Fermionic realisations of simple Lie algebras
and their invariant fermionic operators

J.A. de Azcárraga\textsuperscript{1} and A.J. Macfarlane\textsuperscript{2}

\textsuperscript{1}Dpto. de Física Teórica and IFIC, Facultad de Ciencias,
46100-Burjassot (Valencia), Spain
\textsuperscript{2}Centre for Mathematical Sciences, D.A.M.T.P
Wilberforce Road, Cambridge CB3 0WA, UK

Abstract

We study the representation $D$ of a simple compact Lie algebra $\mathfrak{g}$ of rank $l$
constructed with the aid of the hermitian Dirac matrices of a $(\dim \mathfrak{g})$-dimensional
euclidean space. The irreducible representations of $\mathfrak{g}$ contained in $D$ are found by
providing a general construction on suitable fermionic Fock spaces. We give full
details not only for the simplest odd and even cases, namely $su(2)$ and $su(3)$, but
also for the next $(\dim \mathfrak{g})$-even case of $su(5)$. Our results are far reaching: they apply
to any $\mathfrak{g}$-invariant quantum mechanical system containing $\dim \mathfrak{g}$ fermions. Another
reason for undertaking this study is to examine the role of the $\mathfrak{g}$-invariant fermionic
operators that naturally arise. These are given in terms of products of an odd
number of gamma matrices, and include, besides a cubic operator, $(l-1)$ fermionic
scalars of higher order. The latter are constructed from the Lie algebra cohomology
cocycles, and must be considered to be of theoretical significance similar to the cubic
operator. In the $(\dim \mathfrak{g})$-even case, the product of all $l$ operators turns out to be
the chirality operator $\gamma_q$, $q = (\dim \mathfrak{g} + 1)$.

1 Introduction

Let $G$ be a simple compact Lie group, of dimension $r$ and rank $l$, of Lie algebra $\mathfrak{g}$ defined
by

$$[X_a, X_b] = iC_{abc} X_c,$$

where $a, b, c = 1, 2, \ldots, r$, and the generators $X_a$ are hermitian. We wish to examine here
the realisation $X_a \mapsto S_a$ of $\mathfrak{g}$ given by

$$S_a = -\frac{1}{4} iC_{abc} \gamma_b \gamma_c,$$

where the $\gamma_a$ are the hermitian Dirac matrices

\[ \]
\{\gamma_a, \gamma_b\} = 2\delta_{ab} \quad (3)

of a euclidean space of dimension \( r \equiv \dim \mathfrak{g} \). We wish to describe exactly which representation \( \mathcal{D} \) of \( \mathfrak{g} \) is provided by (2), and how the vector space \( \mathcal{V}_\mathcal{D} \) which carries it may be constructed explicitly. We shall address first (and more fully) the even case in which both \( r \) (and hence the rank \( l \)) are even, contrasting it afterwards with the somewhat different, but also allowed and interesting, odd case in which \( r \) (and \( l \)) are odd.

We have a general reason for undertaking this study and one specific class of applications immediately in mind. The reason is the following: the Fock space of any quantum mechanical system in which there exists \( \dim \mathfrak{g} \) fermions transforming according to \( \mathfrak{g} \) is, necessarily, the carrier space of the representation \( \mathcal{D} \) under investigation here. The specific application that provides much of our motivation is the study of the hidden supercharges \( Q_s \) in \( G \)-invariant supersymmetric quantum mechanical systems involving a set of exactly \( \dim \mathfrak{g} \) fermions. There is one such \( Q_s \) for each Lie algebra cohomology cocycle of \( \mathfrak{g} \) and its construction involves the associated completely antisymmetric invariant tensors fully contracted with the fermionic variables. These matters will be treated in a forthcoming publication.

The first result of this paper (Sec. 2) is a rather intriguing one: in the even case, \( \mathcal{D} \) is the direct sum of \( 2^l \) copies of the irreducible representation (irrep) of \( \mathfrak{g} \) whose highest weight \( \Lambda \) is equal to the Weyl principal vector \( \delta = (1, \ldots, 1) \). Here the Weyl principal vector of \( \mathfrak{g} \), equal to half the sum of the \((r-l)/2\) positive roots of \( \mathfrak{g} \), has been referred to a basis of the fundamental dominant weights of \( \mathfrak{g} \). Below, in Sec. 2, we establish this result, giving a systematic description of \( \mathcal{V}_\mathcal{D} \) as a fermionic Fock space.

Sec. 3 gives full treatment of the \( \mathfrak{g} = A_2 = \text{su}(3) \) case, which clarifies many of the issues typical of the even case. In particular, we exhibit the role of both the chirality operator and of the Kostant fermionic \( \text{SU}(3) \)-invariant cubic operator \( K_3 \). This operator is defined by

\[ K_3 = -\frac{1}{12}iC_{abc}\gamma_a\gamma_b\gamma_c = \frac{1}{3}\gamma_aS_a. \quad (4) \]

For any \( \mathfrak{g} \) of dimension larger than 3, however, \( K_3 \) is not the only relevant fermionic \( G \)-invariant operator. Indeed, for any \( \mathfrak{g} \) of rank \( l \) we may introduce \( l \) fermionic operators by using its \( l \) primitive cocycles. These cocycles are given by skewsymmetric tensors of odd dimension \( 2m_i - 1 \) \((i = 1, \ldots, l)\) where, for each \( i \), \( m_i \) is the order of the symmetric \( G \)-invariant polynomial giving the corresponding primitive Casimir-Racah operator. The structure constants used in (4) simply define the lowest order operator \( K_3 \), for which the \((m_1=2)\)-order tensor is the Killing metric giving the quadratic Casimir. The higher order Casimirs and their associated \((2m_i - 1)\)-cocycles for a simple compact \( \mathfrak{g} \) are all known, and known to be relevant in many areas of physics, as in the mathematical description of anomalies (see \([2],[3]\), current algebras and Schwinger terms (see \( e.g. \,[4],[5]\) and references therein), Wess-Zumino terms and effective actions (\([6],[7],[8]\), \( W \)-algebras (\([9],[10]\)), principal chiral models (see \([11]\) and earlier references therein), and others. For \( \mathfrak{g} = \text{su}(n) \), for example, the Casimir-Racah tensors of order \( m_i = 2, \ldots, n-1 \) give cocycles of orders \( 3, 5, \ldots, (2n-1) \). Writing \( C_{abc} = f_{abc} \) for \( \text{su}(n) \), the five-cocycle is determined by the third-order invariant tensor of coordinates \( d_{abc} \) through

\[ \Omega_{abcd} = f_{xa[b}f_{cd]y}d_{xye}, \quad (5) \]

and is tabulated in \([12]\) for \( n = 3 \) and \( n = 4 \). Such a quantity can be seen \( e.g. \) as a Schwinger term in a two-dimensional chiral \( SU(n) \times SU(n) \) model: see eq. (28) of \([13]\).
and $\mathfrak{g}$. Using (3) we may then construct, for some $k \in \mathbb{R}$
\[ K_5 = -\frac{k}{5!} \Omega_{abcde} \gamma^a \gamma^b \gamma^c \gamma^d \gamma^e. \] (6)

As we shall see, $K_5$ plays a non-trivial role in the understanding of the representation (4) already for $su(3)$, even though, in this case, $K_5$ is related to $K_3$ by Hodge duality as is proved in Sec. 3 and discussed in Sec. 6. For a given $\mathfrak{g}$, higher order $K_{(2m_i-1)}$ operators may be constructed similarly using the corresponding $\Omega_{(2m_i-1)}$ cocycles provided that the rank $l$ is high enough. They all have odd character since they are given by skewsymmetric tensors in the basis of the $r$ different anticommuting $\gamma$’s.

