String baryon model “triangle”:
hypocycloidal solutions

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The considered model of baryon consists of three pointlike masses (quarks) bounded pairwise by relativistic strings forming a curvilinear triangle. Classic analytic solutions for this model corresponding to a planar uniform rotation about the system center of mass are found and investigated. These solutions describe a rotating curve composed of segments of a hypocycloid. The curve is a curvilinear triangle or — a more complicated configuration with a set of internal massless points moving at the speed of light. Different topological types of these motions are classified in connection with different forms of hypocycloids in zero quark mass limit. An application of these solutions to description of baryon states on the Regge trajectories is considered.

INTRODUCTION

The string baryon model “triangle” is genetically connected with the meson model of relativistic string with massive ends [1,2]. The latter model including two pointlike massive quarks bounded by a relativistic string gives possibility to describe the meson orbitally excited states on the Regge trajectories [3].

On a base of this meson model string models of baryon were suggested in some variants [4–7]. These variants differ from each other in the type of spatial junction of three pointlike quarks by relativistic strings: a) the first quark is bounded with the second and the second quark — with the third; b) the “three-string” model or Y-configuration with three strings from three quarks joined in the fourth massless point; c) the quark-diquark model and d) the “triangle” model. The first variant was investigated qualitatively [4], the “three-string” [4–6] and meson-like quark-diquark models [7,8] — in a more detailed way.

In the present paper the “triangle” model of baryon [9] is under consideration. In this model three material points (quarks) are pairwise connected by three relativistic strings forming a curvilinear triangle in space at each instant of time. If tensions of these three strings are equal, such an object could be regarded as a closed string carrying three pointlike masses. From the point of view of describing quark strong interaction in the orbitally excited baryon this model looks rather natural in comparison with three others. Some arguments in favour of the “triangle” baryon model in comparison with Y-configuration are given by Cornwall [10] in the QSD Wilson loop operator approach.

Transformation of the “meson” string with massive ends or the three-string model of baryon to the model “triangle” results in some additional difficulties. In particular, a string world surface in this model has discontinuities of derivatives on quark trajectories; a parametrization with these trajectories as coordinate curves doesn’t exist in general; space-like coordinate lines are not closed in general.

In the present paper these difficulties were overcome and classic analytic solutions were found for a set of motions — uniform planar rotations of the system. This kind of motions is an analog and generalization of well known rotations of a straight relativistic string with massive ends [1–3]. The latter class of motions was a base of applying this model [3,8] and the relativistic tube model [11] to description the meson Regge trajectories.

In this paper rotational motions in the baryonic model “triangle” and their applications are investigated. In Sec. 1 equations of evolution and conditions on the quark trajectories are deduced from the action of the system. In Sec. 2 solutions of these equations corresponding to rotational motions of the system (quarks and the string of hypocycloidal form) are described and classified. In Sec. 3 possibility of description the baryon states on the Regge trajectories by these solutions is discussed.

The string solutions obtained here in Sec. 2 are applicable not only to the particle physics, but to various branches of string or M-brane theory. In particular, the massive points placed on the string (the number of these points is arbitrary) could be regarded as 0-branes.

Note that the rotational solutions of the considered type also take place for a closed massless string. Such a string has a form of a rotating hypocycloid with singular points moving at the speed of light.

I. MODEL AND EQUATIONS

Let us consider the baryon model “triangle” as a closed relativistic string with tension $\gamma$ carrying three pointlike masses $m_1$, $m_2$, $m_3$. The action of this system is [9]

$$S = - \int_{\tau_1}^{\tau_2} \left\{ \frac{\sigma_3(\tau)}{\sigma_{\mu}(\tau)} \sqrt{-g} d\sigma + \sum_{i=1}^{3} m_i \sqrt{V_i^2(\tau)} \right\} d\tau. \quad (1)$$

Here $g = \dot{X}^2 \gamma^2 - (\dot{X}^\gamma)^2$ is a determinant of induced metric on a string world surface $x^\mu = X^\mu(\tau, \sigma)$, $\mu = 0, 1, \ldots$ in $d$-dimensional Minkowski space with signature $+, -, -, \ldots; \ X^\mu = \partial_\tau X^\mu, X^{\mu\nu} = \partial_\sigma X^\mu$, the speed $\dot{X}^\mu = \gamma^\mu \partial_\tau \gamma, \gamma^\mu = \gamma_{\mu\nu} \partial_\sigma \gamma^{\nu}$.
of light in these units \( c = 1 \), \((\tau, \sigma) \in \Delta = \Delta_1 \cup \Delta_2 \cup \Delta_3\), 
\( V_i^{\mu} = \frac{d}{d\tau} X_i^{\mu}(\tau, \sigma_i(\tau)) \) are tangent vector to the i-th quark trajectory with an inner equation \( \sigma = \sigma_i(\tau), i = 0, 1, 2, 3 \) (Fig. 1). The equations \( \sigma = \sigma_0(\tau) \) and \( \sigma = \sigma_3(\tau) \) define

\[
X^{\mu}(\tau, \sigma_0(\tau)) = X^{\mu}(\tau^{*}, \sigma_3(\tau^{*})).
\]  

(2)

Note that the parameters \( \tau \) and \( \tau^{*} \) in these two parametrizations of one curve (2) aren’t equal in general. It means that co-ordinate curves \( \tau = \text{const} \) on the world surface are not closed — the beginning of this curve at \( \sigma = \sigma_0 \) doesn’t coincide spatially with its end at \( \sigma = \sigma_3 \).

The equality \( \tau = \tau^{*} \) may be obtained only by a special choice of \( \tau \) and \( \sigma \), for example, \( \tau = t = X^0 \).

The parametrization of the world surface \( \Sigma^{\mu}(\tau, \sigma) \) is continuous in \( \Delta \), but on the lines \( \sigma_1, \sigma_2 \) and \( \sigma_3 \) its derivatives (except for tangential \( V_i^{\mu} \) and \( \frac{d}{d\tau} V_i^{\mu} \)) have discontinuities in general. Nevertheless, by a local choice of parameters \( \tau \) and \( \sigma \) we can obtain the induced metric \( ds^2 = X^{\mu} d\tau^2 + 2(\dot{X} X') d\tau d\sigma + X' d\sigma^2 \) in these lines. The action (1) is invariant with respect to an arbitrary non-degenerate reparametrization \( \tau = \tau(\tilde{\tau}, \tilde{\sigma}), \sigma = \sigma(\tilde{\tau}, \tilde{\sigma}) \).

