Counting Line-Colored D-ary Trees

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Random tensor models are generalizations of matrix models which also support a 1/N expansion. The dominant observables are in correspondence with some trees, namely rooted trees with vertices of degree at most D and lines colored by a number i from 1 to D such that no two lines connecting a vertex to its descendants have the same color. In this Letter we study by independent methods a generating function for these observables. We prove that the number of such trees with exactly \( p_i \) lines of color \( i \) is

\[
\frac{1}{\prod_{i=1}^{D} p_i+1} \left( \sum_{i=1}^{D} p_i+1 \right) \ldots \left( \sum_{i=1}^{D} p_i+1 \right).
\]

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I. INTRODUCTION

The study of large random matrices in the past thirty years has successfully described measures which can be written as the exponential of single trace invariants perturbing a Gaussian. In addition to the standard Feynman diagrammatic expansion, powerful methods exist to solve such models, including orthogonal polynomials (which rely on eigenvalue decomposition) and more recently the topological expansion developed by Eynard. The latter starts with the Schwinger-Dyson equations (also known as loop equations) written in terms of the resolvent \( \text{Tr}(1-MM^\dagger) \), where \( M \) is the random \( N \times N \) matrix, and provides an intrinsic way to solve them at all orders in the 1/N expansion.

Random tensors are a generalization of random matrices to rank \( D \) objects (having \( D \) indices of size \( N \) each). The study of their large size statistics became possible thanks to the 1/N expansion discovered in [4]. This expansion relies on the construction of multi-unitary invariants, i.e. tensor contractions which are invariant under the external tensor product of \( D \) copies of \( U(N) \), each of them acting independently on each tensor index. In contrast with random matrices, there are many invariant monomials at a fixed degree. These monomials are indexed by colored graphs (these colors correspond to the position of the index in a tensor contraction: in \( T_{a_1 \ldots a_D} \), \( a_i \) has position, hence color, 1, up to \( a_D \) which has color \( D \)). The 1/N expansion generalizes to any measure on a single random tensor that can be written as the exponential of such invariants.

In the diagrammatic approach, the graphs contributing at leading order, known as melonic graphs, are considered in [7] enabling one to solve these models exactly at leading order. At the core of these solutions, a universality theorem, first derived in [8], states that all models are Gaussian at large \( N \) (but with a covariance which crucially depends on the joint distribution). In particular all invariant monomials corresponding to melonic graphs at fixed degree have the same expectation value. However, this certainly does not hold at sub-leading orders in the 1/N expansion, and different melonic graphs have different sub-leading corrections in 1/N.

A method which would be fruitful to adapt to tensor models is the one developed by Eynard. It first requires to introduce an equivalent of the resolvent. Writing the matrix resolvent like \( \text{Tr}(1-MM^\dagger) = \sum_{n \geq 0} z^{-n-1} \text{Tr}(MM^\dagger)^n \), it is tempting to generalize it by changing \( \text{Tr}(MM^\dagger)^n \) with the sum over all invariants of degree \( n \). As \( z \) counts the degree of each invariant, this object does not distinguish different invariants having the same degree. Although sufficient for the study of the leading order, this object should be refined in order to explore the finer structure of sub-leading orders, for which a generating function which probes the structure of each invariant beyond their degree seems better adapted.

This Letter is a modest contribution in this direction. The dominant melonic invariants are in one-to-one correspondence with \( D \)-ary trees with lines colored 1 up to \( D \) such that no two lines connecting a vertex to its descendants have the same color. At leading order only the number of vertices matters, but at subsequent orders one must distinguish between different colored trees having the same number of vertices. As a first step, we study the generating function of these colored trees and find an explicit formula counting how many such trees with \( p_i \) lines of color \( i \) one can build. Countings of colored trees exist in the literature (like [4]) but we could not find the counting which is relevant for our purpose.

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In Section III we introduce the problem and state our main results. The generating function is presented in Section III and the proof of our results, based on a linear recursive sequence related to the generating function, is contained in Section IV.

