Study of Chromatic parameters of Line, Total, Middle graphs and Graph operators of Bipartite graph

R. Nagarathinam¹, N Parvathi²

¹Research scholar, Department of Mathematics, SRM Institute of Science and Technology, Kattankulathur, Chennai.
²Department of Mathematics, Dr. MGR Educational Research Institute, Chennai-95.

Corresponding Author: *parvathi.n@ktr.srmuniv.ac.in

Abstract. Chromatic parameters have been explored on the basis of graph coloring process in which a couple of adjacent nodes receives different colors. But the Grundy and b-coloring executes maximum colors under certain restrictions. In this paper, Chromatic, b-chromatic and Grundy number of some graph operators of bipartite graph has been investigated.

Notations: \( \Gamma \)-Grundy number, \( \varphi \)-Chromatic number, \( \chi \)-b-chromatic number.

1. Introduction.

In combinatorial optimization, clique and graph coloring problems are Scholastic. These problems are widely observed by the research developers in the field of Mathematics, Computer science, Operation research, Engineering, Biology and social science due to their vital role in real world applications. In the world net dictionary Clique is defined as a special natural object of the people with common purpose.

Let \( G=(V,E) \) be simple graph where \( V \) is the set of vertices or nodes \( u_1, u_2, \ldots, u_n \), and \( E \) is the set of edges or arcs \( e_1, e_2, \ldots, e_n \). A graph \( H \) is called a sub graph of \( G \) if \( V(H) \subseteq V(G) \), and \( I_H \) is the restriction of \( I_G \) to \( E(H) \). A Graph \( H \) of \( G \) is defined as an induced sub graph of \( G \) if every arc of \( G \) having it ends in \( V(H) \) is also an arc of \( H \). A clique of \( G \) is a complete sub graph of \( G \). A subset \( S \) of the vertex set \( V \) of a graph \( G \) is independent if a couple of nodes of \( S \) is nonadjacent. Chromatic number of a Graph \( G \) is the minimum number of independent subset that partition the vertex set of \( G \).

A k-vertex coloring is an assignment of \( k \) colors 1, 2, ..., \( k \), to the vertices of \( G \). The coloring is proper if no two distinct adjacent vertices have the same color. Thus a proper k-vertex coloring of loop less graph \( G \) is partition of \( (v_1, v_2, \ldots, v_n) \) of \( V \) into \( k \) independent sets is k-vertex colorable if \( G \) has a proper k-vertex coloring. The chromatic number, \( \varphi(G) \) of \( G \) is the minimum \( k \) for which \( G \) is \( k \)-colorable; if \( \varphi(G) = k \), \( G \) is said to be \( k \)-chromatic. Alkhadeeb states that b-coloring of a graph \( G \) by \( k \)-colors is a
proper vertex coloring such that there is a vertex in each color class, which is adjacent to atleast one vertex in every other color class as in his article in page.no.14 [2]. A Grundy coloring of order k of a graph G is a k-coloring of G with colors 1,2…k such that for each vertex x the color of x is the smallest positive integer not used as a color on any neighbor of x in G. The Grundy number Γ(G) is the largest integer k for which G has a Grundy coloring of order k.[3]

2. PRELIMINARIES:

2.1.DEFINITIONS;

Definition 2.1.1[14]: The subdivision graph S(G) can be defined as the graph acquired from G by altering every arc by a path of length 2 or equivalently by including a new node into each arc of G.

Definition 2.1.2[14]: The operator R(G) is the graph acquired from G by including an additional node equivalent to every arc of G and by connecting every additional node to the end points of the arc correlating to it.

Definition 2.1.3[14]: The operator Q(G) is the graph acquired from G by adding an additional node into every arc of G and by connecting arcs those pairs of these new nodes which lie on the neighbouring arcs of G.

Definition 2.1.4[15]. Middle graph
Middle graph M(G) of a graph G is defined with the node set V(G)∪E(G), in which two components are neighbouring if and only if either both are neighbouring arcs in G or one of the components is a node and the other one is an arc incident to the node in G.

Definition 2.1.5[15]. Total graph
Total graph T(G) of a graph G defined with the node set V(G)∪E(G), in which two componennts are neighbouring if and only if one of the following holds true (i) both areneighbouring arcs or nodes in G (ii) one is a node and other is an arc incident to it in G.

