Boundary value problems of elliptic operators and reduction to the boundary techniques

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ABSTRACT
We study properties of pseudodifferential operators which arise in their use in boundary value problems. Smooth domains as well as intersections of smooth domains are considered.

1. Introduction

The purpose of this article is twofold. First, in the case of smoothly bounded domains, we collect and expand on some of the known machinery involved in the technique of reducing a boundary value problem to the boundary. Second, we study some of the operators involved in the technique in the case of intersection domains. In this case, we introduce some weighted Sobolev spaces which are of use in concluding estimates.

In the case of smooth domains, we follow the work of Hörmander [1] to handle boundary value problems of elliptic equations in Section 2. We provide our own proofs of the mapping properties of operators we study, although several of the proofs can be adapted from those in [2] or [3] or [1]. One advantage in providing our own presentation is the relaxation of some of the assumptions in the above classical works. For instance, we do not consider only symbols which are rational as in [3] or [1] or even symbols with poles given by homogeneous first order tangential symbols as in [2]. Another useful advantage is the immediate recognition of the inverses to elliptic operators we consider in our examples as belonging to the class of operators studied without further reductions or expansions.

It is the case, however, that in our discussion of smooth domains we work with operators which are similar to, or can be reduced to, those of Boutet de Monvel in [2]. We use operators which we define to be decomposable (see Definition 2.8) which, roughly speaking, defines operators with symbols which are meromorphic with respect to the transform variable dual to the defining function, has residues in this variable which themselves are
symbols (in the tangential directions), as well as poles whose imaginary parts are elliptic symbols in the tangential directions. We recall the definition of the transmission property given in [2]. Let \((x, \rho) \in \mathbb{R}^{n+1}\) be coordinates near a boundary point (taken to be the origin) of a smooth domain, where \(\rho\) is a defining function for the domain. Then with \(\eta\) the transform variable dual to \(\rho\) and \(\xi \in \mathbb{R}^n\), the dual to \(x \in \mathbb{R}^n\), an operator of order \(k\) has the transmission property if its symbol (and its derivatives with respect to the \(x\) and \(\rho\) variables) has an expansion of the form

\[
\sum_{j=0}^{k} \alpha_j(x, \xi) \eta^j + \sum_{j=0}^{\infty} \beta_j(x, \xi) \frac{(|\xi| - i\eta)^j}{(|\xi| + i\eta)^{j+1}},
\]

for \(\rho = 0\), with \(\alpha_j\) a symbol of order \(k-j\) and \(\beta_j\) a symbol of order \(k+1\), modulo smoothing operators. Our definition of decomposable allows for inhomogeneous poles in the denominators, for instance, a symbol of the form

\[
\frac{1}{(\eta - i|\xi|b(x, \xi))^2}
\]

for some (non-vanishing) zero order symbol, \(b(x, \xi)\). It is not, however, for this (slight) increase in generality that we introduce our definition but rather because it is immediate that an operator falls under our definition just by looking at its poles without first having to apply contour integrations or a partial fractions decomposition to see if it fits the transmission property. For instance, it is immediately seen that the inverses to the elliptic operators we consider satisfy our definition of decomposable, even if all the analysis with some reduction work could be handled by looking at the mapping properties of the operators with transmission property. Our definition also allows use to treat together both Poisson operators and Green operators, as defined in [2], to handle operators acting on boundary distributions and distributions supported on the entire domain, respectively.

As an example, consider the Laplacian, \(\partial_x^2 + \partial_{\rho}^2\) on \(\mathbb{R}^2\) with symbol \(\xi^2 + \eta^2\) and inverse with symbol

\[
\frac{1}{\eta^2 + \xi^2} = \frac{1}{(\eta + i|\xi|)(\eta - i|\xi|)}.
\]

When operating on a distribution with compact support on \(\rho = 0, f \in \mathcal{E}'(\mathbb{R})\), we have for \(\rho > 0\)

\[
\int \frac{\hat{f}(\xi)}{(\eta + i|\xi|)(\eta - i|\xi|)} e^{ix\xi} e^{i\rho\eta} d\xi d\eta = \pi \int \frac{\hat{f}(\xi)}{|\xi|} e^{ix\xi} e^{-\rho|\xi|} d\xi,
\]

ignoring the singularity at \(\xi = 0\) (we discuss this in detail in Section 2). Thus, the inverse to the Laplacian acting on such a distribution, \(f(x)\), has the same behavior as the operator with symbol given by

\[
\frac{1}{2|\xi|, \frac{1}{\eta - i|\xi|}}.
\]

The second factor is seen to have the transmission property, whereas it was obvious from the beginning the symbol in (1) satisfies our condition that the poles \((\pm i|\xi|)\) are elliptic operators in the \(\partial_x\) direction, ignoring the singularity at \(\xi = 0\) which can be handled by multiplying with functions which vanish identically in a neighborhood of \(\xi = 0\).
We mention here that another approach to boundary value problems is outlined in [4]. The approach there is to factor an elliptic equation with each factor containing a normal derivative and a tangential (pseudodifferential) operator. This allows for some simplifications, in particular, in the calculation of the Dirichlet to Neumann operators. However, the factorization approach is not easily generalized to intersection domains, which we take up in later sections of the article.

It should be noted that we are not interested in developing a full calculus for solving boundary value problems on smoothly bounded domains, as in [2], as the intended use of the article for the author is the application of reduction to the boundary techniques to boundary value problems which reduce to non-elliptic boundary conditions, as in [1] (see also [5] for a discussion of the $\bar{\partial}$-Neumann problem, the type of problem for which the analysis described here is intended). Thus, we do not investigate invertibility or Fredholm properties of operators but rather our focus is to examine operators which arise in a reduction to the boundary.

The main goal of this article is the investigation of operators which arise in the reduction to the boundary techniques in the case of intersection domains. As we shall see many of the operators and the proofs of their properties can be derived from the smooth case, while some operators (the $E_{jk}^{\bar{\partial}}$ operators in Section 3) have no analogues in the smooth case. We setup the required machinery here so that, for instance, properties of a solution can be obtained from a microlocal analysis on the boundary. We provide an example calculation in Section 4, and the full force of the results outlined here will be seen in [6], where this exact scenario is to be played out.

The properties of operators we derive in this article are aimed at a description of the regularity of solutions to boundary value problems in terms of Sobolev spaces. In the smooth case, these are classic results (see the above mentioned references handling the smooth case). In the case of intersections (in fact, on general Lipchitz domains), regularity results have been obtained by applying layer potentials and singular integral operator theory for solutions to elliptic boundary values problems, as in the work of Jerison and Kenig [7] and Verchota [8]. Thus, for instance, from [7] (see also [8,9]),

**Theorem 1.1:** Let $\Omega$ be a bounded domain with Lipschitz boundary. Let $P$ denote the Poisson operator attached to $\Omega$, with the property $P(u_b) \to u_b$ almost everywhere, where the limits are taken non-tangentially.

For $u_b \in W^s(\partial \Omega)$, for $0 \leq s \leq 1$, we have

$$\|P(u_b)\|_{W^{s+1/2}(\Omega)} \lesssim \|u_b\|_{W^s(\Omega)}.$$

Our analysis in this article allows us to (partially) reproduce the results in the above theorem (in the case of intersection domains) but also with some additional information. The spaces we work with are weighted Sobolev spaces. Let $\Omega \subset \mathbb{R}^n$, $\Omega = \cap_{j=1}^m \Omega_j$, where each $\Omega_j$ is a smoothly bounded domain, and $j \leq n$. We also let $\rho_j$ be the defining function for $\Omega_j$, and $\rho = \rho_1 \rho_2 \cdots \rho_m$.

We denote by

$$W^{\alpha,s}(\Omega, \rho) := \{ f \in W^\alpha(\Omega) \mid \rho^rf \in W^{\alpha+r}(\Omega), \forall 0 \leq r \leq s\},$$
with integer $s$ and norm
\[ \|f\|_{W^{s,1}(\Omega,\rho)} = \sum_{r=0}^{s} \|\rho^r f\|_{W^{s+r}(\Omega)}. \]

A consequence of the techniques developed here, we can show

**Theorem 1.2:** Let $\Omega = \bigcap_{j=1}^{m} \Omega_j$ as above. Let $P$ denote the Poisson operator for $\Omega$. Let $0 \leq \alpha < 1/2$, and $u_{bj} \in W^{\alpha,s}(\partial \Omega_j)$. With $u_b \in L^2(\partial \Omega)$ with $u_b \mid_{\partial \Omega_j} = u_{bj}$, we have
\[ \|P(u_b)\|_{W^{\alpha+1/2,s}(\Omega,\rho)} \lesssim \sum_{j} \|u_{bj}\|_{W^{\alpha,s}(\partial \Omega_j \cap \partial \Omega,\rho)}, \]
where $\rho_j := \rho_1 \cdots \rho_j \rho_{j+1} \cdots \rho_m$.

This is a special case of what is proved in [6]. We emphasize here that with our techniques we not only obtain the known $1/2$ gain of regularity, we also obtain information on the degree of singularity of the solution, on how the regularity or singularity is affected upon applying derivatives. This is a subtle issue in the case of the $\bar{\partial}$-Neumann problem (see [6] and [10]).

Section 2 is somewhat expository, reviewing some of the classical results of pseudodifferential operators arising from boundary value problems on smooth domains. There are many works treating the smooth case and we refer the interested reader to [1,2,4,11–14] for detailed treatments of boundary value problems on smooth domains. In addition, [14] handles some non-smooth cases. We provide our own proofs of the operators which we study and which arise in boundary value problems, which will, in addition to providing the above mentioned simplifications, serve as a preparation for the operators which are to arise in the case of intersections, in some cases serving as base cases to which operators in the intersection case will be reduced.

In Section 3, the case of (transversal) intersections of domains is considered with the main results relating to extending estimates obtained in the case of smooth domains via weighted estimates. As far as the author is aware, the technique of reducing to the boundary in the case of intersection domains has not been extensively studied. One reason is certainly some of the troubling operators which mix distributions of the different boundaries do not behave as pseudodifferential operators. Nonetheless, several pseudodifferential operators do arise, and weighted Sobolev spaces are defined and then those operators are studied with respect to the weighted spaces. The above mentioned boundary value operators which are not pseudodifferential operators are also studied with respect to the weighted spaces.

In Section 4, we illustrate how the various operators presented arise in the technique of reduction to the boundary on intersection domains, in the simple case of an intersection of two smoothly bounded domains. We focus both on the boundary value operators which arise as well as those conditions which determine the ellipticity of the boundary conditions (with elliptic highest order operators). The application of the methods and properties of Section 3 to the $\bar{\partial}$-Neumann problem, whose boundary conditions reduce to non-elliptic problems, is the subject of a current study of the author in the above mentioned [6].

With the use of partitions of unity and cutoffs (in a neighborhood of a boundary point of a smooth domain), we can assume coordinates $(x, \rho) \in \mathbb{R}^{n+1}$ for $\rho < 0$ and thus reduce the
study to the case of operators acting on distributions supported in the lower half-space or distributions supported in \( \mathbb{R}^n \). In the case of intersections, the coordinates will be chosen so that near a point on the intersection of several boundaries, the domain looks like the intersection of several lower half-spaces.

2. Analysis on the lower half-space

In this section, we develop some of the properties of pseudodifferential operators on half-spaces. Of particular importance for the reduction to the boundary techniques are the boundary values of pseudodifferential operators on half-spaces, pseudodifferential operators acting on distributions supported on the boundary, as well as pseudodifferential operators on the boundary itself.

We briefly recall the definition of pseudodifferential operators in \( \mathbb{R}^n \). We refer to the book by Treves [4] for a thorough introduction to the subject. Let \( S^m(\mathbb{R}^n) \) denote the space of symbols in \( \mathbb{R}^n \). A symbol \( a(x, \xi) \in S^m(\mathbb{R}^n) \) is a \( C^\infty(\mathbb{R}^n \times \mathbb{R}^n) \) function with estimates on any compact \( K \subset \mathbb{R}^n \)

\[
|\partial_x^\alpha \partial_{\xi}^\beta a(x, \xi)| \leq c_{\alpha, \beta}(K) (1 + |\xi|^m)^{|\alpha|} \quad \forall x \in K, \; \xi \in \mathbb{R}^n,
\]

where \( c_{\alpha, \beta}(K) > 0 \). Note that, for a multi-index, \( \beta = (\beta_1, \beta_2, \ldots, \beta_n) \), we make use of the index notation,

\[
\partial_x^\beta = \partial_{x_1}^{\beta_1} \cdots \partial_{x_n}^{\beta_n},
\]

for both \( x \) and \( \xi \) derivatives.

