Rewriting Systems and Geometric 3-Manifolds

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Abstract: The fundamental groups of most (conjecturally, all) closed 3-manifolds with uniform geometries have finite complete rewriting systems. The fundamental groups of a large class of amalgams of circle bundles also have finite complete rewriting systems. The general case remains open.

§1. Introduction

In this paper we point out that well-known properties of finite complete rewriting systems and well-known facts about geometric 3-manifolds combine to give the following. (See below for definitions.)

Theorem 1. Suppose that \( M \) is a closed 3-manifold bearing one of Thurston’s eight geometries. Suppose further that if \( M \) is hyperbolic, that \( M \) virtually fibers over a circle. Then \( \pi_1(M) \) has a finite complete rewriting system.

According to a conjecture of Thurston ([Th], question 18), every closed hyperbolic 3-manifold obeys the last hypothesis.

We also exhibit a class of non-uniform geometric 3-manifolds whose fundamental groups have finite complete rewriting systems. In particular, suppose that \( M \) is a graph of circle bundles based on a graph \( \Gamma \). We will call an edge of this graph a loop if it has the same initial and terminal vertex. We suppose that when all loops are removed, the resulting graph is a tree. Under certain conditions on the way the vertex manifolds are glued along their boundary tori, the fundamental group \( \pi_1(M) \) has a finite complete rewriting system.
§2. Proof of Theorem 1

We review the appropriate background and definitions.

Let $G$ be a group with finite generating set $A$. We write $A^*$ for the free monoid on $A$. Each element of $A$ evaluates into $G$ under the identity map and this extends to a unique monoid homomorphism of $A^*$ onto $G$ which we denote by $w \mapsto w$.

A rewriting system $R$ over the set $A$ is a subset of $A^* \times A^*$. We write a pair $(u, v) \in R$ as $u \rightarrow v$ and call this a rewriting rule or replacement rule. If $u \rightarrow v$ is a rewriting rule, then for any $xuy \in A^*$, we write $xuy \rightarrow xvy$.

We say a finite set $R = \{ u_i \rightarrow v_i \}$ is a finite complete rewriting system for $(G, A)$ if

1) The monoid presentation $\langle A \mid u_i = v_i \rangle$ is a presentation of the underlying monoid of $G$,
2) For each element $g \in G$ there is exactly one word $w \in A^*$ so that $g = \overline{w}$ and $w$ contains no $u_i$ as a substring (that is, $w$ is irreducible), and
3) There is no word $w_0 \in A^*$ spawning an infinite sequence of rewritings, $w_0 \rightarrow w_1 \rightarrow w_2 \rightarrow \cdots$. Such a system is called Noetherian.

We will say that $G$ has a finite complete rewriting system if there is a generating set $A$ for which there is a finite complete rewriting system for $(G, A)$.

We will need the following facts about finite complete rewriting systems.

Proposition 0. The trivial group has a finite complete rewriting system.

Proposition 1. $\mathbb{Z}$ has a finite complete rewriting system.

Proposition 2 [Hr],[LC]. If $G$ is a surface group, then $G$ has a finite complete rewriting system.

Proposition 3 [GS2]. If $H$ is finite index in $G$ and $H$ has a finite complete rewriting system, then $G$ has a finite complete rewriting system.

Proposition 4 [GS1]. If $1 \rightarrow K \rightarrow G \rightarrow Q \rightarrow 1$ is a short exact sequence, and $K$ and $Q$ have finite complete rewriting systems, then $G$ has a finite complete rewriting system.

A thorough account of Thurston’s eight geometries is given in [Sc]. If $M$ is a Riemannian manifold, then the Riemannian metric on $M$ lifts to a Riemannian metric on the universal cover, $\tilde{M}$. Suppose now that $M$ is a closed 3-manifold with a uniform Riemannian metric. (This means that the isometry group of $\tilde{M}$ acts transitively.) Thurston has shown that up to scaling, there are only eight possibilities for the Riemannian manifold $\tilde{M}$ and that $\pi_1(M)$ is constrained in the following way. (We say that $G$ is virtually $H$ if $G$ contains a finite index copy of $H$.)

