Stability of the line soliton of the Kadomtsev–Petviashvili-I equation with the critical traveling speed

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Abstract

We consider the orbital stability of solitons of the Kadomtsev–Petviashvili-I equation in \( \mathbb{R} \times (\mathbb{R}/2\pi \mathbb{Z}) \) which is one of a high dimensional generalization of the Korteweg–de Vries equation. In [2], Benjamin showed that the Korteweg–de Vries equation possesses the stable one soliton. We regard the one soliton of the Korteweg–de Vries equation as a line soliton of the Kadomtsev–Petviashvili-I equation. Zakharov [39] and Rousset–Tzvetkov [30] proved the orbital instability of the line solitons of the Kadomtsev–Petviashvili-I equation on \( \mathbb{R}^2 \). The orbital instability of the line solitons of the Kadomtsev–Petviashvili-I equation on \( \mathbb{R} \times (\mathbb{R}/2\pi \mathbb{Z}) \) with the traveling speed \( c > 4/\sqrt{3} \) was proved by Rousset–Tzvetkov [31] and the orbital stability of the line solitons with the traveling speed \( 0 < c < 4/\sqrt{3} \) was showed in [33]. In this paper, we prove the orbital stability of the line soliton of the Kadomtsev–Petviashvili-I equation on \( \mathbb{R} \times (\mathbb{R}/2\pi \mathbb{Z}) \) with the critical speed \( c = 4/\sqrt{3} \) and the Zaitsev solitons near the line soliton. Since the linearized operator around the line soliton with the traveling speed \( 4/\sqrt{3} \) is degenerate, we can not apply the argument in [31] [33]. To prove the stability of the line soliton, we investigate the branch of the Zaitsev solitons.

1 Introduction

We consider the Kadomtsev–Petviashvili-I equation:

\[
(u_t + u_{xxx} + uu_x)_x - u_{yy} = 0 \quad (t, x, y) \in \mathbb{R} \times \mathbb{R} \times \mathbb{T},
\] (1.1)
where $T = \mathbb{R}/2\pi \mathbb{Z}$. In [16], Kadomtsev and Petviashvili introduced the equation (1.1) as a model equation for the propagation of long waves weakly modulated in the transverse direction. The local and global well-posedness of the KP-I equation on $\mathbb{R}^2$ was showed by [11, 14, 17, 26, 35]. The Cauchy problem of the KP-I equation for initial datum which are localized perturbations of a non-localized traveling wave solution was studied by [7, 27]. In [13], Ionescu and Kenig proved the global well-posedness of the KP-I equation on $\mathbb{R} \times \mathbb{T}$ for initial data in the second energy space $\mathcal{Z}^2$, where

$$\mathcal{Z}^s = \{ u : \|u\|_{\mathcal{Z}^s} < \infty \}$$

and

$$\|u\|_{\mathcal{Z}^s} = \| (1 + |\xi| + |k\xi^{-1}|)^s \hat{u}(\xi, k) \|_{L^2(\mathbb{R}^2 \times \mathbb{Z}_k)}.$$ 

The conservation laws

$$M(u) = \int_{\mathbb{R} \times \mathbb{T}} |u|^2 dxdy$$

and

$$E(u) = \int_{\mathbb{R} \times \mathbb{T}} \left( |\partial_x u|^2 + |\partial_x^{-1} \partial_y u|^2 - \frac{1}{3} u^3 \right) dxdy$$

of (1.1) identify the space $\mathcal{Z}^1$ as the energy space of (1.1).

Solitons are nontrivial traveling wave solutions of the KP-I equation with the form $u(t, x, y) = Q(x - ct, y)$. The function $Q(x - ct, y)$ is soliton of the KP-I equation on $\mathbb{R} \times \mathbb{T}$ if and only if the function $Q$ is a nontrivial solution to the stationary equation

$$-u_{xx} + \partial_x^{-2} u_{yy} + cu - \frac{1}{2} u^2 = 0, \quad (x, y) \in \mathbb{R} \times \mathbb{T}. \quad (1.2)$$

In [20], Manakov et al. showed that the KP-I equation possesses the lump solitons

$$\phi_c(x - ct, y) = \frac{8c(1 - \frac{c}{3}(x - ct)^2 + \frac{c^2}{3}y^2)}{1 + \frac{c}{3}(x - ct)^2 + \frac{c^2}{3}y^2}.$$ 

In [3, 4, 5], de Bouard and Saut proved the existence and the stability of the ground states of the stationary equation of the generalized KP-I equation on $\mathbb{R}^2$.

The KP-I equation is one of a high dimensional generalization of the Korteweg–de Vries equation:

$$u_t + uu_{xx} + u u_x = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}.$$ 

The KdV equation possesses the one solitons:

$$Q_c(x - ct) = 3c \cosh^{-2} \left( \frac{\sqrt{c}(x - ct)}{2} \right).$$

The orbital stability of the one solitons of the KdV equation was showed by Benjamin [2]. The asymptotic stability of the one solitons of the KdV equation on weighted space was proved by Pego and Weinstein [29] and Mizumachi [23]. In [21, 22], Martel and Merle showed the asymptotic stability of the one solitons of the generalized KdV equation
for perturbations on the energy space $H^1(\mathbb{R})$ by establishing the Liouville theorem. In [18, 27, 35], it is showed that the stationary equation (1.2) has the Zaitsev solitons

$$Z(a)(x, y) = \frac{12(1-a^2)\left(1-a\sqrt{2-a^2}\cosh\left(\frac{\sqrt{1-a^2}}{3^{1/4}}x\right)\cos y\right)}{\sqrt{3}\left(\cosh\left(\frac{\sqrt{1-a^2}}{3^{1/4}}x\right) - a\sqrt{2-a^2}\cos y\right)^2}$$

with the traveling speed

$$c(a) = \frac{4 - 2a^2 + a^4}{\sqrt{3}(1-a^2)}.$$

Then, $Z(0) = Q_{4/\sqrt{3}}$.

