Semantic Characterizations of General Belief Base Revision

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Abstract

The AGM postulates by Alchourrón, Gärdenfors, and Makinson continue to represent a cornerstone in research related to belief change. Katsuno and Mendelzon (K&M) adopted the AGM postulates for changing belief bases and characterized AGM belief base revision in propositional logic over finite signatures. We generalize K&M’s approach to the setting of (multiple) base revision in arbitrary Tarskian logics, covering all logics with a classical model-theoretic semantics and hence a wide variety of logics used in knowledge representation and beyond. Our generic formulation applies to various notions of “base” (such as belief sets, arbitrary or finite sets of sentences, or single sentences). The core result is a representation theorem showing a two-way correspondence between AGM base revision operators and certain “assignments”: functions mapping belief bases to total — yet not transitive — “preference” relations between interpretations. Alongside, we present a companion result for the case when the AGM postulate of syntax-independence is abandoned. We also provide a characterization of all logics for which our result can be strengthened to assignments producing transitive preference relations (as in K&M’s original work), giving rise to two more representation theorems for such logics, according to syntax dependence vs. independence.

1. Introduction

The question of how a rational agent should change her beliefs in the light of new information is crucial to AI systems. It gave rise to the area of belief change, which has been massively influenced by the AGM paradigm of Alchourrón, Gärdenfors, and Makinson (1985). The AGM theory assumes that an agent’s beliefs are represented by a deductively closed set of sentences (commonly referred to as a belief set). A change operator for belief sets is required to satisfy appropriate postulates in order to qualify as a rational change operator. While the contribution of AGM is widely accepted as solid and inspiring foundation, it lacks support for certain relevant aspects: it provides no immediate solution on how to deal with multiple inputs (i.e., several sentences instead of just one), with bases (i.e., arbitrary collections of sentences, not necessarily deductively closed), or with the problem of iterated belief changes.

Katsuno and Mendelzon (1991) – henceforth abbreviated K&M – deal with the issues of belief bases and multiple inputs in an elegant way: as in propositional logic, every set of sentences (including an infinite one) is equivalent to one single sentence, belief states and multiple inputs are considered as such single sentences. In this setting, K&M provide the
following set of postulates, derived from the AGM revision postulates, where \( \varphi, \varphi_1, \varphi_2, \alpha, \) and \( \beta \) are propositional sentences, and \( \circ \) is a base change operator:

\[
\begin{align*}
\text{(KM1)} & \quad \varphi \circ \alpha \models \alpha. \\
\text{(KM2)} & \quad \text{If } \varphi \land \alpha \text{ is consistent, then } \varphi \circ \alpha \equiv \varphi \land \alpha. \\
\text{(KM3)} & \quad \text{If } \alpha \text{ is consistent, then } \varphi \circ \alpha \text{ is consistent.} \\
\text{(KM4)} & \quad \text{If } \varphi_1 \equiv \varphi_2 \text{ and } \alpha \equiv \beta, \text{ then } \varphi_1 \circ \alpha \equiv \varphi_2 \circ \beta. \\
\text{(KM5)} & \quad (\varphi \circ \alpha) \land \beta \models \varphi \circ (\alpha \land \beta). \\
\text{(KM6)} & \quad \text{If } (\varphi \circ \alpha) \land \beta \text{ is consistent, then } \varphi \circ (\alpha \land \beta) \models (\varphi \circ \alpha) \land \beta.
\end{align*}
\]

The postulates (KM1)–(KM6) together are equivalent to the AGM revision postulates, thus they also yield minimal change with respect to the initial beliefs.

While the AGM paradigm is axiomatic, much of its success originated from operationalisations via representation theorems. Yet, most existing characterizations of AGM revision require the underlying logic to fulfil additional assumptions such as compactness, closure under standard connectives, deduction, or supra-classicality (Ribeiro, Wassermann, Flouris, & Antoniou, 2013).

Leaving the safe grounds of these assumptions complicates matters; representation theorems do not easily generalize to arbitrary logics. This has sparked investigations into tailored characterizations of AGM belief change for specific logics, such as Horn logic (Delgrande & Peppas, 2015), temporal logics (Bonanno, 2007), action logics (Shapiro, Pagnucco, Levespérance, & Levesque, 2011), first-order logic (Zhuang, Wang, Wang, & Delgrande, 2019), and description logics (Qi, Liu, & Bell, 2006; Halaschek-Wiener & Katz, 2006; Dong, Duc, & Lamolle, 2017). More general approaches to revision in non-classical logics were given by Ribeiro, Wassermann, and colleagues (Ribeiro et al., 2013; Ribeiro, 2013; Ribeiro & Wassermann, 2014), Delgrande, Peppas, and Woltran (2018), Pardo, Dellunde, and Godo (2009), or Aiguier, Atif, Bloch, and Hudelot (2018).

In this article, we consider (multiple) base revision in arbitrary Tarskian logics, i.e., logics exhibiting a classically defined model theory. We thereby refine and generalise the popular approach by Katsuno and Mendelzon (1991) which was tailored to belief base revision in propositional logic with a finite signature. K&M start out from belief bases, assigning to each a total preorder on the interpretations, which expresses – intuitively speaking – a degree of “modelishness”. The models of the result of any AGM revision then coincide with the preferred (i.e., preorder-minimal) models of the injected information.

Our approach generalises this idea of preferences over interpretations to the general setting, which necessitates adjusting the nature of the “modelishness-indicating” assignments: We have to explicitly require that minimal models always exist (min-completeness) and that they can be described in the logic (min-expressibility). Moreover, we show that demanding preference relations to be preorders is infeasible in our setting; we have to waive transitivity and retain only a weaker property (min-retractivity).
The main contributions of this article are the following:

- We introduce the notion of base logics to uniformly capture all popular ways of defining belief states by certain sets of sentences over Tarskian logics. Among others, this includes the cases where belief states are arbitrary sets of sentences and where belief states are belief sets.
- We extend K&M’s semantic approach from the setting of singular base revision in propositional logic to multiple base revision in arbitrary base logics.
- For this setting, we provide a representation theorem characterizing AGM belief change operators via appropriate assignments.
- We provide a variant of the characterization dealing with the case where the postulate of syntax-independence (KM4) is not imposed.
- We characterize all those logics for which every AGM operator can even be captured by preorder assignments (i.e., in the classical K&M way). In particular, this condition applies to all logics supporting disjunction and hence all classical logics. For those logics, we provide one representation theorem for the syntax-independent and one for the syntax-dependent setting.

The paper is organized as follows. In the following section, we introduce the basic logical setting and notions used in the paper. Section 3 revisits K&M’s key contributions in characterizing AGM revision operators in finite-signature propositional logic. Section 4 investigates the suitability of K&M’s approach for our generalized setting and introduces the necessary adjustments. In Section 5, we present the one-way version of our main representation theorem, providing a characterization for a given change operator satisfying the AGM postulates. Section 6 provides the two-way representation theorem to also capture the existence of the revision operator for any appropriate assignment. Thereafter, in Section 7, we reflect upon what we have achieved so far and motivate the questions subsequently addressed. Section 8 discusses the variation of our general approach when allowing revisions to depend on the syntax of the revised base. Section 9 is dedicated to the question under which circumstances preorder preferences can be salvaged, introducing and leveraging the notion of “critical-loop” in a base logic, followed by the representation theorems. Section 10 addresses further noteworthy aspects, concerning the notion of base, disjunction decomposability, and relation encoding. We discuss connections to related works in Section 11, while Section 12 summarises, concludes, and points out future work.

2. Preliminaries

In this section, we introduce the logical and algebraic notions used in the paper.

2.1 Logics with Classical Model-Theoretic Semantics

We consider logics endowed with a classical model-theoretic semantics. The syntax of such a logic \( L \) is given syntactically by a (possibly infinite) set \( L \) of sentences, while its model theory is provided by specifying a (potentially infinite) class \( \Omega \) of interpretations (also called worlds) and a binary relation \( \models \) between \( \Omega \) and \( L \) where \( \omega \models \varphi \) indicates that \( \omega \) is a model
of \( \varphi \). Hence, a logic \( \mathcal{L} \) is identified by the triple \( (\mathcal{L}, \Omega, \models) \). We let \( \models \varphi = \{ \omega \in \Omega \mid \omega \models \varphi \} \) denote the set of all models of \( \varphi \in \mathcal{L} \). Logical entailment is defined as usual (overloading “\( \models \)” via models: for two sentences \( \varphi \) and \( \psi \) we say \( \varphi \) entails \( \psi \) (written \( \varphi \models \psi \)) if \( \models \varphi \subseteq \models \psi \).

Notions of modelhood and entailment can be easily lifted from single sentences to sets. We obtain the models of a set \( \mathcal{K} \subseteq \mathcal{L} \) of sentences via \( \models \mathcal{K} = \bigcap_{\varphi \in \mathcal{K}} \models \varphi \). For \( \mathcal{K} \subseteq \mathcal{L} \) and \( \mathcal{K}' \subseteq \mathcal{L} \) we say \( \mathcal{K} \) entails \( \mathcal{K}' \) (written \( \mathcal{K} \models \mathcal{K}' \)) if \( \models \mathcal{K} \subseteq \models \mathcal{K}' \). We write \( \mathcal{K} \equiv \mathcal{K}' \) to express \( \mathcal{K} = \models \mathcal{K}' \). A (set of) sentence(s) is called consistent with another (set of) sentence(s) if the two have models in common. Unlike many other belief revision frameworks, we impose no further requirements on \( \mathcal{L} \) (like closure under certain operators or compactness).

The existence of a classical model-theoretic semantics as above is equivalent to the logic being Tarskian \(^1\) (Tarski, 1956; Sernadas, Sernadas, & Caleiro, 1997). Among others, this means that all logics considered here are monotonic, i.e., they satisfy the following condition:

\[
\text{If } \mathcal{K}_1 \models \varphi \text{ and } \mathcal{K}_1 \subseteq \mathcal{K}_2, \text{ then } \mathcal{K}_2 \models \varphi. \quad \text{(monotonicity)}
\]

The notion of Tarskian logic captures many well-known classical logical formalisms and in the following we will give some examples.

### 2.2 Tarskian Logics: Examples

We start by providing an example, where sentences and interpretations are finite sets, which allows us to specify them (as well as the models relation) explicitly. We note that this is an extension of an example given by Delgrande et al. (2018), which will serve as a running example throughout this article.

**Example 2.1** (based on (Delgrande et al., 2018)). Let \( \mathbb{L}_{\text{Ex}} = (\mathcal{L}_{\text{Ex}}, \Omega_{\text{Ex}}, \models_{\text{Ex}}) \) be the logic defined by \( \mathcal{L}_{\text{Ex}} = \{ \psi_0, \ldots, \psi_5, \varphi_0, \varphi_1, \chi, \chi' \} \) and \( \Omega_{\text{Ex}} = \{ \omega_0, \ldots, \omega_5 \} \), with the models relation \( \models_{\text{Ex}} \) implicitly given by:

\[
\begin{align*}
\models_{\text{Ex}} \psi_0 &= \{ \omega_0, \omega_1 \} \\
\models_{\text{Ex}} \varphi_0 &= \{ \omega_0, \omega_1 \} \\
\models_{\text{Ex}} \chi &= \{ \omega_0, \ldots, \omega_5 \} \\
\models_{\text{Ex}} \chi' &= \{ \omega_0, \omega_1, \omega_2, \omega_4, \omega_5 \}
\end{align*}
\]

Since \( \mathbb{L}_{\text{Ex}} \) is defined in the classical model-theoretic way, \( \mathbb{L}_{\text{Ex}} \) is a Tarskian logic. Note that logic \( \mathbb{L}_{\text{Ex}} \) has no connectives. Figure 1 illustrates the logic setting \( \mathbb{L}_{\text{Ex}} \).

Next we turn to propositional logic, observing that the distinction between a finite or infinite set of propositional symbols leads to differences that we will revisit later on.

**Example 2.2** (\( \mathbb{P}_{\text{L}, n} \), propositional logic over \( n \) propositional atoms). The logic is defined by \( \mathbb{P}_{\text{L}, n} = (\mathcal{L}_{\text{PL}, n}, \Omega_{\text{PL}, n}, \models_{\text{PL}, n}) \) in the usual way: Given a finite set \( \Sigma_p = \{ p_1, \ldots, p_n \} \) of atomic propositions, we let \( \mathcal{L}_{\text{PL}, n} \) be the set of Boolean expressions built from \( \Sigma_p \cup \{ \top, \bot \} \) using the usual set of propositional connectives (\( \neg, \land, \lor, \rightarrow, \) and \( \leftrightarrow \)). We then let the set \( \Omega_{\text{PL}, n} \) of interpretations contain all functions from \( \Sigma_p \) to \{true, false\}. The relation \( \models_{\text{PL}, n} \) is then defined inductively over the structure of sentences in the usual way.

Notably, finiteness of \( \Sigma \) implies finiteness of \( \Omega_{\text{PL}, n} \) (more specifically, \( |\Omega_{\text{PL}, n}| = 2^n \)). This in turn ensures that, despite \( \mathcal{L}_{\text{PL}, n} \) being infinite, there are only finitely many (namely \( 2^n \))

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1. Appendix A provides the corresponding formal definition and proofs of this claim.
sentences which are pairwise semantically distinct. Even more so: for every (finite or infinite) set $K$ of $\mathbb{PL}_n$ sentences, there exists some sentence $\varphi \in \mathcal{L}_{\mathbb{PL}_n}$ with $\varphi \equiv K$.

**Example 2.3** ($\mathbb{PL}_\infty$, propositional logic over infinite signature). The basic definitions for $\mathbb{PL}_\infty = (\mathcal{L}_{\mathbb{PL}_\infty}, \Omega_{\mathbb{PL}_\infty}, \models_{\mathbb{PL}_\infty})$ are just like in the previous example, with the notable difference of $\Sigma_p = \{p_1, p_2, \ldots\}$ being countably infinite. This implies immediately that $\Omega_{\mathbb{PL}_\infty}$ is infinite (in fact, even uncountable), implying that there are infinitely many sentences that are pairwise non-equivalent (e.g., all the atomic ones). Also, there exist infinite sets of sentences for which no single equivalent sentence from $\mathcal{L}_{\mathbb{PL}_\infty}$ exists (e.g., $\{p_2, p_4, p_6, \ldots\}$).

Many more (and more expressive) logics are captured by the model-theoretic framework assumed by us, e.g., first-order and second-order predicate logic, modal logics, and description logics. Our considerations do, however, not apply to non-monotonic formalisms, such as default logic, circumscription, or logic programming frameworks using negation as failure.

### 2.3 Relation over Interpretations

For describing belief revision on the semantic level, it is expedient to endow the interpretation space $\Omega$ with some structure. In particular, we will employ binary relations $\preceq$ over $\Omega$ (formally: $\preceq \subseteq \Omega \times \Omega$), where the intuitive meaning of $\omega_1 \preceq \omega_2$ is that $\omega_1$ is “equally good or better” than $\omega_2$ when it comes to serving as a model. We call $\preceq$ total if $\omega_1 \preceq \omega_2$ or $\omega_2 \preceq \omega_1$ for any $\omega_1, \omega_2 \in \Omega$ holds. We write $\omega_1 < \omega_2$ as a shorthand, whenever $\omega_1 \preceq \omega_2$ and $\omega_2 \not\preceq \omega_1$ (the intuition being that $\omega_1$ is “strictly better” than $\omega_2$). For a selection $\Omega' \subseteq \Omega$ of interpretations, an $\omega \in \Omega'$ is called $\preceq$-minimal in $\Omega'$ if $\omega \preceq \omega'$ for all $\omega' \in \Omega'$. 2 We let $\text{min}(\Omega', \preceq)$ denote the set of $\preceq$-minimal interpretations in $\Omega'$. We call $\preceq$ a preorder if it is transitive and reflexive. For a relation $R \subseteq \Omega \times \Omega$, the transitive closure of $R$ is the relation $TC(R) = \bigcup_{i=0}^{\infty} R^i$, where $R^0 = R$ and $R^{i+1} = R^i \cup \{(\omega_1, \omega_3) \mid \exists \omega_2. (\omega_1, \omega_2) \in R^i \text{ and } (\omega_2, \omega_3) \in R^i\}$.

2. If $\preceq$ is total, this definition is equivalent to the absence of any $\omega'' \in \Omega'$ with $\omega'' < \omega$. 

![Figure 1: Illustration of the logic $\mathbb{L}_{\mathbb{Ex}}$, including the modelhood relations where the solid borders represent the set of models.](image)
2.4 Bases

This article addresses the revision of and by bases. In the belief revision community, the term of base commonly denotes an arbitrary (possibly infinite) set of sentences. However, in certain scenarios, other assumptions might be more appropriate. Hence, for the sake of generality, we decided to define the notion of a base on an abstract level with minimal requirements (just as we introduced our notion of logic), allowing for its instantiation in many ways.

Definition 2.4. A base logic is a quintuple \( B = (\mathcal{L}, \Omega, \models, \mathcal{B}, \uplus) \), where

- \( (\mathcal{L}, \Omega, \models) \) is a logic,
- \( \mathcal{B} \subseteq \mathcal{P}(\mathcal{L}) \) is a family of sets of sentences, called bases, and
- \( \uplus: \mathcal{B} \times \mathcal{B} \to \mathcal{B} \) is a binary operator over bases, called the abstract union, satisfying
  \[ [B_1 \uplus B_2] = [B_1] \cap [B_2]. \]

Next, we will demonstrate how, for some logic \( \mathcal{L} = (\mathcal{L}, \Omega, \models) \), a corresponding base logic can be chosen depending on one’s preferences regarding what bases should be.

Arbitrary Sets. If all (finite and infinite) sets of sentences should qualify as bases, one can simply set \( \mathcal{B} = \mathcal{P}(\mathcal{L}) \). In that case, \( \uplus \) can be instantiated by set union \( \cup \), then the claimed behavior follows by definition.

Finite Sets. In some settings, it is more convenient to assume bases to be finite (e.g. when computational properties or implementations are to be investigated). In such cases, one can set \( \mathcal{B} = \mathcal{P}_{\text{fin}}(\mathcal{L}) \), i.e., all (and only) the finite sets of sentences are bases. Again, \( \uplus \) can be instantiated by set union \( \cup \) (as a union of two finite sets will still be finite).

Belief Sets. This setting is closer to the original framework, where the “knowledge states” to be modified were assumed to be deductively closed sets of sentences. We can capture such situations by accordingly letting \( \mathcal{B} = \{ B \subseteq \mathcal{L} \mid \forall \varphi \in \mathcal{L} : B \models \varphi \Rightarrow \varphi \in B \} \). In this case, the abstract union operator needs to be defined via
  \[ [B_1 \uplus B_2] = \{ \varphi \in \mathcal{L} \mid B_1 \cup B_2 \models \varphi \}. \]

Single Sentences. In this popular setting, one prefers to operate on single sentences only (rather than on proper collections of those). For this to work properly, an additional assumption needs to be made about the underlying logic \( \mathcal{L} = (\mathcal{L}, \Omega, \models) \): it must be possible to express conjunction on a sentence level, either through the explicit presence of the Boolean operator \( \land \) or by some other means. Formally, we say that \( \mathcal{L} = (\mathcal{L}, \Omega, \models) \) supports conjunction, if for any two sentences \( \varphi, \psi \in \mathcal{L} \) there exists some sentence \( \varphi \odot \psi \in \mathcal{L} \) satisfying \( [\varphi \odot \psi] = [\varphi] \cap [\psi] \) (if \( \land \) is available within the logic, we would simply have \( \varphi \odot \psi = \varphi \land \psi \)). For such a logic, we can “implement” the single-sentence setting by letting \( \mathcal{B} = \{ \{ \varphi \} \mid \varphi \in \mathcal{L} \} \) and defining
  \[ \{ \varphi \} \uplus \{ \psi \} = \{ \varphi \odot \psi \}. \]

For any of the four different notions of bases, one can additionally choose to disallow or allow the empty set as a base, while maintaining the required closure under abstract union.

In the following, we will always operate on the abstract level of “base logics”; our notions, results and proofs will only make use of the few general properties specified for these. This guarantees that our results are generically applicable to any of the four described (and any other) instantiations, and hence, are independent of the question what the right notion of
bases ought to be. The cognitive overload caused by this abstraction should be minimal; e.g., readers only interested in the case of arbitrary sets can safely assume $\mathfrak{B} = \mathcal{P}(\mathcal{L})$ and mentally replace any $\psi$ by $\cup$.

2.5 Base Change Operators

In this article we use base change operators to model multiple revision, which is the process of incorporating multiple new beliefs into the present beliefs held by an agent, in a consistent way (whenever that is possible). We define change operators over a base logic as follows.

**Definition 2.5.** Let $\mathfrak{B} = (\mathcal{L}, \Omega, \models, \mathfrak{B}, \cup)$ be a base logic. A function $\circ : \mathfrak{B} \times \mathfrak{B} \rightarrow \mathfrak{B}$ is called a multiple base change operator over $\mathfrak{B}$.

We will use multiple base change operators in the “standard” way of the belief change community: the first parameter represents the actual beliefs of an agent, the second parameter contains the new beliefs. The operator then yields the agent’s revised beliefs. The term “multiple” references the fact that the second input to $\circ$ is not just a single sentence, but a belief base that may consist of several sentences. For convenience, we will henceforth drop the term “multiple” and simply speak of base change operators instead.

So far, the pure notion of base change operator is unconstrained and can be instantiated by an arbitrary binary function over bases. Obviously, this does not reflect the requirements or expectations one might have when speaking of a revision operator. Hence, in line with the traditional approach, we will consider additional constraints (so-called “postulates”) for base change operators, in order to capture the gist of revisions.

2.6 Postulates for Revision

We consider multiple revision, focusing on package semantics for revision, which is that all given sentences have to be incorporated, i.e. given a base $\mathcal{K}$ and new information $\Gamma$ (also a base here), we demand success of revision, i.e. $\mathcal{K} \circ \Gamma \models \Gamma$.

Besides the success condition, the belief change community has brought up and discussed several further requirements for belief change operators to make them rational, for summaries see, e.g., (Hansson, 1999; Fermé & Hansson, 2018). This has led to the now famous AGM approach of revision (Alchourrón et al., 1985), originally proposed through a set of rationality postulates, which correspond to the postulates (KM1)–(KM6) by K&M presented in the introduction. In our article, we will make use of the K&M version of the AGM postulates adjusted to our generic notion of a base logic $\mathfrak{B} = (\mathcal{L}, \Omega, \models, \mathfrak{B}, \cup)$:

1. (G1) $\mathcal{K} \circ \Gamma \models \Gamma$.
2. (G2) If $[\mathcal{K} \cup \Gamma] \neq \emptyset$ then $\mathcal{K} \circ \Gamma \equiv \mathcal{K} \cup \Gamma$.
3. (G3) If $[\Gamma] \neq \emptyset$ then $[\mathcal{K} \circ \Gamma] \neq \emptyset$.
4. (G4) If $\mathcal{K}_1 \equiv \mathcal{K}_2$ and $\Gamma_1 \equiv \Gamma_2$ then $\mathcal{K}_1 \circ \Gamma_1 \equiv \mathcal{K}_2 \circ \Gamma_2$.
5. (G5) $(\mathcal{K} \circ \Gamma_1) \cup \Gamma_2 \models \mathcal{K} \circ (\Gamma_1 \cup \Gamma_2)$.
6. (G6) If $[(\mathcal{K} \circ \Gamma_1) \cup \Gamma_2] \neq \emptyset$ then $\mathcal{K} \circ (\Gamma_1 \cup \Gamma_2) \models (\mathcal{K} \circ \Gamma_1) \cup \Gamma_2$.

Together, the postulates implement the paradigm of minimal change, stating that a rational agent should change her beliefs as little as possible in the process of belief revision. We
consider the postulates in more detail: (G1) guarantees that the newly added belief must be a logical consequence of the result of the revision. (G2) says that if the expansion of \( \varphi \) by \( \alpha \) is consistent, then the result of the revision is equivalent to the expansion of \( \varphi \) by \( \alpha \). (G3) guarantees the consistency of the revision result if the newly added belief is consistent. (G4) is the principle of the irrelevance of the syntax, stating that the revision operation is independent of the syntactic form of the bases. (G5) and (G6) ensure more careful handling of (abstract) unions of belief bases. In particular, together, they enforce that
\[
K \circ (\Gamma_1 \cup \Gamma_2) \equiv (K \circ \Gamma_1) \cup \Gamma_2, \text{ unless } \Gamma_2 \text{ contradicts } K \circ \Gamma_1.
\]
We can see that, item by item, (G1)–(G6) tightly correspond to (KM1)–(KM6) presented in the introduction. Note also that further formulations similar to (G1)–(G6) are given in multiple particular contexts, e.g. in the context of belief base revision specifically for Description Logics (Qi et al., 2006), for parallel revision (Delgrande & Jin, 2012) and investigations on multiple revision (Zhang, 1996; Peppas, 2004; Kern-Isberner & Huvermann, 2017). An advantage of the specific form of the postulates (G1)–(G6) chosen for our presentation is that it does not require \( \mathcal{L} \) to support conjunction (while, of course, conjunction on the sentence level is still implicitly supported via (abstract) union of bases).

3. Base Revision in Propositional Logic

A well-known and by now popular characterization of base revision has been described by Katsuno and Mendelzon (1991) for the special case of propositional logic. To be more specific and apply our terminology, K&M’s approach applies to the base logic
\[
\mathbb{PL}_n = (\mathcal{L}_{\mathbb{PL}_n}, \Omega_{\mathbb{PL}_n}, \models_{\mathbb{PL}_n}, \mathcal{P}_{\text{fin}}(\mathcal{L}_{\mathbb{PL}_n}), \cup)
\]
for arbitrary, but fixed \( n \) (cf. Example 2.2). The assumption of the finiteness on the underlying signature of atomic propositions is not overtly explicit in K&M’s paper, but it becomes apparent upon investigating their arguments and proofs – we will see shortly, their characterization fails as soon as this assumption is dropped. K&M’s approach also hinges on other particularities of this setting: As discussed earlier, any propositional belief base \( \mathcal{K} \) can be equivalently written as a single propositional sentence. Consequently, in their approach, belief bases are actually represented by single sentences, without loss of expressivity.

One key contribution of K&M is to provide an alternative characterization of the propositional base revision operators satisfying (KM1)–(KM6) by model-theoretic means, i.e. through comparisons between propositional interpretations. In the following, we present their results in a formulation that facilitates later generalization. One central notion for the characterization is the notion of faithful assignment.

**Definition 3.1** (assignment, faithful). Let \( \mathcal{B} = (\mathcal{L}, \Omega, \models, \mathcal{B}, \psi) \) be a base logic. An assignment for \( \mathcal{B} \) is a function \( \preceq_{(\cdot)}: \mathcal{B} \rightarrow \mathcal{P}(\Omega \times \Omega) \) that assigns to each belief base \( \mathcal{K} \in \mathcal{B} \) a total binary relation \( \preceq_{\mathcal{K}} \) over \( \Omega \). An assignment \( \preceq_{(\cdot)} \) for \( \mathcal{B} \) is called faithful if it satisfies the following conditions for all \( \omega, \omega' \in \Omega \) and all \( \mathcal{K}, \mathcal{K}' \in \mathcal{B} \):

\begin{enumerate}
  \item[(F1)] If \( \omega, \omega' \models K \), then \( \omega \not\prec_{\mathcal{K}} \omega' \) does not hold.
  \item[(F2)] If \( \omega \models K \) and \( \omega' \not\models K \), then \( \omega \prec_{\mathcal{K}} \omega' \).
  \item[(F3)] If \( K \equiv K' \), then \( \preceq_{\mathcal{K}} = \preceq_{\mathcal{K}'} \).
\end{enumerate}

An assignment \( \preceq_{(\cdot)} \) is called a preorder assignment if \( \preceq_{\mathcal{K}} \) is a preorder for every \( \mathcal{K} \in \mathcal{B} \).
Intuitively, faithful assignments provide information about which of the two interpretations is “closer to \( K \)-modelhood”. Consequently, the actual \( K \)-models are \( \preceq_K \)-minimal. The next definition captures the idea of an assignment adequately representing the behaviour of a revision operator.

**Definition 3.2** (compatible). Let \( \mathcal{B} = (\mathcal{L}, \Omega, \models, \mathfrak{B}, \psi) \) a base logic. A base change operator \( \circ \) for \( \mathcal{B} \) is called compatible with some assignment \( \preceq \) for \( \mathcal{B} \) if it satisfies

\[
[\mathcal{K} \circ \Gamma] = \min([\Gamma], \preceq_K)
\]

for all bases \( \mathcal{K} \) and \( \Gamma \) from \( \mathcal{B} \).

With these notions in place, K&M’s representation result can be smoothly expressed as follows:

**Theorem 3.3** (Katsuno and Mendelzon (1991)). A base change operator \( \circ \) for \( \mathbb{PL}_n \) satisfies (G1)-(G6) if and only if it is compatible with some faithful preorder assignment for \( \mathbb{PL}_n \).

In the next section, we discuss and provide a generalization of the overall approach to the setting of arbitrary base logics.

4. Approach for Arbitrary Base Logics

In this section, we prepare our main result by revisiting K&M’s concepts for propositional logic and investigating their suitability for our general setting of base logics. The result by Katsuno and Mendelzon established an elegant combination of the notions of preorder assignments, faithfulness, and compatibility in order to semantically characterize AGM base change operators. However, as we mentioned before and will make more precise in the following, K&M’s characterization hinges on features of signature-finite propositional logic that do not generally hold for Tarskian logics. So far, attempts to find similar formulations for less restrictive logics have made good progress for understanding the nature of AGM revision (cf. Section 11). Here we go further, by extending the K&M approach by novel notions to the very general setting of base logics.

4.1 First Problem: Non-Existence of Minima

The first issue with K&M’s original characterization when generalizing to arbitrary base logics is the possible absence of \( \preceq_K \)-minimal elements in \([\Gamma]\).

**Observation 4.1.** For arbitrary base logics, the minimum from Definition 3.2, required in Theorem 3.3, might be empty.

One way this might happen is due to infinite descending \( \preceq_K \)-chains of interpretations. To illustrate this problem (and to show that it arises even for propositional logic, if the signature is infinite but bases are finite), consider the base logic

\[
\mathbb{PL}_\infty = (\mathcal{L}_{\mathbb{PL}_\infty}, \Omega_{\mathbb{PL}_\infty}, \models_{\mathbb{PL}_\infty}, \mathcal{P}_{\text{fin}}(\mathcal{L}_{\mathbb{PL}_\infty}), \cup),
\]

i.e., propositional logic with finite bases, but countably infinitely many distinct atomic propositions \( \Sigma = \{p_1, p_2, \ldots\} \) (cf. Example 2.3). We will exhibit a base change operator that is compatible with a faithful preorder assignment, yet does violate one of the postulates, due to the problem mentioned above.
Example 4.2. We define $\circ^U$ by simply letting $K \circ^U \Gamma = K \cup \Gamma$. Obviously $\circ^U$ violates (G3) as one can see by picking, say $K = \{p_1\}$ and $\Gamma = \{-p_1\}$. Nevertheless, for this operator, a compatible assignment exists, as we will show next. Assume a base $\Omega$ over the relation $\leq f$ and two propositional interpretations $\omega_1, \omega_2 : \Sigma \rightarrow \{\text{true}, \text{false}\}$. Let $\omega_k^\text{true}$ denote $\{p_1 \in \Sigma \mid \omega_k(p_1) = \text{true}\}$ for $k \in \{1, 2\}$, i.e., the set of atomic symbols that $\omega_k$ maps to true. Then we let $\omega_1 \leq^U \omega_2$ if at least one of the following is the case:

1. $\omega_1 \models K$
2. $\omega_2 \models \neg K$ and $\omega_2^\text{true}$ is infinite
3. $\omega_1, \omega_2 \models \neg K$, both $\omega_1^\text{true}$ and $\omega_2^\text{true}$ are finite, and $|\omega_1^\text{true}| \geq |\omega_2^\text{true}|$

We see that this definition provides a faithful preorder assignment compatible with $\circ$ (see Proposition B.1 in Appendix B for the proof).

Fact 4.3. The base change operator $\circ^U$ for $\mathcal{B}_{L_\infty}$ violates (G3) despite being compatible with the faithful preorder assignment $\leq^U$.

