New $\varepsilon$-regularity criteria and application to the box dimension of the singular set in the 3D Navier-Stokes equations

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Abstract

In this paper, by exploiting the energy hidden in the pressure, we present some new $\varepsilon$-regularity criterion below

$$\|u\|_{L^{p,q}(Q(1))} + \|\Pi\|_{L^1(Q(1))} < \varepsilon, \quad 1 \leq 2/q + 3/p < 2, \quad 1 \leq p, q \leq \infty,$$

then $u \in L^\infty(Q(1/2))$, to suitable weak solutions of the 3D Navier-Stokes equations, which is an improvement of corresponding results recently proved by Guevara and Phuc in [7, Calc. Var. 56:68, 2017]. As an application, we improve the known upper box dimension of the possible interior singular set of suitable weak solutions of this system from $975/758(\approx 1.286)$ [28] to $2400/1903(\approx 1.261)$. 

MSC(2000): 35B65, 35D30, 76D05

Keywords: Navier-Stokes equations; suitable weak solutions; regularity; box dimension;

1 Introduction

We study the following incompressible Navier-Stokes equations in three-dimensional space

$$\begin{align*}
    &u_t - \Delta u + u \cdot \nabla u + \nabla \Pi = 0, \quad \text{div } u = 0, \\
    &u|_{t=0} = u_0,
\end{align*}$$

(1.1)

where $u$ stands for the flow velocity field, the scalar function $\Pi$ represents the pressure. The initial velocity $u_0$ satisfies $\text{div } u_0 = 0$.

In this paper, we are concerned with the regularity of suitable weak solutions originated from Scheffer in [18, 21] to the 3D Navier-Stokes equations (1.1). This kind of weak solutions obeys the local energy inequality (2.1). A point is said to be a regular point to suitable weak solutions of the Navier-Stokes system (1.1) as long as $u$ is bounded in some neighborhood of this point. The remaining points will be called singular points and denoted by $S$. In this direction, the celebrated Caffarelli-Kohn-Nirenberg theorem involving the 3D Navier-Stokes equations is that one dimensional Hausdorff measure of $S$ is zero in [1]. Roughly

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speaking, the regularity of suitable weak solutions strongly rests on the so-called $\varepsilon$-regularity criteria (see, e.g., [1–3, 7–17, 23–28]). Before going further, we give some notations used throughout this paper. For $q \in [1, \infty]$, the notation $L^q((0, T); X)$ stands for the set of measurable functions on the interval $(0, T)$ with values in $X$ and $\|f(t, \cdot)\|_X$ belongs to $L^q(0, T)$. For simplicity, we write

$$\|f\|_{L^{p,q}(Q(r))} = \|f\|_{L^q(Q(-r^2, 0; L^p(B(r))))}$$

where $Q(r) = B(r) \times (t - r^2, t)$ and $B(r)$ denotes the ball of center $x$ and radius $r$. Now, we briefly recall some previous $\varepsilon$-regularity criteria: $u \in L^\infty(Q(1/2))$ provided

- Caffarelli, Kohn and Nirenberg [1],
  $$\|u\|_{L^3(Q(1))} + \|u_p\|_{L^1(Q(1))} + \|\Pi\|_{L^{1,5/4}(Q(1))} \leq \varepsilon.$$  

- Lin [14], Ladyzenskaja and G. Seregin [13],
  $$\|u\|_{L^3(Q(1))} + \|\Pi\|_{L^{3/2}(Q(1))} \leq \varepsilon.$$  

- Vasseur [24], for any $p > 1$,
  $$\|u\|_{L^{2,\infty}(Q(1))} + \|\nabla u\|_{L^2(Q(1))} + \|\Pi\|_{L^{1,p}(Q(1))} \leq \varepsilon.$$  

- Wang and Zhang [25]
  $$\|u\|_{L^{2,\infty}(Q(1))} + \|u\|_{L^{4,2}(Q(1))} + \|\Pi\|_{L^{2,1}(Q(1))} \leq \varepsilon.$$  

- Choi and Vasseur [4], Guevara and Phuc [7]
  $$\|u\|_{L^{2,\infty}(Q(1))} + \|\nabla u\|_{L^2(Q(1))} + \|\Pi\|_{L^1(Q(1))} \leq \varepsilon.$$  

- Guevara and Phuc [7]
  $$\|u\|_{L^{2p,2q}(Q(1))} + \|\Pi\|_{L^{p,q}(Q(1))} \leq \varepsilon, \quad 3/p + 2/q = 7/2 \quad \text{with } 1 \leq q \leq 2.$$  

A special case of $q = 2, p = 6/5$ can be found in [15] by Phuc.

- Gustafson, Kang and Tsai [8]
  $$\limsup_{r \to 0} r^{1-2/p - \frac{2}{q}} \|u\|_{L^{p,q}(Q(r))} \leq \varepsilon, \quad 1 \leq 2/q + 3/p \leq 2, \quad 1 \leq p, q \leq \infty.$$  

See the result [23] by Tian and Xin for the case $p = q = 3$.

Note that the pressure $\Pi$ satisfies $\Delta \Pi = -\text{div}\text{div}(u \otimes u)$, therefore, by the classical elliptic estimate, it seems that $\Pi \sim |u|^2$. Based on this, in comparison with (1.8), it seems that the range $p, q$ in (1.7) is not optimal. Thence, the objective of this paper is to extend this range and relax the integral condition of pressure in (1.7). Our first result for the suitable weak solutions of the Navier-Stokes equations is stated as follows:
**Theorem 1.1.** Let the pair \((u, \Pi)\) be a suitable weak solution to the 3D Navier-Stokes system \((1.1)\) in \(Q(1)\). There exists an absolute positive constant \(\varepsilon\) such that if the pair \((u, \Pi)\) satisfies
\[
\|u\|_{L^{p,q}(Q(1))} + \|\Pi\|_{L^1(Q(1))} < \varepsilon, \quad 1 \leq 2/q + 3/p < 2, 1 \leq p, q \leq \infty,
\] (1.9)
then, \(u \in L^{\infty}(Q(1/2))\).

**Remark 1.1.** Theorem 1.1 is a generalization of \((1.2)-(1.7)\). The pressure \(\Pi\) in terms of \(\nabla \Pi\) in equations \((1.1)\) allows us to replace \(\Pi\) by \(\Pi + 1\) in \((1.9)\) as well as \((1.2)-(1.7)\). At present we are not able to prove \((1.9)\) for \(2/p + 3/q = 2\) and this is still an open problem.

