Deformed oscillators algebra formulation of the Nonlinear Schrödinger hierarchy and of its symmetry

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Abstract

We present a self-contained formulation of the Nonlinear Schrödinger hierarchy and its Yangian symmetry in terms of deformed oscillator algebra (Z.F. algebra). The link between Yangian $Y(gl_N)$ and finite $W(gl_{pN}, N, gl_p)$ algebras is also illustrated in this framework.

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The present text is a summary of the original work [1], done in collaboration with M. Mintchev, P. Sorba and Ph. Zaugg. The proofs and detailed calculations can be found there, as well as a more precise bibliography.

1 The NLS equation

We start with the $N$-vectorial version of the Non Linear Schrödinger equation (NLS) in 1+1 dimension, on the real line:

$$i \frac{∂φ(x, t)}{∂t} + \frac{∂^2 φ}{∂^2 x} = 2g (φ^† ∙ φ)φ$$ with $φ = \begin{pmatrix} φ_1 \\ φ_2 \\ … \\ φ_N \end{pmatrix}$ \hspace{1cm} (1.1)

where $φ_j$ are complex fields, and the coupling constant $g$ is chosen real and negative (repulsive case). The corresponding Hamiltonian is given by

$$H = \int dx \left( \frac{∂φ^†_i}{∂x} \frac{∂φ_i}{∂x} + gφ^†_iφ^†_jφ_iφ_j \right),$$ \hspace{1cm} (1.2)

The solution is given by a series expansion in powers of $g$:

$$φ(x, t) = \sum_{ℓ=0}^{∞} (-g)^ℓ φ^{(ℓ)}(x, t) \text{ where } φ^{(ℓ)} = \begin{pmatrix} φ_{1}^{(ℓ)} \\ φ_{2}^{(ℓ)} \\ … \\ φ_{N}^{(ℓ)} \end{pmatrix} \hspace{1cm} (1.3)$$

$$φ_{j0}^{(ℓ)}(x, t) = \int \frac{d^{ℓ+1}ν d^{ℓ}μ}{(2π)^{2ℓ+1}} \frac{a_{i}^{j1}(μ_1)a_{j2}(μ_2)⋯a_{jℓ}(μ_ℓ)a_{j}(ν_ℓ)⋯a_{j}(ν_1)a_{j0}(ν_0)}{\prod_{j=1}^{ℓ}(μ_j - ν_j - iέ)(μ_j - ν_{j-1} - iέ)} e^{im_ℓ} \hspace{1cm} (1.4)$$

This solution is valid for the classical case [2] ($a$’s are then arbitrary functions) as well as for the quantum case [3], where in that case $a(μ)$ and $a^†(μ)$ generate a deformed oscillator algebra (or ZF algebra [4]):

$$a_i(μ)a_j(ν) = R_{ij}^{kl}(ν - μ)a_k(ν)a_l(μ) \hspace{1cm} (1.5)$$

$$a_i^†(μ)a_j^†(ν) = a_i^†(ν)a_i^†(μ)R_{ij}^{kl}(ν - μ) \hspace{1cm} (1.6)$$

$$a_i(μ)a_j^†(ν) = a_i^†(ν)R_{ij}^{kl}(μ - ν)a_l(μ) + δ_i^j δ(μ - ν) \hspace{1cm} (1.7)$$

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Note that the indices $i, j, \ldots$ run from 1 to $N$, because the vector $\phi$ is in the fundamental representation of $sl(N)$. In the quantum case, the constructed fields obey canonical commutation relations:

$$
[\varphi_j(x, t), \varphi_k(y, t)] = 0 \quad ; \quad [\varphi^\dagger_j(x, t), \varphi^\dagger_k(y, t)] = 0
$$

(1.8)

The R-matrix appearing in this algebra is just the one of $Y(sl(N))$. This is not surprising, since $Y(sl(N))$ is a symmetry of NLS. In fact, it has been shown in [5] that, for $N = 2$, the Yangian generators can be expressed in term of $\phi$ and Pauli matrices $t^a$:

$$
J^a = \int dx \, \phi^\dagger(x) t^a \phi(x)
$$

(1.9)

$$
S^a = \frac{i}{2} \int dx \, \phi^\dagger(x) t^a \partial_x \phi(x) - \frac{ig}{2} \int dx dy \, \text{sgn}(y - x) \left( \phi^\dagger(x) t^a \phi(y) \right) \phi^\dagger(x) \cdot \phi(y)
$$

where $J^a$ and $S^a$ stand for the Yangian generators $Q^a_0$ and $Q^a_1$ in the Drinfel’d presentation of the Yangians [6].