To remedy the problem exposed above, one needs to impose the requirement that minima exist whenever needed, as specified in the notion of min-completeness, defined next.

Definition 4.4 (min-complete). Let $\mathcal{B} = (\mathcal{L}, \Omega, \models, \mathcal{B}, \cup)$ be a base logic. A binary relation $\leq$ over $\Omega$ is called min-complete (for $\mathcal{B}$) if $\min([\Gamma], \leq) \neq \emptyset$ holds for every $\Gamma \in \mathcal{B}$ with $[\Gamma] \neq \emptyset$.

The following example demonstrates that for a binary relation it depends on the base logic whether the relation is min-complete or not.

Example 4.5. Consider two base logics $\mathcal{B}_{\leq}$ and $\mathcal{B}_{\geq}$ with

$\mathcal{B}_{\leq} = (\mathcal{L}_1, \mathbb{Z}, \models, \mathcal{P}_\text{fin}(\mathcal{L}_1) \setminus \emptyset, \cup)$, and  
$\mathcal{B}_{\geq} = (\mathcal{L}_2, \mathbb{Z}, \models, \mathcal{P}_\text{fin}(\mathcal{L}_2) \setminus \emptyset, \cup)$,

where $\mathcal{L}_1 = \{[\leq n] \mid n \in \mathbb{Z}\}$ and $\mathcal{L}_2 = \{[\geq n] \mid n \in \mathbb{Z}\}$. Furthermore let $m \models [\leq n]$ if $m \leq n$ and $m \models [\geq n]$ if $n \leq m$, assuming the usual meaning of $\leq$ for integers. In words, these logics talk about the domain of integers by means of comparisons with a fixed integer. We now define the relation $\leq$ over $\Omega$ by letting $m_1 \leq m_2$ if and only if $m_1 \leq m_2$. It can be verified that the relation is transitive and for any consistent base $\Gamma \in \mathcal{P}_\text{fin}(\mathcal{L}_1)$, respectively for $\Gamma \in \mathcal{P}_\text{fin}(\mathcal{L}_2)$, we have infinitely many models $[\Gamma]$.

Note that for each set of sentences of the form $[\leq n] \in \mathcal{L}_1$, there are no minimal models $\min([\Gamma], \leq)$, and thus, $\leq$ is not min-complete for $\mathcal{B}_{\leq}$. However, for $\mathcal{B}_{\geq}$, the relation $\leq$ is min-complete.

In the special case of $\leq$ being transitive and total, min-completeness trivially holds whenever $\Omega$ is finite (as, e.g., in the case of propositional logic over $n$ propositional atoms; cf. Example 2.2). In the infinite case, however, it might need to be explicitly imposed, as already noted in earlier works (Delgrande et al., 2018) (cf. also the notion of limit assumption by Lewis (1973)). Note that min-completeness does not entirely disallow infinite descending chains (as well-foundedness would), it only ensures that minima exist inside all model sets of consistent belief bases.
4.2 Second Problem: Transitivity of Preorder

When generalizing from the setting of propositional to arbitrary base logics, the requirement that assignments must produce preorders (and hence transitive relations) turns out to be too restrictive.

**Observation 4.6.** Transitivity of the relation produced by the assignment, as required in Theorem 3.3, is too strict property for characterizing arbitrary Tarskian logics.

In fact, it has been observed before that the incompatibility between transitivity and K&Ms approach already arises for propositional Horn logic (Delgrande & Peppas, 2015). The following example builds on Example 2.1 and provides an operator and a belief base for which no compatible transitive assignment exists.

**Example 4.7** (continuation of Example 2.1). Consider the base logic \( B_{Ex} = (L_{Ex}, \Omega_{Ex}, \models_{Ex}, \mathcal{P}(L_{Ex}), \cup) \). Let \( K_{Ex} = \{\psi_3\} \) and let \( o_{Ex} \) be the base change operator defined as follows:

\[
K_{Ex} \circ_{Ex} \Gamma = \begin{cases} 
K_{Ex} \cup \Gamma & \text{if } [K_{Ex} \cup \Gamma] \neq \emptyset, \\
\Gamma \cup \{\psi_4\} & \text{if } [K_{Ex} \cup \Gamma] = \emptyset \text{ and } [\{\psi_4\} \cup \Gamma] \neq \emptyset, \\
\Gamma \cup \{\psi_0\} & \text{if } [K_{Ex} \cup \Gamma] = \emptyset \text{ and } [\{\psi_0\} \cup \Gamma] \neq \emptyset \text{ and } [\{\psi_2\} \cup \Gamma] = \emptyset, \\
\Gamma \cup \{\psi_1\} & \text{if } [K_{Ex} \cup \Gamma] = \emptyset \text{ and } [\{\psi_1\} \cup \Gamma] \neq \emptyset \text{ and } [\{\psi_0\} \cup \Gamma] = \emptyset, \\
\Gamma \cup \{\psi_2\} & \text{if } [K_{Ex} \cup \Gamma] = \emptyset \text{ and } [\{\psi_2\} \cup \Gamma] \neq \emptyset \text{ and } [\{\psi_1\} \cup \Gamma] = \emptyset, \\
\Gamma & \text{if none of the above applies},
\end{cases}
\]

Moreover, for all \( K \) with \( K' \equiv K_{Ex} \) we define \( K' \circ_{Ex} \Gamma = K_{Ex} \circ_{Ex} \Gamma \) and for all \( K' \) with \( K' \neq K_{Ex} \) we define

\[
K' \circ_{Ex} \Gamma = \begin{cases} 
K' \cup \Gamma & \text{if } K' \cup \Gamma \text{ consistent}, \\
\Gamma & \text{otherwise}.
\end{cases}
\]

For all \( K' \) with \( K' \neq K_{Ex} \), there is no violation of the postulates (G1)-(G6) since we obtain a trivial revision, which satisfies (G1)-(G6) (cf. Example 6.5). For the case of \( K' \equiv K_{Ex} \), the satisfaction of (G1)-(G6) can be shown case by case or using Theorem 6.3 in Section 6.

Now assume there were a preorder assignment \( \preceq_{(1)} \) compatible with \( o_{Ex} \). This means that for all bases \( \mathcal{K} \) and \( \Gamma \) from \( \mathcal{P}(L_{Ex}) \), the relation \( \preceq_{\mathcal{K}} \) is a preorder and \( [K \circ_{Ex} \Gamma] = \min([\Gamma], \preceq_{K_{Ex}}) \).

Now consider \( \Gamma_0 = \{\varphi_0\} \), \( \Gamma_1 = \{\varphi_1\} \), and \( \Gamma_2 = \{\varphi_2\} \). From the definition of \( o_{Ex} \) and compatibility, we obtain \( [K_{Ex} \circ_{Ex} \Gamma_0] = \{\omega_0\} = \min([\Gamma_0], \preceq_{K_{Ex}}) \), \( [K_{Ex} \circ_{Ex} \Gamma_1] = \{\omega_1\} = \min([\Gamma_1], \preceq_{K_{Ex}}) \), and \( [K_{Ex} \circ_{Ex} \Gamma_2] = \{\omega_2\} = \min([\Gamma_2], \preceq_{K_{Ex}}) \).

Recall that \( [\Gamma_0] = \{\omega_0, \omega_1\} \), \( [\Gamma_1] = \{\omega_1, \omega_2\} \), and \( [\Gamma_2] = \{\omega_2, \omega_0\} \). Yet, this implies \( \omega_0 \prec_{K_{Ex}} \omega_1, \omega_1 \prec_{K_{Ex}} \omega_2, \) and \( \omega_2 \prec_{K_{Ex}} \omega_0 \), contradicting the assumption that \( \preceq_{K_{Ex}} \) is transitive. Hence it cannot be a preorder.

As a consequence, we cannot help but waive transitivity (and hence the property of the assignment providing a preorder) if we want our characterization result to hold for all Tarskian logics. However, for our result, we need to retain a new, weaker property (which is implied by transitivity) defined next.

**Definition 4.8** (min-retractive). Let \( B = (L, \Omega, \models, \mathcal{B}, \psi) \) be a base logic. A binary relation \( \preceq \) over \( \Omega \) is called min-retractive (for \( B \)) if, for every \( \Gamma \in B \) and \( \omega', \omega \in [\Gamma] \) with \( \omega' \preceq \omega \), \( \omega \in \min([\Gamma], \preceq) \) implies \( \omega' \in \min([\Gamma], \preceq) \).

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We show that \( \min(\text{assignment}) \) and \( \text{total relation} \) interpretations, i.e., \( \Omega \)-interpretations, i.e., Figure 2b. Indeed, we have that scissors beat paper (\( \prec \)), we find that all-three \( \omega \). Let \( \omega \)-equivalent to \( 1 \)-model \( B \) from Example 4.9. Let \( B_{mr} = (\mathcal{L}, \Omega, \models, \mathcal{B}, \models) \) be a base logic with just one base \( \mathcal{B} = \{ \Gamma_{mr} \} \) and four interpretations \( \Omega = \{ \omega_0, \omega_1, \omega_2, \omega_3 \} \) such that \( [\Gamma_{mr}] = \Omega \). Now consider the following total relation \( \preceq_{1}^{mr} \) on \( \Omega \) illustrated in Figure 2a and given by

\[
\begin{align*}
\omega_i &\preceq_{K_{Ex}}^{mr} \omega_i, \quad 0 \leq i \leq 3, \\
\omega_3 &\preceq_{K_{Ex}}^{mr} \omega_i, \quad 0 \leq i \leq 2, \\
\omega_i &\preceq_{K_{Ex}}^{mr} \omega_3, \quad 0 \leq i \leq 2, \\
\omega_0 &\preceq_{K_{Ex}}^{mr} \omega_1,
\end{align*}
\]

We show that \( \preceq_{1}^{mr} \) is not min-retractive for \( B_{mr} \). The \( \preceq_{mr} \)-minimal models of \( \Gamma_{mr} \) are given by \( \min([\Gamma_{mr}], \preceq_{1}^{mr}) = \{ \omega_3 \} \). Observe that \( \omega_0 \) is a non-minimal model of \( \Gamma_{mr} \) while being \( \preceq_{1}^{mr} \)-equivalent to \( \omega_3 \), and in particular \( \omega_0 \preceq_{1}^{mr} \omega_3 \). This is a violation of min-retractivity.

Let \( \preceq_{2}^{mr} \) be the same relation as \( \preceq_{1}^{mr} \), except that \( \preceq_{2}^{mr} \) strictly prefers \( \omega_1 \) over all over interpretations, i.e., \( \preceq_{2}^{mr} = \preceq_{1}^{mr} \setminus \{ (\omega, \omega) \mid \omega \neq \omega_3 \} \). An illustration of \( \preceq_{2}^{mr} \) is given in Figure 2b. Indeed, we have that \( \preceq_{2}^{mr} \) is min-retractive for \( B_{mr} \). In particular, observe that the prior counterexample for \( \preceq_{1}^{mr} \) does not apply to \( \preceq_{2}^{mr} \), as we have \( \omega_0 \preceq_{2}^{mr} \omega_3 \).

As an aside, let us note that, if \( \preceq \) is total but not transitive, min-completeness can be violated even in the setting where \( \Omega \) is finite, by means of strict cyclic relationships.

Example 4.10. Let \( B_{rps} = (\mathcal{L}, \Omega, \models, \mathcal{P}(\mathcal{L}), \cup) \) be the base logic defined by \( \mathcal{L} = \{ \text{ALL-THREE} \} \) and \( \Omega = \{ \text{C,B,O} \} \), with the models relation \( \models \) given by \( [\text{ALL-THREE}] = \Omega \). We now define the relation \( \preceq_{rps} \) as the common game “rock-paper-scissors”: paper beats rock (\( \text{C} \preceq_{rps} \text{B} \)), scissors beat paper (\( \text{C} \preceq_{rps} \text{O} \)), and rock beats scissors (\( \text{B} \preceq_{rps} \text{O} \)). Clearly, the set of interpretations \( \Omega \) is finite and the relation \( \preceq_{rps} \) is total, but not transitive. It is, however vacuously min-retractive. By considering a consistent base \( \Gamma \) containing the only sentence ALL-THREE, we find that \( \min([\Gamma], \preceq_{rps}) = \emptyset \), and hence a violation of min-completeness.

As a last act in this section, we conveniently unite the two identified properties into one notion.

Definition 4.11 (min-friendly). Let \( B = (\mathcal{L}, \Omega, \models, \mathcal{B}, \models) \) be a base logic. A binary relation \( \preceq \) over \( \Omega \) is called min-friendly (for \( B \)) if it is both min-retractive and min-complete. An assignment \( \preceq_{(\cdot)}; \mathcal{B} \to \mathcal{P}(\Omega \times \Omega) \) is called min-friendly if \( \preceq_K \) is min-friendly for all \( K \in \mathcal{B} \).
5. One-Way Representation Theorem

We are now ready to generalize K&M’s representation theorem from propositional to arbitrary Tarskian logics, by employing the notion of compatible min-friendly faithful assignments.

**Theorem 5.1.** Let $\circ$ be a base change operator for some base logic $\mathbb{B}$. Then, $\circ$ satisfies (G1)–(G6) if and only if it is compatible with some min-friendly faithful assignment for $\mathbb{B}$.

We show Theorem 5.1 in three steps. First, we provide a canonical way of obtaining an assignment for a given revision operator. Next, we show that our construction indeed yields a min-friendly faithful assignment that is compatible with the revision operator. Finally, we show that the notion of min-friendly compatible assignment is adequate to capture the class of base revision operators satisfying (G1)–(G6).

5.1 From Postulates to Assignments

Very central for the original result by Katsuno and Mendelzon (1991) is a constructive way to obtain the assignment from a revision operator. In their proof for Theorem 3.3, they provided the following way of extracting the preference relations from the revision operator:

$$\omega_1 \leq_K \omega_2 \text{ if } \omega_1 \models K \text{ or } \omega_1 \models K \circ \text{form}(\omega_1, \omega_2) \quad (1)$$

where $\text{form}(\omega_1, \omega_2) \in \mathcal{L}$ denotes a sentence with $\llbracket \text{form}(\omega_1, \omega_2) \rrbracket = \{\omega_1, \omega_2\}$. Unfortunately, this method for obtaining a canonical encoding of the revision strategy of $\circ$ does not generalize to the general setting here. This is because a belief base $\Gamma$ satisfying $\llbracket \Gamma \rrbracket = \{\omega_1, \omega_2\}$ may not exist.

As a recourse, we suggest the following construction, which we consider one of this article’s core contributions. It realizes the idea that one should (strictly) prefer $\omega_1$ over $\omega_2$ only if there is a witness belief base $\Gamma$ that certifies that $\circ$ prefers $\omega_1$ over $\omega_2$. Should no such witness exist, $\omega_1$ and $\omega_2$ will be deemed equally preferable.

**Definition 5.2.** Let $\mathbb{B} = (\mathcal{L}, \Omega, \models, \mathbb{B}, \mathfrak{w})$ be a base logic, let $\circ$ be a base change operator for $\mathbb{B}$ and let $\mathcal{K} \in \mathbb{B}$ be a belief base. The relation $\sqsubseteq^0$ over $\Omega$ is defined by

$$\omega_1 \sqsubseteq^0_K \omega_2 \text{ if } \omega_2 \models \mathcal{K} \circ \text{implies } \omega_1 \models \mathcal{K} \circ \text{for all } \Gamma \in \mathbb{B} \text{ with } \omega_1, \omega_2 \in \llbracket \Gamma \rrbracket.$$

Definition 5.2 already yields an adequate encoding strategy for many base logics. However, to also properly cope with certain “degenerate” base logics, we have to hard-code that the prior beliefs of an agent are prioritized in all cases, that is, only models of the prior beliefs are minimal. In Section 10.4 we will analyze this in more detail. The following relation builds upon the relation $\sqsubseteq^0_K$ and takes explicit care of handling prior beliefs.

**Definition 5.3.** Let $\mathbb{B} = (\mathcal{L}, \Omega, \models, \mathbb{B}, \mathfrak{w})$ be a base logic, let $\circ$ be a base change operator for $\mathbb{B}$ and let $\mathcal{K} \in \mathbb{B}$ be a belief base. The relation $\sqsubseteq^\circ_K$ over $\Omega$ is then defined by

$$\omega_1 \sqsubseteq^\circ_K \omega_2 \text{ if } \omega_1 \models \mathcal{K} \text{ or } (\omega_1, \omega_2 \not\models \mathcal{K} \text{ and } \omega_1 \sqsubseteq^0_K \omega_2).$$

Let $\sqsubseteq^\circ_K : \mathbb{B} \rightarrow \mathcal{P}(\Omega \times \Omega)$ denote the mapping $\mathcal{K} \mapsto \sqsubseteq^\circ_K$. 

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Yet, one can easily verify that \( \omega \) could appear in the following relation such a belief base we show the latter by contradiction: Assume the contrary, i.e. there are \( J \) \( \omega \) such that \( \omega \) and \( \omega \) Proof. Note that by construction, totality of \( \text{Lemma 5.5} \) always obtaining a relation that is total and reflexive.

Figure 3: The structure of relation \( \preceq_{\text{Ex}} \) on \( \Omega_{\text{Ex}} \), where a solid arrow represents \( \omega \preceq_{\text{K}_{\text{Ex}}} \omega' \) for any \( \omega, \omega' \in \Omega_{\text{Ex}} \).

In the following, we apply the relation encoding given in Definition 5.3 to our running example and show that the relation is not transitive, yet min-friendly.

**Example 5.4** (continuation of Example 4.7). Applying Definition 5.3 to \( K_{\text{Ex}} \) and \( \circ_{\text{Ex}} \) yields the following relation \( \preceq_{\text{K}_{\text{Ex}}} \) on \( \Omega_{\text{Ex}} \) (where \( \omega \preceq_{\text{K}_{\text{Ex}}} \omega' \) denotes \( \omega \preceq_{\text{K}_{\text{Ex}}} \omega' \) and \( \omega' \preceq_{\text{K}_{\text{Ex}}} \omega \)):

\[
\begin{align*}
\omega_1 & \preceq_{\text{K}_{\text{Ex}}} \omega_i, \quad 0 \leq i \leq 5 \\
\omega_3 & \preceq_{\text{K}_{\text{Ex}}} \omega_i, \quad i \in \{0, 1, 2, 4, 5\} \\
\omega_0 & \preceq_{\text{K}_{\text{Ex}}} \omega_1 \quad \omega_4 \preceq_{\text{K}_{\text{Ex}}} \omega_i, \quad i \in \{0, 1, 2, 5\} \\
\omega_1 & \preceq_{\text{K}_{\text{Ex}}} \omega_2 \quad \omega_j \preceq_{\text{K}_{\text{Ex}}} \omega_5, \quad 0 \leq i < 4 \\
\omega_2 & \preceq_{\text{K}_{\text{Ex}}} \omega_0
\end{align*}
\]

Observe that \( \preceq_{\text{K}_{\text{Ex}}} \) is not transitive, since \( \omega_0, \omega_1, \omega_2 \) form a \( \preceq_{\text{K}_{\text{Ex}}} \)-circle (see Figure 3). Yet, one can easily verify that \( \preceq_{\text{K}_{\text{Ex}}} \) is a total and min-friendly relation. In particular, as \( \Omega_{\text{Ex}} \) is finite, min-completeness can be checked by examining minimal model sets of all consistent bases in \( L_{\text{Ex}} \). Moreover, there is no belief base \( \Gamma \in P(L_{\text{Ex}}) \) such that there is some \( \omega \notin \min([\Gamma], \preceq_{\text{K}_{\text{Ex}}}) \) and \( \omega' \in \min([\Gamma], \preceq_{\text{K}_{\text{Ex}}}) \) with \( \omega \preceq_{\text{K}_{\text{Ex}}} \omega' \). Note that such a situation could appear in \( \preceq_{\text{K}_{\text{Ex}}} \) if an interpretation \( \omega \) would be \( \preceq_{\text{K}_{\text{Ex}}} \)-equivalent to \( \omega_0, \omega_1 \) and \( \omega_2 \) and there would be a belief base \( \Gamma \) satisfied in all these interpretations, e.g., if \( \omega = \omega_5 \) would be equal to \( \omega_0, \omega_1 \) and \( \omega_2 \), and \( [\Gamma] = \{\omega_0, \omega_1, \omega_2, \omega_5\} \). However, this is not the case in \( \preceq_{\text{K}_{\text{Ex}}} \) and such a belief base \( \Gamma \) does not exist in \( B_{\text{Ex}} \). Therefore, the relation \( \preceq_{\text{K}_{\text{Ex}}} \) is min-reflective.

As a first insight, we obtain that the construction in Definition 5.3 is strong enough for always obtaining a relation that is total and reflexive.

**Lemma 5.5** (totality). If \( \circ \) satisfies (G5) and (G6), the relations \( \preceq_{\text{K}} \) and \( \subseteq_{\text{K}} \) are total (and hence reflexive) for every \( K \in \mathcal{B} \).

**Proof.** Note that by construction, totality of \( \preceq_{\text{K}} \) is an immediate consequence of totality of \( \subseteq_{\text{K}} \). We show the latter by contradiction: Assume the contrary, i.e. there are \( \subseteq_{\text{K}} \)-incomparable \( \omega_1 \) and \( \omega_2 \). Due to Definition 5.2, there must exist \( \Gamma_1, \Gamma_2 \in \mathcal{B} \) with \( \omega_1, \omega_2 \models \Gamma_1 \) and \( \omega_1, \omega_2 \models \Gamma_2 \), such that \( \omega_1 \models K \odot \Gamma_1 \) and \( \omega_2 \models K \odot \Gamma_1 \) whereas \( \omega_1 \not\models K \odot \Gamma_2 \) and \( \omega_2 \models K \odot \Gamma_2 \). Since \( \omega_1 \in [K \odot \Gamma_1 \cup \Gamma_2] = [(K \odot \Gamma_1) \cup \Gamma_2] \) and thus \( [(K \odot \Gamma_1) \cup \Gamma_2] \neq \emptyset \), (G5) and (G6) jointly entail \( [(K \odot \Gamma_1) \cup \Gamma_2] = [K \odot (\Gamma_1 \cup \Gamma_2)] \). From commutativity of \( \cup \), \( [K \odot (\Gamma_1 \cup \Gamma_2)] = [K \odot (\Gamma_2 \cup \Gamma_1)] \)
follows. Now again, since \( \omega_2 \in [K \circ \Gamma_2] \cap [\Gamma_1] = [(K \circ \Gamma_2) \cup \Gamma_1] \) and hence \( [(K \circ \Gamma_2) \cup \Gamma_1] \neq \emptyset \), (G5) and (G6) together entail \( [K \circ (\Gamma_2 \cup \Gamma_1)] = [(K \circ \Gamma_2) \cup \Gamma_1] \). So, together, we obtain \( \omega_1 \in [(K \circ \Gamma_2) \cup \Gamma_1] = [K \circ \Gamma_2] \cap [\Gamma_1] \) which directly contradicts our assumption \( \omega_1 \not\in [K \circ \Gamma_2] \).

Reflexivity follows immediately from totality. \( \square \)

We proceed with an auxiliary lemma about belief bases and \( \leq^0_K \).

**Lemma 5.6.** Let \( \circ \) satisfy (G2), (G5) and (G6) and let \( K \in \mathcal{B} \). Then the following hold:

(a) If \( \omega_1 \underline{\not}\leq^0_K \omega_2 \) and \( \omega_2 \not\models K \), then there exists some \( \Gamma \) with \( \omega_1, \omega_2 \models \Gamma \) as well as \( \omega_2 \models K \circ \Gamma \) and \( \omega_1 \not\models K \circ \Gamma \).

(b) If there is a \( \Gamma \) with \( \omega_1, \omega_2 \models \Gamma \) such that \( \omega_1 \models K \circ \Gamma \), then \( \omega_1 \underline{\not}\leq^0_K \omega_2 \).

(c) If there is a \( \Gamma \) with \( \omega_1, \omega_2 \models \Gamma \) such that \( \omega_1 \models K \circ \Gamma \) and \( \omega_2 \not\models K \circ \Gamma \), then \( \omega_1 \underline{\not}\leq^0_K \omega_2 \).

**Proof.** For the proofs of all statements, recall that by Lemma 5.5, the relation \( \leq^0_K \) is total.

(a) By totality of \( \leq^0_K \), guaranteed by Lemma 5.5, we obtain \( \omega_2 \underline{\not}\leq^0_K \omega_1 \). By definition of \( \leq^0_K \), this together with \( \omega_2 \not\models K \) entails \( \omega_1 \not\models K \). Therefore, again by definition, we obtain \( \omega_1 \underline{\not}\leq^0_K \omega_2 \). Consequently, in view of Definition 5.2, there must exist some \( \Gamma \in \mathcal{B} \) with \( \omega_1, \omega_2 \models \Gamma \) such that \( \omega_2 \models K \circ \Gamma \) does not imply \( \omega_1 \models K \circ \Gamma \). Yet this can only be the case if \( \omega_2 \models K \circ \Gamma \) and \( \omega_1 \not\models K \circ \Gamma \), as claimed.

(b) Let \( \Gamma \) and \( \omega_1, \omega_2 \) be as assumed. We proceed by case distinction:

\[
\omega_2 \models K. \text{ Then } \omega_2 \in [K] \cap [\Gamma] = [K \cup \Gamma] \text{ and thus } [K \cup \Gamma] \neq \emptyset. \text{ Therefore, by (G2), we obtain } [K \circ \Gamma] = [K \cup \Gamma] = [K] \cap [\Gamma] \text{ and consequently } \omega_1 \models K. \text{ By Definition 5.3, we conclude } \omega_1 \underline{\not}\leq^0_K \omega_2.
\]

\[
\omega_2 \not\models K. \text{ Toward a contradiction, suppose } \omega_1 \underline{\not}\leq^0_K \omega_2. \text{ Then, by part (a) above, there is } \Gamma' \text{ with } \omega_1, \omega_2 \models \Gamma', \omega_1 \not\models K \circ \Gamma' \text{ and } \omega_2 \models K \circ \Gamma'. \text{ Thus } \omega_1 \text{ and } \omega_2 \text{ witness non-emptiness of } [(K \circ \Gamma') \cup \Gamma'] \text{ and } [(K \circ \Gamma') \cup \Gamma'], \text{ respectively. Then, using (G5) and (G6) twice, we obtain } (K \circ \Gamma') \cup \Gamma' \equiv (K \circ \Gamma) \cup \Gamma'. \text{ But this allows to conclude } \omega_1 \in [K \circ \Gamma] \cap [\Gamma] = [(K \circ \Gamma) \cup \Gamma'] = [(K \circ \Gamma') \cup \Gamma'] \subseteq [K \circ \Gamma]', \text{ and thus } \omega_1 \models K \circ \Gamma', \text{ which contradicts } \omega_1 \not\models K \circ \Gamma' \text{ above.}
\]

(c) Let \( \Gamma \) and \( \omega_1, \omega_2 \) be as assumed. We already know \( \omega_1 \underline{\not}\leq^0_K \omega_2 \) due to part (b). It remains to show \( \omega_2 \underline{\not}\leq^0_K \omega_1 \). We proceed by case distinction:

\[
\omega_1 \models K. \text{ Then } \omega_1 \in [K] \cap [\Gamma] = [K \cup \Gamma] \text{ and thus } [K \cup \Gamma] \neq \emptyset. \text{ Therefore, by (G2), we obtain } [K \circ \Gamma] = [K \cup \Gamma] = [K] \cap [\Gamma]. \text{ Since } \omega_2 \not\models K \circ \Gamma \text{ but } \omega_2 \models \Gamma \text{ we can infer } \omega_2 \not\models K. \text{ Consequently, by Definition 5.3, we obtain } \omega_2 \underline{\not}\leq^0_K \omega_1.
\]

\[
\omega_1 \not\models K. \text{ Since we already established } \omega_1 \underline{\not}\leq^0_K \omega_2, \text{ Definition 5.3 ensures } \omega_2 \not\models K. \text{ Yet, by Definition 5.2, the existence of } \Gamma \text{ implies } \omega_2 \underline{\not}\leq^0_K \omega_1, \text{ and thus Definition 5.3 yields } \omega_2 \underline{\not}\leq^0_K \omega_1. \text{ \( \square \)}
\]

We show that our construction indeed yields a compatible assignment.

**Lemma 5.7** (compatibility). If \( \circ \) satisfies (G1)–(G3), (G5), and (G6), then it is compatible with \( \leq^0_K \).
Proof. We have to show that \( [K \circ \Gamma] = \min([\Gamma], \preceq_K) \). In the following, we show inclusion in both directions.

\( \subseteq \) Let \( \omega \in [K \circ \Gamma] \). By (G1), we obtain \( \omega \in [\Gamma] \). But then, using Lemma 5.6(b), we can conclude \( \omega \preceq_K \omega' \) for any \( \omega' \in [\Gamma] \), hence \( \omega \in \min([\Gamma], \preceq_K) \).

\( \supseteq \) Let \( \omega \in \min([\Gamma], \preceq_K) \). Due to \( [\Gamma] \neq \emptyset \) and (G3), there exists an \( \omega' \in [K \circ \Gamma] \). From the \( (\subseteq) \)-proof follows \( \omega' \in \min([\Gamma], \preceq_K) \). Then, by (G1) and Lemma 5.6(b), we obtain \( \omega' \preceq_K \omega \) from \( \omega \in [\Gamma] \) and \( \omega' \in [\Gamma] \) and \( \omega' \in [K \circ \Gamma] \). From \( \omega \in \min([\Gamma], \preceq_K) \) and \( \omega' \in [\Gamma] \) follows \( \omega \preceq_K \omega' \). We proceed by case distinction:

\( \omega \models K \). Then \( \omega \in [K] \cap [\Gamma] = [K \cup \Gamma] \) and thus \( [K \cup \Gamma] \neq \emptyset \). Therefore, by (G2), we obtain \( [K \circ \Gamma] = [K \cup \Gamma] = [K] \cap [\Gamma] \) and hence \( \omega \in [K \circ \Gamma] \).

\( \omega \not\models K \). Then by Definition 5.3, \( \omega \preceq_K \omega' \) requires \( \omega' \not\models K \) and therefore \( \omega \preceq_K \omega' \) must hold. Consequently, by Definition 5.2, \( \omega, \omega' \in [\Gamma] \) and \( \omega' \in [K \circ \Gamma] \) imply \( \omega \in [K \circ \Gamma] \).

For min-friendliness, we have to show that each \( \preceq_K \) is min-complete and min-retractive.

Lemma 5.8 (min-friendliness). If \( \circ \) satisfies (G1)–(G3), (G5), and (G6), then \( \preceq_K \) is min-friendy for every \( K \in \mathcal{B} \).

Proof. Observe that min-completeness is a consequence of (G3) and the compatibility of \( \preceq_K \) with \( \circ \) from Lemma 5.7.

For min-retractivity, suppose towards a contradiction that it does not hold. That means there is a belief base \( \Gamma \) and interpretations \( \omega', \omega \models \Gamma \) with \( \omega' \preceq_K \omega \) and \( \omega \in \min([\Gamma], \preceq_K) \) but \( \omega' \not\in \min([\Gamma], \preceq_K) \). From Lemma 5.7 we obtain \( \omega \models K \circ \Gamma \) and \( \omega' \not\models K \circ \Gamma \). Now, applying Lemma 5.6(c) yields \( \omega \preceq_K \omega' \), contradicting \( \omega' \preceq_K \omega \).

We show that \( \preceq_K \) yields faithful relations for every belief base.

Lemma 5.9 (faithfulness). If \( \circ \) satisfies (G2), (G4), (G5), and (G6), the assignment \( \preceq_K \) is faithful.

Proof. We show satisfaction of the three conditions of faithfulness, (F1)–(F3).

(F1) Let \( \omega, \omega' \in [K] \). Then \( \omega' \preceq_K \omega \) is an immediate consequence of Definition 5.3. This implies \( \omega \not\preceq_K \omega' \).

(F2) Let \( \omega \in [K] \) and \( \omega' \not\in [K] \). By Definition 5.3 we obtain \( \omega \preceq_K \omega' \) and \( \omega' \preceq_K \omega \).

