Parabolic reductions of principal bundles

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Abstract

In this paper we describe the structure of the space of parabolic reductions, and their compactifications, of principal $G$-bundles over a smooth projective curve over an algebraically closed field of arbitrary characteristic. We first prove estimates for the dimensions of moduli spaces of stable maps to the twisted flag varieties $E/P$ and Hilbert schemes of closed subschemes of $E/P$ with same Hilbert polynomials as that of a $P$-reductions of $E$. This generalizes the earlier results of Miheea Popa and Mike Roth to connected reductive groups and the results of Y. I. Holla and M. S. Narasimhan to the case of non-minimal sections. We then prove irreducibility and generic smoothness of the space of reductions for large numerical constraints, using the above result and the methods of G. Harder. We also study these space in more detail for generically stable $G$-bundles. As a consequence we can generalize the lower bound results of H. Lange to $G$.

1 Introduction

Let $C$ be a smooth projective over an algebraically closed field $k$ of arbitrary characteristic. Let $G$ be a connected reductive algebraic group. Let $P$ be a parabolic subgroup. Let $E$ be a principal $G$-bundle over $C$. The main objective of this paper is to understand the structure of the space of $P$-reductions of $E$ and its compactifications. Let $\pi : E/P \to C$ be the associated $G/P$ bundle over $C$. Let $T_\pi$ be the tangent bundle along the fibers of the morphism $\pi$. For a $P$ reduction $\sigma$ of $E$, equivalently a section of $\pi$, we will denote by $T_\sigma = \sigma^*(T_\pi)$ the normal bundle of $\sigma(C)$. Recall that a section $\sigma$ is minimal if $\deg(T_\sigma)$ is minimal among the degrees of the normal bundles for all sections of $\pi$. In Holla-Narasimhan [16] it was proved that the every irreducible component of the Hilbert scheme of closed subschemes of $E/P$ containing a minimal section has a dimension bound of $\dim(G/P)$.

For a $P$ reduction $\sigma$ we define its numerical type $[\sigma] \in \mathcal{A}_\chi(P) = \text{Hom}(\mathcal{A}^*(P), \mathbb{Z})$ by $[\sigma](\chi) = \deg(E_\sigma \times_P \chi)$, where $E_\sigma \times_P \chi$ is the line bundle associated to the $P$-bundle $E_\sigma$ defined by $\sigma$ via the character $\chi$ of $P$.

Fixing a polarization of the curve $C$ we can define a Hilbert scheme $\text{Hilb}^{[\sigma]}_{E/P}$, which parameterizes closed subschemes of $E/P$ whose Hilbert polynomials with respect
to a generating set of polarizations of $G/P$ coincide with that of $[\sigma]$. We also have the open subscheme $\text{Sec}_{E/P}^{[\sigma]}$ of the above Hilbert scheme parameterizing the space of sections of $\pi$. We have a partial ordering on the set $\mathcal{X}_e(P)$ defined by $[\sigma_1] \leq [\sigma_2]$ if for every dominant character $w$ of $P$ we have $([\sigma_2] - [\sigma_1], w) \in \mathbb{Z}_{\geq 0}$ and $([\sigma_2] - [\sigma_1], \chi) = 0$ for every $\chi \in \mathcal{X}(G)$. This defines a notion of a numerically minimal sections (types) that is those section (types) for which $[\sigma]$ is maximal with respect the above ordering among the numerical types of $P$ reductions of $E$. We show that given a $G$-bundle there are only finitely many minimal numerical types and for all these types we have similar dimension bounds as in the case of $[16]$. Our first result is a generalization of the dimension estimates for the Hilbert schemes in the case of non minimal sections.

Let $\gamma_1, \ldots, \gamma_m$ be the set of minimal numerical types for $E$. We show that if $X$ is an irreducible component of $\text{Hilb}_{E/P}^{[\sigma]}$, which contains the reduction of structure group $\sigma$ as a Hilbert point then there exists an $i \in \{1, \ldots, m\}$ such that $[\sigma] \leq \gamma_i$ and

$$\dim(X) \leq \dim(G/P) + d([\sigma]) - d(\gamma_i).$$

The above result is proved using a similar dimension estimates for the moduli space of maps. Let $\overline{M}_g(E/P, \beta_{[\sigma]})$ be the moduli space of stable maps from genus $g$ curves to $E/P$ with $\beta_{[\sigma]} \in H^2(E/P, \mathbb{Q})^*$ be a class determined by the numerical type and the fixed polarization of $C$. We show that if $X$ is an irreducible component of $\overline{M}_g(E/P, \beta_{[\sigma]})$ then there exists an $i \in \{1, \ldots, m\}$ such that $[\sigma] \leq \gamma_i$

$$\dim(X) \leq \dim(G/P) + d([\sigma]) - d(\gamma_i).$$

The above result is a generalization of a similar result of Mihnea Popa and Mike Roth $[25]$ for the case $G = GL_n$ and $P$ a maximal parabolic and the method of proof is similar to this case.

We next address the question of generic smoothness and irreducibility of the space of sections of $\pi$. We fix a root system of $G$ by considering a Borel subgroup $B$ and a maximal torus $T$. We say a numerical type $[\sigma] \in \mathcal{X}_e(P)$ satisfies the property $(\ast)$ for $N$ if $[\sigma](\chi) \leq -N$ for every non-trivial character of $P$ which when restricted to the maximal torus $T$ is a non-negative linear combination of simple roots with respect to the root system. We show that there exists an integer $N$ such that if $E$ admits a $P$ reduction of numerical type $[\sigma]$ satisfying the property $(\ast)$ for $N$ then $\text{Sec}_{E/P}^{[\sigma]}$ is irreducible and generically smooth of expected dimension $d([\sigma]) + (1 - g)\dim(G/P)$. This result is proved for the case $G = GL_n$ and $P$ a maximal parabolic in $[25]$ and for arbitrary $G$ and $P = B$ a Borel subgroup when the curve $C$ is over a finite field in Harder $[14]$. Using the methods in $[14]$ we first derive the above result for Borel subgroups over arbitrary fields and then generalize this to case of parabolic subgroups.

Next we define the notion of a generically stable $G$-bundles extending a similar notion for vector bundles as defined in Example 5.7 of $[25]$. Our main result is the existence of generically stable bundles when the genus of the curve $C$ is at least two. We also study some basic properties enjoyed by these $G$-bundles. For example we show that if
$E$ is a principal $G$ bundle which is generically stable then $d([\sigma]) \geq (g-1)\dim(G/P)$ for every $P$-reduction $\sigma$. This generalizes the lower bound result of Lange (see [21]). Some of these ideas should also lead to the computation of Gromov-Witten invariants for the twisted flag varieties $E/P$ and this will be done elsewhere.

For a fixed $G$-bundle $E$ we have two different compactifications namely the Hilbert scheme and the moduli space of maps. In general one can show that there are no morphisms between them. Also there are several partial Drinfeld compactifications we can define using representations of $G$. It is possible to understand which of these are the images of morphisms from earlier compactifications (see Gaitsgory-Braverman [10] for an account on Drinfeld compactifications). This part will appear elsewhere.

In proving these results we need several basic facts about principle bundles over curves. In Section 2, we prove a technical result about existence of $B$-reductions of $E$ following ideas of Ramanathan [27] and as a consequence we recover several basic properties of principal $G$-bundles. We partially answers a question of Friedman-Morgan on the behavior strata defined by the Harder-Narasimhan reduction. This is in the case when such a reduction is defined by Borel subgroups.

For a principal $G$-bundle $E$ we define a canonical element $c(E) \in \mathcal{X}_* (T)/\hat{Q}$, where $\hat{Q}$ is the coroot lattice of $G$, using a Borel reduction and show that this element $c(E)$ exactly parameterizes the algebraic equivalence classes of $G$-bundles over $C$, thus generalizing the fact that fundamental group of $G$ parameterizes the topological equivalence classes of $G$-bundles over $\mathbb{C}$ to arbitrary characteristic.

For an element $c \in \mathcal{X}_* (T)/\hat{Q}$ and a positive integer $d$, let $M(c,d)$ be the set of isomorphism classes of $G$-bundles of topological type $c$ such that the instability degree $\text{Id}(E) = \max \{ \deg(\text{ad}E_\sigma)(P,\sigma) \} \leq d$, where maximum is taken over all $P$ reductions $\sigma$ of $E$ and all parabolic subgroups. We construct a finite type irreducible smooth scheme $S$ and a family of $G$-bundles $\mathcal{E}$ over $C \times S$ which is versal at every $x \in S$ and such that for each $x \in S$ the bundle $\mathcal{E}_{C \times \{x\}}$ lies in $M(c,d)$ and every member in $M(c,d)$ is an occurs in the above family. We also show existence of stable bundles for curves of genus atleast two.

This paper is organized as follows. In Section 2 we describe the numerical types of $P$-reductions and study some basic properties. Section 3 deals with Borel reductions and here we prove results on algebraic equivalence and versal families and existence of stable bundles. The dimension estimates for the Hilbert schemes and moduli space of stable maps is dealt in Section 4. The irreducibility and generic smoothness is the content of Section 5. Finally in the last Section we prove results about generic stability of $G$-bundles.

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where the work was done.

## 2 Some basic facts about principal bundles

In this section we recall and prove some basic facts about principal $G$ bundles on $C$. Let $k$ be an algebraically closed field. Let $G$ be a connected reductive algebraic group over $k$. Let $T$ be a maximal torus and $B$ a Borel subgroup containing $T$. Let $U$ be the unipotent radical of $B$. Then $B$ is a semi-direct product $U \cdot T$. Let $i : T \hookrightarrow B$ and $j : B \hookrightarrow G$ be the inclusions. and $p_B : B \rightarrow B/U = T$ be the projection. Let $W = N(T)/T$ be the Weyl group and $w_0 \in W$ the element of maximal length in $W$. Let $\mathbb{G}_m$ be the one dimensional torus and $\mathbb{G}_a$ the additive group. We denote by $\mathcal{X}_s(T)$ be the group of 1-parameter subgroups of $T$ (denote by 1-PS). $\mathcal{X}^*(T)$ denotes the group of characters of $T$. We have a perfect pairing $\mathcal{X}_s(T) \otimes \mathcal{X}^*(T) \rightarrow \mathbb{Z}$ which will be denoted by $(\cdot, \cdot)$. Let $\Phi \subset \mathcal{X}^*(T)$ be the root system of $G$, $\Phi^+$ be the set of positive roots and $\Delta = \{\alpha_1, \ldots, \alpha_r\}$ the set of simple roots corresponding to $B$. For any $\alpha \in \Phi$, let $T_\alpha$ the connected component of $\ker(\alpha)$ and $Z_\alpha$ the centralizer of $T_\alpha$ in $G$. Then the derived group $[Z_\alpha, Z_\alpha]$ is of rank one and there is a unique 1-PS $\hat{\alpha} : \mathbb{G}_m \rightarrow T \cap [Z_\alpha, Z_\alpha]$ such that $T = (\text{im } \hat{\alpha}) \cdot T_\alpha$ and $(\hat{\alpha}, \alpha) = 2$. This $\hat{\alpha}$ is the coroot corresponding to $\alpha$. We denote by $\hat{\Phi}$ the set of coroots. The quadruple $\{\mathcal{X}^*(T), \Phi, \mathcal{X}_s(T), \hat{\Phi}\}$ defines a root system. For each $\alpha \in \Phi$ we have the fundamental dominant weight $w_\alpha \in \mathcal{X}^*(T) \otimes \mathbb{Q}$ defined by $(\text{hat } \beta, w_\alpha) = \delta_{\alpha, \beta}$ and $(\gamma, w_\alpha) = 0$ for any 1-parameter group in the connected component of the center of $G$. Let $Q \subset \mathcal{X}^*(T)$ (resp.$\hat{Q} \subset \mathcal{X}_s(T)$) be the (co)-root lattice generated by $\Phi$ (resp $\hat{\Phi}$). We have a partial ordering $\leq$ in $\mathcal{X}_s(T)$ defined by $\mu \leq \lambda$ if and only if $(\lambda - \mu, w_\alpha) \in \mathbb{Z}^{\geq 0}$ and $(\lambda - \mu, \chi) = 0$ for $\chi \in \mathcal{X}(G)$.

Let $P$ be a parabolic subgroup of $G$ containing $B$. Let $R_u(P)$ be its unipotent radical. then there is a subset $I \subset \Delta$ such that $P = P_I$ Let $Z_I = (\cap_{\alpha \in I} \ker(\alpha))^0$ be the connected component of the intersection of the kernels of the roots in $I$. then we have a chosen Levi decomposition $P = R_u P \cdot L$ such that $L$ a Levi subgroup containing $T$ defined to be the centralizer of $Z_I$. We will fix such a splitting $i : L \rightarrow P$.

Let $C$ be an smooth projective curve over $k$. Let $E$ is a principal $G$ bundle over $C$. Let $\sigma$ be a reduction of structure group of $E$ to $P$. By this we mean a pair $\sigma = (E_\sigma, \phi)$ with $E_P$ a principal $P$-bundle and an isomorphism $\phi : E_P \rightarrow E$, equivalently a reduction of structure group is a section $\sigma$ of the fiber bundle $\pi : E/P \rightarrow C$. Here $E/P$ denotes the extended fiber bundle $E \times^G G/P$ over $C$.

Let $T_\pi$ be the tangent bundle along the fibers of the map $\pi$. For a reduction of structure group $\sigma$ we will denote by $T_\sigma$ the vector bundle defined by the pull back of $T_\pi$ under $\sigma$. We will also fix notations for the Lie algebras by putting $\mathfrak{g}$, $\mathfrak{p}$, $\mathfrak{m}$, $\mathfrak{u}$ for Lie algebras of $G$, $P$, $L$ and $R_u P$ respectively. Then we see that $T_\sigma$ is the vector bundle on $C$ associated to $E_\sigma$ for the representation of $P$ on $\mathfrak{g}/\mathfrak{p}$.

First we state a lemma which bounds the degree of the tangent bundle along the fibers of the map $\pi$.
Lemma 2.1 There exists a constant $C$ (independent of $\sigma$) such that for any reduction $\sigma$ of $E$ to $P$ we have $\deg(T_\sigma) \geq C$.

Proof (see Lemma 2.1, [16]). □

Recall that the above lemma enables us to define the notion of a minimal reduction namely those reductions of structure groups for which $\deg(T_\sigma)$ is minimal.

There is a stronger notion of minimality of reductions we will be interested. Suppose $E_\sigma$ is the $P$-bundle associated to $\sigma$ then we define an element $[\sigma]$ of $X^*(P) = \text{Hom}(X^*(L), \mathbb{Z})$ by assigning $[\sigma](\chi) = \deg(\chi^*(E_\sigma))$. This point of $X_\sigma(L)$ actually determines the Atiyah-Bott point in the sense of Friedman-Morgan [7] of the reduction $\sigma$. We say that $[\sigma]$ is the numerical type of the reduction $\sigma$.

Now we can define a partial ordering on the elements of $X^*(L)$ as follows. We say the numerical types $[\sigma_1] \leq [\sigma_2]$ if for every dominant character $w$ of $P$ we have $([\sigma_2] - [\sigma_1], w) \in \mathbb{Z}^{\geq 0}$ and $([\sigma_2] - [\sigma_1], \chi) = 0$ for every $\chi \in X(G)$. We say a reduction of structure group $\sigma$ is numerically minimal if the numerical type $[\sigma]$ is maximal with respect the above ordering among the numerical types of $P$ reductions $E$. Here one observes that the second condition in the definition of the partial ordering is automatically satisfied if the the numerical types corresponds to $P$ reduction of a fixed principal $G$-bundle $E$.

One notes that $\deg(T_\sigma)$ depends only on the numerical type $[\sigma]$ corresponding to a reduction $\sigma$ and not on the reduction itself hence we denote $d([\sigma]) = \deg(T_\sigma)$ for some section $\sigma$ whose numerical type is $[\sigma]$. Also one observes the definition of $d([\sigma])$ can be enlarged to define it for any element of $X_\sigma(L)$ by $d([\sigma]) = [\sigma](\chi_P)$, where $\chi_P$ is the character of $P$, hence $L$, defined by the highest exterior power of the representation of $P$ on $g/p$.

Remark 2.2 A reduction of structure group $\sigma$ is numerically minimal if it is minimal with respect to the degrees of the tangent bundle $T_\sigma$. More generally $\sigma$ is minimal if $\deg(\sigma^*(L))$ is minimal with respect to all reductions for a fixed line bundle $L$ which is ample along the fibers of $\pi$ (this corresponds to negative powers of dominant character of $P$ upto tensoring with a line bundle pulled up from $C$).

The first basic lemma is the existence of numerically minimal sections.

Lemma 2.3 For a given $N$, there are only finitely many numerical types $[\tau]$, defined by $P$-reductions of a fixed $G$-bundle $E$, with the property that $d([\tau]) \leq N$. Moreover given any $P$-reduction $\sigma$ there exists an numerically minimal reduction $\sigma_0$ such that $[\sigma] \leq [\sigma_0]$.

