An Epistemic Interpretation of Tensor Disjunction

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Abstract

This paper aims to give an epistemic interpretation to the tensor disjunction in dependence logic, through a rather surprising connection to the so-called weak disjunction in Medvedev’s early work on intermediate logic under the Brouwer-Heyting-Kolmogorov (BHK)-interpretation. We expose this connection in the setting of inquisitive logic with tensor disjunction $\text{InqB}^\otimes$ discussed by [6], but from an epistemic perspective. More specifically, we translate the propositional formulae of $\text{InqB}^\otimes$ into modal formulae in a powerful epistemic language of knowing how following the proposal by [19, 16]. We give a complete axiomatization of the logic of our full language based on Fine’s axiomatization of S5 modal logic with propositional quantifiers. Finally we generalize the tensor operator with parameters $k$ and $n$, which intuitively captures the epistemic situation that one knows $n$ potential answers to $n$ questions and is sure $k$ answers of them must be correct. The original tensor disjunction is the special case when $k = 1$ and $n = 2$. We show that the generalized tensor operators do not increase the expressive power of our logic, the inquisitive logic and propositional dependence logic, though most of these generalized tensors are not uniformly definable in these logics, except in our dynamic epistemic logic of knowing how.

1 Introduction

As a rapidly growing field of research, Dependence Logic studies reasoning patterns expressed by logical languages extended with (in)dependence atoms (cf. e.g., [10] for a survey). The intuitive meaning of the atomic formulae are best fleshed out formally by the team semantics capturing the (in)dependence between variables. The truth conditions of the logical connectives and other logical constants are also given based on teams, where one usual guideline is to define them in such a way that the language enjoys the property of flatness, i.e., for any formula $\alpha$ without the (in)dependence atoms, it is true w.r.t. a team $X$ ($X \models \alpha$) if it is true on each singleton team $\{s\}$ such that $s \in X$. To some extent, flatness preserves the intuition of the classical logical connectives on possible worlds. In particular, the semantics of the distinct tensor disjunction $\otimes$ in dependence logic can be viewed as a natural lifting of the world-based semantics for classical disjunction to teams, viewed as sets of possible worlds:

$$X \models \alpha \otimes \beta \text{ iff there are } U, V \subseteq X \text{ such that } X \subseteq U \cup V, U \models \alpha \text{ and } V \models \beta$$

Note that a disjunction $\alpha \lor \beta$ is classically true on each world in a set $X$ of possible worlds if and only if there are two subsets jointly covering the whole space of possible worlds such that one subset satisfies $\alpha$ homogeneously and the other satisfies $\beta$ homogeneously. This lifting may
also give the impression that \( \otimes \) can be read more or less as a classical disjunction. However, it is not so straightforward. For example, the truth of the propositional dependence formula \( (p, q) \otimes (p, q) \) over a team is not equivalent to \( (p, q) \). According to the semantics of \( \otimes \), \( (p, q) \otimes (p, q) \) says there are two subteams jointly covering the whole team, and \( q \) depends on \( p \) in each team. However, it is not necessarily that \( q \) depends on \( p \) over the whole team. A natural question arises: how to understand this \( \otimes \) disjunction intuitively and precisely? Our work proposes a possible epistemic understanding of \( \otimes \) (and its generalizations) from a Brouwer-Heyting-Kolmogorov (BHK)-like perspective to be explained below.

The initial idea is based on an unexpected connection between the tensor disjunction and the so-called weak disjunction in Medvedev’s early work [13] on the problem semantics of intuitionistic logic, following Kolmogorov’s problem-solving interpretation [12]. This connection is best exposed in the setting of inquisitive logic with tensor disjunction discussed in [6], since inquisitive logic has intimate connections with both the propositional dependence logic [22] and Medvedev’s logic [8]. More specifically, various versions of propositional dependence logic can be viewed as the disguised intuitionistic logic, e.g., the dependence atom \( (p, q) \) becomes \( (p \lor \neg p) \rightarrow (q \lor \neg q) \) [20, 22, 5]. On the other hand, Medvedev’s logic is the substitution-closed core of inquisitive logic \( \text{InqB} \) that also admits a BHK-like interpretation via resolutions [3, 8]. Another advantage of using inquisitive logic as the “medium” is that we can put classical, intuitionistic, and tensor disjunctions in the same picture to reveal their differences. The last missing piece for an intuitive reading of tensor is an epistemic interpretation that can incorporate the BHK-interpretation. Wang proposed to capture intuitionistic truth using a modality \( Kh \) to express knowing how to prove/solve [19], which reflects Heyting’s often-overlooked early view of intuitionistic logic as an epistemic logic [11]. This also led to an alternative epistemic interpretation of inquisitive logic [16], where a state supports a formula \( \alpha \) is rendered as it is known how to resolve \( \alpha \) (more colloquially, knowing how \( \alpha \) is true) when viewing the state as a set of possible worlds capturing the epistemic uncertainty. This can give us alternative epistemic readings of formulas in inquisitive logic. For example, \( \neg \alpha \) in inquisitive logic is first rendered as \( Kh \neg \alpha \), which can be reduced to \( K \neg \alpha \) (knowing that \( \alpha \) does not have any resolution), reflecting the negation \( \neg \) as the bridge between the intuitionistic and classical worlds. As another example, the excluded middle \( \alpha \lor \neg \alpha \) in inquisitive logic is first rendered as \( Kh \alpha \lor Kh \neg \alpha \) in our system, and eventually can be reduced to the intuitively invalid \( Kh \alpha \lor K \neg \alpha \). When \( \alpha \) is the atomic proposition \( p \), \( p \lor \neg p \) in inquisitive logic is equivalent to the epistemic formula \( KP \lor K \neg p \) in our setting (see [16]).

Now we are ready to give the epistemic interpretation of the tensor disjunction. According to Medvedev’s problem semantics [13], the weak disjunction \( \alpha \sqcup \beta \) captures a composite problem where the solutions are pairs of potential solutions to the problems of \( \alpha \) and \( \beta \) respectively such that at least one solution in each pair is correct. From the epistemic interpretation, Medvedev’s truth concept for a formula \( \gamma \) means it is known how to solve \( \gamma \). In particular, a weak disjunction \( \alpha \sqcup \beta \) is true w.r.t. a set of possible worlds (i.e., a state/team) iff there are two solutions \( r_1 \) and \( r_2 \) such that it is known that one of \( r_1 \) and \( r_2 \) is a correct solution to the corresponding problems. We will show such a truth condition amounts to exactly the team semantics for the tensor.

We first summarize what we actually did in the paper before going into the technical details. After introducing the inquisitive logic with tensor \( \text{InqB} \) in Section 2, we first propose in Section 3 a dynamic epistemic language of know-that and know-how, with extra machinery of announcements and propositional quantifiers, interpreted over epistemic models that are essentially states/teams in the literature. The semantics of the know-how operator is given based on a BHK-like interpretation, with the intention to capture the alternative epistemic meaning of \( \text{InqB} \) formulae, which is formally justified by showing in Section 4 that the valid know-how formulae are exactly theorems in \( \text{InqB} \). Moreover, we also show that the announcements and propositional quantifiers facilitate a recursive process to “open up” the know-how formulae, in particular to decode the \( \otimes \), and eventually translate them into classical ones free of the know-how operator. Based on

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1In [15], it is suggested that the (in)dependence formulae can be viewed as types of teams.
2In the recent literature, inquisitive logic is also viewed as an extension of classical logic [4].
3See [2], for the corresponding Kripke semantics of weak disjunction.
such a process we give a complete axiomatization of our full dynamic epistemic logic in Section 5. Finally, in Section 6 we generalize the idea of the tensor, from our epistemic interpretation, to obtain a spectrum of $n$-ary disjunctions $\otimes^n_k$, which captures the interesting epistemic situation of knowing $n$ potential answers to $n$ questions and being sure at least $k$ of them must be correct. We show that adding the generalized tensor operators does not increase the expressive power of our logic, the inquisitive logic and propositional dependence logic, though most of these generalized tensors are not uniformly definable in these logics, except in our epistemic language.

2 Preliminaries: Inquisitive Logic with Tensor Disjunction

Following [6], we introduce the language and semantics of Inquisitive Logic with Tensor Disjunction ($\text{InqB}^\otimes$). In contrast with [6], we use the symbol $\lor$ for the inquisitive disjunction and adopt the model-based semantics as in [4]. Throughout the paper, we fix a countable set $P$ of proposition letters.

Definition 1 (Language $\text{PL}^\otimes$) The language of propositional logic with tensor ($\text{PL}^\otimes$) is defined as follows:

$$\alpha ::= p | \bot | (\alpha \land \alpha) | (\alpha \lor \alpha) | (\alpha \rightarrow \alpha) | (\alpha \otimes \alpha)$$

where $p \in P$. We write $\neg \alpha$ for $\alpha \rightarrow \bot$, $\top$ and $\alpha \leftrightarrow \beta$ are defined as usual.

Definition 2 (Model and state) A model is a pair $M = \langle W, V \rangle$ where:

- $W$ is a non-empty set of possible worlds\(^4\)
- $V : P \rightarrow \wp(W)$ is a valuation function.

A state $s$ in $M$ is a subset of $W$. We will also view these models as 

epistemic models

for our dynamic epistemic language to be introduced in Section 3.

Given $M$, we refer to its components by $W_M$ and $V_M$. We write $w \in M$ in case that $w \in W_M$, and $M' \subseteq M$ in case that $W_M' \subseteq W_M$. The semantics is defined through the support relation between states (in models) and formulae.

Definition 3 (Support [6]) The support relation $\models$ is defined inductively:

| $\models$ | Description |
|----------|-------------|
| $M, s \models p$ | iff $\forall w \in s, w \in V(p)$ |
| $M, s \models \bot$ | iff $s = \emptyset$ |
| $M, s \models (\alpha \land \beta)$ | iff $M, s \models \alpha$ and $M, s \models \beta$ |
| $M, s \models (\alpha \lor \beta)$ | iff $M, s \models \alpha$ or $M, s \models \beta$ |
| $M, s \models (\alpha \rightarrow \beta)$ | iff $\forall t \subseteq s :$ if $M, t \models \alpha$ then $M, t \models \beta$ |
| $M, s \models (\alpha \otimes \beta)$ | iff there exist two sets $t \subseteq s$ and $t' \subseteq s$ such that $M, t \models \alpha$, $M, t' \models \beta$, and $t \cup t' = s$. |

A formula $\alpha$ is valid if it is supported by any state in any model.

Here are some simple properties.

