Thresholded Lasso for high dimensional variable selection

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Abstract

Given \( n \) noisy samples with \( p \) dimensions, where \( n \ll p \), we show that the multi-step thresholding procedure based on the Lasso – we call it the Thresholded Lasso, can accurately estimate a sparse vector \( \beta \in \mathbb{R}^p \) in a linear model \( Y = X\beta + \epsilon \), where \( X_{n \times p} \) is a design matrix normalized to have column \( \ell_2 \)-norm \( \sqrt{n} \), and \( \epsilon \sim N(0, \sigma^2 I_n) \). We show that under the restricted eigenvalue (RE) condition, it is possible to achieve the \( \ell_2 \) loss within a logarithmic factor of the ideal mean square error one would achieve with an oracle while selecting a sufficiently sparse model – hence achieving sparse oracle inequalities; the oracle would supply perfect information about which coordinates are non-zero and which are above the noise level. We also show for the Gauss-Dantzig selector (Candès-Tao 07), if \( X \) obeys a uniform uncertainty principle, one will achieve the sparse oracle inequalities as above, while allowing at most \( s_0 \) irrelevant variables in the model in the worst case, where \( s_0 \leq s \) is the smallest integer such that for \( \lambda = \sqrt{2\log p/n} \),

\[
\sum_{i=1}^p \min(\beta_i^2, \lambda^2 \sigma^2) \leq s_0 \lambda^2 \sigma^2.
\]

Our simulation results on the Thresholded Lasso match our theoretical analysis excellently.

1 Introduction

In a typical high dimensional setting, the number of variables \( p \) is much larger than the number of observations \( n \). This challenging setting appears in linear regression, signal recovery, covariance selection in graphical modeling, and sparse approximation. In this paper, we consider recovering \( \beta \in \mathbb{R}^p \) in the following linear model:

\[
Y = X\beta + \epsilon,
\]

where \( X \) is an \( n \times p \) design matrix, \( Y \) is a vector of noisy observations and \( \epsilon \) is the noise term. We assume throughout this paper that \( p \geq n \) (i.e. high-dimensional), \( \epsilon \sim N(0, \sigma^2 I_n) \), and the columns of \( X \) are normalized to have \( \ell_2 \) norm \( \sqrt{n} \). Given such a linear model, two key tasks are: (1) to select the relevant set of variables and (2) to estimate \( \beta \) with bounded \( \ell_2 \) loss. In particular, recovery of the sparsity pattern \( S = \text{supp}(\beta) := \{ j : \beta_j \neq 0 \} \), also known as variable (model) selection, refers to the task of correctly identifying the support set, or a subset of “significant” coefficients in \( \beta \), based on the noisy observations. Even in the noiseless case, recovering \( \beta \) (or its support) from \( (X, Y) \) seems impossible when \( n \ll p \) given that we have more variables than observations. Here and in the sequel, we assume that each column of the fixed design matrix \( X \) has length \( \sqrt{n} \).

Over the past two decades, a line of research shows that when \( \beta \) is sparse, that is, when it has a relatively small number of nonzero coefficients, and when the design matrix \( X \) is also sufficiently nice in the sense that it satisfies certain incoherence conditions, it becomes possible to reconstruct \( \beta \). Throughout this paper, we refer to a vector \( \beta \in \mathbb{R}^p \) with at most \( s \) non-zero entries, where \( s \leq p \),
as a \textit{s-sparse} vector. Denote by \([p] = \{1, \ldots, p\}\). One notion of the incoherence which has been formulated in the sparse reconstruction literature bears the name of restricted isometry property (RIP) [7, 8]. It states that for all \(s\)-sparse sets \(T\), the matrix \(X\) restricted to the columns from \(T\) acts as an almost isometry. Let \(X_T\), where \(T \subset [p]\), be the \(n \times |T|\) submatrix obtained by extracting columns of \(X\) indexed by \(T\). For each integer \(s = 1, 2, \ldots\) such that \(s < p\), the \(s\)-restricted isometry constant \(\delta_s\) of \(X\) is defined to be the smallest quantity such that

\[
(1 - \delta_s) \|v\|_2^2 \leq \|X_T v\|_2^2 / \|v\|_2^2 \leq (1 + \delta_s) \|v\|_2^2
\]

(2)

for all \(T \subset [p]\) with \(|T| \leq s\) and coefficients sequences \((v_j)_{j \in T}\) [7], where \(1 + \delta_s\), \(1 - \delta_s\) correspond to the upper and lower \(s\)-sparse eigenvalues of design matrix \(X\) respectively. Under variants of such conditions, the exact recovery or approximate reconstruction of a sparse \(\beta\) using the basis pursuit program [9] has been shown in a series of results [10, 11, 7, 30, 31, 8]. We refer to the books [38, 41] for a complete exposition.

Consider now the linear regression model in (1). For a chosen penalization parameter \(\lambda_n \geq 0\), regularized estimation with the \(\ell_1\)-norm penalty, also known as the Lasso [35] or Basis Pursuit [9] refers to the following convex optimization problem

\[
\widehat{\beta} = \arg\min_{\beta} \frac{1}{2n} \|Y - X\beta\|_2^2 + \lambda_n \|\beta\|_1,
\]

(3)

where the scaling factor \(1/(2n)\) is chosen by convenience. The Dantzig selector [8] is defined as follows: for some \(\lambda_n \geq 0\),

\[
(DS) \quad \arg\min_{\beta \in \mathbb{R}^p} \|\beta\|_1 \quad \text{subject to} \quad \|X^T(Y - X\beta)/\|X^T\|_\infty\|_\infty \leq \lambda_n.
\]

(4)

In the present work, we explore model selection beyond focusing on the notion of exact recovery of the support set \(S\) for \(\beta\), which crucially depends on the so called \(\beta_{\min}\) condition as well as the \textit{Neighborhood stability or irrepresentability condition} [25, 17, 40, 39]. One can not hope that such incoherence conditions always hold in reality.

As pointed out in [37], the \textit{irrepresentability condition} which is essentially a necessary condition for exact recovery of the non-zero coefficients (for which a \(\beta_{\min}\) condition needs to hold) by the Lasso, is much too restrictive in comparison to the Restricted Eigenvalue condition [2], which we now introduce. For some integer \(s \in [p]\) and a number \(k_0 > 0\), we say \(\text{RE}(s,k_0)\) holds with \(K(s,k_0)\) if for all \(v \neq 0\),

\[
1 / K(s,k_0) \triangleq \min_{J \subseteq [p], |J| \leq s, \|v_J\|_1 \leq k_0 \|v_J\|_1} \min_{\|v\|_2} \|Xv\|_2 / \sqrt{n} \|v\|_2 > 0
\]

(5)

where \(v_J\) represents the subvector of \(v \in \mathbb{R}^p\) confined to a subset \(J\) of \([p]\). To be clear, the \(\text{RE}\) condition alone is not sufficient for the Lasso to recover the model \(S\) exactly [41]. Moreover, to ensure variable selection consistency, a \(\beta_{\min}\) condition

\[
\min_{j \in S} |\beta_{j}| \geq C\sigma \sqrt{\frac{2 \log p}{n}}
\]

(6)

has been imposed for some constant \(C > 1/2\), and shown to be crucial to recover the support of \(\beta\) [39, 44] in the information theoretic limit. Such a \(\beta_{\min}\) condition and the corresponding signal-to-noise ratio (SNR) defined as \(\beta_{\min}^2/\sigma^2\), rather than the typical \(\|\beta\|_2^2/\sigma^2\), is shown to be the key quantity that controls subset selection [39].
Ideally, we aim to remove the $\beta_{\min}$ condition, which is rather unnatural for many applications. Toward this end, we define **sparse oracle inequalities** in Section 2 as the new criteria for *model selection consistency* when the *irrepresentability condition* or related mutual incoherence conditions are replaced with the Restricted Eigenvalue (RE) type of conditions [2] and $\beta_{\min}$ condition is eliminated. Roughly speaking, the new criteria ask one to identify a sparse model such that the corresponding least-squares (OLS) estimator based on the selected model achieves an oracle inequality in terms of the $\ell_2$ loss while keeping the selection set small. We deem the bound on the $\ell_2$-loss as a natural criterion for evaluating a sparse model especially when it is not exactly $S$.

In this paper, we achieve this goal by controlling the false positive selection through thresholding initial estimates of $\beta$ obtained via the Lasso or the Dantzig selector at the critical threshold level.

**Contributions.** Our contributions in this work are twofold. From a theoretical point of view, the framework for our analysis is set upon the Restricted Eigenvalue type of condition and an upper sparse eigenvalue condition. These are among the most general assumptions on the design matrix, guaranteeing sparse recovery in the $\ell_2$ loss for the Lasso estimator as well as the Dantzig selector [2]. Moreover, $\text{RE}(s, k_0, X)$ is shown to be a relaxation of the RIP conditions under suitable choices of parameters involved in each condition [2]; cf. (36). From a methodological point of view, we propose to study the **Thresholded Lasso** estimator with the aforementioned goals in mind:

Step 1 First, we obtain an initial estimator $\beta_{\text{init}}$ using the Lasso (3) with $\lambda_n = d_0 \sigma \sqrt{2 \log p/n}$, for some constant $d_0 > 0$, which is allowed to depend on sparse eigenvalues;

Step 2 We then threshold the estimator $\beta_{\text{init}}$ with $t_0$, with the general goal such that, we get a set $I$ with cardinality at most $2s$; in general, we also have $|I \cup S| \leq 2s$, where $I = \{j \in [p] : \beta_{j,\text{init}} \geq t_0\}$ for some $t_0 \asymp \sigma \sqrt{2 \log p/n}$ with hidden constant to be specified;

Step 3 Finally, we feed $(Y, X_I)$ to the ordinary least squares (OLS) estimator to obtain $\hat{\beta}$, where we set $\hat{\beta}_I = (X_I^T X_I)^{-1} X_I^T Y$ and set the other coordinates to zero.

In Theorem 2.1, we show that the critical threshold level for estimating a high dimensional sparse vector $\beta \in \mathbb{R}^p$ should be set at the level $t_0 \asymp \lambda \sigma$ for $\lambda = \sqrt{2 \log p/n}$, to retain signals in $\beta_{\text{init}}$ at or above the level of $\lambda \sigma$, where $\beta_{\text{init}}$ is the solution to the Lasso estimator (3). This result has been presented in an earlier version of the present paper [50]. We show in Theorem 2.1 with exposition in Section 4.1 that $d_0$ and $t_0$ are allowed to depend on sparse and restricted eigenvalue parameters of the design matrix. We deem this result as an important contribution of the current work. Prior to our work, a similar two-step procedure, namely, the Gauss-Dantzig selector, has been proposed and empirically studied in [8]. Oracle bounds in the spirit of Theorem 2.1 are shown in [49] for this estimator, assuming the stronger restricted isometry type of conditions as originally considered in [8]. More precisely, we show the sparse oracle inequalities hold for the Gauss-Dantzig selector in Theorem 3.2 which we include for completeness of our exposition of thresholded estimators.

In summary, this paper contains the proof of Theorem 3.2 with regards to the Gauss-Dantzig selector, as the proof was omitted from [49]. More importantly, we present significant and novel extensions in both theory and numerical simulations, with regards to the thresholded Lasso estimators under the RE and sparse eigenvalue conditions. Compared to the original conference paper [49], we further study the behavior of the Thresholded Lasso estimator in several challenging situations through numerical simulations in Section 6.

We show that the Thresholded Lasso tradeoffs false positives and false negatives nicely in this
case: its advantage in terms of model selection over the Lasso and adaptive Lasso \cite{54, 53} is clearly evident by examining their ROC curves empirically. Moreover, we show examples for which the Thresholded Lasso recovers the support \( S \) exactly with high probability, using a small number of samples per non-zero component in \( \beta \), for which the Lasso would certainly have failed, as predicted by the work of \cite{40, 39}. Our simulation results show that the rates for exact recovery of the support rise sharply for a few types of random matrices once the number of samples passes a certain threshold, using the Thresholded Lasso estimator. In general, our algorithm is robust and adaptive to the overwhelming presence of the weak signals in \( \text{supp}(\beta) \) as shown in our numerical examples.

The paper builds upon the methodology originally developed in a conference paper by the present author \cite{49}. In a subsequent work \cite{52}, bounds developed based on an earlier version of the present paper have been applied to obtain fast rates of convergence for covariance estimation based on a multivariate Gaussian graphical model. There we also show more comprehensive numerical results involving cross-validation to choose both penalty \( \lambda_n \) and thresholding \( t_0 \) parameters. For a complete exposition on prediction errors for thresholded estimators, see \cite{37}.

1.1 Notation and organization of the paper

We now need to define some notation. Let \( s = |S| \). For a matrix \( A \), let \( \Lambda_{\text{min}}(A) \) and \( \Lambda_{\text{max}}(A) \) denote the smallest and the largest eigenvalues respectively. We assume

\[
\Lambda_{\text{min}}(2s) \triangleq \min_{v \neq 0; 2s-\text{sparse}} \|Xv\|_2^2/(n\|v\|_2^2) > 0, \tag{7}
\]

where \( n \geq 2s \) is necessary, as any submatrix with more than \( n \) columns must be singular, and

\[
\Lambda_{\text{max}}(2s) \triangleq \max_{v \neq 0; 2s-\text{sparse}} \|Xv\|_2^2/(n\|v\|_2^2) < \infty. \tag{8}
\]

As a consequence of these definitions, for any subset \( I \) such that \( |I| \leq 2s \), we have

\[
\Lambda_{\text{max}}(2s) \geq \Lambda_{\text{max}}(|I|) \geq \Lambda_{\text{max}}(X_I^TX_I/n) \geq \Lambda_{\text{min}}(X_I^TX_I/n) \geq \Lambda_{\text{min}}(|I|) \geq \Lambda_{\text{min}}(2s). \tag{9}
\]

Occasionally, we use \( \beta_T \in \mathbb{R}^{|T|} \), where \( T \subseteq \{1, \ldots, p\} \), to also represent its 0-extended version \( \beta' \in \mathbb{R}^p \) such that \( \beta'_T = 0 \) and \( \beta'_T = \beta_T \); for example in \cite{11} below. Also relevant is the \( (s, s') \)-restricted orthogonality constant \( \theta_{s,s'} \) \cite{8}, which is defined to be the smallest quantity such that for all disjoint sets \( T, T' \subseteq \{1, \ldots, p\} \) of cardinality \( |T| \leq s \) and \( |T'| \leq s' \), where \( s + s' \leq p \), it holds that

\[
\frac{|\langle X_Tc, X_T'c' \rangle|}{n} \leq \theta_{s,s'} \|c\|_2 \|c'\|_2. \tag{10}
\]

Note that small values of \( \theta_{s,s'} \) indicate that disjoint subsets covariates in \( X_T \) and \( X_{T'} \) span nearly orthogonal subspaces. Moreover, we have

\[
\theta_{s,s'} \leq \frac{\Lambda_{\text{max}}(s + s') - \Lambda_{\text{min}}(s + s')}{2} \quad (\text{cf. Lemma 2.6}). \tag{11}
\]

Technically, each of the entities defined above, namely, \( 1/\Lambda_{\text{min}}(2s) \), \( \Lambda_{\text{max}}(2s) \), \( \theta_{s,s'} \), and \( K(s, k_0) \) as introduced in \cite{5}, is a non-decreasing function of \( s \), \( s' \), and \( k_0 \). Nonetheless, we crudely consider
these as constants following how they are typically treated in the literature [7, 8, 2], as it is to be understood that they grow very slowly with \(s\) and \(s'\). We note that if the \(\text{RE}(s_0, k_0, X)\) condition as defined in (5) is satisfied with \(k_0 \geq 1\) and \(1 \leq s_0 < p\) then (7) must hold for \(s = s_0\) with \(\frac{1}{\Lambda_{\min}(2s_0)} \leq 2K^2(s_0, 1)\). Consider \(2s_0\)-sparse vector \(v\). Let \(T_0\) denote the locations of the \(s_0\) largest coefficients of \(v\) in absolute values. Then \(\|v\|_2^2 \leq 2\|v_{T_0}\|_2^2\); (12)

Thus we have for any \(2s_0\)-sparse vector \(v\),

\[
\frac{\|Xv\|_2^2}{n\|v\|_2^2} \geq \frac{\|Xv\|_2^2}{n2\|v_{T_0}\|_2^2} \geq \frac{1}{2K^2(s_0, 1)}.
\]

Hence by (5), we have \(\frac{1}{\Lambda_{\min}(2s_0)} \leq 2K^2(s_0, 1)\), since

\[
\Lambda_{\min}(2s_0) = \min_{v \neq 0, 2s_0-\text{sparse}} \frac{\|Xv\|_2^2}{n\|v\|_2^2} \geq \frac{1}{2K^2(s_0, 1)}.
\] (13)

The rest of the paper is organized as follows. In Section 2 we describe a thresholding framework for the general setting, and highlight the role thresholding plays in terms of recovering the best subset of variables. Here, we also present new oracle results for the Lasso estimator, which are crucial in proving Theorem 2.1. In Section 3, we show the essential ingredients for proving Theorem 3.2. Section 3.1 discusses related work. Section 4 provides the proof sketch of the main result Theorem 2.1. Section 5 discuss the Type II errors and \(\ell_2\)-loss. Section 6 includes simulation results showing that the Thresholded Lasso is consistent with our theoretical analysis on variable selection and on estimating \(\beta\). We conclude in Section 7. Additional technical proofs are included in the supplement according to the order their corresponding statements appear.