Proposition 5. Suppose $M$ is a closed Riemannian 3-manifold with a uniform metric. Then one of the following holds:
1) $\tilde{M}$ is the 3-sphere and $\pi_1(M)$ is finite, i.e., virtually trivial.
2) $\tilde{M}$ is Euclidean 3-space and $\pi_1(M)$ is virtually $\mathbb{Z}^3$.
3) $\tilde{M}$ is $S^2 \times \mathbb{R}$ and $\pi_1(M)$ is virtually $\mathbb{Z}$.
4) $\tilde{M}$ is $\mathbb{H}^2 \times \mathbb{R}$ and $\pi_1(M)$ is virtually $H \times \mathbb{Z}$, where $H$ is the fundamental group of a closed hyperbolic surface.
5) $\tilde{M}$ is Nil, the Lie group consisting of upper triangular real $3 \times 3$ matrices with one’s on the diagonal, and $\pi_1(M)$ contains a finite index subgroup $G$ which sits in the short exact sequence $1 \to \mathbb{Z} \to G \to \mathbb{Z}^2 \to 1$.
6) $\tilde{M}$ is $\widetilde{\text{PSL}_2}(\mathbb{R})$, the universal cover of the unit tangent bundle of the hyperbolic plane, and $\pi_1(M)$ contains a finite index subgroup $G$ which sits in the short exact sequence $1 \to \mathbb{Z} \to G \to H \to 1$, where $H$ is the fundamental group of a closed hyperbolic surface.
7) $\tilde{M}$ is Sol, a Lie group which is a semi-direct product of $\mathbb{R}^2$ with $\mathbb{R}$, and $\pi_1(M)$ has a finite index group $G$ which sits in the short exact sequence $1 \to \mathbb{Z}^2 \to G \to \mathbb{Z} \to 1$.
8) $\tilde{M}$ is hyperbolic space. Under the further assumption that $M$ virtually fibers over a circle, $\pi_1(M)$ has a finite index group $G$ which sits in the short exact sequence $1 \to H \to G \to \mathbb{Z} \to 1$, where $H$ is the fundamental group of a closed hyperbolic surface.

The proof of Theorem 1 now consists of applying Propositions 1 – 4 to the cases of Proposition 5.

§3. Non-uniform geometric 3-manifolds.

For an arbitrary closed 3-manifold $M$ satisfying Thurston’s geometrization conjecture (see [Sc] for details), but not necessarily admitting a uniform Riemannian metric, finding rewriting systems becomes much more complicated. If $M$ is not orientable, then $M$ has an orientable double cover; Proposition 3 then says that if the fundamental group of the cover has a finite complete rewriting system, then so does $M$. So we may assume $M$ is orientable.

Any closed orientable 3-manifold $M$ can be decomposed as a connected sum $M = M_1 \# M_2 \# \cdots \# M_n$ in which each $M_i$ is either a closed irreducible 3-manifold, or is homeomorphic to $S^2 \times S^1$ ([He]). The fundamental group of $M$, then, can be written as the free product $\pi_1(M) = \pi_1(M_1) \ast \pi_1(M_2) \ast \cdots \ast \pi_1(M_n)$. Another result of [GS1] says that the class of groups with finite complete rewriting systems is closed under free products; therefore, if $\pi_1(M_i)$ has a finite complete rewriting system for each $i$, then so does $\pi_1(M)$. So we may assume that our closed orientable 3-manifold is also irreducible.

Results of [JS] and [Jo] state that a closed irreducible 3-manifold $M$ can also be decomposed in a canonical way. There is a finite graph $\Gamma$ associated to $M$. For each vertex of $v$ of $\Gamma$, there is a compact 3-manifold $M_v \subset M$. For each edge $e$ of $\Gamma$ there is an incompressible torus $T^2_e \subset M$. The boundary of
$M_v$ consists of $\bigsquare_{v \in \partial e} T_e^2$ and $M$ is the union along these boundary tori of the pieces $M_v$. Consequently, the fundamental group of $M$ can also be realized as the group of the graph of groups given by placing the fundamental groups of the vertex manifolds at the corresponding vertices of $\Gamma$, and the fundamental group of a torus on each edge, together with the appropriate injections.

If $M$ satisfies Thurston’s geometrization conjecture, then the interior of each one of these vertex manifolds admits a uniform Riemannian metric.

The simplest case of this type of decomposition occurs when the graph $\Gamma$ consists of a single vertex with no edges; this is dealt with in Proposition 5. In the case where $\Gamma$ has edges, the vertex manifolds are either cusped hyperbolic 3-manifolds or Seifert fibered manifolds with boundaries.

We take up the case in which $M$ is a graph of circle bundles. More specifically, suppose that $\Gamma$ is a finite graph, and suppose that at any vertex $v$ the vertex manifold $M_v$ is a circle bundle over a punctured surface with genus $g_v$. Then $M_v$ will have a torus boundary component for each of the punctures in the base surface; the edges of the graph $\Gamma$ determine how these boundary components will be glued together.

Recall that a loop is an edge with the same initial and terminal vertex. We need to assume that if all of the loops in the graph $\Gamma$ are removed, the resulting graph is a tree. Then the vertices of $\Gamma$ can be colored alternately red and blue, so that each edge that is not a loop joins a blue vertex to a red vertex. Orient all of the non-loop edges in $\Gamma$ by taking the blue vertex to be the initial vertex, so the red vertex is the terminal vertex. Let $V$ be the set of vertices of $\Gamma$, let $E$ be the set of edges in $\Gamma$ that are not loops, and let $L$ be the set of loops in $\Gamma$. For each edge $e \in E$, $\iota(e)$ will denote the initial vertex of $e$, and $\tau(e)$ will denote the terminal vertex; similarly for loops in $L$.

