$D$-dimensions Dirac fermions BEC-BCS crossover thermodynamics

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An effective Proca Lagrangian action is used to address the vector condensation Lorentz violation effects on the equation of state of the strongly interacting fermions system. The interior quantum fluctuation effects are incorporated as an external field approximation indirectly through a fictive generalized Thomson Problem counterterm background. The general analytical formulas for the $d$-dimensions thermodynamics are given near the unitary limit region. In the non-relativistic limit for $d = 3$, the universal dimensionless coefficient $\xi = 4/9$ and energy gap $\Delta/\varepsilon_f = 5/18$ are reasonably consistent with the existed theoretical and experimental results. In the unitary limit for $d = 2$ and $T = 0$, the universal coefficient can even approach the extreme occasion $\xi = 0$ corresponding to the infinite effective fermion mass $m^* = \infty$ which can be mapped to the strongly coupled two-dimensions electrons and is quite similar to the three-dimensions Bose-Einstein Condensation of ideal boson gas. Instead, for $d = 1$, the universal coefficient $\xi$ is negative, implying the non-existence of phase transition from superfluidity to normal state. The solutions manifest the quantum Ising universal class characteristic of the strongly coupled unitary fermions gas.

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I. INTRODUCTION

Since DeMarco and Jin achieved the Fermi degeneracy[1], the property of the strongly coupling fermions system under the ultra-cold extreme conditions serves as a bewitching topic in the fundamental Fermi-Dirac statistical physics.

The universal thermodynamics properties learned from the ultra-cold atomic experiments can be related to many other realistic many-body physics problems[2, 3]. For example, in nuclear physics, the magnitude of the neutron-neutron $S$-wave scattering length $a_{NN} = -18.6$ fm is much larger than the interaction range given by $R \approx 1/m_\pi \sim 1.4$ fm, which will indicate the low energy thermodynamics universal property in the dilute regime[4]. Furthermore, the Feshbach resonance physics can be also related with the strongly interacting $SU(N_c)$ non-Abelian plasma low viscosity physics realized in the ultra-relativistic heavy-ion collisions or existed in the core of neutron star[5] and even the “high-temperature” $T_c$ physics[6].

Across the Feshbach resonance regime, the interaction changes from weakly to strongly attractive according to the magnitude of the applied external magnetic field $B$. At this Bose-Einstein Condensation (BEC)/Bardeen-Cooper-Schrieffer (BCS) crossover point, the scattering length diverges $|a| = \infty$ due to the existence of a zero-energy bound state. This characteristic leads to the universal thermodynamics, i.e., the system details do not contribute to the thermodynamics properties. The intriguing universal unitary fermions thermodynamics properties attract much attention recently in the literature[4–11].

In the dilute unitary limit, the important physical quantities energy density $E/N = \xi (E/N)_{\text{ideal}}$ and energy gap $\Delta$ or critical phase transition temperature $T_c$, are the two key parameters to be determined experimentally and calculated theoretically. The dimensionless proportionality constant $\xi$ is called the universal factor. The $\Delta$ is also proportional to the Fermi kinetic energy. Many growing theoretical and experimental efforts about the universal factor $\xi$ have been made in recent years Refs.[4, 7–11][12]. It is worthy noting that the differences for the critical temperature or the energy gap can be as large as several times among the theoretical or experimental results existed in the literature. Up to now, there is not a reliable analytical method to deal with the unitary limit problem although there are many updating theoretical analytical attempts. Beyond the naive perturbative expansion techniques or the lowest order mean field theory, a novel method is to be explored for the unitary limit fermions thermodynamics[13]. The essential task is how to incorporate the in-medium nonlinear quantum fluctuation/correlation effects into the thermodynamics in a reasonably controllable way.