Proposition 4 (Downward closeness) For any $\alpha \in \text{PL}^\otimes$, if $M, s \models \alpha$ then $M, t \models \alpha$ for any $t \subseteq s$. Moreover, $M, \emptyset \models \alpha$ for all $\alpha \in \text{PL}^\otimes$.

Definition 5 Inquisitive Logic with Tensor Disjunction ($\text{InqB}^\otimes$) is the set of valid $\text{PL}^\otimes$ formulae under the support relation.

\(^4\)In [7], the world set $W$ could be empty. The distinction is not technically significant.
3 A dynamic epistemic language

Definition 6 (Language PALKhII) The language of Public Announcement Logic with Know-how Operator and Propositional Quantifier is defined as:

$$
\varphi ::= p \mid \bot \mid (\varphi \land \varphi) \mid (\varphi \lor \varphi) \mid (\varphi \land \varphi) \mid (\varphi \rightarrow \varphi) \mid K\varphi \mid \forall p\phi \mid [\varphi]\varphi
$$

where $p \in P$ and $\alpha \in PL$. We write $K$ for $\neg \neg \varphi$, $\forall p$ for $\forall \varphi \neg \neg$ for all $p \in P$ and $\langle \varphi \rangle$ for $\neg\neg[\varphi]\neg\neg$ for all $\varphi \in PALKhII$.

Intuitively, $K\varphi$ expresses “the agent knows that $\varphi$”, $K\alpha$ says that “the agent knows how to resolve $\alpha$” or simply “the agent knows how $\alpha$ is true”, $\forall p\varphi$ says that “for any proposition $p$, $\varphi$ holds” and $[\varphi]\psi$ means that “after announcing $\varphi$, $\psi$ holds”. Note that $K$ only allows PL-formulae $\alpha$ in its scope. For instance we can express $K\neg K\alpha$ but not $K\alpha K\alpha$ in PALKhII. We write $\varphi[\psi/\chi]$ for any formula obtained by replacing one or several occurrences of $\psi$ with $\chi$ in $\varphi$.

We view the models in Definition 2 as epistemic models where the implicit epistemic relation is the total relation. The semantics of PALKhII is given on such models, with the notions of resolution space and resolution as below.

Definition 7 (Resolution space) $S$ is a function assigning each $\alpha \in PL^\circ$ its (non-empty) set of potential resolutions:

$$
S(p) = \{p\}, \text{ for } p \in P \\
S(\alpha \lor \beta) = (S(\alpha) \times \{0\}) \cup (S(\beta) \times \{1\}) \\
S(\alpha \land \beta) = S(\alpha) \times S(\beta) \\
S(\alpha \rightarrow \beta) = (S(\beta))^{S(\alpha)} \\
S(\alpha \ominus \beta) = S(\alpha) \times S(\beta)
$$

Resolution spaces reflect the BHK-interpretation, e.g., a possible resolution of an implication is a function transforming a resolution of the antecedent into a resolution of the consequent. Note that resolution spaces for atomic propositions are singletons, based on the assumption in inquisitive semantics that atomic propositions are statements without inquisitiveness. The set of actual resolutions of each formula on each world in a given model is a (possibly empty) subset of the corresponding resolution space, as defined below.

Definition 8 (Resolution in model) Given $M$, $R : W_M \times PL^\circ \rightarrow \bigcup_{\alpha \in PL^\circ} S(\alpha)$ gives the (actual) resolutions for each PL$^\circ$-formula on each world:

$$
R(w, \bot) = \emptyset \quad R(w, p) = \begin{cases} \{p\} & \text{if } w \in V_M(p) \\ \emptyset & \text{otherwise} \end{cases} \\
R(w, \alpha \lor \beta) = (R(w, \alpha) \times \{0\}) \cup (R(w, \beta) \times \{1\}) \\
R(w, \alpha \land \beta) = R(w, \alpha) \times R(w, \beta) \\
R(w, \alpha \rightarrow \beta) = \{f \in S(\beta)^{S(\alpha)} \mid f[R(w, \alpha)] \subseteq R(w, \beta)\} \\
R(w, \alpha \ominus \beta) = (R(w, \alpha) \times S(\beta)) \cup (S(\alpha) \times R(w, \beta))
$$

Important notation For $U \subseteq W_M$, we write $R(U, \alpha)$ for $\bigcap_{w \in U} R(w, \alpha)$.

While $S(\bot) = \{\bot\}$ is non-empty, it never has any actual resolution on specific worlds. For any $p \in P$, $p$ has itself as its resolution iff it is true on $w$. For any implication $\alpha \rightarrow \beta \in PL^\circ$, each of its resolutions on $w$ is a function in $S(\alpha \rightarrow \beta)$ which maps an actual resolution of $\alpha$ to an actual resolution of $\beta$ on $w$. Following the idea of the weak disjunction introduced in [13], each resolution for $\alpha \ominus \beta \in PL^\circ$ on $w$ is a pair of resolutions in $S(\alpha \ominus \beta)$, such that at least one in the pair is actual on $w$ for the corresponding formula.

\footnote{II in the name PALKhII denotes propositional quantifiers as in the literature [9].}
Let $P(\alpha)$ be the set of propositional letters occurring in $\alpha$ and let $V^\alpha_M(w)$ be the collection of $p \in P(\alpha)$ that are true on $w$ in $M$. Proposition 9 is a useful observation on the resolution of negations ($\neg \alpha := \alpha \rightarrow \bot$). Proposition 10 says that $R(w, \alpha)$ only depends on the relevant valuation on $w$ itself.

**Proposition 9 ([16])** For any $M$, $w$, any $\alpha$, $R(w, \neg \alpha)$ is either $\emptyset$ or a fixed singleton set independent from $w$, and $R(w, \neg \alpha) = \emptyset$ iff $R(w, \alpha) \neq \emptyset$.

**Proposition 10** For any $M$, $w$ and $N$, $v$, for all $\alpha \in PL$, if $V^\alpha_M(w) = V^\alpha_N(v)$, then $R(w, \alpha) = R(v, \alpha)$.

Now we are ready to define the satisfaction relation of $PALKhII$ on pointed models, i.e., a model with a designated world, in contrast with the state-based support-semantics. Note that the connectives outside the scope of $K_h$ are classical, in particular $\otimes$ just functions as a classical disjunction. $K$ is the standard epistemic modality of know-that. The semantics for $K_h\alpha$ is defined via resolutions and is intended to capture the know-how interpretation of $InqB^\otimes$. $\forall p$ is a propositional quantifier over the full power set of $W_M$. The semantics of the dynamic operator $[\psi]$ is as in public announcement logic [14].

**Definition 11 (Semantics)** For $\varphi, \psi \in PALKhII$, $\alpha \in PL$ and $M, w$ where $M = (W, V)$, the satisfaction relation is defined as below where $\emptyset \in \{\vee, \otimes\}$:

\[
\begin{align*}
M, w \not\models \bot \\
M, w \models p & \iff w \in V(p) \\
M, w \models (\varphi \otimes \psi) & \iff M, w \models \varphi \text{ or } M, w \models \psi \\
M, w \models (\varphi \land \psi) & \iff M, w \models \varphi \text{ and } M, w \models \psi \\
M, w \models (\varphi \rightarrow \psi) & \iff M, w \models \varphi \text{ implies } M, w \models \psi \\
M, w \models K\varphi & \iff \text{for any } v \in M, M, v \models \varphi \\
M, w \models K_h\alpha & \iff \text{there exists an } x \in S(\alpha) \text{ s.t. for any } v \in M, x \in R(v, \alpha) \\
M, w \models \forall p \varphi & \iff \text{for any } U \in \psi(W_M), M[p \rightarrow U], w \models \varphi \\
M, w \models [\psi]_\varphi & \iff M, w \models \psi \text{ implies } M|_{U}, w \models \varphi 
\end{align*}
\]

where:

- Given $U \in \psi(W_M)$ and $p \in P$, recall that $M[p \rightarrow U] = (W, V')$, where the assignment $V'$ assigns $U$ to $p$ and coincides with $V$ on all other atoms; and

- $[\psi] = \{w \in W_M \mid M, w \models \psi\}$ and $M|_X$ is the submodel of $M$ by restricting to $\emptyset \neq X \subseteq W_M$. Thus $M|_{U}$ is the submodel restricted to the worlds satisfying $\psi$ in $M$. We also write $M|_{v}$ as $M|_{\emptyset}$ for brevity.

Validity and entailment are defined as usual.

In [16], we have a dynamic operator $\square$. $\square \varphi$ says that “given any information updates $\varphi$ holds”. This can be expressed by $\forall p[p] \varphi$ given that $p$ is not free in $\varphi$, which is used to handle the implication in the know-how scope.

We write $M \models \varphi$ iff $M, w \models \varphi$ for all $w \in W_M$. Apparently, $M, w \models K_h\alpha$ iff $M, w \models K\alpha$ and $M, w \models K\varphi$ iff $M, w \models \varphi$. As mentioned in [16], the semantics of $K_h$ is in the $\exists x K$ form as in other know-wh logics [18, 17]. The truth condition of $K_h$ below says that $K_h\alpha$ holds on a (pointed) model as long as there is a uniform resolution for $\alpha$ on that model, where we define $R(U, \alpha)$ as $\bigcap_{w \in U} R(w, \alpha)$.

![Alternative truth condition for PL\textsuperscript{-}formulae can be given via resolutions.](image)

**Proposition 12** For any $\alpha \in PL$ and $M$, $w$, $M, w \models \alpha \iff R(w, \alpha) \neq \emptyset$. 


Proposition 16

For any proof system to be introduced later. First, we have the following observation.

\[ M, w \models (\alpha \rightarrow \beta) \iff M, w \models \alpha \text{ implies } M, w \models \beta \]

\[ \iff R(w, \alpha) \neq \emptyset \text{ implies } R(w, \beta) \neq \emptyset \]

\[ \iff \{ f \in S(\beta)^{S(\alpha)} \} \neq \emptyset \text{ and } f[R(w, \alpha)] \subseteq R(w, \beta) \text{ is possible} \]

\[ \iff R(w, \alpha \rightarrow \beta) \neq \emptyset \]

\[ M, w \models (\alpha \otimes \beta) \iff M, w \models \alpha \text{ or } M, w \models \beta \iff R(w, \alpha) \neq \emptyset \text{ or } R(w, \beta) \neq \emptyset \]

\[ \iff \text{there exists an } x \in R(w, \alpha) \text{ or there exists a } y \in R(w, \beta) \]

\[ \iff \text{there exists a pair } \langle x, x' \rangle \text{ or } \langle y', y \rangle \text{ in } R(w, \alpha \otimes \beta) \]

\[ \text{such that } y' \in S(\alpha) \text{ and } x' \in S(\beta) \]

\[ \iff R(w, \alpha \otimes \beta) \neq \emptyset \]

}\text{□}

From Proposition 12 we see that in propositional formulae, both \( \lor \) and \( \otimes \) collapse to the classical disjunction outside the scope of \( \K \). Yet \( \otimes \) is weaker than \( \lor \) in the way that we can construct a resolution of \( \alpha \otimes \beta \) from that of \( \alpha \lor \beta \). It also follows from Proposition 12 that for any \( \alpha \in \PL^{\otimes} \), \( M, w \models \K \varphi \) if and only if for each \( v \in M \), there is some resolution for \( \alpha \) on \( v \). In contrast, \( M, w \models \K \alpha \) iff there is a uniform resolution for \( \alpha \) on \( M \). The following is immediate.