The fundamental group of the circle bundle at the vertex $v$ will be

$$\pi_1(M_v) = \langle a_{v1}, \ldots, a_{vg_v}, b_{v1}, \ldots, b_{vg_v}, \{p_e \mid e \in E, \iota(e) = v\}, \{q_e \mid e \in E, \tau(e) = v\}, \{r_{l1}, s_l \mid l \in L, \iota(l) = v\} \mid \prod_{\iota(e) = v} p_e \prod_{\tau(e) = v} q_e \prod_{\iota(l) = v} r_l s_l = \prod_{j=1}^{g_v} [a_{vj}, b_{vj}] \rangle \times \langle x_v \rangle.$$

If $v$ is a blue vertex, then, there will be no generators of the form $q_e$, and if $v$ is a red vertex, there will be no generators of the form $p_e$ in the presentation for $\pi_1(M_v)$.

Suppose the edge $e \in E$ has initial vertex $v$ (so $v$ is blue) and terminal vertex $w$ (so $w$ is red). Then the amalgamation along this edge gives relations

$$x_v = q_e^{k_e} x_w^{a_{ve}},$$

$$p_e = q_e^{k'_e} x_w^{n_{ve}}.$$

where the matrix

$$\phi = \begin{pmatrix} k_e & n_e \\ k'_e & n'_e \end{pmatrix} \in \text{SL}_2(\mathbb{Z}).$$
We are able to find finite complete rewriting systems in the case where this matrix is

\[
\phi = \begin{pmatrix} 1 & n_e \\ 0 & 1 \end{pmatrix}.
\]

Our relations in this case are

\[
x_v = q_e x_w^{n_e},
\]

\[
p_e = x_w.
\]

For each red vertex \(w\), define the number \(n_w\) to be the sum over all the edges \(e \in E\), with target \(\tau(e) = w\), of the numbers \(n_e\).

If the loop \(l\) has initial and terminal vertex \(v\), then a generator \(t_l\) is added, along with relations

\[
t_l x_v t_l^{-1} = s_l x_v^{m_l}
\]

\[
t_l r_l t_l^{-1} = s_l x_v^{m_l'}
\]

where, again, the matrix

\[
\phi = \begin{pmatrix} k_l & m_l' \\ k_l' & m_l \end{pmatrix} \in \text{SL}_2(\mathbb{Z}).
\]

As before, in order to find a finite complete rewriting system, we need to assume this matrix is

\[
\phi = \begin{pmatrix} 1 & m_l \\ 0 & 1 \end{pmatrix};
\]

our relations in this case are

\[
t_l x_v t_l^{-1} = s_l x_v^{m_l}
\]

\[
t_l r_l t_l^{-1} = x_v
\]

Replace the generator \(p_e\) with the generator \(x_{\tau(e)}\) in the presentation above, and replace the generator \(q_e\) with the word \(x_{\tau(e)} x_{-n_e}^{n_e}\). The following is a rewriting system for the graph of circle bundles described above, with alphabet \(A = S \cup S^{-1}\), where

\[
S = \{x_v, a_{vj}, b_{vj}, r_l, s_l, t_l \mid v \in V, 1 \leq j \leq g_v, l \in L\}.
\]

- inverse cancellation relators: \(\{zz^{-1} \rightarrow 1, \quad z^{-1}z \rightarrow 1 \mid z \in S\}\)
- blue vertex relators:
  \[
  \{x_v^{\pm 1} a_{vi}^{\pm 1} \rightarrow a_{vi}^{\pm 1} x_v^{\mp 1}, \quad x_v^{\pm 1} b_{vi}^{\pm 1} \rightarrow b_{vi}^{\pm 1} x_v^{\mp 1}, \quad x_{\tau(k)}^{\pm 1} r_{k}^{\mp 1} \rightarrow r_{k}^{\mp 1} x_{\tau(k)}^{\pm 1}, \quad x_{\tau(k)}^{\pm 1} s_{k}^{\mp 1} \rightarrow s_{k}^{\mp 1} x_{\tau(k)}^{\pm 1}\}\]
- red vertex relators:
  \[
  \{a_{wi}^{\pm 1} x_w^{\mp 1} \rightarrow x_w^{\pm 1} a_{wi}^{\pm 1}, \quad b_{wi}^{\pm 1} x_w^{\mp 1} \rightarrow x_w^{\pm 1} b_{wi}^{\pm 1}, \quad r_l^{\pm 1} x_{\tau(l)}^{\mp 1} \rightarrow x_{\tau(l)}^{\pm 1} r_l^{\mp 1}, \quad s_l^{\pm 1} x_{\tau(l)}^{\mp 1} \rightarrow x_{\tau(l)}^{\pm 1} s_l^{\mp 1}\}\]
• edge relators: \( \{ x^{\pm 1}_{\tau(e)} x^{\pm 1}_{\iota(e)} \rightarrow x^{\pm 1}_{\tau(e)} x^{\pm 1}_{\iota(e)} \} \)

