1. Introduction

Let $X$ be a compact simply connected complex irreducible symplectic Kähler manifold of dimension $2n$ (a hyperkähler manifold, for short), that is a compact simply connected Kähler manifold admitting a holomorphic $2-$form $\sigma_X$ which is of maximal rank at every point such that $H^0(\Omega^2_X) = \mathbb{C}\sigma_X$ (hence $\wedge^n \sigma_X$ is a $2n-$form without zeroes). By Fujiki [Fu83], based on Bogomolov’s unobstructedness theorem [Bo78], both projective and non-projective hyperkähler manifolds are dense in the Kuranishi space of $X$.

We are concerned with non-algebraic hyperkähler manifolds, particularly with their algebraic dimensions $a(X) < 2n$ and their algebraic reductions $f : X \to B$, which are unique up to bimeromorphic modification of $B$.

A central role in the theory is played by the Beauville form $q_X : H^2(X, Z) \times H^2(X, Z) \to Z$ of signature $(3, 0, b_2(X) - 3)$ [Be83]. It induces a symmetric bilinear form on the Néron-Severi group $NS(X)$, and the signature is either $(1, 0, \rho(X) - 1)$ in which case we say that $NS(X)$ is hyperbolic, $(0, 1, \rho(X) - 1)$ ($NS(X)$ is parabolic), or $(0, 0, \rho(X))$ ($NS(X)$ is elliptic). Now $X$ is projective (equivalently Moishezon [Moi66]) if and only if $NS(X)$ is hyperbolic (Huybrechts [Hu99]), so we shall be interested in the parabolic case and the elliptic case.

The Beauville form can be seen as a natural higher dimensional version of the intersection form on a compact complex surface $S$. Here we have $a(S) = 0, 1, 2$ according to $NS(S)$ being elliptic, parabolic, hyperbolic. In addition, if $a(S) = 1$, then we have a holomorphic algebraic reduction $f : S \to C$ whose general fiber is an elliptic curve [BHPV04]. Of great importance in the theory of hyperkähler manifolds is the following example due to Beauville [Be83] and Fujiki [Fu83]. Let $S$
be a K3 surface. Then $S^{[n]}$, the Hilbert scheme of $n$ points on $S$, is a hyperkähler manifold of dimension $2n$. We have $a(S^{[n]}) = 0, n, 2n$ according to $a(S) = 0, 1, 2$. In addition, when $a(S) = 1$, the algebraic reduction map $S \rightarrow \mathbb{P}^1$ induces a natural morphism $S^{[n]} \rightarrow \mathbb{P}^n$. This is the algebraic reduction of $S^{[n]}$ and it is also Lagrangian.

This motivates the following [Og07]

1.1. Conjecture. Let $X$ be a hyperkähler manifold of dimension $2n$. Then its algebraic dimension takes only the values $0, n, 2n$. Moreover, if $a(X) = n$, then the algebraic reduction has a holomorphic model $f : X \rightarrow B$ with $B$ a normal projective variety of dimension $n$. Finally $f$ is Lagrangian, that is $\sigma_X|F \equiv 0$ for a general fiber of $f$.

The last statement actually follows from the previous by [Ma99].

The aim of this paper is to establish the following (partial) answers to Conjecture 1.1.

1.2. Theorem. Conjecture 1.1 holds in dimension 4.

In higher dimensions we can determine the algebraic dimension up to the existence of minimal models of Kähler spaces with algebraic dimension and Kodaira dimension 0:

1.3. Theorem. Let $X$ be a hyperkähler manifold of dimension $2n$. Then Conjecture 1.1 holds provided any compact Kähler manifold $Y$ with $\dim Y \leq 2n - 1$, $a(Y) = \kappa(Y) = 0$ has a minimal model.

A general conjecture from minimal model theory says that every compact Kähler manifold of non-negative Kodaira dimension should have a minimal model. Recall that a meromorphic map between compact varieties is almost holomorphic if the general fiber is compact, i.e., does not meet the set of indeterminacies. Then we can state more precisely:

1.4. Theorem. Let $X$ be a hyperkähler manifold of dimension $2n$. Then:

(1) If $NS(X)$ is elliptic, then $a(X) = 0$.
(2) If $NS(X)$ is parabolic, then $0 \leq a(X) \leq n = \dim X/2$.
(3) Assume that $NS(X)$ is parabolic and $a(X) > 0$. Then one can choose an algebraic reduction of one of the following two forms:
   (i) $f : X \rightarrow B$ is holomorphic Lagrangian, in particular, $a(X) = n$, or
   (ii) $f : X \rightarrow B$ is not almost holomorphic and the general fiber $X_b$ ($b \in B$) is isotypically semi-simple, in particular, $a(X_b) = 0$ (see section 2 for the definition of the term “isotypically semisimple”).
(4) Assume that any compact isotypically semi-simple Kähler manifold $Y$ of $\dim Y \leq 2n - 1$, of algebraic dimension $a(Y) = 0$ and of Kodaira dimension $\kappa(Y) = 0$ and with effective canonical divisor $K_Y$, has a minimal model. Then Conjecture 1.1 holds.

One might speculate that a hyperkähler manifold $X$ of dimension $2n$ has algebraic dimension $a(X) = n$ if and only if $NS(X)$ is parabolic. This in turn would be a consequence of a potential semi-ampleness of any nef line bundle on a hyperkähler manifold.
Parts (1) and (2) of Theorem 1.4 will be proved in section 3; parts (3) and (4) in section 4 and Theorem 1.2 finally in section 5. All these sections make essential use of section 2, which contains structure results on meromorphic fibrations on compact Kähler manifolds, in particular on those manifolds admitting a unique holomorphic 2-form which additionally is generically non-degenerate. The final section gives some results on nef line bundles on hyperkähler manifolds.

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2. Fibrations on generalized Hyperkähler Manifolds

In this section we prove some general structure theorems on generalised hyperkähler manifolds.

2.1. Conventions and Notations

(1) By $X, X', X'', \ldots$, we denote $n$-dimensional compact irreducible complex spaces which are bimeromorphic to compact Kähler manifolds. The algebraic dimension ([Ue75]) is denoted by $a(X)$. We have the algebraic reduction map (only defined up to obvious bimeromorphic equivalence)

$$f : X \longrightarrow B.$$ We always take $B$ to be normal projective and often we will choose $B$ smooth.

(2) A fibration $f : X \longrightarrow Y$ is a dominant meromorphic map with connected fibres. The fibration $f$ is said to be almost holomorphic if its generic fibre does not meet the indeterminacy locus of $f$. If $Y$ is not uniruled, then any fibration $f : X \longrightarrow Y$ is automatically almost holomorphic.

The fibration $f$ is said to be trivial if $\dim Y = 0$ or $\dim Y = n$.

(3) A point $y$ in $Y$ is said to be general if it lies outside of a countable union of (suitable) proper closed analytic subsets of $Y$. We denote by $X_y$ the (Chow-theoretic) fibre of $f$ over a generic $y \in Y$.

(4) Recall ([Fu83]) that a compact Kähler manifold $X$ is said to be simple if $X$ is not covered by positive-dimensional irreducible compact proper analytic subsets. If $X$ is simple, then necessarily $a(X) = 0$. Moreover either $q(X) = 0$ or there $X$ is (bimeromorphically) covered by a simple torus, i.e., there exists a simple non-algebraic complex torus $T$ and a meromorphic dominant map $u : T \longrightarrow X$.

