The Spectrum of Triangle-free Graphs

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Abstract

Denote by $q_n(G)$ the smallest eigenvalue of the signless Laplacian matrix of an $n$-vertex graph $G$. Brandt conjectured in 1997 that for regular triangle-free graphs $q_n(G) \leq \frac{4n}{25}$. We prove a stronger result: If $G$ is a triangle-free graph then $q_n(G) \leq \frac{15n}{94} < \frac{4n}{25}$. Brandt’s conjecture is a subproblem of two famous conjectures of Erdős:

1 Sparse-Half-Conjecture: Every $n$-vertex triangle-free graph has a subset of vertices of size $\lceil \frac{n}{2} \rceil$ spanning at most $n^2/50$ edges.
2 Every $n$-vertex triangle-free graph can be made bipartite by removing at most $n^2/25$ edges.

In our proof we use linear algebraic methods to upper bound $q_n(G)$ by the ratio between the number of induced paths with 3 and 4 vertices. We give an upper bound on this ratio via the method of flag algebras.

1 Introduction

We prove a result on eigenvalues of triangle-free graphs which is motivated by the following two famous conjectures of Erdős.

Conjecture 1.1 (Erdős’ Sparse Half Conjecture [9, 10]). Every triangle-free graph on $n$ vertices has a subset of vertices of size $\lceil \frac{n}{2} \rceil$ vertices spanning at most $n^2/50$ edges.

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Erdős offered a $250 reward for proving this conjecture. There has been progress on this conjecture in various directions [4, 12, 14, 15, 17]. Most recently, Razborov [17] proved that every triangle-free graph on \(n\) vertices has an induced subgraph on \(n/2\) vertices with at most \((27/1024)n^2\) edges.

For a graph \(G\), denote by \(D_2(G)\) the minimum number of edges which have to be removed to make \(G\) bipartite.

**Conjecture 1.2** (Erdős [9]). Let \(G\) be a triangle-free graph on \(n\) vertices. Then \(D_2(G) \leq n^2/25\).

There also has been work on this conjecture [1, 3, 11, 13, 18], most recently, Balogh, Clemen and Lidický [3] proved \(D_2(G) \leq n^2/23.5\).

Brandt [5] found a surprising connection between these two conjectures and the eigenvalues of triangle-free graphs. Denote by \(\lambda_n(G) \leq \ldots \leq \lambda_1(G)\) the eigenvalues of the adjacency matrix of an \(n\)-vertex graph \(G\). Brandt [5] proved that

\[
D_2(G) \geq \frac{\lambda_1(G) + \lambda_n(G)}{4} \cdot n
\]

for regular graphs and conjectured the following. 

**Conjecture 1.3** (Brandt [5]). Let \(G\) be a triangle-free regular \(n\)-vertex graph. Then

\[
\lambda_1(G) + \lambda_n(G) \leq \frac{4}{25} \cdot n.
\]

Towards this conjecture, Brandt [5] proved a bound \(\lambda_1(G) + \lambda_n(G) \leq (3 - 2\sqrt{2})n \approx 0.1715n\) for regular triangle-free graphs, which was very recently shown to hold also in the non-regular setting by Csikvári [7]. Brandt also noted that \(\lambda_1(G_{HS}) + \lambda_n(G_{HS}) = 0.14n\) for the so-called Higman-Sims graph \(G_{HS}\), which is the unique strongly regular graph with parameters \((n, d, t, k) = (100, 22, 0, 6)\).

Recall that an \((n, d, t, k)\)-strongly regular graph is an \(n\)-vertex \(d\)-regular graph, where the number of common neighbors of every pair of adjacent vertices is \(t\) and the number of common neighbors of a non-adjacent pair of vertices is \(k\).

The value \(4/25\) is motivated by the fact that if either of Conjectures 1.1 or 1.2 were true, it would imply Conjecture 1.3. As observed by Brandt [5], Conjecture 1.1 implies Conjecture 1.3 by applying the following version of the Expander Mixing Lemma for a set \(S \subset V(G)\) of size \(n/2\) with \(e(S) \leq n^2/50\).