Recently, inspired by the homology between this unitary limit topic and the nonperturbative long range infrared singularity manifested by the hot and dense gauge field theory, we make an analytical detergent to attack this intriguing problem[14]. Our initial observation is that the scattering cross section between the two-body particles at the zero energy bound unitarity is limited as $S$-wave $\sigma = 4\pi/k^2$(with $k$ being the relative wavevector of the colliding particles), while the gauge propagator is $\Delta_{\mu\nu} \sim g_{\mu\nu}/k^2$. The interesting analytical results $\xi = 4/9$ and $\Delta/\varepsilon_f = 5/18$ comparable with other theoretical ones are obtained by taking the still unsolved classical Thomson Problem as a potential nonperturba-
tive quantum many-body arm. In the non-relativistic limit with $T = 0$, the analytical expression for the energy density is the same as that obtained by Steele with the power counting in the effective theory framework[15].

The relativistic continuum quantum field theory formalism provides a natural framework to address the infrared long range singular correlations associated with the density of state in many-body physics according to quantum statistical physics. In terms of the relativistic formalism we can certainly achieve more than those we need in a non-relativistic theory, but we can recover all the non-relativistic limit physical quantities by expanding in powers of $k_f/m$ at any time especially for the $T = 0$ universal unitary gas thermodynamics. In this work, we will continue the relativistic continuum field theory attempts for the strongly interacting Fermi-Dirac statistical physics in terms of the in-medium Lorentz violation. After giving the general analytical expressions for the energy density functional and pressure as well as entropy density according to the density functional theory near the unitary limit, the final analytical results comparable with other theoretical approaches will be obtained by applying the corresponding Taylor series expansion.

Initiated with the recent discussions about the low $d$-dimensions unitary fermions gas[16], we extend the Thomson Problem counterterm analytical method to arbitrary $d$-dimensions with the relativistic continuum field theory formalism. The $d = 2$-dimensions solution reflects the essential interesting physics. The universal dimensionless coefficient $\xi$ can approach to the specific value $\xi = 0$, corresponding to the two-dimensions strongly coupled electrons with the infinity effective mass $m^* = \infty$. This is also very similar to the well-known BEC phenomena for the three-dimensions ideal boson gas with the simultaneous vanishing of both the energy density, pressure and entropy density at $T = 0$. Furthermore, the energy gap can be zero. Instead, for the one-dimension, the $\xi$ can be negative, corresponding to non-existence of the phase transition from the superfluidity to normal state at $T = 0$. The properties of these solutions manifest the strongly coupled quantum Ising universal class characteristic of the unitary fermions thermodynamics realized in the artificial experimental environments.

The present work is organized as follows. In section II, the adopted low energy effective quantum field theory formalism of the relativistic continuum Proca Lagrangian and corresponding Thomson Problem counterterm non-perturbative approach for the strongly interacting unitary gas thermodynamics are described. The effective potential and the general thermodynamics quantities for finite scattering length $a$ as well as the concrete comparisons with existed results are presented in section III. Moreover, the $d$-dimensions universal thermodynamics at unitarity are discussed in section IV and the concluding remarks are made in the final section V.

Throughout this work, the natural units with $c = \hbar = k_B = 1$ are adopted.

II. FORMALISM

A. Effective interaction action

A nonperturbative approach is crucial to understand the strongly correlating dense and hot many-body physics. Near the strongly interacting unitary limit, any form of effective interaction can be used to detect the universal thermodynamics properties in the pseudo-potential spirit of statistical physics. We extend previous work[14] to near unitary limit regime motivated by the successful relativistic quantum many-body formalism[17]. To model the strongly interacting fermions thermodynamics in terms of the fundamental in-medium Lorentz violation effects, let us consider the relativistic Dirac fermions interacting with an auxiliary Proca-like Lorentz massive vector field[18]

$$L_{\text{effective}} = \bar{\psi} i \gamma_{\mu} \partial^\mu - m)\psi - \frac{1}{4} F_{\mu \nu} F^\mu \nu + \frac{1}{2} M_A^2 A_{\mu} A^\mu + A_{\mu} J^\mu + \delta L_{\text{Auxiliary Thomson Counterterm Background}}. \quad (1)$$

