A METHOD FOR THE STUDY OF WHISKERED QUASI-PERIODIC AND ALMOST-PERIODIC SOLUTIONS IN FINITE AND INFINITE DIMENSIONAL HAMILTONIAN SYSTEMS

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Abstract. We describe a method to study the existence of whiskered quasi-periodic solutions in Hamiltonian systems. The method applies to finite dimensional systems and also to lattice systems and to PDE's including some ill posed ones. In coupled map lattices, we can also construct solutions of infinitely many frequencies which do not vanish asymptotically.

1. Introduction

The goal of this note is to describe an approach to the existence of whiskered tori for symplectic maps and Hamiltonian systems. Besides applying to finite dimensional systems, the method applies to lattice systems and to some Hamiltonian partial differential equations, including ill-posed ones. The method also leads to efficient numerical algorithms and can be used to validate numerical computations. These results are developed in detail in [8] (finite dimensional systems), [9] (lattice systems) and [5] (PDE’s) and [15] (numerical algorithms).

Of course, the problem of existence of whiskered tori has a very long history in Hamiltonian mechanics starting with [12, 29]. In this short note we cannot do justice to the extensive literature and compare different methods (the papers above contain more than 20 pages of references). The more modest goal of this note is to give some introduction to the ideas in the above papers so as to serve as a reading guide. Since the detailed estimates are presented in the papers mentioned above we will just describe here the results somehow informally.

Our method to study whiskered tori is an a posteriori treatment of an invariance equation. We show that if there is a function which satisfies some non-degeneracy conditions (including twist, approximate hyperbolicity and that the function is an
embedding) and satisfies the invariance equation up to a sufficiently small error, then there is a true solution of the invariance equation nearby. This exact solution corresponds to a whiskered torus. Up to some obvious symmetries, this exact solution is unique in a neighborhood. One of the key ingredients in our treatment of lattice systems is the introduction of suitable norms which not only define a topology but also have a good algebraic structure. This is accomplished by taking into account the fact that the interactions between nodes of the lattice have some localization properties.

The *a posteriori* approach in KAM theory was emphasized in [22, 21, 23, 28]. In these papers, it was shown that analytic results in an *a posteriori* format together with some quantitative estimates, imply finite differentiability results. In [23] it was shown that *a posteriori* results imply convergence of some perturbation expansions. They also lead rather automatically to results on regularity with respect to parameters.

*A posteriori* results also implied the usual persistence results for quasi-integrable systems (just take the exact solutions of the integrable system as approximate solutions of the quasi-integrable system). But the approximate solutions can be obtained by other methods (numerical calculations, formal expansions, etc.) so that *a posteriori* methods can justify the results of these methods.

We emphasize that the *a posteriori* results presented here requires neither that the system under consideration is close to integrable, nor that it is expressed in action-angle variables, nor that the hyperbolic bundles are trivial and much less that the motion in the hyperbolic directions is reducible to constant coefficients.

One novelty of the method is that we use the geometric properties of the problem very little; the successive corrections do not require transformation theory, but rather are applied additively. This makes the method well adapted for the infinite dimensional situations, provided that we have a good functional analysis framework. Furthermore, the method also leads to very efficient numerical implementations. A Newton step discretizing the function in $N$ Fourier coefficients requires only $O(N)$ storage and $O(N \log N)$ operations. See [15].

The functional analysis framework we use for lattice systems is an extension of a framework developed in [18, 7]. The main idea is to use function spaces that capture the “local effects” but, at the same time, satisfy several Banach algebra-like properties. In this way, the steps in the KAM iteration satisfy estimates very similar to the finite dimensional ones. For PDE’s we adapt the two spaces approach of [16]. It is an important remark that we do not need that the PDE defines an evolution in the whole space. To find solutions of the invariance equation, it suffices that we can define forward evolution in some spaces, backwards evolution in others and that these evolutions have exponential rates of decay. If these evolutions are smoothing, one can deal with unbounded non-linearities.

Taking advantage of several of the features of the method we construct solutions of lattice systems of increasing complexity and consider the limit of the corresponding sequences. In particular we can construct solutions of Klein-Gordon lattices which are almost-periodic. These solutions do not decrease at infinity and the frequencies are all bounded. We think that the role of these solutions in the transport of energy in lattice systems deserves further investigations.
2. The parameterization method for whiskered invariant tori

We recall that a quasi-periodic function of time is a function which can be written in the form
\[ x(t) = \sum_{k \in \mathbb{Z}^\ell} x_k e^{2\pi i k \cdot \omega t}, \]
where \( \omega \in \mathbb{R}^\ell \) is a frequency vector. Equivalently, introducing the function – often called the hull function – \( K(\theta) = \sum_{k \in \mathbb{Z}^\ell} x_k e^{2\pi i k \cdot \theta} \), we have \( x(t) = K(\omega t) \), where \( K \) is a function defined on the torus \( T^\ell \).