Definition 2.1.6. Line graph[11]. According to Balakrishnan.et.al, Line graph L(G) can be established in the following manner: The dot set of L(G) is in 1-1 correlation with the arc set of G and two vertices of L(G) are connected by an arc if and only if the correlating arcs of G are neighboring in G.[11]

Definition 2.1.7.[16]. Bipartite Graph
A Graph G is said to be a bipartite graph, if there is a separation of V(G) into two subsets A and B such that any couple of nodes in the same set are non-adjacent. Example: Ladder graph

Definition 2.1.8:[6] A clique partition of a graph G is any partition of V(G) into subsets say C₁,C₂…Cₙ in such a way that the sub graph of G induced by is a cliqueCi, for each i. We denote by θ (G) the minimum number of subsets in a clique partition of the graph G and call it the clique partition number of G.

2.2.THEOREMS:

Theorem 2.2.1. Let G be a triangle-free graph with matching number m. Then Γ[L(G)]≤Δ(G)+m-1[6]. Theorem 2.2.2. Let G be a graph. Set θ (G) = k and (G) = ω. Then Γ(G) ≤ \frac{k+1}{2} - ω.[2].

2.3.PROPOSITION:
Proposition 2.3.1. For any Graph G, $\chi(G) \leq \left\lfloor \frac{\omega(G) + (t-1)n(G)}{2t-1} \right\rfloor$ where $t = \Theta(G)[2]$.

3. MAIN RESULTS:

Theorem 3.1.
If G is a Cartesian product of two path graphs, then it is a bipartite graph.

Proof:
Let G be Cartesian product of $P_n$ and $P_3$ which contains two vertex sets $X = \{a_1, a_2, \ldots, a_n\}$ and $Y = \{b_1, b_2, \ldots, b_n\}$ such that the vertex $a_i$ in X is connected to $b_i$ in Y by an edge $e_i$ and so on. Also G consists of even cycles. Hence G is a bipartite graph.

Theorem 3.2.
Let G be a path $P_n$ having n vertices and $H = P_2$, line graph of Cartesian product of G and H needs 5 colors for Grundy coloring.

Proof:
Since the line graph of Cartesian product of two graphs have a clique of order 3 and girth of order 3, it is a planar graph and $\Delta(G) = 3.\Delta(L(G\square H)) = 3n-2$. $V(L(G\square H)) = \{ v_1, v_2, \ldots, v_{n-1}, u_1, u_2, \ldots, u_{n-1}, w_1, w_2, \ldots, w_n \}$. $m(L(G\square H)) = 2n-2$.

In accordance with the theorem in [6], $\Gamma[L(G\square H)] \leq \Delta(G) + m(G\square H) - 1 \leq 4 + (2n-2) + 1$ ------- 1

Case 1: When $n = 2k + 1$, where k varies from 1 to n.
According to the equation, $\Gamma[L(G\square H)] \leq 2n + 3 \leq 9$ when $k = 1$.
For Grundy coloring, let us define $f: V \rightarrow \{0, 1, 2, 3, 4\}$ such that $f(v_i) = 0$ when i is odd,
$f(v_i) = 1$ when i is even,
$f(u_i) = 2$ when i is odd,
$f(u_i) = 3$ when i is even,
$f(w_i) = 4$ when i varies from 1 to n.

Case 2: When $n = 2k + 2$, where k varies from 1 to n.
Also in this case $\Gamma[L(G\square H)] \leq \Delta(G) + m(G\square H) - 1 \leq 4 + (2n-2) + 1$ ------- 1

Let us define $f: V \rightarrow \{0, 1, 2, 3, 4\}$ such that $f(v_i) = 0$ when i is odd,
$f(v_i) = 1$ when i is even,
$f(u_i) = 2$ when i is odd,
$f(u_i) = 3$ when i is even,
$f(w_i) = 4$ when i varies from 1 to n which resembles Grundy coloring.

Since the upper restriction of parameter of Grundy coloring is its largest possible degree plus one and the parameter of coloring of planar graph is five. Hence $\Gamma[L(G\square H)] = 5$.

Hence the proof.

Theorem 3.3:
If G is Cartesian product of two path graphs, then $\chi(M(G)) = \chi(Q(G)) = 7 = \Gamma(M(G)) = \Gamma(Q(G))$.

Proof:
Let G be the Cartesian product of two path graphs. Middle graph M(G) and the graph Operator Q(G) of G have equal maximum degree 6, clique of order 3, minimum degree 4 and girth of order 3.

i.e $\omega[M(G)] = \omega[Q(G)] = 3.\Delta[M(G)] = \Delta[G] = 6, \delta[M(G)] = \delta[Q(G)] = 4, \Theta[M(G)] = \Theta[Q(G)] = 3n-2$. $n[M(G)] = n[Q(G)] = 5n-2$.