Proof We first prove the lemma for the case when the parabolic $P$ is maximal. Further we may assume that the parabolic $P$ is of the form $P_{\Delta_{-\alpha}}$ for some $\alpha \in \Delta$ by fixing all the root datum. Now both parts of the lemma follows from the Lemma 2.3 and the fact that $[\sigma]$ is completely determined by $d([\sigma])$. This proves the lemma in this case. For a given $\alpha \in \Delta$ let $n_\alpha$ be a positive integer such that $-n_\alpha w_\alpha$ defines
the character $\chi_{P_{\Delta^-}}$. For the general case, we assume the parabolic $P$ is of the form $P = P/I$ for a subset $I \in \Delta$. The numerical type of $\tau$ is completely determined by what values it takes on the characters $n_\alpha w_\alpha$ for $\alpha \in I$ and the characters of the group $G$. Since any reduction to $P$ automatically determines a reduction to the parabolic $P_{\Delta^- \alpha}$, and using the conclusion of the lemma for the maximal parabolics, we see that $[\sigma](n_\alpha w_\alpha)$ is bounded above as we vary $\sigma$ over the reductions to $P$. Hence we see that subset of $\mathcal{X}_*(L)$ we are interested in is finite. The second assertion in the lemma is a consequence of the first. □

Recall that a principal $G$-bundle is said to be (semi)stable if for any reduction of structure group $\sigma$ of $E$ to any parabolic we have $\deg(T_\sigma) > (\geq)0$. This definition is equivalent to the condition that for any maximal parabolic $P$ and a dominant character $w$ of $P$ we have $\deg(w_*(E_\sigma)) < (\leq)0$.

If $P$ is a Borel subgroup $B$ then the numerical type $[\sigma]$ defines a 1-PS on the maximal torus $T$. Further for any $T$-bundle $E_T$ we can similarly define its numerical type by a 1-PS $[E_T]$ defined by $[E_T](\chi) = \deg(\chi_*(E_T))$.

Fix a polarization of the curve $C$. Recall that for a principal $G$ bundle $\mathcal{E}$ over $C \times S$ for a scheme $S$ of finite type over $k$ there is a projective scheme $\text{Hilb}^p_{\mathcal{E}/P,S}$ over $S$ parameterizing the closed subschemes of $\mathcal{E}/P$ flat over $S$ with a fixed Hilbert polynomial $p$. When the parabolic is not maximal we see that the Hilbert scheme further decomposes into open and closed subschemes owing to the fact that we have many polarizations of $G/P$. Hence we have to fix a finite set of generating polarizations of $G/P$ to set a topological type. One can check that these polarizations can be computed by the numerical type $[\sigma]$ of reductions. Hence we define the Hilbert scheme $\text{Hilb}^{[\sigma]}_{\mathcal{E}/P}$ which parameterizes closed subschemes of $\mathcal{E}/P$ flat over $S$ with all Hilbert polynomials same as that of $\sigma$. This defines an open and closed subschemes of the Hilbert scheme defined above. Also we have an open subscheme of the Hilbert scheme $\text{Hilb}^{[\sigma]}_{\mathcal{E}/P}$ corresponding to the subschemes which are actually sections of the morphism $\pi_S: \mathcal{E}/P \to S$. We will denote this scheme by $\text{Sec}^{[\sigma]}_{\mathcal{E}/P}$. One of the properties of the Hilbert scheme and the space of sections that will be useful to us is their behavior under the base change. Namely for any morphism $S' \to S$ we have natural isomorphisms $\text{Hilb}^{[\sigma]}_{\mathcal{E}/P,S} \cong \text{Hilb}^{[\sigma]}_{\mathcal{E}/P \times S S'}$ and $\text{Sec}^{[\sigma]}_{\mathcal{E}/P,S} \cong \text{Sec}^{[\sigma]}_{\mathcal{E}/P \times S S'}$. Here $\mathcal{E}_{S'}$ denotes the pull back of $\mathcal{E}$ under the morphism $S' \to S$.

we now prove a Lemma about the space of sections which will be used later.

**Lemma 2.4** Let $[\sigma]$ be an numerical type. Let $\mathcal{E}$ be a family of $G$-bundles over $C \times S$ with $S$ a finite type scheme. Then the subset of points in $S$ corresponding to principal $G$-bundles which admits $P$ reduction of numerical type $[\sigma]$ is constructible.

**Proof** The lemma follows from the fact that the subset we are interested in is exactly the image of the morphism $\text{Sec}^{[\sigma]}_{\mathcal{E}/P} \to S$. Since the space of sections are finite type over $S$ hence the image is constructible. □

Let $P_1 \subset P$ be two parabolic subgroups of $G$. Let $L$ (resp. $L_1$) be the Levi quotients of $P$ (resp. $P_1$) and $Z_0(L)$ (resp. $Z_0(L_1)$) be the connected component of the center.
Let $\mathcal{L}$ denote the quotient $L/Z_0(L)$. Let $\mathcal{P}_1$ be the parabolic subgroup of $\mathcal{L}$ defined by the image of $P_1$ under the natural map $p : P \to L$. Let $E$ be a principal $G$-bundle over $C$ and let $\sigma$ be a $P$ reduction of $E$. Let $E_1 = p_*(E_\sigma)$ be the associated principle $\mathcal{L}$-bundle over $C$. We have the following lemma.

**Lemma 2.5** If $[\sigma_1] \in \mathcal{X}_*(\mathcal{P}_1)$ is such that $E_1$ admits a $\mathcal{P}_1$ reductions of numerical type $[\sigma_1]$ then there is a unique $[\sigma_1] \in \mathcal{X}_*(P_1)$ such that there is a bijective correspondence between the $P_1$ reductions of $E_\sigma$ of numerical type $[\sigma_1]$ and $\mathcal{P}_1$ reductions of $E_1$ of numerical type $[\sigma_1]$.

**Proof** The main observation is that we have natural isomorphisms $P/P_1 \cong \mathcal{L}/\mathcal{P}_1$. From here it follows that there is a natural bijection between $P_1$ reductions of $E_\sigma$ and $\mathcal{P}_1$ reductions of $E_1$. Let $\sigma_1$ be a reduction of structure group of $E_1$ to $P_1$ and let $\sigma_1$ be the $P_1$ reduction of $E_\sigma$ which corresponds to $\sigma_1$ under the above bijection. Now the natural morphism $\mathcal{X}_*(P_1) \to \mathcal{X}_*(\mathcal{P}_1)$ takes $[\sigma_1]$ to $[\sigma_1]$ and the natural morphism $\mathcal{X}_*(P_1) \to \mathcal{X}_*(\mathcal{P}_1)$ takes $[\sigma_1]$ to $[\sigma_1]$. The proof of the lemma will be complete once we establish the injectivity of the homomorphism $\mathcal{X}_*(P_1) \to \mathcal{X}_*(P_1) \oplus \mathcal{X}_*(\mathcal{P}_1)$. The last statement follows from the fact that any character of $P_1$ can be uniquely written as a rational linear combination of a character of $P$ and a character of $\mathcal{P}_1$. □

### 3 Borel reductions and algebraic equivalence

In this section prove a result about Borel reductions of $G$-bundles and as a consequence we derive results on algebraic equivalence of principal bundles and irreducibility of the moduli spaces etc.

We will now define a notation which will be used through in the article. We say a point $[\sigma] \in \chi_*(L)(or a reduction of structure group \sigma)$ satisfies the property $(\ast)$ for $N$ if

$$-[\sigma](\chi) \geq N$$

for every non-trivial character $\chi$ which when restricted to $T$ is a non-negative linear combination of simple roots.

Note that a reduction $\sigma$ of $E$ to the Borel subgroup $B$ satisfies $(\ast)$ for $N \geq 2g - 1$ then for each positive root $\alpha$ the line bundle $(-\alpha)_*(E_\sigma)$ is globally generated and satisfies $H^1(C, (-\alpha)_*(E_\sigma)) = 0$.

We now state a basic lemma due to Drinfeld-Simpson [29] about the existence of reductions $\sigma$ satisfying the property $(\ast)$ for any $N$.

**Lemma 3.1** For any $N$ there exists a reduction of structure group $\sigma$ of $E$ to $P$ satisfying the property $(\ast)$ for $N$

**Proof** The result is shown for the case $P = B$ in Drinfeld-Simpson [29] and for the case of arbitrary $P$ one just observes that giving a reduction of structure group
to $B$ automatically gives rise to a reduction of structure group to $P$, by extension and a reduction to $B$ satisfying the property ($\ast$) for $N$ will give rise to a $P$ reduction of $E$ satisfying the same conditions.

\[\square\]

**Remark 3.2** We wish to indicate that the proof in [29] first reduces the problem to the case when $E$ is a trivial bundle by choosing a trivialisation on a Zariski open subset of $C$. Then one can reduce this problem to the case when $G$ is simply connected and $C$ is a projective line. At this stage one can recover the result by using the Theorem 7.4 and Proposition 6.13 of Ramanathan [27].

Let $S$ be a finite type scheme over $k$. Let $E_L$ be a principal $L$-bundle over $C \times S$. There is an conjugation action of $L$ on $\mathfrak{r}_P$ using the splitting of $P \to L$. Hence there is a filtration

\[R_aP = U_1 \supset U_2 \supset \cdots \supset U_k \supset U_{k+1} = \{e\}\]  

such that $U_j$ is normal in $P$, each quotient $U_j/U_{j+1}$ are invariant under the action of $L$, lies in the center of $R_aP/U_{j+1}$ and define irreducible representations of $L$.

Consider the functor $H^1_S(C, R_aP(E_L))$ from $(\text{Sch}/S)_{\text{fppf}} \to \text{Sets}$ which takes a scheme $(s : S' \to S)$ to isomorphism classes of pairs $(E_P, \phi)$ where $E_P$ is a $P$-bundle on $C \times S'$ and $\phi$ is an isomorphism $\phi : p_*(E_P) \cong s^*(E_L)$. We now record the following lemma for later use.

**Lemma 3.3** The functor $H^1_S(C, R_aP(E_L))$ is representable by an affine bundle over $S$ under the assumption $H^0(C, U_j/U_{j+1}(E_L)|_{C \times \{s\}}) = 0$ for each $j = 1, \ldots, k$ and $s \in S$, where $U_j/U_{j+1}(E_L)$ is a vector bundle on $C \times S$ associated to the representation of $L$ on $U_j/U_{j+1}$.

**Proof** See Theorem A.2.6 of Friedman and Morgan [8] for the proof. \[\square\]

The following two lemmas hold when the characteristic of the field $k$ is zero or when the parabolic $P$ is Borel. The method of proof is similar to that of the proof of Lemma 3.6 of Kumar-Narasimhan [18]. We will use these results only for the case of Borel subgroups.

**Lemma 3.4** Suppose $E$ is a $G$ bundle which admits a reduction of structure group $\sigma$ to $P$ such that the property ($\ast$) holds for $N = 1$ and that the associated Levi bundle $E_L = p_*(E_\sigma)$ is semi-stable then $H^0(C, U_j/U_{j+1}(E_L)) = 0$ for each $j$.

**Proof** Since the representation of $L$ on $U_j/U_{j+1}$ is irreducible hence the vector bundle $U_j/U_{j+1}(E_L)$ is semistable. The degree of this vector bundle is recovered from the character of $L$ on the highest exterior power of $U_j/U_{j+1}$ which when restricted to $T$ is non-trivial and is a non-negative linear combination of simple roots hence by condition ($\ast$) for $N = 1$ we have $\deg(U_j/U_{j+1}(E_L)) < 0$. This implies that $H^0(C, U_j/U_{j+1}(E_L)) = 0$. \[\square\]
Lemma 3.5  Let $E$ be a principal $G$-bundle which admits a $P$-reduction $\sigma$ such that $[\sigma]$ satisfies the property $(\ast)$ for $N = 2g - 1$ and that the associated $L$-bundle $E_L$ is semistable. Then $H^1(C, T_\sigma) = 0$

Proof  Let $g$ and $p$ be the lie algebras of $G$ and $P$ respectively. Consider the filtration $0 = V_0 \subset V_1 \subset \ldots \subset V_k = g/p$ of $P$-submodules $V_i$ such that each quotient $V_i/V_{i-1}$ defines an irreducible representation of $P$ (hence of $L$). Hence we get a filtration of the vector bundle $T_\sigma$ for the action of $L$ on these successive quotients. Now again the successive quotients define semistable vector bundles on $C$ and that the character of $T$ defined by restriction of the representation of $L$ on $V_i/V_{i-1}$ is a negative linear combination of simple roots, hence by stability of $E_L$ and the condition $(\ast)$ for $N = 2g - 1$ we see that $H^1(C, T_\sigma) = 0$. 

We record here a corollary of the above lemma about smoothness of the space of sections using the deformation theory of the Hilbert schemes.

Corollary 3.6  If $\mathcal{E}$ is a family of $G$-bundles over a smooth projective curve $C$ over a scheme $S$. If $\mathcal{E}$ is a family of $G$-bundles over a smooth projective curve $C$ over a scheme $S$ of genus $g$. Let $[\sigma] \in \mathcal{X}_s(L)$ be a point satisfying the property $(\ast)$ for $N = 2g - 1$. Let $y \in \text{Sec}^{[\sigma]}_{\mathcal{E}/P}$ be a Hilbert point of a $P$ reduction $\sigma$ of $\mathcal{E}_x$ with $x \in S$ such that the associated $L$-bundle $p_*E_\sigma$ is semistable. Then the natural morphism $\text{Sec}^{[\sigma]}_{\mathcal{E}/P} \longrightarrow S$ is smooth at $y$.

Now prove a result which generalizes the Theorem 7.4 of [27] and has a similar proof.

Proposition 3.7  Let $\sigma$ be a $B$-reduction of a principal $G$-bundle $E$ satisfying the property $(\ast)$ for $N = 2g$. If $\mu$ be a $1$-PS such that $w_0[\sigma] \leq \mu$, then $E$ admits a $P$ reduction $\sigma_0$ with $w_0[\sigma_0] = \mu$.

The following lemma is an extension of the Lemma 7.4.1 of [27] and is a step in the proof of the Proposition 3.7.

Lemma 3.8  With the above notations there exists a sequence $w_0[\sigma] = \mu_1, \mu_2, \ldots, \mu_n = \mu$ of elements of $\mathcal{X}_s(T)$ such that $\mu_{i+1} = \mu_i + \alpha_j$ for some $\alpha_j \in \Delta$ with $(\mu_i, \alpha_j) \geq 2g - 1$.

Proof  We set $w_0[\sigma] = \mu_1$. We prove the lemma by a downward induction. If for each $l > i$, $\mu_l$ has been chosen such that $\mu_1 \leq \mu_i$ and $\mu_{i+1} = \mu_i + \alpha_j$, for some $\alpha_j \in \Delta$ with $(\mu_i, \alpha_j) \geq 2g - 1$ then we want to make a choice for $\mu_i$. Since $\mu_1 \leq \mu_{i+1}$ we can write $\mu_{i+1} = \mu_1 + \sum k_m \alpha_m$ with $\alpha_m \in \Delta$ and $k_m \geq 0$. Now we want to get rid of one of the $\alpha_m$ with $k_m > 0$. First we claim that $(\sum k_m \alpha_m, \alpha) > 0$ for some $\alpha \in \Delta$ such that $k_m > 0$. If not then we would have $k_m = (\sum k_m \alpha_m, w_m) \leq 0$ for every dominant weight $w_m$ as the dominant weights are non-negative rational linear combinations of the simple
roots and the fact that $(\hat{\alpha}_m, \alpha_n) < 0$ for $m \neq n$. For such an $\alpha$, using the condition $(\mu_1, \alpha) \geq 2g$ we get $(\mu_{i+1}, \alpha) > 2g$, hence $(\mu_{i+1} - \hat{\alpha}, \alpha) \geq 2g - 1$ as $(\hat{\alpha}, \alpha) = 2$. Now we define $\mu_i = \mu_{i+1} - \hat{\alpha}$ and the lemma follows by induction. $\square$

The following lemma along with the preceding one implies the Proposition 3.7.

**Lemma 3.9** Let $\mu, \nu$ be 1-PS such that $\mu = \nu + \hat{\alpha}$ for some $\alpha \in \Delta$ and $(\nu, \alpha) \geq 2g - 1$. Let $E$ be a principal $G$-bundle with a reduction of structure group $\sigma$ to $B$ with the property that $w_0[\sigma] = \nu$. Then $E$ admits a $B$ reduction $\sigma_0$ such that $w_0[\sigma_0] = \mu$

**Proof** Let $P_\alpha$ be the minimal parabolic containing $B$ defined by the simple root $\alpha$. Let $P_\alpha = R_\alpha(P_\alpha) L$ be its Levi decomposition. Let $Z_\alpha = (\ker \alpha)^0$ be the connected component of the kernel of $\alpha$. Then $Z_\alpha$ is the connected component of the center and $L = L/Z_\alpha$ is a rank 1 semisimple group. We have the projection $P_\alpha \to L$ which induces an isomorphism $P_\alpha/B \cong \overline{L}/B = \mathbb{P}^1$. The root $\alpha$ induces a $\overline{\alpha} \in \mathcal{X}^*(T/Z_\alpha)$ which is the simple root of the $\overline{L}$. The coroot $\overline{\alpha}$ on $\overline{L}$ is the image of $\hat{\alpha}$ under $X_\alpha(T) \to X_\alpha(T/Z_\alpha)$. Similarly we have $\overline{\alpha}$ and $\overline{\nu}$ as the images of $\alpha$ and $\nu$ under $X_\alpha(T) \to X_\alpha(T/Z_\alpha)$.

