DELTA SHOCK WAVE AND WAVE INTERACTIONS IN A THIN FILM OF A PERFECTLY SOLUBLE ANTI-SURFACTANT SOLUTION

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Abstract. We study the interactions between classical elementary waves and delta shock wave in quasilinear hyperbolic system of conservation laws. This governing system describes a thin film of a perfectly soluble anti-surfactant solution in the limit of large capillary and Péclet numbers. This system is one of the example of non-strictly hyperbolic system whose Riemann solution consists of delta shock wave as well as classical elementary waves such as shock waves, rarefaction waves and contact discontinuities. The global structure of the perturbed Riemann solutions are constructed and analyzed case by case when delta shock wave is involved.

1. Introduction. When the molecules of a dissolved solute are preferentially expelled from the free surface of a solvent, the surface tension of the solution increases. Such solutes, which act in the opposite manner to better-known surfactants, may conveniently be referred to as “anti-surfactants”. Examples of anti-surfactants include many salts, such as sodium chloride, i.e. common table salt, when added to water [19, 18], water when added to short-chain alcohols [11], and certain resins that are included in solvent-based paints [29, 12]. Here, we consider the Riemann problem for a thin film of a perfectly soluble anti-surfactant solution [6] in the limit of large capillary and Péclet numbers and the solution of which belongs to some measure space

\[
\begin{cases}
    h_t + \frac{1}{2}(h^2b)_x = 0, \\
    b_t + \frac{1}{2}(hb^2)_x = 0,
\end{cases}
\quad -\infty < x < \infty, \quad 0 < t < \infty,
\]

with initial data

\[
(h, b) (x, 0) = \begin{cases}
    (h_l, b_l), & x < 0, \\
    (h_r, b_r), & x > 0,
\end{cases}
\]

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in which \( t \) and \( x \) represent time and space coordinates, respectively and \( h(x,t) \) and \( b(x,t) \) are the film thickness and gradient of concentration of the solute and \( h_i, r \geq 0 \), \( b_i, r > 0 \) are given constants.

The system (1) can be derived from the model proposed by Conn et al. [6, 5]. They considered a thin, two-dimensional film of a perfectly soluble anti-surfactant solution of constant viscosity \( \mu^* \) and density \( \rho^* \) lying on top of a flat substrate. (Stars denote dimensional quantities, but dimensionless quantities will be unadorned.) They assumed that the typical depth of the film \( H^* \) is much smaller than the horizontal length scale \( L^* \), i.e., that the aspect ratio \( \epsilon = H^*/L^* \ll 1 \) of the film is small.

Since the anti-surfactant solution is perfectly soluble, the surface concentration of solute is identically zero, and, since the film is thin, gravity effects are neglected. The dimensionless film thickness \( h = h^*/H^* \) is scaled by \( H^* \), and the bulk concentration of solute \( c = c^*/C^* \) is scaled by the typical bulk concentration \( C^* \). The velocity of the fluid is scaled to reflect the fact that flow is driven by gradients in surface tension due to gradients in the concentration of solute, i.e., by the Marangoni effect.

Adopting the natural Cartesian coordinate system and following the reduction of the model proposed by Conn et al. [4], the governing equations for \( h(x,t) \) and \( c(x,t) \) are

\[
\begin{align*}
\frac{\partial h}{\partial t} + \frac{1}{2} \frac{\partial}{\partial x} \left( h^2 \frac{\partial c}{\partial x} \right) &= 0, \\
\frac{\partial c}{\partial t} + \frac{1}{2} h \left( \frac{\partial c}{\partial x} \right)^2 &= 0,
\end{align*}
\]

(3)

where \( h \) and \( c \) are film thickness and bulk concentration of solute, respectively. Also, in equation (3)

\[ Ca = \frac{R^* \eta^* T^* C^*}{\epsilon^2 \sigma^*_{solv}} \text{ and } Pe = \frac{H^* R^* \eta^* T^* C^*}{\mu^* D^*} \]

are a capillary number and a Péclet number, respectively, in which \( R^* \) is the ideal gas constant, \( T^* \) is the (constant) temperature, \( \eta^* \) is the notional thickness of the free surface which is taken to be of the order of a few angstroms [5, 1], \( \sigma^*_{solv} \) is the surface tension of the pure solvent, and \( D^* \) is the diffusivity of the solute.

