Homogenization of Schrödinger equations. Extended Effective Mass Theorems for non-crystalline matter

Vernny Ccajma¹, Wladimir Neves¹, Jean Silva²

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Abstract

This paper concerns the homogenization of Schrödinger equations for non-crystalline matter, that is to say the coefficients are given by the composition of stationary functions with stochastic deformations. Two rigorous results of so-called effective mass theorems in solid state physics are obtained: a general abstract result (beyond the classical stationary ergodic setting), and one for quasi-perfect materials (i.e. the disorder in the non-crystalline matter is limited). The former relies on the double-scale limits and the wave function is spanned on the Bloch basis. Therefore, we have extended the Bloch Theory which was restrict until now to crystals (periodic setting). The second result relies on the Perturbation Theory and a special case of stochastic deformations, namely stochastic perturbation of the identity.

¹Instituto de Matemática, Universidade Federal do Rio de Janeiro, C.P. 68530, Cidade Universitária 21945-970, Rio de Janeiro, Brazil. E-mail: v.ccajma@gmail.com, wladimir@im.ufrj.br.
²Departamento de Matemática, Universidade Federal de Minas Gerais. E-mail: jcs-mat@mat.ufmg.br
1 Introduction

In this work we study the homogenization (asymptotic limit as $\varepsilon \to 0$) of the anisotropic Schrödinger equation in the following Cauchy problem

\[
\begin{aligned}
\frac{i}{\varepsilon} \frac{\partial u_{\varepsilon}}{\partial t} - \text{div} \left( A(\Phi^{-1}(x, \omega), \omega) \nabla u_{\varepsilon} \right) + \frac{1}{\varepsilon^2} V(\Phi^{-1}(x, \omega), \omega) u_{\varepsilon} \\
+ U(\Phi^{-1}(x, \omega), \omega) u_{\varepsilon} = 0, \quad \text{in} \quad \mathbb{R}_{T}^{n+1} \times \Omega,
\end{aligned}
\]

where $\mathbb{R}_{T}^{n+1} := (0, T) \times \mathbb{R}^n$, for any real number $T > 0$, $\Omega$ is a probability space, and the unknown function $u_{\varepsilon}(t, x, \omega)$ is complex-value.

The coefficients in (1.1), that is the matrix-value function $A$, the real-value (potential) functions $V$, $U$ are random perturbations of stationary functions accomplished by stochastic diffeomorphisms $\Phi : \mathbb{R}^n \times \Omega \to \mathbb{R}^n$, (called stochastic deformations). The stationarity property of random functions will be precisely defined in Section 2.2 also the definition of stochastic deformations which were introduced by X. Blanc, C. Le Bris, P.-L. Lions (see [8, 9]). In that paper they consider the homogenization problem of an elliptic operator whose coefficients are periodic or stationary functions perturbed by stochastic deformations.

In particular, we assume that $A = (A_{k\ell})$, $V$ and $U$ are measurable and bounded functions, i.e. for $k, \ell = 1, \ldots, n$

\[
A_{k\ell}, \quad V, \quad U \in L^\infty(\mathbb{R}^n \times \Omega). \tag{1.2}
\]

Moreover, the matrix $A$ is symmetric and uniformly positive defined, that is, there exists $a_0 > 0$, such that, for a.a. $(y, \omega) \in \mathbb{R}^n \times \Omega$, and each $\xi \in \mathbb{R}^n$

\[
\sum_{k,\ell=1}^{n} A_{k\ell}(y, \omega) \xi_k \xi_\ell \geq a_0 |\xi|^2. \tag{1.3}
\]

This paper is the second part of the Project initiated with T. Andrade, W. Neves, J. Silva [3] (Homogenization of Liouville Equations beyond stationary ergodic setting) concerning the study of moving electrons in non-crystalline matter, which justify the form considered for the coefficients in (1.1). We
recall that crystalline materials, also called perfect materials, are described by periodic functions. Thus any homogenization result for Schrödinger equations with periodic coefficients is restricted to crystalline matter. Moreover, perfect materials are rare in Nature, there exist much more non-crystalline than crystalline materials. For instance, there exists a huge class called quasi-perfect materials (see Section 4, also [5]), which are closer to perfect ones. Indeed, the concept of stochastic deformations are very suitable to describe interstitial defects in materials science (see Cances, Le Bris [11], and Myers [21]).

One remarks that, the homogenization of the Schrödinger equation in (1.1), when the stochastic deformation $\Phi(y,\omega)$ is the identity mapping and the coefficients are periodic, were studied by Allaire, Piatnitski [3]. Notably, that paper presents the discussion about the differences between the scaling considered in (1.1) and the one called semi-classical limit. We are not going to rephrase this point here, and address the reader to Chapter 4 in [7] for more general considerations about that. It should be mentioned that, to the best of our knowledge the present work is the first to study the homogenization of the Schrödinger equations beyond the periodic setting, applying the double-scale limits and the wave function is spanned on the Bloch basis.

Last but not least, one observes that the initial data $u_\varepsilon^0$ shall be considered well-prepared, see equation (3.31). This assumption is fundamental for the abstract homogenization result established in Theorem 3.2 where the limit function obtained from $u_\varepsilon$ satisfies a simpler Schrödinger equation, called the effective mass equation, with effective constant coefficients, namely matrix $A^*$, and potential $V^*$. This homogenization procedure is well known in solid state physics as Effective Mass Theorems, see Section 3.

Finally, we stress Section 4 which is related to the homogenization of the Schrödinger equation for quasi-perfect materials, and it is also an important part of this paper. Indeed, a very special case occurs in situations where the amount of randomness is small, more specifically the disorder in the material is limited. In particular, this section is interesting for numerical applications, where specific computationally efficient techniques already designed to deal with the homogenization of the Schrödinger equation in the periodic setting, can be employed to treat the case of quasi-perfect materials.
1.1 Contextualization

Let us briefly recall that the homogenization’s problem for (1.1) has been treated for the periodic case \((A_{\text{per}}(y), V_{\text{per}}(y), U_{\text{per}}(y))\), and \(\Phi(y, \omega) = y\) by some authors. Besides the paper by G. Allaire, A. Piatnitski [3], we already mentioned, we address the following papers for the case of \(A_{\text{per}} = I_{n \times n}\), i.e. isotropic Schrödinger equation in (1.1): G. Allaire, M. Vanninathan [2], L. Barletti, N. Ben Abdallah [6], V. Chabu, C. Fermanian-Kammerer, F. Marcia [14], and we observe that this list is by no means exhaustive. In [2], the authors study a semiconductors model excited by an external potential \(U_{\text{per}}(t, x)\), which depends on the time \(t\) and macroscopic variable \(x\). In [6] the authors treat the homogenization’s problem when the external potential \(U_{\text{per}}(x, y)\) depends also on the macroscopic variable \(x\). Finally, in [14] it was considered an external potential \(U_{\text{per}}(t, x)\) which model the effects of impurities on the otherwise perfect matter.

All the references cited above treat the homogenization’s problem for (1.1), studying the spectrum of the associated Bloch spectral cell equation, that is, for each \(\theta \in \mathbb{R}^n\), find the eigenvalue-eigenfunction pair \((\lambda, \psi)\), satisfying

\[
\begin{aligned}
L_{\text{per}}(\theta)[\psi] &= \lambda \psi, \quad \text{in } [0, 1)^n, \\
\psi(y) &\neq 0, \quad \text{periodic function},
\end{aligned}
\]

where \(L_{\text{per}}(\theta)\) is the Hamiltonian given by

\[
L_{\text{per}}(\theta)[f] = - (\text{div}_y + 2i\pi \theta) \left[ A_{\text{per}}(y)(\nabla_y + 2i\pi \theta)f \right] + V_{\text{per}}(y)f.
\]

The above eigenvalue problem is precisely stated (in the more general context studied in this paper) in Section 2.4. Here, concerning the periodic setting mathematical solutions to (1.1), we address the reader to C. H. Wilcox [25], (see in particular Section 2: A discussion of related literature). Then, once this eigenvalue problem is resolved, the goal is to pass to the limit as \(\varepsilon \to 0\). One remarks that, there does not exist an uniform estimate in \(H^1(\mathbb{R}^n)\) for the family of solutions \(\{u_\varepsilon\}\) of (1.1), due to the scale \(\varepsilon^{-2}\) multiplying the internal potential \(V_{\text{per}}(y)\). To accomplish the desired asymptotic limit, under this lack of compactness, a nice strategy is to use the two-scale convergence, for instance see the proof of Theorem 3.2 in [3].

Let us now focus on the stochastic setting studied here, where the coefficients of the Schrödinger equation in (1.1) are the composition of stationary
functions with stochastic deformations. First, we establish an analogously Bloch spectral cell equation assuming that the solution of equation (1.1) is given by a plane wave. Then, the stochastic spectral Bloch cell equation (2.27) is obtained applying the asymptotic expansion WKB method. More specifically, the Hamiltonian in (2.27) is given by

\[ L^\Phi(\theta)[F] = - (\text{div}_z + 2i\pi\theta) \left( A(\Phi^{-1}(z, \omega), \omega) \left( \nabla_z + 2i\pi\theta \right) F \right) + V(\Phi^{-1}(z, \omega), \omega) F, \]

for each \( F(z, \omega) = f(\Phi^{-1}(z, \omega), \omega) \), where \( f(y, \omega) \) is a stationary function.

To follow, we consider the spectrum of the operator \( L^\Phi(\theta) \), for each \( \theta \in \mathbb{R}^n \) fixed. First, we follow the techniques applied for the periodic setting, and consider a characterization of the Rellich-Kondrachov Theorem on groups given in [13]. One observes that, \( \omega \in \Omega \) can not be treat as a fixed parameter. Then we obtain an abstract theorem, see Theorem 3.2, where it is given the asymptotic limit as \( \varepsilon \to 0 \) of the Cauchy problem (1.1). For that, we apply the two-scale convergence (stochastic setting), developed in Section 2.3, which is beyond the classical stationary ergodic setting. The main difference here with the earlier stochastic extensions of the periodic setting is that, the test functions used are random perturbations of stationary functions accomplished by the stochastic deformations. These compositions are beyond the stationary class, thus we have a lack of the stationarity property in this kind of test functions (see the introduction section in [5] for a deep discussion about this subject). It was introduced a compactification argument that, preserves the ergodic nature of the setting involved and allow us to overcome these difficulties.

The second applied strategy here to study the spectrum of the operator \( L^\Phi(\theta) \) is the Perturbation Theory. Therefore, taking the advantage of the well known spectrum for \( L_{\text{per}}(\theta) \), we consider that the coefficients of the Schrödinger equation in (1.1) are the composition of the periodic functions \( A_{\text{per}}, V_{\text{per}} \) and \( U_{\text{per}} \) with a special case of stochastic deformations, namely stochastic perturbation of the identity (see Definition 4.1), which is given by

\[ \Phi_\eta(y, \omega) := y + \eta Z(y, \omega) + O(\eta^2), \]

where \( Z \) is some stochastic deformation and \( \eta \in (0, 1) \). This concept was introduced by X. Blanc, C. Le Bris, P.-L. Lions [9], and applied to evolutionary equations in T. Andrade, W. Neves, J. Silva [5].
2 Preliminaries and Background

This section introduces the basement theory, which will be used through the paper. To begin we fix some notations, and collect some preliminary results. The material which is well-known or a direct extension of existing work are giving without proofs, otherwise we present them.

We denote by $\mathbb{G}$ the group $\mathbb{Z}^n$ (or $\mathbb{R}^n$), with $n \in \mathbb{N}$. The set $[0, 1)^n$ denotes the unit cube, which is also called the unitary cell and will be used as the reference period for periodic functions. The symbol $\lfloor x \rfloor$ denotes the unique number in $\mathbb{Z}^n$, such that $x - \lfloor x \rfloor \in [0, 1)^n$. Let $H$ be a complex Hilbert space, we denote by $\mathcal{B}(H)$ the Banach space of linear bounded operators from $H$ to $H$.

Let $U \subset \mathbb{R}^n$ be an open set, $p \geq 1$, and $s \in \mathbb{R}$. We denote by $L^p(U)$ the set of (real or complex) $p$–summable functions with respect to the Lebesgue measure (vector ones should be understood componentwise). Given a Lebesgue measurable set $E \subset \mathbb{R}^n$, $|E|$ denotes its $n$–dimensional Lebesgue measure. Moreover, we will use the standard notations for the Sobolev spaces $W^{s,p}(U)$ and $H^s(U) \equiv W^{s,2}(U)$.

2.1 Anisotropic Schrödinger equations

Here we present the well-posedness for the solutions of the Schrödinger equation, and some properties of them. Most of the material can be found in Cazenave, Haraux [12].

First, let us consider the following Cauchy problem, which is driven by a linear anisotropic Schrödinger equation, that is

$$
\begin{cases}
  i \partial_t u(t,x) - \text{div} (A(x) \nabla u(t,x)) + V(x) u(t,x) = 0 & \text{in } \mathbb{R}_T^{n+1}, \\
  u(0,x) = u_0(x) & \text{in } \mathbb{R}^n,
\end{cases}
$$

(2.5)

where the unknown $u(t,x)$ is a complex value function, and $u_0$ is a given initial datum. The coefficient $A(x)$ is a symmetric real $n \times n$-matrix value function, and the potential $V(x)$ is a real function. We always assume that

$$
A(x), V(x) \text{ are measurable bounded functions.}
$$

(2.6)

One recalls that, a matrix $A$ is called (uniformly) coercive, when, there exists $a_0 > 0$, such that, for each $\xi \in \mathbb{R}^n$, and almost all $x \in \mathbb{R}^n$, $A(x)\xi \cdot \xi \geq a_0|\xi|^2$. 


The following definition tells us in which sense a complex function \( u(t, x) \) is a mild solution to (2.5).

**Definition 2.1.** Let \( A, V \) be coefficients satisfying (2.6). Given \( u_0 \in H^1(\mathbb{R}^n) \), a function \( u \in C([0, T]; H^1(\mathbb{R}^n)) \cap C^1((0, T); H^{-1}(\mathbb{R}^n)) \) is called a mild solution to the Cauchy problem (2.5), when for each \( t \in (0, T) \), it follows that
\[
i\partial_t u(t) - \text{div}(A \nabla u(t)) + Vu(t) = 0 \quad \text{in} \quad H^{-1}(\mathbb{R}^n),
\]
and \( u(0) = u_0 \) in \( H^1(\mathbb{R}^n) \).

Then, we state the following

**Proposition 2.2.** Let \( A \) be a coercive matrix-valued function, \( V \) a potential and \( u_0 \in H^1(\mathbb{R}^n) \) a given initial data. Assume that \( A, V \) satisfy (2.6). Then, there exists a unique mild solution of the Cauchy problem (2.5).

**Proof.** The proof follows applying Lemma 4.1.5 and Corollary 4.1.2 in [12].

**Remark 2.3.** It is very important in the homogenization procedure of the Schrödinger equation, when the coefficients \( A \) and \( V \) in (2.5) are constants, the matrix \( A \) is not necessarily coercive, and the initial data \( u_0 \in L^2(\mathbb{R}^n) \). Then, a function \( u \in L^2(\mathbb{R}_T^{n+1}) \) is called a weak solution to (2.5), if it satisfies
\[
i\partial_t u(t) - \text{tr}(AD^2 u(t)) + Vu(t) = 0 \quad \text{in distribution sense}.
\]
Since \( A, V \) are constant, we may apply the Fourier Transform, and obtain the existence of a unique solution \( u \in H^1((0, T); L^2(\mathbb{R}^n)) \). Therefore, the solution \( u \in C([0, T]; L^2(\mathbb{R}^n)) \) after being redefined in a set of measure zero, and we have \( u(0) = u_0 \) in \( L^2(\mathbb{R}^n) \).

### 2.2 Stochastic configuration

Here we present the stochastic context, which will be used thoroughly in the paper. To begin, let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space. For each random variable \( f \) in \( L^1(\Omega; \mathbb{P}) \), \((L^1(\Omega)\) for short), we denote its expectation value by
\[
\mathbb{E}[f] = \int_{\Omega} f(\omega) \, d\mathbb{P}(\omega).
\]
A mapping \( \tau : \mathbb{G} \times \Omega \to \Omega \) is said a \( n \)-dimensional dynamical system if:
(i) (Group Property) \( \tau(0, \cdot) = id_\Omega \) and \( \tau(x + y, \omega) = \tau(x, \tau(y, \omega)) \) for all \( x, y \in G \) and \( \omega \in \Omega \).

(ii) (Invariance) The mappings \( \tau(x, \cdot) : \Omega \to \Omega \) are \( P \)-measure preserving, that is, for each \( x \in G \) and every \( E \in \mathcal{F} \), we have
\[
\tau(x, E) \in \mathcal{F}, \quad P(\tau(x, E)) = P(E).
\]

For simplicity, we shall use \( \tau(k)\omega \) to denote \( \tau(k, \omega) \). Moreover, it is usual to say that \( \tau(k) \) is a discrete (continuous) dynamical system if \( k \in \mathbb{Z}^n \) (\( k \in \mathbb{R}^n \)), but we only stress this when it is not obvious from the context.

A measurable function \( f \) on \( \Omega \) is called \( \tau \)-invariant, if for each \( k \in G \)
\[
f(\tau(k)\omega) = f(\omega) \quad \text{for almost all } \omega \in \Omega.
\]

Hence a measurable set \( E \in \mathcal{F} \) is \( \tau \)-invariant, if its characteristic function \( \chi_E \) is \( \tau \)-invariant. In fact, it is straightforward to show that, a \( \tau \)-invariant set \( E \) can be equivalently defined by
\[
\tau(k)E = E \quad \text{for each } k \in G.
\]

Moreover, we say that the dynamical system \( \tau \) is ergodic, when all \( \tau \)-invariant sets \( E \) have measure \( P(E) \) of either zero or one. Equivalently, we may characterize an ergodic dynamical system in terms of invariant functions. Indeed, a dynamical system is ergodic if each \( \tau \)-invariant function is constant almost everywhere, that is to say
\[
\left(f(\tau(k)\omega) = f(\omega) \quad \text{for each } k \in G \text{ and a.e. } \omega \in \Omega \right) \Rightarrow f(\cdot) = \text{const. a.e.}.
\]

**Example 2.4.** Let \( (\Omega_0, \mathcal{F}_0, P_0) \) be a probability space. For \( m \in \mathbb{N} \) fixed, we consider the set \( S = \{0, 1, 2, \ldots, m\} \) and the real numbers \( p_0, p_1, p_2, \ldots, p_m \) in \((0,1)\), such that \( \sum_{\ell=0}^m p_\ell = 1 \). If \( \{X_k : \Omega_0 \to S\}_{k \in \mathbb{Z}^n} \) is a family of random variables, then it is induced a probability measure from it on the measurable space \( (S^{\mathbb{Z}^n}, \bigotimes_{k \in \mathbb{Z}^n} 2^S) \). Indeed, we may define the probability measure \( P(E) := P_0\{X \in E\}, \ E \in \bigotimes_{k \in \mathbb{Z}^n} 2^S \), where the mapping \( X : \Omega_0 \to S^{\mathbb{Z}^n} \) is given by \( X(\omega_0) = (X_k(\omega_0))_{k \in \mathbb{Z}^n} \).

Now, we denote for convenience \( \Omega = S^{\mathbb{Z}^n} \) and \( \mathcal{F} = \bigotimes_{k \in \mathbb{Z}^n} 2^S \), that is, \( \mathcal{F} = \sigma(\mathcal{A}) \), where \( \mathcal{A} \) is the algebra given by the finite union of sets (cylinders of finite base) of the form \( \prod_{k \in \mathbb{Z}^n} E_k \), where \( E_k \in 2^S \) is different from \( S \) for a finite number of indices \( k \). Additionally we assume that, the family \( \{X_k\}_{k \in \mathbb{Z}^n} \) is independent, and for each \( k \in \mathbb{Z}^n \), we have
\[
P_0\{X_k = 0\} = p_0, \ P_0\{X_k = 1\} = p_1, \ldots, P_0\{X_k = m\} = p_m. \quad (2.8)
\]
Then, we may define an ergodic dynamical system $\tau : \mathbb{Z}^n \times \Omega \to \Omega$, by
$$
(\tau(\ell)\omega)(k) := \omega(k + \ell), \text{ for any } k, \ell \in \mathbb{Z}^n, \text{ where } \omega = (\omega(k))_{k \in \mathbb{Z}^n}.
$$

Now, let $(\Gamma, \mathcal{G}, \mathbb{Q})$ be a given probability space. We say that a measurable function $g : \mathbb{R}^n \times \Gamma \to \mathbb{R}$ is stationary, if for any finite set consisting of points $x_1, \ldots, x_j \in \mathbb{R}^n$, and any $k \in \mathcal{G}$, the distribution of the random vector $(g(x_1 + k, \cdot), \ldots, g(x_j + k, \cdot))$ is independent of $k$. Further, subjecting the stationary function $g$ to some natural conditions it can be showed that, there exists other probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a $n-$dimensional dynamical system $\tau : \mathcal{G} \times \Omega \to \Omega$ and a measurable function $f : \mathbb{R}^n \times \Omega \to \mathbb{R}$ satisfying

- For all $x \in \mathbb{R}^n$, $k \in \mathcal{G}$ and $\mathbb{P}-$almost every $\omega \in \Omega$
  $$f(x + k, \omega) = f(x, \tau(k)\omega). \quad (2.9)$$

- For each $x \in \mathbb{R}^n$ the random variables $g(x, \cdot)$ and $f(x, \cdot)$ have the same law. We recall that, the equality almost surely implies equality in law, but the converse is not true.

One remarks that, the set of stationary functions forms an algebra, and also is stable by limit process. For instance, the product of two stationaries functions is a stationary one, and the derivative of a stationary function is stationary. Moreover, the stationarity concept is the most general extension of the notions of periodicity and almost periodicity for a function to have some "self-averaging" behaviour.

**Example 2.5.** Under the conditions of Example 2.4, we take $m = 1$, and consider the following functions, $\varphi_0 = 0$ and $\varphi_1$ be a Lipschitz vector field, such that, $\varphi_1$ is periodic, $\text{supp } \varphi_1 \subset (0,1)^n$. Consequently, the function

$$f(y, \omega) := \varphi_{\omega(\lfloor y \rfloor)}(y), \quad (y, \omega) \in \mathbb{R}^n \times \Omega$$

satisfies, $f(y, \cdot)$ is $\mathcal{F}$-measurable, $f(\cdot, \omega)$ is continuous, and for each $k \in \mathbb{Z}^n$,

$$f(y + k, \omega) = f(y, \tau(k)\omega).$$

Therefore, $f$ is a stationary function.

Next we present the precise definition of the stochastic deformation as presented in [5].
Definition 2.6. A mapping $\Phi : \mathbb{R}^n \times \Omega \to \mathbb{R}^n$, $(y,\omega) \mapsto z = \Phi(y,\omega)$, is called a stochastic deformation (for short $\Phi_\omega$), when satisfies:

i) For $\mathbb{P}$–almost every $\omega \in \Omega$, $\Phi(\cdot, \omega)$ is a bi–Lipschitz diffeomorphism.

ii) There exists $\nu > 0$, such that
$$\text{ess inf}_{\omega \in \Omega, y \in \mathbb{R}^n} (\det(\nabla \Phi(y,\omega))) \geq \nu.$$ 

iii) There exists a $M > 0$, such that
$$\text{ess sup}_{\omega \in \Omega, y \in \mathbb{R}^n} (|\nabla \Phi(y,\omega)|) \leq M < \infty.$$ 

iv) The gradient of $\Phi$, i.e. $\nabla \Phi(y,\omega)$, is stationary in the sense (2.9).

Now, let us present an interesting example of stochastic deformations.

Example 2.7. Under the conditions of Example 2.5, let us consider (for $\eta > 0$) the following map
$$\Phi(y,\omega) := y + \eta \varphi_{\omega(|y|)}(y), \quad (y,\omega) \in \mathbb{R}^n \times \Omega.$$ 

Then, $\nabla_y \Phi(y,\omega) = I_{\mathbb{R}^n \times \mathbb{R}^n} + \eta \nabla \varphi_{\omega(|y|)}(y)$, and for $\eta$ sufficiently small all the conditions in the Definition 2.6 are satisfied. Then, $\Phi$ is a stochastic deformation.

Given a stochastic deformation $\Phi$, let us consider the following spaces
$$\mathcal{L}_\Phi := \{ F(z,\omega) = f(\Phi^{-1}(z,\omega),\omega); f \in L^2_{\text{loc}}(\mathbb{R}^n; L^2(\Omega)) \text{ stationary} \}$$ (2.10) 
and
$$\mathcal{H}_\Phi := \{ F(z,\omega) = f(\Phi^{-1}(z,\omega),\omega); f \in H^1_{\text{loc}}(\mathbb{R}^n; L^2(\Omega)) \text{ stationary} \}$$ (2.11)
which are Hilbert spaces, endowed respectively with the inner products
$$\langle F, G \rangle_{\mathcal{L}_\Phi} := \int_{\Omega} \int_{\Phi([0,1]^n,\omega)} F(z,\omega) \overline{G(z,\omega)} \, dz \, d\mathbb{P}(\omega),$$
$$\langle F, G \rangle_{\mathcal{H}_\Phi} := \int_{\Omega} \int_{\Phi([0,1]^n,\omega)} F(z,\omega) \overline{G(z,\omega)} \, dz \, d\mathbb{P}(\omega)$$
$$+ \int_{\Omega} \int_{\Phi([0,1]^n,\omega)} \nabla_z F(z,\omega) \cdot \overline{\nabla_z G(z,\omega)} \, dz \, d\mathbb{P}(\omega).$$
Remark 2.8. Under the above notations, when $\Phi = \text{Id}$ we denote $L_{\Phi}$ and $H_{\Phi}$ by $L$ and $H$ respectively. Moreover, a function $F \in H_{\Phi}$ if, and only if, $F \circ \Phi \in H$, and there exist constants $C_1, C_2 > 0$, such that

$$C_1 \| F \circ \Phi \|_H \leq \| F \|_{H_{\Phi}} \leq C_2 \| F \circ \Phi \|_H.$$  

Analogously, $F \in L_{\Phi}$ if, and only if, $F \circ \Phi \in L$, and there exist constants $C_1, C_2 > 0$, such that

$$C_1 \| F \circ \Phi \|_L \leq \| F \|_{L_{\Phi}} \leq C_2 \| F \circ \Phi \|_L.$$  

Indeed, let us show the former equivalence. Applying a change of variables, we obtain

$$\| F \|_{H_{\Phi}}^2 = \int_\Omega \int_{\Phi([0,1]^n,\omega)} |F(z,\omega)|^2 \, dz \, d\mathbb{P}(\omega) + \int_\Omega \int_{\Phi([0,1]^n,\omega)} |\nabla_z F(z,\omega)|^2 \, dz \, d\mathbb{P}(\omega)$$

$$= \int_\Omega \int_{[0,1]^n} |f(y,\omega)|^2 \det[\nabla \Phi(y,\omega)] \, dy \, d\mathbb{P}(\omega)$$

$$+ \int_\Omega \int_{[0,1]^n} |[\nabla \Phi(y,\omega)]^{-1} \nabla_z f(y,\omega)|^2 \det[\nabla \Phi(y,\omega)] \, dy \, d\mathbb{P}(\omega).$$

The equivalence follows from the properties of the stochastic deformation $\Phi$.