If \( \omega \) is rationally independent, the function \( x(t) = K(\omega t) \) is a solution of a given differential equation \( \dot{x} = X(x) \) if and only if \( K \) is a solution of
\[ \partial_\theta K(\theta) = X(K(\theta)), \]
where \( \partial_\theta = \ell \sum_{i=1}^\ell \omega_i \partial_{\theta_i} \).

If \( DK \) has rank \( \ell \), we can think of \( K \) as an embedding of \( T^\ell \) into the phase space \( M \) of \( X \) (which may be infinite dimensional). Equation (2.1) implies that, at a point in \( \text{Range}(K) \) the vector field \( X \) is tangent to \( \text{Range}(K) \). Hence if (2.1) is satisfied, \( \text{Range}(K) \) is a diffeomorphic image of \( T^\ell \) invariant under \( X \).

In the case of a map \( F \), the invariance equation becomes
\[ F(K(\theta)) = K(T^\ell_\omega(\theta)), \]
where \( T^\ell_\omega(\theta) = \theta + \omega \). Equations (2.1) and (2.2) are the centerpieces of our method.

For the sake of brevity in this note, we describe only the case of maps, since the geometric considerations are easier to explain. There are simple arguments which show that the results for flows can be deduced from the results for maps. Nevertheless, giving a direct treatment for the case of flows has the advantage that it can serve as a blueprint for results in PDE’s. This direct treatment for flows can be found in [8] and the treatment for PDE’s can be found in [5].

3. Results for maps in finite dimension

We recall that \( \omega \in \mathbb{R}^\ell \) is Diophantine when there exist \( \kappa > 0, \tau \geq \ell \) such that
\[ |\omega \cdot k - n| \geq \kappa^{-1}|k|^{-\tau}, \quad \forall k \in \mathbb{Z}^\ell \setminus \{0\}, \quad \forall n \in \mathbb{Z}. \]
We denote the set of Diophantine vectors \( \mathcal{D}_\ell(\kappa, \tau) \).

The most important hypothesis of our result is that there is a function \( K_0 : T^\ell \rightarrow \mathcal{M} \) satisfying (2.2) with enough accuracy compared to the rest of the hypotheses. We will assume that \( \mathcal{M} \) is a \( 2n \)-dimensional Euclidean manifold and we will denote by the same symbol its universal covering and its complex extension.

The other hypotheses are roughly that the tangent space at the range of the embedding \( K_0 \) admits an approximately invariant splitting such that the hyperbolic directions have dimension \( 2(n-\ell) \), the center directions have dimension \( 2\ell \) and that there is a twist condition along the center directions.

It is well known that if the torus supports an irrational rotation, the symplectic form restricted to it vanishes (i.e. it is an isotropic manifold). Also the tangent vectors do not grow either in the future or in the past. The preservation of the symplectic structure forces that the vectors symplectically conjugate to tangent vectors do not grow exponentially either. Therefore we are asking that these directions are the only center directions.

To make precise the size of the error, we need some spaces and norms. We will consider functions defined on extensions \( D_{\rho} = \{ \theta \in \mathbb{C}^\ell/\mathbb{Z}^\ell \mid |\text{Im} \theta_i| \leq \rho \} \) of the torus \( T^\ell \) which are analytic in the interior and extend continuously up to the boundary.
We endow the space of such functions with the norm $\|K\|_\rho = \sup_{\theta \in D_\rho} |K(\theta)|$, where $|K(\theta)|$ denotes some norm in the phase space.

**Theorem 1.** Let $(\mathcal{M}, \Omega)$ be an exact symplectic manifold, $U \subset \mathcal{M}$ an open set and assume

$$F : U \subset \mathcal{M} \to \mathcal{M}$$

is an exact symplectic mapping which is real analytic and extends to $\tilde{U}$, a complex extension of $U$. Assume furthermore that $\omega \in D_1(\kappa, \tau)$ is a given Diophantine vector. Let $K_0 : D_\rho \supset \mathbb{T}^\ell \to \tilde{U}$, $\rho > 0$, be a real analytic embedding such that

(H0) For some $\varepsilon$, $\|F \circ K_0 - K_0 \circ T_\omega\|_\rho \leq \varepsilon$.

(H1) There exists $A > 0$ such that $\text{dist}(K_0(D_\rho), \partial \tilde{U}) \geq A$.

(H2) There exists an analytic splitting $T_{K_0(\theta)}\mathcal{M} = \mathcal{E}_{K_0(\theta)}^s \oplus \mathcal{E}_{K_0(\theta)}^c \oplus \mathcal{E}_{K_0(\theta)}^u$ which is approximately invariant in the sense that

$$\text{dist}\left((DF \circ K_0)(\theta)\mathcal{E}_{K_0(\theta)}^{s,c,u}, \mathcal{E}_{K_0(\theta + \omega)}^{s,c,u}\right) \leq \varepsilon, \quad \theta \in D_\rho. \tag{3.2}$$

We furthermore have:

