Vector bundles on quantum conjugacy classes

Andrey Mudrov

University of Leicester,
University Road, LE1 7RH Leicester, UK,

Moscow Institute of Physics and Technology,
9 Institutskiy per., Dolgoprudny, Moscow Region, 141701, Russia,

e-mail: am405@le.ac.uk

Abstract

Let $\mathfrak{g}$ be a simple complex Lie algebra of a classical type and $U_q(\mathfrak{g})$ the corresponding Drinfeld-Jimbo quantum group at $q$ not a root of unity. With every point $t$ of the fixed maximal torus $T$ of an algebraic group $G$ with Lie algebra $\mathfrak{g}$ we associate an additive category $O_q(t)$ of $U_q(\mathfrak{g})$-modules that is stable under tensor product with finite-dimensional quasi-classical $U_q(\mathfrak{g})$-modules. We prove that $O_q(t)$ is essentially semi-simple and use it to explicitly quantize equivariant vector bundles on the conjugacy class of $t$.

Key words: conjugacy classes, vector bundles, quantization, contravariant forms, extremal projector

AMS classification codes: 17B10, 17B37, 53D55.
Contents

1 Introduction 3

2 Preliminaries 5
   2.1 Quantum group basics .......................... 6
   2.2 Poisson Lie structure on conjugacy classes .... 8
   2.3 Definition of generalized parabolic Verma modules. 9
   2.4 Extremal projector ................................ 11

3 Tensor product of highest weight modules 12

4 Quasi-classical limit of Shapovalov elements 16
   4.1 Inverse Shapovalov form and its matrix elements . 17
   4.2 Regularity of Shapovalov elements ................. 19
   4.3 All points of the maximal torus are quantizable . 21
      4.3.1 Orthogonal $g$ ................................ 21

5 Generalized parabolic categories 23
   5.1 Base module ...................................... 25
   5.2 Generalized parabolic Verma modules ............ 28

6 Quantization of associated vector bundles 31
   6.1 Equivariant star product .......................... 31
   6.2 Quantum vector bundles as projective $T^t$-modules . 36

A Induced modules and duality 38
1 Introduction

This paper is devoted to quantization of the category of equivariant vector bundles on a semi-simple conjugacy class of a simple complex algebraic group $G$. This includes quantization of the function algebra as a trivial bundle of rank 1. This work is a continuation of a project started off in [M4, M5, JM] and technically based on [M1, M2]. A complete analysis is done for groups of the four infinite series. With regard to five exceptional types, we believe that the approach is generally applicable as well. The main technical issue to address is the quasi-classical behaviour of Shapovalov elements, which is sorted out for the classical types in this paper.

Semi-simple are the only conjugacy classes that are affine sub-varieties in $G$ [Spr]. By a vector bundle we understand a projective module of global sections over the coordinate ring, in accordance with the Serre-Swan theorem, [S, Sw]. We adopt this point of view in the quantum setting and treat vector bundles over a non-commutative space as projective (one-sided) modules over its quantized coordinate ring. This way the deformation quantization programme for Poisson varieties naturally extends to the realm of vector bundles. In the presence of symmetry, it essentially becomes a part of representation theory.

The Poisson structure underlying the quantization of our concern descents from the Semenov-Tian-Shansky bracket on $G$ related to the standard classical $r$-matrix, [STS]. It makes $G$ a Poisson variety over the Poisson group $G$ with the Drinfeld-Sklyanin bracket generated by $r$, with respect to the conjugation action. The non-trivial Poisson structure on $G$ implies quantization of the symmetry group first. Equivariance is then understood relative to the quantized universal enveloping algebra $U_q(\mathfrak{g})$ of the Lie algebra $\mathfrak{g} = \text{Lie}(G)$.

The Semenov-Tian-Shansky bracket on $G$ is analogous to the $G$-invariant Lie bracket on the Lie algebra $\mathfrak{g} \simeq \mathfrak{g}^*$, which restricts to every adjoint orbit. Equivariant quantization of semi-simple orbits in $\mathfrak{g}$ is of long interest and has been understood some twenty years ago [DGS, DM, EE, AL], in what concerns the conventional point of view restricted to function algebras. The underlying representation theory involves parabolic Verma modules over the classical universal enveloping algebra $U(\mathfrak{g})$.

A representation theoretical approach to equivariant quantization consists in realization of the quantized function algebra on a $G$-space by linear operators on a $U_q(\mathfrak{g})$-module (respectively $U(\mathfrak{g})$-module in the case of a $\mathfrak{g}$-orbit). It is natural to seek for a realization of a more general quantum vector bundle via linear mappings between modules from an appropriate category. They generalize parabolic modules from the Bernstein-Gelfand-Gelfand category of $U_q(\mathfrak{g})$-modules, which we denote by $\mathcal{O}_q$. Such modules form an additive subcategory in $\mathcal{O}_q$ determined, up to an isomorphism,
by a point \( t \) from a fixed maximal torus \( T \subset G \). Modules of highest weight in it are parameterized by finite-dimensional irreducible representations of the subalgebra \( \mathfrak{t} \subset \mathfrak{g} \) centralizing \( t \).

The category under study is stable under tensor product with finite dimensional \( U_q(\mathfrak{g}) \)-modules. It is generated by a base module \( M_\lambda \) of highest weight \( \lambda \) associated with \( t \) and denoted by \( \mathcal{O}_q(t) \). The base weight \( \lambda \) is not uniquely determined by \( t \) but up to an action of the group of \( U_q(\mathfrak{g}) \)-characters known to be \( \cong \mathbb{Z}^{rk_{\mathfrak{g}}} / 2 \mathbb{Z} \), where \( \mathbb{Z}^2 = \mathbb{Z} / 2\mathbb{Z} \). This group acts by isomorphisms on the categories associated with \( t \) via tensor product with the corresponding one-dimensional module.

The base module \( M_\lambda \) supports quantization of the coordinate ring \( \mathbb{C}[O] \) of the conjugacy class \( O \ni t \) as a subalgebra in \( \text{End}(M_\lambda) \). At least for all non-exceptional \( G \), different \( t \) give rise to isomorphic quantizations of \( \mathbb{C}[O] \) but different faithful representations, cf. e.g. [AM]. We expect that be true for all types of \( G \).

If \( t \) is of finite order, then \( \mathcal{O}_q(t) \) is semi-simple for each \( q \) not a root of unity except maybe for a finite set of values. In all cases that we worked out explicitly [M5, JM], the set of exceptional \( q \) is empty. For general \( t \), the category \( \mathcal{O}_q(t) \) is semi-simple for almost all \( q \) away from the roots of the spectrum of the adjoint operator \( \text{Ad}_t \in \text{End}(\mathfrak{g}) \).

The category \( \mathcal{O}_q(t) \) proves to be equivalent to the category of equivariant finitely generated projective modules over the quantized polynomial ring \( \mathbb{C}[O] \), as a module category over the finite-dimensional quasi-classical representations of \( U_q(\mathfrak{g}) \). As an Abelian category, \( \mathcal{O}_q(t) \) is equivalent to that of classical \( \mathfrak{t} \)-modules which are submodules in finite-dimensional \( \mathfrak{g} \)-modules.

In the final section, we construct an equivariant star product on \( \mathbb{C}[O] \) by twisting the multiplication on the RTT algebra of functions on the quantum group, [FRT]. Its restriction to a space of ”\( \mathfrak{t} \)-invariants” delivers a flat associative deformation of \( \mathbb{C}[O] \). This construction is not new for \( \mathfrak{t} \) of Levi type. For a non-Levi \( \mathfrak{t} \), it was done only for even quantum spheres in [M3], with the use of elementary harmonic analysis on the quantum Euclidean space. We further extend this star product to associated vector bundles along the lines of [DM]. Following an approach of [M2] we explicitly express it through the extremal projector of \( U_q(\mathfrak{g}) \).

Our main technical tool is contravariant form on \( U_q(\mathfrak{g}) \)-modules and its relation with extremal projector. Such forms appear in this theory in a few incarnations.

First of all, we use the contravariant form on Verma modules to construct their generalized parabolic quotients. Matrix entries of the inverse form constitute Shapovalov elements \( \phi_{ma} \in U_q(\mathfrak{g}_-) \) of weight \( -m\alpha \) for a simple positive root \( \alpha \) of \( \mathfrak{t} \) and \( m \in \mathbb{N} \). Applied to the highest vector, \( \phi_{ma} \) produce extremal vectors in Verma modules that vanish in the generalized parabolic quotients. We require that \( \phi_{ma} \) turns into the power \( f_{\alpha}^m \) of the root vector \( f_{\alpha} \in \mathfrak{t}_- \) in the classical limit \( q \to 1 \). We check it via a direct analysis of matrix elements of the inverse Shapovalov form.
Another application of contravariant forms is a proof of irreducibility of the base module $M_\lambda$. We approximate its opposite module $M'_\lambda$ of lowest weight $-\lambda$ by a sequence of $U_q(\mathfrak{g}^+)$-submodules in a certain system $\Xi$ of finite-dimensional $U_q(\mathfrak{g})$-modules, using the extremal projector. The set $\Xi$ comprises all counterparts of classical $G$-modules $V$ with an orbit isomorphic to $O$ (exactly those appearing in $\mathbb{C}[O]$). They admit $U_q(\mathfrak{g}^+)$-homomorphisms $M'_\lambda \to V$ whose common kernel over all $V \in \Xi$ is zero. Then we approximate the inverse invariant pairing between $M_\lambda$ and $M'_\lambda$ by certain extremal vectors in $V \otimes M_\lambda$ when $V$ ranges in $\Xi$. An extremal vector determines a map $M_\lambda \to V$ whose injectivity on certain spaces is equivalent to irreducibility of $V$.

The third appearance of contravariant forms in this presentation is a proof of complete reducibility of tensor products. From the base module we proceed to the category $O_q(t)$ it generates. It is found in [M1] that a contravariant form controls complete reducibility of tensor products of modules of highest weights. A relation between the form and extremal projector established in [M2] delivers a practical computational machinery which helps us prove that all modules from the category under study are semi-simple for almost all $q$. The simple objects are generalized parabolic Verma modules $M_{\lambda, \xi}$ of highest weight $\lambda + \xi$, where $\xi$ is a highest weight of a $\mathfrak{k}$-submodule in a finite dimensional $\mathfrak{g}$-module.

Finally, inverted contravariant forms participate in definition of the star product on $\mathbb{C}[O]$ and its actions on vector bundles, like in [AL, DM, EE, EEM, KST].

We prove that the locally finite part of the $U_q(\mathfrak{g})$-module $\text{End}(M_\lambda)$ is a quantization of $\mathbb{C}[O]$, for almost all $q$. An irreducible decomposition of $V \otimes M_\lambda \in O_q(\lambda)$ gives rise to a direct sum decomposition of $V \otimes \text{End}(M_\lambda)$ making the locally finite part of $\text{Hom}(M_{\lambda, \xi}, M_\lambda)$ (respectively, $\text{Hom}(M_\lambda, M_{\lambda, \xi})$) a quantization of the equivariant vector bundle with the $\mathfrak{k}$-submodule of highest weight $\xi$ in $V$ (respectively, its dual) as the fiber. The direct sum decomposition of $V \otimes \text{End}(M_\lambda)$ is quasi-classical and goes to the decomposition of the trivial vector bundle $V \otimes \mathbb{C}[O]$ into a sum of equivariant sub-bundles.

## 2 Preliminaries

Throughout the paper we assume that the deformation parameter $q$ takes values in the set $\mathbb{C} \setminus \sqrt[\lambda]{T}$ of non-zero complex numbers which are not a root of unity. We introduce topology on $\mathbb{C} \setminus \sqrt[\lambda]{T}$ as induced from Zariski topology on $\mathbb{C}$. That is, an open set in $\mathbb{C} \setminus \sqrt[\lambda]{T}$ is the complement to a finite set of points, possibly empty. By all $q$ we understand all from $\mathbb{C} \setminus \sqrt[\lambda]{T}$, and almost all means all from a non-empty Zariski open set.
We mostly work over the ground filed \( \mathbb{C} \) but some topics require consideration over \( \mathbb{C}[q, q^{-1}] \) and further extension to the local ring \( \mathbb{C}_1(q) \) of rational functions in \( q \) regular at the classical point \( q = 1 \).

By deformation of a complex vector space \( A \) we mean an arbitrary \( \mathbb{C}[q, q^{-1}] \)-module \( A_q \) such that \( A_q/(q-1)A_q \cong A \). We call it flat if, upon extension over \( \mathbb{C}_1(q) \), \( A_q \cong A \otimes \mathbb{C}_1(q) \). By quantization of \( A \) we understand its flat deformation, along with additional structures, e.g. algebras, modules etc. Such a structure is preserved by a quantum group in equivariant quantization.

### 2.1 Quantum group basics

Let \( \mathfrak{g} \) be a simple complex Lie algebra of the classical type and \( \mathfrak{h} \subset \mathfrak{g} \) its Cartan subalgebra. Fix a triangular decomposition \( \mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{h} \oplus \mathfrak{g}_+ \) with maximal nilpotent Lie subalgebras \( \mathfrak{g}_\pm \). Denote by \( R = R_\mathfrak{g} \) the root system of \( \mathfrak{g} \), and by \( R^+ = R^+_\mathfrak{g} \) the subset of positive roots with basis \( \Pi = \Pi_\mathfrak{g} \) of simple roots. The weight lattice is denoted by \( \Lambda = \Lambda_\mathfrak{g} \) and the semi-group of dominant weights by \( \Lambda^+ = \Lambda^+_\mathfrak{g} \). We use a similar notation for reductive subalgebras in \( \mathfrak{g} \) and drop the subscript when only the total Lie algebra \( \mathfrak{g} \) is in the context.

Choose an inner product \( (.,.) \) on \( \mathfrak{h} \) as a multiple of a restricted ad-invariant form and transfer it to the space \( \mathfrak{h}^* \) of linear functions on \( \mathfrak{h} \) by duality. For every \( \lambda \in \mathfrak{h}^* \) denote by \( h_\lambda \in \mathfrak{h} \) a unique element such that \( \mu(h_\lambda) = (\mu, \lambda) \), for all \( \mu \in \mathfrak{h}^* \).

By \( U_q(\mathfrak{g}) \) we understand the standard quantum group [D1, ChP] as a complex Hopf algebra with the set of generators \( e_\alpha, f_\alpha, \) and \( q^{\pm h_\alpha} \) labeled with a simple root \( \alpha \) and satisfying relations

\[
q^{h_\alpha} e_\beta = q^{(\alpha, \beta)} e_\beta q^{h_\alpha}, \quad [e_\alpha, f_\beta] = \delta_{\alpha, \beta}[h_\alpha]_q, \quad q^{h_\alpha} f_\beta = q^{-(\alpha, \beta)} f_\beta q^{h_\alpha}, \quad \forall \alpha, \beta \in \Pi.
\]

The elements \( q^{h_\alpha} \) are assumed invertible, with \( q^{h_\alpha} q^{-h_\alpha} = 1 \), while \( \{e_\alpha\}_{\alpha \in \Pi} \) and \( \{f_\alpha\}_{\alpha \in \Pi} \) also satisfy quantized Serre relations, see [ChP] for details. Here and throughout the text we use the notation

\[
[z]_q = \frac{q^z - q^{-z}}{q - q^{-1}} \quad \text{for} \quad z \in \mathfrak{h} + \mathbb{C}.
\]

The complex number \( q \neq 0 \) is not a root of unity.

We fix the Hopf algebra structure on \( U_q(\mathfrak{g}) \) by setting comultiplication on the generators as

\[
\Delta(f_\alpha) = f_\alpha \otimes 1 + q^{-h_\alpha} \otimes f_\alpha, \quad \Delta(q^{\pm h_\alpha}) = q^{\pm h_\alpha} \otimes q^{\pm h_\alpha}, \quad \Delta(e_\alpha) = e_\alpha \otimes q^{h_\alpha} + 1 \otimes e_\alpha.
\]

Then the antipode acts on the generators by the assignment

\[
\gamma(f_\alpha) = -q^{h_\alpha} f_\alpha, \quad \gamma(q^{\pm h_\alpha}) = q^{\mp h_\alpha}, \quad \gamma(e_\alpha) = -e_\alpha q^{-h_\alpha}.
\]

and the counit returns

\[
\epsilon(e_\alpha) = 0, \quad \epsilon(f_\alpha) = 0, \quad \epsilon(q^{h_\alpha}) = 1.
\]
We denote by $U_q(\mathfrak{h})$, $U_q(\mathfrak{g}_+)$, and $U_q(\mathfrak{g}_-)$ the associative unital subalgebras in $U_q(\mathfrak{g})$ generated by $\{q^{\pm h_\alpha}\}_{\alpha \in \Pi}$, $\{e_\alpha\}_{\alpha \in \Pi}$, and $\{f_\alpha\}_{\alpha \in \Pi}$, respectively. The quantum Borel subgroups are defined as $U_q(\mathfrak{b}_\pm) = U_q(\mathfrak{g}_\pm)U_q(\mathfrak{h})$; they are Hopf subalgebras in $U_q(\mathfrak{g})$.

We consider an involutive coalgebra anti-automorphism and algebra automorphism $\sigma$ of $U_q(\mathfrak{g})$ setting it on the generators by the assignment

$$\sigma: e_\alpha \mapsto f_\alpha, \quad \sigma: f_\alpha \mapsto e_\alpha, \quad \sigma: q^{h_\alpha} \mapsto q^{-h_\alpha}.$$  

The involution $\omega = \gamma^{-1} \circ \sigma = \sigma \circ \gamma$ is an algebra anti-automorphism and preserves comultiplication.

With every normal order on $R^+$ (a sum of two positive roots is between the summands) one associates a Lusztig system of root vectors $f_\alpha, e_\alpha$, for all $\alpha \in R^+$. Every such pair defines an associative subalgebra $U_q(\mathfrak{g}_\pm) \subset U_q(\mathfrak{g})$ that is isomorphic to $U_q(\mathfrak{sl}(2))$. Ordered monomials in $e_\alpha$ and $f_\alpha$ deliver a Poincare-Birkhoff-Witt (PBW) basis in $U_q(\mathfrak{g}_+) \text{ and } U_q(\mathfrak{g}_-)$, respectively, [ChP].

By $G$ we denote a simple algebraic group with Lie algebra $\mathfrak{g}$. Unless it is explicitly specified, it can be any group between simply connected and the adjoint group. Let $T \subset G$ denote the maximal torus in $G$ whose Lie algebra is $\mathfrak{h}$. We denote by $T_Q \subset T$ the subset of elements of finite order, i.e. $t \in T_Q$ if and only if $t^m = 1$ for some integer $m$.

For each $t \in T$ we call its centralizer $\mathfrak{k} \subset \mathfrak{g}$ generalized Levi subalgebra. The polarization of $\mathfrak{g}$ induces a polarization $\mathfrak{k} = \mathfrak{k}_- \oplus \mathfrak{h} \oplus \mathfrak{k}_+$ such that $\mathfrak{k}_\pm \subset \mathfrak{g}_\pm$. The subalgebra $\mathfrak{k} \subset \mathfrak{g}$ is called Levi if $\Pi_\mathfrak{k} \subset \Pi_\mathfrak{g}$. We call it pseudo-Levi if it is not isomorphic to a Levi subalgebra via an internal isomorphism, for example, if $\mathfrak{k} \not\subset \mathfrak{g}$ is semi-simple. This terminology is compatible with what is accepted in the literature, see e.g. [Cost]. In general, even for $t$ with a Levi centralizer there may be other points in $T$ from the same conjugacy class whose centralizers are not Levi in the above sense. The Levi type of $\mathfrak{k}$ is special because $U(\mathfrak{k})$ is quantizable as a Hopf subalgebra $U_q(\mathfrak{k}) \subset U_q(\mathfrak{g})$.

