Kinetic Scheme for Solving the M1 Model of Radiative Transfer

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Abstract

We show a numerical scheme to solve the moment equations of radiative transfer, i.e., the M1 model, which follows the evolution of the energy density, $E$, and the energy flux, $F$. In our scheme we reconstruct the intensity from $E$ and $F$, so that it is consistent with the closure relation, $\chi = (3 + 4f^2)/(5 + 2\sqrt{4-3f^2})$. Here, the symbols $\chi$, $f = |F|/(cE)$, and $c$, denote the Eddington factor, the reduced flux, and the speed of light, respectively. We evaluate the numerical flux across the cell surface from the kinetically reconstructed intensity. It is an explicit function of $E$ and $F$ in the neighboring cells across the surface considered. We include absorption and reemission within a numerical cell in evaluating the numerical flux. The numerical flux approaches the diffusion approximation when the numerical cell, itself, is optically thick. Our numerical flux gives a stable solution even when some regions computed are very optically thick. We show the advantages of the numerical flux with examples. They include a flash of beamed photons and irradiated protoplanetary disks.

Key words: radiative transfer — scattering — stars: pre-main sequence

1. Introduction

Radiation plays important roles in many astronomical objects and media. We thus need to include the effects of radiation somehow in a realistic numerical simulation. However it is still an extremely heavy load to solve the full radiative transfer, i.e., taking account of the energy spectrum and angular distribution as well as the spatial distribution and temporal evolution. This is simply because the radiative intensity is a function in 6-dimensional phase space. We need to reduce the load by using some approximations.

There exist several types of ideas for reducing the computation cost. Each of them has both advantages and weak points. First, we can use a symmetry to reduce the computation cost. If spherical symmetry is a good approximation, the intensity is a function in three-dimensional phase space, and the radiative transfer is relatively easy to solve. However, this idea can be applied only to highly symmetric systems.

Second, we can assume that only a few sources are dominant in the radiation fields. If most of the light comes from a restricted relatively small number of sources, solving radiative transfer is relatively easy. We need to solve only the light rays from the sources. The number of light rays to be solved is greatly reduced. It is a good approximation when some stars or black holes are dominant light sources (see, e.g., Susa 2006; Okamoto et al. 2012, and the references therein). However, reflection and reemission from diffuse media cannot be taken into account under this approximation.

Third, we can use the moments of radiative intensity, such as the energy density, by using some approximations of the angular distribution, such as the flux limited diffusion (FLD, see e.g., Levermore 1984). FLD uses only the radiative energy density, which is equivalent to the average radiative intensity over the solid angle of photon traveling, to express the radiation field. The radiative flux is derived from the gradient of the energy density. This approximation is good when the medium is optically thick and radiation field is nearly isotropic. It is not so bad even when the medium is transparent. However, it cannot express a shadow, a dark region behind an opaque object of finite extent. Spurious radiation erases out the shadow in FLD (González et al. 2007).

Scattering is another weak point of FLD. Scattering changes the angular distribution of the intensity, but not the energy density. Thus, FLD cannot evaluate the effects of scattering, since it takes account of only the energy density.

González, Audit, and Huynh (2007) proposed the M1 model in which the radiation field is expressed by the energy density and the flux, i.e., both the 0th and 1st moments of the radiative intensity. It has been demonstrated that the M1 scheme can simulate a shadow by an opaque sphere successfully. The M1 scheme can take account of scattering. Scattering reduces the energy flux while keeping the radiation density constant. When the scattering is isotropic, the effect is correctly taken into account in the M1 model. However, the M1 model also has several weak points and limitations. The M1 model cannot solve the crossing of two beamed lights. They erroneously merge into one beam at the crossing point. This limitation comes from...
the fact that the M1 model evaluates higher moments of radiation intensity from the 0th and 1st moments. In other words, the M1 model has the lowest angular resolution, and the crossing of two beams is beyond its scope. This limitation is compensated by the low computation cost.

M1 model equations are similar to the hydrodynamical equations in the conservation form. Both of the equations are hyperbolic and have source terms. Thus, we can apply numerical methods to integrate the hydrodynamical equations to solve the M1 model equations. However, we have two concerns when solving them. First the characteristic speeds are complex, and are not easy to evaluate, though modern schemes for numerical hydrodynamics rely on them (see, e.g., Toro 2009). We can avoid computing the characteristics by using the HLL-E flux, but the resultant flux is diffusive, and makes a shadow dim, as pointed out by González, Audit, and Huynh (2007).

Second, absorption and emission (the source terms) are dominant when optically thick. On the other hand, the source terms due to gravity are minor contributions in numerical hydrodynamics. Thus, they are simply added after solving the wave propagation. This approach does not work well in M1 model equations when a cell, itself, is optically thick. This difficulty is known as the diffusion-limit behavior, and several solutions are proposed in the literature (see, e.g., Audit et al. 2002; Berthon et al. 2007).

In this paper we propose an idea to construct a numerical flux for the M1 model that is less diffusive, and yet stable in the diffusion limit. First we show a method to evaluate a numerical flux of the M1 model from radiation intensity kinetically reconstructed from the radiation energy density and flux. The reconstructed radiation intensity is consistent with the closure relation, i.e., the formula used to close the moment equations of the radiative transfer. We evaluate the radiative flux and pressure across the boundary between two adjacent computation cells by integrating the reconstructed intensity over the solid angle. We use the intensity of the upwind side, and the numerical flux is subject to causality. Fortunately, the numerical flux is an explicit function of the radiation energy density and the flux. A similar scheme is constructed for gas dynamics, and called the “kinetic scheme” (see, e.g., Pullin 1980; Deschpande 1986 and the references cited in Hauck 2011). We thus use the same terminology in this paper.

Second, we include absorption within a computation cell. The numerical flux is evaluated on the cell surface, while the energy density and flux are evaluated at the cell center. They can be appreciably different if the cell, itself, is optically thick. We propose an interpolation formula that provides a good approximation both in the optically thin and thick limits. It approaches to one obtained from the reconstructed intensity in the optically thin limit, while it also does so to the diffusion approximation in the optically thick limit. We show that this numerical flux gives a stable solution even when the computation box contains both optically thin and thick cells.

This paper is organized as follows. We describe our numerical methods to solve the M1 model in section 2. We provide some simple examples to show the nature of our numerical scheme in section 3. In section 4, we apply the M1 model to irradiated protoplanetary disks. We discuss the accuracy and the stability of our numerical flux in section 5. We also discuss the affinity of the M1 model for massively parallel computing in section 5. Methods for constructing numerical flux of second-order accuracy in space are given in the Appendix.

2. M1 Model

2.1. Basic Equations

First we review the M1 model of González, Audit, and Huynh (2007). We assume that emission is thermal and scattering is isotropic. Then, the radiative-transfer equation for the specific intensity, \( I_\nu(x, t; \nu) \), is expressed as

\[
\left( \frac{1}{c} \frac{\partial}{\partial t} + \nu \cdot \nabla \right) I_\nu(x, t; \nu) = \kappa_{\nu,0} B_\nu \rho(x, t) - \left( \kappa_{\nu,\alpha} + \kappa_{\nu,\sigma} \right) \rho(x, t) I_\nu(x, t; \nu) + \kappa_{\nu,\sigma} \rho(x, t) \int I_\nu(x, t; \nu') \, d\nu',
\]

(1)

where \( \rho \) and \( c \) denote the density and the speed of light, respectively; \( \kappa_{\nu,\alpha} \) and \( \kappa_{\nu,\sigma} \) denote the absorption and scattering opacities at the photon frequency, \( \nu \), respectively. The symbols \( \nu \) and \( \nu' \) denote the angular variable, i.e., the unit vector parallel to the light propagation. The symbol \( B_\nu \) denotes the Planck function, and is a function of the temperature, \( T \).