**Proposition 13**  \( \K \alpha \rightarrow \K \alpha \) is valid for all \( \alpha \in \PL^{\otimes} \).

Since each \( p \in \P \) only has one possible resolution, when each point has a resolution for \( p \), the model has a uniform one. Thus we have Proposition 14.

**Proposition 14**  \( \K p \leftrightarrow \K p \) is valid for all \( p \in \P \).

While the deduction rule replacement of equals by equals is not valid in general, for instance, although \( (p \lor \neg p) \leftrightarrow (p \rightarrow p) \) is valid, \( \K (p \lor \neg p) \leftrightarrow \K (p \rightarrow p) \) is not. However, if we only allow substitution to happen outside the scope of \( \K \) operators, the rule becomes valid. It is not hard to verify the following:

**Proposition 15**  For \( \varphi, \psi, \chi \in \P \K \Pi, \) if \( \varphi \leftrightarrow \psi \) is valid, then \( \chi[\varphi/\psi] \leftrightarrow \chi \) is valid, given that the substitution does not happen in the scope of \( \K \).

### 4 Expressivity

Let \( \P \Pi \) be the \( \K \)-free fragment of \( \P \K \Pi, \) \( \EL \Pi \) be the \( \lceil \cdot \rceil \)-free fragment of \( \P \Pi \) and \( \EL \) be the \( \forall p \)-free fragment of \( \EL \Pi \). In Subsection 4.1 we show \( \K \) and \( \lceil \cdot \rceil \) can be eliminated, thus making \( \P \K \Pi, \) \( \P \Pi \) and \( \EL \Pi \) equally expressive. In Subsection 4.2 we show that the valid \( \K \) formulae of \( \P \K \Pi \) corresponds to \( \InqB^{\otimes} \) precisely.

#### 4.1 Reduction

We introduce the reduction schemata to eliminate the \( \K \) modality, which will also be used in the proof system to be introduced later. First, we have the following observation.

**Proposition 16**  For any \( \alpha, \beta \in \PL^{\otimes} \) where \( p \) does not occur free, for any pointed model \( M, w, \) \( M, w \models \exists p \K(p) \K \alpha \land \neg[p] \K \beta \) iff there is a \( U \subseteq W_M, (U \neq \emptyset \text{ implies } R(U, \alpha) \neq \emptyset) \) and \( \emptyset \neq \emptyset \text{ implies } R(U, \beta) \neq \emptyset \).
PROOF. Given a $U \subseteq W_M$, for any $w \in M$, $M[p \rightarrow U], w \vDash p \iff w \in U$ (*). For brevity, we write $\exists U$ for there exists $U \in W_M$. Recall that $M|U$ denotes the submodel of $M$ restricted to $U$, if $U$ is non-empty (otherwise undefined).

$M, w \vDash \exists pK([p|Kh\alpha \land \neg p|Kh\beta])$
$\iff \exists U, M[p \rightarrow U], w \vDash K([p|Kh\alpha \land \neg p|Kh\beta])$
$\iff \exists U, \forall v \in M, \{p \rightarrow U\}, v \vDash [p|Kh\alpha \land \neg p|Kh\beta]$
(by (*) and the fact that $[\varphi]|v$ holds trivially if $\varphi$ is false, we have: )
$\iff \exists U, \forall v \in U, M[p \rightarrow U], v \vDash [p|Kh\alpha \land \neg p|Kh\beta]$
$\iff \exists U, \forall v \in U, M[p \rightarrow U], v \vDash [p|Kh\alpha \land \neg p|Kh\beta]$

(since $p$ does not occur free in $\alpha$ and $\beta$, we have:)
$\iff \exists U, \forall v \in U, M|_v, v \vDash K h \alpha \land \neg p|Kh\beta$
$\iff \exists U, (U \neq \emptyset \implies R(U, \alpha) \neq \emptyset) \text{ and } (U \neq \emptyset \implies R(\overline{U}, \beta) \neq \emptyset)$.

Together with Proposition[14] and [15] Proposition[17] helps us to first eliminate the $Kh$ modality without changing the expressive power, i.e., each PALKhII-formula is equivalent to a PALII-formula.

**Proposition 17** The following formulae and schemata are valid:

$\mathbf{Kh}: \quad K\alpha \rightarrow \mathbf{Kh}p$
$\mathbf{Kh}: \quad \mathbf{Kh} \alpha \lor \beta \leftrightarrow \mathbf{Kh}\alpha \lor \mathbf{Kh}\beta$
$\mathbf{Kh\alpha}: \quad \mathbf{Kh}(\alpha \land \beta) \leftrightarrow \mathbf{Kh}\alpha \land \mathbf{Kh}\beta$
$\mathbf{Kh\beta}: \quad \mathbf{Kh}(\alpha \rightarrow \beta) \leftrightarrow \mathbf{Kh}[p|\mathbf{Kh}\alpha \rightarrow \mathbf{Kh}\beta], \text{ where } p \text{ does not occur free in } \alpha \text{ or } \beta$
$\mathbf{Kh\circ}: \quad \mathbf{Kh}(\alpha \circ \beta) \leftrightarrow \exists pK([p|\mathbf{Kh}\alpha \land \neg p|\mathbf{Kh}\beta]), \text{ where } p \text{ does not occur free in } \alpha \text{ or } \beta$

**PROOF.** We only show the cases for $\mathbf{Kh\rightarrow}$ and $\mathbf{Kh\circ}$. The rest of the proof can be found in [16].

$\mathbf{Kh\rightarrow}$: Recall that $M, w \vDash \Box \varphi \iff \text{for any } M' \subseteq M \text{ s.t. } w \in M', M', w \vDash \varphi$. We claim that $\Box \varphi$ can be defined by $\forall[p\varphi]$ where $p$ does not occur free in $\varphi$. Then it suffices to show that $\mathbf{Kh}(\alpha \rightarrow \beta) \leftrightarrow K(\mathbf{Kh}\alpha \rightarrow \mathbf{Kh}\beta)$. The following proof comes from [16].

$\implies$: Suppose $M, w \vDash \mathbf{Kh}(\alpha \rightarrow \beta)$, then there is some $f \in R(M, \alpha \rightarrow \beta)$. Towards a contradiction, suppose $M, w \not\vDash K(\mathbf{Kh}\alpha \rightarrow \mathbf{Kh}\beta)$. That is, there is an $v \in M$ and an $M', v \subseteq M, v \in M, M', v \vDash \mathbf{Kh}\alpha \text{ but } M', v \not\vDash \mathbf{Kh}\beta$. So there is an $x \in M, \alpha \rightarrow \beta$. Recall that $f$ is a function with domain $S(\alpha)$, and $S(\alpha) \supseteq R(u, \alpha)$ for all $u \in M'$, thus $x \in Dom(f)$. Moreover, since $f \in R(M, \alpha \rightarrow \beta)$, $f \in R(M', \alpha \rightarrow \beta)$. Let $y = f(x)$. By the definition of $R(M', \alpha \rightarrow \beta)$, $y \in R(u, \beta)$ for each $u \in M'$. Therefore $M', v \vDash \mathbf{Kh}\beta$, a contradiction.

$\iff$: Suppose $M, w \vDash K(\mathbf{Kh}\alpha \rightarrow \mathbf{Kh}\beta)$, then for all $v \in M, M, v \not\vDash \Box(\mathbf{Kh}\alpha \rightarrow \mathbf{Kh}\beta)$. By the semantics of $\Box$, for any $v \in M$ and for any $M', v \subseteq M$, $M', v \vDash \mathbf{Kh}\alpha \text{ and } \mathbf{Kh}\beta$ (*). Since $S(\alpha)$ is finite and non-empty, thus we can assume $S(\alpha) = \{x_0, x_1, \ldots, x_n\}$ for some $n \in \mathbb{N}$. For $i \in \{0, \ldots, n\}$, let $W_i = \{w \mid x_i \in R(w, \alpha)\}$. If $W_i$ is not empty then let $M_i$ be the submodel of $M$ such that $W_{M_i} = W_i$. Clearly $x_i \in R(W_i, \alpha)$, therefore for any $u \in M_i, M_i, u \vDash \mathbf{Kh}\alpha$. By (*) we have $M_i, u \vDash \mathbf{Kh}\beta$ thus there is a $y_i \in R(W_i, \beta)$. Now fix a $y \in S(\beta) \neq \emptyset$, let $f = \{\langle x_i, y_i \rangle \mid i \in \{0, \ldots, n\} \text{ and } W_i \neq \emptyset \} \cup \{\langle x_i, y \rangle \mid i \in \{0, \ldots, n\} \text{ and } W_i = \emptyset\}$. Clearly $f \in S(\alpha) \circ (\beta)$. Now for any $v \in M$ and $i \in \{0, \ldots, n\}$, if $x_i \in R(w, \alpha)$ then $v \in W_i$ by the definition of $W_i$, thus $y_i \in R(v, \beta)$ by the construction of $f$. Therefore $f[R(v, \alpha)] \subseteq R(v, \beta)$ for all $v \in M$. It follows that $M, v \vDash \mathbf{Kh}(\alpha \rightarrow \beta)$ for all $v \in M$ including $w$. Note that the axiom of choice is not needed here.

$\mathbf{Kh\circ}$: $\implies$: Suppose $M, w \vDash \mathbf{Kh}(\alpha \circ \beta)$, then by the semantics, there is some $(x, y) \in R(W_{M_i}, \alpha \circ \beta)$. Let $U = \{v \in M \mid x \in R(v, \alpha)\}$. It is not hard to see $\overline{U} \subseteq \{v \in M \mid y \in R(v, \beta)\}$ by the definition of $R(v, \alpha \circ \beta)$. By Proposition[16] $M, w \vDash \exists pK([p|\mathbf{Kh}\alpha \land \neg p|\mathbf{Kh}\beta])$.