Given a $U_q(\mathfrak{g})$-module $Z$ we denote by $Z[\mu]$ the subspace of weight $\mu \in \mathfrak{h}^*$, i.e. the set of vectors $z \in Z$ satisfying $q^{h_\alpha}z = q^{(\mu,\alpha)}z$ for all $\alpha \in \Pi$. The set of weights of $Z$ is denoted by $\Lambda(Z)$, which notation is also used for general $U_q(\mathfrak{h})$-modules. All modules are assumed $U_q(\mathfrak{h})$-diagonalizable with finite dimensional weight spaces or locally finite over $U_q(\mathfrak{g})$ with finite dimensional isotypic components. For such a module $Z$, the (right or left) restricted dual is denoted by $Z^*$. If $Z$ is a module of highest weight $\lambda$, its opposite module of lowest weight $-\lambda$ is denoted by $Z'$. There is a linear bijection $\hat{\sigma} : Z \to Z'$ intertwining the representation homomorphisms $\pi$ and $\pi'$ via the involution $\sigma: \hat{\sigma} \circ \pi(x) = \pi'(x) \circ \sigma$ for all $x \in U_q(\mathfrak{g})$.

For a diagonalizable $U_q(\mathfrak{h})$-module $V$ with finite dimensional weight spaces we define infinitesimal character as a formal sum $\sum_{\mu \in \Lambda(V)} \dim V[\mu] e^\mu$. We write $\text{ch}(V) \leq \text{ch}(W)$ if $\dim V[\mu] \leq \dim W[\mu]$.
\[ \dim W[\mu] \text{ for all } \mu \text{ and } \text{ch}(V) < \text{ch}(W) \text{ if the inequality is strict for some } \mu. \]

One-dimensional representations of \( U_q(\mathfrak{g}) \) are trivial on \( U_q(\mathfrak{g}_\pm) \) and assign \( \pm 1 \) to every \( q^{h_\alpha}, \alpha \in \Pi \). They form a group of characters isomorphic to \( \mathbb{Z}_{2}^{rk_\mathfrak{g}} \). The category of quasi-classical finite dimensional \( U_q(\mathfrak{g}) \)-modules is denoted by \( \text{Fin}_q(\mathfrak{g}) \). Such modules are diagonalizable with weights from \( q^\Lambda \). General finite dimensional \( U_q(\mathfrak{g}) \)-modules are obtained from \( \text{Fin}_q(\mathfrak{g}) \) by tensoring with a one dimensional module.

### 2.2 Poisson Lie structure on conjugacy classes

In this section we recall the Poisson-Lie structure on the group \( G \) that is a Poisson-Lie analog of the Kostant-Kirillov-Souriau bracket on the (dual of) Lie algebra \( \mathfrak{g} \).

Fix the ad-invariant inner product on \( \mathfrak{g} \). Let \( f_\alpha \) and \( e_\alpha, \alpha \in \Pi_\mathfrak{g} \) be the Chevalley generators of the Lie algebra \( \mathfrak{g} \) satisfying \((e_\alpha, f_\alpha) = 1\) and let \( \{h_i\}_i \) be an orthogonal basis in \( \mathfrak{h} \). The element

\[
 r = \sum_i h_i \otimes h_i + \sum_{\alpha \in \mathbb{R}^+} e_\alpha \otimes f_\alpha \in \mathfrak{g} \otimes \mathfrak{g}
\]

is called classical r-matrix. It satisfies the classical Yang-Baxter equation, cf. [D1].

Consider the left and right invariant vector fields on the group \( G \),

\[
 \xi^l f(g) = \frac{d}{dt} f(ge^{t\xi})|_{t=0}, \quad \xi^r f(g) = \frac{d}{dt} f(e^{t\xi}g)|_{t=0},
\]

generated by \( \xi \in \mathfrak{g} \), where \( f \in \mathbb{C}[G] \) and \( g \in G \). The bivector field \( r^{l,l} - r^{r,r} \) makes \( G \) a Poisson group, cf. [D3].

For each \( \xi \in \mathfrak{g} \), let \( \xi^{\text{ad}} \) denote the vector field \( \xi^l - \xi^r \) on the group \( G \). Put \( r_\pm = \frac{1}{2}(r_{12} \pm r_{21}) \) to be the symmetric and skew symmetric parts of \( r \). The Semenov-Tian-Shansky (STS) bivector field

\[
 r^{\text{ad,ad}} + (r^{r,l} - r^{l,r})
\]

(2.1)

on \( G \) is a Poisson structure, [STS]. It makes \( G \) a Poisson-Lie manifold over the Poisson group \( G \) under the conjugation action. This bivector field is tangent to every conjugacy class making it a homogeneous Poisson-Lie manifold over \( G \), [AlM].

Quantization of the STS bracket gives rise to an algebra \( \mathbb{C}_q[G] \) satisfying reflection equation [KS], and a semi-simple conjugacy class can be quantized as a quotient of \( \mathbb{C}_q[G] \), provided certain technical conditions are fulfilled. This point of view was developed, e.g., in [M7]. In this paper, we view \( O \) as a quotient space \( G/K \), where the subgroup \( K \subset G \) is the centralizer of \( t \). We construct a local star product on sections of equivariant vector bundles on \( O = \text{Ad}_G(t) \) in the spirit of [DM].
Let us describe the restriction of the STS bracket to the class $O$ of a semi-simple element $t \in G$. The Lie algebra $\mathfrak{g}$ splits into the direct sum $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ of vector spaces, where $\mathfrak{m}$ is the $\text{Ad}_t$-invariant subspace where $\text{Ad}_t - \text{id}$ is invertible. This decomposition is orthogonal with respect to the ad-invariant form on $\mathfrak{g}$, and $\mathfrak{m}$ splits to direct sum $\mathfrak{m}^- \oplus \mathfrak{m}^+$ of mutually dual subspaces $\mathfrak{m}_{\pm} = \mathfrak{m} \cap \mathfrak{g}_{\pm}$.

The tangent space to $O$ at the point $t$ is naturally identified with $\mathfrak{m}$ via the action of $G$. Choose a basis $\{e_\mu\} \subset \mathfrak{m}$ of root vectors. We have $(e_\mu, e_\nu) = 0$ unless $\mu + \nu = 0$ and assume the normalization $(e_\mu, e_{-\mu}) = 1$. The restriction of the Poisson bivector (2.1) to tangent space at the point $t$ is the bivector

$$r_{m \wedge m} + \sum_{\mu \in \mathbb{R}^+/t} \frac{\mu(t) + 1}{\mu(t) - 1} e_\mu \otimes e_{-\mu} \in m \wedge m,$$

where the first term is the orthogonal projection of $r$ to $m \wedge m$. The second term is correctly defined since $\text{Ad}_t - \text{id}$ is invertible on $m$.

### 2.3 Definition of generalized parabolic Verma modules.

In this section we introduce the main object of our study: a class of $U_q(\mathfrak{g})$-modules that generalize parabolic Verma modules. We postpone a detailed study of their properties to Section 5 because we need a certain machinery that we develop in Sections 3 and 4.

Every root $\alpha$ is a (multiplicative) character on $T$ returning the eigenvalue of the operator $\text{Ad}_t$ on the $\alpha$-root subspace in $\mathfrak{g}$. By definition of centralizer subalgebra, we have $\alpha(t) = 1$ if and only if $\alpha \in \mathbb{R}_t$. For $t \in T_q$ and $\alpha \in \mathbb{R}_q^+$, the value $\alpha(t) \neq 1$ is a complex root of unity.

Let $\kappa \in \mathfrak{h}^*$ designate the half-sum of positive roots of $\mathfrak{k}$.

**Definition 2.1.** We call $\lambda \in \mathfrak{h}^*$ a base weight associated with $t \in T$ if

$$q^{(\lambda + \rho, \alpha^\vee)} = \pm \sqrt{\alpha(t)}q^{(\kappa, \alpha^\vee)}, \quad \forall \alpha \in \Pi_\mathfrak{g}. \quad (2.2)$$

Here by $\sqrt{\alpha(t)}$ we mean one of two square roots of $\alpha(t) \in \mathbb{C}^\times$. Thus the point $t$ does not determine a multiplicative base weight uniquely but up to the sign in $\pm \sqrt{\alpha(t)}$ for each $\alpha \in \Pi_\mathfrak{g}$.

Recall that an assignment $q^{\alpha \alpha} \mapsto \pm 1$, $f_\alpha \mapsto 0$, $e_\alpha \mapsto 0$ for each $\alpha \in \Pi_\mathfrak{g}$ defines a one-dimensional representation of $U_q(\mathfrak{g})$. They form a group of $U_q(\mathfrak{g})$-characters that is isomorphic to $\mathbb{Z}_2^{rk \mathfrak{g}}$. This group freely acts on the set of base weights of the same $t$. Each base weight will label a category of generalized parabolic modules of our interest. The group of $U_q(\mathfrak{g})$-characters acts on those categories by isomorphisms via tensoring with the corresponding one-dimensional $U_q(\mathfrak{g})$-modules. Thus one can think that $\lambda$ has been fixed for each $t$ in what follows.
By $\Lambda_{t}$ we denote the weight lattice of the semi-simple part of $\mathfrak{k}$. Since $R_{t} \subseteq R_{\mathfrak{g}} \subseteq \mathfrak{h}^{*}$ and the canonical form on $\mathfrak{h}^{*}$ is non-degenerate, we consider $\Lambda_{t}$ as a subset in $\mathfrak{h}^{*}$. A base weight $\lambda$ generates an affine shift $\lambda + \Lambda_{t}^{+} \subseteq \mathfrak{h}^{*}$ of the semi-lattice $\Lambda_{t}^{+}$ of $\mathfrak{t}$-dominant weights. Elements from $\lambda + \Lambda_{t}^{+}$ will be highest weights of the modules of our concern. The set $\Lambda_{t}^{+} \cap \Lambda_{\mathfrak{g}}$ labels irreducible equivariant vector bundles on $O$, thanks to the Frobenius reciprocity.

For each $\xi \in \Lambda_{t}^{+}$ consider a character of the algebra $U_{q}(\mathfrak{h}) \simeq \mathbb{C}[T]$ by the assignment

$$q^{h_{\alpha}} \rightarrow q^{(\lambda + \xi, \alpha)} = \pm \sqrt{\alpha(t)q^{2(\kappa - \rho + \xi, \alpha)}}, \forall \alpha \in \Pi_{\mathfrak{g}},$$

where the signs have been fixed with the choice of $\lambda$. Note with care that we use exponential presentation for the Cartan generators for computational convenience. It means that $q^{(\lambda + \xi, \alpha)}$ is polynomial in $q$. The weight $\lambda + \xi$ satisfies a Kac-Kazhdan condition

$$[(\lambda + \xi + \rho, \alpha^\vee) - m_{\alpha}]q_{\alpha} = 0, \quad m_{\alpha} = (\xi, \alpha^\vee) + 1, \forall \alpha \in \Pi_{t}.$$

Therefore the Verma module $\tilde{M}_{\lambda + \xi}$ of highest weight $\lambda + \xi$ has Verma submodules of highest weights $\lambda + \xi - m_{\alpha} \alpha$ for each $\alpha \in \Pi_{t}$ [DCK].

**Definition 2.2.** The quotient of $\tilde{M}_{\lambda + \xi}$ by $\sum_{\alpha \in \Pi_{t}} \tilde{M}_{\lambda + \xi - m_{\alpha} \alpha}$ is called generalized parabolic Verma module and denoted by $M_{\lambda, \xi}$.

Note that if $\mathfrak{t}$ is a Levi subalgebra in $\mathfrak{g}$ relative to the fixed triangular decomposition, then $M_{\lambda, \xi}$ is a parabolic Verma module induced from $\mathbb{C}_{\lambda} \otimes X_{\xi}$, where $X_{\xi}$ is the finite dimensional $U_{q}(\mathfrak{t})$-module of highest weight $\xi$ and $\mathbb{C}_{\lambda}$ is the one dimensional $U_{q}(\mathfrak{t})$-module of weight $\lambda$. The parabolic case is well studied, and the most interesting situation is when $\Pi_{t} \not\subseteq \Pi_{\mathfrak{g}}$.

In the special case of $\xi = \mathbf{0}$, we denote $M_{\lambda} = M_{\lambda, \mathbf{0}}$ and call it base module. We expect that $\operatorname{ch} M_{\lambda}$ equals the character of the polynomial algebra $\mathbb{C}[\mathfrak{g}+/\mathfrak{k}-]$ up to the factor $e^{\lambda}$. Upon identification $\mathfrak{h}^{*} \simeq \mathfrak{h}$ via the inner product on $\mathfrak{h}^{*}$, one can think of

$$q^{2\lambda} = tq^{2\kappa - 2\rho} \in T$$

as a quantization of the initial point $t$.

The set of eigenvalues $\{\alpha(t)\}_{\alpha \in \mathbb{R}} \cup \{1\}$ of the operator $\operatorname{Ad}_{t} \in \operatorname{End}(\mathfrak{g})$ is an invariant of the class $O \supseteq t$. We call it spectrum of the class/point $t$. Introduce a notation

$$\sqrt{\mathfrak{t}} = \{q \in \mathbb{C}^{\times} | q_{\alpha}^{m} = \alpha(t), \forall m \in \mathbb{Z}, \alpha \in \mathbb{R}^{+}\}.$$ 

This set comprises roots of units and roots of $(\alpha(t))^{\frac{2}{\alpha_{\mathfrak{g}/\mathfrak{k}}}}$ for all $\alpha \in \mathbb{R}^{+}_{\mathfrak{g}/\mathfrak{k}}$. Clearly $\sqrt{\mathfrak{t}}$ depends only on the class of $t$. It will play a role of the exceptional set for the deformation parameter where the properties of modules participating in quantization of the class $O \supseteq t$ may be violated.
2.4 Extremal projector

In this section we recall the $q$-version of extremal projector, [AST, KT], which is the key instrument for this study. We start with the case of $\mathfrak{g} = \mathfrak{sl}(2)$ and normalize the inner product so that $(\alpha, \alpha) = 2$ for its only positive root $\alpha$. Set $e = e_\alpha$, $f = f_\alpha$, and $q^h = q^{\alpha h}$ to be the standard generators of $U_q(\mathfrak{g})$. Extend $U_q(\mathfrak{g})$ to $\hat{U}_q(\mathfrak{g})$ by including infinite sums of elements from $\mathbb{C}[f][e]$ of same weights with coefficients in the field of fractions $\mathbb{C}(q^{\pm h})$. Similar extension works for general semi-simple $\mathfrak{g}$ resulting in an associative algebra $\hat{U}_q(\mathfrak{g})$, see e.g. [KT].

Define $p(s)$ as a rational trigonometric function of $s \in \mathbb{C}$ with values in $\hat{U}_q(\mathfrak{sl}(2))$:

$$p(s) = \sum_{k=0}^{\infty} f^k e^k \frac{(-1)^k q^{k(s-1)}}{[k]_q^! \prod_{i=1}^{k} [h + s + i]_q}.$$  

(2.6)

It is stable under the involution $\omega$.

For every module $V$ with locally nilpotent action of the generator $e$, the element $p(s)$ delivers a rational trigonometric endomorphism of every weight space. On a module of highest weight $\lambda$, it acts by

$$p(s)v = c \prod_{k=1}^{t} \frac{[s - k]_q}{[s + \eta(h) + k]_q} v,$$

(2.7)

where $v$ is a vector of weight $\eta = \lambda - l\alpha$ and $c = q^{l\eta(h)-(l+1)} \neq 0$.

Consider the truncated operator

$$p_m(s) = \sum_{k=0}^{m} f^k e^k \frac{(-1)^k q^{k(s-1)}}{[k]_q^! \prod_{i=1}^{k} [h + s + i]_q}.$$

Lemma 2.3. Suppose that $V$ is a $U_q(\mathfrak{g})$-module and a weight subspace $V[\mu]$ is killed by $e^{m+1}$. If

$$[(\mu, \alpha) + i]_q \neq 0, \quad \text{for} \quad i = 0, \ldots, m + 1,$$

then $p_m(1)V[\mu] \subset \ker(e)$.

Proof. The operator $p_m(s)$ satisfies the relation

$$ep_m(s) = q^{-h-2} \frac{[s - 1]_q}{[h + s - 1]_q} p_{m-1}(s - 1)e + f^m e^{m+1} \frac{(-1)^m q^{m(s-1)}}{[m]_q^! \prod_{i=1}^{m} [h + s + i]_q}.$$  

(2.8)

which implies the statement. \hfill \square
For general $\mathfrak{g}$ fix a normal order on $R^+$ and consider an embedding \( \iota_a : \hat{U}_q(\mathfrak{sl}(2)) \to \hat{U}_q(\mathfrak{g}) \) defined by the Lusztig pair of root vectors \( f_\alpha, e_\alpha \) for every positive root \( \alpha \in R^+ \). Let \( p_\alpha(s) \) denote the image of \( p(s) \) in \( \hat{U}_q(\mathfrak{g}) \) under \( \iota_\alpha \). Put \( \zeta_i = 2(\zeta, \alpha_i^\vee) \alpha_i \in C \) for \( \zeta \in \mathfrak{h}^* \) and \( \alpha_i \in R^+ \) and define

\[
p_g(\zeta) = p_\alpha^{\alpha_1}(\rho_1 + \zeta_1) \cdots p_\alpha^{\alpha_n}(\rho_n + \zeta_n), \quad n = \#R^+,
\]

assuming the product ordered over increasing positive roots. It is independent of the normal ordering and turns to the extremal projector \( p_g \) at \( \zeta = 0 \), which is the only element of zero weight from \( 1 + \mathfrak{g}^- \hat{U}_q(\mathfrak{g})_+ \) satisfying

\[
p_g^2 = p_g, \quad e_\alpha p_g = 0 = f_\alpha p_g, \quad \forall \alpha \in \Pi.
\]

Uniqueness implies that \( p_g \) is \( \omega \)-invariant.

**Proposition 2.4.** For all \( \zeta \in \mathfrak{h}^* \), the operator \( p_g(\zeta) \) is \( \omega \)-invariant.

**Proof.** It is sufficient to prove that \( p_g(\zeta) \) is \( \omega \)-invariant as an operator on every finite dimensional \( U_q(\mathfrak{g}) \)-module \( V \), for generic \( \zeta \). Choose \( \zeta \) such that for all \( \mu \in \Lambda(V) \) and all \( \alpha \in R_+^q \), \( (\zeta + \mu, \alpha^\vee) \not\in -N \). Let \( Z \) be the Verma module of highest weight \( \zeta \). The projector \( p_g \) is well defined as a linear map from \( V \otimes 1_Z \) to the space of \( U_q(\mathfrak{g}_+) \)-invariants in \( V \otimes \tilde{M}_\zeta \). Then for all \( v, w \in V \) the matrix element \( (p_g(\zeta)w, v) \) equals

\[
(p_g(\zeta)w, v) = (p_g(w \otimes 1_Z, v \otimes 1_Z) = (w \otimes 1_Z, p_g(v \otimes 1_Z)) = (w, (p_g(\zeta)b),
\]

as required. The left and right equalities are due to \([M2]\), Proposition 3.1. The middle equality employs \( \omega \)-invariance of the extremal projector. \( \square \)

### 3 Tensor product of highest weight modules

A key issue arising in equivariant quantization of semi-simple conjugacy classes is semi-simplicity of certain tensor product modules. This exposition is utilizing a complete reducibility criterion for tensor products of irreducible modules of highest weight found in \([M1]\). We also modify it to milder restrictions, as a sufficient condition for the tensor product to be a sum of submodules of highest weight, relaxing irreducibility of one tensor factor. Let us remind the finding of \([M1]\) first.