II. STATING THE PROBLEM

We consider a family of rooted trees defined by the following properties

- each vertex has at most \( D \) descendants, \( D \geq 2 \),
- each line receives a color \( i = 1, \ldots, D \) such that no two lines connecting a vertex to its descendants have the same color.

By adding leafs (univalent vertices) appropriately any such tree becomes a rooted \( D \)-ary tree with colored lines. Let \( C_{p_1 \cdots p_D} \) be the number of such trees with exactly \( p_i \) lines of color \( i \) for all \( i = 1, \ldots, D \). The purpose of this note is to derive an explicit formula for \( C_{p_1 \cdots p_D} \).

Our strategy is based on the generating function

\[
F(g_1, \ldots, g_D) = \sum_{p_1, \ldots, p_D = 0}^{\infty} C_{p_1 \cdots p_D} \prod_{i=1}^{D} g_i^{p_i},
\]
and the sequence \( (F_n(g_i))_{n \in \mathbb{N}} \) defined by

\[
F_0 = 1, \quad F_n(g_1, \ldots, g_D) = \sum_{p_1, \ldots, p_D}^n C_{p_1 \cdots p_D} \prod_{i=1}^{D} g_i^{p_i} \quad \text{with} \quad C_{p_1 \cdots p_D} = \frac{n}{\sum_{i=1}^{D} p_i + n} \prod_{j=1}^{D} \left( \sum_{i=1}^{D} p_i + n \right).
\]

We will prove that \( (F_n) \) is a linear recursive sequence whose characteristic equation is an algebraic equation satisfied by \( F \). Examining the roots of this algebraic equation, we will obtain the following proposition.

Proposition 1. The sequence \( (F_n(g_i))_{n \in \mathbb{N}} \) is a geometric sequence with common ratio \( F \),

\[
F_n(g_1, \ldots, g_D) = \left( F(g_1, \ldots, g_D) \right)^n.
\]

This implies that \( F(g_1, \ldots, g_D) = F_1(g_1, \ldots, g_D) \), hence the following corollary.

Corollary 1. The number \( C_{p_1 \cdots p_D} \) of rooted line-colored trees with maximal degree \( D \) and exactly \( p_i \) lines of color \( i = 1, \ldots, D \) is

\[
C_{p_1 \cdots p_D} = C_{p_1 \cdots p_D} = \frac{1}{\sum_{i=1}^{D} p_i + 1} \prod_{j=1}^{D} \left( \sum_{i=1}^{D} p_i + 1 \right).
\]

III. THE GENERATING FUNCTION

The generating function \( F \) satisfies an algebraic equation which is obtained by simply observing that the root of a tree can have \( k \leq D \) descendants connected by lines of colors \( i_1, \ldots, i_k \) all different. Therefore

\[
F(g_1, \ldots, g_D) = 1 + \left( \sum_{1 \leq i_1 \leq D} g_{i_1} \right) F + \left( \sum_{1 \leq i_1 < i_2 \leq D} g_{i_1} g_{i_2} \right) F^2 + \cdots + (g_1 \cdots g_D) F^D,
\]

\[
= \sum_{k=0}^{D} \left( \sum_{1 \leq i_1 < \cdots < i_k \leq D} \prod_{l=1}^{k} g_{i_l} \right) [F(g_1, \ldots, g_D)]^k,
\]

and \( F \) is the root \( x_0(g_1, \ldots, g_D) \) of this polynomial equation such that

\[
\lim_{g_1, \ldots, g_D \to 0} x_0(g_1, \ldots, g_D) = 1.
\]

The following lemma show how to distinguish the desired root \( x_0(g_1, \ldots, g_D) \) from the other roots of the above polynomial.
Lemma 1. For all $R > 1$, there exists $\epsilon_R > 0$ such that for all $|g_1|, \ldots, |g_D| < \epsilon_R$, the polynomial

$$Q(X) = -X + \sum_{k=0}^{D} \left( \sum_{1 \leq i_1 < \cdots < i_k \leq D} g_{i_k} \right) X^k$$

has exactly one root $x_0$ with $|x_0| < R$, all other roots $x_i$ for $i \neq 0$ having norms $|x_i| \geq R$. In particular $x_0 = F$ as it is the only root satisfying $\lim_{y \to 0} x_0 = 1$.