(H2.1) Denoting $\Pi_{K_0(\theta)}^{s,c,u}$ the projections corresponding to the splitting in (H2), we assume moreover that there exist

$$0 < \mu < 1, \quad \eta > 1, \quad \mu\eta < 1$$

and $N \in \mathbb{N}$ such that

$$|DF \circ K_0(\theta + (N - 1)\omega) \cdots DF \circ K_0(\theta)v| \leq \mu^N|v|, \quad v \in \mathcal{E}_{K_0(\theta)}^s,$$

$$|DF^{-1} \circ K_0(\theta - (N - 1)\omega) \cdots DF^{-1} \circ K_0(\theta)v| \leq \mu^N|v|, \quad v \in \mathcal{E}_{K_0(\theta)}^u,$$

$$|DF \circ K_0(\theta + (N - 1)\omega) \cdots DF \circ K_0(\theta)v| \leq \eta^N|v|, \quad v \in \mathcal{E}_{K_0(\theta)}^c,$$

$$|DF^{-1} \circ K_0(\theta - (N - 1)\omega) \cdots DF^{-1} \circ K_0(\theta)v| \leq \eta^N|v|, \quad v \in \mathcal{E}_{K_0(\theta)}^c. \tag{3.3}$$

(H2.2) We have:

$$\text{dist}(\mathcal{E}_{K_0(\theta)}^s, T_{K_0(\theta)}K_0(\mathbb{T}^\ell) \oplus J(K_0(\theta))^{-1} T_{K_0(\theta)}K_0(\mathbb{T}^\ell)) \leq \varepsilon,$$

where $J$ is the matrix representing the symplectic form.

(H3) Denote $N(\theta) = [DK_0(\theta)\mathbb{T} \circ DK_0(\theta)]^{-1}$ and $P(\theta) = DK_0(\theta)N(\theta)$. Assume that the average of the matrix

$$A(\theta) = P(\theta + \omega)^\top \left[DF(K_0)J(K_0)^{-1}P(\theta) - [J(K_0)^{-1}P(\theta + \omega)]\right]$$

on $\mathbb{T}^\ell$ is invertible.

Then, given $0 < \rho' < \rho$, there exist constants $C, \varepsilon_0$ depending explicitly on $|F|_{C^2(\tilde{U})}$, $\kappa$, $\tau$, $\ell$, $\rho'/\rho$, $\|\Pi_{K_0(\theta)}^{s,c,u}\|_\rho$, $\mu$, $\eta$, $|(\text{avg}(A))^{-1}|$, $\|K_0\|_\rho$, $\|N\|_\rho$ such that if $\varepsilon \leq \varepsilon_0$, there exist $K : D_{\rho'} \to \mathcal{M}$ satisfying

$$F \circ K = K \circ T_\omega,$$

$$\|K - K_0\|_{\rho'} \leq C\varepsilon.$$

Furthermore, the invariant torus is whiskered, it has an invariant splitting satisfying (3.3). The distance between the original approximately invariant splitting and the final invariant splitting can also be bounded from above by $C\varepsilon$. 

Furthermore, there is a constant $C_1$ such that if there is another solution $\tilde{K}: D_{\rho'} \to M$ satisfying $\|\tilde{K} - K\|_{\rho'} \leq C_1$, then, there exists $\sigma \in \mathbb{R}^\ell$ such that

$$\tilde{K}(\theta) = K(\theta + \sigma), \quad \theta \in D_{\rho'}.$$

The bundles $E^s, E^u$ could be empty. In this case, the theorem reduces to the standard theorem of existence of maximal Lagrangian tori.

Actually, hypothesis (H2.2) is automatically satisfied under the hypothesis that the dimension of the center space is just $2\ell$ and that the error in the invariance equation is small enough.

In assumption (H2), we assumed that there is an approximately invariant splitting to emphasize that the conditions of Theorem 1 are verifiable in numerical approximations. As is well known in the theory of hyperbolic systems, the existence of approximately invariant hyperbolic splittings implies the existence of exactly invariant hyperbolic splittings which are close to the approximately invariant ones.

The explicit expressions $C$ and $\varepsilon_0$ are slightly cumbersome to write, but they are very easy to program since they are the composition of a few rather easy expressions.

In particular, $\kappa, \rho'/\rho$ enter into $\varepsilon_0$ just as a factor $\kappa^{-4}(\rho'/\rho)^{-4r+2}$. As pointed out in [22, 28] this leads immediately to estimates on the measure of the tori in the quasi-integrable case and also to finite differentiability results. The dependence on the non-degeneracy conditions (twist, hyperbolicity constants, norm of $K$ and of $N$) is also power-like. This leads to “small twist” and “small hyperbolicity” theorems and allows to justify some degenerate perturbation theories. As pointed out in [22, 28], given a finitely differentiable problem one can construct a sequence of analytic problems that approximate it. Taking a solution of a problem as an approximate solution for the next one, and applying the quantitative result, we can obtain a sequence of functions that converge to the solution of the original problem.

4. Sketch of the proof of Theorem 1

The method of proof is an iterative Newton-like method. Having some error $E = F \circ K - K \circ T_\omega$ in the invariance equation (2.2), the Newton method tries to find $\Delta$ in such a way that

$$DF \circ K \Delta - \Delta \circ T_\omega = -E$$

and $\|F \circ (K + \Delta) - (K + \Delta) \circ T_\omega\|' \approx C\|E\|^2$ in some appropriate norms.

