Dynamical $SU(2)$ magnetic monopoles

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Abstract

In our paper Phys. Rev. Lett. 92, 151801 (2004), the oscillations of strongly deformed Bogomolny-Prasad-Sommerfield magnetic monopoles have been studied on a fixed Minkowski background. The purpose of the present article is to provide a more detailed account on the results yielded by our numerical simulations. In particular, an analysis on the dependence on the strength of the initial excitation is carried out in order to distinguish features which are already present for small perturbations from those which are consequences of the nonlinearity of the system.

1 Introduction

The investigation of solitons in particle physics is of fundamental interest (see e.g. [1] for a recent review). In particular, considerable attention has been paid to the study of ’t Hooft-Polyakov magnetic monopole solutions of coupled Yang-Mills–Higgs (YMH) systems [2, 3]. Magnetic monopoles are important because they are present in a wide range of gauge theory models. Moreover, their existence can be used to provide an account on various physical phenomena, including e.g. the quantized nature of electric charge. In a recent paper by the present authors [4] a report is provided on the numerical study on the behavior of strongly excited Bogomolny-Prasad-Sommerfield (BPS) magnetic monopoles [5, 6]. In a related article by Forgács and Volkov [7] the same system is investigated by perturbative methods. There is a striking consistency between the key results of these two independent studies. The aim of the present paper is to provide more detailed description of the system based on the results obtained by the numerical investigations of [4].

The investigated dynamical magnetic monopole is a solution of a coupled $SU(2)$ YMH system. The Yang-Mills field is represented by an $\mathfrak{su}(2)$-valued vector potential $A_a$ and the associated 2-form field $F_{ab}$ reads as

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\[ F_{ab} = \nabla_a A_b - \nabla_b A_a + ig [A_a, A_b] \quad (1) \]

where \([, ,]\) denotes the product in \(su(2)\) and \(g\) stands for the gauge coupling constant. The Higgs field (in the adjoint representation) is given by an \(su(2)\)-valued function \(\psi\) while its gauge covariant derivative reads as \(D_a \psi = \nabla_a \psi + ig[A_a, \psi]\). The dynamics of the investigated YMH system is determined by the Lagrangian

\[ \mathcal{L} = Tr(F_{ef}F^{ef}^\ast) + 2Tr(D_e \psi D^e \psi) + \lambda \left( Tr(\psi \psi) - v^2 \right)^2, \quad (2) \]

where \(\lambda\) is the self interaction constant of the Higgs field. In order to have finite energy solutions the value of \(Tr(\psi \psi)\) must tend to \(v^2\) as the distance from the monopole goes to infinity. The energy-momentum tensor of the YMH field takes the form

\[ T_{ab} = -\frac{1}{4\pi} \left[ Tr(F_{ae}F^e_b) - Tr(D_a \psi D^a_b \psi) + \frac{1}{4} g_{ab} \mathcal{L} \right]. \quad (3) \]

Our considerations are restricted to spherically symmetric configurations yielded by the ‘minimal’ dynamical generalization of the static ’t Hooft-Polyakov magnetic monopole configurations \([2, 3]\) (see also \([8]\)). Accordingly, the evolution takes place on Minkowski spacetime the line element of which, in spherical coordinates \((t, r, \theta, \phi)\), is

\[ ds^2 = dt^2 - dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2), \quad (4) \]

while the Yang-Mills and Higgs fields, in the so called abelian gauge, are assumed to posses the form

\[ A_a = -\frac{1}{g} \left[ w \left( \tau_2 (d\theta)_a - \tau_1 \sin \theta (d\phi)_a \right) + \tau_3 \cos \theta (d\phi)_a \right] \quad (5) \]

\[ \psi = H \tau_3, \quad (6) \]

where the generators \(\{\tau_I\}\) \((I=1,2,3)\) of \(su(2)\) are related to the Pauli matrices \(\sigma_I\) as \(\tau_I = \frac{i}{2} \sigma_I\), moreover, \(w\) and \(H\) are assumed to be smooth functions of \(t\) and \(r\).