A pseudodifferential operator of class \( \Psi^m(\mathbb{R}^n) \) is defined in terms of a symbol of class \( S^m(\mathbb{R}^n) \). A pseudodifferential operator, \( A \in \Psi^m(\mathbb{R}^n) \), can be written as

\[
Af = \frac{1}{(2\pi)^n} \int a(x, \xi) \hat{f}(\xi) e^{ix\xi} \, d\xi.
\]

We write \( A = Op(a) \). Generally, we will consider \( f \) to be a function in a Sobolev space, \( W^s(\mathbb{R}^n) \), defined as the space of functions such that \( (1 + |\xi|)^s \hat{f}(\xi) \in L^2(\mathbb{R}^n) \). We have that \( A : W^s(\mathbb{R}^n) \rightarrow W^{s-m}(\mathbb{R}^n) \) as an operator between Sobolev spaces.

An elliptic operator is a particular type of pseudodifferential operator whose symbol, \( a(x, \xi) \), is such that there exist positive functions \( c(x) \) and \( r(x) \), and for each \( x \)

\[
c(x)|\xi|^m \leq |a(x, \xi)| \quad \forall |\xi| \geq r(x)
\]

(for \( a(x, \xi) \in S^m(\mathbb{R}^n) \)). Then we say \( Op(a) \) is an elliptic operator, see [4].

We now fix some notation to be used throughout this paper. We will reserve the notation, \( A_k \), to refer to some pseudodifferential operator of order \( k \); thus by \( A_k \), we mean \( A_k \in \Psi^k(\mathbb{R}^n) \) (or in the space of operators defined on a given domain, \( \Omega \), depending on context).

We will even allow the specific operator referred to by \( A_k \) to change from one line to the next or within the same line itself. Thus, for instance, we can write \( A_0 \circ A_1 = A_1 \).

We use coordinates \((x, \rho)\) on \( \mathbb{R}^{n+1} \) with \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n \). The full Fourier Transform of a function, \( f(x, \rho) \), will be written as

\[
\hat{f}(\xi, \eta) = \frac{1}{(2\pi)^{n+1}} \int f(x, \rho) e^{-ix\xi} e^{-i\rho \eta} \, dx \, d\rho,
\]
where $\xi = (\xi_1, \ldots, \xi_n)$. For $f$ defined on a subset of $\mathbb{R}^{n+1}$, we define its Fourier Transform as the transform of the function extended by zero to all of $\mathbb{R}^{n+1}$. Thus, for instance, for $f$ defined on $\{(x, \rho) \in \mathbb{R}^{n+1} : \rho < 0\}$, we write

$$\hat{f}(\xi, \eta) = \frac{1}{(2\pi)^{n+1}} \int_{-\infty}^{0} \int_{\mathbb{R}^n} f(x, \rho) e^{-ix\xi} e^{-i\rho \eta} \, dx \, d\rho.$$  

A partial Fourier Transform in the $x$ variables will be denoted by

$$\hat{f}(\xi, \rho) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} f(x, \rho) e^{-ix\xi} \, dx.$$  

We define the half-space $\mathbb{H}^{n+1}_{-} := \{(x, \rho) \in \mathbb{R}^{n+1} : \rho < 0\}$. The space of distributions $\mathcal{E}'(\mathbb{H}^{n+1}_{-})$ is defined as the compactly supported distributions in $\mathcal{E}'(\mathbb{R}^{n+1})$ with support in $\mathbb{H}^{n+1}_{-}$. The topology of $\mathcal{E}'(\mathbb{H}^{n+1}_{-})$ is inherited from that of $\mathcal{E}'(\mathbb{R}^{n+1})$. We endow $C^\infty(\mathbb{R}^{n+1})$ with the topology defined in terms of the semi-norms

$$p_{l,k}(\phi) = \max_{(x, \rho) \in K \subset \subset \mathbb{R}^{m+1}} \sum_{|\alpha| \leq l} |\partial^\alpha \phi(x, \rho)|.$$  

A regularizing operator, $A \in \Psi^{-\infty}(\mathbb{R}^{n+1})$, is a continuous linear map,

$$A : \mathcal{E}'(\mathbb{R}^{n+1}) \to C^\infty(\mathbb{R}^{n+1}). \quad (4)$$  

We will use the term smoothing to describe restrictions of operators as long as smoothness and continuity properties are exhibited. For instance, we say an operator $A : \mathcal{E}'(\mathbb{R}^n) \times \delta(\rho) \to C^\infty(\mathbb{R}^{n+1})$ is smoothing on $\mathcal{E}'(\mathbb{R}^n) \times \delta(\rho)$ if it is a continuous and linear. Note that in this case it may not be true that $A \in \Psi^{-\infty}(\mathbb{R}^{n+1}).$

In working with pseudodifferential operators on half-spaces, with coordinates $(x_1, \ldots, x_n, \rho), \rho < 0$, we can show that by multiplying symbols by smooth cutoffs with compact support, which are functions of transform variables corresponding to tangential coordinates, we produce operators which are smoothing. This is in analogy to the case of $\mathbb{R}^n$ where cutoffs (with compact support) in transform space give rise to regularizing operators: let $a(x, \xi) \in S^k(\mathbb{R}^n)$ and $\chi(\xi) \in C_0^\infty(\mathbb{R}^n)$. Then, the operator with symbol $\chi(\xi)a(x, \xi)$ is regularizing. The reason behind this is that any growth in the $\xi$ variables resulting from differentiation is compensated by the compact support of $\chi(\xi)$. For instance, for $x$ in a compact subset $K$,

$$\left| \partial^\alpha_x \int a(x, \xi) \chi(\xi) \hat{\phi}(\xi) e^{ix\xi} \, d\xi \right| \lesssim \sum_{\alpha_1 + \alpha_2 = \alpha} \left| \partial^\alpha_x a(x, \xi) \chi(\xi) \xi^{\alpha_2} \hat{\phi}(\xi) \right| \, d\xi$$

$$\lesssim \int |\chi(\xi)|(1 + |\xi|)^{|\alpha| + k} |\hat{\phi}(\xi)| \, d\xi$$

$$\lesssim \|\phi\|_{L^2(\mathbb{R}^n)},$$

for all $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $\alpha_i \geq 0$, where the constants of inequality depend only on the compact set $K$. 
Lemma 2.1: Let \( A \in \Psi^{-k}(\mathbb{R}^{n+1}) \), for \( k \geq 1 \) an integer, be such that the symbol, \( \sigma(A)(x, \rho, \xi, \eta) \), is meromorphic (in \( \eta \)) with poles at

\[
\eta = q_1(x, \rho, \xi), \ldots, q_k(x, \rho, \xi),
\]

with \( q_i(x, \rho, \xi) \) themselves symbols of pseudodifferential operators of order 1 (restricted to \( \eta = 0 \)) such that for each \( \rho \), \( \text{Res}_{\eta = q_i} \sigma(A) \in \mathcal{S}^{-k+1}(\mathbb{R}^n) \) with symbol estimates uniform in the \( \rho \) parameter.

Let \( A_\chi \) denote the operator with symbol,

\[
\chi(\xi) \sigma(A),
\]

where \( \chi(\xi) \in C_0^\infty(\mathbb{R}^n) \). Then, \( A_\chi \) is smoothing on distributions supported on the boundary,

\[
A_\chi : \mathcal{E}'(\mathbb{R}^n) \times \delta(\rho) \to C^\infty(\mathbb{R}^n),
\]

where \( \delta \) is the Dirac-delta distribution.

Proof: Without loss of generality, we suppose \( \phi_b(x) \in L^2(\mathbb{R}^n) \) and let \( \phi = \phi_b \times \delta(\rho) \). We estimate derivatives of \( A_\chi(\phi) \). Let \( a(x, \rho, \xi, \eta) \) denote the symbol of \( A \) and \( a_\chi(x, \rho, \xi, \eta) \) that of \( A_\chi \).

We use induction on the (absolute value of the) order, \( k \), of the operator. The base case \( k = 1 \) will follow from the calculations of the induction step. For a given \( k \) we thus assume the lemma has been proven for operators in \( \Psi^{-1}(\mathbb{R}^{n+1}), \ldots, \Psi^{-(k-1)}(\mathbb{R}^{n+1}) \).

We first note that derivatives with respect to the \( x \) variables pose no difficulty due to the \( \chi \) term in the symbol of \( A_\chi \): from

\[
A_\chi \phi = \frac{1}{(2\pi)^{n+1}} \int a_\chi(x, \rho, \xi, \eta) \hat{\phi}(\xi, \eta) e^{ix\xi} e^{i\rho \eta} \, d\xi \, d\eta
\]

we calculate over \( (x, \rho) \in K \subset \subset \mathbb{R}^{n+1} \)

\[
|\partial_\rho^\alpha A_\chi \phi| \lesssim \sum_{\alpha_1 + \alpha_2 = \alpha} \int |\partial_{\alpha_1} a_\chi(x, \rho, \xi, \eta)| |\xi|^{\alpha_2} |\hat{\phi}_b(\xi)| \, d\xi \, d\eta
\]

\[
\lesssim \|\phi_b\|_{L^2} \sum_{\alpha_1 + \alpha_2 = \alpha} \left( \int |\xi|^{2\alpha_2} |\chi^2(\xi)| |\partial_{\alpha_1} a(x, \rho, \xi, \eta)|^2 \, d\xi \, d\eta \right)^{1/2}
\]

\[
\lesssim \|\phi_b\|_{L^2} \left( \int |\xi|^{2\alpha_1} |\chi^2(\xi)| \frac{1}{(1 + \xi^2 + \eta^2)^k} \, d\xi \, d\eta \right)^{1/2}
\]

\[
\lesssim \|\phi_b\|_{L^2} \left( \int |\xi|^{2\alpha_1} |\chi^2(\xi)| \, d\xi \right)^{1/2}
\]

\[
\lesssim \|\phi_b\|_{L^2},
\]

where the constants of inequalities depend on the compact, \( K \). For any mixed derivative \( \partial_\alpha^x \partial_\rho^\beta \), the \( x \) derivatives can be handled in the manner above and so we turn to derivatives of the type \( \partial_\rho^\beta A_\chi \phi \).
We use the residue calculus to integrate over the \( \eta \) variable in

\[
A_\chi \phi = \frac{1}{(2\pi)^{n+1}} \int a_\chi(x, \rho, \xi, \eta) \tilde{\phi}_b(\xi) e^{ix\xi} e^{i\rho\eta} \, d\xi \, d\eta.
\]

Denote the poles of \( a(x, \rho, \xi, \eta) \) which are in the lower half-space (\( \text{Im} \eta < 0 \)) by

\[
\eta = q_1^-(x, \rho, \xi), \ldots, q_l^-(x, \rho, \xi),
\]

where \( l \leq k \). Let

\[
a_{q_j^-}(x, \rho, \xi) = i \text{Res}_{\eta = q_j^-} a_\chi(x, \rho, \xi, \eta)
\]

and

\[
A_{q_j^-} \chi \phi = \frac{1}{(2\pi)^n} \int a_{q_j^-}(x, \rho, \xi) \tilde{\phi}_b(\xi) e^{ix\xi} e^{i\rho q_j^-} \, d\xi.
\]

In particular, we have

\[
A_\chi \phi = \sum_{1 \leq j \leq l} A_{q_j^-} \chi \phi,
\]

modulo terms which are handled by the induction hypothesis, which arise in the case of poles of order higher than one (namely, from the resulting \( \eta \) derivatives landing on the \( e^{i\rho\eta} \) term). Thus estimates (of derivatives) of \( A_\chi \phi \) will be deduced from estimates of \( A_{q_j^-} \chi \phi \). We note that while the above sum in terms of operators with symbols as the residues of \( \sigma(A_\chi) \) is also valid in the case of poles of multiplicity higher than one, the hypothesis that the residues are themselves symbols excludes the case in which two poles merge at a point or neighborhood in the domain but are not identical.