The $A_{\mu}$ is the vector field with the field stress

$$F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \quad (2)$$

In the effective action $L_{\mu \nu \psi} = \frac{1}{2} M_A^2 A_{\mu} A^\mu + A_{\mu} J^\mu$, the vector current contributed by the fermions is

$$J^\mu = g \bar{\psi} \gamma^\mu \psi. \quad (3)$$

In Eq.(1), $m$ is the bare fermion mass with the vector boson mass squared assembling both a bare $m_A^2$ and a fluctuating one $m_{\tilde{l}}^2[19]$, $M_A^2 = m_A^2 + m_{\tilde{l}}^2$. Here, a bare Lorentz violation term $1/2 m^2 A_{\mu} A^\mu$ is included obviously, which determines the interaction strength in vacuum with the bare electric charge $g$. For example, the $S$-wave scattering length $a$ is given through the low energy Born approximation in the language of the low energy quantum mechanics in three-dimensions

$$\frac{g^2}{m^2_A} = \frac{2\pi a}{m}. \quad (4)$$

B. Induced interaction determined by Thomson counterterm background

The induced Lorentz violation term caused by the Thomson background $1/2 m^2_B A_{\mu} A^\mu$ represents the many-body renormalization effect, which has been introduced with a hidden local symmetry manner(\textit{Abelian Higgs} model) by mapping the unitary limit topic to the infrared singularity correlation in dense and hot gauge theory[14]. Taking the coupling constant $g$ to be an “electric” charge of the fermions, makes it possible for us to introduce the generalized renormalization condition, i.e., the fictive Thomson stability condition. With the physical constraint Thomson stability condition, the parameter $m_{\tilde{l}}^2$
is found to be the negative of the gauge invariant Debye mass squared in the unitary limit $|a| = \infty[14, 20]$. The gauge invariant Debye mass associated the density of state (DOS) reflects the essential quantum fluctuation effect on the thermodynamics.

More generally, the analytical expression for the Debye mass squared is the gauge invariant static infrared limit of the 00 ingredient for the vector boson polarization tensor $\Pi_{\mu\nu}$($p_0, p$), which can also be calculated with the full fermion propagator through the Dyson-Schwinger equations or relativistic random phase approximation (RPA)[21]

$$
\Pi_{\mu\nu}(p_0, p) = g^2 T \sum_k \int_k \text{Tr} \left[ \gamma^\mu \frac{1}{k - \mu} \gamma^\nu \frac{1}{(k - \bar{p}) - \mu} \right].
$$

In Eq.(5), the 0-component of the $d+1$-dimensions momentum $k = (k_0, \mathbf{k})$ in the fermion loop is related to temperature $T$ and effective chemical potential $\mu^*$ via $k_0 = (2\pi + 1)\pi T i + \mu^*$ determined by the pole of the full fermion propagator with the relativistic Hartree approximation (RHA) formalism.

Throughout this paper, the shorthand notation $\int_k = \int d^d k/(2\pi)^d$ will be adopted for the $d$-dimensions abstract Dirac phenomenology[22, 23]. With finite temperature field theory, the general analytical expression for $d$-dimensions unitary fermions gas is derived to be

$$
m_B^2 = \Pi_{00}(0, |p| \rightarrow 0) = -\Pi^+_{00}(0, |p| \rightarrow 0).
$$

In the above expressions,

$$
f = \frac{1}{e^{(E_k - \mu^*)}} + 1, \quad \bar{f} = \frac{1}{e^{(E_k + \mu^*)}} + 1,
$$

are the distribution functions for (anti-)particles with $E_k = \sqrt{k^2 + m^2}$ and $\beta = 1/T$ the inverse temperature.

This kind of calculations can be indicated by the comprehensive diagrammatic representation as indicated in Fig.1.

C. Diagrammatic expression for the coupled generalized Dyson-Schwinger equations

It should be stressed the in-medium boson propagator with the negative of the gauge invariant longitudinal component $\Pi_{00}^+(0, |p| \rightarrow 0) = -\Pi_{00}^+(0, |p| \rightarrow 0)$ of the polarization tensor $\Pi_{\mu\nu}^+(0, |p| \rightarrow 0)$ presented in Fig. 1.a. This minus sign implies that the induced interior correlation contribution is realized with the external field approximation formalism indirectly caused by the opposite charged Thomson counterterm background.

