GEOMETRIC REALIZATIONS OF AFFINE KÄHLER CURVATURE MODELS

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Abstract. We show that every Kähler affine curvature model can be realized geometrically.

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Dedicated to Heinrich Wefelscheid
on the occasion of his 70th birthday

1. Introduction

The study of curvature is central in modern differential geometry. One often first considers a problem abstractly in a purely algebraic framework and then subsequently passes to the geometrical context by examining the appropriate geometric realization problem.

The nature of the realization question of course depends on the problem under consideration and its context. Among the realization problems many of them arise in very natural fashions. We shall work in the affine setting and consider the curvature associated to a Kähler affine connection. This curvature operator satisfies some known universal symmetries (see Equations (1.a) and (1.i) below). We study the inverse problem. We shall consider a tensor which satisfies those symmetries and show that it is indeed the curvature tensor associated to an affine connection (Theorem 1.8). We now briefly survey other results in this field to put our result in context.

1.1. Affine geometry. The pair \((M, \nabla)\) is said to be an affine manifold if \(\nabla\) is a torsion free connection on the tangent bundle \(TM\) of a smooth \(m\)-dimensional manifold \(M\). Let

\[ R(x, y) := \nabla_x \nabla_y - \nabla_y \nabla_x - \nabla_{[x, y]} \]

be the associated curvature operator which has the symmetries:

\[ R(x, y) = -R(y, x) \quad \text{and} \quad R(x, y)z + R(y, z)x + R(z, x)y = 0. \quad (1.a) \]

It is convenient to work in a purely algebraic context. Let \(V\) be a real \(m\)-dimensional vector space. We say that \(A \in \text{End}(V) \otimes V^* \otimes V^*\) is a generalized curvature operator if it satisfies the symmetries of Equation (1.a). Let \(\mathfrak{A} = \mathfrak{A}(V)\) be the linear vector space of all generalized curvature tensors on \(V\). If \(A \in \mathfrak{A}\), the structure \((V, A)\) is said to be an affine curvature model. We say that an affine curvature model \((V, A)\) is geometrically realizable by an affine manifold if there exists an affine manifold \((M, \nabla)\), a point \(P \in M\), and an isomorphism \(\Xi : V \to T_PM\) so that \(\Xi^* R_P = A\). The following result [1] permits one to pass from the algebraic to the geometric context. It shows that the relations of Equation (1.a) generate the universal symmetries of the curvature operator associated to a torsion free connection:

\[ Key \text{ words and phrases. geometric realization, affine manifold, curvature operator, affine curvature model, affine Kähler manifold.} \]
The symmetries of Equation (1.a) then become:

\[ R(x, y, z, w) = g(R(x, y)z, w) . \]

The symmetries of Equation (1.a) then become:

\[ R(x, y, z, w) + R(y, x, z, w) = 0, \]
\[ R(x, y, z, w) + R(y, z, x, w) + R(z, x, y, w) = 0 . \]  \hspace{1cm} (1.b)

There are two different traces it is natural to consider in this setting. Let \( \{e_i\} \) be a local orthonormal frame for \( TM \). We adopt the Einstein convention and sum over repeated indices to define:

\[ \rho_{13}(x, y) := R(e_i, e_i, e_i, y) \quad \text{and} \quad \rho_{14}(x, y) := R(e_i, x, y, e_i) . \]  \hspace{1cm} (1.c)

The scalar curvature \( \tau \) is then defined by setting:

\[ \tau(A) := \rho_{14}(e_i, e_i) = -\rho_{13}(e_i, e_i) . \]

The tensor \( \rho_{13} \), the tensor \( \rho_{14} \), and the scalar curvature \( \tau \) are all independent of the particular orthonormal frame which is chosen. Since one may also express \( \rho_{14}(x, y) = \text{Tr}(z \rightarrow R(z, x, y)) \), \( \rho_{14} \) is defined in the affine setting. By contrast, \( \rho_{13} \) and \( \tau \) are dependent on the metric.

The mixed structure \( (M, g, \nabla) \) is said to be a Weyl manifold [15] if there exists a smooth 1-form \( \phi \) so that

\[ \nabla g = -2\phi \otimes g . \]

If \( (M, g, \nabla) \) is a Weyl manifold, then there is an additional curvature symmetry which is satisfied (see [9]):

\[ R(x, y, z, w) + R(x, y, w, z) = \frac{2}{n}(\rho_{14}(y, x) - \rho_{14}(x, y))g(z, w) . \]  \hspace{1cm} (1.d)

We fix a positive definite metric \( \langle \cdot, \cdot \rangle \) on \( V \) henceforth. We say that \( A \in \mathfrak{A} \) is a Weyl generalized curvature tensor if in addition it satisfies the compatibility relationship of Equation (1.d); let \( \mathfrak{M} = \mathfrak{M}(V, \langle \cdot, \cdot \rangle) \) be the space of all such tensors. If \( A \in \mathfrak{M} \), then the structure \( (V, \langle \cdot, \cdot \rangle, \nabla, A) \) is said to be a Weyl curvature model. Such a structure is said to be geometrically realizable by a Weyl manifold if there exists a Weyl manifold \( (M, g, \nabla) \), a point \( P \in M \), and an isomorphism \( \Xi \) from \( V \) to \( T_PM \) so that \( \Xi^*g_P = \langle \cdot, \cdot \rangle \) and so that \( \Xi^*R_P = A \). The following result [7] extends Theorem 1.1 to the Weyl setting; it shows that Equations (1.b) and (1.d) generate the universal symmetries of the curvature tensor in Weyl geometry:

**Theorem 1.2.** Every Weyl curvature model is geometrically realizable by a Weyl manifold.

1.3. Riemannian geometry. If \( g \) is a Riemannian metric on \( M \), let \( R^g \) be the curvature operator and let \( R^g \) be the curvature tensor which are defined by the Levi-Civita connection \( \nabla^g \). There is an additional curvature symmetry in this case:

\[ R^g(x, y, z, w) = R^g(z, w, x, y) . \]  \hspace{1cm} (1.e)

