Let $\Gamma \subseteq \Gamma_1$ be finitely generated subgroups of $\text{GL}_{n_0}(\mathbb{Z}[1/q_0])$. For $i = 1, 2$, let $\Gamma_i$ be the Zariski-closure of $\Gamma_i$ in $(\text{GL}_{n_0})_Q$, $\mathbb{G}_i^0$ be the Zariski-connected component of $\Gamma_i$, and let $G_i$ be the closure of $\Gamma_i$ in $\prod_{p\mid q_0} \text{GL}_{n_0}(\mathbb{Z}_p)$.

In this article we prove that, if $\mathbb{G}_1^0$ is the smallest closed normal subgroup of $G_1^0$ which contains $G_2^0$ and $\Gamma_2 \cap G_2$ has spectral gap, then $\Gamma_1 \cap G_1$ has spectral gap.

1. Introduction and the statement of the main results.

Let $\Gamma$ be a subgroup of a compact, Hausdorff, second countable group $G$. Let $\overline{\Gamma}$ be the closure of $\Gamma$ in $G$. Suppose $\Omega$ is a finite symmetric generating set of $\Gamma$. Let $\mathcal{P}_\Omega$ be the probability counting measure on $\Omega$, and let

$$T_{\Omega} : L^2(\overline{\Gamma}) \to L^2(\overline{\Gamma}), \quad T_{\Omega}(f) := \mathcal{P}_\Omega \ast f := \frac{1}{|\Omega|} \sum_{\omega \in \Omega} L_\omega(f),$$

where $L_\omega(f)(g) := f(\omega^{-1}g)$. Then it is well-known that $T_{\Omega}$ is a self-adjoint operator, $T_{\Omega}(1_{\overline{\Gamma}}) = 1_{\overline{\Gamma}}$ where $1_{\overline{\Gamma}}$ is the constant function on $\overline{\Gamma}$, and the operator norm $\|T_{\Omega}\|$ is 1. So the spectrum of $T_{\Omega}$ is a subset of $[-1, 1]$ and $T_{\Omega}$ sends the space $L^2(\overline{\Gamma})^0$ orthogonal to the constant functions to itself. Let $T_{\Omega}^0$ be the restriction of $T_{\Omega}$ to $L^2(\overline{\Gamma})^0$. Let

$$\lambda(\mathcal{P}_\Omega; G) := \sup\{|c| \mid c \text{ is in the spectrum of the restriction of } T_{\Omega}^0\}.$$ 

We say the left action $\Gamma \curvearrowright G$ of $\Gamma$ on $G$ has spectral gap if $\lambda(\mathcal{P}_\Omega; G) < 1$.

It is worth mentioning that, if $\Omega_1$ and $\Omega_2$ are two generating sets of $\Gamma$ and $\lambda(\mathcal{P}_{\Omega_1}; G) < 1$, then $\lambda(\mathcal{P}_{\Omega_2}; G) < 1$. So having spectral gap is a property of the action $\Gamma \curvearrowright G$, and it is independent of the choice of a generating set for $\Gamma$.

The following is the main theorem of this article.

**Theorem 1.** Let $\Gamma_2 \subseteq \Gamma_1$ be two finitely generated subgroups of $\text{GL}_{n_0}(\mathbb{Z}[1/q_0])$. For $i = 1, 2$, let $\Gamma_i$ be the Zariski-closure of $\Gamma_i$ in $(\text{GL}_{n_0})_Q$ for $i = 1, 2$, and let $\mathbb{G}_i^0$ be the Zariski-connected subgroup of $G_i$. Suppose the smallest closed normal subgroup of $G_i^0$ which contains $G_2^0$ is $G_1^0$. Then, if $\Gamma_2 \cap \prod_{p\mid q_0} \text{GL}_{n_0}(\mathbb{Z}_p)$ has spectral gap, then $\Gamma_1 \cap \prod_{p\mid q_0} \text{GL}_{n_0}(\mathbb{Z}_p)$ has spectral gap.

**Corollary 2.** Let $\Gamma_2 \subseteq \Gamma_1$ be two finitely generated subgroups of $\text{GL}_{n_0}(\mathbb{Z}[1/q_0])$. Let $\Gamma_1$ be the Zariski-closure of $\Gamma_1$ in $(\text{GL}_{n_0})_Q$, and let $\mathbb{G}_1^0$ be the Zariski-connected subgroup of $G_1$. Suppose $\mathbb{G}_1^0$ is an almost $\mathbb{Q}$-simple $\mathbb{Q}$-group, and $\Gamma_2$ is an infinite group. Then, if $\Gamma_2 \cap \prod_{p\mid q_0} \text{GL}_{n_0}(\mathbb{Z}_p)$ has spectral gap, then $\Gamma_1 \cap \prod_{p\mid q_0} \text{GL}_{n_0}(\mathbb{Z}_p)$ has spectral gap.

**Proof.** Since $\Gamma_2$ is infinite, the Zariski-connected component $\mathbb{G}_2^0$ of the Zariski-closure $\mathbb{G}_2$ of $\Gamma_2$ in $(\text{GL}_{n_0})_Q$ is a non-trivial (Zariski-connected) $\mathbb{Q}$-subgroup of $G_1^0$. Since $\mathbb{G}_1^0$ is an almost $\mathbb{Q}$-simple group and $\mathbb{G}_2^0$ is a
non-trivial Zariski-connected $\mathbb{Q}$-subgroup of $G^0$, the smallest normal subgroup of $G^0$ which contains $G^0$ is $G^0$. And so by Theorem 1, claim follows.

Notice that the smallest closed normal subgroup of $G^0$ which contains $G^0$ is $G^0$ if and only if the restriction of any non-trivial representation $\rho : G^0 \to (\text{GL}_m)_{\mathbb{Q}}$ to $G^0$ is still non-trivial.

**Remark 3.** The first result of this kind goes back to the work of Burger and Sarnak [BS91]. Their result implies Theorem 1 when $\Gamma$'s are integral points of (fixed embeddings of) two Zariski-connected semisimple $\mathbb{Q}$-groups $G^0$'s.

Theorem 1 has an immediate application in explicit construction of expanders. Let us quickly recall that a family of $d$-regular graphs $X_i$ is called a family of expanders if the size of vertices $|V(X_i)|$ goes to infinity and there is a positive number $\delta_0$ such that for any subset $A$ of $V(X_i)$ we have

$$\frac{|e(A, V(X_i) \setminus A)|}{\min(|A|, |V(X_i) \setminus A|)} > \delta_0,$$

where $e(A, B)$ is the set of edges that connect a vertex in $A$ to a vertex in $B$. Expanders have a lot of applications in theoretical computer science (see [HLW06] for a survey on such applications).

It is well-known that Theorem 1 is equivalent to the following theorem (see [SG16, Remark 15] or [Lub94, Section 4.3])

**Theorem 1'.** Let $\Omega_1$ and $\Omega_2$ be two finite symmetric subsets of $\text{GL}_m(\mathbb{Z}[1/q_0])$. For $i = 1, 2$, let $\Gamma_i$ be the group generated by $\Omega_i$, and $G^0$ be the Zariski-connected component of the Zariski-closure of $\Gamma_i$ in $(\text{GL}_m)_{\mathbb{Q}}$. Suppose $\Gamma_2 \subseteq \Gamma_1$ and the normal closure of $G^0_2$ in $G^0_1$ is $G^0$. Then if the family of Cayley graphs $\{\text{Cay}(\pi_{\Omega_2}(\Gamma_2), \pi_{\Omega_1}(\Omega_1))\}_{\text{gcd}(q, q_0) = 1}$ is a family of expanders, then $\{\text{Cay}(\pi_{\Omega_1}(\Gamma_1), \pi_{\Omega_1}(\Omega_1))\}_{\text{gcd}(q, q_0) = 1}$ is a family of expanders.

**Definition 4.** Let $\Omega$ be a finite symmetric subset of $\text{GL}_m(\mathbb{Z}[1/q_0])$. We say that the group $\Gamma = \langle \Omega \rangle$ generated by $\Omega$ has super-approximation with respect to a subset $C$ of positive integers if the family of Cayley graphs $\{\text{Cay}(\pi_{\Omega}(\Gamma), \pi_{\Omega}(\Omega))\}_{m \in C}$ is a family of expanders.

We simply say $\Gamma \subseteq \text{GL}_m(\mathbb{Z}[1/q_0])$ has super-approximation if it has super-approximation with respect to $\{q \in \mathbb{Z}^+ | \text{gcd}(q, q_0) = 1\}$.

In the past decade there has been a surge in proving that super-approximation is a Zariski-topological property (see [BG08-a], [BV12] and [SG10, SGV12]). By now we know that

1. A finite generated Zariski-dense subgroup of $\text{SL}_m(\mathbb{Z})$ has super-approximation with respect to $\mathbb{Z}^+$ (see [BV12, Theorem 1]).
2. A finitely generated subgroup $\Gamma$ of $\text{GL}_m(\mathbb{Z}[1/q_0])$ has super-approximation with respect to $\{q^a | q \text{ is a square-free integer, } \text{gcd}(q, q_0) = 1\}$
   if and only if $G^0 = [G^0, G^0]$ where $G^0$ is the Zariski-connected component of the Zariski-closure of $\Gamma$, (see [SGV12, Theorem 1] and [SG, Theorem 1]).
3. A finitely generated subgroup $\Gamma$ of $\text{GL}_m(\mathbb{Z}[1/q_0])$ has super-approximation with respect to $\{p^n | n \in \mathbb{Z}^+, p \text{ is a prime which does not divide } q_0\}$
   if and only if $G^0 = [G^0, G^0]$ where $G^0$ is the Zariski-connected component of the Zariski-closure of $\Gamma$, (see [SG, Theorem 1]).

The following is the main conjecture on this subject.

**Conjecture 5.** A finitely generated subgroup $\Gamma$ of $\text{GL}_m(\mathbb{Z}[1/q_0])$ has super-approximation if and only if $G^0 = [G^0, G^0]$ where $G^0$ is the Zariski-connected component of the Zariski-closure of $\Gamma$. 

Relaxing the condition on the set of possible residues is crucial in some of the results where super-approximation is used in combination with large sieve or thermodynamical techniques (for instance see [BK14, BKM, MOW]).

Theorem 1 is about inducing super-approximation property from a subgroup with large Zariski-closure to the group itself. So since we know (infinite) arithmetic groups in semisimple \( \mathbb{Q} \)-groups, i.e. \( \mathbb{Z}[1/q_0] \)-points in a semisimple \( \mathbb{Q} \)-group, have super-approximation, we get the following corollary.

**Corollary 6.** Let \( \Gamma \) be a finitely generated subgroup of \( \text{GL}_{n_0}(\mathbb{Z}[1/q_0]) \). Suppose the Zariski-closure of \( \Gamma \) in \( (\text{GL}_{n_0})_\mathbb{Q} \) is an almost \( \mathbb{Q} \)-simple group. Suppose further that \( \Gamma \) contains an infinite arithmetic subgroup of some semisimple group. Then \( \Gamma \) has super-approximation.

Using the mentioned result of Bourgain and Varjú [BV12, Theorem 1], we get the following corollary of Theorem 1.

**Corollary 7.** Let \( \Gamma \) be a finitely generated subgroup of \( \text{GL}_{n_0}(\mathbb{Z}[1/q_0]) \). Suppose the Zariski-closure of \( \Gamma \) in \( (\text{GL}_{n_0})_\mathbb{Q} \) is an almost \( \mathbb{Q} \)-simple group. Suppose further that there is a subgroup \( \Lambda \subset \Gamma \cap \text{GL}_{n_0}(\mathbb{Z}) \) whose Zariski-closure is isomorphic to \( (\text{SL}_m)_\mathbb{Q} \) for some positive integer \( m \). Then \( \Gamma \) has super-approximation.

