Equations on knot polynomials and 3d/5d duality

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ABSTRACT

We briefly review the current situation with various relations between knot/braid polynomials (Chern-Simons correlation functions), ordinary and extended, considered as functions of the representation and of the knot topology. These include linear skein relations, quadratic Plucker relations, as well as "differential" and (quantum) \( \hat{A} \)-polynomial structures. We pay a special attention to identity between the \( \hat{A} \)-polynomial equations for knots and Baxter equations for quantum relativistic integrable systems, related through Seiberg-Witten theory to 5d super-Yang-Mills models and through the AGT relation to the \( q \)-Virasoro algebra. This identity is an important ingredient of emerging a 3d−5d generalization of the AGT relation. The shape of the Baxter equation (including the values of coefficients) depend on the choice of the knot/braid. Thus, like the case of KP integrability, where (some, so far torus) knots parameterize particular points of the Universal Grassmannian, in this relation they parameterize particular points in the moduli space of many-body integrable systems of relativistic type.

1 Ordinary and extended HOMFLY polynomials

The HOMFLY polynomials [1]

\[
H^K_R(q|A)_{A=q^N} = \left< \text{Tr}_R P \exp \oint_K A \right>_{SU(N)}
\]

are the analytical continuation in \( N \) of the Wilson-loop averages in 3d Chern-Simons theory [2] with the gauge group \( SU(N) \) and the action \( S = \frac{\kappa}{4\pi} \int d^3x \text{Tr} \left( A dA + \frac{2}{3} A^3 \right) \), where \( q = \exp \frac{2\pi i \kappa}{N} \). They can be lifted to extended polynomials [6, 7]: \( \mathcal{H}^B_R(q|p_k) \), which depend on infinitely many time-variables \( \{p_k\} \) and on particular braid representation \( B \) of the oriented knot \( K \) (the braid arises in projection onto a 2d plane and depends on the particular embedding of the knot and on the choice of the plane), so that

\[
H^K_R(q|A) = \left( q^{-\text{LicH}} A^{-|R|} \right)^{#(B)} \mathcal{H}^B_R(q|p^*_k)
\]

where \( #(B) \) is the writhe number, i.e. the algebraic number of crossings in the braid \( B \), and

\[
p^*_k = \frac{A^k - A^{-k}}{q^k - q^{-k}}
\]

At this topological locus the r.h.s. of (2) does not depend on the choice of \( B \) for a given \( K \).

An \( m \)-strand braid is parameterized by a sequence of integers \( a_{ij} \) with \( i = 1, 2, \ldots, m-1 \) and \( j = 1, \ldots, n \), their meaning can be understood from the picture (in this figure \( a_1 = -2, b_1 = 2, a_2 = -1, b_2 = 3 \): this is knot 8_{10}):

\[
\vspace{1cm}
\]

Representation is parameterized by the Young diagram: an ordered set of \( l(R) \) positive integers \( R = \{ r_1 \geq r_2 \geq \ldots \geq r_{l(R)} > 0 \} = \{ r_1, r_2, \ldots, r_{l(R)} \} \).

One of the hot questions in Chern-Simons theory is to find and investigate the equations, which the HOMFLY and extended HOMFLY polynomials satisfy as functions of \( a_{ij} \) and \( r_k \).

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The main source of information about these relations (besides pure empiricism) is the character expansion separating dependencies on the braid and the time variables:

$$\mathcal{H}_R^B\{q|p_k\} = \sum_{Q\in m|R|} c_{RQ}^B(q) S_Q\{p_k\}$$  \hspace{1cm} (4)$$

where $S_Q$ are the Schur functions, i.e. the characters of $GL(\infty)$ in representations (Young diagrams) $Q$ of the size $|Q| = m|R|$ and $c_{RQ}^B(q)$ are represented as traces in the spaces of intertwining operators of combinations of simple matrices,

$$c_{RQ}^B(q) = \text{Tr}_Q \prod_j \left( \prod_{i=1}^{m-1} \hat{R}^{a_{ij}} \hat{U}_i \right)$$  \hspace{1cm} (5)$$
described in detail in refs.[6, 7, 8, 9]. More artful character expansions (especially in the Hall-Littlewood or MacDonal polynomials) can be even more useful for particular applications [10, 11, 12].

In this paper we briefly review the main known linear and bilinear relations between the HOMFLY polynomials, as well as some closely related topics. The subject is still at empirical stage of development, far less understood than, say, Virasoro constraints, various recursions and integrability in matrix models, though a list of known properties and facts is quite impressive. Therefore, it deserve an attention and further deep investigation.

## 2  Skein relations for ordinary and colored HOMFLY polynomials

The oldest known equation of this type is the skein relation, which for extended polynomials states that

$$\mathcal{H}_R^{(\ldots a_{ij}+1 \ldots)} - \mathcal{H}_R^{(\ldots a_{ij}^{-1} \ldots)} = (q - q^{-1}) \mathcal{H}_R^{(\ldots a_{ij} \ldots)}$$  \hspace{1cm} (6)$$

for any particular parameter $a_{ij}$. The origin of this relation is that the $R$-matrix, acting in the channel $[1] \otimes [1] = [2] \otimes [\Pi]$ has two eigenvalues, $q$ and $-1/q$, associated with the two irreducible representations $[2]$ and $[\Pi]$ respectively, and thus satisfies the Hecke algebra constraint

$$(R - q)(R + q^{-1}) = 0$$  \hspace{1cm} (7)$$

Eq.(6) is then a direct corollary of representation (5).

For bigger representations $R$, the product $R \otimes R = \oplus S$ and $R$-matrix in this channel has many different eigenvalues: as many as there are irreducible representations $S$ in this expansion. These eigenvalues are equal to $\epsilon_S q^{\kappa_S}$, where $\kappa_S$ is an eigenvalue of the cut-and-join operator [13]: $\hat{W}_{[2]}$, associated with its eigenfunction character $S_S\{p_k\}$, and $\epsilon_S = \pm 1$, depending on $R$ and $S$. Therefore, this $R$-matrix satisfies

$$\prod_S \left( R - \epsilon_S q^{\kappa_S} \right) = 0$$  \hspace{1cm} (8)$$

and, as a corollary of (5),

$$\prod_S \left( e^{\partial/\partial a_{ij}} - \epsilon_S q^{\kappa_S} \right) \mathcal{H}_R^{(\ldots a_{ij} \ldots)} = 0$$  \hspace{1cm} (9)$$

for any variable $a_{ij}$.

For example, in the case of the first symmetric representation, one has

$$(R - q^6)(R + q^2)(R - 1) = 0$$  \hspace{1cm} (10)$$
i.e.

$$\mathcal{H}_{[2]}^{(\ldots a_{ij}+3 \ldots)} - (1 + q^2 + q^6) \mathcal{H}_{[2]}^{(\ldots a_{ij}+2 \ldots)} + q^2(1 + q^4 + q^6) \mathcal{H}_{[2]}^{(\ldots a_{ij}+1 \ldots)} - q^8 \mathcal{H}_{[2]}^{(\ldots a_{ij} \ldots)} = 0$$  \hspace{1cm} (11)$$

while for the antisymmetric representation

$$\mathcal{H}_{[11]}^{(\ldots a_{ij}+3 \ldots)} - (1 + q^{-2} + q^{-6}) \mathcal{H}_{[11]}^{(\ldots a_{ij}+2 \ldots)} + q^{-2}(1 + q^{-4} + q^{-6}) \mathcal{H}_{[11]}^{(\ldots a_{ij}+1 \ldots)} - q^{-8} \mathcal{H}_{[11]}^{(\ldots a_{ij} \ldots)} = 0$$  \hspace{1cm} (12)$$

\footnote{This presentation is essentially based on earlier attempts in [3, 4, 5].}
To make use of the skein relations for evaluation of the HOMFLY polynomials one needs to supplement them by "the initial conditions": in the case of \( R = \square \) one can express arbitrary \( \mathcal{H}^{a_{ij}} \) with arbitrary \( \{a_{ij}\} \) through those where all \( a_{ij} = 0 \) or 1. These polynomials should be found by some other means (e.g., at topological locus topological invariance can be used to decrease the number of strands). For higher representations the "boundary" \( a_{ij} \) can take more values.