Now for any $\nu \in X_\alpha(T)$ the integer $(\nu, \alpha)$ is determined by the composite $\alpha \circ \nu : \mathbb{G}_m \to \mathbb{G}_m$ which takes $z \mapsto z^{(\nu, \alpha)}$. Since the map $\alpha : T \to \mathbb{G}_m$ factors through $T/Z_\alpha$, we see that $(\nu, \alpha) = (\overline{\nu}, \overline{\alpha})$.

Let $E_\sigma$ be the $B$-bundle associated to $\sigma$ as in the statement of the lemma satisfying $w_0[\sigma] = \nu$. Then $E_\sigma$ gives rise to a $P_\alpha$-bundle by extension of structure group which we denote by $E_{\sigma, \alpha}$. We also denote $E_{\sigma, \alpha}$ the $\overline{L}$-bundle obtained by the extension of structure group $P_\alpha \to \overline{L}$.

By Lemma 2.5 there is a bijective correspondence between the $B$ reductions of $E_{\sigma, \alpha}$ of numerical type $[\sigma]$ and $\overline{B}$ reductions of $E_{\sigma, \alpha}$ of numerical type $[\overline{\sigma}]$. Let $\sigma_1$ be a $\overline{B}$ reduction of $E_{\sigma, \alpha}$ and $\overline{\sigma}_1$ be the $\overline{B}$ reduction of $E_{\sigma, \alpha}$ under the above correspondence. Now again by arguments of the lemma 2.3 we see that $[\sigma] - \nu$ is a multiple of the coroot $\hat{\alpha}$

Hence we are reduced to proving the Lemma for the case of rank 1 semi-simple groups. This case follows from the following lemma as $\text{PGL}(2)$-bundles on curves come from vector bundles.

**Lemma 3.10** Let $V$ be a rank two vector bundle which can be written as an exact sequence

$$0 \to L_1 \to V \to L_2 \to 0$$

such that $\deg(L_2) - \deg(L_1) = m > 2g - 2$. Then $V$ can be written as an exact sequence

$$0 \to L'_1 \to V \to L'_2 \to 0$$

such that $\deg(L_2) - \deg(L_1) = m + 2$. 

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Proof  Consider the family $V$ of vector bundles on $C \times \mathbb{A}^1$ such that $V_{C \times \{0\}} = L_1 \oplus L_2$ and $V_{C \times \{x\}} = V$ for $x \in \mathbb{A}^1 - 0$. The condition $m > 2g - 2$ ensures that $\text{Sec}_{\nu}^m \to \mathbb{A}^1$ is smooth. Hence it is enough to prove the result for the case $V = L_1 \oplus L_2$. In this case we can construct a sub-bundle $L_1(-x) \hookrightarrow L_1 \oplus L_2$ by choosing a section of $L_2 \otimes L_1^{-1}(x)$ which does not vanish at $x \in C$. 

Now the Proposition 3.7 follows by the Lemma 3.8 and Lemma 3.9. 

Remark 3.11  One observes that the Proposition 3.7 holds even if the ground field is not algebraically closed on the condition that $C$ has a rational point.

In the rest of the section we give some applications of the above result.

We fix the above notation. Let $B_0$ be the opposite Borel subgroup containing $T$ and the negative root spaces.

In the special case when the Harder-Narasimhan reduction of a principal $G$-bundle is defined on a Borel subgroup satisfying some conditions we answer a question of Friedman-Morgan (see [7]) on the behavior of the strata with respect to deformation.

Proposition 3.12  Suppose $\mu$ and $\nu$ are two dominant 1-PS satisfying the conditions $\nu(\alpha) \geq 2g$ for each $\alpha \in \Delta$ and $\nu \leq \mu$. Then there exists a principal $G$-bundle $E$ whose canonical reduction $\sigma_0$ is a reduction on a Borel subgroup $B_0$ satisfying $[\sigma_0] = \nu$ and there exists a family $E$ of $G$-bundles on $C \times \mathbb{A}^1$ such that for $E|_{C \times \{\mu\}} \cong \text{pr}_1^*(E)$ and $E|_{C \times 0}$ is a principal $G$-bundle whose canonical reduction $\tau_0$ is again defined on the Borel subgroup $B_0$ satisfying $[\tau_0] = \mu$.

Proof  Let $\mu$ and $\nu$ be as in the statement of the Proposition. Let $E_T$ be a principal $T$ bundle such that $w_0[E_T] = \nu$. Let $E$ be the principal $G$ bundle obtained by extension to $G$. Let $B$ and $B_0$ be the opposite Borel subgroups intersecting exactly on $T$. We then get by extending to $B$ and $B_0$, reductions of structure groups $\sigma$ and $\sigma_0$ of $E$ to $B$ and $B_0$ respectively satisfying $w_0[\sigma] = [\sigma_0] = \nu$ and $\sigma_0$ being the canonical reduction of $E$ (by the conditions stated in the proposition). Now we have a $G$-bundle $E$ with a reduction $\sigma$ to $B$ and a $\mu$ such that $w_0[\sigma] = \nu \leq \mu$. Hence by Theorem 3.7 we get a $B$ reduction $\tau$ of $E$ such that $w_0[\tau] = \mu$. Now by Proposition 3.7(a) of Kumar-Narasimhan-Ramanathan [19] we find a family $E$ of $G$-bundles on $C \times \mathbb{A}^1$ such that for $E|_{C \times \{\mu\}} \cong \text{pr}_1^*(E)$ and $E|_{C \times 0} \cong E_1$ where Now it is easy to see that $E_1$ is a $G$-bundle whose canonical reduction $\tau_0$ defined on the Borel subgroup $B_0$ satisfying $[\tau_0] = \mu$.

Next we address the question of characterization of algebraic families of $G$-bundles on $C$ and irreducibility of the moduli spaces. This part is actually mentioned in Ramanathan [27] without proofs. We also wish to make reference to Drinfeld-Simpson [28] where the irreducibility of the moduli stack of $G$-bundles are proved. But the basic results here are a little more explicit using the boundedness theorem in [16].

Recall the following definition. Two principal $G$ bundles $E$ and $F$ are algebraically equivalent if there is a connected variety $S$ of finite type over $k$ and a family $E$ of
$G$-bundles on $C \times S$ and two $k$-valued points $s_0$ and $s_1$ such that $\mathcal{E}|C \times s_0 \cong E$ and $\mathcal{E}|C \times s_1 \cong F$.

Let $\hat{Q}$ denote the lattice of coroots in $\mathcal{X}_s(T)$. When $k = \mathbb{C}$, it is well known that $\mathcal{X}_s(T)/\hat{Q}$ classifies principal $G$-bundles topologically. What we prove here is the algebraic classification of principal $G$-bundles in arbitrary characteristic.

We first state a preliminary lemma which is easy to prove and will be used in the sequel.

**Lemma 3.13** If $\mu_1$ and $\mu_2$ are 1-PS in $T$ such that their images in $\mathcal{X}_s(T)/\hat{Q}$ are equal then there is a 1-PS $\mu$ such that $\mu_i \leq \mu$ for $i = 1, 2$ and that $\mu(\alpha) > 2g - 1$ for each $\alpha \in \Phi^+$.

Given a principal $G$-bundle $E$ and a reduction $\sigma$ of $E$ to the Borel $B$ we define $c(E)_\sigma$ to be the image of $[\sigma]$ in $\mathcal{X}_s(T)/\hat{Q}$.

**Lemma 3.14** For any two reductions $\sigma$ and $\tau$ of $E$ to $B$, we have $c(E)_\sigma = c(E)_\tau$. In other words $c(E)_\sigma$ is independent of the reduction $\sigma$.

**Proof** The idea of the proof is already there in Proposition 6.16 of Ramanathan [27]. To prove the lemma we may assume that $G$ is semi-simple. This is because for any character $\chi \in \mathcal{X}_s(G)$ we have $\chi_*(E_\sigma) = \chi_*(E_\tau) = \chi_*(E)$. Let $\tilde{G}$ be the simply connected covering group. Let $Z$ be the kernel of $\tilde{G} \to G$. Let $\tilde{T}$ and $\tilde{B}$ be the maximal torus and the Borel subgroups of $\tilde{G}$ which are the inverse images of $T$ and $B$ respectively. As in the proof of the Proposition 6.16 of [27], we have a commuting diagram with horizontal rows exact in the flat topology on $C$ (and not exact in the category of group schemes)

$$
\begin{array}{cccccc}
1 & \to & Z & \to & \tilde{G} & \to & G & \to & 1 \\
\| & & \uparrow & & \uparrow & & \uparrow & & \\
1 & \to & Z & \to & \tilde{B} & \to & B & \to & 1 \\
\| & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \to & Z & \to & \tilde{T} & \to & T & \to & 1 \\
\end{array}
$$

Then this gives rise to the following commuting diagram of flat cohomologies with horizontal rows exact.

$$
\begin{align*}
H^1(C, \tilde{G}) & \to H^1(C, G) \xrightarrow{\delta_1} H^2(C, Z) \\
\uparrow & \quad \uparrow j_* & \quad \| & \\
H^1(C, \tilde{B}) & \to H^1(C, B) \to H^2(C, Z) \\
\downarrow & \quad \downarrow p_* & \quad \| & \\
H^1(C, \tilde{T}) & \xrightarrow{q_*} H^1(C, T) \xrightarrow{\delta_2} H^2(C, Z)
\end{align*}
$$

Now if $\sigma$ and $\tau$ are two reductions of $E$ to $B$ then the corresponding classes $E_\sigma$ and $E_\tau$ in $H^1(C, B)$ satisfies $j_*(E_\sigma) = j_*(E_\tau)$ hence from the commutativity of the
The exactness of the bottom row implies existence of an element \( a \in H^1(C, T) \) such that \( q_*(a) = p_*(E_\sigma) - p_*(E_\tau) \). Now it is easy to see from the definitions that the following diagram commutes

\[
\begin{array}{ccc}
H^1(C, \tilde{T}) & \xrightarrow{q_*} & H^1(C, T) \\
\downarrow [\cdot] & & \downarrow [\cdot] \\
\mathcal{X}_*(\tilde{T}) & \xrightarrow{q_*} & \mathcal{X}_*(T)
\end{array}
\]

Hence we see that \([\sigma] = [p_*(E_\sigma)]\) and \([\tau] = [p_*(E_\sigma)]\) differ by an element of \( \mathcal{X}_*(\tilde{T}) \). Since \( q_*(\mathcal{X}_*(\tilde{T})) \) is exactly the lattice generated by the coroots we are through with proof of the lemma.

The above lemma enables us to define a map \( c \) for the isomorphism classes of principal \( G \) bundles to \( \mathcal{X}_*(T)/\hat{\mathcal{Q}} \) by assigning \( c(E) = c(E)_\sigma \) for any reduction \( \sigma \) of \( E \) to \( B \). We call \( c(E) \) the topological type of \( E \).

**Proposition 3.15** Two principal \( G \) bundles \( E \) and \( F \) are algebraically equivalent if and only if \( c(E) = c(F) \).

**Proof** To prove the “only if” part of the proposition we need to only consider the case when the principal \( G \)-bundles \( E \) and \( F \) sit in an irreducible family. Let \( S \) be an irreducible finite type scheme over \( k \) with two points \( s_0 \) and \( s_1 \). Let \( \mathcal{E} \) be a family of \( G \)-bundles on \( C \times S \) such that \( \mathcal{E}|_{C \times s_0} \cong E \) and \( \mathcal{E}|_{C \times s_1} \cong F \). Now choose a reduction of structure of structure group \( \sigma \) of \( E \) to \( B \) satisfying the property \((*)\) for \( N = 2g - 1 \). Then by Corollary 3.6 the morphism \( \text{Sec}_{\mathcal{E}/B}^{[\sigma]} \to S \) is smooth and the image contains the point \( s_0 \) defined by \( E \), hence it contains a neighborhood of \( s_0 \).

This gives us an open subset \( U \) of \( S \) with the property that for \( s \in U \) the bundle \( \mathcal{E}|_{C \times S} \) admits a \( B \) reduction of type \([\sigma]\). Similarly by choosing a reduction \( \tau \) of \( F \) satisfying the condition \((*)\) for \( N = 2g - 1 \) we get another open subset \( V \) of \( S \) such that for each \( s' \in V \), the bundle \( \mathcal{E}|_{C \times S} \) admits a \( B \) reduction of type \([\tau]\). Since \( S \) is irreducible, \( U \) and \( V \) have non-trivial intersection, hence we produce a principal \( G \)-bundle \( E' \) which admits a reduction to \( B \) with numerical types \([\sigma]\) and \([\tau]\). Now by Lemma 3.14 we have \( c(E) = c(E)_\sigma = c(E')_\sigma = c(E')_\tau = c(F)_\tau = c(F) \). This proves the only if part of the proposition.

For the other part of the proposition the main idea is the construction of \( T \)-bundles which extend to a fixed \( T \)-bundle. Let \( E \) and \( F \) be principal \( G \) bundles such the \( c(E) = c(F) \). Choose reductions of structure group \( \sigma \) and \( \tau \) of \( E \) and \( F \) respectively satisfying \((*)\) with \( N = 2g \). Since \([\sigma] - [\tau]\) is an integral linear combination of coroots, by Lemma 3.13 there is a dominant 1-PS \( \mu \) with the property that \( w_0[\sigma] \leq \mu \) and \( w_0[\tau] \leq \mu \). Now by Theorem 3.7 there are reductions \( \sigma_0 \) and \( \tau_0 \) of \( E \) and \( F \) respectively with the property that \( w_0[\sigma_0] = w_0[\tau] = \mu \). Let \( S \) be the moduli space of \( T \)-bundles of type \( w_0\mu \). The space \( S \) is essentially a product of Jacobians. There is also a universal family \( \mathcal{E}_T \) over \( C \times S \). The conjugation action of \( T \) on \( U \) using the splitting of \( B \to T \) gives us a filtration as in (II). The
condition $(\ast)$ for $N = 1$ along with Lemma 3.3 and Lemma 3.4 implies that the functor $H^1_S(C, U(E_T))$ is representable by an affine bundle $\mathcal{H}$ over $S$. Let $\mathcal{B}$ be the universal family of $B$ bundles over $C \times \mathcal{H}$. Now this universal family when extended to $G$ gives a family of $G$-bundles over a finite type irreducible scheme $\mathcal{H}$ (in fact smooth) containing both $E$ and $F$. This completes the proof of the proposition. □

In the following Lemma we relate the invariant $c$ with the degree of the $G$-bundle. Consider the natural map $g : \mathcal{X}_s(T) \longrightarrow \mathcal{X}_s(G) := \text{Hom}(\mathcal{X}_s(G), \mathbb{Z})$ defined by the dual of the restriction map $\mathcal{X}_s(G) \rightarrow \mathcal{X}_s(T)$.

**Lemma 3.16** The homomorphism $g$ factors through $\hat{\mathcal{I}}$ to give a homomorphism $g : \mathcal{X}_s(T) / \hat{\mathcal{I}} \rightarrow \mathcal{X}_s(G)$.

**Proof** for the proof we have to show that any coroot $\hat{\alpha}$ with $\alpha \in \Phi$ acts trivially on any character $\chi$ of $G$. This follows from the definition of the coroot as a homomorphism from the maximal torus of $SL_2$ to $G$. Now the fact that any character on $SL_2$ is trivial implies the result. □

**Remark 3.17** Recall that the composition $d \circ c(E)$ is equal to the degree of the principal $G$-bundle as defined in [15].

With the above notations Let $P = P_I$ be the parabolic containing the Borel subgroup $B$ corresponding to the subset $I \subset \Delta$. Let $\mathcal{L} = L / Z_0(L)$ be the quotient of the Levi by the connected component of the center. We denote the projection map $P \rightarrow \mathcal{L}$ by $p$. Let $Q_{\mathcal{T}}$ be the coroot lattice of $\mathcal{T}$. Then $\hat{Q}_{\mathcal{T}}$ is the lattice generated by the coroots $\hat{\alpha}$ with $\alpha \in I$. We also denote by $\mathcal{B}$ (resp. $\mathcal{T}$) the Borel subgroup (maximal torus) of $\mathcal{L}$ defined the images of the respective objects under $p$. We use the notation $c_G$ and $c_L$ for the topological type $c$ to indicate difference when we are working with $G$-bundles and $\mathcal{L}$-bundles respectively.

The following is a lemma which is an algebraic version of the Lemma 2.4 of Friedman-Morgan [4] whose proof is also very similar.

**Lemma 3.18** If $E$ is a principal $G$-bundle which admits a $P$ reduction $\sigma$ then the numerical type $[\sigma]$ and the element $c_G(E)$ determines the element $c_L(p_*(E_\sigma))$.