To develop a deeper view, Sec. 4 is devoted to analyse the next smallest even case, that of $\mathfrak{g} = su(5)$, for which there are four fermionic scalars $K_3, K_5, K_7, K_9$, one for each of the four primitive cocycles of $su(5)$. The higher cocycles give rise to operators which feature in any study of the system essentially on the same footing as the cubic Kostant operator (4). With the aid of some MAPLE programs, we are able to analyse various aspects of the role played by all the fermionic scalar operators $K_{(2m_i-1)}$.

Our treatment will make this fermionic character explicit by replacing $\gamma$ matrices, two at a time, as $e.g.$ in (20) below, by Dirac fermions $A$ such that $\{A, A^\dagger\} = 1$. This is easy to do for $r$ even. A similar approach to the dim $\mathfrak{g} = r$ odd case is evidently complicated by the fact that it leaves over, unpaired, the last matrix $\gamma_r$ or, put otherwise, a single Majorana fermion type entity. This does not mean that the odd case cannot be treated in a satisfactory way. It is however essential in treating the odd case to do so in a fashion that respects fully the fermionic nature of the unpartnered Majorana fermion, and also of the $K_3$ operator and the above generalisations. With this in mind, we treat in Sec.5 the dim $\mathfrak{g} = r = 3$ case of $su(2)$ in detail to indicate how to handle the general odd case. In fact, when dim $\mathfrak{g}$ is odd, $\mathcal{D}$ involves the irrep of $\mathfrak{g}$ with highest weight $\delta$ repeated in direct sum $2^{l-1}/2$ times.

We have referred to $K_3$ above as the Kostant operator. Strictly speaking, however, the operator $K$ recently introduced by Kostant $\mathfrak{g}$ (see also $\mathfrak{g}$) contains, besides the cubic term $K_3$, an additional representation dependent piece that we ignore here by restricting our attention to the purely geometrical part $K_3$ of $K$. Operators such as $K_3$, however, are already familiar in non-relativistic supersymmetric quantum mechanics of particles with spin-$\frac{1}{2}$ (see $e.g.$ [15]) or colour degrees of freedom (see $e.g.$ [16, 17]). In such theories there are Majorana fermion variables $\psi_i$ with anticommutation relations
\[ \{\psi_i, \psi_j\} = \delta_{ij} \] (7)
for which there is a representation $\psi_i \mapsto \gamma_i/\sqrt{2}$ in terms of hermitian Dirac matrices. The supercharges of such theories contain or consist of a term proportional to
\[ iC_{ijk} \psi_i \psi_j \psi_k \] (8)
In the case of $su(2)$, where $C_{ijk} = \epsilon_{ijk}$, then $S_i \mapsto \sigma_i/2$ describes spin one-half $\mathfrak{g}$, $\mathfrak{g}$. In this context, it may be remarked, because it underlies our interest, that more general models of particles with colour can possess supercharges involving the higher fermionic scalars. Demonstrations of how this arises, and its relationship to the mentioned hidden supersymmetries that do not close on the Hamiltonian of the model, will be addressed elsewhere.
2 The representation $\mathcal{D}$

It is easy to use (3) and (4) to show that

$$[S_a, S_b] = iC_{abc}S_c,$$

so that $X_a \mapsto S_a$ is indeed a representation of $g$. Next we define the quadratic Casimir operator

$$C_2 = X_a X_a.$$  \hfill (10)

Then using only (3) and the Jacobi identity for the structure constants of $g$, it follows that for $\mathcal{D}$

$$C_2(\mathcal{D}) = S_a S_a$$  \hfill (11)

has the form

$$C_2(\mathcal{D}) = \frac{1}{8} c_2(ad). \dim g \cdot 1_{\dim \gamma},$$  \hfill (12)

where $c_2(ad)$, the eigenvalue of $C_2$ for the adjoint representation $X_a \mapsto ad(X_a)$ given by $(adX_a)_{bc} = -iC_{abc}$, enters via

$$\Tr (ad X_a ad X_b) = c_2(ad) \delta_{ab}.$$  \hfill (13)

Thus, for $su(n)$,

$$C_2(\mathcal{D}) = \frac{1}{8} n(n^2 - 1). 1_{\dim \gamma}.$$  \hfill (14)

To identify the representation $\mathcal{D}$ in terms of the irreps of $g$ is convenient to discuss first the $r$ even case, even though the answer for the odd-dimensional $g$ case is very similar. Since the irreducible $\gamma_a$ matrices of an $(r = 2s)$-dimensional space have dimension $2^s$, $\mathcal{D}$ must be at least of dimension $2^s$ (larger if we used non-irreducible $\gamma$'s). So we ask, what irreps of $g$ are contained in $\mathcal{D}$? The Weyl formula for the dimension of the irrep of $g$ with highest weight $\Lambda$ is given by the Weyl formula (see, e.g., [19], eq. (5.5)):

$$N(\Lambda) = \prod_{\text{positive roots}} \left(1 + \frac{\langle \Lambda, \alpha \rangle}{\langle \delta, \alpha \rangle}\right),$$  \hfill (15)

where the $\alpha$'s are the positive roots of $g$. For the irrep corresponding to the Weyl principal vector $\Lambda = \delta = (1, \ldots, 1)$ we find, since there are $(r - l)/2$ positive roots,

$$N(\delta) = 2^{(r - l)/2}.$$  \hfill (16)

Further this irrep does have the correct eigenvalue of the Casimir operator (11) to agree with (13). To see this we use the general result (see [19], eq. (5.10))

$$c_2(\Lambda) = (\Lambda, \Lambda + 2\delta), \quad C_2(\Lambda) = c_2(\Lambda). 1_{\dim \gamma},$$  \hfill (17)

to find

$$c_2(\delta) = 3(\delta, \delta) = \frac{1}{8} C_2(ad) r$$  \hfill (18)

upon use of the ‘strange formula’ of Freudenthal and de Vries (see [20], eqs. (7.20) and (14.30)). For $su(n)$, (18) reproduces the scalar in (14) since $c_2(ad) = n$. Moreover, we see from eq. (12) that, for any $g$, $c_2(\delta) = c_2(\mathcal{D})$. Since, for $r$ even, the irreducible gammas that determine the dimension of $\mathcal{D}$ are $p \times p$ matrices with $p = 2^r$, and the irrep $(1, \ldots, 1)$
with the correct value for its Casimir operator involves $q \times q$ matrices with $q = 2^{r_l}$, the
natural conclusion is that $\mathcal{D}$ is the direct sum of exactly $2^{\frac{r_l}{2}}$ copies of $(1,\ldots,1)$. When $\mathfrak{g}$ is
odd-dimensional, $r = 2s + 1$, $\mathcal{D}$ is of dimension $2^s = 2^{\frac{r_l-1}{2}}$ and the same reasoning shows
that the irrep of highest weight $\Lambda = \delta$, of dimension $2^{\frac{r_l}{2}}$ is contained $2^{\frac{l-1}{2}}$ times in $\mathcal{D}$.

We may see this explicitly by constructing the required number of copies of the irrep $(1,\ldots,1)$ in a fermionic Fock space description of $\mathcal{D}$. We first consider the even case $\mathfrak{g} = su(3)$ fully; this not only shows how our realisation works in the simplest non-trivial case but also indicates exactly how the general even case is handled. The odd $r$ case will be discussed in Sec. 5.

