blacksquare \)

**Theorem 2.36.** Let \( S \) be a set, \( f \) and \( g \) - commuting \( S \)-endofunctions: \( fg = gf \).

Let \( P \) be an \( f \)-pseudoline (or \( f \)-pseudoray) with a directed line \( Z = (\ldots, z_0, z_1, \ldots) \) (or a directed ray \( Z = (z_1, \ldots) \)), define \( T_i = T(z_i) \).

Then

1. if \( g|_Z \) is injective, then \( g|_P \) is bijectively defined by the sequence \( \tau = [g|_Z, [\mathcal{A}], [\Phi]] \), where
   a) \( [\mathcal{A}] \) is the sequence \( [A_i]_{i \in Z} \) (or \( [A_i]_{i \in \mathbb{N}} \)), where \( A_i \subseteq V(T_i \setminus z_i) \) such that
for any $a \in \mathcal{V}(P)$ we have that $\phi(g(a)) = 1$ iff $a \in \mathcal{A}_i$ (it is meant with respect to $g(Z)$);

(b) $[\Phi]$ is the sequence $[\Phi_i]_{i \in \mathbb{Z}}$ (or $[\Phi_i]_{i \in \mathbb{N}}$), where $\Phi_i = [\varphi_{i,x}]_{x \in \mathcal{A}_i}$ with $\varphi_{i,x} \in \text{Hom}(\mathcal{T}(x), \mathcal{T}(g(x)))$.

2. If $g|_Z$ is not injective and $g(z_p) = g(z_q)$ with $p < q$, then $g|_P$ is bijectively defined by the sequence $\tau = [g|_{Z_{p,q}}, \mathcal{A}, [\Phi]]$, where

(a) we denote by $Z_{p,q}$ the induced subgraph of $Z$ with the vertex set $\{z_p, \ldots, z_q\}$;

(b) $[\mathcal{A}] = [\mathcal{A}_i]_{i \geq p}$, where $\mathcal{A}_i \subseteq \mathcal{V}(\mathcal{T}(z_i) \setminus z_i)$ such that $\phi(g(a)) = 1$ iff $a \in \mathcal{A}_i$ (it is meant with respect to $g(Z)$);

(c) $[\Phi] = [\Phi_i]_{i \geq p}$ with $\Phi_i = [\varphi_{i,x}]_{x \in \mathcal{A}_i}$, where $\varphi_{i,x} \in \text{Hom}(\mathcal{T}(x), \mathcal{T}(g(x)))$.

Proof. 1. If the injectivity condition holds for $g(Z)$, then for each $z \in Z$ the restriction $g|_{\mathcal{T}(z)}$ is determined by $\mathcal{A} \subseteq \mathcal{V}(\mathcal{T}(z) \setminus z)$ containing vertices $a$ such that $\phi(g(a)) = 1$ and homomorphisms in $\text{Hom}(\mathcal{T}(a), \mathcal{T}(g(a)))$ mapping the remaining subtrees $\mathcal{T}(a)$ for each such $a$.

2. If the injectivity condition does not hold for $g(Z)$, then the cyclic part of $g(Z)$ is determined by $g|_{Z_{p,q}}$, for each $z_n \in Z$ with $n \geq p$ the restriction $g|_{\mathcal{T}(z_n)}$ is determined as in 1.

Use Lemma 2.32 and Lemma 2.34.

2.3. Conclusion. We have described endofunctions $g$ commuting with a given endofunction $f$. Descriptions are given in terms of their functional graphs, as homomorphisms of $f$-graphs, for 4 subcases: 1) permutations commuting with a permutation, in this case weakly connected components of $(f, g)$-graphs can be interpreted as $g$-cycles, which permute $f$-cycles; 2) permutations commuting with an arbitrary function, in this case weakly connected components of $(f, g)$-graphs can be interpreted as $g$-cycles, which permute $f$-cycles; 3) arbitrary functions commuting with a permutation, in this case $(f, g)$-graphs can be interpreted as $g$-pseudoforests with vertices being $f$-cycles; 4) arbitrary functions commuting with an arbitrary function, this is the most complex case: restrictions on $f$-cycles behave as in case 3) and directed trees may be either mapped to cycles or leave cycles and get mapped to directed trees. Results for finite sets can be relatively straightforwardly generalized for arbitrary sets. Future research may be stimulated by questions related to 1) interpretation of graph-theoretic results in terms of purely functional properties, 2) enumerative combinatorics of commuting functions and 3) graph structure of functions satisfying other relations.

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