**Remark 1.2.** The method for Theorem 1.1 presented here can be applicable to suitable weak solutions of the incompressible magnetohydrodynamic equations \([3, 9]\). Here, we omit the detail here.

It should be pointed out that the criterion \((1.3)\) can be applied to the investigation of the Navier-Stokes equations (see eg. \([1, 8, 10, 12, 16, 17, 27]\)). As a consequence of Theorem 1.1, an analogue of \((1.3)\) is the following corollary

**Corollary 1.2.** Assume that the pair \((u, \Pi)\) be a suitable weak solution to the 3D Navier-Stokes system \((1.1)\) in \(Q(1)\). For each \(\delta > 0\), there exists an absolute positive constant \(\varepsilon\) such that \(u \in L^{\infty}(Q(1/2))\) provided
\[
\|u\|_{L^{5/2+2\delta}(Q(1))} + \|\Pi\|_{L^{5/4+\delta}(Q(1))} < \varepsilon. \tag{1.10}
\]

We give some comments on the proof of Theorem 1.1. Inspired by the argument in \([7]\), the idea to prove Theorem 1.1 is to establish an effective iteration scheme via local energy inequality \((2.1)\). In contrast with the work of \([7]\), we will not utilize the Bernoulli (total) pressure \(\Pi + \frac{1}{2}|u|^2\), which plays a critical role in the proof of \((1.7)\). There are two crucial ingredients in the proof of Theorem 1.1. First, roughly, the energy flux in the localized energy inequality is \(\|u\|^3_{L^3((Q(1/2))}\). We note that the following fact is valid
\[
\|u\|^3_{L^3(Q(1/2))} \leq C^2 3^{(\alpha - 1)/2} \|u\|^3_{L^{p,q}(Q(1/2))} \left(\|u\|^2_{L^2(Q(1/2))} + \|\nabla u\|^2_{L^2(Q(1/2))}\right)^{(3-\alpha)/2}, \tag{1.11}
\]
where \(\alpha > 1\) defined in \((2.7)\) allows us to apply the iteration Lemma 2.3. We refer the reader to Lemma 2.2 for its detail and to \([22, 25]\) for slightly different versions. The second ingredient is to exploit the energy hidden in the pressure to bound the term \(\|u\Pi\|_{L^1(Q(1/2))}\). To this end, by choosing appropriate test function in local energy inequality, we utilize the decomposition of pressure to split the term \(\|u\Pi\|_{L^1(Q(1/2))}\) into three parts: \(\Pi_1\) is in terms of \(u\) bounded by the Calderón-Zygmund theorem; \(\Pi_2\) involving \(u\) is a harmonic function, \(\Pi_3\) depending on \(\Pi\) is also harmonic function, which are controlled separately (see Lemma 2.1). In summary, there holds
\[
\|u\Pi\|_{L^1(Q(1/2))} \leq \|\Pi_1 u\|_{L^1(Q(1/2))} + \|\Pi_2 u\|_{L^1(Q(1/2))} + \|\Pi_3 u\|_{L^1(Q(1/2))}
\leq C\|u\|^3_{L^3(Q(1))} + C\|\Pi\|_{L^1(Q(1))}\|u\|_{L^{2,\infty}(Q(1))}.
\]

Finally, the local energy bounds is derived
\[
\|u\|^2_{L^{2,\infty}(Q(1/2))} + \|\nabla u\|^2_{L^2(Q(1/2))} \leq C\|u\|^2_{L^{p,q}(Q(1))} + C\|u\|^{2\alpha/(\alpha - 1)}_{L^{p,q}(Q(1))} + C\|\Pi\|^2_{L^1(Q(1))}. \tag{1.12}
\]
The proof of Theorem 1.1 is an immediate consequence of the above inequality and (1.6).

As an application, we will apply Corollary 1.2 to refine the upper box-counting dimension of the possible interior singular set of suitable weak solutions to the 3D Navier-Stokes equations. Before stating our results, we sketch the known results. There are several works [10–12, 17, 27, 28] trying to show that the upper box dimension of the singular set of suitable weak solutions of the 3D Navier-Stokes system is at most 1 since the Hausdorff dimension of a set is less than its upper box dimension (see eg. [5]). By the backward heat kernel, in two works [11, 12], Kukavica and his co-author Pei proved that this dimension is less than or equal to $\frac{135}{82}(\approx 1.646)$ and $\frac{45}{29}(\approx 1.552)$, respectively. This improved Robinson and Sadowski's [17] result $\frac{5}{3}(\approx 1.667)$. Very recently, a new and efficient iteration approach to calculate the box-dimension is introduced by Koh and Yang in [10], where they proved that the fractal upper box dimension of $S$ is bounded by $\frac{95}{63}(\approx 1.508)$. Shortly afterwards, inspired by Koh and Yang's work, it is shown that this dimension is at most $\frac{360}{277}(\approx 1.300)$ in [27]. It should be noted that a same tool (1.3) was employed in [10, 12, 17, 27]. Very recently, the authors in [28] lower this dimension to $\frac{975}{758}(\approx 1.286)$ via Guevara and Phuc's criterion (1.7) for $p = q = 10/7$. Here, our result reads below

**Theorem 1.3.** The upper box dimension of $S$ in (1.1) is at most $\frac{2400}{1903}(\approx 1.261)$.

**Remark 1.3.** This theorem is an improvement of the known box dimension of $S$ in [10–12, 17, 27, 28].

**Remark 1.4.** From [28] and Theorem 1.3 it seems that the new $\varepsilon$-regularity criteria may yield an improvement of the known upper box dimension of the singular set in the Navier-Stokes equations.

**Remark 1.5.** Although the proof of Theorem 1.3 is close to that of [10, 27, 28], we outline its proof since almost different scaling invariant quantity and associated decay estimate are established here. To make the paper more readable, we will apply (1.10) for $\delta = 0$ to obtain Theorem 1.3.

By Vitali cover lemma and contradiction arguments as in [12, 27], Theorem 1.3 turns out to be a consequence of the following proposition.