We can now estimate \( \partial_\rho^{\beta} A_{q_j^-} \chi \phi \), for some \( j \), by differentiating under the integral. Note that the factor of \( \chi(\xi) \) is contained in \( a_{q_j^-} \). A term \( |\partial_\rho^{\beta} (a_{q_j^-} e^{i\rho q_j^-})| \) is bounded by a sum of terms of the form

\[
s_{\beta_1, \beta_2, \alpha} = \left| \partial_\rho^{\beta_1} a_{q_j^-} \right| (1 + |q_j^-|^\alpha_0) \left| (\partial_\rho q_j^-)^{\alpha_1} \right| \cdots \left| (\partial_\rho^{\beta_2} q_j^-)^{\alpha_{\beta_2}} \right|,
\]

with \( \alpha = (\alpha_0, \alpha_1, \ldots, \alpha_{\beta_2}) \) and

\[
\beta_1 + \beta_2 = \beta,
\]

\[
\alpha_0 + \sum_{j=1}^{\beta_2} j \cdot \alpha_j = \beta_2.
\]

From the properties of a symbol, we can estimate

\[
|s_{\beta_1, \beta_2, \alpha}| \lesssim (1 + |\xi|)^{-k+1+\beta_2}.
\]

For \( \rho < 0 \), over \( (x, \rho) \in K \subset \subset \mathbb{R}^{n+1} \), we have the estimates

\[
|\partial_\rho^{\beta} A_{q_j^-} \chi \phi| \lesssim \sum_{\beta_1, \beta_2, \alpha} s_{\beta_1, \beta_2, \alpha} \int |\tilde{\phi}_b(\xi)| \, d\xi
\]

\[
\lesssim \|\phi_b\|_{L^1},
\]
where the summation is over $\alpha, \beta_1, \beta_2$ which satisfy the conditions in (5). Again, the integral over $\xi$ converges due to the factor of $\chi(\xi)$ contained in $a_{i,j}$.

The base case, $k = 1$, follows the same calculations above. We conclude the proof of the lemma.

The motivation for Lemma 2.1 is from the consideration of inverses to elliptic operators. We recall that an elliptic operator, $B \in \Psi^m(\mathbb{R}^{n+1})$, has an inverse, $A \in \Psi^{-m}(\mathbb{R}^{n+1})$, such that $B \circ A = A \circ B = \text{I}$ modulo $\Psi^{-\infty}(\mathbb{R}^{n+1})$. The symbol of $A$ can be determined using the symbol calculus and can be written as a sum of terms which have powers of $\sigma(B)$ in their denominators. A standard procedure to deal with possible zeros in the denominator is to introduce cutoffs, $\phi_j(x, \rho, \xi, \eta)$, which vanish near the zeros of $\sigma(B)$. For example, the symbol of the inverse, $A$, to the Laplacian operator, $B$, with symbol $\sigma(B) = \xi^2 + \eta^2$, is given by $\sigma(A) = \phi(\xi, \eta)/(\xi^2 + \eta^2)$, where $\phi(\xi, \eta)$ is chosen to be 0 in a small neighborhood of the origin. The problem in applying Lemma 2.1 to such an inverse, $A$, is that the symbol $\sigma(A)$ is no longer meromorphic in $\eta$ due to the use of the cutoff, $\phi(\xi, \eta)$. We therefore mention to the reader interested in applying our analysis that a technique which can be applied to the situation of Lemma 2.1 is to add a zero order operator to $B$ and consider instead the inverse to $B + B_0$, where $B_0$ is chosen such that $\sigma(B + B_0)$ does not vanish. This technique has been applied to the case of intersection domains [6].

We now prove a lemma relating to the zeros of $\eta$, denoted $q_i(x, \rho, \xi)$ above.

**Lemma 2.2:** Let $B \in \Psi^m(\mathbb{R}^{n+1})$ be an elliptic pseudodifferential operator with symbol

$$
\sigma(B) = (\eta - q_1(x, \xi, \rho)) \cdots (\eta - q_m(x, \xi, \rho)),
$$

(6)
a polynomial of order $m$ in the $\eta$ variables. Then, the $q_i(x, \rho, \xi)$ are symbols of elliptic operators of order one (with $\rho$ as a parameter).

**Proof:** We fix $x$ to be in some compact set. We want to show that each $q_i$ satisfies $|q_i| \simeq |\xi|$ for large $\xi$.

Setting $\eta = 0$ in (6), we know from ellipticity,

$$
|q_1 \cdot q_2 \cdots q_m| \simeq |\xi|^m.
$$

If one $q_i$ (suppose it is $q_1$) were such that $|\xi|/|q_1| \to 0$, then with $r_{m-1} = q_2 \cdots q_m$, we would have $r_{m-1}$ is such that $|r_{m-1}|/|\xi|^{m-1} \to 0$. We will use the notation $r_k$ below to denote the coefficient of $\eta^{m-1-k}$ in the polynomial,

$$(\eta - q_2)(\eta - q_3) \cdots (\eta - q_m).$$

Now take $k$ derivatives with respect to the $\eta$ variables of the symbol, $\sigma(B)$, and set $\eta = 0$. We obtain an inequality from (2) of the form

$$
|q_1| \cdot r_{m-1-k} + r_{m-k} \lesssim |\xi|^{m-k}.
$$

At each step, for $k = 1, \ldots, m - 1$, we conclude $|r_{m-1-k}|/|\xi|^{m-1-k} \to 0$ due to the known (slow) growth of $r_{m-k}$ and (fast) growth of $q_1$. 
With \( m - 1 \) derivatives, we have

\[
|q_1 + q_2 + \cdots + q_m| \lesssim |\xi|, \tag{7}
\]

but from the growth of \( r_1 \) from above, we have the property

\[
|q_2 + \cdots + q_m|/|\xi| \to 0.
\]

The relation in (7) thus leads to a contradiction because \( q_1 \) was supposed to have faster growth than \( |\xi| \).

The next theorem is aimed at properties of a finite sum of the first terms of the expansion of the inverse to an elliptic operator of order \( k \geq 1 \). For such operators, the hypotheses of Lemmas 2.1 and 2.2 are satisfied, and so the poles \( \eta = q_i(x, \rho, \xi) \) are elliptic (uniformly in the \( \rho \) parameter) of order 1, as are their imaginary parts. For such operators, without the assumption of the cutoff function \( \chi(\xi) \) in the symbol of \( A \), we can still prove

**Theorem 2.3:** Let \( g \in \mathcal{D}'(\mathbb{R}^{n+1}) \) of the form \( g(x, \rho) = g_b(x) \delta(\rho) \); \( g_b \) is a distribution supported on \( \partial \mathbb{H}^{n+1}_+ = \mathbb{R}^n \). Let \( A \in \Psi^k(\mathbb{R}^{n+1}) \), \( k \leq -1 \) as in Lemma 2.1 with the additional assumption that \( \sigma(A)(x, \rho, \xi, \eta) \) vanishes in a neighborhood of \( \xi = 0 \) and the imaginary parts of the poles, \( q_i(x, \rho, \xi) \) are symbols of elliptic operators (restricted to \( \eta = 0 \)) for order 1.

Then, for all integer \( s \geq 0 \), and \( g_b \in W^{s+k+1/2}(\mathbb{R}^n) \),

\[
\|\varphi A g\|_{W^s(\mathbb{H}^{n+1}_+)} \lesssim \|g_b\|_{W^{s+k+1/2}(\mathbb{R}^n)}
\]

for any \( \varphi \in C_0^\infty(\mathbb{H}^{n+1}_+) \). The estimate also holds for all (non-integer) \( s \geq |k| - 1 \).

**Proof:** We follow and use the notation of the proof of Lemma 2.1, and prove by induction, assuming the theorem holds for operators in \( \Psi^{-j}(\mathbb{R}^{n+1}) \), for \( j = 1, \ldots, k - 1 \).

We again analyze a typical term resulting from a pole at \( \eta = q_j^-(x, \rho, \xi) \) of \( a(x, \rho, \xi, \eta) \).

With

\[
a_{q_j^-}(x, \rho, \xi) = i \text{Res}_{\eta = q_j^-} a(x, \rho, \xi, \eta)
\]

and

\[
A^{q_j^-} g = \frac{1}{(2\pi)^n} \int a_{q_j^-}(x, \rho, \xi) \tilde{g}_b(\xi) e^{ix\xi} e^{i\rho q_j^-} d\xi,
\]

we estimate \( \partial_\alpha^\alpha \partial_\rho^\beta A^{q_j^-} g \) in the case of integer \( s \geq 0 \) and \( |\alpha| + \beta = s \) by differentiating under the integral. As in the proof of Lemma 2.1, we bound \( \partial_\alpha^\alpha \partial_\rho^\beta (a_{q_j^-} e^{ix\xi} e^{i\rho q_j^-}) \) by a sum of terms,

\[
s_{\alpha^\alpha \beta^\beta} = (1 + |\xi|)^{s_1} \left| \partial_\alpha^\alpha \partial_\rho^\beta a_{q_j^-} \right| (1 + |q_j^-|)^{s_0} \left| (D_{x,\rho} q_j^-)^{\gamma_1} \cdots (D_{x,\rho} q_j^-)^{\gamma_{s_3}} \right|,
\]
with $D^j_{x, \rho}$ a derivative of the form $\partial^j_{\rho, x}$ for $i + |l| = j$, $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, $\beta = (\beta_1, \beta_2)$, $\gamma = (\gamma_0, \gamma_1, \ldots, \gamma_{\alpha_3+\beta_2})$ and

$$\alpha_1 + \alpha_2 + \alpha_3 = \alpha,$$

$$\beta_1 + \beta_2 = \beta,$$

$$\gamma_0 + \sum_{j=1}^{\alpha_3+\beta_2} j \cdot \gamma_j = \alpha_3 + \beta_2.$$  \hspace{1cm} (8)

We note the estimates

$$|s_{\alpha \beta \gamma}| \lesssim (1 + |\xi|)^{\alpha_1+k+1+\alpha_3+\beta_2}$$

$$\lesssim (1 + |\xi|)^{k+1+s}.$$  \hspace{1cm} (9)

For $\rho < 0$ we thus have the estimates

$$\left| \partial^\alpha_x \partial^\beta_{\rho} A^j_{\gamma} \bar{g} \right|^2 \lesssim \int (1 + |\xi|^2)^{k+1+s} \left| \widetilde{gb}(\xi) \right|^2 e^{2\rho |\text{Im} \bar{g}|} d\xi,$$

where, as in Lemma 2.1, the estimates are over compact sets, $K, (x, \rho) \in K \subset \subset \mathbb{R}^{n+1}$, with the constants of the inequality depending on $K$.

Since $\text{Im} \bar{g}$ is assumed to be the symbol of an order 1 elliptic operator, we have $|\xi| \lesssim |\text{Im} \bar{g}| \lesssim (1 + |\xi|)$ using the inequalities (2), (3) (for $x$ restricted to a compact set, and the constants of inequalities depending on that set). Thus, we have the property

$$\frac{1}{|\text{Im} \bar{g}|} \sim \frac{1}{|\xi|}.$$  \hspace{1cm} (10)

Thus, when we integrate over $\rho$, we get a factor on the order of $\frac{1}{|\xi|}$ which lowers the order of the norm in the tangential directions by 1/2:

$$\left\| \phi \partial^\alpha_x \partial^\beta_{\rho} A^j_{\gamma} \bar{g} \right\|_{L_2}^2 \lesssim \int (1 + |\xi|^2)^{k+1+s} \left| \widetilde{gb}(\xi) \right|^2 \frac{1}{1 + |\xi|} d\xi$$

$$\lesssim \sum_{\alpha, \beta} \int |1 + |\xi|^2|^{k+1/2+s} \left| \widetilde{gb}(\xi) \right|^2 d\xi$$

$$\lesssim \|gb\|_{W^{s+k+1/2}(\mathbb{R}^n)}^2.$$  \hspace{1cm} (11)

Note that we can use the term $1 + |\xi|$ in the denominator by the assumption that $a(x, \rho, \xi, \eta)$ vanishes near $\xi = 0$.

The base case $k = -1$ is handled by the calculations above, and the rest of the proof, including how to incorporate the induction step, follows that of Lemma 2.1.

This proves the theorem for integer $s \geq 0$. The non-integer case follows by interpolation [15]. \hspace{1cm} \Box

\textbf{Remark 2.4:} In practice, the assumption of the vanishing of the symbol at $\xi = 0$ can be removed by the consideration of symbols with non-vanishing denominators (by modifying
the operators with the addition of zero order terms, for instance, see the discussion after
Lemma 2.1).

Another version of Theorem 2.3 is contained in [3] (see Theorem 5.2.4 (iii)).
The hypotheses of Lemmas 2.1 and 2.2, and Theorem 2.3 are all satisfied, for instance,
in the case of the terms in the expansion of the inverse to an elliptic differential operator
such as the Laplacian.

There are analogue estimates for functions with support in the half-space (as opposed
to support on the boundary):

**Theorem 2.5:** Let $k \leq -1, s \geq |k|$, and $f \in W^{s+k}(\mathbb{H}^{n+1}_-)$. Let $A \in \Psi^k(\mathbb{H}^{n+1})$ be as in
Theorem 2.3. Then,
\[
\|\varphi Af\|_{W^{s}(\mathbb{H}^{n+1}_-)} \lesssim \|f\|_{W^{s+k}(\mathbb{H}^{n+1}_-)},
\]
for any $\varphi \in C_0^\infty(\mathbb{H}^{n+1}_-)$. 