In this paper, we regard the one solitons of the KdV equation as line solitons of the KP-I equation on $\mathbb{R} \times \mathbb{T}$. By using the Lax pair structure of the KP-I equation, Zakharov showed the instability of the line soliton as the KP-I equation on $\mathbb{R}^2$. The spectral stability of the line solitons as the KP equation was obtained by Alexander–Pego–Sachs [1]. Using the method developed by Grenier [9], Rousset–Tzvetkov [30] proved the orbital instability of the line solitons as the generalized KP-I equation for localized perturbations on $\mathbb{R}^2$. In [31, 33], Rousset and Tzvetkov showed that the line soliton $Q_c$ is orbitally stable as the KP-I equation on $\mathbb{R} \times \mathbb{T}$ for $0 < c < \frac{4}{\sqrt{3}}$ and orbitally unstable for $c > \frac{4}{\sqrt{3}}$. To show the instability of line solitons of the generalized KP-I equation, Rousset and Tzvetkov proved the existence of a strongly stable manifolds of line solitons associated to the most negative eigenvalue of the linearized operator around the line soliton in [33]. The spectral instability of periodic traveling wave solutions of the generalized KdV equation as the generalized KP equation was showed by Johnson and Zumbrun [15]. The stability of the line soliton as the KP-II equation was confirmed the heuristic analysis by Kadomtsev and Petviashvili [16]. The stability of the line solitons as the KP-II equation for decaying perturbations was showed by Villarroel and Ablowitz [34]. Mizumachi and Tzvetkov proved the orbital stability and the asymptotic stability of the line solitons as the KP-II equation in $L^2(\mathbb{R} \times \mathbb{T})$ by applying the Bäcklund transformation. The asymptotic stability of the line solitons as the KP-II equation on $\mathbb{R}^2$ has proved by Mizumachi [24, 25].

Now let us introduce our result.

**Theorem 1.1.** There exists $a_0 > 0$ such that for $0 \leq a < a_0$ the soliton $Z(a)(x - c(a)t, y)$ is orbitally stable as a solution of the KP-I equation (1.1). More precisely, for every $\varepsilon > 0$, there exists $\delta > 0$ such that for $u_0 \in Z^2$ with

$$\|u_0 - Z(a)\|_{Z^1} < \delta,$$

the solution $u$ to the KP-I equation (1.1), defined by (1.3) with the initial data $u_0$ satisfies

$$\sup_{t \geq 0} \inf_{(x_0, y_0) \in \mathbb{R} \times \mathbb{T}} \|u(t, \cdot, \cdot) - Z(a)(\cdot - x_0, \cdot - y_0)\|_{Z^1} < \varepsilon.$$

**Remark 1.2.** Since $Z(0) = Q_{4/\sqrt{3}}$ and $c(0) = 4/\sqrt{3}$, Theorem 1.1 yields the orbital stability of the line soliton with the critical speed $4/\sqrt{3}$. 


The proof of the stability of the Zaitsev solitons $Z(a)$ for $0 < a < a_0$ follows the Lyapunov function method in [10]. For $0 < c < 4/\sqrt{3}$, the orbital stability of the line solitons as the KP-I equation follows the coercive type estimate of the linearized operator around the line solitons in [33]. In the case $c = 4/\sqrt{3}$, the linearized operator around the line soliton $Q_{4/\sqrt{3}}$ is degenerate. Therefore, to prove the orbital stability of $Q_{4/\sqrt{3}}$, we can not apply the argument in [10, 33]. In [33], Rousset–Tzvetkov showed the spectral instability of line solitons as the KP-I equation implies the orbital instability of the line solitons. Since the line soliton $Q_{4/\sqrt{3}}$ is spectrally stable, we can not show the instability of $Q_{4/\sqrt{3}}$ by applying the arguments in [8, 33]. To cover the degeneracy of the linearized operator of the stationary equation (1.2), we apply the argument of [36, 37] for the stability of line solitary waves with a critical exponent. The line solitary waves with a critical exponent is a bifurcation point where the branch of line solitary waves and the branch of two dimensional solitary wave connect. In [36, 37], to prove the stability of line solitary waves with a critical exponent, the author showed the positivity of the second order term of $L^2$-norm of the two dimensional solitary waves with respect to the bifurcation parameter. In the case [36, 37], since the second order term of $L^2$-norm of the two dimensional solitary waves is not zero, we can cover the degeneracy of the linearized operator around the line solitary wave with the critical exponent by the forth order term of a Lyapunov function.

Our plan of the present paper is as follows. In Section 2, we show the increase of the $L^2$-norm of the Zaitsev solitons with respect to the bifurcation parameter $a$ and prove the estimate of the Lyapunov function around the Zaitsev solitons and the line soliton $Q_{4/\sqrt{3}}$. In Section 3, we prove the orbital stability of the line soliton $Q_{4/\sqrt{3}}$ with the critical traveling speed. In Section 4, we show the orbital stability of the Zaitsev solitons near the line soliton $Q_{4/\sqrt{3}}$.

