Fault-Tolerant Modular Reconstruction of Rational Numbers

John Abbott
12th March 2013

Abstract

In this paper we present two efficient methods for reconstructing a rational number from several residue-modulus pairs, some of which may be incorrect. One method is a natural generalization of that presented by Wang, Guy and Davenport in [WGD1982] (for reconstructing a rational number from correct modular images), and also of an algorithm presented in [Abb1991] for reconstructing an integer value from several residue-modulus pairs, some of which may be incorrect.

1 Introduction

The problem of intermediate expression swell is well-known in computer algebra, but has been greatly mitigated in many cases by the use of modular methods. There are two principal techniques: those based on the Chinese Remainder Theorem, and those based on Hensel’s Lemma. In this paper we consider only the former approach.

Initially modular methods were used in cases where integer values were sought (e.g., for computing GCDs of polynomials with integer coefficients); the answer was obtained by a direct application of the Chinese Remainder Theorem. Then in 1981 Wang presented a method allowing the reconstruction of rational numbers [Wan1981] from their modular images: the original context was the computation of partial fraction decompositions. Wang’s idea was justified in a later paper [WGD1982] which isolated the rational number reconstruction algorithm from the earlier paper. More recently, Collins and Encarnación [CoEn1994] corrected a mistake in Wang’s paper, and described how to obtain an especially efficient implementation. Wang’s method presupposes that all residue-modulus pairs are correct; consequently, the moduli used must all be coprime to the denominator of the rational to be reconstructed.

A well-known problem of modular methods is that of bad reduction: this means that the modular result is not correct for some reason. Sometimes it will be obvious when the modular result is bad (and these can be discarded), but other times it can be hard to tell. The Continued Fraction Method for the
fault-tolerant reconstruction of integer values when some of the modular images may be bad was presented in [Abb1991].

In this paper we consider the problem of reconstructing a rational number from its modular images allowing for some of the modular images to be erroneous. We combine the corrected version of Wang’s algorithm with the Continued Fraction Method. Our resulting new FTRR Algorithm (see section 4) reconstructs rational numbers from several modular images allowing some of them to be bad. The FTRR Algorithm contains both old methods as special cases: when it is known that all residues are correct we obtain Wang’s method (as corrected in [CoEn1994]), and if the denominator is restricted to being 1 then we obtain the original Continued Fraction Method. Finally, we note that the correction highlighted in [CoEn1994] is a natural and integral part of our method.

Our FTRR Algorithm gives a strong guarantee on its result: if a suitable rational exists then it is unique and the algorithm will find it; conversely if no valid rational exists then the algorithm says so. However, the uniqueness depends on bounds which must be given in input, including an upper bound for the number of incorrect residues. Since this information is often not known, we present also the HRR Algorithm (see section 5) — it is a heuristic reconstruction technique based on the sample principles as FTRR. This heuristic variant is much simpler to apply since it requires only the residue-modulus pairs as input. It will find the correct rational provided the correct modular images sufficiently outnumber the incorrect ones; if this is not the case then HRR will usually return an indication of failure but it may sometimes reconstruct an incorrect rational.

In section 6 we briefly compare our HRR algorithm with the Error Tolerant Lifting Algorithm presented in [BDFP2012] which is based on lattice reduction, and which serves much the same purpose as HRR. We mention also some combinatorial reconstruction schemes (presented in [Abb1991]) which can be readily adapted to perform fault tolerant rational reconstruction.

1.1 Envisaged Setting

We envisage the computation of one or more rational numbers (e.g. coefficients of a polynomial) by chinese remainder style modular computations where not all cases of bad reduction can be detected. If we know upper bounds for numerator and denominator, and also for the number of bad residue-modulus pairs then we can apply the FTRR algorithm of section 4. Otherwise we apply the HRR algorithm of section 5. Naturally, in either case we require that the bad residue-modulus pairs are not too common.

When using FTRR we use the sufficient precondition (inequality (4)) to decide whether more residue-modulus pairs are needed; when we have enough pairs we simply apply the reconstruction algorithm to obtain the answer.