One says that \( A \in \mathfrak{A} \) is a Riemannian curvature tensor if in addition it satisfies the symmetry of Equation (1.e). Let \( \mathfrak{R} = \mathfrak{R}(V, \langle \cdot, \cdot \rangle) \) be the space of all such tensors. If \( R \in \mathfrak{R} \), then Equations (1.b) and (1.e) imply the additional symmetry:

\[ R(x, y, z, w) = -R(x, y, w, z) . \]

It now follows that:

\[ \mathfrak{R} \subset \mathfrak{M} \subset \mathfrak{A} . \]
Thus Weyl geometry is in a certain sense intermediate between affine and Riemannian geometry. If \( A \in \mathcal{R} \), then the structure \((V, \langle \cdot, \cdot \rangle, A)\) is said to be a Riemannian curvature model. Such a structure is said to be geometrically realizable by a Riemannian manifold if there exists a Riemannian manifold \((M, g)\), a point \( P \in M \), and an isomorphism \( \Xi : V \to T_P M \) so that \( \Xi^* g_P = \langle \cdot, \cdot \rangle \) and so that \( \Xi^* R^g_P = A \).

The following result is well known - see, for example, the discussion in [1]. It shows that the universal symmetries of the curvature tensor associated to the Levi-Civita connection are given by Equations (1.a) and (1.e):

**Theorem 1.3.** Every Riemannian curvature model can be realized geometrically by a Riemannian manifold.

1.4. **Almost Hermitian geometry.** Let \( m = 2\bar{m} \) be even henceforth. A triple \((M, g, J)\) is said to be an almost Hermitian manifold if \( g \) is a Riemannian metric on \( M \) and if \( J \) is an endomorphism of \( TM \) satisfying \( J^* g = g \) and \( J^2 = -\text{Id} \). One says an endomorphism \( J \) of \( V \) gives \( V \) a complex structure if \( J^2 = -\text{Id} \). A quadruple \((V, \langle \cdot, \cdot \rangle, J, A)\) is said to be an almost Hermitian curvature model if \( J \) gives \( V \) a complex structure, if \( J^* \langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle \), and if \( A \in \mathcal{R} \). One says such a structure can be geometrically realized by an almost Hermitian manifold if there exists an almost Hermitian manifold \((M, g, J)\), a point \( P \in M \), and an isomorphism \( \Xi : V \to V \) so that \( \Xi^* g_P = \langle \cdot, \cdot \rangle \), \( \Xi^* J_P = J \), and \( \Xi^* R^g_P = A \). There are no additional curvature restrictions and one has the following result [3]:

**Theorem 1.4.** Every almost Hermitian curvature model can be realized geometrically by an almost Hermitian manifold.

1.5. **Hermitian geometry.** If \((M, g, J)\) is an almost Hermitian manifold, then the almost complex structure \( J \) is said to be integrable and \((M, g, J)\) is called a Hermitian manifold if there are local charts \((x_1, \ldots, x_{\bar{m}}, y_1, \ldots, y_{\bar{m}})\) so that

\[
J \partial_{x_i} = \partial_{y_i}, \quad \text{and} \quad J \partial_{y_i} = -\partial_{x_i}.
\]

Equivalently, via the Newlander–Nirenberg Theorem [13], this means that the Nijenhuis tensor

\[
N_J(x, y) := [x, y] + J[Jx, y] + J[x, Jy] - [Jx, Jy]
\]

vanishes. There is an extra curvature restriction in this setting discovered by Gray [8]:

\[
0 = R^g(x, y, z, w) + R^g(Jx, Jy, Jz, Jw) - R^g(Jx, Jy, z, w) - R^g(x, y, Jz, Jw)
- R^g(Jx, y, Jz, w) - R^g(x, y, Jz, Jw) - R^g(Jx, y, z, w) - R^g(x, Jy, Jz, w).
\]

One says that an almost Hermitian curvature model \((V, \langle \cdot, \cdot \rangle, J, A)\) is a Hermitian curvature model if \( A \) also satisfies Equation (1.g). One has [2]:

**Theorem 1.5.** Every Hermitian curvature model is geometrically realizable by a Hermitian manifold.

1.6. **Riemannian Kähler geometry.** The Kähler form of an almost Hermitian manifold \((M, g, J)\) is defined by setting \( \Omega(x, y) := g(x, Jy) \). One says that an almost Hermitian manifold is a Kähler manifold if \( J \) is integrable and \( d\Omega = 0 \) or, equivalently, if \( \nabla(J) = 0 \). This implies a restriction on curvature:

\[
R^g(x, y, z, w) = R^g(x, y, Jz, Jw).
\]

An almost Hermitian curvature model \((V, \langle \cdot, \cdot \rangle, J, A)\) is said to be a Kähler curvature model if Equation (1.h) is satisfied; as necessarily Equation (1.g) is satisfied in this setting, any Kähler curvature model is a Hermitian curvature model. One has [4] that:
Theorem 1.6. Any Kähler curvature model is geometrically realizable by a Kähler manifold.

Remark 1.7. In fact, more is true. One can use the Cauchy-Kovalevskaya Theorem (see, for example, the discussion in Evans [6]) to show that the realizations in Theorems 1.2-1.6 can be chosen to have constant scalar curvature. The arguments work equally in the pseudo-Riemannian setting. Furthermore, Theorem 1.4, Theorem 1.5, and Theorem 1.6 can be extended to the almost para-Hermitian setting, the para-Hermitian setting, and the para-Kähler setting, respectively, [3, 4, 5].

