Improved Bounds on the Restricted Isometry Constant for Orthogonal Matching Pursuit

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In this letter, we will give a counter example to show that for any given positive integer \( K \geq 2 \) and for any \( \frac{1}{\sqrt{K+1}} \leq t < 1 \), there always exist a \( K \)-sparse \( x \) and a matrix \( A \) with the restricted isometry constant \( \delta_{K+1} = t \) such that the OMP algorithm fails in \( K \) iterations.

Our proof is similar to the method used in [8], but the critical idea is different.

**Proof.** For any given positive integer \( K \geq 2 \), let

\[
B = \begin{bmatrix}
K & I \\
\lambda_1 & 0
\end{bmatrix}
\]

where \( 1 \) is a \( K \)-dimensional column vector with all of its entries being 1 and \( I_K \) is the \( K \)-dimensional identity matrix.

By some simple calculations, we can show that the eigenvalues \( \{\lambda_i\}_{i=1}^{K+1} \) of \( B \) are

\[
\lambda_1 = \ldots = \lambda_{K-1} = \frac{K}{K+1}, \quad \lambda_K = 1 - \frac{1}{\sqrt{K+1}}, \quad \lambda_{K+1} = 1 + \frac{1}{\sqrt{K+1}}.
\]

We let \( s = t - \frac{1}{\sqrt{K+1}} \) and \( C = B - sI_{K+1} \). Then by the aforementioned two equations, the eigenvalues \( \{\lambda_i\}_{i=1}^{K+1} \) of \( C \) are

\[
\lambda_1 = \ldots = \lambda_{K-1} = \frac{K}{K+1} - s,
\]

\[
\lambda_K = 1 - \frac{1}{\sqrt{K+1}}, \quad \lambda_{K+1} = 1 + \frac{1}{\sqrt{K+1}} - s.
\]

Since \( \frac{1}{\sqrt{K+1}} \leq t < 1 \), \( C \) is a symmetric positive definite matrix.

Therefore, there exists an upper triangular matrix \( A \) such that \( A^2A = C \).

By the aforementioned inequations and (3), \( \delta_{K+1}(A) = t \).

Let \( x = (1, 1, \ldots, 1) \in \mathbb{R}^{K+1} \), then \( x \) is \( K \)-sparse.

Let \( e_i, 1 \leq i \leq K+1 \), denote the \( i \)-th column of \( I_{K+1} \), then one can easily show that

\[
\frac{K}{K+1} - s = \max_{1 \leq i \leq K} |\langle Ae_i, Ax \rangle| \leq |\langle Ae_{K+1}, Ax \rangle| = \frac{K}{K+1}.
\]

so the OMP fails in the first iteration. Therefore, the OMP algorithm fails in \( K \) iterations for the given vector \( x \) and the given matrix \( A \).

In the following, we will improve the sufficient condition \( \delta_{K+1} < \frac{1}{\sqrt{K+1}} \) of the perfect recovery to \( \delta_{K+1} < \frac{1}{1+\sqrt{K}} \).

**Theorem 1:** Suppose that \( A \) satisfies the restricted isometry property of order \( K+1 \) with the restricted isometry constant

\[
\delta_{K+1} = \frac{1}{\sqrt{K+1}} + \frac{1}{1+\sqrt{K}}
\]

then the OMP algorithm can perfectly recover any \( K \)-sparse signal \( x \) from \( y = Ax \) in \( K \) iteration.

Before proving this theorem, we need to introduce the following two lemmas, where Lemma 2 was proposed in [9].

**Lemma 1:** For each \( x, x' \) supported on disjoint subsets \( S, S' \subseteq \{1, \ldots, n\} \) with \( |S| \leq s, |S'| \leq s' \), we have

\[
|\langle Ax, Ax' \rangle| = \delta_{s+s'} |x||x'| \leq 2(1 - \delta_{s+s'})
\]

and only if

\[
|\langle Ax, Ax' \rangle| = \delta_{s+s'} |x||x'| = 2(1 - \delta_{s+s'})
\]

**Proof.** Let

\[
\bar{x} = x/\|x\|_2, \quad \bar{x}' = x'/\|x'\|_2.
\]

since \( S \cap S' = \emptyset \), we have

\[
|\bar{x} + \bar{x}'|_2^2 = |x - x'|_2^2 = 2(1 - \delta_{s+s'})
\]

