Quantum Hall and Synthetic Magnetic-Field Effects in Ultra-Cold Atomic Systems

Invited Contribution to Encyclopedia of Condensed Matter Physics, 2nd edition

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In this Chapter, we give a brief review of the state of the art of theoretical and experimental studies of synthetic magnetic fields and quantum Hall effects in ultracold atomic gases. We focus on integer, spin, and fractional Hall effects, indicate connections to topological matter, and discuss prospects for the realization of full-fledged gauge field theories where the synthetic magnetic field has its own dynamics. The advantages of these systems over traditional electronic systems are highlighted. Finally, interdisciplinary comparisons with other synthetic matter platforms based on photonic and trapped-ion systems are drawn. We hope this chapter to illustrate the exciting progress that the field has experienced in recent years.

I. KEY POINTS/OBJECTIVES

This chapter aims to

• illustrate how synthetic magnetic fields can effectively be generated in neutral matter;
• highlight breakthroughs in realizing integer and spin quantum Hall effect;
• showcase steps towards strongly interacting systems displaying the fractional quantum Hall effect;
• illustrate some of the rich physics that can be achieved when the atoms act back on the synthetic gauge fields;
• emphasize the fascinating progress of the field and the outstanding main challenges.

II. INTRODUCTION

The discovery of the quantum Hall (QH) effects in two-dimensional electron gases subject to a strong magnetic field has revolutionized our view of quantum condensed-matter physics and has opened a new field of research at the crossroad with quantum information and topology (Tong 2016). While usual phase transitions such as ferromagnetism or even Bose–Einstein condensation are described within the Landau–Ginzburg theory by the onset of a macroscopic order parameter, topological states of matter cannot be revealed by the measurement of any local observable and instead are characterized by subtle quantum correlations between the constituent particles (Kitaev and Preskill 2006, Levin and Wen 2006, Wen 1995, 2013). The most direct signature of integer and fractional quantum Hall states is the quantized value of the transverse conductivity in units of $e^2/h$. Besides opening windows into novel physical phenomena of fundamental interest, such states have been also anticipated for practical applications in quantum information processing, e.g., as a promising platform to transmit quantum information and protect it from decoherence (Nayak et al. 2008).

Beyond the traditional observation of quantum Hall effects in electron gases in solid-state devices, a strong activity is presently being devoted to synthetic quantum matter systems (Ozawa and Price 2019) such as gases of ultracold atoms (Cooper 2008, Pitaevskii and Stringari 2016), spin and phonon systems realized in trapped ions (Bermudez et al. 2011, Geier, Reichstedter and Hauke 2021, Manovitz et al. 2020, Nigg et al. 2014), or fluids of strongly interacting photons in nonlinear topological photonics devices (Carusotto and Ciuti 2013, Carusotto et al. 2020). In all these systems, the charge neutrality of the constituent particles requires introducing a so-called synthetic magnetic field in order to observe QH physics (Dalibard 2015, Dalibard et al. 2011, Goldman et al. 2014); at the same time, such systems allow for a broader range of techniques for the manipulation and diagnostics well beyond the standard transport and optical probes of solid state systems (Yoshioka 2002), which opens exciting new perspectives to experimental studies. While a number of phenomena related to the integer quantum Hall (IQH) effect have been observed in the last years for non-interacting particles in suitably designed band structures for either atoms or photons, fractional quantum Hall (FQH) effects require strong interactions among particles and low temperatures, and are

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still a subject of intense work.

In what follows, we give a timely account of the state of the art of experimental and theoretical research towards the realization of FQH liquids of ultracold atoms. We also discuss relatively novel directions, where the magnetic vector potential acquires its own dynamics, leading to interesting backaction effects of matter particles onto the synthetic magnetic field and even to full-fledged gauge theories reminiscent of quantum electrodynamics. This chapter thus complements other reviews on synthetic gauge fields (Dalibard 2015, Dalibard et al. 2011, Goldman et al. 2014), on rotating atomic clouds (Cooper 2008), and on topological effects in synthetic quantum matter (Cooper et al. 2019, Ozawa and Price 2019, Zhang et al. 2018). While the main focus of our discussion here will be on ultracold atoms, connections to other promising platforms such as trapped ions or photonic devices will also be drawn.

