Phenomenological Quantum Gravity

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Abstract. These notes summarize a set of lectures on phenomenological quantum gravity which one of us delivered and the other attended with great diligence. They cover an assortment of topics on the border between theoretical quantum gravity and observational anomalies. Specifically, we review non-linear relativity in its relation to loop quantum gravity and high energy cosmic rays. Although we follow a pedagogic approach we include an open section on unsolved problems, presented as exercises for the student. We also review varying constant models: the Brans-Dicke theory, the Bekenstein varying $\alpha$ model, and several more radical ideas. We show how they make contact with strange high-redshift data, and perhaps other cosmological puzzles. We conclude with a few remaining observational puzzles which have failed to make contact with quantum gravity, but who knows... We would like to thank Mario Novello for organizing an excellent school in Mangaratiba, in direct competition with a very fine beach indeed.

WHY QUANTUM GRAVITY?

The subject of quantum gravity emerged as part of the unification program that led to electromagnetism and the electroweak model. We’d like to unify all forces of Nature. Forces other than gravity are certainly of a quantum nature. Thus we cannot hope to have a fully unified theory before quantizing gravity.

To come clean about it right from the start, we should stress that there is no compelling experimental reason for quantizing gravity. For all we know, gravity could stand alone with respect to all other forces, and simply be exactly classical in all regimes. There is no evidence at all that the gravitational field ever becomes quantum. Yet this hasn’t deterred a large number of physicists from devoting lifetimes to this pursuit.

Assaults on the problem currently follow two main trends: string/M theory [1, 2] and loop quantum gravity [3, 4]. Both have merits and deficiencies, commented extensively elsewhere. As a poor third we mention Regge-calculus (and lattice techniques), non-commutative geometry, and several other methods none of which has fared better or worse than the two main strands.

This course is not about those theories. Rather it’s about the question: Where might experiment fit into these theoretical efforts of quantizing gravity? A middle ground has recently emerged – phenomenological quantum gravity. The requirements are simple: a phenomenological formalism must provide a believable approximation limit for more sophisticated approaches; it must also make clear contact with experimental anomalies that don’t fit into our current understanding of the world. The following argument illustrates what we mean by this.

When physicists find themselves at a loss they often turn to dimensional analysis. Following this simplistic philosophy we estimate the scales where quantum gravity effects may become relevant by building quantities with dimensions of energy, length and time from $\hbar$ (the quantum), $c$ (relativity) and $G$ (gravity). These are called the Planck energy $E_P$, the Planck length $l_P$ and the Planck time $t_P$. For instance $E_P = \sqrt{\hbar c^3/G} \approx 1.2 \times 10^{19}$GeV $\approx 2.2 \times 10^{-5}$g. Quantum gravitational effects are expected to kick in for energies above $E_P$ or lengths and durations smaller than $l_P$ and $t_P$. Beware: dimensional analysis can be too naive.

We expect quantization of gravity to take the form of a theory in which space and time are discretized. General relativity is a theory of curved space-time. Thus, quantum gravity should quantize not only curvature but actual

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1 Exercise for the student: discuss the gravitational field of a photon undergoing the double slit experiment. Could you collapse the wave function by measuring the gravitational field? Lay out all possibilities in the form of thought-experiments. You may find it interesting to make contact with the old problem of how classical and quantum systems interact. Then repeat the exercise with a double slit experiment where gravitons are used instead of photons.
durations and lengths. Here quantization means to replace a continuum by a discrete structure. This may be done, as a first approximation, just as Planck did in the first quantum theory, when he simply proposed that the energy of a harmonic oscillator would be a multiple of a fixed energy, $\hbar \omega$. The same could be true for space and time: space is granular, time has an atomic structure. This is, however, an approximation. As we know, what actually happens in quantum mechanics is that observables are replaced by operators with a discrete spectrum, whose eigenvalues represent the possible outcomes of a measurement. Similarly, in loop quantum gravity, area and volume become operators with a discrete spectrum and the geometry becomes a quantum state (a spin foam, more specifically). Still, we may use the “Planck-style” of quantization as a basis for a phenomenological model.

The point of this simplified quantization approach is that now we have a springboard for contact with experiment. The argument is based on a paradox similar in flavor to the Zeno paradox in ancient Greek philosophy (which incidentally concerned the absurd apparent conflict between discrete and continuum). Whatever quantum gravity may turn out to be, it is expected to agree with special relativity when the gravitational field is weak or absent, and also with all experiments probing the nature of space-time on scales much larger than $l_p$ (or energy scales smaller than $E_p$). The granules of space-time should be invisible unless we examine these scales with a powerful “microscope”.

This immediately gives rise to a simple question: In whose reference frame is $l_p$ the threshold for new phenomena? For suppose that there is a physical length scale which measures the size of spatial structures in quantum space-times, such as the discrete area and volume predicted by loop quantum gravity. Then if this scale is $l_p$ in one inertial reference frame, special relativity suggests it will be different in another observer’s frame – a straightforward implication of Lorentz-Fitzgerald contraction, easily derived from the Lorentz transformations. In other words the border between classical and quantum gravity is not invariant or well defined. Similar arguments can be made with energy and time.

There are two obvious answers to the problem. On the one hand, Lorentz transformations may be correct on all scales, such that the Planck length is sensitive to Lorentz contraction. In this case, quantum gravity picks up a preferred scale, non-linear relativity is “doubly special” because it fixes, in addition, a relativistic energy scale. This is, however, an approximation. As we know, what actually happens in quantum mechanics is that observables are replaced by operators with a discrete spectrum, whose eigenvalues represent the possible outcomes of a measurement. Similarly, in loop quantum gravity, area and volume become operators with a discrete spectrum and the geometry becomes a quantum state (a spin foam, more specifically). Still, we may use the “Planck-style” of quantization as a basis for a phenomenological model.

The strength of this last approach is that a relation to observations is quickly obtained in this way. Indeed, DSR explains ultra high energy cosmic ray anomalies. This illustrates what is meant by phenomenological quantum gravity. The theoretical problem is too hard. Perhaps it needs a bit of fresh air, called experiment. A simplified formalism could then be set up with the flavors of attempts at a full solution. That is, such a formalism can act as a target for low-energy approximations to the the full solution, and can also make immediate contact with experiment. A bridge between theory and experiment has been set up.

**NONLINEAR RELATIVITY**

"Doubly Special Relativity" (DSR) is a semiclassical theory, formulated in flat space-time, yet significant at extremely high energy scales. It is based on a non-linear extension of the laws of Special Relativity. Just as Einstein’s relativity theory is "special" because it holds invariant the speed of light ($c$) as a fundamental relativistic scale, non-linear relativity is "doubly special" because it fixes, in addition, a relativistic energy scale. We stress the word "relativistic" in order to highlight what type of fundamental scale we are dealing with. For instance, $\hbar$ is not a relativistic scale since it does not affect the transformations between inertial observers like $c$ does at very high velocities. The new fixed energy scale will play a role, independent of $c$, in relativistic transformations at very high energies.

Part of the justification for the introduction of an invariant energy scale into Special Relativity can be found in the lineage of Einstein’s theory. Galileo’s original expression for the energy of a fundamental particle, $E = p^2 / 2m$, is linearly invariant under the transformations he defined circa 1600 A.D.. These Galilean Transformations describe the relativity of inertial motion. They state that position and time coordinates measured in a "primed" frame moving at velocity $v$ in the $x$-direction with respect to a lab frame at rest are expressed as $x' = x - vt$ and $t' = t$, respectively. A length $\ell$, a time interval $t$ and the velocity of a particle moving at speed $v_1$ in the moving frame would then be written

$$v'_1 = v_1 - v \tag{1}$$
Roughly 200 years later Maxwell formulated his equations describing electricity and magnetism and introduced the notion of a constant value for the speed of light. In order for this invariance to hold whilst satisfying Galilean transformations, the notion of a "preferred observer” had to be introduced. The preference was made manifest by introducing a uniform cosmic background called the “ether”. In other words, the relativity of inertial observers was lost.

While Michelson and Morley worked to prove the constancy of the speed of light proposed by Maxwell, Einstein’s work in the early 1900s demolished the concept of the ether by unifying the previously disjoint notions of time and space in new laws describing the relativity of inertial motion. His discovery of the equivalence of mass and energy led him to a revised definition of particle energy, the quadratic "dispersion relation" \( E^2 = p^2 c^2 + m^2 c^4 \). This was no longer, however, invariant under Galilean transformations. In order to reestablish observer independent laws, new relativistic transformations, the Lorentz transformations, had to be formulated. With respect to a frame at rest, a particle will have coordinates in a frame moving at velocity \( v \) in the \( x \)-direction given by,

\[
t' = \gamma (t - \frac{vx}{c^2}) \tag{4}
\]
\[
x' = \gamma (x - vt) \tag{5}
\]
\[
y' = y \tag{6}
\]
\[
z' = z, \tag{7}
\]

where \( \gamma = 1/\sqrt{1 - v^2/c^2} \). With respect to energy and momenta coordinates these transformations become,

\[
E' = \gamma (E - vp_x) \tag{8}
\]
\[
p'_x = \gamma (p_x - \frac{Ev}{c^2}) \tag{9}
\]
\[
p'_y = p_y \tag{10}
\]
\[
p'_z = p_z \tag{11}
\]

Galileo’s linear velocity addition laws, invariant lengths and invariant time intervals were accordingly transformed into nonlinear velocity addition laws, length contraction and time dilation formulae given by,

\[
v'_1 = \frac{v_1 + v}{1 + v_1 v} \tag{12}
\]
\[
\ell' = \frac{\ell}{\gamma} \tag{13}
\]
\[
\Delta t' = \gamma \Delta t. \tag{14}
\]

With these new laws governing inertial motion, Einstein enforced the equality of all observers.

In this vein, in order to preserve the relativity of inertial observers, DSR theories deform the Lorentz transformations such that the modified dispersion relation, \( E^2 = p^2 c^2 + m^2 c^4 + f(E, p^2; E_0) \), is non-linearly invariant under their action. \( E_0 \) is the invariant energy scale. The \( f \)-function in this expression represents a generalized way of introducing the energy-dependence into the Lorentz transformations, as we will see below.

