Asymptotic behavior of Cardassian cosmologies with exponential potentials

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In this paper we analyze the asymptotic behavior of Cardassian cosmological models filled with a perfect fluid and a scalar field with an exponential potential. Cardassian cosmologies arise from modifications of the Friedmann equation, and among the different proposals within that framework we will choose those of the form $3H^2 - \rho \propto \rho^n$ with $n < 1$. We construct two three dimensional dynamical systems arising from the evolution equations, respectively adapted for studying the high and low energy limits. Using standard dynamical systems techniques we find the fixed points and characterize the solutions they represent. We pay especial attention to the properties inherent to the modifications and compare with the (standard) unmodified scenario. Among other interesting results, we find there are no late-time tracking attractors.

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I. INTRODUCTION

Observations indicate that type Ia high redshift supernovae (SNIa) are dimmer than expected (see [1] for most recent results), and the mainstream interpretation of this result is that the universe is currently undergoing accelerated expansion. Further observations, like those of the Cosmic Microwave Background (CMB) [2] or Large Scale structures (LSS) [3], suggest that two thirds of the energy density of the universe correspond to dark energy.

Even though the most popular explanation to the late-time acceleration in the universe is the existence of some kind of dark energy (perhaps a scalar field), this is not the only possibility. Recently, Freese and Lewis [4] proposed the so-called Cardassian models as an alternative explanation which involves only matter and radiation and does not invoke either vacuum energy or a cosmological constant. In these models the universe has a flat geometry, as required by measurements of the cosmic background radiation [5], and it is filled only with radiation and matter (baryonic or not). The Friedmann equation is modified with respect to its usual form by the addition of a term in its right hand side, specifically

$$3H^2 = \rho + \sigma \rho^n,$$  \hspace{1cm} (1.1)

in units such that $8\pi/m^2_{pl} = 1$ and with $\sigma > 0$ being an arbitrary constant.

For $n < 1$ the second term becomes important if $z < O(1)$. From there on it dominates the Friedmann equation and yields $a \propto t^{2/3n}$ for ordinary matter, so there will be acceleration provided $n < 2/3$. There are two main (possibly unrelated) motivations for the $\rho^n$ modifications: (1) As shown in [6], terms of that form typically appear in the Friedmann equation when the universe is embedded as a three-dimensional surface (3-brane) in higher dimensions. (2) Alternatively, these functions may appear in a purely four dimensional theory in which the modified right hand side of the Friedmann equation is due to an extra contribution to the total energy density. One will then regard the right hand side of the Friedman equation as corresponding to a single fluid, with an extra contribution to the energy-density tensor in the (ordinary four dimensional) Einstein equations.

The interpretation of the Cardassian expansion as due to an interacting dark matter fluid with negative pressure was developed in [7]. The Cardassian term on the right hand side of the Friedman equation is interpreted as the interacting term and gives rise to the effective negative pressure which drives the cosmological acceleration.

Interestingly, Cardassian models survive several observational tests, the most significant being that it allows for a universe consisting of just matter and radiation. The energy density giving a closed universe $\rho_c$ is much smaller that its counterpart standard cosmology $\rho_{c, \text{old}}$ (specifically $\rho_c = \rho_{c, \text{old}} [1 + (1 + \Delta q)^{3/2}(1 - n)]^{-1}$), and matter alone is enough to provide a flat geometry.

With regard to the observational tests that depend on the scale or the Hubble factor, in the late-time regime Cardassian models filled with just matter ($\rho \propto a^{-3}$) are indistinguishable from perfect fluid models with a $p = (\gamma - 1)\rho$ equation of state under the identification $n \equiv \gamma$. These perfect fluid models are in turn kinematically equivalent to scalar field ( quintessence) models with an exponential potential. In this way, the Cardassian model can make contact with quintessence with regard to observational tests. Interestingly, observational tests seem to favor $n < 0$, so that asymptotically one would get a phantom equation of state [8].