This review chapter is organized as follows. In Sec. III, we recapitulate approaches to imbibing ultracold atomic gases with synthetic magnetic fields, a key prerequisite for observing QH physics. Sections IV and V discuss recent advances in cold atom systems in observing the IQH and FQH, respectively. In Sec. VI, we highlight some recent developments in inducing backaction of the matter system onto the synthetic magnetic field, leading to the generation of dynamical Peierls phases and even to pioneering steps in the creation of lattice gauge theories. Section VII contains our conclusions and some further outlooks.

III. SYNTHETIC MAGNETIC FIELDS

The motion of quantum mechanical particles of mass $m$ and charge $q$ in a classical background magnetic field is described via the minimal coupling Hamiltonian

$$H_{\text{kin}} = \frac{(\mathbf{p} - q \mathbf{A})^2}{2m}$$

in terms of the vector potential $\mathbf{A}(\mathbf{r})$, related to the magnetic field by $\mathbf{B} = \text{rot} \mathbf{A}$. In a discrete, spatially-periodic lattice geometry, this Hamiltonian can be reformulated in terms of a non-trivial hopping phase in the tight-binding Hamiltonian,

$$H_{\text{kin}} = -J \sum_{\langle i,j \rangle} \psi_i^{\dagger} U_{i,j} \psi_j,$$  

where $J$ is the bare hopping amplitude, $\psi_i^{\dagger}$ and $\psi_i$ are fermionic or bosonic creation and annihilation operators at lattice site $i$, respectively, and $U_{i,j} = \exp(i \frac{q}{\hbar c} \int_{t_i}^{t_j} \mathbf{A}(\mathbf{r}) \cdot d\mathbf{r})$ is the Peierls phase, corresponding to the Aharonov–Bohm phase accumulated by the particle when moving from site $i$ to site $j$. Starting from these formulations, it is straightforward to derive the celebrated features of quantum mechanical motion in magnetic fields, such as cyclotron motion and Landau levels (Girvin 1999, Tong 2016).

Since atoms are charge neutral ($q = 0$), it is impossible to observe this physics using standard magnetic fields, whose effect on atoms typically reduces to Zeeman shifts of the internal levels. More subtle manipulations are therefore necessary to generate effective vector potentials affecting the orbital atomic motion along the lines of Eqs. (1) or (2). Over the years, a number of strategies to generate such synthetic magnetic fields have been developed. Here, we will briefly review the most promising and/or physically transparent ones (Dalibard 2015, Dalibard et al. 2011, Goldman et al. 2014). As the basic object realized in the experiments is typically the vector potential, gauge-invariance of the framework raises interesting subtleties in the conceptual analysis of the experiments (LeBlanc et al. 2015). Though synthetic gauge fields give rise also to other intriguing effects, e.g., supersolid behavior (Geier, Martone, Hauke and Stringari 2021, Li et al. 2017, Putra et al. 2020), our attention in this review will be focused on the family of integer and fractional Quantum Hall effects.

Historically, the first strategy was based on the mathematical analogy between the Coriolis force $\mathbf{F} = \mathbf{v} \times \Omega$ in a rotating frame and the Lorentz force $\mathbf{F} = q \frac{\mathbf{v}}{c} \times \mathbf{B}$ in a magnetic field. By putting the atom trap into mechanical rotation around some axis, the atomic cloud gets to equilibrate in a rotating frame where the angular speed plays the role of the magnetic field. Over the years, this approach has led to pioneering advances for weakly interacting Bose–Einstein condensates, such as the observation of periodic arrays of quantized vortices (Abo-Shaeer et al. 2001, Bretin et al. 2004, Schweikhard et al. 2004) resembling the Abrikosov vortex arrays generated by magnetic fields in type-II superconductors.

![FIG. 1. Quantized vortices nucleated in an atomic BEC under the effect of a synthetic magnetic field. Figure adapted from (Lin et al. 2009).](image-url)
A conceptually different approach to generate a synthetic magnetic field is based on the Berry phase that moving atoms experience when they are dressed by a suitable combination of optical, micro- and radio-wave, and static electromagnetic fields (Dum and Olsahanii 1996, Juzeliūnas and Ūhberg 2004, Ruseckas et al. 2005): in this framework, the roles of the vector potential and of the magnetic field are played by the Berry connection and the Berry curvature, respectively. Remarkable outcomes of this approach include the nucleation of quantized vortices under the effect of strong enough synthetic fields (Lin et al. 2009), as highlighted in Fig. 1.