In the Bogomolny-Prasad-Sommerfield (BPS) limit, i. e. when \(\lambda \to 0\), the mass of the Higgs boson becomes much less than the mass of the vector boson. The field equations in the \(\lambda = 0\) case become

\[ r^2 \partial^2_r w - r^2 \partial^2_\theta w = w \left( (w^2 - 1) + g^2 r^2 H^2 \right) \quad (7) \]

\[ r^2 \partial^2_\theta H + 2r \partial_r H - r^2 \partial^2_\phi H = 2w^2 H, \quad (8) \]

with the restriction that

\[ \lim_{r \to \infty} H = v, \quad (9) \]

in order to represent the BPS limit of finite energy solutions. We note that the time independence of the limit value of \(H\) at both spacelike and null infinity follows from the field equations \(\mathcal{E}\) and \(\mathcal{S}\) (see \([9]\)).

The system of equations \(\mathcal{E}\) and \(\mathcal{S}\) with condition \(\mathcal{B}\) has an analytic solution, the static BPS monopole \([5, 6]\)

\[ u_0 = \frac{gvr}{\sinh(gvr)}, \quad H_0 = v \left[ \frac{1}{\tanh(gvr)} - \frac{1}{gvr} \right]. \quad (10) \]
In the rest of the paper considerations will be restricted to the \( \lambda = 0 \) case and strong impulse type excitations of the BPS monopole will be investigated.

In the BPS limit the Higgs field becomes massless and the only scale parameter of the system is the vector boson mass \( m_w = g v \). Since in the case considered here \( v \neq 0 \), the rescalings \( t \to t = t m_w, \ r \to \tilde{r} = r m_w \) and \( H \to \tilde{H} = H / v \) transform the parameters to the value \( g = v = 1 \). This implies that it suffices to consider the evolution of the \( g = v = 1 \) system numerically and to study merely the dependence of the evolution on the various initial conditions for this system.

To have a computational grid covering the full physical spacetime – ensuring thereby that the outer grid boundary will not have an effect on the time evolution – the technique of conformal compactification, along with the hyperboloidal initial value problem, is used. This way it is possible to study the asymptotic behavior of the fields close to future null infinity, as well as, the inner region for considerably long physical time intervals.

The conformal transformation we use is a slight modification of the static hyperboloidal conformal transformation applied by Moncrief [10]. It is defined by introducing first the new coordinates \( T \) and \( R \) instead of \( t \) and \( r \) as
\[
T = \omega t + 1 - \sqrt{\omega^2 r^2 + 1} \quad \text{and} \quad R = \frac{\sqrt{\omega^2 r^2 + 1} - 1}{\omega r},
\]
where \( \omega \) is an arbitrary positive constant. The Minkowski spacetime is covered by the coordinate domain given by the inequalities \(-\infty < T < +\infty \) and \( 0 \leq R < 1 \). The \( R = \text{const.} \) lines (same as the \( r = \text{const.} \) lines) represent world-lines of ‘static observers’, while the \( T = \text{const.} \) hypersurfaces are hyperboloids of the Minkowski spacetime satisfying the relation \((\omega t + 1 - T)^2 - \omega^2 r^2 = 1\). The constant \( \omega \) is introduced here in order to have control on the size of the monopole region in terms of the \( R \) coordinate, which is useful for the optimization of the numerical code.

In the rest of the paper we present results obtained by setting \( \omega = 0.05 \).

The line element of the conformally rescaled metric \( \tilde{g}_{ab} = \Omega^2 g_{ab} \) in the coordinates \((T, R, \theta, \phi)\) takes the form
\[
d\tilde{s}^2 = \frac{\Omega^2}{\omega^2} dT^2 + 2 R dT dR - dR^2 - R^2 \left( d\theta^2 + \sin^2 \theta \, d\phi^2 \right),
\]
where the conformal factor is
\[
\Omega = \omega (1 - R^2)/2.
\]
The \( R = 1 \) coordinate line represents \( \mathcal{I}^+ \) through which the conformally rescaled metric \( \tilde{g}_{ab} \) smoothly extends to the unphysical coordinate domain with \( R > 1 \). The simple relation \( r \Omega = R \) holds between the old and new radial coordinates.