**Proof.** Let \( H \) be the Zariski-closure of \( \Lambda \) in \( (\text{GL}_{n_0})_\mathbb{Q} \). By the assumption, there is a \( \mathbb{Q} \)-isomorphism \( \rho : H \to (\text{SL}_m)_\mathbb{Q} \). By [SG, Lemma 13], there is \( g \in \text{SL}_m(\mathbb{Q}) \) such that \( \rho(\Lambda) \subseteq g \text{SL}_m(\mathbb{Z})g^{-1} \). And so \( g^{-1}\rho(\Lambda)g \) is a finitely generated Zariski-dense subgroup of \( \text{SL}_m(\mathbb{Z}) \). Hence by [BV12, Theorem 1] and [SG16, Remark 15] we have that \( g^{-1}\rho(\Lambda)g \cap \prod_p \text{SL}_m(\mathbb{Z}_p) \) has spectral gap.

Now let \( \overline{\Lambda} \) be the closure of \( \Lambda \) in \( \prod_p \text{GL}_{n_0}(\mathbb{Z}_p) \), and let \( \Lambda \) be the ring of adeles of \( \mathbb{Q} \). Since \( \rho \) induces a topological isomorphism \( \rho : H(\Lambda) \to SL_m(\Lambda) \), \( g^{-1}\rho(\Lambda)g \) is the closure of \( g^{-1}\rho(\Lambda)g \) in \( \prod_p \text{SL}_m(\mathbb{Z}_p) \). Since \( g^{-1}\rho(\Lambda)g \cap \prod_p \text{SL}_m(\mathbb{Z}_p) \) has spectral gap, we get that \( \rho(\Lambda) \cap \rho(\overline{\Lambda}) \) has spectral gap. And so \( \Lambda \cap \overline{\Lambda} \) has spectral gap. Therefore by Theorem 1 and [SG16, Remark 15] the claim follows.

This kind of result was first obtained by Varjú in the appendix of [BK14] where he proved a special case of Corollary 6 for the group of symmetries of an Apollonian packing. More recently in [FSZ, Theorem 1.3] a special case of Corollary 7 is proved where the Zariski-closure of \( \Gamma \) is assumed to be isomorphic to the restriction of scalars \( R_{k/\mathbb{Q}}((\text{SL}_2)_\mathbb{Q}) \) of \( (\text{SL}_2)_\mathbb{Q} \) for a finite extension \( k/\mathbb{Q} \).

### 1.1. **Outline of the proof.**

Here is an outline of the main ideas of the proof.

**Step 0.** (Initial preparation) Preliminary reductions: it is showed that one can essentially work under the extra assumptions that the Zariski-closure \( G_i \) of \( \Gamma_i \) in \( (\text{GL}_{n_0})_\mathbb{Q} \) is Zariski-connected for \( i = 1, 2 \); and \( G_1 \) is simply-connected (see Section 5).

**Step 1.** (Reduction to an Adelic Bounded Generation) Using Varjú’s Lemma [BK14, Lemma A.2] and [SG16, Lemma 16], the proof of Theorem 1 is reduced to an adelic bounded generation statement (see Theorem 27). The following is a variant of [BK14, Lemma A.2].

**Lemma 8** (Varjú’s Lemma). Suppose \( G \) is a finite group, \( H \) is a subgroup of \( G \), and

\[
G = g_1 H g_1^{-1} \cdot g_2 H g_2^{-1} \cdot \ldots \cdot g_n H g_n^{-1}
\]

for some \( g_i \in G \). Let \( \Omega \) be a symmetric generating set of \( H \). Then

\[
\lambda(\mathcal{P}_\Omega; G) \leq f(|\Omega|, \lambda(\mathcal{P}_{\Omega^1}; H), n) < 1,
\]

where \( \Omega' := \bigcup_{i=1}^n g_i \Omega g_i^{-1} \) and \( f : \mathbb{Z}^+ \times [0, 1] \times \mathbb{Z}^+ \to [0, 1) \).

By [SG16, Lemma 16], it is enough to get a spectral gap for a subgroup of finite index. Hence altogether it is enough to prove the following (see Theorem 27):
Adelic Bounded Generation: there are \( \gamma_1, \ldots, \gamma_m \in \Gamma_1 \) such that \( \gamma_1 \Gamma_2 \gamma_1^{-1} \cdot \ldots \cdot \gamma_m \Gamma_2 \gamma_m^{-1} \) is an open subgroup of \( \hat{\Gamma}_1 \) where \( \hat{\Gamma}_i \) is the closure of \( \Gamma_i \) in \( \prod_{p \neq q} \text{GL}_{n_0}(\mathbb{Z}_p) \).

To get the above-mentioned Adelic Bounded Generation result, we prove many bounded generation results from various angles: Lie algebras; Zariski topology; and \( p \)-adic topology (with certain uniformity on \( p \)). Here is a bit more detailed description of these steps.

**Step 2. (Generating the Lie algebra)** We prove that \( \text{Lie}(\mathbb{G}_2)(\mathbb{Q}) \) generates \( \text{Lie}(\mathbb{G}_1)(\mathbb{Q}) \) as a \( \Gamma_1 \)-module.

**Step 3. (Generating the Lie algebra; Zariski topology)** The infinitesimal result proved in the previous step shows that \( (g_1, \ldots, g_n) \mapsto \gamma_1 g_1 \gamma_1^{-1} \cdot \ldots \cdot g_n \gamma_n g_1^{-1} \) is a geometrically dominant morphism from \( \mathbb{G}_2 \times \cdots \times \mathbb{G}_2 \) to \( \mathbb{G}_1 \) for suitable \( \gamma_i \)’s in \( \Gamma_1 \). Looking at the scheme theoretic closure of \( \Gamma_i \), we deduce that for almost all the geometric fibers \( \mathbb{G}_i^{[p]} \) we still get dominant morphisms (see Proposition 23).

**Step 4. (Generating the Lie algebra; \( p \)-adic topology)** Based on a quantitative open function theorem for \( p \)-adic analytic functions proved in [SC] Lemma 45\(^{1}\), we show the following \( p \)-adic topological bounded generation with certain uniformity on \( p \) (see Proposition 20):

there are a positive integer \( N \) and \( \gamma_1, \ldots, \gamma_n \in \Gamma_1 \) such that \( \Gamma_{1,p}[p^N] \subseteq \gamma_1 \Gamma_{2,p} \gamma_1^{-1} \cdot \ldots \cdot \gamma_n \Gamma_{2,p} \gamma_n^{-1} \) where \( \Gamma_{1,p} \) is the closure of \( \Gamma_i \) in \( \text{GL}_{n_0}(\mathbb{Z}_p) \) and \( \Gamma_{1,p}[p^N] := \Gamma_{1,p} \cap 1 + p^N \text{gl}_{n_0}(\mathbb{Z}_p) \).

**Step 5. (Generating the Lie algebra; modulo \( p \), for large primes)** Step 3 gave us certain dominant morphisms. A result of Pink and R"{u}tsche [PR09] Proposition 2.5 helps us to deduce that the image of these morphisms applied to the \( f_p \)-points of the underlying varieties is large\(^{3}\). And so by a result of Gowers [Gow08] (also see [NP11] Corollary 1) we get that the three fold multiple of the image of such a morphism is the entire \( f_p \)-points of the considered group. So altogether using Nori’s strong approximation [Nor87] Theorem 5.4 we get the following (see Lemma 24).

there are \( \gamma_1, \ldots, \gamma_n \in \Gamma_1 \) such that for large enough \( p \) we have \( \pi_p(\gamma_1 \Gamma_{2,p} \gamma_1^{-1} \cdot \ldots \cdot \gamma_n \Gamma_{2,p} \gamma_n^{-1}) = \pi_p(\Gamma_{1,p}) \), where \( \pi_p \) is the group homomorphism induced by the quotient map \( \mathbb{Z}_p \to f_p \).

**Step 6. (Proving the Adelic Bounded Generation)** Using the truncated logarithmic maps (see Lemma 25), Step 2, taking multiple commutators, and Step 5, we can generate the first \( N \) \( p \)-adic layers of \( \Gamma_{1,p} \) for large enough \( p \). This result together with Step 4 give us the Adelic Bounded Generation.

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## 2. Preliminary results.

In this section, we gather some of the needed well-known results and adapt them to our setting.

### 2.1. Recalling basic analytic properties of \( \mathbb{Q}_p \)-points of an algebraic group.

For the convenience of the reader, in this section some of the well-known analytic properties of the \( \mathbb{Q}_p \)-points \( \mathbb{G}(\mathbb{Q}_p) \) of a linear algebraic \( \mathbb{Q}_p \)-group \( \mathbb{G} \subset (\text{GL}_{n_0})_{\mathbb{Q}_p} \) is recalled. For instance, we will present a detailed argument of why the logarithmic function induces bijection between \( G_c := \mathbb{G}(\mathbb{Q}_p) \cap (I + p^c \text{gl}_{n_0}(\mathbb{Z}_p)) \) and \( \text{Lie}(\mathbb{G})(\mathbb{Q}_p) \cap p^c \text{gl}_{n_0}(\mathbb{Z}_p) \) if \( c \) is a positive integer and \( c > 1 \) when \( p = 2 \).

---

1. In this note \( f_q \) denotes the finite field with \( q \) elements.
We start by summarizing the basic properties of the logarithmic and the exponential functions. In what follows \( c_0 = 1 \) if \( p \) is odd and it is \( 2 \) if \( p = 2 \). For any prime \( p \), the exponential function

\[
\exp : p^{c_0}\mathfrak{gl}_{n_0}(\mathbb{Z}_p) \to \text{GL}_{n_0}(\mathbb{Z}_p)[p^{c_0}], \quad \exp(x) := \sum_{i=0}^{\infty} x^i / i!
\]

and the logarithmic function

\[
\log : \text{GL}_{n_0}(\mathbb{Z}_p)[p^{c_0}] \to p^{c_0}\mathfrak{gl}_{n_0}(\mathbb{Z}_p), \quad \log g := - \sum_{i=1}^{\infty} (I - g)^i / i
\]

are well-defined analytic functions, and inverse of each other, where

\[
\text{GL}_{n_0}(\mathbb{Z}_p)[p^{c_0}] := \ker(\text{GL}_{n_0}(\mathbb{Z}_p) \xrightarrow{\exp} \text{GL}_{n_0}(\mathbb{Z}/p^{c_0}\mathbb{Z}));
\]

moreover we have

\[
\| \exp(x) - I \|_p = \|x\|_p, \quad \text{and} \quad \| \log(g) \|_p = \| g - I \|_p,
\]

for any \( x \in p^{c_0}\mathfrak{gl}_{n_0}(\mathbb{Z}_p) \) and \( g \in \text{GL}_{n_0}(\mathbb{Z}_p)[p^{c_0}] \). Therefore for any integer \( c \geq c_0 \) the exponential and the logarithmic functions induce bijections between \( \text{GL}_{n_0}(\mathbb{Z}_p)[p^{c}] \) and \( p^{c}\mathfrak{gl}_{n_0}(\mathbb{Z}_p) \).

The next lemma is an easy corollary of the chain rule, but a direct detailed argument is presented.

