On equivalence of star-products in arbitrary canonical coordinates

Ziemowit Domański and Maciej Błaszak

Faculty of Physics, Division of Mathematical Physics, Adam Mickiewicz University
Umultowska 85, 61-614 Poznań, Poland
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We present a proof that certain star-products defined on Poisson manifolds and written in a given classical and quantum canonical coordinate system are uniquely equivalent with a Moyal product associated with this coordinate system. The equivalence is assumed to satisfy some additional conditions which guarantee the uniqueness of the equivalence. Moreover, the systematic construction of such equivalence is presented.

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One of the admissible methods of quantization of a classical Hamiltonian system is a deformation quantization procedure. In this procedure one deforms a classical Poisson algebra $A = (C^\omega(M), \cdot, \{\cdot, \cdot\})$ to an appropriate noncommutative algebra $A_Q = (C^\omega(M)[\hbar], \star, \{\cdot, \cdot\})$, where $C^\omega(M)[\hbar]$ is a space of formal power series in $\hbar$ with coefficients from the space $C^\omega(M)$ of analytic functions on $M$, $\star$ is a noncommutative associative product being an $\hbar$-deformation of a point-wise product $\cdot$, and $\{\cdot, \cdot\}$ is a Lie bracket being an $\hbar$-deformation of a Poisson bracket $\{\cdot, \cdot\}$ [1–3]. We will consider only the analytic case, i.e. we will be considering the space of analytic functions $C^\omega(M)$ instead of the space of smooth functions $C^\infty(M)$. The natural question which appears is how to pass to a standard operator representation of quantum mechanics. It is well known how to do this for a Moyal star-product [2, 3] but a case of a general star-product constitutes some problems. However, if for a classical and quantum canonical coordinate system the star-product is equivalent with a Moyal product then the problem reduces to the Moyal case [4, 5]. It can be proved that when the Poisson manifold $(M, \mathcal{P})$ ($\mathcal{P}$ being a Poisson tensor) is contractible to a point, then for a given classical and quantum canonical coordinate system the differential star-product will be equivalent with a Moyal product [6, Proposition 18]. In this paper we will prove that for a Poisson manifold (not necessarily contractible to a point) and with certain additional assumptions on the star-product the resulting star-product will be uniquely equivalent with a Moyal product.

Let us assume that $(M, \mathcal{P})$ is a $2N$-dimensional Poisson manifold. Let us consider the following star-product

$$f \star g = \sum_{k=0}^{\infty} \left( \frac{i\hbar}{2} \right)^k C_k(f, g),$$

(1)

where $C_k$ are bidifferential operators with following properties:

(i) $C_0(f, g) = fg$,

(ii) $C_1(f, g) = \{f, g\}$,

(iii) $C_k(f, g) = (-1)^k C_k(g, f)$,

(iv) $C_k(\bar{f}, \bar{g}) = C_k(f, g)$,

(v) $\sum_{l=0}^{k} (C_l(C_{k-l}(f, g), h) - C_l(f, C_{k-l}(g, h))) = 0$,

(vi) $C_k(f, 1) = C_k(1, f) = 0$ for $k \in \mathbb{Z}_+$.

Let us define the deformed Poisson bracket by the formula

$$[f, g]_\star = \frac{1}{i\hbar}[f, g] - \frac{1}{i\hbar}(f \star g - g \star f).$$

(2)

The $\star$-product and the deformed Poisson bracket have the following properties:

* Electronic address: ziemowit@amu.edu.pl
† Electronic address: blaszakm@amu.edu.pl
(a) \( f \star g = fg + o(\hbar) \),
(b) \([f, g] = \{f, g\} + o(\hbar)\),
(c) \( \overline{f \star g} = \overline{g} \star \overline{f} \),
(d) \( f \star (g \star h) = (f \star g) \star h \) (associativity),
(e) \( f \star 1 = 1 \star f = f \).

