Quantum interference and non-locality of independent photons from disparate sources

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Abstract

We quantitatively investigate the non-classicality and non-locality of a whole new class of mixed disparate quantum and semiquantum photon sources at the quantum–classical boundary. The latter include photon-added thermal and photon-added coherent sources, experimentally investigated recently by Zavatta et al (2009 Phys. Rev. Lett. 103 140406). The key quantity in our investigations is the visibility of the corresponding photon–photon correlation function. We present explicit results on the violations of the Cauchy–Schwarz inequality—which is a measure of non-classicality—as well as of Bell-type inequalities.

1. Introduction

The question of interference between independent photons has attracted considerable attention since the celebrated work of Hong, Ou and Mandel who showed how two photons of the pair emitted in a nonlinear crystal by the process of spontaneous parametric downconversion (SPDC) can interfere on a beam splitter [1]. Many subsequent investigators verified the Hong–Ou–Mandel interference using a variety of SPDC sources [2–4]. Also the interference of photons from truly independent quantum sources, e.g., two distant atoms, has been analysed in great detail [5, 6] and recently observed [7–10]. Note that the interference is displayed in terms of the photon–photon correlation function [11] rather than in measurements of the mean intensity. So far, most works have concentrated on the interference produced by independent but identical sources. However, the questions then arise: what is the extent to which photons from independent but disparate sources can interfere?\textsuperscript{3} Furthermore, what is the nature of the radiation field generated and to what extent are the properties of the field strictly quantum? Can one use such a quantum character to shed light on the non-local character of the field? We provide quantitative answers to these questions.

\textsuperscript{3} For a very recent experiment on interference involving a weak coherent source and a single photon source, cf [12].

Specifically, we investigate the quantum interference of two disparate sources, one being a single-photon source like an excited atom and the other a source which can give rise to properties ranging from completely classical to completely quantum. In the latter category, we specifically examine sources which are either in a state called the photon-added coherent state (PACS) [13] or the photon-added thermal state (PATS) [14]. The quantum properties of these sources arise from the fact that one has added photons to classical states like thermal and coherent states. Such sources have been recently experimentally realized and their properties studied [15–18].

We calculate the interference pattern of the corresponding photon–photon correlation functions, derive their visibilities and discuss the conditions under which the photon–photon correlation functions display strictly quantum behaviour. The latter is formulated quantitatively in terms of the Cauchy–Schwarz (CS) inequality [19] valid for classical fields. After revealing the quantum nature of the light fields generated by disparate sources, we discuss the conditions under which Bell inequalities can be violated [20–23].

The organization of the paper is as follows. In section 2, we derive the photon–photon correlation functions and the corresponding visibilities for combinations of mixed disparate quantum–classical and quantum–semiquantum light sources. As a quantum source we consider throughout the
paper a single-photon source, e.g., an initially excited two-level atom spontaneously emitting a single photon. As a second source we assume either a purely classical source exhibiting Poissonian or thermal statistics [24-26] or a semiquantum source emitting a field which is either in a PACS [13] or in a PATS [14]. The operator $\sigma$ in equation (2) thus denotes the photon annihilation operator for source $B$.

For a better comparison with the following calculations, we start to review the pure classical case where the two independent light sources $A$ and $B$ emit coherent light. The operator $\sigma$ in equation (2) denotes in this case the photon annihilation operator for source $A$. Since the two light fields are in a coherent state, $|\alpha_A\rangle$ and $|\alpha_B\rangle$ with mean photon numbers $\bar{n}_A$ and $\bar{n}_B$, respectively, we calculate from equations (1) and (2) the photon–photon correlation function $G^{(2)}_{\text{class}}(r_1, r_2)$ as

$$G^{(2)}_{\text{class}}(r_1, r_2) = \bar{n}_A^2 + 2\bar{n}_A\bar{n}_B + \bar{n}_B^2 + 2\bar{n}_A\bar{n}_B \cos (\varphi_2 - \varphi_1),$$

where the relative phase $\varphi_i$ is given by $\varphi_i = \phi(r_i) = k d \sin (\xi(r_i))$ (cf figure 1).

The visibility

$$V := \frac{G^{(2)}_{\text{max}} - G^{(2)}_{\text{min}}}{G^{(2)}_{\text{max}} + G^{(2)}_{\text{min}}}$$

of this classical signal is calculated as

$$V_{\text{class}} = \frac{2\bar{n}_A\bar{n}_B}{(\bar{n}_A + \bar{n}_B)^2}.$$
where we have used the fact that the two photon sources are uncorrelated. Different to the pure classical case above, in equation (6) we can omit the term $|A_{1}^{(-)}A_{2}^{(+)}A_{3}^{(+)}A_{4}^{(+)}⟩$ as it vanishes identically for single-photon emitters. However, unlike the pure quantum mechanical signal [28] consisting of, e.g., two single photons scattered by two atoms, we additionally have to take into account, in equation (6), the term $|B_{1}^{(-)}B_{2}^{(-)}B_{3}^{(+)}B_{4}^{(+)}⟩$. This term represents the probability that source B has emitted two photons which are subsequently measured at the two detectors. In the following, we start to discuss the configuration where source B is a standard classical source (section 2.1). Thereafter, we study the configuration where source B is either in a PACS or PATS, i.e. in a state at the quantum–classical border (section 2.2). These states are particularly interesting as they display properties ranging from completely classical to completely quantum.

