Analytic continuation in an annulus and in a Bernstein ellipse

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December 19, 2018

Abstract

Analytic continuation problems are notoriously ill-posed without additional regularizing constraints, even though every analytic function has a rigidity property of unique continuation from every curve inside the domain of analyticity. In fact, well known theorems, guarantee that every continuous function can be uniformly approximated by analytic functions (polynomials or rational functions, for example). We consider several analytic continuation problems with typical global boundedness constraints. All such problems exhibit a power law precision deterioration as one moves away from the source of data. In this paper we demonstrate the effectiveness of our general Hilbert space-based approach for determining these exponents. The method identifies the “worst case” function as a solution of a linear equation with a compact operator. In special geometries, such as the circular annulus this equation can be solved explicitly. The obtained solution is then used to determine the power law exponent for the analytic continuation from an interval between the foci of a Bernstein ellipse to the entire ellipse. In those cases where such exponents have been determined in prior work our results reproduce them faithfully.

1 Introduction

Analytic continuation is a tempting proposition in view of the uniqueness properties of analytic functions. Unfortunately, locally analyticity is “stored” at an infinite depth within the continuum of function values and can be represented by delicate cancellation properties responsible for the validity of Carleman and Carleman type extrapolation formulas [3, 9, 11]. Adding small errors to the exact values of analytic functions destroys these local properties. Instead we want to accumulate the remnants of analyticity and use global properties of analytic functions to achieve analytic continuation. This is only possible under some additional regularizing constraints, such as global boundedness [5, 2, 16, 18]. In the extreme case, any bounded entire function is a constant by Liouville’s theorem, and the effect of boundedness depends strongly on the geometry of the domain of analyticity. In order to quantify the degree to which analytic continuation is possible we consider uniformly bounded analytic functions that are small on a given curve inside the domain of analyticity or on its boundary.
We then want to quantify how large such a function can possibly be at some point $z$ away from the given curve. It is a general property of analytic functions that the regularization by boundedness is never perfect. In general if the function is known with relative precision $\epsilon$ on a curve in its domain of analyticity, it can be reconstructed only up to relative precision $\epsilon^{\gamma(z)}$, where the exponent $\gamma(z) \in (0, 1)$ decreases to 0, as we move further away from the source of data. How fast $\gamma(z)$ decays depends strongly on the geometry of the domain and data source \[18, 10\]. We believe that such power law transition from well-posedness to practical ill-posedness is a general property of analytic continuation, quantifying the tug-of-war between their rigidity (unique continuation property) and flexibility (as in the Riesz theorem). The lower bounds on $\gamma(z)$ can be obtained by exhibiting bounded analytic functions that are small on a curve $\Gamma$ but not quite as small at a particular extrapolation point. The upper bounds are harder to prove but there is ample literature where such results are achieved \[2, 13, 14, 19, 8, 4, 18\]. In fact, it was observed in \[18\] that upper and lower bounds of the form $\epsilon^{\gamma(z)}$ on the extrapolation error do hold for all geometries. However, with few exceptions the upper and lower bounds do not match. In those examples where they do match \[4, 18\] the optimality of the bounds are concluded a posteriori.

In our recent work \[10\] we have developed a new method of characterizing analytic functions in the upper half-plane $H^+$ attaining the optimal upper bound in terms of a solution of an integral equation of the second kind with compact, positive, self-adjoint operator on $L^2(\Gamma)$. The exponent $\gamma(z)$ can be expressed in terms of the rates of exponential decay of eigenvalues of the integral operator and exponential growth of eigenfunctions at the extrapolation point $z \in H^+$. Alternatively, the exponent $\gamma(z)$ can be read off the explicit solution of the integral equation in cases where such an explicit solution is available. In the case of the upper half-plane the set of admissible functions, assumed to be the Hardy space $H^2(H^+)$ can be described in terms of their values on the boundary of $H^+$ by the Cauchy representation formula. Formulating the error maximization problem in terms of quadratic constraints and real-linear objective functional permits us to use convex duality methods to obtain criteria of optimality in the form of a linear integral equation. In this paper we consider functions analytic in an annulus. In this case a similar approach is possible, since such functions can be represented by their Fourier coefficients on the unit circle and the constraints can be formulated in terms of quadratic forms on the Fourier coefficients. The resulting linear system describing optimality can be solved and the exponent $\gamma(z)$ and the maximizing analytic function achieving it can be computed explicitly.

The second part of the paper is a somewhat unexpected application of the annulus result to the analytic continuation problem in a Bernstein ellipse analyzed in \[4\]. Since the annulus is not conformally equivalent to the ellipse one would not expect a relation. Instead, the general results of \[10\] should apply after a conformal map of the ellipse onto the upper half-plane. However, the conformal mapping between the ellipse and the upper half-plane is complicated \[17, 11\] and so would be the image of the interval $[-1, 1]$ between the foci of the ellipse. Hence, there would be little hope of solving the resulting integral equation explicitly. The trick we use, inspired by \[4\], is to map the Bernstein ellipse cut along $[-1, 1]$ onto the annulus using the inverse of the much simpler Joukowski function. The functions analytic in the ellipse are distinguished from the functions analytic in the cut ellipse by continuity along the cut. After the conformal transformation the image of functions analytic in the entire
ellipse would consist of functions analytic in the annulus with a reflection symmetry on the unit circle. This symmetry constraint can be expressed as a linear relation between the Fourier coefficients. Our variational approach based on convex duality can easily incorporate linear constraints, and the resulting problem on the annulus can also be solved explicitly. We then recover the exact same optimal exponent \( \gamma(z) \) obtained in [4] by a different method when \( z \in \mathbb{R} \). Our result extends the formula for the exponent \( \gamma(z) \) to all points \( z \) inside the ellipse.

