SAMPLING FROM $p$-ADIC ALGEBRAIC MANIFOLDS

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Abstract. We give a method for sampling points from an algebraic manifold (affine or projective) over a local field with a prescribed probability distribution. In the spirit of the previous work by Breiding and Marigliano on real algebraic manifolds, our method is based on slicing the given variety with random linear spaces of complementary dimension. We also provide an implementation of our sampling method and discuss a few applications. As an application, we sample from algebraic $p$-adic matrix groups and modular curves.

1. Introduction

Algebraic varieties and manifolds are ubiquitous in many areas of mathematics: In number theory, Shimura varieties arise as complex algebraic varieties that parameterize certain types of Hodge structures; Calabi-Yau manifolds model a number of phenomena in physics generally and superstring theory specifically, and enjoy interesting geometric properties; and many interesting probabilistic models arise as varieties whose algebraic and geometric properties have probabilistic and statistical interpretations.

While the use of $p$-adic numbers has not yet become as common a practice in many domains, they have started to find numerous applications, for example in mathematical physics [38]. Moreover, the interest and research activity addressing probabilistic and statistical questions in the $p$-adic setting have been gaining momentum starting from the early work of Evans [18, 19, 20], Bikulov, Vladimirov, Volovich and Zelenov [38, 3] to the more recent developments [21, 27, 2, 9, 25, 8, 16, 15] to mention a few. In this line of thought, it is quite desirable to have an efficient method of sampling from $p$-adic manifolds.

In this paper, inspired by the work of Breiding and Marigliano [6], we tackle the problem of sampling from a $p$-adic algebraic variety with a prescribed probability distribution by intersecting it with random linear spaces of complementary dimension. Our results provide $p$-adic analogues of the results in [6] and are in line with $p$-adic analogs of the so-called Crofton’s formulas in integral geometry (see [25] and [8] for $p$-adic integral geometry) which provide a link between the volume of a manifold and its expected number of intersection points with random linear spaces.

Figure 1. An illustration of the sampling method. The dotted lines are rejected; the red line intersects in 3 points the curve from which we wish to sample. A point is randomly sampled from the three points and the selected point is colored in green.

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Over the real or complex numbers, sampling from manifolds (especially in the parameterized case) often involves using a Markov chain sampling (Metropolis algorithm, Gibbs sampler, hit-and-run algorithm etc.) [11, 1, 26]. While this method is fairly simple (both computationally and mathematically), it approximates the desired probability measure only asymptotically and requires a study of the mixing time of the Markov chain in question [10, 12]. Moreover, since non-archimedean local fields are totally disconnected topological spaces, the usual Markov chain sampling algorithms are ill suited for such a setting. Our method has the advantage of sampling exactly from the desired probability density. Also, given its geometric nature, it works regardless of the nature of the topology involved.

To set things up, let $K$ be a non-archimedean local field with a normalized discrete valuation map; $\text{val}: K \to \mathbb{Z} \cup \{+\infty\}$. We fix once and for all a uniformizer $\varpi$; that is an element $\varpi \in K$ such that $\text{val}(\varpi) = 1$. We denote by $\mathcal{O} := \{x \in K: \text{val}(x) \geq 0\}$ the valuation ring of $K$; and by $k := \mathcal{O}/\varpi\mathcal{O}$ its residue field. It is known that $k$ is isomorphic to a finite field $\mathbb{F}_q$ with $q$ elements; where $q$ is a power of the characteristic $p := \text{char}(k)$. The valuation $\text{val}$ defines an absolute value $|\cdot|$ on $K$ by setting $|x| := q^{-\text{val}(x)}$ for $x \in K$. We denote by $\mu$ the unique real-valued Haar measure on $K$ such that $\mu(\mathcal{O}) = 1$. For a detailed account on local fields we refer to [31, Part 1].

**Remark 1.1.** Non-archimedean local fields come in two flavors. Those that have the same characteristic as their residue fields are isomorphic (as topological fields) to $\mathbb{F}_q((\varpi))$; the field of Laurent series in one variable $\varpi$ with coefficients in $\mathbb{F}_q$. The second type has characteristic 0 and are finite extensions of the field $\mathbb{Q}_p$ of $p$-adic numbers for some prime $p$.

Let us fix a positive integer $N \geq 2$. We denote by $\mathbb{A}^N = K^N$ the $N$-dimensional affine space over $K$. The space $\mathbb{A}^N$ inherits from $K$ the product Haar measure, which we denote by $\mu_{\mathbb{A}^N}$. When there is no risk of confusion, we simply denote this measure by $dx$. We endow $\mathbb{A}^N$ with the norm

$$
\|x\| = \max_{1 \leq i \leq N} |x_i|, \quad \text{for } x = (x_1, \ldots, x_N) \in \mathbb{A}^N,
$$

and the valuation

$$
\text{val}(x) = \min_{1 \leq i \leq N} \text{val}(x_i) = -\log_q(|\|x\||), \quad \text{for } x = (x_1, \ldots, x_N) \in \mathbb{A}^N.
$$

This makes $\mathbb{A}^N$ a metric space with the metric given by $d(x, y) = \|x - y\|$ for $x, y \in \mathbb{A}^N$. We refer the reader to Section 2.1 for more details on norms, valuation and orthogonality.

An affine algebraic variety in $\mathbb{A}^N$ is the zero set of a system of polynomials $p = (p_1, \ldots, p_r)$ in $K[x_1, \ldots, x_N]$, i.e.

$$
\{x \in \mathbb{A}^N: p_1(x) = \cdots = p_r(x) = 0\}.
$$

We refer to smooth and irreducible\(^1\) varieties over $K$ (affine or projective) as algebraic manifolds.

Although the notions of dimension and degree of an algebraic $K$-manifold $X$ are the usual notions from algebraic geometry, the notion of volume on $X$ is not as standard. Let $X \subset \mathbb{A}^N$ be an affine algebraic $K$-manifold of dimension $n$. For $\epsilon > 0$ and $x \in \mathbb{A}^N$, let us denote by $B_N(x, \epsilon) = \{y \in \mathbb{A}^N: d(x, y) \leq \epsilon\}$ the ball of radius $\epsilon$ and center $x$. The volume measure $\mu_X$ on $X$ is defined as follows, for an open set in $V \subset X$:

$$
\mu_X(V) := \lim_{\epsilon \to 0} \mu_{\mathbb{A}^N} \left( \bigcup_{x \in V} B_N(x, \epsilon) \right) = \lim_{r \to \infty} q^{-(N-n)} \mu_{\mathbb{A}^N} \left( \bigcup_{x \in V} B_N(x, q^{-r}) \right),
$$

This limit exists (see [32, 25]) and the map $\mu_X$ thus defined\(^2\) is a measure on the Borel $\sigma$-algebra of $X$.

**Remark 1.2.** When $X \subset \mathbb{P}^{N-1}$ is a projective manifold, one can still define a volume measure $\mu_X$ in the same manner by replacing the measure $\mu_{\mathbb{A}^N}$ in (1.1) with its normalized push-forward to the projective space $\mathbb{P}^{N-1}$, and by defining balls in $\mathbb{P}^{N-1}$ using Fubini–Study metric. Here we focus on the affine case and delay our treatment of projective manifolds until Section 4.

Given a function $f: X \to \mathbb{R}$ that is integrable with respect to the measure $\mu_X$, we wish to:

1. Estimate the integral $\int_X f(x) \mu_X(dx)$. 

\(^1\)The usual notion of smoothness and irreducibility from algebraic geometry.

\(^2\)We can equivalently define a volume measure on $X$ using local charts and differential forms in the usual way.
(2) Sample a random variable $\xi \in X$ with the probability density $f(x)/\int_X f(x)\mu_X(dx)$, when $f$ is non-negative and $\int_X f(x)\mu_X(dx) > 0$.

Our sampling method entails intersecting the manifold $X$ with affine linear spaces of complementary dimension. For a matrix $A \in K^{n\times N}$ and $b \in K^n$, we denote by $L_{A,b}$ the affine linear space implicitly defined as follows:

$$L_{A,b} := \{ x \in K^N : Ax = b \}.$$  

Such a space is generically\(^3\) of dimension $N - n$. The intersection $L_{A,b} \cap X$, where $A \in K^{n\times N}$ and $b \in K^n$, is generically\(^4\) finite and its size is bounded by the degree of $X$ (see Section 2.2). Loosely speaking, sampling from $X$ is then reduced to sampling a random plane $L_{A,b}$ and then sampling a random point from the finite intersection $L_{A,b} \cap X$. However, given a target probability density on $X$, neither sampling step is entirely straightforward. For that reason, we must introduce a weight function $w_X : X \to \mathbb{R}_{>0}$ on $X$. Before we do so however, we need to define two quantities it involves.

**Definition 1.3.** Let $a, b \geq 1$ be two positive integers, and $M \in K^{a \times b}$ a matrix. Let us write the Smith normal form of $M$ as $M = UDV$, where $U \in \text{GL}(a, \mathcal{O}); V \in \text{GL}(b, \mathcal{O})$; and $D = \text{diag}(\varpi^{v_1}, \ldots, \varpi^{v_{\min(a,b)}}) \in K^{a \times b}$ with $v_1 \geq \cdots \geq v_{\min(a,b)} \in \mathbb{Z} \cup \{ \infty \}$. We then define the absolute determinant of $M$ as follows:

$$\text{N}(M) := |\varpi^{v_1} \cdots \varpi^{v_{\min(a,b)}}| = q^{-v_1 - \cdots - v_{\min(a,b)}}.$$  

If $E, F$ are $K$-vector spaces of respective dimensions $a$ and $\varphi : E \to F$ is $K$-linear, we define

$$\text{N}(\varphi) = \text{N}(A),$$

where $A \in K^{a \times b}$ is a matrix representing $\varphi$ in orthonormal bases\(^5\) of $E$ and $F$.