We integrate equation (1) over a solid angle to obtain

\[
\frac{\partial E_\nu}{\partial t} + \sum_{i=1}^{3} \frac{\partial F_{\nu,i}}{\partial x_i} = \kappa_{\nu,0} \rho (4\pi B_\nu - cE_\nu),
\]

(2)

where

\[
E_\nu(x, t) = \frac{1}{c} \int I_\nu(x, t; \nu) \, d\nu,
\]

(3)

\[
F_{\nu,i}(x, t) = \int (e_i \cdot \nu) I_\nu(\nu) \, d\nu.
\]

(4)

The symbol \( i \) specifies a direction in Cartesian coordinates, and \( e_i \) is the unit vector in that direction. Equation (2) denotes the conservation of radiation energy density, \( E_\nu \), when the right-hand side vanishes. For simplicity we assumed that the emission and absorption are isotropic.
Similarly, we obtain

$$\frac{\partial F_{v,i}}{\partial t} + c^2 \sum_{j=1}^{3} \frac{\partial P_{v,ij}}{\partial x_j} = -c (\kappa_{v,s} + \kappa_{v,t}) \rho F_{v,i} ,$$

(5)

where

$$P_{v,ij}(x, t) = \frac{1}{c} \int (e_i \cdot n) (e_j \cdot n) I_v(n) \, d\Omega ,$$

(6)

by integrating equation (1) multiplied by $n$ over the whole solid angle. The symbol $j$, as well as $i$, specifies one in Cartesian coordinates.

When deriving equation (5), we assumed that the scattering is symmetric with respect to forward and backward. When the scattering is anisotropic, equation (5) should be replaced by

$$\frac{\partial F_{v,i}}{\partial t} + c^2 \sum_{j=1}^{3} \frac{\partial P_{v,ij}}{\partial x_j} = -c [\kappa_{v,s} + \kappa_{v,t} (1 - \langle \cos \theta \rangle)] \rho F_{v,i} ,$$

(7)

where $\langle \cos \theta \rangle$ denotes the scattering asymmetry parameter.

In order to solve equations (2) and (5), we invoke the closure relation,

$$P_{v,ij} = \left[ \frac{1 - \chi_v}{2} + \left( \frac{3\chi_v - 1}{2} \right) \frac{f_{v,ij} f_{v,j}}{|f_v|^2} \right] E_v ,$$

(8)

where

$$f_v = \left( \begin{array}{c} f_{v,1} \\ f_{v,2} \\ f_{v,3} \end{array} \right) = \frac{1}{c E_v} \left( \begin{array}{c} F_{v,1} \\ F_{v,2} \\ F_{v,3} \end{array} \right) ,$$

(9)

$$\chi_v = \frac{3 + 4|f_v|^2}{5 + 2\sqrt{4 - 3|f_v|^2}} .$$

(10)

The above radiative-transfer equation is solved in coupled with the hydrodynamical equations. The solutions of the hydrodynamical equations provide the density, velocity, and temperature. Hence, the opacity and source functions are evaluated as functions of them. In this paper we consider only the change in the temperature, and neglect any changes in the density and velocity. This approximation can be justified when we consider a protoplanetary disk in thermal equilibrium. The frequency-dependent opacity depends little on the density and temperature in a certain regime (see, e.g., Henning & Stognienko 1996), although the Rosseland mean opacity has a dependency.

### 2.2. Hydrodynamics

The gas is heated by absorption and cooled by emission. The heating and cooling are evaluated by

$$\rho T \frac{Ds}{Dt} = \int_0^\infty \sigma_{v,a} [c E_v - 4\pi B_v(T)] \, dv ,$$

(11)

where $D/Dt$ and $s$ denote the Lagrange derivative and the specific entropy, respectively. Then, the hydrodynamical equations are written in the conservation form as

$$\frac{\partial U_H}{\partial t} + \sum_{i=1}^{3} \frac{\partial F_{H,i}}{\partial x_i} = S_H ,$$

(12)

where

$$U_H = \left( \begin{array}{c} \rho \\ \rho v_1 \\ \rho v_2 \\ \rho v_3 \\ \rho E_H \end{array} \right) , \quad F_{H,i} = \left( \begin{array}{c} \rho v_i \\ \rho v_1 v_i + P \delta_{1,i} \\ \rho v_2 v_i + P \delta_{2,i} \\ \rho v_3 v_i + P \delta_{3,i} \\ \rho H_i v_i \end{array} \right) ,$$

(13)

$$S_H = \left\{ \begin{array}{c} 0 \\ \rho g_1 \\ \rho g_2 \\ \rho g_3 \\ \rho v \cdot g + \int_0^\infty \sigma_{v,a} [c E_v - 4\pi B_v(T)] \, dv \end{array} \right\} ,$$

(14)
\[ E_H = \frac{|v|^2}{2} + \frac{1}{\gamma - 1} \frac{P}{\rho} \]  
(15)

\[ H_H = \frac{|v|^2}{2} + \frac{\gamma}{\gamma - 1} \frac{P}{\rho} \]  
(16)

The symbol \( v = (v_1, v_2, v_3) \) denotes the gas velocity; \( g = (g_1, g_2, g_3) \) denotes gravity. The gas is assumed to be ideal, of which the specific heat ratio is \( \gamma \).

2.3. Numerical Scheme

Equations (2) and (5) can be expressed as

\[ \frac{\partial U_v}{\partial t} + \sum_{i=1}^{3} \frac{\partial F_{v,i}}{\partial x_i} = S_v. \]  
(17)

\[ U_v = \begin{pmatrix} E_v \\ F_{v,1} \\ F_{v,2} \\ F_{v,3} \end{pmatrix}, \quad F_{v,i} = \begin{pmatrix} F_{v,i} \\ c^2 P_{v,3i} \\ c^2 P_{v,2i} \\ c^2 P_{v,3i} \end{pmatrix}, \quad S_v = \begin{pmatrix} \sigma_{v,a} (4\pi B_v - c E_v) \\ -c (\sigma_{v,a} + \sigma_{v,j}) F_{v,1} \\ -c (\sigma_{v,a} + \sigma_{v,j}) F_{v,2} \\ -c (\sigma_{v,a} + \sigma_{v,j}) F_{v,3} \end{pmatrix}. \]  
(18)

Equation (18) has the same structure as that of the hydrodynamical equations in the conservation form. Thus, the Godunov-type method, which is often used for solving the hydrodynamical equations (see, e.g., Toro 2009), can be applied to equation (18).

In the Godunov-type method, the time evolution is evaluated based on the characteristics, i.e., the propagation speeds of the signal. The characteristics of equation (18) are rather complex, and their computations take much time. González, Audit, and Huynh (2007) obtained them by interpolating a prepared table rather than by computing them at each time step.

We can avoid detailed computations of the characteristics by employing the HLLE scheme, in which we need only the upper and lower limits on the characteristics (cf. Toro 2009). However, the HLLE scheme gives us a too much diffusive solution if the upper and lower limits are taken to be the speed of light, \( \pm c \) (González et al. 2007).

The ideal characteristics should be evaluated from the radiation fields in the adjacent cells across the surface on which the numerical flux is evaluated. Figure 1 of González, Audit, and Huynh (2007) shows the characteristics for a given radiation field. The ideal characteristics should be appropriately averaged ones when the radiation fields differ appreciably in the two adjacent cells. Roe (1981) obtained such average characteristics for the hydrodynamical equations. Similar average characteristics have not been obtained for the moment equation of the radiative transfer.

2.4. Kinetic Reconstruction of the Intensity

We obtain the numerical flux, not by computing the characteristics, but by reconstructing the intensity consistent with the moments of the radiation field. When the intensity is expressed as

\[ I_v(n) = \frac{3 c (1 - \beta^3_v)}{8\pi (3 + \beta^3_v)} (1 - \beta_v \cdot n)^{-4}. \]  
(19)

\[ \beta_v = \frac{3 f_v}{2 + \sqrt{4 - 3 |f_v|^2}}, \]  
(20)

the moments as well as the closure relation, equation (8), are consistent with the assumed radiation field. Although the closure relation cannot specify the intensity uniquely, equation (19) has a distinctive feature: the entropy is minimum (Dubroca & Feugeas 1999). We can obtain the angular distribution by the Lorentz transform of an isotropic radiation field (Levermore 1984).