If we neglect the capillarity and diffusion effects, i.e., as \( Ca \to \infty \) and \( Pe \to \infty \), then the problem of full thin film (3) simplifies to

\[
\begin{align*}
\frac{\partial h}{\partial t} + \frac{1}{2} \frac{\partial}{\partial x} \left( h^2 \frac{\partial c}{\partial x} \right) &= 0, \\
\frac{\partial c}{\partial t} + \frac{1}{2} h \left( \frac{\partial c}{\partial x} \right)^2 &= 0,
\end{align*}
\]

(4a)

(4b)

From above we can see that equation (4a) and (4b) represents conservation of mass of fluid and solute, respectively. Differentiating (4b) with respect to \( x \) and substitute \( \frac{\partial c}{\partial x} = b \), we obtain the system (1).

Conn et al. [6] discussed exact solution of the Riemann problem of (1) and which contains classical waves only. It can be seen that (1) belongs to Temple class [28], i.e., the rarefaction curves coincide with the shock curve in the phase plane. This system is an example of a symmetric Keyfitz-Kranzer system [16]. The Riemann problem of (1) cannot be solved for all possible Riemann initial data in terms of classical elementary waves. There are some nonclassical situations [13, 23, 22] when for few cases of initial data the Riemann problem fails to contain a weak \( L^\infty \)-solution. For that when we solve the Cauchy problem in this nonclassical situation,
we have to introduce delta shock type singularities, which are solutions of the system of conservation laws.

A delta shock wave is a generalization of an ordinary shock wave; speaking informally, it consists of a discontinuity line \( x = x(t) \) plus a distributed Dirac delta function with the discontinuity line as its support. It is over-compressive which means the number of characteristics entering the discontinuity line of the delta shock is more compared to an ordinary shock. For more details on delta shock, we refer to [10, 17, 24, 21, 2]. Various methods for deriving delta shock solutions are available in the literature. The weak asymptotic method was used by Danilov and Shelkovich [8, 9] in their study of the delta shock wave type solution. Vanishing viscosity method for transonic flow and Riemann solution for a class of hyperbolic systems of conservation laws have been studied in [3, 7]. Delta shock wave as limit of vanishing viscosity for hyperbolic system has been studied by Tan et al. [27]. Stability of the Riemann solution for chromatography equations has been discussed by Sun [26] using split delta function.

In the fundamental work of Conn et. al [5] they discussed the linear stability of fluids. Consequently, Conn et. al [6] derived a set of Riemann solutions to (1) for different initial data and discussed their properties. Recently, Minhajul et. al [20] discussed the interactions of classical elementary waves for the system (1). In contrast to [20], we derive the Riemann solution of (1)-(2) which consists of not only classical waves but also delta shock wave. Further, we discuss the wave interactions between delta shock wave and classical elementary waves. For wave interactions, we deal with Cauchy problem for (1) with initial profile having two jump discontinuities at \( -\varepsilon \) and \( \varepsilon \), in the following way

\[
(h, b)(x, 0) = \begin{cases}
(h_l, b_l), & -\infty < x < -\varepsilon, \\
(h_m, b_m), & -\varepsilon < x < \varepsilon, \\
(h_r, b_r), & \varepsilon < x < \infty,
\end{cases}
\]

where \( \varepsilon > 0 \) is arbitrarily small and \( h_{l,r,m} \geq 0 \) and \( b_{l,r,m} > 0 \) are constants. After perturbating Riemann initial data (2) we obtain initial data (5). We construct perturbed solution of the IVP (1) and (5) case by case.

We organize this article as follows. Section 2 deals with some preliminaries for the system (1) -(2) and the Riemann solution with different possible initial data. In section 3, interaction of delta shock with classical elementary waves are discussed for all the possible cases. We construct solution globally and delta shock strength is computed for each case. Finally, we made conclusions in section 4.