2 Preliminaries

We fix \( 0 < \rho < 1 \) and set

\[
A_\rho = \{ \xi \in \mathbb{C} : \rho < |\xi| < 1 \} \\
\Gamma_l = \{ \xi \in \mathbb{C} : |\xi| = l \}, \quad \forall l > 0 \tag{2.1}
\]

Recall the definition of the Hardy space \( H^2(A_\rho) \) of analytic functions in \( A_\rho \): they satisfy

\[
\sup_{\rho < x < 1} \left( \int_0^{2\pi} |f(xe^{it})|^2 dt \right)^{\frac{1}{2}} < \infty \tag{2.2}
\]

It is known (cf. [15, 12]) that such functions have \( L^2 \) boundary data, i.e. are of class \( L^2 \) on the boundary of \( A_\rho \). The quantity \( (2.2) \) defines a norm in \( H^2(A_\rho) \), however we will consider an equivalent norm given by the boundary data

\[
\|f\|_{H^2(A_\rho)} := \|f\|_{H^2(A_\rho)}^2 = \|f\|_{L^2(\Gamma_\rho)}^2 + \|f\|_{L^2(\Gamma_\rho)}^2
\]

We recall that for a domain \( \Omega \) the space \( H^\infty(\Omega) \) is the space of analytic and bounded functions in \( \Omega \) with the norm \( \|f\|_{H^\infty(\Omega)} = \sup_{\Omega} |f| \).

It is clear that \( H^\infty(A_\rho) \subset H^2(A_\rho) \), moreover the equivalence of the above \( H^2 \)-norm with the quantity in \( (2.2) \) shows that there is a constant \( c > 0 \) (depending on \( \rho \)) such that

\[
\|f\|_{H^2} \leq c\|f\|_{H^\infty(A_\rho)} \tag{2.3}
\]

**Notation:** Let us write \( A_n \sim B_n \) as \( n \to \infty \) if \( \lim_{n \to \infty} A_n/B_n = 1 \). Further, let us use the notation \( A \lesssim B \) if there is a constant \( c > 0 \) such that \( A \leq cB \). Finally, let \( A \asymp B \) if \( A \lesssim B \) and \( B \lesssim A \). Throughout this paper all these implicit constants will be independent of \( \epsilon \).

3 Main Results

**Theorem 3.1 (Annulus).** Let \( f \in H^2(A_\rho) \) and \( r \in (\rho, 1) \). Assume \( \|f\|_{H^2} \leq 1 \), \( \|f\|_{L^2(\Gamma_r)} \leq \epsilon \) and let \( z \in A_\rho \setminus \Gamma_r \) be fixed. Then \( \exists C > 0 \) independent of \( \epsilon \) s.t.

\[
|f(z)| \leq C\epsilon^{\gamma(z)} \quad \forall \epsilon > 0 \text{ small enough} \tag{3.1}
\]

where the exponent \( \gamma(z) \in (0, 1) \) is given by
\[ \gamma(z) = \begin{cases} \frac{\ln|z|}{\ln r} & \text{if } r < |z| < 1 \\ \frac{\ln(|z|/\rho)}{\ln(r/\rho)} & \text{if } \rho < |z| < r \end{cases} \] (3.2)

Moreover, the bound (3.1) is asymptotically (in \( \epsilon \)) optimal and the worst case function attaining the bound (up to a constant independent of \( \epsilon \)) is

\[ W(\xi) = \epsilon^{2-\gamma(z)} \sum_{n \in \mathbb{Z}} \frac{\overline{\xi}^n}{\epsilon^2(\rho^{-1} + \rho^2n) + \rho^2n\xi^n}, \quad \xi \in A_\rho \] (3.3)

In addition \( W \) is analytic in the closure of \( A_\rho \) and \( \|W\|_{H^\infty(\overline{A_\rho})} \) is bounded uniformly in \( \epsilon \).

It is a little surprising that the worst case function, which was required to be analytic in \( A_\rho \) is in fact analytic in a larger annulus (cf. (3.14) and Section 4) \( \{ \xi \in \mathbb{C} : |z_\rho^*| < |\xi| < |z_1^*| \} \), where \( z_1^* = 1/\overline{z} \) is the point symmetric to \( z \) w.r.t the circle \( \Gamma_1 \) and \( z_\rho^* = \rho^2/\overline{z} \) is the point symmetric to \( z \) w.r.t the circle \( \Gamma_\rho \). In particular, \( W \in H^\infty(A_\rho) \). Hence, \( W(\zeta) \) also maximizes \( |W(z)| \), asymptotically, as \( \epsilon \to 0 \), if the constraints were given in \( H^\infty \) and \( L^\infty \), instead of \( H^2 \) and \( L^2 \), respectively.

**Theorem 3.2** (Annulus with symmetry). Let \( f \in H^2(A_\rho) \) and \( r = \sqrt{\rho} \). Assume \( \|f\|_{H^2} \leq 1 \), \( \|f\|_{L^2(\Gamma_\rho)} \leq \epsilon \) and \( f(\zeta) = f(\xi) \) for all \( |\xi| = r \). Let \( z \) satisfying \( r < |z| < 1 \) be fixed. Then \( \exists C > 0 \) independent of \( \epsilon \) s.t.

\[ |f(z)| \leq C\epsilon^{\frac{\ln|z|}{\ln r}} \quad \forall \epsilon > 0 \text{ small enough} \] (3.4)

Moreover, this bound is asymptotically (in \( \epsilon \)) optimal and the worst case function attaining the bound (up to a constant independent of \( \epsilon \)) is

\[ W(\zeta) = \epsilon^{2-\gamma(z)} \sum_{n \in \mathbb{Z}} \frac{\overline{\xi}^n + (\rho/\overline{\xi})^n}{\epsilon^2(1 + \rho^2n) + \rho^2n\xi^n}, \quad \xi \in A_\rho \] (3.5)

In addition \( W \) is analytic in the closure of \( A_\rho \) and \( \|W\|_{H^\infty(\overline{A_\rho})} \) is bounded uniformly in \( \epsilon \).

Let \( E_R \) be the open ellipse with foci at \( \pm 1 \) and the sum of semiminor and semimajor axes equal to \( R > 1 \). The axes lengths of such an ellipse are therefore \( (R \pm R^{-1})/2 \).

**Theorem 3.3** (Ellipse). Let \( F \in H^\infty(E_R) \) with \( \|F\|_{H^\infty(E_R)} \leq 1 \) and

\[ \int_{-1}^{1} |F(x)|^2 \frac{dx}{\sqrt{1 - x^2}} \leq \epsilon^2 \] (3.6)

let \( z \in E_R \setminus [-1, 1] \) be an extrapolation point, then \( \exists C > 0 \) independent of \( \epsilon \) s.t.

\[ |F(z)| \leq C\epsilon^{\alpha(z)} \quad \forall \epsilon > 0 \text{ small enough} \] (3.7)

where
\begin{align*}
\alpha(z) & = 1 - \frac{\ln |\kappa(z)|}{\ln R} \in (0, 1), \quad \kappa(z) = z + (z - 1)\sqrt{\frac{z + 1}{z - 1}} \quad (3.8)
\end{align*}

Moreover, the bound (3.7) is asymptotically (in \(\epsilon\)) optimal and the worst case function attaining the bound (up to a constant independent of \(\epsilon\)) is

\begin{align*}
W_{\text{ell}}(\zeta) = \epsilon^{2-\alpha(z)} \sum_{n=1}^{\infty} \frac{\kappa^n(z) + \kappa^{-n}(z)}{\epsilon^2(R^{2n} + R^{-2n}) + 1} T_n(\zeta) \quad (3.9)
\end{align*}

where \(T_n\) is the Chebyshev polynomial of degree \(n\): \(T_n(x) = \cos(n \cos^{-1} x)\) for \(x \in [-1, 1]\).