For superpolynomials [14, 15] the skein relations are non-trivially deformed. Already for the simplest 2-strand torus knots and links only one half of the relations remain intact:

\[
\mathcal{P}_{\sqcup}^{[2,n+1]} - \mathcal{P}_{\sqcup}^{[2,n-1]} = (q - q^{-1}) \mathcal{P}_{\sqcup}^{[2,n]} \tag{13}
\]

for odd \( n \) (so that there are links at one of the sides of the relation), but for even \( n \) this is not true. The origin of this phenomenon is that the character expansion (in MacDonald polynomials) is different for knots and links [10]:

\[
\mathcal{P}_{\sqcup}^{[2,n]} = q^n M^*_{[2]} \{p\} + \begin{cases} - \frac{(1+q^2)(1-q^2)}{1-q^4} \cdot t^{-n} M_{[11]} \{p\} & \text{for } n \text{ odd (knots)} \\ + \frac{4(1-q^4)}{q(1-q^4)} \cdot t^{-n} M_{[11]} \{p\} & \text{for } n \text{ even (links)} \end{cases} \tag{14}
\]

3 Other relations for knot-dependence

Skein relations restrict the dependence of the knot polynomial on the shape of the braid. They are the only universal equations of this kind, known so far. However, it is already clear that they are not the only ones: some traces of a more profound structure are already observed. We give just two examples, both for torus knots and links.

3.1 "Evolution" of knots induced by skein relations

Torus knots/links

Skein relation as it is can not be directly deals with only torus knots, except for the 2-strand case: for the number of strands \( m > 2 \) it mixes torus knots with non-torus ones. However, the simple procedure, described in the previous section, immediately provides another kind of relations which provides a set of skein relations closed on only the \( m \)-strand torus knots. Namely, the HOMFLY polynomial for the \( m \)-strand torus knot/link \([m,n]\) is a weighted trace of the \( n\)-the power of a special combination of \( m-1 \) \( \mathcal{R} \)-matrices,

\[
\mathcal{H}^{[m,n]}_{\mathcal{R}} \{p\} = \text{Tr} \mathcal{R}_m^n, \quad \mathcal{R}_m = \mathcal{R}_{12} \cdots \mathcal{R}_{m-1,m} \tag{15}
\]

If instead of an individual \( \mathcal{R} \) in the standard skein relation, one now diagonalizes \( \mathcal{R}_m \), one can immediately write the analogues of eqs.\((8)\) and \((9)\) for the series of knots \([m,n]\) with given \( m \). For example, for the 3-strand torus links in the fundamental representation one has [7]

\[
\mathcal{H}^{[3,n]}_{\mathcal{R}} \{p\} = q^{2n} S_{[3]} \{p\} + \text{Tr}_{2 \times 2} \left( \mathcal{R} \mathcal{U} \mathcal{R} \mathcal{U}^\dagger \right)^n \cdot S_{[2]} \{p\} + (-1/q)^{2n} S_{[11]} \{p\} \tag{16}
\]

and the \( 2 \times 2 \) matrix \( \mathcal{R} \mathcal{U} \mathcal{R} \mathcal{U}^\dagger \) has eigenvalues \( e^{\pm 2\pi i/3} \). This means that our \( \mathcal{R}_3 \) in this case has four different eigenvalues: \( q^2, e^{\pm 2\pi i/3}, q^{-2} \), i.e. satisfies

\[
\left( \mathcal{R}_3 - q^2 \right) \left( \mathcal{R}_3 - q^{-2} \right) \left( \mathcal{R}_3^2 + \mathcal{R}_3 + 1 \right) = 0 \tag{17}
\]

and this implies the equation for torus HOMFLY polynomial:

\[
\begin{align*}
\left( e^{\partial/\partial n} - q^2 \right) \left( e^{\partial/\partial n} - q^{-2} \right) \left( e^{2\partial/\partial n} + e^{\partial/\partial n} + 1 \right) \mathcal{H}^{[3,n]}_{\mathcal{R}} \{p\} &= \\
&= \left( \mathcal{H}^{[3,n+2]}_{\mathcal{R}} \{p\} + \mathcal{H}^{[3,n+1]}_{\mathcal{R}} \{p\} + \mathcal{H}^{[3,n+1]}_{\mathcal{R}} \{p\} \right) - \\
&- (q^2 + q^{-2}) \left( \mathcal{H}^{[3,n+3]}_{\mathcal{R}} \{p\} + \mathcal{H}^{[3,n+2]}_{\mathcal{R}} \{p\} + \mathcal{H}^{[3,n+1]}_{\mathcal{R}} \{p\} \right) + \\
&+ \left( \mathcal{H}^{[3,n+2]}_{\mathcal{R}} \{p\} + \mathcal{H}^{[3,n+1]}_{\mathcal{R}} \{p\} + \mathcal{H}^{[3,n]}_{\mathcal{R}} \{p\} \right) = 0 \tag{18}
\end{align*}
\]
Thus one obtains a kind of evolution in the number of $\mathcal{R}_m$. This evolution can be described by "an evolution operator" [10], which is nothing but the cut-and-join operator [13]. Moreover, this evolution can be naturally continued to the superpolynomials [10, 11].

If one wants instead an equation for the topological invariant HOMFLY polynomial, it is necessary to restrict time-variables to topological locus (3), $p_k = p_k^*$ and also include the proper normalization factor (2). In this case the evolution can be considered as an evolution in the number of strands.

**Non-torus knots/links**

In fact, in just the same way one can write a version of skein relation for an arbitrary one-parametric family of braids, obtained by iteration of any braid group element: $B^{(n)} = B_0 B_1^2$. If the product $\mathcal{R}_1$ of $R$-matrices, associated with $B_1$, has eigenvalues $\xi_i$, then

$$\prod_i \left(\mathcal{R}_1 - \xi_i(q)\right) = 0 \quad (19)$$

and the corresponding extended HOMFLY polynomials satisfy

$$\prod_i \left(e^{\partial \phi / \partial n} - \xi_i(q)\right) \mathcal{H}^{B^{(n)}}\{p\} = 0 \quad (20)$$

independently of the choice of the "initial" braid element $B_0$. Thus, one can start an evolution with any "initial condition" $B_0$. For examples of such sequences see [10, 7], note that eigenvalues can be somewhat non-trivial functions of $q$.

Generalization of this procedure to superpolynomials seems straightforward, but it is still unclear if it really works in full generality, see [10].

### 3.2 Strand-changing relations

Like the skein relation itself, the modification, considered in the previous subsection does not change the number of strands in the braid. However, the strand-changing is also possible. An example is the following relation, which at the moment is very restricted: only torus knots, only $SU(2)$ gauge group ($A = q^2$, i.e. Jones polynomials), only fundamental representation, but instead it is in a very different direction in the braid-parameter space, and it survives (with minor modifications) a $\beta$-deformation to superpolynomials [16].

