A Class of Nonassociative Algebras Including Flexible and Alternative Algebras, Operads and Deformations

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Abstract

There exists two types of nonassociative algebras whose associator satisfies a symmetric relation associated with a 1-dimensional invariant vector space with respect to the natural action of the symmetric group $\Sigma_3$. The first one corresponds to the Lie-admissible algebras and this class has been studied in a previous paper of Remm and Goze. Here we are interested by the second one corresponding to the third power associative algebras.

Keywords: Nonassociative algebras; Alternative algebras; Third power associative algebras; Operads

Introduction

Recently, we have classified for binary algebras, Cf. [1], relations of nonassociativity which are invariant with respect to an action of the symmetric group on three elements $\Sigma_3$, in particular nonassociative algebras satisfying (2) with nonassociative of Lie-admissible algebras. We will now investigate the second class and in particular nonassociative algebras satisfying (2) with nonassociative relations in correspondence with the subgroups of $\Sigma_3$.

Convention: We consider algebras over a field $\mathbb{K}$ of characteristic zero.

$G_1$-$p^3$-associative Algebras

Definition

Let $\Sigma_3$ be the symmetric group of degree 3 and $\mathbb{K}$ a field of characteristic zero. We denote by $[\mathbb{K}[\Sigma_3]]$ the corresponding group algebra, that is the set of formal sums $\sum a_{\sigma} \sigma, a_{\sigma} \in \mathbb{K}$ endowed with the natural addition and the multiplication induced by multiplication in $\Sigma_3$, $\mathbb{K}$ and linearity. Let $\{G_i\}_{i=1,\ldots,6}$ be the subgroups of $\Sigma_3$. To fix notations we define

$$G_i = \{ Id | \sigma \neq r_2 \otimes r_3 \otimes r_1 \otimes r_2 \otimes r_3 \otimes r_1 \otimes r_2 \otimes r_3 \neq \sigma = \Sigma_3, \}
$$

where $< \sigma >$ is the cyclic group subgroup generated by $\sigma$. To each subgroup $G_i$ we associate the vector $v_{G_i}$ of $[\mathbb{K}[\Sigma_3]]$.

**Lemma 1.** The one-dimensional subspace $\mathbb{K}[v_{G_i}]$ of $[\mathbb{K}[\Sigma_3]]$ generated by

$v_{G_1} = r_2 \otimes r_3 = \sum_{\sigma \in \Sigma_3} \sigma$

is an irreducible invariant subspace of $[\mathbb{K}[\Sigma_3]]$ with respect to the right action of $\Sigma_3$ on $[\mathbb{K}[\Sigma_3]]$.

Recall that there exists only two one-dimensional invariant subspaces of $[\mathbb{K}[\Sigma_3]]$, the second being generated by the vector $\sum_{\sigma \in \Sigma_3} \sigma \sigma \sigma \in [\mathbb{K}[\Sigma_3]]$ where $E(\sigma)$ is the sign of $\sigma$. As we have precised in the introduction, this case has been studied in literature of Remm [1].

**Definition 2.** A $G_i$-$p^3$-associative algebra is a $\mathbb{K}$ -algebra $(A, \mu)$ whose associator

$$A_\mu = \mu (\mu \otimes Id - Id \otimes \mu)$$

satisfies

$$A_\mu = \Phi_{v_{G_i}} = 0,$$

where $\Phi_{v_{G_i}} : A^{3d} \rightarrow A^{3d}$ is the linear map

$$\Phi_{v_{G_i}} (x_1 \otimes x_2 \otimes x_3) = \sum_{\sigma \in \Sigma_3} x_{\sigma^{-1}(1)} \otimes x_{\sigma^{-1}(2)} \otimes x_{\sigma^{-1}(3)}.$$

Let $O(v_{G_i})$ be the orbit of $v_{G_i}$ with respect to the right action

$$\Sigma_3 \times [\mathbb{K}[\Sigma_3]] \rightarrow [\mathbb{K}[\Sigma_3]]$$

$$(\sigma, \sum_{\sigma} a \sigma) \rightarrow \sum_{\sigma} a \sigma^{-1} \sigma$$

Then putting $F_{v_{G_i}} = K(O(v_{G_i}))$ we have

$$\text{dim } F_{G_1} = 6,$$

$$\text{dim } F_{G_2} = \text{dim } F_{V_{G_3}} = \text{dim } F_{V_{G_4}} = 3,$$

$$\text{dim } F_{G_5} = 2,$$

$$\text{dim } F_{G_6} = 0.$$
Proposition 3. Every $G_i$-p$^i$-associative algebra is third power associative.