Several remarks are now in order.

(i) \(\kappa(\zeta)\) is the branch of an inverse of the Joukowski map, that is analytic in the slit ellipse \(E_R \setminus [-1, 1]\) and satisfies the inequalities \(1 < |\kappa(\zeta)| < R\) (cf. Section 6).

(ii) Because Theorem 3.3 is proved by reducing it to the setting of Theorem 3.2 upon the application of the Joukowski map \(J\), analogous result can be formulated for a weighted \(H^2\) space of the ellipse \(E_R\) (instead of \(H^\infty(E_R)\)), where the weight is related to the Joukowski map \(J\).

(iii) In \cite{4} extrapolation problem for \(F \in H^\infty(E_R)\) from the data given on \([-1, 1]\) was considered, where the data was assumed to be discrete and known with a pointwise error (unlike the continuum weighted \(L^2\) error in the above case) and the extrapolation point \(z\) was assumed to be restricted to the interval \([1, \frac{R+R^{-1}}{2}]\) of the real axis. The least squares polynomial approximant of a specific degree was shown to satisfy the estimate (up to log factors)

\[ |F(z) - e(z)| = \mathcal{O}(\epsilon^{\alpha(z)}). \]

Further, the worst case function

\begin{align*}
g(\zeta) = \sum_{n \leq K} R^{-n} T_n(\zeta), \quad K = \left\lfloor \ln(1/\epsilon)/\ln R \right\rfloor \quad (3.10)
\end{align*}

could be taken to be a polynomial, having the properties \(\max_{[-1,1]} |g| = \mathcal{O}(\epsilon)\) and \(|g(z)| \gtrsim \epsilon^{\alpha(z)}\). This goes parallel with the worst case function (3.3), since we can ignore the exponentially small factors \(\kappa^{-n}(z)\) and \(R^{-2n}\). Moreover, if we truncate the series at \(K\), the term \(\epsilon^2R^{2n}\) dominates 1 and so we get the function

\[ \epsilon^{-\alpha(z)} \sum_{n \geq K} R^{-n} \left[R^{-1}\kappa(z)\right]^n T_n(\zeta) \]

where the factor in square brackets balances the term \(\epsilon^{-\alpha(z)}\) in front of the sum, showing that functions (3.9) and (3.10) asymptotically agree.
Our goal is to understand how large $|f(z)|$ can be, under the assumptions $\|f\|_{H^2} \leq 1$ and $\|f\|_{L^2(\Gamma_r)} \leq \epsilon$. Let $f \in H^2(A_\rho)$ have the Laurent expansion

$$f(\xi) = \sum_{n \in \mathbb{Z}} f_n \xi^n, \quad \xi \in A_\rho$$

Let us now formulate this maximization problem using only the coefficients $f_n$. It is known (cf. [15]) that the restrictions of $f$ to the boundary circles $\Gamma_1, \Gamma_\rho$ have Fourier expansions given by the above series with $\xi = e^{it}, \xi = \rho e^{it}$ respectively. This allows us to write

$$\|f\|_{H^2}^2 = 2\pi \rho \sum_{n \in \mathbb{Z}} (\rho^{-1} + \rho^{2n}) |f_n|^2 = 2\pi \rho (Qg, g)$$

$$\|f\|_{L^2(\Gamma_r)}^2 = 2\pi r \sum_{n \in \mathbb{Z}} r^{2n} |f_n|^2 = 2\pi r (Dg, g)$$

where we changed the variables

$$g_n = \begin{cases} f_n & n \geq 0 \\ f_n \rho^n & n < 0. \end{cases}$$

and boldface notation $g$ denotes the collection $\{g_n\}_{n \in \mathbb{Z}} \in l^2(\mathbb{Z})$. Further, $(\cdot, \cdot)$ denotes the inner product of $l^2(\mathbb{Z})$ and $D, Q : l^2(\mathbb{Z}) \mapsto l^2(\mathbb{Z})$ are multiplication operators given by

$$(Dx)_n = d_n x_n, \quad (Qx)_n = q_n x_n, \quad n \in \mathbb{Z}, \ x \in l^2(\mathbb{Z})$$

where

$$d_n = \begin{cases} r^{2n} & n \geq 0 \\ (\frac{\xi}{\rho})^{2n} & n < 0 \end{cases}, \quad q_n = \begin{cases} \rho^{-1} + \rho^{2n} & n \geq 0 \\ 1 + \rho^{-2n-1} & n < 0 \end{cases}$$

Clearly these are bounded operators on $l^2$, because $d_n \leq 1$ and $q_n \leq 2$. Moreover, $Q$ is also invertible with bounded inverse on $l^2$ since $q_n \geq 1$. Finally,

$$f(z) = (g, \Lambda), \quad \Lambda_n = \begin{cases} \overline{z}^n & n \geq 0 \\ \frac{\overline{z}^n}{\overline{\rho}^n} & n < 0 \end{cases}$$