Namely, for torus knots of the special type $[m, m + 1]$ ($m$-strand braids, but a specially restricted $n = m + 1$)

$$\mathcal{H}_\circ^{[m, m+1]}\{p\} = q^{m^2-1} S_{[m]}\{p\} - q^{(m-3)(m+1)} S_{[m-1,1]}\{p\} + \ldots \quad (21)$$

where the sum goes over all hook diagrams of the size $m$. However, on the Jones topological locus,

$$p_k = p_k^* = \frac{A^k - A^{-k}}{q^k - q^{-k}} \quad (22)$$

only $S_{[n]}^* = \left(q^{n+1} \right) \frac{1}{(q)}$ and $S_{[n-1,1]}^* = S_{[n-2]}^* = \left(q^{n-1} \right) \frac{1}{(q)}$ are non-vanishing so that

$$H_\circ^{[m, m+1]}(A = q^2|q) = A^{-(m^2-1)} H_\circ^{[m, m+1]}\{p^*\} = \frac{q^{-m(m+1)} q^{m^2+4} - q^{2m+2} - q^{2m} + q^2}{q^4-q^{-4}} \equiv \frac{q^{-m(m+1)(m+2)} q^{-2m+2} - q^{2m} + q^2}{q^4-q^{-4}} \quad (23)$$

The four-term polynomial $J^{(m)}(q)$ satisfies the 4-term recurrent relation

$$J^{(m)} = q^4 J^{(m-1)} + q^{-2m+8} \left(J^{(m-2)} - J^{(m-3)}\right) \quad (24)$$

For generalizations to Jones unreduced superpolynomials for the same series of knots see [17] (where slightly different normalization is used and $q \to 1/q$; hereafter, such a difference with expressions for the trefoil in the literature arises, because we use the right-hand trefoil described by braid $(1,1|1,1)$, while often the left-hand trefoil $(-1,-1|-1,-1)$ is used instead).
In fact, there is nothing special about the \([m, m + 1]\) series, at least, for the HOMFLY polynomials. For \([m, m + r]\) one similarly gets
\[
H^K_{m,m+r}(A = q^2|q) = q^{-2(m-1)(m+r)} \left( q^{(m-1)(m+r)} S^*_{(m)} - q^{(m-3)(m+r)} S^*_{(m-2)} \right) = \frac{q^{-(m+1)(m+1+r)}}{q - q^{-1}} \mathcal{J}^{(m|r)}
\] (25)
where the 4-term (differently normalized) Jones polynomial
\[
\mathcal{J}^{(m|r)} = q^{4m+2r+2} - q^{2m+2r} - q^{2m} + q^2
\] satisfies
\[
\mathcal{J}^{(m|r)} - q^{4r} \mathcal{J}^{(m-r|r)} = q^{-2m+2r+6} \left( \mathcal{J}^{(m-2|r)} - \mathcal{J}^{(m-2-r|r)} \right)
\] (27)
Note that this relation at some values of \(m\) and \(r\) involves both the Jones polynomials of knots and of links. For instance, at \(m = 7\) and \(r = 3\), \(T[7, 10], T[4, 7], T[5, 8]\) and \(T[2, 5]\) are involved, all of them being knots, while at \(m = 8\) and \(r = 3\), of involved \(T[8, 11], T[5, 8], T[6, 9]\) and \(T[3, 6]\) only the first two are knots, while the second two are links.

3.3 Colored HOMFLY by cabling

Another application of (7) and (17) is to construct colored HOMFLY polynomials from the ordinary ones. To this end, as a first step, one has to use the cabling procedure, i.e. the relation
\[
H^K_{R \otimes m} = H^K_{R}
\] (28)
where \(K^m\) denotes \(m\)-cabling of the knot \(K\) (see, e.g., [18, eq.(1.5b)]). At the same time, the l.h.s. of this formula can be presented as a sum over the irreducible representations,
\[
H^K_{R \otimes m} = \sum_{S} H^K_{S}
\] (29)
Therefore, in order to calculate \(H^K_S\), one has to calculate \(K^m\) in the fundamental representation and then project out the result onto an irreducible representation \(Q\). This is done using proper projectors.

For instance, relation (7) implies that
\[
P_{[2]} = \frac{1 + qR}{1 + q^2} \quad \text{and} \quad P_{[11]} = \frac{q^2 - qR}{1 + q^2}
\] (30)
are projectors correspondingly onto representations \([2]\) and \([11]\), which emerge in the decomposition of the product \([1] \times [1]\). Similarly, (17) implies that the projectors onto irreducible components of the triple product \([1] \times [1] \times [1] = [3] + 2[21] + [111]\) are
\[
P_{[3]} = \frac{1 - q^2 R_3 (R_3^2 + R_3 + 1)}{(1 - q^2)(1 + q^2 + q^4)} , \quad P_{[111]} = q^6 \frac{(R - q^2)(R_3^2 + R_3 + 1)}{(1 - q^4)(1 + q^2 + q^4)}
\] (31)
and
\[
P^\pm_{[21]} = \pm \frac{e^{\pm \pi i / 6}}{\sqrt{3}} \frac{(q^2 R_3 - 1)(R_3 - q^2)(R_3 - e^{\pm 2\pi i / 3})}{1 + q^2 + q^4}
\] (32)
projects on two representations \([21]\) that correspond to the two eigenvalues \(e^{\pm 2\pi i / 3}\) of \(R_3\).

4 Differentials

Another type of relations comes from the trivial observation that
\[
H^K_R(A|q) - H^K_R(q^N|q) = (A - q^N)G^K_R(A|q),
\]
\[
H^K_R(A|q) - H^K_R(-1/q^N|q) = \left( A - (-1/q)^N \right) \tilde{G}^K_R(A|q)
\] (33)
with some polynomials \( G^K_R(A|q) \), \( \tilde{G}^K_R(A|q) \). This naturally leads to the formalism of ”differentials”, widely used in application to superpolynomials. It is effective, when an additional information is used about the HOMFLY polynomials for a particular \( N \), for example, that \( S^*_Q = S_Q \{ p^*_k \} \) in (4) are non-vanishing only for \( l(Q) \leq N \) (in particular, for \( N = 1 \) a single term with \( Q = [m|R] \) survives in the character expansion of \( H^K_R(q|q) \)).

These relations have a nice pictorial representation [15]. Let us plot the HOMFLY polynomials \( H^K_R(A|q) \) as a set of points corresponding to the monomials \( A^k q^l \) on the \((k, l)\)-plane with positive and negative signs denoted by the white and black circles respectively, and with multiplicities (if necessary). Then, all the terms that could cancel under the specialization \( A = q^N \), lie on the lines \( Nk = -l + \text{const} \), and they cancel if the algebraic sum of the terms on each line vanish. Similarly, all the terms that cancel under the specialization \( A = -1/q^N \) lie on the lines \( Nk = l + \text{const} \), and the algebraic sum of the terms on each such line should also vanish. An important point is that this line structure is preserved by a \( \beta \)-deformation, and this provides an important tool to deal with the superpolynomials [15].

As the simplest example, the fundamental HOMFLY polynomials for all the 2-strand knots pictorially look basically the same: since

\[
\frac{H^{[2,2n+1]}_{S^*}}{S^*_S} \sim \frac{q^{2n+2} - q^{-2n-2}}{q^2 - q^{-2}} A - \frac{q^{2n} - q^{-2n}}{q^2 - q^{-2}} A^{-1}
\]

they are sets of \( n+1 \) and \( n \) points on the horizontal lines \( k = +1 \) and \( k = -1 \) respectively, and one has, say, for the \([2,9] \) case, for \((q^6 + q^4 + 1 + q^{-4} + q^{-6})A - (q^6 + q^2 + q^{-2} + q^{-6})A^{-1} \)

The circles not connected by oblique lines survive to describe the specialized polynomials \( H_a(A = q^N|q) \). For even \( N \) and for odd \( N \geq 9 \) no reduction takes place: all the nine items are present.
Similarly, for the figure eight knot $4_1$ with $H^{(1,-1)}_{a} = 1 + \{Aq\} \{A/q\} = A^2 - q^2 + 1 - q^{-2} + A^{-2}$ one has

\begin{equation}
A = q \\
A = -1/q \quad (N = 1)
\end{equation}

and there are no non-trivial reductions, except for $A = q$ and $A = -1/q \quad (N = 1)$.

