Decomposition of $SU(N)$ Connection and Effective Theory of $SU(N)$ QCD

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We give a general decomposition of $SU(N)$ connection and derive a generalized Skyrme-Faddeev action as the effective action of $SU(N)$ QCD in the low energy limit. The result is obtained by separating the topological degrees which describes the non-Abelian monopoles from the dynamical degree of gauge potential, and integrating all the dynamical degrees of $SU(N)$ QCD.

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I. INTRODUCTION

An open question in QCD is to identify the quark confinement mechanism and understand how it works. The monopole condensation is regarded as a possible explanation for the confinement of color through the dual Meissner effect [1,2]. To realize the dual scenario of the confinement, 't Hooft proposed an Abelian projection where the gauge group is broken by a suitable gauge condition to its maximal Abelian subgroup [3]. Since the topology of the $SU(N)$ and that of its maximal Abelian subgroup $U(1)^{N-1}$ are different, any such gauge is singular, meaning that a gauge group element which transforms a generic $SU(N)$ connection onto the gauge fixing surface is not regular everywhere in space-time. The singularities may form worldline that are usually interpreted as worldlines of magnetic monopoles. As a result the original Yang-Mills theory turns into electrodynamics with magnetic monopoles. Recent numerical simulations show that the monopole degrees of freedom in the Abelian projection can indeed form a condensation responsible for the confinement [3]. However, there still has no satisfactory proof how the desired monopole condensation could take place in QCD.

Recently, there are lots of work [4,5] done for searching the mechanism of confinement in Yang-Mills theory. One hopes to find the confinement mechanism based on the first principle. Y. M. Cho et al [4] provide a possible theoretical mechanism of the monopole condensation in $SU(2)$ QCD. Through utilizing the parametrization of the $SU(2)$ gauge potential and choosing proper gauge, they showed that the monopole potential acquires a mass through the quantum correction after integrating out all the dynamical degrees of the non-Abelian potential.

In this paper, from the similar consideration as that of Cho et al, we obtain the effective theory of $SU(N)$ QCD based on the first principles. It is found that there exists mass gaps in the infra-red limit of $SU(N)$ QCD corresponding to different Abelian projection parts of $SU(N)$ connection. Our result gives a connection between the generalized non-linear model of Skyrme-Faddeev type and $SU(N)$ QCD in the low energy limit. We also shows this effective theory may have special feature at large $N$ limit and may relate to the properties of D-brane. In order to achieve the goal desired, we also give a general decomposition of $SU(N)$ connection in terms of $N-1$ orthonormal vectors defined by conjugating the Cartan matrices by a generic $SU(N)$ element. The $SU(N)$ connection we give corresponds to Cho connection and Faddeev-Niemi connection under different condition.

This paper is arranged as follows. In section 2, we give the general decomposition of $SU(N)$ connection. In section 3, the effective theory of $SU(N)$ QCD is given. At last, from the viewpoint of background field, we give a effective lagrangian of Skyrme-Faddeev type at section 4.

II. DECOMPOSITION OF $SU(N)$ GAUGE POTENTIAL

Let $M$ be a $n$-dimensional Riemannian manifolds and $P(\pi, M, G)$ be a principal bundle with the structure group $G = SU(N)$. A smooth vector field $\phi^a$ ($a = 1, 2, ..., N^2 - 1$) can be found on the base manifold $M$ (a section of a
The covariant derivative 1-form of $\phi^a$ is

$$D\phi^a = d\phi^a + f^{abc}A^b\phi^c$$

where $A^a$ is the connection 1-form

$$A^a = A^a_\mu dx^\mu.$$  

The curvature 2-form is defined as

$$F^a = \frac{1}{2}F^a_{\mu\nu} dx^\mu \wedge dx^\nu = dA^a + g f^{abc} A^b \wedge A^c.$$  