However, the equivalence between Cardassian and per-
fect fluid models is not extensible to the dynamical realm, the evolution of perturbations may differ significantly, and this can lead to discrepancies for instance in observational tests associated with the cosmic microwave background.

Nevertheless, as stated in [8], questions of interpretation remain open, because in the Cardassian model matter alone is responsible for the accelerated behavior, and yet the universe can be flat.

Here we perform a dynamical systems analysis of Cardassian models. These techniques have been used for exploring Cardassian models filled with baryonic matter only [21], but we give one step further by allowing as well for a scalar field component (non-baryonic matter). We investigate for the early and late time regime of the models the existence of tracking solutions, i.e., we look for solutions representing universes with non-negligible and proportional fractions of both baryonic and non-baryonic matter. Tracking solutions are particularly interesting because their dynamical effects mimic a decaying cosmological constant (see [10,12] for seminal references). Such solutions would be devoid of the fine-tuning problems posed by a cosmological constant precisely because of the independence on the initial conditions.

Using standard dynamical systems techniques we will investigate the interplay between the modifications to the Friedmann equations and the features and existence conditions of the mentioned tracking solutions. In this first approach to the problem we will choose an exponential self-interaction potential [10,12,15] for the scalar field because only those potentials allow using the evolution equations for constructing an autonomous system of low a dimension as possible. As shown in [10] for any other potential the number of dimensions will be higher if the system is to remain autonomous. In that reference the existence of tracking Cardassian cosmologies was shown within a more general context, but their analysis remained open, because in the Cardassian model matter filled with baryonic matter will be free parameters, whereas in the vicinity of an hypothetical initial singularity physical variables would typically diverge, whereas at late times they commonly tend to zero [4]. Due to physical considerations normalization with the Hubble factor is an appropriate choice in cosmology. Besides, all available mathematical evidence suggests that Hubble-normalized variables are bounded into the past (that is, as the initial singularity is approached), and if there is a cosmological constant (or something that mimics it) it seems those variables will also be bounded into the future. Thus, even though the Hubble-normalized state space is unbounded, it is sensible to expect that the evolution equations will admit a past attractor and a future attractor.

### A. High energy limit

In view of the discussion above, and following the usual practice, we use normalized (dimensionless) variables defined by

\[
x = \frac{\dot{\phi}}{\sqrt{6H}}
\]

\[
y = \frac{\sqrt{V}}{\sqrt{3H}}
\]

\[
z = \frac{\sigma}{3H}
\]

\[
w = \frac{\sqrt{n \rho_{\gamma}}}{\sqrt{3H}}
\]

This coordinates will allow us analyzing the solutions of (2.1)(2.2), and the cosmological models associated with model filled with a scalar field \(\phi\) with self-interaction potential \(V(\phi) = \xi \exp(-\sqrt{6}\lambda \phi)\) and a barotropic perfect fluid with equation of state \(p_{\gamma} = (\gamma - 1)\rho_{\gamma}\) are

\[
\dot{2\dot{H}} + \left(\gamma \rho_{\gamma} + \dot{\phi}^2\right) \left(1 + n\sigma \rho_{\gamma}^{n-1}\right) = 0
\]

\[
\dot{\phi} + 3H \dot{\phi} + \frac{dV(\phi)}{d\phi} = 0
\]

\[
\dot{\rho}_{\gamma} + 3\gamma H \rho_{\gamma} = 0,
\]

where for the total energy density \(\rho_{\text{tot}}\) we have

\[
\rho_{\text{tot}} = \frac{1}{2} \dot{\phi}^2 + V(\phi) + \rho_{\gamma}.
\]

The evolution equations (2.1) (2.2) are in turn subject to the constraint

\[
H^2 = \frac{1}{3} \rho_{\text{tot}} \left(1 + \sigma \rho_{\text{tot}}^{n-1}\right).
\]

Here and throughout \(\sigma, k\) and \(\lambda\) will be free parameters, and we will restrict ourselves to the \(n < 1\) case.