2.1. Source B as a classical photon source

2.1.1. Coherent source. Let us consider source B to be in a coherent state $|α⟩$. Expanded in terms of Fock states $|n⟩$, these states take the well-known form [11]

$$|α⟩ = e^{-|α|^2/2} \sum_{n=0}^{∞} \frac{α^n}{√n!} |n⟩,$$

where $α$ is any complex number. The 4th moment of the photon number operator $a^†a$ in the coherent beam can be calculated as

$$⟨a^4a^4⟩_{C} = ⟨|α|^4a^4a^4|α⟩ = |α|^{4k} = n^4,$$

where $n = ⟨a^2a^2⟩ \propto I_B \propto |E_B|^2$ is the mean photon number, with $E_B$ the amplitude of the electric field of source B. Using equation (6), the photon–photon correlation function of one quantum and one coherent source is then found to be

$$G_C^{(2)}(ϕ_1, ϕ_2) = n^2 + 2n(1 + \cos(ϕ_2 - ϕ_1)),$$

so that the visibility of the mixed quantum–coherent photon–photon correlation function $G_C^{(2)}(ϕ_1, ϕ_2)$ becomes

$$V_C = \frac{1}{1 + \frac{3}{2} n},$$

For a classical coherent plane wave with $n ≪ 1$, we obtain $V_C = 100\%$, for $n = 1$ we arrive at $V_C = 66\%$, and for $n → ∞$ we have $V_C = 0$. In [24], a single photon created by SPDC was entangled with an attenuated laser beam on a beam splitter. The authors obtained a modulation depth of $(84 ± 3.2)\%$. According to equation (10) this corresponds to a mean photon number in the coherent beam of $0.29 < n < 0.48$.

2.1.2. Thermal source. Now, let us assume that source B exhibits thermal statistics. The corresponding density operator expanded in Fock states $|n⟩$ takes the form

$$ρ_T = \sum_{n=0}^{∞} \frac{\tilde{n}^n}{n!} |n⟩⟨n|.$$

For the moments $⟨a^k a^k⟩_T$, one calculates

$$⟨a^k a^k⟩_T = Tr[α^k a^k \tilde{ρ}_T] = k!\tilde{n}^k.$$

With these expressions, equation (6) takes the form

$$G_T^{(2)}(ϕ_1, ϕ_2) = 2n^2 + 2n(1 + \cos(ϕ_2 - ϕ_1)).$$

The visibility of the mixed quantum–thermal signal $G_T^{(2)}(ϕ_1, ϕ_2)$ is thus calculated as

$$V_T = \frac{1}{1 + \frac{3}{2} n},$$

what leads again to $V_T = 100\%$ for $n ≪ 1$ and to $V_T = 0$ for $n → ∞$. If we choose $n = 1$, we obtain $V_T = 50\%$, corresponding to the visibility of a pure classical photon–photon correlation signal exhibited by coherent light fields if $n_A = n_B$ (cf equation (5)). In a recent experiment, a visibility of $82\%$ was measured in this configuration, equivalent to a mean photon number in the thermal beam of $n = 0.22$ [26].

2.2. Source B as a semiquantum photon source

2.2.1. PACS. Now, let us consider the case where the field of source B is in a PACS [13]. In terms of the coherent states $|α⟩$, a normalized single PACS (in the following abbreviated PC) can be written as

$$|α, 1⟩ = \frac{a^†|α⟩}{√1 + n}.$$

Hereby, $n$ corresponds to the mean photon number in the coherent part of the light field. The 4th moments of the photon number operator $a^†a$ then take the form

$$⟨a^4a^4⟩_{PC} = \frac{⟨|α|^4a^4a^4|α⟩}{1 + n},$$

so that the first and the second moments read

$$n_{PC} ≡ ⟨a^2a^2⟩_{PC} = \frac{n^2 + 3n + 1}{1 + n},$$

$$⟨a^4a^4⟩_{PC} = \frac{n^2 + 4n^2}{1 + n},$$

where we have introduced the net photon number $n_{PC}$ of the PACS field. With the help of equation (6), we arrive at

$$G_T^{(2)}(ϕ_1, ϕ_2) = n^2 + 4n + 2 \frac{2n^2 + 3n + 1}{1 + n} (1 + \cos(ϕ_2 - ϕ_1)),$$