Actually we will solve (4.1) up to a quadratic error. As usual in KAM theory, the quadratic estimates will be obtained in a weaker norm than the norm of the original error.

4.1. The use of normal hyperbolicity. The first step is to use standard methods in normal hyperbolicity theory [10, 17, 25] to show that close to the approximately invariant splitting assumed in (H2) there is a truly invariant one.

The standard argument in [17, 25] seeks the invariant space as the graph of a linear map $a(\theta)$ from the approximately invariant space to its complement. One can manipulate the invariance equations into a fixed point equation for $a$ of the form:

$$a(\theta) = A(\theta)a(\theta \pm \omega)B(\theta) + C(\theta) + R(a(\theta)),$$
where $A, B, C$ are linear operators and $R$ is quadratic in $a$. The different rates of contraction assumed in (H2) imply that $\|A(\theta)\| \cdot \|B(\theta)\| \leq \gamma < 1$ and the approximate invariance implies that $C$ is small. Therefore, just using elementary algebra, one can show that if we consider the RHS of (4.2) as an operator acting on $a$, then it is a contraction operator in some ball of a suitable Banach space of analytic functions.

Taking into account that the dynamics on the torus is a rotation, we can obtain that the fixed point $a(\theta)$ depends analytically on $\theta$.

4.2. Solution of the linearized equation. Using the invariant splitting mentioned in the previous section, equation (4.1) can be split into three equations

\[
(4.3) \quad DF \circ K \Delta^{s,c,u} - \Delta^{s,c,u} \circ T_\omega = E^{s,c,u},
\]

where the superscript index indicates the projections onto the truly invariant subspaces.

For the $s, u$ components in (4.3) we can obtain the solutions using the methods mentioned in the previous section.

4.2.1. Solution of the center component of the linearized equation. Solving the equation along the center directions requires the use of symplectic geometry and small divisor equations. The method is an adaptation of the method in [4] which shows that there is an explicit change of variables which allows us to reduce the center component of (4.3) to a constant coefficient equation (up to a quadratic error).

The key observation is that the $2n \times 2\ell$ matrix $M$ formed by the columns of the $2n \times \ell$ matrices $DK$ and $J(K)^{-1}DK N$ satisfies:

\[
(4.4) \quad DF \circ K(\theta) M(\theta) = M(\theta + \omega) \begin{pmatrix} \text{Id} & A(\theta) \\ 0 & \text{Id} \end{pmatrix} + R(\theta),
\]

where $A$ is the matrix introduced in assumption (H3) and $R$ is an explicit algebraic expression involving $DE$, the projections on invariant subspaces and derivatives of $F$. One consequence of (4.4) is that $\text{Range}(M(\theta))$ is close – up to terms that are comparable to the error – to $E^{c}_\theta$.

If we write $\Delta^c = MW$, then (4.3) along the center is equivalent (up to terms which are quadratic in the error $E$) to the following equation for $W$:

\[
(4.5) \quad \begin{pmatrix} \text{Id} & A(\theta) \\ 0 & \text{Id} \end{pmatrix} W(\theta) - W(\theta + \omega) = -(M^T J(K) M)^{-1} M^T J(K) (\theta + \omega) E(\theta).
\]

Taking components in (4.5) gives the two small divisors equations for some RHS $\tilde{E}$

\[
(4.6) \quad W_1(\theta) - W_1(\theta + \omega) = \tilde{E}_1(\theta) - A(\theta) W_2(\theta),
\]

\[
(4.7) \quad W_2(\theta) - W_2(\theta + \omega) = \tilde{E}_2(\theta).
\]

Using an argument inspired by Rüssmann’s translated curve theorem [27, 19] we show that the average of the RHS of (4.7) vanishes. Hence, we can find $W_2$ using the theory of constant coefficient difference equations and determine $W_2$ up to an additive constant. If the twist condition (H3) is met, we can choose the average of $W_2$ so that the average of the RHS of (4.6) is zero. Hence we can find $W_1$ solving a constant coefficient difference equation. We note that $W_1$ is unique up to the
addition of a constant and that this is the only non-uniqueness in the solution of the linearized equation. This is reflected in the uniqueness results.

Once we have found \( W \), that satisfies the linearized equation approximately, it is standard to show that, provided that \( \Delta \) is small enough and that \( K + \Delta \) is in the domain of \( F \) (which is implied by \( \| E \|_\rho (\rho - \rho')^{-2r} \kappa^2 \ll 1 \)), we have

\[
\| F \circ (K + \Delta) - (K + \Delta) \circ T_\omega \|_\rho' \leq C \kappa \frac{1}{\rho} \| E \|_\rho^2.
\]

We also note that the invariant splitting for \( DF \circ K \) is approximately invariant for \( DF \circ (K + \Delta) \) and that the twist condition has only deteriorated by an amount that can be bounded from above by \( \| \Delta \|_\rho' \) and hence by \( \| E \|_\rho \kappa^2 (\rho - \rho')^{-2r} \).