(F3) Let \( K \equiv K' \) (i.e. \( [K] = [K'] \)). From Definition 5.3 and (G4) follows \( \preceq_K = \preceq_{K'} \), i.e., \( \omega_1 \preceq_K \omega_2 \) if and only if \( \omega_1 \preceq_{K'} \omega_2 \).

The previous lemmas can finally be used to show that the construction of \( \preceq_K \) according to Definition 5.3 yields an assignment with the desired properties.

Proposition 5.10. If \( \circ \) satisfies (G1)–(G6), then \( \preceq_K \) is a min-friendly faithful assignment compatible with \( \circ \).

Proof. Assume (G1)–(G6) are satisfied by \( \circ \). Then \( \preceq_K \) is an assignment since every \( \preceq_K \) is total by Lemma 5.5; it is min-friendly by Lemma 5.8; it is faithful by Lemma 5.9; and it is compatible with \( \circ \) by Lemma 5.7.
5.2 From Assignments to Postulates

Now, it remains to show the "if" direction of Theorem 5.1.

**Proposition 5.11.** If there exists a min-friendly faithful assignment $\preceq_{(\cdot)}$ compatible with $\circ$, then $\circ$ satisfies (G1)–(G6).

*Proof.* Let $\preceq_{(\cdot)}: \mathcal{K} \mapsto \mathcal{K}$ be as described. We now show that $\circ$ satisfies all of (G1)–(G6).

(G1) Let $\omega \in [\mathcal{K} \circ \Gamma]$. Since $[\mathcal{K} \circ \Gamma] = \min([\Gamma], \preceq_{\mathcal{K}})$, we have that $\omega \in \min([\Gamma], \preceq_{\mathcal{K}})$. Then, we also have that $\omega \in [\Gamma]$. Thus, we have that $[\mathcal{K} \circ \Gamma] \subseteq [\Gamma]$ as desired.

(G2) Assume $[\mathcal{K} \cup \Gamma] \neq \emptyset$. We prove $[\mathcal{K} \circ \Gamma] = [\mathcal{K} \cup \Gamma]$ by showing inclusion in both directions.

($\subseteq$) Let $\omega \in [\mathcal{K} \circ \Gamma]$. By compatibility, we obtain $\omega \in \min([\Gamma], \preceq_{\mathcal{K}})$ and thus trivially also $\omega \in [\Gamma]$. Since $[\mathcal{K} \cup \Gamma] \neq \emptyset$, there exists some other $\omega' \in [\mathcal{K} \cup \Gamma] = [\mathcal{K}] \cap [\Gamma]$, which implies $\omega' \in [\mathcal{K}]$ and $\omega' \in [\Gamma]$. Therefore, $\omega \in \min([\Gamma], \preceq_{\mathcal{K}})$ implies $\omega \preceq_{\mathcal{K}} \omega'$, which means that $\omega' \preceq_{\mathcal{K}} \omega$ cannot hold and therefore, by contraposition, (F2) ensures $\omega \in [\mathcal{K}]$. Yet then $\omega \in [\mathcal{K}] \cap [\Gamma] = [\mathcal{K} \cup \Gamma]$ as desired.

($\supseteq$) Let $\omega \in [\mathcal{K} \cup \Gamma] = [\mathcal{K}] \cap [\Gamma]$, i.e. $\omega \in [\mathcal{K}]$ and $\omega \in [\Gamma]$. Since $\omega \in [\mathcal{K}]$, we obtain from (F1) and (F2) that $\omega \preceq_{\mathcal{K}} \omega'$ must hold for all $\omega' \in [\Gamma]$. Hence, $\omega \in \min([\Gamma], \preceq_{\mathcal{K}})$, and by compatibility $\omega \in [\mathcal{K} \circ \Gamma]$.

(G3) Assume $[\Gamma] \neq \emptyset$. By min-completeness, we have $\min([\Gamma], \preceq_{\mathcal{K}}) \neq \emptyset$. Since $[\mathcal{K} \circ \Gamma] = \min([\Gamma], \preceq_{\mathcal{K}})$ by compatibility, we obtain $[\mathcal{K} \circ \Gamma] \neq \emptyset$.

(G4) Suppose there exist $\mathcal{K}_1, \mathcal{K}_2, \Gamma_1, \Gamma_2 \in \mathcal{B}$ with $\mathcal{K}_1 \equiv \mathcal{K}_2$ and $\Gamma_1 \equiv \Gamma_2$. Then, $[\mathcal{K}_1] = [\mathcal{K}_2]$ and $[\Gamma_1] = [\Gamma_2]$. From (F3), we conclude $\preceq_{\mathcal{K}_1} \equiv \preceq_{\mathcal{K}_2}$. Now assume some $\omega \in [\mathcal{K}_1 \circ \Gamma_1]$, then by compatibility $\omega \in \min([\Gamma_1], \preceq_{\mathcal{K}_1}) = \min([\Gamma_2], \preceq_{\mathcal{K}_2})$. Therefore, again by compatibility, $\omega \in [\mathcal{K}_2 \circ \Gamma_2]$. Thus, $[\mathcal{K}_1 \circ \Gamma_1] \subseteq [\mathcal{K}_2 \circ \Gamma_2]$ holds. Inclusion in the other direction follows by symmetry. Therefore, we have $\mathcal{K}_1 \circ \Gamma_1 \equiv \mathcal{K}_2 \circ \Gamma_2$.

(G5) Let $\omega \in [(\mathcal{K} \circ \Gamma_1) \cup \Gamma_2] = [\mathcal{K} \circ \Gamma_1] \cap [\Gamma_2]$. This means that $\omega \in [\Gamma_2]$ but – since $[\mathcal{K} \circ \Gamma_1] = \min([\Gamma_1], \preceq_{\mathcal{K}})$ by compatibility – we also obtain $\omega \in \min([\Gamma_1], \preceq_{\mathcal{K}})$, meaning that $\omega \preceq_{\mathcal{K}} \omega'$ holds for all $\omega' \in [\Gamma_1]$. Yet then $\omega \preceq_{\mathcal{K}} \omega'$ holds particularly for all $\omega' \in [\Gamma_1] \cap [\Gamma_2]$ and hence $\omega \in \min([\Gamma_1] \cap [\Gamma_2], \preceq_{\mathcal{K}}) = \min([\Gamma_1 \cup \Gamma_2], \preceq_{\mathcal{K}})$. By compatibility follows $\omega \in [\mathcal{K} \circ (\Gamma_1 \cup \Gamma_2)]$. Thus $[(\mathcal{K} \circ \Gamma_1) \cup \Gamma_2] \subseteq [(\mathcal{K} \circ (\Gamma_1 \cup \Gamma_2)]$ as desired.

(G6) Let $(\mathcal{K} \circ \Gamma_1) \cup \Gamma_2 \neq \emptyset$, thus $\omega' \in [(\mathcal{K} \circ \Gamma_1) \cup \Gamma_2] = [\mathcal{K} \circ \Gamma_1] \cap [\Gamma_2]$ for some $\omega'$. By compatibility, we then obtain $\omega' \in \min([\Gamma_1], \preceq_{\mathcal{K}})$. Now consider an arbitrary $\omega$ with $\omega \in [\mathcal{K} \circ (\Gamma_1 \cup \Gamma_2)]$. By compatibility we obtain $\omega \in \min([\Gamma_1 \cup \Gamma_2], \preceq_{\mathcal{K}})$ and therefore, since $\omega \in [\Gamma_1] \cap [\Gamma_2] = [\Gamma_1 \cup \Gamma_2]$, we can conclude $\omega \preceq_{\mathcal{K}} \omega'$. This and $\omega' \in \min([\Gamma_1], \preceq_{\mathcal{K}})$ imply $\omega \in \min([\Gamma_1], \preceq_{\mathcal{K}})$ by min-retractivity. Hence every $\omega \in [\mathcal{K} \circ (\Gamma_1 \cup \Gamma_2)]$ satisfies $\omega \in \min([\Gamma_1], \preceq_{\mathcal{K}}) = [\mathcal{K} \circ \Gamma_1]$ but also $\omega \in [\Gamma_2]$, whence $[\mathcal{K} \circ (\Gamma_1 \cup \Gamma_2)] \subseteq [\mathcal{K} \circ \Gamma_1] \cap [\Gamma_2] = [(\mathcal{K} \circ \Gamma_1) \cup \Gamma_2]$ as desired. 

The proof of Theorem 5.1 follows from Proposition 5.10 and 5.11.
6. Two-Way Representation Theorem

Theorem 5.1 establishes the correspondence between operators and assignments under the assumption that \( \circ \) is given and therefore known to exist. What remains unsettled is the question if generally every min-friendly faithful assignment is compatible with some base change operator that satisfies (G1)–(G6). It is not hard to see that this is not the case.

Example 6.1. Consider the base logic \( \mathbb{B}_{nb} = (\mathcal{L}, \Omega, \models, \mathcal{P}(\mathcal{L}), \cup) \) with \( \mathcal{L} = \{\text{none, both}\} \) and \( \Omega = \{\omega_1, \omega_2\} \) satisfying \([\text{none}] = \emptyset\) and \([\text{both}] = \{\omega_1, \omega_2\} = \Omega\). There are four bases in this logic, satisfying \{none\} \(\equiv\) \{none, both\} and \(\emptyset \equiv\) \{both\}. Let the assignment \( \preceq_{\{\text{none}\}}^{\mathbb{B}_{nb}} \) be such that \( \preceq_{\{\text{one}\}}^{\mathbb{B}_{nb}} = \emptyset \times \emptyset \) and \( \preceq_{\{\text{none}\}}^{\mathbb{B}_{nb}} = \{\omega_1, \omega_2\} \). It is straightforward to check that \( \preceq_{\{\text{none}\}}^{\mathbb{B}_{nb}} \) is a min-friendly faithful assignment. Note that any \( \circ \) compatible with \( \preceq_{\{\text{none}\}}^{\mathbb{B}_{nb}} \) would have to satisfy \( \{\text{none}\} \circ \{\text{both}\} \) \(=\) \( \min(\{\{\text{both}\}\}, \preceq_{\{\text{none}\}}^{\mathbb{B}_{nb}}) = \{\omega_1\} \), yet, as we have seen, no base with this model set exists, therefore such a \( \circ \) is impossible.

Therefore, toward a full, two-way correspondence, we have to provide an additional condition on assignments, capturing operator existence.

As indicated by the example, for the existence of an operator, it will turn out to be essential that any minimal model set of a belief base obtained from an assignment corresponds to some belief base, a property which is formalized by the following notion.

Definition 6.2 (min-expressible). Let \( \mathbb{B} = (\mathcal{L}, \Omega, \models, \mathfrak{B}, \psi) \) be a base logic. A binary relation \( \preceq \) over \( \Omega \) is called min-expressible if for each \( \Gamma \in \mathfrak{B} \) there exists a belief base \( \mathbb{B}_{\Gamma, \preceq} \in \mathfrak{B} \) such that \( \mathbb{B}_{\Gamma, \preceq} \models \models \Gamma \). An assignment \( \preceq_{\{\text{one}\}} \) will be called min-expressible, if for each \( \mathbb{B} \in \mathfrak{B} \), the relation \( \preceq_{\mathbb{B}} \) is min-expressible. Given a min-expressible assignment \( \preceq_{\{\text{one}\}} \), let \( \circ_{\preceq_{\{\text{one}\}}} \) denote the base change operator defined by \( \mathbb{B} \circ_{\preceq_{\{\text{one}\}}} \Gamma = \mathbb{B}_{\Gamma, \preceq_{\mathbb{B}}} \).

It should be noted that min-expressibility is a straightforward generalization of the notion of regularity by Delgrande et al. (2018) to base logics. By virtue of this extra notion, we now find the following bidirectional relationship between assignments and operators, amounting to a full characterization.

Theorem 6.3. Let \( \mathbb{B} \) be a base logic. Then the following hold:

- Every base change operator for \( \mathbb{B} \) satisfying (G1)–(G6) is compatible with some min-expressible min-friendly faithful assignment.

- Every min-expressible min-friendly faithful assignment for \( \mathbb{B} \) is compatible with some base change operator satisfying (G1)–(G6).

Proof. For the first item, let \( \circ \) be the corresponding base change operator. Then, by Proposition 5.10, the assignment \( \preceq_{\mathbb{B}} \) as given in Definition 5.3 is min-friendly, faithful, and compatible with \( \circ \). As for min-expressibility, recall that, by compatibility, \( \mathbb{B} \circ \Gamma \models \preceq_{\mathbb{B}} \) for every \( \Gamma \). As \( \mathbb{B} \circ \Gamma \) is a belief base, min-expressibility follows immediately.

For the second item, let \( \preceq_{\mathbb{B}} \) be the corresponding min-expressible assignment and \( \circ \preceq_{\mathbb{B}} \) as provided in Definition 6.2. By construction, \( \circ \preceq_{\mathbb{B}} \) is compatible with \( \preceq_{\mathbb{B}} \). Proposition 5.11 implies that \( \circ \preceq_{\mathbb{B}} \) satisfies (G1)–(G6). \( \square \)
As an aside, note that the above theorem also implies that every min-expressible min-friendly faithful assignment is compatible only with AGM base change operators. This is due to the fact that, one the one hand, any such assignment fully determines the corresponding compatible base change operator model-theoretically and, on the other hand, (G1)–(G6) are purely model-theoretic conditions.

Continuing our running example, we observe that $\succeq_{\mathcal{K}}^{\text{Ex}}$ is also a min-expressible relation.

**Example 6.4** (continuation of Example 5.4). Consider again $\succeq_{\mathcal{K}}^{\text{Ex}}$, and observe that $\succeq_{\mathcal{K}}^{\text{Ex}}$ is compatible with $\succeq_{\mathcal{K}}^{\text{Ex}}$, e.g. $[\mathcal{K} \cup \mathcal{K}^{\text{Ex}}] = \min([\mathcal{K}], \succeq_{\mathcal{K}}^{\text{Ex}})$. Thus, for every belief base $\Gamma \in \mathcal{P}(\mathcal{L})$, the minimum $\min(\Gamma, \succeq_{\mathcal{K}}^{\text{Ex}}) \equiv \mathcal{K} \cap \mathcal{K}^{\text{Ex}}$ yields a set expressible by a belief base. Theorem 6.3 guarantees us that $\succeq_{\mathcal{K}}^{\text{Ex}}$ satisfies (G1)–(G6), as $\succeq_{\mathcal{K}}^{\text{Ex}}$ is also a faithful min-expressible and min-friendly assignment.

As a last step of this section, we will apply the theory developed here to demonstrate that the standard operator of trivial revision (Hansson, 1999; Fermé & Hansson, 2018) indeed satisfies (G1)–(G6) in the general setting of base logics.

**Example 6.5.** Let $\mathcal{B} = (\mathcal{L}, \Omega, \models, \mathcal{P}(\mathcal{L}), \cup)$ be an arbitrary base logic. We define the trivial revision operator $\sigma_{\text{fm}}$ for $\mathcal{B}$ by

$$\mathcal{K} \sigma_{\text{fm}} \Gamma = \begin{cases} \mathcal{K} \cup \Gamma & \text{if } [\mathcal{K} \cup \Gamma] \text{ is consistent} \\ \Gamma & \text{otherwise} \end{cases}$$

To show satisfaction of (G1)–(G6) we construct a min-expressible min-friendly faithful assignment $\succeq_{\text{fm}}^{\text{Ex}}$ compatible with $\sigma_{\text{fm}}$. For each $\mathcal{K} \in \mathcal{B}$ let $\omega_1 \succeq_{\text{fm}}^{\text{Ex}} \omega_2$ if $\omega_1 \models \mathcal{K}$ or $\omega_2 \not\models \mathcal{K}$. Obviously, the relation $\succeq_{\text{fm}}^{\text{Ex}}$ is a total preorder where $\omega_1, \omega_2$ are $\succeq_{\text{fm}}^{\text{Ex}}$-equivalent, if either $\omega_1, \omega_2 \models \mathcal{K}$ or $\omega_1, \omega_2 \not\models \mathcal{K}$ holds. Moreover, it is not hard to see that the relation $\succeq_{\text{fm}}^{\text{Ex}}$ is min-complete and min-retractive. By construction of $\succeq_{\text{fm}}^{\text{Ex}}$ we obtain that $\min([\Gamma], \succeq_{\text{fm}}^{\text{Ex}}) = [\Gamma]$ if $\mathcal{K} \cup \Gamma$ is inconsistent. If $\mathcal{K} \cup \Gamma$ is consistent, we obtain $\min([\Gamma], \succeq_{\text{fm}}^{\text{Ex}}) = [\mathcal{K}] \cap [\Gamma] = [\mathcal{K} \cup \Gamma]$.

In summary, the assignment $\succeq_{\text{fm}}^{\text{Ex}}$ is min-expressible and min-friendly, and the base change operator $\sigma_{\text{fm}}$ is compatible with it.

7. Interim Conclusion

In Section 4 to Section 6, we discussed how K&M’s result about semantically characterizing AGM belief revision in finite-signature propositional logic can be generalized to arbitrary base logics. Thereby, we cover all Tarskian logics and support any notion of bases that are closed under “abstract union”. We demonstrated certain central aspects by our running example (see Example 2.1, Example 4.7, Example 5.4, Example 6.4), which can be summarized as follows.

**Fact 7.1.** The operator $\sigma_{\text{Ex}}$ for the base logic $\mathcal{B}_{\text{Ex}}$ satisfies (G1)–(G6) and is compatible with the faithful min-friendly and min-expressible assignment $\succeq_{\mathcal{K}}^{\text{Ex}}$. That is, for any base $\mathcal{K}$ of $\mathcal{B}_{\text{Ex}}$, the relation $\succeq_{\mathcal{K}}^{\text{Ex}}$ is min-friendly and min-expressible. However there is a base $\mathcal{K}_{\text{Ex}}$, such that $\succeq_{\mathcal{K}}^{\text{Ex}}$ is not transitive. In fact, no transitive faithful min-friendly and min-expressible assignment compatible with $\sigma_{\text{Ex}}$ exists, whatsoever.

3. Note, trivial revision is known to coincide with full meet revision in many logical settings.
By now, our rationale has been to cover the most general setting of base logics possible, while sticking to the complete set of the AGM postulates and without adding further conditions.

However, one might remark that the AGM postulates were specifically designed for describing the change of belief sets, i.e., deductively closed theories, which naturally include all syntactic variants. As opposed to this, approaches to describing the change of (not necessarily deductively closed) bases might take the syntax into account (Hansson, 1999). Under such circumstances, the syntax-independence expressed by (G4) might be called into question.

Another aspect is that, for the sake of generality, we had to replace the stronger requirement of transitivity by the weaker notion of min-retractivity inside the assignments. Waiving transitivity (and hence preorders) might be considered unconventional, as a transitive preference relation is often deemed to be the actual motivation behind the postulates (G1) and (G6). This raises the question for which Tarskian logics the existence of a compatible preorder assignment for any AGM revision operator can be guaranteed.

In the following sections we will discuss these aspects as variations of the approach we presented in the preceding sections, showing that exact characterizations exist for these cases as well. Moreover, we will discuss some aspects of the notion of base logic, and the role of disjunctions in decomposability.

8. Base Changes and Syntax-Independence

Up to this point, we have been considering base change operators fulfilling the full set of postulates (G1)–(G6). The research on base changes deals with syntax-dependent changes, and in our approach the postulate (G4) implies that a base change operator yields semantically the same result on all semantically equivalent bases. As consequence, one might conclude that the base change operators considered here have only limited freedom when it comes to taking the syntactic structure into account when changing.

However, note that neither the postulates (G1)–(G6) nor our representation results make assumptions about the specific syntactic structure of a base obtained by a base change operator. Thus, for syntactically different bases $\Gamma_1$ and $\Gamma_2$ that are semantic equivalent, we might obtain syntactically different results after revision, which are semantic equivalent.

Example 8.1. Consider the logic $\mathbb{PL}_2$ (cf. Example 2.2), e.g. propositional logic over the signature $\{p, q\}$ as follows. Given $K_1 = \{p, q\}, K_2 = \{p \land q\}, \Gamma_1 = \{p, p \rightarrow \neg q\}$, and $\Gamma_2 = \{p \land \neg q\}$. We have $K_1$ and $K_2$, as well as $\Gamma_1$ and $\Gamma_2$, which are two semantic equivalent bases with different syntax. By applying the trivial revision operation $\circ^{\text{fm}}$ (cf. Example 6.5) to $K_1$ by $\Gamma_1$ and to $K_2$ by $\Gamma_2$, we obtain $K_1 \circ \Gamma_1 = \{p, p \rightarrow \neg q\}$ and $K_2 \circ \Gamma_2 = \{p \land \neg q\}$. The two revision results are different syntactically, yet semantically equivalent (i.e. $[K_1 \circ \Gamma_1] = [K_2 \circ \Gamma_2] = \{\omega : p \mapsto \text{true}, q \mapsto \text{false}\}$).

Moreover, the semantic viewpoint developed here in this article is flexible and is eligible for further liberation regarding syntax-dependence of a base change operator. In particular, our approach allows us to drop (G4). As an alternative to (G4), consider the following weaker version (Hansson, 1999):

$$(G4w) \text{ If } \Gamma_1 \equiv \Gamma_2, \text{ then } \mathcal{K} \circ \Gamma_1 \equiv \mathcal{K} \circ \Gamma_2.$$
The main difference between (G4w) and (G4) is that by (G4w) a base change operator is not restricted to treat semantically equivalent prior belief bases equivalently. When considering the extended AGM postulates (G5) and (G6) it turns out that postulate (G4w) is a baseline of syntax-independence, as (G1), (G5) and (G6) together already imply (G4w), which is a generalization of a result by Aiguier et al. (2018, Prop. 3).

**Proposition 8.2.** Let $\circ$ be a base change operator for a base logic $B = (L, \Omega, \models, \emptyset, \trianglerighteq)$. If $\circ$ satisfies (G1), (G5) and (G6), then $\circ$ satisfies (G4w).

**Proof.** Let $K, \Gamma_1, \Gamma_2 \in B$ be belief bases such that $\Gamma_1 \equiv \Gamma_2$. By (G1), the postulate (G4w) holds if $\Gamma_1$ is inconsistent. For the remaining parts of the proof, we assume consistency of $\Gamma_1$.

First observe that $(K \circ \Gamma_1) \trianglerighteq \Gamma_2 \equiv K \circ \Gamma_1$ by (G1) and analogously $(K \circ \Gamma_2) \trianglerighteq \Gamma_1 \equiv K \circ \Gamma_2$. By (G5) we obtain $(K \circ \Gamma_1) \trianglerighteq \Gamma_2 \models (K \circ \Gamma_1 \trianglerighteq \Gamma_2)$. Moreover, because $(K \circ \Gamma_1) \trianglerighteq \Gamma_2$ is consistent, we obtain $K \circ (\Gamma_1 \trianglerighteq \Gamma_2) \models (K \circ \Gamma_1) \trianglerighteq \Gamma_2$ by (G6). In summary we obtain $(K \circ \Gamma_1) \trianglerighteq \Gamma_2 \equiv K \circ (\Gamma_1 \trianglerighteq \Gamma_2) \equiv (K \circ \Gamma_2) \trianglerighteq \Gamma_1$.

Using our prior observations this expands to $K \circ \Gamma_1 \equiv K \circ \Gamma_2$.

To obtain a representation theorem for base change operators without (G4), relaxing the constraint on the syntactic side requires the relation of the conditions on the semantic side. For dropping (G4), we weaken the notion of faithfulness to the notion of quasi-faithfulness.

**Definition 8.3 (quasi-faithful).** An assignment $\preceq$ is called quasi-faithful if it satisfies the following conditions:

(F1) If $\omega, \omega' \models K$, then $\omega \prec K \omega'$ does not hold.

(F2) If $\omega \models K$ and $\omega' \not\models K$, then $\omega \prec K \omega'$.

Note that quasi-faithful assignments might assign to every belief base a different order, independent from whether they are semantically equivalent or not. Thus, this enables a base change operator to treat base differently depending on their syntactic structure.

Luckily, our canonical assignment $\preceq$ (cf. Definition 5.3) carries over to the setting where (G4) is not satisfied. The following lemma attests that $\preceq$ yields a quasi-faithful assignment for this cases.

**Lemma 8.4.** If $\circ$ satisfies (G2), (G5), and (G6), then the assignment $\preceq$ is quasi-faithful.

**Proof.** The proof of the two conditions of quasi-faithfulness, (F1) and (F2), is identical to the proof of (F1) and (F2) in Lemma 5.9.

Using the notion of quasi-faithfulness and $\preceq$ (cf. Definition 5.3) we obtain the following characterization result, which is similar to a result already provided by Aiguier et al. (2018, Thm. 2).

**Proposition 8.5.** Let $\circ$ be a base change operator. The operator $\circ$ satisfies (G1)–(G3), (G5), and (G6) if and only if it is compatible with some min-friendly quasi-faithful assignment.

**Proof (Sketch).** The proof is the nearly the same as for Theorem 5.1. Note that the proof of Theorem 5.1, which shows correspondence between (G1)–(G6) and compatible min-friendly faithful assignments uses (G4) and (F3) only in special situations. In particular, observe that
condition (F3) is only used to show satisfaction of (G4) in the proof of Proposition 5.11. Moreover, note that $\preceq^K_\circ$ from Definition 5.3 is a total min-friendly relation due to Lemma 5.5 and Lemma 5.8 for each $K \in B$; compatibility of $\preceq^K_\circ$ with $\circ$ is ensured by Lemma 5.7 while satisfaction of quasi-faithfulness is ensured by Lemma 8.4.

In view of this, we can now present the syntax-dependent version of our two-way representation theorem.

**Theorem 8.6.** Let $B$ be a base logic. Then the following hold:

- Every base change operator for $B$ satisfying (G1)–(G3), (G5), and (G6) is compatible with some min-expressible min-friendly quasi-faithful assignment.
- Every min-expressible min-friendly quasi-faithful assignment for $B$ is compatible with some base change operator satisfying (G1)–(G3), (G5), and (G6).

In research on base revision, various special postulates for the changing of bases have been considered, e.g. in the seminal research on belief revision by Hansson, special postulates for base changes are proposed, e.g., see (Hansson, 1999). Of course, an interesting and open question is, which of them could be characterized or reconstructed by the approach of this article.

### 9. Total-Preorder-Representability

As we have shown, regrettably, not every AGM belief revision operator in every Tarskian logic can be described by a total preorder assignment. Yet, we also saw that, for some logics (like $\mathit{PL}_n$), this correspondence does indeed hold. Consequently, this section is dedicated to find a characterization of precisely those logics wherein every AGM base change operator is representable by a compatible min-complete faithful preorder assignment. The following definition captures the notion of operators that are well-behaved in that sense.

**Definition 9.1** (total-preorder-representable). A base change operator $\circ$ for some base logic is called total-preorder-representable if there is a min-complete quasi-faithful preorder assignment compatible with $\circ$.

Recall that transitivity implies min-retractivity, and thus, every min-complete preorder is automatically min-friendly. Moreover, in view of Section 8, our definition uses the more lenient notion of quasi-faithfulness to accommodate the syntax-dependent setting. However, as the following lemma shows, the same definition of total-preorder-representability is adequate in the syntax-independent setting.

**Lemma 9.2.** For any base change operator $\circ$ that satisfies (G4), total-preorder-representability coincides with the existence of a min-complete faithful preorder assignment compatible with $\circ$.

**Proof.** Any compatible min-complete faithful preorder assignment is also quasi-faithful and hence the existence of such an assignment implies total-preorder-representability. For the other direction, let $\preceq^K_\circ$ be a min-complete quasi-faithful preorder assignment compatible with $\circ$. We then define $\preceq^K_\circ$ as $K \mapsto \preceq^K_\circ(\sigma(K) \equiv)$ where $\sigma$ is a selection function mapping every
Ξ-equivalence class of \( \mathcal{B} \) to one of its elements (i.e., \( \sigma([K]_\equiv) \in [K]_\equiv \)). Then, the property of being a min-complete quasi-faithful preorder assignment compatible with \( \circ \) carries over pointwise from \( \preceq \) to \( \preceq^\text{ff} \), while the construction ensures that \( \preceq^\text{ff} \) also satisfies (F3) from Definition 3.1 and hence is faithful. \( \square \)

In the next section, we will provide a necessary and sufficient criterion for a logic such that universal total-preorder-representability is guaranteed.

9.1 Critical Loops

The following definition describes the occurrence of a certain relationship between several bases. Such an occurrence will turn out to be the one and only reason to prevent total-preorder-representability.

**Definition 9.3** (critical loop). Let \( \mathcal{B} = (\mathcal{L}, \Omega, \models, \mathcal{B}, \sqcup) \) be a base logic. Three or more bases \( \Gamma_0, \Gamma_1, \ldots, \Gamma_n \in \mathcal{B} \) are said to form a critical loop of length \((n + 1)\) for \( \mathcal{B} \) if there exists a base \( \mathcal{K} \in \mathcal{B} \) and consistent bases \( \Gamma_0, \ldots, \Gamma_n \in \mathcal{B} \) such that

1. \( [\mathcal{K} \sqcup \Gamma_i]_\equiv = \emptyset \) for every \( i \in \{0, \ldots, n\} \), where \( \oplus \) is addition mod \((n + 1)\),
2. \( [\Gamma_i]_\equiv \cup [\Gamma_j]_\equiv = \emptyset \) for each \( i, j \in \{0, \ldots, n\} \) with \( i \neq j \), and
3. for each \( \Gamma_\circ \in \mathcal{B} \) that is consistent with at least three bases from \( \Gamma_0, \ldots, \Gamma_n \), there exists some \( \Gamma_\circ \in \mathcal{B} \) such that \( [\Gamma_\circ]_\equiv \neq \emptyset \) and \( [\Gamma_i]_\equiv \subseteq [\Gamma_\circ]_\equiv \setminus ([\Gamma_0]_\equiv \cup \ldots \cup [\Gamma_n]_\equiv) \).

The three conditions in Definition 9.3 describe the canonic situation brought about by some bases \( \Gamma_0, \ldots, \Gamma_n \) allowing for the construction of a revision operator that unavoidably gives rise to a circular compatible relation. Note that due to Condition (3), every three of \( \Gamma_0, \Gamma_1, \ldots, \Gamma_n \) together are inconsistent, but each two of them which have an index in common are consistent, i.e. \( \Gamma_i \sqcup \Gamma_j \) is consistent for each \( i \in \{0, \ldots, n\} \).

In the following, we provide some intuition for the notion of critical loop. The bases \( \Gamma_0, \ldots, \Gamma_n \) provide model sets that are pairwise disjoint (cf. the second part of Condition (2)) and can be thought of as arranged in a circle, while the bases \( \Gamma_0, \ldots, \Gamma_n \) overlap any two adjacent model sets as indicated by their indices (cf. the first part of Condition (2)). Exploiting this situation, we now want to define the result of revising \( \mathcal{K} \) such that the circular arrangement governs the choice of the “\( \mathcal{K} \)-preferred” models as follows: the models of \( \mathcal{K} \circ \Gamma_i \sqcup \Gamma_j \), obtained by revising \( \mathcal{K} \) with \( \Gamma_i \sqcup \Gamma_j \), encompass all models of \( \Gamma_i \), but no model of \( \Gamma_j \). Consequently, for any \( i \), the revision \( \mathcal{K} \circ \Gamma_i \sqcup \Gamma_j \) provides a preference of \( \Gamma_i \) over \( \Gamma_j \). Thus, a relation compatible to \( \circ \) has to contain a “preference-loop” of interpretations. In order to guarantee that this arrangement technique is applicable, Condition (1) and Condition (3) from Definition 9.3 are ruling out all cases, where other bases of \( \mathcal{B} \) together with (G1)–(G6) prevent our intended construction from working:

Condition (1) ensures that none of the bases \( \Gamma_0, \ldots, \Gamma_n \) has models in common with the current belief base \( \mathcal{K} \) (c.f. Figure 4a). If one base \( \Gamma_i \sqcup \Gamma_j \) would have a model in common with \( \mathcal{K} \), then the postulate (G2) would prevent a circular situation. Thus, this condition is necessary for admitting circular situations.