The defining representation of the $SU(N)$ Lie algebra consists of $N^2 - 1$ matrices $T^a$ which satisfy

$$T^a T^b = \frac{1}{2}N \delta^{ab} + \frac{i}{2} f^{abc} T^c + \frac{1}{2} d^{abc} T^c,$$

where $f^{abc}$ are completely antisymmetric and $d^{abc}$ are completely symmetric. In terms of $T^a$, the vector field $\phi^a$ can be represented as matrix form

$$\phi = \phi^a T^a$$

and its covariant derivative as

$$D\phi = d\phi - ig[A, \phi]$$

Conjugating the Cartan matrices $H_i$ ($i = 1, 2, ..., N-1$) by a generic element $g \in SU(N)$, we define $N - 1$ vectors

$$m_i = m^a_i T^a = g H_i g^{-1}$$

which are orthonormal to each other

$$\frac{1}{2}Tr(m_i m_j) = m^a_i m^a_j = \delta_{ij}.$$  

In this way, $m_i$ make up an over determined set of coordinates on the orbit $SU(N)/U(1)^{N-1}$. The covariant derivative of $m^a_i$ are

$$D_\mu m^a_i = dm^a_i + g f^{abc} A_\mu^b m^c_i$$

By making use of a relation

$$f^{acd} f^{bed} m^c_i m^e_j = \delta^{ab} - m^a_i m^b_j$$

it is easy to solve the equation (9) to get gauge potential $A$ expressed in terms of $m_i$ as

$$A_\mu = \frac{1}{g} f^{abc} \partial_\mu m^b_i m^c_i - \frac{1}{g} f^{abc} D_\mu m^b_i m^c_i + A_{i\mu} m^a_i$$

in which $A_{i\mu}$ is the projection of $A_\mu$ on the $m_i$

$$A_{i\mu} = A^a_{i\mu} m^a_i$$

If $m^a_i$ are covariant constants

$$D_\mu m^a_i = 0$$

we get the Cho connection

$$A_\mu = \frac{1}{g} f^{abc} \partial_\mu m^b_i m^c_i + A_{i\mu} m^a_i$$

in which $A_{i\mu}$ is the projection of $A_\mu$ on the $m_i$
On 4-dimensional manifold, by choosing a complete basis of subspace of $su(N)$ Lie algebra which is orthogonal to the subspace make up by $m_i$, the covariant part of the gauge potential \((11)\) can be expressed as

$$-f^{abc}D_\mu m^b_i m^c_i = \rho^{ij} f^{abc} \partial_\mu m^b_i m^c_j + \sigma^{ij} d^{abc} \partial_\mu m^b_i m^c_j$$  \hspace{1cm} (15)$$

and the connection is expressed as

$$A_\mu = A_{i\mu} m^a_i + \frac{1}{g} f^{abc} \partial_\mu m^b_i m^c_i$$

\hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} + \frac{1}{g} \rho^{ij} f^{abc} \partial_\mu m^b_i m^c_j + \frac{1}{g} \sigma^{ij} d^{abc} \partial_\mu m^b_i m^c_j$$  \hspace{1cm} (16)$$

which is just the connection proposed by Faddeev and Niemi [7].

Define a covariant vector 1-form $X^a = X^a_\mu dx^\mu$ as the covariant part of the connection \((11)\)

$$X^a = -\frac{1}{g} f^{abc} D_m m^b_i m^c_i$$  \hspace{1cm} (17)$$

which is orthogonal to $m^a_i$

$$X^a_i m^a_i = 0$$  \hspace{1cm} (18)$$

and denote the non-covariant part of \((11)\) as

$$\hat{A}^a_\mu = A_{i\mu} m^a_i + C^a_\mu$$  \hspace{1cm} (19)$$

where

$$C^a_\mu = \frac{1}{g} f^{abc} \partial_\mu m^b_i m^c_i$$  \hspace{1cm} (20)$$

We can rewrite the gauge potential simply as

$$A^a_\mu = \hat{A}^a_\mu + X^a_\mu$$  \hspace{1cm} (21)$$

The connection we obtained naturally pick out the Abelian parts from the full connection. In the expression \((11)\) $A_{i\mu}$ serve as "electric" potential and $C_\mu$ serve as "magnetic" potential. Under the infinitesimal gauge transformation

$$\delta m^a_i = -f^{abc} \alpha^b m^c_i, \ \ \ \delta A^a_\mu = \frac{1}{g} D_\mu \alpha^a$$  \hspace{1cm} (22)$$

one has

$$\delta A_{i\mu} = \frac{1}{g} m^a_i \partial_\mu \alpha^a, \ \ \ \delta \hat{A}^a_\mu = \frac{1}{g} \hat{D}_\mu \alpha^a, \ \ \ \delta X^a_\mu = -f^{abc} \alpha^b X^c_\mu$$  \hspace{1cm} (23)$$

