INEQUALITIES FOR THE $s$th DERIVATIVE OF A POLYNOMIAL WITH PRESCRIBED ZEROS

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Abstract. Let $P(z)$ be a polynomial of degree $n$ which does not vanish outside the closed disk $|z| < k$, where $k \leq 1$. According to a famous result known as Turan's Theorem for $k=1$, we have

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{2} \max_{|z|=1} |P(z)|$$

In this paper we shall present several interesting generalizations and a refinement of this result which include some results due to Malik, Govil and others. We extend Turan's Theorem for the $s$th derivatives of a polynomial having $t$-fold zeros at origin and thereby obtain another generalization of this beautiful result.

1. Introduction

Let $P(z)$ be a polynomial of degree $n$, then according to a famous result known as Bernstein’s inequality (for reference, see [6, p-531] or [7]),

$$\max_{|z|=1} |P'(z)| \leq n \max_{|z|=1} |P(z)| \quad (1.1)$$

The result is best possible and equality holds for the polynomial having all its zeros at origin. In the reverse direction it was proved by Turan [8] that if $P(z)$ does not vanish in $|z| > 1$, then

$$\max_{|z|=1} |P'(z)| \geq \frac{n}{2} \max_{|z|=1} |P(z)| \quad (1.2)$$

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Inequality (1) was refined by Aziz and Dawood by showing that under the same hypothesis that (1.2) can be replaced by

$$\max_{|z|=1} |P'(z)| \geq \frac{n}{2} \left\{ \max_{|z|=1} |P(z)| + \min_{|z|=1} |P(z)| \right\} \quad (1.3)$$

Both the inequalities (1.2) and (1.3) are sharp and equality holds for $P(z) = \alpha z^n + \beta$ where $|\alpha| = |\beta|$.

As a generalization of inequality (1.2) Malik [5] proved that if all the zeros of $P(z)$ lie in $|z| \leq k$, $k \leq 1$, then

$$\max_{|z|=1} |P'(z)| \geq \frac{n}{1 + k} \max_{|z|=1} |P(z)| \quad (1.4)$$

Equality in (1.4) holds for the polynomial $P(z) = (z + k)^n$, $k \leq 1$.

In the literature there exists several extensions and generalizations of inequalities (1.3) and (1.4)(see [2],[4]). recently Aziz and Shah [3] have proved the following generalization of inequality (1.2).

**Theorem 1.1.** If $P(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree $n$ having all its zeros in the disk $|z| \leq k$, $k \leq 1$ with $s$-fold zeros at origin, then for $|z| = 1$,

$$\max_{|z|=1} |P'(z)| \geq \frac{n + ks}{1 + k} \max_{|z|=1} |P(z)|. \quad (1.5)$$

The result is sharp and extremal polynomial $P(z) = z^s(z + k)^{n-s}$, $0 < s \leq n$.

In this paper we shall first present the following refinement and a generalization of Theorem 1.1.

**Theorem 1.2.** If $P(z)$ is a polynomial of degree $n \geq 1$ having all its zeros in $|z| \leq k$, $k \leq 1$ with $t$-fold zeros at the origin then for $1 \leq s \leq t + 1 \leq n$,

$$\max_{|z|=1} |P^s(z)| \geq \left( \frac{n + kt}{1 + k} \right) \left( \frac{n + kt}{1 + k} - 1 \right) \cdots \left( \frac{n + kt}{1 + k} - (s - 1) \right) \max_{|z|=1} |P(z)|$$

$$+ \mathcal{L}_s \left( \frac{n - t}{1 + k} \right) \frac{1}{k^t} \min_{|z|=k} |P(z)| \quad (1.6)$$
where

\[ L^s = 1 \text{ for } s = 1 \]

\[ = n(n-1) \cdots (n-s+2) \text{ for } s \geq 2. \]

The result is best possible and equality holds for the polynomial \( P(z) = (z + k)^n \), \( k \leq 1 \).

Remark. For \( t=0 \) and \( s=1 \), this reduces to the result due to Malik.

For \( k=1 \), we get the following result.