Property (a) follows from (i), property (b) follows from (ii) and (iii), property (c) follows from (iv) and (iii), property (d) is a result of (v), and property (e) follows from (vi).

An example of the star-product (1) used in a quantization procedure is a product of the form

\[
\begin{align*}
f \star g &= f \exp \left( \frac{1}{2 i \hbar} \mathcal{J}^{ij} \overrightarrow{D}_i \overrightarrow{D}_j \right) g \\
&= \sum_{k=0}^{\infty} \frac{1}{k!} \left( \frac{i \hbar}{2} \right)^k \mathcal{J}^\mu_1 \cdots \mathcal{J}^\mu_k (D^\mu_1 \cdots D^\mu_k f)(D_{\mu_1} \cdots D_{\mu_k} g),
\end{align*}
\]

where

\[
(\mathcal{J}^{ij}) = \begin{pmatrix} 0_N & \mathbb{I}_N \\ -\mathbb{I}_N & 0_N \end{pmatrix}
\]

and \( D_1, \ldots, D_{2N} \) are globally defined pair-wise commuting vector fields such that

\[
\mathcal{P} = \mathcal{J}^{ij} D_i \otimes D_j.
\]

For a Poisson manifold contractible to a point such vector fields always exist. The sequence of vector fields \( D_1, \ldots, D_{2N} \) is not uniquely specified by the condition (5) but there exists the whole family of such sequences. Every such sequence of vector fields defines a star-product of the form (3). Thus there exists the whole family of star-products (3) associated to the same Poisson tensor \( \mathcal{P} \). All these star-products are related to each other by symplectomorphisms of the Poisson manifold \((M, \mathcal{P})\), i.e. if \( \star \) and \( \star' \) are star-product (3) induced by sequences of vector fields \( D_1, \ldots, D_{2N} \) and \( D'_1, \ldots, D'_{2N} \) then there exists a symplectomorphism \( T \) such that

\[
(D_i f) \circ T = D'_i (f \circ T),
\]

from which follows that

\[
(f \star g) \circ T = (f \circ T) \star' (g \circ T).
\]

If, in particular, we choose some classical canonical coordinate system \((z^1, \ldots, z^{2N})\) on \( M \) then a star-product (3) induced by coordinate vector fields \( \partial_{z^1}, \ldots, \partial_{z^{2N}} \) is a Moyal product in these coordinates and all other star-products from the family (3) are related to the Moyal product by a classical canonical coordinate transformation \( T \). More details of the use of the star-product (3) in a quantization procedure will be the topic of our forthcoming paper.

Let \((z^1, \ldots, z^{2N})\) be a coordinate system on \( M \), which is classically and quantum canonical, i.e.

\[
\{z^i, z^j\} = \mathcal{J}^{ij},
\]

\[
[z^i, z^j] = \mathcal{J}^{ij}.
\]

We can write the \( \star \)-product in this coordinates receiving a new product denoted hereafter by \( \star^{(z)} \). In what follows we will prove the main theorem of the paper.

**Theorem 1.** There exists a unique isomorphism \( S \) of \( C^\omega(M)[[\hbar]] \) of the form

\[
S = \text{id} + \sum_{k=1}^{\infty} \hbar^k S_k,
\]

where \( S_k \) are differential operators on \( C^\omega(M)[[\hbar]] \), such that

\[
S(f \star^{(z)} g) = Sf \star^{(z)} Sg,
\]

\[
S_z^i = z^i,
\]

\[
\overline{S_f} = \overline{S_f}.
\]
By \( z^i \) we denote the Moyal product for the coordinates \((z^1, \ldots, z^{2N})\), i.e. a star-product of the form

\[
f \star_M(z^i) g = f \exp \left( \frac{1}{2i} \hbar \mathcal{J}^i \partial_{z^i} \right) g.
\]  

(11)

Before we present the proof of the theorem we will prove a couple of technical lemmas. First let us prove the following lemma:

**Lemma 1.** Equation (10) are equivalent with the following equations

\[
S(z^i \star_M^f) = z^i \star(z^i) Sf,
\]

(12a)

\[
Sz^i = z^i,
\]

(12b)

\[
\bar{S}f = Sf.
\]

(12c)

**Proof.** Indeed, if the conditions (10) are fulfilled then trivially the conditions (12) are fulfilled. Assume now, that the conditions (12) are fulfilled. From (12) it follows that (10a) will be satisfied for every \( f \) in the form of a \( \star^z_M \)-polynomial. For example when \( f = z^i \star_M^z z^j \) then

\[
S(f \star_M^z g) = S(z^i \star_M^z z^j ) = z^i \star(z^i) S(z^j ) = z^i \star(z^i) S(z^j ) Sg = S(z^i \star(z^i) Sg.
\]

(13)

Every polynomial can be written as a \( \star_M^z \)-polynomial. If \( f \) is a general function from \( C^\omega(M)[\hbar] \) then it can be expanded into a power series. Since (10a) holds for every term of the series it will hold for the function \( f \). □

The operator \( z^i \star_M^z \) takes the form

\[
z^i \star_M^z = z^i + \frac{1}{2} i \hbar \partial^i,
\]

(14)

and the operator \( z^i \star(z^i) \) can be written in the form

\[
z^i \star(z^i) = z^i + \frac{1}{2} i \hbar \partial^i + \sum_{k=2}^\infty \left( \frac{i \hbar}{2} \right)^k A_i^k,
\]

(15)

where \( \partial^i = \mathcal{J}^i \partial_{z^i} \) and \( A_i^k f = C_k(z^i, f) \).

**Lemma 2.** Let \( S = \sum_{k=0}^\infty \hbar^k S_k \), where \( S_0 = \text{id} \). Then \( S \) will satisfy (12) iff

\[
[S_{2k}, z^i] = \sum_{l=1}^k \left( -\frac{1}{4} \right)^l A_{2l}^i S_{2(k-l)},
\]

(16a)

\[
[S_{2k}, \partial^i] = \sum_{l=1}^k \left( -\frac{1}{4} \right)^l A_{2l+1}^i S_{2(k-l)},
\]

(16b)

and \( S_{2k-1} = 0 \) for \( k \in \mathbb{Z}_+ \).

**Proof.** Equation (12a) takes the form

\[
\sum_{k=0}^\infty \hbar^k S_k z^i + \sum_{k=0}^\infty \frac{1}{2} i \hbar^{k+1} S_k \partial^i = \sum_{k=0}^\infty \hbar^k z^i S_k + \sum_{k=0}^\infty \frac{1}{2} i \hbar^{k+1} \partial^i S_k + \sum_{k=0}^\infty \sum_{l=2}^\infty \left( \frac{i}{2} \right)^l \hbar^{k+l} A_i^l S_k.
\]

(17)

Regrouping terms with even and odd \( k \) and \( l \) in the last term in the above equation we get

\[
\sum_{k=0}^\infty \hbar^k [S_k, z^i] + \frac{1}{2} i \sum_{k=0}^\infty \hbar^{k+1} [S_k, \partial^i] = \sum_{n=0}^\infty \sum_{l=1}^\infty \hbar^{2n+2l} \left( -\frac{1}{4} \right)^l A_{2l}^i S_{2n} + \sum_{n=0}^\infty \sum_{l=1}^\infty \hbar^{2n+2l+1} \left( -\frac{1}{4} \right)^l A_{2l+1}^i S_{2n+1}
\]

\[
+ \frac{1}{2} i \sum_{n=0}^\infty \sum_{l=1}^\infty \hbar^{2n+2l+1} \left( -\frac{1}{4} \right)^l A_{2l+1}^i S_{2n} + \frac{1}{2} i \sum_{n=0}^\infty \sum_{l=1}^\infty \hbar^{2n+2l+2} \left( -\frac{1}{4} \right)^l A_{2l+1}^i S_{2n+1}.
\]