When using HRR, we envisage that the computation is organized as follows. Many modular computations are made iteratively, and every so often an attempt is made to reconstruct the sought after rational number(s). If the attempt fails,
further iterations are made. If the attempt succeeds then a check is made of the “convincing correctness” of the reconstructed rational; if the rational is not “convincing” then again further iterations are made.

The perfect reconstruction algorithm would require only the minimum number of residue-modulus pairs (thus not wasting “redundant” iterations), and never reconstructs an incorrect rational (thus not wasting time checking “false positives”). Our HRR algorithm comes close to having both characteristics.

2 Notation and Assumptions

We are trying to reconstruct a rational number, \( \frac{p}{q} \), from many residue-modulus pairs: \( x_i \mod m_i \) for \( i = 1, 2, \ldots, s \). For each index \( i \) satisfying \( qx_i \equiv p \mod m_i \), we say that \( x_i \) is a good residue and \( m_i \) is a good modulus; otherwise, if the equivalence does not hold, we call them a bad residue and a bad modulus.

For simplicity, we assume that the moduli \( m_i \) are pairwise coprime: this assumption should be valid in almost all applications. For clarity of presentation, it will be convenient to suppose that the moduli are labelled in increasing order so that \( m_1 < m_2 < \cdots < m_s \). For our algorithms to work well it is best if the moduli are all of roughly similar size; otherwise, in an extreme situation where there is one modulus which is larger than the product of all the other moduli, if that large modulus is bad then reconstruction cannot succeed.

We say that a rational \( \frac{p}{q} \) is normalized if \( q > 0 \) and \( \gcd(p, q) = 1 \).

2.1 Continued Fractions

Here we recall a few facts about continued fractions; proofs and further properties may be found in [HW1979], for instance.

Let \( x \in \mathbb{R} \); then \( x \) has a unique representation as a continued fraction:

\[
x = [a_0, a_1, a_2, \ldots] = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \cdots}}}
\]

where all \( a_j \in \mathbb{Z} \) and \( a_j > 0 \) for all \( j > 0 \).

The integers \( a_1, a_2, \ldots \) are called partial quotients. If \( x \in \mathbb{Q} \) then there are only finitely many partial quotients; otherwise there are infinitely many.

We define the \( k \)-th continued fraction approximant to be the rational \( \frac{r_k}{s_k} \) whose continued fraction is \([a_0, a_1, \ldots, a_k]\). These approximants give ever closer approximations to \( x \), that is the sequence \( |x - \frac{r_k}{s_k}| \) is strictly decreasing. We also have that:

\[
\begin{align*}
a_k r_{k-1} &\leq r_k < (a_k + 1) r_{k-1} \\
a_k s_{k-1} &\leq s_k < (a_k + 1) s_{k-1}
\end{align*}
\]

We recall here Theorem 184 from [HW1979] which will play a crucial role.

**Theorem 2.1** Let \( x \in \mathbb{R} \) and \( \frac{r}{s} \in \mathbb{Q} \). If \( |x - \frac{r}{s}| < \frac{1}{2s^2} \) then \( \frac{r}{s} \) appears as a continued fraction approximant to \( x \).
3 Main Proposition

Our main proposition provides the key to reconstructing a rational from a single residue-modulus pair, $X \mod M$.

**Proposition 3.1** Let $X \mod M$ be a residue-modulus pair; so $X, M \in \mathbb{Z}$ with $M \geq 2$. Let $P, Q \in \mathbb{N}$ be positive; these are bounds for numerator and denominator respectively. Suppose there exists a factorization $M = M_{\text{good}} M_{\text{bad}} \in \mathbb{N}$ such that $2PQM_{\text{bad}}^2 < M$, and suppose also that there exists a rational $p/q \in \mathbb{Q}$ with $|p| \leq P$ and $1 \leq q \leq Q$ which satisfies $p \equiv qX \mod M_{\text{good}}$. Then $p/q$ is unique, and is given by

$$\frac{p}{q} = X - M \cdot \frac{R}{S}$$

where $\frac{R}{S}$ is the last continued fraction approximant to $\frac{X}{M}$ with denominator $\leq QM_{\text{bad}}$; indeed, $\frac{R}{S}$ is also the last approximant with denominator $\leq M_{\text{good}}/2|p|$.

