HAMILTONIAN DIFFEOMORPHISMS OF TORIC MANIFOLDS

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Abstract. We prove that \( \pi_1(\text{Ham}(M)) \) contains an infinite cyclic subgroup, where \( \text{Ham}(M) \) is the Hamiltonian group of the one point blow up of \( \mathbb{C}P^3 \). We give a sufficient condition for the group \( \pi_1(\text{Ham}(M)) \) to contain an infinite cyclic subgroup, when \( M \) is a general toric manifold.

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1. Introduction

A loop \( \psi \) in the group \( \text{Ham}(M, \omega) \) of Hamiltonian symplectomorphisms \([9]\) of the symplectic manifold \( (M^{2n}, \omega) \) determines a Hamiltonian fibration \( E \xrightarrow{\pi} S^2 \) with standard fibre \( M \). On the total space \( E \) we can consider the first Chern class \( c_1(\text{VTE}) \) of the vertical tangent bundle of \( E \). Moreover on \( E \) is also defined the coupling class \( c \in H^2(E, \mathbb{R}) \) \([6]\). This class is determined by the following properties:

1. \( i_p^*(c) \) is the cohomology class of the symplectic structure on the fibre \( \pi^{-1}(p) \), where \( i_p \) is the inclusion of \( \pi^{-1}(p) \) in \( E \) and \( p \) is an arbitrary point of \( S^2 \).
2. \( c^{n+1} = 0 \).

These canonical cohomology classes determine the characteristic number \([7]\)

\[
I_\psi = \int_E c_1(\text{VTE})c^n.
\]

\( I_\psi \) depends only on the homotopy class of \( \psi \). Moreover \( I \) is an \( \mathbb{R} \)-valued group homomorphism on \( \pi_1(\text{Ham}(M, \omega)) \), so the non vanishing of \( I \) implies that the group \( \pi_1(\text{Ham}(M, \omega)) \) is infinite. That is, \( I \) can be used to detect the infinitude of the corresponding homotopy group. Furthermore \( I \) calibrates the Hofer’s norm \( \nu \) on \( \pi_1(\text{Ham}(M, \omega)) \) in the sense that \( \nu(\psi) \geq C|I_\psi| \), for all \( \psi \), where \( C \) is a positive constant \([12]\).

In \([15]\) we gave an explicit expression for the value of the characteristic number \( I_\psi \). This value can be calculated if one has a family of local symplectic trivializations of \( TM \) at one disposal, whose domains cover \( M \) and are fixed by the \( \psi_t \)'s. In fact we proved the following Theorem

**Theorem 1.** Let \( \psi : S^1 \to \text{Ham}(M, \omega) \) be a closed Hamiltonian isotopy. If \( \{B_1, \ldots, B_m\} \) is a set of symplectic trivializations for \( TM \) which covers \( M \) and such that \( \psi_t(B_j) = B_j \), for all \( t \) and all \( j \), then

\[
I_\psi = \sum_{i=1}^m J_i \int_{B_i \setminus \bigcup_{j<i} B_j} \omega^n + \sum_{i<k} N_{ik},
\]

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where
\[ N_{ik} = n \frac{i}{2\pi} \int_{S^1} dt \int_{A_{ik}} (f_t \circ \psi_t) (d \log r_{ik}) \wedge \omega^{n-1}, \]

\[ A_{ik} = (\partial B_i \setminus \cup_{r<k} B_r) \cap B_k, \]

\( J_i \) is the Maslov index of \( (\psi_t)_* \) in the trivialization \( B_i \) and \( r_{ik} \) the corresponding transition function of \( \det(TM) \).