**Proof** Suppose $E$ and $F$ are two principal $G$-bundles which are algebraically equivalent. Let $\sigma$ and $\tau$ be $P$-reductions of $E$ and $F$ respectively. Let $\sigma_0$ and $\tau_0$ be further reductions of the $\mathcal{L}$-bundles $p_*E_\sigma$ and $p_*F_\tau$. Then these determine reductions $\sigma_0$ and $\tau_0$ of $E$ and $F$ to $B$ (by Lemma 2.3). With the conditions $c_G(E) = c_G(F)$ and $[\sigma] = [\tau]$ we have to show that $c_L(p_*(E_\sigma)) = c_L(p_*(F_\tau))$. The condition $[\sigma] = [\tau]$ implies that the difference $[\sigma_0] - [\tau_0]$ is in the kernel of the natural map $\mathcal{X}_s(T) \otimes \mathbb{Q} \longrightarrow \mathcal{X}_s(P) \otimes \mathbb{Q}$ which is exactly equal to $\hat{Q}_L \otimes \mathbb{Q}$. Hence $[\sigma_0] - [\tau_0]$ is a rational linear combination of elements of $\hat{\alpha}$ with $\alpha \in I$. Now the condition $c_G(E) = c_G(F)$ implies that $[\sigma_0] - [\tau_0]$ is an integral linear combination of elements $\hat{\alpha}$ with $\alpha \in \Delta$. This proves the lemma. □
Recall the definition of the instability degree of a principal $G$-bundle $E$

$$\text{Ideg}_G(E) = \text{Max}\{\text{deg(ad}_E\sigma)(P, \sigma)\},$$

where the maximal is taken over all parabolic reductions of the principal $G$-bundle $E$. It follows from Behrend [3] that if $(P, \sigma)$ is the Harder-Narasimhan reduction of $E$ then $\text{Ideg}_G(E) = -d(\sigma)$ (also see Mehta-Subramanian [23] or Holla-Biswas [6] for an account).

Let $c \in X_*(T)/\hat{Q}$ be a fixed class. Let $M_G(c, d)$ be the set of isomorphism classes of principal $G$ bundles over $C$ such that the instability degree is bounded by $d$. We need the following proposition.

**Proposition 3.19** $M_G(c, d)$ is bounded. In other words there exists a finite scheme $S$ and a principal $G$-bundle $E$ over $S$ such that every member in $M_G(c, d)$ occurs in $S$.

This is exactly the Theorem 1.2 of [16] when $d = 0$. For $d > 0$ the same method works.

The following is a version of Proposition 4.1, Ramanathan [20] about openness of stable bundles ($d < 0$ case) which is proved in the analytic setup but the proof goes through in any characteristic.

**Proposition 3.20** In any family of bundles $G$-bundles $\mathcal{E} \to C \times S$ the subset $S^d$ corresponding points $x \in S$ for which $\mathcal{E}|_{C \times \{x\}}$ has instability degree less than $d$ is open.

The above can also be proved using the Lemma [2.3], the properness of the Hilbert schemes and the Lemma [4.4] (to be proved later).

The idea of the proof of the Proposition [3.13] gives something stronger which will be used later.

**Theorem 3.21** There exists an irreducible smooth variety $S$ and a family of $G$-bundles on $C \times S$ such that every element in $M_G(c, d)$ occurs in the family $\mathcal{E}$.

**Proof** By Lemma [3.19] there is a finite type scheme $X$ with family of $G$-bundles $\mathcal{E}$ over $C \times X$ such that for each $x \in X$ the bundle $\mathcal{E}|_{C \times \{x\}}$ lies in $M_G(c, d)$ and every $G$-bundle in $M_G(c, d)$ is isomorphic to a member in the family $\mathcal{E}$.

Consider the defining morphism $\bigcup \text{Sec}_{\mathcal{E}|_{B}}^{[\sigma]} \to X$, where the union is taken over all $[\sigma]$ satisfying the property $(\ast)$ for $N = 2g$. We see that this is a smooth morphism by Corollary [3.6]. Since $X$ is of finite type there exists finitely many open sets $\{U_i\}_{i=1,...,m}$ which cover $X$ and elements $[\tau_i]$ for $i = 1 \ldots m$ such that for each $x \in U_i$ the principal $G$-bundle $\mathcal{E}|_{C \times \{x\}}$ admits a $B$ reduction with numerical type $[\tau_i]$ and such that each $[\tau_i]$ satisfies the condition $(\ast)$ for $N = 2g$. By Lemma [3.13] we can find a 1-PS $\mu$ with $\mu(\alpha) \geq 2g - 1$ for each $\alpha \in \Phi^+$ satisfying $w_0[\tau_i] \leq \mu$ for each $i$. 

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Now by Theorem 3.7 we see that every \( G \)-bundle which corresponds to a point in \( X \) admits a \( B \) reduction with numerical type \( w_0 \mu \).

Take \( S_0 \) to be the moduli space of \( T \)-bundles of type \( w_0 \mu \) and continue with the steps of the “if” part of the proposition 3.15 to obtain an affine bundle \( S \) over \( S_0 \) which satisfies the conditions of the statement of the theorem. □

Using similar arguments and the remark 3.11, one can also obtain a slight generalization of the Theorem 1 of Drinfeld and Simpson [29].

**Corollary 3.22** If \( E \) is a principal \( G \)-bundle over smooth curve \( C \) over a finite type scheme \( S \). Then for any \( N \) there is a \( \sigma \in X^*_n(T) \) and a surjective étale cover \( S' \to S \) such that \( \sigma \) satisfies the property (\( * \)) for \( N \), and \( E \) admits a \( B \)-reduction over \( S' \) with numerical type \( \sigma \).

Suppose \( P \) is a connected linear algebraic group which is written as \( R_u P \cdot L \) with the property that \( L \) is reductive and \( R_u P \) being the unipotent radical. We have the natural projection \( p : P \to L \). We record here a fact which will be used later.

**Proposition 3.23** Let \( E \) be a principal \( P \)-bundle over \( C \). There is an algebraic versal family of deformations of \( E \) and the tangent space to the deformation functor is \( H^1(C, \text{ad} E) \).

**Proof** First we observe that the deformation functor \( DE \) satisfies the properties \( H_1, H_2 \) and \( H_3 \) of Schlessinger [28]. Hence a formal versal hull exists. To verify that the there is a an algebraic versal hull we use the results of Artin [4]. Also the second condition of Artin namely that \( DE(\hat{A}) \to \lim \leftarrow D_E(A/m^n) \) is injective for a local ring of an algebraic scheme with residue field \( k \), is easy to check using a faithful representation of \( P \) in \( GL_N \) and considering the principal \( P \)-bundle as a vector bundle with a section on some tensor power of it and using the Grothendieck existence theorems [12] III 5. This part is done in [27] Proposition 8.4. So it is enough to check that there is a principal \( P \)-bundle \( V \) on \( C \times S \) for some \( S \) smooth scheme of finite type such that the infinitesimal deformation map is surjective. When the group \( R_u P = \{ e \} \) that is when \( P \) is reductive this follows from Proposition 3.21 and Corollary 3.0.

In the case \( P \) is not reductive we use the filtration (1) on \( R_u P \). Now the proposition follows from the following lemma by the induction on the length of the filtration and by writing the exact sequence \( \{ e \} \to U_k \to P \to P/U_k \to \{ e \} \) and finally reducing this to the case when the group \( P \) is reductive.

**Lemma 3.24** Let \( e \to U \to P \to M \to e \) be an exact sequence of connected algebraic groups such that \( U = G^n \) and that \( P \) action of \( U \) factors through \( M \) to give a representation of \( M \). Let \( E \) be a \( P \)-bundle and let \( E_M \) be the associated \( M \)-bundle under the surjection \( p : P \to M \). Let \( V_1 \to C \times S_1 \) be a family of \( M \)-bundles parameterizes by a finite type smooth scheme \( S_1 \) which is versal at \( E_M \). Then there exists a family \( \mathcal{V} \to C \times S \) of \( P \)-bundles again parameterized by a finite type smooth scheme \( S \) which is versal at \( E \) and there is a surjective morphism \( S \to S_1 \).
Proof The unipotent group scheme \( U(V_1) \) over \( C \times S_1 \) actually comes from a vector bundle \( W \) over \( C \times S_1 \). If \( K^0 \rightarrow K^1 \) is a complex of vector bundles defined by the semi-continuity theorem which computes \( R^i(pr_2)_*W \) then one can check that on the vector bundle \( K^1 \) we have a family of \( P \)-bundles which has the properties mentioned in the lemma.

Another fact which we will need is the following.

**Proposition 3.25** If the genus of the curve \( C \) is at least two then for any \( c \in \mathcal{X}_s(T)/\hat{Q} \) there exists a stable \( G \) bundle whose topological type is \( c \).

**Proof** Let \( E \) be any principal \( G \)-bundle over \( C \). By Proposition 3.23 we have a family \( E \) of \( G \)-bundles parameterized by a finite type smooth and integral scheme \( S \) which is miniversal at every point of \( S \) and there is an \( x \in S \) such that \( E = E_x \). If \( E \) admits a \( P \)-reduction \( \sigma \) with \( d([\sigma]) \leq 0 \) then we have to show that the image of the natural map \( f : Y = \text{Sec}_E^{[\sigma]}P \rightarrow S \) does not contain an open subset of \( S \).

Let \( E_P \) be the family of \( P \)-bundles on \( Y \) obtained by the universal property of \( Y \) and let \( y \in Y \) be such that \( E_{P,y} = E_\sigma \). Let \( Y \rightarrow C \times S_P \) be a family of \( P \)-bundles, parameterized by finite type smooth scheme \( S_P \), which is miniversal at the point \( y' = E_\sigma \in S_P \). Now by going to an étale neighborhood \( V \) of \( y \in Y \) and of \( S_P \), and an automorphism of \( U \) in a neighborhood of \( x \in U \) we may assume that the restriction of \( f \) (again denoted by \( f \)) defines a morphism \( f : V \rightarrow S \) which can be written as \( j \circ g \), where \( g : V \rightarrow S_P \) is defined by the versal property of \( S_P \) at \( E_\sigma \) and \( j : S_P \rightarrow S \) by the versal property of \( U \) at \( E \). Hence it is enough to show that \( H^1(C, \text{ad}E_\sigma) \rightarrow H^1(C, \text{ad}E) \) is not surjective. The last statement follows because \( H^1(C, T_\sigma) \neq 0 \), as \( d([\sigma]) \leq 0 \) and \( g \geq 2 \). \( \square \)

## 4 Hilbert schemes and Moduli space of maps

In this section we use the compactifications of the space of sections to estimate their dimensions. For a given reduction \( \sigma \) of \( E \) to \( P \) we consider the Hilbert scheme \( \text{Hilb}_{E/P}^{[\sigma]} \) and the open subscheme \( \text{Sec}_{E/P}^{[\sigma]} \) as defined before. The following theorem was proved in Holla-Narasimhan \[16\] about the dimension estimates of the Hilbert schemes corresponding to a minimal section \( \sigma \).

**Theorem 4.1** Let \( \sigma \) be a minimal section. Let \( X \) be an irreducible component of the Hilbert scheme \( \text{Hilb}_{E/P}^{[\sigma]} \), which contains \( \sigma \) as a Hilbert point. Then every point in \( X \) corresponds to a Hilbert point of a section. In other words \( X \subset \text{Sec}_{E/P}^{[\sigma]} \). Moreover \( \dim(X) \leq \dim(G/P) \) and \( \text{deg}(T_\sigma) \leq g \dim(G/P) \).

**Remark 4.2** The dimension bound \( \dim(X) \leq \dim(G/P) \) follows from the first assertion of the above theorem by a rigidity argument which implies that the evaluation morphism \( X \times C \rightarrow E/P \) is finite. (see Lemma 2.4 of \[16\]). The last assertion
follows from above by a deformation theoretic argument (see Proposition 3 of Mori [24]).

In our proof of the fact that the minimal section satisfies \( \deg(T_\sigma) \leq g \dim(G/P) \), we only needed the fact that the highest exterior power of \( T_\pi \) is ample over the fibers of the map \( \pi : E/P \to X \). More generally the proof of the above theorem would go through assuming \( \sigma \) to be numerically minimal. This is the content of the proposition below.

**Proposition 4.3** In the above set up, if \( X \) is an irreducible component of \( \text{Hilb}^{[\sigma]}_{E/P} \) which contains a Hilbert point of an numerically minimal section \( \sigma \) then every element in \( X \) corresponds to a Hilbert point of a section. In other words \( X \subset \text{Sec}^{[\sigma]}_{E/P} \). Moreover we have \( \dim(X) \leq \dim(G/P) \) and \( \deg(T_\sigma) \leq g \cdot \dim(G/P) \).

**Proof** We need the following lemma for the proof.

**Lemma 4.4** If the Hilbert point of a closed subscheme \( Y \) of \( E/P \) is in \( \text{Hilb}^{[\sigma]}_{E/P} \) then \( Y \) has a unique irreducible component \( C_0 \) which maps isomorphically onto \( C \) under the composition \( C_0 \to E/P \to C \), hence defines a section \( \sigma_0 \) of \( \pi \) which satisfies the inequality \([\sigma] \leq [\sigma_0]\). Moreover if the above is an equality then the subscheme \( Y \) coincides with \( C_0 \).

**Proof** of the lemma: We just follow the arguments of the Proposition 2.3 of [16]. By choosing a line bundle \( L \) on \( C \) of degree one we see that

\[
\chi(Y, \mathcal{O}_Y) = \chi(C, \mathcal{O}_C) \quad \text{and} \quad \chi(Y, f^*(L)) = \chi(C, L),
\]

By applying Lemma 2.2 (i) of [16] we get the unique irreducible component \( C_0 \) mapping isomorphically onto \( C \). The third part of the Lemma 2.2 (iii) now implies that for any line bundle \( \xi \) on \( E/P \) which is ample along the fibers of \( \pi \) we have \( \deg(C_0, \xi) \leq \deg(Y, \xi) \) and this is an equality if and only if there are no other one dimensional components. This proves the Lemma 4.4 by taking the line bundle \( \xi \) to associated to anti-dominant characters of \( P \). The final remark is that the zero dimensional components automatically disappear once there are no other one dimensional components by Lemma 2.2 (ii) of [16].

The above lemma immediately implies the first part of the Proposition 4.3 and the other parts follow form Remark 4.2.

**Remark 4.5** Let \([\sigma]\) be a numerical type not necessarily minimal. Suppose that we have an irreducible component \( X \) of \( \text{Hilb}^{[\sigma]}_{E/P} \) which is contained in \( \text{Sec}^{[\sigma]}_{E/P} \), in other words every element of \( X \) is a Hilbert point of a section. Then we see again by Remark 4.2 that \( \dim(X) \leq \dim(G/P) \) and \( d([\sigma]) \leq g \cdot \dim(G/P) \). In some sense the minimality condition for the section \( \sigma \) is used only to get components of Hilbert schemes which do not have any boundaries.
Corollary 4.6 There are only finitely many numerically minimal points in $X_\sigma(P)$ corresponding to $P$ reductions of the principal $G$-bundle $E$.

Proof This follows from Lemma 2.3 and the last inequality in the Proposition 4.3. □

We have the following result on the dimension estimates for the irreducible components of the Hilbert scheme containing the Hilbert point of a section.

Theorem 4.7 Let $\gamma_1, \gamma_2, \ldots, \gamma_m$ be the numerically minimal types in $X_\sigma(L)$ for the principal $G$-bundle $E$. If $X$ is an irreducible component of $\text{Hilb}_{E/P}^[[\sigma]]$ which contains the reduction of structure group $\sigma$ as a Hilbert point then there exists an $i$ with $[\sigma] \leq \gamma_i$ such that

$$\dim(X) \leq \dim(G/P) + d([\sigma]) - d(\gamma_i).$$

The above result is proved using a similar result on the dimension estimates of the moduli space of stable maps. Now we recall the basic facts about the moduli space of stable maps.

Let $X$ be a smooth projective variety. We consider the Kontsevich space of stable maps $\overline{M}_{g,n}(X, \beta_d)$. As a functor, its $S$-valued points parameterizes stable families over $S$ of maps from $n$-pointed, genus $g$ curves to $X$ representing the class $\beta$ with isomorphisms. More precisely equivalence class of triples $(\pi : C \to S, \{p_i\}_{1 \leq i \leq n}, \mu : C \to X)$ satisfying the following properties.

1. A family of $n$-pointed, genus $g$ curves $\pi : C \to S$ which is flat and projective.

2. $n$ sections $\{p_1, \ldots, p_n\}$ of $\pi : C \to S$ such that each geometric fiber $(C_s, p_1(s), \ldots, p_n(s))$ is an $n$-pointed genus $g$ curve which is projective, connected, reduced, nodal curve of arithmetic genus $g$ with $n$ distinct, nonsingular, marked points.

3. For each geometric point $s \in S$ the morphism $\mu : C \to X$ restricted to the fiber $C_s$ satisfies the following.

   (a) If a rational component $F$ of $C_s$ is mapped to a point, then $F$ must contain at least three special points (marked points or nodes).

   (b) If a component $F$ of arithmetic genus 1 is mapped to a point, then it must contain at least one special point.

   (c) $\beta = (\mu|_{C_s})_*[C_s]$.

Here the last equality holds in the homology groups $H_2(X, K)$, where $K$ is a characteristic zero field (mostly $\mathbb{Q}$ or $\mathbb{Q}_l$). If the variety lives in characteristic $p$ then we replace the above homology by the étale cohomology group $H^{2n-2}(X, \mathbb{Q}_l)$ (or equivalently $H^2(C, \mathbb{Q}_l)^*$) for a prime $l$ different from $p$.

$\overline{M}_{g,n}(X, \beta_d)$ is known to have a structure of a proper Artin algebraic stack, with finite automorphism at $k$-valued points, which admits a projective coarse moduli
space $\overline{M}_{g,n}(X, \beta_d)$ in all characteristics. For proofs see Fulton-Pandharipande [9], Harris-Morrison [13], and mostly Abramovich-Oort [1].

