Real-space construction of crystalline topological superconductors and insulators in 2D interacting fermionic systems

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The construction and classification of crystalline symmetry protected topological (SPT) phases in interacting bosonic and fermionic systems have been intensively studied in the past few years. Crystalline SPT phases are not only of conceptual importance, but also provide great opportunities towards experimental realization since space group symmetries naturally exist for any realistic material. In this paper, we systematically classify the crystalline topological superconductors (TSC) and topological insulators (TI) in 2D interacting fermionic systems by using an explicit real-space construction. In particular, we discover an intriguing fermionic crystalline topological superconductor that can only be realized in interacting fermionic systems (i.e., not in free-fermion or bosonic SPT systems). Moreover, we also verify the recently conjectured crystalline equivalence principle for generic 2D interacting fermionic systems.

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identified. Moreover, gapless edge states or anomalous fluxes/flux lines, different SPT phases can be uniquely identifying the braiding statistics of the corresponding gauge internal (unitary) symmetry [25–37] and investigating rotational symmetry) for interacting electronic systems complete understanding of “integer” TSC/TI protected by bosonic and fermionic systems recently [4–19], a complete construction and classification of TSC/TI in interacting fermionic systems become a very important but challenging problem. It turns out that a large class of TSC/TI require certain symmetry protection and they are actually the simplest examples of symmetry-protected topological (SPT) phases [3].

Thanks to the cutting-edge breakthrough in the classification and construction of SPT phases for interacting bosonic and fermionic systems recently [4–19], a complete understanding of “integer” TSC/TI protected by internal symmetry (e.g., time reversal symmetry or spin rotational symmetry) for interacting electronic systems has been achieved [11, 20–24]. In general, by “gauging” the internal (unitary) symmetry [25–37] and investigating the braiding statistics of the corresponding gauge fluxes/flux lines, different SPT phases can be uniquely identified. Moreover, gapless edge states or anomalous surface topological orders have also been proposed as another very powerful way to characterize different SPT phases in interacting systems [23, 38–45].

In recent years, the notion of SPT phases was further extended to systems with crystalline symmetry protection and the so-called crystalline SPT phases have been intensively studied [46–67]. Crystalline SPT phases are not only of conceptual importance, but also provide great opportunities towards experimental realization since space group symmetries naturally exist for any realistic material. Crystalline TI is the simplest example of crystalline SPT phases, and it has already been realized in many different materials [68–71]. For free fermionic systems, there are two systematic methods for classifying and characterizing the crystalline TI: one is the so-called “symmetry indicators” [53, 67, 72–75], which classifies and characterizes the crystalline TI by symmetry representations of band structures at high-symmetry momenta; another is a real-space construction based on the concept of topological crystal [49, 60]. Very recently, boundary modes [76–78] of the so-called higher-order TSC/TI [79–84] protected by crystalline symmetry (with additional time reversal symmetry in certain cases) also attracts a lot of interest in both 2D and 3D. In general, an nth-order TSC/TI protects gapless modes at the system boundary of codimension n. For example, a second-order 3D TI has gapless states on its hinges, while its surfaces are gapped, and a third-order 3D TI has gapless states on its corners, while both its surfaces and hinges are gapped. Nevertheless, most of these studies are still focusing on free fermionic systems and it is not quite clear whether the corresponding gapless boundary modes are stable or not against interactions. On the other hand, for interacting bosonic systems, it was pointed out that the classification of crystalline SPT phases is closely related to the SPT phases with internal symmetry. In Ref. [51], a “crystalline equivalence principle” was proposed with a rigorous mathematical proof: i.e., crystalline topological phases with space group symmetry G are in one-to-one correspondence with topological phases protected by the same internal symmetry G, but acting in a twisted way, where if an element of G is a mirror reflection (orientation-reversing symmetry), it should be regarded as a time-reversal symmetry (antiunitary symmetry). This principle indicates the profound relationship between crystalline SPT phases and SPT phases protected by internal symmetry. Thus, the classification of crystalline SPT phases for free-fermion and interacting bosonic systems can be computed systematically.

Despite to the huge success in understanding crystalline SPT phases for free-fermion and interacting bosonic systems, a systematical understanding of crystalline SPT phases for interacting fermionic systems is still lacking. Although it has been believed that the strategy of classification schemes [51, 59–61] should still work and some simple examples have been studied [62, 64, 85], most studies are focusing on the systems with point group

I. INTRODUCTION

A. The goal of this paper

In the past decade, a lot of efforts have been made on the theoretical prediction and experimental searching for topological superconductors (TSC) and topological insulators (TI) in non-interacting or weakly-interacting systems [1, 2]. However, in realistic materials, strong electronic interactions typically play very important role and can not be neglected or treated as perturbations, especially in low dimensional systems. Therefore, a complete construction and classification of TSC/TI in interacting fermionic systems become a very important but challenging problem. It turns out that a large class of TSC/TI require certain symmetry protection and they can be connected to a trivial disorder phase (e.g. an s-wave BCS-superconductor or an atomic insulator) in the absence of global symmetry. Such kind of “integer” TSC/TI are short-range entangled quantum states and they are actually the simplest examples of symmetry-protected topological (SPT) phases [3].

References
symmetry only and the generic cases are unclear. Recent study on the generalizing of “crystalline equivalence principle” into interacting fermionic systems shed new light towards a complete understanding of crystalline SPT phases for interacting fermion. In Ref. 85, by some explicit calculations for both crystalline SPT phases and SPT phases protected by internal symmetry, it has been demonstrated that the crystalline equivalence principle is still valid for 2D crystalline SPT phases protected by point group symmetry, but in a twisted way, where spinless (spin-1/2) fermions should be mapped into spin-1/2 (spinless) fermions.

In this paper, we aim at systematically constructing and classifying crystalline TSC/TI for 2D interacting fermionic systems and establishing a general paradigm of real-space construction for interacting fermionic crystalline SPT phases. We will consider both spinless and spin-1/2 fermionic systems. In particular, we obtain an intriguing fermionic TSC that cannot be realized in spin-1/2 fermionic systems. We will consider both spinless and classifying crystalline TSC/TI for 2D interacting fermionic systems.

For a specific wallpaper group, the following three major steps:

Cell decomposition: For a specific wallpaper group, firstly we can divide it into an assembly of unit cells; then we divide each unit cell into an assembly of lower-dimensional blocks.

Block-state decoration: For a specific wallpaper group with cell decomposition, we can decorate lower-dimensional block-state on different blocks. A gapped assembly of block-states is called obstruction free decoration.

Bubble equivalence: For a specific obstruction-free decoration, we need to further examine whether such a decoration can be trivialized or not. Finally, the obstruction and trivialization free block state decoration corresponds to a 2D fermionic crystalline SPT phase.

B. Space group symmetry for spinless and spin-1/2 systems

Here we also would like to clarify the precise meaning of “spinless” and “spin-1/2” fermions for systems with and without $U^f(1)$ charge conservation.

For a fermionic system with total symmetry group $G_f$, there is always a subgroup $\mathbb{Z}^G_f = \{1, P_f = (-1)^F\}$, where $F$ is the total number of fermions. $\mathbb{Z}^G_f$ is the center of $G_f$ because all physical symmetries commute with $P_f$, i.e., cannot change fermion parity of the system. In particular, for systems without $U^f(1)$ charge conservation, we can define the bosonic (physical) symmetry group by a quotient group $G_b = G_f/\mathbb{Z}^G_f$. In reverse, for a given physical symmetry group $G_b$, there are many different fermionic symmetry groups $G_f$ which are the central extension of $G_b$ by $\mathbb{Z}_2$. It can be expressed by the following short exact sequence:

$$0 \to \mathbb{Z}_2^G \to G_f \to G_b \to 0$$

and different extensions $G_f$ are characterized by different factor systems of Eq. (1) that are 2-cocycles $\omega_2 \in H^2(G_b, \mathbb{Z}_2)$. Consequently, we denote $G_f$ as $\mathbb{Z}_2^G \times \omega_2 G_b$.

For systems with additional $U^f(1)$ charge conservation, the element of $U(1)$ is $U_g = e^{i\theta F}$. Aforementioned fermion parity operator $P_f = U_g$ is the order 2 element of $U(1)$, hence we denote this charge conservation symmetry by $U^f(1)$ with a superscript $f$. It is easy to notice that $U^f(1)$ charge conservation is a normal subgroup of the total symmetry group $G_f$, which can be expressed by the following short exact sequence:

$$0 \to U^f(1) \to G_f \to G \to 0$$

where $G := G_f/U^f(1)$. In reverse, for a given physical symmetry group $G$, we can define $G_f = U^f(1) \times \omega_2 G$. Here $\omega_2$ is related to the extension of the physical symmetry group $G$. The multiplication of the total symmetry group $G_f$ is defined as:

$$(1, g) \times (1, h) = (e^{2\pi i \omega_2 (g, h) F}, gh) \in G_f$$

with $\omega_2 \in \mathbb{R}/\mathbb{Z} = [0, 1]$ as a $U(1)$ phase, associated with $g, h \in G$. Therefore $\omega_2$ is a 2-cocycle in $H^2(G, \mathbb{R}/\mathbb{Z})$.

The spin of fermions (spinless or spin-1/2) is characterized by different 2-cocycles $\omega_2$ for both cases, and the spinless/spin-1/2 fermions correspond to trivial/nontrivial $\omega_2$. For example, consider even-fold dihedral group $D_{2n}$ symmetry with two generators $R$ and $M$ satisfying $R^{2n} = M^2 = 1$ ($n \in \mathbb{Z}$ and $I$ is identity). Different extensions of fermion parity are characterized by different 2-cocycles $\omega_2$:

$$\omega_2 \in H^2(D_{2n}, \mathbb{Z}_2) = \mathbb{Z}_2^3$$

In particular, the spinless fermions corresponding to the 2-cocycle $\omega_2$ satisfy:

$$\begin{align*}
R^{2n} &= 1 \\
M^2 &= 1
\end{align*}$$

Hence we can simply choose the trivial 2-cocycle $\omega_2(a_g, b_h) = 1$ for $a_g, b_h \in D_{2n}$ for spinless fermions.
The spin-1/2 fermions corresponding to the 2-cocycle $\omega_2$ satisfy:

\[
\begin{align*}
R^{2n} &= P_f \\
M^2 &= P_f \\
\text{MRM}^{-1}R &= 1
\end{align*}
\]  

(6)

To satisfy these conditions, we choose the 2-cocycle $\omega_2$ as follows. For $\forall a, b \in D_{2n}$, we have:

\[
\omega_2(a, b) = \left(\frac{[(-1)^{a+h}b]_{2n} + [(-1)^{h}b]_{2n}}{2n}\right) + (1 - \delta_a)(a + 1)h + g \cdot h
\]  

(7)

where we define $[x]_n \equiv x (\mod n)$, $[x]$ as the greatest integer less than or equal to $x$, and

\[
\delta_a = \begin{cases} 
1 & \text{if } a = 0 \\
0 & \text{otherwise}
\end{cases}
\]  

(8)

Here we notice that the translation operations are not relevant to the spin of fermions.

C. Summary of main results

We summarize all classification results of 2D crystalline TSC for both spinless and spin-1/2 fermionic systems. We label the classification attributed to $\rho$-dimensional block-state decorations by $E_\rho^{\text{2D}}$. For the systems with spinless fermions, the classification results are summarized in Table. I, and the classification data are listed layer by layer, i.e., classification contributed by 0D/1D block-state decorations, respectively. For the systems with spin-1/2 fermions, the classification results are summarized in Table. II layer by layer. Furthermore, we also study the group structure of the classifications by explicitly investigating the possible nontrivial stacking relation between 1D and 0D block-states: For certain cases, stacking of several 1D block-states can be deformed into a 0D block-state, hence the total group could be a nontrivial extension between 1D and 0D block-states, thus stacking two root phases will become another fermionic crystalline TSC. In particular, 1D block-state of the $p4$ case is an intriguing fermionic SPT phase that cannot be realized by free-fermion and interacting bosonic systems.

For 2D crystalline TI protected by both wallpaper group 14 and $U^f(1)$ charge conservation symmetry, we generalize the procedures of real-space construction highlighted in Sec. II to include the internal $U^f(1)$ symmetry. It turns out that 1D block-state decoration does not contribute any nontrivial crystalline topological phase because of the absence of nontrivial 1D root phase in the presence of $U^f(1)$ symmetry. All results of classification are summarized in Table III, and we label the classification indices with fermionic root phase by red, and the classification indices with bosonic root phase by blue.

The rest of the paper is organized as follows: In Sec. II, we introduce the general paradigm of the real-space construction of crystalline SPT phases protected by wallpaper group in 2D interacting fermionic systems. In Sec. III, we explicitly show how to construct and classify the wallpaper group SPT phase in 2D interacting fermionic systems for five different crystallographic systems by using real-space construction, for both spinless and spin-1/2 fermions. All classification results are summarized in Tables I and II. Furthermore, we also classify the crystalline TI in 2D interacting fermionic systems with additional $U^f(1)$ charge conservation by using similar real-space construction scheme in Sec. IV, and the results are summarized in Table III. In Sec. V, by comparing these results with the classification results of 2D FSPT phases.
| $G_b$ | $E^{1D}_{1/2}$ | $E^{0D}_{1/2}$ | $G_{1/2}$ |
|-------|----------------|----------------|----------|
| $p1$  | $Z_2$          | $Z_2$          | $Z_4 \times Z_2$ |
| $p2$  | $Z_2$          | $Z_4$          | $Z_4 \times Z_2^2$ |
| $pm$  | $Z_2$          | $Z_4$          | $Z_4 \times Z_2 \times Z_2$ |
| $pg$  | $Z_2$          | $Z_2$          | $Z_4 \times Z_2$ |
| $cm$  | $Z_2$          | $Z_4$          | $Z_4 \times Z_4$ |
| $pmm$ | 0              | $Z_2^2$        | $Z_2^4$ |
| $pgg$ | $Z_2$          | $Z_4 \times Z_2^2$ | $Z_4 \times Z_2 \times Z_2^2$ |
| $cmm$ | $Z_2$          | $Z_4 \times Z_2$ | $Z_4 \times Z_2 \times Z_2$ |
| $p$   | $Z_2$          | $Z_4 \times Z_2^2$ | $Z_4 \times Z_2^2$ |
| $p4$  | 0              | $Z_2^4$        | $Z_2^8$ |
| $p4g$ | $Z_2$          | $Z_8 \times Z_2^2$ | $Z_2 \times Z_8 \times Z_2^2$ |
| $p3$  | 0              | $Z_2 \times Z_4$ | $Z_2 \times Z_4$ |
| $p3m1$| 0              | $Z_4$          | $Z_4$ |
| $p31m$| 0              | $Z_4 \times Z_4$ | $Z_4 \times Z_4$ |
| $p6$  | $Z_2$          | $Z_{12} \times Z_2 \times Z_4$ | $Z_{12} \times Z_8 \times Z_2$ |
| $p6m$ | 0              | $Z_2^3$        | $Z_2^3$ |

TABLE II. Interacting classification of 2D crystalline TSC for spin-1/2 fermionic systems. The results are listed layer by layer, together with their group structure (represented by $G_{1/2}$). We label the classification indices with fermionic/bosonic root phases with red/blue. For $p4$ case, two of three $Z_4$ fermionic indices are from 4-fold rotation, thus stacking two root phases will obtain a bosonic SPT phase; similar for the fermionic $Z_8$ index of the $p4g$ case. All other fermionic $Z_4$ indices are obtained from nontrivial extension between 1D and 0D block-states, and for these cases, stacking two fermionic root phases will become another fermionic crystalline TSC. In addition, the $Z_{12}$ index of $p6$ case is also obtained from 6-fold rotation, and stacking two fermionic root phases will lead to a bosonic phase.

protected by the corresponding on-site symmetry groups, we verify the crystalline equivalence principle for generic 2D interacting fermionic systems. Finally, conclusions and discussions about further applications of real-space construction and experimental implications are presented in Sec. VI. In Supplementary Materials, we first discuss the 2D crystalline TI protected by point group symmetry and compare the results with the classifications of 2D FSPT phases protected by the corresponding internal symmetry, then we discuss the real space construction of TSC and TI for all remaining cases of wallpaper groups [86].

II. GENERAL PARADIGM OF REAL-SPACE CONSTRUCTION

In this section, we highlight the general paradigm of real-space construction of crystalline SPT phases for 2D interacting fermionic systems. There are three major steps: Firstly, we decompose the whole system into an assembly of unit cells, each unit cell is composed by several lower-dimensional blocks; secondly we decorate some proper lower-dimensional block-states on them and check their validity (for SPT phases, we require a fully gapped bulk ground state without ground-state degeneracy), that is, if the bulk state of a block-state construction cannot be fully gapped, we call such a decoration as obstructed; finally we consider the so-called bubble equivalence to investigate all possible trivializations (We note that certain block-states decorations actually lead to a trivial crystalline SPT phase). An obstruction-free and trivialization-free decoration corresponds to a nontrivial crystalline SPT phase. Below we demonstrate these procedures in full details by using the #14 wallpaper group $p3m1$ as an example.

A. Cell decomposition

For a 2D system with an arbitrary wallpaper group symmetry, we can divide the whole system into an assembly of unit cells, where different unit cells are identical and related by translation symmetries, as illustrated in the left panel of Fig. 1. Therefore, we should only specify the physics in each unit cell because of the presence of translational symmetry.

Then we decompose a specific unit cell of the wallpaper group $p3m1$ into an assembly of lower-dimensional blocks (see the right panel of Fig. 1). Here $R_{W_3}$ represents 3-fold rotational symmetry operation centred at the 0D block labeled by $\mu_3$, and $M_4$ represents the reflection symmetry operation with the axis (indicated by the vertical dashed line in right panel of Fig. 1) coincided with the 1D block labeled by $\tau_1$.

The physical background of the “intra-cell” decomposition is the “extended trivialization” in each cell [49]. Suppose $|\psi\rangle$ is an SPT state that cannot be trivialized by a symmetric finite-depth local unitary transformation. Due to the translational symmetry, $|\psi\rangle$ can be expressed in terms of a direct product of the wavefunctions of all cells:

$$|\psi\rangle = \bigotimes_c |\psi_c\rangle$$ (9)

Because of the presence of the translational symmetry, investigation of a specific $|\psi_c\rangle$ in a cell is enough for understanding the SPT state $|\psi\rangle$, since different $|\psi_c\rangle$’s are related by translational symmetries. As a consequence, $|\psi_c\rangle$ will inherit the property that cannot be trivialized by a symmetric finite-depth local unitary transformation $O^{loc}$. Nevertheless, we can still define an alternative local unitary to extensively trivialize $|\psi_c\rangle$. First we can trivialize the region $\sigma$ (see the right panel of Fig. 1): restrict $O^{loc}$ to $\sigma$ as $O^{loc}_\sigma$ and act it on $|\psi_c\rangle$:

$$O^{loc}_\sigma|\psi_c\rangle = |T_\sigma\rangle \otimes |\psi_c^\sigma\rangle$$ (10)

where the system is in the product state $|T_\sigma\rangle$ in region $\sigma$ and the remainder of the system $\bar{\sigma}$ is in the state $|\psi_c^\bar{\sigma}\rangle$. 

To trivialize the system symmetrically, we denote that $V_g O_{\sigma}^{\text{loc}} V_g^{-1}$ trivializes the region $g \sigma$, where $g$ is the group element of dihedral group $D_3$ generated by 3-fold rotation $R_{\mu_3}$ and reflection $M_{\tau_1}$. Therefore, we act on $|\psi_c\rangle$ with:

$$O^{\text{loc}} = \bigotimes_{g \in D_3} V_g O_{\sigma}^{\text{loc}} V_g^{-1}$$  \hspace{1cm} (11)
which results in an extensively trivialized wavefunction:
\[
|\psi_{c}\rangle = O_{R}^{\text{loc}}|\psi_{c}\rangle = \bigotimes_{g \in D_3} T_{g \sigma} \otimes \bigotimes_{j=1, h \in D_3} 3 |\psi_{h \tau_j}\rangle \otimes \bigotimes_{k=1, p \in D_3} 3 |\psi_{p \kappa_k}\rangle
\]
(12)
where \(\tau_j, j = 1, 2, 3\) and \(\mu_k, k = 1, 2, 3\) label the 1D and 0D blocks as illustrated in the right panel of Fig. 1. Now all nontrivial topological properties of \(|\psi_c\rangle\) are encoded in lower-dimensional block-states \(|\psi_{h \tau_j}\rangle\) and \(|\psi_{p \kappa_k}\rangle\), hence all nontrivial properties of \(|\psi\rangle\) are encoded in lower-dimensional blocks in different unit cells.

B. Block-state decoration

Subsequently, with cell decompositions, we can decorate some proper lower-dimensional block-states on the corresponding lower-dimensional blocks. Some symmetry operations act internally on some lower-dimensional blocks, hence the lower-dimensional block-states should respect the corresponding on-site symmetry on which they decorated. As an example, we still consider the #14 wallpaper group \(p3m1\) with the cell decomposition as illustrated in Fig. 1, the 3-fold rotational symmetry operations act on \(g \mu_j\) \((g \in D_3 \text{ and } j = 1, 2, 3)\) internally, and reflection symmetry operations act on \(h \tau_k\) \((h \in D_3 \text{ and } k = 1, 2, 3)\) internally, hence the root phases decorated on 0D and 1D blocks are 0D FSPT phases protected by \(Z_3 \times Z_2\) on-site symmetry and 1D FSPT phases protected by \(Z_2\) on-site symmetry, respectively. All dD block-states form the group \(G_{dD}\), and all block-states form the following group:
\[
\{\text{BS}\} = \bigotimes_{d=0}^2 G_{dD}
\]
(13)
Here “BS” is the abbreviation of “block-states”.

Furthermore, the decorated states should respect the no-open-edge condition. Once we decorate some lower-dimensional block-states on the corresponding blocks, they might leave several gapless modes on the edge of the corresponding blocks, and there are several gapless edge modes coinciding near the lower-dimensional blocks with lower dimension. Repeatedly consider the wallpaper group \(p3m1\) as an example, if we decorate a Majorana chain on the 1D block labeled by \(\tau_1\) (because of the rotational symmetry, there are also two Majorana chains decorated at the 1D blocks labeled by \(R_{g_3} \tau_1\) and \(R_{g_3}^2 \tau_1\), respectively), leaving 3 dangling Majorana modes near the 0D block labeled by \(\mu_3\). In order to contribute an SPT state, the bulk of the system should be fully gapped, hence the aforementioned gapless modes should be gapped out (by some proper interactions, mass terms, entanglement pairs, etc.) in a symmetric way. If the bulk of the system cannot be fully gapped (i.e., several aforementioned 0D modes cannot be gapped in a symmetric way), we call the corresponding decoration obstructed. Equivalently, an obstruction-free decoration should satisfy the no-open-edge condition. All obstruction-free dD block-states form the group \(\tilde{G}_{dD} \subset G_{dD}\) as a subgroup of \(G_{dD}\), and all obstruction-free block-states form the following group:
\[
\{\text{OFBS}\} = \bigotimes_{d=0}^2 \tilde{G}_{dD} \subset \{\text{BS}\}
\]
(14)
Here “OFBS” is the abbreviation of “obstruction-free block-states”, and \{OFBS\} is a subgroup of \{BS\}.

C. Bubble equivalence

In order to obtain a nontrivial SPT state from obstruction-free block state decorations, we should further consider possible trivializations. For blocks with dimension larger than 0, we can further decorate some codimension 1 degree of freedom that could be trivialized when they shrink to a point. This construction is called bubble equivalence, and we demonstrate it for different dimensions:

a. 2D bubble equivalence For 2D blocks, we can consider a 1D chain which can be shrunk to a point inside each 2D block, and there is no on-site symmetry on them for all possible cases. In fermionic systems, the only possible state we can decorate is Majorana chain. There are two distinct boundary conditions: periodic boundary condition (PBC) with odd fermion parity and anti-periodic boundary condition (anti-PBC) with even fermion parity, see Fig. 2. According to the definition of
bubble equivalence, we only choose the “Majorana bubbles” with anti-PBC because it can be trivialized if we shrink it into a point: if we decorate a Majorana chain with anti-PBC on a 2D block, we can shrink it to a smaller one by a 2D local unitary (LU) transformation without breaking any symmetry. Repeatedly apply this LU transformation on “Majorana” bubble, we can shrink it to a point and eliminate it (because a Majorana chain with anti-PBC has even fermion parity) by a symmetric finite-depth circuit.

Technically, it is well known that for two Majorana fermions $\gamma_j$ and $\gamma_k$, their entanglement pair $i\gamma_j\gamma_k$ can be created by the following projection operator [18, 19]:

$$P_{j,k} = \frac{1}{2} (1 - i\gamma_j\gamma_k)$$

and the direction is from $\gamma_j$ to $\gamma_k$. Consequently the creation operator of a Majorana chain containing $2N$ Majorana fermions with anti-PBC on the 2D block $\sigma$ can be generated by an assembly of these projection operators:

$$A_\sigma = \prod_{i=1}^{N-1} P_{2i, 2i+1} \times \frac{1}{2} (1 + i\gamma_{2N}\gamma_1)$$

Here the last bracket shows the direction of the Majorana entanglement pair $\langle \gamma_1, \gamma_{2N} \rangle$ is from $\gamma_1$ to $\gamma_{2N}$, and it explicitly shows the anti-PBC of the Majorana chain we have created. Finally the operator of creating a 2D “Majorana” bubble in the entire lattice is:

$$A = \bigotimes_\sigma A_\sigma$$

b. 1D bubble equivalence For 1D blocks, we can consider two 0D FSPT modes protected by the corresponding on-site symmetry. These two 0D FSPT modes have the following geometry:

$$a_l^\dagger \quad a_r^\dagger$$

Where yellow and red dots represent two 0D FSPT modes $a_l^\dagger$ and $a_r^\dagger$, and that should be trivialized when they are fused, i.e., $a_l^\dagger a_r^\dagger|0\rangle$ should be a trivial state. We demonstrate that this 1D bubble can be shrunk to a point and trivialized by a finite-depth circuit: if we decorate a 1D bubble, we can enclose $a_l^\dagger$ and $a_r^\dagger$ by an LU transformation. Repeatedly apply this LU transformation, we can shrink this two modes to a point. Equivalently, we can trivialize them by a finite-depth circuit. Therefore, the creation operator of the 1D bubbles in the entire lattice is:

$$B_j = \bigotimes_\tau (a_l^\dagger)^\dagger (a_r^\dagger)^\dagger$$

Enlarge these bubbles and proximate to the nearby lower-dimensional blocks, the FSPT phases decorated on the bubble can be fused with the original states on the nearby lower-dimensional blocks, which leads to some possible trivializations of lower-dimensional block-state decorations.

Suppose there are $m$ different kinds of 1D bubble constructions, labeled by $B_j, j = 1, ..., m$. With this notation we can label an arbitrary bubble construction by an operator:

$$A^{n_0} \prod_{j=1}^\beta B_j^{n_j}, \quad n_0, n_j \in \mathbb{Z}$$

here $n_0/n_j$ means that we take 2D/1D bubble construction $A/B_j$ by $n_0/n_j$. According to the definition of the bubble construction, taking an arbitrary bubble construction on the trivial state will lead to another trivial state, and all these trivial states form another group as following:

$$\{\text{TBS}\} = \left\{ A^{n_0} \prod_{j=1}^\beta B_j^{n_j} | 0 \rangle \mid n_0, n_j \in \mathbb{Z} \right\}$$

Here “TBS” is the abbreviation of “trivial block-states”, and $\{\text{TBS}\} \subset \{\text{OFBS}\}$ because all trivial block-states are obstruction-free block-states. Therefore, an obstruction and trivialization free block-state can be labeled by a group element of the following quotient group:

$$G = \{\text{OFBS}\}/\{\text{TBS}\}$$

and all group elements in $G$ are not equivalent because we have already divided all trivial states connected by bubble constructions. Equivalently, group $G$ gives the classification of the corresponding crystalline topological phases.

In the following, we explicitly apply these procedures to calculate the classification of crystalline TSC and TI by several representative examples for each crystallographic systems.

### III. CONSTRUCTION AND CLASSIFICATION OF CRYSTALLINE TSC

In this section, we describe the details of real-space construction for crystalline TSC in 2D interacting fermionic systems by analyzing several typical examples. It is well known that all 17 wallpaper groups can be divided into five different crystallographic systems:

- **Square lattice:** with rotational symmetry of order 4, including $p4, p4m, p4g$.
- **Parallelogrammatic lattice:** with only rotational symmetry of order 2, and no other symmetry than translational, including $p1, p2$.
- **Rhombic lattice:** with reflection combined with glide reflection, including $cm, cmmn$.
**Rectangle lattice**: with reflection or glide reflection, but not both, including \( pm, pg, pm\tilde{m}, pmg, pgg \).

**Hexagonal lattice**: with rotational symmetry of order 3 or 6, including \( p3, p\bar{3}m1, p\bar{3}m, p6, p6m \).

The key distinction between different crystallographic systems is 0D blocks as centers of different point group.

In particular, we apply the general paradigm of real-space construction highlighted in Sec. II to investigate five representative cases that all of them belong to different crystallographic systems:

1. square lattice: \( p4m \);
2. parallelogrammatic lattice: \( p2 \).
3. rhombic lattice: \( cmm \);
4. rectangle lattice: \( pgg \);
5. hexagonal lattice: \( p6m \);

And all other cases are assigned in Supplementary Materials [86]. The classification results are summarized in Table I and II, for spinless and spin-1/2 fermions, respectively.

Furthermore, there is an intrinsic interacting fermion crystalline TSC obtained by real-space construction that cannot be realized by free fermionic systems or interacting bosonic systems: a 2D spinless system with \( p4m \) wallpaper group symmetry.

**A. Square lattice: \( p4m \)**

For square lattice, we demonstrate the TSC protected by \( p4m \) symmetry as an example. In the remainder of this paper, we use the same label of \( p \)-dimensional blocks that can be related by symmetry actions for abbreviation.

The corresponding point group is dihedral group \( D_4 \), and for 2D blocks \( \sigma \), there is no on-site symmetry group; for 1D blocks \( \tau_1, \tau_2, \tau_3 \), the on-site symmetry group is \( Z_2 \) via the reflection symmetry acting internally; for 0D blocks \( \mu_1 \) and \( \mu_3 \), the on-site symmetry group is \( Z_4 \times Z_2 \); and for 0D blocks \( \mu_2 \), the on-site symmetry group is \( Z_2 \times Z_2 \) via the \( D_4 \) symmetry acting internally, as seen in Fig. 3.

We discuss systems with spinless and spin-1/2 fermions separately. The “spinless”/“spin-1/2” fermion means that the point subgroup is extended trivially/nontrivially by fermion parity \( \mathbb{Z}_2^f \) [85].

For spinless systems, we consider the 0D block-state decoration, for 0D blocks \( \mu_1 \) and \( \mu_3 \), and the classification data of the corresponding 0D block-states can be characterized by different 1D irreducible representations of the full symmetry group \( \mathbb{Z}_2^f \times (Z_4 \times Z_2) \):

\[
\mathcal{H}^1 \left[ \mathbb{Z}_2^f \times (Z_4 \times Z_2), U(1) \right] = \mathbb{Z}_2^3
\]  (22)

For 0D blocks \( \mu_2 \), the classification data of the corresponding 0D block-states can also be characterized by different 1D irreducible representations of the full symmetry group \( \mathbb{Z}_2^f \times (Z_4 \times Z_2) \):

\[
\mathcal{H}^1 \left[ \mathbb{Z}_2^f \times (Z_4 \times Z_2), U(1) \right] = \mathbb{Z}_2^3
\]  (23)

For arbitrary 0D block [whose classification data are calculated in Eqs. (22) and (23)], three \( \mathbb{Z}_2 \) have different physical meanings: the first \( \mathbb{Z}_2 \) represents the parity of complex fermion (even or odd), the second \( \mathbb{Z}_2 \) represents the rotation eigenvalue \(-1\), and the third \( \mathbb{Z}_2 \) represents the reflection eigenvalue \(-1\). So at each 0D block, the block-state can be labeled by \((\pm, \pm, \pm)\), where these three \( \pm \) represent the fermion parity, rotation and reflection eigenvalues, respectively. We should note that even-fold dihedral group can also be generated by two independent reflection operations: for 0D blocks \( \mu_1/\mu_3 \), \( D_4 \) symmetry can be generated by reflection operations \( M_{\tau} \) and \( M_{\sigma} \), \( (M_{\tau}, M_{\sigma}, M_{\tau}) \) represent the reflection operation with respect to the axis which coincide with the 1D block labeled by \( \tau_1, \tau_2, \tau_3 \); for 0D block \( \tau_2, D_2 \) symmetry can be generated by reflection operations \( M_{\sigma} \) and \( M_{\tau} \). Hence the last two \( \pm \)'s can also represent the eigenvalues of two independent reflections. According to this notation, the obstruction-free 0D block-states form the following group:

\[
\{\text{OFBS}\}^{{\text{0D}}}_{p4m,0} = \mathbb{Z}_2^9 \]  (24)

where the group elements can be labeled by:

\[
[(\pm, \pm, \pm), (\pm, \pm, \pm), (\pm, \pm, \pm)]
\]

here three brackets represent the block-states at \( \mu_1, \mu_2 \) and \( \mu_3 \), respectively.

Subsequently we consider the 1D block-state decoration. For \( \tau_1, \tau_2 \) and \( \tau_3 \), the total symmetry group is
$\mathbb{Z}_2 \times \mathbb{Z}_2$, so there are two possible 1D block-states: Majorana chain and 1D FSPT state, and all 1D block-states form a group:

$$\{ \text{BS}\}^{1D}_{j,m,0} = \mathbb{Z}_2^6$$  \hspace{1cm} (25)

Below we discuss the decorations of these two root phases separately.

a. Majorana chain decoration Consider Majorana chain decoration on 1D blocks labeled by $\tau_j$, which leaves 4 dangling Majorana fermions at each 0D block $\mu_1/\mu_3$, and 2 dangling Majorana fermions at each 0D block $\mu_2$. Near $\mu_1$, Majorana fermions have the following rotation and reflection symmetry (all subscripts are taken with modulo 4):

$$R_{\mu_1} : \gamma_j \mapsto \gamma_{j+1}, \quad M_{\tau_2} : \gamma_j \mapsto \gamma_{4-j}$$  \hspace{1cm} (26)

The local fermion parity operator and its symmetry properties read:

$$P_f = -\prod_{j=1}^4 \gamma_j, \quad R_{\mu_1}, M_{\tau_2} : \quad P_f \mapsto -P_f$$  \hspace{1cm} (27)

Hence these four Majorana modes form a projective representation of the symmetry group $(\mathbb{Z}_4 \times \mathbb{Z}_2) \times \mathbb{Z}_2^2$ on 0D block $\mu_1$, and a non-degenerate ground state is forbidden. Thus Majorana chain decoration on $\tau_1$ does not contribute nontrivial crystalline TSC because of the violation of the no-open-edge condition. It is similar for 1D blocks $\tau_3$ and $\tau_4$, and all types of Majorana chain decoration are obstructed.

b. 1D FSPT state decoration The 1D FSPT state decoration on $\tau_1$, $\tau_2$ and $\tau_3$ will leaves 8 dangling Majorana fermions ($\xi_j, \xi'_j, j = 1, 2, 3, 4$) at each 0D block labeled by $\mu_1/\mu_3$ and 4 dangling Majorana fermions ($\eta_j, \eta'_j, j = 1, 2$) at each 0D block labeled by $\mu_2$. At $\mu_1/\mu_3$ (we discuss $\mu_1$ as an example), the corresponding 8 Majorana fermions have the following rotation and reflection symmetry properties (all subscripts are taken under modulo 4, e.g., $\xi_5 \equiv \xi_1$ and $\xi'_5 \equiv \xi'_1$):

$$R_{\mu_1} : \xi_j \mapsto \xi_{j+1}, \quad \xi'_j \mapsto \xi'_{j+1}, \quad j = 1, 2, 3, 4$$  \hspace{1cm} (28)

We can define four complex fermions from these eight dangling Majorana fermions:

$$c_j^\dagger = \frac{1}{2} (\xi_j + i \xi'_j), \quad j = 1, 2, 3, 4$$  \hspace{1cm} (29)

And from the point group symmetry properties (28), we can obtain the point group symmetry properties of the above complex fermions as:

$$R_{\mu_1} : \quad \begin{pmatrix} c_1^\dagger, c_2^\dagger, c_3^\dagger, c_4^\dagger \end{pmatrix} \mapsto \begin{pmatrix} c_2^\dagger, c_3^\dagger, c_4^\dagger, c_1^\dagger \end{pmatrix}$$  \hspace{1cm} (30)

We denote the fermion number operators $n_j = c_j^\dagger c_j, j = 1, 2, 3, 4$. Firstly we consider the Hamiltonian with Hubbard interaction ($U > 0$) that can gap out these dangling Majorana fermions:

$$H_U = U \left[ \left( n_1 - \frac{1}{2} \right) \left( n_3 - \frac{1}{2} \right) + \left( n_2 - \frac{1}{2} \right) \left( n_4 - \frac{1}{2} \right) \right]$$  \hspace{1cm} (31)

And it can also be expressed in terms of Majorana fermions with symmetry properties as shown in Eq. (28):

$$H_U = -\frac{U}{4} (\xi_1 \xi'_3 \xi_3' \xi_3' + \xi_2 \xi'_2 \xi_4 \xi_4')$$  \hspace{1cm} (32)

It is easy to verify that $H_U$ respects all symmetries. There is a 4-fold ground-state degeneracy from $(n_1, n_3)$ and $(n_2, n_4)$, which can be viewed as two spin-1/2 degrees of freedom:

$$\tau^\mu_{13} = \begin{pmatrix} c_1^\dagger, c_3^\dagger \end{pmatrix} \sigma^\mu \begin{pmatrix} c_1 \end{pmatrix}$$  \hspace{1cm} (33)

and

$$\tau^\mu_{24} = \begin{pmatrix} c_2^\dagger, c_4^\dagger \end{pmatrix} \sigma^\mu \begin{pmatrix} c_2 \end{pmatrix}$$  \hspace{1cm} (34)

where $\sigma^\mu, \mu = x, y, z$ are Pauli matrices. In order to lift this ground-state degeneracy (GSD), we should further consider the interactions between these two spins. The symmetry properties of these two spins can be easily obtained from (30):

$$R_{\mu_1} : \quad \begin{pmatrix} \tau_{13}^x, \tau_{13}^y, \tau_{13}^z \end{pmatrix} \mapsto \begin{pmatrix} \tau_{24}^x, \tau_{24}^y, \tau_{24}^z \end{pmatrix}$$

$$M_{\tau_1} : \quad \begin{pmatrix} \tau_{13}^x, \tau_{13}^y, \tau_{13}^z \end{pmatrix} \mapsto \begin{pmatrix} -\tau_{13}^x, -\tau_{13}^y, -\tau_{13}^z \end{pmatrix}$$  \hspace{1cm} (35)

Then we can further add a spin Hamiltonian ($J > 0$):

$$H_J = J \left( \tau_{13}^x \tau_{24}^x + \tau_{13}^y \tau_{24}^y - \tau_{13}^z \tau_{24}^z \right)$$  \hspace{1cm} (36)

According to the symmetry properties of spin operations (35), we can easily verify that the spin Hamiltonian $H_J$ is symmetric under all symmetries. We can also verify the symmetry properties in Majorana representations: express $H_J$ in terms of Majorana fermions as:

$$H_J = -\frac{J}{4} (\xi_1 \xi'_3 - \xi_1' \xi_3) (\xi_2 \xi'_4 - \xi_2' \xi_4)$$

$$-\frac{J}{4} (\xi_1 \xi_3 + \xi_1' \xi'_3) (\xi_2 \xi_4' - \xi_2' \xi_4')$$

$$+ \frac{J}{4} (\xi_1 \xi'_1 - \xi_1' \xi_1) (\xi_2 \xi'_2 + \xi_2' \xi_2')$$  \hspace{1cm} (37)

and it is invariant under the symmetry properties defined in Eq. (28). The GSD is lifted by a symmetric Hamiltonian $H_U + H_J$, and the non-degenerate ground-state is:

$$|\psi\rangle_{0D} = -\frac{1}{2} (|\uparrow, \uparrow\rangle + i|\uparrow, \downarrow\rangle - i|\downarrow, \uparrow\rangle - |\downarrow, \downarrow\rangle)$$  \hspace{1cm} (38)
where $\uparrow$ and $\downarrow$ represent spin-up and spin-down of two spin-1/2 degrees of freedom ($\vec{\tau}_3\eta$ and $\vec{\tau}_4\eta$), and the ground-state energy is $-3J$. It is easy to verify that this state is invariant under arbitrary symmetry actions because $|\psi\rangle_{0D}$ is the eigenstate of the operators $R_{\mu_1}$ and $M_{\tau_1}$ as two generators of $D_4$ group at each $\mu_1$:  
\begin{align}
R_{\mu_1}|\psi\rangle_{0D} &= i|\psi\rangle_{0D} \\
M_{\tau_1}|\psi\rangle_{0D} &= -|\psi\rangle_{0D} \tag{39}
\end{align}

Thus the corresponding 8 Majorana fermions are gapped out by interactions in a symmetric way.

Next we consider the dangling Majorana fermions from the 1D FSPT decorations on $\tau_1$ at $\mu_2$ with the rotation and reflection symmetry properties:

\begin{align}
R_{\mu_2} &\colon (\eta_1, \eta'_1, \eta_2, \eta'_2) \mapsto (\eta_2, \eta'_2, \eta_1, \eta'_1) \\
M_{\tau_1} &\colon (\eta_1, \eta'_1, \eta_2, \eta'_2) \mapsto (\eta_1, -\eta'_1, \eta_2, -\eta'_2) \tag{40}
\end{align}

We can define two complex fermions from these four dangling Majorana fermions:

\begin{align}
c' &= \frac{1}{2}(\eta_1 + i\eta_2), \quad c'' = \frac{1}{2}(\eta_1' + i\eta_2') \tag{41}
\end{align}

and from the symmetry properties (40), we can obtain the point group symmetry properties of the above complex fermions:

\begin{align}
R &\colon (c', c'') \mapsto (ic', ic'') \\
M &\colon (c', c'') \mapsto (c', -c'') \tag{42}
\end{align}

We denote the fermion number operators $n = c'^\dagger c$ and $n' = c''^\dagger c'$. First we consider the Hamiltonian with Hubbard interaction ($U' > 0$) that can gap out these dangling Majorana fermions:

\begin{align}
H'_U &= U' \left(n - \frac{1}{2}\right) \left(n' - \frac{1}{2}\right) \tag{43}
\end{align}

And it is easy to verify that $H'_U$ respects all symmetries according to the symmetry properties of defined complex fermions (42). There is a 2-fold ground-state degeneracy from $(n, n')$ that can be viewed as a spin-1/2 degree of freedom:

\begin{align}
\tau^\mu = (c', c'') \sigma^\mu \left(\begin{array}{c} c \\ c' \end{array}\right) \tag{44}
\end{align}

In order to investigate that whether the degenerate ground states can be gapped out, we concentrate on the projective Hilbert space spanned by two states $c'^\dagger|0\rangle$ and $c''^\dagger|0\rangle$. In this projective Hilbert space, two generators of $D_2$ symmetry on each $\mu_2$, $R_{\mu_2}$ and $M_{\tau_1}$ can be represented by two $2 \times 2$ matrices:

\begin{align}
R_{\mu_2} &= \left(\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right) = \sigma^x \\
M_{\tau_1} &= \left(\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array}\right) = \sigma^z \tag{45}
\end{align}

It is obvious that these two generators are anticommutic:

\begin{align}
R_{\mu_2}M_{\tau_1} &= -M_{\tau_1}R_{\mu_2} \\
i.e., a sufficient condition shows that the projective Hilbert space is a projective representation of the symmetry group $D_2$ at each 0D block labeled by $\mu_2$. Hence, the two-fold ground-state degeneracy cannot be lifted.

We demonstrate this conclusion in Majorana representation and elucidate that all possible mass terms are not compatible with symmetries. A mass term is formed by two Majorana operators, and all possible mass terms are:

\begin{align}
\eta_1\eta_2, \quad \eta_1\eta'_1, \quad \eta_2\eta'_2, \quad \eta_2\eta'_1, \quad \eta_1\eta'_2, \quad \eta_1\eta'_1, \quad -\eta'_1\eta'_2
\end{align}

and their linear combinations. Under 2-fold rotation $R_{\mu_2}$, these mass terms will be transformed to:

\begin{align}
-\eta_1\eta_2, \quad \eta_2\eta'_1, \quad \eta_1\eta'_2, \quad \eta_1\eta'_1, \quad -\eta'_1\eta'_2
\end{align}

so there are only two mass terms that are symmetric under $R_{\mu_2}: \eta_1\eta'_1 + \eta_2\eta'_2$ and $\eta_1\eta'_2 + \eta_2\eta'_1$ and their linear combinations. Subsequently, under the reflection $M_{\tau_1}$, these terms are not symmetric:

\begin{align}
-(\eta_1\eta'_1 + \eta_2\eta'_2), \quad - (\eta_1\eta'_2 + \eta_2\eta'_1)
\end{align}

Therefore, there is no symmetric mass term to lift the GSD. Accordingly, 1D FSPT state decoration on $\tau_1$ is obstructed because of the degenerate ground state, similar arguments can also be held on 1D blocks labeled by $\tau_2$ (and the obstruction also happens at 0D block $\mu_2$, as the center of $D_2$ symmetry). 1D FSPT state decoration on $\tau_3$ is obstruction-free because this decoration leaves eight dangling Majorana fermions at each 0D block labeled by $\mu_1$ and $\mu_3$, and both of them are centers of $D_4$ symmetry.

There is one exception: if we decorate a 1D FSPT phase on each 0D block labeled by $\tau_1$ and $\tau_2$ simultaneously, it leaves eight dangling Majorana fermions at each 0D block $\mu_2 (\eta_j, \eta'_j, j = 1, 2, 3, 4)$, with the following rotation and reflection symmetry properties:

\begin{align}
R_{\mu_2} &\colon \left\{(\eta_1, \eta'_1, \eta_2, \eta'_2) \mapsto (\eta_2, \eta'_2, \eta_1, \eta'_1)\right\} \\
M_{\tau_1} &\colon \left\{(\eta_1, \eta'_1, \eta_2, \eta'_2) \mapsto (\eta_1, -\eta'_1, \eta_2, -\eta'_2)\right\} \tag{46}
\end{align}

This problem is quite similar with aforementioned gapping problem at each 0D block labeled by $\mu_1$ or $\mu_3$, with lower point group symmetry ($D_2 \subset D_4$). Thus eight dangling Majorana fermions at each 0D block $\mu_2$ from decorating a 1D FSPT state on each $\tau_1$ and $\tau_2$ can be gapped by previously discussed interactions $H_U + H_J$ [cf. Eqs. (31) and (36)] in a symmetric way, and the 1D FSPT state decoration on $\tau_1$ and $\tau_2$ simultaneously is obstruction-free. We should note that this block-state has no free-fermion realization because as aforementioned, we should introduce some interactions to satisfy the no-open-edge condition, as non-interacting mass terms cannot gap them out. Hence the crystalline TSC realized above is an intriguing fermionic SPT phase. As a consequence, all obstruction-free 1D block-states are:
• 1D FSPT state decoration on $\tau_1$ and $\tau_2$ simultaneously;

• 1D FSPT state decoration on $\tau_3$.

and they form the following group:

$$\text{OFBS}_{p4m,0}^{1D} = \mathbb{Z}_2^2$$  \hspace{1cm} (47)

where the group elements can be labeled by:

$$[n_1 = n_2, n_3]$$

here $n_j = 0, 1$ ($j = 1, 2, 3$) represents the number of decorated 1D FSPT states on $\tau_j$, respectively. According to aforementioned discussions, a necessary condition of an obstruction-free block-state is $n_1 = n_2$.

So far we have already obtained all obstruction-free block-states, and they form the following group:

$$\text{OFBS}_{p4m,0}^{1D} = \text{OFBS}_{p4m,0}^{1D} \times \text{OFBS}_{p4m,0}^{0D} = \mathbb{Z}_2^2 \times \mathbb{Z}_2^9 = \mathbb{Z}_2^{11}$$  \hspace{1cm} (48)

With all obstruction-free block-states, subsequently we discuss all possible trivializations. First we consider the 2D bubble equivalences: we decorate a “Majorana bubble” on each 2D block $\sigma$ (see Fig. 4), and then demonstrate that they can be deformed into double Majorana chains at each nearby 1D block, and this is exactly the definition of the nontrivial 1D FSPT phase protected by on-site $\mathbb{Z}_2$ symmetry. Figs. 4(b) shows that if we cut Majorana bubbles near each 0D block, these “Majorana bubbles” can be deformed to double Majorana chains. For $p4m$ case, all 1D blocks are lying on the reflection axis, and reflection operation acting on them internally: reflection operation (on-site $\mathbb{Z}_2$ symmetry on 1D blocks) exchanges two Majorana chains deformed from “Majorana” bubble constructions, and this is exactly the definition of the nontrivial 1D FSPT phase protected by on-site $\mathbb{Z}_2$ symmetry. Equivalently, 1D FSPT state decorations on all 1D blocks can be deformed to a trivial state via both types of 2D bubble equivalence. Furthermore, we demonstrate that 2D bubble equivalence has no effect on 0D block: Figs. 4(c) shows an alternative type of deformation: at each 1D block, this deformation is identical with aforementioned one, but at each 0D block, there is an alternative Majorana chain with odd fermion parity surrounding it. Nevertheless, for spinless fermions, Majorana chain is not compatible with reflection symmetry [62], hence this deformation is forbidden by reflection symmetry. Therefore, the overall effect of 2D “Majorana” bubble equivalence is deforming the 1D FSPT phase (protected by on-site $\mathbb{Z}_2$ symmetry) decorations on all 1D blocks to a trivial state, see Fig. 5.