2.2.1 Ergodic theorems

We begin this section with the concept of mean value, which is in connection with the notion of stationarity. A function $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ is said to possess a mean value if the sequence $\{f(\cdot/\varepsilon)\}_{\varepsilon>0}$ converges in the duality with $L^\infty$ and compactly supported functions to a constant $M(f)$. This convergence is equivalent to

$$\lim_{t \to \infty} \frac{1}{t^n |A|} \int_{A_t} f(x) \, dx = M(f), \quad (2.12)$$

where $A_t := \{x \in \mathbb{R}^n : t^{-1}x \in A\}$, for $t > 0$ and any $A \subset \mathbb{R}^n$, with $|A| \neq 0$.

Remark 2.9. Unless otherwise stated, we assume that the dynamical system $\tau : \Gamma \times \Omega \to \Omega$ is ergodic and we will also use the notation

$$\int_{\mathbb{R}^n} f(x) \, dx \quad \text{for } M(f).$$
Now, we state the result due to Birkhoff, which connects all the notions considered before, see [20].

**Theorem 2.10 (Birkhoff Ergodic Theorem).** Let $f \in L^1_{\text{loc}}(\mathbb{R}^n; L^1(\Omega))$ be a stationary random variable. Then, for almost every $\tilde{\omega} \in \Omega$ the function $f(\cdot, \tilde{\omega})$ possesses a mean value in the sense of (2.12). Moreover, the mean value $M(f(\cdot, \tilde{\omega}))$ as a function of $\tilde{\omega} \in \Omega$ satisfies for almost every $\tilde{\omega} \in \Omega$:

i) **Discrete case (i.e. $\tau : \mathbb{Z}^n \times \Omega \rightarrow \Omega$);**

$$\int_{\mathbb{R}^n} f(x, \tilde{\omega}) \, dx = E \left[ \int_{[0,1)^n} f(y, \cdot) \, dy \right].$$

ii) **Continuous case (i.e. $\tau : \mathbb{R}^n \times \Omega \rightarrow \Omega$);**

$$\int_{\mathbb{R}^n} f(x, \tilde{\omega}) \, dx = E \left[ f(0, \cdot) \right].$$

The following lemma shows that, the Birkhoff Ergodic Theorem holds if a stationary function is composed with a stochastic deformation.

**Lemma 2.11.** Let $\Phi$ be a stochastic deformation and $f \in L^\infty_{\text{loc}}(\mathbb{R}^n; L^1(\Omega))$ be a stationary random variable in the sense (2.9). Then for almost $\tilde{\omega} \in \Omega$ the function $f(\Phi^{-1}(\cdot, \tilde{\omega}), \tilde{\omega})$ possesses a mean value in the sense of (2.12) and satisfies:

i) **Discrete case;**

$$\int_{\mathbb{R}^n} f(\Phi^{-1}(z, \tilde{\omega}), \tilde{\omega}) \, dz = \frac{E \left[ \int_{\Phi([0,1)^n, \cdot)} f(\Phi^{-1}(z, \cdot), \cdot) \, dz \right]}{\det \left( E \left[ \int_{[0,1)^n} \nabla y \Phi(y, \cdot) \, dy \right] \right)} \quad \text{for a.a. } \tilde{\omega} \in \Omega.$$

ii) **Continuous case;**

$$\int_{\mathbb{R}^n} f(\Phi^{-1}(z, \tilde{\omega}), \tilde{\omega}) \, dz = \frac{E \left[ f(0, \cdot) \det (\nabla \Phi(0, \cdot)) \right]}{\det (E [\nabla \Phi(0, \cdot)])} \quad \text{for a.a. } \tilde{\omega} \in \Omega.$$

**Proof.** See Blanc, Le Bris, Lions [8], also Andrade, Neves, Silva [5].

\[ \square \]
2.2.2 Analysis of stationary functions

In the rest of this paper, unless otherwise explicitly stated, we assume discrete dynamical systems and therefore, stationary functions are considered in this discrete sense. We begin the analysis of stationary functions with the concept of realization.

**Definition 2.12.** Let $f : \mathbb{R}^n \times \Omega \to \mathbb{R}$ be a stationary function. For $\omega \in \Omega$ fixed, the function $f(\cdot, \omega)$ is called a realization of $f$.

Due to Theorem 2.10, almost every realization $f(\cdot, \omega)$ possesses a mean value in the sense of (2.12). On the other hand, if $f$ is a stationary function, then the mapping

$$y \in \mathbb{R}^n \mapsto \int_{\Omega} f(y, \omega) \, d\mathbb{P}(\omega)$$

is a $[0, 1)^n$-periodic function.

In fact, it is enough to consider the realizations to study some properties of stationary functions. For instance, the following theorem will be used more than once through this paper.

**Theorem 2.13.** For $p > 1$, let $u, v \in L^1_{\text{loc}}(\mathbb{R}^n; L^p(\Omega))$ be stationary functions. Then, for any $i \in \{1, \ldots, n\}$ fixed, the following sentences are equivalent:

\begin{align}
(A) \quad & \int_{[0,1)^n} \int_{\Omega} u(y, \omega) \frac{\partial \zeta}{\partial y_i}(y, \omega) \, d\mathbb{P}(\omega) \, dy = - \int_{[0,1)^n} \int_{\Omega} v(y, \omega) \zeta(y, \omega) \, d\mathbb{P} \, dy, \\
& \text{for each stationary function } \zeta \in C^1(\mathbb{R}^n; L^q(\Omega)), \text{ with } 1/p + 1/q = 1. \tag{2.13}
\end{align}

\begin{align}
(B) \quad & \int_{\mathbb{R}^n} u(y, \omega) \frac{\partial \varphi}{\partial y_i}(y) \, dy = - \int_{\mathbb{R}^n} v(y, \omega) \varphi(y) \, dy, \tag{2.14}
\end{align}

for any $\varphi \in C^1_c(\mathbb{R}^n)$, and almost sure $\omega \in \Omega$.

**Proof.** See Blanc, Le Bris, Lions [8].

Similarly to the above theorem, we have the characterization of weak derivatives of stationary functions composed with stochastic deformations, given by the following
Theorem 2.14. Let $u, v \in L^1_{\text{loc}}(\mathbb{R}^n; L^p(\Omega))$ be stationary functions, $(p > 1)$. Then, for any $i \in \{1, \ldots, n\}$ fixed, the following sentences are equivalent:

\[(A) \quad \int_{\Omega} \int_{\Phi((0,1)^n, \omega)} u(\Phi^{-1}(z, \omega), \omega) \frac{\partial(\zeta(\Phi^{-1}(z, \omega), \omega))}{\partial z_k} \, dz \, d\mathbb{P}(\omega) = - \int_{\Omega} \int_{\Phi((0,1)^n, \omega)} v(\Phi^{-1}(z, \omega), \omega) \zeta(\Phi^{-1}(z, \omega), \omega) \, dz \, d\mathbb{P}(\omega),\]

for each stationary function $\zeta \in C^1(\mathbb{R}^n; L^q(\Omega))$, with $1/p + 1/q = 1$.

\[(B) \quad \int_{\mathbb{R}^n} u(\Phi^{-1}(z, \omega), \omega) \frac{\partial \varphi}{\partial z_k}(z) \, dz = - \int_{\mathbb{R}^n} v(\Phi^{-1}(z, \omega), \omega) \varphi(z) \, dz,\]

for any $\varphi \in C^1(\mathbb{R}^n)$, and almost sure $\omega \in \Omega$.

2.3 $\Phi_\omega$—Two-scale Convergence

In this subsection, we shall consider the two-scale convergence in a stochastic setting that is beyond of the classical stationary ergodic setting. The classical concept of two-scale convergence was introduced by Nguetseng [22] and further developed by Allaire [1] to deal with periodic problems.

The notion of two-scale convergence has been successfully extended to non-periodic settings in several papers as in [17, 16] in the ergodic algebra setting and in [10] in the stochastic setting. The main difference here with the earlier studies is that the test functions used here are random perturbations accomplished by stochastic diffeomorphisms of stationary functions. The main difficulty brought by this kind of test function is the lack of the stationarity property (see [5] for a deep discussion about that) which makes us unable to use the results described in [10] and the lack of a compatible topology with the probability space considered. This is overcome by using a compactification argument that preserves the ergodic nature of the setting involved. For this, we will make use of the following lemma, whose simple proof can be found in [4].

Lemma 2.15. Let $X_1, X_2$ be compact spaces, $R_1$ a dense subset of $X_1$ and $W : R_1 \to X_2$. Suppose that for all $g \in C(X_2)$ the function $g \circ W$ is the restriction to $R_1$ of some (unique) $g_1 \in C(X_1)$. Then $W$ can be uniquely extended to a continuous mapping $\tilde{W} : X_1 \to X_2$. Further, suppose in addition
that $R_2$ is a dense set of $X_2$, $W$ is a bijection from $R_1$ onto $R_2$ and for all $f \in C(X_1)$, $f \circ W^{-1}$ is the restriction to $R_2$ of some (unique) $f_2 \in C(X_2)$. Then, $W$ can be uniquely extended to a homeomorphism $\overline{W} : X_1 \to X_2$.

Now, we can prove the following result.

**Theorem 2.16.** Let $S \subset L^\infty(\mathbb{R}^n \times \Omega)$ be a countable set of stationary functions. Then there exists a compact (separable) topological space $\tilde{\Omega}$ and one-to-one function $\delta : \Omega \to \tilde{\Omega}$ with dense image satisfying the following properties:

(i) The probability space $\left(\tilde{\Omega}, \mathcal{F}, \tilde{P}\right)$ and the ergodic dynamical system $\tau : \mathbb{Z}^n \times \Omega \to \Omega$ acting on it extends respectively to a Radon probability space $\left(\tilde{\Omega}, \mathcal{B}, \tilde{P}\right)$ and to an ergodic dynamical system $\tilde{\tau} : \mathbb{Z}^n \times \tilde{\Omega} \to \tilde{\Omega}$.

(ii) The stochastic deformation $\Phi : \mathbb{R}^n \times \Omega \to \mathbb{R}^n$ extends to a stochastic deformation $\tilde{\Phi} : \mathbb{R}^n \times \tilde{\Omega} \to \mathbb{R}^n$ satisfying

$$\Phi(x, \omega) = \tilde{\Phi}(x, \delta(\omega)),$$

for a.e. $\omega \in \Omega$.

(iii) Any function $f \in S$ extends to a $\tilde{\tau}$-stationary function $\tilde{f} \in L^\infty(\tilde{\Omega} \times \mathbb{R}^n)$ satisfying

$$\int_{\mathbb{R}^n} f(\Phi^{-1}(z, \omega), \omega) \, dz = \int_{\mathbb{R}^n} \tilde{f}(\tilde{\Phi}^{-1}(z, \delta(\omega)), \delta(\omega)) \, dz,$$

for a.e. $\omega \in \Omega$.

**Proof.** 1. Let $S$ be the set of the lemma. Given $f \in S$, define

$$f_j(y, \omega) := \int_{\mathbb{R}^n} f(y + x, \omega) \, \rho_j(x) \, dx,$$

where $\rho_j$ is the classical approximation of the identity in $\mathbb{R}^n$. Note that for a.e. $y \in \mathbb{R}^n$, we have that $f_j(y, \cdot) \to f(y, \cdot)$ in $L^1(\Omega)$ as $j \to \infty$. Define $\mathcal{A}$ as the closed algebra with unity generated by the set

$$\left\{ f_j(y, \cdot); j \geq 1, y \in \mathbb{Q}^n, f \in S \right\} \cap \left\{ \partial_{y_i} \Phi_i(y, \cdot); 1 \leq j, i \leq n, y \in \mathbb{Q}^n \right\}.$$
Since $[-1, 1]$ is a compact set, by the well known Tychonoff’s Theorem, the set

$$[-1, 1]^A := \{\text{the functions } \gamma : A \to [-1, 1]\}$$

is a compact set in the product topology. Define $\delta : \Omega \to [-1, 1]^A$ by

$$\delta(\omega)(g) := \begin{cases} \frac{g(\omega)}{\|g\|_{\infty}}, & \text{if } g \neq 0, \\ 0, & \text{if } g = 0. \end{cases}$$

We may assume that the algebra $A$ distinguishes between points of $\Omega$, that is, given any two distinct points $\omega_1, \omega_2 \in \Omega$, there exists $g \in A$ such that $g(\omega_1) \neq g(\omega_2)$. In the case that it is not true we may replace $\Omega$ by its quotient by a trivial equivalence relation, in a standard way, and we proceed correspondingly with the $\sigma$–algebra $\mathcal{F}$ and with the probability measure $\mathbb{P}$. Thus, the function $\delta$ is one-to-one. Define

$$\tilde{\Omega} := \overline{\delta(\Omega)}.$$

Then, we can see that the set $\Omega$ inherits all topological features of the compact space $\tilde{\Omega}$ in a natural way which allows us to identify it homeomorphically with the image $\delta(\Omega)$.

2. Define the mapping $i : A \to C(\delta(\Omega))$ by

$$i(g)(\delta(\omega)) := g(\omega).$$

We claim that there exists a continuous function $\tilde{g} : \tilde{\Omega} \to \mathbb{R}$ such that

$$i(g) = \tilde{g} \text{ on } \delta(\Omega).$$

In fact, take $g \in A$ and $Y := g(\Omega)$. Define the function $f^* : C(Y) \to A$ by

$$f^*(h) := h \circ g \text{ (the algebra structure is used!) }$$

Hence, we can define $f^{**} : [-1, 1]^A \to [-1, 1]^{C(Y)}$ by $f^{**}(h) := h \circ f^*$. Note that the function $f^{**}$ is a continuous function. In order to see that, we highlighted that it is known that a function $H$ from a topological space to a product space $\otimes_{\alpha \in I} X_\alpha$ is continuous if and only if each component $\pi_\alpha \circ H := H_\alpha$ is continuous. Hence, if $\alpha \in C(Y)$ then the projection function $f^{**}_\alpha$ must satisfy

$$f^{**}_\alpha(h) := (\pi_\alpha \circ f^{**})(h) = \pi_\alpha \circ (f^{**}(h)) = \pi_\alpha (h \circ f^*) = (h \circ f^*)(\alpha) = h(f^*(\alpha)) = h(\alpha \circ g) = \pi_{\alpha \circ g}(h).$$
Now, consider the function \( \tilde{\delta} : Y \to [-1, 1]^C(Y) \) given by

\[
\tilde{\delta}(y)(h) := \begin{cases} 
\frac{b(y)}{\|h\|_\infty}, & \text{if } h \neq 0, \\
0, & \text{if } h = 0.
\end{cases}
\]

Since the algebra \( C(Y) \) has the following property: If \( F \subset Y \) is a closed set and \( y \notin F \) then \( f(y) \notin f(F) \) for some \( f \in C(Y) \), we can conclude that the function \( \tilde{\delta} \) is a homeomorphism onto its image. Furthermore, given \( \omega \in \Omega \) we have that \((f^{**} \circ \delta)(\omega) = f^{**}(\delta(\omega)) = \delta(\omega) \circ f^*\). Hence, if \( 0 \neq h \in C(Y) \) it follows that

\[
(f^{**} \circ \delta)(\omega)(h) = (\delta(\omega) \circ f^*)(h) = \delta(\omega)(f^*(h))
\]

\[
= \delta(\omega)(h \circ g) = \frac{(h \circ g)(\omega)}{\|h \circ g\|_\infty} = \tilde{\delta}(g(\omega))(h).
\]

Thus, we see that \( \tilde{\delta}^{-1} \circ f^{**} = i(g) \). Defining \( \tilde{g} := \tilde{\delta}^{-1} \circ f^{**} \) we have clearly that \( i : \mathcal{A} \to C(\tilde{\Omega}) \) is a one-to-one isometry satisfying \( i(g) = \tilde{g} \) and our claim is proved. Moreover, it is easy to see that \( i(\mathcal{A}) \) is an algebra of functions over \( C(\tilde{\Omega}) \) containing the unity. As before, if \( i(\mathcal{A}) \) does not separate points of \( \tilde{\Omega} \), then we may replace \( \tilde{\Omega} \) by its quotient \( \tilde{\Omega}/\sim \), where

\[
\tilde{\omega}_1 \sim \tilde{\omega}_2 \iff \tilde{g}(\tilde{\omega}_1) = \tilde{g}(\tilde{\omega}_2) \forall g \in \mathcal{A}.
\]

Therefore, we can assume that \( i(\mathcal{A}) \) separates the points of \( \tilde{\Omega} \). Hence by the Stone’s Theorem we must have \( i(\mathcal{A}) = C(\tilde{\Omega}) \).

3. Define \( \tilde{\tau} : \mathbb{Z}^n \times \delta(\Omega) \to \delta(\Omega) \) by \( \tilde{\tau}_k(\delta(\omega)) := \delta(\tau_k \omega) \). It is easy to see that

\[
\tilde{\tau}_{k_1+k_2}(\delta(\omega)) = \tilde{\tau}_{k_1}(\tilde{\tau}_{k_2}(\delta(\omega))),
\]

for all \( k_1, k_2 \in \mathbb{Z}^n \) and \( \omega \in \Omega \). Since \( \tilde{g} \circ \tilde{\tau}_k = \tilde{g} \circ \tilde{\tau}_k \) for all \( g \in \mathcal{A} \) and \( k \in \mathbb{Z}^n \), the lemma 2.15 allows us to extend the mapping \( \tilde{\tau}_k \) from \( \delta(\Omega) \) to \( \tilde{\Omega} \) satisfying the group property \( \tilde{\tau}_{k_1+k_2} = \tilde{\tau}_{k_1} \circ \tilde{\tau}_{k_2} \). Given a borelian set \( \tilde{\mathcal{A}} \subset \tilde{\Omega} \) and defining \( \tilde{\mathbb{P}}(\mathcal{A}) := \mathbb{P}(\delta^{-1}(\mathcal{A} \cap \delta(\Omega))) \), we can deduce that \( \tilde{\mathbb{P}} \circ \tilde{\tau}_k = \tilde{\mathbb{P}} \). Thus, the mapping \( \tilde{\tau}_k \) is an ergodic dynamical system over the Radon probability space \( (\tilde{\Omega}, \mathcal{B}, \tilde{\mathbb{P}}) \). Thus, we have concluded the proof of the item (i).

4. Now, note that for each \( \omega \in \tilde{\Omega} \) and each integer \( j \geq 1 \), the function \( f_j(\cdot, \omega) \) is uniformly continuous over \( \mathbb{Q}^n \). Hence, it can be extended uniquely
to a function $\tilde{f}_j(\cdot, \omega)$ defined in $\mathbb{R}^n$ that satisfies
\[
\limsup_{j,l \to \infty} \int_{[0,1)^n \times \tilde{\Omega}} |\tilde{f}_j(y, \omega) - \tilde{f}_l(y, \omega)| \, d\tilde{P}(\omega) \, dy = 0.
\]
Therefore, there exists a $\tilde{\tau}$-stationary function $\tilde{f} \in L^1_{\text{loc}} (\mathbb{R}^n \times \tilde{\Omega})$, such that $\tilde{f}_j \to \tilde{f}$ as $j \to \infty$ in $L^1_{\text{loc}} (\mathbb{R}^n \times \tilde{\Omega})$. Since $\|\tilde{f}_j\|_{\infty} \leq \|f\|_{\infty}$, for all $j \geq 1$ we have that $\tilde{f} \in L^\infty (\mathbb{R}^n \times \tilde{\Omega})$. In the same way, the stochastic deformation $\Phi : \mathbb{R}^n \times \Omega \to \mathbb{R}^n$ extends to a stochastic deformation $\tilde{\Phi} : \mathbb{R}^n \times \tilde{\Omega} \to \mathbb{R}^n$ satisfying $\Phi(y, \omega) = \tilde{\Phi}(y, \delta(\omega))$ for all $\omega \in \Omega$ and
\[
\int_{\mathbb{R}^n} f \left( \Phi^{-1}(z, \omega), \omega \right) \, dz = \int_{\mathbb{R}^n} \tilde{f} \left( \tilde{\Phi}^{-1}(z, \delta(\omega)), \delta(\omega) \right) \, dz,
\]
for a.e. $\omega \in \Omega$. This, completes the proof of the theorem (2.16).

In practice, in the present context, the set $\mathcal{S}$ shall be a countable set generated by the coefficients of our equation (1.1) and by the eigenfunctions of the spectral equation associated to it. Thus, the Theorem (2.16) allow us to suppose without loss of generality that our probability space $\left( \Omega, \mathcal{F}, P \right)$ is a separable compact space. Using the Ergodic Theorem, given a stationary function $f \in L^\infty (\mathbb{R}^n \times \Omega)$ there exists a set of full measure $\Omega_f \subset \Omega$ such that
\[
\int_{\mathbb{R}^n} f \left( \Phi^{-1}(z, \omega), \omega \right) \, dz = c_{\Phi}^{-1} \int_{\Omega} \int_{\Phi([0,1)^n, \omega)} f \left( \Phi^{-1}(z, \omega), \omega \right) \, dz \, dP(\omega),
\]
(2.15)
for almost all $\tilde{\omega} \in \Omega_f$. Due to the separability of the probability compact space $\left( \Omega, \mathcal{F}, P \right)$, we can find a set $\mathcal{D} \subset C^0_{\text{b}} (\mathbb{R}^n \times \Omega)$ such that:

- Each $f \in \mathcal{D}$ is a stationary function.
- $\mathcal{D}$ is a countable and dense set in $C^0_0 ([0,1)^n \times \Omega)$.

In this case, there exists a set $\Omega_0 \subset \Omega$ of full measure such that the equality (2.15) holds for any $\tilde{\omega} \in \Omega_0$ and $f \in \mathcal{D}$.

Now, we proceed with the definition of the two-scale convergence in this scenario of stochastically deformed. In what follows, the set $O \subset \mathbb{R}^n$ is an open set.
Definition 2.17. Let $1 < p < \infty$ and $v_\varepsilon : O \times \Omega \to \mathbb{C}$ be a sequence such that $v_\varepsilon(\cdot, \tilde{\omega}) \in L^p(O)$. The sequence $\{v_\varepsilon(\cdot, \tilde{\omega})\}_{\varepsilon > 0}$ is said to converge to a stationary function $V_{\tilde{\omega}} \in L^p(O \times [0,1)^n \times \Omega)$, when for a.e. $\tilde{\omega} \in \Omega$ holds the following

$$\lim_{\varepsilon \to 0} \int_O v_\varepsilon(x, \tilde{\omega}) \varphi(x) \Theta \left( \Phi^{-1} \left( \frac{x}{\varepsilon}, \tilde{\omega} \right), \tilde{\omega} \right) dx = c_\Phi^{-1} \int_{O \times \Omega} \int_{\Phi([0,1)^n \omega)} V_{\tilde{\omega}}(x, \Phi^{-1}(z, \omega), \omega) \varphi(x) \Theta(\Phi^{-1}(z, \omega), \omega) dz d\mathbb{P}(\omega) dx,$$

for all $\varphi \in C_c^\infty(O)$ and $\Theta \in L^q_{\text{loc}}(\mathbb{R}^n \times \Omega)$ stationary. Here, $p^{-1} + q^{-1} = 1$ and $c_\Phi := \det \left( \int_{[0,1)^n \times \Omega} \nabla \Phi(y, \omega) d\mathbb{P}(\omega) dy \right)$.

Remark 2.18. From now on, we shall use the notation

$$v_\varepsilon(x, \tilde{\omega}) \overset{2-s}{\underset{\varepsilon \to 0}{\rightharpoonup}} V_{\tilde{\omega}}(x, \Phi^{-1}(z, \omega), \omega),$$

to indicate that $v_\varepsilon(\cdot, \tilde{\omega}) \Phi_\omega$–two-scale converges to $V_{\tilde{\omega}}$.

The most important result about the two-scale convergence needed in this paper is the following compactness theorem which generalizes the corresponding one for the deterministic case in [16] (see Theorem 4.8) and the corresponding one for the stochastic case in [10] (see Theorem 3.4).

