Integral Geometric Dual Distributions of Multilinear Models
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Abstract—We propose an integral geometric approach for computing dual distributions for the parameter distributions of multilinear models. The dual distributions can be computed from, for example, the parameter distributions of conics, multiple view tensors, homographies, or as simple entities as points, lines, and planes. The dual distributions have analytical forms that follow from the asymptotic normality property of the maximum likelihood estimator and an application of integral transforms, fundamentally the generalised Radon transforms, on the probability density of the parameters. The approach allows us, for instance, to look at the uncertainty distributions in feature distributions, which are essentially tied to the distribution of training data, and helps us to derive conditional distributions for interesting variables and characterise confidence intervals of the estimates.

Keywords: Duality, Radon Transform, Uncertainty, Confidence Intervals, Integral Geometry, Computer Vision.

1 INTRODUCTION

An essential part of geometric computer vision are multilinear models which include e.g. homographies, multiple view tensors, quadric surfaces, as well as the simple entities of points, lines and planes. There are numerous works related to the estimation of these kinds of multilinear relations, see e.g. [1], [2], [3], [4], [5], [6], [7], [8], [9], [10]. When a statistical approach is selected for the estimation, one should be able to compute the geometric model parameters along with their uncertainty distribution. This paper considers how these parameter distributions of multilinear geometric entities can be dualised. The simplest form of this dualisation is the transformation of the line-probability-density into a point-probability-density as proposed in [12]. This paper generalises this dualisation approach for general multilinear models. An early, conference version of this paper is [11].

By the way of an example, consider fitting a conic section to a set of points using maximum likelihood estimation. We would be interested in the confidence intervals of the conic, but the normal distribution assumption can be made only for the MLE in the parameter space. However, we will show that by the dualisation of the parameter distribution we will obtain a distribution of points that can be used to plot the selected confidence intervals for the estimated conic section. As the second example, let us consider the trifocal point transfer. Given a point match in two views, it would be interesting to compute the conditional position distribution of the transferred point in the third view if we had the uncertainty information of the trifocal tensor available. In fact, by dualising the trifocal tensor parameter distribution, an exact form for this conditional distribution can be computed, as will be shown later in this paper.

Our approach is closely related to the branch of integral geometry in mathematics. There are two main schools of integral geometry of which the traditional is that of Santaló and Blaschke [13]. The classical example is that the length of a plane curve is the probability of random lines intersecting it. The more recent meaning is the school of Gelfand [14]. It studies integral transforms, modelled with the Radon transform, that relates the underlying geometrical incidence relations by incidence graphs. Our approach seems to be somewhat in between these two schools as we compute (generalised) Radon transforms for the probability densities in such a way that the probability measure is preserved. The dualisation is constructed from the fact that the probability of an element (e.g. a point) is the total probability of all the geometric entities (e.g. planes) that coincide with the element. Then by integrating the distribution of the entity over the affine subspaces corresponding to the selected incidence relation, Radon like integral transforms follow.

As the principal assumption we use the normal distribution assumption for the multilinear model parameters. This is reasonable due to the asymptotic normality property of the maximum likelihood estimator, i.e., due to the fact that, with certain general regularity conditions, the distribution of the maximum likelihood parameter estimator converges in distribution to the normal distribution with the (pseudo)inverse of the Fisher information as the parameter covariance matrix [15], [2]. This makes the approach taken here fundamentally different from the work in [16] where an algebraic linear system and
Gaussian approximation in the feature space were used. In contrast to the work in [16], the dual distributions considered here have analytic forms and are exact with the assumptions above.

This paper is organised as follows. In Section 2 we introduce the multilinear models considered in this paper. In Section 4 we derive the dual distributions by first assuming a single constraint equation and then generalise the approach for multiple constraint equations. In Section 5 we show how interesting conditional distributions can be extracted. In Section 6, we compute confidence intervals for conics and compute the point transfer density from two views into the third view. Conclusions are in Section 8.

2 MULTILINEAR MODEL

We first need to define what we mean by a multilinear model.

Definition 2.1: A function \( f : V_1 \times \cdots \times V_k \to W \), where \( V_1, \ldots, V_k \) and \( W \) are real vector spaces, is \( k \)-linear if it is linear in each of its \( k \) arguments:

\[
f(\ldots, \alpha x + \beta y, \ldots) = \alpha f(\ldots, x, \ldots) + \beta f(\ldots, y, \ldots),
\]

for all \( \alpha, \beta \in \mathbb{R} \) and \( x, y \in V \).

Definition 2.2: Let \( f \) be a \((n + 1)\)-linear function \( f : \mathbb{P}^m \times \cdots \times \mathbb{P}^m \times \mathbb{P}^{N-1} \to \mathbb{R}^L \). The multilinear model is defined as the relation

\[
f(x_1, x_2, \ldots, x_n; \theta) = 0,
\]

where \( x_i \) are feature vectors and \( \theta \) is the parameter vector, where \( x_i \in \mathbb{P}^m, \theta \in \mathbb{R}^{N-1} \) and \( i = 1, 2, \ldots, n \).

The definition is to be taken in the general sense so that any of the \( n \) arguments may be repeated arbitrary many times so that, e.g., quadratic forms are included.

With a fixed \( \theta \), the multilinear model defines a multilinear equation system, where each equation is equivalent to a linear subspace, with co-dimension one, in the space of the joint feature vector.