**Proposition 1.4.** Suppose that the pair $(u, \Pi)$ is a suitable weak solution to (1.7). Then, for any $\gamma < \frac{2315}{5709}$, $(x, t)$ is a regular point provided there exists a sufficiently small universal positive constant $\varepsilon_1$ and $0 < r < 1$ such that

$$\int \int_{Q(r)} |\nabla u|^2 + |u|^{10/3} + |\Pi - \Pi_{B(r)}|^{5/3} + |\nabla \Pi|^{5/4} |dxds| \leq r^{5/3 - \gamma_1}. \tag{1.13}$$

**Remark 1.6.** Proposition 1.4 is an improvement of corresponding results obtained in [12, 27, 28].

**Remark 1.7.** Compared with the regularity criteria (1.2)-(1.10), the regularity condition (1.13) is not scale invariant. Note that $r^{-1} \int \int_{Q(r)} |\nabla u|^2 |dxds|, r^{-5/3} \int \int_{Q(r)} (|u|^{10/3} + |\Pi|^{5/3}) |dxds|$ and $r^{-5/4} \int \int_{Q(r)} |\nabla \Pi| |dxds|$ are dimensionless quantities, hence, it seems that $(5/3 - \gamma)(2400/1903)$ in (1.13) can be seen as an interpolation between 1, 5/4 and 5/3.

**Remark 1.8.** It is worth mentioning that the criterion similar to (1.3) plays an important role not only in the upper box-counting dimension mentioned above but also in the investigation of results concerning the classical Caffarelli-Kohn-Nirenberg theorem by a logarithmic factor in [2, 3, 16, 28]. Indeed, in the spirit of [28], by Corollary 1.2, one can also improve the previous corresponding results in [2, 3, 16, 28]. We leave this for the interested readers.
The remainder of this paper is divided into four sections. In Section 2, we first recall the definitions of the upper box-counting dimension and suitable weak solutions to the Navier-Stokes equations. Then, we present the decomposition of pressure and establish some crucial bounds for the scaling invariant quantities. The third section is devoted to the proof of Theorem 1.1. Section 4 is concerned with the box-counting dimension of the possible singular set of suitable weak solutions.

Notations: Throughout this paper, the classical Sobolev norm \( \| \cdot \|_{H^s} \) is defined as
\[
\|f\|_{H^s}^2 = \int_{\mathbb{R}^n} (1 + |\xi|)^{2s} |\hat{f}(\xi)|^2 d\xi, \quad s \in \mathbb{R}.
\]
We denote by \( \dot{H}^s \) homogeneous Sobolev spaces with the norm \( \|f\|_{\dot{H}^s}^2 = \int_{\mathbb{R}^n} |\xi|^{2s} |\hat{f}(\xi)|^2 d\xi \). Denote the average of \( f \) on the set \( \Omega \) by \( \bar{f}_\Omega \). For convenience, \( f_r \) represents \( f_{B(r)} \). \( |\Omega| \) represents the Lebesgue measure of the set \( \Omega \). We will use the summation convention on repeated indices. \( C \) is an absolute constant which may be different from line to line unless otherwise stated in this paper.

2 Preliminaries

First, we begin with the definitions of the upper box-counting dimension and suitable weak solutions of Navier-Stokes equations (1.1), respectively.

Definition 2.1. The upper box dimension of a set \( X \) is usually defined as
\[
d_{\text{box}}(X) = \limsup_{\varepsilon \to 0} \frac{\log N(X, \varepsilon)}{\log \varepsilon},
\]
where \( N(X, \varepsilon) \) is the minimum number of balls of radius \( \varepsilon \) required to cover \( X \).

Materials on box dimension and Hausdorff dimension can be found in [5].

Definition 2.2. A pair \((u, \Pi)\) is called a suitable weak solution to the Navier-Stokes equations (1.1) provided the following conditions are satisfied,

(1) \( u \in L^\infty(-T, 0; L^2(\mathbb{R}^3)) \cap L^2(-T, 0; \dot{H}^1(\mathbb{R}^3)), \Pi \in L^{3/2}(-T, 0; L^{3/2}(\mathbb{R}^3)) \);

(2) \((u, \Pi)\) solves (1.1) in \( \mathbb{R}^3 \times (-T, 0) \) in the sense of distributions;

(3) \((u, \Pi)\) satisfies the following inequality, for a.e. \( t \in [-T, 0] \),
\[
\int_{\mathbb{R}^3} |u(x, t)|^2 \phi(x, t) dx + 2 \int_{-T}^{t} \int_{\mathbb{R}^3} |\nabla u|^2 \phi dx ds \\
\leq \int_{-T}^{t} \int_{\mathbb{R}^3} u^2 (\partial_s \phi + \Delta \phi) dx ds + \int_{-T}^{t} \int_{\mathbb{R}^3} u \cdot \nabla \phi(|u|^2 + 2\Pi) dx ds,
\]
where non-negative function \( \phi(x, s) \in C^\infty_0(\mathbb{R}^3 \times (-T, 0)) \).

Now, we present the decomposition of the pressure \( \Pi \), which plays an important roles in the proof of Theorem 1.1.
Lemma 2.1. Denote the standard normalized fundamental solution of Laplace equation by \( \Gamma \) and suppose that \( 0 < r < \rho < \infty \). Let \( \eta \in C_0^\infty(B(\rho)) \) such that \( 0 \leq \eta \leq 1 \) in \( B(\rho) \), \( \eta \equiv 1 \) in \( B(\frac{r+\rho}{2}) \) and \( |\nabla^k \eta| \leq C/(\rho - r)^k \) with \( k = 1, 2 \) in \( B(\rho) \). Then we can decompose pressure \( \Pi \) in (1.1) as follows

\[
\Pi(x) := \Pi_1(x) + \Pi_2(x) + \Pi_3(x), \quad x \in B\left(\frac{r+\rho}{2}\right),
\]

where

\[
\Pi_1(x) = - \partial_i \partial_j \Gamma \ast (\eta(u_j u_i)), \\
\Pi_2(x) = 2\partial_i \Gamma \ast (\partial_j \eta(u_j u_i)) - \Gamma \ast (\partial_i \eta u_j u_i), \\
\Pi_3(x) = 2\partial_i \Gamma \ast (\partial_\eta \eta) - \Gamma \ast (\partial_\eta \eta \Pi).
\]

Moreover, we have the following estimates

\[
\|\Pi_1\|_{L^{3/2}(Q(\frac{r+\rho}{2}))} \leq C\|u\|_{L^3(Q(\rho))}^2; \quad (2.3)
\]

\[
\|\Pi_2\|_{L^{3/2}(Q(\frac{r+\rho}{2}))} \leq \frac{CR_2}{(\rho - r)^3} \|u\|_{L^3(Q(\rho))}^2; \quad (2.4)
\]

\[
\|\Pi_3\|_{L^{2,1}(Q(\frac{r+\rho}{2}))} \leq \frac{CR_3^{3/2}}{(\rho - r)^3} \|\Pi\|_{L^1(Q(\rho))}. \quad (2.5)
\]

