Empirical constraints on vacuum decay in the stringy landscape

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Abstract

It is generally considered as self evident that the lifetime of our vacuum in the landscape of string theory cannot be much shorter than the current age of the universe. Here I show why this lower limit is invalid. A certain type of “parallel universes” is a necessary consequence of the string-landscape dynamics and might well allow us to “survive” vacuum decay. As a consequence our stringy vacuum’s lifetime is empirically unconstrained and could be very short. Based on this counter-intuitive insight I propose a novel type of laboratory experiment that searches for an apparent violation of the quantum-mechanical Born rule by gravitational effects on vacuum decay. If the lifetime of our vacuum should turn out to be shorter than $6 \times 10^{-13}$ seconds such an experiment is sufficiently sensitive to determine its value with state-of-the-art equipment.

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1 Introduction

1.1 The stringy landscape and its decay

The set of ideas that is usually designated as “landscape of string theory” [1, 2] (abbreviated as “stringy landscape” or “stringscape”), has been hailed “as a possible radical change in what we accept as a legitimate foundation for a physical theory” [3].

This synthesis of string and inflationary theory describes the global universe as an eternally inflating “megaverse” that is continually producing “pocket universes” by tunnelling events between different vacua [4]. String theory predicts and describes a huge number (typically $10^{100}$) of vacua. We inhabit one pocket universe (from now on designated as “our universe”). Its vacuum (from now on designated as “our vacuum”) has a positive cosmological constant, and as a consequence our universe is entering a quasi de Sitter state presently.

Within string theory our vacuum (a “false vacuum”) must eventually decay to some other “true-vacuum” state that has a smaller energy density. This true vacuum features different low-energy theories and parameters. Therefore the true vacuum does not support life, i.e. humans die upon vacuum decay. Various authors explored selected, special decay channels and estimated the respective vacuum-decay rates (e.g. [5, 6, 7]). Their results form one of the most direct string-theoretical prediction of an empirically accessible parameter. The predicted lifetimes lie typically between the current age (13.6 billion years) and the recurrence time (about $10^{460}$ years) of our universe. It has been argued informally (and non-conclusively) that a summation over all possible stringscape decay channels possible might predict vacuum lifetimes that are much shorter than the current age of the universe [8].

1.2 Aims and structure of this paper

My aim is to develop concepts to constrain the rate of vacuum decay in the stringy landscape empirically and reliably. Below I assume that the landscape exists in the form currently laid out in the literature - not as an expression of faith but only as a purely heuristic basis for further conclusions and test proposals. My motto is taken from Joseph Polchinski who paraphrased Dirac
by writing: 
One should take serious all solutions of one’s equation.

The current age of our universe is widely believed to be a conservative lower limit on the lifetime of our de Sitter vacuum (e.g. [7, 10]). In section 2 I explain why - contrary to what common sense might suggest at first sight - this belief is wrong in the stringy landscape. Sections 3.1 (with appendix 5 that presents an auxiliary calculation) and 3.2 contain the main result of this paper, a proposal for an experimental procedure to determine either the value of the vacuum’s decay rate or to place a firm upper limit to it. Based on the insight that vacuum decay might be extremely fast, section 3.3 estimates the maximal mean lifetime of the vacuum that could still be determined in a laboratory experiment with current technology. Section 4 concludes.

2 The lifetime of our vacuum can be much smaller than the current age of the universe in the stringy landscape

After reviewing the concept “parallel universe” (section 2.1) and its inevitability in the stringy landscape (section 2.2), I demonstrate in subsection 2.3 why the (admittedly counter-intuitive) title of this section is true.

2.1 Overview - What are “parallel universes”?

Parallel universes are defined as an infinity of distinct universes that are completely identical to ours until a random decision makes subsets of them different in the random-decision results. In a review of this concept Tegmark [11] distinguished between different types (“levels”) of parallel universes. “Level I” denote universes parallel in Minkowski space whereas e.g. “level III” universes coexist in Hilbert space. The next subsection 2.2 reviews why the existence of “level I” parallel universes is inevitable in the stringy landscape. Other types of parallel universes, like the quantum-mechanical “many worlds” (“level III” [11]) are not considered in this paper.

2.2 Level I parallel universes (a.k.a. “parallel Hubble volumes”) are inevitable in the stringy landscape

Coleman & de Luccia [13] showed that pocket universes born in a tunnelling event are negatively curved i.e. they are spatially infinite. Pocket universes are formed in this way and are therefore our universe is spatially infinite in the stringy landscape.

