HIGGS BUNDLES OVER ELLIPTIC CURVES FOR REAL GROUPS

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Abstract. We study topologically trivial $G$-Higgs bundles over an elliptic curve $X$ when the structure group $G$ is a connected real form of a complex semisimple Lie group $G^\mathbb{C}$. We achieve a description of their (reduced) moduli space, the associated Hitchin fibration and the finite morphism to the moduli space of $G^\mathbb{C}$-Higgs bundles.

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1. Introduction

This is the third of a series of papers dedicated to the description of the moduli spaces of $G$-Higgs bundles over an elliptic curve $(X, x_0)$ (usually denoted simply by $X$). In the first paper [FGN1] we dealt with the classical complex Lie groups and in the second [FGN2] with arbitrary connected complex reductive Lie groups, where we extended the description of the normalization of the moduli space given by Thaddeus [T] to arbitrary degree. In this paper we address the case of real semisimple Lie groups.

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To state our main result we first give some background. Let $G$ be a real semisimple Lie group, and let $H \subset G$ be a maximal compact subgroup. One has the Cartan decomposition $g = h \oplus m$ of the Lie algebra of $G$, where $h$ is the Lie algebra of $H$ and $m$ is the orthogonal subspace to $h$ with respect to the Killing form. The group $H$ acts on $m$ via the isotropy representation and this action extends to the complexification. A $G$-Higgs bundle over the Riemann surface $\Sigma$ is a pair $(E, \Phi)$, where $E$ is a principal $H_C$-bundle over $\Sigma$ and $\Phi$ (the Higgs field) is a holomorphic section of $E(m_C) \otimes \Omega^1_\Sigma$ — the bundle associated to the isotropy representation twisted by the canonical bundle of the curve. We say that $(E, \Phi)$ is topologically trivial if the topological class of the principal bundle $E$ is trivial. We denote the moduli space of topologically trivial $G$-Higgs bundles over $\Sigma$ by $\mathcal{M}_G(\Sigma)$, and by $\mathcal{M}_{G,\text{red}}(\Sigma)$, its reduced subscheme.

The main result of this paper is the following (see Section 3.3 for details).

**Theorem 1.1 (Theorem 3.18).** Suppose that $X$ is an elliptic curve. Let $G$ be a connected real form of a complex semisimple Lie group, let $\mathcal{M}_X(G)$ be the moduli space of topologically trivial $G_C$-Higgs bundles over $X$ and let $\mathcal{M}_{X,\text{red}}(G)$ be its reduced subscheme. Then there exists an isomorphism

$$
\mathcal{M}_{X,\text{red}}(G) \cong \Xi_G / W,
$$

where $W$ is the Weyl group for the action of $G_C$ on $m_C$, and $\Xi_G$ is a quasiprojective variety described as a fibration of (finite quotients of) abelian Lie algebras over $X \otimes \Lambda$, where $\Lambda$ is the cocharacter lattice of $H_C$.

The (reduced) moduli space $\mathcal{M}_{X,\text{red}}(G)$ is not irreducible in general and we describe its irreducible components (see Remark 3.19).

Higgs bundles were introduced by Hitchin in the context of vector bundles in [Hi1] and generalised to arbitrary complex reductive Lie groups in [Hi2]. Simpson [Si2, Si3] constructed the moduli space $\mathcal{M}_\Sigma(G_C)$ of topologically trivial $G_C$-Higgs bundles for any complex reductive Lie group. A major result of the theory of Higgs bundles is the non-abelian Hodge theory correspondence, proved by Hitchin [Hi1], Donaldson [Dn], Simpson [Sh1, Sh2, Sh3] and Corlette [Co]. This states the existence of a homeomorphism, $\mathcal{M}_\Sigma(G_C) \cong \mathcal{C}_\Sigma(G_C)$, between the moduli spaces of $G_C$-Higgs bundles and flat $G_C$-connections.

Higgs bundles for real groups were already considered by Hitchin in [Hi1, Hi3] and Simpson [Si1]. An intrinsic approach and systematic study has been done by the second author in collaboration with Bradlow, Gothen and Mundet i Riera (see e.g. [BGG, BGG2, BGG3, GGM1, GGM2]). The existence of the moduli space $\mathcal{M}_\Sigma(G)$ for any real semisimple Lie group $G$ follows from Schmitt [Sm].

If $G$ is a real form of a complex semisimple Lie group $G_C$, a $G$-Higgs bundle extends naturally to a $G_C$-Higgs bundle, defining a finite morphism (see [GR, Prop. 5.8]).

$$
\mathcal{M}_\Sigma(G) \rightarrow \mathcal{M}_\Sigma(G_C).
$$

For classical complex Lie groups, using spectral curves, Hitchin [Hi2] showed that $\mathcal{M}_\Sigma(G_C)$ fibres over a vector space $B_{CC}$ with abelian varieties as generic fibres, becoming an algebraically completely integrable system. This is the so-called Hitchin fibration. A more canonical definition of the Hitchin fibration was provided.
by Donagi [Do] giving an intrinsic definition of the Hitchin base $B_{CG}$ as the space $H^0(\Sigma, g^C \otimes \Omega^1_{/\Sigma} / G^C)$ of cameral covers of $\Sigma$. One can generalize the Hitchin fibration to the case of real groups (see [P, Sc, HS, GPR])

(1.3) $\mathcal{M}_X(G) \rightarrow B_G$,

where, using Donagi’s approach, $B_G$ is $H^0(\Sigma, m^C \otimes \Omega^1_{/\Sigma})$. As shown in [Fr, Sc, P, HS], the generic fibre of (1.3) is no longer an abelian variety for certain groups.

The canonical bundle of an elliptic curve $X$ is trivial, $\Omega^1_{/X} \cong \mathcal{O}_X$, and therefore, in this case, the Higgs field is simply an element of $H^0(X, E(m^C))$. Working with elliptic curves allows a greater level of explicitness. Atiyah [At] described the moduli space of vector bundles, $M_X(\text{GL}(n, \mathbb{C})) \cong \text{Sym}^n(X)$, while Laszlo [La] and Friedman, Morgan and Witten [FM, FMW] gave a description of the moduli space $M_X(G^C)$ of principal $G^C$-bundles for a complex reductive Lie group:

(1.4) $M_X(G^C) \cong (X \otimes_{\mathbb{Z}} \Lambda) / W$,

where $\Lambda$ is the cocharacter lattice and $W$ is the Weyl group of $G^C$. In [T] Thaddeus studied the case of $G^C$-Higgs bundles, showing that the normalization of the moduli space is

$\tilde{M}_X(G^C) \cong (T^* X \otimes_{\mathbb{Z}} \Lambda) / W$.

In [FGN2] the authors studied the case of $G^C$-Higgs bundles, extending this description to arbitrary degree. Since in the case of real groups, the moduli spaces of $G$-Higgs bundles are no longer bijective with their normalizations, we pursue the description of the reduced subscheme $\mathcal{M}^{red}_X(G) \subset \mathcal{M}_X(G)$, which we call the reduced moduli space.

The results of this paper are structured as follows. In Section 2 we provide the preliminaries needed for our work, a review on Lie theory, $G$-Higgs bundles and principal bundles over elliptic curves. We show in Section 3.1 that, if a $G$-Higgs bundle is semistable (resp. polystable), then the underlying principal $H^C$-bundle is semistable (resp. polystable), and this allows us to describe these objects explicitly. In Section 3.2 we generalize to real groups the existence of what Simpson [Si2, Si3] calls representation space for Higgs bundles and relate it to the moduli space $\mathcal{M}_X(G)$. We also prove some properties of the representation spaces specific to the elliptic case which, together with the explicit description of polystable $G$-Higgs bundles given in Section 3.1, allow us to prove the main result of the article, Theorem 1.1 (Theorem 3.18). If $G$ is complex we recover the description (1.4) given in [T]. In Section 4 we study in detail the involution $\iota_G$ (Proposition 4.2) and the morphism $f_2$ (Proposition 4.9). In Section 5 we provide a complete description of the Hitchin fibration (1.3) and its generic fibre, which is isomorphic to a finite quotient of a certain subset of the abelian variety $X \otimes_{\mathbb{Z}} \Lambda$ (Corollary 5.3). In particular, when $G = \text{SU}^+(4)$ we obtain that the generic Hitchin fibre is $\mathbb{P}^1 \times \mathbb{P}^1$, illustrating our comment above regarding the fact that the generic fibre is not always an abelian variety. Finally, we study the Hitchin equation in Section 6 showing that it decouples into one equation for the metric and another equation for the Higgs field (Proposition 6.1). We observe, then, that the Hitchin–Kobayashi correspondence follows in the elliptic case from the Narasimhan–Seshadri–Ramanathan Theorem [NS, Ra], and the stability results of Section 3.1.
We work in the category of algebraic schemes over \( \mathbb{C} \). Unless otherwise stated, all the bundles considered are algebraic bundles. By bijective morphism we understand an algebraic morphism between schemes that induces a bijection on the sets of \( \mathbb{C} \)-points.

2. Preliminaries

2.1. Lie groups and Lie algebras.

2.1.1. The Cartan decomposition. Let \( G \) be a connected semisimple real form of a connected complex semisimple Lie group \( G^\mathbb{C} \). Let \( H \subset G \) be a maximal compact subgroup and let \( g = h \oplus m \) be the associated Cartan decomposition, where \( h \) is the Lie algebra of \( H \) and \( m \) is the orthogonal subspace to \( h \) with respect to the Killing form. This defines an involution \( \theta : g \to g \) — the Cartan involution —, by \( \theta|_h = \text{id}_h \) and \( \theta|m = -\text{id}_m \), which can be naturally extended to the complexification \( g^\mathbb{C} \), giving the decomposition \( g^\mathbb{C} = h^\mathbb{C} \oplus m^\mathbb{C} \).

This satisfies
\[
[h, h] \subset h, \quad [h, m] \subset m, \quad [m, m] \subset h.
\]

Therefore, the restriction to \( H \) of the adjoint representation reduces to an action on \( m \), which extends to what we will call the isotropy representation of \( H^\mathbb{C} \) on \( m^\mathbb{C} \)
\[
t : H^\mathbb{C} \to \text{GL}(m^\mathbb{C}).
\]

Consider a Cartan subalgebra \( c^\mathbb{C} \) of \( g^\mathbb{C} \) and denote by \( R(g^\mathbb{C}, c^\mathbb{C}) \) the set of roots associated to \( c^\mathbb{C} \). One has the root-space decomposition
\[
g^\mathbb{C} = c^\mathbb{C} \oplus \bigoplus_{\alpha \in R(g^\mathbb{C}, c^\mathbb{C})} (g^\mathbb{C})^{\alpha}.
\]

Recall that the choice of a lexicographic order on some basis of \( c^\mathbb{C} \) defines a notion of positivity for the roots. We write \( R^+(g^\mathbb{C}, c^\mathbb{C}) \) for the set of positive roots with respect to a given lexicographic order. A positive root \( \alpha \in R(g^\mathbb{C}, c^\mathbb{C}) \) is simple if it cannot be written as a sum of any two other positive roots. We denote the set of simple roots by \( \Delta(g^\mathbb{C}, c^\mathbb{C}) \subset R^+(g^\mathbb{C}, c^\mathbb{C}) \).

A Cartan subalgebra of the real Lie algebra \( g \) is a subalgebra whose complexification is a Cartan subalgebra of \( g^\mathbb{C} \). It is always possible to find a \( \theta \)-stable Cartan subalgebra \( \epsilon \) of \( g \). In that case, if \( \alpha \in R(g^\mathbb{C}, c^\mathbb{C}) \) then \( \theta \alpha := \alpha \circ \theta \) is also in \( R(g^\mathbb{C}, c^\mathbb{C}) \). Also we have that \( \epsilon = (\epsilon \cap h) \oplus (\epsilon \cap m) \). The number \( \dim_{\mathbb{R}}(\epsilon \cap h) \) is the compact dimension of \( \epsilon \), while \( \dim_{\mathbb{R}}(\epsilon \cap m) \) is the non-compact dimension of \( \epsilon \). A Cartan subalgebra is said to be maximally compact or maximally non-compact if it maximizes the compact or the non-compact dimension among \( \theta \)-stable Cartan subalgebras.

Remark 2.1. One can always construct a maximally compact Cartan subalgebra.

Take the Lie algebra \( t \) of a maximal torus \( T \) of \( H \). Taking \( a_0 \) to be a maximal abelian subspace of \( \mathfrak{z}(t) \), one has that \( \epsilon_0 = t \oplus a_0 \) is a maximally compact \( \theta \)-stable Cartan subalgebra.

Remark 2.2. Given a maximally compact \( \theta \)-stable Cartan subalgebra \( \epsilon_0 \) of \( g \), one can always find a lexicographic order such that \( \theta \) preserves the set of positive roots, and therefore the set of simple roots. It suffices to define a lexicographic order in terms of a basis of \( it = i(\epsilon_0 \cap h) \) followed by a basis of \( a_0 = (\epsilon_0 \cap m) \). By [Kn Proposition 6.70] there are no roots that vanish entirely on \( t = (\epsilon_0 \cap h) \) when the
Cartan subalgebra is maximally compact. Since $\theta$ is 1 on $c_0 \cap h$ and $-1$ on $c_0 \cap m$, we see that the positivity will be preserved under $\theta$.

We write $R(g, c) = R(g^C, c^C)$ for the set of roots associated to the $\theta$-stable Cartan subalgebra $c^C$ restricted to $c$. One can check that the evaluation of a root $\alpha \in R(g, c)$ is imaginary on $(c \cap h)$, and real on $(c \cap m)$. Consequently, a root is real if it vanishes on $(c \cap h)$, imaginary if it vanishes on $(c \cap m)$, and complex otherwise. Recall that, for every root $\alpha \in R(g, c)$, one has that $\theta\alpha \in R(g, c)$ is a root too. If $\alpha$ is imaginary, $\theta\alpha = \alpha$, so $\theta(g^C)^\alpha = (g^C)^{\theta\alpha} = (g^C)^\alpha$ and the root-space $(g^C)^\alpha$ is $\theta$-stable. Since the root-spaces are 1-dimensional, either $(g^C)^\alpha \subset h^C$ and $\alpha$ is said to be compact, or $(g^C)^\alpha \subset m^C$ and $\alpha$ is said to be non-compact. We write $I_{cp}(g, c), I_{nc}(g, c) \subset R(g, c)$ for the sets of imaginary compact and non-compact roots, $R_{xe}(g, c) \subset R(g, c)$ for the subset of real roots and $R_{xc}(g, c)$ for the subset of pairs $\{\alpha, \theta\alpha\}$ of complex roots. If $c_0$ is maximally compact, one can prove that $R_{re}(g, c_0) = 0$ (see Section 2.1.2), and therefore one has the decomposition

$$R(g, c_0) = I_{cp}(g, c_0) \sqcup I_{nc}(g, c_0) \sqcup R_{xe}(g, c_0).$$

Let $c_0 = t \oplus a_0$ be a maximally compact Cartan subalgebra. Let $C_0^c$ be the Cartan subgroup of $G^C$ associated to $c_0^C$ and let $T^C$ be the Cartan subgroup of $H^C$ associated to $t^C$. One should consider two Weyl groups,

$$Y := W(G^C, C_0^c) = W(g^C, c_0^C)$$

and

$$W := W(H^C, T^C) = W(h^C, t^C).$$

Since $a_0^C = 3m^C(t^C)$, the normalizer $N_{H^C}(t^C)$ also normalizes $a_0^C$ and therefore $c_0^C$. This implies that

$$N_{H^C}(T^C) = N_{C_0^c}(C_0^c) \cap H^C$$

where $C_0^c$ is the Cartan subgroup of $G^C$ with Lie algebra $c_0^C$. Then there is a well defined map of Weyl groups,

$$W \rightarrow Y,$$

which is injective since its kernel is the projection of $N_{H^C}(c_0^C) \cap \text{exp}(a_0^C)$ and therefore it is trivial.

**Remark 2.3.** Considered as a subgroup of $Y$, $W$ is the group that preserves the splitting $c_0^C = t^C \oplus a_0^C$, and therefore, $W$ can be described as the subgroup of $Y$ that preserves the decomposition (2.1).

2.1.2. **Strongly orthogonal roots and real Cartan subalgebras.** Take an imaginary non-compact root $\alpha \in I_{nc}(g, c)$ and let $\{x_\alpha : x_\alpha \in (g^C)^\alpha\}_{\alpha \in R(g, c_0)}$ be a set of (non-zero) representatives of the root-spaces closed under the Lie bracket (i.e. for every two $x_{\alpha_1}, x_{\alpha_2}$ contained in our set, one has that $[x_{\alpha_1}, x_{\alpha_2}]$ is contained in the set too). Since $\alpha$ is imaginary, the complex conjugate $\bar{\alpha}$ is contained in $(g^C)^{-\alpha}$. For $x_\alpha \in (g^C)^\alpha$, one can define the first Cayley transform associated to $\alpha$ as

$$\text{cay}_{1, \alpha} := \text{Ad}(\exp \frac{\pi}{4}(\bar{\alpha} - x_\alpha)) : g^C \rightarrow g^C.$$

Given a Cartan subalgebra $c$ of $g$ with compact dimension $n$, the Cayley transform gives us a new Cartan subalgebra $c'$ of compact dimension $n - 1$

$$c' := g \cap \text{cay}_{1, \alpha}(c^C) = \ker(\alpha|_c) \oplus \mathbb{R}(x_\alpha + \bar{\alpha}).$$
Lemma 2.4. Let \( \alpha \in I_{nc}(g, c_0) \) and take \( x_\alpha \in (g^C)^\alpha \). Each choice of \( \pm e^{i\theta} \in U(1) \) gives a different Cartan subalgebra \( \ker(\alpha|_{c_0}) \oplus \mathbb{R}(e^{i\theta}x_\alpha + e^{-i\theta}x_\alpha) \). All these Cartan subalgebras are conjugate under the action of \( T \).