Recall that a third power associative algebra is an algebra $(A, \mu)$ whose associator satisfies $A_p(x, x, x) = 0$. Linearizing this relation, we obtain

$$A_p(x, x, x) = 0.$$

Since each of the invariant spaces $F_{ij}$ contains the vector $v_{ij}$, we deduce the proposition.

Remark. An important class of third power associative algebras is the class of power associative algebras, that is, algebras such that any element generates an associative subalgebra.

What are $G_i$-p$^i$-associative algebras?

(1) If $i = 1$, since $v_{ij} = Id$, the class of $G_i$-p$^i$-associative algebras is the full class of associative algebras.

(2) If $i = 2$, the associator of a $G_i$-p$^i$-algebra $A$ satisfies

$$A_p(x, x, x) + A_p(x, x, x) = 0,$$

and this is equivalent to

$$A_p(x, y, y),$$

for all $x, y \in A$.

(3) If $i = 3$, the associator of a $G_i$-p$^i$-algebra $A$ satisfies

$$A_p(x, x, x) + A_p(x, x, x) = 0,$$

that is,

$$A_p(x, y, y),$$

for all $x, y \in A$.

Sometimes $G_i$-p$^i$-associative algebras are called left-alternative algebras, $G_i$-p$^i$-associative algebras are right-alternative algebras. An alternative algebra is an algebra which satisfies the $G_i$ and $G_i$-p$^i$-associativity.

(4) If $i = 4$, we have $A_i(x, y, x)$ for all $x, y \in A$, and the class of $G_i^4$-p$^i$-associative algebras is the class of flexible algebras.

(5) If $i = 5$, the class of $G_i$-p$^i$-associative algebras corresponds to $G_i$-associative algebras [2].

(6) If $i = 6$, the associator of a $G_i$-p$^i$-algebra $A$ satisfies

$$A_p(x, x, x) + A_p(x, x, x) = 0 + A_p(x, x, x) + A_p(x, x, x) = 0,$$

If we consider the symmetric product $x * y = \mu(x, y) + \mu(y, x)$ and the skew-symmetric product $[x, y] = \mu(x, y) - \mu(y, x)$, then the $G_i$-p$^i$-associative identity is equivalent to

$$[x * y, z] + [y * z, x] = [z * x, y] = 0.$$

Definition 4. A $([, ]+, +)$-admissible-algebra is a K-vector space $A$ provided with two multiplications:

(a) a symmetric multiplication $*$,

(b) a skew-symmetric multiplication $[,]$ satisfying the identity

$$[x * y, z] + [y * z, x] + [z * x, y] = 0$$

for any $x, y, z \in A$.

Then a $G_i$-p$^i$-associative algebra can be defined as $([, ]+, +)$-admissible algebra.

Remark: Poisson algebras. A $K$-Poisson algebra is a vector space $P$ provided with two multiplications, an associative commutative one $x \cdot y$ and a Lie bracket $[x, y]$, which satisfy the Leibniz identity

$$[x \cdot y, z] + [y \cdot z, x] + [z \cdot x, y] = 0.$$

In studies of Remm [3], it is shown that these conditions are equivalent to provide $P$ with a nonassociative multiplication $x * y$ satisfying

$$x * (y * z) = (x * y) * z - \frac{1}{3} (x * (y * z) - (z * y) * x - (y * z) * x),$$

If we denote by $A(x, y, z) = x * (y * z)$ and $A'(x, y, z) = (x * y) * z$ then the previous identity is equivalent to

$$A''' \circ A''^{p^3} + A'' \circ A''^{p^3} = 0,$$

where $w = 3Id$ and $w_3 = 3Id + e_3 - e_3 - e_3$. In fact the class of Poisson algebras is a subclass of a family of nonassociative algebras defined by conditions on the associator. The product satisfies

$$A(x, y, z) + A(y, z, x) + A(z, x, y) = 0,$$

and

$$A(x, y, z) + A(z, y, x) = 0,$$

so it is a subclass of the class of algebras which are flexible and $G_i$-p$^i$-associative [1].