and the visibility $V_T$ thus takes the form

$$V_T = \frac{1}{1 + \frac{3}{2} n_{PC} + \frac{n}{2} (n_{PC} + 1)}.$$

In the case of a mixed quantum–coherent or mixed quantum–thermal photon–photon correlation signal (cf equations (9) and (13)), the mean photon number $n$ in the coherent or thermal beam of source B trivially corresponds to the net photon number of the field, $n_{PC} = n$, and $n_T = n$, respectively. However, if we want to compare the visibility $V_{PC}$ of the mixed quantum–PACS photon–photon correlation signal to the foregoing results (cf equations (10) and (14)), we have to express $V_{PC}$ in terms of the net photon number $n_{PC}$. We can rewrite equation (19) by using the identity (cf equation (17))

$$n = \frac{3}{2} + \frac{1}{2} \sqrt{5 - 2n_{PC} + n_{PC}^2},$$

$$\tilde{n} = \frac{3}{2} + \frac{1}{2} \sqrt{5 - 2n_{PC} + n_{PC}^2},$$

$$n_{PC} = \frac{3}{2} + \frac{1}{2} \sqrt{5 - 2n + n^2},$$

$$n_T = n.$$
and arrive at a visibility \( V_{PC} \) depending on \( n_{PC} \) rather than \( \bar{n} \). Since the analytic expression is rather complex, we do not present the explicit result; however, the corresponding outcome is plotted in figure 2.

For \( n_{PC} \to 1 \), we obtain \( V_{PC} = 100\% \), whereas for \( n_{PC} \to \infty \), we have \( V_{PC} = 0 \), like in the mixed quantum–classical cases above. However, the visibility \( V_{PC} \) of the mixed quantum–PACS photon–photon correlation function always displays higher values for a given net photon number \( \langle a^\dagger a \rangle \) than the visibility \( V_{C} \) for a mixed quantum–coherent source. For example, for a net photon number \( n_{PC} = 1 \), we obtain a visibility \( V_{PC} = 100\% \) in contrast to a visibility \( V_{C} = 66\% \) for a net photon number \( n_{C} = 1 \) in the case of a mixed quantum–coherent source.

2.2.2. PATS. Next, we consider the field of source \( B \) to be in a PATS [14]. In the case of a single PATS (in the following abbreviated as PT), the normalized density operator \( \hat{\rho}_{PT} \) can be written in the Fock basis as

\[
\hat{\rho}_{PT} = \frac{1}{\bar{n} + 1} a^\dagger \hat{\rho} a
= \frac{1}{\bar{n} + 1} \sum_{n=0}^{\infty} \left( \frac{\bar{n}}{\bar{n} + 1} \right)^n n|n\rangle\langle n|,
\]

where \( \bar{n} \) corresponds to the mean photon number of the thermal part of the light field. In contrast to the thermal density operator of equation (11), it is obvious that the vacuum term is missing and higher excited terms are rescaled. For the first and second moments of the photon number operator \( a^\dagger a \), we obtain

\[
\langle a^\dagger a \rangle_{PT} = 2\bar{n} + 1,
\]

\[
\langle a^\dagger a^2 \rangle_{PT} = 6\bar{n}^2 + 4\bar{n}
\]

where we have introduced again the net photon number \( n_{PT} \) of the PATS field of source \( B \). Thus, equation (6) is calculated as

\[
G_{PT}^{(2)}(\varphi_1, \varphi_2) = \frac{6\bar{n}^3 + 10\bar{n}^2 + 4\bar{n}}{\bar{n} + 1} + 2(2\bar{n} + 1)(1 + \cos(\varphi_2 - \varphi_1)),
\]

and for the visibility \( V_{PT} \), we derive

\[
V_{PT} = \frac{1}{1 + \frac{6\bar{n}^2 + 10\bar{n} + 4\bar{n}}{2(\bar{n} + 1)(2\bar{n} + 1)}}.
\]

Again, we have to formulate the visibility \( V_{PT} \) in terms of the net photon number \( n_{PT} \) in order to compare it to the previous mixed quantum–classical results. Inverting equation (22) we obtain

\[
\bar{n} = \frac{n_{PT} - 1}{2}
\]

From equation (26), we can see that for \( n_{PT} \to 1 \) the visibility \( V_{PT} \) goes to 100% whereas for \( n_{PT} \to \infty \) we have \( V_{PT} = 0 \). Moreover, again the visibility \( V_{PT} \) is taking higher values for any net photon number than the \( V_{T} \) of the mixed quantum–thermal state (cf equation (14)). If we choose for example \( n_{PT} = 2 \) we obtain \( V_{PT} = 53.3\% \), whereas for \( n_{PT} = 2 \) we obtain \( V_{T} = \frac{1}{3} \) (cf equations (12) and (14)).