1.7. Affine Kähler geometry. We now return to the affine setting. One says \((M, J, \nabla)\) is an affine Kähler manifold if \(J\) is an integrable almost complex structure on \(M\), if \(\nabla\) is a torsion free connection on \(M\), and if \(\nabla(J) = 0\). This then implies the curvature symmetry
\[
\mathcal{R}(x, y)J = J\mathcal{R}(x, y).
\] (1.i)

Let \(J\) give \(V\) a complex structure and let \(\mathfrak{R} = \mathfrak{R}(V, J) \subset \mathfrak{A}\) be the subspace of generalized algebraic curvature operators satisfying Equation (1.i). If \(A \in \mathfrak{R}\), then the triple \((V, J, A)\) is said to be a Kähler affine curvature model. We say such a structure is geometrically realizable by an affine Kähler manifold if there exists an affine Kähler manifold \((M, J, \nabla)\), a point \(P \in M\), and an isomorphism \(\Xi : V \rightarrow T_PM\) so \(\Xi^*\mathcal{R} = A\) and \(\Xi^*J_P = J\). The following is the main result of this paper and generalizes the geometric realization results described above to this setting:

Theorem 1.8. Every Kähler affine curvature model is geometrically realizable by an affine Kähler manifold.

1.8. Outline of the paper. Theorems of this type usually use curvature decomposition theory. For example, the proof of Theorem 1.2 rests upon work of Higa [9]. The proof of Theorem 1.5 and the proof of Theorem 1.6 rest upon the decomposition of Tricerri and Vanhecke [14]. The proof of Theorem 1.8 will rest upon the decomposition of \(\mathfrak{R}(V, J)\) as a unitary module given in [10, 11, 12]. We shall review this decomposition in Section 2. In Section 3, we let \(J\) be the standard complex structure on \(\mathbb{R}^m\) of Equation (1.f) and construct affine Kähler connections. The curvature decomposition of Section 2 and the construction of Section 3 will then be applied in Section 4 to complete the proof of Theorem 1.8.

2. The decomposition of \(\mathfrak{R}(V, J)\) as a unitary module

Let \(J\) define a complex structure on \(V\). Introduce an auxiliary positive definite inner product \(\langle \cdot, \cdot \rangle\) so \(J^*\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle). The associated unitary group is defined by
\[
\mathcal{U} := \{T \in \text{GL}(V) : T^*\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle \quad \text{and} \quad TJ = JT\}.
\]

Pullback makes \(\mathfrak{R}(V, J)\) into a \(\mathcal{U}\) module. In this section, we shall describe the decomposition of \(\mathfrak{R}\) into \(\mathcal{U}\) modules given in [10, 11].

2.1. Ricci tensors. We may use the inner product to identify \(\mathfrak{R}\) with the set of all \(A \in \otimes V^*\) so that:
\[
A(x, y, z, w) = -A(y, x, z, w),
\]
\[
A(x, y, z, w) + A(y, z, x, w) + A(z, x, y, w) = 0,
\]
\[
A(x, y, z, w) = A(x, y, Jz, Jw).
\]

We may decompose
\[
\mathfrak{R}^\pm := \{A \in \mathfrak{R} : A(Jx, Jy) = \pm A(x, y)\} = \{A \in \mathfrak{R} : A(Jx, Jy, Jz, Jw) = \pm A(x, y, z, w)\}.
\]
We begin by deriving some basic facts concerning the Ricci tensors $\rho_{13}$ and $\rho_{14}$ defined in Equation (1.c):

**Lemma 2.1.**

1. If $A \in \mathbb{R}$, then $\rho_{13}(x, y) = \rho_{13}(Jx, Jy)$.
2. If $A \in \mathbb{R}^+$, then $J^*\rho_{13} = \rho_{13}$ and $J^*\rho_{14} = \rho_{14}$.
3. If $A \in \mathbb{R}^-$, then $\rho_{13} = 0$ and $J^*\rho_{14} = -\rho_{14}$.

**Proof.** Let $\{e_i\}$ be an orthonormal basis for $V$; $\{J e_i\}$ also is an orthonormal basis for $V$. Let $A \in \mathbb{R}$. We sum over $i$ and compute:

$$2\rho_{13}(Jx, Jy) = A(e_i, Jx, e_i, Jy) + A(Je_i, Jx, Je_i, Jy)$$

$$= -A(e_i, Jx, Je_i, y) + A(Je_i, Jx, e_i, y)$$

$$= A(Jx, Je_i, e_i, y) + A(Je_i, e_i, Jx, y) - A(Jx, Je_i, e_i, y)$$

$$= A(e_i, x, Je_i, Jy) + A(x, Je_i, e_i, Jy)$$

$$= A(e_i, x, e_i, y) + A(Je_i, x, Je_i, y)$$

$$= 2\rho_{13}(x, y)$$

This proves Assertion (1). Let $A \in \mathbb{R}^\pm$. We compute:

$$\rho_{14}(Jx, Jy) = A(e_i, Jx, y, e_i) = \pm A(Je_i, Jx, Jy, Je_i)$$

$$= \pm A(Je_i, x, y, Je_i) = \pm \rho_{14}(x, y),$$

$$\rho_{13}(Jx, Jy) = A(e_i, Jx, e_i, Jy) = \pm A(Je_i, Jx, Je_i, Jy)$$

$$= \pm A(Je_i, x, Je_i, y) = \pm \rho_{13}(x, y).$$

Assertions (2) and (3) follow from these computations and from Assertion (1). \qed

### 2.2. Linear and quadratic representations

The trivial representation appears with multiplicity 2 in $\mathbb{R}$. We define two scalar invariants by setting:

$$\tau := A(e_i, e_j, e_j, e_i) \quad \text{and} \quad \bar{\tau}_j := A(e_i, Je_j, e_j, e_i).$$

Decompose $\otimes^2 V^* = \Lambda^2 \oplus S^2$ as the direct sum of the alternating and the symmetric 2-tensors. We can use $J$ and the metric to further decompose $S^2 = S^2_{0,2} \oplus \mathcal{R} \oplus S^2_{0,+}$ and $\Lambda^2 = \Lambda^2_{0,-} \oplus \mathcal{R} \oplus \Lambda^2_{0,+}$ into irreducible inequivalent $U$ modules where:

$$S^2_{0,2} := \{ \theta \in \otimes^2 V^* : J^*\theta = -\theta, \quad \theta(x, y) = \theta(y, x) \},$$

$$S^2_{0,+} := \{ \theta \in \otimes^2 V^* : J^*\theta = \theta, \quad \theta(x, y) = \theta(y, x) \},$$

$$\Lambda^2_{0,-} := \{ \theta \in \otimes^2 V^* : J^*\theta = -\theta, \quad \theta(x, y) = -\theta(y, x), \quad \theta \perp \Omega \},$$

$$\Lambda^2_{0,+} := \{ \theta \in \otimes^2 V^* : J^*\theta = \theta, \quad \theta(x, y) = -\theta(y, x), \quad \theta \perp \Omega \}.$$