We evaluate the flux, \( F_{v,i} \), across a cell boundary between two adjacent cells by using the reconstructed intensity. We call the two cells left (L) and right (R) for later convenience. The cell surface is assumed to be normal to the \( i \)-th direction, i.e., \((x_R - x_L) \times e_i = 0\). Then, the intensity on the cell surface is evaluated to be

\[ I_v^*(n) = \begin{cases} I_{v,L}(n) & (n \cdot e_i \geq 0) \\ I_{v,R}(n) & (n \cdot e_i < 0). \end{cases} \]  
(21)

where \( I_{v,L}(n) \) and \( I_{v,R}(n) \) denote the radiation field reconstructed in L and R cells, respectively. Equation (21) is based on the fact that photons transmit the surface from L to R when \( n \cdot e_i \geq 0 \) and from R to L otherwise. In other words, the intensity on the upwind side is identified as that on the surface. Thus, the energy flux,

\[ F_{v,i} = \int (e_i \cdot n) I_v^*(n) \, dn. \]  
(22)

is expected to inherit the “upwind” nature, and accordingly to be an alternative to the Godunov-type numerical flux. Similarly, the momentum flux is evaluated to be
Note that cell L. We used computer software, Mathematica, to obtain the above integral. Similarly, we obtain the other half, \( F_{v,R} \), which is expressed by an explicit function of \( E_{v,L} \), \( F_{v,L} \), \( E_{v,R} \), and \( F_{v,R} \). For later convenience, the numerical flux is decomposed into two components:

\[
F_{v,j}^* = F_{v,j,L}^* + F_{v,j,R}^* ,
\]

\[
F_{v,j,L}^* = \int_{\omega \geq 0} (e_i \cdot n) I_{v,L}(n) \, dn .
\]

\[
F_{v,j,R}^* = \int_{\omega < 0} (e_i \cdot n) I_{v,R}(n) \, dn .
\]

The former is evaluated to be

\[
F_{v,j,L}^* = \int_{\omega \geq 0} (e_i \cdot n) I_{v,L}(n) \, dn .
\]

\[
F_{v,j,R}^* = \int_{\omega < 0} (e_i \cdot n) I_{v,R}(n) \, dn .
\]

where the angular variables, \( \theta, \varphi, \) and \( \psi \), are defined to satisfy

\[
(e_i \cdot n) = \cos \theta ,
\]

\[
(e_j \cdot n) = \sin \theta \cos \varphi ,
\]

\[
(e_k \cdot n) = \sin \theta \sin \varphi .
\]

\[
(e_i \cdot \beta_L) = \beta_L \cos \psi .
\]

The symbols \( e_j \) and \( e_k \) denote unit vectors perpendicular to \( e_i \). The suffix, \( L \), indicates that the variables are evaluated in the cell \( L \). We used computer software, Mathematica, to obtain the above integral. Similarly, we obtain the other half,

\[
F_{v,j,R}^* = \int_{\omega \geq 0} (e_i \cdot n) I_{v,R}(n) \, dn .
\]

\[
F_{v,j,L}^* = \int_{\omega < 0} (e_i \cdot n) I_{v,L}(n) \, dn .
\]

\[
F_{v,j,R}^* = \int_{\omega < 0} (e_i \cdot n) I_{v,R}(n) \, dn .
\]

\[
F_{v,j,L}^* = \int_{\omega \geq 0} (e_i \cdot n) I_{v,L}(n) \, dn .
\]

\[
F_{v,j,R}^* = \int_{\omega < 0} (e_i \cdot n) I_{v,R}(n) \, dn .
\]

The numerical fluxes evaluated from the reconstructed intensity for \( (E, F_x, F_y, F_z) = (1/c, 0, 0, f) \). The black solid and dashed curves denote \( F_{x}^{(+)} \) and \( F_{x}^{(-)} \) as functions of \( f \), respectively. Both \( F_{x}^{(+)} \) and \( F_{x}^{(-)} \) approach to 1/4 in the limit of \( f \rightarrow 0 \) (isotropic). They have asymptotic forms of \( F_{x}^{(+)} \rightarrow f \) and \( F_{x}^{(-)} \rightarrow \sqrt{(1 - f)/8} \) in the limit of \( f \rightarrow 1 \).

Similarly, the radiation pressure is evaluated to be

\[
P_{v,j}^* = P_{v,j,L}^* + P_{v,j,R}^* ,
\]

\[
P_{v,j,L}^* = \int_{\omega \geq 0} \int_0^{2\pi} I_{v,L}(\theta, \varphi) \cos^2 \theta \sin \theta \, d\theta \, d\varphi.
\]

\[
P_{v,j,R}^* = \int_{\omega < 0} \int_0^{2\pi} I_{v,R}(\theta, \varphi) \cos^2 \theta \sin \theta \, d\theta \, d\varphi.
\]

\[
P_{v,j,L}^* = \int_{\omega \geq 0} \int_0^{2\pi} I_{v,L}(\theta, \varphi) \cos \theta \sin \theta \, d\theta \, d\varphi.
\]

\[
P_{v,j,R}^* = \int_{\omega < 0} \int_0^{2\pi} I_{v,R}(\theta, \varphi) \cos \theta \sin \theta \, d\theta \, d\varphi.
\]
The values of $P_{zz}^{(+)}$ and $P_{xx}^{(+)}$ are denoted by the grey solid and dashed curves, respectively, in figure 1.

Remember that our numerical flux is obtained by integrating the intensity over the left and light hemispheres separately. Dubroca et al. (2003) proposed a similar idea, named the “half space moment approximation.” They integrated the radiative-transfer equation over the half hemisphere, and obtained two unknowns and two equations for each moment. Their equations are closed by two closure relations, each of which is applied for each half hemisphere. Thus, their scheme is different from ours in many respects, although the idea, integration over the half hemisphere, is common.

### 2.5. Inclusion of Absorption and Emission within a Cell

In the previous subsection we implicitly assumed that the radiation field is uniform within a cell. This is not a good approximation when the cell under consideration, itself, is optically thick. The intensity differs appreciably on the cell surface from that at the cell center. Taking account of the absorption, emission, and scattering within a numerical cell, we modify equation (24) into

\[
F_{v;L} = \eta_{v,L} \left[ \xi_{v,L} F_{v,L}^{(+)} + \left( 1 - \xi_{v,L} \right) \frac{cE_{v,L}}{4} \right] + \left( 1 - \eta_{v,L} \right) \pi B_{v,L} \\
+ \eta_{v,R} \left[ \xi_{v,R} F_{v,R}^{(+)} - \left( 1 - \xi_{v,R} \right) \frac{cE_{v,R}}{4} \right] - \left( 1 - \eta_{v,R} \right) \pi B_{v,R},
\]

Equation (40) denotes an approximation to the formal solution of the radiative transfer in the limit of $\Delta x \to 0$. On the other hand, in the limit of $\Delta x \to \infty$, it describes the state in which the radiation is in the thermal equilibrium in each cell.

Similarly, equations (34) and (37) are modified into

\[
P_{v,ii}^{*} = \eta_{v,L} \left[ \xi_{v,L} P_{v,L}^{(+)} + \left( 1 - \xi_{v,L} \right) \frac{E_{v,L}}{6} \right] + \left( 1 - \eta_{v,L} \right) \frac{2\pi}{3c} B_{v,L} \\
+ \eta_{v,R} \left[ \xi_{v,R} P_{v,R}^{(+)} - \left( 1 - \xi_{v,R} \right) \frac{E_{v,R}}{6} \right] + \left( 1 - \eta_{v,R} \right) \frac{2\pi}{3c} B_{v,R},
\]

\[
P_{v,ij}^{*} = \eta_{v,L} \xi_{v,L} P_{v,L}^{(+)} + \eta_{v,R} \xi_{v,R} P_{v,R}^{(+)}.
\]

We prove that our modified numerical flux is asymptotic preserving, and reproduces the diffusion approximation in the limit of $\kappa \rho \Delta x \to \infty$. For later convenience, we define the symbol
\[
\Delta F_v^* \equiv \sum_{i} \frac{1}{\Delta x_i} \left[ F_{v,i}^* \left( x + \frac{\Delta x_i}{2} e_i \right) - F_{v,i}^* \left( x - \frac{\Delta x_i}{2} e_i \right) \right]
\]
\[\text{(47)}\]

to denote the “divergence” of the numerical flux evaluated on the cell. Similarly, we define
\[
\Delta P_{v,ij}^* \equiv \sum_{k} \frac{1}{\Delta x_k} \left[ P_{v,ij}^* \left( x + \frac{\Delta x_k}{2} e_k \right) - P_{v,ij}^* \left( x - \frac{\Delta x_k}{2} e_k \right) \right]
\]
\[\text{(48)}\]
to denote the divergence of the pressure tensor. The discretized M1 equations reduce to
\[
\Delta F_v^* = -\kappa_v \rho \left( c E_v - 4\pi B_v \right),
\]
\[\text{(49)}\]
\[
c^2 \sum_i \Delta P_{v,ij}^* = - \left( \kappa_{v,\alpha} + \kappa_{v,\gamma} \right) \rho c F_{v,j}
\]
\[\text{(50)}\]
in radiative equilibrium.