Condition (3) comes into play if a belief base \( \Gamma_\circ \) “covers” three or more elements of the circle, meaning that three or more interpretations of a circle are models of this base \( \Gamma_\circ \).
(a) By Condition (2), the models of each base $\Gamma_i, i \not\subseteq 1$ encompass the models of $\Gamma_i$ and of $\Gamma_i \oplus 1$, while by Condition (1), all these model sets are disjoint from the models of $\mathcal{K}$.

(b) By Condition (3), for each $\Gamma_\gamma$ that is consistent with at least three distinct elements of the circle (e.g. $\Gamma_i, \Gamma_j, \Gamma_k \in \{\Gamma_0, \ldots, \Gamma_n\}$), there exists a base $\Gamma'_\gamma$ that is subsumed by $\Gamma_\gamma$ but inconsistent with all $\Gamma_{0,1}, \ldots, \Gamma_{n-1,n}, \Gamma_{n,0}$.

Figure 4: Illustrations of the Conditions (1)–(3) of a critical loop given in Definition 9.3.

For any such $\Gamma_\gamma$, there is a consistent belief base $\Gamma'_\gamma$ which shares all of its models with $\Gamma_\gamma$ but no model with any of the $\Gamma_i, i \not\subseteq 1$ (c.f. Figure 4b). This is crucial for the presence of circles: if no such $\Gamma'_\gamma$ would exist, the operator would (by min-completeness and min-expressibility) choose models of the circle, e.g., the bases $\Gamma_i \cup \Gamma_\gamma$, as the result of the revision by $\Gamma_\gamma$. In the end, this would give one base $\Gamma_i$ preference over $\Gamma_i \oplus 1, \ldots, \Gamma_i \oplus n$ and thus, would prevent creation of a circle. Therefore, Condition (3) rules out the cases where min-completeness and min-expressibility and non-existence of such a $\Gamma'_\gamma$ together would prevent formation of a circle.

Definition 9.3 is inspired by our running example. Before explicating this link, we continue with the presentation of the general results.

The next theorem is the central result of this section, stating that the notion of critical loop captures exactly those base logics for which some operator exists that is not total-preorder-representable. By contraposition, this just means that for all base logics $\mathcal{B}$, the absence of critical loops from $\mathcal{B}$ is a necessary and sufficient criterion for universal total-preorder-representability and hence for the existence of a characterization result for $\mathcal{B}$ that is based on total preorders. This characterization result will not only hold for base change operators that satisfy (G1)–(G6), but also for operators that does not satisfy (G4), but the remaining postulates (G1)–(G3), (G5), and (G6). To provide a result applicable for both groups of postulates, we will show for the necessary and sufficient direction the respectively stronger result, i.e., if our base logic exhibits a critical loop we provide a construction for a non-total-preorder-representable base change operator that satisfies (G1)–(G6), and for the
other direction, we show that in the absence of critical loops every operator that satisfies (G1)–(G3), (G5), and (G6) is total-preorder-representable.

**Theorem 9.4.** For all base logics $B$, the following statements hold:

(I) If $B$ exhibits a critical loop, then there exists a base change operator for $B$ that satisfies (G1)–(G6) and is not total-preorder-representable.  

(II) If $B$ does not admit a critical loop, then every base change operator for $B$ that satisfies (G1)–(G3), (G5), and (G6) is total-preorder-representable.

We dedicate Section 9.2 to the first statement of Theorem 9.4 while the second statement is shown in Section 9.3.

### 9.2 Total-Preorder-Representability Implies Absence of Critical Loops

We proceed to show (by contraposition) that the absence of critical loops is necessary for total-preorder-representability of all AGM change operators. To this end, we will provide a construction which, given a critical loop $C$ in some base logic $B$, yields an AGM change operator $\circ_C$ for $B$ that is demonstrably not total-preorder-representable.

**Definition 9.5.** Let $B = (\mathcal{L}, \Omega, \models, \mathfrak{B}, \cup)$ be a base logic with a critical loop $C = (\Gamma_0, 1, \Gamma_1, 2, \ldots, \Gamma_n, 0)$ and let $\Gamma_0, \ldots, \Gamma_n \in \mathfrak{B}$ and $\mathcal{K}$ as in Definition 9.3.

Let $\mathcal{C}$ denote the set of all $\Gamma'_\mathcal{C}$ guaranteed by Condition (3) from Definition 9.3, i.e. $\Gamma'_\mathcal{C} \in \mathcal{C}$ if there is some $\Gamma_\mathcal{C}$ with $\emptyset \neq [\Gamma'_\mathcal{C}] \subseteq [\Gamma_\mathcal{C}] \setminus ([\Gamma_0, 1] \cup \ldots \cup [\Gamma_n, 0])$ and $\Gamma_\mathcal{C}$ is consistent with three (or more) bases from $\{\Gamma_0, \ldots, \Gamma_n\}$. Now let $\mathcal{C}' = \{\Gamma'_\mathcal{C} \in \mathcal{C} | [\Gamma'_\mathcal{C} \cup \mathcal{K}] = \emptyset\}$, i.e., all belief bases from $\mathcal{C}$ that are inconsistent with $\mathcal{K}$. Let $\leq_{\mathcal{C}'}$ be an arbitrary linear order on $\mathcal{C}'$ with respect to which every non-empty subset of $\mathcal{C}'$ has a minimum.\(^4\)

We now define $\circ_C$ as follows: for every $\mathcal{K}' \neq \mathcal{K}$ and any $\Gamma$, let $\mathcal{K}' \circ_C \Gamma = \mathcal{K}' \cup \Gamma$ if $\mathcal{K}' \cup \Gamma$ is consistent, otherwise $\mathcal{K}' \circ_C \Gamma = \Gamma$. For $\mathcal{K}' \equiv \mathcal{K}$, we define:

$$\mathcal{K}' \circ_C \Gamma = \begin{cases} 
\Gamma \cup \mathcal{K}' & \text{if } [\mathcal{K}' \cup \Gamma] \neq \emptyset, \\
\Gamma \cup \mathcal{K}_{\text{min}}' & \text{if } [\mathcal{K}' \cup \Gamma] = \emptyset, \text{ and } [\Gamma \cup \Gamma'] \neq \emptyset \text{ for some } \Gamma' \in \mathcal{C}', \\
\Gamma \cup \Gamma_i & \text{if none of the above applies, } [\Gamma \cup \Gamma_i] \neq \emptyset, \text{ and } \bigcup_{j \in \{0, \ldots, n\} \setminus \{i, i + 1\}} [\Gamma_j \cup \Gamma] = \emptyset, \\
\Gamma & \text{if none of the cases above apply,}
\end{cases}$$

where $\mathcal{K}_{\text{min}}' = \min\{\{\Gamma'_\mathcal{C} \in \mathcal{C}' | [\Gamma'_\mathcal{C} \cup \Gamma] \neq \emptyset\}, \leq_{\mathcal{C}'}\}$.

In the following, we show that $\circ_C$ from Definition 9.5 is indeed an AGM revision, but not total-preorder-representable.

**Proposition 9.6.** For a base logic $B$ with a critical loop $C$, the operator $\circ_C$ for $B$ satisfies (G1)–(G6) and is not total-preorder-representable.

**Proof.** We will first show that $\circ_C$ satisfies (G1)–(G6). For $\mathcal{K}' \neq \mathcal{K}$ we obtain a trivial revision which satisfies (G1)–(G6) (cf. Example 6.5). Consider the remaining case of $\mathcal{K}$ (and any equivalent base):

\(^4\) Such a $\leq_{\mathcal{C}'}$ exists due to the well-ordering theorem, by courtesy of the axiom of choice (Zermelo, 1904).
Postulates (G1)–(G4). The satisfaction of (G1)–(G3) follows directly from the construction of $\circ \mathcal{C}$. For (G4) observe that, when computing $\mathcal{K} \circ \mathcal{C} \Gamma$, the case distinction above only considers the model sets of the participating bases rather than their syntax. Thus, for $\mathcal{K} \equiv \mathcal{K}'$ and $\Gamma_1 \equiv \Gamma_2$ we always obtain $\mathcal{K} \circ \mathcal{C} \Gamma_1 \equiv \mathcal{K}' \circ \mathcal{C} \Gamma_2$.

Postulate (G5) and (G6). Consider two belief bases $\Gamma_1$ and $\Gamma_2$. If $\Gamma_2$ is inconsistent with $\mathcal{K} \circ \mathcal{C} \Gamma_1$, then we obtain satisfaction of (G5) immediately. For the remaining case of (G5) and for (G6) we assume $\mathcal{K} \circ \mathcal{C} \Gamma_1$ to be consistent with $\Gamma_2$, i.e., $\mathcal{K} \circ \mathcal{C} \Gamma_1 \cup \Gamma_2 \neq \emptyset$. Consequently, there exists some interpretation $\omega$ such that $\omega \in \mathcal{K} \circ \mathcal{C} \Gamma_1$ and $\omega \in \Gamma_2$. The postulate (G1) implies that $\omega \in \Gamma_2$, and hence $\Gamma_1 \cup \Gamma_2$ is satisfied. We now inspect all different cases from the definition of $\circ \mathcal{C}$ above that may apply when revising $\mathcal{K}$ by $\Gamma_1$:

If $\Gamma_1$ is consistent with $\mathcal{K}$, then we obtain from $\mathcal{K} \circ \mathcal{C} \Gamma_1 \cup \Gamma_2 \neq \emptyset$ and (G2) that $\mathcal{K}$ is consistent with $\Gamma_1 \cup \Gamma_2$. This implies $(\mathcal{K} \circ \mathcal{C} \Gamma_1) \cup \Gamma_2 \equiv (\mathcal{K} \cup \Gamma_1) \cup \Gamma_2 \equiv \mathcal{K} \cup (\Gamma_1 \cup \Gamma_2) \equiv \mathcal{K} \circ \mathcal{C} (\Gamma_1 \cup \Gamma_2)$; yielding satisfaction of (G5) and (G6).

Next, consider the second case of the definition, where $\Gamma_1$ is inconsistent with $\mathcal{K}$, but consistent with some $\Gamma_i \in \mathcal{C}'$ and assume $\Gamma_i$ is the $\mathcal{C}'$-minimal such base, i.e., $\Gamma_i = (\Gamma_i)^{C'}$. Then, from the construction of $\circ \mathcal{C}$ and the consistency of $(\mathcal{K} \circ \mathcal{C} \Gamma_1) \cup \Gamma_2$ we obtain $\mathcal{K} \circ \mathcal{C} \Gamma_1 \cup \Gamma_2 = \Gamma_i \cup \Gamma_2 \not\equiv \emptyset$. Consequently, the set $\Gamma_1 \cup \Gamma_2$ is also consistent with $\Gamma_i$, which, together with $\Gamma_i = (\Gamma_i)^{C'}$, implies $\Gamma_i = (\Gamma_1 \cup \Gamma_2)^{C'}$. For determining $\mathcal{K} \circ \mathcal{C} (\Gamma_1 \cup \Gamma_2)$, note that from $\mathcal{K}$ being inconsistent with $\Gamma_1$, it follows that $\mathcal{K}$ must also be inconsistent with $\Gamma_1 \cup \Gamma_2$, therefore, due to the existence of $\Gamma_i$, the second line of the definition of $\circ \mathcal{C}$ must apply. We obtain $(\mathcal{K} \circ \mathcal{C} \Gamma_1) \cup \Gamma_2 \equiv (\Gamma_1)^{C'} \cup \Gamma_2 \equiv \Gamma_i \cup \Gamma_2 \equiv \mathcal{K} \circ \mathcal{C} (\Gamma_1 \cup \Gamma_2)$; establishing (G5) and (G6) for this case.

We now inspect the third case from the definition, i.e., we consider some $\Gamma_1$ that is inconsistent with $\mathcal{K}$ and with all elements from $\mathcal{C}'$. If $\Gamma_1$ is consistent with $\Gamma_i$ and inconsistent with all $\Gamma_j$, where $j \in \{0, \ldots, n\} \setminus \{i, i \oplus 1\}$, then by the construction of $\circ \mathcal{C}$ and the consistency of $(\mathcal{K} \circ \mathcal{C} \Gamma_1) \cup \Gamma_2$ we have $\mathcal{K} \circ \mathcal{C} \Gamma_1 \cup \Gamma_2 = \Gamma_i \cup \Gamma_2 \not\equiv \emptyset$. Then, likewise $\Gamma_i \cup \Gamma_2$ is consistent with $\Gamma_i$ and inconsistent with all $\Gamma_j$ with $j \in \{0, \ldots, n\} \setminus \{i, i \oplus 1\}$. Moreover, if $\Gamma_1$ is inconsistent with $\mathcal{K}$ and with all elements from $\mathcal{C}'$, then so is $\Gamma_i \cup \Gamma_2$, i.e., when determining $\mathcal{K} \circ \mathcal{C} (\Gamma_i \cup \Gamma_2)$, the third case of the definition applies. Hence, by the definition of $\circ \mathcal{C}$ we obtain $(\mathcal{K} \circ \mathcal{C} \Gamma_1) \cup \Gamma_2 \equiv \Gamma_i \cup \Gamma_2 \equiv \mathcal{K} \circ \mathcal{C} (\Gamma_1 \cup \Gamma_2)$.

If none of the conditions above applies to $\Gamma_1$, then they also do not apply to $\Gamma_1 \cup \Gamma_2$. From the construction of $\circ \mathcal{C}$ we obtain $\mathcal{K} \circ \mathcal{C} (\Gamma_1 \cup \Gamma_2) \equiv (\mathcal{K} \circ \mathcal{C} \Gamma_1) \cup \Gamma_2 \equiv \Gamma_1 \cup \Gamma_2$.

In summary, we obtain that $\circ \mathcal{C}$ satisfies (G5) and (G6) in all cases. It remains to show that $\circ \mathcal{C}$ is not total-preorder-representable. Towards a contradiction suppose the contrary, i.e., there is a min-complete faithful preorder assignment $\preceq_{(\_)}$, such that $\circ \mathcal{C}$ is compatible with $\preceq_{(\_)}$. Transitivity and min-completeness imply that $\preceq_{(\_)}$ is min-friendly. As all $\Gamma_0, \ldots, \Gamma_n$ are consistent, there are $\omega_i \in [\Gamma_i]$ for all $i \in \{0, \ldots, n\}$. By construction of $\circ \mathcal{C}$ and Condition (2) of Definition 9.3, we have $\mathcal{K} \circ \mathcal{C} \Gamma_{i,i+1} = \Gamma_{i,i+1} \cup \Gamma_i \equiv \Gamma_i$, and consequently $\omega_i = \mathcal{K} \circ \mathcal{C} \Gamma_{i,i+1} = \mathcal{K} \circ \mathcal{C} \Gamma_{i,i+1} \not\equiv \mathcal{K} \circ \mathcal{C} \Gamma_{i,i+1}$ for each $i \in \{0, \ldots, n\}$. As $\circ \mathcal{C}$ is compatible with $\preceq_{(\_)}$, we obtain $\mathcal{K} \circ \mathcal{C} \Gamma_{i,i+1} = \min([\Gamma_{i,i+1}], \preceq_{\mathcal{K}})$. In particular, the definition of $\circ \mathcal{C}$ yields $\omega_i \in \min([\Gamma_{i,i+1}], \preceq_{\mathcal{K}})$ and $\omega_i, \omega_{i+1} = \Gamma_{i,i+1} \not\equiv \min([\Gamma_{i,i+1}], \preceq_{\mathcal{K}})$. We obtain thereof
the strict relationship \( \omega_1 \prec_K \omega_{i+1} \). In summary, we get \( \omega_0 \prec_K \omega_1 \prec_K \ldots \prec_K \omega_n \prec_K \omega_0 \), which contradicts the presumed transitivity of \( \preceq_K \).

This establishes that the absence of critical loops is a necessary condition for universal total-preorder-representability in any Tarskian logic, because Theorem 9.4 (I) an immediate consequence of Proposition 9.6.

9.3 Absence of Critical Loops Implies Total-Preorder-Representability

We will now show that the identified criterion of critical loop (Definition 9.3) is also sufficient, even in the more general, syntax-dependent setting. That is, we will demonstrate in the following that Theorem 9.4 (II) holds. To this end, we need to argue that any base change operator \( \circ \) that satisfies (G1)–(G3), (G5), and (G6) for any critical-loop-free base \( \mathcal{B} \) gives rise to a compatible min-complete quasi-faithful assignment \( \preceq_\circ \). We will show how to obtain \( \preceq_\circ \) via a step-wise transformation of the assignment \( \preceq_\circ \) from Definition 5.3.

The transformation from \( \preceq_\circ \) to \( \preceq_\circ \) consists of three steps. For the start, recall that \( \preceq_\circ \) is a min-complete quasi-faithful assignment compatible with \( \circ \) by Proposition 8.5. This means that \( \preceq_\circ \) is a total relation for each \( \mathcal{K} \), whence transitivity is the only condition that \( \preceq_\circ \) fails to meet to qualify as a total preorder.

For the first step, we will identify a group of interpretation pairs \( \mathcal{D}_\mathcal{K}^\circ \subseteq \preceq_\circ \) such that at least one pair from \( \mathcal{D}_\mathcal{K}^\circ \) is involved whenever \( \preceq_\circ \) violates transitivity. The first step then consists in drastically removing all \( \mathcal{D}_\mathcal{K}^\circ \) from \( \preceq_\circ \), resulting in \( \preceq_\circ' \). The relation \( \preceq_\circ' \) will be a non-transitive and non-total relation, but minima of models of bases will be preserved. We will then extend \( \preceq_\circ' \) to a transitive relation \( \preceq_\circ'' \) in the second step, by taking the transitive closure. We will show that only elements from \( \mathcal{D}_\mathcal{K}^\circ \) can be added back by the transitive closure, which guarantees that, again, minima of models of bases are preserved. In a last step, we obtain the final result \( \preceq_\circ \) by “linearizing” \( \preceq_\circ' \) to a total preorder in a way that minima of models of bases are again preserved.

Step I: Removing detached pairs. Let \( \circ \) be a base change operator that satisfies (G1)–(G3), (G5), and (G6). Then, for any two bases \( \mathcal{K}, \Gamma \in \mathcal{B} \), all quasi-faithful assignments \( \preceq_\circ \) compatible with \( \circ \) yield the same set of minimal interpretations of \( [\Gamma] \) with respect to \( \preceq_\circ \). This property already stipulates much of \( \preceq_\circ \) for each \( \mathcal{K} \) (for some base logics \( \preceq_\circ \) is even complete determined by that property). Still, in the general case, when forming a compatible assignment, there is certain freedom on relating those interpretations for which the given base change operator gives no hint about how to order them. The following notion formally defines such pairs of interpretations.

**Definition 9.7.** Let \( \circ \) be a base change operator for \( \mathcal{B} \) and \( \mathcal{K} \) a base of \( \mathcal{B} \). A pair \( (\omega, \omega') \in \Omega \times \Omega \) is called detached from \( \circ \) in \( \mathcal{K} \), if \( \omega, \omega' \not\models \mathcal{K} \circ \Gamma \) for all \( \Gamma \in \mathcal{B} \) with \( \omega, \omega' \models \Gamma \). With \( \mathcal{D}_\mathcal{K}^\circ \), we denote the set of all pairs \( (\omega, \omega') \) which are detached from \( \circ \) in \( \mathcal{K} \) and satisfy \( \omega \neq \omega' \).

Note that detachment is a symmetric property, i.e., \( (\omega, \omega') \) is detached if and only if \( (\omega', \omega) \) is. It so happens that \( \preceq_\circ \) may contain too many of such detached pairs, i.e., in some cases, \( \preceq_\circ \) is not a total preorder even if the base change operator \( \circ \) is total preorder-representable (see also Section 10.4). In the following, we show that every violation of transitivity in \( \preceq_\circ \) involves a detached pair (as illustrated in Figure 5).
Lemma 9.8. Assume $\mathbb{B}$ is a base logic which does not admit a critical loop and $\circ$ a base change operator for $\mathbb{B}$ which satisfies $(G1)$–$(G3)$, $(G5)$, and $(G6)$. If $\omega_0 \preceq_K^\circ \omega_1$ and $\omega_1 \preceq_K^\circ \omega_2$ with $\omega_0 \not\preceq_K^\circ \omega_2$, then $(\omega_0, \omega_1)$ or $(\omega_1, \omega_2)$ is detached from $\circ$ in $\mathcal{K}$.

A full, detailed proof of Lemma 9.8 can be found in Appendix C. Here, we present only the main outline of the proof.

Proof (outline). The proof is by contradiction. Therefore, assume that neither $(\omega_0, \omega_1)$ nor $(\omega_1, \omega_2)$ is detached from $\circ$ in $\mathcal{K}$. Because $\preceq_K^\circ$ is a total relation, we obtain $\omega_2 \preceq_K^\circ \omega_0$ from $\omega_0 \not\preceq_K^\circ \omega_2$. From Definition 5.3 and $\omega_2 \preceq_K^\circ \omega_0$, we obtain $\omega_0, \omega_1, \omega_2 \not\models \mathcal{K}$. By employing Definition 9.7 and Lemma 5.6, we obtain that for each $i \in \{0, \ldots, 2\}$ there exists a base $\Gamma_{i,i; \in\{1\}}$ such that $\omega_i, \omega_{i; \in\{1\}} \models \Gamma_{i,i; \in\{1\}}$ and $\omega_i \models \mathcal{K} \circ \Gamma_{i,i; \in\{1\}}$ holds. When setting $\Gamma_i = (\mathcal{K} \circ \Gamma_{i,i; \in\{1\}}) \Psi \Gamma_{i; \in\{1\}}$, for each $i \in \{0, 1, 2\}$, one can show that $\Gamma_{0,1}, \Gamma_{1,2}, \Gamma_{2,0}$ is forming a critical loop.

Lemma 9.8 provides the rationale for the first transformation step: For every $\mathcal{K} \in \mathbb{B}$, we obtain $\preceq_{\mathcal{K}'}$ by removing all non-reflexive detached pairs from $\preceq_{\mathcal{K}}$, that is, $\preceq_{\mathcal{K}'} = \preceq_{\mathcal{K}} \setminus \mathcal{D}_{\mathcal{K}}$. The resulting $\preceq_{\mathcal{K}'}$ is not guaranteed to be total anymore, and it is not necessarily transitive. But we will show that $\preceq_{\mathcal{K}'}$ inherits other important properties from $\preceq_{\mathcal{K}}$.

Lemma 9.9. Let $\mathbb{B} = (\mathcal{L}, \Omega, \models, \mathbb{B}, \Psi)$ be a base logic which does not admit a critical loop, let $\circ$ be a base change operator satisfying $(G1)$–$(G3)$, $(G5)$, and $(G6)$ and let $\preceq_{\mathcal{K}}$ be a quasi-faithful min-friendly assignment compatible with $\circ$. For each $\mathcal{K}, \Gamma \in \mathbb{B}$ holds $\min([\Gamma], \preceq_{\mathcal{K}'}) = \min([\Gamma], \preceq_{\mathcal{K}})$ and $\preceq_{\mathcal{K}'}$ is min-complete and reflexive.

Proof. By definition of $\preceq_{\mathcal{K}'}$ we have $\omega \preceq_{\mathcal{K}'} \omega'$ if and only if $\omega \preceq_{\mathcal{K}} \omega'$ for all $(\omega, \omega') \in \Omega \times \Omega$ which are not detached pairs. Because for every $\omega, \omega' \in [\Gamma]$ with $\omega \in \min([\Gamma], \preceq_{\mathcal{K}})$ we have $\omega \models \mathcal{K} \circ \Gamma$ by compatibility of $\preceq_{\mathcal{K}}$ with $\circ$, Consequently, the pair $(\omega, \omega')$ is not detached and thus $\min([\Gamma], \preceq_{\mathcal{K}'}) = \min([\Gamma], \preceq_{\mathcal{K}})$. The latter implies that min-completeness of $\preceq_{\mathcal{K}}$ carries over to $\preceq_{\mathcal{K}'}$. Reflexivity of $\preceq_{\mathcal{K}'}$ is obtained by construction, the reflexivity of $\preceq_{\mathcal{K}}$ and by the definition of $\mathcal{D}_{\mathcal{K}'}$. □

Step II: Taking the transitive closure. In this step, for every $\mathcal{K} \in \mathbb{B}$, we obtain $\preceq_{\mathcal{K}''}$ by taking the transitive closure of $\preceq_{\mathcal{K}'}$, i.e., we have $\preceq_{\mathcal{K}''} = TC(\preceq_{\mathcal{K}'}) = TC(\preceq_{\mathcal{K}} \setminus \mathcal{D}_{\mathcal{K}})$. The resulting $\preceq_{\mathcal{K}''}$ is still not guaranteed to be total, but it is reflexive and transitive by construction, and it inherits further important properties from $\preceq_{\mathcal{K}'}$. It will turn out that the transitive closure will only add pairs to $\preceq_{\mathcal{K}'}$ that are detached pairs. This means that
Figure 6: Illustration of a critical loop-situation of length \( n \) on the semantic side. This situation is due to Lemma 9.10 impossible for \( \preceq^0_K \) if \( \mathcal{B} \) does not exhibit a critical loop. If \( \mathcal{B} \) does not exhibit a critical loop, then this situation is due to Lemma 9.10 only possible when there is some \( i \in \{1, \ldots, n\} \) such that \( (\omega, \omega_i \oplus 1) \) is a detached pair.

\( \preceq^0_K \) contains only elements from \( \preceq^0_K \) and from \( \mathcal{D}^0_K \). Because adding detached pairs does not influence minimal sets of models of a base \( \Gamma \) with respect to \( \preceq_K \), we will obtain that these sets are preserved when taking the transitive closure.

If the transitive closure would (hypothetically) add non-detached pairs to \( \preceq^0_K \), then the relation \( \preceq^0_K \) would contain a circle of interpretations consisting only of non-detached pairs (such as the circle illustrated in Figure 6). The following lemma shows that for base logics without critical loops such circles do not exist in \( \preceq^0_K \).

Lemma 9.10. Let \( \mathcal{B} = (\mathcal{L}, \Omega, \models, \mathcal{B}, \cup) \) a base logic which does not admit a critical loop, let \( \mathcal{K} \in \mathcal{B} \) be a base, and let \( \circ \) be a base change operator for \( \mathcal{B} \) that satisfies (G1)–(G3), (G5), and (G6). If there are three or more interpretations \( \omega_0, \ldots, \omega_n \in \Omega \), i.e. \( n \geq 2 \), such that

(a) \( \omega_0 \preceq^0_K \omega_1 \),
(b) \( \omega_i \preceq^0_K \omega_{i+1} \) for all \( i \in \{1, \ldots, n\} \), where \( \oplus \) is addition \( \text{mod}(n+1) \),

then there is some \( i \in \{1, \ldots, n\} \) such that \( (\omega, \omega_{i+1}) \) is a detached pair.

We will only present a proof outline here and defer the full, detailed proof to Appendix C.

Proof (outline). The proof is by induction. The case of \( n = 2 \) is an direct consequence of Lemma 9.8. For \( n > 2 \), we strive for a contradiction. Therefore, we assume that \( (\omega, \omega_{i+1}) \) for all \( i \in \{0, \ldots, n\} \) is not a detached pair. From Definition 5.3 and Condition (a), obtain that \( \omega_0, \ldots, \omega_n \not\models \mathcal{K} \). Employing Definition 9.7 and Lemma 5.6, we obtain for each \( i \in \{0, \ldots, n\} \) that there exists a base \( \Gamma_{i+1} \) such that \( \omega_i \models \Gamma_{i+1} \) and \( \omega_i \models \mathcal{K} \circ \Gamma_{i+1} \). One can show that \( \Gamma_0, \ldots, \Gamma_n \) is forming a critical loop, when setting \( \Gamma_i = (\mathcal{K} \circ \Gamma_{i+1}) \cup \Gamma_{i+1} \) in Definition 9.3 for each \( i \in \{0, \ldots, n\} \).

By employing Lemma 9.10 we show now that transformation of \( \preceq^0_K \) to \( \preceq^0_K \) by taking the transitive closure only adds detached pairs.
Lemma 9.11. Let \( \mathcal{B} = (\mathcal{L}, \Omega, \models, \mathcal{B}, \triangleright) \) a base logic which does not admit a critical loop, let \( K \in \mathcal{B} \) be a base, and let \( \circ \) be a base change operator for \( \mathcal{B} \) that satisfies (G1)–(G3), (G5), and (G6). The following holds:

\[
\preceq_{K}^{o} \subseteq \preceq_{K}^{o} \subseteq \preceq_{K}^{o}
\]

Proof. By construction of \( \preceq_{K}^{o} \), we have \( \preceq_{K}^{o} \subseteq \preceq_{K}^{o} \), and by construction of \( \preceq_{K}^{o} \) we have \( \preceq_{K}^{o} \subseteq \preceq_{K}^{o} \). To show that \( \preceq_{K}^{o} \subseteq \preceq_{K}^{o} \) holds, we assume the contrary, i.e., there exists a pair \((\omega_{1}, \omega_{0}) \in \preceq_{K}^{o} \) such that \( \omega_{1} \not\preceq_{K}^{o} \omega_{0} \). From \( \preceq_{K}^{o} \subseteq \preceq_{K}^{o} \) we obtain \( \omega_{1} \not\preceq_{K}^{o} \omega_{0} \) and because \( \preceq_{K}^{o} \) is a total relation, we have that \( \omega_{0} \not\preceq_{K}^{o} \omega_{1} \) holds. By the definition of transitive closure (cf. Section 2.3), there exists \( \omega_{2}, \ldots, \omega_{n} \in \Omega \), for some \( n \in \mathbb{N} \), such that \( \omega_{1} \preceq_{K}^{o} \omega_{2} \) and \( \omega_{n} \preceq_{K}^{o} \omega_{0} \) and \( \omega_{i} \preceq_{K}^{o} \omega_{i+1} \) for each \( i \in \{2, \ldots, n-1\} \). From \( \preceq_{K}^{o} \subseteq \preceq_{K}^{o} \), we obtain \( \omega_{0} \preceq_{K} \omega_{1} \preceq_{K} \omega_{2} \cdots \preceq_{K} \omega_{n} \not\preceq_{K} \omega_{0} \). We obtain a contradiction, because \( \preceq_{K}^{o} \) does not contain any detached pairs, but due to Lemma 9.10 there is some \( i \in \{2, \ldots, n-1\} \) such that \((\omega_{i}, \omega_{i+1})\) is a detached pair. Consequently, we obtain \( \preceq_{K}^{o} \subseteq \preceq_{K}^{o} \). \(\square\)

Combining Lemma 9.9 and Lemma 9.11 yields that \( \preceq_{K}^{o} \) is a (possibly non-total) preorder with useful properties. In particular, the sets of minimal models for every base \( \Gamma \in \mathcal{B} \) coincide for \( \preceq_{K}^{o} \) and \( \preceq_{K}^{o} \).

Lemma 9.12. Let \( \mathcal{B} = (\mathcal{L}, \Omega, \models, \mathcal{B}, \triangleright) \) be a base logic which does not admit a critical loop, let \( K \in \mathcal{B} \) and let \( \circ \) be a base change operator for \( \mathcal{B} \) which satisfies (G1)–(G3), (G5), and (G6). Then \( \preceq_{K}^{o} \) is a min-complete preorder and for any \( \Gamma \in \mathcal{B} \) holds \( \min([\Gamma], \preceq_{K}^{o}) = \min([\Gamma], \preceq_{K}^{o}) \).

Proof. Because of Lemma 9.11 we have \( \min([\Gamma], \preceq_{K}^{o}) = \min([\Gamma], \preceq_{K}^{o}) \) for any \( \Gamma \in \mathcal{B} \), since \( \preceq_{K}^{o} \setminus \mathcal{D}_{\circ} = \preceq_{K}^{o} \setminus \mathcal{D}_{\circ} \). Recall that by Lemma 9.9 we have that \( \preceq_{K}^{o} \) is min-complete and reflexive. Consequently, the transitive closure \( \preceq_{K}^{o} \) of \( \preceq_{K}^{o} \) is a preorder. Moreover, as in the proof of Lemma 9.9, from \( \min([\Gamma], \preceq_{K}^{o}) = \min([\Gamma], \preceq_{K}^{o}) \) we obtain that min-completeness carries over from \( \preceq_{K}^{o} \) to \( \preceq_{K}^{o} \). \(\square\)

Step III: Linearizing. As last step, we extend \( \preceq_{K}^{o} \) to a total relation without losing transitivity. In order to obtain totality, we make use of the following result. Note that this theorem requires the axiom of choice.