The following Lemma will follow from computations we shall perform in Section 3:

**Lemma 2.2.**

1. $\tau \oplus \bar{\tau}_j : \mathbb{R}^+ \to \mathbb{R} \oplus \mathbb{R} \to 0$.
2. $\rho_{14} : \mathbb{R}^- \to S^2_{0,2} \oplus \Lambda^2 \to 0$.
3. $\rho_{14} \oplus \rho_{13} : \mathbb{R}^+ \to S^2_{0,2} \oplus \Lambda^2_{0,+} \oplus S^2_{0,+} \oplus \Lambda^2_{0,+} \to 0$.

All short exact sequences of unitary modules split naturally. Let $\rho_{ij}^\perp := \rho_{ij}$ restricted to $\mathbb{R}^\perp$. We use Lemma 2.2 to define 8 mutually orthogonal submodules of $\mathcal{U}$ by requiring that:

$$\tau \oplus \bar{\tau}_j : W_0 \oplus W_6 \overset{\approx}{\to} \mathbb{R} \oplus \mathbb{R},$$

$$\rho_{14} : W_2 \oplus W_4 \overset{\approx}{\to} S^2_{0,2} \oplus \Lambda^2,$$

$$\rho_{14} \oplus \rho_{13} : W_1 \oplus W_3 \oplus W_7 \oplus W_8 \overset{\approx}{\to} S^2_{0,2} \oplus \Lambda^2_{0,+} \oplus S^2_{0,+} \oplus \Lambda^2_{0,+}.$$
2.3. Other representations. There are 4 other submodules of $\mathfrak{r}$ which appear in the decomposition of [10, 11]. Let:

\[
\begin{align*}
W_9 & := \{ A \in \mathfrak{r}^+ : A(x, y, z, w) = -A(x, y, w, z) \} \cap \ker(\rho_{14}), \\
W_{10} & := \{ A \in \mathfrak{r}^+ : A(x, y, z, w) = A(x, y, w, z) \} \cap \ker(\rho_{14}), \\
W_{11} & := \mathfrak{r}^+ \cap W_9^\perp \cap W_{10}^\perp \cap \ker(\rho_{13}) \cap \ker(\rho_{14}), \\
W_{12} & := \mathfrak{r}^- \cap \ker(\rho_{14}).
\end{align*}
\]

One then has the fundamental result [10, 11]:

**Theorem 2.3.**

1. If $\dim(V) = 4$, then $\mathfrak{r} = W_1 \oplus \ldots \oplus W_{10}$, $W_{11} = \{0\}$, and $W_{12} = \{0\}$. The spaces $W_1$, ..., $W_{10}$ are mutually orthogonal irreducible $\mathcal{U}$ modules.
2. If $\dim(V) \geq 6$, then $\mathfrak{r} = W_1 \oplus \ldots \oplus W_{12}$. The spaces $W_1$, ..., $W_{12}$ are mutually orthogonal irreducible $\mathcal{U}$ modules.

**Remark 2.4.** The dimensions of these modules are given by:

| Module | Dimension |
|--------|-----------|
| $W_1$  | $m^2 - 1$ |
| $W_2$  | $m(m + 1)$ |
| $W_3$  | $m^2 - 1$ |
| $W_4$  | $m(m - 1)$ |
| $W_5$  | $W_6$ |
| $W_7$  | $m^2 - 1$ |
| $W_8$  | $m^2 - 1$ |
| $W_9$  | $\frac{1}{2}m^2(m + 1)(m - 3)$ |
| $W_{10}$ | $\frac{1}{2}m^2(m - 1)(m + 3)$ |
| $W_{11}$ | $\frac{1}{2}(m - 1)(m + 1)(m - 2)(m + 2)$ |
| $W_{12}$ | $\frac{1}{2}m^2(m - 2)(m + 2)$ |

Clearly $W_5 \approx W_6$. Let $T\Theta(x, y) := \Theta(x, Jy)$. The map $\Theta \rightarrow T\Theta$ induces an isomorphism between $A_{0,+}$ and $S_{0,+}$. Consequently $W_1 \approx W_3 \approx W_7 \approx W_8$. Similarly the correspondence $TA(x, y, z, w) = A(x, y, z, Jw)$ defines an isomorphism $W_9 \approx W_{10}$. However otherwise these modules are distinct on dimensional grounds where, if $n = 4$, we ignore $W_{11}$ and $W_{12}$. We note

\[\dim \mathfrak{r} = \frac{1}{2}m^2(m + 1)(5m - 2).\]

3. Constructing Kähler connections

**Definition 3.1.** Let $J$ be the canonical complex structure on $\mathbb{R}^m = \mathbb{C}^\bar{m}$ of Equation (1.1). Let indices $i, j, k$ range from 1 to $\bar{m}$. Let $e_i := \partial_{x_i}$ and $f_i := \partial_{y_i} = Je_i$. Let \( \{u_{ijk}, v_{ijk}\} \) be smooth functions on $\mathbb{R}^m$ where $u_{ijk} = u_{ijk}$ and $v_{ijk} = v_{ijk}$. Let $\Theta := u + \sqrt{-1}v$. Define a torsion free connection $\nabla$ by defining:

\[
\begin{align*}
\nabla_{e_i}e_j &= -\nabla_{f_j}f_i = u_{ijk}e_k + v_{ijk}f_k, \\
\nabla_{f_j}e_i &= \nabla_{e_i}f_j = -v_{ijk}e_k + u_{ijk}f_k.
\end{align*}
\]

The following simple observation will be crucial to our study.