Theorem 2.19. Let $1 < p < \infty$ and $v_\varepsilon : O \times \Omega \to \mathbb{C}$ be a sequence such that

$$\sup_{\varepsilon > 0} \int_O |v_\varepsilon(x, \tilde{\omega})|^p dx < \infty,$$

for almost all $\tilde{\omega} \in \Omega$. Then, for almost all $\tilde{\omega} \in \Omega_0$, there exists a subsequence $\{v_{\varepsilon'}(\cdot, \tilde{\omega})\}_{\varepsilon' > 0}$, which may depend on $\tilde{\omega}$, and a stationary function $V_{\tilde{\omega}} \in L^p(O \times [0,1)^n \times \Omega)$, such that

$$v_\varepsilon(x, \tilde{\omega}) \overset{2-s}{\underset{\varepsilon' \to 0}{\rightharpoonup}} V_{\tilde{\omega}}(x, \Phi^{-1}(z, \omega), \omega).$$

Proof. 1. We begin by fixing $\tilde{\omega} \in \Omega_0$. Due to our assumption, there exists $c(\tilde{\omega}) > 0$, such that for all $\varepsilon > 0$

$$\|v_\varepsilon(\cdot, \tilde{\omega})\|_{L^p(O)} \leq c(\tilde{\omega}).$$
Now, taking $\phi \in \Xi \times D$ with $\Xi \subset C^\infty_c(O)$ dense in $L^q(O)$, we have after applying the H"older inequality and the Ergodic Theorem,

$$\limsup_{\epsilon \to 0} |\int_O v_\epsilon(x, \tilde{\omega}) \phi(x, \Phi^{-1}\left(\frac{x}{\epsilon}, \tilde{\omega}\right)) \, dx|$$

$$\leq c(\tilde{\omega}) \left[ \limsup_{\epsilon \to 0} \int_O |\phi(x, \Phi^{-1}\left(\frac{x}{\epsilon}, \tilde{\omega}\right))|^q \, dx \right]^{1/q}$$

$$= c(\tilde{\omega}) \left[ c_{\Phi}^{-1} \int_O \int_\Omega \int_{\Phi(0,1)^n, \omega} |\phi(x, \Phi^{-1}(z, \omega), \omega)|^q \, dz \, d\mathbb{P}(\omega) \, dx \right]^{1/q}. \quad (2.16)$$

Thus, the use of the enumerability of the set $\Xi \times D$ combined with a diagonal argument yields us a subsequence $\{\epsilon'\}$ (maybe depending of $\tilde{\omega}$) such that the functional $\mu: \Xi \times D \rightarrow \mathbb{C}$ given by

$$\langle \mu, \phi \rangle := \lim_{\epsilon' \to 0} \int_O v_{\epsilon'}(x, \tilde{\omega}) \phi(x, \Phi^{-1}\left(\frac{x}{\epsilon'}, \tilde{\omega}\right)) \, dx \quad (2.17)$$

is well-defined and bounded with respect to the norm $\| \cdot \|_q$ defined as

$$\|\phi\|_q := \left[ c_{\Phi}^{-1} \int_O \int_\Omega \int_{\Phi(0,1)^n, \omega} |\phi(x, \Phi^{-1}(z, \omega), \omega)|^q \, dz \, d\mathbb{P}(\omega) \, dx \right]^{1/q}$$

by $(2.16)$. Since the set $\Xi \times D$ is dense in $L^q(O \times [0,1)^n \times \Omega)$, we can extend the functional $\mu$ to a bounded functional $\tilde{\mu}$ over $L^q(O \times [0,1)^n \times \Omega)$. Hence, we find $V_{\tilde{\omega}} \in L^p(O \times [0,1)^n \times \Omega)$ which can be extended to $O \times \mathbb{R}^n \times \Omega$ in a stationary way by setting

$$V_{\tilde{\omega}}(x, y, \omega) = V_{\tilde{\omega}}(x, y - [y], \tau([y], \omega),)$$

and satisfying for all $\phi \in L^q(O \times [0,1)^n \times \Omega)$,

$$\langle \tilde{\mu}, \phi \rangle := c_{\Phi}^{-1} \int_{O \times \Omega} \int_{\Phi(0,1)^n, \omega} V_{\tilde{\omega}}(x, \Phi^{-1}(z, \omega), \omega) \phi(x, \Phi^{-1}(z, \omega), \omega) \, dz \, d\mathbb{P}(\omega) \, dx.$$

2. Now, take $\varphi \in C^\infty_c(O)$ and $\Theta \in L^q_{10\epsilon}(\mathbb{R}^n \times \Omega)$ a $\tau$-stationary function. Since the set $\Xi \times D$ is dense in $L^q(O \times [0,1)^n \times \Omega)$, we can pick up a sequence $\{((\varphi_j, \Theta_j))\}_{j \geq 1} \subset \Xi \times D$ such that

$$\lim_{j \to \infty} (\varphi_j, \Theta_j) = (\varphi, \Theta) \quad \text{in} \quad L^q\left(O \times [0,1)^n \times \Omega\right).$$
Then, observing that
\[
\limsup_{\varepsilon' \to 0} \left| \int_O v_{\varepsilon'}(x, \tilde{\omega}) \varphi(x) \theta \left( \Phi^{-1} \left( \frac{x}{\varepsilon'}, \tilde{\omega} \right) , \tilde{\omega} \right) \right| dx
\]
\[
- \int_O v_{\varepsilon'}(x, \tilde{\omega}) \varphi_j(x) \theta_j \left( \Phi^{-1} \left( \frac{x}{\varepsilon'}, \tilde{\omega} \right) , \tilde{\omega} \right) \right| dx
\]
\[
\leq c \| \varphi - \varphi_j \|_{L^q(O)} \left[ c^{-1} \int_{\theta_j} \left( \Phi - 1 \left( \frac{x}{\varepsilon'}, \tilde{\omega} \right) , \tilde{\omega} \right) \right] d\mathbb{P}(\omega),
\]
where \( c = c(\tilde{\omega}) \) is a positive constant. Then, combining the previous equality with the (2.17), we concluded the proof of the theorem.

Let us remember the following space (see Remark 2.8)
\[
\mathcal{H} := \{ w \in H^1_{\text{loc}}(\mathbb{R}^n; L^2(\Omega)); w \text{ is a stationary function} \},
\]
which is a Hilbert space with respect to the following inner product
\[
\langle w, v \rangle_{\mathcal{H}} := \int_{(0,1)^n \times \Omega} \nabla_y w(y, \omega) \cdot \nabla_y v(y, \omega) d\mathbb{P}(\omega) dy
\]
\[
+ \int_{(0,1)^n \times \Omega} w(y, \omega) v(y, \omega) d\mathbb{P}(\omega) dy.
\]
The next lemma will be important in the homogenization’s process.

**Lemma 2.20.** Let \( O \subset \mathbb{R}^n \) be an open set and assume that \( \{ u_{\varepsilon} (\cdot, \tilde{\omega}) \}_{\varepsilon > 0} \) and \( \{ \varepsilon \nabla u_{\varepsilon} (\cdot, \tilde{\omega}) \}_{\varepsilon > 0} \) are bounded sequences in \( L^2(O) \) and in \( L^2(O; \mathbb{R}^n) \) respectively for a.e. \( \tilde{\omega} \in \Omega \). Then, for a.e. \( \tilde{\omega} \in \Omega \), there exists a subsequence \( \{ \varepsilon' \} \) (it may depend on \( \tilde{\omega} \)) and \( u_{\tilde{\omega}} \in L^2(O; \mathcal{H}) \), such that
\[
u_{\varepsilon'} (\cdot, \tilde{\omega}) \quad \overset{2-s}{\underset{\varepsilon \to 0}{\rightharpoonup}} \quad u_{\tilde{\omega}},
\]
and
\[
\varepsilon' \nabla u_{\varepsilon'} (\cdot, \tilde{\omega}) \quad \overset{2-s}{\underset{\varepsilon \to 0}{\rightharpoonup}} \quad [\nabla_y \Phi]^{-1} \nabla_y u_{\tilde{\omega}}.
\]

**Proof.** Applying the Theorem 2.19 for the sequences
\[
\{ u_{\varepsilon} (\cdot, \tilde{\omega}) \}_{\varepsilon > 0}, \quad \{ \varepsilon \nabla u_{\varepsilon} (\cdot, \tilde{\omega}) \}_{\varepsilon > 0}
\]

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for a.e. \( \tilde{\omega} \in \Omega \), we can find a subsequence \( \{ \varepsilon' \} \), and functions

\[
\varepsilon' \to 0 \quad \varepsilon' \\
\] 

such that

\[
u_{\varepsilon'}(\cdot, \tilde{\omega}) \rightarrow \nu_{\tilde{\omega}}, \quad (2.18)
\]

and

\[
\varepsilon' \frac{\partial \nu_{\varepsilon'}}{\partial x_k} \rightarrow \nu_{\tilde{\omega}}^{(k)} . \quad (2.19)
\]

Hence for each \( k \in \{1, \ldots, n\} \) and performing an integration by parts we have

\[
\int_{O} \varepsilon' \frac{\partial \nu_{\varepsilon'}}{\partial x_k}(x, \tilde{\omega}) \varphi(x) \Theta \left( \Phi^{-1} \left( \frac{x}{\varepsilon'}, \tilde{\omega} \right), \tilde{\omega} \right) dx \\
= -\varepsilon' \int_{O} \nu_{\varepsilon'}(x, \tilde{\omega}) \frac{\partial \varphi}{\partial x_k}(x) \Theta \left( \Phi^{-1} \left( \frac{x}{\varepsilon'}, \tilde{\omega} \right), \tilde{\omega} \right) dx \\
- \int_{O} \nu_{\varepsilon'}(x, \tilde{\omega}) \varphi(x) \left[ \nabla_y \Phi \right]^{-1} \left( \Phi^{-1} \left( \frac{x}{\varepsilon'}, \tilde{\omega} \right), \tilde{\omega} \right) \nabla_y \Theta \left( \Phi^{-1} \left( \frac{x}{\varepsilon'}, \tilde{\omega} \right), \tilde{\omega} \right) \cdot e_k dx,
\]

for every \( \varphi \in C^\infty_c(O) \) and \( \Theta \in C^\infty_c([0,1)^n; L^\infty(\Omega)) \) extended in a stationary way to \( \mathbb{R}^n \). Then, using the relations (2.18)-(2.19) and a density argument in the space of the test functions, we arrive after letting \( \varepsilon' \rightarrow 0 \)

\[
\int_{\Omega} \int_{\Phi([0,1)^n, \omega)} \nu_{\omega}^{(k)}(x, \Phi^{-1}(z, \omega), \omega) \Theta \left( \Phi^{-1}(z, \omega), \omega \right) d\mathbb{P} \, dz \\
= -\int_{\Omega} \int_{\Phi([0,1)^n, \omega)} u_{\omega}(x, \Phi^{-1}(z, \omega), \omega) \frac{\partial}{\partial z_k} \left( \Theta \left( \Phi^{-1}(z, \omega), \omega \right) \right) d\mathbb{P} \, dz,
\]

for a.e. \( x \in O \) and for any \( \Theta \in C^\infty_c([0,1)^n; L^\infty(\Omega)) \).

Hence applying Theorem 2.14 we obtain

\[
\int_{\mathbb{R}^n} \nu_{\omega}^{(k)}(x, \Phi^{-1}(z, \omega), \omega) \varphi(z) \, dz = -\int_{\mathbb{R}^n} u_{\omega}(x, \Phi^{-1}(z, \omega), \omega) \frac{\partial \varphi}{\partial z_k}(z) \, dz,
\]

for all \( \varphi \in C^\infty_c(\mathbb{R}^n) \) and a.e. \( \omega \in \Omega \). This completes the proof of our lemma.

\[\square\]
2.4 Bloch waves analysis. The WKB method

Here we formally obtain the Bloch spectral cell equation (stochastic setting). To this end, we apply the asymptotic Wentzel-Kramers-Brillouin (WKB for short) expansion method, that is, we assume that the solution of equation (1.1) is given by a plane wave, see Definition 2.27. More precisely, for each \( \varepsilon > 0 \) let us assume that, the solution \( u_\varepsilon(t, x, \omega) \) of the equation (1.1) has the following asymptotic expansion

\[
u_\varepsilon(t, x, \omega) = e^{2\pi i S_\varepsilon(t, x)} \sum_{k=1}^{\infty} \varepsilon^k u_k(t, x, \Phi^{-1}
\left(\frac{x}{\varepsilon}, \omega\right), \omega), \tag{2.20}
\]

where the functions \( u_k(t, x, y, \omega) \) are conveniently stationary in \( y \), and \( S_\varepsilon \) is a real valued function to be established a posteriori (not necessarily a polynomial in \( \varepsilon \)), which take part of the modulated plane wave (2.20) from \( e^{2\pi i S_\varepsilon(t, x)} \).

The spatial derivative of the above ansatz (2.20) is

\[
\nabla u_\varepsilon(t, x, \omega) = e^{2\pi i S_\varepsilon(t, x)} \left( 2i\pi \nabla S_\varepsilon(t, x) \sum_{k=0}^{\infty} \varepsilon^k u_k(t, x, \Phi^{-1}
\left(\frac{x}{\varepsilon}, \omega\right), \omega) \right)
+ \sum_{k=0}^{\infty} \varepsilon^k \left\{ (\partial_x u_k) \left(t, x, \Phi^{-1}
\left(\frac{x}{\varepsilon}, \omega\right), \omega\right) + \frac{1}{\varepsilon}(\nabla \Phi)^{-1}
\left(\Phi^{-1}
\left(\frac{x}{\varepsilon}, \omega\right), \omega\right) (\partial_y u_k) \left(t, x, \Phi^{-1}
\left(\frac{x}{\varepsilon}, \omega\right), \omega\right) \right\}
= e^{2i\pi S_\varepsilon(t, x)} \left( \sum_{k=0}^{\infty} \varepsilon^k \left( \frac{\nabla_x u_k}{\varepsilon} + 2i\pi \nabla S_\varepsilon(t, x) \right) u_k(t, x, \Phi^{-1}
\left(\frac{x}{\varepsilon}, \omega\right), \omega) \right)
+ \sum_{k=0}^{\infty} \varepsilon^k \left( \nabla_x u_k \right) \left(t, x, \Phi^{-1}
\left(\frac{x}{\varepsilon}, \omega\right), \omega\right) .
\]

Now, computing the second derivatives of the expansion (2.20) and writing
as a cascade of the power of $\varepsilon$, we have

$$
e^{-2i\pi S_\varepsilon(t,x)} \text{div} \left( A(\Phi^{-1}\left(\frac{x}{\varepsilon}, \omega\right), \omega) \nabla u_\varepsilon(t,x, \omega) \right)
= \frac{1}{\varepsilon^2} \left( \text{div}_z + 2i\pi\varepsilon \nabla S_\varepsilon(t,x) \right) \left( A(\Phi^{-1}(\cdot, \omega), \omega) \left( \nabla_z + 2i\pi\varepsilon \nabla S_\varepsilon(t,x) \right) \right)
\left. u_0(t,x,\Phi^{-1}(\cdot,\omega),\omega) \right|_{z=x/\varepsilon}
+ \frac{1}{\varepsilon} \left( \text{div}_z + 2i\pi\varepsilon \nabla S_\varepsilon(t,x) \right) \left( A(\Phi^{-1}(\cdot, \omega), \omega) \left( \nabla_z + 2i\pi\varepsilon \nabla S_\varepsilon(t,x) \right) \right)
\left. u_1(t,x,\Phi^{-1}(\cdot,\omega),\omega) \right|_{z=x/\varepsilon} + I_\varepsilon,
$$

(2.21)

where

$$I_\varepsilon = \sum_{k=0}^{\infty} \varepsilon^k \left( \text{div}_z + 2i\pi\varepsilon \nabla S_\varepsilon(t,x) \right) \left( A(\Phi^{-1}(\cdot, \omega), \omega) \left( \nabla_z + 2i\pi\varepsilon \nabla S_\varepsilon(t,x) \right) \right)
\left. u_{k+2}(t,x,\Phi^{-1}(\cdot,\omega),\omega) \right|_{z=x/\varepsilon}
+ \frac{1}{\varepsilon} \left( \text{div}_z + 2i\pi\varepsilon \nabla S_\varepsilon(t,x) \right) \left( A(\Phi^{-1}(\cdot, \omega), \omega) \nabla_x u_0(t,x,\Phi^{-1}(\cdot,\omega),\omega) \right)
\left. \right|_{z=x/\varepsilon}
+ \sum_{k=0}^{\infty} \varepsilon^k \left( \text{div}_z + 2i\pi\varepsilon \nabla S_\varepsilon(t,x) \right) \left( A(\Phi^{-1}(\cdot, \omega), \omega) \nabla_x u_{k+1}(t,x,\Phi^{-1}(\cdot,\omega),\omega) \right)
\left. \right|_{z=x/\varepsilon}
+ \frac{1}{\varepsilon} \text{div}_x \left( A(\Phi^{-1}(\cdot, \omega), \omega) \left( \nabla_z + 2i\pi\varepsilon \nabla S_\varepsilon(t,x) \right) u_0(t,x,\Phi^{-1}(\cdot,\omega),\omega) \right)
\left. \right|_{z=x/\varepsilon}
+ \sum_{k=0}^{\infty} \varepsilon^k \text{div}_x \left( A(\Phi^{-1}(\cdot, \omega), \omega) \left( \nabla_z + 2i\pi\varepsilon \nabla S_\varepsilon(t,x) \right) u_{k+1}(t,x,\Phi^{-1}(\cdot,\omega),\omega) \right)
\left. \right|_{z=x/\varepsilon}
+ \sum_{k=0}^{\infty} \varepsilon^k \text{div}_x \left( A(\Phi^{-1}(\cdot, \omega), \omega) \nabla_x u_k(t,x,\Phi^{-1}(\cdot,\omega),\omega) \right)
\left. \right|_{z=x/\varepsilon}.
$$

(2.22)

Proceeding in the same way with respect to the temporal derivative, we
have
\[ e^{-2\pi S_\varepsilon(t,x)} \frac{\partial_t u_\varepsilon}{\varepsilon^2} \]
\[ = \frac{1}{\varepsilon^2} \left( 2i\pi \varepsilon^2 \partial_t S_\varepsilon(t,x) \right) u_0 \left( t, x, \Phi^{-1} \left( \frac{x}{\varepsilon}, \omega \right), \omega \right) \]
\[ + \frac{1}{\varepsilon} \left( 2i\pi \varepsilon^2 \partial_t S_\varepsilon(t,x) \right) u_1 \left( t, x, \Phi^{-1} \left( \frac{x}{\varepsilon}, \omega \right), \omega \right) \]
\[ + \left( 2i\pi \varepsilon^2 \partial_t S_\varepsilon(t,x) \right) \sum_{k=0}^{\infty} \varepsilon^k u_{k+2} \left( t, x, \Phi^{-1} \left( \frac{x}{\varepsilon}, \omega \right), \omega \right) \]
\[ + \sum_{k=0}^{\infty} \varepsilon^k \partial_t u_k \left( t, x, \Phi^{-1} \left( \frac{x}{\varepsilon}, \omega \right), \omega \right). \]  
(2.23)

Thus, if we insert the equations (2.21) and (2.23) in (1.1) and compute the \( \varepsilon^{-2} \) order term, we arrive at
\[ L_\Phi(\varepsilon \nabla S_\varepsilon(t,x)) u_0 \left( t, x, \Phi^{-1}(\cdot, \omega), \omega \right) = 2\pi \left( \varepsilon^2 \partial_t S_\varepsilon(t,x) \right) u_0 \left( t, x, \Phi^{-1}(\cdot, \omega), \omega \right), \]
where for each \( \theta \in \mathbb{R}^n \), the linear operator \( L_\Phi(\theta) \) is defined by
\[ L_\Phi(\theta)[\cdot] := - \left( \text{div}_z + 2i\pi \theta \right) \left( A(\Phi^{-1}(z, \omega), \omega) \nabla_z + 2i\pi \theta \right)[\cdot] \]
\[ + V(\Phi^{-1}(z, \omega), \omega)[\cdot]. \]  
(2.24)

Therefore, \( 2\pi \left( \varepsilon^2 \partial_t S_\varepsilon(t,x) \right) \) is an eigenvalue of \( L_\Phi(\varepsilon \nabla S_\varepsilon(t,x)) \). Consequently, if \( \lambda(\theta) \) is any eigenvalue of \( L_\Phi(\theta) \) (which is sufficiently regular with respect to \( \theta \)), then the following (eikonal) Hamilton-Jacobi equation must be satisfied
\[ 2\pi \varepsilon^2 \partial_t S_\varepsilon(t,x) - \lambda(\varepsilon \nabla S_\varepsilon(t,x)) = 0. \]

Thus, if we suppose for \( t = 0 \) (companion to (2.20)) the modulated plane wave initial data
\[ u_\varepsilon(0, x, \omega) = e^{2\pi \theta \cdot x \varepsilon} \sum_{k=1}^{\infty} \varepsilon^k u_k \left( 0, x, \Phi^{-1} \left( \frac{x}{\varepsilon}, \omega \right), \omega \right), \]  
(2.25)
then the unique solution for the above Hamilton-Jacobi equation is, for each parameter \( \theta \in \mathbb{R}^n \),
\[ S_\varepsilon(t, x) = \frac{\lambda(\theta) t}{2\pi \varepsilon^2} + \frac{\theta \cdot x}{\varepsilon}. \]  
(2.26)

To sum up, the above expansion, that is the solution \( u_\varepsilon \) of the equation (1.1) with initial data given respectively by (2.20) and (2.25), suggests the following
Definition 2.21 (Bloch or shifted spectral cell equation). Let $\Phi$ be a stochastic deformation. For any $\theta \in \mathbb{R}^n$ fixed, the following time independent asymptotic equation

$$\left\{ \begin{array}{ll}
L^\Phi(\theta)[\Psi(z,\omega)] = \lambda \Psi(z,\omega), & \text{in } \mathbb{R}^n \times \Omega, \\
\Psi(z,\omega) = \psi(\Phi^{-1}(z,\omega),\omega), & \psi \text{ is a stationary function},
\end{array} \right. \tag{2.27}$$

is called Bloch’s spectral cell equation companion to the Schrödinger equation in (1.1), where $L^\Phi(\theta)$ is given by (2.24). Moreover, each $\theta \in \mathbb{R}^n$ is called a Bloch frequency, $\lambda(\theta)$ is called a Bloch energy and the corresponded $\Psi(\theta)$ is called a Bloch wave. If $\Phi$ is well understood in the context, then $L \equiv L^\Phi$.

The unknown $(\lambda, \Psi)$ in (2.27), which is an eigenvalue-eigenfunction pair, is obtained by the associated variational formulation, that is

$$\langle L(\theta)[F], G \rangle = \int_\Omega \int_{\Phi([0,1]^n,\omega)} A(\Phi^{-1}(z,\omega))(\nabla z + 2i\pi\theta)F(z,\omega) \cdot (\nabla z + 2i\pi\theta)G(z,\omega) \, dz \, d\mathbb{P}(\omega)$$

$$+ \int_\Omega \int_{\Phi([0,1]^n,\omega)} V(\Phi^{-1}(z,\omega),\omega) F(z,\omega) G(z,\omega) \, dz \, d\mathbb{P}(\omega). \tag{2.28}$$

Moreover, it is required that for some $\theta^* \in \mathbb{R}^n$, $\lambda(\theta^*) \in \mathbb{R}$ satisfies

$$\left\{ \begin{array}{l}
\lambda(\theta^*) \text{ is a simple eigenvalue of } L^\Phi(\theta^*), \\
\theta^* \text{ is a critical point of } \lambda(\cdot), \text{ that is, } \nabla_{\theta} \lambda(\theta^*) = 0.
\end{array} \right. \tag{2.29}$$

It is not clear the existence of a pair $(\theta^*, \lambda(\theta^*))$, in general stochastic environments, satisfying the two above conditions. However, there are concrete situations in the periodic settings where such conditions take place (see for instance [3, 6, 7]). Section 4 presents realistic stochastic models, whose spectral nature is inherited from the periodic ones and may satisfy (2.29).

2.5 Perturbations of bounded operators

To finish this preliminary section, we consider the perturbation theory for bounded operators with isolated eigenvalues of finite multiplicity. The precisely result is stated in the following theorem, (see for instance Kato [19], Rellich [23]).
Theorem 2.22. Let $H$ be a Hilbert space, and a sequence of operators $\{A_\alpha\}$, $A_\alpha \in \mathcal{B}(H)$ for each $\alpha \in \mathbb{N}^n$. Consider the power series of $n$-complex variables $z = (z_1, \ldots, z_n)$ with coefficients $A_\alpha$, which is absolutely convergent in a neighborhood $\mathcal{O}$ of $z = 0$. Define, the holomorphic map $A : \mathcal{O} \to \mathcal{B}(H)$,

$$A(z) := \sum_{\alpha \in \mathbb{N}^n} z^\alpha A_\alpha$$

and assume that, it is symmetric. If $\lambda$ is an eigenvalue of $A_0 \equiv A(0)$ with finite multiplicity $h$ (and respective eigenvectors $\psi_i$, $i = 1, \ldots, h$), then there exist a neighborhood $U \subset \mathcal{O}$ of $0$, and holomorphic functions

$$z \in U \mapsto \lambda_1(z), \lambda_2(z), \ldots, \lambda_h(z) \in \mathbb{R},$$

$$z \in U \mapsto \psi_1(z), \psi_2(z), \ldots, \psi_h(z) \in H \setminus \{0\},$$

satisfying for each $z \in U$ and $i \in \{1, \ldots, h\}$:

(i) $A(z)\psi_i(z) = \lambda_i(z)\psi_i(z)$,

(ii) $\lambda_i(z = 0) = \lambda$,

(iii) $\dim\{w \in H ; A(z)w = \lambda_i(z)w\} \leq h$.

Moreover, if there exists $d > 0$ such that $\sigma(A_0) \cap (\lambda - d, \lambda + d) = \{\lambda\}$, then for each $d' \in (0, d)$ there exists a neighborhood $W \subset U$ of $0$, such that

$$\sigma(A(z)) \cap (\lambda - d', \lambda + d') = \{\lambda_1(z), \ldots, \lambda_h(z)\}, \quad \text{for all } z \in W. \quad (2.30)$$

3 On Schrödinger Equations Homogenization

In this section, we shall describe the asymptotic behaviour of the family of solutions $\{u_\varepsilon\}_{\varepsilon > 0}$ of the equation (1.1), this is the content of Theorem 3.2 below. It generalizes the similar result of Allaire, Piatnitski [3] where they consider the similar problem in the periodic setting. Our scenario is much different from one considered by them. Here, the coefficients of equation (1.1) are random perturbations accomplished by stochastic diffeomorphisms of stationary functions. Since the two-scale convergence technique is the best tool to deal with asymptotic analysis of linear operators, we make use of it in
analogous way done in [3]. Although, the presence of the stochastic deformation in the coefficients brings out several complications, which we were able to overcome.

To begin, some important and basic a priori estimates of the solution of the Schrödinger equation (1.1) are needed. Then, we have the following

**Lemma 3.1 (Energy Estimates).** Assume that the conditions (1.2), (1.3) hold and let $u_\varepsilon \in C([0,T); H^1(\mathbb{R}^n))$ be the solution of the equation (1.1) with initial data $u_0^0$. Then, for all $t \in [0, T]$ and a.e. $\omega \in \Omega$, the following a priori estimates hold:

(i) (Energy Conservation.) $\int_{\mathbb{R}^n} |u_\varepsilon(t,x,\omega)|^2 dx = \int_{\mathbb{R}^n} |u_0^0(x,\omega)|^2 dx$.

(ii) ($\varepsilon \nabla -$ Estimate.)

$$\int_{\mathbb{R}^n} |\varepsilon \nabla u_\varepsilon(t,x,\omega)|^2 dx \leq C \int_{\mathbb{R}^n} \left\{ |\varepsilon \nabla u_0^0(x,\omega)|^2 + |u_0^0(x,\omega)|^2 \right\} dx,$$

where $C := C(\Lambda, \|A\|_\infty, \|V\|_\infty, \|U\|_\infty)$ is a positive constant which does not depend on $\varepsilon > 0$.

It’s important to remember the following facts that will be necessary in this section:

- The initial data of the equation (1.1) is assumed to be well-prepared, that is, for $(x, \omega) \in \mathbb{R}^n \times \Omega$, and $\theta^* \in \mathbb{R}^n$

$$u_\varepsilon^0(x, \omega) = e^{2i\pi \frac{x}{\varepsilon}} \psi \left( \Phi^{-1}(x/\varepsilon, \omega, \theta^*) \right) v^0(x), \quad (3.31)$$

where $v^0 \in C^\infty_c(\mathbb{R}^n)$, and $\psi(\theta^*)$ is an eigenfunction of the cell problem (2.27).