Discrete, spatially periodic geometries realized by imposing optical lattice potentials offer additional tools to generate synthetic magnetic fields via non-trivial hopping phases. Complex hopping phases induced by intersite Raman processes were first proposed in Ref. (Jaksch and Zoller 2003, Mueller 2004, Osterloh et al. 2005, Sørensen et al. 2005) and then implemented to realize, e.g., an atomic Harper–Hofstadter model (Aidelsburger et al. 2013, Miyake et al. 2013).

Alternatively, non-trivial band geometries and topologies have been realized in a Floquet framework based on periodic shaking or time-modulation of the lattice potential (Bukov et al. 2015, Eckardt and Anisimovas 2015). By suitably choosing the shaking protocol, e.g., such that the periodic shaking function explicitly breaks time-reversal symmetry, one can obtain—in a time-averaged or stroboscopic Floquet description—an effective Hamiltonian that has a non-trivial Peierls phase (Goldman and Jotzu et al. 2014) as a lateral displacement of atomic context in Ref. (Dudarev et al. 2004) and (Hauke et al. 2010, 2011, Kolovsky 2011, Oka and Aoki 2009, Struck et al. 2012). This approach has led to the realization of the Haldane (Jotzu et al. 2014) and the Harper–Hofstadter model (Aidelsburger et al. 2015).

An exciting new frontier of these investigations is to combine the internal and orbital dynamics of atoms into a single geometric entity so to realize configurations with a larger number of effective spatial dimensions than the usual three spatial ones (Celi et al. 2014), where to observe new magnetic and topological effects that are peculiar of high dimensionalities (Price et al. 2015). Pioneering steps in the direction of realizing such synthetic dimensions were reported in the experiments of Refs. (Mancini et al. 2015, Stuhl et al. 2015).

**IV. INTEGER QUANTUM HALL EFFECT**

In electronics, the main signature of the Integer Quantum Hall (IQH) effect consists in a quantized value of the transverse conductance to integer multiples of $e^2/h$ whenever electrons completely fill an integer number of Landau levels (Tong 2016). Soon after the discovery of the IQH, it was realized that the quantized conductance can be related to integer-valued topological invariants characterizing the geometry of the electronic bands (Thouless et al. 1982). In a modern understanding, IQH systems are then interpreted as an example of the wider class of topological insulators, where the bulk-boundary correspondence translates topological properties of the bands of the insulating bulk into chiral currents around the edges of the sample (Hasan and Kane 2010), see Fig. 2(a). Since these features are related to integer-valued topological invariants, they are resilient against disorder and similar types of small perturbations.

Based on such insights, the IQH effect could be studied in a number of atomic systems: even though conductivity experiments analogous to electronic ones are not naturally performed in atomic systems, many other observables are available to obtain an even deeper information on the properties of this state of matter.

**A. IQH in cold atoms**

Anomalous Hall effects were first predicted in the atomic context in Ref. (Dudarev et al. 2004) and experimentally observed in honeycomb-like lattices in Ref. (Jotzu et al. 2014) as a lateral displacement of the atoms whenever their $k$-space distribution is pushed across a finite-Berry-curvature region. A quantized displacement analogous to the IQH was then observed for an atomic cloud uniformly filling a band (Aidelsburger et al. 2015). A precursor of this was the measurement of the Zak phase characterizing topological Bloch bands in a one-dimensional optical lattice (Atala et al. 2013).

While chiral edge states have been the major workhorse to characterize topological photonic systems (Ozawa et al. 2019), a relatively limited number of experiments have investigated edge currents in atomic systems. On the one hand, this is due to the difficulty of directly measuring currents in cold-atom systems, though significant theoretical and experimental advances have been achieved in recent years (Brown et al. 2019, Geier,
FIG. 3. Experimental observation of edge-cyclotron orbits in a synthetic three-leg ladder system based on a spatial $x$ dimension and a synthetic dimension encoded in the atomic spin. Figure adapted from (Mancini et al. 2015).