**Proof:** We prove by induction on the order of the class $\Psi^k(\mathbb{H}^{n+1})$.

Let $a(x, \rho, \xi, \eta)$ be the symbol of the operator $A$,
\[
Af = \frac{1}{(2\pi)^{n+1}} \int a(x, \rho, \xi, \eta) \hat{f}(\xi, \eta) e^{i\rho \eta} d\xi \, d\eta,
\]
In the case $k = -1$, we can write $a(x, \rho, \xi, \eta) = \frac{1}{\eta - q(x, \rho, \xi)}$ and calculate
\[
\partial_\rho Af = \frac{i}{(2\pi)^{n+1}} \int \eta a(x, \rho, \xi, \eta) \hat{f}(\xi, \eta) e^{i\rho \eta} d\xi \, d\eta
\]
\[
= \frac{i}{(2\pi)^{n+1}} \int (\eta - q(x, \rho, \xi)) a(x, \rho, \xi, \eta) \hat{f}(\xi, \eta) e^{i\rho \eta} d\xi \, d\eta
\]
\[
+ \frac{i}{(2\pi)^{n+1}} \int a(x, \rho, \xi, \eta) q(x, \rho, \xi) \hat{f}(\xi, \eta) e^{i\rho \eta} d\xi \, d\eta
\]
\[
= i f(x, \rho) + \frac{i}{(2\pi)^{n+1}} \int a(x, \rho, \xi, \eta) q(x, \rho, \xi) \hat{f}(\xi, \eta) e^{i\rho \eta} d\xi \, d\eta.
\]
In this way, we can relate $\rho$ derivatives to derivatives in the tangential directions. The
second term on the right can be estimated by $\|f\|_{L^2(\mathbb{H}^{n+1})}$. This can be repeated to show any
$\rho$ derivative of order $s$ can be estimated by $\|f\|_{W^{s-1}(\mathbb{H}^{n+1})}$.
Estimates for derivatives with
respect to $x$ are handled directly as with the second term above and the base case of $k = -1$
is proved.

Lower order operators (for $k < -1$) are handled similarly. Let $k < -1$. We let $q_1, \ldots, q_{|k|}$
denote the poles of $a(x, \rho, \xi, \eta)$ (in $\eta$) counted with multiplicity. We have
\[
\partial_\rho Af = \frac{i}{(2\pi)^{n+1}} \int \eta a(x, \rho, \xi, \eta) \hat{f}(\xi, \eta) e^{i\rho \eta} d\xi \, d\eta
\]
\[
= \frac{i}{(2\pi)^{n+1}} \int (\eta - q_1(x, \rho, \xi)) a(x, \rho, \xi, \eta) \hat{f}(\xi, \eta) e^{i\rho \eta} d\xi \, d\eta.
\]
Theorem 2.6: Let \( g \) still derive estimates, up to certain order. \( \sigma(\text{supp} A \subseteq \phi) \| g \|_\mathcal{W}^s \leq |k| \) for any \( s \geq |k| \). This handles the case of \( s \geq |k| \). The general case follows by Sobolev interpolation.

In the case \( \Lambda \in \Psi^k(\mathbb{R}^{n+1}) \) for \( k \leq -1 \) without additional assumptions on the symbol, \( \sigma(A)(x, \rho, \xi, \eta) \), which for instance arises from error terms in a symbol expansion, we can still derive estimates, up to certain order.

**Theorem 2.7:** Let \( f \in \mathcal{D}'(\mathbb{R}^{n+1}) \) of the form \( g(x, \rho) = g_b(x) \delta(\rho) \) for \( g_b \in W^{-1/2}(\mathbb{R}^n) \). Let \( A \in \Psi^k(\mathbb{R}^{n+1}), k \leq -1 \). Then, for integer \( s \leq |k| - 1 \),

\[
\| \varphi A g \|_{W^s(\mathbb{R}^{n+1})} \lesssim \| g_b \|_{W^{s+k+1/2}(\mathbb{R}^n)}
\]

for any \( \varphi \in C_0^\infty(\mathbb{H}^{n+1}_-) \).

**Proof:** As in the proof of Theorem 2.3, we write

\[
Ag = \frac{1}{(2\pi)^{n+1}} \int a(x, \rho, \xi, \eta) \tilde{g}_b(\xi) e^{ix\xi} e^{i\rho \eta} \, d\xi \, d\eta
\]

and estimate

\[
\| \varphi \partial_\alpha \partial_\beta A g \|_{L^2}^2 \lesssim \int \frac{\xi^{2|\alpha|} \eta^{2\beta}}{(1 + \eta^2 + \xi^2)^{|k|} |\tilde{g}_b(\xi)|^2} \, d\eta \, d\xi
\]

\[
\lesssim \int \frac{\xi^{2(|\alpha|+\beta)}}{(1 + |\xi|)^2|k|} |\tilde{g}_b(\xi)|^2 \, d\xi
\]

\[
\lesssim \| g_b \|_{W^{s+k+1/2}(\mathbb{R}^n)}^2.
\]

This handles the case of \( s \geq 0 \). Negative values of \( s \) can be handled by writing

\[
\| \varphi A g \|_{W^s(\mathbb{H}^{n+1}_-)} \simeq \| \varphi \Lambda^{-|s|} \circ \varphi A g \|_{L^2(\mathbb{H}^{n+1}_-)}
\]

where \( \varphi \in C_0^\infty(\mathbb{H}^{n+1}_-) \) is such that \( \varphi = 1 \) on \( \text{supp}(\varphi) \) and \( \Lambda^{-|s|} \) is a pseudodifferential operator with symbol

\[
\sigma(\Lambda^{-|s|}) = \frac{1}{(1 + \xi^2 + \eta^2)^{|s|/2}}
\]

and then applying the theorem to \( \Lambda^{-|s|} \circ \varphi A \).

In the case of a distribution supported on the half-space, we have the following theorem (see also Proposition 3.8 in [2] or, for the analogue in the case of global regularity, Theorem 5.2.5 in [3]).

**Theorem 2.7:** Let \( f \in L^2(\mathbb{H}^{n+1}_-) \). Let \( A \in \Psi^k(\mathbb{R}^{n+1}), k \leq -1 \). Then, for \( 0 \leq s \leq |k| \),

\[
\| \varphi A f \|_{W^s(\mathbb{H}^{n+1}_-)} \lesssim \| f \|_{L^2(\mathbb{H}^{n+1}_-)}
\]

for any \( \varphi \in C_0^\infty(\mathbb{H}^{n+1}_-) \).
Proof: We write

\[ Af = \frac{1}{(2\pi)^{n+1}} \int a(x, \rho, \xi, \eta) \hat{f}(\xi, \eta) e^{i\xi \rho} e^{i\rho \eta} \, d\xi \, d\eta \]

and estimate

\[ \| \varphi \partial_x^\alpha \partial_\rho^\beta A g \|_{L^2}^2 \lesssim \int \frac{\xi^{2|\alpha|} \eta^{2\beta}}{(1 + \eta^2 + \xi^2) |k|} |\hat{f}(\xi, \eta)|^2 \, d\eta \, d\xi \lesssim \|f\|_{L^2(\mathbb{R}^{n+1})}^2. \]

We can combine Theorems 2.3 and 2.6 (respectively, Theorems 2.5 and 2.7) and apply them to operators which can be decomposed into an operator satisfying the hypothesis of Theorem 2.3 and a remainder term.

Definition 2.8: We say an operator \( B \in \Psi^{-k}(\mathbb{R}^{n+1}) \) for \( k \geq 1 \) is decomposable if for any \( N \geq k \) it can be written in the form

\[ B = A + A_{-N}, \]

where \( A \in \Psi^{-k}(\mathbb{R}^{n+1}) \) is an operator satisfying the hypothesis of Theorem 2.3.

We recall the discussion in the Introduction and remark here that for our purposes we could have used the definition of operators with the transmission property as defined by Boutet de Monvel [2] as in the applications which are to appear, all our decomposable operators can be reduced to or can be replaced with such operators with the transmission property (with perhaps some trivial modifications for the order of the operator). This would require, however, a significant amount of explanation and some not so enlightening calculations to show how they can be reduced to the case of Boutet de Monvel.

As we mentioned in the Introduction, on a smoothly bounded domain, \( \Omega \), we can localize by considering a covering of \( \Omega \) so that in each set of the covering there exist local coordinates, \((x, \rho)\), and then apply the above analysis on half-spaces to the domain with (smooth) boundary, \( \partial \Omega \). We can then define \( \Psi^k(\Omega) \) by using local coordinate charts and defining \( \Psi^k(\mathbb{R}^{n+1}) \) as the restriction of an operator in \( \Psi^k(\mathbb{R}^{n+1}) \) to \( \rho < 0 \).

We use the notation \( \Psi_b^k(\mathbb{R}^n) \), respectively, \( \Psi_\rho^k(\partial \Omega) \) in the case of pseudodifferential operators on a domain \( \Omega \subset \mathbb{R}^{n+1} \), to denote the space of pseudodifferential operators of order \( k \) on \( \mathbb{R}^n = \partial \mathbb{H}^{n+1} \), respectively, \( \partial \Omega \). Further following our use of the notation \( A_k \) to denote any operator belonging to the family \( \Psi^k(\mathbb{H}^{n+1}) \) (respectively, \( \Psi^k(\Omega) \)) when acting on distributions \( \phi \in \mathcal{E}'(\mathbb{H}^{n+1}) \) (respectively, in \( \mathcal{E}'(\Omega) \)) we write for \( \phi_b \in \mathcal{E}'(\mathbb{R}^n) \) (respectively, in \( \mathcal{E}'(\partial \Omega) \)) \( A_{k,b} \phi_b, A_{k,b} \) denoting a pseudodifferential operator of order \( k \) on the appropriate boundary of a domain.

With coordinates \((x_1, \ldots, x_n, \rho)\) in \( \mathbb{R}^{n+1} \), let \( R \) denote the restriction operator, \( R : \mathcal{D}'(\mathbb{R}^{n+1}) \rightarrow \mathcal{D}'(\mathbb{R}^n) \), given by \( R \phi = \phi|_{\rho=0} \).

Lemma 2.9: Let \( g \in \mathcal{D}'(\mathbb{R}^{n+1}) \) of the form \( g(x, \rho) = g_\rho(x) \delta(\rho) \) for \( g_\rho \in \mathcal{D}'(\mathbb{R}^n) \). Let \( A \in \Psi^k(\mathbb{R}^{n+1}) \) be an operator of order \( k \) for \( k \leq -2 \). Then, \( R \circ A \) induces a pseudodifferential operator in \( \Psi^k_{b+1}(\mathbb{R}^n) \) acting on \( g_\rho \) via

\[ R \circ A g = A_{k+1,b} g_\rho. \]
**Proof:** Denote the symbol of $A$ with $a(x, \rho, \xi, \eta)$. The symbol
\[
\alpha(x, \rho, \xi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} a(x, \rho, \xi, \eta) \, d\eta
\]
(for any fixed $\rho$) belongs to the class $S^{k+1}(\mathbb{R}^n)$, which follows from the properties of $a(x, \rho, \xi, \eta)$ as a member of $S^k(\mathbb{R}^{n+1})$ and differentiating under the integral. The composition $R \circ A g$ is given by
\[
\frac{1}{(2\pi)^{n+1}} \int a(x, 0, \xi, \eta) \tilde{g}_b(\xi) e^{ix \cdot \xi} \, d\xi \, d\eta
\]
\[
= \frac{1}{(2\pi)^n} \int \left[ \frac{1}{2\pi} \int_{-\infty}^{\infty} a(x, 0, \xi, \eta) \, d\eta \right] \tilde{g}_b(\xi) e^{ix \cdot \xi} \, d\xi
\]
\[
= \frac{1}{(2\pi)^n} \int a(x, 0, \xi) \tilde{g}_b(\xi) \, e^{ix \cdot \xi} \, d\xi
\]
\[
= A_{k+1, b} g_b.
\]

**Remark 2.10:** We can generalize Lemma 2.9 to decomposable operators of order $k = -1$ by using the residue calculus to integrate out the $\eta$ variable. See also Theorem 5.2.4 (ii) of [3].

We work directly with inverses to elliptic operators and as such we consider symbols which are also dependent on the $\rho$ variable. Even if we were to reduce our operators to the case handled by the transmission property, we would need a way to deal with the $\rho$ dependence. The following lemma is useful in illustrating the effect of multiplication by a factor of $\rho$ with an operator (while operating on a boundary distribution).