From (4.8) it is standard in KAM theory that, given sufficiently small initial conditions, the procedure can be iterated and that it converges.

The vanishing of the average in the second component of (4.6) is discussed in [4]. The papers [8, 9] use two different methods inspired by the translated curve theorem of [27].

5. Coupled map lattices and coupled oscillators

In this section, we discuss the adaptation of the method of Section 2 in an infinite dimensional framework. Therefore, we will need to define appropriate spaces of embeddings and spaces of diffeomorphisms. We will also need to make sure that the (rather minimal) geometry used in the iterative step also makes sense in the infinite dimensional setting.

In our treatment, a very important role is played by the fact that the interaction between oscillators decays rapidly with the separation among them. Following [18, 7] we will formulate these localization properties using decay functions.

A prototype of the systems we deal with is the coupled Klein-Gordon model

\[
\ddot{x}_i = -V'(x_i) + \delta \sum_{j \in \mathbb{Z}^d} W_j (x_j - x_i), \quad i \in \mathbb{Z}^d,
\]

with \( \| W_j \| \) having a suitable decay with respect to \( j \) (see later). We assume that \( V \) is a potential such that the equation \( \dot{x} = -V'(x) \) admits both an elliptic and a hyperbolic fixed point and we also assume that \( \delta \) is small enough depending on the non-degeneracy properties of the KAM tori in the one particle system.

Our main result is that the systems under consideration admit solutions which are quasi-periodic whiskered breathers. These solutions can be described as follows: most of the sites are close to the hyperbolic fixed point but there are oscillations close to the elliptic fixed point.

The solutions we construct are at the boundary between solutions which are localized and solutions that are propagating energy. We think it would be interesting to study transitions among these solutions and their role in Arnold diffusion.

5.1. Spaces of localized functions. The main technique we will use is the introduction of some Banach spaces of embeddings which capture the notion that the embeddings are essentially localized around a finite number of centers. These adapted norms are chosen in such a way that composition and multiplication satisfy the same estimates as in the finite dimensional space. Therefore, the proof of Theorem 1 presented in Section 3 goes through when we replace the finite dimensional norms by the adapted ones on lattices.
5.1.1. Decay functions. Following [18], we say that \( \Gamma : \mathbb{Z}^d \to \mathbb{R}^+ \) is a decay function when
\[
(a) \quad \sum_{i \in \mathbb{Z}^d} \Gamma(i) \leq 1, \quad (b) \quad \sum_{j \in \mathbb{Z}^d} \Gamma(i - j) \Gamma(j - k) \leq \Gamma(i - k).
\]

For example, the function \( \Gamma_{\alpha, \theta} \) defined by \( \Gamma_{\alpha, \theta}(i) = a |i|^{-\alpha} e^{-\theta |i|} \), if \( i \neq 0 \) and \( \Gamma_{\alpha, \theta}(0) = a \) is a decay function for \( \theta \geq 0, \alpha > d \) provided that \( a \leq a_*(\alpha, \theta, d) \). However, the function \( \exp(-\theta |i|) \) is not a decay function (see [18]).

5.1.2. Phase space of lattice systems. The phase space of the lattice map system will be
\[
\mathcal{M} = \ell^\infty = \{ x \in M^{\mathbb{Z}^d} \mid \sup_{i \in \mathbb{Z}^d} |x_i| < \infty \},
\]
where \( M = \{ (x, y) \in \mathbb{C}^k / \mathbb{Z}^k \times \mathbb{C}^J \mid |\text{Im } x|, |\text{Im } y| \leq \rho \} \) is the complex extension of an Euclidean exact symplectic manifold, with a symplectic form \( \Omega \).

We consider \( \mathcal{M} \) endowed with the \( \ell^\infty \) topology.

5.1.3. Diffeomorphisms with decay properties. Given a decay function \( \Gamma \) we define the space \( C^1_\Gamma \) of \( C^1 \) diffeomorphisms \( F \) in \( \mathcal{M} \) such that they are real analytic and their derivatives admit the representation
\[
DF_i(x)v = \sum_{j \in \mathbb{Z}^d} \frac{\partial F_i}{\partial x_j}(x)v_j
\]
for all \( i \in \mathbb{Z}^d \) and \( v \in \mathcal{M} \), and \( \| F \|_{C^1_\Gamma} := \sup_{i, j \in \mathbb{Z}^d} \frac{|\partial F_i|}{|\partial x_j|} \Gamma(i - j)^{-1} < \infty \).

We also denote by \( C^1_\Gamma \) the set of vector fields \( X \) that satisfy the previous conditions. We will refer to the functions in these spaces as diffeomorphisms with decay or vector fields with decay.

Remark. Prof. L. Sadun remarked that the condition (5.2) for the interactions has a very transparent physical interpretation. Indeed, note that \( |\partial_j F^i| \) can be interpreted as a bound of the direct effect of system \( j \) on the system \( i \). On the other hand, \( |\partial_k F^i| |\partial_j F^k| \) is a bound on the effect of the system \( j \) on the system \( i \) interacting through the system \( k \). Hence \( \sum_{k \in \mathbb{Z}^d} |\partial_k F^i| |\partial_j F^k| \) is an upper estimate of the effect of \( j \) upon \( i \) through modifications of the medium. Hence condition (5.2) ensures that the bound on the direct interaction between two systems also dominates the interactions through the medium.