When \( \Delta x \) is large and the temperature difference is small, equations (40), (45), and (46) reduce to
\[
F_{v,j}^* \approx \pi [B_v(T_L) - B_v(T_R)] \approx \pi \frac{\partial B_v}{\partial T} (T_L - T_R),
\]
\[\text{(51)}\]
\[
P_{v,ij}^* \approx \frac{2\pi}{3} [B_v(T_L) + B_v(T_R)],
\]
\[\text{(52)}\]
\[
P_{v,ij}^* \approx 0 \quad \text{if} \quad i \neq j.
\]
\[\text{(53)}\]
respectively.

Substituting equations (51) through (53) into equations (49) and (50), we obtain
\[
E_v \approx \frac{4\pi}{c} B_v,
\]
\[\text{(54)}\]
\[
F_{v,j} \approx \frac{4\pi}{3\rho} \frac{1}{\kappa_{v,\alpha} + \kappa_{v,\gamma}} \frac{\partial B_v}{\partial T} \frac{\partial T}{\partial x_j},
\]
\[\text{(55)}\]
where only the most dominant terms are taken into account for simplicity. These equations are equivalent to the diffusion approximation (see, e.g., Castor 2004, for frequency dependent diffusion approximation). Thus, our numerical flux gives a good approximation in the optically thick limit. We can expect that our numerical flux gives a reasonable approximation at any optical depth.

### 2.6. Second-Order Accuracy in Space

The numerical flux given in the previous subsection is of first-order accuracy in space. A numerical flux of second-order accuracy in space can be obtained by applying the Monotone Upwind-centered Scheme for Conservation Laws (MUSCL, see e.g., Hirsch 1990). MUSCL evaluates the physical states on the cell boundaries from the left and right-hand sides by extrapolation. MUSCL applies a limiter, such as a minmod function, in order to avoid spurious extrema from the values obtained by simple extrapolation. We need some additional care to avoid any unphysical extrapolation when applying MUSCL to the M1 equations. Otherwise, the energy density can be smaller than the radiative energy flux divided by the speed of light (\( E < |F|/c \)). We use the formulae given in the Appendix to achieve the second-order accuracy in space.

The numerical flux given in the Appendix is constructed on the assumption that both the emission and scattering change linearly along the line from the cell center to the boundary. The emission and scattering on the cell boundary are evaluated by linear extrapolation with the minmod limiter, i.e., by MUSCL. Therefore, the contribution of the emission and scattering within the cell is expressed as a linear combination of those evaluated at the cell center and boundary. Further details are given in the Appendix.

### 2.7. Time Evolution

Using the notation given in the previous subsection, we can rewrite the M1 equations in the form
\[
\frac{\partial E_v}{\partial t} = \kappa_{v,\alpha} \rho \left( c E_v - 4\pi B_v \right) - \Delta F_v.
\]
\[\text{(56)}\]
\[
\frac{\partial F_{v,j}}{\partial t} = - \left( \kappa_{v,\alpha} + \kappa_{v,\gamma} \right) \rho c F_{v,j} - c^2 \Delta P_{v,j}.
\]
\[\text{(57)}\]
We use two different methods to integrate equations (56) and (57) to first-order accuracy in time, depending on the problem. We use the forward difference of the first order in time in a flash test, in which we solve the propagation of radiation from a sphere into the vacuum. In the remainder of test problems, we use a formal solution for forwarding the energy density and flux in time.

The formal solutions of equations (56) and (57) are expressed as
Fig. 2. Two-dimensional shadow test. The upper and lower panels show results obtained by the numerical flux of the first-order accuracy in space and that of the second order, respectively. The dashed line denotes the absorber. The brightness denotes the radiation energy density in the logarithmic scale. The solid curves denote the contours of $\log E = -3, 2,$ and $-1$. See the text for more details. The arrows denote the vector $F / |F|$.

\[
E_v(t + \Delta t) = e^{-\kappa_\nu \alpha \Delta x} E_v(t) + \left(1 - e^{-\kappa_\nu \alpha \Delta x}\right) \left(B_v + \frac{\Delta F_v}{\kappa_\nu \alpha \rho}\right),
\]

(58)

\[
F_{\nu,j}(t + \Delta t) = e^{-\left(\kappa_\nu \alpha + \kappa_\nu \gamma\right) \Delta x} F_{\nu,j}(t) + \left[1 - e^{-\left(\kappa_\nu \alpha + \kappa_\nu \gamma\right) \Delta x}\right] \frac{c^2 \Delta P_{\nu,j}}{(\kappa_\nu \alpha + \kappa_\nu \gamma) \rho},
\]

(59)

where all of the physical variables are evaluated at time, $t$. Thus, our scheme is explicit in the sense that $E_v$ and $F_{\nu,j}$ are obtained without iteration. Nevertheless, the formal solutions allow to use a much longer time step than the simple forward difference in time when $\kappa_\nu \alpha \rho \Delta x \gg 1$. The time step should be smaller than $\Delta t \leq 1 / (\kappa_\nu \alpha \rho)$ in the simple forward difference. When $\kappa_\nu \alpha \rho \Delta x \gg 1$, the constraint is serious, since it is much shorter than the time necessary for propagation of the signal, $\Delta s / c$.

We integrate the hydrodynamical equations by an explicit scheme. The time step is taken to be the minimum of the thermal and hydrodynamical timescales.

A solution of the second-order accuracy in time can be obtained by a two-step Runge–Kutta method. We used the average of time derivatives evaluated at $t = t_0$ and $t_0 + \Delta t$ when integrating equations from $t = t_0$ to $t_0 + \Delta t$.

3. Monochromatic Test Problems

3.1. Shadow Test

We performed a shadow test to illustrate the effects of including absorption within the cell. We exposed uniform monochromatic radiation to a square absorber of $\kappa_\nu \alpha = 50$. Scattering and emission were neglected for simplicity. The computation box covered a square of $0 \leq x \leq 12$ and $0 \leq y \leq 6$. The absorber occupied the square region of $2 \leq x \leq 3$ and $0 \leq y \leq 2$. The spatial resolution was $\Delta x = \Delta y = 0.1$. We imposed uniform radiation, $(E, F_x, F_y) = (1.0, 0.999 c, 0)$, from $x = 0.0$. A reflection boundary was applied to the upper and lower boundaries of $y = 6$ and $0$. The outgoing boundary was placed on the right boundary, $x = 12.0$, so that no radiation would enter from the boundary. We realized the outgoing boundary by vanishing the flux from outside the boundary. It is a virtue of the kinetic flux that the outgoing boundary is easily constructed. We obtained the equilibrium state by solving the time-dependent M1 model with a time step of $\Delta t = 0.5 \Delta x / c$.

Figure 2 shows the result of the shadow test in the equilibrium state. The numerical flux of the first-order accuracy in space is used in the upper panel, while that of the second-order accuracy is used in the lower panel. The brightness denotes the energy density on the logarithmic scale. Note that the energy density drops very sharply behind the left side of the absorber in the time step. The arrows denote $F / (c E)$. 

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Fig. 3. Result of flash test for $f = 0$. The upper panel shows the solution by the HLLE scheme, and the lower panel shows that by our numerical flux. Both panels denote the solutions of the first-order accuracy in space and time.

Fig. 4. Same as figure 3, but for solutions of the second-order accuracy in space and time.

