Sharp Estimates of the Solutions to Bézout’s Polynomial Equation and a Corona Theorem

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Abstract

The main result of this paper is a corona theorem for the multipliers of a class of de Branges–Rovnyak spaces. A key to this involves estimates for the solutions to the classical Bézout equation that are analogous to Carleson’s solution to the corona theorem.

Keywords

Bézout’s equation · De Branges–Rovnyak spaces · Corona theorem

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1 Introduction

A well-known theorem of Étienne Bézout (1730-1783) says that if $A, B \in \mathbb{C}[z]$ (the vector space of polynomials in the complex variable $z$ with coefficients in $\mathbb{C}$) have no common roots, then there are $R, S \in \mathbb{C}[z]$ with

$$\deg R \leq \deg B - 1 \quad \text{and} \quad \deg S \leq \deg A - 1,$$

such that

$$A(z)R(z) + B(z)S(z) = 1 \quad \text{for all } z \in \mathbb{C}.$$  

Moreover, if $A$ and $B$ are not both constant polynomials, then $R$ and $S$ are uniquely determined by the degree condition in (1.1) and will be called the minimal solutions of (1.2). Here we use the standard convention that the degree of the null polynomial is $-\infty$.

The proof of Bézout’s theorem often presented to algebra students uses a version of the classical Euclidean algorithm for polynomials (noting of course that the lack of common roots for $A$ and $B$ implies that the greatest common divisor of $A$ and $B$ is the constant polynomial one). A result of Sylvester gives an explicit formula for the minimal solutions $R$ and $S$ in terms of a certain matrix equation involving the Sylvester resultant (see Sect. 2.2).

The first goal of this paper is to estimate the coefficients of the polynomials $S$ and $R$. More precisely, we want to estimate $\|R\|$ and $\|S\|$, where

$$\|p\| := \max_{0 \leq j \leq n} |p_j|$$

represents the height of a polynomial $p(z) = \sum_{j=0}^{n} p_j z^j$. To understand what we are aiming towards, consider the following example.

**Example 1.4** If $n \in \mathbb{N}$ and $0 < \delta < 1$, set $A(z) = z^n$ and $B(z) = z - \delta$. One can work out the corresponding minimal polynomials $R$ and $S$ that satisfy (1.1) and (1.2) to be

$$R(z) = \frac{1}{\delta^n} \quad \text{and} \quad S(z) = -\frac{1}{\delta^n} \sum_{j=0}^{n-1} \delta^{n-1-j} z^j.$$  

Then $\|A\| = \|B\| = 1$, $|B(0)| = \delta$, and $\|R\| = \|S\| = \delta^{-n}$. Notice how $n$ is the maximal order of the zeros of $A$ (which in this case is a single zero at the origin of order $n$) while the condition $|B(0)| = \delta$ can be interpreted as a lower bound for $B$ at the zero of $A$.

Our first main theorem proves that the type of phenomenon presented in Example 1.4 always occurs. For $N \in \mathbb{N} \cup \{0\}$, let $\mathcal{P}_N$ denote the $N + 1$ dimensional subspace of polynomials of degree at most $N$. 

\[ Springer\]}
Theorem 1.5  Let $A \in \mathcal{P}_N$ be of the form

$$A(z) = \prod_{j=1}^{n} (z - \alpha_j)^{m_j},$$

where $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{C}$ are distinct and $N = \sum_{j=1}^{n} m_j \geq 1$. Fix $K \in \mathbb{N}$. Then there is a $C > 0$, depending only on $A$ and $K$, such that if $B \in \mathcal{P}_K$ satisfies

(a) $\|B\| \leq 1$, and
(b) $|B(\alpha_j)| \geq \delta > 0$ for all $1 \leq j \leq n$,

then the minimal solutions $R$ and $S$ of the Bézout equation (1.2) satisfy

$$\left(\|R\|^2 + \|S\|^2\right)^{1/2} \leq \frac{C}{\delta \max m_j}. \quad (1.6)$$

Note that condition (a) of the theorem above is essential. Indeed, if

$$B(z) = 1 - \lambda A(z),$$

then $S \equiv 1$ and $R \equiv \lambda$ and letting $|\lambda| \to \infty$ contradicts the estimate in (1.6).

Theorem 2.6 below extends this result to multiple polynomials $A$ and $B_1, \ldots, B_L$.

As shown in Example 1.4, the estimate in (1.6) is sharp in that the exponent $\max\{m_j : 1 \leq j \leq n\}$ on $\delta$ can not be lowered. It seems rather surprising to us that such estimates are lacking in the literature, though there is a connection to results from [30].

The statement of Theorem 1.5 is asymmetric in the polynomials $A$ and $B$: $A$ is fixed (and determines the constant $C$), while $B$ is chosen freely in the finite dimensional space $\mathcal{P}_K$. We will discuss a symmetric version of this in Sect. 2.2. As we will see, the estimate from the symmetric version can be much less precise in terms of the exponent on $\delta$ than the asymmetric one presented in Theorem 1.5.

Theorem 1.5 is connected to a related problem, known as the corona problem [8] for $H^\infty$, the space of bounded analytic functions on the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. In a way, the corona theorem can be viewed as a Bézout theorem for $H^\infty$.

In 1962, Carleson [2] answered a conjecture proposed by Kakutani [16] in 1941 and proved that if $(f_j)_{j=1}^{N}$ is a finite sequence in $H^\infty$ which satisfies

$$0 < \delta \leq \left(\sum_{j=1}^{N} |f_j(z)|^2\right)^{1/2} \leq 1 \quad \text{for all } z \in \mathbb{D},$$

then there is a finite sequence $(g_j)_{j=1}^{N}$ in $H^\infty$ such that

$$\sum_{j=1}^{N} f_j(z) g_j(z) = 1 \quad \text{for all } z \in \mathbb{D}$$
and

\[ \left( \sum_{j=1}^{N} |g_j(z)|^2 \right)^{\frac{1}{2}} \leq C \quad \text{for all } z \in \mathbb{D}. \]

In Carleson’s original result, the constant \( C \) depended on \( \delta \) and \( N \). Later, Tolokonnikov [27] and Rosenblum [25] independently proved that the constant is independent on \( N \) and the corona theorem holds true for infinite sequences \( (f_j)_{j=1}^{\infty} \). Uchiyama obtained the sharpest known estimate in [31], namely

\[ C(\delta) \leq C \frac{1}{\delta^2} \log \frac{1}{\delta}, \quad (1.7) \]

where \( C \) is an absolute constant. On the other hand, Treil [28] proved that one can do no better than

\[ C \frac{1}{\delta^2} \log \log \frac{1}{\delta}. \]

There are also many generalizations of the corona theorem to matrix and operator-valued functions that are important to control theory and similarity problems. For example, Nikolski’s book [23, Ch.9, Sec. 2] relates the operator-valued corona problem to one-sided invertibility of Toeplitz operators.

We plan to apply our Bézout estimates to pursue another direction of inquiry which stems from the fact that \( H^\infty \) is the multiplier algebra for the Hardy \( H^2 \) (see Sect. 3). This inspires one to prove corona type theorems for sequences in the multiplier algebra of other reproducing kernel Hilbert space of analytic functions [20, 21, 29]. A particular class of such reproducing kernel Hilbert spaces, which has received quite a lot of attention over the past several decades, are the de Branges–Rovnyak spaces \( \mathcal{H}(b) \) [10, 11, 26]. These spaces are an important class of linear submanifolds of the classical Hardy space \( H^2 \), that have originally been introduced in [6, 7] for the purpose of modeling Hilbert space contractions (see also [22]). In Theorem 4.5, the second main result of this paper, we use our Bézout estimates to obtain a corona theorem for the multiplier algebras of de Branges–Rovnyak spaces \( \mathcal{H}(b) \) in the case where \( b \) is any rational function (but not a finite Blaschke product) in the closed unit ball of \( H^\infty \). Our results are related to those from [20, 21].