The homotopy type of \( \text{Ham}(M, \omega) \) is completely known in a few particular cases \([8, 12]\) only. When \( M \) is a surface, \( \text{Diff}_0(M) \) (the arc component of the identity map in the diffeomorphism group of \( M \)) is homotopy equivalent to the symplectomorphism group of \( M \), so the topology of the groups \( \text{Ham}(M) \) in dimension 2 can be deduced from the description of the diffeomorphism groups of surfaces given in \([8]\) (see \([12]\)). On the other hand, positivity of the intersections of \( J \)-holomorphic spheres in 4-manifolds have been used in \([4, 11, 2]\) to prove results about the homotopy type of \( \text{Ham}(M) \), when \( M \) is a ruled surface. But these arguments which work in dimension 2 or dimension 4 cannot be generalized to higher dimensions.

Let \( O \) be a coadjoint orbit of a Lie group \( G \). If \( G \) is semisimple and acts effectively on \( O \), McDuff and Tolman have proved that the inclusion \( G \to \text{Ham}(O) \) induces an injection from \( \pi_1(G) \) to \( \pi_1(\text{Ham}(O)) \) \([11]\). This result answers a question posed in \([16]\). In \([14]\) we gave a lower bound for \( \sharp \pi_1(\text{Ham}(O)) \), when \( O \) is a quantizable coadjoint orbit of a compact Lie group. In particular we proved that \( \sharp \pi_1(\text{Ham}(\mathbb{C}P^n)) \geq n + 1 \).

In this note we use Theorem \([11]\) to prove that \( \pi_1(\text{Ham}(M)) \) contains an infinite cyclic subgroup, when \( M \) is a particular toric manifold. More precisely, when \( M \) is the 6-manifold associated to the polytope obtained truncating the tetrahedron of \( \mathbb{R}^3 \) with vertices \((0,0,0), (\tau,0,0), (0,\tau,0), (0,0,\tau)\) by a horizontal plane \([5]\); that is, when \( M \) is the one point blow up of \( \mathbb{C}P^3 \). Moreover we will give a sufficient condition for \( \pi_1(\text{Ham}(M)) \) to contain an infinite cyclic subgroup, when \( M \) is a general toric manifold.

The paper is organized as follows. Section 2 is concerned with the determination of \( \Gamma_\psi \) for a natural circle action on the one point blow up of \( \mathbb{C}P^3 \). In Section 3 we generalize the arguments developed in Section 2 to toric manifolds. From this generalization it follows the aforesaid sufficient condition for the existence of an infinite subgroup in \( \pi_1(\text{Ham}(M)) \), when \( M \) is a toric manifold. Finally we check that this sufficient condition does not hold for \( \mathbb{C}P^n \) with \( n = 1, 2 \). This is consistent with the fact that \( \pi_1(\text{Ham}(\mathbb{C}P^n)) \) is finite for \( n = 1, 2 \).

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2. Hamiltonian group of the one point blow up of \( \mathbb{C}P^3 \)

Given \( \tau, \mu \in \mathbb{R}_{>0} \), with \( \mu < \tau \), let \( M \) be the following manifold

\[ M = \{ z \in \mathbb{C}^5 : |z_1|^2 + |z_2|^2 + |z_3|^2 + |z_5|^2 = \tau/\pi, \ |z_3|^2 + |z_4|^2 = \mu/\pi \} / \mathbb{T}^2, \]

where the action of \( \mathbb{T}^2 \) is defined by

\[ (a,b)(z_1, z_2, z_3, z_4, z_5) = (az_1, az_2, abz_3, bz_4, a^5z_5), \]

for \( a, b \in S^1 \).

\( M \) is a toric 6-manifold; more precisely, it is the toric manifold associated to the polytope obtained truncating the tetrahedron of \( \mathbb{R}^3 \) with vertices

\((0,0,0), (\tau,0,0), (0,\tau,0), (0,0,\tau)\)
by a horizontal plane through the point \((0, 0, \lambda)\), with \(\lambda := \tau - \mu\).