Very weak radiation shines in from the upper-right corner behind the absorber because the incident radiation is not perfectly beamed [$F/(cE) = 0.999$]. The FWHM of the beam was $\theta_{\text{FWHM}}/2 = 1:58$, since it was evaluated to be $\theta_{\text{FWHM}}/2 \approx 0.870 \sqrt{1 - f}$ from equations (19) and (20) for $1 - \beta \approx 2(1 - f) \ll 1$. The intensity decreases by a factor 100 at $\theta = 5\frac{\pi}{2}$. The inclination of the contour of $\log E = -2$ is $6^\circ$ and $5^\circ$ from the horizontal line in the solutions of the first and second-order accuracy, respectively, and is constant with an intensity distribution of $f = 0.999$. The solutions are different only in dark area of $\log E < -2$. The contour of $\log E = -3$ is more inclined in the solution of the first-order accuracy than in that of the second-order accuracy.

3.2. Flash Test

We performed the following test for studying the propagation of radiation in a vacuum. The initial radiation field at $t = 0$ is set to be

\[
(E, F_x, F_y, F_z) = \begin{cases} 
(1, c f, 0, 0) & \text{if } |r| \leq r_0, \\
(0, 0, 0, 0) & \text{if } |r| > r_0,
\end{cases} 
\]

so that uniform radiation is confined within sphere of $r \leq r_0$. No absorption and emission are assumed in this test. Then, the radiation is expected to be confined in a spherical shell of $r_0 - ct \leq r \leq r_0 + ct$.

Figure 3 denotes the radiation energy density for $f = 0.0$ and $r_0 = 0.5$ at $t = 6.0/c$. The spatial resolution and time step are taken to be $\Delta x = 0.1$ and $\Delta t = 0.3\Delta x/c$, respectively. The upper panel denotes the result obtained by the simple HLLE
flux, while the lower panel denotes that obtained with our numerical flux. Both of them denote the solutions of the first-order accuracy in space and time. The brightness and the contours denote log $E$. It should be noted that HLLE does not work at $\Delta t = 0.35\Delta x/c$, while our method works at $\Delta t = 0.5\Delta x/c$. A longer time step can be safely taken if we apply the kinetically reconstructed numerical flux.

If we solve the flash problem perfectly by taking the full angular dependence, the radiation should be confined within an expanding spherical shell of $ct - r_0 \leq r \leq ct + r_0$. It should be isotropic around the origin, and the flux should be radial.

Our model shows a higher contrast than does the simple HLLE model. However, a clear anisotropy of numerical origin appears in our model. This anisotropy is due to the fact that the change in the flux direction in a cell is not taken into account.

Figure 4 is the same as figure 3, but for solutions of the second-order accuracy in space and time. The anisotropy has been removed in the solutions of the second-order accuracy. The low contrast of the simple HLLE model is also improved, although it is still higher in our model.

Figure 5 is the same as figure 3, but for $f = 0.9$. When $f$ is close to 1.0, the forward radiation is much stronger than the backward component. However, the shell of the high radiation energy density should expand spherically, as in the case of $f = 0$, if we take the full angular dependence of the radiation. It expands as expected in the lower panel (kinetically reconstructed numerical flux), while the center of the sphere shows a spurious shift in the upper panel (HLLE). This is due to the characteristic speeds of waves in the M1 model equations. Their absolute values are smaller than the speed of light, and all of the characteristic speeds have the same sign when $f > 0.69$ [see e.g., figure 1 of González, Audit, and Huynh (2007) for the characteristic speeds as a function of $f$]. Thus, the center of the radiation shell shifts rightward in the HLLE solution.

One might think that the shift would be the nature of the M1 model equations, and should be reproduced in numerical simulations. However, we should realize that the characteristic speed changes according to that in $f = F/(cE)$. The ratio, $f$, decreases...
on the left edge of the radiation sphere, since radiation moving rightward has a higher $f$ than the initial value. Once it decreases to the critical value on the edge, the radiation begins to propagate leftward, and approaches to $f = -1$. It should take only an instant for the left edge of the radiation sphere to propagate at the speed of $-c$, if our spatial resolution is extremely high. However, it takes a few time steps for $f$ to decrease to the critical value in a cell close to the left edge. The propagation rightward is delayed a few time steps in the HLLE model.

Figure 6 shows second-order accurate solutions for a flash test of $f = 0.9$. The bright shell is more sharply captured in both of the second-order accurate solutions. The contrast is still higher in our model than in the HLLE model. The central dark hole is shifted leftward in the HLLE model.

4. Application to Irradiated Protoplanetary Disks

Young stars are often associated with gaseous disks, called protoplanetary disks (see, e.g, a review by Williams & Cieza 2011). They are irradiated by radiation from the central stars to shine at various wavelengths. They reflect optical and near-infrared stellar lights, and emit mid and far-infrared lights. Both absorption and scattering are dominated by dust, although the optical properties remain somewhat unknown. It is essential to take account of the frequency-dependent opacity when modeling protoplanetary disks. Optical and near-infrared radiation heat up the disks from outside, while mid and far-infrared emission cool down them from inside. The mid and far-infrared flux from inside balances the optical and near-infrared one from outside in equilibrium.

We applied our numerical method to an irradiated protoplanetary disk as a test for the multi-color problem. We used the opacity table of Draine (2003) in which $\kappa_{\nu,d}$, $\kappa_{\nu,s}$, and $(\cos \theta)$ are given as functions of the wavelength. Applying the spline fit to the table, we obtained the values at the wavelengths

$$\log \left( \frac{\lambda_m}{1.0 \mu m} \right) = 0.02m, \quad (61)$$

where $m$ is an integer in the range of $-50 \leq m \leq 150$. Figure 7 shows $\kappa_{\nu,d}$, $\kappa_{\nu,s}$, and $\kappa_{\nu,d} + \kappa_{\nu,s} (1 - \langle \cos \theta \rangle)$ as functions of $\lambda$. The opacity was obtained under the assumption that the dust occupies 1% of the total mass. In other words, we did not take account of any sedimentation of dust for simplicity.

We used two sets of the M1 model; one denotes direct radiation from the central star and the other for radiation scattered by and reemitted from the protoplanetary disks. Then, the radiation energy density and flux are expressed as

$$E_{\nu} = E'_{\nu} + E''_{\nu},$$
$$F_{\nu} = F'_{\nu} + F''_{\nu}, \quad (62)$$

where the symbols with a prime denote the values of the direct stellar radiation, and those with a double prime concern the total of scattered radiation and emission from the disk. We apply the closure relation separately for the two components. The M1 model equations are expressed as:
Fig. 8. Model spectra for a protoplanetary disk irradiated by a star of $M = 2.0 \, M_\odot$, $R = 2.5 \, R_\odot$, and $T_{\text{eff}} = 9500 \, K$. Each curve denotes the flux from the disk surface at the designated radius.

The separation of the direct stellar light from the remaining radiation avoids any spurious beam collision on the disk surface. The radiation energy densities were evaluated at the wavelengths given by equation (61), i.e., 201 bands in the range of $0.1 \, \mu m < \lambda < 1 \, nm$. Thus, our model gives a spectral resolution of $\Delta \lambda = 4.61 \times 10^{-2} \lambda$. This spectral resolution is good enough to study the spectral features of the dust opacity.

We solved the above moment equation of radiation and the hydrodynamical equations simultaneously. In this paper we restrict ourselves to describing the disk in equilibrium in which the emission from the disk balances the heating by irradiation. We ignored any self gravity of the disk and viscous heating by accretion for simplicity. The radial component of the gravity was assumed to be balanced with the centrifugal force due to disk rotation.

We assumed that the central star has mass and radius of $M_\star$ and $R_\star$, respectively. The stellar radiation was assumed to be the blackbody curve of $T_{\text{eff}}$. We made both one and two-dimensional models of the protoplanetary disk with the numerical flux of first-order accuracy in space. They are described in subsections 4.1 and 4.2, respectively.

4.1. 1D Model Based on the Grazing Recipe

Our 1D model describes the vertical structure of the protoplanetary disk at a given radius, $R$, from the central star. The stellar radiation, $E'_\nu$, is evaluated to be

$$E'_\nu(r, z) = \frac{\pi}{c} \left( \frac{R_\star}{R} \right)^2 B_\nu(T_{\text{eff}}) \exp \left[ -\frac{\tau_\nu(z)}{\alpha} \right],$$

$$\tau_\nu(z) = \left[ \kappa_{\nu,\alpha} + \kappa_{\nu,\lambda} (1 - \langle \cos \theta \rangle) \right] \int_z^\infty \rho(r, z') \, dz',$$

$$\alpha = \left[ 0.4 \frac{R_\star}{R} + R \frac{d}{dR} \left( \frac{H_\star}{R} \right) \right],$$

according to the grazing-angle recipe (Chiang & Goldreich 1997). Here, $H_\star$ denotes the height of the “photosphere” at which the stellar radiation is attenuated by a factor of $e^{-1}$. We evaluated the photosphere at $\lambda = 0.302 \, \mu m$. We evaluated $d(H_\star/R)/dR$.
consistently by solving the vertical structure at slightly different radii, $0.891 R$ and $1.122 R$.