Theorem 9.13 (Hansson (1968), Lemma 3). For every preorder \( \leq \) on a set \( X \) there exists a total preorder \( \leq_{\text{lin}} \) on \( X \) such that

- if \( x \leq y \), then \( x \leq_{\text{lin}} y \), and
- if \( x \leq y \) and \( y \leq x \), then \( x \leq_{\text{lin}} y \) and \( y \leq_{\text{lin}} x \).

As stated in Lemma 9.12, the relation \( \preceq_{K}^{o} \) is a preorder. Thus, we can safely applying Theorem 9.13 to obtain \( \preceq_{K}^{o} \) from \( \preceq_{K}^{o} \) through extension, i.e., \( \preceq_{K}^{o} = (\text{TC}([\Gamma], \preceq_{K}^{o}))_{\text{lin}} \). The resulting relation \( \preceq_{K}^{o} \) is then a total preorder, while it still coincides with \( \preceq_{K}^{o} \) regarding the relevant properties. Combining Theorem 9.13, Lemma 9.9 and Lemma 9.12 we obtain the desired result.

Proposition 9.14. If \( \mathcal{B} \) does not admit a critical loop, then, for any given base change operator \( \circ \) for \( \mathcal{B} \) satisfying (G1)–(G3), (G5), and (G6), the mapping \( \preceq_{o}: K \mapsto \preceq_{K}^{o} \) is a min-friendly quasi-faithful preorder assignment compatible with \( \circ \).
Proof. From Lemma 9.12 we obtain that \( \preceq^{(\text{B})}_E \) is a min-complete preorder assignment. Application of Theorem 9.13 yields a total preorder \( \preceq^{(\text{B})}_E \). Observe that linearization by Theorem 9.13 retains strict relations, i.e., if \( \omega_1 \preceq^{(\text{B})}_E \omega_2 \) and \( \omega_2 \preceq^{(\text{B})}_E \omega_1 \), then \( \omega_1 \preceq^{(\text{B})}_E \omega_2 \) and \( \omega_2 \preceq^{(\text{B})}_E \omega_1 \). Thus, we have \( \omega_1 \in \min(\Gamma_i, \preceq^{(\text{B})}_E) \) and only if \( \omega_1 \in \min(\Gamma_i, \preceq^{(\text{B})}_E) \), which yields \( \min(\Gamma_i, \preceq^{(\text{B})}_E) = \min(\Gamma_i, \preceq^{(\text{B})}_E) \) for each base \( \Gamma \in \mathcal{B} \). Consequently, min-completeness carries over from \( \preceq^{(\text{B})}_E \) to \( \preceq^{(\text{B})}_E \). Moreover, by Lemma 9.9 and Lemma 9.12 we obtain \( \min(\Gamma_i, \preceq^{(\text{B})}_E) = \min(\Gamma_i, \preceq^{(\text{B})}_E) \) for each base \( \Gamma \) of \( \mathcal{B} \). As every \( \preceq^{(\text{B})}_E \) is transitive and total, we obtain that \( \preceq^{(\text{B})}_E \) is min-retractive and thus, \( \preceq^{(\text{B})}_E \) is a min-friendly assignment. Because \( \preceq^{(\text{B})}_E \) is a quasi-faithful assignment which is compatible with \( \circ \) and we have \( \min(\Gamma_i, \preceq^{(\text{B})}_E) = \min(\Gamma_i, \preceq^{(\text{B})}_E) \) for each \( \mathcal{E} \in \mathcal{B} \), we also obtain that \( \preceq^{(\text{B})}_E \) is a quasi-faithful assignment which is compatible with \( \circ \).

This completes the argument regarding the correspondence between the absence of critical loops and total-preorder-representability, by establishing that the former is also sufficient for the latter. Obviously, Theorem 9.4 (II) is a direct consequence of Proposition 9.14.

9.4 Characterization Theorems and Example

Combining the two arguments presented in Section 9.2 and Section 9.3, we establish that the absence of critical loop coincides with universal total-preorder-representability, i.e., Theorem 9.4 holds.

We will now employ the novel notion of critical loop (cf. Definition 9.3) and our representation theorem for total-preorder-representability (Theorem 9.4), to show that there is no (total) preorder assignment for the operator \( \circ \) from our running example.

Example 9.15 (continuation of Example 5.4). We will now see that the base logic \( \mathcal{B}_{\text{Ex}} = (\mathcal{L}_{\text{Ex}}, \circ_{\text{ex}}, \models_{\text{Ex}}, \mathcal{P}(\mathcal{L}_{\text{Ex}}), \cup) \) from Example 4.7 constructed from \( L_{\text{Ex}} \) exhibits a critical loop.

For this, choose \( \Gamma_i, i \in \{0, 1, 2\} \), and \( \mathcal{E} = \mathcal{E}_{\text{Ex}} = \{\psi_3\} \) (as in Example 4.7) and \( \Gamma_i = \{\psi_i\} \) for \( i \in \{0, 1, 2\} \), where \( \oplus \) denotes addition mod 3. We consider each of the three conditions of Definition 9.3 as a separate case:

Condition (1). Observe that \( \mathcal{E}_{\text{Ex}} \) is inconsistent with \( \Gamma_{0,1}, \Gamma_{1,2} \) and \( \Gamma_{2,0} \). Thus, Condition (1) is satisfied.

Condition (2). For each \( i \in \{0, 1, 2\} \), the models of bases \( \Gamma_i \) and \( \Gamma_i \oplus 1 \) are contained in \( [\Gamma_{i,i \oplus 1}] \), but \( \Gamma_i \) is inconsistent with \( \Gamma_j \) with \( i \neq j \), e.g. \( \{\{\psi_0\}\} \cup \{\{\psi_1\}\} \subseteq \{\{\psi_0\}\} \) and \( \{\psi_0\} \) is not consistent with neither \( \{\psi_1\} \) nor \( \{\psi_2\} \).

Condition (3). The belief base \( \Gamma_3' = \{\chi\} \) is the only belief base consistent with \( \Gamma_0, \Gamma_1, \) and \( \Gamma_2 \). For the satisfaction of Condition (3) observe that \( \Gamma_3' = \{\psi_4\} \) fulfills the required condition \( \emptyset \neq [\Gamma_3'] \subseteq [\Gamma_3'] \setminus ([\Gamma_{0,1}] \cup [\Gamma_{1,2}] \cup [\Gamma_{2,0}]) \).

In summary \( \Gamma_{0,1}, \Gamma_{1,2}, \) and \( \Gamma_{2,0} \) form a critical loop for \( \mathcal{B}_{\text{Ex}} \) (see Figure 7). As given by Theorem 9.4 (I), every min-complete faithful preorder assignment compatible with \( \circ_{\text{ex}} \) is not transitive. To illustrate this, we use here the assignment \( \preceq_{\text{Ex}} \) defined in Example 5.4 and sketched in Figure 3 for \( \mathcal{E}_{\text{Ex}} \) (see also Figure 7). Consider the revisions \( \mathcal{E}_{\text{Ex}} \circ_{\text{ex}} \Gamma_{0,1}, \mathcal{E}_{\text{Ex}} \circ_{\text{ex}} \Gamma_{1,2}, \) and \( \mathcal{E}_{\text{Ex}} \circ_{\text{ex}} \Gamma_{2,0} \). From the construction of \( \circ_{\text{Ex}} \) given in Definition 9.5 and
Figure 7: Critical loop situation in $\mathbb{B}_{\text{Ex}}$ presented in Example 9.15. The solid borders represent the sets of models and each arrow depicts the relation $\triangleleft_{K_{\text{Ex}}}$ between models.

Compatibility of $\trianglerighteq_{\text{Ex}}$ with $c_{\text{Ex}}$, we have

- $\omega_0 \in \min([\Gamma_0,1], \trianglerighteq_{K_{\text{Ex}}})$, but $\omega_1 \notin \min([\Gamma_0,1], \trianglerighteq_{K_{\text{Ex}}})$,
- $\omega_1 \in \min([\Gamma_1,2], \trianglerighteq_{K_{\text{Ex}}})$, but $\omega_2 \notin \min([\Gamma_1,2], \trianglerighteq_{K_{\text{Ex}}})$, and
- $\omega_2 \in \min([\Gamma_2,0], \trianglerighteq_{K_{\text{Ex}}})$, but $\omega_0 \notin \min([\Gamma_2,0], \trianglerighteq_{K_{\text{Ex}}})$,

showing $\omega_0 \triangleleft_{K_{\text{Ex}}} \omega_1 \triangleleft_{K_{\text{Ex}}} \omega_2 \triangleleft_{K_{\text{Ex}}} \omega_0$, which is impossible for a transitive relation.

Moreover, observe that the construction of $c_{\text{Ex}}$ presented in Example 4.7 illustrates the construction given by Definition 9.5 and used in the proof of Proposition 9.6. In particular, for the example presented here one would obtain $C' = \{\Gamma'_0\} = \{\{\psi_4\}\}$ when following the outline of the construction.

Having established the necessary and sufficient criterion for total-preorder-representability, we can now provide two more versions of the two-way representation theorem. The first representation theorem is one where the base change operator satisfies (G4), thus abstracting from the syntactic form of the belief bases. Note that transitivity implies min-retractivity, and thus a transitive min-complete relation is automatically min-friendly.

**Theorem 9.16.** Let $\mathbb{B}$ be a base logic which does not admit a critical loop. Then the following hold:

- Every base change operator for $\mathbb{B}$ satisfying (G1)–(G6) is compatible with some min-expressible min-complete faithful preorder assignment.
- Every min-expressible min-complete faithful preorder assignment for $\mathbb{B}$ is compatible with some base change operator satisfying (G1)–(G6).
Proof. The first statement is a consequence of statement (II) of Theorem 9.4 together with Lemma 9.2. The second statement is an immediate consequence of the second statement of Theorem 6.3.

The second representation theorem is for base change operators which do not necessarily satisfy (G4), and thus might be sensitive to the syntax of the prior belief base.

**Theorem 9.17.** Let $\mathbb{B}$ be a base logic which does not admit a critical loop. Then the following hold:

- Every base change operator for $\mathbb{B}$ satisfying (G1)–(G3), (G5), and (G6) is compatible with some min-expressible min-complete quasi-faithful preorder assignment.
- Every min-expressible min-complete quasi-faithful preorder assignment for $\mathbb{B}$ is compatible with some base change operator satisfying (G1)–(G3), (G5), and (G6).

Proof. The first statement is a consequence of statement (II) of Theorem 9.4. The second statement is an immediate consequence of the second statement of Theorem 8.6.

We close this section with an implication of Theorem 9.4. A base logic $\mathbb{B} = (\mathcal{L}, \Omega, \models, \mathcal{B}, \triangleright)$ is called disjunctive, if for every two bases $\Gamma_1, \Gamma_2 \in \mathcal{B}$ there is a base $\Gamma_1 \triangleright \Gamma_2 \in \mathcal{B}$ such that $[\Gamma_1 \triangleright \Gamma_2] = [\Gamma_1] \cup [\Gamma_2]$. This includes the case of any (base) logic allowing disjunction to be expressed on the sentence level, i.e., when for every $\gamma, \delta \in \mathcal{L}$ there exists some $\gamma \otimes \delta \in \mathcal{L}$ with $[\gamma \otimes \delta] = [\gamma] \cup [\delta]$, such that $\Gamma_1 \triangleright \Gamma_2$ can be obtained as $\{\gamma \otimes \delta \mid \gamma \in \Gamma_1, \delta \in \Gamma_2\}$.

**Corollary 9.18.** In a disjunctive base logic, every belief change operator satisfying (G1)–(G6) is total-preorder-representable.

Proof. A disjunctive base logic never exhibits a critical loop; Condition (3) would be violated by picking $\Gamma = ((\Gamma_0 \triangleright \Gamma_1) \ldots) \triangleright \Gamma_n$.

As a consequence, for a vast amount of well-known logics, including all classical logics such as first-order and second order predicate logic, one directly obtains total-preorder-representability of every AGM base change operator by Corollary 9.18.

10. Further Discussion

In this section, we discuss some more specific aspects and noteworthy implications of our approach. First, we will discuss the notion of base logics and demonstrate that the decision how to define bases affects the applicability of certain notions. Next, we explore the novel notion of min-retractivity (in comparison to transitivity) and discuss its relationship to decomposability of disjunctions. In the last subsection, we give additional insights into our way of encoding operators as assignments.

10.1 Dependency on the Notion of Base

In order to capture different conceptualisations of the notion of a “base” in a generic way, we introduced base logics. In this section, we would like to highlight that the question if our characterization applies depends on the particular notion of base applied, even if the
underlying logic is the same. In particular, we would like to make the point that the case of
finite bases cannot just be seen as a "special case" of arbitrary bases, as demonstrated in the
following example.

Example 10.1. Consider the simplistic base logic \( \mathbb{B}_\Pi = (\mathcal{L}, \Omega, \models, \mathcal{B}, \cup) \) where \( \mathcal{L} \) contains
two types of sentences: \([ \geq q] \), and \([ = q] \), for each \( q \in \mathbb{Q}^+ \) (i.e., \( q \) ranges over the nonnegative
rational numbers). Moreover, let \( \Omega = \mathbb{R}^+ \), i.e., the interpretations are just nonnegative real
numbers. Then we let \( r \models [ \geq q] \) if \( r \geq q \) and we let \( r \models [ = q] \) if \( r = q \) (we assume the usual
meaning of \( \leq, \geq \), and = for real numbers).

For \( \mathcal{B} = \mathcal{P}_{\text{fin}}(\mathcal{L}) \), we now define a finite base revision operator \( \circ^{\Pi} \) as follows:

\[
\mathcal{K} \circ^{\Pi} \Gamma = \begin{cases}
\mathcal{K} \cup \Gamma & \text{if } [\mathcal{K} \cup \Gamma] \neq \emptyset, \\
\Gamma & \text{if } [\Gamma] = \emptyset, \\
\{[=q]\} & \text{otherwise, where } \{q\} = \min([\Gamma], \leq).
\end{cases}
\]

Note that the usage of finite bases ensures that the minimum in the third case exists and
is indeed a rational number. It turns out that \( \circ^{\Pi} \) satisfies (G1)–(G6). This could be proven
directly without much effort, but we can also use our result just established, exploiting the fact
that we can easily come up with a corresponding assignment: Let \( \leq^{\Pi} \) be defined by letting
\( r_1 \leq^{\Pi} r_2 \) exactly if either \( r_1 \in [\mathcal{K}] \), or both \( \{r_1, r_2\} \cap [\mathcal{K}] = \emptyset \) and \( r_1 \leq r_2 \). Then we find
that \( \leq^{\Pi} \) is a min-expressible, min-friendly faithful assignment compatible with \( \circ^{\Pi} \) for finite
bases.

The attempt to generalize this operator to arbitrary bases (i.e. \( \mathcal{B} = \mathcal{P}(\mathcal{L}) \)) fails, despite
the fact that our previous definition of \( \leq^{\Pi} \) as such seamlessly carries over to arbitrary bases
and we obtain a min-friendly faithful assignment. However, this assignment is no longer
min-expressible as soon as infinite bases are permitted. To see this, consider \( \mathcal{K} = \{[0]\} \)
and the infinite base \( \Gamma = \{[r] \mid q \in \mathbb{Q}^+, q^2 \leq 2\} \). We obtain \([\Gamma] = \{r \mid r \in \mathbb{R}^+, r \geq \sqrt{2}\} \)
and hence \( \min([\Gamma], \leq^{\Pi}) = \{\sqrt{2}\} \). As \( \sqrt{2} \) is an irrational number, this model set cannot be
characterized by any finite or infinite base of our logic (it can be readily checked, that the only
characterizable model sets are intervals of the form \([r, \infty)\) for any \( r \in \mathbb{R}^+ \) or \([q, q] \) for any
\( q \in \mathbb{Q}^+ \)).

In view of the above example, one might now surmise that the case of finite bases is the
more restrictive one and any assignment that works for arbitrary bases will work for finite
ones as well. The next example shows that this is not the case.

Example 10.2. Consider the simplistic base logic \( \mathbb{B}_\Pi = (\mathcal{L}, \Omega, \models, \mathcal{B}, \cup) \) where \( \mathcal{L} \) contains
two types of sentences: \([ \geq q] \), and \([ \leq q] \), for each \( q \in \mathbb{Q}^+ \), and additionally the sentence \([ \geq \sqrt{2}] \).
Again, let \( \Omega = \mathbb{R}^+ \). The relation \( \models \) is defined in exactly the same way as before.

For \( \mathcal{B} = \mathcal{P}(\mathcal{L}) \), we now define a finite base revision operator \( \circ^{\Omega} \) as follows:

\[
\mathcal{K} \circ^{\Omega} \Gamma = \begin{cases}
\mathcal{K} \cup \Gamma & \text{if } [\mathcal{K} \cup \Gamma] \neq \emptyset, \\
\Gamma & \text{if } [\Gamma] = \emptyset, \\
\{[q^-] \mid q^- \leq r\} \cup \{[q^+] \mid q^+ \geq r\} & \text{otherwise, where } \{r\} = \min([\Gamma], \leq).
\end{cases}
\]

Note that the usage of infinite bases ensures that, in the third case, \([\mathcal{K} \circ^{\Omega} \Gamma] = \{r\} \) even if
\( r \) is irrational. Note that \( \circ^{\Omega} \) satisfies (G1)–(G6). In fact we can define the assignment in
exactly the same way as in the previous example to show that. Thanks to the change of the logic, we now find that $\preceq_{(j)}^{fin}$ is a min-expressible, min-friendly faithful assignment compatible with $\circ$ for arbitrary bases.

However, now the case of finite bases (i.e. $\mathcal{B} = \mathcal{P}_{\text{fin}}(\mathcal{L})$) fails to work, although $\preceq_{(j)}^{fin}$ remains a min-friendly faithful assignment. However, this assignment is no longer min-expressible as soon as finiteness of bases is imposed. To see this, consider $K = \{[=0]\}$ and $\Gamma = \{[\geq \sqrt{2}]\}$. We obtain $[\Gamma] = \{r \mid r \in \mathbb{R}^+, r \geq \sqrt{2}\}$ and hence $\min([\Gamma], \preceq_{(j)}^{fin}) = \{\sqrt{2}\}$. As $\sqrt{2}$ is an irrational number, this model set cannot be characterized by any finite base of our logic (it can be readily checked, that the only characterizable model sets are intervals of the form $[q_1, q_2]$ for any for any $q_1 \in \mathbb{Q}^+ \cup \{\sqrt{2}\}$ and $q_2 \in \mathbb{Q}^+$).

These two examples highlight that notions that we define relative to a base logic (like the postulates or the properties of assignments) may crucially depend on $\mathcal{B}$ and cease to apply when the notion of base is changed, even if the underlying logic and the considered assignment stay the same.

10.2 On the Notion of Min-Retractivity

As a primary ingredient to our results, the novel notion of min-retractivity has been introduced and motivated in Section 4.2 and proven to serve its purpose later on. As noted earlier, it is immediate that any preorder over $\Omega$ is min-retractive, irrespective of the choice of the other components of the underlying base logic. On the other hand, we have exposed examples of min-retractive relations that are not transitive for certain base logics. This raises the question if there are conditions, under which the two notions do coincide, at least when presuming min-completeness (a condition already known and generally accepted). We start by formally defining this notion of coincidence.

**Definition 10.3.** A base logic $\mathcal{B} = (\mathcal{L}, \Omega, \models, \mathcal{B})$ is called preorder-enforcing, if every binary relation over $\Omega$ that is total and min-friendly for $\mathcal{B}$ is also transitive (and hence a total preorder).

As an aside, we note that being preorder-enforcing implies the absence of critical loops.

**Proposition 10.4.** Every preorder-enforcing base logic does not have critical loops.

**Proof.** Let $\mathcal{B} = (\mathcal{L}, \Omega, \models, \mathcal{B})$ be a preorder-enforcing base logic. By Theorem 9.4, absence of critical loops would follow from the fact that every base change operator for $\mathcal{B}$ satisfying (G1)–(G6) is total-preorder-representable. To show the latter, consider an arbitrary base change operator $\circ$ of that kind. By Proposition 5.10, $\preceq_{(j)}^{\circ}$ is a min-friendly faithful assignment compatible with $\circ$. In particular, for every $\mathcal{K} \in \mathcal{B}$, the corresponding $\preceq_{\mathcal{K}}^{\circ}$ is total and min-friendly for $\mathcal{B}$. Yet then, by assumption, any such $\preceq_{\mathcal{K}}^{\circ}$ is also a preorder, and therefore $\preceq_{(j)}^{\circ}$ is even a preorder assignment. Hence, $\circ$ is total-preorder-representable.

The question remains, which base logics are actually preorder-enforcing. We will next present a simple criterion and then show that it is indeed necessary and sufficient for being preorder-enforcing.

**Definition 10.5.** A base logic $\mathcal{B} = (\mathcal{L}, \Omega, \models, \mathcal{B})$ is called trio-expressible, if for any three interpretations $\omega_1, \omega_2, \omega_3 \in \Omega$ there is a base $\Gamma_{\omega_1, \omega_2, \omega_3}$ satisfying $[\Gamma_{\omega_1, \omega_2, \omega_3}] = \{\omega_1, \omega_2, \omega_3\}$. 

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Theorem 10.6. A base logic is preorder-enforcing if and only if it is trio-expressible.

Proof. We show the “if” direction followed by the “only if” one.

“⇐” Let \( \preceq \) be a min-friendly total relation over \( \Omega \). Toward a contradiction, assume \( \preceq \) is not transitive, i.e., there exist interpretations \( \omega, \omega', \omega'' \in \Omega \) such that \( \omega \preceq \omega' \) and \( \omega' \preceq \omega'' \) but \( \omega \not\preceq \omega'' \). By totality, the latter implies \( \omega'' \prec \omega \). Now consider \( \Gamma_{\omega \omega' \omega''} \) and note that \( \min([\Gamma_{\omega \omega' \omega''}], \preceq) \neq \emptyset \) thanks to min-completeness, but then, by min-retractivity, \( \min([\Gamma_{\omega \omega' \omega''}], \preceq) = \{\omega, \omega', \omega''\} \) follows. However, \( \omega \in \min([\Gamma_{\omega \omega' \omega''}], \preceq) \) contradicts \( \omega \not\preceq \omega'' \).

“⇒” We actually show the contraposition: starting from a base logic \( \mathcal{B} = (\mathcal{L}, \Omega, |\cdot|, \mathcal{B}, \emptyset) \) that is not trio-expressible, we show violation of being preorder-enforcing by exhibiting a total, min-friendly relation over \( \Omega \) that is not transitive. From non-trio-expressibility, the existence of \( \omega_1, \omega_2, \omega_3 \in \Omega \) with \( [\Gamma] \neq \{\omega_1, \omega_2, \omega_3\} \) follows for every \( \Gamma \in \mathcal{B} \). Let now \( \preceq^- \) be an arbitrary well-order\(^5\) over \( \Omega^- = \Omega \setminus \{\omega_1, \omega_2, \omega_3\} \), i.e., it is total, transitive (hence min-retractive) and min-complete, therefore also min-friendly. We now define

\[
\preceq = \preceq^- \cup (\Omega^- \times \{\omega_1, \omega_2, \omega_3\}) \cup \{(\omega_1, \omega_2), (\omega_1, \omega_3), (\omega_3, \omega_1), (\omega_2, \omega_3)\}.
\]

It is easy to see that \( \preceq \) is still a total relation. It is min-complete (for \( \mathcal{B} \)) by case distinction: on one hand, if \( [\Gamma] \not\subseteq \{\omega_1, \omega_2, \omega_3\} \), then \( \min([\Gamma], \preceq) \neq \emptyset \) follows from min-completeness of \( \preceq^- \), on the other hand, for any two- or one-element subset of \( \{\omega_1, \omega_2, \omega_3\} \) also a minimum clearly exists (note that by assumption \( [\Gamma] = \{\omega_1, \omega_2, \omega_3\} \) cannot occur). We proceed to show min-retractivity of \( \preceq \), again by case-distinction: If \( [\Gamma] \not\subseteq \{\omega_1, \omega_2, \omega_3\} \) then, due to min-completeness and antisymmetry of \( \preceq^- \), the set \( \min([\Gamma], \preceq) \) contains exactly one element and is strictly smaller than any other element from \( [\Gamma] \), thus min-retractivity is vacuously satisfied. If \( [\Gamma] \subset \{\omega_1, \omega_2, \omega_3\} \), min-retractivity is easily verified case by case. We finish our argument by showing that \( \preceq \) is not transitive: we have \( \omega_2 \preceq \omega_3 \) as well as \( \omega_3 \preceq \omega_1 \), but \( \omega_2 \not\preceq \omega_1 \) fails to hold. \( \square \)

The preceding theorem provides yet another argument why preorders can be used as preference relations for finite-signature propositional logic (as in fact, they are the only preference relations arising in that setting). However, note that the result also applies to more complex logics such as first-order logic under the finite model semantics.\(^6\)

10.3 Decomposability of Disjunctions

In the belief revision literature, the postulates (G5) and (G6) are strongly associated with a decomposability of revisions for disjunctive beliefs. More specifically, under the AGM assumptions (Ribeiro et al., 2013), which includes closure under disjunction, it is known that (G5) and (G6) are together equivalent to disjunctive factoring\(^7\) (Gärdenfors, 1988) in the

---

5. As discussed earlier, existence of such a \( \preceq^- \) is assured by the well-ordering theorem, depending on the axiom of choice.

6. Strictly speaking, this requires a slightly non-standard (but semantically equivalent) model theory which abstracts from the domains used and considers isomorphic models as equal.

7. Here given in a semantic reformulation for (disjunctive) base logics.
presence of (G1)–(G4):

\[ [\mathcal{K} \circ (\Gamma_1 \odot \Gamma_2)] = \begin{cases} [\mathcal{K} \circ \Gamma_1] & \text{or} \\ [\mathcal{K} \circ \Gamma_2] & \text{or} \\ [\mathcal{K} \circ \Gamma_1] \cup [\mathcal{K} \circ \Gamma_2] & \text{or} \end{cases} \]

Factoring conditions like disjunctive factoring are useful and particularly important, as they are commonly used in many generalizations of AGM revision, e.g. (Fermé & Hansson, 1999; Hansson, Fermé, Cantwell, & Falappa, 2001).

We will see that the revision operators considered in this article satisfy a similar decomposability property. Of course, in the unrestricted setting of arbitrary base logics there might be no way to express disjunctions. We propose the following alternative postulates, which we call expressible disjunctive factoring:

For all \( \Gamma, \Gamma_1, \Gamma_2 \in \mathcal{B} \) with \( [\Gamma] = [\Gamma_1] \cup [\Gamma_2] \) holds \( [\mathcal{K} \circ \Gamma] = \begin{cases} [\mathcal{K} \circ \Gamma_1] & \text{or} \\ [\mathcal{K} \circ \Gamma_2] & \text{or} \\ [\mathcal{K} \circ \Gamma_1] \cup [\mathcal{K} \circ \Gamma_2] & \text{or} \end{cases} \).

Clearly, for disjunctive base logics, expressible disjunctive factoring and disjunctive factoring are the same. Now consider the following observation on min-retractivity for the unrestricted setting of arbitrary base logics.

**Proposition 10.7.** Let \( \mathcal{B} = (\mathcal{L}, \Omega, \models, \mathcal{B}, \psi) \) be a base logic, let \( \Gamma_1, \ldots, \Gamma_n, \Gamma \in \mathcal{B} \) be belief bases with \( [\Gamma] = [\Gamma_1] \cup \ldots \cup [\Gamma_n] \) and let \( \preceq \) be a relation over \( \Omega \) which is min-retractive for \( \mathcal{B} \). Then there exists some set \( I \subseteq \{1, \ldots, n\} \) such that \( \min([\Gamma], \preceq) = \bigcup_{i \in I} \min([\Gamma_i], \preceq) \).

**Proof.** Let \( I \) be a minimal set such that \( \min([\Gamma], \preceq) \subseteq \bigcup_{j \in I} [\Gamma_i] \) holds, i.e., there is no \( j \in I \) such that \( \min([\Gamma], \preceq) \subseteq \bigcup_{j \in I} [\Gamma_i] \) holds. Note that the assumption \( [\Gamma] = [\Gamma_1] \cup \ldots \cup [\Gamma_n] \) guarantees the existence of \( I \). Because of \( [\Gamma_i] \subseteq [\Gamma] \), we have that \( \omega \in \min([\Gamma], \preceq) \cap [\Gamma_i] \) implies \( \omega \in \min([\Gamma_i], \preceq) \). This shows that \( \min([\Gamma], \preceq) \subseteq \bigcup_{i \in I} \min([\Gamma_i], \preceq) \) holds.

We show that \( \bigcup_{i \in I} \min([\Gamma_i], \preceq) \subseteq \min([\Gamma], \preceq) \) holds. Because \( I \) is a minimally chosen, for every \( i \in I \) the set \( [\Gamma_i] \cap \min([\Gamma], \preceq) \) is non-empty. If \( \omega \in \min([\Gamma], \preceq) \) and \( \omega \in [\Gamma_i] \), then we have for each \( \omega' \in \min([\Gamma_i], \preceq) \) that \( \omega' \preceq \omega \) holds. Since \( \preceq \) is min-retractive, we conclude that \( \omega' \in \min((\Gamma], \preceq) \). Consequently, we obtain \( \min([\Gamma], \preceq) \subseteq \min([\Gamma], \preceq) \). \( \square \)

By Proposition 10.7, min-retractivity of a relation \( \preceq \) guarantees a decomposition property on the relation with respect to minima and bases. Thus, if a base change operator \( \circ \) is compatible with an assignment \( \preceq(\_\_\_\_) \), which yields always min-retractive relations, then the decomposition property of \( \preceq(\_\_\_\_) \) carries over to the operator. In the following, we use Proposition 10.7 to obtain for arbitrary base logics that (G1)–(G6) imply expressible disjunctive factoring.

**Proposition 10.8.** Let \( \mathcal{B} = (\mathcal{L}, \Omega, \models, \mathcal{B}, \psi) \) be a base logic. If \( \circ \) is a base change operator for \( \mathcal{B} \) that satisfies (G1)–(G6), then \( \circ \) satisfies expressible disjunctive factoring.

**Proof.** If \( \circ \) satisfies (G1) – (G6), then by Proposition 10.7 there is a min-friendly assignment \( \preceq(\_\_\_\_) \) compatible with \( \circ \). Now let \( \Gamma_1, \Gamma_2, \Gamma_3 \in \mathcal{B} \) such that \( [\Gamma_3] = [\Gamma_1] \cup [\Gamma_2] \). By compatibility and Proposition 10.7 we obtain that expressible disjunctive factoring is satisfied. \( \square \)
It turns out that, in the general setting of base logics, (G5) and (G6) are not implied by expressible disjunctive factoring. In particular, the following proposition shows the even stronger statement that (G5) and (G6) are not equivalent to disjunctive factoring even when restricting to disjunctive base logics.

**Proposition 10.9.** There is a disjunctive base logic $\mathcal{B}$ and a base change operator $\circ$ for $\mathcal{B}$ which satisfies (G1)–(G4) and disjunctive factoring, yet both (G5) and (G6) are violated.