**Lemma 1.4** (Bussemaker-Cvetković-Seidel [6], Alon-Chung [2]). Let \(G\) be an \(n\)-vertex \(d\)-regular graph. Then, for every \(S \subseteq V(G)\), we have

\[
e(S) \geq |S| \cdot \frac{|S|d + (n - |S|)\lambda_n(G)}{2n}.
\]

Given a graph \(G\), denote by \(Q = A + D\) the signless Laplacian matrix of \(G\), where \(D\) is the diagonal matrix of the degrees of \(G\) and \(A\) is the adjacency matrix of \(G\). Let \(q_n(G) \leq \ldots \leq q_1(G)\) be the eigenvalues of \(Q\). By considering the signless Laplacian matrix, De Lima, Nikiforov and Olivera [8] extended (1) beyond regular graphs as follows.

**Theorem 1.5** (De Lima, Nikiforov and Olivera [8]). For every \(n\)-vertex graph \(G\) we have

\[
D_2(G) \geq \frac{q_n(G)}{4} \cdot n.
\]
By Theorem 1.5, if Conjecture 1.2 holds then \( q_n(G) \leq \frac{4n}{25} \) for every triangle-free \( n \)-vertex graph \( G \). Motivated by this observation De Lima, Nikiforov and Olívera [8] proposed investigating upper bounds on \( q_n(G) \), and proved \( q_n(G) \leq \frac{2n}{9} \) for \( n \)-vertex triangle-free graphs \( G \). Our main result is an improvement of this bound, which solves Conjecture 1.3.

**Theorem 1.6.** If \( G \) is a triangle-free \( n \)-vertex graph, then

\[
q_n(G) \leq \frac{15}{94} n < 0.1596n.
\]

Note that, if \( G \) is \( d \)-regular, then \( \lambda_1(G) = d \) and \( q_n(G) = \lambda_n(G) + d = \lambda_n(G) + \lambda_1(G) \). Thus Theorem 1.6 implies that \( \lambda_1(G) + \lambda_n(G) < 0.1596n < \frac{4n}{25} \) for every regular triangle-free \( n \)-vertex graph \( G \), confirming Conjecture 1.3 in strong form.

It remains open to determine a sharp upper bound for \( q_n(G)/n \) for triangle-free \( n \)-vertex graph \( G \). While we only prove Theorem 1.6 with the constant \( \frac{15}{94} \approx 0.1596 \), a larger flag algebra computation yields \( q_n(G) < 0.15467n \). Also, one can additionally assume that \( G \) is regular and use flag algebras to show a slightly stronger bound \( q_n(G) = \lambda_1(G) + \lambda_n(G) < 0.15442n \). As we believe neither of these two bounds are sharp (see Section 3), we omit presenting their proofs.

## 2 Proof of Theorem 1.6

Our proof is based on bounding the ratio between the number of induced paths with 3 and 4 vertices in triangle-free graphs. On one hand, we upper bound \( q_n(G) \) in terms of this ratio in Lemma 2.1 and Corollary 2.2. On the other hand, Lemma 2.3, which is proved using flag algebras, gives a sufficiently good bound on the ratio.

For an edge \( e = xy \) of a graph \( G \), let \( m_{xy} \) be the number of edges \( uw \in E(G) \) such that \( ux, vy \in E(G) \). For a vertex \( x \in V(G) \), let \( w_x \) to be the number of walks of length two starting in \( x \), i.e. \( w_x \) is the number of edges \( uv \in E(G) \) such that \( xu \in E(G) \).