Remark that if $X$ is a complex torus with $a(X) = 0$, $X$ is simple in the sense above if and only if it is simple in the classical sense, i.e., $X$ has no nontrivial complex subtorus. However, if $X$ is an abelian variety, it is not simple in the above sense, but can be simple in the classical sense.

(5) Two complex spaces $X$ and $X'$ are commensurable if there exist a complex space $X''$ and generically finite surjective holomorphic maps $X'' \rightarrow X$ and $X'' \rightarrow X'$. This is (easily seen to be) an equivalence relation. Notice that two projective varieties are commensurable if and only if they have the same dimension. But for non algebraic $X$, this equivalence relation is very
restrictive. If $X$ is simple, and if $X'$ is commensurable to $X$, then $X'$ is simple, too. Thus either $q(X') = 0$ for any $X'$ commensurable to $X$, or $X$ is covered by a simple torus.

(6) We say (after [Fu83]) that $X$ is semi-simple if it is commensurable to a product of simple manifolds, and that $X$ is isotypically semi-simple if it is commensurable to a product $S^k$ for some simple $S$ and some $k > 0$. Remark that if $f : X \rightarrow S$ is a fibration with $S$ semi-simple, then $f$ is almost holomorphic, since $S$ is not uniruled.

2.2. Definition. Let $f : X \rightarrow B$ be a fibration from a compact (connected) Kähler manifold $X$.

We say that $f = h \circ g$ is a factorisation of $f$ if $g : X \rightarrow S$ and $h : S \rightarrow B$ are fibrations with $f = h \circ g$.

This factorisation is said to be trivial if $\dim S = \dim X$ or $\dim S = \dim B$. The fibration $f$ is said to be minimal if any factorisation of $f$ is trivial. The variety itself $X$ is minimal if the constant fibration $X \rightarrow \{pt\}$ is minimal.

The following easy observation is essential:

2.3. Theorem. Let $X$ be a compact Kähler manifold of dimension $2n$ and suppose that $h^{2,0}(X) = 1$, and the corresponding holomorphic 2–form $\sigma$ (which is unique up to a scalar) satisfies $\sigma^n \neq 0$. Then the following assertions hold.

(1) If $f : X \rightarrow Y$ is a fibration with $\dim Y < \dim X$, then $Y$ is Moishezon.
(2) The algebraic reduction $f : X \rightarrow B$ is minimal.
(3) If $a(X) = 0$, then $X$ is minimal.
(4) In particular hyperkähler manifolds $X$ with $a(X) = 0$ (see section 3 below) are minimal.

Proof. Only (1) needs to be proved; the other statements are trivial consequences. Suppose that $f : X \rightarrow Y$ is a nontrivial fibration with $Y$ a non-algebraic manifold (we may assume $Y$ smooth and Kähler). Then $h^{2,0}(Y) > 0$, by [Ke54]. Any non-zero holomorphic 2–form on $Y$ lifts to the holomorphic 2–form $\sigma$ on $X$, which is generically of maximal rank, so that $\dim Y = \dim X$.

The main result of this section is:

2.4. Theorem. Let $X$ be a compact Kähler manifold, $f : X \rightarrow B$ be a minimal fibration. Suppose $\dim X > \dim B$. Let $X_b$ be a general fibre of $f$. Then

(1) $X_b$ is either Moishezon or isotypically semi-simple, in which case $a(X_b) = 0$.
(2) Furthermore, if $X$ is not projective and if $B$ and $X_b$ are Moishezon, then $f$ is almost holomorphic and $X_b$ is an abelian variety.

2.4 will be proved at the end of this section.

Applying 2.3(2), we get:

2.5. Corollary. Let $X$ be a compact Kähler manifold of dimension $2n$ with $h^{2,0}(X) = 1$, carrying a holomorphic two-form $\sigma$ such that $\sigma^n \neq 0$. Assume $X$ is nonprojective. Then the following assertions hold.

(1) Let $f : X \rightarrow B$ be the algebraic reduction and $X_b$ be a general fibre of $f$. Then either
2a. $X_b$ is isotypically semi-simple or
2b. $f$ is almost holomorphic, and $X_b$ is an abelian variety.

(2) In particular if $a(X) = 0$, then $X$ is isotypically semi-simple.

2.6. Corollary. Let $X$ be a simply-connected compact Kähler manifold of dimension $2n$ with $h^{2,0}(X) = 1$, carrying a holomorphic two-form $\sigma$ such that $\sigma^n \neq 0$. Assume moreover that $X$ does not contain any effective divisor. Then $X$ is simple.

In particular, any simply connected irreducible hyperkähler manifold without effective divisors is simple and so does the general member of the Kuranshi family of any hyperkähler manifold without effective divisors is simple.

Proof. Since $X$ has no effective divisors, we have $a(X) = 0$. Thus $X$ is isotypically semi-simple by [2.5]. Specifically there exist generically finite meromorphic maps $u : Z \to X$ and $v : Z \to S^k$, with $Z$ smooth, and $S$ simple. Our claim comes down to prove that $u$ is bimeromorphic and that $k = 1$. Since $X$ has no divisor, $u$ is unramified, hence bimeromorphic, $X$ being simply-connected. Thus $h^{2,0}(X) = h^{2,0}(Z) = 1$. Since $S$ is non-algebraic, one has $h^{2,0}(S) \geq 1$, hence necessarily $k = 1$.

The assumption that $X$ does not contain any divisor cannot be removed in (2.6).

In fact, the hyperkähler 4-fold $S^{[2]}$ with $S$ a K3 surface, $a(S) = 0$ is not simple but $a(S^{[2]}) = 0$.

The rest of the section is devoted to the proof of [2.4]. We shall need the following two elementary lemmas, which are relative versions of results similar to those in [Fu83] in a simplified form. We first recall some notions needed in the proof.

A covering family of $X$ will be a compact irreducible analytic subset $S \subset \mathcal{C}(X)$ of the Chow (or cycle, or Barlet)-space $\mathcal{C}(X)$ (see [Ba75]) of $X$, such that if $Z \subset S \times X$ is its incidence graph, with natural projections $p : Z \to X$ and $q : Z \to S$, then $p$ is surjective, and the generic fibre of $q$ is irreducible. In other words, $X$ is covered by the (generically irreducible) cycles $Z_s$, $s \in S$. We call $m = \dim Z_s$ the dimension of the family $S$. If $f : X \to Y$ is a fibration, we denote by $\mathcal{C}(X/Y)$ the closed analytic subset of $\mathcal{C}(X)$ consisting of those points $s \in \mathcal{C}(X)$ such that the corresponding analytic compact pure-dimensional cycle $Z_s$ of $X$ has support contained in one fibre $X_y$ of $f$. If $S \subset \mathcal{C}(X/Y)$ is a covering family of $X$, the map $f_s$ sending $s$ to $y = f(Z_s)$ is a meromorphic dominant map $f_s : S \to Y$.

2.7. Lemma. Let $X$ be a compact Kähler manifold and $f : X \to B$ be any fibration with $a(X_b) = 0$. Let $S \subset \mathcal{C}(X/B)$ be a nontrivial covering family of $X$ over $B$. Assume that $\dim Z_s = m$ is maximal among the dimensions of nontrivial covering families of $X$ over $B$. Then

1. $\dim Z = \dim X$ (hence only finitely many of the $Z_s$’s pass through the generic point of $X$).
2. $S_b$ is simple, and no proper closed analytic subset of $S_b$ is a covering family of $X_b$.
3. $S_b$ is the union of finitely many irreducible components of $\mathcal{C}(X_b)$.