Proof. To establish this lemma we use Rouche's theorem whose statement is now recalled. Let $f, g$ be two holomorphic functions and $S$ a closed contour which does not contain zeros of $f$ and $g$. If for all $z \in S$

$$|f(z) - g(z)| < |g(z)|,$$

then the number of zeros of $f$ and the number of zeros of $g$ encircled by $S$ (counted with multiplicities) are the same.

We now exploit this theorem to get bounds on the absolute value of the roots of the polynomial $Q$. Take

$$f(z) = Q(z), \quad \text{and} \quad g(z) = -z.$$

Let $R > 1$, and $S = \{z \in \mathbb{C} | |z| = R\}$ be the circle of radius $R$. Note that on $S$, $|g(z)| = R$. Set $\epsilon_R > 0$ with $\epsilon_R < \frac{R^{1/D} - 1}{R}$ so that on $S$,

$$|f(z) - g(z)| \leq \sum_{k=0}^{D} \left( \sum_{1 \leq i_1 < \cdots < i_k \leq D} |g_{i_k}| \cdots |g_{i_k}| \right)z^k < \sum_{k=0}^{D} \binom{D}{k}(\epsilon_R R)^k \leq (1 + \epsilon_R R)^D \leq |g(z)|.$$

As $g(z)$ has an unique root $z = 0$ inside the circle of radius $R$, we conclude that $f(z)$ also has exactly one root, which we denote $x_0$, with $|x_0| < R$. As this is the only root which can go to one when $g_i \to 0$, it is identified with the generating function $F$.

Remark 1. At $D = 2$, $Q(X) = 1 + (g_1 + g_2 - 1)X + g_1 g_2 X^2$ and $Q(x) = 0$ is easily solved,

$$x_0 = \frac{1 - g_1 - g_2 - \sqrt{(1 - g_1 - g_2)^2 - 4g_1 g_2}}{2g_1 g_2} \quad \text{and} \quad x_1 = \frac{1 - g_1 - g_2 + \sqrt{(1 - g_1 - g_2)^2 - 4g_1 g_2}}{2g_1 g_2}.$$

IV. THE LINEAR RECURSIVE SEQUENCE

First we show that the sums defining each $F_n(g_i)$ in (2) converge absolutely when $|g_i| < \frac{(D-1)^{D-1}}{D^D}$ for all $i = 1, \ldots, D$. Let $\epsilon > 0$ such that $\epsilon < \frac{(D-1)^{D-1}}{D^D}$, then for all complex $g_1, \ldots, g_D$ with norm $|g_i| < \epsilon$

$$|F_n(g_1, \ldots, g_D)| \leq \sum_{p=0}^{\infty} e^p \frac{n}{p+n} \sum_{\{p_1, \ldots, p_D\}} \binom{p+n}{p_1} \cdots \binom{p+n}{p_D} (Dp + Dn) \epsilon^p.$$

The sums over $p_i$ are computed by equating the coefficients of $x^p$ in $(1 + x)^{Dp + Dn}$ and $(1 + x)^{p_1 + \cdots + \epsilon} x^{p_1 + \cdots + \epsilon}$, hence

$$|F_n(g_1, \ldots, g_D)| \leq \sum_{p=0}^{\infty} \frac{n}{p+n} \binom{Dp + Dn}{p} \epsilon^p.$$

One finds using the Stirling formula that the radius of convergence of the above series is $\frac{(D-1)^{D-1}}{D^D} > \epsilon$, hence $F_n(g_1, \ldots, g_D)$ converges absolutely.