Using the $SU(N)$ connection obtained above, the gauge field can be expressed as

$$F^a_\mu = m^a_i (\partial_\mu A_{i\nu} - \partial_\nu A_{i\mu}) - \frac{1}{g} m^a_i f^{bcd} m^b_i \partial_\mu m^c_j \partial_\nu m^d_j$$

\hspace{1cm} \hspace{1cm} + \hat{D}_\mu X^a_\nu - \hat{D}_\nu X^a_\mu + g f^{abc} X^b_\mu X^c_\nu$$  \hspace{1cm} (24)$$

in which

$$\hat{D}_\mu X^a_\nu = \partial_\mu X^a_\nu + g f^{abc} \hat{A}_\mu X^b_\nu$$  \hspace{1cm} (25)$$
The Yang-Mills Lagrangian can be expressed in terms of $A_{i\mu}$ and $X^a_{\mu}$

$$\mathcal{L} = -\frac{1}{4}(F_{\mu\nu}^a)^2$$
$$= -\frac{1}{4} F_{\mu\nu}^2 - \frac{g}{2} f^{abc} \hat{F}_{\mu\nu}^a X^b_{\mu} X^c_{\nu} - \frac{g^2}{4} (f^{abc} X^b_{\mu} X^c_{\nu})^2 - \frac{1}{4} (\hat{D}_{\mu} X^a_{\nu} - \hat{D}_{\nu} X^a_{\mu})^2$$

(26)

where $\hat{F}_{\mu\nu}$ is defined as

$$\hat{F}_{\mu\nu} = \partial_{\mu} A_{i\nu} - \partial_{\nu} A_{i\mu} - \frac{1}{g} f^{bcd} m_i^b \partial_{\mu} m_i^c \partial_{\nu} m_i^d$$

(27)

To get (26), we assume

$$f^{abc} X^b_{\mu} X^c_{\nu} = m_i^a f^{bcd} m_i^b X^d_{\mu} X^c_{\nu}$$

(28)

which means the Lie product of $X_{\mu}$ is belong to the subspace made up by $m_i$. This condition is satisfied intrinsically at $SU(2)$ gauge theory. The further analysis of this assumption will be showed in the forth coming paper.

In order to get the effective action expressed solely by the topological fields $m_i$, we need to integrate out all the dynamical degrees. Consider the generating function for (26)

$$W[J_{i\mu}, J_{i\nu}] = \int DA_{i\mu} DX^a_{\mu} \exp\left[-i \int \left( \frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} + A_{i\mu} J_{i\nu} + X^a_{\mu} J_{i\nu}^a \right) d^4 x \right].$$

(29)

To integrate out all the dynamical degrees we need to choose a proper gauge to leave $m_i$ as a background. The techniques we use here are similar to that were used by Y. M.. Cho et al [4].

Firstly, we fix $m_i$ as a background such that under the infinitesimal gauge transformation

$$\delta m_i^a = 0, \quad \delta A_{i\mu} = \frac{1}{g} D_\mu \alpha^a$$

(30)

from which one has

$$\delta A_{i\mu} = \frac{1}{g} m_i^a D_\mu \alpha^a, \quad \delta \hat{A}_{i\mu} = \frac{1}{g} (m_i^b D_\mu \alpha^b) m_i^a, \quad \delta X^a_{\mu} = \frac{1}{g} (D_\mu \alpha^a - m_i^a m_i^b D_\mu \alpha^b)$$

(31)

Fix the gauge with the condition

$$F^a_{\mu} = \hat{D}_\mu X^a_{\mu} - f_1 = 0$$
$$\mathcal{L}_{gf_1} = -\frac{1}{2\kappa} (\hat{D}_\mu X^a_{\mu})^2$$

(32)

(33)