We also have an open substack of above giving rise to a coarse moduli scheme $M_{g,n}(X, \beta_d)$, which parameterizes maps from smooth curves. Furthermore there is a “forgetful” map $\overline{M}_{g,n}(X, \beta_d) \to \overline{M}_{g,n-1}(X, \beta_d)$ which extends the natural forgetful map of the open moduli spaces, with one dimensional fibers.

We will be only interested in the case when the space $X$ is of the form $E/P$ where $E$ is a principal $G$ bundle on a smooth projective curve $C$, and $P$ is a parabolic subgroup of $G$. Also in our case we will mostly assume $g = g_C$ and $\beta_d$ is a class in $H^2(X, K)^*$ determined by a reduction of structure group $\sigma$ of $E$ to $P$.

The basic relation between the homology classes $\beta$ and the numerical type $\sigma$ can be described as follows.

Given a $\beta \in H^2(E/P, K)^*$ we define a point $[\beta] \in \mathcal{X}_*(P)$ by $[\beta](\chi) = \beta(c_1(L_\chi))$ where $L_\chi$ is the line bundle on $E/P$ defined by $\chi$. This defines a homomorphism $H^2(E/P, K)^* \to \mathcal{X}_*(P) \otimes K$.

Since $H^2(E/P, K)$ is generated by the first Chern classes of the line bundles (as $H^2(E/P, \mathcal{O}_{E/P}) = 0$) and the fact that every line bundle on $E/P$ is uniquely of the form $L_\chi \otimes \pi^*(L)$ with a line bundles $L$ over $C$ and $L_\chi$ over $E/P$ defined by a character $\chi$ of $P$. Hence we have a well defined set theoretic splitting of the above homomorphism defined by $\beta_\sigma = [\sigma](\chi) + \deg(L)$.

When $\sigma$ is a reduction of structure group of a principal $G$-bundle $E$ then $\sigma$ defines a point of both the spaces $M_g(E/P, \beta_\sigma)$ and $\text{Sec}^{[\sigma]}_{E/P}$. In the following lemma we prove that the above correspondence defines an isomorphism between the two.

**Lemma 4.8** There is a natural isomorphism between $M_g(E/P, \beta_\sigma)$ and $\text{Sec}^{[\sigma]}_{E/P}$ which takes a point corresponding to the irreducible curve to the section defined by it.

**Proof** We first construct the map from the algebraic stack $\mathcal{M}_g(E/P, \beta_\sigma) \to \text{Sec}^{[\sigma]}_{E/P}$, this will automatically define the map from the coarse moduli space. Given a tuple $(q : C \to S, f : C \to E/P)$ with $C$ being a flat family of smooth curves over $S$, we see that $(q, \pi \circ f) : C \to S \times C$ defines an isomorphism over $S$. Hence this setup automatically gives rise to flat family of sections. This defines the morphism. Notice that all the isomorphisms in $\mathcal{M}_g(E/P, \beta_\sigma)$ are automatically collapsed in $\text{Sec}^{[\sigma]}_{E/P}$. The inverse of the above morphism can be obtained by simply inverting the above operation and then composing it with the natural morphism $\mathcal{M}_g(E/P, \beta_\sigma) \to M_g(E/P, \beta_\sigma)$. Hence the lemma is proved.

**Remark 4.9** Suppose there is a component $X$ of $\overline{M}_g(E/P, \beta_\sigma)$ which has no points of the boundary, in other words, every point of $X$ corresponds to a map from an irreducible curve. Then such a component will map isomorphically on to an irreducible component of $\text{Sec}^{[\sigma]}_{E/P} \subset \text{Hilb}^{[\sigma]}_{E/P}$. Since $X$ is proper we see that
X defines an irreducible component of $\text{Hilb}_{E/P}^{[\sigma]}$. Hence by Remark 4.3, we obtain $\dim(X) \leq \dim(G/P)$ and $d([\sigma]) \leq g \cdot \dim(G/P)$.

Now we state the main result of this section

**Theorem 4.10** Let $\gamma_1, \gamma_2, \ldots, \gamma_m$ be the numerically minimal types in $X_*(P)$ for $E$. Let $[\sigma]$ be a numerical type. If $X$ is an irreducible component of $\mathcal{M}_{g}(E/P, \beta_{[\sigma]})$ then there exists an $i$ with $[\sigma] \leq \gamma_i$ such that

$$\dim(X) \leq \dim(G/P) + d([\sigma]) - d(\gamma_i).$$

The above result is a generalization of a result of Mihnea Popa and Mike Roth [25] on the dimension estimates of the moduli space of stable maps in the case of $G = GL_n$ and its proof follows similar ideas but one has to take care of the numerical types of the sections.

Suppose $X$ is a component of $\mathcal{M}_{g}(E/P, \beta_{[\sigma]})$ whose generic point corresponds to a map with irreducible domain. Let $Y$ be the boundary in $X$ corresponding to maps with reducible domain. Let $Y'$ be an irreducible component of $Y$.

The following basic lemma which is a stronger analogue of Lemma 4.4 for the moduli space of maps and which does not hold for the case of Hilbert schemes.

**Lemma 4.11** A generic element of $Y'$ corresponds to a map from a connected reduced nodal curve $C'$ with irreducible components $C_0, C_1, \ldots, C_k$ such that

1. $C_0$ maps isomorphically onto $C$, every other $C_i$ is isomorphic to $\mathbb{P}^1$.

2. Only singularities of $C'$ are ordinary nodes with each $C_i$, for $i = 1 \ldots k$, intersecting $C_0$ at a point $x_i$ with $x_i \neq x_j$ for $i \neq j$. And these are only intersections between the irreducible components.

3. $C_0$ gives rise to section $\sigma_0$ of $\pi$ whose numerical type $[\sigma_0]$ satisfies $[\sigma] \leq [\sigma_0]$. Moreover for any line bundle $\xi$ on $E/P$ ample along the fibers of $\pi$, we have

$$\deg(C_0, \xi|_{C_0}) + \sum_{i=1}^{k} \deg(C_i, \xi|_{C_i}) = \deg(C', \xi|_{C'}),$$

with $\deg(C_i, \xi|_{C_i}) > 0$ for $i = 1 \ldots k$.

**Proof** The first part of the proof of this lemma is same as the proof of the Lemma 4.4, but we have to take care of rational tails. Since $C'$ is a connected reduced curve with only nodal singularities, the arithmetic genus of $C'$ coincides with that of $C$ and $C_0$. This forces the other components to be isomorphic to $\mathbb{P}^1$. Moreover we see that these rational components form trees living at the fibers of the morphism $\pi$. To show that such a generic curve $C'$ is a comb we need to smoothen the rational curves that we encounter. This follows from a general statement proved in Theorem 7.6 II, p. 155 of Kollar [17], which uses the fact that $f^*T_\pi$ when restricted to each of the components of the tails is semi-positive. This is always the case as the tangent bundles of the flag varieties are globally generated and the tails lie in the fibers of the morphism $\pi$. \[\square\]
Remark 4.12 The Lemma 4.11 also hold in the situation where $Y = Y'$ is an irreducible component of $\overline{M}_g(E/P, \beta_{[\sigma]})$ consisting of only reducible curves. This will also be used in the proof of the Theorem 4.10.

One of the steps in the proof of Theorem 4.10 is the following lemma which is essentially follows from deformation theory of the nodal curves.

Lemma 4.13 Let $X$ be an irreducible component of $\overline{M}_g(E/P, \beta_{[\sigma]})$ whose general element corresponds to a irreducible curve. Let $Y' \subset X$ be an irreducible component of the boundary $Y$ corresponding to the reducible curves. Then the codimension of $Y'$ in $X$ is at most $k$, where $k$ is the number of nodes in a curve corresponding to a generic element of $Y'$.

Proof This is a standard statement about deformation theory. It is enough to show that in the algebraic miniversal deformation of such a curve, the boundary is of codimension $l$. The construction of the deformation is done in Vistoli [30]. The tangent space to the deformation functor is $\text{Def}_1(C') = \text{Ext}^1_{\mathcal{O}_C}(\mathcal{O}_{C'}, \mathcal{O}_{C'})$ and it is calculated by the following exact sequence.

$$0 \rightarrow H^1(C', \theta_{C'}) \rightarrow \text{Ext}^1_{\mathcal{O}_C}(\mathcal{O}_{C'}, \mathcal{O}_{C'}) \rightarrow H^0(C', \text{Ext}^1_{\mathcal{O}_C}(\mathcal{O}_{C'}, \mathcal{O}_{C'})) \rightarrow H^2(C', \theta_{C'})$$

In our situation we have $H^2(C', \theta_{C'}) = 0$. An easy computation using the normalization of $C'$ shows that $\dim H^1(C', \theta_{C'}) = 3g_{C'} - 3 - k$ and $\text{Ext}^1_{\mathcal{O}_C}(\mathcal{O}_{C'}, \mathcal{O}_{C'})$ is sky scraper sheaf supported on the nodes and has length 1 at each of the nodes of $C'$. These facts imply that $\text{Def}_1(C')$ has dimension $3g_{C'} - 3$ containing the subspace $\dim H^1(C', \theta_{C'})$ which parameterizes the deformations of the curve $C'$ preserving the singularities. Note that in the case these dimensions are negative one has to put additional markings to ensure that the nodal curves have only finite automorphisms and then work over $\overline{M}_{g,n}(E/P, \beta_{[\sigma]})$ instead of $\overline{M}_g(E/P, \beta_{[\sigma]})$. Using this the Lemma follows.

Proof of the Theorem 4.10: We first prove the result when the generic element in $X$ corresponds to a map from an irreducible domain. Let $Y' \subset Y$ be an irreducible component of the boundary $Y$ in $X$. If $(C', f)$ is a curve which corresponds to a general element of $Y'$, then by Lemma 4.11 we can write $C' = \bigcup_{i=0}^{k} C_i$ where $C_0$ defines a section $\sigma_0$ of $\pi$ and the other $C_i$'s are isomorphic to $\mathbb{P}^1$. Hence the curve $C_0$ along with the points $\{x_i|i = 1 \ldots k\}$ defines a point in $M_{g,k}(E/P, \beta_{[\sigma_0]})$ and the curves $C_i$ along with the point $x_i$ defines an element of $M_{0,1}(G/P, \alpha_i)$, where $\alpha_i$ is the second homology class defined by $(f|_{C_i})_*[C_i]$. We then can estimate the dimension of such curves by first taking irreducible curves of type $[\sigma_0]$ and $k$ copies of rational curves each with types $\alpha_i$ for each $i$ such that sum $[\sigma_0] + \sum_{i=1}^{k} [\alpha_i] = [\sigma]$. The dimension of $\overline{M}_{0,1}(G/P, \alpha_i)$ can be computed by deformation theory to be exactly equal to $\dim(G/P) + d([\alpha_i]) - 2$ (see Theorem 2, Section 1, [3]). From here it follow that $\overline{M}_{0,1}(E/P, \alpha_i)$ has dimension equal to $\dim(G/P) + d([\alpha_i]) - 1$. Now the moduli space $\overline{M}_{g,k}(E/P, \beta_{[\sigma_0]})$ and $\overline{M}_{0,1}(E/P, \alpha_i)$ have natural evaluation
maps $ev_i : M_{g,k}(E/P, \beta_{[\sigma_0]}) \rightarrow E/P$ and $ev_i' : M_{0,1}(G/P, \alpha_i) \rightarrow E/P$ for each $i = 1 \ldots k$.

Consider the space

$$Z([\sigma_0], \alpha_1, \ldots, \alpha_k) = \prod_{i=1}^k M_{g,k}(E/P, \beta_{[\sigma_0]}) \times \prod_{i=1}^k M_{0,1}(E/P, \alpha_i).$$

This space has dimension

$$\dim(M_g(E/P, \beta_{[\sigma_0]})) + k + \sum_{i=1}^k \dim(M_{0,1}(G/P, \alpha_i)) - k \dim(E/P)$$

$$= \dim(M_g(E/P, \beta_{[\sigma_0]})) + k + k \dim(G/P) + \sum_{i=1}^k d([\alpha_i]) - k - k \dim(G/P) - k$$

$$= \dim(M_g(E/P, \beta_{[\sigma_0]})) + \sum_{i=1}^k d([\alpha_i]) - k - k \dim(G/P) - k$$

The dimension of the above will be related to the dimension of $Y'$ using the natural morphism $h_{([\sigma_0], \alpha_1, \ldots, \alpha_k)} : Z([\sigma_0], \alpha_1, \ldots, \alpha_k) \rightarrow M_g(E/P, \beta_{[\sigma]})$ which is defined by $\{C_i\} \mapsto C''$. The existence of such a morphism is easy to see at the level of Algebraic stacks by taking the valued groupoids but such a morphism would automatically descent to the coarse moduli spaces.

Now by Lemma 2.3 and the fact that there are only finitely many possibilities for the collection $\{[\alpha_1], \ldots, [\alpha_k]\}$ as they are squeezed between $[\sigma]$ and $[\sigma_0]$ and generically every element of $Y'$ is in the image of one such map hence we obtain an irreducible component $X_0$ of some $M_g(E/P, \beta_{[\sigma_0]})$ and a collection $\{\alpha_1, \ldots, \alpha_k\}$ of elements in $H^2(E/P, K)^*$ such that the image of the morphism $h_{([\sigma_0], \alpha_1, \ldots, \alpha_k)}$ contains an open subset of $Y'$. Hence we have

$$\dim(Y') \leq \dim(X_0) + \dim([\sigma]) - \dim([\sigma_0]) - k.$$ 

Also the above construction implies that the generic point of $X_0$ corresponds to map from an irreducible domain. Now by Lemma 4.13 we see that the dimension of $X$ is bounded by

$$\dim(X) \leq \dim(X_0) + \dim([\sigma]) - \dim([\sigma_0]).$$

The above is the main step for the induction in the proof of the Theorem. When the induction stops we have two possible cases to take care.

So starting with the numerical type $[\sigma]$ we produce an numerical type $[\sigma_0]$ and an irreducible component $X_0$ of $M_g(E/P, \beta_{[\sigma_0]})$ such that the above inequalities hold. Now by induction we continue this process to get a sequence of numerical types $[\sigma_1], \ldots, [\sigma_l]$, · · · and irreducible components $X_i$ of $M_g(E/P, \beta_{[\sigma_i]})$ for each $i$ such that

$$\dim(X_i) \leq \dim(X_{i+1}) + \dim([\sigma_i]) - \dim([\sigma_{i+1}])$$

This process goes on until we reach an $l$ for which $X_l$ has no boundary or $[\sigma_l]$ is numerically minimal (hence no boundary points). Now the problem reduces to estimating the dimensions of the irreducible components which contain no boundary
points. In this case, by Remark 4.9 we have \( \dim X \leq \dim(G/P) \). Hence combining this with (3) and (2) we obtain
\[
\dim(X) \leq \dim(G/P) + d([\sigma]) - d([\sigma_0]).
\]
Now if the component \( X \) of \( \overline{\text{M}}_g(E/P, \beta_{[\sigma]}) \) consists completely of boundary points, even then we can carry out the above induction by taking \( Y' = X \) and we would get a sequence of numerical types \([\sigma_0]\ldots[\sigma_m]\) and irreducible components \( X_i \) of \( \overline{\text{M}}_g(E/P, \beta_{[\sigma_i]}) \) such that
\[
\dim(X) \leq \dim(X_0) + d([\sigma]) - d([\sigma_0]) - k,
\]
and the inequality (3) holds for \( i = 0 \cdots m - 1 \). Here \( k \) is the number of nodes in a curve corresponding to a generic point of \( X \). Hence the above argument gives
\[
\dim(X) \leq \dim(G/P) + d([\sigma]) - d([\sigma_0]) - k,
\]
which is stronger than the bound for an irreducible component whose generic point to a map with irreducible domain. The Theorem 4.10 now follows because \( d([\sigma]) - d([\sigma_0]) \leq d([\sigma]) - d(\gamma_j) \) for some numerically minimal point \( \gamma_j \).

**Proof** of the Theorem 4.7: Let \( X \) be an irreducible component of \( \text{Hilb}^{[\sigma]}_{E/P} \), which contains the Hilbert point of a section \( \sigma \). Let \( X' \) be the open subset of \( X \) defined by \( X' = X \cap \text{Sec}^{[\sigma]}_{E/P} \). Then \( X' \) is in the image of the isomorphism defined in the Lemma 4.8. Hence Theorem 4.7 follows from the Theorem 4.10.

**Remark 4.14** The proof of the Theorem 4.10 shows that if \( X \) is an irreducible component of \( \overline{\text{M}}_g(E/P, \beta_{[\sigma]}) \) for some \( [\sigma] \) then there exists a \( P \) reduction \( \sigma_0 \) of \( E \) with \( [\sigma_0] \geq [\sigma] \) and \( d([\sigma_0]) \leq g \cdot \dim(G/P) \) such that
\[
\dim(X) \leq \dim(X_0) + d([\sigma]) - d([\sigma_0]) - k.
\]
Here \( k \) is the number of nodes in the curve corresponding to general point of \( X \) and \( X_0 \) is an irreducible component of \( \overline{\text{M}}_g(E/P, \beta_{[\sigma_0]}) \) containing \( \sigma_0 \). This remark will be useful later.