**Lemma 3.2.**

1. $\nabla$ is torsion free and $\nabla(J) = 0$.
2. If $\Theta$ is holomorphic, then $R \in \mathfrak{r}^-$. If $\Theta$ is anti-holomorphic and if $\Theta(0) = 0$, then $R(0) \in \mathfrak{r}^+$.

**Proof.** Since $u_{ijk} = u_{ijk}$ and $v_{ijk} = v_{ijk}$, $\nabla$ is torsion free. We may use $J$ to identify $\mathbb{R}^m$ with $\mathbb{C}^\bar{m}$. Let $\nabla^c e_i = \Theta_{ijk}e_k$. If we extend $\nabla^c$ to be complex linear, then $\nabla$ is the underlying real connection. Thus $\nabla J = J\nabla$ so the connection is Kähler.

If $\Theta(0) = 0$, then the curvature is given by $d\Theta$. By making a complex linear change of basis if necessary, we may assume without loss of generality that the basis is orthonormal at the origin. Let $A = R(0)$. We compute:
Let $R \in \mathbb{R}^-$. If $\Theta$ is holomorphic, then $e_i u = f_i v$ and $e_i v = -f_i u$; it then follows $A \in \mathbb{R}^-$. If $\Theta$ is anti-holomorphic, then $e_i u = -f_i v$ and $e_i v = f_i u$; it then follows that $A \in \mathbb{R}^+$. We complete the proof by assuming that $d\Theta(0) = 0$ and by studying the quadratic terms. Let $R^c$ be the complex curvature. Then:

$$R^c(e_i, e_j) = \sum_{a, \lambda} \left\{ \Theta(e_i, e_a, e_l) - \Theta(e_j, e_a, e_l) \right\}. $$

From this it is clear that $R^c(f_i, f_j) = -R^c(e_i, e_j)$ and $R^c(e_i, f_j) = -R^c(J e_i, J f_j)$. Disentangling the real and imaginary parts of these operators then shows that $R \in \mathbb{R}^-$. \hfill \square

This construction provides Kähler connections whose curvature tensors lie in $\mathbb{R}^+$ at a single point, but not globally. It also provides Kähler connections whose curvature tensors always lie in $\mathbb{R}^-$. 

4. The proof of Theorem 1.8

Let $\mathcal{J}$ be the linear subspace of $\mathbb{R}$ consisting of all curvature tensors which arise from the construction given in Lemma 3.2 where $\Theta(0) = 0$, and let $\mathcal{J}^+$ (resp. $\mathcal{J}^-$) be the subspaces defined by $\Theta$ anti-holomorphic (resp. holomorphic). These are clearly $\mathcal{U}$ sub-modules. We will complete the proof of Theorem 1.8 by showing $\mathbb{R}^\pm = \mathcal{J}^\pm$. We begin by establishing the following Lemma:

**Lemma 4.1.**

1. $\tau \oplus \tau : \mathcal{J}^+ \to \mathbb{R} \oplus \mathbb{R} \to 0$.
2. $\rho_{14} : \mathcal{J}^- \to S_{0,4}^2 + A^2 \to 0$.
3. $\rho_{14} \oplus \rho_{13} : \mathcal{J}^- \to S_{0,4}^1 + S_{0,4}^2 + S_{0,4}^2 + \Lambda_{0,4} \to 0$.
4. $W_1 + W_3 + W_5 + W_6 + W_7 + W_8 \in \mathcal{J}^+$ and $W_2 + W_4 \in \mathcal{J}^-.$

**Proof.** We apply Lemma 3.2. We establish Assertion (1) by taking:

$$\Theta_{111} = \Theta(x_1 - \sqrt{-1} y_1) \quad \text{and} \quad \Theta_{122} = \Theta(x_1 + \sqrt{-1} y_1).$$

Let $\mathcal{A} := \mathcal{R}(0)$. We may then establish Assertion (1) by computing:

$$\nabla_{e_1} e_1 = -\nabla_{f_1} f_1 = \Theta(x_1 e_1 - y_1 f_1), \quad \nabla_{e_1} f_1 = \Theta(f_1 e_1 + x_1 f_1),$$

$$\nabla_{e_2} e_2 = -\nabla_{f_2} f_2 = \Theta(y_1 e_2 + x_1 f_2), \quad \nabla_{e_2} f_2 = \Theta(f_2 e_2 + x_1 f_2),$$

$$\mathcal{A}(e_1, f_1) = 2\Theta_{1} f_1, \quad \mathcal{A}(e_1, f_2) = -\Theta_{1} e_1,$$

$$\mathcal{A}(e_2, e_1) = -\mathcal{A}(e_1, f_2) f_1 = \Theta_{2} f_2, \quad \mathcal{A}(e_2, f_1) = \Theta_{2} e_2,$$

$$\mathcal{A}(f_1, e_2) = -\mathcal{A}(f_2, f_1) f_1 = \Theta_{2} e_2, \quad \mathcal{A}(f_1, f_2) = \Theta_{2} f_2,$$

$$\rho_{14}(e_1, f_1) f_1 = \rho_{14}(f_1, f_1) = -\Theta_{1}, \quad \rho_{14}(e_2, f_1) = \rho_{14}(f_1, e_2) = \Theta_{2},$$

