Existence of solutions of some boundary value problems with stochastic volatility

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1. Introduction

The aim of this paper is to study the existence of solutions of some boundary value problems for a partial differential equation whose value of any asset $u(s,v,t)$, including accrued payment satisfy the form

$$
\frac{1}{2} \sigma^2 \frac{\partial^2 u}{\partial s^2} + \rho \sigma \gamma \frac{\partial u}{\partial s} \frac{\partial v}{\partial t} + \frac{1}{2} \sigma^2 \frac{\partial^2 u}{\partial v^2} + r s \frac{\partial u}{\partial s} + \left[ k \theta v - \lambda(s,v,t) \right] \frac{\partial u}{\partial s} - ru + \frac{\partial u}{\partial t} = 0.
$$

(1.1)

The unspeciﬁed term $\lambda(s,v,t)$ represents the price of volatility risk (the risk premium) and must be independent of the particular asset. In theory the parameter $\lambda$ could be determined by one volatility-dependent asset and used to price all other volatility-dependent assets. This suggest that the pricing results are obtained by arbitrage and do not depend on a particular asset. Since $\lambda$ is the risk premium, it could be estimated by using average returns on option positions that are hedged against the risk of changes in the spot asset. $r$ is the constant risk-free rate, $k$ is the mean reversion speed for the variance, $\theta$ is the mean reversion level for the variance while $S(t)$ and $v(t)$ are the price and volatility process respectively at time $t$. The volatility of volatility is $\sigma$. To ensure that zero is an unattainable boundary for the process $v(t)$, then $2k\theta > \sigma^2$. The volatility $v(t)$ process follows the process [1] as

$$
dv(t) = k\theta - v(t))dt + 2k\theta dv_w(t).
$$

(1.2)

and the assumption herein is that the spot asset at time $t$ follows the diffusion

$$
ds(t) = \mu dt + \sqrt{v(t)}d\omega(t).
$$

(1.3)

$\rho \in [-1,1]$ is the correlation coeﬃcients between $w_1(t)$ and $w_2(t)$. Where $w_1(t)$ and $w_2(t)$ are Wiener processes which take account of the leverage effect, stock returns and implied volatility which are negatively correlated. Partial differential equations (PDE.s) are used to model and analyse dynamic systems in ﬁelds as diverse as physics, biology, economics, and ﬁnance. The linear parabolic ones (LPDE.s) are one class of PDE.s which has received particular attention. The LPDEs make up a large class of PDEs which is of a succinctly simple structure such that a thorough analysis of them is possible. In [2] and [3] any interested reader can ﬁnd an introduction and detailed analysis of their properties.

In ﬁnance, for a contingent claim on a single asset, the generic PDE can be written as

$$
\frac{\partial u}{\partial t} + a(x,t) \frac{\partial^2 u}{\partial x^2} + b(x,t) \frac{\partial u}{\partial x} + c(x,t)u = 0,
$$

(1.4)

where $t$ either represents calendar time or time-to-expiry, $x$ represents either the value of the underlying asset or some monotonic function of it (e.g. log($S$); log-spot) and $u$ is the value of the claim (as a function of $x$ and $t$). The terms $a(\cdot)$, $b(\cdot)$ and $c(\cdot)$ are the \textit{diﬀusion}, \textit{convection} and \textit{reaction} coefficients respectively, and this type of PDE is known as a convection-diffusion PDE. This type of PDE can also be written in the form [4].

$$
\frac{\partial u}{\partial t} + a(x,t) \frac{\partial }{\partial x} \left( a(x,t) \frac{\partial u}{\partial x} + b(x,t) \frac{\partial (b(x,t)u)}{\partial x} + c(x,t)u \right) = 0.
$$

(1.5)
This form occurs in the Fokker-Planck (Kolmogorov forward) equation that describes the evolution of the transition density of a stochastic quantity (e.g. a stock value). This can be put in the form of Eq. (1.4) if the functions $\alpha$ and $\beta$ are both once differentiable in $x$ - although it is usually better to directly discretise the form given. A simple application in finance for this PDE can be found in [5]. Through the celebrated Feynman-Kac representation of solutions to PDEs, LPDEs and discussion models are closely linked [6]. This leads to the representation of
In the two-step estimation method of stochastic volatility models was proposed. In the first step, the non-parametrically estimated the (unobserved) instantaneous volatility process. But with the obtained a closed form solution for a European call option the satisfaction of solution PDEs with stochastic volatility. In [8], a study of sensitivities (Greeks) of Asian options with Heston stochastic volatility model parameters. Several techniques have been used by many authors to study the existence of solution PDEs for fully observed diffusion processes are employed, in the second step. This method is an extension of the method in [9], [7] and [10] also obtained a closed form solution for a European call option the satisfies the PDE (1.1) while [11] focused on analytical approximations and a study of sensitivities (Greeks) of Asian options with Heston stochastic volatility model parameters.