Proof. Note that, unless \( e^{i\theta} = \pm 1 \), \( (x_\alpha + \bar{x}_\alpha) \) and \( (e^{i\theta}x_\alpha + e^{-i\theta}x_\alpha) \) do not commute:

\[
[x_\alpha + \bar{x}_\alpha, e^{i\theta}x_\alpha + e^{-i\theta}x_\alpha] = [x_\alpha + \bar{x}_\alpha, e^{i\theta}x_\alpha + e^{-i\theta}x_\alpha] = [x_\alpha, e^{i\theta}x_\alpha] = e^{i\theta}(1 - e^{-2i\theta})[\bar{x}_\alpha, x_\alpha] \neq 0,
\]

so each gives a different Cartan subalgebra.

The elements of \( T \) have the form \( g = \exp s \) with \( s \in \mathfrak{t} \). Since \( \alpha(s) = i\theta \) is an imaginary number, one has that \( \operatorname{ad}_\alpha(x_\alpha) = e^{i\theta}x_\alpha \), so all these Cartan subalgebras are conjugate by some element of \( T \).

\[\square\]

Remark 2.5. From the proof of Lemma 2.4 we observe that the only automorphisms of \( \mathfrak{c} = g \cap \text{cay}_{1,\alpha}(c^C) \) given by elements of \( T \) are \( \{1, -1\} \) acting on \( \mathbb{R}(x_\alpha + \bar{x}_\alpha) \).

On the other hand, given a real root \( \alpha \in R_{\mathfrak{r}}(g, c) \) of \( \mathfrak{c} \) and a non-zero element \( x_\alpha \in (g^C)^\alpha \) of its associated root-space one has that \( \theta x_\alpha \) is contained in \( (g^C)^{-\alpha} \). Taking \( x_\alpha \), one can define the second Cayley transform

\[\text{cay}_{2,\alpha} := \operatorname{Ad}(\exp \frac{\pi i}{4}(\theta x_\alpha - x_\alpha)): g^C \to g^C.\]

Given a Cartan subalgebra \( \mathfrak{c} \) of \( g \) with compact dimension \( n \), the Cayley transform gives us a new Cartan subalgebra \( \mathfrak{c}'' \) of compact dimension \( n + 1 \)

\[\mathfrak{c}'' := g \cap \text{cay}_{2,\alpha}(c^C) = \ker(\alpha|_{\mathfrak{c}}) \oplus \mathbb{R}(x_\alpha + \theta x_\alpha).\]

See [Su] Section VI.7] for a detailed description of the Cayley transform.

We say that \( \alpha, \beta \in R(g^C, h^C) \) are strongly orthogonal if

\[\alpha + \beta \notin R(g^C, h^C) \quad \text{and} \quad \alpha - \beta \notin R(g^C, h^C).\]

The importance of this definition lies on the fact that one can repeatedly apply the Cayley transform \( \text{cay}_1 \) if we have a set of mutually strongly orthogonal roots. In order to make this statement clearer, set \( I_{nc}^+(g, c_0) := I_{nc}(g, c_0) \cap R^+(g, c_0) \) to be the set of imaginary non-compact roots that are positive with respect to a certain lexicographic order. Following [Su], we say that the subset \( B \subset I_{nc}^+(g, c_0) \) is an admissible root system if any two roots \( \beta_1, \beta_2 \in B \) are strongly orthogonal.

Fix a maximally compact \( \theta \)-stable Cartan subalgebra \( c_0 \) and a non-zero element \( x_\beta \in (g^C)^\beta \) for each \( \beta \in \Delta(g, c_0) \), such that, for each element of the Weyl group \( \omega \in W(g^C, c_0^C) \), one has that \( \omega \cdot x_\beta = x_{\omega \cdot \beta} \) for each \( \beta \in \Delta(g, c_0) \). For the admissible root system \( B = \{\beta_1, \ldots, \beta_t\} \), we define

\[\text{cay}_B := \text{cay}_{1,\beta_t} \circ \cdots \circ \text{cay}_{1,\beta_2} \circ \text{cay}_{1,\beta_1}.\]

Using \( \text{cay}_B \), we set

\[\mathfrak{c}_B := g \cap \text{cay}_B(c^C),\]

which can be described as

\[\mathfrak{c}_B = \bigcap_{\beta \in B} \ker(\beta|_{c_0}) \oplus \bigoplus_{\beta \in B} \mathbb{R}(x_\beta + \bar{x}_\beta).\]
One can check that a permutation of the sequence of Cayley transforms gives the same Cartan subalgebra. We write
\[(2.6) \quad t_B := c_B \cap h = \bigcap_{\beta \in B} \ker(\beta|_t)\]
and
\[(2.7) \quad a_B := c_B \cap m = a_0 \oplus \bigoplus_{\alpha \in B} \mathbb{R}(x_\beta + \overline{\tau}_\beta).\]

Associated to a fixed $\theta$-stable maximally compact Cartan subalgebra $c_0$, we denote by $\Upsilon$ the set of all admissible root systems, that is
\[(2.8) \quad \Upsilon = \{ B \subset I^+_nc(g, c_0) \mid \text{every } \beta_i, \beta_j \in B \text{ are strongly orthogonal} \}.\]
We include the zero set in our definition, $\{0\} \in \Upsilon$.

Take the Weyl groups $Y$ and $W$ given in $(2.2)$ and $(2.3)$, and recall from Remark 2.3 that $W$ can be understood as the subgroup of $Y$ that preserves the set of non-compact roots $I^nc(g, c_0)$. We say that two admissible root systems $B_1$ and $B_2$ are conjugate if there exists an element $\omega \in W$ such that
\[B_2 = \omega \cdot B_1.\]
Conjugacy classes of admissible root systems classify real Cartan subalgebras.

**Lemma 2.6** ([Su] Corollary 2 to Theorem 3 and Theorem 6). Every $\theta$-stable Cartan subalgebra of $g$ is conjugate by $H$ to $c_B$ for some admissible root system $B \in \Upsilon$. Furthermore, the set of conjugacy classes of admissible root systems $B \in \Upsilon$ is in one to one correspondence with the set of conjugacy classes of Cartan subalgebras of $g$.

**Remark 2.7.** We find in [Su] a case by case description of admissible root systems for all real semisimple Lie algebras.

Denoting by $|A|$ the cardinality of a set $A$, we say that the admissible root system $D$ is maximal in $\Upsilon$ if $|D| \geq |B|$ for every $B \in \Upsilon$. Taking $D$ maximal, one has that $c = t \oplus a$ is a maximally non-compact $\theta$-stable Cartan subalgebra and $a$ is a maximal abelian subspace of $m$.

**2.1.3. Some results on Weyl groups.** Given an admissible root system $B \in \Upsilon$ and its associated Cayley transform $\text{cay}_B$, we set $Y_B$ to be the Weyl group $W(g^c, c^c_B)$ associated to $c^c_B$. Note that
\[(2.9) \quad Y_B = \text{cay}_B \circ Y \circ \text{cay}^{-1}_B,\]
and, obviously, $Y_B$ is isomorphic to $Y$.

**Remark 2.8.** Recall that $W$ can be understood as the subgroup of $Y$ that preserves the splitting $c^c_0 = t^c \oplus a^c_0$. Then, the subgroup of $Y_B$ that preserves the splitting $c^c_B = t^c_B \oplus a^c_B$ is contained in the image of $W$. In other words,
\[N_{Y_B}(t^c_B) = N_{Y_B}(a^c_B) \subset \text{cay}_B \circ W \circ \text{cay}^{-1}_B.\]

For any admissible root system $B \in \Upsilon$, define the group
\[(2.10) \quad \Gamma_B := \prod_{\alpha \in B} \alpha \cdot \{1, -1\},\]
Parabolic subgroups and antidominant characters.

We call the connected subgroup $B$ and $cay_\alpha$ elements have the form $h \in H$ where $H$ is a complex reductive Lie group with Lie algebra $\mathfrak{h}$ contained in $Z$, and we refer to any subgroup conjugate to $B$ as a parabolic subgroup.

Define $\Gamma_B = \{ \gamma \in N_B \mid \gamma \in Z \}$ and note that one can define an action of $\Gamma_B$ on $\mathfrak{a}_B$ given by conjugation by $B$, where

$$\mathfrak{a}_B = \mathfrak{a}_0 \oplus \bigoplus_{\alpha \in \Phi_B, \beta \neq \alpha} \mathbb{R}(x_\beta + \overline{\alpha}_\beta)$$

and sending the ray $\mathbb{R}(x_\alpha + \overline{\alpha}_\alpha)$ to the ray $\mathbb{R}(x_{\overline{\alpha}} + \overline{x}_\alpha)$.

Lemma 2.10. We can identify the semidirect product $\Gamma_B \rtimes Z_W(B)$ with a subgroup of $N_{Y_B}(a_B)$ and the action of $N_{Y_B}(a_B)$ on $a_B$ is completely determined by the action of this subgroup.

Proof. Note that $\Gamma_B$ is a commutative normal subgroup of $Y_B$, since it is given by the reflections associated to strictly-orthogonal (and therefore orthogonal) roots $\beta_1, \ldots, \beta_i \in B$ and also $\Gamma_B$ preserves $\mathfrak{a}_B$, so $\Gamma_B \subset N_{Y_B}(a_B)$. Then, it is clear that the subgroup generated by $\Gamma_B$ and $cay_B \circ Z_W(B) \circ cay_B^{-1}$ is a subgroup of $N_{Y_B}(a_B)$. We can identify this subgroup with $\Gamma_B \rtimes Z_W(B)$.

By Remark 2.8, every element of $N_{Y_B}(a_B)$ is of the form $cay_B \circ \omega \circ cay_B^{-1}$ with $\omega \in W$. The action of this element sends the rays of the form $\mathbb{R}(x_\alpha + \overline{\alpha}_\alpha)$ with $\alpha \in B$ to the rays $\mathbb{R}(x_{\overline{\alpha}} + \overline{x}_\alpha)$, possibly changing the orientation of the ray. Then, $\omega$ is contained in $Z_W(B)$ since $\omega \cdot a \in B$ as well. Then the subgroup generated by $\Gamma_B$ and $cay_B \circ Z_W(B) \circ cay_B^{-1}$ is the whole $N_{Y_B}(a_B)$, and the result follows. \qed

2.1.4. Parabolic subgroups and antidominant characters. Let $\mathfrak{h}^C$ be a complex reductive Lie group with Lie algebra $\mathfrak{h}$ being the complexification of some compact Lie algebra $\mathfrak{h}$ and take a Cartan subalgebra $\mathfrak{t}^C = \mathfrak{h}^C = \mathfrak{h}^C \oplus \mathfrak{t}^C$ of $\mathfrak{h}^C$. Let $(\cdot, \cdot)$ be the Killing form extended to $\mathfrak{h}^C$.

For any subset $A \subset \Delta(\mathfrak{h}^C, \mathfrak{t}^C)$ we define $R_A$ to be the subset of $R(\mathfrak{h}^C, \mathfrak{t}^C)$ whose elements have the form $\alpha = \sum_{\beta \in \Delta} m_\beta \beta$ with $m_\beta \geq 0$ for all $\beta \in A$. We define the subalgebra

$$\mathfrak{p}_A := \mathfrak{t}^C \oplus \bigoplus_{\delta \in R_A} (\mathfrak{h}^C)^\delta.$$
The Lie connected subgroup $L_A \subset P_A$ with Lie algebra $\mathfrak{l}_A$ is called the Levi subgroup of $P_A$.

Let $\Lambda_Z$ be the kernel of the exponential map restricted to $\mathfrak{z}_{\mathbb{C}}(\mathfrak{h})$. Define $\mathfrak{z}_{\mathbb{R}}$ to be $\Lambda_Z \otimes_{\mathbb{Z}} \mathbb{R} \subset \mathfrak{z}_{\mathbb{C}}(\mathfrak{h})$ and consider the dual space $\text{Hom}_{\mathbb{R}}(\mathfrak{z}_{\mathbb{R}}, i\mathbb{R})$. For every $\alpha \in \mathcal{R}(\mathfrak{h}, \mathfrak{t})$ we define its coroot as $\hat{\alpha} := \frac{2\alpha^*}{(\alpha, \alpha)}$, where $\alpha^* \in \mathfrak{t}^*$ is such that $\alpha = \langle \cdot, \alpha^* \rangle$. Taking the dual with respect to the Killing form of $\hat{\alpha}$, we define $\lambda_\alpha$ to be the fundamental weight associated to $\alpha$. Note that the fundamental weights are elements of $(\mathfrak{t}^*)^*$ and, by construction, $\lambda|_{\mathfrak{z}_{\mathbb{C}}(\mathfrak{h})} = 0$.

Let $A$ be a subset of $\Delta(\mathfrak{h}, \mathfrak{t})$ and let $\mathfrak{p}_A$ be the standard parabolic subalgebra associated to it. An antidominant character of $\mathfrak{p}_A$ is any element of $(\mathfrak{t}^*)^*$ of the form

$$\chi = \delta + \sum_{\alpha \in A} n_\alpha \lambda_\alpha,$$

where $\delta \in \text{Hom}(\mathfrak{z}_{\mathbb{R}}, i\mathbb{R})$ and each $n_\alpha$ is non-positive. If further we have $n_\alpha < 0$ for every $\alpha \in A$, the character is strictly antidominant.

Remark 2.11. Since $L_A$ is a connected complex reductive Lie group, denoting by $Z_L$ the connected component of its centre, and $L_A^s = [L_A, L_A]$ its semisimple part, one has $L_A \cong Z_L \times F L_A^{ss}$, where $F$ is some finite group. This gives a map $L_A \to Z_L/F$, and composing with $P_A \to L_A$, one has the morphism of Lie groups $\pi_A : P_A \to Z_L/F$.

It the follows that not every character of the Lie algebra $\mathfrak{p}_A$ exponentiates to the associated parabolic group $P_A$, but, since $F$ is finite, we know that for every character $\chi$ of $\mathfrak{p}_A$, there exists $n \in \mathbb{Z}$ such that $\chi^n$ exponentiates to a character of the group $P_A$. Then, the characters of $\mathfrak{p}_A$ which exponentiate generate (as a subset of a vector space) the space of all characters of $\mathfrak{p}_A$.

To any character $\chi$, we associate $s_\chi \in \mathfrak{t}_{ss}^*$, its representative via the Killing form. Note that the roots of $\mathfrak{h}_C$ take pure imaginary values on $\mathfrak{h}$ since $\text{ad} \, h$ with $h \in \mathfrak{t}$ is skew-symmetric with respect to the Killing form. This ensures that $s_\chi$ belongs to $i\mathfrak{h}_{ss}$.

Lemma 2.12. Let $s \in i\mathfrak{h}_{ss}$. Define the sets

$$\mathfrak{p}_s := \{ x \in \mathfrak{h}_C \text{ such that } \text{Ad}(e^{ts})x \text{ remains bounded as } \mathbb{R} \ni t \to \infty \},$$

$$\mathfrak{l}_s := \{ x \in \mathfrak{h}_C \text{ such that } [x, s] = 0 \},$$

$$P_s := \{ g \in H^C \text{ such that } e^{ts}ge^{-ts} \text{ is bounded as } \mathbb{R} \ni t \to \infty \},$$

$$L_s := \{ g \in H^C \text{ such that } \text{Ad}(g)(s) = s \}.$$

The following properties hold:

1. Both $\mathfrak{p}_s$ and $\mathfrak{l}_s$ are Lie subalgebras of $\mathfrak{h}_C$ and $P_s$ and $L_s$ are connected subgroups of $H^C$.

2. Let $\tilde{\chi}$ be an antidominant character of $P_A$. Then there exists $s_\chi$ for which we have inclusions $\mathfrak{p}_A \subset \mathfrak{p}_{s_\chi}$, $\mathfrak{l}_A \subset \mathfrak{l}_{s_\chi}$, $P_A \subset P_{s_\chi}$ and $L_A \subset L_{s_\chi}$, with equality if $\chi$ is strictly antidominant. Furthermore $\chi$ is a strictly antidominant character of $\mathfrak{p}_{s_\chi}$.

3. For any $s \in i\mathfrak{h}$ there exists $h \in H$ and a standard parabolic subgroup $P_A$ such that $P_s = hP_A h^{-1}$ and $L_s = hL_A h^{-1}$. Furthermore, there is a strictly antidominant character $\chi$ of $P_A$ such that $s = hs_\chi h^{-1}$.
Proof. This result is contained in [GGMI] Lemma 2.5, although we provide a proof for the sake of completeness.

One has from the definitions that \( p_s \) and \( l_s \) are subalgebras and \( P_s \) and \( L_s \) groups. Take \( T_s \) to be the closure of \( \{ e^{its} \mid t \in \mathbb{R} \} \). Then \( L_s \) is the centralizer of \( Z_{H^C}(T_s) \) so it is connected by \([Bo1]\) Th. 13.2]. To prove that \( P_s \) is connected note that, if \( g \in P_s \) (i.e. \( e^{its}g e^{-its} \) bounded as \( t \to \infty \)) then the limit exists, and we denote it by \( \pi_s(g) \). Since it is a limit, it follows that \( \pi_s(g) \in L_s \). This gives a morphism of Lie groups \( \pi_s : P_s \to L_s \) that can be identified with the projection \( P_s \to P_s/U_s \cong L_s \), where

\[
U_s := \{ g \in H^C \text{ such that } e^{its}ge^{-its} \text{ converges to 1 as } t \to \infty \} \subset P_S
\]

is the unipotent radical of \( P_s \). Then, for every \( g \in P_s \), the map \( \gamma : [0, \infty) \to H^C \), defined as \( \gamma(t) = e^{its}ge^{-its} \), extends to give a path from \( g \) to \( L_s \). Since \( L_s \) is connected, it follows that \( P_s \) is connected as well. This proves the first statement.

