ARE BANACH SPACES MONADIC?

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ABSTRACT. We will show that Banach spaces are monadic over complete metric spaces via the unit ball functor. For the forgetful functor, one should take complete pointed metric spaces.

1. INTRODUCTION

We will present some results about monadicity of the category $\text{Ban}$ of (complex) Banach spaces and linear maps of norm $\leq 1$. Of course, the question depends on a forgetful functor we choose. We cannot consider the underlying set functor $V_0 : \text{Ban} \to \text{Set}$ because it does not preserve products. One should consider the unit ball functor $U_0 : \text{Ban} \to \text{Set}$ whose left adjoint is $l_1 : \text{Set} \to \text{Ban}$. There is well known that $U_0$ is not monadic and its monadic completion is the category of totally convex spaces (see [15]). However, using [15], we show that the unit ball functor $U : \text{Ban} \to \text{CMet}$ is monadic where $\text{CMet}$ is the category of complete metric spaces and nonexpanding maps. The forgetful functor $V : \text{Ban} \to \text{CMet}$ does not preserve products again.

The category $\text{CMet}$ has many deficiencies and it is natural to replace it by the category $\text{CMet}_\infty$ of generalized complete metric spaces by allowing distances to be $\infty$ while keeping all other requirements, as well as the type of morphisms (see, e.g., [17]). In the same way, we can generalize Banach spaces by allowing norms to be $\infty$. We will show that the forgetful functor $V_\infty : \text{Ban}_\infty \to \text{CMet}_\infty$ is now monadic. The reason is that $\text{CMet}_\infty$ is symmetric monoidal closed and generalized Banach spaces coincide with monoids in $\text{CMet}_\infty$ equipped with scalar multiplication.

Another modification of $\text{CMet}$ is the category $\text{CMet}^*$ of pointed complete metric spaces. Here, the forgetful functor $V^* : \text{Ban} \to \text{CMet}^*$ has a left adjoint (given by Lipschitz-free spaces, see, e.g., [7]). We will show that $V^*$ is monadic, which was suspected by T. Fritz in [9]. Finally, we will touch the question of monadicity of the category $\text{CAlg}$ of $C^*$-algebras over $\text{Ban}$.

We recall that a category $\mathcal{K}$ is locally $\lambda$-presentable, where $\lambda$ is a regular cardinal, if it is cocomplete and has a set $\mathcal{A}$ of $\lambda$-presentable objects such that very object of $\mathcal{K}$ is a $\lambda$-directed colimit of objects fro $\mathcal{A}$. Here, $\lambda$-directed colimits are colimits over $\lambda$-directed posets and an object $A$ is $\lambda$-presentable if its hom-functor $\mathcal{K}(A, -) : \mathcal{K} \to \text{Set}$
preserves \(\lambda\)-directed colimits. A category is locally presentable if it is locally \(\lambda\)-presentable for some regular cardinal \(\lambda\). All needed facts about locally presentable categories can be found in [3].

In what follows, forgetful functors on various categories of Banach spaces will be denoted by \(V\) with needed decorations and, similarly, unit ball functors will be denoted by \(U\).

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2. GENERALIZED BANACH SPACES

The category \(\text{Met}\) of metric spaces and nonexpanding maps is neither complete nor cocomplete, and the tensor product \(X \otimes Y\), which puts the +-metric
\[
d \otimes d((x, y), (x', y')) = d(x, x') + d(y, y')
\]
on \(X \times Y\), fails to make \(\text{Met}\) monoidal closed. (Note that \(X \otimes Y\) must not be confused with the Cartesian product \(X \times Y\) in \(\text{Met}\), which is given by the max-metric.) One therefore enlarges \(\text{Met}\) to the category \(\text{Met}_\infty\) of generalized metric spaces, by allowing distances to be \(\infty\) while keeping all other requirements, as well as the type of morphisms. Then \(\text{Met}_\infty\) is complete and cocomplete and monoidal closed, with the internal hom providing the hom-set \(\text{Met}_\infty(X, Y)\) with the sup-metric
\[
d(f, g) = \sup \{d(f x, g x) \mid x \in X\}.
\]
Moreover, \(\text{Met}_\infty\) is locally \(\aleph_1\)-presentable (see [11] 4.5(3)). The category \(\text{CMet}_\infty\) of complete generalized metric spaces is locally \(\aleph_1\)-presentable too (see [4] 2.3(2)).