In this paper, a set of functions is constructed herein that transforms the problem of Eq. (1.1) into a Laplace equation and a heat equation. Analytical solutions are obtained, and sensitivity analysis given in a concrete setting by the assistance of some boundary conditions.

In what follows, we construct functions $\gamma(s)$, $a(v)$ and $\beta(t)$ so that the set of equations

$$u(s, v, t) = w(s, t)e^{-\phi(s,v)}$$

$$z = z(s) = \gamma(s)$$

$$\tau = \tau(v) = a(v)$$

$$\theta = \theta(t) = \beta(t)$$

transform the financial partial differential Eq. (1.3) into a Laplace equation of the form (when $\lambda = 0$):

$$\frac{\partial^2 w}{\partial z^2} + \frac{\partial^2 w}{\partial \tau^2} = 0,$$  \hspace{1cm} (1.10)

or a heat equation with different diffusivity in $z$ and $\tau$ directions of the form (when $\lambda \neq 0$):

$$k_z \frac{\partial^2 w}{\partial z^2} + k_{\tau} \frac{\partial^2 w}{\partial \tau^2} = \frac{\partial w}{\partial \tau}.$$  \hspace{1cm} (1.11)

2. Result

Given Eq. (1.1) implies that

$$\frac{dw}{dt} = \frac{d}{dt}(we^{-\phi(s,v,t)}) = \frac{\partial w}{\partial s}e^{-\phi(s,v,t)} + w \frac{\partial e^{-\phi(s,v,t)}}{\partial s}$$

$$= \frac{\partial w}{\partial s}e^{-\phi(s,v,t)} - w e^{-\phi(s,v,t)} \frac{\partial \phi}{\partial s}$$

By chain rule for $w = w(s, \tau, t)$ w.r.t gives

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial s} \frac{\partial s}{\partial t} + \frac{\partial w}{\partial \tau} \frac{\partial \tau}{\partial t} + \frac{\partial w}{\partial \theta} \frac{\partial \theta}{\partial t} = \beta(t) \frac{\partial w}{\partial \theta}.$$  \hspace{1cm} (2.1)

Therefore

$$\frac{\partial u}{\partial s} = \frac{\partial w}{\partial s} e^{-\phi(s,v,t)} - w e^{-\phi(s,v,t)} \frac{\partial \phi}{\partial s}.$$  \hspace{1cm} (2.1)

On the other hand

$$\frac{\partial u}{\partial s} = \frac{\partial}{\partial s}(w e^{-\phi(s,v,t)}) = \frac{\partial w}{\partial s}e^{-\phi(s,v,t)} + w \frac{\partial e^{-\phi(s,v,t)}}{\partial s}$$

$$= \frac{\partial w}{\partial s}e^{-\phi(s,v,t)} - w e^{-\phi(s,v,t)} \frac{\partial \phi}{\partial s}.$$
But

\[ \frac{\partial v}{\partial s} = \frac{\partial v}{\partial z} \frac{\partial z}{\partial s} + \frac{\partial v}{\partial \tau} \frac{\partial \tau}{\partial s} + \frac{\partial v}{\partial \theta} \frac{\partial \theta}{\partial s} = \gamma'(s) \frac{\partial v}{\partial z} \]

so that

\[ \frac{d\gamma}{ds} = \gamma'(s) \frac{\partial \gamma}{\partial z} e^{-\theta(s)} - we^{e^{-\theta(s)} \frac{\partial \phi}{\partial s}} \]

Also,

\[ \frac{d\gamma}{ds} = \frac{\partial}{\partial v} (we^{-\theta(s)\gamma}) = \frac{\partial \gamma}{\partial z} e^{-\theta(s)\gamma} + w \frac{\partial}{\partial v} (e^{-\theta(s)\gamma}) \]

\[ = \frac{\partial \gamma}{\partial z} e^{-\theta(s)\gamma} - we^{-\theta(s)\gamma} \frac{\partial \phi}{\partial v} \]