Let \( \chi = \delta + \sum n_{\alpha} \alpha \) be an antidominant character of \( P_A \). Let \( \beta = \sum m_{\alpha} \alpha \) be a root and take \( u \in h_{\alpha} \). One has \( [s_{\chi}, u] = (s_{\chi}, \beta)u = (\chi, \beta)u = \langle \sum n_{\alpha} \alpha, (\alpha, \alpha)/2 \rangle u \). Hence \( \text{Ad}(e^{its})(u) = \langle \sum \exp(\sum n_{\alpha} \alpha, (\alpha, \alpha)/2) \rangle u \), so this remains bounded as \( t \to \infty \) if \( m_{\alpha} \geq 0 \) for any \( \alpha \) such that \( n_{\alpha} \leq 0 \). This implies that \( p_A \subset p_s \) and \( l_A \subset l_s \), the inclusions being equalities when \( \chi \) is strictly antidominant. The analogous results for \( P_A \subset P_s \) and \( L_A \subset L_s \) follow from this and the fact that they are connected. This finishes the proof of the second statement.

To prove the third statement take a maximal torus \( T_s \) containing \( \{ e^{its} \mid t \in \mathbb{R} \} \) and choose \( h \in H \) such that \( h^{-1}T_s h = T \) and \( \text{Ad}(h^{-1})(\xi) \) belongs to the Weyl chamber in \( t \) corresponding to the choice of \( \Delta(h^C, t^C) \). The proof follows from \([2]\). \hfill \square

Lemma 2.13. Take a maximally compact \( \theta \)-stable Cartan subalgebra and a lexicographic order as in Remark \( \text{[2.5]} \). Let \( p \) be a standard parabolic subalgebra of \( h^C \) and let \( \chi \) be an antidominant character of \( p \). If \( p_A \) is preserved by \( \theta \), then \( \theta \chi := \chi \circ \theta \) is an antidominant character of \( p_A \).

Proof. Note that in the context of Remark \( \text{[2.5]} \) \( \theta \) preserves the set of simple roots \( \Delta(h^C, t^C) \). If \( p_A \) is preserved by \( \theta \), then \( \theta(A) \subset A \). Then it is trivial to see that \( \theta \chi \) is antidominant as well. \hfill \square

2.2. \( G \)-Higgs bundles on Riemann surfaces. Let \( \Sigma \) be a compact Riemann surface and denote by \( \Omega^1_{\Sigma} \) its canonical bundle. Let \( G \) be a connected real form of the complex semisimple Lie group \( G^C \). Let \( H \subset G \) be a maximal compact subgroup. Note that \( H^C \) is a connected complex reductive Lie group.

A \( G \)-Higgs bundle over \( \Sigma \) is a pair \(( E, \Phi) \) where \( E \) is a holomorphic \( H^C \)-bundle over \( \Sigma \) and \( \Phi \), called the Higgs field, is a holomorphic section of \( E(m^C) \otimes \Omega^1_{\Sigma} \), where \( E(m^C) \) is the vector bundle associated to the isotropy representation. Two \( G \)-Higgs bundles \(( E, \Phi) \) and \(( E', \Phi') \) are isomorphic if there exists an isomorphism of \( H^C \)-bundles \( f : E \to E' \) such that \( (f \otimes \text{id})^* \Phi' = \Phi \).

Remark 2.14. The Cartan decomposition of the Lie algebra \( \mathfrak{h} \) of a compact Lie group \( H \) is \( \mathfrak{h} = \mathfrak{h} \oplus 0 \), so \( m^C = 0 \). It follows that a \( H \)-Higgs bundle is the same thing as a principal \( H^C \)-bundle.

Remark 2.15. The Cartan decomposition of the Lie algebra \( \mathfrak{h}^C \) of a complex reductive Lie group \( H^C \) is \( \mathfrak{h}^C = \mathfrak{h} \oplus i\mathfrak{h} \), so \( m^C = \mathfrak{h}^C \). A \( H^C \)-Higgs bundle is a pair \(( E, \Phi) \) where \( E \) is a principal \( H^C \)-bundle and \( \Phi \in H^0(\Sigma, E(\mathfrak{h}^C) \otimes \Omega^1_{\Sigma}) \).
Let $S$ be an affine scheme $S$ and denote by $p : \Sigma \times S \rightarrow \Sigma$ the natural projection. We say that an $S$-family of $G$-Higgs bundle is a pair $(E_S, \Phi_S)$, where $E_S$ is a principal $H^C$-bundle over $\Sigma \times S$ (i.e. an $S$-family of principal $H^C$-bundles) and $\Phi_S$ is an element of $H^0(\Sigma \times S, E_S(m^C) \otimes p^*\Omega^1_S)$. Two $S$-families of $G$-Higgs bundles $(E_S, \Phi_S)$ and $(E'_S, \Phi'_S)$ are isomorphic if there exists an isomorphism of $H^C$-bundles $f_S : E_S \rightarrow E'_S$ such that $(f_S \otimes id)^*\Phi'_S = \Phi_S$.

As is well known, in order to define a good moduli problem for the classification of $G$-Higgs bundles, one needs to introduce the notion of semistability.

The Killing form on $\mathfrak{g}$ induces a Hermitian structure on $m^C$ which is preserved by the action of $H^C$. This allows us to define the complex subspace

\[(2.13) \quad (m^C)^- := \{ x \in m^C \text{ such that } \nu(e^{t\alpha})x \text{ remains bounded as } \mathbb{R} \ni t \rightarrow \infty \}.
\]

Let $E$ be a holomorphic $H^C$-bundle and $\sigma$ a holomorphic section of $E(H^C/P_A)$, i.e. a reduction of the structure group giving the $P_A$-bundle $E_\sigma$. We see that $(m^C)_\chi^-$ is invariant under the action of $P_{s_{\chi}}$ and by Lemma 2.12 we have that $P_A \subset P_{s_{\chi}}$. Then we define

\[E(m^C)^-_{\sigma,\chi} := E_\sigma \times_{P_A} (m^C)^-_\chi^-.\]

Now define

\[(m^C)^0_{\chi} := \{ x \in m^C \text{ such that } [s_{\chi}, x] = 0 \}.
\]

This subspace is invariant under $L_{s_{\chi}}$ and hence under $L_A$ by Lemma 2.12. Suppose that $\sigma_L$ is a reduction of the structure group of $E_\sigma$ giving the $L_A$-bundle $E_{\sigma_L}$. Let us set

\[E(m^C)^0_{\sigma_L,\chi} := E_{\sigma_L} \times_{L_A} (m^C)^0_{\chi} \subset E(m^C)^-_{\sigma_L,\chi}.
\]

Given a $H^C$-bundle $E$ with a reduction of the structure group $\sigma$ to the parabolic subgroup $P_A$ and an antidominant character $\chi$ of $p_A$, we define the degree of $E$ with respect to $\sigma$ and $\chi$ as in [GPR, Section 5],

\[\deg_{\sigma,\chi}(E) := \frac{1}{\eta} \deg(\bar{\chi}^{\eta}E_\sigma),\]

where, following Remark 2.11, $\alpha$ is a character of $p_A$ that exponentiates to $P_A$.

We say that the $G$-Higgs bundle $(E, \Phi)$ is semistable (resp. stable) if for any parabolic subgroup $P_A \subset H^C$, any antidominant character $\chi$ of $p_A$, and a reduction of the structure group $\sigma$ to the parabolic subgroup $P_A$ such that $\Phi \in H^0(\Sigma, E(m^C)^-_{\sigma,\chi} \otimes \Omega^1_S)$, we have

\[\deg_{\sigma,\chi}(E) \geq 0 \quad \text{ (resp. } \deg_{\sigma,\chi}(E) > 0).\]

Also, we say that $(E, \Phi)$ is polystable if it is semistable and for any $P_A, \chi$ and $\sigma$ as above, such that $\Phi \in H^0(X, E(m^C)^-_{\sigma,\chi})$, $P_A \neq H^C$ and $\chi$ is strictly antidominant, and such that

\[\deg_{\sigma,\chi}(E) = 0,
\]

there is a holomorphic reduction of the structure group $\sigma_L$ to the associated Levi subgroup $L_A$ and $\Phi$ is contained in $H^0(\Sigma, E(m^C)^0_{\sigma_L,\chi} \otimes \Omega^1_S)$.

Remark 2.16. A principal $H^C$-bundle $E$ is semistable or polystable if the $H$-Higgs bundle $(E, 0)$ is respectively semistable or polystable.

Given a semisimple subgroup $L \subset G$ preserved by the Cartan decomposition, its Lie algebra $\mathfrak{l}$ decomposes into $\mathfrak{l}_\mathfrak{h} \oplus \mathfrak{l}_\mathfrak{m}$, where $\mathfrak{l}_\mathfrak{h} = \mathfrak{l} \cap \mathfrak{h}$ and $\mathfrak{l}_\mathfrak{m} = \mathfrak{l} \cap \mathfrak{m}$. The subgroup $L_H = L \cap H$ is the maximal compact subgroup of $L$. We say that $(E, \Phi)$ reduces to
L if there exists a reduction of structure group $\sigma$ of $E$ to $L^\sigma_H$, giving the $L^\sigma_H$-bundle $E_\rho$ and $\Phi(E_\rho(t_m^\sigma)) \subset E_\rho(t_m^\sigma) \otimes \Omega^1_{\Sigma}$.

Once we have defined the notion of semistability and polystability, it is possible to construct the moduli functor for the classification problem of $G$-Higgs bundles,

$$\mathcal{M}_\Sigma(G) : (\text{Aff}) \longrightarrow (\text{Sets})$$

$$(2.14) \quad S \mapsto \begin{cases} \text{Isomorphism classes of } S\text{-families} \text{ of semistable } G\text{-Higgs bundles,} \\ \text{with trivial characteristic class.} \end{cases}$$

We denote by $\mathcal{M}_\Sigma(G)$ the moduli space of $G$-Higgs bundles. Its existence follows, in full generality, from the work of Schmitt [Sm].

**Theorem 2.17** ([Sm] Theorem 2.8.1.2). There exists a scheme $\mathcal{M}_\Sigma(G)$ corepresenting the moduli functor $\mathcal{M}_\Sigma(G)$. The points of $\mathcal{M}_\Sigma(G)$ correspond to isomorphism classes of polystable $G$-Higgs bundles.

**Remark 2.18.** We have not given any formal definition of $S$-equivalence for $G$-Higgs bundles. This is done in [Sm] and also in [GGM1] Section 2.10, where Jordan–Hölder filtrations are defined. For our purposes, it is sufficient to say that two semistable $G$-Higgs bundles are $S$-equivalent if they determine the same point of the moduli space $\mathcal{M}_\Sigma(G)$.

Following Simpson [Si2, Si3], one can give a rigidification of the moduli functor that provides a fine moduli space. For a fixed geometric point $x_0 \in \Sigma$, we define a **framing** of the $H^C$-bundle $E$ to be an isomorphism $\xi : E|_{\{x_0\}} \xrightarrow{\sim} H^C$. Given an $S$-family $(E_S, \Phi_S)$ of $G$-Higgs bundles, we say that a framing for the family is an isomorphism $\xi_S : E|_{\{x_0\} \times S} \xrightarrow{\sim} H^C \otimes \mathcal{O}_S$. Two $(S\text{-families of}) G$-Higgs bundles with framing, $(E_S, \Phi_S, \xi_S)$ and $(E'_S, \Phi'_S, \xi'_S)$, are isomorphic if there exists an isomorphism of $(S\text{-families of}) G$-Higgs bundles $f : (E_S, \Phi_S) \xrightarrow{\sim} (E'_S, \Phi'_S)$ such that $\xi_S = \xi'_S \circ f|_{\{x_0\} \times S}$. Let us define as follows the moduli functor for the classification of $G$-Higgs bundles with framing,

$$\mathcal{F}_\Sigma(G, x_0) : (\text{Aff}) \longrightarrow (\text{Sets})$$

$$(2.15) \quad S \mapsto \begin{cases} \text{Isomorphism classes of } S\text{-families} \text{ of semistable } G\text{-Higgs bundles,} \\ \text{with framing at } x_0 \text{ and trivial characteristic class.} \end{cases}$$

**Proposition 2.19** ([Si3] Theorem 9.6 and Proposition 9.7 for the case $G = H^C$). There exists a scheme $\mathcal{F}_\Sigma(G, x_0)$ representing the functor $\mathcal{F}_\Sigma(G, x_0)$. Furthermore, there exists an $H^C$-action on $\mathcal{F}_\Sigma(G, x_0)$ and

$$(2.15) \quad \mathcal{M}_\Sigma(G) \cong \mathcal{F}_\Sigma(G, x_0) \parallel H^C.$$

The closed orbits of (2.15) are those given by $G$-Higgs bundles with framing whose underlying $G$-Higgs bundles are polystable.

**Proof.** With minor changes, we can use [FGN2, 3.13] to extend [Si3, Theorem 9.6 and Proposition 9.7] to the case of a real semisimple group $G$. \hfill $\square$

In agreement with Simpson [Si2, Si3], we refer to the scheme $\mathcal{F}_\Sigma(G, x_0)$ as the representation space of $G$-Higgs bundles.
2.3. Principal bundles over elliptic curves. From now on, \((X, x_0)\) (or just \(X\)) will denote an elliptic curve. We write \(X\) for the variety \(\text{Pic}^0(X)\). The Abel–Jacobi map \(x \mapsto O(x) \otimes O(x_0)^{-1}\) gives an isomorphism \(X \cong \hat{X}\), which induces an abelian group structure on \(X\). Having in mind the isomorphism \(X \cong \hat{X}\), we will maintain the use of \(\hat{X}\) through this section in order to clarify the exposition.

Let \(\rho : \pi_1(X) \to H\) be a representation of the fundamental group into a compact Lie group \(H\), we shall refer to such a representation as a unitary representation. Let \(\overline{\rho} : \pi_1(X) \to H/Z_H(H)\) be the induced representation. We say that \(\rho\) is topologically trivial if \(\overline{\rho}\) can be lifted to a representation into the universal cover of \(H/Z_H(H)\). Given a topologically trivial representation of the fundamental group \(\rho\), one can define a holomorphic \(H^C\)-bundle that we denote by \(E_\rho\). If \(\hat{X} \to X\) is the \(\pi_1(X)\)-bundle defined by the universal cover of \(X\), we define \(E_\rho := \rho_* (\hat{X})\), where \(\rho_*\) denotes the extension of structure group associated to the representation. This construction is shown in [AB, Section 6] and [Ra] where is also stated that

- the bundle \(E_\rho\) is polystable,
- \(E_\rho\) has trivial characteristic class,
- two bundles \(E_{\rho_1}\) and \(E_{\rho_2}\) are isomorphic if and only if \(\rho_1\) and \(\rho_2\) are conjugate, and
- every polystable \(H^C\)-bundle is isomorphic to \(E_\rho\) for some topologically trivial unitary representation \(\rho : \pi_1(X) \to H\).

For every element \(y \in \mathfrak{h}(\rho)\), one can define

\[
\eta_y : \mathbb{C} \to Z_{H^C}(\rho)
\]

\[
t \mapsto \exp_H(y \cdot t).
\]

Using \(\eta_y\), one can define

\[
(\eta_y)_* : H^1(X, \mathcal{O}_X) \to H^1(X, Z_{H^C}(\rho)),
\]

and we construct the \(Z_{H^C}(\rho)\)-bundle

\[
L_y := (\eta_y)_* \xi,
\]

where \(\xi\) is a fixed non-zero element of the 1-dimensional space \(H^1(X, \mathcal{O}_X)\). Due to the commutativity, the following map is a morphism of groups,

\[
\mu_\rho : \text{im}(\rho) \times Z_{H^C}(\rho) \to H^C
\]

\[
(a, b) \mapsto ab.
\]

Note that \(E_\rho\) is naturally a \(\text{im}(\rho)\)-bundle while \(L_y\) is a \(Z_{H^C}(\rho)\)-bundle and therefore we can consider the associated extension of structure groups giving a \(H^C\)-bundle,

\[
(2.16) \quad E_{\rho, y} := (\mu_\rho)_*(E_\rho \times_X L_y).
\]

Fix a Stein cover \(\{U_i\}\) on \(X\) such that each \(U_i\) is simply connected and each \(U_i \cap U_j\) is either connected or empty for all \(i \neq j\). Let \(\{f_{ij}\}\) be a non-zero 1-cocycle for the cover \(\{U_i\}\) associated to \(\xi \in H^1(X, \mathcal{O}_X)\). Let \(\{h_{ij}\}\) be the normalized transition functions of \(E_\rho\) with respect to the open cover \(\{U_i\}\). Then, the transition functions of \(E_{\rho, y}\) for the cover \(\{U_i\}\) are

\[
(2.17) \quad \{h_{ij} \exp_H(f_{ij} y)\}.
\]

Since \(y\) and \(\rho\) commute, one can check that \((2.17)\) satisfies the cocycle condition.

Remark 2.20. Note that when \(y \in \mathfrak{h}(\rho)\) is nilpotent, \(E_{\rho, y}\) is semistable and \(S\)-equivalent to \(E_\rho\).
Remark 2.21. Let $V$ be a complex vector space on which $\mathcal{H}_C$ acts. Then, for any $E_{\rho,y}$,

$$H^0(X, E_{\rho,y}(V)) = Z_V(\text{im} \rho) \cap Z_V(y).$$

Over an elliptic curve, every semistable $\mathcal{H}_C$-bundle can be expressed in these terms.