For \(0 \neq z_j \in \mathbb{C}\) we put \(z_j = \rho_j e^{i\theta_j}\), with \(|z_j| = \rho_j\). On the set of points \([z] \in M\) with \(z_i \neq 0\) for all \(i\) one can consider the coordinates

\[
(\rho_1^2, \varphi_1, \rho_2^2, \varphi_2, \rho_3^2, \varphi_3),
\]

where the angle coordinates are defined by

\[
\varphi_1 = \theta_1 - \theta_5, \quad \varphi_2 = \theta_2 - \theta_5, \quad \varphi_3 = \theta_3 - \theta_4 - \theta_5.
\]

Then the standard symplectic structure on \(\mathbb{C}^5\) induces the following form \(\omega\) on this part of \(M\)

\[
\omega = \sum_{j=1}^{3} d\left(\frac{\rho_j^2}{2}\right) \wedge d\varphi_j.
\]

Let \(0 < \epsilon << 1\), we write

\[B_0 = \{[z] \in M : |z_j| > \epsilon, \text{ for all } j\}.\]

For a given \(j \in \{1, 2, 3, 4, 5\}\) we set

\[B_j = \{[z] \in M : |z_j| < 2\epsilon \text{ and } |z_i| > \epsilon, \text{ for all } i \neq j\}\]

The family \(B_0, \ldots, B_5\) is not a covering of \(M\), but if \([z] \notin \bigcup B_k\), then there are \(i, j\) with \(i \neq j\) and \(|z_i| < \epsilon > |z_j|\).

On \(B_0\) we will consider the well-defined Darboux coordinates \((2.3)\). On \(B_1\), \(\rho_j \neq 0\) for \(j \neq 1\); so the angle coordinates \(\varphi_2\) and \(\varphi_3\) of \((2.4)\) are well-defined. We put \(x_1 + iy_1 := \rho_1 e^{i\varphi_1}\). In this way we take as symplectic coordinates on \(B_1\)

\[(x_1, y_1; \rho_2^2, \varphi_2, \rho_3^2, \varphi_3).\]

We will also consider the following Darboux coordinates: On \(B_2\)

\[
(\rho_1^2, \varphi_1, x_2, y_2; \rho_3^2, \varphi_3), \text{ with } x_2 + iy_2 := \rho_2 e^{i\varphi_2}.
\]

On \(B_3\)

\[
(\rho_1^2, \varphi_1, \rho_2^2, \varphi_2, x_3, y_3), \text{ where } x_3 + iy_3 := \rho_3 e^{i\varphi_3}.
\]

On \(B_4\)

\[
(\rho_1^2, \varphi_1, \rho_2^2, \varphi_2, x_4, y_4), \text{ with } x_4 + iy_4 := \rho_4 e^{i\varphi_4} \text{ and } \varphi_4 = \theta_4 - \theta_3 + \theta_5.
\]

On \(B_5\)

\[
(x_5, y_5; \rho_2^2, \chi_2, \rho_3^2, \chi_3),
\]

where

\[
x_5 + iy_5 := \rho_5 e^{i\varphi_5}, \chi_2 = \theta_2 - \theta_1, \chi_3 = \theta_3 - \theta_1 - \theta_4, \chi_5 = \theta_5 - \theta_4.
\]

If \([z_1, \ldots, z_5]\) is a point of

\[M \setminus \bigcup_{i=0}^{5} B_i,\]

then there are \(a \neq b \in \{1, \ldots, 5\}\) such that \(|z_a|, |z_b| < \epsilon\). We can cover the set \(M \setminus \bigcup B_i\) by Darboux charts denoted \(B_6, \ldots, B_q\) similar to the preceding \(B_i\)’s.
satisfying the following condition: The image of each $B_a$, with $a = 6, \ldots, q$, is contained in a prism of $\mathbb{R}^6$ of the form

$$\prod_{i=1}^{6} [c_i, d_i],$$

where at least four intervals $[c_i, d_i]$ have length of order $\epsilon$.