### 3 The fermionic Fock space $\mathcal{V}_\mathcal{D}$ for $su(3)$

We build the fermionic Fock space of $\mathcal{D}$ for $su(3)$ by constructing four Dirac fermions out of the eight available gamma matrices $\gamma_a$, where the index $a = 1,\ldots,8$ also labels the $su(3)$ generators (say, the Gell-Mann $\lambda_a$ matrices, $[\lambda_a,\lambda_b] = 2if_{abc}\lambda_c$). However not all four Dirac fermions enter on the same footing, nor do they have the same role to play. So, on the one hand, we set

$$2B = \gamma_3 + i\gamma_8 \quad 2B^\dagger = \gamma_3 - i\gamma_8 \quad (19)$$

corresponding to the $su(3)$ Cartan subalgebra generators $\lambda_3$ and $\lambda_8$, and, on the other, corresponding to the positive roots, we define

$$2A_5 = \gamma_4 - i\gamma_5 \quad 2A_2 = \gamma_6 + i\gamma_7 \quad 2A_1 = \gamma_1 + i\gamma_2 \quad (20)$$

(the somewhat strange labelling is adapted to the $su(5)$ case in next section). In this way, (19) translates into

$$\{A_i, A_j^\dagger\} = \delta_{ij} \quad \{A_i, A_j\} = 0 \quad \{B, B^\dagger\} = 1 \quad B^2 = 0 \quad \{B, A_j\} = 0 \quad (21)$$

which exhibit the fermionic nature of our realisation. Defining the fermion number operators

$$N_i = A_i^\dagger A_i \quad (i = 5, 2, 1) \quad N_B = B^\dagger B \quad (22)$$

we find

$$\gamma_4\gamma_5 = i(2N_5 - 1) \quad \gamma_6\gamma_7 = -i(2N_2 - 1) \quad \gamma_1\gamma_2 = -i(2N_1 - 1) \quad \gamma_3\gamma_8 = -i(2N_B - 1) \quad (23)$$

$$I_z = S_3 = \frac{1}{2}(N_2 + N_5 - 2N_1) \quad (24)$$

$$Y = \frac{2}{\sqrt{3}}S_8 = N_5 - N_2 \quad (25)$$

independently of $N_B$.

It is now easy to observe the correspondence between the weights of an $su(3)$ octet (irrep $(1,1) = \{8\}$) and fermionic states labelled by occupation numbers

$$|n_5, n_2, n_1\rangle \quad (I_z, Y) \quad |0, 0, 0\rangle \quad (0, 0)$$
Thus, apart from the eigenvalue $I$ of the $su(2)$ Casimir, which is used to distinguish the octet central states, the $|n_5, n_2, n_1\rangle$ part of the full Fermi Fock space determines the bulk of the expected $su(3)$ characteristics. To complete the characterization we may use the $su(3)$ standard $I$, $U$, $V$-spin raising and lowering operators. These are

\[
I_+ = S_1 - iS_2 = -(B + B^\dagger)A_1^\dagger + A_2 A_5 \ ,
\]

\[
V_- = S_4 - iS_5 = (\omega B + \omega^2 B^\dagger)A_5 + A_2^\dagger A_1^\dagger \ ,
\]

\[
U_+ = S_6 + iS_7 = (\omega^2 B + \omega B^\dagger)A_2 + A_1^\dagger A_5^\dagger \ ,
\]

where $\omega = -\frac{1}{2}(1 - i\sqrt{3})$, $\omega^3 = 1$, and $I_+ = I_+^\dagger$, $V_+ = V_+^\dagger$, $U_+ = U_+^\dagger$. The way $B$ enters here points the way directly towards the construction of two sets of orthogonal octet states in the fermionic Fock space of $D$. We write $|n_B, n_5, n_2, n_1\rangle$ to denote the $V_D$ Fock space states that are simultaneous eigenstates of the number operators $N_B$ and the $N_i$; the standard $su(3)$ octet states of chirality $\pm$ are labelled $|8, \pm, I, I_z, Y\rangle$. We define chirality by

\[
\gamma_9 \equiv -\prod_{a=1}^{8} \gamma_a = (2N_B - 1) \prod_{k=5,2,1} (2N_k - 1) = (-1)^{N_B + N_5 + N_2 + N_1} = (-1)^{\text{tot. Fermi}#} \ ,
\]

which commutes with the generators $S_a$ of the representation $D$ of $su(3)$. To complete the construction we label the states of the two octets of different chiralities $|8, \pm, I, I_z, Y\rangle$. To relate these to the Fock $|n_B, n_5, n_2, n_1\rangle$ states we refer to (25), and identify the highest weight $\pm$ octet states with the Fock space states of the correct chirality using

\[
|8, \pm, I = 1, I_z = 1 Y = 0\rangle \equiv |n_B = 0, n_5 = 1, n_2 = 1, n_1 = 0\rangle \ .
\]

The other octet states are constructed using $V_-$ and $U_+$ each once and then $I_-$ as needed to get all other states except the central state with $I = I_z = Y = 0$. One gets this last state by orthogonality with the state with $I = 1, I_z = Y = 0$. The results that give the states $|8, \pm, I, I_z, Y\rangle$ in terms of $|n_B, n_5, n_2, n_1\rangle$ with the correct phases thus are

\[
|8, \pm, \frac{1}{2}, \frac{1}{2}, 1\rangle = U_+ |0, 1, 1, 0\rangle = \frac{\omega^2}{\omega^3} |1, 0, 1, 0\rangle ,
\]

\[
|8, \pm, \frac{1}{2}, \frac{1}{2}, -1\rangle = V_- |0, 1, 1, 0\rangle = \frac{\omega}{\omega^3} |0, 1, 0\rangle ,
\]

\[
|8, \pm, \frac{1}{2}, -\frac{1}{2}, 1\rangle = I_- |8, \pm, \frac{1}{2}, \frac{1}{2}\rangle = \frac{\omega}{\omega^3} |1, 0, 1, 0\rangle ,
\]

\[
|8, \pm, \frac{1}{2}, -\frac{1}{2}, -1\rangle = I_- |8, \pm, \frac{1}{2}, -\frac{1}{2}\rangle = -\frac{\omega^2}{\omega^3} |0, 1, 1\rangle ,
\]

\[
|8, \pm, 1, 0, 0\rangle = \frac{1}{\sqrt{2}} I_+ |0, 1, 1, 0\rangle = \frac{1}{\sqrt{2}} (|1, 0, 1, 1\rangle + |0, 0, 0\rangle) ,
\]

\[
|8, \pm, 1, -1, 0\rangle = \frac{1}{\sqrt{2}} I_- |8, \pm, 1, 0, 0\rangle = |0, 0, 0, 1\rangle .
\]
together with
\[ |8, \pm, 0, 0, 0\rangle = \frac{1}{\sqrt{2}} |\frac{1}{0}, 1, 1, 1\rangle \pm |\frac{0}{0}, 0, 0, 0\rangle. \] (31)

Since the \( I, U \) and \( V \) spin operators are all even, as any generator should be, all states within a given octet have the same chirality by [28]. Thus, chirality distinguishes the two octets \( \{8, \pm\} \) of even/odd total fermion number that span \( D \) in this case.