The Operads $G_i$-p$^i$Ass and their Dual

For each $i \in \{1, \ldots, 6\}$, the operad for $G_i$-p$^i$-associative algebras will be denoted by $G_i$-p$^i$Ass. The operads $\{G_i$-p$^i$Ass, $\}$ are binary quadratic operads, that is, operads of the form $P = \Gamma(E^i) / (R)$, where $\Gamma(E)$ denotes the free operad generated by a $\Sigma$-module placed in arity 2 and $R$ is the operadic ideal generated by a $\Sigma$-invariant subspace $R$ of $\Gamma(E)(3)$. Then the dual operad $P^{*}$ is the quadratic operad $P^{*} = \Gamma(E^{*}) / (R^{*})$, where $R^{*} \supset \Gamma(E^{*})(3)$ is the annihilator of $R \subset \Gamma(E)(3)$ in the pairing

$$< (x_1, x_2, x_3, x_4, x_5), (y_1, y_2, y_3, y_4, y_5) > = 0, \text{ if } [i, j, k] = [i', j', k'],$$

$$< (x_1, x_2, x_3, x_4, x_5), (y_1, y_2, y_3, y_4, y_5) > = -< (y_1, y_2, y_3, y_4, y_5), (x_1, x_2, x_3, x_4, x_5) >, \text{ if } [i, j, k] = [i', j', k'],$$

with

$$\sigma = \begin{cases} i & j & k \\ i' & j' & k' \end{cases},$$

and

$$< (x_1, x_2, x_3, x_4, x_5), (y_1, y_2, y_3, y_4, y_5) > = -< (y_1, y_2, y_3, y_4, y_5), (x_1, x_2, x_3, x_4, x_5) >, \text{ if } [i, j, k] = [i', j', k'],$$

with

$$\sigma = \begin{cases} i & j & k \\ i' & j' & k' \end{cases},$$

and

$$(R^{*})$$ is the operadic ideal generated by $R^{*}$. For the general notions of binary quadratic operads [4,5]. Recall that a quadratic operad $\mathcal{P}$ is Koszul if the free $\mathcal{P}$-algebra based on a $K$-vector space $V$ is Koszul, for any vector space $V$. This property is conserved by duality and can be studied using generating functions of $\mathcal{P}$ and of $\mathcal{P}^{*}$ [4,6]. Before studying the Koszulness of the operads $G_i$-p$^i$Ass, we will compute the homology of an associative algebra which will be useful to look if $G_i$-p$^i$Ass are Koszul or not.

Let $A_i$ the two-dimensional associative algebra given in a basis $\{e_1, e_2\}$ by $e_1 e_1 = e_1, e_2 e_2 = e_2, e_1 e_2 = e_2 e_1 = 0$. Recall that the Hochschild homology of an associative algebra is given by the complex $(C_n(A, A), d_n)$ where $C_n(A, A) = A \otimes A^n$ and the differentials $d_n : C_n(A, A) \to C_{n-1}(A, A)$ are given by

\[ d_n(a_1, \ldots, a_n) = \sum_{i} (\sum_{j<k} a_j a_k a_{i-1} a_{i} a_{i+1} \ldots a_n) - \sum_{i} (\sum_{j<k} a_1 a_2 \ldots a_{i-2} a_j a_k a_{i} a_{i+1} \ldots a_n) \]
...


\[
\begin{align*}
1 - d_1(v_1, v_2) &= v_2 - v_1, \\
2 - d_1(v_1, v_3) &= v_3 - v_1, \\
3 - d_1(v_2, v_3) &= v_3 - v_2,
\end{align*}
\]

and \( \ker d_i \) is of dim 64. The space \( \ker d_i \) doesn’t contain in particular the vectors \( (v_i, v_j) \) for \( i = 1, 2 \) because these vectors \( v_i \) are not in the derived subalgebra. Since these vectors are in \( \ker d_i \) we deduce that the second space of homology is not trivial.

