THE INTRINSIC METRIC ON THE UNIT SPHERE OF A NORMED SPACE

MIEK MESSERSCHMIDT AND MARTEN WORTEL

Abstract. Let $S$ denote the unit sphere of a real normed space. We show that the intrinsic metric on $S$ is strongly equivalent to the induced metric on $S$. Specifically, for all $x, y \in S$,

$$\|x - y\| \leq d(x, y) \leq \sqrt{2\pi} \|x - y\|,$$

where $d$ denotes the intrinsic metric on $S$.

Correction

The authors would like to thank the anonymous referee who pointed out the result [2, Theorem 3.5] to them. This result solves the problem posed in this manuscript and far predates it, (it actually proves Conjecture 1.2 stated below).

The authors take some solace in the following: That the result is true, with optimal constant 2, as conjectured. That their initial this-really-should-exist-somewhere-in-the-literature gut feeling was actually correct. And finally, that this result is now much easier to find by googling for the obvious “intrinsic metric unit sphere.”

1. Introduction

Consider the following problem which arose in other questions under investigation by the first author:

Question. For the unit sphere of a real normed space, is the induced metric strongly equivalent to the sphere’s intrinsic metric?

This paper will answer this question in the affirmative.

More precisely: For a unit sphere $S$ of a real normed space, the length of a path $\rho : [0, 1] \to S$ (assumed to be continuous), is given by

$$L(\rho) := \sup \left\{ \sum_{j=0}^{n-1} \|\rho(t_j) - \rho(t_{j+1})\| \bigg| n \in \mathbb{N}, \quad 0 = t_0 < \ldots < t_n = 1 \right\},$$

and the intrinsic metric on $S$ is defined by taking the infimum of the above quantity over all paths between two points. I.e., for $x, y \in S$, we define the intrinsic metric on $S$ by

$$d(x, y) := \inf \left\{ L(\rho) \bigg| \rho : [0, 1] \to S \text{ continuous}, \quad \rho(0) = x, \quad \rho(1) = y \right\}.$$
The above question is then rephrased as: For the unit sphere $S$ in a normed space, do there exist constants $A, B > 0$ such that, for all $x, y \in S$,

\[ A \| x - y \| \leq d(x, y) \leq B \| x - y \|. \]

To the authors’ knowledge, the answer to this question does not appear in the literature\(^1\), a fact that is perhaps more surprising than the positive solution to the problem which will be presented in this paper.

The problem can essentially be reduced to one in two dimensions and apart from relying on John Ellipsoids—a fundamental structure from local theory—the result follows from entirely elementary (albeit somewhat technical) arguments.

We will now describe the structure of the paper.

After introducing the needed notation, definitions and preliminary results in Section 2, in Section 3 our goal will be to prove Theorem 3.6:

**Theorem 3.6.** For any norm $\| \cdot \|$ on a real vector space $V$, let $d$ denote the intrinsic metric on the unit sphere $S$ of $V$. For all $x, y \in S$,

\[ \| x - y \| \leq d(x, y) \leq \sqrt{2} \pi \| x - y \|. \]

A crucial ingredient is that of the John Ellipsoid: The largest ellipsoid (Euclidean ball of largest volume) that can be contained inside a unit ball of a finite dimensional normed space.

**Theorem 1.1** (John’s Theorem [1, Theorem 12.1.4]). Let $W$ be any normed space of dimension $n > 1$. With $\| \cdot \|_E$ denoting the Euclidean norm on $\mathbb{R}^n$, there exists a norm one isomorphism $T : (\mathbb{R}^n, \| \cdot \|_E) \rightarrow W$ with inverse $T^{-1} : W \rightarrow (\mathbb{R}^n, \| \cdot \|_E)$ whose norm is at most $\sqrt{n}$.

Specifically, all two-dimensional subspaces of a normed space has a Banach-Mazur distance of at most $\sqrt{2}$ from two-dimensional Euclidean space. Of course, the intrinsic metric on a normed space $V$’s unit sphere is bounded above by the “planar” intrinsic metric: where all paths in the defining infimum are taken to live in any two-dimensional subspace of $V$. This allows us to reduce the question to $\mathbb{R}^2$, where $S$ lies between a Euclidian unit sphere $S_E$ and $\sqrt{2}S_E$.