History aside, there are two significant motivational areas for introducing a fixed energy scale into Einstein’s relativistic transformations, one motivated by issues in quantum gravity theories (e.g.\[8\]) and one by anomalous observations of the cosmos (e.g.\[9\]). Included in the first is the idea of a fundamental energy (or length) scale, such as would arise in a theory of quantum gravity. This scale would take on a certain value in one frame of reference, but when boosted to another frame of reference it would assume another value according to the Lorentz transformations. In fact, there would exist such a frame in which the given energy scale would appear to surpass the limiting energy value predicted by quantum gravity. It is thus paradoxical for a "fundamental" scale to be relativistic, and so, in principle, we want to modify the transformations such that they hold this fundamental scale fixed while preserving observer independence\[8\]. This modification would manifest itself only as we approach extremely high energy scales. Otherwise, by the correspondence principle, if we set this energy scale to infinity (or length scale to zero), which is
equivalent to not having a limiting scale at all, we recover Special Relativity and its Lorentz transformations. This is reflective of the \( c \to \infty \) limit taking Lorentz transformations to Galilean transformations, and the \( \hbar \to 0 \) limit taking quantum mechanics to classical mechanics.

Another theoretical motivation within the realm of quantum gravity, is the fact that theories of Non-Commutative Geometries, Stringy Space-Time Foams, and Loop Quantum Gravity all predict modified dispersion relations [3]. So, the hope would be to come to a deformed DSR dispersion relation that is in accord with one of these theoretical predictions. But, naturally, observational support is needed. As we will show below, deformed dispersion relations, manifest at high energy scales, can be used to clarify anomalous observations of high energy particle interactions. In particular, an initial goal for DSR theories was to explain the Ultra High Energy Cosmic Rays (UHECRs) that are observed to collide with the Earth’s atmosphere, but which are theoretically predicted not to exist due to threshold interactions of cosmic rays with the Cosmic Microwave Background (CMB) as they travel through space [11, 12, 13, 9] (see also [14, 15, 16]).

Throughout these notes we will choose the Planck Energy, \( E_p \approx \sqrt{\hbar c^5/G} = 10^{19} \text{GeV} \), as our fundamental energy scale, where \( \hbar \) is Planck’s constant, \( c \) is the speed of light, and \( G \) is Newton’s gravitational constant. This energy is the fundamental scale for many quantum gravity theories and is chosen as an explicit example here for simplicity. We do not ignore the fact that it may not be precisely this value that should come into play in the DSR equations. The exact value must be predicted uniquely and consistently by both quantum gravity and observation.

### The Lorentz group

Before launching into the construction of DSR theories we will review some basics of the structure of the Lorentz Group, which is an example of the general class of Lie groups, pivotal to understanding the relativistic transformations to hand. The "generators" of the Lorentz group actions are the infinitesimal transformations

\[
L_{ab} = p_a \frac{\partial}{\partial p^b} - p_b \frac{\partial}{\partial p^a},
\]

where \( p \) denotes the energy-momentum 4-vector such that the lettered indexes run over the values \((0,1,2,3)\) and Latin indexes, \( i, j, \) etc. run over the spatial indexes \((1,2,3)\). The metric signature we will use throughout the notes is \((-;+++)\). The \( L_{ij} \) are rotations about the \( j \)-axis, and the \( L_{0j} \) are boosts in the \( j \)-direction.

The commutation relations of the Lorentz generators form the algebra of the Lorentz group. The generators, in general, do not commute, which allows for the algebra to be closed. This means that the action of any two generators applied in sequence and then replied in reverse sequence will result in an action that is an element of the original group. Defining the boosts and rotations to be \( B_j \equiv L_{0j} \) and \( R_j \equiv L_{ij} \), respectively, the Lorentz group algebra can be written,

\[
[B_i, B_j] = -R_k \quad (16)
\]

\[
[R_i, R_j] = iR_k \quad (17)
\]

\[
[B_i, R_j] = iB_k \quad (18)
\]

\[
[B_i, R_i] = 0. \quad (19)
\]

Exponentiating these generators gives us the full finite Lorentz transformations, \( p'_a = e^{\theta L_{ab}} p_b \). Consider, for instance, the component \( L_{01} = p_0 \frac{\partial}{\partial p^0} - p_1 \frac{\partial}{\partial p^1} \). Exponentiating it and letting it act on the energy-momenta coordinates will give the finite Lorentz transformation shown above. We do this by tailor expanding the exponential function. For the case of energy, \( p_0 \), the expansion reads \( p'_0 = (1 + \theta L_{01} + \frac{\theta^2}{2!} L_{01}^2 + \frac{\theta^3}{3!} L_{01}^3 + \ldots) p_0 \). It can be shown that the action of \( L_{01} \) on energy and momentum is given by \( L_{01} p_1 = p_0 \) and \( L_{01} p_0 = p_1 \), respectively. Thus, \( \theta \)

\[
p'_0 = p_0 + \theta p_1 + \frac{\theta^2}{2!} p_0 + \frac{\theta^3}{3!} p_1 + \ldots \quad (20)
\]

\[
=p_0 \cosh \theta + p_1 \sinh \theta. \quad (21)
\]

Similarly, \( p'_1 = p_1 \cosh \theta + p_0 \sinh \theta \). These expressions tell us that Lorentz transformations are rotations in hyperbolic space. We can express the boost factor as \( \gamma = \frac{1}{\sqrt{1 - v^2/c^2}} = \cosh \theta \), where the velocity is given by \( v = \tanh \theta \).
A final group property that we will need in the construction of our DSR theory is a realization. A realization of the Lorentz group is a set of generators that has the same algebra as the original group. It is comprised of the generators \( K_i = U^{-1} B_i U \) and \( J_i = U^{-1} R_i U \), for any function \( U \) such that the Lorentz algebra relations hold. For instance, we show

\[
\begin{align*}
[K_i, K_j] &= U^{-1} B_i U U^{-1} B_j U - [i \mapsto j] \\
&= U^{-1} [B_i, B_j] U \\
&= -iU^{-1} R_k U \\
&= -iJ_k.
\end{align*}
\]

Let us now look explicitly at what happens to the Lorentz algebra in DSR theory.

**Implications of a Non-Quadratic Invariant**

The DSR theory we will be discussing keeps the value of the fundamental Planck energy scale, \( E_p \), constant. In so doing, it preserves the Lorentz algebra, such that the DSR generators constitute a realization of the Lorentz Group \([8]\). Recall that the deformed dispersion relation introduced above, \( E^2 = p^2 c^2 + m^2 c^4 + f(E, p^2; E_p) \), incorporates an extra term representing a new dependence on a fundamental energy, \( E_p \). This function, \( f \), is a map from energy-momentum space into itself. We can thus write this as a function acting on energy-momentum coordinates. In particular, we can write the modified boosts of DSR as

\[
K_i = U^{-1}(E_p)B_i U(E_p),
\]

where \( B_i \) are the special relativistic linear boost generators. The \( E_p \)-dependence now appears in the map, \( U \), and must be non-linear in order to keep \( E_p \) invariant \([9]\) (see also \([17, 18, 19]\)). The rotation transformations in DSR remain unchanged under this action, such that the deformed boost generators are all that is necessary to satisfy the aims that DSR looks to achieve \([8]\). We do not wish to modify the rotations anyways since rotational invariance has been proved accurate to extreme precision. The transformed boosts would then hold invariant the nonlinear expression, \( m^2 = \eta^{ab} U_\omega(p) U_\nu(p) \).

To clearly illustrate the effects that DSR has on Special Relativity, we will introduce the \( E_p \) energy scale into the \( U \)-map via a particular example \([8]\). Consider the deformed boost generator given by

\[
K_i = L_{0i} + \frac{p_i}{E_p} D,
\]

where \( D = p_a \frac{\partial}{\partial p_a} \) is the dilatation generator. It can be shown that this simple form leaves \( E_p \) invariant as well as the Lorentz algebra unchanged. The corresponding \( U \)-map, the modification to the original Lorentz boost generator, can thus be written

\[
U = e^{\frac{E}{E_p} D},
\]

where \( E = p_0 \) always. Expanding this in a tailor series, we can find its action on the energy-momenta coordinates. The exact expression is

\[
U(p_\alpha) = \frac{p_\alpha}{1 - \frac{E}{E_p}},
\]

where we have a new factor in the denominator that, by the correspondence principle, reduces to unity if we let \( E_p \rightarrow \infty \). If, on the other hand, \( E = E_p \), the expression blows up, creating the invariant behaviour of \( E_p \) we desire.

Exponentiation of the \( K_i \) generator gives us the finite form for the transformed boosts. However, we need not perform this calculation. Instead, we can derive their form in a much simpler manner by applying our \( U \)-map to the regular special relativistic boost equation \([8]\). That is,

\[
\begin{align*}
U(E') &= \gamma(U(E) - vU(p)) \\
U(p') &= \gamma(U(p) - vU(E)),
\end{align*}
\]

where we have set \( c = 1 \) for simplicity. By tailor expansion it can be proved that indeed \( e^{\theta U^{-1} B' U} = U^{-1} e^{\theta B' U} U \). To make this explicit, \( U \) linearizes the physical momentum, which is then boosted by \( e^{\theta B' U} \) and then converted back to a physical momentum via \( U^{-1} \):

\[
P_{\text{physical}}' \xleftarrow{U^{-1}} P_{\text{linear}}' \xleftarrow{e^{\theta B' U}} P_{\text{linear}} \xleftarrow{U} P_{\text{physical}}.
\]
Using the particular form for the $U$-map shown above, we can solve for $E'$ on the left hand side. This is equivalent to applying the $U^{-1}$ map, thus giving us our deformed transformations. These now hold invariant the mass-squared expression

$$m^2 = \frac{\eta_{ab} p_a p_b}{(1 - \frac{E}{E_p})^2}$$

(33)

For a boost in the $x$-direction, the transformations assume the form:

$$E' = \frac{\gamma(E - vp_x)}{1 + (\gamma - 1) \frac{E_p}{E} - \gamma^2 \frac{v^2}{E_p}}$$

$$p'_{x} = \frac{\gamma(p_x - vE)}{1 + (\gamma - 1) \frac{E_p}{E} - \gamma^2 \frac{v^2}{E_p}}$$

$$p'_{y} = \frac{p_y}{1 + (\gamma - 1) \frac{E_p}{E} - \gamma^2 \frac{v^2}{E_p}}$$

$$p'_{z} = \frac{p_z}{1 + (\gamma - 1) \frac{E_p}{E} - \gamma^2 \frac{v^2}{E_p}}$$

(34)

The numerators of these expressions are the familiar special relativistic ones, whereas the denominators fully represent the deformation. Note that in letting $E_p \to \infty$, we recover the special relativistic boosts in accordance with the correspondence principle.