Definition 2.3: The joint feature vector is defined as the vector \( y \in \mathbb{P}^{N-1} \), such that

\[
y \doteq x_1 \otimes x_2 \otimes \ldots \otimes x_n
\]

containing the elements of the tensor product, up to scale, where the repeating elements have been dropped; \( \doteq \) denotes the correspondence between the expressions, \( \otimes \) is the tensor product, and \( x_i \in \mathbb{P}^m, i = 1, 2, \ldots, n \).

For the case where the feature vectors are distinct, the mapping from the feature vectors to the joint feature vector is known as Segre embedding.

Without a loss of generality, we assume that each equation \( l \) of the multilinear system is in the form of

\[
\theta^T T_l y \equiv \theta^T y_l = 0,
\]

where \( T_l \) is a matrix defined by the multilinear relation. For instance, the well known point and line incidence relations \([1], [17]\) characterising multiple projective views of a scene can be written in this form.

3 STATISTICAL MODEL

Let us assume that we have the maximum likelihood estimate \( \theta_0 \), or the corresponding robust estimate \([4]\), for the parameter vector and its the covariance matrix \( C_\theta \) available, where \( \theta_0 \) is constrained to lie on the unit hypersphere \( S^{N-1} \) in \( \mathbb{R}^N \). Using the asymptotic normality property of the MLE, we assume that \( \theta \sim N(\theta_0, C_\theta) \). Since \( \theta_0 \in \text{Ker} \{C_\theta\} \), \( \theta \) has the density function

\[
p(\theta) \propto \exp \left(-\frac{1}{2} \theta^T C_\theta^{-1} \theta \right),
\]

where \( \cdot^T \) is the Moore–Penrose pseudoinverse. That is, we use the tangential, normal approximation for the variation of \( \theta \) on the unit hypersphere at \( \theta_0 \).

4 DUALS OF THE PARAMETER DISTRIBUTIONS

In this section, we dualise the normal parameter uncertainty distributions starting from the models defined by a single constraint equation and generalise the point–line case \([12]\) to the point–hyperplane duality (Section 4.1). The general case of multiple constraint equations is considered in Section 4.2.

4.1 Single Constraint Equation

Let \( \theta \) be a parameter vector that defines the multilinear model by a single constraint equation

\[
\theta^T y = 0,
\]

with the joint feature vector \( y \in \mathbb{P}^{N-1} \).

Our intention is to compute the dual of \( p(\theta) \). The duality between the parameter vector \( \theta \) and the feature vector \( y \) is illustrated in Fig. 1. As (6) represents the hyperplane \( \theta \) in the space of \( y \), it can also be seen as the hyperplane \( y \) in the dual space or space of \( \theta \). Now, as the dual distribution of \( \theta \), which is a distribution for \( y \), we will identify the total probability (density) of all the models \( \theta \) lying on the hyperplane \( y \) in the dual space. To construct the dual pdf \( p(y) \), we first make the whitening transformation for variables. Then we compute the Radon transform, i.e., transform the whitened dual domain by computing the integrals of \( p(\theta) \) over all the hyperplanes in the dual space. The subsets of these integrals corresponding to parallel hyperplanes form the conditional probability density (marginal density) conditioned on a fixed normal direction of the hyperplane. Multiplying this conditional density with the pdf of normal directions, which is the uniform density on the half of the unit hypersphere with the whitened Gaussian model, we obtain a valid probability density which can be interpreted in the parameter space of \( y \). The details are below.
where the eigenvectors of $\theta_0$ represents the line $\theta_0 = 0$. The dual distribution $p(\theta)$ by making Radon like integral transform to it so that the probability measure is preserved.

Let us make the whitening transform from the eigenvalue decomposition of $C_\theta$, which has been constructed so that $\theta_0$ corresponds to the last eigenvector. We may write

$$\begin{align*}
\theta^T C^T_\theta \theta &= \theta^T U \Lambda U^T \theta \\
&= \theta^T \begin{pmatrix} \Lambda^{-1/2} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \Lambda^{-1/2} & 0 \\ 0 & 1 \end{pmatrix} U^T \theta \\
&= \theta^T \tilde{I} \theta',
\end{align*}$$

where the diagonal matrix $\tilde{\Lambda}$ contains the $M-1$ non-zero eigenvalues of $C_\theta$, sorted in the descending order, and $\tilde{U}$ contains the corresponding eigenvectors and the eigenvector representing $\theta_0$. Now, $\theta' \sim N(e_M, \tilde{I})$, where $e_M$ is the standard basis vector $(0, \ldots, 0, 1) \in \mathbb{R}^M$. Furthermore, for all the feature vectors that are consistent with the model $\theta$, lying on the tangent space, we may write

$$0 = \theta^T y = \theta^T \tilde{U} \begin{pmatrix} \Lambda^{-1/2} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \Lambda^{-1/2} & 0 \\ 0 & 1 \end{pmatrix} \tilde{U}^T y = \theta'^T y',$$

where $y' \in \mathbb{R}^{M-1}$ represents a reduced joint feature vector. For simplicity, we now investigate the reduced, transformed model $\theta'$. Now we are ready to state our main theorem for the case of the single constraint equation.

**Theorem 4.1:** Let $y, \theta \in \mathbb{P}^{N-1}$, normalised so that the joint feature vector $y$ has homogeneous scaling of unity and $\theta$ lies in the tangent space $T_{\theta_0}(S^{N-1})$. Moreover, let $\theta \sim N(\theta_0, C)$ with the probability density function $p(\theta)$, where the eigenvectors of $C$ corresponding to the $M-1$ non-zero eigenvalues span $T_{\theta_0}$, and $y'$ is the reduced joint feature vector corresponding to $y$. The dual distribution of $p(\theta)$ has the analytic form

$$p(\rho, \phi) = \Gamma(M/2 - 1/2)e^{-\frac{1}{2}\rho^2} \prod_{i=1}^{M-3} \sin^{M-2-i}(\phi_i),$$

where the reduced joint feature vector $y'$ is parameterized by the modified hyperspherical coordinates $(\rho, \phi)$ in $\mathbb{R}^{M-1}$.