**Definition 1.4.** Let $X \subset K^N$ be an affine algebraic manifold of dimension $n$, and $x$ be a point on $X$. Let $U \in \text{GL}(N, \mathcal{O})$ such that $Ux = (0, \ldots, \varpi^{\text{val}(x)})^\top$, and let $W \in \mathcal{O}^{N \times n}$ be a matrix whose columns form an orthonormal basis of the tangent space $T_xX$. Finally, let us set $S_x = \text{diag}(1, \ldots, 1, \varpi^{\text{max}(0, -\text{val}(x))}) \in K^{N \times N}$. Then we define

$$\text{Nr}(X, x) := \text{N}(S_xUW).$$

It is not so clear that this definition does not depend on the choice of $W$ and $U$, but we shall see in Lemma 3.1 that this is the case. The quantity $\text{Nr}(X, x)$ can be interpreted as a “measure” of how far the point $x$ is from being an $\mathcal{O}$-point of $X$.

Now we are ready to define a weight function $w_X$ on $X$.

**Definition 1.5.** Let $X \subset K^N$ be an affine algebraic manifold of dimension $n$ over $K$. For a point $x \in X$, we define the weight of $x$ in $X$ as follows

$$w_X(x) = \frac{1 - q^{-\langle n+1 \rangle}}{1 - q^{-1}} \frac{\text{max}(1, \|x\|^n)}{\text{Nr}(X, x)}.$$  

**Remark 1.6.**

1. Notice that, on the $\mathcal{O}$-points $X \cap \mathcal{O}^N$ of the manifold $X$, we have $\text{Nr}(X, x) = 1$ so the weight function $w_X$ is constant and takes the value

$$w_X(x) = \frac{1 - q^{-\langle n+1 \rangle}}{1 - q^{-1}}, \quad x \in X \cap \mathcal{O}^N.$$  

2. The weight function $w_X$ is not intrinsic. As we shall see in Proposition 3.2, it depends on how the random linear space $L_{A,b}$ is distributed, or more precisely on the distribution of the random element $(A, b) \in K^{n \times N} \times K^n$.

Given a real valued function $f$ on $X$, we define the following function on $K^{n \times N} \times K^n$:

$$\tilde{f}(A, b) := \sum_{x \in X \cap L_{A,b}} w_X(x)f(x), \quad \text{for } (A, b) \in K^{n \times N} \times K^n,$$

where the sum is 0 by convention when $X \cap L_{A,b}$ is empty or infinite. Our first result deals with integrating a real-valued integrable function $f$ on a manifold $X$. Namely, we show that the integral can be expressed as the expectation of a real-valued random variable that we can sample.

\(^3\)The set of $(A, b) \in K^{n \times N} \times K^n$ for which $L_{A,b}$ has dimension $N - n$ is non-empty and Zariski-open.

\(^4\)The set of $(A, b) \in K^{n \times N} \times K^n$ for which the intersection is finite is non-empty and Zariski-open.

\(^5\)In the sense of Section 2.1.
Theorem 1.7. Let \( X \subset A^N \) be an \( n \)-dimensional affine algebraic manifold defined over \( K \). Let \((A, b)\) be a random variable in \( K^{n\times N} \times K^n \) with distribution \( 1_{A \in O^{n\times N}, b \in O^n} dA db \). Then we have:

\[
\int_X f(x) \mu_X(dx) = E[f(A, b)].
\]

With this theorem in hand, one can evaluate integrals, up to a certain confidence interval, using Monte-Carlo methods. We discuss this in more detail in Section 5.

Our second result deals with sampling a random point \( \xi \) from a manifold \( X \) with a prescribed probability density \( f \) with respect to the natural volume measure \( \mu_X \) on \( X \):

Theorem 1.8. Let \( X \subset A^N \) be an \( n \)-dimensional affine algebraic manifold defined over \( K \). Let \( f : X \to \mathbb{R}_{\geq 0} \) be a probability density with respect to \( \mu_X \). Let \( (\tilde{A}, \tilde{b}) \) be the random variable in \( K^{n\times N} \times K^n \) with distribution \( f(A, b) 1_{A \in O^{n\times N}, b \in O^n} dA db \).

Let \( \xi \) be the random variable obtained by intersecting \( X \) with the random space \( L_{\tilde{A}, \tilde{b}} \) and choosing a point \( x \) in the finite set \( X \cap L_{\tilde{A}, \tilde{b}} \) with probability

\[
\frac{w_X(x) f(x)}{f(\tilde{A}, \tilde{b})}.
\]

Then \( \xi \) has density \( f \) with respect to the volume measure \( \mu_X \) on \( X \).

We give similar results for projective manifolds, namely Theorems 4.5 and 4.6, in Section 4. We provide an implementation (in SageMath [35]) of the sampling method we describe in this article (in particular cases) in following repository:

(1.2) https://mathrepo.mis.mpg.de/SamplingpAdicManifolds/index.html

Remark 1.9. Although most of our results are stated for algebraic manifolds, there is no issue working with an irreducible variety \( X \) (affine or projective) with potential singularities. This is because the singular locus \( X^{\text{sing}} \) is lower dimensional in \( X \) and we can work with the algebraic manifold \( X \setminus X^{\text{sing}} \). Our sampling method will then produce a point in \( X \) that is smooth with probability 1.

Notation

For clarity, we write random variables in bold. The following table summarizes some of the notation used throughout this paper.

\[
\begin{align*}
K & \quad \text{A non-archimedean local field.} \\
\text{val}(\cdot) & \quad \text{The valuation on } K \text{ and also, depending on context, the valuation on } K^N. \\
O & \quad \text{The valuation ring of } K. \\
\varpi & \quad \text{A uniformizer of } K; \text{ that is } \text{val}(\varpi) = 1. \\
k & \quad \text{The residue field } O/\varpi O. \\
| \cdot | & \quad \text{The absolute value on } K \text{ with } |\varpi| = q^{-1}. \\
\| \cdot \| & \quad \text{The standard norm of a vector in } K^N \text{ (associated with the standard lattice } O^N). \\
X & \quad \text{An algebraic } K\text{-manifold (either affine or projective).} \\
w_X & \quad \text{The weight function on the manifold } X. \\
\mu_X & \quad \text{The volume measure on } X. \\
1_S & \quad \text{The indicator of a set } S; \text{ that is } 1_{x \in S} = 1_S(x) = 1 \text{ if } x \in S \text{ and } 0 \text{ otherwise.}
\end{align*}
\]

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2. Background and preliminaries

In this section we elaborate on some of the concepts involved in the statements of Theorems 1.7 and 1.8 and collect some of the tools necessary throughout this text.

2.1. Norms, lattices and Gaussian measures

Let $E$ be a vector space over $K$ and let $N = \dim_K(E)$. A lattice in $E$ is an $O$-submodule $\Lambda$ of $E$ of rank $N$. More explicitly, a lattice $\Lambda$ is a module generated by a basis $(e_1, \ldots, e_N)$ of $E$ over $O$, i.e.

$$\Lambda = Oe_1 \oplus \cdots \oplus Oe_N.$$  

Any lattice $\Lambda$ induces a valuation map $\operatorname{val}_\Lambda$ on $E$ defined as follows

$$\operatorname{val}_\Lambda(x) := \sup \{ m \in \mathbb{Z} : -m x \in \Lambda \}, \quad x \in E.$$  

This valuation satisfies the following properties

1. $\operatorname{val}_\Lambda(x) = +\infty \iff x = 0$, for all $x \in E$,
2. $\operatorname{val}_\Lambda(\alpha x) = \operatorname{val}(\alpha) + \operatorname{val}_\Lambda(x)$, for all $\alpha \in K, x \in E$,
3. $\operatorname{val}_\Lambda(x + y) \geq \min(\operatorname{val}_\Lambda(x), \operatorname{val}_\Lambda(y))$, for all $x, y \in E$.

This allows us to define an ultrametric norm $\| \cdot \|_\Lambda$ on the vector space $E$ as follows

$$\|x\|_\Lambda := q^{-\operatorname{val}_\Lambda(x)}, \quad x \in E.$$  

Notice that the lattice $\Lambda$ is then the unit ball with respect to the norm $\| \cdot \|_\Lambda$, i.e.

$$\Lambda = \{ x \in E : \|x\|_\Lambda \leq 1 \}.$$  

So there is a one to one correspondence between norms on $E$ and lattices in $E$.

Given a lattice $\Lambda$ in $E$ (or a norm on $E$), there is a natural notion of orthogonality on $(E, \Lambda, \| \cdot \|_\Lambda)$ for which a family of vectors $(e_i)_{i \in I}$ is orthogonal if

$$\left\| \sum_{j \in J} \alpha_j e_j \right\|_\Lambda = \max_{j \in J} |\alpha_j| \|e_j\|_\Lambda, \quad \text{for any finite set } J \subset I \text{ and any } \alpha_j \in K.$$  

Fixing an orthonormal basis of $e_1, \ldots, e_N$ of $E$, the group of linear automorphisms of $E$ preserving the norm $\| \cdot \|_\Lambda$, that is

$$G_\Lambda := \{ g \in \operatorname{GL}(E) : \|g(x)\|_\Lambda = \|x\|_\Lambda \}$$

is isomorphic the group of matrices $\operatorname{GL}(N, O)$.

When $E = K^N$ we call $O^N$ the standard lattice in $K^N$, i.e. the lattice generated by the standard basis of $K^N$ over $O$, and we denote by $\| \cdot \|$ its associated norm. Explicitly, this norm and its corresponding valuation are given by

$$\operatorname{val}(x) := \min_{1 \leq i \leq N} \operatorname{val}(x_i) \text{ and } \|x\| := \max_{1 \leq i \leq N} |x_i|, \quad \text{for } x = (x_1, \ldots, x_N) \in K^N.$$  

From now on, unless mentioned explicitly, the norm on $K^N$ is the norm given by Equation (2.1).

The group $\operatorname{GL}(N, O)$ is the group of matrices with orthonormal rows and columns. This is also the group of linear isometries of $E = K^N$ and the Haar measure $\mu_A$ is invariant under its natural action $A^N$, i.e.,

$$\mu_A(g \cdot A) = \mu_A(A), \quad \text{for any } g \in \operatorname{GL}(N, O) \text{ and Borel set } A \subset K^N.$$  

The uniform probability measure on the standard lattice $O^N$ (with respect to the Haar measure) is called the standard Gaussian distribution on $E$.