But

\[ \frac{\partial v}{\partial s} = \frac{\partial v}{\partial z} \frac{\partial z}{\partial s} + \frac{\partial v}{\partial \tau} \frac{\partial \tau}{\partial s} + \frac{\partial v}{\partial \theta} \frac{\partial \theta}{\partial s} = \alpha'(v) \frac{\partial v}{\partial \tau} \]

therefore

\[ \frac{d\alpha}{ds} = \alpha'(v) \frac{\partial \gamma}{\partial \tau} e^{-\theta(s)\gamma} - we^{e^{-\theta(s)\gamma} \frac{\partial \phi}{\partial s}} \]

Again

\[ \frac{d^2 \gamma}{ds^2} = \frac{d^2}{dx^2} \gamma(x) = \frac{\partial^2}{\partial s^2} (\gamma(s) \frac{\partial \gamma}{\partial z} e^{-\theta(s)\gamma}) - \frac{\partial}{\partial s} \left( we^{\theta(s)\gamma} \frac{\partial \phi}{\partial s} \right) \]

\[ = \gamma'(s) \frac{\partial \gamma}{\partial z} e^{-\theta(s)\gamma} + \gamma'(s) e^{-\theta(s)\gamma} \frac{\partial}{\partial s} \left( \frac{\partial \gamma}{\partial z} \right) + \gamma'(s) \frac{\partial \gamma}{\partial z} e^{-\theta(s)\gamma} \frac{\partial\phi}{\partial s} \]

\[ + \gamma'(s) \frac{\partial}{\partial s} \left( - e^{-\theta(s)\gamma} \frac{\partial\phi}{\partial s} \right) - \gamma'(s) \frac{\partial \gamma}{\partial z} e^{-\theta(s)\gamma} \frac{\partial\phi}{\partial s} - \frac{\partial^2 \phi}{\partial s^2} e^{-\theta(s)\gamma} + we^{-\theta(s)\gamma} \frac{\partial^2 \phi}{\partial s^2} \]

\[ = \gamma'(s) \frac{\partial \gamma}{\partial z} e^{-\theta(s)\gamma} + \gamma'(s) e^{-\theta(s)\gamma} \frac{\partial}{\partial s} \left( \frac{\partial \gamma}{\partial z} \right) + \gamma'(s) e^{-\theta(s)\gamma} \frac{\partial}{\partial s} \left( \frac{\partial \gamma}{\partial z} \right) \]

\[ + \gamma'(s) e^{-\theta(s)\gamma} \frac{\partial}{\partial s} \frac{\partial \gamma}{\partial z} - \gamma'(s) e^{-\theta(s)\gamma} \frac{\partial}{\partial s} \frac{\partial \gamma}{\partial z} \]

\[ = \gamma'(s) e^{-\theta(s)\gamma} \frac{\partial}{\partial s} \left( \frac{\partial \gamma}{\partial z} \right) - \gamma'(s) e^{-\theta(s)\gamma} \frac{\partial}{\partial s} \frac{\partial \gamma}{\partial z} \]

\[ + \gamma'(s) e^{-\theta(s)\gamma} \frac{\partial}{\partial s} \frac{\partial \gamma}{\partial z} - \gamma'(s) e^{-\theta(s)\gamma} \frac{\partial}{\partial s} \frac{\partial \gamma}{\partial z} \]

\[ = \gamma'(s) e^{-\theta(s)\gamma} \frac{\partial}{\partial s} \frac{\partial \gamma}{\partial z} - \gamma'(s) e^{-\theta(s)\gamma} \frac{\partial}{\partial s} \frac{\partial \gamma}{\partial z} \]

\[ - \gamma'(s) e^{-\theta(s)\gamma} \frac{\partial}{\partial s} \frac{\partial \gamma}{\partial z} - \gamma'(s) e^{-\theta(s)\gamma} \frac{\partial}{\partial s} \frac{\partial \gamma}{\partial z} \]

Figure 4. Profiles of Eq. (2.27), Substituting different Values of the Parameters $v, t, \alpha, \theta$ for 2D Graph.
Hence
\begin{equation}
\frac{\partial^2 u}{\partial t^2} = \gamma (s) \frac{\partial w}{\partial z} - \frac{\partial}{\partial z} \left( \gamma' (s) e^{-\Phi (s, t)} \frac{\partial w}{\partial z} - \gamma' (s) e^{-\Phi (s, t)} \frac{\partial w}{\partial t} \right)
- \omega \frac{\partial^2 \phi}{\partial t^2} e^{-\Phi (s, t)} + we^{-\Phi (s, t)} \frac{\partial^2 \phi}{\partial t^2}
\end{equation}
(2.4)