1I would add: this is not a categorical imperative but a heuristic suggestion!
2Their existence would strengthen the conclusion of subsection 2.3. However, it is not yet clear whether they are a necessary element of the stringy landscape [12, 13]
There must be parallel universes in a spatially infinite universe [14] [15]. An infinite pocket universe contains a countably infinite ensemble (i.e., a set of power $\aleph_0$ [16]) of “Hubble volumes” [17]. “Hubble volumes” are defined here as the interior of light cones that extend back from spacetime points in Minkowski spacetime at the present age of the universe to the “time of recombination”, before which the universe was opaque to electromagnetic radiation. Because the number of possible histories in a Hubble volume is finite there must be an infinite number of Hubble volumes with completely identical histories up to the present, that need not be identical in the future [15]. In the following I will call these: “parallel Hubble volumes”.

2.3 We survive rapid vacuum decay if parallel universes exist

Here I demonstrate that the existence of parallel universes allows us to “survive” vacuum decay. I recently argued [18] that the Standard-Model vacuum could have a much shorter lifetime than our (pocket) universe’s age, if parallel universes exist. Here I generalize this argument and apply it to the stringy landscape.

It is well known that vacuum tunnelling events are exponentially unlikely to take place “all over the universe” [1]. Rather they are typically seeded in “critical bubbles” which finite size depends on details of the transition [13, 1]. This bubble then expands with a velocity that I will assume to be very close or equal to $c$ in this paper.

Let “$q$” be the probability that there is no critical bubble in a given Hubble volume, i.e., vacuum decay did not “took place”, yet. For vacuum lifetimes that are much shorter than the present age of the universe, $q$ can be extremely small, but it always remains finite. The number of “parallel universes” since the big bang is reduced by vacuum decay from a number of $\aleph_0$ to $\aleph_0 \times q$ Hubble volumes. This “reduction process” cannot be perceived by human observers, because, under the assumptions stated above, relativistic causality prevents any indication of an impending decay to reach her on a time scale longer than the minimum one necessary for becoming conscious of the impending doom [18].

A basic assumption of the stringy landscape is that the perceived properties our Hubble volume do not have to be likely. Anthropic reasoning asserts that experience only requires that Hubble volumes with such properties exist somewhere in our pocket universe. In the same sense the stability of our vacuum does not need to be likely. It is enough if un-decayed parallel Hubble volumes exist somewhere in our universe. In other words: From the fact that we perceive our vacuum to be stable we can only infer that stable parallel Hubble volumes

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3I generalized the “quantum suicide” concept [19], abstracting the “suicide” and extending it to all types of parallel universes. The fact that the existence of parallel universes allow us to survive vacuum decay was first pointed out in a science-fiction short story [20].

4To a fractional precision of at least $10^{-20}$. This condition is not necessarily but possibly fulfilled in a vacuum transition (it is e.g. for the decay of the standard model (SM) vacuum) [19].
exist somewhere in our pocket universe.
The relation
\[ \aleph_0 = \aleph_0 q \]  
(1)
is true for all finite q. Vacuum decay, no matter how fast, can therefore not reduce the number of \( \aleph_0 \) parallel Hubble volumes. Because \( \aleph_0 \) stable Hubble volumes do exist in our pocket universe even in the presence of arbitrarily fast vacuum decay, one cannot derive any lower limit on the lifetime of our stringy de Sitter vacuum from the mere fact “that we still exist”. In reality vacuum decay could be extremely fast.

3 How to experimentally determine or constrain the lifetime of our vacuum

Subsections (3.1, 3.2) propose that the decay probability of a quantum mechanical system can be made to depend on the decay rate of the vacuum in a suitably designed laboratory experiment. Subsection 3.3 asserts that such an experiment might be sensitive enough to determine a sufficiently large vacuum-decay rate with state-of-the-art methods.