**Proof.** In the following, we denote with $\leq$ the usual relation on the natural numbers. Let $\mathcal{B}_{\text{four}} = (\mathcal{L}, \Omega, \models, \mathcal{B}, \U)$ be the base logic such that $\mathcal{L} = \{[\geq i] \mid 0 \leq i \leq 4\}$ and $\Omega = \{0, 1, 2, 3\}$. The bases of $\mathcal{B}_{\text{four}}$ are all sets with exactly one sentence, i.e., we have $\mathcal{B} = \{\{[\geq i]\} \mid [\geq i] \in \mathcal{L}\}$. For every $[\geq i] \in \mathcal{L}$ and every interpretation $\omega \in \Omega$, we let $\omega \models [\geq i]$ if $i \leq \omega$. We will use min and max with respect to the usual ordering on natural numbers, but slightly abuse notions, and define $\min(\Gamma) = \min\{\{i \mid [\geq i] \in \Gamma\}\}$ and $\max(\Gamma) = \max\{\{i \mid [\geq i] \in \Gamma\}\}$ for $\Gamma \subseteq \mathcal{L}$. Using these notions, we define the operations $\uplus$ and $\otimes$ on bases as follows:

$$
\Gamma_1 \uplus \Gamma_2 = \max(\Gamma_1 \cup \Gamma_2) \quad \Gamma_1 \otimes \Gamma_2 = \min(\max(\Gamma_1) \cup \max(\Gamma_2))
$$

We obtain that $\mathcal{B}_{\text{four}}$ is a disjunctive base logic, as for every two bases $\Gamma_1, \Gamma_2 \in \mathcal{B}$ we have $[\Gamma_1 \uplus \Gamma_2] = [\Gamma_1] \cap [\Gamma_2]$ and we have $\Gamma_1 \otimes \Gamma_2 = [\Gamma_1] \cup [\Gamma_2]$. Note that we have $[[4]] = \emptyset$ and that $[\Gamma_1 \uplus \Gamma_2] \neq \emptyset$ for every two bases $\Gamma_1, \Gamma_2 \in \mathcal{B}$ whenever $[4] \notin \Gamma_1 \cup \Gamma_2$.

Now let $\circ_{\text{four}}$ be the base change operator for $\mathcal{B}_{\text{four}}$ such that:

$$
\mathcal{K} \circ_{\text{four}} \Gamma = \begin{cases} 
\mathcal{K} \uplus \Gamma & \text{if } [\mathcal{K} \uplus \Gamma] \neq \emptyset, \\
\{[\geq 3]\} & \text{if } [\mathcal{K} \uplus \Gamma] = \emptyset \text{ and } \Gamma = \{[\geq 1]\}, \\
\Gamma & \text{otherwise.}
\end{cases}
$$

Satisfaction of (G1)–(G4) is immediate due to the construction of $\circ_{\text{four}}$. We show that $\circ_{\text{four}}$ satisfies (expressible) disjunctive factoring. Let $\mathcal{K}, \Gamma_1, \Gamma_2 \in \mathcal{B}$ be three belief bases. We consider two cases:

- If $[\mathcal{K} \uplus (\Gamma_1 \otimes \Gamma_2)] \neq \emptyset$. This implies that $[4] \notin \mathcal{K} \cup (\Gamma_1 \cap \Gamma_2)$. Therefore, we have $[\mathcal{K} \uplus \Gamma_1] \neq \emptyset$ or $[\mathcal{K} \uplus \Gamma_2] \neq \emptyset$. In particular, we have either $\max(\mathcal{K} \uplus \Gamma_1) = \max(\mathcal{K} \uplus (\Gamma_1 \otimes \Gamma_2))$ or $\max(\mathcal{K} \uplus \Gamma_2) = \max(\mathcal{K} \uplus (\Gamma_1 \otimes \Gamma_2))$. In the first case, we have $[\mathcal{K} \uplus (\Gamma_1 \otimes \Gamma_2)] = [\mathcal{K} \uplus \Gamma_1]$; in the latter we have $[\mathcal{K} \uplus (\Gamma_1 \otimes \Gamma_2)] = [\mathcal{K} \uplus \Gamma_2]$.

  If $[\mathcal{K} \cup (\Gamma_1 \otimes \Gamma_2)] = \emptyset$. We have either $\max(\Gamma_1 \otimes \Gamma_2) = \max(\Gamma_1)$ or $\max(\Gamma_1 \otimes \Gamma_2) = \max(\Gamma_2)$.

In the first case, we obtain $[\mathcal{K} \cup (\Gamma_1 \otimes \Gamma_2)] = \max(\mathcal{K} \circ_{\text{four}} \Gamma_1)$; in the latter case, we obtain $[\mathcal{K} \cup (\Gamma_1 \otimes \Gamma_2)] = \max(\mathcal{K} \circ_{\text{four}} \Gamma_2)$.

For a violation of (G6) observe that by (G4), satisfaction of (G6) implies satisfaction of:

$$
\text{if } \Gamma_1 \models \Gamma_2 \text{ and } [((\mathcal{K} \circ_{\text{four}} \Gamma_2) \uplus \Gamma_1) \neq \emptyset, \text{ then } \mathcal{K} \circ_{\text{four}} \Gamma_1 \models (\mathcal{K} \circ_{\text{four}} \Gamma_2) \uplus \Gamma_1\]}
$$

Now choose $\Gamma_1 = \{[\geq 2]\}$ and $\Gamma_2 = \{[\geq 1]\}$, and let $\mathcal{K} = \{[\geq 4]\}$. Observe that $\Gamma_1 \models \Gamma_2$, as $[\Gamma_1] = \{2, 3\}$ and $[\Gamma_2] = \{1, 2, 3\}$. Moreover, $[\mathcal{K} \circ_{\text{four}} \Gamma_1] = \{2, 3\}$ and $[\mathcal{K} \circ_{\text{four}} \Gamma_2] = \{3\}$. Thus, we have $[\mathcal{K} \circ_{\text{four}} \Gamma_2] \uplus \Gamma_1] = \{3\} \neq \emptyset$. We obtain a contradiction to Equation (2), because $[\mathcal{K} \circ_{\text{four}} \Gamma_1] = \{2, 3\} \not\subseteq [\mathcal{K} \circ_{\text{four}} \Gamma_2] = \{3\} = [((\mathcal{K} \circ_{\text{four}} \Gamma_2) \uplus \Gamma_1]$.

To demonstrate a violation of (G5), let $\Gamma_1 = \{[\geq 0]\}$ and $\Gamma_2 = \{[\geq 1]\}$. Then by definition we have $\Gamma_1 \uplus \Gamma_2 = \{[\geq 1]\}$. Now for the base $\mathcal{K} = \{[\geq 4]\}$ observe that $\mathcal{K} \circ_{\text{four}} \Gamma_1 = \{[\geq 0]\}$ and
Moreover let \( K \circ \text{four}(\Gamma_1 \psi \Gamma_2) = \{[\geq 3]\} \). Moreover, we obtain that \((K \circ \text{four}(\Gamma_1) \psi \Gamma_2) = \{[\geq 1]\} \). In summary we obtain \([\{[\geq 1]\}] = \{1, 2, 3\} \not\subseteq \{3\} = \text{Mod}(\{[\geq 3]\})\), which shows that \( \circ \text{four} \) does not satisfy (G5).

### 10.4 On the Encoding of Operators

We will now discuss how revision operators are encoded into a preference relation. Recall that for K&M’s encoding, presented in Equation (1), the existence of a sentence \( \text{form}(\omega_1, \omega_2) \) satisfying \([\text{form}(\omega_1, \omega_2)] = \{\omega_1, \omega_2\}\) is required for any interpretations \( \omega_1, \omega_2 \) in the considered logic. The problem in a general Tarskian logical setting is that there might not be such a sentence or base.

Therefore, generalizing this idea to our case (using just the bases that do exist) bears the danger that the relation between certain pairs of elements is left undetermined: depending on the shape of the logic (and its model theory) as well as the operator, there might be no preference between certain elements (because there is no revision which provides information on the preference). We called these pairs of interpretations detached (cf. Definition 9.7). In particular, when one wants to obtain a total relation, these elements have to be ordered in a certain way, and the appropriate selection of a “preference” between these two interpretations is a “non-local” choice (as it may have ramifications for other “ordering choices”). As a solution, we came up with Definition 5.3, providing an encoding \( \preceq^{\circ_f} \) different from the approach by Katsuno and Mendelzon. This definition solves the problem with the detached pairs of interpretations by treating them as equally preferable.

This uniform treatment of all detached pairs may produce a non-preorder assignment even in cases where an encoding by means of a preorder assignment were actually possible as demonstrated next.

**Example 10.10.** Let \( \mathcal{B} = (\mathcal{L}, \Omega, \models, \mathcal{B}, \psi) \) with \( \mathcal{L} = \{\bot, \varphi, \psi, \gamma_1, \ldots, \gamma_4\} \) and \( \Omega = \{\omega_1, \ldots, \omega_4\} \), such that:

\[
\begin{align*}
\models \bot & = \{\} & \models \varphi & = \{\omega_1, \omega_2, \omega_4\} & \models \psi & = \{\omega_1, \omega_3\} & \models \gamma_i & = \{\omega_i\}
\end{align*}
\]

Moreover let \( \mathcal{B} = \{\{\chi\} \mid \chi \in \mathcal{L}\} \) and let \( \psi \) be the idempotent, commutative binary function over \( \mathcal{B} \) satisfying \( \{\varphi\} \psi \{\psi\} = \{\varphi\} \psi \{\gamma_1\} = \{\varphi\} \psi \{\gamma_1\} = \{\gamma_1\} \) and producing \( \bot \) in all other cases. Let \( \circ \) be as defined in the following operator table.

| \( \circ \) | \( \{\bot\} \) | \( \{\varphi\} \) | \( \{\psi\} \) | \( \{\gamma_1\} \) | \( \{\gamma_2\} \) | \( \{\gamma_3\} \) | \( \{\gamma_4\} \) |
|-------|---------|---------|---------|---------|---------|---------|---------|
| \( \{\bot\} \) | \( \{\bot\} \) | \( \{\varphi\} \) | \( \{\psi\} \) | \( \{\gamma_1\} \) | \( \{\gamma_2\} \) | \( \{\gamma_3\} \) | \( \{\gamma_4\} \) |
| \( \{\varphi\} \) | \( \{\bot\} \) | \( \{\varphi\} \) | \( \{\gamma_1\} \) | \( \{\gamma_1\} \) | \( \{\gamma_2\} \) | \( \{\gamma_3\} \) | \( \{\gamma_4\} \) |
| \( \{\psi\} \) | \( \{\bot\} \) | \( \{\psi\} \) | \( \{\gamma_1\} \) | \( \{\gamma_1\} \) | \( \{\gamma_2\} \) | \( \{\gamma_3\} \) | \( \{\gamma_4\} \) |
| \( \{\gamma_1\} \) | \( \{\bot\} \) | \( \{\gamma_1\} \) | \( \{\gamma_1\} \) | \( \{\gamma_2\} \) | \( \{\gamma_2\} \) | \( \{\gamma_3\} \) | \( \{\gamma_4\} \) |
| \( \{\gamma_2\} \) | \( \{\bot\} \) | \( \{\gamma_2\} \) | \( \{\gamma_1\} \) | \( \{\gamma_1\} \) | \( \{\gamma_2\} \) | \( \{\gamma_3\} \) | \( \{\gamma_4\} \) |
| \( \{\gamma_3\} \) | \( \{\bot\} \) | \( \{\gamma_3\} \) | \( \{\gamma_1\} \) | \( \{\gamma_1\} \) | \( \{\gamma_2\} \) | \( \{\gamma_3\} \) | \( \{\gamma_4\} \) |
| \( \{\gamma_4\} \) | \( \{\bot\} \) | \( \{\gamma_4\} \) | \( \{\gamma_1\} \) | \( \{\gamma_1\} \) | \( \{\gamma_2\} \) | \( \{\gamma_3\} \) | \( \{\gamma_4\} \) |

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In particular, for $\mathcal{K} = \{ \gamma_4 \}$, we thus obtain

\[
\begin{align*}
\mathcal{K} \circ \{ \varphi \} &= \{ \gamma_4 \} & [\mathcal{K} \circ \{ \varphi \}] &= \{ \omega_4 \} \\
\mathcal{K} \circ \{ \psi \} &= \{ \gamma_3 \} & [\mathcal{K} \circ \{ \psi \}] &= \{ \omega_3 \} \\
\mathcal{K} \circ \{ \gamma_i \} &= \{ \gamma_i \} & [\mathcal{K} \circ \{ \gamma_i \}] &= \{ \omega_i \}
\end{align*}
\]

Figure 8a shows that the assignment $\preceq_\circ$ derived from $\circ$ is not a preorder assignment, while Figure 8b demonstrates that such an assignment for $\circ$ indeed exists.

Still, while failing to yield preorder assignments whenever possible, Definition 5.3 is unique in another respect: $\preceq_\circ$ turns out to be the (set-inclusion-)maximal canonical representation for the preferences of an operator – a property the encoding approaches given by Equation (1) do not have.

**Proposition 10.11.** Let $\circ$ be a base change operator satisfying (G1)-(G6). If $\preceq_\circ$ is a min-friendly faithful assignment compatible with $\circ$, then $\omega_1 \preceq_\circ \omega_2$ implies $\omega_1 \preceq_\circ \omega_2$ for every $\omega_1, \omega_2 \in \Omega$ and every belief base $\mathcal{K} \in \mathcal{B}$.

**Proof.** Toward a contradiction, assume there were $\omega_1, \omega_2 \in \Omega$ such that $\omega_1 \preceq_\circ \omega_2$ but $\omega_1 \not\preceq_\circ \omega_2$ (hence, by totality $\omega_2 \prec_\circ \omega_1$).

Let us first consider the case $\omega_2 \models \mathcal{K}$. Then $\omega_2 \prec_\circ \omega_1$ and faithfulness of $\preceq_\circ$ imply $\omega_1 \not\models \mathcal{K}$. But this contradicts $\omega_1 \preceq_\circ \omega_2$, as $\preceq_\circ$ is also faithful by assumption.

It remains to consider the case $\omega_2 \not\models \mathcal{K}$. Then, by Lemma 5.6(a), there is a belief base $\Gamma$ with $\omega_1, \omega_2 \models \Gamma$ such that $\omega_2 \models \mathcal{K} \circ \Gamma$ and $\omega_1 \not\models \mathcal{K} \circ \Gamma$. Therefore, by compatibility, $\omega_2 \in \min([\Gamma], \preceq_\mathcal{K}) = [\mathcal{K} \circ \Gamma]$ and $\omega_1 \notin \min([\Gamma], \preceq_\mathcal{K}) = [\mathcal{K} \circ \Gamma]$, a contradiction to $\omega_1 \preceq_\mathcal{K} \omega_2$ due to min-retractivity.

As a last discussion item in this section, we would like to point out that the more smoothly and economically defined relation $\preceq_\circ$ (Definition 5.2) is very close to already serving the purpose of the somewhat more “tinkered” $\preceq_\circ$ (Definition 5.3). In fact, the very natural

\[
(\text{a) Relation } \preceq_\mathcal{K} \text{ for } \mathcal{K} = \{ \gamma_4 \}. \text{ No preorder as } \omega_1 \preceq_\mathcal{K} \omega_2 \text{ and } \omega_2 \preceq_\mathcal{K} \omega_3, \text{ yet } \omega_1 \not\preceq_\mathcal{K} \omega_3.
\]

\[
(\text{b) Appropriate preorder encoding } \preceq_\mathcal{K} \text{ of preference relation with respect to } \mathcal{K} = \{ \gamma_4 \} \text{ for } \circ.
\]

\[
\text{Figure 8: Illustrations of the relations used in Example 10.10.}
\]
assumption of the existence of an “non-constraining” base that covers all interpretations
makes the two relation encodings coincide. In most logics, such a base is trivially available
(for instance, the empty base).

**Proposition 10.12.** Let \( \mathcal{B} = (\mathcal{L}, \Omega, \models, \mathcal{B}, \psi) \) be a base logic and \( \circ \) be a base change operator
for \( \mathcal{B} \) satisfying (G1)-(G6). If there exists a base \( \Gamma_\Omega \in \mathcal{B} \) such that \([\Gamma_\Omega] = \Omega\), then \( \lhd_K = \preceq_K \)
for any \( K \in \mathcal{B} \), i.e. \( \omega_1 \lhd_K \omega_2 \) if and only if \( \omega_1 \preceq_K \omega_2 \) for any \( \omega_1, \omega_2 \in \Omega \).

**Proof.** Let \( \omega_1, \omega_2 \) be two interpretations and assume there exists a base \( \Gamma_\Omega \in \mathcal{B} \) such that
\([\Gamma_\Omega] = \Omega\). Then, for any \( K \in \mathcal{B} \), we have \([K \cup \Gamma_\Omega] = [K] \cap \Omega = [K]\). We show the equivalence of
\( \lhd_K \) and \( \preceq_K \) in two directions:

\(\Rightarrow\) Let \( \omega_1 \lhd_K \omega_2 \). Assume for a contradiction that \( \omega_1 \npreceq_K \omega_2 \). From Definition 5.3, we have
\( \omega_1 \models K \) and three cases: \( \omega_1 \models K \) or \( \omega_1 \npreceq_K \omega_2 \). The case \( \omega_1 \models K \) immediately
contradicts \( \omega_1 \npreceq K \) and the third case \( \omega_1 \npreceq_K \omega_2 \) contradicts our assumption \( \omega_1 \lhd_K \omega_2 \).
For the remaining case \( \omega_2 \models K \), since \( \omega_2 \in [K] = [K \cup \Gamma_\Omega] \), from postulate (G2) we
obtain \([K \circ \Gamma_\Omega] = [K \cup \Gamma_\Omega] \). Now consider from Definition 5.2 we have two subcases:

\(\omega_1 \models K \circ \Gamma_\Omega \). Since \([K \circ \Gamma_\Omega] = [K \cup \Gamma_\Omega] \), we have \( \omega_1 \in [K \cup \Gamma_\Omega] = [K] \), which
contradicts \( \omega_1 \nmodels K \).

\(\omega_2 \nmodels K \circ \Gamma_\Omega \). Since \([K \circ \Gamma_\Omega] = [K \cup \Gamma_\Omega] \), we have \( \omega_2 \nmodels [K \cup \Gamma_\Omega] \), and hence \( \omega_2 \nmodels [K] \),
which contradicts our case assumption \( \omega_2 \models K \).

\(\Leftarrow\) Let \( \omega_1 \preceq_K \omega_2 \). In view of Definition 5.3, we consider two cases: \( \omega_1 \models K \) or \( \omega_1, \omega_2 \nmodels K \)
and \( \omega_1 \lhd_K \omega_2 \). The second case immediately yields the desired \( \omega_1 \lhd_K \omega_2 \). For the
former case, \( \omega_1 \models K \), assume for a contradiction \( \omega_1 \npreceq_K \omega_2 \). Then, there exists \( \Gamma \in \mathcal{B} \)
with \( \omega_1, \omega_2 \in [\Gamma] \) such that \( \omega_1 \nmodels K \circ \Gamma \) and \( \omega_2 \models K \circ \Gamma \). Since \( \omega_1 \models [K] \cap [\Gamma] = [K \cup \Gamma] \),
from postulate (G2) we have \([K \circ \Gamma] = [K \cup \Gamma] \). This implies \( \omega_1 \in [K \circ \Gamma] \), which
contradicts \( \omega_1 \nmodels K \circ \Gamma \). \(\square\)

11. Related Work

In settings beyond propositional logic, we are aware of three closely related approaches
that propose model-based frameworks for revision of belief bases (or sets) without fixing a
particular logic or the internal structure of interpretations, and characterize revision operators
via minimal models à la K&M with some additional assumptions. In the following, we discuss
these results and their relationship to our approach. Table 1 summarizes the three approaches
and compares them with K&M’s result and our approach, which comes in 4 variants.

**Grove (1988).** One semantic-based approach related to the one of K&M was proposed by
Grove (1988) in the setting of Boolean-closed logics. Instead of a preorder relation \( \preceq_K \), he
originally characterized AGM revision operators via *systems of spheres*, collections \( \mathcal{S} \) of sets
of interpretations satisfying certain conditions. The notion of a system of spheres is closely
related to that of a faithful preorder relation \( \preceq_K \) as the latter can be generated from the
former (Gärdenfors & Rott, 1995). Given a faithful preorder \( \preceq_K \), for each \( \omega \in \Omega \), one can
define \( S_\omega \) as a set of interpretations \( \omega' \) such that \( \omega' \preceq_K \omega \). However, the set \( \mathcal{S} \) of all such sets
might not satisfy min-completeness in general (Grove (1988) denotes min-completeness with
| logic setting | belief bases | assignment | postulates | relation encoding | notes |
|---------------|--------------|------------|------------|------------------|-------|
| propositional logic over finite signature | $\mathcal{P}(\mathcal{L})$, $\mathcal{P}_{\text{fin}}(\mathcal{L})$, $\mathcal{L}$ | preorder, faithful | (G1)-(G6) | Eq. (1) | logic natively free of critical loops; natively min-friendly; two-way representation theorem |
| Boolean-closed logics that are $\Omega$-expressible | $\mathcal{P}(\mathcal{L})$ | preorder, faithful, min-complete | (G1)-(G6) | Eq. (1) | all such logics natively free of critical loops; any assignment min-expressible; two-way representation theorem |
| Tarskian logics with finite $\Omega$, any $\omega, \omega'$ distinguishable by some sentence | $\mathcal{P}(\mathcal{L})$ | preorder, faithful, min-complete | (G1)-(G6), (Acyc) | Eq. (3) | extra postulate (Acyc) rules out “non-preorder operators”; two-way representation theorem |
| Tarskian logics | $\mathcal{P}(\mathcal{L})$ | quasi-faithful, min-complete | (G1)-(G3), (G5)-(G6) | Eq. (4) | non-standard notion of inconsistency; additional ad-hoc constraint on compatibility; one-way representation theorem |
| our approach | | faith, min-complete, min-retractive, min-expressible | (G1)-(G6) | Def. 5.3 | most general (syntax-independent version); two-way representation theorem (Theorem 6.3) |
| | | quasi-faithful, min-complete, min-retractive, min-expressible | (G1)-(G3), (G5)-(G6) | Def. 5.3 | most general (syntax-dependent version); two-way representation theorem (Theorem 8.6) |
| | | preorder, faithful, min-complete, min-expressible | (G1)-(G6) | Def. 5.3 | preorder preference (syntax-independent version); two-way representation theorem (Theorem 9.16) |
| | | preorder, quasi-faithful, min-complete, min-expressible | (G1)-(G3), (G5)-(G6) | Def. 5.3 | preorder preference (syntax-dependent version); two-way representation theorem (Theorem 9.17) |

Table 1: Overview of our characterization results and comparison with related work.
Delgrande and colleagues (2018) then reformulated Grove’s representation theorem stating that (expressed in our terminology) any AGM revision operator can be obtained from a compatible min-complete faithful preorder assignment, provided the set of interpretations is $\Omega$-expressible, i.e. for any subset $\Omega' \subseteq \Omega$ there exists a base $\Gamma$ such that $[\Gamma] = \Omega'$. In this formulation, Grove’s result also holds for logics with infinite $\Omega$.

Grove’s result constitutes a special case of our representation theorem: from the assumption of Boolean-closedness, it follows that the considered logics are disjunctive and therefore free of critical loops (cf. Theorem 9.4 and Corollary 9.18). The assumption of $\Omega$-expressibility immediately implies min-expressibility for all relations. In the light of these observations, Grove’s result turns out to be a special case of the third variant of our result in Table 1.

Delgrande et al. (2018). The representation result of Delgrande et al. (2018) confines the considered logics to those where the set $\Omega$ of interpretations (or possible worlds) is finite and where any two different interpretations $\omega, \omega' \in \Omega$ can be distinguished by some sentence $\varphi \in \mathcal{L}$, i.e., $\omega \in [\varphi]$ and $\omega' \notin [\varphi]$. Moreover, they extend the AGM postulates by the following extra one, denoted (Acyc):

For any base $\mathcal{K}$ and all $\Gamma_1, \ldots, \Gamma_n \in \mathcal{P}(\mathcal{L})$ with $[\Gamma_i \cup \mathcal{K} \circ \Gamma_{i+1}] \neq \emptyset$ for each $1 \leq i < n$ as well as $[\Gamma_n \cup \mathcal{K} \circ \Gamma_1] \neq \emptyset$ holds $[\Gamma_1 \cup \mathcal{K} \circ \Gamma_n] \neq \emptyset$.

With these ingredients in place, Delgrande et al. (2018) establish that, for the logics they consider, there is a two-way correspondence between those AGM revision operators satisfying (Acyc) and min-expressible faithful preorder assignments. Instead of the term “min-expressible”, they use the term regular.

The approach of Delgrande et al. can be seen as complementary to ours. As we saw before, in logics exhibiting critical loops, one cannot hope for a characterization of all AGM revision operators by means of assignments producing preorders. Our proposal is to relinquish the requirement of using preorders, giving up transitivity and merely retaining min-retractivity. As an alternative to this approach, one might argue that those AGM revision operators not corresponding to some preorder assignment are somewhat “unnatural” and should be ruled out from the consideration. The additional postulate (Acyc) serves precisely this purpose: it allows for a preorder characterization even in logics with critical loops, by disallowing some “unnatural” AGM revision operators.

Aiguier et al. (2018). The approach of Aiguier et al. (2018) considers AGM belief base revision in logics with a possibly infinite set $\Omega$ of interpretations. Notably, they propose to consider certain bases, that actually do have models, as inconsistent (and thus in need of revision). While, in our view, this is at odds with the foundational assumptions of belief revision (revision should be union/conjunction unless facing unsatisfiability), this appears to be a design choice immaterial to the established results. To avoid confusion, we will ignore it in our presentation. As far as the postulates are concerned, Aiguier et al. decide to rule out (KM4)/(G4), arguing in favor of syntax-dependence. Consequently, they re-define the notion of faithfulness of assignments, eliminating (F3), and arriving at the notion that we call

8. Note that this precondition excludes more complex logics such as first-order or modal logics and most of their fragments, but also propositional logic with infinite signature. On the positive side, this choice guarantees min-completeness of any preorder.
quasi-faithfulness. Like us, Aiguier et al. propose to drop the requirement that assignments have to yield preorders. Also, their representation result (Theorem 1) features a condition that corresponds to min-completeness (second bullet point). In addition to the standard notion of compatibility, their result hinges on the following additional correspondence between the assignment and the preorder (third bullet point), for every $\mathcal{K}, \Gamma_1, \Gamma_2 \in \mathcal{P}(L)$:

$$\text{If } (\mathcal{K} \circ \Gamma_1) \cup \Gamma_2 \text{ is consistent, then } \min([\Gamma_1], \preceq_{\mathcal{K}}) \cap [\Gamma_2] = \min([\Gamma_1 \cup \Gamma_2], \preceq_{\mathcal{K}}).$$

A closer inspection of this extra condition shows that it is essentially a translation of a combination of the postulates (G5) and (G6) into the language of assignments and minima. It remains somewhat unclear to us what the intuitive justification of this (arguably rather technical and unwieldy) extra condition should be, beyond providing the missing ingredient to make the result work. Possibly, this is the reason why the presented result is just one-way: it does not provide a characterization of exactly those assignments for which a compatible AGM revision operator exists. Rather it pre-assumes existence of a revision operator under consideration.

We think that our approach provides improvements regarding ways to construct an appropriate assignment from a given belief revision operator. For comparison, Delgrande et al. (2018) solve this problem by simultaneously revising with all sentences satisfying $\omega_1$ and $\omega_2$, in order to “simulate” the revision by the desired formula $\text{form}(\omega_1, \omega_2)$: for every base $\mathcal{K}$,

$$\preceq_{\mathcal{K}} \text{ is the transitive closure of } \{(\omega_1, \omega_2) \mid \omega_1 \models \mathcal{K} \circ \{t(\omega_1) \cap t(\omega_2)\}\}$$

where $t(\{\omega\})$ is the set of all sentences satisfied by $\omega$. Aiguier et al. (2018) use a similar approach by revising with all sentences at once: they let, for every base $\mathcal{K}$,

$$\omega_1 \preceq_{\mathcal{K}} \omega_2 \text{ if } \omega_1 \models \mathcal{K} \text{ or } \omega_1 \models \mathcal{K} \circ \{\omega_1, \omega_2\}^*$$

where $\{\omega_1, \omega_2\}^*$ is the set of all sentences satisfied by both $\omega_1$ and $\omega_2$. In summary, neither Aiguier et al. nor Delgrande et al. use an encoding approach in the spirit of Katsuno and Mendelzon, attempting to establish a relation between two interpretations, whenever the revision operator provides evidence. As discussed in Section 10.4, we take a somewhat dual approach: we establish a relation between any two worlds, unless the considered revision operator provides evidence to the contrary.

12. Summary and Conclusion
The central objective of our treatise was to provide an exact model-theoretic characterization of AGM belief revision in the most general reasonable sense, i.e., one that uniformly applies

- to every logic with a classical model theory (i.e., every Tarskian logic),
- to any notion of bases that allows for taking some kind of “unions” (including the cases of belief sets, sets of sentences, finite sets of sentences, and single sentences), and
- to all base change operators adhering to the unaltered AGM postulates (without imposing further restrictions through additional postulates).
To this end, we followed the well-established approach of using assignments: functions that map every base $K$ to a binary relation $\preceq_K$ over the interpretations $\Omega$ of the considered logic, where $\omega_1 \preceq_K \omega_2$ intuitively expresses a preference, i.e., that $\omega_1$ is “closer” than (or at least as close as) $\omega_2$ to being a model of $K$ (even if it is not a proper model). With this notion in place, the compatibility between a revision operator and an assignment then requires that the result of revising a base $K$ by some base $\Gamma$ yields a base whose models are the $\preceq_K$-minimal interpretations among the models of $\Gamma$.

The original result of K&M for signature-finite propositional logic, establishing a two-way correspondence between AGM revision operators and faithful assignments that yield total preorders. We showed that in the general case considered by us, this original result fails in many ways and needs substantial adaptations. In particular, aside from delivering total relations and being faithful, the assignment now needs to satisfy

- **min-expressibility**, guaranteeing existence of a describing base for any model set obtained by taking minimal interpretations among some base’s models,
- **min-completeness**, ensuring that minimal interpretations exist in every base’s model set, and
- **min-retractivity** instead of transitivity, making sure that minimality is inherited to more preferable elements.

While the first two adjustments have been recognized and described in prior work, the notion of min-retractivity (and the decision to replace transitivity by this weaker notion and thus give up on the requirement that preferences be preorders) seems to be novel. Yet, it turns out to be the missing piece for establishing the desired two-way compatibility-correspondence between AGM revision operators and preference assignments of the described kind (cf. Theorem 6.3).

In view of the fact that the requirement of syntax-independence – as expressed in Postulate (G4) – may be (and has been) put into question, we also established a syntax-dependent version of our characterization (cf. Theorem 8.6). Crucial to this result is the observation that (G4) is exactly mirrored by the third faithfulness condition on the semantic side; thus removing it (going from faithfulness to quasi-faithfulness) yields the class of assignments compatibility-corresponding to revision operators satisfying the postulates (G1)–(G3), (G5), and (G6).

Conceding that transitivity is a rather natural choice for preferences and preorder assignments might be held dear by members of the belief revision community, we went on to investigate for which logics our general result holds even if assignments are required to yield preorders. We managed to pinpoint a specific logical phenomenon (called critical loop), the absence of which in a logic is necessary and sufficient for total-preorder-representability. While the criterion by itself maybe somewhat technical and unwieldy, it can be shown to subsume all logics featuring disjunction and therefore all classical logics. This justifies to formulate these findings in two theorems: a syntax independent version (Theorem 9.16) and a syntax-dependent one (Theorem 9.17).

As one of the avenues for future work, we will consider the largely untreated problem of iterated base revision. To this end, our aim is to combine the general setting of base logics
presented here with work in the line of research by Darwiche and Pearl (1997). In particular, we will explore the specific role of a non-transitive, yet min-retractive, relation, in the process of iteration.

Finally, we will also be working on concrete realisations of the approach presented here in popular KR formalisms such as ontology languages, including Description Logics (DLs), both under standard and non-standard semantics. With our results in place, we will be able to scrutinize the underlying logics for the existence of critical loops and find uniform ways of describing AGM revision by means of appropriate assignments. We also plan to arrive at more fine-grained representation theorems that take computability and complexity aspects into account.

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Appendix A. Tarskian Logics and Model Theory

In the following, we show that the class of logics defined model-theoretically (as laid out in Section 2.1) and the class of Tarskian logics coincide. We start by providing the definition of Tarskian logics.