2. The Riemann problem. Here, we discuss some results on the Riemann solution of (1) and (2). The eigenvalues of the system (1) are \( \lambda_1 = \frac{hb}{3} \) and \( \lambda_2 = \frac{3hb}{2} \). So \( \lambda_1 < \lambda_2 \) when \( hb > 0 \) and \( \lambda_1 = \lambda_2 \) when \( hb = 0 \). Therefore (1) is strictly hyperbolic in the quarter \((h, b)\) state space and nonstrictly hyperbolic on \( h = 0 \) line. The corresponding right eigenvectors are \( \overrightarrow{e_1}=(h, -b)^t \) and \( \overrightarrow{e_2}=(h, b)^t \), respectively. Since \( \nabla \lambda_1 \cdot \overrightarrow{e_1} = 0 \) and \( \nabla \lambda_2 \cdot \overrightarrow{e_2} = 3hb \neq 0 \), the first characteristic field is always linearly degenerate and second one is genuinely nonlinear for \( hb 
eq 0 \). Therefore the waves corresponding to the first family represent contact discontinuities and rarefaction waves or shock waves for the second family. The Riemann invariants associated with the first and second characteristic fields are \( z_1 = hb \) and \( z_2 = \frac{h}{b} \), respectively.

Let \((h_l, b_l)\) be left and \((h, b)\) be right-hand states of either classical elementary waves or delta shock wave. Let us fix \((h_l, b_l)\) and calculate the state \((h, b)\) which is
connected on the right by either classical elementary waves or delta shock wave as given below (see Figure 1).

The curves associated with 2-rarefaction waves are:

\[ R(h_l, b_l) = \begin{cases} 
\frac{dx}{dt} = \lambda_2 = \frac{3hb}{2}, \\
\frac{h}{b} = \frac{h_l}{b_l}, \\
h_l < h, b_l < b,
\end{cases} \]

and the curves associated with 2-shock waves are:

\[ S(h_l, b_l) = \begin{cases} 
C = \frac{h^2b - h_l^2b_l}{2(h - h_l)}, \\
\frac{h}{b} = \frac{h_l}{b_l}, \\
h < h_l, b < b_l,
\end{cases} \]

where \( C \) is the shock speed.

The desirable states which connect \((h_l, b_l)\) by a contact discontinuity on right are given as follows:

\[ J(h_l, b_l) = \begin{cases} 
\mu = \frac{hb}{2} = \frac{h_l b_l}{2}, \\
hb = h_l b_l,
\end{cases} \]

where \( \mu \) is the contact discontinuity speed.

![Figure 1. Wave curves in \((h, b)\) phase plane](image)

Depending on initial data, there are three possible wave patterns for the Riemann solution of (1) and (2) which are described below.

Case i: If \(0 < h_l b_l < h_r b_r\), the Riemann solution consists of a contact discontinuity \(J\) and a rarefaction wave \(R\) (see Figure 2)

\[ (h, b)(x, t) = \begin{cases} 
(h_l, b_l), & x < \mu t, \\
(h_2, b_2), & \mu t < x < \frac{3h_l b_l}{2} t, \\
(h, b), & \frac{3h_l b_l}{2} t < x < \frac{3h_r b_r}{2} t, \\
(h_r, b_r), & x > \frac{3h_r b_r}{2} t,
\end{cases} \]
Figure 2. $J + R$ when $0 < h_l b_l < h_r b_r$

in which the state $(h, b)$ in $R$ is given by $(h, b) = \left( \frac{\sqrt{2x_l h_l}}{b_l}, \frac{\sqrt{2x_l b_l}}{h_l} \right)$ and intermediate state between $J$ and $R$ is $(h_2, b_2) = \left( \frac{h_l b_l b_r}{h_r}, \frac{h_l b_l b_r}{b_r} \right)$.

Case ii: If $0 < h_r b_r < h_l b_l$, the Riemann solution consists of a contact discontinuity $J$ and a shock wave $S$ (see Figure 3)

Figure 3. $J + S$ when $0 < h_r b_r < h_l b_l$

$$(h, b)(x, t) = \begin{cases} (h_l, b_l), & x < \mu t, \\ (h_s, b_s), & \mu t < x < Ct, \\ (h_r, b_r), & x > Ct, \end{cases}$$

where $(h_s, b_s) = \left( \sqrt{\frac{h_l b_l b_r}{b_r}}, \sqrt{\frac{h_l b_l b_r}{h_r}} \right)$ and propagation speed of shock is

$$C = \frac{h_r b_r \sqrt{h_r b_r - h_l b_l \sqrt{h_l b_l}}}{2(\sqrt{h_r b_r - \sqrt{h_l b_l}})}.$$