Reichstetter and Hauke 2021, Hauke et al. 2014, Jepsen et al. 2020, Keßler and Marquardt 2014, Krinner et al. 2015, Laflamme et al. 2017, Nichols et al. 2019, Scherg et al. 2018). On the other hand, an obstacle is the absence of sharp boundaries in standard, harmonically trapped atomic clouds. Also for this reason, the remarkable studies of edge transport in Ref. (Mancini et al. 2015, Stuhl et al. 2015) illustrated in Fig. 3 made use of a synthetic dimension set-up combining a spatially one-dimensional lattice with an additional dimension formed by the internal spin degrees.

A great advantage of atomic systems is the possibility of combining macroscopic transport measurements with a direct microscopic insight into the quantum many-body wavefunction (Gebert et al. 2020). In the IQH context, a remarkable example in this direction was the experimental tomography of the Berry phase (Fläschner et al. 2016, Hauke et al. 2014), with follow-ups such as the identification of dynamical topological invariants in the time evolution of an atomic system (Tarnowski et al. 2019).

B. Extensions

An important extension of the IQH effect is obtained when a spin degree of freedom of the particles is included, $\sigma = \uparrow, \downarrow$, such that Eq. (2) becomes $H_{\text{kin}} = -J \sum_{\langle i,j \rangle} \sum_{\sigma} \psi_{i,\sigma}^\dagger U_{i,j}^{\sigma,\sigma'} \psi_{j,\sigma'}$. In the so-called spin-Hall effect (Hasan and Kane 2010), particles with different spin orientations see opposite magnetic fields, $(U_{i,j}^{\uparrow,\downarrow})^\dagger = U_{i,j}^{\downarrow,\uparrow}$ and time-reversal symmetry is restored. The result is a system without any net charge current but chiral spin edge currents that run around the sample in opposite directions, see Fig. 2(b). Pioneering proposals have been based on extending schemes for light-induced vector potentials to several hyperfine levels (Zhu et al. 2006) and on state-dependent, slowly time-varying magnetic potentials engineered in an atom chip on a micron scale (Goldman et al. 2010). The spin Hall effect has been observed in free space, where spin-dependent Lorentz forces have been measured using a spatially inhomogeneous spin-orbit coupling field (Beeler et al. 2013), as well as in an optical lattice using laser-assisted tunneling induced in a tilted optical lattice via periodic driving (Aidelsburger et al. 2013).

This physics can be generalized even further by replacing the spin-dependent Peierls phase by a matrix, $H_{\text{kin}} = -J \sum_{\langle i,j \rangle} \sum_{\sigma,\sigma'} \psi_{i,\sigma}^\dagger U_{i,j}^{\sigma,\sigma'} \psi_{j,\sigma'}$, where $U_{i,j}$ mixes the different spin states during hopping. Genuinely non-Abelian effects occur if the product of the $U_{i,j}^{\sigma,\sigma'}$ around a closed path (the non-Abelian Wegner–Wilson loop) does not reduce to a simple phase factor $e^{i\phi}$. Such a situation can be obtained through periodically shaken optical superlattices where an internal degree of freedom of the atoms plays the role of the (pseudo-)spin (Hauke et al. 2012). As a striking consequence of the non-Abelian Hall effect, novel fractal features have been predicted to appear in the phase diagram (Bermudez et al. 2010, Goldman et al. 2009).

Another exciting direction of research concerns the extension of IQH effects to higher-dimensional, 4+1 geometries as first proposed in Ref. (Price et al. 2015) using the concept of synthetic dimensions mentioned above. Pioneering experimental evidence of the role of higher-dimensional topological invariants in the atomic transport properties under crossed synthetic-electric and magnetic fields in 4-dimensional models was reported in Ref. (Lohse et al. 2018).