### 2 Bézout’s Equation and Related Estimates

#### 2.1 Proof of Theorem 1.5

Without loss of generality, we assume that \( 0 < \delta < 1 \). Fix \( N \in \mathbb{N} \) and \( A \in \mathcal{P}_N \). For fixed \( K \in \mathbb{N} \), assume \( B \in \mathcal{P}_K \) satisfies the condition of the theorem, namely

\[ \| B \| \leq 1 \quad \text{and} \quad |B(\alpha_j)| \geq \delta > 0 \quad \text{for all } 1 \leq j \leq n, \]
where $\alpha_j$ are the zeros of $A$. Since $\deg A = N$, the Bézout equation $AR + BS = 1$ admits unique solutions $R \in \mathcal{P}_{K-1}$ and $S \in \mathcal{P}_{N-1}$. Looking at the zeros $\alpha_j$, $1 \leq j \leq n$ of $A$, the Bézout equation implies that

$$
B(\alpha_j)S(\alpha_j) = 1 \quad \text{and} \quad (BS)^{(k)}(\alpha_j) = 0 \quad \text{for all } 1 \leq k < m_j.
$$

Leibniz’ rule allows to write out

$$
(BS)^{(k)}(\alpha_j) = \sum_{m=0}^{k} \binom{k}{m} B^{(k-m)}(\alpha_j) S^{(m)}(\alpha_j),
$$

which yields a triangular system in $S^{(\ell)}(\alpha_j)$ whose solution can be expressed as:

$$
S(\alpha_j) = \frac{1}{B(\alpha_j)},
$$

$$
S^{(k)}(\alpha_j) = -\frac{1}{B(\alpha_j)} \sum_{m=0}^{k-1} \binom{k}{m} B^{(k-m)}(\alpha_j) S^{(m)}(\alpha_j).
$$

Solving this system recursively (the computation of $S^{(k)}(\alpha_j)$ involves all the preceding terms $S^{(m)}(\alpha_j)$ for $0 \leq m < k$), we see that at each step we increase the power of $B(\alpha_j)$ in the denominator. Note that every map $B \mapsto B^{(\ell)}(\alpha_i)$ is continuous on $\mathcal{P}_K$, for all $1 \leq \ell < m_j$ and $1 \leq j \leq n$, and so there is a constant $C_1$ depending only on $A$ such that $|B^{(\ell)}(\alpha_j)| \leq C_1 \|B\| \leq C_1$. We therefore obtain a constant $C$, depending only on $A$ and $K$, such that

$$
|S^{(k)}(\alpha_j)| \leq \frac{C}{|B(\alpha_j)|^{k+1}} \leq \frac{C}{\delta^{k+1}} \leq \frac{C}{\delta \max m_j}, \quad 0 \leq k < m_j.
$$

We introduce a new norm on $\mathcal{P}_{N-1}$ defined by

$$
\|P\|_A = \sup_{1 \leq j \leq n} \sup_{0 \leq m < m_j} |P^{(m)}(\alpha_j)|.
$$

By equivalence of norms in the $N$-dimensional space $\mathcal{P}_{N-1}$, there is $C_A \geq 1$ such that

$$
\|P\|_A C_A^{-1} \leq \|P\| \leq C_A \|P\|_A.
$$

Thus,

$$
\|S\| \leq C_A \|S\|_A \leq \frac{CC_A}{\delta \max m_j}, \quad (2.1)
$$

which achieves the estimate for $S$. 

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To estimate the polynomial $R$, we introduce another norm as follows. Let $L = \max\{N, K\}$ and let $E$ denote a finite subset of $\mathbb{C}$ with at least $L$ points, none of which is a zero of the polynomial $A$. Then

$$\|P\|_A^* := \max_{z \in E} |P(z)|$$

is an equivalent norm on $\mathcal{P}_{L-1}$ and thus there is $C_A^* \geq 1$ such that

$$\|P\|_A^{* \cdot C_A^* - 1} \leq \|P\| \leq C_A^* \|P\|_A^*.$$

Let

$$\gamma = \min_{z \in E} |A(z)| > 0.$$  

Then, for every $z \in E$,

$$|R(z)| = \frac{|1 - B(z)S(z)|}{|A(z)|} \leq \frac{1 + \|B\|_A^* \|S\|_A^*}{\gamma}.$$  

Hence, by equivalence of norms on $\mathcal{P}_{K-1}$, and since $\|B\| \leq 1$,

$$\|R\| \leq C_A^* \|R\|_A^* \leq C_A^* \frac{1 + \|B\|_A^* \|S\|_A^*}{\gamma} \leq C_A^* 3 \frac{1 + \|S\|}{\gamma}.$$  

By (2.1), the last term is bounded by a suitable constant divided by $\delta^{\max mj}$, completing the proof.

\[\square\]

### 2.2 Comments on Theorem 1.5

One can obtain a more symmetric version of Theorem 1.5 with the cost of a worse estimate at the end. To see what we mean here, the Bézout equation $AR + BS = 1$ can be rewritten in terms of the (known) coefficients of $A$ and $B$ and the (unknown) coefficients of $R$ and $S$. Then the coefficient equation translates to a matrix equation where the matrix entries are determined by the coefficients of $A$ and $B$ and an unknown vector with coefficients given by $R$ and $S$. The matrix appearing here is a so-called Sylvester matrix for which the modulus of the determinant can be computed as

$$|B_K|^N |A(\beta_1) \cdots A(\beta_K)| = |A_N|^K |B(\alpha_1) \cdots B(\alpha_N)|,$$

where $A_N$ and $B_K$ are respectively the dominant coefficients of $A$ and $B$, and the $\alpha_j$’s and the $\beta_j$’s are the roots of $A$ and $B$ respectively, counted with multiplicities. We refer to [17, p. 200] or [4, p. 77] for the necessary tools on Sylvester matrices. The following more symmetric result can be deduced from Cramer’s rule.

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Proposition 2.2 Let $\|A\|, \|B\| \leq 1$ be two polynomials of respective degree $N$ and $K$, and
\[
\min\{|A(\beta_j)|, |B(\alpha_i)|, 1 \leq i \leq N, 1 \leq j \leq K\} = \delta > 0.
\]
Then
\[
\left( \|R\|^2 + \|S\|^2 \right)^{\frac{1}{2}} \leq \frac{\sqrt{2}(N + K - 1)!}{\min\{|A_N|^K \delta^N, |B_K|^N \delta^K\}}.
\]
(2.3)

Compared to Theorem 1.5, the estimate given in (2.3) may be worse, as seen in the following example.

Example 2.4 Fix $0 < \eta < 1$ and define

\[
A(z) = z(z - 1) \quad \text{and} \quad B(z) = (z - \eta)(z - 1 + \eta).
\]

Then
\[
\delta = \min\{|A(\eta)|, |A(1 - \eta)|, |B(0)|, |B(1)|\} = \eta(1 - \eta).
\]

A computation using the identity $B(z) = A(z) + \eta(1 - \eta)$ leads to

\[
R(z) = -\frac{1}{\delta} \quad \text{and} \quad S(z) = \frac{1}{\delta}.
\]

Hence $\left( \|R\|^2 + \|S\|^2 \right)^{\frac{1}{2}} = \sqrt{2}\delta^{-1}$, as expected from Theorem 1.5, while the right hand side of (2.3) yields $\left( \|R\|^2 + \|S\|^2 \right)^{\frac{1}{2}} \leq 6\sqrt{2}\delta^{-2}$.

We emphasize that the asymmetric version of Theorem 1.5 is precisely what is needed to discuss our corona theorem for the de Branges-Rovnyak spaces in Theorem 4.5 where $A$ is part of the definition of the space.