Figure 2 displays the visibility of the various photon–photon correlation functions as a function of the net photon numbers \( n_j \) for the investigated combinations of mixed quantum–classical and mixed quantum–semiquantum sources. It thus shows a summary of the results discussed in this section. For comparison, the constant visibility \( V = 1 \) obtained for a pure quantum signal, e.g., two photons from two single-photon emitters, and the constant visibility obtained for a pure classical signal (\( V_{Class} = \frac{1}{2} \)) are also plotted.

3. Test of the quantum character of the mixed radiation fields

After deriving the various photon–photon correlation signals of the mixed quantum–classical and quantum–semiquantum sources in section 2, the natural question arises of how to characterize and specify the non-classicality [29, 30] of these sources. For this purpose, we introduce a particular version of the CS inequality for our photon–photon correlations, valid for classical fields. If this inequality is violated, the underlying radiation field is non-classical. This is because the key ingredient in the derivation of the CS inequality is the assumption that the Glauber–Sudarshan \( P \)-function [11, 31] behaves like a classical probability distribution. In its simplest form it reads [19]

\[
\langle a^\dagger a^2 \rangle \langle b^\dagger b^2 \rangle \geq |\langle a^\dagger b^\dagger ab \rangle|^2,
\]
where $a$ and $b$ are the mode variables for the classical fields. For quantum fields, $a$ and $b$ are the annihilation operators. In this case, the inequality can be violated since the Glauber–Sudarshan $P$-function can be negative, singular or need not exist. However, for the two point photon–photon correlations of our sources, we have to derive an alternate form of the CS inequality. As discussed in appendix A, a direct application of equation (27) does not work.

Let us consider the net intensity $I(\psi)$ of two sources $A$ and $B$ at the point $\psi(\mathbf{r})$. If the underlying probability distribution (represented by the $P$-function) is positive, then the expression

$$
(\alpha(\tilde{I}(\psi_1)) - (\tilde{I}(\psi_2))) + \beta(\tilde{I}(\psi_2) - (\tilde{I}(\psi_2)))^2
$$

(28)

should be positive for arbitrary $\alpha$ and $\beta$ (where $:\cdots:\$ denotes normal ordering). As shown in appendix A, the corresponding photon–photon correlation functions must satisfy this inequality provided that the $P$-function is positive. Any violation of this inequality then implies that the underlying radiation field is a nonclassical field. From equation (28), we arrive at the inequality (see appendix A)

$$
\mathcal{S} := \frac{|\tilde{G}^{(2)}(\psi_1, \psi_2)|^2}{\tilde{G}^{(2)}(\psi_1, \psi_1) \tilde{G}^{(2)}(\psi_2, \psi_2)} \leq 1,
$$

(29)

where the variance $\tilde{G}^{(2)}(\psi_1, \psi_2)$ is given by

$$
\tilde{G}^{(2)}(\psi_1, \psi_2) := \tilde{G}^{(2)}(\psi_1, \psi_2) - \langle I_A + I_B \rangle^2,
$$

(30)

and $\tilde{G}^{(2)}(\psi_1, \psi_2)$ ($j = \{T, PT, C, PC\}$) corresponds to the photon–photon correlation functions given in equations (3), (9), (13), (18) and (23), respectively. Hereby, $I_A(I_B)$ denotes the intensity of the photon sources located at $\mathbf{r}_A(\mathbf{r}_B)$ (cf figure 1).

Keeping in mind that we assume uncorrelated sources and that we have $\langle I_A \rangle_{QM} = 1$, and taking into account the corresponding moments given in equations (8), (12), (16) and (22), the variances $\tilde{G}^{(2)}(\psi_1, \psi_2)$ are calculated as

$$
\tilde{G}^{(2)}_{\text{Class}}(\psi_1, \psi_2) = 2n_{\text{ckl}},
$$

(31)

$$
\tilde{G}^{(2)}_{\text{C}}(\psi_1, \psi_2) = 2n_{\text{ckl}} - 1,
$$

(32)

$$
\tilde{G}^{(2)}_{\text{PT}}(\psi_1, \psi_2) = -\frac{3n^2 + 4n + 2}{(n + 1)^2} \tilde{G}^{(2)}_{\text{C}} + \frac{2n^3 + 8n^2 + 8n + 2}{(n + 1)^2} c_{\text{ckl}},
$$

(33)

$$
\tilde{G}^{(2)}_{\text{PC}}(\psi_1, \psi_2) = 2\tilde{G}^{(2)}_{\text{C}} + 2(2n + 1)c_{\text{ckl}} - 2,
$$

(34)

$$
\tilde{G}^{(2)}_{\text{T}}(\psi_1, \psi_2) = 2c_{\text{ckl}} - 2,
$$

(35)

(25) into equations (34) and (35), respectively. If we now plug the variances of equations (31)–(36) in the CS inequality, equation (29), we obtain a violation for every mixed quantum–classical and mixed quantum–semiquantum signal below a certain net photon number $\langle a^\dagger a \rangle_j$ ($j = \{T, PT, C, PC\}$), after an optimization with respect to $\psi_1$ and $\psi_2$, i.e. with respect to the detector positions. Note that according to figure 2, reducing the net photon number $n_j$ in the field of source $B$ increases the visibility $V_j$ of the corresponding photon–photon correlation signal ($j = \{T, PT, C, PC\}$).