- Using the Ergodic Theorem, it is easily seen that the sequences

$$\{ u_\varepsilon^0(\cdot, \omega) \}_{\varepsilon > 0} \quad \text{and} \quad \{ \varepsilon \nabla u_\varepsilon^0(\cdot, \omega) \}_{\varepsilon > 0}$$

are bounded in $L^2(\mathbb{R}^n)$ and $[L^2(\mathbb{R}^n)]^n$, respectively.
3.1 The Abstract Theorem.

In the next, we establish an abstract homogenization theorem for Schrödinger equations.

**Theorem 3.2.** Let $\Phi(y, \omega)$ be a stochastic deformation, and $\tau : \mathbb{Z}^n \times \Omega \to \Omega$ an ergodic $n-$dimensional dynamical system. Assume the conditions (1.2), (1.3), and there exists a Bloch frequency $\theta^* \in \mathbb{R}^n$ such that (2.29) holds, with $\lambda(\theta^*)$ associated to the eigenfunction $\Psi(z, \omega, \theta^*) \equiv \psi(\Phi^{-1}(z, \omega), \omega, \theta^*)$. Also, assume that the initial data $u^0$ satisfies (3.31). If $u_\varepsilon \in C([0, T); H^1(\mathbb{R}^n))$ is the solution of (1.1) for each $\varepsilon > 0$ fixed, then the sequence $v_\varepsilon$ defined by

$$
v_\varepsilon(t, x, \omega) := e^{-\left(\frac{\lambda(\theta^*)t}{\varepsilon^2} + 2im\theta^* \cdot \frac{x}{\varepsilon}\right)}u_\varepsilon(t, x, \omega), \ (t, x) \in \mathbb{R}^{n+1}, \ \omega \in \Omega,
$$

$\Phi_{\omega}-$two-scale converges to the function $v(t, \omega) \psi(\Phi^{-1}(z, \omega), \omega, \theta^*)$, and satisfies for a.e. $\omega \in \Omega$

$$
\lim_{\varepsilon \to 0} \int_{\mathbb{R}^{n+1}} \left| v_\varepsilon(t, x, \omega) - v(t, x) \psi\left(\Phi^{-1}\left(\frac{x}{\varepsilon}, \omega\right), \omega, \theta^*\right)\right|^2 \, dx \, dt = 0,
$$

where $v \in C([0, T); L^2(\mathbb{R}^n))$ is the unique solution of the homogenized Schrödinger equation

$$
\begin{cases}
  i\frac{\partial v}{\partial t} - \text{div}(A^* \nabla v) + U^* v = 0, & \text{in } \mathbb{R}^{n+1}, \\
  v(0, x) = v^0(x), & x \in \mathbb{R}^n,
\end{cases}
$$

(3.32)

with effective (constant) coefficients: matrix $A^* = D^2_\theta \lambda(\theta^*)$, and potential

$$
U^* = c^-_\psi \int_\Omega \int_{\Phi([0, 1]^n, \omega)} U\left(\Phi^{-1}(z, \omega), \omega\right) \left| \psi\left(\Phi^{-1}(z, \omega), \omega, \theta^*\right) \right|^2 \, dz \, d\mathbb{P}(\omega),
$$

where

$$
c^-_\psi = \int_\Omega \int_{\Phi([0, 1]^n, \omega)} \left| \psi\left(\Phi^{-1}(z, \omega), \omega, \theta^*\right) \right|^2 \, dz \, d\mathbb{P}(\omega).
$$

Proof. In order to better understand the main difficulties brought by the presence of the stochastic deformation $\Phi$, we split our proof in five steps.

1. *(A priori estimates and $\Phi_{\omega}-$two-scale convergence.*) First, we define

$$
v_\varepsilon(t, x, \omega) := e^{-\left(\frac{\lambda(\theta^*)t}{\varepsilon^2} + 2im\theta^* \cdot \frac{x}{\varepsilon}\right)}u_\varepsilon(t, x, \omega), \ (t, x) \in \mathbb{R}^{n+1} \times \Omega.
$$

(3.33)
Then, computing the first derivatives with respect to the variable $x$, we get
\[
e^{-\frac{(i\omega_0^2)\varepsilon^2 + 2i\pi \theta^*}{\varepsilon}} = (\varepsilon + 2i\pi \theta^*)v_{\varepsilon}(t, x, \omega).
\] (3.34)

Applying Lemma 3.1 yields:

1. \[\int_{\mathbb{R}^n} |v_{\varepsilon}(t, x, \omega)|^2 dx = \int_{\mathbb{R}^n} |u^0_{\varepsilon}(x, \omega)|^2 dx,\]
2. \[\int_{\mathbb{R}^n} |\varepsilon \nabla v_{\varepsilon}(t, x, \omega)|^2 dx \leq \tilde{C} \int_{\mathbb{R}^n} \left( |\varepsilon \nabla u^0_{\varepsilon}(x, \omega)|^2 + |u^0_{\varepsilon}(x, \omega)|^2 \right) dx\]

for all $t \in [0, T)$ and a.e. $\omega \in \Omega$, where the constant $\tilde{C}$ depends on $||A||_\infty$, $||V||_\infty$, $||U||_\infty$ and $\theta^*$. Then, from the uniform boundedness of the sequences $\{u^0_{\varepsilon}(\cdot, \omega)\}_{\varepsilon > 0}$ and $\{\varepsilon \nabla u^0_{\varepsilon}(\cdot, \omega)\}_{\varepsilon > 0}$, we deduce that the sequences

\[
\{v_{\varepsilon}(\cdot, \cdot, \omega)\}_{\varepsilon > 0}\quad\text{and}\quad\{\varepsilon \nabla v_{\varepsilon}(\cdot, \cdot, \omega)\}_{\varepsilon > 0}
\]

are bounded, respectively, in $L^2(\mathbb{R}^{n+1}_T)$ and $[L^2(\mathbb{R}^{n+1}_T)]^n$ for a.e. $\omega \in \Omega$. Therefore, applying Lemma 2.20, there exists a subsequence $\{\varepsilon'\}$ (which may depend on $\omega$), and a stationary function $v^*_{\omega} \in L^2(\mathbb{R}^{n+1}_T, \mathcal{H})$, for a.e. $\omega \in \Omega$, such that

\[
v_{\varepsilon'}(t, x, \omega) \xrightarrow{2-s_{\varepsilon'}} \v^*_{\omega}(t, x, \Phi^{-1}(z, \omega), \omega),
\]

and

\[
\varepsilon' \frac{\partial v_{\varepsilon'}}{\partial x_k}(t, x, \omega) \xrightarrow{2-s_{\varepsilon'}} \frac{\partial}{\partial z_k}\left(v^*_{\omega}(t, x, \Phi^{-1}(z, \omega), \omega)\right),
\]

which means that, for $k \in \{1, \ldots, n\}$, we have

\[
\begin{align*}
\lim_{\varepsilon' \to 0} \int_{\mathbb{R}^{n+1}_T} v_{\varepsilon'}(t, x, \omega) \varphi(t, x) \Theta \left(\Phi^{-1}(\frac{x}{\varepsilon'}, \omega), \omega\right) dx dt &= c^{-1}_\Phi \int_{\mathbb{R}^{n+1}_T} \int_{\Omega} \int_{\Phi([0,1]^n, \omega)} v^*_{\omega}(t, x, \Phi^{-1}(z, \omega), \omega) \\
&\times \varphi(t, x) \Theta \left(\Phi^{-1}(z, \omega), \omega\right) dz d\mathbb{P} dx dt \tag{3.35}
\end{align*}
\]

and

\[
\begin{align*}
\lim_{\varepsilon' \to 0} \int_{\mathbb{R}^{n+1}_T} \varepsilon' \frac{\partial v_{\varepsilon'}}{\partial x_k}(t, x, \omega) \varphi(t, x) \Theta \left(\Phi^{-1}(\frac{x}{\varepsilon'}, \omega), \omega\right) dx dt &= c^{-1}_\Phi \int_{\mathbb{R}^{n+1}_T} \int_{\Omega} \int_{\Phi([0,1]^n, \omega)} \frac{\partial}{\partial z_k}\left(v^*_{\omega}(t, x, \Phi^{-1}(z, \omega), \omega)\right) \\
&\times \varphi(t, x) \Theta \left(\Phi^{-1}(z, \omega), \omega\right) dz d\mathbb{P} dx dt, \tag{3.36}
\end{align*}
\]
for all functions $\varphi \in C^\infty_c((0, T) \times \mathbb{R}^n)$ and $\Theta \in L^2_{\text{loc}}(\mathbb{R}^n \times \Omega)$ stationary. Moreover, the sequence $\{v^0_\epsilon(\cdot, \tilde{\omega})\}_{\epsilon > 0}$ defined by,

$$v^0_\epsilon(x, \omega) := \psi(\Phi^{-1}(x, \omega), \omega, \theta^*) v^0(x), \quad (x, \omega) \in \mathbb{R}^n \times \Omega,$$

satisfies

$$v^0_\epsilon(\cdot, \tilde{\omega}) \xrightarrow{2-s \epsilon \to 0} v^0(x) \psi(\Phi^{-1}(z, \omega), \omega, \theta^*),$$

for each stationary function $\psi(\theta^*)$.

2. (The Split Process.) We consider the following

Claim: There exists $v^s_\omega \in L^2(\mathbb{R}^{n+1})$, such that

$$v^s_\omega(t, x, \Phi^{-1}(z, \omega), \omega) = v^s_\omega(t, x) \psi(\Phi^{-1}(z, \omega), \omega, \theta^*)$$

$$\equiv v^s_\omega(t, x) \Psi(z, \omega, \theta^*).$$

Proof of Claim: First, for any $\tilde{\omega} \in \Omega$ fixed, we take the function

$$Z_\epsilon(t, x, \tilde{\omega}) = \epsilon^2 e^{i \frac{\lambda^0 \epsilon t}{\epsilon^2} + 2i \frac{\theta^* \epsilon x}{\epsilon} \varphi(t, x) \Theta(\Phi^{-1}(\frac{x}{\epsilon}, \tilde{\omega}), \tilde{\omega})}$$

(3.39)

as a test function in the associated variational formulation of the equation (3.1), where $\varphi \in C^\infty_c((0, T) \times \mathbb{R}^n)$ and $\Theta \in L^\infty(\mathbb{R}^n \times \Omega)$ stationary. Therefore, we obtain

$$- i \int_{\mathbb{R}^{n+1}} u_\epsilon(t, x, \tilde{\omega}) \frac{\partial Z_\epsilon}{\partial t}(t, x, \tilde{\omega}) dx dt + i \int_{\mathbb{R}^n} u^0_\epsilon(x, \tilde{\omega}) Z_\epsilon(0, x, \tilde{\omega}) dx dt$$

$$+ \int_{\mathbb{R}^{n+1}} A \left( \Phi^{-1} \left( \frac{x}{\epsilon}, \tilde{\omega} \right), \tilde{\omega} \right) \nabla u_\epsilon(t, x, \tilde{\omega}) \cdot \nabla Z_\epsilon(t, x, \tilde{\omega}) dx dt$$

$$+ \frac{1}{\epsilon^2} \int_{\mathbb{R}^{n+1}} V \left( \Phi^{-1} \left( \frac{x}{\epsilon}, \tilde{\omega} \right), \tilde{\omega} \right) u_\epsilon(t, x, \tilde{\omega}) Z_\epsilon(t, x, \tilde{\omega}) dx dt$$

$$+ \int_{\mathbb{R}^{n+1}} U \left( \Phi^{-1} \left( \frac{x}{\epsilon}, \tilde{\omega} \right), \tilde{\omega} \right) u_\epsilon(t, x, \tilde{\omega}) Z_\epsilon(t, x, \tilde{\omega}) dx dt = 0,$$

and since

$$\frac{\partial Z_\epsilon}{\partial t}(t, x, \tilde{\omega}) = i \lambda(\theta^*) e^{i \frac{\lambda(\theta^*) t}{\epsilon^2} + 2i \frac{\theta^* x}{\epsilon} \varphi(t, x) \Theta(\Phi^{-1}(\frac{x}{\epsilon}, \tilde{\omega}), \tilde{\omega}) + O(\epsilon^2),$$

$$\nabla Z_\epsilon(t, x, \tilde{\omega}) = \epsilon e^{i \frac{\lambda(\theta^*) t}{\epsilon^2} + 2i \frac{\theta^* x}{\epsilon} (\epsilon \nabla + 2i \pi \theta^*) (\varphi(t, x) \Theta(\Phi^{-1}(\frac{x}{\epsilon}, \tilde{\omega}), \tilde{\omega}))},$$

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it follows that

\[- \lambda(\theta^*) \int_{\mathbb{R}^{n+1}} v_\varepsilon(t, x, \tilde{\omega}) \varphi(t, x) \Theta(\Phi^{-1}(\frac{x}{\varepsilon}, \tilde{\omega}), \tilde{\omega}) \, dx \, dt \]

\[+ \int_{\mathbb{R}^{n+1}} A(\Phi^{-1}(\frac{x}{\varepsilon}, \tilde{\omega}), \tilde{\omega}) (\varepsilon \nabla + 2i\pi \theta^*) v_\varepsilon(t, x, \tilde{\omega}) \cdot \varphi(t, x) \Theta(\Phi^{-1}(\frac{x}{\varepsilon}, \tilde{\omega}), \tilde{\omega}) \, dx \, dt \]

\[+ \int_{\mathbb{R}^{n+1}} v_\varepsilon(t, x, \tilde{\omega}) \varphi(t, x) \Theta(\Phi^{-1}(\frac{x}{\varepsilon}, \tilde{\omega}), \tilde{\omega}) \, dx \, dt = O(\varepsilon^2), \]

where we have used (3.33), (3.34), (3.37), and (3.39). Although, it is more convenient to rewrite as

\[- \lambda(\theta^*) \int_{\mathbb{R}^{n+1}} v_\varepsilon(t, x, \tilde{\omega}) \varphi(t, x) \Theta(\Phi^{-1}(\frac{x}{\varepsilon}, \tilde{\omega}), \tilde{\omega}) \, dx \, dt \]

\[+ \int_{\mathbb{R}^{n+1}} (\varepsilon \nabla + 2i\pi \theta^*) v_\varepsilon(t, x, \tilde{\omega}) \cdot A(\Phi^{-1}(\frac{x}{\varepsilon}, \tilde{\omega}), \tilde{\omega}) (\varepsilon \nabla + 2i\pi \theta^*) \varphi(t, x) \Theta(\Phi^{-1}(\frac{x}{\varepsilon}, \tilde{\omega}), \tilde{\omega}) \, dx \, dt \]

\[+ \int_{\mathbb{R}^{n+1}} v_\varepsilon(t, x, \tilde{\omega}) \varphi(t, x) V(\Phi^{-1}(\frac{x}{\varepsilon}, \tilde{\omega}), \tilde{\omega}) \Theta(\Phi^{-1}(\frac{x}{\varepsilon}, \tilde{\omega}), \tilde{\omega}) \, dx \, dt = O(\varepsilon^2). \]

(3.40)

Now, making \( \varepsilon = \varepsilon' \), letting \( \varepsilon' \to 0 \) and using the Definition 2.17, we have for a.e. \( \tilde{\omega} \in \Omega \), for all \( \varphi \in C_0^\infty((-\infty, T) \times \mathbb{R}^n) \), \( \Theta \in L^\infty(\mathbb{R}^n \times \Omega) \) stationary
and \( \Theta(\cdot, \omega) \) smooth,

\[
- \lambda(\theta^*) \left[ \mathcal{B}^{-1} \int_{\mathbb{R}_+^{n+1}} \int_{\Omega} \int_{\Phi((0,1)^n,\omega)} v_\omega^*(t, x, \Phi^{-1}(z, \omega), \omega) \right.
\]
\[
\times \phi(t, x) \Theta(\Phi^{-1}(z, \omega), \omega) \, dz \, d\mathbb{P}(\omega) \, dx \, dt
\]
\[+ \mathcal{B}^{-1} \int_{\mathbb{R}_+^{n+1}} \int_{\Omega} \int_{\Phi((0,1)^n,\omega)} (\nabla_z + 2i\pi\theta^*) \left( v_\omega^*(t, x, \Phi^{-1}(z, \omega), \omega) \right) \]
\[
\times \phi(t, x) A(\Phi^{-1}(z, \omega), \omega) \left( (\nabla_z + 2i\pi\theta^*) \Theta(\Phi^{-1}(z, \omega), \omega) \right) \, dz \, d\mathbb{P}(\omega) \, dx \, dt
\]
\[+ \mathcal{B}^{-1} \int_{\mathbb{R}_+^{n+1}} \int_{\Omega} \int_{\Phi((0,1)^n,\omega)} v_\omega^*(t, x, \Phi^{-1}(z, \omega), \omega) \]
\[
\times \phi(t, x) V(\Phi^{-1}(z, \omega), \omega) \Theta(\Phi^{-1}(z, \omega), \omega) \, dz \, d\mathbb{P}(\omega) 
\]
\[= 0.
\]

Therefore, due to an argument of density in the test functions (thanks to the topological structure of \( \Omega \)), we can conclude that

\[
- \lambda(\theta^*) \left[ \int_{\Omega} \int_{\Phi((0,1)^n,\omega)} v_\omega^*(t, x, \Phi^{-1}(z, \omega), \omega) \right.
\]
\[
\times \phi(t, x) \Theta(\Phi^{-1}(z, \omega), \omega) \, dz \, d\mathbb{P}(\omega)
\]
\[+ \int_{\Omega} \int_{\Phi((0,1)^n,\omega)} A(\Phi^{-1}(z, \omega), \omega) (\nabla_z + 2i\pi\theta^*) \left( v_\omega^*(t, x, \Phi^{-1}(z, \omega), \omega) \right) \]
\[
\times (\nabla_z + 2i\pi\theta^*) \Theta(\Phi^{-1}(z, \omega), \omega) \, dz \, d\mathbb{P}(\omega)
\]
\[+ \int_{\Omega} \int_{\Phi((0,1)^n,\omega)} V(\Phi^{-1}(z, \omega), \omega) v_\omega^*(t, x, \Phi^{-1}(z, \omega), \omega) \]
\[
\times \Theta(\Phi^{-1}(z, \omega), \omega) \, dz \, d\mathbb{P}(\omega) = 0,
\]

for a.e. \((t, x) \in \mathbb{R}_+^{n+1}\) and for all \(\Theta\) as above. Thus, the simplicity of the eigenvalue \(\lambda(\theta^*)\) assures us that for a.e. \(\mathbb{R}_+^{n+1}\), the function

\[(z, \omega) \mapsto v_\omega^*(t, x, \Phi^{-1}(z, \omega), \omega)\]

(which belongs to the space \(\mathcal{H}\)) is parallel to the function \(\Psi(\theta^*)\), i.e., we can find \(v_\omega(t, x) \in \mathbb{C}\), such that

\[
v_\omega^*(t, x, \Phi^{-1}(z, \omega), \omega) = v_\omega(t, x) \Psi(z, \omega, \theta^*)
\]
\[
\equiv v_\omega(t, x) \psi(\Phi^{-1}(z, \omega), \omega, \theta^*).
\]
Finally, since \( v_\varepsilon^* \in L^2(\mathbb{R}^{n+1}; \mathcal{H}) \), we conclude that \( v_\varepsilon \in L^2(\mathbb{R}_T^{n+1}) \), which completes the proof of our claim.

3. (Homogenization Process.) Let \( \Lambda_k(\theta^*) \), for any \( k \in \{1, \ldots, n\} \), be the function defined by

\[
\Lambda_k(z, \omega, \theta^*) = \frac{1}{2i\pi} \frac{\partial \psi}{\partial \theta_k}(z, \omega, \theta^*) = \frac{1}{2i\pi} \frac{\partial \psi}{\partial \theta_k}(\Phi^{-1}(z, \omega), \omega, \theta^*), \quad (z, \omega) \in \mathbb{R}^n \times \Omega,
\]

where the function \( \psi(z, \omega, \theta^*) = \psi(\Phi^{-1}(z, \omega), \omega, \theta^*) \) is the eigenfunction of the spectral cell problem \((2.27)\). Then, we consider the following test function

\[
Z_\varepsilon(t, x, \tilde{\omega}) = e^{i \frac{\lambda \varepsilon^* n}{\varepsilon} + 2i \pi \frac{\theta^*}{\varepsilon}} (\varphi(t, x) \Psi_\varepsilon(x, \tilde{\omega}, \theta^*) + \varepsilon \sum_{k=1}^{n} \frac{\partial \varphi}{\partial x_k}(t, x) \Lambda_k,\varepsilon(x, \tilde{\omega}, \theta^*)),
\]

where \( \varphi \in C_0^\infty((\infty, \infty) \times \mathbb{R}^n) \) and

\[
\Psi_\varepsilon(x, \tilde{\omega}, \theta^*) = \Psi\left(\frac{x}{\varepsilon}, \tilde{\omega}, \theta^*\right), \quad \Lambda_k,\varepsilon(x, \tilde{\omega}, \theta^*) = \Lambda_k\left(\frac{x}{\varepsilon}, \tilde{\omega}, \theta^*\right).
\]

Using the function \( Z_\varepsilon \) as test function in the variational formulation of the equation \((1.1)\), we obtain

\[
\begin{align*}
\left[i \int_{\mathbb{R}^n} u_\varepsilon^0(x, \tilde{\omega}) Z_\varepsilon(0, x, \tilde{\omega}) \, dx - i \int_{\mathbb{R}_T^{n+1}} u_\varepsilon(t, x, \tilde{\omega}) \frac{\partial Z_\varepsilon}{\partial t}(t, x, \tilde{\omega}) \, dx \, dt \right] \\
+ \left[ \int_{\mathbb{R}_T^{n+1}} A(\Phi^{-1}\left(\frac{x}{\varepsilon}, \tilde{\omega}\right), \tilde{\omega}) \nabla u_\varepsilon(t, x, \tilde{\omega}) \cdot \nabla Z_\varepsilon(t, x, \tilde{\omega}) \, dx \, dt \right] \\
+ \left[ \frac{1}{\varepsilon^2} \int_{\mathbb{R}_T^{n+1}} V(\Phi^{-1}\left(\frac{x}{\varepsilon}, \tilde{\omega}\right), \tilde{\omega}) u_\varepsilon(t, x, \tilde{\omega}) Z_\varepsilon(t, x, \tilde{\omega}) \, dx \, dt \right] \\
+ \int_{\mathbb{R}_T^{n+1}} U\left(x, \Phi^{-1}\left(\frac{x}{\varepsilon}, \tilde{\omega}\right), \tilde{\omega}\right) u_\varepsilon(t, x, \tilde{\omega}) Z_\varepsilon(t, x, \tilde{\omega}) \, dx \, dt \right] = 0.
\end{align*}
\]

(3.41)

In order to simplify the manipulation of the above equation, we shall denote by \( I_1^\varepsilon(k = 1, 2, 3) \) the respective term in the \( k \)th brackets, so that we can rewrite the equation \((3.41)\) as \( I_1^\varepsilon + I_2^\varepsilon + I_3^\varepsilon = 0 \).

