Analytical canonical partition function of a quasi-one dimensional system of hard disks

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The exact canonical partition function of a hard disk system in a narrow quasi-one dimensional pore of given length and width is derived analytically in the thermodynamic limit. As a result the many body problem is reduced to solving two transcendental equations which can be easily done numerically. The longitudinal and transverse pressures in the whole density range are presented for three different pore widths. The transition from the solidlike zigzag to the liquidlike state is found to be quite sharp in the density scale but shows no genuine singularity. This transition is quantitatively described by the distribution of zigzag’s windows through which disks exchange their positions across the pore.

Introduction. Over more than a century the idea to model molecules as hard spheres has been widely used in the theory of liquids [1,2]. In spite of apparent simplicity, the behavior of hard sphere systems is so complex mathematically that no exact analytical result has been obtained in 3 and even in 2 dimensions (2D). Under these circumstances, the numerical Monte Carlo and molecular dynamics approaches have become the main tools in the study of 3D hard sphere and 2D hard disk (HD) systems. The numerical results however are restricted to systems of a finite number of particles whereas such effects as, for instance, phase transitions, are related to systems in the thermodynamic limit when the number of particles $N$ is infinite. As this limit can be studied only theoretically, analytical results are of great importance. The first exact analytical result was obtained in 1936 by Tonks for the purely 1D system of HDs. This system is much simpler than any 2D system, nevertheless Tonks’ solution has become the analytical platform for further expansion into the world of 2D HD systems via moving to certain quasi-1D models. Barker was the first to point to the general possibility that the 1D case is amenable to a solvable generalization to quasi-1D case of HDs in narrow pores [2]. The simplest quasi-1D system (from now on just q1D) is such that each disk can touch no more neighbors than one from both sides (the so-called single-file system); the width of such q1D pore must be below $\sqrt{3}$ times HD diameter. The analytical theory of HDs in q1D pore was presented by Wojciechowski et al. [5] and ten years later was further developed by Kofke and Post [6] who have elegantly shown that the problem can be reduced to solving certain integral equation. In general however the integral equation of this, now known as the transfer matrix method, cannot be solved analytically. A density expansion [8] and simplified model [9] to approximate solve this equation analytically have been proposed. The peculiarity of this method is that it is essentially related to the pressure-based $(N, P, T)$ ensemble which does not directly predict pressure as a function of the system’s width $D$ and length $L$. In this letter I present exact analytical derivation of the canonical $(N, L, D, T)$ partition function (PF) in the thermodynamic limit. As a result, finding the thermodynamic properties of a q1D HD system for given $L$ and $D$ is reduced to solving two transcendental equations which can be easily done numerically. The longitudinal and transverse pressures are presented for three pore widths and disks’ arrangement for different densities $N/L$ is discussed. The system shows a sharp crossover from solidlike to liquidlike state, but the thermodynamics does not show any genuine discontinuity.

Partition function. Consider a pore of the width $D$ and length $L$ filled with $N$ HDs of diameter $d=1$, Fig. 1. We assume the thermodynamic limit $N \rightarrow \infty, L \rightarrow \infty$ while $N/L = \text{const}$; the terms which vanish in this limit (e.g., the end effects) will be omitted. The width parameter $\Delta = (D-d)/d$ in the quasi 1D case ranges from 0 in the 1D case to the maximum $\sqrt{3}/2 \approx 0.866$. The $i$-th disk has two coordinates, $x_i$ along and $y_i$ across the pore; $y$ is in units $D-d$ so that $0 \leq y \leq 1$; the pore volume is $LD$. The vertical center-to-center distance between two neighbors, $\delta y_i = y_{i+1} - y_i$, determines the contact distance $\sigma_i$ between them along the pore:

$$\sigma_i = \min |x_{i+1}(y_{i+1}) - x_i(y_i)|,$$

$$\sigma_i = \sqrt{d^2 - \Delta^2 \delta y_i^2},$$

$$\sigma_m = \sqrt{d^2 - \Delta^2} \leq \sigma \leq d = 1.$$

The minimum possible $\sigma, \sigma_m$, obtains for $\delta y = 1$ when the two disks are in contact with the opposite walls. Thus, each set of coordinates \{\{y\} = y_1, y_2, ..., y_N\} determines the correspondent densely packed state of the total length $L'\{y\} = \sum_{i=1}^{N-1} \sigma_i(\delta y_i)$, Fig. 1, which we call condensate and which plays the central role in our theory. The minimum condensate length is $\sigma_m N$, the maximum length can be as large as $Nd$, but it cannot exceed $L \geq L'_{\text{max}} \leq L'_{\text{max}} = \min(Nd, L)$. From now on $d = 1$.

We will use the notation $D^{N-1} x = dx_1...dx_{N-1}$ for the product measure. Omitting unimportant factors, the exact configurational canonical $(N, L, D)$ PF of the q1D...
The HD system has the following form:

\[ Z = \int_0^1 D^{N-1} \delta y \times \theta \left( L'_\text{max} - \sum_{j=1}^{N-1} \sigma_j \right) \int_X D^N x, \quad (2) \]

The \( x \) integration domain \( X \), which was formulated by Tonks [4] for the 1D case, ensures that under the single-file condition the disks do not intersect. In turn, the step function \( \theta \left( L'_\text{max} - \sum \sigma_j \right) \) restricts the \( y \) integration domain to those \{ \( y \) \} for which \( \sigma \)'s are in the allowed range, \( \sigma_m \leq \sigma_j \leq 1 \), but their sum does not exceed \( L'_\text{max} \). Wojciechowski et al have shown that the \( x \) integral can be solved the same way Tonks did the \( x \) integration in the 1D case [3], i.e.,

\[ Z = \int_0^1 D^{N-1} \delta y \theta \left( L'_\text{max} - \sum_{j=1}^{N-1} \sigma_j \right) \left( L - \sum_{j=1}^{N-1} \sigma_j \right)^N. \quad (3) \]

Now we turn to the \( \delta y \) integration. Rather than integrating over the entire \( \delta y \) domain defined by the \( \theta \) function, it is convenient first to fix the condensate’s length at some \( L' \) and then integrate over its possible values. This can be done by introducing following representation of the step function \( \theta \):

\[ \theta \left( L'_\text{max} - \sum \sigma_j \right) = \int_{N\sigma_m}^{L'_\text{max}} dL' \delta \left( L' - \sum \sigma_j \right). \quad (4) \]

Next we change the integration over \( \delta y \) to that over \( \sigma \), \( d\delta y_i = d\delta \sigma_i = \sigma_i d\sigma_i / \sqrt{1 - \sigma_i^2} \). Then, in the context of \[ 4 \], \( Z \) becomes

\[ Z = \int_{N\sigma_m}^{L'_\text{max}} dL' (L - L')^N I, \quad (5) \]

\[ I = \int_{\sigma_m}^1 D^{N-1} \delta (L' - \sum \sigma). \quad (6) \]

The integration in \( I \) goes over all possible sequences \( \{ \sigma_j \} = \{ \sigma_1, \sigma_2, ..., \sigma_{N-1} \} \), \( \sigma_m < \sigma_j \leq 1 \), such that the total sum of \( \sigma \)'s is equal to \( L' \), a possible condensate length. As the \( \sigma \) integration domain depends on \( L' \), it is essential to not reverse the integration order in \( \sigma \) and \( L' \).