• blue amalgam relators:
  \[ a_{v_1} b_{v_1} \rightarrow \Lambda_v \prod_{j=g_v}^2 [b_{v_j}, a_{v_j}] b_{v_1} a_{v_1}, \]
  \[ a_{v_1} b_{v_1}^{-1} \rightarrow b_{v_1}^{-1} \prod_{j=g_v}^2 [a_{v_j}, b_{v_j}] \Lambda_v^{-1} a_{v_1}, \]
  \[ a_{v_1}^{-1} \Lambda_v \prod_{j=g_v}^2 [b_{v_j}, a_{v_j}] b_{v_1}^{-1} \rightarrow b_{v_1}^{-1} a_{v_1} \Lambda_v \prod_{j=g_v}^2 [b_{v_j}, a_{v_j}] \]

• red amalgam relators:
  \[ a_{w_1} b_{w_1} \rightarrow x^{-n_w} \Omega_w \prod_{j=g_w}^2 [b_{w_j}, a_{w_j}] b_{w_1} a_{w_1}, \]
  \[ a_{w_1} b_{w_1}^{-1} \rightarrow x^{-n_w} b_{w_1}^{-1} \prod_{j=g_w}^2 [a_{w_j}, b_{w_j}] \Omega_w^{-1} a_{w_1}, \]
  \[ a_{w_1}^{-1} \Omega_w \prod_{j=g_w}^2 [b_{w_j}, a_{w_j}] b_{w_1}^{-1} \rightarrow x^{-n_w} b_{w_1}^{-1} a_{w_1}^{-1} \Omega_w \prod_{j=g_w}^2 [b_{w_j}, a_{w_j}] \]

• blue HNN relators:
  \[ \{ x_{i(k)} t_k \rightarrow t_k r_k^{-1}, \quad r_k^{-1} \rightarrow x^{-1}_{i(k)} t_k \}, \]
  \[ \{ r_k t_k \rightarrow t_k r_k^{-1} x_{i(k)}^{-1}, \quad s_k t_k \rightarrow t_k r_k^{-m_k} x_{i(k)}^{-1} \}, \]
  \[ \{ x_{i(k)} t_k^{-1} \rightarrow t_k^{-1} s_k x_{i(k)}^{-m_k}, \quad x_{i(k)}^{-1} t_k^{-1} \rightarrow t_k^{-1} s_k^{-1} x_{i(k)}^{-m_k} \}. \]


• red HNN relators:
  \[ \{ t_l r_l \rightarrow x_{i(l)} t_l, \quad t_l r_l^{-1} \rightarrow x_{i(l)}^{-1} t_l, \}, \]
  \[ \{ t_l^{-1} x_{i(l)} \rightarrow r_l t_l^{-1}, \quad t_l^{-1} x_{i(l)}^{-1} \rightarrow r_l^{-1} t_l^{-1}, \}, \]
  \[ \{ t_l x_{i(l)} \rightarrow x_{i(l)} t_l^{-1} s_l t_l, \quad t_l x_{i(l)}^{-1} \rightarrow x_{i(l)}^{-1} s_l^{-1} t_l, \}, \]
  \[ \{ t_l^{-1} s_l \rightarrow x_{i(l)} r_l^{-m_l} t_l^{-1}, \quad t_l^{-1} s_l^{-1} \rightarrow x_{i(l)}^{-1} r_l^{-m_l} t_l^{-1} \}. \]

These rules range over all blue vertices \( v \), red vertices \( w \), and edges \( e \in E \), as well as all loops \( k \) at blue vertices and all loops \( l \) at red vertices. The letter \( \Lambda_v \) denotes the string of letters

\[
\Lambda_v = \prod_{\iota(e)=v} x_{\tau(e)} \prod_{\iota(k)=v} r_k s_k.
\]

\( \Lambda_v^{-1} \), then, denotes the formal inverse of \( \Lambda_v \), taking the letters in the string \( \Lambda_v \) in the opposite order with their signs changed. The letter \( \Omega_w \) denotes the string of letters

\[
\Omega_w = \prod_{\tau(e)=w} x_{\iota(e)} \prod_{\iota(l)=w} r_l s_l,
\]

and \( \Omega_w^{-1} \) is its formal inverse.

Denote this set of rules to be \( R \); the generators \( A \) together with our rewriting rules \( R \) give a presentation for the fundamental group of the graph of circle bundles.

**Theorem 2.** The rewriting system \( R \) on the set \( A \) is a finite complete rewriting system for the fundamental group of the graph of circle bundles described above.
Proof. In order to show that this rewriting system is complete, we will first show that a subset of the rules give rise to a complete rewriting system. Let
\[ A' = A - \{ t_{k}^{\pm 1} \mid k \in L, \nu(k) \text{ is blue} \}, \]
and define \( R' \) to be the rewriting system consisting of all of the rules above except the blue HNN relators and the inverse cancellation relators involving the letters of \( A - A' \).