The corresponding Faddeev-Popov determinant can be written as

$$M_{1}^{ab} = \frac{\delta F^a_{\mu}}{\delta \alpha^b} = \frac{1}{g} (\hat{D}_\mu D_\mu)^{ab}$$

(34)

With this the generating functional take the form with $\kappa = 1$,

$$W[J_{a\mu}] = \int DX^a_{\mu} \det ||M_1|| \exp\left\{ -i \int \left( \frac{1}{4} \hat{F}_{\mu\nu}^a - \frac{1}{2} X^a_{\mu} \hat{D}_\mu X^a_{\mu} + g f^{abc} \hat{F}_{\mu\nu}^a X^b_{\mu} X^c_{\nu} + \frac{g^2}{4} (f^{abc} X^b_{\mu} X^c_{\nu})^2 + X^a_{\mu} J_{a\mu} \right) d^4 x \right\}$$

(35)

where $\hat{F}_{\mu\nu}^a$ is defined as

$$\hat{F}_{\mu\nu}^a = m_i^a (\partial_{\mu} A_{i\nu} - \partial_{\nu} A_{i\mu}) - \frac{1}{g} m_i^a f^{bcd} m_i^b \partial_{\mu} m_i^c \partial_{\nu} m_i^d$$

(36)
To remove the quartic term of $X_\mu^a$, we introduce $N^2 - 1$ auxiliary antisymmetric tensor field $\chi^a_{\mu\nu}$ and express the quartic term as

$$\exp\left[ -\frac{i}{4} \int (f^{abc} X_\mu^a X_\nu^b X_\sigma^c) d^4x \right] = \int D\chi^a_{\mu\nu} \exp\left[ -\frac{i}{4} \int (\chi^a_{\mu\nu} \chi^a_{\mu\nu} - 2i\chi^a_{\mu\nu} f^{abc} X_\mu^b X_\nu^c) d^4x \right]$$

(37)

The integration over $X_\mu^a$ results in the functional determinant

$$\det^{-\frac{1}{2}} K_{\mu\nu}^{ab} = \det^{-\frac{1}{2}} \left[ g_{\mu\nu} \delta^{ab} D \bar{D} - 2gf^{abc} \bar{F}_{\mu\nu}^c - i f^{abc} \chi^a_{\mu\nu} \right]$$

so that the generating function is given by

$$W[J^a_\mu] = \int D\chi^a_{\mu\nu} \det ||M_1|| \det^{-\frac{1}{2}} ||K|| \exp\left[ -i \int \left( \frac{1}{4} \bar{F}_{\mu\nu}^2 + \frac{1}{4} (\chi^a_{\mu\nu})^2 + \frac{1}{2} f^{a\mu}_{\nu} K^{-1} J^a_\mu \right) d^4x \right]$$

(38)

(39)

Take the trivial classical configurations $\chi^a_{\mu\nu} = 0$, we can perform the integrate over the auxiliary antisymmetric field. In one loop approximation, by making use of the dimensional regulation, we can calculate the determinants and those terms involving only divergent parts are

$$\ln \det ||M_1|| = \frac{iNg^2}{96\pi^2 \varepsilon} \int \bar{F}_{\mu\nu}^2 d^4x$$

$$\ln \det^{-\frac{1}{2}} ||K|| = \frac{i10Ng^2}{96\pi^2 \varepsilon} \int \bar{F}_{\mu\nu}^2 d^4x$$

(40)

So the effective Lagrangian for the restricted QCD is written as

$$\mathcal{L}_{(R)eff} = -\frac{1}{4} - \frac{11Ng^2}{96\pi^2 \varepsilon} \bar{F}_{\mu\nu}^2$$

(41)

This restricted QCD action contains only the Abelian projection which nevertheless has the full non-Abelian gauge degrees of freedom.