2 Preliminaries

In this section, we show the estimate of the Lyapunov function around the Zaitsev solitons and the line soliton $Q_{4/\sqrt{3}}$.

We show the increase to the $L^2$-norm of the Zaitsev solitons with respect to the parameter $a$.

Lemma 2.1. The following hold.

\[ \partial_a \| Z(a) \|_{L^2}^2 \big|_{a=0} = \partial_a^2 \| Z(a) \|_{L^2}^2 \big|_{a=0} = \partial_a^3 \| Z(a) \|_{L^2}^2 \big|_{a=0} = 0 \]

and

\[ \partial_a^4 \| Z(a) \|_{L^2}^2 \big|_{a=0} = 4^3 \cdot 3^{13/4} \pi \int_{\mathbb{R}} f^{-4} dx, \]

where

\[ \partial_a^n \| Z(a) \|_{L^2}^2 \big|_{a=0} = \lim_{a \to 0} \partial_a^n \| Z(a) \|_{L^2}^2. \]
Proof. Let $\beta = a\sqrt{2 - a^2}$; $f(x) = \cosh x$ and $g(y) = \cos y$. Then, we have

$$1 - a^2 = \sqrt{1 - \beta^2},$$

$$2k \int_{\mathbb{R}} f^{-2k}(x)dx = (2k + 1) \int_{\mathbb{R}} f^{-2(k+1)}(x)dx \quad (2.1)$$

and

$$\|Z(a)\|_{L^2}^2 = \left(\frac{12^2(1 - \beta^2)^{3/4}}{3^{3/4}}\right) \int_{\mathbb{R} \times \mathbb{T}} \frac{(1 - \beta f(x)g(y))^2} {(f(x) - \beta g(y))^4} dxdy.$$

By the elementally calculation, we obtain the following equations.

$$\frac{3^{3/4}}{12^2} \partial_\beta\|Z(a)\|_{L^2}^2 = - \frac{3\beta}{2(1 - \beta^2)^{1/4}} \int_{\mathbb{R} \times \mathbb{T}} \frac{(1 - \beta f g)^2} {(f - \beta g)^4} dxdy + (1 - \beta^2)^{3/4} \int_{\mathbb{R} \times \mathbb{T}} \frac{4g - 2f^2 g + 2\beta f g^2 - 6\beta f g^2 + 2\beta^2 f^2 g^3} {(f - \beta g)^5} dxdy.$$

Thus, by the equation (2.1) we have

$$\frac{3^{3/4}}{12^2} \partial_\beta^2\|Z(a)\|_{L^2}^2|_{\beta = 0} = \pi \int_{\mathbb{R}} \left(- \frac{3}{f^4} + \frac{20}{f^6} + \frac{2}{f^2} - \frac{16}{f^4}\right) dx = 0.$$

Similarly, we obtain that

$$\frac{3^{3/4}}{12^2} \partial_\beta^2\|Z(a)\|_{L^2}^2 = - \frac{3(6\beta - \beta^3)}{8(1 - \beta^2)^{9/4}} \int_{\mathbb{R} \times \mathbb{T}} \frac{(1 - \beta f g)^2} {(f - \beta g)^4} dxdy - \frac{9(2 - \beta^2)}{4(1 - \beta^2)^{5/4}} \int_{\mathbb{R} \times \mathbb{T}} \frac{4g - 2f^2 g + 2\beta f g^2 - 6\beta f g^2 + 2\beta^2 f^2 g^3} {(f - \beta g)^5} dxdy - \frac{9\beta}{2(1 - \beta^2)^{1/4}} \int_{\mathbb{R} \times \mathbb{T}} \frac{20g^2 + 2f^4 g^2 - 16f^2 g^2 + 12\beta f g^3 - 24\beta f g^3 + 6\beta^2 f^2 g^4} {(f - \beta g)^6} dxdy + (1 - \beta^2)^{3/4} \int_{\mathbb{R} \times \mathbb{T}} \frac{120g^3 + 24f^4 g^3 - 120f^2 g^3 + 72\beta f g^4 - 120\beta f g^4 + 24\beta^2 f^2 g^5} {(f - \beta g)^7} dxdy.$$
and
\[
\frac{3^{3/4}}{12} \partial^3_Z \|Z(a)\|_{L^2}^2 = -\frac{9(4 + 12 \beta^2 - \beta^4)}{16(1 - \beta^2)^{13/4}} \int_{R \times T} \frac{(1 - \beta g)^2 dx dy}{(f - \beta g)^4} - \frac{3(6 \beta - \beta^3)}{2(1 - \beta^2)^{9/4}} \int_{R \times T} \frac{4 g f^2 + 2 \beta^3 g^2 - 6 \beta f g^2 + 2 \beta^2 f^2 g^3}{(f - \beta g)^5} dx dy - \frac{9(2 - \beta^2)}{2(1 - \beta^2)^{5/4}} \int_{R \times T} \frac{20 g^2 + 2 f^4 g^2 - 16 f^2 g^2 + 12 \beta f^2 g^3 - 24 \beta f g^3 + 6 \beta^2 f^2 g^4}{(f - \beta g)^6} dx dy - \frac{6 \beta}{(1 - \beta^2)^{3/4}} \int_{R \times T} \frac{120 g^3 + 24 f^4 g^3 - 120 f^2 g^3 + 72 \beta f^2 g^4 - 120 \beta f g^4 + 24 \beta^2 f^2 g^5}{(f - \beta g)^7} dx dy + (1 - \beta^2)^{3/4} \int_{R \times T} \frac{840 g^4 + 240 f^4 g^4 - 960 f^2 g^4 + 480 \beta f^2 g^5 - 720 \beta f g^5 + 120 \beta^2 f^2 g^6}{(f - \beta g)^8} dx dy.
\]