Thus we arrive at a convex maximization problem with two quadratic constraints. Since the constraints are invariant with respect to the choice of the constant phase factor for the function $f$, instead of maximizing $|f(z)|$ we consider the equivalent problem of maximizing a real linear functional $\Re f(z)$ and by rescaling we drop the constants $2\pi \rho, 2\pi r$ from (4.1):

$$\begin{cases} \Re (g, \Lambda) \rightarrow \max \\ (Qg, g) \leq 1 \\ (Dg, g) \leq \epsilon^2 \end{cases} \quad (4.2)$$
For every $g$, satisfying (4.2)(b) and (4.2)(c) and for every nonnegative numbers $\mu$ and $\nu$ ($\mu^2 + \nu^2 \neq 0$) we have the inequality
\[
((\mu Q + \nu D)g, g) \leq \mu + \nu \epsilon^2
\]
obtained by multiplying (4.2)(b) by $\mu$ and (4.2)(c) by $\nu$ and adding. Also, for any uniformly positive definite, self-adjoint invertible operator $M$ on $l^2$ we have
\[
\Re(a, b) \leq \frac{1}{2}(Ma, a) + \frac{1}{2}(Ma, a)
\]
valid for all $a, b \in l^2$ (expand $(M(M^{-1}b - a), (M^{-1}b - a)) \geq 0$). The uniform positivity of $M$ ensures that $M^{-1}$ is defined on all of $l^2$. This is an example of convex duality (cf. [6]) applied to the function $x \mapsto \frac{1}{2}(Mx, x)$. Then we also have for $\mu > 0$
\[
\Re(g, \Lambda) \leq \frac{1}{2}((\mu Q + \nu D)^{-1}\Lambda, \Lambda) + \frac{1}{2}(\mu Q + \nu D)g, g)
\]
and therefore,
\[
\Re(g, \Lambda) \leq \frac{1}{2}((\mu Q + \nu D)^{-1}\Lambda, \Lambda) + \frac{1}{2}(\mu + \nu \epsilon^2)
\]
which is valid for every $g$, satisfying (4.2)(b) and (4.2)(c) and all $\mu > 0, \nu \geq 0$. In order for the bound to be optimal we must have equality in (4.3), which holds if and only if
\[
\Lambda = (\mu Q + \nu D)g
\]
giving the formula for optimal vector $g$:
\[
g = (\mu Q + \nu D)^{-1}\Lambda
\]
The goal is to choose the Lagrange multipliers $\mu$ and $\nu$ so that the constraints in (4.2) are satisfied by $g$ given by the above formula.
• We did not consider the case $\mu = 0$, since the operator $(\mu Q + \nu D)^{-1}$ is not defined on all of $l^2$. It is however defined on a dense subspace of $l^2$. Even so, the choice $\mu = 0$ cannot be optimal since the optimal vector $g$ would satisfy $Dg = \frac{1}{\nu} \Lambda$. This equation has no solution in $l^2$, since $g_n = \frac{1}{\nu} \Lambda_n$.
• if $\nu = 0$, then $g = \frac{Q^{-1}\Lambda}{\sqrt{(Q\Lambda, \Lambda)}}$, which does not depend on the small parameter $\epsilon$. This leads to a contradiction, because the second constraint $(Dg, g) \leq \epsilon^2$ is violated if $\epsilon$ is small enough.

Thus we are looking for $\mu > 0, \nu > 0$, so that equalities in (4.2) hold (these are complementary slackness relations in Karush-Kuhn-Tucker conditions):
\[
\begin{cases}
(Q(\mu Q + \nu D)^{-1}\Lambda, (\mu Q + \nu D)^{-1}\Lambda) = 1 \\
(D(\mu Q + \nu D)^{-1}\Lambda, (\mu Q + \nu D)^{-1}\Lambda) = \epsilon^2
\end{cases}
\]
Let $\eta = \frac{\epsilon}{\nu}$ and write $g = \frac{1}{\nu} g_n$, where
\[ g_\eta = (\eta Q + D)^{-1} \Lambda \] (4.5)

let us solve the second equation for \( \mu \):

\[ \nu^2 = \frac{1}{\epsilon^2} (D g_\eta, g_\eta) \]

and substitute it into the first one:

\[ \Phi(\eta) := \frac{(Q g_\eta, g_\eta)}{(D g_\eta, g_\eta)} = \frac{1}{\epsilon^2} \] (4.6)

It is straightforward to check that one can write

\[ \Phi(\eta) = \sum_{n \in \mathbb{Z}} \frac{p_n}{(\eta + \lambda_n)^2} =: \frac{N(\eta)}{D(\eta)} \] (4.7)

where

\[ \lambda_n = \frac{\nu^{2n}}{\rho^{-1} + \rho^{2n}}, \quad \pi_n = \frac{\nu^{2n}|z|^{2n}}{(\rho^{-1} + \rho^{2n})^2}, \quad p_n = \frac{|z|^{2n}}{\rho^{-1} + \rho^{2n}} \] (4.8)

**Lemma 4.1.** The equation (4.6) has a unique solution \( \eta \) (when \( \epsilon \) is sufficiently small) which satisfies \( \eta \simeq \epsilon^2 \).

**Proof.** Uniqueness of the solution will follow if we show that \( \Phi(\eta) \) is monotone. Using the formula

\[ g'_\eta = -(\eta Q + D)^{-1} Q g_\eta \]

we can write the numerator of \( \Phi'(\eta) \) as

\[ 2 \Re \left[ (Q g'_\eta, g_\eta) (D g_\eta, g_\eta) - (Q g_\eta, g_\eta) (D g'_\eta, g_\eta) \right] \]

For the ease of notation let us drop the subscript \( \eta \) from the notation. From the identity \( D(\eta Q + D)^{-1} = \text{Id} - \eta Q (\eta Q + D)^{-1} \), the expression in the square brackets can be simplified to

\[ (Q g, g)^2 - ((\eta Q + D) g, g) (Q(\eta Q + D)^{-1} Q g, g) =: (x, Ay)^2 - (Ax, x)(y, Ay) \]

where \( A = (\eta Q + D)^{-1}, \ x = Q g \) and \( y = A^{-1} g \). Since \( A \) is self-adjoint and positive definite, real part of the above expression is non-positive by the Cauchy-Schwartz inequality. Thus, \( \Phi(\eta) \) is non-increasing.

Now let us turn to the existence. First, observe that
\[
\lim_{\eta \to \infty} \Phi(\eta) = \frac{(Q^{-1}\Lambda, \Lambda)}{(DQ^{-1}\Lambda, Q^{-1}\Lambda)} < \infty
\]

On the other hand invoking Theorems 7.1 and 7.2 (in the Appendix) we find

\[
\Phi(\eta) \approx \frac{1}{\eta}, \quad \text{for } \eta \text{ small enough}
\]

In particular \( \lim_{\eta \to 0} \Phi(\eta) = \infty \). Thus, \( \frac{1}{\eta} \) lies between \( \Phi(0) \) and \( \Phi(\infty) \), if \( \epsilon \) is small enough, and this establishes existence of the solution to (4.6). Moreover, (4.9) shows that that solution must satisfy \( \eta \approx \epsilon^2 \).