One can also consider "unreduced" HOMFLY "polynomials", not divided by $S^*_R$. If $A$ is considered as an independent variable, they are actually series, not polynomials in $q$ and their pictorial representation contains (semi)infinitely many vertices, of which finitely many survive after specializations $A = q^N$ with arbitrary $N = 1, 2, 3, \ldots$ For example, already for the trefoil $3_1$ one has

\begin{equation}
H^{3_1}_{\circ}(A|q) = -q^A A^{-1} \frac{q^2 + q^{-2}}{1 - q^2} (A - A^{-1})
\end{equation}

and

Here we manifestly indicated non-zero sums $\Sigma$ along the lines, which, hence, survive after the specialization.

Deformation to superpolynomials dictated by these pictures by the rules of [15] is actually different for the reduced and unreduced HOMFLY polynomials, what provides two different families of superpolynomials: the reduced and unreduced ones. The former are much simpler and have more profound structures (in particular, possess character decompositions and can be extended to arbitrary time variables [10, 6]), however, the latter seem to be related to somewhat simpler versions of Khovanov-Rozhansky homological descriptions. Of course, the two versions are not independent, for the same example of the trefoil in the fundamental representation one has

\begin{equation}
H^{3_1}_{\circ}(q) = \begin{cases}
\text{reduced} : & P^{3_1}_{\circ}(a, q, t) = 1 + a^2 t q^{-2} + a^2 q^2 t^3 \\
\text{unreduced} : & \hat{P}^{3_1}_{\circ}(a, q, t) = \frac{a - a^{-1}}{q - q^{-1}} a^2 q^2 t^3 + \left( at + \frac{1}{a} \right) \left( \frac{a^2}{q^2} - 1 \right)
\end{cases}
\end{equation}

and

\begin{equation}
\hat{P}^{3_1}_{\circ}(a, q, t) = \frac{a - a^{-1}}{q - q^{-1}} q^2 a^2 t^3 + \frac{a + a^{-1} t^{-1}}{q - q^{-1}} \frac{1 - q^2 a^{-2}}{1 + q^2 a^{-2} t^{-1}} \left( P^{3_1}_{\circ}(a, q, t) - q^2 a^2 t^3 \right)
\end{equation}

One can see that the unreduced superpolynomial $\hat{P}^{3_1}_{\circ}(a, q, t)$, in contrast with the reduced one, contains non-positive coefficients and only after specialization $a = q^N$ becomes a polynomial with positive coefficients. For instance, at $a = q^2$ (Jones)

\begin{equation}
\hat{P}^{3_1}_{\circ}(a, q, t) = q^3 t + \frac{1}{q} + q^2 t^3 + q^5 t^3
\end{equation}
which, indeed, coincides with the unreduced Khovanov homology [19].

A relation similar to (37) persists for other knots. For instance, for any "thin" knot [15]

$$P_\sigma^K(a, q, t) = \frac{a - a^{-1}}{q - q^{-1}} \left( \frac{q}{a} \right)^{S_K} + \frac{a + a^{-1}t^{-1}}{q - q^{-1}} \left[ \frac{1 - q^2a^{-2}}{1 + q^2a^{-2}t^{-1}} \right] \left( P_\sigma^K(a, q, t) - \left( \frac{q}{a} \right)^{S_K} \right)$$

(39)

where \( S_K \) is the signature of the knot (for the torus knot \( T[m, n] \), e.g., \( S_K = (m-1)(n-1) \)). To make formula simpler, we used here slightly different normalization of superpolynomials.

5 Plücker relations (KP integrability)

The third important kind of equations is related to integrability. For example, for the generating function of the extended HOMFLY polynomials

$$Z_R^B(p_k|\bar{p}_k) = \begin{cases} \sum_R H_R^B(q|p_k) S_R(\bar{p}_k) & \text{for knots} \\ \sum_{R_1 \ldots R_l} H_{R_1 \ldots R_l}^B(q|p_k) \prod_{i=1}^l S_{R_i}(\bar{p}_k^{(i)}) & \text{for links} \end{cases}$$

(40)

to be a \( \tau \)-function of the KP hierarchy in the \( p_k/k \)-variables [20], the coefficients \( \xi_Q = \sum_R C^B_R(q) S_R(q|\bar{p}_k) \) should satisfy the infinite set of quadratic Plücker relations:

$$\xi_{[22]}[ix][0] - \xi_{[21]}[xi][1] + \xi_{[2]}[\xi][11] = 0,$$

$$\xi_{[32]}[0][1] - \xi_{[3]}[\xi][i][1] + \xi_{[3]}[\xi][11] = 0,$$

$$\xi_{[2]}[2][i][0] - \xi_{[2]}[\xi][11][1] + \xi_{[2]}[\xi][1111] = 0,$$

$$\ldots$$

(41)

This indeed happens for one special class [6]: the torus knots/links with all \( a_{ij} = 1 \) (1 \( i \leq m-1, 1 \leq j \leq n \)) but not in general.

Another possibility is that \( H_R^B \) themselves satisfy the Plücker relations as functions of \( R \), i.e. that \( Z_R^B \) is a \( \tau \)-function w.r.t. times \( p_k/k \). Then, in particular, the Ooguri-Vafa partition function \( \sum_R H_R^K(q|A) S_R(\bar{p}_k) \) would be a KP \( \tau \)-function in these time variables. Again, this is not true in general, but an example is known when this is true (besides the trivial example of the unknot): in the limit \( q \to 1 \), when the HOMFLY polynomials turn into the "special" polynomials

$$\mathcal{G}_K^R(A) = \lim_{q \to 1} \frac{H_R^K(q|A)}{S_R(q|A)}$$

(42)

which have a very simple dependence on \( R \):

$$\mathcal{G}_K^R(A) = \left( \mathcal{G}_0^K(A) \right)^{|R|}$$

(43)

The full HOMFLY polynomial is restored by action of a combination of the cut-and-join operators [21]

$$\sum_R H_R^K(A) S_R(\bar{p}_k) = \exp \left( \sum_Q (q - q^{-1})^{c_Q} \frac{\sigma_Q^K(A)}{\sigma_Q^K(A)} W_Q(\bar{p}_k) \right) \exp \left( \sum_k \frac{1}{k} \left( \mathcal{G}_0^K(A) \right)^k \bar{p}_k p_k \right)$$

(44)

with

$$\sigma_Q^K(A) \equiv \mathcal{G}_Q^K(A)$$

(45)

and resembles very much the Hurwitz partition function [22, 23], which is also in general not a KP/Toda \( \tau \)-function in the variables \{\( p, \bar{p} \)\}. In the Hurwitz case the way is known to cure this problem [24] (by switching to times associated with a special basis in the space of cut-and-join operators), perhaps, this analogy can show a way to proceed in the case of HOMFLY polynomials as well. The integer coefficients of the special polynomials \( \sigma_Q^K(A) \) are related to the Ooguri-Vafa numbers [25, 8], but in a somewhat complicated way for \( R \neq \square \).
6 $\hat{A}$-polynomials

The fourth, currently most interesting type of relations are difference equations in $R$ [26], so far found empirically mostly for symmetric representations $R = [p]$ [27, 28, 29] or for the small groups with $N = 2, 3$ [30, 31].