Subsequently we consider the 1D bubble equivalences. For instance, we decorate a pair of complex fermions [cf. Eq. (18)] on each 1D block $\tau_1$: Near each 0D block $\mu_1$, there are 4 complex fermions forming the following atomic insulator:

$$|\psi\rangle_{\mu_1}^{\mu_1} = c_1^\dagger c_2^\dagger c_3^\dagger c_4^\dagger |0\rangle$$  \hspace{1cm} (49)

with two independent reflection properties:

$$M_{\tau_1} |\psi\rangle_{\mu_1}^{\mu_1} = c_1^\dagger c_2^\dagger c_3^\dagger c_4^\dagger |0\rangle = - |\psi\rangle_{\mu_1}^{\mu_1}$$  \hspace{1cm} (50)

$$M_{\tau_3} |\psi\rangle_{\mu_1}^{\mu_1} = c_1^\dagger c_3^\dagger c_2^\dagger c_4^\dagger |0\rangle = |\psi\rangle_{\mu_1}^{\mu_1}$$  \hspace{1cm} (50)

i.e., at 0D blocks $\mu_1$, 1D bubble construction on $\tau_1$ changes the reflection eigenvalue of $M_{\tau_1}$, and leaves the reflection eigenvalue of $M_{\tau_3}$ invariant. Near each 0D block $\mu_2$, there are two complex fermions forming another atomic insulator:

$$|\psi\rangle_{\mu_2}^{\mu_2} = c_1^\dagger c_2^\dagger |0\rangle$$  \hspace{1cm} (51)

with two independent reflection properties:

$$M_{\tau_1} |\psi\rangle_{\mu_2}^{\mu_2} = c_1^\dagger c_2^\dagger |0\rangle = |\psi\rangle_{\mu_2}^{\mu_2}$$  \hspace{1cm} (52)

$$M_{\tau_3} |\psi\rangle_{\mu_2}^{\mu_2} = c_1^\dagger c_2^\dagger |0\rangle = - |\psi\rangle_{\mu_2}^{\mu_2}$$  \hspace{1cm} (52)

i.e., at 0D blocks $\mu_2$ 1D bubble construction on $\tau_1$ changes the reflection eigenvalue of $M_{\tau_2}$, and leaves the reflection eigenvalue of $M_{\tau_3}$ invariant. Similar 1D bubble constructions can be held on 1D blocks $\tau_2$ and $\tau_3$, and we summarize the effects of 1D bubble constructions as following:

1. 1D bubble construction on $\tau_1$: simultaneously changes the eigenvalues of $M_{\tau_1}$ at $\mu_1$ and $M_{\tau_3}$ at $\mu_2$;

2. 1D bubble construction on $\tau_2$: simultaneously changes the eigenvalues of $M_{\tau_2}$ at $\mu_2$ and $M_{\tau_3}$ at $\mu_3$;

3. 1D bubble construction on $\tau_3$: simultaneously changes the eigenvalues of $M_{\tau_3}$ at $\mu_1$ and $M_{\tau_2}$ at $\mu_3$;

With all possible trivializations, we are ready to study the trivial states. Start from the original 0D trivial block-state (nothing is decorated on arbitrary 0D blocks):

$$[(+, +, +), (+, +, +), (+, +, +)]$$

If we take 1D bubble constructions on $\tau_j$ by $n_j$ times ($j = 1, 2, 3$), above trivial 0D block-state will be transformed to a new 0D block-state labeled by:

$$[(+, (-1)^{n_1}, (-1)^{n_2}), (+, (-1)^{n_2}, (-1)^{n_1}), (+, (-1)^{n_2}, (-1)^{n_1})]$$  \hspace{1cm} (53)

According to the definition of bubble equivalence, all these states should be trivial. It is easy to see that there are only three independent quantities ($n_j, j = 1, 2, 3$) in Eq. (53). Together with the 2D “Majorana” bubble construction that deforms the vacuum 1D block-state to 1D FSPT states decorated on all 1D blocks, all these trivial states form the group:

$$\{\text{TBS}\}_{p4m,0}^{1D} = \{\text{TBS}\}_{p4m,0}^{1D} \times \{\text{TBS}\}_{p4m,0}^{0D} = \mathbb{Z}_2 \times \mathbb{Z}_2^9 = \mathbb{Z}_2^4$$  \hspace{1cm} (54)

where $\{\text{TBS}\}_{p4m,0}^{1D}$ represents the group of trivial states with non-vacuum 1D blocks (i.e., 1D FSPT phase decorations on all 1D blocks), and $\{\text{TBS}\}_{p4m,0}^{0D}$ represents the group of trivial states with non-vacuum 0D blocks.
FIG. 4. “Majorana” bubble equivalence near each 0D block \( \mu_2 \). (a): “Majorana” bubble construction: decorate a Majorana chain with anti-PBC (indicated by red arrow) on each 2D block. (b): We cut Majorana bubbles near each 0D block (for instance, \( \mu_2 \) as illustrated), near each nearby 1D block, “Majorana bubbles” are deformed to double Majorana chains that is exactly the definition of nontrivial 1D FSPT state protected by on-site \( \mathbb{Z}_2 \) symmetry. (c): Alternative way to cut the Majorana bubbles: Near each nearby 1D block, “Majorana bubbles” are deformed to double Majorana chains again; and near the 0D block labeled by \( \mu_2 \), Majorana fermions surrounded by green dashed circle are deformed toward an enclosed Majorana chain. Nevertheless, Majorana chain is not compatible with the reflection symmetry for spinless fermions [62], hence the deformation illustrated in this panel is forbidden by symmetry. Here ellipses represent the physical sites, and the solid oriented line represents the entanglement pair of two Majorana fermions.

Therefore, all independent nontrivial block-states are labeled by the group elements of the following quotient group:

\[
E_{p4m,0} = \{\text{OFBS}\}_{p4m,0}/\{\text{TBS}\}_{p4m,0} = \mathbb{Z}_2^1/\mathbb{Z}_2^4 = \mathbb{Z}_2^7
\]  

(55)

here one \( \mathbb{Z}_2 \) is from the nontrivial 1D block-state, and other six \( \mathbb{Z}_2 \) are from the nontrivial 0D block-states.

With all nontrivial block-states, we consider the group structure of the ultimate classification. The physical meaning of the group structure investigation is finding whether 1D block-state extends 0D block-state or not. We argue that there is no stacking between block-states with different dimensions, and the group structure of classification data (55) have already been accurate. In order to investigate the possible stacking, we consider two identical 1D block-states: for example, we decorate two copies of 1D FSPT states on each 1D block labeled by \( \tau_3 \), which leaves 16 dangling Majorana fermions at each 0D block labeled by \( \mu_1/\mu_3 \). It is easy to verify that two copies of 1D FSPT states should be a trivial 1D block-state because the root phase has a \( \mathbb{Z}_2 \) structure. First of all, according to previous discussions, these decoration cannot be deformed to a Majorana chain surrounding the 0D block to change the corresponding fermion parity because the Majorana chain is not compatible with the reflection symmetry. Subsequently at each 0D block \( \mu_1/\mu_3 \), we can treat these 16 Majorana fermions as 8 complex fermions: \( c_j \) and \( c'_j \) \((j = 1, 2, 3, 4)\) that form two
atomic insulators:

\[
\left| \phi \right> = a_1^\dagger a_2^\dagger a_3^\dagger a_4^\dagger |0\rangle
\]
\[
\left| \phi' \right> = a_1^{\prime\dagger} a_2^{\prime\dagger} a_3^{\prime\dagger} a_4^{\prime\dagger} |0\rangle
\]

and the wavefunction of these 8 complex fermions are direct product of $|\phi\rangle$ and $|\phi'\rangle$:

\[
|\Phi\rangle = |\phi\rangle \otimes |\phi'\rangle
\]  

(57)

$|\phi\rangle$ and $|\phi'\rangle$ are eigenstates of two generators of $D_4$ symmetry, $M_{r_1}$ and $M_{r_3}$:

\[
M_{r_1}|\phi\rangle = a_2^\dagger a_3^\dagger a_4^\dagger a_1^\dagger |0\rangle = |\phi\rangle
\]
\[
M_{r_2}|\phi'\rangle = a_2'^\dagger a_3'^\dagger a_4'^\dagger a_1'^\dagger |0\rangle = |\phi'\rangle
\]
\[
M_{r_3}|\phi\rangle = a_1^\dagger a_2^\dagger a_3^\dagger a_4^\dagger |0\rangle = -|\phi\rangle
\]
\[
M_{r_3}|\phi'\rangle = a_1'^\dagger a_2'^\dagger a_3'^\dagger a_4'^\dagger |0\rangle = -|\phi'\rangle
\]  

(58)

Then the eigenvalues of $|\Phi\rangle$ under $M_{r_1}$ and $M_{r_3}$ is trivial:

\[
M_{r_1} |\Phi\rangle = |\Phi\rangle
\]
\[
M_{r_3} |\Phi\rangle = |\Phi\rangle
\]  

(59)

Therefore, 1D block-state cannot extend 0D block-state, and the group structure of classification data (55) have already been accurate.

Now we turn to discuss systems with spin-1/2 fermions. We first consider the 0D block-state decoration. For each 0D block labeled by $\mu_1$ or $\mu_3$, the classification data can also be characterized by different 1D irreducible representations of the full symmetry group $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ (the symbol $\times \mathbb{Z}_2$ means that the physical symmetry group is nontrivially extended by fermion parity $\mathbb{Z}_2$, which is characterized by a 2-cocycle $\omega_2$, see Sec. 1B):

\[
\mathcal{H}^1 \left[ \mathbb{Z}_2^4 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2, U(1) \right] = \mathbb{Z}_4^2
\]  

(60)

similar for each 0D block $\mu_2$:

\[
\mathcal{H}^1 \left[ \mathbb{Z}_2^2 \times \mathbb{Z}_2 \times \mathbb{Z}_2, U(1) \right] = \mathbb{Z}_2^2
\]  

(61)

To calculate this, we should firstly calculate the following two cohomologies:

\[
\begin{align*}
\mathcal{H}^0 (\mathbb{Z}_n \times \mathbb{Z}_2, \mathbb{Z}_2) &= \mathbb{Z}_2 \\
\mathcal{H}^1 (\mathbb{Z}_n \times \mathbb{Z}_2, U(1)) &= \mathbb{Z}_2
\end{align*}
\]

(62)

But the 0-cocycle $n_0 \in \mathcal{H}^0 (\mathbb{Z}_n \times \mathbb{Z}_2, \mathbb{Z}_2)$ does not contribute a nontrivial 0D block-state: a specific $n_0$ is obstructed if and only if $(-1)^{\omega_2 - n_0} \in \mathcal{H}^2 (\mathbb{Z}_2 \times \mathbb{Z}_2, U(1)) = \mathbb{Z}_2^2$ is a nontrivial 2-cocycle with $U(1)$ coefficient. From Refs. [19] and [85] we know that nontrivial 0-cocycle $n_0 = 1$ (fermion parity odd) leads to a nontrivial 2-cocycle $(-1)^{\omega_2 - 1} \in \mathcal{H}^2 (\mathbb{Z}_2 \times \mathbb{Z}_2, U(1))$, and the 0D block-states at $\mu_1$ and $\mu_3$ with odd fermion parity are obstructed, similar for 0D block $\mu_2$. Hence different $\mathbb{Z}_2$'s in the classification data represent the rotation and reflection eigenvalues at each $D_4$ or $D_2$ center. As a consequence, all obstruction-free 0D block-states form the following group:

\[
\{ \text{OFBS} \}_{\text{p4m,1/2}}^{\text{0D}} = \mathbb{Z}_2^6
\]  

(63)

And it is straightforward to see that there is no more trivialization ($\{ \text{TS} \}_{\text{p4m,1/2}}^{\text{0D}} = 0$), hence the classification attributed to 0D block-states is:

\[
E_{\text{p4m,1/2}}^{\text{0D}} = \mathbb{Z}_2^6
\]  

(64)

Subsequently we consider the 1D block-state decoration. For arbitrary 1D blocks, the total symmetry group is $\mathbb{Z}_4$, hence there is no nontrivial 1D block-state due to the trivial classification of the corresponding 1D FSPT.
phases (i.e., there is no obstruction-free 1D block-state), and the classification attributed to 1D block-state decorations is trivial:
\[ E_{p4m,1/2}^{1D} = \{\text{OFBS}\}_{p4m,1/2}^{1D} = 0 \]  

Therefore it is obvious that there is no stacking between 1D and 0D block-states because of the trivial contribution from 1D block-state. The ultimate classification with accurate group structure is:
\[ G_{p4m,1/2} = E_{p4m,1/2}^{0D} \times E_{p4m,1/2}^{1D} = Z_2^6 \]

**B. Parallelogrammatic lattice: p2**

For parallelogrammatic lattice, we demonstrate the crystalline TSC protected by p2 symmetry as an example. The corresponding point group of p2 is rotation group C2. For 1D and 2D blocks, there is no on-site symmetry group, but the rotational subgroup C2 acts on each 0D blocks internally, just identical with on-site Z2 symmetry, see Fig. 6.

We discuss systems with spinless and spin-1/2 fermions separately. Consider the 0D block-state decoration, for 0D blocks labeled by \( \mu_j, j = 1, 2, 3, 4 \), the total symmetry group of each of them is \( Z_4^f \times Z_2 \), and the classification data can be characterized by different 1D irreducible representations of the symmetry group \( Z_4^f \times Z_2 \):
\[ \mathcal{H}^1 \left[ Z_4^f \times Z_2, U(1) \right] = Z_2^2 \]  

One \( Z_2 \) is from the fermion parity, and the other is from the rotation eigenvalue \(-1\). So at each 0D block, the block-state can be labeled by \((\pm, \pm)\), here these two \( \pm \)'s represent the fermion parity and rotation eigenvalue, respectively. According to this notation, the obstruction-free 0D block-states form the following group:
\[ \{\text{OFBS}\}_{p2,0}^{0D} = Z_2^8 \]  

and the group elements can be labeled by (four brackets represent the block-states at \( \mu_j, j = 1, 2, 3, 4 \)):
\[ [(\pm, \pm), (\pm, \pm), (\pm, \pm), (\pm, \pm)] \]

Then we consider possible trivializations via bubble construction. First of all, we consider the 2D bubble equivalence: as illustrated in Fig. 7, we decorate a Majorana chain with anti-PBC on each 2D block which can be trivialized if it shrinks to a point. At each nearby 1D block, we can see that these “Majorana” bubbles can be deformed into double Majorana chains. But distinct with the p4m case, there is no on-site symmetry at arbitrary 1D block, hence the only root phase at each 1D block should be Majorana chain as the only invertible topological phase, and double of them should be trivialized because of the \( Z_2 \) classification of this phase. Hence “Majorana bubble” construction has no effect on 1D blocks. At each nearby 0D block (\( \mu_2 \) as an example, see Fig. 7), these “Majorana” bubbles can be deformed into an alternative Majorana chain with odd fermion parity surrounding it. Distinct with the p4m case, this Majorana chain respects all symmetry actions of p2 because there is no reflection operation on this Majorana chain, so this “Majorana” bubble construction can change the fermion parities of all 0D blocks simultaneously.

Furthermore, consider 1D bubble equivalence on \( \tau_1 \): on each 1D block labeled by \( \tau_1 \), we decorate a pair of complex fermions [cf. Eq. (18)]: Near each 0D block \( \mu_2 \), there are 2 complex fermions which form an atomic insulator:
\[ |\psi\rangle_{p2}^{\mu_2} = c_1^\dagger c_2^\dagger |0\rangle \]
with rotation property as \((R_{\mu_2}) \) represents the 2-fold rotation operation centred at \(\mu_2\):

\[
R_{\mu_2}\ket{\psi}_{\mu_2}^f = c_1^2 c_1^1 \ket{0} = -\ket{\psi}_{\mu_2}^f
\]  

(70)

Hence the rotation eigenvalue \(-1\) can be trivialized by atomic insulator \(\ket{\psi}_{\mu_2}^f\). Similar for \(\mu_1\), and we can conclude that rotation eigenvalues at 0D blocks labeled by \(\mu_1\) and \(\mu_2\) are not independent. Similar bubble equivalences can be held on arbitrary 1D blocks \(\tau_j\), \(j = 1, 2, 3, 4\), and rotation eigenvalues at all 0D blocks are not independent as a consequence.

With all possible bubble constructions, we are ready to study the trivial states. Start from the original trivial state (nothing is decorated on arbitrary 0D block):

\[
[(+, +), (+, +), (+, +), (+, +)]
\]

if we take 2D bubble construction \(n_z\) times, and take 1D bubble constructions on \(\tau_j\) by \(n_j\) times, above trivial state will be transformed to a new 0D block-state labeled by:

\[
[((-1)^{n_1}, (-1)^{n_2}), ((-1)^{n_0}, (-1)^{n_3}), ((-1)^{n_0}, (-1)^{n_2}), ((-1)^{n_0}, (-1)^{n_3})]
\]  

(71)

According to the definition of bubble equivalence, all these states should be trivial. Alternatively, all 0D block-states can be viewed as a vector of an 8-dimensional \(\mathbb{Z}_2\) valued vector space, and all trivial 0D block-states with the form as Eq. (71) can be viewed as a vector of the subspace of aforementioned vector space. The dimensionality of this subspace can be determined by calculating the rank of the following transformation matrix:

\[
\begin{pmatrix}
1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

(72)

Here different rows of this matrix represent different bubble constructions. Hence the vector subspace containing all trivial 0D block-states is 4, and all trivial 0D block-states form the group:

\[
\text{TBS}^{0D}_{p_{2,0}} = \mathbb{Z}_2^4
\]  

(73)

Therefore, all independent nontrivial 0D block-states are labeled by different group elements of the following quotient group:

\[
E^{0D}_{p_{2,0}} = \{\text{OFBS}^{0D}_{p_{2,0}} \} / \{\text{TBS}^{0D}_{p_{2,0}}\} = \mathbb{Z}_2^4
\]  

(74)

Subsequently we consider the 1D block-state decorations. The unique possible 1D block-state is Majorana chain due to the absence of on-site symmetry on arbitrary 1D block, and all 1D block-states form a group:

\[
\text{BS}^{1D}_{p_{2,0}} = \mathbb{Z}_2^5
\]  

(75)

Then we consider about the possible obstruction: Majorana chain decoration on \(\tau_1\) leaves 2 dangling Majorana fermions at each 0D block labeled by \(\mu_2\) which can be glued by an entanglement pair \(i\gamma_1\gamma_2\). Nevertheless, this entanglement pair breaks \(C_2\) symmetry:

\[
R_{\mu_2}: \ i\gamma_1\gamma_2 \mapsto -i\gamma_2\gamma_1 = i\gamma_1\gamma_2
\]  

(76)

hence this decoration is obstructed, and does not contribute nontrivial crystalline TSC because of the violation of the no-open-edge condition. It is similar for all other 1D blocks. As a consequence, 1D block-state decorations do not contribute any nontrivial crystalline topological state because all block-states are obstructed:

\[
E^{1D}_{p_{2,0}} = \{\text{OFBS}^{1D}_{p_{2,0}}\} = 0
\]  

(77)

Hence it is obvious that there is no stacking between 1D and 0D block-states, and the ultimate classification with accurate group structure is:

\[
\mathcal{G}_{p_{2,0}} = E^{1D}_{p_{2,0}} \times E^{0D}_{p_{2,0}} = \mathbb{Z}_2^4
\]  

(78)

Now we turn to discuss systems with spin-1/2 fermions. First we consider the 0D block-state decoration, whose candidate states can be characterized by different 1D irreducible representations of the symmetry group \(Z_2^f\) (non-trivial \(Z_2^f\) extension of \(Z_2\) on-site symmetry):

\[
\mathcal{H}^1[Z_2^f, U(1)] = \mathbb{Z}_4
\]  

(79)

All root phases are characterized by eigenvalues \(\{i, -1, -i, 1\}\) of 2-fold rotation operation composed by fermion parity. So at each 0D block, the block-state can be labeled by \(\nu \in \{i, -1, -i, 1\}\). According to this notation, the obstruction-free 0D block-states form the following group:

\[
\text{OFBS}^{0D}_{p_{2,1/2}} = \mathbb{Z}_2^4
\]  

(80)

and different group elements can be labeled by:

\[
[\nu_1, \nu_2, \nu_3, \nu_4]
\]

where \(\nu_j\) labels the 0D block-state at \(\mu_j\) \((j=1,2,3,4)\). It is easy to see that there is no trivialization on 0D block-states (i.e., \(\text{OFBS}^{0D}_{p_{2,1/2}} = 0\)), so the classification attributed to 0D block-state decoration is:

\[
E^{0D}_{p_{2,1/2}} = \{\text{OFBS}^{0D}_{p_{2,1/2}}\} / \{\text{TBS}^{0D}_{p_{2,1/2}}\} = \mathbb{Z}_2^4
\]  

(81)

Subsequently consider the 1D block-state decorations. The unique possible 1D block-state is still Majorana chain due to the absence of on-site symmetry on each 1D block. The Majorana chain decoration on \(\tau_1\) leaves 2 dangling Majorana fermions at each 0D block labeled by \(\mu_2\) which can be glued by an entanglement pair \(i\gamma_1\gamma_2\), and it respects rotational symmetry:

\[
R_{\mu_2}: \ i\gamma_1\gamma_2 \mapsto -i\gamma_2\gamma_1 = i\gamma_1\gamma_2
\]  

(82)
Hence Majorana chain decoration on $\tau_1$ is an obstruction-free block-state because of the satisfaction of the no-open-edge condition. It is similar for 1D blocks labeled by $\tau_2$ and $\tau_3$. Hence all obstruction-free 1D block-states form the following group:

$$\{\text{OFBS}\}_{p,2,1/2}^{1D} = \mathbb{Z}_2^3$$  \hspace{1cm} (83)

It is obvious that there is no trivialization (i.e., $\{\text{TBS}\}_{p,2,1/2}^{1D} = 0$), so the classification attributed to 1D block-state decorations is:

$$E_{p,2,1/2}^{1D} = \{\text{OFBS}\}_{p,2,1/2}^{1D}/\{\text{TBS}\}_{p,2,1/2}^{1D} = \mathbb{Z}_2^3$$  \hspace{1cm} (84)

With the classification data as Eqs. (81) and (84), we consider the group structure of the corresponding classification. Equivalently, we investigate that if 1D block-state extends 0D block-state. As an example, we decorate two copies of Majorana chains on each 1D block labeled by $\tau_1$ which leaves four dangling Majorana fermions at each 0D block labeled by $\mu_1/\mu_2$. Similar with Ref. 62, these Majorana chains can be smoothly deformed to another assembly of Majorana chains surrounding 0D blocks labeled by $\mu_1$ and $\mu_2$ as follows (each yellow ellipse represents a physical site):

$$\gamma_2 \gamma_1 \gamma'_1 \gamma'_2 \gamma_3 \gamma'_4 \gamma'_4 \gamma_3$$  \hspace{1cm} (85)

with rotational symmetry properties: $\gamma_j \mapsto \gamma'_j$ and $\gamma'_j \mapsto -\gamma_j$. The gapped Hamiltonian corresponding to the graph in Eq. (85) is:

$$H = -i\gamma_1\gamma'_1 - i\gamma_4\gamma'_4 - i\gamma_2\gamma'_3 - i\gamma'_2\gamma_3$$  \hspace{1cm} (86)

We can further define four complex fermions according to eight Majorana fermions in Eq. (85) as follows:

$$c_1 = (\gamma_2 + i\gamma_1)/2 \quad c_2 = (\gamma_3 + i\gamma_4)/2$$
$$c'_1 = (\gamma'_2 + i\gamma'_1)/2 \quad c'_2 = (\gamma'_3 + i\gamma'_4)/2$$  \hspace{1cm} (87)

It is easy to find the ground state of Eq. (86):

$$|\phi\rangle_{0D} = (c_1^\dagger - c_2^\dagger i + ic'_2 - c'_1 c'_1 c'_2 + c'_2 c_1 c_2^\dagger + i c_1 c_2 c'_1 - ic'_2 c'_2 c'_1)|0\rangle$$

with the 2-fold rotation property:

$$R_{\mu_1}|\phi\rangle_{0D} = i|\phi\rangle_{0D}$$  \hspace{1cm} (89)

If a 0D block-state with eigenvalue $e^{i\pi/2}$ under 2-fold rotation is attached to each 1D block-state near each 0D block labeled by $\mu_1$, the rotation eigenvalue $r$ of the obtained 0D block-state becomes:

$$r = e^{i\pi/2 + i\pi q}$$  \hspace{1cm} (90)

And there is no solution to the formula $r = 1$. Therefore, near each 0D block labeled by $\mu_1/\mu_2$, 1D block-states extend 0D block-states, hence the 0D block-states at $\mu_1/\mu_2$ have the group structure $\mathbb{Z}_2$ as the nontrivial extension of $\mathbb{Z}_4$ and $\mathbb{Z}_2$ which should be attributed to 0D and 1D block-state decorations, respectively.

Similar for other 1D and 0D block-states, we can obtain that the 0D block-states have the group structure $\mathbb{Z}_8$ for an arbitrary 0D block. Nevertheless, stacking between 1D and 0D block-states at different 0D blocks are not independent. For instance, if we decorate two copies of Majorana chain on 1D blocks $\tau_1$, these two Majorana chains extend the 0D block-states at both $\mu_1$ and $\mu_2$. It is not hard to verify that only three 0D blocks have independent stacking between 1D and 0D block-states, hence the ultimate classification with accurate group structure is:

$$G_{p,2,1/2} = E_{p,2,1/2}^{1D} \times \mathbb{Z_2} \quad E_{p,2,1/2}^{0D} = \mathbb{Z}_4 \times \mathbb{Z}_2^3$$  \hspace{1cm} (91)

here the symbol $\times \mathbb{Z_2}$ means that independent nontrivial 1D and 0D block-states $E_{p,2,1/2}^{1D}$ and $E_{p,2,1/2}^{0D}$ have nontrivial extension, described by the following short exact sequence:

$$0 \rightarrow E_{p,2,1/2}^{1D} \rightarrow G_{p,2,1/2} \rightarrow E_{p,2,1/2}^{0D} \rightarrow 0$$  \hspace{1cm} (92)

C. Rhombic lattice: cmm

For rhombic lattice, we demonstrate the crystalline TSC protected by cmm symmetry as an example. The corresponding point group of cmm is dihedral group $D_2$, and for 2D blocks $\sigma$ and 1D blocks $\tau_1$, there is no on-site symmetry; and for 1D blocks $\tau_2/\tau_3$ and 0D blocks $\mu_1$, the on-site symmetry is $\mathbb{Z}_2$ via the reflection symmetry acting internally; for 0D blocks $\mu_2$ and $\mu_3$, the on-site symmetry group is $\mathbb{Z}_2 \times \mathbb{Z}_2$ via the $D_2$ symmetry acting internally. The cell decomposition of cmm is illustrated in Fig. 8.

We discuss systems with spinless and spin-1/2 fermions separately. Consider the 0D block-state decoration: For 0D blocks $\mu_1$, the total symmetry group of each is $\mathbb{Z}_2^2 \times \mathbb{Z}_2$, and candidate states can be characterized by different 1D irreducible representations of the symmetry group:

$$\mathcal{H}^1_{\mathbb{Z}_2^2 \times \mathbb{Z}_2, U(1)} = \mathbb{Z}_2^2$$  \hspace{1cm} (93)

where the first $\mathbb{Z}_2$ is the complex fermion, and the other is eigenvalue $-1$ of rotational symmetry operation. So at each 0D block labeled by $\mu_1$, the block-state can be labeled by $(\pm, \pm)$, where these two $\pm$’s represent the fermion and rotation eigenvalue, respectively. For 0D blocks $\mu_2$ and $\mu_3$, the classification data can be characterized by different irreducible representations of the full symmetry group $\mathbb{Z}_2^3 \times (\mathbb{Z}_2 \times \mathbb{Z}_2)$:

$$\mathcal{H}^1_{\mathbb{Z}_2^3 \times (\mathbb{Z}_2 \times \mathbb{Z}_2), U(1)} = \mathbb{Z}_2^3$$  \hspace{1cm} (94)
and three $\mathbb{Z}_2$ have different physical meanings: the first $\mathbb{Z}_2$ represents the complex fermion, the second $\mathbb{Z}_2$ represents the rotation eigenvalue $-1$, and the third $\mathbb{Z}_2$ represents the reflection eigenvalue $-1$. So at each 0D block, the block-state can be labeled by $(\pm, \pm, \pm)$, where these three $\pm$’s represent the fermion parity, rotation and reflection eigenvalues (or two independent reflection eigenvalues, because even-fold dihedral group can also be generated by two independent reflections), respectively. According to this notation, the obstruction-free 0D block-states form the following group:

$$\{\text{OFBS}\}_{\text{cmm},0}^{\text{0D}} = \mathbb{Z}_2^8$$  \hspace{1cm} (95)

where the group elements can be labeled by:

$$[(\pm, \pm), (\pm, \pm, \pm), (\pm, \pm, \pm)]$$  \hspace{1cm} (96)

here three brackets represent the block-states at $\mu_1$, $\mu_2$ and $\mu_3$, respectively.

Subsequently we consider the 1D block-state decoration. For 1D block $\tau_1$, the total symmetry group is just fermion parity $\mathbb{Z}_2$, so the only nontrivial 1D block-state is Majorana chain; for 1D blocks $\tau_2$ and $\tau_3$, the total symmetry group is $\mathbb{Z}_2^f \times \mathbb{Z}_2$, so there are two possible 1D block-states: Majorana chain and 1D FSPT state (composed by double Majorana chains), so all 1D block-states form a group:

$$\{\text{BS}\}_{\text{cmm},0}^{\text{1D}} = \mathbb{Z}_2^5$$  \hspace{1cm} (97)

Then we discuss the decorations of these two root phases separately.

a. Majorana chain decoration  First we consider the Majorana chain decoration on 1D blocks $\tau_1$ which leaves two/four dangling Majorana fermions at each 0D block $\mu_1/\mu_2$. Near $\mu_1$, Majorana fermions have following rotational symmetry properties:

$$R_{\mu_1} : \gamma_1 \leftrightarrow \gamma_2$$  \hspace{1cm} (98)

with local fermion parity and its symmetry property:

$$P_f = i\gamma_1 \gamma_2, \ R_{\mu_1} : P_f \mapsto -P_f$$  \hspace{1cm} (99)

Hence these two Majorana fermions form a projective representation of the total symmetry group $\mathbb{Z}_2^f \times \mathbb{Z}_2$ on 0D block $\mu_1$, hence a non-degenerate ground state is forbidden. Thus Majorana chain decoration on 1D block $\tau_1$ is obstructed because of the violation of the no-open-edge condition.

Then we consider the Majorana chain decoration on 1D blocks $\tau_2$ which leaves two Majorana fermions at each 0D block $\mu_2/\mu_3$. Near $\mu_2$, Majorana fermions have following reflection symmetry properties:

$$M_{\tau_2} : \gamma_1 \leftrightarrow \gamma_2$$  \hspace{1cm} (100)

with local fermion parity and its symmetry property:

$$P_f = i\gamma_1 \gamma_2, \ M_{\tau_2} : P_f \mapsto -P_f$$  \hspace{1cm} (101)

Hence these two Majorana fermions form a projective representation of the total symmetry group $(\mathbb{Z}_2 \times \mathbb{Z}_2^f) \times \mathbb{Z}_2$ on 0D block $\mu_2$, and a non-degenerate ground state is forbidden. Thus Majorana chain decoration on $\tau_2$ is obstructed because of the violation of the no-open-edge condition. Majorana chain decoration on 1D blocks $\tau_3$ is similar, hence all types of Majorana chain decorations are obstructed.

b. 1D FSPT state decoration  1D FSPT state can only be decorated on 1D blocks $\tau_2$ and $\tau_3$ because 1D block $\tau_1$ does not have $\mathbb{Z}_2$ on-site symmetry. First we consider the 1D FSPT state decoration on 1D blocks $\tau_2$ which leaves four dangling Majorana fermions ($\xi_j, \xi'_j, j = 1, 2$) at each 0D block $\mu_2/\mu_3$. Near $\mu_2/\mu_3$, the corresponding 4 Majorana fermions have the following symmetry properties ($j = 1, 2$):

$$M_{\tau_2} : \gamma_j \mapsto \gamma_j', \ \gamma'_j \mapsto -\gamma'_j$$  \hspace{1cm} (102)

Similar with the 1D block-state decorations for $p4m$ case (point group of 0D block $\mu_2$ in the cell decomposition of $p4m$ symmetry is $D_2$), these 4 Majorana fermions cannot be gapped out because they form a projective representation of $D_2$ group at each corresponding 0D block, and a non-degenerate ground state is forbidden. Accordingly, the 1D FSPT state decoration on $\tau_2$ or $\tau_3$ is obstructed because of the degenerate ground state.

There is one exception: if we decorate 1D FSPT phases on 1D blocks $\tau_2$ and $\tau_3$ simultaneously, it leaves eight dangling Majorana fermions at each $\mu_2$ and $\mu_3$. We demonstrate that these Majorana fermions can be gapped out: we have shown that if we decorate a 1D FSPT phase.
solely on each 1D block $\tau_2$ or $\tau_3$, four dangling Majorana fermions at each 0D block $\mu_2/\mu_3$ form a projective representation of $D_2$ symmetry group (acting internally, identical with $\mathbb{Z}_2 \times \mathbb{Z}_2$ on-site symmetry), so for the present situation, eight dangling Majorana fermions at each $\mu_2$ and $\mu_3$ form two projective representations of $D_2$ symmetry group. Nevertheless, there is only one nontrivial projective representation of $D_2$ which can be obtained by the following 2-cohomology:

$$\mathcal{H}^2 [\mathbb{Z}_2 \times \mathbb{Z}_2, U(1)] = \mathbb{Z}_2$$

(103)

So these two projective representations can form a linear representation of $D_2$ symmetry group, and the corresponding eight Majorana fermions can be gapped out. As a consequence, the only nontrivial obstruction-free 1D block-state is 1D FSPT state decorations on $\tau_2$ and $\tau_3$ simultaneously, and all obstruction-free 1D block-states form a group:

$$\{\text{OFBS}\}_{\text{cmm},0}^{1D} = \mathbb{Z}_2$$

(104)

where the group elements can be labeled by $n_2 = n_3$ ($n_2/n_3$ represents the number of decorated 1D FSPT states on $\tau_2/\tau_3$). According to aforementioned discussions, a necessary condition of an obstruction-free block-state is $n_2 = n_3$.

So far we have already obtained all obstruction-free block-states, and they form the following group:

$$\{\text{OFBS}\}_{\text{cmm},0} = \{\text{OFBS}\}_{\text{cmm},0}^{1D} \times \{\text{OFBS}\}_{\text{cmm},0}^{0D} = \mathbb{Z}_2 \times \mathbb{Z}_2^8 = \mathbb{Z}_2^9$$

(105)

With all obstruction-free block-states, subsequently we discuss all possible trivializations. First we consider the 2D bubble equivalences: we decorate a Majorana chain with anti-PBC on each 2D block and enlarge all “Majorana bubbles”, near each 1D block labeled by $\tau_1$, it can be deformed to double Majorana chains which can be trivialized because there is no on-site symmetry on $\tau_1$ and the classification of 1D invertible topological phases (i.e., Majorana chain) is $\mathbb{Z}_2$; near each 1D block labeled by $\tau_2/\tau_3$, it can also be deformed to double Majorana chains, nevertheless, these double Majorana chains cannot be trivialized because there is an on-site $\mathbb{Z}_2$ symmetry on each $\tau_2/\tau_3$ by internal action of reflection symmetry, and this $\mathbb{Z}_2$ action exchanges these two Majorana chains, and this is exactly the definition of the nontrivial 1D FSPT phase protected by on-site $\mathbb{Z}_2$ symmetry. Equivalently, 1D FSPT state decorations on 1D blocks $\tau_2$ and $\tau_3$ can be deformed to a trivial state via 2D “Majorana” bubble equivalence. Furthermore, similar with the $p4m$ case, there is no effect on 0D blocks labeled by $\mu_2$ and $\mu_3$ by taking 2D “Majorana” bubble equivalence; nevertheless, similar with the $p2$ case, 2D “Majorana bubble” construction changes the fermion parity of each 0D block labeled by $\mu_1$ because there is no reflection operation on 0D block $\mu_1$, and the alternative Majorana chain surrounding each $\mu_1$ is compatible with all other symmetry operations.

Subsequently we consider the 1D bubble equivalences. For instance, we decorate a pair of complex fermions [cf. Eq. (18)]: near each 0D block $\mu_1$, there are 2 complex fermions forming the following atomic insulator:

$$|\psi\rangle_{\text{cmm}}^{\mu_1} = c_1^{\dagger} c_2^{\dagger} |0\rangle$$

(106)

with rotation property:

$$R_{\mu_1} |\psi\rangle_{\text{cmm}}^{\mu_1} = c_2^{\dagger} c_1^{\dagger} |0\rangle = -|\psi\rangle_{\text{cmm}}^{\mu_1}$$

(107)

i.e., 1D bubble construction on $\tau_1$ changes the rotation eigenvalue at each 0D block $\mu_1$. Near each 0D block $\mu_2$, there are 4 complex fermions forming another atomic insulator:

$$|\psi\rangle_{\text{cmm}}^{\mu_2} = c_1^{\dagger} c_2^{\dagger} c_3^{\dagger} c_4^{\dagger} |0\rangle$$

(108)

with two independent reflection symmetry properties ($D_2$ symmetry at 0D block $\mu_2$ can also be generated by two independent reflections $M_{\tau_2}$ and $M_{\tau_3}$):

$$M_{\tau_3} |\psi\rangle_{\text{cmm}}^{\mu_2} = c_3^{\dagger} c_4^{\dagger} c_1^{\dagger} c_2^{\dagger} |0\rangle = |\psi\rangle_{\text{cmm}}^{\mu_2}$$

$$M_{\tau_2} |\psi\rangle_{\text{cmm}}^{\mu_2} = c_2^{\dagger} c_1^{\dagger} c_4^{\dagger} c_3^{\dagger} |0\rangle = |\psi\rangle_{\text{cmm}}^{\mu_2}$$

(109)

i.e., 1D bubble construction on $\tau_1$ does not change anything on $\mu_2$. Similar 1D bubble constructions can be held on 1D blocks $\tau_2$ and $\tau_3$, and we summarize the effects of 1D bubble constructions as following:

1. 1D bubble construction on $\tau_1$: changes the eigenvalue of $R_{\mu_1}$ at $\mu_1$;
2. 1D bubble construction on $\tau_2$: simultaneously changes the eigenvalues of $M_{\tau_2}$ at $\mu_2$ and $\mu_3$;
3. 1D bubble construction on $\tau_3$: simultaneously changes the eigenvalues of $M_{\tau_3}$ at $\mu_2$ and $\mu_3$.

With all possible trivializations, we are ready to study the trivial states. Start from the original trivial 0D block-state (nothing is decorated on arbitrary 0D blocks):

$$[(+,+),(+,+,+),(+,+,+)]$$

If we take 2D “Majorana bubble” construction $n_0$ times, and take 1D bubble equivalences on $\tau_j$ by $n_j$ times ($j = 1, 2, 3$), above trivial state will be deformed to a new 0D block-state labeled by:

$$[(−1)^{n_0},(−1)^{n_1}),(+,−(1)^{n_3}),(−1)^{n_2})$$

$$[(+,−(1)^{n_3}),(−1)^{n_2})]$$

(110)

According to the definition of bubble equivalence, all these states should be trivial. It is easy to see that there are only four independent quantities ($n_j = 0, 1, 2, 3$) in Eq. (110), hence all these trivial states form the following group:

$$\{\text{TBS}\}_{\text{cmm},0} = \{\text{TBS}\}_{\text{cmm},0}^{1D} \times \{\text{TBS}\}_{\text{cmm},0}^{0D} = \mathbb{Z}_2 \times \mathbb{Z}_2^3 = \mathbb{Z}_2^4$$

(111)
here \( \{\text{TBS}\}_{\text{cmm},0}^{\text{1D}} \) represents the group of trivial states with non-vacuum 1D blocks (i.e., 1D FSPT phase decorations on \( \tau_2 \) and \( \tau_3 \) simultaneously that is obtained from 2D “Majorana” bubble construction on the vacuum block-state), and \( \{\text{TBS}\}_{\text{cmm},0}^{\text{1D}} \) represents the group of trivial states with non-vacuum 0D blocks.

Therefore, all independent nontrivial block-states are labeled by the group elements of the following quotient group:

\[
E_{\text{cmm},0}^{\text{1D}} = \{\text{OFBS}\}_{\text{cmm},0}^{\text{1D}}/\{\text{TBS}\}_{\text{cmm},0}^{\text{1D}} = \mathbb{Z}_2^3/\mathbb{Z}_2 = \mathbb{Z}_2^5
\]

where all \( \mathbb{Z}_2 \)'s are from the nontrivial 0D block-states. It is obvious that there is no nontrivial group extension because of the absence of nontrivial 1D block-state, and the group structure of \( E_{\text{cmm},0} \) has already been correct.

Now we turn to discuss systems with spin-1/2 fermions. First we consider the 0D block-state decorations. For 0D blocks labeled by \( \mu_1 \), the 2-fold rotational symmetry acts on each of them internally, hence the total symmetry is \( \mathbb{Z}_2^4 \): nontrivial \( \mathbb{Z}_2^4 \) extension of on-site \( \mathbb{Z}_2 \) symmetry. And all different 0D block-states at which can be characterized by different 1D irreducible representations of the corresponding symmetry group are:

\[
\mathcal{H}^1 \left[ \mathbb{Z}_4^4, U(1) \right] = \mathbb{Z}_4
\]  

And there is no trivialization on them. Furthermore, for 0D blocks labeled by \( \mu_2 \) and \( \mu_3 \), the dihedral group symmetry \( D_2 \) acts on each of them internally, and similar with the \( p4m \) case, the classification of corresponding 0D block-states can be characterized by different 1D irreducible representations of the full symmetry group:

\[
\mathcal{H}^1 \left[ \mathbb{Z}_2 \times \mathbb{Z}_2, U(1) \right] = \mathbb{Z}_2^2
\]

Here different \( \mathbb{Z}_2 \)'s represent the rotation and reflection eigenvalues at each \( D_2 \) center. As a consequence, all obstruction-free 0D block-states form the following group:

\[
\{\text{OFBS}\}_{\text{cmm},1/2}^{\text{0D}} = \mathbb{Z}_4 \times \mathbb{Z}_2^4
\]

and there is no trivialization on them \((\{\text{TBS}\}_{\text{cmm},1/2}^{\text{0D}} = 0)\). As a consequence, the classification attributed to 0D block-state decorations is:

\[
E_{\text{cmm},1/2}^{\text{0D}} = \mathbb{Z}_4 \times \mathbb{Z}_2^4
\]

Subsequently we investigate the 1D block-state decoration. On \( \tau_1 \), the unique possible 1D block-state is Majorana chain because of the absence of the on-site symmetry; on \( \tau_2 \) and \( \tau_3 \), the total symmetry group is \( \mathbb{Z}_4^4 \), hence there is no candidate block-state due to the trivial classification of the corresponding 1D FSPT phases. The Majorana chain decoration on \( \tau_1 \) leaves 2 dangling Majorana fermions at each \( \mu_1 \), and 4 dangling Majorana fermions at each \( \mu_2 \). At \( \mu_1 \), the 2 dangling Majorana fermions at which can be gapped out by an entanglement pair without breaking any symmetry are:

\[
R_{\mu_1} : i\gamma_1 \gamma_2 \mapsto -i\gamma_2 \gamma_1 = i\gamma_1 \gamma_2
\]

at \( \mu_2 \), the 4 Majorana fermions have the following reflection symmetry properties \((D_2 \text{ symmetry can also be generated by two independent reflections } M_{\tau_2} \text{ and } M_{\tau_3})\):

\[
M_{\tau_2} : (\eta_1, \eta_2, \eta_3, \eta_4) \mapsto (\eta_2, -\eta_1, -\eta_4, -\eta_3)
\]

\[
M_{\tau_3} : (\eta_1, \eta_2, \eta_3, \eta_4) \mapsto (\eta_4, \eta_3, -\eta_2, -\eta_1)
\]

Consider the following Hamiltonian containing two entanglement pairs of these four Majorana fermions:

\[
H_{\mu_2} = -i\eta_1 \eta_3 - i\eta_2 \eta_4
\]

It is easy to verify that \( H_{\mu_2} \) is invariant under the symmetry actions \((\text{118})\). As a consequence, all obstruction-free 1D block-states form the following group:

\[
\{\text{OFBS}\}_{\text{cmm},1/2}^{\text{1D}} = \mathbb{Z}_2
\]

And it is easy to see that there is no trivialization \(\text{(i.e., } \{\text{TBS}\}_{\text{cmm},1/2}^{\text{1D}} = 0\)\). So the classification attributed to 1D block-state decorations is:

\[
E_{\text{cmm},1/2}^{\text{1D}} = \mathbb{Z}_2
\]

With the classification data as Eqs. \((\text{116})\) and \((\text{121})\), we consider the group structure of the corresponding classification. Equivalently, we investigate that if 1D block-state extends 0D block-state. The only possible case of stacking should happen on 1D blocks labeled by \( \tau_1 \) because according to discussions about \( p4m \) and \( p2 \) cases, other 1D blocks have no nontrivial block-state. We decorate two copies of Majorana chains on \( \tau_1 \), which will leave 2 dangling Majorana fermions at each 0D block labeled by \( \mu_1 \) and 4 dangling Majorana fermions at each 0D block labeled by \( \mu_2 \). At \( \mu_1 \), these Majorana chains can be smoothly deformed to the state described by Eqs. \((\text{85})\) and \((\text{88})\), with the symmetry properties as Eq. \((\text{89})\). So similar with \( p2 \) case, near each 0D block labeled by \( \mu_1 \), 1D block-states extend 0D block-state, and 0D block-states at \( \mu_1 \) have the group structure \( \mathbb{Z}_8 \) as a consequence. At \( \mu_2 \), these Majorana chains can be smoothly deformed to two copies of the state described by Eqs. \((\text{85})\) and \((\text{88})\), and have eigenvalue \(-1\) under 2-fold rotational symmetry. The classification data of 0D block-states at \( \mu_2 \) is determined by Eq. \((\text{114})\), hence if a 0D block-state with eigenvalue \(-1\) under 2-fold rotation is attached to each 1D block-state near each 0D block labeled by \( \mu_2 \), the rotation eigenvalue \( s \) of the obtained 0D block-state becomes:

\[
s = -1 \times -1 = 1
\]

Therefore, near 0D block \( \mu_2 \) there is an appropriate 1D block-state which itself form a \( \mathbb{Z}_2 \) structure under stack-
respectively. So at each 0D block, the block-state can have 2-fold rotational symmetry on each 0D block, these two Z's represent the fermion parity and rotation eigenvalue, respectively. According to this notation, the obstruction-free 0D block-states form the following group:

$$\{\text{OFBS}\}^{0\text{D}}_{\mu_0, \mu_2} = \mathbb{Z}_2$$  \hspace{1cm} (126)

and the group elements can be labeled by (two brackets represent the block-states at \(\mu_1\) and \(\mu_2\)):

\[
[(\pm, \pm), (\pm, \pm)]
\]

Then we consider possible trivializations via bubble construction. First of all, we consider the 2D bubble equivalence: we decorate a Majorana chain with anti-PBC on each 2D block which can be trivialized if it shrinks to a point. Similar with the \(p\) case, by some proper local unitary transformations, this assembly of bubbles can be deformed to an assembly of Majorana chains with odd fermion parity surrounding each of 0D block, and the fermion parities of all 0D blocks are changed simultaneously. Equivalently, the fermion parities of 0D blocks labeled by \(\mu_1\) and \(\mu_2\) are not independent.

Then we study the role of rotational symmetry. Consider the 1D bubble equivalence on \(\tau_2\): we decorate a pair of complex fermions [cf. Eq. (18)]: Near \(\mu_2\), there are 2 complex fermions which form an atomic insulator:

$$|\psi\rangle^{\mu_2}_{\text{pgg}} = c_1^\dagger c_2^\dagger |0\rangle$$  \hspace{1cm} (127)

with rotation property as \(R_{\mu_2}\) represents the rotation operation centred at the 0D block labeled by \(\mu_2\):

$$R_{\mu_2} |\psi\rangle^{\mu_2}_{\text{pgg}} = c_2^\dagger c_1^\dagger |0\rangle = -|\psi\rangle^{\mu_2}_{\text{pgg}}$$  \hspace{1cm} (128)

i.e., \(|\psi\rangle^{\mu_2}_{\text{pgg}}\) can trivialize the rotation eigenvalue \(-1\) at each 0D block labeled by \(\mu_2\), similar for the 0D block labeled by \(\mu_1\). Hence the rotation eigenvalues at \(\mu_1\) and \(\mu_2\) are not independent; and we further consider the 1D bubble equivalence on \(\tau_1\): Near each 0D block labeled by \(\mu_1\), there are 4 complex fermions which form another atomic insulator:

$$|\psi\rangle^{\mu_1}_{\text{pgg}} = c_1 c_2 c_3^\dagger c_4^\dagger$$  \hspace{1cm} (129)

with rotation property as \(R_{\mu_1}\) represents the rotation operation centred at the 0D block labeled by \(\mu_1\):

$$R_{\mu_1} |\psi\rangle^{\mu_1}_{\text{pgg}} = c_3 c_4 c_1^\dagger c_2^\dagger = |\psi\rangle^{\mu_1}_{\text{pgg}}$$  \hspace{1cm} (130)

So there is no trivialization from this bubble construction.