Experience has demonstrated that dynamical systems methods can be used to describe the evolution of cosmological models by means of past and future attractors. In order to cast our set of equations as a dynamical system, it is convenient to normalize the variables, because in the vicinity of an hypothetical initial singularity physical variables would typically diverge, whereas at late times they commonly tend to zero [4]. Due to physical considerations normalization with the Hubble factor is an appropriate choice in cosmology. Besides, all available mathematical evidence suggests that Hubble-normalized variables are bounded into the past (that is, as the initial singularity is approached), and if there is a cosmological constant (or something that mimics it) it seems those variables will also be bounded into the future. Thus, even though the Hubble-normalized state space is unbounded, it is sensible to expect that the evolution equations will admit a past attractor and a future attractor.

### II. PHASE-SPACE

The evolution equations for a flat Friedmann-Robertson-Walker (FRW) Cardassian cosmological

\[
\dot{x} = \frac{\dot{\phi}}{\sqrt{6H}}
\]

\[
y = \frac{\sqrt{V}}{\sqrt{3H}}
\]

\[
z = \frac{\sigma}{3H}
\]

\[
w = \frac{\sqrt{n \rho_{\gamma}}}{\sqrt{3H}}
\]
As will become clear below, this choice of coordinates leads to a dynamical system which is best adapted for studying high energy limit solutions, so we will refer to it as our high energy limit adapted (HELA) dynamical system as opposed to a low energy limit adapted (LELA) that will emerge below after an alternative choice of coordinates.

In addition, the variables will be related among them through

$$S(w, x, y) + S^n(w, x, y)z^{1-n} = 1,$$  \hspace{1cm} (2.10)

where $S(w, x, y) = w^2 + x^2 + y^2$.

The constraint (2.10) lets us “forget” about the evolution of one of the coordinates. Here we will choose the discarded coordinate to be $z$. Using the variables \textbf{2.2} and equation \textbf{2.10}, and the conservation equations \textbf{2.2} and \textbf{2.3} we get the equations

$$x' = \frac{1}{2S} \left( x \left( 2 (n - 1) x^2 - 2 y^2 + (\gamma n - 2) w^2 \right) - \left( n - 1 \right) x \left( 2x^2 + \gamma xw^2 - 2\lambda y^2 \right) S \right),$$  \hspace{1cm} (2.11)

$$y' = \frac{y}{2S} \left( n \left( 2x^2 + \gamma w^2 \right) - \left( 2\lambda x + (n - 1) \left( 2x^2 + \gamma w^2 \right) \right) S \right),$$  \hspace{1cm} (2.12)

$$w' = \frac{w}{2S} \left( x^2 (2n - \gamma) + \gamma (n - 1) w^2 - \gamma y^2 - (n - 1) \left( 2x^2 + \gamma w^2 \right) S \right),$$  \hspace{1cm} (2.13)

Table I: Location and existence conditions of the critical points of the HELA dynamical system

| Name | $x$ | $y$ | $w$ | Existence |
|------|-----|-----|-----|-----------|
| $W_\pm$ | 0 | 0 | $\pm 1$ | All $\gamma$ and $\lambda$ |
| $X_\pm$ | $\pm 1$ | 0 | 0 | All $\gamma$ and $\lambda$ |
| $XY_\pm W_+$ | $\frac{\gamma}{2\lambda}$ | $\pm \frac{1}{2\lambda} \sqrt{(2 - \gamma)\gamma}$ | $\sqrt{1 - \frac{\gamma^2}{2\lambda^2}}$ | $1 \geq \lambda^2 \geq \frac{\gamma}{2}$ |
| $XY_\pm W_-$ | $\frac{\gamma}{2\lambda}$ | $\pm \frac{1}{2\lambda} \sqrt{(2 - \gamma)\gamma}$ | $-\sqrt{1 - \frac{\gamma^2}{2\lambda^2}}$ | $1 \geq \lambda^2 \geq \frac{\gamma}{2}$ |
| $XY_\pm$ | $\lambda$ | $\pm \sqrt{1 - \lambda^2}$ | 0 | $\lambda^2 \leq 1$ |
| $O$ | 0 | 0 | 0 | All $\gamma$ and $\lambda$ |