Also
\begin{equation}
\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial t} \left( \partial u \partial t - we^{-\Phi (s, t)} \partial \phi \partial t \right)
= \frac{\partial}{\partial t} \left( \partial u \partial t - we^{-\Phi (s, t)} \partial \phi \partial t \right)
= \partial^2 \frac{\partial u}{\partial t^2} + we^{-\Phi (s, t)} \partial \phi \partial t
\end{equation}
(2.5)

Hence
\begin{equation}
\frac{\partial^2 u}{\partial t^2} = \partial^2 \frac{\partial u}{\partial t^2} + we^{-\Phi (s, t)} \partial \phi \partial t
\end{equation}
(2.6)

Substituting for Eqs. (2.1), (2.2), (2.3), (2.4), (2.5) and (2.6) in Eq. (1.3) gives
\begin{equation}
\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial t} \left( \partial u \partial t - we^{-\Phi (s, t)} \partial \phi \partial t \right)
= \partial^2 \frac{\partial u}{\partial t^2} + we^{-\Phi (s, t)} \partial \phi \partial t
\end{equation}
(2.7)
Factorizing and dividing Eq. (2.7) by \((e^{-\phi(s,v)})\) gives

\[
\frac{1}{2} \sqrt{\gamma(s)} \frac{\partial \nu}{\partial \xi} + \frac{1}{2} \sqrt{\gamma(s)} \frac{\partial \phi}{\partial \eta} + \frac{1}{2} \sqrt{\gamma(s)} \frac{\partial \phi}{\partial \eta} = \frac{1}{2} \sqrt{\gamma(s)} \frac{\partial \phi}{\partial \eta} = \frac{1}{2} \sqrt{\gamma(s)} \frac{\partial \phi}{\partial \eta} - \frac{1}{2} \sqrt{\gamma(s)} \frac{\partial \phi}{\partial \eta},
\]

\[
\frac{1}{2} \sqrt{\gamma(s)} \frac{\partial \phi}{\partial \eta} = \frac{1}{2} \sqrt{\gamma(s)} \frac{\partial \phi}{\partial \eta} - \frac{1}{2} \sqrt{\gamma(s)} \frac{\partial \phi}{\partial \eta},
\]

\[
\frac{1}{2} \sqrt{\gamma(s)} \frac{\partial \phi}{\partial \eta} = \frac{1}{2} \sqrt{\gamma(s)} \frac{\partial \phi}{\partial \eta} - \frac{1}{2} \sqrt{\gamma(s)} \frac{\partial \phi}{\partial \eta}.
\]

\[
\left\{ \begin{array}{l}
\gamma(s) - 2 \gamma(s) \frac{\partial \phi}{\partial \eta} = 0, \\
\alpha(v) - 2 \alpha(v) \frac{\partial \phi}{\partial \eta} = 0, \\
-rs \frac{\partial \phi}{\partial \eta} + \lambda(s,v,t) \frac{\partial \phi}{\partial \eta} - rw + \beta(t) \frac{\partial \phi}{\partial \eta} = 0.
\end{array} \right.
\]

(2.9)

Equivalently from Eq. (2.8), for \(k = 1, \rho = 0\) and \(\theta = \nu(t)\), one gets;

\[
\left\{ \begin{array}{l}
\gamma(s) - 2 \gamma(s) \frac{\partial \phi}{\partial \eta} = 0, \\
\alpha(v) - 2 \alpha(v) \frac{\partial \phi}{\partial \eta} = 0, \\
-rs \frac{\partial \phi}{\partial \eta} + \lambda(s,v,t) \frac{\partial \phi}{\partial \eta} - rw + \beta(t) \frac{\partial \phi}{\partial \eta} = 0.
\end{array} \right.
\]

(2.10)

From Eq. (2.9), whenever \(\lambda = 0\), one gets

\[
s^2 \phi(s,v) = 0.
\]

(2.11)

In (2.11), set

\[
s^2 \phi^2 = \sigma^2 \alpha^2 \Rightarrow \sigma = \frac{s}{\phi}.
\]

(2.12)

Now let

\[
s^2 \phi^2 = 1 \Rightarrow \sigma = \frac{s}{\phi} \Rightarrow \gamma \equiv \ln s.
\]

(2.13)

Thus,

\[
\alpha \equiv \frac{\nu}{\sigma}
\]

(2.14)

Therefore, Eq. (2.11) is a 2-dimensional Laplace equation of the form as required in Eq. (1.8).