The analysis of the \( I_1^\varepsilon \) term is triggered by the following computation

\[
\frac{\partial Z_\varepsilon}{\partial t}(t, x, \tilde{\omega}) = e^{i \frac{\lambda \varepsilon^* n}{\varepsilon} + 2i \pi \frac{\theta^*}{\varepsilon}} \left[ i \frac{\lambda(\theta^*)}{\varepsilon^2} (\varphi(t, x) \Psi_\varepsilon(x, \tilde{\omega}, \theta^*) \\
+ \varepsilon \sum_{k=1}^{n} \frac{\partial \varphi}{\partial x_k}(t, x) \Lambda_k,\varepsilon(x, \tilde{\omega}, \theta^*) \right) + \frac{\partial \varphi}{\partial t}(t, x) \Psi_\varepsilon(x, \tilde{\omega}, \theta^*) + \varepsilon \sum_{k=1}^{n} \frac{\partial^2 \varphi}{\partial t \partial x_k}(t, x) \Lambda_k,\varepsilon, \right],
\]

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therefore we have

\[ I_1^\varepsilon = i \int_{\mathbb{R}^n} v_\varepsilon^0(\varphi(0, x) \Psi_\varepsilon(x, \bar{\omega}, \theta^*) + \varepsilon \sum_{k=1}^{n} \frac{\partial \varphi}{\partial x_k}(0, x) \Lambda_{k, \varepsilon}(x, \bar{\omega}, \theta^*)) dx \]

\[- \frac{\lambda(\theta^*)}{\varepsilon^2} \int_{\mathbb{R}^{n+1}} v_\varepsilon(\varphi(t, x) \Psi_\varepsilon(x, \bar{\omega}, \theta^*) + \varepsilon \sum_{k=1}^{n} \frac{\partial \varphi}{\partial x_k}(t, x) \Lambda_{k, \varepsilon}(x, \bar{\omega}, \theta^*)) dx \, dt \]

\[- i \int_{\mathbb{R}^{n+1}} v_\varepsilon(\frac{\partial \varphi}{\partial t}(t, x) \Psi_\varepsilon(x, \bar{\omega}, \theta^*) + \varepsilon \sum_{k=1}^{n} \frac{\partial^2 \varphi}{\partial t \partial x_k}(t, x) \Lambda_{k, \varepsilon}(x, \bar{\omega}, \theta^*)) \]

For the analysis of the term \( I_2^\varepsilon \), we need to make the following computations

\[ \nabla Z_\varepsilon(t, x, \bar{\omega}) = e^{i \frac{\lambda(\theta^*)}{\varepsilon^2} + 2i\pi \frac{\theta^*}{\varepsilon}} \left[ \nabla \varphi(t, x) \Psi_\varepsilon(x, \bar{\omega}, \theta^*) + \varphi(t, x) \nabla \Psi_\varepsilon(z, \bar{\omega}, \theta^*) \right. \]

\[ + \varepsilon \sum_{k=1}^{n} \nabla \left( \frac{\partial \varphi}{\partial x_k}(t, x) \right) \Lambda_{k, \varepsilon}(x, \bar{\omega}, \theta^*) + \varepsilon \sum_{k=1}^{n} \frac{\partial \varphi}{\partial x_k}(t, x) \nabla \Lambda_{k, \varepsilon}(x, \bar{\omega}, \theta^*) \]

\[ + 2i\pi \frac{\theta^*}{\varepsilon} \left[ \varphi(t, x) \Psi_\varepsilon(x, \bar{\omega}, \theta^*) + \varepsilon \sum_{k=1}^{n} \frac{\partial \varphi}{\partial x_k}(t, x) \Lambda_{k, \varepsilon}(x, \bar{\omega}, \theta^*) \right] \]

\[ = e^{i \frac{\lambda(\theta^*)}{\varepsilon^2} + 2i\pi \frac{\theta^*}{\varepsilon}} \left[ \varphi(t, x) \left( \nabla + 2i\pi \frac{\theta^*}{\varepsilon} \right) \Psi_\varepsilon(x, \bar{\omega}, \theta^*) \right. \]

\[ + \varepsilon \sum_{k=1}^{n} \frac{\partial \varphi}{\partial x_k}(t, x) \left( \nabla + 2i\pi \frac{\theta^*}{\varepsilon} \right) \Lambda_{k, \varepsilon}(x, \bar{\omega}, \theta^*) + \nabla \varphi(t, x) \Psi_\varepsilon(x, \bar{\omega}, \theta^*) \]

\[ + \varepsilon \sum_{k=1}^{n} \nabla \left( \frac{\partial \varphi}{\partial x_k}(t, x) \right) \Lambda_{k, \varepsilon}(x, \bar{\omega}, \theta^*) \right], \]
and from this, we have
\[
A\left(\Phi^{-1}\left(\frac{x}{\varepsilon}, \tilde{\omega}\right), \tilde{\omega}\right) \nabla u_{\varepsilon}(t, x, \tilde{\omega}) \cdot \nabla Z_{\varepsilon}(t, x, \tilde{\omega})
\]
\[
= A\left(\Phi^{-1}\left(\frac{x}{\varepsilon}, \tilde{\omega}\right), \tilde{\omega}\right) \left[\nabla u_{\varepsilon}(t, x, \tilde{\omega}) e^{-\left(i\lambda(\theta^*)T + 2\pi \theta^* \varepsilon \frac{\lambda}{2}\right)}\right]
\]
\[
\cdot \left[\Theta(t, x) (\nabla - 2i\pi \frac{\theta^*}{\varepsilon})\Psi_{\varepsilon}(x, \tilde{\omega}, \theta^*) + \varepsilon \sum_{k=1}^{n} \frac{\partial \varphi}{\partial x_k}(t, x) (\nabla - 2i\pi \frac{\theta^*}{\varepsilon})\Lambda_{k, \varepsilon}(x, \tilde{\omega}, \theta^*)
\]
\[
+ \nabla \Theta(t, x) \Psi_{\varepsilon}(x, \tilde{\omega}, \theta^*) + \varepsilon \sum_{k=1}^{n} \nabla \left(\frac{\partial \varphi}{\partial x_k}(t, x)\right) \Lambda_{k, \varepsilon}(x, \tilde{\omega}, \theta^*)\right].
\]

Then, from equation (3.34) and using the terms
\[
(\nabla + 2i\pi \frac{\theta^*}{\varepsilon})(v_{\varepsilon}(t, x, \tilde{\omega}) \Theta(t, x)), \quad (\nabla + 2i\pi \frac{\theta^*}{\varepsilon})(v_{\varepsilon}(t, x, \tilde{\omega}) \frac{\partial \varphi}{\partial x_k}(t, x)),
\]

it follows from the above equation that
\[
A\left(\Phi^{-1}\left(\frac{x}{\varepsilon}, \tilde{\omega}\right), \tilde{\omega}\right) \nabla u_{\varepsilon} \cdot \nabla Z_{\varepsilon} = \sum_{k=1}^{n} \left(I_{2,1}^{\varepsilon, k} + I_{2,2}^{\varepsilon, k} + I_{2,3}^{\varepsilon, k}\right)(t, x, \tilde{\omega}),
\]

where
\[
I_{2,1}^{\varepsilon, k}(t, x, \tilde{\omega}) := \varepsilon A\left(\Phi^{-1}\left(\frac{x}{\varepsilon}, \tilde{\omega}\right), \tilde{\omega}\right) \left[(\nabla + 2i\pi \frac{\theta^*}{\varepsilon})(v_{\varepsilon}(t, x, \tilde{\omega}) \frac{\partial \varphi}{\partial x_k}(t, x))\right]
\]
\[
\cdot \left[(\nabla - 2i\pi \frac{\theta^*}{\varepsilon})\Lambda_{k, \varepsilon}(x, \tilde{\omega}, \theta^*)\right]
\]
\[
- \varepsilon A\left(\Phi^{-1}\left(\frac{x}{\varepsilon}, \tilde{\omega}\right), \tilde{\omega}\right) \left[v_{\varepsilon}(t, x, \tilde{\omega}) \frac{\partial \varphi}{\partial x_k}(t, x) e_k\right] \cdot \left[(\nabla - 2i\pi \frac{\theta^*}{\varepsilon})\Psi_{\varepsilon}(x, \tilde{\omega}, \theta^*)\right]
\]
\[
+ A\left(\Phi^{-1}\left(\frac{x}{\varepsilon}, \tilde{\omega}\right), \tilde{\omega}\right) \left[(\nabla + 2i\pi \frac{\theta^*}{\varepsilon})(v_{\varepsilon}(t, x, \tilde{\omega}) \frac{\partial \varphi}{\partial x_k}(t, x))\right] \cdot [e_k \Psi_{\varepsilon}(x, \tilde{\omega}, \theta^*)],
\]
\[
I_{2,2}^{\varepsilon, k}(t, x, \tilde{\omega}) := \frac{1}{n} A\left(\Phi^{-1}\left(\frac{x}{\varepsilon}, \tilde{\omega}\right), \tilde{\omega}\right) \left[(\nabla + 2i\pi \frac{\theta^*}{\varepsilon})(v_{\varepsilon}(t, x, \tilde{\omega}) \varphi(t, x))\right]
\]
\[
\cdot \left[(\nabla - 2i\pi \frac{\theta^*}{\varepsilon})\Psi_{\varepsilon}(x, \tilde{\omega}, \theta^*)\right]
\]
\[
- A\left(\Phi^{-1}\left(\frac{x}{\varepsilon}, \tilde{\omega}\right), \tilde{\omega}\right) \left[v_{\varepsilon}(t, x, \tilde{\omega}) \nabla \frac{\partial \varphi}{\partial x_k}(t, x)\right] \cdot [e_k \Psi_{\varepsilon}(x, \tilde{\omega}, \theta^*)],
\]
\[
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\]
and

\[ I_{2,3}^\varepsilon(t, x, \tilde{\omega}) := A(\Phi^{-1}(\frac{x}{\varepsilon}, \tilde{\omega}), \tilde{\omega}) \left[ (\varepsilon \nabla + 2i\pi \theta^*) v_\varepsilon(t, x, \tilde{\omega}) \right] \cdot \nabla \left( \frac{\partial \varphi}{\partial x_k}(t, x) \right) \Lambda_k,\varepsilon(x, \tilde{\omega}, \theta^*) \]

\[ - A(\Phi^{-1}(\frac{x}{\varepsilon}, \tilde{\omega}), \tilde{\omega}) \left[ v_\varepsilon(t, x, \tilde{\omega}) \nabla \frac{\partial \varphi}{\partial x_k}(t, x) \right] \cdot \left[ (\varepsilon \nabla - 2i\pi \theta^*) \Lambda_k,\varepsilon(x, \tilde{\omega}, \theta^*) \right]. \]

Thus integrating in \( \mathbb{R}^{n+1}_T \) we recover the \( I_{2}^\varepsilon \) term, that is

\[ I_{2}^\varepsilon = \int_{\mathbb{R}^{n+1}_T} \int A(\Phi^{-1}(\frac{x}{\varepsilon}, \tilde{\omega}), \tilde{\omega}) \nabla u_\varepsilon(t, x, \tilde{\omega}) \cdot \nabla Z_\varepsilon(t, x, \tilde{\omega}) \, dx \, dt \]

\[ = \sum_{k=1}^n \int_{\mathbb{R}^{n+1}_T} \left( I_{2,1}^{\varepsilon,k} + I_{2,2}^{\varepsilon,k} + I_{2,3}^{\varepsilon,k} \right)(t, x, \tilde{\omega}) \, dx \, dt. \quad (3.42) \]

Now, we intend to simplify the expression of \( I_{2}^\varepsilon \) to a more comely one. For this aim, we shall take \( v_\varepsilon(t, \cdot, \tilde{\omega}) \varphi(t, \cdot), t \in (0, T) \), as a test function. Then, we obtain

\[ \int_{\mathbb{R}^n} A(\Phi^{-1}(\frac{x}{\varepsilon}, \tilde{\omega}), \tilde{\omega}) \left[ (\nabla + 2i\pi \theta^*) (v_\varepsilon(t, x, \tilde{\omega}) \varphi(t, x)) \right] \cdot \left[ (\nabla - 2i\pi \theta^*) \Psi_\varepsilon(x, \tilde{\omega}, \theta^*) \right] \, dx \]

\[ = \frac{\lambda(\theta^*)}{\varepsilon^2} \int_{\mathbb{R}^n} (v_\varepsilon(t, x, \tilde{\omega}) \varphi(t, x)) \Psi_\varepsilon(x, \tilde{\omega}, \theta^*) \, dx \]

\[ - \frac{1}{\varepsilon^2} \int_{\mathbb{R}^n} V(\Phi^{-1}(\frac{x}{\varepsilon}, \tilde{\omega}), \tilde{\omega}) (v_\varepsilon(t, x, \tilde{\omega}) \varphi(t, x)) \Psi_\varepsilon(x, \tilde{\omega}, \theta^*) \, dx. \]

Therefore, comparing with \( I_{2,2}^{\varepsilon,k}(t, x, \tilde{\omega}) \) term obtained before, we have

\[ \int\int_{\mathbb{R}^{n+1}_T} I_{2,2}^{\varepsilon,k}(t, x, \tilde{\omega}) \, dx \, dt \]

\[ = \frac{\lambda(\theta^*)}{n\varepsilon^2} \int\int_{\mathbb{R}^{n+1}_T} (v_\varepsilon(t, x, \tilde{\omega}) \varphi(t, x)) \Psi_\varepsilon(x, \tilde{\omega}, \theta^*) \, dx \, dt \]

\[ - \frac{1}{n\varepsilon^2} \int\int_{\mathbb{R}^{n+1}_T} V(\Phi^{-1}(\frac{x}{\varepsilon}, \tilde{\omega}), \tilde{\omega}) (v_\varepsilon(t, x, \tilde{\omega}) \varphi(t, x)) \Psi_\varepsilon(x, \tilde{\omega}, \theta^*) \, dx \, dt \]

\[ - \int\int_{\mathbb{R}^{n+1}_T} A(\Phi^{-1}(\frac{x}{\varepsilon}, \tilde{\omega}), \tilde{\omega}) [v_\varepsilon(t, x, \tilde{\omega}) \nabla \frac{\partial \varphi}{\partial x_k}(t, x)] \cdot e_k \Psi_\varepsilon(x, \tilde{\omega}, \theta^*) \, dx \, dt. \quad (3.43) \]
Analogously, taking \( v_\varepsilon(t, \cdot, \tilde{\omega}) \frac{\partial \varphi}{\partial x_k}(t, \cdot) \), \( t \in (0, T) \), with \( k \in \{1, \ldots, n\} \) as a test function, taking into account that \( \nabla \theta \lambda(\theta^*) = 0 \) and comparing this expression with \( I_{2,1}^{\varepsilon,k}(t, x, \tilde{\omega}) \), we deduce that

\[
\begin{align*}
\int \int_{\mathbb{R}_{n+1}^T} I_{2,1}^{\varepsilon,k}(t, x, \tilde{\omega}) \, dx \, dt &= \frac{\lambda(\theta^*)}{\varepsilon} \int \int_{\mathbb{R}_{n+1}^T} (v_\varepsilon(t, x, \tilde{\omega}) \frac{\partial \varphi}{\partial x_k}(t, x)) \Lambda_{k,\varepsilon}(x, \tilde{\omega}, \theta^*) \, dx \, dt \quad (3.44) \\
&- \frac{1}{\varepsilon} \int \int_{\mathbb{R}_{n+1}^T} V(\Phi^{-1}(\frac{x}{\varepsilon}, \tilde{\omega})), (v_\varepsilon(t, x, \tilde{\omega}) \frac{\partial \varphi}{\partial x_k}(t, x)) \Lambda_{k,\varepsilon}(x, \tilde{\omega}, \theta^*) \, dx \, dt.
\end{align*}
\]

Therefore, summing equations (3.43), (3.44), we arrive at

\[
\begin{align*}
\sum_{k=1}^{n} \int \int_{\mathbb{R}_{n+1}^T} (I_{2,1}^{\varepsilon,k} + I_{2,2}^{\varepsilon,k})(t, x, \tilde{\omega}) \, dx \, dt &= \frac{\lambda(\theta^*)}{\varepsilon^2} \int \int_{\mathbb{R}_{n+1}^T} v_\varepsilon(t, x, \tilde{\omega}) \varphi(t, x) \Psi_\varepsilon(x, \tilde{\omega}, \theta^*) + \varepsilon \sum_{k=1}^{n} \frac{\partial \varphi}{\partial x_k}(t, x) \Lambda_{k,\varepsilon}(x, \tilde{\omega}, \theta^*) \, dx \, dt \\
&- \frac{1}{\varepsilon^2} \int \int_{\mathbb{R}_{n+1}^T} V(\Phi^{-1}(\frac{x}{\varepsilon}, \tilde{\omega})), (v_\varepsilon(t, x, \tilde{\omega}) \frac{\partial \varphi}{\partial x_k}(t, x)) \Lambda_{k,\varepsilon}(x, \tilde{\omega}, \theta^*) \, dx \, dt \\
&\times \varphi(t, x) \Psi_\varepsilon(x, \tilde{\omega}, \theta^*) + \varepsilon \sum_{k=1}^{n} \frac{\partial \varphi}{\partial x_k}(t, x) \Lambda_{k,\varepsilon}(x, \tilde{\omega}, \theta^*) \, dx \, dt \\
&- \sum_{k=1}^{n} \int \int_{\mathbb{R}_{n+1}^T} A(\Phi^{-1}(\frac{x}{\varepsilon}, \tilde{\omega})), (v_\varepsilon(t, x, \tilde{\omega}) \nabla \frac{\partial \varphi}{\partial x_k}(t, x)) [e_k \Psi_\varepsilon(x, \tilde{\omega}, \theta^*)] \, dx \, dt.
\end{align*}
\]

Moreover, expressing the \( I_3^{\varepsilon} \) term as

\[
\begin{align*}
I_3^{\varepsilon} &= \int \int_{\mathbb{R}_{n+1}^T} \frac{1}{\varepsilon^2} V(\Phi^{-1}(\frac{x}{\varepsilon}, \tilde{\omega})), (v_\varepsilon(t, x, \tilde{\omega}) \varphi(t, x) \Psi_\varepsilon(x, \tilde{\omega}, \theta^*) + \varepsilon \sum_{k=1}^{n} \frac{\partial \varphi}{\partial x_k}(t, x) \Lambda_{k,\varepsilon}(x, \tilde{\omega}, \theta^*) \, dx \, dt \\
&\quad + \int \int_{\mathbb{R}_{n+1}^T} U(\Phi^{-1}(\frac{x}{\varepsilon}, \tilde{\omega})), (v_\varepsilon(t, x, \tilde{\omega}) \varphi(t, x) \Psi_\varepsilon(x, \tilde{\omega}, \theta^*) + \varepsilon \sum_{k=1}^{n} \frac{\partial \varphi}{\partial x_k}(t, x) \Lambda_{k,\varepsilon}(x, \tilde{\omega}, \theta^*) \, dx \, dt,
\end{align*}
\]
adding with (3.45) and $I_1^\varepsilon$, we obtain

$$I_1^\varepsilon + \sum_{k=1}^{n} \iint_{R^{n+1}_T} \left( I_{2,1}^{\varepsilon,k} + I_{2,2}^{\varepsilon,k} \right) (t, x, \tilde{\omega}) \, dx \, dt + I_3^\varepsilon$$

$$= i \int_{R^n} v_0^0(x, \tilde{\omega}) \varphi(0, x) \Psi(x, \tilde{\omega}, \theta^*) dx - i \int \int_{R^{n+1}_T} v_\varepsilon(t, x, \tilde{\omega}) \frac{\partial \varphi}{\partial t}(t, x) \Psi_\varepsilon(x, \tilde{\omega}, \theta^*) dx \, dt$$

$$- \sum_{k, \ell=1}^{n} \iint_{R^{n+1}_T} v_\varepsilon(t, x, \tilde{\omega}) e_\ell \frac{\partial^2 \varphi}{\partial x_\ell \partial x_k}(t, x) \cdot A(\Phi^{-1}(\frac{x}{\varepsilon}, \tilde{\omega}), \tilde{\omega}) e_k \Psi_\varepsilon(x, \tilde{\omega}, \theta^*) \, dx \, dt$$

$$+ \iint_{R^{n+1}_T} U(\Phi^{-1}(\frac{x}{\varepsilon}, \tilde{\omega}), \tilde{\omega}, \Psi_\varepsilon(t, x, \varphi(t, x), \Psi_\varepsilon(x, \tilde{\omega}, \theta^*)) \, dx \, dt + O(\varepsilon).$$

For $\varepsilon = \varepsilon'(\tilde{\omega})$ and due to Step 2, i.e. $v_{\varepsilon'}(t, x, \tilde{\omega}) \xrightarrow{\varepsilon' \to 0} v_{\varepsilon}(t, x) \Psi(z, \omega, \theta^*)$, we obtain after letting $\varepsilon' \to 0$, from the previous equation

$$\lim_{\varepsilon' \to 0} \left( I_1^{\varepsilon'} + \sum_{k=1}^{n} \iint_{R^{n+1}_T} \left( I_{2,1}^{\varepsilon',k} + I_{2,2}^{\varepsilon',k} \right) (t, x, \tilde{\omega}) \, dx \, dt + I_3^{\varepsilon'} \right)$$

$$= i \int_{R^n} \left( \iint_{\Omega} \Phi((0,1)^n, \omega) \right) |\Psi(z, \omega, \theta^*)|^2 \, dz \, dP \right) v^0(x) \varphi(0, x) \, dx$$

$$- i \int \int_{R^{n+1}_T} \left( \iint_{\Phi((0,1)^n, \omega)} \right) |\Psi(z, \omega, \theta^*)|^2 \, dz \, dP \right) v_{\varphi}(t, x) \frac{\partial \varphi}{\partial t}(t, x) \, dx \, dt$$

$$- \sum_{k, \ell=1}^{n} \iint_{R^{n+1}_T} \left( \iint_{\Phi((0,1)^n, \omega)} A(\Phi^{-1}(z, \omega), \omega) (e_\ell \Psi) \cdot (e_k \Psi) \, dz \, dP \right)$$

$$\times v_{\varphi}(t, x) \frac{\partial^2 \varphi}{\partial x_\ell \partial x_k}(t, x) \, dx \, dt$$

$$+ \iint_{R^{n+1}_T} \left( \iint_{\Phi((0,1)^n, \omega)} U(\Phi^{-1}(z, \omega), \omega) |\Psi|^2 \, dz \, dP \right) v_{\varphi}(t, x) \varphi(t, x) \, dx \, dt.$$

(3.46)
Proceeding in the same way with respect to the term $I_{2,3}^\varepsilon(t, x, \omega)$, we obtain

$$
\lim_{\varepsilon' \to 0} \sum_{k=1}^{n} \int_{\mathbb{R}^{n+1}} I_{2,3}^\varepsilon(t, x, \omega) \, dx \, dt
= \sum_{k, \ell=1}^{n} \int_{\mathbb{R}^{n+1}} \left( \int_{\Omega} A(\Phi^{-1}(z, \omega), \omega) \left( (\nabla z + 2i\pi\theta^*) \Psi(z, \omega, \theta^*) \right) \right)
\cdot (e_\ell \overline{\Lambda}_k(z, \omega, \theta^*)) \, dz \, d\mathcal{P}(t, x) \frac{\partial^2 \overline{\varphi}}{\partial x_\ell \partial x_k}(t, x) \, dt
- \sum_{k, \ell=1}^{n} \int_{\mathbb{R}^{n+1}} \left( \int_{\Omega} A(\Phi^{-1}(z, \omega), \omega) (e_\ell \Psi(z, \omega, \theta^*)) \right)
\cdot (\nabla z - 2i\pi\theta^*) \overline{\Lambda}_k(z, \omega, \theta^*) \, dz \, d\mathcal{P}(t, x) \frac{\partial^2 \overline{\varphi}}{\partial x_\ell \partial x_k}(t, x) \, dx \, dt.
$$

Therefore, since $I_1^\varepsilon + I_2^\varepsilon + I_3^\varepsilon = 0$, (see (3.41)), combining the two last equations we conclude that, the function $v_\omega$ is a distribution solution of the following homogenized Schrödinger equation

$$
\left\{ \begin{array}{l}
\frac{i}{\varepsilon} \frac{\partial v_\omega}{\partial t}(t, x) - \text{div} (B^* \nabla v_\omega(t, x)) + U^* v_\omega(t, x) = 0, \quad (t, x) \in \mathbb{R}^{n+1}, \\
v_\omega(0, x) = \psi_0(x), \quad x \in \mathbb{R}^n,
\end{array} \right.
$$

(3.48)

where the effective tensor

$$
B_{k, \ell}^* = \frac{1}{c_\psi} \int_{\Omega} \int_{\Phi((0,1)^n, \omega)} \left\{ A(\Phi^{-1}(z, \omega), \omega) (e_\ell \Psi(z, \omega, \theta^*)) \cdot (e_k \overline{\Psi}(z, \omega, \theta^*)) \right\}
+ A(\Phi^{-1}(z, \omega), \omega) (e_\ell \Psi(z, \omega, \theta^*)) \cdot ((\nabla z - 2i\pi\theta^*) \overline{\Lambda}_k(z, \omega, \theta^*))
- A(\Phi^{-1}(z, \omega), \omega) ((\nabla z + 2i\pi\theta^*) \Psi(z, \omega, \theta^*)) \cdot (e_\ell \overline{\Lambda}_k(z, \omega, \theta^*)) \, dz \, d\mathcal{P}(\omega),
$$

(3.49)

for $k, \ell \in \{1, \ldots, n\}$, and the effective potential

$$
U^* = c_\psi^{-1} \int_{\Omega} \int_{\Phi((0,1)^n, \omega)} U(\Phi^{-1}(z, \omega), \omega) |\Psi(z, \omega, \theta^*)|^2 \, dz \, d\mathcal{P}(\omega)
$$

with

$$
c_\psi = \int_{\Omega} \int_{\Phi((0,1)^n, \omega)} |\Psi(z, \omega, \theta^*)|^2 \, dz \, d\mathcal{P}(\omega)
\equiv \int_{\Omega} \int_{\Phi((0,1)^n, \omega)} \left| \psi(\Phi^{-1}(\frac{x}{\varepsilon}, \omega), \omega) \right|^2 \, dz \, d\mathcal{P}(\omega).
$$

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Moreover, we are allowed to change the tensor $B^*$ in the equation (3.48) by the corresponding symmetric part of it, that is $A^* = (B^* + (B^*)^t)/2$.