By the infinitesimal “size” of the $B_j$, for $j \geq 1$, it turns out

$$(2.5) \int_{B_j} \omega^3 = O(\epsilon), \quad \text{for } j \geq 1.$$  

Let $\psi$ be the symplectomorphism of $M$ defined by

$$(2.6) \psi_t[z] = [z_1 e^{2\pi t}, z_2, z_3, z_4, z_5].$$

Then $\{\psi_t\}$ is a loop in the group $\text{Ham}(M)$ of Hamiltonian symplectomorphisms of $M$. By $f$ is denoted the corresponding normalized Hamiltonian function. Hence

$$f = \pi^2 - \kappa$$

with $\kappa \in \mathbb{R}$ such that

$$\int_M f \omega^3 = 0.$$  

In the coordinates (2.3) of $B_0$, $\psi$ is the map $\varphi_1 \mapsto \varphi_1 + 2\pi t$. So the Maslov index $J_{B_0} = 0$. It follows from (2.5) and Theorem 1

$$(2.7) I_\psi = \sum_{i<k} N_{ik} + O(\epsilon),$$

with

$$N_{ik} = \frac{3i}{2\pi} \int_{A_{ik}} f \log r_{ik} \wedge \omega^2.$$  

If $[z] \in A_{ik} \subset \partial B_i \cap B_k$, with $1 \leq i < k$, then at least the modules $|z_a|$ and $|z_b|$ of two components of $[z]$ are of order $\epsilon$; so $N_{ik}$ is of order $\epsilon$ when $1 \leq i < k$. Analogously $N_{0k}$ is of order $\epsilon$, for $k = 6, \ldots, q$. Hence (2.7) reduces to

$$(2.8) I_\psi = \sum_{k=1}^{5} N_{0k} + O(\epsilon).$$

If we put

$$(2.9) N_{0k}' = \frac{3i}{2\pi} \int_{A_{0k}'} f \log r_{ik} \wedge \omega^2,$$  

with

$$A_{0k}' = \{[z] \in M : |z_k| = \epsilon, |z_r| > \epsilon \text{ for all } r \neq k\}$$

then

$$(2.10) N_{0k} = N_{0k}' + O(\epsilon).$$

Next we determine the value of $N_{01}'$. To know the transition function $r_{01}$ one needs the Jacobian matrix $R$ of the transformation

$$\begin{pmatrix} x_1, y_1, \rho_2^2/2, \varphi_2, \rho_3^2/2, \varphi_3 \end{pmatrix} \rightarrow \begin{pmatrix} \rho_1^2/2, \varphi_1, \rho_2^2/2, \varphi_2, \rho_3^2/2, \varphi_3 \end{pmatrix}$$

in the points of $A_{01}'$; where $\rho_1^2 = x_1^2 + y_1^2$, $\varphi_1 = \tan^{-1}(y_1/x_1)$. The function $r_{01} = \rho(R)$, where $\rho : \text{Sp}(6, \mathbb{R}) \rightarrow U(1)$ is the map which restricts to the determinant on $U(3)$ [13]. The non trivial block of $R$ is the diagonal one

$$\begin{pmatrix} x_1 & y_1 \\ r & s \end{pmatrix},$$
with \( r = -y_1(x_1^2 + y_1^2)^{-1} \) and \( s = x_1(x_1^2 + y_1^2)^{-1} \). The non real eigenvalues of \( R \) are
\[
\lambda_{\pm} = \frac{x_1 + s \pm \sqrt{4 - (s + x_1)^2}}{2}.
\]

On \( A'_{01} \) these non real eigenvalues occur when \( (s + x_1)^2 < 2 \), that is, if \( |\cos \varphi_1| < 2\epsilon(\epsilon^2 + 1)^{-1} =: \delta \). If \( y_1 > 0 \) then \( \lambda_- \) of the first kind (see [13]) and \( \lambda_+ \) is of the first kind, if \( y_1 < 0 \).

