Modified cosmology through spacetime thermodynamics and Barrow horizon entropy

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Abstract: We present modified cosmological scenarios that arise from the application of the “gravity-thermodynamics” conjecture, using the Barrow entropy instead of the usual Bekenstein-Hawking one. The former is a modification of the black hole entropy due to quantum-gravitational effects that deform the black-hole horizon by giving it an intricate, fractal structure. We extract modified cosmological equations which contain new extra terms that constitute an effective dark-energy sector, and which coincide with the usual Friedmann equations in the case where the new Barrow exponent acquires its Bekenstein-Hawking value. We present analytical expressions for the evolution of the effective dark energy density parameter, and we show that the universe undergoes through the usual matter and dark-energy epochs. Additionally, the dark-energy equation-of-state parameter is affected by the value of the Barrow deformation exponent and it can lie in the quintessence or phantom regime, or experience the phantom-divide crossing. Finally, at asymptotically large times the universe always results in the de-Sitter solution.

Keywords: Modified gravity, Dark energy, First law of thermodynamics, Barrow entropy
1 Introduction

From cosmological observations of various origins we deduce that the universe has experienced two phases of accelerated expansion, one at late and one at early times. In order to describe this behavior one can either construct modified gravitational theories, whose richer structure provides the extra degrees of freedom (for reviews see [1–3]), or alter the content of the universe, by introducing new fields such as the inflaton [4, 5] or the dark energy concept [6, 7]. Concerning the first direction, the usual approach is to start from the action of general relativity and add correction terms, resulting to \( f(R) \) gravity [8–10], \( f(G) \) gravity [11], Lovelock gravity [12], etc. Alternatively, one can construct gravitational modifications using torsion, such as in \( f(T) \) gravity [13–15], non-metricity [16, 17], Finsler corrections [18] or other geometrical structures.

Additionally, there is a well-known conjecture that gravity and thermodynamics are related [19–21], and in particular one can show that the cosmological Friedmann equations can be expressed as the first law of thermodynamics, if we consider the universe as a thermodynamical system bounded by the apparent horizon [22–24]. Similarly, performing the procedure in a reverse way, one can extract the Friedmann equations by applying the first law of thermodynamics. The “gravity-thermodynamics” conjecture is applied very efficiently in a variety of modified theories of gravity, with the important step being the use of the modified entropy relation which is valid in each theory [25–34]. Hence, although an interesting way to investigate gravity, as long as the modified entropy relation is needed, the above procedure cannot provide new gravitational theories, since the modified gravity needs to be known a priori.

Recently, Barrow [35] was inspired by the Covid-19 virus illustrations and considered the possibility that the black-hole surface might have intricate structure down to arbitrarily small scales, due to quantum-gravitational effects. Such a fractal structure for the horizon leads to finite volume but with infinite (or finite) area. Hence, due to the basic principle
of black hole thermodynamics, the above possible effects of the quantum-gravitational spacetime foam on the horizon area will lead to a new black hole entropy relation, namely

\[ S_B = \left( \frac{A}{A_0} \right)^{1+\Delta/2}, \tag{1.1} \]

where \( A \) is the standard horizon area and \( A_0 \) the Planck area. The new exponent \( \Delta \) quantifies the quantum-gravitational deformation, and it is bounded as \( 0 \leq \Delta \leq 1 \), with \( \Delta = 1 \) corresponding to the most intricate and fractal structure, while \( \Delta = 0 \) corresponds to the simplest horizon structure in which case the standard Bekenstein-Hawking entropy is restored (note that the above formula is similar with Tsallis nonextensive entropy \([36–38]\), although the physical principles and interpretation is completely different).

In the present manuscript we are interested in applying the “gravity-thermodynamics” conjecture in a reverse way, in order to construct new modified gravities, but using the Barrow entropy instead of the usual one. In particular, we will obtain modified Friedmann equations, whose extra terms disappear in the case where Barrow entropy becomes the standard Bekenstein-Hawking one.