As in the case of [25] we have a dimension estimate for the lower bound and this will be used later.

**Proposition 4.15** Let \( [\sigma] \) and \( [\tau] \) be two numerical types in \( \mathcal{X}_*(L) \) such that \( [\sigma] \leq [\tau] \). Then we have the following inequality
\[
\overline{\text{M}}_g(E/P, \beta_{[\sigma]}) \geq \overline{\text{M}}_g(E/P, \beta_{[\tau]}) + d([\sigma]) - d([\tau]) - 1
\]

**Proof** Let \( \alpha = [\sigma] - [\tau] \). Firstly \( \alpha \) defines a canonical element of \( H^2(E/P, K)^* \) (independent of the chosen line bundle \( L \) over \( C \)). Now one can check from the definitions that the \( h_{[\tau], \alpha} : Z([\tau], \alpha) \to \overline{\text{M}}_g(E/P, \beta_{[\sigma]}) \), defined in the proof of the Theorem 4.10, is generically injective. Hence the proposition follows from the dimension estimate for \( Z([\tau], \alpha) \) as obtained in Theorem 4.10. □
Remark 4.16 One observes that if $[\sigma]$ is such that $\overline{M}_g(E/P, \beta_{[\sigma]})$ is irreducible and that the generic element corresponds to map from an irreducible curve then the proof of the Proposition \[4.13\] actually shows that $\overline{M}_g(E/P, \beta_{[\sigma]}) \geq \overline{M}_g(E/P, \beta_{[\tau]}) + d([\sigma]) - d([\tau])$, as the image of $h_{(\tau, \alpha)}$ will not be generically surjective. This remark will be used later.

5 Irreducibility and Generic Smoothness of reductions

We make a temporary change in the notations. Let $C$ be a smooth projective curve over an algebraically closed field $k$. Let $G_k$ be a semisimple simply connected algebraic group over $k$. Let $E$ be a principal $G_k$-bundle over $C$. In this section we want to prove that when the numerical type is large enough the space of reductions of $E$ to a fixed parabolic subgroup is irreducible and generically smooth. Our method of proof is to derive this from a similar result for the case when the parabolic is a Borel subgroup, and use the results in the previous sections to prove it for the parabolic case. The result for the Borel subgroups was proved by Harder [14] for the case when the curve is defined over the algebraic closure of a finite field. From here it does not directly follow for arbitrary fields but the method of proof works. This will be the first part of this section.

Let $B_k$ be a Borel subgroup of $G_k$. Let $T_k \subset B_k$ be a maximal torus. The first result we want to prove in this section is the following.

Theorem 5.1 There exists an integer $N$ such that if $E$ admits a $B$ reduction of numerical type $[\sigma]$ satisfying the property $(\ast)$ for $N$ then $\text{Sec}^{[\sigma]}_E$ is an irreducible smooth variety of dimension $d([\sigma]) + (1 - g)\dim(G/B)$.

Proof Our method of proof is to reduce the problem to the case where we can apply the methods of Harder. We make some first reductions.

We will denote by $F$ the prime field $\mathbb{F}_p$ or $\mathbb{Z}$ depending on whether our curve is in characteristic $p$ or $0$. Since the curve $C$ and the principal $G$-bundle is defined by finitely many equation, we may assume that there is a finite type affine integral scheme $S$ over $F$, a semi simple simply connected split algebraic group scheme $G = G_S$ over $S$, a curve $C \to S$ which is smooth and proper over $S$ with geometrically integral fibers, and there is a principal $G$-bundle $\mathcal{E}$ over $C$ such that over the generic point $\text{Spec}K \to S$, the principal $G_K$-bundle $E_K$ over the curve $C_K$ extends to the $G_K$-bundle $E$ over $C$ via the field extension $\text{Spec}(k) \to \text{Spec}(K)$.

Recall the definition of the instability degree $\text{Id}_{G}(E)$ of a principal $G$-bundle $E$. We now prove some basic lemmas needed in the proof of the theorem.

Lemma 5.2 Let $E$ be a principal $G$-bundle over a smooth projective geometrically connected curve $C$ over a perfect field $F_0$. Let $L$ be a finite extension of $F_0$. Let $C_L$
be the curve \( C \otimes_{K_0} L \) obtained by base change and \( E_L \) be the corresponding \( G_L \)-bundle over \( C_L \). Then \( \text{Ideg}_{G_L}(E_L) = \text{Ideg}(E) \).

**Proof** Let \( \overline{F}_0 \) be a fixed algebraic closure of \( F_0 \). Let \( E_{\overline{F}_0} \) be the principal \( G_{\overline{F}_0} \)-bundle over \( C_{\overline{F}_0} \). One first observes that if \( (P, \sigma) \) is a pair with the property that \( P \) is maximal among all the parabolic subgroup \( P' \) containing \( B \) for which there are reductions \( \sigma' \) satisfying \( \text{Ideg}_{C_{\overline{F}_0}}(E_{\overline{F}_0}) = \deg(E_{\overline{F}_0,\sigma'}) \) then the pair \( (P, \sigma) \) defines a Harder-Narasimhan reduction (see Behrend [3]). Now by uniqueness of the Harder-Narasimhan filtration and the Galois descent argument that This parabolic reduction is defined over \( F_0 \). If \( M \) is a Line bundle over \( C \) and if \( M_L \) is its pull back over \( C_L \) then we have \( \deg(M_L) = \deg(M) \), hence the instability degree does not change when we take a finite field extension.

Let \( E \) be a principal \( G \)-bundle over \( C \to S \) as above. We will denote by \( x \) a finite field valued point \( x : \text{Spec}(k(x)) \to S \), and by \( q_x = |k(x)| \). Here \( k(x) \) is not necessarily the residue field but a finite extension of it. Here we fix our notation for \( x \). Whenever we say \( x \) we mean a finite field valued point of \( S \). We also denote by \( E_x \) the principal \( G_x \)-bundle over the curve \( C_x \) which is the pullback of the corresponding objects over \( S \) via \( x \). The property we have for \( C \) ensures that \( C_x \) is a smooth projective geometrically connected curve over \( k(x) \).

We need the following lemma.

**Lemma 5.3** There exists an integer \( N \) such that for each \( x \) as above we have \( \text{Ideg}(E_x) \leq N \).

**Proof** Let \( \text{ad}(E) \) be the adjoint bundle of \( E \). For an \( x \) and a parabolic reduction \( (P, \sigma) \) of \( E_x \) we have an inclusion \( \text{ad}(E_x)_{P,\sigma} \hookrightarrow \text{ad}(E_x) \) hence by Riemann-Roch Theorem it is enough to bound the degree of \( \text{ad}(E_x) \) independent of \( x \). But this follows from the semi-continuity theorems.

Let \( G \) be a semisimple simply connected split algebraic group scheme over \( S \). Let \( B \) be a fixed Borel subgroup and \( T \) be a maximal torus contained in \( B \). Let \( \mathcal{X}_s(T) \) be the group of one parameter subgroups of \( T \). Let \([\sigma] \in \mathcal{X}_s(T)\) be a numerical type. Let \( E \) be a principal \( G \)-bundle over \( C \). Then the scheme \( \text{Sec}_{[\sigma]/B} \) is quasiprojective over \( S \). By Corollary 3.23 it follows that \( \text{Sec}_{[\sigma]/B} \) is smooth over \( S \) if \([\sigma] \) is a numerical type satisfying the property \((*)\) for \( N > 2g - 2 \).

We now want to define the Eisenstein series for \( E \). For this we need a fixed Borel reduction of \( E \) over all of \( S \). By Corollary 3.23 there is a \( B \)-reduction of \( E \) with some numerical type \([\sigma_0] \) by going to a surjective étale extension of \( S \). Hence by we may assume that our affine scheme \( S \) is such that there is a section of the map \( \text{Sec}_{[\sigma_0]/B} \to S \) for \([\sigma_0] \in \mathcal{X}_s(T)\). We fix such a Borel reduction \( \sigma_0 \) once and for all. This has the implication that for any choice of \( x \) we have a \( B_x \) reduction \( \sigma_{0,x} \) of \( E_x \) of numerical type \([\sigma_0] \). Now for any \( B_x \) reduction of \( E_x \) of numerical type \([\sigma] \) we see that \([\sigma] - [\sigma_0] \in \mathcal{X}_s(T)\) has the property that it lies in \( \tilde{Q} \) (by Lemma 3.14). Hence if \( \{w_\alpha\} \) are the fundamental dominant weights then \(([\sigma] - [\sigma_0])(w_\alpha) \) are integral.
Recall from Harder [14] the definition of $d_\alpha([\sigma]) = ([\sigma] - [\sigma_0])(w_\alpha)$. For simplicity we write $p[\sigma] = [\sigma] - [\sigma_0]$. Let $\gamma_x(p[\sigma])$ be the cardinality of Spec$(k(x))$ rational points of $\text{Sec}_{\mathcal{E}_x/B_x}^p$. Recall the definition of the Normalized Eisenstein Series

$$E(x, \mathcal{E}, \tau) = \sum_{[\sigma] \in X_\sigma(T)} \gamma_x(p[\sigma])q_x^{-\sum_{\alpha \in \Delta} d_\alpha([\sigma])} \prod_{\alpha \in \Delta} \tau^d_\alpha([\sigma]).$$

Here $\tau_\alpha$, for $\alpha \in \Delta$ are being thought as variables. We have suppressed the dependence of the above series on the Borel subgroup $B$ and the reduction $\sigma_0$. Now it follows from Lemma 5.3 that there exists an $N_0$ such that for all $x$, $\gamma_x(p[\sigma]) = 0$ when $d_\alpha([\sigma]) < -N_0$. Now we state some of the basic properties of this series which is proved in [14].

**Proposition 5.4** The Laurent series $E(x, \mathcal{E}, \tau)$ is a rational function on the variables $\tau$. Moreover it can be written as $E(x, \mathcal{E}, \tau) = P(x, \mathcal{E}, \tau)/Q(x, \tau)$, where $P$ is a polynomial in $\tau_\alpha$’s and $\tau^{-1}_\alpha$’s for $\alpha \in \Delta$. And

$$Q(x, \tau) = \prod_{\gamma \in \Phi^+} (1 - q_\gamma \tau^\gamma) \prod_{i=1}^{2g} (1 - w_i(x)q_x^{-1} \tau^\gamma),$$

where $w_i(x)$ are the eigen values of the Frobenius $Fr_x$ on the first cohomology of the curve $\mathcal{C}_x$, and $\tau_\gamma = \prod_{\alpha \in \Delta} \tau_\alpha^{\nu_\alpha}$ where $\gamma = \sum_{\alpha \in \Delta} \nu_\alpha w_\alpha$.

This proposition is a consequence of Theorem 1.6.6 of [14] for case $x = 1$ and $\omega = 1$. From the above proposition it follows that the polynomial $P(x, \mathcal{E}, \tau)$ has its negative powers of $\tau_\alpha$ bounded by $N_0$ for each $\alpha$ as it holds for the Eisenstein series $E(x, \mathcal{E}, \tau)$.

There is a second part of the Theorem 1.6.6 of [14] about the functional equation satisfied by the Eisenstein series using which we get the following upper bound for the degree of the polynomial $P(x, \mathcal{E}, \tau)$.

**Proposition 5.5** There exists a constant $N_1$ such that for each $x$ we have

$$-N_0 \leq \deg_\alpha P(x, \mathcal{E}, \tau) \leq N_1$$

The above proposition is essentially Theorem 1.6.10 of [14]. One observes that the constants $N_0$ and $N_1$ depend on the Borel $B$ and the reduction $\sigma_0$. We write the polynomial $P(x, \mathcal{E}, \tau) = \sum_{\mathcal{d}} a(x, \mathcal{d}) \tau^\mathcal{d}$ where $\mathcal{d} = \{d_\alpha\}_{\alpha \in \Delta}$ and $\tau^\mathcal{d} = \prod_{\alpha \in \Delta} \tau_\alpha^{d_\alpha}$.

The next Lemma we need is about the bound for the coefficients $|a(x, \mathcal{d})| \leq q_x^M$.

**Lemma 5.6** There exists a constant $M$ independent of $x$ such that for each $x$ we have $|a(x, \mathcal{d})| \leq q_x^M$.

**Proof** For a fixed $[\sigma]$, we know that $\text{Sec}_{\mathcal{E}_x/B}^{p[\sigma]}$ is a finite type quasi-projective scheme over $S$ hence we can find an constant $M_{p[\sigma]}$ such that for each $x$ we have a bound $\gamma_x(p[\sigma]) \leq q_x^{M_{p[\sigma]}}$. It follows from Lemma 2.3 that there are only finitely many $[\sigma]$ with the property that

$$- N \leq d_\alpha([\sigma]) \leq N_1 \tag{5}$$
for each $\alpha$. Hence we can find a single constant $M_0$ such that $\gamma_x(p[\sigma]) \leq q_x^{M_0}$ for all $[\sigma]$ satisfying $(\mathcal{I})$. Now we use the Proposition [5,4] to write $E(x, E, \tau) = P(x, E, \tau)/Q(x, \tau)$. Expanding both sides of the series using the power series expansion of $1/Q(x, \tau)$, the coefficients $a(x, d)$ can be computed as a linear combinations of $\gamma_x(p[\sigma])$ for $[\sigma]$ satisfying $(\mathcal{I})$ (by inverting an upper triangular matrix) with coefficients as powers of $q_x$ and $w_i(x)$. This proves the Lemma.

Now we follow the arguments of the Theorem 2.3.1 and Theorem 2.3.2 of [14] we get the following.

**Proposition 5.7** There exists a constants $N$ and $C$ independent of $x$ such that if $[\sigma]$ satisfies the property $(\ast)$ for $N$ then we have

$$\left| \gamma_x([\sigma]) \right| = q_x^{d([\sigma])+(1-g)\dim(G/B)} + R_x([\sigma])$$

where

$$R_x([\sigma]) \leq C q_x^{d([\sigma])+(1-g)\dim(G/B)} - 1/2$$

The basic idea of proof of the above proposition is that we know explicitly the poles of the Eisenstein series namely the zeros of the polynomial $Q(x, \tau)$. The next step is the explicit computation of the residue of the Eisenstein series at the point $\{\tau_\alpha\} = \{1/q_x\}$. Now one writes the series $E = E_1(1/\Pi_{\alpha\in\Delta}(1-q_x\tau_\alpha)) + E_2$ where $E_1$ is essentially the residue of $E$. Then one observes that the series $E_2$ has better radius of convergence. From here it follow that for large $N$ the coefficients of the Eisenstein series are dominated by the coefficients of $E_1$ and the explicit residue computation now yield the Proposition.

Now by applying the results of Lang and Weil [20] we see that if $[\sigma]$ satisfies $(\ast)$ for $N$ then after a finite base change of $x$, $\text{Sec}^{[\sigma]}_{E/B}$ has a unique irreducible component of maximal dimension and this dimension is equal to $d([\sigma]) + (1-g)\dim(G/B)$. By deformation theoretic lower bounds we see that every component is of this dimension. Hence we see that $\text{Sec}^{[\sigma]}_{E/B}$ is absolutely irreducible for each $x$.

Now we have a finite type smooth morphism $\text{Sec}^{[\sigma]}_{E/B} \to S$. By taking an affine open subscheme $S_0 = \text{Spec}(A)$ of $S$ we may assume that there is a dense affine open subscheme $U = \text{Spec}(R)$ of $\text{Sec}^{[\sigma]}_{E/B} \to S$ such that $U$ surjects onto $S_0$. Hence we obtain a finite type faithfully flat morphism $U \to S_0$ with the property that for each $x$ in $S_0$ the fiber $R \otimes_A k(x)$ is irreducible. To show that $\text{Sec}^{[\sigma]}_{E/K}$ is irreducible it is enough to show that $R \otimes_A K$ has no non trivial (not equal to $0$ or $1$) idempotent elements as $R$ is a smooth $A$ algebra. If $R \otimes_A K$ has an idempotent element then there is an $f \in A$ such that $R_f = R \otimes_A A_f$ contains a non-trivial idempotent element. Hence we may assume by replacing $A$ by $A_f$ that $R$ contains a non trivial idempotent element $e$. Since $R$ is reduced, there is an open subscheme $U_1$ of $\text{Spec}(R)$ such that for $p \in U_0$ the image of $1 - e$ in $R_p/pR_p$ is non zero. Since $R$ is smooth over $A$, the image of $U_1$ in $\text{Spec}(A)$ is open. Hence by shrinking $A$ to another $A_f$ we may assume that the non-trivial idempotent element $e \in R$ has the property that the image of $1 - e$ in $R \otimes_A k(x)$ is non-zero for each $x$. Now by irreducibility of $R \otimes_A k(x)$. We
see that $e$ maps to 0 for each $x$. Hence $e \in \bigcap mR$ where the intersection is over all maximal ideals of $A$.

**Lemma 5.8** Let $A$ be a finite type algebra over $F (= \mathbb{Z}$ or $\mathbb{F}_q$). If $R$ is finite type over $A$ then $\bigcap mR = \text{rad}(R)$.