From above we can see that when $0 < h_r b_r < h_l b_l$, the Riemann solution of (1) and (2) is $J$ and $S$. The intermediate state between $J$ and $S$ is $(h_s, b_s) = \left( \sqrt{\frac{h_l b_l h_r}{b_r}}, \sqrt{\frac{h_l b_l b_r}{h_r}} \right)$. When $h_r \to 0^+$, we can see that $b_s = \sqrt{\frac{h_l b_l b_r}{h_r}} \to \infty$. At the same time, the shock wave speed $C = \frac{h_r b_r \sqrt{h_r b_r - h_l b_l \sqrt{h_l b_l}}}{2(\sqrt{h_r b_r - \sqrt{h_l b_l}})}$ tends to the speed of the contact discontinuity $\mu = \frac{h_l b_l}{2}$ as $h_r \to 0^+$, which indicates that $J$ and $S$ coincide to form a new type of nonlinear hyperbolic wave which is called delta shock wave $\delta S$ (see Figure 4). The total quantity of $b$ between $J$ and $S$ when $h_r \to 0^+$ is
Figure 4. $\delta S$ when $b_{l,r} > 0$, $h_l > 0$ and $h_r = 0$ given in the following way

$$\lim_{h_r \to 0^+} \int_{\mu - 0}^{C + 0} bd\xi = \lim_{h_r \to 0^+} \int_{\mu - 0}^{C + 0} \frac{h_l b_l b_r}{h_r} d\xi,$$

in which $\mu = \frac{h_l b_l}{2}$ and $C = \frac{h_r b_r \sqrt{h_r b_r} - b_l b_r \sqrt{h_l b_l}}{2(\sqrt{h_r b_r} - \sqrt{h_l b_l})}$.

By integrating, $-\xi b + \left(\frac{h b^2}{2}\right) = 0$ with respect to $\xi$ from $\mu - 0$ to $C + 0$, we get

$$0 = -\int_{\mu - 0}^{C + 0} \xi b d\xi + \int_{\mu - 0}^{C + 0} d\left(\frac{h b^2}{2}\right)$$

$$= -\xi b_{\mu - 0}^{C + 0} + \int_{\mu - 0}^{C + 0} bd\xi + \left(\frac{h b^2}{2}\right)_{\mu - 0}^{C + 0}.$$

From (7), we obtain

$$\lim_{h_r \to 0^+} \int_{\mu - 0}^{C + 0} bd\xi = \lim_{h_r \to 0^+} \left( C b_r - \mu b_l + \frac{h_l b_l^2}{2} - \frac{h_r b_r^2}{2} \right) = \frac{h_l b_l b_r}{2}.$$

Therefore when $h_r = 0$, the superposition of $J$ and $S$ leads to the singularity for $b$, on the line $x = \frac{h_l b_l}{2} t$, as a weighted Dirac delta function which is called delta shock wave.

For measure solution, we have the following definitions.

**Definition 2.1.** The two-dimensional weighted $\delta$-measure $\beta(s)\delta_\Gamma$ that has support on a smooth curve $\Gamma = \{(x(s), t(s)) : c < s < d\}$ can be defined by:

$$\langle \beta(s)\delta_\Gamma, \phi(x, t) \rangle = \int_c^d \beta(s)\phi(x(s), t(s)) ds,$$

for every test function $\phi \in C_0^\infty(\mathbb{R} \times \mathbb{R}_+)$.

Now, let us use the definition of delta shock wave solution which was introduced by Danilov and Shelkovich [8, 9] and improved by Kalisch and Mitrovic [14, 15] below.

Suppose that $\Gamma = \{\gamma_i | i \in I\}$ is a graph in the closed upper half-plane $\{(x, t) | x \in \mathbb{R}, t \geq 0\}$ which contains smooth arcs $\gamma_i$, where $i \in I$ and $I$ is the finite index set. Let $I_0$ be a subset of $I$ which contains all indices of arcs starting from the $x-$axis and $\Gamma_0 = \{x^0_j : j \in I_0\}$ is the set of initial points of the arcs $\gamma_j$ with $j \in I_0$. 