The category \(\text{Ban}\) of (complex) Banach spaces and linear maps of norm \(\leq 1\) is locally \(\aleph_1\)-presentable (see [3] 1.48). We will also consider the category \(\text{Ban}_\infty\) of generalized Banach spaces, by allowing norms to be \(\infty\) while keeping all other requirements, as well as the type of morphisms. Similarly, \(\text{Norm}_\infty\) will be the category generalized normed spaces and linear maps of norm \(\leq 1\).

Proposition 2.1. The category \(\text{Ban}_\infty\) is locally \(\aleph_1\)-presentable.

Proof. Consider the single-sorted signature with unary relation symbols \(R_r\) for each \(0 \leq r \in \mathbb{R}\), constant 0, binary operation + and unary operations \(c \cdot -\) for \(c \in \mathbb{C}\). Let \(T\) consist of complex vector space axioms and axioms
\[
(\forall x)(R_0(x) \leftrightarrow x = 0)
\]
for all \(r \leq s\)
\[
(\forall x)(R_r(x) \rightarrow R_s(x))
\]
for all \(r, s\)
\[
(\forall x, y)(R_r(x) \land R_s(y) \rightarrow R_{r+s}(x + y))
\]
for all \(r\)
\[
(\forall x)R_r(x) \leftrightarrow R_{|c|r}(c \cdot x)
\]
for \( r_0 \geq r_1 \geq \ldots r_n \geq \ldots \) with \( r = \lim r_n \)

\[
(\forall x) \bigg( \bigwedge_n R_{r_n}(x) \rightarrow R_r(x) \bigg)
\]

Since \( T \) is a universal Horn theory in \( L_{\omega_1, \omega} \), the category \( \text{Mod}(T) \) of \( T \)-models and homomorphisms is locally \( \aleph_1 \)-presentable (see [3] 5.30). If we interpret \( R_r(a) \) as \( \| a \| \leq r \), \( \text{Mod}(T) \) is isomorphic to the category \( \text{Norm}_\infty \).

**Ban\(_\infty\)** is a reflective subcategory of \( \text{Norm}_\infty \) closed under \( \aleph_1 \)-directed colimits (reflection is given by the completions). Hence \( \text{Ban}_\infty \) is locally \( \aleph_1 \)-presentable (see [3] 1.39).

**Theorem 2.2.** The forgetful functor \( V_\infty : \text{Ban}_\infty \rightarrow \text{CMet}_\infty \) is monadic.

**Proof.** Since \( V_\infty \) preserves limits and \( \aleph_1 \)-directed colimits, it has a left adjoint \( F_\infty \) (see [3] 1.66). Let \( T_\infty = V_\infty F_\infty \) be the induced monad. Given a generalized Banach space \( A \), the operation \( \cdot : V_\infty A \otimes V_\infty A \rightarrow V_\infty A \) is nonexpanding because

\[
d((x, y), (x', y')) = d(x, x') + d(y, y') = \| x - x' \| + \| y - y' \| \geq \| x - x' + y - y' \|
\]

\[
= d(x + y, x' + y').
\]

Hence \( V_\infty A \) is a monoid in \( \text{CMet}_\infty \), i.e., a complete metric space \( M \) equipped with operations \( + : M \otimes M \rightarrow M \) and \( 0 : I \rightarrow M \), where \( I \) is the one-point metric space, satisfying the monoid axioms. Hence \( V = V_2 V_1 \) where \( V_1 : \text{Ban}_\infty \rightarrow \text{MonCMet}_\infty \) and \( V_2 : \text{MonCMet}_\infty \rightarrow \text{CMet}_\infty \) are forgetful functors. Following [14], the category \( \text{MonCMet}_\infty \) of monoids in \( \text{CMet}_\infty \) is monadic over \( \text{CMet}_\infty \).