**Fixed \( L' \).** As \( N \) is very large, possible values of \( \sigma \) in the condensate repeat many times. Let a possible \( \sigma_j \) in the sequence \( \{ \sigma_j \} \) occurs \( \nu_j \) times, \( 0 < \nu_j \leq N - 1 \). Then \( I \) is the sum over all possible distributions of \( \sigma \)'s with different \( \nu \)'s for which the total aggregate length \( \sum \sigma \nu = L' \) while the total sum of \( \nu \)'s, \( \sum \nu = N - 1 \):

\[ I = \sum_{\{ \sigma_j, \nu_j \}} \int \frac{D\bar{\sigma}}{(N-1)!} w(\nu_j) \times \delta \left( L' - \sum_{j=1}^{N-1} \sigma_j \nu_j \right) \times \delta \left( N - \sum_{j=1}^{N-1} \nu_j \right). \quad (7) \]

The factor \((N-1)!\) in the denominator excludes permutations of similar \( \sigma \)'s with different \( \nu \)'s. The statistical weight \( w(\nu_j) = w(v_1, ..., v_k) \) is the number of ways to divide \( N - 1 \) objects into groups of \( \nu_j \) objects:

\[ w = \frac{(N-1)!}{\nu_1! \nu_2! ... \nu_k!}, \quad (8) \]

where we made use of Stirling’s formula.

Now we are going to change the summation over \( \nu \) to integration. To this end we first introduce the reduced variables \( \nu_j' = \nu_j / N \). Second, in the continuous limit Kroner’s delta symbol in \[ 7 \] goes over into the delta function. Now there are two delta functions in \( I \) which we take in the form

\[ \delta \left( L' - \sum \sigma \nu \right) = \int_{-\infty}^{\infty} d\alpha e^{i\alpha N (L' / N - \sum \sigma \nu')}, \quad (9) \]

\[ \delta \left( N - \sum \nu \right) = \int_{-\infty}^{\infty} d\beta e^{i\beta N (1 - \sum \nu')} . \]

Finally, we introduce the per disk lengths \( l' = L' / N \), \( l'_\text{max} = L'_\text{max} / N, l = L / N \), and employ formulas \[ 5 \] and \[ 6 \] in \( I \) \[ 7 \] which gives (see Appendix A for details):

\[ I = \int d\alpha d\beta e^{iN(\alpha l' + \beta)} \times \prod_{j=1}^{N-1} \left( 1 + N \int_{\sigma_m}^1 d\bar{\sigma} \int_0^1 d\nu' \exp N \varphi_j \right), \quad (10) \]
where
\[ \varphi_j = -i\alpha \nu'_j \sigma_j - i\beta \nu'_j - \nu'_j \ln \nu'_j + \nu'_j + \nu'_j \ln N. \]  \hspace{1cm} (11)

The integrals over \( \nu', \alpha, \beta, \) and \( l' \) have the large factor \( N \) in the exponents of their integrands which allows us to calculate the PF by the steepest descent method. As the result, in the limit \( N \rightarrow \infty, \) the principal contribution to the PF comes from the saddle point. We shall see that it is convenient to introduce \( a = -i\alpha \) and \( b = -i\beta \) which are real since \( \alpha \) and \( \beta \) at the saddle point both lie on the imaginary axes and both integration contours has to be (and can be) properly deformed. Then the \( \nu' \) integral in \( I \) above is determined by the real \( \nu'_{j, \text{max}} = e^{a\sigma_j + b}/N \) which maximizes \( \varphi_j. \) The factor \( N \) cancels out because the pre-exponential factor \( \sqrt{2\pi/N} \partial^2 \varphi_j / \partial \nu'_j^2 \ll 1/N, \) whence (Appendix B)
\[ N \int_0^1 d\nu' e^{N\varphi_j} = \sqrt{2\pi e^{(a\sigma_j + b)/2+\exp(a\sigma_j + b)}}. \]  \hspace{1cm} (12)

**Free energy.** On substituting this \( I \) to \( [3] \) we integrate over \( L' \) and get the PF in the following form:
\[ Z = \int_{\sigma_m}^{\nu'_m} dl' \int da \int d\nu' e^{N s}, \]  \hspace{1cm} (13)
where the factor \( N^{N+1} \) is omitted and
\[ s = -al' - b + \ln(l - l') \]  \hspace{1cm} (14)
\[ + \ln \left[ 1 + \sqrt{2\pi} \int_{\sigma_m}^{\nu'_m} d\sigma e^{(a\sigma + b)/2+\exp(a\sigma + b)} \right] . \]