**Lemma 9.** Let \( p \) be a prime, and \( x \in p^{c_0}\mathfrak{gl}_{n_0}(\mathbb{Z}_p) \) where \( c_0 = 2 \) when \( p = 2 \) is odd and \( c_0 = 1 \) otherwise. Suppose \( f \in \mathbb{Q}[X_{ij}] \) is a polynomial on the entries of \( n_0 \)-by-\( n_0 \) matrices. Then

\[
\lim_{n \to \infty} \frac{f(\exp(p^n x)) - f(I)}{p^n} = (df)_I(x),
\]

where

\[
(df)_I(Y_{ij}) := \sum_{ij} \frac{\partial f}{\partial X_{ij}}(I) Y_{ij}.
\]

**Proof.** After multiplying \( f \) by a suitable non-zero integer, we can and will assume that \( f \) has integer coefficients. To make the symbols a bit more clear, we view \( f \) as a polynomial of \( n_0^2 \) variables \( T_i \). Here \( J \) denotes a multi-index, i.e. \( J = (j_1, \ldots, j_{n_0^2}) \) where \( j_i \) are non-negative integers. We denote the symbolic higher-order partial derivatives of a polynomial \( f \) by \( \partial_J f \). For a multi-index \( J \), let \( T^J \) be the monomial \( \prod_i T_i^{j_i} \). By Taylor expansion we have

\[
f(I + [T_i]) = \sum_J \partial_J f(I) T^J = f(I) + (df)_I(T_i) + \sum_{J, \|J\|_1 > 1} \partial_J f(I) T^J,
\]

where \([T_i]\) is the \( n_0 \)-by-\( n_0 \) matrix whose \( ij \)-entry is \( T_{n_0(i-1)+j} \) and \( \|J\|_1 := \sum_i |j_i| \). For a positive integer \( n \), let \( t_i := T_i(\exp(p^n x) - I) \) be the \( i \)-th component of \( \exp(p^n x) - I \) in the above ordering. So

\[
\||f(\exp(p^n x)) - f(I) - (df)_I(t_i)||_p^p = \left\| \sum_{J, \|J\|_1 > 1} \partial_J f(I) T^J \right\|_p \leq \|\exp(p^n x) - I\|_p^2 \|p^n\|_p \leq \|p^n\|_p.
\]

On the other hand, \( t_i = \sum_{k=1}^{\infty} (p^{kn}/k!) T_i(x^k) \) where \( T_i(\cdot) \) is the function which gives the \( i \)-th component of a matrix in the above ordering. Thus we get

\[
\left\| \frac{(df)_I(t_i)}{p^n} - (df)_I(x) \right\|_p = \| \sum_{k=2}^{\infty} (p^{(k-1)n}/k!)(df)_I(x^k) \|_p \leq \|p^n\|_p.
\]

Hence by (2) and (3) we get

\[
\left\| \frac{f(\exp(p^n x)) - f(I)}{p^n} - (df)_I(x) \right\|_p \leq \|p^n\|_p.
\]
which implies our claim. \qed

**Corollary 10.** Let $G \subseteq (GL_{n_0})_{Q_p}$ be a given embedding of a Zariski-connected $Q_p$-group $G$. Then for any positive integer $c$ which is more than 1 for $p = 2$, the logarithmic function

$$\log : G(Q_p) \cap GL_{n_0}(Z_p)[p^c] \rightarrow Lie(G)(Q_p) \cap p^c gl_{n_0}(Z_p)$$

is a well-defined injection.

**Proof.** We already know that $\log : GL_{n_0}(Z_p)[p^c] \rightarrow p^c gl_{n_0}(Z_p)$ is a bijection. So it is enough to show that for any $g \in G_c := G(Q_p) \cap GL_{n_0}(Z_p)[p^c]$ we have $\log g \in Lie(G)(Q_p)$. Next we notice that the natural embedding $g \mapsto \text{diag}(g, (\det g)^{-1})$ of $(GL_{n_0})_{Z_p}$ into $(SL_{n_0+1})_{Z_p}$ sends $GL_{n_0}(Z_p)[p^c]$ to $SL_{n_0+1}(Z_p)[p^c]$ and commutes with the logarithmic function. So we can and will consider $G$ as a subgroup of $(SL_{n_0+1})_{Q_p}$. So $G$ as a closed subset of $(SL_{n_0+1})_{Q_p}$ is given by relations $f_i$, where $f_i$ are polynomials on the entries of $(n_0 + 1)$-by-$(n_0 + 1)$ matrices. Hence

$$\text{Lie}(G)(Q_p) = \{ x \in gl_{n_0+1}(Q_p) | (df_i)_I(x) = 0 \}.$$  

By Lemma [10] we have

$$(df_i)_I(\log g) = \lim_{n \to \infty} \frac{f_i(exp(p^n \log g)) - f_i(I)}{p^n} = \lim_{n \to \infty} \frac{f_i(g^{p^n}) - f_i(I)}{p^n} = 0,$$

which implies $\log g \in Lie(G)(Q_p)$. \qed

**Proposition 11.** Let $G \subseteq (GL_{n_0})_{Q_p}$ be a given embedding of a Zariski-connected $Q_p$-group $G$. Then for a positive integer $c$, which is at least 2 if $p = 2$, the logarithmic function

$$\log : G(Q_p) \cap GL_{n_0}(Z_p)[p^c] \rightarrow Lie(G)(Q_p) \cap p^c gl_{n_0}(Z_p)$$

is a bijection; and so the restriction of the exponential function is its inverse.

**Proof.** First we prove the claim when $c$ is large depending on the embedding of $G$.

By [Mar91] Chapter I, Lemma 2.5.1 (i), we have that the dimension of $G(Q_p)$ as a $Q_p$-analytic manifold is the same as $\dim G$. Since, by Lemma [10] the restriction of the logarithmic function is an analytic immersion of $G_2 := G(Q_p) \cap GL_{n_0}(Z_p)[p^2]$ into $\text{Lie}(G)(Q_p)$ and as $p$-adic analytic manifolds $G_2$ has the same dimension as $\text{Lie}(G)(Q_p)$, we have that for some positive integer $c_1$

$$p^{c_1} gl_{n_0}(Z_p) \cap \text{Lie}(G)(Q_p) \subseteq \log(G_2).$$

This implies that $\exp(p^{c_1} gl_{n_0}(Z_p) \cap \text{Lie}(G)(Q_p)) \subseteq G(Q_p)$. Hence for any $c \geq c_1$ we have

(4) \hspace{1cm} \exp(p^c gl_{n_0}(Z_p) \cap \text{Lie}(G)(Q_p)) \subseteq G_c,

So by (4) and Lemma [10] we get that $\log : G_c \rightarrow \text{Lie}(G)(Q_p) \cap p^c gl_{n_0}(Z_p)$ is a bijection.

Suppose the closed immersion of $G$ is given by the ideal $I_G \subset Q[GL_{n_0}]$. For any $f \in I_G$ the composite function $f \circ \exp$ defines a $p$-adic analytic function on $\text{Lie}(G)(Q_p) \cap p^c gl_{n_0}(Z_p)$ where $c_0 = 1$ if $p > 2$ and $c_0 = 2$ if $p = 2$. We have proved that this analytic function is identically zero on the open set $\text{Lie}(G)(Q_p) \cap p^c gl_{n_0}(Z_p)$ if $c$ is large enough. Hence it is zero, which implies

$$\exp(\text{Lie}(G)(Q_p) \cap p^c gl_{n_0}(Z_p)) \subseteq G(Q_p).$$

Therefore $\log : G(Q_p) \cap GL_{n_0}(Z_p)[p^c] \rightarrow \text{Lie}(G)(Q_p) \cap p^c gl_{n_0}(Z_p)$ is a bijection. \qed
2.2. A remark on certain flat models of an algebraic group. In this work we need to work with certain group schemes over either \( \mathbb{Z}[1/q_0] \) or \( \mathbb{Z}_p \). In order to treat them at the same time, only in this section, we let \( A \) be a PID and \( F \) be its quotient field.

Let \( G \) be a linear algebraic group defined over \( F \). For a fixed \( F \)-embedding \( \rho : G \to (\text{GL}_{n_0})_F \), \( G_\rho \) denotes the image of \( \rho \). Now we view \((\text{GL}_{n_0})_F\) as the generic fiber of the group scheme \((\text{GL}_{n_0})_A\) and let \( G_\rho \) be the Zariski-closure of \( G_\rho \) in \((\text{GL}_{n_0})_A\).

To clarify the previous paragraph, let \( F[\text{GL}_{n_0}] \) be the ring of regular functions of \((\text{GL}_{n_0})_F\) and \( I_{G_\rho} \) be the defining ideal of \( G_\rho \) (which means the scheme structure of \( G_\rho \) is \( \text{Spec}(F[\text{GL}_{n_0}]/I_{G_\rho}) \)); then the ring of regular functions \( A[\text{GL}_{n_0}] \) of \( G_\rho \) is \( A[\text{GL}_{n_0}]/(A[\text{GL}_{n_0}] \cap I_{G_\rho}) \) where \( A[\text{GL}_{n_0}] \) is the ring of regular functions of \((\text{GL}_{n_0})_A\).

**Lemma 12.** In the above setting, the generic fiber of \( G_\rho \) is isomorphic to \( G_\rho \); and, for any \( A \)-algebra \( R \) with free \( A \)-module structure, \( G_\rho(R) \) can be naturally identified with \( G_\rho(R \otimes_A F) \cap \text{GL}_{n_0}(R) \). (Here, since \( R \) is a free \( A \)-modules, we can and will identify \( R \) with a subring of \( R \otimes_A F \) through \( x \mapsto x \otimes 1 \).) In particular, \( \text{Lie}(G_\rho)(R) \) can be naturally identified with \( \text{Lie}(G_\rho)(R \otimes_A F) \cap \text{gl}_{n_0}(R) \).

**Proof.** Let \( S := A \setminus \{0\} \) and \( A := A[\text{GL}_{n_0}] \). Then \( F = S^{-1}A \) and \( A \otimes_A F \simeq S^{-1}A = F[\text{GL}_{n_0}] \). Let \( I := A \cap I_{G_\rho} \). Then \( S^{-1}I = I_{G_\rho} \). So \( (A/I) \otimes_A F \simeq S^{-1}(A/I) \simeq S^{-1}A/S^{-1}I = F[\text{GL}_{n_0}]/I_{G_\rho} = F[\text{G}_\rho] \).

For any \( \phi \in G_\rho(R) := \text{Hom}_{A\text{-alg}}(A[\text{G}_\rho], R) \) and \( \sigma \in A \setminus \{0\} \), we have \( \phi(a) = a \phi(1) \neq 0 \) as \( R \) is a free \( A \)-module. Hence \( S^{-1} \phi : S^{-1}A(\text{G}_\rho) \to S^{-1}R \) is well-defined. As we discussed above \( S^{-1}A(\text{G}_\rho) = F[\text{G}_\rho] \); and \( S^{-1}R \simeq R \otimes_A F \). On the other hand, \( \phi \) can be lifted to an \( A \)-algebra homomorphism from \( A \) to \( R \) such that we get the desired point in \( G_\rho(R \otimes_A F) \cap \text{GL}_{n_0}(R) \).

Now suppose \( \tilde{\phi} \in G_\rho(R \otimes_A F) \cap \text{GL}_{n_0}(R) \). This means \( \tilde{\phi} : F[\text{GL}_{n_0}] \to R \otimes_A F \), \( \tilde{\phi}(x) := \tilde{\phi}(x + I_{G_\rho}) \), sends the generators \( t_{ij} \), \( t \) of \( F[\text{GL}_{n_0}] \) to \( t \).

As \( A \) is a PID, \( A[\text{G}_\rho] \) is a free \( A \)-module; and so we can and will identify \( A[\text{G}_\rho] \) by a subring of \( S^{-1}A[\text{G}_\rho] = F[\text{G}_\rho] \). Now let \( \phi \) be the \( A \)-algebra homomorphism which is the restriction of \( \tilde{\phi} \) to \( A[\text{G}_\rho] \). Since the standard generators of \( F[\text{GL}_{n_0}] \) are sent to \( R \) by \( \phi \), we can let \( R \) to be the codomain of \( \phi \). Hence we get an \( A \)-algebra homomorphism \( \phi : A[\text{G}_\rho] \to R \), i.e. \( \phi \in G_\rho(R) \).

To get the last part, we first notice that \( R[T]/(T^2) = R \oplus R[T] \) is a free \( A \)-module. Hence by the previous part and the definition of Lie algebra we get the following commutative diagram

\[
\begin{array}{cccc}
1 & \rightarrow & \text{Lie}(G_\rho)(R) & \rightarrow & G_\rho(R[T]) & \rightarrow & G_\rho(R) & \rightarrow & 1 \\
1 & \rightarrow & \text{Lie}(G_\rho)(R \otimes_A F) \cap \text{gl}_{n_0}(R) & \rightarrow & G_\rho(R[T] \otimes_A F) \cap \text{GL}_{n_0}(R[T]) & \rightarrow & G_\rho(R \otimes_A F) \cap \text{GL}_{n_0}(R) & \rightarrow & 1,
\end{array}
\]

which implies the last part.

\[\square\]

3. Generating the Lie algebra as a module under the adjoint action.

In this section, we study perfect algebraic groups and their adjoint representation. The main result of this section is Proposition 13.

Let’s recall that the normal closure of an algebraic subgroup \( G_2 \) of \( G_1 \) is the smallest normal closed subgroup of \( G_2 \) in \( G_1 \).