Then we consider

$$W[J_{\mu\nu}] = \int DA_{\mu\nu} \exp\left[ -i \int \left( \frac{1}{4} - \frac{11Ng^2}{96\pi^2 \varepsilon} \bar{F}_{\mu\nu}^2 + A_{\mu\nu} J_{\mu\nu} \right) d^4x \right]$$

(42)

We have to integrate out the electric degrees $A_{\mu\nu}$ from the restricted QCD to obtain the effective action for $m_i$. We can choose the gauge condition as

$$F_2^a = \partial_\mu A_{\mu i} m_i^a + \partial_\mu C_\mu^a - f_2$$

$$= \partial_\mu A_{\mu i} m_i^a + \frac{1}{g} f^{abc} \partial_\mu m_i^b m_i^c - f_2 = 0$$

(43)

$$\mathcal{L}_{gf2} = -\frac{1}{2\lambda} \left[ (\partial_\mu A_{\mu i})^2 + (\partial_\mu C_\mu^a)^2 \right]$$

(44)

One can easy to obtain the Faddeev-Popov determinant as

$$M_2^{ab} = \partial_\mu \delta^{ab} + (m_i^a \partial_\mu m_i^b - 2f^{ace} f^{db} m_i^c \partial_\mu m_i^d) \partial_\mu$$

$$+ f^{ace} f^{db} (\partial_\mu m_i^c m_i^d - m_i^c \partial_\mu m_i^d) - f^{abc} \partial_\mu A_{\mu i} m_i^c$$

(45)

Calculate the determinant in one-loop approximation, one can get

$$\ln \det ||M_2|| = i \int \left[ -\frac{N+2}{32\pi^2 \varepsilon} \mu_0^2 g^2 (C_\mu^a)^2 + \frac{1}{192\pi^2 \varepsilon} \left( \frac{N}{4} - \frac{11}{8} \right) g^2 (\partial_\mu C_\mu^a)^2 \right]$$

$$+ \frac{3}{4} \frac{N^2 - 1}{32\pi^2 \varepsilon} \mu_0^2 g^2 (C_\mu^a)^2$$

$$+ \frac{5}{8} \frac{N^2 - 1}{32\pi^2 \varepsilon} \mu_0^2 g^2 (\partial_\mu C_\mu^a)^2$$

$$- 3\partial_\mu m_i^a \partial_\mu m_i^a - 7\partial_\mu m_i^b \partial_\mu m_i^b \partial_\mu m_i^a \partial_\mu m_i^d - \frac{1}{2} \partial_\mu m_i^a \partial_\mu m_i^b \partial_\mu m_i^c \partial_\mu m_i^d$$

$$- \frac{3}{4} \partial_\mu m_i^a \partial_\mu m_i^a \partial_\mu m_i^b \partial_\mu m_i^d$$

(46)
where $\mu_0$ is a mass scale. Integrate $A_\mu$ from (12) with (14) and (46), with $\lambda = 1$. The effective Lagrangian is obtained

$$\mathcal{L}_{eff} = -\frac{N + 2}{32\pi^2\varepsilon} \mu_0^2 \xi^2 (\partial_\mu m_i^a)^2 - \frac{1}{4}(1 - \frac{1}{8\pi^2 \varepsilon}) \frac{181N}{384\pi^2 \varepsilon} g^2 (m_i^a \partial_\mu C_\mu^a)^2$$

$$+ \left( -\frac{1}{2g^2} + \frac{1}{192\pi^2} \frac{11}{4} \ln \left( \frac{N - 3}{\varepsilon} \right) \right) g^2 (\partial_\mu C_\mu^a)^2 + \frac{1}{192\pi^2 \varepsilon} g^2 (\partial_\mu C_\mu^a)^2$$

$$- \frac{N}{128\pi^2 \varepsilon} g^4 Tr(C_\mu C_\nu C_\mu C_\nu) - \frac{1}{64\pi^2 \varepsilon} \frac{1}{2} \partial^2 m_i^a \partial^2 m_i^a - \frac{1}{512\pi^2 \varepsilon} (\partial_\mu m_i^b \partial_\mu m_i^b)^2$$

$$- \frac{1}{384\pi^2 \varepsilon} (\partial_\mu m_i^b \partial_\mu m_i^b)^2 - \frac{1}{128\pi^2 \varepsilon} (\partial_\mu m_i^b \partial_\mu m_i^b)^2 - \frac{1}{512\pi^2 \varepsilon} (\partial_\mu m_i^b \partial_\mu m_i^b)^2$$

(47)