The usual strategy to derive these equations [30] (at least at $N = 2$, when $R = [r-1]$ is given by the single positive integer) is to

- convert the Jones polynomial (with $A = q^2$) into a form of a proper $q$-hypergeometric series [32, 33]

\[
J_r(q) = \sum_{k \in \mathbb{Z}^+} \prod_{i,j} (B^+_i, q^2)_{b^+(r,k)} q^{C(r,k)} Y^k
\]  

(46)

where $b^\pm(r,k)$ and $C(r,k)$ are linear forms and a quadratic form accordingly, which, as well as $B^+_i$ and the $r$-vector $Y$, are just parameters which describe the concrete knot and the $q$-Pochhammer symbol

\[
(B,q)_r = \prod_{i=0}^{r-1} (1 - Bq^i)
\]

(47)

- write down a difference equation, which always exists for the proper $q$-hypergeometric series.

For example, in the simplest case of the trefoil ($3_1$ knot) [35]

\[
J_r^{3_1}(q) = q^{2-2r} \sum_{k=0}^{\infty} q^{-2kr} \left( q^{2-2r}, q^2 \right)_k
\]

(48)

the equation is [30, 34]:

\[
J_r^{3_1}(q) - U_r(q)J_{r-1}^{3_1}(q) = V_r(q),
\]

\[
U_r(q) = -q^{4-6r} \frac{1 - q^{2r-2}}{1 - q^{2r}}, \quad V_r(q) = q^{4(1-r)} \frac{1 - q^{4r-2}}{1 - q^{2r}}
\]

(49)

For arbitrary torus knot $[m,n]$ (trefoil is either $[2,3]$ or $[3,2]$), the equation is of the second order, however, for the whole series $T[2,2s+1$] [36]

\[
J_r^{T[2,2s+1]}(q) = q^{2s(1-r)} \sum_{k_m \geq \cdots \geq k_1 \geq 0} q^{-2k_m r} \left( q^{2-2r}, q^2 \right)_{k_m} \times \left( \prod_{i=1}^{s-1} q^{2k_i (k_i+1-2r)} \right) \frac{\left( q^2, q^2 \right)_{k_{i+1}}}{\left( q^2, q^2 \right)_{k_i} \left( q^2, q^2 \right)_{k_{i+1-k_i}}}
\]

(50)

it reduces to the first order and looks similar to (49) with [36]

\[
U_r(q) = -q^{2(s+1)-2(2s+1)} \frac{1 - q^{2r-2}}{1 - q^{2r}}, \quad V_r(q) = q^{2(s+1)(1-r)} \frac{1 - q^{4r-2}}{1 - q^{2r}}
\]

(51)

For non-torus knots the formulas get longer, and it is convenient to write them in terms of the operator $\hat{M}$: $\hat{M}f(q,r) = q^{-2r} f(q,r)$ and the shift operator $\hat{L}$: $\hat{L}f(q,r) = f(q,r + 1)$, $\hat{L}\hat{M} = q^2 \hat{M}\hat{L}$. Then, for knot $4_1$

\[
J_r^{4_1}(q) = \sum_{k=0}^{\infty} (-1)^k q^{-k(k+1)} \left( q^{2-2r}, q^2 \right)_k \left( q^{2+2r}, q^2 \right)_k
\]

(52)

the equation is [30, 34]:

\[
\hat{A}^{4_1}(\hat{L}, \hat{M})J_r^{4_1}(q) = \mathcal{B}(\hat{M}),
\]

\[
\hat{A} = q^4 \hat{M}^2 (1 - \hat{M})(1 - q^6 \hat{M}^2) - (q^6 \hat{M} + 1)(1 - q^2 \hat{M} - q^2 \hat{M}^2 - q^6 \hat{M}^2 - q^6 \hat{M}^4 + q^8 \hat{M}^4) (1 - q^2 \hat{M})^2 \hat{L} + q^2 \hat{M}^2 (1 - q^2 \hat{M}^2)(1 - q^4 \hat{M}) \hat{L}^2,
\]

\[
\hat{B} = q^2 \hat{M} (1 - q^6 \hat{M}^2)(1 - q^6 \hat{M}^2)(1 + q^2 \hat{M})
\]

In the limit of $q \to 1$, $\hat{A}(\hat{L}, \hat{M})$ can be considered as an ordinary c-number function $\mathcal{A}(l,m)$ of $\hat{L} \to l$ and $\hat{M} \to m$ and coincides (this is called the AJ-conjecture [26]) with the A-polynomial [37] (this is why $\mathcal{A}^K(\hat{L}, \hat{M})$ is called the non-commutative A-polynomial), which defines a ”spectral curve”, associated with the knot $K$:

\[
\Sigma^K : \quad \mathcal{A}^K(l,m) = 0
\]

(54)
which is naturally equipped with the Seiberg-Witten differential

\[ dS^K = \log l \, d \log m \] (55)

According to the standard dictionary of the Seiberg-Witten theory [38, 39, 40], this form of the SW differential is a clear sign of the relation to 5d YM theories.

7 The large-\(|R|\) limit

Given a difference equation (53), a natural question to ask is about various kinds of the loop (quasiclassical) expansions for its solutions. Loop expansion arises when one introduces the 't Hooft coupling constant \(u\) by putting \(q = e^h = e^{\theta^2} = e^{u/|R|}\) and \(|R| = |y| - \frac{u}{g^2}\).

7.1 Special polynomial expansion

To begin with one can look at the anzatz, provided by the special polynomial expansion (44). It would imply that the solution behaves like

\[ \mathcal{H}_R^K \sim \sigma_{[2]}^K(A)^{|R|} = \exp \left( |R| \cdot \log \sigma_{[2]}^K(A) \right), \] (56)

i.e. grows exponentially with \(|R|\), in accordance with the volume conjecture [41]. However, things are not so simple and strongly depends on the range of values of \(u\).

First of all, corrections to this formula could also contribute to the exponential growth. Indeed, the next correction is [21]

\[ \exp \left( \epsilon \sum_R \frac{\kappa_R \sigma_{[2]}^K(A)}{\sigma(A)} \right) \] (57)

where \(\kappa_R \sim \frac{|R|^2}{2} = \frac{e^2}{2}\) at large \(|R| = r\) and \(\epsilon = q - q^{-1}\). Since the latter is equal to \(2 \sinh h = 2 \sinh \theta^2 \sim \frac{u}{|R|}\), the exponent in this formula is proportional to \(\epsilon \kappa_R \sim |R|\) at large \(|R|\), so that one should add \(\frac{u \sigma_{[2]}^K(A)}{\sigma(A)}\) to \(\log \sigma(A)\) in (56). The next correction will be proportional to \(\frac{u^2 \sigma_{[3]}^K(A)}{\sigma^2(A)}\) and so on: there is an infinite series in powers of \(u\) of contributions to the leading (\(\sim |R|\)) ”volume” behavior of the exponent.

Second, for \(N\) fixed, i.e. for \(A \to 1\) (for example, for the Jones polynomial with \(A = q^2\)), this volume contribution is in fact vanishing term by term, because \(\sigma_{[2]}^K(A = 1) = 1\) and all other \(\sigma_{[2]}^K(A = 1) = 0\) [21]. In fact this is in accordance with the well known fact that for small enough \(u\) there is no exponential growth in \(J_r(q = e^{u/r})\). Instead (this formula was realized for the figure eight in [42] and for generic knots in [43] basing on the Melvin-Morton-Rozansky conjecture [44])

\[ J_r(q = e^{u/r}) = \exp \left( \sum_{k \geq 0} \left(\frac{u}{r}\right)^{2k} f_k(u) \right) = \frac{1}{\text{Alexander}(q = e^u)} + \sum_{k \geq 1} r^{-k} \frac{w_k(q = e^u)}{\text{Alexander}(q = e^u)^k} \] (58)

with some polynomials \(w_k\). In particular,

\[ f_0(u) = -\log \left( \text{Alexander}(q = e^u) \right) \] (59)

and this is in a nice accordance with the special polynomial expansion [21].

Only when \(e^u\) exceeds the smallest root of the Alexander polynomial, another solution appears, with \(f_{-1}(u) \neq 0\).