Proposition 2.22 ([FM] Theorem 3.6 and Theorem 4.1).

1. Let $E$ be a semistable principal $\mathcal{H}_C$-bundle over the elliptic curve $X$. Then there exist a central unitary representation $\rho$ and a nilpotent element $y \in \mathfrak{h}(\rho)$ such that $E \cong E_{\rho,y}$.

2. The group of automorphisms of $E_{\rho,y}$ is identified with

$$\text{Aut}_{\mathcal{H}_C}(E_{\rho,y}) = Z_{\mathcal{H}_C}(\rho, y) = Z_{\mathcal{H}_C}(\rho) \cap Z_{\mathcal{H}_C}(y).$$

3. $E_{\rho,y}$ and $E_{\rho',y'}$ are isomorphic if and only if $\rho$ and $\rho'$ are conjugate by an element $h \in \mathcal{H}$ sending $y$ to $y'$.

The fundamental group of an elliptic curve is abelian, $\pi_1(X) \cong \mathbb{Z} \times \mathbb{Z}$. Then, the representations associated to a polystable $\mathcal{H}_C$-bundle of trivial characteristic class are completely determined by commuting pairs $(a, b) = (\rho(\alpha), \rho(\beta))$ such that $[a, b] = \text{id}$ and such that the projections $\pi, \overline{\pi}$ of $a$ and $b$ to $H/Z_H(H)$ can be lifted to a commuting pair in the universal cover of this last group. By a result of Borel [Bo2], such $a$ and $b$ are contained in the same maximal torus $T \subset H$ (up to conjugation by $H$). From the Narasimhan–Seshadri–Ramanathan Theorem [NS, Ra], one obtains the following result.

Proposition 2.23 ([FMW], [La]). Let $H$ be a compact group and let $T$ be a maximal torus. Every topologically trivial polystable $\mathcal{H}_C$-bundle over the elliptic curve $X$ admits a reduction of structure group to $T$.

Given a torus $T$, denote its cocharacter lattice by

$$\Lambda_T := \text{Hom}(U(1), T) = \text{Hom}(\mathbb{C}^*, \mathcal{T}_C),$$

which is a lattice in $\mathfrak{t}$. Note that the fundamental group is $\pi_1(T) = \Lambda_T$. One has the natural isomorphism of groups

$$\mathbb{C}^* \otimes_\mathbb{Z} \Lambda_T \xrightarrow{\cong} \mathcal{T}_C \sum_i u_i \otimes_\mathbb{Z} \lambda_i \mapsto \Pi_i \lambda_i(u_i).$$

Take the Poincaré bundle $\mathcal{P}_{\mathcal{C}_T} \to X \times \hat{X}$. For a given torus $T$, using the isomorphism (2.19) and fibre products of the Poincaré bundle, one can construct a family of $\mathcal{T}_C$-bundles with trivial characteristic class,

$$\mathcal{P}_T \to X \times (\hat{X} \otimes_{\mathbb{Z}} \Lambda_T).$$

By [Se3 Theorem 9.6] (among other references), $\mathcal{P}$ is a universal family for the classification problem for $\mathcal{T}_C$-bundles of characteristic class 0 with framing.

Recall Proposition 2.23. Let $i : T \hookrightarrow H_C$ be the natural injection and denote by $i_*$ the extension of structure group associated to it. As a consequence, the family

$$\mathcal{E}_H := i_*(\mathcal{P}_T) \to X \times (\hat{X} \otimes_{\mathbb{Z}} \Lambda_T)$$

induces a surjective morphism from its parametrizing space to the moduli space $M_X(H_C)$ of topologically trivial $H_C$-bundles:

$$\hat{X} \otimes_{\mathbb{Z}} \Lambda_T \to M_X(H_C).$$
There is a standard action of the Weyl group $W = W(H^c, T^c) = N_H(t)/Z_H(t)$ on $A_T$, which extends naturally to an action on $X \otimes \Lambda_T$. The previous surjection factors through this action giving a bijection. Since the moduli space $M_X(H^c)$ is a normal variety, this is enough to prove the following.

**Theorem 2.24** ([FMW] Theorem 2.6, [La] Theorem 4.16). Let $H^c$ be a connected complex reductive Lie group and let $T \subset H$ be a maximal torus. Then

(2.22) \[ M_X(H^c) \cong (\hat{X} \otimes \Lambda_T) / W. \]

3. G-Higgs bundles over elliptic curves

Over an elliptic curve $X$, one has $\Omega_X^1 \cong \mathcal{O}_X$. Therefore, a $G$-Higgs bundle over $X$ is a pair $(E, \Phi)$, where $E$ is a principal holomorphic $H^c$-bundle and $\Phi \in H^0(X, E(m^c))$.

3.1. Stability in terms of the underlying principal bundle. We have the non-canonical isomorphism $H^0(X, \mathcal{O}_X) \cong \mathbb{C}$. To simplify the presentation of the results in this section, we pick (non-canonically) a non-zero element of this space, $s \in H^0(X, \mathcal{O}_X)$.

**Proposition 3.1.** Let $(E, \Phi)$ be a semistable $G$-Higgs bundle. Then $E$ is a semistable $H^c$-bundle.

**Proof.** Fix a maximally compact $\theta$-stable Cartan subalgebra $a_0$ and a lexicographic order as in Remark 2.22.

Suppose that $E$ is an unstable $H^c$-bundle. We know by the Harder-Narasimhan Theorem [AB, Section 10] that $E$ has a reduction $\sigma$ to some parabolic subgroup $P_{HN} \subset H^c$ giving the $P_{HN}$-bundle $E_{\sigma}$. Since $P_{HN}$ is defined up to conjugation, one can assume that it is a standard parabolic subgroup associated to the subset $A \subset \Delta(h^c, t^c)$.

We know that $E_{G^c}$ is an unstable $G^c$-bundle and so we can apply again the Harder–Narasimhan Theorem to obtain a reduction $\gamma$ to the parabolic subgroup $Q_{HN} \subset G^c$ giving the $Q_{HN}$-bundle $(E_{G^c}, \gamma)$. We take $Q_{HN}$ to be a standard parabolic. The theorem also ensures the existence of an antidominant character $\tau$ of $Q_{HN}$ such that $\deg \gamma, \tau(E_{G^c}) < 0$, and implies that the holomorphic sections of the adjoint bundle are contained in the reductions to the Harder–Narasimhan parabolics

\[ H^0(X, E(h^c)) = H^0(X, E_{\sigma}(p_{HN})) \]

and

\[ H^0(X, E_{G^c}(g^c)) = H^0(X, (E_{G^c}, \gamma)(q_{HN})). \]

By [AB, Proposition 10.4], the Harder–Narasimhan reduction is functorial with respect to group homomorphisms, so $H^c \to G^c$ implies that $P_{HN} \subset Q_{HN}$ and therefore the Lie algebra $q_{HN}$ is preserved by the Cartan involution. As a consequence

(3.1) \[ H^0(X, E(m^c)) = H^0(X, E_{\sigma}(q_{HN} \cap m^c)). \]

Recall that $\theta$ denotes the Cartan involution. Since $q_{HN}$ is preserved by $\theta$, by Lemma 2.13 we know that there exists an antidominant character $\eta = \frac{1}{2}(\tau + \tau \circ \theta)$ of $q_{HN}$. Therefore, one has an antidominant character $\chi$ of $p_{HN}$ such that $\chi = |\gamma|_{h^c}$ and then the representatives via the Killing form of $\chi$ and $\eta$ are equal, $s_\chi = s_\eta$. 


Since \( \tau \) restricted to \( \mathfrak{p}_{\text{HN}} \) is equal to our character \( \chi \) and \( \gamma_*(E_{G^c}) = (\sigma_*E)Q_{\text{HN}} \), one has that
\[
\deg_{\gamma_*(\tau)}(E) = \deg_{\gamma_*(\tau)}(E_{G^c}) < 0.
\]

(3.2) 
Recall from (2.13) the linear subspaces \( (m^c)_\chi^- \) and \( q^\eta_- \). Since \( s_\chi = s_\eta \) we know that \( (m^c)_\chi^- = q^\eta_- \cap m^c \). By Lemma (2.14) we have \( (q_{\text{HN}}) \subset q^\eta_- \), so \( q_{\text{HN}} \cap m^c \subset (m^c)_\chi^- \). The parabolic subgroup \( P \) acts on both subalgebras so
\[
E_\sigma(q_{\text{HN}} \cap m^c) \subseteq E_\sigma((m^c)_\chi^-).
\]
Due to (3.1) and the statement above, we have
\[
H^0(X, E((m^c)_\chi^-)) = H^0(X, E(m^c)).
\]
The existence of an antidominant character \( \chi \) of \( \mathfrak{p} \) satisfying (3.2) and (3.3) implies that every \( G \)-Higgs bundle of the form \( (E, \Phi) \) is unstable. \( \square \)

From Propositions 3.1 and 2.22 and Remark 2.21 one has the following description of semistable \( G \)-Higgs bundles up to isomorphism.

**Corollary 3.2.** Let \( X \) be an elliptic curve.

1. Every semistable \( G \)-Higgs bundle over \( X \) is isomorphic to \( (E_{\rho,y}, z \otimes s) \) for some topologically trivial unitary representation \( \rho : \pi_1(X) \to H \), \( y \in \mathfrak{z}_{H^c}(\rho) \) nilpotent, \( z \in \mathfrak{z}_{H^c}(\rho) \cap \mathfrak{z}_{H^c}(y) \).
2. If \( \rho \) is a topologically trivial unitary representation of \( \pi_1(X) \), \( y \) a nilpotent element of \( \mathfrak{z}_{H^c}(\rho) \) and \( z \in \mathfrak{z}_{H^c}(\rho) \cap \mathfrak{z}_{H^c}(y) \), then the \( G \)-Higgs bundle \( (E_{\rho,y}, z \otimes s) \) is semistable.
3. The group of automorphisms of \( (E_{\rho,y}, z \otimes s) \) is identified with
\[
\text{Aut}_G(E_{\rho,y}, z \otimes s) = Z_{H^c}(\rho, y, z) = Z_{H^c}(\rho) \cap Z_{H^c}(y) \cap Z_{H^c}(z).
\]
4. The \( G \)-Higgs bundles \( (E_{\rho,y}, z \otimes s) \) and \( (E_{\rho',y'}, z' \otimes s) \) are isomorphic if and only if \( \rho \) and \( \rho' \) are conjugate by an element \( h \in H^c \) sending \( y \) to \( y' \) and \( z \) to \( z' \).

We continue with our study of stability.

**Proposition 3.3.** Let \( (E, \Phi) \) be a polystable \( G \)-Higgs bundle. Then \( E \) is a polystable \( H^c \)-bundle.

**Proof.** Suppose that \( (E, \Phi) \) is a polystable \( G \)-Higgs bundle. By Corollary 3.2 one can assume with no loss of generality that \( (E, \Phi) = (E_{\rho,y}, z \otimes s) \) where \( y \in \mathfrak{z}_{H^c}(\rho) \) is nilpotent. Since \( z \) belongs to \( \mathfrak{z}_{H^c}(\rho) \cap \mathfrak{z}_{H^c}(y) \) we can construct the semistable \( G \)-Higgs bundle \( (E_{\rho}, z \otimes s) \). Using Remark 2.20 we see that \( (E_{\rho,y}, z \otimes s) \) and \( (E_{\rho}, z \otimes s) \) are S-equivalent.

In each S-equivalence class, there is only one isomorphy class of polystable \( G \)-Higgs bundle. So, if \( (E_{\rho}, z \otimes s) \) is polystable as well, we would have that \( (E_{\rho,y}, z \otimes s) \) and \( (E_{\rho'}, z \otimes s) \) are necessarily isomorphic and the proof would be completed since, in that case
\[
E = E_{\rho,y} \cong E_{\rho}
\]
is a polystable \( H^c \)-bundle.

Take a parabolic subgroup \( P \) and a strictly antidominant character \( \chi \) such that \( \text{im}(\rho) \subset P \) (giving a reduction \( \sigma \) of \( E_{\rho} \) to \( P \)), \( z \in (m^c)_{\sigma,\chi} \) and
\[
\deg_{\sigma,\chi}(E_{\rho}) = 0.
\]
We claim that there exists a reduction of $E\gamma\sim\gamma$ (reduction of $E\gamma\gamma(y)$) and only if $E\gamma\gamma(z\otimes s)$ is polystable and therefore $(E\gamma\gamma,y\gamma,z\otimes s)\cong (E\gamma\gamma,z\otimes s)$, so the proof is follows from this claim.

Let us prove the polystability of $(E\gamma\gamma,z\otimes s)$. Take $p'$ to be the minimal parabolic subalgebra containing $y$ and the parabolic subalgebra $p' = \text{Lie}(P)$. Let $P'$ be the parabolic subgroup associated with $p'$ and let $\gamma' : P' \rightarrow C$ be the antidominant character determined by $s\chi$ as in Lemma 2.12 (therefore we have $s\chi = s\chi$). By construction $\text{im}(\rho) \times U_y$ is contained in $P'$ (we take $U_y$ to be the unipotent group generated by $y$), so there is a reduction $\sigma'$ of the structure group of $E\gamma\gamma,y$ to $P'$. Note that we have $z \in (m^\gamma)_{\sigma',\gamma}$, since $P \subset P'$ and $s\chi = s\chi$.

Let $n$ be a positive integer such that $(\gamma')^n$ exponentiates to a character of the group $(\gamma')^n : P' \rightarrow C^*$. By construction of $E\gamma\gamma,y$, one has that $(\gamma')^n \cdot e_{\gamma\gamma} \cong \gamma^n \cdot e_{\gamma\gamma} \otimes (\gamma')^n L_y$, where the transition functions of $(\gamma')^n L_y$ are $\{e^{ad}\gamma(\gamma')f_{ij}\}$. This line bundle is topologically trivial since we can give a connected path $\gamma$ on the moduli space of line bundles connecting $(\gamma')^n L_y$ with the trivial bundle. We have

$$\deg_{\sigma',\gamma}(E\gamma\gamma,y) = \frac{1}{n} \deg((\gamma')^n L_y) = 0.$$
can consider \( \lambda \) to be a 1-parameter subgroup of \( H^C \) and we let \( \lambda \) act on \( \mathcal{F}_X(G, x_0) \). Since the image of \( \lambda \) is contained in \( Z_{HC}(\rho) \) its action on \( E_\rho \) is the identity. The previous discussion implies, trivially, that
\[
\lim_{t \to 0} \lambda(t) \cdot (E_\rho, z \otimes s)
\]
exists but does not belong to the \( H^C \)-orbit of \((E_\rho, z \otimes s)\) inside \( \mathcal{F}_X(G, x_0) \). Then, the \( H^C \)-orbit of \((E_\rho, z \otimes s)\) is not closed and \((E_\rho, z \otimes s)\) is not polystable by Proposition 2.19.

Now, we suppose that there exists a maximal abelian subalgebra \( a^C_\rho \) of \( \mathfrak{m}^C(\rho) \) containing \( z \). Then, the subalgebra \( \mathfrak{h}_C(\rho, z) = \mathfrak{h}(\rho, z)^C \) is reductive. Take an abelian subalgebra \( \mathfrak{s} \) of \( \mathfrak{h}(\rho, z) \) and let \( S \subset H \) be the torus with Lie algebra \( \mathfrak{s} \). Note that \( \text{im}(\rho) \subset Z_{HC}(S) \) and \( z \in \mathfrak{m}^C_S \), by construction. The \( G \)-Higgs bundle \((E_\rho, z \otimes s)\) reduces to a \( Z_G(S) \)-Higgs bundle. Recall the definitions in Lemma 2.12 and (2.13) for any \( s' \in i\mathfrak{h}(S) \). Note that, if one has
\[
\text{im}(\rho) \subset P_{s'}
\]
and
\[
z \in (m^C)_{s'}
\]
then, by the maximality of \( \mathfrak{s} \) inside \( \mathfrak{h}(\rho, z) \), this implies that \( s' \in i\mathfrak{s} \). Then, \((E_\rho, z \otimes s)\) is a stable \( Z_G(S) \)-Higgs bundle and by [BGM] it gives a solution of the Hitchin equations, so it is a polystable \( G \)-Higgs bundle.

Using Lemma 3.5 and Corollary 3.2, one can complete the description of polystable \( G \)-Higgs bundles that we started in Corollary 3.3.

**Corollary 3.6.** Let \( X \) be an elliptic curve.

1. Every polystable \( G \)-Higgs bundle over \( X \) is isomorphic to \((E_\rho, z \otimes s)\) for some topologically trivial unitary representation \( \rho : \pi_1(X) \to H \) and \( z \in a^C_\rho \), where \( a^C_\rho \) is a maximal abelian subalgebra of \( \mathfrak{m}^C(\rho) \).
2. Let \( \rho : \pi_1(X) \to H \) be a topologically trivial unitary representation, let \( a^C_\rho \) be a maximal abelian subalgebra of \( \mathfrak{m}^C(\rho) \) and take \( z \in a^C_\rho \). Every \( G \)-Higgs bundle of the form \((E_\rho, z \otimes s)\) is polystable.
3. The group of automorphisms of \((E_\rho, z \otimes s)\) is identified with
\[
\text{Aut}_G(E_\rho, z \otimes s) = Z_{HC}(\rho, z) = Z_{HC}(\rho) \cap Z_{HC}(z),
\]
and is a complex reductive subgroup of \( H^C \).
4. The polystable \( G \)-Higgs bundles \((E_\rho, z \otimes s)\) and \((E_{\rho'}, z' \otimes s)\) are isomorphic if and only if \( \rho \) and \( \rho' \) are conjugate by an element \( h \in H^C \) sending \( z \) to \( z' \).