**Lemma 2.11:** Let $g \in \mathcal{D}'(\mathbb{R}^{n+1})$ of the form $g(x, \rho) = g_b(x) \delta(\rho)$ for $g_b \in \mathcal{D}'(\mathbb{R}^n)$. Let $A \in \Psi^k(\mathbb{R}^{n+1})$ be a pseudodifferential operator of order $k$. Let $\rho$ denote the operator of multiplication with $\rho$. Then, $\rho \circ A$ induces a pseudodifferential operator of order $k-1$ on $g$:
\[
\rho A g = A_{k-1} g.
\]

**Proof:** We write the symbol of the operator $A$ symbol as $a(x, \rho, \xi, \eta)$: $A = \text{Op}(a)$. Since $a(x, \rho, \xi, \eta)$ is of order $k$, $\rho \cdot a(x, \rho, \xi, \eta)$ is also of order $k$, and
\[
\rho \circ A(g) = \int \rho a(x, \rho, \xi, \eta) \tilde{g}_b(\xi) e^{ix \cdot \xi} \, e^{ip \eta} \, d\xi \, d\eta
\]
\[
= -i \int a(x, \rho, \xi, \eta) \tilde{g}_b(\xi) e^{ix \cdot \xi} \frac{\partial}{\partial \eta} \, e^{ip \eta} \, d\xi \, d\eta
\]
\[
= i \int \frac{\partial}{\partial \eta} (a(x, \rho, \xi, \eta)) \tilde{g}_b(\xi) e^{ix \cdot \xi} \, e^{ip \eta} \, d\xi \, d\eta
\]
\[
= A_{k-1} g,
\]
as $\frac{\partial}{\partial \eta} (a(x, \rho, \xi, \eta))$ is a symbol of class $S^{k-1}(\mathbb{R}^{n+1})$. ■
Lemma 2.9 concerned itself with the restrictions of pseudodifferential operators (applied to distributions supported on the boundary) to the boundary, while Theorem 2.3 allows us to consider pseudodifferential operators applied to restrictions of distributions. A special case of Theorem 2.3 is

**Lemma 2.12:** Let $A \in \Psi^k(\mathbb{R}^{n+1})$, for $k \leq -1$, be a decomposable operator. Then,

$$A \circ R \circ A^{-\infty} : \mathcal{E}'(\mathbb{H}^{n+1}) \to C^\infty(\mathbb{H}^{n+1}),$$

i.e. $A \circ R \circ A^{-\infty} = A^{-\infty}$.

**Proof:** Let $f \in \mathcal{E}'(\mathbb{H}^{n+1})$ and apply Theorem 2.3 (for decomposable operators) with $gb = R \circ A^{-\infty} f$. Then, for all $s$,

$$\|A \circ R \circ A^{-\infty} f\|_{W^s(\mathbb{H}^{n+1})} \lesssim \|R \circ A^{-\infty} f\|_{W^{s+k+1/2}(\mathbb{R}^n)} \lesssim \|A^{-\infty} f\|_{W^{\max(1,s+1)}(\mathbb{R}^{n+1})} \lesssim \|f\|_{W^{-\infty}(\mathbb{H}^{n+1})}.$$ 

The lemma thus follows from the Sobolev Embedding Theorem. ■

Similarly proven is the

**Lemma 2.13:** Let $A \in \Psi^k(\mathbb{R}^{n+1})$, for $k \leq -1$, be a decomposable operator. Then,

$$A \circ A^{-\infty,b} : \mathcal{E}'(\mathbb{R}^n) \to C^\infty(\mathbb{R}^{n+1}).$$

### 3. Analysis on intersections of half-spaces

Another situation in which the above analysis of pseudodifferential operators can be applied, and the situation which is our main interest in writing this article, is on an (non-degenerate) intersection of smooth domains. Localizing the problem in analogy to the localizations of Section 2 allows us to represent each domain composing the intersection as a separate half-space. With appropriate choice of metric the domain can be modeled by the intersection of several half-spaces. In this section, we study some properties of pseudodifferential operators on such spaces. The motivation for this study is the application of the following results to the study of elliptic operators on intersection domains and, in particular, to be able to obtain weighted estimates for solutions to elliptic problems on the intersection of smooth domains.

In this section, $\rho$ is a variable in $\mathbb{R}^m$ for $m \leq n$: $\rho = (\rho_1, \ldots, \rho_m)$, and $x = (x_1, \ldots, x_{n-m})$ is an $n-m$ dimensional variable. We define the half-spaces,

$$\mathbb{H}^n_j = \{(x, \rho) \in \mathbb{R}^n : \rho_j < 0\}.$$ 

With a multi-index $I = (i_1, \ldots, i_k)$, we denote the intersection of half-spaces,

$$\mathbb{H}^n_I = \bigcap_{j \in I} \mathbb{H}^n_j.$$ 

The convention used here is $\mathbb{H}^n_I = \mathbb{R}^n$ when $I = \emptyset$. For this section, we will fix $I$ with $|I| = m$. Without loss of generality, $I = (1, \ldots, m)$.
We use the multi-index notation:
\[ \rho_{\alpha} = \prod_{j \in J} \rho_{\alpha_j}^j, \]
for \( \alpha = (\alpha_1, \ldots, \alpha_{|J|}) \) a multi-index. To indicate a missing index, \( j \), we use the notation \( \hat{j} \). Thus, we write
\[ I_\hat{j} := I \setminus \{j\}. \]
For ease of notation, in place of \( \rho_{\alpha_j}^j \), we write
\[ \rho_{\alpha_j}^j = \rho_{\alpha_1}^1 \cdots \rho_{\alpha_{j-1}}^{j-1} \rho_{\alpha_{j+1}}^{j+1} \cdots \rho_{\alpha_m}^m. \]
Similarly, we write
\[ \rho_{\alpha_k}^j = \prod_{i \in I \setminus \{i \neq j, k\}} \rho_{\alpha_i}^i. \]
In the case we have equal powers,
\[ \alpha_1 = \cdots = \alpha_{j-1} = \alpha_{j+1} = \cdots = \alpha_m = r, \]
we write
\[ \rho_j^{r \times (m-1)} := \rho_{\alpha_j}^j. \]
We now define the weighted Sobolev norms on the half-spaces, for \( \alpha \in \mathbb{R} \), and \( s, k \in \mathbb{N} \):
\[ W^{\alpha,s}(\mathbb{H}_I^n, \rho, k) = \left\{ f \in W^\alpha(\mathbb{H}_I^n) \mid \rho^{(sk-rk)x+m} f \in W^{\alpha+s-r}(\mathbb{H}_I^n) \text{ for each } 0 \leq r \leq s \right\} \]
with norm
\[ \|f\|_{W^{\alpha,s}(\mathbb{H}_I^n, \rho, k)} = \sum_{0 \leq j \leq s} \|\rho^{(sk-rk)x+(m-1)} f\|_{W^{\alpha+s-j}(\mathbb{H}_I^n)}. \]
Similar weighted spaces can be defined with one (or more) \( \rho_j \) terms missing. For example,
\[ W^{\alpha,s}(\mathbb{H}_I^n, \rho_j, k) = \left\{ f \in W^\alpha(\mathbb{H}_I^n) \mid \rho_j^{(sk-rk)x+(m-1)} f \in W^{\alpha+s-r}(\mathbb{H}_I^n) \text{ for each } 0 \leq r \leq s \right\} \]
with norm
\[ \|f\|_{W^{\alpha,s}(\mathbb{H}_I^n, \rho_j, k)} = \sum_{0 \leq j \leq s} \|\rho_j^{(sk-rk)x+(m-1)} f\|_{W^{\alpha+s-j}(\mathbb{H}_I^n)}. \]
In the case \( k = 1 \), we shall use the notation,
\[ W^{\alpha,s}(\mathbb{H}_I^n, \rho) := W^{\alpha,s}(\mathbb{H}_I^n, \rho, 1), \]
which has the norm
\[ \|f\|_{W^{\alpha,s}(\mathbb{H}_I^n, \rho)} = \sum_{r=0}^s \|\rho^{r \times m} f\|_{W^{\alpha+r}(\mathbb{H}_I^n)}. \]
As there are several domains whose boundaries make up the boundary of an intersection domain, we use a subscript to indicate a pseudodifferential operator in the class of operators on a specific boundary. For example, if \( A \in \Psi^\alpha(\partial \mathbb{H}^n_j) \), we write \( A = A_{\alpha, bj} \). We adhere to the convention that \( A_k \) denotes an operator in \( \Psi_k(\mathbb{R}^n) \), and that \( A_{k, bj} \) denotes an operator in \( \Psi^k(\partial \mathbb{H}^n_j) \).

To denote extensions by 0 across \( \rho_j = 0 \) to \( \rho_j > 0 \), we use the superscript \( E_j \): let \( g \in L^2(\mathbb{H}^n_I) \); then \( g^{E_j} \in L^2(\mathbb{H}^n_{I, j}) \) is defined by

\[
g^{E_j} = \begin{cases} g & \text{if } \rho_j < 0 \\ 0 & \text{if } \rho_j \geq 0. \end{cases}
\]

Similarly, for a multi-index, \( J \), we define \( g^{E_J} \in L^2(\mathbb{H}^n_I \setminus J) \) by \( g^{E_J} = g \) on \( \mathbb{H}^n_I \) and 0 elsewhere (that is for any \( (x, \rho) \in \mathbb{H}^n_{I, J} \) for which any \( \rho_j \geq 0 \) for \( j \in J \)).

One of the (equivalent) definitions of Sobolev spaces on Lipschitz domains (which applies to our case of intersection domains) relies on first defining the Sobolev spaces in \( \mathbb{R}^n \) and then restricting functions defined in all of \( \mathbb{R}^n \) to a bounded Lipschitz domain. The next lemmas show that multiplication by factors of the defining functions allows one to consider extensions by zero as the functions on which to apply restrictions.

We establish

**Lemma 3.1:** Let \( g \in W^s(\mathbb{H}^n_I) \) for some integer \( s \geq 0 \). Then \( \rho_j^s g^{E_j} \in W^s(\mathbb{H}^n_{I, j}) \).

**Proof:** We only need to check the derivatives with respect to \( \rho_j \). We have

\[
\partial_{\rho_j}^s \left( \rho_j^s g^{E_j} \right) = \sum c_k \rho_j^{s-k} \partial_{\rho_j}^{s-k} g^{E_j}. \tag{9}
\]

\( \partial_{\rho_j}^{s-k} g^{E_j} \) itself is a sum of terms of (derivatives of) delta functions, \( \delta^{(i)}, i \leq s - k - 1 \), in addition to the extension of terms \( \left( \partial_{\rho_j}^{s-k-i} g \right)^{E_j} \mid_{\mathbb{H}^n_I} \):

\[
\partial_{\rho_j}^{s-k} g^{E_j} = \sum_{i=0}^{s-k} d_i \delta^{(i)}(\rho_j) \left( \partial_{\rho_j}^{s-k-i} g \right)^{E_j},
\]

where we consider \( \delta^{-1} \equiv 1 \), and the \( d_i \) are constants with \( d_0 = 1 \). Inserting this into (9), the delta functions combine with the powers of \( \rho_j \) to yield zero, and we have

\[
\partial_{\rho_j}^s \left( \rho_j^s g^{E_j} \right) = \sum c_k \rho_j^{s-k} \left( \partial_{\rho_j}^{s-k} g \right)^{E_j}.
\]

The lemma now follows (in the case \( s \) is an integer) by the assumption on the regularity of \( g \) in \( \mathbb{H}_I \). \( \blacksquare \)

A similar proof shows

**Lemma 3.2:** Let \( s \geq 0 \) be an integer, \( k \in \mathbb{N} \), and \( \rho_j^k g \in W^{r-\alpha}(\mathbb{H}^n_I) \) for integers \( r \leq s \) and \( \alpha \geq 0 \). Then \( \rho_j^k g^{E_j} \in W^{s-\alpha}(\mathbb{H}^n_{I, j}) \).
For a multi-index, $J$, let us denote
\[ \mathbb{H}_{J,k}^{n-1} := \partial \mathbb{H}_{k}^{n} \cap \mathbb{H}_{J}^{n}, \]
with the convention $J_{k} = J$ in the case $k \notin J$.

We present the following theorem which is a weighted analogue of Theorem 2.3. In the following theorem, we use the notation $\delta_{j} := \delta(\rho_{j})$. A pseudodifferential operator will be applied to a distribution supported on the boundary $\mathbb{H}_{I,bj}^{n-1}$ and to apply Theorem 2.3 we look at the hypotheses with respect to $\eta_{j}$, the dual variable to $\rho_{j}$; for instance, the symbol of the operator will be meromorphic with respect to $\eta_{j}$ with poles giving symbols of order 1 operators. We say in this case that the operator satisfies the hypotheses of Theorem 2.3 with respect to $\mathbb{H}_{I,bj}^{n-1}$.