## 2 Sparse oracle inequalities

In this section, we define **sparse oracle inequalities** as the new criteria for model selection consistency when some of the signals in \(\beta\) are relatively weak, for example, well below the information theoretic detection limit (6) for high dimensional sparse recovery. The current paper answers the following question: Is there a good thresholding rule that enables us to obtain a sufficiently sparse estimator \(\hat{\beta}\) that satisfies an oracle inequality in the sense of (17), when some components of \(\beta_S\) are well below \(\sigma/\sqrt{n}\)? Such oracle results are accomplished without any knowledge of the significant coordinates or parameter values of \(\beta\). Both Theorem 2.1 and 3.2 answer this question positively, where we elaborate upon the sparse recovery properties of the Lasso and Dantzig selector in combination with thresholding and refitting.

We mention in passing that while the idea of thresholding and refitting is widely used in statistical theory and applications in various contexts, for example [15, 13, 52, 17], we quantify the threshold level based on the oracle \(\ell_2\) loss for the Lasso and Dantzig selector respectively in the present work, with the following goals. Specifically, (a) we wish to obtain \(\hat{\beta}\) such that \(|\text{supp}(\hat{\beta}) \setminus S|\) (and sometimes the set difference between \(S\) and \(\text{supp}(\hat{\beta})\) denoted by \(|S \setminus \text{supp}(\hat{\beta})|\)) also is small, with high probability; (b) while at the same time, we wish to bound \(\|\hat{\beta} - \beta\|_2^2\) within logarithmic factor of
the ideal mean squared error one would achieve with an oracle that would supply perfect information about which coordinates are non-zero and which are above the noise level (hence achieving the oracle inequality as studied in [12, 8]). Our results and methodology are indeed inspired by an oracle result on the $\ell_2$ loss for the Dantzig selector as derived in [8]; cf. Proposition 3.3. The crucial theoretical and methodological idea in the current paper originated from [49].

Formally, we evaluate the selection set through the following criterion. Consider the least squares estimators $\hat{b}_I = (X_I^T X_I)^{-1}X_I^T Y$, where $I \subset [p]$ and $|I| \leq s$. Consider the ideal least-squares estimator $\beta^\diamond$ based on a subset $I$ of size at most $s$, which minimizes the mean squared error:

$$\beta^\diamond = \arg\min_{I \subseteq \{1, \ldots, p\}, |I| \leq s} \mathbb{E} \|\beta - \hat{\beta}_I\|_2^2.$$  \hfill (14)

It follows from the analysis in [8] that for $\Lambda_{\max}(s) < \infty$ (cf. supplementary Section B),

$$\mathbb{E}\|\beta - \beta^\diamond\|_2^2 \geq \min(1, 1/\Lambda_{\max}(s)) \sum_{i=1}^p \min(\beta_i^2, \sigma^2/n)$$  \hfill (15)

where

$$\sum_{i=1}^p \min(\beta_i^2, \sigma^2/n) = \min_{I \subseteq \{1, \ldots, p\}} \|\beta - \hat{\beta}_I\|_2^2 + |I|\sigma^2/n$$  \hfill (16)

represents the squared bias and variance. Now we check if (17) holds with high probability,

$$\|\hat{\beta} - \beta\|_2^2 = O(\lambda^2\sigma^2 + \sum_{i=1}^p \min(\beta_i^2, \lambda^2\sigma^2)) = O(\log p) \sum_{i=1}^p \min(\beta_i^2, \sigma^2/n)$$  \hfill (17)

so that the following holds in view of (15):

$$\|\hat{\beta} - \beta\|_2^2 = O(\log p) \max(1, \Lambda_{\max}(s)) \mathbb{E} \|\beta^\diamond - \beta\|_2^2.$$  \hfill (18)

Here the $\ell_2$ loss in (17) is optimal up to a log $p$ factor. We note that (18) is not the tightest upper bound that we could derive due to a relaxation we have on the lower bound as stated in (15). Nevertheless, we use it for its simplicity. Previously, it has been shown that (17) holds with high probability for the Dantzig selector under a Uniform Uncertainty Principle (UUP), where the UUP states that for all $s$-sparse sets $J$, the columns of $X$ corresponding to $J$ are almost orthogonal (cf. Assumption 3.1). Denote by $s_0$ the smallest integer such that the following holds:

$$\sum_{i=1}^p \min(\beta_i^2, \lambda^2\sigma^2) \leq s_0\lambda^2\sigma^2,$$  \hfill (19)

where $\lambda = \sqrt{2\log p/n}$.

The parameter $s_0$ is relevant especially when we do not wish to impose any lower bound on $\beta_{\min}$. For a given pair of $(n, p)$ values, $s_0$ characterizes more accurately than $s$ the number of significant coefficients of $\beta$ with respect to the noise level $\sigma$ that we should (could) try to recover. To make this statement precise, we have as a consequence of the definition in (19) [8]

$$|\beta_j| < \lambda\sigma$$  \hfill (20)

for all $j > s_0$, if we order $|\beta_1| \geq |\beta_2| \geq \ldots \geq |\beta_p|$. For simplicity of presentation, in both Theorems 2.1 and 3.2 we set $|I| < 2s_0$ while achieving the oracle inequality as in (17). One could aim to bound $|I| < cs_0$ for some other constant.
between the noise and covariates of $X$ setting $c$ before leaving this section, we define a quantity $\lambda$ selection based on Gaussian graphical models. We use cross-validation to choose both penalty and threshold parameters in the context of covariance paper. In practical settings, one can choose is even more flexible, which we elaborated in [49, 50] and hence details are omitted from the current 

In the general settings, we allow $t_0$ to be chosen from a reasonably wide range, where we tradeoff the width of the range with the tightness of the upper bound on the $\ell_2$ loss for $\hat{\beta}$; See Theorem 2.1 and Section 2.3 for discussions. This saves us from having to estimate incoherence parameters in a refined (and tedious) manner. When $\beta_{\min}$ is sufficiently large, the range of thresholding parameters is even more flexible, which we elaborated in [49, 50] and hence details are omitted from the current paper. In practical settings, one can choose $\lambda_n$ using cross-validation; See for example [52], where we use cross-validation to choose both penalty and threshold parameters in the context of covariance selection based on Gaussian graphical models.

Before leaving this section, we define a quantity $\lambda_{\sigma,a,p}$, which bounds the maximum correlation between the noise and covariates of $X$; For each $a \geq 0$, let

$$T_a := \left\{ \epsilon : \|X^T \epsilon / n\|_\infty \leq \lambda_{\sigma,a,p}, \text{ where } \lambda_{\sigma,a,p} = \sigma \sqrt{1 + a} \sqrt{2 \log p / n} \right\}; \quad (21)$$

Then, we have $P(T_a) \geq 1 - (\sqrt{\pi \log pp^2})^{-1}$ when $X$ has column $\ell_2$ norms bounded by $\sqrt{n}$.

### 2.1 The thresholded Lasso estimator

We now state Theorem 2.1 that such sparse oracle inequalities as elaborated in Section 2 hold for the Thresholded Lasso under no restriction on $\beta_{\min}$.

**Theorem 2.1. (Ideal model selection for the Thresholded Lasso)** Suppose RE$(s_0, 4, X)$ holds with $K(s_0, 4)$, and sparse eigenvalue conditions (7) and (8) hold. Let $\beta_{\text{init}}$ be an optimal solution to the Lasso (3) with $\lambda_n = d_0 \sqrt{2 \log p / n} / \sigma \geq 2 \lambda_{\sigma,a,p}$, where $d_0 \geq 2 \sqrt{1 + a}$ for $a \geq 0$. Suppose that we choose $t_0 = C_4 \lambda \sigma \geq 2 \sqrt{1 + a} \lambda \sigma$, for some constant $C_4 \geq D_1$ for $D_1$ as in (30).

Set $I = \{ j \in [p] : \beta_{j,\text{init}} \geq t_0 \}$. We have for $\hat{\beta}_I = (X_I^T X_I)^{-1} X_I^T Y$,

$$|I| \leq s_0(1 + D_1/C_4) < 2s_0, \quad |I \cup S| \leq s + s_0 \quad \text{and}$$

$$\|\hat{\beta} - \beta\|_2^2 \leq D_1^2 s_0 \Lambda^2 \sigma^2 \quad (22)$$

with probability at least $1 - P(T_a)$, where for $D'_0$ as defined in (28),

$$D_1^2 \leq \left( (D'_0 + C_4)^2 + 1 \right) \left( \frac{3}{2} + \frac{\Lambda_{\max}(2s) - \Lambda_{\min}(2s)}{2 \Lambda_{\min}^2(2s)} \right) \quad (23)$$

Theorem 2.1 is the key contribution of this paper and is proved in the supplementary Section A. We do not optimize constants in this paper. Theorem 2.1 relies on new oracle results that we prove for the Lasso estimator under a RE condition in Theorem 2.3. The Lasso estimator achieves essentially the same bound in terms of $\ell_2$ loss as stated in (17), which adapts nearly ideally not only to the uncertainty in the support set $S$ but also the “significant” set.
2.2 The thresholding rules

Suppose that we aim to target the set of variables of size at least \( \sigma \), where \( \sigma \geq 2\log(p/n) \). The tight analysis in Theorem 2.1 for the Thresholded Lasso estimator is motivated by the sparse oracle inequalities on the Gauss-Dantzig selector (cf. Theorem 3.2), which was originally developed in a conference paper \([49]\). It is clear from our analysis in \([49]\) that, showing an oracle inequality as in (17) for the initial Lasso estimator is motivated by the sparse oracle inequalities for the Thresholded Lasso. We now elaborate on these results. Lemma 2.2 is a deterministic result.

**Lemma 2.2. (A deterministic result on the bias.)** Let \( \beta_{\text{init}} \) be an initial estimator of \( \beta \) in \((1)\), where \( \epsilon \sim N_n(0, \sigma^2I) \) and \( \|X_i\|_2 = \sqrt{n} \). Let \( s_0 \) be as in \((19)\). Let \( T_0 \) denote the largest \( s_0 \) coordinates of \( \beta \) in absolute values. Let \( h = \beta_{\text{init}} - \beta_{T_0} \). Suppose that for \( \lambda := \sqrt{2\log(p/n)} \),

\[
\|h_{T_0}\|_2 \leq D \lambda \sigma \sqrt{s_0} \quad \text{and} \quad \|h_{T_0^c}\|_1 \leq D \lambda \sigma s_0.
\]

Suppose that we choose \( t_0 = C_4 \lambda \sigma \) for some positive constant \( C_4 \). Let \( I = \{j : |\beta_{j,\text{init}}| \geq t_0\} \) and \( D := \{1, \ldots, p\} \setminus I \). Then the set \( I \) satisfies

\[
|I| \leq s_0(1 + D_1/C_4), \quad |I \cup S| \leq s + s_0D_1/C_4, \quad \text{and} \quad \|\beta_D\|_2 \leq \sqrt{(D_0 + C_4)^2 + 1} \lambda \sigma \sqrt{s_0}, \text{ where } D_0, D_1 \text{ are as defined in } (24).
\]

Intuitively, a tighter bound on \( \|h_{T_0^c}\|_1 \) or \( \|h_{T_0}\|_2 \) will decrease the threshold \( t_0 \) while a tighter bound on \( \|h_{T_0}\|_2 \leq D_0 \lambda \sigma \sqrt{s_0} \) will tighten the upper bound in (26) on the bias component through the triangle inequality. Theorem 2.3 may be of independent interest.

**Theorem 2.3. (Oracle inequalities of the Lasso)** Let \( Y = X \beta + \epsilon \), for \( \epsilon \) being i.i.d. \( N(0, \sigma^2) \) and \( \|X_i\|_2 = \sqrt{n} \). Let \( s_0 \) be as in \((19)\) and \( T_0 \) denote locations of the \( s_0 \) largest coefficients of \( \beta \) in absolute values. Suppose that \( \text{RE}(s_0, 4, X) \) holds with \( K(s_0, 4) \) and condition (8) holds. Let \( \beta_{\text{init}} \) be an optimal solution to the Lasso \((3)\) with \( \lambda_n = d_0 \lambda \sigma \geq 2\lambda_{\sigma,a,p} \), where \( a \geq 0 \) and \( d_0 \geq 2\sqrt{1 + a} \). Let \( h = \beta_{\text{init}} - \beta_{T_0} \). Then on \( T_0 \) as in \((21)\), we have

\[
\|\beta_{\text{init}} - \beta\|_2^2 \leq 2\lambda^2 \sigma^2 s_0 (D_0^2 + D_1^2 + 1),
\]

\[
\|h_{T_0}\|_1 + \|\beta_{T_0}\|_1 \leq D_2 \lambda \sigma s_0,
\]

\[
\|X \beta_{\text{init}} - X \beta\|_2 / \sqrt{n} \leq D_3 \lambda \sigma \sqrt{s_0},
\]

where \( D_0, \ldots, D_3 \) are defined in \((28)\) to \((32)\). Moreover, for any subset \( I_0 \subset S \), by assuming that \( \text{RE}(|I_0|, 4, X) \) holds with \( K(|I_0|, 4) \), we have

\[
\|X \beta_{\text{init}} - X \beta\|_2 / \sqrt{n} \leq 2 \|X \beta - X \beta_{I_0}\|_2 / \sqrt{n} + 3K(|I_0|, 4)\lambda \sqrt{|I_0|}.
\]

The proof of Theorem 2.3 appears in the supplementary Section \([49]\), which yields the following:

\[
D_0 = \max \left\{ D, 2K(s_0, 4)\sqrt{2\Lambda_{\max}(s - s_0) + 3\sqrt{2d_0K^2(s_0, 4)}} \right\}, \quad \text{for } (28)
\]

\[
D_0' = \max \left\{ D, 2K(s_0, 4)\sqrt{\Lambda_{\max}(s - s_0) + 3d_0K^2(s_0, 4)} \right\}, \quad \text{for } (29)
\]

\[
D = \frac{3\sqrt{\Lambda_{\max}(s - s_0)}}{\sqrt{\Lambda_{\min}(2s_0)}} + \frac{4\theta_{s_0, 2s_0} \Lambda_{\max}(s - s_0)}{d_0 \Lambda_{\min}(2s_0)},
\]
where recall $1/\Lambda_{\min}(2s_0) < 2K^2(s_0, 1) \leq 2K^2(s_0, 4) < \infty$ by \cite{13} and \(\theta_{s_0,2s_0} < \infty\) (cf. \cite{43}), and

\[
D_1 = \max \left\{ \frac{4\Lambda_{\max}(s-s_0)}{d_0}, \left( \frac{\Lambda_{\max}(s-s_0)/d_0 + 3\sqrt{d_0}K(s_0, 4)/2}{d_0} \right)^2 \right\} , \quad (30)
\]
\[
D_2 = 2\Lambda_{\max}(s-s_0)/d_0 + \max \left\{ 8d_0K^2(s_0, 4), 3\Lambda_{\max}(s-s_0)/d_0 \right\} , \quad (31)
\]
\[
D_3 = \sqrt{\Lambda_{\max}(s-s_0) + \max \left\{ \Lambda_{\max}(s-s_0), 3d_0K(s_0, 4) \right\}} . \quad (32)
\]

The proof of Theorem 2.3 draws upon techniques from a concurrent work in \cite{37}. We compare it with a well known $\ell_p$ error result in \cite{2} (cf. Theorem 7.2) in Section 4.1. The sparse oracle properties of the Thresholded Lasso in terms of variable selection, $\ell_2$ loss, and prediction error then follow from Theorem 2.3 Lemmas 2.2 and 2.5. To help build intuition, we first state in Lemma 2.4 a general result on the $\ell_2$ loss for the OLS estimator when a subset of relevant variables is missing from the fixed model $I$. Lemma 2.4 is also an important technical contribution of this paper, which may be of independent interest. We note that Lemma 2.4 applies to $X$ so long as sparse eigenvalue conditions \cite{7} and \cite{8} hold. The assumption on $I$ being fixed is then relaxed in Lemma 2.5.

**Lemma 2.4. (OLS estimator with missing variables)** Suppose that the sparse eigenvalue conditions \cite{7} and \cite{8} hold. Given a deterministic set $I \subset \{1, \ldots, p\}$, set $D := \{1, \ldots, p\} \setminus I$ and $S_D = D \cap S = S \setminus I$. Let $|I| = m \leq (c_0s_0) \land s$ for some absolute constant $c_0$. Suppose $|I \cup S| \leq 2s$. Then, for $\hat{\beta}_I = (X_I^T X_I)^{-1}X_I^T Y$, it holds with probability at least $1 - 2\exp(-3m/64)$,

\[
\left\| \hat{\beta}_I - \beta \right\|_2^2 \leq \left( \frac{2\sigma^2 I_{|I|,|S_D|}}{\Lambda_{\min}(m)} + 1 \right) \left\| \beta_D \right\|_2^2 + \frac{3m\sigma^2}{n\Lambda_{\min}(m)}
\]

Lemma 2.4 implies that even if we miss some columns of $X$ in $S$, we can still hope to get the $\ell_2$ loss bounded as in Theorem 2.1 (and Theorem 3.2) so long as $\left\| \beta_D \right\|_2$ is bounded by $O_P(\lambda \sigma \sqrt{s_0})$ while $|I|$ is sufficiently small. Both conditions are guaranteed to hold by our choices of the thresholding parameters as shown in Lemma 2.2. Although the tight analysis of Lemma 2.4 depends on the fact that the selection set $I$ is deterministic, a simple variation of the statement makes it work well with the thresholded estimators as considered in the present paper, with $\ell_2$ error bounded essentially at the same order of magnitude as in \cite{17} so long as $|I| = O(s_0)$; Lemma 2.5 is presented in \cite{49}.