Proof. (1) Assume that $\dim Z > \dim X$. Then also $\dim Z_b > \dim X_b$ so that we may assume $\dim B = 0$. The fibres of $p : Z \to X$ are Moishezon by [Ca81]. Thus
we can find a covering family of $S$ by curves $(C_v)_{v \in V}$. For general $v \in V$ we define

$$W_v := p(q^{-1}(C_v)).$$

This is an irreducible compact analytic subset of $X$ and defines a covering family $(W_v)_{v \in V}$ of $X$ with $\dim W_v = m+1$. In order to show that this family is non-trivial, we prove that $m+1 < \dim X$. In fact, if $\dim X \leq m+1$, then $\dim X = m+1$ so that the $Z_v$ are divisors in $X$. This contradicts $a(X) = 0$. Thus the family $(W_v)$ is non-trivial, contradicting the maximality of $m$.

(2) The same argument shows that $S_b$ is simple. In fact, if $S_b$ were not simple, then in every fiber we find a covering family of proper subvarieties and in total we obtain a nontrivial covering family $(C_v)_{v \in V}$ of $S$ and define $W_v$ as above. By the maximality of $m$, we must have $W_v = X$ for all $v \in V$. But then $(Z_s)_{s \in C_v}$ is a nontrivial covering family of $X$. Let $Z' := q^{-1}(C_v)$ be the graph of this covering family. Then $\dim Z' = \dim X = \dim Z$. Thus by irreducibility we obtain $Z' = Z$ and $C_v = S$, a contradiction.

(3) The third assertion is an obvious consequence of the second. \qed

2.8. Lemma. Let $k \in \mathbb{N}$ and let $S_j$ be simple manifolds for $1 \leq j \leq k$. Put

$$S = S_1 \times \cdots \times S_k$$

with projections $p_j : S \to S_j$. More generally, for a given subset $J = \{j_1, \ldots, j_h\} \subset \{1, 2, \ldots, k\}$, let

$$p_J : S \to S_J = S_{j_1} \times \cdots \times S_{j_h}$$

be the projection. Let $Y \subset S$ be an irreducible compact analytic subset such that $p_J(Y) = S_J$ for all $J$.

There exists $J$ such that $p_J : Y \to S_J$ is surjective and generically finite. In particular, $Y$ is commensurable to $S_J$ and is therefore semi-simple. In particular, if $S_j \approx S_k$ for all $j, k$, then $Y$ is isotypically semi-simple.

Proof. Let $K = \{1, \ldots, k-1\}$. If $p_K(Y) \neq S_K$, we proceed by induction on $k$. Thus we may assume that $p_K(Y) = S_K$ for $K = \{1, \ldots, k-1\}$. If $p_K : Y \to S_K$ is not generically finite, let $S'_K$ be its Stein factorisation with map $p'_K : Y \to S'_K$, and define a meromorphic map

$$\varphi : S'_K \to C(S_K)$$

by sending a general $s \in S'_K$ to $p_k(p'_K^{-1}(s))$. The image of $\varphi$ gives a covering family of $S_K$. Because $S_K$ is simple, we must have $\varphi(s) = S_k$ for all $s$. Thus $Y = S$ (in which case we take $J = \{1, \ldots, k\}$). \qed

Proof. of [2.4] Let $a_H : X \to Y$, $h : Y \to B$ with $f = h \circ a_H$ be the relative algebraic reduction [Ca81, Fu83] of $f$ (so that the restriction of $a_H$ to $X_b$ is the algebraic reduction of $X_b$, for $b$ general in $B$). Since $f$ is minimal, either $Y = X$ up to bimeromorphic equivalence and $X_b$ is Moishezon, or $Y = B$ up to bimeromorphic equivalence and $a(X_b) = 0$.

(1) In the first step we assume that $a(X_b) = 0$ and need to show that $X_b$ is isotypically semi-simple. If $X_b$ is simple, we are already done. If $X_b$ is not simple, let $S \subset C(X/B)$ be a nontrivial covering family of $X$ with $m = \dim Z_a$ maximal. By [2.7], $S_b$ is simple and $p : Z \to X$ is generically finite onto $X$. Let $\delta$ be the degree of $p$ and

$$\varphi : X \to \text{Sym}^\delta(S/B)$$
be the meromorphic map sending a general \( x \in X \) to \( q_*(p^{-1}(x)) \) (here \( \text{Sym}^\delta(S/B) \) denotes the subspace of \( \text{Sym}^\delta(S) \) consisting of \( \delta \)-tuples of \( S \) contained in some fibre of \( S \) over \( B \). We adopt a similar convention for \((S^k/B)\)).

Since \( f \) is minimal, this map is generically finite onto its image \( Y_0 \subset \text{Sym}^\delta(S/B) \). Let \( Y \subset (S^k/B) \) be a main component of the inverse image of \( Y_0 \) under the natural map \((S^k/B) \to \text{Sym}^\delta(S/B)\). Then \( Y \) maps surjectively onto \( S \) under all projections from \((S^k/B)\) to \( S \) (otherwise there would exist some irreducible proper compact analytic subset \( S' \subset S \) parametrising a covering family of \( X \), contradicting (2.7)).

From (2.8) we conclude that \( Y \), and hence so \( X \), is commensurable to \((S^k/B)\) for some \( k \leq \delta \). The first assertion of (2.4) is thus established.

Thus we now assume that \( X_b \) is Moishezon. By the minimality assumption, it follows that \( f \) is the algebraic reduction of \( X \). From [Ca81] we deduce that \( f \) is almost holomorphic. Moreover by [Fu83] the general fiber \( X_b \) is abelian or a unirational manifold.

If \( \kappa(X) \geq 0 \) - and this is sufficient for all our applications - \( X_b \) cannot be uniruled and we conclude. But if \( X_b \) is unirational, then [Fu83], Prop.2.5 implies that \( X \) is projective, which is excluded by assumption. \( \square \)

We now consider the restriction of holomorphic 2-forms to fibers.

**2.9. Corollary.** Let \( X \) be a compact Kähler manifold and \( f : X \to B \) be a fibration. Assume that the restriction of any holomorphic 2-form on \( X \) to the fiber \( X_b \) for generic \( b \in B \) vanishes. Then

1. \( X_b \) is Moishezon for all \( b \).
2. If \( f \) is the algebraic reduction of \( X \) and a minimal fibration, then \( f \) is almost holomorphic, and \( X_b \) is an abelian variety.
3. In particular, suppose that \( h^{2,0}(X) = 1 \), given by a holomorphic 2-form \( \sigma \) such that \( \sigma^n \neq 0 \). Then the algebraic reduction \( f \) is almost holomorphic Lagrangian provided \( \sigma \) restricted to \( X_b \) vanishes.

**Proof.** The first statement is a lemma due to C. Voisin (see [Ca04]). The second follows from (2.4). \( \square \)

**3. Basics on Hyperkähler Manifolds and first results**

We begin by fixing some notations. For the rest of the paper we fix an irreducible simply connected compact complex symplectic Kähler manifold \( X \) of dimension \( 2n \), that is a simply connected compact Kähler manifold admitting a holomorphic 2-form \( \sigma \) which is of maximal rank at every point (hence \( \sigma^{2n} \) is a \( n \)-form without zeroes), such that \( H^0(X, \Omega_X^2) = \mathbb{C} \sigma \). We say for short that \( X \) is a hyperkähler manifold.