After a proper renormalization the final effective Lagrangian can be written as

$$\mathcal{L}_{eff} = -\frac{\mu^2}{2} (\partial_\mu m_i^a)^2 - \frac{1}{4}(m_i^a \partial_\mu C_\mu^a)^2 - \frac{1}{2} \alpha_1 (\partial_\mu C_\mu^a)^2 - \frac{1}{2} \alpha_2 (\partial_\mu C_\mu^a)^2$$

$$- \frac{1}{2} \alpha_3 Tr(C_\mu C_\nu C_\mu C_\nu) - \frac{1}{2} \alpha_4 (\partial^2 m_i^a)^2 - \frac{1}{2} \alpha_5 (\partial_\mu m_i^a \partial_\mu m_i^a)^2$$

$$- \frac{1}{2} \alpha_6 (\partial_\mu m_i^b \partial_\mu m_i^b)^2 - \frac{1}{2} \alpha_7 (\partial_\mu m_i^b \partial_\mu m_i^b)^2$$

(48)

where $\mu$ and $\alpha$ are the renormalized coupling constants. This is nothing but a generalized Skyrme-Faddeev Lagrangian to $SU(N)$. One can find it is much complex than the case of $SU(2)$ QCD. We have obtained the effective action of $SU(N)$ QCD from the first principles.

**IV. SKYRME-FADDEEV-LIKE EFFECTIVE LAGRANGIAN OF SU(N) QCD**

Actually, in the above procedure, we have integrated out $X_\mu^a$ and $A_\mu$ separately in two steps to emphasize the importance of the restricted theory QCD [3]. But certainly we could integrate them simultaneously in one step to obtain the desired effective action. For example, such a gauge

$$F^a = \partial_\mu A_\mu^a m_i^a + \tilde{D}_\mu X_\mu^a + \partial_\mu C_\mu^a - f^a = 0$$

(49)

is a kind of suitable choice. In this case, the final result is basically similar. Furthermore, we could study our problem conveniently, seeking a suitable gauge is very meaningful as will be shown in the following.

In our decomposition, one could regard $C_\mu^a$ either as a covariant multiplet or simply as a fixed background. The first point of view provides an active type by (48), but the second point of view gives the following passive type.

Fix the gauge with the condition

$$F^a = \tilde{D}_\mu (A_\mu^a m_i^a - C_\mu^a) - f^a = 0$$

(50)

which $\tilde{D}_\mu$ is defined only by $C_\mu^a$. The Lagrangian of ghost is

$$L_{gf} = -\frac{1}{2}\xi [(\partial_\mu A_\mu)^2 + (\tilde{D}_\mu X_\mu^a)^2]$$

(51)

so with () the generating function is

$$W[J_\mu, J_\mu^a] = \int DA_\mu DX_\mu^a DC_\mu^a \exp \left\{ i \int \frac{1}{4} (F_{\mu\nu})^2 + c^{\alpha} \tilde{D}_\mu D_\mu c^\alpha - \frac{1}{2\xi} (\partial_\mu A_\mu)^2$$

$$- \frac{1}{2\xi} (\tilde{D}_\mu X_\mu^a)^2 - A_\mu^a J_\mu^a - X_\mu^a J_\mu^a \right\}$$

(52)

where $c$ and $c^\ast$ are ghost fields. In one loop approximation, the $X_\mu^a$ and the ghost fields integrations give the following functional determinations (with $\xi = 1$)

$$\ln Det^{-\frac{1}{2}} K_{\mu\nu}^{ab} \simeq \ln Det^{-\frac{1}{2}} [g_{\mu\nu}(\tilde{D}\tilde{D})^{ab} - 2gH_{\mu\nu} f^{abc}m_i^c]$$

$$\simeq -\frac{i5g^2N}{48\pi^2\varepsilon} \int d^4x (\tilde{H}_{\mu\nu})^2$$

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\[ \ln \text{Det} M_p^b \simeq \frac{i g^2 N}{24 \pi^2 \varepsilon} \int d^4 x (\tilde{H}_{\mu \nu})^2. \]  

(53)