A simple analysis of these equations lends immediate insight into how they are different from regular special relativistic boosts. Most importantly, plugging $(E_p, 0, 0, 0)$ in for $(E, p_x, p_y, p_z)$ gives $(E_p, -vE_p, 0, 0)$, demonstrating that $E_p$ is indeed an invariant quantity. The regular boosts would have given us extra $\gamma$-factors in front of the $E$ and $p_x$ values, which are now cancelled by the new factor in the denominator of the deformed boosts. Because of this limit, $E < E_p$ will always hold in the single particle sector. We will discuss the multi-particle sector in the next section. Similarly, rest masses will always be finite due to this energy limit, which we can see by solving for $E$ in the deformed dispersion relation with $p = 0$. This gives

$$E = \frac{mc^2}{1 + \frac{m^2}{E_p}},$$

(35)

where we have reinserted $c$ for clarity. This expression, in turn, implies that the classical momentum, $p = mv$, is now given by

$$p = \frac{mv}{1 + \frac{m}{E_p}}$$

(36)

The invariance of $E_p$ is signalled by a singularity in the $U$-map at $U(E_p) = \infty$. The equations have another invariant, for $E = 0$, however, this does not give the singularities necessary for an invariant $E_p$ scale.

Now, consider the case of a Planck energy photon. Though perhaps unphysical, this extreme case displays the limiting behavior of these transformations. We see that $(E_p, E_p, 0, 0) \to (E_p, E_p, 0, 0)$. Complete invariance. That is to say that as $E \to E_p$, boosts become increasingly unproductive, until ultimately, at the Planck scale, they do no work at all. As it turns out, this variable boost property becomes ironically unproductive for accomplishing the original goal of solving the UHECR anomaly, which is discussed below.

We can also read off the new redshift formula for a photon by setting $m = 0$ in the deformed dispersion relation. This gives $E = p$, such that the boost formula gives,

$$\frac{E'}{E} = \frac{\gamma(1 - v)}{1 + (\gamma - 1) \frac{E_p}{E}}$$

(37)

In the limit $E_p \to \infty$, this reduces to the standard Doppler redshift formula, $E'/E = \sqrt{(1 - v)/(1 + v)}$. The blueshift formula is just as above, but with the signs in front of $v$ switched. The deformed Doppler formula thus implies that blueshifting a sub-Planckian photon up to $E_p$ is impossible, because $\Delta E \to 0$ as $E_p$ energies are approached. This reduced Doppler effect reflects the decrease in boost productivity as we approach Planck energies. By using the principle of equivalence, we can get an expression for the gravitational redshift. Namely,

$$\Delta E/E = \Delta \phi (1 - E/E_p).$$

(38)
THE PHYSICAL CONTENT OF DSR

We now describe how the considerations above may be used to make contact with some anomalies and upcoming experiments. This is expected to provide a window into quantum gravity, dependent upon the formalism encoded in the choice of the $U$-map. There are two main physical implications to consider.

Ultra high energy cosmic rays

UHECRs are believed to be highly relativistic protons that collide with the Earth’s atmosphere \(^{20}\). Due to their apparent isotropic distribution \(^{21}\), they are assumed not to be of galactic origin and thus to have traversed large distances through the universe, interacting with the Cosmic Microwave background (CMB) en route. There exists a theoretical bound on the energies that the protons can have due to their threshold interactions with the CMB photons \(^{20}\). This bound, or the GZK cutoff, occurs at a threshold energy of \(6 \times 10^{10} \text{ GeV} \), due to the \( p^+ + \gamma_{\text{CMB}} \rightarrow p^+ \pi^0 \) interaction. Note that the cross-section for the \( p^+ + \gamma_{\text{CMB}} \rightarrow p^+ \pi^0 \) interaction does not lend a significant contribution.

The anomaly rests in the observations of cosmic rays far above this threshold energy \(^{22, 23}\). DSR, in altering the laws of relativistic interactions, looks to raise the GZK cutoff in order to account for the observed ultra high energy samples.

To see how DSR can alter the threshold energy, consider the proton-photon interaction first in the center of mass frame such that the proton and pion produced are at rest \(^{9}\). In this case, the interaction is not relativistic, so we can use regular energy and momentum conservation laws. These give \( E_p + E_\gamma = m_p + m_\pi \) and \( E_\gamma - p_p = 0 \), respectively. Boosting to the cosmological frame, the photon is redshifted to the temperature of the CMB radiation, \( E_\gamma' \equiv E_{\text{CMB}} \sim 2.7K \). The boosted proton energy is the threshold energy, expressed \( E_p' = E_{\text{th}} = \gamma (E_p + v p_p) \). here we have used the approximation \( v \approx 1 \) because \( c = 1 \) in our equations and we are dealing with a highly relativistic situation such that \( v \rightarrow c \).

Using the energy and momentum conservation equations we can rewrite the threshold energy as

\[
E_{\text{th}} = \frac{(m_p + m_\pi)^2 - m_p^2}{4E_{\text{CMB}}}. \tag{39}
\]

In general \( U_u(p) \approx p_u \) for the non-relativistic particles. But, for the highly relativistic proton in the cosmological frame, we must use our new DSR transformation laws,

\[
E_{\text{th}}^{\text{DSR}} = U^{-1}(E_{\text{th}}^{\text{SR}}), \tag{40}
\]

where \( E_{\text{th}}^{\text{SR}} \) is the linear threshold energy and \( E_{\text{th}}^{\text{DSR}} \) is the physical threshold energy \(^{9}\). Clearly, the nonlinear DSR effects are relevant only for the energy coordinate of the relativistic particle in the interaction, not the momenta coordinates.

Unfortunately, the simple \( U \)-map used in this calculation ends up lowering the threshold energy \(^{9}\), which is precisely the opposite of what we need in order to solve the anomaly. There is, however, an alternate form of \( U \)-transformation which does provide a solution \(^{9}\). Let us first write the deformed dispersion relation in a form that splits up the action of the \( U \)-map on energy and momentum coordinates, such that \( E^2 f_2^2(E, p; E_p) + p^2 f_2^2(E, p; E_p) = m^2 \) \(^{9}\). Choosing a \( U \)-map, thus corresponds to choosing the functions, \( f_1 \) and \( f_2 \), because \( U_u(p) = (E f_1, p f_2) \). In this form, we may consider particular functions that do not necessarily act on energy and momenta in the same manner. The anomaly-resolving map is then given by

\[
f_1 = \frac{1}{(1 + E_{\text{th}})(1 - \frac{E}{E_p})}. \tag{41}
\]

where \( E_{\text{th}} \) is the energy at which the particle threshold interaction occurs, and \( f_2 \) can be any function. If you recall that the “strength”, or efficiency, of the boosts are energy-dependent in DSR, what the introduction of this second energy scale, \( E_{\text{th}} \), does is to increase the boost efficiency around this energy in order to raise the threshold and thus resolve the anomaly. Above these energies the boosts will again become increasingly unproductive as \( E_p \) is approached.

Knowing that the fundamental Planck energy scale, \( E_p \sim 10^{19} \text{ GeV} \), and the cosmic ray threshold energy scale, \( E_{\text{th}} \sim 10^{10} \text{ GeV} \), are many orders of magnitude apart, DSR is essentially trying to kill two birds with one stone. It is
trying to solve the UHECR anomaly whilst holding fixed the Planck energy scale so as to resolve the paradox of a fundamental scale being relativistic. Again, we mention that the particular form for the $U$-transformation needed will be up to experiment, especially since the UHECR data is not plentiful and conclusive [25, 26]. In particular, two caveats to consider are, for one, the fact that the primary shower used to detect the UHECRs is never directly seen. Rather, it is from the analysis of the secondary shower particles that we draw conclusions. Thus, it could be that the cosmic rays are not in fact protons, but some other form of highly energetic dark matter, for instance. Second, further data may prove that the arrival of UHECRs is not actually isotropic [21], implying that they may originate from within the galaxy. Needless to say, this would crucially alter our threshold energy calculation above, since the CMB would not play the same role.

Invariance of the Speed of Light and its energy dependence

The general invariant in DSR is

$$\eta^{ab}U_a(p)U_b(p) = m^2$$  \hspace{0.5cm} (42)

where $U_a(p) = p_a^{\text{linear}}$. Given a map of the form $U_a(p) = p_a/(1 - E/E_p)$, the invariant reads $\frac{E^2 - p^2}{(1 - E/E_p)^2} = m^2$. We have also just seen how we can rewrite the dispersion relation in the even more general form, $E^2 f_1^2 + p^2 f_2^2 = m^2$, where the new $E_p$ dependence now appears in the functions multiplying the energy and momenta coordinates, while the rest mass, $m$, stands alone as the new invariant. Setting $m = 0$ in the dispersion relation, we find that the velocity for a massless particle is given by

$$c(E) = \frac{E}{p} = \frac{f_2}{f_1}.$$  \hspace{0.5cm} (43)

Note that this is not the regular Hamiltonian expression $\frac{dE}{dp}$. We thus have an energy-dependent speed of light if $U$ does not act in the same way on energy and momentum coordinates, that is if $f_1 \neq f_2$. This variability introduces a new notion of invariance. Our invariant speed of light is now $c = c(E)$, whose value in a boosted frame, $c(E')$, will be predicted uniquely by the invariant form of the new deformed boosts. What this says is that the laws governing our universe are not fixed in the classical sense, but rather, are allowed to change. It is the manner in which they change that is now invariant and dictated by physical law.

PROBLEMS FOR THE STUDENT: ISSUES ARISING FROM NONLINEAR RELATIVITY

We now collect a few unresolved problems within the formalism, hoping to motivate further work.

Fixing a U-Map

We have said nothing so far about the reason why we have chosen the $U$-map that we did. The uniqueness of this map, or lack thereof, is a key outstanding issue in DSR theories. Requesting that $E_p$ be invariant does not point to a unique theory. That is, $f_1$ and $f_2$ in the general deformed dispersion relation are not uniquely specified. This is a undesirable freedom that cannot be resolved until further observational results are achieved. More observations of particle threshold interactions is just one experimental source of information for DSR. There is also the GLAST satellite looks to measure any energy dependence of the speed of light [27], which would lend insight into whether the functions $f_1$ and $f_2$ are the same or not. Hence, we leave the issue of the uniqueness of the $U$-transformation to future experiment.