The non-degenerate symmetric bilinear form, constructed by Beauville [Be83], will be denoted

\[
q = q_X : H^2(X, \mathbb{Z}) \times H^2(X, \mathbb{Z}) \to \mathbb{Z}.
\]

We shall use the shorthand \( q(a) = q(a, a) \). The Picard number of \( X \) is denoted by \( \rho \). Then the signature of \( q_X \) on the Néron-Severi group

\[
NS(X) = H^{1,1}(X) \cap H^2(X, \mathbb{Z})
\]

is one of the following:

- \((1, 0, \rho - 1)\) (hyperbolic case):
• \((0,1,\rho-1)\) (parabolic case);
• \((0,0,\rho)\) (elliptic case).

Moreover \(X\) is projective if and only if \(NS(X)\) is hyperbolic \([\text{Hu}99]\), so we are interested in the parabolic case and the elliptic case.

3.1. Theorem.

(1) If \(NS(X)\) is elliptic, then \(a(X) = 0\).

(2) If \(0 < a(X) < 2n\), then \(NS(X)\) is parabolic. Let \(\ell \in NS(X)\) be the unique primitive isotropic vector of \(NS(X)\) with \(q_X(\ell, \eta) > 0\) for a Kähler class \(\eta\). Then there is a line bundle \(L\) whose linear system defines the algebraic reduction, such that \(c_1(L) \in \mathbb{Z}_{>0}\). In particular \(q_X(L) = 0\).

Proof. Let \(f : X \dashrightarrow B\) be the algebraic reduction with \(B\) normal projective. If \(\dim B > 0\), i.e., \(a(X) > 0\), then there is a line bundle \(L\) with \(D_1, D_2 \in |L|\) such that \(D_1\) and \(D_2\) have no common component. By definition of the Beauville form, one has (up to positive constant multiple):

\[q_X(L, L) = \int_X c_1(D_1)c_1(D_2)(\sigma \wedge \overline{\sigma})^{n-1} = \int_{D_1 \cap D_2} (\sigma \wedge \overline{\sigma})^{n-1} \geq 0.\]

Thus \(NS(X)\) is not elliptic if \(a(X) > 0\). This proves (1). Since \(X\) is projective if \(NS(X)\) is hyperbolic, we have from the same inequality that \(q_X(L) = 0\). Hence \(NS(X)\) is parabolic if \(0 < a(X) < 2n\). Moreover, \(q_X(L, \eta) > 0\) by the shape of Beauville form above. This proves (2). \(\square\)

3.2. Setup. (1) We shall assume that \(0 < a(X) < 2n\). Thus \(X\) is not projective and \(NS(X)\) is parabolic. We consider “the” algebraic reduction

\[f : X \dashrightarrow B.\]

From the previous section we recall that the general fiber is isotypically semi-simple or that \(f\) is almost holomorphic and the general fiber is abelian. We always take \(B\) to be normal projective and often we will choose \(B\) smooth, too. Let

\[\pi : \tilde{X} \rightarrow X\]

be a resolution of indeterminacies of \(f\) so that the induced map \(\tilde{f} : \tilde{X} \rightarrow B\) is holomorphic.

(2) We fix a very ample line bundle \(A\) on \(B\) and set

\[L = \pi_*(f^*(A))^{**}.\]

This is a holomorphic line bundle on \(X\) and we find an effective divisor \(E\) on \(\tilde{X}\) such that

\[\pi^*(L) = \tilde{f}^*(A) + E.\]

We set \(\tilde{L} = \pi^*(L)\).

(3) In all what follows \(\eta\) will always denote a Kähler form on \(X\). We set \(\tilde{\eta} = \pi^*(\eta)\).

By the results of section 2 we may already state:

3.3. Proposition. If the algebraic reduction \(f : X \rightarrow B\) is holomorphic (with \(B\) projective and \(\dim B > 0\)) then \(f\) is Lagrangian, in particular all smooth fibers are abelian.
Proof. By [Ma99], \( f \) is Lagrangian and \( \dim B = n \) - his argument works in the Kähler case as well. Then there is no holomorphic 2-form with non-zero restriction to the general fiber. Therefore we conclude by (2.9).

3.4. Theorem. Assume that \( 0 < a(X) < 2n \). Then \( c_1(L) \in \mathcal{K}(X) \), the closure of the Kähler cone, i.e., \( L \) is (analytically) nef. Moreover, \( (L.C) = 0 \) for all curves \( C \subset X \).

Proof. Let \( \mathcal{P}(X) \subset H^{1,1}(X, \mathbb{R}) \) be the closure of the positive cone of \( X \). By Theorem 3.1 (2), \( c_1(L) \in \mathcal{P}(X) \). Thus by [Hu99]

\[
c_1(L) \in \mathcal{K}(X)
\]

if \( L \cdot C \geq 0 \) for all curves \( C \subset X \). As, the Beauville form \( q_X \) is non-degenerate and defined over \( H^2(X, \mathbb{Q}) \), the map \( x \mapsto q_X(\ast, x) \) gives an isomorphism

\[
\iota : H^2(X, \mathbb{Q}) \simeq H^{4n-2}(X, \mathbb{Q}).
\]

Here we identify \( H^{4n-2}(X, \mathbb{Q}) \) with \( H^2(X, \mathbb{Q})^\ast \) by the intersection pairing. Moreover, by the shape of the Beauville form, \( \iota \) induces an isomorphism

\[
\iota : H^{1,1}(X, \mathbb{Q}) \simeq H^{2n-1,2n-1}(X, \mathbb{Q}).
\]

Since \( [C] \in H^{2n-1,2n-1}(X, \mathbb{Q}) \), there is an element \( \alpha \in H^{1,1}(X, \mathbb{Q}) \) such that \( \iota(\alpha) = [C] \). Hence

\[
L \cdot C = q_X(L, \alpha) = 0
\]

by Theorem 3.1 (2). This proves our claim.

3.5. Lemma.

1. \( L^n \cdot \eta^n > 0 \).
2. \( L^{n+1} \alpha_1 \ldots \alpha_{n-1} = 0 \) for all \( \alpha_i \in H^{1,1}(X) \).
3. \( L^{n+1} = 0 \) in \( H^{2n+2}(X, \mathbb{R}) \), hence the numerical dimension \( \nu(L) = n \).
4. For all \( a, b \geq 0 \) with \( a + b > n \) we have

\[
\tilde{f}^*(A)^a \cdot \tilde{L}^b \cdot \tilde{\eta}^{2n-(a+b)} = 0.
\]
5. For all \( k \geq 0 \) we have

\[
\tilde{f}^*(A)^k \cdot \tilde{L}^{2n-k} = 0.
\]

Proof. Let \( \alpha \in H^{1,1}(X) \). Then we have via \( q(L) = 0 \), using Fujiki’s relation [Fu87]

\[
(L + t\alpha)^{2n} = c q(L + t\alpha)^n = c (2t q(L, \alpha) + t^2 q(\alpha))^n
\]

with \( c > 0 \). Comparing coefficients, this shows

\[
L^{2n-k} \cdot \alpha^k = 0
\]

for \( n > k \geq 1 \), hence (2) by polarization.

The first claim is just (up to a positive multiple) \( L^n \cdot \eta^n = c q(L, \eta)^n > 0 \) by Theorem 3.1 (2).