(18)
Regrouping terms with even and odd \( k \) in the left hand side of the above formula and replacing the summation over \( n \) and \( l \) by a summation over \( k = n + l \) and \( l \) we receive

\[
\sum_{k=0}^{\infty} \hbar^{2k} [S_{2k}, z^j] + \sum_{k=0}^{\infty} \hbar^{2k+1} [S_{2k+1}, z^j] + \frac{1}{2} \sum_{k=0}^{\infty} \hbar^{2k+1} [S_{2k}, \partial^i] + \frac{1}{2} \sum_{k=0}^{\infty} \hbar^{2k+2} [S_{2k+1}, \partial^i] =
\]

\[
= \sum_{k=1}^{\infty} \hbar^{2k} \sum_{l=1}^{k} \left( -\frac{1}{4} \right)^l A_{2l}^i S_{2(k-l)} + \sum_{k=1}^{\infty} \hbar^{2k+1} \sum_{l=1}^{k} \left( -\frac{1}{4} \right)^l A_{2l+1}^i S_{2(k-l)+1} + \frac{1}{2} \sum_{k=1}^{\infty} \hbar^{2k+1} \sum_{l=1}^{k} \left( -\frac{1}{4} \right)^l A_{2l+1}^i S_{2(k-l)+1}.
\]

Comparing terms with the same order in \( \hbar \) and using (12c) we get the following recursive equations for \( S_k \)

\[
[S_{2k}, z^j] = \sum_{l=1}^{k} \left( -\frac{1}{4} \right)^l A_{2l}^i S_{2(k-l)}, \quad (20a)
\]

\[
[S_{2k}, \partial^i] = \sum_{l=1}^{k} \left( -\frac{1}{4} \right)^l A_{2l+1}^i S_{2(k-l)}, \quad (20b)
\]

and

\[
[S_{2k+1}, z^j] = \sum_{l=1}^{k} \left( -\frac{1}{4} \right)^l A_{2l}^i S_{2(k-l)+1}, \quad (21c)
\]

\[
[S_{2k+1}, \partial^i] = \sum_{l=1}^{k} \left( -\frac{1}{4} \right)^l A_{2l+1}^i S_{2(k-l)+1}, \quad (21d)
\]

for \( k \in \mathbb{Z}_+ \). From (21a) and (21b) we get that \( S_1 = \text{const} \), and by virtue of (12b) this implies that \( S_1 = 0 \). Thus, from (21c) and (21d), we get that \( S_{2k+1} = 0, \ k \in \mathbb{Z}_+ \).

Equations (16) can be used to recursively calculate the isomorphism \( S \) order by order in \( \hbar \). Note that if the system of equations (16) has a solution then this solution is unique up to an additive constant (this can be seen from the fact that, since \( A_k \) and \( S_k \) are differential operators, the solution of (16a) is specified up to an additive function which is determined by (16b) up to an additive constant). By virtue of (12b) this constant have to be equal 0.

Let us derive the condition on a coordinate system \((z^1, \ldots, z^{2N})\), which has to be satisfied to make it a classical and quantum canonical coordinate system.

**Lemma 3.** A coordinate system \((z^1, \ldots, z^{2N})\) is classical and quantum canonical iff

\[
C_1(z^i, z^j) = \mathcal{J}^{ij}, \quad (22a)
\]

\[
C_k(z^i, z^j) = 0, \quad k = 3, 5, \ldots, \quad (22b)
\]

for every \( i, j = 1, \ldots, 2N \).