**Proof**

By hypothesis we have $p = qX - kM_{\text{good}}$ for some $k \in \mathbb{Z}$. Dividing by $qM$ we obtain:

$$\frac{p}{qM} = \frac{X}{M} - \frac{k}{qM_{\text{bad}}}$$

We shall write $\frac{R}{S}$ for the normalized form of $\frac{k}{qM_{\text{bad}}}$; thus $0 < S \leq qM_{\text{bad}} \leq QM_{\text{bad}}$.

Now, the relationship between $P, Q$ and $M_{\text{bad}}$ implies that $|p| < \frac{M_{\text{good}}}{2QM_{\text{bad}}}$.

We use this to estimate how well $\frac{R}{S}$ approximates $\frac{X}{M}$:

$$\left| \frac{X}{M} - \frac{R}{S} \right| = \left| \frac{p}{qM} \right| < \frac{1}{2QM_{\text{bad}}^2} \leq \frac{1}{2S^2}$$

We may now apply Theorem 184 from [HW1979] to see that $\frac{R}{S}$ is one of the continued fraction approximants for $\frac{X}{M}$. We show that $\frac{R}{S}$ is the last approximant with denominator not exceeding $QM_{\text{bad}}$.

We start by showing that if $\frac{r}{s}$ is any other rational number with $1 \leq s \leq \frac{M_{\text{good}}}{2|p|}$ then $\left| \frac{X}{M} - \frac{r}{s} \right| = \left| \frac{X}{M} - \frac{r}{s} \right| > \left| \frac{X}{M} - \frac{r}{s} \right|$. First note that:

$$\left| \frac{r}{s} - \frac{R}{S} \right| \geq \frac{1}{sS} \geq \frac{2|p|}{M_{\text{good}}} \cdot \frac{1}{qM_{\text{bad}}} = \frac{2|p|}{qM}$$

Whence $\left| \frac{X}{M} - \frac{r}{s} \right| = \left| \frac{X}{M} - \frac{r}{s} \right| - \left| \frac{r}{s} - \frac{R}{S} \right| \geq \left| \frac{R}{S} \right| = \left| \frac{X}{M} - \frac{R}{S} \right|$. Now, any approximant coming after $\frac{R}{S}$ will be closer to $\frac{X}{M}$, so it must have denominator greater than $M_{\text{good}}/2|p|$.

The claim that $\frac{p}{q} = X - M \cdot \frac{R}{S}$ follows immediately from equation (3). □
Corollary 3.2 Let $j$ be the index of the approximant $\frac{R}{S}$ in Proposition 3.1. Let $M_\text{good}^* = \gcd(p - qX, M)$, and $M_\text{bad}^* = \frac{M}{M_\text{good}^*}$. Let $Q_{\text{max}}$ be the greatest integer strictly less than $\frac{M_\text{good}^*}{2|p|M_\text{bad}^*}$. Then the $(j+1)$-th partial quotient is at least $Q_{\text{max}} - 1$.

If $M_\text{good}^* \geq 2|p|qM_\text{bad}^*(\max(2|p|, q)M_\text{bad}^* + 2)$ then the $(j+1)$-th partial quotient is the largest of all.

Proof

Observe that $M_\text{good}^* \geq M_\text{good}$ and $M_\text{bad}^* \leq M_\text{bad}$ regardless of the original factorization $M = M_\text{good}M_\text{bad}$ used in the proposition.

By applying the proposition with $P = |p|$ and $Q = q$, and using the factorization $M = M_\text{good}^*M_\text{bad}^*$ we see that $S \leq qM_\text{bad}^*$; furthermore the $(j+1)$-th approximant has denominator greater than $M_\text{good}^*/2|p| > Q_{\text{max}}M_\text{bad}^*$. Thus by the final inequality of formula (2) the $(j+1)$-th partial quotient must be at least $Q_{\text{max}} - 1$.

Since $S$, the denominator of the $j$-th approximant, is at most $qM_\text{bad}^*$ no partial quotient with index less than or equal to $j$ can exceed $qM_\text{bad}^*$. Also, since the denominator of the $(j+1)$-th approximant is greater than $M_\text{good}^*/2|p|$ and the denominator of the final approximant is at most $M$, every partial quotient with index greater than $j + 1$ is less than $2|p|/M_\text{bad}^*$.