2 The NLS hierarchy

The Yangian symmetry of the NSL equation (and in fact of its whole hierarchy) is nicely formulated using the Quantum Inverse Scattering Method (QISM).

We start with the Lax operator of the QISM

$$
L(x|\lambda) = i\frac{\lambda}{2} \Sigma + \Omega(x), \quad \text{with} \quad \Sigma = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \\ 0 & \cdots & 0 & -1 \\ \end{pmatrix},
$$

(2.10)

and

$$
\Omega(x) = i\sqrt{g} \begin{pmatrix} \varphi_1(x) \\ \varphi_2(x) \\ \vdots \\ -\varphi^\dagger_1(x) \\ -\varphi^\dagger_2(x) \\ \cdots \\ -\varphi^\dagger_N(x) \end{pmatrix}
$$

(2.11)

and look at the quantum monodromy matrix $T(x, y|\lambda)$, defined by the equations

$$
\frac{\partial}{\partial x} T(x, y|\lambda) = :L(x|\lambda)T(x, y|\lambda):, \quad T(x, y|\lambda)|_{x=y} = I_{N+1},
$$

(2.12)
where \( I_{N+1} \) is the identity matrix. The infinite volume monodromy matrix \( T(\lambda) \) is then formally defined by

\[
T(\lambda) = \lim_{x \to \infty, y \to -\infty} E(-x|\lambda)T(x,y|\lambda)E(y|\lambda), \quad \text{where} \quad E(x|\lambda) = \exp(i\lambda x \Sigma/2).
\]

The commutation relations of \( T(\lambda) \) are computed using the canonical commutators of the \( \varphi \)’s and are encoded in the exchange relation

\[
R^+(\lambda - \mu) \ T(\lambda) \otimes T(\mu) = T(\mu) \otimes T(\lambda) \ R^-(\lambda - \mu) \quad (2.13)
\]

with the \( R \)-matrices

\[
R^\pm(\mu) = \begin{pmatrix}
-i\frac{g}{\mu - ig \mu} & \frac{1}{\mu - ig} E_{jj} \otimes E_{kk} + \frac{1}{\mu - ig} E_{aj} \otimes E_{j\alpha} \\
\frac{\mu + ig}{(\mu + i0)^2} E_{j,N+1} \otimes E_{N+1,j} + \frac{1}{\mu} E_{N+1,N+1} \otimes E_{N+1,N+1} \\
\pm \frac{i\pi}{(\mu + i0)^2} (E_{jj} \otimes E_{N+1,N+1} - E_{N+1,N+1} \otimes E_{jj}),
\end{pmatrix}
\]

where the latin (resp. greek) indices run from 1 to \( N \) (resp. \( N + 1 \)), and \( E_{ab} \) is the \((N + 1) \times (N + 1)\) matrix with 1 at position \((a, b)\). The term \( i\pi \) is a consequence of the principal value regularisation adopted when \( \mu \) goes to zero.

It will be convenient to rename some elements of the monodromy matrix such as \( D(\lambda) = T_{N+1,N+1}(\lambda) \) and \( b^i(\lambda) = T_{N+1,j}(\lambda) \). Further examination of some components of (2.13) yields the following relations

\[
D(\lambda) D(\mu) = D(\mu) D(\lambda), \quad \text{(2.15)}
\]

\[
D(\lambda) b^i(\mu) = \frac{\lambda - \mu + ig}{\lambda - \mu + i0} b^i(\mu) D(\lambda), \quad \text{(2.16)}
\]

\[
b^i(\lambda) b^k(\mu) = \frac{\lambda - \mu}{\lambda - \mu - ig} b^k(\mu) b^i(\lambda) - \frac{ig}{\lambda - \mu - ig} b^i(\mu) b^k(\lambda). \quad \text{(2.17)}
\]