**Proof.** From (8a) and (ii) we get (22a). In accordance with (8b) a coordinate system \((z^1, \ldots, z^{2N})\) is a quantum canonical coordinate system iff

\[
[z^i, z^j] = z^i \star z^j - z^j \star z^i = i\hbar \mathcal{J}^{ij}.
\]

The above condition can be written in the form

\[
\sum_{k=0}^{\infty} \left( \frac{i\hbar}{2} \right)^k (C_k(z^i, z^j) - C_k(z^j, z^i)) = i\hbar \mathcal{J}^{ij}.
\]
The above equation is equivalent with the following system of equations

\[
\frac{1}{2} \left( C_1(z^i, z^j) - C_1(z^j, z^i) \right) = J^{ij}, \tag{25a}
\]

\[
C_k(z^i, z^j) = C_k(z^j, z^i), \quad k = 2, 3, \ldots \tag{25b}
\]

Equation (25a) is satisfied due to classical canonicity of the coordinate system. Equation (25b) due to (iii) can be rewritten in the form

\[
C_k(z^i, z^j) = (-1)^k C_k(z^i, z^j). \tag{26}
\]

The above formula is automatically satisfied for even \(k\), and for odd \(k\) we get the condition (22b). \(
\)

To prove Theorem 1 we have to prove that (16) have a solution. Before doing this let us prove the following lemmas.

**Lemma 4.** A system of equations

\[
[B, z^i] = F^i, \quad i = 1, 2, \ldots, 2N, \tag{27a}
\]

\[
[B, \partial^i] = G^i, \quad i = 1, 2, \ldots, 2N, \tag{27b}
\]

where \(F^i = \sum_{n \geq 0} f_{n, \mu_1, \ldots, \mu_n}(z) \partial^{\mu_1} \cdots \partial^{\mu_n}\) and \(G^i = \sum_{n \geq 0} g_{n, \mu_1, \ldots, \mu_n}(z) \partial^{\mu_1} \cdots \partial^{\mu_n}\) are some differential operators, have a solution \(B\) iff

\[
[F^i, z^j] = [F^j, z^i], \tag{28a}
\]

\[
[F^i, \partial^j] = [G^j, z^i], \tag{28b}
\]

\[
[G^i, \partial^j] = [G^j, \partial^i] \tag{28c}
\]

for all \(i, j = 1, 2, \ldots, 2N\).

**Proof.** First let us assume that (27) have a solution. From Jacobi’s identity we have that

\[
[[B, z^i], \partial^j] + [[z^i, \partial^j], B] + [[\partial^j, B], z^i] = 0, \tag{29}
\]

from which follows that

\[
[[B, z^i], \partial^j] = [[B, \partial^j], z^i]. \tag{30}
\]

Using (27) from this we receive (28b). Equations (28a) and (28c) can be received analogically.

Now, let us assume that (28a) is satisfied. From the form of \(F^i\) it can be easily seen that (27a) for \(i = 1\) have a solution. Assume that for some \(m \geq 1\) (27a) have a solution for all \(i \leq m\). This solution is not unique but there exists a family of solutions such that if \(B\) and \(B'\) are solutions of (27a) for all \(i \leq m\) then there exists an operator \(H^{(m)}\) such that \(B' = B + H^{(m)}\) and \([H^{(m)}, z^i] = 0\) for all \(i = 1, 2, \ldots, m\). From (28a) and (27a) we have that

\[
[[B, z^i], z^{m+1}] = [F^{m+1}, z^i], \quad i = 1, 2, \ldots, m. \tag{31}
\]

Using Jacobi’s identity the above equation takes the form

\[
[[B, z^{m+1}], z^i] = [F^{m+1}, z^i], \quad i = 1, 2, \ldots, m. \tag{32}
\]

From this follows that

\[
[B, z^{m+1}] = F^{m+1} + H^{(m)}, \quad i = 1, 2, \ldots, m \tag{33}
\]

for some operator \(H^{(m)}\) such that \([H^{(m)}, z^i] = 0\) for all \(i = 1, 2, \ldots, m\). From the freedom of the solution \(B\) there exists \(B\) for which \(H^{(m)} = 0\). Hence for all \(i \leq m + 1\) (27a) have a solution. Thus we inductively proved that (27a) have a solution for all \(i = 1, 2, \ldots, 2N\).