Let us now look for the role played by the two fermionic scalars \( K_3, K_5 \). For \( K_3 \) we may use the results given so far to find its complete operator expression in the full fermionic Fock space \( V_D \). This gives
\[ K_3 = L_{125} - B(N_1 + \omega N_5 + \omega^2 N_2) - B^\dagger(N_1 + \omega^2 N_5 + \omega N_2) \] , (32)
where \( L_{125} = A_1 A_2 A_5 + A_5^\dagger A_2^\dagger A_1^\dagger \). \( L_{125} \) in eq. (32) affects only the two central octet states, while the other terms affect only the states of the hexagonal rim of the octet (i.e. the states of the orbit of the state \( |29\rangle \) under the Weyl group of \( su(3) \)). Eq. (30) leads easily to the result
\[ K_3 |8, \pm, II_z Y\rangle = |8, \mp, II_z Y\rangle. \] (33)
Thus, \( K_3 \) changes the chirality of the state while respecting the \( su(3) \) labels. This does not depend on \( g \) being \( su(3) \): it is a consequence of \( K_3 \) being realized by a three-cocycle of a \( g \), since a cocycle is both odd (so that \( \{K_3, \gamma_9\} = 0 \)) and \( G \)-invariant. Thus, similar remarks apply also for the higher order fermionic operators \( K_{(2m_i - 1)} \) associated with the various primitive cocycles of any \( g \), since they are all odd and \( G \)-invariant and, in particular, to \( K_5 \) for \( su(3) \).

Since for larger \( g \) and higher cocycles the task of finding complete operator expressions for quantities like \( K_3, K_5 \ldots \) becomes rapidly more time-consuming, it is better to seek expressions only for their parts that are non-trivial in their action on highest weight states. We illustrate this for \( K_3 \) first. We have
\[ K_3 = -\frac{1}{12} i \sum_{\text{triples}} f_{abc} \gamma_a \gamma_b \gamma_c, \] (34)
over non-trivial triples such that \( a < b < c \), so that
\[
K_3 = -\frac{1}{2} i (\gamma_{123} + \frac{1}{2} (\gamma_{147} + \gamma_{365} + \gamma_{257} + \gamma_{246}) + \frac{1}{2} (\gamma_{345} + \gamma_{376}) + \frac{\sqrt{3}}{2} (\gamma_{458} + \gamma_{678}))
\]
\[ = \frac{1}{2} (N_2 + N_5 - 2N_1) \gamma_3 + \frac{\sqrt{3}}{2} (N_5 - N_2) \gamma_8 + \ldots \]
\[ = \gamma_3 S_3 + \gamma_8 S_8 + \ldots , \] (35)
by (24), (25). Here the dots indicate terms (coming from the second bracket of (33)) that give vanishing contribution to the action of \( K_3 \) on the Weyl orbit of \( |29\rangle \). Hence on \( |29\rangle \)
\[ K_3 = \gamma_3 = B + B^\dagger \] and \[ K_3^2 = 1 \] , (36)
in agreement with (32).

Similarly, for actions on \( |29\rangle \) we have
\[ K_5 = -\frac{k}{5!} \sum_{\text{pentuples}} \Omega_{abcde} \gamma_a \gamma_b \gamma_c \gamma_d \gamma_e = -k \sum_{\text{pentuples}} \Omega_{abcde} \gamma_a \gamma_b \gamma_c \gamma_d \gamma_e \] , (37)
over pentuples such that \( a \in \{3, 8\}, \ b = b + 1, \ c < d, \ e = d + 1 \). This is justified: the total antisymmetry of \( \Omega_{abcd} \) allows us to order the indices in any convenient way and sum over pentuples that both respect this order and give terms that fail to annihilate \((29)\). We find

\[
K_5 = k\gamma_3(\Omega_{31245} - \Omega_{31267}) + k\gamma_8(\Omega_{81245} - \Omega_{81267} - \Omega_{84567}) = k\frac{\sqrt{3}}{12}\gamma_8(1 + 1 + 2) \ , \quad (38)
\]

using data from table 3 of \([12]\) and the results

\[
i\gamma_1\gamma_2 = 2N_1 - 1 \rightarrow -1 , \ i\gamma_4\gamma_5 = 1 - 2N_5 \rightarrow -1 , \ i\gamma_6\gamma_7 = 2N_2 - 1 \rightarrow 1 \ , \quad (39)
\]

for actions on \((29)\). Thus

\[
K_5 = \frac{k}{\sqrt{3}}\gamma_8 \quad \text{and} \quad K_5^2 = \frac{1}{3}k^2 \quad (40)
\]
on \((29)\). It was also checked explicitly that non-zero coefficients of the type \( \Omega_{38cde} \) do not give rise to any non-vanishing contributions to \((40)\).

Thus, setting \( k = \sqrt{3} \), and acting on the vector space spanned by the two highest weight vectors of \((29)\),

\[
|n_b = 0, \text{HW}\rangle \quad , \quad |n_b = 1, \text{HW}\rangle \quad , \quad (41)
\]
we find \( K_3 \leftrightarrow \sigma_1, \) and \( K_5 \leftrightarrow \sigma_2, \) where \( 2B = \gamma_3 + i\gamma_8 \). Hence \( \chi = -iK_3K_5 \leftrightarrow -i\sigma_1\sigma_2 = \sigma_3 \).

We may also use Hodge duality to give a direct proof

\[
K_5 = \left(-\frac{\sqrt{3}}{5!}\right)\Omega_{pqrst}\gamma_{pqrst} = \left(-\frac{1}{12.5!}\right)\gamma_{[pqrst]}\epsilon_{pqrstxyz}f_{xyz} = \frac{1}{12}f_{xyz}\gamma_{[xyz]}\gamma_9 = iK_3\gamma_9 \ . \quad (42)
\]
Here, we have used Eq.(8.14) of \([12]\), the definition \((28)\) of \( \gamma_9 \) and the result

\[
\epsilon_{pqrstxyz}\gamma_{[pqrst]} = -5!\gamma_{[xyz]}\gamma_9 \ . \quad (43)
\]

Eq. \((42)\) implies

\[
- iK_3K_5 = \gamma_9 \quad (44)
\]
known to be represented by \( \sigma_3 \). As a final remark on the \( \text{su}(3) \) case, we notice that insertion of \((24)\) to \((27)\) into \((11)\) gives rise to the result

\[
C_2(D) = 311_6 \ , \quad (45)
\]
as it should by eq. \((14)\), upon cancellation of all number operator terms.

4 The general \( g \) even case

4.1 General remarks

The method of Sec. 3 extends directly to a general even-dimensional \( g \) and indeed, without much modification, to the odd case (Sec. 5). For \( r \) (\( l \)) even, we define \( \frac{1}{2} \) operators
The possible degeneracy. Since $D$ contains $(1, \ldots, 1)$ $2^{l/2}$ times (Sec. 2), we still need $l/2$ labels taking two possible values to distinguish the states of the different copies of $(1, \ldots, 1)$ in $D$. This means, in all, $r/2$ labels, provided by the $l/2$ operators $N_{B_{\mu}}$ and the $(r - l)/2$ operators $N_{A_{l}}$. There is thus one $A$ for each of the positive roots and we can establish of a one to one correspondence like (22) between the weights of the irrep $(1, \ldots, 1)$ of $\mathfrak{g}$ and the simultaneous eigenstates of the commuting number operators $N_{A_{l}}$.

As seen in already for the $su(3)$ case (eq. (27)), however, both the $B_{\mu}$’s and the $A_{i}$’s appear in the definition of the various ladder operators that generate the states of the $(1, \ldots, 1)$ representation. The eigenvalues $n_{1}, \ldots, n_{l/2}$ of the $N_{B_{\mu}}$ can be used as in (18) to define the Fock space states equal to the highest weight states (24 of them) of the different copies of $(1, \ldots, 1)$ in $D$. If one builds the rest of the states of each copy by application of lowering operators, one will find that any one state of any copy is orthogonal to all states of any other copy, as well, of course, to all of the states of its own copy.

The details of the next simplest even $\mathfrak{g}$ of higher rank, $su(5)$, offers further insight into the situation surrounding the fermionic $SU(5)$-invariant operators $K_{3}, K_{5}, K_{7}, K_{9}$ built with the aid of its four non-trivial cocycles. Our discussion for $su(5)$ proceeds along lines similar to those followed for $su(3)$.