Consequently, for \(\lambda \neq 0\) Eq. (2.9) gives birth to a heat equation with different diffusivity in \(z\) and \(r\) directions of the form as in Eq. (1.9), where \(k_1, k_2 = \frac{c_{01}^2}{\partial^2\gamma(\xi)}\).

Using the first two of Eq. (2.10), it is not difficult to see that

\[
\phi(s,v,t) = \left( \ln s + \Theta(t) + 2 \zeta(v) \right),
\]

and that

\[
\frac{\ln s + \Theta(t)}{2} = \zeta(v).
\]

(2.15)

Also by the third of Eqs. (2.10) and (2.15) we easily see that

\[
rsw' + rw + w \Theta(t) = 0 \Rightarrow \Theta(t) = -\frac{5r}{4},
\]

(2.16)

(2.17)

so that

\[
\Theta(t) = -\frac{5r}{4} + v.
\]

(2.18)

Combining Eqs. (2.16) and (2.18) shows that;

\[
2\zeta(v) = \left( \ln s - \frac{5r}{4} + v \right).
\]

(2.19)

Hence Eq. (2.16) becomes
\[ \phi(s, v, t) = \frac{1}{8} (5t - 2(\ln s + v)). \] \hspace{1cm} (2.20)

**Proposition (2.1):** The solution of Eq. (1.8) is

\[ w(\tau, z) = \frac{2}{L} \sum_{n=1}^{\infty} \left( \int_{0}^{L} f(u) \sin \left( \frac{n \pi u}{L} \right) du \right) \sin \left( \frac{n \pi \tau}{L} \right) \cos \left( \frac{n \pi c}{L} z \right). \] \hspace{1cm} (2.21)

**Proof:** By the method of separation of variables, we assume a solution of the form

\[ w(\tau, z) = T(\tau) Z(z). \]

Substituting for \( w \) in (1.10) yields

\[ \frac{d^2(T(\tau) Z(z))}{dt^2} = \frac{d^2(T(\tau) Z(z))}{dz^2} \Rightarrow T \frac{d^2Z}{dt^2} = \frac{d^2T}{dz^2} Z \]

Thus,

\[ \frac{1}{Z} \frac{d^2Z}{dt^2} = \frac{1}{T} \frac{d^2T}{dz^2} \]

and for constant \( -\xi \), say, we have

\[ \frac{d^2Z}{dt^2} + \xi Z = 0 \Rightarrow \frac{d^2\xi}{dt^2} + \xi T = 0, \]

with bounded solutions (for \( \xi = \lambda^2 \)) given as;

\[ Z(z) = C \cos(\lambda z) + D \sin(\lambda z), \] \hspace{1cm} (2.22)

\[ T(\tau) = A \cos(\lambda \tau) + B \sin(\lambda \tau). \] \hspace{1cm} (2.23)

The boundary condition imposed for the displacement function \( w(\tau, z) \) is for

\[ 0 < \tau < L; \quad z > 0, \quad w(0, z) = w(L, z) = 0 \]

for all \( z \geq 0, \ w(\tau, 0) = f(\tau); \ 0 \leq \tau \leq L, \ \frac{\partial w}{\partial z} \text{ at } (\tau, 0) = 0; \ 0 \leq \tau < L. \]

For the boundary condition \( w(0, z) = w(L, z) = 0 \) for \( z \geq 0 \) we have

\[ w(0, z) = 0 \Rightarrow T(0) Z(z) = 0 \quad \forall \quad z \geq 0, \ T(0) = 0 \Rightarrow T = 0 = A. \]

Also, \( w(L, z) = 0 \Rightarrow T(L) = 0 \Rightarrow T(L) = 0 \Rightarrow B \sin(\lambda L) = 0. \]

\[ \sin(\lambda L) = 0 \Rightarrow \lambda_n = \frac{n \pi}{L} \quad n \in \mathbb{Z} \]

But

\[ T(\tau) = B \sin(\lambda \tau) \Rightarrow T_n(\tau) = B \sin \left( \frac{n \pi \tau}{L} \right) \quad \text{for } n = 1, 2, 3, 4, \ldots \]

For

\[ \frac{\partial v}{\partial t} \text{ at } (\tau, z) = T(\tau) Z'(z), \quad \frac{\partial v}{\partial z} \text{ at } (\tau, 0) = T(\tau) Z(z) = 0 \Rightarrow Z(z) = 0 \]