V. FRACTIONAL QUANTUM HALL EFFECT IN COLD ATOMS

In the previous section, we have seen the intriguing effects that appear in the presence of completely occupied Landau levels or topological bands, but where interactions play at most a subdominant role. Even more subtle physics occurs when the microscopic degeneracy of partially filled Landau levels is lifted by strong inter-particle interactions and the lowest energy state acquires a non-trivial topology for the many-body wavefunction (Kitaev and Preskill 2006, Levin and Wen 2006, Wen 1995, 2013). One of the most celebrated examples of such topological many-body phases of matter is the so-called fractional quantum Hall (FQH) effect, first detected in two-
dimensional electron gases under strong magnetic fields as a precise quantization of the transverse conductivity at rational values proportional to the electron filling, i.e., the number of electrons per unit magnetic flux piercing the system (Girvin 1999, Tong 2016, Yoshioka 2002). As even more intriguing signatures of the many-body topology of the FQH state, excitations with a fractional charge and fractional statistics have been anticipated for these systems, the so-called anyons (Stern 2008). While fractionally charged edge excitations have been observed in shot-noise measurements (De-Picciotto et al. 1998, Saminadayar et al. 1997), the observation of fractional statistical properties interpolating between bosonic and fermionic behaviours is still the subject of intense experimental work (Bartolomei et al. 2020, Nakamura et al. 2020).

First steps towards the realization of FQH fluids using ultracold atomic clouds were made by pushing the rotation frequency towards the trapping frequency, so that the atomic cloud expands in space under the effect of the centrifugal potential and reaches the required magnetic flux per particle values. For sufficiently low temperatures, it is then predicted to enter a FQH state (Cooper 2008). In spite of early concerns about heating due to spurious asymmetries of the trap potential and the need for a fine tuning of the rotation speed, remarkable advances on condensates in rotating traps have been recently reported (Fletcher et al. 2021, Mukherjee et al. 2022). Besides these works on a mean-field-interacting dynamics, hints of correlated FQH physics have been reported in small atomic clusters trapped at the rotating minima of a temporally modulated optical lattice potential (Gemelke et al. 2010).

A different strategy to realize a strongly interacting Harper–Hofstadter lattice using a combination of optical lattice potentials and Raman transitions was adopted in Ref. (Tai et al. 2017) and led to the observation of a correlated two-particle dynamics, a precursor of fractional quantum Hall effects, see Fig. 4. In the meanwhile, an intense theoretical activity has been devoted to the study of alternative protocols to realize FQH states using, e.g., adiabatic ramps along suitably designed paths in parameter space (Popp et al. 2004, Sørensen et al. 2005) or sequences of adiabatic flux insertion and hole replenishing (Grusdt et al. 2014).

The main advantage of FQH fluids of ultracold atoms over solid-state systems is the much wider variety of manipulation and diagnostics tools compared to the transport and optical probes traditionally available for electronic systems. While waiting for macroscopic FQH clouds to be experimentally realized, theoretical proposals to investigate their intriguing properties include interferometric techniques (Grusdt et al. 2016, Paredes et al. 2001), quasi-hole dynamics in response to external potentials (Graß et al. 2012), microscopic imaging of the quasi-hole density profile (Macaluso et al. 2020), quantum mechanics of anyonic molecules formed by an impurity bound to a FQH quasi-hole (Baldelli et al. 2021, de las Heras et al. 2020, Lundholm and Rougerie 2016, Yakaboylu et al. 2020, Zhang et al. 2014), the center-of-mass motion of the cloud (Repellin et al. 2020), its circular–dichroic response to circular excitations in the bulk (Repellin and Goldman 2019), the linear and nonlinear response of the edge modes (Nardin and Carusotto 2022).

Together with direct measurements of the entanglement entropy (Jiang et al. 2012, Li and Haldane 2008) as experimentally pioneered in Ref. (Islam et al. 2015), we anticipate that these proposals will eventually be a most valuable tool to obtain a deeper insight in the basic physics of FQH fluids and synthetic topological matter, and to exploit the benefits of their atomic implementation in view of quantum information tasks (Nayak et al. 2008).

VI. DYNAMICAL GAUGE FIELDS

In all of the above discussions, the vector potential appearing in Eqs. (1) and (2) was assumed to have a constant value, corresponding to a classical, externally imposed background field. Entirely new effects can appear once the vector potential is imbued with its own dynamics.

A. Dynamical vector potentials and Peierls phases

In the first proposal for continuum systems Edmonds et al. (2013), a density-dependent vector potential was anticipated to arise from interaction-induced energy shifts of the internal atomic states involved in the optical transitions generating the Berry phase. Among the observable consequences, density-dependent persistent currents in ring geometries and chiral solitons were pointed out. Along similar lines, (Ballantine et al. 2017) proposed to replace one of the optical fields generating the Berry phase by a multimode cavity field. The density of the atoms can then act back on the cavity field, which—due to its multimode character—can acquire a spatial dependence. In this way, the equivalent of the celebrated Meissner effect can be generated, whereby a magnetic field is expelled from the atom-cloud sample, see Fig. 5.