Hence, on \( A'_{01} \),
\[
\rho(R) = \begin{cases} 
\lambda_+|\lambda_+|^{-1} = x + iy, & \text{if } |\cos \varphi_1| < \delta \text{ and } y_1 < 0; \\
\lambda_-|\lambda_-|^{-1} = x - iy, & \text{if } |\cos \varphi_1| < \delta \text{ and } y_1 > 0; \\
\pm 1, & \text{otherwise.}
\end{cases}
\]
where \( x = \delta^{-1}\cos \varphi_1, \) and \( y = \sqrt{1 - x^2}. \)

If we put \( \rho(R) = e^{i\gamma} \), then \( \cos \gamma = \delta^{-1}\cos \varphi_1 \) (when \( |\cos \varphi_1| < \delta \), and
\[
\sin \gamma = \begin{cases} 
-\sqrt{1 - \cos^2 \gamma}, & \text{if } \sin \varphi_1 > 0; \\
\sqrt{1 - \cos^2 \gamma}, & \text{if } \sin \varphi_1 < 0.
\end{cases}
\]

So when \( \varphi_1 \) runs anticlockwise from 0 to \( 2\pi \), \( \gamma \) goes round clockwise the circumference; that is, \( \gamma = h(\varphi_1) \), where \( h \) is a function such that
\[
(2.11) \quad h(0) = 2\pi, \text{ and } h(2\pi) = 0.
\]
As \( r_{01} = \rho(R) \), then \( d\log r_{01} = idh. \)

On \( A'_{01} \) the form \( \omega \) reduces to \((1/2)\, d\rho_1^2 \wedge d\varphi_2 + d\rho_2^2 \wedge d\varphi_3 \). From (2.9) one deduces
\[
(2.12) \quad N'_{01} = \frac{3i}{4\pi} \int_{A'_{01}} if \frac{\partial h}{\partial \varphi_1} \, d\varphi_1 \wedge d\rho_2^2 \wedge d\varphi_2 \wedge d\rho_3^2 \wedge d\varphi_3.
\]

The submanifold \( A'_{01} \) is oriented as a subset of \( \partial B_0 \) and the orientation of \( B_0 \) is the one defined by \( \omega^3 \), that is, by
\[
d\rho_1^2 \wedge d\varphi_1 \wedge d\rho_2^2 \wedge d\varphi_2 \wedge d\rho_3^2 \wedge d\varphi_3.
\]

Since \( \rho_1 > \epsilon \) for the points of \( B_0 \), then \( A'_{01} \) is oriented by \(-d\varphi_1 \wedge d\varphi_2^2 \wedge d\varphi_2 \wedge d\varphi_3\).

On the other hand, the Hamiltonian function \( f = -\kappa + O(\epsilon) \) on \( A'_{01} \). Then it follows from (2.12) together with (2.11)
\[
N'_{01} = 6\pi^2\kappa \int_0^{\mu/\pi} d\rho_1^3 \int_0^{\tau/\pi - \rho_1^3} d\rho_2^3 + O(\epsilon).
\]
that is,
\[
(2.13) \quad N'_{01} = 3\kappa(\tau^2 - \lambda^2) + O(\epsilon).
\]

The contributions \( N'_{02}, N'_{03}, N'_{04}, N'_{05} \) to \( I_\psi \) can be calculated in a similar way. One obtains the following results up to addends of order \( \epsilon \)
\[
(2.14) \quad N'_{02} = N_{05} = -(\tau^3 - \lambda^3) + 3\kappa(\tau^2 - \lambda^2), \quad N'_{03} = \tau^2(3\kappa - \tau), \quad N'_{04} = \lambda^2(3\kappa - \lambda).
\]

As \( I_\psi \) is independent of \( \epsilon \), it follows from (2.8), (2.10), (2.13) and (2.14)
\[
(2.15) \quad I_\psi = 6\kappa(2\tau^2 - \lambda^2) + \lambda^3 - 3\tau^3.
\]
On the other hand, straightforward calculations give
\[ \int_M \omega^3 = (\tau^3 - \lambda^3), \]
and
\[ \int_M \pi \rho_1^2 \omega^3 = \frac{1}{4} (\tau^4 - \lambda^4). \]
So
\[ (2.16) \kappa = \frac{1}{4} \left( \frac{\tau^4 - \lambda^4}{\tau^3 - \lambda^3} \right). \]

It follows from (2.15) and (2.16)
\[ (2.17) I_\psi = \frac{\lambda^2}{2} \left( -3 \tau^4 + 8 \tau^3 \lambda - 6 \tau^2 \lambda^2 + \lambda^4 \right). \]

Hence \( I_\psi \) is a rational function of \( \tau \) and \( \lambda \). It is easy to check that its numerator does not vanish for \( 0 < \lambda < \tau \). So we have

**Proposition 2.** If \( \psi \) is the closed Hamiltonian isotopy defined in (2.6), then the characteristic number \( I_\psi \neq 0 \).