**Lemma 2.5. (OLS estimator with missing variables)** \cite{49} Suppose that \cite{7} and \cite{8} hold. Given an arbitrary set $I \subset \{1, \ldots, p\}$, possibly random, set $D := \{1, \ldots, p\} \setminus I$ and $S_D = D \cap S$. Suppose on event $\mathcal{T}_a$, we have $|I| = m \leq (c_0s_0) \land s$ for some absolute constant $c_0$ and $|I \cup S| \leq 2s$. Then, for $\hat{\beta}_I = (X_I^T X_I)^{-1}X_I^T Y$, it holds on $\mathcal{T}_a$ that

\[
\left\| \hat{\beta}_I - \beta \right\|_2^2 \leq \left( \frac{2\sigma^2 I_{|I|,|S_D|}}{\Lambda_{\min}(|I|)} + 1 \right) \left\| \beta_D \right\|_2^2 + \frac{2|I|(1+a)\sigma^2\lambda^2}{\Lambda_{\min}(|I|)}
\]

\[
(33)
\]

### 2.3 Discussions

We now state Lemma 2.6, which follows from \cite{7} (Lemma 1.2). However, the sparse eigenvalue conditions are considerably relaxed from the UUP in Assumption 3.1, which requires that $\theta_{s,2s} < \Lambda_{\min}(2s)$. Moreover, the lower sparse eigenvalue condition \cite{7} can be replaced by

\[
\Lambda_{\min}(|I|) \geq \Lambda_{\min}(cs_0) > 0, \text{ in case } |I| \leq cs_0 \text{ for some } c > 0.
\]

\[
(34)
\]
It is worth mentioning that for disjoint sets $I$ and $S_D$ as mentioned in Lemmas 2.4 and 2.5

$$|I| + |S_D| = |I \cup S| \leq 2s$$

and thus $\theta_{|I|,|S_D|}$ is bounded by $(\Lambda_{\max}(2s) - \Lambda_{\min}(2s))/2$, where it is understood that $\Lambda_{\min}(2s) = 0$ is also permitted (cf. Proof of Lemma 2.6). Without (7), we still obtain (35) following the arguments as in [43].

**Lemma 2.6.** (7) Let $\delta_s$ be as in (2). Suppose that sparse eigenvalue conditions (7) and (8) hold. Then for all disjoint sets $I, S_D \subseteq \{1, \ldots, p\}$ of cardinality $|S_D| < s$ and $|I| + |S_D| \leq 2s$,

$$\theta_{|I|,|S_D|} \leq (\Lambda_{\max}(2s) - \Lambda_{\min}(2s))/2;$$

In particular, if $\delta_s < 1$, we have $\theta_{|I|,|S_D|} \leq \delta_{|I|+|S_D|} \leq \delta_s < 1$. Moreover, under (8),

$$\theta_{|I|,|S_D|} \leq \Lambda_{\max}(s) \wedge \Lambda_{\max}(2s)/2 \quad \text{if} \quad |I| \vee |S_D| \leq s,$$

which holds under conditions as stated in Lemma 2.5.

In summary, Lemmas 2.2 and 2.5 ensure that the general thresholding rules with threshold level at about $\lambda \sigma$ achieve the following property: although we cannot guarantee the presence of variables indexed by $S_R = \{j : |\beta_j| < \sigma \sqrt{\log p/n}\}$ to be included due to their lack of strength, we will include in $I$ most variables in $S \setminus S_R$ such that the OLS estimator based on model $I$ achieves the oracle bound (17). This goal is accomplished despite some variables from the support set $S$ are missing from the model $I$, since their overall $\ell_2$-norm $||\beta_D||_2$ is bounded in (26). As mentioned, Proposition 5.1 (by setting $c = 1$) shows that the number of variables in $\beta$ that are larger than or equal to $\sqrt{\log p/n} \sigma$ in magnitude is bounded by $2s_0$. In hindsight, it is clear that we wish to retain most of them by keeping $2s_0$ variables in the model $I$.

Indeed, suppose that $D_1 s_0 < (c_0 s_0) \wedge (s - s_0)$. Then, by choosing $t_0 \propto \lambda \sigma$ on event $T_a$, we are guaranteed to obtain under the settings of Theorem 2.1 (and Lemma 2.2),

$$|I| \leq s_0 (1 + D_1) < s, \quad |I \cup S| \leq 2s \quad \text{and} \quad ||\hat{\beta} - \beta||_2^2 \leq D_1^2 s_0^2 \lambda^2 \sigma^2.$$ 

Here, we assume that $D_1$ as in (30) will grow only mildly with the parameters $s_0, s$, under conditions (5) and (8), and it is not necessary to set $C_4 > D_1$. The set of missing variables in $D$ is the price we pay in order to obtain a sparse model when some coordinates in the support are well below $\sigma \sqrt{\log p/n}$. Note that when we allow the model size to increase by lowering $t_0$, the variance term $\propto |I| \lambda^2 / \Lambda_{\min}(|I|)$ becomes correspondingly larger. Since the increased model $I$ may not include more true variables, the size of $S_D = S \setminus I$ may remain invariant; if so, the overall interaction term $\theta ||\beta_D||_2 / \Lambda_{\min}(|I|)$ can still increase due to the increased orthogonality coefficient $\theta := \theta_{|I|,|S_D|}$, even though $||\beta_D||_2$ is a non-increasing function of the model size.

This argument favors the selection of a small (yet sufficient) model as stated in Theorem 2.1 rather than blindly including extraneous variables in the set $I$. We mention in passing that by setting an upper bound on the desired size of $I$ to $2s_0$, we are able to make some interesting connections between the thresholded estimators as studied in the present paper and the $\ell_0$ penalized least squares estimators. In particular, we show that the prediction error, $||X_{\beta_{\text{init}}} - X\beta||_2$, and a complexity-based penalty term $\sigma \sqrt{|I| \log p}$ on the chosen model $I$ are both bounded by $O_P(\sigma \sqrt{s_0 \log p})$ in case $|I| \propto 2s_0$ for the thresholded estimators 50. Proofs of Lemmas 2.4, 2.5 and 2.6 are provided in the supplementary Sections A.1, A.2 and A.3 respectively, for completeness.
3 Nearly ideal model selections under the UUP

In [49], we show that thresholding of an initial Dantzig selector $\hat{\beta}_{\text{init}}$ at the level of $\sqrt{2 \log p / n} \sigma$ followed by OLS refitting, achieves the sparse oracle inequalities under a UUP.

**Assumption 3.1. (A Uniform Uncertainty Principle)** For some integer $1 \leq s < n/3$, assume $\delta_{2s} + \theta_{s,2s} < 1 - \tau$ for some $\tau > 0$, which implies that $\Lambda_{\text{min}}(2s) > \theta_{s,2s}$.

**Remark 3.1.** It is clear that $\delta_{2s} < 1$ implies that the sparse eigenvalues condition (7) and (8) hold. Moreover, Assumption 3.1 implies that $\text{RE}(s_0, k_0, X)$ as in (5) hold for $s_0 \leq s$ with

$$K(s_0, k_0) \leq K(s, k_0) = \frac{\sqrt{\Lambda_{\text{min}}(2s)}}{\Lambda_{\text{min}}(2s) - \theta_{s,2s}} \leq \frac{\sqrt{\Lambda_{\text{min}}(2s)}}{1 - \delta_{2s} - \theta_{s,2s}} \leq \frac{\sqrt{\Lambda_{\text{min}}(2s)}}{\tau} \quad \text{(36)}$$

for $k_0 = 1$, as $K(s, k_0)$ is nondecreasing with respect to $s$ for the same $k_0$; see [2].

**The Gauss-Dantzig Procedure:** Assume $\delta_{2s} + \theta_{s,2s} < 1 - \tau$, where $\tau > 0$:

Step 1 First obtain an initial estimator $\hat{\beta}_{\text{init}}$ using the Dantzig selector in (4) with $\lambda_n = (\sqrt{1 + a + \tau^{-1}})\sqrt{2 \log p / n} \sigma =: \lambda_{p,\tau} \sigma$, where $a \geq 0$; then threshold $\hat{\beta}_{\text{init}}$ with $t_0$, chosen from the range $(C_1 \lambda_{p,\tau} \sigma, C_4 \lambda_{p,\tau} \sigma]$, for $C_1$ as defined in (39); set $I := \{j \in \{1, \ldots, p\} : \beta_{j,\text{init}} \geq t_0\}$.

Step 2 Given a set $I$ as above, construct the estimator $\hat{\beta}_I = (X_I^T X_I)^{-1} X_I^T Y$ and set $\hat{\beta}_j = 0, \forall j \notin I$.

**Theorem 3.2. (Variable selection under UUP)** Choose $\tau, a > 0$ and set $\lambda_n = \lambda_{p,\tau} \sigma$, where $\lambda_{p,\tau} := (\sqrt{1 + a + \tau^{-1}})\sqrt{2 \log p / n}$, in (4). Suppose $\beta$ is $s$-sparse with $\delta_{2s} + \theta_{s,2s} < 1 - \tau$. Let threshold $t_0$ be chosen from the range $(C_1 \lambda_{p,\tau} \sigma, C_4 \lambda_{p,\tau} \sigma]$ for some constants $C_1, C_4$ to be defined. Then the Gauss-Dantzig selector $\hat{\beta}$ selects a model $I := \text{supp}(\hat{\beta})$ such that we have

$$|I| \leq 2s_0, \quad |I \setminus S| \leq s_0 \leq s \quad \text{and} \quad \|\hat{\beta} - \beta\|_2^2 \leq C_2^2 \lambda^2 \sigma^2 s_0 \quad \text{(37)}$$

with probability at least $1 - (\sqrt{\pi \log p} \rho^a)^{-1}$, where $C_1$ is defined in (39) and $C_3$ depends on $a, \tau, \delta_{2s}, \theta_{s,2s}$ and $C_4$; see (40).

Our analysis for Theorem 3.2 builds upon Proposition 3.3 [8], which shows the Dantzig selector achieves the oracle inequality as stated in (17) under Assumption 3.1. We note that, in Assumption 3.1, the sparsity level is fixed at $s$ rather than $s_0$. Hence it is stronger than the conditions we impose in Theorem 2.1 for the Thresholded Lasso estimator. We now show the oracle inequalities for the Dantzig selector. We then show in the supplementary Lemma D.1 that thresholding at the level of $\sigma \lambda$ as elaborated in Step 1 in the Gauss-Dantzig Procedure selects a set $I$ of at most $2s_0$ variables, among which at most $s_0$ are from the complement of the support set $S$ as required in (37).

**Proposition 3.3.** [8] Let $Y = X \beta + \epsilon$, for $\epsilon$ being i.i.d. $N(0, \sigma^2)$ and $\|X_j\|_2^2 = n$. Choose $\tau, a > 0$ and set $\lambda_n = (\sqrt{1 + a + \tau^{-1}})\sqrt{2 \log p / n}$ in (4). Then if $\beta$ is $s$-sparse with $\delta_{2s} + \theta_{s,2s} < 1 - \tau$, the Dantzig selector obeys with probability at least $1 - (\sqrt{\pi \log p} \rho^a)^{-1}$,

$$\|\hat{\beta} - \beta\|_2^2 \leq C_2^2 (\sqrt{1 + a + \tau^{-1}})^2 s_0 \lambda^2 \sigma^2$$
From this point on we let \( \delta := \delta_2 \) and \( \theta := \theta_2 \); Analysis in \cite{8} (Theorem 2) and the current paper yields the following constants,

\[
C_2 = 2C'_0 + \frac{1 + \delta}{1 - \delta - \theta} \quad \text{where} \quad C' = \frac{C_0}{1 - \delta - \theta} + \frac{\theta(1 + \delta)}{(1 - \delta - \theta)^2},
\]

where \( C_0 = 2\sqrt{2} \left( 1 + \frac{1}{1 - \delta^2} \right) + (1 + 1/\sqrt{2})\frac{(1+\delta^2)}{1-\delta^2} \). We now define

\[
C_1 = C'_0 + \frac{1 + \delta}{1 - \delta - \theta} \quad \text{and}
C_2^3 = 3((\sqrt{1 + a + \tau^{-1}})^2(1) + 4(1 + a)/A_{\min}(2s_0),
\]

where \( C_3 \) is used in (37) and has not been optimized in our analysis.

### 3.1 Background and related work

In this section, we briefly discuss related work. Lasso and the Dantzig selector are both computational efficient and shown with provable nice statistical properties; see for example \cite{25, 16, 40, 47, 8, 5, 6, 21, 20, 45, 26, 2}. We refer to the books for a comprehensive survey of related results \cite{4, 41}, and \cite{36} a comprehensive comparison between Assumption 3.1 and other incoherence conditions.

As mentioned, a series of recent papers \cite{29, 32, 22, 33, 51} show that a broader class of random matrices with complex row/columnwise (or both) dependencies satisfy the RE condition introduced in the present paper, once the sample size is sufficiently large.

The sparse recovery problem under arbitrary noise is also well studied, see for example \cite{28, 27}. Moreover, greedy algorithms in \cite{28, 27} require \( s \) to be part of the input, while algorithms in the present paper do not have such a requirement, and hence adapt to the unknown level of sparsity well. Before this work, hard thresholding idea has been shown in \cite{8} (via Gauss-Dantzig selector) as a method to correct the bias of the initial Dantzig selector. For the Lasso, \cite{26} has also shown in theoretical analysis that thresholding is effective in obtaining a two-step estimator \( \tilde{\beta} \) that is consistent in its support with \( \beta \) when \( \beta_{\min} \) is sufficiently large. See also \cite{54, 24, 55, 46}. As pointed out in \cite{2}, a weakening of the incoherence condition in \cite{26} is still sufficient for \( \beta_{\min} \) to hold. A more general framework on multi-step variable selection was explored in \cite{43}. They control the probability of false positives at the price of false negatives, similar to what we aim for here; their analysis is constrained to the case when \( s \) is a constant.

**Subsequent development.** This choice of the threshold parameter identified in \cite{49, 50} and the current paper has deep connection with the classic and current literature on model selection \cite{15, 13, 43, 44}. \cite{44} proved the minimax concave penalty (MCP) procedure is selection consistent under a sparse Riesz condition and an information requirement in the sense of \cite{6}. Nonconvexity of the minimization problem cause computational and analytical difficulties; Compared to the elegant yet complex method in MCP, thresholded Lasso procedures in \cite{49, 50} provide a much simpler framework, which is desirable from the practical point of view, with overall good performance. This is confirmed in a subsequent study \cite{42}.

Part of this work was presented in a conference paper \cite{49}. The current version expands the original idea and show new results on the sparse oracle inequalities under the RE conditions (cf. Theorem 2.1, Theorem 2.3). Not included in the present work are Theorem 2.1 and the Iterative
Procedure in [49], where we show conditions under which one can recover a subset of strong signals; cf. Section 2 therein. See also Theorem 3.1 [50]. In [37], we revisit the adaptive lasso [54] [18] [53] as well as the thresholded Lasso with refitting [49] [50], in a high-dimensional linear model, and study prediction error and bound the number of false positive selections. Empirically, in the present work, we compare the performance in terms of variable selections between the adaptive and the thresholding methods in our simulation study. While the focus of the present paper is on variable selection and oracle inequalities for the $\ell_2$ loss, prediction errors are also explicitly derived in [50]. We refer to interested readers to [15] [1] [3] [34] for related work on complexity regularization criteria.

4 Proof Sketch for the main result

We will now describe the main ideas of our analysis in this section. Combining Theorem 2.3 with Lemmas 2.4 and 2.2 allows us to prove Theorem 2.1 which we will elaborate in more details in Section A. For now, we highlight the important differences between our results and a previous result on the Lasso [2] (cf. Theorem 7.2), which we refer to as the BRT results. While a bound of $O_P(\lambda \sigma \sqrt{s})$ on the $\ell_2$ loss as obtained in [2] makes sense when all signals are strong, significant improvements are needed for the general case where $\beta_{\text{min}}$ is not bounded from below. In the present work, it is of our first interest to investigate sufficient conditions under which we could achieve a bound of $O_P(\lambda \sigma \sqrt{s_0})$ on the $\ell_2$ loss for both the Lasso and Thresholded Lasso estimators. Given such error bound, thresholding of an initial estimator $\beta_{\text{init}}$ at the level of $\approx \sigma \sqrt{2 \log p / n}$ will select nearly the best subset of variables in the spirit of Theorem 2.1 and 3.2. Some more comments.

(a) As stated in Theorem 2.1, we use $\text{RE}(s_0, 4, X)$, for which we fix the sparsity level at $s_0$ and $k_0 = 4$, and sparse eigenvalue conditions (7) and (8). While the constants in association with the BRT results depend on $K^2(s, 3)$, the constants in association with the Lasso and the Thresholded Lasso crucially depend on $K^2(s_0, 4)$, $\Lambda_{\text{max}}(2s)$, $\Lambda_{\text{min}}(2s_0)$, and $\theta_{s_0, 2s_0}$ (cf. (43)).