Let $V(M(G)) = \{ u_i / i \rightarrow 1 to u \ n \ v_j / j \rightarrow 1 to u \ n \ x_i / i \rightarrow 1 to n \ y_j / j \rightarrow 1 to n \}$ and that the edge $u_i u_{i+1}$ is subdivided by the vertex $x_i$ for $i \rightarrow 1 to n-1$, the edge $v_i v_{i+1}$ is subdivided by the vertex $y_i$ for $i \rightarrow 1 to n-1$, the edge $u_1 v_1$ is subdivided by the vertex $w_1$ for $i \rightarrow 1 to n$ and

$V(Q(G)) = \{ w_i / i \rightarrow 1 to u \ n \ v_j / j \rightarrow 1 to u \ n \ x_i / i \rightarrow 1 to n \ y_j / j \rightarrow 1 to n \}$ in which near the edge $w_1 w_{i+1}$ vertex $x_i$ is assumed for $i \rightarrow 1 to n-1$. Also near the edge $v_1 v_{i+1}$ vertex $y_i$ has assumed for $i \rightarrow 1 to n$.
n-1 and form a clique $v_1 v_{i+1} w_1$ of order 3 and near the edge the $w_1 v_i$ vertex $u_i$'s are assumed for $i = 1$ to $n$.

Since there exists more number of cliques in the above graphs clique number can be computed for the determination of $b$-coloring. In view of the Proposition explored by Alkhadeeb in [2],

$$\chi(G) \leq \left[ \frac{t \omega(G) + (t-1) \omega(G)}{2t-1} \right] \leq \left[ \frac{(3n-2)3 + (3n-2-1)(5n-2)}{2(3n-2)-1} \right] \leq \left[ \frac{15n^2 - 12n}{6n - 5} \right]$$

Here $\chi(G)$ varies according to various $n$ values. But Irving and Manlove determined that a graph should contain $n$ vertices of degree $n-1$ for $b$-coloring. Here we have 7 vertices of degree 6. Hence $\chi[M(G)]=\chi[Q(G)]=7$.

For $b$-coloring, the color set $C= \{1,2,3,4,5,6,7\}$ have assigned the vertices of $M(G)$ and $Q(G)$ such that $C(w_i)=(i+1)(\mod 7)$ for $i = 1$ to $n$

$C(u_i)=(i+1)(\mod 7)$ for $i = 1$ to $n$

$C(x_i)=(i+3)(\mod 7)$ for $i = 1$ to $n-1$

$C(y_i)=(i+6)(\mod 7)$ for $i = 1$ to $n-1$ and the $b$-vertices are $w_2, w_3, w_4, w_5, w_6, w_7$ and $w_8$.

As per the Theorem 2 in [2],

$$\Gamma(G) \leq k+1 \leq \frac{3n+1}{2} X \leq \frac{3n-3}{2}$$

which gives different values of $\Gamma(G)$ according to the values of $n$.

But the upper bound of the Grundy number is its maximum degree plus one.

For Grundy coloring, nodes of $T(G)$ are allotted by the color set $S = \{1,2,3,4,5,6,7\}$ such that $S(u_i) = 1$ when $i$ is odd and $i = 1$ to $n$

$S(u_i) = 3$ when $i$ is even and $i = 1$ to $n$

$S(v_i) = 4$ when $i$ is odd and $i = 1$ to $n$

$S(v_i) = 6$ when $i$ is even and $i = 1$ to $n$

$S(w_i) = 7$ when $i = 1$ to $n$

$S(x_i) = 2$ when $i = 1$ to $n$

$S(y_i) = 5$ when $i = 1$ to $n$

Also vertices of $Q(G)$ are assigned by the same color set in the following manner,

$S(w_i) = 1$ when $i$ is odd and $i = 1$ to $n$

$S(w_i) = 3$ when $i$ is even and $i = 1$ to $n$

$S(v_i) = 4$ when $i$ is odd and $i = 1$ to $n$

$S(v_i) = 6$ when $i$ is even and $i = 1$ to $n$

$S(u_i) = 7$ when $i = 1$ to $n$

$S(x_i) = 2$ when $i = 1$ to $n-1$

$S(y_i) = 5$ when $i$ varies from 1 to $n-1$.

Hence $\Gamma[M(G)]=\Gamma[Q(G)]=7$.

**Theorem 4**

Let $G$ be a bipartite graph, then $\chi[T(G)], \chi[R(G)], \Gamma[T(G)]$ and $\Gamma[R(G)]$ are equal to 7.