The hypothesis relating $M_\text{good}^*$ to $M_\text{bad}^*$ thus guarantees that the $(j+1)$-th partial quotient is the largest.

Example

Let $X = 7213578109$ and $M = 101 \times 103 \times 105 \times 107 \times 109$. Let $P = Q = 100$. By magic we know that $M_\text{bad} = 101$, so we seek the last approximant to $\frac{X}{M}$ with denominator at most $QM_\text{bad} = 10100$. It is the 10-th approximant and has value $\frac{R}{S} = 2116/3737$. Hence the candidate rational is $\frac{R}{q} = X - M \cdot \frac{R}{S} = 13/37$ which does indeed satisfy the numerator and denominator bounds. The next approximant has denominator 9701939 > $M_\text{good}^*/2|p|$. And the 11-th partial quotient is 2596 > $\frac{Q_{\text{max}}}{q} - 1$ as predicted by the corollary.

4 The Fault Tolerant Rational Reconstruction Algorithm

We present our first algorithm for reconstructing rational numbers based on Proposition 3.1. The algorithm expects as inputs:

- a set of residue-modulus pairs $\{x_i \mod m_i : i = 1, \ldots, s\}$,
- upper bounds $P$ (for the numerator), and $Q$ (for the denominator) of the rational to be reconstructed,
- an upper bound $e$ for the number of bad residue-modulus pairs.
We recall that the moduli $m_i$ are coprime, and are ordered so that $m_1 < m_2 < \cdots < m_s$. We define $M_{\text{max}} = m_{s-e+1}m_{s-e+2}\cdots m_s$. So that we can apply the proposition we require that

$$M = m_1m_2\cdots m_s > 2PQ M_{\text{max}}^2$$

Comparing this with the condition given in [WGD1982] we see that an extra factor of $M_{\text{max}}^2$ appears: this is to allow for a loss of information “up to $M_{\text{max}}$”, and to allow for an equivalent amount of redundancy requisite for proper reconstruction. If the denominator bound $Q = 1$ then the precondition (1) simplifies to that for the Continued Fraction Method [Abb1991].

### 4.1 The FTRR Algorithm

The main loop of this algorithm is quite similar to that in [WGD1982]: it just runs through the continued fraction approximants for $X/M$, and selects the last one with “small denominator”; there is a simple final computation to produce the answer.

1. **Input** $e$, $P$, $Q$, and $\{x_i \mod m_i : i = 1, \ldots, s\}$
2. If $x_i \equiv 0 \mod m_i$ for at least $s - e$ indices $i$ then return 0.
3. Set $M = \prod m_i$. Use Chinese remaindering to compute $X \mod M$ from $x_i \mod m_i$.
4. Compute $M_{\text{max}} = m_{s-e+1}m_{s-e+2}\cdots m_s$.
5. If $\gcd(X, M) > PM_{\text{max}}$ then return failure.
6. Put $u = (1, 0, M)$ and $v = (0, 1, X)$.
7. While $|v_2| \leq QM_{\text{max}}$ do
   7.1 $q = \lfloor u_3/v_3 \rfloor$
   7.2 $u = u - qv$; swap $u \leftrightarrow v$
8. Set $r = X + M \cdot \frac{u_2}{u_3}$ as a normalized rational.
9. Check whether $r$ is a valid answer:
   \[i.e. |\text{num}(r)| \leq P\] and $\text{den}(r) \leq Q$ and at most $e$ bad moduli.
10. If $r$ is valid, return $r$; otherwise return failure.

**Note** that in the algorithm the successive values of $-\frac{u_2}{u_3}$ at the end of each iteration around the main loop are just the continued fraction approximants to $X/M$.

**Example** For some inputs to algorithm $FTRR$ there is no valid answer. If the input parameters are $e = 0$, $P = Q = 1$ and $x_1 = 2$ with modulus $m_1 = 5$ then with the given bounds the only possible valid answers are $\{-1, 0, 1\}$ but 2 mod 5 does not correspond to any of these.
4.2 Correctness of FTRR Algorithm

We show that FTRR finds the right answer if it exists, and otherwise it produces \textit{failure}.