**Remark 2.1.** For readers not familiar with non-archimedean orthogonality and Gaussian measures, we refer to [20] and [16, 15] for recent accounts. For a more detailed treatment of non-archimedean functional analysis, we refer to [37, Chapter 5].
2.2. A pinch of intersection theory

In this section we recall some facts from intersection theory of algebraic varieties (see [22, Section 18] and [14] for more details). The reader may skip this and come back to it when necessary.

Let \( X \subset \mathbb{A}^N \) be an affine algebraic manifold of dimension \( n \) and let \( d \) be its degree. Then there exists a variety \( V_X \in K^{n \times N} \times K^n \) of lower dimension such that

\[
\#(L_{A,b} \cap X) \leq d \quad \text{for any } (A,b) \in (K^{n \times N} \times K^n) \setminus V_X.
\]

Since \( V_X \) is a lower dimensional variety in \( K^{n \times N} \times K^n \) it has measure 0 with respect to the volume measure \( dAdb \). So if \( (A,b) \) is a random variable that has a density with respect to \( dAdb \), then with probability 1, the intersection \( L_{A,b} \cap X \) contains at most \( d \) points. So, given a real valued function \( f : X \to \mathbb{R} \) the function

\[
\overline{f} : (A,b) \mapsto \sum_{x \in L_{A,b} \cap X} w_X(x)f(x),
\]

is well defined on the Zariski open set \( K^{n \times N} \times K^n \setminus V_X \).

Similarly, if \( X \subset \mathbb{P}^{N-1} \) is a projective algebraic \( K \)-manifold of dimension \( n \) and degree \( d \) the set of matrices \( A \in K^{n \times N} \) such that the intersection \( L_A \cap X \) is infinite, where

\[ L_A = \{ x \in \mathbb{P}^{N-1} : Ax = 0 \}, \]

is a lower dimensional algebraic variety \( W_X \) in \( K^{n \times N} \). Hence \( W_X \) has measure 0 with respect to the Haar measure \( dA \) and

\[
\#(L_A \cap X) \leq d \quad \text{for any } A \in K^{n \times N} \setminus W_X.
\]

Moreover, given a real valued function \( f : X \to \mathbb{R} \) the function

\[
\overline{f} : A \mapsto (1 + q^{-1}) \cdots (1 + q^{-n}) \sum_{x \in L_A \cap X} f(x)
\]

is well defined on the Zariski open set \( K^{n \times N} \setminus W_X \).

2.3. The \( p \)-adic co-area formula

In this section we recall a few notions on \( p \)-adic integration on manifolds. We refer the reader to [8, 25, 30, 29] and references therein for a more detailed account.

Let \( X \) be a smooth algebraic (affine or projective) manifold defined over \( K \). One can then endow the variety \( X \) with the structure of a \( K \)-analytic manifold in the sense of [8] and a volume measure \( \mu_X \). A definition of the latter is given in Equation (1.1) for the affine case and in Equation (4.1) for the projective case.

**Definition 2.2.** Let \( X \) and \( Y \) be two \( K \)-analytic manifolds, \( x \in X \) and \( \varphi : X \to Y \) be a \( K \)-analytic map. We define the absolute Jacobian of \( \varphi \) at \( x \) as

\[
J(\varphi, x) := N(D_x \varphi),
\]

the absolute determinant of the \( K \)-linear map \( D_x \varphi : T_x X \to T_{\varphi(x)} Y \).

The following is the \( p \)-adic coarea formula from [8].

**Theorem 2.3 ([8, Theorem 6.2.1]).** Let \( X \) and \( Y \) be two analytic \( K \)-manifolds with \( \dim(X) \geq \dim(Y) \) and let \( \varphi : X \to Y \) be a \( K \)-analytic map. Then, for any function \( f : X \to \mathbb{R} \) that is integrable with respect to the volume measure on \( X \), we have

\[
\int_X J(\varphi, x)f(x)\mu_X(dx) = \int_Y \left( \int_{\varphi^{-1}(y)} f(z)\mu_{\varphi^{-1})(y)}(dz) \right) \mu_Y(dy).
\]

**Corollary 2.4.** Let \( X \) and \( Y \) be two \( K \)-manifolds and \( \varphi : X \to Y \) an analytic map from \( X \) to \( Y \).

1. Suppose that \( \xi \) is an \( X \)-valued random variable with density \( f \) with respect to \( \mu_X \). Then the density \( g \) of \( \eta = \varphi(\xi) \) with respect to \( \mu_Y \) is

\[
g(y) = \int_{\varphi^{-1}(y)} f(z) J(\varphi, z)\mu_{\varphi^{-1}(y)}(dz).
\]
(ii) Let \( \eta \) be a \( Y \)-valued random variable with density \( g \) with respect to \( \mu_Y \) and let \( \xi \) be the \( X \)-valued random variable such that, conditioned on \( (Y = y) \), the variable \( \xi \) has density \( f_y \) on \( \varphi^{-1}(Y) \) with respect to \( \mu_{\varphi^{-1}(y)} \). Then \( \xi \) has density

\[
f(x) = J(\varphi, x) g(\varphi(x)) f_{\varphi(x)}(x).
\]

Proof. (i) Let \( V \) be a Borel set in \( Y \). Applying Theorem 2.3 we get

\[
P(\eta \in V) = \int_X 1_V(\varphi(x)) f(x) \mu_X(dx)
= \int_Y \left( \int_{\varphi^{-1}(y)} 1_V(\varphi(z)) f(z) \mu_{\varphi^{-1}(y)}(dz) \right) \mu_Y(dy)
= \int_Y \left( \int_{\varphi^{-1}(y)} f(z) \frac{1_V(\varphi(z))}{J(\varphi, z)} \mu_{\varphi^{-1}(y)}(dz) \right) \mu_Y(dy)
= \int_Y g(y) \mu_Y(dy).
\]

(ii) Let \( U \) be a Borel set in \( X \). Then, applying Theorem 2.3 we get

\[
P(\xi \in U) = \mathbb{E}[P(\xi \in U|\eta)]
= \int_Y \left( \int_{\varphi^{-1}(y)} f_y(z) 1_U(z) \mu_{\varphi^{-1}(y)}(dz) \right) g(y) \mu_Y(dy)
= \int_Y \left( \int_{\varphi^{-1}(y)} g(\varphi(z)) f_{\varphi(z)}(z) 1_U(z) \mu_{\varphi^{-1}(y)}(dz) \right) \mu_Y(dy)
= \int_X J(\varphi, x) g(\varphi(x)) f_{\varphi(x)}(x) 1_U(x) \mu_X(dx)
= \int_X f(x) 1_U(x) \mu_X(dx).
\]

We denote by \( \text{Gr}(n, K^m) \) the Grassmannian variety parametrizing \( n \)-dimensional vector subspaces of \( K^m \). The orthogonal group \( \text{GL}(m, O) \) has a natural action on \( \text{Gr}(n, K^m) \).

**Lemma 2.5.** Let \( m \geq n \geq 1 \) be two integers. There exists a unique orthogonally invariant probability distribution on the Grassmannian \( \text{Gr}(n, K^m) \).

**Proof.** Since \( \text{GL}(m, O) \) acts transitively on \( \text{Gr}(n, K^m) \) and the stabilizer of the subspace generated by the first \( n \) vectors of the standard basis of \( K^m \) is

\[
H = \left\{ \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} : A \in \text{GL}(k, O), B \in \text{GL}(m - n, O) \text{ and } C \in O^{n \times (m-n)} \right\},
\]

we can write \( \text{Gr}(n, K^m) \) as a homogeneous space as follows

\[
\text{Gr}(n, K^m) = \text{GL}(m, O)/H.
\]

Let \( \nu \) be a probability measure on \( \text{GL}(m, O)/H \) that is \( \text{GL}(m, O) \)-invariant. Then its pullback \( \nu^* \) to \( \text{GL}(m, O) \) is also \( \text{GL}(m, O) \) invariant, so it is a Haar measure on \( \text{GL}(m, O) \) with \( \nu^*(\text{GL}(m, O)) = 1 \). We then conclude since \( \text{GL}(m, O) \) is a compact topological group, there is a unique Haar measure on \( \text{GL}(m, O) \) up to scaling. \( \square \)

We end this section with the following simple but useful lemma.

**Lemma 2.6.** Let \( n \geq 1 \) be a positive integer, then we have

\[
\int_{C \in O^{n \times n}} |\det(C)| dC = \frac{1 - q^{-1}}{1 - q^{-(n+1)}}.
\]
Proof. Let $C$ be a random matrix in $K^{n \times n}$ whose entries are independent and uniform in $\mathcal{O}$. Then the integral in question is the expectation $E[|\det(C)|]$. We can compute this expectation using the distribution of $|\det(C)|$ from [21, Theorem 4.1]. We then have

$$E[|\det(C)|] = (1 - q^{-1}) \cdots (1 - q^{-n}) \sum_{m=0}^{\infty} \binom{n + m - 1}{m} q^{-2m},$$

where $\binom{n}{k}_{q^{-1}}$ denotes the usual $q^{-1}$-binomial coefficient (also known as the Gaussian binomial coefficient):

$$\binom{n}{k}_{q^{-1}} := \frac{(1 - q^{-1}) \cdots (1 - q^{-n})}{(1 - q^{-1}) \cdots (1 - q^{-k}) \cdots (1 - q^{-n-k})}$$

for $n \geq k \geq 0$.

Then using the well known generating series

$$\sum_{m=0}^{\infty} \binom{n + m - 1}{m} q^{-2m} t^m = \prod_{k=0}^{n-1} \frac{1}{1 - q^{-k}t},$$

we get

$$\sum_{m=0}^{\infty} \binom{n + m - 1}{m} q^{-2m} = \prod_{k=0}^{n-1} \frac{1}{1 - q^{-k-2}} = \frac{1}{(1 - q^{-2}) \cdots (1 - q^{-(n+1)})}.$$ 

So we deduce that

$$E[|\det(C)|] = \int_{C \in \mathcal{O}^{n \times n}} |\det(C)| dC = \frac{1 - q^{-1}}{1 - q^{-(n+1)}}. \qed$$

Remark 2.7. We shall see in Section 4.1 that the integral in Lemma 2.6 is exactly the total measure projective pace $\mu_{PN}(\mathbb{P}^n)$ i.e.

$$\mu_{PN}(\mathbb{P}^n) = \frac{1 - q^{-(n+1)}}{1 - q^{-1}}.$$

3. Sampling from affine manifolds

In this section, we proceed to proving the main results of this article, namely Theorems 1.7 and 1.8. Similar results for projective manifolds are stated and proved in Section 4. We start with the following:

Lemma 3.1. Let $X \subset \mathbb{A}^N$ be an affine algebraic $K$-manifold of dimension $n$. The definition of $\text{Nr}(X,x)$ in Definition 1.4 does not depend on the choice of $U$ and $W$.