Thus
Therefore, the general solution is

\[ w(t, z) = T(t)Z(z) = \sum_{n=1}^{\infty} \left( a_n \cos \left( \frac{n\pi z}{L} \right) + b_n \sin \left( \frac{n\pi z}{L} \right) \right) \cos \left( \frac{n\pi z}{H} \right) \]  

Since the solution is expressed in terms of \( n \), one gets,

\[ w(t, z) = C_n \sin \left( \frac{n\pi z}{L} \right) \cos \left( \frac{n\pi z}{H} \right) \tag{2.24} \]

Where \( C_n = BC \). For \( w(t, 0) = f(t) \), the solution is of the form

\[ \sum_{n=1}^{\infty} C_n \sin \left( \frac{n\pi z}{L} \right) = f(t) \tag{2.25} \]

Notice that Eq. (2.25) is Fourier sine series expansion of \( f(t) \) on \([0, L] \).

That is \( f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\pi t + b_n \sin n\pi t) \) on \([-\pi, \pi]\).

Since sine is an odd function, then \( a_0 = 0 \), \( a_n = 0 \), \( b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin nt \, dt \).

For even function \( f(t) \), \( \int_{-\pi}^{\pi} f(t) \cos nt \, dt \), hence \( a_n = \frac{2}{\pi} \int_{0}^{\pi} f(t) \cos nt \, dt \).

Thus \( C_n = \frac{2}{\pi} \int_{0}^{\pi} f(t) \sin \left( \frac{n\pi t}{L} \right) \, dt \).

Therefore, our complete solution is as in Eq. (2.21)

**Proposition 2.2:** The solution of the Heat equation on a rectangle with different diffusivities in the z- and r-directions is given by

\[ w(z, r, v) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{nm} \sin \left( \frac{n\pi z}{L} \right) \cos \left( \frac{m\pi r}{H} \right) e^{-\eta \tau} \tag{2.26} \]

Proof: We solve the heat equation where the diffusivity is different in the z- and r-directions:

On a rectangle \( \{0 < L, 0 < y < H\} \) subject to the BCs

\[ w(0, r, v) = 0, \quad \frac{\partial w}{\partial r}(z, 0, v) = 0, \quad w(L, r, v) = 0, \quad \frac{\partial w}{\partial r}(z, H, v) = 0, \]

and the initial condition

\[ w(z, r, 0) = f(z, r) \]

Separating variables gives,

\[ w(z, r, v) = Z(z)T(r)V(v) \tag{2.27} \]

So that the PDE (1.11) becomes

\[ Z(z)T(r)V'(v) = k_1 Z(z)T'(r)V(v) + k_2 Z(z)T'(r)V(v) \tag{2.28} \]

Dividing by \( Z(z)T(r)V(v) \) gives;

\[ \frac{V'(v)}{V(v)} = k_1 \frac{Z(z)}{Z(z)} + k_2 \frac{T'(r)}{T'(r)} \tag{2.29} \]

Since the left hand side depends on \( v \) and the right hand side depends on \( (z, r) \), both sides must be equal to a constant \(-\eta\) (that is separation of constant), thus;

\[ \frac{V'(v)}{V(v)} = k_1 \frac{Z(z)}{Z(z)} + k_2 \frac{T'(r)}{T'(r)} = -\eta \tag{2.30} \]

Hence,
\( V(v) = ce^{-\nu} \).

Separating the Boundary Conditions gives

\[
\begin{align*}
V(0, \tau, v) &= 0 \Rightarrow Z(0) = 0, \\
V(L, \tau, v) &= 0 \Rightarrow Z(L) = 0,
\end{align*}
\]

\[
\frac{\partial V}{\partial Z}(z, 0, v) = 0 \Rightarrow T'(0) = 0,
\]

\[
\frac{\partial V}{\partial Z}(z, L, v) = 0 \Rightarrow T'(L) = 0.
\]

Rearranging the other part of the equation gives

\[
k_2^2 T(\tau) = \frac{V}{C_0} \eta \frac{V}{C_0} k_1 Z(z)
\]

(2.31)

Since the l.h.s depends on \( y \) and the r.h.s on \( r \) both sides must equal a constant.