$$\tau = -\Theta_{1}, \quad \tau = \Theta_{2}.$$

Next take $\Theta_{111} = \Theta(x_2 + \sqrt{-1} y_2)$ and $\Theta_{222} = \Theta(x_1 + \sqrt{-1} y_1)$. Again, let $\mathcal{A} := \mathcal{R}(0)$. Then:
Thus we complete the proof of Assertion (3) by constructing an example where
\[ \mathcal{A}(e_2, e_1) e_1 = -\mathcal{A}(e_2, f_1) f_1 = \varrho_1 e_1, \quad \mathcal{A}(f_2, e_1) e_1 = -\mathcal{A}(f_2, f_1) f_1 = \varrho_1 f_1, \]
\[ \mathcal{A}(e_2, e_1) f_1 = \mathcal{A}(e_2, f_1) e_1 = \varrho_1 e_1, \quad \mathcal{A}(f_2, e_1) f_1 = \mathcal{A}(f_2, f_1) e_1 = \varrho_1 f_1, \]
\[ \mathcal{A}(e_1, e_2) e_2 = -\mathcal{A}(e_1, f_2) f_2 = \varrho_2 e_2, \quad \mathcal{A}(f_1, e_2) e_2 = -\mathcal{A}(f_1, f_2) f_2 = \varrho_2 f_2, \]
\[ \rho_{14}(e_2, e_1) = -2 \varrho_1, \quad \rho_{14}(f_2, f_1) = 2 \varrho_1, \quad \rho_{14}(f_1, f_2) = 2 \varrho_2. \]
If we take \( \varrho_1 = \varrho_2 \), then \( \rho_{14} \in S^2 \); if we take \( \varrho_1 = -\varrho_2 \), then \( \rho_{14} \in \Lambda^2 \). This proves Assertion (2).

We begin the proof of Assertion (3) by taking:
\[ \Theta_{111} = \varrho_1 (x_1 - \sqrt{-1} y_1) + \varrho_2 (x_2 - \sqrt{-1} y_2), \]
\[ \Theta_{222} = \varrho_3 (x_2 - \sqrt{-1} y_2) + \varrho_4 (x_1 - \sqrt{-1} y_1). \]

We then have
\[ \nabla_{e_1} e_1 = -\nabla_{f_1} f_1 = (\varrho_1 x_1 + \varrho_2 x_2) e_1 - (\varrho_1 y_1 + \varrho_2 y_2) f_1, \]
\[ \nabla_{e_1} f_1 = \nabla_{f_1} e_1 = (\varrho_1 y_1 + \varrho_2 y_2) e_1 + (\varrho_1 x_1 + \varrho_2 x_2) f_1, \]
\[ \nabla_{e_2} e_2 = \nabla_{f_2} f_2 = (\varrho_3 x_2 + \varrho_4 x_1) e_2 - (\varrho_3 y_2 + \varrho_4 y_1) f_2, \]
\[ \nabla_{e_2} f_2 = \nabla_{f_2} e_2 = (\varrho_3 y_2 + \varrho_4 y_1) e_2 + (\varrho_3 x_2 + \varrho_4 x_1) f_2. \]

Let \( \mathcal{A} := \mathcal{R}(0) \). Then:
\[ \mathcal{A}(e_1, f_1) f_1 = -2 \varrho_1 e_1, \quad \mathcal{A}(e_1, f_1) e_1 = 2 \varrho_1 f_1, \]
\[ \mathcal{A}(e_2, e_1) e_1 = -\mathcal{A}(e_2, f_1) f_1 = \varrho_2 e_1, \quad \mathcal{A}(e_2, e_1) f_1 = \mathcal{A}(e_2, f_1) e_1 = \varrho_2 f_1, \]
\[ \mathcal{A}(f_2, f_1) f_1 = -\mathcal{A}(f_2, f_1) e_1 = \varrho_2 f_1, \quad \mathcal{A}(f_2, f_1) e_1 = \mathcal{A}(f_2, f_1) f_1 = \varrho_2 e_1, \]
\[ \rho_{14}(e_1, e_1) = \rho_{14}(f_1, f_1) = -2 \varrho_1, \quad \rho_{14}(e_1, e_2) = \rho_{14}(f_1, f_2) = -2 \varrho_4, \]
\[ \rho_{14}(e_2, e_2) = \rho_{14}(f_2, f_2) = -2 \varrho_3, \quad \rho_{13}(e_1, e_1) = \rho_{13}(f_1, f_1) = 2 \varrho_1, \]
\[ \rho_{13}(e_1, e_2) = \rho_{13}(f_1, f_2) = 2 \varrho_3, \quad \rho_{13}(e_2, e_2) = \rho_{13}(f_2, f_2) = 2 \varrho_4. \]

Note that \( \tau = -4 \varrho_1 - 4 \varrho_3 \) and \( \tau_J = 0 \).

(1) Take \( \tilde{\varrho} = (0, 1, 0, 1) \) to construct \( \nabla \) with \( 0 \neq \rho_{14}(A) \in S^2_{0,3} \) and \( \rho_{13} = 0 \).

(2) Take \( \tilde{\varrho} = (0, 1, 0, -1) \) to construct \( \nabla \) with \( 0 \neq \rho_{14}(A) \in \Lambda^2_{0,3} \) and \( \rho_{13} = 0 \).

(3) Take \( \tilde{\varrho} = (1, 0, -1, 0) \) to construct \( \nabla \) with \( 0 \neq \rho_{13}(A) \in S^2_{3,0} \).

Thus we complete the proof of Assertion (3) by constructing an example where \( \rho_{13} \) has a non-zero component in \( \Lambda^2_{0,3} \). We take
\[ \Theta_{122} = \Theta_{212} = \varrho_5 (x_2 - \sqrt{-1} y_2). \]

Set \( \mathcal{A} := \mathcal{R}(0) \). We have:
\[ \nabla_{e_1} e_2 = -\nabla_{f_1} f_2 = \varrho_5 (x_2 e_2 - y_2 f_2), \quad \nabla_{e_1} f_2 = \nabla_{f_1} e_2 = \varrho_5 (y_2 e_2 + x_2 f_2), \]
\[ \nabla_{e_2} e_1 = -\nabla_{f_2} f_1 = \varrho_5 (x_2 e_2 - y_2 f_2), \quad \nabla_{e_2} f_1 = \nabla_{f_2} e_1 = \varrho_5 (y_2 e_2 + x_2 f_2), \]
\[ \mathcal{A}(e_2, e_1) e_2 = -\mathcal{A}(e_2, f_1) f_2 = \varrho_5 e_2, \quad \mathcal{A}(e_2, e_1) f_2 = \mathcal{A}(e_2, f_1) e_2 = \varrho_5 f_2, \]
\[ \mathcal{A}(e_2, f_2) e_1 = 2 \varrho_5 f_2, \quad \mathcal{A}(e_2, f_2) f_1 = -2 \varrho_5 e_2, \]
\[ \rho_{13}(e_1, e_2) = 2 \varrho_5, \quad \rho_{13}(f_1, f_2) = 2 \varrho_5. \]
This belongs to $S^n_{\theta,+} \oplus \Lambda^n_{\theta,+}$. It is not symmetric and thus has a non-zero component in $\Lambda^n_{\theta,+}$. Assertion (3) follows.