There are a couple of reasons for using the shown form for the map [9]. For one, it can be easily embedded in a conformal group. Second, it is in keeping with the original nonlinear Special Relativistic Fock-Lorentz transformations proposed in 1964 [17, 18]. The difference there is that the invariant scale in the theory was a large cosmological distance scale as opposed to our large fundamental energy scale, $E_p$ (equivalent to a tiny distance scale, $\ell_p$). The map we have been working with is also simple enough in form to enable us to work through and concisely display results, such as time dilation and length contraction formulae, which we will show in the next section.
Position Space and Field Theory

DSR theories have always been formulated in momentum space (however see [17, 18, 19]). But, as we know, we live in and perceive, not momentum and energy coordinates, but position and time coordinates. How to retrieve a position space formulation of DSR from the existing momentum space ones, however, is highly non-trivial. There are two paths one can follow in accomplishing this task [28]. The first is to begin anew in position space, following a nonlinear special relativistic set-up just like that which we did in momentum space. The second is to use the pre-existing theory in momentum space to somehow fix the position space formulation of the transformations.

The deciding factor between these two approaches is, for the most part, the importance of field theory. In order to have a kinematic theory where excitations of a field have an invariant that correspond to that of DSR, then one must use the latter of the two approaches above – formulating a position space DSR theory using the pre-existing theory in momentum space [28]. To explain why, consider first the former approach of formulating a theory directly in position space. A map analogous to the $U$-map used in momentum space would have to be postulated, this time holding the Planck length, inverse to the Planck energy, invariant. But, this implies that we have now a two-fold freedom from the ambiguity corresponding to the $U$-map in momentum space, plus the ambiguity for the new map in position space, not to mention the fact that these two maps would be completely uncorrelated. The fact that these maps are non-linear and would be unrelated implies that the contraction between momentum and position coordinates, $U^a(x)U_a(p)$, would be non-linear.

The formulation of a field theory is dependent upon the linearity of this contraction in order for the wave solutions of a free field theory, $\phi \sim Ae^{-iE\tau}p_a$, to be planar. Maintaining this linearity within DSR, we can use the momentum space map to uniquely fix the position space one [28]. We can make the standard field theoretic operator association, $\hat{p}_a = i\partial_a$, such that $\hat{p}_a e^{-iE\tau}p_a = p_a e^{-iE\tau}p_a$. Applying this to our deformed dispersion relation we find that the deformed Klein Gordon Equation whose solutions are plane waves would be of the form [28]

$$\eta^{ab} \frac{\partial_a}{1 - \frac{iE_p}{E_d}} \frac{\partial_b}{1 - \frac{iE_p}{E_d}} + m^2 \phi = 0. \quad (44)$$

This can be proved by using Taylor expansions of the fractional operators.

Note that this theory preserves causality with respect to the non-linear Lorentz group because it dictates that light cones will themselves be deformed [28]. This is due to the mixing of space-time and energy-momentum coordinates, which we will discuss shortly. Also, the theory does not raise the issue of renormalizability because the theory has a natural cutoff and is therefore strictly finite. That is, $E < E_p$ by construction, such that there exists a frame-independent energy acting as a natural cutoff to divergences. There exists an alternative approach, independent of the entire formulation we have presented here, which was proposed by Kowalski-Glikman and uses non-commuting space-time operators [6, 7, 29]. Our approach, on the other hand, will lead us to an energy-dependent space-time structure [28].

Linearity of the contraction between position and momentum coordinates ensures that the duals, $x_a$ and $p_a$, mimic one another [28]. Relativity tells us that if $x^p p_a$ is linear, then their transformed contraction, $U^a(x)U_a(p)$, should be linear as well. The map acting on position coordinates therefore must be inverse to that acting on momentum coordinates in order for the nonlinear factors to cancel one another. In general, if $U_a(p) = (E f_1, p f_2)$, then $U_a(X) = (t f_1, x f_2)$. For $U \sim e^{(E/E_p)D}$, we have

$$U_a(x) = x_a \left(1 - \frac{E}{E_p}\right). \quad (45)$$

This equation introduces a mixing of space-time coordinates with energy-momentum coordinates due to the explicit presence of the energy, $E$. The corresponding invariant is energy dependent,

$$s^2 = (-t^2 + x^2) \left(1 - \frac{E}{E_p}\right)^2, \quad (46)$$

which shows that we are dealing with an energy-dependent Minkowski metric.

Applying the above action onto the special relativistic boost equation in position space gives us the deformed boosts [28].

$$t' = \gamma(t - vx) \left(1 + (\gamma - 1) \frac{E}{E_p} - \gamma \frac{vp}{E_p}\right) \quad (47)$$
\[
\begin{align*}
\gamma' &= \gamma(x - vt) \left( 1 + (\gamma - 1) \frac{E}{E_p} - \gamma \frac{vp_x}{E_p} \right) \\
\gamma' &= y \left( 1 + (\gamma - 1) \frac{E}{E_p} - \gamma \frac{vp_x}{E_p} \right) \\
\zeta' &= z \left( 1 + (\gamma - 1) \frac{E}{E_p} - \gamma \frac{vp_x}{E_p} \right).
\end{align*}
\]

The term that appeared in the denominator of the deformed boosts in momentum space now appears in the numerator, which guarantees that the linearity of the contraction of duals holds in every inertial frame. Again, the mixing of space-time and energy-momentum coordinates is apparent here.

We mentioned before that light cones are deformed, allowing causality to be preserved in DSR. This is because of the space-time-energy-momentum mixing occurring. The speed of light will be energy-dependent, but will always be the maximum velocity and thus respect causality. Setting \( ds^2 = 0 \) in the space-time dispersion relation (just as we set \( m = 0 \) in the energy-momentum dispersion relation above) we find that the velocity for a massless particle is given by

\[
c(E) = \frac{dx}{dt} = f_2 \frac{f_2}{f_1}.
\]

In order to measure or see space-time, a probe is needed, and every probe has an energy. It is the energy of this probe that appears in the space-time metric \[28\]. What this is saying is that particles with different energies will transform differently and thus feel different space-time metrics. We have a "running" of geometry with energy. The geometrical interpretation into space-time energy-dependence is an 8-dimensional phase space \[28\], \((E, p, t, x)\), as opposed to separate 4-dimensional position and momentum spaces. What we are dealing with geometrically is a twisted bundle. Imagine the base space of the bundle as position space with an energy-dependent lift into momentum space, which has a natural cutoff at \( E_p \). Near this cutoff, there is a twist in the bundle, such that the projection down into position space is not one to one, but rather introduces a mixing of coordinates, corresponding to deformed topologies, or equivalently, a "running" geometry \[30\].

Before discussing the next issue in DSR, let us turn briefly to a few familiar special relativistic formulae and what they look like in DSR theory \[28\]. The addition of velocities is the same as in Special Relativity, which can be calculated by boosting \( x = v_0 t \) to get

\[
\gamma' = \gamma(x - vt) \left( 1 + (\gamma - 1) \frac{E}{E_p} - \gamma \frac{vp_x}{E_p} \right).
\]

The classic formulae for time dilation and length contraction, however, introduce new factors. The time dilation formula can be read off from the deformed space-time boost equations,

\[
\Delta t' = \gamma \Delta t \left( 1 + (\gamma - 1) \frac{E}{E_p} \right),
\]

where we have set \( p_x = 0 \) and \( x = 0 \), consistent with a clock at the origin in the "lab" frame. For the length contraction formula, we boost a rod of length \( L \) from an unprimed frame at rest to a primed frame moving at velocity \( v \) to get \( x' = L' - vt' \), such that

\[
L' = \frac{L}{\gamma} \left( 1 + (\gamma - 1) \frac{E_0}{E_p} \right).
\]

\( E_0 = \frac{m}{1 + \frac{m}{E_p}} \) is the particle’s rest energy gotten by setting \( p = 0 \) in the deformed dispersion relation. Note that the boost parameter is equal to the origin’s velocity here. For \( E = E_p \) the length contraction formula gives invariant lengths, recalling the inefficiency of boosts that we discussed above as Planck energy scales are approached.

The "Soccer Ball" Problem

The so-called "Soccer Ball Problem" is that of describing composite systems in DSR theories. The goal now is to extend our discussion of the single particle sector of DSR to the multi-particle sector, a highly non-trivial task. With a non-linear realization of the dispersion relation and Lorentz group, our laws for adding momenta and energy must also be non-linear \[30\]. The straightforward and classical expression, \( p_{12} = p_1 + p_2 \), will no longer suffice for it would
immediately imply a loss of relativity since in the boosted frame the same law would not hold, \( p'_{12} \neq p'_1 + p'_2 \). A simple nonlinear composition law can be written as [30]

\[
p_{12} = p_1 \oplus p_2 = U^{-1}(U(p_1) + U(p_2)). \tag{55}
\]

In this case, the composite momentum, \( p_{12} \), is covariant, transforming as \( e^{iK} \). This implies that the rest energy of the system must satisfy \( E_0 < E_p \). But, \( E_p \) corresponds to a mass of \( 10^{-5} \), which is clearly not the mass limit in our world of composite systems. A soccer ball cannot have this energy, except in some unusual football leagues.

An alternative and only slightly more complicated approach is modelled by the association \( U^{(N)}_1 = U^{(1)}(E \rightarrow NE_p) \), such that 2 particles would scale as \( 2E_p \) and so on up for any number of particles. The multi particle dispersion relation, or the momenta and energy addition laws, would then read [30]

\[
\frac{p_{a}^{(N)}}{1 - \frac{E^{(N)}}{NE_p}} = \Sigma_i \frac{p_{a}^{(i)}}{1 - \frac{E^{(i)}}{E_p}}. \tag{56}
\]

This addition law implies that while the system is still sub-Planckian in the single particle sector, \( E < E_p \), it can now be super-Planckian for composites, \( E_{tot} < nE_p \), as physically expected. The addition law is non-associative, meaning that all particles in a system must be accounted for at once. That is, the laws of physics would have insight into whether a system was elementary or composite and adjust its addition laws accordingly. In a sense, the addition laws would scale with particle number. Note that if the particles of the system all have the same energy, then the addition reduces to the regular linear law.

**Gravity and Kinetic Theory**

Not much to report along here. A generalisation of the idea of an energy dependent metric for curved space-times was proposed in [30]. Statistical physics results assuming thermal equilibrium were first examined in [31, 32]; this approach may well be wrong because of the non-linearities required to solve the soccer-ball problem invalidate most of the partition function arguments.

These areas are still very much an exercise for the student.

**VARYING "CONSTANTS"**

Having already glimpsed the notion of the variability of a fundamental "constant" of nature, specifically the energy-dependence of the speed of light as predicted by certain formulations of DSR, we are prepared to consider other cosmological theories where the conventional "constants" of Nature may actually vary in space and time. There is renewed interest in understanding the physical implications of these theories motivated both theoretically and observationally.

