Complex Instantons and Charged Rotating Black Hole Pair Creation

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Abstract

We consider the general process of pair-creation of charged rotating black holes. We find that instantons which describe this process are necessarily complex due to regularity requirements. However their associated probabilities are real, and fully consistent with the interpretation that the entropy of a charged rotating black hole is the logarithm of the number of its quantum states.
Black hole pair production is by now a well-established process in semi-classical quantum gravity. Pair-production is a tunnelling process in which the mass-energy of the created pair of black holes is balanced by their negative potential energy in some background field. Studies of this phenomenon for non-rotating black holes have repeatedly provided us with evidence that the exponential of the entropy of a black hole does indeed correspond to the number of its quantum states. It is quite robust, with many different background fields – an electromagnetic field, a positive cosmological constant, a cosmic string, or a domain wall – yielding qualitatively similar features for the pair production process. The probability amplitude for the process is approximated by $e^{-I}$, where $I$ is the action of the relevant instanton i.e. a Euclidean solution to the field equations which interpolates between the states before and after a pair of black holes is produced.

In this paper we consider this problem for the rotating case and show that we must necessarily modify the definition of instanton beyond that which is given by the usual periodic-imaginary time formalism. This formalism, originally developed to enable the computation of tunnelling processes and canonical partition functions has (apparently) led to the belief that the relevant instanton action must always be obtained from real, positive-definite (i.e. Euclidean) metrics. For rotating black holes, the real Euclidean metric is then obtained by supplementing the analytic continuation $t \rightarrow it$ with the transformation $J \rightarrow iJ$, where $J$ is the real angular momentum. We find that this prescription cannot consistently define an instanton that will mediate the creation of a pair of charged rotating black holes in a background with a positive cosmological constant, and that complex instantons must be employed to describe the pair-production of rotating black holes. To our knowledge this is the first example in which naive analytic continuation of parameters in a stationary black hole metric will not correctly describe the relevant physical process.

The relevance of complex metrics for black hole thermodynamics was discussed in ref., and our results are consistent with this interpretation of the functional integral formalism. Although there is no real Euclidean instanton that describes the pair production of rotating black holes, the action associated with the complex instanton is real, and describes the probability of the pair creation of rotating black holes in a manner fully consistent with the interpretation of the entropy as the logarithm of the number of quantum states of the black hole, as in the non-rotating case.

To illustrate this we shall take our background energy to be that provided by a cosmological constant, and consider two oppositely charged black holes undergoing uniform acceleration in this background. This physical situation is described by the following metric:

$$ds^2 = \frac{1}{(p-q)^2} \left[ \frac{1+p^2 q^2}{P} dp^2 + \frac{P}{1+p^2 q^2} (d\sigma - q^2 d\tau)^2 - \frac{1+p^2 q^2}{Q} dq^2 + \frac{Q}{1+p^2 q^2} (d\tau + p^2 d\sigma)^2 \right],$$

generalizing the $C$ metric which describes the spacetime associated with a pair of uniformly accelerating neutral spinless black holes. The accompanying electromagnetic field is
defined by the vector potential

\[ A = -\frac{e_0 q (d\tau + p^2 d\sigma)}{1 + p^2 q^2} + \frac{g_0 p (d\sigma - q^2 d\tau)}{1 + p^2 q^2}, \]

(2)

where \( p, q, \tau, \) and \( \sigma \) (which is periodically identified with some period \( T \)) are coordinate functions, \( P(p) = (-\frac{\Lambda}{6} - q_0^2 + \gamma) - ep^2 + 2mp^3 + (-\frac{\Lambda}{6} - e_0^2 - \gamma)p^4, \) and \( Q(q) = P(q) + \frac{\Lambda}{3}(1+q^4). \) \( \Lambda \) is the cosmological constant (assumed positive), \( \gamma, \epsilon \) are constants connected in a non-trivial way with rotation and acceleration, \( e_0 \) and \( g_0 \) are linear multiples of electric and magnetic charge, and \( m \) is the mass parameter. This solution may be analytically extended across the singularity at \( p = q. \) Then the two accelerating black holes are on opposite sides of that \( p = q \) hypersurface.