- With $b, h_l, h_r > 0$, we have $\delta S > 0$.
- With $b, h_l > 0, h_r = 0$, we have $\delta S$ increases by $\delta S = \frac{h_l b_l h_r}{2}$.
- With $h_l > 0, b, h_r = 0$, we have $\delta S$ decreases by $\delta S = \frac{h_l b_l h_r}{2}$.
- With $h_l = 0, b, h_r > 0$, we have $\delta S$ decreases by $\delta S = \frac{h_l b_l h_r}{2}$.
Definition 2.2. Let \((h, b)\) be a pair of distributions where \(b\) has the form
\[
b(x, t) = \tilde{b}(x, t) + \beta(x, t)\delta(\Gamma),
\]
in which \(h, \tilde{b} \in L^\infty(\mathbb{R} \times \mathbb{R}_+)\) and the singular part is given by
\[
\beta(x, t)\delta(\Gamma) = \sum_{i \in I} \beta_i(x, t)\delta(\gamma_i).
\]

Let us consider the delta shock wave type initial data
\[
(h, b)(x, 0) = (h_0(x), \tilde{b}_0(x) + \sum_{j \in J_0} \beta_j(x^0_j, 0)\delta(x - x^0_j)),
\]
in which \(h_0, \tilde{b}_0 \in L^\infty(\mathbb{R})\), then the pair of distributions \((h, b)\) are called as generalized
delta shock wave solution for (1) with the above delta shock type initial data if the
following integral identities
\[
\int_{-\infty}^{\infty} \int_{0}^{\infty} \left(h \phi_t + \frac{h^2 b}{2} \phi_x\right) dx dt + \int_{-\infty}^{\infty} h_0(x)\phi(x, 0)dx = 0,
\]
and
\[
\int_{-\infty}^{\infty} \int_{0}^{\infty} \left(\tilde{b} \phi_t + \frac{\tilde{b}^2}{2} \phi_x\right) dx dt + \sum_{i \in I} \int_{\gamma_i} \beta_i(x, t)\frac{\partial\phi(x, t)}{\partial l} dx dt + \int_{-\infty}^{\infty} \tilde{b}_0(x)\phi(x, 0)dx + \sum_{k \in K_0} \beta_k(x^0_k, 0)\phi(x^0_k, 0) = 0,
\]
hold for all test functions \(\phi \in C^0_0(\mathbb{R} \times \mathbb{R}_+)\), where \(\frac{\partial\phi(x, t)}{\partial l}\) is the tangential derivative
of \(\phi\) on the curve \(\gamma_i\) and \(\int_{\gamma_i} dl\) is the line integral over the arc \(\gamma_i\).

With the above definition, a family of \(\delta\)-measure solutions \((h, b)\) of (1) and (2)
with parameter \(\sigma\), when \(b_{l, r} > 0, h_l > 0\) and \(h_r = 0\), is written as:
\[
h(x, t) = h_l + |b| H(x - \sigma t), \quad b(x, t) = b_l + |b| H(x - \sigma t) + \beta(t)\delta_{\Gamma},
\]
where \(\Gamma = \{(\sigma t, t) : t \geq 0\}\), \(H(x)\) is Heaviside function and \(|h| = h(x(t) + 0) - h(x(t) - 0)\) denotes jump in \(h\) across the discontinuity \(x = x(t)\).

In fact, the \(\delta\)-measure solution \((h, b)\) constructed above to the Riemann problem
(1) and (2) must hold
\[
\langle b, \phi_t \rangle + \langle \frac{h^2 b}{2}, \phi_x \rangle = 0, \quad \langle b, \phi_t \rangle + \langle \frac{\tilde{b}^2}{2}, \phi_x \rangle = 0,
\]
for every test function \(\phi \in C^0_0(\mathbb{R} \times \mathbb{R}_+)\). From (8), as in [27, 25], we have
\[
\langle b, \phi \rangle = \int_{0}^{\infty} \int_{-\infty}^{\infty} h_0 \phi dx dt + \langle \beta(t)\delta_{\Gamma}, \phi \rangle,
\]
\[
\langle \frac{\tilde{b}^2}{2}, \phi \rangle = \int_{0}^{\infty} \int_{-\infty}^{\infty} \tilde{b}_0 \phi dx dt + \langle \sigma \beta(t)\delta_{\Gamma}, \phi \rangle,
\]
in which \(h_0 = h_l + |b| H(x - \sigma t), b_0 = b_l + |b| H(x - \sigma t)\) and
\[
\frac{h_0 b_0^2}{2} = \frac{h_l b_l^2}{2} + |\frac{\tilde{b}^2}{2}| H(x - \sigma t).
\]

For delta shock wave we propose the following theorem.
Theorem 2.3. When \( b_{r,r} > 0, h_1 > 0 \) and \( h_r = 0 \), then the Riemann solution (1) and (2) is delta shock, which can be written as follows:

\[
(h, b) (x, t) = \begin{cases} 
(h_1, b_1), & x < \sigma t, \\
(0, \beta(t) \delta(x - \sigma t)), & x = \sigma t, \\
(0, b_r), & x > \sigma t,
\end{cases}
\]

where

\[
\sigma = \frac{h_1 b_1}{2} \quad \text{and} \quad \beta(t) = \frac{h_1 b_0 r}{2} t
\]
denote speed and strength of delta shock wave, respectively.

