A NOTE ON THE UNIQUENESS FROM SETS OF POSITIVE MEASURE FOR TIME DEPENDENT PARABOLIC OPERATORS

NICOLAS BURQ AND CLAUDE ZUILY

Abstract. The purpose of this short note is to show how it is possible to combine existing results in the literature to get the unique continuation from sets of positive measure for time dependent parabolic equations with Lipschitz principal part and bounded lower order terms, result which was known in the case of analytic coefficients in [3].

1. Introduction

The problem of unique continuation for solutions to parabolic equations has received a lot of attention starting from the pioneering work of T. Carleman in 1939. Since that time a huge number of papers have been devoted to this question. The interested reader may consult the review paper [9] and its bibliography. Most of these works deal with the question of weak continuation from open sets [8], strong continuation from a point [4], quantitative estimates etc..

Recently, motivated by control theory, the question of unique continuation from sets of positive measure has been discussed and several new results for elliptic equations (see [6]), and for parabolic equations with analytic coefficients have been proved (see [3]).

The purpose of this note is to handle the case of time dependent equations with Lipschitz coefficients and to show that known results in the literature can be recombined to give a positive answer to this question.

1.1. Notations. For $T > 0, R > 0$ let $I = (-T, T)$ and $B_R = \{ x \in \mathbb{R}^d : |x| < R \}$. We set $Q = I \times B_R$.

We shall consider the parabolic operator,

\begin{equation}
P = \partial_t - \text{div}(A(t, x) \nabla x) + \sum_{j=1}^{d} b_j(t, x) \partial_x j + c(t, x),
\end{equation}

where $A(t, x) = (a_{jk}(t, x))_{1 \leq j, k \leq d}$ is a symmetric matrix with real valued entries satisfying the ellipticity condition,

\begin{equation}
\exists \kappa > 0 : \text{Re} \langle A(t, x) \zeta, \zeta \rangle \geq \kappa |\zeta|^2, \quad \forall (t, x) \in Q, \quad \forall \zeta \in \mathbb{C}^d,
\end{equation}

the $(a_{jk})$ are uniformly Lipschitz continuous in $\overline{I} \times \overline{B_R}$ and $b_j, c$ belong to $L^\infty(\overline{I} \times \overline{B_R})$.

Let $(t_0, x_0) \in (-T, T) \times B_R$. We shall set, for small $r > 0$,

\[ I_r(t_0) = \{ t \in \mathbb{R} : |t - t_0| < r^2 \}, \quad B_r(x_0) = \{ x \in \mathbb{R}^d : |x - x_0| < r \}, \]

\[ Q_r(t_0, x_0) = I_r(t_0) \times B_r(x_0). \]

Eventually, we shall denote by $\mu$ the Lebesgue measure on $\mathbb{R}^d$. 

2010 Mathematics Subject Classification. 35KXX, 35B60.
1.2. Main result. The purpose of this note is to prove the following result.

**Theorem 1.1.** Let \( u \in L^2((-T, T), H^2(B_R)) \) satisfying \( Pu = 0 \) in \( Q \). Assume that there exists \( E \subset B_R \) with \( \mu(E) > 0 \) such that \( u(t, x) = 0 \) for all \( (t, x) \in (-T, T) \times E \). Then \( u = 0 \) on \( Q \).

1.3. Preliminaries. This result will be a consequence of several results that we recall now. The first one is the well known Cacciopoli inequality.

**Proposition 1.2.** Let \( u \in L^2((-T, T), H^2(B_R)) \) satisfying \( Pu = 0 \) in \( Q \). One can find a constant \( C > 0 \) depending only on \( d \) and the ellipticity constant \( \kappa \) such that for every \( r > 0 \) such that \( Q_{2r}(x_0, t_0) \subset Q \) we have,

\[
\iint_{Q_r(t_0, x_0)} |\nabla_x u(t, x)|^2 \, dx \, dt \leq \frac{C}{r^2} \iint_{Q_{2r}(t_0, x_0)} |u(t, x)|^2 \, dx \, dt.
\]