$\iff$: Suppose $M, w \vDash \exists pK([p|\mathbf{Kh}\alpha \land \neg p|\mathbf{Kh}\beta])$, by Proposition[16] there is a $U$ satisfying the desired property. If $U \neq \emptyset$ and $\overline{U} \neq \emptyset$, pick $(x, y)$ as the witness for $R(W_{M_i}, \alpha \circ \beta)$ such that...
Proposition 18
The following formulae and schemata are valid:

\[ \begin{align*}
\llbracket \lnot p \rrbracket & \equiv (\chi \rightarrow p), \ p \in P \cup \{\bot\} \\
\llbracket \top \rrbracket & \equiv [\chi] \lor [\chi] \lor [\chi] \\
\llbracket K \rrbracket & \equiv (\chi \rightarrow K([\chi] \varphi)) \\
\llbracket 
\end{align*} \]

Proof The only non-trivial case is \[ \llbracket \varphi \lor \psi \rrbracket \] and we only show \[ [\chi](\varphi \lor \psi) \leftrightarrow [\chi] \varphi \lor [\chi] \psi \] as example. By Proposition 16 we further eliminate the \( \llbracket \cdot \rrbracket \) operator (without \( K_\alpha \))

4.2 \( \text{KgL} = \text{InqB}^\circ \)

Now we show that \( \text{KgL} = \{ \alpha \in \text{PL}^\circ \mid \vdash K_\alpha \} \) is exactly \( \text{InqB}^\circ \).

Lemma 19 For any \( \alpha \in \text{PL}^\circ \), \( M, w \models K_\alpha \) iff \( M, W_M \models \alpha \). As a consequence, for any non-empty state \( s \) in \( M \), \( M, s \models \alpha \) iff \( M|_s \models K_\alpha \).

Proof Note that \( M, w \models K_\alpha \) iff \( M \models K_\alpha \) by the semantics, so we simply show \( M \models K_\alpha \) iff \( M, W_M \models \alpha \) inductively on the structure of \( \alpha \). We only prove the case for \( \otimes \) and the rest are the same as in [16]. By Proposition 17, \( M \models K_\alpha(\otimes \beta) \) amounts to \( \exists U, (U \neq \emptyset \text{ implies } R(U, \alpha) \neq \emptyset) \) and \( (U \neq \emptyset \text{ implies } R(U, \beta) \neq \emptyset) \). We show this is exactly \( M, W_M \models \alpha \otimes \beta \).

\[ \Rightarrow: \] If both \( U \) and \( \overline{U} \) are non-empty, then \( M \models K_\alpha(\otimes \beta) \) amounts to \( M|_U \models K_\alpha \) and \( M|_{\overline{U}} \models K_\beta \). By IH, it is equivalent to \( M|_U, U \models \alpha \) and \( M|_{\overline{U}}, \overline{U} \models \beta \), which implies \( M, W_M \models \alpha \otimes \beta \) since \( U \cup \overline{U} = W_M \). If one of \( U \) and \( \overline{U} \) is empty, suppose w.l.o.g. \( U = \emptyset \), then we can also show \( M, U \models \beta \) (as before), and \( M, U \models \alpha \), for the empty state support all formulae by Proposition 4.

\[ \Leftarrow: \] Suppose \( M, W_M \models \alpha \otimes \beta \), then there are states \( t \) and \( t' \) such that \( t \cup t' = W_M \) and \( t \models \alpha \) and \( t' \models \beta \). Now at least one of \( t \) and \( t' \) is nonempty since \( W_M \) is non-empty. W.l.o.g., suppose \( t \neq \emptyset \). Note that since \( t = (W_M \setminus t) \subseteq t' \), then \( M, t \models \beta \) by Proposition 4. Now we take \( U = t \), then by IH, \( M|_U \models K_\alpha \) and if \( U \neq \emptyset \) then \( M|_{\overline{U}} \models K_\beta \). Therefore, \( R(U, \alpha) \neq \emptyset \) and \( (U \neq \emptyset \text{ implies } R(U, \beta) \neq \emptyset) \). Thus, \( M, W_M \models K_\alpha(\otimes \beta) \) by Proposition 16. This concludes the first part of the proposition.

For the consequence, \( M|_s, w \models K_\alpha \) iff \( M|_s, s \models \alpha \) iff \( M, s \models \alpha \), and the last step is due to the fact the \( \alpha \) only rely on the state in the support semantics. \[^6\]
Remark 1 Note that the proof for the $\otimes$ case above actually established the equivalence between our semantics based on the idea of weak disjunction by Medvedev and the team/support semantics in dependence/inquisitive logics. In our settings, the formula $=(p, q)$ mentioned in the introduction says that there is a pair of dependence functions $(f_1, f_2)$ s.t. you know that one of these functions captures how $q$ depends on $p$.

Based on the lemma above, we can establish the relation between InqB and KhL, where $K\Gamma = \{K\alpha | \alpha \in \Gamma\}$.

Theorem 20 Given any $\alpha \cup \Gamma \subseteq PL^\otimes$, $\Gamma \vdash \alpha$ iff $K\Gamma \vdash K\alpha$. As a consequence when $\Gamma = \emptyset$, InqB$^\otimes = KhL$.

Proof Suppose $\Gamma \vdash \alpha$ and $M, w \vdash K\Gamma$. Now we have $M, W_M \vdash \Gamma$ by Lemma [19] thus $M, W_M \vdash \alpha$, therefore $M, w \vdash \alpha$. For the other way around, if $K\Gamma \vdash K\alpha$ and $M, s \vdash \Gamma$, then $M|s \vdash K\Gamma$ by Lemma [19] thus $M|s \vdash K\alpha$. By Lemma [19] again, $M, s \vdash \alpha$. ■

5 Axiomatization of PALKhΠ

We first introduce the proof system S5KhPALΠ$^+$ as below.

System S5KhPALΠ$^+$

| Axioms | Rules |
|--------|-------|
| TAUT   | MP    |
| R↓$\otimes$ | $\varphi, \varphi \rightarrow \psi$ |
| DIST$^k$ | NEC$^k$ |
| $\bot_p$ | $\vdash \varphi$ |
| $\otimes$ | $\vdash \varphi \rightarrow \psi$ |
| $\otimes K\varphi$ | $\vdash \varphi \rightarrow K\varphi$ |
| $\otimes K\rightarrow K\varphi$ | $\vdash \varphi \rightarrow K\varphi$ |
| $\otimes$ | GEN$^\varphi$ |
| $\otimes K\alpha \rightarrow K\alpha$ | $\vdash \varphi \rightarrow \forall \varphi'$ |
| $\otimes$ | rRE |
| $\otimes$ | $\vdash \chi[\varphi/\psi] \leftrightarrow \chi$, given that the substitution does not happen in the scope of $K\varphi$. |

where $p \in P$, $\alpha, \beta \in PL^\otimes$, $\varphi, \psi, \chi \in PALKh\Pi$, $\otimes \subseteq \{\land, \lor, \rightarrow\}$; $p$ does not occur free in $\alpha$ and $\beta$ in $K\varphi$ and $K\psi$.

Together with rRE, R↓$\otimes$ states the fact that $\otimes$ behaves exactly like $\lor$ when it occurs outside $K\varphi$. S5 axiom schemata/rules for $K$ together with TAUT, DIST$^\chi$, SUB$^\chi$, SU and rule GEN$^\varphi$ form a complete axiomatization S5$\Pi^+$ of S5 logic with propositional quantifiers [9], where SU states the existence of atoms. Operators $\otimes$, $\otimes K$, $\otimes_\land$ and $\otimes_\lor$ are reduction axioms for $\otimes$ [14, 11] $\otimes$ KKh$\varphi$, $K\varphi_\bot$, $K\varphi_\land$, $K\varphi_\lor$ and $K\varphi_\otimes$ are the reduction axioms decoding the PL$^\otimes$ formulae, whose usages are shown in Lemma [23] Barcan Formula BC, introspection schemata 4$k$, 4$\varphi$, and 5$\varphi$ can be proved from the rest of the system. In particular, 4$\varphi$, requires an inductive proof on the structure of $\alpha$. We include them for their intuitive meanings.

7The original form of $\otimes$ in [11] is $|\chi\varphi| \leftrightarrow (\chi \rightarrow \forall \varphi[\chi])$ (p not is in $\chi$).
In order to show the power of S5KhPALΠ⁺, we give some examples of provable formulae in the system.

**Proposition 21** The following are provable in S5KhPALΠ⁺:

\[ \square_p \varphi \leftrightarrow (x \rightarrow [x]\varphi) \]

\[ \exists_p \exists_p \varphi \leftrightarrow \exists_p [x]\varphi, \ p \ \text{is not in} \ x \]

**Proof** For \([\square_3]_p\): Following Lemma 28 We first change each PALKhII-formula \(\varphi\) into the PALII-formula \(\varphi'\) such that \(\varphi'\) is provably equivalent to \(\varphi\). With Rule rRE, we only need to construct the proof of \([x]\varphi' \leftrightarrow (x \rightarrow [x]\varphi')\).

We prove by induction on \(\varphi'\) to show that there is always a proof for \([\square_3]_p\) in S5KhPALΠ⁺.