**Proposition 7.** The current operad of \((G_p, p)\)-Ass is

\[
G_{p - p} - Ass = \mathbb{P}(Perm).
\]

This is directly deduced of the definition of the current operad \([7]\). 

**The operad \((G_p, p)\)-Ass**

It is defined by the module of relations generated by the vector

\[
(x_1, x_2, x_3) - x_1(x_2, x_3) + (x_1, x_2, x_3) - x_2(x_1, x_3),
\]

and \( R \) is the linear span of

\[
(x_1, x_2, x_3), \quad (x_1, x_2, x_3), \quad (x_1, x_3, x_2).
\]

**Proposition 8.** A \((G_p, p)\)-algebra is an associative algebra \( A \) satisfying

\[
abc = -acb,
\]

for any \( a, b, c \in A \).

Since \((G_p, p)\)-Ass is basically isomorphic to \((G_p, p)\)-Ass we deduce that \((G_p, p)\)-Ass is not Koszul.

**The operad \((G_p, p)\)-Ass**

Remark that a \((G_p, p)\)-algebra is generally called flexible algebra.

The relation

\[
A_p(x_1, x_2, x_3) + A_p(x_2, x_1, x_3) = 0
\]

is equivalent to \( A_p(x, y, x) = 0 \) and this denotes the flexibility of \((A, \mu)\).

**Proposition 9.** A \((G_p, p)\)-algebra is an associative algebra satisfying

\[
abc = -cba.
\]

This implies that \( \dim \ (G_p, p)\)-Ass = 3. We have also

\[
\begin{align*}
&\exists_{x_1 x_2 x_3 \in \mathbb{K}} \left( (-1)^{\left| \sigma \right|} x_{\sigma(1)} x_{\sigma(2)} x_{\sigma(3)} \right) \text{ for any } \sigma \in \Sigma_3, \\
&\text{This gives dim } \ (G_p, p)\)-Ass = 4. Similarly
\end{align*}
\]

\[
\begin{align*}
&x_{x_1 x_2 x_3 x_4} = x_{x_3 x_1 x_2 x_4} = x_{x_1 x_2 x_3 x_4} = x_{x_1 x_2 x_3 x_4}, \\
&x_{x_1 x_2 x_3 x_4} = x_{x_1 x_2 x_3 x_4} = x_{x_1 x_2 x_3 x_4} = x_{x_1 x_2 x_3 x_4},
\end{align*}
\]

The algebra is associative so we put some parenthesis just to explain how we pass from one expression to the other. We deduce \((G_p, p)\)-Ass \( (5) = \{0\} \) and more generally \((G_p, p)\)-Ass \( (a) = \{0\} \) for \( a \geq 5 \).

The generating function of \((G_p, p)\)-Ass is

\[
f(x) = x + x^2 + \frac{x^3}{2} + \frac{x^4}{12}.
\]

Let \( F(G_p, p)_{Ass}(V) \) be the free \((G_p, p)\)-Ass\( (V) \)-algebra based on the vector space \( V \). In this algebra we have the relations

\[
\begin{align*}
&[a^2 = 0, \\
&[a b c = 0, \\
&[a b a = 0].
\end{align*}
\]

for any \( a, b, c \in \mathbb{K} \). Assume that \( \dim V = 1 \). If \( \{e_i\} \) is a basis of \( V \), then \( e_i^3 = 0 \) and \( F(G_p, p)_{Ass}(V) = F(G_p, p)_{Ass}(V) \). We deduce that \( F(G_p, p)_{Ass}(V) \) is not a Koszul algebra.

**Proposition 10.** The operad for flexible algebra is not Koszul.

Let us note that, if \( \dim V = 2 \) and \( \{e_i, e_j\} \) is basis of \( V \), then \( F(G_p, p)_{Ass}(V) \) is generated by \( \{e_1, e_2, e_1^2, e_1 e_2, e_2 e_1, e_2 e_2, e_1 e_1 e_2, e_2 e_1 e_1, e_2^2 e_1, e_1 e_2 e_2\} \) and is of dimension 12.