**Lemma 2.1.** If \( G \) is an \( n \)-vertex triangle-free graph and \( xy \in E(G) \), then

\[
(deg(x) + deg(y)) \cdot q_n(G) \leq w_x + w_y - 2m_{xy}.
\]

**Proof.** Define a vector \( z = (z_v)_{v \in V(G)} \in \mathbb{R}^{V(G)} \) by

\[
z_v = \begin{cases} +1, & \text{if } xv \in E(G), \\ -1, & \text{if } yv \in E(G), \\ 0, & \text{otherwise.} \end{cases}
\]

The vector \( z \) is well-defined since \( G \) is triangle-free. Also note that \( \|z\|^2 = \deg(x) + \deg(y) \). Let \( Q \) be the signless Laplacian matrix of \( G \). We have

\[
z^T Q z = \sum_{u,v \in V(G)} Q_{uv} z_u z_v = \sum_{u \in V(G)} (z_u)^2 \deg(u) + 2 \cdot \sum_{uv \in E(G)} z_u z_v
\]

\[
= w_x + w_y + 2 \cdot \sum_{uv \in E(G)} z_u z_v = w_x + w_y - 2m_{xy},
\]

3
where in the last equality we used that $G$ is triangle-free. Since $Q$ is symmetric, $q_n(G)$ is upper bounded by the Rayleigh-Ritz quotient of $z$, i.e.

$$q_n(G) \leq \frac{z^T Q z}{\|z\|^2} = \frac{w_x + w_y - 2m_{xy}}{\deg(x) + \deg(y)},$$

as desired. \hfill \Box

A map $\varphi : V(H) \to V(G)$ is a strong homomorphism from a graph $H$ to a graph $G$ if for every pair of vertices $u, v \in V(H)$ we have $uv \in E(H)$ if and only if $\varphi(u)\varphi(v) \in E(G)$. Let $\text{hom}_s(H, G)$ denote the number of strong homomorphisms from $H$ to $G$. Let $P_k$ denote the $k$-vertex path. Summing the bound from Lemma 2.1 over all the edges of $G$ yields the following.

**Corollary 2.2.** If $G$ is an $n$-vertex triangle-free graph, then

$$\text{hom}_s(P_3, G) \cdot q_n(G) \leq \text{hom}_s(P_4, G). \quad (3)$$

**Proof.** First, note that

$$\sum_{xy \in E(G)} (\deg(x) + \deg(y)) = \sum_{x \in V(G)} \deg(x) = \text{hom}_s(P_3, G), \quad (4)$$

where in the last equality we used that $G$ is triangle-free. Meanwhile, $\sum_{xy \in E(G)} (w_x + w_y)$ is equal to the number of walks of length three in $G$, i.e. the number of maps $\phi : \{1, 2, 3, 4\} \to V(G)$ such that $\{\phi(1)\phi(2), \phi(2)\phi(3), \phi(3)\phi(4)\} \subset E(G)$. Similarly, the expression $2\sum_{xy \in E(G)} m_{xy}$ is equal to the number of maps $\phi : \{1, 2, 3, 4\} \to V(G)$ such that $\{\phi(1)\phi(2), \phi(2)\phi(3), \phi(3)\phi(4), \phi(4)\phi(1)\} \subset E(G)$.

It follows that $\sum_{xy \in E(G)} (w_x + w_y - 2m_{xy})$ counts the maps $\psi : \{1, 2, 3, 4\} \to V(G)$ such that $\{\psi(1)\psi(2), \psi(2)\psi(3), \psi(3)\psi(4)\} \subset E(G)$ and $\psi(4)\psi(1) \notin E(G)$, i.e.,

$$\sum_{xy \in E(G)} (w_x + w_y - 2m_{xy}) = \text{hom}_s(P_4, G). \quad (5)$$

Summing (2) over all $xy \in E(G)$ and using (4) and (5), we obtain (3). \hfill \Box

Theorem 1.6 is an immediate consequence of the above corollary and the following lemma which is proved using standard, albeit computer-assisted flag-algebra calculation.