2.3 Extension to Several Polynomials

Extending Theorem 1.5 to several polynomials involves the plank theorem [1].

Lemma 2.5 For vectors $v_1, v_2, \ldots, v_n$ in a Hilbert space $\mathcal{H}$ that satisfy
\[
\|v_i\|_{\mathcal{H}} \geq 1 \quad \text{for all} \quad 1 \leq i \leq n,
\]
there exists a unit vector $y \in \mathcal{H}$ such that
\[
|\langle v_i, y \rangle_{\mathcal{H}}| \geq \frac{1}{\sqrt{n}} \quad \text{for all} \quad 1 \leq i \leq n.
\]

Here is our extension of Theorem 1.5 to several polynomials.
Theorem 2.6 Let $A \in \mathcal{P}_N$ be of the form

$$A(z) = \prod_{j=1}^{n} (z - \alpha_j)^{m_j},$$

where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are distinct and $N = \sum_{j=1}^{n} m_j \geq 1$. Fix $K \in \mathbb{N}$. Then there is a $C > 0$, depending only on $A$ and $K$, such that, if $B_1, \ldots, B_L \in \mathcal{P}_K$ satisfy the conditions

(a) $\sum_{j=1}^{L} \|B_j\|^2 \leq 1$ and

(b) $\sum_{j=1}^{L} |B_j(\alpha_i)|^2 \geq \delta^2 > 0$ for all $1 \leq i \leq n$,

then there is an $R \in \mathcal{P}_{K-1}$, and $S_1, \ldots, S_L \in \mathcal{P}_{N-1}$, such that

$$RA + S_1 B_1 + \cdots + S_L B_L \equiv 1,$$  \hspace{1cm} (2.7)

and

$$\left( \|R\|^2 + \sum_{j=1}^{L} \|S_j\|^2 \right)^{\frac{1}{2}} \leq \frac{C}{\delta \max_{m_i}}.$$  \hspace{1cm} (2.8)

Proof Consider the vectors $v_1, \ldots, v_n \in \mathbb{C}^L$ defined by

$$v_i = \frac{1}{\delta} (B_1(\alpha_i), B_2(\alpha_i), \ldots, B_L(\alpha_i))$$

and note that $\|v_i\|_{\mathbb{C}^L} \geq 1$ for all $1 \leq i \leq n$. Lemma 2.5 produces a unit vector $y = (y_1, \ldots, y_L) \in \mathbb{C}^L$ such that

$$\frac{1}{\sqrt{n}} \leq |\langle v_i, y \rangle_{\mathbb{C}^L}| = \frac{1}{\delta} \left| \sum_{j=1}^{L} B_j(\alpha_i) \overline{y_j} \right| \text{ for all } 1 \leq i \leq n.$$

If $B(z) = \sum_{j=1}^{L} \overline{y_j} B_j(z)$, it follows that

$$|B(\alpha_i)| \geq \frac{\delta}{\sqrt{n}} \geq \frac{\delta}{\sqrt{N}}.$$ 

Furthermore,

$$\|B\| \leq \left( \sum_{j=1}^{L} |y_j|^2 \right)^{\frac{1}{2}} \left( \sum_{j=1}^{L} \|B_j\|^2 \right)^{\frac{1}{2}} \leq 1.$$
By Theorem 1.5 there is an \( R \in \mathcal{P}_{K-1} \) and an \( S \in \mathcal{P}_{N-1} \) that satisfy the conditions

\[
RA + SB \equiv 1 \quad \text{and} \quad (\|R\|^2 + \|S\|^2)^{\frac{1}{2}} \leq \frac{C\sqrt{N\max m_i}}{\delta_{\max m_i}}. \tag{2.9}
\]

The Bézout identity \( RA + SB \equiv 1 \) can be written as

\[
RA + \sum_{j=1}^{L} \tilde{y}_j SB_j \equiv 1.
\]

Moreover, if we define \( S_j = \tilde{y}_j S \), then (2.7) holds along with

\[
\|R\|^2 + \sum_{j=1}^{L} \|S_j\|^2 = \|R\|^2 + \sum_{j=1}^{L} |y_j|^2 \|S\|^2 = \|R\|^2 + \|S\|^2.
\]

The estimate in (2.8) now follows from the estimate in (2.9). \( \square \)

**Remark 2.10** In Theorem 1.5 the polynomials \( R, S \) are uniquely determined by the degree condition from (1.1). However, Theorem 2.6 only yields the existence of *some* polynomials \( R_1, \ldots, R_L \) that satisfy the desired estimates.

### 3 de Branges–Rovnyak Spaces

The second main theorem of this paper (Theorem 4.5) extends results from [21] and establishes a corona theorem for the multipliers of certain de Branges–Rovnyak spaces. In fact, this corona theorem is what originally drew us to investigate coefficient estimates of Bézout’s identity. In this section we present some of the basics of de Branges–Rovnyak spaces [10, 11, 26], along with some additional results which seem to be interesting on their own. The next section will contain our corona theorem.

Let

\[
\text{ball}(H^\infty) := \{b \in H^\infty : \|b\|_\infty = \sup_{z \in \mathbb{D}} |b(z)| \leq 1\}
\]

denote the closed unit ball in \( H^\infty \). For \( b \in \text{ball}(H^\infty) \), the *de Branges–Rovnyak space* \( \mathcal{H}(b) \) is the reproducing kernel Hilbert space associated with the positive definite kernel

\[
k^b_\lambda(z) := \frac{1 - b(\lambda)b(z)}{1 - \lambda z}, \quad \lambda, z \in \mathbb{D}. \tag{3.1}
\]

It is known that \( \mathcal{H}(b) \) is contractively contained in the well-studied Hardy space \( H^2 \) of analytic functions \( f \) on \( \mathbb{D} \) for which

\[
\|f\|_{H^2} := \left( \sup_{0 < r < 1} \int_{\mathbb{T}} |f(r\xi)|^2 dm(\xi) \right)^{\frac{1}{2}} < \infty,
\]
where \( m \) is normalized Lebesgue measure on the unit circle \( \mathbb{T} = \{ \xi \in \mathbb{C} : |\xi| = 1 \} \) \cite{9,15}. For \( f \in H^2 \), the radial limit \( \lim_{r \to 1^-} f(r\xi) =: f(\xi) \) exists for \( m \)-almost every \( \xi \in \mathbb{T} \) and

\[
\| f \|_{H^2} = \left( \int_{\mathbb{T}} |f(\xi)|^2 \, dm(\xi) \right)^{1/2}.
\]

Furthermore, Parseval’s theorem says that if \( f = \sum_{k=0}^{\infty} a_k \xi^k \in H^2 \), then \( \| f \|_{H^2}^2 = \sum_{k=0}^{\infty} |a_k|^2 \). Though \( \mathcal{H}(b) \) is contractively contained in \( H^2 \), it is generally not closed in the \( H^2 \) norm. In fact, for the \( b \) explored in this section, \( \mathcal{H}(b) \) is dense in \( H^2 \).

Throughout this section, we will assume that \( b \in \text{ball}(H^\infty) \) is a rational function that is not a finite Blaschke product. We exclude the finite Blaschke products from our discussion since we will be exploring a corona theorem for the multiplier algebra of \( \mathcal{H}(b) \). When \( b \) is a finite Blaschke product, \( \mathcal{H}(b) \) becomes the usual model space \( H^2 \ominus bH^2 \) and in that case it is well-known that the multiplier algebra of \( \mathcal{H}(b) \) becomes the constant functions \cite[p. 138]{14}. Thus, when \( b \) is a finite Blaschke product, any corona theorem concerning the multipliers of \( \mathcal{H}(b) \) becomes a triviality.