**Theorem 3.3:** Let $A$ be a pseudodifferential operator (of order $-\alpha \leq -1$) satisfying the hypotheses of Theorem 2.3 with respect to $\mathbb{H}_{I,bj}^{n-1}$. Let $0 \leq \gamma < 1/2$ and $g_{b} \in W^{\gamma,s}(\mathbb{H}_{I,bj}^{n-1}, \rho_{j}, k)$ with compact support in $\mathbb{H}_{I,bj}^{n-1}$. Then, for $\beta - 1/2$ a non-negative integer with $\beta - \alpha \leq \gamma$, $\rho^{rk \times m}A \left( \frac{E_{j}}{g_{b}} \times \delta_{j} \right) \in W^{r+\beta}(\mathbb{R}^{n})$

for all $r \leq s$, and

\[ \left\| A \left( \frac{E_{j}}{g_{b}} \times \delta_{j} \right) \right\|_{W^{\beta-1/2,s}(\mathbb{H}_{I,b}^{n}, \rho_{j}, k)} \lesssim \| g_{b} \|_{W^{\beta-\alpha}(\mathbb{H}_{I,bj}^{n-1}, \rho_{j}, k)}. \]

The estimates also hold for all $\beta \geq 1/2$ with the property $-1/2 \leq \beta - \alpha \leq \gamma$.

**Proof:** We prove the Theorem in the case $\beta = \alpha + \gamma$. The general case follows the same steps.

We use an operator $\Lambda_{bj}$ on $\partial \mathbb{H}_{j}^{n}$, which is defined in analogy to the operator $\Lambda$:

\[ \sigma \left( \Lambda_{bj}^{k} \right) = (1 + \xi^{2} + \eta_{j}^{2})^{k/2}. \]

With our notation, $\eta_{j}$ is understood to denote $(\eta_{1}, \ldots, \eta_{j-1}, \eta_{j+1}, \ldots, \eta_{m})$ and

\[ \eta_{j}^{2} = \eta_{1}^{2} + \cdots + \eta_{j-1}^{2} + \eta_{j+1}^{2} + \cdots + \eta_{m}^{2}. \]

We recall that for a bounded Lipschitz domain, $\Omega$, the operator defined by extension by zero outside of $\Omega$ to all of $\mathbb{R}^{n}$ is bounded on $W^{\gamma}(\Omega)$ (see Theorem 3.33 in [16]). Thus, we have

\[ \frac{E_{j}}{g_{b}} \in W^{\gamma}(\partial \mathbb{H}_{j}^{n}) \]

or

\[ \Lambda_{bj}^{\gamma} \frac{E_{j}}{g_{b}} \in L^{2}(\partial \mathbb{H}_{j}^{n}). \]
By assumption,
\[ D^{r}_{b} \rho_{j}^{r \times (m-1)} g_{b} \in W^{\gamma} \left( \mathbb{H}_{I,bj}^{n-1} \right), \]  
(10)
for \( 0 \leq r \leq s \), and \( D^{r}_{b} \) a differential operator of order \( r \) on \( \partial \mathbb{H}_{j}^{n} \). Using extensions by zero, we see
\[ D^{r}_{b} \rho_{j}^{r \times (m-1)} E_{l}^{j} g_{b} \in W^{\gamma} \left( \partial \mathbb{H}_{j}^{n} \right) \]
and thus
\[ \rho_{j}^{r \times (m-1)} E_{l}^{j} g_{b} \in W^{r+\gamma} \left( \partial \mathbb{H}_{j}^{n} \right). \]

Write \( g_{j} = g_{b} \times \delta_{j} \). We use Lemma 2.11 to write
\[ \rho_{j}^{r} A g_{j} = A_{-r-k} g_{j}. \]
We have
\[ \rho^{r \times m} A g_{j} = \rho_{j}^{r \times (m-1)} \rho_{j}^{r} A_{-a} g_{j} = \rho_{j}^{r \times (m-1)} A_{-r-a} g_{j} = \sum_{l=0}^{r} A_{-r-r-(r-l)} \left( \rho_{j}^{l \times (m-1)} g_{j} \right). \]
The \( A_{-r-r-(r-l)} \) operators above satisfy the hypotheses of Theorem 2.3, while
\[ \rho_{j}^{l \times (m-1)} E_{l}^{j} g_{b} \in W^{l+\gamma} \left( \partial \mathbb{H}_{j}^{n} \right). \]

Therefore, Theorem 2.3 applies to give the estimates
\[ \sum_{l} \left\| A_{-r-r-(r-l)} \left( \rho_{j}^{l \times (m-1)} g_{j} \right) \right\|_{W^{r+\alpha+\gamma-1/2}(\mathbb{R}^{n})} \ll \sum_{l} \left\| A_{-r-r-(r-l)} \left( \rho_{j}^{l \times (m-1)} g_{j} \right) \right\|_{W^{r+k+r+\alpha+\gamma-1/2}(\mathbb{R}^{n})} \ll \sum_{l} \left\| \rho_{j}^{l \times (m-1)} g_{b} \right\|_{W^{l+\gamma} \left( \mathbb{H}_{I,bj}^{n-1} \right)} \ll \left\| g_{b} \right\|_{W^{\gamma \cdot r} \left( \mathbb{H}_{I,bj}^{n-1} \rho_{j}^{k} \right)}. \]

The assumption that \( g_{b} \) has compact support in \( \mathbb{H}_{I,bj}^{n-1} \) was used in order to apply extensions by zero. Such an assumption will not be needed when the analysis on bounded domains is applied.

For operators which do not satisfy the hypotheses of Theorem 2.3 we can derive estimates in a similar manner to the method of Theorem 2.6.
**Theorem 3.4:** Let $A \in \Psi^{-\alpha}(\mathbb{R}^n)$, $-\alpha \leq -1$. Then for $\beta - 1/2$ an integer with $\beta \leq \alpha - 1/2$, and $g_b \in W^{0,s}(\mathbb{H}^{n-1}_{I,j}, \rho_j, k)$ with compact support in $\mathbb{H}^{n-1}_{I,j}$,

$$\left\| A \left( g_b \chi_{\tilde{\delta}}^j \right) \right\|_{W^{\beta-1/2,s}(\mathbb{H}^{n-1}_{I,j}, \rho_j, k)} \lesssim \| g_b \|_{W^{\beta-\alpha,s}(\mathbb{H}^{n-1}_{I,j}, \rho_j, k)}.$$  

**Proof:** The proof of Theorem 3.3 applies up to the last estimate where Theorem 2.6 is to be applied as opposed to Theorem 2.3. For this case, we need to ensure $\beta \leq \alpha - 1/2$. 

In the case of operators acting on functions supported on all of $\mathbb{H}^n_I$, we have the following weighted estimates

**Theorem 3.5:** Let $A \in \Psi^{-\alpha}(\mathbb{R}^n)$ for $\alpha \geq 1$. Let $f \in W^{0,s}(\mathbb{H}^n_I, \rho, k)$. Then,

$$\| Af \|_{W^{\alpha,s}(\mathbb{H}^n_I, \rho, k)} \lesssim \| f \|_{W^{0,s}(\mathbb{H}^n_I, \rho, k)}.$$  

**Proof:** We have for $r \leq s$,

$$\rho^{rk \times m} Af = \sum_{l=0}^r A_{-\alpha-(r-l)} \rho^{lk \times m} f.$$  

By Lemma 3.2, we have $\rho^{lk \times m} f \in W^l(\mathbb{R}^n)$ from which the estimates

$$\left\| \rho^{rk \times m} Af \right\|_{W^{\alpha+r}(\mathbb{R}^n)} \lesssim \sum_{l=0}^r \left\| \rho^{lk \times m} f \right\|_{W^l(\mathbb{H}^n_I)} \lesssim \| f \|_{W^{0,r}(\mathbb{H}^n_I, \rho, k)}$$

follow. Summing over all $r \leq s$ finishes the proof. 

We can improve the above theorem by removing one of the $\rho$ components and using Theorem 2.7.

We say an operator $B \in \Psi^{-k}(\mathbb{R}^{n+1})$ for $k \geq 1$ is decomposable with respect to $\mathbb{H}^{n-1}_{I,j}$ if for any $N \geq k$ it can be written in the form

$$B = A + A_{-N},$$

where $A \in \Psi^{-k}(\mathbb{R}^{n+1})$ is an operator satisfying the hypothesis of Theorem 2.3 with respect to $\mathbb{H}^{n-1}_{I,j}$.

**Theorem 3.6:** Let $A \in \Psi^{-\alpha}(\mathbb{R}^n)$ for $\alpha \geq 1$ be decomposable with respect to $\mathbb{H}^{n-1}_{I,j}$, for some $j \in I$. Let $f \in W^{0,s}(\mathbb{H}^n_I, \rho_j, k)$. Then,

$$\| Af \|_{W^{\alpha,s}(\mathbb{H}^n_I, \rho_j, k)} \lesssim \| f \|_{W^{0,s}(\mathbb{H}^n_I, \rho_j, k)}.$$
Proof: The proof is almost the same as that of Theorem 3.5. We have for \( r \leq s \),

\[
\rho_j^{rk \times (m-1)} Af = \sum_{l=0}^r A_{-\alpha-(r-l)} \left( \rho_j^{lk \times (m-1)} f \right).
\]

By Lemma 3.2 we have \( \rho_j^{lk \times (m-1)} f \in W^l(\mathbb{H}^n_j) \). Thus, an application of Theorem 2.5 yields

\[
\left\| \rho_j^{rk \times (m-1)} Af \right\|_{W^\alpha+r(\mathbb{R}^n)} \lesssim \sum_{l=0}^r \left\| \rho_j^{lk \times (m-1)} f \right\|_{W^l(\mathbb{H}^n_j)} \\
\lesssim \|f\|_{W^{0,r}(\mathbb{H}^n_j,\rho_j,k)}.
\]

Summing over all \( r \leq s \) finishes the proof.

When working with boundary value problems on intersection domains, or intersections of half-spaces, restrictions to one boundary of an operator applied to a distribution with support on a different boundary arise.

Let \( R_j \) denote the operator of restriction to the boundary, \( \rho_j = 0 \). To deal with restrictions to one boundary of an operator acting on a distribution supported on another boundary, we introduce some notation: for \( \alpha + 1/2 \in \mathbb{N}, \alpha \geq 1/2 \), and \( j \neq k \),

\[
\mathcal{E}_{-\alpha}^{jk} : W^s \left( \mathbb{H}^{n-1}_{I,j}, \rho_j, \lambda \right) \to W^{s+\alpha} \left( \mathbb{H}^{n-1}_{I,k}, \rho_k, \lambda \right)
\]

(with some restriction on \( s \) to be introduced), where \( \mathcal{E}_{-\alpha}^{jk} \) is of the form

\[
\mathcal{E}_{-\alpha}^{jk} g_j = R_k \circ B_{-\alpha-1/2} g_j,
\]

where, as above \( g_j := g_{\mathcal{E}_j} \times \delta_j \), and where \( B_{-\alpha-1/2} \in \Psi^{-\alpha-1/2}(\mathbb{R}^n) \) is decomposable with respect to \( \mathbb{H}^{n-1}_{I,j} \).

For some crude estimates in the case \( 1/2 \leq \beta \leq \alpha - 1 \), we could write

\[
\left\| \rho_j^{rk \times (m-1)} \mathcal{E}_{-\alpha}^{jk} g_j \right\|_{W^{\beta-1/2} \left( \mathbb{H}^{n-1}_{I,k} \right)} \lesssim \left\| \rho_j^{rk \times (m-1)} R_k \circ A_{-\alpha-1/2} g_j \right\|_{W^{\beta-1/2} \left( \mathbb{H}^{n} \right)} \\
\lesssim \left\| \rho_j^{rk \times (m-1)} A_{-\alpha-1/2} g_j \right\|_{W^{\beta} \left( \mathbb{H}^{n} \right)} \\
\lesssim \left\| g_j \right\|_{W^{\beta-\alpha} \left( \mathbb{H}^{n-1}_{I,j}, \rho_j \lambda \right)},
\]

where in the last step we use the estimates from Theorem 3.4 (with some restrictions on which sets, such as the integers, \( \alpha \) and \( \beta \) belong to). And after summing over \( r \leq s \), we would have the estimates

\[
\left\| \mathcal{E}_{-\alpha}^{jk} g_j \right\|_{W^{\beta-1/2,s} \left( \mathbb{H}^{n-1}_{I,k}, \rho_k \right)} \lesssim \left\| g_j \right\|_{W^{\beta-\alpha,s} \left( \mathbb{H}^{n-1}_{I,j}, \rho_j \right)}
\]

for \( \beta \leq \alpha - 3/2 \).
However, we can improve the estimates for the operators with meromorphic symbols in two ways. First, the order of the Sobolev spaces can be increased by making use of relations between elliptic operators acting on distributions supported on the boundary. Second, there is a loss of a factor of $\rho$ in the estimate above; as $g_b \times \delta_j$ is supported on $\mathbb{H}^n_{I,j}$, a weighted estimate, using $\rho_j$ is desired on the right (as opposed to $\rho_k^* j$). These improvements will be made in the following Corollary of Theorems 3.3 and 3.4.