**Proposition 13.** Let \( G_2 \subseteq G_1 \) be Zariski-connected \( \mathbb{Q} \)-groups. Suppose \( G_2 \) is a perfect \( \mathbb{Q} \)-subgroup of \( G_1 \), and further assume that the normal closure of \( G_2 \) in \( G_1 \) is \( G_1 \). Let \( g_1 := \text{Lie}(G_1)(\mathbb{Q}) \) be the \( \mathbb{Q} \)-structure of the Lie algebras of \( G_1 \). Then \( g_2 \) generates \( g_1 \) as an \( G_1(\mathbb{Q}) \)-module under the adjoint representation.
Lemma 14. Let $G_2 \subseteq G_1$ be Zariski-connected $\mathbb{Q}$-groups. Suppose $G_2$ is a perfect $\mathbb{Q}$-subgroup of $G_1$, and further assume that the normal closure of $G_2$ in $G_1$ is $G_1$. Then $G_1$ is perfect.

Proof. Let $f : G_1 \to G_1/[G_1, G_1]$ be the natural quotient map. Since $G_2$ is perfect, $G_2$ is a subgroup of the kernel of $f$. Since $\ker(f)$ is a normal subgroup of $G_1$ and the normal closure of $G_2$ in $G_1$ is $G_1$, we get that $\ker(f) = G_1$. Hence $G_1$ is perfect. \hfill $\square$

The following lemma has been proved in [SG] Lemma 20] for large finite fields. It is reproved here for the convenience of the reader. Lemma 14 enables us to reduce the proof of Proposition 13 to the case where the unipotent radical of $G_1$ is abelian.

Lemma 15. Let $H$ be a Zariski-connected semisimple $\mathbb{Q}$-group and $U$ be a unipotent $\mathbb{Q}$-group. Suppose $H$ acts on $U$, and let $G := H \ltimes U$. Let $A_G$ be the $\mathbb{Q}$-span of $\operatorname{Ad}(G(\mathbb{Q})) \subseteq \operatorname{End}_\mathbb{Q}(\operatorname{Lie}(G)(\mathbb{Q}))$ and $a_U$ be the ideal of $A_G$ generated by $\{\operatorname{Ad}(u) - 1 | u \in U(\mathbb{Q})\}$. Then

1. The Jacobson radical $J(A_G)$ of $A_G$ is equal to $a_U$.
2. $[u, u] \subseteq J(A_G)\mathfrak{g}$ where $u := \operatorname{Lie}(U)(\mathbb{Q})$ and $G := \operatorname{Lie}(G)(\mathbb{Q})$.

Proof. Since $U$ is unipotent, $a_U$ is a nilpotent ideal. Hence $a_U \subseteq J(A_G)$. Let $A_H$ be the $\mathbb{Q}$-span of $\operatorname{Ad}(H(\mathbb{Q}))$ as a subset of $\operatorname{End}_\mathbb{Q}(\mathfrak{g})$. Since $H$ is semisimple, $A_H$ is a semisimple algebra. Moreover, as a $\mathbb{Q}$-algebra, we have $A_G/a_U \simeq A_H$. Overall we get $J(A_G) = a_U$.

Since $U$ is unipotent, log and exp define $\mathbb{Q}$-morphisms between $U$ and its Lie algebra. For any $n \in \mathbb{Z}$, $x, y \in \mathfrak{u}$ we have

$$\operatorname{Ad}(\exp(nx))(y) = \exp(n \operatorname{ad}(x))(y).$$

Thus for $n \in \mathbb{Z}^+$, $x, y \in \mathfrak{u}$ we have

$$n^{-1}(\operatorname{Ad}(\exp(nx))(y) - y) = [x, y] + \sum_{i=1}^{\dim \mathfrak{u}} \frac{n^i}{i!} \operatorname{ad}(x)^i(y) \in J(A_G)\mathfrak{g}.$$

Therefore by the Vandermonde determinant we have that $[x, y] \in J(A_G)\mathfrak{g}$. \hfill $\square$

Lemma 16. Let $H$ be a Zariski-connected, semisimple $\mathbb{Q}$-group, and $\mathfrak{h} = \operatorname{Lie}(H)(\mathbb{Q})$. Let $M$ be a subspace of $\mathfrak{h}$ which is $\overline{H} := \operatorname{Ad}(\overline{H}(\mathbb{Q}))$-invariant. Then there is a normal subgroup $N$ of $H$ such that $\operatorname{Lie}(N)(\mathbb{Q}) = M$; in particular, if $M$ is a proper subspace of $\mathfrak{h}$, then $N$ is a proper normal subgroup of $H$.

Proof. Without loss of generality we can and will assume that $H$ is simply-connected. So there are Zariski-connected, simply-connected, semisimple, $\mathbb{Q}$-simple groups $\overline{H}_i$ such that $H \simeq \oplus_i \overline{H}_i$. Without loss of generality we can and will identify $H$ with $\oplus_i H_i$. Therefore $\mathfrak{h}_i := \operatorname{Lie}(H_i)(\mathbb{Q})$ are simple $\overline{H}_i$-modules, and $M$ can be identified with a subspace of $\oplus \mathfrak{h}_i$.

Claim. Let $pr_j : \oplus \mathfrak{h}_i \to \mathfrak{h}_j$ be the projection onto the $j$-th component. Let $J := \{j | pr_j(M) \neq 0\}$. Then $M = \oplus_{j \in J} \mathfrak{h}_j$.

Proof of Claim. For $j \in J$, there is $x = (x_i)_i \in \oplus \mathfrak{h}_i$ such that $x_j \neq 0$. So there is $\mathfrak{h}_j \in H_j(\mathbb{Q})$ such that $\operatorname{Ad}(h_j)(x_j) \neq x_j$. Hence

$$\operatorname{Ad}(h_j)(x) - x = \operatorname{Ad}(h_j)(x_j) - x_j \in (M \cap \mathfrak{h}_j) \setminus \{0\}.$$

So $M \cap \mathfrak{h}_j \neq 0$. Since $\mathfrak{h}_j$ is a simple $\overline{H}_j$-module, we get $\mathfrak{h}_j \subseteq M$. Thus $M = \oplus_{j \in J} \mathfrak{h}_j$.

Now it is clear that the subgroup $N := \oplus_{j \in J} H_j$ satisfies all the mentioned conditions. \hfill $\square$

The next lemma is tightly related to [SGV12] Lemma 13], where normal subgroups of a perfect algebraic group is described. And we closely follow the proof of the mentioned result.
Lemma 17. Let $H$ be a Zariski-connected semisimple $\mathbb{Q}$-group. Let $\rho : H \rightarrow \mathbb{G}(\mathbb{V})$ be a $\mathbb{Q}$-representation of $H$. Suppose $\mathbb{V}(\mathbb{Q})$ has no non-zero $H(\mathbb{Q})$-fixed point. Let $G := H \ltimes V$, and $\mathfrak{g} := \text{Lie}(G)(\mathbb{Q})$. Let $M$ be a subspace of $\mathfrak{g}$ which is $G := \text{Ad}(G(\mathbb{Q}))-\text{invariant}$. Then there is a normal subgroup $N$ of $G$ such that $M = \text{Lie}(N)(\mathbb{Q})$.

Proof. Let $M' := M \cap V$, and $M''$ be the $\mathbb{Q}$-subgroup of $V$ which is induced by $M'$. Since $H$ is semisimple, $H(\mathbb{Q})$ is Zariski-dense in $H$. Hence $M''$ is invariant under the action of $H$.

Passing to $H \ltimes \mathbb{V}/M'$ and $M/M' \subseteq \mathfrak{h} \oplus V/M'$, we can and will assume that $V \cap M = \{0\}$. Thus projection to $\mathfrak{h}$ induces an embedding and we get an $\text{Ad}(H(\mathbb{Q}))-\text{module homomorphism } \phi : \text{pr } M \rightarrow V$, where $\text{pr} : \mathfrak{h} \oplus V \rightarrow \mathfrak{h}$ is the projection map, such that $M = \{(x, \phi(x)) \mid x \in \text{pr } M\}$.

For any $v \in V$ and $x \in \text{pr } M$, we have

$$(7) \quad \text{Ad}(1, v)(x, \phi(x)) - (x, \phi(x)) = (0, [x, v]) \in M.$$ 

Hence by (7) we have $[\text{pr } M, V] = 0$. On the other hand, by Lemma 10 there is a normal subgroup $H_M$ of $H$ such that $\text{pr } M = \text{Lie}(H_M)(\mathbb{Q})$. In particular, $H_M$ is a semisimple $\mathbb{Q}$-group. So we have that $H_M$ acts trivially on $V$. Thus $H_M$ is a normal subgroup of $G$.

For any $u$ in the centralizer $C_H(\mathbb{Q})(H_M(\mathbb{Q}))$ of $H_M(\mathbb{Q})$ in $H(\mathbb{Q})$ and $x \in \text{pr } M$, we have $\text{Ad}(u)(\phi(x)) = \phi(\text{Ad}(u)(x)) = \phi(x)$. On the other hand, since $H$ is a semisimple $\mathbb{Q}$-group and $H_M$ is a normal $\mathbb{Q}$-subgroup of $H$, we have $\text{Lie}(H(\mathbb{Q})) = \text{Lie}(M(\mathbb{Q})C(H_M)(\mathbb{Q})).$ Overall we get that $H(\mathbb{Q})$ acts trivially on $\phi(\text{pr } M)$. Hence $\phi = 0$, and we get $M = (\text{Lie}(H)(\mathbb{Q}))$. \hfill \Box

Proof of Proposition 13. By Lemma 14, $G_1$ is perfect. So $G_1 \simeq H \ltimes U$ where $H$ is a semisimple $\mathbb{Q}$-group $\mathbb{H}$ and $U$ is a unipotent $\mathbb{Q}$-group. Moreover $\mathbb{V}(\mathbb{Q})$ has no non-zero $\text{Ad}(H(\mathbb{Q}))-\text{fixed element}$ where $V = U/\mathbb{U}$.

Let $M$ be the $\text{Ad}(G_1(\mathbb{Q}))-\text{module generated by } \mathfrak{g}_2$. We want to show that $M = \mathfrak{g}_1$. By Lemma 15 and Nakayama’s lemma, it is enough to prove $[u, u] + M = \mathfrak{g}_1$ where $u := \text{Lie}(U)(\mathbb{Q})$.

By applying Lemma 17 for $H \ltimes \mathbb{V}$ and

$$(M + [u, u])/[u, u] \subseteq \text{Lie}(H)(\mathbb{Q}) \oplus \text{Lie}(\mathbb{V})(\mathbb{Q}) = \mathfrak{h} \oplus [u, u],$$

we get that there is a normal $\mathbb{Q}$-subgroup $N$ of $G_1$ such that $M + [u, u] = \text{Lie}(N)(\mathbb{Q}) \supseteq \text{Lie}(G_2)(\mathbb{Q})$. Hence $N \supseteq G_2$. Since the normal closure of $G_2$ in $G_1$ is $G_1$, we get that $N = G_1$. And so $M + [u, u] = \mathfrak{g}_1$, which finishes the proof as it was explained above. \hfill \Box

4. Simultaneous bounded generation of almost all the fibers: Zariski-topology.

In this section we prove a bounded generation statement at the level of Zariski-topology (see Proposition 13). To avoid recalling the definition of some well-known terms within the statements, they are mentioned here. Along the way, some needed notation is introduced.

Let $\text{Spec}(\mathbb{Z}[1/q_0])$ be the affine scheme of the ring $\mathbb{Z}[1/q_0]$; in particular, the set of its points is

$$\text{Spec}(\mathbb{Z}[1/q_0]) = \{0\} \cup \{p\mathbb{Z}[1/q_0] \mid p \text{ a prime integer}, p \nmid q_0\}.$$ 

The point $(0)$ is called the generic point (as it is dense). For any $p \in \text{Spec}(\mathbb{Z}[1/q_0])$, the residue field over $p\mathbb{Z}[1/q_0]$ is denoted by $k(p)$; that means $k(p)$ is either the finite field with $p$ elements if $p \neq 0$, or $\mathbb{Q}$ if $p = 0$. For any field $F$, its algebraic closure is denoted by $\overline{F}$.