Now, let us assume that (28) is satisfied. As was shown above (27a) have a solution. This solution is not unique but there exists a family of solutions such that if \(B\) and \(B'\) are solutions of (27a) then there exists an operator \(H\) such that \(B' = B + H\) and \([H, z^i] = 0\) for all \(i = 1, 2, \ldots, 2N\). From (28b) and (27a) we have that

\[
[[B, z^i], \partial^j] = [G^j, z^i], \quad i = 1, 2, \ldots, 2N. \tag{34}
\]
Using Jacobi’s identity the above equation takes the form
\[
[[B, \partial^1], z^i] = [G^1, z^i], \quad i = 1, 2, \ldots, 2N. \tag{35}
\]
From this follows that
\[
[B, \partial^1] = G^1 + H^{(1)}, \tag{36}
\]
for some operator \(H^{(1)}\) such that \([H^{(1)}, z^i] = 0\) for all \(i = 1, 2, \ldots, 2N\). From the freedom of the solution \(B\) there exists \(B\) for which \(H^{(1)} = 0\). Hence we have shown that there exists a solution to the system of equations
\[
[B, z^i] = F^i, \quad i = 1, 2, \ldots, 2N, \tag{37a}
\]
\[
[B, \partial^1] = G^1. \tag{37b}
\]
This solution is specified up to an operator \(H\) such that \([H, z^i] = 0\) for all \(i = 1, 2, \ldots, 2N\) and \([H, \partial^j] = 0\). Assume now that for \(m \geq 1\) there exists a solution \(B\) to the system of equations
\[
[B, z^i] = F^i, \quad i = 1, 2, \ldots, 2N, \tag{38a}
\]
\[
[B, \partial^j] = G^j, \quad j = 1, 2, \ldots, m, \tag{38b}
\]
specified up to an operator \(H\) such that \([H, z^i] = 0\) for \(i = 1, 2, \ldots, 2N\) and \([H, \partial^j] = 0\) for \(j = 1, 2, \ldots, m\). From (28b) and (27a) we have that
\[
[[B, z^i], \partial^{m+1}] = [G^{m+1}, z^i], \quad i = 1, 2, \ldots, 2N. \tag{39}
\]
Using Jacobi’s identity the above equation takes the form
\[
[[B, \partial^{m+1}], z^i] = [G^{m+1}, z^i], \quad i = 1, 2, \ldots, 2N. \tag{40}
\]
From this follows that
\[
[B, \partial^{m+1}] = G^{m+1} + H^{(m)}, \tag{41}
\]
for some operator \(H^{(m)}\) such that \([H^{(m)}, z^i] = 0\) for all \(i = 1, 2, \ldots, 2N\). Moreover, \(H^{(m)}\) satisfies: \([H^{(m)}, \partial^j] = 0\) for all \(j = 1, 2, \ldots, m\). Indeed,
\[
[[B, \partial^{m+1}], \partial^j] = [G^{m+1}, \partial^j] + [H^{(m)}, \partial^j], \quad j = 1, 2, \ldots, m, \tag{42}
\]
from which follows, by virtue of Jacobi’s identity and (28c), that
\[
[[B, \partial^j], \partial^{m+1}] = [G^j, \partial^{m+1}] + [H^{(m)}, \partial^j], \quad j = 1, 2, \ldots, m. \tag{43}
\]
Since \(B\) satisfies (38b) we receive that
\[
[H^{(m)}, \partial^j] = 0, \quad j = 1, 2, \ldots, m. \tag{44}
\]
From the freedom of the solution \(B\) there exists \(B\) for which \(H^{(m)} = 0\). Hence (38) have a solution for \(j \leq m + 1\). Thus we inductively proved that (27) have a solution for all \(i = 1, 2, \ldots, 2N\).