In order to show that this system \( R' \) is Noetherian, we will show that there is a well-founded ordering on the words in \( A'^* \) so that whenever a word is rewritten, the resulting word is smaller with respect to this order. This ordering is a recursive path ordering.

**Definition [De].** Let \( > \) be a partial well-founded ordering on a set \( A' \). The recursive path ordering \( >_{\text{rpo}} \) on \( A'^* \) is defined recursively from the ordering on \( A' \) as follows. Given \( s_1, \ldots, s_m, t_1, \ldots, t_n \in A' \), \( s_1 \ldots s_m >_{\text{rpo}} t_1 \ldots t_n \) if and only if one of the following holds.

1) \( s_1 = t_1 \) and \( s_2 \ldots s_m >_{\text{rpo}} t_2 \ldots t_n \).
2) \( s_1 > t_1 \) and \( s_1 \ldots s_m >_{\text{rpo}} t_2 \ldots t_n \).
3) \( s_2 \ldots s_m \geq_{\text{rpo}} t_1 \ldots t_n \).

The recursion is started from the ordering \( > \) on \( A' \) and from \( s >_{\text{rpo}} 1 \) for all \( s \in A' \), where 1 is the empty word in \( A'^* \).

Recursive path ordering is a well-founded ordering which is compatible with concatenation of words [De]. The following lemma is proved by inspection of the rules in the set \( R' \).

**Lemma.** Let \( > \) be the recursive path ordering induced by
\[
\begin{align*}
t_{l}^{-1} & > t_{l} > a_{w_{1}}^{-1} > a_{w_{1}} > b_{w_{1}}^{-1} > b_{w_{1}} > a_{w_{2}}^{-1} > \cdots > b_{w_{g_{w}}} > x_{v}^{-1} > r_{l}^{-1} > r_{l} > s_{l}^{-1} > s_{l} > x_{v} > a_{v_{1}}^{-1} > a_{v_{1}} > b_{v_{1}}^{-1} > b_{v_{1}} > a_{v_{2}}^{-1} > \cdots > b_{v_{g_{v}}} > r_{k}^{-1} > r_{k} > s_{k}^{-1} > s_{k} > x_{w}^{-1} > x_{w},
\end{align*}
\]
where \( v \) is any blue vertex, \( w \) is any red vertex, \( k \) is any loop at a blue vertex, and \( l \) is any loop at a red vertex. Then for each of the rules \( u \to v \) in \( R' \), we have \( u > v \).

It follows from the Lemma that this system \( R' \) is Noetherian. In order to show that \( R' \) is also complete, it suffices to show that in the monoid presented by \((A', R')\), for each element \( m \) in this monoid, there is exactly one word in \( A'^* \) representing \( m \) that cannot be rewritten.

The Knuth-Bendix algorithm [KB] is a computational procedure for checking that a Noetherian rewriting system is complete. This algorithm checks for overlapping for overlapping pairs of rules either of the form \( r_{1}r_{2} \to s, r_{2}r_{3} \to t \in R' \) with \( r_{2} \neq 1 \), or of the form \( r_{1}r_{2}r_{3} \to s, r_{2} \to t \in R' \), where each \( r_{i} \in A'^* \); these are called critical pairs. In the first case, the word \( r_{1}r_{2}r_{3} \) rewrites to both \( sr_{3} \) and \( r_{1}t \); in the second, it rewrites to both \( s \) and \( r_{1}tr_{3} \). If there is a word \( z \in A'^* \) so that \( sr_{3} \) and \( r_{1}t \) both rewrite to \( z \) in a finite number of steps in the first case, or so that \( s \) and \( r_{1}tr_{3} \) both rewrite to \( z \) in the second case, then the critical pair
is said to be resolved. The Knuth-Bendix algorithm checks that all of the critical pairs of the system are resolved; if this is the case, then the rewriting system is complete. We have used this procedure to check that the rewriting system $R'$ is complete.

Since the rewriting system $R'$ is complete, for each word $u \in A^*$, there is a bound on the lengths of all sequences of rewritings $u \rightarrow w_1 \rightarrow \cdots \rightarrow w_n$ (where the length of this sequence is defined to be $n$). The maximum of the lengths of all of the possible rewritings of $u$ is called the disorder of $u$, denoted $d_{R'}(u)$. We will use these numbers in order to show that the larger rewriting system $R$ is Noetherian.