Recall that a module of highest weight \( Z \) with highest vector \( 1_Z \) has a unique contravariant form such that \( (1_Z, 1_Z) = 1 \). The module is irreducible if and only if its contravariant form is non-degenerate. We denote by \( \epsilon_Z \) a linear map \( Z \to \mathbb{C} \) acting by \( \epsilon_Z(z) = (z, 1_Z) \) for all \( z \in Z \).
Tensor product of highest weight modules $V \otimes Z$ is equipped with a canonical contravariant form that is the product of contravariant forms on the tensor factors. Regard $Z$ as a cyclic $U_q(\mathfrak{g}_-)$-module generated by the highest vector $1_Z$ and let $J^- \subset U_q(\mathfrak{g}_-)$ denote a $U_q(\mathfrak{h})$-graded finitely generated left ideal lying in the annihilator of $1_Z$. If $J^-$ exhausts all of the annihilator, then we have an isomorphism $Z \simeq U_q(\mathfrak{g}_-)/J^-$. Similarly we introduce a left ideal $J^+ = \sigma(J^-) \subset U_q(\mathfrak{g}_+)$. It kills the lowest vector in the opposite module $Z'$ of lowest weight that is negative highest weight of $Z$.

Denote by $V^{J_+} \subset V$ the kernel of $J^+$. It is the annihilator of the vector space $\omega(J^+)V = \gamma^{-1}(J^-)V$ (and vice versa thanks to finite dimensionality of weight subspaces) with respect to the contravariant form. If $Z$ is irreducible, then $Z^* \simeq Z'$. In that case, if $J^+$ is the annihilator of the lowest vector in $Z'$, then $V^{J_+} \simeq \text{Hom}_{U_q(\mathfrak{g}_+)}(Z^*,V)$.

Now suppose that $Z$ is irreducible and $J^-$ is the entire annihilator of $1_Z$. Let $(V \otimes Z)^+$ denote the span of extremal vectors in $V \otimes Z$ (the subspace of $U_q(\mathfrak{g}_+)$-invariants). There is a linear isomorphism $\delta_V: V^{J^+} \rightarrow (V \otimes Z)^+$ that is the inverse to $\text{id} \otimes \epsilon_Z$ restricted to $(V \otimes Z)^+$. The pullback of the canonical form from $(V \otimes Z)^+$ to $V^{J^+}$ via $\delta_V$ defines a linear map $\theta: V^{J^+} \mapsto V/\omega(J^+)V$ (the space of coinvariants of the right ideal $\omega(J^+)$).

**Theorem 3.1 ([M1]).** Let $V$ and $Z$ be irreducible $U_q(\mathfrak{g})$-modules of highest weight. Then the following assertions are equivalent:

1. $V \otimes Z$ is completely reducible,
2. $\theta$ is bijective,
3. all submodules of highest weight in $V \otimes Z$ are irreducible,
4. $V \otimes Z$ is the sum of submodules of highest weight.

Relaxing the irreducibility assumption on $Z$ we are looking for a sufficient condition for $V \otimes Z$ to be a sum of submodules of highest weight. We would like to mimic the above criterion in a situation when we do not know the annihilator of the highest vector of $Z$ but only a "part" of it. The new input ingredient that compensates this deficit of information is the extremal projector and its relation with the extremal twist [M2].

For modules whose weights $\mu$ are in $-\Gamma_+ + \nu$ for some $\nu \in \mathfrak{h}^*$ (e.g. modules of highest weight and their tensor products), we define height of $\mu$ as the number of simple roots in $\nu - \mu \in \Gamma_+$. Height of a weight vector is defined as the height of its weight. If $V$ is equipped with a contravariant form, then extremal vectors of different heights and the modules they generate are orthogonal to
each other. For a module $V$ equipped with height function let $V_k$ denote its submodule generated by vectors of height $\leq k$. It is known that $(V \otimes Z)_k$ is generated by tensors of height $\leq k$ from $V \otimes 1_Z$, [M1], Corollary 5.2.

We still assume that $V$ is irreducible but we do not require the left ideal $J^- \subset U_q(\mathfrak{g}_-)$ be the entire annihilator of $1_Z$. We define $J^+ = \sigma(J^-) \subset U_q(\mathfrak{g}_+)$ as before.

Suppose that $V^{J^+}$ is in the range of id $\otimes \epsilon_Z$ restricted to $(V \otimes Z)^+$ and define a $U_q(\mathfrak{h})$-affine (preserving weights up to a constant summand) section $\delta_V: V^{J^+} \to (V \otimes Z)^+$ of id $\otimes \epsilon_Z$. Consider the pull-back to $V$ of the canonical form via the map $\delta_V$ and define the extremal twist $\theta: V^{J^+} \to V/\omega(J^+)V$ via ($\theta(v), w) = (\delta_V(v), \delta_V(w))$ for all $v, w \in V^{J^+}$, as before. Clearly $\theta$ commutes with the action of $U_q(\mathfrak{h})$.

The map $\delta_V$ preserves height because it shifts weights by the highest weight of $Z$.

**Proposition 3.2.** Suppose that the $\delta_V$-pullback of the canonical form is non-degenerate on $V^{J^+}$. Then $V \otimes Z$ is a sum of submodules of highest weights whose highest vectors are from $\delta_V(V^{J^+})$.

**Proof.** Denote by $V \boxtimes Z$ the sum of submodules generated by extremal vectors from $\delta_V(V^{J^+})$. Clearly $(V \boxtimes Z)_k \subset (V \otimes Z)_k$. The assertion will be proved if we demonstrate the reverse inclusion.

Suppose we proved the required inclusion for $k \geq 0$ (it is obviously true for $k = 0$). Pick up $v \in \omega(J^+)V$ of height $k + 1$ and present it as $v = \sum_i \omega(e_i)v_i$, where $e_i \in J^+$ and $v_i \in V$ are some vectors of height $\leq k$. By Lemma 5.1 from [M1], $v \otimes 1_Z = \sum_i v_i \otimes \sigma(e_i)1_Z = 0$ modulo $(V \otimes Z)_k$, that is $v \otimes 1_Z \in (V \boxtimes Z)_k$ by the induction assumption.

Furthermore, if $v' \in V^{J^+}$ of height $k + 1$, then there is $v \in V$ of height $k + 1$ such that $\delta_V(v') = v \otimes 1_Z$ modulo $(V \otimes Z)_k$ by [M1], Lemma 5.1. The vector $\theta(v')$ is the projection of $v$ along $\omega(J^+)V$ because all $\delta_V(w')$ with $ht(w') = k + 1$ are orthogonal to extremal vectors of smaller heights and therefore to all $(V \otimes Z)_k$ by the induction assumption:

$$(\theta(v'), w') = (\delta_V(v'), \delta_V(w')) = (v \otimes 1_Z, \delta_V(w')) = (v, w').$$

By the hypothesis, the map $\theta: V^{J^+} \to V/\omega(J^+)V$ is surjective (and preserves heights because it preserves weights). Then each tensor $v \otimes 1_Z$ from $V \otimes 1_Z$ of height $k + 1$ can be presented as $\delta_V(v')$ modulo $(V \otimes Z)_k$ plus a tensor from $\omega(J^+)V \otimes 1_Z$ of height $k + 1$, which is also in $(V \otimes Z)_k$ as already proved. Therefore the tensor $v \otimes 1_Z$ is in $(V \boxtimes Z)_{k+1}$ for all $v$ of height $k + 1$, as required. This implies $(V \boxtimes Z)_{k+1} \supset (V \otimes Z)_{k+1}$. Induction on $k$ completes the proof. 

We will construct the map $\delta_V$ with the help of extremal projector provided it can be regularized on an appropriate subspace, cf. [M2]. Let $\widehat{V^{J^+}} \subset V$ denote the sum of weight subspaces in $V$ whose weights are in $\Lambda(V^{J^+})$. Let $\zeta$ denote the highest weight of $Z$.
Lemma 3.3. Suppose that projector \( p_\theta \) is a regular map \( \hat{V}^{J^+} \otimes 1_Z \rightarrow (V \otimes Z)^+ \). Then \( V^{J^+} \) contains the range \( p_\theta(\zeta) \hat{V}^{J^+} \), and the subspace \( \hat{V}^{J^+} \cap \omega(J^+)V \) is in its kernel.

\[ \text{Proof.} \] It is proved in [M2], Proposition 3.1, that the operator \( p_\theta(\zeta) \) is well defined on \( \hat{V}^{J^+} \) and \( p_\theta(\zeta)v = (\text{id} \otimes \epsilon_Z) (p_\theta(v \otimes 1_Z)) \) for all \( v \in \hat{V}^{J^+} \). Then for all \( w \in \hat{V}^{J^+} \) and \( e \in J^+ \):

\[ (w, ep_\theta(\zeta)v) = (w \otimes 1_Z, (e \otimes 1)p_\theta(v \otimes 1_Z)) = (w \otimes 1_Z, (1 \otimes \gamma^{-1}(e))p_\theta(v \otimes 1_Z) = 0, \]

because \( (1_Z, \gamma^{-1}(e)z) = (\sigma(e)1_Z, z) = 0 \) for all \( z \in Z \). Therefore the range of \( p_\theta(\zeta) \) restricted to \( \hat{V}^{J^+} \) is in \( V^{J^+} \). That \( V^{J^+} \cap \omega(J^+)V \) is in \( \ker p_\theta(\zeta) \) follows from \( \omega \)-invariance of \( p_\theta(\zeta) \), cf. Proposition 2.4.

Under the assumption of Lemma 3.3, we can think that \( p_\theta(\zeta) \) is defined on entire \( V \) taking zero value on \( V[\mu] \) if \( \mu \notin \Lambda(V^{J^+}) \). Such weight subspaces are in \( \omega(J^+)V \), and so defined operator is \( \omega \)-invariant. The following result is analogous to Theorem 3.2 from [M2].

Proposition 3.4. Under the assumptions of Lemma 3.3, suppose that \( p_\theta(\zeta) \) is surjective onto \( V^{J^+} \). Then there is a \( U_q(\mathfrak{g}) \)-affine section \( \delta_V \) of \( \text{id} \otimes \epsilon_Z \) such that \( \theta \) is invertible and its inverse \( V/\omega(J^+)V \rightarrow V^{J^+} \) is the map induced by \( p_\theta(\zeta) \).

\[ \text{Proof.} \] Define \( \delta_V \) as a composition \( V^{J^+} \rightarrow V \rightarrow (V \otimes Z)^+ \), where the left arrow is any \( U_q(\mathfrak{g}) \)-linear section of the map \( p_\theta(\zeta) : \hat{V}^{J^+} \rightarrow V^{J^+} \), and the right map is \( v \mapsto p_\theta(v \otimes 1_Z) \) for \( v \in \hat{V}^{J^+} \). By [M2], Proposition 3.1, it is indeed a section of \( \text{id} \otimes \epsilon_Z \).

By definition of \( \theta \), the matrix element \( (\delta \circ p_\theta(\zeta))(v), (\delta \circ p_\theta(\zeta))(w) \) equals \( (\theta \circ p_\theta(\zeta)(v), p_\theta(\zeta)(w)) \) for all \( v, w \in V^{J^+} \). On the other hand, it is equal to

\[ (p_\theta(v \otimes 1_Z), p_\theta(w \otimes 1_Z)) = (v \otimes 1_Z, \omega(p_\theta) p_\theta(w \otimes 1_Z)) = (v, (\text{id} \otimes \epsilon_Z)(p_\theta(w \otimes 1_Z))) = (v, p_\theta(\zeta)(w)). \]

Since the image of \( p_\theta(\zeta) \) is \( V^{J^+} \), one arrives at \( \theta \circ p_\theta(\zeta) = \text{id} \) on \( V/\omega(J^+)V \).

Corollary 3.5. Suppose that assumptions of Proposition 3.4 are fulfilled. Then \( V \otimes Z \) is a sum of submodules of highest weight. If \( Z \) is irreducible, then \( V \otimes Z \) is completely reducible and \( V^{J^+} = \text{Hom}_{U_q(\mathfrak{g}_+)}(Z', V) \).

\[ \text{Proof.} \] The map \( \delta_V \) constructed from the extremal projector and the operator \( \theta \) it defines fulfil conditions of Proposition 3.2, hence the first part of the statement. The second assertion holds by virtue of Theorem 3.1.
Remark that the shifted extremal projector was considered as a form on coinvariants in the special case of Verma modules in [KN]. Corollary 3.5 can be viewed as a generalization of Proposition 2.3 in [KN].

We denote by \( p_{\mathfrak{g}}^{-1}(\zeta) \) an arbitrary \( U_q(\mathfrak{h}) \)-linear section of the map \( p_{\mathfrak{g}}(\zeta) : \hat{V}^{J^+} \to V^{J^+} \). Note that, for irreducible \( Z \), extremal vectors in \((V \otimes Z)^+\) can be alternatively constructed via the extremal projector or via a lift \( S \in U_q(\mathfrak{g}_+) \otimes U_q(\mathfrak{g}_-) \) of the inverse invariant pairing \( Z \otimes Z' \to \mathbb{C} \), see the next section. The relation is given by the formula

\[
S(v \otimes 1_Z) = p_{\mathfrak{g}}(p_{\mathfrak{g}}^{-1}(\zeta)v \otimes 1_Z)
\]

(3.10)

for each weight vector \( v \in V^{J^+} \), [M2]. We will use this relation for construction of star product on conjugacy classes in Section 6.

It is easy to calculate \( \theta_{V,Z} \) if the weights of \( V \) are multiplicity free. Suppose that \( \zeta \) is a highest weight of module \( Z \). Then \( \theta_{V,Z} \) acts on \( V[\mu] \) as multiplication by a scalar, that is proportional to \( \prod_{\alpha \in R^+} \theta_{\alpha}^{\mu} \), where

\[
\theta_{\alpha}^{\mu} = \frac{l_{\mu,\alpha} [(\zeta + \rho + \mu, \alpha^\vee) + k]_{q_{\alpha}}}{[(\zeta + \rho, \alpha^\vee) - k]_{q_{\alpha}}}
\]

(3.11)

Here \( l_{\mu,\alpha} \) stands for the maximal integer \( k \) such that \( e^k_{\alpha}V^+[\mu] \neq \{0\} \).

## 4 Quasi-classical limit of Shapovalov elements

Recall that a generalized parabolic Verma module \( M_{\lambda,\xi} \), where \( \lambda \) is a base weight for \( t \in T \) and \( \xi \in \Lambda^+_t \) is a dominant weight for its centralizer subalgebra \( \mathfrak{k} \), is defined as a quotient of the Verma module \( \tilde{M}_{\lambda+\xi} \). Consider the extremal vector (defined up to a scalar factor) in \( \tilde{M}_{\lambda+\xi} \) of weight \( \lambda + \xi - m_\alpha \alpha \), where \( m_\alpha = (\xi, \alpha^\vee) + 1 \). Let \( \phi_{m_\alpha} \in U_q(\mathfrak{g}_-) \) denote its lift under the linear isomorphism \( U_q(\mathfrak{g}_-) \to \tilde{M}_{\lambda+\xi} \). It is called Shapovalov element corresponding to the positive root \( \alpha \) and a positive integer \( m_\alpha \). With fixed \( \lambda \) and \( \xi \), it is a rational \( U_q(\mathfrak{g}_-) \)-valued function of \( q \), cf. a remark after (2.3). Since \( \phi_{m_\alpha} \) is defined up to a scalar multiple, we assume that it is regular in a neighbourhood of 1 and does not vanish at \( q = 1 \).

A key assumption about the initial point \( t \in T \) that facilitates our approach to quantization is that it features certain behaviour in the classical limit in the following sense.

**Definition 4.1.** We call the point \( t \in T \) quantizable if \( \lim_{q \to 1} \phi_{m_\alpha} = f_\alpha^m \) for all \( m \in \mathbb{N} \) and all \( \alpha \in \Pi_t \), where \( f_\alpha \in \mathfrak{k}_- \) is the classical root vector of root \(-\alpha\).
Let \( \mathfrak{l} \) denote the maximal Lie subalgebra in \( \mathfrak{k} \) that is Levi in \( \mathfrak{g} \), so that \( \Pi_{\mathfrak{l}} = \Pi_{\mathfrak{k}} \cap \Pi_{\mathfrak{g}} \). It is clear that if \( \Pi_{\mathfrak{k}} = \Pi_{\mathfrak{l}} \) then \( M_{\lambda, \xi} \) is just the parabolic Verma module, and the point \( t \) is quantizable. Thus this property is questionable only when the set \( \Pi_{\mathfrak{k}/\mathfrak{l}} = \Pi_{\mathfrak{k}} \setminus \Pi_{\mathfrak{l}} \) is not empty.

We conjecture that all \( t \) are quantizable for all simple \( G \) and prove that for non-exceptional \( G \) in this section. We do it by a direct analysis of Shapovalov elements using their explicit construction from the inverse Shapovalov form (as certain rescaled Shapovalov matrix elements).

## 4.1 Inverse Shapovalov form and its matrix elements

In this section we give an explicit construction of Shapovalov elements relative to generalized parabolic Verma modules, following [M8, M6].

For each weight \( \mu \in \Gamma_+ \) put

\[
\eta_{\mu} = h_\mu + (\mu, \rho) - \frac{1}{2}(\mu, \mu) \in \mathfrak{h} \oplus \mathbb{C}.
\] (4.12)

Regard it as an affine function on \( \mathfrak{h}^* \) by the assignment \( \eta_{\mu} : \zeta \mapsto (\mu, \zeta + \rho) - \frac{1}{2}(\mu, \mu), \zeta \in \mathfrak{h}^* \).

Let \( \{ h_i \}_{i=1}^{rk_{\mathfrak{g}}} \in \mathfrak{h} \) be an orthonormal basis. The element \( q^{\sum_i h_i \otimes h_i} \) belongs to a completion of \( \mathcal{U}_q(\mathfrak{h}) \otimes \mathcal{U}_q(\mathfrak{h}) \) in the \( \mathfrak{h} = \ln q \)-adic topology. Choose an \( R \)-matrix of \( \mathcal{U}_q(\mathfrak{g}) \) such that \( \tilde{R} = q^{-\sum_i h_i \otimes h_i} \mathcal{R} \in \mathcal{U}_q(\mathfrak{g}_+) \otimes \mathcal{U}_q(\mathfrak{g}_-) \) and set \( \mathcal{C} = \frac{1}{q-q^{-1}}(\tilde{R} - 1 \otimes 1) \). Sending the left tensor leg of \( \mathcal{C} \) to a representation produces a matrix \( \mathcal{C} \) with entries in \( \mathcal{U}_q(\mathfrak{g}_-) \) which will be used for construction of Shapovalov elements.

Note that, in the classical limit, the tensor \( \mathcal{C} \) tends to \( \sum_{\alpha \in \mathbb{R}^+} e_\alpha \otimes f_\alpha \), where \( e_\alpha, f_\alpha \) are classical root vectors normalized to \( (e_\alpha, f_\alpha) = 1 \) by the ad-invariant form on \( \mathfrak{g} \). This fact will be used in the proof of Proposition 4.11 below.

