2-Local derivations on matrix algebras over semi-prime Banach algebras and on $AW^*$-algebras

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Abstract. The paper is devoted to 2-local derivations on matrix algebras over unital semi-prime Banach algebras. For a unital semi-prime Banach algebra $A$ with the inner derivation property we prove that any 2-local derivation on the algebra $M_{2n}(A)\,;\, n \geq 2$ is a derivation. We apply this result to $AW^*$-algebras and show that any 2-local derivation on an arbitrary $AW^*$-algebra is a derivation.

1. Introduction

Given an algebra $\mathcal{A}$, a linear operator $D : \mathcal{A} \to \mathcal{A}$ is called a derivation, if $D(xy) = D(x)y + xD(y)$ for all $x, y \in \mathcal{A}$ (the Leibniz rule). Each element $a \in \mathcal{A}$ implements a derivation $D_a$ on $\mathcal{A}$ defined as $D_a(x) = ad(a)(x) = ax - xa, \, x \in \mathcal{A}$. Such derivations $D_a$ are said to be inner derivations.

In 1997, P. Semrl [10] introduced the concepts of 2-local derivations and 2-local automorphisms. Recall that a map $\Delta : \mathcal{A} \to \mathcal{A}$ (not linear in general) is called a 2-local derivation if for every $x, y \in \mathcal{A}$, there exists a derivation $D_{x,y} : \mathcal{A} \to \mathcal{A}$ such that $\Delta(x) = D_{x,y}(x)$ and $\Delta(y) = D_{x,y}(y)$. In particular, he has described 2-local derivations on the algebra $B(H)$ of all bounded linear operators on the infinite-dimensional separable Hilbert space $H$. In [8] S.O.Kim and J.S.Kim gave a short proof that every 2-local derivation on a finite-dimensional complex matrix algebras is a derivation. The methods of the proofs above mentioned results from [8] and [10] are essentially based on the fact that the algebra $B(H)$ is generated by two elements for separable Hilbert space $H$. Later J. H. Zhang and H. X. Li [11] have extended the above mentioned result of [8] for arbitrary symmetric digraph matrix algebras and constructed an example of 2-local derivation which is not a derivation on the algebra of all upper triangular $2 \times 2$-matrices.

As it was mentioned above, the proofs of the papers [8] and [10] are essentially based on the fact that the algebra $B(H)$ is generated by two elements for separable Hilbert space $H$. Since the algebra $B(H)$ is not generated by two elements for non separable $H$, one cannot directly apply the methods of the above papers in this case. In [2] the authors suggested a new technique and have generalized the above mentioned results of [8] and [10] for arbitrary Hilbert spaces. Namely, we considered 2-local derivations on the algebra $B(H)$ of all bounded linear operators on an arbitrary (no separability is assumed) Hilbert space $H$ and proved that every 2-local derivation on $B(H)$ is a derivation. A similar result for 2-local derivations on finite von
Neumann algebras was obtained in [5]. In [1] the authors extended all above results and give a short proof of this result for arbitrary semi-finite von Neumann algebras. Finally, in [3], using the analogue of Gleason Theorem for signed measures, we have extended this result to type III von Neumann algebras. This implies that on arbitrary von Neumann algebra each 2-local derivation is a derivation.

In the present paper we consider 2-local derivations on matrix algebras over unital semi-prime Banach algebras. Let $A$ be a unital semi-prime Banach algebra with the inner derivation property. We prove that any 2-local derivation on the algebra $M_{2^n}(A)$, $n \geq 2$, is a derivation. We also apply this result to $AW^*$-algebras and prove that any 2-local derivation on an arbitrary $AW^*$-algebra is a derivation.

2. 2-local derivations on matrix algebras

If $\Delta : A \to A$ is a 2-local derivation, then from the definition it easily follows that $\Delta$ is homogenous. At the same time,

$$\Delta(x^2) = \Delta(x)x + x\Delta(x) \quad (1)$$

for each $x \in A$.

In [6] it is proved that any Jordan derivation (i.e. a linear map satisfying the above equation) on a semi-prime algebra is a derivation. So, in the case semi-prime algebras in order to prove that a 2-local derivation $\Delta : A \to A$ is a derivation it is sufficient to prove that $\Delta : A \to A$ is additive.

We say that an algebra $A$ has the inner derivation property if every derivation on $A$ is inner. Recall that an algebra $A$ is said to be semi-prime if $aa = 0$ implies that $a = 0$.

The following theorem is the main result of this section.

**Theorem 2.1** Let $A$ be a unital semi-prime Banach algebra with the inner derivation property and let $M_{2^n}(A)$ be the algebra of $2^n \times 2^n$-matrices over $A$. Then any 2-local derivation $\Delta$ on $M_{2^n}(A)$ is a derivation.

The proof of Theorem 2.1 consists of two steps. In the first step we shall show additivity of $\Delta$ on the subalgebra of diagonal matrices from $M_{2^n}(A)$.