For the proof of (3) notice that \( c_1(L)^{n+1} \) is represented by a closed positive current \( T \) of bidimension \( (n+1, n+1) \) (approximate e.g. \( c_1(L) \) by Kähler forms). Choosing \( \alpha_i \) in (2) to be represented by Kähler forms \( \omega_i \), we conclude that

\[
0 = L^{n+1} \cdot \alpha_1 \ldots \alpha_{n-1} = T(\omega_1 \wedge \ldots \wedge \omega_{n-1}).
\]
Thus $T = 0$ and therefore $c_1(L)^{n+1} = 0$ in $H^2(X, \mathbb{R})$. Hence $\nu(L) = n$.

Let $c > n$ and $a + b = c$. By (2) we have
\[ \hat{L}^c \cdot \tilde{\eta}^{2n-c} = 0. \]

Hence
\[ (\hat{f}^* (A) + E) \cdot \hat{L}^{c-1} \cdot \tilde{\eta}^{2n-c} = 0. \]

Since $\check{L}$ is nef, this gives
\[ \check{f}^* (A) \cdot \check{L}^{c-1} \cdot \tilde{\eta}^{2n-c} = E \cdot \check{L}^{c-1} \cdot \tilde{\eta}^{2n-c} = 0. \]

Continuing in this way, we obtain
\[ \check{f}^* (A)^a \cdot \check{L}^{c-a} \cdot \tilde{\eta}^{2n-c} = 0 \]

proving (4).

Claim (5) is the following special case of (4): $a = k$ and $b = 2n - k$. \qed

**3.6. Theorem.** Let $X$ be a non-algebraic hyperkähler manifold of dimension $2n$. Then $a(X) \leq n$.

**Proof.** Recall that $a(X) = \dim B$ and suppose $k = \dim B > n$. Then we can take $a = k$ and $b = 0$ in Lemma 3.5, i.e., $(\check{f}^* A)^k \tilde{\eta}^{2n-k} = 0$. The class $(\check{f}^* A)^k$ is represented by a positive multiple of general fiber $\tilde{F}$ of $\check{f}$. So, if we put $F = \pi_* \tilde{F}$, then $F$ is a $2n - k$-dimensional non-zero effective cycle on $X$, but
\[ F^k \tilde{\eta}^{2n-k} = \tilde{F} \tilde{\eta}^{2n-k} = 0, \]

a contradiction. \qed

**3.7. Theorem.** Let $X$ be a hyperkähler manifold of dimension $2n$. Suppose $a(X) = n$. Then any nef line bundle $D$ on $X$ is semi-ample. In particular its algebraic reduction can be taken holomorphic.

**Proof.** Since $\text{NS}(X) = \mathbb{Z} \ell \oplus V$ where $q_X$ is negative definite on $V$, the line bundle $D = L$ up to a multiple. By (3.5) we know $\nu(L) = n$, on the other hand $\kappa(L) = n$. Hence [Na85] (see also [Fi07]) applies and $D = L$ is semi-ample. \qed

**4. Almost holomorphic algebraic reductions**

We use the same notations as in section 3 and first prove that an almost holomorphic algebraic reduction has in fact a holomorphic model.

**4.1. Theorem.** Let $X$ be a hyperkähler manifold of dimension $2n$ such that $0 < a(X) < 2n$. If the algebraic reduction $f$ is almost holomorphic, then $f$ has a Lagrangian holomorphic model.

**Proof.** It suffices to show that $L$ is semi-ample. By Hironaka’s flattening theorem [Hi75] (applied to $\check{f} : \check{X} \rightarrow B$) and the normalization (for the resulting source space), we have an equi-dimensional modification $\hat{f} : \hat{X} \rightarrow \hat{B}$ of $\check{f} : \check{X} \rightarrow B$. More precisely, there are a normal space $\check{X}$, a proper bimeromorphic morphism
\[ \mu = \hat{\mu} \circ \pi : \check{X} \rightarrow \hat{X} \rightarrow X, \]

a smooth projective manifold $\hat{B}$, a birational morphism $\mu_B : \hat{B} \rightarrow B$ and an equi-dimensional morphism $\check{f} : \check{X} \rightarrow \check{B}$ such that $\mu_B \circ \check{f} = \check{f} \circ \hat{\mu}$ and $\mu_B \circ \check{f} = f \circ \mu$.

Note that we can make $\check{B}$ smooth as flatness is preserved under base change. Note also that we can make $\check{X}$ normal as the normalization map is a finite map.
Put $\hat{A} = \mu_\nu^*(A)$ so that $\hat{A}$ is big, nef and semi-ample and set $\hat{L} = \mu^*L = \hat{f}^*\hat{A} + \sum a_i E_i$, where $\cup E_i$ are the exceptional divisors of $\mu$ and $a_i$ are non-negative integers. As $\hat{D}$ and $\hat{f}^*\hat{A}$ are Cartier, so is $\sum a_i E_i$. Since $f$ is almost holomorphic and $\hat{f}$ is equi-dimensional, $\hat{f}(E_i)$ is a divisor on $\hat{B}$. As $\hat{B}$ is smooth, it is not only a Weil divisor but also Cartier. Let $C$ be a sufficiently general ample complete intersection curve on $\hat{B}$. Let

$$\hat{V} = \hat{X} \times_B C$$

and $\nu : Z \to \hat{V} \subset \hat{X}$ be a resolution of $\hat{V}$. Let $\varphi : Z \to C$ be the induced morphism. Let $\eta_Z$ be a Kähler class of $Z$. Then, as $\varphi^*(\hat{A}|C)$ and $\nu^*(\sum a_i E_i)$ are supported in the fibers of $\varphi$, we have:

$$\int_Z \nu^*L \wedge \varphi^*(\hat{A}|C) \wedge \eta_Z^{n-1} = \int_Z (\varphi^*(\hat{A}|C) + \nu^*(\sum a_i E_i)) \wedge \varphi^*(\hat{A}|C) \wedge \eta_Z^{n-1} = 0. \tag{1.5}$$

As $\nu^*L$ and $\varphi^*(\hat{A}|C) \in K(Z)$, they are proportional in $NS(Z)$ by the Hodge index theorem. So are $\varphi^*(\hat{A}|C)$ and $\nu^*(\sum a_i E_i)$. Consequently

$$\nu^*(N(\sum a_i E_i)) = \varphi^*(\Theta)$$

for some positive integer $N$ and an effective divisor $\Theta$ on $C$. As $f$ is equi-dimensional and $C$ is a general ample complete intersection curve, the Cartier divisor $N(\sum a_i E_i)$ on $\hat{X}$ is then of the form $\hat{f}^*\Delta$ for some effective Cartier divisor $\Delta$ on $\hat{B}$. Thus, replacing $L$ by some positive multiple, we have $\mu^*\hat{L} = \hat{f}^*(\hat{A} + \Delta)$ for some semi-ample big divisor $\hat{A}$ on $\hat{B}$ and an effective Cartier divisor $\Delta$ on $\hat{B}$. As $\mu^*\hat{L} \in K(\hat{X})$ (strictly speaking after passing to a resolution of $\hat{X}$ which does not matter, as we do not need equi-dimensionality any longer), it follows that $\hat{A} + \Delta \in K(\hat{B})$. As $\hat{B}$ is projective, this implies that the divisor $\hat{A} + \Delta$ is nef. As $\hat{A}$ is big and $\Delta$ is effective, the divisor $\hat{A} + \Delta$ is also big. Thus,

$$\kappa(\hat{L}) = \dim B = \nu(\hat{A} + \Delta) = \nu(\hat{L}). \tag{1.6}$$

As $\hat{L} = \mu^*L$, we have $\kappa(L) = \nu(L) > 0$ as well. Thus $L$ is semi-ample by [Na85] (see also [Pa07]). The morphism given by $|mL|$ is then Lagrangian fibration by a result of Matsushita [Ma99]. This proves Theorem 4.1.