The minus sign in front of $\Pi_{00}^+(0, |p| \rightarrow 0)$ is crucial, which ensures the theoretical thermodynamics self-consistency and the unitarity of the final physical results in the unitary limit[20]. The comprehensive Landau dynamical screening effects are automatically incorporated while the concrete physical contents are quite different from the conventional weak coupling resummation techniques. With these two Feynman diagrams, the relevant algebra equations formalism instead of the coupled integral equations one makes it possible for us to deal with the low energy density and temperature fluctuation/concentration effects on the strongly interacting fermions thermodynamics in an accurate way. In other words, the in-medium density correlation effects are included through the effective chemical potential instead of directly through the effective fermion mass. The spirit hidden in this kind of approach is quite similar to the Kohn-Sham density functional theory[17, 24]. Especially, the integral momentum divergence is naturally gauged by the relativistic kinematic factors due to taking the density functional theory formalism to approach the universal unitary gas thermodynamics.

That the correlation effects are embodied through the fictive opposite electric charged Thomson counterterm background can also avoid the theoretical double counting trouble, which leads to the key minus sign difference as found in previous works[20]. This minus sign difference is inspired by the anomalous thermodynamics discussions of neutron-star nuclear matter with either short or long-range force interactions[25, 26].
D. Parameters $m_B^2$ at $T = 0$

At $T = 0$, from the general expression Eq.(6), one can have

\[ \frac{m_B^2}{g^2} = - \frac{2d}{(2\sqrt{\pi})^d} \frac{k_f^{d-2}E_f}{\Gamma(\frac{d}{2} + 1)}, \]

which can be reduced to

\[ \frac{m_B^2}{g^2} = - \frac{2d}{(2\sqrt{\pi})^d} \frac{k_f^{d-2}m}{\Gamma(\frac{d}{2} + 1)} \]

for the non-relativistic limit. The fermion density in $d$-dimensions is

\[ n = \frac{2}{(2\sqrt{\pi})^d} \frac{k_f^{d}}{\pi^2}. \]

In above expressions, the $\Gamma(\frac{d}{2} + 1)$ is the Gamma function.

For example, with $d = 3$, the parameter $m_B^2$ is

\[ \frac{m_B^2}{g^2} = - \frac{k_fE_f}{\pi^2}, \]

with $k_f$ being the Fermi momentum and $E_f = \sqrt{k_f^2 + m^2}$ the relativistic Fermi energy. In the non-relativistic limit, the corresponding result is \( m_B^2/g^2 = -k_f m \).

For $d = 2$, one can have \( m_B^2/g^2 = -E_f \pi \) with the expression Eq.(6). The corresponding non-relativistic result is \( \frac{m_B^2}{g^2} = -\frac{m}{\pi} \).

For $d = 1$, the result is \( \frac{m_B^2}{g^2} = -\frac{2E_f}{\pi k_f} \), while the non-relativistic limit is \( \frac{m_B^2}{g^2} = -\frac{2m}{\pi k_f} \). In the unitary limit, they are negative of the gauge invariant Debye/Thomas-Fermi mass squared[23].

Before making the following calculations, let us discuss the physical meanings of the coupling constants represented by the vector boson mass $m_A$ and electric charge $g$. They are the parameters introduced to characterize the bare vacuum interaction strength (scattering length $a$ in three-dimensions) between particles with relativistic continuum Dirac formalism. Only in the three-dimensions, the coupling constant $g$ can be of dimensionless, which can be clearly seen from Eq.(4). Usually, it will have the mass dimensional $\mu = \frac{g}{g'}$, where $g'$ is dimensionless and $\mu$ is an arbitrary mass scale of the problem.