Let $M_n(A)$ be the algebra of $n \times n$-matrices over $A$ and let $\{e_{i,j}\}_{i,j=1}^n$ be the system of matrix units in $M_n(A)$. For $x \in M_n(A)$ by $x_{i,j}$ we denote the $(i, j)$-entry of $x$, i.e. $x_{i,j} = e_{i,i}xe_{i,j}$, where $1 \leq i,j \leq n$. We shall, when necessary, identify this element with the matrix from $M_n(A)$ whose $(i, j)$-entry is $x_{i,j}$, other entries are zero.

Further in Lemmata 2.2–2.6 we assume that $n \geq 2$.

**Lemma 2.2** Let $A$ be a unital Banach algebra with the inner derivation property. Then the algebra $M_n(A)$ also has the inner derivation property.

**Proof.** Let $D$ be a derivation on $M_n(A)$. Set

$$a = \sum_{i=1}^n D(e_{i,1})e_{1,i}.$$ 

We have

$$\sum_{i=1}^n D(e_{i,1})e_{1,i} + \sum_{i=1}^n e_{i,1}D(e_{1,i}) = \sum_{i=1}^n D(e_{i,1}e_{1,i}) = \sum_{i=1}^n D(e_{i,i}) = D\left(\sum_{i=1}^n e_{i,i}\right) = D(1) = 0,$$

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where 1 is the unit matrix. Therefore \( a = -\sum_{i=1}^{n} e_{i,1}D(e_{i,1}). \)

From the above equalities by direct calculations we obtain that

\[
 ae_{k,s} - e_{k,s}a = D(e_{k,1})e_{1,s} + e_{k,1}D(e_{1,s}) = D(e_{k,s}),
\]

i.e. \( D(e_{k,s}) = [a, e_{k,s}] \) for all \( k, s \in \{1, n\}. \)

Put \( D_0 = D - \text{ad}(a). \) Since \( D_0(e_{1,1}) = 0, \) it follows that \( D_0 \) maps \( e_{1,1}M_n(\mathcal{A})e_{1,1} \equiv \mathcal{A} \) into itself. Therefore the restriction \( D_0|_{\mathcal{A}} \) of \( D_0 \) onto \( \mathcal{A} \) is a derivation. Since \( \mathcal{A} \) has the inner derivation property there exists an element \( a_{1,1} \in \mathcal{A} \) such that \( D_0(x) = [a_{1,1}, x] \) for all \( x \in \mathcal{A}. \)

Set \( b = \sum_{i=1}^{n} e_{i,1}a_{1,1}e_{1,i}. \) Let \( x \) be a matrix such that \( x = e_{k,k}xe_{s,s}. \) Then

\[
 bx - xb = e_{k,1}a_{1,1}e_{1,k}x - x e_{s,1}a_{1,1}e_{1,s} = e_{k,1}a_{1,1, e_{1,k}xe_{s,1}]}e_{1,s},
\]

and

\[
 D_0(x) = D_0(e_{k,1}e_{1,k}xe_{s,1}e_{1,s}) = D_0(e_{k,1}) e_{1,k}xe_{s,1}e_{1,s} + e_{k,1}D_0(e_{1,k}xe_{s,1}) e_{1,s} + e_{k,1}e_{1,k}xe_{s,1}D_0(e_{1,s}) = 0 + e_{k,1}D_0(e_{1,k}xe_{s,1}) e_{1,s} + 0 = e_{k,1}a_{1,1, e_{1,k}xe_{s,1}]}e_{1,s},
\]

i.e. \( D_0(x) = [b, x] \) for all \( x \) of the form \( x = e_{k,k}xe_{s,s}. \) By linearity of \( D_0 \) we have that \( D_0 = \text{ad}(b). \) So, \( D = \text{ad}(a + b). \) The proof is complete. □

Consider the following two matrices:

\[
 u = \sum_{i=1}^{n} \frac{1}{2^n} e_{i,i}, \quad v = \sum_{i=2}^{n} e_{i-1,i}.
\]

It is easy to see that an element \( x \in M_n(\mathcal{A}) \) commutes with \( u \) if and only if it is diagonal, and if an element \( a \) commutes with \( v, \) then \( a \) is of the form

\[
 a = \begin{pmatrix}
 a_1 & a_2 & a_3 & \cdots & a_n \\
 0 & a_1 & a_2 & \cdots & a_{n-1} \\
 0 & 0 & a_1 & \cdots & a_{n-2} \\
 \vdots & \vdots & \vdots & \ddots & \vdots \\
 0 & 0 & \cdots & a_1 & a_2 \\
 0 & 0 & \cdots & 0 & a_1
\end{pmatrix}
\]

A result, similar to the following one, was proved in [4, Lemma 4.4] for matrix algebras over commutative regular algebras.