The matrix element \( D(\lambda) \) serves as a generating operator-function for the commuting integrals of motion of the NLS model: \( D(\lambda) = 1 + \sum_{n=0}^{\infty} H_{(n)} \lambda^{-n-1} \). Consequently, eq. (2.15) implies that the \( H_{(n)} \)'s are all in involution. Indeed, one can compute that \( H_{(0)} \) is proportional to \( \int \, dx \, \phi^{\dagger} \phi \) (particle number operator), \( H_{(1)} \) to (up to \( H_{(0)} \) terms) \( -i \int \, dx \, \phi^{\dagger} \partial \phi \) (momentum) and \( H_{(2)} \) is the NLS Hamiltonian. The other \( H_{(n)} \)'s (\( n > 2 \)) define higher Hamiltonians which define the NLS hierarchy. The operators \( b^i(\mu) \) just correspond to the creation operators of theZF algebra previously mentioned. More precisely, one has \( a^{ij}(\lambda) = \frac{1}{\sqrt{g}} b^i(\lambda) D^{-1}(\lambda) \), and the commutation relations (2.16-2.17) ensure the right exchange relation for the \( a^{ij}(\lambda) \)'s. The operators \( a_j(\lambda) \) are the adjoint operators of these \( a^{ij}(\lambda) \)'s.
Finally, considering $\tilde{T}(\lambda)$, the $N \times N$ submatrix of $T(\lambda)$, and examining the appropriate components of (2.13), one deduces the following relations

$$\tilde{R}(\lambda - \mu) \tilde{T}(\lambda) \otimes \tilde{T}(\mu) = \tilde{T}(\mu) \otimes \tilde{T}(\lambda) \tilde{R}(\lambda - \mu)$$  \hspace{1cm} (2.18)

with yet another $R$-matrix

$$\tilde{R}(\lambda - \mu) = (\lambda - \mu) E_{jk} \otimes E_{kj} - igI_N \otimes I_N.$$  \hspace{1cm} (2.19)

This coincides precisely with the defining relation of the Yangian $Y(gl_N)$.

The fact that the Yangian algebra commutes with the Hamiltonians of the NLS hierarchy is a consequence of the exchange relation as well, since one extracts from (2.13) that

$$[\tilde{T}_{ij}(\lambda), D(\mu)] = 0$$  \hspace{1cm} (2.20)

3 Z.F. formulation

The two previous sections have both made appear the ZF algebra: the former as a "building block" for the quantum solutions of the NLS equation, and the latter as the "remains" of the matrix $T(\lambda)$ when one has picked up the Yangian $\tilde{T}(\lambda)$ and the Hamiltonians $D(\lambda)$. It is thus natural to wonder whether this ZF algebra can be the central element of the NLS hierarchy and of its symmetry. Indeed, since the field $\phi$ is built on it and as the Yangian generators are constructed on $\phi$ (at least for $N = 2$), one could think that the task is not hard. However, $\phi$ is given as a series expansion in $a$'s and $a^\dagger$'s, and the Yangian generators are polynomial in $\phi$, so that a direct calculation is almost impossible. Alternatively, we construct operators which have the right commutation relations with the $a$'s and $a^\dagger$'s: since the Fock space spanned by these latter is dense in the Hilbert space of the NLS model, this will be enough to identify the operators.

3.1 Hamiltonians of the hierarchy

We introduce the operators

$$\tilde{H}^{(n)} = \int d\mu \mu^n a^{\dagger j}(\mu) a_j(\mu)$$  \hspace{1cm} (3.21)

Using the relations (1.5-1.7), it is easy to show that

$$[\tilde{H}^{(n)}, a^{\dagger j}(\mu)] = \mu^n a^{\dagger j}(\mu) \quad ; \quad [\tilde{H}^{(n)}, a_j(\mu)] = -\mu^n a_j(\mu)$$  \hspace{1cm} (3.22)
which is just the definition of the Hamiltonians in the quantum NLS hierarchy.

Thanks to the simple expression (3.21), it is easy to deduce the general solution to the NLS hierarchy. In fact, the local fields \( \phi \) which has a time evolution given by the Hamiltonian \( H \sim \), have the same expansion \((\ref{eq:3.2})\) with now

\[
\Omega^{(n)}_t = \sum_{j=0}^\ell (x \nu_j - t\nu^\prime_j) - \sum_{j=1}^\ell (x \mu_j - t\mu^\prime_j) \quad (3.23)
\]