4. (The Form of the Matrix $A^*$.) Now, we show that the homogenized tensor $A^*$ is a real value matrix, and it coincides with the hessian matrix of the function $\theta \mapsto \lambda(\theta)$ in the point $\theta^*$. In fact, due to $\nabla_\theta \lambda(\theta^*) = 0$ we can write

$$
\frac{1}{4\pi^2} \frac{\partial^2 \lambda(\theta^*)}{\partial \theta_\ell \partial \theta_k} \epsilon \psi
= \int_\Omega \int_{\Phi((0,1)^n,\omega)} \{ A(\Phi^{-1}(z,\omega), \omega)(e_\ell \Psi(z,\omega,\theta^*)) \cdot (e_k \overline{\Psi}(z,\omega,\theta^*)) \\
+ A(\Phi^{-1}(z,\omega), \omega) (e_\ell \Psi(z,\omega,\theta^*)) \cdot ((\nabla_z - 2i\pi \theta^*) \Lambda_k(z,\omega,\theta^*)) \\
- A(\Phi^{-1}(z,\omega), \omega) [(\nabla_z + 2i\pi \theta^*) \overline{\Psi}(z,\omega,\theta^*)] \cdot (e_k \overline{\Lambda}(z,\omega,\theta^*)) \\
+ A(\Phi^{-1}(z,\omega), \omega) (e_k \Psi(z,\omega,\theta^*)) \cdot ((\nabla_z - 2i\pi \theta^*) \Lambda_\ell(z,\omega,\theta^*)) \\
+ A(\Phi^{-1}(z,\omega), \omega) (e_k \Psi(z,\omega,\theta^*)) \cdot ((\nabla_z + 2i\pi \theta^*) \Psi(z,\omega,\theta^*)) \\
- A(\Phi^{-1}(z,\omega), \omega) ((\nabla_z + 2i\pi \theta^*) \Psi(z,\omega,\theta^*)) \\
\cdot (e_k \overline{\Lambda_\ell}(z,\omega,\theta^*)) \} \, dz \, d\mathbb{P}(\omega),
$$

from which we obtain

$$
A^* = \frac{1}{8\pi^2} D_\theta^2 \lambda(\theta^*).
$$

Therefore, from Remark 2.3 we deduce the well-posedness of the homogenized Schrödinger (3.48). Hence the function $v_\omega \in L^2(\mathbb{R}_T^{n+1})$ does not depend on $\tilde{\omega} \in \Omega$. Moreover, denoting by $v$ the unique solution of the problem (3.48), we have that the sequence $\{v_\varepsilon(t,x,\tilde{\omega})\}_{\varepsilon > 0} \subset L^2(\mathbb{R}_T^{n+1})$ $\Phi_\omega$--two-scale converges to the function

$$
v(t,x) \Psi(z,\omega,\theta^*) \equiv v(t,x) \psi(\Phi^{-1}(z,\omega),\omega,\theta^*).
$$

5. (A Corrector-type Result.) Finally, we show the following corrector type result, that is, for a.e. $\tilde{\omega} \in \Omega$

$$
\lim_{\varepsilon \to 0} \int_{\mathbb{R}_T^{n+1}} \left| v_\varepsilon(t,x,\tilde{\omega}) - v(t,x) \psi(\Phi^{-1}\left(\frac{x}{\varepsilon},\tilde{\omega}\right),\tilde{\omega},\theta^*) \right|^2 dx \, dt = 0.
$$
We begin by the simple observation

\[
\begin{align*}
\int_\mathbb{R}^{n+1}_T |v_\varepsilon(t, x, \tilde{\omega}) - v(t, x) \psi\left(\Phi^{-1}\left(\frac{x}{\varepsilon}, \tilde{\omega}\right)\right)|^2 dx dt \\
= \int_\mathbb{R}^{n+1}_T |v_\varepsilon(t, x, \tilde{\omega})|^2 dx dt \\
- \int_\mathbb{R}^{n+1}_T v_\varepsilon(t, x, \tilde{\omega}) v(t, x) \psi\left(\Phi^{-1}\left(\frac{x}{\varepsilon}, \tilde{\omega}\right)\right) dx dt \\
- \int_\mathbb{R}^{n+1}_T \tilde{v}_\varepsilon(t, x, \tilde{\omega}) v(t, x) \psi\left(\Phi^{-1}\left(\frac{x}{\varepsilon}, \tilde{\omega}\right)\right) dx dt \\
+ \int_\mathbb{R}^{n+1}_T |v(t, x) \psi\left(\Phi^{-1}\left(\frac{x}{\varepsilon}, \tilde{\omega}\right)\right)|^2 dx dt.
\end{align*}
\]

(3.51)

From Lemma 3.1 we see that, the first integral of the right hand side of the above equation satisfies, for all \(t \in [0, T]\) and a.e. \(\tilde{\omega} \in \Omega\)

\[
\int_\mathbb{R}^n |v_\varepsilon(t, x, \tilde{\omega})|^2 dx = \int_\mathbb{R}^n |u_\varepsilon(t, x, \tilde{\omega})|^2 dx = \int_\mathbb{R}^n |v_\varepsilon^0(x, \tilde{\omega})|^2 dx = \int_\mathbb{R}^n |v_\varepsilon^0(x, \tilde{\omega})|^2 dx.
\]

Using the elliptic regularity theory (see E. De Giorgi [15], G. Stampacchia [24]), it follows that \(\psi(\theta) \in H^\infty(\mathbb{R}^n; L^2(\Omega))\) and we can apply the Ergodic Theorem to obtain

\[
\begin{align*}
\lim_{\varepsilon \to 0} \int_\mathbb{R}^{n+1}_T |v_\varepsilon(t, x, \tilde{\omega})|^2 dx dt \\
= \lim_{\varepsilon \to 0} \int_\mathbb{R}^{n+1}_T |v_\varepsilon^0(x, \tilde{\omega})|^2 dx \\
= c_{\Phi}^{-1} \int_\mathbb{R}^{n+1}_T \int_\Omega \int_{\Phi([0,1]^n, \omega)} |v_\varepsilon^0(x) \psi(\Phi^{-1}(z, \omega))|^2 dz dP dx dt.
\end{align*}
\]

Similarly, we have

\[
\begin{align*}
\lim_{\varepsilon \to 0} \int_\mathbb{R}^{n+1}_T |v(t, x) \psi(\Phi^{-1}\left(\frac{x}{\varepsilon}, \tilde{\omega}\right)\right)|^2 dx dt \\
= c_{\Phi}^{-1} \int_\mathbb{R}^{n+1}_T \int_\Omega \int_{\Phi([0,1]^n, \omega)} |v(t, x) \psi(\Phi^{-1}(z, \omega))|^2 dz dP dx dt.
\end{align*}
\]

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Moreover, seeing that for a.e. $\tilde{\omega} \in \Omega$

$$\lim_{\varepsilon \to 0} \iint_{\mathbb{R}_{T}^{n+1}} v_{\varepsilon}(t, x, \tilde{\omega}) \psi (\Phi^{-1} \left( \frac{x}{\varepsilon}, \tilde{\omega} \right), \tilde{\omega}, \theta^{*}) \, dxdt$$

$$= c_{\Phi}^{-1} \iint_{\mathbb{R}_{T}^{n+1}} \int_{\Omega} \int_{\Phi ([0,1]^{n}, \omega)} v(t, x) \psi (\Phi^{-1} (z, \omega), \omega, \theta^{*}) \times \psi (\Phi^{-1} (z, \omega), \omega, \theta^{*}) \, dz \, d\mathbb{P} \, dxdt,$$

we can make $\hat{A} \varepsilon \to 0$ in the equation (3.51) to find

$$\lim_{\varepsilon \to 0} \iint_{\mathbb{R}_{T}^{n+1}} |v_{\varepsilon}(t, x, \tilde{\omega}) - v(t, x) \psi (\Phi^{-1} \left( \frac{x}{\varepsilon}, \tilde{\omega} \right), \tilde{\omega}, \theta^{*})|^{2} \, dxdt$$

$$= c_{\Phi}^{-1} \left( \int_{\Omega} \int_{\Phi ([0,1]^{n}, \omega)} |\psi (\Phi^{-1} (z, \omega), \omega, \theta^{*})|^{2} \, dz \, d\mathbb{P} (\omega) \right)$$

$$\times \left( \iint_{\mathbb{R}_{T}^{n+1}} |v^{0}(x)|^{2} \, dx \, dt - \iint_{\mathbb{R}_{T}^{n+1}} |v(t, x)|^{2} \, dx \, dt \right),$$

for a.e. $\tilde{\omega} \in \Omega$. Therefore, using the energy conservation of the homogenized Schrödinger equation (3.32), that is, for all $t \in [0, T]$

$$\int_{\mathbb{R}^{n}} |v(t, x)|^{2} \, dx = \int_{\mathbb{R}^{n}} |v^{0}(x)|^{2} \, dx,$$

we obtain that, for a.e. $\tilde{\omega} \in \Omega$

$$\lim_{\varepsilon \to 0} \iint_{\mathbb{R}_{T}^{n+1}} |v_{\varepsilon}(t, x, \tilde{\omega}) - v(t, x) \psi (\Phi^{-1} \left( \frac{x}{\varepsilon}, \tilde{\omega} \right), \tilde{\omega}, \theta^{*})|^{2} \, dxdt = 0,$$

completing the proof of the theorem.

\[\square\]

### 3.2 Radom Perturbations of the Quasiperiodic Case

In this section, we shall give a nice application of the framework introduced in this paper, which can be used to homogenize a model beyond the periodic settings considered by Allaire and Piatnitski in [3].

Let $n, m \geq 1$ be integers numbers and $\lambda_{1}, \cdots, \lambda_{m}$ be vectors in $\mathbb{R}^{n}$ linearly independent over the set $\mathbb{Z}$ satisfying the condition

$$\{ k \in \mathbb{Z}^{m}; |k_{1}\lambda_{1} + \cdots + k_{m}\lambda_{m}| < d \}$$

(3.52)
is a finite set for any \( d > 0 \). Therefore, given a stochastic deformation \( \Phi \), we have \( \mathcal{H}_\Phi \subset \subset \mathcal{L}_\Phi \) under condition (3.52), and it follows a solution of the Bloch’s spectral cell equation, see [13].

Let \( (\Omega_0, \mathcal{F}_0, \mathbb{P}_0) \) be a probability space and \( \tau_0 : \mathbb{Z}^n \times \Omega_0 \to \Omega_0 \) be a discrete ergodic dynamical system and \( \mathbb{R}^m/\mathbb{Z}^m \) be the \( m \)-dimensional torus which can be identified with the cube \([0, 1]^m\). For \( \Omega := \Omega_0 \times [0, 1]^m \), consider the following continuous dynamical system \( T : \mathbb{R}^n \times \Omega \to \Omega \), defined by

\[
T(x)(\omega_0, s) := \left( \tau_{[s + Mx]} \omega_0, s + Mx - \lfloor s + Mx \rfloor \right),
\]

where \( M \) is the matrix \( M = (\lambda_i \cdot e_j)_{i=1,j=1}^{m,n} \) and \( \lfloor y \rfloor \) denotes the unique element in \( \mathbb{Z}^m \) such that \( y - \lfloor y \rfloor \in [0, 1)^m \). Now, we consider \([0, 1)^m\)-periodic functions \( A_{per} : \mathbb{R}^m \to \mathbb{R}^2 \), \( V_{per} : \mathbb{R}^m \to \mathbb{R} \) and \( U_{per} : \mathbb{R}^m \to \mathbb{R} \) such that

- There exists \( a_0, a_1 > 0 \) such that for all \( \xi \in \mathbb{R}^n \) and for a.e \( y \in \mathbb{R}^m \) we have
  \[
a_0 |\xi|^2 \leq A_{per}(y) \xi \cdot \xi \leq a_1 |\xi|^2.
\]

- \( V_{per}, U_{per} \in L^\infty(\mathbb{R}^m) \).

Let \( B_{per} : \mathbb{R}^m \to \mathbb{R}^{n^2} \) be a \([0, 1)^m\)-periodic matrix and \( \Upsilon : \mathbb{R}^n \times [0, 1)^m \to \mathbb{R}^n \) be any stochastic diffeomorphism satisfying

\[
\nabla \Upsilon(x, s) = B_{per}(T(x)(\omega_0, s)).
\]

Thus, we define the following stochastic deformation \( \Phi : \mathbb{R}^n \times \Omega \to \mathbb{R}^n \) by \( \Phi(x, \omega) = \Upsilon(x, s) + X(\omega_0), \) where we have used the notation \( \omega \) for the pair \((\omega_0, s) \in \Omega \) and \( X : \Omega_0 \to \mathbb{R}^n \) is a random vector. Now, taking

\[
A(x, \omega) := A_{per}(T(x)\omega), \ V(x, \omega) := V_{per}(T(x)\omega), \ U(x, \omega) := U_{per}(T(x)\omega)
\]

in the equation (1.1), it can be seen after some computations that the spectral equation correspondent is

\[
\begin{cases}
- \left( \text{div}_{QP} + 2i\pi \theta \right) \left[ A_{per}(\cdot) \left( \nabla_{QP} + 2i\pi \theta \right) \Psi_{per}(\cdot) \right] \\
\quad + V_{per}(\cdot) \Psi_{per}(\cdot) = \lambda \Psi_{per}(\cdot) \quad \text{in } [0, 1)^m, \\
\Psi_{per}(\cdot) \psi \text{ is a } [0, 1)^m\text{-periodic function},
\end{cases}
\]

where the operators \( \text{div}_{QP} \) and \( \nabla_{QP} \) are defined as
\( (\nabla^{\mathrm{QP}} u_{\text{per}})(y) := B^{-1}_{\text{per}}(y) M^* (\nabla u_{\text{per}})(y); \)

\( (\text{div}_{\mathrm{QP}} a)(y) := \text{div} \left( MB^{-1}_{\text{per}}(\cdot) a(\cdot) \right)(y). \)

Assume that for some \( \theta^* \in \mathbb{R}^n \), the spectral equation (3.53) admits a solution \( (\lambda(\theta^*), \Psi_{\text{per}}(\theta^*)) \in \mathbb{R} \times H^1((0,1)^m) \), such that (2.29) holds. Then, we consider the problem (1.1) with new coefficients as highlighted above and with well-prepared initial data, that is,

\[
u_\varepsilon(x,\omega) := e^{2\pi i \frac{\theta^* \cdot x}{\varepsilon}} \Psi_{\text{per}} \left( T \left( \Phi^{-1} \left( \frac{x}{\varepsilon}, \omega \right) \right) \omega, \theta^* \right) v^0(x),
\]

for \((x,\omega) \in \mathbb{R}^n \times \Omega \) and \( v^0 \in C^\infty_c(\mathbb{R}^n) \). Applying Theorem 3.2, the function

\[
u_\varepsilon(t,x,\omega) := e^{-i \frac{\lambda(\theta^*) t}{\varepsilon} + 2\pi i \frac{\theta^* \cdot x}{\varepsilon}} u_\varepsilon(t,x,\omega), \quad (t,x) \in \mathbb{R}^{n+1} \times \Omega,
\]

\( \Phi_{\omega} \)-two-scale converges strongly to \( \nu(t,x) \Psi_{\text{per}} \left( T \left( \Phi^{-1}(z,\omega) \right) \omega, \theta^* \right) \), where \( \nu \in C([0,T],L^2(\mathbb{R}^n)) \) is the unique solution of the homogenized Schrödinger equation

\[egin{cases}
\frac{i}{\varepsilon} \partial v \partial t - \text{div}(A^* \nabla v) + U^* v = 0, & \text{em } \mathbb{R}^{n+1}, \\
v(0,x) = v^0(x), & x \in \mathbb{R}^n,
\end{cases}
\]

with effective matrix \( A^* = D^2_{\phi^*} \lambda(\theta^*) \) and effective potential

\[U^* = c_\psi^{-1} \int_{[0,1]^m} U_{\text{per}}(y) \left| \Psi_{\text{per}}(y,\theta^*) \right|^2 \left| \det(B_{\text{per}}(y)) \right| dy,
\]

where

\[c_\psi = \int_{[0,1]^m} \left| \Psi_{\text{per}}(y,\theta^*) \right|^2 \left| \det(B_{\text{per}}(y)) \right| dy.
\]

It is worth highlighting that this singular example encompasses the settings considered by Allaire-Piatnitski in [3]. For this, it is enough to take

\[n = m, \lambda_j = e_j, \gamma(\cdot, s) \equiv I_{n \times n}, \text{ and } X(\cdot) \equiv 0.
\]

## 4 Homogenization of Quasi-Perfect Materials

We consider in this final section an interesting context, which is the small random perturbation of the periodic setting. To begin, we recall that the
homogenization analysis (see Theorem 3.2) of the equation (1.1) rely on the spectral study of the operator $L^\Phi(\theta)(\theta \in \mathbb{R}^n)$ posed in the dual space $H^*$ and with domain $D(L^\Phi(\theta)) = \mathcal{H}$, defined by

$$L^\Phi(\theta)[f] := -(\text{div} + 2i\pi \theta) \left[ A\left(\Phi^{-1}(\cdot, \cdot), \cdot\right) (\nabla + 2i\pi \theta) f\left(\Phi^{-1}(\cdot, \cdot), \cdot\right) \right] + V\left(\Phi^{-1}(\cdot, \cdot), \cdot\right) f\left(\Phi^{-1}(\cdot, \cdot), \cdot\right),$$

(4.54)

where $\Phi : \mathbb{R}^n \times \Omega \to \mathbb{R}^n$ is a stochastic deformation, $A : \mathbb{R}^n \times \Omega \to \mathbb{R}^{n^2}$ and $V : \mathbb{R}^n \times \Omega \to \mathbb{R}$ are stationary functions.

### 4.1 Perturbed Periodic Case: Spectral Analysis

We shall study the spectral properties of the operator $L^\Phi(\theta)$, when the diffeomorphism $\Phi$ is a stochastic perturbation of the identity. This concept was introduced in [9], and well-developed by T. Andrade, W. Neves, J. Silva [5] for modelling quasi-perfect materials.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $\tau : \mathbb{Z}^n \times \Omega \to \Omega$ a discrete dynamical system, and $Z$ any fixed stochastic deformation. Then, we consider the concept of stochastic perturbation of the identity given by the following

**Definition 4.1.** Given $\eta \in (0, 1)$, let $\Phi_\eta : \mathbb{R}^n \times \Omega \to \mathbb{R}^n$ be a stochastic deformation. Then $\Phi_\eta$ is said a stochastic perturbation of the identity, when it can be written as

$$\Phi_\eta(y, \omega) = y + \eta Z(y, \omega) + O(\eta^2),$$

(4.55)

for some stochastic deformation $Z$.

We emphasize that the equality (4.55) is understood in the Lip$_{\text{loc}}(\mathbb{R}^n; L^2(\Omega))$ sense, i.e. for each bounded open subset $\mathcal{O} \subset \mathbb{R}^n$, there exist $\delta, C > 0$, such that for all $\eta \in (0, \delta)$

$$\sup_{y \in \mathcal{O}} \|\Phi_\eta(y, \cdot) - y - \eta Z(y, \cdot)\|_{L^2(\Omega)} + \text{ess sup}_{y \in \mathcal{O}} \|\nabla_y \Phi_\eta(y, \cdot) - I - \eta \nabla_y Z(y, \cdot)\|_{L^2(\Omega)} \leq C \eta^2.$$

Moreover, after some computations, we have

$$\begin{cases} 
\nabla_y^{-1}\Phi_\eta = I - \eta \nabla_y Z + O(\eta^2), \\
\det (\nabla_y \Phi_\eta) = 1 + \eta \text{div}_y Z + O(\eta^2).
\end{cases}$$

(4.56)
Now, we consider the $[0, 1)^n$–periodic functions $A_{\text{per}} : \mathbb{R}^n \to \mathbb{R}^{n^2}$, $V_{\text{per}} : \mathbb{R}^n \to \mathbb{R}$ and $U_{\text{per}} : \mathbb{R}^n \to \mathbb{R}$, such that

- There exists $a_0, a_1 > 0$ such that for all $\xi \in \mathbb{R}^n$ and for a.e $y \in \mathbb{R}^n$ we have
  \[ a_0|\xi|^2 \leq A_{\text{per}}(y)\xi \cdot \xi \leq a_1|\xi|^2. \]

- $V_{\text{per}}, U_{\text{per}} \in L^\infty(\mathbb{R}^n)$.

The following lemma is well-known and it is stated explicitly here only for reference.

**Lemma 4.2.** For $\theta \in \mathbb{R}^n$ and $f \in H^1_{\text{per}}([0, 1)^n)$, let $L_{\text{per}}(\theta)$ be the operator defined by

$$L_{\text{per}}(\theta)[f] := -(\text{div}_y + 2\pi \theta)[A_{\text{per}}(y)(\nabla_y + 2\pi \theta)f(y)] + V_{\text{per}}(y)f(y),$$

with variational formulation

$$\langle L_{\text{per}}(\theta)[f], g \rangle := \int_{[0, 1)^n} A_{\text{per}}(y)(\nabla_y + 2\pi \theta)f(y) \cdot (\nabla_y + 2\pi \theta)\overline{g(y)} dy$$

$$+ \int_{[0, 1)^n} V_{\text{per}}(y)f(y)\overline{g(y)} dy,$$

for $f, g \in H^1_{\text{per}}([0, 1)^n)$. Then $L_{\text{per}}(\theta)$ has the following properties:

(i) There exist $\gamma_0, b_0 > 0$, such that $L_{\gamma_0} := L_{\text{per}}(\theta) + \gamma_0 I$ satisfies for all $f \in H^1_{\text{per}}([0, 1)^n)$,

$$\langle L_{\gamma_0}[f], f \rangle \geq b_0\|f\|_{H^1_{\text{per}}([0, 1)^n)}^2.$$

(ii) The point spectrum of $L_{\text{per}}(\theta)$ is not empty and their eigenspaces have finite dimension, that is, the set

$$\sigma_{\text{point}}(L_{\text{per}}(\theta)) = \{ \lambda \in \mathbb{C} : \lambda \text{ an eigenvalue of } L_{\text{per}}(\theta) \}$$

is not empty and for all $\lambda \in \sigma_{\text{point}}(L_{\text{per}}(\theta))$ fixed,

$$\dim\{ f \in H^1_{\text{per}}([0, 1)^n) : L_{\text{per}}(\theta)[f] = \lambda f \} < \infty.$$

(iii) Every point in $\sigma_{\text{point}}(L_{\text{per}}(\theta))$ is isolated.
In what follows, we are interested in the study of spectral properties of the operator \( L_{\Phi} \) whose variational formulation is given by

\[
\langle L_{\Phi} \Phi_{\eta}(\theta) \rangle[f,g] := \int_{\Omega} \int_{\Phi_{\eta}(0,1) \times \omega} \frac{A_{\text{per}}(\Phi_{\eta}^{-1}(z,\omega))(\nabla z + 2i\pi \theta)f(\Phi_{\eta}^{-1}(z,\omega),\omega)}{(\nabla z + 2i\pi \theta)g(\Phi_{\eta}^{-1}(z,\omega),\omega)} \, dz \, d\mathbb{P}(\omega)
\]

\[
+ \int_{\Omega} \int_{\Phi_{\eta}(0,1) \times \omega} V_{\text{per}}(\Phi_{\eta}^{-1}(z,\omega)) f(\Phi_{\eta}^{-1}(z,\omega),\omega) g(\Phi_{\eta}^{-1}(z,\omega),\omega) \, dz \, d\mathbb{P}(\omega),
\]

for \( f, g \in H \). As we shall see in the next theorem, some of the spectral properties of the operator \( L_{\Phi_0}(\theta) \) are inherited from the periodic case.

**Theorem 4.3.** Let \( \Phi_{\eta}, \eta \in (0,1) \) be a stochastic perturbation of identity and \( \theta_0 \in \mathbb{R}^n \). If \( \lambda_0 \) is an eigenvalue of \( L_{\text{per}}(\theta_0) \) with multiplicity \( k_0 \in \mathbb{N} \), that is,

\[
\dim\{ f \in H^1_{\text{per}}([0,1)^n) ; L_{\text{per}}(\theta_0)[f] = \lambda_0 f \} = k_0,
\]

then there exist a neighbourhood \( U \) of \((0,\theta_0)\), \( k_0 \) real analytic functions

\[
(\eta,\theta) \in U \mapsto \lambda_k(\eta,\theta) \in \mathbb{R}, \ k \in \{1,\ldots,k_0\},
\]

and \( k_0 \) vector-value analytic maps

\[
(\eta,\theta) \in U \mapsto \psi_k(\eta,\theta) \in \mathcal{H} \setminus \{0\}, \ k \in \{1,\ldots,k_0\},
\]

such that, for all \( k \in \{1,\ldots,k_0\} \),

(i) \( \lambda_k(0,\theta_0) = \lambda_0 \),

(ii) \( L_{\Phi_0}(\theta)[\psi_k(\eta,\theta)] = \lambda_k(\eta,\theta) \psi_k(\eta,\theta), \forall (\eta,\theta) \in U, \)

(iii) \( \dim\{ f \in \mathcal{H} ; L_{\Phi_0}(\theta)[f] = \lambda_k(\eta,\theta) f \} \leq k_0, \forall (\eta,\theta) \in U. \)

**Proof.** 1. The aim of this step is to rewrite the operator \( L_{\Phi_0}(\theta) \in \mathcal{B}(\mathcal{H},\mathcal{H}^*) \), for \( \eta \in (0,1) \) and \( \theta \in \mathbb{R}^n \) as an expansion in the variable \((\eta,\theta)\) of operators in \( \mathcal{B}(\mathcal{H},\mathcal{H}^*) \) around the point \((\eta,\theta) = (0,\theta_0)\). For this, using the variational formulation (4.58), a change of variables, and the expansions (4.56) we obtain

\[
L_{\Phi_0}(\theta) = L_{\text{per}}(\theta_0) + \sum_{|\alpha,\beta| = 1}^3 ((\eta,\theta) - (0,\theta_0))^{(\alpha,\beta)} L_{(\alpha,\beta)} + O(\eta^2), \quad (4.59)
\]
in $\mathcal{B}(\mathcal{H}, \mathcal{H}^*)$ as $\eta \to 0$, where $L_{(\alpha, \beta)} \in \mathcal{B}(\mathcal{H}, \mathcal{H}^*)$ and $|(\alpha, \beta)| = \alpha + \sum_{k=1}^{n} \beta_k$. Here, for $(\alpha, \beta) \in \mathbb{N} \times \mathbb{N}^n$ and $\beta = (\beta_1, \ldots, \beta_n)$, we are using the multi-index notation $((\eta, \theta) - (0, \theta_0))^{(\alpha, \beta)} = \eta^{\alpha} \prod_{k=1}^{n} (\theta_k - \theta_{0k})^{\beta_k}$. Clearly, we can consider the parameters $(\eta, \theta)$ in the set $B(0, 1) \times \mathbb{C}^n$.

2. In this step, we shall modify the expansion (4.59) conveniently in order to obtain an holomorphic invertible operator in the variable $(\eta, \theta)$. For this, remember that according to the item $(i)$ in Lemma 4.2, there exists $\gamma_0 > 0$ such that the operator $L_{\text{per}}(\theta_0) + \gamma_0 I$ is invertible. Then there exists $\delta > 0$ such that the expansion

$$L^{\Phi_0}(\theta) + \gamma_0 I = L_{\text{per}}(\theta_0) + \gamma_0 I + \sum_{|(\alpha, \beta)| = 1} (\eta, \theta) - (0, \theta_0))^{(\alpha, \beta)} L_{(\alpha, \beta)} + O(\eta^2) \quad (4.60)$$

in $\mathcal{B}(\mathcal{H}, \mathcal{H}^*)$ as $\eta \to 0$, is invertible for all $(\eta, \theta) \in B(0, \delta) \times B(\theta_0, \delta)$, since the set of invertible bounded operators $GL(\mathcal{H}, \mathcal{H}^*)$ is an open subset of $\mathcal{B}(\mathcal{H}, \mathcal{H}^*)$. Now, we denote by $S(\eta, \theta)$ the inverse operator of $L^{\Phi_0}(\theta) + \gamma_0 I$, $(\eta, \theta) \in B(0, \delta) \times B(\theta_0, \delta)$. Since the map $L \in GL(\mathcal{H}, \mathcal{H}^*) \mapsto L^{-1} \in \mathcal{B}(\mathcal{H}^*, \mathcal{H})$ is continuous, the map

$$(\eta, \theta) \in B(0, \delta) \times B(\theta_0, \delta) \mapsto S(\eta, \theta) \in \mathcal{B}(\mathcal{H}^*, \mathcal{H})$$

is continuous. As a consequence of this, for $(\tilde{\eta}, \tilde{\theta}) \in B(0, \delta) \times B(\theta_0, \delta)$ fixed, the limit of

$$\frac{S(\eta, \tilde{\theta}) - S(\tilde{\eta}, \tilde{\theta})}{\eta - \tilde{\eta}} = -S(\eta, \tilde{\theta}) \left[ \frac{(L^{\Phi_0}(\tilde{\theta}) + \gamma_0 I) - (L^{\Phi_0}(\tilde{\theta}) + \gamma_0 I)}{\eta - \tilde{\eta}} \right] S(\tilde{\eta}, \tilde{\theta}),$$

as $\tilde{\eta} \neq \eta \to 0$, exists. Thus, $\eta \in B(0, \delta) \mapsto S(\eta, \tilde{\theta})$ is an holomorphic map. In analogy with it, for $j \in \{1, \ldots, n\}$, we can prove that

$$\theta_j \mapsto S(\tilde{\eta}, \tilde{\theta}_1, \ldots, \tilde{\theta}_{j-1}, \theta_j, \tilde{\theta}_{j+1}, \ldots, \tilde{\theta}_n)$$

is an holomorphic map. Therefore, by Osgood’s Lemma, see for instance [?], we conclude that

$$(\eta, \theta) \in B(0, \delta) \times B(\theta_0, \delta) \mapsto S(\eta, \theta) \in \mathcal{B}(\mathcal{H}^*, \mathcal{H}) \quad (4.61)$$

is a holomorphic function.
3. Finally, we are in conditions to prove items (i), (ii) and (iii) (the spectral analysis of the operator $S(\eta, \theta)$). First, we shall note that for $(\eta, \theta)$ in a neighbourhood of $(0, \theta_0)$, the map $(\eta, \theta) \mapsto S(\eta, \theta)$ satisfies the assumptions of the Theorem 2.22. We begin recalling that the restriction operator $T \in \mathcal{B}(\mathcal{H}^*, \mathcal{H}) \mapsto T|_{\mathcal{L}} \in \mathcal{B}(\mathcal{L}, \mathcal{L})$ is continuous and it satisfies

$$\|T\|_{\mathcal{B}(\mathcal{L}, \mathcal{L})} \leq \|T\|_{\mathcal{B}(\mathcal{H}^*, \mathcal{H})}, \quad \forall T \in \mathcal{B}(\mathcal{H}^*, \mathcal{H}).$$

Then, by (4.61), the map $(\eta, \theta) \in B(0, \delta) \times B(\theta_0, \delta) \mapsto S(\eta, \theta) \in \mathcal{B}(\mathcal{L}, \mathcal{L})$ is holomorphic. Since holomorphic maps are, locally, analytic maps there exists a neighbourhood $U$ of $(0, \theta_0), (0, \theta_0) \in U \subset \mathbb{C} \times \mathbb{C}^n$, and a family $\{S_\sigma\}_{\sigma \in \mathbb{N} \times \mathbb{N}}$ contained in $\mathcal{B}(\mathcal{L}, \mathcal{L})$, such that

$$S(\eta, \theta) = S_0 + \sum_{\sigma \in \mathbb{N} \times \mathbb{N}^n \setminus \{0\}} (\eta, \theta)^\sigma S_\sigma, \quad \forall (\eta, \theta) \in U. \quad (4.63)$$

Using (4.60) and (4.63), it is easy to see that $S_0 = (L_{\text{per}}(\theta_0) + \gamma_0 I)|_{\mathcal{L}}$. Notice also that $\mu_0 := (\lambda_0 + \gamma_0)^{-1}$ is an eigenvalue of $S_0$ if and only if $\lambda_0$ is an eigenvalue of $L_{\text{per}}(\theta_0)$ that is

$$g \in \{f \in \mathcal{L} : S_0[f] = \mu_0 f\} \iff g \in \{f \in \mathcal{L} : L_{\text{per}}(\theta_0)[f] = \lambda_0 f\}.$$

The final part of the proof is a direct application of the Theorem 2.22. Due to our assumption, $\mu_0$ is a real eigenvalue of the operator $S_0$ with multiplicity $k_0$. Hence, by the Theorem 2.22 there exists a neighbourhood $\tilde{U}$ of $(0, \theta_0)$, with $\tilde{U} \subset U$ and analytic maps

$$(\eta, \theta) \in \tilde{U} \mapsto \mu_{01}(\eta, \theta), \mu_{02}(\eta, \theta), \ldots, \mu_{0k_0}(\eta, \theta) \in (0, \infty),$$

$$(\eta, \theta) \in \tilde{U} \mapsto \psi_{01}(\eta, \theta), \psi_{02}(\eta, \theta), \ldots, \psi_{0k_0}(\eta, \theta) \in \mathcal{L} - \{0\},$$

such that

- $\mu_{0e}(0, \theta_0) = \mu_0$,
- $S(\eta, \theta)[\psi_{0e}(\eta, \theta)] = \mu_{0e}(\eta, \theta)\psi_{0e}(\eta, \theta), \quad \forall (\eta, \theta) \in \tilde{U}$,
- $\dim\{f \in \mathcal{L} : S(\eta, \theta)[f] = \mu_{0e}(\eta, \theta)f\} \leq k_0, \quad \forall (\eta, \theta) \in \tilde{U},$
for all $\ell \in \{1, \ldots, k_0\}$. Thus, the proof of the item $(i)$ is clear.