Definition A.1. Let \( \mathcal{L} \) be a set. A function \( Cn : \mathcal{P}(\mathcal{L}) \to \mathcal{P}(\mathcal{L}) \) is a called a Tarskian consequence operator (on \( \mathcal{L} \)) if it is a closure operator, i.e., it satisfies the following properties for all subsets \( \mathcal{K}, \mathcal{K}_1, \mathcal{K}_2 \subseteq \mathcal{L} \):

\[
\begin{align*}
\mathcal{K} &\subseteq Cn(\mathcal{K}) \quad \text{(extensive)} \\
\text{if } \mathcal{K}_1 \subseteq \mathcal{K}_2, \text{ then } Cn(\mathcal{K}_1) &\subseteq Cn(\mathcal{K}_2) \quad \text{(monotone)} \\
Cn(\mathcal{K}) &= Cn(Cn(\mathcal{K})) \quad \text{(idempotent)}
\end{align*}
\]

Any Tarskian consequence operator \( Cn : \mathcal{P}(\mathcal{L}) \to \mathcal{P}(\mathcal{L}) \) gives rise to a Tarskian consequence relation \( \models \subseteq \mathcal{P}(\mathcal{L}) \times \mathcal{L} \) defined by \( \mathcal{K} \models \varphi \) if \( \varphi \in Cn(\mathcal{K}) \). Each \( (\mathcal{L}, \models) \) obtained from a Tarskian consequence operator \( Cn : \mathcal{P}(\mathcal{L}) \to \mathcal{P}(\mathcal{L}) \) will be called a Tarskian logic here.

We proceed to show that the existence of a model-theoretically defined semantics is sufficient and necessary for a logic being Tarskian.

Proposition A.2. For every model theory \( (\mathcal{L}, \Omega, \models) \) there exists a Tarskian logic \( (\mathcal{L}, \models) \) with \( \mathcal{K} \models \varphi \) if and only if \( \mathcal{K} \models \varphi \) for all \( \varphi \in \mathcal{L} \) and \( \mathcal{K} \in \mathcal{P}(\mathcal{L}) \).

Proof. Given \( (\mathcal{L}, \Omega, \models) \), let \( Cn : \mathcal{P}(\mathcal{L}) \to \mathcal{P}(\mathcal{L}) \) be defined by \( \mathcal{K} \mapsto \{ \varphi \in \mathcal{L} \mid [\mathcal{K}] \subseteq [\varphi] \} \). We will show that \( Cn \) is a Tarskian consequence operator.

For extensivity, consider some arbitrary \( \psi \in \mathcal{K} \). Then we obtain \([\mathcal{K}] = \bigcap_{\varphi \in \mathcal{K}} [\varphi] \subseteq [\psi]\) and hence \( \psi \in Cn(\mathcal{K}) \). Hence, since \( \psi \) was chosen arbitrarily, we obtain \( \mathcal{K} \subseteq Cn(\mathcal{K}) \).

For monotonicity, suppose \( \mathcal{K}_1 \subseteq \mathcal{K}_2 \). Then \([\mathcal{K}_2] = \bigcap_{\varphi \in \mathcal{K}_2} [\varphi] \subseteq \bigcap_{\varphi \in \mathcal{K}_1} [\varphi] = [\mathcal{K}_1]\). Therefore, we obtain \( Cn(\mathcal{K}_1) = \{ \varphi \in \mathcal{L} \mid [\mathcal{K}_1] \subseteq [\varphi] \} \subseteq \{ \varphi \in \mathcal{L} \mid [\mathcal{K}_1] \subseteq [\varphi] \} = Cn(\mathcal{K}_2) \).

For idempotency, we show bidirectional inclusion. \( Cn(\mathcal{K}) \subseteq Cn(Cn(\mathcal{K})) \) is an immediate consequence of extensivity already shown. For the other direction, consider an arbitrary \( \psi \in Cn(Cn(\mathcal{K})) \). Then, we obtain \([Cn(\mathcal{K})] \subseteq [\psi] \). On the other hand, we have

\[
[Cn(\mathcal{K})] = \bigcap_{\varphi \in \mathcal{L}} [\varphi] = \bigcap_{\varphi \in \mathcal{K}} [\varphi] = [\mathcal{K}],
\]

and therefore, we obtain \([\mathcal{K}] \subseteq [\psi] \) and finally \( \psi \in Cn(\mathcal{K}) \). Hence, since \( \psi \) was chosen arbitrarily, we obtain \( Cn(Cn(\mathcal{K})) \subseteq Cn(\mathcal{K}) \).

Let now \( \models \) denote the Tarskian consequence relation induced by \( Cn \). Then we obtain for all \( \mathcal{K} \subseteq \mathcal{L} \) and \( \varphi \in \mathcal{L} \) the following:

\[
\mathcal{K} \models \varphi \iff \varphi \in Cn(\mathcal{K}) \iff [\mathcal{K}] \subseteq [\varphi] \iff \mathcal{K} \models \varphi.
\]

As last step, we show that for each Tarskian logic there is a canonical model-theoretic semantics for this Tarskian logic.
**Proposition A.3.** For every Tarskian logic \((\mathcal{L}, \models)\) there exists a model theory \((\mathcal{L}, \Omega, \models)\) such that \(\mathcal{K} \models \varphi\) if and only if \(\mathcal{K} \models \varphi\) holds for all \(\varphi \in \mathcal{L}\) and \(\mathcal{K} \in \mathcal{P}(\mathcal{L})\).

**Proof.** Let \((\mathcal{L}, \models)\) be a Tarskian logic and let \(\text{Cn} : \mathcal{P}(\mathcal{L}) \to \mathcal{P}(\mathcal{L})\) be the corresponding Tarskian consequence operator. We now define an appropriate \((\mathcal{L}, \Omega, \models)\) as follows: Let \(\Omega = \{\text{Cn}(T) \mid T \subseteq \mathcal{L}\}\). Define the models relation \(\models \subseteq \Omega \times \mathcal{L}\) such that some \(\text{Cn}(T) \in \Omega\) is a model of some \(\varphi \in \mathcal{L}\) whenever \(\varphi \in \text{Cn}(T)\).

Then we obtain for all \(\mathcal{K} \subseteq \mathcal{L}\) and \(\varphi \in \mathcal{L}\) the following:

\[
\begin{align*}
\mathcal{K} \models \varphi & \iff [\mathcal{K}] \subseteq [\varphi] \\
& \iff \bigcap_{\kappa \in \mathcal{K}} [\kappa] \subseteq [\varphi] \\
& \iff \{\text{Cn}(T) \mid T \subseteq \mathcal{L}, \mathcal{K} \subseteq \text{Cn}(T)\} \subseteq \{\text{Cn}(T) \mid T \subseteq \mathcal{L}, \varphi \in \text{Cn}(T)\} \\
& \iff \forall T \subseteq \mathcal{L} : \mathcal{K} \subseteq \text{Cn}(T) \Rightarrow \varphi \in \text{Cn}(T) \quad \text{(\#)}
\end{align*}
\]

Moreover, we obtain

\[
\begin{align*}
(\#) & \Rightarrow \mathcal{K} \subseteq \text{Cn}(\mathcal{K}) \Rightarrow \varphi \in \text{Cn}(\mathcal{K}) & \text{instantiate } T = \mathcal{K} \\
& \Rightarrow \varphi \in \text{Cn}(\mathcal{K}) & \text{extensivity of Cn} \\
& \Rightarrow \mathcal{K} \models \varphi,
\end{align*}
\]

and on the other hand:

\[
\begin{align*}
\mathcal{K} \models \varphi & \Rightarrow \varphi \in \text{Cn}(\mathcal{K}) \\
& \Rightarrow \forall S \subseteq \mathcal{L} : \text{Cn}(\mathcal{K}) \subseteq S \Rightarrow \varphi \in S \\
& \Rightarrow \forall T \subseteq \mathcal{L} : \text{Cn}(\mathcal{K}) \subseteq \text{Cn}(T) \Rightarrow \varphi \in \text{Cn}(T) & \text{restriction to closed sets} \\
& \Rightarrow \forall T \subseteq \mathcal{L} : \text{Cn}(\mathcal{K}) \subseteq \text{Cn}(\text{Cn}(T)) \Rightarrow \varphi \in \text{Cn}(T) & \text{idempotency of Cn} \\
& \Rightarrow (\#) & \text{monotonicity of Cn}
\end{align*}
\]

Concluding, we have established that for all \(\mathcal{K} \subseteq \mathcal{L}\) and \(\varphi \in \mathcal{L}\) the following holds:

\[
\mathcal{K} \models \varphi \iff (\#) \iff \mathcal{K} \models \varphi.
\]

\(\square\)
Appendix B. Proof for Example 4.2

We take up again Example 4.2 and show that $\preceq_{K}^{\omega}$ is a faithful preorder assignment that is compatible with $\omega^{\cup}$. This shows that Theorem 3.3 by K&M does not straightforwardly generalize to $\mathbb{P}_{\omega_{\infty}}$, i.e., propositional logic with countably infinite many atoms.

**Proposition B.1.** The relation $\preceq_{K}^{\omega}$ is a faithful preorder assignment and is compatible with the base change operator $\omega^{\omega}$ for $\mathbb{P}_{\omega_{\infty}}$, yet $\omega^{\omega}$ does not satisfy (G3).

**Proof.** We show that $\preceq_{K}^{\omega}$ is a preorder assignment.

*(Totality)* For totality, assume the contrary, i.e. there are two interpretations $\omega_{1}, \omega_{2}$ with $\omega_{1} \not\preceq_{K}^{\omega} \omega_{2}$ and $\omega_{2} \not\preceq_{K}^{\omega} \omega_{1}$. From the definition of $\preceq_{K}^{\omega}$, we have $\omega_{1}, \omega_{2} \not\models K$ where both $\omega_{1}$ and $\omega_{2}$ are finite with $|\omega_{1}^{\text{true}}| \not\geq |\omega_{2}^{\text{true}}|$ and $|\omega_{2}^{\text{true}}| \not\geq |\omega_{1}^{\text{true}}|$. Since $\geq$ is total over integers, this is a contradiction. Reflexivity follows from totality.

*(Transitivity)* For transitivity, suppose $\omega_{1} \preceq_{K}^{\omega} \omega_{2}$ and $\omega_{2} \preceq_{K}^{\omega} \omega_{3}$. We make a case distinction by $\omega_{1} \preceq_{K}^{\omega} \omega_{2}$ and the definition of $\preceq_{K}^{\omega}$:

1. The case of $\omega_{1} \models K$. Then $\omega_{1} \preceq_{K}^{\omega} \omega_{3}$ follows immediately.
2. The case of $\omega_{2} \not\models K$ and $\omega_{2}^{\text{true}}$ is infinite. As $\omega_{2} \preceq_{K}^{\omega} \omega_{3}$, we consider three subcases:
   1. $\omega_{2} \models K$. This contradicts the prior assumption, and hence this case is not possible.
   2. $\omega_{3} \not\models K$ with infinite $\omega_{3}^{\text{true}}$. Then $\omega_{1} \preceq_{K}^{\omega} \omega_{3}$ follows.
   3. $\omega_{2}^{\text{true}}$ and $\omega_{3}^{\text{true}}$ are finite. This is also impossible due to immediate contradiction.
3. The case of $\omega_{1}, \omega_{2} \not\models K$, both $\omega_{1}^{\text{true}}$ and $\omega_{2}^{\text{true}}$ are finite and $|\omega_{1}^{\text{true}}| \geq |\omega_{2}^{\text{true}}|$. From $\omega_{2} \preceq_{K}^{\omega} \omega_{3}$ we consider three subcases:
   1. $\omega_{2} \models K$. This is not possible, immediate contradiction.
   2. $\omega_{3} \not\models K$ with infinite $\omega_{3}^{\text{true}}$. This implies $\omega_{1} \preceq_{K}^{\omega} \omega_{3}$.
   3. $\omega_{2}, \omega_{3} \not\models K$, both $\omega_{2}^{\text{true}}$ and $\omega_{3}^{\text{true}}$ are finite with $|\omega_{2}^{\text{true}}| \geq |\omega_{3}^{\text{true}}|$. Since $|\omega_{1}^{\text{true}}| \geq |\omega_{2}^{\text{true}}|$ and $|\omega_{2}^{\text{true}}| \geq |\omega_{3}^{\text{true}}|$, from transitivity of $\geq$ over integers, we have $|\omega_{1}^{\text{true}}| \geq |\omega_{3}^{\text{true}}|$ and finally $\omega_{1} \preceq_{K}^{\omega} \omega_{3}$.

We show that $\preceq_{K}^{\omega}$ is faithful and that $\preceq_{(\omega)}$ is compatible with $\omega^{\omega}$.

*(Faithfulness)* The first condition of faithfulness, the Condition (F1), follows from the assumption $\omega_{1}, \omega_{2} \models K$ and case (1) of the definition of $\preceq_{K}^{\omega}$, given in Example 4.2.

For (F2), let $\omega_{1} \models K$ and $\omega_{2} \not\models K$. From the case (1) of the definition, $\omega_{1} \preceq_{K}^{\omega} \omega_{2}$ holds. Now for contradiction assume that $\omega_{2} \preceq_{K}^{\omega} \omega_{1}$. Following the definition of $\preceq_{K}^{\omega}$, we consider three cases. (Case 1) $\omega_{2} \models K$ contradicts our assumption. The (Case 2) and (Case 3) are not applicable because they require $\omega_{1} \not\models K$. Hence, $\omega_{2} \not\preceq_{K}^{\omega} \omega_{1}$ and therefore $\omega_{1} \not\preceq_{K}^{\omega} \omega_{2}$ holds.

For (F3), assume $K \equiv K'$ (i.e. $[K] = [K']$) and let $\omega_{1} \preceq_{K}^{\omega} \omega_{2}$. We consider three cases. (Case 1) $\omega_{1} \models K$. Then it also holds $\omega_{1} \models K'$, and hence $\omega_{1} \preceq_{K'}^{\omega} \omega_{2}$. (Case 2) $\omega_{2} \not\models K$ and $\omega_{2}^{\text{true}}$ is infinite. Then $\omega_{2} \not\preceq_{K'}^{\omega}$ and hence $\omega_{1} \preceq_{K'}^{\omega} \omega_{2}$. (Case 3) where $\omega_{1}, \omega_{2} \not\models K$ we also
have $\omega_1, \omega_2 \not\models K'$ and consequently $\omega_1 \preceq_{K'} \omega_2$. Therefore, we have $\preceq^u_K = \preceq_{K'}^u$ (i.e. $\omega_1 \preceq^u_K \omega_2$ if and only if $\omega_1 \preceq_{K'} \omega_2$).

(Compatibility with $\circ$) For the compatibility with $\circ^u$, we show that $[K \circ^u \Gamma] = \min([\Gamma], \preceq_{K}^u)$. For any inconsistent $\Gamma$, we have $[K \circ^u \Gamma] = 0 = \min([\Gamma], \preceq_{K}^u)$. If $K \cup \Gamma$ is consistent, then we have $[K \circ \Gamma] = [K \cup \Gamma]$. Because $\preceq_{K}^u$ is faithful, we directly obtain $[K \circ \Gamma] = \min([\Gamma], \preceq_{K}^u)$.

Thus, for the remaining steps of the proof, we assume that $K \cup \Gamma$ is inconsistent and $\Gamma$ is consistent.

We show in the following that $\min([\Gamma], \preceq_{K}^u) = \emptyset$ holds by contradiction, i.e., there exists some $\omega_1 \in \min([\Gamma], \preceq_{K}^u)$. This means, that $\omega_1 \in [\Gamma]$ and there is no other $\omega_2 \in [\Gamma]$ such that $\omega_2 \prec_{K}^u \omega_1$. Note that from the definition of $\circ^u$ and our case assumption, we have $[K \circ^u \Gamma] = [K \cup \Gamma] = 0$, and hence $\omega_1, \omega_2 \not\models K$. Let $\Sigma_{\Gamma} \subseteq \Sigma$ be a set of atomic symbols occurring in $\Gamma$. Clearly, because $\Gamma$ is finite, we have that $\Sigma_{\Gamma}$ contains finitely many atoms.

We have two cases: $\omega_1^{\text{true}}$ can be finite or infinite.

($\omega_1^{\text{true}}$ is finite) Then, there exists an atomic symbol $q$ such that $q \in \Sigma \setminus (\omega_1^{\text{true}} \cup \Sigma_{\Gamma})$ (as both $\omega_1^{\text{true}}$ and $\Sigma_{\Gamma}$ are finite and $\Sigma$ is infinite). Then we could define another interpretation $\omega_2$ such that $\omega_2(q) = \text{true}$ and $\omega_2(p_i) = \omega_1(p_i)$ for all $p_i \in \Sigma \setminus \{q\}$. Since $q$ does not occur in $\Gamma$, we have $\omega_2 \in [\Gamma]$ and $|\omega_2^{\text{true}}| = |\omega_1^{\text{true}}| + 1$. Hence, $\omega_2 \prec_{K}^u \omega_1$, a contradiction to the minimality of $\omega_1$.

($\omega_1^{\text{true}}$ is infinite) We define another interpretation $\omega_2$ such that for all $p_i \in \Sigma$ we set $\omega_2(p_i) = \text{true}$ if $p_i \in (\Sigma_{\Gamma} \cap \omega_1^{\text{true}})$ and $\omega_2(p_i) = \text{false}$ otherwise. As $\omega_1$ and $\omega_2$ coincide on $\Sigma_{\Gamma}$, we obtain $\omega_2 \in [\Gamma]$. Since $\omega_2^{\text{true}}$ is finite while $\omega_1^{\text{true}}$ is infinite, we have $\omega_2 \prec_{K}^u \omega_1$, which again is a contradiction to the minimality of $\omega_1$. \qed
Appendix C. Detailed Proofs for Section 9.3

In this section, we present full proofs for Lemma 9.8 and Lemma 9.10 from Section 9.3. We start with Lemma 9.8.

**Lemma 9.8.** Assume \( B \) is a base logic which does not admit a critical loop and \( \circ \) a base change operator for \( B \) which satisfies (G1)-(G3), (G5), and (G6). If \( \omega_0 \preceq_K^o \omega_1 \) and \( \omega_1 \preceq_K^o \omega_2 \) with \( \omega_0 \preceq_K^o \omega_2 \), then \( (\omega_0, \omega_1) \) or \( (\omega_1, \omega_2) \) is detached from \( \circ \) in \( K \).

*Proof.* Let \( \omega_0, \omega_1, \omega_2 \) such that a violation of transitivity is obtained as given above, i.e. \( \omega_0 \preceq_K \omega_1 \) and \( \omega_1 \preceq_K \omega_2 \) with \( \omega_0 \preceq_K \omega_2 \). By Definition 5.3, we have that \( \omega_0 \preceq_K \omega_2 \) is only possible if \( \omega_0 \not\models K \). From Definition 5.3 and \( \omega_0 \preceq_K \omega_1 \), we obtain \( \omega_1 \not\models K \). By an analogue argument we obtain \( \omega_2 \not\models K \). Thus, for the rest of the proof we have \( \omega_0, \omega_1, \omega_2 \not\models K \).

Towards a contradiction, assume that \( (\omega_0, \omega_1) \) and \( (\omega_1, \omega_2) \) are both not detached from \( \circ \) in \( K \). By Lemma 5.5 the relation \( \preceq_K^o \) is total, and thus we have that \( \omega_2 \preceq_K \omega_0 \). As \( \omega_2 \not\models K \) and \( \omega_0 \preceq_K \omega_2 \), due to Lemma 5.6(a), there is a base \( \Gamma_{2,0} \in B \) with \( \omega_0, \omega_2 \models \Gamma_{2,0} \) such that \( \omega_2 \models K \circ \Gamma_{2,0} \) and \( \omega_0 \not\models K \circ \Gamma_{2,0} \). By \( \omega_0, \omega_1, \omega_2 \not\models K \) and Definition 5.3 we obtain \( \omega_0 \preceq_K \omega_1 \) and \( \omega_1 \preceq_K \omega_2 \) (cf. Definition 5.2). Because \( (\omega_0, \omega_1) \) is not detached, there is some \( \Gamma_{0,1} \in B \) with \( \omega_0, \omega_1 \models \Gamma_{0,1} \) such that \( \omega_0 \models K \circ \Gamma_{0,1} \) or \( \omega_1 \models K \circ \Gamma_{0,1} \). By Definition 5.2 and \( \omega_0 \preceq_K \omega_1 \) we obtain that \( \omega_0 \models K \circ \Gamma_{0,1} \). Using an analogue argumentation, there exist \( \Gamma_{1,2} \in B \) satisfying \( \omega_1, \omega_2 \models \Gamma_{1,2} \) and \( \omega_1 \models K \circ \Gamma_{1,2} \).

Recall that \( \preceq_K^o \) is compatible, min-retractive and quasi-faithful by Lemma 5.7 and by the proof of Lemma 8.4. Let \( \Gamma_i = (K \circ \Gamma_{i,i} \uplus \Gamma_{i,j=1,2}) \) for each \( i \in \{0,1,2\} \). Note that each \( \Gamma_i \) is a consistent base, since we have \( \omega_i \in [\Gamma_i] \). We now show that Conditions (1) and Condition (2) from Definition 9.3 are satisfied:

1. Towards a contradiction, assume that \( K \) is consistent with some \( \Gamma_{i,i} \). From (G2) we obtain \( [K \circ \Gamma_{i,i} \uplus \Gamma_{i,j=1,2}] = [K \uplus \Gamma_{i,j=1,2}] \) for some \( i \in \{0,1,2\} \). Since \( \omega_i \in [\Gamma_i] \), by the definition of \( \Gamma_i \) we have \( \omega_i \in [K \circ \Gamma_{i,i} \uplus \Gamma_{i,j=1,2}] = [K \uplus \Gamma_{i,j=1,2}] \) and obtain \( \omega_i \in [K] \), which contradicts \( \omega_0, \omega_1, \omega_2 \not\models K \).

2. By the postulate (G1) we have \( [K \circ \Gamma_{i,i} \uplus \Gamma_{i,j=1,2}] \subseteq [\Gamma_{i,j=1,2}] \) for each \( i \in \{0,1,2\} \). The definition of \( \Gamma_j \) yields \( [\Gamma_j] \subseteq [\Gamma_{i,j=1} \uplus \Gamma_{i,j=2}] \) for each \( i \in \{0,1,2\} \). Substituting \( i \) by \( i \pm 1 \) yields \( [\Gamma_{i+1}] \subseteq [\Gamma_{i+1,j=1} \uplus \Gamma_{i+1,j=2}] \); showing that \( [\Gamma_i] \cup [\Gamma_{i+1}] \subseteq [\Gamma_{i,j=1,2}] \) holds for each \( i \in \{0,1,2\} \).

We show that each \( \Gamma_i \uplus \Gamma_j \) is inconsistent, by assuming the contrary, i.e., there are some \( i, j \in \{0,1,2\} \) such that \( i \neq j \) and \( \Gamma_i \uplus \Gamma_j \) is consistent, i.e. there exists some \( \omega^* \in [\Gamma_i] \cap [\Gamma_j] \). From the definition of \( \Gamma_i \) and the definition of \( \Gamma_j \), we obtain \( \omega^* \in [K \circ \Gamma_{i,i} \uplus \Gamma_{i,j} \downarrow \Gamma_{i,j=2}] \cap [K \circ \Gamma_{j,j} \downarrow \Gamma_{i,j=2} \downarrow \Gamma_{i,j=2,j}] \). Hence, we obtain \( \omega^* \in [\Gamma_{i,j=1} \uplus \Gamma_{i,j=2,j}] \) from compatibility of \( \preceq_K^o \) with \( \circ \), we obtain \( \omega^* \in \min([\Gamma_{i,j=1} \downarrow \Gamma_{i,j=2}]) \) and \( \omega^* \in \min([\Gamma_{j,j=2} \downarrow \Gamma_{i,j=2}]) \). Now observe that \( \omega_0, \omega_1, \omega_2 \in [\Gamma_{i,j=1} \uplus \Gamma_{i,j=2,j}] \) holds; this is because by the definition of \( \Gamma_{i,j} \) we have \( [\Gamma_{i,j}] \subseteq [\Gamma_{k,k}] \cup [\Gamma_{k,j} \downarrow \Gamma_{k,k}] \) for each \( k \in \{0,1,2\} \). Hence, independent of the specific \( i \) and \( j \), we obtain \( \omega^* \preceq_K^o \omega_k \) from \( \omega^* \in [K \circ \Gamma_{i,i} \downarrow \Gamma_{i,j} \downarrow \Gamma_{i,j=2,j}] \) for each \( k \in \{0,1,2\} \). Together, \( \omega_i \in [K \circ \Gamma_{i,i} \downarrow \Gamma_{i,j} \downarrow \Gamma_{i,j=2,j}] \), \( \omega_j \in [K \circ \Gamma_{i,j} \downarrow \Gamma_{i,j=2,j}] \), and compatibility, imply \( \omega_i \preceq_K^o \omega^* \) and \( \omega_j \preceq_K^o \omega^* \). Because of \( [\Gamma_i] \cup [\Gamma_{i,j}] \subseteq [\Gamma_{i,j=1,2}] \), we have that \( \omega_i, \omega_j, \omega^* \in [\Gamma_{i,j=1,2}] \) or \( \omega_i, \omega_j, \omega^* \in [\Gamma_{i,j=1,2}] \). For the case \( \omega_i, \omega_j, \omega^* \in [\Gamma_{i,j=1,2}] \), since \( \omega_j \preceq_K \omega^* \) and \( \omega^* \in \min([\Gamma_{i,j=1,2}]) \), from min-retractivity we obtain \( \omega_j \in \min([\Gamma_{i,j=1,2}]) \).
\( \min(\{\Gamma_{i,\ominus 1}\}, \ominus_{K}) \). As \( \omega_i \in \min(\{\Gamma_{i,\ominus 1}\}, \ominus_{K}) \), we obtain \( \omega_i \preceq_{K} \omega_j \) and \( \omega_j \preceq_{K} \omega_i \). By an analogue argumentation, we obtain for the case of \( \omega_i, \omega_j, \omega^* \in [\Gamma_{j,\ominus 1}] \) the same conclusion, i.e., \( \omega_i \preceq_{K} \omega_j \) and \( \omega_j \preceq_{K} \omega_i \). This shows that \( \omega_i \preceq_{K} \omega_j \) and \( \omega_j \preceq_{K} \omega_i \) must hold in general.

We consider in the following all possible choices for \( i \) and \( j \). For the case of \( i = 0 \) and \( j = 2 \), we obtain a contradiction to \( \omega_2 \prec_{K} \omega_0 \). We next consider the case of \( i = 1 \) and \( j = 2 \). Because of \( [\Gamma_0] = [K \circ \Gamma_{0,1}] \cap [\Gamma_{2,0}] = \min([\Gamma_{0,1}], \ominus_{K}) \cap [\Gamma_{2,0}] \), we have that \( \omega_0, \omega_2, \omega^* \in [\Gamma_{2,0}] \) holds. As \( \omega_0 \preceq_{K} \omega^* \) and \( \omega^* \in \min([\Gamma_{2,0}], \ominus_{K}) \) holds, min-retractivity of \( \ominus_{K} \) yields \( \omega_0 \in \min([\Gamma_{2,0}], \ominus_{K}) \). Consequently, we obtain that \( \omega_0 \preceq_{K} \omega_2 \) holds, which is a contradiction to \( \omega_2 \preceq_{K} \omega_0 \). The proof for the case of \( i = 2 \) and \( j = 1 \) is analogous to the case of \( i = 1 \) and \( j = 2 \). We therefore obtain that Condition (2) from Definition 9.3 is satisfied.

Recall that by assumption, the base logic \( B \) does not exhibit a critical loop. Yet \( \Gamma_{0,1}, \Gamma_{1,2}, \Gamma_{2,0} \) satisfy Conditions (1) and Condition (2) of a critical loop, hence Condition (3) of Definition 9.3 must be violated. This means that there exists some \( \Gamma_{\tau} \in B \) such that \( [\Gamma_{\tau}] \) is not the empty set for every \( i \in \{0, 1, 2\} \), but no required base \( \Gamma_{\tau} \in B \) such that Condition (3) is satisfied. Consequently, for all \( \Gamma \in B \) holds

\[
[\Gamma] \neq \emptyset \text{ implies } [\Gamma] \not\subseteq [\Gamma_\tau] \setminus ([\Gamma_{0,1}] \cup [\Gamma_{1,2}] \cup [\Gamma_{2,0}]).
\]  

(\ast 1)

For the remaining parts of the proof, let \( \omega_\tau \in \Omega \) be an interpretation with \( \omega_\tau \in [\Gamma_\tau] \cap [\Gamma_\tau] \) for each \( i \in \{0, 1, 2\} \). Because \( \circ \) satisfies (G1) and (G3), we obtain \( [K \circ \Gamma_{\tau}] \subseteq [\Gamma_\tau] \) and consistency of \( K \circ \Gamma_{\tau} \). Together with (\ast 1) we obtain that there exists \( k \in \{0, 1, 2\} \) with \( [K \circ \Gamma_{\tau}] \cap [\Gamma_{k,\oplus 1}] \neq \emptyset \). We consider each of the two cases \( [K \circ \Gamma_{\tau}] \cap [K \circ \Gamma_{k,\oplus 1}] \neq \emptyset \) and \( [K \circ \Gamma_{\tau}] \cap [K \circ \Gamma_{k,\oplus 1}] = \emptyset \) independently.

The case of \( [K \circ \Gamma_{\tau}] \cap [K \circ \Gamma_{k,\oplus 1}] \neq \emptyset \). As first step, we show that

\[
\omega_0 \preceq_{K} \omega_2 \quad \text{and} \quad \omega_2 \preceq_{K} \omega_1 \quad \text{and} \quad \omega_2 \preceq_{K} \omega_0
\]

(\ast 2)

holds for this case. Clearly, \( [K \circ \Gamma_{\tau}] \cap [K \circ \Gamma_{k,\oplus 1}] \neq \emptyset \) implies that there exists some \( \omega_2 \in \Omega \) such that \( \omega_2 \in [K \circ \Gamma_{\tau}] \) and \( \omega_2 \in [K \circ \Gamma_{k,\oplus 1}] \). From the compatibility of \( \circ \) with \( \preceq_{K} \), we obtain \( \omega_2 \in \min([\Gamma_{k,\oplus 1}], \preceq_{K}) \), implying that \( \omega_2 \preceq_{K} \omega_2 \) holds. Remember that \( \omega_2, \omega_2 \in [\Gamma_{\tau}] \) and \( \omega_2 \in \min([\Gamma_{\tau}], \preceq_{K}) \), by min-retractivity we obtain \( \omega_2 \in \min([\Gamma_{\tau}], \preceq_{K}) \). From this last observation and from \( \omega_2 \in [\Gamma_{\tau}] \), we obtain \( \omega_2 \preceq_{K} \omega_2 \). Thus, we obtain \( \omega_2 \preceq_{K} \omega \). By a symmetric argument, we have \( \omega_2 \preceq_{K} \omega_2 \) and \( \omega_2 \preceq_{K} \omega_2 \). Thus, we obtain \( \omega_2 \preceq_{K} \omega_2 \) from Lemma 5.6(b). By combination of these observations with \( \omega_2 \in [\Gamma_{\tau}] \) and \( \omega_2 \in [\Gamma_{\tau}] \), we obtain \( \omega_2 \in [\Gamma_{\tau}], \omega_2 \in [\Gamma_{\tau}] \) from min-retractivity. As direct consequence, we obtain that (\ast 2) holds.

We will now show that a contradiction with \( \omega_2 \preceq_{K} \omega_0 \) is unavoidable. Recall that \( \omega_0, \omega_2 \in [\Gamma_{2,0}] \) and \( \omega_2 \not\subseteq [K \circ \Gamma_{2,0}] \), but \( \omega_0 \not\subseteq [K \circ \Gamma_{2,0}] \). The last observation together with the compatibility of \( \preceq_{K} \) with \( \circ \) implies that \( \omega_2 \in [\Gamma_{2,0}] \) holds. Because (\ast 2) holds, we obtain \( \omega_0 \in [\Gamma_{2,0}], \omega_2 \in [\Gamma_{2,0}] \) from min-retractivity of \( \preceq_{K} \). Similarly, we
obtain $\omega_0, \omega_0^\sigma \in \min([\Gamma_{0,1}], \preceq^\omega_K)$ from compatibility and $\omega_0, \omega_0^\sigma \in [K \circ \Gamma_{0,1}]$; showing that $\omega_0 \preceq^\omega_K \omega_0^\sigma$ holds. Because of $\omega_0^\sigma, \omega_0, \omega_2 \in [\Gamma_{2,0}]$, we obtain $\omega_0 \in \min([\Gamma_{2,0}], \preceq^\omega_K)$ from $\omega_0^\sigma \in \min([\Gamma_{2,0}], \preceq^\omega_K)$ and min-retractivity, and consequently, we obtain the contradiction $\omega_0 \preceq^\omega_K \omega_2$.