Since
\[
\partial_a \|Z(a)\|_{L^2}^2 |_{a=0} = \partial^4_a \|Z(a)\|_{L^2}^2 |_{a=0} = \partial^3_a \|Z(a)\|_{L^2}^2 |_{a=0} = 0,
\]
from the equation (2.1) we obtain
\[
\partial^4_a \|Z(a)\|_{L^2}^2 |_{a=0} = \left( \frac{d^4}{da^4} \right) |_{a=0} \partial^2_a \|Z(a)\|_{L^2} \mid_{\beta=0} = \frac{2 \cdot 12^2 \pi}{3^{3/4}} \int_R \left( -\frac{36}{f^2} + \frac{639}{f^4} - \frac{1800}{f^6} + \frac{1260}{f^8} \right) dx = \frac{2 \cdot 12^2 \cdot 3^2 \pi}{3^{3/4}} \int_R f^{-4} dx.
\]

Let \(v_s(x) \cos y = \partial_a Z(a) |_{a=0}(x, y)\) and
\[
Z(a, \gamma)(x, y) = \gamma Z(|a|)(\sqrt{\gamma}x, y + \theta(a)),
\]
where \((\cos \theta(a), \sin \theta(a)) = (\frac{a_1}{|a|}, \frac{a_2}{|a|})\) for \(a = (a_1, a_2) \in \mathbb{R}^2 \setminus \{(0, 0)\}\). Then, we have
\[
\partial^{-1}_x v_s(x) = \frac{12 \sqrt{2} \sinh \left( \frac{x}{2 \sqrt{\gamma}} \right)}{3^{1/4} \cosh^2 \left( \frac{x}{2 \sqrt{\gamma}} \right)}.
\]

We define the action as
\[
S_c(u) = E(u) + cM(u).
\]
Let \((\cdot, \cdot)_{L^2(X)}\) be the inner product of \(L^2(X)\). The coerciveness of \(S_{4/\sqrt{3}}(Q_{4/\sqrt{3}})\) on a subspace of \(Z^1\) yields the following lemma.

**Lemma 2.2.** There exist \(k_1 > 0\) and \(\varepsilon_0 > 0\) such that for \(a_1, a_2 \in (-\varepsilon_0, \varepsilon_0)\) and \(\gamma \in (1 - \varepsilon_0, 1 + \varepsilon_0)\), if \(w \in Z^1\) satisfies
\[
(w, Z(a, \gamma))_{L^2(R \times T)} = (w, \partial_a Z(a, \gamma))_{L^2(R \times T)} = (w, \partial_{a_2} Z(a, \gamma))_{L^2(R \times T)} = 0,
\]
then
\[
(w, Z(a, \gamma))_{L^2(R \times T)} = 0.
\]
From Lemma 2.1 and Lemma 2.3 in [33], there exists $k_1 \geq 0$ such that
\[ \langle S''_{4/\sqrt{3}}(Z(a, \gamma))w, w \rangle_{Z^{-1}, Z^1} \geq k_1 \| w \|_{Z^1}^2, \]
where $\langle \cdot, \cdot \rangle_{Z^{-1}, Z^1}$ denotes the pairing between $Z^{-1}$ and $Z^1$ such that
\[ \langle w, v \rangle_{Z^{-1}, Z^1} = (w, v)_{L^2(\mathbb{R} \times \mathbb{T})} \]
for $w, v \in Z^1$.

**Proof.** By the Fourier expansion in the transverse direction $y$, we have
\[ S''_{4/\sqrt{3}}(Q_{4/\sqrt{3}})w = \sum_{n \in \mathbb{Z}} \left( -\partial_x^2 w_n - n^2 \partial_x^{-2} w_n + \frac{4}{\sqrt{3}} w_n - Q_{4/\sqrt{3}} w_n \right) e^{iny}, \]
where
\[ w(x, y) = \sum_{n \in \mathbb{Z}} w_n(x) e^{iny}. \]

From Lemma 2.1 and Lemma 2.3 in [33], there exists $C > 0$ such that for $n \in \mathbb{Z} \setminus \{-1, 1\}$, $w_0 \in H^1(\mathbb{R})$ and $w_n \in Z^1_x$ with $(w_0, Q_{4/\sqrt{3}})_{L^2(\mathbb{R})} = (w_0, \partial_x Q_{4/\sqrt{3}})_{L^2(\mathbb{R})} = 0$ we have
\[ \left\langle \left( -\partial_x^2 + \frac{4}{\sqrt{3}} - Q_{4/\sqrt{3}} \right) w_0, w_0 \right\rangle_{H^{-1}(\mathbb{R}), H^1(\mathbb{R})} \geq C \| w_0 \|_{H^1(\mathbb{R})}^2 \tag{2.2} \]
and
\[ \left\langle \left( -\partial_x^2 - n^2 \partial_x^{-2} + \frac{4}{\sqrt{3}} - Q_{4/\sqrt{3}} \right) w_n, w_n \right\rangle_{Z^{-1}_x, Z^1_x} \geq C(\| w_n \|_{H^1(\mathbb{R})}^2 + n^2 \| \partial_x^{-1} w_n \|_{L^2(\mathbb{R})}^2) \tag{2.3} \]
where
\[ Z^*_x = \{ u : \| (|\xi| + |\xi|^{-1})^s \hat{u}(\xi) \|_{L^2_x(\mathbb{R})} < \infty \} \]
and $\langle \cdot, \cdot \rangle_{Z^{-1}_x, Z^1_x}$ denote the pairing between $Z^{-1}_x$ and $Z^1_x$. Let $u_1 = \partial_x^{-1} w_1$ and $L_1 = -\partial_x^2 - \partial_x^{-2} + \frac{4}{\sqrt{3}} - Q_{4/\sqrt{3}}$. Then, we have
\[ \langle L_1 w_1, w_1 \rangle_{Z^{-1}_x, Z^1_x} = \langle -\partial_x L_1 \partial_x u_1, u_1 \rangle_{H^{-2}(\mathbb{R}), H^2(\mathbb{R})}. \]