Here we have to work with affine, finite type, reduced, flat group schemes $G$ over $\mathbb{Z}[1/q_0]$; that means, as a scheme $G = \text{Spec}(A)$ where $A$ is a finitely generated $\mathbb{Z}[1/q_0]$-algebra with no non-zero nilpotent element and no additively torsion element, and in addition $A$ has a Hopf algebra structure. For an affine group scheme $G = \text{Spec}(A)$ over $\mathbb{Z}[1/q_0]$, its fiber over $p \in \text{Spec}(\mathbb{Z}[1/q_0])$ is denoted by $G^{(p)}$; that means $G^{(p)} =\ldots$
Proposition 18. Let $G_2 \subseteq G_1$ be subgroups of $\text{GL}_n(\mathbb{Z}[1/q_0])$. For $i = 1, 2$, let $G_i$ be the Zariski-closure of $\Gamma_i$ in $(\text{GL}_n)_{\mathbb{Z}[1/q_0]}$, and suppose the geometric generic fiber $G_i$ of $G_i$ is irreducible. Suppose the normal closure of $G_2$ in $G_1$ is $G_2$. Then there are $\gamma_1, \ldots, \gamma_c$ in $\Gamma_1$ such that the following is a surjective morphism

$$f_1(p) = G_2^p \times \cdots \times G_2^p \rightarrow G_1^p,$$

where $p\mathbb{Z}[1/q_0]$ ranges in a non-empty open subset of $\text{Spec}(\mathbb{Z}[1/q_0])$; that means $p\mathbb{Z}[1/q_0]$ can be any prime ideal except for finitely many non-zero ones. Furthermore, $G_i^p$ are Zariski-connected.

Let us recall parts of Proposition 9.6.1, Theorem 9.7.7 and Theorem 12.2.4 from [EGA66] where constructibility of being a dominant morphism (see [EGA66], Chapter I.3.3) for definition of a constructible set) and genericity of dimension, being smooth, and being geometrically irreducible for the fibers of a $\mathbb{Z}[1/q_0]$-scheme are proved (see [EGA66], Chapter I.4.4) for definition of a smooth morphism and see [SGV12], Theorem 40, Lemma 42 for an effective version of the former results).

Theorem 19. (1) Let $V$ be an affine $\mathbb{Z}[1/q_0]$-scheme of finite type. Suppose the generic fiber $V^{(0)}$ of $V$ is smooth and geometrically irreducible; that is to say $V^{(0)} := V^{(0)} \times_{\text{Spec}(\mathbb{Q})} \text{Spec}(\mathbb{Q})$ is irreducible. Then there is $q_1 \in \mathbb{Z} \setminus \{0\}$ such that

(a) For any $p \nmid q_0 q_1$, the fiber $V^{(p)}$ of $V$ over $p\mathbb{Z}[1/q_0]$ is a smooth $k(p)$-scheme,

(b) For any $p \nmid q_0 q_1$, $V^{(p)}$ is geometrically irreducible; that is to say $V^{(p)} := V^{(p)} \times_{\text{Spec}(k(p))} \text{Spec}(k(p))$ is irreducible,

(c) For any $p \nmid q_0 q_1$, $\dim V^{(0)} = \dim V^{(p)} = \dim V^{(p)}$.

(2) Let $V$ and $W$ be two affine, of finite type, $\mathbb{Z}[1/q_0]$-schemes. Let $f : V \rightarrow W$ be a $\mathbb{Z}[1/q_0]$-morphism. Let $V^{(p)}$ and $W^{(p)}$ be the geometric fibers over $p\mathbb{Z}[1/q_0] \in \text{Spec}(\mathbb{Z}[1/q_0])$ of $V$ and $W$, respectively. Suppose $V^{(0)}$ and $W^{(0)}$ are irreducible and $f^{(0)} : V^{(0)} \rightarrow W^{(0)}$ is dominant. Then there is $q_1 \in \mathbb{Z} \setminus \{0\}$ such that

$$f^{(p)} : V^{(p)} \rightarrow W^{(p)}$$

is dominant for any $p \nmid q_0 q_1$.

Proof. As it is mentioned earlier, there are special cases of [EGA66], Proposition 9.6.1, Theorem 9.7.7, Theorem 12.2.4.

Proof of Proposition 18. We start by getting a dominant map on the geometric fiber of the generic point.

Claim 1. For a given $(\gamma_i)_{i=1}^c \subseteq \Gamma_1$, if $f_{(\gamma_i)_{i=1}^c}^{(0)}$ is not dominant, then there are $\gamma_{c+1}, \ldots, \gamma_c \in \Gamma_1$ such that

$$\dim \text{Im}(f^{(0)}_{(\gamma_i)_{i=1}^c}) > \dim \text{Im}(f^{(0)}_{(\gamma_i)_{i=1}^c}).$$

Proof of Claim 1. Since $G_2$ is irreducible, $V := \text{Im}(f^{(0)}_{(\gamma_i)_{i=1}^c})$ is irreducible. Therefore $\overline{V} \cdot \overline{V}^{-1} \cdot \overline{V}$ is irreducible, too. If $\overline{V} \cdot \overline{V}^{-1} \cdot \overline{V} \neq V$, then $\dim \overline{V} \cdot \overline{V}^{-1} \cdot \overline{V} > \dim V$ and we get the desired claim as

$$\overline{V} \cdot \overline{V}^{-1} \cdot \overline{V} = \text{Im}(f^{(0)}_{(\gamma_i)_{i=1}^c}).$$
where \( c_2 := 3c_1 \) and \( \gamma_{k_1+i} := \gamma_i \) for \( 0 \leq k \leq 2 \) and \( 1 \leq i \leq c_1 \).

So we can and will assume that \( V \cdot V^{-1} = V \), which implies that \( V \) is a subgroup of \( G_1 \). Since the normal closure of \( G_2 \) in \( G_1 \) is \( G_1 \), \( V \) cannot be a normal subgroup of \( G_1 \). Since \( G_1 \) is Zariski-dense in \( G_1 \), there is \( \gamma \in \Gamma_1 \) such that \( \gamma V \gamma^{-1} \cdot V \neq V \). Again by irreducibility of \( V \) and \( \gamma V \gamma^{-1} \cdot V \) we get the desired claim.

**Claim 2.** There are \( \gamma_1, \ldots, \gamma_{c_1} \in \Gamma_1 \) such that \( f^{(0)}_{(\gamma_i)} \) is a dominant map.

**Proof of Claim 2.** Applying Claim 1 for at most \( \text{dim} \ G_1 \) many times, we get \( \gamma_1, \ldots, \gamma_{c_1} \in \Gamma_1 \) such that \( f^{(0)}_{(\gamma_i)} \) is dominant.

**Claim 3.** There are \( q_1 \in \mathbb{Z} \setminus \{0\} \) and \( \gamma_1, \ldots, \gamma_{c_1} \in \Gamma_1 \) such that for any \( p \nmid q_0 q_1 \) (including \( p = 0 \)) we have that \( f^{(p)}_{(\gamma_i)} \) is dominant and \( G^{(p)}_i \) is irreducible.

**Proof of Claim 3.** Let \( \gamma_i \)'s be as in Claim 2. Now we can get the rest using Theorem 19 parts 1(b) and (2).

**Claim 4.** There are \( \gamma_1, \ldots, \gamma_c \in \Gamma_1 \) and \( q_1 \in \mathbb{Z} \setminus \{0\} \) such that for any \( p \nmid q_0 q_1 \) (including \( p = 0 \)) we have that \( f^{(p)}_{(\gamma_i)} \) is surjective.

**Proof of Claim 4.** Since \( k(p) \) is algebraically closed, it is enough to find \( \gamma_i \) such that \( f^{(p)}_{(\gamma_i)} \) induces a surjection from \( G_2^{(p)}(k(p)) \times \cdots \times G_2^{(p)}(k(p)) \) to \( G_1^{(p)}(k(p)) \) (see [DGS80 Corollary 11, Chapter I.3.6]). Let \( \gamma_1, \ldots, \gamma_{c_1} \) be as in Claim 3. So by Chevalley’s theorem [Hum95 Chapter 4.4], there is an open and dense subset \( U_p \subseteq G_1^{(p)}(k(p)) \) of \( \text{Im}(f^{(p)}_{(\gamma_i)}) \). Hence by [Hum95 Chapter 7.4] we have \( G_1^{(p)}(k(p)) = U_p \cdot U_p \). Therefore \( f^{(p)}_{(\gamma_i)} \) is surjective where \( c := 2c_1 \) and \( \gamma_{c_1+i} := \gamma_i \) for any \( 1 \leq i \leq c_1 \).

\[ \square \]

5. LARGE CONGRUENCE SUBGROUP: FOR AN ARBITRARY PLACE.

In this section, we will prove a bounded generation statement in \( p \)-adic setting for arbitrary \( p \) (see Proposition 20).

Besides some well-known terms from algebraic geometry that has been already introduced in Section 4, we need to recall some terms from algebraic group theory. An algebraic group \( G \) is called perfect if there is no non-trivial homomorphism to an abelian algebraic group; that is to say the derived subgroup \( [G, G] \) is equal to \( G \). A perfect group \( G \) is isomorphic to \( \mathbb{L} \times \mathbb{U} \) where \( \mathbb{L} \) is a semisimple group and \( \mathbb{U} \) is a unipotent group.

We say a perfect group \( G \) is simply connected if its semisimple part is simply connected (see [SGV12 Section 3] for relevant basic properties of perfect algebraic groups).

To avoid repeating a series of assumptions for the statements of this section, we record them here for the future reference.

- (A1) \( \Gamma_2 \subseteq \Gamma_1 \) are finitely generated subgroups of \( \text{GL}_{n_0}(\mathbb{Z}[1/q_0]) \).
- (A2) \( G_i \) is the Zariski-closure of \( \Gamma_i \) in \( \text{GL}_{n_0}(\mathbb{Z}[1/q_0]) \).
- (A3) For any \( p \in \text{Spec}(\mathbb{Z}[1/q_0]) \), \( G_i^{(p)} \) is the fiber over \( p \) of \( G_i \).
- (A4) For any \( p \in \text{Spec}(\mathbb{Z}[1/q_0]) \), \( G_i^{(p)} \) is the geometric fiber of \( G_i^{(p)} \).
- (A5) \( G_i := G_i^{(0)} \) are perfect and \( G_1 \) is simply-connected.
- (A6) For \( \gamma_i \in \Gamma_1 \) and \( p \in \text{Spec}(\mathbb{Z}[1/q_0]) \), \( f_{p, (\gamma_i)} : G_2^{(p)} \times \cdots \times G_2^{(p)} \rightarrow G_1^{(p)} \) and \( f^{(p)}_{(\gamma_i)} : G_2^{(p)} \times \cdots \times G_2^{(p)} \rightarrow G_1^{(p)} \) be the morphisms that are given by \((h_1) \mapsto \pi_p(\gamma_1) h_1 \pi_p(\gamma_1)^{-1} \cdots \pi_p(\gamma_c) h_1 \pi_p(\gamma_c)^{-1} \).
- (A7) For some \( \gamma_1, \ldots, \gamma_{n_0} \in \Gamma_1 \), \( f^{(p)}_{(\gamma_i)} \) is surjective if \( p \) is large enough.
Proposition 20. In the setting of (A1)-(A7), there are a positive integer $N$ and $\gamma_{m_0+1}, \ldots, \gamma_{m_2} \in \Gamma$, such that for any prime $p \mid q_0$ we have
\[ \Gamma_{1,p}[p^N] \subseteq \gamma_1 \Gamma_{2,p} \gamma_1^{-1} \cdots \gamma_{m_1} \Gamma_{2,p} \gamma_{m_1}^{-1}, \]
where $\Gamma_{i,p}$ is the closure of $\Gamma_i$ in $\text{GL}_{n_0}(\mathbb{Z}_p)$ and $\Gamma_{1,p}[p^N] := \Gamma_{1,p} \cap \ker(G_1(\mathbb{Z}_p) \rightarrow G_1(\mathbb{Z}/p^N\mathbb{Z}))$.

Lemma 21. In the setting of (A1)-(A7), there are $\dim \mathbb{G}_1$-many elements $\gamma'_i$ in $\Gamma_1$ such that
\[ \text{Lie}(\mathbb{G}_1)(\mathbb{Q}) = \sum_{i=1}^{\dim \mathbb{G}_1} \text{Ad}(\gamma'_i)(\text{Lie}(\mathbb{G}_2)(\mathbb{Q})). \]
And so $\sum_{i=1}^{\dim \mathbb{G}_1} \text{Ad}(\gamma'_i)(\text{Lie}(\mathbb{G}_2)(\prod_{q_0/p} \mathbb{Z}_p))$ is an open subgroup of $\text{Lie}(\mathbb{G}_1)(\prod_{q_0/p} \mathbb{Z}_p)$.