Let \( \tilde{M}_\zeta \) be an irreducible Verma module of highest weight \( \zeta \) and \( \mathcal{S} \in \mathcal{U}_q(\mathfrak{g}_+) \otimes \mathcal{U}_q(\mathfrak{g}_-) \) the lifted inverse of the invariant pairing \( \tilde{M}_\zeta \otimes \tilde{M}_\zeta^\prime \to \mathbb{C} \). Pick up \( V \in \text{Fin}_q(\mathfrak{g}) \) and denote by \( S \) the image of \( \mathcal{S} \) in \( \text{End}(V) \otimes \mathcal{U}_q(\mathfrak{g}_-) \).

The module \( \tilde{M}_\zeta \) becomes reducible at certain \( \zeta \), which results in poles of \( \mathcal{S} \) (they may not appear in \( S \) for a particular \( V \)). So we can relax the assumption that \( \tilde{M}_\zeta \) is irreducible and work with \( S \) independently regarding its entries as rational trigonometric \( \mathcal{U}_q(\mathfrak{g}_-) \)-valued functions of \( \zeta \).

Every pair of vectors \( v, w \in V \) define a matrix element \( (v, S_1 w)S_2 \in \hat{U}_q(\mathfrak{g}_-) \) (Sweedler notation for \( S \)) with respect to the contravariant form on \( V \). An explicit expression for the matrix entries \( s_{ij} = (v_i, S_1 v_j)S_2 \) of \( S \) in an orthonormal weight basis \( \{ v_i \}_{i \in I} \subset V \) can be formulated using the language of Hasse diagrams associated with partially ordered sets. We introduce such an order on \( \{ v_i \}_{i \in I} \) (equivalently on \( I \)) by writing \( v_i \succ v_j \) if \( v_i - v_j \in \Gamma_+ \setminus \{0\}, i = 1, \ldots, k \). The matrix
$S$ is triangular: $s_{ii} = 1$ and $s_{ij} = 0$ if $i \neq j$. The entry $s_{ij}$ is a rational trigonometric function $\mathfrak{h}^* \to U_q(\mathfrak{g}_-)$, its value carries weight $\nu_j - \nu_i \in -\Gamma_+$.

Set $\tilde{s}_{ab} = -(v_b, S_1 v_a) S_2 [\eta_{\nu_b - \nu_a}] q^{-\nu_b - \nu_a} \in U_q(\mathfrak{b}_-)$ for each $v_b \succ v_a$. An explicit formula for $\tilde{s}_{ab}$ in terms of the image $C = \sum_{ij} e_{ij} \otimes c_{ij} \in \text{End}(V) \otimes U_q(\mathfrak{g}_-)$ of the element $C$ can be extracted from [M6]:

$$
\tilde{s}_{ab} = c_{ba} + \sum_{k \geq 1} \sum_{v_b \succ v_{b'} \succ \ldots \succ v_1 \succ v_a} c_{kk} \cdots c_{1a} \frac{(-1)^k q^{\eta_{\mu_k}} \cdots q^{\eta_{\mu_1}}}{[\eta_{\mu_k}]_q \cdots [\eta_{\mu_1}]_q},
$$

where $\mu_l = \nu_l - \nu_a \in \Gamma_+, l = 1, \ldots, k$. One can see that every node $v_l$ between $v_b$ and $v_a$ contributes $-\frac{q^{\eta_{\mu_l}}}{[\eta_{\mu_l}]_q} \in \hat{U}_q(\mathfrak{h})$ to the products which we call node factor. These node factors may produce singularities when evaluated at a particular weight.

Let $\beta \in \Pi$ be a composite positive root and $\Pi_\beta \subset \Pi$ be the set of simple roots entering the expansion of $\beta$ over the basis $\Pi$ with positive coefficients. Recall that a simple Lie subalgebra, $\mathfrak{g}(\beta) \subset \mathfrak{g}$ generated by $e_\alpha, f_\alpha$ with $\alpha \in \Pi_\beta$ is called support of $\beta$. Its universal enveloping algebra is quantized as a Hopf subalgebra in $U_q(\mathfrak{g})$.

**Definition 4.2.** Let $V$ be a finite dimensional $U_q(\mathfrak{g})$-module and $v_a, v_b \in V$ a pair of vectors of weights $\nu_a, \nu_b$, respectively. We call a triple $(V, v_a, v_b)$ admissible $\beta$-representation if $e_\alpha v_b = 0$ for all $\alpha \in \Pi_\beta$, $v_a = f_\beta v_b$, and $(\beta^\vee, \nu_b) = 1$.

In other words, a triple $(V, v_a, v_b)$ is admissible if the vector $v_b \in V$ is extremal for $U_q(\mathfrak{g}(\beta))$ and generates a submodule $\simeq \mathbb{C}^2$ of the subalgebra $U_q(\mathfrak{g}^\beta)$. Assuming a triple $(V, v_a, v_b)$ admissible $\beta$-representation we put $\phi_\beta(\zeta) = \tilde{s}_{ba}(\zeta)$.

**Proposition 4.3** ([M8]). Suppose that $(V, v_a, v_b)$ is an admissible $\beta$-representation. For $\zeta \in \mathfrak{h}^*$ and $m \in \mathbb{N}$ set $\zeta_0 = \zeta, \zeta_k = \zeta_{k-1} + \nu_a, k = 1, \ldots, m$. Then the product

$$
\phi_{m\beta}(\zeta) = \phi_\beta(\zeta_{m-1}) \cdots \phi_\beta(\zeta_0) \in U_q(\mathfrak{g}_-)
$$

is a Shapovalov element for generic $\zeta$ satisfying $q^{2(\zeta + \rho, \beta)} = q^{m(\beta, \beta)}$.

Next we point out admissible representations for all composite roots of non-exceptional Lie algebras. Their simple roots are written below in terms of an orthonormal system $\{\varepsilon_k\}_{k=1}^n$:

$$
\Pi_{\mathfrak{sl}(n)} = \{\varepsilon_1 - \varepsilon_2, \ldots, \varepsilon_{n-1} - \varepsilon_n\}, \quad \Pi_{\mathfrak{sp}(2n)} = \{\varepsilon_1 - \varepsilon_2, \ldots, \varepsilon_{n-1} - \varepsilon_n, 2\varepsilon_n\},
$$

$$
\Pi_{\mathfrak{so}(2n+1)} = \{\varepsilon_1 - \varepsilon_2, \ldots, \varepsilon_{n-1} - \varepsilon_n, \varepsilon_n\}, \quad \Pi_{\mathfrak{so}(2n)} = \{\varepsilon_1 - \varepsilon_2, \ldots, \varepsilon_{n-1} - \varepsilon_n, \varepsilon_n + \varepsilon_n\}.
$$

We will enumerate them from left to right.
Proposition 4.4. For each composite root \( \beta \in \mathbb{R}_0^+ \) there is an admissible \( \beta \)-representation.

Proof. In all cases except for short roots of \( \mathfrak{so}(2n+1) \) we take for \( V \) the natural \( \mathfrak{g} \)-module of minimal dimension. Then

\[
\begin{align*}
\mathfrak{g} &= \mathfrak{sl}(n): v_b = v_{\varepsilon_i}, v_a = v_{\varepsilon_j}, \beta = \varepsilon_{\varepsilon_i} - \varepsilon_{\varepsilon_j}, i < j. \\
\mathfrak{g} &= \mathfrak{sp}(2n), \mathfrak{so}(2n), \mathfrak{so}(2n+1): v_b = v_{\varepsilon_i}, v_a = v_{\pm \varepsilon_j} \text{ for } \beta = (\varepsilon_i \mp \varepsilon_j), i < j. \\
\mathfrak{g} &= \mathfrak{sp}(2n): v_b = v_{\varepsilon_i}, v_a = v_{-\varepsilon_i} \text{ for } \beta = 2\varepsilon_i.
\end{align*}
\]

For short roots \( \beta = \varepsilon_i \) of \( \mathfrak{so}(2n+1) \) we take for \( V \) the fundamental spin module with \( v_b = \frac{1}{2} \sum_{l=1}^n \varepsilon_l \) and \( v_a = \frac{1}{2} \sum_{l=1}^n \varepsilon_l - \varepsilon_i \). \( \square \)

In what follows we assume that the admissible triples \((V,v_a,v_b)\) are fixed as in Proposition 4.4. Observe that in all cases the dimension of weight spaces in \( V \) is 1. For each simple root \( \alpha \in \Pi \),

\[
e_\alpha \phi_{m\beta}(\zeta)_1 \psi(\zeta) = \delta_{\nu_0} - \nu_0,\alpha(2^{2(\zeta,\nu,\beta)} - q^{m(\beta,\beta)}) \psi(\zeta) \psi(\zeta) \psi(\zeta) \psi(\zeta) \psi(\zeta) \psi(\zeta),
\]

where \( \psi(\zeta) \in U(\mathfrak{g}_-)[M_0] \). The vector \( \phi_{m\beta}(\zeta)_1 \psi(\zeta) \) is extremal if \( \zeta \) satisfies \( q^{2(\zeta,\nu,\beta)} = q^{m(\beta,\beta)} \), and the elements \( \phi_{m\beta}(\zeta) \) and \( \psi(\zeta) \neq 0 \). Generically these conditions are fulfilled but we are interested in very special \( \zeta \) that is a sum of base weight \( \lambda \) and \( \xi \in \Lambda_\ell^+ \). We require that \( \phi_{m\beta}(\zeta) \) is regular in \( q \) in a neighbourhood of \( q = 1 \) for every such \( \zeta \). Moreover, the element \( \phi_{m\beta} \) should have a proper classical limit \( q \to 1 \) with fixed \( \zeta \).

Factorization of Shapovalov elements reduces the problem of regularity to the question about \( \phi_\beta(\zeta) \). Observe from (4.13) that the node factors tend to zero for generic \( \zeta \), whence \( \phi_\beta(\zeta_k) \) tends to the classical root vector \( f_\beta \). However \( \phi_\beta(\zeta_k) \) may have poles at special \( \zeta_k \), and regularized \( \phi_\beta(\zeta_k) \) may fail to tend to \( f_\beta \). In the next section we demonstrate that admissible triples from Proposition 4.4 guarantee the proper classical limit of \( \phi_\beta(\zeta_k) \) for all \( t \).

Once the triple \((V,v_a,v_b)\) has been fixed, the sequence of weights \((\zeta_k)_{k=0}^m\) from Proposition 4.3 depends only on \( \zeta = \zeta_0 \). Abusing notation we will write \( \phi_\beta^m(\zeta) = \phi_\beta(\zeta_{m-1}) \ldots \phi_\beta(\zeta_0) \) or simply \( \phi_\beta^m \) when the weight \( \zeta \) is clear from the context. This convention will unify notation with the case of \( \beta \in \Pi_0 \), when the Shapovalov element \( \phi_\beta^m \) is a true power of the Chevalley generator \( f_\beta \), which is of course independent of \( \zeta \).

4.2 Regularity of Shapovalov elements

In the previous section we presented a construction of extremal vectors in Verma modules from matrix elements of the Shapovalov form. We discussed that it apparently works for “generic” weight satisfying a particular Kac-Kazhdan condition. When it comes to a special weight from
λ + Λ_g, some node factors may get singular, and properties of regularized matrix entries are not obvious. This problem is solved in this section for all initial points and their base weights.

Fix \( t \in T \) with the centralizer \( \mathfrak{t} \), a base weight \( \lambda \), and pick up \( \beta \in \Pi_\mathfrak{t} \).

**Definition 4.5.** We call an admissible \( \beta \)-representation \((V,v_a,v_b)\) \( t \)-regular if for each \( v_c \in V \) of weight \( \nu_c \) such that \( v_b \succ v_c \succ v_a \) and all \( \zeta \in \lambda + \Lambda_g \) the \( U_q(\mathfrak{g}_-) \)-valued function \( q \mapsto s_{ca}(\zeta,q) \) is regular at \( q = 1 \) and \( s_{ca}(\zeta,1) = 0 \).

It follows from (4.13) that being regular depends only on \( t \) and not on a choice of base weight \( \lambda \) because the node factors essentially involve squared \( q^{(\lambda,\mu)} \) with \( \mu \in \Gamma \), cf. (2.3).

In particular, regularity implies that specialization at \( \zeta \) makes \( s_{ca} \) a well defined rational function of \( q \). Clearly if the root \( \beta \) is simple, then its any representation is \( t \)-regular because there is no node between \( v_b \) and \( v_a \) to violate the conditions.

We call a node \( v_c \) between \( v_b \) and \( v_a \) \( t \)-singular if \( \nu_c(t) = \nu_a(t) = \nu_b(t) \), that is, if \( \mu_c(t) = 1 \) for \( \mu_c = \nu_c - \nu_a \). Then the node factor \( -q^{\nu_{mc}}_{\nu_{nc}|q} \) in (4.13) evaluated at \( \zeta \in \lambda + \Lambda_g \) may not vanish in the classical limit \( q \to 1 \). As a consequence, the matrix element \( s_{ca} \) may not vanish as \( q \to 1 \). On the contrary, if all nodes between \( v_b \) and \( v_a \) are non-singular relative to \( t \), then the \( \beta \)-representation \((V,v_a,v_b)\) is \( t \)-regular because all node factors between the end nodes go to zero at \( q = 1 \).

A special case of regular representation of \( \beta \in \Pi_\mathfrak{t} \) is realized when both weight differences \( \nu_b - \nu_c \) and \( \nu_c - \nu_a \) are roots for all \( v_c \) between \( v_b \) and \( v_a \). We call such nodes \( v_c \) root splitting.

Then the \( \beta \)-representation is \( t \)-regular for any \( t \) for which \( \beta \in \Pi_\mathfrak{t} \). Indeed, there is no \( t \)-singular node \( v_c \) between \( v_b \) and \( v_a \) since otherwise \( \beta = (\nu_b - \nu_c) + (\nu_c - \nu_a) \) is a sum of two other roots from \( R_{\mathfrak{t}}^+ \), which is impossible.

**Proposition 4.6.** Suppose that there is a \( t \)-regular \( \beta \)-representation \((V,v_a,v_b)\) for each \( \beta \in \Pi_\mathfrak{t} \). Then \( t \) is quantizable.

**Proof.** Element \( \phi_{m\beta} \) admits a factorization

\[
\theta_{\beta,m}(\zeta) = \theta_{\beta}(\zeta_{m-1}) \ldots \theta_{\beta}(\zeta_0) \in U_q(\mathfrak{g}_-) \quad (4.15)
\]

where \( \zeta_0 = \zeta \) and \( \zeta_k = \zeta_{k-1} + \nu_a \), \( k = 1, \ldots, m \), [M8]. Therefore it is sufficient to consider the case \( m = 1 \). Up to a non-zero scalar multiplier, the summation formula (4.13) can be rewritten as

\[
\tilde{s}_{ba} = c_{ba} + \sum_{v_b \succ v_c \succ v_a} c_{bc}s_{ca}. \quad (4.16)
\]

Only the first term survives in the classical limit, and it goes to the classical root vector \( f_\beta \). \( \square \)
4.3 All points of the maximal torus are quantizable

In this section, we prove that all points from the maximal torus are quantizable for $G$ of type $A$, $B$, $C$, and $D$. Specifically, we will show that the admissible $\beta$-representations from Proposition 4.4 are $t$-regular for each $\beta \in \Pi_{t/1}$ and all $t \in T \subset G$.

A node factor in (4.13) evaluated at weight $\zeta = \lambda + \xi$, $\xi \in \Lambda_g$ reads

$$-\frac{q^{\eta(\zeta)}}{[\eta(\zeta)]_q} \propto \frac{q - q^{-1}}{\mu(t)q^{2(\xi + \eta, \mu) - \frac{1}{2}(\mu, \mu) - 1}},$$

(4.17)

If the node is not singular, i.e. $\mu(t) \neq 1$, it may have at most a finite set of poles, as a function of $q$. In the special case of $t \in T_Q$, there are no poles at all because $\mu(t)$ is a root of unity while $q$ is not. Such factors tend to zero in the classical limit. If the node is $t$-singular, then the analysis is more delicate. We will come across it when doing orthogonal $g$.

**Proposition 4.7.** All $t \in T$ are quantizable for $g = \mathfrak{sl}(n)$ or $g = \mathfrak{sp}(2n)$.

**Proof.** For each composite root, the module $V$ in the triple $(V, v_a, v_b)$ from Proposition 4.4 is the natural representation of minimal dimension. The partial ordering in $V$ is total, and the difference between any pair of distinct weights from $\Lambda(V)$ is a root. So every node between $v_b$ and $v_a$ is root splitting, and no node factor is singular. Now the proof follows because all factors (4.17) go to zero in the classical limit. \qed

4.3.1 Orthogonal $g$

There are natural $\mathfrak{sl}(n)$-subalgebras in orthogonal $g$ of rank $n$. Their composite roots have been treated in the previous section. We will consider only complementary roots below.

First suppose that $g = \mathfrak{so}(2n + 1)$ and $\beta = \varepsilon_i$ is a short root. The admissible triple is realized in the spin module $V$ with $v_b$ of weight $\frac{1}{2} (\varepsilon_1 + \ldots + \varepsilon_n)$ and $v_a = v_b - \varepsilon_i$. The Hasse sub-diagram between these points is linear:

Every node between $v_b$ and $v_a$ splits $\varepsilon_i$ to the sum of two roots $\varepsilon_i = (\varepsilon_i - \varepsilon_k) + \varepsilon_k$ for some $i < k < n$. Therefore this $\varepsilon_i$-representation is $t$-regular for all $t$.

Let us turn to the case of long roots. Set $N$ to be the dimension of the minimal fundamental representation of $g$.

**Lemma 4.8.** If $\beta = \varepsilon_i + \varepsilon_j \in \Pi_t$ and $\alpha = \varepsilon_i - \varepsilon_j \in \mathbb{R}_t^+$ for $i < j$, then $\varepsilon_i(t) = \varepsilon_j(t) = -(\pm 1)^{N+1}$.
Proof. The inclusion $\alpha, \beta \in R^+_t$ implies $\varepsilon_i(t) = \varepsilon_j(t) = \pm 1$. Now suppose that $g = so(2n + 1)$ and $\varepsilon_i(t) = \varepsilon_j(t) = 1$. Then $\varepsilon_i, \varepsilon_j \in R^+_t$ and $\beta = \varepsilon_i + \varepsilon_j$ is not simple in $R^+_t$ which is a contradiction. □

Denote $d_j = q^{\nu_{2j}} + 1 = q^{2\nu_{2j}} - 1$ if $g = so(2n + 1)$ and $d_j = q^{2\nu_{2j}} - 1$ if $g = so(2n)$. If the point $t$ underlying a base weight $\lambda$ satisfies $\varepsilon_j(t) = -(\pm 1)^{N+1}$, then $d_j(\lambda)$ tends to zero as $q \to 1$.

Now let $V$ be the fundamental $U_q(g)$-module of minimal dimension, with the set of weights $\Lambda(V) = \{\pm \varepsilon_i\}_{i=1}^n$ in the even and $\Lambda(V) = \{\pm \varepsilon_i\}_{i=1}^n \cup \{0\}$ in the odd cases.

Lemma 4.9. Suppose that $\beta = \varepsilon_i + \varepsilon_j \in \Pi_t$, with $i < j$. Then the triple $(V, v_i, v_{-j})$ is a $t$-regular $\beta$-representation.