Remark. The axiomatic definition of the space \( C^1_\Gamma \) is nontrivial because we have given the \( \ell^\infty \) topology on the lattice. As emphasized in [7] there are linear operators in \( \ell^\infty \) which are not determined by their matrix elements. For example, the linear operator \( A : \ell^\infty(\mathbb{Z}) \to \mathbb{R} \) defined by \( Ax = \lim_{n \to -\infty} x_i \) on a closed subspace (then extended by Hahn-Banach theorem) satisfies \( \frac{\partial (Ax)}{\partial x_i} = 0 \) for all \( i \in \mathbb{Z} \) so that the matrix of partial derivatives is zero, but clearly \( A \) is not identically zero. The existence of these observables with vanishing partial derivatives has the physical meaning of existence of “observables at infinity”.

In \( \ell^\infty \) we can consider linear operators \( A \) which are given by their matrix representation, i.e.
\[
(Av)_i = \sum_{j \in \mathbb{Z}^d} A_{i,j}v_j
\]
with \( ||A||_\Gamma = \sup_{i,j \in \mathbb{Z}^d} |A_{i,j}| \Gamma(i-j)^{-1} < \infty \). We call them decay operators. By (5.2) we have the Banach algebra property \( ||AB||_\Gamma \leq ||A||_\Gamma ||B||_\Gamma \).

An important consequence of the properties of decay functions is that if \( F, G \in C^1_\Gamma \) then \( F \circ G \in C^1_\Gamma \). More generally we define \( C^r_\Gamma \) maps as the maps \( F : \mathcal{B} \subset \mathcal{M} \to \mathcal{M} \) such that \( F \in C^r(\mathcal{B}) \) and \( D^jF(x) \in C^1_\Gamma(\mathcal{B}, \ell^\infty) \), for \( 0 \leq j \leq r - 1 \).

Another result, whose proof can be found in [9], is the following lemma.

Lemma 2. If \( X \) is a \( C^r_\Gamma \) vector field, the problem

\[
\frac{d}{dt} S_t = X \circ S_t, \quad S_0 = \text{Id},
\]

admits a unique solution \( S_t \in C^1([0,T], C^r_\Gamma) \).

Using Lemma 2, we can obtain results for vector fields from results for maps. Of course, coupled map lattices have been studied extensively on their own [1].

We will consider maps \( F \) of the form \( F = F_0 + \varepsilon F_1 \) with \( F_0 = \bigotimes_{i \in \mathbb{Z}} f \), where \( f : \mathcal{M} \to \mathcal{M} \) is an analytic, exact symplectic map with a positive measure set of KAM tori with twist condition and a hyperbolic fixed point. By assumption, the map \( F_0 \) has a hyperbolic fixed point. We will choose the origin of coordinates in such a way that it corresponds to 0.

5.1.4. Embeddings with decay properties. Given \( c = (c_1, \ldots, c_L) \) with \( c_j \in \mathbb{Z}^d \) and \( \rho > 0 \), we define the space of embeddings from \( D_\rho \supset \mathbb{T}^d \) into \( \mathcal{M} \) which are centered around \( c_1, \ldots, c_L \)

\[
A_{\rho,c,\Gamma} = \left\{ K : D_\rho \to \mathcal{M} \mid \text{continuous embedding, analytic in } \hat{D}_\rho, ||K||_{\rho,\Gamma} < \infty \right\},
\]

where the norm \( ||K||_{\rho,\Gamma} \) is defined by \( \sup_{i \in \mathbb{Z}^d} ||K_i||_\rho \left( \min_{j=1,\ldots,L} \Gamma(i - c_j) \right)^{-1} \).

The key observation is that for all \( i \in \mathbb{Z}^d \), and a fixed sequence \( \rho \) whose centers are very separated, we have:

\[
||D\Gamma\Delta||_\rho \leq \sum_{j \in \mathbb{Z}^d} ||\frac{\partial F_i}{\partial x_j}||_\rho ||\Delta_j||_\rho \leq \sum_{j \in \mathbb{Z}^d} ||DF||_\rho \Gamma(i-j) ||\Delta||_{\rho,\Gamma} \min_{1 \leq m \leq L} \Gamma(j-c_m)
\]

\[
\leq ||DF||_\rho ||\Delta||_{\rho,\Gamma} C \min_{1 \leq m \leq L} \Gamma(i-c_m),
\]

where \( C \) and \( D \) depend on \( \Gamma \). Then we obtain, when the centers are well separated:

\[
(5.3) \quad ||DF\Delta||_{\rho,\Gamma} \leq C ||DF||_\Gamma ||\Delta||_{\rho,\Gamma}.
\]

We have estimates similar to (5.3) for any decay operator in place of \( DF \).