**Proof:** On the basis of the construction above, the variation of $\theta'$ occurs only in the $M-1$ dimensional affine subspace $\pi'$ perpendicular to $e_M$ in the dual space. Since every point on this tangent hyperplane can be identified with a unique one-dimensional linear subspace of $\mathbb{R}^M$, we may regard the tangent hyperplane as a projective space $\mathbb{P}^{M-1}$. Moreover, the points on the tangent plane can be considered to be already in the homogeneous form. Hence, in the following we assume that $\theta' \in \mathbb{P}^{M-1}$.

In the dual space, the $M-2$ dimensional hyperplanes $\pi' \theta' = 0$ embedded in the $M-1$ dimensional affine subspace $\pi'=\mathbb{R}^{M-1}$, may be parameterised by the signed distance $s$ from the origin and by the unit vector $v \in S^{M-2}$, as Fig 2(b) illustrates. The origin can be represented by $e_M \in \mathbb{P}^{M-1}$ and we assume that $v = v(\phi)$ where $\phi$ parameterises the normal direction of the $M-2$ dimensional hyperplane $y'$ in $\mathbb{R}^{M-1}$. We choose the sign of $s$ to be equal to sign of the intercept of the hyperplane in the dual space, hence,

$$s = \frac{y_N}{\sqrt{y_N^T y' - y_N'^2}}.$$  

On the other hand, in the reduced joint feature space, the homogeneous vector $y'$ can be parameterised by the signed distance $\rho$ from the origin $e_M$ and the direction $v$ (Fig 2(a)). The sign of $\rho$ is identified as the last variable sign of the inhomogeneous representation of $y'$. Then we have

$$s = -\frac{1}{\rho}.$$  

As we assume that a $\theta' \sim N(e_M, \tilde{I})$, the probability of the hyper plane $y_N' \theta' = 0$, conditioned on the direction $v(\phi)$ in the dual space, is simply the marginal probability of the Gaussian over the hyperplane, i.e.,

$$p(s|\phi) = \int y_N'^T \theta' = 0 \, p_G(\theta'; e_M, \tilde{I}) \, dS = p(s)$$  

which is a mean zero, 1-D Gaussian with unity variance $[12]$, where $dS$ denotes the volume differential in the hyperplane.
By using the modified spherical coordinates $y' = y'(\rho, \phi)$ in the reduced joint feature space (see Appendix A) and the fact that $s = s(\rho)$, we get

$$p(\rho|\phi) = \frac{1}{\sqrt{2\pi\rho^2}} \exp \left( -\frac{\rho^2}{2} \right).$$

(13)

On the other hand, since we have an isotropic Gaussian distribution, the distribution of normal directions $\mathbf{W}$ and marginalising the Gaussian distribution, the distribution of normal directions $\mathbf{W}$ in $\mathbf{M}$—it could give certain algebraic benefits that are to be investigated in future.

In this paper, we study Gaussians in the reduced coordinate frame that suggests a simple parameterisation, generalising the result on the single coordinate equation. We represent an affine subspace by the offset vector $\mathbf{w}$ and parameters of the parallel linear subspace $V$. To parameterise $V$, we use the fact that the related orthogonal projection matrix

$$\mathbf{P} = \mathbf{I} - \mathbf{A}^\dagger \mathbf{A},$$

(19)

projecting onto the subspace is unique. It is well known that a matrix represents an orthogonal projection onto a linear subspace if and only if it is idempotent, $\mathbf{P}^2 = \mathbf{P}$, and self-adjoint, $\mathbf{P}^T = \mathbf{P}$. Now, to uniquely parameterise the matrix $\mathbf{P}$ we need the following lemma.

Lemma 4.2: Let $\mathbf{P}$ be an idempotent and symmetric $(M - 1) \times (M - 1)$ matrix. If $\mathbf{p}_i = \mathbf{P}e_i$, $i = 1, \ldots, K$ are linearly independent and span the range of $\mathbf{P}$, where $K \leq \frac{M}{2}$, is the dimension of the range, then there is a unique lower triangular $K \times (M - 1)$ matrix $\mathbf{L}$ with orthonormal columns and strictly positive diagonal so that $\mathbf{P} = \mathbf{L}\mathbf{L}^T$.

The proof is in Appendix B.

The lemma suggests that we may parameterise the orthogonal projection matrix $\mathbf{P}$ by parameterising the elements of $\mathbf{L} = \mathbf{L}(\Phi)$ when we may define

$$\mathbf{P}(\Phi) = \begin{cases} \mathbf{L}(\Phi)\mathbf{L}(\Phi)^T, & \text{if } K \leq M/2 \\ \mathbf{I} - \mathbf{L}(\Phi)\mathbf{L}(\Phi)^T, & \text{otherwise}. \end{cases}$$

(20)

The matrix $\mathbf{L}$ can be parameterised by the modified spherical coordinates (see Appendix A) with the radial parameter equal to unity. The first column vector is on the half of the unit sphere $S^{M-2}$, the second column vector is orthogonal to the first and it has one element less and is hence on $S^{M-4}$, to parameterise $S^{M-4}$, we form an orthogonal basis in the orthogonal complement of the first column vector, by creating the orthogonal projection matrix that projects onto the orthogonal complement, and employ the Gram–Schmidt orthonormalisation procedure, as shown in Appendix B, and form the unit sphere in the subspace spanned by the orthonormal vectors. The other columns $l \leq K$ can be similarly parameterised on the half of the unit sphere $S^{M-2l}$, whereas $K = K$, if $K \leq M/2$, and $K = M - K - 1$, otherwise. Let $\Phi_k$ denote the vector of modified spherical coordinates of the column $k$ in $\mathbf{L}$. We collect the parameters in
the vector \( \Phi = (\Phi_1, \Phi_2, \ldots, \Phi_K) \). In total, there are \( K' = (M - 1)K - K^2 \) free parameters in \( L \).