Although, for a general \( b \in \text{ball}(H^\infty) \) the contents of \( \mathcal{H}(b) \) seem mysterious, when \( b \in \text{ball}(H^\infty) \) is a rational function (and not a finite Blaschke product) the description of \( \mathcal{H}(b) \) is quite explicit. For such a \( b \) there exists a unique nonconstant rational function \( a \) with no zeros on \( \mathbb{D} \) such that \( a(0) > 0 \) and \( |a(\xi)|^2 + |b(\xi)|^2 = 1 \) for all \( |\xi| = 1 \). This function \( a \) is called the Pythagorean mate of \( b \). In fact, one can obtain \( a \) from the Fejér–Riesz theorem (see \cite{12}). Let \( \xi_1, \ldots, \xi_n \) denote the distinct roots of \( a \) on \( \mathbb{T} \), with corresponding multiplicities \( m_1, \ldots, m_n \), and define the polynomial \( a_1 \) by

\[
a_1(z) := \prod_{j=1}^{n} (z - \xi_j)^{m_j}.
\]

Results from \cite{3,12} show that \( \mathcal{H}(b) \) has an explicit description as

\[
\mathcal{H}(b) = a_1 H^2 \oplus \mathcal{D}_{N-1},
\]

where \( N = m_1 + \cdots + m_n \) and \( \oplus \) above denotes the algebraic direct sum in that \( a_1 H^2 \cap \mathcal{D}_{N-1} = \{0\} \). Moreover, if \( f \in \mathcal{H}(b) \) is decomposed with respect to (3.4) as

\[
f = a_1 \widetilde{f} + p, \quad \text{where} \quad \widetilde{f} \in H^2 \text{ and } p \in \mathcal{D}_{N-1},
\]

a norm on \( \mathcal{H}(b) \), equivalent to the natural one induced by the positive definite kernel \( k^b_\lambda(z) \) above, is

\[
\| a_1 \widetilde{f} + p \|_b^2 := \| \widetilde{f} \|_{H^2}^2 + \| p \|_{H^2}^2.
\]

It is important to note that since \( \| \cdot \|_b \) is only equivalent to the original norm corresponding to the kernel in (3.1), its scalar product as well as the reproducing kernels and the adjoints of operators defined on \( \mathcal{H}(b) \) will be different. With the norm \( \| \cdot \|_b \) and the corresponding inner product in mind, we need to introduce a new notation for the associated reproducing kernels, different from (3.1), namely \( k^b_\lambda \) (note the bold face). By the term reproducing kernel we mean that \( k^b_\lambda \in \mathcal{H}(b) \) for all \( \lambda \in \mathbb{D} \) and

\[\mathcal{H}(b)\] Springer
(f, k^b_\lambda)_b = f(\lambda) \quad \text{for all } f \in \mathcal{H}(b) \text{ and } \lambda \in \mathbb{D}.

Using (3.5) and the standard estimate that any \( g \in H^2 \) satisfies

\[ |g(z)|^2 \leq \frac{\|g\|_{L^2}^2}{1 - |z|^2} \quad \text{for all } z \in \mathbb{D}, \tag{3.7} \]

we see that for fixed \( 1 \leq k \leq n \) and for each \( f \in \mathcal{H}(b) \) we have

\[ f(\xi_k) = \lim_{r \to 1^-} f(r \xi_k) = p(\xi_k), \tag{3.8} \]

where \( f = a_1 \tilde{f} + p \) with \( \tilde{f} \in H^2 \) and \( p \in \mathcal{P}_{N-1} \). In the spirit of (3.7), the next lemma (interesting in its own right and useful later) yields more precise information on the boundary behavior of \( \mathcal{H}(b) \) functions. In particular, it shows that \( \mathcal{H}(b) \) functions admit tangential limits in suitable approach regions at each point \( \xi_k \) (see Remark 3.12 below).

**Lemma 3.9** For each fixed \( 1 \leq k \leq n \), there is a \( c_k > 0 \), depending only on \( b \), such that for each \( f \in \mathcal{H}(b) \), \( \eta > 0 \), and \( z \in \mathbb{D} \), we have

\[ |f(z)|^2 \leq (1 + \eta)|f(\xi_k)|^2 + c_k \left( 1 + \frac{1}{\eta} \right) \frac{|z - \xi_k|^2}{1 - |z|^2} \|f\|_{L^2}, \tag{3.9} \]

**Proof** For fixed \( 1 \leq k \leq n \), remembering from (3.3) that \( \xi_k \) is a root of \( a_1 \), we define the polynomial \( a^#(z) \) by

\[ a^#_k(z) := \frac{a_1(z)}{z - \xi_k}. \]

Write \( f \in \mathcal{H}(b) \) as \( f = a_1 \tilde{f} + p \) as in (3.5). By (3.8) we have \( f(\xi_k) = p(\xi_k) \), and so

\[
\begin{align*}
f(z) &= (z - \xi_k) a^#_k(z) \tilde{f}(z) + (p(z) - p(\xi_k)) + p(\xi_k) \\
&= (z - \xi_k) \left( a^#_k(z) \tilde{f}(z) + \frac{p(z) - p(\xi_k)}{z - \xi_k} \right) + p(\xi_k) \\
&= (z - \xi_k) f_k(z) + f(\xi_k),
\end{align*}
\]

where

\[ f_k(z) = a^#_k(z) \tilde{f}(z) + \frac{p(z) - p(\xi)}{z - \xi_k} \in H^2. \tag{3.10} \]

Given \( \eta > 0 \), for any \( a, b > 0 \) we have

\[ 2ab \leq \eta a^2 + \frac{1}{\eta} b^2 \]
and so

\[ |f(z)|^2 \leq |(z - \xi_k) f_k(z) + f(\xi_k)|^2 \]
\[ \leq |f(\xi_k)|^2 + |z - \xi_k|^2 |f_k(z)|^2 + 2|z - \xi_k| f_k(z) ||f(\xi_k)| \]
\[ \leq (1 + \eta) |f(\xi_k)|^2 + \left(1 + \frac{1}{\eta}\right)|z - \xi_k|^2 |f_k(z)|^2 \]
\[ \leq (1 + \eta) |f(\xi_k)|^2 + \left(1 + \frac{1}{\eta}\right)|z - \xi_k|^2 \| f_k \|_H^2. \]

In the last inequality above, note the use of (3.7).

To finish the proof, it suffices to show there exists a \( c_k > 0 \), depending only on \( b \) and \( k \), such that

\[ \| f_k \|_H^2 \leq c_k \| f \|_H^2. \]  \hspace{1cm} (3.11)

The definition of \( f_k \) from (3.10) says that

\[ \| f_k \|_H^2 \leq 2 \left( \| a_k^\# \tilde{f} \|_H^2 + \left\| \frac{p - p(\xi_k)}{z - \xi_k} \right\|_H^2 \right) \]
\[ \leq 2 \left( \| a_k^\# \|_\infty \| \tilde{f} \|_H^2 + \left\| \frac{p - p(\xi_k)}{z - \xi_k} \right\|_H^2 \right). \]

Since the map

\[ p(z) \mapsto \frac{p(z) - p(\xi_k)}{z - \xi_k} \]

is a linear transformation from \( P_{N-1} \) to itself and \( P_{N-1} \) is a finite dimensional space (and hence all norms on \( P_{N-1} \) are equivalent), this map is continuous and hence there is a constant \( \tilde{c}_k > 0 \) such that

\[ \left\| \frac{p - p(\xi_k)}{z - \xi_k} \right\|_H^2 \leq \tilde{c}_k \| p \|_H^2 \]  \hspace{1cm} for all \( p \in P_{N-1}. \)