(b) We note that the lower sparse eigenvalue condition $\Lambda_{\text{min}}(2s) > 0$ (7) is implied by, and hence is weaker than the $\text{RE}(s, 3, X)$ condition. Moreover, it is possible to prove Theorem 2.1 even if we leave condition (7) out. In particular, we note that so long as $|I| \leq 2s_0$, then $\text{RE}(s_0, 4, X)$ condition already implies that (34) holds for $c = 2$ [2]. We will not pursue such optimizations in the present work, as in general, our goal is to bound the size of model to be: $|I| \leq cs_0$ for some constant $c > 0$ which need not to be upper bounded by 2.

(c) We note that in $\text{RE}(s, 3, X)$ condition as required in the BRT results to achieve the $\ell_2$ loss of $O_P(\lambda \sigma \sqrt{s})$: while $k_0 = 3$ is chosen, they fix sparsity at $s$ instead of $s_0$, which is not ideal when $s_0$ is much smaller than $s$. We emphasize that in the $\text{RE}(s_0, 4, X)$ condition that we impose, $k_0 = 4$ is rather arbitrarily chosen; in principle, it can be replaced by any number that is strictly larger than 3. Results in [32] reveal that for $\text{RE}$ conditions with a smaller $s_0$, we need correspondingly smaller sample size $n$ in order for the random design matrix $X$ of dimension $n \times p$ to satisfy such a condition, when the independent row vectors of $X_i$, $i = 1, \ldots, n$ have covariance $\Sigma(X_i) = E X_i \otimes X_i = E X_i X_i^T$ satisfying $\text{RE}(s_0, (1 + \varepsilon)k_0, \Sigma^{1/2})$ condition in the sense that (41) holds for any $\varepsilon > 0$, for all $\nu \neq 0$,

$$
\frac{1}{K(s_0, (1 + \varepsilon)k_0, \Sigma^{1/2})} \triangleq \min_{J_0 \subset \{1, \ldots, p\}} \min_{|J_0| \leq s_0} \|v_{J_0}\|_1 \leq k_0 \|v_{J_0}\|_1 \\|\Sigma^{1/2}\nu\|_2 > 0.
$$

(41)
In the context of compressed sensing, RE conditions can also be taken as a way to guarantee recovery for anisotropic measurements \cite{[22]}. We impose an explicit upper bound on $\Lambda_{\text{max}}(2s)$, which is absent from \cite{[2]}, in order to obtain the tighter bounds in the present work for both the Lasso and the thresholded Lasso estimators. This condition is required by our OLS refitting procedure as stated in Lemmas \cite{[26], [27]}. This is consistent with the fact that known oracle inequalities for the Dantzig and Gauss-Dantzig selectors are proved under the UUP, which impose tighter upper and lower sparse eigenvalue bounds in the sense that $\theta_{s,2s} + \delta_{2s} < 1$. The maximum sparse eigenvalue condition \cite{[8]} guarantees that:

$$\theta_{s_0,2s_0} \leq \Lambda_{\text{max}}(2s_0) \wedge \frac{1}{2}(\Lambda_{\text{max}}(3s_0) - \Lambda_{\text{min}}(3s_0)) \leq \Lambda_{\text{max}}(2s_0) < \infty; \tag{42}$$

To see \cite{[22]}, let disjoint sets $J, J' \subset [p]$ satisfy $|J| \leq s_0$ and $|J'| \leq 2s_0$ and vectors $c, c'$ satisfy $\|c\|_2 = 1$ and $\|c'\|_2 = 1$. Thus following the proof of Lemma \cite{[27]} we have by the parallelogram identity and the Cauchy–Schwarz inequality,

$$\left| \frac{\langle X_{jc}, X_{jc'} \rangle}{n} \right| = \left| \frac{n}{2}(\Lambda_{\text{max}}(3s_0) - \Lambda_{\text{min}}(3s_0)) \right| \leq \frac{1}{2}(\Lambda_{\text{max}}(s_0) - \Lambda_{\text{min}}(s_0))$$

and hence

$$\theta_{s_0,2s_0} \leq \sqrt{\Lambda_{\text{max}}(s_0)\Lambda_{\text{max}}(2s_0)} \wedge \frac{1}{2}(\Lambda_{\text{max}}(3s_0) - \Lambda_{\text{min}}(3s_0)), \tag{43}$$

where $0 \leq 2\Lambda_{\text{min}}(3s_0) \leq \|X_{jc} - X_{jc'}\|_2^2 / n, \|X_{jc} + X_{jc'}\|_2^2 / n \leq 2\Lambda_{\text{max}}(3s_0)$.

### 4.1 Relaxed thresholding rules

We now sketch a proof of Theorem \cite{[23]} where we elaborate on the $\ell_p, p = 1, 2$ loss on $h_{T_0}$ and $h_{T_0}'$, and their implications on variable selection, where recall $h = \beta_{\text{init}} - \beta_{T_0}$. Improving the bounds on each component will result in a tighter upper bound on controlling the bias. We specify parameters $\lambda_n$ and $t_0$ in Theorem \cite{[23]} and Lemma \cite{[22]}, respectively. Specifically, these bounds ensure that under RE and sparse eigenvalue conditions, both the $\ell_2$ loss on the set $T_0$ of significant coefficients and the $\ell_1$ norm of the estimated coefficients on $T_0$ are tightly bounded with respect to the size of $T_0$ for the Lasso estimator: hence achieving an oracle inequality on the $\ell_2$ loss in the sense of \cite{[17]}. Specifically, we have shown that for some constants $D'_0, D_1$, on $T_0$,

$$\|h_{T_0}\|_2 = \|((\beta_{\text{init}} - \beta)_{T_0})\|_2 \leq D'_0 \lambda_n \sqrt{s_0} \quad \text{and} \quad \|h_{T_0}\|_1 = \|\beta_{\text{init},T_0}\|_1 \leq D_1 \lambda_n s_0$$

First, by definition of $h$, $T_0$, we aim to keep variables in $T_0$, while for variables outside of $T_0$, we may need to trim these off. It is also clear by Lemma \cite{[22]} that we cannot cut too many “significant” variables in $T_0$ by following the thresholding rules in our proposal; for example, for those that are $\geq \lambda \sigma \sqrt{s_0}$, we can cut at most a constant number of them. Let $T_1$ denote the $s_0$ largest positions of $h$ in absolute values outside of $T_0$. So what do we do with those in $T_1$? The fate of these variables is pretty much up to the choice of the threshold for a given $\beta_{\text{init}}$, knowing these are the largest in magnitude in $h$ (and $\beta_{\text{init}}$) outside of $T_0$ and hence most likely to be included in model $I$. Moreover, even if we were able to retain all variables in $T_0$, we may include at least some variables in $T_1$ when
Lemma 4.2. Suppose all conditions in Theorem 2.3 hold. Then bounded ℓ1 norm of order 1 may not satisfy the cone constraint, both components of h have bounded ℓ1 norm of order ∝ \lambda_n s_0. First we state Lemma 4.2

Lemma 4.2. Suppose all conditions in Theorem 2.3 hold. Then

$$\|Xh_{0}\|_2 / \sqrt{n} \leq \|Xh\|_2 / \sqrt{n} + (\sqrt{\Lambda_{\text{max}}(s_0)} / \sqrt{s_0}) \|h_{T_0}\|_1.$$
Moreover, suppose that \( \theta \) and the bounds immediately above,
\[
\|h_{T_0}\|_2 \leq \|h_{T_0'}\|_2 \leq \frac{1}{\sqrt{\lambda_{\min}(2s_0)}} \left( \|Xh\|_2 / \sqrt{n} + \sqrt{\Lambda_{\max}(s_0)} \|h_{T_0}'\|_1 / \sqrt{s_0} \right)
\]
\[
\leq \lambda_n \sqrt{2s_0} \lambda(s_0, 1) (3 + 4 \sqrt{\Lambda_{\max}(s_0)}),
\]
where the last step holds by (13), since \( 1/\sqrt{\lambda_{\min}(2s_0)} \leq \sqrt{2} \lambda(s_0, 1) \). This allows us to bound \( \|h\|_2 \) in view of Lemma 4.1.

**Case 2.** In the second case, we have \( h \) satisfying the cone constraint in the sense that
\[
\|h_{T_0}\|_1 \leq 4 \|h_{T_0}\|_1; \quad \text{and hence}
\]
\[
\|h_{T_0}\|_1 \leq \lambda_n s_0 (1 + 3K/2)^2 =: D_1 \lambda_n s_0, \quad \text{where } K = K(s_0, 4),
\]
and \( \|Xh\|_2 / \sqrt{n} \leq 2 \|X\beta_{T_0}\|_2 / \sqrt{n} + 3K \lambda_n \sqrt{s_0} \leq (3K + 2) \lambda_n \sqrt{s_0} \).

Now by the RE condition, we have \( \|h_{T_0}\|_2 \leq K(s_0, 4) \|Xh\|_2 / \sqrt{n} \) and
\[
\|h_{T_0}\|_2 \leq \sqrt{2} \lambda(s_0, 4) \|Xh\|_2 / \sqrt{n} \leq (3K + 2) \lambda_n \sqrt{2s_0}
\]
Then we have the following bounds on selections from variables in \( T_0' \): for some \( f_0 > 0 \),
\[
|I \cap T_0'| \leq \|h_{T_0}\|_1 / t_0 \leq D_1 \lambda_n s_0 / (f_0 \lambda_n) \leq (D_1 / f_0)s_0 \quad \text{in case } t_0 = f_0 \lambda_n.
\]
Combining the two cases, we have
\[
D_0' \leq (3K^2(s_0, 4) + 2K(s_0, 4)) \lor (3K(s_0, 1) + 4K(s_0, 1) \sqrt{\Lambda_{\max}(s_0)}) .
\]
Moreover, suppose that \( \theta_{s_0,2s_0} < \Lambda_{\min}(2s_0) \), which holds under UUP (44), one can get rid of the factor \( \sqrt{\Lambda_{\max}(s_0)} \) in (45) and (46), by bounding \( \|h_{T_0'}\|_2 \) for **Case 1** following Lemma 4.1 instead. The proof of Lemma 4.1 is included in supplementary Section C.1 for the purpose of for self-containment.

4.2 Proof of Lemma 2.2

Without loss of generality, we order \( |\beta_1| \geq |\beta_2| \geq \ldots \geq |\beta_p| \). Then \( T_0 = \{1, \ldots, s_0\} \). Let \( T_1 \) be the largest \( s_0 \) positions of \( \beta_{\text{init}} \) outside of \( T_0 \). Then
\[
|I \cap T_0'| \leq \frac{\|\beta_{\text{init}, T_0'}\|_1}{(C_4 \lambda)} = \frac{\|h_{T_0'}\|_1}{(C_4 \lambda)} \leq s_0 D_1 / C_4.
\]
Thus \( |I| = |I \cap T_0| + |I \cap T_0'| \leq s_0 + s_0 D_1 / C_4 \); Now (25) holds since \( T_0 \subseteq S \) and hence
\[
|I \cup S| = |S| + |I \cap S'| \leq s + |I \cap T_0'| \leq s + s_0 D_1 / C_4.
\]
We now bound \( \|\beta_{D'}\|_2^2 \). Denote by
\[
\beta^{(1)}_j = \beta_j \cdot 1_{j \leq s_0} \quad \text{and} \quad \beta^{(2)}_j = \beta_j \cdot 1_{j > s_0},
\]

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Let \( \beta_D = \beta_D^{(1)} + \beta_D^{(2)} \), where \( \beta_D^{(1)} := (\beta_j)_{j \in T_0 \cap D} \) consists of coefficients that are significant relative to \( \lambda \sigma \), but are dropped as \( \beta_{j, \text{init}} < t_0 \), and \( \beta_D^{(2)} \) consists of those below \( \lambda \sigma \) in magnitude that are dropped. Hence

\[
\|\beta_D\|_2^2 = \|\beta_D^{(1)}\|_2^2 + \|\beta_D^{(2)}\|_2^2.
\]

(47)

Now it is clear \( \beta_D^{(2)} \) is bounded given (20), indeed, we have for \( \lambda = \sqrt{2 \log p/n} \),

\[
\|\beta_D^{(2)}\|_2^2 \leq \sum_{j > s_0} \beta_j^2 = \sum_{j > s_0} \min(\beta_j^2, \lambda^2 \sigma^2) \leq s_0 \lambda^2 \sigma^2,
\]

(48)

where the second equality is by (20) and the last inequality is by definition of \( s_0 \) as in (19). We now focus on \( \beta_D^{(1)} \), where \( |D_1| < s_0 \); we have by the triangle inequality,

\[
\|\beta_D^{(1)}\|_2 \leq \|\beta_{D_1, \text{init}}\|_2 + \|\beta_{D_1, \text{init}} - \beta_D^{(1)}\|_2 \leq t_0 \sqrt{|D_1|} + \|h_{T_0}\|_2 \leq (C_4 + D_0) \lambda \sigma \sqrt{s_0},
\]

(49)

where we used the fact that \( \|\beta_{D_1, \text{init}} - \beta_D^{(1)}\|_2 := \|h_{D_1}\|_2 \leq \|h_{T_0}\|_2 \leq D_0 \lambda \sigma \sqrt{s_0} \) by (24). Hence (26) holds given (47), (49), and (48).

\[\square\]

5 On Type II errors and \( \ell_2 \)-loss optimality

So far, we have focused on controlling Type I errors, which would be meaningless if significant variables are all missing. In this section, we briefly discuss possibilities of recovering a subset of strong signals via thresholding, despite the existence of (or interference from) other relatively weak signals. Again order \( \beta_j \)'s in decreasing order of magnitude: \( |\beta_1| \geq |\beta_2| \geq \ldots \geq |\beta_p| \). Let \( T_0 = \{1, \ldots, s_0\} \). Recall that as a consequence of the definition in (19), we have \( |\beta_j| < \lambda \sigma \) for all \( j \in T_0 \) (cf. (20)). The goal of this section is to demonstrate the remarkable properties of the Lasso and the thresholded Lasso estimators: while exact recovery of all non-zero variables requires very stringent incoherence and \( \beta_{\min} \) conditions, we can significantly relax both conditions when we only require a subset of active variables, namely, those in \( A_0 \), to be included in our selection set. We loosely refer to \( A_0 \) or its superset \( A_0 \subseteq T_0 \subset S \) which we aim to identify as an active set. More precisely, we decompose \( T_0 = \{1, \ldots, s_0\} \) into two sets: \( A_0 \) and \( T_0 \setminus A_0 \), where \( A_0 \) contains the set of coefficients of \( \beta \) strictly larger than \( \lambda \sigma \), for which we define a constant \( \beta_{\min,A_0} \):

\[A_0 = \{j : |\beta_j| > \lambda \sigma\} = \{1, \ldots, a_0\}; \quad \beta_{\min,A_0} := \min_{j \leq a_0} |\beta_j| > \lambda \sigma.\]

(50)

**Proposition 5.1.** Let \( A_0 := \{j : |\beta_j| > \sqrt{2 \log p/n} \lambda \sigma\} \). Let \( T_0 \) denote positions of the \( s_0 \) largest coefficients of \( \beta \) in absolute values, where \( s_0 \) is defined in (19). Let \( a_0 = |A_0| \) denote the cardinality of \( A_0 \) (see also (50)). Then \( \forall \epsilon' > 1/2 \), we have

\[|\{j \in T_0^c : |\beta_j| \geq \sqrt{\log p/(\epsilon' n) \sigma}\}| \leq (2\epsilon' - 1)(s_0 - a_0).
\]

One could choose another target set: for example \( \{j : |\beta_j| \geq \sqrt{\log p/(\epsilon' n) \sigma}\} \), for some \( \log p/2 > \epsilon' > 1/2 \). Moreover, we consider the consequence of setting \( t_0 \in [\sigma \sqrt{2/(n)}, \sigma \sqrt{2 \log p/(n)}] \). In particular, when we set \( \epsilon' = \log p/2 \), we have

\[|\{j \in T_0^c : |\beta_j| \geq \sigma \sqrt{2/(n)}\}| \leq (\log p - 1)(s_0 - a_0).
\]
Our goal is to show when $\beta_{\text{min}, A_0}$ is sufficiently large, we have $A_0 \subset I$ while achieving the sparse oracle inequalities. This result is shown in Theorem 6.3 under the RE and sparse eigenvalue conditions [50], which is a corollary of Lemma 5.3. We first show in Lemma 5.2 that under no restriction on $\beta_{\text{min}}$, we achieve an oracle bound on the $\ell_2$ loss, which depends only on the $\ell_2$ loss of the initial estimator on the set $T_0$. Bounds in Lemma 2.2 is a special case of (51) as we state now.