Proof. Thanks to \( \partial_i \partial_j \Pi = - \partial_i \partial_j (u_i u_j) \) and Leibniz’s formula, we conclude that

\[
\partial_i \partial_j (\Pi \eta) = - \eta \partial_i \partial_j (u_i u_j) + 2\partial_i \eta \partial_i \Pi + \Pi \partial_i \partial_j \eta.
\]

This enables us to write, for \( x \in B\left(\frac{r+\rho}{2}\right) \),

\[
\Pi(x) = \Gamma \ast (- \eta \partial_i \partial_j (u_i u_j) + 2\partial_i \eta \partial_i \Pi + \Pi \partial_i \partial_j \eta)
\]

\[
= - \partial_i \partial_j \Gamma \ast (\eta(u_j u_i)) + 2\partial_i \Gamma \ast (\partial_j \eta(u_j u_i)) - \Gamma \ast (\partial_i \partial_j \eta u_j u_i)
\]

\[
- 2\partial_i \Gamma \ast (\partial_\eta \eta \Pi) - \Gamma \ast (\partial_\eta \eta \Pi)
\]

\[
:= \Pi_1(x) + \Pi_2(x) + \Pi_3(x),
\]

where we have used integrating by parts.

The classical Calderón-Zygmund theorem ensures that

\[
\|\Pi_1\|_{L^{3/2}(B(\frac{r+\rho}{2}))} \leq C\|u\|_{L^3(B(\rho))}^2.
\]

In view of the property of the cut-off function, there is no singularity in \( p_2 \) and \( p_3 \), in \( B\left(\frac{r+\rho}{2}\right) \). So, a straightforward computation gives

\[
|\Pi_2(x)| \leq \frac{C}{(\rho - r)^3} \int_{B(\rho)} |u(y)|^2 \, dy, \quad x \in B\left(\frac{r+\rho}{2}\right),
\]

and

\[
|\Pi_3(x)| \leq \frac{C}{(\rho - r)^3} \int_{B(\rho)} |\Pi(y)| \, dy, \quad x \in B\left(\frac{r+\rho}{2}\right).
\]

Then applying Hölder’s inequality gives the desired estimates (2.3), (2.4) and (2.5). This completes the proof of the lemma.
Before we turn our attentions to the proof of the inequality (1.11), we have to introduce

$$\alpha = \frac{2}{\frac{3}{p} + \frac{2}{q}} > 1. \quad (2.7)$$

**Lemma 2.2.** Let $1 \leq 2/q + 3/p < 2, 1 \leq p, q \leq \infty$ and $\alpha$ be defined as above. There is an absolute constant $C$ such that

$$\|u\|_{L^2(Q(\rho))}^3 \leq C\rho^{3(\alpha - 1)/2}\|u\|_{L^{p,q}(Q(\rho))}^2 \left(\|u\|_{L^{2,\infty}(Q(\rho))}^2 + \|\nabla u\|_{L^2(Q(\rho))}^2\right)^{(3-\alpha)/2}. \quad (2.8)$$

**Proof.** First, we recall an interpolation inequality. For each $2 \leq l \leq \infty$ and $2 \leq k \leq 6$ satisfying $\frac{2}{l} + \frac{2}{k} = \frac{2}{2}$, according to the Hölder inequality and the Young inequality, we know that

$$\|u\|_{L^k,1(Q(\mu))} \leq C\|u\|^{1-\frac{2}{k}}_{L^2,\infty}(Q(\mu))\|u\|^\frac{2}{k}_{L^6,2(Q(\mu))} \leq C\|u\|^\frac{1}{2}_{L^2,\infty}(Q(\mu))\|u\|^\frac{1}{2}_{L^6,2}(Q(\mu)) \quad (2.9)$$

It is clear that $q/\alpha \geq 1$ and $p/\alpha \geq 1$. Let $(\frac{2}{p})^\star$ and $(\frac{2}{q})^\star$ be the Hölder dual of $\frac{2}{p}$ and $\frac{2}{q}$. An elementary computation gives that $2 \leq 2(\frac{2}{p})^\star \leq 6$ and

$$\frac{3}{2\left(\frac{2}{p}\right)^\star} + \frac{2}{2\left(\frac{2}{q}\right)^\star} = \frac{3}{2}.$$ 

Hence, taking advantage of Hölder’s inequality and (2.9), we have

$$\int\int_{Q(\rho)} |u|^3 dx dt = \int\int_{Q(\rho)} |u|^\alpha |u|^{3-\alpha} dx dt$$

$$\leq \|u\|^\alpha_{L^{p,q}(Q(\rho))} \|u\|^{3-\alpha}_{L^{(3-\alpha)(\frac{2}{p})^\star, (3-\alpha)(\frac{2}{q})^\star}(Q(\rho))}$$

$$\leq C\rho^{3(\alpha - 1)/2}\|u\|_{L^{p,q}(Q(\rho))}^2 \left(\|u\|_{L^{2,\infty}(Q(\rho))}^2 + \|\nabla u\|_{L^2(Q(\rho))}^2\right)^{(3-\alpha)/2}. \quad (2.9)$$

The proof of this lemma is completed. \(\square\)

For the convenience of the reader, we recall the following well-known iteration lemma.

**Lemma 2.3.** [6, Lemma V.3.1, p.161] Let $I(s)$ be a bounded nonnegative function in the interval $[r, R]$. Assume that for every $\sigma, \rho \in [r, R]$ and $\sigma < \rho$ we have

$$I(\sigma) \leq A_1(\rho - \sigma)^{-\alpha_1} + A_2(\rho - \sigma)^{-\alpha_2} + A_3 + \ell I(\rho)$$

for some non-negative constants $A_1, A_2, A_3$, non-negative exponents $\alpha_1 \geq \alpha_2$ and a parameter $\ell \in [0, 1)$. Then there holds

$$I(r) \leq c(\alpha_1, \ell)[A_1(R - r)^{-\alpha_1} + A_2(R - r)^{-\alpha_2} + A_3].$$
Note that if the pair \((u(x,t),\Pi(x,t))\) is a solutions of (1.1), then, for any \(\lambda > 0\), the pair \((\lambda u(\lambda x, \lambda^2 t), \lambda^2 \Pi(\lambda x, \lambda^2 t))\) also solves (1.1). Based on this, as [1], we introduce the following dimensionless quantities:

\[
E(r) = r^{-1}\|u\|^2_{L^\infty L^2(Q(r))}, \quad E_*(r) = r^{-1}\|\nabla u\|^2_{L^2(Q(r))},
\]

\[
E_p(r) = r^{p-5}\|u\|^p_{L^p(Q(r))}, \quad P_{5/4}(r) = r^{-5/4}\|\nabla \Pi\|^5_{L^{5/4}(Q(r))},
\]

\[
P_{5/3}(r) = r^{-5/3}\|\Pi - \Pi B\|^{5/3}_{L^{5/3}(Q(r))}.
\]

In the spirit of [28], we derive some decay estimates involving the scaling invariant quantity, which is helpful in the proof of Proposition 1.4.