Now, we are ready to state our theorem about the probability density for the affine subspace \( W \).

**Theorem 4.3:** Let \( Y \in \mathbb{R}^{N \times L} \) and \( \theta \in \mathbb{R}^{N-1} \), normalised so that \( \theta \) lies in the tangent space \( \mathbb{T}_{\theta_0}(S^{N-1}) \). Moreover, let \( \theta \sim N(\theta_0, C) \) with the probability density function \( p(\theta) \), where the eigenvectors of \( C \) corresponding to the \( M - 1 \) non-zero eigenvalues span \( \mathbb{T}_{\theta_0} \), and \( Y' \) is the reduced matrix corresponding to \( Y \). The dual distribution of \( p(\theta) \), corresponding to the affine subspace \( W = W(s, \Phi) \), has the analytic form

\[
p(s, \Phi) = \frac{e^{-\frac{1}{2}s^T s}}{2^{(M-K-1)/2} \pi^{(M-K+KM-K^2-1)/2}} \prod_{k=1}^{K} \Gamma((M-2k+1)/2) \prod_{i=1}^{M-2k-1} \sin^{M-2k-i}(\phi_i^k).
\]

(21)

where \( (s, \Phi) \) are the parameters of the affine subspace \( W \).

**Proof:** We create an orthogonal basis \( u_1, u_2, \ldots, u_{M-1} \) for the subspace \( V \) and its orthogonal complement \( V^\perp \) using the Gram-Schmidt orthonormalization procedure for the projections \( p_i = \Pi e_i \), \( i = 1, 2, \ldots, K \) onto the subspace \( V \) (see Appendix B) and similarly for the projections \( p_i = (I - \Pi) e_i \), \( i = K + 1, K + 2, \ldots, M - 1 \) on its orthogonal complement \( V^\perp \). We marginalise the \( M - 1 \) dimensional Gaussian over all the parallel affine subspaces of \( V \), i.e., those which are of the form \( W = s + V \). We obtain the conditional probability density of \( w = s \in \mathbb{R}^{M-K-1} \)

\[
p(s|\Phi) = \int_V p_c(\theta'; e_M, \mathbf{1}) dS = p(s),
\]

which is a mean zero, \( M-K-1 \) dimensional Gaussian with the identity matrix as the covariance matrix.

As to the distribution of directions \( \Phi \), isotropic Gaussians imply uniform directions on half hyperspheres. We decompose the vector as \( \Phi = (\Phi_1, \Phi_2, \ldots, \Phi_K) \) and further \( \Phi_k = (\phi_1^k, \phi_2^k, \ldots, \phi_{M-2k}^k) \). Thus we have

\[
p(\Phi) = p(\Phi_1, \Phi_2, \ldots, \Phi_K)
\]

\[
= p(\Phi_1)p(\Phi_2|\Phi_1)p(\Phi_3|\Phi_1, \Phi_2) \cdots p(\Phi_K|\Phi_1, \Phi_2, \ldots, \Phi_{K-1})
\]

(23)

where

\[
p(\Phi_k|\Phi_1, \Phi_2, \ldots, \Phi_{k-1}) = \frac{\Gamma((M-2k+1)/2)}{\pi^{(M-2k+1)/2}} \prod_{i=1}^{M-2k-1} \sin^{M-2k-i}(\phi_i^k),
\]

(24)

\[k = 1, 2, \ldots, K.\]

So finally,

\[
p(W) = p(s, \Phi) = p(s|\Phi)p(\Phi),
\]

(25)

and the claim follows.

We have thus derived an analytic form for the probability density of the affine subspace \( W = W(s, \Phi) \) or, equivalently, the probability density of the left nullspace of the reduced coefficient matrix \( Y' \), assuming the model \cite{16} and Gaussian distributed parameter vector \( \theta \).

## 5 Mappings to Feature Distributions

In the previous section, we derived the general form for the dual distributions in the function of the parameters of the corresponding affine subspace. Now we discuss how we can extract interesting feature distributions from the dual distributions. The interesting feature distribution often has less parameters than the affine subspace or one may be interested only in certain conditional feature distributions, conditioned on some fixed a subset of the features. In these cases the mapping from the feature distribution to the affine subspace \( W \) is not necessarily one-to-one and onto. However, we may form the conditional distribution

\[
p(s, \Phi|R) = \frac{p(s, \Phi)}{\int_R p(s, \Phi) d\Phi} \propto p(s, \Phi).
\]

(26)

conditioned on the restriction \( R \subseteq \Omega \), where \( R \) is parameterised by the interesting part \( \bar{x} \in \mathbb{R}^n \) of the feature distribution.