Lemma 5.2. Let $\beta_{\text{init}}$ be an initial estimator. Let $h = \beta_{\text{init}} - \beta_{T_0}$ and $\lambda := \sqrt{2 \log p/n}$. Suppose that we choose a thresholding parameter $t_0$ and set

$$I = \{j : |\beta_{j, \text{init}}| \geq t_0\}.$$ 

Then for $D := \{1, \ldots, p\} \setminus I$, we have for $D_{11} := D \cap A_0$ and $a_0 = |A_0|$, 

$$\|\beta_D\|_2^2 \leq (s_0 - a_0)\lambda^2 \sigma^2 + (t_0\sqrt{a_0} + \|h_{D_{11}}\|_2)^2.$$ 

(51)

Suppose that $t_0 < \beta_{\text{min}, A_0}$ as defined in [50]. Then (51) can be replaced by 

$$\|\beta_D\|_2^2 \leq (s_0 - a_0)\lambda^2 \sigma^2 + \|h_{D_{11}}\|_2^2 (\beta_{\text{min}, A_0}/(\beta_{\text{min}, A_0} - t_0))^2.$$ 

(52)

We prove Lemma 5.2 in the supplementary Section E. In Lemma 5.3, we impose a lower bound on $\beta_{\text{min}, A_0}$ as in [53] in order to recover the subset of variables in $A_0$, while achieving the nearly ideal $\ell_2$ loss with a sparse model $I$. Discussions are to follow the proof of Lemma 5.3.

Lemma 5.3. (Oracle Ideal MSE with $\ell_\infty$ bounds) Suppose that (7) and (8) hold. Let $\beta_{\text{init}}$ be an initial estimator. Let $h = \beta_{\text{init}} - \beta_{T_0}$ and $\lambda := \sqrt{2 \log p/n}$. Suppose on some event $Q_c$, for $\beta_{\text{min}, A_0}$ as defined in [50], it holds that 

$$\beta_{\text{min}, A_0} \geq \|h_{A_0}\|_\infty + \min \left\{ (s_0)^{-1/2} \|h_{T_0}\|_2, (s_0)^{-1} \|h_{T_0^c}\|_1 \right\}. $$

(53)

Now we choose a thresholding parameter $t_0$ such that on $Q_c$, for some $\tilde{s}_0 \in [s_0, s)$,

$$\beta_{\text{min}, A_0} - \|h_{A_0}\|_\infty \geq t_0 \geq \min \left\{ (\tilde{s}_0)^{-1/2} \|\beta_{\text{init}, T_0}\|_2, (\tilde{s}_0)^{-1} \|\beta_{\text{init}, T_0^c}\|_1 \right\}$$

(54)

holds and set $I = \{j : |\beta_{j, \text{init}}| \geq t_0\}$. Then we have on $T_0 \cap Q_c$,

$$A_0 \subset I \quad \text{and} \quad |I \cap T_0^c| \leq \tilde{s}_0, \quad \text{and hence}$$

$$|I| \leq s_0 + \tilde{s}_0 \quad \text{and} \quad \|\beta_D\|_2 \leq (s_0 - a_0)\lambda^2 \sigma^2.$$ 

(56)

For $\tilde{\beta}_I$ being the OLS estimator based on $(X_I, Y)$ and $\tilde{s}_0 \leq s$, we have on $T_0 \cap Q_c$,

$$\|\tilde{\beta}_I - \beta\|_2^2 \leq \left( 1 + \frac{(\Lambda_{\text{min}}(2s) - \Lambda_{\text{min}}(2s))^2 + 8(1 + a)}{2\Lambda_{\text{min}}^2(|I|)} \right) \tilde{s}_0 \lambda^2 \sigma^2.$$ 

(57)

5.1 Proof of Proposition 5.1

Recall that $|\beta_j| \leq \lambda \sigma$ for all $j > a_0$ as defined in [50]; hence for $\lambda = \sqrt{2 \log p/n}$, we have by (96),

$$\sum_{j > a_0} \min(\beta_j^2, \lambda^2 \sigma^2) = \sum_{j > a_0} \beta_j^2 \leq (s_0 - a_0)\lambda^2 \sigma^2; \quad \text{hence}$$

$$\left\{ j \in A_0^c : |\beta_j| \geq \sqrt{\log p/(c'n)\sigma} \right\} \leq 2c'(s_0 - a_0) \text{ where } |T_0 \setminus A_0| = s_0 - a_0.$$ 

Now given that $\beta_i \geq \beta_j$ for all $i \in T_0, j \in T_0^c$, the proposition holds. \qed
5.2 Proof of Lemma 5.3

Suppose \( T_a \cap Q_c \) holds. It is clear by the choice of \( t_0 \) in (54) and by (53) that

\[
\min_{i \in A_0} \beta_{\text{init},i} \geq \beta_{\text{min},A_0} - \|h_{A_0}\|_\infty \geq t_0, \quad \text{and} \quad D_{11} = \emptyset.
\]

Thus by (54), we can bound \( |I \cap T_0^c| \), depending on which one is applicable, by either

\[
|I \cap T_0^c| \leq \|\beta_{T_0^c,\text{init}}\|_1/t_0 \leq \hat{s}_0 \quad \text{or} \quad |I \cap T_0^c| \leq \|\beta_{T_0^c,\text{init}}\|_2^2/t_0^2 \leq \hat{s}_0.
\]

Moreover, we have by Lemma 2.5, for \( s_0 \leq \hat{s}_0 \leq s \), on event \( T_a \),

\[
\|\hat{\beta}_I - \beta\|_2^2 \leq \left( \frac{2\theta^2_{|I|,|S_D|} + 1}{\Lambda_{\min}^2(|I|)} \right) \|\beta_D\|_2^2 + \frac{2(1 + a)|I|\lambda^2\sigma^2}{\Lambda_{\min}^2(|I|)} \\
\leq \left( 2\theta^2_{|I|,|S_D|} + \Lambda_{\min}^2(|I|) + 4(1 + a) \right) \hat{s}_0 \lambda^2\sigma^2/\Lambda_{\min}^2(|I|),
\]

where \( |I| \leq s_0 + \hat{s}_0 \), and \( \theta^2_{|I|,|S_D|} \) is bounded in Lemma 2.6 given \( |I| + |S_D| \leq s + |I \cap T_0^c| \leq s + \hat{s}_0 \leq 2s \). \( \square \)

5.3 Discussions

Lemma 5.3 explains our results in Theorem 1.1 in [48] as well as the numerical results. We also introduce \( s_0 \) so that the dependency of \( t_0 \) on the knowledge of \( s_0 \) is relaxed; in particular, it can be used to express a desirable level of sparsity for the model \( I \) that one wishes to select. Thus one can increase \( t_0 \) as \( \beta_{\text{min},A_0} \) increases in order to reduce the number of false positives while not increasing the number of false negatives from the active set \( A_0 \). Compared with the almost exact sparse recovery result in Theorem 1.1 [49], we have relaxed the restriction on \( \beta_{\text{min}} \); rather than requiring all non-zero entries to be large in absolute values, we only require those in a subset \( A_0 \) to be recovered to be large. In this case, it is also possible to remove variables from \( T_0^c \) entirely by increasing the threshold \( t_0 \) while strengthening the lower bound on \( \beta_{\text{min},A_0} \) by a constant factor.

We omit such results from the present paper.

We note that in the statement of Lemma 5.3, we assume the knowledge of the bounds on various norms of \( \beta_{\text{init}} - \beta \) implicitly (hence the name of “oracle”). In Theorem 6.3 [50], we show that one can indeed recover a subset \( A_0 \) of variables accurately, for \( A_0 \) as defined above, when \( \beta_{\text{min},A_0} := \min_{j \in A_0} |\beta_j| \) is large enough (relative to the \( \ell_2 \) loss of an initial estimator under the RE condition on the set \( A_0 \)); in addition, a small number of extra variables from \( T_0^c := \{1, \ldots, p\} \setminus T_0 \) are also possibly included in the model \( I \). We mention in passing that changing the coefficients of \( \beta_{A_0} \) will not change the values of \( s_0 \) or \( a_0 \), so long as their absolute values stay strictly larger than \( \lambda \sigma \).

Choosing the set \( A_0 \) is rather arbitrary; one could for example, consider the set of variables that are strictly above \( \lambda \sigma/2 \). Bounds on \( \|h_{A_0}\|_\infty \) are in general harder to obtain than \( \|h_{A_0}\|_2 \); Under stronger incoherence conditions, such bounds can be obtained; see for example [23, 40, 6]. In general, we can still hope to bound \( \|h_{A_0}\|_\infty \) by \( \|h_{A_0}\|_2 \) as done in Theorem 6.3 in [50]. Having a tight bound on \( \|h_{T_0}\|_2 \) (or \( \|h_{T_0}\|_\infty \)) and \( \|\beta_D\|_2^2 \) naturally helps relaxing the requirement on \( \beta_{\text{min},A_0} \) for Lemma 5.3, while as shown in Lemma 5.2, such tight upper bounds will help us to control both the size of \( I \) and \( \|\beta_D\| \) and therefore achieve a tight bound on the \( \ell_2 \) loss in the expression.
Figure 1: Illustrative example: i.i.d. Gaussian ensemble; \( p = 256, n = 72, s = 8, \) and \( \sigma = \sqrt{s}/3. \) (a) compare with the Lasso estimator \( \hat{\beta} \) which minimizes \( \ell_2 \) loss. Here \( \hat{\beta} \) has only 3 FPs, but \( \rho^2 \) is large with a value of 64.73. (b) Compare with the \( \beta_{\text{init}} \) obtained using \( \lambda_n. \) The dotted lines show the thresholding level \( t_0. \) The \( \beta_{\text{init}} \) has 15 FPs, all of which were cut after the 2nd step; resulting \( \rho^2 = 12.73. \) After refitting with OLS in the 3rd step, for the \( \hat{\beta}, \rho^2 \) is further reduced to 0.51.

In general, when the strong signals are close to each other in their strength, then a small \( \beta_{\min,A_0} \) implies that we are in a situation with low signal to noise ratio (low SNR); one needs to carefully tradeoff false positives with false negatives; this is shown in our experimental results in Section 6. We refer to [39] and references therein for discussions on information theoretic limits on sparse recovery where the particular estimator is not specified.

6 Numerical experiments

In this section, we present results from numerical simulations designed to validate the theoretical analysis presented in previous sections. In our Thresholded Lasso implementation (we plan to release the implementation as an R package), we use a Two-step procedure as described in Section 1: we use the Lasso as the initial estimator, and OLS in the second step after thresholding. Specifically, we carry out the Lasso using procedure \( \text{LARS}(Y, X) \) that implements the LARS algorithm [14] to calculate the full regularization path. We then use \( \lambda_n, \) whose expression is fixed throughout the experiments as follows,

\[
\lambda_n = 0.69 \lambda \sigma, \text{ where } \lambda = \sqrt{2 \log p/n}, \text{ in [5]}
\]

(58)

to select a \( \beta_{\text{init}} \) from this output path as our initial estimator. We then threshold the \( \beta_{\text{init}} \) using a value \( t_0 \) typically chosen between 0.5\( \lambda \sigma \) and \( \lambda \sigma. \) See each experiment for the actual value used. Given that columns of \( X \) being normalized to have \( \ell_2 \) norm \( \sqrt{n}, \) for each input parameter \( \beta, \) we compute its SNR as follows: \( \text{SNR} := \|\beta\|_2^2/\sigma^2. \) To evaluate \( \beta, \) we use metrics defined in Table 1, we also compute the ratio between squared \( \ell_2 \) error and the ideal mean squared error, known as the \( \rho^2; \) see Section 6.3 for details.
6.1 Illustrative example

In the first example, we run the following experiment with a setup similar to what was used in [8] to conceptually compare the behavior of the Thresholded Lasso with the Gauss-Dantzig selector:

1. Generate an *i.i.d. Gaussian ensemble* $X_{n \times p}$, where $X_{ij} \sim N(0, 1)$ are independent, which is then normalized to have column $\ell_2$-norm $\sqrt{n}$.

2. Select a support set $S$ of size $|S| = s$ uniformly at random, and sample a vector $\beta$ with independent and identically distributed entries on $S$ as follows, $\beta_i = \mu_i (1 + |g_i|)$, where $\mu_i = \pm 1$ with probability $1/2$ and $g_i \sim N(0, 1)$.

3. Compute $Y = X\beta + \epsilon$, where the noise $\epsilon \sim N(0, \sigma^2 I_n)$ is generated with $I_n$ being the $n \times n$ identity matrix. Then feed $Y$ and $X$ to the Thresholded Lasso with thresholding parameter being $t_0$ to recover $\beta$ using $b\beta$.

In Figure 1, we set $p = 256$, $n = 72$, $s = 8$, $\sigma = \sqrt{s}/3$ and $t_0 = \lambda \sigma$. We compare the Thresholded Lasso estimator $\hat{\beta}$ with the Lasso, where the full LARS regularization path is searched to find the optimal $\hat{\beta}$ that has the minimum $\ell_2$ error.

6.2 Type I/II errors

We now evaluate the Thresholded Lasso estimator by comparing Type I/II errors under different values of $t_0$ and SNR. We consider Gaussian random matrices for the design $X$ with both diagonal and Toeplitz covariance. We refer to the former as *i.i.d. Gaussian ensemble* and the latter as *Toeplitz ensemble*. In the Toeplitz case, the covariance is given by $T(\gamma)_{i,j} = \gamma^{|i-j|}$ where $0 < \gamma < 1$.

We run under two noise levels: $\sigma = \sqrt{s}/3$ and $\sigma = \sqrt{s}$. For each $\sigma$, we vary the threshold $t_0$ from $0.01\lambda \sigma$ to $1.5\lambda \sigma$. For each $\sigma$ and $t_0$ combination, we run the following experiment: First we generate $X$ as in Step 1 above. After obtaining $X$, we keep it fixed and then repeat Steps 2 – 3 for 200 times with a new $\beta$ and $\epsilon$ generated each time and we count the number of Type I and II errors in $b\beta$. We compute the average at the end of 200 runs, which will correspond to one data point on the curves in Figure 2 (a) and (b).

For both types of designs, similar behaviors are observed. For $\sigma = \sqrt{s}/3$, FNs increase slowly; hence there is a wide range of values from which $t_0$ can be chosen such that FNs and FPs are both zero. In contrast, when $\sigma = \sqrt{s}$, FNs increase rather quickly as $t_0$ increases due to the low SNR. It is clear that the low SNR and high correlation combination makes it the most challenging situation for variable selection, as predicted by our theoretical analysis and others. In (c) and (d), we run additional experiments for the low SNR case for Toeplitz ensembles. The performance is improved by increasing the sample size or lowering the correlation factor.

6.3 $\ell_2$ loss

We now compare the performance of the Thresholded Lasso with the ordinary Lasso by examining the metric $\rho^2$ defined as follows: $\rho^2 = \frac{\sum_{i=1}^{p} (\hat{\beta}_i - \beta_i)^2}{\sum_{i=1}^{p} \min(\hat{\beta}_i^2, \sigma^2/n)}$.

We first run the above experiment using i.i.d. Gaussian ensemble under the following thresholds: $t_0 = \lambda \sigma$ for $\sigma = \sqrt{s}/3$, and $t_0 = 0.36\lambda \sigma$ for $\sigma = \sqrt{s}$. These are chosen based on the desire to have low errors of both types (as shown in Figure 2 (a)). Naturally, for low SNR cases, small $t_0$ will
Figure 2: $p = 256$ $s = 8$. (a) (b) Type I/II errors for i.i.d. Gaussian and Toeplitz ensembles. Each vertical bar represents ±1 std. The unit of x-axis is in $\lambda \sigma$. For both types of design matrices, FPs decrease and FNs increase as the threshold increases. For Toeplitz ensembles, in (c) with fixed correlation $\gamma$, FNs decrease with more samples, and in (d) with fixed sample size, FNs decrease as the correlation $\gamma$ decreases. (e) (f) Histograms of $\rho^2$ under i.i.d Gaussian ensembles from 500 runs.
Table 1: Metrics for evaluating $\hat{\beta}$

| Metric                              | Definition                                                                 |
|-------------------------------------|---------------------------------------------------------------------------|
| Type I errors or False Positives (FPs) | # of incorrectly selected non-zeros in $\hat{\beta}$                     |
| Type II errors or False Negatives (FNs)      | # of non-zeros in $\beta$ that are not selected in $\hat{\beta}$     |
| True positives (TPs)                | # of correctly selected non-zeros                                        |
| True Negatives (TNs)                | # of zeros in $\hat{\beta}$ that are also zero in $\beta$              |
| False Positive Rate (FPR)          | $FPR = FP/(FP + TN) = FP/(p - s)$                                      |
| True Positive Rate (TPR)           | $TPR = TP/(TP + FN) = TP/s$                                              |

reduce Type II errors. In practice, we suggest using cross-validations to choose the exact constants in front of $\lambda \sigma$; See, for example, a subsequent work [52] for details. We plot the histograms of $\rho^2$ in Figure 2 (e) and (f). In (e), the mean and median are 1.45 and 1.01 for the Thresholded Lasso, and 46.97 and 41.12 for the Lasso. In (f), the corresponding values are 7.26 and 6.60 for the Thresholded Lasso and 10.50 and 10.01 for the Lasso. With high SNR, the Thresholded Lasso performs extremely well; with low SNR, the improvement of the Thresholded Lasso over the ordinary Lasso is less prominent; this is in close correspondence with the Gauss-Dantzig selector’s behavior as shown by [8].

Next we run the above experiment under different sparsity values of $s$. We again use i.i.d. Gaussian ensemble with $p = 2000$, $n = 400$, and $\sigma = \sqrt{s/3}$. The threshold is set at $t_0 = \lambda \sigma$. The SNR for different $s$ is fixed at around 32.36. Table 2 shows the mean of the $\rho^2$ for the Lasso and the Thresholded Lasso estimators. The Thresholded Lasso performs consistently better than the ordinary Lasso until about $s = 80$, after which both break down. For the Lasso, we always choose from the full regularization path the optimal $\hat{\beta}$ that has the minimum $\ell_2$ loss.