Thus,

\[ \| f_k \|_H^2 \leq 2(\| a_k^\# \|_\infty \| \tilde{f} \|_H^2 + \tilde{c}_k \| p \|_H^2) \]
\[ \leq c_k (\| \tilde{f} \|_H^2 + \| p \|_H^2) \]
\[ = c_k \| f \|_H^2, \]

where \( c_k = 2 \max(\| a_k^\# \|_\infty, \tilde{c}_k) \). This verifies (3.11) and thus completes the proof. \( \square \)

**Remark 3.12** In the spirit of the above proof, one can write

\[ f(z) = (z - \xi_k)^m a_k^\dagger(z) \tilde{f}(z) + p(z), \]
where
\[ a_k^*(z) = \prod_{j \neq k} (z - \xi_j)^{m_j}, \]
to prove that each \( f \in \mathcal{H}(b) \) admits a boundary limit at \( \xi_k \) in the approach regions
\[
\left\{ z \in \mathbb{D} : \frac{|z - \xi_k|}{1 - |z|} \leq c \right\}, \quad c > 1,
\]
which are larger than the standard nontangential (Stolz) regions
\[
\left\{ z \in \mathbb{D} : \frac{|z - \xi_k|}{1 - |z|} \leq c \right\}, \quad c > 1.
\]

Let
\[ \mathcal{M}(\mathcal{H}(b)) := \{ \varphi \in \mathcal{H}(b) : \varphi \mathcal{H}(b) \subseteq \mathcal{H}(b) \} \]
denote the multiplier algebra of \( \mathcal{H}(b) \). Standard results for multiplier algebras, true for any reproducing kernel Hilbert space of analytic functions on \( \mathbb{D} \), say that if \( \varphi \in \mathcal{M}(\mathcal{H}(b)) \), then \( \varphi \in H^\infty \), and the multiplication operator \( M_{\varphi} f = \varphi f \) is bounded on \( \mathcal{H}(b) \) and satisfies
\[
M_{\varphi}^* k_\lambda^b = \overline{\varphi(\lambda)} k_\lambda^b \quad \text{for all } \lambda \in \mathbb{D}. \tag{3.13}
\]

For general \( \mathcal{H}(b) \) spaces, the multiplier algebra \( \mathcal{M}(\mathcal{H}(b)) \) lacks a complete description [5, 18, 19]. In our case, where \( b \in \text{ball}(H^\infty) \) is rational and not a finite Blaschke product, things again become much more explicit. Indeed, [13, Proposition 3.1] says that
\[
\mathcal{M}(\mathcal{H}(b)) = \mathcal{H}(b) \cap H^\infty, \tag{3.14}
\]
and (3.4) implies that \( \varphi \in \mathcal{M}(\mathcal{H}(b)) \) if and only if
\[
\varphi = a_1 \tilde{\varphi} + r, \quad \text{where } \tilde{\varphi} \in H^2, r \in \mathcal{P}_{N-1}, \text{ and } a_1 \tilde{\varphi} \in H^\infty.
\]

In particular, it follows easily from (3.5) that every polynomial is a multiplier of \( \mathcal{H}(b) \) (this is also a consequence of more general facts from [26, Ch. IV]). Since all norms on the finite dimensional space \( \mathcal{P}_{N-1} \) are equivalent, we fix a \( C_1 > 0 \) that satisfies
\[
\max\{ \| p \|, \| p \|_{\mathcal{M}(\mathcal{H}(b))} \} \leq C_1 \| p \|_b = C_1 \| p \|_{H^2}, \quad p \in \mathcal{P}_{N-1}. \tag{3.15}
\]
4 A Corona Theorem for de Branges–Rovnyak Spaces

Our corona theorem for $\mathcal{M}(\mathcal{H}(b))$ will be stated in terms of column multipliers. For a sequence $\Phi = (\varphi_j)_{j \geq 1}$ of functions in $\mathcal{M}(\mathcal{H}(b))$, define the column multiplier

$$M_\Phi : \mathcal{H}(b) \to \bigoplus_{j=1}^{\infty} \mathcal{H}(b), \quad M_\Phi f = (\varphi_j f)_{j \geq 1}. \quad (4.2)$$

When $M_\Phi$ is bounded, its adjoint is given by $M_\Phi^* = (M_{\varphi_1}^*, M_{\varphi_2}^*, \ldots)$. As is standard,

$$\bigoplus_{j=1}^{\infty} \mathcal{H}(b) := \{(f_j)_{j \geq 1} : f_j \in \mathcal{H}(b), \sum_{j=1}^{\infty} \|f_j\|^2_b < \infty\},$$

with

$$\|(f_j)_{j \geq 1}\|_{\bigoplus_{j=1}^{\infty} \mathcal{H}(b)} := \left(\sum_{j=1}^{\infty} \|f_j\|^2_b\right)^{\frac{1}{2}}$$

(recall the norm $\|\cdot\|_b$ on $\mathcal{H}(b)$ from (3.6)). The next lemma generalizes [20, Lemma 3.2.3].

**Lemma 4.2** $M_\Phi$ is a bounded (column) operator if and only if

(a) $C_2 := \left(\sum_{j=1}^{\infty} \|\varphi_j\|^2_b\right)^{\frac{1}{2}} < \infty$, and

(b) $C_3 := \sup_{z \in \mathbb{D}} \left(\sum_{j=1}^{\infty} |\varphi_j(z)|^2\right)^{\frac{1}{2}} < \infty$.

Furthermore, $\max(C_2, C_3) \leq \|M_\Phi\| \leq \sqrt{2} \max(C_3, C_1 C_2)$, where $C_1$ was defined in (3.15).

**Proof** Suppose that $M_\Phi$ is a bounded operator. Since $1 \in \mathcal{H}(b)$ and (3.6) shows that $\|1\|_b = 1$, we conclude that

$$C_2^2 = \sum_{j=1}^{\infty} \|\varphi_j\|^2_b = \|M_\Phi 1\|^2_{\bigoplus_{j=1}^{\infty} \mathcal{H}(b)} \leq \|M_\Phi\|^2 \cdot \|1\|^2_b = \|M_\Phi\|^2.$$ 

This proves (a) and that $C_2 \leq \|M_\Phi\|$. To prove (b), let $N \in \mathbb{N}$ and $(\gamma_j)_{j \geq 1}$ be a complex sequence such that $\gamma_j = 0$ when $j \geq N + 1$. For every $z \in \mathbb{D}$ and $N \in \mathbb{N}$, it follows from (3.13) that

$$M_\Phi^*((\gamma_j k^b_z)_{j \geq 1}) = \sum_{j=1}^{N} M_{\varphi_j}^*(\gamma_j k^b_z) = \left(\sum_{j=1}^{N} \gamma_j \varphi_j(z)\right) k^b_z.$$
and so

\[ \left| \sum_{j=1}^{N} \gamma_j \overline{\varphi}_j(z) \right| k_z^b \leq \| M_\Phi^* \| \left\| (\gamma_j k_z^b)_{j \geq 1} \right\|_{\mathcal{H}(b)} \]

\[ = \| M_\Phi \| \left( \sum_{j=1}^{N} |\gamma_j|^2 \right)^{\frac{1}{2}} k_z^b \|_{b} . \]

Therefore,

\[ \left| \sum_{j=1}^{N} \gamma_j \overline{\varphi}_j(z) \right| \leq \| M_\Phi \| \left( \sum_{j=1}^{N} |\gamma_j|^2 \right)^{\frac{1}{2}} \text{ for all } N \in \mathbb{N}. \]

Since the inequality above is true for any \( (\gamma_j)_{j \geq 1} \), one can pick \( \gamma_j = \varphi_j(z) \) to obtain

\[ \sum_{j=1}^{N} |\varphi_j(z)|^2 \leq \| M_\Phi \|^2 \text{ for all } z \in \mathbb{D}. \]

The inequality above is true for all \( N \), which proves (b) and that \( C_3 \leq \| M_\Phi \| \).