**Theorem 5.1:** Let us consider a smooth submanifold \( R \) of the parameter space \( \Omega \) so that there is a differentiable bijection between the parameters \( \bar{x} \in \mathbb{R}^n \) and \( (s, \Phi) \in R \) almost everywhere. Then

\[
p(\bar{x}|X) = \frac{\sqrt{\det(J^T J)}}{\int_R \sqrt{\det(J^T J)}} p(s(\bar{x}), \Phi(\bar{x})) d\Phi,
\]

(27)

where \( J = \frac{\partial s}{\partial \bar{x}} \).

**Proof:** Let \( \varphi : X \rightarrow R \) so that \( (s, \Phi) = \varphi(\bar{x}) \) is continuous and invertible almost everywhere on \( X \subset \mathbb{R}^N \). Given an orthonormal basis \( B = \{u_1, u_2, \ldots, u_N\} \) on the tangent space \( T_{\varphi_0}(R) \)

\[
\varphi - \varphi_0 = (u_1 \ u_2 \ \cdots \ u_N)^T (\xi_1 \ \xi_2 \ \cdots \ \xi_N)^T,
\]

(28)

where \( \varphi_0 = \varphi(\bar{x}_0) \) and \( \varphi \in T_{\varphi_0}(R) \). The Jacobian of the mapping \( \bar{x} \mapsto \xi \) is the \( N \times N \) matrix

\[
J_\xi = \frac{\partial \xi}{\partial \bar{x}} = \frac{\partial \xi}{\partial \varphi} J_0 = (u_1 \ u_2 \ \cdots \ u_N)^T J_0,
\]

(29)

where we have used the property \( \xi_i = u_i^T (\varphi - \varphi_0), \ i = 1, \ldots, N \). Thus

\[|\det J_\xi| = \sqrt{|\det J_\xi|^2} = \sqrt{|\det(J_0^T J_0)|},
\]

(30)

which holds almost everywhere and is independent of the choice of the orthonormal bases. Using the substitution rule for integrals, the conditional distribution of \( \bar{x} \) takes the form

\[
p(\bar{x}|X) \propto \sqrt{\det(J^T J)} p(s(\bar{x}), \Phi(\bar{x})).
\]

(31)
As \( p(\mathbf{x}|X) \) is easily computed up to a global constant, we may draw samples from it by generating a MCMC chain with the Metropolis–Hastings sampling rule. Moreover, at least in cases where the interesting distribution covers a linear manifold, direct sampling methods can be applied similar to those derived in [12].

## 6 Experiments

In this section we show two application examples of the dual distributions. We create confidence intervals for conics (Section 6.1) and show how probabilistic point transfer can be constructed by using the covariance information of the estimated trifocal tensor (Section 6.2).

### 6.1 Dual Distributions for Conics

The points on a conic satisfy the homogeneous quadratic equation

\[ \mathbf{x}^T \mathbf{A} \mathbf{x} = 0, \quad (32) \]

which is a bilinear equation in \( \mathbf{x} \) and \( \mathbf{A} \) is a symmetric \( 3 \times 3 \) matrix. This equation can be written in the form, using the Veronese embedding,

\[ \theta^T \mathbf{y} = 0, \quad (33) \]

where \( \theta \doteq (a_{11}, a_{22}, a_{33}, 2a_{12}, 2a_{23}, 2a_{13}) \) and \( \mathbf{y} \doteq (x_1^2, x_2^2, x_3^2, x_1x_2, x_2x_3, x_1x_3) \), i.e., a conic forms a 5-dimensional linear subspace in the six dimensional joint feature space. We assume that the parameter vector estimate \( \hat{\theta} = \theta_0 \) and its covariance matrix \( \mathbf{C}_\theta \) are available.

If we dualise the relationship (33) above, we see that a fixed point on the image plane determines a 5-dimensional linear subspace in the dual space, i.e., the space of all conics that intersect the point on the image plane, see Fig. 3. Moreover, to characterise the probability of the point on the image we may construct the total probability (density) of all those conics containing the point. In other words, the dual distribution characterises the confidence of the estimated conic by illustrating what has been learned from the locations of the points on the true conic.

In this way, we estimated the maximum likelihood estimate and its covariance matrix for the conic containing 25 points, shown in Fig. 4b, assuming i.i.d. Gaussian noise in the 2D measurements. To evaluate the dual density at the selected location \((x_1, x_2)\) on the 2D plane, we need to evaluate (15) as well as compute the right magnification factor. As we parameterised the reduced joint feature vector \( \mathbf{y}' \) by the modified spherical coordinates, we may construct the mapping \((x_1, x_2) \mapsto (\rho, \phi)\) and its Jacobian to evaluate the dual pdf using (31). The dual pdf for the estimated conic is illustrated in Fig. 4b. The contours visualise the fact that we have a strong belief about the true conic points near the training data but extrapolation beyond the training data contains a substantial risk.

### 6.2 Probabilistic Point Transfer with an Uncertain Trifocal Tensor

The geometry of three projective views is characterised by several incidence relations or trilinearities, which are collected into Table 1. According to our preferences, we could use any of the trilinearities to create dual distributions or conditional dual distributions. As an example, we now illustrate how the trifocal point transfer can be “probabilised” by constructing a dual distribution for the trifocal ten-

![Fig. 3](image-url)  
(a) The dual distribution as the point distribution is here generated from the total probability of all the conics containing the point. (b) The subset of conics containing a given point is a five-dimensional linear subspace in the six-dimensional dual space, thus the probability of a point can be identified as the total probability of the corresponding linear subspace.

![Fig. 4](image-url)  
(a) The training data and the maximum likelihood estimate for the conic. (b) Contours of the dual pdf (point density) of the estimated conic characterising the current evidence where the points of the true conic locate.