Instantons associated with the metric (1) can then be interpreted as describing the pair-creation of rotating charged black holes in a cosmological vacuum provided that the periodic identification of \( \sigma \) does not force the inclusion of additional sources of stress-energy (e.g. struts). These are required if there are conical singularities in (1). By matching the accelerations of the holes to the source of background energy, such conical singularities (located at the roots of \( P \)) may be eliminated from (1) as follows. Suppose \( P \) has two real roots at \( p_0 \pm \alpha \) and two complex roots at \( \hat{p} \pm i \beta, \) where \( p_0, \alpha, \hat{p}, \beta \in \mathbb{R}. \) If \( P(p) \) has an axis of symmetry along the line \( \hat{p} = p_0 \) then conical singularities at \( p_0 \pm \alpha \) may be simultaneously eliminated by identifying \( \sigma \) with period \( T = \frac{4\pi}{P'(p_0 - \alpha)}, \) where \( P' = \frac{dp}{dp}. \) Making a series of \( \beta \) dependent parameter rescalings and coordinate transformations (which up to linear factors associate \( p \leftrightarrow p_0 + \alpha \cos \theta, q \leftrightarrow \frac{1}{r}, \sigma \leftrightarrow \phi, \) and \( \tau \leftrightarrow t), \) and then taking the \( \beta \to 0 \) limit, we obtain the well-known Kerr-Newmann-deSitter (KNDS) solution [11]:

\[ ds^2 = -\frac{Q}{G\chi^4} (dt - a \sin^2 \theta d\phi)^2 + \frac{G}{Q} dr^2 + \frac{G}{\mathcal{H}} d\theta^2 + \frac{\mathcal{H} \sin^2 \theta}{G\chi^4} (adt - [r^2 + a^2] d\phi)^2, \]

(3)

where

\[ G \equiv r^2 + a^2 \cos^2 \theta, \quad \mathcal{H} = 1 + \frac{\Lambda}{3} a^2 \cos^2 \theta, \quad \chi^2 = 1 + \frac{\Lambda}{3} a^2, \quad \text{and} \]

\[ Q = -\frac{\Lambda}{3} r^4 + \left(1 - \frac{\Lambda}{3} a^2 \right) r^2 - 2Mr + \left(a^2 + E_0^2 + G_0^2 \right). \]

and where

\[ F = -\frac{1}{G^2 \chi^2} \left\{ X dr \wedge (dt - a \sin^2 \theta d\phi) + Y \sin \theta d\theta \wedge (adt - (r^2 + a^2) d\phi) \right\}, \]

(4)

is the electromagnetic field, with \( X = E_0 \Gamma + 2aG_0 r \cos \theta, \) \( Y = G_0 \Gamma - 2aE_0 r \cos \theta, \) and \( \Gamma = r^2 - a^2 \cos^2 \theta, M \) is the black hole mass, \( a \) is the rotation parameter, and \( E_0 \) and \( G_0 \) the respective electric and magnetic charges.

Hence cosmological pair creation of charged rotating black holes reduces to a consideration of the nonsingular instantons that can be constructed from the KNDS metric. We have
checked that most of the other methods of eliminating conical singularities from the metric \( \Box \) also lead to this class (although a few yield a broader class of metrics), and so we shall only consider KNDS instantons in the sequel.

We consider cases where the polynomial \( \mathcal{Q} \) has four real roots, three of which are positive and correspond (in decreasing order) to the cosmological, outer and inner black hole horizons. Under appropriate periodic identifications, the maximal extension yields a space-time whose \( t = \) constant hypersurfaces each contain only a single pair of black holes with equal but opposite charge and rotation. To construct regular instantons which mediate rotating black hole pair-creation, we foliate space-time with a set of space-like hypersurfaces \( \Sigma \) labeled by a time coordinate \( t \) (whose forward pointing unit normal is \( u_a = -N dt \)), so that the metric is

\[
 ds^2 = -N^2 dt^2 + h_{ij}(dx^i + V^i dt)(dx^j + V^j dt),
\]

where \( h_{ij} \) is the induced metric on the hypersurfaces, \( N \) is the lapse function, and \( V^i \) is the shift vector field (a three vector field defined on each hypersurface). The extrinsic curvature \( K_{ij} \) and the 3-vectors \( E^i \) and \( B^j \) that describe the electric and magnetic fields defined on \( \Sigma \) must satisfy the constraints

\[
 \mathcal{H} \equiv (3)R + K^2 - K^{ij}K_{ij} - 2(E^2 + B^2) = 0
\]

\[
 \mathcal{H}_i \equiv D_jK_{ij} - D_iK - 2\varepsilon_{ijk}E^jB^k = 0
\]

\[
 \mathcal{F}_{el} \equiv D_jE^j = 0
\]

\[
 \mathcal{F}_{mag} \equiv D_jB^j = 0,
\]