Next we consider the loop \( \hat{\psi} \) defined by
\[ (2.18) \hat{\psi}_t[z] = [z_1, z_2, z_3 e^{2\pi i t}, z_4, z_5]. \]
The corresponding normalized Hamiltonian function is \( \hat{f} = \pi \rho_3^2 - \hat{\kappa} \), where
\[ (2.19) \hat{\kappa} = \frac{1}{4} \left( \frac{\lambda^4 - 4 \tau \lambda^3 + 3 \lambda^4}{\tau^3 - \lambda^3} \right). \]

As in the preceding case
\[ (2.20) I_{\hat{\psi}} = \sum_{j=1}^{5} \hat{N}_{0j} + O(\epsilon), \]
where
\[ \hat{N}_{0j} = \frac{3i}{2\pi} \int_{A_0,j} \hat{f} d \log r_0 \wedge \omega^2. \]

The expression for \( \hat{N}_{01} \) can be obtained from (2.12) substituting \( f \) for \( \hat{f} \); so
\[ (2.21) \hat{N}_{01} = -3(\tau - \hat{\kappa})(\tau^2 - \lambda^2) + 2(\tau^3 - \lambda^3) + O(\epsilon). \]

Similar calculations give the following values for the \( \hat{N}_{0j} \)'s, up to summands of order \( \epsilon \),
\[ (2.22) \hat{N}_{02} = \hat{N}_{05} = -3(\tau - \hat{\kappa})(\tau^2 - \lambda^2) + 2(\tau^3 - \lambda^3), \quad \hat{N}_{03} = 3\hat{\kappa} \tau^2, \quad \hat{N}_{04} = 3\lambda^2(\hat{\kappa} - \mu). \]

It follows from (2.22), (2.21) and (2.20)
\[ (2.23) I_{\hat{\psi}} = 6\hat{\kappa}(2\tau^2 - \lambda^2) - 3(\tau^3 - 2\tau \lambda^2 + \lambda^3). \]

After (2.19) we obtain
\[ I_{\hat{\psi}} = -3I_\psi. \]

In the definition of \( M \) the variables \( z_1, z_2, z_5 \) play the same role. However we can consider the following \( S^1 \) action on \( M \)
\[ (2.24) \hat{\psi}_t[z] = [z_1, z_2, z_3, e^{2\pi it} z_4, z_5]. \]
Its Hamiltonian is \( \hat{f} = \pi \rho_4^2 - \hat{\kappa} \), with
\[ (2.25) \hat{\kappa} = \frac{1}{4} \left( \frac{\lambda^4 - 4 \tau \lambda^3 + 3 \tau^4}{\tau^3 - \lambda^3} \right). \]
The corresponding $\hat{N}'_{ij}$ have the following values up summand of order $\epsilon$

(2.26)

$\hat{N}'_{01} = \hat{N}'_{02} = \hat{N}'_{05} = 3(\lambda + \kappa)(\tau^2 - \lambda^2) - 2(\tau^3 - \lambda^3)$, $\hat{N}'_{03} = 3\tau^2(\hat{\kappa} - \mu)$, $\hat{N}'_{04} = 3\hat{\kappa}\lambda^2$.

From the preceding formulae one deduces

$I_{\hat{\psi}} = -I_{\tilde{\psi}} = 3I_{\psi}$.