Corollary. If \( P(z) \) is a polynomial of degree \( n \) having all its zeros in \(|z| \leq 1\), with \( t \)-fold zeros at the origin then for \( 1 \leq s \leq t + 1 \leq n \),

\[
\max_{|z|=1} |P^s(z)| \geq \left( \frac{n+t}{2} \right) \left( \frac{n+t}{2} - 1 \right) \cdots \left( \frac{n+t}{2} - (s-1) \right) \max_{|z|=1} |P(z)|
\]

\[ + L^s \left( \frac{n-t}{2} \right) \min_{|z|=1} |P(z)|, \]

where

\[ L^s = 1 \text{ for } s = 1 \]

\[ = n(n-1) \cdots (n-s+2) \text{ for } n \geq 2. \]

Next we prove the following result which extends inequality (1.4) to the \( s \)th derivative.

Theorem 1.3. If \( P(z) \) is a polynomial of degree \( |z| \leq k \), \( k \leq 1 \), having all its zeros in \(|z| \leq k\), \( k \leq 1 \), then

\[
\max_{|z|=1} |P^s(z)| \geq \frac{n(n-1) \cdots (n-s+2)}{(1+k)^s} \max_{|z|=1} |P(z)|. \tag{1.7}
\]

The result is best possible with equality in (1.7) for the polynomial \( P(z) = (z + k)^n \).
2. Lemmas

For the proofs of these theorems, we need the following Lemmas.

**Lemma 2.1.** If \( P(z) \) is a polynomial of degree \( n \) having all its zeros in \( |z| \leq k \), then

\[
\min_{|z|=1} |P'(z)| \geq n \min_{|z|=1} |P(z)|
\]  

(2.1)

The result is best possible with equality for the polynomial \( P(z) = me^{i\alpha}z^n \), \( m > 0 \).

Lemma 2.1 is due to Aziz and Dawood [1].

**Lemma 2.2.** If \( P(z) \) is a polynomial of degree \( n \) having all its zeros in \( |z| \leq k \), \( k \geq 1 \), then

\[
\min_{|z|=k} |P^s(z)| \geq \frac{n(n-1)\cdots(n-s+1)}{k^s} \min_{|z|=k} |P(z)|
\]  

(2.2)

The result is best possible and equality in (2.2) holds for the polynomial \( P(z) = (z+k)^n \).

**Proof.** Since \( P(z) \) has all its zeros in \( |z| \leq 1 \). Let \( F(z) = P(kz) \) then \( F(z) \) has all its zeros in \( |z| \leq 1 \). Applying Lemma 2.1 to the polynomial \( F(z) \), we get

\[
\min_{|z|=1} |F'(z)| \geq n \min_{|z|=1} |F(z)|
\]

Equivalently,

\[
\min_{|z|=1} |P'(kz)| \geq \frac{n}{k} \min_{|z|=1} |P(kz)|
\]

or

\[
\min_{|z|=k} |P'(z)| \geq \frac{n}{k} \min_{|z|=k} |P(kz)|
\]  

(2.3)

\( P'(z) \) is a polynomial of degree \( n-1 \), therefore by (2.3), we have

\[
\min_{|z|=k} |P''(z)| \geq \frac{n(n-1)}{k^2} \min_{|z|=k} |P(z)|
\]

Proceeding in a similar way it follows that
\[
\min_{|z|=k} |P^s(z)| \geq \frac{n(n-1) \cdots (n-s+1)}{k^s} \min_{|z|=k} |P(z)|
\]

This completes the proof of Lemma 2.2. \qed

**Lemma 2.3.** If \( P(z) \) is a polynomial of degree \( n \) having all its zeros in \( |z| \leq k, \ k \geq 1 \), with \( t \)-fold zeros at the origin, then

\[
\max_{|z|=1} |P'(z)| \geq \frac{n+kt}{1+k} \max_{|z|=1} |P(z)| + \frac{n-t}{(1+k)k^t} \min_{|z|=k} |P(z)| \tag{2.4}
\]

The result is sharp and equality holds for the polynomial \( P(z) = z^t(z+k)^{n-t} \), \( 0 < t \leq n \).