### 4.2 Basic definitions for $su(5)$

We use an explicit and essentially standard (cf. [22]) set of Gell-Mann lambda matrices for $su(5)$. For the diagonal matrices $\lambda_{(p^{2}-1)}$, ($p = 2, 3, 4, 5$), we have

\[
\lambda_{3} = \text{diag} (1, -1, 0, 0, 0) \quad , \quad \sqrt{3}\lambda_{8} = \text{diag} (1, 1, -2, 0, 0) \quad , \\
\sqrt{6}\lambda_{15} = \text{diag} (1, 1, 1, -3, 0) \quad , \quad \sqrt{10}\lambda_{24} = \text{diag} (1, 1, 1, 1, -4) \quad .
\]

(46)

The index pairs

\[
(1, 2), (4, 5), (6, 7), (9, 10), (11, 12), (13, 14), (16, 17), (18, 19), (20, 21), (22, 23) \quad ,
\]

(47)

are associated with the remaining $\lambda$’s in a way that can be inferred from the following array:

\[
\begin{pmatrix}
- & (1, 2) & (4, 5) & (9, 10) & (16, 17) \\
- & - & (6, 7) & (11, 12) & (18, 19) \\
- & - & - & (13, 14) & (20, 21) \\
- & - & - & - & (22, 23)
\end{pmatrix}
\]

(48)

For example, $\lambda_{6}$ is a symmetric matrix whose only non-zero element above the main diagonal is a 1 at the place marked $(6, 7)$ in (18), while $\lambda_{7}$ is antisymmetric with a single entry $-i$ in the same place. Next, follow the successive ‘diagonals’ of the array to define the $su(5)$ Dirac fermions for the various pairs as follows.

\[
\begin{align*}
2A_{1} = \gamma_{1} + i\gamma_{2} & \quad , \quad 2A_{2} = \gamma_{6} + i\gamma_{7} \quad , \\
2A_{3} = \gamma_{13} - i\gamma_{14} & \quad , \quad 2A_{4} = \gamma_{22} + i\gamma_{23} \quad , \\
2A_{5} = \gamma_{4} - i\gamma_{5} & \quad , \quad 2A_{6} = \gamma_{11} + i\gamma_{12} \quad , \\
2A_{7} = \gamma_{20} + i\gamma_{21} & \quad , \quad 2A_{8} = \gamma_{9} + i\gamma_{10} \quad , \\
2A_{9} = \gamma_{18} - i\gamma_{19} & \quad , \quad 2A_{10} = \gamma_{16} - i\gamma_{17} \quad ,
\end{align*}
\]

(49)
plus
\[ 2B_1 = \gamma_3 + i\gamma_8 \quad , \quad 2B_2 = \gamma_{15} - i\gamma_{24} \quad . \] (50)

This leads to relations with the number operators \( N_\alpha = A_\alpha^\dagger A_\alpha \), like
\[ i\gamma_1\gamma_2 = 2N_1 - 1 \quad \text{and similarly for} \quad N_\alpha \quad \text{when} \quad \alpha \in \{1, 2, 4, 6, 7, 8\} \]
\[ -i\gamma_4\gamma_5 = 2N_5 - 1 \quad \text{and similarly for} \quad N_\alpha \quad \text{when} \quad \alpha \in \{3, 5, 9, 10\} \quad . \] (51)

Then \( S_a = -\frac{1}{4}if_{abc}\gamma_b\gamma_c \), with the aid of MAPLE output for structure constants, allows us to derive
\[
2S_3 = 2I_3 = N_2 + N_5 - 2N_1 - N_8 + N_{10} - N_9 + N_6 \quad ,
\]
\[
S_8 = \frac{\sqrt{3}}{2}Y \quad , \quad 3Y = (3N_5 - 3N_2 - N_8 - N_6 + N_9 + N_{10} - 2N_3 + 2N_7) \quad ,
\]
\[
S_{15} = \frac{\sqrt{6}}{3}Z_3 \quad , \quad 4Z_3 = \begin{pmatrix} 4N_3 - 4N_6 - 4N_8 + N_9 + N_{10} - N_7 + 3N_4 \end{pmatrix} \quad ,
\]
\[
S_{24} = \frac{\sqrt{10}}{4}Z_4 \quad , \quad Z_4 = N_9 + N_{10} - N_4 - N_7 \quad ,
\] (52)

which reproduce the \( su(3) \) expressions when only the (1,2 and 5)-labelled quantities are retained. Our choice of signs in (49) depends upon the signs of the structure constants for \( su(5) \) – our choice of lambda-matrices was made above to yield agreement with the tables given in [22], upon our wish of avoiding constant terms in the definitions (52) and upon our desire of having all entries in (57) below equal to +1.

We define the highest weight state of any irrep of \( su(5) \) by taking first the highest \( Z_4 \) eigenvalue, then the highest \( Z_3 \) eigenvalue that can arise for that \( Z_4 \) eigenvalue. Next the highest \( Y \) and finally the highest \( I_3 \). Thus we get \( Z_4 = 2 \) for \( N_4 = N_7 = 0, N_9 = N_{10} = 1 \). Hence \( Z_3 = -N_8 - N_6 + N_3 + \frac{1}{2} = \frac{3}{2} \) for \( N_8 = N_6 = 0, N_3 = 1 \), and \( Y = N_5 - N_2 = 1 \) for \( N_2 = 0, N_5 = 1 \). Finally \( I_3 = \frac{1}{2} \) for \( N_1 = 0 \). Hence our highest weight state for any of the four possible irreps \((1, 1, 1, 1)\) of \( su(5) \) in \( D \) includes the \((r - l)/2 = 10 \) labels
\[
|0, 0, 1, 0, 1, 0, 0, 0, 1, 1, 0) \quad ,
\] (53)

where we have written the eigenvalues \( n_\alpha \) of the \( N_\alpha \) with the \( \alpha \) in the standard order \( 1, \ldots, 10 \). We remark that our method of definition corresponds to the use of the subgroup chain in which \((1, 1, 1, 1)\) of \( su(5) \) reduces to \((1, 1, 1)_2 \) of \( su(4) \times u(1) \), plus irreps of lower \( Z_4 \), and then \((1, 1, 1)\) of \( su(4) \) reduces to \((1, 1)_2 \) of \( su(3) \times u(1) \), plus irreps of \( su(3) \) of lower \( Z_3 \), and so on.

For \( g = su(5) \), there are two operators \( N_{B_1} \) and \( N_{B_2} \), providing the \( l/2 = 2 \) additional labels \(|n_{B_1}, n_{B_2}| \) that give rise to the possibilities
\[
|0, 0), |1, 1), |0, 1), |1, 0) \quad ,
\] (54)

available to construct the highest weight states of chiralities +, +, −, − of the four different copies of the \((1, 1, 1, 1)\) irrep contained in \( D \). The complete irreps can be built from these by lowerings.