Using the second equality above, we obtain

$$
(L^{\Phi_\eta}(\theta) + \gamma_0 I)[\psi_{\ell \theta}(\eta, \theta)] = \frac{1}{\mu_{\ell \theta}(\eta, \theta)}(L^{\Phi_\eta}(\theta) + \gamma_0 I)\{S(\eta, \theta)[\psi_{\ell \theta}(\eta, \theta)]\} = \frac{1}{\mu_{\ell \theta}(\eta, \theta)} \psi_{\ell \theta}(\eta, \theta),
$$

which implies that $L^{\Phi_\eta}(\theta)[\psi_{\ell \theta}(\eta, \theta)] = \lambda_{\ell \theta}(\eta, \theta) \psi_{\ell \theta}(\eta, \theta)$, for $(\eta, \theta) \in \tilde{U}$, $\ell \in \{1, \ldots, m_0\}$ and $\lambda_{\ell \theta}(\eta, \theta) := [\mu_{\ell \theta}(\eta, \theta)]^{-1} - \gamma_0$. This finish the proof of the item $(ii)$.

Finally, note that $S(\eta\theta)[\mathcal{L}] \subset \mathcal{H}$ and

$$
g \in \{f \in \mathcal{H}; S(\eta, \theta)[f] = \mu_{\ell \theta}(\eta, \theta) f\} \iff g \in \{f \in \mathcal{H}; L^{\Phi_\eta}(\theta)[f] = \lambda_{\ell \theta}(\eta, \theta) f\},
$$

which concludes the proof of the item $(iii)$. Hence the proof is completed. □

### 4.2 Homogenization Analysis of the Perturbed Model

In this section, we shall investigate in which way the stochastic perturbation of the identity characterize the form of the coefficients, during the asymptotic limit of the Schrödinger equation

$$
\begin{aligned}
&\frac{i}{\varepsilon} \frac{\partial u_{\eta \varepsilon}}{\partial t} - \text{div} \left( A_{\text{per}} \left( \Phi^{-1}_\eta \left( \frac{x}{\varepsilon}, \omega \right) \right) \nabla u_{\eta \varepsilon} \right) \\
&+ \left( \frac{1}{\varepsilon^2} V_{\text{per}} \left( \Phi^{-1}_\eta \left( \frac{x}{\varepsilon}, \omega \right) \right) + U_{\text{per}} \left( \Phi^{-1}_\eta \left( \frac{x}{\varepsilon}, \omega \right) \right) \right) u_{\eta \varepsilon} = 0 \quad \text{in } \mathbb{R}^{n+1}_T \times \Omega, \\
u_{\eta \varepsilon}(0, x, \omega) = u_{\eta \varepsilon}^0(x, \omega), \quad (x, \omega) \in \mathbb{R}^n \times \Omega,
\end{aligned}
$$

(4.64)

where $0 < T < \infty$, $\mathbb{R}^{n+1}_T = (0, T) \times \mathbb{R}^n$. The coefficients are accomplishing of the periodic functions $A_{\text{per}}(y), V_{\text{per}}(y), U_{\text{per}}(y)$ (as defined in the last subsecion) with a stochastic perturbation of identity $\Phi_\eta$; $\eta \in (0, 1)$, presenting an rate of oscillation $\varepsilon^{-1}, \varepsilon > 0$. The function $u_{\eta \varepsilon}^0(x, \omega)$ is a well prepared initial data (see (4.72)) and this well-preparedness is triggered by natural periodic conditions on the existence of a pair $(\theta^*, \lambda_{\text{per}}(\theta^*)) \in \mathbb{R}^n \times \mathbb{R}$ such that

$$(i) \quad \lambda_{\text{per}}(\theta^*) \text{ is a simple eigenvalue of } L_{\text{per}}(\theta^*),
\quad (ii) \quad \theta^* \text{ is a critical point of } \lambda_{\text{per}}(\cdot), \text{ that is, } \nabla_\theta \lambda_{\text{per}}(\theta^*) = 0.
$$

(4.65)
By the condition $(i)$ and the Theorem 4.3, there exists a neighborhood $U$ of $(0, \theta^*)$ and the analytic maps
\[
(i) \quad (\eta, \theta) \in U \mapsto \lambda(\eta, \theta) \in \mathbb{R},
(ii) \quad (\eta, \theta) \in U \mapsto \psi(\eta, \theta) \in \mathcal{H} \setminus \{0\},
\]
(4.66)
such that $\lambda(0, \theta^*) = \lambda_{\text{per}}(\theta^*)$, $L^{\Phi_\eta}(\theta)[\psi(\eta, \theta)] = \lambda(\eta, \theta) \psi(\eta, \theta)$ and
\[
\dim \{f \in \mathcal{H} : L^{\Phi_\eta}(\theta) = \lambda(\eta, \theta) f \} = 1, \quad \forall (\eta, \theta) \in U.
\]

Thus,
\[
\lambda(\eta, \theta) \text{ is a simple eigenvalue of } L^{\Phi_\eta}(\theta), \forall (\eta, \theta) \in U. \quad (4.67)
\]

Additionally, as $\lambda(0, \theta^*) = \lambda_{\text{per}}(\theta^*)$ is an isolated point of $\sigma_{\text{point}}(L_{\text{per}}(\theta^*))$ (any point has this property ), $\lambda(\eta, \theta)$ is an isolated point of $\sigma_{\text{point}}(L^{\Phi_\eta}(\theta^*))$ for each $(\eta, \theta) \in U$. Thus, we have $\lambda(0, \cdot) = \lambda_{\text{per}}(\cdot)$ in a neighbourhood of $\theta^*$. We now denote $\psi_{\text{per}}(\cdot) := \psi(0, \cdot)$. Without loss of generality, we assume $\int_{[0,1]^n} |\psi_{\text{per}}(\theta^*)|^2 \, dy = 1$. Moreover, we shall assume that the homogenized (periodic) matrix $A^*_{\text{per}} = D^2_{\theta} \lambda_{\text{per}}(\theta^*)$ is invertible which happens if $\theta = \theta^*$ is a point of local minimum or local maximum strict of $\mathbb{R}^n \ni \theta \mapsto \lambda_{\text{per}}(\theta)$.

Thus, an immediate application of the Implicit Function Theorem gives us the following lemma:

**Lemma 4.4.** Let the condition $(4.65)$ be satisfied and $A^*_{\text{per}}$ be an invertible matrix. Then, there exists a neighborhood $\mathcal{V}$ of $0$, $0 \in \mathcal{V} \subset \mathbb{R}$, and a $\mathbb{R}^n$-value analytic map
\[
\theta(\cdot) : \eta \in \mathcal{V} \mapsto \theta(\eta) \in \mathbb{R}^n,
\]
such that $\theta(0) = \theta^*$ and
\[
\nabla_\theta \lambda(\eta, \theta(\eta)) = 0, \quad \forall \eta \in \mathcal{V}. \quad (4.68)
\]

By the analytic structure of the functions in (4.66) and the Lemma 4.4, there exists a neighborhood $\mathcal{V}$ of $0$, $0 \in \mathcal{V} \subset \mathbb{R}$, such that
\[
(i) \quad \eta \in \mathcal{V} \mapsto \lambda(\eta, \theta(\eta)) \in \mathbb{R},
(ii) \quad \eta \in \mathcal{V} \mapsto \psi(\eta, \theta(\eta)) \in \mathcal{H} \setminus \{0\},
(iii) \quad \eta \in \mathcal{V} \mapsto \xi_k(\eta, \theta(\eta)) \in \mathcal{H}, \forall \{1, \ldots, n\},
\]
(4.69)
are analytic functions, where 
\[ \xi_k(\eta, \theta) := (2i\pi)^{-1}\partial_\theta \psi(\eta, \theta), \]
for \( k \in \{1, \ldots, n\} \).
We also consider \( \xi_{k, \text{per}}(\cdot) = \xi_k(0, \cdot) \).
Furthermore, by (4.67) and (4.68), for each fixed \( \eta \in \mathcal{V} \) we have that the pair \( (\theta(\eta), \lambda(\eta, \theta(\eta))) \in \mathbb{R}^n \times \mathbb{R} \) satisfies:

(i) \( \lambda(\eta, \theta(\eta)) \) is a simple eigenvalue of \( L^{\Phi_\eta}(\theta(\eta)) \),
(ii) \( \theta(\eta) \) is a critical point of \( \lambda(\eta, \cdot) \), that is, \( \nabla_\theta \lambda(\eta, \theta(\eta)) = 0 \).

This means that the Theorem 3.2 can be used. Before, we establish a much simplified notations for the functions in (4.69) as follows:

(i) \( \theta_{\eta} := \theta(\eta) \),
(ii) \( \lambda_{\eta} := \lambda(\eta, \theta(\eta)) \),
(iii) \( \psi_{\eta} := \psi(\eta, \theta(\eta)) \),
(iv) \( \xi_{k, \eta} := \xi_k(\eta, \theta(\eta)) \), \( k \in \{1, \ldots, n\} \).

Finally, from (4.70), for each fixed \( \eta \in \mathcal{V} \), the notion of well-preparedness for the initial data \( u_{\eta_\varepsilon}^0 \) is given as below.

\[
\begin{align*}
\Phi_\omega - \text{two-scale converges to the limit } v_\eta(t, x, \tilde{\omega}) \psi_\eta(\Phi_\eta^{-1}(z, \omega), \omega)
\end{align*}
\]

with

\[
\lim_{\varepsilon \to 0} \int_{\mathbb{R}_T^{n+1}} \left| v_{\eta_\varepsilon}(t, x, \tilde{\omega}) - v_\eta(t, x) \psi_\eta(\Phi_\eta^{-1}(z, \omega), \omega) \right|^2 \, dx \, dt = 0,
\]

for a.e. \( \tilde{\omega} \in \Omega \), where \( v_\eta \in C([0, T], L^2(\mathbb{R}^n)) \) is the unique solution of the homogenized Schrödinger equation

\[
\begin{align*}
\left\{ \begin{array}{ll}
\begin{align*}
i \frac{\partial v_\eta}{\partial t} - \text{div}(A_\eta \nabla v_\eta) + U_\eta^* v_\eta &= 0, & \text{in } \mathbb{R}^{n+1}, \\
v_\eta(0, x) &= v^0(x), & x \in \mathbb{R}^n,
\end{align*}
\end{array} \right.
\end{align*}
\]

(4.73)
with effective coefficients $A^*_\eta = D^2_\theta \lambda(\eta, \theta(\eta))$ and

$$U^*_\eta = c_{\eta}^{-1} \int_{\Omega} \int_{\Phi(0,1)^n,\omega} U_{\text{per}}(\Phi^{-1}_\eta(z, \omega)) \left| \psi_\eta(\Phi^{-1}_\eta(z, \omega), \omega) \right|^2 dz \, d\mathbb{P}(\omega), \quad (4.74)$$

where

$$c_{\eta} = \int_{\Omega} \int_{\Phi(0,1)^n,\omega} \left| \psi_\eta(\Phi^{-1}_\eta(z, \omega), \omega) \right|^2 dz \, d\mathbb{P}(\omega). \quad (4.75)$$

**Remark 4.5.** We remember that, using the equality (3.49), we have for each $\eta$ fixed that the matrix $B_\eta \in \mathbb{R}^{n \times n}$ must satisfy for $k, \ell \in \{1, \ldots, n\}$

$$(B_\eta)_{k\ell} := c_{\eta}^{-1} \left[ \int_{\Omega} \int_{\Phi(0,1)^n,\omega} A_{\text{per}}(\Phi^{-1}_\eta(z, \omega)) \left( e_\xi \psi_\eta(\Phi^{-1}_\eta(z, \omega), \omega) \right) \right. \left. \cdot \left( e_k \psi_\eta(\Phi^{-1}_\eta(z, \omega), \omega) \right) \right. \left. \cdot \left( \nabla z + 2i \pi \theta_\eta(z, \omega) \right) \psi_\eta(\Phi^{-1}_\eta(z, \omega), \omega) \right] dz \, d\mathbb{P}(\omega)$$

$$+ \int_{\Omega} \int_{\Phi(0,1)^n,\omega} A_{\text{per}}(\Phi^{-1}_\eta(z, \omega)) \left( e_\xi \psi_\eta(\Phi^{-1}_\eta(z, \omega), \omega) \right) \cdot \left( \nabla z + 2i \pi \theta_\eta(z, \omega) \right) \psi_\eta(\Phi^{-1}_\eta(z, \omega), \omega) \left. \cdot \left( e_k \psi_\eta(\Phi^{-1}_\eta(z, \omega), \omega) \right) \right] dz \, d\mathbb{P}(\omega)$$

$$- \int_{\Omega} \int_{\Phi(0,1)^n,\omega} A_{\text{per}}(\Phi^{-1}_\eta(z, \omega)) \left( \nabla z + 2i \pi \theta_\eta(z, \omega) \right) \psi_\eta(\Phi^{-1}_\eta(z, \omega), \omega) \left. \cdot \left( e_k \psi_\eta(\Phi^{-1}_\eta(z, \omega), \omega) \right) \right] dz \, d\mathbb{P}(\omega), \quad (4.76)$$

and the homogenized matrix can be written as $A^*_\eta = 2^{-1}(B_\eta + B_\eta^t)$.

### 4.2.1 Expansion of the effective coefficients

As a consequence of the formula of the effective coefficients of the homogenized equation (4.73), we have the following proposition:

**Proposition 4.6.** The maps $\eta \mapsto A^*_\eta, \, B_\eta \in \mathbb{R}^{n \times n}$ and $\eta \mapsto U^*_\eta \in \mathbb{R}$ are analytic in a neighbourhood of $\eta = 0$.

**Proof.** Let us assume $U$ and $V$ as in (4.66) and (4.69), respectively. For each $\eta \in V$, the above arguments give us the formula $A^*_\eta = D^2_\theta \lambda_\eta(\theta(\eta))$. Thus, as $(\eta, \theta) \in U \mapsto D^2_\theta \lambda(\eta, \theta) \in \mathbb{R}^{n \times n}$ and $\eta \in V \mapsto \theta(\eta) \in \mathbb{R}^n$ are analytic maps, we conclude that $\eta \in V \mapsto D^2_\theta \lambda(\eta, \theta(\eta)) \in \mathbb{R}^{n \times n}$ is also an analytic map. This means that $\eta \mapsto A^*_\eta$ is an analytic map.
From (4.74) and (4.75), making a change of variables, we have
\[ U^*_\eta = c^{-1}_\eta \int_\Omega \int_{[0,1]^n} U_{\text{per}}(y) |\psi_\eta(y,\omega)|^2 \det[\nabla_y \Phi_\eta(y,\omega)] \, dz \, dP(\omega) \]
and
\[ c_\eta = \int_\Omega \int_{[0,1]^n} |\psi_\eta(y,\omega)|^2 \det[\nabla_y \Phi_\eta(y,\omega)] \, dz \, dP(\omega) \neq 0. \]

Then, as the map \( \eta \mapsto \psi_\eta \in \mathcal{H} \setminus \{0\} \) is analytic, the map \( \eta \mapsto c_\eta \neq 0 \) is also analytic. Hence the map \( \eta \mapsto c^{-1}_\eta \) is analytic. Therefore, \( \eta \mapsto U^*_\eta \) is analytic.

As a consequence of this proposition, there exist \( \{ A^{(j)} \}, B^{(j)} \}_{j \in \mathbb{N}} \subset \mathbb{R}^{n \times n} \) and \( \{ U^{(j)} \}_{j \in \mathbb{N}} \subset \mathbb{R} \) such that
\[
\begin{align*}
A^*_\eta &= A^{(0)} + \eta A^{(1)} + \eta^2 A^{(2)} + \ldots, \\
U^*_\eta &= U^{(0)} + \eta U^{(1)} + \eta^2 U^{(2)} + \ldots, \\
B^*_\eta &= B^{(0)} + \eta B^{(1)} + \eta^2 B^{(2)} + \ldots.
\end{align*}
\] (4.77)

Now, the object of our interest is determine the terms of order \( \eta^0 \) and \( \eta \) of these homogenized coefficients. For this purpose, guided by (4.56) and by the formulas (4.76), (4.74) and (4.75), we shall analyse the expansion of the analytic functions in (4.71). By analytic property, there exist the sequences \( \{ \theta^{(j)} \}_{j \in \mathbb{N}} \subset \mathbb{R}^n, \{ \lambda^{(j)} \}_{j \in \mathbb{N}} \subset \mathbb{R}, \{ \psi^{(j)} \}_{j \in \mathcal{H}} \subset \mathcal{H} \) and \( \{ \xi^{(j)}_k \}_{j \in \mathcal{H}} \subset \mathcal{H}, k \in \{1, \ldots, n\} \), such that, for \( k \in \{1, \ldots, n\} \)
\[
\begin{align*}
\theta_\eta &= \theta^{(0)} + \eta \theta^{(1)} + \eta^2 \theta^{(2)} + \ldots = \theta^{(0)} + \eta \theta^{(1)} + O(\eta^2), \\
\lambda_\eta &= \lambda^{(0)} + \eta \lambda^{(1)} + \eta^2 \lambda^{(2)} + \ldots = \lambda^{(0)} + \eta \lambda^{(1)} + O(\eta^2), \\
\psi_\eta &= \psi^{(0)} + \eta \psi^{(1)} + \eta^2 \psi^{(2)} + \ldots = \psi^{(0)} + \eta \psi^{(1)} + O(\eta^2), \\
\xi_{k,\eta} &= \xi^{(0)}_k + \eta \xi^{(1)}_k + \eta^2 \xi^{(2)}_k + \ldots = \xi^{(0)}_k + \eta \xi^{(1)}_k + O(\eta^2). \end{align*}
\] (4.78-4.81)

At first glance, in order to determine the coefficients of the expansions in (4.77) we should solve, a priori, auxiliary problems that involves both, the deterministic and stochastic variables. This can be a disadvantage from the point of view of numerical analysis. Our aim hereafter is to prove that, we can simplify the computations of these coefficients working in a periodic environment which is computationally cheaper. In order to do this, note that
\[ \theta^{(0)} = \theta^* , \lambda^{(0)} = \lambda_{\text{per}}(\theta^*) \], \[ \psi^{(0)} = \psi_{\text{per}}(\theta^*) \] and \[ \xi_k^{(0)} = \xi_{k, \text{per}}(\theta^*) , k \in \{1, \ldots, n\} \],

which satisfy

\[
\begin{cases}
(L_{\text{per}}(\theta^*) - \lambda_{\text{per}}(\theta^*))[\psi_{\text{per}}(\theta^*)] = 0 \text{ in } [0, 1]^n , \\
\psi_{\text{per}}(\theta^*) \text{ } [0, 1]^n\text{-periodic},
\end{cases}
\]

(4.82)

\[
\begin{cases}
(L_{\text{per}}(\theta^*) - \lambda_{\text{per}}(\theta^*))[\xi_{k, \text{per}}(\theta^*)] = \mathcal{X}[\psi_{\text{per}}(\theta^*)] \text{ in } [0, 1]^n , \\
\xi_{k, \text{per}}(\theta^*) \text{ } [0, 1]^n\text{-periodic},
\end{cases}
\]

(4.83)

where for \( f \in \mathcal{H} \)

\[
\mathcal{X}[f] := (\text{div}_y + 2i\pi \theta^*)\{A_{\text{per}}(y)(e_k f)\} + (e_k)\{A_{\text{per}}(y)(\nabla_y + 2i\pi \theta^*)f\}.
\]

The equation (4.82) is the spectral cell equation and (4.83) is the first auxiliary cell equation related to the periodic case (see Section ??).

The following theorem show us that, the terms \( \psi^{(1)} \) and \( \xi_k^{(1)} , k \in \{1, \ldots, n\} \), given by (4.80) and (4.81) respectively, satisfy auxiliary type cell equations.

**Theorem 4.7.** Let \( \psi^{(1)} \) and \( \xi_k^{(1)} , k \in \{1, \ldots, n\} \), be as above. Then these functions satisfy the following equations:

\[
\begin{cases}
(L_{\text{per}}(\theta^*) - \lambda_{\text{per}}(\theta^*))[\psi^{(1)}] = \mathcal{Y}[\psi_{\text{per}}(\theta^*)] \text{ in } [0, 1]^n \times \Omega , \\
\psi^{(1)} \text{ stationary},
\end{cases}
\]

(4.84)

\[
\begin{cases}
(L_{\text{per}}(\theta^*) - \lambda_{\text{per}}(\theta^*))[\xi_k^{(1)}] = \mathcal{X}[\psi^{(1)}] \\
+ \mathcal{Y}[\xi_{k, \text{per}}(\theta^*)] + \mathcal{Z}_k[\psi_{\text{per}}(\theta^*)] \text{ in } [0, 1)^n \times \Omega ,
\end{cases}
\]

(4.85)

\( \xi_k^{(1)} \) stationary,
where the operators $Y$ and $Z_k$, $k \in \{1, \ldots, n\}$, are defined by

$$
Y[f] := (\text{div}_y + 2i\pi\theta^*)\{A_{\text{per}}(y)(- [\nabla_y Z](y, \omega) \nabla_y f + 2i\pi\theta^{(1)} f)\}
$$

$$
- \text{div}_y \{[\nabla_y Z]^t(y, \omega) A_{\text{per}}(y)(\nabla_y + 2i\pi\theta^*) f\}
$$

$$
+ (2i\pi\theta^{(1)})\{A_{\text{per}}(y)(\nabla_y + 2i\pi\theta^*) f\}
$$

$$
+ (\text{div}_y + 2i\pi\theta^*)\{[\nabla_y Z(y, \omega) A_{\text{per}}(y)](\nabla_y + 2i\pi\theta^*) f\} + \lambda^{(1)} f
$$

$$
+ \{\text{div}_y Z(y, \omega) [\text{per}(\theta^*) - V_{\text{per}}(y)]\} f,
$$

$$
Z_k[f] := (\text{div}_y + 2i\pi\theta^*)\{[\text{div}_y Z(y, \omega) A_{\text{per}}(y)](e_k f)\}
$$

$$
- \text{div}_y \{[\nabla_y Z]^t(y, \omega) A_{\text{per}}(y)(e_k f)\}
$$

$$
+ (2i\pi\theta^{(1)})\{A_{\text{per}}(y)(e_k f)\}
$$

$$
+ (e_k)\{[\text{div}_y Z(y, \omega) A_{\text{per}}(y)](\nabla_y + 2i\pi\theta^*) f\}
$$

$$
- (e_k)\{A_{\text{per}}(y)\nabla_y Z(y, \omega) \nabla_y f\} + (e_k)\{A_{\text{per}}(y)(2i\pi\theta^{(1)} f)\},
$$

for $f \in \mathcal{H}$.

For the proof of this theorem, we shall use essentially the structure of the spectral cell equation (2.27) with periodic coefficients accomplished by stochastic deformation of identity $\Phi_{\eta}$ together with the identities (4.56).