We first observe that if the correct result is 0 and at most \(e\) residue-modulus pairs are faulty then step (2) detects this, and rightly returns 0. We may henceforth assume that the correct answer, if it exists, is a non-zero rational \(\frac{p}{q} \in \mathbb{Q}\) with \(|p| \leq P\) and \(1 \leq q \leq Q\).

\textbf{Lemma 4.1} If there is a valid non-zero solution \(p/q\) then \(\gcd(X, M) \leq PM_{\text{max}}\).

\textbf{Proof} As the \(m_i\) are coprime \(\gcd(X, M) = \prod \gcd(x_i, m_i)\). If the modulus \(m_i\) is good then \(\gcd(x_i, m_i)|p\); conversely if \(\gcd(x_i, m_i)|p\) then \(m_i\) is a bad modulus. Hence \(\prod_{m_i, \text{good}} \gcd(x_i, m_i)|p\); while \(\prod_{m_i, \text{bad}} \gcd(x_i, m_i) \leq \prod_{m_i, \text{bad}} m_i \leq M_{\text{max}}\). It is now immediate that \(\gcd(X, M) \leq PM_{\text{max}}\). \hfill \(\Box\)

From the lemma we deduce that the check in step (5) eliminates only \((X, M)\) pairs which do not correspond to a valid answer. We also observe that for all \((X, M)\) pairs which pass the check in step (5) the denominator of the normalized form of \(X/M\) is at least \(2QM_{\text{max}}\), so the loop exit condition in step (7) will eventually trigger.

The values \(X\) and \(M\) computed in step (3) are precisely the corresponding values in the statement of Proposition \textbf{3.1}. However, we do not know the correct factorization \(M = M_{\text{good}}M_{\text{bad}}\); but since there are at most \(e\) bad residue-modulus pairs we do know that \(M_{\text{bad}} \leq M_{\text{max}}\), and this inequality combined with the requirement \textbf{4} together imply that \(2PQM_{\text{bad}}^2 < M\) so we may apply the proposition. Thus the algorithm simply has to find the last continued fraction approximant \(\frac{R}{S}\) with denominator not exceeding \(QM_{\text{max}}\), which is precisely what the main loop does: at the end of each iteration \(-\frac{u_2}{v_2}\) and \(-\frac{u_1}{v_1}\) are successive approximants to \(\frac{X}{M}\), and the loop exits when \(|v_2| > QM_{\text{max}}\).

So when execution reaches step (8), the fraction \(-\frac{u_1}{v_1}\) is precisely the approximant \(\frac{R}{S}\) of the proposition. Thus step (8) computes the candidate answer in \(r\), and step (9) checks that the numerator and denominator lie below the bounds \(P\) and \(Q\), and that there are no more than \(e\) bad moduli. If the checks pass, the result is valid and is returned; otherwise the algorithm reports \textit{failure}.

4.3 Which Residues were Faulty?

Assume the algorithm produced a normalized rational \(p/q\), and we want to determine which moduli (if any) were faulty. We could simply check which images of \(p/q\) modulo each \(m_i\) are correct. However, there is another, more direct way of identifying the bad moduli: we show that the bad \(m_i\) are exactly those which have a common factor with \(S\), that is the final value of \(u_2\).

If \(m_i\) is a good modulus then we have \(\gcd(m_i, q) = 1\) because otherwise if the gcd, \(h\), were greater than 1 then \(p \equiv qx_i \mod m_i\) implies that \(h\) divides \(p\), contradicting the assumption that \(p\) and \(q\) are coprime.
Multiplying equation 3 from the proof of Proposition 3.1 by $qM$ we obtain $p = qX - M \cdot \frac{2}{S}$ whence $M \cdot \frac{2}{S}$ is an integer. By definition of a bad modulus $m_i$ we must have $M \cdot \frac{2}{S} \not\equiv 0 \mod m_i$. Since $m_i | M$, we must have $\gcd(m_i, S) > 1$.

5 The Heuristic Algorithm

The main problem with the FTRR Algorithm is that we do not generally know good values for the input bounds $P, Q$ and $e$. In this heuristic variant the only inputs are the residue-modulus pairs; the result is either a rational number or an indication of failure. The algorithm is heuristic in that it may (rarely) produce an incorrect result, though if sufficiently many residue-pairs are input (with fewer than $\frac{1}{3}$ of them being bad) then the result will be correct.