Proof. Set $S_x = \text{diag}(1, \ldots, 1, \varpi_{\text{max}(0,-\text{val}(x))})$ and let $U_1, U_2 \in \text{GL}(N, \mathcal{O})$ and $W_1, W_2 \in \mathcal{O}^{N \times n}$ be such that

1. $U_1 x = U_2 x = (0, \ldots, 0, \varpi_{\text{val}(x)})^\top$,
2. columns of $W_1, W_2$ are two orthonormal bases of the tangent space $T_x X$.

Then there exists $V \in \text{GL}(n, \mathcal{O})$ such that $W_2 = W_1 V$. Let $A = U_2 U_1^{-1}$ and $B = S_x A S_x^{-1}$ so that we have $B(S_x U_1 W_1)V = S_x U_3 W_2$. We claim that $B \in \text{GL}(N, \mathcal{O})$ is an orthogonal matrix. To see why, notice that, thanks to condition (1), the matrix $A$ is of the form

$$A = \begin{pmatrix} A' & 0 \\ \vdots & \vdots \\ z_1 & \cdots & z_{N-1} \\ 0 & \vdots & \vdots \\ 1 & \end{pmatrix},$$

where $A' \in \text{GL}(N - 1, \mathcal{O})$ and $z_1, \ldots, z_{N-1} \in \mathcal{O}$. We then deduce that $B$ is of the form

$$B = \begin{pmatrix} A' & 0 \\ \vdots & \vdots \\ \alpha z_1 & \cdots & \alpha z_{N-1} \\ 0 & \end{pmatrix},$$

where $\alpha \in \text{GL}(1, \mathcal{O})$.
where \( \alpha = \varpi^{\max(0, -\val(x))} \in \mathbb{O} \). So we deduce that \( B \in \text{GL}(N, \mathbb{O}) \). Now, since \( V \) and \( B \) are both orthogonal, from Definition 1.3 we can see that

\[
N(S_x U_1 W_1) = N(S_x U_2 W_2) = N(BS_x U_1 W_1 V),
\]

which finishes the proof. \( \Box \)

This means that the weight function \( w_X \) in Definition 1.5 is indeed well defined.

**Proposition 3.2.** Let \( X \subset \mathbb{A}^N \) be an affine algebraic \( K \)-manifold of dimension \( n \) and \( x \) a point on \( X \). We then have

\[
w_X(x) = \left( \int_{A \in K^{n \times N}} |\det(A|_{T_x X})| 1_{A \in \mathbb{O}^{n \times N}, \|Az\| \leq 1} \right)^{-1}.
\]

**Proof.** Let \( U \in \text{GL}(N, \mathbb{O}) \) such that \( y := Ux = (0, \ldots, 0, \varpi^{\val(x)})^T \). Let \( W \) be a matrix whose columns form an orthonormal basis of \( T_y X \). Let us fix the matrix \( R_x = \text{diag}(1, \ldots, 1, \varpi^{v(x)}) \in K^{N \times N} \). Let us denote by \( I_X(x) \) the following integral

\[
I_X(x) = \int_{A \in K^{n \times N}} |\det(A|_{T_x X})| 1_{A \in \mathbb{O}^{n \times N}, 1_{Ax \in \mathbb{O}^n}} \, dA.
\]

Then, by a change of variable \( BU = A \), we have

\[
I_X(x) = \int_{B \in K^{n \times N}} |\det((BU)|_{T_x X})| 1_{B \in \mathbb{O}^{n \times N}, B_1 \in \mathbb{O}^n} \, dB
= \int_{B \in K^{n \times N}} |\det(BU)| 1_{B \in \mathbb{O}^{n \times N}, B_1 \in \mathbb{O}^n} \, dB
= \int_{B \in \mathbb{O}^{n \times N} \cap (\mathbb{O}^{n \times N}R^{-1})} |\det(BU)| \, dB.
\]

Notice the following equality

\[
\mathbb{O}^{n \times N} \cap \mathbb{O}^{n \times N} R^{-1} = \mathbb{O}^{n \times N} S_x,
\]

where \( S_x = \text{diag}(1, \ldots, 1, \varpi^{\max(0, -\val(x))}) \in K^{N \times N} \). So, using the change of variables \( B = B'S_x \), we deduce that

\[
I_X(x) = \int_{B \in \mathbb{O}^{n \times N} S_x} |\det(BU)| \, dB
= \left( \frac{1}{\max(1, \|x\|)} \right)^n \int_{B' \in \mathbb{O}^{n \times N}} |\det(B'S_x UW)| \, dB'.
\]

Let us write the Smith normal form of the matrix \( S_x UW \), i.e.

\[
S_x UW = V_1 DV_2,
\]

where \( V_1 \in \text{GL}(N, \mathbb{O}), V_2 \in \text{GL}(n, \mathbb{O}) \) and \( D = \text{diag}(\varpi^{v_1}, \ldots, \varpi^{v_n}) \in K^{N \times n} \). So, by the change of variables \( B'V_1 = C \), we get

\[
I_X(x) = \frac{1}{\max(1, \|x\|^n)} \int_{C \in \mathbb{O}^{n \times N}} |\det(CD)| \, dC
= \frac{1}{\max(1, \|x\|^n)} \int_{C \in \mathbb{O}^{n \times N}} |\det(C)| \, dC.
\]

Combining the previous equation with Definition 1.4, Definition 1.5 and Lemma 2.6, we get

\[
I_X(x) = \frac{N_r(X, x)}{\max(1, \|x\|^n)} \frac{1 - q^{-1}}{1 - q^{-(n+1)}} = \frac{1}{w_X(x)}
\]

as desired. \( \Box \)

**Remark 3.3.**

(i) Proposition 3.2 gives another proof of the fact that \( N_r(X, x) \) does not depend on the choice of \( U \) and \( W \) in Definition 1.4.
(ii) Recall, from Remark 1.6, that the weight function is constant on \( X \cap O^N \). Unwinding the definition of \( w_X \) we can also see that for \( U \in \text{GL}(N, O) \) and \( x \in X \) we have \( w_{Ux}(Ux) = w_X(x) \) where \( UX = \{ Ux : x \in X \} \).

(iii) If the probability density \( f \) we wish to sample from is supported on \( X \cap \omega^{-r}O^N \), we can scale \( X \) by \( \omega^r \) and sample \( \xi' \) from \( \omega^{-r}X \cap O^N \) (where the weight function is constant) with density \( f(\omega^{-r}) \).

We can then obtain a random variable \( \xi \) on \( X \) with density \( f \) by taking \( \xi = \omega^{-r}\xi' \).

We are now ready to prove our main theorems.

**Proof of Theorem 1.7.** By definition we have

\[
E(\mathcal{J}(A, b)) = \int_{K^n \times K^n} \mathcal{J}(A, b) 1_{A \in O^n \times N, b \in O^n} \, dAdb.
\]

Let us define the following map:

\[
\varphi : K^n \times X \to K^n \times K^n, \quad (A, x) \mapsto (A, Ax).
\]

The map \( \varphi \) is analytic and its differential is given by

\[
D(\varphi) = \begin{pmatrix}
\varphi & 0
\end{pmatrix} = \begin{pmatrix}
I & A
\end{pmatrix},
\]

where \( A \) acts trivially on the first component of the product. Hence, \( D(\varphi) \) is analytic. By Definition 1.6 we have

\[
\int_{K^n \times X} |\det(A_{|T_x})| w_X(x) f(x) 1_{A \in O^n \times N, A z \in O^n} \, dA \, \mu_X(dx)
\]

\[
= \int_{K^n \times X} \mathcal{J}(\varphi, (A, x)) w_X(x) f(x) 1_{A \in O^n \times N, A z \in O^n} \, dA \, \mu_X(dx)
\]

\[
= \int_{K^n \times X} \left( \int_{\varphi^{-1}(A, y)} w_X(z) f(z) \mu_{\varphi^{-1}(A, y)}(dz) \right) 1_{A \in O^n \times N, A z \in O^n} \, dA \, \mu_{\varphi^{-1}(A, y)}(dz)
\]

\[
= \int_{K^n \times K^n} \left( \int_{\varphi^{-1}(A, y)} w_X(z) f(z) \mu_{\varphi^{-1}(A, y)}(dz) \right) 1_{A \in O^n \times N, y \in O^n} \, dAdy.
\]

But, for \( A \in K^n \times N \) and \( y \in K^n \) we have

\[
\varphi^{-1}((A, y)) = \{ (A, z) \in K^n \times X : Az = y \text{ and } z \in X \},
\]

and this is a finite set for almost every \( A \) and \( y \). So for almost every \( A \) and \( y \), the measure \( \mu_{\varphi^{-1}(A, y)} \) equals the counting measure on the finite set \( \varphi^{-1}((A, y)) \) and we then have

\[
= \int_{K^n \times N} |\det(A_{|T_x})| w_X(x) f(x) 1_{A \in O^n \times N, A z \in O^n} \, dA \, \mu_X(dx)
\]

\[
= \int_{K^n \times N} \left( \sum_{x \in X, A x = y} w_X(x) f(x) \right) 1_{A \in O^n \times N, y \in O^n} \, dAdy
\]

\[
= \int_{K^n \times N} \mathcal{I}(A, y) 1_{A \in O^n \times N, y \in O^n} \, dAdy
\]

\[
= E[\mathcal{I}(A, b)].
\]

Hence the equation

\[
E[\mathcal{I}(A, b)] = \int_X \left( \int_{K^n} |\det(A_{|T_x})| 1_{A \in O^n \times N, A z \in O^n} \, dA \right) w_X(x) f(x) \mu_X(dx).
\]
Then, combining Equation (3.3) and Proposition 3.2, we conclude that
\[ \mathbb{E} \left[ \mathcal{J}(A, b) \right] = \int_X f(x) \mu_X(dx). \]