The equations of motion and the boundary conditions on the quark trajectories in this model are deduced by variation and minimization of action (1). This procedure is partially similar to that for the model of relativistic string with massive ends [2] and results in the same equations of motion

\[
\frac{\partial}{\partial \tau} \frac{\partial L}{\partial \dot{X}^{\mu}} + \frac{\partial}{\partial \sigma} \frac{\partial L}{\partial X^{\mu}} = 0, \quad (\tau, \sigma) \in \Delta, \quad L = \sqrt{g}.
\]  

(3)

But to derive boundary conditions in the model “triangle” we are to take into account the discontinuities of \( \dot{X}^{\mu}, X^{\mu} \) on the lines \( \sigma = \sigma_i(\tau) \). Thereby the term \( \int_{\Delta} \left[ \frac{\partial}{\partial \tau} \frac{\partial L}{\partial \dot{X}^{\mu}} + \frac{\partial}{\partial \sigma} \frac{\partial L}{\partial X^{\mu}} \right] d\tau d\sigma \) in \( \delta S[X^{\mu}] \) transformed using the Green’s formula equals the sum of three curvilinear integrals of internal boundary values along the borders of the domains \( \Delta_1, \Delta_2, \Delta_3 \) and therefore — in the following boundary conditions:

\[
m_i \frac{d}{d\tau} \frac{V_i^{\mu}}{|V_i|} - \gamma \left[ \frac{\partial L}{\partial \dot{X}^{\mu}} - \frac{\partial L}{\partial X^{\mu}} \sigma_i'(\tau) \right] \bigg|_{\sigma = \sigma_i, 0} + \gamma \left[ \frac{\partial L}{\partial \dot{X}^{\mu}} - \frac{\partial L}{\partial X^{\mu}} \sigma_i'(\tau) \right] \bigg|_{\sigma = \sigma_i, -0} = 0, \quad i = 1, 2, 3.
\]  

(4)

For the third quark \( i = 3 \) in the first two summands we are to put \( \sigma = \sigma_0(\tau) \), and in the third we put \( \sigma = \sigma_3(\tau^*) \) in accordance with the closure condition (2).

From the physical point of view equations (4) are the 2-nd Newtonian law for the material points \( m_i \), moduli of the applied tension forces equal to \( \gamma \).

Let the induced metric on the world surface be conformally flat, i.e., conditions of orthonormality be tied:

\[
\dot{X}^2 + X'^2 = 0, \quad (\dot{X} X') = 0.
\]  

(5)

These equalities in \( \Delta \) may always be obtained by the reparametrizations \( \tau = \tau(\tilde{\tau}, \tilde{\sigma}), \sigma = \sigma(\tilde{\tau}, \tilde{\sigma}) \) (new co-ordinate lines \( \tilde{\tau} \pm \tilde{\sigma} \) on the world sheet are integral curves of equations \( \dot{X}^2 d\tau + [(\dot{X} X') \pm L] d\sigma = 0 \). We use the same notation for \( \tau \) and \( \sigma \) below and suppose that equalities (5) are satisfied.

Under conditions (5) the equations of motion (3) become linear

\[
\ddot{X}^{\mu} - X'^{\mu} = 0,
\]  

(6)

Eqs. (5) and (6) are invariant with respect to reparametrizations \( \tau \pm \sigma = f_{\pm}(\tilde{\tau} \pm \tilde{\sigma}) \) [2]. Choosing these two arbitrary functions \( f_{\pm} \) one can fix two (of four) functions \( \sigma_i(\tau) \), for example, in the form

\[
\sigma_1(\tau) = 0, \quad \sigma_2(\tau) = \pi.
\]  

(7)

The boundary equations (4) on these lines under conditions (5) and (7) take the form

\[
m_i \frac{d}{d\tau} \frac{\dot{X}^{\mu}(\tau, \sigma_i)}{|\dot{X}^{\mu}(\tau, \sigma_i)|^{1/2}} - \gamma X'^{\mu}(\tau, \sigma_i + 0) = 0, \quad \gamma X'^{\mu}(\tau, \sigma_i - 0) = 0, \quad i = 1, 2.
\]  

(8)

Reparametrizations of the mentioned type with \( f_{+}(\eta) = f_{-}(\eta) = \eta + \phi(\eta), \phi(\eta + 2\pi) = \phi(\eta), \phi'(\eta) < 1 \) [12] preserving the form of equations (7) don’t permit to fix \( \sigma_3 = \text{const} \) (or \( \sigma_0 = \text{const} \)) for all \( \tau \) in general.

Thus choosing \( \tau \) and \( \sigma \) one can’t fix three functions \( \sigma_0(\tau), \sigma_3(\tau) \) and \( \tau^{*}(\tau) \) for an arbitrary motion in a convenient form. The necessity of determining these functions from initial data essentially sophisticates the initial-boundary-value problem for the model “triangle” in comparison with the string model of meson [13].

In the present paper the functions \( \sigma_0(\tau), \sigma_3(\tau) \) and \( \tau^{*}(\tau) \) are defined from properties of symmetry for a class of uniform planar rotations of the system.

II. ROTATIONAL MOTIONS

Let the closed string with three material points uniformly rotate (preserving its form in time) in a plane \( xy \) around the origin of coordinates. The quark trajectories in \((2 + 1)\)-dimensional Minkowski space are the screw lines. For this motion one can choose on the world surface a parametrization with screw lines \( \sigma = \text{const} \) and with an uniform growth of \( \tau \) along these lines. In these coordinates on the quark trajectories

\[
\sigma_i(\tau) = \text{const}, \quad i = 0, 1, 2, 3,
\]  

\[
|V_i| = \sqrt{X^2} \bigg|_{\sigma = \sigma_i} = C_i = \text{const}, \quad i = 1, 2, 3.
\]  

(9)
All four functions $\sigma_i(\tau)$ are fixed simultaneously, $\sigma_1$ and $\sigma_2$ — in the form (7) non-limiting a generality.

Let coordinate curves $\tau = \text{const}$ be orthogonal trajectories to the specified lines $\sigma = \text{const}$ and conditions (5) be satisfied. These lines $\tau = \text{const}$ (don’t coinciding with sections $t = \text{const}$) are not closed, but a connection between $\tau$ and $\tau^*$ in the closure condition (2) is very simple: $\tau^* = \tau + \text{const}$. It is a consequence of the symmetry of this motion — the world surface in Minkowski space coincide with itself after a rotation about $t = x^0$ - axis with simultaneous translation along this axis.

Under these circumstances and conditions (5), (9) the third Eq. (4) takes the form

$$m_3 c_3^{-1} \ddot{X}^\mu(\tau, \sigma_0) - \gamma X^{\mu}(\tau, \sigma_0 + 0) + \gamma X^{\mu}(\tau^*, \sigma_3 - 0) = 0, \quad \tau^* = \tau + \text{const}. \quad (10)$$

A solution of the string oscillatory equation (6) satisfying the conditions (5), (7) – (9) may be found by the Fourier method: $X^\mu = \sum_k e_k^\mu u_k(\sigma) T_k(\tau)$. The functions $u_k(\sigma)$ and $T_k(\tau)$ with the same $k$ as a consequence of Eq. (6) are linear functions or harmonics with the same frequency $\omega$. Taking into account the described above properties of the rotational motion and its parametrization one can find the Fourier series for $X^\mu$ in (2 + 1) Minkowski space (with the unique frequency $\omega$) in the form

$$X^\mu = \{t_0 + a \tau + b \sigma; \ u(\sigma) \cos \omega \tau - \ddot{u}(\sigma) \sin \omega \tau; \ u(\sigma) \sin \omega \tau + \ddot{u}(\sigma) \cos \omega \tau\}. \quad (11)$$