Proof. The only node between $v_i$ and $v_{-j}$ whose Cartan factor may be $t$-singular is $v_j$, because all other nodes split $\beta$ into sum of two roots. Therefore $s_{l,-j}$ are regular at almost all $q$ (at all if $t \in T_Q$) and tend to zero as $q \to 1$ for all $l$ such that $j \succ l \succ -j$. Let us prove that for $s_{j,-j}$ assuming $\varepsilon_j(t) = -(\pm 1)^{N+1}$, as in Lemma 4.8. This matrix element has an apparent singularity in the classical limit because its denominator includes $[\eta_{2j}]_q$ divisible by $d_j$. We will show that the singularity is removable. Without loss of generality we will assume $j = 1$ (otherwise we should proceed to the quantum subgroup with roots $\alpha_1, \ldots, \alpha_n$).

Notice that $d_1(\zeta) = 0$ does not imply the $q$-version of Kac-Kazhdan condition for any root, [DCK]. Therefore we can choose an irreducible Verma module $\tilde{M}_\zeta$ of generic highest weight $\zeta$ such that $d_1(\zeta) = 0$. The extremal vector

$$-v_{-1} \otimes q^{-\eta_{2j}} [\eta_{2j}]_q 1_\zeta + \ldots + v_1 \otimes s_{1,-1} 1_\zeta \in V \otimes \tilde{M}_\zeta$$

vanishes at such $\zeta$ because the leftmost term does (otherwise its $\tilde{M}_\zeta$-components span a $U_q(g_+)$-submodule which contains an extremal vector in $\tilde{M}_\zeta$ distinct from $1_\zeta$). Therefore $s_{1,-1}$ is divisible by $d_1$ and $s_{1,-1} = \psi d_1$, where $\psi$ is regular at $d_1 = 0$ for almost all $q$ (all if $t$ is of finite order) including $q = 1$. That is also true for $s_{1,-1} = \tilde{s}_{1,-1}(q - q^{-1})/(q^{-2\eta_{2j}} - 1)$ at $\zeta \in \lambda + \Lambda_g$, thanks to Lemma 4.8. Moreover, $s_{1,-1} = (q - q^{-1})\psi d_1/(q^{-2\eta_{2j}} - 1)$ vanishes in the classical limit $q \to 1$.

Using the presentation (4.16) for all nodes $l$ between $i$ and $j$ we conclude that $(v_i, v_{-j})$ is indeed a $t$-regular representation of $\varepsilon_i + \varepsilon_j$. The element $s_{i,-j}$ is regular at all $q$ if $t$ is of finite order, because the only possible singularity in $s_{j,-j}$ is canceled by the factor $d_j$. □

Summarizing the findings of this section, we conclude that

Proposition 4.10. All $t \in T$ are quantizable for orthogonal $g$.

We conclude this section with a refinement of Proposition 4.6, which will be needed in a study of the whole collection of modules $M_{\lambda, \zeta}$ with fixed $\lambda$. 22
Proposition 4.11. Suppose that $\beta \in \Pi_{\mathfrak{k}}$ and $m \in \mathbb{N}$ and pick up a base weight $\lambda$ for $t$. Then

1. the Shapovalov element $\phi_{m\beta}(\zeta) = \check{s}_{ba}(\zeta)$ evaluated at $\zeta \in \lambda + \Lambda_{\mathfrak{t}}$ is regular as a function of $q$ at all $q \in \mathbb{C} \setminus \sqrt{t}$,

2. $\phi_{\beta}(\zeta)$ does not vanish at all $q \in \mathbb{C} \setminus \sqrt{t}$.

Proof. For $g$ of the classical type, each positive root $\beta$ contains a simple root $\alpha$ with multiplicity 1. By [M8] Proposition 4.4, one can assume that $\phi_{\beta} \in U_{q}(\mathfrak{b}-)$ is such that $\phi_{m\beta} = \phi_{m}^{\beta}$. Then $\theta_{\beta,m}(\zeta) = \prod_{k=0}^{m-1} \theta_{\beta}(\zeta - k\beta)$ for all $\zeta$. Since the shift by $\beta \in \Pi_{\mathfrak{k}}$ preserves $\Lambda_{\mathfrak{t}}$, it suffices to consider only $\phi_{\beta}$. Furthermore, $\phi_{\beta}$ can be constructed as a matrix element $\check{s}_{ba}$ via a Hasse diagram of the form

$$v_{b} \xleftarrow{c_{a}} f_{a}v_{b} \ldots v_{a},$$

where the suppressed part is independent of $\alpha$. It can be realized as a sub-diagram in the Hasse diagram of the quantized adjoint $g$-module where $v_{b}$ is of zero weight, hence weight differences $\nu_{b} - \nu_{c}$ are positive roots for all $c \prec b$, cf. [M8] Theorem 5.3.

We saw that $t$-singular nodes in $s_{ca}$ may appear only for orthogonal $g$ but the corresponding node factors cancel. Thus the only node (4.17) may contribute to singularity of $\phi_{\beta}(\zeta)$ is one whose weight $\mu$ is a positive root, and this root is in $R_{\mathfrak{g}/t}^{+}$ (otherwise the node will split a simple root $\beta \in \Pi_{\mathfrak{k}}$ to a sum of two roots from $R_{\mathfrak{t}}^{+}$, which is impossible). Then every denominator in (4.17) is proportional to $\mu(t)q^{2(\xi,\mu) - (\mu,\mu)} - 1$, and $\mu(t) \in \text{Spec}(t)$. Therefore this factor does not turn zero at $q \in \mathbb{C} \setminus \sqrt{t}$. Thus we have proved 1).

To prove 2), observe that we have chosen the Hasse diagram so that $s_{ca}$ in (4.16) are independent of the genrator $f_{a}$ because the only $\alpha$-arrow in the subdiagram enters the first term $c_{bc}$. Hence the first term is independent of the sum thanks to the PBW theorem. This proves the second statement.

□

5 Generalized parabolic categories

In this section we continue our study of generalized parabolic modules introduced in Section 2.3.

Fix a point $t \in T$ with its centralizer $\mathfrak{k}$ and the maximal Levi subalgebra $\mathfrak{l} \subset \mathfrak{k}$. Pick up a base weight $\lambda \in \mathfrak{h}^{*}$, a $\mathfrak{k}$-dominant weight $\xi \in \Lambda_{\mathfrak{t}}^{+}$, and put $\zeta = \lambda + \xi$. We have a system of Kac-Kazhdan conditions (2.4) and the set $\{\phi^{m_{\alpha}}_{\alpha}1_{\zeta}\}_{\alpha \in \Pi_{t}}$ of extremal vectors in the Verma module $\tilde{M}_{\zeta}$. Denote by $\tilde{M}_{\lambda,\xi}$ the quotient of $\tilde{M}_{\zeta}$ by the sum of submodules generated by $\{\phi^{m_{\alpha}}_{\alpha}1_{\zeta}\}_{\alpha \in \Pi_{t}}$. There is a sequence of epimorphisms

$$\tilde{M}_{\zeta} \rightarrow \tilde{M}_{\lambda,\xi} \rightarrow M_{\lambda,\xi}.$$
The parabolic Verma module $\hat{M}_{\lambda, \xi}$ is locally finite over $U_q(\mathfrak{l})$, [M4]. Therefore $M_{\lambda, \xi}$ is also locally finite when restricted to $U_q(\mathfrak{l})$. This fact is of importance for our further study.

Pick up a composite root $\beta \in \Pi_{\ell}/\mathfrak{t}$. It follows from factorization (4.14) and Proposition 4.11 that the Shapovalov elements $\phi_{\beta}^{m, \beta}$ are regular once $q \in \mathbb{C} \setminus \sqrt{q}$. Since the leading term in $\phi_{\beta}(\zeta)$ (the first summand in (4.13)) is the only one that contains a generator $f_{\beta}$ of the PBW-basis, it is independent of the other terms. Thus we conclude that $\phi_{\beta}^{m, \beta} \xi$ does not vanish in $\hat{M}_{\lambda, \xi}$ and is an extremal vector, provided $q \in \mathbb{C} \setminus \sqrt{q}$.

As a $U_q(\mathfrak{gl})$-module, $M_{\lambda, \xi}$ is isomorphic to $U_q(\mathfrak{gl})/J_{\xi}^-$, where $J_{\xi}^-$ is the left ideal annihilating the highest vector in $M_{\lambda, \xi}$. It is generated by $\phi_{\alpha}^{m, \alpha}$ with all $\alpha \in \Pi_{\ell}$. We will also consider the opposite module $M_{\lambda, \xi}'$ of lowest weight $-\lambda - \xi$. It is isomorphic to the $U_q(\mathfrak{gl})$-module $U_q(\mathfrak{gl})/J_{\xi}^+$, where $J_{\xi}^+ = \sigma(J_{\xi}^-)$.

By $V_{\xi}^+$ we denote the kernel $V^J_{\xi}$ of $J_{\xi}^+$. In the classical limit, weight vectors in $V_{\xi}^+$ are in bijection with irreducible submodules in the $\mathfrak{g}$-module $V \otimes X_{\xi}$, so $\Lambda(V_{\xi}^+) + \xi$ is in $\Lambda_{t}^+$. We denote by $\Lambda V_{\xi}$ any $U_q(\mathfrak{h})$-invariant subspace in $V$ that is transversal to $\omega(J_{\xi}^+)V$, thus we can identify it with $V/\omega(J_{\xi}^+)V$, the dual to $V_{\xi}^+$ with respect to the contravariant form on $V$. One can prove that in the classical limit the form is non-degenerate on $V_{\xi}^+$, so one can choose $\Lambda V_{\xi} = V_{\xi}^+$ for almost all $q$. Then the external twist $\theta$ responsible for complete reducibility of $V \otimes M_{\lambda, \xi}$ becomes an operator from $\text{End}(V_{\xi}^+)$. Much of our further analysis relies on elementary technical facts about base weights which we arrange as a separate proposition for further convenience.

**Lemma 5.1.** Let $t \in T$, $\mathfrak{t}$ be its centralizer, and $\lambda$ a $t$-base weight. Then

1. For each $\alpha \in \mathbb{R}_{\mathfrak{g}/\mathfrak{t}}^+$ and all $c \in \mathbb{Q}$ the number $[(\lambda + \rho, \alpha^\vee) + c]_{q_{\alpha}}$ is not zero at all $q \in \mathbb{C} \setminus \sqrt{q}$. Its reciprocal tends to zero as $q \to 1$.

2. For each $\alpha \in \mathbb{R}_{\mathfrak{t}}^+$ and any $c \in \mathbb{Q}$ the function $q \mapsto [(\lambda + \rho, \alpha^\vee) + c]_{q_{\alpha}}$ is either identically zero or does not vanish at all $q \in \mathbb{C} \setminus \frac{1}{\sqrt{q}}$.

**Proof.** By definition of base weight (2.2) we write

$$[(\lambda + \rho, \alpha^\vee) + c]_{q_{\alpha}} = q_{\alpha}^{-(\lambda + \rho, \alpha^\vee) - c} \times \frac{\alpha(t)q_{\alpha}^{2(\kappa, \alpha^\vee)}}{q_{\alpha} - q_{\alpha}^{-1}} - 1.$$  

If $\alpha \in \mathbb{R}_{\mathfrak{g}/\mathfrak{t}}^+$ then 1) holds true because $\alpha(t) \neq 1$. Finally, the inverse fraction goes to zero as $q_{\alpha} - q_{\alpha}^{-1}$ does. This proves the first assertion.

Now suppose that $\alpha \in \mathbb{R}_{\mathfrak{t}}^+$ and therefore $\alpha(t) = 1$. Then $\alpha(t)q_{\alpha}^{2(\kappa, \alpha^\vee) + 2c} = 1$ if and only if $c = -(\kappa, \alpha^\vee)$ as $q$ is not a root of unity. This proves the second assertion.  

24
Proposition 5.2. Let $Z$ be a module of highest weight $\zeta \in \lambda + \Lambda$. Suppose that $Z$ is locally finite over $U_q(\mathfrak{l})$. Then for each $V \in \text{Fin}_q(\mathfrak{g})$ and $\eta \in \Lambda^+_t$, the map $p_\eta: (V \otimes Z)[\lambda + \mu] \to (V \otimes Z)^+$ is well defined for $q \in \mathbb{C}\setminus \sqrt[+]{t}$.

Proof. Without loss of generality, we can assume that the base weight $\lambda$ defines the trivial representation of $U_q(\mathfrak{l})$, that is $(\lambda, \alpha) = 0$ for all $\alpha \in \mathfrak{r}_t$.

If $\alpha \in \mathfrak{r}_t^+$, then all Cartan denominators in $p_\alpha$ entering factorization (2.9) do not turn zero at $q \in \mathbb{C}\setminus \sqrt{t}$, and $p_\alpha$ is regular on $(V \otimes Z)[\lambda + \eta]$. In particular, for simple $\alpha$, the operator $p_\alpha$ sends $(V \otimes Z)[\lambda + \eta]$ to $\ker e_\alpha$ because $V \otimes Z$ is locally nilpotent over $U_q(\mathfrak{g}_\alpha^u)$, cf. Lemma 2.3.

Now suppose that $\alpha \in \mathfrak{r}_t^+$. The denominator in (2.6) returns $[(\kappa + \eta, \alpha^\vee) + i]_{q_\eta}$ on the weight space $(V \otimes Z)[\lambda + \eta]$. It never vanishes because $(\kappa + \eta, \alpha^\vee)$ and $i$ are positive integers. Since $Z$ is locally finite over $U_q(\mathfrak{l})$, the tensor product $V \otimes Z$ is locally finite too. Then $\lambda + \eta$ is $t$-dominant, and $p_\alpha$ is well defined on $(V \otimes Z)[\lambda + \eta]$. Moreover, $(V \otimes Z)[\lambda + \eta]$ is mapped to $\ker e_\alpha$ by $p_\eta$ if $\alpha \in \Pi_t$, see [M2], Proposition 3.6.

Thus the map $p_\eta: (V \otimes Z)[\lambda + \eta] \to V \otimes Z$ is well defined and independent of a normal ordering on $\mathfrak{r}_t^+$. We are left to show that it ranges in $(V \otimes Z)^+$. Fix $\alpha \in \Pi_\eta$ and order positive roots with $\alpha$ in the left-most position. Then $p_\eta$-image of $(V \otimes Z)[\lambda + \eta]$ is killed by $e_\alpha$ as argued and hence by all $e_\alpha$ with $\alpha \in \Pi_\eta$ since $p_\eta$ is independent of normal ordering. \hfill \Box

Remark that the operator $p_\eta$ can be factorized as $p_{\eta/\lambda} \times p_t$ as there is a normal ordering with roots from $\mathfrak{r}_t^+$ on the right. The right factor is the extremal projector to the subspace of $U_q(\mathfrak{l}_+)$-invariants in $V \otimes Z$.

5.1 Base module

We start our analysis with the base module $M_\lambda$ and prove that it is irreducible for almost all $q$. Our interest in $M_\lambda$ is motivated by an idea to represent quantized polynomial ring $\mathbb{C}[O]$ as a subalgebra of linear operators on $M_\lambda$. A neighborhood of the initial point $t \in O$ can be parameterized by the tangent space $\mathfrak{g}/\mathfrak{t} \simeq \mathfrak{g}_-/\mathfrak{t}_- \oplus \mathfrak{g}_+/\mathfrak{g}_+$, which facilitates an embedding of $\mathbb{C}[O]$ in $\mathbb{C}[\mathfrak{g}/\mathfrak{t}] \simeq \mathbb{C}[\mathfrak{g}_+/\mathfrak{t}_+] \otimes \mathbb{C}[\mathfrak{g}_-/\mathfrak{t}_-]$. This tensor product looks like a matrix algebra if there is a suitable pairing between factors making them dual vector spaces. This observation suggests to seek for a deformation of $\mathbb{C}[\mathfrak{g}_-/\mathfrak{t}_-]$ as a module of highest weight of the same functional dimension while a deformation of $\mathbb{C}[\mathfrak{g}_+/\mathfrak{t}_+]$ as its opposite module of lowest weight. The required duality will be secured if the modules are irreducible.

It is easy to meet the requirement on the size of $M_\lambda$ if $\mathfrak{t}$ is Levi thanks to a PBW basis in $U_q(\mathfrak{g}_-)$ with all monomials from $U_q(\mathfrak{l}_-)$ on the right. The complementary monomials deliver a
basis on the scalar parabolic Verma module $M_\lambda$ of highest weight $\lambda$. Finding a basis in $M_\lambda$ is challenging for non-Levi $\mathfrak{t}$, so we have to resort to deformation arguments. We regard $M_\lambda$ as a module over $U_q(\mathfrak{g}_-)$ and extend the ring of scalars to $\mathbb{C}_1(q)$. Then $M_\lambda$ is a deformation of a classical $U_q(\mathfrak{g}_-)$-module, and the deformation respects the weight spaces, up to the shift by $\lambda$.

**Lemma 5.3.** The character of the $\mathbb{C}_1(q)$-extension of $M_\lambda$ equals $\text{ch}(M_\lambda) = \prod_{\alpha \in R^+_{\mathfrak{g}/\mathfrak{t}}} (1-e^{-\alpha})^{-1} \times e^\lambda$.

**Proof.** The Lusztig generators of the PBW basis are deformations of classical root vectors. We redefine $f_\alpha$ for $\alpha \in R^+_{\mathfrak{g}/\mathfrak{t}}$ as follows. The classical root vector $f_\alpha$ is a composition of commutators among $f_\beta$, $\beta \in \Pi_{\mathfrak{t}/\mathfrak{l}}$ and $f_\nu$, $\mu \in \Pi_\mathfrak{t}$. Define its quantum counterpart by replacing $f_\beta$ with $\phi_\beta(\lambda)$, simple root vectors from $\mathfrak{l}$ with Chevalley generators of $U_q(\mathfrak{l})$, and classical commutators with $q$-commutators. By construction, they are deformations of classical $f_\alpha$ for all $\alpha \in R^+_{\mathfrak{t}}$.

Order roots so that $R^+_\mathfrak{t}$ is on the right of $R^+_{\mathfrak{g}/\mathfrak{t}}$ and consider the PBW system of ordered monomials in the negative root vectors modified as above. The cardinality of this system in every weight space equals its dimension, thanks to the presence of a standard PBW basis in $U_q(\mathfrak{g}_-)$. In the classical limit, this system delivers a PBW basis in $U(\mathfrak{g}_-)$, therefore it is a basis in every weight subspace of $U_q(\mathfrak{g}_-)$, for almost all $q$. Then the monomials in the root vectors with roots from $R^+_{\mathfrak{g}/\mathfrak{t}}$ deliver a basis in every weight subspace of $M_\lambda$, for almost all $q$.

The rest of the section is devoted to the question of irreducibility of $M_\lambda$. Suppose that $G$ is a connected simply connected group with Lie algebra $\mathfrak{g}$. Let $K$ be its closed subgroup with the Lie algebra $\mathfrak{k}$. It is known that $K$ is connected, [Hum]. Let $\Xi$ denote the set of isomorphism classes of $\mathfrak{g}$-modules $V \in \Xi$ with non-zero space of $\mathfrak{t}$-invariants. Such modules are those appearing in $\mathbb{C}[G/K]$ thanks to Peter-Weyl decomposition, [GW]. Since $\mathfrak{t}$ is reductive, every module enters $\Xi$ along with its dual. We will use the same notation for classes of $U_q(\mathfrak{g})$-modules whose quasi-classical counterparts are in $\Xi$.