In this setting, the crucial ingredients are Lemmas 3.3 and 3.4 which allows us to conclude the local bi-Lipschitzness of the map $\sigma : S \rightarrow S_E$ defined by $\sigma(x) := x/\|x\|_E$ where both $S$ and $S_E$ are endowed with the Euclidean induced metric. Since the Euclidean induced- and intrinsic metrics on $S_E$ are easily calculated and related (Lemma 2.1), our main results, Theorems 3.5 and 3.6 then easily follow.

We note that the constant $\sqrt{\pi}$ obtained in our main result (Theorem 3.6) is likely not optimal. In two dimensions, the $\| \cdot \|_\infty$-norm provides a worst case for the John Ellipsoid. Let $S_\infty$ be the unit sphere of this norm, with intrinsic metric $d_\infty$, then for $x, y \in S_\infty$, it is easily seen that $\|x - y\|_\infty \leq d_\infty(x, y) \leq 2 \|x - y\|_\infty$. This prompts the following conjecture:

**Conjecture 1.2.** For any norm $\| \cdot \|$ on a real vector space $V$, let $d$ denote the intrinsic metric on the unit sphere $S$ of $V$. For all $x, y \in S$,

\[ \| x - y \| \leq d(x, y) \leq 2 \| x - y \|. \]

\(^1\)False! See the correction above!

\(^2\)This conjecture is proven true in [2, Theorem 3.5].
2. Definitions, notation and preliminary results

This section will explicitly define all notation used in this paper. Since we will translate between many different metrics on many different sets, we will take extreme care to make our notation as explicit as possible.

Let $V$ be a real vector space. Let $A$ be an arbitrary index symbol and $\| \cdot \|_A$ any norm on $V$. We will denote unit sphere, closed unit ball and open unit ball with respect to $\| \cdot \|_A$ respectively by $S_A$, $B_A$ and $R_A$.

For any subset $M \subseteq V$, we define the induced metric $d_A: M \to \mathbb{R}_{\geq 0}$ on $M$ by $d_A(x,y) := \| x - y \|_A$ for $x, w \in M$.

We define the $A$-$M$-path-space by

$$\mathcal{P}_A(M) := \{ \rho | \rho : [0,1] \to M, \| \cdot \|_A \text{ - continuous} \},$$

and the planar $A$-$M$-path-space by

$$\mathcal{P}^{\text{planar}}_A(M) := \{ \rho \in \mathcal{P}_A(M) | \dim(\text{span}(\text{Im}\rho)) = 2 \}.$$

We define the $A$-path-length operator $L_A: \mathcal{P}_A(V) \to \mathbb{R}_{\geq 0} \cup \{ \infty \}$ by

$$L_A(\rho) := \sup \left\{ \sum_{j=0}^{n-1} \| \rho(t_j) - \rho(t_{j+1}) \|_A \left| \begin{array}{c} n \in \mathbb{N}, \\ 0 = t_0 < \ldots < t_n = 1 \end{array} \right. \right\} (\rho \in \mathcal{P}_A(V)).$$

We define the (extended) $A$-$M$-intrinsic metric $d_{A,M}: M \to \mathbb{R}_{\geq 0} \cup \{ \infty \}$, by

$$d_{A,M}(v,w) := \inf \{ L_A(\rho) | \rho \in \mathcal{P}_A(M), \rho(0) = v, \rho(1) = w \} (v, w \in M)$$

and the (extended) $A$-$M$-planar-intrinsic metric $d_{A,M}^{\text{planar}}: M \to \mathbb{R}_{\geq 0} \cup \{ \infty \}$ by

$$d_{A,M}^{\text{planar}}(v,w) := \inf \{ L_A(\rho) | \rho \in \mathcal{P}^{\text{planar}}_A(M), \rho(0) = v, \rho(1) = w \} (v, w \in M).$$

We introduce the following abbreviated notation that will aid in readability of the paper: For (extended) metrics $d$ and $d'$ on some set $D$, subset $M \subseteq D$, and constant $K > 0$, by

$$d \leq K d' \text{ on } M$$

we will mean $d(a,b) \leq K d'(a,b)$ for all $a, b \in M$.