The \( K_3 \) operator commutes with all \( su(5) \) actions and changes chiralities. This does not furnish a complete picture; there are three other fermionic scalar operators that deserve to be treated on the same footing as the simplest one \( K_3 \). We turn to them next.
4.3 $su(5)$ fermionic operators

We define the 5th order fermionic operator for $su(5)$ via the five-cocycle $\Omega_{abcd}$ by

$$K_5 = -\frac{k_5}{5!} \Omega_{abcd} \gamma_s \gamma_a \gamma_b \gamma_c \gamma_d \quad , \quad k_5 \in \mathbb{R} ,$$

$$= k_5 \sum_{\text{pentuples}} \Omega_{abcd} (-i \gamma_a \gamma_b) (-i \gamma_c \gamma_d) \gamma_s , \quad (55)$$

over pentuples such that $s \in \{3, 8, 15, 24\}$, and $a \in \{1, 4, 6, 9, 11, 13, 16, 18, 20\}$, and $b = a + 1$ , $c > b$, $d = b + 1$. These are the only ones that can fail to annihilate the highest weight state. In fact, for the action of $K_5$ on (53), it follows that

$$(-i \gamma_s \gamma_{x+1}) = 1 \quad \text{for all} \quad x \in \{1, 4, 6, 9, 11, 13, 16, 18, 20, 22\} \quad . \quad (56)$$

Thus if define an array $N_{ab}$ whose only non-vanishing entries on (53) are

$$N_{12} = N_{45} = N_{67} = N_{9,10} = N_{11,12} = N_{13,14} = N_{16,17} = N_{18,19} = N_{20,21} = N_{22,23} = +1 , \quad (57)$$

we may do a computation of the coefficients in $K_5$ of the $\gamma_s$, for each of $s \in \{3, 8, 15, 24\}$, by a separate MAPLE run. This yields a result valid for the action of $K_5$ on highest weight states:

$$2K_5 = k_5 (3 \gamma_3 + \frac{5}{3} \sqrt{3} \gamma_8 + \frac{1}{3} \sqrt{6} \gamma_{15} - \sqrt{10} \gamma_{24}) \quad . \quad (58)$$

The same procedure works for the 7th order fermionic scalar $K_7$ of $su(5)$, given by

$$K_7 = i \frac{k_7}{7!} \Omega_{abcdef} (-i \gamma_{ab}) (-i \gamma_{cd}) (-i \gamma_{ef}) \gamma_s$$

$$= k_7 \sum_{\text{heptuples}} \Omega_{abcdef} (-i \gamma_{ab}) (-i \gamma_{cd}) (-i \gamma_{ef}) \gamma_s , \quad (59)$$

over heptuples with $b = a + 1$, $d = c + 1$, $f = e + 1$, $c > b$, $e > d$ and

$$a \in \{1, 4, 6, 9, 11, 13, 16, 18\} \quad .$$

This leads, by four runs of the corresponding MAPLE program, to

$$K_7 = k_7 \left( \frac{2}{5} \gamma_3 - \frac{2}{45} \sqrt{3} \gamma_8 - \frac{7}{45} \sqrt{6} \gamma_{15} + \frac{1}{15} \sqrt{10} \gamma_{24} \right) \quad . \quad (60)$$

on the HW states (53).

It is of course possible to derive (by MAPLE program) that

$$2K_3 = \gamma_3 + \sqrt{3} \gamma_8 + \sqrt{6} \gamma_{15} + \sqrt{10} \gamma_{24} \quad (61)$$

on (53). But it is easier to derive this directly from

$$K_3 = S_3 \gamma_3 + S_8 \gamma_8 + S_{15} \gamma_{15} + S_{24} \gamma_{24} \quad , \quad (62)$$

and insertion of the known eigenvalues of the $S_s$ for (53) gives back (51). To see that (62) for $K_3$ is correct, think in terms of triples $s, x, x + 1$ with $s \in \{3, 8, 15, 24\}$ and

$$x \in \{1, 4, 6, 9, 11, 13, 16, 18, 20, 22\} \quad . \quad (63)$$

The relevant $f_{abc}$ are in the tables of (22) and we used data from MAPLE, which agreed with these tables.
4.4 Discussion of \( su(5) \) results

By giving a direct evaluation that uses Jacobi identities, it may be shown that the different fermionic scalars \( K_3 \), etc., anticommute. It is wise to verify that the results (58), (60), (63) are in agreement with this. This requires only that the numbers that come from terms like \( \gamma_3^2 = 1 \), etc. sum to zero, which they indeed do. Our results also give the squares of the fermionic scalars

\[
K_3^2 = 5, \quad K_5^2 = 7k_5^2, \quad K_7^2 = \frac{16}{45}k_7^2.
\]

Fortunately we know sufficient cocycle identities valid for all \( su(n) \) to complete a worthwhile check upon our MAPLE computation methods.

We have \( K_3 = -\frac{1}{12}if_{abc}\gamma^a\gamma^b\gamma^c \), and also \( K_3^2 = xI \) for some \( x \in \mathbb{R} \), since \( K_3^2 \) is a scalar. Hence

\[
(dim \gamma)x = \text{tr}K_3^2 = \frac{1}{24}(dim \gamma)f_{abc}f_{abc},
\]

using Dirac trace methods with \( \text{tr}I = dim \gamma \). Since \( f_{abc}f_{abc} = n(n^2 - 1) = 5.24 \), we find \( x = 5 \), as required. To confirm the answers (64) for the cases of \( K_5 \) and \( K_7 \), with \( k_5 \) and \( k_7 \) set equal to 1, we need the identities which are the \( n = 5 \) special cases of

\[
\Omega_{abcde}\Omega_{abcde} = \frac{1}{3}(n^2 - 1)(n^2 - 4) \quad ,
\]

\[
\Omega_{abcdefg}\Omega_{abcdefg} = \frac{2}{45}n(n^2 - 1)(n^2 - 4)(n^2 - 9) \quad .
\]

Thus, setting \( K_5^2 = yI \) for some \( y \in \mathbb{R} \), we obtain

\[
(dim \gamma)y = \text{tr}K_5^2 = \frac{1}{51}\Omega_{abcde}\Omega_{abcde}(dim \gamma)
\]

upon evaluating the trace in a fashion that takes full advantage of antisymmetries. Hence, using (66) at \( n = 5 \), we find \( y = 7 \), as expected.

We note the right hand sides of (66) and (67) are zero for low enough \( n \) as consistency requires. We note also that (60) evaluated for \( n = 3 \) allows us to see agreement with the result (40), when one puts \( k_5 = 1 \) in (40). Also, we check from (61),(68), and (60) that \( K_3, K_5 \) and \( K_7 \) anticommute with each other. Next, by asking for the unique linear combination that anticommutes with \( K_3, K_5 \) and \( K_7 \), we can show that on highest weight states (53)

\[
K_9 = k_9 (10\gamma_3 - 10\sqrt{3}\gamma_8 + 5\sqrt{6}\gamma_{15} - \sqrt{10}\gamma_{24})
\]

so that

\[
K_9^2 = 560k_9^2.
\]

Setting \( k_5, k_7, k_9 = 1 \), we can compute the effect on (53) of the product \( K_3K_5K_7K_9 \). It is found that, as expected, only terms containing permutations of \( \gamma_3\gamma_8\gamma_{15}\gamma_{24} \) survive, so that

\[
K_3K_5K_7K_9 = \frac{16}{3}7\sqrt{5}\gamma_3\gamma_8\gamma_{15}\gamma_{24}.
\]

If we define \( L_3 \) so that \( L_3 = c_3K_3 \) and \( L_3^2 = 1 \), and so on, then above results allow the choice

\[
c_3 = \sqrt{5}, \quad c_5 = \sqrt{7}, \quad c_7 = \frac{3}{4}\sqrt{5}, \quad c_9 = \frac{1}{4\sqrt{35}},
\]

so that
so that
\[ \chi = L_3 L_5 L_7 L_9 = \gamma_3 \gamma_8 \gamma_15 \gamma_{24} \]    \hspace{1cm} (73)
and \( \chi^2 = 1 \).