Proof. The first part is a corollary of Proposition 13 and the assumption that $\Gamma_1$ is Zariski-dense in $\mathbb{G}_1$. The second part is consequence of Chinese remainder theorem and the first part. \hfill \Box

Lemma 22. In the setting of (A1), (A2), and (A5), for any $p \mid q_0$, let $\Gamma_{1,p}$ be the closure of $\Gamma_1$ in $\text{GL}_{n_0}(\mathbb{Z}_p)$, and $\tilde{\Gamma}_1$ be the closure of $\Gamma_1$ in $\prod_{p \not| q_0} \text{GL}_{n_0}(\mathbb{Z}_p)$. Then the following holds:

1. (Strong Approximation: simply-connected) $\tilde{\Gamma}_1$ is an open subgroup of $\prod_{p \not| q_0} \mathbb{G}_1(\mathbb{Z}_p)$; in particular, there is a positive integer $c_0$ such that for any prime $p \mid q_0$ and any integer $c \geq c_0$, we have $\Gamma_{1,p}[p^c] = \mathbb{G}_1(\mathbb{Z}_p)[p^c]$ where as before these are principal congruence subgroups.

2. (Strong Approximation: general) There is a positive integer $c_1$ such that for any integer $c \geq c_1$ we have $\mathbb{G}_2(\mathbb{Z}_p)[p^c] = \Gamma_{2,p}[p^c]$. Moreover, if $p$ is large enough, then $\mathbb{G}_2(\mathbb{Z}_p)[p] = \Gamma_{2,p}[p]$.

3. (Local charts) There is a positive integer $c_2$ such that for any integer $c \geq c_2$, the exponential and the logarithmic maps induce bijections between $\mathbb{G}_1(\mathbb{Z}_p)[p^c]$ and $\text{Lie}(\mathbb{G}_1)(\mathbb{Z}_p) \cap p^c \mathfrak{gl}_{n_0}(\mathbb{Z}_p)$ for $i = 1$ or 2.

Proof. The first part is Nori’s strong approximation [Nor87 Theorem 5.4]. The third part is a direct consequence of Proposition 13 and Lemma 12.

Let $\tilde{\mathbb{G}}_2$ be the simply-connected cover of $\mathbb{G}_2$, and $\iota : \tilde{\mathbb{G}}_2 \rightarrow \mathbb{G}_2$ be the $\mathbb{Q}$-central isogeny. Let $\Lambda := \iota^{-1}(\mathbb{G}_2) \cap \tilde{\mathbb{G}}_2(\mathbb{Q})$. Then, as in [SGS13 Lemma 24], we have that $\Lambda$ is a finitely generated, Zariski-dense subgroup of $\tilde{\mathbb{G}}_2$. Hence, by Nori’s strong approximation, we have that the closure $\Lambda_p$ of $\Lambda$ in $\iota^{-1}(\mathbb{G}_2(\mathbb{Z}_p))$ is open; and moreover, for large enough $p$, $\Lambda_p = \iota^{-1}(\mathbb{G}_2(\mathbb{Z}_p))$. Since $|\mathbb{G}_2(\mathbb{Q}_p)/\iota(\tilde{\mathbb{G}}_2(\mathbb{Q}_p))| \leq p^{c_3}$ where $c_3$ just depends on the dimension of $\mathbb{G}_2$, we have that $\iota(\Lambda_p) \supseteq \mathbb{G}_2(\mathbb{Z}_p)[p^{c_1}]$ where $c_1$ just depends on $\mathbb{G}_2 \subseteq \text{GL}_{n_0}(\mathbb{Z}/[1/q_0])$. Therefore $\Gamma_{2,p} \supseteq \mathbb{G}_2(\mathbb{Z}_p)[p^{c_1}]$.

To show the last claim of part (2), we notice that $\mu := \ker(\iota)$ is a central subgroup of $\tilde{\mathbb{G}}_2$ which is a $\mathbb{Q}_p$-group. Using Galois cohomology, we get the following exact sequence:
\[ 1 \rightarrow \mu(\mathbb{Q}_p) \rightarrow \tilde{\mathbb{G}}_2(\mathbb{Q}_p) \rightarrow \mathbb{G}(\mathbb{Q}_p) \rightarrow H^1(\mathbb{Q}_p, \mu). \]

Hence $\mathbb{G}(\mathbb{Q}_p)/\iota(\tilde{\mathbb{G}}_2(\mathbb{Q}_p))$ can be embedded into $H^1(\mathbb{Q}_p, \mu)$, which is an abelian $m$-torsion group for some $m$ that can be bounded by the dimension of $\mathbb{G}_2$. In particular, for large enough $p$, $\mathbb{G}(\mathbb{Q}_p)/\iota(\tilde{\mathbb{G}}_2(\mathbb{Q}_p))$ does not have any $p$-element. As $\mathbb{G}_2(\mathbb{Z}_p)[p]/\iota(\tilde{\mathbb{G}}_2(\mathbb{Z}_p))[p]$ can be embedded into $\mathbb{G}(\mathbb{Q}_p)/\iota(\tilde{\mathbb{G}}_2(\mathbb{Q}_p))$, we get that $\mathbb{G}_2(\mathbb{Z}_p)[p]/\iota(\tilde{\mathbb{G}}_2(\mathbb{Z}_p))[p]$ has no $p$-element. On the other hand, $\mathbb{G}_2(\mathbb{Z}_p)[p]$ is a pro-$p$ group, and so all of its finite quotients are $p$-groups. Therefore we get
\[ (8) \quad \mathbb{G}_2(\mathbb{Z}_p)[p] = \iota^{-1}(\mathbb{G}_2(\mathbb{Z}_p))[p]. \]

As we said earlier, $\Lambda_p = \iota^{-1}(\mathbb{G}_2(\mathbb{Z}_p))$, for large enough $p$. Thus by (8) we have
\[ \mathbb{G}_2(\mathbb{Z}_p)[p] = \Gamma_{2,p}[p]. \]

\hfill \Box
Proof of Proposition 22. Let \( m_1 := m_0 + \dim G_1 \) and \( \gamma_{n_0+i} := \gamma_i' \) where \( \gamma_i' \) are the elements given by Lemma 21. Let \( c \) be an integer larger than \( \max\{c_0, c_1, c_2\} \) where \( c_i \)'s are given in Lemma 22. Let \( \mathfrak{g}_{1,p} := \text{Lie}(G_1)(\mathbb{Z}_p) \). Let \( F : \mathfrak{g}_{2,p} \times \cdots \times \mathfrak{g}_{2,p} \to \mathfrak{g}_{1,p} \) be the composite of the following p-adic analytic functions:

\[
\exp_{\mathfrak{g}_{2,p}} \times \cdots \times \exp_{\mathfrak{g}_{2,p}}, \quad \text{Conj} : G_2(\mathbb{Z}_p)[[p]] \to G_1(\mathbb{Z}_p)[[p]], \quad \text{Prod} : G_1(\mathbb{Z}_p)[[p]] \to G_1(\mathbb{Z}_p)[[p]], \quad \log : G_1(\mathbb{Z}_p)[[p]] \to G_1(\mathbb{Z}_p)[[p]].
\]

So by the chain rule we have \( dF(0) : \mathfrak{g}_{2,p} \times \cdots \times \mathfrak{g}_{2,p} \to \mathfrak{g}_{1,p} \) is the composite of the following maps:

\[
g_{2,p} \times \cdots \times \mathfrak{g}_{2,p} \xrightarrow{d(\exp(0))} \mathfrak{g}_{2,p} \times \cdots \times \mathfrak{g}_{2,p} \xrightarrow{d(\text{Conj}(I))} \mathfrak{g}_{1,p} \times \cdots \times \mathfrak{g}_{1,p} \xrightarrow{d(\text{Prod}(I))} \mathfrak{g}_{1,p} \xrightarrow{d(\log(I))} \mathfrak{g}_{1,p}.
\]

Hence by Lemma 21 and Lemma 22 we have that, for large enough \( p \), \( dF(0) \) is onto, and for any \( p \nmid q_0 \) the image of \( dF(0) \) is open in \( \mathfrak{g}_{1,p} \). Hence for some non-zero integer \( a_0 \) we have \( N(dF(0)) \geq |a_0|_p \) where, for a \( d \)-by-\( m \) matrix \( X = [v_1 \cdots v_m] \), \( N(X) := \max_{1 \leq i_1 \leq \cdots \leq i_d \leq m} |\det[v_{i_1} \cdots v_{i_d}]| \).

By definition of \( F \), we have \( F(x_1, \ldots, x_m) = \log(\exp(\text{Ad}(\gamma_1)(x_1)) \cdots \exp(\text{Ad}(\gamma_m)(x_m))) \). Hence by Baker-Campbell-Hausdorff formula \([Jac79, \text{Chapter V.5}]\) we have

\[
F(x_1, \ldots, x_m) = \sum_i \text{Ad}(\gamma_i)(x_i) + \sum_i c_i L_i(\text{Ad}(\gamma_1)(x_1), \ldots, \text{Ad}(\gamma_m)(x_m)),
\]

where \( L_i(y_1, \ldots, y_m) \) is a Lie monomial with multi-index \( i \) and \( c_i \in \mathbb{Q} \); moreover using the explicit Baker-Campbell-Hausdorff formula we see that

\[
|c_i|_p \leq p^{m_2 ||i||},
\]

where \( ||i|| = \sum_i i_j \) and \( m_2 \) is a constant which depends on \( m_1 \) (independent of \( p \)). Let \( F_1 : \mathfrak{g}_1 \times \cdots \times \mathfrak{g}_1 \to \mathfrak{g}_1 \) be \( F(\mathfrak{g}^{m_2}x) \). Choosing \( \mathbb{Z}_p \)-basis for \( \mathfrak{g}_1 \) and \( \mathfrak{g}_2 \), we identify them by \( \mathbb{Z}_p^{d_1} \) and \( \mathbb{Z}_p^{d_2} \), where \( d_i = \dim G_i \). Now using the mentioned Baker-Campbell-Hausdorff formula, writing the Taylor expansion of \( F' \) at \( 0 \) with respect to the chosen coordinates we get \( F_1(x) = \sum_i (c_{i,1} x^1, \ldots, c_{i,d_1} x^1) \) and \( |c_{i,j}|_p \leq 1 \) for any \( i \) and \( j \). We also have \( N(dF_1(0)) \geq |a_0 p^{m_2 d_1}|. \) Therefore by [SG, Lemma 45] there is \( l_0 \) which is independent of \( p \) such that

\[
F_1(0) + p^{l_0} \mathbb{Z}_p^{d_1} \subseteq F_1(\mathbb{Z}_p^{d_2 l_1}),
\]

which implies that

\[
p^l \mathfrak{g}_{1,p} \subseteq \log(\gamma_1 \mathcal{G}_2(\mathbb{Z}_p)[[p]][\gamma_1^{-1} \cdots \gamma_m \mathcal{G}_2(\mathbb{Z}_p)[[p]][\gamma_1^{-1}]].
\]

By Lemma 22 and Equation (10) we have

\[
\Gamma_{1,p}[p^l] \subseteq \gamma_1 \mathcal{G}_2(\mathbb{Z}_p)[[p]][\gamma_1^{-1} \cdots \gamma_m \mathcal{G}_2(\mathbb{Z}_p)[[p]][\gamma_1^{-1}] \subseteq \gamma_1 \Gamma_{2,p}[p^l] \gamma_1^{-1} \cdots \gamma_m \Gamma_{2,p} \gamma_1^{-1} \cdots \gamma_m \Gamma_{2,p} \gamma_1^{-1}.
\]

\[\square\]

6. Bounded generation: Large primes.

In this section, we will prove a bounded generation statement in \( p \)-adic setting for large \( p \) (see Proposition 23).