In a monoid \( M \) over \( \text{CMet}_\infty \), we define \( \| x \| = d(x, 0) \). We have

\[
\| x + y \| = d(x + y, 0) \leq d((x, y), (0, 0)) = d(x, 0) + d(y, 0) = \| x \| + \| y \|.
\]

Hence generalized Banach spaces coincide with monoids in \( \text{CMet}_\infty \) equipped with unary operations \( c \cdot \) — satisfying the appropriate axioms. These operations are nonexpanding iff \( |c| \leq 1 \). Since \( c \cdot \) is inverse for \( c^{-1} \cdot - \), it is easy to see that the category \( \text{Ban}_\infty \) is equivalent to the category of monoids in \( \text{CMet}_\infty \) equipped with nonexpanding operation \( c \cdot \) for \( |c| \leq 1 \) satisfying the appropriate axioms.

Consider the functor \( H : \text{CMet}_\infty \rightarrow \text{CMet}_\infty \) sending \( X \) to \( (X \otimes X) \amalg I \amalg \bigsqcup c X \) where \( c \in \mathbb{C} \), \( |c| \leq 1 \). Then \( H \)-algebras are generalized complete metric spaces equipped with operations \( +, 0 \) and \( c \cdot - \) for \( |c| \leq 1 \). Since \( H \) preserves directed colimits, [3] Remark 2.75 implies that the category \( H\text{-Alg} \) of \( H \)-algebras is locally presentable. Following [2] 5.6, the forgetful functor to \( H\text{-Alg} \rightarrow \text{CMet}_\infty \) creates all colimits that \( H \) preserves. In particular, it creates absolute coequalizers and, following Beck’s theorem, it is monadic (see [12]). Like in [14], the category of \( H \)-algebras satisfying the appropriate axioms is closed in \( H\text{-Alg} \) under directed and \( V_\infty \)-absolute colimits. Thus \( \text{Ban}_\infty \) is monadic. \( \square \)
Proof. Like in the proof of 2.2, the functor $V_\infty$ even preserves directed colimits. Indeed, if $x = \lim_n x_n$ and $y = \lim_n y_n$ then $x + y = \lim_n (x_n + y_n)$. Hence the monad $T_\infty$ preserves directed colimits.

(2) The value $F_\infty(1)$ of the left adjoint $F_\infty : \text{CMet}_\infty \to \text{Ban}_\infty$ is the generalized Banach space $C_\infty$ of complex numbers where all non-zero elements have norm $\infty$.

3. Complete pointed metric spaces

A pointed generalized metric space $(X, 0)$ is a generalized metric space $X$ with a chosen element $0 \in X$. Morphisms of pointed generalized metric spaces are nonexpanding maps preserving $0$. Let $\text{Met}^*_\infty$ be the category of pointed generalized metric spaces and $\text{CMet}^*_\infty$ the category of pointed generalized complete metric spaces. The categories $\text{Met}^*_\infty$ and $\text{CMet}^*_\infty$ are locally $\aleph_1$-presentable. They are also symmetric monoidal closed where the tensor product is the smash product $X \wedge Y$. Recall that $X \wedge Y$ is the pushout

$$
\begin{array}{ccc}
X \otimes 1 & \otimes & Y \\
\downarrow & & \downarrow \\
1 & \otimes & X \wedge Y
\end{array}
$$

where $1 = \{0\}$ is the zero object in $\text{Met}^*_\infty$. The internal hom provides the hom-set $\text{Met}^*_\infty(X, Y)$ with the sup-metric.

The category $\text{Met}^*$ of pointed metric spaces is a coreflective full subcategory of $\text{Met}^*_\infty$ where the coreflector assigns to a pointed generalized metric space $A$ its subspace consisting of all elements $a$ such that $d(0, a) < \infty$. Like in the proof of 2.1 $\text{Met}^*$ is locally $\aleph_1$-presentable. Similarly, the category $\text{CMet}^*$ of pointed complete metric spaces is locally $\aleph_1$-presentable.