The functions $u(\sigma)$ and $\ddot{u}(\sigma)$ are continuous in $[\sigma_0, \sigma_3]$, may have discontinuities of derivatives at $\sigma = 0, \sigma = \pi$ and in the segments $[\sigma_{i-1}, \sigma_i]$: are:

$$u(\sigma) = \begin{cases} A_0 \cos \omega \sigma + B_0 \sin \omega \sigma, & \sigma \in [\sigma_0, 0], \\ A_1 \cos \omega \sigma + B_1 \sin \omega \sigma, & \sigma \in [0, \pi], \\ A_2 \cos \omega \sigma + B_2 \sin \omega \sigma, & \sigma \in [\pi, \sigma_3]; \end{cases} \quad (12)$$

$$\ddot{u}(\sigma) = \dot{A}_i \cos \omega \sigma + \dot{B}_i \sin \omega \sigma, \quad \sigma \in [\sigma_i, \sigma_{i+1}].$$

Let the functions $e^{\mu} u(\sigma) T(\tau)$ and $e^{\mu} \ddot{u}(\sigma) T(\tau)$ (with $T = \cos \omega \tau$ or $T = \sin \omega \tau$) satisfy the boundary conditions (8) independently. With the continuity conditions at $\sigma = 0$ and $\sigma = \pi$ it results in 4 equations both for $u$ and $\ddot{u}$ which may be presented in the form solved with respect to $A_1 \equiv A$ and $B_1 \equiv B$ (the same formulae express $A_1, B_1$ by $A_1 \equiv \dot{A}$ and $B_1 \equiv \dot{B}$):

$$A_0 = A, \quad B_0 = h_1 A + B,$$

$$A_2 = (1 + h_2 c_1 s_1) A + h_2 s_1^2 B,$$

$$B_2 = -h_2 c_1 A + (1 - h_2 c_1 s_1) B.$$

Here $c_1 = \cos \pi \omega, \ s_1 = \sin \pi \omega, \ h_1 = \omega m_i/\gamma C_i$.

Under relations (13) solution (11) - (12) satisfies conditions (8). Substitution Eqs. (11) – (12) into the second of the orthonormality conditions (5) results in three equations

$$A_i \ddot{B}_i - \dddot{A}_i B_i = ab/\omega^2, \quad i = 0, 1, 2. \quad (14)$$

But among Eqs. (14) only one is independent, for example, with $i = 1$. It’s satisfied and the relations (13) take place — two other conditions (14) are satisfied too. And v.v. substitution (11) – (12) into the first condition (5) results in $A_i^2 + B_i^2 + A_i^2 + B_i^2 = (a^2 + b^2)/\omega^2$ — three independent equations. Transform this system with taking into account Eqs. (13) in the following equivalent form:

$$A_i^2 + B_i^2 + \dot{A}_i^2 + \dot{B}_i^2 = (a^2 + b^2)/\omega^2, \quad (15)$$

$$h_1 (A_1^2 + B_1^2) + 2 (AB + \dot{A}\dot{B}) = 0, \quad (16)$$

$$\lambda_1 (A_1^2 + \dot{A}_1^2) = \lambda_2 (B_1^2 + \dot{B}_1^2). \quad (17)$$

Here $\lambda_1 = (h_1 b_2 - 2) c_1 s_1 + h_1 (1 - 2 c_1) - h_2 c_1^2$ and $\lambda_2 = h_2 s_1^2 - 2 c_1 s_1$.

Expression (11) is a solution of the given problem if the last necessary conditions (2) and (10) are satisfied. Denote $-\theta/\omega$ the constant in Eq. (10):

$$\tau^* = \tau - \theta/\omega, \quad \theta = (\sigma_3 - \sigma_0) \omega b/\omega. \quad (18)$$

The expression for $\theta$ results from the substitution of $X^0 = t_0 + a \tau + b \sigma$ into the closure condition (2). The angle $\theta$ has the following geometrical sense: $\theta$ is the phase shift on a screw trajectory of the third quark between the beginning (at $\sigma = \sigma_0$) and the end (at $\sigma = \sigma_3$) of an unclosed coordinate line $\tau = \text{const}$.

Substitute Eqs. (11) – (13), (18) into the closure (2) and boundary (10) conditions with $\mu = 1, 2$.

Values of $u, \ddot{u}$ and their derivatives at $\sigma = \sigma_0$ and $\sigma_3$ express through $A, B, \dot{A}, \dot{B}$ by Eqs. (13), for example: $u(\sigma_3) = \{\cos \omega \sigma_3 - h_2 c \sin \omega (\sigma_3 - \pi)\} A + \{\sin \omega \sigma_3 - h_2 s \sin \omega (\sigma_3 - \pi)\} B$.

Equating the similar terms with $\cos \omega \tau$ and $\sin \omega \tau$ in 4 Eqs. (2), (10) with $\mu = 1, 2$ we obtain 8 homogeneous equations with respect to $A, B, \dot{A}, \dot{B}$ which reduce to 4 pairs of coinciding ones. For the sake of simplicity and explication of its intrinsic structure we write this homogeneous system with the matrix notation:

$$M_1 \alpha = M_2 \beta, \quad M_3 \alpha = M_4 \beta. \quad (19)$$

Here $\alpha = \begin{pmatrix} A \\ \dot{A} \end{pmatrix}, \quad \beta = \begin{pmatrix} B \\ \dot{B} \end{pmatrix}$ and matrices

$$M_1 = (h_1 s_0 - c_0) I + (c - h_2 c_1 s_2) U,$$

$$M_2 = -s_0 I - (s - h_2 s_1 s_2) U,$$

$$M_3 = [(1 - h_1 h_3) s_0 + (h_1 + h_3) c_0] I + (s + h_2 c_1 c_2) U,$$

$$M_4 = (h_3 s_0 - c_0) I + (c - h_2 s_1 c_2) U$$

are linear combinations of the identity matrix $I$ and the matrix $U(\theta) = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$.

The coefficients are:

$$c_i = \cos \omega d_i, \quad s_i = \sin \omega d_i, \quad c = \cos \omega \sigma_3, \quad s = \sin \omega \sigma_3;$$

$$d_i = \sigma_{i+1} - \sigma_i, \quad d_0 = -\sigma_0, \quad d_1 = \pi, \quad d_2 = \sigma_3 - \pi; \quad D = \sigma_3 - \sigma_0 = d_0 + d_1 + d_2.$$

Taking into account mutual commutability of $M_k$ one can exclude $\alpha$ or $\beta$ from the system (19)

$$M_1 \alpha = 0, \quad M_2 \beta = 0. \quad (20)$$

$$M = M_1 M_4 - M_2 M_3 = I + U^2 - FU = (2 \cos \theta - F) U \text{ (an equality } I + U^2(\theta) = 2 \cos \theta \cdot U \text{ is used).}$$

The parameter $F$ may be transformed to the simple form
\[ F = 2 \cos \omega D - \sum_i h_i \sin \omega D + \sum_{i<j} h_i h_j s_i \sin \omega (d_{i-1} + d_j) - h_i h_2 h_3 s_i s_2 s_0 = G_1 + G_2 + G_3 - G_1 G_2 G_3 \]

through the following notation:

\[ G_i = \frac{h_i s_{i-1} s_i - \sin \omega (d_{i-1} + d_i)}{s_{i+1}}. \quad (21) \]

The notation here is cyclically equivalent: \( d_{i+3} = d_i, \ s_{i+3} = s_i, \ G_{i+3} = G_i \), for example, \( d_3 \equiv d_0, \ s_4 \equiv s_1 \).