Say \( \mu \),

\[
k_2^2 T(\tau) + \eta - \frac{k_1}{k_1} Z(z) = \mu
\]

The problem for \( Z(z) \) is now

\[
Z'(z) + \frac{\mu}{k_1} Z(z) = 0, \quad Z(0) = 0 = Z(L)
\]

The solution is,

\[
Z_n(z) = \sin \left( \frac{n\pi z}{L} \right), \quad \mu_n = k_1 \left( \frac{n\pi}{L} \right)^2, \quad n = 1, 2, 3, \ldots
\]

(2.32)

The problem for \( T(\tau) \) is

\[
T'(\tau) + \frac{\eta}{k_2} T(\tau) = 0 \rightarrow T'(0) = 0 = T'(L).
\]

The solution is

\[
T_m(\tau) = \cos \left( \frac{m\pi \tau}{H} \right), \quad \mu_m = \frac{\eta}{k_2} \mu_n = \frac{m\pi}{H} \frac{m\pi}{H}, \quad m = 1, 2, 3, \ldots
\]

(2.33)

Thus

\[
\eta_m = \mu_m + k_2 \left( \frac{m\pi}{H} \right)^2 = \frac{k_1}{H} \left( \frac{m\pi}{H} \right)^2 + \frac{k_1}{H} \frac{m^2}{H}, \quad m, n = 1, 2, 3, \ldots
\]

and the solution to Eq. (2.26) and boundary conditions is

\[
w_m(z, \tau, v) = \sin \left( \frac{n\pi z}{L} \right) \cos \left( \frac{m\pi \tau}{H} \right) ce^{-\nu v}, \quad m, n = 1, 2, 3, \ldots
\]

(2.34)

Summing over all \( m, n \) to obtain the general solution gives;

\[
w(z, \tau, v) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} w_m(z, \tau, v)
\]

\[
= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \left( \frac{n\pi z}{L} \right) \cos \left( \frac{m\pi \tau}{H} \right) e^{-\nu v},
\]

the constants \( A_{mn} \) are found by imposing the initial condition
\[ w(z, t, \theta) = f(z, \tau) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{nm} \sin \left( \frac{n\pi z}{L} \right) \cos \left( \frac{m\pi \tau}{H} \right). \]  

(2.35)

By the Orthogonality of sines and cosines, multiply by \( \sin k\pi x/L \cos \frac{\lambda t}{H} \) and integrate in \( z \) from 0 to \( L \) and \( \tau \) from 0 to \( H \) to obtain

\[
\int_0^L \int_0^H f(z, \tau) \sin \left( \frac{k\pi z}{L} \right) \cos \left( \frac{\lambda \tau}{H} \right) d\tau dz = \int_0^L \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{nm} \sin \left( \frac{n\pi z}{L} \right) \cos \left( \frac{m\pi \tau}{H} \right) d\tau dz \]

\[
= \int_0^L \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{nm} \int_0^H \sin \left( \frac{k\pi z}{L} \right) \sin \left( \frac{n\pi z}{L} \right) \cos \left( \frac{\lambda \tau}{H} \right) d\tau dz \]

\[
= \int_0^L \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{nm} \int_0^H \frac{L}{2} \delta_{m,n} H \frac{\sin \frac{(k+n)\pi z}{L}}{\frac{(k+n)\pi z}{L}} \cos \left( \frac{\lambda \tau}{H} \right) d\tau dz \]

\[
= \int_0^L \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{nm} \int_0^H \frac{L}{2} \delta_{m,n} H \frac{\sin \frac{(k+n)\pi z}{L}}{\frac{(k+n)\pi z}{L}} \cos \left( \frac{\lambda \tau}{H} \right) d\tau dz \]

\[
= \frac{A_{0}}{4LH} \int_0^L \int_0^H f(z, \tau) \sin \left( \frac{n\pi z}{L} \right) \cos \left( \frac{m\pi \tau}{H} \right) d\tau dz ,
\]

so that

\[
u(s, v, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{nm} \sin \left( \frac{n\pi z}{L} \right) \cos \left( \frac{m\pi \tau}{H} \right) e^{-\sigma z^2 + \gamma (5\tau - 2\ln \sigma)}
\]

(2.37)

In the sequel we state;

**Theorem 2.1:** The set of functions

\[
u(s, v, t) = w(z, \tau, \theta) e^{\phi(s, \tau)}
\]

\[
\begin{align*}
\phi(s, \tau) &= \ln s \\
\theta(t) &= \frac{\lambda}{\sigma} \\
\phi(s, v, t) &= \frac{1}{8} (5\tau - 2\ln s + \sigma)
\end{align*}
\]

transforms the financial PDE of Eq. (1.1) into a Laplace equation of the form (for \( \lambda = 0 \)) of Eq. (1.10) into a heat equation with different diffusivity in \( z \) and \( \tau \) directions of the form (for \( \lambda \neq 0 \)) of Eq. (1.11).