### 3.2. The representation space.

Proposition 3.1 allows us to describe \( \mathcal{F}_X(G, x_0) \) in terms of \( \mathcal{F}_X(H, x_0) \). Recall that \( \mathcal{F}_X(H, x_0) \) is a fine moduli space and let \( \mathcal{U}_H : X \times \mathcal{F}_X(H, x_0) \to \mathcal{O}_X \) be the corresponding universal bundle. Take the obvious projection
\[
q : X \times \mathcal{F}_X(H, x_0) \to \mathcal{F}_X(H, x_0).
\]
If \( \mathcal{U}_H(m^C) \) is the vector bundle induced from \( \mathcal{U}_H \) under the isotropy action of \( H^C \) on \( m^C \) and \( R^1q_*\mathcal{U}_H(m^C) \) the 1-cohomology direct image sheaf under \( q \). This is a sheaf over \( \mathcal{F}_X(H, x_0) \) whose stalk over \((E, \xi)\) coincides with \( H^1(X, E(m^C)) \). Take the symmetric algebra \( \text{Sym}^*(R^1q_*\mathcal{U}_H(m^C)) \) associated to this sheaf and consider the
scheme \( \text{Spec}(\text{Sym}^\bullet(R^1q_*\mathcal{U}_H(m^C))) \). Note that this scheme projects naturally to \( \mathcal{F}_X(H, x_0) \).

\[(3.4)\quad p : \text{Spec}(\text{Sym}^\bullet(R^1q_*\mathcal{U}_H(m^C))) \to \mathcal{F}_X(H, x_0),\]

and the fibre over \((E, \xi) \in \mathcal{F}_X(H, x_0)\) is \(H^1(X, E(m^C))^*\).

**Proposition 3.7.** The scheme \( \mathcal{F}_X(G, x_0) \) represents the moduli functor \( \mathcal{F}_X(G, x_0) \) and one has an isomorphism of schemes

\[(3.5)\quad \mathcal{F}_X(G, x_0) \cong \text{Spec}(\text{Sym}^\bullet(R^1q_*\mathcal{U}_H(m^C))).\]

Furthermore, the representation space of \(G\)-Higgs bundles projects to the representation space of \(H^C\)-bundles,

\[(3.6)\quad \mathcal{F}_X(G, x_0) \to \mathcal{F}_X(H, x_0)
(\xi, \varphi, \xi) \mapsto (\xi, \varphi),\]

and the fibre of \((3.6)\) over \((E, \xi) \in \mathcal{F}_X(H, x_0)\) is \(H^0(X, E(m^C))\).

**Proof.** Recall the Cartan decomposition \(\mathfrak{g}^C = \mathfrak{h}^C \oplus m^C\), where \(m^C\) is orthogonal to \(\mathfrak{h}^C\) under the Killing form. Since the adjoint bundle \(E(\mathfrak{g}^C)\) is naturally self dual, this orthogonality implies that \(E(m^C)\) is self-dual as well. Thanks to Serre duality and the triviality of the canonical bundle, one has a canonical identification

\[H^1(X, E(m^C))^* \cong H^0(X, E(m^C)).\]

Let \(\tau : X \times \text{Spec}(\text{Sym}^\bullet(R^1q_*\mathcal{U}_H(m^C)))\) be the tautological section and take the family

\[U_G : = ((\text{id} \times p)^*U_H, \tau) \to X \times \text{Spec}(\text{Sym}^\bullet(R^1q_*\mathcal{U}_H(m^C))).\]

It follows, by the universal properties of \(U_H\), that \(U_G\) is a universal family for the moduli functor \(\mathcal{F}_X(G, x_0)\). Since \(\mathcal{F}_X(G, x_0)\) corepresents this functor, one necessarily obtains the isomorphism \((3.5)\).

Let \(\mathcal{F}_\Sigma(G, x_0)^{ps}\) denote the subset of \(\mathcal{F}_\Sigma(G, x_0)\) given by the polystable \(G\)-Higgs bundles. In general, this subset is not open or closed inside \(\mathcal{F}_\Sigma(G, x_0)\). The purpose of this section is to show that, in the case of an elliptic curve \(\Sigma = X\), one can prove that \(\mathcal{F}(G, x_0)^{ps} \subset \mathcal{F}_X(G, x_0)\) is closed.

Recall from \(2.21\) the family of polystable \(H^C\)-bundles \(\mathcal{E}_H \to X \times (\hat{X} \otimes \Lambda_T)\) and fix a framing \(\xi\) at \(x_0\) for it. The family \((\mathcal{E}_H, \xi) \to X \times (\hat{X} \otimes \Lambda_T)\) induces, by moduli theory, a morphism to the representation space

\[(3.7)\quad \nu_H : \hat{X} \otimes \Lambda_T \to \mathcal{F}_X(H, x_0).\]

**Lemma 3.8.** Let \(H\) be a compact Lie group. The polystable locus \(\mathcal{F}_X(H, x_0)^{ps}\) is closed inside \(\mathcal{F}_X(H, x_0)\). Furthermore,

\[(3.8)\quad \mathcal{F}_X(H, x_0)^{ps} = H^C \cdot \nu_H(\hat{X} \otimes \Lambda_T),\]

and

\[(3.9)\quad \nu_H(\hat{X} \otimes \Lambda_T) \cong \hat{X} \otimes \Lambda_T.\]

**Proof.** The map \(\nu_H\) in \((3.7)\) is closed since \(\hat{X} \otimes \Lambda_T\) is compact. The image of \(\nu_H\) is contained in \(\mathcal{F}_X(H, x_0)^{ps}\). In fact \(H^C \cdot \nu_H(\hat{X} \otimes \Lambda_T)\) is clearly contained in \(\mathcal{F}_X(H, x_0)^{ps}\). Furthermore, since every polystable \(H^C\)-bundle is isomorphic to one parametrized by \(\mathcal{E}_H\), one has that

\[(3.10)\quad H^C \cdot \nu_H(\hat{X} \otimes \Lambda_T) / H^C \to M_X(H^C)\]
is surjective and therefore an isomorphism since $H^C \cdot \nu_H(\hat{X} \otimes \Lambda_T)$ injects into $\mathcal{F}_X(H, x_0)$. This implies \cite{3.8} and therefore $\mathcal{F}_X(H, x_0)^{ps}$ is closed inside $\mathcal{F}_X(H, x_0)$.

Finally, recall that $M_X(H^C)$ is described in Theorem \cite{2.22} as the finite quotient \cite{2.22}. Note that this, together with the surjection \cite{3.10}, implies \cite{3.9}. \qed

**Proposition 3.9.** Let $H$ be a maximal compact subgroup of $G$ and let $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ be the Cartan decomposition of its Lie algebra. Then, the polystable locus $\mathcal{F}_X(G, x_0)^{ps}$ is closed inside $\mathcal{F}_X(G, x_0)$ and isomorphic to a closed subset of the direct product $\mathcal{F}_X(H, x_0)^{ps} \times (m^C \otimes H^0(X, \mathcal{O}_X))$.

**Proof.** Recall the family of polystable $H^C$-bundles $\mathcal{E}_H \to X \times (\hat{X} \otimes \Lambda_T)$ and take the projection

$$q : X \times (\hat{X} \otimes \Lambda_T) \to \hat{X} \otimes \Lambda_T.$$ 

Consider a construction analogous to (3.4), giving a natural projection

$$\pi : \Sigma_G := \text{Spec} (\text{Sym}^*(R^1q_*\mathcal{E}_H(m^C))) \to \hat{X} \otimes \Lambda_T.$$ 

Note that the fibre over $t \in \hat{X} \otimes \Lambda_T$ is $H^0(X, \mathcal{E}_H|_t(m^C))$. Using the tautological section

$$\tau : X \times \Sigma_G \to \mathcal{E}_H(m^C),$$

one can construct the family of $G$-Higgs bundles with framing

$$((\text{id} \times \pi)^*\mathcal{E}_H, \tau, (\text{id} \times \pi)^*\xi) \to X \times \Sigma_G.$$ 

Thanks to Corollary \cite{3.2} we know that this family parametrizes all polystable $G$-Higgs bundles of characteristic class $d$ (although, as we know by Lemma \cite{3.5} some $G$-Higgs bundles parametrized by this family might be strictly semistable). By moduli theory, there exists a map from the parametrizing space of this family to the moduli space $\mathcal{F}_X(G, x_0)$,

$$\nu_G^e : \Sigma_G \to \mathcal{F}_X(G, x_0).$$

By Remark \cite{2.21} and the construction of the family $\mathcal{E}_H$, for every $t \in \hat{X} \otimes \Lambda_T$ one has

(3.11) $H^0(X, \mathcal{E}_H|_t(m^C)) \subset H^0(X, m^C \otimes \mathcal{O}_X) \cong m^C \otimes H^0(X, \mathcal{O}_X).$

Indeed, the elements of $H^0(X, \mathcal{E}_H|_t(m^C))$ have the form $z \otimes s$ where $z \in m^C$ commutes with the transition functions of $\mathcal{E}_H|_t = \mathcal{P}|_t \otimes \mathcal{E}^0_{|_t}$. Note that the conjugation of $\mathcal{E}_H|_t$ by any $h \in H^C$ preserves the previous inclusion

(3.12) $H^0(X, \text{ad}_h(\mathcal{E}_H|_t)(m^C)) \subset H^0(X, m^C \otimes \mathcal{O}_X).$

By (3.11), one has that $\nu_G^e(\Sigma_G)$ is isomorphic to a closed subset $S_0$ of $\nu_H(\hat{X} \otimes \Lambda_T) \times (m^C \otimes H^0(X, \mathcal{O}_X))$. Furthermore, thanks to \cite{3.12}, one has that

(3.13) $H^C \cdot \nu_G^e(\Sigma_G) \cong H^C \cdot S_0,$

where $H^C \cdot S_0$ is a closed subset of $(H^C \cdot \nu_H(\hat{X} \otimes \Lambda_T)) \times (m^C \otimes H^0(X, \mathcal{O}_X))$.

By \cite{3.8}, $H^C \cdot S_0$ is a closed subset of $\mathcal{F}_X(H, x_0)^{ps} \times (m^C \otimes H^0(X, \mathcal{O}_X))$ and it corresponds to the restriction of $R^0q_*\mathcal{U}_H(m^C)$ to $\mathcal{F}_X(H, x_0)^{ps}$. Recalling that not every $G$-Higgs bundle parametrized by $H^C \cdot S_0$ is polystable, we consider the closed subset

$$S := (H^C \cdot S_0) \cap (\mathcal{F}_X(H, x_0)^{ps} \times (m^C)^{ss} \otimes H^0(X, \mathcal{O}_X)), $$
where \((\mathfrak{m}^C)^{ss}\) is the closed subset of semisimple elements of \(\mathfrak{m}^C\) given by
\[
(\mathfrak{m}^C)^{ss} := H^C \cdot \mathfrak{a}_B^C,
\]
with \(\mathfrak{a}_B^C \subset \mathfrak{m}^C\) a maximal abelian subalgebra (recall that all the maximal abelian subalgebras of \(\mathfrak{m}^C\) are conjugate under \(H^C\)). Thanks to Lemma 3.5, we see that \(\mathcal{S}\) corresponds under \((3.13)\) with those elements of \(H^C \cdot \nu'_G(\Sigma_G)\) that are polystable, and this coincides with \(\mathcal{F}_X(G, x_0)^{ps}\).

3.3. The moduli space. Using Proposition 3.7 one can describe \(\mathcal{M}_X(G)\) in terms of a fibration over \(M_X(H^C)\).

**Corollary 3.10.** The moduli space of \(G\)-Higgs bundles projects onto the moduli space of \(H^C\)-bundles,
\[
(3.14) \quad \mathcal{M}_X(G) \longrightarrow M_X(H^C), \quad (E, \Phi) \longrightarrow E.
\]

Let \(\rho : \pi_1(X) \to H\) be a topologically trivial unitary representation and let \(E_\rho\) be the polystable \(H^C\)-bundle associated to it. The fibre of the surjection \((3.14)\) over the isomorphism class of \(E_\rho\) is identified with the vector space \(\mathfrak{z}_{m^C}(\rho)/Z_{H^C}(\rho)\).

**Proof.** Since \((3.6)\) is \(H^C\)-equivariant, it descends to
\[
\mathcal{M}_X(G) \cong \mathcal{F}_X(G, x_0)/H^C \longrightarrow M_X(H^C) \cong \mathcal{F}_X(H, x_0)/H^C.
\]
The fibre over the isomorphism class of \(E_\rho\) is \(H^0(X, E_\rho(m^C))/\text{Aut}_{H^C}(E_\rho)\). By Proposition 2.22 and Remark 2.21 this is identified with \(\mathfrak{z}_{m^C}(\rho)/Z_{H^C}(\rho)\). \(\square\)

Let \(\theta\) be a Cartan involution of \(\mathfrak{g}\) whose associated Cartan decomposition is \(\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}\) and fix a maximally compact \(\theta\)-stable Cartan subalgebra \(\mathfrak{a}_0\) of \(\mathfrak{g}\). We recall that \(\mathfrak{a}_0 = t \oplus \mathfrak{a}_0\), where \(t \subset \mathfrak{h}\) is the Lie algebra of a maximal torus \(T\) of \(H\) and \(\mathfrak{a}_0 \subset \mathfrak{m}\). Fix once for all \(x_0 \in (\mathfrak{g}^C)^{\alpha}\) for each \(\alpha \in \Delta(\mathfrak{g}, \mathfrak{a}_0)\) and recall that every admissible root system \(B\) defines as in \((2.2)\) a Cartan subalgebra \(\mathfrak{c}_B\) of \(\mathfrak{g}\), that splits into \(\mathfrak{t}_B \oplus \mathfrak{a}_B\) as we observe in \((2.6)\) and \((2.7)\). Let us denote by \(T_B \subset T\) the torus with Lie algebra \(\mathfrak{t}_B\). Recall from \((2.2)\) that \(\Upsilon\) is the set of all possible admissible systems.

**Lemma 3.11.** Let \((E, \Phi)\) be a polystable \(G\)-Higgs bundle over an elliptic curve. Then there exists an admissible root system \(B \in \Upsilon\) such that \((E, \Phi) \cong (E_\rho, z \otimes s)\) where \(\rho : \pi_1(X) \to T_B\) and \(z \in \mathfrak{a}_B^C\).

**Proof.** Since \(z\) is semisimple by Lemma 3.5 there exists a \(\theta\)-stable Cartan subalgebra \(\mathfrak{c}\) that contains it. By construction \(\text{im}\rho \subset \exp(\mathfrak{c} \cap \mathfrak{h})\) and \(z \in (\mathfrak{c} \cap \mathfrak{m})^C\). Finally, note that \(\mathfrak{c}\) is conjugate to some \(\mathfrak{c}_B\) by Lemma 2.6. \(\square\)

Every \(\alpha \in I_{nc}(\mathfrak{g}, \mathfrak{a}_0)\) (the set of imaginary non-compact roots) is by definition an element of the dual space \(\text{Hom}(t, i\mathbb{R})\). Since \(\Lambda_T = \text{Hom}(\mathbb{C}^*, T^C)\) can be seen as a lattice inside \(t \subset \mathfrak{c}_B^C\) and \(\alpha\) is a root of \(R(\mathfrak{g}^C, \mathfrak{c}_B^C)\), we have that \(\alpha(\Lambda_T) \subset i\mathbb{Z}\). Therefore, for each \(\alpha\) one has a well defined projection
\[
\eta_\alpha : \hat{X} \otimes Z\Lambda_T \longrightarrow \hat{X}, \quad \sum_j L_j \otimes \lambda_j \longrightarrow \bigotimes_i L_j^{\otimes (\alpha_i)(i)}.
\]
Given an admissible root system \(B \in \Upsilon\) with \(B = \{\alpha_1, \ldots, \alpha_t\}\) we define
\[
(3.15) \quad (\hat{X} \otimes \Lambda_T)|_B := \bigcap_{\alpha \in B} \eta_\alpha^{-1}(O_{\alpha X})
\]
which is a closed subset of $\hat{X} \otimes_{\mathbb{Z}} \Lambda_T$. Note that $(\hat{X} \otimes_{\mathbb{Z}} \Lambda_T)|_0 = \hat{X} \otimes_{\mathbb{Z}} \Lambda_T$.

**Remark 3.12.** Recall the definition of $t_B$ in (2.4) and the construction of $\mathcal{P}_T$ (2.20), where we made use of the isomorphism of groups (2.19). Note that, by the construction (3.15), the restriction of $\mathcal{P}_T$ to $(\hat{X} \otimes_{\mathbb{Z}} \Lambda_T)|_B$ reduces its structure group to the compact torus $T_B$.

**Remark 3.13.** Let $B \in \Upsilon$ be an admissible root system and take $t \in (\hat{X} \otimes_{\mathbb{Z}} \Lambda_T)|_B$ and $z \in a_B^\circ$. By Remark 3.12 and Lemma 3.5, we have that $(\mathcal{E}_H|_{X \times \{t\}}, z \otimes s)$ is a polystable $G$-bundle of trivial characteristic class.