With all possible bubble constructions, we are ready to study the trivial states. Start from the original trivial state (nothing is decorated on arbitrary 0D block):

\[
[(+, +), (+, +)]
\]

if we take 2D bubble construction \(n_0\) times and 1D bubble construction on \(\tau_2\) by \(n_2\) times, above trivial state will be deformed to a new 0D block-state labeled by:

\[
[((-1)^{n_0}, (-1)^{n_2}), ((-1)^{n_0}, (-1)^{n_2})] \hspace{1cm} (131)
\]
According to the definition of bubble equivalence, all these states should be trivial. It is easy to see that there are only two independent quantities in the state \((131)\), hence all trivial states form the group:

\[
\{\text{TBS}\}_{p,0}^{0D} = \mathbb{Z}_2 \tag{132}
\]

Therefore, all independent nontrivial 0D block-states are labeled by different group elements of the following quotient group:

\[
E_{p,0}^{0D} = \{\text{OFBS}\}_{p,0}^{0D}/\{\text{TBS}\}_{p,0}^{0D} = \mathbb{Z}_2^2 \tag{133}
\]

Subsequently we investigate the 1D block-state decoration. Due to the absence of the on-site symmetry, the unique possible 1D block-state is Majorana chain. So all 1D block-states form a group:

\[
\{\text{BS}\}_{\gamma,0}^{1D} = \mathbb{Z}_2^2 \tag{134}
\]

Then we discuss the possible obstructions: we discuss the 1D block-state decorations on \(\tau_1\) and \(\tau_2\) separately.

\[\text{a. Majorana chain decoration on } \tau_1\]

Majorana chain decoration on \(\tau_1\) leaves 4 dangling Majorana fermions at each corresponding 0D block \(\mu_1\) with the following rotational symmetry properties:

\[
R_{\mu_1} : \gamma_j \mapsto \gamma_{j+2} \tag{135}
\]

where \(R_{\mu_1}\) is the generator of \(C_2\) rotational symmetry centred at each 0D block labeled by \(\mu_1\) and all subscripts are taken with modulo 4 (i.e., \(\gamma_5\) represents the Majorana mode labeled by \(\gamma_1\)). Consider the local fermion parity and its symmetry property:

\[
P_f = -\prod_{j=1}^{4} \gamma_j, \quad R_{\mu_1} : P_f \mapsto P_f
\]

Hence these 4 dangling Majorana fermions can be gapped by some proper interactions in a symmetric way, equivalently the no-open-edge condition is satisfied. More precisely, we consider the following Hamiltonian near each 0D block \(\mu_1\):

\[
H = i\gamma_1\gamma_2 + i\gamma_3\gamma_4 \tag{136}
\]

It is obvious that \(H\) is symmetric under \((135)\), and it can gap out the four Majorana fermions at each \(\mu_1\).

\[\text{b. Majorana chain decoration on } \tau_2\]

Majorana chain decoration on \(\tau_2\) leaves 2 dangling Majorana fermions at each corresponding 0D block which can be gapped out by an entanglement pair. Nevertheless this entanglement pair breaks the rotational symmetry, and the no-open-edge condition is violated.

As a consequence, all obstruction-free 1D block-states form the following group:

\[
\{\text{OFBS}\}_{p,0}^{1D} = \mathbb{Z}_2 \tag{137}
\]

and it is easy to verify that there is no trivialization (i.e., \(\{\text{OFBS}\}_{p,0}^{1D} = 0\)). Therefore, all independent nontrivial 1D block-states are labeled by different group elements of the following group:

\[
E_{p,0}^{1D} = \{\text{OFBS}\}_{p,0}^{1D}/\{\text{TBS}\}_{p,0}^{1D} = \mathbb{Z}_2 \tag{138}
\]

It is straightforward to see that there is no stacking between 1D and 0D block-states, and the ultimate classification with accurate group structure is:

\[
\mathcal{G}_{p,0}^{1D} = E_{p,0}^{1D} \times E_{p,0}^{0D} = \mathbb{Z}_2^3 \tag{139}
\]

Then we turn to the systems with spin-1/2 fermions. First we investigate the 0D block-state decorations. All 0D blocks are 2-fold rotation centers, hence the total symmetry group of each 0D block is \(\mathbb{Z}_4^f\); nontrivial \(\mathbb{Z}_2^f\) extension of the on-site symmetry \(\mathbb{Z}_2\), and different 0D block-states at which can be characterized by different 1D irreducible representations of the corresponding symmetry group are:

\[
\mathcal{H}^1\left[\mathbb{Z}_4^f, U(1)\right] = \mathbb{Z}_4 \tag{140}
\]

All root phases at each 0D block are characterized by group elements of \(\{1, i, -1, -i\}\). So at each 0D block, the block-state can be labeled by \(\nu \in \{1, i, -1, -i\}\). According to this notation, the obstruction-free 0D block-states form the following group:

\[
\{\text{OFBS}\}_{\nu,1/2}^{0D} = \mathbb{Z}_2^2 \tag{141}
\]

and different group elements can be labeled by:

\[
[\nu_1, \nu_2]
\]

where \(\nu_1\) and \(\nu_2\) label the 0D block-state at \(\mu_1\) and \(\mu_2\). It is easy to see that there is no trivialization on 0D block-states (i.e., \(\{\text{BS}\}_{\nu,1/2}^{0D} = 0\)), so the classification attributed to 0D block-state decorations is:

\[
E_{\nu,1/2}^{0D} = \{\text{OFBS}\}_{\nu,1/2}^{0D}/\{\text{TBS}\}_{\nu,1/2}^{0D} = \mathbb{Z}_4^2 \tag{142}
\]

Subsequently we investigate the 1D block-state decoration. The unique possible 1D block-state is Majorana chain because of the absence of on-site symmetry.

\[\text{c. Majorana chain decoration on } \tau_1\]

Majorana chain decoration on \(\tau_1\) leaves 4 dangling Majorana fermions at each 0D blocks labeled by \(\mu_1\) with identical symmetry properties with the spinless fermions [cf. Eq. \(135\)], hence these 4 Majorana fermions can be gapped out by some proper interactions in a symmetric way, and the no-open-edge condition is satisfied.

\[\text{d. Majorana chain decoration on } \tau_2\]

Majorana chain decoration on \(\tau_2\) leaves 2 dangling Majorana fermions at each 0D block \(\mu_2\) which can be gapped out by an entanglement pair in a symmetric way. Therefore the no-open-edge condition is satisfied. Consequently, all obstruction-free 1D block-states form the following group:

\[
\{\text{OFBS}\}_{\nu,1/2}^{1D} = \mathbb{Z}_2^2 \tag{143}
\]
and it is obvious that there is no trivialization from bubble constructions (i.e., \(\{\text{TBS}\}_{\text{pgg},1/2}^{\text{ID}} = 0\)). Therefore, the classification of 2D FSPT phases with \(\text{pgg}\) symmetry attributed to 1D block-state decoration is:

\[
E_{\text{pgg},1/2}^{\text{ID}} = \{\text{OFBS}\}_{\text{pgg},1/2}^{\text{ID}} / \{\text{TBS}\}_{\text{pgg},1/2}^{\text{ID}} = \mathbb{Z}_2^2
\]

Then we study the possible stacking between 1D and 0D block-states. If we decorate two Majorana chains on each 1D block labeled by \(\tau_1\), similar with \(\text{cmcm}\) case, there is no stacking between 1D and 0D block-states; if we decorate two Majorana chains on each 1D block labeled by \(\tau_2\), similar with \(\rho 2\) case, it can be smoothly deformed to an assembly of 0D root phases at 0D blocks \(\mu_2\). Therefore, the ultimate classification with accurate group structure is:

\[
\mathcal{G}_{\text{pgg},1/2} = E_{\text{pgg},1/2}^{\text{ID}} \times \omega_2 E_{\text{pgg},1/2}^{\text{0D}} = \mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_8
\]

here the symbol “\(\times \omega_2\)” means that 1D and 0D block-states \(E_{\text{cmcm},1/2}^{\text{ID}}\) and \(E_{\text{cmcm,1/2}}^{\text{0D}}\) have nontrivial extension, and are described by the following short exact sequence:

\[
0 \rightarrow E_{\text{pgg},1/2}^{\text{ID}} \rightarrow \mathcal{G}_{\text{pgg},1/2} \rightarrow E_{\text{pgg},1/2}^{\text{0D}} \rightarrow 0
\]

E. Hexagonal lattice: \(\rho 6m\)

For hexagonal lattice, we demonstrate the crystalline TSC protected by \(\rho 6m\) symmetry as an example. The corresponding point group of \(\rho 6m\) is dihedral group \(D_6\), and for 2D blocks labeled by \(\sigma\), there is no on-site symmetry; for arbitrary 1D block, the on-site symmetry is \(\mathbb{Z}_2\) which is attributed to the reflection symmetry acting internally; for 0D blocks \(\mu_1\), the on-site symmetry group is \(\mathbb{Z}_6 \times \mathbb{Z}_2\) which is attributed to the \(D_6\) group acting internally; for 0D blocks \(\mu_2\), the on-site symmetry group is \(\mathbb{Z}_2 \times \mathbb{Z}_2\) which is attributed to the \(D_2 \subset D_6\) acting internally; for 0D blocks \(\mu_3\), the on-site symmetry group is \(\mathbb{Z}_3 \times \mathbb{Z}_2\), which is attributed to the \(D_3 \subset D_6\) acting internally. The cell decomposition is shown in Fig. 10.

We discuss systems with spinless and spin-1/2 fermions separately. Consider the 0D block-state decorations, for \(\mu_j\), \(j = 1, 2, 3\), the classification data can be characterized by different 1D irreducible representations of the full symmetry groups, respectively:

\[
\mathcal{H}_1^1 \left[ \mathbb{Z}_2^f \times (\mathbb{Z}_6 \times \mathbb{Z}_2), U(1) \right] = \mathbb{Z}_2^3
\]

\[
\mathcal{H}_1^1 \left[ \mathbb{Z}_2^f \times (\mathbb{Z}_2 \times \mathbb{Z}_2), U(1) \right] = \mathbb{Z}_2^3
\]

\[
\mathcal{H}_1^1 \left[ \mathbb{Z}_2^f \times (\mathbb{Z}_3 \times \mathbb{Z}_2), U(1) \right] = \mathbb{Z}_2^2
\]

For 0D blocks labeled by \(\mu_1\) and \(\mu_2\), three \(\mathbb{Z}_2\) in the classification data [cf. first two rows in Eq. (147)] have different physical meanings: the first \(\mathbb{Z}_2\) represents the complex fermion, the second \(\mathbb{Z}_2\) represents the rotation eigenvalue \(-1\), and the third \(\mathbb{Z}_2\) represents the reflection eigenvalue \(-1\); for 0D blocks labeled by \(\mu_3\), two \(\mathbb{Z}_2\) in the classification data [cf. the last row in Eq. (147)] have different physical meanings: the first \(\mathbb{Z}_2\) represents the complex fermion, and the second \(\mathbb{Z}_2\) represents the reflection eigenvalue \(-1\) (i.e., the rotational symmetry plays no role in the 0D block-state decorations). So the 0D block-states at \(\mu_1\) and \(\mu_2\) can be labeled by \((\pm, \pm, \pm)\), here these three \(\pm\)’s represent the fermion parity, 2-fold rotation and reflection symmetry eigenvalues (alternatively, the last two \(\pm\)’s can also represent the eigenvalues of two independent reflection operations because even-fold dihedral group can also be generated by two independent reflections); the 0D block-states at \(\mu_3\) can be labeled by \((\pm, \pm)\), here these two \(\pm\)’s represent the fermion parity and reflection symmetry eigenvalues. According to this notation, the obstruction-free 0D block-states form the following group:

\[
\{\text{OFBS}\}_{\text{rho6m}}^{0D} = \mathbb{Z}_2^8
\]

and the group elements can be labeled by (three brackets represent the block-states at \(\mu_1\), \(\mu_2\) and \(\mu_3\)):

\[
[(\pm, \pm, \pm), (\pm, \pm, \pm), (\pm, \pm)]
\]

Subsequently we investigate the 1D block-state decoration. For all 1D blocks, the total symmetry group is \(\mathbb{Z}_2^f \times \mathbb{Z}_2\), and the candidate 1D block-state is Majorana chain and 1D FSPT state. So all 1D block-states form a group:

\[
\{\text{BS}\}_{\text{rho6m,0}}^{1D} = \mathbb{Z}_2^6
\]

Then we discuss the decorations of these two root phases separately.

a. Majorana chain decoration. Consider Majorana chain decorations on 1D blocks labeled by \(\tau_1\), which
leaves 6 dangling Majorana fermions at each \( \mu_1 \) and 2 dangling Majorana fermions at each \( \mu_2 \). Near each 0D block \( \mu_1 \), six dangling Majorana fermions have the following rotational symmetry properties (all subscripts are taken with modulo 6):

\[
R_{\mu_1}: \gamma_j \mapsto \gamma_{j+1}, \quad j = 1, \ldots, 6. \tag{150}
\]

Then we consider the local fermion parity and its rotational symmetry property:

\[
P_f = i \prod_{j=1}^{6} \gamma_j, \quad R_{\mu_1}: P_f \mapsto -P_f \tag{151}
\]

Thus these 6 dangling Majorana fermions form a projective representation of the symmetry group \( p6m \times Z_2^f \), and a non-degenerate ground state is forbidden. Thus Majorana chain decoration on 1D blocks \( \tau_1 \) is obstructed because of the violation of the no-open-edge condition. On \( \tau_2 \), the Majorana chain decoration leaves 6 dangling Majorana fermions at each \( \mu_1 \) and 3 dangling Majorana fermions at each \( \mu_3 \). It is well-known that odd number of Majorana fermions cannot be gapped out, hence Majorana chain decoration on \( \tau_2 \) is obstructed. On \( \tau_3 \), Majorana chain decoration leaves 2 dangling Majorana fermions at each \( \mu_2 \) and 3 dangling Majorana fermions at each \( \mu_3 \). Similar with the \( \tau_2 \) case, Majorana chain decoration is obstructed. Note that if we consider all 1D blocks together and decorate a Majorana chain on each, it leaves 12 dangling Majorana fermions at each \( \mu_1 \), 4 dangling Majorana fermions at each \( \mu_2 \) and 6 dangling Majorana fermions at each \( \mu_3 \). Consider Majorana fermions as edge modes of the decorated Majorana fermions at each \( \mu_2 \), with the following rotation and reflection symmetry properties (all subscripts are taken with modulo 4):

\[
R_{\mu_3}: \gamma_j' \mapsto \gamma_{j+2}' \quad M_{\tau_3}: \gamma_j' \mapsto \gamma_{6-j}' \tag{152}
\]

Then we consider the local fermion parity and its rotation and reflection symmetry properties:

\[
P_f' = -i \prod_{j=1}^{4} \gamma_j', \quad \begin{cases} 
R_{\mu_3}: P_f' \mapsto P_f' \\
M_{\tau_3}: P_f' \mapsto -P_f'
\end{cases} \tag{153}
\]

Thus these Majorana fermions cannot be gapped out in a symmetric way, and Majorana chain decoration on all 1D blocks is obstructed. As a consequence, Majorana chain decoration does not contribute a nontrivial crystalline TSC.

b. 1D FSPT state decoration 1D FSPT state decoration on \( \tau_1 \) leaves 12 dangling Majorana fermions at each \( \mu_1 \) and 4 dangling Majorana fermions at each \( \mu_2 \). Similar with the \( p4m \) and \( cmm \) cases, four Majorana fermions at each \( \mu_2 \) form a projective representation of \( D_2 \) symmetry group, and non-degenerate ground-state is forbidden. Thus the 1D FSPT state decoration on \( \tau_1 \) is obstructed.

1D FSPT state decoration on \( \tau_2 \) leaves 12 dangling Majorana fermions at each \( \mu_1 \) and 6 dangling Majorana fermions at each \( \mu_3 \). Consider the Majorana fermions at each \( \mu_3 \), with the following rotation and reflection symmetry properties (all subscripts are taken with modulo 3):

\[
R_{\mu_3}: \eta_j \mapsto \eta_{j+1}, \quad \eta_j' \mapsto \eta_{j+1}' \quad M_{\tau_3}: \eta_j \mapsto -\eta_{6-j}, \quad \eta_j' \mapsto \eta_{6-j}' \tag{154}
\]

Then we consider the local fermion parity with its rotation and reflection symmetry properties:

\[
P_f'' = i \prod_{j=1}^{3} \eta_j, \quad \begin{cases} 
R_{\mu_3}: P_f'' \mapsto P_f'' \\
M_{\tau_3}: P_f'' \mapsto -P_f''
\end{cases} \tag{155}
\]

Hence these 6 Majorana fermions form a projective representation of the symmetry group \( Z_2^f \times p6m \) that cannot be gapped out in a symmetric way, and the corresponding 1D FSPT state decoration is obstructed because of the violation of the no-open-edge condition. 1D FSPT state decoration on \( \tau_3 \) leaves 4 dangling Majorana fermions at each \( \mu_2 \) and 6 dangling Majorana fermions at each \( \mu_3 \). Similar with the 1D FSPT state decoration on \( \tau_2 \) case, 6 Majorana fermions at each \( \mu_3 \) cannot be gapped out in a symmetric way: consider the Majorana fermions as the edge modes of decorated Majorana chains on \( \tau_3 \) at each \( \mu_3 \), with the following rotation and reflection symmetry properties (all subscripts are taken with modulo 3):

\[
R_{\mu_3}: \zeta_j \mapsto \zeta_{j+1}, \quad \zeta_j' \mapsto \zeta_{j+1}' \quad M_{\tau_3}: \zeta_j \mapsto -\zeta_{6-j}, \quad \zeta_j' \mapsto \zeta_{6-j}' \tag{156}
\]

with the local fermion parity and its rotation and reflection symmetry properties:

\[
P_f'' = i \prod_{j=1}^{3} \zeta_j, \quad \begin{cases} 
R_{\mu_3}: P_f'' \mapsto P_f'' \\
M_{\tau_3}: P_f'' \mapsto -P_f''
\end{cases} \tag{158}
\]

Thus 1D FSPT state decoration on \( \tau_3 \) is obstructed, and it does not contribute nontrivial crystalline TSC because of the violation of the no-open-edge condition.

Note that if we consider 1D blocks labeled by \( \tau_2 \) and \( \tau_3 \) together and decorate a 1D FSPT state on each of them, this decoration leaves 12 dangling Majorana fermions at each \( \mu_1 \) and \( \mu_3 \), and 4 dangling Majorana fermions at each \( \mu_2 \). For the Majorana fermions as the edge modes of the decorated 1D FSPT states at each \( \mu_2 \), as aforementioned, they can be gapped out in a symmetric way: For Majorana fermions as the edge modes of the decorated 1D FSPT states at each \( \mu_2 \), the local fermion parity is the product of \( P_f'' \) and \( P_f''' \), with the following symmetry properties:

\[
P_f'' = P_f'' P_f''', \quad \begin{cases} 
R_{\mu_3}: P_f'' \mapsto P_f'' \\
M_{\tau_3}: P_f'' \mapsto -P_f''
\end{cases} \tag{159}
\]

Hence any symmetry operations commute with the fermion parity. Furthermore, there is no nontrivial
projective representation of the $D_3$ group acting internally (identical with the internal symmetry group $\mathbb{Z}_3 \times \mathbb{Z}_2$), it can be obtained by calculating the following 2-cohomology of the symmetry group:

$$\mathcal{H}^2 [\mathbb{Z}_3 \times \mathbb{Z}_2, U(1)] = 0$$ (160)

Therefore, these 12 dangling Majorana fermions form a linear representation of the symmetry group, and can be gapped out by some proper interactions in symmetry way. Nevertheless, four Majorana fermions at each 0D block labeled by $\mu_2$ form a projective representation of the $D_2$ symmetry group that forbids the non-degenerate ground-state, so the 1D FSPT state decoration on $\tau_2$ and $\tau_3$ is still obstructed because of the violation of no-open-edge condition at each 0D block $\mu_2$.

There is one exception: If we decorate a 1D FSPT phase on each 1D block (including $\tau_j$, $j = 1, 2, 3$), the dangling Majorana fermions at each 0D block can be gapped out in a symmetric way. In the aforementioned discussions we have elucidated that at each $\mu_3$, there are 12 dangling Majorana fermions via 1D FSPT state decorations that can be gapped in a symmetric way; and at each $\mu_2$, there are 8 dangling Majorana fermions and similar with the $p\bar{4}m$ and $cmcm$ case, they can be gapped out in a symmetric way because they form a linear representation of the corresponding symmetry group. Near each 0D block labeled by $\mu_1$, this decoration leaves 24 dangling Majorana fermions as the edge modes of decorated 1D FSPT phases. Consider half of them from 1D FSPT state decorations on $\tau_1$ with the following rotation and reflection symmetry properties (all subscripts are taken with modulo 6):

$$R_{\mu_1} : \gamma_j \mapsto \gamma_{j+1}, \quad \gamma'_j \mapsto \gamma'_{j+1}$$
$$M_{\tau_1} : \gamma_j \mapsto \gamma_{-j}, \quad \gamma'_j \mapsto \gamma'_{-j}$$ (161)

Then we consider the local fermion parity and its rotation and reflection symmetry properties:

$$P_{f_{\tau_1}} = -\prod_{j=1}^{6} \gamma_j \gamma'_j, \quad R_{\mu_1}, M_{\tau_1} : P_{f_{\tau_1}} \mapsto P_{f_{\tau_1}}$$ (162)

Hence arbitrary symmetry actions commute with the fermion parity of these 12 Majorana fermions, and they form either a linear representation or a projective representation of the $D_6$ symmetry. Similar arguments can be held for other 12 Majorana fermions. We should note that there is only one nontrivial projective representation of the $D_6$ symmetry group acting internally (i.e., $\mathbb{Z}_6 \times \mathbb{Z}_2$ on-site symmetry) that can easily to be verified by the following 2-cohomology:

$$\mathcal{H}^2 [\mathbb{Z}_6 \times \mathbb{Z}_2, U(1)] = \mathbb{Z}_2$$ (163)

So these 24 Majorana fermions together form a linear representation of the $D_6$ symmetry at each 0D block labeled by $\mu_1$, and they can be gapped out in a symmetric way. Thus the 1D FSPT state decorations on all 1D blocks simultaneously is obstruction-free, and all obstruction-free 1D block-states form the following group:

$$\{\text{OFBS}\}_\mu^{1D} = \{\text{OFBS}\}_\mu^{1D} = \{\text{OFBS}\}_\mu^{1D} \times \{\text{OFBS}\}_\mu^{1D}$$ (164)

and the group elements can be labeled by $n_1 = n_2 = n_3$. Here $n_j = 0, 1$ ($j = 1, 2, 3$) represents the number of decorated 1D FSPT states on $\tau_j$, respectively. According to aforementioned discussions, a necessary condition of an obstruction-free block-state is $n_1 = n_2 = n_3$.

So far we have already obtained all obstruction-free block-states, and they form the following group:

$$\{\text{OFBS}\}_\mu^{1D} = \{\text{OFBS}\}_\mu^{1D} \times \{\text{OFBS}\}_\mu^{1D}$$ (165)

With all obstruction-free block-states, subsequently we discuss all possible trivializations. First we consider about the 2D bubble equivalence: Similar with the $p\bar{4}m$ case, “Majorana bubbles” can be deformed to double Majorana chains at each nearby 1D block, and this is exactly the definition of the nontrivial 1D FSPT phase protected by on-site $\mathbb{Z}_2$ symmetry (by reflection symmetry acting internally). As a consequence, 1D FSPT state decorations on all 1D blocks can be deformed to a trivial state via 2D “Majorana” bubble equivalences. Furthermore, repeatedly similar with the $p\bar{4}m$ case, “Majorana bubble” constructions have no effect on 0D blocks.

Subsequently we consider the 1D bubble equivalences. For example, on each 1D block labeled by $\tau_2$, we decorate a pair of complex fermions [cf. Eq. (18)]: Near each 0D block labeled by $\mu_1$, there are 6 complex fermions which form an atomic insulator with even fermion parity:

$$|\psi\rangle_{\mu_1}^{\mu_1} = \prod_{j=1}^{6} c_j^{\dagger} |0\rangle$$ (166)

hence $|\psi\rangle_{\mu_1}^{\mu_1}$ cannot change the fermion parity of the 0D block labeled by $\mu_1$: Near each 0D block labeled by $\mu_3$, there are 3 complex fermions which form another atomic insulator with odd fermion parity:

$$|\psi\rangle_{\mu_3}^{\mu_3} = c_1^{\dagger} c_2^{\dagger} c_3^{\dagger} |0\rangle$$ (167)

and it can change the fermion parity at each 0D block labeled by $\mu_3$. Then we consider the symmetry properties of these atomic insulators: the eigenvalues of $|\psi\rangle_{\mu_3}^{\mu_3}$ at $\mu_1$ under two independent reflection operations is:

$$M_{\tau_1} |\psi\rangle_{\mu_3}^{\mu_3} = c_1^{\dagger} c_2^{\dagger} c_3^{\dagger} = -|\psi\rangle_{\mu_3}^{\mu_3}$$ (168)

$$M_{\tau_2} |\psi\rangle_{\mu_3}^{\mu_3} = c_1^{\dagger} c_2^{\dagger} c_3^{\dagger} = -|\psi\rangle_{\mu_3}^{\mu_3}$$ (169)

i.e., 1D bubble construction on $\tau_2$ can change the eigenvalue of $M_{\tau_2}$ and leave the eigenvalue of $M_{\tau_3}$ invariant. The eigenvalue of $|\psi\rangle_{\mu_3}^{\mu_3}$ at $\mu_3$ under reflection $M_{\tau_2}$ is:

$$M_{\tau_2} |\psi\rangle_{\mu_3}^{\mu_3} = c_1^{\dagger} c_2^{\dagger} c_3^{\dagger} |0\rangle = -|\psi\rangle_{\mu_3}^{\mu_3}$$ (169)
i.e., 1D bubble construction on $\tau_2$ can change the eigenvalue of $M_{\tau_2}$. Similar 1D bubble constructions can be held on other 1D blocks, and we summarize the effects of 1D bubble constructions as following:

1. 1D bubble construction on $\tau_2$: simultaneously changes the eigenvalue of $M_{\tau_2}$ at $\mu_1$ and $M_{\tau_2}$ at $\mu_2$;

2. 1D bubble construction on $\tau_3$: simultaneously changes the eigenvalue of $M_{\tau_3}$ at $\mu_1$, $M_{\tau_3}$ at $\mu_3$, and the fermion parity of $\mu_3$;

3. 1D bubble construction on $\tau_3$: simultaneously changes the eigenvalues of $M_{\tau_3}$ at $\mu_2$, $M_{\tau_2}$ at $\mu_3$, and the fermion parity of $\mu_3$.

There is another type of 1D bubble construction on $\tau_2$ and $\tau_3$ (we denote the above 1D bubble construction by “type-I” and this 1D bubble construction by “type-II”): we decorate an Eq. (18) on each $\tau_2$ (here both yellow and red dots represent the 0D FSPT mode with reflection eigenvalue $-1$), near $\mu_1$, there are six 0D FSPT modes with reflection eigenvalue $-1$ that change nothing; near $\mu_3$, there are three 0D FSPT modes with reflection eigenvalues $-1$ that changes the reflection eigenvalue at $\mu_3$. Similar for 1D bubble constructions on $\tau_3$.

With all possible bubble constructions, we are ready to investigate the trivial states. Start from the original 0D trivial block-state (nothing is decorated on arbitrary 0D blocks):

$$[(+, +, +), (+, +, +), (+, +)]$$

if we take type-I 1D bubble constructions on $\tau_j$ by $n_j$ times ($j = 1, 2, 3$), and type-II 1D bubble constructions on $\tau_2$ and $\tau_3$ by $n_2'$ and $n_3'$ times, above trivial state will be deformed to a new block-state labeled by:

$$[(+, (-1)^{n_2}, (-1)^{n_3}), (+, (-1)^{n_3}, (-1)^{n_3}), (-1)^{n_2 + n_3}, (-1)^{n_2 + n_3}].$$

(170)

According to the definition of bubble equivalence, all these states should be trivial. Alternatively, all 0D block-states can be viewed as vectors of an 8-dimensional $\mathbb{Z}_2$-valued vector space $V$, and all trivial 0D block-states with the form as Eq. (170) can be viewed as vectors of the subspace of $V$. The dimensionality of this subspace can be determined by calculating the rank of the following transformation matrix:

$$\begin{pmatrix}
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0
\end{pmatrix} = 4$$

(171)

Here different rows of this matrix represent different bubble constructions. Hence the dimensionality of the vector subspace containing all trivial 0D block-states is 4. Together with the 2D bubble equivalence, all trivial states form the group:

$$\{\text{TBS}\}_{p6m,0} = \{\text{TBS}\}_{p6m,0}^{1D} \times \{\text{TBS}\}_{p6m,0}^{0D}$$

$$\mathbb{Z}_2 \times \mathbb{Z}_2^3 = \mathbb{Z}_2^5$$

(172)

here $\{\text{TBS}\}_{p6m,0}^{1D}$ represents the group of trivial states with non-vacuum 1D blocks (i.e., 1D FSPT phase decorations on all 1D blocks simultaneously), and $\{\text{TBS}\}_{p6m,0}^{0D}$ represents the group of trivial states with non-vacuum 0D blocks.

Therefore, all independent nontrivial block-states are labeled by the group elements of the following quotient group:

$$G_{p6m,0} = \{\text{OFBS}\}_{p6m,0}^{1D} / \{\text{TBS}\}_{p6m,0}^{0D}$$

$$\mathbb{Z}_2^3 / \mathbb{Z}_2^5 = \mathbb{Z}_2^4$$

(173)

here all $\mathbb{Z}_2$’s are from the nontrivial 0D block-states. It is obvious that there is no nontrivial group extension because of the absence of nontrivial 1D block-state, and the group structure of $E_{p6m,0}$ has already been correct.

Now we turn to discuss systems with spin-1/2 fermions. Consider the 0D block-state decoration, and similar with the $p4m$ case, the classification data can also be characterized by different 1D irreducible representations of alternative symmetry groups:

$$\mathcal{H}^1 \left[ \mathbb{Z}_2^f \times \omega_2 (\mathbb{Z}_6 \cong \mathbb{Z}_2), U(1) \right] = \mathbb{Z}_2^4$$

$$\mathcal{H}^1 \left[ \mathbb{Z}_2^f \times \omega_2 (\mathbb{Z}_3 \cong \mathbb{Z}_2), U(1) \right] = \mathbb{Z}_2^4$$

$$\mathcal{H}^1 \left[ \mathbb{Z}_2^f \times \omega_3 (\mathbb{Z}_3 \cong \mathbb{Z}_2), U(1) \right] = \mathbb{Z}_2^4$$

(174)

For $D_6$ and $D_2$ centers, the physical meaning of two $\mathbb{Z}_2$’s in the classification data are rotation and reflection eigenvalues, respectively. Furthermore, the group structure of the classification of 0D FSPT phases protected by $\mathbb{Z}_3 \times \mathbb{Z}_2$ on-site symmetry for systems with spin-1/2 fermions is $\mathbb{Z}_4$. Equivalently, we can label different 0D block-states by the group elements of the 4-fold cyclic group:

$$\mathbb{Z}_4 = \{1, i, -1, -i\}$$

(175)

So the 0D block-states at $\mu_1$ and $\mu_2$ can be labeled by $(\pm, \pm)$, here these two $\pm$’s represent the 2-fold rotation and reflection symmetry eigenvalues (alternatively, they can also represent the eigenvalues of two independent reflection operations because even-fold dihedral group can also be generated by two independent reflections); the 0D block-states at $\mu_3$ can be labeled by $\nu \in \{1, i, -1, -i\}$ as the eigenvalues of $\mathbb{Z}_4^f$ symmetry. According to this notation, all obstruction-free 0D block-states form the following group:

$$\{\text{OFS}\}_{p6m,1/2} = \mathbb{Z}_2^4 \times \mathbb{Z}_4$$

(176)
and the group elements can be labeled by (three brackets represent the block-states at $\mu_1$, $\mu_2$ and $\mu_3$):

$$[(\pm, \pm), (\pm, \pm), \nu]$$

Then we investigate the possible trivializations. Consider the 1D bubble equivalence on 1D blocks labeled by $\tau$: on each $\tau$, the total on-site symmetry is $\mathbb{Z}_4^f$: nontrivial $\mathbb{Z}_2^f$ extension of the on-site symmetry $\mathbb{Z}_2$. Next we decorate an Eq. (175) onto each of them, here the yellow/red dots represent the 0D FSPT modes protected by $\mathbb{Z}_4^f$ symmetry which are labeled by $i \& -i \in \mathbb{Z}_4$, cf. Eq. (175), and they can be trivialized if they shrink to a point. Near each 0D block labeled by $\mu_3$, there are three 0D FSPT modes labeled by $i \in \mathbb{Z}_4$ and they can change the label of 0D block-state decorated at each 0D block $\mu_3$ by $-i \in \mathbb{Z}_4$. Therefore, the 0D block-state on each $\mu_3$ can be trivialized by this bubble construction. Near 0D block $\mu_1$, this 1D bubble construction changes nothing because there is no $\mathbb{Z}_4^f$ on-site symmetry on $\mu_1$. Similar 1D bubble construction can be held on $\mu_3$.

With all possible bubble constructions, we are ready to investigate the trivial states. Start from the original trivial state (nothing decorated on arbitrary 0D block):

$$[(+, +), (+, +), 1]$$

if we take above 1D bubble constructions on $\tau_2$ and $\tau_3$ by $n_2$ and $n_3$ times, above trivial state will be deformed to a new 0D block-state labeled by:

$$[(+, +), (+, +), (-i)^{3(n_2+n_3)}]$$

(177)

According to the definition of bubble equivalence, all these states should be trivial and all trivial states form the group:

$$\{\text{TBS}\}_{p6m, 1/2}^{0D} = \mathbb{Z}_4$$

(178)

Therefore, all independent nontrivial 0D block-states are labeled by different group elements of the following quotient group:

$$E_{p6m, 1/2}^{0D} = \{\text{OFBS}\}_{p6m, 1/2}^{0D}/\{\text{TBS}\}_{p6m, 1/2}^{0D} = \mathbb{Z}_2^4$$

(179)

Subsequently we consider the 1D block-state decoration. For arbitrary 1D blocks, the total on-site symmetry on them is $\mathbb{Z}_4^f$: nontrivial $\mathbb{Z}_2^f$ extension of $\mathbb{Z}_2$ on-site symmetry, hence there is no nontrivial 1D block-state due to the trivial classification of the corresponding 1D FSPT phases, and the classification attributed to 1D block-state decorations is trivial:

$$E_{p6m, 1/2}^{1D} = \{\text{OFBS}\}_{p6m, 1/2}^{1D} = 0$$

(180)

Therefore, it is obvious that there is no stacking between 1D and 0D block-states, and the ultimate classification with accurate group structure is:

$$G_{p6m, 1/2} = \mathbb{Z}_2^4$$

(181)

IV. CONSTRUCTION AND CLASSIFICATION OF CRYSSTALLINE TI

So far we have discussed the construction and classification of crystalline TSC in 2D interacting fermionic systems. In this section, we will discuss the crystalline TI with additional $U^f(1)$ by generalizing the real-space construction highlighted in Sec. II. We demonstrate that 1D block-state decoration has no contribution and all nontrivial crystalline TI in 2D interacting fermionic systems can be constructed by 0D block-state decoration.

For 1D blocks, there are two different cases: symmetry group with/without the reflection symmetry operation. Since bosonic and fermionic systems can be mapped to each other by Jordan-Wigner transformation, the classification data of 1D SPT phases for bosonic and fermionic systems are identical: by calculating the different projective representations of the symmetry group. (However, the group structure of the classification data could be different in general as staking operation has different physical meaning for boson and fermion systems.)

For symmetry groups without reflection symmetry operation, the on-site symmetry group of an arbitrary 1D blocks should be $U^f(1)$ charge conservation only, and the corresponding classification for 2D systems with spinless/spin-1/2 fermions can be calculated by the following group cohomology:

$$H^2[U^f(1), U(1)] = 0$$

(182)

Thus, there is no nontrivial 1D block-state for this case.

For the symmetry group with reflection symmetry operation, the on-site symmetry group of some 1D blocks should be $U^f(1)$ charge conservation only, and the corresponding classification for 2D systems with spinless/spin-1/2 fermions can be calculated by the following group cohomology:

$$H^2[U^f(1) \times \mathbb{Z}_2, U(1)] = 0$$

(183)

Again, there is also no nontrivial 1D block-state for this case.

Below we will again study five representative cases belonging to different crystallographic systems:

1. square lattice: $p4m$;
2. parallelogrammatic lattice: $p2$;
3. rhombic lattice: $cmm$;
4. rectangle lattice: $pgg$;
5. hexagonal lattice: $p6m$;

And all other cases are assigned in Supplementary Materials [86]. The classification results are summarized in Table III.
A. Square lattice: \( p4m \)

Again, we begin with the cell decomposition of \( p4m \) as illustrated in Fig. 3. For 0D blocks labeled by \( \mu_1 \) and \( \mu_3 \), different 0D block-states are characterized by different irreducible representations of the corresponding onsite symmetry group:

\[
\mathcal{H}^1[U^f(1) \times (Z_4 \times Z_2), U(1)] = Z \times Z_2^2
\]  

(184)

Here \( Z \) represents the \( U^f(1) \) charge carried by complex fermion, and first \( Z_2 \) represents the rotation eigenvalue \(-1\) and the second \( Z_2 \) represents the reflection eigenvalue \(-1\). Similarly, for 0D blocks labeled by \( \mu_2 \), different 0D block-states are characterized by different irreducible representations of the corresponding onsite symmetry group:

\[
\mathcal{H}^1[U^f(1) \times (Z_2 \times Z_2), U(1)] = Z \times Z_2^2
\]  

(185)

Again, here \( Z \) represents the \( U(1) \) charge carried by the complex fermion, and first \( Z_2 \) represents the rotation eigenvalue \(-1\) and the second \( Z_2 \) represents the reflection eigenvalue \(-1\). Therefore, for systems with spinless fermions, the 0D block-states at \( \mu_1 \) (\( j = 1, 2, 3 \)) can be labeled by \((n_1, \pm, \pm), (n_2, \pm, \pm), (n_3, \pm, \pm)\) (187)

Nevertheless, this is not the final classification data and we must further consider possible trivializations. For systems with spinless fermions, we first consider the 1D and we must further consider possible trivializations. For

\[
\tau
\]

systems with spinless fermions, we first consider the 1D and we must further consider possible trivializations. For

\[
\mu
\]

Again, the obstruction-free 0D block-states form the following group:

\[
\text{OFBS}^U_{p4m,0} \times Z_2^3 \times (Z_2^2) \]  

(186)

and the group elements can be labeled by (three brackets represent the block-states at \( \mu_1, \mu_2 \) and \( \mu_3 \)):

\[
[(n_1, \pm, \pm), (n_2, \pm, \pm), (n_3, \pm, \pm)]
\]

(187)

With the help of above discussions, we consider the 1D bubble equivalence. Start from the trivial state:

\[
[(0, +, +), (0, +, +), (0, +, +)]
\]  

(192)

Take aforementioned 1D bubble constructions on \( \tau_2 \) by \( n_j \in Z \) times, it will lead to a new 0D block-state labeled by:

\[
((n_1 + n_3, (-1)^{n_1}, (-1)^{n_3}), (-2n_1 + 2n_2, (-1)^{n_2}, (-1)^{n_1}), (-n_2 - 2n_3, (-1)^{n_2}, (-1)^{n_3})]
\]  

(193)

And this state should be trivial according to the definition of bubble equivalence. Alternatively, all 0D block-states can be viewed as vectors of a 9-dimensional vector space \( V \), where the complex fermion components are \( Z \)-valued and all other components are \( Z_2 \)-valued attributed to rotation and reflection eigenvalues. Then all trivial 0D block-states with the form as Eq. (193) can be viewed as a vector subspace \( V' \) of \( V \). It is easy to see that there are only three independent quantities in Eq. (193): \( n_1, n_2 \) and \( n_3 \), so the dimensionality of the vector subspace \( V' \) should be 3. For the \( U^f(1) \) charge sector, we have the following relationship:

\[-(4n_1 + 4n_3) - 2(-2n_1 + 2n_2) = -4n_2 - 4n_3 \]  

(194)

i.e., there are only two independent quantities which serves a \( 2Z \times 4Z \) trivialization. The remaining one degree of freedom of the vector subspace \( V' \) should be attributed to the eigenvalues of point group symmetry action, and serves a \( Z_2 \) trivialization. Therefore, all trivial states with the form as shown in Eq. (193) compose the following group:

\[
\text{TBS}^U_{p4m,0} \times 2Z \times 4Z \times Z_2
\]  

(195)
and different independent nontrivial 0D block-states can be labeled by different group elements of the following quotient group:

$$\mathcal{G}^{U(1)}_{p4m,0} = \{\text{OFBS}\}^{U(1)}_{p4m,0}/\{\text{TBS}\}^{U(1)}_{p4m,0} = \mathbb{Z} \times \mathbb{Z}_4 \times \mathbb{Z}_6$$

(196)

For systems with spin-1/2 fermions, the rotation and reflection properties of $|\phi\rangle^{p4m}_m$ and $|\phi\rangle^{p4m}_m$ at $\mu_1$ and $\mu_2$ are changed by an additional $-1$ and it leads to no trivialization. Furthermore, like the cases without $U(1)$ charge conservation, the classification data of the corresponding 0D block-states can be characterized by different irreducible representations of the corresponding on-site symmetry group ($n = 2, 4$):

$$\mathcal{H}^1[U(1) \times \omega_2 (\mathbb{Z}_n \times \mathbb{Z}_2), U(1)] = 2\mathbb{Z} \times \mathbb{Z}_2^2$$

(197)

To calculate this classification data, we should firstly calculate the following two cohomologies [86]:

$$\begin{align*}
&n_0 \in \mathcal{H}^0(\mathbb{Z}_n \times \mathbb{Z}_2, \mathbb{Z}) = \mathbb{Z} \\
&n_1 \in \mathcal{H}^1[\mathbb{Z}_n \times \mathbb{Z}_2, U(1)] = \mathbb{Z}_2^2
\end{align*}$$

(198)

Here $\mathbb{Z}$ represents the $U(1)$ charge carried by complex fermions, and two $\mathbb{Z}_2$’s represent the rotation and reflection eigenvalues. We demonstrate that the odd number of the $U(1)$ charge at each 0D block is not allowed: a specific $n_0$ is obstructed if and only if $(-1)^{n_2-n_0} \in \mathcal{H}^2[\mathbb{Z}_n \times \mathbb{Z}_2, U(1)]$ is a nontrivial 2-cocycle with $U(1)$-coefficient. From Refs. 19 and 85 we know that for cases without $U(1)$ charge conservation, nontrivial 0-cocycle $n_0 = 1, n_0 \in \mathcal{H}^0(\mathbb{Z}_n \times \mathbb{Z}_2, \mathbb{Z}_2)$ leads to nontrivial 2-cocycle $(-1)^{n_2-n_0} \in \mathcal{H}^2[\mathbb{Z}_n \times \mathbb{Z}_2, U(1)]$. So for $U(1)$ charge conserved cases, odd $n_0 \in \mathcal{H}^0(\mathbb{Z}_n \times \mathbb{Z}_2, \mathbb{Z})$ leads to nontrivial 2-cocycle $(-1)^{n_2-n_0} \in \mathcal{H}^2[\mathbb{Z}_n \times \mathbb{Z}_2, U(1)]$. As a consequence, for systems with spin-1/2 fermions, we can only decorate even number of complex fermions on each 0D block. As consequence, all obstruction-free block-states form a group:

$$\{\text{OFBS}\}^{U(1)}_{p4m,1/2} = (2\mathbb{Z})^3 \times \mathbb{Z}_2^6$$

(199)

Then we consider the possible trivializations via 1D bubble constructions. Repeatedly consider aforementioned “particle-hole” bubble, and above atomic insulators have alternative symmetry properties:

$$\begin{align*}
&\mathcal{M}_\tau |\phi\rangle^{p4m}_{\mu_1} = -p_1 \mathcal{M}_\tau p_3 |\phi\rangle^{p4m}_{\mu_2} = |\phi\rangle^{p4m}_{\mu_1} \\
&\mathcal{M}_\tau |\phi\rangle^{p4m}_{\mu_2} = p_1^4 \mathcal{M}_\tau |\phi\rangle^{p4m}_{\mu_2} = |\phi\rangle^{p4m}_{\mu_2}
\end{align*}$$

(200)

and

$$\begin{align*}
&\mathcal{M}_\tau |\phi\rangle^{p4m}_{\mu_1} = h_1^4 h_2^4 |\phi\rangle^{p4m}_{\mu_1} = |\phi\rangle^{p4m}_{\mu_1} \\
&\mathcal{M}_\tau |\phi\rangle^{p4m}_{\mu_2} = h_1^4 h_2^4 |\phi\rangle^{p4m}_{\mu_2} = |\phi\rangle^{p4m}_{\mu_2}
\end{align*}$$

(201)

i.e., atomic insulators $|\phi\rangle^{p4m}_{\mu_1}$ and $|\phi\rangle^{p4m}_{\mu_2}$ do not change the eigenvalues of any symmetry. The discussion of $U(1)$ charge sector is identical, so again we start from the original trivial state (192), take above 1D bubble constructions on $\tau_j$, by $n_j \in \mathbb{Z}$ times, and it will lead to a new 0D block-state labeled by:

$$\begin{align*}
&[(4n_1 + 4n_4, 0, 0), \\
&(−2n_1 + 2n_2, 0, 0), \\
&(−4n_2 - 4n_3, 0, 0])
\end{align*}$$

(202)

Similar with the spinless case, all states with the form (202) are trivial, and forming the following group:

$$\{\text{TBS}\}^{U(1)}_{p4m,1/2} = 2\mathbb{Z} \times 4\mathbb{Z}$$

(203)

because the $U(1)$ charge sector for systems with spinless and spin-1/2 fermions is identical. Different independent nontrivial 0D block-states can be labeled by different group elements of the following quotient group:

$$\mathcal{G}^{U(1)}_{p4m,1/2} = \{\text{OFBS}\}^{U(1)}_{p4m,1/2}/\{\text{TBS}\}^{U(1)}_{p4m,1/2} = 2\mathbb{Z} \times \mathbb{Z}_2^7$$

(204)

Here $2\mathbb{Z}$ means that we can only decorate even number of complex fermions on each 0D block.

**B. Parallelogrammatic lattice: p2**

Repeatedly consider the cell decomposition of $p2$ as illustrated in Fig. 6. For arbitrary 0D blocks, different 0D block-states are characterized by different irreducible representations of the symmetry group as:

$$\mathcal{H}^1[U(1) \times \mathbb{Z}_2, U(1)] = \mathbb{Z} \times \mathbb{Z}_2$$

(205)

Here $\mathbb{Z}$ represents the complex fermion and $\mathbb{Z}_2$ represents the rotation eigenvalues. So 0D block-states at $\mu_j$ ($j = 1, 2, 3, 4$) can be labeled by $(n_j, \pm)$, here $n_j \in \mathbb{Z}$ represents the number of complex fermions decorated on $\mu_j$ and $\pm$ represents the eigenvalue of 2-fold rotation operation. According to this notation, all obstruction-free 0D block-states form the following group:

$$\{\text{OFBS}\}^{U(1)}_{p2,0} = \mathbb{Z}^2 \times \mathbb{Z}_2^4$$

(206)

We should further consider possible trivializations: for systems with spinless fermions, consider the 1D bubble equivalence on 1D blocks labeled by $\tau_1$: we decorate a 1D “particle-hole” bubble [cf. Eq. (18), here yellow and red dots represent particle and hole, respectively] on each $\tau_1$, and they can be trivialized if we shrink them to a point. Near each 0D block labeled by $\mu_1$, there are two particles forming the following atomic insulator:

$$|\xi\rangle^{p2}_\mu = p_1 |p_2^1|_0$$

(207)

with following rotation property:

$$\mathcal{R}_{\mu_1} |\xi\rangle^{p2}_\mu = p_1^2 p_1^1 |0\rangle = -|\xi\rangle^{p2}_\mu$$

(208)
i.e., rotation eigenvalue $-1$ at each 0D block $\mu_1$ can be trivialized by atomic insulator $|\xi \rangle_{\mu_2}^\dagger$. Near $\mu_2$, there are two holes forming another atomic insulator:

$$|\xi \rangle_{\mu_2}^\dagger = h_1^R h_2^L |0\rangle$$  \hspace{1cm} (209)

with following rotation property:

$$R_{\mu_2} |\xi \rangle_{\mu_2}^\dagger = h_2^R h_1^L |0\rangle = - |\xi \rangle_{\mu_2}^\dagger$$  \hspace{1cm} (210)

i.e., rotation eigenvalue $-1$ at each 0D block $\mu_2$ can be trivialized by atomic insulator $|\xi \rangle_{\mu_2}^\dagger$. Therefore, aforementioned 1D bubble construction leads to the nonindependence of rotation eigenvalues at $\mu_1$ and $\mu_2$ (can be changed simultaneously).