**Lemma 2.3** For every 2-local derivation \( \Delta \) on \( M_n(\mathcal{A}) \), there exists a derivation \( D \) such that \( \Delta|_{M_n(Z(\mathcal{A}))} = D|_{M_n(Z(\mathcal{A}))} \), where \( Z(\mathcal{A}) \) is the center of the algebra \( \mathcal{A}. \) In particular, \( \Delta|_{sp(e_{i,j})}_{i,j=1}^{n} = D|_{sp(e_{i,j})}_{i,j=1}^{n} \), where \( sp(e_{i,j})_{i,j=1}^{n} \) is the linear span of the set \( \{e_{i,j}\}_{i,j=1}^{n}. \)

**Proof.** By Lemma 2.2 there exists an element \( a \in M_n(\mathcal{A}) \) such that

\[
 \Delta(u) = [a, u], \quad \Delta(v) = [a, v],
\]

where \( u, v \) are the elements from (2). Replacing \( \Delta \) by \( \Delta - \text{ad}(a) \), if necessary, we can assume that \( \Delta(u) = \Delta(v) = 0. \)
Let $i, j \in \{1, n\}$. Take a matrix $h$ such that
\[ \Delta(e_{i,j}) = [h, e_{i,j}], \quad \Delta(u) = [h, u]. \]
Since $\Delta(u) = 0$, it follows that $h$ has diagonal form, i.e. $h = \sum_{i=1}^{n} h_{i,i}$. So we have
\[ \Delta(e_{i,j}) = he_{i,j} - e_{i,j}h. \]
In the same way, but starting with the element $v$ instead of $u$, we obtain
\[ \Delta(e_{i,j}) = be_{i,j} - e_{i,j}b, \]
where $b$ has the form (3), depending on $e_{i,j}$. So
\[ \Delta(e_{i,j}) = he_{i,j} - e_{i,j}h = be_{i,j} - e_{i,j}b. \]
Since $\Delta(e_{i,j}) = 0$, it follows that $\Delta(e_{i,j}) = 0$.

Now let us take a matrix $x = \sum_{i,j=1}^{n} f_{i,j}e_{i,j} \in M_n(Z(A))$. Then
\[ e_{i,j}\Delta(x)e_{i,j} = e_{i,j}De_{i,j}x(e_{i,j}) = D(e_{i,j})xe_{i,j}e_{i,j} - e_{i,j}xe_{i,j}D(e_{i,j}) = D(e_{i,j})f_{j,i}e_{i,j} - 0 - 0 = f_{j,i}D(e_{i,j})e_{i,j} = 0, \]
i.e. $e_{i,j}\Delta(x)e_{i,j} = 0$ for all $i, j \in \{1, n\}$. This means that $\Delta(x) = 0$. The proof is complete. \hfill \Box

Further in Lemmata 2.4–2.9 we assume that $\Delta$ is a 2-local derivation on the algebra $M_n(A)$, such that $\Delta|_{sp(e_{i,j})}^{n}_{i,j=1} = 0$.

Let $\Delta_{i,j}$ be the restriction of $\Delta$ onto $M_{i,j} = e_{i,i}M_n(A)e_{j,j}$, where $1 \leq i, j \leq n$.

**Lemma 2.4** $\Delta_{i,j}$ maps $M_{i,j}$ into itself.

**Proof.** First, let us show that
\[ \Delta_{i,j}(x) = e_{i,i}\Delta(x)e_{j,j} \tag{4} \]
for all $x \in M_{i,j}$.

Let $x = x_{i,j} \in M_{i,j}$. Take a derivation $D$ on $M_n(A)$ such that
\[ \Delta(x) = D(x), \quad \Delta(e_{i,j}) = D(e_{i,j}). \]
It is suffices to consider the following two cases.

Case 1. Let $i = j$. Then
\[ \Delta_{i,i}(x) = \Delta(x) = D(x) = D(e_{i,i}xe_{i,i}) = D(e_{i,i})xe_{i,i} + e_{i,i}D(x)e_{i,i} + e_{i,i}xD(e_{i,i}) = 0 + e_{i,i}\Delta(x)e_{i,i} + 0 = e_{i,i}\Delta(x)e_{i,i}. \]
Case 2. Let \( i \neq j \). Denote by \( 1 \) the unit matrix. Since \( e_{i,i}x(1 - e_{i,i}) = x \) and \( (1 - e_{j,j})xe_{j,j} = x \), we obtain that

\[
e_{i,i}\Delta(x)(1 - e_{i,i}) = e_{i,i}De_{i,i,x}(x)(1 - e_{i,i}) = De_{i,i,x}(e_{i,i})x(1 - e_{i,i}) - e_{i,i}xD_{e_{i,i},x}(1 - e_{i,i}) = De_{i,i,x}(x) - 0 - 0 = \Delta(x)
\]

and

\[
(1 - e_{j,j})\Delta(x)e_{j,j} = (1 - e_{j,j})De_{j,j,x}(x)e_{j,j} = De_{j,j,x}(1 - e_{j,j})xe_{j,j} - (1 - e_{j,j})xD_{e_{j,j},x}(e_{j,j}) = De_{j,j,x}(x) - 0 = \Delta(x).
\]

Hence

\[
e_{i,i}\Delta(x)e_{j,j} = (1 - e_{j,j})e_{i,i}\Delta(x)(1 - e_{i,i})e_{j,j} = (1 - e_{j,j})\Delta(x)e_{j,j} = \Delta(x).
\]

The proof is complete. \( \square \)

**Lemma 2.5** Let \( x = \sum_{i=1}^{n} x_{i,i} \) be a diagonal matrix. Then

\[
e_{k,k}\Delta(x)e_{k,k} = \Delta(x_{k,k})
\]

for all \( k \in \overline{1,n} \).