Theorem 3.1. The forgetful functor $V^* : \text{Ban} \to \text{CMet}^*$ is monadic.

Proof. Like in the proof of 2.2, the functor $V^*$ preserves limits and $\aleph_1$-directed colimits, thus it has a left adjoint $F^*$. Let $T^* = V^*F^*$ be the induced monad and $W : \text{CMet}^* \to \text{CMet}$ the forgetful functor. Then $WV^* = V$ is the forgetful functor $\text{Ban} \to \text{CMet}$ which is the domain-codomain restriction of $V_\infty$.

Consider a pair $f, g : A \to B$ of morphisms in $\text{Ban}$ such that $V^*f, V^*g$ has a split coequalizer in $\text{CMet}^*$ given by $h : V^*B \to C$, $s : C \to V^*B$ and $t : V^*B \to V^*A$. Then $V_\infty f, V_\infty g : V_\infty A \to V_\infty B$ has a split coequalizer in $\text{CMet}$ given by $Wh : V_\infty B \to WC$, $Ws$ and $Wt$. Following 2.2 and Beck's theorem, there is a unique $\bar{C}$ and a unique $\bar{h} : B \to \bar{C}$ such that $V_\infty \bar{C} = WC$ and $V_\infty \bar{h} = Wh$ and, moreover, $\bar{h}$ is a coequalizer of $f$ and $g$. We have $WV^*\bar{C} = V_\infty \bar{C} = WC$. This means that $V^*W\bar{C}$ and $WC$ are the same metric spaces. Since $0$ in $V_\infty \bar{C}$ is $0 \in \bar{C}$ and $0$ in $C$ is $0 \in \bar{C}$, $V^*\bar{C} = C$. Since $W$ is faithful and $WV^*\bar{h} = V_\infty h = Wh$, we have $V^*\bar{h} = h$. Thus $V^*$ creates the coequalizer of $V^*f$ and $V^*g$. Following Beck's theorem, $V^*$ is monadic. \qed

Remark 2.3. (1) The functor $V_\infty$ even preserves directed colimits. Indeed, if $x = \lim_n x_n$ and $y = \lim_n y_n$ then $x + y = \lim_n (x_n + y_n)$. Hence the monad $T_\infty$ preserves directed colimits.

(2) The value $F_\infty(1)$ of the left adjoint $F_\infty : \text{CMet}_\infty \to \text{Ban}_\infty$ is the generalized Banach space $C_\infty$ of complex numbers where all non-zero elements have norm $\infty$. 

Remark 3.2. (1) Banach spaces are not monoids in $\text{CMet}^\bullet$ because $+$ is not a morphism $V^\bullet A \times V^\bullet A \to V^\bullet A$.

(2) The left adjoint $F^\bullet$ sends a pointed complete metric space $X$ to its Lipschitz-free space (see, e.g., [7]), they are also called Arens-Eells spaces.

(3) Like in 2.3, $T^\bullet$ preserves directed colimits.

(4) In the same way as in 3.1 we show that the forgetful functor $V^\bullet : \text{Ban}_{\infty} \to \text{CMet}^\bullet_{\infty}$ is monadic.

4. Banach spaces

The forgetful functor $V : \text{Ban} \to \text{CMet}$ does not preserve products and we have to take the unit ball functor $U_0 : \text{Ban} \to \text{Set}$ is not monadic and its monadic completion is the category of totally convex spaces (see [15]). These are algebras with operations indexed by sequences $(c_i)_{i=0}^\infty$ of complex numbers satisfying $\sum_{i=0}^\infty |c_i| \leq 1$. These operations are denoted by $\sum_{i=0}^\infty c_i x_i$ and are called totally convex operations. They satisfy some equations (see [15], or [3] 1.48). Over $\text{CMet}$, it suffices to take only finitary totally convex operations indexed by sequences $c_1, \ldots, c_n$ such that $\sum_{i=0}^n |c_i| \leq 1$. Their algebras are called finitely totally convex spaces.