**Proof.** If \( m = \min_{|z|=k} |P(z)| \), then \( m \leq |P(z)| \) for \( |z| = k \), which gives \( m |z|^t \leq |P(z)| \) for \( |z| = k \). Since all the zeros of \( P(z) \) lie in \( |z| \leq k \leq 1 \), with \( t \)-fold zeros at the origin, therefore for every complex number \( \alpha \) such that \( |\alpha| < 1 \), it follows (by Rouches Theorem for \( m > 0 \)) that the polynomial \( G(z) = P(z) + \frac{\alpha m}{1} z^t \) has all its zeros in \( |z| \leq k, \ k \leq 1 \), with \( t \)-fold zeros at the origin. So that we can write

\[
G(z) = z^t H(z) \tag{2.5}
\]

Where \( H(z) \) is a polynomial of degree \( n-t \) having all its zeros in \( |z| \leq k, \ k \leq 1 \).

From (2.5), we get

\[
\frac{zG'(z)}{G(z)} = t + \frac{zH'(z)}{H(z)} \tag{2.6}
\]

If \( z_1, z_2, \cdots, z_{n-t} \) are the zeros of \( H(z) \), then \( |z_j| \leq k \leq 1 \) and from (2.6), we have

\[
\text{Re} \left\{ \frac{e^{i\theta} G'(e^{i\theta})}{G(e^{i\theta})} \right\} = t + \text{Re} \left\{ \frac{e^{i\theta} H'(e^{i\theta})}{H(e^{i\theta})} \right\}
\]

\[
= t + \text{Re} \sum_{j=1}^{n-t} \frac{e^{i\theta}}{e^{i\theta} - z_j}
\]
\[ = t + \sum_{j=1}^{n-t} \text{Re} \left( \frac{1}{1 - z_j e^{-i\theta}} \right) \tag{2.7} \]

for points \( e^{i\theta}, \ 0 \leq \theta < 2\pi \), which are not the zeros of \( H(z) \).

Now if \( |w| \leq k \leq 1 \), then it can be easily verified that

\[ \text{Re} \left( \frac{1}{1 - w} \right) \geq \frac{1}{1 + k} \]

Using this fact in (2.7), we see that

\[ \left| \frac{G'(e^{i\theta})}{G(e^{i\theta})} \right| \geq \text{Re} \left( \frac{e^{i\theta}G'(e^{i\theta})}{G(e^{i\theta})} \right) \]

\[ = t + \sum_{j=1}^{n-t} \text{Re} \left( \frac{1}{1 - z_j e^{-i\theta}} \right) \]

\[ \geq t + \frac{n - t}{1 + k}, \]

which gives,

\[ |G'(e^{i\theta})| \geq \frac{n + tk}{1 + k} |G(e^{i\theta})| \tag{2.8} \]

for points \( e^{i\theta}, \ 0 \leq \theta < 2\pi \), which are not the zeros of \( G(z) \). Since inequality (2.8) is trivially true for points \( e^{i\theta}, \ 0 \leq \theta < 2\pi \), which are the zeros of \( P(z) \), it follows that

\[ |G'(z)| \geq \frac{n + tk}{1 + k} |G(z)| \text{ for } |z| = 1 \tag{2.9} \]

Replacing \( G(z) \) by \( P(z) + \frac{\alpha m}{k^t} z^t \) in (2.9), then we get

\[ |P'(z) + \alpha \frac{m}{k^t} z^{t-1}| \geq \frac{n + tk}{1 + k} |P(z) + \frac{\alpha m}{k^t} z^t| \text{ for } |z| = 1 \tag{2.10} \]
and for every $\alpha$ with $|\alpha| < 1$. Choosing the argument of $\alpha$ such that

$$|P(z) + \frac{\alpha m}{k^t} z^t| = |P(z)| + |\alpha| \frac{m}{k^t} \text{ for } |z| = 1,$$

it follows from (2.10), that

$$|P'(z)| + \frac{t|\alpha|m}{k^t} \geq n + tk \left\{ \frac{|P(z)| + |\alpha|m}{k^t} \right\} \text{ for } |z| = 1,$$