Homogeneous systems (20) have a desirable non-trivial solution if and only if \( \det M = (2 \cos \theta - F)^2 = 0 \), i.e.,

\[ 2 \cos \theta = G_1 + G_2 + G_3 - G_1 G_2 G_3. \quad (22) \]

Under condition (22) the matrix \( M = 0 \) and an arbitrary non-zero column \( \alpha \) is its eigenvector. It is connected with the rotational symmetry of the problem. So one can choose an optional pair \( A \& \hat{A}, \ B \& \hat{B} \) or \( A \& B \) and determine two other constants from Eq. (19) (under condition (22) two systems (19) are equivalent), in particular:

\[ \hat{A} = -K (h_1 A + 2B), \quad \hat{B} = K (2HA + h_1 B), \quad (23) \]

where

\[ K = \frac{s_0 s_1 (G_2 G_3 - 1)}{2 s_2 \sin \theta}, \quad H = \frac{1 + h_1^2 K^2}{4 K^2}. \quad (24) \]

Values (23) must obey conditions (14) – (17) descending from the orthonormality conditions (5). Substitution of (23) in Eqs. (16) and (17) after transformations results in relations

\[ \frac{G_{i+1} - G_i}{G_i G_{i+1} - 1} = \frac{\sin \omega (d_{i-1} - d_{i+1})}{s_i}, \quad i = 1, 2, 3. \quad (25) \]

One of these equations \((i = 2)\) is a consequence of (16), the second — of (17) and the third — of the previous two.

Substitution of (23) in Eqs. (14) and (15) after transformations with taking into account (18), (21) – (25) results in two equations which may be written in the form

\[ a^2 = 2K D \omega^3 \theta - 1 (H A^2 + h_1 A B + B^2), \quad (26) \]

\[ \frac{D \omega \theta}{D^2 \omega^2 + \theta^2} = \frac{2K}{1 + (4 + h_1^2) K^2} \quad (27) \]

with \( K \) from (24).

The latter equation determines a set of acceptable frequencies \( \omega \) if the parameters \( G_i, \ d_i \) and \( \theta \) are given. All these parameters defining a rotational motion of the model (except for translations and a scale factor) are related by the system of non-linear equations (21), (22), (24), (25) and (27).

The simplest way to construct solutions of the considered problem is to start with fixing three parameters \( G_1, \ G_2, \ G_3 \) as initial data. On the next step we determine the angle \( \theta \) by Eq. (22). The result of this procedure isn’t unique — for an every triplet \( G_i \) one can find a countable set of values \( \theta = \theta_j \). Further, the lengths \( d_i \) are defined from Eqs. (25) by the following two steps (the value \( d_1 = \pi \) was already chosen):

\[ \delta = d_0 - d_2 = \frac{1}{\omega} \left[ (1 - j_2^2) \arcsin s \frac{G_2 - G_1}{G_1 G_2 - 1} + \pi j_2 \right] \]

\[ d_0 = \frac{1}{\omega} \left[ \arctan \frac{\sin \theta (\delta + \pi)}{\cos \theta (\delta + \pi) - \frac{G_1 - G_2}{G_1 G_2 - 1} + \pi j_3} \right] \]

and \( d_2 = d_0 - \delta \) with arbitrary integer \( j_2, j_3 \). Substitution of \( G_i, \ d_i, \ \theta \) and \( K \) into Eq. (27) results in a countable set of frequencies \( \omega \). The latter equation is solved numerically by the secant method [14].

After a choice of the amplitudes \( A \) and \( B \) one can determine the values \( \hat{A}, \ B, \ a, \ b \) correspondingly by Eqs. (23), (26), (18) and through Eqs. (12) – (13) — the world surface (11).

To investigate the constructed world surface one can consider its section \( t = t_0 = \text{const} \) — a “photograph” of the string position at time moment \( t_0 \). These sections (curvilinear triangles) are shown in Fig. 2. Without limiting generality \( A = 0 \) is supposed in these examples — a transition to another “gauge” with \( A \neq 0 \) doesn’t change the form of such a curve, but only rotates it.

A parametrization of these curves is

\[ x = u(\sigma) \cdot \cos \frac{\sigma}{D} \sigma + \tilde{u}(\sigma) \cdot \sin \frac{\sigma}{D} \sigma, \]

\[ y = -u(\sigma) \cdot \sin \frac{\sigma}{D} \sigma + \tilde{u}(\sigma) \cdot \cos \frac{\sigma}{D} \sigma, \]

in particular, for two sides of the “triangle” \((A = 0)\)

\[ u = B \sin \omega \sigma, \quad \tilde{u} = BK (h_1 \sin \omega |s| - 2 \cos \omega \sigma), \quad \sigma \in [\sigma_0, \pi]. \]

The curve (28) is composed of three segments of a hypocycloid joined at non-zero angles in three points (the quark positions). Hypocycloid is the curve drawing by a point of a circle (with radius \( r \)) that is rolling in another fixed circle with larger radius \( R \) [15]. In the case (28) a relation of these radii

\[ R/r = 2/(1 - |b|/a) = 2/(1 - |\theta|/(D\omega)) \quad (29) \]

is irrational in general.

Differentiating Eqs. (28) results in the following fact: the curve (28) (for its smooth segments) is the hypocycloid if and only if the parameters of the curve are bounded by Eq. (27).

The curves in Fig. 2 rotate in the \( xy \) plane at the angular velocity \( \Omega = \omega/a \) where \( a \) is determined by Eq. (26); three quarks move at speeds

\[ v_i = \sqrt{\frac{\theta s_{i-1} s_i (G_{i-1} G_{i+1} - 1)}{\omega D s_{i+1} \sin \theta}}, \quad i = 1, 2, 3. \quad (30) \]

along circles with radii \( R_i = v_i/\Omega = av_i/\omega. \)

A free choice of the mentioned integer parameters \( j_1, j_2, j_3, l \) results in a very large number of different motions of the system distinguishing from each other by their topological structure. A motion or state of the system we’ll denote as “simple” if the position of the string (section \( t = \text{const} \)) is a curvilinear triangle with smooth
sides (Fig. 2). In the opposite case if there are come singular massless points on the sides of the “triangle” we’ll denominate the state “exotic”. These singular points move at the speed of light.

The motion of the system is simple if its parameters satisfy the following conditions:

\[ |\theta| < \pi, \quad |\omega|d_i < \pi, \quad G_i > 1, \quad i = 1, 2, 3, \]
\[ G_1 + G_2 + G_3 - G_1G_2G_3 > -2. \] (31)

In particular, if two quark masses are equal, for example, \( m_2 = m_3 \), the conditions (31) for \( G_i \) take the form \( G_i > 1, \quad 1 < G_2 = G_3 < 1 + 2/G_1 \). And in a symmetric case \( m_1 = m_2 = m_3 \) the limitation (31) for the simple states is \( 1 < G_1 = G_2 = G_3 < 2 \).