By applying the Weyl Lemma, we obtain that the essential spectrum of the operator $-\partial_x L_1 \partial_x$ is $[n^2, \infty)$. Thus, to show the coerciveness of $L_1$ on a subspace, we investigate the eigenvalue problem
\[ -\partial_x L_1 \partial_x u_1 = \lambda u_1, \quad u_1 \in H^4(\mathbb{R}) \tag{2.4} \]
From the argument in [1, 33], it was proved that the problem (2.4) has no negative eigenvalue. Moreover, $-\partial_x L_1 \partial_x u_n = 0$ has linearly independent solutions
\[ \partial_x g_{3r}, \quad \partial_x g_{-3r}, \quad \partial_x g_1, \quad \lim_{\mu \to 1} \frac{\partial_x (g_{\mu} + g_{-\mu})}{\mu - 1}, \]
where
\[ g_\mu(x) = e^{3^{-1/4}\mu x} \left( \mu^3 + 2\mu - 3\mu^2 \tanh(3^{-1/4}x) \right). \]
Then,
\[ \partial_x g_1(x) = -\frac{6 \sinh \frac{x}{3^{1/4}}}{3^{1/4} \cosh^2 \frac{x}{3^{1/4}}} = \frac{1}{2\sqrt{2}} \partial_x^{-1} v_x. \]
Since the growth rates of \( \partial_x g_\sqrt{x}, \partial_x g_{-\sqrt{x}} \) and \( \lim_{\mu \to 1} \frac{\partial_x(g_\mu + g_{-\mu})}{\mu - 1} \) at the infinity are different and do not belong to \( L^2 \), we obtain
\[ L^2(\mathbb{R}) \cap \left\{ c_1 \partial_x g_{\sqrt{3}} + c_2 \partial_x g_{-\sqrt{3}} + c_3 \lim_{\mu \to 1} \frac{\partial_x(g_\mu + g_{-\mu})}{\mu - 1} : c_1, c_2, c_3 \in \mathbb{R} \right\} = \{0\}. \]
Therefore, the zero eigenvalue of \(-\partial_x L_1 \partial_x\) is simple and there exists \( C > 0 \) such that
\[ \langle L_1 w_1, w_1 \rangle_{Z^1, Z^1} = \langle -\partial_x L_1 \partial_x u_1, u_1 \rangle_{H^{-2}(\mathbb{R}), H^2(\mathbb{R})} \geq C' \|u_1\|_{H^2}^2 \geq C(\|w_1\|_{H^1}^2 + \|\partial_x^{-1} w_1\|_{L^2}^2) \quad (2.5) \]
for \( w_1 \in H^1(\mathbb{R}) \) with \( \partial_x^{-1} w_1 = u_1 \in L^2(\mathbb{R}) \) and \( \langle w_1, v_\ast \rangle_{L^2(\mathbb{R})} = 0 \). From (2.2), (2.3) and (2.5), we obtain that there exists \( C_0 > 0 \) such that
\[ \langle S''_{4/\sqrt{3}}(Q_{4/\sqrt{3}})w, w \rangle_{Z^1, Z^1} \geq C_0 \|w\|_{Z^1}^2. \]
for \( w \in Z^1 \) with
\[ \left( w, Q_{4/\sqrt{3}} \right)_{L^2(\mathbb{R} \times T)} = \left( w, \partial_x Q_{4/\sqrt{3}} \right)_{L^2(\mathbb{R} \times T)} = \left( w, v \cos y \right)_{L^2(\mathbb{R} \times T)} = \left( w, v \sin y \right)_{L^2(\mathbb{R} \times T)} = 0. \]
By a continuity argument and the above inequality, we obtain the conclusion. \( \square \)

In the following lemma, we set a modulation of the scaling in \( x \) direction and show the expansion of the modulation.

**Lemma 2.3.** Let \( B_1 = \{ b \in \mathbb{R}^2 : |b| < 1 \} \) and
\[ \gamma_l(a) = \left( M(Z(l))M(Z(a))^{-1} \right)^{\frac{2}{3}} \]
for \( l \geq 0, a \in \mathbb{R} \). For \( a \in B_1 \)
\[ M(Z(a, \gamma_l(|a|))) = M(Z(l)), \]
\[ \gamma_0(|a|) - 1 = -\frac{\partial_a^4 M(Z(a))|a=0}{36 M(Q_{4/\sqrt{3}})} |a|^4 + o(|a|^4) \text{ as } |a| \to 0 \quad (2.6) \]
and
\[ \gamma_l(|a|) - 1 = -\frac{2 \partial_a M(Z(a))|a=l}{3 M(Z(l))} (|a| - l) + O((|a| - l)^2) \text{ as } |a| \to l. \quad (2.7) \]
Proof. The equation (2.7) follows the Taylor expansion. Since
\[ M(Z(a)) = M(Q_{4/\sqrt{3}}) + \frac{\partial_a^4 M(Z(a))}{4!} + o(a^4) \]
and
\[ \gamma_0(|a|) = 1 - \frac{M(Z(|a|))}{M(Z(|a|))} = 1 - \frac{\partial_a^4 M(Z(a))}{36 M(Q_{4/\sqrt{3}})} |a|^4 + o(|a|^4), \]
we have the equation (2.6).