Now we move to the $U^J(1)$ charge sector. Repeatedly consider the aforementioned 1D bubble construction on $\tau_1$: it adds two complex fermions at each 0D block $\mu_1$ and removes two complex fermions at each 0D block $\mu_2$ (by adding two holes), hence the number of complex fermions at $\mu_1$ and $\mu_2$ are not independent. More specifically, suppose there are $a$ complex fermions on each $\mu_1$, $b$ complex fermions on each $\mu_2$ and the total number of complex fermions on $\mu_1$ and $\mu_2$ within a certain unit cell is $c = a + b$. Take above manipulation $n$ times ($n \in \mathbb{Z}$), the number of complex fermions on each $\mu_1/\mu_2$ is $a + 2n/b - 2n$, and the total number of complex fermions on $\mu_1$ and $\mu_2$ remains invariant. So for a specific $c$, there are only two independent cases: $c = a + b$ and $c = (a + 1) + (b - 1)$. It is similar for other 1D blocks, and we summarize effects of all possible 1D bubble constructions:

1. 1D bubble construction on $\tau_1$: Add two complex fermions on $\mu_1$, eliminate two complex fermions on $\mu_2$, and simultaneously change the rotation eigenvalues of $\mu_1$ and $\mu_2$;

2. 1D bubble construction on $\tau_2$: Add two complex fermions on $\mu_1$, eliminate two complex fermions on $\mu_2$, and simultaneously change the rotation eigenvalues of $\mu_1$ and $\mu_3$;

3. 1D bubble construction on $\tau_3$: Add two complex fermions on $\mu_2$, eliminate two complex fermions on $\mu_4$, and simultaneously change the rotation eigenvalues of $\mu_2$ and $\mu_4$;

With the help of above discussions, we consider the 1D bubble equivalence. Start from the original trivial state (nothing is decorated on all blocks):

$$[(0, +), (0, +), (0, +), (0, +)]$$  \hspace{1cm} (211)

Take aforementioned 1D bubble constructions on $\tau_j$ by $n_j \in \mathbb{Z}$ times ($j = 1, 2, 3$), this trivial state will be deformed to a new 0D block-state labeled by:

$$[(2n_1 + 2n_2, (-1)^{n_1+n_2}), (-2n_1 + 2n_3, (-1)^{n_1+n_3}), (-2n_2, (-1)^{n_2}), (-2n_3, (-1)^{n_3})]$$  \hspace{1cm} (212)

According to the definition of bubble equivalence, this state should be trivial. Alternatively, all 0D block-states can be viewed as vectors of an 8-dimensional vector space $V$, where the complex fermion components are $\mathbb{Z}$-valued, and all other components are $\mathbb{Z}_2$-valued. Then all trivial 0D block-states with the form as Eq. (212) can be viewed as a vector space $V'$ of $V$. It is easy to see that there are only three independent quantities in Eq. (212): $n_1$, $n_2$, and $n_3$. So the dimensionality of the vector subspace $V'$ should be 3. For the $U^J(1)$ charge sector, it is easy to notice that there are 3 independent quantities in the following 4 variables:

$$2n_1 + 2n_2, -2n_1 + 2n_3, -2n_2, -2n_3$$

Hence all 1D bubble constructions serve a $(2\mathbb{Z})^3$ trivialization in $U^J(1)$ charge sector, and all trivial states form the following group:

$$\{\text{TBS}\}^U_{p_2,0} = (2\mathbb{Z})^3$$  \hspace{1cm} (213)

and different independent nontrivial 0D block-states can be labeled by different group elements of the following quotient group:

$$\mathcal{G}^U_{p_2,0} = \{\text{OFBS}\}^U_{p_2,0} / \{\text{TBS}\}^U_{p_2,0}$$

$$= \mathbb{Z}^4 \times \mathbb{Z}_2^4 / (2\mathbb{Z})^3 = \mathbb{Z} \times \mathbb{Z}_2^7$$  \hspace{1cm} (214)

For systems with spin-1/2 fermions, 0D obstruction-free block-states are identical with spinless case:

$$\{\text{OFBS}\}^U_{p_2,1/2} = \mathbb{Z}^4 \times \mathbb{Z}_2^4$$  \hspace{1cm} (215)

then repeatedly consider the aforementioned 1D bubble constructions: rotation properties of $|\xi \rangle_{\mu_2}^\dagger$ and $|\xi \rangle_{\mu_3}^\dagger$ at $\mu_1$ and $\mu_2$ are changed by an additional $-1$ and it leads to no trivialization. Furthermore, it is easy to verify that the complex fermion decorations for spinless and spin-1/2 fermions are identical. So again we start from the original trivial state (211), take above 1D bubble constructions on $\tau_j$ by $n_j \in \mathbb{Z}$ times ($j = 1, 2, 3$), and it will lead to a new 0D block-state labeled by:

$$[(2n_1 + 2n_2, +), (-2n_1 + 2n_3, +), (-2n_2, +), (-2n_3, +)]$$  \hspace{1cm} (216)

Similar with the spinless case, all states with this form are trivial, and forming the following group:

$$\{\text{TBS}\}^U_{p_2,1/2} = (2\mathbb{Z})^3$$  \hspace{1cm} (217)

and different independent nontrivial 0D block-states can be labeled by different group elements of the following quotient group

$$\mathcal{G}^U_{p_2,1/2} = \{\text{OFBS}\}^U_{p_2,1/2} / \{\text{TBS}\}^U_{p_2,1/2}$$

$$= \mathbb{Z}^4 \times \mathbb{Z}_2^4 / (2\mathbb{Z})^3 = \mathbb{Z} \times \mathbb{Z}_2^7$$  \hspace{1cm} (218)
We notice that the classifications of 2D crystalline topological phases protected by $p2$ symmetry for systems with both spinless and spin-1/2 fermions are identical. Now we give a comprehension of this issue: for systems with both spinless and spin-1/2 fermions (for 0D blocks, $R^2 = 1$ and $R^2 = -1$, respectively), the group structure of the symmetry group on 0D blocks are identical: direct product of $U^f(1)$ charge conservation and $\mathbb{Z}_2$ on-site symmetry (by 2-fold rotational symmetry acting internally): $U^f(1) \times \mathbb{Z}_2$. We explicitly formulate the $U^f(1)$ charge conservation and $\mathbb{Z}_2$ on-site symmetry as:

$$Z_2 = \{E, R\}, \quad U^f(1) = \{e^{i\theta} | \theta \in [0, 2\pi]\}$$ (219)

For systems with spinless fermions, $R^2 = 1$. Nevertheless, we can twist the group elements of $Z_2$ by a $U^f(1)$ phase factor as:

$$R' = R e^{i\pi/2}, \quad e^{i\pi/2} \in U^f(1)$$ (220)

then we reformulate the total symmetry group with the twisted operators:

$$Z_2 = \{E, R'\}, \quad U^f(1) = \{e^{i\theta} | \theta \in [0, 2\pi]\}$$ (221)

But $R^2 = -1$ for this case. Therefore, the symmetry group for both spinless and spin-1/2 fermions are identical, and can be deformed to each other by Eq. (220). We stress that such a statement is true for all wall paper groups with a single reflection axis.

C. Rhombic lattice: $cmm$

Repeatedly consider the cell decomposition of $cmm$ as illustrated in Fig. 8. For 0D blocks labeled by $\mu_1$, different 0D block-states are characterized by different irreducible representations of the symmetry group as:

$$\mathcal{H}^i[U^f(1) \times Z_2, U(1)] = \mathbb{Z} \times Z_2$$ (222)

Here $Z$ represents the complex fermion and $Z_2$ represents the rotation eigenvalue $-1$. For 0D blocks labeled by $\mu_2$ and $\mu_3$, different 0D block-states are characterized by different irreducible representations of the symmetry group as:

$$\mathcal{H}^i[U^f(1) \times (Z_2 \times Z_2), U(1)] = \mathbb{Z} \times Z_2^2$$ (223)

Here $Z$ represents the complex fermion, first $Z_2$ represents the rotation eigenvalue $-1$ and the second $Z_2$ represents the reflection eigenvalue $-1$. So 0D block-states at $\mu_1$ can be labeled by $(n_1, \pm)$, here $n_1 \in \mathbb{Z}$ represents the number of complex fermions at each $\mu_1$ and $\pm$ represents the eigenvalues of 2-fold rotation operation $R_{n_1}$; and 0D block-states at $\mu_2$ and $\mu_3$ can be labeled by $(n_2/n_3, \pm, \pm)$, here $n_2/n_3 \in \mathbb{Z}$ represents the number of complex fermions at each $\mu_2/\mu_3$, and two $\pm$’s represent the eigenvalues of two independent reflection operations (because even-fold dihedral group can also be generated by two independent reflections). According to this notation, all obstruction-free 0D block-states form the following group:

$$\{\text{OFBS}\}_{cmm, 0}^U(1) = \mathbb{Z}^3 \times \mathbb{Z}^2_2$$ (224)

We should further consider possible trivializations: for systems with spinless fermions, consider the 1D bubble equivalence on 1D blocks labeled by $\tau_1$: we decorate a 1D “particle-hole” bubble [cf. Eq. (18), here yellow and red dots represent particle and hole, respectively] on each $\tau_1$, and they can be trivialized if we shrink them to a point. Near each 0D block labeled by $\mu_1$, there are two particles forming atomic insulator:

$$|\xi\rangle_{cmm}^{\mu_1} = p_1^\dagger p_2^\dagger |0\rangle$$ (225)

with rotation property:

$$R_{\mu_1} |\xi\rangle_{cmm}^{\mu_1} = p_2^\dagger p_1^\dagger |0\rangle = -|\xi\rangle_{cmm}^{\mu_1}$$ (226)

i.e., rotation eigenvalue $-1$ at each 0D block $\mu_1$ can be trivialized by atomic insulator $|\xi\rangle_{cmm}^{\mu_1}$. Near $\mu_2$, there are four holes forming another atomic insulator:

$$|\xi\rangle_{cmm}^{\mu_2} = h_1^\dagger h_2^\dagger h_3^\dagger h_4^\dagger |0\rangle$$ (227)

with rotation property:

$$R_{\mu_2} |\xi\rangle_{cmm}^{\mu_2} = h_4^\dagger h_3^\dagger h_1^\dagger h_2^\dagger |0\rangle = |\xi\rangle_{cmm}^{\mu_2}$$ (228)

i.e., rotation eigenvalue $-1$ at each 0D block $\mu_2$ cannot be trivialized by atomic insulator $|\xi\rangle_{cmm}^{\mu_2}$. Therefore, aforementioned 1D bubble construction leads to the trivialization of rotation eigenvalues at $\mu_1$. Then we consider the 1D bubble equivalence on 1D blocks labeled by $\tau_2$: we decorate an identical 1D “particle-hole” bubble as aforementioned on each $\tau_2$. Near each 0D block labeled by $\mu_2$, there are two particles forming the following atomic insulator:

$$|\eta\rangle_{cmm}^{\mu_2} = p_1^\dagger h_1^\dagger p_2^\dagger |0\rangle$$ (229)

with rotation and reflection properties as:

$$R_{\mu_2} |\eta\rangle_{cmm}^{\mu_2} = p_2^\dagger p_1^\dagger |0\rangle = -|\eta\rangle_{cmm}^{\mu_2}$$ (230)

$$M_{\tau_2} |\eta\rangle_{cmm}^{\mu_2} = p_1^\dagger p_2^\dagger |0\rangle = -|\eta\rangle_{cmm}^{\mu_2}$$ (231)

i.e., rotation and reflection eigenvalues $-1$ at each 0D block $\mu_2$ can be trivialized by atomic insulator $|\eta\rangle_{cmm}^{\mu_2}$. Near $\mu_3$, there are two holes forming another atomic insulator:

$$|\eta\rangle_{cmm}^{\mu_3} = h_1^\dagger h_2^\dagger |0\rangle$$ (232)

with rotation and reflection properties as:

$$R_{\mu_3} |\eta\rangle_{cmm}^{\mu_3} = h_2^\dagger h_1^\dagger |0\rangle = -|\eta\rangle_{cmm}^{\mu_3}$$ (233)

$$M_{\tau_3} |\eta\rangle_{cmm}^{\mu_3} = h_2^\dagger h_1^\dagger |0\rangle = -|\eta\rangle_{cmm}^{\mu_3}$$ (234)
i.e., rotation and reflection eigenvalues \(-1\) at each 0D block \(\mu_3\) can be trivialized by atomic insulator \(|\eta|_{\text{cmm}}^{\mu_3}\). Therefore, aforementioned 1D bubble construction leads the nonindependence of rotation and reflection eigenvalues of \(\mu_2\) and \(\mu_3\) (can be changed simultaneously).

Subsequently we consider the \(U^f(1)\) charge sector. First of all, as shown in Fig. 8, we should identify that within a specific unit cell, there are two 0D blocks labeled by \(\mu_1\) and one 0D block labeled by \(\mu_2/\mu_3\). Repeatedly consider above 1D bubble construction on \(\tau_j\): it adds two complex fermions on each 0D block \(\mu_1\) and removes four complex fermions at each 0D block \(\mu_2\) (by adding four holes), hence the numbers of complex fermions at \(\mu_1\) and \(\mu_3\) are not independent. More specifically, suppose there are \(a\) complex fermions on each \(\mu_1\), \(b\) complex fermions on each \(\mu_2\), and the total number of complex fermions on \(\mu_1\) and \(\mu_2\) within a certain unit cell is \(c = 2a + b\). Take above manipulation \(n\) times \((n \in \mathbb{Z})\), and the number of complex fermions on each \(\mu_1/\mu_2\) is \(a + 2n/b - 4n\), and the total number of complex fermions on \(\mu_1\) and \(\mu_2\) within a certain unit cell remains invariant. So for a specific \(c\), there are only two independent cases: \(c = 2a + b\) and \(c = 2(a + 1) + (b - 2)\).

Then we consider aforementioned 1D bubble equivalence on 1D blocks \(\tau_2\) repeatedly: it adds two complex fermions at each 0D block \(\mu_2\) and removes two complex fermions at each 0D block \(\mu_3\) (by adding two holes), hence the number of complex fermions at \(\mu_2\) and \(\mu_3\) are not independent. More specifically, suppose there are \(a'\) complex fermions on each \(\mu_2\), \(b'\) complex fermions on each \(\mu_3\), and the total number of complex fermions on \(\mu_2\) and \(\mu_3\) within a certain unit cell is \(c' = a' + b'\). Take above manipulation \(n'\) times \((n' \in \mathbb{Z})\), the number of complex fermions on each \(\mu_2/\mu_3\) is \(a' + 2n'/b' - 2n'\), and the total number of complex fermions on \(\mu_2\) and \(\mu_3\) within a certain unit cell remains invariant. So for a specific \(c'\), there are only two independent cases: \(c' = a' + b'\) and \(c' = (a' + 1) + (b' - 1)\).

With the help of above discussions, we consider the 0D block-state decorations. Start from the original trivial state (nothing is decorated on all blocks):

\[
[(0, +), (0, +, +), (0, +, +)]
\]  
(233)

Take aforementioned 1D bubble construction on \(\tau_j\) by \(n_j \in \mathbb{Z}\) times \((j = 1, 2, 3)\), it will lead to a new 0D block-state labeled by:

\[
[(2n_1, (-1)^{n_1}), (-2n_1 + 2n_2 + 2n_3, (-1)^{n_2 + n_3}, (-1)^{n_2}) \\
(-2n_2 - 2n_3, (-1)^{n_2 + n_3}, (-1)^{n_2})]
\]  
(234)

According to the definition of bubble equivalence, all states with this form should be trivial. Alternatively, all 0D block-states can be viewed as vectors of an 8-dimensional vector space \(V\), where the complex fermion components are \(\mathbb{Z}\)-valued and all other components are \(\mathbb{Z}_2\)-valued. Then all trivial 0D block-states with the form as Eq. (234) can be viewed as a vector subspace \(V'\) of \(V\). It is easy to see that there are only three independent quantities in Eq. (234): \(n_1, n_2\) and \(n_3\). So the dimensionality of the vector subspace \(V'\) should be 3. For the \(U^f(1)\) charge sector, we have the following relationship:

\[-2n_1 - (-2n_1 + 2n_2 + 2n_3) = -2n_2 - 2n_3\]  
(235)

i.e., there are only two independent quantities which serves a \((2\mathbb{Z})^2\) trivialization. The remaining one degree of freedom of the vector subspace \(V'\) should be attributed to the eigenvalues of point group symmetry action, and serves a \(\mathbb{Z}_2\) trivialization. Therefore, all trivial states (234) form the following group:

\[
\{\text{TBS}\}_{\text{cmm}, 0}^{U(1)} = (2\mathbb{Z})^2 \times \mathbb{Z}_2
\]  
(236)

and different independent nontrivial 0D block-states can be labeled by different group elements of the following quotient group:

\[
\mathcal{G}_{\text{cmm}, 0}^{U(1)} = \{\text{OFBS}\}_{\text{cmm}, 0}^{U(1)}/\{\text{TBS}\}_{\text{cmm}, 0}^{U(1)}
\]  
\[= \mathbb{Z}^3/((2\mathbb{Z})^2 \times \mathbb{Z}_2) = \mathbb{Z} \times \mathbb{Z}_2^2
\]  
(237)

For systems with spin-1/2 fermions, like the cases without \(U^f(1)\) charge conservation, the classification data of the 0D block-states of 0D blocks labeled by \(\mu_2\) and \(\mu_3\) can be characterized by different irreducible representations of the corresponding on-site symmetry group (the meaning of and \(\omega_2\) are refer to Sec. 1B):

\[
\mathcal{H}^{U(1)}[U(1) \times \omega_2 (\mathbb{Z}_2 \times \mathbb{Z}_2), U(1)] = 2\mathbb{Z} \times \mathbb{Z}_2^2
\]  
(238)

Here \(2\mathbb{Z}\) represents the \(U^f(1)\) charge carried by complex fermion, and two \(\mathbb{Z}_2\)’s represent the rotation and reflection eigenvalues (similar with the \(p4m\) case, we can only decorate even number of \(U^f(1)\) charge on each 0D block). So all obstruction-free 0D block-states form the following group:

\[
\{\text{OFBS}\}_{\text{cmm}, 1/2}^{U(1)} = \mathbb{Z} \times (2\mathbb{Z})^2 \times \mathbb{Z}_2^3
\]  
(239)

Then we discuss possible trivializations. Repeatedly consider aforementioned 1D bubble constructions, and now the rotation properties of \(|\xi|^\mu_3_{\text{cmm}}\), \(|\xi|^\mu_2_{\text{cmm}}\), \(|\eta|^\mu_3_{\text{cmm}}\) and \(|\eta|^\mu_3_{\text{cmm}}\) at \(j = 1, 2, 3\) are changed by an additional \(-1\); the reflection properties of \(|\eta|^\mu_3_{\text{cmm}}\) and \(|\eta|^\mu_3_{\text{cmm}}\) at \(\mu_2\) and \(\mu_3\) are also changed by an additional \(-1\). All of them lead to no trivialization. Furthermore, it is easy to see that all arguments about the \(U^f(1)\) charge sector are identical. So again we start from the original trivial state (233), take above 1D bubble constructions on \(\tau_j\) by \(n_j\) times \((j = 1, 2, 3)\), it will lead to a new 0D block-state labeled by:

\[
[(2n_1, +), (-2n_1 + 2n_2 + 2n_3, +), (-2n_2 - 2n_3, +)]
\]  
(240)

Similar with the spinless case, all states with this form are trivial and forming the following group:

\[
\{\text{TBS}\}_{\text{cmm}, 1/2}^{U(1)} = (2\mathbb{Z})^2
\]  
(241)
and all different independent nontrivial 0D block-states can be labeled by different group elements of the following quotient group:

$$\mathcal{G}^{U(1)}_{cmm,1/2} = \{\text{OFBS}\}^{U(1)}_{cmm,1/2}/\{\text{TBS}\}^{U(1)}_{cmm,1/2}$$

$$= \mathbb{Z} \times (2\mathbb{Z})^2 \times \mathbb{Z}_2^2 / (2\mathbb{Z})^2 = 2\mathbb{Z} \times \mathbb{Z}_2^2$$  \(\text{(242)}\)

We should notice that the group structure of the classification should be $\mathbb{Z} \times \mathbb{Z}_2^2$ rather than $\mathbb{Z} \times \mathbb{Z}_2$: two independent quantities are $n_1$ and $n_2 + n_3$, hence the classification contributed from complex fermion decorations on $\mu_1$ should be $\mathbb{Z} / 2\mathbb{Z} = \mathbb{Z}_2$. Equivalently, 0D block-state $(1,+)$ at $\mu_1$ is nontrivial.

### D. Rectangle lattice: $pgg$

Repeatedly consider the cell decomposition of $pgg$ as illustrated in Fig. 9. For an arbitrary 0D block, different 0D block-states are characterized by different irreducible representations of symmetry group as:

$$\mathcal{H}^{U'(1)}\{[U'(1) \times \mathbb{Z}_2, U(1)] = \mathbb{Z} \times \mathbb{Z}_2$$  \(\text{(243)}\)

Here $\mathbb{Z}$ represents the complex fermion and $\mathbb{Z}_2$ represents the eigenvalues of 2-fold rotational symmetry operation. So the 0D block-state decorated on $\mu_j$ ($j = 1, 2$) can be labeled by $(n_j \pm, \pm)$, where $n_j \in \mathbb{Z}$ represents the number of complex fermions decorated on $\mu_j$ and $\pm$ represents the eigenvalues of 2-fold rotational symmetry on $\mu_j$. According to this notation, all obstruction-free 0D block-states form the following group:

$$\{\text{OFBS}\}^{U(1)}_{pgg,0} = \mathbb{Z}^2 \times \mathbb{Z}_2^2$$  \(\text{(244)}\)

We should further consider the possible trivialization. For systems with spinless fermions, consider the 1D bubble equivalence on $\tau_2$: we decorate a 1D “particle-hole” bubble [cf. Eq. (18), here yellow and red dots represent particle and hole, respectively] on each $\tau_2$, and can be trivialized if we shrink them to a point: to add two complex fermions at each 0D block and then remove two complex fermions at each 0D block ($\mu_2$) by adding two holes, hence the numbers of complex fermions at $\mu_1$ and $\mu_2$ are independent. More specifically, consider there are $a$ complex fermions at each $\mu_1$ and $b$ complex fermions at each $\mu_2$, and suppose $c = a + b$. Take above manipulation $n$ times ($n \in \mathbb{Z}$), the number of complex fermions on each $\mu_1 / \mu_2$ is $n + 2n / b - 2n$, and their summation remains invariant. So for a specific $c$, there are only two independent cases: $c = a + b$ and $c = (a + 1) + (b - 1)$.

With the help of above discussions, we consider the 0D block-state decorations. Start from the following trivial state:

$$[0, +, (0, +)]$$  \(\text{(249)}\)

Take aforementioned 1D bubble construction on $\tau_2$ by $n \in \mathbb{Z}$ times will obtain the following group containing all trivial states:

$$\{\text{TBS}\}^{U(1)}_{pgg,0} = \left\{ [(2n, (−1)^n), (−2n, (−1)^n)] \big| n \in \mathbb{Z} \right\}$$

$$= 2\mathbb{Z}$$  \(\text{(250)}\)

Therefore, the ultimate classification of crystalline topological phases protected by $pgg$ symmetry for 2D systems with spinless fermions is:

$$\mathcal{G}^{U(1)}_{pgg,0} = \{\text{OFBS}\}^{U(1)}_{pgg,0}/\{\text{TBS}\}^{U(1)}_{pgg,0}$$

$$= \mathbb{Z}^2 \times \mathbb{Z}_2^2 / 2\mathbb{Z} = \mathbb{Z} \times \mathbb{Z}_2^3$$  \(\text{(251)}\)

For systems with spin-$1/2$ fermions, 0D obstruction-free block-states are identical with spinless case:

$$\{\text{OFBS}\}^{U(1)}_{pgg,1/2} = \mathbb{Z}^2 \times \mathbb{Z}_2^2$$  \(\text{(252)}\)

Then repeatedly consider the aforementioned 1D bubble constructions: the rotation properties of $|\phi\rangle^{\mu_1}_{pgg}$ and $|\phi\rangle^{\mu_2}_{pgg}$ are changed by an additional $−1$, which leads to non trivialization. It is easy to verify that the complex fermion decorations for spinless and spin-$1/2$ fermions are identical. Repeatedly consider the 1D bubble construction on $\tau_2$ and it will lead to the following group containing all trivial states:

$$\{\text{TBS}\}^{U(1)}_{pgg,1/2} = \left\{ [(2n, +), (−2n, +)] \big| n \in \mathbb{Z} \right\} = 2\mathbb{Z}$$  \(\text{(253)}\)
Therefore, the ultimate classification of crystalline topological phases protected by $pgg$ symmetry for 2D systems with spin-1/2 fermions is:

$$G_{pgg,1/2}^{U(1)} = \{ \text{OFBS} \}^{U(1)}_{pgg,1/2} / \{ \text{TBS} \}^{U(1)}_{pgg,1/2} = \mathbb{Z}^2 \times \mathbb{Z}_2^2 / 2\mathbb{Z} = \mathbb{Z} \times \mathbb{Z}_2^2$$ (254)

E. Hexagonal lattice: $p6m$

Repeatedly consider the cell decomposition of $p6m$ as illustrated in Fig. 10. For 0D blocks labeled by $\mu_1$, different 0D block-states are characterized by different irreducible representations of the symmetry group as:

$$\mathcal{H}^1[U^f(1) \times (\mathbb{Z}_6 \times \mathbb{Z}_2), U(1)] = \mathbb{Z} \times \mathbb{Z}_2^2$$ (255)

Here $\mathbb{Z}$ represents the complex fermion, first $\mathbb{Z}_2$ represents the rotation eigenvalue $\pm 1$ and the second $\mathbb{Z}_2$ represents the reflection eigenvalue $\pm 1$. For 0D blocks labeled by $\mu_2$, different 0D block-states are characterized by different irreducible representations of the symmetry group as:

$$\mathcal{H}^1[U^f(1) \times (\mathbb{Z}_2 \times \mathbb{Z}_2), U(1)] = \mathbb{Z} \times \mathbb{Z}_2^2$$ (256)

Here $\mathbb{Z}$ represents the complex fermion, first $\mathbb{Z}_2$ represents the rotation eigenvalue $\pm 1$ and the second $\mathbb{Z}_2$ represents the reflection eigenvalue $\pm 1$. For 0D blocks labeled by $\mu_3$, different 0D block-states are characterized by different irreducible representations of the symmetry group as:

$$\mathcal{H}^1[U^f(1) \times (\mathbb{Z}_3 \times \mathbb{Z}_2), U(1)] = \mathbb{Z} \times \mathbb{Z}_2^2$$ (257)

Here $\mathbb{Z}$ represents the complex fermion and $\mathbb{Z}_2$ represents the reflection eigenvalue $\pm 1$. So the 0D block-states on $\mu_1$ and $\mu_2$ can be labeled by $(n_1/n_2, \pm, \pm)$, where $n_1/n_2$ represents the number of complex fermions decorated on $\mu_1/\mu_2$ and two $\pm$’s represent the eigenvalues of two independent reflection operations (because even-fold dihedral group can also be generated by two independent reflections); the 0D block-states on $\mu_3$ can be labeled by $(n_3, \pm)$, where $n_3$ represents the number of complex fermions decorated on $\mu_3$ and $\pm$ represents the eigenvalue of reflection operation. According to this notation, all obstruction-free 0D block-states form the following group:

$$\{ \text{OFBS} \}^{U(1)}_{p6m,0} = \mathbb{Z}^3 \times \mathbb{Z}_2^5$$ (258)

We should further consider possible trivializations: for systems with spinless fermions, consider the 1D bubble equivalence on 1D blocks labeled by $\tau_1$: we decorate a 1D “particle-hole” bubble [cf. Eq. (18), here yellow and red dots represent particle and hole, respectively] one each $\tau_1$, and they can be trivialized if we shrink them to a point. Near each 0D block labeled by $\mu_1$, there are six particles forming the following atomic insulator:

$$|\xi\rangle_{p6m}^{\mu_1} = p_1^+ p_2^+ p_3^+ p_4^+ p_5^+ p_6^+ |0\rangle$$ (259)

with rotation and reflection properties as:

$$R_{\mu_1} |\xi\rangle_{p6m}^{\mu_1} = p_2^+ p_3^+ p_4^+ p_5^+ p_6^+ p_1^+ |0\rangle = -|\xi\rangle_{p6m}^{\mu_1}$$ (260)

$$M_{\tau_2} |\xi\rangle_{p6m}^{\mu_1} = p_6^+ p_5^+ p_4^+ p_3^+ p_2^+ p_1^+ |0\rangle = -|\xi\rangle_{p6m}^{\mu_1}$$ (261)

i.e., rotation and reflection eigenvalues $-1$ at each 0D block $\mu_1$ can be trivialized by atomic insulator $|\xi\rangle_{p6m}^{\mu_1}$. Near $\mu_2$, there are two holes forming another atomic insulator:

$$|\eta\rangle_{p6m}^{\mu_2} = h_1^+ h_2^+ |0\rangle$$ (262)

with rotation and reflection properties as:

$$R_{\mu_2} |\eta\rangle_{p6m}^{\mu_2} = p_2^+ p_1^+ |0\rangle = -|\eta\rangle_{p6m}^{\mu_2}$$ (263)

$$M_{\tau_2} |\eta\rangle_{p6m}^{\mu_2} = p_2^+ p_1^+ |0\rangle = -|\eta\rangle_{p6m}^{\mu_2}$$ (264)

i.e., rotation and reflection eigenvalues $-1$ at each 0D block $\mu_2$ can be trivialized by atomic insulator $|\eta\rangle_{p6m}^{\mu_2}$. Therefore, aforementioned 1D bubble construction leads the nonindependence of rotation and reflection eigenvalues of $\mu_1$ and $\mu_2$ (can be changed simultaneously). Then we consider the 1D bubble equivalence on 1D blocks labeled by $\tau_3$: we decorate an identical 1D “particle-hole” bubble as aforementioned on each $\tau_3$. Near each 0D block labeled by $\mu_2$, there are two particles forming the following atomic insulator:

$$|\eta\rangle_{p6m}^{\mu_2} = p_1^+ p_2^+ |0\rangle$$ (265)

with rotation and reflection properties as:

$$R_{\mu_3} |\eta\rangle_{p6m}^{\mu_2} = p_1^+ p_2^+ |0\rangle = -|\eta\rangle_{p6m}^{\mu_2}$$ (266)

$$M_{\tau_3} |\eta\rangle_{p6m}^{\mu_2} = p_1^+ p_2^+ |0\rangle = -|\eta\rangle_{p6m}^{\mu_2}$$ (267)

i.e., rotation and reflection eigenvalues $-1$ at each 0D block $\mu_3$ can be trivialized by atomic insulator $|\eta\rangle_{p6m}^{\mu_3}$. Hence the reflection eigenvalues $-1$ at all 0D blocks are not independent, and the rotation eigenvalues $-1$ at $\mu_1$ and $\mu_2$ can be totally trivialized. Furthermore, we investigate an alternative 1D bubble equivalence on 1D blocks $\tau_2$ (we label above 1D bubble construction by “type-I,” and label this 1D bubble construction by “type-II”): we decorate an alternative 1D bubble on each 1D block labeled by $\tau_2$ [cf. Eq. (18), here both yellow and red dots represent the 0D FSPT modes characterized by eigenvalues $-1$ of reflection symmetry, and they can be trivialized if we shrink them to a point]. According to this 1D bubble
construction, the reflection eigenvalue at each 0D block $\mu_3$ is changed by $-1$ while the reflection eigenvalue at each 0D block $\mu_2$ remains invariant. Another type-II 1D bubble construction can also be constructed on $\tau_3$.

Subsequently we consider the $U^I(1)$ charge sector. First of all, as shown in Fig. 10, we should identify that within a specific unit cell, there is one 0D block labeled by $\mu_1$, two 0D blocks labeled by $\mu_3$ and three 0D blocks labeled by $\mu_2$. Repeatedly consider the aforementioned 1D bubble construction on $\tau_j$: it adds six complex fermions at each 0D block $\mu_1$ and removes two complex fermions at each 0D block $\mu_2$ (by adding two holes), hence the number of complex fermions at $\mu_1$ and $\mu_2$ are not independent.

With the help of above discussions, we consider the 0D block-state decorations. Start from the original trivial state (nothing is decorated on all blocks):

$$[(0, +, +), (0, +, +), (0, +)] \quad (267)$$

Take aforementioned type-I 1D bubble constructions on $\tau_j$ by $n_j$ times ($j = 1, 2, 3$), and type-II 1D bubble constructions on $\tau_2/\tau_3$ by $n'_2/n'_3$ times, it will lead to a new 0D block-state labeled by:

$$[(6n_1 + 6n_2, (-1)^{n_1}, (-1)^{n_2}),$$
$$(-2n_1 + 2n_3, (-1)^{n_1}, (-1)^{n_2}),$$
$$(-3n_2 - 3n_3, (-1)^{n_2+n_3+n'_2+n'_3})] \quad (268)$$

According to the definition of bubble equivalence, all states with this form should be trivial. Alternatively, all 0D block-states can be viewed as vectors of an 8-dimensional vector space $V$, where the complex fermion components are $\mathbb{Z}$-valued and all other components are $\mathbb{Z}_2$-valued. Then all trivial 0D block-states with the form as Eq. (268) can be viewed as a vector subspace $V'$ of $V$.

We notice that $n'_2$ and $n'_3$ should appear simultaneously, hence they are not independent. As a consequence, there are only four independent quantities in Eq. (268): $n_1, n_2, n_3$ and $n'_2/n'_3$. So the dimensionality of the vector subspace $V'$ should be 4. For the $U^I(1)$ charge sector, we have the following relationship:

$$-(6n_1 + 6n_2) - 3(-2n_1 + 2n_3) = 2(-3n_2 - 3n_3) \quad (269)$$

i.e., there are only two independent quantities which serves a $\mathbb{Z} \times 3\mathbb{Z}$ trivialization. The remaining two degrees of freedom of the vector subspace $V'$ should be attributed to the eigenvalues of point group symmetry actions, and serves a $\mathbb{Z}_2^2$ trivialization. Therefore, all trivial states with form as shown in Eq. (268) compose the following group:

$${\text{TBS}}^{U(1)}_{p6m,0} = 2\mathbb{Z} \times 3\mathbb{Z} \times \mathbb{Z}_2^2 \quad (270)$$

and different independent nontrivial 0D block-states can be labeled by different group elements of the following quotient group:

$${\mathcal{G}}^{U(1)}_{p6m,0} = \{\text{OFBS}\}^{U(1)}_{p6m,0}/\{\text{TBS}\}^{U(1)}_{p6m,0}$$

$$= \mathbb{Z} \times \mathbb{Z}_2^5/2\mathbb{Z} \times 3\mathbb{Z} \times \mathbb{Z}_2^2 = \mathbb{Z} \times \mathbb{Z}_3 \times \mathbb{Z}_2^4 \quad (271)$$

For systems with spin-1/2 fermions, the rotation properties of $|\xi\rangle^{\mu_1}_{p6m}$, $|\xi\rangle^{\mu_2}_{p6m}$ and $|\eta\rangle^{\mu_2}_{p6m}$ at $\mu_1$ and $\mu_2$ are changed by an additional $-1$; the reflection properties of $|\xi\rangle^{\mu_1}_{p6m}$, $|\xi\rangle^{\mu_3}_{p6m}$, $|\eta\rangle^{\mu_2}_{p6m}$ and $|\eta\rangle^{\mu_3}_{p6m}$ at $\mu_1$, $\mu_2$ and $\mu_3$ are also changed by an additional $-1$. All of them lead to no trivialization. Furthermore, like the cases without $U^I(1)$ charge conservation, the classification data of the corresponding 0D block-states on $\mu_1$ and $\mu_2$ can be characterized by different irreducible representations of the corresponding on-site symmetry group (the meaning of $\omega_2$ are refer to Sec. 1B):

$$\mathcal{H}^1 U^I(1) \times \omega_2 (Z_6 \times Z_2), U(1) = 2\mathbb{Z} \times \mathbb{Z}_2^2 \quad (272)$$

Here each $2\mathbb{Z}$ represents the $U^I(1)$ charge carried by complex fermion, and different $\mathbb{Z}_2$’s represent the rotation and reflection eigenvalues at each 0D block labeled by $\mu_1$ and $\mu_2$ (similar with the $p4m$ case, we can only decorate even number of $U^I(1)$ charge on each 0D block). So all obstruction-free 0D block-states form the following group:

$${\text{OFBS}}^{U(1)}_{p6m,1/2} = \mathbb{Z} \times (2\mathbb{Z})^2 \times \mathbb{Z}_2^2 \quad (273)$$

then repeatedly consider the aforementioned 1D bubble constructions, the reflection properties of the atomic insulators: $|\xi\rangle^{\mu_1}_{p6m}$, $|\xi\rangle^{\mu_3}_{p6m}$, $|\eta\rangle^{\mu_1}_{p6m}$, $|\eta\rangle^{\mu_3}_{p6m}$ and $|\eta\rangle^{\mu_3}_{p6m}$ are changed by an additional $-1$, and all of them lead to no trivialization. Other 1D bubble constructions are identical. So again we start from the original trivial state (267), take above type-I 1D bubble constructions on $\tau_j$ by $n_j$ times ($j = 1, 2, 3$), and type-II 1D bubble constructions on $\tau_2/\tau_3$ by $n'_2/n'_3$ times, it will lead to a new 0D block-state labeled by:

$$[(6n_1 + 6n_2, +, +),$$
$$(-2n_1 + 2n_3, +, +),$$
$$(-3n_2 - 3n_3, (-1)^{n_2+n_3+n'_2+n'_3})] \quad (274)$$

The $U^I(1)$ charge sector is identical with spinless case, and there is one independent nonzero reflection eigenvalue $(−1)^{n_2+n_3}$. Therefore, all trivial states with form as shown in Eq. (274) compose the following group:

$${\text{TBS}}^{U(1)}_{p6m,1/2} = 2\mathbb{Z} \times 3\mathbb{Z} \times \mathbb{Z}_2 \quad (275)$$

and different independent nontrivial 0D block-states can be labeled by different group elements of the following group:

$${\mathcal{G}}^{U(1)}_{p6m,1/2} = \{\text{OFBS}\}^{U(1)}_{p6m,1/2}/\{\text{TBS}\}^{U(1)}_{p6m,1/2}$$

$$= \mathbb{Z} \times (2\mathbb{Z})^5 \times 2\mathbb{Z} \times 3\mathbb{Z} \times \mathbb{Z}_2$$

$$= 2\mathbb{Z} \times \mathbb{Z}_3 \times \mathbb{Z}_2^4 \quad (276)$$
V. GENERALIZED CRYSSTALLINE EQUIVALENCE PRINCIPLE

In this section, we discuss how to generalize the crystalline equivalence principle that is rigorously proven for interacting bosonic systems [51]. By comparing the classification results of the topological crystalline TSC summarized in Table I, Table II and the classification results of crystalline TI summarized in Table III with the classification results of the 2D FSPT phases protected by the corresponding on-site symmetry [87, 88], we verify the fermionic crystalline equivalence principle for all TSC and TI (for both spinless and spin-1/2 cases) constructed in this paper.

In particular, we should map the space group symmetry to on-site symmetry according to the following rules:

1. Subgroup of translational symmetry along a particular direction should be mapped to the on-site symmetry group $\mathbb{Z}$. Equivalently, the total translational subgroup should be mapped to the on-site symmetry group $\mathbb{Z}^2$;

2. $n$-fold rotational symmetry subgroup should be mapped to the on-site symmetry group $\mathbb{Z}_n$;

3. Reflection symmetry subgroup should be mapped to the time-reversal symmetry group $\mathbb{Z}_2^T$ which is antiunitary.

4. Spinless (spin-1/2) fermionic systems should be mapped into spin-1/2 (spinless) fermionic systems.

The additional twist on spinless and spin-1/2 fermions can be naturally interpreted as the spin rotation of fermions: a $2\pi$ rotation of a fermion around a specific axis results in a $-1$ phase factor [86]. We conjecture that such crystalline equivalence principle is also correct for 3D crystalline SPT phases as well.

VI. CONCLUSION AND DISCUSSION

In this paper, we derive the classification of crystalline TSC and TI in 2D interacting fermionic systems by using the explicit real-space constructions. For a 2D system with a specific wallpaper group symmetry, we first decompose the system into an assembly of unit cells. Then according to the so-called extensive trivialization scheme, we can further decompose each unit cell into an assembly of lower-dimensional blocks. After cell decompositions, we can decorate some lower-dimensional block-states on them, and investigate the obstruction and trivialization for all block states by checking the no-open-edge condition and bubble equivalence. An obstruction/trivialization free decoration corresponds to a non-trivial crystalline SPT phases. We further investigated the group structures of the classification data by considering the possible stacking between 1D and 0D block-states. Finally, with the complete classification results, we compare our results with classification of 2D FSPT phases protected by the corresponding on-site symmetry, we verify the crystalline equivalence principle for generic 2D interacting fermionic systems.

We believe that the real-space construction scheme for crystalline SPT are also applicable to 3D interacting fermionic systems, with similar procedures discussed in this work. In future works, we will try to construct and fully classify the crystalline TSC/TI in 3D interacting fermionic systems.

We stress that the method in this paper can also be applied for cases with mixture of internal and space group symmetries, i.e. when considering about the lower-dimensional block-states, we should also include the internal symmetry together with the space group symmetry acting internally that leads to different lower-dimensional root phases and bubbles. Then based on these root phases, we can further discuss possible obstructions and trivializations by using the general paradigms highlighted in Sec. II.

Moreover, we also predict an intriguing fermionic crystalline TSC (that cannot be realized in both free-fermion and interacting bosonic systems) with $p4m$ wall paper group symmetry. The iron-based superconductor could be a natural strongly correlation electron system to realize such a new phase, especially the monolayer iron selenide/pnictide [89]. Since the spin-orbit interactions in FeSe is relatively small [distinct from Fe(Se,Te) because of the absence of tellurium], we can effectively treat fermions in this system as spinless.

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[89] The space group of 3D iron selenide/pnictide is *P4/nmm*, and according to recent STM experiments, there is no nematicity for monolayer iron selenide/pnictide grown on STO substrate by MBE, distinguished from the 3D crystals. So the corresponding wallpaper group of monolayer iron selenide/pnictide is *p4m*. 
### Supplemental Materials

**Appendix S-1: Classifications of FSPT states in 0D**

In the main text, we have characterized the 0D block-states by different 1D irreducible representations of the full symmetry group, for both crystalline TSC and TI. In Ref. [1], the crystalline TSC cases are systematically studied, hence we mainly focus on the crystalline TI in this section.

It is well-known that the 0D BSPT states with symmetry group $G$ are classified by 1D linear representations of $G$: $\mathcal{H}^1(G, U(1))$ [2]. This is because the SPT state should be both symmetric and non-degenerate. In 0D, there are essentially no difference between bosonic and fermionic systems, except that there is an additional $U^f(1)$ for fermionic systems with charge conservation. We can treat a fermionic system with total fermionic symmetry group $G_f = U^f(1) \times \omega_2 G$ as a bosonic system with total symmetry group $G_f$. Therefore, we can conclude:

- 0D FSPT phases with symmetry group $U^f(1) \times \omega_2 G$ are classified by 1D irreducible representations of $G_f$. i.e., $\mathcal{H}^1(G_f, U^f(1))$. 

Equivalently, we can unpack the above result and show that:

- 0D FSPT phases with symmetry group $U^f(1) \times \omega_2 G$ are classified by 0-cocycle $\nu_0$ and 1-cocycle $\nu_1$, with some symmetry conditions and consistency equations.

Here $\nu_0 \in \mathcal{H}^0(G, \mathbb{Z})$ is the number of $U^f(1)$ charge carried by complex fermions, and $\nu_1 \in \mathcal{H}^1(G, U^f(1))$ is the usual 0D BSPT classification.

In general, for a given 1D representation $\hat{U}$ of $G_f$, we can always separate $\hat{U}(e^{i\theta n_0}, g)$ with $g \in G$ and $e^{i\theta n_0} \in U^f(1)$, $\theta \in [0, 2\pi]$ into three parts:

$$\hat{U}(e^{i\theta n_0}, g) = e^{i\theta n_0} \nu_1(g) K_{\theta n_0}(g) \quad (S1)$$

where $\nu_1(g)$ is a $U(1)$ phase factor, $K$ is the complex conjugation operator and $s_1(g)$ represents whether $g$ contains time-reversal or not [if so, $s_1(g) = 1$, otherwise $s_1(g) = 0$]. Using the multiplication rule of $G_f$ (defined in the main text), the representation condition $\hat{U}(e^{i\theta n_0}, g) \hat{U}(e^{i\theta' n_0}, h) = \hat{U}(e^{i\theta n_0}, g) \cdot e^{i\theta' n_0, h}$ becomes:

$$\nu_1(g) \nu_1(h) e^{-2s_1(g)} = e^{2\pi i \omega_2 (g, h) n_0} \nu_1(gh) \quad (S2)$$

and can be summarized as:

$$(d\nu_1)(g, h) := \frac{\nu_1(h) e^{-2s_1(g)} \nu_1(g)}{\nu_1(gh)} = e^{2\pi i (\omega_2 - n_0)(g, h)} \quad (S3)$$

which means that the cocycle equation of $\nu_1$ is twisted by $\omega_2 \sim n_0$.

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**Appendix S-2: Point group protected TI in 2D interacting fermion systems**

In Ref. 3 and 4, the authors systematically constructed and classified the point group SPT phases without $U^f(1)$ charge conservation in 2D interacting fermionic systems. In this section, we construct and classify the 2D point group FSPT phases with $U^f(1)$ charge conservation. As demonstrated in the main text, we do not need to investigate the 1D block-state decoration because of the absence of 1D root phase.

Here we explicitly construct and classify the point group SPT phases with $U^f(1)$ charge conservation in 2D interacting fermionic systems by real-space construction. We demonstrate the $D_4$ case as an example. Inherit the definitions and terminologies of Ref. 4, we decorate some block-states on the corresponding lower-dimensional blocks. We only need to consider the 0D block-state decoration: different 0D block-states are characterized by different irreducible representations of the symmetry group $(\mathbb{Z}_4 \times \mathbb{Z}_2$, by $D_4$ acting internally on the 0D block):

$$\mathcal{H}^1[U^f(1) \times (\mathbb{Z}_4 \times \mathbb{Z}_2), U^f(1)] = \mathbb{Z} \times \mathbb{Z}_2^2 \quad (S1)$$

Here $\mathbb{Z}$ represents the complex fermion, the first $\mathbb{Z}_2$ represents the rotation eigenvalue $-1$ and another $\mathbb{Z}_2$ represents the reflection eigenvalue $-1$. We demonstrate that the 0D block-state with four complex fermions can be trivialized. We label these four complex fermions as $(c_1, c_2, c_3, c_4)$. As illustrated in Fig. S1, we can manipulate these complex fermions in a symmetric way and trivialize the corresponding 0D block-state: move the complex fermion labeled by $c_j$ outward the origin along the 1D block labeled by $j$ ($j = 1, 2, 3, 4$) toward infinite far away from the origin, and the system is trivialized to a vacuum under this symmetric manipulation. As a consequence, the classification attributed to the complex fermion decoration on 0D block is trivialized from $\mathbb{Z}$ to $\mathbb{Z}_4$.

Then we consider the trivializations beyond this. For systems with spinless fermions, we repeatedly consider Fig. S1, we decorate 4 complex fermions at the rotation center (which can be trivialized as aforementioned), they...
can form an atomic insulator $|\psi\rangle^{U(1)}_{D_1} = c_1^+ c_2^+ c_3^+ c_4^+ |0\rangle$, with the following rotation and reflection symmetry properties:

\begin{align}
R |\psi\rangle^{U(1)}_{D_1} &= c_2^+ c_3^+ c_4^+ c_1^+ |0\rangle = -|\psi\rangle^{U(1)}_{D_1} \\
M |\psi\rangle^{U(1)}_{D_1} &= c_1^+ c_3^+ c_2^+ c_4^+ |0\rangle = -|\psi\rangle^{U(1)}_{D_1}
\end{align}

(S2)

eigenvalue \(-1\) of rotation and reflection symmetry can be trivialized by $|\psi\rangle^{U(1)}_{D_1}$, and the classification for systems with spinless fermions is:

$$G^{U(1)}_{D_1,0} = \mathbb{Z}_4$$

(S3)

For systems with spin-1/2 fermions, The minus signs in Eq. (S2) disappear, hence there is no trivialization beyond Fig. S1. But there is an obstruction: a specific $n_0$ is obstructed if and only if $(−1)^{\nu_2−n_0} \in \mathcal{H}^4[\mathbb{Z}_2 \times \mathbb{Z}_2, U(1)]$ is a nontrivial 2-cocycle with $U(1)$-coefficient. From Refs. [1] and [4] we know that for cases where $U^f(1)$ charge conservation, nontrivial 0-cocycle $n_0 = 1$, $n_0 \in \mathcal{H}^0[\mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_2]$ leads to nontrivial 2-cocycle $(−1)^{\nu_2−n_0} \in \mathcal{H}^2[\mathbb{Z}_2 \times \mathbb{Z}_2, U(1)]$. So for $U^f(1)$ charge conserved cases, all odd $n_0 \in \mathcal{H}^0[\mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_2]$ lead to nontrivial 2-cocycle $(−1)^{\nu_2−n_0} \in \mathcal{H}^2[\mathbb{Z}_2 \times \mathbb{Z}_2, U(1)]$. As a consequence, for systems with spin-1/2 fermions, we can only decorate even number of complex fermions on the center of $D_4$ symmetry. Therefore, the classification for systems with spin-1/2 fermions is:

$$G^{U(1)}_{D_4,1/2} = \mathbb{Z}_2 \times \mathbb{Z}_2^2$$

(S4)

Similar discussions can be held for other point group FSPT phases, and we summarize all results of classification in Table IV.

Finally, we compare our results with the classification of SPT phases protected by corresponding internal group and find a perfect agreement.

**Appendix S-3: Other cases of crystalline topological phases with wallpaper group symmetry**

In the main text, we explicitly constructed and classified five examples of 2D interacting fermionic crystalline topological phases protected by wallpaper group which belongs to different crystal systems. In this section we summarize all other cases, and the results of classification are summarized in Table I and II in the main text. Furthermore, we also discuss about the systems with $U^f(1)$ charge conservation whose general principles are highlighted in the main text, and all results of classification are summarized in Table III in the main text.