At the same time, if one chooses \(u = 2\pi i\), there is the exponential behaviour (56) [45] and the resulting coefficient in front of \(|R|\), for the Jones polynomial \(A = q^2\), is equal to the hyperbolic volume of the knot [41] (volume conjecture).
7.2 Loop expansion

It is described by the standard loop expansion:

$$J_r(q = e^{g^2}) = \exp \left( \sum_{k \geq -1} g^{2k} f_k(v) \right)$$  \hfill (60)

where $v = u - 2\pi i$, i.e. $v = 0$ is the volume conjecture point. The shift operations are

$$J_{r \pm 1}(q = e^{g^2}) = \exp \left( \sum_{k \geq -1} g^{2k} f_k(v \pm g^2) \right) = \exp \left( \sum_{k \geq -1} (-1)^j g^{2k+2j} \partial_v^j f_k(v) \right)$$  \hfill (61)

Therefore, the difference equation (53), if expanded in powers of $g^2$, turns into an infinite system of recurrent relations for infinite set of functions $f_k(v)$. Remarkably this system can be related [46] to a similar system of equations for multdensities $\rho_{p,k}(v_1, \ldots, v_k)$, which constitutes the essence of the AMM/EO topological recursion [47]. More exactly, the knot polynomial is associated with $\langle \log \det (M - v) \rangle$, where the brackets denote averaging over the “matrix” $M$ in the would-be matrix model underlying the recursion.

Recursion itself means that, given $f_{-1}(v)$, usually described in terms of the spectral Riemann surface (54) with a Seiberg-Witten differential (55), one can recursively reconstruct all the functions $f_k(v)$.

8 3d/5d AGT relation

8.1 $A$-polynomials, state integral model and Baxter equations

As soon as $f_{-1} \neq 0$ appears, one can neglect the r.h.s. in equation (53) and substitute it by the homogeneous $A$-polynomial equation. This one has a simpler solution, in the form of integrals of the double-sine ratios instead of sums of the $q$-Pochhammer ratios [48]. This solution is known as partition function of the state integral model [49].

According to [34] one can interpret this homogeneous difference equation as a Baxter equation for some relativistic integrable system (such systems are associated through Seiberg-Witten theory [50, 38, 39, 40] to 5d SYM theories, what explains the possible name for such a relation).

In fact, this should not come as a surprise due to the well-known connection between 2d CFT and Chern-Simons theory. As soon as the former one is related via the AGT correspondence [51, 52, 53] with the 4d SW systems, one would naturally expect a similar relation in the dimensions increased by unit: the one between Chern-Simons theory and 5d SW theory\footnote{Within a different context, a 5d/3d duality emerged in [54].}. On the other hand, the Baxter equation is an evident object to look at, since it encodes changing traces of the group element with representation (fusion), and there is a counterpart of this on the knot theory side: the difference $A$-polynomial equation describes exactly changing the traces (of the braid group element) with representation. The technical similarity of these two things is however appealing.

To be more concrete, one can notice that eq.(53) has the form of the Baxter equation for the relativistic Toda chain, which, in accordance with [39], describes the 5d SW system. Instead, one could definitely start from the classical limit of the Baxter equation (relativistic Toda spectral curve) noticing that it coincides with the ”classical” $A$-polynomial, and then apply the topological recursion with the 5d SW differential $dS_{SW}$ (55). This was done in [46] and the result reproduced the expected asymptotics of the Jones polynomial (see also [55] for a more generic context).

This is depicted at the scheme:

$$\hat{A}H = 0 \quad \longrightarrow \quad 5d \text{ Baxter}(Q) = 0$$

$$dS = d \log \Psi$$

$$\uparrow \quad H = Q \quad \downarrow \quad h = 0$$

$$\hat{H} \quad \text{AMM/EO topological recursion} \quad \Sigma : \quad A = 0$$

$$\downarrow \quad dS_{SW}$$

$$\uparrow \quad \hat{A}H = 0$$
This scheme implies that, at the quantum level, there is a relativistic integrable system/5d SW system described by the Baxter equation for the Baxter operator $Q(\mu)$. The Fourier transform of it, $\Psi(x)$ satisfies a relativistic Schrödinger equation, and one can construct the 5d Nekrasov functions (in the Nekrasov-Shatashvili limit [56]) as the Bohr-Sommerfeld integrals of this equation [57], which are A- and B-periods of the differential $dS = d \log \Psi(x)$. On the other hand, the same $Q(\mu)$ is the partition function of the state integral model, since this latter satisfies the same difference equation.

At the classical level, the system is described by the Riemann surface (the spectral curve of the classical integrable system) of the 5d SW system with the differential given by the classical limit of $dS$: $dS \xrightarrow{\hbar \to 0} dS_{SW}$. This Riemann surface coincides with the classical $A$-polynomial (which describes the $SL(2)$ representation of the knot complement to $S^3$, as viewed from the boundary [37]) and $dS_{SW}$ (55) allows one to apply the topological recursion and to restore the partition function of the state integral model.

### 8.2 Universal group element

Let us now briefly describe the origin of the Baxter equation as the fusion relation of the universal group element. First of all, one has to construct the group element given over the non-commutative ring, the algebra of functions on the quantum group (see a review in [58]). That is, one constructs such an element $g \in U_q(G) \otimes A(G)$ of the tensor product of the Universal Enveloping Algebra (UEA) $U_q(G)$ and its dual algebra of functions $A(G)$ that

$$\Delta_U(g) = g \otimes_U g \in A(G) \otimes U_q(G) \otimes U_q(G)$$

(63)

To construct this element [59, 58], we fix some basis $T^{(\alpha)}$ in $U_q(G)$. There exists a non-degenerated pairing between $U_q(G)$ and $A(G)$, which we denote $< \ldots >$. We also fix the basis $X^{(\beta)}$ in $A(G)$ orthogonal to $T^{(\alpha)}$ w.r.t. this pairing. Then, the sum

$$T \equiv \sum_\alpha X^{(\alpha)} \otimes T^{(\alpha)} \in A(G) \otimes U_q(G)$$

(64)

is exactly the group element we are looking for. It is called the universal $T$-matrix (as it is intertwined by the universal $R$-matrix) or the universal group element.

In order to prove that (64) satisfies formula (63) one should note that the matrices $M_{\alpha \beta}$ and $D_{\beta \gamma}^{\alpha}$ giving respectively the multiplication and co-multiplication in $U_q(G)$

$$T^{(\alpha)} \cdot T^{(\beta)} \equiv M^{\alpha \beta}_{\gamma} T^{(\gamma)}, \quad \Delta(T^{(\alpha)}) \equiv D^{\alpha}_{\beta \gamma} T^{(\beta)} \otimes T^{(\gamma)}$$

(65)

give rise to, inversely, co-multiplication and multiplication in the dual algebra $A(G)$:

$$D^{\alpha}_{\beta \gamma} = \left< \Delta(T^{(\alpha)}), X^{(\beta)} \otimes X^{(\gamma)} \right> \equiv \left< T^{(\alpha)}, X^{(\beta)} \cdot X^{(\gamma)} \right>$$

$$M^{\alpha \beta}_{\gamma} = \left< T^{(\alpha)} T^{(\beta)}, X^{(\gamma)} \right> = \left< T^{(\alpha)} \otimes T^{(\beta)}, \Delta(X^{(\gamma)}) \right>$$

(66)

Then,

$$\Delta_U(T) = \sum_\alpha X^{(\alpha)} \otimes \Delta_U(T^{(\alpha)}) = \sum_{\alpha, \beta, \gamma} D^{\alpha}_{\beta \gamma} X^{(\alpha)} \otimes T^{(\beta)} \otimes T^{(\gamma)} = \sum_{\beta, \gamma} X^{(\beta)} X^{(\gamma)} \otimes T^{(\beta)} \otimes T^{(\gamma)} = T \otimes_U T$$