The corresponding behaviour of the Schwarz function $S_{\text{max}}^n$ as a function of the visibilities $V_j$ ($j = \{T, PT, C, PC\}$) is displayed in figure 3. With the pure classical signal it is not possible to violate the CS inequality as $S_{\text{max}}^n = 1$ for all $V_{\text{Class}}$. For a mixed quantum–thermal signal (T), the CS inequality is only violated for a visibility $V_T > 50\%$, i.e. if the net photon number in the thermal beam fulfills $\bar{n} < 1$ (cf equation (14)). Note that a visibility $> 50\%$ is just above the maximal visibility of the pure classical coherent signal (cf equation (5)). Moreover, for $V_T = 1/\sqrt{2}$, the CS function $S_{\text{max}}^n$ diverges. By contrast, if we consider the mixed quantum–PATS signal (PT), the CS inequality can be violated already for a visibility $V_{PT} > 37.5\%$, corresponding to a net photon number $n_{PT} < 3$ of source $B$ (cf equation (26)). Furthermore, we have $S_{\text{max}}^n \to \infty$ for $V_{PT} \to 0$.

For comparison we have also included the pure quantum mechanical case, i.e. two single-photon emitters, where

$$
\tilde{G}^{(2)}(\psi_1, \psi_2) = 2(1 + \cos(\psi_2 - \psi_1))
$$

(37)

with $\mathcal{Y} = 1$ [6].
> 0 (see figure 2 and equations (10), (19) and (20)). For the mixed quantum–coherent signal (C), we find $\mathcal{S}_C^{\text{max}} \to \infty$ for a visibility $\nu_C = 80\%$, corresponding to a net photon number of $\tilde{n} = 1/2$ in the coherent beam (cf equation (10)). For the mixed quantum–PACS signal (PC), the CS function goes to $\infty$ for $\nu_{\text{PT}} \to 1$ (see figure 2). In this way, we can pin down quantitatively the transgression to the non-classical regime for all our investigated mixed quantum–classical and mixed quantum–semiquantum signals. As expected, a signal of two single-photon sources also violates the CS inequality, which is not depicted in figure 3. Finally, it should be kept in mind that with no violation of the CS inequality no conclusion can be drawn regarding the quantum nature of the radiation field.

4. Test of the non-local character of the mixed radiation fields

In the past there have been many tests of Bell inequalities using photons obtained from parametric downconverters (e.g., [32–36]). Furthermore, Bell inequalities have been studied using photons obtained either from two independent downconverters [34] or one photon from a downconverter and one photon from a coherent source [24]. In all these experiments, Bell inequalities were tested using polarization (spin like) correlations between the photons. We have already seen in the previous section that the radiation field produced by two disparate sources, where one source is a single-photon emitter, is strictly quantum in certain ranges of the net photon number of source B. Hence, we would expect a violation of Bell-type inequalities. However, in order to see this in detail we need to normalize our photon–photon correlation function. We use the following normalization (see appendix B):

$$\tilde{g}_{j}^{(2)}(\varphi_1, \varphi_2) = \frac{g_{j}^{(2)}(\varphi_1, \varphi_2)}{\mathcal{N}},$$

where $\mathcal{N}$ is an appropriate scaling factor. This scaling factor can be obtained by physical considerations—we take it to be the integrated photon–photon correlation function obtained by fixing one detector and moving the other detector. Using the moments given in equations (8), (12), (16) and (22), and recalling that we assume uncorrelated sources and that we have $I_{j}^{(2)}_{\text{QM}} = 0$, we can express the normalized photon–photon correlation functions in the collective form

$$\tilde{g}_{j}^{(2)}(\varphi_1, \varphi_2) = \frac{g_{j}^{(2)}(\varphi_1, \varphi_2)}{2\pi^{-1} \int g_{j}^{(2)}(\varphi_1, \varphi_2) d\varphi_1 |_{\varphi_1 = \text{const}}} - 1,$$

$$= \frac{(I_A + I_B)^2}{(I_A + I_B)^2} + 2(I_A) (I_B) \cos(\varphi_2 - \varphi_1) - 1$$