where primes denote differentiation with respect to $\tau \equiv \ln a^3$. Equations (2.11–2.13) form the HELA dynamical system, which is defined on the phase space

$$\Psi = \{(x, y, w) : S \leq 1\}.$$  \hspace{1cm} (2.14)

To begin with the study of the evolution of our the dynamical system we have to find its fixed (or critical) points. The fixed points $(x^*, y^*, w^*)$ are given by the conditions

$$x' (x^*, y^*, w^*) = 0,$$  \hspace{1cm} (2.15)

$$y' (x^*, y^*, w^*) = 0,$$  \hspace{1cm} (2.16)

$$w' (x^*, y^*, w^*) = 0.$$  \hspace{1cm} (2.17)

In order to analyze the stability of the fixed points $(x^*, y^*, w^*)$ one studies the linearized dynamical system obtained by expanding equations (2.11–2.13) about those fixed points (see, e.g. [17]). Then one tries solutions in the form $(x, y, w) = (c_1, c_2, c_3) e^{\lambda t}$ in the linear approximation, and finds that the characteristic exponent $\lambda$ and the constant vector $(c_1, c_2, c_3)$ must be respectively an eigenvalue and an eigenvector of the matrix

$$\begin{pmatrix} \partial x' & \partial x' & \partial x' \\ \partial y' & \partial y' & \partial y' \\ \partial w' & \partial w' & \partial w' \end{pmatrix} \begin{pmatrix} x \\ y \\ w \end{pmatrix} = (x^*, y^*, w^*)$$  \hspace{1cm} (2.18)

If the real part of the three characteristic exponents is negative, the fixed point is asymptotically stable, i.e., an attractor. On the other hand, it is enough to have (at least) one characteristic exponent with positive real part to make the fixed point asymptotically unstable, i.e., a repeller. This repeller is a saddle point if at least one of the other characteristic exponents has a negative real part, in which case there is, apart from the unstable man-
if, a stable manifold containing the exceptional orbits that converge to the fixed point. The case in which the largest real part is precisely zero must be analyzed using other methods, for the linear analysis in un conclusive. The geometric form of the orbits near the fixed point is determined by the imaginary part of the characteristic exponents. If the three are real the fixed point is a node.

A couple of complex conjugate exponents leads, except in degenerate cases, to an spiral point (the orbits are helices near the fixed point). In addition, when one of the exponents is null the point is not hyperbolic and therefore structural stability cannot be guaranteed (the geometric form of the trajectories may change under small perturbations).