5.2. The analogue of Theorem 1 for lattice systems. The version of Theorem 1 for lattice systems requires the following additional hypotheses

- The map \( F \) is a decay diffeomorphism and is such that \( F(0) = 0 \).
- The approximate invariant embedding is a decay embedding with respect to some centers \( c \).
- The error of the invariance equation is small in the norm \( ||\cdot||_{c,\rho,\Gamma} \).
- The projections over the approximately invariant hyperbolic splitting have decay properties. That is, the operators \( \Pi^\sigma, \sigma = s, c, u \), are bounded in \( ||\cdot||_{c,\rho,\Gamma} \).
We get in the conclusions that the true invariant embedding is also a decay
embedding with respect to the centers, that the distance between the exact solution
of (2.1) and the initial approximation is bounded in \( \| \cdot \|_{c,\rho',\Gamma} \) and that the projections
on the invariant splittings also have decay.

If we take \( K \) to be the embedding corresponding to an uncoupled torus, i.e.
\[
K_i(\theta) = K^i(\theta_i), \quad i = 1, \ldots, \ell,
\]
where \( K^i \) is a parameterization of a torus in the node \( i \), and \( K_i(\theta) = 0 \) otherwise,
the assumptions are satisfied for \( \varepsilon \) sufficiently small.

The modifications on the norms of the embeddings are very straightforward.
Let us discuss only the decay properties of the whiskers. We assume that the
projections \( \Pi^\sigma (\sigma = s, c, u) \) associated to the approximately invariant splitting are
decay operators. That is, they are linear operators that can be represented by their
matrix elements and that
\[
\| \Pi^\sigma \|_\Gamma < \infty.
\]

The assumption that the spaces are quasi-invariant should be modified to assume
that whenever \( \sigma' \neq \sigma \), \( \sigma, \sigma' = s, c, u \in D_p \), we have
\[
\| \Pi^{\sigma'}_{K(\theta + \omega')}DF \circ K(\theta + (N-1)\omega) \cdots DF \circ K(\theta)\Pi^{\sigma}_{K(\theta)}\|_\Gamma \leq \varepsilon.
\]
Similarly, the hyperbolicity condition needs to be modified respectively to
\[
\| \Pi^{\sigma'}_{K(\theta + \omega')}DF \circ K(\theta + (N-1)\omega) \cdots DF \circ K(\theta)\Pi^{\sigma}_{K(\theta)}\|_\Gamma \leq \mu^N \| \Pi^{\sigma}_{K(\theta)}\|_\Gamma
\]
\[
\| \Pi^{u}_{K(\theta - \omega')}DF^{-1} \circ K(\theta - (N-1)\omega) \cdots DF^{-1} \circ K(\theta)\Pi^{u}_{K(\theta)}\|_\Gamma \leq \eta^N \| \Pi^{u}_{K(\theta)}\|_\Gamma.
\]

After these modifications in the hypotheses and the conclusions, the proof of
the result goes through line by line provided we change the norms in the above
calculations to be the norms in the decay spaces.

In particular, the proof of the perturbation result for invariant spaces is based
on the contraction properties which only depend on the Banach algebra properties
of \( \| \cdot \|_\Gamma \) and the assumptions in (5.5).

The equations on the center space are finite dimensional, so that the treatment
in Section 3 does not need any change. As a matter of fact, we just need that
the pull-back of the formal symplectic form \( \Omega_\infty = \sum_{i \in \mathbb{Z}^d} \Omega \) on \( \mathcal{M} \) to the center
subspace (which is isomorphic to the cotangent bundle of a torus) makes sense.
Since this space is finite dimensional and since we take the pull-back through decay
functions, the pull-back is a well-defined symplectic form on the torus.

5.3. Solutions with infinitely many frequencies. Using the a posteriori format
of the theorem, the decay properties of the whiskered tori and the translation
invariance, we can construct increasingly complicated solutions. The idea is that,
given two exact solutions (with two frequencies \( \omega_1 \) and \( \omega_2 \)) localized around two
collections of sites \( c_1, c_2 \), if we translate one of them far away so that the two
solutions interact weakly with each other, then we can consider them as an
approximately invariant solution localized around \( \tilde{c} \equiv c_1 \cup (c_2 + k) \), where \( k \in \mathbb{Z}^d \) is
the translation on the lattice. Then the existence result shows that, if the frequencies are jointly Diophantine, and the solutions satisfy some non-degeneracy and hyperbolicity conditions, there is a true solution localized around \( \tilde{c} \).

The process can be repeated indefinitely. In each step of the procedure, the hyperbolicity and non-degeneracy conditions deteriorate only slightly and we take some slightly weaker decay properties of the interaction. Of course, as we increase the number of frequencies, the Diophantine conditions become worse. The key point is that any smallness condition on the error (in a space corresponding to a slightly slower decay) can be accomplished by translating the trial solutions sufficiently far apart (namely increasing \(|k|\)).

The sequence of solutions constructed by iterating the procedure converges in the sense that the motion on each of the sites (i.e. componentwise on the lattice) converges. Of course, each step produces a severe change in some of the sites, so that the convergence is very non-uniform. Nevertheless, the convergence on the sites is enough to guarantee that the limit satisfies the invariance equation.

