Geodesic Motion on Closed Spaces

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Two manifolds with genus $g = 0$ are given. In one the geodesic motion is apparently integrable. In the second example the geodesic flow, shows the presence of chaotic regions.

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I. INTRODUCTION

Anosov’s famous result, states that the geodesic flow is chaotic in a compact manifold of constant negative curvature [1]. Anosov flows are very chaotic being not only mixing, but even Bernoullian [2]. For instance, the Poincaré section shows the absence of KAM tori [3]. For a review, see [4].

On the other hand integrability or not of a given mechanical system remains as an non answered question. Since Krylov’s work [5] many researchers have transformed the mechanical problem of the motion of a particle in a given potential into a billiard problem [6]. This is achieved by writing the Jacobi metric associated to the given potential. In this approach, the motion is geodesic. The Jacobi manifold is specifically obtained to incorporate the effects of the fields.

In the cosmological context it is the geometry itself the more fundamental field. The geodesic motion of particles, follow directly from the covariant divergence of the energy momuntum tensor [7]. A few years ago Cornish et al suggested that the chaotic motion of particles in a closed negatively curved manifold as a possible mechanism responsible for the homogenization of the Universe [8].

In 1839 it was discovered by Jacobi itself, that the geodesic motion on an ellipsoid is integrable [9]. Jacobi used a particular coordinate system [10], now known as Jacobi elliptical

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coordinates, and obtained an independent, and involutive constant of the motion. In 1994, Knieper and Weiss \[11\] proved that there are many smooth Riemannian metrics on $S^2$ with chaotic geodesic flows. The authors consider arbitrary small deviations of the metric on the ellipsoid. Then, the Melnikov method is used to prove the existence of an homoclinic point.

The purpose of this article is to provide a concrete example of two manifolds topologically equivalent to a sphere. The geodesics are investigated using the technique of the Poincaré surface of section. Section \[\text{III}\] includes a very brief discussion of the Gauss-Bonnet theorem, for more details see \[12\]. In section \[\text{II}\] the geodesic motion of the given manifold is apparently integrable. In section \[\text{III}\] the geodesic flow on the manifold shows the presence of chaotic regions, which are related to the domains of negative curvature.

Anyway, in the conclusions it is stressed that a closed space with domains of negative curvature is not a mandatory condition for chaotic geodesic motion.

II. AN EVERYWHERE POSITIVE CURVATURE SPACE

This manifold is obtained as the immersion of a closed surface in the Euclidean space $E^3$. As is well known, the spherical harmonics $Y_i^m(\theta, \phi)$ form a complete base for any function defined on $S^2$. In this work, the following class of surfaces

$$r = 5 + aY_3^3(\theta, \phi)$$

$$r = 5 - a\frac{\sqrt{35}\sin(3\phi)\sin(\theta)^3}{8\sqrt{\pi}} \quad (1)$$

is considered. When $a = 0$, it corresponds to the usual $S^2$. For small values of $a$, it provides deformations of $S^2$. In this section the particular value $a = 1$ is chosen, as shown in Fig. \[\text{I}\]

The line element is the usual one

$$ds^2 = dr^2 + r^2(d\theta^2 + \sin(\theta)^2 d\phi^2)$$

$$ds^2 = \left(\frac{\partial}{\partial \theta} rd\theta + \frac{\partial}{\partial \phi} rd\phi\right)^2 + \left(5 - a\frac{\sqrt{35}\sin(3\phi)\sin(\theta)^3}{8\sqrt{\pi}}\right)^2 (d\theta^2 + \sin(\theta)^2 d\phi^2),$$
FIG. 1: The manifold, the parameter $a = 1$ in (1) is chosen.

and the induced metric on the surface

$$g = \begin{bmatrix}
\frac{315}{64} a^2 (\sin(\theta))^3 (\sin(3\phi))^2 (\cos(\theta))^2 + \left(\frac{40\sqrt{\pi} - a\sqrt{35} (\sin(\theta))^3 \sin(3\phi)}{8\sqrt{\pi}}\right)^2 \\
315 a^2 (\sin(\theta))^5 \sin(3\phi) \cos(\theta) \cos(3\phi) \\
315 a^2 (\sin(\theta))^5 \sin(3\phi) \cos(\theta) \cos(3\phi) \\
\frac{315}{64} a^2 (\sin(\theta))^5 \sin(3\phi) \cos(\theta) \cos(3\phi)
\end{bmatrix}$$

(2)

The Riemann scalar curvature for the metric (2), with $a = 1$ is shown in Fig. 2.