Denote by $N^\pm$ the quotients $U(\mathfrak{g}_\pm)/U(\mathfrak{g}_\pm)\mathfrak{t}_\pm$. It follows from Lemma 5.3 that each subspace in $M_\lambda$ of weight $\mu + \lambda$ has dimension of $N^{-}[\mu]$ for almost all $q \in \mathbb{C}\setminus \sqrt[2]{T}$. Every $v \in V^\mathfrak{t}$ defines a $U(\mathfrak{g}_+)$-invariant map $\tilde{\varphi}_v: U(\mathfrak{g}_+) \rightarrow V$, $x \mapsto x \triangleright v$, that factors through a map $\varphi_v: N^+ \rightarrow V$.

**Lemma 5.4.** The intersection of $\ker \varphi_v$ over all $v \in V^\mathfrak{t}$ and all $V \in \Xi$ is zero.

**Proof.** Pick up a module $V \in \Xi$ and a vector $v \in V^K$ such that $K$ is the isotropy subgroup of $v$. Such a pair $(V,v)$ does exist by [GW], Theorem 11.1.13. The coset space $G/K$ is embedded in $V$ via the map $g \mapsto gv$, hence $\varphi_v$ yields an embedding $\mathfrak{g}_+/\mathfrak{t}_+ \rightarrow V$. Fix an order on positive roots and consider a PBW monomial $\prod_{\alpha \in R^+_{\mathfrak{g}/\mathfrak{t}}} e^{m_\alpha}$. Apply it to the $K$-invariant tensor $v^{\otimes m}$ in the symmetrized tensor power $\text{Sym}(V^{\otimes m})$. The result is a symmetrized tensor...
\( \alpha \text{Sym}(\otimes_{\alpha \in R^+_{\mathbb{Q}}} (e_\alpha v)^{\otimes m_\alpha}) \) plus terms \( \alpha \text{Sym}(v^{\otimes k} \otimes (\ldots)) \) with \( k > 0 \). The first term is independent of the remainder, and furthermore, the images of the PBW monomials of degree \( m \) are independent in \( V^{\otimes m} \).

\[ \square \]

It follows that for every weight \( \mu \in \Lambda(N^+) \) there is \( V \in \Xi \) and a \( g_+ \)-invariant map \( N^+ \rightarrow V \) generated by \( v \in V^t = V^{\otimes 0} \) that is injective on \( N^+[\mu] \). We will mimic this situation in the quantum setting.

Set \( Z = M_\lambda \). Suppose that \( u \in V \otimes Z \) is an extremal vector such that \( v = (\text{id} \otimes \epsilon_Z)(u) \in V \) is not zero. Define a linear map \( \psi_v : Z \rightarrow V \) as \( \psi_v(z) = u^1(u^2, z) \) (in Sweedler notation), for all \( z \in Z \). It factors through a composition \( Z \rightarrow \ast Z \rightarrow V \), where the first arrow is the contravariant form regarded as a linear map from \( Z \) to its restricted \((U_q(\mathfrak{h})\)-locally finite\) right dual \( \ast Z \).

**Lemma 5.5.** For every element \( f \in U_q(\mathfrak{g}_-) \) of weight \(-\alpha \), the mapping \( \psi_v \) acts by the assignment \( \psi_v(f 1_Z) = q^{-\langle \lambda + \mu, \alpha \rangle} \sigma(f)v \), where \( \mu \) is the weight of \( v \).

**Proof.** It is sufficient to prove the equality for \( f \) a monomial in Chevalley generators. For simple \( \alpha \in \Pi \) one has

\[
(1 \otimes \omega(f_\alpha))u = -(1 \otimes q^{-h_\alpha} e_\alpha)u = -(\gamma(e_\alpha) \otimes q^{-h_\alpha})u = (e_\alpha q^{-h_\alpha} \otimes q^{-h_\alpha})u = (\sigma(f_\alpha) \otimes 1)q^{-h_\alpha}u.
\]

This implies \( (1 \otimes \omega(f))u = q^{-\langle \lambda + \mu, \alpha \rangle} (\sigma(f) \otimes 1)u \) for all \( \alpha \) and all monomial \( f \), by induction on degree of \( f \). Now the proof is immediate as \( \psi_v(f 1_Z) = (\text{id} \otimes \epsilon_Z)((1 \otimes \omega(f_\alpha))(u)) \). \( \square \)

In other words, \( \psi_v \) is a homomorphism of \( U_q(\mathfrak{g}_-) \)-modules. Remark that the vector \( v \) is not an arbitrary element from \( \text{Hom}_{U_q(\mathfrak{g}_-)}(Z, V) \), but one originating from an extremal vector in \( V \otimes Z \). This fact is crucial because only then the homomorphism \( Z \rightarrow V \), \( f 1_Z \mapsto q^{-\langle \lambda + \mu, \alpha \rangle} \sigma(f)v \), factors through a homomorphism \( \ast Z \rightarrow V \). Irreducibility of \( Z \) will be proved if the map \( Z \rightarrow \ast Z \) is shown to have no kernel and therefore be an isomorphism. It is sufficient to check it only for those weight spaces where extremal vectors in \( Z \) may appear. That is, on the orbit of highest weight \( \zeta \) of \( Z \) under the affine action \( \zeta \mapsto w.\zeta = w(\zeta + \rho) - \rho \) of the Weyl group \( W \ni w \).

The following theorem is one of our main results. We rely in its proof on the special case of Proposition 5.7 for \( \xi = 0 \).

**Theorem 5.6.** The base module \( M_\lambda \) is irreducible for almost all \( q \in \mathbb{C} \setminus \sqrt{q} \).

**Proof.** It is sufficient to prove that \( M_\lambda \) has no extremal vectors or, equivalently, the contravariant form of \( M_\lambda \) is non-degenerate on \( M_\lambda[\mu] \) with \( \mu \in W.\lambda \). Let \( N^+_q \simeq M'_\lambda \) denote the quotient of \( U_q(\mathfrak{g}_+) \) by the left ideal \( J^+ \) annihilating the lowest vector in module \( M_\lambda \). By Lemma 5.3,

\[
\dim N^+_q[\lambda - \mu] = \dim N^+[\lambda - \mu] = \dim M'_\lambda[-\mu] = \dim M_\lambda[\mu]
\]

27
for almost all \( q \). Pick up a module \( V \in \Xi \) such that the weight space \( N^+[\lambda - \mu] \) is embedded in \( V \) in the classical limit via some \( \varphi_{v_0} \). Proposition 5.7 for \( \xi = 0 \) below guarantees that, for almost all \( q \), there is a non-zero vector \( v = p_\xi(\lambda) v' \), \( v' \in V[0] \), deforming \( v_0 \), such that \( u = p_\xi(v' \otimes 1_\lambda) \) is extremal and \( v = (\text{id} \otimes \epsilon_{M_\lambda})(u) \). Then \( N^+_q [\lambda - \mu] \) is embedded in \( V \) and the map \( \psi_v : M_\lambda[\mu] \rightarrow N^+_q [\lambda - \mu] v \) is an isomorphism for almost all \( q \). Since \( \mu \) runs over a finite set \( W(\lambda) \), the assertion follows. \( \square \)

### 5.2 Generalized parabolic Verma modules

In this section we study the \( \text{Fin}_q(\mathfrak{g}) \)-module category generated by the base module. As before, \( t \in T \) and \( \mathfrak{k} \subset \mathfrak{g} \) is the centralizer of \( t \). Pick up a base weight \( \lambda \) for \( t \), a \( \mathfrak{k} \)-dominant weight \( \xi \in \Lambda^+_\mathfrak{k} \) and set \( \xi = \lambda + \xi \). Denote \( m_\alpha = (\xi, \alpha^\vee) + 1 \in \mathbb{N} \), for all \( \alpha \in \Pi_{\mathfrak{k}} \).

Let \( V \) be a \( \mathfrak{g} \)-module considered as a \( \mathfrak{k} \) module by restriction. For an irreducible finite dimensional \( \mathfrak{k} \)-module \( X \) set \( V_X^+ \) to be \( \text{Hom}_{\mathfrak{k}}(X^*, V) \).

**Proposition 5.7.** Suppose that all weight of \( V \in \text{Fin}_q(\mathfrak{g}) \) are multiplicity free. Then for almost all \( q \in \mathbb{C} \setminus \sqrt{t} \), the \( U_q(\mathfrak{h}) \)-modules \( V^+_\xi \) are flat deformations of \( V_X^+ \) and the operator \( p_\xi(\zeta) : \widehat{V}^+_\xi \rightarrow V^+_\xi \) is surjective.

**Proof.** Denote by \( W = \sum_{\mu \in \Lambda(V_X^+)} V[\mu] \subset V \) the sum of all weight spaces whose weights belong to \( \Lambda(V_X^+) \). Thanks to Proposition 5.2 the operator \( p_{\alpha}(\zeta) \) is well defined in \( W \) and ranges in \( V^+_\xi \) due to Lemma 3.3.

Consider the eigenvalues of the foot factors \( p_{\alpha}(\zeta) \), given by (3.11), on the subspace of weight \( \mu \in \Lambda(W) \). If \( \alpha \in R^+_q/t \), then neither enumerators \( \alpha \{ (\lambda + \xi + \mu + \rho, \alpha^\vee) + k \}_{q_\alpha} \), nor denominators \( \alpha \{ (\lambda + \xi + \rho, \alpha^\vee) + k \}_{q_\alpha} \) turn zero at \( q \in \mathbb{C} \setminus \sqrt{t} \) By Lemma 5.1, 1). We are left to work out the case of \( \alpha \in R^+_t \).

In the classical limit, all factors \( p_{\alpha}(\zeta) \) with \( \alpha \in R^+_q/t \) turn to identical operator on \( V^+_X \). Then only the factors with \( \alpha \in R^+_t \) are left, and the operator \( p_{\xi}(\zeta) \) turns to \( p_{\xi}^{\mathfrak{k}}(\xi) \) (a normal order on \( R^+_q \) induces a normal order on \( R^+_t \subset R^+_g \)). But this is the inverse extremal twist \( \Theta_{V,X} \) for the finite dimensional \( \mathfrak{k} \)-module \( X \), which is invertible because of complete reducibility of finite dimensional \( \mathfrak{k} \)-modules. Thus all factors (3.11) are regular in \( q \) and do not vanish in some neighbourhood of \( q = 1 \), and thence for all \( q \in \mathbb{C} \setminus \sqrt{t} \), by Lemma 5.1, 2).

We are left to prove that \( W = V^+_\xi \) except maybe for a finite subset of \( q \in \mathbb{C} \setminus \sqrt{t} \) independent of \( \xi \). Since kernels do not increase in deformation, \( V^+_\xi \subset W \) for almost all \( q \in \mathbb{C} \setminus \sqrt{t} \). On the other hand, as we proved, \( V^+_\xi \supset p_\xi(\zeta) W = W \) for all \( q \in \mathbb{C} \setminus \sqrt{t} \), whence \( V^+_\xi \supset W \) for all and \( V^+_\xi = W \) for almost all \( q \in \mathbb{C} \setminus \sqrt{t} \). As the set of weights in \( V \) is finite, \( W \) can be smaller than
V only for a finite number of \( \xi \) (Shapovalov elements of high degree annihilate all V). Hence the finite set of \( q \in \mathbb{C}\backslash\sqrt[3]{t} \), where equality \( W = V^{\xi} \) is violated can be chosen independent of \( \xi \).

Recall that the category \( \mathcal{O}_q \) consists of finitely generated \( U_q(\mathfrak{g}) \)-modules that are \( U_q(\mathfrak{h}) \)-diagonalizable and \( U_q(\mathfrak{g}^-) \)-locally finite. Tensor product with finite dimensional modules preserves \( \mathcal{O}_q \) and makes it a \( \text{Fin}_q(\mathfrak{g}) \)-module category. Our goal is to study a \( \text{Fin}_q(\mathfrak{g}) \)-module subcategory \( \mathcal{O}_q(t) \subset \mathcal{O}_q \) associated with a point \( t \in O \cap T \). Objects of \( \mathcal{O}_q(t) \) will be interpreted as "representations" of vector bundles over the quantized \( \mathbb{C}[O] \) provided \( \mathcal{O}_q(t) \) is semi-simple, which is the main question to answer.

**Definition 5.8.** A full subcategory \( \mathcal{O}_q(t) \) of the category \( \mathcal{O}_q \) whose objects are \( U_q(\mathfrak{g}) \)-submodules in \( V \otimes M_\lambda \), where \( V \in \text{Fin}_q(\mathfrak{g}) \), is called generalized parabolic category of the point \( t \).

Category \( \mathcal{O}_q(t) \) is obviously additive and stable under tensor product with modules from \( \text{Fin}_q(\mathfrak{g}) \) by construction. Although \( \mathcal{O}_q(t) \) apparently depends on \( \lambda \), a different choice of \( \lambda \) results in an isomorphic category, so we suppress the base weight from notation.

Note that for different points \( t \in T \cap O \) the categories \( \mathcal{O}_q(t) \) are different although they will be shown equivalent for almost all \( q \in \mathbb{C}\backslash\sqrt[3]{t} \).

Fix \( V \) to be the fundamental module of minimal dimension for special linear and symplectic \( \mathfrak{g} \). For orthogonal \( \mathfrak{g} \) let \( V \) be a fundamental spin module. In all cases, weight subspaces in \( V \) have dimension 1. The simply connected group \( G \) with Lie algebra \( \mathfrak{g} \) is faithfully represented in End(\( V \)), and all equivariant vector bundles on \( O \) are generated by the vector sub-bundles appearing in \( O \times V \).

Let \( L(\mu) \) denote the irreducible \( U_q(\mathfrak{g}) \)-module of highest weight \( \mu \in \mathfrak{h}^* \).

**Theorem 5.9.** For each \( t \in T \) and almost all \( q \in \mathbb{C}\backslash\sqrt[3]{t} \) the following holds true:

1. The category \( \mathcal{O}_q(t) \) is semi-simple.

2. Simple objects in \( \mathcal{O}_q(t) \) are exactly \( L(\lambda + \xi) \) with \( \xi \in \Lambda^+_t \cap \Lambda_\mathfrak{g} \).

**Proof.** Since every finite-dimensional \( U_q(\mathfrak{g}) \)-module is a submodule in a tensor power of \( V \), we will prove 1) if we do it for all \( V^\otimes m \otimes M_\lambda \) using induction on \( m \in \mathbb{Z}_+ \) (for \( m = 0 \) this is Theorem 5.6). Assuming that \( V^\otimes m \otimes M_\lambda \) is completely reducible and its all simple submodules are \( L(\lambda + \xi) \) such that \( X_\xi \subset V^\otimes m \) we will prove 1) for each \( V \otimes L(\zeta), \zeta = \lambda + \xi \), as the induction transition.

Suppose that we did it for some \( m \geq 0 \). Let \( V^\otimes m \otimes M_\lambda = \bigoplus_i L(\zeta_i) \) with \( \zeta_i = \lambda + \xi_i \) be an irreducible decomposition and set \( \zeta \) to one of \( \zeta_i \). Then complete reducibility of \( V \otimes L(\zeta) \) is a consequence of Proposition 5.7, 1) and Corollary 3.5.
It is clear from Proposition 5.7.1) that every module $L(\lambda + \xi)$ with $\xi \in \Lambda^+_t \cap \Lambda_g$ appears in $V^\otimes m \otimes M$ for some $m$ because $X_\xi$ appears in some $V^\otimes m$. This proves 2).\vspace{12pt}

It follows that the category $Q_q(t)$ is semi-simple at those $q \in \mathbb{C}\setminus \sqrt[3]{7}$ where the dimensionality of the generalized extremal spaces equals classical and the base module is irreducible. We denote by $\Omega_t \subset \mathbb{C}\setminus \sqrt[3]{7}$ the set of such $q$. It is open with respect to the topology induced by the Zariski topology on $\mathbb{C}$.

**Proposition 5.10.** For all $q \in \Omega_t$ and all $\xi \in \Lambda^+_t \cap \Lambda_g$, $\text{ch}(L(\lambda + \xi)) = \text{ch}(X_\xi)\text{ch}(M_\lambda)$.

**Proof.** Consider $M_{\lambda, \xi}$ as a $U_q(g_-)$-module. It the classical limit, it goes to the quotient of $U(g_-)$ by the left ideal generated by the annihilator of the highest vector in $X_\xi$. Therefore

$$\text{ch}(L(\lambda + \xi)) \leq \text{ch}(M_{\lambda, \xi}) \leq \text{ch}(X_\xi)\text{ch}(g_-/t_-)e^\lambda = \text{ch}(X_\xi)\text{ch}(M_\lambda)$$

over $\mathbb{C}_1(q)$ meaning inequality of each weight space dimension for $q$ from a punctured neighbourhood of 1 (which depends on the weight space in general).

Suppose that the statement is true for all $L(\lambda + \xi) \subset V^m \otimes M_\lambda$ with some $m \geq 0$ (that is obviously so for $m = 0$). The direct sum decomposition $V \otimes L(\lambda + \xi) = \sum_i L(\lambda + \xi_i)$ implies

$$\text{ch}(V)\text{ch}(L(\lambda + \xi)) = \sum \text{ch}(L(\lambda + \xi_i)) \leq \sum \text{ch}(M_{\lambda, \xi_i}) \leq \sum \text{ch}(X_\xi_i)\text{ch}(M_\lambda) = \text{ch}(V)\text{ch}(X_\xi)\text{ch}(M_\lambda)$$ (5.18)

over $\mathbb{C}_1(q)$ because $V \otimes X_\xi = \sum_i X_\xi_i$. Therefore the inequalities are all equalities. Furthermore, for each $i$ and each weight $\mu$ an equality

$$\dim L(\zeta_i)[\mu] = \dim M_{\lambda, \xi_i}[\mu] = \dim(X_\xi_i \otimes M_\lambda)[\mu]$$ (5.19)

holds for almost all $q \in \Omega_t$. But then $\text{ch}(L(\zeta_i)) \geq \text{ch}(X_\xi)\text{ch}(M_\lambda)$ for all $q \in \Omega_t$ as $L(\zeta_i)$ is a quotient of a Verma module, which is flat at all $q \in \mathbb{C}\setminus \sqrt[3]{7} \supset \mathbb{C}\setminus \sqrt[3]{7}$. If the inequality is strict for some $i$, then

$$\text{ch}(V)\text{ch}(L(\lambda + \xi)) = \sum \text{ch}(L(\lambda + \xi_i)) > \sum \text{ch}(X_\xi_i)\text{ch}(M_\lambda).$$

But this is impossible because the left- and right-most terms are equal to $\text{ch}(V)\text{ch}(X_\xi)\text{ch}(M_\lambda)$.

Therefore $\text{ch}(L(\zeta_i)) = \text{ch}(X_\xi_i)\text{ch}(M_\lambda)$ in $\Omega_t$. This is true for all $\xi_i$ such that $X_\xi_i \subset V^\otimes (m+1)$ and therefore for all $\xi \in \Lambda^+_t \cap \Lambda_g$, by induction. This completes the proof.\vspace{12pt}
Now we describe the irreducible submodules in $O_q(t)$ and show that they are essentially general-
ized parabolic Verma modules.

**Corollary 5.11.** For each $q \in \mathbb{C} \setminus \sqrt{t}$ and every $\xi \in \Lambda^+_t \cap \Lambda_q$ the module $M_{\lambda, \xi}$ is irreducible, and $\text{ch}(M_{\lambda, \xi}) = \text{ch}(X_\xi)\text{ch}(M_\lambda)$.

