Internal Structure of Einstein–Yang–Mills Black Holes

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(Received 7 July 2018)

It is shown that a generic black hole solution of the SU(2) Einstein–Yang–Mills (EYM) equations develops a new type of an infinitely oscillating behavior near the singularity. Only for certain discrete values of the event horizon radius exceptional solutions exist possessing an inner structure of the Schwarzschild or Reissner–Nordström type.

04.20.Jb, 97.60.Lf, 11.15.Kc

Discovered soon after the regular Bartnik–McKinnon (BK) solutions [1], EYM black holes (BH) [2] provided new insights into the BH physics related to the no–hair and uniqueness theorems [3]. They show spherical properties of the BK particle–like solutions [1] and also exhibit an unusual discreteness (‘quantization’ of the YM field on the event horizon) due to a singular non–linear boundary value problem in the domain between the horizon and the asymptotically flat (AF) infinity. Still, the existing knowledge of the EYM BH’s (unlike the BK objects) is incomplete since only external solutions have been constructed so far (though a qualitative discussion of inner solutions is available [3]). Here we present brief results of our investigation of the interior structure of the EYM BH’s which reveal new surprising features due to coupling of non–linear fields to gravity.

Assume the static spherically symmetric magnetic ansatz for the YM potential

\[ A = (W(r) - 1)(T_\varphi d\theta - T_\theta \sin \theta d\varphi), \]

\( (T_\varphi, T_\theta) \) are spherical projections of the SU(2) generators and the following parametrization of the metric

\[ ds^2 = \frac{(\Delta/r^2)\sigma^2 dt^2 - (r^2/\Delta)dr^2 - r^2 d\Omega^2,} \]

where \( d\Omega^2 = d\theta^2 + \sin^2 \theta d\varphi^2 \), and \( \Delta, \sigma \) depend on \( r \).

The field equations include a coupled system for \( W, \Delta \)

\[ \Delta(W'/r)' + FW' = WV/r, \]

\[ (\Delta/r)' + 2\Delta(W'/r)^2 = F, \]

where \( V = (W^2 - 1), F = 1 - V^2/r^2 \), and a decoupled equation for \( \sigma \):

\[ (\ln \sigma)' = 2W'^2/r. \]

These equations admit BH solutions in the domain \( r \geq r_h \) for any radius of the event horizon \( r_h \). The solutions are specified by the number \( n \in N \) of nodes of \( W \) thus forming a discrete set for each \( r_h \). Although it is not guaranteed a priori that the chart \( \text{\textcopyright} \) is extendible to the full region \( r < r_h \), for AF solutions we did not meet any singularity in the interior region unless the genuine one \( r = 0 \) is reached. In terms of coordinates \( \text{\textcopyright} \) one can find three distinct local power series solutions. The first one is Schwarzschild–like (S), it corresponds to the vacuum value of the YM field \( W(0) = 1 \). Using the mass function \( m(r), \Delta = r^2 - 2m, \) one gets

\[ W = -1 + br^2 + b^2(3 - 8b)^5/(3m_0) + O(r^6), \]

\[ m = m_0(1 - 4b^2r^2 + 8b^4r^4 + 2b^2r^4 + O(r^5)), \]

where \( m_0, b \) are (the only) free parameters.

The second is the Reissner–Nordström (RN) type of solution which can be found assuming the leading term of \( \Delta \) to be a positive constant. This requires \( W(0) = W_0 \neq \pm 1, 0 \) and gives

\[ W = W_0 - W_0r^2/(2V_0) + cr^3/(2W_0V_0) + O(r^4), \]

\[ \Delta = V_0^2 - 2m_0r + 2(c + m_0W_0^2/V_0^2)r^3 + O(r^4), \]

what corresponds to the RN metric of the mass \( m_0 \) and the (magnetic) charge \( P^2 = V_0^2, V_0 = V(W_0) \). The expansion contains three free parameters \( W_0, m_0, c \).