Theoretically, the low energy limit of string and M quantum gravity theories are so-called dilaton theories. Dilaton fields, incorporated into these quantum gravity theories for the purpose of consistency, are massless and gauge-neutral scalar fields that couple to matter with a strength proportional to that of the gravitational force. Varying "constant" theories, in which a field is used to represent a variable parameter, can be formulated in such a way that they adopt the form of a dilaton theory, at least in a few cases. For parameters that are classically constant, we expect this field to settle to a fixed value at late times – a behavior perhaps due to that of space-time itself.

Dirac was the first to question the constancy of the laws of nature. In the 1930s, he also questioned the constancy of the constants themselves by postulating that the gravitational parameter, \( G \), may have assumed different values throughout the evolution of the universe. This postulate was based on the observed evolution of the Hubble parameter, \( H = \dot{a}/a \), where \( a \) is the scale factor of the universe, and its apparent, yet unexplained, proportionality to other fundamental parameters of nature, namely \( (H_0h^2/cG)^{1/3} \). He thus proclaimed that the tiny value of \( G \) today was simply a product of the old age of the universe since his theory posited \( G \propto t^{-1} \). He chose to vary \( G \) as opposed to another one of the parameters in order not to have to reformulate atomic and nuclear physics [38]. The simplicity of his time varying construction made limited predictions that ran quickly into problems. A more successful theory was the first "varying constant" field theory, Brans-Dicke theory, which described both temporal and spatial variations of \( G \) due to a dynamic cosmic scalar field, \( \phi \).
Variable-$G$ theories aside, Bekenstein’s model of a changing electromagnetic constant, $e$, is the next simplest theory and is constrained not by the observational predictions of general relativity, but rather by observations that support a variable fine structure constant, $\alpha = e^2/4\pi hc$. [34]. Given the definition of $\alpha$, it is apparent that the variability of $\alpha$ may be attributed not to the behavior of $e$, but to a changing speed of light, $c$, or a variable quantum scale, $\hbar$. These based on these latter two constants [35] are, however, more complex. Quantum gravity (noncommutative geometry) in general leads to deformed dispersion relations, which may imply a frequency-dependent speed of light, and in turn, a varying alpha value. Since it is merely a matter of convenience whether a varying alpha theory is formulated in terms of a varying $e$ or $c$, the difference being in the choice of units, there is clearly a connection drawn between the fundamental principles of a quantum gravity theory and a varying alpha theory, regardless of the particular choice of varying constant. Herein also lies the connection to the DSR theories, which are based on deformed dispersion relations, discussed above. The simultaneous exploration of DSR and varying "constant" theories is thus mutually beneficial in that a discovery about the variation of constants in one theory could lend insight into the variability determined by the other, and visa versa. Say, for instance, that experimental data helped us determine the form of the $U$-map in DSR. This would imply whether or not the speed of light was a constant or an energy-dependent quantity, which would in turn guide us towards choosing the "correct" constant to vary in the definition of alpha.

We will begin our discussion by looking at the simplest "varying constant" theory, Brans-Dicke theory, which predicts the variability of the gravitational constant, $G$, within the constraints of general relativistic data. Then we will turn to the simplest model in the class of "varying alpha" theories, the Bekenstein model of an electromagnetic description of the variation of the fine structure constant, $\alpha$. Finally, we will look at Varying Speed of Light (VSL) theories.

### Brans-Dicke Theory

Today, the theoretical motivation for Brans-Dicke theory [36] stems, in part, from the appearance of scalar fields coupled to gravity in quantum gravity theories. The hope is that the theory would be a low energy limit of a particular quantum gravity formulation, thus linking Planck scale physics to a notion of testability, based on the observational evidence for variation of the "constant", $G$. To see if such a connection is valid one must look at the equations of motion governing the dynamics of the scalar field, $G$. Let us look first at the classical gravity action to understand the origin of the Brans-Dicke action and the equations of motion derived from it.

The classical action for gravity is the Einstein-Hilbert action,

$$S_{EH} = \frac{1}{16\pi G} \int d^4x \sqrt{-g} R.$$  \hfill (57)

Variation of this with respect to the metric gives rise to Einstein’s equations of motion,

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}. \hfill (58)$$

where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$. The integrability condition that arises in the derivation of these equations from the variational principle is $\nabla_{\mu}G_{\nu}^{\mu} = 0$ (the so-called Bianchi identities), and henceforth $\nabla_{\mu}T_{\nu}^{\mu} = 0$. The volume element is $\sqrt{-g}d^4x$, where $\sqrt{-g}$ is invariant and $d^4x$ is the tensor density. To get the gravitational equations of motion, we must vary the Einstein-Hilbert action with respect to the metric, $\delta S_{EH} = \int d^4x [\delta(\sqrt{-g}g^{\mu\nu}R_{\mu\nu} + \sqrt{-g}g^{\mu\nu}\delta R_{\mu\nu})]$. Substituting $R = g^{\mu\nu}R_{\mu\nu}$ we find a full divergent term, which can be thrown away according to classical General relativity theory.

To calculate the equations of motion rigorously, which, for instance, requires the calculation of $\delta R_{\mu\nu} = \delta \Gamma_{\mu\nu,\alpha} - \delta \Gamma_{\mu\alpha,\nu} - \delta \Gamma_{\nu\alpha,\mu}$, one must go through the entire process of calculating $g_{\mu\nu}$, then the connection, $\Gamma$, then the curvature tensor, $R_{\mu\nu}$, and so on. It is complex and actually not necessary here. Instead, we can go to the freely falling frame where $\Gamma_{\mu\beta}^{\alpha} = 0$. Calculations in this frame will be far simpler. For instance, it can be shown that $\delta g = g^{\alpha\beta}\delta g_{\alpha\beta}$, using the identity $det M = e^{Tr\ln M}$.

The general action is

$$S = S_{EH} + S_M, \hfill (59)$$

where $S_M$ is the matter action characterized by the energy momentum tensor $T_{\mu\nu} = -(2/\sqrt{-g})(\delta S_m/\delta g^{\mu\nu})$. Note that we have assumed in these equations that the cosmological constant, $\Lambda$, is zero. If one wishes to include it, the correct equations come from the replacement of $R$ with $R - 2\Lambda$. 
We can now extend this classical theory to the case of a variable gravitational constant. We promote $G$ to be a function of a scalar field, $\phi$, which produces another long-range force like gravity. Specifically,
\[
\phi = \frac{1}{G}.
\]  
(60)

This form comes from the simplest generally covariant scalar field equation, $\nabla^\rho \nabla_\rho \phi \sim T_{\mu\nu}^\phi$, where $T_{\mu\nu}^\phi$ is the energy-momentum tensor for matter, not including gravity and the scalar field. A rough estimate of the average cosmic value of $\phi$ produces a value, $10^{27}$ g cm$^{-3}$, close to $1/G = 10^{28}$ g cm$^{-3}$. The gravitational action describing Brans-Dicke theory will thus be
\[
S_{BD} = \int d^4x \sqrt{-g} \frac{R}{16\pi},
\]  
(61)

the action for $\phi$
\[
S_\phi = \int d^4x \sqrt{-g} \left( -\frac{\omega}{\phi} \partial_\mu \phi \partial^\mu \phi \right),
\]  
(62)

where $\omega$ is a dimensionless coupling parameter. The full action is the sum of these plus the matter action,
\[
S = S_{BD} + S_\phi + S_M.
\]  
(63)

As $\omega \rightarrow \infty$, Brans-Dicke cosmology approaches Einstein’s gravity. The matter action is not a function of $\phi$ in this theory. Varying the Brans-Dicke action leads to a boundary term that is no longer a full divergence as in the classical Einstein-Hilbert action. Therefore, new terms will appear in the field equations. Variation with respect to the metric gives
\[
G_{\mu\nu} = -8\pi \phi T_{\mu\nu}^\phi - \frac{\omega}{\phi} (\phi_{,\mu} \phi_{,\nu} - \frac{1}{2} g_{\mu\nu} \phi_{,\alpha} \phi_{,\alpha}) - \frac{1}{\phi} \left( \nabla_\nu \nabla_\mu \phi - g_{\mu\nu} \nabla^\alpha \nabla_\alpha \phi \right).
\]  
(64)

The usual conservation law (i.e. the integrability condition) holds, $\nabla_\mu T^\mu_{\nu} = 0$, which implies the conservation of the standard stress-energy tensor, and henceforth the weak equivalence principle and geodesic motion. Variation of the action with respect to the scalar field gives the $\phi$ equation of motion
\[
\nabla^\mu \nabla_\mu \phi = \frac{8\pi T}{(3 + 2\omega)},
\]  
(65)

where $T = T_{\mu}^\mu$.

Brans-Dicke Theory can be phrased in two different frames that are conformally related. The formulation presented above is in the so-called Jordan frame. The conformal, not coordinate, transformation that takes the theory into the Einstein frame involves the following metric replacement,
\[
g_{\mu\nu} \rightarrow g_{\mu\nu} (16\pi \phi) = \tilde{g}_{\mu\nu}.
\]  
(66)

We then reexpress the scalar field as
\[
\sigma = \frac{\ln (16\pi \phi)}{\omega + \frac{3}{2}}.
\]  
(67)

These transformations dress the Brans-Dicke action such that it looks like the original Einstein-Hilbert action, solidifying the idea that Brans-Dicke theory is just Einstein’s theory with a scalar field. An important fact regarding this conformal transformation is that $\phi$ does not appear in the matter action in the Jordan frame, telling us that the scalar field does not interact with matter in this frame. It does, however, in the Einstein frame since the transformation $S_\sigma \rightarrow S_M(\tilde{g}_{\mu\nu} e^{\sigma})$ introduces non-minimal metric coupling to $\sigma$. It is important to remember when making calculations that the Einstein frame is not the physical frame, it is merely a conformal frame in which the equations and calculations are simplified.

Let us enter the "stringy" realm of quantum gravity for a moment. The basic string theory action has an additional multiplicative factor,
\[
S_{\text{string}} = \int d^4x \sqrt{-g} e^{-\chi} [R/16\pi + \mathcal{L}_M + ...],
\]  
(68)

where $\chi$ is a scalar field, the dilaton field mentioned above. The form of this action displays a clear scalar field resemblance to the Brans-Dicke action, indeed it’s its formulation in yet another conformal frame, the string frame.
The hope is to connect varying constant theories with quantum gravity in this way. That is, to successfully formulate a varying constant theory that can be proven to be a limit of a quantum gravity theory. In this particular case, we would hope that the dilaton field was the quantum gravitational manifestation of the field causing the variability of the gravitational constant.