- If \(\varphi' \in P \cup \{\perp\}\), then we construct the following proof.
  \[
  \vdash [x]\varphi' \leftrightarrow (x \rightarrow \varphi') \quad ([\square_3]_p) \\
  \vdash (x \rightarrow \varphi') \leftrightarrow (x \rightarrow (x \rightarrow \varphi')) \quad \text{TAUT} \\
  \vdash [x]\varphi' \leftrightarrow (x \rightarrow [x]\varphi') \quad (1)(2) rRE
  \]

- If \(\varphi'\) is \(\varphi_1 \circ \varphi_2, \circ \in \{\&, \vee, \circ, \rightarrow\}\), we construct the following proof.
  \[
  \vdash [x](\varphi_1 \circ \varphi_2) \leftrightarrow [x]\varphi_1 \circ [x]\varphi_2 \quad ([\square_3]_{\bigcirc}) \\
  \vdash [x](\varphi_1 \circ \varphi_2) \leftrightarrow (x \rightarrow [x]\varphi_1) \circ (x \rightarrow [x]\varphi_2) \quad (1) rRE, IH \\
  \vdash [x](\varphi_1 \circ \varphi_2) \leftrightarrow x \rightarrow ([x]\varphi_1 \circ [x]\varphi_2) \quad (2) TAUT \\
  \vdash [x](\varphi_1 \circ \varphi_2) \leftrightarrow x \rightarrow [x](\varphi_1 \circ \varphi_2) \quad (3)(1) rRE
  \]

- If \(\varphi'\) is \(K\psi\), we construct the following proof.
  \[
  \vdash [x]K\psi \leftrightarrow (x \rightarrow K[x]\psi) \quad ([\square_3]_{K}) \\
  \vdash (x \rightarrow K[x]\psi) \leftrightarrow (x \rightarrow (x \rightarrow K[x]\psi)) \quad \text{TAUT} \\
  \vdash [x]K\psi \leftrightarrow (x \rightarrow (x \rightarrow K[x]\psi)) \quad (1)(2) rRE \\
  \vdash [x]K\psi \leftrightarrow (x \rightarrow [x]K\psi) \quad (3) rRE
  \]

- If \(\varphi'\) is \(\forall \psi\), construct the following proof. Let \(q \in P\) be the first propositional variable that is not in \([x]\varphi\).
  \[
  \vdash \forall \psi \leftrightarrow \forall \psi[q/p] \quad \text{SUB}_v, \text{GEN}_v \\
  \vdash [x]\forall \psi \leftrightarrow \forall \psi[q/x] \psi \quad ([\square_3]_{\forall}) \\
  \vdash [x]\forall \psi \leftrightarrow \forall \psi(x \rightarrow [x]\psi) \quad (2) rRE, IH \\
  \vdash [x]\forall \psi \leftrightarrow (\forall \psi x \rightarrow \forall \psi[x]\psi) \quad (3) TAUT, \text{DIST}_v, \text{SUB}_v, \text{GEN}_v \\
  \vdash \forall \psi x (q \ is \ not \ in \ x) \quad \text{SUB}_v, \text{GEN}_v \\
  \vdash [x]\forall \psi \leftrightarrow \forall \psi[x]\psi \quad (4) rRE \\
  \vdash [x]\forall \psi \leftrightarrow (x \rightarrow \forall \psi[x]\psi) \quad (5)(4) rRE \\
  \vdash [x]\forall \psi \leftrightarrow (x \rightarrow [x]\forall \psi) \quad (6)(1) rRE \\
  \vdash [x]\forall \psi \leftrightarrow (x \rightarrow [x]\forall \psi) \quad (7)(1) rRE
  \]

For \([\exists_3]_p\): By definition of \(\exists p\) and \(\neg\) we only have to prove \([x](\forall \psi (\varphi \rightarrow \perp) \rightarrow \perp) \leftrightarrow \forall \psi([x]\varphi \rightarrow \perp) \rightarrow \perp\), where \(p\) is not in \(x\).

\[
\vdash [x](\forall \psi (\varphi \rightarrow \perp) \rightarrow \perp) \leftrightarrow [x]\forall \psi (\varphi \rightarrow \perp) \rightarrow [x] \perp \quad ([\exists_3]_{\bigcirc}) \\
\vdash [x](\forall \psi (\varphi \rightarrow \perp) \rightarrow \perp) \leftrightarrow \forall \psi[x](\varphi \rightarrow \perp) \rightarrow (x \rightarrow \perp) \quad (1) rRE, ([\exists_3]_p, [\exists_3]_v) \\
\vdash [x](\forall \psi (\varphi \rightarrow \perp) \rightarrow \perp) \leftrightarrow (\forall \psi[x](\varphi \rightarrow \perp) \land \chi) \rightarrow \perp \quad (2) TAUT
  \]
same result by referring to the soundness of SS5KhPALΠ.

5.1 Provable equivalence

In Section 4.1 we showed PALKhII is expressively equivalent to ELII. Now we show the same result by referring to the soundness of SS5KhPALΠ (Theorem 22) and that each PALKhII-formula $\phi$ is provably equivalent to a ELII-formula $\phi'$ (Lemma 27). Meanwhile we provide a translation from $\phi$ to $\phi'$.

**Theorem 22 (Soundness)** SS5KhPALΠ is sound over the class of all models.

**Proof** The validity of $[\psi]$ are given in Proposition 18. DISTv, SUBv, GENv and rule GENv are given in 9. KKh, Kh, $K_{\downarrow}$, $K_{\rightarrow}$, $K_{\leftarrow}$, and $K_{\otimes}$ are given in Proposition 17. The rest are trivial.

To prove the completeness we first prove Lemmata 23 and 26 with the two sets of reduction axioms. Recall that PALII is the $K$-free fragment of PALKhII, and ELII is the $[\psi]$-free fragment of PALII.

**Lemma 23** Each PALKhII-formula is provably equivalent to a $K$-free PALII formula in SS5KhPALΠ.

**Proof** We use rRE and Axioms $K_{\downarrow}$, $K_{\rightarrow}$, $K_{\leftarrow}$, $K_{\otimes}$ repeatedly to reduce $K\alpha$ to some formula with $K\beta$ only. With $\vdash K\beta \leftrightarrow K\beta$ from KKh and KKh, we can eliminate all $K$ modalities.

To eliminate the announcement operator, we need a notion of complexity.

**Definition 24 (Announcement rank)** For each $\phi \in$ PALII, we define its announcement rank $ar(\phi)$ inductively as follows:

- If $\phi = p$ or $\phi = \bot$, then $ar(\phi) = 0$.
- If $\phi = \psi \circ \psi_2$ where $\circ = \land, \lor, \otimes, or \rightarrow$, then $ar(\phi) = \max\{ar(\psi_1), ar(\psi_2)\}$.
- If $\phi = K\psi$, $p \in P$, then $ar(\phi) = ar(\psi)$.
- If $\phi = \forall p\psi$, $p \in P$, then $ar(\phi) = ar(\psi)$.
- If $\phi = [\chi]\psi$, then $ar(\phi) = ar(\psi) + ar(\chi) + 1$.

**Lemma 25** Each PALII-formula of the form $[\chi]\psi$ is provably equivalent to a PALII-formula $\phi$ in SS5KhPALΠ such that $ar(\phi) < ar([\chi]\psi)$.

**Proof** We prove by induction on $n = ar([\chi]\psi)$. By definition, $n \geq 1$. In the induction base, suppose $n = 1$, then $ar(\chi) = ar(\psi) = 0$. We prove by induction on $\psi$ that there is a $\phi$ such that $\phi \leftrightarrow [\chi]\psi$ and $ar(\phi) < n$. 

$\top$
1. If \( \psi = p \) or \( \psi = \bot \), then by axiom \([\psi]\), \([\chi]\psi \leftrightarrow \chi \rightarrow \psi\). Hence \( \varphi = \chi \rightarrow \psi \) is what we need.

2. If \( \psi = \psi_1 \land \psi_2 \) where \( \land = \land, \lor, \land, \rightarrow \), then by \([\psi]\), \([\chi]\psi \leftrightarrow [\chi]\psi_1 \land [\chi]\psi_2\). By IH, there are \( \varphi_1 \leftrightarrow [\chi]\psi_1 \) and \( \varphi_2 \leftrightarrow [\chi]\psi_2 \) such that \( \text{ar}(\varphi_1) < \text{ar}([\chi]\psi_1) \) and \( \text{ar}(\varphi_1) < \text{ar}([\chi]\psi_1) \). \( \varphi = [\chi]\psi \) is what we need.

3. If \( \psi = K\varphi' \), then by \([\chi], [\chi]\psi \leftrightarrow \chi \rightarrow [\chi]\psi'\). Note that \( \text{ar}(\chi) < \text{ar}([\chi]\psi) \) and \( \text{ar}([\chi]\psi) = \text{ar}(K[\chi]\psi') \) by definition. By IH, we find \( \varphi' \leftrightarrow [\chi]\psi'\). \( \varphi = \chi \rightarrow \psi' \) is what we need.

4. If \( \psi = \forall p\psi' \) where \( p \in P \), we consider two subcases. 1) if \( p \) is not in \( \chi \), we use \([\chi]\) and the proof is similar to the above cases. 2) if \( p \) is in \( \chi \), replace \( p \) with the first letter \( q \in P \) which is not in \( \chi \) and then go to 1).

In the induction step, suppose \( n > 1 \). Since \( \text{ar}([\chi]\psi) = \text{ar}(\chi) + \text{ar}(\psi) + 1 \), either \( 1 \leq \text{ar}(\chi) \leq n \) or \( 1 \leq \text{ar}(\psi) \leq n \). Assume that \( 1 \leq \text{ar}(\chi) \leq n \). By IH, we find a \( \chi' \leftrightarrow \chi \) s.t. \( \text{ar}(\chi') < \text{ar}(\chi) \). And \( \varphi = [\chi']\psi \) has the desired properties. The other case is similar.

The idea is that we start from the innermost subformulae, and replace them with equivalent \( \text{ELII} \)-formulae using the reduction axioms and \( \text{rRE} \). In this way, we can always get an equivalent formula with lower announcement rank. Since the announcement rank is finite, we can decrease the rank till zero eventually by repeating the process above. Therefore we have the following Lemma 26.

**Lemma 26** Each \( \text{PALII} \)-formula is provably equivalent to an \( \text{ELII} \)-formula in \( \text{SSKhPAL}^\perp \).

Combining Lemmata 23 and 26 we immediately have.

**Lemma 27** Each \( \text{PALK}^\perp \)-formula is provably equivalent to an \( \text{ELII} \)-formula in \( \text{S5KhPAL}^\perp \).

Theorem 28 follows naturally from Lemma 27 and Theorem 22.

**Theorem 28** \( \text{PALK}^\perp \) is equally expressive as \( \text{ELII} \) over all models.

Note that \( \text{ELII} \) is more expressive than \( \text{EL} \) [9].

### 5.2 Completeness

With Lemma 27 and Theorem 28 the completeness of System \( \text{S5KhPAL}^\perp \) can be reduced to that of \( \text{S5}^\perp \), which is given in [9]. \( \text{S5}^\perp \) is a variety of second order modal logic, containing all the axiom schema/rules of \( S5 \) as well as those concerning propositional quantifiers in \( \text{S5KhPAL}^\perp \).

**Theorem 29 (Completeness of \( \text{S5}^\perp \) [9])** \( \text{S5}^\perp \) is a complete axiomatization with regard to the class of models.

**Theorem 30 (Completeness)** System \( \text{S5KhPAL}^\perp \) is a complete axiomatization of \( \text{InqKhL} \).

**Proof** We first use Lemma 23 and Lemma 26 to translate each \( \text{PALK}^\perp \)-formula \( \varphi \) into an equivalent \( \text{ELII} \)-formula \( \varphi' \) and then use the completeness of \( \text{S5}^\perp \). Note that \( \vdash \psi \) below means \( \varphi \) is in \( \text{S5KhPAL}^\perp \).

\[
\begin{align*}
\vdash \varphi & \quad \text{expressive equivalence} \\
\text{Theorem 28} & \\
\vdash \varphi' & \quad \text{completeness of \( \text{S5}^\perp \)} \\
\text{Theorem 29} & \\
\vdash \text{S5}^\perp & \quad \varphi' \\
\text{S5}^\perp \subseteq \text{S5KhPAL}^\perp & \\
\vdash & \quad \text{provably equivalence} \\
\text{Lemma 27} & \\
\vdash & \quad \psi
\end{align*}
\]
6 Generalization of Tensor Disjunction

Inspired by our epistemic interpretation, we generalize the binary $\otimes$ to $n$-ary operators for any $n \geq 2$ with another parameter $k \leq n$.