We have also found the third local power series solution assuming a negative value for \( \Delta(0) \) (i.e. imaginary \( P \)):

\[ W = W_0 \pm r - W_0r^2/(2V_0) + O(r^3), \]

\[ \Delta = -V_0^2 - 8W_0r/V_0 + O(r^2), \]

\[ \sigma = \sigma_1(r^2 + 4W_0^3/V_0^2) + O(r^3). \]

Here there is only one free parameter \( (W_0) \) for \( W, \Delta \). The corresponding space–time near the singularity is conformal to the cylinder (after a time rescaling):

\[ ds^2 = r^2(dt^2 - dr^2 - d\theta^2 - \sin^2 \theta d\varphi^2). \]

However, one may suspect that such asymptotics cannot correspond to a generic BH. Imposing ‘boundary conditions’ in the singularity we obtain the same kind of the
singular boundary value problem as one for external solutions where a similar role is played by the AF condition. The latter is known to result in the ‘quantization’ of the allowed values $W_n(r_h)$. Internal ‘boundary value’ problem leads to the second ‘quantization’, now of the event horizon radius $r_h$. Therefore, EYM BH’s with S and RN interiors may constitute only the set of zero measure in the whole EYM BH solution space. For a generic $r_h$ the interior metric is strikingly different.

The system (2–3) was integrated numerically in the region $0 < r < r_h$ using an adaptive step size Runge–Kutta method for various $r_h = 10^{-8}, ..., 10^6$ starting at the left vicinity of the event horizon $r_h$ with one free parameter $W_h = W(r_h)$ satisfying inequalities $|W_h| < 1$ and $1 - W_h^2 < r_h$ which are the necessary conditions for asymptotic flatness. For given $r_h$, the interior solutions meeting the expansions (5–7) may exist only for some appropriate $W_h$. A numerical strategy used to find such $W(r_h)$ consisted in detecting the change of sign of the derivative $W'$. In the S-case we found the curve $W(r_h)$ which starts at $-1$ as $r_h \to 0$ and approaches $-0.1424125$ for large $r_h$ (Fig. 1) (without loss of generality we choose $W_h < 0$). Our S-curve intersects the $n = 1$ member of the family $W^n(r_h)$, corresponding to the set of external AF solutions. Parameters of this BH solution are shown in Tab. 1, its global behavior is depicted in Fig. 2.

*In later publication by Breitenlohner et al. (gr-qc/9703047) S-solution with $n = 3$ was found, so our S-solution is not unique as was claimed in the first version of this paper. We also eliminate here an irrelevant for BH solutions ‘microstructure’ of the RN curve near the boundary (shown previously in Figs. 1,4) which was perhaps a numerical artefact.

**Fig. 1.** $W(r_h)$ for the S– and RN–type interior solutions. Dashed lines – $W^n(r_h)$ for $n = 1, 2$ (higher–$n$ curves lie between the $n = 2$ one and the boundary $r_h = 1 - W_h^2$, dotted line). Note that S and RN curves $W(r_h)$ do not merge.

**Fig. 2.** The $n = 1$ EYM BH (S–type).

**Fig. 3.** The $n = 2$ EYM BH (RN–type).

**Fig. 4.** The $n = 3$ EYM BH (RN–type).

Interior solutions of the RN–type, meeting the expansions (5–7) in the singularity, were found for $r_h > r^*_{h} = 0.990288617$. The corresponding curve $W(r_h)$ (also shown in Fig. 1) intersects the curves $W^n(r_h)$ for all
n ≥ 2. These solutions possess an inner Cauchy horizon at some \( r_c < r_h \) with \( |W(r_c)| > 1 \) (Figs. 3, 4).

Solutions of the third type (7) were studied numerically starting from the vicinity of the origin. The unique solution has been found for the horizon data subject to conditions \( |W_h| < 1 \) and \( 1 - W_h^2 < r_h \) for the upper sign in (6) and \( W(0) = -0.9330656 \), corresponding to \( r_h = 1.889088 \) (Fig. 3). This solution, however, does not meet any value \( W^n(r_h) \) and thus does not present a BH.