2 Modified cosmology through Barrow horizon entropy

Since we are interested in constructing modified Friedmann equations through the cosmological application of the “gravity-thermodynamics” conjecture, using Barrow entropy, in this section we will first present the basic application and then we will extend it using the latter.

2.1 Friedmann equations from the first law of thermodynamics

We start by presenting the above procedure in the basic case of general relativity. We consider a homogeneous and isotropic universe, described by the Friedmann-Robertson-Walker (FRW) metric

\[ ds^2 = -dt^2 + a^2(t) \left( \frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right), \tag{2.1} \]

with \( a(t) \) the scale factor, and where \( k = 0, +1, -1 \) corresponds respectively to flat, closed and open spatial geometry. Additionally, we assume that the universe is filled with the perfect fluid. According to the “gravity-thermodynamics” conjecture the first law can be interpreted in terms of energy flux through local Rindler horizons, i.e. it is applied on the universe horizon itself, considered as a thermodynamical system separated by a causality barrier \([19–21]\). This horizon is generally considered to be the apparent one \([22, 23, 39]\)

\[ r_A = \frac{1}{\sqrt{H^2 + \frac{k}{a^2}}}, \tag{2.2} \]

where \( H = \frac{\dot{a}}{a} \) is the Hubble parameter and dots representing time-derivatives.

In order to apply the first law of thermodynamics we need to attribute to the universe horizon an entropy and a temperature. These are provided by black hole thermodynamics,
replacing the black hole horizon with the cosmological apparent horizon. For the black-hole temperature it is known that it is inversely proportional to its horizon radius, independently of the underlying gravitational theory, and thus for the universe horizon temperature we obtain [21]

\[ T_h = \frac{1}{2\pi r_A}. \tag{2.3} \]

The black-hole entropy, which as we mentioned depends on the underlying gravitational theory, in the case of general relativity is the usual Bekenstein-Hawking relation \( S = A/(4G) \), with \( A = 4\pi r_A^2 \) the area of the black hole horizon and \( G \) the gravitational constant (we use units where \( \hbar = k_B = c = 1 \)). Therefore, the apparent horizon entropy is

\[ S_h = \frac{1}{4G}A. \tag{2.4} \]

In an expanding universe, during a time interval \( dt \) the heat flow through the horizon is easily found to be [23]

\[ \delta Q = -dE = A(\rho_m + p_m)Hr_A dt, \tag{2.5} \]

with \( \rho_m \) and \( p_m \) the energy density and pressure of the matter fluid that fills the universe. In order to apply the first law of thermodynamics \( -dE = TdS \), we need to know \( T \) and \( dS \). The first is given by (2.3), while the second is calculated from (2.4) as \( dS = 2\pi \dot{r}_A dt/G \), where \( \dot{r}_A \) can be straightforwardly calculated using (2.2). Assembling everything we obtain

\[ -4\pi G(\rho_m + p_m) = \dot{H} - \frac{k}{a^2}, \tag{2.6} \]

Finally, imposing the matter conservation equation

\[ \dot{\rho}_m + 3H(\rho_m + p_m) = 0, \tag{2.7} \]

integration of (2.6) leads to the first Friedmann equation

\[ \frac{8\pi G}{3}\rho_m = H^2 + \frac{k}{a^2} - \frac{\Lambda}{3}, \tag{2.8} \]

where the cosmological constant arises as an integration constant. We mention that in the above procedure one applies the reasonable equilibrium assumption that the universe horizon has the same temperature as the universe fluid, which is true for the late-time universe [21–24, 31, 41]. Lastly, as we mentioned in the Introduction, the above steps can be extended to modified gravity theories too, as long as one uses not the general relativity relation (2.4), but the modified one of each theory [25–34].