Table 2: $\rho^2$ under different sparsity and fixed SNR. Average over 100 runs for each $s$.

| $s$ | 5   | 18  | 20  | 40  | 60  | 80  | 100 |
|-----|-----|-----|-----|-----|-----|-----|-----|
| SNR | 34.66 | 32.99 | 32.29 | 32.08 | 32.28 | 32.56 | 32.54 |
| Lasso | 17.42 | 22.01 | 44.89 | 52.68 | 31.88 | 29.40 | 47.63 |
| Thresholded Lasso | 1.02 | 0.96 | 1.11 | 1.54 | 10.32 | 29.38 | 53.81 |

6.4 Linear Sparsity

We next present results demonstrating that the Thresholded Lasso recovers a sparse model using a small number of samples per non-zero component in $\beta$ when $X$ is a subgaussian ensemble. We run under three cases of $p = 256, 512, 1024$; for each $p$, we increase the sparsity $s$ by roughly equal steps from $s = 0.2p/\log(0.2p)$ to $p/4$. For each $p$ and $s$, we run with different sample size $n$. For each tuple $(n, p, s)$, we run an experiment similar to the one described in Section 6.2 with an i.i.d. Gaussian ensemble $X$ being fixed while repeating Steps 2 – 3 100 times. In Step 2, each randomly selected non-zero coordinate of $\beta$ is assigned a value of $\pm 0.9$ with probability 1/2. After each run, we compare $\hat{\beta}$ with the true $\beta$; if all components match in signs, we count this experiment as a success. At the end of the 100 runs, we compute the percentage of successful runs as the probability of success. We compare with the ordinary Lasso, for which we search over the full regularization path of LARS and choose the $\hat{\beta}$ that best matches $\beta$ in terms of support.
We experiment with $\sigma = 1$ and $\sigma = \sqrt{s}/3$. For $\sigma = 1$, we set $t_0 = f_t \sqrt{|\hat{S}_0|} \lambda \sigma$, where

$$\hat{S}_0 = \{ j : \beta_{j, \text{init}} \geq 0.5 \lambda_n = 0.34 \lambda \sigma \}$$

for $\lambda_n$ as in [58], and $f_t$ is chosen from the range of $[0.12, 0.24]$. See Section 5 [49]. For $\sigma = \sqrt{s}/3$, we set $t_0 = 0.7 \lambda \sigma$ with SNR being fixed. The results are shown in Figure 3. We observe that under both noise levels, the Thresholded Lasso estimator requires much fewer samples than the ordinary lasso in order to conduct exact recovery of the sparsity pattern of the true linear model when all non-zero components are sufficiently large. When $\sigma$ is fixed as $s$ increases, the SNR is increasing; the experimental results illustrate the behavior of sparse recovery when it is close to the noiseless setting. Given the same sparsity, more samples are required for the low SNR case to reach the same level of success rate. Similar behavior was also observed for Toeplitz and Bernoulli ensembles with i.i.d. $\pm 1$ entries.

6.5 ROC comparison

We now compare the performance of the Thresholded Lasso estimator with the Lasso and the Adaptive Lasso by examining their ROC curves. Our parameters are $p = 512$, $n = 330$, $s = 64$ and we run under two cases: $\sigma = \sqrt{s}/3$ and $\sigma = \sqrt{s}$. In the Thresholded Lasso, we vary the threshold level from 0.01$\lambda \sigma$ to 1.5$\lambda \sigma$. For each threshold, we run the experiment described in Section 6.2 with an i.i.d. Gaussian ensemble $X$ being fixed while repeating Steps 2–3 100 times. After each run, we compute the FPR and TPR of the $b$, and compute their averages after 100 runs as the FPR and TPR for this threshold. For the Lasso, we compute the FPR and TPR for each output vector along its entire regularization path. For the Adaptive Lasso, we use the optimal output $\hat{\beta}$ in terms of $\ell_2$ loss from the initial Lasso penalization path as the input to its second step, that is, we set $\beta_{\text{init}} := \hat{\beta}$ and use $w_j = 1/\beta_{\text{init}, j}$ to compute the weights for penalizing those non-zero components in $\beta_{\text{init}}$ in the second step, while all zero components of $\beta_{\text{init}}$ are now removed. We then compute the FPR and TPR for each vector that we obtain from the second step’s LARS output. We implement the algorithms as given in [54], the details of which are omitted here as its implementation has become standard. The ROC curves are plotted in Figure 4. The Thresholded Lasso performs better than both the ordinary Lasso and the Adaptive Lasso; its advantage is more apparent when the SNR is high.

7 Conclusion

In this paper, we show that the thresholding method is effective in variable selection and accurate in statistical estimation. It improves the ordinary Lasso in significant ways. For example, we allow very significant number of non-zero elements in the true parameter, for which the ordinary Lasso would have failed. On the theoretical side, we show that if $X$ obeys the RE condition and if the true parameter is sufficiently sparse, the Thresholded Lasso achieves the $\ell_2$ loss within a logarithmic factor of the ideal mean square error one would achieve with an oracle, while selecting a sufficiently sparse model $I$. This is accomplished when threshold level is at about $\sqrt{2 \log p/n} \sigma$, assuming that columns of $X$ have $\ell_2$ norm $\sqrt{n}$. When the SNR is high, almost exact recovery of the non-zeros in $\beta$ is possible as shown in our theory; exact recovery of the support of $\beta$ is shown in our simulation study when $n$ is only linear in $s$ for several Gaussian and Bernoulli random ensembles. When the SNR is relatively low, the inference task is difficult for any estimator. In this case, we show that
Figure 3: (a) (b) Compare the probability of success for $p = 256$ and $p = 512$ under two noise levels. The Thresholded Lasso estimator requires much fewer samples than the ordinary Lasso. (c) (d) (e) show the probability of success of the Thresholded Lasso under different levels of sparsity and noise levels when $n$ increases for $p = 512$ and 1024. (f) The number of samples $n$ increases almost linearly with $s$ for $p = 1024$. More samples are required to achieve the same level of success when $\sigma = \sqrt{s}/3$ due to the relatively low SNR.
Thresholded Lasso tradeoffs Type I and II errors nicely: we recommend choosing the thresholding parameter conservatively. These findings not only validate our theoretical analysis excellently but also indicate that in practical applications, this approach could be made very effective and relevant.

Acknowledgement

The author thanks Peter Bühlmann, Emmanuel Candès, Sara van de Geer, John Lafferty, Po-Ling Loh, Richard Samworth, Xiaotong Shen, Martin Wainwright, Larry Wasserman, Dana Yang and Cun-Hui Zhang for helpful discussions. The author thanks Bin Yu warmly for hosting her visit at UC Berkeley while she finishes this work in Spring 2010, and my family for their encouragements. Part of this work has appeared in a conference paper with title: Thresholding Procedures for High Dimensional Variable Selection and Statistical Estimation, in Proceedings of Advances in Neural Information Processing Systems 22.

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A Proof of Theorem 2.1

It holds by definition of $S_D$ that $I \cap S_D = \emptyset$. One can check via the proof of Theorem 2.3 that (24) holds for $D'_0, D_1$ as defined in (29) and (30) respectively. Hence by Lemma 2.2, we have on event $T_\alpha$ for $C_4 \geq D_1$, $|I| \leq s_0$ and $|I \cup S_D| \leq |I \cup S| \leq s + s_0 \leq 2s$, given that $|S_D| < s$ and moreover

$$\|\beta_D\|_2 \leq \sqrt{(D'_0 + C_4)^2 + 1}\lambda \sigma \sqrt{s_0}.$$  

We have by Lemma 2.5, on event $T_\alpha$, for $\lambda = \sqrt{2\log p/n}$, $|I| < 2s_0$,

$$\|ar{\beta}_I - \beta\|_2^2 \leq \|\beta_D\|_2^2 \left(1 + \frac{2\theta_2^2 |I| \min |I|}{\Lambda_\min^2 (|I|)} + \frac{2|I|(1 + a)\sigma^2 \lambda^2}{\Lambda_\min^2 (|I|)} \right)$$

$$\leq \|\beta_D\|_2^2 \left(1 + \frac{4(1 + a)}{\Lambda_\min^2 (2s_0)} \right) \sigma^2 s_0$$

$$\leq \lambda^2 \sigma^2 s_0 (D'_0 + C_4)^2 + 1 \left(1 + \frac{2\theta_2^2 |I| \min |I|}{\Lambda_\min^2 (|I|)} + \frac{4}{9} \right),$$

where we use the fact that $4(1 + a)/\Lambda_\min^2 (2s_0) \leq \frac{4}{9}(D'_0)^2$ since

$$(D'_0)^2 \geq 9a_0^2 K^4(s_0, 4) \geq 9(1 + a)/\Lambda_\min^2 (2s_0),$$

where

$$K^4(s_0, 4) \geq K^4(s_0, 1) \geq 1/(4\Lambda_\min^2 (2s_0))$$

in view of (13). Now (22) clearly holds with

$$D_4^2 = (D'_0 + C_4)^2 + 1 \left(1 + \frac{2\theta_2^2 |I| \min |I|}{\Lambda_\min^2 (2s_0)} + \frac{4}{9} \right).$$

Now (23) holds by Lemma 2.6.

A.1 Proof of Lemma 2.4

Recall that the random variable $\|Q\|_2^2 \sim \chi_m^2$ is distributed according to the chi-square distribution where $\|Q\|_2^2 = \sum_{i=1}^m Q_i^2$ with $Q_i \sim N(0, 1)$ that are independent and normally distributed. By (19),

$$\Pr \left( \frac{\chi_m^2}{m} - 1 \leq -\varepsilon \right) \leq \exp(-m\varepsilon^2/4) \quad \text{for} \ 0 \leq \varepsilon \leq 1,$$  

$$\Pr \left( \frac{\chi_m^2}{m} - 1 \geq \varepsilon \right) \leq \exp(-3m\varepsilon^2/(16)) \quad \text{for} \ 0 \leq \varepsilon \leq 1/2.$$  

Although we need to bound the bad event only on one side, we provide a tight bound on the norm of $\|Q\|_2$ with (59) and (60). Let $|I| = m$. Thus we have for $Q = (Q_1, \ldots, Q_m)$ where
\( Q_j \sim \text{i.i.d } N(0, 1) \), for \( \delta < 1/2 \),

\[
\mathbb{P} \left( \left| \frac{1}{m} \sum_{j=1}^{m} Q_j^2 - 1 \right| \geq \delta \right) =: \mathbb{P}(Q) =
\]

\[
\mathbb{P} \left( \frac{1}{m} \sum_{j=1}^{m} Q_j^2 - 1 \geq \delta \right) + \mathbb{P} \left( \frac{1}{m} \sum_{j=1}^{m} Q_j^2 - 1 \leq -\delta \right)
\]

\[
= \mathbb{P} \left( \frac{\lambda^2}{m} - 1 \geq \delta \right) + \mathbb{P} \left( \frac{\lambda^2}{m} - 1 \leq -\delta \right)
\]

\[
\leq \exp(-3m\delta^2/(16)) + \exp(-m\delta^2/4).
\]

Note that \( X_I; \beta_I = X_{S_D} \beta_{S_D} \). We have

\[
\hat{\beta}_I = (X_I^T X_I)^{-1} X_I^T Y = (X_I^T X_I)^{-1} X_I^T (X_I \beta_I + X_I; \beta_I + \epsilon)
\]

\[
= \beta_I + (X_I^T X_I)^{-1} X_I^T X_{S_D} \beta_{S_D} + (X_I^T X_I)^{-1} X_I^T \epsilon;
\]

Hence by the triangle inequality,

\[
\|\hat{\beta}_I - \beta_I\|_2 \leq \|(X_I^T X_I)^{-1} X_I^T X_{S_D} \beta_{S_D} + (X_I^T X_I)^{-1} X_I^T \epsilon\|_2
\]

\[
\leq \|(X_I^T X_I)^{-1} X_I^T X_{S_D} \beta_{S_D}\|_2 + \|(X_I^T X_I)^{-1} X_I^T \epsilon\|_2,
\]

(61)

where the two terms are bounded below as follows.

First notice that \( w/\sigma = (X_I^T X_I)^{-1} X_I^T \epsilon/\sigma \) is a mean zero Gaussian random vector with covariance \((X_I^T X_I)^{-1}\), since \( \epsilon_i/\sigma \sim N(0, 1) \) and

\[
\frac{1}{\sigma^2} \mathbf{E}(ww^T) = \mathbf{E}((X_I^T X_I)^{-1} X_I^T (\epsilon/\sigma) \otimes \epsilon/\sigma | X_I(X_I^T X_I)^{-1})
\]

\[
= (X_I^T X_I)^{-1} X_I^T \frac{1}{\sigma^2} \mathbf{E}(\epsilon\epsilon^T) X_I(X_I^T X_I)^{-1} = (X_I^T X_I)^{-1}.
\]

Then on event \( Q^c \), which holds with probability at least \( 1 - 2\exp(-3m/64) \)

\[
\|(X_I^T X_I)^{-1} X_I^T \epsilon/\sigma\|_2^2 = Q^T (X_I^T X_I)^{-1} Q \leq \Lambda_{\max} ((X_I^T X_I)^{-1}) \|Q\|_2^2
\]

\[
\leq \frac{3m}{2\Lambda_{\min} (X_I^T X_I)} \leq \frac{3m}{2n\Lambda_{\min}(|I|)},
\]

(62)

(63)

where we used an upper bound on \( \|Q\|_2^2 \leq 3m/2, \|Q\|_2^2 \sim \chi^2_m \), and the fact that

\[
\Lambda_{\max} ((X_I^T X_I)^{-1}) = \frac{1}{\Lambda_{\min} ((X_I^T X_I))} \leq \frac{1}{n\Lambda_{\min}((X_I^T X_I)/n)} \leq \frac{1}{n\Lambda_{\min}(|I|)}.
\]

We now focus on bounding the first term in (61). Let \( P_I \) denote the orthogonal projection onto \( I \). Clearly, \( I \cap S_D = \emptyset \). Let

\[
c = (X_I^T X_I)^{-1} X_I^T X_{S_D} \beta_{S_D} \quad \text{ and } \quad X_I c = P_I X_{S_D} \beta_{S_D}.
\]
By the disjointness of $I$ and $S_D$, we have for $P_l X_{S_D} \beta_{S_D} := X_I c$,
\[
\|P_l X_{S_D} \beta_{S_D}\|_2 = \langle P_l X_{S_D} \beta_{S_D}, X_{S_D} \beta_{S_D} \rangle = \langle X_I c, X_{S_D} \beta_{S_D} \rangle \\
\leq n\theta |I| |S_D| \|c\|_2 \|\beta_{S_D}\|_2 \leq n\theta |I| |S_D| \|\beta_{S_D}\|_2 \sqrt{n\Lambda_{\min}(|I|)} ,
\]
where $\|c\|_2 \leq \frac{\|X_I c\|_2}{\sqrt{n\Lambda_{\min}(|I|)}} \leq \frac{\|P_l X_{S_D} \beta_{S_D}\|_2}{\sqrt{n\Lambda_{\min}(|I|)}} . \tag{64}$

Hence
\[
\|P_l X_{S_D} \beta_{S_D}\|_2 \leq \frac{\sqrt{n\theta |I| |S_D|}}{\sqrt{\Lambda_{\min}(|I|)}} \|\beta_{S_D}\|_2 , \quad \text{where} \quad \|\beta_{S_D}\|_2 = \|\beta_D\|_2 \tag{65}
\]
and $\|c\|_2 \leq \frac{\theta |I| |S_D| \|\beta_D\|_2}{\Lambda_{\min}(|I|)} \Lambda_{\min}(|I|). \tag{66}$

Now we have on $T_a$, by \[\[62] \]
\[
\|\hat{\beta}_I - \beta_I\|_2 \leq \|\left( X_I^T X_I \right)^{-1} X_I^T X_{S_D} \beta_{S_D}\|_2 + \|\left( X_I^T X_I \right)^{-1} X_I^T \epsilon\|_2 \\
\leq \frac{\theta |I| |S_D| \|\beta_D\|_2}{\Lambda_{\min}(|I|)} \|\beta_D\|_2 + \frac{\sqrt{3|I| \sigma}}{2n\Lambda_{\min}(|I|)} .
\]

Now the lemma holds given
\[
\|\hat{\beta}_I - \beta\|_2^2 = \|\hat{\beta}_I - \beta_I\|_2^2 + \|\beta_I - \beta\|_2^2 \\
\leq \frac{2\theta^2 |I| |S_D|}{\Lambda_{\min}(|I|)} \|\beta_D\|_2^2 + \frac{3|I| \sigma^2}{2n\Lambda_{\min}(|I|)} + \|\beta_{S_D}\|_2^2 . \tag{67}
\]

Remark A.1. Notice that we can also derive the lower bound on event $Q^c$ for $|I| = m$
\[
\|\left( X_I^T X_I \right)^{-1} X_I^T \epsilon / \sigma\|_2^2 = Q^T (X_I^T X_I)^{-1} Q \geq \Lambda_{\min} \left( (X_I^T X_I)^{-1} \right) \|Q\|_2^2 \tag{68}
\]
\[
\geq \frac{m}{2\Lambda_{\max}(X_I^T X_I)} = \frac{m}{2n\Lambda_{\max}(|I|)} , \tag{69}
\]
where we used an upper bound on $\|Q\|_2^2 \geq m/2$, $\|Q\|_2^2 \sim \chi_m^2$, and the fact that
\[
\Lambda_{\min} \left( (X_I^T X_I)^{-1} \right) = \frac{1}{\Lambda_{\max}(X_I^T X_I)} = \frac{1}{n\Lambda_{\max}(X_I^T X_I)/n} = : \frac{1}{n\Lambda_{\max}(|I|)} .
\]

Now suppose that $|I| \asymp 2s_0$, then
\[
\|\left( X_I^T X_I \right)^{-1} X_I^T \epsilon / \sigma\|_2^2 \geq \frac{m}{2n\Lambda_{\max}(2s_0)} . \tag{70}
\]

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A.2 Proof of Lemma 2.5

The only part we make modification is the following: On event $T_\alpha$, we have by (9), for an arbitrary set $I$ of size $|I| \leq 2s$,

$$\| (X^T X_I)^{-1} X_I^T e \|_2 \leq \| (X^T X_I/n)^{-1} X_I^T e/n \|_2 \leq \frac{\sqrt{|I| \lambda_{\sigma,\alpha,p}}}{\Lambda_{\min}(|I|)}.$$  \hfill (71)

Now we have on $T_\alpha$, we have by (71), (64), and (65),

$$b_\beta I - b_\beta I_2^2 \leq \| (X^T X_I)^{-1} X_I^T X_{SD} \beta_{SD} \|_2^2 + \| (X^T X_I)^{-1} X_I^T e/n \|_2 \leq \theta |I| |I| \lambda_{\sigma,\alpha,p} \Lambda_{\min}(|I|) \| \beta_{SD} \|_2^2 .$$

and thus (33) holds since $b_\beta I - b_\beta I_2^2 = b_\beta I_2^2 + \| \beta_{SD} \|_2^2$.