The case of $[K \circ \Gamma_i] \cap [K \circ \Gamma_{k,k \oplus 1}] = \emptyset$. Using $[K \circ \Gamma_i] \cap [\Gamma_{k,k \oplus 1}] \neq \emptyset$ yields that there exist some $\omega^* \in [K \circ \Gamma_i] \cap [\Gamma_{k,k \oplus 1}]$. From Lemma 5.6(c) and $\omega^* \in [\Gamma_{k,k \oplus 1}]$ and $\omega_K^\sigma \in [K \circ \Gamma_{k,k \oplus 1}]$ and $\omega^* \in [K \circ \Gamma_{k,k \oplus 1}]$ we obtain $\omega_K^\sigma \preceq^\omega_K \omega^*$. Because (G1) is satisfied by $\circ$, we have that $\omega^* \in [K \circ \Gamma_i]$ implies $\omega^* \in [\Gamma_i]$. We obtain the contradiction $\omega^* \preceq^\omega_K \omega_K^\sigma$ from $\omega^*, \omega_K^\sigma \in [\Gamma_i]$ and $\omega^* \in [K \circ \Gamma_i]$ by using Lemma 5.6(b).

In summary, this shows that Conditions (1)–(3) from Definition 9.3 are satisfied, i.e., $\Gamma_{0,1}, \Gamma_{1,2}, \Gamma_{2,0}$ form a critical loop. This contradicts the assumption that $B$ does not exhibit a critical loop and consequently, $(\omega_0, \omega_1)$ or $(\omega_1, \omega_2)$ is detached from $\circ$ in $K$. \hfill\Box

We continue with some preparations for the proof of Lemma 9.10. The proof will make use of circles of interpretations which are violating the situation given in Lemma 9.10. To make such situations more easy to handle, we introduce the following notion which make implicit use of $\preceq^\omega_K$, defined in Definition 5.3.

**Definition C.1.** Let $B = (L, \Omega, \models, \exists, \psi)$ a base logic, let $K \in B$ be a base, and let $\circ$ be a base change operator for $B$ that satisfies (G1)–(G3), (G5), and (G6). A sequence of interpretations $\circ = \omega_0, \ldots, \omega_n, \omega_0$ from $\Omega$ is said to form a strict circle of length $n + 1$ (with respect to $\circ$ and $K$) if

- $\omega_0, \ldots, \omega_n$ are satisfying Condition (a) and Condition (b) from Lemma 9.10, and
- $(\omega_i, \omega_{i+1})$ is not a detached pair for each $i \in \{0, \ldots, n\}$, where $\oplus$ is addition mod($n + 1$).

We will also substitute elements in a strict circle $\circ$ and use therefore the following notation. For a substitution $\sigma = \{\omega_{i_1} \mapsto x_1, \omega_{i_2} \mapsto x_2, \ldots\}$, we denote by $\circ[\sigma]$ the simultaneously replacement of $\omega_{i_j}$ by $x_j$ in $\circ$ for all $\omega_{i_j} \mapsto x_j \in \sigma$.

The following lemma will be useful, and describes situations like in Figure 9.
Lemma C.2 (cross lemma). Let $\mathcal{B} = (\mathcal{L}, \Omega, |=, \mathcal{B}, \mathcal{W})$ a base logic with no critical loop, let $K \in \mathcal{B}$ be a base, and let $\circ$ be a base change operator for $\mathcal{B}$ that satisfies (G1)–(G3), (G5), and (G6). If there are $\omega_0, \ldots, \omega_n \in \Omega$, with $n > 3$, and pairwise distinct $\lambda, a, b, c \in \{0, \ldots, n\}$, such that

(a) $\omega_0, \omega_1, \ldots, \omega_n, \omega_0$ is a strict circle of length $n + 1$,

(b) there exists an interpretation $\omega^*$ such that

\[
\omega^* \leq_K^0 \omega_a \quad \omega^* \leq_K^0 \omega_b \quad \omega^* \leq_K^0 \omega_c \quad \omega^* \leq_K \omega^*,
\]

and

(c) every pair of $\leq_K^0$ considered in (b) is not detached from $\circ$ in $K$,

then there is a strict circle of length $m$ with $3 \leq m \leq n$.

Proof. We assume $a < b < c$, and we assume that the path $\omega_c, \ldots, \omega_\lambda$ does not contain $\omega_a$ and $\omega_b$ (when seeing $\leq_K^0$ as graph). All other cases will follow by symmetry. We continue by consider several cases:

The case of $\omega_\lambda \prec_K \omega^*$. We obtain $\omega_\lambda \prec_K \omega^* \prec_K \omega_c \prec_K \omega \cdots \prec_K \omega_\lambda$, which yields that $\circ_{\lambda c} = \omega_\lambda, \omega^*, \omega_c, \ldots, \omega_\lambda$ is a strict circle. Note that because $\circ_{\lambda c}$ contains $\omega^*$ and in addition only elements of $\{\omega_0, \ldots, \omega_n\} \setminus \{\omega_a, \omega_b\}$, we have that $\circ_{\lambda c}$ has a length of at most $n$.

The case of $\omega^* \prec_K \omega_c$ and no prior case applies. If $\omega^* \prec_K \omega_c$, then we obtain $\omega^* \prec_K \omega_c \prec_K \omega \cdots \prec_K \omega_\lambda \prec_K \omega^*$, yielding that $\circ_{\lambda c} = \omega^*, \omega_c, \ldots, \omega_\lambda, \omega^*$ is a strict circle. Note that because $\circ_{\lambda c}$ contains $\omega^*$ and in addition only elements of $\{\omega_0, \ldots, \omega_n\} \setminus \{\omega_a, \omega_b\}$, we have that $\circ_{\lambda c}$ has a length of at most $n$.

The case of $\omega^* \prec_K \omega_b$ and no prior case applies. In this case we have $\omega_c \leq_K \omega^*$. We obtain $\omega^* \prec_K \omega_b \prec_K \omega_c \prec_K \omega^* \prec_K \omega^*$, which yields that $\circ_{bc} = \omega^*, \omega_b, \ldots, \omega_c, \omega^*$ is a strict circle. Note that because $\circ_{bc}$ contains, beside of $\omega^*$, only elements of $\{\omega_0, \ldots, \omega_n\} \setminus \{\omega_a, \omega_\lambda\}$, we have that $\circ_{bc}$ has a length of at most $n$.

The case of $\omega^* \prec_K \omega_a$ and no prior case applies. In this case we have $\omega_b \leq_K \omega^*$. We obtain $\omega^* \prec_K \omega_a \prec_K \omega_b \prec_K \omega^* \prec_K \omega^*$, which yields that $\circ_{ab} = \omega^*, \omega_a, \ldots, \omega_b, \omega^*$ is a strict circle. Note that because $\circ_{ab}$ contains, beside of $\omega^*$, only elements of $\{\omega_0, \ldots, \omega_n\} \setminus \{\omega_c, \omega_\lambda\}$, we have that $\circ_{ab}$ has a length of at most $n$.

If none of the cases above applies, then we have that $\omega^* \leq_K \omega_\lambda$ and $\omega_a \leq_K \omega^*$ and $\omega_b \leq_K \omega^*$ and $\omega_c \leq_K \omega^*$ holds. For the following line of arguments, recall that $a < b < c$ holds. We consider the case of $0 < \lambda < a$; for all other cases (where $0 < \lambda < a$ does not hold), the line of arguments is symmetric to the proof we present here in the following for the case of $0 < \lambda < a$. Because $\omega_0, \omega_1, \ldots, \omega_n, \omega_0$ is a strict circle of length $n + 1$, we obtain that $\omega_0 \prec_K \omega_1 \prec_K \omega_\lambda \prec_K \omega^* \prec_K \omega_c \prec_K \omega_0$. This show that $\circ_{0\lambda c} = \omega_0, \omega_1, \ldots, \omega_\lambda, \omega^*, \omega_c, \ldots, \omega_0$ is a strict circle. Because $\circ_{0\lambda c}$ contains $\omega^*$ and additionally only elements from $\omega_0, \ldots, \omega_n$, but not $\omega_a$ and $\omega_b$, we obtain that $\circ_{0\lambda c}$ has a length of at most $n$.

In summary, we obtain a strict circle of length $m$ with $3 \leq m \leq n$ for each case. \(\square\)

Note that it is not necessary to assume that $\omega^*$ is distinct from $\omega_0, \ldots, \omega_n$ in Lemma C.2. We now recall Lemma 9.10 and give a full proof thereof.
Lemma 9.10. Let $\mathcal{B} = (\mathcal{L}, \Omega, \models, \mathcal{B}, \psi)$ be a base logic which does not admit a critical loop, let $\mathcal{K} \in \mathcal{B}$ be a base, and let $\circ$ be a base change operator for $\mathcal{B}$ that satisfies (G1)–(G3), (G5), and (G6). If there are three or more interpretations $\omega_0, \ldots, \omega_n \in \Omega$, i.e. $n \geq 2$, such that

1. $\omega_0 \prec^\mathcal{K} \omega_1,
2. \omega_i \prec^\mathcal{K} \omega_{i+1}$ for all $i \in \{1, \ldots, n\}$, where $\oplus$ is addition $\mod(n + 1),$

then there is some $i \in \{1, \ldots, n\}$ such that $(\omega_i, \omega_{i+1})$ is a detached pair.

Proof. Let $\omega_0, \ldots, \omega_n \in \Omega$ such that Condition (a) and Condition (b) of Lemma 9.10 are satisfied. With $\oplus$ we denote addition $\mod(n + 1)$. The proof will be by induction. Note that for $n = 2$ we obtain the result by Lemma 9.8. We proceed the proof for the case of $n > 2$ and assume that Lemma 9.10 already holds for all $m$ with $2 \leq m < n$. A consequence of the induction hypothesis is that there is no strict circle of length $c$ for $3 \leq c \leq n$.

We are striving for a contradiction. Therefore, we assume \( \circ_{0n} = \omega_0, \ldots, \omega_n, \omega_0 \) is a strict circle of length $n + 1$, which is, due to Condition (a) and Condition (b) from Lemma 9.10, equivalent to assume that $(\omega_1, \omega_{i+1})$ is not a detached pair for each $i \in \{1, \ldots, n\}$. The remaining parts of the proof show that the existence of the strict circle $\circ_{0n}$ implies existence of a critical loop.

As first step, we show that $\omega_0, \ldots, \omega_n \not\in [\mathcal{K}]$ holds. If $\omega_1 \in [\mathcal{K}]$, then due Definition 5.3, we obtain $\omega_1 \preceq^\mathcal{K} \omega_0$, which contradicts Condition (a). If $\omega_i \in [\mathcal{K}]$ for some $i \in \{0, 2, 3, \ldots, n\}$, then, because of Condition (b), there is some $j$ with $\omega_j \preceq^\mathcal{K} \omega_{j+1}$ and $\omega_j \not\in [\mathcal{K}]$ and $\omega_{j+1} \in [\mathcal{K}]$, which is again impossible due to Definition 5.3. Thus, we have $\omega_0, \ldots, \omega_n \not\in [\mathcal{K}]$ for the remaining parts of the proof.

We continue by showing the existence of several bases, which will form a critical loop. Definition 5.3 and Definition 9.7 together implies that for each $i \in \{1, \ldots, n\}$ exists a base $\Gamma_{i,i+1} \in \mathcal{B}$ such that

$$\omega_i, \omega_{i+1} \models \Gamma_{i,i+1} \text{ and } \omega_1 \models \mathcal{K} \circ \Gamma_{i,i+1}. \quad (#1)$$

holds. Moreover, by $\omega_0 \prec^\mathcal{K} \omega_1$ from Condition (1) and $\omega_1 \not\models \mathcal{K}$ and Lemma 5.6(a), there exists a base $\Gamma_{0,1} \in \mathcal{B}$ such that the following holds:

$$\omega_0, \omega_1 \models \Gamma_{0,1} \text{ and } \omega_0 \models \mathcal{K} \circ \Gamma_{0,1} \text{ and } \omega_1 \not\models \mathcal{K} \circ \Gamma_{0,1}. \quad (#2)$$

We show that $\Gamma_{0,1}, \Gamma_{1,2}, \ldots, \Gamma_{n,0}$ is forming a critical loop. To this end we are setting $\Gamma_i = (\mathcal{K} \circ \Gamma_{i,i+1}) \cup \Gamma_{i+1,n,i}$ for each $i \in \{0, \ldots, n\}$. By (#1) and (#2) each $\Gamma_i$ is a consistent base with $\omega_i \in [\Gamma_i]$. We continue by verifying that Conditions (1)–(3) from Definition 9.3 are satisfied.

1. If $\mathcal{K}$ is inconsistent, then Condition (1) is immediately satisfied. We consider the case where $\mathcal{K}$ is consistent and $[\mathcal{K} \cup \Gamma_{i,i+1}] \neq \emptyset$ for some $i \in \{0, \ldots, n\}$. From (G2) we obtain $[\mathcal{K} \circ \Gamma_{i,i+1}] = [\mathcal{K} \cup \Gamma_{i,i+1}]$. From $\omega_i \in [\Gamma_i]$ and the definition of $\Gamma_i$, we obtain $\omega_i \in [\mathcal{K} \circ \Gamma_{i,i+1}] \cap [\Gamma_{i+1,n,i}]$. As $[\mathcal{K} \circ \Gamma_{i,i+1}] = [\mathcal{K} \cup \Gamma_{i,i+1}]$, we obtain $\omega_i \in [\mathcal{K} \cup \Gamma_{i,i+1}] \cap [\Gamma_{i+1,n,i}]$. Consequently, we there exists some $i \in \{0, \ldots, n\}$ such that $\omega_i \in [\mathcal{K}]$, yielding a contradiction to $\omega_0, \ldots, \omega_n \not\in [\mathcal{K}]$.

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(2) By the postulate (G1) we have \([K \circ \Gamma_{i,i\otimes1}] \subseteq [\Gamma_{i,i\otimes1}]\) for each \(i \in \{0, \ldots, n\}\). The definition of \(\Gamma_i\) yields \([\Gamma_i] \subseteq [\Gamma_{i,i\otimes1} \cup \Gamma_{i,i\otimes0}] \subseteq [\Gamma_{i,i\otimes1}]\) for each \(i \in \{0, 1, 2\}\). Substitution of \(i\) by \(i \oplus 1\) yields \([\Gamma_{i\oplus1}] \subseteq [\Gamma_{i,i\otimes2} \cup \Gamma_{i,i\otimes1}] \subseteq [\Gamma_{i,i\otimes1}]\); showing that \([\Gamma_i] \cup [\Gamma_{i,i\otimes1}] \subseteq [\Gamma_{i,i\otimes1}]\) holds for each \(i \in \{0, \ldots, n\}\).

We show that each \(\Gamma_i \cup \Gamma_j\) is inconsistent, by assuming the contrary, i.e. there are some \(i,j \in \{0, \ldots, n\}\) such that \(i \neq j\) and \(\Gamma_i \cup \Gamma_j\) is consistent. Because of the commutativity of \(\cup\), we assume \(i < j\) without loss of generality. By compatibility and definition of \(\Gamma_i\) and by definition of \(\Gamma_j\), there exists some \(\omega^* \in [\Gamma_{i\otimes n,i} \cup \Gamma_{j\otimes n,j}]\) with \(\omega^* \in \min([\Gamma_{i,i\otimes1}], \varsigma_0^K)\) and \(\omega^* \in \min([\Gamma_{j,j\otimes1}], \varsigma_0^K)\). Recall that \(\omega_i, \omega_{i\otimes1} \in [\Gamma_{i,i\otimes1}]\) and \(\omega_j, \omega_{j\otimes1} \in [\Gamma_{j,j\otimes1}]\). Consequently, for all \(k \in \{i, i\oplus1, j, j\oplus1\}\) holds \(\omega^* \leq_k^0 \omega_k\). Moreover, because of (\#1) and (\#2) we obtain \(\omega_i \leq_k^0 \omega^*\) and \(\omega_j \leq_k^0 \omega^*\) from Lemma 5.6(b).

From \(\omega^* \in [\Gamma_{i\otimes n,i} \cup \Gamma_{j\otimes n,j}]\) we obtain, by an analogue argumentation, that \(\omega_{i\otimes n} \leq_k^0 \omega^*\) and \(\omega_{j\otimes n} \leq_k^0 \omega^*\) holds. In summary, we have:

\[
\begin{align*}
\omega^* & \leq_k^0 \omega_i \quad \omega^* \leq_k^0 \omega_j \quad \omega^* \leq_k^0 \omega_{i\otimes1} \quad \omega_{i\otimes n} \leq_k^0 \omega^* \\
\omega_i & \leq_k^0 \omega^* \quad \omega_j \leq_k^0 \omega^* \quad \omega_{i\otimes1} \leq_k^0 \omega^* \quad \omega_{j\otimes n} \leq_k^0 \omega^*
\end{align*}
\]

(\#1)

Note that all pairs \((\omega^*, \omega_\xi), (\omega^*, \omega_{\xi\otimes1})\) and \((\omega_{\xi\otimes n}, \omega^*)\) with \(\xi \in \{i, j\}\) are not detached.

We are now striving for a contradiction by showing the existence of a strict circle with a length of at most \(n\). Recall that \(\circ_{0n} = \omega_0, \ldots, \omega_n, \omega_0\) is a strict circle of length \(n+1\).

At first, we consider two particular cases:

\((\omega_i = \omega_{j\otimes1})\) We obtain a strict circle of length of at most \(n\) from Lemma C.2 by using \(\circ_{0n}\) and setting \(\lambda = i, a = j, b = i \oplus 1,\) and \(c = j \oplus n\). Note that \(\lambda, a, b, c\) are pairwise distinct indices.

\((\omega_j = \omega_{i\otimes1})\) We obtain a strict circle of length of at most \(n\) from Lemma C.2 by using \(\circ_{0n}\) and setting \(\lambda = i, a = j, b = j \oplus 1,\) and \(c = i \oplus n\). Note that \(\lambda, a, b, c\) are pairwise distinct indices.

For all situations not covered by the cases above, we obtain that \(i, i \oplus 1, j, j \oplus 1\) are pairwise distinct. Because of (\#1), we can apply Lemma C.2 by using \(\circ_{0n}\) and setting \(\lambda = i, a = i \oplus 1, b = j,\) and \(c = j \oplus n\). This yields a strict circle with a length of at most \(n\).

In summary, for every possible case we obtain a contradiction, which shows that Condition (2) of critical loops (cf. Definition 9.3) is satisfied.

(3) We show Condition (3) from Definition 9.3 by contradiction. Therefore, assume there is a base \(\Gamma_{\psi} \in \mathfrak{B}\) such that for

\[
B = \{ \Gamma_i \mid [\Gamma_{\psi} \cup \Gamma] \neq \emptyset \} \subseteq \{ \Gamma_0, \ldots, \Gamma_n \}
\]

holds \(|B| \geq 3\) and there exists no base \(\Gamma_{\psi} \in \mathfrak{B}\) as required by Condition (3). Consequently, for each base \(\Gamma \in \mathfrak{B}\) we have

\[
[\Gamma] \neq \emptyset \text{ implies } [\Gamma] \not\subseteq [\Gamma_{\psi}] \setminus ([\Gamma_{0,1}] \cup \ldots \cup [\Gamma_{n,0}]).
\]

(\star2)
From (\(*1\)) we obtain that \(\Gamma_\theta\) is consistent, and thus, by (G3), that \(\mathcal{K} \circ \Gamma_\theta\) is consistent and from satisfaction of (G1) we obtain \([\mathcal{K} \circ \Gamma_\theta] \subseteq [\Gamma_\theta]\). Consequently, because of (\(*2\)), we have \([\mathcal{K} \circ \Gamma_\theta] \not\subseteq [\Gamma_\theta] \setminus ([\Gamma_{0,1}] \cup \ldots \cup [\Gamma_{n,0}])\). This implies that \([\mathcal{K} \circ \Gamma_\theta] \cap [\Gamma_{k,k+1}] \not= \emptyset\) for some \(k \in \{0,\ldots,n\}\). Let \(\omega^*_k\) be an interpretation with \(\omega^*_k \in [\mathcal{K} \circ \Gamma_\theta] \cap [\Gamma_{k,k+1}]\).

From (G5) and (G6), \(\omega^*_k \in [\mathcal{K} \circ \Gamma_\theta] \cap [\Gamma_{k,k+1}]\) and commutativity of \(\circ\) we obtain

\[
\omega^*_k \in \min([\Gamma_{0,1}] \cup [\Gamma_{k,k+1}]) = \min([\Gamma_{k,k+1} \cup [\Gamma_{0,1}]], \omega^*_k) \mbox{ and } \omega_k \leq \omega^*_k, \quad (\star 3)
\]

whereby the latter is a direct consequence of \(\omega^*_k \in [\Gamma_{k,k+1}] \) and \(\omega_k \in \min([\Gamma_{k,k+1}], \omega^*_k)\).

Furthermore, let \(\omega^*_\ell\) be some interpretation with \(\omega^*_\ell \in [\Gamma_{\ell,\ell+1}]\) for each \(\Gamma_i \in B\). We show as next for each \(\Gamma_\xi \in B\) and for each \(\omega^*_{\xi,n} \in [\Gamma_{\xi,n}]\) and for each \(\omega^*_{\xi,1} \in [\Gamma_{\xi,1}]\) that we have

\[
\omega^*_\ell \leq \omega^*_k \mbox{ and } \omega^*_{\xi,1} \leq \omega^*_k \mbox{ and } \omega^*_{\xi,n} \leq \omega^*_k \quad \omega^*_\ell \leq \omega^*_k \quad \omega_k \leq \omega^*_k \quad (\star 4)
\]

We obtain \(\omega^*_k \leq \omega^*_\ell \mbox{ and } \omega^*_\ell \leq \omega^*_k \) from Lemma 5.6(b), because \(\omega^*_k, \omega^*_\ell \in [\Gamma_\theta]\) and \(\omega^*_k, \omega^*_\ell \in [\mathcal{K} \circ \Gamma_\theta]\) holds. From \(\omega^*_\ell, \omega^*_n \in \min([\Gamma_{\xi,n}], \omega^*_\ell)\) we directly obtain \(\omega^*_\ell \leq \omega^*_\ell \mbox{ and } \omega^*_n \leq \omega^*_n\).

Compatibility of \(\leq\) with \(\circ\), together with the definitions of \(\Gamma_\xi\) and Condition (2), yields the remaining statements of (\(\star 4\)).

Moreover, as next step, we show for each \(\Gamma_\xi \in B\) and for each \(\omega^*_{\xi,n} \in [\Gamma_{\xi,n}]\) and for each \(\omega^*_{\xi,1} \in [\Gamma_{\xi,1}]\) the following holds:

\[
\omega^*_{\xi,n} \leq \omega^*_\ell \mbox{ if and only if } \omega^*_{\xi,n} \leq \omega^*_k \mbox{ and } \omega^*_\ell \leq \omega^*_k \quad \omega_k \leq \omega^*_k \quad (\star 5)
\]

Observe that (\(\star 5\)) holds, otherwise, we would obtain a strict circle of length 3. These strict circles are directly obtainable from (\(\star 4\)): if \(\omega^*_{\xi,n} \leq \omega^*_k \mbox{ and } \omega^*_\ell \leq \omega^*_k \mbox{ and } \omega^*_{\xi,n} \leq \omega^*_\ell\), obtain the strict circle \(\omega^*_{\xi,n} \leq \omega^*_k \mbox{ and } \omega^*_k \leq \omega^*_n\) with a length of 3. For all other cases, we obtain analogously a strict circle of length 3.

Now let \(\ell_{\min}, \ell_{\med}, \ell_{\max}\) be integers with \(0 \leq \ell_{\min} < \ell_{\med} < \ell_{\max} \leq n\) such that \(\Gamma_{k,\ell_{\min}}, \Gamma_{k,\ell_{\med}}, \Gamma_{k,\ell_{\max}} \in B\) and \(\ell_{\min}\) is the smallest number from \(\{0,\ldots,n\}\) with \(\Gamma_{k,\ell_{\min}} \in B\), and \(\ell_{\max}\) is the greatest number from \(\{0,\ldots,n\}\) with \(\Gamma_{k,\ell_{\max}} \in B\). For convenience, we will sometimes write \(\ell_x\) and \(\omega_x\), instead of \(\ell_{k,\ell_x}\) and \(\omega_{k,\ell_x}\), respectively, for any \(x \in \{\min, \med, \max\}\).

We now establish that replacing \(\omega_i\) in \(\mathcal{O}_n\) by \(\omega^*_i\) for some \(\Gamma_i \in B\) yields again a strict circle. Remember that each pair given in (\(\star 4\)) and (\(\star 5\)) is a non-detached pair. Because of this and because \(\mathcal{O}_n\) is a strict circle of length \(n+1\), we obtain from (\(\star 4\)) and (\(\star 5\)) that \(\mathcal{O}_n[\sigma]\) is also a strict circle of length \(n+1\) for each substitution \(\sigma\) with

\[
\sigma \subseteq \{\omega_{\min} \mapsto \omega^*_{\min}, \omega_{\med} \mapsto \omega^*_{\med}, \omega_{\max} \mapsto \omega^*_{\max}\},
\]

i.e., substituting each \(\omega_x\) by \(\omega^*_x\) in \(\omega_0, \omega_1, \ldots, \omega_n\), for some of \(x \in \{\min, \med, \max\}\), yields a strict circle of length \(n+1\).

We consider two cases, the case where \(\Gamma_k \cup \Gamma_\theta\) is inconsistent and the case where \(\Gamma_k \cup \Gamma_\theta\) is consistent.
The case of $\Gamma_k \cup \Gamma_\tau$ is inconsistent. For this case we have that $\Gamma_k \not\in B$ holds. Remember that by $(\ast 3)$ and $(\ast 4)$ the following holds:

\[
\begin{align*}
\omega_k &\prec^c_k \omega_k^* \\
\omega_k^* &\prec^0_k \omega^*_k \\
\omega^*_k &\prec^c_k \omega^*_{\min} \\
\omega^*_{\min} &\prec^c_k \omega^*_{med} \\
\omega^*_{med} &\prec^c_k \omega^*_{\max} \\
\omega^*_{\max} &\prec^c_k \omega^*_{med}
\end{align*}
\]

We obtain that there exists a strict circle with a length of at most $n$ by using Lemma C.2 when setting $\lambda = k$, $a = k \oplus \ell_{\min}$, $b = k \oplus \ell_{\med}$ and $c = k \oplus \ell_{\max}$, using the strict circle $\circ_{0n}[\omega_a \mapsto \omega^*_p, \omega_b \mapsto \omega^*_r, \omega_c \mapsto \omega^*_q]$.

The case of $\Gamma_k \cup \Gamma_\tau$ is consistent. This case is equivalent to having $\ell_{\min} = 0$, i.e., $\Gamma_{\min} = \Gamma_k \in B$. Consequently, we have that $\omega^*_k \in [\Gamma_k \cup \Gamma_\tau]$. From the definition of $\Gamma_k$, and from $\omega^*_k, \omega^*_h \in [\Gamma_k, k \oplus 1]$ with $\omega^*_k \in [K \circ \Gamma_k, k \oplus 1]$, and from compatibility and min-retractivity we also obtain $\omega^*_k, \omega^*_h \in [K \circ \Gamma_k, k \oplus 1] \cap [\Gamma_\tau] = \min([\Gamma_k, k \oplus 1], \prec^c_k) \cap [\Gamma_\tau]$. Consequently, all observations for $\omega^*_k$ do also hold for $\omega^*_h$; in particular, this applies to $(\ast 3)$–$(\ast 5)$. Thus, we assume $\omega^*_k = \omega^*_h$ in the following.

Together with $(\ast 4)$ and $(\ast 5)$ we can summarize as follows:

\[
\begin{align*}
\omega^*_k &\prec^c_k \omega^*_{\max} \\
\omega^*_h &\prec^c_k \omega^*_{\min} \\
\omega^*_h &\prec^c_k \omega^*_{med} \\
\omega^*_k &\prec^c_k \omega^*_{\med} \\
\omega^*_k &\prec^c_k \omega^*_{\max} \\
\omega^*_h &\prec^c_k \omega^*_{med}
\end{align*}
\]

We are striving for a contradiction by showing the existence of a strict circle of length that is strictly smaller than $n + 1$. Therefore, we will make use of Lemma C.2, whenever that is possible. We consider three cases in the following, depending on the values of $\ell_{\med}$ and $\ell_{\max}$. Recall that $1 \leq \ell_{\med} < \ell_{\max} \leq n$ holds.

$(\ell_{\med} \neq 1)$ Because of $(\ast 6)$, we can directly apply Lemma C.2 by setting $\lambda = k$, $a = k \oplus 1$, $b = k \oplus \ell_{\med}$ and $c = k \oplus \ell_{\max}$, and by using the strict circle $\circ_{0n}[\omega_p \mapsto \omega^*_p, \omega_q \mapsto \omega^*_q]$ for Lemma C.2, which yields a strict circle with a length of at most $n$.

$(\ell_{\max} \neq n)$ We apply Lemma C.2 by setting $\lambda = k$, $a = k \oplus \ell_{\med}$, $b = k \oplus \ell_{\max}$ and $c = k \oplus n$, and by using the strict circle $\circ_{0n}[\omega_p \mapsto \omega^*_p, \omega_r \mapsto \omega^*_r]$, which yields again a strict circle with a length of at most $n$.

$(\ell_{\med} = 1$ and $\ell_{\max} = n)$ Because of $\ell_{\max} = n$, we have that $\omega^*_k \in [\Gamma_k]$. Together with $\omega^*_k \in [\Gamma_k]$, we obtain from $(\ast 4)$ that $\omega^*_{\max} \preceq^c_k \omega^*_k$. From the min-retractivity of $\preceq^c_k$ and $\Gamma_{k \oplus n} \subseteq B$, we obtain $\omega^*_{\max} \in \min([\Gamma_\tau], \preceq^c_k)$, which implies $\omega^*_{\max} \preceq^c_k \omega^*_k$.

If $\omega^*_{\max} \preceq^c_k \omega^*_{med}$, then we obtain the strict circle $\omega^*_{\max}, \omega^*_{med}, \omega_{k \oplus (\ell_{\med} + 1)}, \ldots, \omega_{k \oplus (\ell_{\max} + n)}$, which has a length of at most $n$. Due to the induction hypothesis there is no such strict circle, and thus, $\omega^*_{\max} \preceq^c_k \omega^*_{med}$ is impossible. From totality of $\preceq^c_k$ we obtain that $\omega^*_{\med} \preceq^c_k \omega^*_{\max}$ holds. If $k \oplus \ell_{\max} = 0$, then we obtain the strict circle $\omega^*_{\max}, \omega_{k \oplus (\ell_{\med} + 1)}, \omega^*_{med}, \omega_{k \oplus (\ell_{\med} + 2)}$, of length 3. If $k = 0$, then we obtain the strict circle $\omega_{k \oplus (\ell_{\med} + 1)}, \omega^*_{med}, \omega_{k \oplus (\ell_{\med} + 2)}$ of length 3. If none of the prior cases applies, then $\circ = \omega_q, \omega_1, \ldots, \omega^*_{\med}, \omega^*_{med}, \ldots, \omega_0$ is a strict circle. Note that $\omega_k$ is not part of the strict circle $\circ$, and consequently the length of $\circ$ is bounded by $n$.

We obtain a contradiction in every case, which shows that Condition (3) of Definition 9.3 is satisfied.
In summary, assuming that each \((\omega_i, \omega_{i\oplus 1})\) is not a detached pair leads to formation of a critical loop by \(\Gamma_{0,1}, \Gamma_{1,2}, \ldots, \Gamma_{n,0}\); contradicting the critical loop-freeness of \(\mathbb{B}\). Consequently, at least one \((\omega_i, \omega_{i\oplus 1})\) has to be a detached pair.