We will be using all the mentioned terms from algebraic geometry and algebraic group theory; in particular, we will operate under the assumptions (A1)-(A7) which are mentioned in Section 3. In addition, we assume \( \gamma_i \)'s satisfy Proposition 20 and Lemma 21 which means
(A8) We have $\text{Lie}(G_1) (\mathbb{Q}) = \sum_{i=1}^{c} \text{Ad} (\gamma_i) (\text{Lie}(G_2)(\mathbb{Q}))$.

(A9) For a positive integer $N$ and any prime $p \mid q_0$, we have $\Gamma_1.p [p^N] \subseteq \gamma_1 \Gamma_2.p \gamma_1^{-1} \cdots \gamma_c \Gamma_2.p \gamma_c^{-1}$.

It is worth mentioning that, as we have seen in the proof of Proposition 20, (A1)-(A8) implies (A9).

**Proposition 23.** In the setting of (A1)-(A9), there is a positive integer $C$ such that for any large enough $p$ we have

\[ \prod_{i \in \mathbb{C}} (\gamma_1 \Gamma_2.p \gamma_1^{-1} \cdots \gamma_c \Gamma_2.p \gamma_c^{-1}) = \Gamma_1.p, \]

where $\Gamma_1.p$ is the closure of $\Gamma_1$ in $\text{GL}_{n_0}(\mathbb{Z}_p)$ and $\prod_{i \in \mathbb{C}} X = \{x_1 \cdots x_C \mid x_i \in X\}$.

It is enough to prove that the equality holds modulo all the powers of $p$. We start with proving that (11) holds modulo $p$ for $C = 3 \dim G_1$.

**Lemma 24.** In the setting of (A1)-(A8), for large enough $p$, we have

\[ \prod_{i \in \mathbb{C}} \pi_p (\gamma_1 \Gamma_2.p \gamma_1^{-1} \cdots \gamma_c \Gamma_2.p \gamma_c^{-1}) = \pi_p (\Gamma_1.p), \]

where $\Gamma_1.p$ is the closure of $\Gamma_1$ in $\text{GL}_{n_0}(\mathbb{Z}_p)$ and $\prod_{i \in \mathbb{C}} X = \{x_1 \cdots x_C \mid x_i \in X\}$.

**Proof.** By (A7), for large enough $p$, $f_{p, (\gamma_i)} \mid_{\gamma_i \leq 1}$ is dominant. By (A5) and Proposition 18 for large enough $p$, $G_2^{(p)}$ is irreducible. So by [PR09, Proposition 2.4] fibers of $f_{p, (\gamma_i)} \mid_{\gamma_i \leq 1}$ have dimension at most $d_1(c d_2 - 1)$, where $d_1 = \dim G_1$, $d_2 = \dim G_2$, $\gamma_{r+c} = \gamma_i$ for $0 \leq r < d_1$ and $1 \leq i \leq c$.

Clearly $f_p := f_{p, (\gamma_i)} \mid_{\gamma_i \leq 1}$ is dominant. Hence, by [PR09, Proposition 2.5], there is a positive constant $C_1$ such that for any prime $p$ and any $y \in G_1^{(p)} (k(p))$ we have

\[ |f_p^{-1}(y)(k(p))| \leq C_1 p^{\dim(f_p^{-1}(y))}. \]

Since any fiber of $f$ has dimension at most $d_1(c d_2 - 1)$, by (12) we get

\[ |\pi_p (\Gamma_2.p)|^{c d_1} \leq \sum_{y \in f_p (\pi_p (\Gamma_2.p) \times \cdots \times \pi_p (\Gamma_2.p))} (c d_1 \text{ times}) \leq C_1 p^{d_1(c d_2 - 1)} |f_p(\Gamma_2.p)|. \]

Since $G_2$ is perfect, by Nori’s strong approximation (see [SGV12, Section 3]) and the well-known order of finite quasi-simple groups of Lie type we have that there is a positive constant $C_2$ such that $|\pi_p (\Gamma_2.p)| \geq C_2^{-1} p^{d_2}$ for any large enough prime $p$. So together with (13) we get

\[ C_2^{-c d_1} C_1^{-1} p^{d_2} \leq |f_p (\pi_p (\Gamma_2.p) \times \cdots \times \pi_p (\Gamma_2.p))| = |\prod_{d_1} \pi_p (\gamma_1 \Gamma_2.p \gamma_1^{-1} \cdots \gamma_c \Gamma_2.p \gamma_c^{-1})|. \]

Since $G_1$ is perfect and $G_2$ is simply-connected, again by Nori’s strong approximation (see [SGV12, Section 3]) and the well-known order of finite quasi-simple groups of Lie type we have that there is a constant $C_3$ such that $|\pi_p (\Gamma_1.p)| \geq C_3^{-1} p^{d_2}$ for any large enough prime $p$. Hence by (14) we get

\[ C_4^{-1} |\pi_p (\Gamma_1.p)| \leq |\prod_{d_1} \pi_p (\gamma_1 \Gamma_2.p \gamma_1^{-1} \cdots \gamma_c \Gamma_2.p \gamma_c^{-1})|, \]

for some positive constant $C_4$. By Nori’s Strong Approximation (see [Nor87, Theorem 5.4] or [SGV12, Theorem A]) we have that $\pi_p (\Gamma_1.p) = L_p (k(p)) \times U_p (k(p))$ for large enough $p$, where $L_p$ is a semisimple, simply connected, Zariski connected $k(p)$-group with dimension at most $d_1$ and $U_p$ is a unipotent Zariski connected $k(p)$-group with dimension at most $d_1$. Let $\rho$ be an irreducible (complex) representation of $\pi_p (\Gamma_1.p)$. By [SGV12, Corollary 4], the restriction of $\rho$ to $L_p (k(p))$ is not trivial. Hence by [LS74]

\[ \dim \rho \geq |L_p (k(p))|^{1/C_5} \geq |\pi_p (\Gamma_1.p)|^{1/C_6}. \]
for some positive constants $C_5$ and $C_6$ (depending only on $d_1$). By [15], [16], and a theorem of Gowers [Gow08] (see [NP11] Corollary 1), it follows that
\[ \prod_{i=1}^{3d_i} \pi_p(\gamma_1 \Gamma_{2,p} \gamma_1^{-1} \cdots \gamma_c \Gamma_{2,p} \gamma_c^{-1} - 1) = \pi_p(\Gamma_{1,p}) \]
for large enough $p$. \hfill \Box

Next we generate the $k$-th grade $\Gamma_{1,p}[p^k]/\Gamma_{1,p}[p^{k+1}]$ using $\gamma_i$’s. Let us recall the connection between the $k$-th grade and the Lie algebra of the group scheme $G_i$.

**Lemma 25.** In the setting of (A1)-(A5), for large enough prime $p$ and any positive integer $k$,
\[ \Psi_k : G_i(Z_p)[p^k] / G_i(Z_p)[p^{k+1}] \rightarrow g_{i,p}/p g_{i,p}, \quad \Psi_k((I + p^k x)G_i(Z_p)[p^{k+1}]) := \pi_p(x), \]
where $g_{i,p} := \text{Lie}(G_i)(Z_p)$, is an isomorphism; moreover for any $g \in G_i(Z_p)$ and $g' \in G_i(Z_p)[p^k]$ we have
\[ \Psi_k(gg'g^{-1}G_i(Z_p)[p^{k+1}]) = \text{Ad}(\pi_p(g)) (\Psi_k(g'G_i(Z_p)[p^{k+1}])). \]

**Proof.** By the discussion in [SC10 Section 2.9], we get that
\[ \tilde{\Psi}_k : G_i(Z_p)[p^k] / G_i(Z_p)[p^{k+1}] \rightarrow \text{Lie}(G_i(Z_p/pZ_p)), \quad \tilde{\Psi}_k((I + p^k x)G_i(Z_p)[p^{k+1}]) := \pi_p(x), \]
is a well-defined injective group homomorphism.

For large $p$, $G_i \times_{Z_p} Z_p$ is a smooth $Z_p$-group scheme, and so $\text{Lie}(G_i)(Z_p/pZ_p)$ is naturally isomorphic to $g_{i,p}/p g_{i,p}$. So we get that $\Psi_k$ is a well-defined injective group homomorphism.

For any $x \in g_{i,p}$ and $p > 2$, by Proposition 11, $\exp(p^k x) \in G_i(Z_p)[p^k]$. By the definition of the exponential function we can see that
\[ \Psi_k(\exp(p^k x)G_i(Z_p)[p^{k+1}]) = \pi_p(x), \]
which implies that $\Psi_k$ is a group isomorphism. The other part of Lemma is easy to check. \hfill \Box

**Lemma 26.** In the setting of (A1)-(A8), for large enough $p$ and any positive integer $k$, we have
\[ (\gamma_1 \Gamma_{2,p}[p^k] \gamma_1^{-1} \cdots \gamma_c \Gamma_{2,p}[p^k] \gamma_c^{-1})\Gamma_{1,p}[p^{k+1}] = \Gamma_{1,p}[p^k]. \]

**Proof.** By (A8), for large enough $p$, we have $g_{1,p} = \sum_{i=1}^c \text{Ad}(\gamma_i)(g_{2,p})$ where $g_{i,p} := \text{Lie}(G_i)(Z_p)$. Hence, by Lemma 25 we have
\[ (\gamma_1 \Gamma_{2,p}[p^k] \gamma_1^{-1} \cdots \gamma_c \Gamma_{2,p}[p^k] \gamma_c^{-1})G_i(Z_p)[p^{k+1}] = G_i(Z_p)[p^k]. \]
On the other hand, by the second part of Lemma 22 we have that, for large enough $p$, $\Gamma_{i,p}[p^k] = G_i(Z_p)[p^k]$, which together with (17) implies the claim. \hfill \Box

**Proof of Proposition 23.** Using Lemma 25 for $k = 1, \ldots, N - 1$, we get that
\[ \prod_{i=1}^{N-1} (\gamma_1 \Gamma_{2,p}[p^k] \gamma_1^{-1} \cdots \gamma_c \Gamma_{2,p}[p^k] \gamma_c^{-1})\Gamma_{1,p}[p^N] = \Gamma_{1,p}[p]. \]

Lemma 24 and Equation (18), we get
\[ \prod_{i=1}^{3 \dim G_i + N} (\gamma_1 \Gamma_{2,p}[p^k] \gamma_1^{-1} \cdots \gamma_c \Gamma_{2,p}[p^k] \gamma_c^{-1})\Gamma_{1,p}[p^N] = \Gamma_{1,p}. \]
Condition (A9) together with Equation (19) imply that
\[ \prod_{i=1}^{3 \dim G_i + N} (\gamma_1 \Gamma_{2,p}[p^k] \gamma_1^{-1} \cdots \gamma_c \Gamma_{2,p}[p^k] \gamma_c^{-1}) = \Gamma_{1,p}. \]
7. Bounded generation: Adelic version.

In this section, we will prove an adelic bounded generation statement (see Theorem 27).

We will be working under the assumptions (A1)-(A5) mentioned at the beginning of Section 5.

**Theorem 27.** In the setting of (A1)-(A5), let \( \widehat{\Gamma}_i \) be the closure of \( \Gamma_i \) in \( \prod_{p \neq q_0} \text{GL}_{m_i}(\mathbb{Z}_p) \). Then there are \( \gamma_1, \ldots, \gamma_m \in \Gamma_1 \) such that

\[
\gamma_1 \widehat{\Gamma}_2 \gamma_1^{-1} \cdots \gamma_m \widehat{\Gamma}_2 \gamma_m^{-1} \subseteq \widehat{\Gamma}_1
\]

is an open subgroup of \( \widehat{\Gamma}_1 \).

**Proof.** As in the proof of Lemma 22 and [SGS13 Lemma 24], let \( \tilde{G}_2 \) be the simply-connected cover of \( G_2 \), and \( \iota : \tilde{G}_2 \to G_2 \) be the \( \mathbb{Q} \)-central isogeny. Let \( \Lambda := \iota^{-1}(G_2(\mathbb{Q})) \cap G_2(\mathbb{Q}) \) and \( \Gamma_2' := \iota(\Lambda) \). Then, as in [SGS13 Lemma 24], \( \Lambda \) is a finitely generated Zariski-dense subgroup of \( G_2 \); and so \( \Gamma_2' \) is a finitely generated Zariski-dense subgroup of \( G_2 \). We also notice that \( \Gamma_2' \subseteq \Gamma_2 \subseteq \Gamma_1 \).