6.1 Generalizing the tensor operator

Consider the following scenario: You completed an exam with $n$ questions with one point each, and get a total score of $m$ without knowing which of your answers were correct. What is your epistemic state? The original tensor actually captures the special case when $m = 1$ and $n = 2$: you have two resolutions for $\alpha$ and $\beta$ respectively, and you are sure at least one of them must be an actual resolution for the corresponding formula. For any $n \geq 2$ and $1 \leq m \leq n$, we now define an $n$-ary connective $\otimes^k_n$.

Definition 31 (Language $\text{PL}^{\otimes^k_n}$) The propositional language with general tensor ($\text{PL}^{\otimes^k_n}$) is as follows:

$$\alpha ::= p \mid \bot \mid (\alpha \land \alpha) \mid (\alpha \lor \alpha) \mid (\alpha \rightarrow \alpha) \mid \otimes^k_n(\alpha_1, \ldots, \alpha_n)$$

where $p \in \mathbb{P}$ and $n \geq 2$, $1 \leq k \leq n$.

Definition 32 (Language $\text{PALKhIIG}$) The Public Announcement Logic with Know-how and General Tensor ($\text{PALKhIIG}$) is as follows:

$$\varphi ::= p \mid \bot \mid (\varphi \land \varphi) \mid (\varphi \lor \varphi) \mid (\varphi \rightarrow \varphi) \mid \otimes^k_n(\varphi_1, \ldots, \varphi_n) \mid \mathcal{K}_\alpha \mid \mathcal{K}_h \alpha \mid \forall p \varphi \mid [\varphi]\varphi$$

where $p \in \mathbb{P}$ and $\alpha \in \text{PL}^{\otimes^k_n}$.

Now, we introduce the semantics of new connectives $\otimes^k_n$ via resolutions.

Definition 33 For any positive integer $n \geq 2$ and $1 \leq k \leq n$, we define the resolution space and resolution of $\otimes^k_n$ as follow:

$$S(\otimes^k_n(\alpha_1, \ldots, \alpha_n)) = S(\alpha_1) \times \cdots \times S(\alpha_n)$$

$$R(w, \otimes^k_n(\alpha_1, \ldots, \alpha_n)) = \{ (r_1, \ldots, r_n) \mid k \leq |\{i \in [1, n] \mid r_i \in R(w, \alpha_i)\}| \}$$

The truth condition for $\mathcal{K}_h$ is as before in Definition[11] In particular, $M, w \models \mathcal{K}_h \otimes^k_n(\alpha_1, \ldots, \alpha_n)$ iff $R(W_M, \otimes^k_n(\alpha_1, \ldots, \alpha_n)) \neq \emptyset$.

By Definition[11] and [33] it is not hard to see the following.

Proposition 34 $M, w \models \mathcal{K}_h \otimes^k_n(\alpha_1, \ldots, \alpha_n)$ if and only if there is an $n$-tuple $(r_1, \ldots, r_n)$ such that for any $v \in W_M$, $|\{ i \mid r_i \in R(v, \alpha_i)\}| \geq k$, i.e., there are at least $k$ indexes $i \in [1, n]$ such that $r_i \in R(v, \alpha_i)$.

Note that based on the above proposition, the truth condition for $\otimes^1_n$ is exactly as the one for the standard $\otimes$ defined earlier.

$\otimes^k_n$ can also appear out of $\mathcal{K}_h$. Hence we define its semantics as below.

Definition 35 (Semantics)

$$M, w \models \otimes^k_n(\varphi_1, \ldots, \varphi_n) \iff M, w \models \bigvee_{I \subseteq \{1, 2, \ldots, n\} \atop |I| = k} \bigwedge_{i \in I} \varphi_i$$

The semantics is guided by Proposition[12] with the desired property below.

Proposition 36 For any $\alpha \in \text{PL}^{\otimes^k_n}$ and $M, w$, $M, w \models \alpha \iff R(w, \alpha) \neq \emptyset$.  


\[ M, w \models \otimes_n^k (\alpha_1, \ldots, \alpha_n) \]
\[ \iff M, w \models \bigvee_{I \subseteq \{1, 2, \ldots, n\} \atop |I| = k} \bigwedge_{i \in I} \alpha_i \]
\[ \iff \exists I \subseteq \{1, 2, \ldots, n\} \text{ with } |I| = k \text{ s.t. } M, w \models \bigwedge_{i \in I} \alpha_i \]
\[ \iff \exists I \subseteq \{1, 2, \ldots, n\} \text{ with } |I| = k \text{ s.t. } \forall i \in I, R(w, \alpha_i) \not= \emptyset \text{ (by IH)} \quad (\dagger) \]

And it is easy to see that \( R(w, \otimes_n^k (\alpha_1, \ldots, \alpha_n)) \) is nonempty iff at least \( k \) of \( R(w, \alpha_i) \) is nonempty. Hence, \((\dagger)\) implies that \( R(w, \otimes_n^k (\varphi_1, \ldots, \varphi_n)) \not= \emptyset \).

Next, we show how to reduce the general tensors in \( \text{PALKH}^{\Pi} \).

**Proposition 37** The following schemata are valid:
\[ R^d \otimes_n^k \leftrightarrow \bigvee_{I \subseteq \{1, 2, \ldots, n\} \atop |I| = k} \bigwedge_{i \in I} \varphi_i \]
\[ \text{Kh} \otimes_n^k \leftrightarrow \exists p_1 \ldots \exists p_n (K \otimes_n^k (p_1, \ldots, p_n) \land \bigwedge_{i=1}^n K[p_i] \varphi_i) \]
(where all the \( p_i \) do not occur free in all the \( \alpha_i \)).

**Proof**

For \( \text{Kh} \otimes_n^k \):

\[ \implies \text{By Proposition 34, } M, w \models \text{Kh} \otimes_n^k (\alpha_1, \ldots, \alpha_n) \text{ iff there is a } n\text{-tuple } (r_1, \ldots, r_n) \text{ s.t. for any } v \in W_M \text{, there are at least } k \text{ indexes } i \in [1, n] \text{ such that } r_i \in R(v, \alpha_i). \]

Let \( U_i = \{ v \in W_M \mid r_i \in R(v, \alpha_i) \} \), then consider \( M[p \mapsto U] = (W, V') \) such that \( V' \) assigns \( U_i \) to \( p_i \) for \( i \in \{1, \ldots, n\} \) and coincides with \( V \) on all other atoms. Then, for any \( v \in W_M \), there are at least \( k \) indexes \( i \in [1, n] \) s.t. \( M[p \mapsto U], v \models p_i \), so \( M[p \mapsto U], w \models \text{Kh} \otimes_n^k (p_1, \ldots, p_n) \). And since for any \( v \in U_i \) we have \( r_i \in R(v, \alpha_i) \), so we have for any \( v \in W_M, M[p \mapsto U], w \models \text{Kh} \otimes_n^k (p_1, \ldots, p_n) \).

\[ \iff \text{Suppose } M, w \models \exists p_1 \ldots \exists p_n (K \otimes_n^k (p_1, \ldots, p_n) \land \bigwedge_{i=1}^n K[p_i] \varphi_i), \text{ then there are } U_i \subseteq W_M \text{ such that } M[p \mapsto U], w \models K \otimes_n^k (p_1, \ldots, p_n) \land \bigwedge_{i=1}^n K[p_i] \varphi_i. \]

For the first conjunct: \( M[p \mapsto U], w \models K \otimes_n^k (p_1, \ldots, p_n) \) means that for any \( v \in W_M \) we have \( M[p \mapsto U], v \models \otimes_n^k (p_1, \ldots, p_n) \). So at least \( k \) of \( p_i \) is true in \( v \), which means that \( v \) belongs to at least \( k \) of \( U_i \). For the second conjunct: \( M[p \mapsto U], w \models \bigwedge_{i=1}^n K[p_i] \varphi_i \) means that for any \( v \in W_M, v \in U_i \) implies that \( R(U_i, \alpha_i) \not= \emptyset \). So, if \( U_i \not= \emptyset \), choose an element from \( R(U_i, \alpha_i) \) and denote it as \( r_i \). If \( U_i = \emptyset \), choose an arbitrary element from \( S(\alpha_i) \) and denote it as \( r_i \).

Combining the meaning of the two conjuncts, we know that for any \( v \in W_M, v \) belongs to at least \( k \) of \( U_i \) and \( U_i \not= \emptyset \) implies \( r_i \in R(U_i, \alpha_i) \) for every \( i \). Hence, \((r_1, \ldots, r_n)\) is a \( n\)-tuple such that for any \( v \in W_M, \) there are at least \( k \) indexes \( i \in [1, n] \) such that \( r_i \in R(v, \alpha_i) \), by Proposition 34, we have \( M, w \models \text{Kh} \otimes_n^k (\alpha_1, \ldots, \alpha_n) \).

By using the reduction axioms above, all general tensors can be eliminated semantically, and thus \( \text{PALKH}^{\Pi} \) are equally expressive.

Let \( S5\text{KH}^{\Pi} \) be \( S5\text{KH}^{\Pi} \) extended with \( R^d \otimes_n^k \) and \( \text{Kh} \otimes_n^k \) for any \( n \geq 2 \) and \( 1 \leq k \leq n \). Similar to Theorem 30, it is straightforward to show:

**Theorem 38** (Soundness and completeness) Proof system \( S5\text{KH}^{\Pi} \) is sound and complete over the class of all models.
6.2 Support semantics for $\otimes^n_k$

We can now go back to the support semantics for $\otimes^n_k$.

**Definition 39 (Support for $\otimes^n_k$)** $\mathcal{M}, s \models \otimes^n_k(\alpha_1, \ldots, \alpha_n)$ iff there exist $n$ subsets $t_1, \ldots, t_n$ of $s$ such that for any $i \in [1, n]$, $\mathcal{M}, t_i \models \alpha_i$ and any $w \in s \subseteq W_M$ belongs to at least $k$ of $t_i$.