| Name       | Eigenvalues                                                                 | Dynamical Character                                                   |
|------------|-----------------------------------------------------------------------------|-----------------------------------------------------------------------|
| $W_{\pm}$  | $\left(\frac{1}{2}(\gamma - 2), \frac{\gamma}{2}, \gamma(1 - n)\right)$   | non-hyperbolic for $\gamma = 0, 2$, repeller for $0 < \gamma \leq 2$ (saddle for $0 < \gamma < 2$) |
| $X_{\pm}$  | $\left(2(1 - n), 1 - \frac{\gamma}{2}, 1 \mp \lambda\right)$               | non-hyperbolic for $\gamma = 2$ or $\lambda = 1$, always repeller     |
| $XY_{\pm}W_+$ | $\left(\gamma(1 - n), \frac{(\gamma - 2)\lambda - \beta}{4\lambda^2}, \frac{(\gamma - 2)\lambda + \beta}{4\lambda}\right)$ | non-hyperbolic for $\gamma = 0, 2$, repeller for $\gamma > 0$ (saddle or spiral-out saddle for $0 < \gamma < 2$) |
| $XY_{\pm}W_-$ | $\left(\gamma(1 - n), \frac{(\gamma - 2)\lambda - \beta}{4\lambda}, \frac{(\gamma - 2)\lambda + \beta}{4\lambda}\right)$ | non-hyperbolic for $\gamma = 0, 2$, repeller for $\gamma > 0$ (saddle or spiral-out saddle for $0 < \gamma < 2$) |
| $XY_{\pm}$  | $\left(\lambda^2 - 1, \lambda^2 - \frac{\gamma}{2}, 2(1 - n)\lambda^2\right)$ | non-hyperbolic for $\lambda^2 = 1, \gamma/2$, always repeller (saddle for $\lambda^2 < 1$ or $\lambda^2 < \gamma/2$) |
| $O$        | $(0, 0, 0)$                                                                | undefined                                                             |

Depending on the values of $\lambda$ and $\gamma$ the HELA system above has up to eleven critical points. We will denote them as $W_{\pm}$, $X_{\pm}$, $XY_{\pm}W_-$, $XY_{\pm}W_+$, $XY_{\pm}$, $O$, and the first ten are closely related those found in Ref. [10], where the $n = 0$ case of our system was studied.

Observe that the equations are invariant under the variable changes $y \rightarrow -y$, $z \rightarrow -z$, but not under $x \rightarrow -x$, and so in our numerical examples we will concentrate on the region $\{x^2 + y^2 + z^2 \geq 1, -1 \leq x \leq 1, y \geq 0, w \geq 0\}$, which is a fourth of the unit sphere. This is equivalent to saying we are just considering expanding universes ($H > 0$). However, an analytic description of all eleven points is presented in the lines below and in tables [10] and [11]. We will also set restrictions $0 \leq \gamma \leq 2$ and $\lambda \leq 1$ so that neither the barotropic fluid nor the scalar field have supraluminal sound speeds and the fluid satisfies the weak energy condition. We will also assume $n < 1$, as this is the case of interest.

Although the location of the critical points of this dynamical system does not depend on $n$ [23], the same is not true for their dynamical character. Note as well that all the critical points but the point $O$ are located on the hypersurface $S = 1$, that is, they correspond to $z = 0$ and $H = \infty$, which in turn implies those are solutions to be obtained under the high energy limit approximation. Interestingly, the fact that $H = \infty$ means those are singular (big-bang) cosmologies.

In what follows, and in order to complete the information provided in the tables, we will characterize the cosmological models represented by the fixed point living in the above mentioned fourth of the unit sphere.

The first point, called $W_+$, represents a solution completely dominated by the fluid. The unstable character of these solutions agrees with what one might have antic-
ipated, are they only expected to be relevant at early times.

The second point, called \( X_+ \), represents a solution completely dominated by the scalar field, more specifically by its kinetic energy. Figs. 1 and 2 strongly suggest it is a past attractor.

The third point, called \( XY_+ \), represents a solution dominated by the scalar field, specifically by its kinetic energy. Figs. 1 and 2 show it is a past attractor.

The fifth point, called \( XY \), represents a scalar field dominated solution, which is inflationary if \( \lambda^2 < 1/3 \). Figs. 3 and 4 show it is an unstable saddle in the asymptotic future.