**Proof** Firstly the conditions of the lemma ensures that $R$ is a Jacobson ring hence rad$(R)$ is the intersection of all maximal ideals of $R$. Again since $R$ is finitely generated over $F$ ensures that for every maximal ideal $n$ in $R$ the field $R/n$ is finite. Hence we see that the morphism $\text{Spec}(R) \to \text{Spec}(A)$ takes maximal ideals to maximal ideals of $A$. This implies the lemma. □

Now by above lemma, since $R$ is reduced, it follows that $e = 0$. Hence the proof of the theorem 5.1 is complete. □

**Remark 5.9** Note that we need to actually show that Sec$^{[\sigma]}_{E_K/B}$ is absolutely irreducible. For this we have to further show that if $L$ is a finite separable extension of $K$ then Sec$^{[\sigma]}_{E_K/B}$ remains irreducible. We take the normalization $A'$ of $A$ in $L$. Then the whole setup pulls back to the setup over $A'$. Since Spec$(A') \to \text{Spec}(A)$ takes maximal ideals to maximal ideals hence above proof actually shows that for the same $N$, the Proposition 5.7 works in the setup for $A'$.

We return to our notations where the objects are over an algebraically closed field $k$. Now we extend the above results to the case of parabolic subgroups.

We need the following lemma about the behavior of the Harder-Narasimhan reduction of a principal $G$-bundle which vary over a family.

**Lemma 5.10** Let $\mathcal{E} \to C \times S$ be a family of $G$-bundles with $S$ an irreducible scheme of finite type over $k$. Then there is a non-empty open subset $U$ of $S$, a parabolic $P$ and reduction $\sigma_u$ of $\mathcal{E}|_{C \times \{u\}}$ to $P$ for each $u \in U$ such that $\sigma_u$ is the Harder-Narasimhan reduction of $\mathcal{E}|_{C \times \{u\}}$ for each $u \in U$ and the numerical types $[\sigma_u] = [\sigma_v]$ for every $u, v \in U$.

**Proof** Consider the family $\text{ad}(\mathcal{E})$ over $C \times S$, since the degrees of the subbundles of each of the vector bundles occurring in the above family is bounded above by an integer which depends only on $S$, there are only finitely many choices for the $(P, [\sigma])$ with $P \supset B$ where the $G$-bundles in the family $\mathcal{E}$ can admit a Harder-Narasimhan reduction to these parabolics with numerical types $[\sigma]$. Using the uniqueness of the Harder-Narasimhan reduction we produce a finite collection of constructible subsets of $S$ (by Lemma 2.4) corresponding to $G$-bundles in the above family whose Harder-Narasimhan reduction is defined by the pair $(P, [\sigma])$. The union of these constructible subsets is $S$. Hence one of them contains a non-empty open subset of $S$. This proves the lemma. □

Let $P$ be a fixed parabolic subgroup of $G$ containing $B$. Let $L$ be the Levi quotient of $P$. We denote by $\mathcal{T}$ the quotient $L/Z^0(L)$ where $Z^0(L)$ is the connected component.
of the center of \( L \). Let \( \overline{B} \) (resp. \( \overline{T} \)) be the Borel subgroup (resp. maximal torus) be defined by image of \( B \) (resp. \( T \)). This defines an induced root system for \( \overline{L} \). Let \( \overline{Q}_L \) be the coroot lattice of this root system. Let \( \sigma \) be a \( P \) reduction of \( E \) with the numerical type \([\sigma]\). We will denote by \( c_L([\sigma]) \) the element \( c_L(p_*(E_\sigma)) \). This is well defined by Lemma \( 3.18 \). Recall the definition of \( M_G(c, d) \) from the paragraph above Proposition \( 3.19 \).

**Proposition 5.11** Let \( E \) be a principal \( G \)-bundle over \( C \). There exists a constant \( D \) with the property if \( E \) admits a \( P \) reduction of numerical type \([\sigma]\) then the scheme of sections \( \text{Sec}_{E/P}^{[\sigma]} \) has an open dense subscheme \( U^{[\sigma]} \) with the property that the \( \overline{L} \)-bundles associated to every point of \( U^{[\sigma]} \) is a member of \( M_{\overline{L}}(c_L([\sigma]), D) \).

**Proof** Let \( X \) be an irreducible component of \( \text{Sec}_{E/P}^{[\sigma]} \). Let \( \mathcal{E}_P \) be the restriction of the universal \( P \)-bundle over \( C \times X \) Let \( p \) be the natural surjection from \( P \) to \( \overline{L} \). Then we have a family \( p_*\mathcal{E}_P \) of \( \overline{L} \)-bundles over \( C \times X \). By Lemma \( 5.11 \), there is a parabolic subgroup \( \overline{P}_1 \) of \( \overline{L} \) and an open subscheme \( U \) of \( X \) such that \( (p_*\mathcal{E}_P)_x \) admits Harder-Narasimhan reduction \( \overline{\sigma}_{1,x} \) to the parabolic \( \overline{P}_1 \) with a fixed numerical type \([\overline{\sigma}_x] = [\overline{\sigma}_1] \) for each \( x \in U \). Let \( P_1 \) be the parabolic subgroup of \( G \) contained in \( P \) whose image in \( \overline{L} \) is \( \overline{P}_1 \). Now by Lemma \( 2.3 \) the reduction of structure group \( \overline{\sigma}_{1,x} \) canonically defines a reduction of structure group \( \sigma_{1,x} \) of \((E_P)_x\) to \( P \) with the property that \([\sigma_{1,x}] = [1] \) for each \( x \in U \) and such that \([\sigma_1] \) maps to \([\sigma]\) (resp. \([\overline{\sigma}_1]\)) under the homomorphism \( \mathcal{X}_*(P_1) \to \mathcal{X}_*(P) \) (resp. \( \mathcal{X}_*(P_1) \to \mathcal{X}_*(\overline{P}_1) \) is \([\overline{\sigma}_1]\)). By definitions the instability degree \( \text{Ideg}(p_*\mathcal{E}_P)_x \) at \( x \in U \) is exactly computed to be equal to \(-d([\overline{\sigma}_1])\). Using the exact sequence

\[
0 \to T_{\overline{\sigma}_{1,x}} \to T_{\sigma_{1,x}} \to T_\sigma \to 0
\]

we conclude that \( d([\overline{\sigma}_1]) = d([\sigma_1]) - d([\sigma]) \). Now we have a natural morphism \( \text{Sec}_{E/P}^{[\sigma_{1,x}]} \to \text{Sec}_{E/P}^{[\sigma]} \) whose image contains \( U \). This implies that \( \dim(X) \leq \dim(\text{Sec}_{E/P}^{[\sigma_{1,x}]}) \). Now we use the deformation theoretical lower bounds for the dimension of \( X \) and the upper bound Theorem \( 4.10 \) we obtain for each \( x \in U \)

\[
\text{Ideg}(p_*\mathcal{E}_P)_x \leq (g-1)\dim(G/P) + \dim(G/P_1) - \text{Min}_{i=1\ldots m}\{d(\gamma_i)\}
\]

Where \( \gamma_i \)'s are the minimal numerical types for \( E \) with respect to \( P_1 \). Now the proposition follows as the right hand side of the above expression is independent of \([\sigma]\). \( \square \)

We will fix the constant \( D \) prescribed by the above proposition once and for all. This will allow us to work with the open dense subscheme \( U^{[\sigma]} \) of \( \text{Sec}_{E/P}^{[\sigma]} \).

We also need the following lemma Let \( c \in \mathcal{X}_*(T)/\hat{Q} \) be fixed.

**Lemma 5.12** For every positive integer \( n \) there exists an integer \( M_c(n) \) such that every member in \( M_G(c, D) \) admits a reduction to \( B \) with numerical type \([\sigma]\) satisfying \( n \leq [\sigma](\alpha) \leq M_c(n) \) for each \( \alpha \in \Delta \).
Let \( U \) be the open dense subscheme prescribed by the Proposition 5.11 the \( n \) we show that
\[
Hence if \( X \) (resp. \( M \) negative linear combination of simple roots then some positive integral multiple of \( \chi \beta \) extends to a character of \( L \) (hence \( P \)). This implies that there exists a character \( \chi _{\beta} \) of \( P \) such that
\[
\chi _{\beta}|T = n_{\beta \beta} + \sum _{\alpha \in I} n_{\beta, \alpha } \alpha . \tag{6}
\]
We note that if \( \chi \) is any non-trivial character of \( L \) which when restricted to \( T \) is a non-negative linear combination of simple roots then some positive integral multiple of \( \chi \) is a non-negative linear combination of \( \chi _{\beta} \) for \( \beta \not\in I \). Let \( m _{I} = \max _{\beta \not\in I} \{ \sum _{\alpha \in I} |n_{\beta, \alpha }| \} \)
Hence if \( [\sigma ] \) satisfies (\( * \)) for \( N \) then \( [\sigma ](\chi _{\beta}) \geq N \) and conversely if \( [\sigma ](\chi _{\beta}) \geq N \) for each \( \beta \not\in I \) then \( [\sigma ] \) satisfies (\( * \)) for \( N _{1} \) where \( N _{1} = N/n _{I} \) and \( n _{I} = \max _{\beta \not\in I} \{ n _{\beta }\} \).

By Theorem 5.1 there exists an integer \( N _{B} \) such that for each \( [\sigma _{0}] \in \mathcal{X} _{s}(T) \) satisfying (\( * \)) for \( N _{B} \) the space of sections \( \text{Sec}^{[\sigma ]}_{E/B} \) is irreducible and smooth of dimension \( d([\sigma ]) + (1-g)\dim(G/B) \).

Let \( N _{P} \) be defined by \( N _{P} = n _{I}N _{B} + m _{I}M _{D} \), where \( D \) is prescribed by the Proposition 5.11 and \( M _{D} = \max _{x \in \mathcal{X} _{s}(\overline{T})/\mathbb{Q} _{L}} \{ M _{c}(N _{B}) \} \). Here \( M _{c}(N _{B}) \)'s are the constants prescribed by the Lemma 5.12 for the choice of \( D \) and \( n = N _{B} \). The constant \( M _{D} \) is finite because \( \overline{T} \) is semisimple algebraic group, hence there are only finitely many classes in \( \mathcal{X} _{s}(\overline{T})/\mathbb{Q} _{L} \).

Let \( [\sigma ] \in \mathcal{X} _{s}(P) \) be such that \( [\sigma ] \) satisfies (\( * \)) for \( N _{P} \) then we claim that \( \text{Sec}^{[\sigma ]}_{E/P} \) satisfies the properties mentioned in the statement of the Theorem. Infact if \( U^{[\sigma ]} \) is the open dense subscheme prescribed by the Proposition 5.11 then we show that \( U^{[\sigma ]} \) is irreducible and smooth of expected dimension.

Let \( \mathcal{E} \) be the family of \( P \)-bundles over \( C \times U^{[\sigma ]} \) defined by the universal properties of the space of sections. Then by Lemma 5.12 for each \( x \in U^{[\sigma ]} \) the \( \overline{T} \)-bundle \( p _{x} \mathcal{E} _{x} \) admits \( \overline{B} \) reductions with numerical type \( [\overline{\sigma } _{1}] \) satisfying \( N _{B} \leq [\overline{\sigma } _{1}]|\alpha ) \leq M _{D} \).

Now by Lemma 2.3 these reductions canonically defines \( B \)-reductions of \( \mathcal{E} _{x} \) with numerical type \( [\sigma _{1}] \) such that the image of \( [\sigma _{1}] \) under the map \( \mathcal{X} _{s}(B) \to \mathcal{X} _{s}(P) \) (resp. \( \mathcal{X} _{s}(B) \to \mathcal{X} _{s}(\overline{B}) \)) is \( [\sigma ] \) (resp. \( [\overline{\sigma } _{1}] \)). Hence \( U^{[\sigma ]} \) lies in the image of the natural morphism \( f : \text{Sec}^{[\sigma ]}_{E/B} \to \text{Sec}^{[\sigma ]}_{E/P} \).

The Theorem now follows if we show that \( [\sigma _{1}] \) satisfies the property (\( * \)) for \( N _{B} \). This is because the irreducibility of \( \text{Sec}^{[\sigma _{1}]}_{E/B} \) would imply irreducibility of \( U^{[\sigma ]} \). Moreover
for a reduction $\sigma \in U[^\sigma]$ if $\sigma_1 \in \text{Sec}[^\sigma_{E/B}]$ is such that $f(\sigma_1) = \sigma$ then the $B$-bundle $E_{\sigma_1}$ extends to $E_{\sigma}$. Since the map $H^1(C, T_{\sigma_1}) \to H^1(C, T_{\sigma})$ is surjective, the fact that $H^1(C, T_{\sigma_1}) = 0$ implies that $U[^\sigma]$ is smooth at $\sigma$.

Hence we have to only show that $[\sigma_1](\alpha) \geq N_B$ for each $\alpha \in \Delta$. If $\alpha \in I$ then already we have $[\sigma_1](\alpha) = ([\sigma_1])(\alpha) \geq N_B$. If $\beta \notin I$ then the inequality $[\sigma_1](\beta) \geq N_B$ follows from the above inequality for $\alpha \in I$ and the Equation (6) by observing that $[\sigma_1](\chi_{\beta}|_{T}) = [\sigma](\chi_{\beta}) \geq N_P$. This completes the proof of Theorem 5.13.

\[ \square \]

Remark 5.14 The results of this section holds for any connected reductive algebraic groups. To do this firstly we may assume that the group $G$ has no center as the map $G \to G/Z$ has the property that $P$-reduction of a principal $G$-bundle $E$ is in bijection with $P/Z$-reductions of the associated $G/Z$-bundle. This reduces the problem to semisimple algebraic group. Let $f : \tilde{G} \to G$ be the simply connected cover. Then one checks that all the proofs go through if we take semisimple simply connected group schemes over the curve instead of the principal bundles. Now one proves that for any principal $G$-bundle there is a $\tilde{G}$ group scheme whose quotient modulo center in the flat topology is the group scheme over $C$ associated to the $G$-bundle $E$. The last statement follows from the existence of a connected reductive algebraic group $H$, and a surjective homomorphism (in the fppf topology) $g : H \to G$ such that $\ker(g) = (\mathbb{G}_m)^n$ and $[H, H] = \tilde{G}$. This is because $E$ comes from a $H$-bundle $E_1$ (as $H^2(C, \mathbb{G}_m) = 0$) hence the group scheme we are interested is the associated fiber space $E_1(\tilde{G})$ for the conjugation action of $H$ on $\tilde{G}$. Since there exists an embedding of the group scheme $\ker f \subset (\mathbb{G}_m)^n$ for some $n$, we check that the quotient $(\tilde{G} \times (\mathbb{G}_m)^n)/\ker f$ (in the fppf topology) for the diagonal action of $\ker f$ defines such a choice of $H$.

6 Generic stable bundles

In this section we prove the main results about the structure of the moduli spaces of stable maps $\overline{M}_g(E/P, \beta[^\sigma])$ under stronger assumptions on the principal $G$-bundle $E$. In this section we will also assume that the genus of the curve $C$ is at least two. This will ensure that there are stable $G$ bundles of every topological type by Proposition 3.25.

Following the Example 5.7 of [23] we define the notion of a generically stable $G$-bundles as follows.

**Definition 6.1** We say a principal $G$-bundle $E$ is generically stable if for any parabolic $P$ and a $P$ reduction $\sigma$ of $E$ the following holds.

\[
\dim(\overline{M}_g(E/P, \beta[^\sigma])) = d([\sigma]) + (1 - g)\dim(G/P)
\]  

(7)

Note from the definition that generically stable is actually stable. Our main result is the following.
Theorem 6.2 Let \( C \) be a curve of genus at least two. Let \( c \in \mathcal{X}_c(T)/\hat{Q} \) be fixed. Then there exists a generically stable \( G \)-bundle with topological type \( c \).

Proof Let \( c \in \mathcal{X}_c(T)/\hat{Q} \) be fixed. Let \( \mathcal{E} \) be a family of \( G \)-bundles on the curve \( C \) parameterized by a finite type smooth scheme \( S \) which is miniversal at every point of \( S \) and such that \( \mathcal{E}_x = \mathcal{E}|_{C \times \{x\}} \) is stable for each \( x \in S \). Such a family exists by Proposition 3.23 and 3.24.

Let \( \Gamma_1 \) be the set of numerical types satisfying the property that \( d([\sigma]) \leq g \cdot \text{dim}G/P \) and there exists a \( y \in S \) such that \( \mathcal{E}_y \) admits a \( P \) reduction with numerical type \([\sigma]\). By Lemma 2.3 and Lemma 5.3 it follows that \( \Gamma_1 \) is a finite set.

Lemma 6.3 There exists an non-trivial open subset \( U_P \subset S \) with the property that if for some \( x \in U_P \) the \( G \)-bundle \( \mathcal{E}_x \) admits a \( P \) reduction of numerical type \([\sigma] \in \Gamma \) then for every \( y \in U_P \) the bundle \( \mathcal{E}_y \) admits a \( P \) reduction of numerical type \([\sigma] \).

Proof Let \([\sigma] \in \Gamma_1 \) be an numerical type. Let \( V_\sigma \) be the constructible subset of\( S \) consisting of bundles in \( \mathcal{E}_y \) which admit \( P \) reductions of numerical type \([\sigma]\) (by Lemma 2.4). Let \( \Gamma \subset \Gamma_1 \) be the subset of numerical types with the property that for each \([\sigma] \in \Gamma \) the constructible set \( V_\sigma \) contains a non-empty open subset of \( S \).