(67)

This is the first defining property of the universal $T$-operator, which coincides with the classical one. The second property that allows one to consider $T$ as an element of the "true" group is the group multiplication law $g \cdot g' = g''$ given by the map:

$$g \cdot g' \equiv T \otimes_A T \in A(G) \otimes A(G) \otimes U_q(G) \to g'' \in A(G) \otimes U_q(G)$$

(68)

This map is canonically given by the co-multiplication and is again the universal $T$-operator:

$$T \otimes_A T = \sum_{\alpha, \beta} X^{(\alpha)} \otimes X^{(\beta)} \otimes T^{(\alpha)} T^{(\beta)} = \sum_{\alpha, \beta, \gamma} M^{\alpha \beta}_{\gamma} X^{(\alpha)} \otimes X^{(\beta)} \otimes T^{(\gamma)} = \sum_{\alpha} \Delta(X^{(\alpha)}) \otimes T^{(\alpha)}$$

(69)

i.e.

$$g \equiv T(X, T), \quad g' \equiv T(X', T), \quad g'' \equiv T(X'', T)$$

$$X \equiv \{X^{(\alpha)} \otimes I\} \in A(G) \otimes I, \quad X' \equiv \{I \otimes X^{(\alpha)}\} \in I \otimes A(G) \quad X'' \equiv \{\Delta(X^{(\alpha)})\} \in A(G) \otimes A(G)$$

(70)
8.3 Baxter equation

The most essential property of the group elements is that they are intertwined by the universal $R$-matrix [60], i.e.

$$ R \ T \otimes_U \ T' = T' \otimes_U \ T \ R $$

(71)

This means that their traces taken at any representations of the UEA, $T_V$, commute:

$$ [T_V, T_V'] = 0 $$

(72)

The equation that connects these traces in different representations is called fusion and its specific limit is the Baxter equation. As a simple instance, let us consider the case of quantum affine algebra $\widehat{SL}(2)_q$ and the evaluation representation of this latter $V_j(\lambda)$ labeled by the spin $j$ and evaluation parameter $\lambda$ (see, e.g., [61] for a review). The product of two such representations $(j, \lambda)$ and $(k, \mu)$ is an irreducible representation unless

$$ \frac{\lambda - \mu}{p} = q^{j+k-p+1} $$

(73)

where $p = 1, ..., \min(2j, 2k)$. In this latter case, the product is partly reducible:

$$ 0 \rightarrow V_{j-p/2}(q^{p/2}\lambda) \otimes V_{k-p/2}(q^{-p/2}\mu) \rightarrow V_j(\lambda) \otimes V_k(\mu) \rightarrow V_{j+k-p/2+1/2}(q^{p/2-k-1/2}\lambda) \otimes V_{p/2-1/2}(q^{k+1/2-p/2}\mu) \rightarrow 0 $$

(74)

In terms of traces of the group element, this (fusion) relation can be written in the form

$$ T_j(\lambda)T_k(\mu) = T_{j-p/2}(q^{p/2}\lambda)T_{k-p/2}(q^{-p/2}\mu) + T_{j+k-p/2+1/2}(q^{p/2-k-1/2}\lambda)T_{p/2-1/2}(q^{k+1/2-p/2}\mu) $$

(75)

Choosing $k = 1/2$, $p = 1$ and $\lambda = q^{j+1/2}\mu$, one immediately obtains

$$ T_j(q^{j+1/2}\mu)T_{1/2}(\mu) = T_{j-1/2}(q^{j+1/2}\mu) + T_{j+1/2}(q^j\mu) $$

(76)

It turns out [62] that (upon the proper normalization of $T_j(\lambda)$) there is a finite limit

$$ \lim_{j \rightarrow \infty} T_j(q^{j+1/2}\mu) = Q(\mu) $$

(77)

and (76) turns into the standard Baxter equation

$$ T_{1/2}(\mu)Q(\mu) = Q(q\mu) + Q(q^{-1}\mu) $$

(78)

Here $T_{1/2}(\mu)$ is a Laurent polynomial of degree 1 both in $\mu$ and $\mu^{-1}$. One can generate Laurent polynomials of higher degrees considering in (76) traces of products of the group elements, (68), the product of $n$ group elements giving rise to degree $n$. Thus, generically one would get instead of (78)

$$ P_n[\mu, \mu^{-1}]Q(\mu) = Q(q\mu) + Q(q^{-1}\mu) $$

(79)

which coincides with the Baxter equation for the $n$-particle relativistic Toda chain, which corresponds at the SW level to the 5d pure gauge theory. The transition to four dimensions and the standard Toda system is done by replacing the affine algebra with the Yangian. This leads to the same equation with $P_n$ being a degree $n$ polynomial of $\mu$.

8.4 Knots, Baxter equations and 5d SW theory

One can also choose $p = 2$ and $k = 3/2$. Then,

$$ Q(\mu)T_{3/2}(\mu) = T_{1/2}(q^{-1}\mu)Q(q^2\mu) + T_{1/2}(q\mu)Q(q^{-2}\mu) $$

(80)

This equation is another version of the Baxter equation, it gives rise to the same spectral curve in the classical limit.

Remind that these representations can be treated as either finite dimensional but reducible, or infinite dimensional irreducible representations [61]. We prefer here the latter point of view.
Let us consider, for instance, the first non-trivial example of the figure eight knot $4_1$ which $\mathcal{A}$-polynomial equation is the homogeneous part of (53), i.e.
\[ \hat{A}^{4_1}(\hat{L}, \hat{M}) \mathcal{J}^{4_1}(M) = 0 \]  
We use the basis where $\hat{M}$ acts on $\mathcal{J}$ diagonally so that (81) reduces to
\begin{align*}
(1 - q^6 \mu^2)Q(q^{-2} \mu) + (1 - q^2 \mu^2)Q(q^2 \mu) &= P_2[\mu, \mu^{-1}]Q(\mu) , \\
P_2[\mu, \mu^{-1}] &= \left( q^4 \mu^2 + \frac{1}{q^4 \mu^2} \right) - \left( q^2 \mu + \frac{1}{q^2 \mu} \right) - \left( q^2 + \frac{1}{q^2} \right)
\end{align*}
where $Q(\mu) = (1 - \mu)\mathcal{J}^{4_1}(\mu)$ with identification $M = \mu$. (82) is nothing but (80) with
\begin{align*}
T_{1/2}(\mu) &= q^2 \mu - \frac{1}{q^2 \mu} , \\
T_{3/2}(\mu) &= \left( q^2 \mu - \frac{1}{q^2 \mu} \right) P_2[\mu, \mu^{-1}]
\end{align*}
The coefficients of $T_{1/2}(\mu)$ and $T_{3/2}(\mu)$ can be related with concrete values of the energies (integrals of motion) in the periodic 2-particle relativistic Toda chain, and the corresponding SW curve ($5d$ pure gauge theory) is
\[ l + \frac{1}{l} = \left( \mu^2 + \frac{1}{\mu^2} \right) - \left( \mu + \frac{1}{\mu} \right) - 2 \]
In contrast with the generic SW curve, where there is an arbitrary degree two Laurent polynomial at the r.h.s. of (84), a peculiar polynomial with all coefficients fixed corresponds to the figure eight knot. Similarly, (83) specifies very concrete polynomials, while in the generic integrable system these are arbitrary ones (of fixed degree).