For any real normed space $(V, \| \cdot \|_A)$ and subset $M \subseteq V$, since $\mathcal{P}^{\text{planar}}_A(M) \subseteq \mathcal{P}_A(M) \subseteq \mathcal{P}_A(V)$, the chain of inequalities

$$d_A = d_{A,V} \leq d_{A,M} \leq d_{A,M}^{\text{planar}} \leq \infty \text{ on } M$$

is easy to verify.

An elementary calculation establishes the following lemma:

**Lemma 2.1.** With $\| \cdot \|_E$ denoting the Euclidean norm on $\mathbb{R}^2$,

$$d_E \leq d_{E,S_E} \leq \frac{\pi}{2} d_E \text{ on } S_E.$$

If $\mathbb{R}^2$ is endowed with the euclidean norm $\| \cdot \|_E$ arising from an inner product $\langle \cdot, \cdot \rangle$, for elements $x, y \in \mathbb{R}^2$ the ray from $x$ through $y$ is denoted by $r_{x,y}$ and defined by $r_{x,y} := \{ (1 - t)x + ty | t \geq 0 \}$. For a point $x$ and points $y, z \in \mathbb{R}^2$ distinct from $x$, when referring to the size of the angle between $r_{x,y}$ and $r_{x,z}$ we will mean the quantity

$$\arccos \left( \frac{\langle y - x | z - x \rangle}{\| y - x \|_E \| z - x \|_E} \right) \in [0, \pi].$$
For points $v,w,x,y \in \mathbb{R}^2$, we will say the ray $r_{x,y}$ lies between the rays $r_{x,v}$ and $r_{x,w}$ if $v$ and $w$ and $x$ are in general position and $r_{x,y} \cap \{(1-t)v + tw \mid t \in [0,1]\} \neq \emptyset$, or, $r_{x,y} = r_{x,v} = r_{x,w}$.

3. The intrinsic metric on unit spheres in $\mathbb{R}^2$

In this section we will prove our main results. Although somewhat technical, our results follow mostly from elementary trigonometry and Euclidian plane geometry.

Let $\|\cdot\|_E$ denote the Euclidean norm on $\mathbb{R}^2$ and let $\|\cdot\|_X$ be any norm on $\mathbb{R}^2$ satisfying $B_E \subseteq B_X \subseteq K B_E$ for some $K \geq 1$. A large part of our attention will be devoted to proving that the map $\sigma : S_X \to S_E$ defined by $\sigma(x) := x/\|x\|_E$ is locally bi-Lipschitz when both $S_X$ and $S_E$ are both endowed with the Euclidean induced metrics. Once this has been achieved through Lemmas 3.3 and 3.4, a straightforward calculation will prove our main results Theorems 3.5 and 3.6.

Let $\|\cdot\|_E$ denote the Euclidean norm on $\mathbb{R}^2$ and let $\|\cdot\|_X$ be any other norm on $\mathbb{R}^2$ satisfying $B_E \subseteq B_X$. We will first relate points on $S_X$ to lines tangent to $S_E$. Specifically, for any point $x \in S_X$ that is not in $S_E$, the two lines through $x$ that are tangent to $S_E$ are such that points in $S_X$ “close to” $x$ are “wedged between” the tangent lines. Also, if $x \in S_X \cap S_E$, then the whole of $S_X$ lies on the same side of the line $\{y \in \mathbb{R}^2 \mid \langle x \mid y \rangle = 1\}$.