From Lemma 4 we get:

**Lemma 5.** The system of equations (16) for \(k \in \mathbb{Z}_+\) have a solution iff

\[
\sum_{l=0}^{k} [A^i_{2l}, A^j_{2(k-l)}] = 0, \tag{45a}
\]
\[
\sum_{l=0}^{k} [A^i_{2l+1}, A^j_{2(k-l)}] = 0, \tag{45b}
\]
\[
\sum_{l=0}^{k} [A^i_{2l+1}, A^j_{2(k-l)+1}] = 0, \tag{45c}
\]

for all \(i, j = 1, 2, \ldots, 2N\).
Proof. We will prove the lemma by induction. Directly from Lemma 4 follows that for \( k = 1 \) the assumption of the lemma is true. Assume that for \( k = 1, 2, \ldots, K \) where \( K \geq 1 \) the assumption of the lemma holds. From Lemma 4 the system of equations (16) for \( k = K + 1 \) have a solution iff

\[
\sum_{l=1}^{K+1} \left( -\frac{1}{4} \right)^l A^i_{2l} S_{2(2K+1-l), A^j_0} = \sum_{l=1}^{K+1} \left( -\frac{1}{4} \right)^l A^j_{2l} S_{2(2K+1-l), A^i_0},
\]

(46a)

\[
\sum_{l=1}^{K+1} \left( -\frac{1}{4} \right)^l A^i_{2l+1} S_{2(2K+1-l), A^j_0} = \sum_{l=1}^{K+1} \left( -\frac{1}{4} \right)^l A^j_{2l+1} S_{2(2K+1-l), A^i_0},
\]

(46b)

\[
\sum_{l=1}^{K+1} \left( -\frac{1}{4} \right)^l A^i_{2l+1} S_{2(2K+1-l), A^j_0} = \sum_{l=1}^{K+1} \left( -\frac{1}{4} \right)^l A^j_{2l+1} S_{2(2K+1-l), A^i_0}.
\]

(46c)

Equation (46a), by virtue of the Leibniz’s rule, is equivalent with the following equation

\[
\sum_{l=1}^{K+1} \left( -\frac{1}{4} \right)^l [A^i_{2l}, A^j_0] S_{2(2K+1-l)} + \sum_{l=1}^{K+1} \left( -\frac{1}{4} \right)^l A^i_{2l} [S_{2(2K+1-l)}, A^j_0] = \sum_{l=1}^{K+1} \left( -\frac{1}{4} \right)^l [A^j_{2l}, A^i_0] S_{2(2K+1-l)}
\]

\[
+ \sum_{l=1}^{K+1} \left( -\frac{1}{4} \right)^l A^j_{2l} [S_{2(2K+1-l)}, A^i_0].
\]

(47)

Using (16a) we have that

\[
\sum_{l=1}^{K+1} \left( -\frac{1}{4} \right)^l A^i_{2l} [S_{2(2K+1-l)}, A^j_0] = \sum_{l=1}^{K+1} \sum_{r=1}^{K+1-l} \left( -\frac{1}{4} \right)^l A^j_{2l} A^i_{2r} S_{2(2K+1-l-r)}
\]

\[
= \sum_{n=2}^{K+1} \sum_{r=1}^{n-1} \left( -\frac{1}{4} \right)^n A^j_{2(n-r)} A^i_{2r} S_{2(2K+1-n)}.
\]

(48)

Using (48) and (45a) for \( k = 1 \), (47) can be rewritten in the form

\[
\sum_{l=2}^{K+1} \left( -\frac{1}{4} \right)^l \left( [A^j_l, A^i_{2l}] + \sum_{r=1}^{l-1} [A^j_{2r}, A^i_{2(l-r)}] + [A^j_{2l}, A^i_0] \right) S_{2(2K+1-l)} = 0,
\]