**Proof.** Without loss of generality one may assume that \((t_0, x_0) = (0, 0)\). Let \( \theta_1 \in C_0^\infty(\mathbb{R}), \theta_1(t) = 1 \) for \( |t| \leq 1, \supp \theta_1 \subset \{|t| \leq 2\} \) and \( \theta_2 \in C_0^\infty(\mathbb{R}^d), \theta_2(x) = 1 \) for \( |x| \leq 1, \supp \theta_2 \subset \{|x| \leq 2\} \), \( 0 \leq \theta_j \leq 1 \). We set for small \( r \), \( \chi(t, x) = \theta_1(\frac{t}{r}), \theta_2(\frac{x}{r}) \). Then,

\[
|\partial_\chi(X)| + |\nabla_x \chi(X)|^2 \leq \frac{C}{r^2}, \quad X = (t, x).
\]

Denoting by \((\cdot, \cdot)\) the scalar product in \( L^2((-T, T) \times \mathbb{R}^d) \) we have,

\[
((Pu, \chi^2 u)) = (F, \chi^2 u), \quad |F| \leq C(|u| + |\nabla_x u|).
\]

Now and using the fact that \( \chi \) has compact support in \((-T, T) \times B_R\) we have,

\[
\Re((\partial_t u, \chi^2 u)) = \frac{1}{2} \iint \chi^2(X) \frac{d}{dt}|u(X)|^2 \, dX = -\iint \chi(X)(\partial_t \chi)(X)|u(X)|^2 \, dX,
\]

and,

\[
-\Re((\text{div}(A \nabla_x u), \chi^2 u))
= \Re \iint A(X) \nabla_x u(X) \left(\chi^2(X) \nabla_x u(X) + 2 \chi(X) \nabla_x \chi(X) u(X)\right) \, dX.
\]

Using (1.4), the hypothesis (1.2) and the Cauchy-Schwarz inequality we obtain,

\[
k||\chi|\nabla_x u||^2 \leq \frac{C}{r^2} \iint \chi(X)|u(X)|^2 \, dX + 2\|A\|_{L^\infty(Q)} \left( \iint |\chi(X)|^2 |\nabla_x u(X)|^2 \, dX \right)^\frac{1}{2}
\cdot \left( \iint |\nabla_x \chi(X)|^2 |u(X)|^2 \, dX \right)^\frac{1}{2}
+ C \iint \chi^2(X)|u(X)|^2 \, dX + C \left( \iint |\chi(X)|^2 |\nabla_x u(X)|^2 \, dX \right)^\frac{1}{2}
\cdot \left( \iint |\chi(X)|^2 |u(X)|^2 \, dX \right)^\frac{1}{2}.
\]

Using the inequality \( ab \leq \varepsilon a^2 + \frac{1}{4\varepsilon} b^2 \), the estimates (1.3), and the fact that \( r \) is small we obtain,

\[
\frac{1}{2}k \iint |\chi(X)|^2 |\nabla_x u(X)|^2 \, dX \leq \frac{C}{r^2} \iint_{I_{2r} \times Q_{2r}} |u(X)|^2 \, dX.
\]

The conclusion follows from the fact that \( \chi(X) = 1 \) on \( I_r \times Q_r \). \( \square \)
Notice that this Proposition appears in the work by P.Auscher, S.Bortz, M.Egert, O.Saari ([1] Proposition 4.3).

The second one is a result by L.Escauriaza, F.J.Fernandez, S.Vessela ([2] Theorem 3).

**Theorem 1.3.** ([2]) Let $P$ be defined in (1.1) and let $u \in L^2((-T,T),H^2(B_R))$ be a solution of $Pu = 0$ in $Q$. Set,

$$\Theta = \frac{\iint_{Q_r} |u(t,x)|^2 \, dx \, dt}{\iint_{B_t(x_0)} |u(0,x)|^2 \, dx}.$$ 

Then there exists $N = N(d,\kappa) > 0$ such that the following holds when $0 < r < (N\log(N\Theta))^{-\frac{1}{2}},$

$$\iint_{Q_{2r}((t_0,x_0))} |u(t,x)|^2 \, dx \, dt \leq D(N,u) \iint_{Q_r((t_0,x_0))} |u(t,x)|^2 \, dx \, dt$$

where $D(N,u) = \exp(N\log(N\Theta)\log(N\log(N\Theta))$.

The third result is the following lemma, a kind of Poincaré inequality, which can be found in the book by O. Ladyzenskaya, N.Ural’tseva [5].