2.2 Modified Friedmann equations through Barrow entropy

We are now ready to apply the procedure the “gravity-thermodynamics” conjecture in the case of Barrow entropy, i.e. extending the procedure described in subsection 2.1. In particular, the first law of thermodynamics is \(-dE = TdS\), where \(-dE\) is still given by (2.5), \( T \) is again given by (2.3), but now the entropy relation will be different, namely it
is the Barrow entropy (1.1) (for the generalized second law of thermodynamics in the case of Barrow entropy see [42]). Hence, we now have

\[ dS = (4\pi)^{(1+\Delta/2)} \frac{(2 + \Delta)}{A_0^{(1+\Delta/2)}} r_A^{\Delta+1} \dot{r}_A dt, \tag{2.9} \]

where we have used that \( A = 4\pi r_A^2 \). Inserting these relations into the first law of thermodynamics, and substituting \( \dot{r}_A \) using (2.2), we finally result to

\[ - (4\pi)^{(1-\Delta/2)} A_0^{(1+\Delta/2)} (\rho_m + p_m) = 2(2 + \Delta) \frac{\dot{H} - \frac{k}{a^2}}{(H^2 + \frac{k}{a^2})^{\Delta/2}}. \tag{2.10} \]

Lastly, inserting (2.7) and integrating, for the validity region \( 0 \leq \Delta \leq 1 \) we acquire

\[ \frac{(4\pi)^{(1-\Delta/2)} A_0^{(1+\Delta/2)}}{6} \rho_m = \frac{2 + \Delta}{2 - \Delta} \left( H^2 + \frac{k}{a^2} \right)^{1-\Delta/2} - \frac{C}{3} A_0^{(1+\Delta/2)}, \tag{2.11} \]

with \( C \) the integration constant.

As we can see, the use of Barrow entropy in the first law of thermodynamics resulted to the modified Friedmann equations (2.10) and (2.11), with extra terms comparing to general relativity (note that this is a completely different theory and cosmology comparing to the application of Barrow entropy in a holographic context [43, 44]). Focusing on the flat case \( k = 0 \) for simplicity, we can re-express them as

\[ H^2 = \frac{8\pi G}{3} (\rho_m + \rho_{DE}) \tag{2.12} \]

\[ \dot{H} = -4\pi G (\rho_m + p_m + \rho_{DE} + p_{DE}), \tag{2.13} \]

where

\[ \rho_{DE} = \frac{3}{8\pi G} \left\{ \frac{\Lambda}{3} + H^2 \left[ 1 - \frac{\beta(\Delta + 2)}{2 - \Delta} H^{-\Delta} \right] \right\}, \tag{2.14} \]

\[ p_{DE} = \frac{1}{8\pi G} \left\{ \Lambda + 2\dot{H} \left[ 1 - \beta \left( 1 + \frac{\Delta}{2} \right) H^{-\Delta} \right] + 3H^2 \left[ 1 - \frac{\beta(2 + \Delta)}{2 - \Delta} H^{-\Delta} \right] \right\} \tag{2.15} \]

respectively are the energy density and pressure of the effective dark energy sector, and with \( \Lambda \equiv 4CG(4\pi)^{\Delta/2} \) a parameter with dimensions \([L^{-2}]\), and \( \beta = \frac{4(4\pi)^{\Delta/2} G}{A_0^{1+\Delta/2}} \) a parameter with dimensions \([L^{-\Delta}]\) (we use units where \( \hbar = k_B = c = 1 \)). Hence, the effective equation of state reads

\[ w_{DE} \equiv \frac{p_{DE}}{\rho_{DE}} = -1 - \frac{\dot{H} \left[ 1 - \beta \left( 1 + \frac{\Delta}{2} \right) H^{-\Delta} \right]}{\Lambda + 3H^2 \left[ 1 - \frac{\beta(2 + \Delta)}{2 - \Delta} H^{-\Delta} \right]} \tag{2.16} \]

As expected, in the case \( \Delta = 0 \) the modified Friedmann equations (2.12),(2.13) reduce to \( \Lambda \text{CDM} \) paradigm, i.e.

\[ H^2 = \frac{8\pi G}{3} \rho_m + \frac{\Lambda}{3} \]

\[ \dot{H} = -4\pi G (\rho_m + p_m). \tag{2.17} \]
3 Cosmological implications

In the previous section we applied the “gravity-thermodynamics” conjecture with the Barrow entropy and we resulted in a modified cosmology, characterized by the modified Friedmann equations (2.12) and (2.13). These equations coincide with ΛCDM paradigm in the limit ∆ = 0, in which case Barrow entropy becomes the standard one, however in the general case they give rise to an effective dark energy sector. In the present section we will investigate the cosmological evolution, extracting analytical solutions.