3.1 How to slow down vacuum decay in the laboratory

Here I calculate how many vacuum-decay events “\( \Delta n \) less occur at a given point in space within a time period \( \Delta t \)” as the consequence of “blowing up” a massive sphere mechanically. The expected number of vacuum-decay events “\( n \)” that affect a given point P in space within the time period “\( \Delta t \)” is given as:
\[ n = \rho \times \left( 4/3 \pi c^3 \times (\Delta t)^3 - \Delta V_4 \right). \]  
(2)
Here \( \rho \) is the density of “critical vacuum-decay bubbles” (see section 2.3) in 4-space. \( \Delta V_4 \) is a small general-relativistic correction that depends on the distribution of masses in space (see below). The mean lifetime of our vacuum “\( t_{vac} \)” is defined as
\[ t_{vac} = \left( \frac{1}{\rho c^3} \right)^\frac{1}{2} \]  
(3)
Let us evaluate \( \Delta V_4 \) for a sphere centred at a point P and filled with a perfect fluid of constant density with an initial radius \( r_{f1} \) and empty space beyond it. It can then be shown within general relativity (GR) (see appendix 5) that:
\[ \Delta V_4(r_{f1}) = -\frac{2}{5} \pi r_S \Delta tr_{f1}^2. \]  
(4)
Here \( r_S = 2 G m/c^2 \) is the Schwarzschild radius of the sphere, m its mass and G Newton’s constant of gravitation. If the sphere is “mechanically blown up”

\(^5\text{Affect means that the transition front of a vacuum decay event eventually reaches P.}\)
Figure 1: Illustration of the proposed setup, described in section 3.2, in the initial state, before the measurement is performed (assuming that vacuum decay has not taken place). The space-time diagram depicts part of our pocket universe. The full cone-shaped lines symbolize - not to scale - the spatial confines of a parallel Hubble volume that extends back from the location of the experiment. Today one can look back to a distance of $H \approx 10^{10}$ light years. The lower dot-dashed line delineates the time of recombination (380000 years after the big bang), the universe is transparent to light only above the line. The small circle with a diameter of $D \approx 10^4$ km stands for the earth today (13.6 billion years after the big bang). The mechanical sphere and the measured system A are on one side of the earth, and the observer’s worldline - marked by the thick dashed line - on the other. The quantum state of A is indicated here and in fig.2.
Figure 2: Illustration of the experiment in its three possible final states after a just performed experiment (that is described in subsection 3.2.1). The distance between the location of three final states in parallel Hubble volumes has been estimated to $L \approx 10^{10^{29}}$ m[11]. The sphere is inflated in final state 2 upon the measurement result “1” of system A. For final state 3 the part of spacetime in the true-vacuum state is hatched. The vacuum transition began in a bubble of diameter “B”. The value of B depends on the precise nature of the transition. E.g. for a SM transition $B \approx 10^{-33}$ m.
The “blow up” thus “destroys” a volume of 4-space

$\Delta\Delta V_4^4 = \frac{2}{5}\pi r S \Delta t (r_{f2}^2 - r_{f1}^2).$  

(6)

The blow up of the sphere therefore decreases the total number of vacuum decays during a time $\Delta t$ by:

$\Delta n = \rho \times \Delta \Delta V_4.$  

(7)

decays. Vacuum decay can be slowed down by changing the distribution of mass in space.

3.2 Proposal for an experiment that probes vacuum stability via apparent violations of the Born rule

This section proposes an experimental setup to determine the effect of a reduced number of vacuum-decay events. A schematic picture of the initial and final state of this experiment is provided in figs. 1 and 2 respectively.

3.2.1 Detailed description of the experiment

At a time “$t_m$” (the origin $t=0$ is set to the recombination time) the state of a quantum-mechanical system $A$ is measured (see fig 2 for the initial state). With a certain probability “1-p”, that is given be the Born rule, the outcome be ”0”, with the probability $p$ it be “1”. If, and only if, the outcome is “1”, a massive sphere that is independent of $A$ but located at the same position in space, is mechanically and automatically “blown up” (defined in the previous section 3.1). The blow up is to be concluded within a time period $\Delta t_{bu}$. An inertial observer “$O$” is located at a large enough distance $d$ from the sphere, so that $d/c \gg \Delta t_{bu}$. The earliest time that $O$ can measure the outcome of the experiment is a time $t_o \approx t_m + d/c$. To avoid (bound to be controversial) questions concerning the nature of quantum-mechanical measurements, I assume that $O$’s measurement takes place instantaneously at time $t_o$. This is the most conservative possible option, because - as we will see below - it is under this assumption that the proposed experiment leads to the least restrictive upper limits on the vacuum-decay rate.