By (2), we have

\[
\frac{1}{4} |\langle A\bar{x}, A\bar{x}' \rangle| = \frac{1}{4} |\langle A(x + x') \rangle|^2 \leq 2(1 - \delta_{s+s'})
\]

By (3), (4) holds if and only if the equality in (8) holds. By (7), the equality in (8) holds if and only if

\[
|A(x + x')|_2^2 = 2(1 - \delta_{s+s'})\cdot |A(x - x')|_2^2 = 2(1 + \delta_{s+s'})
\]
or
\[ \|A(x - x')\|^2 = 2(1 - \delta_{x',x}), \|A(x + x')\|^2 = 2(1 + \delta_{x',x}). \]

Therefore, (3) holds if and only if
\[ \|A(x - x')\|^2 + \|A(x + x')\|^2 = 4. \]

Obviously, the aforementioned equation is equivalent to (5), so the lemma holds.

**Lemma 2:** For \( S \subseteq \{1, 2, \ldots, n\}, \) if \( \delta_{|S|} < 1, \) then
\[ (1 - \delta_{|S|}) \|x\|_2 \leq \|A^T S Ax\|_2 \leq (1 + \delta_{|S|}) \|x\|_2 \]
for any vector \( x \) supported on \( S. \)

We will prove it by induction. Our proof is similar to the method used in (3), but the critical idea is different.

**Proof of Theorem 2** Firstly, we prove that if (3) holds, then the OMP can choose a correct index in the first iteration.

Let \( S \) denote the support of the \( K \)-sparse signal \( x \) and let \( \alpha = \max_{i \in S} \|A e_i, Ax\|. \) Then
\[ |\langle A e_i, Ax \rangle| = \sum_{j \in S} x_j |\langle A e_i, A e_j \rangle| \leq \alpha \|x\|_2 \leq \alpha \sqrt{K} \|x\|_2. \]

By (2), it holds that
\[ |\langle A e_j, Ax \rangle| \geq (1 - \delta_{K+1}) \|x\|_2^2. \] (9)

By the aforementioned two inequations, we have
\[ \max_{i \in S} |\langle A e_i, Ax \rangle| \geq \frac{(1 - \delta_{K+1}) \|x\|_2^2}{\sqrt{K}}. \] (10)

and if the equality in (10) holds, then the equality in (9) must also hold.

By Lemma 2.1 in (1), for each \( j \notin S, \) it holds
\[ |\langle A e_j, Ax \rangle| \leq \delta_{K+1} \|x\|_2. \] (11)

So if (3) holds and at least there is one equality in (9) or (11) does not hold, then for each \( j \notin S, \) it holds
\[ |\langle A e_j, Ax \rangle| < \max_{i \in S} |\langle A e_i, Ax \rangle|. \]

Therefore, it suffices to show that the equality in (2) and the equation in (11) can not hold simultaneously.

Suppose both the equality in (9) and the equation in (11) hold, then by Lemma (1) \( \|A e_j\|^2 = 1 + \delta_{K+1}. \) Let \( C = (A_{S \cup J})^T A_{S \cup J}, \) then \( C_{ij} = \|A e_j\|^2 = 1 + \delta_{K+1}, \) thus for each \( i \in S, C_{ij} = 0. \) In fact, suppose there exists one \( i \in S \) such that \( C_{ij} \neq 0, \) then
\[ \|A^2_{S \cup J} A_{S \cup J} e_j\|_2 \geq \sqrt{\sum_{i \in S} C_{ij}^2} > 1 + \delta_{K+1} \]
which contradicts Lemma (2). Therefore, for each \( i \in S, C_{ij} = 0. \) However, in this case, we have
\[ |\langle A e_j, Ax \rangle| = 0 \]
which contradicts the equality in (11). Thus the equality in (9) and the equation in (11) can not hold simultaneously. Therefore, if (3) holds, then the OMP can choose a correct index in the first iteration.

By applying the method used in (3) or (2) and the aforementioned proof, one can similarly show that if (3) holds, then the OMP can choose a correct index in the latter iterations, so the theorem is proved.

**Future Work:** In the future, we will prove or disprove whether \( \frac{1}{\sqrt{K} + 1} \) is a sufficient condition for the OMP to recover every \( K \)-spars signal \( x \) from \( y = Ax \) in \( K \) iterations.

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