The dependence of a form of the curvilinear triangle on its rotational speed is shown in Fig. 2 for the case of the simple motion of the system with fixed quark masses \( m_i \). In this figure the “photographs” of the same system in various rotational states are placed, the higher quark speeds — the larger size of the curvilinear triangle. The dependence \( R_i(v_i) \) for \( m_i/\gamma = \text{const} \) is too sharp so the smallest (inner) triangle in Fig. 2 is 5 times magnified, and the largest is 7 times diminished in comparison with the natural size (natural — is in the case \( m_i/\gamma = \text{const} \)). For the middle curve \( \gamma = 1 \) is taken.

The speed of rotation could be measured by anyone of the parameters \( v_i, \omega, \Omega, \theta, G_i, \) the energy \( E \), the angular momentum \( J \) (Sec. 3), etc. Some rounded values of these and others parameters (in particular, the minimal \( v_1 \) and maximal \( v_2 \) quark speeds in the case of different \( m_i \) for the simple motions in Fig. 2 are presented in the following table:

| \( G_1 \) | \( G_2 \) | \( \omega \) | \( \theta \) | \( d_2 \) | \( v_1 \) | \( v_2 \) | \( E \) | \( J \) |
|---|---|---|---|---|---|---|---|---|
| 1.15 | 1.7 | 2.2 | | | | | | |
| 1.041 | 1.312 | 1.69 | | | | | | |
| 0.143 | 0.386 | 0.752 | | | | | | |
| 0.139 | 0.923 | 2.328 | | | | | | |
| 3.955 | 3.493 | 3.222 | | | | | | |
| 0.145 | 0.497 | 0.909 | | | | | | |
| 0.354 | 0.711 | 0.954 | | | | | | |
| 9.832 | 16.38 | 95.61 | | | | | | |
| 0.231 | 7.005 | 520.0 | | | | | | |

In the symmetric case that is considered in Ref. [9] for the simple states the following parameters are equal: \( G_1 = G_2 = G_3 = G, \quad d_1 = d_2 = d_3 = \pi, \quad v_1 = v_2 = v_3 = v \), and a value of the parameter \( G \) in the interval \((1, 2)\) is the measure of rotational rate.

In the case with different masses \( m_i \) given as initial data (Fig. 2) the mentioned procedure needs some complement. The given quark masses are connected with the others parameters of the system by the expressions

\[ m_i = \frac{\gamma}{\omega C_i h_i} = \frac{\gamma}{\omega} h_i a \sqrt{1 - v_i^2}, \] (32)

The values of the parameters for the rotational states in Fig. 2 were calculated as follows: a value of \( G_1 \) was chosen as a measure of rotational rate, \( G_2 \) and \( G_3 \) were taken as tentative at the first step of an iteration. After realization of the mentioned procedure of determination of \( \theta, \omega, G_i \), etc., the masses (32) (or relations \( m_i/m_1 = (h_i/h_1)\sqrt{(1-v_1^2)/(1-v_i^2)} \) were found and compared with the given values. The two-dimensional secant method [14] was applied in this iterative process.

The simple states in Fig. 2 demonstrate the following asymptotics in a non-relativistic and an ultrarelativistic limits. If the quark velocities \( v_i \), the system energy \( E \), the momentum \( J \) and the values \( \omega \) and \( \theta \) decrease, the curvilinear triangle tends to a rectilinear triangle. A form of the latter depends on an answer the question: is the triangle inequality for the quark masses \( m_1, m_2, m_3 \) satisfied?

If this inequality is satisfied, i.e., each of the quark masses \( m_i \) is less then a sum of two others (Fig. 2) in the non-relativistic limiting case the parameters \( \omega, \theta, v_i, R_i \) tend to 0, \( G_i \rightarrow 1 + 0 \) for all \( i = 1, 2, 3 \); the triangle tends to the rectilinear one, and lengths of its sides \( l_{ij} \) (between \( i \)-th and \( j \)-th quark) in this limit are proportional to associated \( d_i \) and opposite quark masses:

\[ \frac{l_{12}}{d_1} = \frac{l_{23}}{d_2} = \frac{l_{31}}{d_3} = \frac{d_1}{m_1} = \frac{d_2}{m_2} = \frac{d_0}{m_0}, \quad v_i \rightarrow 0. \] (33)

If one of these masses is larger then a sum of two others, for example \( m_1 > m_2 + m_3 \), in the low energy limit \( \theta \rightarrow 0 \) the obtuse angle at the corner with the largest mass \( m_1 \) tends to \( \pi \) and the triangle tends to a rectilinear segment. This limit is attained with \( \theta = 0, \quad R_1 = 0, \quad v_1 = 0, \quad G_2 = G_3 = 1 + 0 \) and non-zero values of \( \omega, v_2, v_3, G_1 > 1 \). These limiting values are connected by the equations resulting
from Eqs. (21), (24) − (27), (30), (32):

\[ v_2 = s_1, \quad d_2 = d_0 + d_1, \quad G_1 = 1 + 2d_2ωs_0s_1s_2^{-1}, \]
\[ v_3 = s_0, \quad \frac{m_2}{m_3} = \frac{c_1^2s_0}{c_3^2s_1}, \quad \frac{m_1}{m_3} = d_2ωs_0s_1 + s_2. \quad (34) \]

If \( v_2 \) and \( v_3 \) become less then the limiting values (34), the heaviest quark occupies a position at the rotational centre and the string rotates as the rectilinear segment. It looks like the string mesonic model with two light quarks, bounded by two relativistic strings (details in Sec. 3) with a supplement — the heavy quark at rest.

In the ultrarelativistic limit \( v_i \to 1 \) for the simple states the values \( d_0 \) and \( d_2 \) tend to \( d_1 = \pi, |ω| \to 1 - 0, |θ| \to π - 0 \), and the curvilinear triangle tends to a hypocycloid with three arcs (deltoid)

\[ x = B(2\sin \frac{2π}{3}σ - \sin \frac{4π}{3}σ), \quad y = -B(2\cos \frac{2π}{3}σ + \cos \frac{4π}{3}σ), \quad σ \in [-π, 2π]. \quad (35) \]

A form of the limiting curve (35) doesn’t depend on the (fixed) values \( m_1, m_2, m_3 \). So one can deduce Eqs. (35) by the simplest way in the symmetric case \( m_1 = m_2 = m_3 \). In this case the ultrarelativistic limit \( v_i \to 1 \) corresponds to a limit \( G_i = G \to 2 - 0 \). Substitution of expressions \( ω = 1 - δ, G = 2 - g^2 \) with infinitesimals \( δ, g \) into Eqs. (24), (27) results in the limiting relation

\[ \frac{3}{10} = \lim \frac{πδ(2π^2δ^2)}{g^2} - 1. \]

The root \( \frac{πδ}{g^2} = 3 \) of this equation corresponds to the desirable physical case \( m_i > 0 \). The following terms of expansion \( ω \) and \( θ \) in (27) are:

\[ ω \simeq 1 - \frac{3}{8}\pi g + \frac{15}{8}\pi g^2, \quad θ \simeq π - 3g, \quad g \to +0. \quad (36) \]

Substitution of (36) in Eqs. (22) and (13) results in limiting expressions at \( g \to +0 \) (in the case \( A = 0 \))

\[ u(σ) = B\sin σ, \quad \bar{u}(σ) = -3BCos σ, \quad σ \in [-π, 2π] \]

and the world surface (11)

\[ X^μ = B\{nτ + kσ; \sin σ \cos τ + n \cos σ \sin τ; \sin σ \sin τ - n \cos σ \cos τ\}. \quad (37) \]

A section of this surface \( t = const \) is the hypocycloid (35).