Using the substitution \( H(t, r) = h(t, r)/r + v \) the field equations \( \Psi \) and \( \Phi \) in the new coordinates read as
\[
\Psi w = w \left[ (w^2 - 1) + g^2 (h + v r \Omega^{-1})^2 \right], \quad \Phi = 2 (h + v r \Omega^{-1}) w^2.
\]
where the differential operator $\mathcal{P}$ is defined as

$$\mathcal{P} = \frac{4R^2}{(R^2+1)^2} \left[ \frac{\Omega^2}{\omega^2} \partial_R^2 - \partial_T^2 - 2R\partial_R\partial_T \right.
\left. - \frac{2\Omega}{\omega(R^2+1)} \partial_T - \frac{\Omega R(R^2+3)}{\omega(R^2+1)} \partial_R \right].$$

(16)

These equations can be put into the form of a first order strongly hyperbolic system [9]. The initial value problem for such a system is known to be well-posed [11]. In particular, we solved this first order system numerically by making use of the ‘method of line’ in a fourth order Runge-Kutta scheme following the recipes proposed by Gustafsson et al [11]. All the details related to the numerical approach, including representations of derivatives, treatment of the grid boundaries are to be published in [9]. The convergence tests justified that our code provides a fourth order representation of the selected evolution equations. Moreover, the monitoring of the energy conservation and the preservation of the constraint equations, along with the coincidence between the field values which can be deduced by making use of the Green’s function and by the adaptation of our numerical code to the case of massive Klein-Gordon fields, made it apparent that the phenomena described below have to be, in fact, physical properties of the magnetic monopoles.

In each of the numerical simulations initial data on the $T = 0$ hypersurface was specified for the system of our first order evolution equations. In particular, a superposition of the data associated with the BPS monopole, (10), and of an additional pulse of the form

$$ (\partial_T \omega)_{\alpha} = \begin{cases} 
  c \exp \left[ \frac{d}{(r-a)^2-b^2} \right], & \text{if } r \in [a-b, a+b]; \\
  0, & \text{otherwise},
\end{cases}$$

(17)

with $a \geq b > 0$, which is a smooth function of compact support, was used. This choice, providing non-zero time derivative for $\omega$, corresponds to “hitting” the static monopole configuration between two concentric shells at $r = a - b$ and $r = a + b$ with a bell shape distribution. Basically the same type of evolution occurs when instead of $(\partial_T \omega)_{\alpha}$ we prescribe $(\partial_T h)_{\alpha}$ in a similar fashion. The only distinction is that in this later case the energy of the pulse is given directly to the massless Higgs field which carries immediately a considerably large portion of it to $\mathcal{F}^+$ before the Yang-Mills field could take it over to feed it into the breathing state of the monopole and into the expanding oscillations of the massive Yang-Mills field.

All the simulations shown below refer to the same pulse shape (17) corresponding to the choice of the parameters $a = 2$, $b = 1.5$ and $d = 10$, only the amplitude $c$ is varied. It is important to keep in mind that the energy of the pulse is proportional to $c^2$. For example, for the case $c = 70$, presented in [11] it is 55.4% of the energy of the static monopole, which is clearly not just a simple perturbation. We would also like to emphasize that the figures shown below are typical in the sense that for a wide range of the parameters characterizing the exciting pulse, qualitatively, and in
certain cases even quantitatively, the same type of responses are produced by the monopole [9].

In order to provide a better picture about the size and distribution of the monopole and its relation to the initial data it is instructive to plot the energy density distribution on the $T = 0$ initial hypersurface. The conserved energy current vector $j^a = T^a_{\ b} u^b$, where $u = \frac{\partial}{\partial t}$ is the velocity vector of the the static observers characterized by the constant value of $r$ along their worldline. The projection $\varepsilon = j^a n_a = \omega T^0_{\ 0}(R^2 + 1)\Omega^{-1}/2$, where $n_a$ is the future pointed unit normal to the constant $T$ hypersurfaces, gives a gauge invariant quantity characterizing the energy density on these surfaces. The energy “density” associated to shells of radius $R$ can be calculated as $E = \int \varepsilon \sqrt{|h|} d\theta d\phi = 2\pi \omega T^0_{\ 0}(R^2 + 1)R^2\Omega^{-4}$, where $h$ is the determinant of the induced metric on the constant $T$ hypersurfaces. On Fig. 1 we plot the value of $\varepsilon$ and $E$ for the static monopole both with and without an initial deformation of the form of (17). The plot of $E$ is more instructive because its integral from the center $R = 0$ to null infinity $R = 1$ (i. e. the area below the curve) gives the total energy of the system along the chosen constant $T$ hypersurface. As the time passes, the value of this integral decreases exactly by the energy radiated to null infinity.