Hence, we have found that the EYM BH’s with the ‘standard’ interiors of the S and RN types exist only for certain discrete values of \( r_h \). For all other (continuously varying) \( r_h \) one observes oscillations of \( \Delta \) in the lower half-plane with an infinitely growing amplitude near the singularity. The oscillation region starts with an exponential fall of \( \Delta \) which typically occurs after passing a local maximum \( r^{\text{max}} \) (Fig. 3). In this regime the right hand side of (2) becomes comparatively small with respect to other terms, and one can simplify the system analytically. Neglecting the right side of (2), one obtains the following (approximate) first integral of the system (2):

\[
Z = \Delta W^\sigma/r^2 = \text{const.} \quad (9)
\]

In what follows we will describe in more detail the lower–n case (qualitatively the behavior of \( \Delta \) in the oscillating regime is the same for all \( n \)). Starting from some point \( r_0 \) close to \( r^{\text{max}} (r_0 \lesssim r^{\text{max}}) \) the quantity \( U = W'/r \) becomes approximately constant \( U = U_0 \gg 1/r^{\text{max}} \gg 1 \) (\( r^{\text{max}} \ll 1 \) for lower \( n \)). Then, using the Eqs. (3) and (1), one finds the following estimate for \( \Delta \) valid in the region of the exponential fall (7):

\[
\Delta = -Cr \exp \left[-(U_0 r^{\text{max}})^2\right], \quad (10)
\]

where \( C \) is a positive constant. From this expression it is clear that the fall must stop at the local minimum

\[
r^{\text{min}} = (\sqrt{2} |U_0|)^{-1}.
\]

FIG. 5. Interior solution meeting expansions (7) at \( r = 0 \).

FIG. 6. The beginning of \( \Delta \)-oscillations for \( n = 1, r_h = 2, W(r_h) = -0.342072 \).

FIG. 7. |\( \Delta \)|\(^{-1} \) for the second oscillation cycle of \( n = 1 \) BH solution with \( r_h = 2, W_h = -0.342072 \).

According to (3, 10), \( \sigma \) decreases exponentially during the fall of \( \Delta \):

\[
\sigma^{\text{min}} \sim \sigma^{\text{max}} \exp \left[-(U_0 r^{\text{max}})^2\right], \quad U_0 r^{\text{max}} \gg 1,
\]

(typically \( r^{\text{min}} \ll r^{\text{max}} \)). After \( r^{\text{min}} \) is passed, \( U \) still remains almost unchanged, and hence the exponential factor in (10) becomes irrelevant. It follows that after passing the minimum, \( \Delta \) starts to grow linearly so that \( |\Delta/r| \approx 2n \) is constant. This regime breaks down when \( |\Delta| \) becomes comparable with \( V^2 \). Then a rapid growth of \( U \) takes place, while \( \sigma \) is still changing slowly. At this stage, according to (3), the product \( U \Delta \) remains almost constant. Finally \( \Delta \) reaches the next local maximum (asymptotically points of local maxima approach zero), and then a new oscillation period starts with a grown up amplitude (Tab. 4, Fig. 6). It is clear from (4) that \( \sigma \) monotonically decreases exhibiting rapid falls during exponential falls of \( \Delta \) and keeping almost constant values while \( \Delta \) is growing up. Thus, \( \sigma \) tends to (but does not reach) a zero limit as \( r \to 0 \).
Although the derivative \( W' \) takes rather large absolute values on some intervals, the corresponding variation of \( W \) is still small because these intervals are also extremely small. All the above features (small variation of \( Z \) and \( W \), constancy of \( U \) while \( |\Delta| \) is falling down) become more pronounced while oscillations progress implying that both \( Z \) and \( W \) have finite limits as \( r \to 0 \). Then neglecting the right side of (2), omitting 1 in \( F \), and replacing \( W \) by its limiting value \( W_0 \), one arrives at the following two-dimensional autonomous dynamical system
\[
\dot{q} = p, \\
\dot{p} = (3e^{-q} - 1)p + 2e^{-2q} - 1/2, \\
\]
where \( \Delta = -(V_0^2/2) \exp(q) \), and a dot stands for derivatives with respect to \( \tau = 2\ln(r_h/r) \). This system has one (focal) fixed point \( (p = 0, q = \ln 2) \) with eigenvalues \( \lambda = (1 \pm i\sqrt{15})/4 \), its phase portrait is shown in Fig. 3 together with an invariant set \( p = -e^{-q} - 1/2 \) corresponding to the RN-type solution. The oscillating solutions lie above this curve. The phase motion in this region is unbounded, and there are no limiting circles. The limit \( q = -\infty (\Delta = 0) \) can not be reached, \( \Delta \) remains negative valued as \( r \to 0 \) and passes an infinite sequence of local maxima and minima (Fig. 3).