As for the sixth point, called \( O \), the linear analysis is not sufficient for determining its dynamical character, because it has three null eigenvalues. One possibility is resorting to numerical inspection. As shown in the figures below, this approach tells us the fifth point is the late time attractor of the system. Another tricky aspect of the analysis of this point is the sort of solutions it represents. Clearly, in the vicinity of \((0,0,0)\) we have solutions in which \( \dot{\phi} \ll H, V \ll H, \rho \ll H^2 \), but nothing can be said about the ratios \( \dot{\phi}^2/V, V/\rho, \) and \( \dot{\phi}/\rho \). This means we cannot say whether the solution represented by \( O \) is scalar field dominated, fluid dominated or tracking. Note as well that in the vicinity of \( O \) we have \( S^n(w,x,y)z_{1-n} \approx 1 \), which in turn implies \( 3H^2 \approx \sigma \rho_{\text{tot}}^n \), that is, the points in that region represent solutions in which the corrections terms in the Friedmann equation dominate, and this obviously happens in the low energy regime. We have seen this system is good for the description of the points far away from that region (i.e. associated with solutions in the high energy regime), so we say the system is adapted to a high energy limit description and thus dub it HELA system, as opposed to the system adapted to a low energy limit description which will be used in the next subsection, which we dub LELA system. By reformulating the whole analysis using the LELA system as an alternative to the HELA one, we will be able to unveil the fine structure of the \( O \) point, and more importantly we will be allowed to tell what sort of solution is the late-time attractor.

\section*{B. Low energy limit}

We consider now dimensionless variables defined by

\begin{align*}
x &= \sqrt{2} \sigma^{\frac{1}{2n}} \left( \sqrt{3}H \right)^{-\frac{1}{2n}} \hat{t}, \\
y &= \sigma^{\frac{1}{2n}} \left( \sqrt{3}H \right)^{-\frac{1}{2n}} \sqrt{V}, \\
w &= \sigma^{\frac{1}{2n}} \left( \sqrt{3}H \right)^{-\frac{1}{2n}} \sqrt{\rho}, \\
z &= \sigma^{\frac{1}{2n(1-n)}} \left( \sqrt{3}H \right)^{-\frac{1}{2n}},
\end{align*}

where the above variables are related by

\begin{equation*}
S(w,x,y)^n + S(w,x,y)z^{-2(1-n)} = 1. \tag{2.23}
\end{equation*}

Using variables (2.19-2.22), the relation (2.23), and the field equations (2.3-2.5) we get the equations

\begin{align*}
x' &= \frac{1}{2nS} \left( 2(n-1)(S^n - 1)x^3 - 2nx^2y^2 + w^2x((1 + (n-1)S^n)\gamma - 2n) + \\
y' &= \frac{1}{2nS} \left( 2(n-1)(S^n - 1)y^3 - 2nx^2y^2 + w^2y((1 + (n-1)S^n)\gamma - 2n) + \\
w' &= \frac{1}{2nS} \left( 2(n-1)(S^n - 1)w^3 - 2nx^2y^2 + w^2((1 + (n-1)S^n)\gamma - 2n) + \\
z' &= \frac{1}{2nS} \left( 2(n-1)(S^n - 1)z^3 - 2nx^2y^2 + w^2z((1 + (n-1)S^n)\gamma - 2n) + 
\end{align*}
FIG. 4: XW projection of some phase-space trajectories for $\gamma = \lambda = 1$ and $n = 0.2$. All trajectories cross the hypersurface $w = 0.69$.

\[
2n\sqrt{S}\sqrt{1 - S^n y^2 \lambda},
\]

(2.24)

\[
y' = \frac{y}{2nS} \left(1 + (n - 1) S^n \right) \left(2x^2 + \gamma w^2\right) - 2n\lambda x \sqrt{S}\sqrt{1 - S^n},
\]

(2.25)

\[
w' = \frac{1}{2nS} \left((n - 1) (S^n - 1) w^3 \gamma - w (n\gamma(y^2 - 1) - x^2 (2 (1 + (n - 1) S^n)))\right),
\]

(2.26)

Equations (2.24)–(2.26) form a dynamical system defined on the state space

\[
\Psi = \{(x, y, w) : S(w, x, y) \leq 1\}.
\]

(2.27)

As discussed above this will be our LELA system, and like in the HELA one, the location of the critical points of this dynamical system does depend on $n$, but their the dynamical character does.