![Figure 1: Energy density distributions $\varepsilon$ and $E$ on the $T = 0$ initial hypersurface for the static BPS monopole ($c = 0$) and for two different choices of the initial amplitude $c$.](image)

Choosing temporarily $c = 70$, we consider next the quantity $w$, corresponding to the massive Yang-Mills field, plotted on succeeding $T = \text{const.}$ hypersurfaces, providing thereby a spacetime picture of its time evolution (see Fig. 2). One can see the formation of expanding shells of high frequency oscillations in the asymptotic region [12]. These shells take out energy from the central monopole region, although they never reach null infinity because of their massive character. The time evolution of the
other gauge variable, the massless Higgs field $h$ is strikingly different (see Fig. 4). There is no sign of the high frequency oscillations in $h$, justifying that the two fields effectively decouple in the asymptotic region. Since the oscillations in $h$ are nonzero at null infinity, the direct energy transport to $\mathcal{I}^+$ by the Higgs field, with the velocity of light, is apparent.

From Fig. 2 and Fig. 3 it appears that the lower frequency oscillations are not concentrated only at the monopole but they are present basically everywhere. However, if one considers the clearly physically meaningful quantity, the energy density $\mathcal{E}$ corresponding to spherical shells of radius $R$, it is apparent that the oscillations are concentrated at the central region, forming a long lasting ‘breathing state’ of the monopole. In Fig. 4 we show the energy density difference $\mathcal{E} - \mathcal{E}_0$ in a large central region, where $\mathcal{E}_0$ is the energy density of the static monopole. (See Fig. 1 of [1] for the plot of $\mathcal{E}$ up to null infinity.) Although Fig. 4 covers a relatively early time period, the picture remains essentially the same for the rest of the time evolution when taking time intervals of the same length, with the only difference of an overall decrease of its amplitude proportional to $T^{-5/6}$ (see Fig. 5 of [1]).

Next we consider the dependence of the energy density oscillations on the amplitude of the initial deformation. The energy density $\rho = T_{ab}u^au^b$ observed by static Minkowski observers cannot be used to form a conserved quantity when integrating on the hyperbolic $T = \text{const.}$ hypersurfaces. However, for our specific choice $\omega = 0.05$ the $T = \text{const.}$ hyperboloids are so close to the flat $t = \text{const.}$ surfaces in the inner monopole region that plots of the quantities $\varepsilon$ and $\rho$ are almost indistinguishable.
Figure 3: Spacetime diagram showing the time evolution of the variable $h$ corresponding to the massless Higgs component of the gauge field for the initial amplitude choice $c = 70$. At the center $h = 0$ for all time, while at infinity $h$ remains $-1$ until the radiation arrives there.

Figure 4: Spacetime diagram showing the time evolution of the energy density difference $\mathcal{E} - \mathcal{E}_0$ from $T = 4.97$ to $T = 5.58$.

Furthermore, for the static BPS solution $\varepsilon = \rho \equiv \rho_0$. On Fig. 4 we plot the energy density fluctuation $\rho - \rho_0$ around the static monopole value $\rho_0$ for four different choices of the amplitude $c$. Since for small initial per-

turbations the deviations of the density $\rho$ from the static value $\rho_0$ appear to be proportional to the amplitude of the perturbation, we rescale the amplitude of the four graphs accordingly, by plotting the value $(\rho - \rho_0)/c$.

Figure 5: Evolution of the rescaled energy density difference $(\rho - \rho_0)/c$ as seen by a constant radius observer at $R = 0.0254$ (i.e. $r = 1.018$) for four different values of $c$ and two different time intervals.