Now the upper bound (4.4) reads

\[
|f(z)| \leq \frac{\epsilon(g_\eta, \Lambda)}{2\sqrt{(Dg_\eta, g_\eta)}} + \frac{\sqrt{(Dg_\eta, g_\eta)}}{2\epsilon} (\eta + \epsilon^2)
\]

Motivated by Lemma 4.1 let us take \( \eta = \epsilon^2 \), then the upper bound becomes (we suppress \( \epsilon \) dependence from the notation)

\[
|f(z)| \leq \frac{\epsilon}{2\sqrt{(Dg, g)}} [(g, \Lambda) + 2(Dg, g)] \leq \frac{3\epsilon}{2} \frac{(g, \Lambda)}{\sqrt{(Dg, g)}}
\]

where in the last step we used the inequality \( (Dg, g) \leq (g, \Lambda) \), which holds true since \( Q \) is positive. Introducing

\[
u(\xi) = \sum_{n \in \mathbb{Z}} u_n \xi^n, \quad u_n = \frac{\psi^n}{\epsilon^2(\rho^{-1} + \rho^{2n}) + r^{2n}} \quad \xi \in A_{\rho}
\]

we rewrite the upper bound as

\[
|f(z)| \leq 3\sqrt{\frac{\pi r}{2}} \frac{\epsilon u(z)}{\|u\|_{L^2(\Gamma_r)}}
\]

It remains to analyze the asymptotic behavior of the right-hand side of (4.11). We first observe that

\[
u(\xi) = \sum_{n \geq 0} a_n (\zeta \xi)^n + \sum_{n \geq 1} b_n \left( \frac{\rho^2}{\zeta \xi} \right)^n
\]

where

\[
a_n^{-1} = \epsilon^2(\rho^{-1} + \rho^{2n}) + r^{2n}, \quad b_n^{-1} = \epsilon^2(1 + \rho^{2n-1}) + r^{-2n} \rho^{2n}
\]

in particular \( a_n, b_n \leq \epsilon^{-2} \) and so \( u(\xi) \) is analytic in the annulus \( \{|z_0^\rho| < |\xi| < |z_1^\rho|\} \), where \( z_1^\star = 1/\zeta \) is the point symmetric to \( z \) w.r.t. the circle \( \Gamma_1 \) and \( z_0^\star = \rho^2/\zeta \) is the point symmetric to \( z \) w.r.t. the circle \( \Gamma_\rho \). This annulus contains \( A_\rho \) so \( u \) is analytic in the closure \( \overline{A_\rho} \) and therefore \( u \in H^\infty(A_\rho) \). Note that
\[ \|u\|_{L^2(\Gamma_r)}^2 = 2\pi r \sum_{n \in \mathbb{Z}} \frac{\pi_n}{(\varepsilon^2 + \lambda_n)^2} = 2\pi r D(\varepsilon^2) \approx \varepsilon^{2(\gamma(z)-1)} \quad (4.12) \]

where \( \gamma(z) \) is given by \((3.2)\) and the last step follows from Theorem \(7.1\). On the other hand

\[ u(z) = \sum_{n \in \mathbb{Z}} \frac{p_n}{\varepsilon^2 + \lambda_n} \approx \varepsilon^{2(\gamma(z)-1)} \quad (4.13) \]

where the last assertion follows from analogous considerations to the proof of Theorem \(7.2\). Thus the upper bound \((4.11)\) now reads

\[ |f(z)| \lesssim \varepsilon^{\gamma(z)}, \quad \text{for } \varepsilon \text{ small enough} \]

which proves \((3.1)\).

Finally let us prove optimality of the above bound. Take \( \rho \leq |\xi| \leq 1 \), following the steps of the proof of Theorem \(7.1\) we find

\[ |u(\xi)| \lesssim \varepsilon^{\gamma(z) - 2} \]

where the implicit constant is independent of \( \xi \) and \( \varepsilon \). Taking into account \((4.12)\) we get

\[ \frac{\varepsilon |u(\xi)|}{\|u\|_{L^2(\Gamma_r)}} \lesssim 1, \quad \forall \xi \in \overline{A}_\rho \]

Thus, the function \( \xi \mapsto \varepsilon \frac{u(\xi)}{\|u\|_{L^2(\Gamma_r)}} \) has \( L^2(\Gamma_r) \)-norm equal to \( \varepsilon \), has \( H^\infty(A_\rho) \)-norm (and hence also \( H^2(A_\rho) \)-norm, due to \((2.3)\)) bounded uniformly in \( \varepsilon \), and reaches the bound \((4.11)\) (or equivalently \((3.1)\)), up to a constant independent of \( \varepsilon \), at the point \( z \). Therefore, this describes the worst case function, or equivalently by replacing \( \|u\|_{L^2(\Gamma_r)} \) with its asymptotic relation we obtain the maximizer function

\[ W(\xi) = \varepsilon^{2-\gamma(z)} u(\xi), \quad \xi \in A_\rho \quad (4.14) \]

This concludes the proof of Theorem \(3.1\).

5 Proof of Theorem \(3.2\)

Let us use the boldface notation \( \mathbf{f} \) to denote the collection of all Laurent coefficients of \( f \) with nonnegative indices: \( \{ f_n \}_{n \in \mathbb{N}_0} \). Using similar considerations to Section \(4\) we arrive at the maximization problem

\[
\begin{align*}
\begin{cases}
\Re(\mathbf{f}, \Lambda) & \longrightarrow \max \\
(Qf, f) & \leq 1 \\
(Df, f) & \leq \varepsilon^2
\end{cases}
\end{align*}
\]

where \((\cdot, \cdot)\) denotes the inner product of \( l^2(\mathbb{N}_0) \), \( D, Q : l^2(\mathbb{N}_0) \mapsto l^2(\mathbb{N}_0) \) are multiplication operators given by \((Dx)_n = d_n x_n\), \((Qx)_n = q_n x_n\) for \( n \in \mathbb{N}_0 \) with
\[ d_n = \begin{cases} 
1 & n = 0 \\
 r^{2n} + (\frac{\rho}{r})^{2n} & n > 0 
\end{cases}, \quad q_n = \begin{cases} 
1 + \rho^{-1} & n = 0 \\
 (1 + \rho^{-1})(1 + \rho^{2n}) & n > 0 
\end{cases} \]

and

\[ \Lambda_n = \begin{cases} 
1 & n = 0 \\
 \frac{\bar{z}^n + \rho^n}{\lambda} & n > 0 
\end{cases} \]