$$= \mathcal{V}_j \cos(\varphi_2 - \varphi_1),$$

with the visibilities $\mathcal{V}_j$ given in equations (5), (10), (14), (19) and (26), respectively. We want to emphasize that our proposed rescaled correlation functions are independent of experimental inefficiencies, e.g., the quantum efficiency of the detectors and the restricted angle subtended by the detector surfaces (see appendix B). Equation (39) is the celebrated correlation commonly used in testing violations of Bell-type inequalities [39]. This is also the correlation for two spins in a Werner state which is a mixed entangled state [41]. It is well known that such a correlation violates Bell’s inequalities in the CHSH formulation [22]

$$|\tilde{g}_{j}^{(2)}(\varphi_1, \varphi_2) - \tilde{g}_{j}^{(2)}(\varphi_1', \varphi_2) + \tilde{g}_{j}^{(2)}(\varphi_1, \varphi_2') + \tilde{g}_{j}^{(2)}(\varphi_1', \varphi_2')| \leq 2$$

if

$$\mathcal{V}_j > \frac{1}{\sqrt{2}}.$$ (41)

Down to the present day this particular version of Bell’s inequalities is tested in most experiments (cf, e.g., [24, 33–35]). From figure 2 and equations (10), (14), (19), (20) and (26), we can derive the net photon numbers $n_j$ of source B for which the CHSH inequalities are violated. The net photon number must be less than (a) $\sqrt{2} - 1$ for a thermal source, (b) $\approx 1.45$ for a photon-added thermal source ($\hbar \approx 0.22$), (c) $2(\sqrt{2} - 1)$ for a coherent source and (d) $\approx 1.52$ for a photon-added coherent source ($\hbar \approx 0.29$). Clearly, in order to see the violations of locality, the condition on $n_j$ for photon-added sources is more relaxed.

5. Summary

In conclusion we studied position-dependent photon–photon correlations of photons emitted from disparate mixed quantum–classical and mixed quantum–semiquantum sources in free space. As a quantum source we considered a spontaneously emitting atom; however, we want to emphasize that our results are not limited to this case as any single-photon source can act as a quantum source in the presented scheme. The classical sources were represented either by thermal or coherent beams. These two classical sources have been studied in the past by overlapping photons at a beam splitter [24, 26]. As a semiquantum source we investigated both PATS and PACS what corresponds to a whole new class of disparate mixed quantum photon sources at the quantum–classical boundary. We introduced the CS inequality and obtained quantitative results for the non-classicality of all our mixed radiation fields, depending on the visibility $\mathcal{V}_j$ of the corresponding position–photon correlations (see figure 3), what relates also to the net photon number $n_j$ in the field of source B (see figure 2). By normalizing the photon–photon correlation function in a physical way, we obtained a violation of Bell-type inequalities for all considered mixed fields, again dependent on the visibility $\mathcal{V}_j$ and the net photon number $n_j$ of source B.

The violation of Bell-type inequalities requires more stringent conditions on the visibility of the photon–photon correlations than the violation of the CS inequality. This is reminiscent of the behaviour of Werner states for spins.

6 The experiments of [24–26] use a single photon obtained from a downconverter.
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Appendix A. CS inequality for photon–photon correlations

Let us investigate equation (28) in the form

\[ \text{Tr}\{\rho [\delta \hat{I}(\varphi_1) + \beta \delta \hat{I}(\varphi_2)]^2\} \geq 0 \quad \forall \alpha, \beta, \]  
(A.1)

with \( \delta \hat{I}(\varphi) = \hat{I}(\varphi) - \langle \hat{I}(\varphi) \rangle \) and the density operator \( \rho \) in the Glauber–Sudarshan \( P \)-representation:

\[ P(\alpha) = \int P(\alpha|\alpha) |\alpha| \text{d} \text{Re}(\alpha) \text{d} \text{Im}(\alpha). \]  
(A.2)

\( P(\alpha) \) denotes a (quasi-) probability function depending on the state of the investigated light field. For classical states, \( P(\alpha) \) is a classical probability distribution assuming only positive values. Since for any \( \alpha \) and \( \beta \) the quantity \( |\delta \hat{I}(\varphi_1) + \beta \delta \hat{I}(\varphi_2)|^2 \) is also positive, inequality (A.1) holds for any classical light field. However, for nonclassical states, \( P(\alpha) \) is a quasi-probability distribution and the \( P \)-function can be negative, singular or need not exist. Thus, the inequality can be violated.