In order to define a well-founded ordering on the set $A^*$, note that every word $w \in A^*$ can be written uniquely in the form
\[ w = u_1 t_1 u_2 t_2 \cdots u_j t_j u_{j+1}, \]
where each $u_i$ is a (possibly empty) word in $A^*$ and each $t_i$ is a letter in $A - A'$. Define functions $\psi_i$ from $A^*$ to the nonnegative integers by
\[
\begin{align*}
\psi_0(w) &= j, \\
\psi_{2i}(w) &= d_{R'}(u_i), \text{ and} \\
\psi_{2i+1}(w) &= \text{length}(u_i),
\end{align*}
\]
where $i$ ranges from 1 to $j + 1$, and \text{length} denotes the word length over $A'$. In order to compare words of different length, define $\psi_i(w) = 0$ if $i > 2j + 3$. For two words $w_1$ and $w_2$ in $A^*$, define $w_1 > w_2$ if $\psi_0(w_1) > \psi_0(w_2)$ or if $\psi_i(w_1) = \psi_i(w_2)$ for all $i < k$ and $\psi_k(w_1) > \psi_k(w_2)$. We claim this defines a well-founded ordering on $A^*$.

To check the claim, suppose $w \in A^*$. If a rule in $R'$ is applied to $w$, the rule must to be applied to one of the subwords $u_i$, so the value of $\psi_{2i}$ is reduced without altering the values of $\psi_k$ for any $0 \leq k \leq 2i - 1$. Suppose an inverse cancellation relator involving the letters of $A - A'$ is applied to $w$; in this case, the value of $\psi_0$ is reduced. Finally, if a blue HNN relator is applied to $w$, the rule must be applied to a subword $u_i t_i$ of $w$. Then the values of $\psi_k$ for any $0 \leq k \leq 2i - 1$ are not altered; the value of $\psi_{2i}$ either decreases or remains unchanged; and the value of $\psi_{2i+1}$ is reduced. So each time a word is rewritten, the resulting word is smaller with respect to this ordering. Therefore the rewriting system $R$ is also Noetherian.

Since $R$ is Noetherian, we have again applied the Knuth-Bendix procedure to check that the rewriting system $R$ is complete. \qed
§4. An example

Rather than give the details of the Knuth-Bendix computation, we will give a description of the normal forms that these rewriting rules produce in the case where $M$ decomposes into two circle bundles. In this case, $G$ is a free product with amalgamation and the graph $\Gamma$ consists of two vertices $v$ and $w$ joined by a single edge. We assume that $v$ is blue and $w$ is red. $M_v$ and $M_w$ are circle bundles, each with a single torus boundary component and $M$ is formed by gluing along these tori. Thus we have

\[ A = \pi_1(M_v) = \langle a_1, \ldots, a_g, b_1, \ldots, b_g \rangle \times \langle x \rangle \]
\[ C = \pi_1(M_w) = \langle c_1, \ldots, c_h, d_1, \ldots, d_h \rangle \times \langle y \rangle \]
\[ X = \langle x \rangle \times \langle y \rangle \]

so $G = \pi_1(M) = A \ast_X C$, where the gluing along $X$ is given by

\[ x = \prod_{i=1}^{h} [c_i, d_i] y^n \]

and

\[ \prod_{i=1}^{g} [a_i, b_i] = y. \]

The generating set for the fundamental group $G$ of $M$ will be $S \cup S^{-1}$ where

\[ S = \{ a_1, \ldots, a_g, b_1, \ldots, b_g, x, c_1, \ldots, c_h, d_1, \ldots, d_h, y \}. \]

The rewriting rules are:

- inverse cancellation relators: \{ $zz^{-1} \to 1, z^{-1}z \to 1$ | $z \in S$ \}
- blue vertex relators: \{ $x^\pm a_1^\pm b_1 = a_1^\pm x^\pm, x^\pm b_1^\pm a_1 = b_1^\pm x^\pm$ \}
- red vertex relators: \{ $c_1^\pm y^\pm = y^\pm c_i^\pm, d_1^\pm y^\pm = y^\pm d_i^\pm$ \}
- edge relators: \{ $x^\pm y^\pm = y^\pm x^\pm$ \}
- blue amalgam relators:
  \{ $a_1 b_1 \to y p^{-1} b_1 a_1, a_1 b_1^{-1} \to b_1^{-1} p y^{-1} a_1, a_1^{-1} y p^{-1} b_1 \to b_1 a_1^{-1}, a_1^{-1} b_1^{-1} \to b_1^{-1} a_1^{-1} y p^{-1}$ \}
- red amalgam relators:
  \{ $c_1 d_1 \to y^n x q^{-1} d_1 c_1, c_1^{-1} d_1^{-1} \to y^n d_1^{-1} c_1 x q^{-1}, c_1^{-1} x q^{-1} d_1 \to y^n d_1^{-1} c_1^{-1} x q^{-1}$ \}

Here we use the letter $p$ to denote the string of letters

\[ a_2 b_2 a_2^{-1} b_2^{-1} \ldots a_g b_g a_g^{-1} b_g^{-1}, \]

so $p^{-1}$ denotes

\[ b_g a_g b_g^{-1} a_g^{-1} \ldots b_2 a_2 b_2^{-1} a_2^{-1}. \]

Similarly the letter $q$ denotes the string

\[ c_2 d_2 c_2^{-1} d_2^{-1} \ldots c_h d_h c_h^{-1} d_h^{-1}, \]
and $q^{-1}$ denotes
\[ d hdc h^{-1} c h^{-1} \cdots d 2 c 2 d^{-1} c 2^{-1}. \]

To understand the normal forms that these produce, we consider several sublanguages.