We focus on the case of dust matter (ρ_m ≈ 0), in which case as usual (2.7) leads to
\[ ρ_m = \frac{8\pi G}{3H^2}ρ_m \]
for the matter energy density at the current scale factor a_0 = 1 (from now on the subscript “0” denotes the value of a quantity at present time). Furthermore, we introduce the density parameters as
\[ Ω_m = \frac{8\pi G}{3H^2}H^2ρ_m \]
\[ Ω_{DE} = \frac{8\pi G}{3H^2}H^2ρ_{DE} \]
for the matter and effective dark energy sector respectively. Hence, (3.1) gives
\[ Ω_m = Ω_{m0}H_0^2/a^3H^2, \]
which knowing that Ω_m + Ω_{DE} = 1 leads to
\[ H = \sqrt{Ω_{m0}H_0/a^3(1-Ω_{DE})}. \]

It proves convenient to use the redshift 1 + z = 1/a as the independent variable. Differentiating (3.3) we acquire
\[ \dot{H} = -\frac{H^2}{2(1-Ω_{DE})}[3(1-Ω_{DE}) + (1+z)Ω'_{DE}], \]
where primes mark z-derivatives. Inserting (2.14) into (3.2) and using (3.3) and (3.4) we acquire a simple differential equation for Ω_{DE}(z), which can be easily solved as
\[ Ω_{DE}(z) = 1 - H_0^2Ω_{m0}(1+z)^3\left\{\frac{(2-Δ)}{β(2+Δ)}\left[H_0^2Ω_{m0}(1+z)^3 + \frac{Λ}{3}\right]\right\}^{\frac{2-Δ}{2}}, \]
This is the analytical solution for the dark energy density parameter, in the case of dust matter in a flat universe. Lastly, applying it at z = 0 we obtain
\[ Λ = \frac{3β(2+Δ)}{2-Δ}H_0^{2-Δ} - 3H_0^2Ω_{m0}, \]
which leaves the scenario with two free parameters since it can be used to eliminate one of Λ, Δ and β in terms of the observationally determined quantities Ω_{m0} and H_0. We mention that for Δ = 0 we re-obtain ΛCDM concordance scenario.

Concerning the important observable quantity, the dark-energy equation-of-state parameter given in (2.16), inserting ˙H from (3.4) we find that
\[ w_{DE}(z) = -1 + \frac{\left\{3[1-Ω_{DE}(z)] + (1+z)Ω'_{DE}(z)\right\} - \beta(1+Δ/2)\left[H_0^2Ω_{m0}(1+z)^3\right]^{Δ/2}}{[1-Ω_{DE}(z)]\left[H_0^2Ω_{m0}(1+z)^3\right]^{Δ/2} + 3\left[1 - \frac{3β(2+Δ)}{2-Δ}\left[H_0^2Ω_{m0}(1+z)^3\right]^{Δ/2}\right]}}, \]

where $\Omega'_{DE}(z)$ is calculated from (3.5) as

\[
\Omega'_{DE}(z) = \left\{ \frac{(2 - \Delta)}{\beta(2 + \Delta)} \left[ 1 + \frac{\Lambda}{3 \Omega_{m0}H_0^2(1+z)^3} \right] \right\}^{\frac{3}{2(2 + \Delta)}} \\
\times \frac{1}{\beta(2 + \Delta)} \left[ \Omega_{m0}H_0^2(1+z)^3 \right]^{\frac{3}{2(2 + \Delta)}} \left[ 3\Delta \Omega_{m0}H_0^2(1+z)^2 + (\Delta - 2) \frac{\Lambda}{1+z} \right].
\]

(3.8)

In conclusion, we were able to extract analytical solutions for the effective dark energy density and its equation of state, for the modified cosmology arisen from Barrow entropy.