Next, we investigate the expansion of the Lyapunov function along the bifurcation parameter \( a \).

**Lemma 2.4.** Let \( \epsilon_0 > 0 \). For \( a \in B_1 \setminus \{(0,0)\} \) and \( 0 < l < \epsilon_0 \),
\[
S_{4/\sqrt{3}}(Z(a, \gamma_0(|a|))) - S_{4/\sqrt{3}}(Q_{4/\sqrt{3}}) = \frac{5\epsilon''(0)\partial_a^4 M(Z(a))}{6!} |a|^6 + o(|a|^6) \tag{2.8}
\]
and
\[
S_{c(l)}(Z(a, \gamma_l(|a|))) - S_{c(l)}(Z(l)) = \left( \frac{\partial_a^4 M(Z(a))}{9 M(Z(l))^2} \partial_c Q_{4/\sqrt{3}}^2 \right) |a| - l)^2 + O((\epsilon_0 + ||a| - l|)(|a| - l)^2). \tag{2.9}
\]

**Proof.** By the expansion
\[
Z(a, \gamma_l(|a|)) = Z(a, 1) + (\gamma_l(|a|) - 1) \partial_c Q_{4/\sqrt{3}}
+ O((l + ||a| - l| + (\gamma_l(|a|) - 1))(\gamma_l(|a|) - 1)), \tag{2.10}
\]
we have
\[
S_{c(l)}(Z(a, \gamma_l(|a|))) - S_{c(l)}(Z(l)) = S_{c(|a|)}(Z(|a|)) - S_{c(l)}(Z(l)) + (c(l) - c(|a|)) M(Z(l))
+ \frac{1}{2} (\gamma_l(|a|) - 1)^2 (S''_{4/\sqrt{3}}(Q_{4/\sqrt{3}}) \partial_c Q_{4/\sqrt{3}}, \partial_c Q_{4/\sqrt{3}})_{L^2} + O((l + ||a| - l|)(\gamma_l(|a|) - 1)^2)
\]
for \( l \geq 0 \). Since
\[
\partial_a^n \left( S_{c(a)}(Z(a)) - S_{c(l)}(Z(l)) + (c(l) - c(a)) M(Z(l)) \right)
= - c^{(n)}(a) M(Z(l)) + \sum_{j=0}^{n-1} n-1 C_j c^{(j+1)}(a) \partial_a^{n-1-j} M(Z(a)),
\]
we obtain the results.
we obtain
\[ S_c(|a|)(Z(|a|)) - S_{4/(\sqrt{3})}(Q_{4/(\sqrt{3})}) + (4/\sqrt{3} - c(|a|))M(Q_{4/(\sqrt{3}}) \]
\[ = \frac{5}{6!} c''(0) \partial^4_a M(Z(a))|_{a=0}|a|^6 + o(|a|^6) \]  
(2.11)
and
\[ S_c(|a|)(Z(|a|)) - S_c(l)(Z(l)) + (c(l) - c(|a|))M(Z(l)) \]
\[ = \frac{1}{2} c'(l) \partial^2_a M(Z(a))|_{a=1}|a| - l)^2 + o(|a| - l)^2) \]  
(2.12)
for \( l \geq 0 \). The equations (2.8) and (2.9) follow (2.11) and (2.12).

Let a distance
\[ \text{dist}_t(u) = \inf_{(x_0,y_0) \in \mathbb{R}^T} \|u(\cdot,\cdot) - Z(l)(\cdot - x_0, \cdot - y_0)\|_{L^1} \]
and neighborhoods
\[ N_{\varepsilon,l} = \{ u \in Z^1 : \text{dist}_t(u) \leq \varepsilon \}, \]
\[ N_{\varepsilon,k}^l = \{ u \in N_{\varepsilon,l} : M(u) = M(Z(k)) \}. \]

In the following lemma, we define the modulation parameters to control the degenerate direction of the Lyapunov function. The following lemma follows the implicit function theorem.

Lemma 2.5. Let \( \varepsilon > 0 \) sufficiently small. Then, there exist \( K_1 > 0 \), \( C_2 \) functions \( \rho : N_{\varepsilon,0} \to \mathbb{R} \), \( \gamma : N_{\varepsilon,0} \to \mathbb{R} \), \( a = (a_1, a_2) : N_{\varepsilon,0} \to B_1 \) and \( \eta : N_{\varepsilon,0} \to Z^1 \) such that for \( u \in N_{\varepsilon,0} \)
\[ u(\cdot + \rho(u), \cdot) = Z(a(u), \gamma(u))(\cdot, \cdot) + \eta(u)(\cdot, \cdot), \]
\[ |\gamma(u) - 1| + |a(u)| + \|\eta(u)\|_{L^1} \leq K_1 \text{dist}_0(u) \]  
(2.13)
and \( (\eta(u), Z(a(u), \gamma(u)))_{L^2} = (\eta(u), \partial_x Z(a(u), \gamma(u)))_{L^2} = (\eta(u), \partial_{a_1} Z(a(u), \gamma(u)))_{L^2} = (\eta(u), \partial_{a_2} Z(a(u), \gamma(u)))_{L^2} = 0. \) Moreover, \( \gamma, \rho \) and \( a = (a_1, a_2) \) satisfy
\[ G(u, \gamma(u), \rho(u), a_1(u), a_2(u)) = 0, \]
where
\[ G(u, \gamma, \rho, a_1, a_2) = \begin{pmatrix}
(u(\cdot - \rho, \cdot) - Z(a, \gamma), Z(a, \gamma))_{L^2} \\
(u(\cdot - \rho, \cdot) - Z(a, \gamma), \partial_x Z(a, \gamma))_{L^2} \\
(u(\cdot - \rho, \cdot) - Z(a, \gamma), \partial_{a_1} Z(a, \gamma))_{L^2} \\
(u(\cdot - \rho, \cdot) - Z(a, \gamma), \partial_{a_2} Z(a, \gamma))_{L^2}
\end{pmatrix}. \]