FIG. 2: Riemann scalar curvature for the metric in (2) against $\theta, \phi$ for the manifold given in Fig. 1 with $a = 1$.

Given a geodesic Lagrange function

$$L = \frac{1}{2} g_{ab} \dot{x}^a \dot{x}^b$$

the Hamiltonian follows

$$H = \frac{1}{2} g^{ab} p_a p_b,$$
where $p_a$ are the momenta conjugate to the velocity $\dot{x}^a$. For the case of interest we have

$$H = \left\{ \frac{315}{64} \frac{a^2 (\sin (\theta))^6 (\cos (3 \phi))^2}{\pi} + \frac{r^2 (\sin (\theta))^2}{\pi} \right\} p_\theta^2 - \frac{315}{32} \frac{a^2 (\sin (\theta))^5 \sin (3 \phi) \cos (\theta) \cos (3 \phi) p_\theta p_\phi}{\pi} + \left( \frac{315}{64} \frac{a^2 (\sin (\theta))^4 (\sin (3 \phi))^2 (\cos (\theta))^2}{\pi} + r^2 \right) p_\phi^2 \right\} - \frac{99225}{4096} \frac{a^4 (\sin (\theta))^10 (\sin (3 \phi))^2 (\cos (\theta))^2 (\cos (3 \phi))^2}{\pi^2} -1,$$

(3)

where $a = 1$ and $r$ is given by the equation (1).

Following A. Saa and J. Szczesny and T. Dobrowolski [6] we quote here the geodesic deviation equations

$$u^a \nabla_a n^b = R^b_{\ klm} u^k u^l n^m,$$

where $R^b_{\ klm}$ is the Riemann tensor for the metric in (2). Which in Fermi coordinates $E_1 = u$, $E_1$ is in the direction of the velocity vector $u$, and $E_2 = x n$, where $x$ is the orthogonal separation between neighboring geodesics

$$\frac{d^2 x}{ds^2} = -K x,$$

(4)

where $K$ is Gauss’s curvature of the surface. The Gaussian curvature $K$ of a surface, is given by the ratio of the determinant of the second fundamental form, by the first fundamental form. The relation between the Gaussian curvature and the Riemann scalar is very simple

$$R = 2K.$$

If the curvature is positive $K > 0$, we can see that the solutions of (4) do not diverge.

The Poincaré section for the Hamiltonian (3) with $a = 1$, is shown in Fig 3. The intersection surface is set to $\phi = 5.0$ and the constant Hamiltonian, with $H = 1.98765$ shows a cumulative error in one part in $\sim 10^{11}$. Apparently the motion is integrable.
FIG. 3: Poincaré section for the geodesic flow given by (3) with \( a = 1 \). The intersection surface is set to \( \phi = 5.0 \) and the constant Hamiltonian shows a cumulative error in one part in \( 10^{11} \).

III. A MANIFOLD WITH DOMAINS OF NEGATIVE CURVATURE

The immersed surface is very similar as the one in the last section, given by (1); the difference is that now \( a = 2 \). The metric is the same (2), with \( a = 2 \), and the Hamiltonian also, (3), with \( a = 2 \).

FIG. 4: The manifold. The parameter \( a = 2 \) in (1) is chosen.

The Gauss-Bonnet theorem relates the integrated Gaussian curvature to the Euler number of the surface \( \chi = 2 - 2g \), for more details see [12]. It is connected to the degree of the
application \( f : N \to M \), where \( M, N \) have the same dimension \( n \) and \( x_i \in M \) and \( y_i \in N \)

\[
\deg f = \sum_{f(x_i) = y_0} \text{sgn} \det \left( \frac{\partial y_0^\alpha}{\partial x_i^\beta} \right),
\]

where the \( x_i \) are the pre images of \( y_0 \), \( y_0 = f(x_i) \), and \( \alpha, \beta = 1, 2, ..., n \). In this work, \( f(\theta, \phi, 1) = (\theta, \phi, r(\theta, \phi)) \), where \( r(\theta, \phi) \) is given by (1).