Let $x \in \mathbb{R}^2 \setminus B_E$, and let $\tau(x) \in S_E$ be a point on a tangent line to $S_X$ through $x$. Then the angle between $r_{0,x}$ and $r_{0,\tau(x)}$ equals $\arccos(\|x\|_E^{-1})$. Let $x^\perp \in S_X \cap \{x\}^\perp$ be such that $\langle \tau(x) \mid x^\perp \rangle \geq 0$. If we now define

$$a(x) := \cos \left( \arccos \left( \frac{1}{\|x\|_E} \right) \right) \frac{x}{\|x\|_E} = \frac{x}{\|x\|_E^2}$$

and

$$b(x) := \sin \left( \arccos \left( \frac{1}{\|x\|_E} \right) \right) x^\perp = \sqrt{1 - \frac{1}{\|x\|_E^2}} x^\perp,$$

then $\tau(x) = a(x) + b(x)$.

**Lemma 3.1.** Let $\langle \cdot \mid \cdot \rangle$ be an inner product and $\|\cdot\|_E$ be the associated Euclidean norm on $\mathbb{R}^2$. Let $\|\cdot\|_X$ be any other norm on $\mathbb{R}^2$ such that $B_E \subseteq B_X$. For all $x \in S_X$,

1. For all $t \in \mathbb{R}$, $\langle tx + (1-t)\tau(x) \mid \tau(x) \rangle = 1$.
2. For all $t \in [0,1]$, $\|tx + (1-t)\tau(x)\|_X \leq 1$.
3. For all $t > 1$, $\|tx + (1-t)\tau(x)\|_X \geq 1$.
4. If $x \in S_X \cap S_E$ and $y \in S_X$, then $\langle x \mid y \rangle \leq 1$.

**Proof.** We prove (1). Let $x \in S_X$. For all $t \in \mathbb{R}$,

$$\langle tx + (1-t)\tau(x) \mid \tau(x) \rangle = t \langle x \mid \tau(x) \rangle + (1-t) \langle \tau(x) \mid \tau(x) \rangle = t \frac{x}{\|x\|_E^2} + (1-t) = t + (1-t) = 1.$$
We prove (3). Let \( x \in S_X \). Since \( \|x\|_X = 1 \), if \( \tau(x) = x \), then \( \|tx + (1 - t)\tau(x)\|_X = 1 \), and the result is trivial. We therefore assume \( \tau(x) \neq x \). Since \( \tau(x) \in S_E \), so that \( \|\tau(x)\|_X \leq 1 \), by the reverse triangle inequality and intermediate value theorem there exists some \( t_0 \leq 0 \) such that \( 1 = \|t_0 x + (1 - t_0)\tau(x)\|_X = \|1x + (1-1)\tau(x)\|_X \) (here we used \( \tau(x) \neq x \)). Since the map \( t \mapsto \|tx + (1-t)\tau(x)\|_X \) is convex, we cannot have that \( \|tx + (1-t)\tau(x)\|_X < 1 \) for any \( t > 1 \), as this would contradict \( 1 = \|t_0 x + (1 - t_0)\tau(x)\|_X = \|1x + (1-1)\tau(x)\|_X \). We conclude that \( \|tx + (1-t)\tau(x)\|_X \geq 1 \) for all \( t > 1 \).

We prove (4). Let \( x \in S_X \cap S_E \) and \( y \in S_X \), but suppose \( \langle x \mid y \rangle > 1 \).

If \( y \) and \( x \) are linearly dependent, then \( \|y\|_X > 1 \), contradicting \( y \in S_X \), and we therefore may assume that \( y \) and \( x \) are linearly independent.

Let \( L \) denote the line through \( x \) and \( y \), parameterized by the affine map \( \eta(t) := (1 - t)y + tx \) for \( t \in \mathbb{R} \). The line \( L \) is not tangent to \( S_E \) (else we would have \( \langle x \mid y \rangle = 1 \)). Therefore \( L \) intersects \( S_E \) in two distinct points, one being \( x \); let \( t_0 \in \mathbb{R} \) be such that \( \eta(t_0) \in S_E \cap L \) is the other. We must have \( t_0 > 1 \), since \( 1 < \langle x \mid \eta(t) \rangle \) for \( t \in [0, 1) \). Since \( \eta \) is an affine map and \( (\mathbb{R}^2, \|\cdot\|_E) \) is a strictly convex space,

\[
\left\| \eta \left( \frac{1 + t_0}{2} \right) \right\|_X = \frac{\|\eta(t_0) + \eta(1)\|_X}{2} \leq \frac{\|\eta(t_0) + \eta(1)\|_E}{2} < 1.
\]