**Lemma 2.4.** For \(0 < r \leq \frac{1}{2} \rho\) and \(8/3 \leq b \leq 6\), there is an absolute constant \(C\) independent of \(r\) and \(\rho\), such that

\[
E_{5/2}(r) \leq C \left( \frac{\rho}{r} \right)^{5/4} E^{2b-5}_{4(b-2)}(\rho) E_*^{4(b-2)}(\rho) + C \left( \frac{\rho}{r} \right)^{5/2} E^{5/4}(\rho).
\] (2.10)

**Proof.** Taking advantage of the Hölder inequality and the Poincaré-Sobolev inequality, for any \(5/2 < b \leq 6\), we see that

\[
\int_{B(r)} |u - \bar{u}_{B(\rho)}|^5 dx \leq C \left( \int_{B(r)} |u - \bar{u}_{B(\rho)}|^2 dx \right)^{2b-5} \left( \int_{B(r)} |\nabla u|^b dx \right) \frac{1}{b^{4(b-2)}} \leq C r^{(6-\lambda)/b} \left( \int_{B(\rho)} |u|^2 dx \right)^{2b-5} \left( \int_{B(\rho)} |\nabla u|^2 dx \right) \frac{1}{b^{4(b-2)}}.
\]

By means of the triangle inequality and the last inequality, we know that

\[
\int_{B(r)} |u|^{5/2} dx \leq C \int_{B(r)} |u - \bar{u}_{B(\rho)}|^{5/2} dx + C \int_{B(r)} |\bar{u}_{B(\rho)}|^{5/2} dx \leq C r^{(6-\lambda)/b} \left( \int_{B(\rho)} |u|^2 dx \right)^{2b-5} \left( \int_{B(\rho)} |\nabla u|^2 dx \right) \frac{1}{b^{4(b-2)}} + \frac{r^3 C}{\rho^{1/4}} \left( \int_{B(\rho)} |u|^2 dx \right)^{5/4}.
\]

Integrating with respect to \(s\) from \(t - \mu^2\) to \(t\) and utilizing the Hölder inequality again, for any \(b \geq 8/3\), we get

\[
\int_{Q(r)} |u|^{5/2} dx ds \leq C r^2 \left( \sup_{t-\mu^2 \leq s \leq t} \int_{B(\rho)} |u|^2 dx \right)^{2b-5} \left( \int_{Q(r)} |\nabla u|^2 dx ds \right) \frac{1}{b^{4(b-2)}} + C \frac{r^5}{\rho^{1/4}} \left( \sup_{t-\mu^2 \leq s \leq t} \int_{B(\rho)} |u|^2 dx \right)^{5/4}.
\] (2.11)

Therefore,

\[
E_{5/2}(r) \leq C \left( \frac{\rho}{r} \right)^{5/4} E^{2b-5}_{4(b-2)}(\rho) E_*^{4(b-2)}(\rho) + C \left( \frac{\rho}{r} \right)^{5/2} E^{5/4}(\rho).
\]
In the spirit of [17, Lemma 2.1, p.222], we can apply the interior estimate of harmonic function to establish the following decay estimate of pressure \( \Pi - \Pi_{B(r)} \). Since the pressure \( \Pi \) is in terms of \( \nabla \Pi \) in equations (1.1), as said before, we can employ this lemma in the proof of Theorem 1.3 and Proposition 1.4.

**Lemma 2.5.** For \( 0 < r \leq \frac{1}{8}\rho \), there exists an absolute constant \( C \) independent of \( r \) and \( \rho \) such that

\[
P_{5/4}(r) \leq C \left( \frac{\rho}{r} \right)^{5/2} E_{5/2}(\rho) + C \left( \frac{r}{\rho} \right)^{7/4} P_{5/4}(\rho). \tag{2.12}
\]

**Proof.** Fix a smooth function \( \phi \) supported in \( B(\rho/2) \) and with value 1 on the ball \( B(\frac{3}{8}\rho) \). Moreover, there holds \( 0 \leq \phi \leq 1 \) and \( |\nabla \phi| \leq C\rho^{-1}, \ |\nabla^2 \phi| \leq C\rho^{-2} \).

As in Lemma 2.1, we have

\[
\partial_i \partial_j (\Pi \phi) = -\phi \partial_i \partial_j [u_i u_j] + 2\phi \partial_i \partial_j \Pi + \Pi \partial_i \partial_j \phi.
\]

For any \( x \in B(\frac{3}{8}\rho) \), we deduce from integrations by parts that

\[
\Pi(x) = \Gamma \{ -\phi \partial_i \partial_j [u_i u_j] + 2\phi \partial_i \partial_j \Pi + \Pi \partial_i \partial_j \phi \} \\
= -\partial_i \partial_j \Gamma \{ \phi [u_i u_j] \} \\
+ 2\partial_i \Gamma \{ \partial_j \phi [u_i u_j] \} - \Gamma \{ \partial_i \partial_j \phi [u_i u_j] \} \\
+ 2\partial_i \Gamma \{ \partial_j \phi \Pi \} - \Gamma \{ \partial_i \partial_j \phi \Pi \} \\
=: \Pi_1(x) + \Pi_2(x) + \Pi_3(x), \tag{2.13}
\]

Thanks to \( \phi(x) = 1 \ (x \in B(\rho/4)) \), we discover that

\[
\Delta(\Pi_2(x) + \Pi_3(x)) = 0.
\]

By virtue of the interior estimate of harmonic function and the Hölder inequality, we know that, for every \( x_0 \in B(\rho/8) \),

\[
|\nabla(\Pi_2 + \Pi_3)(x_0)| \leq \frac{C}{\rho^4} \|(\Pi_2 + \Pi_3)\|_{L^1(B_{x_0}(\rho/8))} \\
\leq \frac{C}{\rho^4} \|(\Pi_2 + \Pi_3)\|_{L^1(B(\rho/4))} \\
\leq \frac{C}{\rho^{17/5}} \|(\Pi_2 + \Pi_3)\|_{L^{5/4}(B(\rho/4))}. 
\]