Furthermore, the delta shock wave solution (10) together (11) must satisfy the generalized Rankine-Hugoniot jump conditions

\[
\begin{cases}
\frac{dx(t)}{dt} = \sigma, \\
\sigma[h] = \left[ \frac{h^2 b}{2} \right], \\
\frac{d\beta(t)}{dt} = (\sigma[b] - \left[ \frac{hb^2}{2} \right]),
\end{cases}
\]

where \([h] = h(x(t) + 0) - h(x(t) - 0)\) indicates jump in \( h \) across the discontinuity \( x = x(t) \).

For uniqueness, the delta entropy conditions \( \lambda_2(h_r, b_r) = \lambda_1(h_r, b_r) = 0 < \sigma \leq \lambda_1(h_1, b_1) = \frac{h_1 b_1}{2} < \lambda_2(h_1, b_1) = \frac{3h_1 b_1}{2} \) are imposed, which means that all the characteristics on both sides of the delta shock are not outgoing.

Proof. In order to prove (10) is the solution of (1) and (2) in the sense of distributions, it is enough to prove the following integral identities

\[
\int_0^{\infty} \int_{-\infty}^{\infty} \left( h \phi_t + \left( \frac{h^2 b}{2} \right) \phi_x \right) dx dt = 0, \quad \int_0^{\infty} \int_{-\infty}^{\infty} \left( b \phi_t + \left( \frac{hb^2}{2} \right) \phi_x \right) dx dt = 0
\]

hold for every test function \( \phi \in C_0^\infty(\mathbb{R} \times \mathbb{R}_+) \).

Without loss of generality, let \( \sigma > 0 \), then we have

\[
I = \int_0^{\infty} \int_{-\infty}^{\infty} \left( h \phi_t + \left( \frac{h^2 b}{2} \right) \phi_x \right) dx dt
\]

\[
= \int_0^{\infty} \int_{-\infty}^{\sigma t} \left( h_1 \phi_t + \left( \frac{h_1^2 b_1}{2} \right) \phi_x \right) dx dt + \int_0^{\infty} \int_{\sigma t}^{\infty} \left( h_r \phi_t + \left( \frac{h_r^2 b_r}{2} \right) \phi_x \right) dx dt
\]

\[
= \int_0^{\infty} \int_{0}^{\sigma t} h_1 \phi_t dx dt + \int_0^{\infty} \int_{0}^{\sigma t} h_r \phi_t dx dt + \int_0^{\infty} \int_{0}^{\sigma t} \left( \frac{h_1^2 b_1}{2} - \frac{h_r^2 b_r}{2} \right) \phi(\sigma t, t) dt
\]

\[
= 0.
\]

Similarly,

\[
II = \int_0^{\infty} \int_{-\infty}^{\infty} \left( b \phi_t + \left( \frac{hb^2}{2} \right) \phi_x \right) dx dt
\]

\[
= \int_0^{\infty} \int_{-\infty}^{\sigma t} \left( b_1 \phi_t + \left( \frac{b_1^2 b_1}{2} \right) \phi_x \right) dx dt + \int_0^{\infty} \int_{\sigma t}^{\infty} \left( b_r \phi_t + \left( \frac{b_r^2 b_r}{2} \right) \phi_x \right) dx dt
\]