In lattice systems, a complementary direction is taken by turning the Peierls phase into a quantum-mechanical operator. This can happen, e.g., through density-dependent tunneling, where the complex hopping phase
U_{i,j}$ becomes a function $U_{i,j}(\hat{n}_i, \hat{n}_j)$ of the atom number operators $\hat{n}_\ell = a_\ell^\dagger a_\ell$ at adjacent sites $\ell = i, j$.

In other schemes, the role of the Peierls phase can also be taken over by a second, independent atomic species. E.g., in an atomic mixture, an itinerant species can be coupled to the (pseudo-)spin degree of freedom of a second, kinetically frozen species (Kasper et al. 2021, Mil et al. 2020). By replacing the static hopping matrix by an effective $U_{i,j}$, one can study the dynamics of the combined system without the need to enforce the $U(1)$ gauge symmetry. This approach has been successfully implemented in Rydberg quantum simulators (Bernien et al. 2017, Surace et al. 2020), using Floquet techniques in optical lattices (Schweizer et al. 2019), or even in optical superlattices (Schweizer et al. 2019, Datta et al. 2020) (proposed, e.g., in Refs. (Stamper-Kurn et al. 2014, Zache et al. 2018, Zohar et al. 2013)).

Most of these implementations focused on salient physical phenomena in one spatial dimension, and it is a current challenge to advance ultracold-atom lattice gauge theories into higher spatial dimensions (Zohar 2022). In the future, these may enable, e.g., the investigation of strong-field effects on anomalous currents (Ott et al. 2020) as well as a range of new anomalous transport phenomena including the chiral magnetic effect (Fukushima et al. 2008, Kharzeev 2014), the chiral vortical effect (Son and Zhitnitsky 2004), or the conformal magnetic edge effect (Chernodub et al. 2019, Chu and Miao 2018).

### VII. CONCLUSIONS AND PERSPECTIVES

This review chapter summarized our point of view on the rich physics that can be obtained by synthetic static and dynamical gauge fields in cold-atom setups. We have reviewed the way synthetic magnetic fields can be engineered in such systems, discussed progress in controlled implementations of the integer, spin, and fractional quantum Hall effects, as well as some of the fascinating phenomena that appear when the atoms act back on the synthetic magnetic fields, without or with respecting gauge symmetry given by Gauss’s law. These are, however, just some of the many facets of ongoing research in the broad context of synthetic gauge fields.
Promising implementations have been designed and realized in other synthetic quantum matter platforms. For example, topological photonic systems are presently one of the most active fields of research in optics, with potential technological applications such as efficient optical isolation and large-area, fabrication-disorder-robust topological lasers (Price et al. 2022). Synthetic magnetic field for photons are one of the key building blocks in this context and, analogously to the atomic case, can be realized in many different ways, from twisted optical cavities leading to an effective overall rotation (Schine et al. 2016), to static or time-dependent complex hopping phases (Hafezi et al. 2013, Roushan et al. 2017), to Floquet-like temporal modulations (Rechtsman et al. 2013). In combination with sizable photon-photon interactions arising from the optical nonlinearity of the underlying optical medium, a synthetic magnetic field for light may lead to the generation of novel states of photonic matter (Carusotto et al. 2020); baby versions of a quantum Hall fluid of light of a few photons have been recently realized (Clark et al. 2020). Other promising platforms are, e.g., analog implementations in trapped ions, where techniques similar to those for optical lattices have been developed to imprint Peierls phases (Bermudez et al. 2011, Manovitz et al. 2020), as well as universal quantum computers, where proposals for realizing fractional quantum Hall states (Rahmani et al. 2020) as well as experimental realizations of topological matter exist (Nigg et al. 2014, Satzinger et al. 2021, Semeghini et al. 2021).

All together, the examples discussed in this review illustrate the on-going exciting progress in the study of quantum Hall effects in cold atomic fluids as well as in related synthetic quantum matter platforms (Ozawa and Price 2019). Beyond the remarkable achievements obtained so far, there are a lot of equally exciting open questions ahead of the community, as well as a number of potentially game-changing applications in quantum technologies (Nayak et al. 2008).

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