The support semantics for other connectives stays the same as in Definition 3. Let InqB$^\otimes_k$ be the set of valid PL$^\otimes_k$ formulae by the support semantics. We can show KhL$^\otimes_k = \{ \alpha \in PL^\otimes_k \models \kappa \alpha \}$ is exactly InqB$^\otimes_k$, based on the following generalization of Lemma 19.

**Proposition 40** For any $\alpha \in PL^\otimes_k$, $\mathcal{M}, w \models \kappa \alpha \iff \mathcal{M}, W_M \models \alpha$.

**Proof** Based on Lemma 19 we only consider the case of $\otimes^n_k(\alpha_1, \ldots, \alpha_n)$ and write $\exists U \subseteq W_M$ for brevity, similarly for $\forall$.

$\mathcal{M}, w \models \kappa(\otimes^n_k(\alpha_1, \ldots, \alpha_n))$

$\iff \mathcal{M}, w \models \exists p_1 \cdots \exists p_n (K \otimes^n_k (p_1, \ldots, p_n) \land \bigwedge_{i=1}^n K[p_i] \kappa \alpha_i)$ (by Proposition 37).

$\iff \exists U_1, \ldots, U_n, \forall v \in W_M, v \text{ belongs to at least } k \text{ of } U_i$ and

$\forall i \in [1, n], \text{ if } U_i \neq \emptyset \text{ then } R(U_i, \alpha_i) \neq \emptyset$.

$\iff \exists t_1, \ldots, t_n, \forall i \in [1, n], t_i \models \alpha_i \text{ and } \forall v \in W_M, v \text{ belongs to at least } k \text{ of } t_i$.

$\iff \mathcal{M}, W_M \models \otimes^n_k(\alpha_1, \ldots, \alpha_n)$.

As shown in [21], adding tensor does not increase the expressive power of inquisitive logic. In fact, adding all the general tensors also does not increase the expressive power of inquisitive logic.

First, we extend the definition of realization in [8] to our new connectives.

**Definition 41 (Realizations)**

- $RL(p) = \{ p \}$ for $p \in P$
- $RL(\bot) = \{ \bot \}$
- $RL(\alpha \lor \beta) = RL(\alpha) \cup RL(\beta)$
- $RL(\alpha \land \beta) = \{ \rho \land \sigma \mid \rho \in RL(\alpha) \text{ and } \sigma \in RL(\beta) \}$
- $RL(\alpha \rightarrow \beta) = \{ \lambda_{p \in RL(\alpha)}(\rho \rightarrow f(\rho)) \mid f : RL(\alpha) \rightarrow RL(\beta) \}$
- $RL(\otimes^n_k(\alpha_1, \ldots, \alpha_n)) = \{ \neg \bigwedge_{i \in [1, 2, \ldots, n]} \Delta_{i \in I} \rho_i \mid \text{ for all } i, \rho_i \in RL(\alpha_i) \}$

Then we can generalize the Inquisitive normal form in [5][8].

**Proposition 42 (Normal form)** For any $\alpha \in PL^\otimes_k$, $s \models \alpha$ iff $s \models \bigvee_{\rho \in RL(\alpha)} \rho$.

**Theorem 43** The languages of InqB and InqB$^\otimes_k$ are equally expressive.

**Proof** By Proposition 42 for any $\alpha \in PL^\otimes_k$, $\alpha$ is equivalent to a disjunction of some $\rho$ without general tensors.

In [22], it is shown that the variants of propositional dependence logics PD, PD$^\vee$, PID, InqB are all equally expressive. Similarly, adding general tensors to these logics will also not increase the expressive power.

**Corollary 44** Adding general tensors to PD, PD$^\vee$, PID or InqB does not increase their expressive power.
6.3 Uniform Definability of general tensors

It is natural to ask whether the generalized tensors are uniformly definable by the standard binary tensor $\otimes$. In [8], it is proved that $\otimes$ is not uniformly definable in $\text{InqB}$. Inspired by the techniques in [8], we will show in Theorem [51] that all the $\otimes^k_n$ are not uniformly definable in $\text{InqB}^\otimes$ except $\otimes^1_n$ and $\otimes^2_n$, where $1 \leq k \leq n$ and $2 \leq n$.

First, we show that $\otimes^2_n$ is a trivial conjunction, $\otimes^1_n$ can be uniformly defined by $\otimes^1_2$, and by using $\top$ or $\bot$, some general tensor can be uniformly defined by others.

**Proposition 45** For any $\alpha_1, \cdots, \alpha_n \in \text{InqB}^{\otimes^k_n}$, there are following properties:

1. For any $n \geq 2$ and any state $s$, $s \models \otimes^1_n(\alpha_1, \cdots, \alpha_n) \iff s \models \lor_{i=1}^n \alpha_i$.
2. For any $n \geq 3$ and any state $s$, $s \models \otimes^2_n(\alpha_1, \cdots, \alpha_n) \iff s \models \otimes^1_2(\otimes_{n-1}^{\alpha_1} \alpha_1, \cdots, \alpha_{n-1}, \alpha_n, \alpha_n)$.
3. For any $n \geq 3$, $1 \leq k \leq n$ and any state $s$, $s \models \otimes^k_n(\alpha_1, \cdots, \alpha_{n-1}, \bot) \iff s \models \otimes_{n-1}^k(\alpha_1, \cdots, \alpha_{n-1})$.
4. For any $n \geq 3$, $1 \leq k \leq n-1$ and any state $s$, $s \models \otimes^k_n(\alpha_1, \cdots, \alpha_{n-1}, \bot) \iff s \models \otimes_{n-1}^k(\alpha_1, \cdots, \alpha_{n-1})$.

**Proof**

(1) For any $n \geq 2$ and any state $s$, $s \models \otimes^1_n(\alpha_1, \cdots, \alpha_n)$ iff $\exists t_1, \cdots, t_n \subseteq s, \forall i \in [1, n], t_i \models \alpha_i$, and for any $w \in s$, $w$ belongs to $n$ of $t_i$. So $w$ belongs to all the $t_i$, which means that $t_i = s$ for all $i \in [1, n]$. Hence, for all $i \in [1, n]$ we have $s \models \alpha_i$, which is equivalent to $s \models \lor_{i=1}^n \alpha_i$.

(2) For any $n \geq 3$, $1 \leq k \leq n-1$ and any state $s$, $s \models \otimes^k_n(\alpha_1, \cdots, \alpha_n)$ iff $\exists t_1, \cdots, t_n \subseteq s, \forall i \in [1, n], t_i \models \alpha_i$, and $\bigcup_{i=1}^n t_i = s$. Then it is obvious that $\bigcup_{i=1}^n t_i \models \otimes^k_{n-1}(\alpha_1, \cdots, \alpha_{n-1})$ and $t_n \models \alpha_n$, hence $s \models \otimes^1_2(\otimes_{n-1}^{\alpha_1} \alpha_1, \cdots, \alpha_{n-1}, \alpha_n)$.

(3) For any $n \geq 3$ and any state $s$, $s \models \otimes^k_n(\alpha_1, \cdots, \alpha_{n-1}, \bot)$ iff $\exists t_1, \cdots, t_{n-1}, t_n \subseteq s, \forall i \in [1, n-1], t_i \models \alpha_i$, and for any $w \in s$, $w$ belongs to at least $k$ of $t_i$. Since $s \models \bot$ is trivially true, we can assume $t_n = s$, then the condition is equivalent to $\exists t_1, \cdots, t_{n-1} \subseteq s, \exists i \in [1, n-1], t_i \models \alpha_i$, and for any $w \in s$, $w$ belongs to at least $k$ of $t_i$.

Hence, it is equivalent to $s \models \otimes_{n-1}^k(\alpha_1, \cdots, \alpha_{n-1})$.

(4) For any $n \geq 3$, $1 \leq k \leq n$ and any state $s$, $s \models \otimes^k_n(\alpha_1, \cdots, \alpha_{n-1}, \bot)$ iff $\exists t_1, \cdots, t_{n-1}, t_n \subseteq s, \forall i \in [1, n-1], t_i \models \alpha_i$ and $t_n \models \bot$, and for any $w \in s$, $w$ belongs to at least $k$ of $t_i$.

Since only $\bot \models \bot$, so we can assume $t_n = \emptyset$, then the condition is equivalent to $\exists t_1, \cdots, t_{n-1} \subseteq s, \forall i \in [1, n-1], t_i \models \alpha_i$, and for any $w \in s$, $w$ belongs to at least $k$ of $t_i$. Hence, it is equivalent to $s \models \otimes_{n-1}^k(\alpha_1, \cdots, \alpha_{n-1})$.

There are some definitions about uniform definability from [21] as below.

**Definition 46 (Context)** A context for a propositional logic $\mathcal{L}$ is an $\mathcal{L}$-formula $\varphi(p_1, \cdots, p_n)$ with distinguished atoms $p_1, \cdots, p_n$, and it is also allowed to contain other atoms besides $p_1, \cdots, p_n$. For any $\mathcal{L}$-formulae $\psi_1, \cdots, \psi_n$, we write $\varphi(\psi_1, \cdots, \psi_n)$ for the formula $\varphi(\psi_1/p_1, \cdots, \psi_n/p_n)$.

**Definition 47 (Uniform definability)** In a language $\mathcal{L}$, we say that an $n$-ary connective $\otimes$ is uniformly definable if there exists a context $\zeta(p_1, \cdots, p_n)$ such that for all $\chi_1, \cdots, \chi_n \in \mathcal{L}$: $\otimes(\chi_1, \cdots, \chi_n)$ is equivalent to $\zeta(\chi_1, \cdots, \chi_n)$.

In order to show that $\otimes^2_3$ is not uniformly definable, we consider equivalence relativized to a state $s$. 
Definition 48 (Relativized equivalence [6]) Let $s$ be a state in $M$ and $\varphi, \psi \in \text{PL}^O$. We say that $\varphi$ and $\psi$ are equivalent relativized to $s$, $\varphi \equiv_s \psi$ iff for all states $t \subseteq s$, $t \Vdash \varphi \iff t \Vdash \psi$.

Note that if $\varphi$ and $\psi$ are equivalent then they are equivalent relativized to any state $s$.

Consider $\psi = p_1 \lor p_2 \lor p_3 \lor p_4$ and $s = \{w_{12}, w_{13}, w_{14}, w_{23}, w_{24}, w_{34}\}$ where only $p_i, p_j$ are true in $w_{ij}$ and all of other propositional letters are false. Now, we show that relativized to this state $s$, $\otimes_2$ can't be uniformly defined by any context in InqB$^O$.

Lemma 49 For any context $\varphi(p_0)$, with $\varphi \in \text{PL}^O$ not containing $p_1, p_2, p_3, p_4$, $\varphi(\psi/p_0)$ would be equivalent to $\bot, \psi, \otimes_2^{\bot}(\psi, \psi)$ or $\top$ relativized to $s$.