FIG. 8. Phase portrait of (11), RN – an invariant set corresponding to the RN-type solution (dashed – zero slope lines).

Thus, the inner behavior of the EYM black holes shows the following two unexpected features. The first is ‘second quantization’ of solutions with the ‘S’ and ‘RN’ type interiors: such solutions exist only for discrete values of the horizon radius. The second is an infinitely oscillating nature of the generic interior metric and an associated new type of singularity. No Cauchy horizon is formed in such a regime, although the blueshift at the local minima of the mass function is extremely large and tends to infinity as singularity is approached. Mass function inflates exponentially during the first stage of each oscillation cycle and after some stabilization period comes back to nearly zero values. The upper bound of mass is infinitely growing at the singularity which is space-like conformably with common expectations. However the singularity is not of the mixmaster type. It is described effectively by the second order dynamical system and hence is not chaotic, although huge metric oscillations are encountered.

D. V. G. thanks the Theory Division, CERN for hospitality while the work was in progress. Stimulating discussions with G. A. Alekseev, I. Bakas, G. Clément, I. G. Dymnikova, P. S. Letelier, O. I. Mokhov, M. S. Volkov, and technical assistance of R. N. Zhukov are gratefully acknowledged. The research was supported in part by the RFBR grants 96–02–18899, 18126.

\[ \begin{array}{|c|c|c|}
\hline
\text{S-type, } n = 1 & \text{RN-type, } n = 2 & \text{RN-type, } n = 3 \\
\hline
r_h & 0.613861419 & 1.273791 & 1.0318420 \\
W(r_h) & -0.8478649145 & -0.113763994 & -0.10185163 \\
r_0 & -0.02171654 & 0.08948446 \\
W(0) & -1 & -1.212296124 & -1.3566052 \\
\sigma(0) & 0.2263801 & 5.991210 \cdot 10^{-3} & 1.751928 \cdot 10^{-3} \\
Mass & 0.8807931 & 1.018002 & 1.000277 \\
\hline
\end{array} \]
TABLE II. Oscillations parameters for $r_h = 2$, $W_h = -0.342072$ (n = 1 BH).

| $r_{\text{max}}$ | $r$  | $W(r)$ | $W'(r)/r$ | $\Delta(r)$ |
|------------------|------|--------|------------|-------------|
| $r_1^{\text{max}}$ | 4.32 $\cdot$ 10^{-2} | -0.881410 | 27.89 | -2.52 $\cdot$ 10^{-2} |
| $r_1^{\text{min}}$ | 1.12 $\cdot$ 10^{-2} | -0.922056 | 67.03 | -1.68 $\cdot$ 10^{-1} |
| $r_2^{\text{max}}$ | 6.63 $\cdot$ 10^{-4} | -0.926862 | 7.30 $\cdot$ 10^{3} | -4.16 $\cdot$ 10^{-4} |
| $r_2^{\text{min}}$ | 4.73 $\cdot$ 10^{-5} | -0.930103 | 1.49 $\cdot$ 10^{4} | -8.12 $\cdot$ 10^{36} |
| $r_3^{\text{max}}$ | 6.44 $\cdot$ 10^{-44} | -0.930120 | 4.10 $\cdot$ 10^{81} | -1.36 $\cdot$ 10^{-79} |
| $r_3^{\text{min}}$ | 8.81 $\cdot$ 10^{-83} | -0.930136 | 8.01 $\cdot$ 10^{81} | $-2 \cdot 10^{1.16 \cdot 10^{77}}$ |
| $r_4^{\text{max}}$ | 2.10 $\cdot$ 1.16 $\cdot 10^{77}$ | -0.930136 | 0.5 $\cdot$ 10^{2.32 \cdot 10^{77}} | $-2 \cdot 10^{-2.32 \cdot 10^{77}}$ |