**Proof:**

Let $G$ be a bipartite graph. Graph operators $R(G), Q(G), M$ and $T(G)$ of $G$ induces a clique of order 3. Hence $\chi(G) \geq \omega(G) \geq 3$

Suppose $\chi(G)=3$ . It is not possible that the maximum degree of above graphs are equal to 6 and there are 7 vertices of degree 6 for $b$-coloring. To determine $b$-coloring, consider the color set $S= \{1,2,3,4,5,6,7\}$ have assigned to the vertices of $M(G)$ and $Q(G)$ such that $w_i$ for $i = 1$ to $n$ are allocated by the color set $S$ in the increasing order $1234567 \ldots u_i$ for $i = 1$ to $n$ are allocated by the color set $S$ in the sequence $23456712345671 \ldots$ and $x_i$, for $i = 1$ to $n-1$ have allocated with the color set in the sequence $45671234567123 \ldots$. Finally $y_i$, for $i = 1$ to $n-1$ and $v_i$ for $i = 1$ to $n$ are allocated in the decreasing order $7654321 \ldots$
For Grundy coloring of $T(G)$ the color set $S =\{1,2,3,4,5,6,7\}$ have been considered and the colors 1&3 have assigned to $u_i (1 \leq i \leq n)$ where 1 corresponds to even numbers of $i$ and 3 corresponds to odd numbers of $i$. Vertices $w_i, x_i$ and $y_i$ are assigned by the colors 7,2 & 5 respectively. From the color set $S$ the remaining colors 4 & 5 are assigned to $v_i (1 \leq i \leq n)$ for the odd and even numbers of $i$ respectively.

Also, for Grundy coloring of $R(G)$, the identical color set as previously assumed are well chosen. Above chosen colors are assigned to the vertices as $w_i (1 \leq i \leq n) \rightarrow 1 & 3$ for the odd and even numbers of $i$ respectively, $u_i (1 \leq i \leq n) \rightarrow 7, v_i \rightarrow 4 & 6$ for the odd and even values of $i$ which varies from 1 to $n$ and 2 & 5 $\rightarrow x_i$ and $y_i$ respectively where $i$ varies from 1 to n-1.

Hence $\Gamma[T(G)]= \Gamma[R(G)]=7$.

Hence the proof.

4. CONCLUSION:
In accordance with the above facts proved it is concluded that parameter of Grundy and $b$-coloring of Middle graph and Graph operator $Q(G)$ of a bipartite graph are equal. Also parameter of Grundy and $b$-coloring of Total graph and graph operator $R(G)$ of a bipartite graph are equal.

REFERENCES
[1] Douglas B 1999. west, Introduction to Graph Theory, Prentice-Halof India private Limited, New Delhi.
[2] Alkhateeb 2012, On $b$-colorings and $b$-continuity of graphs, Ph.D. Thesis, Technische University Bergakademie, Freiberg, Germany.
[3] T R. Jensen 1995. B. Toft, Graph Coloring problems, Wiley, New York.
[4] Christen 1979, S.M. Selkow, Some perfect coloring properties of graphs, J. Combin. Theory Ser. B, 27, 49-59.
[5] A. Kohl, I. Schiermeyer, Some results on Reed’s conjecture about $\lambda$, $\Delta$ and $\chi$ with respect to $\chi$, Discrete Mathematics, 310 (2009), 1429-1438.
[6] M. Zaker, Inequalities for the Grundy chromatic number of graphs, Discrete Applied Mathematics, 155, 2567-2572.
[7] Zaker 2006, Results on the Grundy chromatic number of graphs, Discrete Mathematics, 306 3166-3173.
[8] Victor Campos 2010. et. al, New bounds on the Grundy number of products of graphs, HAL.
[9] Irving, D.F. Man love 1991, The $b$-chromatic number of a graph, Discrete. Appl. Math., 91, 127-141.
[10] S.K. Vaidya et.al., $b$-Chromatic number of some path related graphs, International Journal of Mathematics and Scientific Computing, ISSN(2231-5330), 4, No. 1 (2014).
[11] Balakrishnan, K. Ranganathan, A textbook of Graph Theory, Springer, New York (2012).
[12] Purna Chandra Biswa 2015, Discrete Mathematics and Graph theory, Eastern Economy Edition, PHI Learning Private Ltd., Delhi, Page 275.
[13] Weigen Yan, Bo-Yin Yang and Yeong-Nan Yeh, Wiener indices and Polynomials of Fivograph Operators, precision.moscito.org/M/publ/frentest/operator arch 2, 2007.
[14] Venkatachalam et.al, The $b$-chromatic number of star graph families, LE MATEMATICHE, Vol. LXV (2010) – Fasc. I, pp. 119–125.
[15] Choudum 2011, “A First course in Graph Theory”, Macmillan Publishers India Limited.