Proof of Theorem 1.8. Let \((\bar{A}, \xi)\) be the random variable, with values in \(K^{n \times N} \times X\), obtained by first sampling \((\bar{A}, \bar{b}) \in K^{n \times N} \times K^n\) from distribution \(f(A, b)1_{A \in \mathcal{O}^{n \times N}, b \in \mathcal{O}^{n}}\) and then choosing a point \(\xi\) from \(\mathcal{L}_{\bar{A}, \bar{b}} \cap X\) with probability
\[ w_X(x) f(x), \]
then applying Corollary 2.4-(ii) to the map \(\varphi\) from Equation (3.1), we deduce that \((\bar{A}, \xi)\) has density
\[ g_{(\bar{A}, \xi)}(A, x) = f(\varphi(A, x)) 1_{\varphi(A, x) \in \mathcal{O}^{n \times N}} \frac{w_X(x) f(x)}{f(\varphi(A, x))} J(\varphi, (A, x)) \]
with respect to the volume measure \(dA \mu_X(dx)\) on \(K^{n \times N} \times X\). Computing the second marginal of this joint distribution, we deduce that the density \(g_\xi\) of \(\xi\) is
\[ g_\xi(x) = \int_{A \in K^{n \times N}} w_X(x) f(x) J(\varphi, (A, x)) 1_{\varphi(A, x) \in \mathcal{O}^{n \times N}} dA \]
\[ = w_X(x) f(x) \int_{A \in K^{n \times N}} |\det(A_{1|x})| 1_{A \in \mathcal{O}^{n \times N}} dA \]
\[ = f(x). \]
The second (resp. third) equation follows from Equation (3.2) (resp. Proposition 3.2). So, as desired, \(\xi\) has density \(f\) with respect to \(\mu_X\) on \(X\).

4. Sampling from projective manifolds

This section deals with sampling from projective algebraic manifolds. More precisely, we shall state and prove analogs of Theorem 1.7 and Theorem 1.8 in projective space.

Let \(N \geq 2\) be an integer. We denote by \(\mathbb{P}^{N-1}\) the projective space of dimension \(N - 1\) over \(K\). Let us denote by \(S^{N-1}\) the unit sphere in \(K^N\), i.e.,
\[ S^{N-1} := \{ x \in K^N : \|x\| = 1 \}. \]
We warn the reader that, unlike the Euclidean setting, the unit sphere is actually an open set in \(K^N\) and has dimension \(N\) (as a topological space). Consider the Hopf fibration
\[ \psi : S^{N-1} \to \mathbb{P}^{N-1}, \quad (x_1, \ldots, x_N) \mapsto (x_1 : \cdots : x_N). \]
The projective space \(\mathbb{P}^{N-1}\) can be endowed with a metric \(d\) defined as follows:
\[ d(x, y) = \|\hat{x} \wedge \hat{y}\|, \quad x, y \in \mathbb{P}^{N-1} \]
where \(\hat{x} \in \psi^{-1}(x), \hat{y} \in \psi^{-1}(y)\) and the norm \(\|\hat{x} \wedge \hat{y}\|\) is the standard norm in \(\wedge^2 K^N\) associated to its standard lattice \(\mathcal{O}^N\). This metric is called the Fubini-Study metric. For \(x \in \mathbb{P}^{N-1}\) and \(\epsilon > 0\) let us denote by
\[ \mathbb{B}_{N-1}(x, \epsilon) := \{ y \in \mathbb{P}^{N-1} : d(x, y) \leq \epsilon \} \]
the ball of radius \(\epsilon\) around \(x\).

Endowed with the metric \(d\), the projective space \(\mathbb{P}^{N-1}\) is a compact metric space on which we define a volume measure \(\mu_{\mathbb{P}^{N-1}}\) as follows
\[ \mu_{\mathbb{P}^{N-1}} := \frac{1}{1 - q^{-1}} \psi_* \mu_{S^{N-1}}, \]
that is the normalized push-forward of \(\mu_{S^{N-1}}\) by \(\psi\). Note that \(\mu_{S^{N-1}}(S^{N-1}) = 1 - q^{-N}\), so the measure \(\mu_{\mathbb{P}^{N-1}}\) is finite and we have
\[ \mu_{\mathbb{P}^{N-1}}(\mathbb{P}^{N-1}) = \frac{1 - q^N}{1 - q^{-1}}. \]
Remark 4.1. Notice that from Lemma 2.6, we have

\[ \int_{C \in \mathcal{O}^{\times n}} |\det(C)|dC = \frac{1}{\mu_{\mathbb{P}^n}(\mathbb{P}^n)}. \]

A projective algebraic variety in \(\mathbb{P}^{N-1}\) is the zero set of a system of homogeneous polynomials \(p = (p_1, \ldots, p_r)\) in \(K[x_1, \ldots, x_N]\); that is

\[ \{ x \in \mathbb{P}^{N-1} : p_1(x) = \cdots = p_r(x) = 0 \}. \]

We refer to irreducible and smooth projective varieties as projective algebraic manifolds.

Let \(X \subset \mathbb{P}^{N-1}\) be an algebraic projective manifold of dimension \(n \geq 1\). Similar to the affine case (1.1), we can define a volume measure on \(X\) as follows:

\[ \mu_X(V) := \lim_{\epsilon \to 0} \frac{\mu_{\mathbb{P}^{N-1}}\left( \bigcup_{x \in V} \mathbb{B}_{N-1}(x, \epsilon) \right)}{\mu_{\mathbb{P}^{N-1}}(\mathbb{B}_{N-1}(0, \epsilon))}, \quad \text{for } V \subset X \text{ open}. \]

The limit in (4.1) exists (see [32, 25] for more details) and this defines a volume measure \(\mu_X\) on the projective manifold \(X\).

Remark 4.2. For our purposes, the main difference between the affine and projective spaces is that the projective space is a compact topological space (with the quotient topology induced by the Hopf fibration \(\psi\)). So, unlike the affine case, a projective algebraic manifold admits a uniform probability density. Also, loosely speaking, there are no “far” points in the projective space, so as we shall see, the weight function is constant or, in other words, no point gets more weight than another. We can say that the space is, in some sense, “isotropic”.

Before we state our results for projective manifolds, we recall a few facts and establish a couple of preliminary results.

4.1. Preliminaries Suppose that \(X \subset \mathbb{P}^{N-1}\) is a projective algebraic manifold of dimension \(n\) defined by homogeneous polynomials \(p_1, \ldots, p_r \in K[x_1, \ldots, x_N]\) and let \(x\) be a point in \(X\). The tangent space \(T_xX\) can be defined in many ways, and one way to do so is the following. The cone \(\tilde{X} \subset A^{N+1}\) over \(X\) defined as follows

\[ \tilde{X} = \{ (\lambda y_1, \ldots, \lambda y_N) \in A^{N} : \lambda \in K \text{ and } (y_1 : \cdots : y_N) \in X \}. \]

This is an affine algebraic variety which is smooth at every non-zero point \(x \in \tilde{X} \setminus \{0\}\) and has dimension \(n+1\). The tangent space \(T_x\tilde{X}\) is a linear subspace in \(K^{N}\) of dimension \(n+1\) and \(x \in T_x\tilde{X}\). The tangent space \(T_xX\) can then be defined as an orthogonal complement\(^6\) of the line \(K \cdot x\) in \(T_x\tilde{X}\) and we thus view \(T_xX\) as a linear subspace\(^7\) of \(K^{N}\) of dimension \(n\).

Proposition 4.3. Let \(X \subset \mathbb{P}^{N-1}\) be a projective algebraic manifold of dimension \(n\) and let us define \(\tilde{X} \subset A^{N+1} \times \mathbb{P}^{N-1}\) as follows:

\[ \tilde{X} = \{ (A, x) \in A^{n \times N} \times X : Ax = 0 \}. \]

Then \(\tilde{X}\) is a manifold, and for \((A, x) \in \tilde{X}\) we have

\[ T_{(A,x)}\tilde{X} = \{ (H, h) \in K^{n \times N} \times T_xX : Hx + Ah = 0 \}. \]

Moreover, if \(\varphi, \phi\) are the projections from \(\tilde{X}\) to \(A^{n \times N}\) and \(\mathbb{P}^{N-1}\) respectively, then we have

\[ \frac{\partial p_i}{\partial x_j}(A, x) = |\det(A)T_xX)|, \]

for \((A, x) \in \tilde{X}\) such that \(A|T_xX\) is an isomorphism.

Proof. Let \((p_1, \ldots, p_r) \in K[x_1, \ldots, x_N]\) be homogeneous polynomials generating the ideal of \(X\). Let \((A, x) \in \tilde{X}\) and let \(J_x\) be the following Jacobian matrix

\[ J_x = \left( \frac{\partial p_i}{\partial x_j}(x) \right)_{1 \leq i \leq r, 1 \leq j \leq N}. \]

\(^6\) All such vector spaces are isomorphic to one another.