Lemma 2. The sequence $(F_n)$ respects the recursion

$$\forall n \geq 0 \quad F_{n+1} = F_n + \sum_{k=1}^{D} \left( \sum_{1 \leq i_1 < \cdots < i_k \leq D} \prod_{i=1}^{k} g_{i_k} \right) F_{n+k}.$$
Proof: The recursion translates into
\[ \forall p_i \geq 1 \quad C_{p_1 \ldots p_D}^{n+1} = C_{p_1 \ldots p_D}^n + \sum_{k=1}^{D} \sum_{1 \leq i_1 < \cdots < i_k \leq D} C_{p_{i_1} \ldots p_{i_k} - 1 \ldots p_D}^{n+k} \cdot \quad (15) \]
The boundary cases, when some \( p_i = 0 \), just reproduce the recursion at level \( D - 1 \). Let us denote \( P = \sum_{i=1}^{D} p_i \). The right hand side of (15) is
\[ \frac{(P + n)^{-1}}{\prod_{i=1}^{D} p_i [(P - p_i + n + 1)!]} \left[ n \prod_{i=1}^{D} (P - p_i + n + 1) + \sum_{k=1}^{D} (n + k) \sum_{1 \leq i_1 < \cdots < i_k \leq D} \prod_{j \neq i_1, \ldots, i_k} (P - p_j + n + 1) \right]. \quad (16) \]
We write \( n \prod_{i=1}^{D} (P - p_i + n + 1) = (n + 1) \prod_{i=1}^{D} (P - p_i + n + 1) - \prod_{i=1}^{D} (P - p_i + n + 1) \) so as to re-arrange the square bracket above as
\[ n \prod_{i=1}^{D} (P - p_i + n + 1) + \sum_{k=1}^{D} (n + k) \sum_{1 \leq i_1 < \cdots < i_k \leq D} \prod_{j \neq i_1, \ldots, i_k} (P - p_j + n + 1) \]
\[ = (n + 1) \prod_{i=1}^{D} (P + n + 1) - \prod_{i=1}^{D} (P - p_i + n + 1) + \sum_{k=1}^{D} (k - 1) \sum_{1 \leq i_1 < \cdots < i_k \leq D} \prod_{j \neq i_1, \ldots, i_k} (P - p_j + n + 1) \]
\[ = (n + 1)(P + n)(P + n + 1)^{D-1} + (n + 1)(P + n + 1)^{D-1} - \prod_{i=1}^{D} (P - p_i + n + 1) \]
\[ + \sum_{k=1}^{D} (k - 1) \sum_{1 \leq i_1 < \cdots < i_k \leq D} \prod_{j \neq i_1, \ldots, i_k} (P - p_j + n + 1). \quad (17) \]
The first term of the last equality is exactly what is needed to form \( C_{p_1 \ldots p_D}^{n+1} \). Therefore we focus now on the sum of the three other contributions,
\[ (n + 1)(P + n + 1)^{D-1} - \prod_{i=1}^{D} (P + n + 1 - p_i) + \sum_{k=2}^{D} (k - 1) \sum_{1 \leq i_1 < \cdots < i_k \leq D} \prod_{j \neq i_1, \ldots, i_k} (P + n + 1 - p_j) \]
\[ = (n + 1)(P + n + 1)^{D-1} - (P + n + 1)^D + (P + n + 1)^{D-1} - \prod_{i=1}^{D} (P + n + 1 - p_i) \]
\[ + \sum_{k=2}^{D} (k - 1) \sum_{1 \leq i_1 < \cdots < i_k \leq D} \prod_{j \neq i_1, \ldots, i_k} (P + n + 1)^{D-k-m} (-1)^m \sum_{1 \leq j_1 < \cdots < j_m \leq D} \prod_{l=1}^{m} p_{j_l} \cdot \quad (18) \]
The first three terms cancel. For the remaining terms
\[ \sum_{k=2}^{D} (-1)^{k+1} \sum_{1 \leq i_1 < \cdots < i_k \leq D} \prod_{l=1}^{k} p_{i_l} (P + n + 1)^{D-k} \]
\[ + \sum_{k=2}^{D} (k - 1) \sum_{1 \leq i_1 < \cdots < i_k \leq D} \prod_{m=0}^{D-k} (P + n + 1)^{D-k-m} (-1)^m \sum_{1 \leq j_1 < \cdots < j_m \leq D} \prod_{l=1}^{m} p_{j_l}, \quad (19) \]
we take into account that \( k + m \) ordered integers can be partitioned into \( \binom{k+m}{k} \) ways into two subsets of \( k \) and \( m \) ordered integers. Thus the second sum rewrites as a sum over \( q = k + m \)
\[ \sum_{q=2}^{D} \sum_{1 \leq i_1 < \cdots < i_q \leq D} p_{i_1} \cdots p_{i_q} (-1)^q (P + n + 1)^{D-q} \sum_{k=2}^{q} (-1)^k (k-1) \binom{q}{k} \cdot \quad (20) \]
But
\[ \sum_{k=2}^{q} (-1)^k (k-1) \binom{q}{k} = 1 + \sum_{k=0}^{q} (-1)^k (k-1) \binom{q}{k} = 1 - (1 - 1)^q - q(1 - 1)^{q-1} = 1, \quad (21) \]
hence the whole quantity displayed in (19) is zero and the lemma follows.