### Table 1

The trilinearities, adopted from [1].

| Correspondence | Relation | dof |
|---------------|---------|-----|
| three points  | \( x^3x'^3x''^3 \epsilon_{qrs}\epsilon_{krt}T_{q''r} = 0 \) | 4   |
| two points, one line | \( x^2x'^2x''\epsilon_{qrs}T_{q''r} = 0 \) | 2   |
| one point, two lines | \( x^2x'^2T_{q''r} = 0 \) | 1   |
| three lines   | \( x^2x'^2x'' \epsilon_{qrs}T_{q''r} = 1 \) | 2   |
Fig. 5. Training images for estimating the trifocal tensor.

Fig. 6. Probabilistic point transfer. (a) Three equi-
probability contours of the point transfer pdf in the
first view, at the levels $10^{-1}, 10^{-2},$ and $10^{-3}$ times
the maximum value, conditioned on the two-view point
match in the views (b) two and (c) three. The circle
indicates the transferred point using the (deterministic)
point transfer [1] with only the ML estimate for the
trifocal tensor, and the two points in the other views;
(d) the same contours shown after scaling of the y-
axis and superimposing the ML epipolar lines (dashed)
corresponding to the points given in the second and
third view. The contours are closed curves surrounding
the most likely match locations and illustrate the fact
there the trifocal transfer is more than the epipolar
transfer from the views two and three.

1. The diagonal Tikhonov regulariser was adequate here but a
more elegant way would have been investigating the numerical
rank of the covariance matrix and developing an automatic model
selection scheme to determine the dimension of the affine subspace.
Degenerate configurations could be also handled in this way.

assuming i.i.d. Gaussian noise in the measurements.

Then, by using a novel point correspondence
in the views two (Fig. 6b) and three (Fig. 6c), we
computed the conditional probability density in the
first view, as reported above. To visualise the shape
of the pdf, we selected, three pdf values at the levels
$10^{-1}, 10^{-2},$ and $10^{-3}$ times the maximum value and
show the corresponding contours in Fig. 6a and 6d.
It can be seen that the transferred density is non-
Gaussian while it indicates the feasible locations for
the correspondence. The pdf has its maximum close to
the point where the ML epipolar lines meet whereas
the local shape of the peak is oriented towards the
mean axis of the two epipolar lines. The pdf shape
also seems to illustrate the well known fact that the
trifocal constraint is more versatile than the mere
epipolar geometries between the three views.

7 Discussion

The dual distributions are a tool for Bayesian inference with uncertain multilinear models, used e.g. in
geometric image analysis. According to the Bayesian

Given the point match $m' \leftrightarrow m''$ in the views two
and three, the construction of the conditional proba-
bility density $p(m'|m'', T, C_T)$ is as follows. The
trilinear relations for three points define the elements
of $\mathbf{Y}$ in the model (16). After constructing $\mathbf{Y}$, the
whitened matrix $\mathbf{Y}'$ is obtained from (17). However,
we regularised the whitening transform by replacing
$\tilde{\Lambda}$ by $\tilde{\Lambda} + \lambda I$, where $\lambda = 10^{-8}$ since the number
of data points was limited in our experiment and
hence, due to overfitting of the covariance matrix, the
smallest ones of the eighteen non-zero eigenvalues
were (assuming a non-degenerate configuration) rel-
atively small. The probability density for the affine
subspace, corresponding to $\mathbf{Y}'$, is given by (25) and
the desired conditional probability density values are
finally obtained from (31), up to scale. In short, to
compute the conditional probability density value for
a location of interest in the first view, the correspond-
ing affine subspace parameters are computed from $\mathbf{Y}'$
and the probability density value of the corresponding
affine subspace is weighted by the term containing the
Jacobian of the transformation.

To illustrate the conditional point transfer density,
we estimated the maximum likelihood trifocal tensor
and its covariance matrix from 274 outlier free point
correspondences for the image triplet shown in Fig. 5.

In this case, the dual distribution is the conditional
position distribution of the transferred point in the
third view. We use the nine point–point–point con-
straint equations (the first relation in Table 1) of which
only four equations are independent. Therefore the
dimension of the affine subspace of this example is
$K = (M - 1) - 4 = 14$.

The dual distributions are a tool for Bayesian infer-
ence with uncertain multilinear models, used e.g. in
geometric image analysis. According to the Bayesian
paradigm, we handle geometric relationships as probability distributions in contrast to the traditional approaches, which are based on deterministic estimates. Moreover, the theory provides tools for propagating the geometric uncertainty information to the dual variables. This approach helps especially in the cases when there is no explicit form for the mapping but a multilinear incidence relation which indirectly connects the parameter space and the observations.

As in the point–line case there are certain points and lines with a special role [12], likewise there are special subspaces in the general case to be investigated in future. They arise from the affine subspaces spanned by the eigenvectors of the covariance matrix—the deterministic trifocal point transfer, for instance, should be constructed by identifying the most likely affine subspace in the Gaussian model using the point–point–point incidence relation. In addition, there are many other ways to utilise the special subspaces arising from the multiple constraint equations. However, one should be careful in the interpretations as covariance estimates are not generally invariant to the selection of the coordinate frame.

This work is fundamentally a study of the generalised Radon transform for probability measures. Since the transform is based on integration over all the affine subspaces, a suitable parameterisation for affine subspaces is required. A natural algebraic choice would be the Grassmannian parameterisation, and it is likely that it would provide certain algebraic benefits that are to be investigated in future. For instance, one should study the transforms with respect to a natural Riemannian metric, invariant to motions of the coordinate system, to make the approach invariant to Gauge transforms. Proceeding in this direction would clarify the role of the dual distributions in between Santaló’s and Gelfand’s schools of integral geometry.