\(^7\) The projective tangent space is also often defined as the projectivisation of \(T_x\tilde{X}\).
Then, considering $\mathcal{X}$ as the variety in $K^{n \times N} \times \mathbb{P}^{N-1}$ cut out by the equations $Ax = 0$ and $p_1(x) = \cdots = p_r(x) = 0$ we can compute the Jacobian matrix of $\mathcal{X}$ at the point $(A, x)$. This matrix represents the linear map

$$K^{n \times N} \times K^N \to K^n \times K^r, \quad (H, h) \mapsto (Hx + Ah, J_x h).$$

The tangent space of $\mathcal{X}$ at $(A, x)$ is the kernel of this map, so

$$T_{(A,x)} \mathcal{X} = \{(H, h) \in K^{n \times N} \times T_x \mathcal{X} : Hx + Ah = 0\}.$$ 

The projection maps $\varphi, \phi$ are clearly analytic, and for any $(A, x) \in \mathcal{X}$ we have

$$d_{(A,x)} \varphi : T_{(A,x)} \mathcal{X} \to K^{n \times N} \quad \text{and} \quad d_{(A,x)} \phi : T_{(A,x)} \mathcal{X} \to T_x \mathcal{X}$$

$$\quad (H, h) \mapsto H \quad \quad \quad \quad \quad (H, h) \mapsto h.$$

Suppose that $(A, x) \in \mathcal{X}$ is such that $A|_{T_x \mathcal{X}}$ is an isomorphism. Fix $U \in \text{GL}(N, \mathcal{O})$ such that $Ux = (1 : \cdots : 0)^T$ and define the maps

$$\pi_1 : K^{n \times N} \to T_{(A,x)} \mathcal{X} \quad \quad \quad \quad \quad \pi_2 : T_x \mathcal{X} \to T_{(A,x)} \mathcal{X}$$

$$H \mapsto (H, -(A|_{T_x \mathcal{X}})^{-1} Hx) \quad \quad \quad \quad \quad h \mapsto (-Ah(0), U, h)$$

where $(-Ah(0)) \in K^{n \times N}$. Notice that $d_{(A,x)} \varphi \circ \pi_1 = \text{Id}_{K^{n \times N}}$ and $d_{(A,x)} \phi \circ \pi_2 = \text{Id}_{T_x \mathcal{X}}$. Since $A \in \mathcal{O}^{n \times N}$ we have

$$\mathcal{N}(\pi_2) = 1,$$

because $\pi_2$ sends any orthonormal basis of $T_x \mathcal{X}$ to an orthonormal family in $T_{(A,x)} \mathcal{X} \subset K^{n \times N} \times K^N$. Also, since $A \in \mathcal{O}^{N \times N}$, the singular values of $A|_{T_x \mathcal{X}}$ are all in $\mathcal{O}$ so the singular values of $A|_{T_x \mathcal{X}}$ have negative or zero valuation. From this we can see that

$$\mathcal{N}(\pi_1) = \frac{1}{|\det(A|_{T_x \mathcal{X}})|}.$$ 

We deduce that

$$\frac{\mathcal{J}(\varphi, (A, x))}{\mathcal{J}(\phi, (A, x))} = \frac{\mathcal{N}(\pi_2)}{\mathcal{N}(\pi_1)} = \frac{1}{|\det(A|_{T_x \mathcal{X}})|}. \quad \Box$$

**Lemma 4.4.** Let $X$ be a projective manifold of dimension $n$ in $\mathbb{P}^{N-1}$ and $x$ be a point on $X$. Set $M_x := \{ A \in K^{n \times N} : Ax = 0 \}$. Then

$$\int_{M_x} |\det(A|_{T_x \mathcal{X}})| \cdot 1_{A \in \mathcal{O}^{n \times N}} \cdot \mu_{M_x}(dA) = \frac{1 - q^{-1}}{1 - q^{-(n+1)}}.$$ 

**Proof.** Let $W \in \mathcal{O}^{N \times n}$ be a matrix whose columns form an orthonormal basis of $T_x \mathcal{X}$ and let $U \in \text{GL}(N, \mathcal{O})$ such that $Ux = e_1 = (1 : 0 : \cdots : 0)^T$. The space $M_x$ is a vector space of dimension $(N-1) \times n$ and $M_e U = M_x$. So, with the change of variable $BU = A$, we get

$$\int_{M_x} |\det(A|_{T_x \mathcal{X}})| \cdot 1_{A \in \mathcal{O}^{n \times N}} \cdot \mu_{M_x}(dA) = \int_{A \in M_x} |\det(AW)| \cdot 1_{A \in \mathcal{O}^{n \times N}} \cdot \mu_{M_x}(dA)$$

$$= \int_{B \in M_{n+1}} |\det(BUW)| \cdot 1_{B \in \mathcal{O}^{N \times n}} \cdot \mu_{M_x}(dA)$$

$$= \int_{C \in \mathcal{O}^{N \times (N-1)}} |\det((0 | C)UW)| \cdot dC.$$ 

Let $\tilde{W} \in K^{(N-1) \times n}$ be the matrix obtained from $UW$ by deleting the first row and let us write the Smith normal form of $\tilde{W}$ as

$$\tilde{W} = V_1 DV_2,$$

where $V_1 \in \text{GL}(N-1, \mathcal{O}), V_2 \in \text{GL}(n, \mathcal{O})$ and $D = \text{diag}(\varpi^{v_1}, \ldots, \varpi^{v_n}) \in K^{(N-1) \times n}$. We then deduce that

$$\int_{M_x} |\det(A|_{T_x \mathcal{X}})| \cdot 1_{A \in \mathcal{O}^{n \times N}} \cdot \mu_{M_x}(dA) = \int_{C \in \mathcal{O}^{n \times (N-1)}} |\det(C\tilde{W})| \cdot dC$$

$$= q^{-(v_1 + \cdots + v_n)} \int_{C \in \mathcal{O}^{n \times N}} |\det(C)| dC$$

$$= \mathcal{N}(\tilde{W}) \frac{1 - q^{-1}}{1 - q^{-(n+1)}}.$$
Since $X$ is a projective manifold, the tangent space $T_xX$ is orthogonal to $x$ (see Section 4.1). We deduce that the columns of $UW$ are orthogonal to $(1,\ldots,0)^\top$ so $\tilde{W}$ has orthonormal columns. Hence $N(\tilde{W}) = 1$ which finishes the proof.

Similarly to the affine case given a real valued function $f : X \to \mathbb{R}$, we define the weighted average function of $f$ as follows:

$$\overline{f}(A) = \sum_{x \in \mathcal{L}_A \cap X} f(x), \quad \text{for } A \in K^{n \times N}.$$ 

By convention, the sum is taken to be 0 whenever $\mathcal{L}_A \cap X$ is empty or infinite.

### 4.2. Sampling from projective manifolds

Now we state and prove the analogues of Theorems 4.2 and 4.3 for the projective case.

**Theorem 4.5.** Let $X \subset \mathbb{P}^{N-1}$ be an $n$-dimensional projective algebraic manifold defined over $K$. Let $A$ be a random variable in $K^{n \times N}$ with distribution $1_{A \in \mathcal{O}^{n \times N}}dA$. Then we have

$$\int_X f(x)\mu_X(dx) = \mu_{\mathbb{P}^n}(\mathbb{P}^N)\mathbb{E}[\overline{f}(A)].$$

**Proof.** Let $\mathcal{X} \subset K^{n \times N} \times \mathbb{P}^{N-1}$ be the algebraic variety defined by

$$\mathcal{X} := \{(A,x) \in K^{n \times N} \times X : Ax = 0\}.$$ 

Let us denote by $\varphi$ and $\phi$ the natural projections from $\mathcal{X}$ onto $K^{n \times N}$ and $X$ respectively, and, for a point $x \in X$, set $M_x := \{A \in K^{n \times N} : Ax = 0\}$. We apply Theorem 2.3 on $\varphi$ and then on $\phi$ to get the following

$$\mathbb{E}[\overline{f}(A)] = \int_{K^{n \times N}} \left(\sum_{x \in \mathcal{X}, Ax = 0} f(x)\right) 1_{A \in \mathcal{O}^{n \times N}}dA$$

$$= \int_{K^{n \times N}} \left(\int_{(A,x) \in \varphi^{-1}(A)} f(z)1_{A \in \mathcal{O}^{n \times N}}\mu_{\varphi^{-1}(A)}(dz)\right)dA$$

$$= \int_X J(\varphi, (A,x)) f(x) 1_{A \in \mathcal{O}^{n \times N}}\mu_X(dA, dx)$$

$$= \int_X \left(\int_{(A,x) \in \phi^{-1}(x)} \frac{J(\varphi, (A,x))}{J(\phi, (A,x))} 1_{A \in \mathcal{O}^{n \times N}}\mu_{\phi^{-1}(A)}(dz)\right)f(x)\mu_X(dx)$$

$$= \int_X \left(\int_{A \in M_x} |\det(A)|_{T_xX}|1_{A \in \mathcal{O}^{n \times N}}\mu_{\phi^{-1}(x)}(dA)\right)f(x)\mu_X(dx)$$

$$= \frac{1 - q^{-1}}{1 - q^{-(n+1)}} \int_X f(x)\mu_X(dx)$$

$$= \frac{1}{\mu_{\mathbb{P}^n}(\mathbb{P}^N)} \int_X f(x)\mu_X(dx).$$

The last equality follows from Lemma 4.4. □

**Theorem 4.6.** Let $X \subset \mathbb{P}^N$ be an $n$-dimensional projective algebraic manifold defined over $K$. Let $f : X \to \mathbb{R}_{\geq 0}$ be a probability density with respect to the volume measure $\mu_X$ on $X$. Let $\tilde{A}$ be the random variable in $K^{n \times N}$ with distribution $1 - q^{-(n+1)}\overline{f}(A)1_{A \in \mathcal{O}^{n \times N}}dA$.

Let $\xi$ be the random variable obtained by intersecting $X$ with the random space $\mathcal{L}_{\tilde{A}}$ and choosing a point $x$ in the finite set $X \cap \mathcal{L}_{\tilde{A}}$ with probability $\frac{f(x)}{\overline{f}(A)}$.

Then $\xi$ has density $f$ with respect to $\mu_X$. 
Proof. Let \((\tilde{A}, \xi)\) be the random variable with values in \(X\) (as defined in Proposition 4.3) such that \(\tilde{A}\) has distribution
\[
\hat{f}(A)1_{A \in O^\times N}dA
\]
and, given \(\tilde{A}, \xi\) is a random point in \(L_{\tilde{A}} \cap X\) with probability
\[
P(\xi = x|\tilde{A}) = \frac{f(x)}{\hat{f}(A)}.
\]
Then, by virtue of Corollary 2.4-(ii) applied to the projection map \(\varphi : X \to K^{n \times N}\), we deduce that the density of \((\tilde{A}, \xi)\), with respect to \(\mu_X\), is given by
\[
f_{\tilde{A}, \xi}(A, x) = \frac{1 - q^{-(n+1)}}{1 - q^{-1}} \hat{f}(A)1_{A \in O^\times N} f(x) \varphi(A, x)) 
= \frac{1 - q^{-(n+1)}}{1 - q^{-1}} f(x) \varphi(A, x)),
\]
for \((A, x) \in X\). Applying Corollary 2.4-(i) of to the projection map \(\phi : X \to X\), we then deduce that the density of \(\xi\) is
\[
f_\xi(x) = \int_{\varphi^{-1}(x)} \frac{1 - q^{-(n+1)}}{1 - q^{-1}} f(x) \varphi(A, x)) \mu_{\phi^{-1}(x)}(dA) 
= \frac{1 - q^{-(n+1)}}{1 - q^{-1}} f(x) \int_{\varphi^{-1}(x)} \mu_{\phi^{-1}(x)}(dA) 
= \frac{1 - q^{-(n+1)}}{1 - q^{-1}} f(x) \sum_{\phi^{-1}(x)} |\text{det}(A|T_x X)| \mu_{\phi^{-1}(x)}(dA) 
= f(x).
\]
The last equation follows from the Lemma 4.4. This concludes the proof. \(\square\)

5. Sampling linear spaces in practice

In this section we explain how to sample the random planes \(L_{A,b}\) and \(L_A\) explicitly. We also explain how to sample the random planes \(L_{\tilde{A}, b}\) from Theorem 1.8 and \(L_{\tilde{A}}\) from Theorem 4.6 by rejection sampling, and we give bounds on how efficient this sampling method is.