Since the modules in question are irreducible, Assertions (1)-(3) show the maps of Lemma 2.2 define surjective maps

$$\rho_{14} : \mathcal{J}^- \to W_2 \oplus W_4 \to 0,$$

$$\tau \oplus \tau_j \oplus \rho_{14} \oplus \rho_{13} : \mathcal{J}^+ \to W_1 \oplus W_3 \oplus W_5 \oplus W_6 \oplus W_7 \oplus W_8 \to 0.$$

We consider the collection of modules

$$\mathcal{C}_1 := \{W_2, W_4\},$$

$$\mathcal{C}_2 := \{W_1, W_3, W_5, W_6, W_7, W_8\},$$

$$\mathcal{C}_3 := \{W_9, W_{10}, W_{11}, W_{12}\}$$

where we omit $W_{11}$ and $W_{12}$ if $m = 4$. By Remark 2.4, no module from collection $\mathcal{C}_i$ is isomorphic to any module from $\mathcal{C}_j$ for $i \neq j$ on dimensional grounds. Assertion (4) now follows from Theorem 2.3.

We complete the proof of Theorem 1.8 by establishing:

**Lemma 4.2.**

1. If $m \geq 6$, then $W_{12} \cap \mathcal{J}^- \neq \{0\}$.
2. $W_9 \cap \mathcal{J}^+ \neq \{0\}$.
3. $W_{10} \cap \mathcal{J}^+ \neq \{0\}$.
4. If $m \geq 6$, then $W_{11} \cap \mathcal{J}^+ \neq \{0\}$.

**Proof.** Set $\Theta_{112} := x_3 + \sqrt{-1}y_3$. We then have

$$\nabla_{e_1} e_1 = -\nabla_{f_1} f_1 = x_3 e_2 + y_3 f_2, \quad \nabla_{e_1} f_1 = \nabla_{f_1} e_1 = -y_3 e_2 + x_3 f_2,$$

$$\mathcal{A}(e_3, e_1) e_1 = -\mathcal{A}(e_3, f_1) f_1 = e_2, \quad \mathcal{A}(f_3, e_1) e_1 = -\mathcal{A}(f_3, f_1) f_1 = f_2,$$

$$\mathcal{A}(e_3, e_1) f_1 = \mathcal{A}(e_3, f_1) e_1 = f_2, \quad \mathcal{A}(f_3, e_1) f_1 = \mathcal{A}(f_3, f_1) e_1 = -e_2.$$

Since $\rho_{14} = 0$ and $\mathcal{A} \in \mathcal{J}^-, 0 \neq \mathcal{A} \in W_{12}$, this establishes Assertion (1).

We clear the previous notation and take:

$$\Theta_{112} = g_1(x_1 - \sqrt{-1}y_1), \quad \Theta_{111} = g_3(x_2 - \sqrt{-1}y_2),$$

$$\Theta_{211} = \Theta_{211} = g_2(x_1 - \sqrt{-1}y_1).$$

Consequently we have:

$$\nabla_{e_1} e_1 = -\nabla_{f_1} f_1 = g_1(x_1 e_2 - y_1 f_2) + g_3(x_2 e_1 - y_2 f_1),$$

$$\nabla_{f_1} e_1 = \nabla_{e_1} f_1 = g_1(y_1 e_2 + x_1 f_2) + g_3(y_2 e_1 + x_2 f_1),$$

$$\nabla_{e_1} e_2 = -\nabla_{f_1} f_2 = \nabla_{e_2} e_1 = -\nabla_{f_2} f_1 = g_2(x_1 e_1 - y_1 f_1),$$

$$\nabla_{f_1} e_2 = \nabla_{e_1} f_2 = \nabla_{e_2} f_1 = \nabla_{f_2} e_1 = g_2(y_1 e_1 + x_1 f_1).$$

Set $\mathcal{A} := \mathcal{R}(0)$. Then:

$$\mathcal{A}(e_1, f_1) e_1 = 2g_1 f_2, \quad \mathcal{A}(e_1, f_1) f_1 = -2g_1 e_2,$$

$$\mathcal{A}(e_1, e_2) e_1 = 2g_2 f_1, \quad \mathcal{A}(e_1, f_1) f_2 = -2g_2 e_1,$$

$$\mathcal{A}(e_1, e_2) f_1 = -g_2 e_1 - g_3 e_2, \quad \mathcal{A}(e_1, e_2) f_1 = g_2 f_1 - g_3 f_1,$$

$$\mathcal{A}(e_1, f_2) e_1 = g_2 f_1 + g_3 f_1, \quad \mathcal{A}(e_1, f_2) f_1 = -g_2 e_1 - g_3 e_1,$$

$$\mathcal{A}(f_1, f_2) e_1 = g_2 e_1 - g_3 e_2, \quad \mathcal{A}(f_1, f_2) f_1 = g_2 f_1 - g_3 f_1,$$

$$\mathcal{A}(f_1, e_2) e_1 = -g_2 f_1 - g_3 f_1, \quad \mathcal{A}(f_1, e_2) f_1 = g_2 e_1 + g_3 e_1.$$