**Corollary 3.7:** Let $E_{-\alpha}^j$ as above, and $g_b \in W^{0,s}(\mathbb{H}^n_{I,j}, \rho_j, \lambda)$. Then, for $0 \leq \beta < \alpha$

$$
\left\| E_{-\alpha}^j g_b \right\|_{W^{\beta,s} \left( \mathbb{H}^n_{I,j}, \rho_k^* \lambda \right)} \lesssim \left\| g_b \right\|_{W^{\beta-a,s} \left( \mathbb{H}^n_{I,j}, \rho_j^* \lambda \right)}.
$$

**Proof:** For given $\alpha, s$, we choose $N$ large, $\alpha + s \ll N$, and we write

$$
E_{-\alpha}^j g_b = R_k \circ A_{-\alpha-\frac{1}{2}}g_j + R_k \circ A_{-N}g_j,
$$

where $A_{-\alpha-\frac{1}{2}}$ satisfies the hypotheses of Theorem 2.3 with respect to $\mathbb{H}^n_{I,j}$.

We first estimate $R_k \circ A_{-N}g_j$,

$$
\left\| R_k \circ A_{-N}g_j \right\|_{W^{\beta,s} \left( \mathbb{R}^{n-1} \right)} \lesssim \int \frac{\left( \eta^2 + \xi^2 \right)^{\beta+s}}{(1 + \eta^2 + \xi^2)^N} |\tilde{g}_b(\xi, \eta_j)|^2 \, d\xi \, d\eta.
$$

Integrating over $\eta_j$ yields

$$
\left\| R_k \circ A_{-N}g_j \right\|_{W^{\beta,s} \left( \mathbb{R}^{n-1} \right)} \lesssim \int \frac{\left( \eta_j^2 + \xi^2 \right)^{\beta+s}}{(1 + \eta_j^2 + \xi^2)^{N-1/2}} |\tilde{g}_b(\xi, \eta_j)|^2 \, d\xi \, d\eta
$$

$$
\lesssim \left\| g_b \right\|_{W^{\beta-a, \left( \mathbb{H}^n_{I,j} \right)}}^2,
$$

from the assumption $\alpha + s \ll N$.

To show

$$
\left\| R_k \circ A_{-\alpha-\frac{1}{2}}g_j \right\|_{W^{\beta \left( \mathbb{R}^{n-1} \right)}} \lesssim \left\| g_b \right\|_{W^{\beta-a, \left( \mathbb{H}^n_{I,j} \right)}}^2,
$$

for $\beta < \alpha$, we estimate

$$
\left\| R_k \circ A_{-\alpha-\frac{1}{2}}g_j \right\|_{W^{\beta \left( \mathbb{R}^{n-1} \right)}} \lesssim \int \frac{\left( \eta_j^2 + \xi^2 \right)^{\beta}}{(1 + \eta_j^2 + \xi^2)^{\alpha+1/2}} |\tilde{g}_b(\xi, \eta_j)|^2 \, d\xi \, d\eta
$$

$$
\lesssim \int \frac{\left( \eta_j^2 + \xi^2 \right)^{\beta}}{(1 + \eta_j^2 + \xi^2)^{\alpha}} |\tilde{g}_b(\xi, \eta_j)|^2 \, d\xi \, d\eta
$$

$$
\lesssim \left\| g_b \right\|_{W^{\beta-a, \left( \mathbb{H}^n_{I,j} \right)}}^2.
$$
When calculating the weighted estimates on $\mathbb{H}_{l,k}^{1}$, with weights which do not include factors of $\rho_k$, we use the identity

$$\rho_k^{\delta \times (m-1)} R_k \circ A_{-\alpha} g_j \simeq R_k \circ \frac{\partial^s}{\partial \rho_k^s} \rho_k^{\delta \times m} A_{-\alpha} g_j,$$

with $\alpha \geq 1$. Note that the regularity provided by the order, $-\alpha \leq -1$ of the pseudodifferential operator ensures that derivatives with respect to $\rho_k$ produce no $\delta$ distributions (or derivatives of $\delta$) terms. Thus, we write

$$\rho_k^{r \lambda \times (m-1)} R_k \circ A_{-\alpha-\frac{1}{2}} g_j \simeq R_k \circ \frac{\partial^{r \lambda}}{\partial \rho_k^{r \lambda}} \rho_k^{r \lambda \times m} A_{-\alpha-\frac{1}{2}} g_j,$$

and further, we use the relation

$$\rho_k^{r \lambda \times m} A_{-\alpha-\frac{1}{2}} g_j = \rho_j^{r \lambda \times (m-1)} A_{-\alpha-1/2-r \lambda} g_j$$

$$= \sum_{l=0}^{r} A_{-\alpha-1/2-r \lambda-(r-l)} \left( \rho_j^{l \lambda \times (m-1)} g_j \right)$$

to show

$$\frac{\partial^{r \lambda}}{\partial \rho_k^{r \lambda}} \rho_k^{r \lambda \times m} A_{-\alpha-\frac{1}{2}} g_j = \sum_{l=0}^{r} A_{-\alpha-1/2-(r-l)} \left( \rho_j^{l \lambda \times (m-1)} g_j \right).$$

Therefore, we have

$$\rho_k^{r \lambda \times (m-1)} R_k \circ A_{-\alpha-\frac{1}{2}} g_j = R_k \circ \sum_{l=0}^{r} A_{-\alpha-1/2-(r-l)} \left( \rho_j^{l \lambda \times (m-1)} g_j \right).$$

We note the symbols of the $A_{-\alpha-1/2-(r-l)}$ operators can be bounded by

$$\frac{(1 + \eta_j^2 \eta_l^{r \lambda/2})}{(1 + \eta^2 + \xi^2)^{1/2(\alpha + 1/2+(r-l)+r \lambda)}}$$

modulo lower order symbols with bounds of the same form.

The calculation of the estimates then follow those above. In particular, for $\beta < \alpha$, we have

$$\left\| \rho_k^{r \lambda \times (m-1)} R_k \circ A_{-\alpha-\frac{1}{2}} g_j \right\|_{W^{\beta+r}(\mathbb{R}^{n-1})}$$

$$\lesssim \sum_{l=0}^{r} \int \left( \eta^2 + \xi^2 \right)^{\beta+r} \frac{(1 + \eta_j^2 \eta_l^{r \lambda})}{(1 + \eta^2 + \xi^2)^{\alpha + 1/2+(r-l)+r \lambda}} |\widehat{g_j}(\xi, \eta_j)|^2 \ d\xi \ d\eta$$

$$\lesssim \sum_{l=0}^{r} \int \left( \eta_j^2 + \xi^2 \right)^{\beta+r+\alpha} \frac{1}{(1 + \eta^2 + \xi^2)^{\alpha + (r-l)+r \lambda}} |\widehat{g_j}(\xi, \eta_j)|^2 \ d\xi \ d\eta$$

$$\lesssim \|g_j\|_{W^{\beta-a,r}(\mathbb{H}_{l,k}^{1})}^2.$$
where $h_j^l = \rho_j^{\frac{1}{l} \times (m-1)} g_j$.

Summing over $r \leq s$ give the weighted estimates in terms of $\|g_b\|_{W^\beta-a,s(\mathbb{H}^{n-1},_b)}^2$, as in the statement of the corollary.

For boundary operators mapping $\mathbb{H}^{n-1}_{I,bj}$ to itself we use the notation $\mathcal{E}^{ji}_{-\alpha}$ to denote operators of the form

$$\mathcal{E}^{ji}_{-\alpha} = R_j \circ B_{-\alpha-1},$$

where $B_{-\alpha-1} \in \Psi^{\alpha-1}(-\mathbb{R}^n)$ is decomposable with respect to $\mathbb{H}^{n-1}_{I,bj}$ for $\alpha \geq 1$, and thus of the form

$$\mathcal{E}^{ji}_{-\alpha} = A_{-\alpha,bj},$$

using Lemma 2.9, for $\alpha \geq 1$. We also use the notation to denote compositions:

$$\mathcal{E}^{ji}_{-\alpha} = \mathcal{E}^{kj}_{-\alpha_1} \circ \mathcal{E}^{jk}_{-\alpha_2},$$

where $\alpha = \alpha_1 + \alpha_2$ and $\alpha_1, \alpha_2 \geq 1/2$.

As a corollary of Theorem 3.5, we have

$$\| R_j \circ B_{-\alpha-1} g_j \|_{W^\beta,s\left([\partial\mathbb{H}^{n-1}_{j},\rho^\lambda_j]\right)} \lesssim \| g_b \|_{W^\beta-a,s\left(\mathbb{H}^{n-1}_{I,bj}\right)},$$

for $\beta < \alpha + 1/2$, while Corollary 3.7 applied (twice) to $\mathcal{E}^{ji}_{-\alpha} = \mathcal{E}^{kj}_{-\alpha_1} \circ \mathcal{E}^{jk}_{-\alpha_2}$ yields

$$\| \mathcal{E}^{kj}_{-\alpha_1} \circ \mathcal{E}^{jk}_{-\alpha_2} g_b \|_{W^{\mu_1-\epsilon,s}\left([\partial\mathbb{H}^{n-1}_{j},\rho^\lambda_j]\right)} \lesssim \| g_b \|_{W^{\mu_2-\alpha_2,s}\left(\mathbb{H}^{n-1}_{I,bj}\right)},$$

for $\epsilon > 0$.

4. Example on an intersection of two domains

We close this paper with a mention of how the weighted estimates in the previous section may be applied. The motivation for the introduction of the weighted estimates is the study of estimates for operators of boundary value problems such as the Poisson and Green operators. Let $\Omega_1, \ldots, \Omega_m \subset \mathbb{R}^n$ be smoothly bounded domains which intersect real transversely. That is to say, if $\rho_j$ is a smooth defining function for $\Omega_j$, $|d\rho_j| \neq 0$ on $\partial\Omega_j$, then for all $1 \leq i_1 < \cdots < i_l \leq m$, we have

$$d\rho_{i_1} \wedge \cdots \wedge d\rho_{i_l} \neq 0$$

on $\bigcap_{j=1}^l \partial\Omega_{i_j} \cap \partial\Omega$, where $\rho_j$ is a defining function for $\Omega_j$. We say in this case $\Omega$ is an intersection domain. An intersection domain is an example of a piecewise smooth domain (see [17], from which we base our definition of intersection domains). Then using a suitable metric locally near a point on $\partial\Omega$ the intersection can be modeled by the intersection of $m$ half-spaces.
To illustrate some of the reduction to the boundary techniques on intersection domains, we consider $\Omega = \Omega_1 \cap \Omega_2$, with $\Omega_j$ a bounded smooth domain for $j = 1, 2$. We take $\Gamma$ in this section to be a second order elliptic operator, and consider a Dirichlet problem

$$\Gamma v = 0 \quad \text{in } \Omega$$
$$v = g \quad \text{on } \partial \Omega.$$

The boundary condition can be expressed on the individual boundaries as

$$v_j = g_j \quad \text{on } \partial \Omega_j,$$

where $v_j = v|_{\partial \Omega_j}$ for $j = 1, 2$ (with a similar notation holding for $g_j$). As in the previous section, we will use $R_j$ to denote the operator of restriction to the boundary, $\partial \Omega_j$. Then with this operator we can also write $g_j = R_jv$.

Using a partition of unity we assume $v$ and $g$ have support in a neighborhood of a point of intersection (at which $\rho_1 = \rho_2 = 0$) which we take to be the origin. We assume $\Gamma$ has a local expression in a neighborhood containing the support of $v$ and $g$ of the form

$$\Gamma = -\partial_{\rho_1}^2 - \partial_{\rho_2}^2 - \sum_{j=1}^{n-2} \partial_{x_j}^2 + \sum_{ij} c_{ij}(x) \partial_{x_i} \partial_{x_j} + \sum_i b_i(x, \rho) \partial_{x_i} + O(\rho_1)A_2 + O(\rho_2)A_2 + s_1(x, \rho) \partial_{\rho_1} + s_2(x, \rho) \partial_{\rho_2},$$

(15)

where $\rho_j$ is a defining function for $\Omega_j$ for $j = 1, 2$, with $c_{ij}(x) = O(x)$ and smooth $b_i (1 \leq i \leq n - 2)$, $s_1$ and $s_2$.