**Lemma 2.3.** If $G$ is an $n$-vertex triangle-free graph, then

$$\text{hom}_s(P_4, G) \leq \frac{15n}{94} \cdot \text{hom}_s(P_3, G). \quad (6)$$

**Proof.** Suppose the lemma is false, and let $G$ be an $n$-vertex triangle-free graph such that

$$\text{hom}_s(P_4, G) = \frac{15n}{94} \cdot \text{hom}_s(P_3, G) + \varepsilon n^4, \quad (7)$$

for some $\varepsilon > 0$. Let $G^{(b)}$ be the $b$-blowup of $G$, obtained by replacing every vertex of $G$ by $b$ pairwise non-adjacent vertices. Then $\text{hom}_s(P_k, G^{(b)}) = \text{hom}_s(P_k, G) \cdot b^k$ for $k = 3, 4$. In particular, for every $b \in \mathbb{N}$, the graph $G^{(b)}$ satisfies the analogue of (7) as well.

Let us now reformulate (7) in the flag algebra language [16]. Given a graph $H$, let $p\left(\bigwedge, H\right)$ be the probability that a 3-vertex subset of $V(H)$ chosen uniformly at random induces exactly
two edges. Analogously, let \( p\left(\bigcup, H\right) \) be the probability that a randomly chosen 4-vertex subset induces a path of length 3.

For every fixed \( \ell \)-vertex graph \( F \) and a \( k \)-vertex graph \( H \), only \( O\left(k^{\ell-1}\right) \) maps \( V(F) \to V(H) \) are non-injective. Therefore, \( \text{hom}_s(F, H) = |\text{Aut}(F)| \cdot p(F, H) \cdot \binom{k}{\ell} + O(k^{\ell-1}) \), so in particular every \( k \)-vertex triangle-free graph \( H \) satisfies

\[
\text{hom}_s(P_4, H) = \frac{k^4}{12} \cdot p\left(\bigcup, H\right) + O\left(k^3\right) \quad \text{and} \quad \text{hom}_s(P_3, H) = \frac{k^3}{3} \cdot p\left(\bigwedge, H\right) + O\left(k^2\right).
\]

Recall that \( G^{(b)} \) satisfies (7). Multiplying it by \( 564/(bn)^4 \) and rearranging yields that

\[
\lim_{b \to \infty} 30 \cdot p\left(\bigwedge, G^{(b)}\right) - 47 \cdot p\left(\bigcup, G^{(b)}\right) = -564 \varepsilon^4.
\]

In order to derive a contradiction, we present a flag algebra computation proving that an inequality

\[
30 \cdot \bigwedge - 47 \cdot \bigcup \geq 0 \tag{8}
\]

asymptotically holds in the theory of triangle-free graphs. To see that, consider the following 6 flag-algebra expressions, which are all non-negative:

1) \( 13 \cdot \left( \frac{1}{1_{123}^4} + \frac{1}{1_{123}^2} + \frac{1}{1_{123}} \right) - 52 \cdot \left( \frac{1}{1_{123}^4} + \frac{1}{1_{123}^2} + \frac{1}{1_{123}} \right) + 84 \cdot \frac{1}{1_{123}} \)

2) \( 31 \cdot \left( \frac{1}{1_{123}^3} + \frac{1}{1_{123}^2} \right) - 63 \cdot \left( \frac{1}{1_{123}^3} + \frac{1}{1_{123}^2} \right) + 3 \cdot \frac{1}{1_{123}} \)

3) \( 94 \cdot \frac{1}{1_{123}} - 55 \cdot \frac{1}{1_{123}} - 14 \cdot \frac{1}{1_{123}} + 58 \cdot \frac{1}{1_{123}} \)

4) \( 1 \cdot \frac{1}{1_{123}} \times \left( 2 \cdot \frac{1}{1_{123}} + 10 \cdot \frac{1}{1_{123}} - 24 \cdot \frac{1}{1_{123}} \right) \)

5) \( 1 \cdot \frac{1}{1_{123}} \times \left( 14 \cdot \frac{1}{1_{123}} + 19 \cdot \frac{1}{1_{123}} - 44 \cdot \frac{1}{1_{123}} \right) \)

6) \( 1 \cdot \frac{1}{1_{123}} \times \left( 9 \cdot \frac{1}{1_{123}} - 14 \cdot \frac{1}{1_{123}} - 3 \cdot \frac{1}{1_{123}} \right) \)