Conversely, assume that conditions (a) and (b) are satisfied. For any \( f \in \mathcal{H}(b) \) we have

\[ \| M_\Phi f \|^2_{\mathcal{H}(b)} = \sum_{j=1}^{\infty} \| \varphi_j f \|^2_{b}. \]

From (3.5) we can write \( f = a_1 \tilde{f} + p \), with \( \tilde{f} \in H^2 \) and \( p \in \mathcal{P}_{N-1} \). The definition of the norm on \( \mathcal{H}(b) \) from (3.6) yields

\[ \| \varphi_j f \|^2_{b} = \| a_1 \tilde{f} \varphi_j + \varphi_j p \|^2_{b} \]

\[ \leq 2(\| a_1 \tilde{f} \varphi_j \|^2_{b} + \| \varphi_j p \|^2_{b}) \]

\[ = 2(\| \tilde{f} \varphi_j \|^2_{H^2} + \| \varphi_j p \|^2_{b}). \]

Hence,

\[ \| M_\Phi f \|^2_{\mathcal{H}(b)} \leq 2 \left( \sum_{j=1}^{\infty} \| \tilde{f} \varphi_j \|^2_{H^2} + \sum_{j=1}^{\infty} \| \varphi_j p \|^2_{b} \right). \quad (4.3) \]

To estimate the first sum on the right hand side of (4.3), we can use (3.2) and Fubini’s theorem to obtain
\[
\sum_{j=1}^{\infty} \| \tilde{f} \varphi_j \|_{L^2}^2 = \sum_{j=1}^{\infty} \int_{T} |\varphi_j(\xi)|^2 |\tilde{f}(\xi)|^2 \, dm(\xi) \\
= \int_{T} \sum_{j=1}^{\infty} |\varphi_j(\xi)|^2 |\tilde{f}(\xi)|^2 \, dm(\xi).
\] (4.4)

From the facts that \( \varphi_j(r \xi) \to \varphi_j(\xi) \) for almost every \( \xi \in \mathbb{T} \) as \( r \to 1^- \), and

\[
\sum_{j=1}^{N} |\varphi_j(r \xi)|^2 \leq C_3^2
\]
for all \( N \in \mathbb{N} \),

one sees that

\[
\sum_{j=1}^{N} |\varphi_j(\xi)|^2 \leq C_3^2
\]
for almost every \( \xi \in \mathbb{T} \) and every \( N \in \mathbb{N} \).

Now let \( N \to \infty \) to conclude that

\[
\sum_{j=1}^{\infty} |\varphi_j(\xi)|^2 \leq C_3^2
\]
for almost every \( \xi \in \mathbb{T} \).

Thus, continuing the estimate from (4.4), we obtain

\[
\sum_{j=1}^{\infty} \| \tilde{f} \varphi_j \|_{L^2}^2 \leq C_3^2 \int_{T} |\tilde{f}(\xi)|^2 \, dm(\xi) = C_3^2 \| \tilde{f} \|_{L^2}^2.
\]

To estimate the second term on the right hand side of (4.3), observe that

\[
\| \varphi_j p \|_b \leq \| p \|_{\mathfrak{H}(b)} \| \varphi_j \|_b \leq C_1 \| p \|_{L^2} \| \varphi_j \|_b,
\]
where \( C_1 \) is defined by (3.15). Therefore,

\[
\sum_{j=1}^{\infty} \| \varphi_j p \|_b^2 \leq C_1^2 \| p \|_{L^2}^2 \left( \sum_{j=1}^{\infty} \| \varphi_j \|_b^2 \right) = C_1^2 C_2^2 \| p \|_{L^2}^2.
\]

It follows from (4.3) that

\[
\| M_{\Phi} f \|_{\mathfrak{H}(b)}^2 \leq 2 \left( C_3^2 \| \tilde{f} \|_{L^2}^2 + C_1^2 C_2^2 \| p \|_{L^2}^2 \right).
\]

Since \( \| f \|_b^2 = \| \tilde{f} \|_{L^2}^2 + \| p \|_{L^2}^2 \) (see (3.6)), we see that

\[
\| M_{\Phi} f \|_{\mathfrak{H}(b)}^2 \leq 2 \max(C_3^2, C_1^2 C_2^2) \| f \|_b^2.
\]
Therefore, $M_\Phi$ is bounded and $\|M_\Phi\| \leq \sqrt{2} \max(C_3, C_1 C_2)$, which finishes the proof of the lemma. \qed

Here is the second main result of this paper, a corona theorem for $M(H(b))$. We remind the reader that for rational $b \in \text{ball}(H^\infty)$ (and not a finite Blaschke product), there exists a Pythagorean mate $a$ to which we can associate the polynomial $a_1(z) = \prod_{j=1}^n (z - \xi_j)^{m_j}$ as explained in (5.4).

**Theorem 4.5** Let $b \in \text{ball}(H^\infty)$ be rational, but not a finite Blaschke product. Suppose that $\Phi = (\varphi_j)_{j \geq 1}$ is a sequence in $M(H(b))$ that satisfies the conditions

(i) $\|M_\Phi\| \leq 1$, and
(ii) $0 < \delta^2 \leq \sum_{j=1}^\infty |\varphi_j(z)|^2$ for all $z \in \mathbb{D}$.

Then there is a sequence $B = (b_j)_{j \geq 1}$ in $M(H(b))$ such that

(a) $\sum_{j=1}^\infty \varphi_j(z)b_j(z) = 1$ for all $z \in \mathbb{D}$, and

(b) $\|MB\| \leq \frac{C}{\delta \max m_j} \left( 1 + \frac{1}{\delta^2} \log \frac{1}{\delta} \right)$,

where $C > 0$ depends only on $b$.

**Proof** We begin by decomposing each $\varphi_j \in M(H(b))$ as

$$\varphi_j = a_1 \tilde{\varphi}_j + p_j, \quad \tilde{\varphi}_j \in H^2, \quad p_j \in P_{N-1},$$

where $a_1$ is the polynomial associated with the Pythagorean mate of $b$ as defined by (3.3). Furthermore,

$$a_1 \tilde{\varphi}_j \in H^\infty \quad \text{for all } j \geq 1.$$

Conditions (i), (ii), and Lemma 4.2 imply that

$$\delta^2 \leq \sum_{j=1}^\infty |\varphi_j(z)|^2 \leq 1 \quad \text{for all } z \in \mathbb{D}. \quad (4.8)$$

Apply Lemma 3.9 to see that for fixed $1 \leq k \leq n$ there is a $c_k > 0$, depending only on $b$, such that for every $\eta > 0$, every $z \in \mathbb{D}$, and every $j \geq 1$,

$$|\varphi_j(z)|^2 \leq (1 + \eta)|\varphi_j(\xi_k)|^2 + \left( 1 + \frac{1}{\eta} \right)c_k \frac{|z - \xi_k|^2}{1 - |z|^2} \|\varphi_j\|^2_b.$$
By summing over $j$ in the previous inequality, it follows from Lemma 4.2 that

$$
\delta^2 \leq (1 + \eta) \sum_{j=1}^{\infty} |\varphi_j(\xi_k)|^2 + \left(1 + \frac{1}{\eta}\right)c_k \frac{|z - \xi_k|^2}{1 - |z|^2} \sum_{j=1}^{\infty} \|\varphi_j\|_B^2
$$

$$
\leq (1 + \eta) \sum_{j=1}^{\infty} |\varphi_j(\xi_k)|^2 + \left(1 + \frac{1}{\eta}\right)c_k \frac{|z - \xi_k|^2}{1 - |z|^2}.
$$