Now we can compute the PF \([13] \) by the steepest descent method. The saddle point is determined by the maximum of the function \( s \) \([13] \) which, for given \( l, \) depends on \( a, b, \) and \( l'. \) The saddle point can be found from the equations \( \partial s / \partial a = \partial s / \partial b = \partial s / \partial l' = 0. \) To present these equations we introduce the two functions of \( \sigma: \)
\[ u = a\sigma + b, \]  \hspace{1cm} (15)
\[ f_\sigma = \frac{\sqrt{2\pi} e^{u/2+\exp u} (e^u + \frac{1}{2}) (1 - \sigma)^{-1/2}}{1 + \sqrt{2\pi} \int_{\sigma_m}^{\nu'_m} \frac{d\sigma}{\sqrt{1 - \sigma^2}} e^{u/2+\exp u}}. \]  \hspace{1cm} (16)

Then the equations of saddle point reduce to the two equations which read:
\[ \int_{\sigma_m}^{\nu'_m} d\sigma f_\sigma(\sigma, a, b) = 1, \]  \hspace{1cm} (17)
\[ \int_{\sigma_m}^{\nu'_m} d\sigma f_\sigma(\sigma, a, b) \sigma = l + 1/a. \]  \hspace{1cm} (18)

The solution \( \sigma, \nu'_m \) of the system \([17] - [18] \) depends on the per disk pore length \( l \) and, via \( \sigma_m, \) on the pore width \( D, \) and fully determines the FE. The FE \( F \) per disk, which therefore is the function of the length \( l, \) width \( D, \) and the temperature \( T, \) is \( F(l, D, T) = -TS(\sigma, \nu'_m) = -TS \) where \( S \) is system’s per disk entropy (up to terms independent of \( L \) and \( D)):\n\[ S = -\overline{\sigma} - \overline{\nu} + 2 \ln(l - \overline{\sigma}) + \ln \left[ 1 + \sqrt{2\pi} \int_{\sigma_m}^{\nu'_m} \frac{d\sigma}{\sqrt{1 - \sigma^2}} e^{(\overline{\sigma} + \overline{\nu})/2+\exp(\overline{\sigma} + \overline{\nu})} \right] , \]  \hspace{1cm} (19)
where we introduced the important quantity \( \overline{\sigma} = l + 1/\pi. \)

**Pressure and disks’ arrangement in the pore.** The system is anisotropic and has two different pressures: the longitudinal, \( P_L = -(\partial F/\partial l)/D/DT, \) and the transverse, \( P_D = -(\partial F/\partial D)/DT. \) These pressures were obtained numerically from the system \([17] - [19] \) as functions of the linear density \( \rho = N/L \) for three different pore widths: \( \Delta = 0.141 \) close to the 1D case, 0.5, and \( \sqrt{3}/2 \approx 0.866, \) the maximum width in the q1D system; the quantities \( P_L \) and \( P_D \) are shown in Fig.2. In Figs.2a and 2b also presented is the contribution of the term \( \ln(l - \overline{\sigma}) \) alone which is the Tonks’ 1D longitudinal pressure with \( \overline{\sigma} \) in place of the unity (see below).

The function \( f_\sigma(\sigma) \) \([16] \) presents the distribution of the longitudinal contact distances \( \sigma, \) eq. \([11] \), and eq. \([18] \) gives its mean value \( \overline{\sigma}. \) This \( \overline{\sigma} \) is growing with \( l \) and for \( l \sim 3 \) practically attains its maximum limiting value \( \overline{\sigma}_\infty = \overline{\sigma}(l \rightarrow \infty). \) The limiting value \( \overline{\sigma}_\infty \) is larger for smaller \( \Delta \) but remains below 1 for all \( \Delta > 0 \) \( (\overline{\sigma}_\infty = 0.854, 0.957, 0.997 \) for \( \Delta = 0, 0.5, 0.141 \) respectively), and only for \( \Delta = 0, \) i.e., in the 1D case, \( \overline{\sigma} = \overline{\sigma}_\infty = 1. \) This shows that the discontinuity in \( L' \) at the upper \( L' \) integration limit \( L'_{\text{max}} \) is never attained and thus does not manifest itself. At the same time \( \overline{\nu} \) is the mean per disk length of the relevant aggregate by which the system’s per disk length \( l \) is reduced in the term \( \ln(l - \overline{\sigma}) \) in \( S. \) It turns out that as \( l \rightarrow \infty, \) \( \overline{\sigma} \) and \( \overline{\nu} \) attain certain constant values so that in this limit \( S \) tends to the sum of this term plus certain \( l \) independent constant. As a result, for \( l \gg 1, \) \( P_L D \sim (l - \overline{\sigma}_\infty)^{-1} \) and \( P_D L \) becomes \( l \) independent, Fig.2.