5.1. Sampling linear spaces explicitly When the codimension of the manifold \(X\) is small (hypersurfaces for example), for computational reasons, it is easier to find the intersection of \(X\) with a linear space \(E\) of complementary dimension \(N - n\) when the latter has an explicit form. That is writing \(E\) in the form
\[
E = u + \text{span}_K(x_1, \ldots, x_{N-n}),
\]
where \(u \in K^N\) and \(x_1, \ldots, x_{N-n} \in K^N\) are linearly independent.

**Lemma 5.1.** Let \(A \in K^{n \times N}\), \(b \in K^n\) and \(B \in K^{(N+1) \times (N-n+1)}\) be matrices with random i.i.d entries uniformly distributed in \(O\), and \(u, x_1, \ldots, x_{N-n} \in K^N\) be such that
\[
\begin{pmatrix} u \\ 1 \end{pmatrix}, \begin{pmatrix} x_1 \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} x_{N-n} \\ 0 \end{pmatrix}
\]
form an orthonormal basis of \(\text{columnspan}(B)\).

The random affine space \(E_{u,x_1,\ldots,x_{N-n}} := u + \text{span}(x_1, \ldots, x_{N-n})\) has the same probability distribution as \(L_{A,b}\).

**Proof.** Notice that the linear space \(L_{A,b}\) can be written as
\[
\mathcal{L}_{A,b} = \left\{ x \in K^N : (A-b)x = 0 \right\}.
\]
So it suffices to show that \(\text{columnspan}(B)\) and \(\text{Ker}(A-b)\) have the same distribution in the Grassmannian \(\text{Gr}(N-n+1, K^{N+1})\). Thanks to Lemma 2.5, it is enough to notice that the distributions of \(\text{columnspan}(B)\)
and \( \text{Ker}((A - b)) \) are both orthogonally invariant. This is indeed the case since for any \( U \in \text{GL}(N + 1, \mathcal{O}) \) we have

\[
(A - b)U \overset{d}{=} (A - b) \quad \text{and} \quad UB \overset{d}{=} B.
\]

### 5.2. Rejection sampling

Let \( X \subset \mathbb{A}^N \) be an affine algebraic manifold of dimension \( n \) and degree \( d \) and let \( f: X \to \mathbb{R}_{\geq 0} \) be a probability density function with respect to \( \mu_X \). We recall that the average function \( \overline{f} \) in the affine case is defined as

\[
\overline{f}(A, b) = \frac{\sum_{x \in \mathcal{L}_{A,b} \cap X} w_X(x)f(x)}{w_X(A, b)}, \quad \text{for } (A, b) \in K^{n \times N} \times K^n,
\]

where, by convention, the sum is 0 whenever the intersection \( \mathcal{L}_{A,b} \cap X \) is empty or infinite.

**Proposition 5.2** (Rejection sampling). Suppose that there exists a constant \( M > 0 \) such that \( \overline{f}(A, b) < M \) almost everywhere with respect to \( dAdb \). Let \( (A, b) \) be the random variable with distribution \( 1_{A \in \mathcal{O}^{n \times N}, b \in \mathcal{O}^n} \) \( dAdb \) and let \( \eta \) be a random variable such that

\[
P(\eta = 1|(A, b)) = \frac{\overline{f}(A, b)}{M} \quad \text{and} \quad P(\eta = 0|(A, b)) = \frac{M - \overline{f}(A, b)}{M}.
\]

Then, conditioned on the event \( (\eta = 1) \), the random variable \((A, b)\) has distribution

\[
\overline{f}(A, b)1_{A \in \mathcal{O}^{n \times N}, b \in \mathcal{O}^n} \, dAdb.
\]

**Proof.** This follows directly from Bayes’ rule as follows

\[
P\left((A, b) \in (dA, db)|\eta = 1\right) = \frac{P\left(\eta = 1|(A, b) \in (dA, db)\right)P\left((A, b) \in (dA, db)\right)}{P(\eta = 1)}
\]

\[
= \frac{P\left(\eta = 1|(A, b) \in (dA, db)\right)}{P(\eta = 1)} \cdot 1_{A \in \mathcal{O}^{n \times N}, b \in \mathcal{O}^n} \, dAdb
\]

\[
= \frac{P(\eta = 1)/M}{E[\overline{f}(A, b)]/M} \cdot 1_{A \in \mathcal{O}^{n \times N}, b \in \mathcal{O}^n} \, dAdb
\]

\[
= \overline{f}(A, b).
\]

The last equation follows from Theorem 1.7 and the equality

\[
P(\eta = 1) = \mathbb{E}[P(\eta = 1|(A, b))] = \frac{1}{M} \mathbb{E}[\overline{f}(A, b)] = \frac{1}{M}.
\]

**Lemma 5.3.** Let \( f: X \to \mathbb{R}_{\geq 0} \) be a probability density function supported on \( X \cap \varpi^{-r}\mathcal{O}^N \) for some integer \( r \geq 0 \). Suppose that \( \kappa := \sup_{x \in X} \right] f(x) < \infty \). Then we have

\[
\overline{f}(A, b) \leq dq^{(n+1)r} \frac{1 - q^{-(n+1)}}{1 - q^{-1}} \kappa.
\]

In particular, if \( f \) is the uniform probability density on \( X \cap \mathcal{O}^N \) then

\[
\overline{f}(A, b) \leq d \frac{1 - q^{-(n+1)}}{1 - q^{-1}}.
\]

**Proof.** Let \( x \in X \) and \( U, W, S_x \) as in Definition 1.4. Then, since the columns of \( UW \) are orthonormal in \( K^N \), its rows are in \( \mathcal{O}^n \) and, modulo \( \varpi \), they span \( k^n \). So we deduce that

\[
\text{Nr}(X, x) = \text{N}(S_x UW) \geq \min(1, \|x\|^{-1}).
\]

Hence, from Definition 1.5, we get

\[
w_X(X, x) \leq \frac{1 - q^{-(n+1)}}{1 - q^{-1}} \max(1, \|x\|^{n+1}).
\]

---

8By \( \overset{d}{=} \) we mean equality in distribution.
Then for \((A, b) \in K^{n \times N} \times K^n\) we get
\[
\overline{T}(A, b) \leq \#(X \cap L_{A,b}) \frac{1 - q^{-\nu(n+1)}}{1 - q^{-1}} \sup_{x \in X} f(x).
\]
Since the number of intersection points \#(X \cap L_{A,b}) is at most \(d = \deg(X)\) (except for a measure zero set of \((A, b) \in K^{n \times N} \times K^n\), see Section 2.2), we deduce the desired result. The second statement is an immediate consequence of the first one.

**Remark 5.4.** The bound given for \(\overline{T}(A, b)\) is far from being sharp. Moreover, when one wishes to sample from \(X \cap \omega^{-1} O^N\), this bound is not very practical for rejection sampling. In this case, it is better to use Remark 3.3 (iii).

Let \(h : X \to \mathbb{R}\) be an integrable function on \(X\) supported on \(X \cap O^N\) and let \((A_i, b_i)_{i \geq 0}\) be a sequence of i.i.d random variables such that \((A_i, b_i)\) has the uniform distribution on \(O^{n \times N} \times O^n\) for all \(i \geq 0\). Finally, set
\[
S_m(h) := \overline{T}(A_1, b_1) + \overline{T}(A_2, b_2) + \cdots + \overline{T}(A_m, b_m).
\]
Then we have the following:

**Proposition 5.5.** The random variable \(S_m(h)/m\) converges almost surely to the integral \(I(h) := \int_X h(x) \mu_X(dx)\) as \(m \to \infty\). Moreover, if \(\kappa := \sup_{x \in X} |h(x)| < \infty\), then
\[
P\left(\left|\frac{S_m(h)}{m} - I(h)\right| \geq \epsilon\right) \leq \frac{\kappa^2 d^2}{\epsilon^2 m} \left(1 - q^{-\nu(n+1)}\right)^2, \quad \text{for} \ m \geq 1.\]

**Proof.** The first statement is an immediate application of the law of large numbers. The second follows from Lemma 5.3 and Chebychev’s inequality.

**Remark 5.6.** While this section focuses on affine manifolds, the results discussed within can be restated and proved for projective manifolds without much difficulty.

### 6. Applications and examples

In this section we discuss a few concrete examples and applications. The first case of interest is when the algebraic manifold \(X\) is an algebraic group.

**6.1. Measures on algebraic groups** Let \(G\) be an algebraic group defined over \(K\), by which we mean a smooth (either affine or projective) algebraic variety together with

1. (identity element) an element \(e \in G\),
2. (multiplication) a morphism \(m : G \times G \to G, (x, y) \mapsto xy\),
3. (inverse) a morphism \(\iota : G \to G, x \mapsto x^{-1}\),

with respect to which \(G\) is a group (see [28] or [4] for a detailed account). In our discussion, \(m\) and \(\iota\) are \(K\)-morphisms and we are interested in the group \(G(K)\) of \(K\) points of \(G\) which we also denote by \(G\) for simplicity and, for our purposes, \(G\) is embedded in some affine or projective space over \(K\).