**Proof.** Take an element \( a \) from \( M_n(A) \) such that

\[
\Delta(x) = [a, x], \quad \Delta(x_{k,k}) = [a, x_{k,k}].
\]

Since \( x \) is a diagonal matrix, the equality (4) implies that

\[
\Delta(x_{k,k}) = e_{k,k}\Delta(x_{k,k})e_{k,k} = e_{k,k}[a, x_{k,k}]e_{k,k} = [a_{k,k}, x_{k,k}]
\]

and

\[
e_{k,k}\Delta(x)e_{k,k} = e_{k,k}[a, x]e_{k,k} = [a_{k,k}, x_{k,k}].
\]

Thus \( e_{k,k}\Delta(x)e_{k,k} = \Delta(x_{k,k}) \). The proof is complete. \( \square \)

**Lemma 2.6** Let \( x = x_{i,i} \in M_{i,i} \). Then

\[
e_{j,i}\Delta_{i,i}(x)e_{i,j} = \Delta_{j,j}(e_{j,i}xe_{i,j}).
\]

**Proof.** For \( i = j \) we have already proved (see Lemma 2.5).

Suppose that \( i \neq j \). For arbitrary element \( x = x_{i,i} \in M_{i,i} \), consider \( y = x + e_{j,i}xe_{i,j} \in M_{i,i} + M_{j,j} \). Take an element \( a \in A \) such that

\[
\Delta(x) = [a, y] \quad \text{and} \quad \Delta(v) = [a, v],
\]
where \( v \) is the element from (2). Since \( \Delta(v) = 0 \), it follows that \( a \) has the form (3). By Lemma 2.5 we obtain that

\[
e_{j,i}
abla_{i,i}(x)e_{i,j} = e_{j,i}e_{i,i}
abla(y)e_{i,i}e_{i,j} = e_{j,i}[a, y]e_{i,j} = e_{j,i}[a_1, x]e_{i,j}
\]

and

\[
\Delta_{j,j}(e_{j,i}x e_{i,j}) = e_{j,j}\Delta(y)e_{j,j} = e_{j,j}[a, y]e_{j,j} = e_{j,j}[a, x + e_{j,i}xe_{i,j}]e_{j,j} = e_{j,i}[a_1, x]e_{i,j}.
\]

The proof is complete. \( \Box \)

Further in Lemmata 2.7–2.9 we assume that \( n \geq 3 \).

**Lemma 2.7** \( \nabla_{i,i} \) is additive for all \( i \in \overline{1, n} \).

*Proof.* Let \( i \in \overline{1, n} \). Since \( n \geq 3 \), we can take different numbers \( k, s \) such that \( k \neq i, s \neq i \).

For arbitrary \( x, y \in M_{i,i} \), consider \( z = x + y + e_{k,i}xe_{i,k} + e_{s,i}ye_{i,s} \in M_{i,i} + M_{k,k} + M_{s,s} \). Take a matrix \( a \in M_{n}(\mathcal{A}) \) such that

\[
\Delta(z) = [a, z] \text{ and } \Delta(v) = [a, v],
\]

where \( v \) is the element from (2). Since \( \Delta(v) = 0 \), it follows that \( a \) has the form (3). Using Lemma 2.5 we obtain that

\[
\Delta_{i,i}(x + y) = e_{i,i}\Delta(z)e_{i,i} = e_{i,i}[a, z]e_{i,i} = [a_1, x + y],
\]

\[
\Delta_{i,i}(x) = e_{i,k}\Delta(e_{k,i}xe_{i,k})e_{k,i} = e_{i,k}e_{k,k}\Delta(z)e_{k,k}e_{k,i} = e_{i,k}[a, z]e_{k,i} = [a_1, x],
\]

\[
\Delta_{i,i}(y) = e_{i,s}\Delta(e_{s,i}ye_{i,s})e_{s,i} = e_{i,s}e_{s,s}\Delta(z)e_{s,s}e_{s,i} = e_{i,s}[a, z]e_{s,i} = [a_1, y].
\]

Hence

\[
\Delta_{i,i}(x + y) = \Delta_{i,i}(x) + \Delta_{i,i}(y).
\]

The proof is complete. \( \Box \)

As it was mentioned in the beginning of the section any additive 2-local derivation on a semi-prime algebra is a derivation. Since \( M_{i,i} \equiv \mathcal{A} \) is semi-prime, Lemma 2.7 implies the following result.