Let \( \lambda := 2(1 + t_0)^{-1} \in (0, 1) \), so that \( \lambda \left( \frac{1 + t_0}{2} \right) + (1 - \lambda)0 = 1 \). Then, again since \( \eta \) is affine,

\[
1 &= \|x\|_X \\
&= \|\eta(1)\|_X \\
&= \left\| \lambda \left( \frac{1 + t_0}{2} \right) + (1 - \lambda)0 \right\|_X \\
&= \lambda \left\| \frac{1 + t_0}{2} \right\|_X + (1 - \lambda) \|\eta(0)\|_X \\
&= \lambda \|\eta(1)\|_X + (1 - \lambda) \|\eta(0)\|_X \\
&< \lambda \cdot 1 + (1 - \lambda) \cdot 1 \\
&= 1,
\]

which is absurd. We conclude that \( \langle x \mid y \rangle \leq 1 \) for all \( x \in S_X \cap S_E \) and all \( y \in S_X \).

Next, we show that for points \( x, y \in S_X \) that are “sufficiently close”, the size of the angle formed by the rays \( r_{0,x} \) and \( r_{0,y} \) bounds the size of the acute angle formed by the ray \( r_{x,y} \) and the perpendicular line to \( r_{0,x} \) through \( x \).

**Lemma 3.2.** Let \( \|\cdot\|_E \) denote the Euclidean norm on \( \mathbb{R}^2 \) and \( \|\cdot\|_X \) be any norm on \( \mathbb{R}^2 \) such that \( B_E \subseteq B_X \). Let \( x, y \in S_X \) and let \( x^{\perp} \in S_E \cap \{x\}^{\perp} \) be such
that \( \langle x^\perp \mid y \rangle \geq 0 \) and define \( v := x + x^\perp \). If \( K \geq 1 \) and \( x, y \in S_X \) are such that \( \|x\|_E \leq K \) and the size of the angle between the rays \( r_{0,x} \) and \( r_{0,y} \) is at most \( \arccos(K^{-1}) \), then \( \alpha \), the size of the angle between the rays \( r_{x,v} \) and \( r_{x,y} \), is also at most \( \arccos(K^{-1}) \).

**Proof.** As a visual aid, the reader is referred to Figure 3.1.

Let \( \beta := \arccos(K^{-1}) \) and \( u \in S_E \) be such that \( \langle x^\perp \mid u \rangle > 0 \) and that the size of the angle formed by the rays \( r_{0,x} \) and \( r_{0,u} \) equals \( \beta \) (i.e., \( \langle x \mid u \rangle = \|x\|_E \cos \beta \)). Let \( \tau_1(x), \tau_2(x) \in S_E \) be the point(s) on the lines through \( x \) that are tangent to \( S_E \), such that \( \langle x^\perp \mid \tau_1(x) \rangle \geq 0 \). Let

\[
  w := \begin{cases} 
    v & \text{if } x = \tau_1(x) = \tau_2(x) \\
    2x - \tau_2(x) & \text{otherwise},
  \end{cases}
\]