**Proof.** The module $L(\lambda + \xi)$ is a quotient of $M_{\lambda, \xi}$, so it suffices to check that their infinitesimal characters are equal. We have checked equality of weight spaces in (5.19) for each weight and almost all $q \in \Omega_t$ and then extended it for all $q \in \Omega_t$. This proves irreducibility of $M_{\lambda, \xi}$, the the character formula follows from Proposition 5.10. □

## 6 Quantization of associated vector bundles

A construction of equivariant star product on homogeneous spaces with Levi isotropy subgroups was discovered about twenty years ago [AL, DM, EE, EEM]. It was employing dynamical twist, or equivalently, the inverse invariant pairing between parabolic base module $M_\lambda$ and its opposite $M'_\lambda$. A lift of the form to $U_q(\mathfrak{g}+) \otimes U_q(\mathfrak{g}^-)$ delivers a quasi-Hopf algebra twist of $U_q(\mathfrak{g})$, [D2]. A coherent twist of its dual algebra of functions on the quantum group turns out to be associative on the subspace of $U_q(\mathfrak{t})$-invariants.

Algebraically this construction works in a more general setting than parabolic Verma modules, [KST], however there is problem of the size of "$\mathfrak{t}$-invariants" in the absence of the quantum subgroup $U_q(\mathfrak{t}) \subset U_q(\mathfrak{g})$. We worked it out for even quantum spheres in [M3] through harmonic analysis on the quantum Euclidean plane. In this section we extend that result for all conjugacy classes $O(t)$, $t \in T$. Moreover, we put it in a more general context of quantum vector bundles, in development of the approach of [DM]. Furthermore, we obtain an explicit presentation of the star product by expressing it through the extremal projectors, similarly to [M2].

### 6.1 Equivariant star product

Let $\mathcal{T}$ be the Hopf algebra quantization of $\mathbb{C}[G]$ along the Drinfeld-Sklyanin Poisson bracket, [FRT]. It is known to be a local star product (the multiplication is delivered by a bi-differential operator) [T, EK]. The quantum group $U_q(\mathfrak{g})$ enjoys a two-sided action on $\mathcal{T}$ by left and right translations. The Peter-Weyl decomposition splits $\mathcal{T}$ in the direct sum $\bigoplus_{[V]} V^* \otimes V$ over all equivalence classes of irreducible finite-dimensional $U_q(\mathfrak{g})$-modules. The structure of a $U_q(\mathfrak{g})$-bimodule descends from a realization of $\mathcal{T}$ by matrix elements of representations: $x \triangleright (v^* \otimes v) = \ldots$
(v^* \otimes xv) and (v^* \otimes v) \triangleleft x = (v^* x \otimes v), where we assume the natural right action on the dual space by transposition. In terms of Hopf pairing between \( T \) and \( U_q(g) \) they can be written as
\[
x \triangleright a = a^{(1)}(a^{(2)}, x), \quad a \triangleleft x = (a^{(1)}, x)a^{(2)}
\]
for all \( a \in T \) and \( x \in U_q(g) \). This implies that \( T \) is a module algebra with respect to the left action \( \triangleright \). The opposite multiplication on \( T \) is equivariant with respect to the left action \( x \triangleright a = a \triangleleft \gamma(x) \).

With every finite-dimensional irreducible \( \mathfrak{k} \)-module \( X \in \text{Fin}(\mathfrak{k}) \) one associates an equivariant vector bundle over the base \( O \) with fiber \( X \). With a realization of \( C[O] \) as the subalgebra of \( \mathfrak{k} \)-invariants in \( C[G] \) under the action \( \triangleright \) (the classical limit of), the \( C[O] \)-module of global sections can be realized as \( \text{Hom}_{\mathfrak{k}}(X^*, T) \), under the left multiplication in \( C[G] \). This picture will be quantized in this section.

**Proposition 6.1.** For every \( V \in \text{Fin}_q(g) \) and for all \( \xi, \eta \in \Lambda^+_\mathfrak{t} \), an isomorphism
\[
\text{Hom}_{U_q(g)}(M_{\lambda, \eta}, V \otimes M_{\lambda, \xi}) \simeq \text{Hom}_{\mathfrak{k}}(X_\eta, V \otimes X_\xi)
\]
holds at all \( q \in \Omega_t \).

**Proof.** Corollary 5.11 implies an equality \( \text{ch}(V \otimes M_{\lambda, \xi}) = \sum_{\eta \in I} \text{ch}(M_{\lambda, \eta}) \), where the summation is over weights in \( V^\vee_{\mathfrak{t}, \xi} \) counted with multiplicities (they parameterize an irreducible decomposition of \( V \otimes M_{\lambda, \xi} \)). This equality implies \( \text{ch}(V \otimes X_\xi) = \sum_{\eta \in I} \text{ch}(X_\eta) \), by the same corollary. Therefore the \( \mathfrak{k} \)-module \( \bigoplus_{\eta \in I} X_\eta \) is isomorphic to \( V \otimes X_\xi \) and the assertion follows.

It follows that for \( t \) of finite order the set of exceptional \( q \) can be chosen independent of \( \xi, \eta \) and \( V \).

We derive the following description of isotypic components in \( \text{Hom}(M_{\lambda, \eta}, M_{\lambda, \xi}) \).

**Corollary 6.2.** For every \( V \in \text{Fin}_q(g) \) and for all \( \xi, \eta \in \Lambda^+_\mathfrak{t} \),
\[
\text{Hom}_{U_q(g)}(V^*, \text{Hom}(M_{\lambda, \eta}, M_{\lambda, \xi})) \simeq \text{Hom}_{\mathfrak{k}}(X_\eta, V \otimes X_\xi)
\]
holds at all \( q \in \Omega_t \).

**Proof.** Since \( M_{\lambda, \xi} \) and \( M_{\lambda, \eta} \) are irreducible along with their dual modules of lowest weight, equivariant maps from \( V^* \) to \( \text{Hom}(M_{\lambda, \eta}, M_{\lambda, \xi}) \) are in bijection with equivariant maps from \( \text{Hom}(M'_{\lambda, \xi}, M'_{\lambda, \eta}) \) to \( V \), for every \( V \in \text{Fin}_q(g) \). We have a version of Proposition 6.1 for dual modules of lowest weights:
\[
\text{Hom}_{U_q(g)}(\text{Hom}(M'_{\lambda, \xi}, M'_{\lambda, \eta}), V) \simeq \text{Hom}_{U_q(g)}(M'_{\lambda, \eta}, V \otimes M'_{\lambda, \xi}) \simeq \text{Hom}_{\mathfrak{k}}(X_\eta^*, V \otimes X_\xi^*).
\]
The rightmost term is isomorphic to \( \text{Hom}_{\mathfrak{k}}(V^* \otimes X_\xi, X_\eta) \simeq \text{Hom}_{\mathfrak{k}}(X_\eta, V^* \otimes X_\xi) \) as \( V^* \otimes X_\xi \) is completely reducible over \( \mathfrak{k} \).
For every locally finite $U_q(\mathfrak{g})$-module $A$ with finite dimensional isotypic components and for every pair of weights $\eta, \xi \in \Lambda_+^*$, define

$$A^{(\eta, \xi)} = \text{Hom}_{U_q(\mathfrak{g})}(M_{\lambda, \eta}, A \otimes M_{\lambda, \xi}) \simeq \text{Hom}_{U_q(\mathfrak{g})}(M_{\lambda, \eta} \otimes M_{\lambda, \xi}, A).$$

In the case of $\xi = 0 = \eta$ we will write $A^\ell = A^{(0,0)}$. It follows from the right isomorphism that $A^{(\xi, \eta)} \simeq (\ker J^{+}_\xi \cap \ker J^{-}_\eta)[\eta - \xi]$ and therefore $(A^{\ast})^{(\xi, \eta)} \simeq A^{(\eta, \xi)}$. We have a quasi-classical isomorphism $A^{(\xi, \eta)} \simeq \text{Hom}_\mathfrak{g}(X^n, X^\xi \otimes A)$, by Proposition 6.1. Note that $A^+_0 = \ker J^+_0$ is the sum of $A^{(0, \xi)}$ over all $\xi \in \Lambda(A^+_0)$ because such weights are highest for finite dimensional $\mathfrak{f}$-submodules in $V$ and therefore $\mathfrak{f}$-dominant.

**Lemma 6.3.** For any pair of modules $V, W \in \text{Fin}_q(\mathfrak{g})$,

$$(V \otimes W)^\mathfrak{f} \simeq \oplus_{\xi \in \Lambda(W^+_0)} W^{(0, \xi)} \otimes V^{(\xi, 0)}.$$

**Proof.** This readily follows from complete reducibility:

$$\text{Hom}(M_\lambda, V \otimes W \otimes M_\lambda) \simeq \oplus_{\xi \in \Lambda(W^+_0)} W^{(0, \xi)} \otimes \text{Hom}(M_\lambda, V \otimes M_{\lambda, \xi}) \simeq \oplus_{\xi \in \Lambda(W^+_0)} W^{(0, \xi)} \otimes V^{(\xi, 0)},$$

and from the decomposition $W^+_0 \simeq \oplus_{\xi \in \Lambda(W^+_0)} W^{(0, \xi)}$. \hfill $\Box$

Now suppose that $A$ is an associative $U_q(\mathfrak{g})$-module algebra with multiplication $\cdot$. Define a multiplication on $\text{Hom}_{U_q(\mathfrak{g})}(M_\lambda, A \otimes M_\lambda) \simeq A^\ell$ by assigning the composition

$$f_2 * f_1: M_\lambda \xrightarrow{f_1} A \otimes M_\lambda \xrightarrow{f_2} A \otimes (A \otimes M_\lambda) \longrightarrow A \otimes M_\lambda,$$

to a pair of morphisms $f_1, f_2$. Clearly $\ast$ is associative.

Furthermore, it extends to a right $A^\ell$-action on $\text{Hom}_{U_q(\mathfrak{g})}(M_\lambda, A \otimes M_{\lambda, \xi}) \simeq A^{(\xi, 0)}$ by

$$h \triangleleft f: M_\lambda \xrightarrow{f} A \otimes M_\lambda \xrightarrow{id \otimes h} A \otimes A \otimes M_{\lambda, \xi} \longrightarrow A \otimes M_{\lambda, \xi},$$

and a left $A^\ell$-action on $\text{Hom}_{U_q(\mathfrak{g})}(M_{\lambda, \xi}, A \otimes M_\lambda) \simeq A^{(0, \xi)}$ by

$$f \triangleright g: M_{\lambda, \xi} \xrightarrow{g} A \otimes M_\lambda \xrightarrow{f} A \otimes A \otimes M_\lambda \longrightarrow A \otimes M_\lambda,$$

Again associativity readily follows from associativity of composition and of the multiplication $\cdot$.

Denote by $S^\nu \in U_q(\mathfrak{g}_+) \otimes U_q(\mathfrak{g}_-)$ a lift of the inverse form of the irreducible module of highest weight $\nu$. The operations $\triangleright$ and $\triangleleft$ can be written with the help of the inverse forms:

$$h \triangleleft f = (S^1 \triangleright f) \cdot S^2 \triangleright h), \quad h \in A^{(\xi, 0)}, \quad f \in A^\ell,$$

$$f \triangleright g = (S^1 \triangleright g) \cdot S^2 \triangleright f), \quad f \in A^\ell, \quad g \in A^{(0, \xi)}.$$
Explicit expressions for $\mathcal{S}^\nu$ are known only for some special cases, e.g. Verma modules [M6] and base modules for quantum spheres [M3]. With the use of the relation between the inverse forms and extremal projectors (3.10), the introduced operations can be presented in an explicit although more cumbersome form. Setting $\zeta = \lambda + \xi$, we write

$$h \triangleleft f = (\cdot \otimes \epsilon^\zeta)\left(p_\theta(0)\left(p_\theta^{-1}(\lambda)f \otimes p_\theta(0)\left(p_\theta^{-1}(\zeta)h \otimes 1_{\lambda, \xi}\right)\right)\right), \quad h \in A^{(\xi, 0)}, \quad f \in A^t,$$

$$f \triangleright g = (\cdot \otimes \epsilon_\lambda)\left(p_\theta(0)\left(p_\theta^{-1}(\lambda)g \otimes p_\theta(0)\left(p_\theta^{-1}(\lambda)f \otimes 1_\lambda\right)\right)\right), \quad f \in A^t, \quad g \in A^{(0, \xi)}.$$

The map $\epsilon_\zeta$ is a linear functional that acts by pairing with the highest vector via the contravariant form.

Next we study the classical limit of the introduced operations. Let $U_h(\mathfrak{g})$ be the extension of $U_q(\mathfrak{g})$ over the ring of formal power series in $\hbar = \ln q$. Let $N_0^\pm \subset U_h(\mathfrak{g}_\pm)$ be $\mathbb{C}[[\hbar]]$-submodules that are $U_h(\mathfrak{h})$-affine lifts of $M_\lambda$ and $M'_\lambda$, respectively.

Consider a regular $\hat{U}_h(\mathfrak{h})$-submodule in $\hat{U}_h(\mathfrak{g})$, generated by $\gamma^{-1}(N_0^\pm)N_0^+$. It contains the extremal twist $\Theta^\lambda$.

**Proposition 6.4.** The classical limit of $\mathcal{S}^\lambda$ is $1 \otimes 1 \in U(\mathfrak{k}_+) \otimes U(\mathfrak{k}_-)$.

**Proof.** Let $\theta_0 \in N_0^-, N_0^+$ denote the classical limit of $\theta_\lambda$. Pick up a module $V \in \text{Fin}_q(\mathfrak{g})$ and a pair of vectors $w, v \in \hat{V}_k^\pm$. In the classical limit, one has $(\theta_\lambda^{-1}v, w) = (p_\theta(\lambda)v, w) \rightarrow (p_\theta(\lambda)v, w)$. Therefore, $\theta_\lambda$ tends to identity on $V^*_\pm$ for every $V$. Hence $\theta_0 = 1$ by Proposition A.1.

Now regard $S^\lambda_{21}$ as an element of $N_0^+N_0^-$ under the linear isomorphism $M'_{\lambda} \otimes M_\lambda \rightarrow N_0^+N_0^- \subset \hat{U}_h(\mathfrak{g})$ facilitated by the triangular decomposition of $U_h(\mathfrak{g})$. Consider a linear automorphism $\hat{U}_h(\mathfrak{g}) \rightarrow \hat{U}_h(\mathfrak{g})$ defined by

$$efh \mapsto \gamma^{-1}(f)e\hbar, \quad f \in U_h(\mathfrak{g}_-), \quad h \in U_h(\mathfrak{h}), \quad e \in U_h(\mathfrak{g}_+).$$

This map takes $S^\lambda_{21}$ to $\theta_\lambda = \gamma^{-1}(S^\lambda_2)S^\lambda_1$. Therefore $\mathcal{S}^\lambda \rightarrow 1 \otimes 1$ in the classical limit as required. □

Now suppose that $A$ is a flat deformation of a $U(\mathfrak{g})$-module algebra $A_0$. Then $A^t$ is a flat deformation (as a vector space) of the subspace of $\mathfrak{k}$ invariants $A_0^t$.

**Corollary 6.5.** The associative algebra $A^t$ with multiplication $\star$ is a flat deformation of the algebra $A_0^t$ with opposite multiplication.

**Proof.** The formula (6.20) for $h \in A_0^{0, 0}$ turns to $h \star f = S^\lambda_1 f \cdot S^\lambda_2 h$. By Lemma 6.4, $\mathcal{S}^\lambda \rightarrow 1 \otimes 1$ in the classical limit $q \rightarrow 1$, and the assertion follows. □
We take $\mathcal{T}$ with the $U_q(\mathfrak{g})$-action $\rhd$ for the module-algebra $A$. Endow the dual vector space $V^*$ of a $U_q(\mathfrak{g})$-module $V$ with a left action $x \circ v^* = v^* \lhd \gamma(x)$. When applied to $\mathcal{T}$, it gives rise to a left action, which is compatible with the opposite multiplication and commutes with $\lhd$.

Note that $\star$-product is often defined as opposite to the one introduced above. That version is equivariant with respect to the right translation action $\lhd$ while the present one respects $\triangleright$.

**Theorem 6.6.**

1. The associative algebra $\mathcal{T}^t$ with multiplication $\star$ is an equivariant flat deformation of $\mathbb{C}[O]$.

2. The $\mathcal{T}^t$-modules $\mathcal{T}^{(0,\xi)}$ and $\mathcal{T}^{(\xi,0)}$ are $U_q(\mathfrak{g})$-equivariant flat deformations of the associated vector bundles on $O$ with fibers $X_\xi$ and its dual, respectively.

**Proof.** First of all note that the multiplication $\cdot$ is a star-product deformation of the classical multiplication on $\mathbb{C}[G]$, $[T]$. Furthermore, by Corollary 6.2, the $U_q(\mathfrak{g})$-module structure of $\mathcal{T}^{0,\xi}$ (resp. $\mathcal{T}^{\xi,0}$) under the action $\circ$ is similar to the $U(\mathfrak{g})$-module structure of the associated vector bundles with fiber $X_\xi$ (resp. $X^*_\xi$). So $\mathcal{T}^{0,\xi}$ and $\mathcal{T}^{\xi,0}$ are flat deformations as $U_q(\mathfrak{g})$-modules. Finally, 1) is a special case of Corollary 6.5 for $A = \mathcal{T}^t$ and 2) directly follows from Lemma 6.4. □

Up to now we viewed sections of quantized vector bundles as equivariant linear maps from $\text{Hom}_{U_q(\mathfrak{g})}(-, T \otimes -)$. Let us give them an alternative interpretation in terms of $\mathbb{C}$-linear maps between objects from $\mathcal{O}_q(t)$. They are natural $U_q(\mathfrak{g})$-modules whose locally finite parts also have a natural algebraic structure.

We endow the vector space $\text{Hom}(A, B)$ between two $U_q(\mathfrak{g})$-modules $A$ and $B$ with a left $U_q(\mathfrak{g})$-action $(x \triangleright f)(a) = x^{(1)} f(\gamma(x^{(2)}a))$.

**Proposition 6.7.** Upon extension over the field $\mathbb{C}(q)$, the algebra $\mathcal{T}^t$ is isomorphic to the locally finite part of $\text{End}(M_\lambda)$. The isomorphism commutes with the action of $U_q(\mathfrak{g})$. The $\mathcal{T}^t$-actions $\triangleright$ and $\blacktriangleright$ go over to the natural $\text{End}(M_\lambda)$-actions on $\text{Hom}(M_{\lambda, \xi}, M_\lambda)$ and $\text{Hom}(M_\lambda, M_{\lambda, \xi})$, respectively.

**Proof.** Define a map $\mathcal{T}^{(0,\xi)} \to \text{Hom}(M_{\lambda, \xi}, M_\lambda)$ for each $\xi \in \Lambda_+^*$ as follows. Every matrix element $g = v^* \otimes v \in V^* \otimes V^{(0,\xi)} \subset \mathcal{T}^{(0,\xi)}$ goes to a linear map

$$g: M_{\lambda, \xi} \supset x_{1, \lambda, \xi} \mapsto (v^*, x^{(1)} S_1^\lambda v) \otimes x^{(2)} S_2^\lambda 1_{\lambda} \in M_\lambda, \quad x \in U_q(\mathfrak{g}).$$

This assignment is equivariant and its image in $\text{Hom}(M_{\lambda, \xi}, M_\lambda)$ is the isotypic $V^*$-component, hence it is an isomorphism by Proposition 6.1. Furthermore, for $f = w^* \otimes w \in W^* \otimes W^{(0,0)}$ we
have
\[
M_\lambda \leftarrow M_\lambda \leftarrow M_{\lambda,\xi} \quad M_\lambda \leftarrow \leftarrow M_{\lambda,\xi}
\]
\[
f \uparrow \quad g \uparrow \quad = \quad \uparrow f \triangleright g
\]
\[
W^* \otimes V^* = (V \otimes W)^*
\]
The equality holds because the morphism in the right diagram is \(S^\lambda(g \otimes f)\), and the tensor product is the multiplication \(\cdot\) of matrix elements of representations constituting \(T\). This proves the statement with regard to \(\triangleright\) (and \(\triangleright\) as a special case). The case of \(\blacktriangleleft\) is proved similarly. \(\square\)

Remark that for \(t\) of finite order, the above assertion can be specialized at almost all \(q \neq 1\).