Suppose $0 < a(X) < 2n$. In order to prove that always $a(X) = n$, we are reduced to the case that for the general fiber $F$ has algebraic dimension $a(F) = a(\hat{F}) = 0$ and moreover that $\hat{F}$ is isotypically semi-simple. Unfortunately not much is known about compact Kähler manifolds $\hat{F}$ with $a(\hat{F}) = 0$. If however $\hat{F}$ has a minimal model, things work out:

### 4.2. Proposition. If every isotypically semi-simple compact Kähler manifold $Z$ of dimension at most $2n - 1$ with $h^0(K_Z) = 1$ has a minimal model with numerically trivial canonical bundle (i.e. a bimeromorphically equivalent normal Kähler variety which is $\mathbb{Q}_-$-Gorenstein with numerically trivial canonical class), then every compact hyperkähler manifold $X$ of dimension $2n$ has algebraic dimension $a(X) = 0, n, 2n$.

**Proof.** We must rule out that $1 \leq a(X) \leq n - 1$. We argue by contradiction, hence we are in situation (3.2). We write more precisely

$$\hat{L} = \hat{f}^*(A) + \sum a_i E_i$$

and
with \( a_i \geq 0 \). Furthermore we have
\[
K_X = \sum b_i E_i
\]
with \( b_i > 0 \). Let \( D_i = E_i \cap \tilde{F} \). Then \( \tilde{L}_F = \sum_{i'} a_i D_i \) with \( I' \subset I, I' \neq \emptyset \), the set of all \( i \) such that \( E_i \cap \tilde{F} \neq \emptyset \) so that by adjunction
\[
K_F = \sum b_i D_i.
\] (1)
Moreover
\[
\tilde{L}_F = \sum_{i'} a_i D_i.
\] (2)

Let \( h : \tilde{F} \to F' \) be a minimal model of \( \tilde{F} \); then \( K_{F'} \equiv 0 \). Choose a modification \( \tau : \tilde{F} \to \tilde{F} \) from a compact Kähler manifold \( \tilde{F} \) such that the bimeromorphic map \( h : \tilde{F} \to F' \) induces a holomorphic map \( \tilde{h} : \tilde{F} \to F' \). Let
\[
L' = (\tilde{h}_*(\tau^*(\tilde{L}_F)))^*.
\] Since \( K_{F'} \equiv 0 \), by (1) every \( D_i, i \in I' \) is contracted by \( h \). Moreover by (1) and (2)
\[
mK_F = \tilde{L}_F + D'
\] with \( D' \) effective and supported in \( \bigcup_{i'} D_i \). Therefore \( \tau^*(\tilde{L}_F) \) is on one hand nef, on the other hand effective with support necessarily in the exceptional locus of \( \tilde{h} \). This is only possible when \( \tilde{L}_F = O_{\tilde{F}} \). Hence all \( a_i = 0 \) for \( i \in I' \), a contradiction. \( \square \)

5. The 4-dimensional Case

In this section we settle Conjecture 1.1 in dimension 4 completely. What still needs to be proved is

5.1. Theorem. Let \( X \) be a 4-dimensional hyperkähler manifold. Then \( a(X) \neq 1 \).

Proof. Assume to the contrary that \( a(X) = 1 \). Let \( f : X \to B \simeq \mathbb{P}_1 \) be the algebraic reduction with the setup (3.2); we set specifically \( A = O_B(1) \). By (2.4) and (4.1) we know that \( a(F) = a(\tilde{F}) = 0 \); moreover \( \tilde{F} \) is isotypically semi-simple. But since \( \dim \tilde{F} = 3 \) necessarily \( \tilde{F} \) must be simple. We may also assume that \( q(\tilde{F}) = 0 \); otherwise the Albanese map of \( \tilde{F} \) must be birational onto \( \text{Alb}(\tilde{F}) \), so that \( \tilde{F} \) has a minimal model, and we conclude by (4.2). Since \( A = O_V(1) \), we have \( h^0(L) = 2 \); we take \( F_1, F_2 \in |L| \), both necessarily irreducible and set
\[
S = F_1 \cap F_2
\]
as complex spaces. Hence \( S \) is a possibly non-reduced complete intersection. Notice that
\[
\pi_*(\mathcal{I}_E) = \mathcal{I}_S.
\]
In fact, we have on the level of analytic preimages (complex subspaces)
\[
\pi^*(S) = \pi^*(F_1) \cap \pi^*(F_2) = (\tilde{F}_1 + E) \cap (\tilde{F}_2 + E) = E.
\]
In other words
\[
\pi^*(\mathcal{I}_S) \cdot O_{\tilde{X}} = \mathcal{I}_E,
\]
where the left hand side denotes the image of \( \pi^*(\mathcal{I}_S) \) in \( O_{\tilde{X}} \). Therefore the canonical monomorphism \( \mathcal{I}_S \to \pi_*(\mathcal{I}_E) \) must be an isomorphism.

We first show
5.2 Claim. $H^q(X, L) = 0$ for $q = 1, 3, 4$ and dim $H^2(X, L) = 1$.

Proof. We proceed in several steps. (1) $H^1(X, L \otimes I_S) = 0.$
To verify this vanishing, we deduce from (*)
$$\pi^*(\tilde{f}^*(\mathcal{O}_B(1))) = \pi^*(I_E \otimes \tilde{L}) = I_S \otimes L.$$ Thus our claim (1) certainly holds if we can show
$$H^1(\tilde{X}, \tilde{f}^*(\mathcal{O}_B(1))) = 0.$$ By the Leray spectral sequence (and the projection formula for $\tilde{f}$), this in turn comes down to
$$R^1\tilde{f}_*(\mathcal{O}_{\tilde{X}}) = 0.$$ Since $q(\tilde{F}) = 0$, the sheaf $R^1\tilde{f}_*(\mathcal{O}_{\tilde{X}})$ is torsion, supported on a finite set. Thus if the sheaf would not be 0, again the Leray spectral sequence would yield $H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}) \neq 0$, which is absurd.
(2) $\chi(L_S) = 0$.
Now there is a constant $K$ such that
$$a^2 \cdot c_2(X) = K q(a)$$ for all $(1,1)$-classes $a$ ([Fu82, GHJ03, 23.17]). Hence
$$L^2 \cdot c_2(X) = 0$$ and via Riemann-Roch we obtain
$$\chi(X, mL) = \chi(\mathcal{O}_X) = 3$$ for all $m \in \mathbb{Z}$. The Koszul complex
$$0 \rightarrow L^* \rightarrow \mathcal{O}_X \oplus \mathcal{O}_X \rightarrow L \otimes I_S \rightarrow 0$$ gives $\chi(L \otimes I_S) = 2\chi(\mathcal{O}_X) - \chi(L^*) = 3$, so that
$$\chi(L_S) = \chi(L) - \chi(L \otimes I_S) = 0.$$ This establishes (2).
(3) The vanishing (1) and the isomorphism $H^0(I_S \otimes L) \rightarrow H^0(L)$ gives
$$H^0(L_S) = 0.$$ Finally we obtain
$$H^3(L) = H^4(L) = 0 ; \ h^2(L) = 1.$$ Concerning $H^2$ we calculate using the adjunction formula $K_S = 2L_S$:
$$H^2(L_S) = H^0(L_S^* \otimes 2L_S) = H^0(L_S) = 0.$$ Hence by (2):
$$H^1(L_S) = 0.$$ Therefore
$$H^1(X, L) = 0.$$ Next
$$H^3((X, L) = 0.$$
for $q = 4, 3$ by Serre duality resp. by a Kodaira vanishing theorem in the Kähler case [DP03] (observe $L^2 \neq 0$). Hence by Riemann-Roch
\[
\dim H^2(X, L) = 1.
\]
Thus we completely determined the cohomology of $L$ and Claim (5.2) is established.