E. Effective interaction strength

Physically, the interaction between the fermions is renormalized by the Lorentz violating many-body environment and the induced interaction strength between the particles is characterized by the three parameters $m_A^2$, $m_B^2$ and $g^2$. The in-medium effective scattering length formalism can give relevant concise expressions to describe the strongly correlating effect in a nonperturbative manner

\[ a_{eff} = \frac{m}{2\pi M_A^2} = a \frac{1}{1 + \frac{m_B^2}{m_A^2}}. \]

It is worthy noting that the $m_B^2/g^2$ and $g^2$ appear in the analytical expressions as a fraction ratio. Actually, the magnitude of this fraction ratio $m_B^2/g^2$ characterizes the remarkable low energy long range infrared quantum fluctuating effects associated with the DOS on the universal thermodynamics according to the quantum statistical physics. Therefore, there are not any remained adjustable or expansion parameters, i.e., the physical vacuum interaction strength parameter/S-wave scattering length $a$ controls completely the final thermodynamics quantities as given below.

III. EFFECTIVE POTENTIAL THROUGH LORENTZ VECTOR CONDENSATION WITH EXTERNAL FIELD APPROXIMATION

In methodology, the quantum many-body correlation/DOS effects are incorporated as an external field approximation to ensure the theoretical thermodynamics self-consistency, from which the effective potential can be obtained with the standard path integral by taking the relativistic Hartree instantaneous approximation. The external field approximation formalism implies that the vector current conservation/gauge invariance can be guaranteed by the Lorentz transversality condition realized by RHA

\[ \partial_{\mu} A^\mu = 0, \]

from which the effective potential reads[17, 27]

\[ \Omega/V = -T \ln Z = -M_A^2 A_0^2 - T \int_k \sum_i \left[ \ln(1 + e^{-\beta(E_i - \mu_i)}) \right] \]

\[ + \ln(1 + e^{-\beta(E_{0i} + \mu_i)}) \]

\[ = -M_A^2 A_0^2 - 2T \sum_k \left[ \ln(1 + e^{-\beta(E_{+})}) \right] \]

\[ + \ln(1 + e^{-\beta(E_{-})}), \]

where “2” represents the hyperfine-spin degenerate factor of the fermions system. Furthermore, the self-consistency condition or the tadpole diagram with the boson self-energy for the full fermion propagator leads to

\[ A_0 = -\frac{g}{M_A^2} \mu = -\frac{2\pi a_{eff}}{g m} \mu, \]

from which the effective (local) chemical potential $\mu^*$ is defined with a gauge invariant manner

\[ \mu^* = \mu + \mu_t = \mu - \frac{2\pi a_{eff}}{m} \mu, \]

where $\mu$ is the global chemical potential. The fermions particle number (electric charge number) density is

\[ n = 2 \int_k [f - \bar{f}] . \]
A. Thermodynamics near the strongly interacting unitary limit regime

With the external field counterterm instantaneous approximation formalism of the interior correlating effects, one obtains the general analytical expressions for the energy density functional from the effective potential Eq.(13) with the thermodynamics relation $\epsilon = \frac{\pi}{m} + \mu n + T \frac{\partial \epsilon}{\partial T}$

$$\epsilon = \frac{\pi a_{eff}}{m} n^2 + 2 \int_k E_k [f + \bar{f}], \quad (17)$$

and pressure $P = -\Omega/V$

$$P = \frac{\pi a_{eff}}{m} n^2 + \frac{2}{d} \int_k \frac{k^2}{E_k} [f + \bar{f}], \quad (18)$$

with the entropy density

$$\frac{S}{V} = -2 \int_k [f \ln f + (1 - f) \ln (1 - f) + (f \rightarrow \bar{f})]$$

$$= \frac{1}{T} \left(2 \int_k E_k (f + \bar{f}) + \frac{2}{d} \int_k \frac{k^2}{E_k} (f + \bar{f}) - \mu^* n \right). \quad (19)$$

In Eq.(17) and Eq.(18), the first terms are directly related to the interaction and/or quantum fluctuation contributions. In the weak coupling limit, the bare vacuum interaction strength/scattering length $a_{eff} \rightarrow a$ can recover the lowest order conventional mean field theory which neglects the quantum fluctuation contributions. The second terms in Eq.(17) and Eq.(18) as well as the entropy density Eq.(19) appear as very much the analytical formalisms for the free Fermi-Dirac gas. However, the correlating effects are also implicitly included through the effective chemical potential esp. for $T \neq 0$.