so that \( w \in r_{\tau_2(x),x} \) is distinct from \( x \), and is such that \( r_{x,w} \subseteq r_{\tau_2(x),x} \). Let \( P_u \) denote the orthogonal projection onto the span of \( u \). Then size of the angle formed between the rays \( r_{x,P_u,x} \) and \( r_{x,v} \) is exactly \( \beta \). Since \( \|x\|_E \leq K \), the point \( P_u x \) lies on the line segment \( \{tu \mid t \in (0,1)\} \) (if \( \|x\|_E = K \), then \( P_u x = u = \tau_1(x) \)), and therefore the size of angle between rays \( r_{x,u} \) and \( r_{x,v} \) is at most \( \beta \). Since \( r_{x,\tau_1(x)} \) is between the rays \( r_{x,u} \) and \( r_{x,v} \), and since \( r_{x,v} \) bisects the angle formed by the rays \( r_{x,\tau_1(x)} \) and \( r_{x,w} \), the size of the angle formed by \( r_{x,v} \) and \( r_{x,w} \) is also at most \( \beta \). Finally, by Lemma 3.1 (2), (3) and (4) and the fact that \( B_E \subseteq B_X \subseteq K B_E \), the ray \( r_{x,y} \) lies either between the rays \( r_{x,u} \) and \( r_{x,v} \) or the rays \( r_{x,v} \) and \( r_{x,w} \) (The point \( y \) can only lie in the shaded area in Figure 3.1). We conclude that \( \alpha \), the size of the angle between rays \( r_{x,v} \) and \( r_{x,y} \), is at most \( \beta = \arccos(K^{-1}) \).

\[ \square \]

**Lemma 3.3.** Let \( \|\cdot\|_E \) denote the Euclidean norm on \( \mathbb{R}^2 \) and \( \|\cdot\|_X \) be any norm on \( \mathbb{R}^2 \) such that \( B_E \subseteq B_X \subseteq K B_E \) for some \( K \geq 1 \). If \( x, y \in S_X \) is such that \( \theta \),
the size of angle between the rays \( r_{0,x} \) and \( r_{0,y} \), is at most \( \arccos(K^{-1}) \), then

\[
\|x - y\|_E \leq K^2 \left\| \frac{x}{\|x\|_E} - \frac{y}{\|y\|_E} \right\|_E.
\]

**Proof.** As a visual aid, the reader is referred to Figure 3.2.

![Figure 3.2.](image)

Let \( x^\perp \in S_E \cap \{x\}^\perp \) be such that \( \langle x^\perp | y \rangle > 0 \) and define \( v := x + x^\perp \). Let \( P_x \) and \( P_y \) be the orthogonal projections onto the span of \( x \) and \( y \) respectively. Define \( u := P_y(x/\|x\|_E) \), and \( \lambda := \|P_y y\|_E^{-1} \) so that \( P_x(\lambda y) = x/\|x\|_E \). Let \( \alpha \) denote the size of the angle formed by the rays \( r_{x,y} \) and \( r_{x,v} \). We note that then size of the angle formed between the rays \( r_{x/\|x\|_E,\lambda y} \) and \( r_{x/\|x\|_E,u} \) also equals \( \theta \). Elementary trigonometry will establish

\[
\|x - y\|_E = \frac{1}{\cos \alpha} \|y - P_x y\|_E
\]

\[
= \frac{1}{\lambda \cos \alpha} \|\lambda y - P_x(\lambda y)\|_E
\]

\[
= \frac{1}{\lambda \cos \alpha} \|\lambda y - \frac{x}{\|x\|_E}\|_E
\]

\[
= \frac{1}{\lambda \cos \alpha \cos \theta} \|u - \frac{x}{\|x\|_E}\|_E.
\]

Now we note that \( \|u - \frac{x}{\|x\|_E}\|_E \leq \left\| \frac{y}{\|y\|_E} - \frac{x}{\|x\|_E} \right\|_E \), since \( u \) is the closest point (with respect to \( \|\cdot\|_E \)) in the span of \( y \) to the point \( x/\|x\|_E \). Also, by Lemma 3.2 we have \( \alpha \leq \arccos(K^{-1}) \), so that \( \cos \alpha \geq K^{-1} \). Furthermore \( \lambda^{-1} = \|P_x y\|_E = \)
\[ \|y\|_E \cos \theta \leq K \cos \theta. \] Finally we conclude

\[ \|x - y\|_E = \frac{1}{\lambda \cos \alpha \cos \theta} \left\| u - \frac{x}{\|x\|_E} \right\|_E \leq \frac{K \cos \theta}{\cos \alpha \cos \theta} \left\| \frac{y}{\|y\|_E} - \frac{x}{\|x\|_E} \right\|_E \]

\[ = K^2 \left\| \frac{y}{\|y\|_E} - \frac{x}{\|x\|_E} \right\|_E. \]