Analogously, we can obtain the bound

\[ |f(z)| \leq \frac{3\epsilon}{2} \frac{u(z)}{\|u\|_{L^2(\Gamma_r)}}, \quad u(\xi) = \sum_{n \in \mathbb{Z}} u_n \xi^n, \quad \xi \in \mathbb{A}_\rho \]

where

\[ u_n = \frac{\bar{z}^n + (\rho/\bar{z})^n}{\epsilon^2(1 + \rho^{-1})(1 + \rho^{2n}) + r^{2n} + (\rho/r)^{2n}} \]

Observe that \( u_{-n} = \rho^n u_n \) for \( n \geq 0 \). To have a simpler expression for the worst case function let us replace \( 1 + \rho^{-1} \) in the denominator of the above formula by 2. This changes the asymptotic behavior of \( u_n \) by a multiplicative constant, which changes just the constant in the upper bound for \( |f(z)| \) that multiplies \( \epsilon^\gamma \), but we don’t keep track of this constant and are rather interested in the exponent \( \gamma \). Let us now use the assumption \( \rho = r^2 \), so that

\[ u_n = \frac{\bar{z}^n + (r/\bar{z})^n}{\epsilon^2(1 + r^{-1})(1 + r^{2n}) + r^{2n} + (r/r)^{2n}} \]

Recall that \(|z| > r\), it is straightforward to check that \( u \) is analytic in \( \{|z|\rho < |\xi| < |z|^{-1}\} \supset \mathbb{A}_\rho \), hence \( u \in H^\infty(\mathbb{A}_\rho) \). We then observe that

\[ \frac{1}{2\pi r} \|u\|^2_{L^2(\Gamma_r)} = |u_0|^2 + \sum_{n=1}^{\infty} \frac{\pi_n}{(\epsilon^2 + \lambda_n)^2}, \quad \lambda_n = \frac{r^{2n}}{(1 + r^{-4n})} \]

and

\[ \pi_n := \frac{(r|z|)^{2n}|1 + (r/|z|)^{2n}|^2}{2(1 + r^{-4n})^2} \sim \frac{1}{2} (r|z|)^{2n}, \quad n \to +\infty \]

Note that \( \lambda_n \) and \( \pi_n \) have the same asymptotic behavior as the ones defined in \((4.8)\), hence the same proof of Section \( 4 \) carries over and we obtain

\[ \|u\|_{L^2(\Gamma_r)} \approx \epsilon^{\frac{\ln(|z|/r)}{\ln r}}, \quad u(z) \approx \epsilon^{\frac{\ln(|z|/r)}{\ln r}} \]

consequently

\[ |f(z)| \lesssim \epsilon^{\frac{\ln|z|}{\ln r}} \]
Finally, in the same way we see that the function \( \xi \mapsto \frac{\epsilon}{\|u\|_{L^2(\Gamma_r)}} u(\xi) \) is of class \( H^2(A_\rho) \), satisfies the symmetry assumption, has \( L^2(\Gamma_r) \)-norm equal to \( \epsilon \), has \( H^2(A_\rho) \) and \( H^\infty(A_\rho) \) norms bounded uniformly in \( \epsilon \), and reaches the bound \( (4.11) \) up to a constant independent of \( \epsilon \), at the point \( z \). Therefore, this describes the worst case function, or equivalently, replacing \( \|u\|_{L^2(\Gamma_r)} \) with its asymptotic relation we obtain the maximizer function

\[
W(\xi) = \epsilon^{2 - \frac{\ln |z|}{\ln R}} u(\xi), \quad \xi \in A_{\rho}
\]

6 Proof of Theorem 3.3

The Joukowski map \( J(\omega) = \frac{\omega + \omega^{-1}}{2} \) maps the annulus \( \{ R^{-1} < |\omega| < R \} \) onto \( E_R \) in 2-1 fashion, meaning that each point in \( E_R \) has two preimages in the annulus (note that \( J(\omega) = J(\omega^{-1}) \)). Moreover, the unit circle gets mapped onto \([-1, 1] \subset E_R \) under \( J \). So the function \( f(\xi) := F(J(\xi)) \) is analytic in \( A_\rho \) (cf. \( (2.1) \)), with \( \rho = R^{-2} \) and satisfies the symmetry \( f_{-n} = f_{n\rho^n} \) for any \( n \geq 0 \). Now

\[
\|f\|_{L^2(F_{\frac{1}{R}})}^2 = \frac{1}{R} \int_0^{2\pi} |F(J(e^{it}))|^2 dt = \frac{1}{R} \int_0^{2\pi} |F(\cos t)|^2 dt = \frac{2}{R} \int_{-1}^1 |F(x)|^2 \sqrt{1 - x^2} dx
\]

Note that \( J : \{ 1 < |\omega| < R \} \mapsto E_R \setminus [-1, 1] \) is bijective, similarly so is \( J : \{ R^{-1} < |\omega| < 1 \} \mapsto E_R \setminus [-1, 1] \). Hence \( J^{-1} \) has two branches analytic in \( E_R \setminus [-1, 1] \), one taking values that have modulus larger than 1 and the other one - smaller.

Let \( \zeta \in E_R \setminus [-1, 1] \), consider the branch of \( J^{-1} \) that takes values in the annulus \( \{ 1 < |\omega| < R \} \), for that we solve \( J(\omega) = \zeta \), i.e. \( \omega^2 - 2\zeta \omega + 1 = 0 \) and obtain

\[
\omega = \zeta + (\zeta - 1) \sqrt{\frac{\zeta + 1}{\zeta - 1}} =: \kappa(\zeta)
\]

where \( \sqrt{\cdot} \) denotes the principal branch of square root. Indeed, this is the required branch since as a function of \( \zeta \) it is analytic in \( E_R \setminus [-1, 1] \) and when \( \zeta \) is real and larger than one we see that \( \omega = \zeta + \sqrt{\zeta^2 - 1} > \zeta > 1 \), so that \( (6.1) \) is the branch of \( J^{-1} \) taking values in \( \{ 1 < |\omega| < R \} \). Thus, the extrapolation point corresponding to \( \zeta \) in the annulus \( A_\rho \) is \( z = \frac{\omega}{R} \) and \( |z| > \frac{1}{R} =: r \).