We continue with the proof of Theorem 5.1. Notice that $h^{1,1}(X) \geq 2$, otherwise $X$ would be projective. Hence by [Ca83], [Fu83-2] there is a smooth hyperkähler deformation $p : \mathcal{X} \to \Delta$ over the unit disk with the following properties

- $X \cong X_0$;
- there is a line bundle $\mathcal{L}$ over $\mathcal{X}$ such that $\mathcal{L}|X_0 \cong L$;
- there is a sequence $(t_k)$ in $T$ converging to 0 such that $X_{t_k}$ is projective;
- the set $\Delta_1$ of all $t$ such that $X_t$ is not projective is dense in $\Delta$ with countable complement.

Here $X_t = p^{-1}(t)$, the fiber of $p$ over $t \in \Delta$. Let $L_t = \mathcal{L}|X_t$. From the knowledge of the cohomology of $L_0$, semi-continuity and the constancy of $\chi(L_t)$ we obtain immediately (possibly after shrinking $\Delta$) for all $t$:
\[
h^0(L_t) = 2, \quad h^2(L_t) = 1
\]
and
\[
h^q(L_t) = 0
\]
for $q = 1, 3, 4$. Therefore $a(X_t) \geq 1$ for all $t$. Notice that using (2.7), $a(X_t)$ takes the values 1, 2 and 4; the set $\Delta_2$ of all $t$ such that $a(X_t) = 1$ is moreover dense in $\Delta$ with countable complement (otherwise we conclude via a relative algebraic reduction that $a(X_0) \geq 2$). We consider the meromorphic map
\[
f_t : X_t \dasharrow B_t \cong \mathbb{P}_1
\]
defined by $|L_t|$. Our plan is to apply [AC05] to $X_1$: [AC05] gives a composition of flops $X_{t_k} \dasharrow \mathcal{X}'$ to some other projective hyperkähler manifold $X'$ such that the induced rational map $\mathcal{X}' \dasharrow B$ is actually a morphism. But then by [Ma99], $B_t$ cannot have dimension 1 (a projective hyperkähler 4-fold does not admit a surjective morphism to a curve). However in order to be able to apply [AC05] we need to check that
\[
\kappa(F_{t_k}) \leq 0;
\]
where $\kappa(F_t)$ is the Kodaira dimension of a desingularisation of a general fiber of $f_t$. The maps $f_t$ fits together in a family
\[
\mathcal{X} \dasharrow \mathbb{P}_1 \times \Delta.
\]
We introduce a resolution of indeterminacies
\[
\varphi : \tilde{\mathcal{X}} \to \mathbb{P}_1 \times \Delta.
\]
Then we can consider a family $(\tilde{F}_t)$ of general fibers of $\varphi_t$. After possibly shrinking $\Delta$, we may assume that all $\tilde{F}_t$ are smooth except for $t = 0$. Here we have an abuse of language and $\tilde{F}_0$ may split in a component which was called $\tilde{F}_0$ formerly, and possibly some other components. This possible imprecision is avoided by considering instead of $X_0$ some $X_s$ with $s \in \Delta_1, s \neq 0$ and by treating this $X_s$ as our new $X_0$. Thus we may assume that $p : \tilde{\mathcal{X}} \to \Delta$ is a submersion and that $\tilde{F}_0$ is smooth. Now choose a universal number $M$ such that $|MKZ|$ defines the Iitaka fibration.
for all smooth projective threefolds $Z$. This number exists by [FM00] and [VZ07]
(including references for the general type case and the case of Kodaira dimension
0, which actually are not needed here). Now by semi-continuity [Gr60] there is a
neighborhood $U \subset \Delta$ of 0 such that
\[ h^0(MK_{\tilde{F}_t}) \geq h^0(MK_{\tilde{F}_0}) \]
for all $t \in U$. Thus
\[ h^0(MK_{\tilde{F}_{t_k}}) \leq 1 \]
for all $t_k \in U$. Therefore $\kappa(\tilde{F}_{t_k}) = \kappa(F_{t_k}) \leq 0$ for all $t_k \in U$. \hfill \Box

6. NEF LINE BUNDLES ON HYPERKÄHLER MANIFOLDS

If $X$ is a non-algebraic hyperkähler manifold, then $NS(X)$ is parabolic if and
only if $X$ carries a nef non-trivial line bundle $L$, which is then unique up to a
multiple. We expect that $L$ is actually semi-ample. In this section we give some
results pointing in this direction.

A line bundle $L$ on a compact complex manifold is hermitian semi-positive if there
exists a (smooth) hermitian metric on $L$ whose curvature form is semi-positive.
Equivalently, there exists a semi-positive $(1,1)$–form $\omega$ such that
\[ c_1(L) = [\omega]. \]
A hermitian semi-positive line bundle is nef, but the converse is in general not true, see [DPS01].

6.1. Theorem. Let $X$ be a non-projective hyperkähler manifold of dimension $2n$.
Let $L$ be a non-trivial hermitian semi-positive line bundle on $X$. Then $a(X) = \kappa(L)$;
in particular $\kappa(L) \geq 0$.

Proof. We use Riemann-Roch in the following form (see e.g. [GHJ03])
\[ \chi(mL) = \sum_{i=0}^{n} b_i q_X(L)^i, \]
where $b_i$ are some numbers which do not depend on $L$. Since $X$ is assumed to be
non-algebraic, we have $q_X(L) = 0$ and Riemann-Roch reads
\[ \chi(mL) = b_0 = \chi(O_X) = n + 1. \]
If $h^0(mL) \geq n + 1$ for all $m >> 0$, then $\kappa(L) \geq 1$, in particular $a(X) \geq 1$. Since $L$ defines the algebraic reduction in the sense of (3.2) (recall that $NS(X)$ must
be parabolic and that we have only one nef line bundle up to scalars), we obtain
$\kappa(L) = a(X)$.
So we may assume that there is a sequence $(m_k)$ converging to $\infty$ and some number
$q > 0$ (actually even) such that
\[ H^q(X, m_k L) \neq 0 \]
for all $m_k$. Fix a Kähler form $\omega$. By the Hard Lefschetz Theorem in the semi-positive
case [Mo99, Ta97], see also [DPS01], the canonical morphism
\[ \wedge_\omega^q : H^0(X, \Omega_X^{2n-q} \otimes m_k L) \to H^q(X, m_k L) \]
is surjective. Thus
\[ H^0(X, \Omega_X^{2n-q} \otimes m_k L) \neq 0 \]
for all $k$. Now we apply [DPS01], (2.15): one has $a(X) \geq 1$ or $\kappa(L) \geq 0$. In both cases we argue as above and conclude $\kappa(L) = a(X)$. □

The arguments of Theorem 6.1 actually sometimes work also in the nef case, namely when the zero locus of a suitable multiplier ideal is not too large. This leads to the following

**6.2. Theorem.** Let $X$ be a parabolic hyperkähler manifold of dimension $2n \geq 4$. Then $X$ contains a positive dimensional compact subvariety of dimension at least 2.