**Proof.** 1. For beginning, let us consider the set $V$ as in (4.69). Then, making change of variables in the spectral cell equation (2.27) adapted to this context, we find

$$
\int_{[0,1]^n} \int_{\Omega} \left\{A_{\text{per}}(y)\left([\nabla_y \Phi_{\eta}]^{-1} \nabla_y \psi_\eta + 2i\pi\theta_\eta \psi_\eta\right) \cdot \left([\nabla_y \Phi_{\eta}]^{-1} \nabla_y \zeta + 2i\pi\theta_\eta \zeta\right)
$$

$$
+ (V_{\text{per}}(y) - \lambda_\eta) \psi_\eta \zeta\right\} \det[\nabla_y \Phi_{\eta}] d\mathbb{P}(\omega) dy = 0, \quad (4.86)
$$

for all $\eta \in V$ and $\zeta \in \mathcal{H}$. If we insert the equations (4.56), (4.78), (4.79) and (4.80) in equation (4.86) and compute the term $\eta$, we arrive at
\[
\int_{(0,1)^n} \int_{\Omega} \left\{ A_{\text{per}}(y) (\nabla_y \psi^{(1)} + 2i\pi \theta^* \psi^{(1)}) \cdot (\nabla_y \zeta + 2i\pi \theta^* \zeta) + (V_{\text{per}}(y) - \lambda_{\text{per}}(\theta^*)) \psi^{(1)} \bar{\zeta}
\right. \\
+ A_{\text{per}}(y) \left( -[\nabla_y Z] \nabla_y \psi_{\text{per}}(\theta^*) + 2i\pi \theta^{(1)} \psi_{\text{per}}(\theta^*) \right) \cdot (\nabla_y \zeta + 2i\pi \theta^* \zeta)
\]
\\
+ A_{\text{per}}(y) (\nabla_y \psi_{\text{per}}(\theta^*) + 2i\pi \theta^* \psi_{\text{per}}(\theta^*)) \cdot (\nabla_y \zeta + 2i\pi \theta^* \zeta) \text{div}_y Z
\\
- \lambda^{(1)} \psi_{\text{per}}(\theta^*) \bar{\zeta} + (V_{\text{per}}(y) - \lambda_{\text{per}}(\theta^*)) \psi^{(0)} \bar{\zeta} \text{div}_y Z \right \} d\mathbb{P}(\omega) dy = 0,
\]
for all \( \eta \in \mathcal{V} \) and \( \zeta \in \mathcal{H} \). This equation is the variational formulation of the equation (4.84), which concludes the first part of the proof.

2. For the second part of the proof, we have

\[
\int_{(0,1)^n} \int_{\Omega} \left\{ A_{\text{per}}(y) \left( [\nabla_y \Phi_\eta]^{-1} \nabla_y \xi_{k,\eta} + 2i\pi \theta_\eta \xi_{k,\eta} \right) \cdot (\nabla_y \Phi_\eta)^{-1} \nabla_y \zeta + 2i\pi \theta_\eta \zeta)
\right. \\
+ A_{\text{per}}(y) (e_k \psi_\eta) \cdot (\nabla_y \Phi_\eta)^{-1} \nabla_y \zeta + 2i\pi \theta_\eta \zeta)
\]
\\
- A_{\text{per}}(y) ([\nabla_y \Phi_\eta]^{-1} \nabla_y \psi_\eta + 2i\pi \theta_\eta \psi_\eta) \cdot (e_k \zeta)
\\
+ (V_{\text{per}}(y) - \lambda_\eta) \xi_{k,\eta} \bar{\zeta} - \frac{1}{2i\pi \partial\theta_k} (\eta, \theta(\eta)) \psi_\eta \bar{\zeta} \right \} \det[\nabla_y \Phi_\eta] d\mathbb{P}(\omega) dy = 0,
\]
(4.87)
for all \( \eta \in \mathcal{V} \), \( \zeta \in \mathcal{H} \) and \( k \in \{1, \ldots, n\} \). Hence, taking into account the Lemma 4.4 and inserting the equations (4.56), (4.78), (4.79), (4.80) and (4.81) in equation (4.87), a computation of the term of order \( \eta \) lead us to
\[
\int_{[0,1]^n} \int_{\Omega} \left\{ A_{\text{per}}(y) \left( \nabla_y \xi_k^{(1)} + 2i\pi \theta^* \xi_k^{(1)} \right) \cdot \left( \nabla_y \zeta + 2i\pi \theta^* \zeta \right) + (V_{\text{per}}(y) - \lambda_{\text{per}}(\theta^*)) \xi_k^{(1)} \cdot \zeta \\
+ A_{\text{per}}(y) \left( e_k \psi(y) \right) \cdot \left( \nabla_y \zeta + 2i\pi \theta^* \zeta \right) - A_{\text{per}}(y) \left( \nabla_y \psi(y) + 2i\pi \theta^* \psi(y) \right) \cdot (e_k \cdot \zeta) \\
+ A_{\text{per}}(y) \left( - [\nabla_y Z] \nabla_y \xi_{k,\text{per}}(\theta^*) + 2i\pi \theta^{(1)} \xi_{k,\text{per}}(\theta^*) \right) \cdot \left( \nabla_y \zeta + 2i\pi \theta^* \zeta \right) \\
+ A_{\text{per}}(y) \left( \nabla_y \xi_{k,\text{per}}(\theta^*) + 2i\pi \theta^* \xi_{k,\text{per}}(\theta^*) \right) \cdot \left( \nabla_y \zeta + 2i\pi \theta^* \zeta \right) \cdot \left( V_{\text{per}}(y) - \lambda_{\text{per}}(\theta^*) \right) \xi_{k,\text{per}}(\theta^*) \cdot \zeta \cdot \text{div}_y Z \\
- \lambda^{(1)} \xi_{k,\text{per}}(\theta^*) \cdot \zeta + (V_{\text{per}}(y) - \lambda_{\text{per}}(\theta^*)) \xi_{k,\text{per}}(\theta^*) \cdot \zeta \cdot \text{div}_y Z \\
+ A_{\text{per}}(y) \left( e_k \psi_{\text{per}}(\theta^*) \right) \cdot \left( \nabla_y \zeta + 2i\pi \theta^* \zeta \right) \cdot \text{div}_y Z - \left( [\nabla_y Z] \nabla_y \zeta + 2i\pi \theta^{(1)} \zeta \right) \\
- A_{\text{per}}(y) \left( \nabla_y \psi_{\text{per}}(\theta^*) + 2i\pi \theta^* \psi_{\text{per}}(\theta^*) \right) \cdot (e_k \cdot \zeta) \cdot \text{div}_y Z \\
- A_{\text{per}}(y) \left( - [\nabla_y Z] \nabla_y \psi_{\text{per}}(\theta^*) + 2i\pi \theta^{(1)} \psi_{\text{per}}(\theta^*) \right) \cdot (e_k \cdot \zeta) \right\} d\mathbb{P}(\omega) dy = 0,
\]

for all $\zeta \in \mathcal{H}$. Noting that this is the variational formulation of the equation (4.85), we conclude the proof. \qed

We remember the reader that if $f : \mathbb{R}^n \times \Omega \to \mathbb{R}$ is a stationary function, then we shall use the following notation

\[
\mathbb{E}[f(x, \cdot)] = \int_{\Omega} f(x, \omega) d\mathbb{P}(\omega),
\]

for any $x \in \mathbb{R}^n$. Roughly speaking, the theorem below tell us that the homogenized matrix of the problem (4.64) can be obtained by solving periodic problems.

**Theorem 4.8.** Let $A^*_\eta$ be the homogenized matrix as in (4.73). Then

\[
A_{\text{per}}^* = A^*_{\text{per}} + \eta A^{(1)} + O(\eta^2).
\]

Moreover, the term of order $\eta^0$ is given by the homogenized matrix of the periodic case, that is, $A_{\text{per}}^* = 2^{-1} (B^{(0)} + (B^{(0)})^t)$, where the matrix $B^{(0)}$ is
the term of order $\eta^0$ in (1.77) and it is defined by

$$(B^{(0)})_{k\ell} := \int_{[0,1]^n} A_{\text{per}}(y)(e_\ell \psi_{\text{per}}(\theta^*)) \cdot (e_k \psi_{\text{per}}(\theta^*)) \, dy$$

$$+ \int_{[0,1]^n} A_{\text{per}}(y)(e_\ell \psi_{\text{per}}(\theta^*)) \cdot (\nabla_y \xi_{k,\text{per}}(\theta^*) + 2i\pi \theta^* \xi_{k,\text{per}}(\theta^*)) \, dy$$

$$- \int_{[0,1]^n} A_{\text{per}}(y) (\nabla_y \psi_{\text{per}}(\theta^*) + 2i\pi \theta^* \psi_{\text{per}}(\theta^*)) \cdot (e_\ell \xi_{k,\text{per}}(\theta^*)) \, dy.$$

The term of order $\eta$ is given by $A^{(1)} = 2^{-1}(B^{(1)} + (B^{(1)})^t)$, where the matrix $B^{(1)}$ is the term of order $\eta$ in (1.77) and it is defined by

$$(B^{(1)})_{k\ell} = \left[ \int_{[0,1]^n} A_{\text{per}}(y)(e_\ell \psi_{\text{per}}(\theta^*)) \cdot (e_k \psi_{\text{per}}(\theta^*)) \, dy \right.$$

$$+ \int_{[0,1]^n} A_{\text{per}}(y)(e_\ell \psi_{\text{per}}(\theta^*)) \cdot (e_k \mathbb{E}[\psi^{(1)}(y, \cdot)]) \, dy$$

$$+ \int_{[0,1]^n} A_{\text{per}}(y)(e_\ell \mathbb{E}[\psi^{(1)}(y, \cdot)]) \cdot (e_k \psi_{\text{per}}(\theta^*)) \, dy$$

$$+ \int_{[0,1]^n} A_{\text{per}}(y)(e_\ell \psi_{\text{per}}(\theta^*)) \cdot (\nabla_y \xi_{k,\text{per}}(\theta^*) + 2i\pi \theta^* \xi_{k,\text{per}}(\theta^*)) \mathbb{E}[\text{div}_y Z(y, \cdot)] \, dy$$

$$+ \int_{[0,1]^n} A_{\text{per}}(y)(e_\ell \psi_{\text{per}}(\theta^*)) \cdot (\nabla_y \mathbb{E}[\xi^{(1)}_k(y, \cdot)] + 2i\pi \theta^* \mathbb{E}[\xi^{(1)}_k(y, \cdot)])$$

$$\frac{+ 2i\pi \theta^{(1)} \xi_{k,\text{per}}(\theta^*) - \mathbb{E}[\nabla_y Z(y, \cdot)] \nabla_y \xi_{k,\text{per}}(\theta^*)}{\sqrt{d\theta^*}} \, dy$$

$$+ \int_{[0,1]^n} A_{\text{per}}(y)(e_\ell \mathbb{E}[\psi^{(1)}(y, \cdot)]) \cdot (\nabla_y \xi_{k,\text{per}}(\theta^*) + 2i\pi \theta^* \xi_{k,\text{per}}(\theta^*)) \, dy$$

$$- \int_{[0,1]^n} A_{\text{per}}(y)(\nabla_y \psi_{\text{per}}(\theta^*) + 2i\pi \theta^* \psi_{\text{per}}(\theta^*)) \cdot (e_\ell \xi_{k,\text{per}}(\theta^*)) \mathbb{E}[\text{div}_y Z(y, \cdot)] \, dy$$

$$- \int_{[0,1]^n} A_{\text{per}}(y)(\nabla_y \psi_{\text{per}}(\theta^*) + 2i\pi \theta^* \psi_{\text{per}}(\theta^*)) \cdot (e_\ell \mathbb{E}[\xi^{(1)}_k(y, \cdot)]) \, dy.$$
\[-\int_{[0,1]^n} A_{\text{per}}(y) \left( \nabla_y \mathbb{E} \left[ \psi^{(1)}(y, \cdot) \right] + 2i\pi \theta^* \mathbb{E} \left[ \psi^{(1)}(y, \cdot) \right] \right) dy + 2i\pi \theta^{(1)} \psi_{\text{per}}(\theta^*) - \mathbb{E} \left[ [\nabla_y Z](y, \cdot) \right] \nabla_y \psi_{\text{per}}(\theta^*) \cdot (e_\ell \xi_{k,\text{per}}(\theta^*)) dy \]

\[-\left[ \int_{[0,1]^n} |\psi_{\text{per}}(\theta^*)|^2 \mathbb{E} \left[ \text{div}_y Z(y, \cdot) \right] dy + \int_{[0,1]^n} \psi_{\text{per}}(\theta^*) \mathbb{E} \left[ \psi^{(1)}(y, \cdot) \right] dy + \int_{[0,1]^n} A_{\text{per}}(y)(e_\ell \psi_{\text{per}}(\theta^*) \cdot (e_\ell \psi_{\text{per}}(\theta^*))) dy \\ + \int_{[0,1]^n} A_{\text{per}}(y)(e_\ell \psi_{\text{per}}(\theta^*) \cdot (\nabla_y \xi_{k,\text{per}}(\theta^*) + 2i\pi \theta^* \xi_{k,\text{per}}(\theta^*))) dy - \int_{[0,1]^n} A_{\text{per}}(y)[(\nabla_y \psi_{\text{per}}(\theta^*) + 2i\pi \theta^* \psi_{\text{per}}(\theta^*)) \cdot (e_\ell \xi_{k,\text{per}}(\theta^*))) \right] dy \right].

Proof. 1. Taking into account V as in (4.69), we get from (4.76), for \( \eta \in V \), that the homogenized matrix is given by \( A_{\eta}^* = 2^{-1}(B_{\eta} + B_{\eta}^t) \). Thus, in order to describe the terms of the expansion of \( A_{\eta}^* \), we only need to determine the terms in the expansion of \( B_{\eta} \).

2. Using the equations (4.56) and (4.80), the map \( \eta \mapsto c_{\eta} \in (0, +\infty) \) has an expansion about \( \eta = 0 \). Remembering that \( \int_{[0,1]^n} |\psi_{\text{per}}(\theta^*)|^2 dy = 1 \), we have

\[
c_{\eta}^{-1} = 1 - \eta \left[ \int_{\Omega} \int_{[0,1]^n} |\psi_{\text{per}}(\theta^*)|^2 \text{div}_y Z(y, \omega) \ dy \ d\mathbb{P} + \int_{\Omega} \int_{[0,1]^n} \psi_{\text{per}}(\theta^*) \psi^{(1)}(y, \omega) \ dy \ d\mathbb{P} + \int_{\Omega} \int_{[0,1]^n} \psi^{(1)}(y, \omega) \psi_{\text{per}}(\theta^*) \ dy \ d\mathbb{P} \right] + O(\eta^2),
\]

in \( \mathbb{C} \) as \( \eta \to 0 \). Thus, using the expansions (4.56), (4.78), (4.80) and (4.81) in the formula (4.76), the computation of the resulting term of order \( \eta^0 \) of \( B_{\eta} \) give us the desired expression for \( (B^{(0)})_{k,\ell} \). The same reasoning with a little more computations, which is an exercise that we leave to the reader, allow us to obtain the expression for \( (B^{(1)})_{k,\ell} \).

Remark 4.9. Next we stress that, the computation of the coefficients of \( A_{\text{per}}^* \) is performed by solving the equations (4.82) and (4.83), which are equations with periodic boundary conditions. In order to compute the coefficients of \( A^{(1)} \), we need to know the functions \( \psi^{(1)} \) and \( \xi^{(1)}_k \), \( k \in \{1, \ldots, n\} \), which are
a priori stochastic in nature (see the equations (4.84) and (4.85), respectively). But Theorem 4.8 shows that, we only need their expectation values, \( E[\psi_1(y, \cdot)] \) and \( E[\xi_k^{(1)}(y, \cdot)] \), \( k \in \{1, \ldots, n\} \), which are \([0, 1]^n\)-periodic functions and, respectively, solutions of the following equations

\[
\begin{cases}
  \left( L_{\text{per}}(\theta^*) - \lambda_{\text{per}}(\theta^*) \right) E[\psi_1(y, \cdot)] = Y_{\text{per}}[\psi_{\text{per}}(\theta^*)] \text{ in } [0, 1]^n, \\
  E[\psi_1(y, \cdot)] \text{ is } [0, 1]^n\text{-periodic},
\end{cases}
\]

\[
\begin{cases}
  \left( L_{\text{per}}(\theta^*) - \lambda_{\text{per}}(\theta^*) \right) E[\xi_k^{(1)}(y, \cdot)] = X_{\text{per}}[\psi_{\text{per}}(\theta^*)] + Z_{k,\text{per}}[\psi_{\text{per}}(\theta^*)] \text{ in } [0, 1]^n, \\
  E[\xi_k^{(1)}(y, \cdot)] \text{ is } [0, 1]^n\text{-periodic},
\end{cases}
\]

where for \( f \in H^1_{\text{per}}([0, 1]^n) \)

\[
Y_{\text{per}}[f] := (\text{div} + 2i\pi\theta^*) \left\{ A_{\text{per}}(y) \left( -E[\nabla_y Z(y, \cdot)] \nabla_y f + 2i\pi\theta^{(1)} f \right) \right\}
- \text{div} \left\{ E[\nabla_y Z(y, \cdot)] A_{\text{per}}(y) (\nabla_y + 2i\pi\theta^*) f \right\}
+ \left( 2i\pi\theta^{(1)} \right) \left\{ A_{\text{per}}(y)(\nabla_y + 2i\pi\theta^*) f \right\}
+ \left( \text{div} + 2i\pi\theta^* \right) \left\{ \left[ E[\text{div}_y Z(y, \cdot)] A_{\text{per}}(y) \right] (\nabla_y + 2i\pi\theta^*) f \right\} + \lambda^{(1)} f
+ \left\{ E[\text{div}_y Z(y, \cdot)] \left( \lambda_{\text{per}}(\theta^*) - V_{\text{per}}(y) \right) \right\} f,
\]

\[
Z_{k,\text{per}}[f] := (\text{div} + 2i\pi\theta^*) \left\{ \left[ E[\text{div}_y Z(y, \cdot)] A_{\text{per}}(y) \right] (e_k f) \right\}
- \text{div} \left\{ E[\nabla_y Z(y, \cdot)] A_{\text{per}}(y)(e_k f) \right\}
+ \left( 2i\pi\theta^{(1)} \right) \left\{ A_{\text{per}}(y)(e_k f) \right\}
+ (e_k) \left\{ \left[ E[\text{div}_y Z(y, \cdot)] A_{\text{per}}(y) \right] (\nabla_y + 2i\pi\theta^*) f \right\}
- (e_k) \left\{ A_{\text{per}}(y) E[\nabla_y Z(y, \cdot)] \nabla_y f \right\} + (e_k) \left\{ A_{\text{per}}(y)(2i\pi\theta^{(1)} f) \right\}.
\]

Summing up, the determination of the homogenized coefficients for (1.1) is a stochastic problem in nature. However, when we consider the interesting context of materials which have small deviation from perfect ones (modeled by periodic functions), this problem, in the specific case (1.1) reduces, at the
first two orders in \( \eta \), to the simpler solution to the two periodic problems above. Both of them are of the same nature. Importantly, note that \( Z \) in (4.1) is only present through \( \mathbb{E} \left[ \text{div}_y Z(y, \cdot) \right] \) and \( \mathbb{E} \left[ \nabla_y Z(y, \cdot) \right] \).

In the theorem below, we assume that the homogenized matrix of the periodic case satisfies the uniform coercive condition, that is,
\[
A^*_\text{per} \xi : \xi \geq \Lambda |\xi|^2,
\]
for some \( \Lambda > 0 \) and for all \( \xi \in \mathbb{R}^n \), which has experimental evidence for metals and semiconductors. Therefore, due to Theorem 4.8 the homogenized matrix of the perturbed case \( A^*_\eta \) has similar property for \( \eta \sim 0 \).

**Theorem 4.10.** Let \( v_\eta \) be the solution of homogenized equation (4.73). Then
\[
v_\eta(t, \sqrt{A^*_\eta x}) = v_{\text{per}}(t, \sqrt{A^*_{\text{per}} x}) + \eta v^{(1)}(t, \sqrt{A^*_{\text{per}} x}) + O(\eta^2),
\]
weakly in \( L^2(\mathbb{R}^n_T) \) as \( \eta \to 0 \), that means,
\[
\int_{\mathbb{R}^n_T} \left( v_\eta(t, \sqrt{A^*_\eta x}) - v_{\text{per}}(t, \sqrt{A^*_{\text{per}} x}) - \eta v^{(1)}(t, \sqrt{A^*_{\text{per}} x}) \right) h(t, x) \, dx \, dt = O(\eta^2),
\]
for each \( h \in L^2(\mathbb{R}^n_T) \), where \( v_{\text{per}} \) is the solution of the periodic homogenized problem
\[
\begin{align*}
\left\{ 
\begin{array}{l}
i \frac{\partial v_{\text{per}}}{\partial t} - \text{div} \left(A^*_{\text{per}} \nabla v_{\text{per}}\right) + U^*_{\text{per}} v_{\text{per}} = 0, \quad \text{in } \mathbb{R}^{n+1}_T, \\
v_{\text{per}}(0, x) = v_0(x), \quad x \in \mathbb{R}^n,
\end{array}
\right.
\end{align*}
\]
and \( v^{(1)} \) is the solution of
\[
\begin{align*}
\left\{ 
\begin{array}{l}
i \frac{\partial v^{(1)}}{\partial t} - \text{div} \left(A^*_{\text{per}} \nabla v^{(1)}\right) + U^*_{\text{per}} v^{(1)} = \text{div} \left(A^*_{\text{per}} \nabla v_{\text{per}}\right) - U^{(1)} v_{\text{per}}, \quad \text{in } \mathbb{R}^{n+1}_T, \\
v^{(1)}(0, x) = v^1_0(x), \quad x \in \mathbb{R}^n,
\end{array}
\right.
\end{align*}
\]
where \( U^{(1)} \) is the coefficient of the term of order \( \eta \) of the expansion \( U^*_\eta \) and \( v^1_0 \in C^\infty_c(\mathbb{R}^n) \) is given by the limit
\[
v^1_0 \left( \sqrt{A^*_{\text{per}} x} \right) := \lim_{\eta \to 0} \frac{v_0 \left( \sqrt{A^*_\eta x} \right) - v_0 \left( \sqrt{A^*_{\text{per}} x} \right)}{\eta}.
\]
Proof. 1. Taking into account the set $\mathcal{V}$ as in (4.69), we have for $\eta \in \mathcal{V}$ and from the conservation of energy of the homogenized Schrödinger equation (4.73), that the solution $v_\eta : \mathbb{R}_T^n \to \mathbb{C}$ satisfies
\[
\|v_\eta\|_{L^2(\mathbb{R}_T^{n+1})} = T \|v_0\|_{L^2(\mathbb{R}^n)}, \quad \forall \eta \in \mathcal{V}.
\]
Thus, after possible extraction of a subsequence, we have the existence of a function $v(0) \in L^2(\mathbb{R}_T^{n+1})$ such that
\[
v_\eta \xrightarrow{\eta \to 0} v(0) \text{ em } L^2(\mathbb{R}_T^{n+1}).
\]
By the variational formulation of the equation (4.73), we find
\[
0 = i \int_{\mathbb{R}^n} v_0(x) \overline{\varphi}(0, x) \, dx - i \int_{\mathbb{R}_T^n} v_\eta(t, x) \frac{\partial \overline{\varphi}}{\partial t}(t, x) \, dx \, dt
\]
\[
+ \int_{\mathbb{R}_T^n} \left\{- \langle A_\eta^* v_\eta(t, x), D^2 \varphi(t, x) \rangle + U_\eta^* v_\eta(t, x) \overline{\varphi}(t, x) \right\} \, dx \, dt,
\]
for all $\varphi \in C_c^1((-\infty, T)) \otimes C^2(\mathbb{R}^n)$. Recall that $\langle P, Q \rangle := \text{tr}(PQ)$, for $P, Q$ in $\mathbb{C}^{n \times n}$. Then, using (4.90), the Theorem 4.6, making $\eta \to 0$ and invoking the uniqueness property of the equation (4.88), we conclude that $v(0) = v_{\text{per}}$.

2. Now, using that $U_\eta^* = U_{\text{per}}^* + \eta U^{(1)} + O(\eta^2)$ as $\eta \to 0$, defining $V_\eta(t, x) := v_\eta(t, \sqrt{A_\eta^* x})$ and using the homogenized equation (4.73), we arrive at
\[
\begin{cases}
  i \frac{\partial V_\eta}{\partial t} - \Delta V_\eta + U_{\text{per}}^* V_\eta = - \left( \eta U^{(1)} + O(\eta^2) \right) V_\eta, \quad \text{in } \mathbb{R}_T^{n+1}, \\
  V_\eta(0, x) = v^0 \left( \sqrt{A_\eta^* x} \right), \quad x \in \mathbb{R}^n,
\end{cases}
\]
(4.92)

Proceeding similarly with respect to $V(t, x) := v_{\text{per}}(t, \sqrt{A_{\text{per}}^* x})$, we obtain
\[
\begin{cases}
  i \frac{\partial V}{\partial t} - \Delta V + U_{\text{per}}^* V = 0, \quad \text{in } \mathbb{R}_T^{n+1}, \\
  V(0, x) = v^0 \left( \sqrt{A_{\text{per}}^* x} \right), \quad x \in \mathbb{R}^n.
\end{cases}
\]
(4.93)
Now, the difference between the equations (4.92) and (4.93) yields,

\[
\begin{cases}
  i \frac{\partial (V_\eta - V)}{\partial t} - \Delta (V_\eta - V) + U^*_\text{per}(V_\eta - V) = - \left( \eta U^{(1)} + O(\eta^2) \right) V_\eta, & \text{in } \mathbb{R}^{n+1}, \\
  (V_\eta - V)(0, x) = v_0 \left( A^*_\eta x \right) - v_0 \left( \sqrt{A^*_\text{per} x} \right), & x \in \mathbb{R}^n.
\end{cases}
\]

Hence, multiplying the last equation by \( V_\eta - V \), integrating over \( \mathbb{R}^n \) and taking the imaginary part yields

\[
\frac{d}{dt} \| V_\eta - V \|_{L^2(\mathbb{R}^n)} \leq O(\eta), \quad \text{for } \eta \in \mathcal{V}.
\]

Defining

\[
W_\eta(t, x) := \frac{V_\eta(t, x) - V(t, x)}{\eta}, \quad \eta \in \mathcal{V},
\]

the last inequality provides, \( \sup_{\eta \in \mathcal{V}} \| W_\eta \|_{L^2(\mathbb{R}^{n+1})} < +\infty \). Thus, taking a subsequence if necessary, there exists \( v^{(1)} \in L^2(\mathbb{R}^{n+1}) \) such that

\[
W_\eta(t, x) \xrightarrow{\eta \to 0} v^{(1)}(t, \sqrt{A^*_\text{per} x}), \quad \text{in } L^2(\mathbb{R}^{n+1}).
\]

Hence, multiplying the equation (4.94) by \( \eta^{-1} \), letting \( \eta \to 0 \) and performing a change of variables, we reach the equation (4.89) finishing the proof of the theorem.

\[\square\]

**Conflict of Interest**

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