Because of the assumptions on \( F \), we can apply Theorem 3.2 (with \( r = R^{-1} \)) and conclude

\[
|f(z)| \leq C \epsilon^{\frac{\ln |z|}{\ln R}} \quad \Rightarrow \quad |F(\zeta)| \leq C \epsilon^{\alpha(\zeta)}
\]

where \( \alpha \) is given by \( (3.8) \). Let us now rewrite the worst case function \( W \) for the annulus \( A_\rho \) (cf. \( (3.3) \));

\[
\epsilon^{\frac{\ln |z|}{\ln R}} W(\xi) = \frac{2}{2\epsilon^2 + 1} + \sum_{n=1}^{\infty} \frac{(R\xi)^n + (R\xi)^{-n}}{\epsilon^2(R^{2n} + R^{-2n}) + 1} T_n(J(R\xi))
\]

where \( T_n \) is the Chebyshev polynomial of degree \( n \), and we used the identity
\[ T_n(J(\xi)) = \frac{\xi^n + \xi^{-n}}{2}, \quad \forall \xi \neq 0 \]

which obviously holds for \( \xi = e^{it}, \ t \in [0, 2\pi] \), and for general \( \xi \neq 0 \) we use that both sides of the equality are analytic functions. Since \( \frac{2}{2^\xi+1} = \Theta(1) \) we can drop this term from the above sum and the resulting function

\[ \tilde{W}(\xi) := e^{2-\ln|z|} \sum_{n=1}^{\infty} \frac{(R\xi)^n + (R\xi)^{-n}}{e^2(R^{2n} + R^{-2n}) + 1} T_n(J(R\xi)) \]

still will be a worst case function for \( A_\rho \) so we consider \( W_{ell} \) defined by (3.9), i.e. \( \tilde{W}(\xi) = W_{ell}(J(R\xi)) \). Clearly \( W_{ell} \) is analytic in \( E_R \), because the series of Chebyshev polynomials in its definition converges uniformly on compact subsets of \( E_R \), further

\[ \int_{-1}^{1} |W_{ell}(x)|^2 \frac{dx}{\sqrt{1-x^2}} = \frac{R}{2} \| \tilde{W} \|^2_{L^2(\Gamma_1)} = \Theta(e^2) \]

\[ \| W_{ell} \|_{H^\infty(E_R)} \leq \| \tilde{W} \|_{H^\infty(A_\rho)} = \Theta(1) \]

\[ W_{ell}(\zeta) = \tilde{W}(z) \geq e^{-\ln|\zeta|} = \epsilon^{\alpha(\zeta)} \]

Acknowledgments. This material is based upon work supported by the National Science Foundation under Grant No. DMS-1714287.

7 Appendix

The goal of this section is to understand the behavior of \( \Phi(\eta) = \frac{N(\eta)}{D(\eta)} \), defined by (4.7), for small positive values of \( \eta \). This is done in the two theorems below, where the numerator and the denominator are analyzed separately.

Theorem 7.1. Let \( D(\eta) \) be defined by (4.7), then

\[ D(\eta) \cong \eta^{\gamma(z)-1}, \quad \text{for } \eta \text{ small enough} \]

where \( \gamma(z) \) is given by (3.2) and the implicit constants do not depend on \( \eta \).

Proof. Introduce the switch-over index \( J = J(\eta) \) defined by

\[ \lambda_n \geq \eta \quad \forall \ 0 \leq n \leq J \]

\[ \lambda_n < \eta \quad \forall \ n > J \]

where \( \lambda_n \) is given by (4.8). Let us assume that \( \eta \) is small enough, which implies that \( J \) is large enough. Consider the first piece of the sum in the definition of \( D(\eta) \), corresponding to nonnegative \( n \), it is easy to see that
\[
\sum_{n \geq 0} \frac{\pi_n}{(\eta + \lambda_n)^2} \approx \sum_{n=0}^{J} \frac{\pi_n}{\lambda_n^2} + \frac{1}{\eta^2} \sum_{n=J+1}^{\infty} \pi_n
\]

Observe that
\[
\sqrt{\frac{\pi_{n+1}}{\pi_n}} = r|z|\frac{\rho^{-1} + \rho^{2n}}{\rho^{-1} + \rho^{2n+2}}
\]
and the limit of the above expression as \( n \to +\infty \) is equal to \( r|z| < 1 \), hence \( \exists \sigma_1, \sigma_2 \in (0, 1) \) (depending on \( z \) and \( r \)) s.t.
\[
\sigma_1 \pi_n \leq \pi_{n+1} \leq \sigma_2 \pi_n, \quad \forall n \geq 0 \text{ large enough} \quad (7.3)
\]
which in particular implies
\[
\sum_{n=J+1}^{\infty} \pi_n \leq \pi_{J+1} \sum_{n=0}^{\infty} \sigma_2^n = \frac{\pi_{J+1}}{1 - \sigma_2}
\]
and we conclude
\[
\sum_{n=J+1}^{\infty} \pi_n \approx \pi_{J+1}
\]

We also note that
\[
\frac{\pi_n}{\lambda_n^2} = q^n, \quad q = \frac{|z|^2}{r^2} \quad (7.4)
\]
let us assume first \( q > 1 \), then we have the comparison \( \frac{\pi_n}{\lambda_n^2} \leq \frac{1}{q} \frac{\pi_{n+1}}{\lambda_{n+1}^2} \) and therefore
\[
\sum_{n=0}^{J} \frac{\pi_n}{\lambda_n^2} \approx \frac{\pi_J}{\lambda_J^2}
\]

Now consider the sum corresponding to negative values of \( n \): again introduce the switch-over index \( I = I(\eta) \)
\[
\begin{align*}
\lambda_{-n} &\geq \eta & 1 \leq n \leq I \\
\lambda_{-n} &< \eta & n > I \quad (7.5)
\end{align*}
\]
After analogously treating the term \( \sum_{n \geq 1} \frac{\pi_{-n}}{(\eta + \lambda_{-n})^2} \), we conclude that
\[
D(\eta) \approx \frac{1}{\eta^2} \left( \pi_{J+1} + \pi_{-(I+1)} \right) + \frac{\pi_J}{\lambda_J^2} + \mathcal{O}(1) \quad (7.6)
\]
Clearly \( \lambda_{n+1}/\lambda_n \geq r^2 \) for any \( n \geq 0 \), hence from the definition of \( J \) we get \( \lambda_J \approx \eta \), then (7.3) implies that \( \pi_{J+1} \approx \pi_J \) and analogously \( \pi_{-(I+1)} \approx \pi_{-I} \). Thus (7.6) reads
\[ D(\eta) \cong \frac{1}{\eta^2} (2\pi J + \pi_{-I}) + O(1) \]  \hspace{1cm} (7.7)