For $c = 0.7$ and $c = 7$ the corresponding curves almost entirely overlap, showing that these excitations are still in the linear domain. Indeed, the energy provided by the initial perturbation in those two cases are $0.00554$ and $0.554$ percent, respectively, of the energy $4\pi$ of the static monopole. For the $c = 70$ case the energy given to the system is $6.966$, which is $55.4$ percent of the energy of the initially static monopole. In this case the behavior is clearly nonlinear initially, but after the direct pulses have left the system the quasinormal oscillation of the monopole is still not too far from the linear regime. For $c = 280$ the additional energy is $111.46$, which is more than $8$ times the energy of the static monopole. In this case even the monopole oscillations show a different character at the beginning, having a significantly lower frequency. However, in all cases, the frequency of the monopole oscillations tend as $t^{-2/3}$ to the asymptotic value $1$ corresponding to the vector boson mass in our rescaled coordinates (see Fig. 5 of [4]).

After the initial period, i.e. for $t > 5$, the monopole starts a quasi-periodic oscillation with amplitude decreasing as $t^{-5/6}$. It is visible that the mean value of these oscillations, which represents the average energy
density at the given radius is negative for large initial deformations. This means that the energy contained in the oscillating monopole region is actually smaller than the energy of the static monopole which is a clear manifestation of the nonlinear character of the evolution. As it was pointed out to us by Michael Volkov, this situation is similar to that where a large stone is dropped into a lake, and the water level decreases because of the submerging of the stone. This analogy, however, does not provide a completely satisfactory explanation in the monopole case, because the average energy remains negative during the entire evolution, approaching merely in the asymptotic limit the static value from below. This might lead to the interpretation that the monopole behaves like an extremely viscous fluid.

In the last part of the paper we consider the energy radiated to null infinity up to the time $T$. This energy can be calculated as $E_r = \int_0^T SdT$ where $S = \lim_{r \to \infty} j^a k_a$ with $k_a$ being the unit normal to the constant $r$ hypersurfaces. The possibility that we could study this radiation, in particular the evolution near $\mathcal{I}^+$, is due to the fact that the hyperboloidal initial value problem was used in our numerical simulations. The radiation arrives to $\mathcal{I}^+$ when the light cone emanating from the outer edge of the initial perturbation reaches null infinity, i.e. approximately at $T = 0.86$. For small perturbations the radiated energy is proportional to the square of the initial amplitude. Hence we plot the rescaled energy $E_r/c^2$ on Fig.6 for various initial data specifications.

The energy given to the system by the initial deformation is approximately $0.0014216 \cdot c^2$. For small initial perturbations the total radiated energy is $0.00068273 \cdot c^2$, which is 48.02 percent of the energy given to the system. The rest of the energy is in the self-similarly expanding shells formed by the massive Yang-Mills field $w$. These shells get arbitrarily far from the monopole region, but because of their massive character their velocity is smaller than the speed of light, whence they never reach null infinity. Actually, the energy in the shells is initially a little bit larger than the remaining 51.98 percent, since as we have seen, the average energy in the central monopole region is smaller than the energy of the static monopole solution in the same region. As the time passes the amplitude of the monopole oscillations decreases and the mean value of the energy density approaches the static value from below. Accordingly, there is, although a low scale, energy transfer back to the monopole from the outer regions ensuring, in turn, that in the inner region the system will settle down to the static monopole solution.

We would also like to emphasize that – as it is intuitively expected – the ratio between the energy carried by the shells and the energy radiated to $\mathcal{I}^+$ depends on the profile of the initial excitation. It is found that in case of the specific choice this ratio depends on $a$, $b$ and $d$, but for small amplitudes in the linear regime it is apparently independent of the parameter $c$.

Various new features of the evolution of excited $SU(2)$ BPS monopoles have been presented here. The results contained in the last part of this paper indicate that there remained a lot of interesting issues to be studied both numerically and analytically in the dynamical behavior of the t'Hooft-Polyakov monopoles.
Figure 6: The rescaled energy $E_r/c^2$ radiated by the system until time $T$ for various initial amplitudes $c$. On the lower figure the logarithmic plot of the difference from the asymptotic value $E_f$ is displayed for a longer time period, indicating that the energy tends to $E_f$ as $T^{-2/3}$.

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