A totally convex space $A$ is separated if for $a_1, a_2 \in A$ and $c \in \mathbb{C}$, $0 < |c| < 1$, $ca_1 = ca_2$ implies $a_1 = a_2$ (see [16] 11.2).

Theorem 4.1. The unit ball functor $U : \text{Ban} \to \text{CMet}$ is monadic.

Proof. Like in the proof of [2.2] the functor $U$ preserves limits and $\aleph_1$-directed colimits, thus it has a left adjoint $F$. Let $T = UF$ be the corresponding monad. In the same way as in [15] 1.3, we prove that the induced functor $\text{Ban} \to \text{Alg}(T)$ is fully faithful. We have to prove that it is essentially surjective.

Let $(A, h)$ be a $T$-algebra. Since $h \eta_A = \text{id}_A$, $\eta_A$ isometrically embeds $A$ to the unit ball $UFA$. Unit balls of Banach spaces are totally convex and totally convex operations $(UB)^\infty \to UB$ have norm $\leq 1$. They are also preserved by morphisms in $\text{Ban}$. We define totally convex operations on $A$ by

$$\sum_{i=0}^\infty c_i a_i = h \sum_{i=0}^\infty c_i \eta_A(a_i).$$

Following the equation $hT(h) = h \mu_A$, where $\mu$ is the multiplication of $T$, $(A, h)$ is a totally convex space.

We will show that $A$ is separated. Assume that $a_1, a_2 \in A$ and $ca_1 = ca_2$ for all $0 < |c| < 1$. Hence

$$h(c \eta_A(a_1)) = ca_1 = ca_2 = h(c \eta_A(a_2))$$

for all $0 < |c| < 1$. Since $1 = \lim_n c_n$, $0 < c_n < 1$ and $h$ is continuous,

$$a_1 = h \eta_A(a_1) = h(\lim_n c_n \eta_A(a_1)) = h(\lim_n c_n \eta_A(a_2)) = h \eta_A(a_2) = a_2.$$
The unit ball functor makes $\text{Ban}$ a full reflective subcategory of the category of totally convex spaces (see [15], 7.7). Let $\sigma_A : A \to UA^*$ be the unit of this reflection. Since $A$ is separated, $\sigma_A : A \to UA^*$ is an isometry (see [16] 11.3). Following [15], 7.3, the image of $\sigma_A$ contains the open unit ball of $A^*$.

Assume that there is $a \in A^*$ which does not belong to this image $\sigma_A(A)$. Put $s_n = \sum_{k=1}^n \frac{1}{2^k}$ and $a_n = s_na$. Then $a = \lim_n a_n$ and $a_n \in \sigma_A(A)$. Thus $a_n = \sigma_A(b_n)$ where $b_n \in A$. Since $\frac{a_n}{s_n} = a = \frac{a_m}{s_m}$, we have $a_m = \frac{s_m}{s_n}a_n$. Since $\frac{s_m}{s_n} \leq 1$ for $m \leq n$ and $\sigma_A$ is a morphism of totally convex spaces, we have

$$h\eta_A(b_m) = b_m = \frac{s_m}{s_n}b_n = h\left(\frac{s_m}{s_n}\eta_A(b_n)\right)$$

for $m \leq n$. Since $h$ is nonexpanding, we have

$$d(b_m, b_n) = d(h\eta_A(b_m), h\eta_A(b_n)) = d(h\left(\frac{s_m}{s_n}\eta_A(b_n)\right), h\eta_A(b_n)) \leq d\left(\frac{s_m}{s_n}\eta_A(b_n), \eta_A(b_n)\right)$$

$$= \left\|\frac{s_n - s_m}{s_n}\eta_A(b_n)\right\| \leq \frac{s_n - s_m}{s_n}.$$

Hence $b_1, b_2, \ldots, b_n, \ldots$ is a Cauchy sequence in $A$ and, since $A$ is complete, it is converging to $b \in A$. Since $a = \sigma_A(b) \in \sigma_A(A)$, we get a contradiction.