**Lemma 1.4.** ([5]) Let $x_0 \in \mathbb{R}^d$ and $r > 0$, Let $F$ be a measurable subset of $B_r(x_0)$ with $\mu(F) > 0$. Then for every $v \in H^1(B_r(x_0))$ and for every measurable set $A \subset B_r(x_0)$ we have,

$$\mu(F) \int_A |v(x)|^2 \, dx \leq 2\mu(A) \int_F |v(x)|^2 \, dx + C r^{d+1} \mu(A)^{\frac{1}{2}} \int_{B_r(x_0)} |\nabla_x v(x)|^2 \, dx,$$

where $C = \frac{2d+3}{4d+1} (\mu(S^{d-1}))^{1-\frac{1}{d}}$.

For the reader’s convenience we give the proof of this lemma.

**Proof of Lemma 1.4.** It is of course enough to prove this lemma for $x_0 = 0$ and for $v \in C^\infty(\overline{B_r})$ where $B_r = B_r(0)$. Let $y \in F$ and $x \in A, x \neq y$. We have,

$$v(y) - v(x) = \int_0^{[y-x]} \frac{d}{dp} \left( v(x + \rho \frac{y-x}{|y-x|}) \right) \, dp,$$

$$= \int_0^{[y-x]} \nabla_x v \left( x + \rho \frac{y-x}{|y-x|} \right) \cdot \frac{y-x}{|y-x|} \, dp.$$ 

Using the Cauchy-Schwarz inequality we obtain,

$$|v(x)|^2 \leq 2|v(y)|^2 + 2|y-x| \int_0^{[y-x]} \left| \nabla_x v \left( x + \rho \frac{y-x}{|y-x|} \right) \right|^2 \, dp.$$ 

Integrating this inequality for $y \in F$ we get,

$$\mu(F)|v(x)|^2 \leq 2 \int_F |v(y)|^2 \, dy + \int_{B_r} \int_0^{[y-x]} \left| \nabla_x v \left( x + \rho \frac{y-x}{|y-x|} \right) \right|^2 \, dp \, dy.$$ 

We would like to set, in the second integral in the right hand side, $y = x + t\omega, t > 0, \omega \in S^{d-1}$. We first prove that, for $x \in B_r$,

$$y = x + t\omega \in B_r, t > 0, \omega \in S^{d-1} \iff y = x + t\omega, 0 < t < t^*(x,\omega),$$

with $0 < t^* \leq 2r$.

Indeed $y \in B_r$ is equivalent to $|x + t\omega|^2 < r^2$ thus to $t^2 + 2(x \cdot \omega)t + |x|^2 - r^2 < 0$. The reduced discriminant of this second order polynomial in $t$ is equal to $\Delta = (x \cdot \omega)^2 + r^2 - |x|^2$ which is strictly positive. Therefore this polynomial has two real roots and since their product is
Since \( |x|^2 - r^2 < 0 \), there are one positive root \( t^* \) and one negative root \( t_* \). Therefore this polynomial is strictly negative if and only if \( 0 < t < t^* \). Moreover \( t^* = -x \cdot \omega + \sqrt{(x \cdot \omega)^2 + r^2 - |x|^2} \) therefore \( 0 < t^* \leq |x| + r \leq 2r \).

Making the change of variable \( y = x + t\omega \) in the second integral in the right hand side of (1.5) we obtain,

\[
\mu(F)|v(x)|^2 \leq 2 \int_F |v(y)|^2 \, dy + 2 \int_{S^{d-1}} \left( \int_{t=0}^{t^*} |\nabla_x v(x + t\omega)|^2 dt \right) \, d\rho \, dt \, d\omega.
\]

By the Fubini Theorem one can write,

\[
\mu(F)|v(x)|^2 \leq 2 \int_F |v(y)|^2 \, dy + \int_{S^{d-1}} \left( \int_{t=0}^{t^*} \int_{\rho=0}^{\rho_*} \left( t^d \, |\nabla_x v(x + \rho\omega)|^2 \right) \, d\rho \, dt \right) \, d\omega.
\]

Since \( t^* < 2r \) we obtain,

\[
\mu(F)|v(x)|^2 \leq 2 \int_F |v(y)|^2 \, dy + \frac{2^{d+2}}{d+1} r^{d+1} \int_{S^{d-1}} \left( \int_{\rho=0}^{\rho_*} \frac{\left| \nabla_x v(x + \rho\omega) \right|^2}{\rho^{d-1}} \, d\rho \right) \, d\omega.
\]