The Eq.(17) and Eq.(18) with $m_B^2 = -\Pi^I_A(0, |p| \rightarrow 0) = \Pi_{\Delta}^0(0, |p| \rightarrow 0)$ are the final results. The remaining task is to give the numerical results for given temperature $T$ and density $n$ with the fermion mass $m$. In this work, we limit ourselves to the $T = 0$ universal thermodynamics attempting to obtain the analytical results.

B. Thermodynamics universality at $T = 0$ for three-dimensions

At $T = 0$ and from the above general analytical expressions, one can obtain the energy density and pressure for finite scattering length $a$

$$\epsilon = \frac{\pi a_{eff}}{m} n^2 + \frac{(2k_f^2 + m^2)k_f E_f - m^4 \ln \frac{k_f + E_f}{m}}{8\pi^2},$$

$$P = \frac{\pi a_{eff}}{m} n^2 + \frac{(2k_f^2 - 3m^2)k_f E_f + 3m^4 \ln \frac{k_f + E_f}{m}}{24\pi^2}. \quad (20)$$

In the non-relativistic limit with $T = 0$, the analytical expression for the energy density can recover Steele’s main result obtained within the effective theory framework[15]. In deed, the non-relativistic negative Debye/Thomas-Fermi mass squared can readily approach the non-relativistic power counting result of Steele with the generalized coupled Dyson-Schwinger equations (through the full anti-screened vector boson propagator) as indicated by Fig.1. The non-trivial physics occurs at the pole of the first term in Eq.(20), i.e., in the repulsive BEC regime with $m - 2k_f E_f a/\pi = 0 \rightarrow 2k_f a/\pi \approx 1$ as pointed out in Refs. [13, 15].

Due to the universal properties at $T = 0$ in the dilute unitary limit, the energy density can be scaled as[9]

$$\varepsilon(n) = \frac{3}{5}\varepsilon_f F(\frac{1}{k_f a}),$$

$$= \frac{3}{5} \varepsilon_f \left(\frac{\xi - \zeta}{k_f a} - \frac{5}{3} \frac{v}{k_f^2 a^2} + 0(\frac{1}{(k_f a)^3})\right), \quad (21)$$

where $\varepsilon_f = k_f^2/(2m)$ is the Fermi kinetic energy. Applying the Taylor series expansion of $\epsilon$ Eq.(20) according to $1/(ak_f)$ as well as $k_f$ by keeping only up to the lowest order of $k_f/m$, the corresponding universal coefficients $\xi = 4/9 \approx 0.44, \zeta = 5\pi/18 \approx 0.87, v = \pi^2/12 \approx 0.82$ are exactly consistent with these $\xi \approx 0.44, \zeta \approx 1, v \approx 1$ of Refs.[7, 9] obtained with Monte Carlo calculation. Especially, one of them with much attention is $\xi = 4/9$. The recent quantum Monte Carlo calculation also gives the same result[28].

In the unitary limit, the strong fluctuation and correlation effects not only do manifest themselves in the bulk properties but also modify the quasi-particle properties in a substantial way from the viewpoint of the universal thermodynamics. At unitary $|a| = \infty$, the energy gap $\Delta$ is derived from the total energy density with the odd-even staggering (OES)

$$\Delta = f_{sw}\frac{2k_f^2 + 3m^2 - 3mE_f}{6E_f}, \quad (22)$$

with $f_{sw}$ being the Fermi-Dirac statistical weight factor. In the non-relativistic limit with the factor $f_{sw} = 5/3$, the analytical result is $\Delta/\varepsilon_f = 5/18$, which is reasonably consistent with the theoretical result $\Delta/\varepsilon_f \sim 0.40$ obtained through the BCS model but with the effective scattering length. The corresponding phase transition critical temperature is still approximated by the BCS relation $T_c = \frac{\varepsilon}{\Delta} \Delta/\pi$ with $\gamma$ being the Euler-Gamma constant

$$T_c \approx 0.157T_f, \quad (23)$$

as given in Ref.[14]. This is in reasonable agreement with the updating results[10, 28–30].