Let $L(A/X)$ be the set of irreducible words on $\{a_i, b_i, y\}^{\pm1}$ which do not end in $y^\pm1$. Similarly, we let $L(X\setminus C)$ be the set of irreducible words on $\{c_i, d_i, x\}^{\pm1}$ which do not begin in $x^\pm1$. We take $L(X) = \{y^mx^n \mid m, n \in \mathbb{Z}\}$.

**Lemma.**
1) $L(A/X)$ bijects to $A/X$.
2) $L(X\setminus C)$ bijects to $X\setminus C$.
3) $L(X)$ bijects to $X$.

**Proof.** Clearly $L(A/X)$ surjects to $Z/X$, for the set of reduced words on these letters surjects to $A$, and deleting any trailing $y^\pm1$ does not change the coset. Thus, to prove 1) we must show that if $u, u' \in L(A/X)$ with $\overline{\pi}X = \overline{w}X$ then $u = u'$. Here $u$ and $u'$ both evaluate into the free group on $\{a_i, b_i\}$, so in this case we have $\overline{uy^m} = \overline{w}$ for some $m$. Observe that no rewriting rule has a left hand side consisting of letters of $\{a_i, b_i, y\}^{\pm1}$ and ending in $y^\pm1$ (other than free reduction). Thus, if $u$ is irreducible, then so is $uy^m$ for any $m$. If $m \neq 0$ we have two distinct irreducible words representing the same group element. However, it is not hard to carry out the Knuth-Bendix procedure on the set of rules evaluating into $A$. This ensures that for any element of $A$ there is a unique irreducible word and thus $u$ and $u'$ are identical as required.

The proof of 2) is similar. Once again it is easy to see that $L(X\setminus C)$ surjects to $X\setminus C$. Now we suppose that $X \overline{\pi} = X \overline{w'}$ with $w$ and $w'$ in $L(X\setminus C)$. We then have $\overline{y^mw^n}\overline{w} = \overline{w'}$ for some $m$ and $n$. Once again, these are both irreducible as there is no rewriting rule beginning in $y^m x^n$ that can be applied. Appealing to the Knuth-Bendix procedure in $C$ forces $m = n = 0$ and thus $w = w'$ as required.

The proof of 3) is immediate. $\Box$

Now observe that any irreducible word $\theta$ has the form
\[ \theta = u_1 v_1 w_1 \ldots u_k v_k w_k \]

where
- For each $i$, $v_i = y^m_i x^n_i \in L(X)$.
- For each $i$, $u_i$ is a maximal subword lying in $L(A/X)$.
- For each $i$, $w_i$ is a maximal subword lying in $L(X\setminus C)$.

The maximality of each $u_i$ ensures that no $v_i$ consists solely of $y^\pm1$’s directly preceding $v_{i+1}$. Likewise the maximality of each $w_i$ ensures that no $v_i$ consists solely of $x^\pm1$’s directly following $v_{i−1}$.

We call $k$ the **length** of $\theta$. Let $L_k$ be the set of all irreducible words of length $k$. For each $g \in G$ the **$AC$-length** of $g$ is the minimal $k$ such that $g \in (AC)^k_k$. Let $G_k$ be the set of all group elements of $AC$-length $k$. That is, $g \in G_k$ if $k$ is minimal such that $g = A_1 C_1 \ldots A_k C_k$ with $A_i \in A$, $C_i \in C$. 

Claim. $L_k$ bijects to $G_k$.

Proof. This is an induction on $k$.

We check the case $k = 1$. Clearly $L_1$ surjects to $AC$, since any element of $A$ has the form $u_1v_1$ and each element of $C$ has the form $v_1'w_1$. Multiplying these together and applying the replacement rules produces a word of the form $u_1v_1w_1$ as required.

We now check that the map from $L_1$ to $AC$ is injective. Suppose $g \in AC$ and $g = u_1v_1w_1$. Notice that $A/X$ bijects to $AC/C$. Thus $g$ determines a coset $gC$ in $AC/C$ and thus a unique element of $A/X$. Consequently, $g$ determines $u_1$.

On the other hand, having determined the coset representative $u_1$ of $gC$ in $AC/C$, there is a unique $c \in C$ so that $g = u_1c$ and this, in turn, determines $v_1w_1$.

We now assume by induction that $L_k$ bijects to $G_k$ and check that $L_{k+1}$ bijects to $G_{k+1}$. It is easy to see that $L_{k+1}$ surjects to $G_{k+1}$. For suppose $g \in G_{k+1}$. Then $g$ has the form $gkh$ with $g_k \in G_k$, $h \in G_1$. We represent $g_k$ by a word of $L_k$ and $h$ by a word of $L_1$. We concatenate these words and apply our rewriting rules. The resulting word $\theta$ lies in $\cup_{i=1}^{k+1} L_i$. Since $\cup_{i=1}^k L_i$ misses $G_{k+1}$, it follows that $\theta \in L_{k+1}$.