Consequently,

\[
\|\nabla(\Pi_2 + \Pi_3)\|_{L^\infty(B(\rho/8))}^{5/4} \leq C\rho^{-17/4} \|(\Pi_2 + \Pi_3)\|_{L^{5/4}(B(\rho/4))}^{5/4} \leq C \|\nabla(\Pi_2 + \Pi_3)\|_{L^\infty(B(\rho/8))}^{10/7} \|(\Pi_2 + \Pi_3)\|_{L^{5/4}(B(\rho/4))}^{10/7} 
\]

This together with the mean value theorem gives that, for each \( r \leq \frac{1}{8}\rho \),

\[
\|((\Pi_2 + \Pi_3) - (\Pi_2 + \Pi_3)_{B(r)})\|_{L^{5/4}(B(r))}^{5/4} \leq C r^3 \|((\Pi_2 + \Pi_3) - (\Pi_2 + \Pi_3)_{B(r)})\|_{L^{5/4}(B(r))}^{5/4} \\
\leq C r^{17/4} \|\nabla((\Pi_2 + \Pi_3))\|_{L^\infty(B(\rho/8))}^{10/7} \|(\Pi_2 + \Pi_3)\|_{L^{5/4}(B(\rho/4))}^{10/7} \\
\leq C \left( \frac{r}{\rho} \right)^{17/4} \|(\Pi_2 + \Pi_3)\|_{L^{5/4}(B(\rho/4))}^{5/4}. 
\]
Sine \((\Pi_2 + \Pi_3) - (\Pi_2 + \Pi_3)_{B(\rho/4)}\) is also a harmonic function on \(B(\rho/4)\), we see that
\[
\|(\Pi_2 + \Pi_3) - (\Pi_2 + \Pi_3)_{B(\rho/4)}\|_{L^{5/4}(B(\rho/4))}^{5/4} 
\leq C \left( \frac{r}{\rho} \right)^{17/4} \|(\Pi_2 + \Pi_3) - (\Pi_2 + \Pi_3)_{B(\rho/4)}\|_{L^{5/4}(B(\rho/4))}^{5/4}.
\]
The triangle inequality implies that
\[
\|(\Pi_2 + \Pi_3) - (\Pi_2 + \Pi_3)_{B(\rho/4)}\|_{L^{5/4}(B(\rho/4))}^{5/4} 
\leq \|\Pi - \Pi_{B(\rho/4)}\|_{L^{5/4}(B(\rho/4))}^{5/4} + \|\Pi_1 - \Pi_{B(\rho/4)}\|_{L^{5/4}(B(\rho/4))}^{5/4} 
\leq C \|\Pi - \Pi_{B(\rho)}\|_{L^{5/4}(B(\rho/4))}^{5/4} + C \|\Pi_1\|_{L^{5/4}(B(\rho/4))},
\]
which means that
\[
\|(\Pi_2 + \Pi_3) - (\Pi_2 + \Pi_3)_{B(\rho/4)}\|_{L^{5/4}(B(\rho/4))}^{5/4} 
\leq C \left( \frac{r}{\rho} \right)^{17/4} \left( \|\Pi - \Pi_{B(\rho/4)}\|_{L^{5/4}(B(\rho/4))}^{5/4} + \|\Pi_1\|_{L^{5/4}(B(\rho/4))}^{5/4} \right). \tag{2.14}
\]
In view of classical Calderón-Zygmund theorem, we thus infer that
\[
\int_{B(\rho/4)} |\Pi_1(x)|^{5/4} dx \leq C \int_{B(\rho/2)} |u|^{5/2} dx. \tag{2.15}
\]
This also yields, for any \(r \leq \frac{1}{8}\rho\),
\[
\int_{B(r)} |\Pi_1(x)|^{5/4} dx \leq C \int_{B(\rho/2)} |u|^{5/2} dx. \tag{2.16}
\]
Integrating in time on \((t - r^2, t)\) and using the triangle inequality, we conclude using (2.14)-(2.16) that
\[
\int\int_{Q(r)} |\Pi - \Pi_{B(r)}|^{5/4} dx\,ds 
\leq C \left( \frac{r}{\rho} \right)^{17/4} \left( \|\Pi - \Pi_{B(\rho/4)}\|_{L^{5/4}(B(\rho/4))}^{5/4} + \|\Pi_1\|_{L^{5/4}(B(\rho/4))}^{10/7} \right)
\]
which means (2.12) \(\square\). The proof of this lemma is completed.

3 Proof of Theorem 1.1

The goal of this section is to prove Theorem 1.1. As said before, it is enough to show (1.12). We state precise proposition involving local energy energy bound (1.12) below.
Proposition 3.1. Let $\alpha$ be defined as in (2.7). Suppose that $(u, \Pi)$ is a suitable weak solution to the Navier-Stokes equations in $Q(R)$. Then there holds, for any $R > 0$

$$
\|u\|_{L^2(Q(R/2))}^2 + \|\nabla u\|_{L^2(Q(R/2))}^2 \leq CR^{(3\alpha-4)/\alpha}\|u\|_{L^{p,q}(Q(R))}^2
+ CR^{(3\alpha-5)/(\alpha-1)}\|u\|_{L^{p,q}(Q(R))}^2 + CR^{-6}\|\Pi\|_{L^1(Q(R))}^3.
$$

(3.1)

Proof. Consider $0 < R/2 \leq r < \frac{3\rho+\mu}{4} < \frac{\rho}{2} \leq \rho \leq R$. Let $\phi(x, t)$ be non-negative smooth function supported in $Q(\frac{r+\rho}{2})$ such that $\phi(x, t) \equiv 1$ on $Q(\frac{3\rho+\mu}{4})$, $|\nabla \phi| \leq C/(\rho - r)$ and $|\nabla^2 \phi| + |\partial_t \phi| \leq C/(\rho - r)^2$.