**Remark 3.14.** Since $T_B$ is a subtorus of $T$, one has that the cocharacter lattice $\Lambda_{T_B} = \text{Hom}(\mathbb{C}^*, T_B^\circ)$ is a sublattice of $\Lambda_T = \text{Hom}(\mathbb{C}^*, T^\circ)$. We can describe

$$(\hat{X} \otimes_{\mathbb{Z}} \Lambda_T)|_B = \hat{X} \otimes_{\mathbb{Z}} \Lambda_{T_B} \subset \hat{X} \otimes_{\mathbb{Z}} \Lambda_T.$$  

As a consequence, we observe that the $(\hat{X} \otimes_{\mathbb{Z}} \Lambda_T)|_B$ are irreducible.

Let us define the closed subset of $(\hat{X} \otimes_{\mathbb{Z}} \Lambda_T) \times \mathfrak{m}^C$ given by the union of the irreducible subvarieties,

$$(3.16) \quad \Xi'_G := \bigcup_{B \in \Upsilon} \left( (\hat{X} \otimes_{\mathbb{Z}} \Lambda_T)|_B \times a_B^C \right).$$  

Recalling the action of $\Gamma_B$ on $a_B^C$ defined in (2.11), set also

$$(3.17) \quad \Xi_G := \bigcup_{B \in \Upsilon} \left( (\hat{X} \otimes_{\mathbb{Z}} \Lambda_T)|_B \times \left( a_B^C \backslash \Gamma_B \right) \right).$$  

By construction of $\Xi_G$, one has the obvious projection

$$(3.18) \quad \Xi_G \quad \longrightarrow \quad \hat{X} \otimes_{\mathbb{Z}} \Lambda_T$$  

$$(t, z) \quad \longmapsto \quad t.$$  

Let $\Upsilon_t$ be the set of admissible root systems $\{B_1, \ldots, B_\ell\}$, where $B_i \in \Upsilon_t$ if and only if

$t \in (\hat{X} \otimes_{\mathbb{Z}} \Lambda_T)|_{B_i}.$

For every $B_i \in \Upsilon_t$, there is only one admissible root system $D_j \in \Upsilon_t$ which contains $B_i$ as a subset and is not contained in any other element of $\Upsilon_t$. We say that such an element is maximal in $\Upsilon_t$, and we denote the set of all of them by

$\Upsilon_t^{max} = \{D_1, \ldots, D_k \in \Upsilon_t : D_j \text{ is maximal in } \Upsilon_t\}.$

**Remark 3.15.** Every $a_{B_i}^\circ$ with $B_i \in \Upsilon_t$ is contained in some $a_{D_j}^C$ for some $D_j \in \Upsilon_t^{max}$. Thus, the fibre of (3.18) over $t$ is precisely the union $\bigcup_{D_j \in \Upsilon_t^{max}} a_{D_j}^\circ$.

We define the family of $G$-Higgs bundles parametrized by $\Xi'_G$

$$(3.19) \quad \mathcal{H} \longrightarrow X \times \Xi'_G,$$

setting for every $(t, z) \in \Xi'_G$,

$$\mathcal{H}|_{(t, z)} := (\mathcal{E}_H|_{t}, z \otimes s).$$

Recall that $t \in (\hat{X} \otimes_{\mathbb{Z}} \Lambda_T)|_B$ and $z \in a_B^C$ for some $B \in \Upsilon$. 


Remark 3.16. By Remark 3.13, $\mathcal{H}$ is a well-defined family of polystable $G$-Higgs bundles. By Proposition 3.11, every polystable $G$-Higgs bundle of trivial characteristic class is isomorphic to $\mathcal{H}|_{(t,z)}$ for some $(t,z) \in \Xi_G$.

Recall from Remark 2.3 that $W$ preserves $I_{nc}(g,0)$ and therefore $\Upsilon$. Then, $\omega \in W$ sends $a_B^c$ to $a_B^c$, and if $\omega$ lies in $Z_W(B)$ and therefore $\omega$ normalizes $a_B^c$, we set
$$\omega \cdot (x^\alpha + y^\alpha) = x^{\omega \cdot \alpha} + y^{\omega \cdot \alpha}.$$This allows us to extend the action of $W$ on $\hat{X} \otimes_\mathbb{Z} \Lambda_T$ to $\Xi'_G$ and further to $\Xi'_G$. Two points of $\Xi_G$ related by the action of $W$ are conjugate by some element of $H^c$.

The family $\mathcal{H} : X \times \Xi'_G$ of topologically trivial polystable $G$-Higgs bundles comes naturally with a framing at $x_0$. Take the map to the representation space induced by $\mathcal{H}$ and this framing,
$$\nu_G : \Xi'_G \rightarrow \mathcal{F}_X(G,x_0).$$

In the next lemma we identify a closed subscheme of the moduli space containing the reduced subscheme.

Lemma 3.17. The image of $\nu_G$ is closed inside $\mathcal{F}_X(H,x_0)$. Furthermore,
$$\nu_G(\Xi'_G) \cong \Xi'_G.$$Also, the polystable locus is $\mathcal{F}_X(G,x_0)^{ps} = H^c \cdot \nu_G(\Xi'_G)$ and
$$\mathcal{M}' := H^c \cdot \nu_G(\Xi'_G) \parallel H^c$$is a closed subscheme of $\mathcal{M}_X(G)$ such that
$$\mathcal{M}^c_{X}(G) \subset \mathcal{M}' \subset \mathcal{M}_X(G).$$

Proof. Recall from Proposition 3.9 that $\mathcal{F}_X(G,x_0)^{ps}$ is isomorphic to a closed subset of the direct product $\mathcal{F}_X(H,x_0)^{ps} \times (m_c \otimes H^0(X,\mathcal{O}_X))$. After this and (3.9), we see that the projection of $\nu_G(\Xi'_G)$ to $\mathcal{F}_X(H,x_0)^{ps}$ is isomorphic to $\hat{X} \otimes_\mathbb{Z} \Lambda_T$. It also follows from this decomposition that the fibre over $\nu_G((\hat{X} \otimes_\mathbb{Z} \Lambda_T)|_{B})$ is precisely $a_B^c$. This proves (3.21) and the closedness of $\nu_G(\Xi'_G)$.

The bundles parametrized by $\mathcal{H}$ are polystable, so the image of $\nu_G$ is contained in $\mathcal{F}_X(G,x_0)^{ps}$ as well as $H^c \cdot \nu_G(\Xi'_G)$. In fact, since $H^c \cdot \nu_G(\Xi'_G)$ is closed and $H^c$-invariant, one has that
$$\mathcal{M}' \subset \mathcal{M}_X(G)$$is a closed subscheme. But by Remark 3.14, every polystable $G$-Higgs bundle is contained in $H^c \cdot \nu_G(\Xi'_G)$. Then, $\mathcal{M}'$ contains every closed point of $\mathcal{M}_X(G)$ so it contains the reduced subscheme of $\mathcal{M}_X(G)$. □

We can now address the main theorem of the article.

Theorem 3.18. Let $G$ be a connected real form of the complex semisimple Lie group $G^c$ and let $X$ be an elliptic curve. The reduced moduli space of topologically trivial $G$-Higgs bundles over $X$ is
$$\mathcal{M}^c_{X}(G) \cong \Xi'_G / W.$$Proof. Recalling that the action of $H^c$ over the polystable locus is free, we have from the description of (3.22), that
$$\mathcal{M}' = H^c \cdot \nu_G(\Xi'_G) \parallel H^c = H^c \cdot \nu_G(\Xi'_G) / H^c.$$
The action of the $\Gamma_B$ on (3.17) comes from the conjugation by elements in $T^C$. Also, since two points of $\Xi_G$ related by the action of $W$ are conjugate by some element of $H^C$, the morphism (3.20) induces

$$\Xi_G / W \rightarrow H^C \cdot \nu_G(\Xi_G) / H^C = \mathcal{M}'.$$

By (3.9), the construction of $\Xi'_G$ and Proposition 3.9 it follows that

$$\nu_G(\Xi'_G) \cong \Xi'_G.$$

Then, due to the universality of the quotients, the morphism (3.24) is an isomorphism provided it is bijective.

The next part of the proof is devoted to proving bijectivity of (3.24). Thanks to the projection (3.18), one can construct

$$\Xi_G / W \rightarrow \hat{X} \otimes _Z \Lambda_T / W,$$

and the following diagram commutes

$$\begin{array}{ccc}
\Xi_G / W & \rightarrow & H^C \cdot \nu_G(\Xi'_G) / H^C \\
\downarrow & & \downarrow \\
\hat{X} \otimes _Z \Lambda_T / W & \rightarrow & H^C \cdot \nu_H(\hat{X} \otimes _Z \Lambda_T) / H^C.
\end{array}$$

Since the bottom row morphism is an isomorphism by (3.8) and Theorem 2.24 it is enough to prove bijectivity on the fibres.

Take a point $t \in \hat{X} \otimes _Z \Lambda_T$ and take $\rho : \pi_1(X) \rightarrow T$ such that $\nu(t)$ corresponds to the polystable bundle $E_\rho$. By Remark 3.13 the fibre of the left column morphism over $t$ is $\bigcup_{D_j \in T_{i_{max}}^+} a_{D_j}^C$ quotted by $Z_W(t)$, those elements that fix $t$. On the other column, by Proposition 3.10 the fibre associated to $E_\rho$ is $\mathfrak{m}_{iC}(\rho) / Z_{H^C}(\rho)$.

Then, the previous commuting diagram restricts to

$$\begin{array}{ccc}
\bigcup_{D_j \in T_{i_{max}}^+} (a_{D_j}^C / \Gamma_{D_j}) / Z_W(t) & \rightarrow & \mathfrak{m}_{iC}(\rho) / Z_{H^C}(\rho) \\
\downarrow & & \downarrow \\
[t]_W & \rightarrow & [\nu(t)]_{H^C}.
\end{array}$$

Fixing $D_1 \in T_{i_{max}}^+$, one has that

$$\bigcup_{D_j \in T_{i_{max}}^+} (a_{D_j}^C / \Gamma_{D_j}) / Z_W(t) = (a_{D_1}^C / \Gamma_{D_1}) / Z_W(D_1) \cap Z_W(t).$$

By the choice of $\rho$, one has that $Z_W(t) = Z_W(\rho)$ and the maximal abelian subalgebra of $\mathfrak{m}_{iC}(\rho)$ is conjugate to $a_{D_1}^C$ (and to every $a_{D_j}$ with $D_j \in T_{i_{max}}^+$). Recall that $a_B^C$ is the maximal abelian subalgebra of $\mathfrak{m}_B^C$. The Real Chevalley Theorem (see for instance [Kn] Theorem 6.57) allows us to express the GIT quotient in terms of a quotient of the maximal abelian subalgebra by the Small Weyl group

$$\mathfrak{m}_{iC}(\rho) / Z_{H^C}(\rho) \cong a_{D_1}^C / W_{sm}(Z_{H^C}(\rho), a_{D_1}^C).$$

Note that $a_{D_1}^C = \hat{v}_{D_1}^C \oplus a_{D_1}^C$ is a Cartan subalgebra of $Z_{G^C}$. Let $Y_D^\rho$ be the Weyl group $W(Z_{G^C}(\rho), a_{D_1}^C)$. By [Ko] Theorem 3], the small Weyl group is generated by
the normalizer of $a_{D_1}^C$ in the Weyl group, that is
\[ a_{D_1}^C \bigg/ W_{sm}(Z_{H^C}(\rho), a_{D_1}^C) \cong a_{D_1}^C \bigg/ N_{Y_{D_1}}(a_{D_1}^C). \]

Since $a_{D_1}^C$ is contained in $\mathfrak{m}^C(\rho)$, we have that $\text{im} \rho$ is contained in $T_{D_1} \subset T$ and therefore the Weyl group of $Z_{G^C}(\rho)$ is the subgroup of the Weyl group of $G^C$ that centralizes $\rho$, \[ Y_{D_1}^\rho = Z_{Y_{D_1}}(\rho). \]

Then,
\[ N_{Y_{D_1}}(a_{D_1}^C) = N_{Y_{D_1}}(a_{D_1}^C) \cap Z_{Y_{D_1}}(\rho). \]

By Lemma 3.10 one has
\[ a_{D_1}^C \bigg/ N_{Y_{D_1}}(a_{D_1}^C) \cap Z_{Y_{D_1}}(\rho) \cong a_{D_1}^C \bigg/ (\Gamma_{D_1} \times Z_W(D_1)) \cap Z_W(\rho), \]
and therefore
\[ (3.29) \quad a_{D_1}^C \bigg/ W_{sm}(Z_{H^C}(\rho), a_{D_1}^C) \cong (a_{D_1}^C / \Gamma_{D_1}) \bigg/ Z_W(D_1) \cap Z_W(\rho). \]

Combining (3.27), (3.28) and (3.29) one concludes that the top row morphism of (3.26) is an isomorphism. Then (3.24) is an isomorphism,
\[ \mathcal{M}' \cong \Xi_G / W. \]

Hence, $\mathcal{M}'$ is reduced.

Finally, since $\mathcal{M}_{X}^{\text{red}}(G) \subset \mathcal{M}' \subset \mathcal{M}_X(G)$ by Lemma 3.17, and $\mathcal{M}'$ is reduced, one has that $\mathcal{M}_{X}^{\text{red}}(G) = \mathcal{M}'$ and the proof is complete. \[ \square \]

Remark 3.19. Consider the family $\mathcal{H}_G \to X \times \Xi_G$ and denote by $\mathcal{H}_B$ the restriction to $(\check{X} \otimes \Lambda_T)|_B \times a_B^C$. Denote by $p_B$ the morphism to the moduli space induced by $\mathcal{H}_B$, given by moduli theory. Since $\Gamma_B \times Z_W(B)$ normalizes $(\check{X} \otimes \Lambda_T)|_B \times a_B^C$, after Theorem 3.18, one has
\[ (3.30) \quad p_B \left( (\check{X} \otimes \Lambda_T)|_B \times a_B^C \right) \cong \left( (\check{X} \otimes \Lambda_T)|_B \times a_B^C \right) / \Gamma_B \times Z_W(B). \]

Each of the $p_B \left( (\check{X} \otimes \Lambda_T)|_B \times a_B^C \right)$ is irreducible, since $(\check{X} \otimes \Lambda_T)|_B \times a_B^C$ is irreducible. We observe that $\mathcal{M}_{X}^{\text{red}}(G)$ has the following decomposition into irreducible components,
\[ \mathcal{M}_{X}^{\text{red}}(G) = \bigcup_{B \in \mathcal{T}} p_B \left( (\check{X} \otimes \Lambda_T)|_B \times a_B^C \right). \]

Remark 3.20. In the case of complex semisimple Lie groups $H^C$ the Cartan decomposition is $\mathfrak{h}^C = \mathfrak{h} \oplus i\mathfrak{h}$. In that case, Cartan subalgebras have the form $\mathfrak{t} \oplus i\mathfrak{t}$, i.e. the non-compact part of a Cartan subalgebra is $\mathfrak{a}_0 = i\mathfrak{t}$. Also, we note that there are no imaginary non-compact roots, since for every root $\alpha$, one has that $(\mathfrak{h}_C)^\alpha$ is contained in the complexification of the compact subalgebra $\mathfrak{h}_C$ (which in this case coincides with the total Lie algebra). Then
\[ \Xi_{H^C} = (\check{X} \otimes \Lambda_T) \times \mathfrak{a}_0^C = (\check{X} \otimes \Lambda_T) \times \mathfrak{t}^C \cong (T^* \check{X} \otimes \Lambda_T), \]
where we recall that $T^* \check{X} \cong \check{X} \otimes \mathbb{C}$ and $\mathfrak{t}^C \cong \mathbb{C} \otimes \Lambda_T$, by the differential of (2.19).

We also observe that, in this case, $\Xi_{H^C} = \Xi_{H^C}^\rho$ by construction. Then, (3.28) becomes
\[ \mathcal{M}_{X}^{\text{red}}(H^C) \cong (T^* \check{X} \otimes \Lambda_T) / W. \]
and we recover the description of $[1]$.

**Remark 3.21.** For the group $G = \text{SU}(1, 1)$ one has that $H = \text{S}(\text{U}(1) \times \text{U}(1)) \cong \text{U}(1)$ and therefore $W = \{1\}$. The complexification of SU(1, 1) is $\text{SL}(2, \mathbb{C})$, which has a single (imaginary non-compact) root $\alpha$. Since $T = H = \text{U}(1)$ we have that $\Lambda_T \cong \lambda \cdot \mathbb{Z}$, where $\lambda = 2i\alpha^*$ is the generator of $\Lambda_T$. Then $\hat{X} \otimes \mathbb{Z} \Lambda_T \cong X$. Also, one has that $\eta_\alpha: \hat{X} \otimes \mathbb{Z} \Lambda_T \longrightarrow \hat{X}$.

The complexification of SU(1, 1) is SL(2, $\mathbb{C}$), which has a single (imaginary non-compact) root $\alpha$. Since $T = \text{U}(1)$ we have that $\Lambda_T \cong \lambda \cdot \mathbb{Z}$, where $\lambda = 2i\alpha^*$ is the generator of $\Lambda_T$. Then $\hat{X} \otimes \mathbb{Z} \Lambda_T \cong X$. Also, one has that $\eta_\alpha: \hat{X} \otimes \mathbb{Z} \Lambda_T \longrightarrow \hat{X}$, and we recover the description of $[1]$. $\lambda \cdot \mathbb{Z}$.

so the preimage $\eta_\alpha^{-1}(O_X)$ is the subset $\hat{X}[2] \subset \hat{X}$ of points of order 2 (square roots of $O_X$). To obtain $\Xi_{\text{SU}(1, 1)}$ we glue a copy of $\mathbb{C}$ over the points of this set. Therefore $\Xi_{\text{SU}(1, 1)}$ is $\hat{X} \cup (\hat{X}[2] \times \mathbb{C}/\pm)$ and, since $W$ is trivial, Theorem 3.18 gives the following isomorphism,

$$\mathcal{M}^\text{red}_{\text{SU}(1, 1)} \cong \hat{X} \cup (\hat{X}[2] \times \mathbb{C}/\pm).$$

This moduli space is not normal since the singular locus has codimension 1.