**Lemma 2.8** \( \nabla_{i,i} \) is a derivation for all \( i \in \overline{1, n} \).

Since \( \Delta_{1,1} \) is a derivation on \( e_{1,1}M_{n}(\mathcal{A})e_{1,1} \equiv \mathcal{A} \) and \( \mathcal{A} \) has the inner derivation property, it follows that there exists an element \( a_{1,1} \in \mathcal{A} \) such that \( \Delta_{1,1} = \text{ad}(a_{1,1}) \). Set \( \tilde{a} = \sum_{i=1}^{n} e_{i,1}a_{1,1}e_{1,i} \).

Denote by \( \mathcal{D}_n \) the set of all diagonal matrices from \( M_{n}(\mathcal{A}) \), i.e. the set of all matrices of the following form

\[
x = \begin{pmatrix}
x_1 & 0 & 0 & \ldots & 0 \\
0 & x_2 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & x_{n-1} & 0 \\
0 & 0 & \ldots & 0 & x_n
\end{pmatrix}.
\]

Let \( x \in \mathcal{D}_n \). Then

\[
[a, x]_{i,i} = e_{i,i}\tilde{a}e_{i,i} - e_{i,i}x\tilde{a}e_{i,i} = e_{i,1}a_{1,1}e_{1,i}x_{i,i} - x_{i,i}e_{1,1}a_{1,1}e_{1,i} = \\
e_{i,1}a_{1,1}e_{1,i}x_{i,i}e_{1,1} = e_{i,1}\Delta_{1,1}(e_{1,1}x_{i,i}e_{1,1})e_{1,i} = \\
\Delta_{i,i}(x_{i,i}) = \Delta_{i,i}(x)e_{i,i} = \Delta(x)_{i,i},
\]
We also denote by $\Delta(\| \cdot \|)$ the unit of the algebra $B$. Then, it is identically zero on the whole algebra. In order to prove this we first consider the 2-local derivation $\Delta$ on a matrix algebra equals to zero on all diagonal matrices and on the linear span of matrix units, then it is identically zero on the whole algebra. In order to prove this we first consider the $2 \times 2$-matrix algebras case.

\[ \Delta(x)_{1,1} = \Delta(x)_{2,2} = 0 \] for all $x \in M_2(B)$.

**Lemma 2.10** \( \Delta(x)_{1,1} = \Delta(x)_{2,2} = 0 \) for all $x \in M_2(B)$.

**Proof.** Let $x = \begin{pmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{pmatrix}$. Since $\Delta$ is homogeneous, we can assume that $\|x_{1,1}\| < 1$, where $\| \cdot \|$ is the norm on $B$. Set $y = \begin{pmatrix} e + x_{1,1} & 0 \\ 0 & 0 \end{pmatrix}$. Since $\|x_{1,1}\| < 1$, it follows that $e + x_{1,1}$ is invertible in $B$. Take an element $a \in M_2(B)$ such that

\[ \Delta(x) = [a, x], \quad \Delta(y) = [a, y]. \]

Since $y \in D_2$ we have that $0 = \Delta(y) = [a, y]$, and therefore

\[ 0 = \Delta(y)_{1,1} = a_{1,1}(e + x_{1,1}) - (e + x_{1,1})a_{1,1} = 0, \]

\[ 0 = \Delta(y)_{2,1} = a_{2,1}(e + x_{1,1}) = 0, \]

and

\[ 0 = \Delta(y)_{1,2} = -(e + x_{1,1})a_{1,2} = 0. \]

Thus

\[ a_{1,1}x_{1,1} - x_{1,1}a_{1,1} = 0 \]

and

\[ a_{2,1} = a_{1,2} = 0. \]

The above equalities imply that

\[ \Delta(x)_{1,1} = a_{1,1}x_{1,1} - x_{1,1}a_{1,1} = 0. \]

In a similar way we can show that $\Delta(x)_{2,2} = 0$. The proof is complete. \( \square \)
Lemma 2.11 Let $x$ be a matrix with $x_{k,s} = \lambda e$, where $\lambda \in \mathbb{C}$. Then $\Delta(x)_{k,s} = 0$.

Proof. 
\[
e_{s,k} \Delta(x)e_{s,k} = e_{s,k} D_{e_{s,k}x}(x)e_{s,k} = \Delta(x)_{s,k} = \lambda D_{e_{s,k}x}(e_{s,k}) - D_{e_{s,k}x}(e_{s,k})x - e_{s,k}x D_{e_{s,k}x}(e_{s,k}) = \lambda x D_{e_{s,k}x}(e_{s,k}) - 0 = 0.
\]
Thus
\[
\Delta(x)_{k,s} = e_{k,s} \Delta(x)e_{s,k} = e_{k,s} e_{s,k} \Delta(x)e_{s,k} = 0.
\]
The proof is complete. $\square$

Lemma 2.12 Let $x = \begin{pmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{pmatrix}$ and $y = \begin{pmatrix} x_{1,1} & x_{1,2} \\ 0 & x_{2,2} \end{pmatrix}$. Then $\Delta(x)_{1,2} = \Delta(y)_{1,2}$.