Let’s consider a situation, where the condition of “simplicity” (31) are not satisfied. Such a motion was denominated as exotic. Its world surface has peculiarities \( X^2 = X'^2 = 0 \) on the world lines of singular points (cusps) of the hypocycloid (28) which move at the speed of light.

There are many types of exotic motion differing from each other by the number and positions of these peculiarities. These topological configurations of the exotic states may be classified by investigation of the massless \( m_i \to 0 \) or ultrarelativistic \( v_i \to 1 \) limit. In this limit for the exotic states Eqs. (22) – (27), (30) result to expressions

\[ \lim_{m_i \to 0} \frac{|ω|d_i}{π} = 1 + n_i, \quad \lim_{m_i \to 0} h_i = 0, \quad \lim_{m_i \to 0} 2K = \frac{n}{k}, \]

where

\[ n = \lim_{m_i \to 0} \frac{|ω|D}{π} = n_1 + n_2 + n_3 + 3, \quad k = \lim_{m_i \to 0} \frac{θ}{π}. \quad (38) \]

Here \( n_1 \) is the number of singular points between the 1-st and the 2-nd quark, \( n_2 \) — between 2 and 3, \( n_3 \) — between 3 and 1, \( k \) is an integer.

Substitution of these expressions into Eq. (23) with \( A = 0 \) results in the following limiting form of the world surface for all parts of the string as a generalization of Eq. (37):

\[ X^μ = B\{nτ + kσ; k \sin σ \cos τ + n \cos σ \sin τ; k \sin σ \sin τ - n \cos σ \cos τ\}. \quad (39) \]

Here \( σ \in [0, πn] \), the integer parameters (38) \( n \) and \( k \) are restricted by the conditions

\[ n \geq 2, \quad |k| \leq n - 2, \quad n - k \quad \text{is even.} \quad (40) \]

For the simple states (31) \( n = 3, |k| = 1 \).

Note that world surfaces (39) describe motions of a closed massless relativistic string. Expression (39) is a solution of Eq. (6), satisfies the orthonormality conditions (5) and the closure condition \( X^μ(τ, 0) = X^μ(τ - πk, πn) \) with \( n \geq 2 \) and \( k \) restricted by (40).

A section \( t = const \) of world surface (39) is a closed hypocycloid with rational relation of the two radii

\[ R/r = 2n/(n - |k|) \]

(compare with Eq. (29)). If \( |k| = n - 2 \), this relation equals \( n \) and the curve has no selfintersections. If \( |k| \leq n - 4 \), the hypocycloid is starlike. The singular points of these hypocycloids move at the speed of light.

Topological types of rotational motions of the considered system may be exhaustively classified by pointing out a set of the mentioned integer parameters \((n, k, n_2, n_2, n_3)\) which are connected by Eq. (38) and satisfy the inequalities (40).

The states of the system differing from each other only by changing \( k \) to \( -k \) should be interpreted as the same topological type. It results from the fact that replacement \( θ \to -θ \) in Eqs. (18) − (28) changes only the bypass direction of the curvilinear “triangle”.

In the case \( k = 0 \) (it’s possible for even \( n \)) the exotic state has a form of uniformly rotating rectilinear string that is folded some times. The simplest of these states \( n = 2, k = 0 \) is the case of coincidence of two quarks (one of \( d_i \) equals 0). In this state the model “triangle” practically reduces to the quark-diquark one with the quark and diquark, connected by the double string with the tension \( 2γ \). This rectilinear segment is the particular case of the hypocycloid with \( R/r = 2 \).

If \( n \geq 4, k = 0 \) the quarks and the massless peculiarities \( X^2 = 0 \) are situated at the fold points. In this case in Eq. (11) \( b = ω = 0, \bar{u}(σ) = u(σ) \cdot const. \) These states have analogs in the meson string model with massive ends. A solution [16]

\[ X^μ = \{ατ; \tilde{B}u_n(σ) \cos ω_nτ; Bu_n(σ) \cos ω_nτ\} \quad (41) \]

describes a rotation of \( n - 1 \) times folded rectilinear open string. Here \( u_n(σ) = \cos ω_nσ - ω_nQ_1^{-1} \sin ω_nσ \),
\[ \sigma \in [0, \pi], \quad Q_i = \gamma m_i^{-1} \sqrt{X^2} \left|_{\sigma = \sigma_i} \right. = \text{const and } \omega_n \text{ is } \left. \right. \text{n-th positive root of the transcendental equation } \tan \pi \omega = (Q_1 + Q_2) \omega / (\omega^2 - Q_1 Q_2). \]

III. ENERGY AND ANGULAR MOMENTUM OF ROTATIONAL STATES

In this section possibility of application of considered solutions for description of baryon states on the Regge trajectories is briefly discussed. The Regge trajectory includes states of baryons with the same quark composition and almost the same set of quantum numbers. This trajectory is linear (without satisfactory theoretical explanation) between square of mass or rest energy of the particle \( M^2 = E^2 \) and its spin or angular momentum \( J = \alpha E^2 + \alpha_0 \).

Let’s find a connection between the energy \( E \) and angular momentum \( J \) of the rotational state (11) of the baryonic model “triangle” on the classical level. The problem for the string model of meson is solved in Ref. [3,8].

In accordance with [2,3] consider new parameters \( t, \sigma \) on the world surface, where \( t = X^0 \) is time, \( \sigma \) is the former parameter. The Lagrangian in action (1) is \( \Lambda = -\gamma \int_{\sigma_0}^{\sigma_3} \mathcal{L}(X_t, \dot{X}_t) \, d\sigma - \sum_{i=1}^{3} m_i \sqrt{1 - \dot{X}^2(t, \sigma_i)}, \)

where \( \mathcal{L} = ((X_t \dot{X}_\sigma)^2 + \dot{X}_t^2 (1 - \dot{X}_t^2))^{1/2}, \quad \dot{X}_t = \frac{\partial X_t}{\partial \sigma}, \quad \dot{X}_\sigma = \frac{\partial X_\sigma}{\partial \sigma}, \quad X = (X(t, \sigma), Y(t, \sigma)) - \text{2D-vector, the scalar product is Euclidean.} \)

In the co-ordinates \( t, \sigma \) the orthonormal conditions (5) aren’t satisfied, so the canonical momentum \( \dot{\mathcal{P}}(t, \sigma) = \delta \lambda / \delta \dot{X}_t \) and \( \delta \mathcal{L} / \delta \dot{X}_\sigma \) are non-linear with respect to \( \dot{X}_t \).

The energy of the system \( E = \int_{\sigma_0}^{\sigma_3} \mathcal{L} \, d\sigma + \sum_{i=1}^{3} m_i \sqrt{1 - \dot{v}_i^2} \) has the form

\[ E = \gamma D \frac{a^2 - b^2}{a} + \sum_{i=1}^{3} m_i \sqrt{1 - v_i^2}, \]

where \( \dot{v}_i^2 = \dot{X}_i^2 (t, \sigma_i). \)

The following expressions resulting from Eqs. (11) – (17) were used in the calculations: \( \dot{X}_t^2 = \dot{X}_t^2 (1 - \dot{X}_t^2) = (a^2 - b^2) \alpha^{-1}, \) \( \dot{X}_\sigma^2 = (a^2 - b^2) \alpha^{-1} \).