The LELA system has always seven critical points $W_\pm$, $X_\pm$, $Y_\pm$, $O$, no matter the values of $\lambda$ and $\gamma$. Once again, given the invariance under the transformations $y \to -y$, $z \to -z$, combined with the lack of invariance under the transformation $x \to -x$, we will concentrate on the region \( \{x^2 + y^2 + z^2 \geq 1, -1 \leq x \leq 1, y \geq 0, w \geq 0\} \) in our numerical examples, and as explained above this will mean we will address expanding universes only. However, an analytic description of all seven points is presented in the lines below and in tables III and IV. We will also set the same restrictions on $\gamma$, $\lambda$ and $n$ as above.

We have seen that the HELA system is not suitable for the description of solutions satisfying $3H^2 \approx \sigma\rho^n$. Under the definitions used to construct our LELA system such solutions live in the region $S \approx 1$, so the solutions

\[
y \to -y, z \to -z,
\]

combined with the lack of invariance under the transformation $x \to -x$, we will concentrate on the region \( \{x^2 + y^2 + z^2 \geq 1, -1 \leq x \leq 1, y \geq 0, w \geq 0\} \) in our numerical examples, and as explained above this will mean we will address expanding universes only. However, an analytic description of all seven points is presented in the lines below and in tables III and IV. We will also set the same restrictions on $\gamma$, $\lambda$ and $n$ as above.

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FIG. 8: XW projection of some phase-space trajectories for \(\gamma = \lambda = 1\) and \(n = 0.2\). All trajectories cross the hypersurface \(y = 0.01\).

represented by the fixed points of \(W_\pm, X_\pm\) and \(Y_\pm\) of our LELA system are also solutions where corrections are important. From (2.23) we see those points are characterized by \(z = \infty\), which is a condition equivalent to \(H = 0\). Since, in an expanding universe \(H\) goes to zero at late times, the previous conclusion is consistent with the requirement that at late times acceleration appears due to the domination of corrections.

In contrast the fixed point \(O\) represents solutions for which \(3H^2 \approx \rho\), that is, solutions in a regime in which corrections are negligible, but since these solutions were already investigated during our analysis of the HELA system [24], we will not bother to study the features of this \(O\) point.

In such limit, the critical points are \(X_\pm\), \(Y_\pm\), and \(W_\pm\). In Table IV we summarize the information about these points.

The first point, called \(X_+\), is located at \((x, y, w) = (1, 0, 0)\). It is saddle if \(\gamma \neq 2\) and repeller otherwise. It represents a solution completely dominated by the scalar field, and more specifically by its kinetic energy.

The second point, called \(Y_+\), is located at \((x, y, w) = (0, 1, 0)\). Its eigenvalues are undefined because when the point is approached from different directions different results are obtained. Numerical analysis suggests it is the late time attractor, it represents a solution completely dominated by the scalar field, and more specifically by its potential energy.

The third point, called \(W_+\), is located at \((x, y, w) = (0, 0, 1)\). It is a saddle if \(\gamma \neq 0, 2\), a repeller if \(\gamma = 2\) and it is undefined if \(\gamma = 0\). It represents a solution completely dominated by the fluid.

Finally the fourth point, called \(O\) is located at \((x, y, w) = (0, 0, 1)\). Its dynamical character is also undefined, but numerical analysis indicates it is a repeller.

The choice of variables leading to the LELA system does not allow for an investigation of this fixed point, but it must be noted that it comprises all the points in the HELA system except the \(O\) point, so there is closes the circle, because the character of those points was investigated in the previous section.

Note that the problems to define the dynamical character of some points in the HELA and the LELA systems should not come as a surprise, because this is a commonplace problem in points lying outside the variety (points at which the equations or their derivatives blow up).