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6 This result is intuitive: Near a massive sphere space is dilated in the radial direction and time is compressed. Outside the sphere the two effects cancel, so the 4-volume is exactly the same as in Minkowski space. Within a filled perfect-fluid sphere time is compressed to a greater degree so that the spacetime volume within the sphere becomes smaller upon inflation. The “destruction” of spacetime is mainly due to general-relativistic time dilation. There are fewer vacuum-decay events upon inflation because “time ticks slower within a massive sphere”.

7 The value of $\Delta t$ in subsection 3.3 would be longer under different assumptions, and this would decrease $t_{crit}^{vac}$. 

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3.2.2 Determination of the effective probability $p_e$

My aim is to calculate the “effective” probability “$p_e$” (rather than $p$) with which observer O finds the outcome “1” due to the effect of vacuum decay. The basic idea is that there are “more” parallel universes with outcome “1” because their vacuum-decay proceeds slower according to the arguments in section 3.1. Intuitively this should mean that the $p_e$ increases with the vacuum-decay rate. But we run into two problems:

Firstly, according to the argument of subsection 2.3 once a quantum mechanical measurement is performed, the vacuum decay rate can have no more effect on its outcome probability. We addressed this problem by delaying the earliest time at which the system can be measured by $\Delta t \approx d/c$ in the experimental design. At least during this time period the outcome-dependent vacuum-decay rates can “take effect”.

Secondly there is an infinite number of parallel Hubble volumes. Therefore the ratio of outcomes “0” and “1” is a quotient of infinite numbers and therefore ill defined. A gauge-invariant regulator for the countably infinite number of Hubble volumes is needed. This problem is closely related to the problem of counting pocket universes with different vacuum types in the megaverse [22]. A method proposed by Easther et al. [23] to attack this problem can be readily adapted to the present case. Easther et al. consider a collection of a finite number $N$ of initial points. In our case these points are chosen at the time of recombination in $N$ different “parallel Hubble volumes” in such a way that their world lines pass up to the location of observer O at time $t_o$.

These world lines can end in three final states (see fig. 2):

1. $|\text{our vacuum, outcome } 0\rangle$
2. $|\text{our vacuum, outcome } 1, \text{ sphere blown up} \rangle$
3. $|\text{true vacuum, no observer}, \text{ in which our vacuum has decayed, and no human life is possible} \rangle$

Easther et al. [23] propose to estimate $p_e$ as the ratio of worldlines that end in state 2 to the total number of world lines that support “observers”. They further argue anthropically (and in full accordance with the argument of our section 2.3) that final state 3, in which no human observers exist, must be, quote, “dropped from consideration”.

The number of world lines ending in state 1 is given as:

$$N_1 = N \times P_{d1} \times (1 - p)$$

(8)

Here $P_{d1}$ is the probability for no vacuum decay on the world line within the time period $t_o \approx 13.6$ billion years since the big bang:

$$P_{d1} = e^{-n_0}$$

(9)

Here $n_0$ is the expected value of vacuum decay events $n$ (eq. 2) for $\Delta t = t_0$. The number of world lines ending in state 2 is given as:

$$N_2 = N \times P_{d2} \times p$$

(10)

8In this case the world line passes up to the position where the observer O would have been had the vacuum not decayed.
The probability for vacuum decay is smaller for this case due to the smaller amount of 4-space from the “blow up” of the sphere ($\Delta n$ is given in eq.(7)).

\[
P_{d2} = P_{d1} \times e^{\Delta n}
\]

Finally

\[
N_3 = N - N_1 - N_2
\]

The probability $p_e$ to find result “1” in a decaying vacuum can then be derived from eqs.(9,11) as (analogous to eq. (4.2) in Easther et al. [23])

\[
p_e = \frac{N_2}{N_1 + N_2} = \frac{pe^{\Delta n}}{(1 - p) + pe^{\Delta n}}
\]

and the effective probability for result “0” is $1 - p_e$. For a stable vacuum $\Delta n = 0$, and one gets the usual result $p_e = p$. If $\Delta n$ is finite, $p_e$ is larger than $p$. In this case the lifetime of the vacuum $t_{\text{vac}}$ can be inferred from eqs.(13,3,7).

The implied violation of the Born rule is only apparent because the increased value of $p_e$ is a subjective consequence of the increased probability of survival to vacuum decay when result “1” was obtained.

### 3.3 Sensitivity of the experiment - achievable limit on the vacuum-decay lifetime

I roughly estimate the sensitivity of a reasonably sized experiment as described in the previous subsection 3.2.