1. $p1$

There is no more on-site symmetry on each block for arbitrary dimensions (cf. Fig. S2, and we use the same label for $p$-dimensional blocks who can be related by symmetry actions), so the only significant decoration is Majorana chain decorations on 1D blocks and complex fermion decoration on 0D blocks ($G_{0D} = \mathbb{Z}_2$). Spin of fermions is irrelevant for this case because there is no symmetry operation which rotates fermions in the system. Consider the Majorana chain decorations on $\tau_1$ which leave two dangling Majorana fermions at each 0D block. Add the term $i\gamma_1\gamma_2$ at each 0D block in order to glue these dangling Majorana fermions toward an entanglement pair and the no-open-edge condition [5] is satisfied as a consequence. Similar for 1D blocks labeled by $\tau_2$. So the eventual classification is $G_{1D} = \mathbb{Z}_2^3$, which is attributed to complex fermion decoration:

$$E^{0D}_{p1} = \mathbb{Z}_2$$

(S1)

and Majorana chain decoration:

$$E^{1D}_{p1} = \mathbb{Z}_2^2$$

(S2)

Similar with the $p2$ case, decorate two copies of Majorana chains on each 1D block labeled by $\tau_1$ or $\tau_2$ can be deformed to an alternative Majorana chain surrounding each 0D block $\mu$, and changes the fermion parity of each 0D block. Equivalently, the 1D block-state decorations

| $G$      | spinless | spin-1/2 |
|----------|----------|----------|
| $C_{2m−1}$ | $\mathbb{Z}_2^2 m−1$ | $\mathbb{Z}_2^2 m−1$ |
| $C_{2n}$  | $\mathbb{Z}_2^2 m$  | $\mathbb{Z}_2^2 m$   |
| $D_{2m−1}$ | $\mathbb{Z}_2 m−1$  | $\mathbb{Z}_2 m−1$   |
| $D_{2m}$  | $\mathbb{Z}_2 m$    | $\mathbb{Z}_2 m$     |

**Table IV.** The classification of interacting 2D FSPT phases with point group symmetry and $U^f(1)$ charge conservation, $m = 1, 2, 3$.  

FIG. S2. #1 wallpaper group $p1$ and its cell decomposition.
extend the 0D block-state decorations, and the ultimate classification with accurate group structure is:

$$G_{p1}^0 = G_{p1}^{1/2} = E_{p1}^{0D} \times \omega E_{p1}^{1D} = \mathbb{Z}_2 \times \mathbb{Z}_4$$  \hspace{0.5cm} (S3)

where the symbol “$\times \omega$” means that 1D and 0D block-states $E_{p1}^{1D}$ and $E_{p1}^{0D}$ have nontrivial extension, and described by an nontrivial factor system of the following short exact sequence in mathematical language:

$$0 \rightarrow E_{p1}^{1D} \rightarrow G_{p1}^0 = G_{p1}^{1/2} \rightarrow E_{p1}^{0D} \rightarrow 0$$  \hspace{0.5cm} (S4)

Then we consider the systems with $U^f(1)$ charge conservation. We note that in the main text, we have demonstrated that 1D block-state decorations does not contribute any nontrivial crystalline topological phase because of the absence of nontrivial 1D root phase, and the construction and classification are equivalent for systems with spinless fermions and spin-1/2 fermions. For an arbitrary 0D block, different 0D block-states are characterized by different irreducible representations of the symmetry group as:

$$\mathcal{H}^1[U(1), U(1)] = \mathbb{Z}$$  \hspace{0.5cm} (S5)

here $\mathbb{Z}$ represents the complex fermion, and there is no trivialization. Hence the classification for the systems with $U^f(1)$ charge conservation is:

$$G_{p1}^{U(1)} = \mathbb{Z}$$  \hspace{0.5cm} (S6)

2. pm

For 2D blocks and 1D blocks labeled by $\tau_1$, there is no on-site symmetry, and for 1D blocks labeled by $\tau_2/\tau_3$, the on-site symmetry is $\mathbb{Z}_2$ because the reflection symmetry acts internally, and all 0D blocks have on-site $\mathbb{Z}_2$ symmetry via the reflection symmetry acts internally, see Fig. S3. We discuss systems with spinless and spin-1/2 fermions separately.

a. Spinless fermions

First, we consider the 0D block-state decorations, the classification data can be characterized by different 1D irreducible representations of the symmetry group $\mathbb{Z}_2^f \times \mathbb{Z}_2$:

$$\mathcal{H}^1[\mathbb{Z}_2^f \times \mathbb{Z}_2, U(1)] = \mathbb{Z}_2^2$$  \hspace{0.5cm} (S7)

One $\mathbb{Z}_2$ is from the complex fermion, and the other is from the reflection eigenvalue $-1$. So all obstruction-free 0D block-states form the following group:

$$\{\text{OFBS}\}^{0D}_{\text{pm},0} = \mathbb{Z}_2^4$$  \hspace{0.5cm} (S8)

Then consider the 1D block-state decorations: For 1D blocks labeled by $\tau_1$, there is no on-site symmetry, so the unique possible 1D block-state is Majorana chain; for 1D blocks labeled by $\tau_2$ and $\tau_3$, the total symmetry group is $\mathbb{Z}_2^f \times \mathbb{Z}_2$, and the possible 1D block-states are Majorana chain and 1D FSPT states (which can be realized by 2 Majorana chains with different eigenvalues of $\mathbb{Z}_2$ on-site symmetry). So all 1D block-states form the following group:

$$\{\text{BS}\}^{1D}_{\text{pm},0} = \mathbb{Z}_2^5$$  \hspace{0.5cm} (S9)

Then we discuss about the decorations of these two root phases separately.

a. Majorana chain decoration First, we consider the Majorana chain decoration on $\tau_1$, which leaves 2 dangling Majorana fermions at each 0D block labeled by $\mu_2$ who can be glued by an entanglement pair $i\gamma_1 \gamma_2$. Nevertheless, this entanglement pair breaks the reflection symmetry, so the no-open-edge condition is violated.

Subsequently we consider the Majorana chain decoration on $\tau_2/\tau_3$, which leaves 2 dangling Majorana fermions at $\mu_1/\mu_2$ and can be glued by an entanglement pair without breaking any symmetry. Hence the no-open-edge condition is preserved.

b. 1D FSPT state decoration The 1D FSPT state decoration can only be decorated on $\tau_2$ and $\tau_3$ due to the on-site symmetry, and this decoration leaves 4 dangling Majorana fermions at $\mu_1/\mu_2$ with the following symmetry properties:

$$\mathbb{Z}_2 : \begin{cases} \gamma^1_j \mapsto \gamma^1_j, & j = A, B, \\ \gamma^2_j \mapsto -\gamma^2_j \end{cases}$$  \hspace{0.5cm} (S10)

Consider the local fermion parity $P_f = -\gamma^1_A \gamma^2_A \gamma^1_B \gamma^2_B$ which is invariant under $\mathbb{Z}_2$ symmetry, so these 4 dangling Majorana fermions can be gapped out by some proper interactions in a symmetric way. Equivalently, the no-open-edge condition is satisfied. Therefore, the obstruction-free 1D block-state decorations form the group:

$$\{\text{OFBS}\}^{1D}_{\text{pm},0} = \mathbb{Z}_2^4$$  \hspace{0.5cm} (S11)
and all obstruction-free block-states for arbitrary dimensions form the group:
\[ \{\text{OFBS}\}_{pm,0} = \{\text{OFBS}\}^{1D}_{pm,0} \times \{\text{OFBS}\}^{0D}_{pm,0} = Z_2^4 \times Z_2^2 = Z_2^6 \]  
(S12)

With all obstruction-free block-states, subsequently we discuss about all possible trivializations. Decorate a Majorana chain with anti-PBC on each 2D block and enlarge all “Majorana bubbles”, near each 1D block labeled by \( \tau_1 \), “Majorana bubble” construction can be deformed to double Majorana chains which can be trivialized because there is no on-site symmetry on \( \tau_1 \) and the classification of 1D invertible topological phases (i.e., Majorana chain with anti-PBC on each 2D block and enlargement) has no obstruction-free block-states for arbitrary dimensions.

Furthermore, consider the 1D bubble equivalence on 1D blocks \( \tau_1 \): on each 1D block labeled by \( \tau_1 \), we decorate a 1D bubble onto it. Here both yellow and red dots represent the complex fermions. Near each 0D block labeled by \( \tau_2 \), there is 2 complex fermions which form an atomic insulator:
\[ |\psi\rangle^{\mu_2}_{pm} = c_1^\dagger c_2^\dagger |0\rangle \]  
(S13)

with reflection symmetry property as (\( M_{\tau_1} \) represents the reflection operation with the axis coincide with the 1D block labeled by \( \tau_3 \)):
\[ M_{\tau_3} |\psi\rangle^{\mu_2}_{pm} = c_2^\dagger c_1^\dagger |0\rangle = - |\psi\rangle^{\mu_2}_{pm} \]  
(S14)

Hence the reflection eigenvalue \(-1\) can be trivialized by the atomic insulator \( |\psi\rangle^{\mu_2}_{pm} \). Similar for \( \mu_1 \), and we can conclude that reflection eigenvalues at 0D blocks labeled by \( \mu_1 \) and \( \mu_2 \) are not independent.

With all possible trivializations, we are ready to study the trivial states. Start from the original 0D trivial state (nothing is decorated on arbitrary blocks):
\[ [(+,+),(+,+)] \]

If we take 1D bubble equivalences on \( \tau_1 \) by \( n_1 \) times, above trivial state will be deformed to a new 0D block-state labeled by:
\[ [(+,-1)^{n_1}),(+,-1)^{n_1}] \]  
(S15)

According to the definition of bubble equivalence, all these states should be trivial. It is easy to see that there are only one independent quantities \( (n_1) \) in Eq. (S15).

Together with the 2D bubble equivalence, all these trivial states form the following group:
\[ \{\text{TBS}\}_{pm,0} = \{\text{TBS}\}^{1D}_{pm,0} \times \{\text{TBS}\}^{0D}_{pm,0} = Z_2 \times Z_2 = Z_2^2 \]  
(S16)

here \( \{\text{TBS}\}^{1D}_{pm,0} \) represents the group of trivial states with non-vacuum 1D blocks (i.e., 1D FSPT phase decorations on \( \tau_2 \) and \( \tau_3 \) simultaneously), and \( \{\text{TBS}\}^{0D}_{pm,0} \) represents the group of trivial states with non-vacuum 0D blocks.

Therefore, all independent nontrivial block-states are labeled by the group elements of the following quotient group:
\[ E_{pm,0} = \{\text{OFBS}\}_{pm,0}/\{\text{TBS}\}_{pm,0} = Z_2^6/Z_2^2 = Z_2^4 \]  
(S17)

here three \( Z_2 \)'s are from the nontrivial 0D block-states, and other three \( Z_2 \)'s are from the nontrivial 1D block-states.

Similar with \( p4m \) case, there is no stacking between 1D and 0D block-states, and the ultimate classification with accurate group structure is:
\[ G^0_{pm} = Z_2^6 \]  
(S18)

b. Spin-1/2 fermions

First, we consider the 0D block-state decorations. The total symmetry of each 0D block labeled by \( \mu_1 \) and \( \mu_2 \) is the nontrivial \( Z_2^4 \) extension of the on-site symmetry \( Z_2 \): \( Z_2^4 \), at which different 0D block-states can be characterized by different 1D irreducible representations of the corresponding symmetry group:
\[ H^1 \left[ Z_2^4, U(1) \right] = Z_4 \]  
(S19)

and there is no trivialization on the classification data. Hence the classification attributed to 0D block-state decorations is:
\[ E^{0D}_{pm,1/2} = Z_4^2 \]  
(S20)

Then consider the 1D block-state decorations: for 1D blocks labeled by \( \tau_1 \), there is no on-site symmetry, hence the unique possible 1D block-state is Majorana chain; for 1D blocks labeled by \( \tau_2 \) and \( \tau_3 \), the total symmetry group is \( Z_2^4 \), and there is no nontrivial block state due to the trivial classification of the corresponding 1D FSPT phases. Subsequently we consider the Majorana chain decoration on \( \tau_1 \) who leaves 2 dangling Majorana fermions at each 0D block labeled by \( \mu_2 \) who can be glued by an entanglement pair \( \tau_1 \) and \( \tau_2 \) and preserve the reflection symmetry. Hence the no-open-edge condition is fulfilled and the classification attributed to 1D block-state decorations is:
\[ E^{1D}_{pm,1/2} = Z_2 \]  
(S21)
Similar with $p2$ case, there is a stacking between 1D and 0D block-states which leads to the classification data $Z_4 \times Z_2$ to $Z_8$, and the ultimate classification with accurate group structure is:

$$G^{1/2}_{pm} = Z_4 \times Z_8 \quad (S22)$$

c. With $U^I(1)$ charge conservation

Then we consider the systems with $U^I(1)$ charge conservation. For an arbitrary 0D block, different 0D block-states are characterized by different irreducible representations of the symmetry group as:

$$\mathcal{H}^I[U(1) \times Z_2, U(1)] = Z \times Z_2 \quad (S23)$$

Here $Z$ represents the complex fermion, and $Z_2$ represents different reflection eigenvalues on each 0D block. Then we consider possible trivializations. For systems with spinless fermions, consider the 1D bubble equivalence on 1D blocks labeled by $\tau_1$ [cf. 1D bubble, here yellow and red dots represent particle and hole, respectively, and they can be trivialized if we shrink them to a point]. Near each 0D block labeled by $\mu_1$, there are 2 fermionic particles ($p_{1}^{\dagger}, p_{2}^{\dagger}$) that form an atomic insulator:

$$|\phi_{0D}^{\mu_1}\rangle = p_{1}^{\dagger}p_{2}^{\dagger}|0\rangle \quad (S24)$$

with reflection property as ($M_{\tau_2}$ represents the reflection operation with the axis coincide with the 1D block labeled by $\tau_2$):

$$M_{\tau_2} |\phi_{0D}^{\mu_1}\rangle = p_{2}^{\dagger}p_{1}^{\dagger}|0\rangle = -|\phi_{0D}^{\mu_1}\rangle \quad (S25)$$

i.e., the reflection eigenvalue $-1$ at $\mu_1$ can be trivialized by atomic insulator $|\phi_{0D}^{\mu_1}\rangle$. Near each 0D block labeled by $\mu_2$, there are 2 fermionic holes ($h_{1}^{\dagger}, h_{2}^{\dagger}$) that form another atomic insulator:

$$|\phi_{0D}^{\mu_2}\rangle = h_{1}^{\dagger}h_{2}^{\dagger}|0\rangle \quad (S26)$$

with reflection property as:

$$M_{\tau_1} |\phi_{0D}^{\mu_2}\rangle = h_{2}^{\dagger}h_{1}^{\dagger}|0\rangle = -|\phi_{0D}^{\mu_2}\rangle \quad (S27)$$

i.e., the reflection eigenvalue $-1$ at $\mu_2$ can be trivialized by atomic insulator $|\phi_{0D}^{\mu_2}\rangle$. Thus the 1D bubble construction on $\tau_1$ can change the reflection eigenvalues of $\mu_1$ and $\mu_2$ simultaneously. Equivalently, the reflection eigenvalues of 0D blocks $\mu_1$ and $\mu_2$ are not independent. Furthermore, above particle-hole construction on 1D blocks labeled by $\tau_1$ will add two complex fermions at each 0D block labeled by $\mu_1$ and delete two complex fermions at each 0D block labeled by $\mu_2$ (by adding two holes), hence the number of complex fermions at $\mu_1$ and $\mu_2$ are not independent. More specifically, consider there are $a$ complex fermions on each $\mu_1$ and $b$ complex fermions on each $\mu_2$, and suppose $a + b = c$. Take above manipulation $n$ times ($n \in \mathbb{Z}$), the number of complex fermions on each $\mu_1/\mu_2$ is $a + 2n/b - 2n$, and their summation remains invariant. Therefore for a specific $c$, there are only two independent cases: $c = a + b$ and $c = (a + 1) + (b - 1)$.

With the help of above discussions, we consider the 0D blocks $\mu_1$ and $\mu_2$ with 1D block $\tau_1$ connecting them. The 0D block-state decorated on $\mu_1$ can be characterized by $(a, \pm)$, where $a \in \mathbb{Z}$ represents the number of complex fermions decorated at $\mu_1$, and $\pm$ represents the eigenvalues of reflection symmetry. Hence all candidates of 0D block-states on $\mu_1$ and $\mu_2$ can be labeled by elements of the group $(Z \times Z_2)^2$. Then we consider the trivial state labeled by $[(0, +), (0, +)]$, take aforementioned 1D bubble construction on $\tau_1$ one time will lead to a new 0D block-state labeled by $[(2, -), (2, -)]$ which is also trivial. By parity of reasoning, all trivial 0D block-states will form the following group:

$$Z \times Z_2^3 \quad (S28)$$

And they can reduce the classification of 0D block-states on $\mu_1$ and $\mu_2$ from $Z^2 \times Z_2^2$ to $Z \times Z_2^3$. Then the classification of crystalline topological phases protected by $pm$ symmetry for 2D systems with spinless fermions is (where the subscript 0 represents the spinless fermions):

$$G^{U(1)}_{pm,0} = Z \times Z_2^3 \quad (S29)$$

Subsequently we investigate systems with spin-1/2 fermions. Repeatedly consider the atomic insulators $|\phi_{0D}^{\mu_1}\rangle$ and $|\phi_{0D}^{\mu_2}\rangle$, with the following reflection properties:

$$\begin{align*}
M_{\tau_2} |\phi_{0D}^{\mu_1}\rangle &= -p_{1}^{\dagger}p_{2}^{\dagger}|0\rangle = |\phi_{0D}^{\mu_1}\rangle \\
M_{\tau_2} |\phi_{0D}^{\mu_2}\rangle &= p_{2}^{\dagger}p_{1}^{\dagger}|0\rangle = |\phi_{0D}^{\mu_2}\rangle
\end{align*} \quad (S30)$$

Hence for systems with spin-1/2 fermions, reflection symmetry eigenvalues on different 0D blocks are independent. We repeatedly consider the 0D blocks $\mu_1$ and $\mu_2$ with 1D block $\mu_1$ connecting them. Take aforementioned 1D bubble construction on $\tau_1$ for the trivial state $[(0, +), (0, +)]$ will lead to a new 0D block-state labeled by $[(2, +), (2, +)]$ which is also trivial. By parity of reasoning, all trivial 0D block-states will form the following group:

$$Z \times Z_2^3 \quad (S31)$$

And they can reduce the classification of 0D block-states from $Z^2 \times Z_2^2$ to $Z \times Z_2^3$. Finally the classification of crystalline topological phases protected by $pm$ symmetry for 2D systems with spin-1/2 fermions is (where the subscript 1/2 represents the spin-1/2 fermions):

$$G^{U(1)}_{pm,1/2} = Z \times Z_2^3 \quad (S32)$$

All results are summarized in the main text.
There is no on-site symmetry on blocks with arbitrary dimensions, because the reflection operation is accomplished with a translation operation toward a “glide reflection” operation, see Fig. S4. The spin of fermions is irrelevant for this case, hence we only need to discuss about the systems with spinless fermions.

### a. Spinless and spin-1/2 fermions

First, we consider the 0D block-state decorations. The only relevant on-site symmetry is the local fermion parity $\mathbb{Z}_2^f$. Consider the 2D bubble equivalence: we decorate a Majorana chain with anti-PBC on each 2D block which can be trivialized if it shrinks to a point. By some proper local unitary transformations, this assembly of bubbles can be deformed to an assembly of Majorana chains with anti-PBC surrounding each of 0D block. Nevertheless, the fermion parities of 0D blocks cannot be changed by 2D bubble equivalence because the fermion parity of Majorana chain with anti-PBC is even. As a consequence, the classification contributed by complex fermion decorations on 0D blocks is $G^{0}_{pg} = \mathbb{Z}_2$, and subsequently, the classification via 0D block-state decorations is:

$$E^{0D}_{pg} = \mathbb{Z}_2$$ (S33)

Then consider the 1D block-state decorations: the unique possible 1D block-state is Majorana chain due to the absence of the on-site symmetry. Majorana chain decoration on $\tau_1/\tau_2$ leaves 2 dangling Majorana fermions on each 0D block $\mu$ which can be gapped out by an entanglement pair, hence the no-open-edge condition is satisfied. Thus the classification attributed to 1D block-state decoration is:

$$E^{1D}_{pg} = \mathbb{Z}_2^2$$ (S34)

Similar with the p1 case, decorate two copies of Majorana chains on each 1D block labeled by $\tau_1$ or $\tau_2$ can be deformed to an alternative Majorana chain surrounding each 0D block $\mu$, and changes the fermion parity of each 0D block. Equivalently, the 1D block-state decorations extend the 0D block-state decorations, and the ultimate classification with accurate group structure is:

$$G_{pg}^0 = G^{1/2}_{pg} = E^{0D}_{pg} \times \omega_2 E^{1D}_{pg} = \mathbb{Z}_2 \times \mathbb{Z}_4$$ (S35)

Here the symbol “$\times \omega_2$” means that 1D and 0D block-states $E^{1D}_{pg}$ and $E^{0D}_{pg}$ have nontrivial extension, and described by a nontrivial factor system of the following short exact sequence in mathematical language:

$$0 \to E^{1D}_{pg} \to G_{pg}^0 = G^{1/2}_{pg} \to E^{0D}_{pg} \to 0$$ (S36)

### b. With $U^f(1)$ charge conservation

Then we consider the systems with $U^f(1)$ charge conservation. For an arbitrary 0D block, different 0D block-states are characterized by different irreducible representations of the symmetry group as:

$$H^1[U(1), U(1)] = \mathbb{Z}$$ (S37)

Here $\mathbb{Z}$ represents the complex fermion. There is no trivialization, hence the classification of the systems with $U^f(1)$ charge conservation is:

$$\mathcal{G}^{U(1)}_{pg,0} = \mathcal{G}^{U(1)}_{pg,1/2} = \mathbb{Z}$$ (S38)

### 4. $cm$

There is no on-site symmetry on 2D block $\sigma$ and 1D blocks $\tau_1$; and there is an on-site $\mathbb{Z}_2$ symmetry on 1D block $\tau_2$ and 0D block $\mu$ because the reflection symmetry action on the corresponding blocks are identical with an on-site $\mathbb{Z}_2$ symmetry, see Fig. S5.
a. Spinless fermions

First, we consider the 0D block-state decorations. The classification data can be characterized by different 1D irreducible representations of the symmetry group $Z_2^f \times Z_2^e$:

$$
\mathcal{H}^1\left[Z_2^f \times Z_2^e, U(1)\right] = Z_2^2
$$

(S39)

One $Z_2$ represents the complex fermions, and the other represents the reflection eigenvalue $-1$. So all obstruction-free 0D block-states form the group:

$$
\{\text{OFBS}\}^{0D}_{cm,0} = Z_2^2
$$

(S40)

Subsequently consider the 1D block-state decorations, and the decorations on $\tau_1$ and $\tau_2$ are different: the unique possible block-state on $\tau_1$ is Majorana chain due to the absence of the on-site symmetry, but for $\tau_2$, the total on-site symmetry is $Z_2^f \times Z_2^e$, hence the possible 1D block-states are Majorana chain and 1D FSPT state.

b. Decoration on $\tau_2$

First, we consider the Majorana chain decoration on $\tau_2$ which leaves 2 dangling Majorana fermions at each 0D block $\mu$, with the same $Z_2$-eigenvalue because of the translational symmetry. So these 2 Majorana fermions can be symmetrically gapped out by an entanglement pair, and the no-open-edge condition is satisfied. Subsequently we consider the 1D FSPT state decoration on $\tau_2$, which leaves 4 dangling Majorana fermions on each $\mu$, with the $Z_2$ symmetry properties as Eq. (S10). Thus the local fermion parity satisfies the symmetry, and these 4 Majorana fermions can be gapped by interaction in a symmetric way. Therefore, the no-open-edge condition is satisfied. Thus all obstruction-free 0D block-states form the group:

$$
\{\text{OFBS}\}^{1D}_{cm,0} = Z_2^3
$$

(S42)

and all obstruction-free block-states for arbitrary dimensions form the following group:

$$
\{\text{OFBS}\}_{cm,0} = \{\text{OFBS}\}^{1D}_{cm,0} \times \{\text{OFBS}\}^{0D}_{cm,0}
$$

$$
= Z_2^3 \times Z_2^2 = Z_2^5
$$

(S43)

and different group elements can be labeled by:

$$
[\pm; n_1; n_2, n_2']
$$

(S44)

here $\pm$ is the reflection eigenvalues at each $\mu$, $n_1$ and $n_2$ represent the number of decorated Majorana chains on the 1D block $\tau_1$ and $\tau_2$, and $n_2'$ represents the number of decorated 1D FSPT phases on the 1D block $\tau_2$.

With all obstruction-free block-states, subsequently we discuss about all possible trivializations. Decorate a Majorana chain with anti-PBC on each 2D block and enlarge all “Majorana bubble” construction can be deformed to double Majorana chains at each $\tau_1$ that can be trivialized because there is no on-site symmetry and the classification of 1D invertible topological phases (i.e., Majorana chain) is $Z_2^2$; near each 1D block labeled by $\tau_2$, “Majorana bubble” construction can also be deformed to double Majorana chains. Nevertheless, these double Majorana chains cannot be trivialized because there is an on-site $Z_2$ symmetry on each $\tau_2$ by reflection symmetry acting internally, and this $Z_2$ action exchanges these two Majorana chains, and this is exactly the definition of the nontrivial 1D FSPT phase protected by on-site $Z_2$ symmetry. Furthermore, similar with the $p4m$ case, 2D “Majorana bubble” construction has no effect on arbitrary 0D blocks.

Subsequently we consider the 1D bubble equivalence on 1D blocks $\tau_1$ [cf. 1D bubble, here both yellow and red dots represent the complex fermions]. near each 0D block, there are 4 complex fermions which can form an atomic insulator:

$$
|\psi\rangle_{cm} = c_1^\dagger c_2^\dagger c_3^\dagger c_4^\dagger |0\rangle
$$

(S45)

with reflection property as ($M_{\tau_2}$ represents the reflection operation with the axis coincide with the 1D block labeled by $\tau_2$):

$$
M_{\tau_2} |\psi\rangle_{cm} = c_4^\dagger c_3^\dagger c_2^\dagger c_1^\dagger |0\rangle = |\psi\rangle_{cm}
$$

(S46)

Hence there is no trivialization from 1D bubble equivalence, and all trivial states form the following group:

$$
\{\text{TBS}\}_{cm,0} = \{\text{TBS}\}^{1D}_{cm,0} = Z_2
$$

(S47)

Therefore, all independent nontrivial block-states are labeled by the group elements of the following quotient group:

$$
E_{cm,0} = \{\text{OFBS}\}_{cm,0} / \{\text{TBS}\}_{cm,0}
$$

$$
= Z_2^5 / Z_2 = Z_2^4
$$

(S48)

Here two $Z_2$'s are from the nontrivial 0D block-states, and other two $Z_2$'s are from the nontrivial 1D blockstates.

Similar with $p4m$ case, there is no stacking between 1D and 0D block-states, hence the group structure of the classification data (S48) has already been accurate.
b. Spin-1/2 fermions

First, we consider the 0D block-state decorations. The total symmetry of each 0D block labeled by $\mu_1$ and $\mu_2$ is the nontrivial $\mathbb{Z}_2^f$ extension of the on-sity symmetry $\mathbb{Z}_2$: $\mathbb{Z}_2^f$, at which different 0D block-states can be characterized by different 1D irreducible representations of the corresponding symmetry group:

$$\mathcal{H}^1[\mathbb{Z}_4^f, U(1)] = \mathbb{Z}_4$$ (S49) 

and there is no obstruction and trivialization on the classification data. Hence the classification attributed to 0D block-state decorations is:

$$E^0_{\text{cm},1/2} = \mathbb{Z}_4$$ (S50)

Subsequently we consider the 1D block-state decorations. The unique possible block-state for 1D blocks labeled by $\tau_1$ is Majorana chain because of the absence of on-site symmetry. On $\tau_2$, the total symmetry group is $\mathbb{Z}_2^f$ (i.e., nontrivial $\mathbb{Z}_2^f$ extension of $\mathbb{Z}_2$), hence there is no candidate block-state for 1D blocks labeled by $\tau_2$ because of the trivial classification. So the only possible 1D block-state is Majorana chain decoration on $\tau_1$. The Majorana chain decoration on $\tau_1$ leaves 4 dangling Majorana fermions at each 0D block which can be gapped out in a symmetric way by some proper interactions because the local fermion parity keeps invariant under arbitrary symmetry actions. Equivalently the no-open-edge condition is satisfied, and the classification attributed to 1D block-state decorations is:

$$E^{1\text{D}}_{\text{cm},1/2} = \mathbb{Z}_2$$ (S51)

Similar with the pgg case, there is no stacking between 1D and 0D block-states, and the ultimate classification with accurate group structure is:

$$\mathcal{G}^{1/2}_{\text{cm}} = \mathbb{Z}_4 \times \mathbb{Z}_2$$ (S52)

c. With $U^f(1)$ charge conservation

Then we consider the systems with $U^f(1)$ charge conservation. For an arbitrary 0D block, different 0D block-states are characterized by different irreducible representations of the symmetry group as:

$$\mathcal{H}^1[U(1) \times \mathbb{Z}_2, U(1)] = \mathbb{Z} \times \mathbb{Z}_2$$ (S53)

Here $\mathbb{Z}$ represents the complex fermion, and $\mathbb{Z}_2$ represents the eigenvalues of reflection symmetry operation. There is no trivialization, hence the classification of the systems with $U^f(1)$ charge conservation is:

$$\mathcal{G}^{U(1)}_{\text{cm},0} = \mathcal{G}^{U(1)}_{\text{cm},1/2} = \mathbb{Z} \times \mathbb{Z}_2$$ (S54)

Then we consider the systems with $U^f(1)$ charge conservation. For an arbitrary 0D block, different 0D block-states are characterized by different irreducible representations of the symmetry group as:

$$\mathcal{H}^1[U(1) \times \mathbb{Z}_2, U(1)] = \mathbb{Z} \times \mathbb{Z}_2$$ (S53)

Here $\mathbb{Z}$ represents the complex fermion, and $\mathbb{Z}_2$ represents the eigenvalues of reflection symmetry operation. There is no trivialization, hence the classification of the systems with $U^f(1)$ charge conservation is:

$$\mathcal{G}^{U(1)}_{\text{cm},0} = \mathcal{G}^{U(1)}_{\text{cm},1/2} = \mathbb{Z} \times \mathbb{Z}_2$$ (S54)

FIG. S6. #7 wallpaper group pmm and its cell decomposition.

5. pmm

The corresponding point group for this case is 2-fold dihedral group $D_2$. For 2D blocks $\sigma$, 1D blocks labeled by $\tau_3$ and $\tau_4$, there is no on-site symmetry; for 1D blocks $\tau_1$ and $\tau_2$, 0D blocks labeled by $\mu_1$ and $\mu_2$, the on-site symmetry is $\mathbb{Z}_2$ via the reflection symmetry acting internally; for 0D blocks $\mu_3$ and $\mu_4$, the on-site symmetry is $\mathbb{Z}_2$ via the 2-fold rotational symmetry acting internally, see Fig. S7.

a. Spinless fermions

First, we investigate the 0D block-state decorations. For an arbitrary 0D block, the classification data can be characterized by different 1D irreducible representations of the full symmetry group $\mathbb{Z}_2 \times \mathbb{Z}_2$:

$$\mathcal{H}^1[\mathbb{Z}_2^f \times (\mathbb{Z}_2 \times \mathbb{Z}_2), U(1)] = \mathbb{Z}_2^3$$ (S55)

here three $\mathbb{Z}_2$ in the classification data have different physical meanings: the first $\mathbb{Z}_2$ represents the complex fermion, the second $\mathbb{Z}_2$ represents the rotation eigenvalue $-1$, and the third $\mathbb{Z}_2$ represents the reflection eigenvalue $-1$. So the 0D block-states at $\mu_j$ ($j = 1, 2, 3, 4$) can be labeled by $(\pm, \pm, \pm)$, here these three $\pm$’s represent the fermion parity, 2-fold rotation and reflection symmetry eigenvalues (alternatively, the last two $\pm$’s can also represent the eigenvalues of two independent reflection operations because even-fold dihedral group can also be generated by two independent reflections). According to this notation, the obstruction-free 0D block-states form the following group:

$$\{\text{OFBS}\}^{0\text{D}}_{\text{pmm}} = \mathbb{Z}_2^{12}$$ (S56)
and the group elements can be labeled by (three brackets represent the block-states at \( \mu_1, \mu_2 \) and \( \mu_3 \)):

\[
[(\pm, \pm, \pm), (\pm, \pm, \pm), (\pm, \pm, \pm), (\pm, \pm, \pm)]
\]

Subsequently we investigate the 1D block-state decoration. For all 1D blocks, the total symmetry group is \( \mathbb{Z}_4 \times \mathbb{Z}_2 \), and the candidate 1D block-state is Majorana chain and 1D FSPT state. So all 1D block-states form a group:

\[
\{\text{BS}\}_{pmm,0}^{1D} = \mathbb{Z}_2^8
\]  

(S57)

Then we discuss about the decorations of these two root phases separately.

a. Majorana chain decoration Consider Majorana chain decorations on 1D blocks labeled by \( \tau_1 \) as an example, which leaves 2 dangling Majorana fermions at each 0D block labeled by \( \mu_1, \mu_3 \). Near each \( \mu_1 \), two dangling Majorana fermions have the following rotational symmetry properties:

\[
R_{\mu_1} : \gamma_1 \leftrightarrow \gamma_2
\]  

(S58)

Then consider the local fermion parity and its rotational symmetry property:

\[
P_f = i^\gamma_1\gamma_2, \quad R_{\mu_1} : P_f \mapsto -P_f
\]  

(S59)

Thus these two dangling Majorana fermions form a projective representation of the symmetry group \( pmm \times \mathbb{Z}_4 \mathbb{Z}_2 \), and a non-degenerate ground state is forbidden. Thus Majorana chain decoration on 1D blocks \( \tau_1 \) is obstructed because of the violation of the no-open-edge condition. Similar arguments can be held on all other 1D blocks, and we can conclude that the Majorana chain decorations on arbitrary 0D blocks are obstructed.

b. 1D FSPT state decoration Then we consider about the 1D FSPT state decorations. As an example, 1D FSPT state decoration on \( \tau_1 \) leaves four dangling Majorana fermions at each \( \mu_1, \mu_3 \). Similar with arguments about the \( p4m \) case, these four Majorana fermions form a projective representation of the \( D_z \) symmetry group at each 0D block \( \mu_1, \mu_3 \), and the non-degenerate ground state is forbidden as a consequence, and similar for all other 1D blocks. Therefore, the 1D FSPT state decorations solely on \( \tau_j \) (\( j = 1, 2, 3, 4 \)) is obstructed because of the violation of the no-open-edge condition.

There is one exception: If we decorate a 1D FSPT phase on each 1D block (including \( \tau_j, j = 1, 2, 3, 4 \)), the dangling Majorana fermions at each 0D block can be gapped out in a symmetric way. For this case, there are two nontrivial projective representations of \( D_z \) symmetry group at each 0D block that can be deformed to a linear representation, because there is only one nontrivial projective representation of the \( D_z \) symmetry group (acting internally at each 0D block, which is identical with \( \mathbb{Z}_4 \times \mathbb{Z}_2 \) on-site symmetry group). This fact can be checked by the following 2-cohomology:

\[
\mathcal{H}^2[\mathbb{Z}_4 \times \mathbb{Z}_2, U(1)] = \mathbb{Z}_2
\]  

(S60)

As a consequence, the eight dangling Majorana fermions at each 0D block due to the decoration can be gapped out in a symmetric way, and the corresponding 1D block-state is obstruction-free. Thus all obstruction-free 1D block-states form the following group:

\[
\{\text{OFBS}\}_{pmm,0}^{1D} = \mathbb{Z}_2
\]  

(S61)

and the group elements can be labeled by \( n_1 = n_2 = n_3 = n_4 \). Here \( n_j = 0, 1 \) (\( j = 1, 2, 3, 4 \)) represents the number of decorated 1D FSPT states on \( \tau_j \), respectively. According to aforementioned discussions, a necessary condition of an obstruction-free block-state is \( n_1 = n_2 = n_3 = n_4 \).

So far we have already obtained all obstruction-free block-states, and they form the following group:

\[
\{\text{OFBS}\}_{pmm,0} = \{\text{OFBS}\}_{pmm,0}^{1D} \times \{\text{OFBS}\}_{pmm,0}^{0D}
\]

(S62)

With all obstruction-free block-states, subsequently we discuss about all possible trivializations. First, we consider about the 2D bubble equivalence: as we discussed in the main text, both types of “Majorana bubble” constructions are allowed because all 0D blocks are the centers of \( D_z \) point group symmetry, including “Majorana bubbles” with both PBC and anti-PBC. Similar with the \( p4m \) case, both types of “Majorana bubbles” can be deformed to double Majorana chains at each nearby 1D block, and this is exactly the definition of the nontrivial 1D FSPT phase protected by on-site \( \mathbb{Z}_2 \) symmetry (by reflection symmetry acting internally). As a consequence, 1D FSPT state decorations on all 1D blocks can be deformed to a trivial state via 2D “Majorana bubble” equivalence. Furthermore, repeatedly similar with the \( p4m \) case, both types of “Majorana bubble” constructions have no effect on 0D blocks, and effects of both type of “Majorana bubble” constructions are equivalent, so take one of them into account is enough.

Subsequently we consider the 1D bubble equivalences. 1D bubble equivalence on 1D blocks \( \tau_1 \) [cf. 1D bubble, here both yellow and red dots represent the complex fermions]: near the 0D block labeled by \( \mu_3 \), there are 2 complex fermions which form an atomic insulator:

\[
|\psi\rangle_{pmm}^{\mu_3} = c_1^\dagger c_2^\dagger|0\rangle
\]  

(S63)

with rotation property as (\( R_{\mu_3} \) represents the rotation operation centred at the 0D block labeled by \( \mu_3 \)):

\[
R_{\mu_3} |\psi\rangle_{pmm}^{\mu_3} = c_2^\dagger c_1^\dagger|0\rangle = -|\psi\rangle_{pmm}^{\mu_3}
\]  

(S64)

Hence the rotation eigenvalue \(-1\) at \( \mu_3 \) can be trivialized by the atomic insulator \( |\psi\rangle_{pmm}^{\mu_3} \), and the rotation eigenvalues of 0D blocks \( \mu_1, \mu_3 \) are independent. Similar discussion can be held on 1D blocks \( \tau_2, \tau_3, \tau_4 \), therefore rotation eigenvalues on all 0D blocks \( \mu_j \) (\( j = 1, 2, 3, 4 \)) are independent.

Then we repeatedly consider the atomic insulator \( |\psi\rangle_{pmm} \), with reflection property as (\( M_{\tau_3} \) represents the
reflection operation with the axis coincide with the 1D block labeled by $\tau_4$:

$$M_{\tau_4}|\psi\rangle_{pmm}^{\mu_3} = c_2^{-1}c_1|0\rangle = -|\psi\rangle_{pmm}$$

(S65)

Hence the reflection eigenvalue $-1$ at $\mu_3$ can be trivialized by the atomic insulator $|\psi\rangle_{pmm}^{\mu_3}$, and the reflection eigenvalues of 0D blocks $\mu_1$ and $\mu_3$ are not independent. Similar discussion can be held on 1D blocks $\tau_2$, $\tau_3$ and $\tau_4$, therefore reflection eigenvalues on all 0D blocks $\mu_j$ ($j = 1, 2, 3, 4$) are not independent.

As we mentioned before, $D_2$ symmetry can also be generated by two independent reflection operations, we summarize the effects of 1D bubble constructions in terms of the changes on reflection eigenvalues as following:

1. 1D bubble construction on $\tau_1$: simultaneously change the eigenvalue of $M_{\tau_2}$ at $\mu_1$ and $M_{\tau_4}$ at $\mu_3$;

2. 1D bubble construction on $\tau_2$: simultaneously change the eigenvalue of $M_{\tau_1}$ at $\mu_1$ and $M_{\tau_3}$ at $\mu_2$;

3. 1D bubble construction on $\tau_3$: simultaneously change the eigenvalues of $M_{\tau_2}$ at $\mu_2$ and $M_{\tau_4}$ at $\mu_4$;

4. 1D bubble construction on $\tau_4$: simultaneously change the eigenvalues of $M_{\tau_1}$ at $\mu_3$ and $M_{\tau_3}$ at $\mu_4$;

With all possible bubble constructions, we are ready to investigate the trivial states. Start from the original trivial state (nothing is decorated on arbitrary lower-dimensional blocks):

$$[[+, +, +], (+, +, +), (+, +, +), (+, +, +)]$$

if we take 1D bubble constructions on $\tau_j$ by $n_j$ times ($j = 1, 2, 3, 4$), above trivial state will be deformed to a new block-state labeled by:

$$[(+, (-1)^{n_2}, (-1)^{n_1})], (+, (-1)^{n_3}, (-1)^{n_2}),$$

$$(+, (-1)^{n_1}, (-1)^{n_4}), (+, (-1)^{n_4}, (-1)^{n_3})]$$

(S66)

According to the definition of bubble equivalence, all these states should be trivial. It is easy to see that there are only four independent quantities in Eq. (S66): $n_i, i = 1, 2, 3, 4$. Together with the 2D bubble equivalence, all trivial states form the group:

$$\{\text{TBS}\}_{pmm,0} = \{\text{TBS}\}_{pmm,0}^{1D} \times \{\text{TBS}\}_{pmm,0}^{0D}$$

$$= \mathbb{Z}_2 \times \mathbb{Z}_2^5 = \mathbb{Z}_2^6$$

(S67)

here $\{\text{TBS}\}_{pmm,0}^{1D}$ represents the group of trivial states with non-vacuum 1D blocks (i.e., 1D FSPT phase decorations on all 1D blocks simultaneously), and $\{\text{TBS}\}_{pmm,0}^{0D}$ represents the group of trivial states with non-vacuum 0D blocks.

Therefore, all independent nontrivial block-states are labeled by the group elements of the following quotient group:

$$E_{pmm,0} = \{\text{OFBS}\}_{pmm,0}^{1D} / \{\text{TBS}\}_{pmm,0}$$

$$= \mathbb{Z}_2^{13} / \mathbb{Z}_2^5 = \mathbb{Z}_2^8$$

(S68)

here all $\mathbb{Z}_2$’s are from the nontrivial 0D block-states. It is obvious that there is no nontrivial group extension because of the absence of nontrivial 1D block-state, and the group structure of the classification data $E_{pmm,0}$ has already been accurate.

b. Spin-1/2 fermions

Now we turn to discuss systems with spin-1/2 fermions. Consider the 0D block-state decoration, similar with the $p4m$ case, the classification data can be characterized by different 1D irreducible representations of the full symmetry group $\mathbb{Z}_2^4 \times \mathbb{Z}_2$:

$$\mathcal{H}^1[\mathbb{Z}_2^4 \times \mathbb{Z}_2, U(1)] = \mathbb{Z}_2^2$$

(S69)

Hence nontrivial 2D FSPT states attributed to 0D block-state decoration are classified as:

$$E_{pmm,1/2}^{0D} = \mathbb{Z}_2^8$$

(S70)

Then we consider the 1D block-state decorations. The total on-site symmetry on each 1D block is $\mathbb{Z}_4$ (nontrivial $\mathbb{Z}_4$ extension of on-site symmetry $\mathbb{Z}_2$, the mathematical meaning of spin-1/2 fermions). Hence in this situation, 1D block-state decorations do not contribute any 2D FSPT state due to the trivial classification of the corresponding block-state:

$$E_{pmm,1/2}^{1D} = \mathbb{Z}_4$$

(S71)

Therefore, it is obvious that there is no stacking because of the trivial contribution of 1D block-state decoration, and the ultimate classification with accurate group structure of crystalline topological phases protected by $pmm$ symmetry in 2D interacting fermionic systems with spin-1/2 fermions is:

$$\mathcal{G}_{pmm}^{1/2} = \mathbb{Z}_2^8$$

(S72)

where the superscript 1/2 expresses "spin-1/2".

c. With $U(1)$ charge conservation

For an arbitrary 0D block $\mu_j, j = 1, 2, 3, 4$, different 0D block-states are characterized by different irreducible representations of the symmetry group as:

$$\mathcal{H}^1[U(1) \times (\mathbb{Z}_2 \times \mathbb{Z}_2), U(1)] = \mathbb{Z} \times \mathbb{Z}_2^2$$

(S73)

Here $\mathbb{Z}$ represents the complex fermion, first $\mathbb{Z}_2$ represents the rotation eigenvalue $-1$ and the second $\mathbb{Z}_2$ represents the reflection eigenvalue $-1$. We should further consider possible trivialization: for systems with spinless fermions, consider the 1D bubble equivalence on 1D blocks labeled by $\tau_1$: we decorate a 1D bubble on each $\tau_1$, here yellow and red dots represent the particle and hole,
respectively, and they can be trivialized if we shrink them to a point. Near each 0D block labeled by $\mu_1$, there are two particles forming an atomic insulator:

$$|\phi\rangle_\text{pmm}^{\mu_1} = p_{14}^{\dagger} p_{21}^{\dagger} |0\rangle$$  \hspace{2cm} (S74)

with rotation and reflection properties as:

$$R_{\mu_1} |\phi\rangle_\text{pmm}^{\mu_1} = p_{24}^{\dagger} p_{12}^{\dagger} |0\rangle = -|\phi\rangle_\text{pmm}^{\mu_1}$$

$$M_{\tau_2} |\phi\rangle_\text{pmm}^{\mu_1} = p_{24}^{\dagger} p_{12}^{\dagger} |0\rangle = -|\phi\rangle_\text{pmm}^{\mu_1}$$  \hspace{2cm} (S75)

i.e., rotation and reflection eigenvalues $-1$ at each 0D block $\mu_1$ can be trivialized by the atomic insulator $|\phi\rangle_\text{pmm}^{\mu_1}$. Near $\mu_3$, there are two holes that form another atomic insulator:

$$|\phi\rangle_\text{pmm}^{\mu_3} = h_{12}^{\dagger} h_{21}^{\dagger} |0\rangle$$  \hspace{2cm} (S76)

with rotation and reflection properties as:

$$R_{\mu_3} |\phi\rangle_\text{pmm}^{\mu_3} = h_{24}^{\dagger} h_{14}^{\dagger} |0\rangle = -|\phi\rangle_\text{pmm}^{\mu_3}$$

$$M_{\tau_4} |\phi\rangle_\text{pmm}^{\mu_3} = h_{24}^{\dagger} h_{14}^{\dagger} |0\rangle = -|\phi\rangle_\text{pmm}^{\mu_3}$$  \hspace{2cm} (S77)

i.e., rotation and reflection eigenvalues $-1$ at each 0D block $\mu_3$ can be trivialized by the atomic insulator $|\phi\rangle_\text{pmm}^{\mu_3}$. Therefore, according to aforementioned 1D bubble construction on $\tau_1$, the reflection and rotation eigenvalues on 0D blocks $\mu_1$ and $\mu_3$ are not independent (can be changed simultaneously). Similar 1D bubble construction can be held for all 1D blocks $\tau_j$, $j = 1, 2, 3, 4$, which leads to the nonindependence of rotation and reflection eigenvalues of all 0D blocks $\mu_j$.

Subsequently we consider the complex fermion sector. Repeatedly consider the aforementioned 1D bubble construction on $\tau_1$: it adds two complex fermions at each 0D block $\mu_1$ and removes two complex fermions at each 0D block $\mu_3$ (by adding two holes), hence the number of complex fermions at $\mu_1$ and $\mu_3$ are not independent. More specifically, suppose there are $a$ complex fermions at each $\mu_1$, $b$ complex fermions at each $\mu_2$ and there summation $c = a + b$. Take above manipulation $n$ times ($n \in \mathbb{Z}$), the number of complex fermions on each $\mu_1/\mu_3$ is $a + 2n/b - 2n$, and their summation remains invariant. So for a specific $c$, there are only two different cases: $c = a + b$ and $c = (a+1) + (b-1)$. Similar arguments can be held for all 1D blocks $\tau_j$, $j = 1, 2, 3, 4$, which leads to the nonindependence of the number of complex fermions at all 0D blocks $\mu_j$.