We must show that $L_{k+1}$ injects to $G_{k+1}$. Notice that $g_k$ and $h$ are determined by $g$ up to an element of $X$. Thus, if $g = g_kh = g_k' h'$ then there is $z \in X$ so that $g_k = g_k z$ and $h' = z^{-1}h$. Suppose then that $g$ is represented by two irreducible words

\[
\theta = u_1v_1w_1 \ldots u_kv_kw_ku_{k+1}v_{k+1}w_{k+1} \\
\theta' = u_1'v_1'w_1' \ldots u_k'v_k'w_k'u_{k+1}'v_{k+1}'w_{k+1}'
\]

We take $g_k$ and $g_k'$ to be the group elements represented by the $L_k$ portions of $\theta$ and $\theta'$. Thus $h$ and $h'$ are represented by the remaining portions of these two words.

Notice that if $w_k$ ends in $x^\pm 1$ then $u_{k+1}$ and $v_{k+1}$ are both empty, for otherwise, the final $x^\pm 1$'s of $w_k$ would have moved right through $u_{k+1}$ and any $y^\pm 1$'s of $v_{k+1}$. This cannot happen, since each $w_i$ was chosen to be maximal. In the same manner, we do not have $w_k$ empty and $v_k$ ending in $x^\pm 1$.

The same argument applies to $\theta'$. Suppose the element $z \in X$ is represented by the word $y^m x^n \in L(X)$. Then if the word $u_1v_1w_1 \ldots u_kv_kw_ky^m x^n$ is reduced using the rules of the rewriting system, for the resulting irreducible word

\[u_1v_1w_1 \ldots u_{k-1}v_{k-1}w_{k-1} \tilde{u}_k \tilde{v}_k \tilde{w}_k,\]

we have that either $\tilde{w}_k = w_kx^n$ or $\tilde{w}_k = w_k$ is empty and $\tilde{v}_k = v_kx^n$. Now this irreducible word represents the same element of $G_k$ as $u_1'v_1'w_1' \ldots u_k'v_k'w_k'$. Therefore our induction hypothesis says that $\tilde{v}_k = v_k'$ and $\tilde{w}_k = w_k'$. So either $\tilde{w}_k = w_kx^n = w_k'$ or else $\tilde{w}_k$ and $w_k'$ are both empty and $\tilde{v}_k = v_kx^n = v_k'$. Since $w_k'$ cannot end with $x^{\pm 1}$ and if $w_k'$ is empty then $v_k'$ cannot end with $x^{\pm 1}$, this shows that $n$ must be zero. Consequently, $u_1v_1w_1 \ldots u_kv_kw_k$ and $u_1'v_1'w_1' \ldots u_k'v_k'w_k'$ differ by at most a power of $y$ in $G$; that is, we have $z = \bar{y}^m$. 
On the other hand, if \( u_{k+1} \) begins in \( y^{\pm 1} \), we must have \( v_k \) and \( w_k \) empty, for any leading \( y^{\pm 1} \)'s of \( w_{k+1} \) would have had to move left through \( w_k \) and any \( x^{\pm 1} \)'s of \( v_k \). Maximal of the words \( u_i \) does not allow this to happen. Also, we cannot have \( u_{k+1} \) is empty and \( v_{k+1} \) beginning with \( y^{\pm 1} \). It follows by a similar argument, then, that \( u_{k+1}v_{k+1}w_{k+1} \) and \( u'_{k+1}v'_{k+1}w'_{k+1} \) differ in \( G \) by at most a power of \( x \); that is, \( z = x^n \). Since \( z \) is now both a power of \( x \) and a power of \( y \), that power is plainly 0, so \( g_k = g'_k \) and \( h = h' \). By induction

\[
 u_1v_1w_1 \ldots u_kv_kw_k = u'_1v'_1w'_1 \ldots u'_kv'_kw'_k
\]

and

\[
 u_{k+1}v_{k+1}w_{k+1} = u'_{k+1}v'_{k+1}w'_{k+1},
\]

so \( \theta = \theta' \) as required.

Since \( G = \coprod G_k \) and the language of irreducible words is \( \coprod L_k \) it follows that the language of irreducible words is a normal form which bije cts to \( G \).

\[\]5. A question

When the gluings of the circle bundles at the vertices of \( \Gamma \) are more complicated, or when the circle bundles themselves are replaced by more general Seifert-fibered spaces, we were unable to find finite complete rewriting systems. So we end with the following.

**Question.** Does every fundamental group of a closed 3-manifold satisfying Thurston’s geometrization conjecture have a finite complete rewriting system?

**Acknowledgement**

In the course of our work we have used Rewrite Rule Laboratory [KZ], a software package for performing the Knuth-Bendix algorithm, to check completeness of our rewriting systems for many examples.

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