The parameters \( D, a, b = ab / (\Omega d\omega) \) and \( v_i \) are defined by Eqs. (18), (26) and (30).

The angular momentum \( J = \int_{\sigma_0}^{\sigma_3} (XP_y - YP_x) \, d\sigma \) of the state (11) is calculated by the similar way

\[ J = \frac{a}{2\omega} \left( E - \sum_{i=1}^{3} m_i \sqrt{1 - v_i^2} \right). \]

The latter relation between \( E, J \) and the angular frequency \( \omega = \omega / a \) has almost the same form as in the string model of meson [3].

Expressions (42) and (43) set an implicit non-linear connection between \( E \) and \( J \) of the considered system. A form of this connection depends on the topological type \( (n, k; n_1, n_2, n_3) \) of the state of the system.

In Fig. 3 the results of numerical calculation the dependence \( J \) on \( E^2 \) are shown for various states of the systems with fixed \( m_2 = m_3 = 0.3, \gamma = 3/(16\pi). \) Such a choice of \( \gamma \) approximately corresponds to the experimental value \( \alpha' \approx 1 \text{ GeV}^{-2}. \) One can suppose conditionally that \( E^2 \) in Fig. 6 is measured in GeV\(^2\) and \( J \) — in units \( \hbar. \)

Curves 1, 2, 3 and 4 describe the simple motions of the system correspondingly with \( m_1 = 0.05, m_1 = 0.3, m_1 = 0.6 \) and \( m_1 = 1; \) curve 5 — the exotic state of the symmetric system with equal masses \( m_1 = m_2 = m_3 = 0.3 \) and with topological type of this state (6, 2; 1, 1, 1).

The symbol “T” on curve 4 shows the point of transformation of the triangular configuration of this system with \( m_1 = 1 > m_2 + m_3 = 0.6 \) to the rectilinear configuration. This point of transformation corresponds to satisfying of Eqs. (34).

The analysis shows that in the non-relativistic limit the asymptotic behaviour of the function \( J(E) \) depends on satisfaction of the triangle inequality between three masses \( m_i. \) If each mass is less than the sum of two others, the limiting relations (33) for the simple state take place, and energy (42) of this state has the form \( E = \sum_{i=1}^{3} m_i + 3/2 \gamma m_1 m_2 m_3 \pi^2 \omega^2 + o(\omega^2), \) where \( \omega \) is an infinitesimal. The considered asymptotic relation in this case is

\[ J \simeq \left( \frac{2}{3} \right)^{3/2} \frac{1}{2\gamma} \sqrt{m_1 m_2 m_3} \left( E - \sum_{i=1}^{3} m_i \right)^{3/2}, \quad v_i \ll 1. \]

If one of the masses, for example, \( m_1 \) is larger than the sum of two others (curve 4 in Fig. 3), then the non-relativistic asymptotic case describes a rotation of rectilinear double string with two masses \( m_2 \) and \( m_3 \) at the ends and the mass \( m_1 \) at the rotational center. In this limit for the simple state the following expression take place

\[ J \simeq \left( \frac{2}{3} \right)^{3/2} \frac{1}{2\gamma} \sqrt{m_2 m_3} \left( E - \sum_{i=1}^{3} m_i \right)^{3/2}, \quad v_i \ll 1. \]

It looks like the formula in Ref. [3] for the string model
of meson and may be deduced from solution (41), but the tension of the string equals $2\gamma$.

The exponent $3/2$ is the same for both cases considered. So graphs 1–4 in Fig. 3 have similar forms and curve 4 in the vicinity of the transformation point is rather smooth.

In the opposite ultrarelativistic limit $v_i \rightarrow 1$, $E \rightarrow \infty$, $J \rightarrow \infty$ the analysis of dependence $J(E)$ includes substituting limiting formulae (38) and expressions $s_d = \pi(n_i + 1) - \delta_i$, $\theta = \pi k(1 - \delta_0)$, $\sqrt{1 - v^2} = \varepsilon_i$ with the infinitesimals $\delta_i$, $\delta_0$, $\varepsilon_i$ (generalization of Eq. (36)) into Eqs. (22)–(30), (42) and (43). Expansion in series in Eqs. (25), (27) and (30) results in the following relations between the infinitesimals:

$$h_i \simeq 2\sqrt{n^2 - k^2} \varepsilon_i (1 + \frac{n^2 - 2k^2}{2(n^2 - k^2)}\varepsilon_i^2),$$

$$\frac{\delta_i}{\sqrt{m_i^2 + \delta^{i+1}}} \simeq \frac{\delta_i}{\sqrt{m + \delta^{j+1}}} \simeq \frac{n}{\sqrt{n^2 - k^2} \sqrt{m_i}},$$

$$\sum_{i=1}^{3} \frac{\delta_i}{2} \frac{2m_{i+1}^{-1/2} (\sum_{i=1}^{3} m_{i}^{1/2})\varepsilon_i + \frac{mm_{i+1}^{-3/2}}{(n^2 - k^2)^{3/2}} \times (\frac{n^2 - 2k^2}{2} m_{i+1}^{1/2} - \frac{n^2 - 6k^2}{6} \sum_{i=1}^{3} \frac{m_{i}^{3/2}}{2}) \varepsilon_i^3.}$$

By substitution of these and analogous relations into Eqs. (32), (42) and (43) we obtain the ultrarelativistic asymptotic dependence for a state with an arbitrary type $(n, k; n_1, n_2, n_3)$

$$J \simeq \alpha' E^2 + \alpha_1 E^{1/2}, \quad v_i \rightarrow 1,$$ (44)

where

$$\alpha' = \frac{1}{2\pi}\left(\frac{n}{n^2 - k^2}\right), \quad \alpha_1 = \frac{1}{8 \pi\alpha' ^{-2}} \sum_{i=1}^{3} \frac{m_{i}^{3/2}}{2}.$$

That is close to the standard linear form $J = \alpha' E^2 + \alpha_0$.

The slope coefficient in Eq. (44) differs from the Nambu value for the mesonic model $\alpha' = (1/2\pi\gamma)$ by the factor $n/(n^2 - k^2)$. This factor equals 3/8 for the simple motions and attains the maximal value 1/2 (under admissible $n$ and $k$) for the “quark-diquark” motions with $n = 2, k = 0$. The latter case differs from the quark-diquark baryon model only by the substitution $\gamma \rightarrow 2\gamma$.

The “quark-diquark” state is preferable, if we assume the principle of minimal energy — the string system with given $J$ chooses the configuration with the minimal energy [7,8].

The first summand in Eq. (42) that could be interpreted as the “string energy” or “gluon energy” in the limit $v_i \rightarrow 1$ or $\varepsilon_i \rightarrow 0$ grows as $\varepsilon_i^{-2}$, but the last summands — “quark kinetic energy” $\sum m_{i}/\varepsilon_i$ grow as $\varepsilon_i^{-1}$. So in the ultrarelativistic limit the “string energy” dominates, and the slope coefficient $\alpha'$ in (44) doesn’t depend on quark masses $m_i$.