Therefore, $F_{n+D}$ is obtained recursively from the set $(F_p)_{p<n+D}$. The characteristic polynomial of this recursion is exactly $Q(X)$ (Equation 4). It means that the solution $x(0)$ is one of the common ratios of $(F_n)$. We denote the others $x_j$, and assuming they are all different,

$$F_n = a x_n(0) + \sum_j b_j x_n(j),$$

(22)

for some functions $a(g_1, \ldots, g_D)$, $b_j(g_1, \ldots, g_D)$ which can in principle be determined by $D$ initial conditions. However, we cannot use the initial conditions (remember we want to prove that $F = F_1$) so we have to proceed differently. Each common ratio in the sum (22) is controlled thanks to the Lemma 1, as $x(0)$ is bounded from above and each $x_j$ is bounded from below. Now we need to control the sequence $(F_n)$ independently of its common ratios. This is done through the following lemma.

**Lemma 3.** For all $K > 1$, there exists $\epsilon_K > 0$ such that for all $g_1, \ldots, g_D \in \mathbb{C}$ with $|g_1|, \ldots, |g_D| < \epsilon_K$, $F_n(g_1, \ldots, g_D)$ is polynomially bounded by $K$,

$$\forall n \geq 0 \quad |F_n(g_1, \ldots, g_D)| \leq K^n.$$  

(23)

**Proof.** For $n = 0$, this is trivial as $F_0 = 1$. Let thus be $n \geq 1$ and $K > 1$. It is enough to chose $\epsilon_K$ such that

$$\epsilon_K < \frac{(D-1)^{D-1}}{D^D}, \quad \epsilon_K < \frac{1}{2De} \quad \text{and} \quad e^{2De\epsilon_K} + \frac{1}{\sqrt{2\pi}} 1 - 2De\epsilon_K < K.$$  

(24)

With $\epsilon_K < \frac{(D-1)^{D-1}}{D^D}$, one can use Equation (13) which implies

$$|F_n| \leq \sum_{p=0}^{\infty} \frac{n^p (Dn + Dp)^p}{p!} \epsilon_K^p \leq \sum_{p=0}^{\infty} \frac{(n + p)^p}{p!} (D\epsilon_K)^p.$$  

(25)

We use the fact that $(n + p)^p \leq (2n)^p$ when $p \leq n$ and $(n + p)^p \leq (2p)^p$ when $p \geq n$ to obtain the bound

$$|F_n| \leq \sum_{p=0}^{n} n^p \frac{(2D\epsilon_K)^p}{p!} + \sum_{p=n}^{\infty} \frac{p^p}{p!} (2D\epsilon_K)^p \leq e^{2Dn\epsilon_K} + \sum_{p=n}^{\infty} \frac{p^p}{p!} (2D\epsilon_K)^p.$$  

(26)