### 3.2 Yangian generators

In the same way, we define the operators

\[
J^a = \sum_{\ell=1}^\infty \frac{(-1)^{\ell+1}}{\ell!} J_{(\ell)}^a \quad \text{and} \quad S^a = \sum_{\ell=1}^\infty \frac{(-1)^{\ell+1}}{\ell!} S_{(\ell)}^a \quad (3.24)
\]

with

\[
J_{(\ell)}^a = \int d^\mu \, a^{ij_1}(\mu_1) a^{ij_2}(\mu_2) \ldots a^{ij_\ell}(\mu_\ell) \left( T^a_{j_1j_2 \ldots j_\ell} \right) a_{k_1}(\mu_1) \ldots a_{k_\ell}(\mu_\ell) \quad (3.25)
\]

\[
S_{(\ell)}^a = \int d^\mu \, a^{ij_1}(\mu_1) a^{ij_2}(\mu_2) \ldots a^{ij_\ell}(\mu_\ell) \left( \tilde{T}^a_{j_1j_2 \ldots j_\ell} \right) a_{k_1}(\mu_1) \ldots a_{k_\ell}(\mu_\ell) \quad (3.26)
\]

The tensor \( T^a \) is given by

\[
(T^a)_{j_1j_2 \ldots j_\ell}^{k_1k_2 \ldots k_\ell} = \sum_{m=1}^\infty (-1)^{m-1} \left( \frac{\ell - 1}{m - 1} \right) (t^a)_{j_m}^{k_m} \delta_{j_1}^{k_1} \ldots \delta_{j_{m-1}}^{k_{m-1}} \delta_{j_m}^{k_{m+1}} \ldots \delta_{j_\ell}^{k_{\ell}}. \quad (3.27)
\]

where \( t^a \) are the generators of \( sl_N \) in the fundamental representation \((t^a, t^b) = if^{abc} t^c\). With this form, \( J^a \) satisfies

\[
[J^a, a^{ik}(\mu)] = (a^i(\mu) t^a)^k, \quad [J^a, a_k(\mu)] = -(t^a a(\mu))_k \quad (3.28)
\]

\[
[J^a, J^b] = if^{abc} J^c \quad (3.29)
\]

Similarly, omitting the obvious \( \delta_{k_c}^b \) symbols, we write

\[
(T^a)_{j_1j_2 \ldots j_\ell}^{k_1k_2 \ldots k_\ell} = \sum_{m=1}^\ell (-1)^{m-1} \left( \frac{\ell - 1}{m - 1} \right) \left\{ \mu_m (t^a)_{j_m}^{k_m} - \frac{g}{2} f^a_{bc} \sum_{i=1}^{m-1} (t^b)_{j_i}^{k_i} (t^c)_{j_m}^{k_m} \right\}. \quad (3.30)
\]

With this definition, one gets

\[
[S^a, a^{ik}(\mu)] = \mu (a^i(\mu) t^a)^k - \frac{g}{2} f^a_{bc} (a^i(\mu) t^c)^k J^b, \quad (3.31)
\]

\[
[S^a, a_k(\mu)] = -\mu (t^a a(\mu))_k - \frac{g}{2} f^a_{bc} J^b (t^c a(\mu))_k, \quad (3.32)
\]

\[
[J^a, S^b] = if^{abc} S^c \quad (3.33)
\]
which are the required commutators for the Yangian generators $Q_a^0$ and $Q_a^1$.

Note that the Yangian generators are defined through a series expansion, not in the coupling constant, but rather in the number of creation operators $a^\dagger$. In the case of the $\phi$ fields, these two types of series indeed coincide.

Let us also remark that the form (3.25-3.26) clearly shows (using the relations (3.22)) that $J^a$ and $S^a$ commute with the Hamiltonians $H_{(n)}$, so that the Yangian is a manifest symmetry of the whole NLS hierarchy.

4 Fock space and finite $W(gl_{pN}, N.gl_p)$ algebras

It is now known [1] that there is an algebra homomorphism between the Yangian $Y(gl_N)$ and the finite $W(gl_{pN}, N.gl_p)$ algebras (for any $p$). This links can be illustrated in the framework of NLS hierarchy, using the ZF algebra.

Indeed, if one considers the Fock space $\mathcal{F}$ spanned by the $a^\dagger$’s, it is easy to see that it is built on subspaces $\mathcal{F}_p$ with fixed particle number (ie number of $a^\dagger$). Since the Yangian generators commute with the particle number operator, one can consider their restriction to $\mathcal{F}_p$. In that case, the series expansion defining the Yangian generators troncates at level $p$, and we are left with a polynomial (finite dimensional) algebra. This algebra is nothing but the $W(gl_{pN}, N.gl_p)$ algebra (in a special representation).

Thus, while on the full space $\mathcal{F}$ we have the complete Yangian symmetry, on each subspace $\mathcal{F}_p$ the Yangian troncates to a finite $W(gl_{pN}, N.gl_p)$ algebra which leaves $\mathcal{F}_p$ (globally) invariant.

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