In view of the direct computation (42) for \( su(3) \) and of (71), we might have expected to find
\[ \chi = \gamma_{25} = -\prod_{\alpha=1}^{24} \gamma_\alpha \] \hspace{1cm} (74)
rather than (73) for (factors apart) the product of the \( K \)'s in (71). However, recalling from (56) that for action upon (53)
\[ \gamma_1 \gamma_2 = i \text{ etc.,} \]
we see that all the \( \gamma \)'s absent from (73) can be smuggled back (73) at the sole cost of a factor \( i^{10} = -1 \), so that the two
equations for \( \chi \) (73), (74) coincide. We shall come back to this point in Sec. 6.

Since the \( L \) operators commute with the \( su(5) \) action, we can consider the effect on the vector of \( su(5) \) highest weight states
\[ |00, HW\rangle , |11, HW\rangle , |01, HW\rangle , |10, HW\rangle \] \hspace{1cm} (75)
where the first two labels provide the eigenvalues of the number operators \( N_{B1} \) and \( N_{B2} \), associated with the Dirac fermions \( 2B_1 \) and \( 2B_2 \) in (20), and \( HW \) indicates the specification given previously (23) of the highest weight state of the \( su(5) \) irrep \((1,1,1,1,1)\).

Thus the four \( L \)-operators may be represented by a set of four Dirac matrices \( \Gamma_\mu \), where \( \mu \in \{1,2,3,4\} \) with \( \chi \) represented by \( \Gamma_5 = \Gamma_1 \Gamma_2 \Gamma_3 \Gamma_4 \). While the \( \Gamma_\mu \) have complicated looking actions, although with the correct anti-commutation relations
\[ \{ \Gamma_\mu , \Gamma_\nu \} = 2\delta_{\mu\nu} \] \hspace{1cm} (76)
the bosonic scalar \( \chi \) is given by
\[ \chi = (2N_{B1} - 1)(2N_{B2} - 1) \rightarrow I \otimes \sigma_3 \] \hspace{1cm} (77)
To compare with the action of \( K_3, K_5 \) and chirality on the highest weight states (29), we recall that from Sec. 3 that these actions are given simply by Pauli matrices. Also the eigenvalues of chirality distinguish between the two octets that comprise the Brink-Ramond representation of \( su(3) \). While we have found a fairly natural analogue of this picture for \( su(5) \), no nice representation of the \( \Gamma \) matrices just discussed emerges.

5 The odd case

Again doing the simplest odd case of \( su(2) \) with \( r = 3 \) and \( l = 1 \) with sufficient care indicates the pattern that governs the general odd case quite clearly.

Given \( S_c = -\frac{1}{2}i\epsilon_{abc}\gamma_a\gamma_b \), we have
\[ 2A = \gamma_1 - i\gamma_2 \text{ , } 2A^\dagger = \gamma_1 + i\gamma_2 \text{ , } N_A = A^\dagger A \text{ , } -i\gamma_1 \gamma_2 = 2(N_A - 1) \] \hspace{1cm} (78)
\[ S_3 = -\frac{1}{2}i\gamma_1 \gamma_2 = N_A - \frac{1}{2} \text{ , } S_+ = \gamma_3 A^\dagger \text{ , } S_- = A \gamma_3 \] \hspace{1cm} (79)
It then follows algebraically that \( S^2 = \frac{3}{4}I \), without raising the question as to whether or not we use Pauli matrices \( \sigma_1 \) and \( \sigma_2 \) for \( \gamma_1 \) and \( \gamma_2 \). The \( K_3 \) operator is now given by
\[ K_3 = -\frac{i}{12}\epsilon_{abc}\gamma_a\gamma_b\gamma_c \] \hspace{1cm} (80)
One way to get a complete Fock space description introduces \( \phi \) such that \( \phi^2 = 1 \), \( \phi^i = \phi \), \( \{ \phi, \gamma_a \} = 0 \), for \( a \in \{1, 2, 3\} \). Defining \( B \) by

\[
2B = \gamma_3 - i\phi ,
\]

so that \( N_B = B^\dagger B, -i\gamma_3\phi = 2(N_B - 1) \), we find the representation

\[
S_+ = (B + B^\dagger)A^\dagger , \quad S_- = (S_+)^\dagger .
\]

Now in the \( \mathbb{C}^2 \otimes \mathbb{C}^2 \) Fock space of the \( B \) and \( A \) fermions, we build two equivalent \( j = \frac{1}{2} \) irreps of \( su(2) \) of chiralities \( \pm \), using the definition of chirality

\[
\gamma_5 \equiv -\gamma_1\gamma_2\gamma_3\phi = (-1)^{N_B + N_A} = (-1)^{\text{tot. Fermi}#} ,
\]

Labelling the Fock states \( |n_B, n_A\rangle \), we set

\[
|\pm, j = \frac{1}{2} m = -\frac{1}{2} \rangle = |0, 0\rangle .
\]

Then,

\[
|\pm, j = m = \frac{1}{2} \rangle = S_+|\pm, j = \frac{1}{2} m = -\frac{1}{2} \rangle = \pm|1, 1\rangle
\]

with a typical and essential fermionic minus sign making its appearance. It is clear that we have two orthogonal \( j = \frac{1}{2} \) irreps. It is easy too to compute the matrices of our Fock space representation \( \sigma_0 = 1_2 \)

\[
2S_3 \mapsto \sigma_0 \otimes \sigma_3 \quad 2S_1 \mapsto \sigma_3 \otimes \sigma_1 \quad 2S_2 \mapsto \sigma_3 \otimes \sigma_2
\]

\[
\gamma_1 \mapsto \sigma_2 \otimes \sigma_2 \quad \gamma_2 \mapsto -\sigma_2 \otimes \sigma_1 \quad \gamma_3 \mapsto \sigma_1 \otimes \sigma_0
\]

\[
\phi = \gamma_4 \mapsto \sigma_2 \otimes \sigma_3 \quad \gamma_5 \mapsto \sigma_3 \otimes \sigma_0 \quad 2K_3 \mapsto \sigma_1 \otimes \sigma_3
\]

with all the expected properties. The action of \( K_3 \) (eq. (81)) on any state of our Fock space basis is to produce a state of opposite chirality. We note that all quantities that should anticommute with other fermionic quantities actually do so.

To achieve all the features noted above, a certain price has had to be paid. In case it may not be thought worthwhile to pay it, especially in simple contexts, we offer the following argument. To reach a Fock space description of \( D \) for \( su(2) \), we chose to introduce a dynamical variable \( \phi \) of Majorana type not present in the original formalism. This led us to use \( 4 \times 4 \) rather than \( 2 \times 2 \) gammas. Had we used Pauli matrices, \( \gamma_a = \sigma_a \), the Kostant operator (81) would have read

\[
K_3 = -\frac{1}{12} i\epsilon_{abc}\sigma_a\sigma_b\sigma_c = \frac{1}{2} 1_2
\]

losing sight of the dynamical role of \( K_3 \), and of its fermionic nature as well. Using the irreducible two-dimensional picture is much the same as using only the chirality plus piece of the system, and forgetting about the negative chirality piece. This does represent \( D \) itself more or less satisfactorily, but fails, as our formalism does not, in the treatment of a composite system of two independent systems of type \( D \) [21].

Passing to the general odd case, we see that it is only the last gamma \( \gamma_r \), viewed as a lone Majorana fermion, that makes us do more than we have already done for the even case. And it is to be treated just as we described for \( (\gamma_3, \phi) \) in the \( su(2) \) example. Thus we have \((r - l)/2\) fermions of \( A \)-type as well as \((l + 1)/2\) of \( B \)-type, allowing for \((l + 1)/2\) copies of the irrep \((1, , !, , 1)\). The latter irrep has dimension \( 2^{(r-l)/2} \) and uses \( r = 2s + 1 \).
gammas that may represented minimally by $2^s \times 2^s$ matrices. This itself would imply that $\mathcal{D}$ contains $2^p$ copies of $(1, \ldots, 1)$ with $p = s - (r - l)/2 = (l - 1)/2$. But we have in our Fock space realisation of $\mathcal{D}$ twice as many copies in virtue of the fact that we chose to adjoin to the original dynamical system one additional gamma, thereby doubling the size of all the gammas. We have the choice for $\mathcal{D}$ itself of accepting this, or else, of using only the positive chirality half of the full fermionic Fock space that carries $2^{(l-1)/2}$ copies of $(1, \ldots, 1)$ for $\mathfrak{g}$. As above, this fails to bring into clear focus the full fermionic nature of all the relevant fermionic quantities for the system $\mathcal{D}$.