Note that choosing $k = 1$, $p = 1$ and $\lambda = q^{1+1} \mu$ (or $p = 2$ and $\lambda = q^3 \mu$), one obtains from (75) a third order difference equation (or its counterpart with $q \to 1/q$)
\[ T_1(q^{1/2} \mu)Q(\mu) = Q(q^2 \mu) + T_{1/2}(q \mu)Q(q^{-1} \mu) \]

The procedure can be continued further to generate higher order difference equations which can be compared with the $\mathcal{A}$-polynomial equations for other, more complicated knots. In accordance with [30], such an equation exists for any knot. The coefficients of the polynomial $P_n[\mu, \mu^{-1}]$ are fixed by choosing the representation of the algebra of functions on the quantum group. Indeed, due to (72) we know that all the coefficients of this polynomial are commuting. They are nothing but the Casimir functions that determine the representation of $A(G)$.

In integrable systems these coefficients are just integrals of motion, which can have arbitrary values. On the contrary, in knot theory their values are fixed to be very concrete numbers, i.e. the representations of $A(G)$ in knot theory are fixed. It still remains unclear what is the condition that fixes them in group theory.

Now let us stress out that all these Baxter equations or quantum $\mathcal{A}$-polynomials correspond in the classical limit to $5d$ SW theories, generically of the quiver type [63, 64]. On the integrable side, these theories are described by various degenerations of the $sl(p)$ XXZ spin chain [40] (or corresponding Gaudin type systems, see [65]). Their quantization is immediate, either as integrable systems which goes exactly through the Baxter equation [57], or as $\mathcal{A}$-polynomials [46, 55].

Even more general Baxter equations are described by the generic $sl(p)$ XXZ spin chain, in the classical limit giving rise to the $5d$ SW curves that are plane complex curves given by sets of arbitrary Laurent polynomials $P_n[\mu, \mu^{-1}]$ [40]
\[ \sum_{i=1}^{p} w^i P_n[\mu, \mu^{-1}] = 0 \] (86)
However, obtaining these Baxter equations directly from the representation theory as in the previous subsection is not that immediate in this case, and the whole construction becomes less elegant and much more involved [66].

Note that one can similarly consider the Baxter equation describing $\widetilde{SL(N)}_q$. In terms of knot theory, it should describe the difference equations for the HOMFLY polynomials with specialization $A = q^N$. 

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8.5 State integral model and open relativistic Toda chain

Let us note that solutions to the Baxter equations are usually quite involved, and are unknown in a generic case. On the contrary, the partition function of the state integral model, which solves the $A$-polynomial (i.e. Baxter) equation is often known. The reason is that with peculiar values of the integral of motions, i.e. with the specific form of traces $T_j$ corresponding to knots, the equation is explicitly solved. The answer is presented as a multiple integral of product of the quantum dilogarithms [67]. This answer is in fact associated with the wave function of the open relativistic Toda chain (which, within the SW context, corresponds to the perturbative limit of the gauge theory). More exactly, this wave function solves the Schrödinger equation given by the Hamiltonian of the system dual [68, 69, 70] to the relativistic Toda chain. In other words, this is the (difference) equation that describes the dependence of the wave function on the energies.

It happens, since the Baxter equation, which is a difference equation for the separated variables in the periodic chain, is simultaneously (at peculiar values of energies) a Schrödinger equation for the (dual) open chain (this is possible definitely only due to the fact that chain is relativistic and, hence, both the Schrödinger and the Baxter equations are difference ones).

Let us see how it works in already discussed simplest non-trivial example of the figure eight knot. Remind that the Hamiltonian of the $k$-particle relativistic Toda chain is given by [71]

$$h\{x_i; p_i\} = \sum_{i=1}^{k} \left[ e^{x_i p_i} + q^{-1} g^{2 \omega_1} e^{\frac{2 \pi i}{\omega} (x_i - x_{i+1}) + \frac{1}{2} \omega_1 p_i + \frac{1}{2} \omega_1 p_{i+1}} \right]$$  \hspace{1cm} (87)

with the boundary condition $x_{k+1} = x_1$ in the periodic case and $x_{k+1} = \infty$ in the open case. Here $q = e^{\pi \omega_1/\omega_2}$ is the relativistic parameter ($q \to 1$ as the speed of light goes to infinity), $\epsilon = 0, \pm 1$ parameterizes different relativistic Toda systems and $g$ is a coupling constant, which can be removed by redefinitions of coordinates $x_i$ only in the open case. In the 2-particle open case, the Schrödinger equation for $\epsilon = 0$ can be reduced (upon neglecting the $U(1)$-factor) to the equation

$$\psi_\gamma (q^{-1} \mu) + \psi_\gamma (q \mu) + g^{2 \omega_1} \mu \psi_\gamma (\mu) = (e^{\pi \gamma} + e^{-\pi \gamma}) \psi_\gamma (\mu)$$ \hspace{1cm} (88)

where $\gamma$ parameterizes the energy and we fixed $\omega_2 = 1$, $\omega_1 = \hbar / \pi i$ and denoted $\mu = \exp(2 \pi x)$, $x = x_1 - x_2$. A solution to this eigenvalue problem is of the form [67]

$$\psi_\gamma (\mu) \propto \int dt \frac{S \left( - it - ix + 1 + \frac{\hbar}{\pi i} \frac{\hbar}{2 \pi} \log g \left| \frac{\hbar}{\pi i}, 1 \right. \right)}{S \left( it + i \frac{2}{\pi} - \frac{\hbar}{\pi i} \log g \left| \frac{\hbar}{\pi i}, 1 \right. \right)} e^{- \pi \gamma (2 \gamma - 1)}$$ \hspace{1cm} (89)

where $S(x|\hbar / \pi i, 1)$ [67] is related with the quantum dilogarithm [49] by the formula

$$\Phi_h(z) = S \left( \frac{z}{2 \pi i} + \frac{\pi i + \hbar}{2 \pi i} \left| \frac{\hbar}{\pi i}, 1 \right. \right)$$ \hspace{1cm} (90)

and the contour $C$ is defined in [67, (2.26)]. Now, if one puts $\mu g^{2 \omega_1} = 1$, integral (89) reduces to ($\frac{2}{2 \pi} = t - \frac{\hbar}{2 \pi i} \log g$)

$$\psi_\gamma (g^{-2 \omega_1}) \propto \int dp \frac{\Phi_h(p + i \pi + \hbar)}{\Phi_h(-p - \pi \gamma - i \pi - \hbar)} e^{- \pi \gamma (p + \pi \gamma / 2 - \pi \gamma / 2)}$$ \hspace{1cm} (91)

which is exactly the partition function of the state model for the figure eight knot as a function of $u = \pi \gamma / 2$ [49]

$$\Psi^{4i} (e^u) \propto \psi_{2u / \pi} (g^{-2 \omega_1})$$ \hspace{1cm} (92)

Since the dependence on $\gamma$ is regulated by the dual Toda Hamiltonian, we come to the claim above.

In the case of more complicated knots one has instead of (89) a multiple integral of the product of the quantum dilogarithms [49]. This integral could be associated with multi-particle relativistic open Toda wave functions [67] and their further generalizations.
9 Conclusion

To conclude, in development of the program outlined in [34, 6], we presented a brief review of the new and very interesting research field: the study of interrelations between various HOMFLY polynomials, probably reflecting the powerful underlying integrable structure, which still remains to be fully revealed. The relations involve different knots/links and different representations, they are rather diverse and form a somewhat strange intertwined set, and partly unexpected parallels are seen with the Hurwitz theory (i.e. with that of symmetric group characters), quantum integrable systems and AGT relations. Even more interesting should be generalizations from the HOMFLY to superpolynomials [14], however, what structures survive this $\beta$-deformation [72] remains an open question.

A way to systematical study of these problems is now open by the powerful HOMFLY calculus, developed in [10, 6, 7, 8, 9], exploiting the simple and universal structure of $SU_q$ $R$-matrices and Racah coefficients, and allowing one to obtain the parameter-dependent expressions for knot polynomials. One now expects a fast and fruitful development of this old and intriguing field.

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