We obtained the steady-state solution by integrating the M1 model equations and equations for hydrostatic balance simultaneously. We used the Lagrangian coordinate in this 1D model.

Figure 8 shows the model spectra at $R = 50$, 100, and 200 AU from a star of $M_*= 2.0 M_\odot$, $R_*= 2.5 R_\odot$, and $T_{\mathrm{eff}} = 9500$ K. These parameters are taken to be similar to those of AB Aur (van den Ancker et al. 1997) for which the inner hole and spirals structure are seen in optical (Grady et al. 1999) and near infrared (Fukagawa et al. 2004; Hashimoto et al. 2011).

The surface density is assumed to be

$$\Sigma = 0.35 \left( \frac{R}{100 \, \text{AU}} \right)^{-1} \, \text{g cm}^{-2}. \quad (71)$$

The model spectra are consistent with those obtained by D’Alessio et al. (1998) and Dullemond, Dominki, and Natta (2001), who solved the angle-dependent radiative transfer by using the Monte Carlo simulation. The ratio of the grazing angle to the disk aspect is $d/d \ln R[\ln (H_*/R)] = 0.251$ at $R = 100$ AU, which is close to the standard value, $2/7 = 0.282$ (Chiang & Goldreich 1997). The value depends rather on the opacity and surface-density distribution. The standard value was obtained by a simplified analytical model. The difference is not numerical.

Figure 9 confirms that the heating is balanced with the cooling correctly in our solution. Each curve denotes the energy flux, $\nu F_{\nu z}$ or $\nu F''_{\nu z}$, at a given height as a function of the wavelength. The value is taken to be positive when the energy flows outward from the disk. The solid curves denote $F_{\nu z}$ (thin) and $F''_{\nu z}$ (thick) at the disk (not the photosphere). The dashed curves denote $F_{\nu z}$ (thin) and $F''_{\nu z}$ (thick) at the level above which the disk has a surface density of $3.9 \times 10^{-4}$ g cm$^{-2}$. The thick dash dotted curve indicates $F''_{\nu z}$ at $\Sigma = 3.5 \times 10^{-2}$ g cm$^{-2}$. It is clearly shown that the net flux vanishes at any height. The mid-infrared radiation is the main heating source in the layer below $\Sigma > 3.5 \times 10^{-2}$ g cm$^{-2}$.

### 4.2. 2D Axisymmetric Model

The 1D model assumes implicitly that the surface density gradually changes with the radius. However, some protoplanetary disks may have holes, and the surface density may rise sharply at some radius. The transition disks are thought to be the case (see, e.g., a review by Williams & Cieza 2011 and the references therein). We constructed a two-dimensional model of a transitional disk assuming the symmetry around the axis and that on the mid plane.

Following Honda et al. (2012), who made spectral energy distribution (SED) model for a transitionl disk around HD 169142, we made a model in which the surface density is expressed as

$$\ln \Sigma = \frac{1}{2} \left\{ \ln \left( \frac{\Sigma_{\mathrm{in}}}{\Sigma_{\mathrm{out}}} \right) + \ln \left( \frac{\Sigma_{\mathrm{out}}}{\Sigma_{\mathrm{in}}} \right) \right\} \tanh \left[ \Gamma \ln \left( \frac{r}{r_0} \right) \right] - \ln \left( \frac{r}{r_0} \right). \quad (72)$$

where $\Sigma_{\mathrm{in}}$, $\Sigma_{\mathrm{out}}$, $\Gamma$, and $r_0$ are model parameters to be chosen. We obtained the radiation and gas in the region of $40 \, \text{AU} \leq r \leq 160 \, \text{AU}$ and $|z| \leq 80 \, \text{AU}$, with a resolution of $\Delta r = 0.3 \, \text{AU}$ and $\Delta z = 0.4 \, \text{AU}$ in cylindrical coordinates. We placed the reflection boundary on $z = 0$ and the outgoing boundary on $z = 80 \, \text{AU}$. The incoming flux from $R = 40 \, \text{AU}$ was fixed. The flux incoming from $r = 160 \, \text{AU}$ was assumed to balance with the outgoing one in the disk, and to vanish outside the disk. The mass, radius, and effective temperature of the central star are assumed to be $M = 2.0 M_\odot$, $R_e = 1.25 \times 10^{-2} \, \text{AU}$, and $T_{\mathrm{eff}} = 9000 \, \text{K}$, respectively.

Figure 10 shows the density and temperature distribution in equilibrium for $r_0 = 100 \, \text{AU}$, $\Gamma = \infty$, $\Sigma_{\mathrm{in}} = 3.5 \times 10^{-3} \, \text{g cm}^{-3}$, and $\Sigma_{\mathrm{out}} = 3.5 \times 10^{-1} \, \text{g cm}^{-2}$. The wall at $r = 100 \, \text{AU}$ is heated up to $T \simeq 140 \, \text{K}$, and hence expands more. The brightness denotes the temperature, and the curves denote the contours of $\log \rho$ in the interval of $\Delta \log \rho = 0.5$. Note that we solved the vertical hydrodynamic balance consistently, while the vertical density distribution is given in Honda et al. (2012). The radiation

![Fig. 10. Temperature and density distributions in the model having a sharp rise in the surface density at $r = 100 \, \text{AU}$. The brightness denotes the temperature, and the curves denote the contours of $\log \rho$ in the interval of $\Delta \log \rho = 0.5$.](https://academic.oup.com/pasj/article-abstract/65/4/72/1530072)
from the wall heats up the inner disk by 15 K in the range of 70 AU ≤ r ≤ 100 AU compared with the model without the wall, i.e., $\Sigma = \Sigma_{\infty}(r/r_0)^{-1}$.

Figure 11 shows the total energy density, $E_{\nu}$, at $\lambda = 0.316 \mu$m, 1.58 $\mu$m ($H$-band), 20 $\mu$m ($Q$-band), and 501 $\mu$m from top left to bottom right. The brightness denotes log $E_{\nu}$ in units of erg cm$^{-3}$ Hz$^{-1}$, as indicated in the right bar. The arrows denote the vector $\mathbf{F}_{\nu}/(cE_{\nu})$.

At $\lambda = 0.316 \mu$m, stellar radiation is absorbed or scattered on an upper layer of low density, and does not penetrate into protoplanetary disks. Near-infrared radiation penetrates a little deeper into the interior of the disk, but does not reach the mid plane. The upper part of the wall is bright at $\lambda = 20 \mu$m. The wall is heated up by stellar light, and emits mid and far-infrared radiation. The disk is more transparent at longer wavelengths as shown in the bottom right panel of figure 11. The direction of the energy flux also depends on the wavelength. The energy flows from the central star in the optical and near-infrared bands, while it flows from the wall and upper surface in the mid infrared and from the disk in the far-infrared region.

We obtained simulated images of the protoplanetary disk by integrating equation (1) along the line of sight. The source terms were evaluated from $T$ and $E_{\nu}$, obtained by our M1 model. The upper panels of figure 12 show the simulated images at $\lambda = 1.58 \mu$m (left, $H$-band) and 20 $\mu$m (right, $Q$-band). The line of sight is assumed to be inclined by 15° from the axis normal to the disk. Both images show bright rings at $r \simeq 100$ AU. These images are similar to those of Honda et al. (2012), who solved the radiative transfer by a Monte-Carlo simulation.

We made another model by assuming $\Gamma = 20$, while keeping the other model parameters unchanged. The temperature and density distributions are shown in figure 13. The density change around the wall is smooth, and more likely than that shown in figure 10.

Figure 14 is the same as figure 11, but for $\Gamma = 20$. The result is almost the same, but the wall boundary is less sharps, as expected. The photospheres are located just behind the wall in the model of $\Gamma = \infty$ in a broad range of wavelengths. However, the location of the photosphere depends on the wavelength in the model of $\Gamma = 20$.