found by projecting the Einstein-Maxwell equations into the hypersurface, so that the specified field configuration is one admitted by the full field equations. Here \( (3)R \) is the Ricci scalar for \( (\Sigma, h) \), \( K = h^{ij}K_{ij} \), \( E^2 = h^{ij}E^iE^j \), \( B^2 = h^{ij}B^iB^j \), \( D_j \) is the covariant derivative on \( \Sigma \) that is compatible with \( h_{jk} \), and \( \varepsilon_{ijk} \) is a Levi-Cevita tensor. Given a full solution \( \{ M, g_{\alpha\beta}, F_{\alpha\beta} \} \) to the equations of motion and a surface \( \Sigma \), the surface fields \( h_{ij}, K_{ij}, E_i, \) and \( B_i \) on \( \Sigma \) are induced via the relations \( h_{ij} = e^\alpha_i e^\beta_j g_{\alpha\beta}, K_{ij} = -e^\alpha_i e^\beta_j \nabla_\alpha u_\beta, E_i = e^\alpha_i F_{\alpha\beta} u^\beta, \) and \( B^i = -\frac{1}{2}e^\alpha_i e^\beta_k e^\gamma_k u_\beta F_{\gamma\delta} \) where the \( e^\alpha_i/e_i^\alpha \) are the projection operators from the tangent/cotangent spaces to points in \( M \) (that are also in \( \Sigma \)) into the intrinsic tangent/cotangent spaces of \( \Sigma \).

To find the probability amplitude for black hole pair-creation we must match the instanton smoothly onto a Lorentzian solution describing the charged rotating pair of black holes across the hypersurface \( \Sigma \), which implies that the induced metric field \( h_{ij} \) and extrinsic curvature \( K_{ij} \) must be continuous there, along with the induced electric and magnetic fields. Using \( \{ x^1 = \phi, x^2 = \theta, x^3 = r \} \) as coordinates on \( \Sigma \) we obtain

\[
 h_{ij} \equiv e^\alpha_i e^\beta_j (g_{\alpha\beta} + u_\alpha u_\beta) = \text{diag}[h_{\phi\phi}, h_{\theta\theta}, h_{rr}],
\]

\[
 K_{ij} \equiv e^\alpha_i e^\beta_j u_{\alpha;\beta} = \begin{bmatrix}
 0 & \frac{h_{\phi\phi}}{2N} & \frac{h_{\phi\theta}}{2N} \\
 \frac{h_{\phi\theta}}{2N} & 0 & 0 \\
 \frac{h_{\phi\theta}}{2N} & 0 & 0
\end{bmatrix},
\]
\[ E_i \equiv e^\alpha_i F_{\alpha\beta} u^\beta = \left[ 0, \frac{F_{\theta t} - F_{t \theta} V^\phi}{N}, \frac{F_{rt} - F_{r \theta} V^\phi}{N} \right], \text{ and } \]

\[ B_i \equiv -\frac{1}{2} e^\alpha_i g_{\alpha\beta} \varepsilon^{\beta\gamma\delta\gamma} u_c F_{\delta\epsilon} = \left[ 0, -\frac{h_{\theta \theta} F_{\phi r}}{\sqrt{h_{\phi \phi} h_{\theta \theta} h_{rr}}}, \frac{h_{rr} F_{\phi \theta}}{\sqrt{h_{\phi \phi} h_{\theta \theta} h_{rr}}} \right] \]

for these quantities.

To construct an instanton that can match onto \( \Sigma_t \) and these fields, we map \( N \rightarrow iN \), \( V^j \rightarrow iV^j \), \( F_{ij} \rightarrow iF_{ij} \), \( F_{jt} \rightarrow iF_{jt} \) (for \( j \in \{ \phi, \theta, r \} \)), which is equivalent to sending \( t \rightarrow i\tau \).

This map preserves the matching quantities \( h_{ij} \), \( K_{ij} \), \( E_i \), and \( B_i \), the constraint equations (6 – 9) and the dynamical equations of motion \([8]\). We then identify the \( \tau \) coordinate with some period \( T \) and further identify this “time” coordinate as a single time at the horizons.

Ensuring regularity is a little more difficult given that the metric as a whole is now complex (easily seen by setting \( t \rightarrow i\tau \) in (3)), but if we examine the hypersurfaces of the instanton where \( \theta = 0 \) and \( \theta = \pi \) we may again work with a real two dimensional Euclidean metric on the surface \( \theta = 0 \) (or \( \pi \)), and eliminate conical singularities by making an appropriate choice of values for the physical parameters \( \Lambda \), \( M \), \( a \), \( E_0 \), and \( G_0 \) (for a full description see ref. [12]).

The resultant complex solutions may be sliced in half by making cuts along the \( \tau = 0 \) and \( \tau = \frac{T}{2} \) hypersurfaces. This gives a hemisphere which may then be attached to a Lorentzian KNdS space-time along a \( t = \) constant hypersurface without difficulty, providing us with an instanton that describes the creation of a space-time containing two rotating charged black holes. Several admissible classes of instantons arise (generalizing earlier cases [2]) which are categorized by their relative black hole and cosmological horizon temperatures. We refer to the class in which both temperatures are equal as the lukewarm class. Here the \( t = \) constant spatial sections consist of two non-degenerate black holes in thermal equilibrium on opposite sides of the cosmological horizon; the analogous case where the black hole horizons are degenerate is called the cold class. The non-degenerate limit in which the cosmological and outer black hole horizons coalesce is called the charged-rotating Nariai class. Although it does not contain any black holes, for non-rotating cases the Nariai case is unstable with respect to quantum tunnelling into a standard RNdS space-time [13], and so is relevant to consider. The classes in which all three horizons are coincident are referred to as ultracold; it is not clear that these space-times have anything to do with black hole pair creation.