**Proposition 11.** We have

\[
G_{p - p} - Ass = (G_{p - p} - Ass).
\]

This means that \( G_{p - p} - Ass \) is an associative algebra \( A \) satisfying \( abc = cba \) for any \( a, b, c \in A \).

**The operad \((G_p, p)\)-Ass**

It coincides with \((G_p, -p)\)-Ass and this last has been studied in studies of Remm \([2]\).

**The operad \((G_p, p)\)-Ass**

A \((G_p, p)\)-algebra \((A, \mu)\) satisfies the relation

\[
A_p(x_1, x_2, x_3) + A_p(x_1, x_3, x_2) + A_p(x_2, x_1, x_3) + A_p(x_2, x_3, x_1) + A_p(x_3, x_1, x_2) + A_p(x_3, x_2, x_1) = 0.
\]

The dual operad \((G_p, p)\)-Ass is generated by the relations

\[
(x_1 x_2 x_3, x_1 x_2 x_3) = (x_1 x_2 x_3, x_1 x_2 x_3),
\]

and \( \lambda \in \Sigma_3 \).

We deduce

**Proposition 12.** A \((G_p, p)\)-algebra is an associative algebra \( A \) which satisfies

\[
abc = -bac = -cgb = cba = cba,
\]

for any \( a, b, c \in A \). In particular

\[
[a^2 = 0, \\
[aba = aab = bab = 0].
\]

**Lemma 13.** The operad \((G_p, p)\)-Ass satisfies \((G_p, p)\)-Ass \( (4) = \{0\} \).

**Proof.** We have in \((G_p, p)\)-Ass \( (4) \) that

\[
(x_1 x_2 x_3 x_4) = x_1 x_2 x_3 x_4 = x_1 x_2 x_3 x_4 = x_1 x_2 x_3 x_4 = x_1 x_2 x_3 x_4,
\]

so \( x_1 x_2 x_3 x_4 = 0 \). We deduce that the generating function of \((G_p, p)\)-Ass is

\[
f(x) = x + x^2 + \frac{x^3}{6}.
\]

If this operad is Koszul the generating function of the operad \((G_p, p)\)-Ass should be of the form

\[
f(x) = x + x^2 + \frac{11}{6} x^3 + \frac{25}{6} x^4 + \frac{127}{12} x^5 + \cdots.
\]

But if we look the free algebra generated by \( V \) with \( \dim V = 1 \), it satisfies \( a^2 = 0 \) and coincides with \( F(G_p, p)_{Ass}(V) \). Then \((G_p, p)\)-Ass is not Koszul.
Proposition 14. We have
\[ G \to p^* Ass \to \text{LieAdm} \]
that is the binary quadratic operad whose corresponding algebras are associative and satisfying
\[ abc = acb = bac. \]

Cohomology and Deformations

Let \((A,\mu)\) be a \(K\)-algebra defined by quadratic relations. It is attached to a quadratic linear operad \(P\). By deformations of \((A,\mu)\), we mean [10]

- A \(K\) non archimedian extension field of \(K\), with a valuation \(v\) such that, if \(A\) is the ring of valuation and \(\mathcal{M}\) the unique ideal of \(A\), then the residual field \(A/\mathcal{M}\) is isomorphic to \(K\).
  - The \(A/\mathcal{M}\) vector space \(\mathcal{A}\) is \(K\)-isomorphic to \(A\).
  - For any \(a, b \in A\) we have that
    \[ \mu(a,b) = \mu(a,b) \]
    belongs to the \(A\)-module \(\mathcal{A}\) (isomorphic to \(A \otimes \mathcal{M}\)).

The most important example concerns the case where \(A\) is \(K[[t]]\), the ring of formal series. In this case \(\mathcal{M} = \{ \sum_{i} a_i t^i, a_i \in K \}, \ K = K((t))\) the field of rational fractions. This case corresponds to the classical Gerstenhaber deformations. Since \(A\) is a local ring, all the notions of valued deformations coincides [11].