We take $\vec{a} = (-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$ to create an element $A_1 \in \mathcal{J}^+$ with:
$A_1(e_1, f_1, e_1, f_2) = A_1(f_1, e_1, f_1, e_2) = -1,$
$A_1(f_2, e_1, f_1, e_1) = A_1(e_2, f_1, e_1, f_1) = -1,$
$A_1(e_1, f_1, f_2, e_1) = A_1(f_1, e_1, e_2, f_1) = 1,$
$A_1(f_2, e_1, e_1, f_1) = A_1(e_2, f_1, e_1, f_1) = 1,$
$\rho_{14}(A_1)(e_1, e_2) = \rho_{14}(A_1)(e_2, e_1) = 1,$
$\rho_{14}(A_1)(f_1, f_2) = \rho_{14}(A_1)(f_2, f_1) = 1.$

Interchanging the roles of the indices ‘1’ and ‘2’ then creates a tensor $A_2 \in \mathcal{J}^+$ such that
$A_2(e_2, f_2, e_2, f_1) = A_2(f_2, e_2, f_2, e_1) = -1,$
$A_2(f_1, e_2, f_2, e_2) = A_2(e_1, f_2, e_2, f_2) = -1,$
$A_2(e_2, f_2, f_1, e_2) = A_2(f_2, e_2, e_1, f_2) = 1,$
$A_2(f_1, e_2, f_2, f_2) = A_2(e_1, f_2, e_2, f_2) = 1,$
$\rho_{14}(A_2)(e_1, e_2) = \rho_{14}(A_2)(e_2, e_1) = 1,$
$\rho_{14}(A_2)(f_1, f_2) = \rho_{14}(A_2)(f_2, f_1) = 1.$

These tensors are anti-symmetric in the last two indices so $\rho_{13} = -\rho_{14}$. We verify that $0 \neq A_1 - A_2 \in W_9 \cap \mathcal{J}^+$ which establishes Assertion (2).

Next, we take $\varrho = (\frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$ to create a tensor such that:
$A_3(e_1, f_1, e_1, f_2) = A_3(e_1, f_1, f_2, e_1) = 1,$
$A_3(e_1, f_1, f_1, e_2) = A_3(e_1, f_1, e_2, f_1) = A_3(e_1, e_2, f_1, f_1) = -1,$
$A_3(e_1, e_2, e_1, e_1) = A_3(f_1, f_2, e_1, e_1) = A_3(f_1, f_2, f_1, f_1) = -1,$
$\rho_{14}(A_3)(e_1, e_2) = \rho_{14}(A_3)(f_1, f_2) = 1,$
$\rho_{14}(A_3)(e_2, e_1) = \rho_{14}(A_3)(f_2, f_1) = -1.$

We interchange the roles of the indices “1” and “2” to create an element of $\mathcal{J}^+$ such that:
$A_4(e_2, f_2, e_2, f_1) = A_4(e_2, f_2, f_1, e_2) = 1,$
$A_4(e_2, f_2, f_1, e_1) = A_4(e_2, f_2, e_1, f_2) = A_4(e_2, e_1, f_2, f_2) = -1,$
$A_4(e_2, e_1, e_2, e_2) = A_4(f_2, f_1, e_2, f_2) = A_4(f_2, f_1, f_2, f_2) = -1,$
$\rho_{14}(A_4)(e_2, e_1) = \rho_{14}(A_4)(f_2, f_1) = 1,$
$\rho_{14}(A_4)(e_1, e_2) = \rho_{14}(A_4)(f_1, f_2) = -1.$

These two tensors are symmetric in the last two indices so $\rho_{13} = \rho_{14}$. We then have $0 \neq A_3 + A_4 \in W_{10} \cap \mathcal{J}^+$ which establishes Assertion (3).

We clear the previous notation and set $\Theta_{112} = x_3 - \sqrt{-1} y_3$. Let $\mathcal{A} = \mathcal{R}(0)$. Then:
$\nabla_{e_1} e_1 = -\nabla_{f_1} f_1 = x_3 e_2 - y_3 f_2,$
$\nabla_{e_1} f_1 = \nabla_{f_1} e_1 = y_3 e_2 + x_3 f_2,$
$A(e_3, e_1)e_1 = -A(e_3, f_1)f_1 = e_2,$
$A(f_3, e_1)e_1 = -A(f_3, f_1)f_1 = -f_2,$
$A(e_3, f_1)f_1 = A(e_3, f_1)e_1 = f_2,$
$A(f_3, e_1)f_1 = A(f_3, e_1)e_1 = e_2,$
$\rho_{13}(\mathcal{A}) = \rho_{14}(\mathcal{A}) = 0.$

We have $\mathcal{A} \in \mathcal{J}^+ \cap \ker(\rho_{13}) \cap \ker(\rho_{14}) = W_9 \oplus W_{10} \oplus W_{11}$. To prove Assertion (4), it suffices to show $\mathcal{A}$ has a non-zero component in $W_{11}$. Suppose to the contrary that $\mathcal{A} \in W_9 \oplus W_{10}$. We may then decompose $\mathcal{A} = \mathcal{A}_9 + \mathcal{A}_{10}$ where $\mathcal{A}_9 \in W_9$ and $\mathcal{A}_{10} \in W_{10}$. We lower indices to define $A$, $A_9$, and $A_{10}$. We then have
$A(x, y, z, w) + A(x, y, w, z) = A_9(x, y, z, w) - A_9(x, y, z, w)$
$+ A_{10}(x, y, z, w) + A_{10}(x, y, w, z)$
$= 2A_{10}(x, y, z, w).$
We now compute that:

\[ A_{10}(f_3, f_1, e_2, e_1) + A_{10}(f_1, e_2, f_3, e_1) + A_{10}(e_2, f_3, f_1, e_1) = \frac{1}{2} + 0 + 0 \neq 0. \]

This shows that the Bianchi identity; the Lemma now follows from this contradiction. \( \square \)

**Remark 4.3.** In fact, we have proved just a bit more. We have shown that if \( A \in \mathfrak{R}^\sim \), then \( (V, J, A) \) is geometrically realizable by an affine Kähler manifold \( (M, J, \nabla) \) where \( R \in \mathfrak{R} \) at all points of \( M \).

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