With the help of above discussions, we consider the 0D block-state decorations. The 0D block-state decorated on $\mu_1$ can be labeled by $(n_1, \pm, \pm)$, where $n_1 \in \mathbb{Z}$ represents the number of complex fermions decorated on $\mu_1$ and two $\pm$’s represent the eigenvalues of 2-fold rotational symmetry and reflection symmetry, respectively. Then we consider the trivial state labeled by:

$$[(0, +, +), (0, +, +), (0, +, +), (0, +, +)]$$

Take aforementioned 1D bubble constructions on $\tau_j$ by $n_j \in \mathbb{Z}$ times, and it will lead to a new 0D block-state labeled by:

$$[(2(n_1 + n_2), (−1)^{n_1+n_2}, (−1)^{n_1}), (2(n_2 + n_3), (−1)^{n_2+n_3}, (−1)^{n_2}), (2(n_1 + n_4), (−1)^{n_1+n_4}, (−1)^{n_1}), (−2(n_3 + n_4), (−1)^{n_3+n_4}, (−1)^{n_4})]  \hspace{2cm} (S78)$$

And this state should be trivial. First, we consider the sector of complex fermion decoration. Alternatively, all 0D block-states can be viewed as vectors of a 12-dimensional vector space $V$, where the complex fermion components are $\mathbb{Z}$-valued and all other components are $\mathbb{Z}_2$-valued. Then all trivial 0D block-states with the form as Eq. (S78) can be viewed as a vector subspace $V'$ of $V$. It is easy to see that there are only four independent quantities in Eq. (S78): $n_j, j = 1, 2, 3, 4$. So the dimensionality of the vector subspace $V'$ should be 4. For the complex fermion sector, we have the following relation:

$$\begin{align*}
-2(n_1 + n_2) & - 2(n_2 + n_3) - 2(n_1 + n_4) \\
& = -2(n_3 + n_4)
\end{align*}  \hspace{2cm} (S79)$$

i.e., there are only three independent quantities which serves a $(2\mathbb{Z})^3$ trivialization. The remaining one degree of freedom of the vector subspace $V'$ should be attributed to the eigenvalues of point group symmetry actions, and serves a $\mathbb{Z}_2$ trivialization. Therefore, the ultimate classification of crystalline topological phases protected by $\text{pmm}$ symmetry for 2D systems with spinless fermions is:

$$\mathcal{G}_{\text{pmm}, 0}^{U(1)} = \mathbb{Z}^4 \times \mathbb{Z}_2^4 / (2\mathbb{Z})^3 \times \mathbb{Z}_2 = \mathbb{Z} \times \mathbb{Z}_2^7  \hspace{2cm} (S80)$$

For systems with spin-1/2 fermions, the rotation and reflection properties of $|\phi\rangle_\text{pmm}^{\mu_1}$ and $|\phi\rangle_\text{pmm}^{\mu_3}$ at each 0D blocks $\mu_1$ and $\mu_3$ are changed by $-1$, respectively, which leads to no trivialization. Furthermore, like the $p4m$ case, the classification data of the corresponding 0D block-states can be characterized by different 1D irreducible representations of the full symmetry group:

$$\mathcal{H}^1 \left[ U_f(1) \ltimes \omega_2 (\mathbb{Z}_2 \times \mathbb{Z}_2) = 2\mathbb{Z} \times \mathbb{Z}_2^2 \right]  \hspace{2cm} (S81)$$

we should notice that for systems with spin-1/2 fermions, we can only decorate even number of complex fermions on each 0D block. Now we repeatedly consider the complex fermion decorations on 0D blocks $\mu_1$ and $\mu_3$: for systems with spin-1/2 fermions, $c = (a + 1) + (b - 1)$ case is not valid because of the aforementioned obstruction for complex fermion decorations. Therefore, the ultimate classification of crystalline topological phases protected by $\text{pmm}$ symmetry for 2D systems with spin-1/2 fermions is:

$$\mathcal{G}_{\text{pmm}, 1/2}^{U(1)} = 2\mathbb{Z} \times \mathbb{Z}_2^8  \hspace{2cm} (S82)$$
Subsequently we investigate the 1D block-state decoration. For 1D blocks labeled by $\tau_1$ and $\tau_2$, the total symmetry group is $\mathbb{Z}_2^4 \times \mathbb{Z}_2$, hence the candidate 1D block-states are Majorana chain and 1D FSPT state (double Majorana chains); for 1D blocks labeled by $\tau_3$ and $\tau_4$, the only possible 1D block-state is Majorana chain because of the absence of on-site symmetry. So all 1D block-states form a group:

$$\{\text{BS}\}_{1\text{D},0}^{\text{pmg},0} = \mathbb{Z}_2^6$$  \hspace{1cm} (S85)

Then we discuss the decorations on $\tau_1/\tau_2$ and $\tau_3/\tau_4$ separately due to the different on-site symmetry on the corresponding 1D blocks.

a. Decorations on $\tau_3$ and $\tau_4$  Majorana chain decorations on 1D blocks $\tau_3$ will leave two dangling Majorana fermions at each 0D block labeled by $\mu_2/\mu_4$. Near each 0D block $\mu_4$, the nearby two Majorana fermions violate the no-open-edge condition: the fermion parity $P_f = i\gamma_1\gamma_2$ is not invariant under the 2-fold rotation $R_{\mu_4}$:

$$R_{\mu_4} : i\gamma_1\gamma_2 \mapsto -i\gamma_1\gamma_2$$  \hspace{1cm} (S86)

Similar for 1D blocks $\tau_4$. Therefore, there is no obstruction-free block-state on 1D blocks $\tau_3$ and $\tau_4$.

b. Decorations on $\tau_1$ and $\tau_2$  From Fig. S7 we can see that different 1D blocks $\tau_1/\tau_2$ are not connected, hence if we want to get an obstruction-free block-state, we should consider $\tau_1$ and $\tau_2$ together and decorate identical block-states on them. First, we consider the Majorana chain decoration, it leaves two dangling Majorana fermions at each 0D block labeled by $\mu_1$ and $\mu_2$. Near each $\mu_1$, these two Majorana fermions can be gapped out by an entanglement pair $i\gamma_1\gamma_2$ without breaking any symmetry, so do $\mu_2$. So Majorana chain decoration satisfy the no-open-edge condition, and contribute an obstruction-free block-state. Subsequently we consider the 1D FSPT state decoration which leaves four dangling Majorana fermions at each 0D block labeled by $\mu_1$ and $\mu_2$. Similar with the pm case, these four Majorana fermions can be gapped out in a symmetric way, and the no-open-edge condition is fulfilled. Overall, the obstruction-free 1D block-states form the following group:

$$\{\text{OFBS}\}_{1\text{D},0}^{\text{pmg},0} = \mathbb{Z}_2^6$$  \hspace{1cm} (S87)

and the group elements can be labeled by $n_1 = n_2$ and $n'_1 = n'_2$. Here $n_1/n_2$ represents the number of decorated Majorana chains on $\tau_1/\tau_2$, and $n'_1/n'_2$ represents the number of decorated 1D FSPT states on $\tau_1/\tau_2$. According to aforementioned discussions, a necessary condition of an obstruction-free block-state is $n_1 = n_2$ and $n'_1 = n'_2$.

So far we have already obtained all obstruction-free block-states, and they form the following group:

$$\{\text{OFBS}\}_{cmm,0} = \{\text{OFBS}\}_{cmm,0}^{1\text{D}} \times \{\text{OFBS}\}_{cmm,0}^{0\text{D}} = \mathbb{Z}_2^2 \times \mathbb{Z}_2^8 = \mathbb{Z}_2^{10}$$  \hspace{1cm} (S88)
and the different group elements (i.e., different obstruction-free block-states) can be labeled by:

\[ [(\pm, \pm), (\pm, \pm), (\pm, \pm), (\pm, \pm); n_1 = n_2; n_1' = n_2'] \] (S89)

here the first four brackets represent the 0D block-states at \( \mu_j \) (\( j = 1, 2, 3, 4 \)), and the last two quantities represent the number of Majorana chains and 1D FSPT states at \( \tau_1/\tau_2 \).

With all obstruction-free block-states, subsequently we discuss about all possible trivializations. First, we consider about the 2D bubble equivalences: as we discussed in the main text, only type-II (i.e., “Majorana bubbles” with anti-PBC) 2D bubble equivalence is valid because there is no 0D block as the center of even-fold dihedral group symmetry. Enlarge all “Majorana bubble” construction can be deformed to double Majorana chains at each \( \tau_3 \) and \( \tau_4 \) that can be trivialized because there is no on-site symmetry and the classification of 1D invertible topological phases (i.e., Majorana chain) is \( \mathbb{Z}_2 \); near each 1D block labeled by \( \tau_1 \) and \( \tau_2 \), “Majorana bubble” construction can also be deformed to double Majorana chains. Nevertheless, these double Majorana chains cannot be trivialized because there is an on-site \( \mathbb{Z}_2 \) symmetry on each \( \tau_1/\tau_2 \) by reflection symmetry acting internally, and this \( \mathbb{Z}_2 \) action exchanges these two Majorana chains, and this is exactly the definition of the nontrivial 1D FSPT phase protected by on-site \( \mathbb{Z}_2 \) symmetry. Furthermore, similar with the \( p4m \) case, there is no effect on 0D blocks labeled by \( \mu_3 \) and \( \mu_4 \) by taking 2D “Majorana” bubble equivalence, because the alternative Majorana chain surrounding each \( \mu_1/\mu_2 \) is not compatible with the reflection operations; nevertheless, similar with the \( p2 \) case, 2D “Majorana bubble” construction changes the fermion parity of each 0D block labeled by \( \mu_3/\mu_4 \) because there is no reflection operation on 0D block \( \mu_3/\mu_4 \), and the alternative Majorana chain surrounding each \( \mu_3/\mu_4 \) is compatible with all other symmetry operations.

Subsequently we consider the 1D bubble equivalences. For instance, we decorate a 1D bubble on each 1D block labeled by \( \tau_4 \) (here both yellow and red dots represent the complex fermions). Near each 0D block \( \mu_3 \), there are two complex fermions forming the following atomic insulator:

\[
|\psi\rangle^\mu_3_{\text{pmg}} = c_1^\dagger c_2 |0\rangle
\] (S90)

with rotation property as:

\[
R_{\mu_3} |\psi\rangle^\mu_3_{\text{pmg}} = c_2^\dagger c_1 |0\rangle = -|\psi\rangle^\mu_3_{\text{pmg}}
\] (S91)

i.e., the rotation eigenvalue \(-1\) at each 0D block \( \mu_3 \) can be trivialized by the atomic insulator \( |\psi\rangle^\mu_3_{\text{pmg}} \). Near \( \mu_1 \), there are another two complex fermions forming another atomic insulator:

\[
|\psi\rangle^\mu_1_{\text{pmg}} = c_1^\dagger c_2^\dagger |0\rangle
\] (S92)

with reflection symmetry as:

\[
M_{\tau_1} |\psi\rangle^\mu_1_{\text{pmg}} = c_2^\dagger c_1^\dagger |0\rangle = -|\psi\rangle^\mu_1_{\text{pmg}}
\] (S93)

i.e., the reflection eigenvalue \(-1\) at \( \mu_1 \) can be trivialized by the atomic insulator \( |\psi\rangle^\mu_1_{\text{pmg}} \). Therefore, 1D bubble construction on \( \tau_4 \) leads to the nonindependence of rotation eigenvalues of \( \mu_3 \) and reflection eigenvalues of \( \mu_1 \). Similar 1D bubble construction can be held on \( \tau_1 \) and \( \tau_2 \): it will change the fermion parities of 0D blocks \( \mu_1 \) and \( \mu_2 \) simultaneously by adding a complex fermion on each of them.

There is another type of 1D bubble construction (we denote above type of 1D bubble construction by “type-I”, this type of 1D bubble construction by “type-II”): we decorate a 1D bubble on each 1D block labeled by \( \tau_1 \) (here both yellow and red dots represent the 0D FSPT modes characterized by eigenvalue \(-1\) of reflection symmetry acting internally). This 1D bubble construction changes the reflection eigenvalues of 0D blocks \( \mu_1 \) and \( \mu_2 \) simultaneously.

With all possible 2D and 1D bubble constructions, we are ready to study the trivial block-states. Start from the original trivial state (nothing is decorated on arbitrary blocks):

\[
[(+, +), (+, +), (+, +), (+, +)]
\]

If we take 2D bubble construction \( n_0 \) times, take 1D bubble constructions (complex fermions) on \( \tau_3/\tau_4 \) by \( n_3/n_4 \) times, and take type-I 1D bubble constructions (0D \( \mathbb{Z}_2 \) FSPT modes) on \( \tau_1/\tau_2 \) by \( n_1/n_2 \) times and type-II 1D bubble constructions on \( \tau_1/\tau_2 \) by \( n_1'/n_2' \) times, above trivial state will be deformed to a new 0D block-state labeled by:

\[
\left[((-1)^{n_1'+n_2'}, (-1)^{n_1+n_2+n_4}), \right.
\left((-1)^{n_1'+n_2}, (-1)^{n_1+n_2+n_3}\right]
\]

According to the definition of bubble equivalence, all these 0D block-states should be trivial. Alternatively, all 0D block-states can be viewed as vectors of an 8-dimensional \( \mathbb{Z}_2 \)-valued vector space, and all trivial 0D block-states with the form as Eq. (S94) can be viewed as a vector of the subspace of aforementioned vector space. The dimensionality of this subspace can be determined by calculating the rank of the following transformation matrix:

\[
\begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
\text{rank} = 5 \] (S95)

Here different rows of this matrix represent different bubble constructions. Therefore, all trivial block-states form
the following group:
\[ \{\text{TBS}\}_{\text{pmg},0} = \{\text{TBS}\}_{\text{pmg},0}^{1D} \times \{\text{TBS}\}_{\text{pmg},0}^{0D} \]
\[ = \mathbb{Z}_2 \times \mathbb{Z}_2^5 = \mathbb{Z}_2^6 \]  
(S96)

here \( \{\text{TBS}\}_{\text{pmg},0}^{1D} \) represents the group of trivial states with non-vacuum 1D blocks (i.e., 1D FSPT phase decorations on \( \tau_1 \) and \( \tau_2 \) simultaneously), and \( \{\text{TBS}\}_{\text{pmg},0}^{0D} \) represents the group of trivial states with non-vacuum 0D blocks.

Therefore, all independent nontrivial block-states are labeled by the group elements of the following quotient group:
\[ G_{\text{cmm},0} = \{\text{OFBS}\}_{\text{cmm},0}/\{\text{TBS}\}_{\text{cmm},0} \]
\[ = \mathbb{Z}_2^{10}/\mathbb{Z}_2^5 = \mathbb{Z}_2^5 \]  
(S97)

here one \( \mathbb{Z}_2 \) is from the Majorana chain decorations on 1D blocks \( \tau_1 \) and \( \tau_2 \) simultaneously, and all other \( \mathbb{Z}_2 \)'s are from the nontrivial 0D block-states. Similar with the \( pm \) case, there is no nontrivial group extension because of the absence of nontrivial 1D block-state, and the group structure of \( E_{\text{cmm},0} \) has already been accurate.

b. Spin-1/2 fermions

Now we turn to discuss about the systems with spin-1/2 fermions. For arbitrary 0D blocks, the on-site symmetry should be \( \mathbb{Z}_4^f \) by nontrivial \( \mathbb{Z}_4^f \) (fermion parity) extension of on-site \( \mathbb{Z}_4^f \) symmetry. And different 0D block-states at a certain \( \mu_j \) (\( j = 1, 2, 3, 4 \)) can be characterized by different 1D irreducible representations of \( \mathbb{Z}_4^f \) that can be labeled by:
\[ \mathcal{H}_4^{1D} \left[ \mathbb{Z}_4^f, U(1) \right] = \mathbb{Z}_4 = \{1, i, -1, -i\} \]  
(S98)

So at each 0D block, the block-state can be labeled by an index \( \nu \in \{1, i, -1, -i\} \). According to this notation, the obstruction-free 0D block-states form the following group:
\[ \{\text{OFBS}\}_{\text{pmg},1/2}^{0D} = \mathbb{Z}_4^4 \]  
(S99)

and the group elements can be labeled by (four indices represent the block-states at \( \mu_j, j = 1, 2, 3, 4 \)):
\[ \nu_1, \nu_2, \nu_3, \nu_4 \]

Then we consider possible trivializations. For example, we decorate a 1D bubble on each 1D block labeled by \( \tau_1 \), here the yellow and red dots represent the 0D FSPT mode characterized by \( i \in \mathbb{Z}_4 \) and \( -i \in \mathbb{Z}_4 \), respectively, and they can be trivialized by shrinking them to a point. According to this bubble construction, the eigenvalue of \( \mathbb{Z}_4^f \) at each 0D block labeled by \( \mu_1/\mu_2 \) is changed by \( i/-i \). 1D bubble construction on \( \tau_2 \) is similar. And for 1D blocks \( \tau_3 \) and \( \tau_4 \), the unique possible 1D bubble construction is “complex fermion” bubble (i.e., both yellow and red dots in 1D bubble represent the complex fermions). Near each 0D block \( \mu_3 \), it leaves 2 dangling complex fermions forming the following atomic insulator:
\[ |\phi\rangle_{\text{pmg}}^{\mu_3} = a_1^\dagger a_1^\dagger |0\rangle \]  
(S100)

with following rotational symmetry property:
\[ R_{\mu_3} |\phi\rangle_{\text{pmg}}^{\mu_3} = -a_1^\dagger a_1^\dagger |0\rangle = |\phi\rangle_{\text{pmg}} \]  
(S101)
i.e., 1D bubble constructions on \( \tau_3 \) and \( \tau_4 \) lead to no trivialization.

With all possible bubble constructions, we are ready to study the trivial states. Start from the original trivial state (nothing is decorated on arbitrary 0D block):
\[ [1, 1, 1, 1] \]

if we take 1D bubble construction on \( \tau_1 \) and \( \tau_2 \) by \( n_1 \) and \( n_2 \) times, above trivial state will be deformed to a new 0D block-state labeled by:
\[ [i^{n_1+n_2}, (-i)^{n_1+n_2}, 1, 1] \]  
(S102)

According to the definition of bubble equivalence, all these states should be trivial. It is easy to see that there is only one independent quantity in the state (S102), hence all these trivial states form the following group:
\[ \{\text{TBS}\}_{\text{pmg},1/2}^{0D} = \mathbb{Z}_4 \]  
(S103)

Therefore, all independent nontrivial 0D block-states are labeled by different group elements of the following quotient group:
\[ E_{\text{pmg},1/2}^{0D} = \{\text{OFBS}\}_{\text{pmg},1/2}^{0D}/\{\text{TBS}\}_{\text{pmg},1/2}^{0D} = \mathbb{Z}_4^3 \]  
(S104)

Then we consider the 1D block-state decorations. There is no nontrivial candidate block-state on 1D blocks labeled by \( \tau_1 \) and \( \tau_2 \) because of the trivial classification of 1D FSPT phases with \( \mathbb{Z}_4^f \) symmetry, and the unique possible block-state for 1D blocks labeled by \( \tau_3 \) and \( \tau_4 \) is Majorana chain because of the absence of on-site symmetry. So as an example, we consider the Majorana chain decoration on \( \tau_3 \): it leaves two dangling Majorana fermions at each 0D block \( \mu_2/\mu_3 \), and can be gapped out by an entanglement pair in a symmetric way: near \( \mu_2 \), the entanglement pair \( i\gamma_1\gamma_2 \) has the following reflection symmetry property:
\[ M_{\tau_1} : i\gamma_1\gamma_2 \leftrightarrow -i\gamma_2\gamma_1 = i\gamma_1\gamma_2 \]  
(S105)

similar for another two Majorana fermions near \( \mu_4 \). Hence the Majorana chain decorations on 1D blocks \( \tau_3 \) and \( \tau_4 \) are obstruction free, and forming the group containing all obstruction-free 1D block-states:
\[ \{\text{OFBS}\}_{\text{pmg},1/2}^{1D} = \mathbb{Z}_2^2 \]  
(S106)
And it is obvious that there is no trivialization (i.e., \( \{\text{TBS}\}^{1D}_{\text{pmg}, 1/2} = \mathbb{Z}_1 \)). As a consequence, the classification attributed to 1D block-state decorations is:

\[
E^{1D}_{\text{pmg}, 1/2} = \{\text{OFBS}\}^{1D}_{\text{pmg}, 1/2}/\{\text{TBS}\}^{1D}_{\text{pmg}, 1/2} = \mathbb{Z}^2
\]  
(S107)

With all classification data, we consider the group structure of the corresponding classification. Similar to the \( p2 \) case, the Majorana chain decorations on \( \tau_3 \) and \( \tau_4 \) have nontrivial extension with 0D block-state decorations on \( \mu_4 \) and \( \mu_3 \), and the ultimate classification with accurate group structure is:

\[
\mathcal{G}^{1/2} = E^{1D}_{\text{pmg}, 1/2} \times \omega_2 E^{0D}_{\text{pmg}, 1/2} = \mathbb{Z}_4 \times \mathbb{Z}_8
\]  
(S108)

here the symbol \( \times \omega_2 \) means that independent nontrivial 1D and 0D block-states \( E^{1D}_{\text{pmg}, 1/2} \) and \( E^{0D}_{\text{pmg}, 1/2} \) have nontrivial extension, and described by an nontrivial factor system of the following short exact sequence in mathematical language:

\[
0 \rightarrow E^{1D}_{\text{pmg}, 1/2} \rightarrow \mathcal{G}^{1/2} \rightarrow E^{1D}_{\text{pmg}, 1/2} \rightarrow 0
\]  
(S109)

c. With \( U(1) \) charge conservation

Then we consider the systems with \( U(1) \) charge conservation. For a 0D block labeled by \( \mu_1 \) or \( \mu_2 \), different 0D block-states are characterized by different irreducible representations of symmetry group as:

\[
\mathcal{H}^{1}[U(1) \times \mathbb{Z}_2, U(1)] = \mathbb{Z} \times \mathbb{Z}_2
\]  
(S110)

Here \( \mathbb{Z} \) represents the complex fermion, and \( \mathbb{Z}_2 \) represents the eigenvalues of reflection symmetry operation. For a 0D block labeled by \( \mu_3 \) and \( \mu_4 \), different 0D block-states are also characterized by different irreducible representations of symmetry group as:

\[
\mathcal{H}^{1}[U(1) \times \mathbb{Z}_2, U(1)] = \mathbb{Z} \times \mathbb{Z}_2
\]  
(S111)

Here \( \mathbb{Z} \) represents the complex fermion, and \( \mathbb{Z}_2 \) represents the eigenvalues of rotational symmetry operation. We should further investigate the possible trivializations. For systems with spinless fermions, consider the 1D bubble equivalence on 1D blocks labeled by \( \tau_3 \) [cf. 1D bubble, here yellow and red dots represent particle and hole, respectively, and they can be trivialized if we shrink them to a point]: it adds one complex fermion at each 0D block \( \mu_3 \) and removes one complex fermion at each 0D block \( \mu_4 \) (by adding one hole), hence the number of complex fermions at \( \mu_3 \) and \( \mu_4 \) are not independent. More specifically, consider there are complex fermions at each \( \mu_3 \) and \( \mu_4 \) complex fermions on each \( \mu_2 \), and suppose \( a + b = c \). Take above manipulation \( n \) times \((n \in \mathbb{Z})\), the number of complex fermions on each \( \mu_3 \) is \( a + n/b - n \), and their summation remains invariant. So for a specific \( c \), all possible cases are equivalent. Then consider 1D bubble equivalence on 1D blocks \( \tau_4 \) [cf. 1D bubble, here yellow and red dots represent particle and hole, respectively, and they can be trivialized if we shrink them to a point]: it adds two complex fermions at each 0D block \( \mu_2 \) and removes two complex fermions at each 0D block \( \mu_4 \) (by adding two holes), hence the number of complex fermions at \( \mu_2 \) and \( \mu_4 \) are not independent. More specifically, consider there are complex fermions at each \( \mu_2 \) and \( \mu_4 \) complex fermions on each \( \mu_3 \), and suppose \( a + b = c \). Take above manipulation \( n \) times \((n \in \mathbb{Z})\), the number of complex fermions on each \( \mu_3 \) is \( a + 2n/b - 2n \), and their summation remains invariant. So for a specific \( c \), there are only two independent cases: \( c = a + b \) and \( c = (a + 1) + (b - 1) \).

With the help of above discussions, we consider the 0D block-state decorations. The 0D block-state decorated on \( \mu_1/\mu_3 \) can be labeled by \( (n_1/n_3, \pm) \), where \( n_1/n_3 \in \mathbb{Z} \) represents the number of complex fermions decorated on \( \mu_1/\mu_3 \) and \( \pm \) represents the eigenvalues of reflection/2-fold rotational symmetry on \( \mu_1/\mu_3 \). Then we consider by \( \mu_4 \), there are two fermionic holes that form another atomic insulator:

\[
|\phi\rangle^{\mu_4}_{\text{pmg}} = h^\dagger_1 h^\dagger_2 |0\rangle
\]  
(S114)

with the rotation property as:

\[
R_{\mu_4} |\phi\rangle^{\mu_4}_{\text{pmg}} = h^\dagger_2 h^\dagger_1 |0\rangle = -|\phi\rangle^{\mu_4}_{\text{pmg}}
\]  
(S115)

i.e., the rotation eigenvalue \( -1 \) can be trivialized by atomic insulator \( |\phi\rangle^{\mu_4}_{\text{pmg}} \). Hence, this 1D bubble construction can change the reflection eigenvalue of \( \mu_2 \) and rotation eigenvalue of \( \mu_4 \) simultaneously. Similar for 1D blocks labeled by \( \tau_4 \), and 1D bubble construction can change the reflection eigenvalue of \( \mu_1 \) and rotation eigenvalue of \( \mu_3 \) simultaneously. We further consider the 1D bubble equivalence on 1D blocks labeled by \( \tau_1 \) [cf. 1D bubble, here both yellow and red dots represent the 0D FSPT mode characterized by eigenvalue \( -1 \) of reflection symmetry operation, and they can be trivialized if we shrink them to a point]: this bubble construction can change the reflection eigenvalue of 0D blocks \( \mu_1 \) and \( \mu_2 \) simultaneously. Summarize all above trivializations, we know that rotation/reflection eigenvalues at 0D blocks \( \mu_j \), \( j = 1, 2, 3, 4 \) are not independent.

Subsequently consider the complex fermion sector: consider 1D bubble equivalence on 1D blocks \( \tau_3 \) [cf. 1D bubble, here yellow and red dots represent particle and hole, respectively, and they can be trivialized if we shrink them to a point]: it adds one complex fermion at each 0D block \( \mu_3 \) and removes one complex fermion at each 0D block \( \mu_4 \) (by adding one hole), hence the number of complex fermions at \( \mu_3 \) and \( \mu_4 \) are not independent. More specifically, consider there are complex fermions at each \( \mu_3 \) and \( b \) complex fermions on each \( \mu_2 \), and suppose \( a + b = c \). Take above manipulation \( n \) times \((n \in \mathbb{Z})\), the number of complex fermions on each \( \mu_3 \) is \( a + n/b - n \), and their summation remains invariant. So for a specific \( c \), all possible cases are equivalent. Then consider 1D bubble equivalence on 1D blocks \( \tau_4 \) [cf. 1D bubble, here yellow and red dots represent particle and hole, respectively, and they can be trivialized if we shrink them to a point]: it adds two complex fermions at each 0D block \( \mu_2 \) and removes two complex fermions at each 0D block \( \mu_4 \) (by adding two holes), hence the number of complex fermions at \( \mu_2 \) and \( \mu_4 \) are not independent. More specifically, consider there are a complex fermions at each \( \mu_2 \) and \( b \) complex fermions on each \( \mu_3 \), and suppose \( a + b = c \). Take above manipulation \( n \) times \((n \in \mathbb{Z})\), the number of complex fermions on each \( \mu_3 \) is \( a + 2n/b - 2n \), and their summation remains invariant. So for a specific \( c \), there are only two independent cases: \( c = a + b \) and \( c = (a + 1) + (b - 1) \).
the trivial state labeled by:

$$[(0,+),(0,+),(0,+),(0,+)]$$  \(\text{(S16)}\)

We should note that there are two different types of 1D bubble constructions on \(\tau_1\) and \(\tau_2\): particle-hole construction and 0D FSPT mode construction. Take 0D FSPT mode/particle-hole construction on \(\tau_1\) and \(\tau_2\) by \(n_{1,j}/n_{2,j}\) \((j=1,2)\) times, and particle-hole construction on \(\tau_3\) and \(\tau_4\) by \(n_3\) and \(n_4\) times, and it will lead to a new 0D block-state labeled by:

$$[(n_{1,1} + n_{2,2} + 2n_4, (-1)^{n_1,n_2,n_4})$$
$$(-n_{1,2} - n_{2,2} + 2n_3, (-1)^{n_1,n_2,n_3})$$
$$(-2n_4, (-1)^{n_1})], \ (-2n_3, (-1)^{n_1})]$$  \(\text{(S17)}\)

And this state should be trivial. Alternatively, all 0D block-states can be viewed as vectors of an 8-dimensional vector space \(\mathcal{V}\), where the complex fermion components are \(\mathbb{Z}_2\)-valued and all other components are \(\mathbb{Z}_2\)-valued. Then all trivial 0D block-states with the form as Eq. (S117) can be viewed as a vector subspace \(\mathcal{V}'\) of \(\mathcal{V}\).

We note that \(n_{1,1}/n_{2,2}\) should appears together with \(n_{1,1}/n_{1,2}\), hence they are not independent. As a consequence, there are only four independent quantities in Eq. (S117): \(n_{1,1}, n_{1,2}, n_3\) and \(n_4\). So the dimensionality of the vector subspace \(\mathcal{V}'\) should be 4. For the complex fermion sector, we have the following relation:

$$- (n_{1,2} + n_{2,2} + 2n_4) - (-n_{1,2} - n_{2,2} + 2n_3)$$
$$- (-2n_4) = -2n_3$$  \(\text{(S18)}\)

i.e., there are only three independent quantities which serves a \(\mathbb{Z} \times (2\mathbb{Z})^2\) trivialization. The remaining one degree of freedom of the vector subspace \(\mathcal{V}'\) should be attributed to the eigenvalues of point group symmetry actions, and serves a \(\mathbb{Z}_2\) trivialization. Therefore, all trivial states with the form as shown in Eq. (S117) compose the following group:

$$\{\text{TBS}\}^{U(1)}_{p4m,0} = \mathbb{Z} \times (2\mathbb{Z})^2 \times \mathbb{Z}_2$$  \(\text{(S19)}\)

Finally, the ultimate classification of crystalline topological phases protected by \(\text{pmg}\) symmetry for 2D systems with spinless fermions is:

$$\mathcal{G}^{U(1)}_{\text{pmg},0} = \mathbb{Z}^4 \times \mathbb{Z}_2^4 / \mathbb{Z} \times (2\mathbb{Z})^2 \times \mathbb{Z}_2 = \mathbb{Z} \times \mathbb{Z}_2^5$$  \(\text{(S20)}\)

For systems with spin-1/2 fermions, the reflection property of \(|\phi\rangle^{\mu_2}_{\text{pmg}}\) and rotation property of \(|\phi\rangle^{\mu_2}_{\text{pmg}}\) are changed by an additional \(-1\), which leads to no trivialization. 1D bubble equivalence on 1D blocks \(\tau_1\) is identical with spinless fermions which leads to the nonindependence of reflection eigenvalues of 0D blocks \(\mu_1\) and \(\mu_2\).

It is straightforward to see that the complex fermion sector is identical with the spinless case, and the 1D bubble construction can only serve a \(\mathbb{Z}_2\) trivialization on reflection/rotation eigenvalues because there are only one independent integer \(n_{1,1} + n_{2,1}\). Therefore, the ultimate classification of crystalline topological phases protected by \(\text{pmg}\) symmetry for 2D systems with spin-1/2 fermions is:

$$\mathcal{G}^{U(1)}_{\text{pmg},1/2} = \mathbb{Z}^4 \times \mathbb{Z}_2^4 / \mathbb{Z} \times (2\mathbb{Z})^2 \times \mathbb{Z}_2 = \mathbb{Z} \times \mathbb{Z}_2^5$$  \(\text{(S21)}\)

7. \(p4\)

The corresponding point group of this case is 4-fold rotational group \(C_4\). For 2D blocks \(\sigma\), 1D blocks \(\tau_1\) and \(\tau_2\), there is no on-site symmetry; for 0D blocks \(\mu_1\) and \(\mu_3\), the on-site symmetry group is \(\mathbb{Z}_4\) via the 4-fold rotational symmetry group \(C_4\) acting internally, and for 0D blocks \(\mu_2\), the on-site symmetry is \(\mathbb{Z}_2\) via the 2-fold rotational symmetry group \(C_2 \subset C_4\) acting internally, see Fig. S8.

a. Spinless fermions

Firstly, we consider the 0D block-state decorations. For 0D blocks labeled by \(\mu_1\) and \(\mu_3\), the total symmetry
group of each of them is $Z_2^f \times Z_4$, and the classification data can be characterized by different 1D irreducible representations of the symmetry group $Z_2^f \times Z_4$:

$$\mathcal{H}^1 \left[ Z_2^f \times Z_4, U(1) \right] = Z_2 \times Z_4 \quad (S122)$$

Here $Z_2$ is from the fermion parity, and $Z_4$ is from the rotation eigenvalues. So the corresponding 0D block-state can be labeled by $(i_1/i_2, \pm)$, here $i_1/i_2 \in \{1, i, -1, -i\}$ represents the rotation eigenvalue, and $\pm$ represents the fermion parity. For 0D blocks labeled by $\mu_2$, the total symmetry group of each of them is $Z_2^f \times Z_2$, and the classification data can be characterized by different 1D irreducible representations of the symmetry group $Z_2^f \times Z_2$:

$$\mathcal{H}^1 \left[ Z_2^f \times Z_2, U(1) \right] = Z_2^2 \quad (S123)$$

One $Z_2$ is from the fermion parity, and the other is from the rotation eigenvalue $-1$, and at each 0D block, the block-state can be labeled by $(\pm, \pm)$, here these two $\pm$'s represent the fermion parity and rotation eigenvalue, respectively. According to this notation, the obstruction-free 0D block-states form the following group:

$$\{ \text{OFBS} \}_{p4,0}^{0D} = Z_2^4 \times Z_4^2 \quad (S124)$$

and the group elements can be labeled by (three brackets represent the block-states at $\mu_j$, $j = 1, 2, 3$):

$$[(\nu_1, \pm), (\pm, \pm), (\nu_3, \pm)]$$

Subsequently we investigate the 1D block-state deco-rotation. The unique possible 1D block-state is Majorana chain because of the absence of the on-site symmetry. On 1D blocks labeled by $\tau_1$, the 1D block-state deco-rotation leaves 4 dangling Majorana fermions on each $\mu_1$, with the following rotational symmetry properties:

$$R_{\mu_1} : \gamma_j \mapsto \gamma_{j+1}, \quad j = 1, 2, 3, 4. \quad (S125)$$

where $R_{\mu_1}$ is the generator of $C_4$ group: 4-fold rotation operation centred at each 0D block labeled by $\mu_1$, and all subscripts are taken with modulo 4. Consider the local fermion parity and its symmetry property:

$$P_f = -\prod_{j=1}^4 \gamma_j, \quad R_{\mu_1} : P_f \mapsto -P_f \quad (S126)$$

Hence these 4 Majorana fermions cannot be gapped out in a symmetric way, and the no-open-edge condition is violated. Similar for $\tau_2$, and there is no nontrivial 1D block-state:

$$\{ \text{BS} \}_{p4,0}^{1D} = Z_4 \quad (S127)$$

With all obstruction-free block-states, subsequently we discuss about all possible trivializations. First, we consider about the 2D bubble equivalences: as we discussed in the main text, only type-II (i.e., “Majorana bubbles” with anti-PBC) 2D bubble equivalence is valid because there is no 0D block as the center of even-fold dihedral group. Enlarge all “Majorana bubble” construction can be deformed to double Majorana chains at arbitrary 0D block that can be trivialized because there is no on-site symmetry and the classification of 1D invertible topological phases (i.e., Majorana chain) is $Z_2$. Furthermore, similar with the $p2$ case, 2D “Majorana bubble” construction changes the fermion parities of all 0D blocks simultaneously.

Subsequently we consider the 1D bubble equivalences. We study the role of rotational symmetry. Consider the 1D bubble equivalence on 1D blocks labeled by $\tau_1$ [cf. 1D bubble, here both yellow and red dots represent the complex fermions]: near $\mu_1$, there are 4 complex fermions which form an atomic insulator:

$$|\psi\rangle_{p4}^{\mu_1} = c_1^\dagger c_3^\dagger c_4^\dagger|0\rangle \quad (S128)$$

with rotation property as ($R_{\mu_1}$ represents the 4-fold rotation operation centred at the 0D block labeled by $\mu_1$):

$$R_{\mu_1}|\psi\rangle_{p4}^{\mu_1} = c_2^\dagger c_3^\dagger c_4^\dagger|0\rangle = -|\psi\rangle_{p4}^{\mu_1} \quad (S129)$$

i.e., the rotation eigenvalue $-1$ can be trivialized by the atomic insulator $|\psi\rangle_{p4}^{\mu_1}$. Near $\mu_2$, there are 2 complex fermions which form another atomic insulator:

$$|\psi\rangle_{p4}^{\mu_2} = c_1^\dagger c_2^\dagger|0\rangle \quad (S130)$$

with rotation property as ($R_{\mu_2}$ represents the 2-fold rotation operation centred at the 0D block labeled by $\mu_2$):

$$R_{\mu_2}|\psi\rangle_{p4}^{\mu_2} = c_2^\dagger c_1^\dagger|0\rangle = -|\psi\rangle_{p4}^{\mu_2} \quad (S131)$$

i.e., the rotation eigenvalue $-1$ can be trivialized by the atomic insulator $|\psi\rangle_{p4}^{\mu_2}$. Hence the rotation eigenvalues of $\mu_1$ and $\mu_2$ are not independent. 1D bubble equivalence on $\tau_2$ is similar, and therefore the rotation eigenvalues at $\mu_j$, $j = 1, 2, 3$ are not independent.

With all possible 2D and 1D bubble constructions, we are ready to study the trivial block-states. Start from the original trivial state (nothing is decorated on arbitrary blocks):

$$[(1, +), (+, +), (1, +)] \quad (S132)$$

If we take 2D bubble construction $n_0$ times, take 1D bubble equivalences on $\tau_1$ and $\tau_2$ by $n_1$ and $n_2$ times, above trivial state will be deformed to a new 0D block-state labeled by:

$$[((-1)^{n_1}, (-1)^{n_2}), ((-1)^{n_1+n_2}, (-1)^{n_0}), ((-1)^{n_2}, (-1)^{n_0})] \quad (S133)$$

According to the definition of bubble equivalence, all these 0D block-states should be trivial. It is straightforward to check that there are only three independent quantities $(n_j, j = 0, 1, 2)$ in Eq. (S133), hence all trivial block-states form the following group:

$$\{ \text{TBS} \}_{p4,0} = \{ \text{TBS} \}_{p4,0}^{0D} = Z_2^2 \quad (S134)$$
Therefore, all independent nontrivial block-states are labeled by the group elements of the following quotient group:

$$E_{p4,0} = \{\text{OFBS}\}_{p4,0}/\{\text{TBS}\}_{p4,0} = \mathbb{Z}_4 \times \mathbb{Z}_2^2 / \mathbb{Z}_2^3 = \mathbb{Z}_4 \times \mathbb{Z}_2^d$$  \hspace{1cm} (S135)

We should notice that the group structure should be $\mathbb{Z}_4 \times \mathbb{Z}_2^2$ rather than $\mathbb{Z}_8$, because rotation eigenvalue $i$ and $-i$ at each 0D block labeled by $\mu_1$ should be a nontrivial 0D block-state. There is no stacking between 1D and 0D block-states because there is no nontrivial 1D block-state, and the group structure of the classification data $E_{p4,0}$ has already been accurate.

b. Spin-1/2 fermions

First, we investigate the 0D block-state decorations. For 0D blocks labeled by $\mu_1$ and $\mu_3$, the total symmetry group is $\mathbb{Z}_4^I$: nontrivial $\mathbb{Z}_2^d$ extension of the on-site symmetry $\mathbb{Z}_4$. All different 0D block-states can be characterized by different 1D irreducible representations of the corresponding symmetry group:

$$\mathcal{H}^1 \left[ \mathbb{Z}_4^I , U(1) \right] = \mathbb{Z}_8$$  \hspace{1cm} (S136)

And there is no more trivialization. For the 0D blocks labeled by $\mu_2$, the total symmetry group is $\mathbb{Z}_4^I$ nontrivial $\mathbb{Z}_2^d$ extension of the on-site symmetry $\mathbb{Z}_2$. All different 0D block-states can be characterized by different 1D irreducible representations of the corresponding symmetry group:

$$\mathcal{H}^1 \left[ \mathbb{Z}_4^I , U(1) \right] = \mathbb{Z}_4$$  \hspace{1cm} (S137)

And there is no trivialization. As a consequence, the classification attributed to 0D block-state decorations is:

$$E_{p4,1/2}^{0D} = \{\text{OFBS}\}_{p4,1/2}^{0D} = \mathbb{Z}_8 \times \mathbb{Z}_2$$  \hspace{1cm} (S138)

Subsequently we investigate the 1D block-state decoration. The unique possible block-state is Majorana chain because of the absence of on-site symmetry. Majorana chain decoration on $\tau_1/\tau_3$ leaves 4 dangling Majorana fermions at each 0D block $\mu_1/\mu_3$, and 2 dangling Majorana fermions at $\mu_2$. The 4 Majorana fermions at $\mu_1/\mu_3$ have the following rotational symmetry properties:

$$R_{\mu_1/\mu_3} : (\gamma_1, \gamma_2, \gamma_3, \gamma_4) \rightarrow (\gamma_2, \gamma_3, \gamma_4, -\gamma_1)$$  \hspace{1cm} (S139)

Where $R_{\mu_1/\mu_3}$ is the generator of $C_4$ group: 4-fold rotation operation centred at each 0D block labeled by $\mu_1/\mu_3$. Then consider the local fermion parity and its symmetry property:

$$P_f = -\prod_{j=1}^{4} \gamma_j, \hspace{0.5cm} R_{\mu_1/\mu_3} : P_f \rightarrow P_f$$  \hspace{1cm} (S140)

Therefore, these 4 Majorana fermions can be gapped out by some proper interactions in a symmetric way. Next the 2 Majorana fermions at $\mu_2$ can be gapped out by an entanglement pair which respect all symmetry operations, identical with p2 case. Therefore the no-open-edge condition is satisfied, and finally the classification attributed to 1D block-state decorations is:

$$E_{p4,1/2}^{1D} = \{\text{OFBS}\}_{p4,1/2}^{1D} = \mathbb{Z}_2$$  \hspace{1cm} (S141)

With full classification data, we investigate the possible stacking between 1D and 0D blocks. If we decorate two Majorana chains on each 1D block labeled by $\tau_1$, it can be smoothly deformed to two copies of 0D state (88) at each 0D block labeled by $\mu_1$ and one at each 0D block labeled by $\mu_2$. Near each $\mu_1$, the deformed 0D block-state has the following rotational property [3]:

$$R_{\mu_1} |\phi\rangle_{0D} = e^{-i\pi/2}$$  \hspace{1cm} (S142)

Hence if an 0D block-state with eigenvalue $e^{i\pi q/4}$ under 4-fold rotation is attached to each 1D block state (single Majorana chain decoration) near each 0D block labeled by $\mu_1$, the rotation eigenvalue $r$ of the obtained 0D block-state becomes:

$$r = e^{-i\pi + i\pi q/2}$$  \hspace{1cm} (S143)

And $r = 0$ if $q = 2$. Therefore, there is an appropriate 1D block-state which itself form a $\mathbb{Z}_2$ structure under stacking, and there is no stacking between 1D and 0D block-states as a consequence. Near each $\mu_2$, similar with the p2 case, there is a stacking between 1D and 0D block-states. Therefore, the ultimate classification with accurate group structure is:

$$G_{p4}^{1/2} = E_{p4,1/2}^{1D} \times \mathbb{Z}_2 \times E_{p4,1/2}^{0D} = \mathbb{Z}_2 \times \mathbb{Z}_8^2$$  \hspace{1cm} (S144)

Here the symbol “\times” means that 1D and 0D block-states $E_{p4,1/2}^{1D}$ and $E_{p4,1/2}^{0D}$ have nontrivial extension, and described by a nontrivial factor system of the following short exact sequence in mathematical language:

$$0 \rightarrow E_{p4,1/2}^{1D} \rightarrow G_{p4}^{1/2} \rightarrow E_{p4,1/2}^{0D} \rightarrow 0$$  \hspace{1cm} (S145)

\text{c. With } U^I(1) \text{ charge conservation}

Then we consider the systems with $U^I(1)$ charge conservation. For each 0D block labeled by $\mu_1$ and $\mu_3$, different 0D block-states are characterized by different irreducible representations of symmetry group as:

$$\mathcal{H}^1 [U(1) \times \mathbb{Z}_4, U(1)] = \mathbb{Z} \times \mathbb{Z}_4$$  \hspace{1cm} (S146)

Here $\mathbb{Z}$ represents the complex fermion and $\mathbb{Z}_4$ represents the eigenvalues of 4-fold rotational symmetry operation. For each 0D block labeled by $\mu_2$, different 0D block-states
are also characterized by different irreducible representations of symmetry group as:

$$\mathcal{H}^T[U(1) \times \mathbb{Z}_2, U(1)] = \mathbb{Z} \times \mathbb{Z}_2 \quad \text{(S147)}$$

Here $\mathbb{Z}$ represents the complex fermion and $\mathbb{Z}_2$ represents the eigenvalues of 2-fold rotational symmetry operation. We should further consider possible trivializations. For systems with spinless fermions, consider the 1D bubble equivalence on 1D blocks, we decorate a 1D bubble on each 1D block labeled by $\tau_1$, here yellow and red dots represent the fermionic particle and hole, respectively, and can be trivialized if we shrink them to a point. Near each 0D block labeled by $\mu_1$, there are four particles that form an atomic insulator:

$$|\xi\rangle_{\mu_1}^0 = p_1^1 p_2^1 p_3^1 p_4^1 |0\rangle \quad \text{(S148)}$$

with rotation property as:

$$R_{\mu_1} |\xi\rangle_{\mu_1}^0 = p_2^1 p_3^1 p_4^1 p_1^1 |0\rangle = -|\xi\rangle_{\mu_1}^0 \quad \text{(S149)}$$

i.e., rotation eigenvalue $-1$ can be trivialized by the atomic insulator $|\xi\rangle_{\mu_1}^0$ at each 0D block labeled by $\mu_1$. Near each 0D block labeled by $\mu_2$, there are two holes that form another atomic insulator:

$$|\xi\rangle_{\mu_2}^{\mu_2} = h_1^1 h_2^1 |0\rangle \quad \text{(S150)}$$

with rotation property as:

$$R_{\mu_2} |\xi\rangle_{\mu_2}^{\mu_2} = h_2^1 h_1^1 |0\rangle = -|\xi\rangle_{\mu_2}^{\mu_2} \quad \text{(S151)}$$

i.e., rotation eigenvalue $-1$ can be trivialized by the atomic insulator $|\xi\rangle_{\mu_2}^{\mu_2}$ at each 0D block labeled by $\mu_2$. Thus the 1D bubble construction on $\tau_1$ can change the rotation eigenvalues of $\mu_1$ and $\mu_2$ simultaneously. Similar for 1D bubble equivalence on 1D blocks $\tau_2$ which can change the rotation eigenvalues of 0D blocks $\mu_2$ and $\mu_3$ simultaneously. Therefore, rotation eigenvalues of 0D blocks $\mu_j, j = 1, 2, 3$ are not independent.

Subsequently we consider the complex fermion sector. First of all, as shown in Fig. S8, we should identify that there is only one 0D block labeled by $\mu_1/\mu_3$ per unit cell, but there are two 0D blocks labeled by $\mu_2$ per unit cell. Then consider 1D bubble equivalence on $\tau_2$ (cf. 1D bubble, here yellow and red dots represent particle and hole, respectively, and they can be trivialized if we shrink them to a point): it adds four complex fermions at each 0D block $\mu_3$ and removes two complex fermions at each 0D block $\mu_2$ (by adding two holes), hence the number of complex fermions at $\mu_2$ and $\mu_3$ are not independent. Similar arguments can be held for the complex fermion decorations on 0D blocks $\mu_1$ and $\mu_2$.

With the help of above discussions, we consider the 0D block-state decorations. The 0D block-state decorated on $\mu_j, j = 1, 2, 3$ can be labeled by $(n_j, \phi_j)$, where $n_j \in \mathbb{Z}$ represents the number of complex fermions decorated on $\mu_j$ and $\phi_j$ represent the eigenvalues of rotational symmetry on $\mu_j$. Start from the following trivial state:

$$[(0, 1), (0, 1), (0, 1)] \quad \text{(S152)}$$

Take aforementioned 1D bubble construction on $\tau_j$ by $n_j$ times ($j = 1, 2$), and it will lead to the following 0D block-state:

$$[(4n_1, (-1)^n_1), (-2n_1 + 2n_2, (-1)^{n_1+n_2})$$

$$(-4n_2, (-1)^n_2)] \quad \text{(S153)}$$

And this state should be trivial. Alternatively, all 0D block-states can be viewed as vectors of an 6-dimensional vector space $V$, where the complex fermion components are $\mathbb{Z}$-valued, two components attributed to $C_4$ centers $\mu_1$ and $\mu_3$ are $\mathbb{Z}_4$-valued, and one component attributed to $C_2$ center $\mu_2$ is $\mathbb{Z}_2$-valued. Then all trivial 0D block-states with the form as Eq. (S153) can be viewed as a vector subspace $V'$ of $V$. It is easy to see that there are only two independent quantities in Eq. (S153): $n_1$ and $n_2$. So the dimensionality of the vector subspace $V'$ should be 4. For the complex fermion sector, we have the following relation:

$$-4n_1 - 2(-2n_1 + 2n_2) = -4n_2 \quad \text{(S154)}$$

i.e., there are only two independent quantities which serves a $2\mathbb{Z} \times 4\mathbb{Z}$ trivialization. Hence all trivial 0D block-states form the group:

$$\{\text{TBS}_{\tau_1}^{U(1)}\} = 2\mathbb{Z} \times 4\mathbb{Z} \quad \text{(S155)}$$

Therefore, the ultimate classification of crystalline topological phases protected by $p4$ symmetry for 2D systems with spinless fermions is:

$$G^{U(1)}_{p4} = \mathbb{Z}^3 \times \mathbb{Z}_4^2 \times \mathbb{Z}/2\mathbb{Z} \times 4\mathbb{Z} = \mathbb{Z} \times \mathbb{Z}_4^3 \times \mathbb{Z}_2^2 \quad \text{(S156)}$$

For systems with spin-1/2 fermions, the rotation properties of $|\xi\rangle_{\mu_1}^0$ and $|\xi\rangle_{\mu_2}^{\mu_2}$ are changed by an additional $-1$, which leads to no trivialization. It is easy to verify that the complex fermion decorations for spinless and spin-1/2 fermions are identical for $p4$ symmetric case. For this case, Take aforementioned 1D bubble construction on $\tau_j$ by $n_j$ times ($j = 1, 2$) on trivial state (S152) will lead to another state:

$$[(4n_1, 1), (-2n_1 + 2n_2, 1), (-4n_2, 1)] \quad \text{(S157)}$$

and the trivialization is almost identical with spinless case. Therefore, the ultimate classification of crystalline topological phases protected by $p4$ symmetry for 2D systems with spin-1/2 fermions is:

$$G^{U(1)}_{p4, 1/2} = \mathbb{Z}^3 \times \mathbb{Z}_4^2 \times \mathbb{Z}_2/2\mathbb{Z} \times 4\mathbb{Z}$$

$$= \mathbb{Z} \times \mathbb{Z}_4^3 \times \mathbb{Z}_2^2 \quad \text{(S158)}$$

8. $p4g$

The corresponding point group of this case is 4-fold dihedral group. For 2D blocks $\sigma$ and 1D blocks $\tau_1$, there is
no on-site symmetry; for 1D blocks $\tau_2$, the on-site symmetry is $\mathbb{Z}_2$ which is attributed to the reflection symmetry acting internally; for 0D blocks $\mu_1$, the on-site symmetry is $\mathbb{Z}_4$ via the symmetry $C_4$ acting internally, and for 0D blocks $\mu_2$, the on-site symmetry is $\mathbb{Z}_2 \times \mathbb{Z}_2$ via the symmetry $D_2 \in D_4$ acting internally, see Fig. S9.