Upon result “1” a sphere with a mass of 1000 kg be blown up from a radius of 0.5 m to 10 m within $\Delta t_{\text{bu}} = 5$ msecs. This could be approximated e.g. by a chemical explosion of 1 kt of TNT - within air. Observer O is located at a distance $d=10000$ km (corresponding to two distant sites on earth) from the sphere so that $d/c = 30$ msec and vacuum can “take effect” (see second paragraph of subsection 3.2.2) for $\Delta t = 25$ msec. Inserting these values into eq.(6) one finds via eqs.(3,7) that $\Delta n$ becomes $> 1$ for a “critical” mean lifetime of our vacuum $t_{\text{vac}}$ of:

\[
t_{\text{vac}}^{\text{crit}} < 6 \times 10^{-13} \text{ seconds}
\]

Should such an experiment find that $p_e$ (eq.(13)) is equal to $p$ within errors the lifetime of the vacuum must be $> 6 \times 10^{-13}$ seconds. Shorter lifetimes increase $p_e$. I stress again the counter-intuitive fact that such short lifetime are not empirically ruled by the fact that we are “still alive” (see section 2.3).

While $10^{-13}$ sec is about 30 orders of magnitude smaller than the current age of the universe, it is still about 31 orders of magnitude larger than the Planck time, so there is a lot of “room at the bottom” for possibly “detectable” lifetimes of our vacuum.

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9 The velocity of TNT debris is estimated $\approx 2$ km/sec until it is slowed down to much smaller velocities at $\approx 10$ m due to the swept-up air mass.
4 Conclusion and outlook

Because the landscape firmly predicts an infinity of “parallel Hubble volumes”, some “copies” of our world will always survive, even if the probability to survive vacuum decay in any individual world is arbitrarily small. Experience only tells us that at least one surviving copy exists, and cannot be the basis for an empirical limit on the mean lifetime of the vacuum. Therefore it is definitely incorrect to consider the current age of the universe as an upper limit to the lifetime of our vacuum within the stringy landscape.

I proposed a novel type of laboratory experiment to set a less restrictive but more accurate upper limit to the lifetime of our vacuum. If the mean lifetime of the vacuum is shorter than about 0.6 picoseconds (and such a short lifetime is not empirically ruled out) vacuum-decay could be detected with a technologically feasible experiment.

The proposed experiment is unlikely to be the “last word” on this problem. It is not fully understood how to calculate probabilities or how to use transfinite numbers in the landscape, yet. But one conclusion seems fairly definite: the lifetime of our vacuum - a quantity calculable with string theory (even if a calculation of the total decay rate is still outstanding) - can be subjected to a direct laboratory test.

A detection of vacuum decay would lend support to crucial concepts of the stringy landscape like the existence of parallel universes and the correctness of anthropic reasoning. It seems of course more likely that the experiment proposed in section 3.2 will merely yield an upper limit to the decay rate of our stringy vacuum. But even this will be a genuine experimental constraint on a parameter of the stringy landscape.

Polchinski’s motto, quoted in section 1.2 urges us to take the $10^{\text{hundreds}}$ vacuum solutions of string theory serious. This paper argues to do the same with the $\aleph_0$ Hubble volumes in our pocket universe.

References

[1] A. Linde, Particle physics and inflationary cosmology, Harwood, Chur, 1990.

[2] L. Susskind, The anthropic landscape of string theory, hep-th/0302219v1.

[3] S. Weinberg, Living in the Multiverse, hep-th/0511037v1.

[4] B. Freivogel, M. Kleban, M. Rodriguez Martinez, L. Susskind, Observational Consequences of a Landscape, hep-th/0505232v2.

[5] A. R. Frey, M. Lippert, B. Williams, The fall of stringy de Sitter, Phys.Rev. D68 (2003) 046008.

[6] A. Ceresole, G. Dall’ Agata, A. Giryavets, R. Kallosh, A. Linde, Domain walls, near-BPS bubbles and probabilities in the landscape, hep-th/0605266v2.

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[7] S. Kachru, R. Kallosh, A. Linde, S.P. Trivedi, *de Sitter vacua in string theory* Phys.Rev. D68 (2003) 046005.

[8] L. Motl, *Landscape decay channels*, motls.blogspot.com/2005/11/landscape-decay-channels.html, 2005.

[9] J. Polchinski, *The Cosmological Constant And the String Landscape*, hep-th/0603249v2.

[10] F. Denef, M.R. Douglas, *Computational complexity of the landscape I*, hep-th/0602072v2.