The coefficient $\alpha_1$, otherwise, is determined by the combination $\sum m_{i}^{3/2}$. This fact gives possibility of estimating (in the model frameworks) the mentioned sum and the quark masses $m_i$. This estimation will be accurate only for baryons which satisfy two conditions: the quark motion is to be relativistic and close to classic (the model is classic with spinless quarks). The latter condition is equivalent to the standard inequality $J/h \gg 1$ and in particular, results from the comparison of typical sizes of the “triangle” system in the relativistic case (if $E \gg m_i \sqrt{1 - v_i^2}$ in Eq. (43))

$$R_i = \frac{a}{\omega} v_i \simeq \frac{2J}{E} v_i \simeq 3.95 \cdot 10^{-14} \frac{J}{h} \frac{1}{E} \frac{1}{c} \text{ cm}$$

with the corresponding length $h/p \simeq h/E$. Furthermore, the motion is relativistic if the quarks are not very heavy: $m_i \ll M = E$.

Express the combination $\sum m_{i}^{3/2}$ from Eq. (44)

$$\sum_{i=1}^{3} m_{i}^{3/2} \simeq \frac{3(n^2 - k^2)^{-1/4}}{2^{3/2}\sqrt{\pi}} \left(E^{3/2} - \frac{J}{\alpha' E} \right).$$ (45)

It is natural to suppose that the states of the model are simple $(n = 3, k = 1)$ or “quark-diquark” $(n = 2, k = 0)$. For these two cases the mass estimations differ from each other by the small factor $\simeq 2^{1/6}$.

The expression in the parentheses in r.h.s. of Eq. (45) is a small difference of two large values. So it is very sensitive to errors in $J$ and $E$. The simplest quantum correction to these values due to quark spins implies an addition $S = \sum_{i=1}^{3} s_i$ (quark spin projection) to the classic angular momentum (43) and $\Delta E = \Delta E_{SS} + \Delta E_{SO}$ — to the energy (42). The latter correction results from spin-spin ($\Delta E_{SS}$) and spin-orbit ($\Delta E_{SO}$) interaction of quarks. The value $\Delta E_{SO}$ is supposed to be due to pure Thomas precession of quark spins [8,17], but there are some doubts as to the form and the sign of this correction. A precise form of $\Delta E$ is to be found only from a consecutive quantum theory of this baryon model that isn’t constructed yet.

In the examples below the spin correction wasn’t made. The results of using (45) for estimating quark masses on examples of two Regge trajectories (nucleonic and for strange A-particles) are shown in the following table. Masses of $u$ and $d$-quarks are assumed to be equal. Here $m_{ud}$ and $m_s$ are effective quark masses measured in GeV; $J$ — in units $h$.

| $J$ | 1/2 | 5/2 | 9/2 |
|-----|-----|-----|-----|
| Particle | $n$, $p$ | $N$(1680) | $N$(2220) |
| $m_{ud}$ | 0.138 ± 0.015 | 0.105 ± 0.03 | 0.11 ± 0.02 |
| Particle | $\Lambda$ | $\Lambda$(1815) | $\Lambda$(2350) |
| $m_s$ | 0.41 ± 0.03 | 0.345 ± 0.07 | 0.35 ± 0.055 |
| $m_s^*$ | 0.34 ± 0.035 | 0.26 ± 0.07 | 0.27 ± 0.06 |

The values $m_s$ were calculated under the assumption that $m_{ud} \ll m_s$, and $m_s^*$ — under the assumption $m_{ud} \approx 0.1$ GeV.

The error ranges for $m_i$ are due to error ranges in particle masses $M$ which influence the value $\alpha'$. The small difference between the results for the simple and “quark-diquark” configurations is also included in the error ranges.

Note that the considered model (and other mentioned string models) is applicable only to the orbitally excited
baryon states (resonances) with $J \geq 5/2$ and isn’t adequate for $p$, $n$ and $\Lambda$-particles.

We may conclude that calculated by Eq. (45) quark masses are steady with growing $J$. But the found values $m_{ud} \simeq 100 \text{ MeV}$ and $m_s \simeq 250 \text{ MeV}$ (larger then other data for free quark masses [18] and less then the constituent masses [8]) are preliminary and depend on the spin correction.

The necessity of the spin correction is demonstrated by the following fact. For the Regge trajectory with $\Delta$-resonances ($S = 3/2$) Eq. (45) results in small negative values of $\sum m_{1/2}^2$ (error boxes include some positive range). But with substituting $J - 1/2$ instead of $J$ the formula (45) gives the steady value $m_{ud} \simeq 0.1 \text{ GeV}$ for heavy $\Delta$-resonances.

With growing $E$ and $J$ the influence of the unknown $\Delta E$ on the values $m_i$ in Eq. (45) diminishes, but too slowly — as $E^{-1/2}$ or $J^{-1/4}$. So for the available baryon mass range $1 - 3 \text{ GeV}$ the spin correction in Eq. (45) is required for valid estimating the quark masses in the frameworks of the considered model.

**CONCLUSION**

In the present paper a set of rotational motions of the baryon model “triangle” is investigated on the classic level. Quantization in this model as in the string model of meson with massive ends [2,12] encounters some problems connected with non-linear form of the boundary conditions (8). Progress in this direction, for example, description of quark spins will give possibility of precise model prediction of the effective quark masses through comparison of calculated dependence $J(E^2)$ with the experimental Regge trajectories. But the problem of quantization needs special research which is beyond the present paper.

On the other hand, the slope $\alpha'$ is finally determined by Eq. (44) on the classic level. It was mentioned that this coefficient in the baryonic model “triangle” differs from the mesonic slope $\alpha' = 1/(2\pi \gamma)$ by the factor $1/2$ for the “quark-diquark” states and by the factor $3/8$ for the simple states. The experimental value $\alpha' \simeq 1 \text{ GeV}^{-2}$ is approximately equal for mesons and baryons. So an effective value of string tension $\gamma$ in the model “triangle” is to be about $1/2$ or $3/8$ of the tension in the model of meson. It is probably connected with different energies of QCD interaction in the pairs: quark and quark-antiquark.

For the sake of comparison note that in the three-string model [4–7] this factor equals $2/3$, i.e., the Regge slope in the ultrarelativistic limit is $\alpha' = \frac{2}{3}(2\pi \gamma)^{-1}$, and the effective string tension is to differ by the same factor from the mesonic one.

In the quark-diquark model and in the linear (---) configuration the Regge slope $\alpha' = 1/(2\pi \gamma)$ equals the mesonic one. So these models explain the equality of values $\alpha'$ for mesons and baryons by the natural way (the rotational motions of these models are meson-like). But this advantage is balanced an explicit dissymmetry of the quarks in the both models. Furthermore, the “triangle” and $Y$ configurations unlike two others string baryon models are QSD-motivated in the Wilson loop operator approach [10].

For description of baryons on the Regge trajectories the “quark-diquark” states and the simple states (Fig. 2) of the model “triangle” were used. Under the assumption that the energy of the orbitally excited string state for the given angular momentum $J$ is minimal [7,8] these configurations are preferable.

The exotic states naturally emerging in this model are probably too exotic for applications in particle physics, (except for hybrids).

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