\[
= 0.
\]
where \( + \) means followed by. Since the speed, \( C \)
(see Figure 5) problem, i.e., for a given \((h, J, \sigma, b, S)\) of the first Riemann problem and
\(-\) delta shock wave with classical waves to occur from \((\text{Case i})\):
the process of interaction, the strength of delta shock wave is computed.
other. A new Riemann problem forms at the time of interaction. Further, during
which are coming from first and second Riemann problem may interact with each
The interactions of a delta shock wave with the classical elementary waves
as in (5). For any \(\epsilon > 0\), we choose \((h, b, m)\) and \((h, b, r)\) in terms of \((h, b, l)\).
Hence, for this initial data (5), we have two Riemann problems locally. The waves
which are coming from first and second Riemann problem may interact with each
other. A new Riemann problem forms at the time of interaction. Further, during
the process of interaction, the strength of delta shock wave is computed.
Depending upon the initial data, there are two possibilities for an interaction of
delta shock wave with classical waves to occur from \((-\epsilon, 0)\) and \((\epsilon, 0)\) as follows:
(i) \(J + S\) and \(\delta S\); (ii) \(J + R\) and \(\delta S\).

**Case i:** \(J + S\) and \(\delta S\). We consider that \((h, l, b)\) is connected to \((h, m, b)\) by a
contact discontinuity, \(J\), followed by shock wave, \(S\), of the first Riemann problem and
\((h, l, b)\) is connected to \((h, r, b)\) by a delta shock wave, \(\delta S_1\), of second Riemann
problem, i.e., for a given \((h, l, b)\), we choose \((h, m, b)\) and \((h, r, b)\) such that \(0 < h_m b_m < h_l b_l\)
and \(h = 0\).

The solution of the IVP (1) and (5), when \(t\) is small can be written as follows
(see Figure 5)
\[
(h, l, b) + J + (h, b_*) + S + (h, m, b) + \delta S_1 + (h, r, b),
\]
where \(+\) means followed by. Since the speed, \(C = \frac{h_m b_m \sqrt{h_m b_m - h_l b_l \sqrt{h_l b_l}}}{2(\sqrt{h_m b_m - h_l b_l})}\) of shock
wave of first Riemann problem is greater than the speed, \(\sigma_1 = \frac{h_m b_m}{2}\), of delta shock
wave \(\delta S_1\) of second Riemann problem, so the shock wave \(S\) overtakes \(\delta S_1\) and the
interaction will take place at, say, \((x_1, t_1)\) (see Figure 5).
The interaction point \((x_1, t_1)\) can be calculated as follows:

\[
\begin{cases}
(x_1 + \varepsilon) = Ct_1, \\
(x_1 - \varepsilon) = \sigma_1 t_1,
\end{cases}
\]

which implies that

\[
(x_1, t_1) = \left(\varepsilon(C + \sigma_1), \frac{2\varepsilon}{C - \sigma_1}\right).
\] (12)

The strength of \(\delta S_1\) at \((x_1, t_1)\) is given by

\[
\beta(t_1) = \left(\frac{h_m b_m b_r}{2}\right) t_1.
\] (13)

At the point of interaction \((x_1, t_1)\), new initial data is formed as follows:

\[
h|_{t=t_1} = \begin{cases} h_*, & x < x_1, \\
h_r, & x > x_1, \end{cases}, \quad b|_{t=t_1} = \begin{cases} b_*, & x < x_1, \\
b_r, & x > x_1, \end{cases} + \beta(t_1)\delta(x_1, t_1),
\]

where \((h_*, b_*) = \left(\sqrt{\frac{h_l b_l h_m}{b_m}}, \sqrt{\frac{h_l b_l b_m}{h_m}}\right)\) is the state between J and S.

A new delta shock \(\delta S_2\) is generated after interaction, which is written as

\[
h(x,t) = \begin{cases} h_*, & x < x(t), \\
h_r, & x > x(t), \end{cases}, \quad b(x,t) = \begin{cases} b_*, & x < x(t), \\
b_r, & x > x(t), \end{cases} + \beta(t)\delta(x-x(t)),
\]

where delta function have support on \(x(t) = x_1 + \sigma_2(t - t_1)\). Since the propagation speed, \(\sigma_2 = \frac{h_l b_r}{\mu} = \frac{h_l b_r}{2}\), of \(\delta S_2\) is same as the speed, \(\mu = \frac{h_l b_r}{2}\), of contact discontinuity J, it follows that the delta shock \(\delta S_2\) will not interact J. After time \(t_1\), the delta shock wave is parallel to J with the speed \(\sigma_2\) and its strength after time \(t_1\) is

\[
\beta(t) = \beta(t_1) + \left(\frac{h_l b_r}{2}\right)(t - t_1), \text{ for } t > t_1,
\]

where \(\beta(t_1)\) can be calculated from (13).
Case ii: $J + R$ and $\delta S$. We consider that $(h_l, b_l)$ is connected to $(h_m, b_m)$ by a contact discontinuity, $J$, followed by rarefaction wave, $R$, of the first Riemann problem and $(h_m, b_m)$ is connected to $(h_r, b_r)$ by a delta shock, $\delta S_1$, of second Riemann problem, i.e., for a given $(h_l, b_l)$, we choose $(h_m, b_m)$ and $(h_r, b_r)$ such that $0 < h_l b_l < h_m b_m$ and $h_r = 0$.