[11] M. Tegmark, *Parallel universes*, In: *Science and the Ultimate Reality: From Quantum to Cosmos*, J.D. Barrow, P.C.W. Davies, C.L. Harper eds., Cambridge University Press, Cambridge, 2003; astro-ph/0302131v1.

[12] L. Susskind, priv. comm. (2006).

[13] S.R. Coleman, F. DeLuccia *Gravitational Effects On and Of Vacuum Decay* Phys.Rev. D 21 (1980) 3305.

[14] Ellis G.F.R., Brundrit G.B., 1979, *Life in the Infinite Universe*, Q.Jl.R.astr.Soc. 20, 37-41.

[15] Garriga J., Vilenkin A., 2001, *Many worlds in one*, Phys.Rev. D64, 043511; gr-qc/0102010.

[16] G. Cantor, 1895, *Beiträge zur Begründung der transfiniten Mengenlehre*, Math.Ann. 46, 481

[17] J.D. North, *The measure of the universe*, Dover, New York, 1999; Chapt. 17.

[18] R. Plaga, *New physics beyond the standard model of particle physics and parallel universes*, Phys.Lett. B634 (2006) 116.

[19] E. Squires, *The Mystery of the Quantum World*, Institute of Physics Publishing, Bristol, 1986.

[20] J. Gribbin, *The doomsday device*, Analog 105 (1985) 120.

[21] Einstein A., 1956, *Grundzüge der Relativitätstheorie*, (Vieweg & Sohn, Braunschweig).

[22] A. Vilenkin, *Probabilities in the landscape*, hep-th/0602264v2.

[23] R. Easther, E.A. Lim, M.R. Martin, *Counting pockets with World Lines in Eternal Inflation*, astro-ph/0511233v3.

[24] Schwarzschild K., 1916, *Über das Gravitationsfeld einer Kugel aus inkompressibler Flüssigkeit nach der Einsteinschen Theorie*, Sitzungber. Preuß. Akad. Wiss. Berlin, Kl. Math.-Phys. Tech. **18**, 424-434.
5 Appendix - The 4-volume of perfect-fluid sphere

The aim of this appendix is to calculate the 4-volume of a perfect-fluid sphere in general relativity to derive eq.(1). Consider a incompressible-fluid sphere. Both in the interior and to the exterior of such a sphere only the diagonal components of the metric tensor are non zero. In the interior the metric components are[24]

(I use a spacelike (−,+,+,+) metric):

\[
\begin{align*}
g_{tt} &= \frac{1}{2} \left( 3 \sqrt{1 - r_S/r_f} - \sqrt{1 - \frac{r_S r^2}{r_f}} \right)^2 \\
g_{rr} &= -\left( 1 - \frac{r_S r^2}{r_f^3} \right)^{-1} \\
g_{\theta\theta} &= -r^2 \\
g_{\phi\phi} &= -(r \sin(\theta))^2
\end{align*}
\]

(15)

\(r, \theta, \phi\) are the usual polar coordinates. \(r_f\) is the radius of the sphere, \(m\) is the total mass of the sphere, \(r_S\) is the Schwarzschild radius \((2 \ G \ m)/c^2\) and \(G\) is the constant of gravitation. The space-time volume \(V_4\) of this 3-sphere extended in time from \(t_1\) to \(t_2\) is the desired result:

\[
V_4 = \int_{t_1}^{t_2} \int_0^{r_f} \int_0^{2\pi} \int_0^\pi \sqrt{- \det(g)} \ dt \ dr \ d\theta \ d\phi = \left( \frac{4}{3} \pi r_f^3 - \frac{2}{5} \pi r_f^2 r_s \right) \Delta t
\]

(16)

Here \(\Delta t = t_2 - t_1\). Because \(r_S/r_f \ll 1\) in all laboratory situations, terms of order 2 and higher in this quotient were neglected for the final expression. At the exterior to the sphere the diagonal components of the metric components are[24]:

\[
\begin{align*}
g_{tt} &= (1 - r_S/r_f) \\
g_{rr} &= (1 - r_S/r_f)^{-1} \\
g_{\theta\theta} &= -r^2 \\
g_{\phi\phi} &= -(r \sin(\theta))^2
\end{align*}
\]

(17)

Because here \(g_{rr} = 1/g_{tt}\), \(\det(-g)\) (and therefore the space-time volume exterior to the sphere) is identical to the one in flat space time.