Proof. Take a matrix $a \in M_2(\mathbb{B})$ such that
\[
\Delta(x) = [a, x] \text{ and } \Delta(y) = [a, y].
\]
Then
\[
\Delta(x)_{1,2} = a_{1,1} x_{1,2} + a_{1,2} x_{2,2} - x_{1,1} a_{1,2} - x_{1,2} a_{2,2} = \Delta(y)_{1,2}.
\]
The proof is complete. $\square$

Lemma 2.13 Let $x = \begin{pmatrix} e + x_{1,1} \\ 0 \end{pmatrix}$, where $x_{1,1} \in \mathbb{B}$, $\|x_{1,1}\| < 1$, $\lambda \in \mathbb{C}$. Then $\Delta(x) = 0$.

Proof. From Lemmata 2.10 and 2.11, it follows that
\[
\Delta(x)_{1,1} = \Delta(x)_{2,2} = \Delta(x)_{2,1} = 0.
\]
Let us to show that $\Delta(x)_{1,2} = 0$.

Case 1. Let $\lambda = 0$. Take a matrix $a \in M_2(\mathbb{B})$ such that
\[
\Delta(x) = [a, x] \text{ and } \Delta(e_{2,1}) = [a, e_{2,1}].
\]
Since the element $a$ commutes with $e_{2,1}$, it follows that $a$ is of the form $a = \begin{pmatrix} a_1 & 0 \\ a_2 & a_1 \end{pmatrix}$. Then
\[
0 = \Delta(x)_{2,1} = a_2 (e + x_{1,1}) = 0.
\]
Since $e + x_{1,1}$ is invertible in $\mathbb{B}$, it follows that $a_2 = 0$. From the last equality we obtain that 
\[
0 = \Delta(x)_{1,1} = a_1 x_{1,1} - x_{1,1} a_1 = \Delta(x)_{1,2}.
\]
i.e. $\Delta(x)_{1,2} = 0$. Therefore, $\Delta(x) = 0$.

Case 2. Let $\lambda \neq 0$. Set $y = \begin{pmatrix} e + x_{1,1} \\ 0 \end{pmatrix}$. By Case 1, $\Delta(y) = 0$. Applying Lemma 2.12 we obtain that
\[
0 = \Delta(y)_{1,2} = \Delta(x)_{1,2}.
\]
i.e. $\Delta(x)_{1,2} = 0$. Thus $\Delta(x) = 0$. The proof is complete. $\square$

Lemma 2.14 If $x = \begin{pmatrix} x_{1,1} & x_{1,2} \\ \lambda e & 0 \end{pmatrix}$, where $x_{1,1}$ is an invertible element in $\mathbb{B}$, then $\Delta(x) = 0$. 

\[8\]
Proof. As in the proof of Lemma 2.13 it is suffices to show that \( \Delta(x)_{1,2} = 0 \). Since \( \Delta \) is homogeneous, we can assume that \( \|x_{1,2}\| < 1 \).

Case 1. Let \( \lambda = 0 \). Set \( y = \begin{pmatrix} e + x_{1,2} & x_{1,2} \\
0 & 0 \end{pmatrix} \). Take a matrix \( a \in M_2(\mathcal{B}) \) such that
\[
\Delta(x) = [a, x] \quad \text{and} \quad \Delta(y) = [a, y].
\]
By Lemma 2.13 we have that \( \Delta(y) = 0 \). Since
\[
0 = \Delta(x)_{2,1} = a_{2,1} x_{1,1},
\]
and \( x_{1,1} \) is invertible in \( \mathcal{B} \), it follows that \( a_{2,1} = 0 \). From
\[
0 = \Delta(y)_{2,2} = a_{2,1} x_{1,2} - a_{1,2},
\]
it follows that \( a_{1,2} = 0 \). So, \( a \) is a diagonal matrix. This implies that
\[
0 = \Delta(y)_{1,2} = a_{1,1} x_{1,2} - x_{1,2} a_{2,2} = \Delta(x)_{1,2},
\]
i.e. \( \Delta(x)_{1,2} = 0 \). Therefore \( \Delta(x) = 0 \).

Case 2. Let \( \lambda \neq 0 \). Set \( y = \begin{pmatrix} x_{1,1} & x_{1,2} \\
0 & 0 \end{pmatrix} \). By Case 1, \( \Delta(y) = 0 \). From Lemma 2.12 we obtain that
\[
0 = \Delta(y)_{1,2} = \Delta(x)_{1,2}.
\]
i.e. \( \Delta(x)_{1,2} = 0 \), and hence \( \Delta(x) = 0 \). The proof is complete. \( \square \)

Lemma 2.15 Let \( x = \begin{pmatrix} x_{1,1} & x_{1,2} \\
x_{2,1} & x_{2,2} \end{pmatrix} \), where \( x_{1,1} \) is an invertible element in \( \mathcal{B} \). Then \( \Delta(x) = 0 \).