**Proof**

First we notice for any state $t$, $t \Vdash \bot \Rightarrow t \Vdash \psi \Rightarrow t \Vdash \otimes_2^{\bot}(\psi, \psi) \Rightarrow t \Vdash \top$ ($\ast$).

Then we prove by induction on $\varphi$. For short, we write $\varphi^*$ for $\varphi(\psi/p_0)$:

- For $\varphi = \bot$ or $\varphi = p$ with $p \neq p_0$: Since we assume that $p_1, p_2, p_3, p_4$ are not in $\varphi$, so $p$ is different from them. Hence, it is obvious that $\varphi^* \equiv_s \bot$.

- For $\varphi = p_0$: It is obvious that $\varphi^* \equiv_s \psi$.

- For $\varphi = \varphi_1 \land \varphi_2$: so $\varphi^* = \varphi_1^* \land \varphi_2^*$ and $t \Vdash \varphi_1^* \land \varphi_2^*$ iff $t \Vdash \varphi_1^*$ and $t \Vdash \varphi_2^*$. By IH, $\varphi_1^*$ and $\varphi_2^*$ are both equivalent to one of $\bot, \psi, \otimes_2^{\bot}(\psi, \psi), \top$. Since we have ($\ast$) and that $t \Vdash \chi_1 \Rightarrow t \Vdash \chi_2$ implies $t \Vdash \chi_1 \land \chi_2 \iff t \Vdash \chi_1$, it is obvious that $\varphi^*$ is also equivalent to one of $\bot, \psi, \otimes_2^{\bot}(\psi, \psi), \top$ in $s$.

- For $\varphi = \varphi_1 \lor \varphi_2$: so $\varphi^* = \varphi_1^* \lor \varphi_2^*$ and $t \Vdash \varphi_1^* \lor \varphi_2^*$ iff $t \Vdash \varphi_1^*$ or $t \Vdash \varphi_2^*$. Similarly, we have ($\ast$) and that $t \Vdash \chi_1 \Rightarrow t \Vdash \chi_2$ implies $t \Vdash \chi_1 \lor \chi_2 \iff t \Vdash \chi_2$. Obviously $\varphi^*$ is equivalent to one of $\bot, \psi, \otimes_2^{\bot}(\psi, \psi), \top$ in $s$.

- For $\varphi = \varphi_1 \rightarrow \varphi_2$: so $\varphi^* = \varphi_1^* \rightarrow \varphi_2^*$, and $t \Vdash \varphi_1^* \rightarrow \varphi_2^*$ iff for any $t' \leq t$, $t' \Vdash \varphi_1^*$ implies $t' \Vdash \varphi_2^*$. Since we have ($\ast$), we could know that:
  
  - $\bot \rightarrow \bot, \bot \rightarrow \psi, \bot \rightarrow \otimes_2^{\bot}(\psi, \psi), \bot \rightarrow \top, \psi \rightarrow \psi, \psi \rightarrow \otimes_2^{\bot}(\psi, \psi), \psi \rightarrow \top, \otimes_2^{\bot}(\psi, \psi) \rightarrow \otimes_2^{\bot}(\psi, \psi), \otimes_2^{\bot}(\psi, \psi) \rightarrow \top$ and $\top \rightarrow \top$ are all equivalent to $\top$ in $s$. Also, if $\psi \leftrightarrow \bot$, then $\psi \rightarrow \bot$ and $\otimes_2^{\bot}(\psi, \psi) \rightarrow \bot$ are equivalent to $\top$ in $s$.
  
  - If $\psi \neq \bot$, $\psi \rightarrow \bot, \otimes_2^{\bot}(\psi, \psi) \rightarrow \bot$ and $\top \rightarrow \bot$ are all equivalent to $\bot$ in $s$.
  
  - $\otimes_2^{\bot}(\psi, \psi) \rightarrow \psi$ and $\top \rightarrow \psi$ are equivalent to $\psi$ in $s$.
  
  - $\top \rightarrow \otimes_2^{\bot}(\psi, \psi)$ is equivalent to $\otimes_2^{\bot}(\psi, \psi)$ in $s$.

Hence, $\varphi^*$ is equivalent to one of $\bot, \psi, \otimes_2^{\bot}(\psi, \psi), \top$ in $s$.

- For $\varphi = \otimes_2^{\bot}(\varphi_1, \varphi_2)$: so $\varphi^* = \otimes_2^{\bot}(\varphi_1^*, \varphi_2^*)$. We consider the following cases:

  - $\varphi_1^* \equiv_s \top$. Then $\otimes_2^{\bot}(\varphi_1^*, \varphi_2^*) \equiv_s \top$.
  
  - $\varphi_1^* \equiv_s \bot$. Then $\otimes_2^{\bot}(\varphi_1^*, \varphi_2^*) \equiv_s \varphi_2^*$.
  
  - $\varphi_1^* \equiv_s \psi$. If $\varphi_2^* \equiv_s \top$ or $\varphi_2^* \equiv_s \bot$, it would be the same as former cases. Then we need to distinguish two sub-cases:

    * $\varphi_2^* \equiv_s \psi$. Then $\otimes_2^{\bot}(\varphi_1^*, \varphi_2^*) \equiv_s \otimes_2^{\bot}(\psi, \psi)$.
    
    * $\varphi_2^* \equiv_s \otimes_2^{\bot}(\psi, \psi)$. Then $t \Vdash \varphi^* \iff$ there are $t_1, t_2 \subseteq t$ and $t_1 \cup t_2 = t$ such that $t_1 \Vdash \psi$ and $t_2 \Vdash \otimes_2^{\bot}(\psi, \psi) \iff$ there are $t_1, t_2 \subseteq t$, $t_1 \cup t_2 = t$ and $p_{i_1}, p_{i_2}, p_{i_3}$ such that $p_i$ is true in any $w \in t_1$ and for any $w \in t_2$, $p_{i_2}$ or $p_{i_3}$ is true in $w \iff$ there are $p_{i_1}, p_{i_2}, p_{i_3}$ such that for any $w \in t$, $p_{i_1}, p_{i_2}$ or $p_{i_3}$ is true in $w$. However, there are only four propositional letters $p_1, p_2, p_3, p_4$ and in each $w \in s$, two of these propositional letters are true. So consider $p_1, p_2$ and $p_3$, we will know that for any $w \in t \subseteq s$, at least one of $p_1, p_2$ and $p_3$ is true in $w$. Hence, $\otimes_2^{\bot}(\psi, \otimes_2^{\bot}(\psi, \psi)) \equiv_s \top$. 

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- \( \varphi_1 \equiv_s \otimes_2^1(\psi, \psi) \). Then if \( \varphi_2 \equiv_s \top, \varphi_2 \equiv_s \bot \) or \( \varphi_2 \equiv_s \psi \), it would be the same as former cases. And if \( \varphi_2 \equiv_s \otimes_2^1(\psi, \psi) \), the proof is similar to the previous case and the result is that \( \otimes_2^1(\otimes_2^1(\psi, \psi), \otimes_2^1(\psi, \psi)) \equiv_s \top. \)

**Lemma 50** \( \otimes_2^3 \) is not uniformly definable in InqB\(^\odot\).

**Proof** If \( \otimes_2^3 \) is uniformly definable in InqB\(^\odot\), there will be a context \( \varphi(p) \) such that for any \( \chi \in \text{InqB}^\odot \): \( \varphi(\chi) \) is equivalent to \( \otimes_2^3(\chi, \chi, \chi) \).

However, as we proved in Lemma [49] for any context \( \varphi(p_0) \in \text{InqB}^\odot \), \( \varphi(\psi/p_0) \) would be equivalent to \( \bot, \psi, \otimes_2^1(\psi, \psi) \) or \( \top \) relativized to \( s \). But it is obvious that \( \otimes_2^3(\psi, \psi, \psi) \) is not equivalent to \( \bot, \psi, \otimes_2^1(\psi, \psi) \) or \( \top \) relativized to \( s \). Hence, \( \otimes_2^3(\psi, \psi, \psi) \) and \( \varphi(\psi/p_0) \) are not equivalent relativized to \( s \), and hence not equivalent in general, which gives rise to a contradiction! ■

**Theorem 51** All the \( \otimes_k^n \) are not uniformly definable in InqB\(^\odot\) except \( \otimes_1^n \) and \( \otimes_n^n \), i.e., for any \( 2 \leq k \leq n-1 \), \( \otimes_k^n \) is not uniformly definable.

**Proof** When \( 2 \leq k \leq n-1 \) (thus \( n \geq 3 \)), by Proposition [45], \( \otimes_3^2 \) can be uniformly defined by \( \otimes_k^n \) in the way of fixing some components as \( \top \) or \( \bot \), so \( \otimes_3^2 \) is not uniformly definable in InqB\(^\odot\) implies that \( \otimes_k^n \) is not uniformly definable in InqB\(^\odot\). ■

### 7 Conclusions and future work

In this paper, we proposed an epistemic interpretation of the tensor disjunction in dependence logic. The interpretation is inspired by the notion of weak disjunction in Medvedev's early work in terms of the BHK-like semantics. The connection between the two disjunctions is exposed in inquisitive logic with tensor disjunction, studied in the literature. We introduce a powerful dynamic epistemic language in which the corresponding know-how formulae of each InqB\(^\odot\) formula can be formulated and reduced to a know-how free formula. In particular, the tensor disjunction can be defined by an epistemic formula using propositional quantifiers. We give the axiomatization of our full logic, and generalize the tensor disjunction to a family of \( n \)-ary operators parametered by a \( k \leq n \), which capture the intuitive epistemic situations that one knows a list of \( n \) possible answers to \( n \) questions such that \( k \) of the \( n \) answers are correct.

Besides further technical questions regarding our logic, the generalized tensors particularly invite further investigations. Its obvious combinatorial features may find applications in cryptographic protocols and game theory. To see the connection with the latter, we end the paper with the following interesting scenario where \( \otimes_2^2 \) makes perfect sense. Consider a badminton match between two teams. Each team has one good player and two other less capable ones. We can measure the abilities of the players by numbers, which will determine the result of the matches in the most obvious way. For team \( A \), it is \( 6, 2, 2 \) for the three players, and for team \( B \) it is \( 5, 3, 3 \). The battle between the two teams consists of three single matches, and the rule of game does not prevent one player from playing two matches if not in a row, although the second time the player will lose \( 1/3 \) of his or her ability due to tiredness. Now, with some reflection, we can see team \( B \) has a unique arrangement of the playing players to make sure they can win at least two out of the three matches no matter how team \( A \) orders their playing players. Do you know which one?

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