**Theorem 3.** Let $M$ be the toric manifold defined by (2.1) and (2.2). If $\psi$, $\tilde{\psi}$ and $\hat{\psi}$ are the Hamiltonian loops in $M$ defined by (2.6), (2.18) and (2.24) respectively, then

$I_{\hat{\psi}} = -I_{\tilde{\psi}} = 3I_{\psi}$, with

$I_{\psi} = \lambda^2(-3\tau^4 + 8\tau^3\lambda - 6\tau^2\lambda^2 + \lambda^4)$

$2(\tau^3 - \lambda^3)$,

$\lambda$ being $\lambda := \tau - \mu$.

**Corollary 4.** Let $(M, \omega)$ be the toric manifold one point blow up of $\mathbb{C}P^2$, then $\pi_1(\text{Ham}(M, \omega))$ contains an infinite cyclic subgroup.

**Proof.** By Proposition 2, $I_{\psi} \neq 0$. As $I$ is a group homomorphism then the class $[\psi^l] \in \pi_1(\text{Ham}(M, \omega))$ does not vanish, for all $l \in \mathbb{Z} \setminus \{0\}$. $\square$

### 3. Hamiltonian group of toric manifolds

In this Section we generalize the calculations carried out in Section 2 for the 6-manifold one point blow up of $\mathbb{C}P^2$ to a general toric manifold.

Let $\mathcal{T}$ be the torus $(S^1)^r$, and $\mathfrak{t} = \mathbb{R} \oplus \cdots \oplus \mathbb{R}$ its Lie algebra. Given $w_j \in \mathbb{Z}^r$, with $j = 1, \ldots, m$ and $\tau \in \mathbb{R}^r$ we put

(3.1)

$M = \{z \in \mathbb{C}^m : \pi \sum_{j=1}^{m} |z_j|^2 w_j = \tau \}/\mathcal{T},$

where the relation defined by $\mathcal{T}$ is

(3.2) $(z_j) \simeq (z'_j)$ iff there is $\xi \in \mathfrak{t}$ such that $z'_j = z_j e^{2\pi i (w_j, \xi)}$ for $j = 1, \ldots, m$.

We will assume that there is an open half space in $\mathbb{R}^r$ which contains all the vectors $w_j$ and that $\{w_j\}$ span $\mathbb{R}^r$. We also assume that $\tau$ is a regular value of the map

$$z \in \mathbb{C}^m \mapsto \pi \sum_{j=1}^{m} |z_j|^2 w_j \in \mathbb{R}^r.$$ 

Then $M$ is a closed toric manifold of dimension $n := 2(m - r)$.

When $0 \neq z_a \in \mathbb{C}$, we write $z_a = \rho_a e^{i\theta_a}$. The standard symplectic form on $\mathbb{C}^m$ gives rise to the symplectic structure $\omega$ on $M$. On

$$\{[z] \in M : z_j \neq 0 \text{ for all } j\}$$

$\omega$ can be written as

$$\omega = \sum_{i=1}^{n} d\left(\frac{\rho^2_{ai}}{2}\right) \wedge d\varphi_{ai},$$

with $\varphi_{ai}$ a linear combination of the $\theta_c$'s.
Given $0 < \epsilon << 1$, we set
\[ B_0 = \{ [z] \in M : |z_j| > \epsilon \text{ for all } j \} \]
\[ B_k = \{ [z] \in M : |z_k| < \epsilon, |z_j| > \epsilon \text{ for all } j \neq k \}, \]
as in Section 2. On $B_0$ we will consider the Darboux coordinates
\[ \{ \frac{\rho_{ai}^2}{2}, \phi_{ai} \}_{i=1,\ldots,n}. \]

Given $k \in \{ 1, \ldots, m \}$ we write $\omega$ in the form
\[ \omega = d \left( \frac{\rho_k^2}{2} \right) \wedge d\varphi_k + \sum_{i=1}^{n-1} d \left( \frac{\rho_{ki}^2}{2} \right) \wedge d\varphi_{ki}, \]
where $\varphi_k$ and $\varphi_{ki}$ are linear combinations of the $\theta_c$'s. Then we consider on $B_k$ the following Darboux coordinates
\[ \{ x_k, y_k, \frac{\rho_{ki}^2}{2}, \varphi_{ki} \}_{i=1,\ldots,n-1}, \]
with $x_k + iy_k := \rho_k e^{i\varphi_k}$.