### Table IV: Eigenvalues and dynamical character of the critical points of the LELA dynamical system

| Name | Eigenvalues | Dynamical Character |
|------|-------------|---------------------|
| \(X_\pm\) | \((-2(1-n), 1, 1 - \frac{\gamma}{2})\) | non-hyperbolic for \(\gamma = 2\), always repeller (saddle for \(0 \leq \gamma < 2\)) |
| \(Y_\pm\) | undefined | undefined |
| \(W_\pm\) | \((-1 - n)\gamma, -1 + \frac{\gamma}{2}, \frac{\gamma}{2}\) | non-hyperbolic for \(\gamma = 0, 2\), repeller for \(\gamma > 0\) (saddle for \(0 < \gamma < 2\)) |
| \(O\) | (0, 0, 0) | undefined |

### III. COSMOLOGICAL CONSEQUENCES

The cosmological consequences of this analysis are simple but important. As compared to the situation in standard cosmology, for the description of Cardassian models we find that the first complication stems from
the necessity of introducing an additional variable, which we call $z$. 

Our numerical analysis tell us that the past attractors correspond to $S(w, x, y) ≡ 1$, and because of the constraint the latter enforces $z^{1-n} = 0$ which with in turn implies the recovery of the usual form of the Friedmann equation. In the case of models expanding from an initial singularity will and for $(n < 1)$ we then conclude that the past attractors corresponds more specifically to to $z = 0$, and from the definition of $z$ we see that those are solutions with an initial singularity. Summarizing, from the perspective of dynamical systems Cardassian models with a fluid and a scalar field with an exponential potential will preferably have a big bang.

More specifically, the early-time attractor is a solution completely dominated by the kinetic energy of the scalar field and satisfying $\rho \propto a^{-6}$, and its evolution is indistinguishable from that of perfect fluid models with a $p = (\gamma - 1)\rho$ equation of state under the identification $n \equiv \gamma/2$, and the condition for inflation is simply $n < 1/3$.

In contrast, the late-time solution attractor is a solution completely dominated by the potential energy of the scalar field and satisfying $\rho \equiv \text{const.}$, that is, the de Sitter solution, and since $H \equiv \text{const.}$ too, inflation proceeds for any value of $n$.

Interestingly, there are no tracking late-time attractors, this is an important difference with respect to the behavior in standard (non-Cardassian) models.

**IV. CONCLUSIONS**

Cardassian models have been proposed as yet one more possible explanation for late-time acceleration. The main interest of the proposal is it involves only matter and radiation and does not invoke either vacuum energy or a cosmological constant. The idea consists in introducing a modification to the Friedmann equation, so that the effects of the modification become important at low redshift.

We have concentrated here on modifications of the form $3H^2 - \rho \propto \rho^n$ with $n < 1$, and we have studied its asymptotic behavior assuming $\rho$ is made up of two contributions: the energy density of a perfect fluid with a $p = (\gamma - 1)\rho$ equation of state and a self-interacting scalar field with an exponential potential.

Our analysis falls mainly on the analytical side, but we have also carried out some numerical investigations. We constructed two dynamical system arising from the evolution equations. The first system is best suited to telling us about the asymptotic behavior in the high energy limit and we have called it our HELA system. The second system, in contrast is best suited for a description of the asymptotic behavior in the low energy limit and we have called it our LELA system. We have found all the fixed points and we have characterized dynamically most of them by analytical means. However, the information regarding the dynamical character of some of the most important points seem to be only accessible by numerical methods.

Our analysis allows us to say that the late-time attractor of the system is the de Sitter solution, and whatever the value of $n$, there will only be inflation. We have also found that the early-time attractor is a solution with a big-bang, and that there are not tracking late-time attractor. Basically, the behavior in the high energy regime is the same as in standard cosmology but the modifications make the structure of late-time attractors rather different.

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[23] This is confirmed by numerical tests.

[24] There they were called $W_\pm$, $X_\pm$, $XY_\mp W_\pm$, $XY_\pm W_\mp$, $XY_\pm$. 