We let $\eta_j$ be the transform variable dual to the $\rho_j$ and $\xi_j$ dual to $x_j$, and $\xi = (\xi_1, \ldots, \xi_{n-2})$ as well as $\eta = (\eta_1, \eta_2)$. In writing (15) as an equation with pseudodifferential operators we use the following notations with Fourier Transforms. The full transform of a function, $w$, will be written with the duals to the defining functions first:

$$\hat{w}(\eta, \xi) = \frac{1}{(2\pi)^n} \int w(\rho, x) e^{-i\rho \eta} e^{-ix \xi} \, d\rho \, dx.$$

Partial Fourier Transforms will be denoted with the $\hat{j}$ notation to indicate which $\rho$ variable is not to be transformed. Thus

$$F.T.1 w(\rho_1, \eta_2, \xi) = \frac{1}{(2\pi)^{n-1}} \int w(\rho, x) e^{-i\rho_2 \eta_2} e^{-ix \xi} \, d\rho_2 \, dx.$$

We also use a non-standard notation to set a specific value, which will always be 0, to a variable. We will use the $\hat{w}$ notation even to indicate a partial Fourier Transform. We then rearrange arguments to write that set value first:

$$F.T.2 w|_{\rho_2=0} = \hat{w}(0_2, \eta_2, \xi),$$
where \( k \neq j \). Thus, for instance,
\[
\hat{w}(0_2, \eta_1, \xi) := \frac{1}{(2\pi)^{n-1}} \int w(\rho_1, 0, x) e^{-i\rho_1 \eta_1} e^{-i\xi \hat{w}} \, d\rho_1 \, dx.
\]
This is not to be confused with \( \hat{w}(\eta_1, 0, \xi) = \hat{w}|_{\eta_2=0} \).

With this notation, we rewrite (15) in terms of transforms as
\[
\frac{1}{(2\pi)^2n} \int \left( \eta^2 + \xi^2 - \sum c_{jk} \xi_j \xi_k + \sum O(\rho_j) \right) \hat{v}(\eta, \xi) e^{i\rho \cdot \eta} e^{i\xi \hat{w}} \, d\eta \, d\xi \\
- \frac{1}{(2\pi)^2n} \sum_{j=1}^{2} \int \left( F.T. j \partial_{\rho_j} v(0_j, \eta_j, \xi) + i\eta_j \hat{g}_j(\eta_j, \xi) \right) e^{i\rho \cdot \eta} e^{i\xi \hat{w}} \, d\eta \, d\xi \\
+ \sum_{j=1}^{2} S_j(\partial_{\rho_j} v) + B v = 0, \tag{16}
\]
where \( S_j \) is the zeroth operator with symbol \( s_j(x, \rho) \), and \( B \) a first order operator with symbol \( i \sum_{j=1}^{n-2} b_j(x, \rho) \xi_j \). Note that \( S_j(\partial_{\rho_j} v) \) can be written
\[
S_j(\partial_{\rho_j} v) = \frac{1}{(2\pi)^2n} \sum_{j} \int s_j(x, \rho) \left( \hat{g}_j(\eta_j, \xi) + i\eta_j \hat{v}(\eta, \xi) \right) e^{i\rho \cdot \eta} e^{i\xi \hat{w}} \, d\eta \, d\xi.
\]

We define the symbol, \( \Xi \), by
\[
\Xi(x, \rho, \xi) = \left( \eta^2 + \xi^2 - \sum c_{jk} \xi_j \xi_k + \sum O(\rho_j) \right)^{1/2},
\]
where the \( O(\rho_j) \) terms are those in (16).

Applying an inverse of the highest order terms in (16) yields
\[
v = \frac{1}{(2\pi)^2n} \sum_{j=1}^{2} \int \frac{F.T. j \partial_{\rho_j} v(0_j, \eta_j, \xi) + i\eta_j \hat{g}_j(\eta_j, \xi)}{\eta^2 + \Xi^2(\rho, x, \xi)} e^{i\rho \cdot \eta} e^{i\xi \hat{w}} \, d\eta \, d\xi \\
+ A_{-3}(\partial_{\rho} v|_{\partial \Omega}) + A_{-2} g + A_{-1} v,
\]
locally.

We now expand \( \Xi(\rho, x, \xi) \) in each \( \rho_j \), writing \( \Xi_{bj} = \Xi|_{\rho_j=0} \):
\[
v = \sum_{j=1}^{2} \frac{1}{(2\pi)^2n} \int \frac{F.T. j \partial_{\rho_j} v(0_j, \eta_j, \xi) + i\eta_j \hat{g}_j(\eta_j, \xi)}{\eta^2 + \Xi^2_{bj}} e^{i\rho \cdot \eta} e^{i\xi \hat{w}} \, d\eta \, d\xi \\
+ A_{-1} v + \sum \rho_j A_{-2} \left( \partial_{\rho_j} u \bigg|_{\partial \Omega} \right) + \sum \rho_j A_{-1} g_j \\
+ A_{-3} \left( \partial_{\rho} v \bigg|_{\partial \Omega} \right) + A_{-2} g.
Using Lemma 2.11 to handle the $\rho_j$ terms multiplied with the $A_{-1}$ and $A_{-2}$ operators, we have

$$
\nu = \sum_{j=1}^{2} \frac{1}{(2\pi)^{2n}} \int \frac{F.T.\partial_{\rho_j}v(0_1, \eta_1, \xi) + in_2\hat{g}(\eta_1, \xi)}{\eta^2 + \Xi_{bj}^2} e^{ip\cdot\eta} e^{ix\xi} \, d\eta \, d\xi
+ A_{-3} \left( \partial_{\rho_j}v \bigg|_{\partial\Omega} \right) + A_{-2}g + A_{-\infty}v.
$$

(17)

In the integral in (17) for $j = 1$, we integrate with respect to $\eta_1$ for $\rho_1 > 0$. We then let $\rho_1 \to 0^+$ to obtain

$$
0 = \frac{i}{(2\pi)^{2n-1}} \int \frac{F.T.\partial_{\rho_1}v(0_1, \eta_2, \xi) - \sqrt{\eta_2^2 + \Xi_{b1}^2}\hat{g}(\eta_2, \xi)}{2i\sqrt{\eta_2^2 + \Xi_{b1}^2}} e^{i\rho_2\cdot\eta_2} e^{ix\xi} \, d\eta_2 \, d\xi
+ \frac{1}{(2\pi)^{2n}} R_1 \circ \int \frac{F.T.\partial_{\rho_2}v(0_1, \eta_1, \xi) + in_2\hat{g}(\eta_1, \xi)}{\eta^2 + \Xi_{b2}^2} e^{ip\cdot\eta} e^{ix\xi} \, d\eta \, d\xi
+ A_{-2,b1} \left( \partial_{\rho_1}v \bigg|_{\partial\Omega_1} \right) + A_{-1,b1}g_1 + R_{b1}^{-\infty}
+ R_1 \circ A_{-3} \left( \partial_{\rho_2}v \bigg|_{\partial\Omega_2} \right) + R_1 \circ A_{-2}g_2,
$$

where $R_{b1}^{-\infty}$ refers to smooth terms on $\partial\Omega_1$.

Note that the second term on the right side above can be written, according to our notation from Section 3, as $E_{-3/2}^{21}(\partial_{\rho_2}v|_{\partial\Omega_2}) + E_{-1/2}^{21}g_2$, as can the last two terms. Inverting the operator with symbol locally given by $1/2\sqrt{\eta_2^2 + \Xi_{b1}^2}$ yields

$$
\partial_{\rho_1}v|_{\partial\Omega_1} = |D_{b1}|g_1 + A_{1,b1} \circ E_{-1/2}^{21}g_2 + A_{0,b1}g_1
+ A_{-1,b1} \partial_{\rho_1}v|_{\partial\Omega_1} + E_{-1/2}^{21} \partial_{\rho_2}v|_{\partial\Omega_2},
$$

(18)

where $|D_{b1}|$ is the operator with symbol (in an neighborhood of the origin) given by $\sqrt{\eta_2^2 + \Xi_{b1}^2}$.

Similar calculations lead to the expression

$$
\partial_{\rho_2}v|_{\partial\Omega_2} = |D_{b2}|g_2 + A_{1,b2} \circ E_{-1/2}^{12}g_2 + A_{0,b2}g_2
+ A_{-1} \partial_{\rho_2}v|_{\partial\Omega_2} + E_{-1/2}^{12} \partial_{\rho_1}v|_{\partial\Omega_1}
$$

(19)

for $\partial_{\rho_2}v|_{\partial\Omega_2}$. We note the occurrence of the $E_{-1/2}^{jk}$ operators, which shows an expression for the Dirichlet to Neumann operator (DNO), giving the boundary values of the normal derivative of $v$, on intersection domains is not as simple as finding an expansion in terms of pseudodifferential operators as in the case of smooth domains.

Before we further explore reduction to the boundary techniques, we use (18) and (19) to obtain the first few principal terms in the Poisson operator. Inserting (18) and (19) in (17), we obtain

$$
\nu = \Theta_1^+g_1 + \Theta_2^+g_2 + \sum_{j,k} A_{-1} \circ E_{-1/2}^{kj}g_{k} + \sum_j A_{-2}g_j
$$
+ \sum_{j,k} A_{-2} \circ E_{j,k}^{k,j} \left( \partial_{\rho_k} v \bigg|_{\partial \Omega_k} \right) + A_{-3} \left( \partial_{\rho} v \bigg|_{\partial \Omega} \right) + R^{-\infty}, \quad (20)

where $R^{-\infty}$ refers to smooth terms, and where the symbol of $\Theta_j^+$ is locally

$$\sigma(\Theta_j^+) = \frac{i}{\eta_j + i \Xi_{bj}}.$$

Such an expression can be used to determine the mapping properties of the Poisson operator using the weighted spaces of Section 3. Thus we see the expression for the Poisson operator on the intersection domain contains a sum of the principal terms of the individual Poisson operators for each domain comprising the intersection. This can be generalized to any number of intersections (with $m \leq n$).

Consider now the boundary value problem of the form

$$\Gamma v = 0 \quad \text{in } \Omega,$$

$$\partial_v v + Bv = g \quad \text{on } \partial \Omega,$$

where $\partial_v$ denotes the normal derivative ($\partial_{\rho_j}$ on $\partial \Omega_j, j = 1, 2$), and $B$ is a tangential pseudodifferential operator.

We use the techniques above to reduce to the boundary. From (18) and (19), the boundary condition gives the two relations:

$$|D_{bj}| v_{bj} + B v_{bj} + A_{1,bj} \circ E_{-1/2}^{k,j} v_{bk} + A_{0,bj} v_{bj} + A_{-1} (\partial_{\rho_j} v \bigg|_{\partial \Omega_j}) + E_{-1/2}^{k,j} (\partial_{\rho_k} v \bigg|_{\partial \Omega_k}) = 0$$

for $j, k = 1, 2, j \neq k$. Solving this system together with (18) and (19), would lead to a solution for the Dirichlet problem. The boundary solutions would be inserted in (20) to obtain an expression for $v$ on the entire domain. In particular, if $|D_{bj}| + B$ forms an elliptic system weighted estimates can be obtained by inverting the highest order term, $|D_{bj}| + B$, and considering the remaining terms as error terms (which lead to estimates in lower level Sobolev spaces). One would of course need estimates for the boundary values of the normal derivatives, estimates which could be obtained with the help of (18) and (19).

We can immediately see the need for weighted estimates due to the $E_{-\alpha}^{k,j}$ operators. The use of weights allows us to consider higher order Sobolev estimates, which would be unobtainable without the introduction of terms which deal with the singularity at the points of intersection. Weights are also needed even for the pseudodifferential operators, since the arguments are considered to be extended by zero. For instance, for a function, $u_{bj} \in W^s(\partial \Omega_j \cap \partial \Omega)$ in a term of the form $A_{0,bj} u_{bj}$, the argument, $u_{bj}$ is considered to be extended by zero to all of $\partial \Omega_j$ and this extension is not in $W^s(\partial \Omega)$.

Of particular interest to the author is the case in which $|D_{bj}| + B$ does not form an elliptic system. Such is the case in the $\bar{\partial}$-Neumann problem on intersection domains. In this case, the zeroth order terms in the expansion of the DNO are required, as well as a method to deal with the $E_{-1/2}^{k,j}$ operators. Such is the subject of a current study, which the author will publish separately.
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