From equation (A.1), it follows that

\[ \left| \begin{array}{cc} \langle \delta \hat{I}(\varphi_1) \delta \hat{I}(\varphi_1) \rangle & \langle \delta \hat{I}(\varphi_1) \delta \hat{I}(\varphi_2) \rangle \\ \langle \delta \hat{I}(\varphi_2) \delta \hat{I}(\varphi_1) \rangle & \langle \delta \hat{I}(\varphi_2) \delta \hat{I}(\varphi_2) \rangle \\ \end{array} \right| \geq 0. \]  
(A.3)

Evaluating the determinant, we arrive at

\[ \sum_{i=1}^{2} \langle \hat{I}(\varphi_i) \hat{I}(\varphi_i) - 2 \langle \hat{I}(\varphi_i) \rangle \hat{I}(\varphi_i) \rangle + \langle \hat{I}(\varphi_i) \rangle \langle \hat{I}(\varphi_i) \rangle \rangle - \sum_{i=1}^{2} \sum_{j=1,i \neq j}^{2} \langle \hat{I}(\varphi_i) \hat{I}(\varphi_j) - \langle \hat{I}(\varphi_i) \rangle \langle \hat{I}(\varphi_j) \rangle \rangle - \langle \hat{I}(\varphi_i) \rangle \langle \hat{I}(\varphi_j) \rangle \langle \hat{I}(\varphi_j) \rangle \rangle = 0. \]  
(A.4)

If we now use the identities

\[ \langle \hat{I}(\varphi_i) \rangle = \langle I_A + I_B \rangle, \]  
\[ \langle \hat{I}(\varphi_i) \rangle \langle \hat{I}(\varphi_j) \rangle = G^{(2)}(\varphi_i, \varphi_j), \]  
(A.5)

we obtain equation (29).

We want to emphasize that if instead of equation (A.1) we use arguments similar to those leading to equation (27), we would obtain a CS inequality for normalized photon–photon correlation functions \( g^{(2)}(\varphi_1, \varphi_2) \) of the form (cf., e.g., [38])

\[ |g^{(2)}(\varphi_1, \varphi_2)|^2 \leq g^{(2)}(\varphi_1, \varphi_1) g^{(2)}(\varphi_2, \varphi_2). \]  
(A.6)

For two quantum sources, this form of the CS inequality is never violated as

\[ g^{(2)}(\varphi_1, \varphi_2) = \frac{1}{4} (1 + \cos (\varphi_2 - \varphi_1)). \]  
(A.7)

Thus, clearly, such an inequality does not enable us to test the quantum nature of our sources.

Appendix B. Derivation of position-dependent Bell correlations

In this appendix, we want to derive equations (38) and (39). Bell inequalities in the CHSH formulation are inequalities based on a double-channel experiment. In order to establish this double-channel character, we have to slightly change our setup. Let us consider that the photon sources \( A \) and \( B \) are connected to both detectors via optical fibres. Each detector shall now consist of two photomultipliers, located at the output ports of a 50–50 beam splitter, representing a transmitting + and a reflecting − channel. Measuring the photon–photon correlation function, we thus consider four photodetectors, say \( D^+_1, D^-_1, D^+_2 \) and \( D^-_2 \). In this configuration, we can make use of the well-known correlation function [39, 40] commonly employed in Bell experiments:

\[ C(\varphi_1, \varphi_2) = \frac{\sum_{\alpha, \beta} \alpha \beta G_{\alpha \beta}(\varphi_1, \varphi_2)}{\left( \sum_{\alpha, \beta} G_{\alpha \beta}(\varphi_1, \varphi_2) \right)^2}, \]  
(B.1)

where \( \alpha, \beta = \{+, -\} \). Hereby, \( G_{\alpha \beta}(\varphi_1, \varphi_2) \) denotes the photon–photon correlation of measuring one photon in the \( \alpha \) channel at position \( \varphi_1 \) (i.e. at detector \( D^+_{\alpha} \)) and the other photon in the \( \beta \) channel at position \( \varphi_2 \) (i.e. at detector \( D^-_{\beta} \)). In an experiment, these correlations would correspond to four twofold coincidence rates. Keeping in mind that due to reflection we obtain a phase shift of \( \pi \) we arrive at

\[ G_{\alpha \beta}(\varphi_1, \varphi_2) = (\langle I_A + I_B \rangle^2 \pm 2 \langle I_A \rangle \langle I_B \rangle \cos (\varphi_2 - \varphi_1)). \]  
(B.2)

where the plus (minus) sign holds for \( \alpha = \beta (\alpha \neq \beta) \). Thus, \( C(\varphi_1, \varphi_2) \) is calculated as

\[ C(\varphi_1, \varphi_2) = V_j \cos (\varphi_2 - \varphi_1), \]  
(B.3)

with the visibility

\[ V_j = \frac{2 \langle I_A \rangle \langle I_B \rangle}{\langle I_A + I_B \rangle^2}. \]  
(B.4)

Equation (B.3) is equivalent to equation (39), i.e. equation (39) integrates the foregoing procedure of deriving correlations suitable for a Bell test. We want to emphasize that this expression holds for every position-dependent mixed photon–photon correlation function with visibility \( V_j \). Due to the normalization on \( \sum_{\alpha, \beta} G_{\alpha \beta}(\varphi_1, \varphi_2) \), the correlation \( C(\varphi_1, \varphi_2) \), like equation (38), becomes independent of experimental inefficiencies of the form \( \eta = \alpha_0 \Delta \Omega \) which we have set to unity in the course of this paper. Hereby, \( \alpha_0 \) stands for the quantum efficiency of the detectors and \( \Delta \Omega \) for the restricted solid angle subtended by the detector surfaces.