Proof. Set \( y = \begin{pmatrix} x_{1,1} & x_{1,2} \\
e & 0 \end{pmatrix} \). Take a matrix \( a \in M_2(\mathcal{B}) \) such that
\[
\Delta(x) = [a, x] \quad \text{and} \quad \Delta(y) = [a, y].
\]
By Lemma 2.14 we have that \( \Delta(y) = 0 \). Since
\[
0 = \Delta(y)_{1,1} = \Delta(x)_{1,1} = a_{1,2},
\]
it follows that \( a_{1,2} = 0 \). From the last equality we obtain that
\[
0 = \Delta(y)_{1,2} = a_{1,1} x_{1,2} - x_{1,2} a_{2,2} = \Delta(x)_{1,2},
\]
i.e. \( \Delta(x)_{1,2} = 0 \). Therefore \( \Delta(x) = 0 \). The proof is complete. \( \square \)

Lemma 2.16 Let \( \Delta \) be a 2-local derivation on \( M_2(\mathcal{B}) \) such that
\[
\Delta|_{sp(e_{ij})^2_{j=1}} \equiv 0, \quad \Delta|_{e_2} \equiv 0.
\]
Then \( \Delta \equiv 0 \).

Proof. Let \( x = \begin{pmatrix} x_{1,1} & x_{1,2} \\
x_{2,1} & x_{2,2} \end{pmatrix} \). By Lemma 2.10, \( \Delta(x)_{1,1} = \Delta(x)_{2,2} = 0 \).

Let us to show that \( \Delta(x)_{1,2} = 0 \). Since \( \Delta \) is homogeneous we can assume that \( \|x_{1,1}\| < 1 \). Further, since \( \Delta(x) = \Delta(1+x) \), replacing, if necessary, \( x \) by \( 1+x \), we may assume that \( x_{1,1} \) is invertible in \( \mathcal{B} \).

Put \( y = \begin{pmatrix} x_{1,1} & x_{1,2} \\
0 & x_{2,2} \end{pmatrix} \). Lemma 2.15 implies that \( \Delta(y) = 0 \). Now from Lemma 2.12 we obtain that
\[
\Delta(x)_{1,2} = \Delta(y)_{1,2} = 0.
\]
In a similar way we can show that \( \Delta(x)_{2,1} = 0 \). The proof is complete. \( \square \)
2.2. The general case

Now we are in position to prove Theorem 2.1.

**Proof of Theorem 2.1.** Let \( \Delta \) be a 2-local derivation on \( M_2^n(A) \), where \( n \geq 2 \). By Lemma 2.3 there exists a derivation \( D \) on \( M_2^n(A) \) such that \( \Delta|_{\text{sp}\{e_{i,j}\}_{i,j=1}^{2^n}} = D|_{\text{sp}\{e_{i,j}\}_{i,j=1}^{2^n}} \). Replacing, if necessary, \( \Delta \) by \( \Delta - D \), we may assume that \( \Delta \) is equal to zero on \( \text{sp}\{e_{i,j}\}_{i,j=1}^{2^n} \). Further, by Lemma 2.9 there exists a diagonal element \( \tilde{a} \) in \( M_2^n(A) \) such that \( \Delta|_{\mathcal{D}_{2^n}} = \text{ad}(\tilde{a})|_{\mathcal{D}_{2^n}} \). Now replacing \( \Delta \) by \( \Delta - \text{ad}(\tilde{a}) \), we can assume that \( \Delta \) is identically zero on \( \mathcal{D}_{2^n} \). So, we can assume that

\[
\Delta|_{\text{sp}\{e_{i,j}\}_{i,j=1}^{2^n}} = 0 \quad \text{and} \quad \Delta|_{\mathcal{D}_{2^n}} = 0.
\]

Let us to show that \( \Delta \equiv 0 \). We proceed by induction on \( n \).

Let \( n = 2 \). We identify the algebra \( M_4(A) \) with the algebra of \( 2 \times 2 \)-matrices \( M_2(B) \), over \( B = M_2(A) \).

Let \( \{e_{i,j}\}_{i,j=1}^{4} \) be a system of matrix units in \( M_4(A) \). Then

\[
p_{1,1} = e_{1,1} + e_{2,2}, \quad p_{2,2} = e_{3,3} + e_{4,4}, \quad p_{1,2} = e_{1,3} + e_{2,4}, \quad p_{2,1} = e_{3,1} + e_{4,2}
\]

is the system of matrix units in \( M_2(B) \). Since \( \Delta|_{\text{sp}\{e_{i,j}\}_{i,j=1}^{4}} = 0 \), it follows that \( \Delta|_{\text{sp}\{p_{i,j}\}_{i,j=1}^{4}} = 0 \).