**Corollary 3.22.** The dimension of the moduli space of topologically trivial $G$-Higgs bundles is

$$\dim_{\mathbb{C}}(\mathcal{M}(G)) = \dim_{\mathbb{C}}(\mathcal{M}^\text{red}(G)) = \text{rk}(G).$$

**Proof.** Take the dense open subset $U \subset (\hat{X} \otimes \mathbb{Z} \Lambda_T)$ defined as the complement of the union of all $(\hat{X} \otimes \mathbb{Z} \Lambda_T)|_B$ for every non-zero $B \in \mathcal{Y}$. By construction, we have that $\Xi_G|_U = U \times a_0^C$, so

$$\dim_{\mathbb{C}}(\Xi_G) = \dim_{\mathbb{C}}(\Xi_G|_U) = \dim_{\mathbb{C}}(U) + \dim_{\mathbb{C}}(a_0^C) = \dim_{\mathbb{C}}(\hat{X} \otimes \mathbb{Z} \Lambda_T) + \dim_{\mathbb{C}}(a_0^C) = \dim_{\mathbb{C}}(t^C) + \dim_{\mathbb{C}}(a_0^C) = \dim_{\mathbb{C}}(a_0^C) = \text{rk}(G).$$

The group $W$ is finite, so taking the quotient by its action preserves the dimension. $\square$

## 4. Fixed points of involutions

Let $\sigma_G: G^C \to G^C$ be the involution defining the connected real form $G$. As stated in [G, GR], this defines a holomorphic involution in the moduli space of $G^C$-Higgs bundles given by

$$\iota_G: \mathcal{M}_X(G^C) \longrightarrow \mathcal{M}_X(G^C) \quad (E, \Phi) \longmapsto ((\sigma_G)_*E, -(d\sigma_G)_*\Phi).$$

Taking the extension of structure group associated to $H^C \to G^C$ and the inclusion $m^C \subset g^C$ we construct an étale morphism

$$j: \mathcal{M}_X(G) \longrightarrow \mathcal{M}_X(G^C)^{m^C} \tag{4.1}$$

Take a maximally compact Cartan subalgebra $\mathfrak{c}_0 = t \oplus a_0$ of $g$ and let $C^C$ be the associated Cartan subgroup of $G^C$. Write $\Lambda_C$ for the cocharacter lattice of $G^C$,
Λ_{C_0} = \text{Hom}(\mathbb{C}^*, G^C)$, and write $Y$ for the Weyl group $W(G^C, C^C)$. Recall from Remark 3.20 that one has the isomorphism

$$(4.2) \quad \mathcal{M}_X^{\text{red}}(G^C) \cong T^*\hat{X} \otimes_{\mathbb{Z}} \Lambda_{C_0} / Y.$$ 

In this section we study the involution $i_G$ in the context of this description.

**Remark 4.1.** Associated to the involution $\sigma_{U(1)}$ on $\mathbb{C}^*$, that sends $z$ to $\overline{z}^{-1}$, one has

$$i_{U(1)} : \quad T^*\hat{X} \quad \rightarrow \quad T^*\hat{X}$$

$$(L, \phi) \quad \mapsto \quad (L, -\phi).$$

For any $\lambda \in \Lambda_{C_0}$, we define the holomorphic cocharacter $\sigma_G \cdot \lambda = \sigma_G \circ \lambda \circ \sigma_{U(1)}$. By abuse of notation, we denote also by $\sigma_G$ the involution induced on $\Lambda_{C_0}$. This involution allows us to define

$$\hat{\sigma}_G : \quad \mathbb{C}^* \otimes_{\mathbb{Z}} \Lambda_{C_0} \quad \rightarrow \quad \mathbb{C}^* \otimes_{\mathbb{Z}} \Lambda_{C_0}$$

$$(\sum z_i \otimes \lambda_i) \quad \mapsto \quad \sum \sigma_{U(1)}(z_i) \otimes \sigma_G(\lambda_i).$$

We observe that $\hat{\sigma}_G$ corresponds with $\sigma_G$ via the isomorphism (2.19). Thus the diagram

$$(4.3) \quad \begin{array}{ccc}
\mathbb{C}^* \otimes_{\mathbb{Z}} \Lambda_{C_0} & \xrightarrow{\cong} & \mathbb{C}^*_0 \\
\hat{\sigma}_G & \downarrow & \sigma_G \\
\mathbb{C}^* \otimes_{\mathbb{Z}} \Lambda_{C_0} & \xrightarrow{\cong} & \mathbb{C}^*_0
\end{array}$$

commutes. One can also check that the action of the Weyl group $Y = W(G^C, C^C_0)$ on $\Lambda_{C^0}$, extended to $\mathbb{C}^* \otimes_{\mathbb{Z}} \Lambda_{C_0}$, commutes with the natural action of $\omega \in Y$ on $\mathbb{C}^*_0$ under the isomorphism given in (2.19),

$$(4.4) \quad \begin{array}{ccc}
\mathbb{C}^* \otimes_{\mathbb{Z}} \Lambda_{C_0} & \xrightarrow{\cong} & \mathbb{C}^*_0 \\
\omega & \downarrow & \omega \\
\mathbb{C}^* \otimes_{\mathbb{Z}} \Lambda_{C_0} & \xrightarrow{\cong} & \mathbb{C}^*_0
\end{array}$$

In accordance with the definition of $\hat{\sigma}_G$, we set

$$(4.5) \quad i_G : \quad T^*\hat{X} \otimes_{\mathbb{Z}} \Lambda_{C_0} \quad \rightarrow \quad T^*\hat{X} \otimes_{\mathbb{Z}} \Lambda_{C_0}$$

$$(\sum (L_i, \phi_i) \otimes \lambda_i) \quad \mapsto \quad \sum i_{U(1)}(L_i, \phi_i) \otimes \sigma_G \cdot \lambda_i$$

The commutativity of (4.3) implies that $i_G$ commutes with the action of $\omega \in Y$. Therefore this involution descends to the quotient by $Y$.

**Proposition 4.2.** Under the isomorphism (4.2), $i_G$ is identified with

$$(4.6) \quad \begin{array}{ccc}
\mathcal{M}_X^{\text{red}}(G^C) & \xrightarrow{\cong} & \mathcal{M}_X^{\text{red}}(G^C) \\
[\omega] & \mapsto & [\omega] \\
T^*\hat{X} \otimes_{\mathbb{Z}} \Lambda_{C_0} / Y \quad \rightarrow \quad T^*\hat{X} \otimes_{\mathbb{Z}} \Lambda_{C_0} / Y \\
[(t, z)]_Y & \mapsto & [i_G(t, z)]_Y
\end{array}$$
we observe that
\[ \sigma \] \hspace{1cm} (4.8) \hspace{1cm} \sigma \]

The automorphisms \( \sigma \)

The fixed point set

**Corollary 4.3.** The fixed point set \( \mathcal{M}_{X}^{\text{red}}(G^C)^{i_G} \) is the union of all the projections of closed subsets \( (T^* \hat{X} \otimes \Lambda_{C_0})^{i_G \circ \omega} \) given by the fixed points of the automorphisms \( i_G \circ \omega \),

\[
\mathcal{M}_{X}^{\text{red}}(G^C)^{i_G} = \bigcup_{\omega \in Y} p_0 \left( (T^* \hat{X} \otimes \Lambda_{C_0})^{i_G \circ \omega} \right).
\]

The next step is to study the fixed points of the automorphisms \( i_G \) and \( i_G \circ \omega \) for every \( \omega \in Y \). To do so, first we have to study the involution \( \sigma_G : \Lambda_C \to \Lambda_C \) and the automorphisms \( \sigma_G \circ \omega : \Lambda_{C_0} \to \Lambda_{C_0} \). Since

\[
(\sigma_G \circ \omega)^2 = \sigma_G \circ \omega \circ \sigma_G \circ \omega = \sigma_G(\omega) \circ \sigma_G \circ \omega = \sigma_G(\omega) \omega,
\]

we observe that \( \sigma_G \circ \omega \) is an involution if and only if

\[
(4.8) \hspace{1cm} \sigma_G(\omega) \omega = \text{id}.
\]

Note that the projection \( p_0 \) preserves the dimension since it is given by a finite quotient. Then, the dimension of each of the components of the fixed locus is

\[
\text{dim} \left( p_0 \left( (T^* \hat{X} \otimes \Lambda_{C_0})^{i_G \circ \omega} \right) \right) = \frac{\text{dim} \left( (T^* \hat{X} \otimes \Lambda_{C_0})^{i_G \circ \omega} \right)}{\text{ord}(\sigma_G \circ \omega)} = \frac{2 \text{rk}(G_C)}{\text{ord}(\sigma_G \circ \omega)}.
\]

When \( \sigma_G \circ \omega \) is an involution, \( \text{ord}(\sigma_G \circ \omega) = 2 \) and therefore the dimension of the fixed locus is \( \text{rk}(G_C) \). If \( \text{4.8} \) is not satisfied, the order of the automorphism \( \sigma_G \circ \omega \) is greater than 2 and then, the dimension of the fixed locus is lower than \( \text{rk}(G_C) \). We consider the union of all the components with maximal dimension, i.e. those components given by \( \sigma_G \circ \omega \) satisfying \( \text{4.8} \)

\[
\mathcal{M}_{X}^{\text{red}}(G^C)^{i_G}_{\text{max}} := \bigcup_{\sigma_G(\omega) = \text{id}} p_0 \left( (T^* \hat{X} \otimes \Lambda_{C_0})^{i_G \circ \omega} \right).
\]

**Remark 4.4.** When \( \text{4.8} \) is satisfied, \( \sigma_G \circ \omega \) is an involution of \( C_0^C \), and by Lemma 2.6, the fixed point set \( (C_0^C)^{\sigma_G \circ \omega} \) is \( H^\omega \)-conjugate to \( C_B \), for some \( B \in Y \), which is the fixed point set of \( \sigma_G : C_B \to C_B^C \).

One can prove the converse as well.

**Lemma 4.5.** For every \( B \in Y \) there exists \( \omega \in Y \) satisfying \( \text{4.8} \), such that the involution \( \sigma_G : C_B^C \to C_B^C \) is conjugate to the involution \( \sigma_G \circ \omega : C_0^C \to C_0^C \).
Proof. Since \( C_B^C \) and \( C_0^C \) are conjugate by some element \( g \in G^C \), one has that \( \sigma_G : C_B^C \to C_0^C \) is conjugate to \( \text{ad}_g \circ \sigma_G \text{ad}_g^{-1} : C_0^C \to C_0^C \). Note that

\[
\text{ad}_g \circ \sigma_G \text{ad}_g^{-1} = \sigma_G \circ \text{ad}_{\sigma_G(g)} \circ \text{ad}_g^{-1} = \sigma_G \circ \text{ad}_{\sigma_G(g)g^{-1}}.
\]

By hypothesis, \( \sigma_G \) preserves \( C_0^C \), so \( \sigma_G(g)g^{-1} \) belongs to the normalizer \( N_{G^C}(C_0^C) \) and therefore it defines an element of the Weyl group \( Y = W(G^C, C_0^C) \). Furthermore,

\[
\sigma_G(\sigma_G(g)g^{-1}) \cdot (\sigma_G(g)g^{-1}) = \sigma_G^2(g)\sigma_G(g^{-1})\sigma_G(g)g^{-1} = gg^{-1} = \text{id},
\]

and (4.8) is satisfied. \( \square \)

Denote the cocharacter lattice of \( C_B^C \) by \( \Lambda_B := \text{Hom}(\mathbb{C}^*, C_B^C) \) and construct the families of \( G^C \)-Higgs bundles \( \mathcal{H}_0 \to T^*X \otimes \mathbb{Z} \Lambda_{C_0} \) and \( \mathcal{H}_B \to T^*X \otimes \mathbb{Z} \Lambda_{C_B} \) as we did in (3.19) and Remark 3.20. Denote by \( q_0 \), or \( q_B \), the corresponding morphism from the parametrizing space of the family to the reduced moduli space \( \mathcal{M}_{X}^\text{rel}(G^C) \).

**Lemma 4.6.** Take \( \omega \in Y \) satisfying (4.8). Then we have the identification

\[
q_0 \left( (T^*X \otimes \mathbb{Z} \Lambda_{C_0})^{\text{ric}} \right) = q_B \left( (T^*X \otimes \mathbb{Z} \Lambda_{C_B})^{\text{ric}} \right).
\]

**Proof.** Since \( i_G \) is induced by \( \sigma_G \), the lemma follows from Remark 4.4, Lemma 4.5, and the fact that conjugation gives an isomorphism of \( G^C \)-Higgs bundles. \( \square \)

Thus, to study \( \mathcal{M}_X(G^C)_{\text{max}} \), we can reduce ourselves to the study of \( i_G \) acting on \( T^*X \otimes \mathbb{Z} \Lambda_{C_B} \).

**Corollary 4.7.**

\[
\mathcal{M}_{X}^\text{rel}(G^C)_{\text{max}} = \bigcup_{B \in \mathcal{T}} q_B \left( (T^*X \otimes \mathbb{Z} \Lambda_{C_B})^{\text{ric}} \right).
\]

Recall that \( C_B^C \cong \mathbb{C} \otimes \mathbb{Z} \Lambda_{C_B} \) and \( C_0^C \cong \mathbb{C} \otimes \mathbb{Z} \Lambda_{C_B} \), for the sublattice \( \Lambda_{T_B} \subset \Lambda_{C_B} \). Consider also

\[
\Lambda_{ab} := \text{Hom}(\mathbb{C}^*, \exp(a_B^C)),
\]

and note that \( a_B^C \cong \mathbb{C} \otimes \mathbb{Z} \Lambda_{ab} \). It follows that \( \Lambda_{T_B} \oplus \Lambda_{ab} \) has the same rank as \( \Lambda_{C_B} \), although it might be a proper sublattice of \( \Lambda_{C_B} \).

The involution \( \sigma_G \) leaves \( \Lambda_{T_B} \) invariant,

\[
\sigma_G|_{\Lambda_{T_B}} = \text{id}_{\Lambda_{T_B}},
\]

while \( \sigma_G \) inverts \( \Lambda_{ab} \),

\[
\sigma_G|_{\Lambda_{ab}} = -\text{id}_{\Lambda_{ab}}.
\]

We now study the étale morphism (4.1). Recall from Theorem 3.18 that \( \mathcal{M}_X(G) \) is described in terms of \( \Xi_G \), defined in (3.16) as the union of the irreducible components \( (X \otimes \mathbb{Z} \Lambda_T)|_B \times a_B^C \).

**Lemma 4.8.** \( (X \otimes \mathbb{Z} \Lambda_T)|_B \times a_B^C \) is an irreducible component of \( (T^*X \otimes \mathbb{Z} \Lambda_{C_B})^{\text{ric}} \).

**Proof.** Recalling that \( a_B^C \cong \mathbb{C} \otimes \mathbb{Z} \Lambda_{ab} \) and \( (X \otimes \mathbb{Z} \Lambda_T)|_B \cong (X \otimes \mathbb{Z} \Lambda_{T_B}) \), take the natural identification

\[
(4.11) \quad (X \otimes \mathbb{Z} \Lambda_T)|_B \times a_B^C \cong \left( \left( X \times \{0\} \right) \otimes \mathbb{Z} \Lambda_{T_B} \right) \oplus ((O_X) \times \mathbb{C}) \otimes \mathbb{Z} \Lambda_{ab}.
\]
By (4.9), (4.10) and the definition of $i_G$ given in (4.9), one has that $i_G$ restricted to $T^*\hat{X} \otimes_{\mathbb{Z}} (\Lambda_{T_B} \oplus \Lambda_{a_B})$ is

$$i_G : T^*\hat{X} \otimes_{\mathbb{Z}} (\Lambda_{T_B} \oplus \Lambda_{a_B}) \rightarrow T^*\hat{X} \otimes_{\mathbb{Z}} (\Lambda_{T_B} \oplus \Lambda_{a_B})$$

$$\sum (L_i, \phi_i) \otimes_{\mathbb{Z}} (\lambda_i \otimes \lambda'_i) \mapsto \sum (L_i, -\phi_i) \otimes_{\mathbb{Z}} (\lambda_i \otimes -\lambda'_i).$$

Note that

$$i_G \left( \sum (L_i, \phi_i) \otimes_{\mathbb{Z}} (\lambda_i \oplus 0) \right) = \sum (L_i, -\phi_i) \otimes_{\mathbb{Z}} (\lambda_i \oplus 0)$$

and

$$i_G \left( \sum (L_i, \phi_i) \otimes_{\mathbb{Z}} (0 \oplus \lambda'_i) \right) = \sum (L_i, -\phi_i) \otimes_{\mathbb{Z}} (0 \oplus -\lambda'_i)$$

$$= \sum (L^*_i, \phi_i) \otimes_{\mathbb{Z}} (0 \oplus \lambda'_i).$$

Thus, we see that (4.11) is fixed by $i_G$. □

Since $(\hat{X} \otimes_{\mathbb{Z}} \Lambda_T)|_B \times a_B^C$ is irreducible, $q_B(\hat{X} \otimes_{\mathbb{Z}} \Lambda_T)|_B \times a_B^C) \subset \mathcal{M}_X(G_C)$ is irreducible as well. By Corollary 4.7 and Lemma 4.8, the image of the étale map $j$ from (4.1) is

$$j \left( \mathcal{M}_X^{red}(G) \right) = \bigcup_{B \in \mathcal{Y}} q_B \left( \hat{X} \otimes_{\mathbb{Z}} \Lambda_T \right)_B \times a_B^C$$

$$\mathcal{M}_X^{red}(G_C)_{max} = \bigcup_{B \in \mathcal{Y}} q_B \left( (T^*\hat{X} \otimes_{\mathbb{Z}} \Lambda_C)_B \right)^{i_G}.$$  

Recall from (2.10) that $Y_B$ is the Weyl group associated to $C_B^C$ and note that the centralizer of $(\hat{X} \otimes_{\mathbb{Z}} \Lambda_T)|_B \times a_B^C$ in $Y_B$ coincides with $N_{Y_B}(a_B^C)$. Then, the projection of $(\hat{X} \otimes_{\mathbb{Z}} \Lambda_T)|_B \times a_B^C$ to $\mathcal{M}_X(G_C)_{0}^{i_G}$ is $q_B \left( (\hat{X} \otimes_{\mathbb{Z}} \Lambda_T)|_B \times a_B^C \right) \simeq (\hat{X} \otimes_{\mathbb{Z}} \Lambda_T)|_B \times a_B^C / N_{Y_B}(a_B^C).$

Recall from Remark 3.19 that $\mathcal{M}_X(G)$ decomposes as the union of irreducible components $p_B \left( (\hat{X} \otimes_{\mathbb{Z}} \Lambda_T)|_B \times a_B^C \right)$, where each of these components is described in (3.30) as $p_B \left( (\hat{X} \otimes_{\mathbb{Z}} \Lambda_T)|_B \times a_B^C \right) \simeq (\hat{X} \otimes_{\mathbb{Z}} \Lambda_T)|_B \times a_B^C / \Gamma_B \times Z_W(B).$

Recall from Lemma 2.10 that $\Gamma_B \times Z_W(B)$ can be identified as a subgroup of $N_{Y_B}(a_B^C)$.