By means of Hölder’s inequality, we arrive at

$$
\int_{-T}^{t} \int_{\mathbb{R}^3} |u|^2 (\partial_t \phi + \Delta \phi) dx ds \leq \frac{C}{(\rho - r)^2} \int_{Q(\frac{r+\rho}{2})} |u|^2 dx ds
\leq \frac{C\rho^{5/3}}{(\rho - r)^2} \left( \int_{Q(\rho)} |u|^3 dx ds \right)^{2/3}
=: L_1.
$$

(3.2)

Thanks to the local energy inequality (2.1) and the decompose of pressure in Lemma 2.1, we know that

$$
\int_{B(\frac{r+\rho}{2})} |u(x, t)|^2 \phi(x, t) dx + 2 \int_{Q(\frac{r+\rho}{2})} |\nabla u|^2 \phi dx ds \leq L_1 + L_2 + L_3 + L_4 + L_5,
$$

(3.3)

where

$$
L_2 = \frac{C}{(\rho - r)} \int_{Q(\frac{r+\rho}{2})} |u|^3 dx ds;
L_3 = \frac{C}{(\rho - r)} \int_{Q(\frac{r+\rho}{2})} u\Pi_1 dx ds;
L_4 = \frac{C}{(\rho - r)} \int_{Q(\frac{r+\rho}{2})} u\Pi_2 dx ds;
L_5 = \frac{C}{(\rho - r)} \int_{Q(\frac{r+\rho}{2})} u\Pi_3 dx ds.
$$}

By Hölder inequality and (2.3)-(2.5), we find that

$$
L_3 \leq \frac{C}{(\rho - r)} \|\Pi_1\|_{L^{3/2}(Q(\frac{r+\rho}{2}))} \|u\|_{L^3(Q(\frac{r+\rho}{2}))} \leq \frac{C}{(\rho - r)} \|u\|_{L^3(Q(\rho))}^3;
$$

(3.4)

$$
L_4 \leq \frac{C}{(\rho - r)} \|\Pi_2\|_{L^{3/2}(Q(\frac{r+\rho}{2}))} \|u\|_{L^3(Q(\frac{r+\rho}{2}))} \leq \frac{C\rho^3}{(\rho - r)^2} \|u\|_{L^3(Q(\rho))}^3;
$$

(3.5)

$$
L_5 \leq \frac{C}{(\rho - r)} \|\Pi_3\|_{L^{2,1}(Q(\frac{r+\rho}{2}))} \|u\|_{L^{2,\infty}(Q(\frac{r+\rho}{2}))} \leq \frac{C\rho^{3/2}}{(\rho - r)^2} \|\Pi\|_{L^1(Q(\rho))} \|u\|_{L^{2,\infty}(Q(\rho))}.
$$

(3.6)

From (3.3)-(3.6), we see that it is enough to bound $\|u\|_{L^3(Q(\rho))}^3$. To this end, plugging (2.8) into (3.2), (3.4) and (3.5) respectively, by the Young inequality, we conclude that

$$
L_1 \leq \frac{C\rho^{3+2/\alpha}}{(\rho - r)^6} \|u\|_{L^{p,q}(Q(\rho))}^2 + \frac{1}{5} \left( \|u\|_{L^{2,\infty}(Q(\rho))}^2 + \|\nabla u\|_{L^2(Q(\rho))}^2 \right),
$$
In addition, utilizing the Young inequality again, we get

\[ L_2 + L_3 \leq \frac{C\rho^3}{(\rho - r)^{2/(\alpha - 1)}} \|u\|_{L^{p,q}(Q(\rho))}^{2\alpha/(\alpha - 1)} + \frac{1}{5} \left( \|u\|_{L^{2,\infty}(Q(\rho))}^2 + \|\nabla u\|_{L^2(Q(\rho))}^2 \right), \]

\[ L_4 \leq \frac{C\rho^{3/(\alpha - 1)/(\alpha - 1)}}{(\rho - r)^{8/(\alpha - 1)}} \|u\|_{L^{p,q}(Q(\rho))}^{2\alpha/(\alpha - 1)} + \frac{1}{5} \left( \|u\|_{L^{2,\infty}(Q(\rho))}^2 + \|\nabla u\|_{L^2(Q(\rho))}^2 \right). \]

Collecting all the above estimates, we know that

\[ L_5 \leq \frac{C\rho^3}{(\rho - r)^8} \|\Pi\|_{L^1(Q(\rho))}^2 + \frac{1}{5} \|u\|_{L^{2,\infty}(Q(\rho))}^2. \]

The iteration Lemma 2.3 allows us to derive (3.1) from the last inequality.

\[ \boxed{} \]

4 Proof of Proposition 1.3

The main part of this section is the proof of Proposition 1.3.

Proof of Proposition 1.3. Along the lines of [10, 27, 28], under the hypotheses of (1.13), we select \( 2\rho < 1 \) such that \( \rho^2 < 1/2 \), where parameter \( \beta \) is to be determined later and

\[ \int\int_{Q(2\rho)} |\nabla u|^2 + |u|^{10/3} + \|\Pi - \Pi_{B(2\rho)}\|^{5/3} + |\nabla \Pi|^{5/4} \, dx \, ds \leq (2\rho)^{5/3 - \gamma} e. \] (4.1)

We will make use of the following result

\[ E(p) \leq C_1^{3/5} \rho^{3/5} - \rho^{3/\gamma}, \quad (\gamma \leq 5/12), \] (4.2)

which is shown in [27]. Here we omit its details, the reader is referred to [27, Theorem 1.2, p.1768-1769] for a proof. Second, iterating (2.12) in Lemma 2.5, we see that

\[ P_{5/4}(\theta^{N} r) \leq C \sum_{k=1}^{N} \theta^{-\frac{7}{4} + \frac{7(k-1)}{4}} E_{5/2}(\theta^{N-k} r) + C \theta^{7N/4} P_{5/4}(r). \] (4.3)

In view of the Poincaré-Sobolev inequality and the Hölder inequality, we know that

\[ \|\Pi - \Pi_{B(r)}\|_{L^{5/4}(Q(r))} \leq \|\Pi - \Pi_{B(r)}\|_{L^{5/4,15/7}(Q(r))}^{3/8} \|\Pi - \Pi_{B(r)}\|_{L^{5/4,1}(Q(r))}^{5/8} \]

\[ \leq C r \|\nabla \Pi\|_{L^{5/4}(Q(r))}^{3/8} \|\Pi - \Pi_{B(r)}\|_{L^{5/3}(Q(r))}^{5/8} \] (4.4)

which in turn implies that

\[ P_{5/4}(r) \leq C P_{5/4}(r) P_{5/3}^{15/32}(r). \]
Inserting this inequality into (4.3), we have

$$P_{5/4}(\theta^N r) \leq C \sum_{k=1}^{N} \theta^{-\frac{5}{2} + \frac{7(k-1)}{4}} E_{5/2}(\theta^{N-k} r) + C \theta^{7N/4} P_{5/4}^{3/8}(r) P_{5/3}^{15/32}(r). \quad (4.5)$$