To compute the probability of creating a pair of black holes within one of these classes we must carry out a proper calculation of the associated instanton action. This requires the appropriate inclusion of boundary terms, which are necessary to hold physical quantities constant so that the boundary data (10–13) on \( \Sigma \) are fixed [8, 16]. For a given instanton action functional only “paths” that meet these criteria are considered in the full path integral. For the present calculation we must add boundary terms to the action \( I = \frac{1}{16\pi} \int d^4x \sqrt{g}(R - 2\Lambda - F_{\mu\nu}F^{\mu\nu}) \) that fix both the angular momentum and electric charge of our space-times (no additional boundary term is required to fix the magnetic charge). A straightforward but
tedious calculation of the action for each class yields

\[ I_{lw,c,n} = -\frac{1}{8} \sum A_{lw,c,n} \]  

which is the sum of the areas of the non-degenerate horizons in each of the lukewarm (lw), cold (c) and Nariai (n) cases. Since the pair-creation probability is proportional to \( e^{-2I} \), the creation rate of these space-times relative to that of a deSitter space with the same cosmological constant is \( e^{2I_{dS} - 2I} \). Pure deSitter space is always more likely to be created than a black hole spacetime as is illustrated in figure 1 which plots \( I/I_{dS} \) for the instantons. A careful analysis indicates that, for a given value of the mass, as the rotation parameter increases (and the charge correspondingly decreases) the creation rate of lukewarm holes is slightly enhanced, whereas for the cold and Nariai cases it is slightly suppressed. Note also that the creation rate for cold black holes is suppressed relative to the non-extreme (lukewarm) black holes by a factor of \( \exp A_h \) (where \( A_h \) is the outer horizon), as in the non-rotating cases [1, 2, 3].

In cases where we may regard a space-time as being in thermodynamic equilibrium we may reinterpret the path integral as a partition function and thereby use the instantons to calculate its entropy. For each case a direct calculation yields \( S = -2I \), and so the predicted entropies of the space-times are equal to one-quarter the sum of the areas of the non-degenerate horizons, in agreement with non-rotating cases [1, 2, 3].
Our approach is contrary to that taken in earlier studies [17, 18, 11, 19] in which real Euclidean instantons are obtained by complexifying physical parameters, although the possible physical relevance of complex instantons was briefly considered in ref. [11]. As noted previously [8], such metrics have little to do with physical black holes. Indeed, using such a prescription (e.g. $a \rightarrow ia$), the fields $h_{ij}$, $E^i$ and $B^j$ induced by the instanton solution will no longer match their Lorentzian counterparts on $\Sigma$; furthermore the root structure of the function $Q$ is altered, implying that certain KNdS solutions have corresponding real instantons that do not even have the correct number of roots necessary to close them up in the manner discussed above. Consequently it is inconsistent to simultaneously demand that instantons be real and Euclidean whilst matching onto their Lorentzian counterparts.

In order for the relationship between the entropy of a black hole and its rate of pair creation to hold, the black hole space-time should be at least quasi-static. However, the complex instantons discussed here mediate the creation of space-times that are in thermal, but not in full thermodynamic equilibrium, as they are unstable to both discharge and spin-down effects, where the former is expected to occur more quickly [20]. However for black holes whose mass is large relative to their charge and rotation, discharge and spin down will occur relatively slowly and so the space-time may be considered quasi-static even if it is not in full thermodynamic equilibrium. These issues are discussed in more detail in [12].

We have argued that in order to successfully discuss stationary-axisymmetric space-times within the path integral formulation of quantum gravity we must either give up the idea that instantons must match onto the Lorentzian space-times that they are creating along a hypersurface or we must allow the existence of complex instantons. Although this work has been carried out in a cosmological background, we expect that these results will apply for any background which can produce rotating black hole pairs. Since the matching conditions are the only criteria that definitively link a given solution with a given instanton it would seem to be essential to include complex instantons in the generic description of black hole pair-production processes. We have seen that complex instantons smoothly match onto Lorentzian solutions according to the standard laws for matching solutions to the Einstein-Maxwell equations. The pair-creation rates are consistent with the interpretation of the entropy as the logarithm of the number of quantum states of the black hole.

We conclude that the pair creation of all standard black holes may reasonably be described using the path-integral formulation of quantum gravity and the no-boundary proposal.

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