As far as the numerical development is concerned, more efficient numerical tools would be advantageous for geometric inference. The study of special subspaces and efficient sampling from the dual distributions in the general case are additionally to be investigated in future. Likewise, the study of marginal distributions, and proper handling of nonlinear geometric constraint manifolds are part of the future work.

8 Conclusions

In this paper we have shown how the parameter distributions of multilinear models can be dualised. The proposed approach provides means for a pure statistical treatment of multilinear relations where the uncertainty of the geometric entity is taken into account. The dual distributions are closely connected to integral geometry and have analytic forms with relatively small assumptions. We demonstrated the applicability of the theory by characterising the confidence of estimated conics and constructed the probabilistic trifocal point transfer by using an uncertain trifocal tensor with its estimated covariance matrix. In future, an interesting research direction is to investigate the dual distributions from the view point of integral geometry to deeply understand the theoretical connections as well as to develop additional numerical methods for applications.

References

[1] R. Hartley and A. Zisserman, Multiple View Geometry in Computer Vision. Cambridge, 2000.
[2] K. Kanatani, Statistical Optimization for Geometric Computation, ser. Machine Intelligence and Pattern Recognition. Amsterdam: Elsevier Science, 1996, vol. 18.
[3] M. Fischler and L. Bolles, “Random sample consensus: A paradigm for model fitting with applications to image analysis and automated cartography,” Commun. ACM, vol. 24, pp. 381–395, 1981.
[4] S. S. Brandt, “Maximum likelihood robust regression by mixture models,” Journal of Mathematical Imaging and Vision, vol. 25, pp. 25–48, 2006.
[5] A. Agarwal, C. V. Jawahar, and P. J. Narayanan, “A survey of planar homography estimation techniques,” International Institute of Information Technology, Hyderabad, Tech. Rep. IIT/ITR/2005/12, 2005.
[6] P. H. S. Torr and A. Zisserman, “Robust parameterization and computation of the trifocal tensor,” Image and Vision Computing, vol. 15, pp. 591–605, 1997.
[7] G. Cross and A. Zisserman., “Quadric surface reconstruction from dual-space geometry.” in Proc. ICCV, Bombay, India, 1998, pp. 25–31.
[8] Z. Zhang, “Determining the epipolar geometry and its uncertainty: A review,” Int. J. Comput. Vis., vol. 27, no. 2, pp. 161–195, 1999.
[9] B. Triggs, P. McLauchlan, R. Hartley, and A. Fitzgibbon, “Bundle adjustment – a modern synthesis,” in Vision Algorithms: Theory and Practice, ser. LNCS, B. Triggs, A. Zisserman, and R. Szeliski, Eds., vol. 1883. Berlin/New York: Springer, 2000, pp. 298–372.
[10] T. Scoleri, “Fundamental numerical schemes for parameter estimation in computer vision,” Ph.D. dissertation, School of Mathematical Sciences, University of Adelaide, 2008.
[11] S. S. Brandt, “Dual distributions for multilinear geometric entities,” in Proc. CVPR, June 2009, pp. 2679–2686.
[12] B. Triggs, “On the probabilistic epipolar geometry,” Image and Vision Computing, vol. 26, no. 3, pp. 405–414, 2008.
[13] L. Santaló, Introduction to integral geometry. Paris: Hermann, 1953.
[14] I. Gelfand, S. Findkin, and M. Graey, Selected Topics in Integral Geometry, ser. Translations of Mathematical Monographs. American Mathematical Society, 2003, vol. 220.
[15] A. F. Karr, Probability. Springer, 1992.
[16] B. Triggs, “Joint feature distributions for image correspondence,” in Proc. ICCV, vol. II, Vancouver, Canada, 2001, pp. 201–208.
[17] O. Faugeras and Q.-T. Luong, Geometry of Multiple Images. The MIT Press, 2001.
[18] B. Triggs, “Matching constraints and the joint image,” in Proc. ICCV, Cambridge, MA, June 1995.
[19] J. R. Weeks, The Shape of Space: How to Visualize Surfaces and Three-Dimensional Manifolds. Marcel Dekker, 1985.

Appendix

A. Modified Spherical Coordinates in $\mathbb{R}^N$

In this paper, we use modified spherical coordinates in $N$-dimensional space. That is, we assign a sign for the radius in the conventional $N$-dimensional spherical
coordinates [19] and parameterise the directions using points only on the other half of the unit hypersphere. This parameterisation is natural for one-dimensional subspaces in $\mathbb{R}^N$ as the directions then uniquely parameterise the linear subspaces almost everywhere and the (signed) radial parameter parameterises the points in the subspace.

Assume that $N \geq 3$ and let $\rho \in \mathbb{R}$ be the radial coordinate and $\phi_1, \phi_2, \ldots, \phi_{N-1}$ be the angular coordinates so that $\phi_i$ takes values between $0$ and $\pi/2$, $\phi_2, \ldots, \phi_{N-2}$ are between $0$ and $\pi$ and $\phi_{N-1}$ is between $0$ and $2\pi$. The Cartesian coordinates $x_i$, $i = 1, \ldots, N$ are then defined as
\[
x_1 = \rho \cos(\phi_1), \\
x_i = \rho \cos(\phi_i) \prod_{k=1}^{i-1} \sin(\phi_k), \quad i = 2, 3, \ldots, N-1, \\
x_N = \rho \prod_{i=1}^{N-1} \sin(\phi_i).
\]  
(34)