The distributions \( f_\sigma(\sigma) \) for \( \Delta = 0.5 \) is shown in Fig.3 for different densities \( \rho = N/L. \) The \( \sigma \) distribution has an important peculiarity: it has two peaks, one at the smallest \( \sigma = \sigma_m, \) and another one at the largest \( \sigma = 1, \) and a flat minimum in between. Consider the case \( \Delta = 0.5 \) recently studied numerically by Huerta et al \([10] \). At a large density \( \sigma_m \) dominates implying that disks contact the opposite walls making a solidlike zigzag. At the same time, \( \sigma = 1 \) indicates that some disks can move across the pore through windows between the zigzags. The second peak appears quite sharply in terms of the density variation, but not abruptly: it is present for any \( \rho, \) but becomes visible at about \( \rho \approx 1.111, \) Fig.4. For \( \rho > 1.111 \) the \( \sigma_m \) peak dominates, at \( \rho \sim 1.06 \) the \( \sigma = 1 \) becomes well visible, then it grows and for \( \rho < 1 \)
FIG. 2: The longitudinal, $P_L$, and transverse, $P_D$, pressures for three different widths $\Delta$: a) 0.5, b) 0.866, c) 0.141. The dash curves in a) and b) show the contribution solely from the term $\ln(L - \bar{m})$ with the relevant $\bar{m}$. It is not visibly deviates from $P_L$ for $\Delta = 0.141$.

FIG. 3: Distribution $f_\sigma$ of $\sigma$ in the condensate for $\Delta = 0.5$ and different densities $\rho$: 1) 1.14, 2) 1.111, 3) 1.056, 4) 1.01, 5) 0.909, 6) 0.79, 7) 0.5. Note that the last curve is very close to the limiting curve attained for $\rho \to 0$.

FIG. 4: Ratio $f_\sigma(\sigma = 1)/f_\sigma(\sigma = \sigma_m)$ as a function of the linear density $\rho$ for $\Delta = 0.5$.

becomes higher than $\sigma_m$ peak. This implies that at $\rho \sim 1.06$ an appreciable fraction of the zigzag arrangement is replaced by strings of disks with close $y$'s. At the same time, the distribution with two peaks indicates that these strings are mostly located close to the walls. Indeed, in that case moving one disk from a string to the opposite wall results in a direct string-to-zigzag transformation and contributes little to the distribution with intermediate $\sigma$'s. At lower $\rho < 1$, the $\sigma$ distribution becomes wide which shows that at low $\rho$ disks freely move between the walls as in an ideal gas. This picture is in a qualitative agreement with the Monte Carlo results of Refs. [9, 10].
The pair distribution function was found to have sharp peak at the contact distance \( \sigma_m \) at high density \( \rho = 1.11 \), then it widens and, for \( \rho \approx 1.05 \), develops second peak at the unit distance which then widens and becomes dominating for \( \rho = 0.91 \). The system behavior for \( \Delta = 0.141 \) and \( \Delta = 0.866 \) is qualitatively similar to that of \( \Delta = 0.5 \).