Denote by $j_B$ the restriction of the étale morphism (4.4) to the corresponding irreducible component. We study $j$ by describing each of the restrictions $j_B$.

**Proposition 4.9.** Denote by $\pi$ the natural projection induced by the identification of $\Gamma_B \times Z_W(B)$ as a subgroup of $N_{Y_B}(a_B^C)$. One has the following commutative
Proof. After the identifications that we have previously studied, the proof follows from the observation that \( j \) is determined by the extension of structure groups given by the natural inclusions \( H^C \subset G^C \) and \( m^C \subset g^C \).

5. The Hitchin fibration

Let us consider the isotropy action of the complex reductive Lie group \( H^C \) on \( m^C \) and take the quotient map \( m^C \to m^C/H^C \). Let \( E \) be any algebraic \( H^C \)-bundle. Since the isotropy action of \( H^C \) on \( m^C/H^C \) is obviously trivial, we note that the fibre bundle induced by \( E \) is trivial, \( E(m^C/H^C) = (m^C/H^C) \otimes O_X \), and so the projection induces a surjective morphism of fibre bundles

\[
E(m^C) \to E(m^C/H^C)
\]

and a morphism on the set of global sections

\[
\left( H^0(X, E(m^C)) \to H^0(X, (m^C/H^C) \otimes O_X) \right) / \Phi \to \Phi/H^C.
\]

One can easily check that the map constructed above is constant along \( S \)-equivalence classes. This allows us to define the Hitchin map

\[
b_G : \mathcal{M}_X(G) \to B_G := H^0(X, (m^C/H^C) \otimes O_X) / \Phi/H^C.
\]

Let \( D \) be a maximal admissible root system in \( \mathcal{Y} \), and let \( c_0 = t + a \) be the maximally non-compact Cartan subalgebra of \( g \) associated to it. Then, \( a^C \) is a maximal (abelian) subalgebra of \( m^C \). Denote by \( W_{sm} (H^C, a^C_D) \) the corresponding small Weyl group. Recall again the Real Chevalley Theorem ([Kn] Theorem 6.57), for instance), that states

\[
m^C/H^C \cong a^C_D / W_{sm} (H^C, a^C_D).
\]

By [Kn] Theorem 3], the small Weyl group is generated by the normalizer of \( a^C_D \) in the Weyl group \( Y \), and thanks to Lemma [2.10] one can identify \( a^C / W_{sm} (H^C, a^C_D) \) with \( a^C / \Gamma \times Z_W(D) \). Since \( X \) is a projective variety, we have

\[
B_G = H^0(X, (m^C/H^C) \otimes O_X) \cong a^C / \Gamma \times Z_W(D).
\]

Recalling that \( \Xi'_G \subset (\hat{X} \otimes \Lambda_T) \times m^C \), we consider the natural projection

\[
\pi_G : \Xi'_G \to \bigcup_{B \in \Gamma} a^C_B
\]
where \( t \in \hat{X} \otimes \mathbb{C} \Lambda_T \) and \( z \in \mathfrak{m}^\mathbb{C} \). Due to Lemma 2.6 all the maximal admissible root systems \( D_1, \ldots, D_r \) are conjugate under \( W \). Therefore, one has
\[
\frac{\mathfrak{a}^\mathbb{C}}{\Gamma \rtimes Z_W(D)} = (\bigcup_{B \in \Upsilon} (\mathfrak{a}_B^\mathbb{C}/\Gamma_B)) \big/ W,
\]
where we recall that every \( B \) is contained in some maximal \( D \). One obtains the projection
\[
\beta_G : \left( \bigcup_{B \in \Upsilon} \mathfrak{a}_B^\mathbb{C} \right) \rightarrow \frac{\mathfrak{a}^\mathbb{C}}{\Gamma \rtimes Z_W(D)}.
\]

It is clear that the following diagram
\[
\begin{array}{ccc}
\Xi_G & \xrightarrow{\pi_G} & \bigcup_{B \in \Upsilon} \mathfrak{a}_B^\mathbb{C} \\
\downarrow{p_G} & & \downarrow{\beta_G} \\
\mathcal{M}_X^{\text{red}}(G) & \xrightarrow{b_G} & \frac{\mathfrak{a}^\mathbb{C}}{\Gamma \rtimes Z_W(D)}
\end{array}
\]

is commutative.

Given \( z \in \mathfrak{a}^\mathbb{C} \), we define
\[
\Upsilon_z := \{ B \in \Upsilon \text{ such that } z \in \mathfrak{a}_B^\mathbb{C} \}.
\]

We say that the admissible system \( F \) is \textit{minimal} in \( \Upsilon_z \) if it does not contain any other admissible system of \( \Upsilon \). Let \( \Upsilon_z^{\text{min}} \) denote the set of minimal elements. Recall that, when \( F \subset B \), one has \( (\hat{X} \otimes \Lambda_T)|_B \subset (\hat{X} \otimes \Lambda_T)|_F \). Therefore, by (3.15), one has that
\[
\pi_G^{-1}(z) = \bigcup_{F_i \in \Upsilon_z^{\text{min}}} (\hat{X} \otimes \Lambda_T)|_{F_i} \times \{ z \}.
\]

We can now describe explicitly the fibres of the Hitchin map restricted to \( \mathcal{M}_X^{\text{red}}(G) \).

**Lemma 5.1.** Let \( z \in \mathfrak{a}^\mathbb{C} \) and take \( F \in \Upsilon_z^{\text{min}} \). Then, the Hitchin fibre in \( \mathcal{M}_X^{\text{red}}(G) \) over \( z \) is
\[
b_G^{-1}(\beta_G(z)) \cong (\hat{X} \otimes \mathbb{C} \Lambda_T)|_F \big/ Z_W(F, z).
\]

**Proof.** If we take any other minimal admissible root system \( F' \in \Upsilon_z^{\text{min}} \), we observe that \( |F| = |F'| \) so, by Lemma 2.6 \( F' \) and \( F \) are conjugate by the action of some element of \( W \).

By (3.3), since all the \( F_i \) are related by the action of \( W \), one has that the image under \( p_G \) of \( \pi_G^{-1}(z) \) is the quotient of one of the components, \( (\hat{X} \otimes \mathbb{C} \Lambda_T)|_F \), by the group that centralizes \( z \) and preserves the component, which is \( Z_W(z) \cap Z_W(F) = Z_W(F, z) \).

**Corollary 5.2.** The Hitchin fibre in \( \mathcal{M}_X^{\text{red}}(G) \) over \( z \in \mathfrak{a}_0^\mathbb{C} \subset \mathfrak{a}^\mathbb{C} \) is
\[
b_G^{-1}(\beta_G(z)) \cong (\hat{X} \otimes \mathbb{C} \Lambda_T) \big/ Z_W(z).
\]

We can see that the set of \( z \in \mathfrak{a}^\mathbb{C} \) such that \( z \notin \mathfrak{a}_B^\mathbb{C} \) for any other \( B \in \Upsilon \) such that \( B \neq D \), is a dense open subset of \( \mathfrak{a}^\mathbb{C} \). If \( z \) lies in this subset, we say that it is a \textit{generic} element of \( \mathfrak{a}^\mathbb{C} \). Note that we have \( \Upsilon_z = \{ D \} \) when \( z \) is generic.

**Corollary 5.3.** The Hitchin fibre in \( \mathcal{M}_X^{\text{red}}(G) \) over a generic element \( z \in \mathfrak{a}^\mathbb{C} \) is
\[
b_G^{-1}(\beta_G(z)) \cong (\hat{X} \otimes \mathbb{C} \Lambda_T)|_D \big/ Z_W(D, z).
\]
Remark 5.4. The group SU*(4) has Sp(4) as maximal compact subgroup and a unique (up to conjugation) Cartan subalgebra c = t ⊕ a. By this uniqueness of the Cartan subalgebra, we know that Y = {0}. If δ_i for i = 1, . . . , 4 are the elements with 1 in the i-th position of the diagonal and 0 elsewhere, we have that

\[ t = R \cdot i(\delta_2 - \delta_3) \oplus R \cdot i(\delta_1 - \delta_2 + \delta_3 - \delta_4). \]

and

\[ a = R \cdot \frac{1}{2}(\delta_1 - \delta_2 - \delta_3 + \delta_4). \]

Note that any non-zero element of a is a generic element. The kernel of expT is the lattice Λ = Z · i(δ_2 - δ_3) ⊕ Z · i(δ_1 - δ_3).

The Weyl group W = W(Sp(4, C), tC) is generated by the reflections of the roots δ_2 - δ_3, 1/2(δ_1 - δ_2 + δ_3 - δ_4) and δ_1 - δ_3. One can check that the centralizer of a generic element \(z \neq 0\) of a is \(Z_W(z) = \langle \sigma_{14}, \sigma_{23} \rangle\) where \(\sigma_{ij}\) is the permutation that sends \(\delta_i\) to \(\delta_j\) and leaves the rest unchanged. Applying Corollary 5.3, we have that the Hitchin fibre over \(z\) is

\[ b_G^{-1}(\beta_G(z)) \cong \hat{X} \otimes \mathbb{Z} \Lambda_T / Z_W(z) \]

\[ = \hat{X} \otimes \mathbb{Z} (i(\delta_2 - \delta_3) \oplus Z \cdot i(\delta_1 - \delta_4)) / \langle \sigma_{14}, \sigma_{23} \rangle \]

\[ \cong \left( \hat{X} \otimes \mathbb{Z} (i(\delta_2 - \delta_3)) / \langle \sigma_{23} \rangle \right) \times \left( \hat{X} \otimes \mathbb{Z} (i(\delta_1 - \delta_4)) / \langle \sigma_{14} \rangle \right) \]

\[ \cong \left( X / \pm \right) \times \left( X / \pm \right) \]

\[ \cong \mathbb{P}^1 \times \mathbb{P}^1. \]

It is remarkable that the generic fibre of the Hitchin fibration for SU*(4) is \(\mathbb{P}^1 \times \mathbb{P}^1\), which is not an abelian variety.

6. The Hitchin Equation and Flat Connections

Let \(G\) be a connected real form of a complex semisimple Lie group \(G^C\). Let \((E, \Phi)\) be a \(G\)-Higgs bundle and let \(h\) be a metric on \(E\), i.e. a \(C^\infty\) reduction of \(E\) to the maximal compact subgroup \(H \subset H^C\) giving the \(H\)-bundle \(E_h\). Hitchin introduced in [H11] the so called Hitchin equation for a metric on a \(G\)-Higgs bundle.

Recall the Cartan involution \(\theta\) and let \(\theta_h : E_h(g^C) \to E_h(g^C)\) be the involution induced fibrewise by the Cartan involution. Let \(\bar{\nabla}_E\) denote the Dolbeault operator of \(E\) and denote by \(A_h\) the Chern connection, which is the unique \(H\)-connection on \(E_h\) compatible with \(\bar{\nabla}_E\). We denote by \(F_h\) the curvature of \(A_h\). Take also \(dx \in \Omega^{1,0}(X, \mathcal{O}_X)\) and \(\sigma \in \Omega^{0,1}(X, \mathcal{O}_X)\). In the case of an elliptic curve, the Hitchin equation reads

\[ F_h + [\Phi \, dx, \theta_h(\Phi \, dx)] = 0. \]

We have seen in [FGN2] Proposition 5.1 that the Hitchin equation splits in the case of elliptic curve and a complex reductive Lie group \(G = H^C\). This result generalizes to semisimple real forms.

**Proposition 6.1.** Let \(G\) be a connected real form of the complex semisimple Lie group \(G^C\) and let \(H \subset G\) be a maximal compact subgroup. Fix a \(G\)-Higgs bundle
(E, Φ). Suppose that either H is semisimple or E has trivial characteristic class. Then (E, Φ) is polystable if and only if there exists a metric h on E that satisfies

\[ F_h = 0 \quad \text{and} \quad \left[ \Phi dx, \theta_h(\Phi) d\pi \right] = 0. \]

Proof. By Proposition 6.3 if the G-Higgs bundle (E, Φ) is polystable, then E is polystable and by the Narasimhan–Seshadri–Ramanathan Theorem there exists a metric for which \( F_h = 0 \).

By Corollary 3.4 (E, Φ) is isomorphic to \((E_p, z \otimes s)\) where \( z \in \mathfrak{a}_r^C \) maximal abelian subalgebra of \( \mathfrak{z}_mC(\rho) \). With no loss of generality, we can take \( \mathfrak{a}_p^C \) to be contained in a \( \theta \)-stable Cartan subalgebra. Then, we have \([z, \theta(z)] = 0\) and

\[ [\Phi dx, \theta_h(\Phi) d\pi] \cong [z, \theta(z)] \otimes s \otimes (dx \wedge d\pi) = 0. \]  

(6.2)

Conversely, it follows from the fact that \( F_h = 0 \) defines a representation \( \rho : \pi_1(X) \to H \) and then \((E, \Phi) \cong (E_p, z \otimes s)\) where \( z \in \mathfrak{z}_mC(\rho) \) by Corollary 3.2. Take \( \mathfrak{a}_p^C \) to be a maximal abelian subalgebra of \( \mathfrak{z}_mC(\rho) \) containing \( z \) and \( \theta(z) \). Finally, by Lemma 3.5 \((E_p, z \otimes s)\) is polystable.

□

Remark 6.2. Note that Proposition 6.1 states the Hitchin–Kobayashi correspondence. In particular, \( \mathcal{M}_X(G) \) is homeomorphic to the moduli space \( C_X(G) \) of \( G \)-bundles with flat \( G^C \)-connections. Note also, that in order to prove the hard implication (“polystable implies existence of solutions of the Hitchin equation”) in the elliptic case, we make use only of the Narasimhan–Seshadri–Ramanathan Theorem and Proposition 3.3.

Proposition 6.3. Let \( G \) be semisimple with maximal compact subgroup \( H \subset G \) and let \((E, \Phi)\) be a semistable \( G \)-Higgs bundle. If \( H \) has non-finite center, there are no solutions of the Hitchin equation (6.1) for \( E \) with non-trivial characteristic class \( d \) in \( \pi_1(H^C) \).

Proof. Note that, thanks to Corollary 3.2, the vanishing in (6.2) still holds. Then, the Hitchin equation for a metric \( h \) on \( E \) forces

\[ F_h = 0, \]

which has no solutions in the cases stated in the hypothesis. □

Remark 6.4. Let us take the semisimple real form \( G = SU(p, q) \) whose maximal compact subgroup \( H = S(U(p) \times U(q)) \) has non-finite center. By Proposition 6.3 there are no solutions of the Hitchin equation unless \( d \in \pi_1(H^C) \) is trivial. In this case \( H^C = S(GL(p, \mathbb{C}) \times GL(q, \mathbb{C})) \), and then a \( H^C \)-bundle \( E \) can be seen as a direct sum of two vector bundles \( V \oplus W \) of rank \( p \) and \( q \) with \( \det(V \oplus W) \cong \mathcal{O}_X \) and degrees \( a = \text{deg}(V) \) and \( b = \text{deg}(W) \) satisfying

\[ pa + qb = 0. \]  

(6.3)

The characteristic class \( d \in \pi_1(H^C) \) is determined by \( a \) and \( b \) and Proposition 6.3 implies that they are solutions of (6.1) only when \( a = 0 \) and \( b = 0 \). We can see that this agrees with the Milnor-Wood inequality for the Toledo invariant [BGG],

\[ -2 \min\{p, q\}(g - 1) \leq \frac{pb - qa}{pq} \leq 2 \min\{p, q\}(g - 1), \]

which in this case reads

\[ 2 \min\{p, q\}(g - 1) \leq \frac{pb - qa}{pq} = 0. \]  

(6.4)

As we can see, \( a = b = 0 \) is the only possible solution of (6.3) and (6.4).
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