A.3 Proof of Lemma 2.6

It is sufficient to show that (72) holds for $\| c \|_2 = \| c' \|_2 = 1$.

$$\left| \frac{\langle X_I c, X_{SD} c' \rangle}{n} \right| \leq \frac{(\Lambda_{\max}(2s) - \Lambda_{\min}(2s)) \| c \|_2 \| c' \|_2}{2}. \hfill (72)$$

Indeed, by (7) and (8), we have $2\Lambda_{\min}(2s) \leq \| X_I c + X_{SD} c' \|_2^2/n \leq 2\Lambda_{\max}(2s)$ and $2\Lambda_{\min}(2s) \leq \| X_I c - X_{SD} c' \|_2^2/n \leq 2\Lambda_{\max}(2s)$. Hence (72) follows from the parallelogram identity:

$$\left| \frac{\langle X_I c, X_{SD} c' \rangle}{n} \right| = \left| \frac{\| X_I c + X_{SD} c' \|_2^2 - \| X_I c - X_{SD} c' \|_2^2}{4} \right|.$$

Moreover, (35) follows from the arguments as in (43), using the Cauchy-Schwarz inequality.

B Proof of the MSE lower bound

We show the proof of (15) for self-containment. Note that due to different normalization of columns of $X$, our expressions are slightly different from those in [8]. Hence we give a complete derivation here. Consider the least square estimator $\hat{\beta}_I = (X^T X_I)^{-1} X_I^T Y$, where $|I| \leq s$ and consider the ideal least-squares estimator $\beta^\circ$ which minimizes the expected mean squared error

$$\beta^\circ = \arg\min_{I \subset \{1, \ldots, p\}, |I| \leq s} \mathbb{E} \left\| \beta - \hat{\beta}_I \right\|_2^2.$$  \hfill (73)

**Proposition B.1.** [8] If $\Lambda_{\max}(s) < \infty$, then

$$\mathbb{E} \left\| \beta - \beta^\circ \right\|_2^2 \geq \min (1, 1/\Lambda_{\max}(s)) \sum_{i=1}^p \min (\beta_i^2, \sigma_i^2/n). \hfill (74)$$

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Proof. Let $I$ be a fixed subset of indices and consider the OLS estimator. The error of this estimator is given by

$$\|\hat{\beta}_I - \beta\|^2_2 = \|\hat{\beta}_I - \beta\|^2_I + \|\beta_I - \beta\|^2_2. \quad (75)$$

The first term is:

$$\hat{\beta}_I - \beta_I = (X_I^T X_I)^{-1}X_I^TY - \beta_I$$

$$= (X_I^T X_I)^{-1}X_I^T(X_I \beta_I + X_{I^c} \beta_{I^c} + \epsilon) - \beta_I$$

$$= (X_I^T X_I)^{-1}X_I^T X_{I^c} \beta_{I^c} + (X_I^T X_I)^{-1}X_I^T \epsilon,$$

and its mean squared error is given by

$$\mathbb{E} \|\hat{\beta}_I - \beta\|^2_2 = \|(X_I^T X_I)^{-1}X_I^T X_{I^c} \beta_{I^c}\|^2_2 + \mathbb{E} \|(X_I^T X_I)^{-1}X_I^T \epsilon\|^2_2,$$

where

$$\mathbb{E} \|(X_I^T X_I)^{-1}X_I^T \epsilon\|^2_2 = \frac{\sigma^2}{n} \text{Tr} \left( \left( \frac{X_I^T X_I}{n} \right)^{-1} \right) \geq \frac{\sigma^2}{n} \frac{|I|}{\Lambda_{\max}(\{I\})},$$

since eigenvalues of $\left( \frac{X_I^T X_I}{n} \right)^{-1}$ are in the range of $\left[ \frac{1}{\Lambda_{\max}(X_I^T X_I/n)}, \frac{1}{\Lambda_{\min}(X_I^T X_I/n)} \right]$. Thus

$$\mathbb{E} \|\hat{\beta}_I - \beta\|^2_2 \geq \frac{\sigma^2}{n} \frac{|I|}{\Lambda_{\max}(\{I\})}. \quad (76)$$

Thus for all sets $I$, such that $|I| \leq s$ and for $\Lambda_{\max}(s) < \infty$, we have

$$\mathbb{E} \|\hat{\beta}_I - \beta\|^2_2 \geq \frac{\sigma^2}{n} \frac{|I|}{\Lambda_{\max}(s)} + \|\beta_{I^c}\|^2_2$$

$$\geq \min(1, 1/\Lambda_{\max}(s)) \left( \sum_{j \in I^c} \beta_j^2 + \frac{\sigma^2}{n} |I| \right),$$

which gives that the ideal mean squared error is bounded below by

$$\mathbb{E} \|\beta - \beta^o\|^2_2 \geq \min(1, 1/\Lambda_{\max}(s)) \min_I \left( \sum_{j \in I^c} \beta_j^2 + \frac{\sigma^2}{n} |I| \right)$$

$$= \min(1, 1/\Lambda_{\max}(s)) \sum_{i=1}^p \min(\beta_i^2, \sigma^2/n).$$
C Proof of Theorem 2.3

We first show Lemma C.1 which gives us the prediction error using $\beta_{T_0}$.

**Lemma C.1.** Suppose that (8) holds. We have for $\lambda = \sqrt{(2 \log p)/n}$.

$$
\|X\beta - X\beta_{T_0}\|_2 / \sqrt{n} \leq \sqrt{\Lambda_{\max}(s-s_0)}\lambda \sigma \sqrt{s_0}.
$$  \hfill (77)

**Proof.** The lemma holds given that $\|\beta_{T_0}\|_2 \leq \lambda \sigma \sqrt{s_0}$, and

$$
\|X\beta - X\beta_{T_0}\|_2 / \sqrt{n} = \|X\beta_{T_0}\|_2 / \sqrt{n} \leq \sqrt{\Lambda_{\max}(s-s_0)}\|\beta_{T_0}\|_2.
$$

We do not focus on obtaining the best constants in the proof of Theorem 2.3.

**Proof of Theorem 2.3.** Throughout this proof, we assume that (7a) holds. We use $\hat{\beta} := \beta_{\text{init}}$ to represent the solution to the Lasso estimator in (3). By the optimality of $\hat{\beta}$, we have

$$
\frac{1}{2n} \left\| Y - X\hat{\beta} \right\|_2^2 - \frac{1}{2n} \left\| Y - X\beta_{T_0} \right\|_2^2 \leq \lambda_n \left\| \beta_{T_0} \right\|_1 - \lambda_n \left\| \hat{\beta} \right\|_1,
$$  \hfill (78)

and similarly, we have for $\beta_0 = \beta_{T_0}$,

$$
\left\| Y - X\beta_0 \right\|_2^2 = \left\| X\beta - X\beta_0 + \epsilon \right\|_2^2 = \left\| X\hat{\beta} - X\beta \right\|_2^2 + 2(\beta - \hat{\beta})^T X^T \epsilon + \|\epsilon\|_2^2.
$$

Let $h = \hat{\beta} - \beta_0$. Thus by (78) and the triangle inequality, we have on $T_a$

$$
\frac{\left\| X\hat{\beta} - X\beta \right\|_2^2}{n} \leq \frac{\left\| X\beta - X\beta_0 \right\|_2^2}{n} + \frac{2h^T X^T \epsilon}{n} + 2\lambda_n (\|\beta_0\|_1 - \|h + \beta_0\|_1) \leq \frac{\left\| X\beta - X\beta_0 \right\|_2^2}{n} + 2\|h\|_1 \left\| \frac{X^T \epsilon}{n} \right\|_\infty + 2\lambda_n (\|h_{T_0}\|_1 - \|h_{T_0}\|_1) \leq \frac{\left\| X\beta - X\beta_0 \right\|_2^2}{n} + 3\lambda_n \|h_{T_0}\|_1 - \lambda_n \|h_{T_0}\|_1,
$$

where we have used the fact that $\lambda_n \geq 2\lambda_{\sigma,a,p}$ for $a \geq 0$. Thus we have on $T_a$,

$$
\left\| X\hat{\beta} - X\beta \right\|_2^2 / n + \lambda_n \|h_{T_0}\|_1 \leq \left\| X\beta - X\beta_0 \right\|_2^2 / n + 3\lambda_n \|h_{T_0}\|_1,
$$  \hfill (79)

which is also the starting point of our analysis on the oracle inequalities of the Lasso estimator. Now we differentiate between two cases.

1. Suppose that on $T_a$, $\left\| X\beta - X\beta_0 \right\|_2^2 / n \geq \lambda_n \|h_{T_0}\|_1$. We then have that

$$
\left\| X\hat{\beta} - X\beta \right\|_2^2 / n + \lambda_n \|h_{T_0}\|_1 \leq 4 \left\| X\beta - X\beta_0 \right\|_2^2 / n
$$  \hfill (80)
and hence for $\lambda_n = d_0 \lambda \sigma$, where $d_0 \geq 2\sqrt{1 + a}$, we have by Lemma C.1 on $T_a$,

$$\|h_T\|_1 \leq \frac{4\Lambda_{\max}(s - s_0)\lambda^2 \sigma^2 s_0}{d_0 \lambda \sigma} \leq \frac{4\Lambda_{\max}(s - s_0)\lambda \sigma s_0}{d_0}.$$  \hspace{1cm} (81)

Now by (79), we have for $d_0 \geq 2\sqrt{1 + a}$,

$$\|h\|_1 \leq \|X\beta - X\beta_0\|_2^2/(n\lambda_n) + 4\|h_{T_0}\|_1 \leq 5\|X\beta - X\beta_0\|_2^2/(n\lambda_n)$$

$$= 5\Lambda_{\max}(s - s_0)\lambda \sigma s_0/d_0,$$  \hspace{1cm} (82)

and clearly $\|Xh\|_2 \leq \|X\beta - X\beta\|_2^2 + \|X\beta - X\beta_0\|_2 \leq 3\|X\beta - X\beta_0\|_2$  \hspace{1cm} (83)

by the triangle inequality and (80). By Lemma 4.1 and (81), we have on $T_a$,

$$\|h_{T_0}\|_2 \leq \frac{1}{\sqrt{n}\sqrt{\Lambda_{\min}(2s_0)}} \|Xh\|_2 + \frac{\theta_{s_0,2s_0}}{\sqrt{s_0\Lambda_{\min}(2s_0)}} \|h_{T_0}\|_1$$

$$\leq 3\lambda \sigma \sqrt{s_0} \frac{\sqrt{\Lambda_{\max}(s - s_0)}}{\sqrt{\Lambda_{\min}(2s_0)} + \frac{\theta_{s_0,2s_0}}{\sqrt{s_0\Lambda_{\min}(2s_0)}} \frac{4\Lambda_{\max}(s - s_0)\lambda \sigma s_0}{d_0}}$$

$$:= D\lambda \sigma \sqrt{s_0}, \text{ for } D = \frac{3\sqrt{\Lambda_{\max}(s - s_0)}}{\sqrt{\Lambda_{\min}(2s_0)}} + \frac{4\theta_{s_0,2s_0}\Lambda_{\max}(s - s_0)}{d_0\Lambda_{\min}(2s_0)}.  \hspace{1cm} (84)$$

2. Otherwise, suppose on $T_a$, we have $\|X\beta - X\beta_0\|_2^2/n \leq \lambda_n \|h_{T_0}\|_1$; thus

$$\left\|X\beta - X\beta\right\|_2^2/(n\lambda_n) + \|h_{T_0}\|_1 \leq 4\|h_{T_0}\|_1$$

and $\|h_{T_0}\|_1 \leq 4\|h_{T_0}\|_1$, which under the RE($s_0, 4, X$) condition immediately implies that

$$\|h_{T_0}\|_2 \leq K(s_0, 4) \|Xh\|_2/\sqrt{n}.  \hspace{1cm} (85)$$

The rest of the proof is devoted to this second case.

We use $K := K(s_0, 4)$ as a shorthand below. By (79) and (85), we have on $T_a$,

$$\left\|X\beta - X\beta\right\|_2^2/n + \lambda_n \|h_{T_0}\|_1 - \|X\beta - X\beta_0\|_2^2/n \leq 3\lambda_n \|h_{T_0}\|_1 \leq 3\lambda_n \sqrt{s_0} \|h_{T_0}\|_2 \leq \frac{3K\lambda_n\sqrt{s_0}}{\sqrt{n}} \|Xh\|_2$$

$$\leq 3K\lambda_n\sqrt{s_0} \left\|X\beta - X\beta\right\|_2^2 + 3K\lambda_n\sqrt{s_0} \|X\beta - X\beta_0\|_2/\sqrt{n}  \hspace{1cm} (86)$$

$$\leq \left\|X\beta - X\beta\right\|_2^2/n + (3K\lambda_n\sqrt{s_0}/2)^2 + 3K\lambda_n\sqrt{s_0} \|X\beta - X\beta_0\|_2/\sqrt{n},  \hspace{1cm} (87)$$

where (86) is by the triangle inequality and (87) is by the Cauchy Schwartz inequality, from which the following immediately follows: for $\lambda_n = d_0 \lambda \sigma \geq 2\lambda_{\sigma,a,p}$,

$$\|h_{T_0}\|_1 \leq \left\|X\beta - X\beta_0\|_2^2/(n\lambda_n) + (3K/2)^2 \lambda_n s_0 + 3K\sqrt{s_0} \|X\beta - X\beta_0\|_2/\sqrt{n} = \left(\|X\beta - X\beta_0\|_2/\sqrt{n\lambda_n} + (3K/2)\sqrt{\lambda_n s_0}\right)^2 := D'_{d_0}d_0\lambda \sigma s_0.  \hspace{1cm} (88)$$
Now recall $X\beta - X_{T_0}\beta = X_{T_0}\beta$ and hence when we take $d_0 \propto \sqrt{\lambda_{\max}(s - s_0)}$, then
\[
\|X_{T_0}\beta\|^2_2 / (n\lambda_n) \leq \lambda_{\max}(s - s_0) \|\beta_{T_0}\|^2_2 / \lambda_n
\]
\[
\leq \lambda_{\max}(s - s_0)\lambda\sigma s_0 / d_0 \propto \sqrt{\lambda_{\max}(s - s_0)\lambda\sigma s_0} \propto \lambda_n s_0,
\]
while $\lambda_n s_0 = d_0 \lambda\sigma s_0$ and hence $D'_1 = (1 + 3K(s_0, 4)/2)^2$.