**Proof.** Assume to the contrary that all compact subvarieties of $X$ have dimension at most 1, in particular $a(X) = 0$. Since $NS(X)$ is parabolic, there is, as already mentioned at the beginning of this section, a non-trivial nef line bundle $L$, unique up to a scalar. On $L^\otimes m$ we introduce a singular metric $h_m$ with multiplier ideal $I_m$ with zero locus $V_m$. We argue similarly as in Theorem 6.1. From Riemann-Roch we deduce the existence of a positive even number $q \geq 2$ such that $H^q(X, m_k L) \neq 0$ for a sequence $(m_k)$ converging to $\infty$. Since $\dim V_{m_k} \leq 1$ by our assumption, we conclude

$$H^q(X, m_k L \otimes I_{m_k}) \neq 0$$

for all $k$. By the Hard Lefschetz Theorem for nef line bundles [Ta97], [DPS01], we obtain the non-vanishing

$$H^0(X, \Omega_X^{2n-q} \otimes m_k L \otimes I_{m_k}) \neq 0.$$ 

Now [DPS01], (2.15) implies $a(X) \geq 1$ or $\kappa(L) \geq 0$. Since the only positive-dimensional subvarieties in $X$ are curves, the first alternative is only possible when $a(X) = 2n - 1$, contradicting (3.6). In the second alternative $X$ contains a divisor, since $L$ cannot be trivial, again a contradiction. □
References

[AC05] Amerik E., Campana F.: Fibrations meromorphes sur certaines varietes de classe canonique triviale, arXiv:math/0510299 to appear in a volume in honour of F. Bogomolov

[Ba75] Barlet, D.: Espace analytique réduit des cycles analytiques complexes compact d’un espace analytique complexe de dimension finie, Lecture Notes in Math. 482, 1–158, Springer (1975)

[BHPV04] Barth, W. P., Hulek, K., Peters, C. A. M., Van de Ven, A.: Compact complex surfaces, Springer (2004).

[Be83] Beauville, A.: Variétés kählériennes dont la premi’ere classe de Chern est nulle, J. Diff. Geom. 18 (1983) 755–782.

[Bo78] Bogomolov, F.: Hamiltonian Kählerian manifolds, Dokl. Akad. Nauk SSSR 243 (1978), 1101–1104

[Ca81] Campana, F.: Coréduction algébrique d’un espace analytique faiblement kählérien compact, Invent. Math. 63 (1981) 187–223.

[Ca81-2] Campana, F.: Réduction algébrique d’un morphisme faiblement Kählérien propre et applications, Math. Ann. 256 (1981) 157–189.

[Ca83] Campana, F.: Densité des variétés hamiltoniennes primitives projectives, C. R. Acad. Sci. Paris Sér. I Math. 297 (1983) 413–416.

[Ca06] Campana, F.: Isotrivialité de certaines familles kählériennes de variétés non projectives, Math. Z. 252 (2006) 147–156.

[DPS01] Demailly, J-P., Peternell, Th., Schneider, M.: Pseudo-effective line bundles on compact Kähler manifolds, Intl. J. Math. 12 (2001), 689–741

[DP03] Demailly, J-P., Peternell, Th.: A Kodaira vanishing theorem on compact Kähler manifolds, J. Diff. Geom. 63 (2003), 231–277.

[Fu81] Fujiki, A.: A theorem on bimeromorphic maps of Kähler manifolds and its applications, Publ. Res. Inst. Math. Sci. 17 (1981) 735–754.

[Fu83] Fujiki, A.: On the structure of compact complex manifolds in C, Adv. Stud. Pure Math. 1, North-Holland, Amsterdam (1983) 231–302.

[Fu83-2] Fujiki, A.: On primitively symplectic compact Kähler V-manifolds of dimension four, Classification of algebraic and analytic manifolds (Katata, 1982), 71–250, Progr. Math., 39, Birkhäuser Boston, Boston, MA, 1983.

[Fu87] Fujiki, A.: On the de Rham cohomology group of a compact Kähler symplectic manifold, Adv. Stud. Pure Math. 10 North-Holland, Amsterdam (1987) 105–165.

[FM00] Fujino, O., Mori, S.: A canonical bundle formula, J. Diff. Geom. 56 (2000), 176-188

[Fu07] Fujino, O.: Base point free theorems - saturation, B-divisors, and canonical bundle formula, arXiv:math/050854v2.

[Gr60] Grauert, H.: Ein Theorem der analytischen Garbentheorie und die Modulräume komplexer Strukturen, Inst. Hautes Études Sci. Publ. Math. 5 (1960).

[GHJ03] Gross, M., Huybrechts, D., and Joyce, D.: Calabi-Yau manifolds and related geometries, Universitext. Springer-Verlag, Berlin (2003).

[Hu99] Huybrechts, D.: Compact hyperkähler manifolds: basic results, Invent. Math. 135 (1999) 63–113; Erratum: ”Compact hyper-Kähler manifolds: basic results” Invent. Math. 152 (2003) 209–212.

[Hu03] Huybrechts, D.: The Kähler cone of a compact hyperkähler manifold, Math. Ann. 326 (2003) 499–513.

[Hi75] Hironaka, H.: Flattening theorem in complex-analytic geometry, Amer. J. Math. 97 (1975) 503–547.

[Hi77] Hironaka, H.: Bimeromorphic smoothing of a complex-analytic space, Acta Math. Vietnam. 2 (1977), 103–168.

[Ko54] Kodaira, K.: On Kähler varieties of restricted type (an intrinsic characterization of algebraic varieties), Ann. of Math. 60 (1954) 28–48.

[Ka85] Kawamata, Y.: Pluricanonical systems on minimal algebraic varieties, Invent. Math. 79 (1985) 567–588.

[Ma99] Matsushita, D.: On fiber space structures of a projective irreducible symplectic manifold, Topology 38 (1999) 79–83; Addendum ibid. 40 (2001) 431–432.
Matsushita, D.: Equidimensionality of Lagrangian fibrations on holomorphic symplectic manifolds, Math. Res. Lett. 7 (2000) 389–391.

Moishezon, B. G.: On n-dimensional compact complex manifolds having n algebraically independent meromorphic functions. 1., Izv. Akad. Nauk SSSR Ser. Mat. 30 (1966) 133–174.

Moishezon, B. G.: On n-dimensional compact complex manifolds having n algebraically independent meromorphic functions. I., Izv. Akad. Nauk SSSR Ser. Mat. 30 (1966) 133–174.

Mourougane, C.: Théorèmes d’annulation génériques pour les fibrés vectoriels semi-négatifs, Bull. Soc. Math. France 127 (1999), 115–133

Nakayama, N.: The lower semicontinuity of the plurigenera of complex varieties, Adv. Stud. Pure Math. 10 North-Holland, Amsterdam (1987) 551–590.

Oguiso, K.: Salem polynomials and birational transformation groups for hyperkähler manifolds, Sugaku 59 (2007) 1–23.

Takegoshi, K.: On cohomology groups of nef line bundles tensorized with multiplier ideal sheaves on compact Kähler manifolds, Osaka J. Math. 34 (1997), 783–802

Ueno, K.: Classification theory of algebraic varieties and compact complex spaces, Lecture Notes in Mathematics 439 Springer-Verlag, Berlin-New York (1975).

Varouchas, J.: Kähler Spaces and Proper Open Morphisms, Math. Ann. 283 (1989) 13–52.

Viehweg, E., Zhang De-Qi: Effective Iitaka fibrations, arXiv:0707.4287

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