In the next section we describe quantized vector bundles as projective modules over \(T^t\). This models local triviality of the classical vector bundles, [S, Sw].

### 6.2 Quantum vector bundles as projective \(T^t\)-modules

We saw in the previous section that the locally finite parts of the \(U_q(\mathfrak{g})\)-modules \(\text{Hom}(M_\lambda, M_{\lambda,\xi})\) and \(\text{Hom}(M_{\lambda,\xi}, M_\lambda)\) are isomorphic to \(T^{(\xi,0)}\) and, respectively, \(T^{(0,\xi)}\). In particular, the locally finite part of \(\text{End}(M_\lambda)\) is a \(U_q(\mathfrak{g})\)-algebra, isomorphic to \(T^t\). The following result is obtained in [JM], Theorem 5.6 for quaternionic projective plain but is valid in general.

**Theorem 6.8.** The \(\text{Fin}_q(\mathfrak{g})\)-module category \(O_q(t)\) is equivalent to the category of equivariant finitely generated projective right \(T^t\)-modules.

Similar statement holds upon replacement of right \(T^t\)-modules with left.

It follows that any invariant projector from \(V \otimes M_\lambda\) to an irreducible submodule is a matrix with entries in the locally finite part of \(\text{End}(M_\lambda)\), which coincides with \(T^t\) if we allow for division by \(q - 1\). The question is if such a projector has classical limit. We did not check it for projective spaces and even spheres in [M4, M5] and we fill that gap here.

**Lemma 6.9.** For each \(W \in \text{Fin}_q(\mathfrak{g})\) there is a quasi-classical isomorphism
\[
W \otimes T^t \simeq \bigoplus_{[V]} V^* \otimes (V \otimes W)^t
\]
of \(U_q(\mathfrak{g})\)-modules.

**Proof.** Decomposing the left-hand side to isotypic components we write it as
\[
\bigoplus_{[Z]} (W \otimes Z^*) \otimes Z^t \simeq \bigoplus_{[V]} V^* \otimes \left( \bigoplus_{[Z]} \text{Hom}_{U_q(\mathfrak{g})}(V^*, W \otimes Z^*) \otimes Z^t \right).
\]
Replacing \(\text{Hom}_{U_q(\mathfrak{g})}(V^*, W \otimes Z^*)\) with \(\text{Hom}_{U_q(\mathfrak{g})}(Z, V \otimes W)\) we find the sum in the brackets equal to \((V \otimes W)^t\) (a consequence of complete reducibility of \(V \otimes W\)), which implies the required isomorphism. It is obviously quasi-classical. \(\square\)
We arrive at the main results of this section.

**Theorem 6.10.** The quantum vector bundles \( \mathcal{T}^{(\xi,0)} \) and \( \mathcal{T}^{(0,\xi)} \) are flat deformations as projective \( \mathcal{T}^\xi \)-modules.

**Proof.** It is sufficient to realize \( \mathcal{T}^{(\xi,0)} \) and \( \mathcal{T}^{(0,\xi)} \) as direct summands in a free \( \mathcal{T}^\xi \)-module and show that such a decomposition is a deformation of the classical decomposition of the corresponding trivial vector bundle. We will do it only for \( \mathcal{T}^{(0,\xi)} \) as the case of \( \mathcal{T}^{(\xi,0)} \) is similar.

Pick up a module \( W \in \text{Fin}_q(\mathfrak{g}) \) such that \( \xi \in \Lambda(W_0^+) \). For almost all \( q \) we have an equivariant diagram of isomorphisms of \( \mathcal{T}^\xi \)-modules from Proposition 6.7:

\[
\begin{array}{ccc}
\oplus_{\xi \in \Lambda(W_0^+)} W^{(0,\xi)} \otimes \mathcal{T}^{(0,\xi)} & \longrightarrow & W^* \otimes \mathcal{T}^\xi \\
\downarrow & & \downarrow \\
\oplus_{\xi \in \Lambda(W_0^+)} W^{(0,\xi)} \otimes \text{Hom}(M_{\lambda, \xi}, M_\lambda) & \longrightarrow & \text{Hom}(W \otimes M_\lambda, M_\lambda)
\end{array}
\]

where \( \longrightarrow \) is determined by the other arrows for almost all \( q \neq 1 \). On passing to the Peter-Weyl expansion \( \mathcal{T}^{(0,\xi)} = \sum [V] V^* \otimes V^{(0,\xi)} \), this map operates by an isomorphism on Hom-s

\[
\begin{array}{ccc}
\oplus_{\xi \in \Lambda(W_0^+)} W^{(0,\xi)} \otimes V^{(0,\xi)} & \overset{\sim}{\longrightarrow} & (V^* \otimes W)^\xi \\
\end{array}
\]

in each isotypic \( V^* \)-component thanks to Lemma 6.9. This is a consequence of isomorphism \( V^{(0,\xi)} \overset{\sim}{\approx} (V^*)^{(\xi,0)} \) and Lemma 6.3.

All terms here are flat deformations of their classical counterparts by Proposition 5.7 and the isomorphism is implemented via extremal projectors which turn to projectors of \( U_q(\mathfrak{g}) \) as \( q \to 1 \). In the classical limit, these isomorphisms turn to direct sum decomposition of the trivial vector bundle \( W \otimes \mathcal{T}^\xi \).

\[\square\]

In conclusion of the section, we present a direct quasi-classical construction of the algebra of \( U_q(\mathfrak{g}) \)-intertwiners splitting trivial quantum vector bundles into direct sum of sub-bundles. Take \( V \in \text{Fin}_q(V) \) and let \( \pi \) denote the representation homomorphism \( U_q(\mathfrak{g}) \to \text{End}(V) \). For a \( U_q(\mathfrak{g}) \)-module \( \mathcal{A} \) with action \( x \otimes a \mapsto x.a \) we call a tensor \( A \in \text{End}(V) \otimes \mathcal{A} \) invariant if \( x.A = \pi(\gamma(x^{(1)})) A \pi(x^{(2)}) \). In the case when \( \mathcal{A} \) is a module algebra, invariant matrices form a subalgebra of tensors in \( \text{End}(V) \otimes \mathcal{A} \times U_q(\mathfrak{g}) \) that commute with \( (\pi \otimes \text{id}) \circ \Delta(x) \) for all \( x \in U_q(\mathfrak{g}) \).

Now consider an injective linear map \( \text{End}(V) \to \text{End}(V) \otimes \mathcal{T}^\xi, A \mapsto \pi(\mathcal{R}_1) T A T^{-1} \pi(\gamma(\mathcal{R}_2) q^{2h_\rho}) \) (the Sweedler notation for R-matrix used). The image of \( A \) is an invariant matrix with respect to the action \( \circ \) on the entries. By dimensional reasons, this map delivers a linear isomorphism between \( (\text{End}(V))^\xi \) and \( \text{End}(V) \otimes \mathcal{T}^\xi \) at almost all \( q \) including the classical point. With the \( \ast \)-product on \( \mathcal{T}^\xi \) the image of \( (\text{End}(V))^\xi \) is the algebra of invariant matrices separating quantum
sub-bundles in $V \otimes T^k$. This algebra is quasi-classical by the mere construction and isomorphic to the subalgebra of classical $\mathfrak{t}$-invariants in $\text{End}(V)$ by Proposition 6.1.

## A Induced modules and duality

In this appendix we establish a fact that is used in the proof of Proposition 6.4. We think it should be a sort of classical but we have not found any references so we present it here. Suppose $\mathfrak{g}$ is a Lie algebra of a linear algebraic group and $\mathfrak{t} \subset \mathfrak{g}$ is a Lie subalgebra of its closed subgroup.

Introduce an equivariant pairing $\text{Ind}_{\mathfrak{g}}^{\mathfrak{t}} \mathbb{C} \otimes \mathbb{C}[G]^t \to \mathbb{C}$, induced by the pairing

$$U(\mathfrak{g}) \otimes \mathbb{C}[G]^t \to \mathbb{C}, \quad u \otimes \phi \mapsto (u.\phi)(1)$$

with $u \in U(\mathfrak{g})$ and $\phi \in \mathbb{C}[G]^t$. Presenting $\phi$ as a matrix element $w^* \otimes w \in W^* \otimes W$, where $w \in W$, we rewrite the pairing as $u \otimes (w^* \otimes w) \mapsto w^*(u.w)$.

**Proposition A.1.** Suppose there is a finite dimensional $\mathfrak{g}$-module $V$ with a vector $v_0 \in V^t$ such that $\dim \mathfrak{g}v_0 = \dim \mathfrak{g}/\mathfrak{t}$ and $v_0 \not\in \mathfrak{g}v_0$. Then the pairing is non-degenerate with respect to both tensor factors.

**Proof.** It is straightforward that the pairing has no kernel in the right factor (that is a consequence that the Hopf pairing between $U(\mathfrak{g})$ and $\mathbb{C}[G]$ is non-degenerate).

Pick up a basis in $\mathfrak{g}$ such that its first $m$ elements $u_1, \ldots, u_m$ span a subspace transversal to $\mathfrak{t}$ and put $v_i = u_i.v_0 \in V$. Put $u^{\vec{l}} = u_1^{l_1} \cdots u_m^{l_m}$, where $\vec{l} = (l_1, \ldots, l_m)$ with $l_i \in \mathbb{Z}_+$, and denote $|\vec{l}| = \sum_{i=1}^m l_i$. Then ordered PBW monomials $u^{\vec{l}}$, form a basis in $\text{Ind}_{\mathfrak{t}}^{\mathfrak{g}} \mathbb{C}$. Suppose that the element $u = \sum_{\vec{l},j} c_{\vec{l}}u^{\vec{l}}$, where $c_{\vec{l}} \in \mathbb{C}$, is in the kernel of the pairing. The sum is finite, so let $n$ be the highest degree $|\vec{l}|$ of its $u^{\vec{l}}$. Take $w = v_0 \otimes \cdots \otimes v_0 \subset V^\otimes n$. As $u$ is in the kernel by assumption, one should have $\sum_{\vec{l}} c_{\vec{l}}u^{\vec{l}}.w = 0$. Let us show that it is impossible.

Denote by Sym the symmetrizing projector in tensor powers of $V_0$. Every such element with $|\vec{l}| = n$ produces $m!\text{Sym}(v_1^{\otimes l_1} \otimes \cdots \otimes v_m^{\otimes l_m}) + \ldots$, where the terms containing $\text{Sym}(v_0 \otimes \ldots)$ are suppressed. Such terms are also resulted from $u^{\vec{l}}.w$ with $|\vec{l}| < n$. But $\text{Sym}(v_1^{\otimes l_1} \otimes \cdots \otimes v_m^{\otimes l_m})$ are linearly independent and independent from the suppressed terms since $\{v_i\}_{i=0}^m$ are independent. Therefore all $c_{\vec{l}}$ with $|\vec{l}| = n$ are zero. Descending induction on $n$ proves that the pairing has no kernel in $\text{Ind}_{\mathfrak{t}}^{\mathfrak{g}} \mathbb{C}$.

Here is a geometrical interpretation of the conditions of Proposition A.1: the local homogeneous space $(\mathfrak{g}, \mathfrak{t})$ is realized as the orbit of the vector $v \in V$, and $v$ is transversal to the orbit. These conditions are fulfilled for semi-simple conjugacy classes of algebraic groups. Such a class can be
realized as an orbit of a vector $v$ in some representation [GW], Theorem 11.1.13. Since the Cartan subalgebra is in the stabilizer, $v$ carries zero weight and is therefore transversal to $gv$.

**Acknowledgement.**
This research was done at the Center of Pure Mathematics, MIPT. It was partly supported by Ministry of Science and Higher Education of the Russian Federation, agreement no. 075-15-2022-289.

**Declarations**

**Funding**
The author received partial research support from Ministry of Science and Higher Education of the Russian Federation, agreement no. 075-15-2022-289.

**Competing interests**
The author have no competing interests to declare that are relevant to the content of this article.

**References**

[ABRR] D. Arnaudon, E. Buffenoir, E. Ragoucy, and P. Roche, Universal solutions of quantum dynamical Yang-Baxter equations, Lett. Math. Phys. 44 no 3 (1998), 201–214.

[AL] Alekseev, A. Lachowska, A.: Invariant $*$-product on coadjoint orbits and the Shapovalov pairing, Comment. Math. Helv. 80 (2005), 795–810.

[AlM] A. Alekseev and Malkin: Symplectic structures associated to Lie-Poisson groups, Comm. Math. Phys. 162 (1994), 147–173.

[AM] Ashton, T., Mudrov, A.: On representations of quantum conjugacy classes of $GL(n)$, Lett. Math. Phys. 103 (2013), 1029–1045.

[AST] Asherova, R. M., Smirnov, Yu. F., and Tolstoy, V. N.: Projection operators for the simple Lie groups, Theor. Math. Phys. 8 (1971), 813–825.

[BFFLS] Bayen, F., Flato, M., Fronsdal, C., Lichnerowicz, A. and Sternheimer, D.: Deformation Theory and Quantization, Ann. Phys. 111 (1978), 61–110.
[ChP] Chari, V. and Pressley, A.: A guide to quantum groups, Cambridge University Press, Cambridge 1994.

[Cost] Costantini, M.: A classification of spherical conjugacy classes, Pass. J. Math. 285, no 1 (2016), 63–91.

[D1] Drinfeld, V.: Quantum Groups. In Proc. Int. Congress of Mathematicians, Berkeley 1986, Gleason, A. V. (eds) pp. 798–820, AMS, Providence (1987).

[D2] Drinfeld, V.: Quasi-Hopf algebras, Leningrad Math. J., 1 no 6 (1990), 1419–1457.

[D3] Drinfeld, V.: On Poisson homogeneous spaces of Poisson-Lie groups, Theor. Math. Phys., 95 (1993), 524–525.

[DCK] De Concini, C., Kac, V. G.: Representations of quantum groups at roots of 1, Operator algebras, unitary representations, enveloping algebras, and invariant theory (Paris, 1989), Progress in Mathematics, 92, Birkhäuser, 1990, pp 471–506.

[DGS] Donin, J., D. Gurevich, G., Shnider, S.: Quantization of function algebras on semi-simple orbits in $g^*$, arXiv:q-alg/9607008.

[DM] J. Donin and A. Mudrov: Dynamical Yang–Baxter equation and quantum vector bundles, Commun. Math. Phys. 254 (2005), 719–760.

[EE] Enriquez, B., Etingof, P.: Quantization of classical dynamical r-matrices with nonabelian base, Commun. Math. Phys. 254 (2005), 603–650.

[EEM] Enriquez, B., Etingof, P., Marshall, I.: Quantization of some Poisson-Lie dynamical r-matrices and Poisson homogeneous spaces, Contemp. Math. 433 (2007), 135–176.

[EK] P. Etingof and D. Kazhdan: Quantization of Lie bialgebras, Selecta Math., 2 # 1 (1996) 1–41.

[FRT] Faddeev, L., Reshetikhin, N., and Takhtajan, L.: Quantization of Lie groups and Lie algebras, Leningrad Math. J., 1 (1990), 193–226.

[GW] Goodman, R., Wallach, N.: Symmetries, Representations, and Invariants, Grad. Texts. in Math. 255, Springer, New York, 2009.

[Hum] Humphreys, J.: Conjugacy classes in semi-simple algebraic groups, AMS, Providence, 1995.
[JM] G. Jones, A. Mudrov: Pseudo-parabolic category over quaternionic projective plane, Algebr. Represent. Theor. (2022), https://doi.org/10.1007/s10468-022-10185-8.

[Khor] Khoroshkin, S.: Extremal Projector and Dynamical Twist, Th. Math. Phys. 139 #1, 582—597 (2004).

[KMST] E. Karolinsky, K. Muzykin, A. Stolin, V. Tarasov: Dynamical Yang-Baxter equations, quasi-Poisson homogeneous spaces, and quantization, Lett.Math.Phys. 71 (2005) 179–197.

[KN] Khoroshkin, S., Nazarov, M.: Mickelsson algebras and representations of Yangians, Trans. Amer. Math.d Soc. 364 (2012), 1293–1367.

[Kolb] Kolb, S.: Quantum symmetric Kac-Moody pairs, Adv. Math. 267 (2014), 395–469.

[KS] Kulish, P. P., Sklyanin, E. K.: Algebraic structure related to the reflection equation, J. Phys. A, 25 (1992), 5963–5975.

[KST] Karolinsky, E., Stolin, A., Tarasov, V.: Equivariant quantization of Poisson homogeneous spaces and Kostant’s problem, J. Alg. 409 (2014), 362–381.

[KT] Khoroshkin, S.M., and Tolstoy, V.N.: Extremal projector and universal R-matrix for quantized contragredient Lie (super)algebras. Quantum groups and related topics (Wroclaw, 1991), 23–32, Math. Phys. Stud., 13, Kluwer Acad. Publ., Dordrecht, 1992.

[M1] Mudrov, A.: Contravariant form on tensor product of highest weight modules, SIGMA 15 (2019), 026, 10 pp.

[M2] Mudrov, A.: Contravariant forms and extremal projectors, J. Pure Appl. Algebra, (2022), 106902, doi:https://doi.org/10.1016/j.jpaa.2021.106902.

[M3] Mudrov, A.: Star-product on complex sphere $S^{2n}$, Lett. Math. Phys. 108 (2018), 1443–1454.

[M4] A. Mudrov, A.: Equivariant vector bundles over projective spaces, J.Theor.Math.Phys., 198 #2 (2019) 284–295.

[M5] Mudrov, A. : Equivariant vector bundles over quantum spheres, J.Noncommut. Geom. 15, no. 1 (2021) 79-111.

[M6] Mudrov, A.: R-matrix and inverse Shapovalov form, J. Math. Phys., 57 (2016), 051706.
[M7] Mudrov, A.: *Quantum conjugacy classes of simple matrix groups*, Commun. Math. Phys., 272 (2007), 635 – 660.

[M8] Mudrov, A.: *Shapovalov elements for classical and quantum groups*, JPAA 228 #7, (2024), 107634.

[S] Serre, J.-P.: *Faisceaux Algebriques Coherents*, Ann. Math., 61 # 2 (1955), 197–278.

[Spr] Springer T.A.: (1986) Conjugacy classes in algebraic groups. In: Tuan HF. (eds) Group Theory, Beijing 1984. Lecture Notes in Mathematics, vol 1185. Springer, Berlin, Heidelberg. https://doi.org/10.1007/BFb0076175.

[STS] M. Semenov-Tian-Shansky: *Poisson-Lie Groups, Quantum Duality Principle, and the Quantum Double*, Contemp. Math., 175 (1994) 219–248.

[Sw] Swan, R.: *Vector Bundles and Projective Modules*, Trans. AMS, 105 # 2 (1962), 264–277.

[T] L. A. Takhtajan, *Introduction to quantum groups*, International Press Inc., Boston 1993. Lecture Notes in Phys., 370 (1989), 3–28.