We know that there exists always a cohomology which parametrizes deformations. If the operad \(P\) is Koszul, this cohomology is the "standard"-cohomology called the operadic cohomology. If the operad \(P\) is not Koszul, the cohomology which governs deformations is based on the minimal model of \(P\) and the operadic cohomology and deformations cohomology differ [12].

In this section we are interested by the case of left-alternative algebras, that is, by the operad \((G_{-p^*} Ass)\) and also by the classical alternative algebras.

Deformations and cohomology of left-alternative algebras

A \(K\)-left-alternative algebra \((A,\mu)\) is a \(K\)-\((G_{-p^*} Ass)\)-algebra. Then satisfies
\[ A_{x}(x_1, x_2, x_3) + A_{y}(x_2, x_1, x_3) = 0. \]
A valued deformation can be viewed as \(K[[t]]\)-algebra \((A \otimes K[[t]], \mu_t)\) whose product \(\mu_t\) is given by
\[ \mu_t = \mu + \sum \phi_i t^i. \]

The operadic cohomology: It is the standard cohomology \(H_{(G_{-p^*} Ass)}(A,\mu)\) of the \((G_{-p^*} Ass)\)-algebra \((A,\mu)\). It is associated to the cochains complex
\[ C^1_p(A,\mu) \to C^2_p(A,\mu) \to C^3_p(A,\mu) \to \cdots\]
where \(P = (G_{-p^*} Ass)\) and
\[ C^1_p(A,\mu) = \text{Hom}(P(p) \otimes A^{\otimes p}, A). \]

Since \((G_{-p^*} Ass)'(4) = 0\), we deduce that
\[ H^p_p(A,\mu) = 0 \quad \text{for} \quad p \geq 4, \]
because the cochains complex is a short sequence
\[ C^p_p(A,\mu) \to C^2_p(A,\mu) \to C^3_p(A,\mu) \to \cdots\]
\[ \delta

The cohomology of \((G_{-p^*} Ass)\) is a homology isomorphism
\[ (G_{-p^*} Ass, 0) \to H(G(E), \delta) \]
of dg-operads such that the image of \(\delta\) consists of decomposable elements of the free operad \(\Gamma(E)\). Since \((G_{-p^*} Ass)(1) = K\), this minimal model exists and it is unique. The deformations cohomology \(H^p_p(A,\mu)\) of \(A\) is the cohomology of the complex
\[ C^p_p(A,\mu) = \text{Hom}(\otimes_p E_p(q) \otimes A^{\otimes p}, A). \]

The Euler characteristics of \(E(g)\) can be read off from the inverse of the generating function of the operad \((G_{-p^*} Ass)\)
\[ g(t) = t + t^2 + \frac{3}{2} t^3 + \frac{5}{2} t^4 + \frac{53}{12} t^5 \]
which is
\[ g(t) = t - t^2 + \frac{13}{3} t^3 + O(t^4). \]
We obtain in particular
\[ \chi(E(p)) = 0. \]
Each one of the modules \(E(p)\) is a graded module \((E_p)\) and
\[ \chi(E(p)) = \dim E_p - \dim E_{p+1} + \dim E_{p+2} \cdots \]
We deduce
- \(E(2)\) is generated by two degree 0 bilinear operation \(\mu_1 : V \times V \to V, \)
- \(E(3)\) is generated by three degree 1 trilinear operation \(\mu_2 : V^{\otimes 3} \to V, \)
- \(E(4) = 0\).

Considering the action of \(\Sigma_p\) on \(E(n)\) we deduce that \(E(2)\) is generated by a binary operation of degree 0 whose differential satisfies
\[ \delta(\mu_1) = 0. \]
\(E(3)\) is generated by a trilinear operation of degree one such that
\[ \delta(\mu_2) = \mu_1 \circ \mu_1 - \mu_1 \circ \mu_1 + \mu_1 \circ \mu_2 \circ \mu_1 - (\mu_1 \circ \mu_2 \circ \mu_1) \circ \mu_1 \]
(we have \((\mu_2 \circ \mu_2)(a,b,c) = b(ac)\))
Since \(E(4) = 0\) we deduce