**Discussion.** As we said above, a very small and extremely narrow peak at \( \sigma = 1 \) exists at any, even very large density, Figs.3,4. This however does not indicate defects of the otherwise solid phase. Instead the above picture suggests that this peak is an essential part of the equilibrium state. At large \( \rho \), the disks choose to move closer to the walls to get compressed into solidlike zigzag array with the interparticle distance somewhat smaller than its average and \( \sigma \) smaller than \( \sigma \) in order to provide windows with \( \sigma \) close to unity. Through these windows the disks can interchange their vertical positions, extend their wondering to the total pore width and bring some entropy gain to the whole system. The two HDs in the window form a bound pair: the disks roll over each other’s surface and their positions are highly correlated. As the density drops, the correlation between the exchangees weakens, the pair dissociates into free disks which can travel across the pore independently, their number rises while the number of HDs at the walls diminishes, Fig.4. This picture invokes similarity with the continuous Kosterlitz-Thouless transition from solidlike to liquidlike phase of a crystal. The similarity is supported by the numerical findings by Huerta et al [10] that above \( \rho \approx 1.111 \) the longitudinal pair correlation drops as a power law whereas below this \( \rho \) it drops exponentially. Thus, our theory shows that the crossover between the solidlike zigzag and the liquidlike intermittence of zigzag and string arrangements is sharp in the scale of density variation, but continuous so that the thermodynamic potentials of the q1D system of HD’s do not have discontinuities. The last conclusion is similar to that achieved by Varga et al [9] based on the numerical study of a q1D HD system. We emphasize that the narrow peak at \( \sigma = 1 \) for any density is the effect which can be lost in a finite system: only an infinite system can provide the window with \( \sigma = 1 \) for whatever density as its size is negligible in the limit \( N \to \infty \).

Recently HDs in q1D geometry have received a great deal of interest and there is an indication that it will last. The transfer matrix method by Kofke and Prost is on the way of incorporating wider pores where the interaction includes more than one next neighbor [11,13]. Moreover, HDs in q1D geometry are nowadays considered in a wider aspect related to the glass transitions and HDs’ dynamics [10,15,16]. Our result derived without simplifying assumptions gives the direct method to get the thermodynamics of a q1D HD system for given \( \rho, L, D \). The \( \sigma \) distribution (10) derived here suggests a novel quantitative analysis of the solidlike-to-liquidlike transformation. The result can be helpful for further development of the physics of HD systems in low and higher dimensions.

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**Appendix A. Summation over different condensate structures and derivation of formula (10).**

Consider the summation over different \( \nu \)'s in (7) and changing it to the integration. One has:

\[
I = \sum_{\{\sigma_j,\nu_j\}} \int D\tilde{\sigma} \times w \times \delta \left( L' - \sum_{j=1}^{N-1} \sigma_j \nu_j \right) \\
\times \delta_{N-1, \sum_{j=1}^{N-1} \nu_j}(A1)
\]

The summation here is over all possible nonzero sequences \( \{\sigma_j,\nu_j\} \) of all lengths from 1 to \( N - 1 \). Namely, this sum runs over condensates with only one \( \sigma \), with two different \( \sigma \)'s, ..., with up to \( N - 1 \) different \( \sigma \)'s:

\[
I = \int d\tilde{\sigma} \sum_{\nu_1,\nu_2}^{N-1} w_1 \delta (L' - \nu_1 \sigma) \delta_{N-1,\nu_1+\nu_2} \\
+ \int d\tilde{\sigma}_1 d\tilde{\sigma}_2 \sum_{\nu_1,\nu_2=1}^{N-1} w_2 \delta (L' - \nu_1 \sigma_1 - \nu_2 \sigma_2) \delta_{N-1,\nu_1+\nu_2} \\
+ \ldots \\
+ \int D^k \tilde{\sigma} \sum_{\nu_1,\nu_2,\ldots,\nu_k=1}^{N-1} w_k \delta \left( L' - \sum_{j=1}^{k} \sigma_j \nu_j \right) \delta_{N-1, \sum_{j=1}^{N-1} \nu_j} \\
+ \ldots \\
+ \int D^{N-1} \tilde{\sigma} \sum_{\nu_1,\nu_2,\ldots,\nu_{N-1}=1}^{N-1} w_{N-1} \delta \left( L' - \sum_{j=1}^{N-1} \sigma_j \nu_j \right) \delta_{N-1, \sum_{j=1}^{N-1} \nu_j} (A2)
\]