The group \(G\) is a locally compact topological group and thus admits a left Haar measure; that is a non-zero measure \(\nu_G\) such that
\[
\nu(gA) = \nu(A), \quad \text{for any Borel measurable set } A \subset G.
\]
which is unique up to scaling. If \(G\) is an algebraic group embedded in a projective space as an algebraic manifold, then \(G\) is compact and the measure \(\mu_G\) is then finite and also right-invariant. In this case we normalize \(\nu\) so that \(\nu(G) = 1\). In the case where \(G\) is affine, the measure \(\nu\) is finite on the set \(G(O)\) of \(O\)-points of \(G\) and we normalise \(\nu\) so that \(\nu(G(O)) = 1\).

**Remark 6.1.** It is not always the case that the points in \(G(O)\) form a subgroup of \(G\). For example, this fails to be the case for \(G = GL(n, K)\).

**Example 6.2.** Let \(n \geq 1\) be a positive integer. If \(G\) is either the special linear group \(SL(n, K)\) or the special orthogonal group \(SO(n, K)\) or the symplectic group \(Sp(n, K)\), the \(O\)-points \(G(O)\) form a compact subgroup of \(G\). Moreover, the normalized Haar measure \(\nu\) on \(G(O)\) coincides with the uniform probability measure on \(G(O)\) with respect to the volume measure \(\mu_G\) (as defined in Equation (1.1)). This is because the measure
\( \mu_{\text{Haar}} \) is invariant under the action of \( \text{GL}(n, \mathcal{O}) \) and in particular under the action of \( G(\mathcal{O}) \subset \text{GL}(n, \mathcal{O}) \) and hence \( \mu_G \) is also \( G(\mathcal{O}) \)-invariant. So using Theorem \ref{thm:haar} we can sample from the Haar measure on the compact matrix groups \( \text{SL}(n, \mathcal{O}), \text{SO}(n, \mathcal{O}) \) and \( \text{Sp}(n, \mathcal{O}) \). For small values of \( n \), we provide examples of this in the repository (1.2).

In general however, the measure \( \mu_G \) (as defined in Equation \ref{eq:mu_G} or Equation \ref{eq:mu_G2}) may not be invariant under the action of \( G \). In other words, the following may fail:

\[
\mu_G(g \cdot A) = \mu_G(A), \quad \text{for any Borel set } A \subset G.
\]

Note that the measure \( \mu_G \) depends on how \( G \) is embedded in its ambient space.

### 6.2. Moduli spaces

Another case of interest is when the algebraic manifold \( X \) is a moduli space parametrizing certain objects. Then sampling from \( X \), we can get an idea of how often a certain property of these objects holds or how rare are objects of certain kind are in \( X \). We give two examples of such a situation.

#### 6.2.1. Modular curves

Let \( N \) be a positive integer and consider the modular curve \( X_1(N) \). This is a smooth projective curve defined over \( \mathbb{Q} \), and it has the following moduli interpretation: for any field \( K \) with characteristic 0, noncuspidal \( K \)-points of \( X_1(N) \) classify isomorphism classes of pairs \( (E, P) \), where \( E \) is an elliptic curve over \( K \) and \( P \) is a point of \( E(K) \) of order \( N \). For the theory of modular curves, see [13]. See also [33, Section C.13] for a quick introduction.

In this example, we will sample uniformly from \( \mathbb{Z}_{31} \)-points of \( X_1(30) \), and compute the Tamagawa numbers of the corresponding elliptic curves over \( \mathbb{Q}_{31} \). For an elliptic curve \( E/\mathbb{Q}_p \), the finite index

\[
c_p = [E(\mathbb{Q}_p) : E^0(\mathbb{Q}_p)]
\]

is referred to as the Tamagawa number of \( E/\mathbb{Q}_p \), where \( E^0(\mathbb{Q}_p) \) is the subgroup of \( E(\mathbb{Q}_p) \) consisting of points that have good reduction. Clearly, if \( E/\mathbb{Q}_p \) has good reduction, then \( c_p \) equals 1. We note that Tamagawa numbers of elliptic curves are important local arithmetic invariants. They arise in the conjecture of Birch and Swinnerton-Dyer, for example; see [33, Section C.16]. Moreover, they can be easily computed using Magma [5].

The following (optimized) equation for \( X_1(30) \) was provided by Sutherland in [34]:

\[
X_1(30) : y^6 + (x^6 - 5x^5 + 6x^4 + 3x^3 - 6x^2 + 7x + 3)y^5 \\
+ (x^7 - 3x^6 - 13x^5 + 44x^4 - 18x^3 + x^2 + 18x + 3)y^4 \\
+ (x^8 - 3x^7 - 13x^6 + 27x^5 + 46x^4 - 32x^3 + 21x^2 + 15x + 1)y^3 \\
+ 2x(x^7 - 8x^6 + 9x^5 + 20x^4 + 6x^3 - 6x^2 + 9x + 2)y^2 \\
- 4x^2(2x^5 - 7x^4 - 3x^3 - 1)y + 8x^6 = 0.
\]

Moreover, if \( (x_0, y_0) \) is a noncuspidal point on \( X_1(30) \), then the corresponding elliptic curve is of the form

\[
y^2 = x^3 + (t^2 - 2qt - 2)x^2 - (t^2 - 1)(qt + 1)^2x,
\]

where

\[
q = y_0 + 1,
\]

\[
t = 4(y_0 + 1)(x_0 + y_0)/(x_0 y_0^3 - 4x_0 y_0 - 4x_0 - 3y_0^3 - 6y_0^2 - 4y_0).
\]

See the table in https://math.mit.edu/~drew/X1_optcurves.html. Table 1 presents the Tamagawa numbers of elliptic curves obtained for a sample of 500,000 \( \mathbb{Z}_{31} \)-points on \( X_1(30) \), and the number of times they occurred.

---

9Modular curves have only finitely many cuspidal points. This will be important for what follows.


| C31 | C31 | C31 |
|-----|-----|-----|
| 1   | 266775 | 8 | 1 | 20 | 382 |
| 2   | 53317  | 9 | 48 | 242 | 2 |
| 3   | 56726  | 10 | 13174 | 30 | 6549 |
| 4   | 1601   | 12 | 1804 | 45 | 16 |
| 5   | 27759  | 15 | 12956 | 60 | 192 |
| 6   | 58623  | 18 | 68 | 90 | 7 |

Table 1. The Tamagawa numbers and their multiplicities that appeared in our sampling.

6.2.2. Hilbert modular surfaces

Here, we will work with Hilbert modular surfaces $Y_-(D)$, with the notation in [17]. These surfaces parametrize abelian surfaces with real multiplication. More precisely, let $d > 1$ be a squarefree integer, and set

$$D = \begin{cases} d & \text{if } d \equiv 1 \mod 4, \\ 4d & \text{if } d \equiv 2, 3 \mod 4. \end{cases}$$

Note that $D$ is nothing but the discriminant of the ring of integers $\mathcal{O}_D$ of the real quadratic field $\mathbb{Q}(\sqrt{D})$. Such a number is called a positive fundamental discriminant. The quotient

$$\text{PSL}_2(\mathcal{O}_D) \setminus (\mathcal{H}^+ \times \mathcal{H}^-)$$

is the coarse moduli space of principally polarized abelian surfaces with real multiplication by $\mathcal{O}_D$. Here, $\mathcal{H}^+$ (resp. $\mathcal{H}^-$) denotes the complex upper (resp. lower) half plane. There is a holomorphic map from this quotient to the moduli space $A_2$ of principally polarized abelian surfaces. The image is the Humbert surface $\mathcal{H}_D$, and the Hilbert modular surface $Y_-(D)$ is a double cover of $\mathcal{H}_D$ branched along a finite union of modular curves. For the theory of Hilbert modular surfaces, see, for example, [36, 7].

The surfaces $Y_-(D)$ have models over $\mathbb{Q}$, and points on these surfaces correspond, generically, to Jacobians of smooth projective curves\footnote{Recall that a principally polarized abelian surface over an algebraically closed field is either the Jacobian variety of a smooth projective curve of genus 2 or the product of two elliptic curves.} of genus 2. Explicit equations for birational models of $Y_-(D)$, as well as the Igusa–Clebsch invariants $I_2, I_4, I_6$ and $I_{10}$ of the corresponding genus-2 curves, were provided by Elkies and Kumar in [17] for all fundamental discriminants $D$ between 1 and 100. In this final example, we will

- sample uniformly from $\mathbb{Z}_5$-points of $Y_-(5)$, and
- compute the minimal skeleta of the Berkovich analytifications of the corresponding genus-2 curves.

It is well known that there are precisely 7 different (graph-theoretical) types, which are depicted in Figure 2. The recent work of Helminck [23] shows that tropical Igusa invariants, which can easily be computed from Igusa–Clebsch invariants, distinguish between the different types; see [23, Theorem 2.11]. See also [24] for a similar result concerning Picard curves.

![Figure 2. Minimal skeleta of the Berkovich analytifications of genus-2 curves.](image-url)

A birational model of the surface $Y_-(5)$ is given by

$$z^2 = 2 \left( 6250h^2 - 4500g^2h - 1350gh - 108h - 972g^5 - 324g^4 - 27g^3 \right).$$
see [17, Theorem 16]. Moreover, the map from \( \mathcal{H}_5 \) to \( \mathcal{A}_2 \) (or, more precisely, to the moduli space \( \mathcal{M}_2 \) of curves of genus 2) is given by

\[
(I_2 : I_4 : I_6 : I_{10}) = (6(4g + 1), 9g^2, 9(4h + 9g^3 + 2g^2), 4h^2)
\]

see [17, Corollary 15]. Table 2 shows how many times the types occurred for a sample of 500000 \( \mathbb{Z}_5 \)-points on \( Y_2 \).

| Type I | Type II | Type III | Type IV | Type V | Type VI | Type VII |
|--------|---------|----------|---------|--------|---------|----------|
| 414900 | 0       | 40338    | 23040   | 21688  | 2       | 32       |

**Table 2.** The multiplicities of the types that appeared in our sampling.

As shown the in table, Types II, VI and VII are quite rare. In fact, we never see Type II, and it is unclear to the authors if there is a theoretical reason behind this.

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