The pressure is $P = 1/6\Omega_{FG}$ in the unitary limit, from which one can find the sound speed is reduced remarkably

$$v = \sqrt{\frac{1}{6}} v_{FG} = \sqrt{\frac{2}{3}} v_f, \quad (24)$$
where \( v_{FG}^2 = 1/3v_T^2 \) is the sound speed squared for the ideal Fermi gas with the Fermi velocity \( v_f = k_f/m \).

For the ideal fermion gas, the ratio of pressure to energy density is well known as 2/3. For unitary fermion gas, this ratio is found to be changed to 1/4 for the non-relativistic occasion due to the strongly correlating effects. This is different from that obtained with the universal hypothesis based on assuming the scaling property in terms of the zero-energy bound state. This difference can be attributed to the implicit pairing correlation contribution to the binding energy in the strongly coupling limit[31]. The simultaneous experimental detection for the universal coefficient \( \xi \) and the sound speed can judge this dilemma. We also note that similar conclusion can be seen in the analytical Dyson-Schwinger attempt of Ref.[30].

IV. D-DIMENSIONS FERMIIONS UNIVERSAL PROPERTIES AT UNITARITY

Although the unitary limit issue is proposed for the three-dimensions physics representing for such as the inner neutron star crust physics in the low energy strongly interacting nuclear many-body theory context, it is very interesting to visit the \( d \)-dimensions occasion of the general Fermi-Dirac statistics as a toy model in the unitary limit by applying the Thomson Problem counterterm analytical method.

Concentrating on the realistic physics realized in the experimental environments, the non-relativistic analytical results reduced from the general expressions Eq.(17) and Eq.(18) in the unitary limit with the Thomson Problem counterterm approach are given below. The kinetic energy density contribution for \( d \)-dimensions non-relativistic fermion gas is

\[
\varepsilon_{\text{ideal}} = \frac{2d}{(2\sqrt{\pi})^d (d+2)} \frac{\varepsilon_f k_f^d}{\Gamma(\frac{d}{2}+1)},
\]

while the pressure is

\[
P_{\text{ideal}} = \frac{4}{(2\sqrt{\pi})^d (d+2)} \frac{\varepsilon_f k_f^d}{\Gamma(\frac{d}{2}+1)}.
\]

The negative correlation energy is derived to be

\[
\varepsilon_{\text{int}} = -\frac{2}{(2\sqrt{\pi})^d d \Gamma(\frac{d}{2}+1)} \frac{\varepsilon_f k_f^d}{\Gamma(\frac{d}{2}+1)}.
\]

Analogously to the discussion for three-dimensions, one can obtain the general universal dimensionless coefficient \( \xi \) expression according to dimension \( d \)

\[
\xi = f_{\text{sw}} \frac{(d-2)(d+1)}{d(d+2)} = \frac{(d-2)(d+1)}{d^2},
\]

With the OES method for the \( d \)-dimensions unitary fermions gas, one can have the extended \( S \)-wave energy gap

\[
\Delta/\varepsilon_f = f'_{\text{sw}} \frac{d-2}{2d} = \frac{(d-2)(d+2)}{2d^2}.
\]

In above expressions, \( f_{\text{sw}} = (d+2)/d \) is the \( d \)-dimensions Fermi-Dirac statistical weight factor in the non-relativistic limit. It is worthy noting that the ratio of pressure to energy density is changed from \( 2/d \) of the ideal fermion gas to \( 1/(1+d) \) of that in the unitary limit.

From Eq.(28), one can find that \( \xi \) will be equal to zero at \( d = 2 \). This corresponds to the infinity mass \( m^* = \infty \) of the effective fermion mass. At this specific two-dimensions, the pressure as well as entropy density is also equal to zero, which is quite similar to the BEC phenomena for the ideal boson gas at three-dimensions. In this specific occasion, the sound speed will approach to zero in terms of the Landau Fermi liquid theory. Furthermore, the energy gap will be zero in the meantime with the OES result Eq.(29) of the two-dimensions. This result indicates the low two-dimensions unitary limit thermodynamics properties are exotic.