In the above, note the use of the fact that $\|M_{\Phi}\| \leq 1$ and so $C_2 \leq 1$. Now let $z \to \xi_k$ radially to see that the second term above goes to zero and thus

$$
\delta^2 \leq (1 + \eta) \sum_{j=1}^{\infty} |\varphi_j(\xi_k)|^2.
$$

Letting $\eta \to 0^+$ yields

$$
\delta^2 \leq \sum_{j=1}^{\infty} |\varphi_j(\xi_k)|^2.
$$

The estimate in (4.8) and Fatou’s lemma yield

$$
\sum_{j=1}^{\infty} |\varphi_j(\xi_k)|^2 \leq 1.
$$

Finally, use $p_j(\xi_k) = \varphi_j(\xi_k)$ to obtain

$$
\delta^2 \leq \sum_{j=1}^{\infty} |p_j(\xi_k)|^2 \leq 1.
$$

In particular, for every $1 \leq k \leq n$ there exists an $\ell_k \in \mathbb{N}$ such that

$$
\sum_{j=\ell_k+1}^{\infty} |p_j(\xi_k)|^2 \leq \frac{\delta^2}{2}.
$$

Define $L = \max\{\ell_k : 1 \leq k \leq n\}$. Then for every $1 \leq k \leq n$, we have

$$
\sum_{j=1}^{L} |p_j(\xi_k)|^2 = \sum_{j=1}^{\infty} |p_j(\xi_k)|^2 - \sum_{j=L+1}^{\infty} |p_j(\xi_k)|^2
$$

\[\geq \delta^2 - \sum_{j=\ell_k+1}^{\infty} |p_j(\xi_k)|^2\]

\[\geq \delta^2 - \frac{\delta^2}{2} = \frac{\delta^2}{2} > 0.\]
On the other hand, by assumption (i) and Lemma 4.2,
\[ \sum_{j=1}^{\infty} \| \varphi_j \|_b^2 = C_2^2 \leq \| M \varphi \|_b^2 \leq 1, \]
whence the norm (3.6) implies that \( \sum_{j=1}^{\infty} \| p_j \|_{H^2} \leq 1 \). Using the constant \( C_1 \) defined in (3.15), we see that
\[ \sum_{j=1}^{L} \| p_j \|_{H^2}^2 \leq C_2^2 \sum_{j=1}^{L} \| p_j \|_{H^2} \leq C_1^2, \]
for some \( C > 0 \), depending only on \( a_1 \) and hence depending only on \( b \). (Since all norms on \( P_{N-1} \) are equivalent, we may replace the norm used in (2.8) by the \( H^2 \)-norm). We set \( q_k \equiv 0 \) for all \( k \geq L + 1 \).

By Tolokonnikov’s theorem [27] (mentioned in the introduction) and the estimate of Uchiyama [31], there is a universal \( C > 0 \) and a sequence \( (e_j)_{j \geq 1} \) in \( H^\infty \) such that
\[ \sum_{j=1}^{\infty} \varphi_j(z)e_j(z) = 1 \text{ for all } z \in \mathbb{D}, \]
\[ \left( \sup_{z \in \mathbb{D}} \sum_{j=1}^{\infty} |e_j(z)|^2 \right)^{\frac{1}{2}} \leq C \frac{1}{\delta^2} \log \frac{1}{\delta}. \]
As in the proof of Lemma 4.2, it follows that
\[ \left( \sum_{j=1}^{\infty} |e_j(\xi)|^2 \right)^{\frac{1}{2}} \leq C \frac{1}{\delta^2} \log \frac{1}{\delta} \text{ for a.e. } \xi \in \mathbb{T}. \]
For each \( j \geq 1 \) define
\[ b_j := q_j + \left( 1 - \sum_{k=1}^{L} \varphi_k q_k \right) e_j. \]
First we check that \( b_j \in \mathcal{M}(\mathcal{H}(b)) \) for all \( j \geq 1 \). Using (4.6) and (4.9), one obtains

\[
(1 - \sum_{k=1}^{L} \varphi_k q_k) e_j = (1 - \sum_{k=1}^{L} q_k (a_1 \tilde{\varphi}_k + p_k)) e_j \\
= (1 - \sum_{k=1}^{L} q_k p_k - a_1 \sum_{k=1}^{L} q_k \tilde{\varphi}_k) e_j \\
= a_1 (q - \sum_{k=1}^{L} q_k \tilde{\varphi}_k) e_j.
\]

Therefore, (4.12) can be written as

\[
b_j = a_1 (q - \sum_{k=1}^{L} q_k \tilde{\varphi}_k) e_j + q_j. \tag{4.13}
\]

Since \( q_j \in \mathcal{P}_{N-1} \) and \( q - \sum_{k=1}^{L} q_k \tilde{\varphi}_k \in H^2 \), this is precisely the decomposition of \( b_j \) from (3.5). Therefore, \( b_j \in \mathcal{H}(b) \) and it follows from (4.7) that \( b_j \in H^\infty \). Thus, from (3.14), \( b_j \in \mathcal{H}(b) \cap H^\infty = \mathcal{M}(\mathcal{H}(b)) \).

Second, we observe that

\[
\sum_{j=1}^{\infty} b_j \varphi_j = \sum_{j=1}^{\infty} \left( q_j + (1 - \sum_{k=1}^{L} \varphi_k q_k) e_j \right) \varphi_j \\
= \sum_{j=1}^{\infty} \varphi_j q_j + \left(1 - \sum_{k=1}^{L} \varphi_k q_k \right) \sum_{j=1}^{\infty} e_j \varphi_j \\
= \sum_{j=1}^{\infty} \varphi_j q_j + \left(1 - \sum_{k=1}^{L} \varphi_k q_k \right) \cdot 1 \quad (\text{by (4.10)}) \\
= \sum_{j=1}^{L} \varphi_j q_j + \left(1 - \sum_{k=1}^{L} \varphi_k q_k \right) \\
= 1.
\]

Thus (a) is proved.

In order to prove (b) of Theorem 4.5, we need to show that \( \mathbf{M}_B \) satisfies inequalities (a) and (b) in Lemma 4.2. Apply (3.6) and (4.13) to obtain

\[
\sum_{j=1}^{\infty} \| b_j \|_B^2 = \sum_{j=1}^{\infty} \| q_j \|_{H^2}^2 + \sum_{j=1}^{\infty} \left\| \left( q - \sum_{k=1}^{L} q_k \tilde{\varphi}_k \right) e_j \right\|_{H^2}^2. \tag{4.14}
\]
By (4.9), the first term on the right hand side of the above is bounded by \( \frac{C}{\delta^2 \max m_j} \). To bound the second term, we have

\[
\sum_{j=1}^\infty \| q - \sum_{k=1}^L q_k \tilde{\varphi}_k \|_{H^2}^2
\]

\[
= \int_T \sum_{j=1}^\infty |q(\zeta) - \sum_{k=1}^L q_k(\zeta) \tilde{\varphi}_k(\zeta)|^2 |e_j(\zeta)|^2 \, dm(\zeta)
\]

\[
= \int_T |q(\zeta) - \sum_{k=1}^L q_k(\zeta) \tilde{\varphi}_k(\zeta)|^2 \left( \sum_{j=1}^\infty |e_j(\zeta)|^2 \right) \, dm(\zeta)
\]

\[
\leq \left( \frac{C}{\delta^2 \log \frac{1}{\delta}} \right)^2 \int_T |q(\zeta) - \sum_{k=1}^L q_k(\zeta) \tilde{\varphi}_k(\zeta)|^2 \, dm(\zeta) \quad \text{(by (4.11))}
\]

\[
= \left( \frac{C}{\delta^2 \log \frac{1}{\delta}} \right)^2 \| q - \sum_{k=1}^L q_k \tilde{\varphi}_k \|_{H^2}^2
\]

\[
\leq 2 \left( \frac{C}{\delta^2 \log \frac{1}{\delta}} \right)^2 \left( \| q \|_{H^2}^2 + \left\| \sum_{k=1}^L q_k \tilde{\varphi}_k \right\|_{H^2}^2 \right).
\]