5.1 Algorithm HRR: Heuristic Rational Reconstruction

1. Input $x_i \mod m_i$ for $i = 1, \ldots, s$. Set $A_{\text{crit}} = 10^6$ (see note below).
2. Put $M = \prod m_i$. Use Chinese remaindering to compute $X \in \mathbb{Z}$ such that $|X| < M$ and $X \equiv x_i \mod m_i$.
3. If $\gcd(X, M)^2 > A_{\text{crit}}M$ then return 0.
4. Let $A_{\text{max}}$ be the largest partial quotient in the continued fraction of $X/M$.
   If $A_{\text{max}} < A_{\text{crit}}$ then return failure.
5. Put $u = (1, 0, M)$ and $v = (0, 1, X)$, and set $q = 0$.
6. While $q \neq A_{\text{max}}$ do
   6.1 $q = \lfloor u_3/v_3 \rfloor$
   6.2 $r = u - qv; u = v; v = r$
7. Return $N/D$ the normalized form of $X + Mu_1/u_2$; we could also return $M_{\text{bad}} = \gcd(M, u_2)$.

The idea behind the algorithm is simply to exploit Corollary 5.2 algorithmically. This corollary tells us that, provided $M_{\text{good}}$ is large enough relative to $M_{\text{bad}}$, we can reconstruct the correct rational from the last approximant before the largest partial quotient. Moreover if the proportion of residue-modulus pairs which are bad is less than $\frac{1}{3}$ then $M_{\text{good}}$ will eventually become large enough.

Since zero requires special handling, there is a special check in step (3) for this case. The heuristic will produce zero if “significantly more than half of the residues” are zero — strictly this is true only if all the moduli are prime and of about the same magnitude.

To avoid producing too many false positives we demand that the largest partial quotient be greater than a certain threshold, namely $A_{\text{crit}}$. The greater the threshold, the less likely we will get a false positive; but too great a value will
delay the final recognition of the correct value. The suggested value $A_{crit} = 10^6$ worked well in our trials.

**Alternative criterion for avoiding false positives**

Our implementation in CoCoALib actually uses a slightly different convincingness criterion in step (4). Let $A_{max}$ be the largest partial quotient, and $A_{next}$ the second largest. Our alternative criterion is to report failure if $A_{max}/A_{next}$ is smaller than a given threshold — in our trials a threshold value of 4096 worked well, but our implementation also lets the user specify a different threshold.

**5.1.1 Complexity of HRR**

Under the natural assumption that each residue satisfies $|x_i| \leq m_i$, we see that the overall complexity of algorithm HRR is $O((\log M)^2)$, the same as for Euclid’s algorithm. Indeed the chinese remaindering in step (2) can be done with a modular inversion (via Euclid’s algorithm) and two products. The computation of the partial quotients in step (4) is Euclid’s algorithm once again. And the main loop in step (6) is just Euclid’s algorithm in reverse.

We note that the overall computational cost depends on how often HRR is called in the envisaged lifting loop (see subsection [1.1]). Assuming that the moduli chosen are all about the same size, a reasonable compromise approach is a “geometrical strategy” where HRR is called whenever the number of main iterations reaches the next value in a geometrical progression. This compromise avoids excessive overlifting and also avoids calling HRR prematurely too often. The overall cost of HRR with such a strategy remains $O((\log M)^2)$ where $M$ here denotes the combined modulus in the final, successful call to HRR.

In practice, if the cost of calling HRR is low compared to the cost of one modular computation in the main loop then it makes sense to call HRR frequently. The geometrical strategy should begin only when (if ever) the cost of a call to HRR is no longer relatively insignificant.

**6 Comparison with Other Methods**

**6.1 Reconstruction via Lattice Reduction**

A reconstruction technique based on 2-dimensional lattice reduction is presented as Algorithm 6 Error Tolerant Lifting (abbr. ETL) in [BDFP2012]. This algorithm is similar in scope to our HRR, and not really comparable to our FTRR algorithm (which needs extra inputs from the user).