(49)

which in turn can be written as

\[
\sum_{l=2}^{K+1} \left( -\frac{1}{4} \right)^l \sum_{r=0}^{l-1} [A^j_{2r}, A^i_{2(l-r)}] S_{2(2K+1-l)} = 0.
\]

(50)

Using the inductive assumption the above equation reduces to

\[
\sum_{r=0}^{K+1} [A^j_{2r}, A^i_{2(K+1-r)}] = 0,
\]

(51)

which proves that (46a) is equivalent with (45a) for \( k = K + 1 \). Analogically we prove that (46b) and (46c) are equivalent with (45b) and (45c). This ends the induction.

Now we are ready to prove Theorem 1.

Proof of Theorem 1. We have to show that the system of equations (16) have a solution. From Lemma 5 it is enough to show that (45) holds. From (v) we get

\[
\sum_{l=0}^{2k} A^i_l (A^j_{2k-l} f) = \sum_{l=0}^{2k} C_l(z^i, C_{2k-l}(z^j, f)) = \sum_{l=0}^{2k} C_l(C_{2k-l}(z^i, z^j), f)
\]

\[
= \sum_{l=0}^{k} C_{2l}(C_{2(k-l)}(z^i, z^j), f) + \sum_{l=0}^{k-1} C_{2l+1}(C_{2(k-l)-1}(z^i, z^j), f).
\]

(52)
The second term in the last equality in (52) vanishes because of the classical and quantum canonicity condition (Lemma 3). Hence, with the use of (iii) equation (52) reduces to

$$
\sum_{l=0}^{2k} A_j^i(A_{2k-l}^j f) = \sum_{l=0}^{k} C_{2l}(C_{2l}(z^i, z^i), f) = \sum_{l=0}^{k} C_{2l}(z^i, z^i), f) = \sum_{l=0}^{2k} A_{2k-l}^j(A_l^i f). \tag{53}
$$

Thus we get that

$$
\sum_{l=0}^{2k} [A_l^i, A_{2k-l}^j] = 0. \tag{54}
$$

Analogically we get that

$$
\sum_{l=0}^{2k+1} [A_l^i, A_{2k-l+1}^j] = 0. \tag{55}
$$

On the other hand from (v) we have that

$$
\sum_{l=0}^{k} C_{l}(C_{k-l}(z^j, f), z^i) = \sum_{l=0}^{k} C_{l}(z^j, C_{k-l}(f, z^i)), \tag{56}
$$

which can be rewritten in the form

$$
\sum_{l=0}^{k} (-1)^l A_j^i(A_{k-l}^j f) = \sum_{l=0}^{k} (-1)^l A_j^i(A_{k-l}^i f) = \sum_{l=0}^{k} (-1)^l A_{k-l}^j(A_l^i f). \tag{57}
$$

Thus we get that

$$
\sum_{l=0}^{2k} (-1)^l [A_l^i, A_{2k-l}^j] = 0, \tag{58a}
$$

$$
\sum_{l=0}^{2k+1} (-1)^l [A_l^i, A_{2k-l+1}^j] = 0. \tag{58b}
$$

By adding (54) to (58a) we receive (45a) and by subtracting them we get (45c). By adding or subtracting (55) to (58b) we receive (45b).

In the paper we have presented a proof that every star-product of the form (1) on a Poisson manifold is, for any classical and quantum canonical coordinate system, uniquely equivalent with a Moyal product for these coordinates, where the equivalence satisfies the relations (10). Furthermore, formulas (16) can be used for a systematic construction of this equivalence order by order in $\hbar$.

We believe that when a coordinate system is not classical canonical but only quantum canonical the resulting equivalence of the star-product (1) with a Moyal product should still exists, be unique, and satisfy (10), but the proof of this fact remains an open problem. In such case the coordinate system will depend on $\hbar$ and the presented procedure cannot be used.

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