It has been established in [7] that a European call option with strike \( K \) and maturity at time \( T \) satisfies the PDE (1.1) subject to the boundary conditions;

\[
u(s, v, T) = \max(0, s - 1),
\]

\[
u(0, v, t) = 0,
\]

\[
\frac{\partial \nu}{\partial s}(s, v, 0, t) = 1,
\]

\[
r s \frac{\partial \nu}{\partial t}(s, 0, t) + \frac{\partial \nu}{\partial s}(s, 0, t) - r u(s, 0, t) + \frac{\partial u}{\partial t}(s, 0, t) = 0,
\]

\[
u(s, \infty, t) = 0.
\]

(2.38)

The boundary conditions of Eq. (2.38) are applied to Eq. (2.37).

### 3. Numerical results

In this section, some numerical illustrations of our results in the above sections in a specific example is presented using the Maple software. This is to enable us to observe the behaviour of the volatility as against the expected final surplus of the worth of investment (see figure 1–26). The values of parameters that considered in this paper, are the following; \( z = 2, t = 2.5 \), \( s = 0.5 \), \( v = 3.7 \), \( r = 0.06 \), \( L = 3 \), \( \eta = 3.5 \), \( t = 10 \), \( A = 10 \), \( m = n = A = \text{const} \), \( z = 3 \), \( L = 4 \), \( H = 6 \) and applied in Eqs. (2.21) and (2.24). While \( z = 2, u = m \tau = 2.5, s = 500, v = 0.0037, r = 0.06, L = 150, H = 3, \eta = 3.5, t = 60, A = 10 \) are applied in Eqs. (2.26) and (2.27).  

### 4. Discussion of results

This study is supported by efficient numerical simulations showing the behaviour of the system using the values of the parameters.

Figure 1 is the numerical illustrations of the results for different values of the parameters. In (a) to (b) there are existence of incomplete cone shape and unequal amplitude with respect to the parameters showing discontinuity in the buying and selling of asset. This invariably causes distortion in stock prices. Unequal amplitude in (c) and (d) shows sensational variations in terms of prices which is highly periodic in nature. The downwall turn in (e) dictates hedging of assets which in turns leads to recovery in the market.

In Figure 2, similar numerical simulations were obtained but with decreasing amplitude below the horizontal direction. These decrease is cause by the increase in the volatility parameter which replicate sinusoidal behaviour in the stock price.

In Figures 3 and 2 dimensional graph were obtained for different values of the parameters. The trajectory is shown using frequency dependent. Jump continuities which arise as a result of volatility parameter was obtained as one move from (a) to (e). The jump continuities continued over time as the frequency is nearly uniform convergence.

In Figure 4, positive trajectories were obtained with high amplitude. This is caused by the decrease in the volatility in relation to income price. Frequency convergence in term of price was obtained which shows positive synergist profit. The wavelike nature is positive due to high convergence rate.

In Figures 5 and 3 dimensional plot of the stochastic parameters were obtained. The wavelike nature was obtained but restricted to quadratic nature. This quadratic nature shows uncertainty, inequality in the values of assets that is held long or protected against a rise in the value of assets held short. In this case, the price may be inconsistent with the value of the option as predicted by the PDE with stochastic volatility.

In Figures 6 and 3 dimensional graph for different parameters were obtained. In (a) to (c) the wavelike form is cubic in nature showing that the PDE is nonlinear. This nonlinearity causes speculations in the market which depend on the volatility rate.

In Figures 7, 3D graph of approximate analytical solution of (2.27) were obtained. The solutions obtained agree with the solutions obtained by reduced differential transform method and radial basis functions as provided in the references.

### 5. Conclusion

In this study, the separation of variables method and differentiation approach were successfully utilized for the existence of unique solution of partial differentiation equation with stochastic volatility. The transform equation was expressed in terms of the volatility parameters which have effects on the system. These effects were critically examined using numerical simulation. The numerical simulation which describes the behaviour of the system gave different trajectories for different values of the parameters. The different trajectories obtained dictates uncertainty in the price history of the stock market which is determined by the stochastic parameter (v). This invariably leads to instability in the stock price.
Declarations

Author contribution statement

B. O. Osu: Conceived and designed the experiments; Wrote the paper.
E. O. Eze: Analyzed and interpreted the data.
U. E. Obasi: Performed the experiments.
H. I. Ukomah: Contributed materials and analysis tools.

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Additional information

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