The lower panels of figure 12 are the same as the upper panels, but for $\Gamma = 20$. The bright ring is broader in the model of $\Gamma = 20$ than in that of $\Gamma = \infty$. The peak brightness is lower in the model of $\Gamma = 20$. This is because the wall is appreciably inclined when $\Gamma = 20$ (see figure 13). The ring corresponds to the region in which the surface density increases with the radius. In other words, the wall is seen as a bright ring in the image. Our model will serve to evaluate the surface density distribution from observed images.
5. Discussions

As demonstrated in the previous sections, the M1 model works well both in a vacuum and in optically thick media, if absorption is taken into account properly in the numerical flux. If we use the formal solution for computing time evolution, the time step can be as large as $\Delta x/(2c)$ irrespectively of the optical depth. Thanks to this improvement, we succeeded in applying the M1 model to protoplanetary disks. They are optically thick in optical bands, while optically thin in far-infrared bands. They are heated by both optical and near-infrared stellar lights, and cooled by mid and far-infrared emission. Thus, it is important to take account of
both optically thin and thick radiation simultaneously. Our numerical scheme will expand the applicability of the M1 model.

The M1 model can also be used for neutrino transfer. The neutrino is a major coolant in compact objects, such as neutron stars and black holes. It should play an important role in the dynamics of core-collapse supernovae and in gamma-ray bursts (see, e.g., a review by Mezzacappa 2005 and the references therein). Both core-collapse supernovae and gamma-ray burst sources are thought to be highly anisotropic, and their dynamics should be ideally studied based on three-dimensional numerical simulations. It is also essential to take account of the neutrino energy in these simulations. The neutrino opacity is proportional to the square of the neutrino energy. Low-energy neutrinos are easy to leak, while high-energy ones are not. Thus the M1 model is a reasonable choice for numerical simulations of these objects. It reduces the computation cost by lowering the angular resolution. Yet, it can express a shadow and beamed radiation.

It should be noted that the M1 model can solve the propagation of a flash in an explicit manner. Radiation at the next time step depends only on those in the neighboring cells. We need neither ray tracing nor global iteration. In other words, the M1 model does not require global communication for proceeding a time step. This is beneficial for massive parallel computation, since global communication between processors is often a bottleneck.

In compact objects, the gas has a high temperature, and the sound speed is comparable to, or only by a factor of ten smaller than, the speed of light. Thus it is not serious that the time step is restricted to the light propagation time, $0.5 \Delta x/c$; a similar small time step is required for hydrodynamical simulations if we integrate them explicitly. If we solve both the radiation and hydrodynamics explicitly, we can take account of heating by a flash of neutrinos. Compact sources may have a highly variable luminosity.

The M1 model may be applied to the dynamics of non-relativistic objects in which the sound speed is much slower than the speed of light. We can reduce the speed-of-light propagation in the M1 model to prolong the time step. If we replace $c$ by $c'$ in equations (58) and (59), the propagation speed is reduced to $c'$. We expect that the reduction does not affect the result seriously for the following reason. Both the hydrodynamical and thermal timescales are much longer than the time for light propagation in most non-relativistic objects. Thus, the light propagation speed is assumed to be infinite in some radiation hydrodynamics. The results are valid since the light speed is much faster than other speeds of waves, and we can neglect the difference between the real light speed and infinity. We think that we can neglect the difference between $c$ and $c'$ as long as $c'$ is much larger than other speeds of wave propagation. Our idea is based on a heuristic experiment by Hotta et al. (2012). They performed numerical simulations of convection in the Sun by reducing the sound speed artificially. The velocity of material convection is much lower than the sound speed. Thus, most of the simulations thus far have applied the anelastic approximation in which the sound speed is infinitely large. However, Hotta et al. (2012) demonstrated that nearly the same results are obtained even when the sound speed is reduced artificially, as long as the reduced sound speed is still much faster than the convection velocity. Their experiment suggests to us that we can reduce the light speed artificially without any loss of quality.
In summary, the M1 model has potential applicability to many problems, including protoplanetary disks, core collapse supernovae, and gamma-ray bursts.

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Appendix. Numerical Flux of the Second Order Accuracy in Space

First we introduce a new variable,

\[ E'_v \equiv \sqrt{E'^2_v - \left( \frac{F'_v}{c} \right)^2}. \]  

(A1)

for later convenience. We evaluate \( E'_{v,j+1/2} \) on the cell boundary by extrapolation with the minmod limiter, and obtain

\[ E'^{\text{(L)}}_{v,j+1/2} = E'_{v,j} + \frac{1}{2} \Delta E'^{\text{(L)}}_{v,j+1/2}, \]  

(A2)

\[ E'^{\text{(R)}}_{v,j+1/2} = E'_{v,j} - \frac{1}{2} \Delta E'^{\text{(R)}}_{v,j+1/2}, \]  

(A3)

\[ \Delta E'^{\text{(L)}}_{v,j+1/2} = \min \left( \left| \Delta E'_{v,j+1/2} \right|, \left| \Delta E'_{v,j-1/2} \right| \right) \times \left[ \text{sgn} \left( \Delta E'_{v,j+1/2} \right) + \text{sgn} \left( \Delta E'_{v,j-1/2} \right) \right], \]  

(A4)

\[ \Delta E'^{\text{(R)}}_{v,j+1/2} = \min \left( \left| \Delta E'_{v,j+1/2} \right|, \left| \Delta E'_{v,j+3/2} \right| \right) \times \left[ \text{sgn} \left( \Delta E'_{v,j+1/2} \right) + \text{sgn} \left( \Delta E'_{v,j+3/2} \right) \right], \]  

(A5)

\[ \Delta E'_{v,j-1/2} = E'_{v,j} - E'_{v,j-1}, \]  

(A6)

\[ \Delta E'_{v,j+1/2} = E'_{v,j+1} - E'_{v,j}, \]  

(A7)

\[ \Delta E'_{v,j+3/2} = E'_{v,j+2} - E'_{v,j+1}, \]  

(A8)

where \( \text{sgn} \) denotes the sign function. Similarly, we evaluate \( F_{x,v,j+1/2} \) on the cell boundary to obtain

\[ F^{\text{(L)}}_{x,v,j+1/2} = F_{x,v,j} + \frac{1}{2} \Delta F^{\text{(L)}}_{x,v,j+1/2}, \]  

(A9)

\[ F^{\text{(R)}}_{x,v,j+1/2} = F_{x,v,j} - \frac{1}{2} \Delta F^{\text{(R)}}_{x,v,j+1/2}, \]  

(A10)

\[ \Delta F^{\text{(L)}}_{x,v,j+1/2} = \min \left( \left| \Delta F_{x,v,j+1/2} \right|, \left| \Delta F_{x,v,j-1/2} \right| \right) \times \left[ \text{sgn} \left( \Delta F_{x,v,j+1/2} \right) + \text{sgn} \left( \Delta F_{x,v,j-1/2} \right) \right], \]  

(A11)

\[ \Delta F^{\text{(R)}}_{x,v,j+1/2} = \min \left( \left| \Delta F_{x,v,j+1/2} \right|, \left| \Delta F_{x,v,j+3/2} \right| \right) \times \left[ \text{sgn} \left( \Delta F_{x,j+1/2} \right) + \text{sgn} \left( \Delta F_{x,j+3/2} \right) \right], \]  

(A12)

\[ \Delta F_{x,v,j-1/2} = F_{x,v,j} - F_{x,v,j-1}, \]  

(A13)

\[ \Delta F_{x,v,j+1} = F_{x,v,j+1} - F_{x,v,j}, \]  

(A14)

\[ \Delta F_{x,v,j+3/2} = F_{x,v,j+2} - F_{x,v,j+1}. \]  

(A15)

We obtain \( F^{\text{(L)}}_{y,v,j+1/2}, F^{\text{(R)}}_{y,v,j+1/2}, F^{\text{(L)}}_{z,v,j+1/2}, \) and \( F^{\text{(R)}}_{z,v,j+1/2} \) by the same procedure. The energy density on the cell boundary is evaluated to be

\[ E'^{\text{(L)}}_{v,j+1/2} = \sqrt{ \left( E'^{\text{(L)}}_{v,j+1/2} \right)^2 + \left( F^{\text{(L)}}_{x,v,j+1/2} \right)^2 + \left( F^{\text{(L)}}_{y,v,j+1/2} \right)^2 + \left( F^{\text{(L)}}_{z,v,j+1/2} \right)^2 } / c. \]  