References

[1] Hong C. K, Ou Z Y and Mandel L 1987 Phys. Rev. Lett. 59 2044
[2] Michler M, Mattle K, Weininfurter H and Zeilinger A 1996 Phys. Rev. A 53 R1209
[3] Weihs G, Reck M, Weininfurter H and Zeilinger A 1996 Phys. Rev. A 54 893
[4] Legero T, Wilk T, Herrnrich M, Rempe G and Kuhn A 2004 Phys. Rev. Lett. 93 070503
[5] Mandel L 1983 Phys. Rev. A 28 929
[6] Agarwal G S, von Zanthier J, Skornia C and Walther H 2002 Phys. Rev. A 65 053826
[7] Beugnon J, Jones M P A, Dingjan J, Darquie B, Messin G, Browaeys A and Grangier P 2006 Nature 440 779
[8] Kaltenbaek R, Blauensteiner B, Zukowski M, Aspelmeyer M and Zeilinger A 2006 Phys. Rev. Lett. 96 240502
[9] Maunz P, Moehring D L, Olmschenk S, Younge K C, Matsukevich D N and Monroe C 2007 Nature Phys. 3 538
[10] Gerber S, Rotter D, Hennrich D, Blatt R, Rohde F, Schuck C, Almendros M, Gehr R, Dubin F and Eschner J 2009 New J. Phys. 11 013032
[11] Glauber R 1963 Phys. Rev. 130 2529
[12] Bennett A J, Patel R B, Nicoll C A, Ritchie D A and Shields A J 2009 Nature Phys. 5 715
[13] Agarwal G S and Tura K 1991 Phys. Rev. A 43 492
[14] Agarwal G S and Tura K 1992 Phys. Rev. A 46 485
[15] Zavatta A, Parigi V, Kim M S, Jeong H and Bellini M 2009 Phys. Rev. Lett. 103 140406
[16] Parigi V, Viciani S and Bellini M 2004 Science 306 660
[17] Parigi V, Zavatta A, Kim M S and Bellini M 2007 Science 317 1890
[18] Zavatta A, Parigi V and Bellini M 2007 Phys. Rev. A 75 052106
[19] Loudon R 1980 Rep. Prog. Phys. 43 913
[20] Bell J S 1964 Physics 1 195
[21] Bell J S 1987 Speakable and Unspeakable in Quantum Mechanics (Cambridge: Cambridge University Press)
[22]Clauser J F, Horne M A, Shimony A and Holt R 1969 Phys. Rev. Lett. 23 880
[23] Reid M D, Drummond P D, Bowen W P, Cavalcanti E G, Lam P K, Bachor H A, Andersen U L and Leuchs G 2009 Rev. Mod. Phys. 81 1727
[24] Pittman T B and Franson J D 2003 Phys. Rev. Lett. 90 240401
[25] Laiho K, Cassemiro K N and Silberhorn C 2009 Opt. Express 17 22823
[26] Li X, Yang L, Cui L, Ou Z Y and Yu D 2008 Opt. Express 16 12505
[27] Agarwal G S 1974 Quantum Optics, Springer Tracts in Modern Physics vol 70 (Berlin: Springer)
[28] Ou Z Y 1988 Phys. Rev. A 37 1607
[29] Vogel W 2008 Phys. Rev. Lett. 100 013605
[30] Rivas A and Luis A 2009 Phys. Rev. A 79 042105
[31] Sudarshan E C G 1963 Phys. Rev. Lett. 10 277
[32] Ghosh R and Mandel L 1987 Phys. Rev. Lett. 59 1903
[33] Shih Y H and Alley C 1988 Phys. Rev. Lett. 61 2921
[34] Zukowski M, Zeilinger A, Horne M A and Ekert A K 1993 Phys. Rev. Lett. 71 4297
[35] Kwiat P G, Mattle K, Weiñfurter H and Zeilinger A 1995 Phys. Rev. Lett. 75 4337
[36] Yarnall T, Abouraddy A F, Saleh B E A and Teich M C 2007 Phys. Rev. Lett. 99 170408
Yarnall T, Abouraddy A F, Saleh B E A and Teich M C 2007 Phys. Rev. Lett. 99 250502
[37] Wiegner R, Thiel C, von Zanthier J and Agarwal G S 2010 Phys. Lett. A 374 3405
[38] Reid M D and Walls D F 1986 Phys. Rev. A 34 1260
[39] Bell J S 1972 Foundations of Quantum Mechanics ed B d’Espagnat (New York: Academic)
[40] Grangier P, Potashek M J and Yu B 1988 Phys. Rev. A 38 3132
[41] Werner R F 1989 Phys. Rev. A 40 4277