In conclusion, the effective action is right the Skyrme-Faddeev type

\[ L_{\text{eff}} = -\mu^2 g^2 N (C_\mu^a)^2 - \frac{N}{16 \pi^2 \varepsilon} (H_{\mu \nu}^a)^2, \]

(54)

where

\[ H_{\mu \nu}^a = \partial_\mu C_\nu^a - \partial_\nu C_\mu^a + f^{abc} C_\mu^b C_\nu^c. \]

(55)

After renormalization the final effective Lagrangian can be written as

\[ L_{\text{eff}} = -\mu^2 (C_\mu^a)^2 - \frac{1}{4} (H_{\mu \nu}^a)^2 \]

which is similar to the Lagrangian Faddeev and Niemi expect [7]. This is just the Skyrme-Faddeev-like lagrangian which allows the topological knot solutions. Our solutions shows there exist connection between the generalized non-linear sigma model of Skyrme-Faddeev type and \( SU(N) \) QCD.

In the above discussion, we construct the effective theory for \( SU(N) \) QCD. Our analysis shows, just like \( SU(2) \) QCD, there exist the mass gaps corresponding to \( m_i \) in the infra-red limit of QCD based on the first principles. The monopole condensation can serve as the physical mechanism for the confinement of color in \( SU(N) \) QCD. However, one can find the mass gotten for different \( m_i \) are the same. The reason is that we choose only one gauge condition (43) for all \( m_i \). Alternatively, one can choose \( N \) gauge condition for every \( m_j = n \) such as

\[ F_{2j} = \partial_\mu A_{\mu j} n^a + f^{abc} \partial^2 m_i b C_\nu^c - f_2 = 0 \]

(56)

Then \( N \) different mass \( \mu_j \) are obtained and the mass term of the effective Lagrangian becomes

\[ -\frac{\mu_j^2}{2} (\partial_\mu m_j^a)^2. \]

(57)

When calculating the Yang-Mills Lagrangian, we assume (28) to remove a cross term. This assumption may be superfluous and do not affect the effective Lagrangian we obtain too much for it involves only the covariant parts of the gauge potential.

The result gotten in section 3 is not the same as that gotten in section 4. The reason is the different choices of gauge condition. We notice an important feature of the former is that when \( N \to \infty \), some special terms of the Lagrangian (48) manifest their importances and the others can be ignored

\[ L_{\text{eff}} = \frac{\mu_0^2}{2} (\partial_\mu m_i^a)^2 + \frac{1}{4} (m_i^a \partial_\nu C_\mu^a)^2 + \alpha_2 (\partial_\mu C_\nu^a)^2 + \alpha_3 (\partial_\nu C_\mu^a)^2 - \alpha_4 Tr(C_\mu^a C_\mu^b C_\nu^c) \]

(58)

We still don’t know what problem it will cause. But we think it is crucial at the large \( N \) cases, and may relate to some properties of \( D \)-brane. And we believe the last term in above Lagrangian will give rise to many interesting physical effects.

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[1] Y. Nambu, Phys. Rev. D10, 4262 (1974); S. Mandelstam, Phys. Rep. 23C, 245 (1976); A. Polyakov, Nucl. Phys. B120, 429 (1977).
[2] G. 't Hooft, Nucl Phys. B190, 455 (1981).
[3] J. Stack, S. Neiman, and R. Wensley, Phys. Rev. D50, 3399 (1994); H. Shiha and T. Suzuki, Phys. Lett. B333, 461 (1994); G. Bali, V. Bornyakov, M. Müller-Preussker, and K. Schilling, Phys. Rev. D54, 2863 (1996).
[4] Y. M. Cho, Haewon Lee and D. G. Pak, hep-th/9905215; Y. M. Cho hep-th/9906195.
[5] S. V. Shabanov, Phys. Lett. B463 (1999) 263-272, hep-th/9907182; Phys. Lett. B458 (1999) 322-330, hep-th/9903223.
[6] Y. M. Cho, Phys. Rev. D21 (1980) 1080; D23 (1981) 2415; Phys. Rev. Lett. 44 (1980) 1115.
[7] L. Faddeev and Antti J. Niemi, Phys.Lett. B464 (1999) 90-93; Phys.Lett. B449 (1999) 214-218.