Let \( F \) be the set of all the 5-vertex triangle-free graphs with at least 2 edges. A case analysis yields \(|F| = 12\); see Figure 1. Now observe that averaging over all choices of the labelled vertices in each of the 6 expressions yields a linear combination of subgraph densities, where every term has
5 vertices and at least 2 edges. Thus a flag algebra argument yields that the average of the $i$-th expression is equal to the $i$-th coordinate of $M \cdot (v_F)^T$, where $v_F = (F_1, \ldots, F_{12})$ and

$$M = \frac{1}{30} \times \begin{pmatrix}
507 & 2028 & 0 & -4056 & -3549 & 0 & 1248 & 8112 & 16224 & -13104 & 0 & 21168 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
12100 & -23688 & -19140 & -23620 & 12172 & 20184 & -37248 & 17486 & 47664 & 2956 & 86730 & -7392 \\
0 & 0 & 6 & 140 & 0 & 0 & -48 & -100 & 0 & 358 & -1200 & 0 \\
196 & 0 & 798 & 196 & -420 & 2166 & 762 & -1036 & -2464 & -702 & -3080 & 792 \\
81 & 0 & -378 & 81 & 54 & 1176 & -165 & 27 & -108 & -87 & -135 & 279 \\
\end{pmatrix}.$$ 

On the other hand, another flag algebra argument yields that the left-hand side of (8) is equal to

$$3 \cdot F_1 + 9 \cdot F_3 + 3 \cdot F_4 - \frac{17}{5} \cdot F_5 + 18 \cdot F_6 - \frac{34}{5} \cdot F_7 - \frac{49}{5} \cdot F_8 + 12 \cdot F_9 - \frac{4}{5} \cdot F_{10} - 32 \cdot F_{11} + 27 \cdot F_{12}.$$ 

A tedious yet straightforward calculation reveals the following coordinate-wise inequality

$$\left( \begin{array}{cccc}
1 & 12 & 3 & 231 \\
33 & 209 & 1147 & 163 \\
84 & 293 & 17 & 12 \\
\end{array} \right) \cdot M < \left( \begin{array}{cccc}
3, 0, 9, 3, -\frac{17}{5}, 18, -\frac{34}{5}, -\frac{49}{5}, 12, -\frac{4}{5}, -32, 27 \end{array} \right),$$

which in turn shows that (8) asymptotically holds in the theory of triangle-free graphs. ☐

The flag algebra calculations used in the proof of Lemma 2.3 can be independently verified by a SAGE script, which is available as an ancillary file of the arXiv version of this manuscript.

### 3 Concluding remarks

As we have already mentioned in the introduction, a significantly larger flag algebra computation than the one used in our proof yields that $q_n(G) < 0.15467n$ for every triangle-free $n$-vertex graph. Similarly, assuming that $G$ is regular allows us to show $\lambda_1(G) + \lambda_n(G) < 0.15442n$. On the other hand, our method will be able to get neither of the coefficients below $42/275 = 0.1527$.

Indeed, consider the Higman-Sims graph $G_{HS}$. It is edge-transitive so $m_{xy} = 21 \cdot 6 + 22$ for every $xy \in E(G_{HS})$, and $w_x = 22^2$ for every $x \in V(G)$, where $m_{xy}$ and $w_x$ are defined as before Lemma 2.1. Therefore,

$$\frac{w_x + w_y - 2m_{xy}}{(\deg(x) + \deg(y)) \cdot |V(G_{HS})|} = \frac{2(22^2 - 21 \cdot 6 - 22)}{2 \cdot 22 \cdot 100} = \frac{42}{275},$$

for every $xy \in E(G_{HS})$, and so Lemma 2.1 only yields $q_n(G_{HS}) \leq \frac{42}{275} \cdot |V(G_{HS})|$. However, we have $q_n(G_{HS}) = \lambda_1(G_{HS}) + \lambda_n(G_{HS}) = 0.14 \cdot |V(G_{HS})|$. It might be that $q_n(G) \leq 0.14n$ holds for every triangle-free graph $G$ on $n$ vertices.

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