Let us examine the scenario in more detail. We start with the case where the explicit cosmological constant $\Lambda \neq 0$. Since for $\Delta = 0$ we re-obtain $\Lambda$CDM paradigm, we are interested in investigating the role of the new Barrow parameter $\Delta$ on the universe evolution. We use relation (3.6) in order to eliminate $\Lambda$, imposing the observed value $\Omega_{m0} \approx 0.3$ [45], and we set $A_0 = 1$. Using (3.5), in the left graph of Fig. 1 we present $\Omega_{DE}(z)$ and $\Omega_m(z) = 1 - \Omega_{DE}(z)$, for the case where $\Delta = 0.2$. Additionally, in the right graph we depict the evolution of the dark-energy equation-of-state parameter from (3.7). As we can see,

![Figure 1](image)

**Figure 1.** Left graph: The evolution of the effective dark energy density parameter $\Omega_{DE}$ (orange-solid) and of the matter density parameter $\Omega_m$ (blue-dashed), as a function of the redshift $z$, for the modified cosmology through Barrow entropy with $\Delta = 0.2$ and $A_0 = 1$. Right graph: The evolution of the corresponding dark-energy equation-of-state parameter $w_{DE}$. We have imposed $\Omega_{m0} \approx 0.3$ at present time.

we are able to obtain the thermal history of the universe in agreement with observations. Moreover, the dark energy equation-of-state parameter can experience the phantom-divide crossing, which is an advantage of the scenario.

In order to examine the effect of the Barrow parameter $\Delta$ on the dark energy features, in Fig. 2 we present $w_{DE}(z)$ for various values of $\Delta$. As expected, for $\Delta = 0$ we acquire $w_{DE} = -1 = \text{const.}$, namely $\Lambda$CDM scenario. Nevertheless, as the Barrow exponent increases, and the quantum-gravitational deformation becomes more important, $w_{DE}$ at larger redshifts acquires larger values, while at small redshifts and current time it acquires algebraically smaller values. Hence, the Barrow parameter $\Delta$, that lies in the core of the modified cosmology at hand, leads the dark energy to have a dynamical nature, departing from $\Lambda$CDM cosmology. Furthermore, dark energy can be quintessence-like, phantom-like,
or experience the phantom-divide crossing during the evolution, which is an advantage of the scenario.

![Figure 2](image)

**Figure 2.** The evolution of $w_{DE}$ as a function of the redshift $z$, for $A_0 = 1$, and for $\Delta = 0$ (blue-dashed), $\Delta = 0.2$ (purple-dotted), $\Delta = 0.4$ (red-dashed-dotted), and $\Delta = 0.6$ (orange-solid). We have imposed $\Omega_{m0} \approx 0.3$ at present time.

Finally, we can calculate analytically the asymptotic value of $w_{DE}$ in the far future. In particular, taking the limit $z \to -1$ in (3.5), (3.8) and (3.7), we respectively find $\Omega_{DE} \to 1$, $\Omega_{DE}' \to 0$, and $w_{DE} \to -1$. This implies that although at intermediate times the dark-energy equation-of-state parameter experiences an interesting behavior which departs from $\Lambda$CDM cosmology, at asymptotically large times it will always stabilize at the cosmological constant value $-1$, and the universe will reach the de-Sitter solution, independently of the Barrow exponent. The fact that the de Sitter solution is a stable late-time attractor independently of the Barrow exponent, is a significant advantage of the modified cosmology through Barrow entropy.