By Nori’s strong approximation, the closure \( \widehat{\Lambda} \) of \( \Lambda \) in \( \prod_{p \neq q_0} \iota^{-1}(G_2(\mathbb{Z}_p)) \) is open. So passing to a finite-index subgroup of \( \Lambda \), if needed, we can and will assume that \( \widehat{\Lambda} = \prod_{p \neq q_0} \Lambda_p \) where \( \Lambda_p \) is the closure of \( \Lambda \) in \( \iota^{-1}(G_2(\mathbb{Z}_p)) \). Therefore the closure of \( \Gamma_2' \) in \( \prod_{p \neq q_0} G_2(\mathbb{Z}_p) \) is \( \iota(\widehat{\Lambda}) = \prod_{p \neq q_0} \Gamma_2',p \), where \( \Gamma_2',p \) is the closure of \( \Gamma_2' \) in \( G_2(\mathbb{Z}_p) \). We notice that \( \Gamma_2' \) is a finitely generated subgroup of \( G_2 \), and it is Zariski-dense in \( G_2 \) since \( G_2 \) is Zariski-connected.

By Proposition 18 applied to \( \Gamma_2' \subseteq \Gamma_1 \), we get that \( \Gamma_2' \subseteq \Gamma_1 \) satisfy (A1)-(A7) for some \( \gamma_i \in \Gamma_1 \). Hence by Proposition 20 for some \( \gamma_i \in \Gamma_1 \) and positive integer \( N \) we have

\[
(20) \quad \Gamma_{1,p}[p^N] \subseteq \gamma_1 \Gamma_{2,p}^{-1} \cdots \gamma_m \Gamma_{2,p}^{-1}.
\]

By Lemma 21 applied to \( \Gamma_2' \subseteq \Gamma_1 \), there are \( \gamma_i \) in \( \Gamma_1 \) such that (A1)-(A9) hold for \( \Gamma_2' \subseteq \Gamma_1 \). Therefore there are \( \gamma_i \) in \( \Gamma_1 \) such that, for any large prime \( p \), we have

\[
(21) \quad \gamma_1 \Gamma_{2,p}^{-1} \cdots \gamma_m \Gamma_{2,p}^{-1} = \Gamma_{1,p}.
\]

By Equations (20) and (21), we have there are \( \gamma_i \) in \( \Gamma_1 \) such that

\[
(22) \quad \gamma_1 \Gamma_{2}^{-1} \cdots \gamma_m \Gamma_{2}^{-1} = \prod_{p \neq q_0} (\gamma_1 \Gamma_{2,p}^{-1} \cdots \gamma_m \Gamma_{2,p}^{-1}) \supseteq \prod_{p \neq q_0, p < q_0} \Gamma_{1,p}[p^N] \cdot \prod_{p \geq p_0} \Gamma_{1,p}.
\]

Therefore \( \gamma_1 \Gamma_{2}^{-1} \cdots \gamma_m \Gamma_{2}^{-1} \) contains an open normal subgroup (of finite index) of \( \widehat{\Gamma}_1 \). So by multiplying this set by itself finitely many times we get an open subgroup of \( \Gamma_1 \). This means there are \( \gamma_i \) in \( \Gamma_1 \) such that

\[
\gamma_1 \widehat{\Gamma}_2 \gamma_1^{-1} \cdots \gamma_m \widehat{\Gamma}_2 \gamma_m^{-1} \subseteq \widehat{\Gamma}_1.
\]

\[\Box\]

8. Proof of Theorem 1 inducing super-approximation.

Let us recall that \( G_1 \) is the Zariski-closure of \( \Gamma_1 \) in \( (\text{GL}_{m_0})_\mathbb{Q} \) and \( G_i^\circ \) is the Zariski-connected component of \( G_i \). Let \( \tilde{G}_1 \) be the simply-connected cover of \( G_i^\circ \) and \( \iota : \tilde{G}_1 \to G_i^\circ \) be its covering map. Let \( \Lambda_1 := \iota^{-1}(G_1(\mathbb{Q})) \cap G_1(\mathbb{Q}) \), \( \Lambda_2 := \iota^{-1}(G_2(\mathbb{Q})) \cap G_2(\mathbb{Q}) \), where \( G_2(\mathbb{Q}) \) is the Zariski-connected component of the Zariski-closure of \( \iota^{-1}(G_2(\mathbb{Q})) \cap G_1 \) in \( \tilde{G}_1 \). Finally let \( \Lambda_1 := \iota(\Lambda_1) \).

Now we will check that \( \Lambda_2 \subseteq \Lambda_1 \) satisfies conditions (A1) and (A5).

(A1) First we notice that \( G_i^\circ(\mathbb{Q}) \cap G_1 \) is a subgroup of finite-index of \( G_1 \), and so it is finitely generated. Since \( G_i^\circ(\mathbb{Q}) \cap G_1 \) is an abelian bounded torsion group and \( G_i^\circ(\mathbb{Q}) \cap G_1 \) is finitely generated, we have that \( \Lambda_1 \) is a subgroup of finite-index in \( G_1 \). Thus \( \Lambda_1 \) is finitely generated. Since \( \ker(\iota(\mathbb{Q})) \) is finite, we get that \( \Lambda_1 \) is finitely-generated.
By a similar argument we get \( t((\Gamma_2) \cap \Lambda_1) \) is a finite-index subgroup of \( \Gamma_2 \cap G_2^\circ(\mathbb{Q}) \). Therefore \( \Lambda_2 \) is a subgroup of finite-index in \( \Gamma_2 \). Hence it is finitely-generated, and so is \( \tilde{\Lambda}_2 \).

By [SG16, Lemma 11], there is a \( \mathbb{Q} \)-embedding \( \tilde{G}_1 \subseteq (GL_{n_1})_\mathbb{Q} \) such that \( \tilde{\Lambda}_1 \subseteq GL_{n_1}(\mathbb{Z}[1/q_0]) \). So we have \( \tilde{\Lambda}_2 \subseteq \Lambda_1 \) are two finitely generated subgroups of \( GL_{n_1}(\mathbb{Z}[1/q_0]) \).

\( \text{(A5)} \) Since \( \Gamma_1 \) is Zariski-dense in \( G_1 \), \( \Gamma_1 \cap G_1^\circ(\mathbb{Q}) \) is Zariski-dense in \( G_1^\circ \). Since \( \Lambda_1 \) is a finite-index subgroup of \( \Gamma_1 \cap G_1^\circ(\mathbb{Q}) \), it is Zariski-dense in \( G_1^\circ \). So the restriction of \( \iota \) to the Zariski-closure of \( \Lambda_1 \) is still surjective. And so \( \Lambda_1 \) is Zariski-dense in \( \tilde{G}_1 \).

Since \( \Lambda_2 \) is a subgroup of finite-index in \( \Gamma_2 \), its Zariski-closure is a subgroup of finite index of \( G_2 \). On the other hand, \( \tilde{\Lambda}_2 \) is Zariski-dense in a Zariski-connected group \( \tilde{G}_2 \). Hence the Zariski-closure of \( \Lambda_2 \) is \( \tilde{\Lambda}_2 \) is a Zariski-connected group. Therefore the Zariski-closure of \( \Lambda_2 \) is \( G_2^\circ \).

Since \( \Gamma_2 \cap \prod_{p|q_0} GL_{n_0}(\mathbb{Z}_p) \) has spectral gap, by [SG16, Proposition 8], \( G_2^\circ \) is perfect. By Lemma 14 we get that \( G_2^\circ \) is perfect. Therefore \( G_2 \)'s are perfect.

Hence by Theorem 27 we have that there are \( \tilde{\lambda}_1 \in \tilde{\Lambda}_1 \) such that \( \tilde{\lambda}_1 \tilde{\lambda}_2 \tilde{\lambda}_1^{-1} \cdots \tilde{\lambda}_m \tilde{\lambda}_2 \tilde{\lambda}_1^{-1} \) is an open subgroup of \( \Lambda_1 \) where \( \tilde{\Lambda}_i \) is the closure of \( \Lambda_i \) in \( \prod_{p|q_0} GL_{n_1}(\mathbb{Z}_p) \). Hence, applying \( \iota \) and letting \( \lambda_i := \iota(\tilde{\lambda}_i) \), we get that \( \lambda_1 \tilde{\lambda}_2 \lambda_1^{-1} \cdots \lambda_m \tilde{\lambda}_2 \lambda_1^{-1} \) is an open subgroup of \( \Lambda_1 \) where \( \Lambda_i \) is the closure of \( \Lambda_i \) in \( \prod_{p|q_0} GL_{n_0}(\mathbb{Z}_p) \).

**Claim 1.** Let \( \Pi_2 \) be a finite symmetric generating set of \( \Lambda_2 \). Let \( \Omega := \bigcup_{i=1}^m \lambda_i \Pi_2 \lambda_i^{-1} \) and \( \Lambda := (\Omega) \). Then \( \Lambda \cap \prod_{p|q_0} GL_{n_0}(\mathbb{Z}_p) \) has spectral gap.

**Proof of Claim 1.** The closure of \( \Lambda \) in \( \prod_{p|q_0} GL_{n_0}(\mathbb{Z}_p) \) is

\[
\tilde{\Lambda} := \lambda_1 \tilde{\lambda}_2 \lambda_1^{-1} \cdots \lambda_m \tilde{\lambda}_2 \lambda_1^{-1}.
\]

By [SG16, Remark 15, (5)], \( \lambda(\mathcal{P}_\Omega; \tilde{\Lambda}) = \sup_{\gcd(q,q_0)=1} \lambda(\mathcal{P}_{\pi_q(\Omega)}; \pi_q(\Lambda)) \).

On the other hand, by [SG16, Section 2.3] we have that \( \Gamma_2 \cap \prod_{p|q_0} GL_{n_0}(\mathbb{Z}_p) \) has spectral gap if and only if \( \Lambda_2 \cap \prod_{p|q_0} GL_{n_0}(\mathbb{Z}_p) \) has spectral gap. Hence

\[
1 > \lambda(\mathcal{P}_{\Pi_2}; \tilde{\Lambda}_2) = \sup_{\gcd(q,q_0)=1} \lambda(\mathcal{P}_{\pi_q(\Pi_2)}; \pi_q(\Lambda_2))
\]

Therefore, by Varjú’s lemma [BK14, Lemma A.4], Equations (23) and (24), we get that \( \lambda(\mathcal{P}_\Omega; \tilde{\Lambda}) < 1 \).

**Claim 2.** \( \Lambda_1 \cap \prod_{p|q_0} GL_{n_0}(\mathbb{Z}_p) \) has spectral gap.

**Proof of Claim 2.** Let \( \overline{\Lambda} := \Lambda_1 \cap \tilde{\Lambda} \) where \( \tilde{\Lambda} \) is given in the proof of Claim 1. Since \( \tilde{\Lambda} \) is a subgroup of finite index of \( \Lambda_1 \), \( \overline{\Lambda} \) is a subgroup of finite index of \( \Lambda_1 \). Therefore by [SG16, Section 2.3] we have that \( \Lambda_1 \cap \prod_{p|q_0} GL_{n_0}(\mathbb{Z}_p) \) has spectral gap if and only if \( \Lambda \cap \prod_{p|q_0} GL_{n_0}(\mathbb{Z}_p) \) has spectral gap. Since \( \Lambda \subseteq \overline{\Lambda} \), both of them are dense in \( \tilde{\Lambda} \), and \( \Lambda \cap \tilde{\Lambda} \) has spectral gap, we get that \( \overline{\Lambda} \cap \Lambda \) has spectral gap. Thus \( \Lambda_1 \cap \prod_{p|q_0} GL_{n_0}(\mathbb{Z}_p) \) has spectral gap.

Since \( \Lambda_1 \) is a subgroup of finite index of \( \Gamma_1 \) and \( \Lambda_1 \cap \prod_{p|q_0} GL_{n_0}(\mathbb{Z}_p) \) has spectral gap, another application of [SG16, Section 2.3] implies that \( \Gamma_1 \cap \prod_{p|q_0} GL_{n_0}(\mathbb{Z}_p) \) has spectral gap.

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