Setting \( z = x + t\omega \) and using again (1.6) we obtain,

\[
\mu(F)|v(x)|^2 \leq 2 \int_F |v(y)|^2 \, dy + \frac{2^{d+2}}{d+1} r^{d+1} \int_{B_r} \frac{\left| \nabla_x v(z) \right|^2}{|z - x|^{d-1}} \, dz.
\]

Integrating both members of this inequality with respect to \( x \) on \( A \) we get,

\[
(1.7) \quad \mu(F) \int_A |v(x)|^2 \, dx \leq 2\mu(A) \int_F |v(y)|^2 \, dy + \frac{2^{d+2}}{d+1} r^{d+1} \int_{B_r} \frac{\left| \nabla_x v(z) \right|^2 g(z)}{|z - x|^{d-1}} \, dz,
\]

where \( g(z) = \int_A \frac{1}{|z - x|^{d-1}} \, dx \).

To compute \( g \) let \( \delta > 0 \) and set \( A_1 = \{ x \in A : |z - x| < \delta \}, A_2 = \{ x \in A : |z - x| > \delta \} \). We have,

\[
\int_{A_1} \frac{1}{|z - x|^{d-1}} \, dx \leq \int_{|z-x|<\delta} \frac{1}{|z - x|^{d-1}} \, dx \leq \mu(S^{d-1}) \delta,
\]

\[
\int_{A_2} \frac{1}{|z - x|^{d-1}} \, dx \leq \delta^{-(d-1)} \mu(A).
\]

We then choose \( \delta \) such that \( \mu(S^{d-1}) \delta = \delta^{-(d-1)} \mu(A) \), that is, \( \delta = \left( \frac{\mu(A)}{\mu(S^{d-1})} \right)^\frac{1}{d} \) and we obtain the estimate,

\[
g(z) \leq 2\left( \mu(S^{d-1}) \right)^{1-\frac{d}{2}} \mu(A)^{\frac{d}{2}}.
\]

Using (1.7) we obtain eventually,

\[
\mu(F) \int_A |v(x)|^2 \, dx \leq 2\mu(A) \int_F |v(y)|^2 \, dy + \frac{2^{d+3}}{d+1} \left( \mu(S^{d-1}) \right)^{1-\frac{d}{2}} r^{d+1} \mu(A)^{\frac{d}{2}} \int_{B_r} \left| \nabla_x v(z) \right|^2 \, dz.
\]

\[\square\]
1.4. **Proof of Theorem 1.1.** We shall use the method described in R. Regbaoui [7].

We know that almost every point of $E$ is of density 1. If $x_0$ is such a point this means that,

$$
\lim_{r \to 0} \frac{\mu(E \cap B_r(x_0))}{\mu(B_r(x_0))} = 1.
$$

Let $\varepsilon \in (0, \frac{1}{2})$ then there exists $r_0 > 0$ such that for $r \leq r_0$,

(1.8) \quad \mu(E \cap B_r(x_0)) \geq (1 - \varepsilon)\mu(B_r(x_0)), \quad \mu(E^c \cap B_r(x_0)) \leq \varepsilon\mu(B_r(x_0)).

We apply Lemma 1.4 to $v(x) = u(t, x), F = E \cap B_r(x_0), A = E^c \cap B_r(x_0)$. Since $u(t, x) = 0$ for $x \in E$, we obtain,

$$
\int_{B_r(x_0)} |u(t, x)|^2 \, dx = \int_{E^c \cap B_r(x_0)} |u(t, x)|^2 \, dx \leq C_d r^{d+1} \frac{\mu(E^c \cap B_r(x_0))}{\mu(E \cap B_r(x_0))} \int_{B_r(x_0)} |\nabla u(t, x)|^2 \, dx.
$$

Using (1.8) we obtain,

$$
\int_{B_r(x_0)} |u(t, x)|^2 \, dx \leq 2C_d r^{d+1} \varepsilon \frac{2}{\mu(B_r(x_0))} \int_{B_r(x_0)} |\nabla u(t, x)|^2 \, dx,
$$

$$
\leq C'_d r^2 \varepsilon^2 \int_{B_r(x_0)} |\nabla u(t, x)|^2 \, dx,
$$

since $\mu(B_r(x_0)) = C_d r^d$.