The solution of the IVP (1) and (5), when $t$ is small can be written as follows (see Figure 6)

$$(h_l, b_l) + J + (h_2, b_2) + R + (h_m, b_m) + \delta S_1 + (h_r, b_r).$$

Since the speed, $\sigma = \frac{h_m b_m}{2}$, of delta shock wave of second Riemann problem is less than the speed, $\xi_a = \frac{3h_m b_m}{2}$, of the wave front of rarefaction wave, $R$, of first Riemann problem, it follows that $\delta S_1$ interact $R$ after a finite time and the interaction will take place at, say, $(x_1, t_1)$.

The interaction point $(x_1, t_1)$ can be calculated as follows

\[
\begin{align*}
(x_1 + \varepsilon) &= \left(3h_m b_m \right) t_1, \\
(x_1 - \varepsilon) &= \frac{h_m b_m}{2} t_1,
\end{align*}
\]

which implies that

\[
(x_1, t_1) = \left(2\varepsilon, \frac{2\varepsilon}{h_m b_m} \right).
\]

(14)

The strength of $\delta S_1$ at $(x_1, t_1)$ is given by

\[
\beta(t_1) = \left(\frac{h_m b_m b_r}{2} \right) t_1.
\]

(15)

After time $t_1$, the delta shock wave starts to penetrate the rarefaction wave, $R$. In the process of penetration, we use $\delta S_2$ to express delta shock wave. The values of $h$ and $b$ on the right-hand side of delta shock wave are $h_r$ and $b_r$, respectively and left-hand side values are determined with the help of rarefaction wave. We use
Γ_0 : \{(x(t), t) : t \geq t_1\} with x(t) to express the curve δS_2. The values of h and b on the left-hand side of Γ_0 are given by
\[
\frac{3hb}{2} = \frac{x + \varepsilon}{t}.
\]
(16)
The speed of δS_2 can be calculated by using Rankine-Hugoniot jump conditions as follows:
\[
\frac{dx(t)}{dt} = \frac{hb}{2}.
\]
(17)
From (16) and (17), we obtain
\[
\frac{dx(t)}{dt} = \frac{x + \varepsilon}{3t}.
\]
The delta shock wave curve Γ_0 with initial condition x(t_1) = x_1 is given by
\[
x(t) = \left(\frac{27}{2} \varepsilon^2 h_m b_m t\right)^{\frac{1}{3}} - \varepsilon.
\]
(18)
After time t_1 the delta shock strength is obtained from the equation \(\frac{d\beta(t)}{dt} = \frac{hb}{2}\) with initial condition \(\beta(t_1)\). The interaction of δS_2 with wave back of R, at the point \((x_2, t_2)\) can be obtained from (16) and (18) combining with the line of the wave back of R given by \(x + \varepsilon t = \frac{3hb}{2}\). After crossing wave back of rarefaction wave, R, we use delta shock wave as δS_3. Now delta shock wave δS_3 speed is \(\frac{hb}{2}\), which is same as the speed of contact discontinuity, J. Therefore no further interaction of delta shock wave is possible with contact discontinuity. Therefore the delta shock wave curve is parallel to contact discontinuity curve after time t_2. The strength of δS_3 after time t_2 is given by \(\beta(t) = \beta(t_2) + \frac{hb}{2} (t - t_2)\) for \(t > t_2\).

4. Conclusions. We considered a thin film model in which the governing system of quasilinear partial differential equations are non-strictly hyperbolic. We have shown that the Riemann problem for this system cannot be solved for all combinations of piecewise constant initial states with classical elementary waves. We studied delta shock wave and generalized Rankine-Hugoniot jump conditions. We discussed possible wave interactions of delta shock wave with classical elementary waves. We constructed the structure of global solution of perturbed Riemann problem (1) and (5).

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