By Theorem 1.1 of Holla-Narasimhan \([10]\), the union of the these finitely many constructible sets \( V_\sigma \) for \([\sigma] \in \Gamma_1 \) is all of \( S \). Hence one of them must contain a non-empty open subset of \( S \). This proves that \( \Gamma \) is non-empty. Let

\[
U = \bigcap_{[\sigma] \in \Gamma} V_\sigma \bigcap_{[\sigma] \in \Gamma_1 - \Gamma} (V_{[\sigma_1]}^c)
\]

where \((V_{[\sigma_1]}^c)^c\) is the complement of the closure of \( V_{[\sigma_1]} \). Then \( U \) contains a non-empty open subset of \( S \) which satisfies the properties mentioned in the lemma. \( \square \)

Let \( P \) be a parabolic subgroup containing \( B \). Let \( \Gamma \) be the finite set numerical types for the family \( \mathcal{E} \) as described before. By Lemma 3.3 we have an open subscheme \( U_P \) of \( S \) satisfying the properties mentioned in the Lemma. Now define \( U = \bigcap U_P \) where the intersection is over the finitely many parabolic subgroups containing the Borel subgroup \( B \). Hence \( U \) is a non-empty open subscheme of \( S \) with the property that the defining morphism \( f : \text{Sec}^{[\sigma]}_{\mathcal{E}_U/P} \to U \) is surjective for each \( P \supset B \). We will show that for any \( x \in U \) the principal bundle \( E = \mathcal{E}_x \) is generically stable.

Let \( \sigma \) be a \( P \) reduction of \( E \) such that \([\sigma] \in \Gamma \). To simply notations we will denote by \( Y \) the space \( \text{Sec}^{[\sigma]}_{\mathcal{E}_U/P} \).

If \( y \in Y \) is the point corresponding to \( \sigma \) then \( \text{Sec}^{[\sigma]}_{E_y/P} \) is the fiber \( Y_x \) of the morphism \( f : Y \to U \) at the point \( x = f(y) \in U \). We denote by \( df : T_yY \to T_xU = H^1(C, \text{ad}E) \) the induced map on the tangent space (Here the schemes are not smooth and by tangent space we mean the vector space of \( k[e]/e^2 \) valued points whose associated \( k \)-valued point is \( y \)). Then we have the following exact sequence

\[
T_y\text{Sec}^{[\sigma]}_{E_y/P} = H^0(C, T_x) \to T_yY \to H^1(C, \text{ad}E),
\]  

(8)
where $T_\sigma$ is the pull back of the tangent bundle of the fibers of the morphism $E/P \to C$ by $\sigma$. Let $V \to C \times S_P$ be a family of $P$-bundles, parameterized by finite type smooth scheme $S_P$, which is miniversal at the point $y' = E_\sigma \in S_P$. Now the family of $P$-bundles defined by $S_P$ when extended to $G$ defines a family of $G$-bundles. There is a stable $G$-bundle in this family namely $E$ which is an extension of $E_\sigma$. Hence there is a non-trivial open subset $S_0$ of $S_P$ which corresponds to points which extend to stable $G$-bundles (by Proposition 3.20). Hence by replacing $S_P$ by an étale neighborhood we may assume there is a morphism $f$ defined by the versal property of $S$ at $P$. Consider the family $E_P$ of $P$-bundles on $C \times Y$ obtained by the universal property of $Y$. Now by applying to an étale neighborhood $V$ of $y \in Y$ and of $S_P$, and an automorphism of $U$ in a neighborhood of $x \in U$ we may assume that the restriction of $f$ (again denoted by $f$) defines a morphism $f : V \to U$ which can be written as $j \circ g$, where $g : V \to S_P$ is defined by the versal property of $S_P$ at $E_\sigma$ and $j : S_P \to U$ by the versal property of $U \subset S$ at $E$.

Now this implies that the map $df : T_y V \to H^1(C, \text{ad}E)$ factors through $T_yS_P = H^1(C, \text{ad}E_\sigma)$. Hence we have the following commuting diagram which has exact horizontal rows

$$
\begin{array}{cccc}
0 & \to & H^0(C, T_\sigma) & \to & T_y V & \xrightarrow{df} & H^1(C, \text{ad}E) \\
\downarrow & & \parallel & & \downarrow dg & & \parallel \\
H^0(C, \text{ad}E) & \xrightarrow{\eta} & H^0(C, T_\sigma) & \xrightarrow{\delta} & H^1(C, \text{ad}E_\sigma) & \xrightarrow{dj} & H^1(C, \text{ad}E)
\end{array}
$$

From the above diagram it follows that $\ker(dg) = \im(\eta)$. This gives us a dimension bound $\dim(T_y V) \leq \dim(H^1(C, \text{ad}E_\sigma)) + \dim(\im(\eta))$. Using the following exact sequence

$$
0 \to H^0(C, \text{ad}E_\sigma) \to H^0(C, \text{ad}E) \xrightarrow{\eta} H^0(C, T_\sigma)
$$

and Riemann-Roch Theorem for $E_\sigma$ we get the dimension bound $\dim(T_y V) \leq d([\sigma]) + (g - 1)\dim(P) + \dim(H^0(C, \text{ad}(E))).$ The Equation (8) now implies that

$$
\dim(\text{Sec}_{E_\sigma}^{[\sigma]}(E/P)) \leq d([\sigma]) + (1 - g)\dim(G/P).
$$

But this is exactly the deformation theoretic lower bound estimate of the dimension of the space of sections. Hence we have proved that the inequality (8) is an equality. This proves the Equation (8) for $[\sigma] \in \Gamma$ assuming there are no pathological components in $\overline{M}_g(E/P, \beta_{[\sigma]})$.

For an arbitrary $[\sigma]$. Let $X$ be any irreducible component of $\overline{M}_g(E/P, \beta_{[\sigma]})$. By Remark 4.14, there exists a $P$ reduction $\sigma_0$ of $E$ with $[\sigma_0] \in \Gamma$ and $[\sigma] \leq [\sigma_0]$ such that $\dim(X) \leq \dim(X_0) + d([\sigma]) - d([\sigma_0]) - k$. Here $k$ is the number of nodes in the curve corresponding to a general point in $X$ and $X_0$ is an irreducible component of $\overline{M}_g(E/P, \beta_{[\sigma_0]})$ containing $\sigma_0$. Now the Theorem 5.2 follows from (8) for $\sigma_0$ and the deformation theoretic lower bounds for the dimension of $X$ (also we get $k = 0$).

**Remark 6.4** Note that even for a stable bundle $E$, in general $H^0(C, \text{ad}(E))$ is not equal to the center of the Lie algebra of $G$ (this is true in characteristic 0). So, one
has to be careful at this stage. Also the use of versal family is again due to the lack of the moduli space of stable bundles in positive characteristic. Even in characteristic zero we need to use the versal family because of the non-representability of the functor $H^1_S(C, R_u P(E))$ for a family of $L$ bundles $E$.

**Remark 6.5** The proof of the above theorem actually show that there exists an open subset of stable bundles in any family which are generically stable.

**Remark 6.6** Let $N \geq g \cdot \dim(G/P)$ be an integer. Let $c \in X(T)/\hat{Q}$ be fixed. Let $\Gamma$ be the finite set of numerical types for $P$ defined by the property that for each $[\sigma] \in \Gamma$ there exists a stable bundle to topological type $c$ which admits a $P$-reduction of numerical type $[\sigma]$ and $d([\sigma]) \leq N$. The proof of the above theorem actually shows that if the genus of the curve is at least two then there exists a stable bundle $E$ (hence an open set of stable bundles) such that if $E$ admits a $P$ reduction of numerical type $[\sigma] \in \Gamma$ then $\text{Sec}^{[\sigma]}_{E/P}$ is smooth. This is because in the proof we actually get the dimension bound for the tangent space of $\text{Sec}^{[\sigma]}_{E/P}$. Hence the smoothness follows from the deformation theoretic lower bounds.

As a consequence of the Theorem 6.2 we get a result which generalizes the lower bound theorem of Lange (Satz 2.2,[22]).

**Corollary 6.7** Let $E$ be a generic stable bundle if $E$ admits a $P$ reduction $\sigma$ then we have $d([\sigma]) \geq (g-1)\dim(G/P)$.

**Proof** This follows from the definition of a generic stable bundle. The main point is the existence of such bundles which is the content of the Theorem 6.2.

Now we prove a result on the structure of the space of parabolic reductions of a generically stable bundle.

**Proposition 6.8** Let $E$ be a generic stable $G$-bundle. Let $[\sigma]$ be an numerical type. Assume that $\overline{M}_g(E/P, \beta_{[\sigma]})$ is non-empty. Then a generic element in every component of $\overline{M}_g(E/P, \beta_{[\sigma]})$ corresponds to reduction of structure group of $E$ to $P$ with the property that the associated Levi $L$-bundle is generically stable.

**Proof** If $X$ is an irreducible component of $\overline{M}_g(E/P, \beta_{[\sigma]})$ and if the generic element of $X$ corresponds to a map from a curve with $k$ nodes then using the Remark 4.14 we see that $\dim(X) \leq \dim(X_0) + d([\sigma]) - d([\sigma_0]) - k$ for some $[\sigma_0]$ and $X_0$ an irreducible component of $\overline{M}_g(E/P, \beta_{[\sigma_0]})$. Since $X_0$ has the expected dimension we get a contradiction to the deformation theoretic lower bound for $X$. This proves that there are no pathological components in $\overline{M}_g(E/P, \beta_{[\sigma]})$. Hence it is enough to prove the proposition for an irreducible component $S$ of $\text{Sec}^{[\sigma]}_{E/P}$. By the universal property we have a family $E_P$ of $P$ bundles over $C \times S$. Let $E_{L_1}^T$ be the associated family of $L_1$ bundles over $C \times S$, where $L_1$ is the quotient of Levi subgroup of $P$ by its
connected component of the center. For any parabolic $P_{1} \subset P$ such that $P_{1}$ is the image of $P_{1}$ under the map $p : P \to \overline{L}$.

Now we continue as in the proof of the Theorem 6.2 with the family over $\overline{L}$ instead of $G$. Using the Equation (7) one checks that the same proof works once we use the following Lemma.

**Lemma 6.9** With above notations, let $[\sigma_{1}] \in X_{\ast}(P_{1})$ be a numerical type which maps to $[\sigma]$ and to a numerical type $[\overline{\sigma}_{1}]$ of $P_{1}$. Then the spaces $\text{Sec}_{E/P_{1}}^{[\sigma_{1}]}$ and $\text{Sec}_{E/P_{1}}^{[\overline{\sigma}_{1}]}$ are naturally isomorphic.

**Proof** This is a simple consequence of the universal properties of the space of sections.

We also get the following result which generalizes the Proposition 3.7 for the case of parabolic subgroups when $E$ is generically stable.

**Corollary 6.10** Suppose $[\sigma]$ and $[\tau]$ are two numerical types with the property $[\tau] \leq [\sigma]$. Suppose $E$ is a generic stable $G$-bundle which admits a reduction of structure group $\sigma$ to $P$ with numerical type $[\sigma]$. Then $E$ admits a reduction of structure group to $P$ with numerical type $[\tau]$.

**Proof** This immediately follows from the Proposition 6.8 once we show that the space $\overline{M}_{g}(E/P, \beta_{[\tau]})$ is non-empty and this is so because we can always attach rational tails to a map from a curve corresponding to reduction $\sigma$. Now we prove result which generalizes the Theorem 6.7 of [25] and gives a characterization of generically stable bundles.

**Proposition 6.11** A principal $G$-bundle $E$ is generically stable if and only if there exists an integer $N$ such that $\overline{M}_{g}(E/P, \beta_{[\sigma]})$ is irreducible for all parabolic subgroups and numerical types $[\sigma]$ satisfying the property $(\ast)$ for $N$.

**Proof** If a $G$-bundle $E$ is generically stable then by Proposition 6.8 it follows that $\overline{M}_{g}(E/P, \beta_{[\sigma]})$ has no pathological components. Now the “only if part” follows from Theorem 6.13. For the other way, using the Theorem 6.13 again we find an $N_{1} \geq N$ such that if $E$ admits a $P$ reduction of numerical type $[\sigma]$ satisfying $(\ast)$ for $N_{1}$ then $\overline{M}_{g}(E/P, \beta_{[\sigma]})$ has expected dimension. Hence by Remark 116 we conclude that there is a $P$ reduction $\sigma_{0}$ of $E$ and a component $X_{0}$ of $\overline{M}_{g}(E/P, \beta_{[\sigma]})$ containing $\sigma_{0}$ such that $X_{0}$ has the expected dimension. Now the proposition follows by the method of proof of the last part of the Theorem 6.2 and a variant of the Lemma 3.1 which states that if $\sigma_{1}$ is a $P$ reduction of $E$ and then there exists $P$ reduction $\sigma_{2}$ of $E$ such that $[\sigma_{2}] \leq [\sigma_{1}]$ and that $[\sigma_{2}]$ satisfies the property $(\ast)$ for $N_{1}$. This variant has a similar proof. This completes the proof of the Proposition 6.11. □
References

[1] Dan Abramovich and Frans oort. Stable maps and Hurwitz schemes in mixed characteristics. In Advances in Algebraic geometry motivated by Physics, volume 276, pages 89–100. Contemp. Math, 2001.

[2] M. Artin. Versal deformations and algebraic stacks. Invent. Math., 27:165–189, 1974.

[3] K. Behrend. Semi-stability of reductive group schemes over curves. Math. Ann., 301(2):281–305, 1995.

[4] A. Bertram. Towards a Schubert calculus for maps from a Riemann surface to a Grassmannian. Internat. J. Math., 5:811–825, 1994.

[5] A. Bertram, G. Daskalopoulos, and R. Wentworth. Gromov invariants for Holomorphic maps from Riemann surfaces to Grassmannians. J. Amer. Math. Soc., 9(2):529–571, 1996.

[6] Indranil Biswas and Yogish I. Holla. Harder-Narasimhan reduction of a principal bundle. Preprint.

[7] Robert Friedman and John W. Morgan. On the Converse to a Theorem of Atiyah and Bott. Algebraic Geometry Preprint, AG-0006086, 2000.

[8] Robert Friedman and John W. Morgan. Holomorphic Principal Bundles over Elliptic Curves ii. Algebraic Geometry Preprint, AG-0006174 v2, 2001.

[9] W. Fulton and R. Pandharipande. Notes on stable maps and quantum cohomology. In Algebraic Geometry—Santa Cruz, volume 62(2), pages 45–96. Proc. Sympos. Pure math., 1995.

[10] D. Gaitsgory and A. Braverman. Geometric Eisenstein series. Algebraic Geometry Preprint, AG-0009007, 2000.

[11] A. Grothendieck and M. Demazure. Schémas en groupes III. Springer-verlag, 1964. Séminaire de Géométrie Algébrique 3.

[12] A. Grothendieck and J. Dieudonné. Elements de Géométrie Algébrique I. Springer-Verlag, 1971.

[13] G. Harder. Halbeinfinite Gruppenschemata über vollständigen Kurven. Invent. Math., 6:107–149, 1968.

[14] G. Harder. Chevalley groups over function fields and automorphic forms. Ann. of Math., 100:249–306, 1974.

[15] J. Harris and I. Morrison. Moduli of Curves. Springer-Verlag, 1998.
[16] Y. I. Holla and M. S. Narasimhan. A Generalisation of Nagata’s Theorem on Ruled Surfaces. *Composito Math.*, 127:321–332, 2001.

[17] János Kollar. *Rational Curves on Algebraic Varieties*. Springer-Verlag, 1996.

[18] Shrawan Kumar and M.S.Narasimhan. Picard group of the moduli spaces of $G$-bundles. *Math. Ann.*, 308(1):155–173, 1997.

[19] Shrawan Kumar, M.S.Narasimhan, and A. Ramanathan. Infinite Grassmannians and moduli spaces of $G$-bundles. *Math. Ann.*, 300:41–75, 1994.

[20] S. Lang and A. Weil. number of points of varieties over finite fields. *Am. J. Math*, 76:819–827, 1954.

[21] H. Lange. Zur Klassifikation von Regelmannigfaltigkeiten. *Math. Ann.*, 262:447–459, 1983.

[22] H. Lange. Some geometrical aspects of vector bundles on curves. *Aportaciones Matematicas*, 5:53–72, 1992.

[23] V.B. Mehta and S. Subramanian. Harder-Narasimhan filtration of principal bundles. Preprint.

[24] S. Mori. Projective manifolds with ample tangent bundles. *Ann. of Math.*, 110:593–606, 1979.

[25] Mihnea Popa and Mike Roth. Stable maps and Quot schemes. Algebraic Geometry Preprint, AG-0012221, 2000.

[26] A. Ramanathan. Stable principal bundles on a compact Riemann surface. *Math. Ann.*, 213:129–152, 1975.

[27] A. Ramanathan. Deformations of Principal Bundles on the Projective Line. *Invent. Math.*, 71:165–191, 1983.

[28] M. Schlessinger. Functors on artin rings. *Trans. Amer. Math. Soc.*, 130:208–222, 1968.

[29] V.G.Drinfeld and Carlos Simpson. $B$-structures on $G$-bundles and local triviality. *Mathematical Research Letters*, 2:823–829, 1995.

[30] Angelo Vistoli. The deformation theory of local complete intersections. Algebraic Geometry Preprint, AG-9703008, 1997.