Hence $(A, h)$ is a unit ball of a Banach space. \(\square\)

**Remark 4.2.** (1) Like in [2,3] $T$ preserves directed colimits.

(2) The Kantorovich monad $K$ on $\text{C Met}$ is given by barycentric operations and its algebras are closed convex subsets of Banach spaces (see [10]). Our monad $T$ has more operations, i.e., less algebras. Following [1], $K$ preserves directed colimits.

(3) The unit ball functor $\text{Ban} \to \text{Met}$ is monadic as well. Its $T$-algebras are complete being retracts of complete metric spaces - given by $h\eta_A = id_A$. The rest is the same as in [11].

**5. $C^*$-algebras**

Let $\text{CAlg}$ be the category of unital $C^*$-algebras and $\text{CCAlg}$ the category of commutative unital $C^*$-algebras. The forgetful functor $G : \text{CAlg} \to \text{Ban}$ preserves limits, isometries and directed colimits (see [8] 6.10). Thus it has a left adjoint $F$. The same holds for the restriction $G_c : \text{CCAlg} \to \text{Ban}$ of $G$ on the category $\text{CCAlg}$. The unit $\eta_B : B \to GFB$ is a linear isometry. Thus $F$ is faithful. In the commutative case, the left adjoint $F_c$ was described in [18] and called the Banach-Mazur functor.

Let $\text{BanAlg}$ be the category of unital Banach algebras and $\text{IBanAlg}$ the category of unital involutive Banach algebras. The forgetful functors $G_0 : \text{BanAlg} \to \text{Ban}$ and $G_1 : \text{BanAlg} \to \text{Ban}$ again preserve limits, isometries and directed colimits.

**Theorem 5.1.** The forgetful functors $G_0 : \text{BanAlg} \to \text{Ban}$ and $G_1 : \text{IBanAlg} \to \text{Ban}$ are monadic.
Proof. Let $T_0 = G_0 F_0$ and $T_1 = G_1 F_1$ be the induced monads. Given a unital Banach algebra $A$, the operation $\cdot : G_0 A \otimes_p G_0 A \to G_0 A$ has norm $\leq 1$. Here, $\otimes_p$ is the projective tensor product on $\text{Ban}$ which satisfies $\| x \otimes_p y \| = \| x \| \| y \|$ (see [6] 2.5.10). Hence $\| x \cdot y \| \leq \| x \| \| y \|$.

Hence $GA$ is a monoid in $\text{Ban}$. In fact, $\text{BanAlg}$ coincides with the category of monoids in $\text{Ban}$. Since $X \otimes -$ has a right adjoint (see [5] 6.1.9h), the category of monoids in $\text{Ban}$ is monadic over $\text{Ban}$ (see [14]).

If $A$ is an unital involutive Banach algebra $G_1 A$ is also equipped with a unary operation $(-)^*$ which is of norm $\leq 1$ again. Hence unital involutive Banach algebras coincide with involutive monoids in $\text{Ban}$. Like in [2.2] $G_1$ is monadic. \hfill \Box

Remark 5.2. (1) The same holds for commutative unital (involutive) Banach algebras.

(2) Unital $C^*$-algebras are unital involutive Banach algebras satisfying

$$\| x^* \cdot x \| = \| x \|^2.$$  

They form a full reflective subcategory of $\text{IBanAlg}$ where the reflector is given by the enveloping $C^*$-algebra. The unit ball functor $\text{CAlg} \to \text{Set}$ is monadic (see [13]). Similarly, the unit ball functor $\text{CCalg} \to \text{Set}$ is monadic (see [19]).

(3) Is $G : \text{CAlg} \to \text{Ban}$ monadic? Similarly, is $G_c : \text{CCAlg} \to \text{Ban}$ monadic?

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