6 The chirality operator $\gamma_{r+1}$ and final remarks

The previous discussion has exhibited the existence of $l$ fermionic anticommuting scalars $K_{(2m_i-1)}$ which are constructed from the Lie algebra cohomology cocycles of a Lie algebra $\mathfrak{g}$ of even dimension $r$, and the associated set of Dirac matrices. In the even case, to which the rest of this section mainly refers, all these fermionic scalar operators change the chirality of the states, since they anticommute with $\gamma_{r+1}$. In the simplest case of $su(3)$, we have seen in eq. (44) that $-i K_3 K_5 = \gamma_9$, while for $su(5)$ we see, from (71)-(74), that for actions on highest weight states

$$K_3 K_5 K_7 K_9 \propto \chi = \gamma_{25} = -\prod_{\alpha=1}^{24} \gamma_{\alpha}.$$  \hspace{1cm} (90)

But, since all $su(5)$ generators and in particular the raising and lowering operators commute with the ($\mathfrak{g}$-invariant) $K$'s, the same applies to actions on all states. In general, we expect, for each even $\mathfrak{g}$, that $\gamma_{r+1}$ is given to within normalisation by the product of all the available $K$'s, $\prod_{i=1}^{l} K_{(2m_i-1)}$; notice that $\sum_{i=1}^{l} (2m_i - 1) = r$.

The above properties of the various $K$'s reflect the underlying group geometry. The $l$ cocycles of the Lie algebra cohomology of $\mathfrak{g}$ may be looked at as invariant forms on the compact group manifold $G$ associated with $\mathfrak{g}$ and, although not all the form properties are transported to the $K$'s (for instance, unlike forms, $K^{2} \propto 1$), some of them are. The product of all the $(2m_i - 1)$-forms associated with the cocycles is the volume $r$-form on $G$, an even form for each even $\mathfrak{g}$. This accounts for duality relations such as (14), which can be read as $K_3(*K_3) \propto \gamma_9$, expressing the fact that $K_3$ and $K_5$ are dual to each other. It also shows that, when defined, the even chirality operator takes over the role of the volume form on the group manifold in the present context.

The original motivation of [1] was to study the invariant cubic Kostant operator on Lie algebra (symmetric) cosets, to understand the physical degrees of freedom of certain supersymmetric theories; these appeared as solutions to the Kostant-Dirac equation associated with specific cosets. The generalisations introduced in this paper retain many of the properties of the representation independent part of the cubic Kostant cubic operator, in particular that their square is given by Casimir invariants. At the same time, however, they have rich geometrical properties that reflect their Lie algebra cohomology origin (as e.g., that the product of the $l$ operators $K_i$ in the even case is represented by the chirality/volume form). It seems worthwhile to extend the geometrical methods of this paper to the coset case, not discussed here, and to the possible full higher order Kostant operators. This, and the analysis of the hidden supersymmetries mentioned in the introduction, will be discussed elsewhere.
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References

[1] L. Brink and P. Ramond, Dirac equations, light cone supersymmetry and superconformal algebras, hep-th/9908208, Contributed article to Golfand’s Memorial Volume, M. Shifman ed., World Scientific

[2] S.B. Treiman, R. Jackiw, B. Zumino and E. Witten (eds.), Current algebra and anomalies, World Sci. (1985)

[3] J.A. de Azcárraga and J.M. Izquierdo, Lie groups, Lie algebras, cohomology and some applications in physics, Cambridge Univ. Press, Cambridge U.K. (1995)

[4] L.D. Faddeev and S.L. Shatashvili, Algebraic and hamiltonian methods in the theory of non-abelian anomalies, Theor. Math. Phys. 60, 770-778 (Teor. i Mat. Fiz. 50, 206-217) (1984)

[5] J.A. de Azcárraga, J.M. Izquierdo and A.J. Macfarlane, Current algebra and Wess-Zumino terms: a unified geometric treatment, Ann. Phys. 202, 1-21 (1990).

[6] E. d’Hoker and S. Weinberg, General effective actions, Phys. Rev. D50, R6050-4 (1994)

[7] E. d’Hoker, Invariant effective actions, cohomology of homogeneous spaces and anomalies, Nucl. Phys. B451, 725-748 (1995)

[8] J.A. de Azcárraga, A.J. Macfarlane and J.C. Perez Bueno. Effective actions, relative cohomology and Chern-Simons terms, Phys. Lett. B419, 186-194 (1998)

[9] E. Bergshoeff, A. Bilal and K. Stelle, W-symmetries: gauging and symmetry, Mod. Phys. Lett. A6, 4959-4984 (1991)

[10] P. Bouwknegt and K. Schoutens, W symmetry in conformal field theory, Phys. Rep. 223, 183-276 (1993)

[11] J.M. Evans, M. Hassan, N.J. MacKay and A.J. Mountain, Local conserved charges and principal chiral models, Nucl. Phys. 561, 385-412 (1999)

[12] J.A. de Azcárraga, A.J. Macfarlane, A.J. Mountain and J.C. Pérez Bueno, Invariant tensors for simple groups, Nucl. Phys. B 510, 657-687 (1998)

[13] A.C. Davis, A.J. Macfarlane and J.A. Gracey, Anomalous current algebras in the Skyrme model and in chiral $G \times G$ with Wess-Zumino term, Phys. Lett. B194, 415-419 (1987).

[14] B. Kostant, A cubic Dirac operator and the emergence of Euler number multiplets of representations for equal rank subgroups, Duke Math. J. 100, 447-501 (1999)

[15] F. de Jonghe, A.J. Macfarlane, K. Peeters and J.W. van Holten, New symmetries of the monopole, Phys. Lett. B359, 114-117 (1995)
[16] M. Tanimoto, *The role of Killing-Yano tensors in supersymmetric quantum mechanics on a curved manifold*, Nucl. Phys. **B442**, 549-559 (1995)

[17] A.J. Macfarlane and A.J. Mountain, *Hidden supersymmetries of particle motion in a Wu-Yang monopole field*, Phys. Lett. **B373**, 125-129 (1996)

[18] J.L. Martin, *Generalized classical dynamics, and the ‘classical analogue’ of a Fermi oscillator*, Proc. Roy. Soc. (London) **251**, 536-542 (1959); *The Feynman principle for a Fermi system*, *ibid.* **251**, 542-559 (1959)

[19] R. Slansky, *Group theory for unified model building*, Phys. Rep. **79**, 1-128 (1981)

[20] J. Fuchs and C. Schwiegert, *Symmetries, Lie algebras and representations*, Cambridge University Press, Cambridge, U.K. (1997)

[21] A.J. Macfarlane and S.H. Majid, *Quantum group structure in a fermionic extension of the quantum harmonic oscillator*, Phys. Lett. **B268**, 71-74 (1991)

[22] H. Hayashi, I. Ishiwata, S. Iwao, M. Shako and S. Takeshita, *Meson mass formula in broken $SU_N$ symmetry and its applications*, Ann. Phys. (N.Y.) **101**, 394-412 (1976)