from the asymptotic relations

\[ \pi_n \sim \rho^2 (r|z|)^{2n}, \quad \pi_{-n} \sim \left(\frac{\rho^2}{r|z|}\right)^{2n}, \quad n \to +\infty \]

\[ \lambda_n \sim \rho^n, \quad \lambda_{-n} \sim \left(\frac{\rho}{r}\right)^{2n} \]

we conclude that for small enough \( \epsilon \)

\[ \pi_J \cong (r|z|)^{2J}, \quad \pi_{-J} \cong \left(\frac{\rho^2}{r|z|}\right)^{2I} \]

and taking into account the relations \( \lambda_J \cong \eta \) and \( \lambda_{-I} \cong \eta \) we find

\[ J = \frac{\ln \eta}{2\ln r} + O(1), \quad I = \frac{\ln \eta}{2\ln(\rho/r)} + O(1) \]  \hspace{1cm} (7.8)

performing some simplifications we get

\[ \pi_J \cong \eta^{1 + \frac{\ln(|z|)}{\ln r}}, \quad \pi_{-I} \cong \eta^{1 + \frac{\ln(|z|/\rho)}{\ln(\rho/r)}} \]

but then

\[ \frac{\pi_{-I}}{\eta^2} \cong \eta^{\frac{\ln(|z|/\rho)}{\ln(\rho/r)}} = o(1) \]

where the last step follows from the assumption that \( q > 1 \), i.e. \( |z| > r \) and therefore the exponent of \( \eta \) is positive, and so this term can be absorbed in \( O(1) \) in (7.7). Hence,

\[ D(\eta) \cong \eta^{\frac{\ln |z|}{\ln r} - 1} + O(1) \Rightarrow D(\eta) \cong \eta^{\frac{\ln |z|}{\ln(\rho/r)} - 1} \]

where \( O(1) \) was absorbed into the implicit constants of \( \cong \). Similarly, when \( q < 1 \) we find

\[ D(\eta) \cong \eta^{\frac{\ln |z|}{\ln(\rho/r)} - 1}, \]

combining these we conclude the proof.

\[ \square \]

**Theorem 7.2.** Let \( N(\eta) \) be defined by (4.7), then

\[ N(\eta) \cong \eta^{\gamma(z)-2}, \quad \text{for } \eta \text{ small enough} \]

where \( \gamma(z) \) is given by (3.2) and the implicit constants do not depend on \( \eta \).
Proof. Let $I, J$ be defined by (7.5) and (7.2), then

$$\sum_{n \geq 0} \frac{p_n}{(\eta + \lambda_n)^2} \approx \sum_{n=0}^{J} \frac{p_n}{\lambda_n^2} + \frac{1}{\eta^2} \sum_{n=J+1}^{\infty} p_n \approx \frac{p_J}{\eta^2} + \sum_{n=0}^{J} \frac{p_n}{\lambda_n^2}$$

and similarly

$$\sum_{n \geq 1} \frac{p_{-n}}{(\eta + \lambda_{-n})^2} \approx \frac{p_{-1}}{\eta^2} + \sum_{n=1}^{l} \frac{p_{-n}}{\lambda_{-n}^2}$$

Note that, as $n \to +\infty$ we have

$$p_n \sim \frac{1}{\rho} \left( \frac{|z|}{\rho^2} \right)^{2n}, \quad p_{-n} \sim \left( \frac{r^2}{|z|\rho} \right)^{2n}$$

- If $r^2 < |z| < r^2/\rho$ we obtain

$$\sum_{n=0}^{J} \frac{p_n}{\lambda_n^2} \approx \frac{p_J}{\lambda_J^2}, \quad \sum_{n=1}^{l} \frac{p_{-n}}{\lambda_{-n}^2} \approx \frac{p_{-1}}{\lambda_{-1}^2}$$

because $\lambda_J \approx \eta$ and $\lambda_{-1} \approx \eta$ we get

$$N(\eta) \approx \frac{1}{\eta^2} (p_J + p_{-1}) \quad (7.9)$$

It can be checked that (7.9) is still valid if $|z| < r^2$ or $|z| > r^2/\rho$. From the asymptotic relations

$$p_n \sim \rho |z|^{2n}, \quad p_{-n} \sim \left( \frac{\rho}{|z|} \right)^{2n}, \quad n \to +\infty$$

and (7.8) we obtain

$$p_J \approx |z|^{\frac{\ln \eta}{\ln r}} = \eta^{\frac{\ln |z|}{\ln r}}, \quad p_{-1} \approx \left( \frac{\rho}{|z|} \right)^{\frac{\ln \eta}{\ln (\rho/r)}} = \eta^{\frac{\ln (\rho/|z|)}{\ln (\rho/r)}} \quad (7.10)$$

If now $|z| > r$ we can check that

$$\eta^{\frac{\ln |z|}{\ln r}} \gg \eta^{\frac{\ln (\rho/|z|)}{\ln (\rho/r)}}, \quad \eta \downarrow 0$$

hence $p_{-1}$ can be ignored, which gives

$$N(\eta) \approx \eta^{\gamma(z)-2} \quad (7.11)$$

where $\gamma$ is given by (3.2). This formula also holds when $|z| < r$.

Thus it only remains to consider the endpoint cases:

- $|z| = r^2$, but in this case $\frac{p_n}{\lambda_n^2} \sim c$ as $n \to \infty$, hence
\[
\sum_{n=0}^{J} \frac{p_n}{\lambda_n^2} \approx J, \quad \sum_{n=1}^{I} \frac{p_{-n}}{\lambda_{-n}^2} \approx \frac{p_{-I}}{\lambda_{-I}^2}
\]

but then

\[N(\eta) \approx \frac{1}{\eta^2} (p_J + p_{-I}) + J\]

since in this case \(|z| < r\) we get that \(p_J \ll p_{-I} \approx \eta^{\gamma(z)}\), but from (7.8)

\[J = \frac{\ln \eta}{2 \ln r} + \Theta(1) \ll \eta^{\gamma(z)-2}, \quad \text{as}\ \eta \downarrow 0\]

because \(\gamma(z) - 2 < 0\). Thus we again conclude that (7.11) holds. Finally, the case \(|z| = r^2/\rho\) can be treated analogously.

\[\square\]

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