Finally, let us briefly address \( \omega \), the new constant introduced via Brans-Dicke theory. It is undesirable, in general, to introduce new parameters into a theory because they are extra constants that need to be set or fine-tuned. Physicists look to constrain theories rather than introduce extra degrees of freedom. From the Brans-Dicke equations of motion, we know that as \( \omega \rightarrow \infty \) the theory becomes indistinguishable from General Relativity. In a sense, experiments that constrain the value of \( \omega \) can be seen to be a measure of just how accurate General Relativity describes our universe given particular experimental probes to hand. In general, measures of \( \omega \) are quite large, with conservative estimates being in the 100s \(^{[37]}\), and recent solar system tests using the Cassini probe to be greater than 40,000 \(^{[38]}\). Such a large range of values is not necessarily mutually exclusive, since the tests used probe vastly different length and time scales.

There are also theories called scalar-tensor theories that allow \( \omega \) to vary along with the changing scalar field, such that \( \omega = \omega(\phi) \) \(^{[39]}\). In these, the general relativistic limit would correspond to values of \( \phi \) such that \( \dot{\phi} \) is negligible and \( \omega \) is very large.

### The Bekenstein Model

The fine structure constant, defined as \( \alpha = e^2/4\pi\hbar c \), is the most observationally sensitive of Nature’s “constants”. Webb, et al. \(^{[40, 5]}\), use a new observational many-multiplet (MM) technique to measure \( \alpha \), by exploiting the extra sensitivity gained in studying relativistic transitions to different ground states. The tests use absorption lines from gas clouds that intersect our line of sight to quasar (QSO) spectra at medium redshift. This method has provided the first evidence that the fine structure constant may change throughout cosmological time \(^{[40, 5, 41]}\). Independent samples yielded imply that the value of \( \alpha \) was lower in the past, such that \( \Delta\alpha/\alpha = -0.72 \pm 0.18 \times 10^{-5} \) for \( z \approx 0.5 - 3.5 \). VLT tests of Iron absorption lines at a redshift of \( z = 1.15 \) back-lit by a QSO and using the standard MM technique result in \( 10^{-6} \) \(^{[42]}\). Another test for the variability of alpha comes from the Oklo natural fission reactor, a uranium mine in Africa, which determines historical values of alpha by measuring isotopes 2000 million years in age. This has set tight constraints on the temporal variation of alpha of the order \( 10^{-7} \) \(^{[43]}\).

Big Bang Nucleosynthesis (BBN) is one of the great experimental successes in cosmology. It describes a phase of the early universe when particles could coalesce to form atoms due to a sufficient decrease in the temperature of the universe. The theory successfully predicts the abundances of light elements in our cosmos today. These abundances depend critically on the number of baryons (neutrons and protons) at the time of nucleosynthesis, which in turn is governed by the particle interaction mediator, the fine structure constant. The idea here is to accommodate the data in support of variable fine structure constant while staying in agreement BBN successes such that we are able to fix the value of alpha to what it is measured to be today. Some investigations into the variability of this "constant" have come to conclude \( \Delta\alpha < 0 \) values in order to best fit Cosmic Microwave Background (CMB) data at \( z \approx 10^3 \) and BBN data at \( z \approx 10^{10} \), respectively \(^{[44, 45]}\).

To address the question of what has made the value of alpha increase throughout cosmological time, one must first ask, to which of the "constants" defining alpha should we attribute the variation? Through which parameter the variation is realized is a matter of convenience, but it is this choice that is crucial to formulating a simple and cohesive theory. The most conservative of the varying alpha theories is one in which the electromagnetic charge, \( e \), varies in space-time, while \( e \) and \( \hbar \) are held fixed \(^{[44]}\). A varying \( e \) theory can be set up in much the same way as in the Brans-Dicke case, by prescribing that \( e \) become a dynamic field, the so-called minimal coupling prescription \(^{[46]}\). This electromagnetic varying \( e \) theory has been thoroughly explored, including a formal rearrangement of these theories done to convert them into dilaton-type theories, in which the dilaton couples to the electromagnetic \( "F^2" \) term in the Lagrangian, but not to the other gauge fields \(^{[44]}\).

In the varying \( \alpha \) theories proposed initially \(^{[44]}\) one takes \( e \) and \( \hbar \) to be constants and attributes variations in \( \alpha \) to changes in \( e \), the permittivity of free space. This is done by letting \( e \) take on the value of a real scalar field which varies in space and time, \( e_0 \rightarrow e(x^\mu) = e_0 e(x^\mu) \), where \( e(x^\mu) \) is a dimensionless scalar field and \( e_0 \) is a constant denoting the present value of \( e(x^\mu) \). One then proceeds to set up a theory based on the principles of local invariance, causality of electromagnetism, and the scale invariance of the \( e \)-field.
Since covariant derivatives in electromagnetic theory take the form \(D_\mu \phi = (\partial_\mu + ieA_\mu)\phi\), for gauge transformations of the form \(\delta \phi = -i\chi \phi\) one must impose \(\delta A_\mu \to \epsilon A_\mu + \chi_\mu\).

\[
\epsilon A_\mu \to \epsilon A_\mu + \chi_\mu.
\]  
(69)

The gauge-invariant electromagnetic field tensor is then

\[
F_{\mu\nu} = \frac{(\epsilon A_\nu)_\mu - (\epsilon A_\mu)_\nu}{\epsilon},
\]  
(70)

which reduces to the usual form for constant \(\epsilon\). The electromagnetic Lagrangian is still

\[
L_{\text{em}} = -\frac{F^{\mu\nu}F_{\mu\nu}}{4},
\]  
(71)

and the dynamics of the \(\epsilon\) field are controlled by the kinetic term

\[
L_\epsilon = -\frac{1}{\omega} \frac{\epsilon_\mu \epsilon^\mu}{\epsilon^2},
\]  
(72)

where the self-coupling constant \(\omega\) is introduced into the lagrangian density for dimensional reasons and is proportional to the inverse square of the characteristic length scale of the theory, \(\omega \sim \ell^{-2}\), such that \(\ell \geq L_p \approx 10^{-33}\text{cm}\) holds \(\ell < 10^{-33}\text{cm}\). This length scale corresponds to an energy scale \(\frac{1}{\ell^2}\), with an upper bound set by experiment. Note that the metric signature used is still \((- , +, +, +, +)\).

A simpler formulation of this theory \(\epsilon A_\mu \equiv \epsilon A_\mu\), and field tensor \(f_{\mu\nu} = \epsilon F_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu\). We are free to do this without physical implications because, at the classical level, it is not \(A_\mu\) that we measure, but rather the electric and magnetic fields, \(E\) and \(B\). The covariant derivative then assumes the familiar form, \(D_\mu = \partial_\mu + ie\partial_\mu\), and the dependence of the Lagrangian on \(\epsilon\) then occurs only in the kinetic term for \(\epsilon\) and in the \(F^2 = f^2/\epsilon^2\) term, not in a kinetic term, \(D_\mu \phi D^\mu \phi\), of the matter lagrangian, \(L_{\text{M}}\). To simplify further, we can redefine the variable, \(\epsilon \to \psi = \frac{1}{\epsilon}\). The total action then becomes

\[
S = \int d^4x \sqrt{-g} \left( L_{\text{M}} + L_\psi + L_{\text{em}} e^{-2\psi} \right),
\]  
(73)

with

\[
L_\psi = -\frac{\omega}{2} \nabla_\mu \psi \nabla^\mu \psi,
\]  
(74)

\[
L_{\text{em}} = -\frac{1}{4} f_{\mu\nu} f^{\mu\nu},
\]  
(75)

where the matter Lagrangian \(L_{\text{M}}\) does not contain \(\psi\). This is a dilaton theory coupling to the electromagnetic \(\epsilon A_\mu\) part of the Lagrangian only \(\epsilon A_\mu\). Note that the scale invariance of the \(\epsilon\) field is such that the action is invariant under the transformation \(\epsilon \to k\epsilon\) for any constant \(k\). The action, written in this simplified form, clearly resembles the dilaton-string action shown above.

Minimizing the variation of the action with respect to the scalar field, \(\delta S/\delta \phi = 0\), gives us the equations of motion governing the temporal and spatial development of \(\phi\).

\[
\nabla^\mu \nabla_\mu \phi = \frac{2}{\omega} \psi e^{-2\psi} L_{\text{EM}}.
\]  
(76)

Ignoring, for the moment, any spatial variation in \(\phi\), we have the Friedman-Robertson-Walker equations governing the time-evolution of \(\phi\)

\[
\ddot{\phi} + \frac{3}{a} \dot{\phi} = -\frac{2}{\omega} \psi e^{-2\psi} L_{\text{EM}},
\]  
(77)

where the right hand side of the equation is the driving force and \((3\dot{a}/a)\dot{\phi}\) on the left hand side is the fluid friction term. It is the electromagnetic lagrangian, \(E^2 - B^2\), therefore, which drives the variation in \(\epsilon\). In a radiation-dominated universe \(E^2 - B^2 = 0\) such that there is no variation in \(\epsilon\) and thus \(\alpha\). Numerical results \(\epsilon A_\mu\) support a variable alpha with the following behavior. The radiation era sees no variation in \(\alpha\), but the driving term is not zero during the matter
era, $E^2 - B^2 \neq 0$, allowing for small variations during this period. In the ensuing epoch of accelerated expansion the fluid friction term dominates, leading to the stability of the field $\psi$, and hence constant $\epsilon$ and $\alpha$.

Unfortunately, the variation in alpha predicted by these equations with a positive energy density for the scalar field (i.e. $\omega > 0$) is precisely opposite to that needed to support observational data. That is, the equations predict a value of $\alpha$ that decreases in time, whereas observations support an increasing alpha [47]. The theory can fit the data, however, for two very different physical scenarios. Firstly, if there exists a type of dark matter such that the $B^2$ contribution to the electromagnetic lagrangian dominates the $E^2$ contribution, then $\alpha$ will increase in time within observational bounds. Some cosmological defect theories, specifically those of superconducting strings, satisfy this requirement, but it is more or less unappealing since most matter types do not have this property.