We close this analysis by investigating the interesting case where an explicit cosmological constant $\Lambda$ is absent, and thus the modified cosmology at hand offers a more radical modification, without possessing $\Lambda$CDM paradigm as a particular limit, which however can still describe the effective dark energy and late-time acceleration. In the case $\Lambda = 0$, relations (2.14), (2.15) become

$$\rho_{DE} = \frac{3}{8\pi G} \left[ H^2 - \Omega_{m0}(H_0/H)^{\Delta} \right]$$

$$p_{DE} = -\frac{1}{8\pi G} \left\{ 3H^2 \left[ 1 - \Omega_{m0}(H_0/H)^{\Delta} \right] + 2H \left[ 1 - \Omega_{m0}(1 - \Delta/2)(H_0/H)^{\Delta} \right] \right\}.$$  \hfill (3.10)

Hence, (3.5) is now written as

$$\Omega_{DE}(z) = 1 - \Omega_{m0}(1 + z)^{\frac{3\Delta}{2 - \Delta}}.$$ \hfill (3.11)

and (3.7) as

$$w_{DE}(z) = \frac{\Delta}{(2 - \Delta)} \left[ 1 - \Omega_{m0}(1 + z)^{\frac{3\Delta}{2 - \Delta}} \right]^{-1}.$$ \hfill (3.12)
Notice that in this case ΛCDM scenario cannot be re-obtained for any parameter values, and therefore one should use a non-trivial value for ∆ in order to suitably acquire agreement with observations. Finally, we mention that from (3.11) we observe that we are able to obtain the usual thermal history of the universe, with the sequence of matter and dark energy eras, while in the asymptotically far future (z → −1) the universe tends to the complete dark-energy domination. Lastly, note that according to (3.11), for high redshifts we obtain either early-time dark energy or Ω_{DE}(z) < 0, cases that are not physically interesting. However, as expected, these are eliminated if one includes the radiation sector too, which changes and regulates the early-time behavior.

4 Conclusions

There is a long-standing conjecture that gravity is related to thermodynamics, which concerning cosmological frameworks implies that the Friedmann equations can arise from the first law of thermodynamics. In this manuscript we constructed modified cosmological scenarios through the application of this conjecture, but using the Barrow entropy, instead of the usual Bekenstein-Hawking one. In particular, as it was recently proposed in [35], the black-hole surface may have intricate, fractal structure, due to quantum-gravitational effects. Hence, the corresponding black-hole entropy will deviate from the Bekenstein-Hawking one, and this deformation is quantified through a new exponent ∆, with the limit ∆ = 0 corresponding to the standard case where Barrow entropy becomes the standard one, while the limit ∆ = 1 corresponds to the case where the deformation is maximal.

Applying the “gravity-thermodynamics” procedure with Barrow entropy we resulted to modified cosmological equations which contain new extra terms, and which coincide with the usual Friedmann equations in the case where the new Barrow exponent acquires its usual value ∆ = 0. In the general case these new terms constitute an effective dark energy sector leading to interesting phenomenological behavior, while in the special case ∆ = 0 ΛCDM concordance model is restored.

Assuming the matter sector to be dust, we extracted analytical expressions for the evolution of the effective dark energy density parameter and its equation of state. As we saw, the scenario at hand can describe the usual thermal history of the universe, with the dark-energy epoch following the matter one. Concerning the dark-energy equation-of-state parameter we saw that in the recent and current universe it is affected by the value of the Barrow deformation exponent. Specifically, $w_{DE}$ at larger redshifts acquires larger values, while at small redshifts and current time it acquires algebraically smaller values, and moreover dark energy can be quintessence-like, phantom-like, or experience the phantom-divide crossing during the evolution. However, at asymptotically large times the universe will reach the de-Sitter solution, independently of the Barrow exponent, which is an additional advantage. Finally, the scenario at hand exhibits interesting cosmological behavior even in the case where an explicit cosmological constant is absent, where the modification is more radical and the effective dark-energy sector is constituted solely by the new terms.
In conclusion, modified cosmology through “gravity-thermodynamics” procedure using Barrow entropy is efficient in quantifying the universe evolution in agreement with observations. It would be interesting to perform a full observational confrontation using data from Supernovae type Ia data (SNIa), Cosmic Microwave Background (CMB) shift parameters, Baryonic Acoustic Oscillations (BAO), growth rate and Hubble data observations, in order to constrain the new Barrow exponent $\Delta$. This works lies beyond the scope of the present work and it is left for a future project.

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