Combining (81) and (88) gives us the expression in (30). Similarly, we can derive a bound on $\|h\|_1$ from (79); we have on $T_a$,
\[
\|X\tilde{\beta} - X\beta\|^2_2 / n + \lambda_n \|h_{T_0}\|_1 + \lambda_n \|X\beta - X\beta_0\|^2_2 / n \leq 4\lambda_n \|h_{T_0}\|_1
\]
\[
\leq 4\lambda_n \sqrt{s_0} \|h_{T_0}\|_2 \leq 4K\lambda_n \sqrt{s_0} \|Xh\|_2 / \sqrt{n} \quad \text{(by (85))}
\]
\[
\leq 4K\lambda_n \sqrt{s_0} \|X\beta - X\beta_0\|_2 / \sqrt{n} + \|X\tilde{\beta} - X\beta\|^2_2 / n + (2K\lambda_n \sqrt{s_0})^2.
\]
Hence it is clear that for $\lambda_n = d_0 \lambda\sigma \geq 2\lambda_{\sigma,ap}$, we have by Lemma 4.1 on $T_a$,
\[
\|h\|_1 \leq \|X\beta - X\beta_0\|^2_2 / (n\lambda_n) + 4K\sqrt{s_0} \|X\beta - X\beta_0\|^2_2 / \sqrt{n} + 4K^2\lambda_n s_0
\]
\[
= \left(\|X\beta - X\beta_0\|^2_2 / \sqrt{n}\lambda_n + 2K\lambda_n \sqrt{s_0}\right)^2 \leq D'_2\lambda s_0 \propto K(s_0, 4)^2\lambda_n s_0,
\]
where $D'_2 = (\sqrt{\lambda_{\max}(s - s_0)/d_0} + 2\sqrt{d_0}K(s_0, 4))^2$. Combining (89) and (82) gives us the bound on $\|h\|_1$ and the expression in (31). Now we derive a bound for $\|X\tilde{\beta} - X\beta\|^2_2 / n$; our starting point is (86), from which by shifting items around and adding $(3K\lambda_n \sqrt{s_0}/2)^2$ to both sides, we obtain
\[
\|X\tilde{\beta} - X\beta\|^2_2 / n - 3K\lambda_n \sqrt{s_0} \|X\tilde{\beta} - X\beta\|^2_2 / \sqrt{n} + (3K\lambda_n \sqrt{s_0}/2)^2 + \lambda_n \|h_{T_0}\|_1
\]
\[
\leq \|X\beta - X\beta_0\|^2_2 / n + 3K\lambda_n \sqrt{s_0} \|X\beta - X\beta_0\|_2 / \sqrt{n} + (3K\lambda_n \sqrt{s_0}/2)^2.
\]
Thus we have for $\lambda_n = d_0 \lambda\sigma \geq 2\lambda_{\sigma,ap}$,
\[
\left(\frac{1}{\sqrt{n}} \|X\tilde{\beta} - X\beta\|^2_2 - \frac{3K\lambda_n \sqrt{s_0}}{2}\right)^2 + \lambda_n \|h_{T_0}\|_1 \leq (\|X\beta - X\beta_0\|^2_2 / \sqrt{n} + 3K\lambda_n \sqrt{s_0}/2)^2
\]
and hence by Lemma 4.1
\[
\|X\tilde{\beta} - X\beta\|^2_2 / \sqrt{n} \leq \|X\beta - X\beta_0\|^2_2 / \sqrt{n} + 3K\lambda_n \sqrt{s_0}
\]
\[
\leq \lambda\sigma \sqrt{s_0} \left(\sqrt{\lambda_{\max}(s - s_0)} + 3d_0K(s_0, 4)\right).
\]
By (80), (77) and (90), we obtain the expression in (32). Now under RE$(s_0, 4, X)$ condition, we have by (90)
\[
\|h_{T_0}\|_2 \leq K(s_0, 4) \|Xh\|_2 / \sqrt{n} \leq \frac{K(s_0, 4)}{\sqrt{n}} \left(\|X\tilde{\beta} - X\beta\|^2_2 + \|X\beta - X\beta_0\|^2_2\right)
\]
\[
\leq K(s_0, 4) \left(2 \|X\beta - X\beta_0\|_2 / \sqrt{n} + 3K(s_0, 4)\lambda_n \sqrt{s_0}\right)
\]
\[
\leq \lambda\sigma \sqrt{s_0} K(s_0, 4) (2\sqrt{\lambda_{\max}(s - s_0)} + 3d_0K(s_0, 4))
\]
\[
= D'_0\lambda \sigma \sqrt{s_0}, \quad \text{where}
\]
\[
D'_0 = (\sqrt{\lambda_{\max}(s - s_0)} \lor d_0)(2 + 3K(s_0, 4))K(s_0, 4).
\]

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Let $T_1$ be the $s_0$ largest positions of $h$ outside of $T_0$; Now by Proposition A.1 as derived in [48], we have
\[ \|h_{T_0}\|_2 \leq \sqrt{2/nK(s_0, 4)} \|Xh\|_2 \leq \sqrt{2D'\sigma_n}. \quad (91) \]
Combining (91) and (84) gives us the expression in (28). Moreover, we have by Lemma 4.1,
\[ \|\hat{\beta} - \beta\|_2^2 \leq 2\|\hat{\beta} - \beta_{T_0}\|_2^2 + 2\|\beta - \beta_{T_0}\|_2^2 \leq 2\|h\|_2^2 + 2\lambda_n^2\|\beta\|_2^2 \]
\[ \leq 2(\|h_{T_0}\|_2^2 + \|h_{T_0}\|_1/s_0) + 2\lambda_n^2\|\beta\|_2^2 \leq 2\lambda_n^2\|\beta\|_2^2(D_0^2 + D_1^2 + 1). \]
We note that (27) holds given (80) and (90).

\[ \square \]

C.1 Proof of Lemma 4.1

Decompose $h_{T_0}$ into $h_{T_2}, \ldots, h_{T_k}$ such that $T_2$ corresponds to locations of the $s_0$ largest coefficients of $h_{T_0}$ in absolute values, and $T_3$ corresponds to locations of the next $s_0$ largest coefficients of $h_{T_0}$ in absolute values, and so on. Let $V$ be the span of columns of $X_j$, where $j \in T_0$, and $P_V$ be the orthogonal projection onto $V$. Decompose $P_VXh$:
\[ P_VXh = P_VXh_{T_0} + \sum_{j \geq 2} P_VXh_{T_j} = Xh_{T_0} + \sum_{j \geq 2} P_VXh_{T_j}, \]
where \[ \forall j \geq 2, \|P_VXh_{T_j}\|_2 \leq \frac{\sqrt{n\theta_{s_0,2s_0}}}{\sqrt{\Lambda_{\min}(2s_0)}} \|h_{T_j}\|_2 \]
and \[ \sum_{j \geq 2} \|h_{T_j}\|_2 \leq \|h_{T_0}\|_1/\sqrt{s_0} ; \]
see [8] for details. Thus we have
\[ \|Xh_{T_0}\|_2 = \|P_VXh - \sum_{j \geq 2} P_VXh_{T_j}\|_2 \leq \|P_VXh\|_2 + \sum_{j \geq 2} \|P_VXh_{T_j}\|_2 \]
\[ \leq \|Xh\|_2 + \sum_{j \geq 2} \|P_VXh_{T_j}\|_2 \leq \|Xh\|_2 + \frac{\sqrt{n\theta_{s_0,2s_0}}}{\sqrt{\Lambda_{\min}(2s_0)}} \|h_{T_0}\|_1 , \]
where we used the fact that $\|P_V\|_2 \leq 1$. Hence the lemma follows given $\|h_{T_0}\|_2 \leq \frac{1}{\sqrt{\Lambda_{\min}(2s_0)}} \|Xh_{T_0}\|_2$. For other bounds, the fact that the $k$th largest value of $h_{T_0}$ obeys $|h_{T_0}^{(k)}| \leq \|h_{T_0}\|_1/k$ has been used; see [3].

\[ \square \]

C.2 Proof of Lemma 4.2

For $Xh_{T_0} = Xh - \sum_{j \geq 2} Xh_{T_j}$, we have by the triangle inequality and sparse eigenvalue condition,
\[ \|Xh_{T_0}\|_2 / \sqrt{n} \leq \|Xh\|_2 / \sqrt{n} + \sum_{j \geq 2} \|Xh_{T_j}\|_2 / \sqrt{n} \]
\[ \leq \|Xh\|_2 / \sqrt{n} + \sqrt{\Lambda_{\max}(s_0)} \sum_{j \geq 2} \|h_{T_j}\|_2 \]
\[ \leq \|Xh\|_2 / \sqrt{n} + \sqrt{\Lambda_{\max}(s_0)} \sum_{j \geq 1} \|h_{T_j}\|_1 / \sqrt{s_0} \]
\[ \leq \|Xh\|_2 / \sqrt{n} + \sqrt{\Lambda_{\max}(s_0)} \|h_{T_0}\|_1 / \sqrt{s_0} . \]

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Thus it follows from the proof Lemma 4.1 that
\[ \| h_{T_0} \|_2 \leq \| X h_{T_0} \|_2 / \sqrt{n} \Lambda_{\min}(2s_0) \leq \frac{1}{\sqrt{\Lambda_{\min}(2s_0)}} \left( \| X h \|_2 / \sqrt{n} + \sqrt{\Lambda_{\max}(s_0)} \| h_{I^0} \|_1 / \sqrt{s_0} \right), \]
where we replace \( \frac{\theta_{s,2s_0}}{\sqrt{\Lambda_{\min}(2s_0)}} \) with \( \sqrt{\Lambda_{\max}(s_0)} \).

\( \square \)

**D Proof of Theorem 3.2**

Now similar to Lemma 2.2, Lemma D.1 bounds the size of \( I \), as well as the bias that we introduce to model \( I \) by thresholding. Theorem 3.2 is an immediate corollary of Lemmas 2.4 and D.1. The proof follows from Lemma D.1 and Proposition 3.3. We include its proof in Section D.1 for self-containment.

**Lemma D.1.** Choose \( \tau > 0 \) such that \( \delta_{2s} + \theta_{s,2s} < 1 - \tau \). Let \( \beta_{\text{init}} \) be the solution to (4) with \( \lambda_n = \lambda_{\rho,\tau} := (1 + \alpha + \tau^{-1}) \sqrt{2 \log p / n \sigma}. \) Given some constant \( C_4 \geq C_1 \), for \( C_1 \) as in (39), choose a thresholding parameter \( t_0 \) such that \( C_4 \lambda_{\rho,\tau} \geq t_0 > C_1 \lambda_{\rho,\tau} \) and set \( I = \{ j : | \beta_{j,\text{init}} | > t_0 \} \). Then with probability at least \( 1 - (\sqrt{\pi \log pp}^3)^{-1} \), we have (37) and \( \| \beta_D \|_2 \leq \sqrt{(C_0 + C_4)^2 + 1 \lambda_{\rho,\tau} \sigma \sqrt{s_0}} \), where \( D := \{ 1, \ldots, p \} \setminus I \) and \( C_0 \) is defined in (38).

**Proof of Theorem 3.2.** It holds by definition of \( S_D \) that \( I \cap S_D = \emptyset \). It is clear by Lemma D.1 that \( |S_D| < s \) and \( |I| \leq 2s_0 \) and \( |I \cup S_D| \leq |I \cup S| \leq s + s_0 \leq 2s \); Thus for \( \tilde{\beta}_I = (X_I^T X_I)^{-1} X_I^T Y \) and \( \lambda = \sqrt{2 \log p / n} \), we have by Lemma 2.5
\[ \| \tilde{\beta}_I - \beta \|_2^2 \leq \| \beta_D \|_2^2 \left( 1 + \frac{2 \theta_{s,2s_0}^2}{\Lambda_{\min}(2s_0)} \right) + \frac{4s_0}{\Lambda_{\min}(2s_0)} \Lambda_{\sigma,a,p}^2 \]
\[ \leq \lambda^2 s_0 \left( \sqrt{1 + \alpha + \tau^{-1}} \right)^2 \left( (C_0^2 + C_4)^2 + 1 + \frac{2 \theta_{s,2s_0}^2}{\Lambda_{\min}(2s_0)} \right) \]
\[ \leq C_3^2 \lambda^2 \sigma^2 s_0 \]
with probability at least \( 1 - (\sqrt{\pi \log pp}^3)^{-1} - \exp(-3m/64) \) where \( m = |I| \). Thus the theorem holds for \( C_3 \) as in (40), where it holds for \( \tau > 0 \) that
\[ \frac{\theta_{s,2s_0}}{\Lambda_{\min}(2s_0)} \leq \frac{\theta_{s,2s}}{\Lambda_{\min}(2s_0)} \leq \frac{1 - \delta_{2s} - \tau}{\Lambda_{\min}(2s)} < 1 \]
given that \( \theta_{s,2s} < 1 - \tau - \delta_{2s} < \Lambda_{\min}(2s) \) for \( \tau > 0 \).

**D.1 Proof of Lemma D.1**

Suppose that \( T_a \) holds. Consider the set \( I \cap T_0^c \equiv \{ j \in T_0^c : | \beta_{j,\text{init}} | > t_0 \}. \) It is clear by definition of \( h = \beta_{\text{init}} - \beta^{(1)} \) and (44) that
\[ |I \cap T_0^c| \leq \| \beta_{T_0^c,\text{init}} \|_1 / t_0 = \| h_{T_0^c} \|_1 / t_0 < s_0, \] (92)
where $t_0 \geq C_1 \lambda_{p,\tau}$. Thus $|I| = |I \cap T_0| + |I \cap T_0^c| \leq 2s_0$; Now (37) holds given (92) and $|I \cup S| = |S| + |I \cap S^c| \leq s + |I \cap T_0^c| < s + s_0$. We now bound $\|\beta_D\|_2^2$. By (93) and (51), where $D_{11} \subset T_0$, we have for $\tau < C_4 \lambda_{p,\tau}$, by the triangle inequality,

$$\|\beta_D\|_2^2 = \|\beta_{D \cap T_0}\|_2^2 + \|\beta_{D \cap T_0^c}\|_2^2 \leq \beta(12) + \|\beta_{I \cap T_0}\|_2^2 \leq s_0 \lambda^2 \sigma^2 + (t_0 \sqrt{s_0} + \|h_{T_0}\|_2^2)^2 \leq ((C_4 + C_0)^2 + 1) \lambda^2 \sigma^2 s_0.$$

The proof of Proposition 3.3 (cf. [5]) yields the following on $T_a$,

$$\|h_{T_0}\|_2 \leq C_0 \lambda_{p,\tau} \sqrt{s_0}, \quad \text{for } C_0 \text{ as in (38)}, \quad (93)$$

$$\|h_{T_0}\|_1 \leq C_1 \lambda_{p,\tau} s_0, \quad \text{where } C_1 = \left( C_0 + \frac{1 + \delta}{1 - \delta - \theta} \right), \quad \text{and} \quad (94)$$

$$\|h_{T_0}\|_2 \leq \|h_{T_0}\|_1 \sqrt{s_0} \leq C_1 \lambda_{p,\tau} \sqrt{s_0}, \quad \text{(cf. Lemma 4.1).} \quad (95)$$

The rest of the proof follows that of Lemma 2.2 and hence omitted.

\[\square\]

E Proof of Lemma [5.2]

We write $\beta = \beta^{(11)} + \beta^{(12)} + \beta^{(2)}$ where

$$\beta^{(11)}_j = \beta_j \cdot 1_{1 \leq j \leq a_0}, \quad \beta^{(12)}_j = \beta_j \cdot 1_{a_0 < j \leq s_0}, \quad \text{and} \quad \beta^{(2)} = \beta_j \cdot 1_{j > s_0}.$$ 

By definition of $s_0$ as in (19), we have $\sum_{i=1}^{p} \min(\beta_i^2, \lambda^2 \sigma^2) \leq s_0 \lambda^2 \sigma^2$. Now it is clear that

$$\sum_{j \leq a_0} \min(\beta_j^2, \lambda^2 \sigma^2) = a_0 \lambda^2 \sigma^2,$$

and hence

$$\sum_{j > a_0} \min(\beta_j^2, \lambda^2 \sigma^2) = \beta(12) + \beta(2) \leq (s_0 - a_0) \lambda^2 \sigma^2. \quad (96)$$

It is clear for $D_{11} = D \cap A_0$, we have $D_{11} \subset A_0 \subset T_0 \subset S$. Let $\beta_{D}^{(11)} := (\beta_j)_{j \in A_0 \cap D}$ consist of coefficients of $\beta$ that are above $\lambda \sigma$ in their absolute values but are dropped as $|\beta_{j, \text{init}}| < t_0$. Now by (96), we have

$$\|\beta_D\|_2^2 \leq \|\beta_{D}^{(11)}\|_2^2 + \|\beta(12) + \beta(2)\|_2^2 \leq \|\beta_{D}^{(11)}\|_2^2 + (s_0 - a_0) \lambda^2 \sigma^2,$$

where $|D_{11}| \leq a_0, \|\beta_{D}^{(11)}\|_\infty < t_0$ and we have by the triangle inequality,

$$\|\beta_{D}^{(11)}\|_2 \leq \|\beta_{D, \text{init}}\|_2 + \|\beta_{D_{11}, \text{init}} - \beta_{D}^{(11)}\|_2 \leq t_0 \sqrt{|D_{11}|} + \|h_{D_{11}}\|_2 \leq t_0 \sqrt{a_0} + \|h_{D_{11}}\|_2. \quad (97)$$
Thus (51) holds. Now we replace the crude bound of $|\mathcal{D}_{11}| \leq a_0$ with

$$|\mathcal{D}_{11}| \leq \frac{\|h_{\mathcal{D}_{11}}\|_2^2}{\beta_{\min,A_0 - t_0}^2}$$

in (97) to obtain

$$\|\beta_{\mathcal{D}}^{(11)}\|_2 \leq t_0 \frac{\|h_{\mathcal{D}_{11}}\|_2}{\beta_{\min,A_0 - t_0}^2} + \|h_{\mathcal{D}_{11}}\|_2 = \|h_{\mathcal{D}_{11}}\|_2 \frac{\beta_{\min,A_0}}{\beta_{\min,A_0 - t_0}},$$

which proves (52).