In practice there are two evident differences between ETL and our HRR. The first is that ETL produces many more false positives than HRR; our refinement (B) below proposes a way to rectify this. The second is that ETL finds balanced rationals more easily than unbalanced ones, i.e. it works best if the numerator and denominator contain roughly the same number of digits. For balanced rationals, ETL and HRR need about the same number of residue-modulus pairs;
for unbalanced rationals ETL usually needs noticeably more residue-modulus pairs than HRR.

6.1.1 Practical Refinements to ETL

We propose two useful refinements to ETL as it is described in [BDFP2012].

A We believe that a final checking step should be added to the ETL algorithm so that it rejects results where half or more of the moduli are bad. Consider the following example: the moduli are 11, 13, 15, 17, 19 and the corresponding residues are \(-4, -4, -4, 1, 1\). The rather surprising result produced by ETL is 1; it seems difficult to justify this result as being correct. Here we see explicitly the innate tendency of ETL to favour “trusting” larger moduli over smaller ones.

B The aim of our other refinement is to reduce the number of false positives which ETL produces. We suggest replacing their acceptance criterion $a_{i+1}^2 + b_{i+2}^2 < N$ by a stricter condition such as $a_{i+1}^2 + b_{i+2}^2 < N/100$. This change may require one or two more “redundant” residue-modulus pairs before ETL finds the correct answer, but it does indeed eliminate most of the false positives.

6.1.2 Comparison of Efficiency

We have implemented HRR and ETL in CoCoALib and CoCoA-5. Using these implementations we compared the efficiency of HRR and ETL by generating a random rational $N/D$ (with a specified number of bits each for the numerator and denominator), and then generating the modular images $x_i \mod m_i$ where the $m_i$ run through successive primes starting from 1013. Note that in this trial there are no bad residue-modulus pairs. We then counted how many residue-modulus pairs were needed by the algorithms before they were able to reconstruct the original rational.

We then repeated the experiment but this time, with probability 10%, each residue was replaced by a random value to simulate the presence of bad residues. As expected, the number of pairs needed by HRR is essentially constant, while ETL matches the efficiency of HRR only for perfectly balanced rationals; as soon as there is any disparity between the sizes of numerator and denominator ETL becomes significantly less efficient.

In each case the successful reconstruction took less than 0.1 seconds.

|            | 2000/0 bits | 1600/400 bits | 1200/800 bits | 1000/1000 bits |
|------------|-------------|---------------|---------------|---------------|
| HRR 0% bad | 190         | 191           | 190           | 190           |
| ETL 0% bad | 361         | 293           | 224           | 189           |
| HRR 10% bad| 244         | 236           | 246           | 244           |
| ETL 10% bad| 457         | 375           | 283           | 242           |
6.2 Combinatorial Methods

It is shown in [Sto1963] that reconstruction of integers by Chinese Remaindering is possible provided no more than half of the redundant residues are faulty. The correct value is identified using a voting system (see [Sto1963] for details). We can extend the idea of a voting system to allow it to perform fault tolerant rational reconstruction: the only difference is that for each subset of residue-modulus pairs we effect an exact rational reconstruction (rather than an exact integer reconstruction). However the problem of poor computational efficiency remains.

An elegant and efficient scheme for fault-tolerant chinese remaindering for integers was given in [Ram1983]; however the method is valid only for at most one bad modulus. Several generalizations of Ramachandran’s scheme were given in [Abb1991]; however, these are practical really only for at most 2 bad moduli. Like the voting system, these schemes could be easily adapted to perform fault-tolerant rational reconstruction, but in the end the Continued Fraction Method (upon which FTRR is based) is more flexible and more efficient.

7 Conclusion

We have presented two new algorithms for solving the problem of fault tolerant rational reconstruction, FTRR and HRR. The former is a natural generalization both of the original rational reconstruction algorithm [WGD1982] and of the fault tolerant integer reconstruction algorithm [Abb1991]. The latter is a heuristic variant which is easier to use in practice since it does not require certain bounds as input.

Our HRR algorithm and the ETL algorithm from [BDFP2012] offer two quite distinct (yet simple) approaches to the same problem. They have comparable practical efficiency when reconstructing balanced rationals, whereas HRR is usefully more efficient when reconstructing unbalanced rationals.

References

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