Take an arbitrary element \( x \in p_{1,1}M_2(B)p_{1,1} \equiv B \). Choose a derivation \( D \) on \( M_2(B) \) such that

\[
\Delta(x) = D(x), \quad \Delta(p_{1,1}) = D(p_{1,1}).
\]

Since \( \Delta(p_{1,1}) = 0 \), we obtain that

\[
p_{1,1}D(x)p_{1,1} = p_{1,1}D(x)p_{1,1} = D(p_{1,1}xp_{1,1}) - D(p_{1,1})xp_{1,1} - p_{1,1}xD(p_{1,1}) = \Delta(x).
\]

This means that the restriction \( \Delta_{1,1} \) of \( \Delta \) onto \( p_{1,1}M_2(B)p_{1,1} \equiv B \) maps \( B = M_2(A) \) into itself. If \( \mathcal{D}_4 \) is the subalgebra of diagonal matrices from \( M_4(A) \), then \( p_{1,1}\mathcal{D}_4p_{1,1} \) is the subalgebra of diagonal matrices in the algebra \( M_2(A) \). Since \( \Delta|_{\mathcal{D}_4} = 0 \), it follows that \( \Delta_{1,1} \) is identically zero on the diagonal matrices from \( M_2(A) \). So,

\[
\Delta_{1,1}|_{\text{sp}\{e_{i,j}\}_{i,j=1}^{4}} = 0 \quad \text{and} \quad \Delta_{1,1}|_{p_{1,1}\mathcal{D}_4p_{1,1}} = 0.
\]

By Lemma 2.16 it follows that \( \Delta_{1,1} \equiv 0 \).

Let \( \mathcal{D}_2 \) be the set of diagonal matrices from \( M_2(B) \). Since

\[
\mathcal{D}_2 = \begin{pmatrix} B & 0 \\ 0 & B \end{pmatrix}
\]

and \( \Delta_{1,1} = 0 \), Lemma 2.5 implies that \( \Delta|_{\mathcal{D}_2} = 0 \). Hence, \( \Delta \) is a 2-local derivation on \( M_2(B) \) such that

\[
\Delta|_{\text{sp}\{p_{i,j}\}_{i,j=1}^{4}} = 0 \quad \text{and} \quad \Delta|_{\mathcal{D}_2} = 0.
\]

Again by Lemma 2.16 it follows that \( \Delta \equiv 0 \).

Now assume that the assertion of the Theorem is true for \( n - 1 \).

Considering the algebra \( M_2^n(A) \) as the algebra of \( 2 \times 2 \)-matrices \( M_2(B) \) over \( B = M_2^{n-1}(A) \) and repeating the above arguments we obtain that \( \Delta \equiv 0 \). The proof is complete. \( \square \)

The condition on the algebra \( A \) to be a Banach algebra was applied only for the invertibility of elements of the forms \( 1 + x \), where \( x \in A \), \( \|x\| < 1 \). In this connection the following question naturally arises.

**Problem 2.17** Does Theorem 2.1 hold for arbitrary (not necessarily normed) algebra \( A \) with the inner derivation property?
3. An application to $AW^*$-algebras

In this section we apply Theorem 2.1 to the description of 2-local derivations on $AW^*$-algebras.

**Theorem 3.1** Let $A$ be an arbitrary $AW^*$-algebra. Then any 2-local derivation $\Delta$ on $A$ is a derivation.

**Proof.** Let us first note that any $AW^*$-algebra is semi-prime. It is also known [9] that $AW^*$-algebra has the inner derivation property.

Let $z$ be a central projection in $A$. Since $D(z) = 0$ for an arbitrary derivation $D$; it is clear that $\Delta(zx) = 0$ for any 2-local derivation $\Delta$ on $A$.

Take $x \in A$ and let $D$ be a derivation on $A$ such that $\Delta(zx) = D(zx) = D(z)x + zD(x) = z\Delta(x)$. This means that every 2-local derivation $\Delta$ maps $zA$ into $zA$ for each central projection $z \in A$.

Let $A$ be an abelian $AW^*$-algebra. It is well-known that any derivation on a such algebra is identically zero. Therefore any 2-local derivation on an abelian $AW^*$-algebra is also identically zero.

If $A$ is an $AW^*$-algebra of type $I_n$, $n \geq 2$, with the center $Z(A)$, then it is isomorphic to the algebra $M_n(Z(A))$. By Lemma 2.3 there exists a derivation $D$ on $A \cong M_n(Z(A))$ such that $\Delta \equiv D$. So, $\Delta$ is a derivation.

Let the $AW^*$-algebra $A$ have one of the types $I_\infty$, II or III. Then the halving Lemma [7, Lemma 4.5] for type $I_\infty$-algebras and [7, Lemma 4.12] for type II or III algebras, imply that the unit of the algebra $A$ can be represented as a sum of mutually equivalent orthogonal projections $e_1, e_2, e_3, e_4$ from $A$. Then the map $x \mapsto \sum_{i,j=1}^4 e_ixe_j$ defines an isomorphism between the algebra $A$ and the matrix algebra $M_4(\mathcal{B})$, where $\mathcal{B} = e_{1,1}Ae_{1,1}$. Therefore Theorem 2.1 implies that any 2-local derivation on $A$ is a derivation. The proof is complete. □

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