It is, on the other hand, possible to explain the observational data supporting an $\alpha$ value that was lower in the past if $\omega < 0$ [42]. A negative scalar coupling implies a negative energy density for $\psi$, which in turn tells us that $\psi$ is a "ghost" field. "Ghost fields" coupled to matter have often been deemed undesirable since they will consistently dump positive energy into matter. However, it is the specific forms of the equations of motion that will dictate just how this interchange is mediated such that it may, in fact, not be problematic. We will ignore the "runaway" pathological behavior of ghosts at the quantum level on the grounds that a scalar field $\phi$ is non-renormalizable and should not be quantized, just as in classical general relativity.

It has been shown [29] that the simplest varying constant theories, such as the Brans-Dicke and Bekenstein models described above, predict a non-singular and cyclic universe given a scalar coupling $\omega < 0$. In particular, the Bekenstein model equations with negative $\omega$ are the same as the Brans-Dicke equations in the Einstein frame. In both cases, a regularly oscillating universe results if the scalar field is uncoupled to matter. On the other hand, a cyclic universe with "bounces" of ever-increasing amplitudes is produced if the field is coupled to matter near the bounce, such that positive energy is transferred into the matter field from the negative energy field. The universe, with this additional positive energy density will be hotter and will thus have to grow to a larger size in order to reach the critical temperature for a turnaround. Note that these models occur strictly within the radiation-dominated epoch, meaning that the scalar fields couple to radiation. Assuming that we are presently in the first matter-dominated era, we can calculate the number of bounces needed to get to the value of the constants that are measured today [49]. In this manner, it is possible to justify the extreme values of the parameters simply given the large age of our universe.

We have so far reviewed a large number of motivations for varying "constant" scenarios. Among them are included quantum gravity theories with dilaton scalar fields resembling the scalar fields that describe varying constants, theories such as DSR which predict deformed dispersion relations, and observations pointing to the variability of the fine structure constant. In all cases, scalar fields play the pivotal role of describing how these fundamental parameters of nature vary throughout space-time. We know that scalar fields appear in other roles in physics, such as the cosmological constant that drives inflation, so it is more than relevant to ask what the crucial role of scalar fields in the universe is in general. Is their primary role to vary the fundamental constants of nature? Are they there simply to source inflation? Or, are they merely an unobservable byproduct of higher dimensional theories such as string theories? Perhaps these theories are intimately connected, but to even begin to address these questions, one must broaden the scope of study. For instance, an extension of the Bekenstein model to the electroweak scenario been done [50] in order to see if there exists any evidence of a scalar field affecting the standard model of particle physics. With respect to cyclic universes, there are other non-singular models [51] sourced other than by negative energy scalar fields, which could help shed light on the necessity and/or importance of scalar fields in bouncing models.

A Variable Speed of Light

Even after the proposal of special relativity in 1905 many varying speed of light theories were considered, most notably by Einstein himself [53]. VSL was then rediscovered and forgotten on several occasions. For instance, in the 1930s, VSL was used as an alternate explanation for the cosmological redshift [54][55, 56] (these theories conflict with fine structure observations). None of these efforts relate to recent VSL theories, which are firmly entrenched in the successes (and remaining failures) of the hot big bang theory of the universe. In this sense the first "modern" VSL theory was J. W. Moffat’s ground breaking paper [57], where spontaneous symmetry breaking of Lorentz symmetry leads to VSL and an elegant solution to the horizon problem.

Since then there has been a growing literature on the subject, with several groups working on different aspects of VSL (for a comprehensive review see [58]). Here we examine the simplest and most conservative implementations currently being considered. The first resembles Brans-Dicke theory in that two metrics are considered, one for gravity,
another for matter (this is comparable to the metrics used in the Einstein and Jordan frame in Brans-Dicke theory). The second implementation is a generalization of Bekenstein’s theory in which all coupling strengths become dynamical.

**Bimetric VSL theories**

This approach was initially proposed by J. W. Moffat and M. A. Clayton \[58\], and by I. Drummond \[59\]. It does not sacrifice the principle of relativity and special care is taken with the damage caused to the second principle of special relativity. In these theories the speeds of the various massless species may be different, but special relativity is still (linearly) realized within each sector. Typically the speed of the graviton is taken to be different from that of massless matter particles. This is implemented by introducing two metrics (or tetrads in the formalism of \[59, 60\]), one for gravity and one for matter. The model was further studied by \[61\] (scalar-tensor model), \[62\] (vector model), and \[63, 64, 65\].

We now sketch the scalar-tensor model. It uses a scalar field \(\phi\) that is minimally coupled to a gravitational field described by the metric \(g_{\mu\nu}\). However the matter couples to a different metric, given by

\[
\hat{g}_{\mu\nu} = g_{\mu\nu} + B\partial_\mu \phi \partial_\nu \phi. \tag{78}
\]

Thus there is a space-time, or graviton metric \(g_{\mu\nu}\), and a matter metric \(\hat{g}_{\mu\nu}\). The total action is

\[
S = S_g + S_\phi + \hat{S}_M, \tag{79}
\]

where the gravitational action is as usual

\[
S_g = \frac{1}{16\pi} \int d^4x \sqrt{-g}(R(g) - 2\Lambda), \tag{80}
\]

(notice that the cosmological constant \(\Lambda\) could also, non-equivalently, appear as part of the matter action). The scalar field action is

\[
S_\phi = \frac{1}{16\pi} \int d^4x \sqrt{-g} \left[ \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right], \tag{81}
\]

leading to the the stress-energy tensor,

\[
T^\mu_\nu = \frac{1}{16\pi} \left[ g^{\mu\alpha} g^{\nu\beta} \partial_\alpha \phi \partial_\beta \phi - \frac{1}{2} g^{\mu\nu} g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi + g^{\mu\nu} V(\phi) \right]. \tag{82}
\]

The matter action is then written as usual, but using the metric \(\hat{g}_{\mu\nu}\). Variation with respect to \(g_{\mu\nu}\) leads to the gravitational field equations,

\[
G^\mu_\nu = \Lambda g^\mu_\nu + 8\pi T^\mu_\nu + \frac{8\pi}{\sqrt{-g}} \hat{T}^\mu_\nu. \tag{83}
\]

In this theory the speed of light is not preset, but becomes a dynamical variable predicted by a special wave equation

\[
\hat{g}^{\mu\nu} \hat{\nabla}_\mu \hat{\nabla}_\nu \phi + KV'(|\phi|) = 0 \tag{84}
\]

where the biscalalar metric \(\hat{g}\) is defined in \[58\].

This model not only predicts a varying speed of light (if the speed of the graviton is assumed to be constant), but also allows solutions with a de Sitter phase that provides sufficient inflation to solve the horizon and flatness problems. This is achieved without the addition of a potential for the scalar field. The model has also been used as an alternative explanation for the dark matter \[60\] and dark energy \[64, 65\].

**“Lorentz invariant” VSL theories**

It is also possible to preserve the essence of Lorentz invariance in its totality and still have a space-time (as opposed to energy dependent) varying \(c\). One possibility is that Lorentz invariance is spontaneously broken, as proposed by J. W. Moffat \[57, 66\] (see also \[67\]). Here the full theory is endowed with exact local Lorentz symmetry; however
the vacuum fails to exhibit this symmetry. For example an $O(3,1)$ scalar field $\phi^a$ (with $a = 0,1,2,3$) could acquire a time-like vacuum expectation value (VEV), providing a preferred frame and spontaneously breaking local Lorentz invariance to $O(3)$ (rotational invariance). Such a VEV would act as the preferred vector $u^a = \phi_0^a$; however the full theory would still be locally Lorentz invariant. Typically in this scenario the speed of light undergoes a first or second order phase transition to a value more than 30 orders of magnitude smaller, corresponding to the presently measured speed of light. Interestingly, before the phase transition the entropy of the universe is reduced by many orders of magnitude, but afterwards the radiation density and entropy of the universe vastly increase. Thus the entropy increase follows the arrow of time determined by the spontaneously broken direction of the timelike VEV $\phi_0^a$. This solves the enigma of the arrow of time and the second law of thermodynamics.

Another example is the covariant and locally Lorentz invariant theory proposed in [63]. In that work definitions were proposed for covariance and local Lorentz invariance that remain applicable when the speed of light $c$ is allowed to vary. They have the merit of retaining only those aspects of the usual definitions which are invariant under unit scalar field transformations, and which can therefore legitimately represent the outcome of an experiment. In the simplest case a scalar field is then defined $\psi = \log(c/c_0)$, and minimal coupling to matter requires that

$$\alpha_t \propto g_t \propto \hbar c \propto c^q$$

with $q$ a parameter of the theory. The action may be taken to be

$$S = \int d^4x \sqrt{-g}(\psi^a R - 2\Lambda + \mathcal{L}_\psi) + 16\pi e^{b\psi} \mathcal{L}_m$$

and the simplest dynamics for $\psi$ derives from:

$$\mathcal{L}_\psi = -\kappa(\psi)\nabla_\mu \psi \nabla^\mu \psi$$

where $\kappa(\psi)$ is a dimensionless coupling function. For $a = 4, b = 0$, this theory is nothing but a unit transformation applied to Brans-Dicke theory. More generally, it’s only when $b + q = 0$ that these theories are scalar-tensor theories in disguise. In all other cases it has been shown that a unit transformation may always be found such that $c$ is a constant but then the dynamics of the theory becomes much more complicated. Thus we should label these theories varying speed of light theories.

In these theories the cosmological constant $\Lambda$ may depend on $c$, and so act as a potential driving $\psi$. Since the vacuum energy usually scales like $c^4$ we may take $\Lambda \propto (c/c_0)^n = e^{n\psi}$ with $n$ an integer. In this case, if we set $a = b = 0$ the dynamical equation for $\psi$ is:

$$\nabla_\mu \nabla^\mu \psi = \frac{32\pi}{c^4}\mathcal{L}_m + \frac{1}{\kappa}n\Lambda$$

Thus it is possible that the presence of Lambda drives changes in the speed of light, a matter examined (in another context) in [68].

Particle production and second quantization for this model has been discussed in [63]. Black hole solutions were also extensively studied [69]. Predictions for the classical tests of relativity (gravitational light deflection, gravitational redshift, radar echo delay, and the precession of the perihelion of Mercury) were also shown to differ distinctly from their Einstein counterparts, while still evading experimental constraints [67]. Other interesting results were the discovery of Fock-Lorentz space-time [18, 19] as the “free” solution, and fast-tracks (tubes where the speed of light is much higher) as solutions driven by cosmic strings [63].

Beautiful as these two theories may be, their application to cosmology is somewhat cumbersome.