Before going further, we introduce some notations $r = \rho^\alpha = \theta^N r$, $\theta = \rho^\beta$, $r_i = r^{-i} r = \rho^{\alpha-i\beta} (1 \leq i \leq N)$, where $\alpha$ and $\beta$ are determined by $\gamma$. Their precise selection will be given in the end. As a consequence, by $E_{5/2}(u, r) \leq C \theta^{-\frac{5}{2}} E_{5/2}(u, \theta^{-1} r)$ and (4.5), we infer that

$$P_{5/4}(r) + E_{5/2}(r) \leq C \sum_{k=1}^{N} \theta^{-\frac{5}{2} + \frac{7(k-1)}{4}} E_{5/2}(r_k) + C \theta^{7N/4} P_{5/4}^{3/8}(r) P_{5/3}^{15/32}(r_N) \quad (4.6)$$

$$:= I + II.$$

In the light of (1.10), it suffices to prove that there exists a constant $\sigma > 0$ such that $P_{5/4}(\sigma) + E_{5/2}(\sigma) < \varepsilon_0$. For this purpose, we employ (2.10) with $b = 8/3$ in Lemma 2.4 (4.2) and (4.1) to conclude

$$E_{5/2}(r_k) \leq C \left( \frac{\theta}{r_k} \right)^{\frac{5}{2}} E_{1/4}(\rho) E_{\ast}(\rho) + C \left( \frac{r_k}{\rho} \right)^{5/2} E_{5/4}(\rho)$$

$$\leq C \varepsilon_1^{3/4} \left( \rho^{-\frac{23}{12} + \frac{2}{3}(\alpha-k\beta) - \frac{23\gamma}{20} + \frac{23}{12} + \rho^{-\frac{17\beta}{4} + \frac{5\gamma}{4} - \frac{5}{2} - \frac{3\alpha}{4} - \frac{3\beta}{4} - \frac{3\gamma}{8}} \right).$$

Inserting this inequality into $I$, we find that

$$I \leq C \varepsilon_1^{6/7} \left( \rho^{-\frac{17\beta}{4} + \frac{5\gamma}{4} - \frac{5}{2} - \frac{3\alpha}{4} - \frac{3\beta}{4} - \frac{3\gamma}{8}} \right).$$

To minimise the right-hand side of this inequality, we choose

$$\alpha = \frac{4}{15} (3\beta + \frac{53}{12} - \frac{2\gamma}{5} + \frac{3N\beta}{4}). \quad (4.7)$$

Hece, for sufficiently large $N$, there holds

$$I \leq C \varepsilon_1^{6/7} \left( \rho^{-\frac{5\beta}{4} - \frac{5\gamma}{4} - \frac{3\alpha}{4} - \frac{3\beta}{4} - \frac{3\gamma}{8}} \right) \leq C \varepsilon_1^{6/7} \rho^{-\frac{5\beta}{4} - \frac{5\gamma}{8} + \frac{N\beta}{4}}. \quad (4.8)$$

To bound $II$, assume for a while there holds $r_N \leq \rho$, that is

$$\rho^\alpha N^\beta \leq \rho. \quad (4.9)$$

Using the bounds (4.1) and (4.7), we have the estimate

$$II \leq C \rho^{\frac{7N\beta}{4} + \frac{3}{10} \rho^{-\frac{3}{10}} \left( \int_{Q(r_N)} |\nabla \Pi|^{5/4} dxds \right)^{3/8} \left( \int_{Q(r_N)} |\nabla \Pi|^{5/4} dxds \right)^{15/32} \leq C \rho^{\frac{7N\beta}{4} + \frac{3}{10} \rho^{-\frac{3}{10}} \left( \int_{Q(2\rho)} |\nabla \Pi|^{5/4} dxds \right)^{3/8} \left( \int_{Q(2\rho)} |\nabla \Pi|^{5/4} dxds \right)^{15/32}$$

$$\leq C \rho^{\frac{11N\beta}{4} - \frac{19}{288} - \frac{3\alpha}{4} - \frac{3\beta}{4} - \frac{3\gamma}{8}} \varepsilon_1^{27/32}. \quad (4.10)$$
To guarantee that \( I + II \leq C \varepsilon^{3/4} \leq \varepsilon_0 \), we need \(-9\beta + 4 - \frac{61\gamma}{10} - \frac{N\beta}{4} \geq 0\) and \(\frac{11N\beta}{4} - \frac{19}{288} - \frac{341\gamma}{480} - \beta \geq 0\). In addition, we have derived (4.10) assuming that (4.9), so, we also need \(\alpha - N\beta - 1 \geq 0\). Now, we collect all the restrictions of \(\gamma\) below

\[
\gamma \leq \min \left\{ \frac{5(16 - 9N\beta - 81\beta)}{183}, \frac{5(2 - 9N\beta + 9\beta)}{6}, \frac{5(792N\beta - 19 - 288\beta)}{1023}, \frac{5}{12} \right\}. \tag{4.11}
\]

Maximising this bound on \(\gamma\) with respect to \(N\beta\), we discover that \(N\beta = 245/1903\). Furthermore, We deduce using (4.11) that

\[
\beta = \frac{245}{1903N} \leq \frac{183}{405} \left( \frac{2315}{5709} - \gamma \right).
\]

Hence, choosing \(\beta\) sufficiently small by selecting \(N\) sufficiently large, we can have any \(\gamma < 2315/5709\). Then, we pick \(\alpha = \frac{4}{15}(3\beta - \frac{2\gamma}{9} + \frac{25766}{5709})\). We derive from (4.6), (4.8) and (4.10) that

\[
P_{10/7}(\sigma) + E_{20/7}(\sigma) \leq C \varepsilon^{3/4} \leq \varepsilon_0,
\]

with \(\sigma = \rho^\alpha\). By (1.10), finally, we see that \(u \in L^\infty(Q(\sigma/2))\). This completes the proof of Proposition 1.3. \(\square\)

### Acknowledgement

The research of Wang was partially supported by the National Natural Science Foundation of China under grant No. 11601492. The research of Zhou is supported in part by the National Natural Science Foundation of China under grant No. 11401176 and Doctor Fund of Henan Polytechnic University (No. B2012-110).

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