Lemma 3.4. Let \( \|\cdot\|_E \) denote the Euclidean norm on \( \mathbb{R}^2 \) and \( \|\cdot\|_X \) be any norm on \( \mathbb{R}^2 \) such that \( B_E \subseteq B_X \). If \( x, y \in S_X \), then

\[ \left\| \frac{x}{\|x\|_E} - \frac{y}{\|y\|_E} \right\|_E \leq \|x - y\|_E. \]

Proof. As a visual aid we refer the reader to Figure 3.3.

Let \( x, y \in S_X \). By exchanging the roles of \( x \) and \( y \) if necessary, we may assume \( \|y\|_E \geq \|x\|_E \geq 1 \). Let \( P_y \) be the orthogonal projection onto the span of \( y \) and let \( u := P_y \left( \frac{x}{\|x\|_E} \right) \). Then \( \|u\|_E \leq 1 \) and \( \|y/\|x\|_E\|_E \geq 1 = \|y/\|y\|_E\|_E \). Then, by the Pythagorean theorem,

\[ \left\| \frac{x}{\|x\|_E} - \frac{y}{\|y\|_E} \right\|_E^2 = \left\| \frac{x}{\|x\|_E} - u \right\|_E^2 + \left\| u - \frac{y}{\|y\|_E} \right\|_E^2 \]

\[ \leq \left\| \frac{x}{\|x\|_E} - u \right\|_E^2 + \left\| u - \frac{y}{\|x\|_E} \right\|_E^2 \]

\[ = \left\| \frac{x}{\|x\|_E} - \frac{y}{\|x\|_E} \right\|_E^2 \]

\[ = \frac{1}{\|x\|_E^2} \|x - y\|_E^2 \]

\[ \leq \|x - y\|_E^2. \]

□
In essence, the previous two Lemmas together establish that the local bi-Lipschitzness of the map \( \sigma : S_X \rightarrow S_E \) defined by \( \sigma(x) := x/\|x\|_E \) when \( B_E \subseteq B_X \subseteq KB_E \).

We will now use the previous results to prove one of our main results which relates the intrinsic metric on \( S_X \) to the induced metric on \( S_X \) when \( B_E \subseteq B_X \subseteq KB_E \) for some \( K \geq 1 \).

**Theorem 3.5.** Let \( \|\cdot\|_E \) denote the Euclidean norm on \( \mathbb{R}^2 \) and \( \|\cdot\|_X \) be any norm on \( \mathbb{R}^2 \) such that \( B_E \subseteq B_X \subseteq KB_E \) for some \( K \geq 1 \). Then

\[
d_X \leq d_{X,S_X} \leq K^3 \frac{\pi}{2} d_X \quad \text{on} \quad S_X.
\]

**Proof.** We have already noted in Section 2 that \( d_X \leq d_{X,S_X} \leq S_X \).

Let \( x, y \in S_X \) be arbitrary. Let \( c : \mathbb{R} \rightarrow \mathbb{R}^2 \) be the map defined by \( c(\theta) := (\cos(\theta), \sin(\theta)) \) for \( \theta \in \mathbb{R} \). Let \( \theta_x, \theta_y \in \mathbb{R} \) be such that \( c(\theta_x) = x \) and \( c(\theta_y) = y \). By switching the roles of \( x \) and \( y \), if necessary, we may assume that \( 0 \leq \theta_y - \theta_x \leq \pi \). Consider the path \( \rho : [\theta_x, \theta_y] \rightarrow S_X \) defined by \( \rho(\theta) := c(\theta)/\|c(\theta)\|_X \) for \( \theta \in [\theta_x, \theta_y] \).