The simplistic cosmological motivation

Like inflation [70], modern VSL theories were motivated by the “cosmological problems” – the flatness, entropy, homogeneity, isotropy and cosmological constant problems of Big Bang cosmology (see [10, 71]). The definition of a cosmological arrow of time was also a strong consideration.

But at its most simplistic, VSL was inspired by the horizon problem. As we go back into our past the present comoving horizon breaks down into more and more comoving causally connected regions. These disconnected early days of the Universe prevent a physical explanation for the large scale features we observe – the “horizon problem”. It does not take much to see that a larger speed of light in the early universe could open up the horizons [57, 24] (see
FIGURE 1. A conformal diagram (in which light travels at $45^\circ$). This diagram reveals that the sky is a cone in 4-dimensional space-time. When we look far away we look into the past; there is an horizon because we can only look as far away as the Universe is old. The fact that the horizon is very small in the very early Universe, means that we can now see regions in our sky outside each others' horizon. This is the horizon problem of standard Big Bang cosmology.

FIGURE 2. Diagram showing the horizon structure in a model in which at time $t_c$ the speed of light changed from $c^-$ to $c^+ << c^-$. Light travels at $45^\circ$ after $t_c$ but it travels at a much smaller angle to the spatial axis before $t_c$. Hence it is possible for the horizon at $t_c$ to be much larger than the portion of the Universe at $t_c$ intersecting our past light cone. All regions in our past have then always been in causal contact. This is the VSL solution of the horizon problem.

More mathematically, the comoving horizon is given by $r_h = c/\dot{a}$, so that a solution to the horizon problem requires that in our past $r_h$ must have decreased in order to causally connect the large region we can see nowadays. Thus

$$\frac{\ddot{a}}{\dot{a}} - \frac{\dot{c}}{c} > 0$$

that is, either we have accelerated expansion (inflation), or a decreasing speed of light, or a combination of both. This argument is far from general: a contraction period ($\dot{a} < 0$, as in the bouncing universe), or a static start for the universe ($\dot{a} = 0$) are examples of exceptions to this rule.

However the horizon problem is just a warm up for the other problems. More recently structure formation has been the leading driving force in these scenarios. This is still very much unaccomplished, in spite of recent efforts and consequently is left as an exercise for the student.
The experimental front of VSL

As explained in \[35\] varying $e$ and $c$ theories in general predict the same $\alpha(z)$ profiles. To distinguish between varying $e$ and varying $c$ theories one must look elsewhere. The status of the equivalence principle in these theories turns out to be a good solution. The varying $e$ theories \[43, 56, 72\] violate the weak equivalence principle, whereas VSL theories do not \[73, 74\]. The Eötvös parameter

$$\eta \equiv \frac{2|a_1-a_2|}{a_1+a_2}$$

(90)

is of the order $10^{-13}$ in varying $e$ theories, just an order of magnitude below existing experimental bounds.

Still, it would be good to find a widely independent confirmation of the Webb results. A very promising area is cold atom clocks \[75\]. The search for the perfect unit of time has led to the quest for very stable oscillatory systems, leading to a gain, every ten years, of about one order of magnitude in timing accuracy. Cold atom clocks may be used as laboratory “table-top” probes for varying $\alpha$, with a current sensitivity of about $10^{-15}$ per year. For all varying alpha theories it is found that at present:

$$\frac{\dot{\alpha}}{\alpha} \approx 2.98 \times 10^{-16} h \text{ year}^{-1}$$

(91)

with $H_0 = 100h \text{ Km sec}^{-1} \text{ Mpc}^{-1}$, $\Omega_\Lambda = 0.71$ and $\Omega_m = 0.29$. For $h = 0.7$ this gives a fractional variation in alpha of about $2 \times 10^{-16}$ per year, which should soon be within the reach of technology. Such an observation would be an incredible vindication of the Webb results. On the other hand this effect would become a further annoyance for those concerned with the practicalities of defining the unit of time.

Spatial variations of $\alpha$ are likely to be significant \[74\] in any varying alpha theory. For any causal theory relative variations in $\alpha$ near a star are proportional to the local gravitational potential. The exact relation between the change in $\alpha$ with redshift and in space (near massive objects) is model dependent. For instance, we have

$$\frac{\delta \alpha}{\alpha} = -\frac{\zeta_s M_s}{\omega \pi r} \approx 2 \times 10^{-4} \frac{\zeta_m M_s}{\pi r}$$

(92)

for a typical varying $e$ theory, but

$$\frac{\delta \alpha}{\alpha} = -\frac{b q M_s}{\omega 4 \pi r} \approx 2 \times 10^{-4} \frac{M_s}{\pi r}$$

(93)

for a VSL theory. Here $M_s$ is the mass of the compact object, $r$ is its radius, and $\zeta$ is the ratio between $E^2 - B^2$ and $E^2 + B^2$. When $\zeta_m$ (for the dark matter) and $\zeta_s$ (for, say, a star) have different signs, for a cosmologically increasing $\alpha$, varying $e$ theories predict that $\alpha$ should decrease on approach to a massive object. And indeed one must have $\zeta_m < 0$ in order to fit the Webb results. In VSL, on the contrary, $\alpha$ increases near compact objects (with decreasing $c$ if $q < 0$, and increasing $c$ if $q > 0$). In VSL theories, near a black hole $\alpha$ could become much larger than 1, so that electromagnetism would become non-perturbative with dramatic consequences for particle physics near black holes. In varying-$e$ theories precisely the opposite happens: electromagnetism switches off.

These effects are in principle observable using similar spectroscopic techniques to those of Webb, but applied to lines formed on the surface of very massive objects near us (in the sense of $z \ll 1$). For that, we need an object with a radius sufficiently close to its Schwarzschild radius, such as an AGN, a pulsar or a white dwarf, for the effect to be non-negligible. Furthermore we need the “chemistry” of such an object to be sufficiently simple, so that line blending does not become problematic.

A LAST EXERCISE FOR THE STUDENT: MOND

It is sadly the case that we can’t finish this review with resounding conclusions. Rather we will peter out in the realm of uncertainties – problems which remain unsolved and that may have something to say about phenomenological quantum gravity. For this purpose we have selected the problem of dark matter in the Universe.

Galactic rotation curves have long puzzled cosmologists. Newtonian theory predicts that they should fall out like $v_r \propto 1/r^{1/2}$. This follows from the simple calculation:

$$F = ma \rightarrow \frac{Mm}{r^2} = m \frac{v^2}{r} \rightarrow v^2 = \frac{M}{r}$$

(94)
Instead we observe them flattening out: $v \rightarrow v_\infty$. A simple solution is that besides the visible matter there is a halo of dark matter which dominates gravity on the outskirts of the galaxy. This halo has the property $M_{DM} = Ar$, i.e. it must have a density profile $\rho_{DM} = B/r^2$. Thus

$$\frac{1}{2} M_{DM} m r^2 = m \frac{v^2}{r} \rightarrow v^2 = A$$

Historically this the first hint of dark matter in the Universe. It is important to stress that there are now many other reasons to invoke dark matter².

This is all very well; however three difficulties are quickly encountered:

1. The halos don’t appear to be stable when left to evolve according to their own gravity. Rather they collapse into a central cusp. This is the drama of every N-body simulation performed so far. Lack of resolution and physical content is usually blamed.

2. The onset of the terminal velocity seems to be triggered not by a length or mass scale but by an acceleration. This has been measured to be $a_0 \approx 10^{-10} m s^{-2}$.

3. An empirical law has been established called the Tully-Fisher relation establishing that $v_\infty^4$ is proportional to the luminosity (which presumably is proportional to the visible mass). This is the equivalent of Kepler’s third law.

It is hard to see how dark matter, even if creating a stable halo, could explain the Tully-Fisher relation. There would have to be a finely tuned correlation between constant $B$ (appearing in the density profile for the halo) and the mass in visible matter. Likewise the emergence of $a_0$ in the dark matter scenario is hard to understand. Still, it is possible that future N-body simulations may solve these problems.

Disconcertingly there is a very simple alternative solution, called MOND (MOdified Newtonian Dynamics). Perhaps galactic rotation curves are simply telling us that gravity has departed from Newton’s equations (and that there is no dark matter). Changing Newton’s gravitational law, however, won’t do because this would trigger novel behaviour at a given length scale rather than at an acceleration scale. Instead MOND posits that the response law, $F = ma$, must be modified. The usual law is only valid at high accelerations; for accelerations smaller than $a_0$ we have instead that

$$F = m \frac{a^2}{a_0}.$$  (96)

Straightforward application of the MOND prescription leads to

$$\frac{Mm}{r^2} = m \frac{v^4}{a_0 r^2} \rightarrow v^4 = Ma_0$$

Thus the Tully-Fisher relation is trivially explained as well as the fact that novel behaviour is triggered by an acceleration.

MOND is an excellent phenomenological description of galactic rotation curves. However it makes no sense whatsoever. Applied crudely it violates energy and momentum conservation in ways that would readily conflict with observations. It also has no relativistic generalization; we need such a relativistic theory in order, for consistency, to do away with dark matter in all regimes. We quote the examples of gravitational lenses, the cosmological expansion, and structure formation. Again, we’re left in the realm of wishful thinking: for dark matter with regards to computer power, here with respect to brain power and essential theoretical developments.

Why might this discussion be relevant to quantum gravity? Most obviously because MOND leads to a scary possibility: in trying to quantize gravity we may have chosen the wrong classical theory. No wonder we’re stuck. Our failures could simply signal that we don’t yet have the correct classical theory of gravity. This is a speculation, but one with dramatic, far-reaching consequences. As the title of this section shows, whatever one makes of it, this is very much exercise for the student³.

Specifically, here’s the problem:

1. Find a relativistic version of MOND consistent with energy and momentum conservation and capable of explaining gravitational lenses, and all the successes of the Big Bang without dark matter.

2. Quantize this theory.

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² It’s not obvious that the required matter is always the same.

³ Warning: this is an exercise for a very keen graduate student hoping to start a high risk, high return project.
The following points may be interesting hints (but then again, they may also be red herrings):

- The Pioneer puzzle, related to the anomalous acceleration suffered by these satellites in their courses outside the Solar system, is associated with acceleration \( a_P = 8.74 \pm 1.33 \times 10^{-10} \text{m/s}^2 \).
- The observed cosmic acceleration is of the same order as \( a_0 \).
- In contradiction with Mach’s principle there appears to be absolute frames in the Universe for acceleration (but not speed). Why?

We’ll leave a single reference on this topic: [76].

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