(A16)

\[ E'^{\text{(R)}}_{v,j+1/2} = \sqrt{ \left( E'^{\text{(R)}}_{v,j+1/2} \right)^2 + \left( F^{\text{(R)}}_{x,v,j+1/2} \right)^2 + \left( F^{\text{(R)}}_{y,v,j+1/2} \right)^2 + \left( F^{\text{(R)}}_{z,v,j+1/2} \right)^2 } / c. \]  

(A17)

The introduction of \( E'_v \) guarantees that the energy density is larger than the energy flux divided by the speed of light on the cell boundary.
The energy density, \( E_{v,j+1/2}^{(L)} \), can exceed \((3/2)E_{v,j}\). If we would apply MUSCL to \( E_{v,j} \), it can not be very high. To avoid such a high-energy density, we reduce both the energy density and flux by multiplying the following factors:

\[
\begin{align*}
\varepsilon_v^{(L)} &= \begin{cases} 
1, & \text{if } E_{v,j+1/2}^{(L)*)} \leq \frac{3}{2}E_{v,j} \\
\frac{3}{2} \frac{E_{v,j}}{E_{v,j+1/2}}, & \text{otherwise}
\end{cases} \\
\varepsilon_v^{(R)} &= \begin{cases} 
1, & \text{if } E_{v,j+1/2}^{(R)*)} \leq \frac{3}{2}E_{v,j+1} \\
\frac{3}{2} \frac{E_{v,j+1}}{E_{v,j+1/2}}, & \text{otherwise}
\end{cases}
\end{align*}
\]

(A18)  
(A19)

Accordingly we obtain

\[
\begin{align*}
E_{v,j+1/2}^{(L)} &= \varepsilon_v^{(L)} F_{v,j+1/2}^{(L)*)}, \\
F_{v,j+1/2}^{(L)} &= \varepsilon_v^{(L)} F_{v,j+1/2}^{(L)*)}, \\
E_{v,j+1/2}^{(R)} &= \varepsilon_v^{(R)} F_{v,j+1/2}^{(R)*)}, \\
F_{v,j+1/2}^{(R)} &= \varepsilon_v^{(R)} F_{v,j+1/2}^{(R)*)}.
\end{align*}
\]

(A20)  
(A21)  
(A22)  
(A23)

We use \( E_{v,j+1/2}^{(L)} \) and \( F_{v,j+1/2}^{(L)} \) for computing the energy flux from cell \( j \) to \( j+1 \), and \( E_{v,j+1/2}^{(R)} \) and \( F_{v,j+1/2}^{(R)} \) for that from \( j+1 \) to \( j \).

Emission and scattering within a numerical cell are included in the second-order numerical flux by the following procedure. First we evaluate the the blackbody emission on the cell boundary, \( B_{v,j+1/2}^{(L)} \) and \( B_{v,j+1/2}^{(R)} \), by using the MUSCL approach:

\[
\begin{align*}
B_{v,j+1/2}^{(L)} &= B_{v,j} + \frac{1}{2} \Delta B_{v,j+1/2}^{(L)}, \\
B_{v,j+1/2}^{(R)} &= B_{v,j+1} - \frac{1}{2} \Delta B_{v,j+1/2}^{(R)}. \\
\Delta B_{v,j+1/2}^{(L)} &= \min \left( \Delta B_{v,j+1/2}, \left| \Delta B_{v,j+1/2} \right| \right) \times \left[ \text{sgn} \left( \frac{1}{2} \Delta B_{v,j+1/2} \right) + \text{sgn} \left( \frac{1}{2} \Delta B_{v,j-1/2} \right) \right], \\
\Delta B_{v,j+1/2}^{(R)} &= \min \left( \Delta B_{v,j+1/2}, \left| \Delta B_{v,j+1/2} \right| \right) \times \left[ \text{sgn} \left( \frac{1}{2} \Delta B_{v,j+1/2} \right) + \text{sgn} \left( \frac{1}{2} \Delta B_{v,j+3/2} \right) \right].
\end{align*}
\]

(A24)  
(A25)  
(A26)  
(A27)

We assume that \( B_{v} \) is a linear function of the optical depth between the cell center and the boundary. We then obtain

\[
\int_{x_j}^{x_{j+1}} \kappa_{v,a} \rho_j B_v(x') \, dx' = B_{v,j+1/2}^{(L)} \left( 1 - \frac{1 - e^{-w_{v,j}}}{w_{v,j}} \right) + B_{v,j} \left( \frac{1 - e^{-w_{v,j}}}{w_{v,j}} - e^{-w_{v,j}} \right),
\]

(A28)

where

\[
w_{v,j} = \kappa_{v,a} \rho \Delta x_j.
\]

(A29)

The symbol \( \Delta x_j \) denotes the cell width. Similarly, we can evaluate the scattering within the cell.

By taking the emission and scattering within the cell, the numerical flux of the second-order accuracy is expressed as

\[
\begin{align*}
F_{v,j+1/2} &= \xi_v F_{v,j+1/2}^{(L)} + \eta_v \pi B_{v,j+1/2}^{(L)} + \eta_v'' B_{v,j} + \eta_v \pi B_{v,j} + \eta_v \left[ \xi_v \frac{c E_{v,j+1/2}^{(L)}}{4} + \xi_v'' \frac{c E_{v,j}}{4} \right] \\
&\quad + \xi_v F_{v,j+1/2}^{(R)} + \eta_v \pi B_{v,j+1/2}^{(R)} - \eta_v'' B_{v,j+1/2} - \eta_v' B_{v,j+1} - \eta_v \pi B_{v,j+1} \left[ \xi_v \frac{c E_{v,j+1/2}^{(R)}}{4} + \xi_v'' \frac{c E_{v,j+1}}{4} \right].
\end{align*}
\]

(A30)

\[
\begin{align*}
P_{v,i,j+1/2} &= \xi_v P_{v,i,j+1/2}^{(L)} + \eta_v \frac{2\pi}{3c} B_{v,i,j+1/2}^{(L)} + \eta_v' \frac{2\pi}{3c} B_{v,j} + \eta_v \frac{2\pi}{3c} B_{v,j} + \eta_v \left[ \xi_v \frac{E_{v,j+1/2}^{(L)}}{6} + \xi_v'' \frac{E_{v,j}}{6} \right] \\
&\quad + \xi_v P_{v,i,j+1/2}^{(R)} + \eta_v \frac{2\pi}{3c} B_{v,i,j+1/2}^{(R)} + \eta_v' \frac{2\pi}{3c} B_{v,j} + \eta_v \frac{2\pi}{3c} B_{v,j} + \eta_v' \left[ \xi_v \frac{E_{v,j+1/2}^{(R)}}{6} + \xi_v'' \frac{E_{v,j+1}}{6} \right].
\end{align*}
\]

(A31)
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\[ P_{v,i,k,j+1/2} = \eta_{v,j} P_{v,i,k,j+1/2}^{(L)} + \eta_{v,j+1} P_{v,i,k,j+1/2}^{(R)} \quad \text{if } i \neq k, \]  
(A32)

\[ \eta_{v,j} = e^{-w_{v,j}}, \]  
(A33)

\[ \eta'_{v,j} = \left(1 - \frac{1 - e^{-w_{v,j}}}{w_{v,j}}\right), \]  
(A34)

\[ \eta''_{v,j} = \frac{1 - e^{-w_{v,j}} - e^{-w_{v,j}}}{w_{v,j}} \]  
(A35)

\[ \zeta_{v,j} = e^{-w_{v,j} - s_{v,j}}, \]  
(A36)

\[ \zeta'_{v,j} = \frac{1 - e^{-s_{v,j}}}{s_{v,j}}, \]  
(A37)

\[ \zeta''_{v,j} = \left(1 - \frac{1 - e^{-s_{v,j}}}{s_{v,j}^2} \right), \]  
(A38)

\[ s_{v,j} = \frac{1}{2} k_{v,x,j} \rho_j \Delta x_j. \]  
(A39)

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