Letting $|\alpha| \to 1$, we obtain

$$|P'(z)| \geq \frac{n + tk}{1 + k} |P(z)| + \left\{ \frac{n + tk}{1 + k} - t \right\} \frac{m}{k^t}$$

$$= \frac{n + tk}{1 + k} |P(z)| + \frac{n - t}{1 + k} \frac{m}{k^t} \text{ for } |z| = 1.$$

This implies,

$$\max_{|z|=1} |P'(z)| \geq \frac{n + tk}{1 + k} \max_{|z|=1} |P(z)| + \frac{n - t}{(1 + k)k^t} \min_{|z|=k} |P(z)|$$

Which is the desired result. □

3. Proof of the theorems

**Proof of Theorem 1.2.** We prove this result with the help of mathematical induction. We use induction on $s$. For $s=1$, the result follows by Lemma 2.3. Assume that inequality (1.6) is true for $s=r$, that is we assume for $1 \leq r \leq t+1$,

$$\max_{|z|=1} |P^r(z)| \geq \left( \frac{n + tk}{1 + k} \right) \left( \frac{n + tk}{1 + k} - 1 \right) \cdots \left( \frac{n + tk}{1 + k} - (r-1) \right) \max_{|z|=1} |P(z)|$$

$$+ \mathcal{L}^r \frac{(n - t)}{(1 + k)k^t} \min_{|z|=k} |P(z)|$$

(3.1)

Where

$$\mathcal{L}^r = 1 \text{ for } r = 1$$
\[= n(n - 1) \cdots (n - r + 2) \text{ for } r \geq 2.\]

We show (1.6) holds for \(s = r + 1\) also. Since \(P(z)\) is a polynomial of degree \(n\) having all its zeros in \(|z| \leq k\), \(k \leq 1\), with \(t\)-fold zeros at the origin, therefore by Gauss-Lucas Theorem \(P^r(z)\) which is a polynomial of degree \(n-r\) has all its zeros in \(|z| \leq k\), \(k \leq 1\), with \(t-r\) fold zeros at the origin. Applying Lemma 2.3 to the polynomial \(P^r(z)\), we get,

\[
\max_{|z|=1} |P^{r+1}(z)| \geq \left\{ \frac{n - r}{1 + k} \right\}^r \max_{|z|=1} |P^r(z)| + \left( \frac{n - r}{1 + k} \right) \min_{|z|=k} |P^r(z)|
\]

(3.2)

Using Lemma 2.2, we get

\[
\max_{|z|=1} |P^{r+1}(z)| \geq \left\{ \frac{n + tk}{1 + k} - r \right\} \max_{|z|=1} |P^r(z)| + \left( \frac{n - t}{1 + k} \right) n(n - 1) \cdots (n - r + 1) \min_{|z|=k} |P(z)|
\]

This implies with the help of Lemma 2.1 that,

\[
\max_{|z|=1} |P^{r+1}(z)| \geq \left( \frac{n + kt}{1 + k} \right) \left( \frac{n + kt}{1 + k} - 1 \right) \cdots \left( \frac{n + kt}{1 + k} - (r - 1) \right) \left( \frac{n + kt}{1 + k} - (r + 1 - 1) \right) \max_{|z|=1} |P(z)| + \left( \frac{n - t}{1 + k} \right) n(n - 1) \cdots (n - (r + 1) + 2) \min_{|z|=k} |P(z)|.
\]

(3.3) shows that the result is true for \(s = r + 1\) also. We conclude with the help of mathematical induction that (1.6) holds for all \(1 \leq s < n\). This completes the proof of Theorem 1.2. \(\square\)

**Proof of Theorem 1.3.** Since \(P(z)\) has all its zeros in \(|z| \leq k\), \(k \leq 1\), therefore by Gauss-Lucas Theorem \(P^r(z)\) has all its zeros in \(|z| \leq k\), \(k \leq 1\), for \(1 \leq s < n\). Applying inequality (1.4) to the polynomial \(P^{s-1}(z)\) and proceeding similarly as in the above Theorem it follows that

\[
\max_{|z|=1} |P^s(z)| \geq \frac{n(n - 1) \cdots (n - s + 1)}{(1 + k)^s} \max_{|z|=1} |P(z)|.
\]
This proves Theorem 1.3. □

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