We denote by $\psi_t$ the map
\[ \psi_t : [z] \in M \mapsto [e^{2\pi it} z_1, z_2, \ldots, z_m] \in M. \]
\{ $\psi_t : t \in [0,1]$ \} is a loop in Ham($M$). By repeating the arguments of Section 2 one obtains
\[ I_{\psi} = \sum_{k=1}^{m} N'_{0k} + O(\epsilon), \]
where
\[ N'_{0k} = \frac{ni}{2\pi} \int_{A'_{0k}} f d \log r_{0k} \wedge \omega^{n-1}, \]
\[ A'_{0k} = \{ [z] \in M : |z_k| = \epsilon, |z_j| > \epsilon \text{ for all } j \neq k \}, \]
and $f = \pi\rho_1^2 - \kappa$, with
\[ \int_M \pi\rho_1^2 \omega^n = \kappa \int_M \omega^n. \]

As in Section 2, on $A'_{0k}$ the exterior derivative $d \log r_{0k} = i h'(\varphi_k) d\varphi_k$, where
\[ h = h(\varphi_k) \text{ is a function such that } h(0) = 2\pi, h(2\pi) = 0. \]
Then
\[ N'_{0k} = -n \int_{\{[z] : z_k = 0\}} f \omega^{n-1} + O(\epsilon). \]

Since $I_{\psi}$ is independent of $\epsilon$, we obtain
\[ (3.3) \quad I_{\psi} = -n \sum_{k=1}^{m} \left( \int_{\{[z] : z_k = 0\}} (\pi\rho_1^2 - \kappa)\omega^{n-1} \right). \]

This formula together with the fact that $I$ is a group homomorphism give the following Theorem

**Theorem 5.** Let $(M, \omega)$ be the toric manifold defined by (3.1) and (3.2). If
\[ \sum_{k=1}^{m} \left( \int_{\{[z] : z_k = 0\}} (\pi\rho_1^2 - \kappa)\omega^{n-1} \right) \neq 0, \]
then $\pi_1(Ham(M, \omega))$ contains an infinite cyclic subgroup.
Examples. We will check the above result calculating \( I_\psi \) by (3.3) in two particular cases: When the manifold is \( \mathbb{C}P^1 \) and when is \( \mathbb{C}P^2 \).

For \( \mathbb{C}P^1 = \{ [z_1, z_2] : |z_1|^2 + |z_2|^2 = \tau/\pi \} / S^1 \) and \( \psi([z_1, z_2]) = [e^{2\pi i t} z_1, z_2] \), the normalized Hamiltonian is \( f = \pi \rho^2 - \tau/2 \), that is, \( \kappa = \tau/2 \). In this case (3.3) reduces to \( I_\psi = -\tau + 2\kappa = 0 \). This is compatible with the fact that \( \pi_1(\text{Ham}(\mathbb{C}P^1)) = \mathbb{Z}/2\mathbb{Z} \).

For \( \mathbb{C}P^2 = \{ [z_1, z_2, z_3] : |z_1|^2 + |z_2|^2 + |z_3|^2 = \tau/\pi \} / S^1 \), the Hamiltonian is \( f = \pi \rho^2 - \tau/3 \). Moreover for \( k \in \{1, 2, 3\} \)

\[
\int_{\{[z]: z_k = 0\}} \omega = \tau.
\]

On the other hand, for \( k = 2, 3 \)

\[
\int_{\{[z]: z_k = 0\}} \pi \rho^2 \omega = \tau^2/2.
\]

After (3.3) \( I_\psi = -2(\tau^2 - 3\kappa \tau) = 0 \). This result is consistent with the finiteness of \( \pi_1(\text{Ham}(\mathbb{C}P^2)) \), since \( \text{Ham}(\mathbb{C}P^2) \) has the homotopy type of \( PU(3) \). [4]

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