Integrating both members of this inequality with respect to $t$ on $I_r(t_0) = (t_0 - r^2, t_0 + r^2)$ where $t_0$ is an arbitrary point of $I$, we obtain,

$$
\int_{Q_r(t_0, x_0)} |u(t, x)|^2 \, dx \, dt \leq C'_d \varepsilon r^2 \int_{Q_r(t_0, x_0)} |\nabla u(t, x)|^2 \, dx \, dt.
$$

Using Theorem 1.2 we deduce that, for $r \leq r_0$,

(1.9) \quad \int_{Q_r(t_0, x_0)} |u(t, x)|^2 \, dx \, dt \leq C''_d \varepsilon \int_{Q_{2r}(t_0, x_0)} |u(t, x)|^2 \, dx \, dt.

By Theorem 1.3 there exists $N = N(d, \kappa)$ such that when $0 < r < \sqrt{N \log(N \Theta)} - \frac{1}{2}$ we have,

$$
\int_{Q_{2r}(t_0, x_0)} |u(t, x)|^2 \, dx \, dt \leq D(N, u) \int_{Q_r(t_0, x_0)} |u(t, x)|^2 \, dx \, dt,
$$

where $D(N, u) = e^{N \log(N \Theta) + \log(N \log(N \Theta))}$.

Combining this inequality with (1.9), we obtain,

$$
\int_{Q_r(t_0, x_0)} |u(t, x)|^2 \, dx \, dt \leq C''_d \varepsilon^3 D(N, u) \int_{Q_{r}(t_0, x_0)} |u(t, x)|^2 \, dx \, dt.
$$

Taking $\varepsilon \in (0, \frac{1}{2})$ so small that, $C''_d \varepsilon^3 D(N, u) < 1$ we deduce from the above inequality that there exists $r > 0$ such that $u = 0$ in $Q_r(t_0, x_0)$. This argument holds for every $t_0 \in (-T, T)$.

Therefore applying Theorem 1.1 in [8] or Theorem 1 in [4] we deduce that $u = 0$ in $Q$. 

REFERENCES

[1] P. Auscher, S. Bortz, M. Egert, O. Saari: On regularity of weak solutions to linear parabolic systems with measurable coefficients. Journ. Math Pures et Appl., 121, pp. 216-243, (2019).
[2] L. Escauriaza, F.J. Fernandez, S. Vessella Doubling properties of caloric functions. Appl. Anal. 85 no. 1-3, pp. 205-223, 85 (2006).
[3] L. Escauriaza, S. Montaner, C. Zhang Analyticity of solutions to parabolic evolutions and applications. Siam. J. Math. Anal. 49, (5), pp. 4064-4092, (2017).
[4] H. Koch, D. Tataru: Carleman estimates and unique continuation for second order parabolic equations with non smooth coefficients. Comm on pde, 34, (4), pp. 305-366, (2009).
[5] O. Ladyzenskaya, N. Uraltseva: Linear and quasilinear elliptic equations Academic press, New York (1968).
[6] A. Logunov, E. Malinnikova: Quantitative propagation of smallness for solutions of elliptic equations. Proceedings of the International Congress of Mathematicians, Rio de Janeiro, World Sci. Publ., Hackensack, NJ, Vol. 3. 2391-2411, (2018).
[7] R. Regbaoui: Unique continuation from sets of positive measure in "Carleman estimates and applications to uniqueness and control theory". Progress in non linear differential equations Vol 46, Birkhäuser pp. 179-190, (2001).
[8] J.C. Saut, B. Scheurer: Unique continuation for some evolution equation. Journ. of diff. equations, 66, pp. 118 -139, (1987).
[9] S. Vessella Unique continuation properties and quantitative estimates of unique continuation for parabolic equations. Handbook of Differential equations, Evolutionary Equations, Elsevier, Volume 5, 423-500, (2009).

Université Paris-Saclay, Mathématiques, UMR 8628 du CNRS, Bât 307, 91405 Orsay Cedex, France, and Institut Universitaire de France
Email address: Nicolas.burq@universite-paris-saclay.fr

Université Paris-Saclay, Mathématiques, UMR 8628 du CNRS, Bât 307, 91405 Orsay Cedex, France,
Email address: Claude.zuily@universite-paris-saclay.fr