Proof. Since \( G(1, 0, 0, 0) = 0 \) and the Jacobian matrix
\[ \frac{\partial G}{\partial (\gamma, \rho, a_1, a_2)}|_{\gamma=1, \rho=0, a_1=0, a_2=0} \]
is regular, there exists $C^2$ functions $\gamma, \rho, a_1, a_2 : N_{\varepsilon, 0} \to \mathbb{R}$ such that

$$G(u, \gamma(u), \rho(u), a_1(u), a_2(u)) = 0.$$ 

Let

$$\eta(u)(\cdot, \cdot) = u(\cdot + \rho(u), \cdot) - Z(a(u), \gamma(u))(\cdot, \cdot).$$

Then, $\eta$ satisfies the orthogonal condition and \[2.13\].

The following lemma shows the smallness of the difference between $Z(a(u), \gamma(|a(u)|))$ and $Z(a(u), \gamma(u))$.

**Lemma 2.6.** Let $\varepsilon > 0$ sufficiently small. There exists $C > 0$ such that for $0 \leq l < 2\varepsilon$ and $u \in N^l_{\varepsilon, 0}$

$$\|Z(a(u), \gamma(|a(u)|)) - Z(a(u), \gamma(u))\|_{L^1} \leq C\|\eta(u)\|_{L^2}^2,$$

$$|\gamma_l(a(u)) - \gamma(u)| \leq C\|\eta(u)\|_{L^2}^2. \quad (2.14)$$

**Proof.** By Lemma 2.5,

$$M(Z(a(u), \gamma_l(a(u)))) = M(Z(l)) = M(\eta(u) + Z(a(u), \gamma(u))) = M(\eta(u)) + M(Z(a(u), \gamma(u)))$$

for $u \in N^l_{\varepsilon, 0}$. Since there exists $K > 0$ such that

$$|\gamma(u) - 1| + |\gamma_l(a(u)) - 1| < K\varepsilon,$$

we obtain

$$M(\eta(u)) = M(Z(a(u), \gamma_l(a(u)))) - M(Z(a(u), \gamma(u))) = (\gamma_l(a(u))^\frac{3}{2} - \gamma(u)^\frac{3}{2})M(Z(a(u))) \geq \gamma_l(a(u)) - \gamma(u) \geq 0.$$ 

Therefore, we have

$$\|Z(a(u), \gamma_l(a(u))) - Z(a(u), \gamma(u))\|_{L^1} \leq \gamma_l(a(u)) - \gamma(u) \leq M(\eta(u)).$$ 

The following lemma shows that the degeneracy of the linearized operator $S''_{4/\sqrt{3}}(Q_{4/\sqrt{3}})$ is covered by the sixth order nonlinearity of $a$.

**Lemma 2.7.** There exists $\varepsilon_1 > 0$ such that for $0 < \varepsilon < \varepsilon_1$ and $u \in N^0_{\varepsilon, 0}$

$$S_{4/\sqrt{3}}(u) - S_{4/\sqrt{3}}(Q_{4/\sqrt{3}}) = \frac{5c''(0)\partial^4 M(Z(a))|_{a=0}|a(u)|^6}{6!} + \frac{1}{2}(S''_{4/\sqrt{3}}(Z(a(u), \gamma_0(|a(u)|)))\eta(u), \eta(u))_{L^1} + \eta(u)_{L^1}^2 + O(\|\eta(u)\|_{L^2}^2 + |a(u)|^6). \quad (2.15)$$
Moreover, for $0 < \varepsilon < \varepsilon_1$, $l > 0$ and $u \in \mathcal{N}_{\varepsilon_0}$

$$
S_{c(l)}(u) - S_{c(l)}(Z(l)) = \left( \frac{\partial_a M(Z(a))|_{a=l}}{9 M(Z(l))} \partial_{\|Q_e\|_{L^2}}^2 \right) \|Q_e\|_{L^2}^2 + \frac{c'(l)\partial_a M(Z(a)|_{a=l})}{2}(\|a\| - l)^2
$$

$$
\frac{1}{2}(S''_{4/\sqrt{3}}(Z(a(u), \gamma_0(\|a(u)\|))) \eta(u), \eta(u))_{Z^{-1},Z^1} + O((l + \|\eta(u)\|_{Z^1} + \|a(u)\| - l)(\|\eta(u)\|_{Z^1}^2 + (\|a(u)\| - l)^2)).
$$

(2.16)