Now we use $p! \geq \sqrt{2\pi p} e^{p \ln p - p}, \forall p \geq 1$ and as $\epsilon_K < \frac{1}{2De}$ we get

$$|F_n| \leq e^{2Dn\epsilon_K} + \frac{1}{\sqrt{2\pi}} \sum_{p=1}^{\infty} \frac{1}{p^{2p}} (2D\epsilon_K)^p \leq e^{2Dn\epsilon_K} + \frac{1}{\sqrt{2\pi}} \sum_{p=1}^{\infty} (2D\epsilon_K)^p \leq e^{2D\epsilon_K} + \frac{1}{\sqrt{2\pi}} 1 - 2D\epsilon_K < K^n.$$  

(27)

We are now in position to prove Proposition 4 by combining Lemmas 2 and 3. Choose $R > K > 1$ and consider $|g_i| < \inf(\epsilon_R, \epsilon_K)$ with $\epsilon_R, \epsilon_K$ as in the lemmas. First we consider the case where all $x(j)$ have different norms and denote $x_{(\max)} \neq 0$ the one with the largest norm. In particular $|x_{(\max)}| \geq R > 1$. At large $n$, the norm of $F_n$ is dominated by $b_{(\max)} x_{(\max)}$. Hence there exist a constant $A > 0$ and an integer $N$ such that for all $n \geq N,$

$$|F_n(g_1, \ldots, g_D)| \geq A |b_{(\max)}| |x_{(\max)}|^n \geq A |b_{(\max)}| R^n.$$  

(28)

However $|F_n| \leq K^n$ with $R > K$. Therefore we conclude $b_{(\max)} = 0$. We can repeat this reasoning with the root $x_{(j)}$ that has the second largest norm, and so on until we get $F_n = ax_{(0)}^n$. The initial condition $F_0 = 1$ for all $g_i$ finally leads to $F_n = x_{(0)}^n$.

The case where some of the roots have the same norm is quite similar. The idea is to extract sub-sequences $(F_{r(n)})$ for which $F_{r(n)}$ behaves at large $n$ like a coefficient times some combination of the roots $x_{(j)}$, $j \neq 0$, where this combination is greater than $R^n$ when $|x_{(j)}| \geq R$.

**Remark 2.** At $D = 2$, the number of line-colored trees with $p_1$ lines of color 1 and $p_2$ lines of color 2 is $C_{p_1p_2} = \frac{1}{p_1+p_2+1} \binom{p_1+p_2+1}{p_1} \binom{p_1+p_2+1}{p_2}$. These numbers are known as the Narayana numbers $N(p_1 + p_2 + 1, p_1 + 1)$. 

Remark 3. By summing the numbers $C_{p_1 \cdots p_D}$ over all possible numbers of lines of each color at a fixed total number of lines $P - 1$,

$$\sum_{\{p_i\}} C_{p_1 \cdots p_D} = \frac{1}{P} \binom{D(P-1)+D}{P-1} = \frac{1}{DP+1} \binom{DP+1}{P},$$

(29)

we obtain the total number of $D$-ary trees on $P$ vertices also known as the $D$-Catalan numbers (pp. 200 in [10], proposition 6.2.2 in [11] and more details in [6]).

Remark 4. Proposition 1 implies that $F_n F_m = F_{n+m}$, corresponding to interesting combinatorial identities,

$$\sum_{\{k_i=0, \ldots, p_i\}} C_{n-k} C_{m-p-k-D} = C_{n+m-p-D}.$$  \hspace{1cm} (30)

For example, when $D = 2$ one gets

$$\sum_{k_1, k_2} \binom{n}{p_1-k_1+p_2-k_2+m} \binom{m}{p_1-k_1+p_2-k_2+k_2+n} \binom{1+n+k_1+k_2}{k_1} \binom{1+n+k_2}{k_2} \binom{1+k_1+k_2+n}{n} \binom{p_1+p_2+m+n}{p_1} \binom{p_1+p_2+m+n}{p_2}.$$  \hspace{1cm} (31)

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