Again (4.9) yields \( \| q \|_{H^2}^2 \leq \frac{C}{\delta^2 \max m_j} \). On the other hand, (3.6) implies that \( \| \tilde{\varphi}_k \|_{H^2} \leq \| \varphi_k \|_b \), which, together with the Cauchy–Schwarz inequality, yield

\[
\left\| \sum_{k=1}^L q_k \tilde{\varphi}_k \right\|_{H^2}^2 \leq \left( \sum_{k=1}^L \| q_k \tilde{\varphi}_k \|_{H^2}^2 \right) \leq \left( \sum_{k=1}^L \| q_k \|_{H^2} \| \varphi_k \|_b \right)^2 \leq C \left( \sum_{k=1}^L \| q_k \|_{H^2} \| \varphi_k \|_b \right)^2 \leq C \left( \sum_{k=1}^L \| q_k \|_{H^2}^2 \right) \left( \sum_{k=1}^L \| \varphi_k \|_b^2 \right).
\]

Using (4.9) once more, we see that the first factor in the last formula is bounded above by \( \frac{C}{\delta^2 \max m_j} \), while, by condition (i) in the statement of the theorem and Lemma 4.2, the second factor is bounded above by 1. Consequently,

\[
\sum_{j=1}^\infty \left\| q - \sum_{k=1}^L q_k \tilde{\varphi}_k \right\|_{H^2}^2 \leq \frac{C}{\delta^2 \max m_j} \left( \frac{1}{\delta^2} \log \frac{1}{\delta} \right)^2.
\]
Returning to (4.14), it follows that

$$\sum_{j=1}^{\infty} \|b_j\|_b^2 \leq \frac{C}{\delta^2 \max m_j} \left(1 + \frac{1}{\delta^2} \log \frac{1}{\delta}\right)^2.$$  \hspace{1cm} (4.15)

proving that the inequality (a) in Lemma 4.2 is satisfied.

In order to prove (b), fix \( z \in \mathbb{D} \). From the definition of \( b_j \) from (4.12), we see that

\[ |b_j(z)|^2 \leq 2 \left( |q_j(z)|^2 + |1 - \sum_{k=1}^{L} \varphi_k(z)q_k(z)|^2 |e_j(z)|^2 \right), \]

whence

$$\sum_{j=1}^{\infty} |b_j(z)|^2 \leq 2 \sum_{j=1}^{\infty} |q_j(z)|^2 + 2 \left| 1 - \sum_{k=1}^{L} \varphi_k(z)q_k(z) \right|^2 \left( \sum_{j=1}^{\infty} |e_j(z)|^2 \right).$$  \hspace{1cm} (4.16)

From (4.10) we obtain the estimate

$$\sum_{j=1}^{\infty} |e_j(z)|^2 \leq \left( \frac{C}{\delta^2} \log \frac{1}{\delta} \right)^2,$$

while from (4.9) we have the estimate

$$\sum_{j=1}^{\infty} |q_j(z)|^2 = \sum_{j=1}^{L} |q_j(z)|^2 \leq \sum_{j=1}^{L} \|q_j\|_\infty^2 \leq C \sum_{j=1}^{L} \|q_j\|_{H^2}^2 \leq \frac{C}{\delta^2 \max m_j}$$

(note that \( q_j \equiv 0 \) for all \( j > L \)). Finally, using condition (b) in Lemma 4.2 (valid for \( M_\Phi \) by assumption) as well as (4.9), we see that

\[
\left| 1 - \sum_{k=1}^{L} \varphi_k(z)q_k(z) \right| \leq 1 + \sum_{k=1}^{L} |\varphi_k(z)q_k(z)| \\
\leq 1 + \left( \sum_{k=1}^{L} |\varphi_k(z)|^2 \right)^{\frac{1}{2}} \left( \sum_{k=1}^{L} |q_k(z)|^2 \right)^{\frac{1}{2}} \\
\leq 1 + \left( \sum_{k=1}^{L} |\varphi_k(z)|^2 \right)^{\frac{1}{2}} \left( \sum_{k=1}^{L} \|q_k\|_{\infty}^2 \right)^{\frac{1}{2}} \\
\leq 1 + C \left( \sum_{k=1}^{L} |\varphi_k(z)|^2 \right)^{\frac{1}{2}} \left( \sum_{k=1}^{L} \|q_k\|_{H^2}^2 \right)^{\frac{1}{2}} \\
\leq 1 + \frac{C}{\delta \max m_j}.\]
Gathering up all the last estimates and plugging them into (4.16) yields
\[ \sum_{j=1}^{\infty} |b_j(z)|^2 \leq \frac{C}{\delta^2 \max m_j} \left( 1 + \frac{1}{\delta^2} \log \frac{1}{\delta} \right)^2. \]  
(4.17)

Together, (4.15) and (4.17) show that $M_B$ satisfies conditions (a) and (b) in Lemma 4.2. It is therefore a bounded operator whose norm satisfies
\[ \|M_B\| \leq \frac{C}{\delta \max m_j} \left( 1 + \frac{1}{\delta^2} \log \frac{1}{\delta} \right), \]  
(4.18)

for some constant $C > 0$, which ends the proof of the theorem.

If $\|b\|_\infty < 1$, it is known that $\mathcal{H}(b) = H^2$, with an equivalent norm. Furthermore, in this case the Pythagorean mate $a$ will have no zeros on $\mathbb{T}$ and so the exponent on $\delta$ in (4.18) will be $\max m_j = 0$. This corresponds to the estimate
\[ \|M_B\| \leq C \left( 1 + \frac{1}{\delta^2} \log \frac{1}{\delta} \right) \]

which is the Uchiyama estimate from (1.7). Of course, Uchiyama’s result was used in our proof.

5 Final Remarks

As noted in the introduction, Theorem 4.5 is related to some of the results in [20, 21]. Here is how one makes the connection. If $\mu$ is a finite positive Borel measure on $\mathbb{T}$ and $P_\mu$ is its Poisson integral
\[ P_\mu(z) := \int_{\mathbb{T}} \frac{1 - |z|^2}{|z - \xi|^2} d\mu(\xi), \quad z \in \mathbb{D}, \]

the harmonically weighted Dirichlet space $\mathcal{D}_\mu$, introduced by Richter [24], is the space of $f \in H^2$ satisfying
\[ \int_{\mathbb{D}} |f'(z)|^2 P_\mu(z) dA(z) < \infty, \]

where $dA$ is area measure. In [3], Costara and Ransford proved that a de Branges–Rovnyak space $\mathcal{H}(b)$, where $b \in \text{ball}(H^\infty)$ is rational and not a finite Blaschke product, coincides (with equivalent norms) with a $\mathcal{D}_\mu$ space if and only if the zeros on $\mathbb{T}$ of the Pythagorean mate $a$ of $b$ are simple, that is, with our notation from (3.3), when all the zeros of $a_1$ are simple. If this happens, then the support of $\mu$ is precisely this set of (simple) zeros.
For this class of $D_\mu$ spaces, Luo [20] proved a corona theorem, including estimates of the norm of the solutions. This turns out to be, when translated in the context of de Branges spaces, the particular case of Theorem 4.5 when $m_j = 1$ for all $1 \leq j \leq n$. One sees that our Theorem 4.5 covers the general case when the roots of $a$ on $\mathbb{T}$ have arbitrary multiplicities, where $\mathcal{H}(b)$ no longer coincides with a $D_\mu$ space. It should also be noted that part of the argument in the proof of Theorem 4.5 is similar to an argument from [20, Theorem 3.2.4].

Finally, note that in [21] Luo obtained a general corona theorem for harmonically weighted Dirichlet spaces $D_\mu$.

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Declarations

Conflict of interest There is no conflict of interest.

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