In the strongly coupling limit, the essential characteristic \( m^* \rightarrow \infty \) is in line with the strongly coupled two-dimensions electrons discussed in the literature[32]. The zero energy gap is so similar to the vanishing of the extended \( S \) pairing correlation etc. with the increase of repulsive interaction strength in the two-dimensions Hubbard Model with quantum Monte Carlo study[33]. It is worthy noting that the behavior of \( \xi \) according to the spatial dimensions \( d \) as indicated by Fig.2 is different from those found in the recent works[16].

![FIG. 2: The universal coefficient \( \xi \) versus dimension \( d \). The lower solid line (a) is for the non-relativistic Eq.(28), while the upper dotted line (b) is for the ultra-relativistic limit Eq.(30). The dashed parts correspond to the instability region.](image)
It is worthy noting that in this opposite limit the corresponding correction of the Fermi kinetic energy is the magnitude of the Fermi momentum $k_f$, i.e., $\varepsilon'_f \rightarrow k_f$. The prime symbol has been adopted in order to avoid the confusion with the non-relativistic results. In Eqs. (30)-(31), the statistical weight factor is $f'_{sw} = (d+1)/d$. There exists a rummy duality relation between the non-relativistic and the ultra-relativistic results seen from above expressions.

From Fig. 2, one can clearly see that these solutions manifest the quantum Ising universal class characteristic of the strongly interacting fermions gas realized in the experimental environments[34]. For the non-relativistic occasion, the universal coefficient is $\xi < 0$ for $d < 2$ with the negative sound speed squared. The negative sound speed squared means that there is no phase transition from the superfluidity state to normal state due to the spinodal instability of the superfluidity state. In the one-dimension, the strong fluctuation effects in the system will make the regions out of phase with each other and reduce the possibility of reaching the thermodynamics equilibrium. This is consistent with the general phase transition theory, i.e., there is not a phase transition for one-dimension.

With the analytical results such as Eq. (28) and Eq. (29) as well as Eq. (30) and Eq. (31), one can also see that the proportionality coefficients do depend only on the spatial dimensions $d$ and are independent of the system details in the unitary limit. This is of the genuine thermodynamics universality characteristic associated with the critical phenomena in terms of the general statistical physics.

V. SUMMARY

In conclusion, we have given the general analytical formulas of the energy density functional and pressure as well as entropy density at both finite temperature and density in terms of the relativistic quantum many-body formalism near the unitary limit regime. The interior non-linear fluctuation/correlation effects on the thermodynamics have been incorporated with an external field approximation scheme through a fictive opposite charged Thomson background. This detour will allow us to go beyond the lowest order mean field theory or the naive perturbative expansions in an accurate way.

The $d$-dimensions universal thermodynamics in the unitary limit are discussed. It is found the thermodynamics properties of low dimension $d = 2$ are very interesting in the unitary limit, i.e., the universal dimensionless coefficient $\xi = 0$ with simultaneous vanishing of energy density, pressure, energy gap and sound speed as well as entropy density. In the non-relativistic limit, the effective fermion mass can approach to $m^* = \infty$, corresponding to the strongly coupled electrons in condensed matter physics. This characteristic is quite similar to the well-known BEC of the ideal boson gas at $T = 0$ for three-dimensions. These $d$-dimensions solutions manifest the quantum Ising universal class characteristic of the strongly interacting low temperature fermions gas physics.

In addition to the exact consistency of the obtained physical results with some existed theoretical analytical or simulation ones, this attempt with the unknown side/Thomson Problem as a potential quantum many-body nonperturbative arm to solve the other unknown side facilitates the concrete comparison of the non-relativistic many-body methods and the Dirac phenomenology of the relativistic continuum formalism in terms of the fundamental in-medium Lorentz violation.

The essential features of the strongly coupled unitary gases may be captured by the Lorentz violation with vector condensation formalism. The key physics may be obtained by the external field realization scheme of the interior correlations between the strongly interacting particles through a fictive Thomson background but still with the standard RHA and RPA techniques (or the generalized coupled Dyson-Schwinger equations).

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