First, we consider the 0D block-state decoration. The different 0D block states at $\mu_1$ can be characterized by different 1-dimensional irreducible representations of the symmetry group:

$$\mathcal{H}^1 \left[ \mathbb{Z}_2^f \times \mathbb{Z}_4, U(1) \right] = \mathbb{Z}_2 \times \mathbb{Z}_4 \quad (S159)$$

Here $\mathbb{Z}_2$ represents the complex fermion, and $\mathbb{Z}_4$ represents the rotation eigenvalues, and different block-states on each $\mu_1$ can be labeled by $(\pm, \nu_1)$, where $\nu_1 \in \{1, i, -1, -i\}$ represents the eigenvalue of 4-fold rotational symmetry, and $\pm$ represents the fermion parity. The classification data of 0D block states at $\mu_2$ can be calculated by general group super-cohomology:

$$\mathcal{H}^0 (\mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_2) \times \mathcal{H}^1 [\mathbb{Z}_2 \times \mathbb{Z}_2, U(1)] = \mathbb{Z}_2 \times \mathbb{Z}_2^3 \quad (S160)$$

Three $\mathbb{Z}_2$ have different physical meanings: the first $\mathbb{Z}_2$ represents the complex fermion, the second $\mathbb{Z}_2$ represents the rotation eigenvalue $-1$, and the third $\mathbb{Z}_2$ represents the reflection eigenvalue $-1$. Different block-states on each $\mu_2$ can be labeled by $(\pm, \pm, \pm)$, where these three $\pm$’s represent the fermion parity, the rotation eigenvalue $-1$ and the reflection eigenvalue $-1$, respectively.

According to this notation, the obstruction-free 0D block-states form the following group:

$$\{\text{OFBS}\}^{0D}_{\text{p4g,0}} = \mathbb{Z}_4 \times \mathbb{Z}_2^2 \quad (S161)$$

and the group elements can be labeled by (two brackets represent the block-states at $\mu_1$ and $\mu_2$):

$$[(\nu_1, \pm), (\pm, \pm, \pm)] \quad (S162)$$

Subsequently we investigate the 1D block-state decoration. For $\tau_1$, the unique candidate 1D block-state is Majorana chain due to the absence of on-site symmetry; for $\tau_2$, the total on-site symmetry group is $\mathbb{Z}_2^f \times \mathbb{Z}_2$, hence the candidate 1D block-states are Majorana chain and 1D FSPT state. So all 1D block-states form a group:

$$\{\text{BS}\}^{1D}_{\text{p4g,0}} = \mathbb{Z}_2^3 \quad (S163)$$

Then we discuss about the decorations of these two root phases separately.

a. Majorana chain decoration

Majorana chain decoration on 1D blocks labeled by $\tau_1$ leaves 4 dangling Majorana fermions at each 0D block labeled by $\mu_1$ and $\mu_2$. At $\mu_1$, these 4 Majorana fermions have the following rotational properties:

$$R_{\mu_1} : \gamma_j \mapsto \gamma_{j+1}, \ j = 1, 2, 3, 4. \quad (S164)$$

Where $R_{\mu_1}$ represents the 4-fold rotation operation centred at each 0D block labeled by $\mu_1$, and all subscripts are taken with modulo 4. Consider the local fermion parity with its symmetry property under rotation:

$$P_f = -\prod_{j=1}^{4} \gamma_j, \quad R_{\mu_1} : P_f \mapsto -P_f \quad (S165)$$

Therefore, these 4 Majorana fermions cannot be gapped out in a symmetric way. Equivalently, the no-open-edge condition cannot be satisfied.

Majorana chain decoration on 1D blocks labeled by $\tau_2$ leaves 4 dangling Majorana fermions at each 0D block labeled by $\mu_2$, with the following rotation and reflection symmetry properties:

$$R_{\mu_2} : (\gamma_1, \gamma_2, \gamma_3, \gamma_4) \mapsto (\gamma_3, \gamma_4, \gamma_1, \gamma_2) \quad (S166)$$

$$M_{\tau_2} : (\gamma_1, \gamma_2, \gamma_3, \gamma_4) \mapsto (\gamma_1, \gamma_4, \gamma_3, \gamma_2)$$

where $R_{\mu_2}$ is the generator of $C_4$ group: 2-fold rotation operation centred at each 0D block labeled by $\mu_2$, and $M_{\tau_2}$ represents the reflection operation with the axis coincide with the 1D blocks labeled by $\tau_2$. Then consider the local fermion parity and the corresponding rotation and reflection symmetry properties:

$$P_f = -\prod_{j=1}^{4} \gamma_j, \quad \begin{cases} R_{\mu_2} : P_f \mapsto P_f \\ M_{\tau_2} : P_f \mapsto -P_f \end{cases} \quad (S167)$$

Hence these 4 Majorana fermions cannot be gapped out in a symmetric way, equivalently the no-open-edge condition cannot be satisfied.
b. 1D FSPT state decoration

1D FSPT state decoration on 1D blocks labeled by \( \tau_2 \) leaves 8 dangling Majorana fermions at each 0D block labeled by \( \mu_2 \), with the following rotation and reflection symmetry properties:

\[
R_{\mu_2}: \gamma_j \mapsto \gamma_{j+2}, \quad \gamma_j' \mapsto \gamma_{j+2}', \quad j = 1, 2, 3, 4. \quad (S168)
\]

\[
M_{\mu_2}: \gamma_j \mapsto -\gamma_{-j}, \quad \gamma_j' \mapsto \gamma_{-j}', \quad j = 1, 2, 3, 4. \quad (S169)
\]

Where all subscripts are taken with modulo 4. Then consider the local fermion parity with the corresponding rotation and reflection symmetry properties:

\[
P_f = \prod_{j=1}^{4} \gamma_j \gamma_j', \quad R_{\mu_2}, M_{\tau_2}: \quad P_f \mapsto P_f \quad (S169)
\]

Thus these 8 Majorana fermions can be gapped out by some proper interactions (just like the \( p4m \) case) in a symmetric way. Equivalently, the no-open-edge condition is satisfied, and the obstruction-free 1D block-states form the following group:

\[
\text{OFBS}_{1D}^{1D} = \mathbb{Z}_2 \quad (S170)
\]

and the group elements can be labeled by \( n_2 = 0, 1 \) that represents the number of decorated 1D FSPT states on \( \tau_2 \). Then we see that all obstruction-free block-states form the following group:

\[
\text{OFBS}_{1D}^{1D} \times \text{OFBS}_{0D}^{1D} = \mathbb{Z}_4 \times \mathbb{Z}_2^5 \quad (S171)
\]

and the group elements can be labeled by:

\[
[(\pm, n_1), (\pm, \pm, \pm); n_2] \quad (S172)
\]

here the first two brackets represent the 0D block-states at \( \mu_1 \) and \( \mu_2 \), and the last quantity represents the 1D block-states at \( \tau_2 \).

With all obstruction-free block-states, subsequently we discuss about all possible trivializations. First, we consider about the 2D bubble equivalence: as we discussed in the main text, both types of “Majorana bubble” constructions are allowed because 0D blocks labeled by \( \mu_2 \) are the centers of \( D_2 \) point group symmetry, including “Majorana bubbles” with both PBC and anti-PBC. Similar with the \( p4m \) case, both types of “Majorana bubbles” can be deformed to double Majorana chains at each nearby 1D block, but the effects of them are distinct: near each 1D block labeled by \( \tau_1 \), these double Majorana chains can be trivialized because there is no on-site symmetry on \( \tau_1 \) and the classification of 1D invertible topological phases (i.e., Majorana chain) is \( \mathbb{Z}_2 \); near each 1D block labeled by \( \tau_2 \), these double Majorana chains cannot be trivialized because there is an on-site \( \mathbb{Z}_2 \) symmetry on each \( \tau_2 \) by internal action of reflection symmetry, and this \( \mathbb{Z}_2 \) action exchanges these two Majorana chains, and this is exactly the definition of the nontrivial 1D FSPT phase protected by on-site \( \mathbb{Z}_2 \) symmetry. Furthermore, similar with the \( p4m \) case, there is no effect on 0D blocks labeled by \( \mu_2 \) by taking 2D “Majorana” bubble equivalence, because the alternative Majorana chain surrounding each \( \mu_2 \) is not compatible with the reflection operations; nevertheless, similar with the \( p2 \) case, 2D “Majorana bubble” construction changes the fermion parity of each 0D block labeled by \( \mu_1 \) because there is no reflection operation on 0D block \( \mu_1 \), and the alternative Majorana chain surrounding each \( \mu_1 \) is compatible with all other symmetry operations. We notice that the “Majorana bubble” constructions with both PBC and anti-PBC are equivalent, so take one of them into account is enough.

Subsequently we consider the 1D bubble equivalences. Consider the 1D bubble equivalence on 1D blocks labeled by \( \tau_1 \) [cf. 1D bubble, here both yellow and red dots represent the complex fermions]: Near each 0D block labeled by \( \mu_1 \), there are 4 complex fermions which form an atomic insulator:

\[
|\psi\rangle^{\mu_1}_{p4g} = c_1^{\dagger} c_2^{\dagger} c_3^{\dagger} c_4^{\dagger} |0\rangle \quad (S173)
\]

with rotation property as \( (R_{\mu_1} \text{ represents the 4-fold rotation operation centred at the 0D block labeled by } \mu_1) : \)

\[
R_{\mu_1} |\psi\rangle^{\mu_1}_{p4g} = c_2^{\dagger} c_3^{\dagger} c_4^{\dagger} c_1^{\dagger} |0\rangle = -|\psi\rangle^{\mu_1}_{p4g} \quad (S174)
\]

i.e., \( |\psi\rangle^{\mu_1}_{p4g} \) can trivialized the rotation eigenvalue \(-1\) at each 0D block labeled by \( \mu_1 \); near each 0D block labeled by \( \mu_2 \), there are 4 complex fermions which form another atomic insulator:

\[
|\psi\rangle^{\mu_2}_{p4g} = c_1^{\dagger} c_2^{\dagger} c_3^{\dagger} c_4^{\dagger} |0\rangle \quad (S175)
\]

with rotation property as \( (R_{\mu_2} \text{ represents the 2-fold rotation operation centred at the 0D block labeled by } \mu_2) : \)

\[
R_{\mu_2} |\psi\rangle^{\mu_2}_{p4g} = c_3^{\dagger} c_4^{\dagger} c_1^{\dagger} c_2^{\dagger} |0\rangle = |\psi\rangle^{\mu_2}_{p4g} \quad (S176)
\]

i.e., there is no trivialization on rotation eigenvalues at each 0D block labeled by \( \mu_2 \). Therefore, the 1D bubble construction on \( \tau_1 \) solely trivializes the rotation eigenvalue \(-1\) at each 0D block labeled by \( \mu_1 \). Then consider the 1D bubble equivalence on 1D blocks labeled by \( \tau_2 \) [cf. 1D bubble, here both yellow and red dots represent the complex fermions]: near each 0D block labeled by \( \mu_2 \), there are 4 complex fermions which further forms an atomic insulator:

\[
|\phi\rangle^{\mu_2}_{p4g} = a_1^{\dagger} a_2^{\dagger} a_3^{\dagger} a_4^{\dagger} |0\rangle \quad (S177)
\]

with the rotation property as \( (R_{\mu_2} \text{ represents the 2-fold rotation operation centred at 0D block labeled by } \mu_2) : \)

\[
R_{\mu_2} |\phi\rangle^{\mu_2}_{p4g} = a_1^{\dagger} a_3^{\dagger} a_2^{\dagger} a_4^{\dagger} |0\rangle = |\phi\rangle^{\mu_2}_{p4g} \quad (S178)
\]

i.e., there is no trivialization at each 0D block labeled by \( \mu_2 \).

And the role of reflection symmetry should also be investigated. Reflection symmetry solely acts on 0D blocks labeled by \( \mu_2 \) internally. Near each 0D block labeled by
\(\mu_2\), again we consider the atomic insulator \(|\phi\rangle_{p4g}\), with the reflection property as \((M_{2z})\) represents the reflection operation with the axis coincide with the 1D blocks labeled by \(\tau_2\):

\[
M_{2z}|\phi\rangle_{p4g} = a_1^1a_2^1a_3^1a_4^1|0\rangle = -|\phi\rangle_{p4g}
\]

i.e., the reflection eigenvalue \(-1\) at \(\mu_2\) can be trivialized by the atomic insulator \(|\phi\rangle_{p4g}\).

With all possible 2D and 1D bubble constructions, we are ready to study the trivial block-states. Start from the original trivial state (nothing is decorated on arbitrary blocks):

\[
[(+1),(+1),(+1),(+1)]
\]

(S180)

If we take 2D bubble construction \(n_0\) times, take 1D bubble equivalences on \(\tau_1\) and \(\tau_2\) by \(n_1\) and \(n_2\) times, above trivial state will be deformed to a new 0D block-state labeled by:

\[
[((-1)^{n_0},(-1)^{n_1}),(+1,(-1)^{n_2})]
\]

(S181)

According to the definition of bubble equivalence, all these 0D block-states should be trivial. It is straightforward to check that there are only three independent quantities \((n_j, j=0,1,2)\) in Eq. (S181), hence all trivial block-states form the following group:

\[
\{\text{TBS}\}_{p4g,0} = \mathbb{Z}_2^3
\]

(S182)

Therefore, all independent nontrivial block-states are labeled by the group elements of the following quotient group:

\[
E_{p4g,0} = \{\text{OFBS}\}_{p4g,0}/\{\text{TBS}\}_{p4g,0}
\]

(S183)

\[= \mathbb{Z}_4 \times \mathbb{Z}_2^3 / \mathbb{Z}_2^3 = \mathbb{Z}_4^2\]

and \(\mathbb{Z}_2^3\)'s are from nontrivial 0D block-states, so it is obvious that there is no stacking between different block-states, and the group structure of \(E_{p4g,0}\) has already been accurate.

**b. Spin-1/2 fermions**

First, we consider the 0D block-state decorations. For 0D blocks labeled by \(\mu_4\), the total symmetry group at each of them is \(\mathbb{Z}_4^2\): nontrivial \(\mathbb{Z}_2^2\) extension of the on-site symmetry \(\mathbb{Z}_4\), and the different block-states on each of them can be characterized by different 1D irreducible representations of the corresponding symmetry group:

\[
\mathcal{H}^1[\mathbb{Z}_4^2, U(1)] = \mathbb{Z}_8
\]

(S184)

For 0D blocks labeled by \(\mu_2\), the classification data can be calculated by general group super-cohomology:

\[
\mathcal{H}^0(\mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_2) \times \mathcal{H}^1[\mathbb{Z}_2 \times \mathbb{Z}_2, U(1)] = \mathbb{Z}_2 \times \mathbb{Z}_2^3
\]

(S185)

and the first \(\mathbb{Z}_2\) who represents the odd fermion parity is obstructed [1]. As a consequence, the classification attributed to 0D block-state decorations is:

\[
E_{p4g,1/2}^{0D} = \mathbb{Z}_8 \times \mathbb{Z}_2^2
\]

(S186)

Subsequently we investigate the 1D block-state decoration. For \(\tau_1\), the unique candidate 1D block-state is Majorana chain due to the absence of on-site symmetry; for \(\tau_2\), the total symmetry group is \(\mathbb{Z}_4^1\), hence there is no nontrivial 1D block-state due to the trivial classification for the corresponding FSPT phases. Majorana chain decoration on \(\tau_1\) leaves 4 dangling Majorana fermions at each 0D block labeled by \(\mu_1\) and \(\mu_2\). At each of them, these 4 Majorana fermions have the following rotational symmetry properties:

\[
R_{\mu_1} : (\gamma_1, \gamma_2, \gamma_3, \gamma_4) \mapsto (\gamma_2, \gamma_3, \gamma_4, -\gamma_1)
\]

(S187)

Then consider the local fermion parity at each 0D block and its symmetry property:

\[
P_f = -\prod_{j=1}^4 \gamma_j, \quad R_{\mu_1} : P_f \mapsto P_f
\]

(S188)

Thus these 4 Majorana fermions can be gapped out by some proper interactions in a symmetric way; at each \(\mu_1\), the corresponding 2 Majorana fermions which can be gapped out by an entanglement pair in a symmetric way. Hence the no-open-edge condition is satisfied. Hence for the system with spin-1/2 fermions, the classification of 2D FSPT phases protected by \(p4g\) symmetry attributed to the 1D block-state decoration is:

\[
E_{p4g,1/2}^{1D} = \mathbb{Z}_2
\]

(S189)

Similar with the \(p4\) case, there is no stacking between 1D and 0D block-states, and the ultimate classification with accurate group structure is:

\[
G_{p4g}^{1/2} = \mathbb{Z}_8 \times \mathbb{Z}_2^3
\]

(S190)

**c. With \(U^f(1)\) charge conservation**

Then we consider the systems with \(U^f(1)\) charge conservation. For each 0D block labeled by \(\mu_1\), different 0D block-states are characterized by different irreducible representations of symmetry group as:

\[
\mathcal{H}^1[U(1) \times \mathbb{Z}_4, U(1)] = \mathbb{Z} \times \mathbb{Z}_4
\]

(S191)

Here \(\mathbb{Z}\) represents the complex fermion and \(\mathbb{Z}_4\) represents the eigenvalues of 4-fold rotational symmetry operation. For each 0D block labeled by \(\mu_2\), different 0D block-states are also characterized by different irreducible representations of symmetry group as:

\[
\mathcal{H}^1[U(1) \times (\mathbb{Z}_2 \times \mathbb{Z}_2), U(1)] = \mathbb{Z} \times \mathbb{Z}_2^2
\]

(S192)
Here $Z$ represents the complex fermion, the first $Z_2$ represents the rotation eigenvalue $-1$ and the second $Z_2$ represents the eigenvalue $-1$ of reflection symmetry operation. We should further consider possible trivializations. For systems with spinless fermions, consider the 1D bubble equivalence on $\tau_1$: we decorate a 1D bubble on each 1D block labeled by $\tau_1$, here yellow and red dots represent the fermionic particle and hole, respectively, and can be trivialized if we shrink them to a point. Near each 0D block labeled by $\mu_1$, there are four particles that can form an atomic insulator:

$$|\xi^{\mu_1}_{p4g}\rangle = p_1^\dagger p_2^\dagger p_3^\dagger p_4^\dagger |0\rangle$$  \hspace{1cm} (S193)

with rotation property as:

$$R_{\mu_1} |\xi^{\mu_1}_{p4g}\rangle = p_2^\dagger p_3^\dagger p_4^\dagger p_1^\dagger |0\rangle = -|\xi^{\mu_1}_{p4g}\rangle$$  \hspace{1cm} (S194)

i.e., rotation eigenvalue $-1$ at each 0D block labeled by $\mu_1$ can be trivialized by the atomic insulator $|\xi^{\mu_1}_{p4g}\rangle$. Near $\mu_2$, there are four holes that form another atomic insulator:

$$|\xi^{\mu_2}_{p4g}\rangle = h_1^\dagger h_2^\dagger h_3^\dagger h_4^\dagger |0\rangle$$  \hspace{1cm} (S195)

with rotation property as:

$$R_{\mu_2} |\xi^{\mu_2}_{p4g}\rangle = h_4^\dagger h_1^\dagger h_2^\dagger h_3^\dagger |0\rangle = |\xi^{\mu_2}_{p4g}\rangle$$  \hspace{1cm} (S196)

i.e., there is no trivialization on rotation eigenvalues at each 0D block labeled by $\mu_2$. Therefore, the 1D bubble construction on $\tau_1$ solely trivializes the rotation eigenvalue $-1$ at each 0D block labeled by $\mu_1$. Then consider the 1D bubble equivalence on $\tau_2$ (cf. 1D bubble, here yellow and red dots represent the fermionic particle and hole, respectively, and can be trivialized if we shrink them to a point): near each 0D block labeled by $\mu_2$, there are two particles and two holes that form an atomic insulator:

$$|\eta^{\mu_2}_{p4g}\rangle = p_1^\dagger p_2^\dagger h_1^\dagger h_2^\dagger |0\rangle$$  \hspace{1cm} (S197)

with rotation and reflection property as:

$$R_{\mu_2} |\eta^{\mu_2}_{p4g}\rangle = p_1^\dagger p_2^\dagger h_1^\dagger h_2^\dagger = |\eta^{\mu_2}_{p4g}\rangle$$

$$M_{\tau_2} |\eta^{\mu_2}_{p4g}\rangle = p_1^\dagger p_2^\dagger h_1^\dagger h_2^\dagger = -|\eta^{\mu_2}_{p4g}\rangle$$  \hspace{1cm} (S198)

i.e., the reflection eigenvalue $-1$ at $\mu_2$ can be trivialized by the atomic insulator $|\eta^{\mu_2}_{p4g}\rangle$.

Subsequently we consider the complex fermion sector: consider 1D bubble equivalence on 1D blocks $\tau_1$ (cf. 1D bubble, here yellow and red dots represent particle and hole, respectively, and they can be trivialized if we shrink them to a point); it adds four complex fermions at each 0D block $\mu_1$ and removes four complex fermions at each 0D block $\mu_2$ (by adding four holes), hence the number of complex fermions at $\mu_1$ and $\mu_2$ are not independent.

With the help of above discussions, we consider the 0D block-state decorations. The 0D block-state decorated on $\mu_1$ can be labeled by $(n_1, \phi_1)$, where $n_1 \in Z$ represents the number of complex fermions decorated on $\mu_1$, $\phi_1$ represents the eigenvalues of 4-fold rotational symmetry on $\mu_1$; the 0D block-state decorated on $\mu_2$ can be labeled by $(n_2, \pm, \pm)$, where $n_2 \in Z$ represents the number of complex fermions decorated on $\mu_2$, two $\pm$'s represent the eigenvalues of 2-fold rotational symmetry and reflection symmetry on $\mu_2$, respectively. Start from the following trivial state:

$$[(0, 1), (0, +, +)]$$  \hspace{1cm} (S199)

Take aforementioned 1D bubble construction on $\tau_1$ by $n_1$ times ($j = 1, 2$), and it will lead to the following 0D block-state:

$$[((4n_1, (−1)^{n_1}), (−4n_1, +, (−1)^{n_1+n_2}))$$  \hspace{1cm} (S200)

And this state should be trivial. First, we consider the sector of complex fermion decoration: there are only one independent quantity $4n_1$, hence the 1D bubble construction serves a $4Z$. Furthermore, once we fix a $n_1$, the rotation eigenvalue $(−1)^{n_1}$ is determined, so there is no trivialization in rotation sector. Subsequently the 1D bubble construction can serve a $Z_2$ in the reflection sector via the phase factor $(−1)^{n_1+n_2}$. Therefore, the ultimate classification of crystalline topological phases protected by $p4g$ symmetry for 2D systems with spinless fermions is:

$$G^{U(1)}_{p4g,0} = Z^2 \times Z_4 \times Z_2^2 / 4Z \times Z_2 = Z \times Z_4^2 \times Z_2$$  \hspace{1cm} (S201)

For systems with spin-$1/2$ fermions, the rotation property of $|\xi^{\mu_1}_{p4g}\rangle$ at each 0D block $\mu_1$ and reflection property of $|\eta^{\mu_2}_{p4g}\rangle$ at each 0D block $\mu_2$ are changed by an additional $−1$, which leads to no trivialization. Furthermore, like the $p4m$ case, the classification data of the 0D blockstates of 0D blocks labeled by $\mu_2$ can be characterized by different 1D irreducible representations of the full symmetry group:

$$\mathcal{H}^1 \{U^f(1) \rtimes \mathbb{Z}_2, (Z_2 \times Z_2), U(1)\} = Z \times Z_2^2$$  \hspace{1cm} (S202)

we should notice that for systems with spin-$1/2$ fermions, we can only decorate even number of complex fermions on each 0D block labeled by $\mu_2$. Repeatedly take aforementioned 1D bubble construction on $\tau_1$ by $n_1$ times on trivial state (S199) will lead to another state:

$$[((4n_1, 1), (−4n_1, +, +))$$  \hspace{1cm} (S203)

Hence the 1D bubble construction serves a $4Z$ in complex fermion sector. Therefore, the ultimate classification of crystalline topological phases protected by $p4g$ symmetry for 2D systems with spin-$1/2$ fermions is:

$$G^{U(1)}_{p4g,1/2} = Z \times 2Z \times Z_4 \times Z_2^2 / 4Z$$

$$= Z \times Z_4 \times Z_2^2$$  \hspace{1cm} (S204)
And the group elements can be labeled by (three brackets represent the block-states at $\mu_j$, $j = 1, 2, 3$):

$$[(\pm, \nu_1), (\pm, \nu_2), (\pm, \nu_3)]$$ (S207)

Subsequently we investigate the 1D block-state decoration. The unique candidate 1D block-state for each 1D block is Majorana chain due to the absence of the on-site symmetry, so all 1D block-states form a group:

$$\{\text{BS}\}^{1\text{D}}_{p^3,0} = \mathbb{Z}_2^3$$ (S208)

Majorana chain decoration on $\tau_1/\tau_2$ leaves 3 dangling Majorana fermions at each corresponding 0D block. It is well-known that odd number of Majorana fermions cannot be gapped out by entanglement pair/interactions, hence the no-open-edge condition cannot be satisfied, and the classification attributed to 1D block-state decorations is trivial:

$$E^{1\text{D}}_{p^3,0} = \{\text{OFBS}\}^{1\text{D}}_{p^3,0} = \mathbb{Z}_1$$ (S209)

With all obstruction-free block-states, subsequently we discuss about all possible trivializations. First, we consider about the 2D bubble equivalences: as we discussed in the main text, only type-II (i.e., “Majorana bubbles” with anti-PBC) 2D bubble equivalence is valid because there is no 0D block as the center of even-fold dihedral group. And from Ref. [3] we know that there is no effects on both 1D and 0D blocks, so 2D bubble equivalence contributes no trivialization.

Then we consider the 1D bubble equivalences. Consider the 1D bubble equivalence on 1D blocks labeled by $\tau_1$ [cf. 1D bubble, here both yellow and red dots represent the complex fermions]: Near each 0D block labeled by $\mu_1$ or $\mu_3$, there are 3 complex fermions which form an atomic insulator:

$$|\psi\rangle^{p^3}_{\mu_3} = c_1^+ \gamma_2^+ c_1^+ |0\rangle$$ (S210)

and it is obvious that the fermion parity of $|\psi\rangle^{p^3}_{\mu_3}$ is odd, hence $|\psi\rangle^{p^3}_{\mu_3}$ can change the fermion parity at each 0D block labeled by $\mu_3$. Similar for each 0D block labeled by $\mu_1$, and the fermion parities of 0D blocks labeled by $\mu_1$ and $\mu_3$ can be changed by 1D bubble construction on $\tau_1$ simultaneously. Similar arguments can also be held on 1D block labeled by $\tau_2$, and we summarize the effects of all 1D bubble equivalences:

1. 1D bubble equivalence on $\tau_1$: change the fermion parities of 0D blocks $\mu_1$ and $\mu_3$ simultaneously;

2. 1D bubble equivalence on $\tau_2$: change the fermion parities of 0D blocks $\mu_1$ and $\mu_2$ simultaneously.

With all possible bubble equivalences, we are ready to study the trivial block-states. Start from the original trivial state (nothing is decorated on arbitrary blocks):

$$[(+, 1), (+, 1), (+, 1)]$$
If we take 1D bubble equivalences on $\tau_1$ and $\tau_2$ by $n_1$ and $n_2$ times, above trivial state will be deformed to a new 0D block-state labeled by:

$$\{((-1)^{n_1+n_2}, 1), ((-1)^{n_2}, 1), ((-1)^{n_1}, 1)\}$$  \hspace{1cm} (S211)

According to the definition of bubble equivalence, all these 0D block-states should be trivial. It is straightforward to check that there are only two independent quantities ($n_1$ and $n_2$) in Eq. (S211), hence all trivial block-states form the following group:

$$\{\text{TBS}\}_{p_3,0} = \mathbb{Z}_2^2$$  \hspace{1cm} (S212)

Therefore, all independent nontrivial block-states are labeled by the group elements of the following quotient group:

$$E_{p_3,0} = \{\text{OFBS}\}_{p_3,0}/\{\text{TBS}\}_{p_3,0} = \mathbb{Z}_2^3/\mathbb{Z}_2^2 = \mathbb{Z}_2 \times \mathbb{Z}_4^2$$ \hspace{1cm} (S213)

It is obvious that there is no stacking between 1D and 0D block-states because of the absence of nontrivial 1D root phase. Therefore, the ultimate classification with accurate group structure is:

$$\mathcal{G}_{p_3}^{0} = \mathbb{Z}_2 \times \mathbb{Z}_4^3$$ \hspace{1cm} (S214)

b. Spin-$1/2$ fermions

First, we investigate the 0D block-state decorations. For each 0D block labeled by $\mu_j$, $j = 1, 2, 3$, the total symmetry group is $\mathbb{Z}_6^J$: nontrivial $\mathbb{Z}_2^J$ extension of on-site symmetry $\mathbb{Z}_3$, and the different 0D block-states can be characterized by different 1D irreducible representations of the corresponding symmetry group:

$$\mathcal{H}^1[\mathbb{Z}_6^J, U(1)] = \mathbb{Z}_6$$ \hspace{1cm} (S215)

Then we investigate the possible trivializations. Consider the 1D bubble equivalence on 1D blocks labeled by $\tau_1$: on each 1D block labeled by $\tau_1$, we decorate a 1D bubble onto it. Here both yellow and red dots represent the complex fermions (Note: the 0D FSPT mode with eigenvalue $-1$ of the symmetry group $\mathbb{Z}_6^J$ is just the complex fermion because $\mathbb{Z}_6^J$ is the nontrivial $\mathbb{Z}_2^J$ extension of the on-site symmetry $\mathbb{Z}_3$). Near each 0D block labeled by $\mu_1$, there are 3 complex fermions which form an atomic insulator with odd fermion parity (equivalently, eigenvalue $-1$ of the symmetry group $\mathbb{Z}_6^J$ on each 0D block labeled by $\mu_1$):

$$|\phi\rangle_{p_3} = a_1^\dagger a_2^\dagger a_3^\dagger |0\rangle$$ \hspace{1cm} (S216)

Hence the eigenvalue $-1$ of the symmetry group $\mathbb{Z}_6^J$ at each 0D block labeled by $\mu_1$ can be trivialized by the atomic insulator $|\phi\rangle_{p_3}$. Near each 0D block labeled by $\mu_3$, there is 3 complex fermions which form an atomic insulator with odd fermion parity (equivalently, eigenvalue $-1$ of the symmetry group $\mathbb{Z}_6^J$ on each 0D block labeled by $\mu_3$):

$$|\phi\rangle_{p_3} = a_1^\dagger a_2^\dagger a_3^\dagger |0\rangle$$ \hspace{1cm} (S217)

Hence the eigenvalue $-1$ of the symmetry group $\mathbb{Z}_6^J$ at each 0D block labeled by $\mu_3$ can be trivialized by the atomic insulator $|\phi\rangle_{p_3}$. Equivalently, the eigenvalue $-1$ of the symmetry group $\mathbb{Z}_6^J$ at 0D block labeled by $\mu_1$ and $\mu_3$ are not independent. Similarly, the 1D bubble equivalence on 1D blocks labeled by $\tau_2$ leads to that the eigenvalue $-1$ of the symmetry group $\mathbb{Z}_6^J$ at 0D block labeled by $\mu_1$ and $\mu_2$ are not independent. Therefore, the eigenvalue $-1$ of the symmetry group $\mathbb{Z}_6^J$ at all 0D blocks are not independent, and the classification attributed to the 0D block-state decorations is:

$$E_{p_3,1/2}^{0D} = \mathbb{Z}_6 \times \mathbb{Z}_4^2$$ \hspace{1cm} (S218)

Subsequently we investigate the 1D block-state decoration, which is similar with the case of spinless fermions: the no-open-edge condition cannot be satisfied, and the classification attributed to 1D block-state decorations is:

$$E_{p_3,1/2}^{1D} = \mathbb{Z}_1$$ \hspace{1cm} (S219)

It is obvious that there is no stacking between 1D and 0D block-states because of the absence of nontrivial 1D root phase. Therefore, the ultimate classification with accurate group structure is:

$$\mathcal{G}_{p_3}^{1/2} = \mathbb{Z}_6 \times \mathbb{Z}_4^2$$ \hspace{1cm} (S220)

c. With $U^J(1)$ charge conservation

Then we consider the systems with $U^J(1)$ charge conservation. For an arbitrary 0D block, different 0D block-states are characterized by different irreducible representations of the symmetry group as:

$$\mathcal{H}^1[U(1) \times \mathbb{Z}_3, U(1)] = \mathbb{Z} \times \mathbb{Z}_3$$ \hspace{1cm} (S221)

Here $\mathbb{Z}$ represents the complex fermion and $\mathbb{Z}_3$ represents the eigenvalues of 3-fold rotational symmetry operation. For systems with spinless fermions, consider 1D bubble equivalence on 1D blocks $\tau_1$ [cf. 1D bubble, here yellow and red dots represent particle and hole, respectively, and they can be trivialized if we shrink them to a point]: it adds three complex fermions at each 0D block $\mu_1$ and removes three complex fermions at each 0D block $\mu_3$ (by adding three holes), hence the number of complex fermions at $\mu_1$ and $\mu_2$ are not independent. More specifically, suppose there are a complex fermions at each $\mu_1$ and $b$ complex fermions at each $\mu_3$, and suppose $c = a + b$. Take above manipulation $n$ times ($n \in \mathbb{Z}$), the number...
of complex fermions on each $\mu_j/\mu_3$ is $a + 3n/b - 3n$, and their summation remains invariant. So for a specific $c$, there are only three independent cases:

\[
\begin{align*}
    c &= a + b \\
    c &= (a + 1) + (b - 1) \\
    c &= (a + 2) + (b - 2)
\end{align*}
\] (S222)

which reduces the classification of complex fermion decorations from $\mathbb{Z}^2$ to $\mathbb{Z} \times \mathbb{Z}_3$. Therefore, the ultimate classification of crystalline topological phases protected by $p3$ symmetry for 2D systems with spinless fermions is:

\[\mathcal{G}_{p3,0}^{U(1)} = \mathbb{Z} \times \mathbb{Z}_3^5 \] (S223)

It is easy to verify that for $p3$ symmetry, there is no difference between systems with spinless and spin-1/2 fermions, hence the ultimate classification of crystalline topological phases protected by $p3$ symmetry for 2D systems with spin-1/2 fermions is:

\[\mathcal{G}_{p3,1/2}^{U(1)} = \mathbb{Z} \times \mathbb{Z}_3^5 \] (S224)

10. $p3m1$

The corresponding point group of this case is 3-fold dihedral symmetry group $D_3$. For 2D blocks $\sigma$, there is no on-site symmetry; for 1D blocks $\tau_j$, $j = 1, 2, 3$, the on-site symmetry group is $\mathbb{Z}_2$ caused by reflection symmetry acting internally on the corresponding 1D blocks; for 0D blocks $\mu_j$, $j = 1, 2, 3$, the on-site symmetry group is $\mathbb{Z}_3 \rtimes \mathbb{Z}_2$ which is attributed to the $D_3$ group acting internally, see Fig. S11.

a. Spinless fermions

First, we consider the 0D block-state decoration. For each 0D block, the classification data can be determined by calculating different 1D irreducible representations of full symmetry group:

\[\mathcal{H}^0 \left[ U^f(1) \times (\mathbb{Z}_3 \times \mathbb{Z}_2), U(1) \right] = \mathbb{Z}_2^6 \] (S225)

Here the two $\mathbb{Z}_2$ have different physical meanings: the first $\mathbb{Z}_2$ represents the complex fermion, and the second $\mathbb{Z}_2$ represents the reflection eigenvalue $-1$ (i.e., the rotational symmetry plays no role in the 0D block-state decorations), and different 0D block-states on each 0D block can be labeled by $(\pm, \pm)$, where these two $\pm$’s represent the fermion parity and eigenvalues of reflection operation, respectively. According to this notation, all obstruction-free 0D block-states form the following group:

\[\{\text{OFBS}\}_{p3m1,0}^{0D} = \mathbb{Z}_2^6 \] (S226)

And the group elements can be labeled by (three brackets represent the block-states at $\mu_j, j = 1, 2, 3$):

\[
[\pm, \pm, \pm], [\pm, \pm, \pm], [\pm, \pm, \pm]
\] (S227)

Subsequently we consider the 1D block-state decoration. For each 1D block, the total symmetry is $\mathbb{Z}_2^6 \times \mathbb{Z}_2$, hence the candidate 1D block-states are Majorana chains and 1D FSPT state, and all 1D block-states form a group:

\[\{\text{BS}\}_{p3m1,0}^{1D} = \mathbb{Z}_2^6 \] (S228)

a. Majorana chain decoration. Majorana chain decoration on $\mu_j, j = 1, 2, 3$ will leaves 3 dangling Majorana fermions at each corresponding 0D block, and it is well-known that odd number of Majorana fermions cannot be gapped out by entanglement pairs/interactions, thus the no-open-edge condition is violated. Nevertheless, there is one exception: if we consider all 1D blocks $\tau_1, \tau_2$ and $\tau_3$ together and decorate a Majorana chain on each of them, it will leaves 6 dangling Majorana fermions at each 0D block (we discuss the 0D block labeled by $\mu_1$ as an example), with the following rotation and reflection symmetry properties:

\[R_{\mu_1} : \gamma_j \mapsto \gamma_{j+1}, \gamma'_j \mapsto \gamma'_{j+1} \]
\[M_{\tau_j} : \gamma_j \mapsto \gamma_{5-j}, \gamma'_j \mapsto \gamma'_{5-j} \] (S229)

Where $R_{\mu_1}$ represents the 3-fold rotation operation centred at each 0D block labeled by $\mu_1$ and $M_{\tau_j}$ represents the reflection operation with the axis coincide with the 1D blocks labeled by $\tau_j$. All subscripts are taken with modulo 4. Consider the Hamiltonian who includes 3 entanglement pairs as:

\[\mathcal{H} = i \sum_{j=1}^{3} \gamma_j \gamma'_j \] (S230)
Which is invariant under arbitrary symmetry operations. So the no-open-edge condition is satisfied for this exception.

b. 1D FSPT state decoration 1D FSPT state decoration on \( \tau_j, j = 1, 2, 3 \) will leaves 6 dangling Majorana fermions at each corresponding 0D block, with the following rotation and reflection symmetry properties:

\[
\begin{align*}
R_{\mu_1} & : \gamma_j \mapsto \gamma_{j+1}, \gamma'_j \mapsto \gamma'_{j+1} \\
M_{\gamma} & : \gamma_j \mapsto -\gamma_{N-j}, \gamma'_j \mapsto \gamma'_{N-j}
\end{align*}
\]

Then consider the local fermion parity at each corresponding 0D block with its symmetry properties:

\[
P_f = i \prod_{j=1}^{3} \gamma_j \gamma'_j,
\begin{align*}
R_{\mu_1} : P_f & \mapsto P_f \\
M_{\gamma} : P_f & \mapsto -P_f
\end{align*}
\]

Thus these 6 Majorana fermions cannot be gapped out in a symmetric way, and the no-open-edge condition is violated. Nevertheless, there is one exception: if we consider all 1D blocks together and decorate a 1D FSPT state on each of them, there is 12 dangling Majorana fermions \( \gamma_j, \gamma'_j, j = 1, ..., 6 \) as the edge modes of the decorated Majorana chains, with the following rotation and reflection symmetry properties:

\[
\begin{align*}
R_{\mu_1} & : \gamma_j \mapsto \gamma_{j+2}, \gamma'_j \mapsto \gamma'_{j+2} \\
M_{\gamma} & : \gamma_j \mapsto -\gamma_{8-j}, \gamma'_j \mapsto \gamma'_{8-j}
\end{align*}
\]

where all subscripts are taken with modulo 6. Then consider the local fermion parity and its symmetry properties:

\[
P_f = i \prod_{j=1}^{6} \gamma_j \gamma'_j,
\begin{align*}
R_{\mu_1}, M_{\gamma} : P_f & \mapsto P_f
\end{align*}
\]

Therefore these 12 Majorana fermions can be gapped out by some proper interactions in a symmetric way, and the no-open-edge is satisfied. All obstruction-free 1D block-states form the following group:

\[
\begin{align*}
\text{OFBS}^{1D}_{p3m1,0} & = \mathbb{Z}_2^2
\end{align*}
\]

and the group elements can be labeled by \( n_1 = n_2 = n_3 \) and \( n'_1 = n'_2 = n'_3 \) that represents the number of decorated Majorana chains/1D FSPT states on 1D block labeled by \( \tau_j, j = 1, 2, 3 \). Then we see that all obstruction-free block-states form the following group:

\[
\begin{align*}
\text{OFBS}^{1D}_{p3m1,0} = \{ \text{OFBS}^{1D}_{p3m1,0} \} \times \{ \text{OFBS}^{0D}_{p3m1,0} \} \\
= \mathbb{Z}_2^6 \times \mathbb{Z}_2^2 = \mathbb{Z}_2^8
\end{align*}
\]

and the group elements can be labeled by:

\[
[\{\pm, \pm\}, \{\pm, \pm\}, \{\pm, \pm\}; n_1 = n_2 = n_3; \\
n'_1 = n'_2 = n'_3]
\]

here the first three brackets represent the 0D block-states at \( \mu_j \), and the last two quantities represent the number of Majorana chain/1D FSPT states at \( \tau_j, j = 1, 2, 3 \).

With all obstruction-free block-states, we discuss about all possible trivializations. First, we consider about the 2D bubble equivalence: as we discussed in the main text, only type-II (i.e., “Majorana bubbles” with anti-PBC) 2D bubble equivalence is valid because there is no 0D block as the center of even-fold dihedral group symmetry. Similar with the \( p4m \) case, “Majorana bubbles” can be deformed to double Majorana chains at each nearby 1D block, and this is exactly the definition of the nontrivial 1D FSPT phase protected by on-site \( \mathbb{Z}_2 \) symmetry (by reflection symmetry acting internally).

As a consequence, 1D FSPT state decorations on all 1D blocks can be deformed to a trivial state via 2D “Majorana” bubble equivalences. Furthermore, repeatedly similar with the \( p4m \) case, both types of “Majorana bubble” constructions have no effect on 0D blocks.

Subsequently we consider the 1D bubble equivalences. Consider the 1D bubble equivalence on 1D blocks labeled by \( \tau_2 \) [cf. 1D bubble, here both yellow and red dots represent the complex fermions]. Near each 0D block labeled by \( \mu_3 \), there are 3 complex fermions which form an atomic insulator with odd fermion parity:

\[
|\psi\rangle^{\mu_3}_{p3m1} = c_1^0 c_2^0 c_3^0 |0\rangle
\]

i.e., fermion parity at each \( \mu_3 \) can be changed by the atomic insulator \( |\psi\rangle^{\mu_3}_{p3m1} \). Similar for the 0D blocks labeled by \( \mu_1 \), we can conclude that the fermion parities of 0D blocks labeled by \( \mu_1 \) and \( \mu_3 \) are not independent. Hence if we further consider the 1D bubble equivalences on 1D blocks labeled by \( \tau_2 \) and \( \tau_3 \), we can obtain that the fermion parities of 0D blocks labeled by \( \mu_j, j = 1, 2, 3 \) are not independent. Furthermore, this 1D bubble construction can change the reflection eigenvalues on 0D block: for instance, consider the reflection property of the atomic insulator \( |\psi\rangle^{\mu_3}_{p3m1} \) is:

\[
M_{\tau_2} |\psi\rangle^{\mu_3}_{p3m1} = c_1^0 c_2^0 c_3^0 |0\rangle = -|\psi\rangle^{\mu_3}_{p3m1}
\]

i.e., the atomic insulator \( |\psi\rangle^{\mu_3}_{p3m1} \) changes the reflection eigenvalue of the 0D block \( \mu_3 \), similar for all other 0D blocks.

Then we study another possible 1D bubble equivalence. Here we consider an alternative 1D bubble equivalence on 1D blocks labeled by \( \tau_2 \) [cf. 1D bubble, here both yellow and red dots represents the 0D mode with reflection eigenvalue \(-1\)]; we call the aforementioned 1D bubble equivalence “type-I” and this 1D bubble equivalence “type-II”): near each 0D block labeled by \( \mu_3 \), there are three 0D modes characterized by reflection eigenvalue \(-1\) which can form another type of “atomic insulator” (here \( d'_j \) is the creation operator of the corresponding 0D mode):

\[
|\phi\rangle^{\mu_3}_{p3m1} = d_1^0 d_2^0 d_3^0 |0\rangle
\]

who can change the reflection eigenvalue of each 0D block labeled by \( \mu_3 \); each 0D mode carries an eigenvalue \(-1\) of reflection symmetry, hence \( |\phi\rangle^{\mu_3}_{p3m1} \) carries an eigenvalue
If we take 2D “Majorana” bubble equivalences \( n \) times, we obtain that the reflection eigenvalues at \( \mu_j \), \( j = 1, 2, 3 \) are not independent. We summarize the effects of all 1D bubble equivalences:

1. Type-I 1D bubble construction on \( \tau_1 \): change the fermion parities and reflection eigenvalues of 0D blocks \( \mu_2 \) and \( \mu_3 \) simultaneously;

2. Type-I 1D bubble construction on \( \tau_2 \): change the fermion parities and reflection eigenvalues of 0D blocks \( \mu_1 \) and \( \mu_2 \) simultaneously;

3. Type-I 1D bubble construction on \( \tau_3 \): change the fermion parities and reflection eigenvalues of 0D blocks \( \mu_1 \) and \( \mu_3 \) simultaneously;

4. Type-II 1D bubble construction on \( \tau_1 \): change the reflection eigenvalues of 0D blocks \( \mu_2 \) and \( \mu_3 \) simultaneously;

5. Type-II 1D bubble construction on \( \tau_2 \): change the reflection eigenvalues of 0D blocks \( \mu_1 \) and \( \mu_3 \) simultaneously;

6. Type-II 1D bubble construction on \( \tau_3 \): change the reflection eigenvalues of 0D blocks \( \mu_1 \) and \( \mu_2 \) simultaneously.

With all possible bubble equivalences, we are ready to study the trivial block-states. Start from the original trivial state (nothing is decorated on arbitrary blocks):

\[
[(+, +), (+, +), (+, +)]
\]

If we take 2D “Majorana” bubble equivalences \( n_0 \) times, take type-I 1D bubble equivalences \( n_j \) times, and type-II 1D bubble equivalences on \( \tau_j \) by \( n_j' \) times \( (j = 1, 2, 3) \), above trivial state will be deformed to a new 0D block-state labeled by:

\[
\left[ \left( (-1)^{n_2+n_3}, (-1)^{n_2+n_3+n_2'+n_3'} \right), \left( (-1)^{n_1+n_3}, (-1)^{n_1+n_3+n_1'+n_2'} \right), \left( (-1)^{n_1+n_2}, (-1)^{n_1+n_2+n_1'+n_2'} \right) \right]
\]

(S241)

According to the definition of bubble equivalence, all these 0D block-states should be trivial. It is straightforward to check that there are only five independent quantities in Eq. (S241) by calculating the rank of the following matrix:

\[
\begin{pmatrix}
0 & 0 & 1 & 1 & 1 & 1 \\
1 & 1 & 0 & 0 & 1 & 1 \\
1 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 & 0
\end{pmatrix}
\]

(S242)

Together with the 2D bubble equivalence, all trivial block-states form the following group:

\[
\{\text{TBS}\}_{\text{p}3\text{m}1,0} = \{\text{TBS}\}_{\text{p}3\text{m}1,0}^{1\text{D}} \times \{\text{TBS}\}_{\text{p}3\text{m}1,0}^{0\text{D}}
\]

\[
= \mathbb{Z}_2 \times \mathbb{Z}_2' = \mathbb{Z}_2^2
\]

(S243)

here \( \{\text{TBS}\}_{\text{p}3\text{m}1,0}^{1\text{D}} \) represents the group of trivial states with non-vacuum 1D blocks, and \( \{\text{TBS}\}_{\text{p}3\text{m}1,0}^{0\text{D}} \) represents the group of trivial states with non-vacuum 0D blocks.