Let \( \varepsilon > 0 \) be arbitrary and \( \theta_x = \theta_0 < \theta_1 < \ldots < \theta_n = \theta_y \) be a partition of \( [\theta_x, \theta_y] \) such that

\[
\sum_{j=0}^{n-1} \|\rho(\theta_j) - \rho(\theta_{j+1})\|_X \geq L_X(\rho) - \varepsilon.
\]

We may assume that \( 0 < \theta_{j+1} - \theta_j \leq \arccos K^{-1} \), since the triangle inequality ensures that every refinement of \( \{\theta_j\}_{j=1}^n \) still satisfies the above inequality.

We note that \( B_E \subseteq B_X \subseteq KB_E \) implies \( \|w\|_X \leq \|w\|_E \leq K \|w\|_X \) for all \( w \in \mathbb{R}^2 \). Then, by Lemmas 3.3, 2.1 and 3.4, we obtain

\[
d_{X,S_X}(x, y)
\leq L_X(\rho)
\leq \sum_{j=0}^{n-1} \|\rho(\theta_j) - \rho(\theta_{j+1})\|_X + \varepsilon
\leq \sum_{j=0}^{n-1} \|\rho(\theta_j) - \rho(\theta_{j+1})\|_E + \varepsilon
\leq \sum_{j=0}^{n-1} K^2 \|c(\theta_j) - c(\theta_{j+1})\|_E + \varepsilon
\leq K^2 \sup \left\{ \sum_{j=0}^{m-1} \|c(\phi_j) - c(\phi_{j+1})\|_E \bigg| \theta_x = \phi_0 < \ldots < \phi_m = \theta_y \right\} + \varepsilon
\leq K^2 d_{X,S_E} \left( \frac{x}{\|x\|_E}, \frac{y}{\|y\|_E} \right) + \varepsilon
\leq K^2 \frac{\pi}{2} \left\| \frac{x}{\|x\|_E} - \frac{y}{\|y\|_E} \right\|_E + \varepsilon
\leq K^2 \frac{\pi}{2} \|x - y\|_E + \varepsilon
\leq K^3 \frac{\pi}{2} \|x - y\|_X + \varepsilon.
\]
Since \( \varepsilon > 0 \) was chosen arbitrarily, the result follows. 

Our final result now follows through an easy application of the previous result and John’s Theorem (Theorem 1.1):

**Theorem 3.6.** For any norm \( \| \cdot \|_X \) on a real vector space \( V \),

\[
d_X \leq d_{X, S_X} \leq \sqrt{2\pi} d_X \quad \text{on} \quad S_X.
\]

**Proof.** We have already noted in Section 2 that \( d_X \leq d_{X, S_X} \leq \max_{S_X} \) on \( S_X \). Let \( x, y \in S_X \) be arbitrary and let \( W \subseteq V \) be any two-dimensional subspace containing \( x \) and \( y \), noting that then \( d_{X, S_X}^{\text{planar}}(x, y) \leq d_{X, S_X} \cap W(x, y) \). By John’s Theorem (Theorem 1.1), there exists a Euclidean norm \( \| \cdot \|_E \) on \( W \) such that \( \|w\|_X \leq \|w\|_E \leq \sqrt{2}\|w\|_X \) for all \( w \in W \), i.e., \( B_E \subseteq B_X \cap W \subseteq \sqrt{2}B_E \). Then, by Theorem 3.5 we may conclude that \( d_X \leq d_{X, S_X} \leq \sqrt{2\pi} d_X \) on \( S_X \). □

**References**

1. F. Albiac and N.J. Kalton, *Topics in Banach space theory*, Graduate Texts in Mathematics, Springer, New York, 2006.
2. J.J. Schäffer, *Inner diameter, perimeter, and girth of spheres*, Math. Ann. 173 (1967), 59–79; addendum, ibid. 173 (1967), 79–82.

MIEK MESSERSCHMITT; UNIT FOR BMI; NORTH-WEST UNIVERSITY; PRIVATE BAG X6001; POTCHEFSTROOM; SOUTH AFRICA; 2520
E-mail address: mmesserschmidt@gmail.com

MARTEN WORTEL; UNIT FOR BMI; NORTH-WEST UNIVERSITY; PRIVATE BAG X6001; POTCHEFSTROOM; SOUTH AFRICA; 2520
E-mail address: marten.wortel@gmail.com