The inverse transform is
\[
\rho = \text{sign}(x_1) \|x\|, \quad \phi_1 = \arctan\left(\sqrt{\frac{\|x\|^2}{x_1^2} - 1}\right), \\
\phi_i = \arctan\left(\frac{\sum_{k=i+1}^{N} x_k^2}{\text{sign}(x_1)x_i}\right), \quad i = 2, 3, \ldots, N-2, \\
\phi_{N-1} = \text{sign}(x_1)x_N, \text{sign}(x_1)x_{N-1},
\]
(35)

and the volume element is obtained as
\[
|\det \frac{\partial (x_1, x_2, \ldots, x_N)}{\partial (\rho, \phi_1, \phi_2, \ldots, \phi_{N-1})}| d\rho d\phi_1 d\phi_2 \cdots d\phi_{N-1} = |\rho|^{N-1} \prod_{k=1}^{N-2} \sin^{N-k-1}(\phi_k) d\rho d\phi_1 d\phi_2 \cdots d\phi_{N-1}.
\]  
(36)

\section*{B. Proof of Lemma 1}

Proof: It is well known that an orthogonal projection matrix $P$ may be decomposed as $P = UU^T$ where the columns of $U$ form an orthogonal basis of the range of $P$. We hence form an orthonormal basis for the range from the basis $p_i = Pe_i$, $i = 1, \ldots, K$ by using the Gram–Schmidt orthonormalisation procedure. Now, let
\[
\begin{align*}
\mathbf{u}_1 & = \frac{\mathbf{Pe}_1}{\|\mathbf{Pe}_1\|}, \\
\mathbf{u}_2 & = \frac{\mathbf{Pe}_2 - (\mathbf{u}_1^T \mathbf{Pe}_2) \mathbf{u}_1}{\|\mathbf{Pe}_2 - (\mathbf{u}_1^T \mathbf{Pe}_2) \mathbf{u}_1\|}, \\
\mathbf{u}_K & = \frac{\mathbf{Pe}_K - \sum_{i=1}^{K-1} (\mathbf{u}_1^T \mathbf{Pe}_K) \mathbf{u}_i}{\|\mathbf{Pe}_K - \sum_{i=1}^{K-1} (\mathbf{u}_1^T \mathbf{Pe}_K) \mathbf{u}_i\|}.
\end{align*}
\]  
(37)

and let $L = (\mathbf{u}_1 \mathbf{u}_2 \ldots \mathbf{u}_K)$, i.e., the columns of $L$ form an orthonormal basis for the range of $P$, and $P = LL^T$.

We need to show that $L$ is additionally a lower triangular matrix which has a strictly positive diagonal. We note that
\[
\begin{align*}
\mathbf{u}_1 & = \frac{\mathbf{Pe}_1}{\|\mathbf{Pe}_1\|}, \\
\mathbf{u}_2 & = \frac{\mathbf{Pe}_2 - (\mathbf{u}_1^T \mathbf{Pe}_2) \mathbf{u}_1}{\|\mathbf{Pe}_2 - (\mathbf{u}_1^T \mathbf{Pe}_2) \mathbf{u}_1\|}, \\
\mathbf{u}_K & = \frac{\mathbf{Pe}_K - \sum_{i=1}^{K-1} (\mathbf{u}_1^T \mathbf{Pe}_K) \mathbf{u}_i}{\|\mathbf{Pe}_K - \sum_{i=1}^{K-1} (\mathbf{u}_1^T \mathbf{Pe}_K) \mathbf{u}_i\|}.
\end{align*}
\]  
(38)

where the second term in the nominator is zero because the vectors $\mathbf{u}_k$, $k = 1, 2, \ldots, K$ are linearly independent by assumption. It follows that $u_{1k} = 0$, $k = 2, \ldots, K$. Thus, $u_{11}/|u_{11}| = 1 \iff u_{11} > 0$. Similarly,
\[
\mathbf{u}_2 = \frac{\mathbf{Pe}_2 - (\mathbf{u}_1^T \mathbf{Pe}_2) \mathbf{u}_1}{\|\mathbf{Pe}_2 - (\mathbf{u}_1^T \mathbf{Pe}_2) \mathbf{u}_1\|},
\]  
(39)

that implies $u_{2k} = 0$, $k = 3, \ldots, K$ and $u_{22} > 0$. Likewise, $u_{lk} = 0$, and $u_{ll} > 0$ when $k = l + 1, \ldots, K$, hence, $L$ is a lower triangular matrix with strictly positive diagonal.

We still need to show that the representation is unique. Let us assume the contrary, i.e., there are two different lower triangular matrices $L_1 = (\mathbf{u}_1 \mathbf{u}_2 \ldots \mathbf{u}_K)$ and $L_2 = (\mathbf{v}_1 \mathbf{v}_2 \ldots \mathbf{v}_K)$ which have the indicated properties. Then $P = L_1 L_1^T = L_2 L_2^T$ and
\[
\mathbf{Pe}_1 = u_{11} \mathbf{u}_1 = v_{11} \mathbf{v}_1.
\]  
(40)

Since $\mathbf{u}_1$ and $\mathbf{v}_1$ are unit vectors, it follows that $|u_{11}| = |v_{11}|$. The diagonal elements are additionally strictly positive so $u_{11} = v_{11}$ that implies $\mathbf{u}_1 = \mathbf{v}_1$. Similarly, we see that $\mathbf{u}_l = \mathbf{v}_l$, when $l = 2, \ldots, K$, i.e., $L_1 \equiv L_2$ that contradicts the assumption, hence, $L$ must be unique.

\[\square\]