Therefore, all independent nontrivial block-states are labeled by different group elements of the following quotient group:

\[
E_{\text{p}3\text{m}1,0} = \{\text{OFBS}\}_{\text{p}3\text{m}1,0}/\{\text{TBS}\}_{\text{p}3\text{m}1,0}
\]

\[
= \mathbb{Z}_2^8/\mathbb{Z}_2^3 = \mathbb{Z}_2^5
\]

(S244)

here one \( \mathbb{Z}_2 \) is from the Majorana chain decorations on all 1D blocks simultaneously, and all other \( \mathbb{Z}_2 \)'s are from the nontrivial 0D block-states. Similar with the \( \text{pm} \) case, there is no stacking between 1D and 0D block-states, and the group structure of the classification data \( E_{\text{p}3\text{m}1,0} \) has already been accurate.

b. Spin-1/2 fermions

First, we investigate the 0D block-state decorations. Similar with the spinless fermions case, the classification data for all 0D blocks can be characterized by different 1D irreducible representations of the full symmetry group:

\[
\mathcal{H}^1 \left[ U/(1) \rtimes \mathbb{Z}_3 \times \mathbb{Z}_2, U(1) \right] = \mathbb{Z}_4
\]

(S245)

Equivalently, we can label different 0D block-states by the group elements of the 4-fold cyclic group:

\[
\mathbb{Z}_4 = \{1, i, -1, -i\}
\]

(S246)

Then we investigate the possible trivialization. Consider the 1D bubble equivalence on 1D blocks labeled by \( \tau_1 \); on each \( \tau_1 \), the total on-site symmetry is \( \mathbb{Z}_4^f \); nontrivial \( \mathbb{Z}_2^f \) extension of the on-site symmetry \( \mathbb{Z}_2 \). Next we decorate a 1D bubble onto each of them, here the yellow/red dots represents the 0D FSPT mode protected by \( \mathbb{Z}_2^f \) symmetry which is labeled by \( i/j = i \in \mathbb{Z}_4 \), cf. Eq. (S246), and they can be trivialized if they shrink to a point. Near each 0D block labeled by \( \mu_3 \), there are three 0D FSPT modes labeled by \( i \in \mathbb{Z}_4 \) and they can change the label of 0D block-state decorated at each 0D block \( \mu_3 \) by \( -i \in \mathbb{Z}_4 \). Meanwhile, near each 0D block labeled by \( \mu_3 \), there are three 0D FSPT modes labeled by \( -i \in \mathbb{Z}_4 \) and they can change the label of 0D block-state decorated at each 0D block \( \mu_2 \) by \( i \in \mathbb{Z}_4 \). Therefore, the block-state decorations on 0D blocks labeled by \( \mu_2 \) and \( \mu_3 \) are not independent. Similar for the 1D bubble equivalence on 1D blocks labeled by \( \tau_2 \), we can conclude that the block-state decorations on 0D blocks labeled by \( \mu_j \), \( j = 1, 2, 3 \)
are not independent. As a consequence, the classification attributed to the 0D block-state decorations is:

\[ E_{p3m1,1/2}^{0D} = Z_4 \quad (S247) \]

Subsequently we investigate the 1D block-state decoration. For all 1D blocks, the total symmetry on each of them is \( Z_3 \), hence there is no nontrivial 1D block-state because of the trivial classification of the corresponding 1D FSPT phases, and the classification attributed to 1D block-state decorations is trivial:

\[ E_{p3m1,1/2}^{1D} = Z_1 \quad (S248) \]

It is obvious that there is no stacking between 1D and 0D block-states because of the absence of nontrivial 1D root phase. Therefore, the ultimate classification with accurate group structure is:

\[ G_{p3m1}^{1/2} = Z_4 \quad (S249) \]

\( c. \) With \( U^f(1) \) charge conservation

Then we consider the systems with \( U^f(1) \) charge conservation. For an arbitrary 0D block, different 0D block-states are characterized by different irreducible representations of the symmetry group as:

\[ H^1[U(1) \times (Z_3 \times Z_2), U(1)] = Z \times Z_2 \quad (S250) \]

Here \( Z \) represents the complex fermion and \( Z_2 \) represents the eigenvalues of reflection symmetry operation. We need to further consider the possible trivializations. For systems with spinless fermions, consider the 1D bubble equivalence: on each 1D block labeled by \( \tau_1 \), we decorate a 1D bubble onto it. Here both yellow and red dots represent the 0D FSPT modes characterized by eigenvalue \(-1\) of reflection symmetry operation, and they can be trivialized when they shrink to a point. According to this bubble construction, the reflection eigenvalues at 0D blocks \( \mu_2 \) and \( \mu_3 \) can be changed simultaneously, hence the reflection eigenvalues at \( \mu_2 \) and \( \mu_3 \) are not independent. Similarly, the 1D bubble constructions on other 1D blocks will lead to the fact that the reflection eigenvalues of all 0D blocks are not independent.

Subsequently we consider the complex fermion sector: consider 1D bubble equivalence on 1D blocks \( \tau_1 \) [cf. 1D bubble, here yellow and red dots represent particle and hole, respectively, and they can be trivialized if we shrink them to a point]: it adds three complex fermions at each 0D block \( \mu_2 \) and removes three complex fermions at each 0D block \( \mu_3 \) (by adding three holes), hence the number of complex fermions at \( \mu_2 \) and \( \mu_3 \) are not independent. More specifically, suppose there are \( a \) complex fermions at each \( \mu_2 \) and \( b \) complex fermions at each \( \mu_3 \), and suppose \( c = a + b \). Take above manipulation \( n \) times \((n \in Z)\), the number of complex fermions on each \( \mu_2/\mu_3 \) is \( a + 3n/b - 3n \), and their summation remains invariant. So for a specific \( c \), there are only three independent cases:

\[
\begin{align*}
    c &= a + b \\
    c &= (a + 1) + (b - 1) \\
    c &= (a + 2) + (b - 2)
\end{align*}
\quad (S251)
\]

which reduces the classification of complex fermion decorations on \( \mu_2 \) and \( \mu_3 \) from \( Z^2 \) to \( Z \times Z_3 \). Similar arguments can be held to 0D blocks labeled by \( \mu_2 \) and \( \mu_3 \). Therefore, the ultimate classification of crystalline topological phases protected by \( p3m1 \) symmetry for 2D systems with spinless fermions is:

\[ G_{p3m1,0}^{U^{(1)}} = Z \times Z_2 \times Z_2 \quad (S252) \]

It is easy to verify that for \( p3m1 \) symmetry, there is no difference between systems with spinless and spin-1/2 fermions, hence the ultimate classification of crystalline topological phases protected by \( p3m1 \) symmetry for 2D systems with spin-1/2 fermions is:

\[ G_{p3m1,1/2}^{U^{(1)}} = Z \times Z_3 \times Z_2 \quad (S253) \]

11. \( p31m \)

The corresponding point group of this case is 3-fold dihedral group \( D_3 \) by quotient out the translations. For 2D blocks \( \sigma \) and 1D blocks \( \tau_3 \), there is no on-site symmetry; for 1D blocks \( \tau_1 \), the on-site symmetry is \( Z_2 \) attributed to the reflection symmetry acting internally; for 0D blocks \( \mu_1 \), the on-site symmetry is \( Z_2 \times Z_2 \) attributed to the \( D_3 \) symmetry acting internally; for 0D blocks \( \mu_2 \), the on-site symmetry is \( Z_3 \) attributed to the \( C_3 \) symmetry acting internally, see Fig. S12.

\( a. \) Spinless fermions

First, we investigate the 0D block-state decoration. The classification data of each 0D block labeled by \( \mu_1 \) can be characterized by different 1D irreducible representations of the full symmetry group:

\[ H^1[U^f(1) \times (Z_3 \times Z_2), U(1)] = Z_2^2 \quad (S254) \]

Here the two \( Z_2 \) have different physical meanings: the first \( Z_2 \) represents the complex fermion, and the second \( Z_2 \) represents the reflection eigenvalue \(-1\), and different 0D block-states on each 0D block \( \mu_1 \) can be labeled by \((\pm, \pm)\), where these two \( \pm \)'s represent the fermion parity and eigenvalues of reflection operation, respectively.

Different 0D block-states on each 0D block labeled by \( \mu_2 \) can be characterized by different 1-dimensional irreducible representations of the symmetry group:

\[ H^1[Z_2 \times Z_3, U(1)] = Z_2 \times Z_3 \quad (S255) \]
We discuss about these two root phases separately.

a. Majorana chain decoration

Majorana chain decoration on $\tau_1$ leaves 6 dangling Majorana fermions at each 0D block labeled by $\mu_1$, with the following rotation and reflection symmetry properties:

$$R_{\mu_1} : \gamma_j \mapsto \gamma_{j+2}, \quad M_{\tau_1} : \gamma_j \mapsto \gamma_{8-j}$$  \hspace{1cm} (S259)

where $R_{\mu_1}$ represents the 3-fold rotation operation centred at each 0D block labeled by $\mu_1$, and $M_{\tau_1}$ represents the reflection operation with the axis coincide with the 1D blocks labeled by $\tau_1$. All subscripts are taken with modulo 6. Then consider the local fermion parity with its symmetry properties:

$$P_f = i \prod_{j=1}^{6} \gamma_j, \quad R_{\mu_1}, M_{\tau_1} : P_f \mapsto P_f$$  \hspace{1cm} (S260)

Hence these 6 Majorana fermions can be gapped out by 3 entanglement pairs in a symmetric way, as shown in the following Hamiltonian who respects arbitrary symmetry actions:

$$\mathcal{H} = i \sum_{j=1}^{3} \gamma_j \gamma_{j+3}$$  \hspace{1cm} (S261)

Therefore the no-open-edge condition is satisfied. And the Majorana chain decoration on $\tau_2$ leaves 3 dangling Majorana fermions at each 0D block labeled by $\mu_2$. It is well-known that odd number of Majorana fermions cannot be gapped out, hence the no-open-edge condition is violated.

b. 1D FSPT state decoration

1D FSPT state decoration on $\tau_1$ leaves 12 dangling Majorana fermions at each 0D block labeled by $\mu_1$, with the following rotation and reflection symmetry properties (all subscripts are taken with modulo 6):

$$R_{\mu_1} : \gamma_j \mapsto \gamma_{j+2}, \quad \gamma'_j \mapsto \gamma'_{j+2}$$
$$M_{\tau_1} : \gamma_j \mapsto -\gamma_{8-j}, \quad \gamma'_j \mapsto \gamma'_{8-j}$$  \hspace{1cm} (S262)

Then consider the local fermion parity and its rotation and reflection symmetry properties:

$$P_f = - \prod_{j=1}^{6} \gamma_j \gamma'_j, \quad R_{\mu_1}, M_{\tau_1} : P_f \mapsto P_f$$  \hspace{1cm} (S263)

hence these 12 Majorana fermions can be gapped out by some proper interactions in a symmetric way, and the no-open-edge condition is satisfied. As a consequence, all obstruction-free 1D block-states form the following group:

$$\{\text{OFBS}\}_{p31m,0}^{1D} = \mathbb{Z}_2^3$$  \hspace{1cm} (S264)

and the group elements can be labeled by $n_1$ and $n'_1$ that represent the number of decorated Majorana chains/1D FSPT states on 1D block labeled by $\tau_1$. Then we see that all obstruction-free block-states form the following group:

$$\{\text{OFBS}\}_{p31m,0} = \{\text{OFBS}\}_{p31m,0}^{1D} \times \{\text{OFBS}\}_{p31m,0}^{0D} = \mathbb{Z}_2^5 \times \mathbb{Z}_3$$  \hspace{1cm} (S265)

and the group elements can be labeled by:

$$[(\pm, \pm), (\pm, \nu_j); n_1; n'_1]$$  \hspace{1cm} (S266)
here the first two brackets represent the 0D block-states at $\mu_1$ and $\mu_2$, and the last two quantities represent the number of Majorana chains/1D FSPT states at $\tau_1$.

With all obstruction-free block-states, we discuss above all possible trivializations. First, we consider about the 2D bubble equivalence: as we discussed in the main text, only type-II (i.e., “Majorana bubbles” with anti-PBC) 2D bubble equivalence is valid because there is no 0D block as the center of even-fold dihedral group symmetry. Similar with the $\text{pgm}$ case, “Majorana bubbles” can be deformed to double Majorana chains at each nearby 1D block, but the effects of them are distinct: near each 1D block labeled by $\tau_2$, these double Majorana chains can be trivialized because there is no on-site symmetry on $\tau_1$ and the classification of 1D invertible topological phases (i.e., Majorana chain) is $\mathbb{Z}_2$; near each 1D block labeled by $\tau_1$, these double Majorana chains cannot be trivialized because there is an on-site $\mathbb{Z}_2$ symmetry on each $\tau_2$ by internal action of reflection symmetry, and this $\mathbb{Z}_2$ action exchanges these two Majorana chains, and this is exactly the definition of the nontrivial 1D FSPT phase protected by on-site $\mathbb{Z}_2$ symmetry. Furthermore, similar with the $\text{pgm}$ case, there is no effect on 0D blocks labeled by $\mu_2$ by taking 2D “Majorana” bubble equivalence.

Subsequently we consider the 1D bubble equivalence on 1D blocks labeled by $\tau_2$ [cf. 1D bubble, here both yellow and red dots represent the complex fermions]: Near each 0D block labeled by $\mu_1$, there are 6 complex fermions which can form an atomic insulator with even fermion parity:

$$|\psi\rangle_{p31m}^{\mu_1} = \prod_{j=1}^{6} c_j^\dagger |0\rangle \quad (S267)$$

So it cannot change the fermion parity at each 0D block labeled by $\mu_1$. Near each 0D block labeled by $\mu_2$, there are 3 complex fermions which form another atomic insulator:

$$|\psi\rangle_{p31m}^{\mu_2} = c_1^\dagger c_2^\dagger c_3^\dagger |0\rangle \quad (S268)$$

and it can change the fermion parity of each 0D block labeled by $\mu_2$.

Then we study the role of the rotational symmetry. We only need to consider the 0D blocks labeled by $\mu_2$ as aforementioned. We consider the atomic insulator $|\psi\rangle_{p31m}^{\mu_2}$ repeatedly, with the rotation property as ($R_{\mu_2}$ represents the 3-fold rotation operation centred at the 0D block labeled by $\mu_2$):

$$R_{\mu_2} |\psi\rangle_{p31m}^{\mu_2} = c_1^\dagger c_2^\dagger c_3^\dagger |0\rangle = |\psi\rangle_{p31m}^{\mu_2} \quad (S269)$$

i.e., there is no trivialization.

And the role of reflection symmetry should also be investigated. Reflection symmetry solely acts on 0D blocks labeled by $\mu_1$ internally. We repeatedly consider the atomic insulator $|\psi\rangle_{p31m}^{\mu_1}$, with the reflection property as ($M_{\tau_1}$ represents the reflection operation with the axis coincide with the 1D blocks labeled by $\tau_1$):

$$M_{\tau_1} |\psi\rangle_{p31m}^{\mu_1} = c_1^\dagger c_2^\dagger c_3^\dagger c_4^\dagger c_5^\dagger c_6^\dagger |0\rangle = -|\psi\rangle_{p31m}^{\mu_1} \quad (S270)$$

i.e., the atomic insulator $|\psi\rangle_{p31m}^{\mu_1}$ can trivialize the reflection eigenvalue $-1$.

With all possible bubble equivalences, we are ready to study the trivial block-states. Start from the original trivial state (nothing is decorated on arbitrary blocks):

$$[(+, +), (+, +), 0; 0]$$

If we take 2D “Majorana” bubble equivalences $n_0$ times and 1D bubble equivalences on $\tau_2$ by $n_2$ times, above trivial state will be deformed to a new 0D block-state labeled by:

$$[(+, +), ((-)^{n_2}, +), 0; n_0] \quad (S271)$$

According to the definition of bubble equivalence, all these 0D block-states should be trivial. It is straightforward to check that there are only two independent quantities in Eq. (S271): $n_0$ and $n_2$. Hence all trivial block-states form the following group:

$$\{\text{TBS}\}_{p31m, 0} = \{\text{TBS}\}_{p31m, 0}^{1D} \times \{\text{TBS}\}_{p31m, 0}^{0D}$$

$$= \mathbb{Z}_2 \times \mathbb{Z}_2 \quad (S272)$$

here $\{\text{TBS}\}_{p31m, 0}^{1D}$ represents the group of trivial states with non-vacuum 1D blocks, and $\{\text{TBS}\}_{p31m, 0}^{0D}$ represents the group of trivial states with non-vacuum 0D blocks.

Therefore, all independent nontrivial block-states are labeled by different group elements of the following quotient group:

$$E_{p31m, 0} = \{\text{OFBS}\}_{p31m, 0}^{2D} \div \{\text{TBS}\}_{p31m, 0}$$

$$= \mathbb{Z}_2^5 \times \mathbb{Z}_3/\mathbb{Z}_2^3 \times \mathbb{Z}_3$$

$$= \mathbb{Z}_2^3 \times \mathbb{Z}_3 \quad (S273)$$

here one $\mathbb{Z}_2$ is from the Majorana chain decorations on 1D blocks labeled by $\tau_1$, and $\mathbb{Z}_2^3 \times \mathbb{Z}_3$ is from the nontrivial 0D block-states. Similar with the $\text{pgm}$ case, there is no stacking between 1D and 0D block-states, and the group structure of the classification data $E_{p31m, 0}$ has already been accurate.

b. Spin-1/2 fermions

First, we investigate the 0D block-state decorations. For each 0D block labeled by $\mu_1$, the classification data can be characterized by different 1D irreducible representations of the full symmetry group:

$$\mathcal{H}^1 [U_f (1) \times \omega_2 (\mathbb{Z}_3 \times \mathbb{Z}_2), U(1)] = \mathbb{Z}_2^3 \quad (S274)$$

For each 0D block labeled by $\mu_2$, the total on-site symmetry group is $\mathbb{Z}_6^f$: nontrivial $\mathbb{Z}_6^f$ extension of the on-site symmetry group $\mathbb{Z}_3$, whereas different 0D block-states
can be characterized by different 1D irreducible representations of the corresponding symmetry group:

\[ \mathcal{H}^1 \left[ \mathbb{Z}_6, U(1) \right] = \mathbb{Z}_6 \]  \hspace{1cm} (S275)

Then we investigate the possible trivialization. Consider the 1D bubble equivalence on 1D blocks labeled by \( \tau_2 \); we decorate a 1D bubble onto each of them, here both yellow and red dots represent the complex fermions. Near each 0D block labeled by \( \mu_1 \), there are 6 complex fermions which can form an atomic insulator with even fermion parity who cannot lead to any trivialization:

\[ |\phi_\mu^{(1)}\rangle_{p31m} = \prod_{j=1}^{6} a_j^\dagger |0\rangle \]  \hspace{1cm} (S276)

Near each 0D block labeled by \( \mu_2 \), there are 3 complex fermions which can form another atomic insulator with odd fermion parity:

\[ |\phi_\mu^{(2)}\rangle_{p31m} = a_1^\dagger a_2^\dagger a_3^\dagger |0\rangle \]  \hspace{1cm} (S277)

On each 0D block labeled by \( \mu_2 \), the eigenvalue \(-1\) of the symmetry group \( \mathbb{Z}_6 \) is exactly the complex fermion because the symmetry group \( \mathbb{Z}_6 \) is the on-site symmetry \( \mathbb{Z}_3 \) extended by fermion parity nontrivially. Therefore, the eigenvalue \(-1\) of the symmetry group \( \mathbb{Z}_6 \) can be trivialized by the atomic insulator \( |\phi_\mu^{(2)}\rangle_{p31m} \). As a consequence, the classification attributed to 0D block-state decorations is:

\[ E_{0D}^{0D,1/2} = \mathbb{Z}_4 \times \mathbb{Z}_3 \]  \hspace{1cm} (S278)

and the index \( \mathbb{Z}_3 \) has the bosonic root phase.

Subsequently we investigate the 1D block-state decoration. For 1D blocks labeled by \( \tau_1 \), the total symmetry group is \( \mathbb{Z}_6 \), so there is no nontrivial block-state due to the trivial classification of the corresponding 1D FSPPT phases; for 1D blocks labeled by \( \tau_2 \), the unique candidate block-state is Majorana chain due to the absence of the on-site symmetry. Majorana chain decoration on \( \tau_2 \) leaves 3 dangling Majorana fermions at each 0D block labeled by \( \mu_2 \). It is well-known that the odd number of Majorana fermions cannot be gapped out in a symmetric way, and the no-open-edge condition is violated. As a consequence, the classification attributed to 1D block-state decorations is:

\[ E_{1D}^{1D,1/2} = \mathbb{Z}_1 \]  \hspace{1cm} (S279)

It is obvious that there is no stacking between 1D and 0D block-states because of the absence of nontrivial 1D root phase. Therefore, the ultimate classification with accurate group structure is:

\[ \mathcal{G}_{p31m}^{1/2} = \mathbb{Z}_4 \times \mathbb{Z}_3 \]  \hspace{1cm} (S280)

c. With \( U^f(1) \) charge conservation

Then we consider the systems with \( U^f(1) \) charge conservation. For each 0D block labeled by \( \tau_1 \), different 0D block-states are characterized by different irreducible representations of the symmetry group as:

\[ \mathcal{H}_0^1 \left[ U(1) \times (\mathbb{Z}_3 \times \mathbb{Z}_2) \right] = \mathbb{Z} \times \mathbb{Z}_2 \]  \hspace{1cm} (S281)

Here \( \mathbb{Z} \) represents the complex fermion and \( \mathbb{Z}_2 \) represents the eigenvalues of reflection symmetry operation. For each 0D block labeled by \( \tau_2 \), different 0D block-states are also characterized by different irreducible representations of the symmetry group as:

\[ \mathcal{H}_0^1 \left[ U(1) \times \mathbb{Z}_3, U(1) \right] = \mathbb{Z} \times \mathbb{Z}_3 \]  \hspace{1cm} (S282)

Here \( \mathbb{Z} \) represents the complex fermion and \( \mathbb{Z}_3 \) represents the eigenvalues of 3-fold rotational symmetry operation. We should further consider possible trivializations. For systems with spinless fermions, consider the 1D bubble equivalence: we decorate a 1D bubble on each 1D block labeled by \( \tau_2 \), here yellow and red dots represent the fermionic particle and hole, respectively, and can be trivialized if we shrink them to a point. Near each 0D block labeled by \( \mu_1 \), there are six particles that can form an atomic insulator:

\[ |\xi_\mu^{(1)}\rangle_{p31m} = \prod_{j=1}^{6} p_j^\dagger |0\rangle \]  \hspace{1cm} (S283)

with reflection property as:

\[ M_{\tau_1} |\xi_\mu^{(1)}\rangle_{p31m} = p_6 p_5 p_4 p_3 p_2 p_1 |0\rangle = -|\xi_\mu^{(1)}\rangle_{p31m} \]  \hspace{1cm} (S284)

i.e., the reflection eigenvalue \(-1\) at each 0D block labeled by \( \mu_1 \) can be trivialized by the atomic insulator \( |\xi_\mu^{(1)}\rangle_{p31m} \).

Subsequently we consider the complex fermion sector. First of all, as shown in Fig. S12, we should identify that there is only one 0D block labeled by \( \mu_1 \) per unit cell, but there are two 0D blocks labeled by \( \mu_2 \) per unit cell. Repeatedly consider 1D bubble equivalence on 1D blocks \( \tau_2 \) [cf. 1D bubble, here yellow and red dots represent particle and hole, respectively, and they can be trivialized if we shrink them to a point]; it adds six complex fermions at each 0D block \( \mu_1 \) and removes three complex fermions at each 0D block \( \mu_2 \) (by adding three holes), hence the number of complex fermions at \( \mu_1 \) and \( \mu_2 \) are not independent. More specifically, suppose there are \( a \) complex fermions on each \( \mu_1 \) and \( b \) complex fermions on each \( \mu_2 \). Suppose the total number of complex fermions within a certain unit cell is \( c = a + 2b \). Take above manipulation \( n \) times \( (n \in \mathbb{Z}) \), the number of complex fermions on each \( \mu_1 / \mu_2 \) is \( a + 6n/b - 3n \), and the total number of complex fermions within a certain unit cell remains invariant. So for a specific \( c \), there are only three independent cases:

\[
\begin{align*}
   c &= a + 2b & \\
   c &= (a + 2) + 2(b - 1) & \quad (S285) \\
   c &= (a + 4) + 2(b - 2) & 
\end{align*}
\]
which reduces the classification from complex fermion decorations from \(Z^2\) to \(Z \times Z_3\). Therefore, the ultimate classification of crystalline topological phases protected by \(\bar{p}31m\) symmetry for 2D systems with spinless fermions is:

\[
\mathcal{G}^{U(1)}_{\bar{p}31m,0} = Z \times Z_3^2
\]

For systems with spin-1/2 fermions, the reflection property of \(|\xi\rangle_{\bar{p}31m}^\mu\) near the 0D block labeled by \(\mu_1\) are changed by an additional \(-1\), which leads to no trivialization. It is easy to verify that the complex fermion decorations for spinless and spin-1/2 fermions are identical. Therefore, the ultimate classification of crystalline topological phases protected by \(p4g\) symmetry for 2D systems with spin-1/2 fermions is:

\[
\mathcal{G}^{U(1)}_{p31m,1/2} = Z \times Z_3^2 \times Z_2
\]

12. \(p6\)

The corresponding point group of this case is 6-fold rotation group \(C_6\) by quotient out the translations. For 2D blocks \(\sigma\), 1D blocks \(\tau_1\) and \(\tau_2\), there is no on-site symmetry; for 0D blocks \(\mu_1\), the on-site symmetry group is \(Z_6\) attributed to the 6-fold rotational symmetry acting internally; for 0D blocks \(\mu_2\), the on-site symmetry group is \(Z_2\) attributed to the 2-fold rotational symmetry acting internally; for 0D blocks \(\mu_3\), the on-site symmetry group is \(Z_3\) attributed to the 3-fold rotational symmetry acting internally, see Fig. S13.

First, we consider the 0D block-state decorations. The different 0D block states at \(\mu_1\) can be characterized by different 1-dimensional irreducible representations of the symmetry group:

\[
\mathcal{H}^1 \left[Z_2^f \times Z_6, U(1)\right] = Z_2 \times Z_6
\]

Here \(Z_2\) represents the complex fermion, and \(Z_6\) represents the 6-fold rotation eigenvalues, and different 0D block-states on each \(\mu_1\) can be labeled by \((\pm, \nu_1)\), where \(\pm\) represents the fermion parity, and \(\nu_1 \in \{e^{i\pi j/3}\}_{j = 0, 1, 2, 3, 4, 5}\) represents the eigenvalues of 6-fold rotation operation.

The different 0D block-states at \(\mu_2\) can be characterized by different 1-dimensional irreducible representations of the symmetry group:

\[
\mathcal{H}^1 \left[Z_2^f \times Z_2, U(1)\right] = Z_2 \times Z_2
\]

Here the first \(Z_2\) represents the complex fermion, and the second \(Z_2\) represents the 2-fold rotation eigenvalues, and different 0D block-states on each \(\mu_1\) can be labeled by \((\pm, \pm)\), where two \(\pm\)’s represent the fermion parity and 2-fold rotation eigenvalue.

The different 0D block-states at \(\mu_3\) can be characterized by different 1-dimensional irreducible representations of the symmetry group:

\[
\mathcal{H}^1 \left[Z_2^f \times Z_3, U(1)\right] = Z_2 \times Z_3
\]

Here the first \(Z_2\) represents the complex fermion, and the second \(Z_3\) represents the 3-fold rotation eigenvalues, and different 0D block-states on each \(\mu_1\) can be labeled by \((\pm, \nu_3)\), where \(\pm\) represents the fermion parity, and \(\nu_3 \in \{1, e^{2\pi i/3}, e^{4\pi i/3}\}\) represents the eigenvalues of 3-fold rotation operation. According to this notation, all obstruction-free 0D block-states form the following group:

\[
\{\text{OFBS}\}_{p6,0}^{0D} = Z_6 \times Z_3 \times Z_2^3
\]

Subsequently we consider the 1D block-state decoration. For arbitrary 1D block, the unique candidate block-state is Majorana chain because of the absence of on-site symmetry. So all 1D block-states form a group:

\[
\{\text{BS}\}_{p6,0}^{1D} = Z_2^3
\]

Majorana chain decoration on \(\tau_1\) leaves 6 dangling Majorana fermions at each \(\mu_1\) and 2 dangling Majorana fermions at each \(\mu_2\). Consider the Majorana fermions as the edge modes of the decorated Majorana chains at each \(\mu_1\), with the following rotational symmetry properties (all subscripts are taken with modulo 6):

\[
R_{\mu_1} : \gamma_j \mapsto \gamma_{j+1}, \quad j = 1, ..., 6
\]
Then consider the local fermion parity and its rotational symmetry property:

\[ P_f = i \prod_{j=1}^{6} \gamma_j, \quad R_{\mu_1} : P_f \mapsto -P_f \]  
(S294)

Thus these 6 dangling Majorana fermions cannot be gapped out in a symmetric way, and the no-open-edge condition cannot be satisfied.

Majorana chain decoration on \( \tau_2 \) leaves 6 dangling Majorana fermions at each \( \mu_1 \) and 3 dangling Majorana fermions at each \( \mu_3 \). Consider the Majorana fermions as the edge modes of the decorated Majorana chains at each \( \mu_3 \), it is well-known that the odd number of Majorana fermions cannot be gapped out, hence the no-open-edge condition cannot be satisfied. Finally, the classification attributed to the 1D block-state decorations is trivial:

\[ E_{p6,0}^{1D} = \{ \text{OFBS} \}_{p6,0}^{1D} = Z_1 \]  
(S295)

With all possible block-states, we discuss all possible trivializations. First, we consider about the 2D bubble equivalence: as we discussed in the main text, only type-II (i.e., “Majorana bubbles” with anti-PBC) 2D bubble equivalence is valid because there is no 0D block as the center of even-fold dihedral group. Enlarge all “Majorana bubble” construction can be deformed to double Majorana chains at arbitrary 0D block that can be trivialized because there is no on-site symmetry and the classification of 1D invertible topological phases (i.e., Majorana chain) is \( Z_2 \). Furthermore, similar with the \( p2 \) case, 2D “Majorana bubble” construction changes the fermion parities of 0D blocks labeled by \( \mu_1 \) and \( \mu_2 \) simultaneously.

Then consider the 1D bubble equivalence on 1D blocks labeled by \( \tau_2 \) [cf. 1D bubble, here both yellow and red dots represent the complex fermions]: Near each 0D block labeled by \( \mu_1 \), there are 6 complex fermions which form an atomic insulator with even fermion parity:

\[ |\psi\rangle^{\mu_1}_{p6} = \prod_{j=1}^{6} c_j^\dagger |0\rangle \]  
(S296)

Hence it cannot change the fermion parity at each 0D block labeled by \( \mu_1 \): Near each 0D block labeled by \( \mu_3 \), there are 3 complex fermions which form another atomic insulator with odd fermion parity:

\[ |\psi\rangle^{\mu_3}_{p6} = c_1^\dagger c_2^\dagger c_3^\dagger |0\rangle \]  
(S297)

Hence it can change the fermion parity at \( \mu_3 \).

Then we study the role of the rotational symmetry. We consider the atomic insulator \( |\psi\rangle^{\mu_1}_{p6} \) and \( |\psi\rangle^{\mu_3}_{p6} \) repeatedly. Near each 0D block labeled by \( \mu_1 \), the atomic insulator \( |\psi\rangle^{\mu_1}_{p6} \) has the following rotation property \( (R_{\mu_1} \text{ represents the 6-fold rotation operation centred at the 0D block labeled by } \mu_1) \):

\[ R_{\mu_1} |\psi\rangle^{\mu_1}_{p6} = c_2^\dagger c_3^\dagger c_4^\dagger c_5^\dagger c_6^\dagger c_1^\dagger |0\rangle = -|\psi\rangle^{\mu_1}_{p6} \]  
(S298)

i.e., the rotation eigenvalue \(-1\) can be trivialized by the atomic insulator \( |\psi\rangle^{\mu_1}_{p6} \); Near each 0D block labeled by \( \mu_3 \), the atomic insulator \( |\psi\rangle^{\mu_3}_{p6} \) has the following rotation property \( (R_{\mu_3} \text{ represents the 3-fold rotation operation centred at the 0D block labeled by } \mu_3) \):

\[ R_{\mu_3} |\psi\rangle^{\mu_3}_{p6} = c_2^\dagger c_3^\dagger c_1^\dagger |0\rangle = |\psi\rangle^{\mu_3}_{p6} \]  
(S299)

i.e., no trivialization. We should further consider the 1D bubble equivalence on 1D blocks labeled by \( \tau_1 \) [cf. 1D bubble, here both yellow and red dots represent the complex fermions]: Near each 0D block labeled by \( \mu_1 \), there are 6 complex fermions that form another atomic insulator:

\[ |\phi\rangle^{\mu_1}_{p6} = \prod_{j=1}^{6} a_j^\dagger |0\rangle \]  
(S300)

with rotation property as:

\[ R_{\mu_1} |\phi\rangle^{\mu_1}_{p6} = a_2^\dagger a_3^\dagger a_4^\dagger a_5^\dagger a_6^\dagger a_1^\dagger |0\rangle = -|\phi\rangle^{\mu_1}_{p6} \]  
(S301)

i.e., the rotation eigenvalue \(-1\) can be trivialized by the atomic insulator \( |\phi\rangle^{\mu_1}_{p6} \). Near each 0D block labeled by \( \mu_2 \), there are 2 complex fermions that form another atomic insulator:

\[ |\phi\rangle^{\mu_2}_{p6} = a_1^\dagger a_2^\dagger |0\rangle \]  
(S302)

with rotation property as:

\[ R_{\mu_2} |\phi\rangle^{\mu_2}_{p6} = a_2^\dagger a_1^\dagger |0\rangle = -|\phi\rangle^{\mu_2}_{p6} \]  
(S303)

i.e., the rotation eigenvalue \(-1\) can be trivialized by the atomic insulator \( |\phi\rangle^{\mu_2}_{p6} \). Hence we can conclude that the rotation eigenvalues of \( \mu_1 \) and \( \mu_2 \) are not independent.

With all possible 2D and 1D bubble equivalences, we are ready to study the trivial block-states. Start from the original trivial state (nothing is decorated on arbitrary blocks):

\[ [(+, 1), (+, +), (+, 1)] \]

If we take 2D bubble construction \( n_0 \) times, take 1D bubble equivalences on \( \tau_1 \) and \( \tau_2 \) by \( n_1 \) and \( n_2 \) times, above trivial state will be deformed to a new block-state labeled by:

\[ [((-1)^{n_0}, (-1)^{n_1+n_2}), ((-1)^{n_0}, (-1)^{n_1}), ((-1)^{n_0}, 1)] \]  
(S304)

According to the definition of bubble equivalence, all these 0D block-states should be trivial. It is straightforward to see that there are only three independent quantities in Eq. (S304), hence all trivial block-states form the following group:

\[ \{ \text{TBS} \}_{p6,0} = \{ \text{TBS} \}_{p6,0}^{1D} = Z_2^3 \]  
(S305)
Therefore, all independent nontrivial block-states are labeled by the group elements of the following quotient group:

$$E_{p6,0} = \{\text{OFBS}\}_{p6,0}/\{\text{TBS}\}_{p6,0} = \mathbb{Z}_6 \times \mathbb{Z}_3 \times \mathbb{Z}_2^4/\mathbb{Z}_3^3 = \mathbb{Z}_2^2 \times \mathbb{Z}_3^2 \quad (S306)$$

We should note that the group structure should be $\mathbb{Z}_2^2 \times \mathbb{Z}_3^2$ rather than $\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_6$ because the eigenvalue $-1$ of the 6-fold rotational symmetry at each 0D block labeled by $\mu_1$ is trivialized by 1D bubble equivalence. It is obvious that there is no stacking between 1D and 0D block-states because there is no nontrivial 1D block-state, and the group structure of the classification data $E_{p6,0}$ has already been accurate.

b. Spin-1/2 fermions

First, we investigate the 0D block-state decoration. For each 0D block labeled by $\mu_1$, the total symmetry group is $\mathbb{Z}_{12}$: on-site $\mathbb{Z}_6$ symmetry with nontrivial extension of fermion parity, and the different block-states on each of them can be characterized by different 1D irreducible representations of the corresponding symmetry group:

$$\mathcal{H}^1 \left[ \mathbb{Z}_{12}^f, U(1) \right] = \mathbb{Z}_{12} \quad (S307)$$

For each 0D block labeled by $\mu_2$, the total symmetry group is $\mathbb{Z}_4$: on-site $\mathbb{Z}_2$ symmetry with nontrivial extension of fermion parity, and the different block-states on each of them can be characterized by different 1D irreducible representations of the corresponding symmetry group:

$$\mathcal{H}^1 \left[ \mathbb{Z}_4^f, U(1) \right] = \mathbb{Z}_4 \quad (S308)$$

For each 0D block labeled by $\mu_3$, the total symmetry group is $\mathbb{Z}_6$: on-site $\mathbb{Z}_3$ symmetry with nontrivial extension of fermion parity, and the different block-states on each of them can be characterized by different 1D irreducible representations of the corresponding symmetry group:

$$\mathcal{H}^1 \left[ \mathbb{Z}_6^f, U(1) \right] = \mathbb{Z}_6 \quad (S309)$$

Then we consider the possible trivializations. Consider the 1D bubble equivalence on the 1D blocks labeled by $\gamma_2$; we decorate a 1D bubble onto each of them, whereas both yellow and red dots represent the complex fermions. Near each 0D block labeled by $\mu_1$, there are 6 complex fermions which can form an atomic insulator with even fermion parity which cannot lead to any trivialization:

$$|\phi\rangle^{\mu_1}_{p6} = \prod_{j=1}^{6} a_j^\dagger |0\rangle \quad (S310)$$

Near each 0D block labeled by $\mu_3$, there are 3 complex fermions which can form another atomic insulator with odd fermion parity:

$$|\phi\rangle^{\mu_3}_{p6} = a_1^\dagger a_2^\dagger a_3^\dagger |0\rangle \quad (S311)$$

On each 0D block labeled by $\mu_3$, the eigenvalue $-1$ of the symmetry group $\mathbb{Z}_6^f$ is exactly the complex fermion because the symmetry group $\mathbb{Z}_6^f$ is the on-site symmetry $\mathbb{Z}_3$ extended by fermion parity nontrivially. Therefore, the eigenvalue $-1$ of the symmetry group $\mathbb{Z}_6^f$ can be trivialized by the atomic insulator $|\phi\rangle^{\mu_3}_{p6}$. As a consequence, the classification attributed to 0D block-state decorations is:

$$E_{p6,0/1/2}^{0D} = \mathbb{Z}_{12} \times \mathbb{Z}_4 \times \mathbb{Z}_3 \quad (S312)$$

Subsequently we consider the 1D block-state decoration. For arbitrary 1D block, the unique candidate 1D block-state is Majorana chain due to the absence of on-site symmetry. Majorana chain decoration on $\tau_1$ leaves 6 dangling Majorana fermions at each $\mu_1$ and 2 dangling Majorana fermions at each $\mu_2$. Consider the Majorana fermions as the edge modes of the decorated Majorana chains at $\mu_1$, with the following rotational symmetry properties (here $R_{\mu_1}$ represents the 6-fold rotation operation centred at each 0D block labeled by $\mu_1$):

$$R_{\mu_1} : \gamma_j \mapsto \gamma_{j+1}, \ j = 1, \ldots, 5; \ \gamma_6 \mapsto -\gamma_1. \quad (S313)$$

Consider the Hamiltonian including 3 entanglement pairs which is invariant under the rotational symmetry:

$$\mathcal{H} = i \sum_{j=1}^{3} \gamma_j \gamma_{j+3} \quad (S314)$$

and these Majorana chains are gapped out in a symmetric way. For Majorana fermions at each $\mu_2$, identical with the $p2$ case, these 2 Majorana fermions can be gapped out by an entanglement pair who respects all symmetry. Therefore, the no-open-edge condition is satisfied.

Majorana chain decoration on $\tau_2$ leaves 6 dangling Majorana fermions at each $\mu_1$ and 3 dangling Majorana fermions at each $\mu_3$. It is well-known that odd number of Majorana fermions cannot be gapped out, hence the no-open-edge condition cannot be satisfied, and finally the classification attributed to 1D block-state decorations is:

$$E_{p6,0/1/2}^{1D} = \mathbb{Z}_2 \quad (S315)$$

With full data of classification, we investigate the possible stacking between 1D and 0D block-states. If we decorate two Majorana chains on each 1D block labeled by $\tau_1$, it can be deformed to an assembly of 0D block-states at 0D blocks labeled by $\mu_1$ and $\mu_2$. Similar with the $pmm$ case, 1D block-states extend 0D block-states at 0D blocks $\mu_1$ and $\mu_2$ simultaneously. Therefore, the ultimate classification with accurate group structure is:

$$E_{p6}^{1/2} = \mathbb{Z}_{12} \times \mathbb{Z}_6 \times \mathbb{Z}_3 \quad (S316)$$
c. With $U^f(1)$ charge conservation

Then we consider the systems with $U^f(1)$ charge conservation. For each 0D block labeled by $\mu_1$, different 0D block-states are characterized by different irreducible representations of the symmetry group as:

$$\mathcal{H}^1[U(1) \times Z_6, U(1)] = Z \times Z_6$$

(S317)

Here $Z$ represents the complex fermion and $Z_6$ represents the eigenvalues of 6-fold rotational symmetry operation. For each 0D block labeled by $\mu_1$, different 0D block-states are also characterized by different irreducible representations of the symmetry group as:

$$\mathcal{H}^1[U(1) \times Z_2, U(1)] = Z \times Z_2$$

(S318)

Here $Z$ represents the complex fermion and $Z_2$ represents the eigenvalues of 2-fold rotational symmetry operation. For each 0D block labeled by $\mu_1$, different 0D block-states are still characterized by different irreducible representations of the symmetry group as:

$$\mathcal{H}^1[U(1) \times Z_3, U(1)] = Z \times Z_3$$

(S319)

Here $Z$ represents the complex fermion and $Z_3$ represents the eigenvalues of 3-fold rotational symmetry operation. We should further consider possible trivializations. For systems with spinless fermions, consider the 1D bubble equivalence: we decorate a 1D bubble on each 1D block labeled by $\tau_1$, here yellow and red dots represent the fermionic particle and hole, respectively, and can be trivialized if we shrink them to a point. Near each 0D block labeled by $\mu_1$, there are 6 particles that can form an atomic insulator:

$$|\xi\rangle_{p_6}^\mu_1 = \prod_{j=1}^6 p_j^f|0\rangle$$

(S320)

with rotation property as:

$$R_{\mu_1} |\xi\rangle_{p_6}^\mu_1 = p_{j_2}^f p_{j_1}^f p_{j_4}^f p_{j_5}^f p_{j_6}^f |0\rangle = -|\xi\rangle_{p_6}^\mu_1$$

(S321)

i.e., the rotation eigenvalue $-1$ at 0D blocks $\mu_1$ are trivialized by the atomic insulator $|\eta\rangle_{p_6}^\mu_1$. Near $\mu_2$, there are two holes that can form another atomic insulator:

$$|\xi\rangle_{p_6}^\mu_2 = h_{j_1}^f h_{j_2}^f |0\rangle$$

(S322)

with rotation property as:

$$R_{\mu_2} |\xi\rangle_{p_6}^\mu_2 = h_{j_2}^f h_{j_1}^f |0\rangle = -|\xi\rangle_{p_6}^\mu_2$$

(S323)

i.e., the rotation eigenvalue $-1$ at 0D blocks $\mu_2$ are trivialized by the atomic insulator $|\eta\rangle_{p_6}^\mu_2$. Equivalently, the rotation eigenvalues of $\mu_1$ and $\mu_2$ are not independent. Moreover, consider the 1D bubble equivalence on $\tau_2$: we decorate a 1D bubble on each of them, here yellow and red dots represent the fermionic particle and hole, respectively, and can be trivialized if we shrink them to a point.

Near each 0D block labeled by $\mu_1$, there are 6 particles that can form an atomic insulator:

$$|\eta\rangle_{p_6}^\mu_1 = \prod_{j=1}^6 p_j^f|0\rangle$$

(S324)

with rotation property as:

$$R_{\mu_1} |\eta\rangle_{p_6}^\mu_1 = p_{j_2}^f p_{j_1}^f p_{j_4}^f p_{j_5}^f p_{j_6}^f |0\rangle = -|\eta\rangle_{p_6}^\mu_1$$

(S325)

i.e., the rotation eigenvalue $-1$ at 0D blocks $\mu_1$ are trivialized by the atomic insulator $|\eta\rangle_{p_6}^\mu_1$. Near $\mu_3$, there are three holes that can form another atomic insulator:

$$|\eta\rangle_{p_6}^\mu_3 = h_{j_1}^f h_{j_2}^f h_{j_3}^f |0\rangle$$

(S326)

with rotation property as:

$$R_{\mu_3} |\eta\rangle_{p_6}^\mu_3 = h_{j_2}^f h_{j_3}^f h_{j_1}^f |0\rangle = |\eta\rangle_{p_6}^\mu_2$$

(S327)

so there is no trivialization near $\mu_3$.

Subsequently we consider the complex fermion sector. First of all, as shown in Fig. S13, we should identify that there is one 0D block labeled by $\mu_1$ per unit cell, two 0D block labeled by $\mu_3$ per unit cell and three 0D block labeled by $\mu_2$ per unit cell. Repeatedly consider 1D bubble equivalence on 1D blocks $\tau_1$ [cf. 1D bubble, here yellow and red dots represent particle and hole, respectively, and they can be trivialized if we shrink them to a point]: it adds six complex fermions at each 0D block $\mu_1$ and removes two complex fermions at each 0D block $\mu_2$ (by adding two holes), hence the number of complex fermions at $\mu_1$ and $\mu_2$ are not independent. More specifically, suppose there are $a$ complex fermions on each $\mu_1$, $b$ complex fermions on each $\mu_2$, and the total number of complex fermions on $\mu_1$ and $\mu_2$ within a certain unit cell is $c = a + 3b$. Take above manipulation $n$ times ($n \in Z$), the number of complex fermions on each $\mu_1/\mu_2$ is $a + 6n/b - 2n$, and the total number of complex fermions on $\mu_1$ and $\mu_2$ within a certain unit cell remains invariant. So for a specific $c$, there are only two independent cases:

$$\begin{cases} c = a + 3b \\ c = (a + 3) + 3(b - 1) \end{cases}$$

(S328)

which reduces the classification from complex fermion decorations on $\mu_1$ and $\mu_2$ from $Z^2$ to $Z \times Z_2$. Then consider 1D bubble equivalence on 1D blocks $\tau_2$: it adds six complex fermions at each 0D block $\mu_1$ and removes three complex fermions at each 0D block $\mu_3$ (by adding three holes), hence the number of complex fermions at $\mu_1$ and $\mu_3$ are not independent. More specifically, suppose there are $a'$ complex fermions on each $\mu_1$, $b'$ complex fermions at each $\mu_3$, and the total number of complex fermions on $\mu_1$ and $\mu_3$ within a certain unit cell is $c' = a' + 2b'$. Take above manipulation $n'$ times ($n' \in Z$), the number of complex fermions on each $\mu_1/\mu_3$ is $a' + 6n'/b' - 3n'$, and the total number of complex fermions on $\mu_1$ and $\mu_3$
within a certain unit cell remains invariant. So for a specific \( c' \), there are only three independent cases:

\[
\begin{align*}
    c' &= a' + 2b' \\
    c' &= (a' + 2) + 2(b' - 1) \\
    c' &= (a' + 4) + 2(b' - 2)
\end{align*}
\]  \quad (S329)

which reduces the classification from complex fermion decorations on \( \mu_1 \) and \( \mu_3 \) from \( \mathbb{Z}^2 \) to \( \mathbb{Z} \times \mathbb{Z}_3 \). Therefore, the ultimate classification of crystalline topological phases protected by \( p6 \) symmetry for 2D systems with spinless fermions is:

\[
\mathcal{G}_{p6,0}^{U(1)} = \mathbb{Z} \times \mathbb{Z}_6 \times \mathbb{Z}_3 \times \mathbb{Z}_2^2 \]  \quad (S330)

For systems with spin-1/2 fermions, the rotation properties of \( |\xi\rangle_{\mu_1} \) and \( |\eta\rangle_{\mu_1} \) at 0D block \( \mu_1 \), \( |\xi\rangle_{\mu_2} \) at 0D block \( \mu_2 \) are changed by an additional \(-1\), which leads to no trivialization. Furthermore, it is easy to verify that the complex fermion decorations for spinless and spin-1/2 fermions are identical. Finally the ultimate classification of crystalline topological phases protected by \( p6 \) symmetry for 2D systems with spin-1/2 fermions is:

\[
\mathcal{G}_{p6,1/2}^{U(1)} = \mathbb{Z} \times \mathbb{Z}_6 \times \mathbb{Z}_3 \times \mathbb{Z}_2^2 \]  \quad (S331)

13. Remarks

In this section, together with other five examples in the main text, we systematically constructed and classified all possible cases of 2D interacting fermionic crystalline TSC and TI by explicit real-space construction (see Sec. II in the main text), for both spinless and spin-1/2 fermions. All results of classification, together with accurate group structures (i.e., possible stacking between block-states with different dimensions) are summarized in three tables in the main text. Compare our results with the classifications of FSPT phases protected by the corresponding internal symmetries, we verify the fermionic crystalline equivalence principle for all cases.

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