Solutions in lattice systems with infinite frequencies were constructed also in [11, 26, 2] based on very different arguments (solutions decreasing [11], frequencies increasing [2]) from the ones used here. The solutions in the above papers are maximal dimensional, rather than whiskered. We also call attention to [24] which produces solutions with infinitely many frequencies in the “beam equations” and to [14, 13] which also consider the existence of quasi-periodic breathers with long range coupling. These solutions have many elliptic normal directions, which we do not consider.

For simplicity, we state the main result for the Klein-Gordon models \((5.1)\).

**Theorem 3.** Consider models of the form \((5.1)\) with analytic potential with decay and assume that the single site equations \( \ddot{x} = -V'(x) \) admit both

- A set \( \Omega_0 \subset D_l(\kappa, \tau) \) of positive measure such that \( \omega \in \Omega_0 \) implies that there is a non-degenerate KAM torus of the single site equations.
- A hyperbolic fixed point.

Assume also that \( \delta \) is sufficiently small.

Then for each infinite frequency \( \omega \in \Omega_0^\infty \) satisfying

\[
\sum_{i=1}^R \omega_i \cdot k_i \geq \kappa_R^{-1} \left( \sum_{i=1}^R |k_i| \right)^{-\tau}, \quad \forall k \in (Z^l)^R \setminus \{0\}
\]

for some \( \kappa_R > 0 \) and \( \tau_R \geq R \ell \), and for every \( \alpha > d, \theta \geq 0 \), there exists a sequence \( c \equiv \{c_j\}_{j \in \mathbb{N}} \subset \mathbb{Z}^d \) and an embedding \( K : (T^l)^\infty \to M \) satisfying \((2.1)\).

Furthermore, the embedding satisfies the estimate for all \( i \in \mathbb{Z}^d \)

\[
\|K_i\|_{(D_\rho)^\infty} \leq C \sup_j \Gamma_{\alpha, \theta}(i - c_j),
\]

where \( \| \cdot \|_{(D_\rho)^\infty} \) is the norm \( \| \cdot \|_\rho \) on the extension of the infinite dimensional torus \((T^l)^\infty\).

We also note that if we equip \( \Omega_0 \) with the probability measure \( \Lambda \) obtained by normalizing the Lebesgue measure on it (the Kolmogorov measure), the set of frequencies satisfying \((5.6)\) has full \( \Lambda^\infty \) probability measure.
6. The result for PDE’s

The theory for whiskered tori described in the previous sections is flexible enough to be used to construct whiskered tori for partial differential equations. Particularly, this allows to obtain results in the case of ill-posed (in the sense of Hadamard) equations.

Two examples to which the methods apply are the Boussinesq equation

\[
\frac{\partial^2}{\partial t^2} u(t, x) = \frac{\partial^2}{\partial x^2} u(t, x) + \alpha \frac{\partial^4}{\partial x^4} u(t, x), \quad \alpha > 0
\]

subject to periodic boundary conditions \( u(t, 0) = u(t, 1), (0 \leq t < \infty) \), and the closely related Boussinesq system

\[
\frac{\partial}{\partial t} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & -\partial_x - \alpha \partial_{xxx} \\ -\partial_x & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} \partial_x (uv) \\ 0 \end{pmatrix}
\]

also subject to periodic boundary conditions.

One can think of (6.1) as a formal evolution equation on pairs \( u, \partial_t u \) and of (6.2) as a formal evolution equation for \( u, v \).

Equation (6.1) (resp. (6.2)) is extremely ill posed, as can be seen by the fact that, when we linearize at \( u = 0 \) (resp. \( u = 0, v = 0 \)), we have the dispersion relation

\[
\lambda_k^\pm (\alpha) = \pm |k| 2\pi i \sqrt{1 - 4\pi^2 \alpha k^2}, \quad k \in \mathbb{Z}.
\]

Note that \( \lambda_k^\pm \approx \pm 4\pi^2 \sqrt{\alpha} |k|^2 \) as \( k \to +\infty \).

Nevertheless, one can prove the existence of analytic finite dimensional quasi-periodic solutions. The existence of center manifolds for (6.1) and (6.2) has been considered in [3]. In the case of PDE’s, one has of course to take into account that the evolution is not given by a continuous vector field but rather by some unbounded operators. Of course, in applications, one has to consider the approximate solutions to initialize the KAM procedure. We note that the linear system is degenerate — no twist — but if we consider Lindstedt series solutions of small amplitude, the degeneracy gets removed at some order in the expansion. The error can be made smaller to a much smaller order so that we can apply the small twist theorem.

The main result in this context follows from an abstract theorem following the proof in the finite dimensional case for evolution equations defined in an abstract Banach space. We need to assume that the hyperbolic splitting defined in (H2) is a splitting into closed subspaces. The evolution in the stable subspace is assumed to define a semigroup for positive time, while the evolution in the unstable space is assumed to define a semigroup for negative time. Similar ideas are very standard and have been used in the literature [6, 20]. In the PDE case, the existence of invariant splitting amounts to spectral properties of the linearization.

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