where \( w_k = w(\nu_1, \ldots, \nu_k) \), eq.(8). Now we introduce \( \nu'_j = \nu_j / N \), change to the \( \nu' \) integration, and make use of the delta functions in the form (9) which gives:

\[
I = \int d\tilde{\sigma} e^{L_0 + iN\tilde{\sigma}} \left\{ 1 + \int d\tilde{\sigma} N \int_0^1 d\nu e^{N\varphi} \\
+ \int d\tilde{\sigma}_1 d\tilde{\sigma}_2 N^2 \int_0^1 d\nu_1 d\nu_2 e^{N(\varphi_1 + \varphi_2)} \\
+ \int D^k \tilde{\sigma} N^k \int_0^1 D^k \nu e^{N(\varphi_1 + \ldots + \varphi_k)} \\
+ \int D^{N-1} \tilde{\sigma} N^{N-1} \int_0^1 D^{N-1} \nu e^{N(\varphi_1 + \varphi_2 + \ldots + \varphi_{N-1})} \right\} 
\] (A3)
where \( \phi_j \) is given in eq. (12); the \( \alpha \) and \( \beta \) integrals with the unity added for convenience give zero. The above sum in the curly brackets is exactly equal to the product in formula (10). The idea of the above representation (A2) of the sum (A1) was formulated in Ref. [18] and is associated with the summation over the subspaces of the system’s phase space.

Appendix B. Calculation of the \( \nu' \) integral in Eq. (12).

Here, in the limit \( N \to \infty \), we calculate the integral

\[
J = N \int_0^1 d\nu' \exp N\phi, 
\]

where

\[
\phi = a\nu' + b\nu' - \nu' \ln \nu' + \nu' + \nu' \ln N. 
\]

The value \( \nu'_{\text{max}} \) of \( \nu' \) at which \( \phi \) has maximum, its value and the second derivative at \( \nu'_{\text{max}} \) are

\[
\nu'_{\text{max}} = \frac{1}{N} e^{a\sigma + b}, 
N\phi_{\text{max}} = e^{a\sigma + b}, 
N\phi''_{\text{max}} = -N^2 e^{-(a\sigma + b)}. 
\]

This shows that the integrand \( \exp N\phi \) has a very narrow maximum of the width \( \sim 1/N \) at \( \nu'_{\text{max}} \) which however lies at a very small distance \( \sim 1/N \) from the left end \( \nu = 0 \) of the integration range. Thus, the integrand is Gaussian, but the integration range is cut from the left at \( \nu' = 0 \).

We now show that this restriction is negligible and the principal contribution to the integral \( J \) is Gaussian.

One has:

\[
J = J_G - J_0 + O(1/N), 
\]

\[
J_G = 2N \int_{\nu'_{\text{max}}}^{\infty} d\nu' (-N\phi''_{\text{max}} \nu'^2/2), 
J_0 = N \int_{-\infty}^{\nu'_{\text{max}}} d\nu' \exp N\phi. 
\]

The first integral is Gaussian, i.e.,

\[
J_G = \sqrt{\frac{2\pi}{N|\phi''_{\text{max}}|}} e^{N\phi_{\text{max}}} = \sqrt{2\pi e^{(a\sigma + b)/2 + \exp(a\sigma + b)}}. 
\]

The second integral is evaluated like that:

\[
J_0 = \frac{1}{\phi'(0)} \left[ e^{N\phi(0)} + O \left( \frac{1}{\phi'(0)} \right) \right], 
\]

where \( \phi'(0) = (d\phi/d\nu')_{\nu = 0} = -\ln(0N) = \infty \). Thus, \( J_0 = 0 \) and \( J = J_G \) which proves formula (12).

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