Proposition 15. The cohomology \(H^p_p(A,\mu)\) which governs deformations of right-alternative algebras is associated to the complex
\[ C^p_p(A,\mu) \to C^2_p(A,\mu) \to C^3_p(A,\mu) \to \cdots \]
with
\[ C^p_p(A,\mu) = \text{Hom}(V^{\otimes p}, V), \]
\[ C^2_p(A,\mu) = \text{Hom}(V^{\otimes 2}, V), \]
\[ C^3_p(A,\mu) = \text{Hom}(V^{\otimes 3}, V), \]
\[ C^4_p(A,\mu) = \text{Hom}(V^{\otimes 4}, V). \]

\[ C^p_p(A,\mu) \to C^2_p(A,\mu) \to C^3_p(A,\mu) \to \cdots \]
In particular any 4-cochains consists of 5-linear maps.

Alternative algebras
Recall that an alternative algebra is given by the relation
\[ A_s(x_1, x_2, x_5) = -A_s(x_2, x_1, x_5) = A_s(x_1, x_5, x_2). \]

**Theorem 16.** An algebra \((A, \mu)\) is alternative if and only if the associator satisfies
\[ A_p + \Phi^4 = 0, \]
with \( p = 2I + r_1 + r_2 + r_3 + c_1. \)

**Proof.** The associator satisfies \( A_s = \Phi^4 \) with \( v = I + r_2 \) and \( v' = I + r_3. \) The invariant subspace of \( \mathbb{K} \Sigma \) generated by \( v \) and \( v' \) is of dimension 5 and contains the vector \( \sum c_\sigma \sigma. \) From literature of Remm [1], the space is generated by the orbit of the vector \( v. \)

**Proposition 17.** Let \( \text{Alt} \) be the operad for alternative algebras. Its operadic cohomology is the cohomology associated to the operad \( \text{Alt} \).

**Remark.** The current operad \( \tilde{\text{Alt}} \) is the operad for associative algebras satisfying \( ab = bac = cba = ach = bca, \) that is, 3-commutative algebras so \( \tilde{\text{Alt}} = \text{LieAdm}. \)

In literature of Dzhumadil’daev and Zusmanovich [9], one gives the generating functions of \( \text{P} = \text{Alt} \) and \( \text{P} = \text{Alt}^1. \)

\[ g_p(x) = x + 2 \left( x^2 + \frac{7}{31} x^3 + \frac{32}{41} x^4 + \frac{175}{61} x^5 + \frac{180}{61} x^6 + O(x^7), \right. \]
\[ g_p(x) = x + 2 \left( x^2 + \frac{5}{31} x^3 + \frac{12}{41} x^4 + \frac{15}{51} x^5, \right. \]
and conclude to the non-Koszulness of \( \text{Alt}. \)

The operadic cohomology is the cohomology associated to the complex
\[ C^0_\alpha(A, A)_0 = \text{Hom}(\text{Alt}(p), \delta_{\text{Alt}}) \]
\[ C^0_\alpha(A, A)_0 \to C^1_\alpha(A, A)_0 \to \cdots \to C^p_\alpha(A, A)_0 \to 0. \]

But if we compute the formal inverse of the function \( -g_{\alpha}(x) \) we obtain
\[ x + x^2 + \frac{5}{6} x^3 + \frac{1}{2} x^4 + \frac{1}{8} x^5 + \frac{11}{72} x^6 + O(x^7). \]

Because of the minus sign it can not be the generating function of the operad \( \text{P} = \text{Alt}. \) So this implies also that both operad are not Koszul. But it gives also some information on the deformation cohomology. In fact if \( \Gamma(E) \) is the free operad associated to the minimal model, then
\[ \dim \gamma(E(2)) = -2, \]
\[ \dim \gamma(E(3)) = -5, \]
\[ \dim \gamma(E(4)) = -12, \]
\[ \dim \gamma(E(5)) = -15, \]
\[ \dim \gamma(E(6)) = +110. \]

Since \( \gamma(E(6)) = \sum (-1)^i \dim E_i, \) the graded space \( E(6) \) is not concentrated in degree even. Then the 6-cochains of the deformation cohomology are 6-linear maps of odd degree.

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