Proof. From the Taylor expansion, we have for $l \geq 0$

$$
S_{c(l)}(u) - S_{c(l)}(Z(l)) = S_{c(l)}(Z(a(u), \gamma(u)) + \eta(u)) - S_{c(l)}(Z(l))
$$

$$
= S_{c(l)}(Z(a(u), \gamma(|a(u)|)) - S_{c(l)}(Z(l))
$$

$$
+ \langle S''_{c(l)}(Z(a(u), \gamma(|a(u)|)), Z(a(u), \gamma(u)) + \eta(u) - Z(a(u), \gamma(|a(u)|)))_{Z^{-1},Z^1}
$$

$$
+ \frac{1}{2}(S''_{c(l)}(Z(a(u), \gamma(|a(u)|))) \eta(u), \eta(u))_{Z^{-1},Z^1} + o(\|\eta(u)\|_{Z^1}^2).
$$

Since $S''_{4/\sqrt{3}}(Q_{4/\sqrt{3}})\partial_{\|Q_e\|_{L^2}} = -Q_{4/\sqrt{3}}$, by (2.14) we have

$$
\langle S''_{c(l)}(Z(a(u), \gamma(|a(u)|)), \eta(u))_{Z^{-1},Z^1}
$$

$$
= \langle (S''_{c(|a(u)|)}(Z(a(u), \gamma(u))) - S''_{4/\sqrt{3}}(Q_{4/\sqrt{3}}))(\gamma(|a(u)|) - \gamma(u))\partial_{\|Q_e\|_{L^2}}, \eta(u)\rangle_{Z^{-1},Z^1}
$$

$$
= (c(l) - c(|a(u)|))(Z(a(u), \gamma(|a(u)|)) - Z(a(u), \gamma(u)), \eta(u))_{L^2}
$$

$$
+ O((l + \|a\| - l + \|\eta(u)\|_{Z^1})|\gamma(|a(u)|) - \gamma(u)|^2 + |\eta(u)|_{Z^1}^2))
$$

$$
= O((l + \|a\| - l + \|\eta(u)\|_{Z^1})(|\gamma(|a(u)|) - \gamma(u)|^2 + |\eta(u)|_{Z^1}^2)).
$$

Therefore, by Lemma [2.4] we obtain (2.15) and (2.16).

3 Proof of the orbital stability of the line soliton with the critical speed

In this section, applying Lemma [2.2] and Lemma [2.4] we show the orbital stability of the line soliton $Q_{4/\sqrt{3}}$ with the critical traveling speed. We prove the orbital stability by the contradiction. We assume the line soliton $Q_{4/\sqrt{3}}$ is unstable. Then, there exist $\varepsilon_0 > 0$, a sequence $\{u_n\}_n$ of solutions to (1.1) and a sequence $\{t_n\}_n$ such that $t_n > 0$, $u_n(0)$ $\rightarrow$ $Q_{4/\sqrt{3}}$ as $n \rightarrow \infty$ in $Z^1$ and $\text{dist}_{0}(u_n(t_n)) > \varepsilon_0$. Let $v_n = M(Q_{4/\sqrt{3}})^{1/2}M(u_n)^{-1/2}u_n(t_n)$. Then, we have $M(v_n) = M(Q_{4/\sqrt{3}})$, $\lim_{n \rightarrow \infty} \|v_n - u_n(t_n)\|_{Z^1} = 0$ and $\lim_{n \rightarrow \infty} S'_{4/\sqrt{3}}(v_n) = S'_{4/\sqrt{3}}(Q_{4/\sqrt{3}})$. By Lemma [2.2] and Lemma [2.7] we obtain that there exists $k_0 > 0$ such that

$$
S'_{4/\sqrt{3}}(v_n) - S'_{4/\sqrt{3}}(Q_{4/\sqrt{3}}) \geq k_0(\|a(v_n)\|^6 + \|\eta(v_n)\|_{Z^1}^2)
$$

(3.1)
for sufficiently large $n$. From the inequality (3.1), we have
\[
\lim_{n \to \infty} a(v_n) = 0, \quad \lim_{n \to \infty} \|\eta(v_n)\|_{Z^1} = 0.
\]
(3.2)

By Lemma 2.6, the equation \(\lim_{n \to \infty} \gamma_0(a(v_n)) = 1\) yields
\[
\lim_{n \to \infty} \gamma(v_n) = 1.
\]
(3.3)

From (3.2) and (3.3), we obtain
\[
\lim_{n \to \infty} \text{dist}_0(u_n(t_n)) = \lim_{n \to \infty} \text{dist}_0(v_n) = 0.
\]
This is a contradiction. Thus, \(Q_{4/\sqrt{3}}\) is orbitally stable.

4 Proof of the orbital stability of the Zaitsev solitons near by the line soliton

In this section, we show the orbital stability of the Zaitsev solitons near by the line soliton \(Q_{4/\sqrt{3}}\). The positivity of \(c'(l)\) and \(\partial_a M(Z(l))\) for sufficiently small \(l > 0\) follows \(c''(0) = 4/\sqrt{3}\) and \(\partial_a^4 M(Z(u))|_{a=0} > 0\). Thus, from Lemma 2.2 and Lemma 2.7, we have that for sufficiently small \(l > 0\) there exist \(k_l, \varepsilon_0 > 0\) such that
\[
S_{c(l)}(u) - S_{c(l)}(Z(l)) \geq k_l((|a(u)| - l)^2 + \|\eta(u)\|_{Z^1}^2)
\]
for \(u \in N_{\varepsilon_0,l}^l\). Therefore, applying the same argument as the argument in the proof of the orbital stability of \(Q_{4/\sqrt{3}}\), we obtain the orbital stability of the Zaitsev soliton \(Z(l)\) for sufficiently small \(l > 0\).

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