First, in an appropriate local coordinate system the metric on any surface \( \sqrt{g} = 1 \), and the Gaussian curvature coincides with the Jacobian \( J = K \). For the unit sphere, \( S^2 \), \( K = 1 \) and

\[
4\pi = \int_{S^2} K d\sigma,
\]

where \( d\sigma \) is the surface element. The above integral is a topological invariant. Remind that the second fundamental form can be diagonalized. For minimum points of the height function the principal curvatures are \( \lambda_1, \lambda_2 > 0 \) and \( K > 0 \), for maximum points of the height function \( \lambda_1, \lambda_2 < 0 \) and \( K > 0 \) while for saddle points \( \lambda_1 > 0, \lambda_2 < 0 \) and \( K < 0 \).

It can be seen from Fig. 5 that the degree of \( f \) for, a)\( \deg f = 2 \), b)\( \deg f = 0 \), c)\( \deg f = -2 \), that is, each hole contributes with a \(-2\) term.

Since (1) is a map from \( S^2 \to Q \)

\[
\deg f \int_{S^2} K d\tau = \int_{Q} K d\sigma
\]

\[
2\pi(2 - 2g) = \int_{Q} K d\sigma,
\]

\[
4\pi(2 - 2g) = \int_{Q} R d\sigma
\]
where the genus \( g \) is the number of holes in the manifold \( Q \), and \( d\tau, d\sigma \) are the surface elements on \( S^2 \) and \( Q \) respectively. Of course, both manifolds in this work, have \( g = 0 \), as can be seen in Fig. 1 and Fig. 4. Anyway this second manifold has some domains with negative curvature as can be seen by Riemann scalar curvature for the metric given in (2) plotted in Fig. 6.

\[
\begin{array}{c}
\text{FIG. 6: Riemann scalar curvature for the metric given in (2), against } \theta, \phi \text{ for the manifold given in Fig. 4 with } a = 2.
\end{array}
\]

According to (4) in these domains, since \( K < 0 \), the geodesics diverge exponentially.

The Poincaré section for the Hamiltonian (3) with \( a = 2 \), is shown in Fig 7. The intersection surface is set to \( \phi = 5.0 \) and the constant Hamiltonian, with \( H = 1.98765 \) shows a cumulative error in one part in \( \sim 10^{11} \). The system is not integrable as the chaotic regions can be seen in Fig. 7.

**IV. CONCLUSIONS**

Anosov’s famous result, states that the geodesic flow is chaotic in a compact manifold of constant negative curvature [1]. Anosov flows are very chaotic being not only mixing, but even Bernoullian [2]. For instance, the Poincaré section shows the absence of KAM tori [3]. For a review, see [4].

Integrability or not of a given mechanical system remains as an non answered question. Since Krylov’s work [5] many researchers have transformed the mechanical problem of the motion of a particle in a given potential into a billiard problem [6]. This is achieved by writing the Jacobi metric associated to the given potential. In this approach, the motion is geodesic. The Jacobi manifold is specifically obtained to incorporate the effects of the fields.
There has been an attempt to establish a local criteria for chaos. In particular A. Saa shows an example of chaos occurring in a strictly positive curvature space, $K > 0$. J. Szczesny and T. Dobrowolski in show an example of an integrable system with $K < 0$, namely the classical scattering Kepler problem. In the above mentioned examples the manifolds are not closed. This result is in contrast to the geodesic motion on $S^2$, $K = 1$ which is integrable, and on the genus $g = 2$ torus, $K = -1$ which is strongly chaotic.

In Fig. 7, we believe that the presence of geodesic chaos in the closed manifold given Fig. 4 occurs due to the regions with $K < 0$. And in Fig. 3 the geodesic flow is apparently integrable, and the curvature $K > 0$ is positive everywhere. The fact that a given closed boundary less space has positive and negative curvature regions does not imply that the geodesic motion is chaotic, as we discuss in the following.

The fundamental region of the genus $g = 1$ torus in Fig. 5 is a rectangle with opposite sides identified. The geodesic flow in the torus with $g = 1$ is integrable. On the other hand in Fig. 5 it can be seen that the torus has regions with negative curvature, saddles; and regions with positive curvature, maximum and minimum.

We stress that having domains of positive and negative curvature in a closed space does
not imply the existence of chaos.

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