An itinerant half-metal spin-density-wave state on the hexagonal lattice

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We consider electrons on a honeycomb or triangular lattice doped to the saddle point of the bandstructure. We assume system parameters are such that spin density wave (SDW) order emerges below a temperature $T_N$ and investigate the nature of the SDW phase. We argue that at $T \leq T_N$ the system develops a uniaxial SDW phase whose ordering pattern breaks $O(3) \times Z_2$ symmetry and corresponds to an eight site unit cell with non-uniform spin moments on different sites. This state is a half-metal – it preserves full original Fermi surface, but has gapless charged excitations in one spin branch only. It allows for electrical control of spin currents and is desirable for nano-science.

Introduction: The electronic properties of single layer graphene have been the subject of considerable experimental and theoretical interest.1 Near half-filling, a description in terms of non-interacting Dirac electrons captures the essential physics, since interactions effects are suppressed by the low density of states (DOS). A sharply different behavior arises when graphene is strongly doped to 3/8 or 5/8 filling.2 At this filling, a divergent density of states and nested Fermi surface (FS) conspire to produce weak coupling instabilities to an extensive buffet of ordered states, including spin density waves (SDW) R, Fermi liquids, Pomeranchuk metals,3 and $d$ wave superconductors (SC) S, 8. 0. A similar situation arises on a triangular lattice at 3/4 filling 11 [11].

It has recently been established using renormalization group (RG) methods that the two most relevant instabilities at weak coupling are towards SDW and a $d$-wave SC. The SDW vertex is the largest at intermediate RG scales, but superconducting vertex eventually overshoots it both at perfect nesting and away from perfect nesting, making $d$-wave superconductivity the leading weak coupling instability. The SC state has a $d + id$ gap structure and breaks time-reversal symmetry.

In this paper we assume that superconductivity is destroyed by an applied magnetic field, or alternatively that system parameters are such that corrections to the RG flow become relevant before SC vertex overshoots SDW vertex. In both cases, the SDW instability becomes dominant, and an SDW order emerges. Previous work argued that the SDW state is non-coplanar and has a non-zero spin chirality.3 0. Such a state gaps out the entire Fermi surface (FS), i.e., is an insulator. The chiral SDW state has also been found in the strong coupling analysis for classical spins of fixed length.

We argue that the situation is more complex than originally thought, and the chiral SDW state is present only at the lowest temperatures. Over a wide intermediate range of temperatures, a different SDW state emerges in which SDW order develops simultaneously at three inequivalent wavevectors $Q_i$, but the three vector order parameters are all aligned along the same axis. This state has a eight site unit cell with non-uniform spin moments and zero net magnetization (Fig. 1b). Such a state cannot be accessed starting from a spin Hamiltonian for local moments with a fixed length, and can only be accessed starting from a model of itinerant fermions. We show that in this state, unlike in any other known SDW state, the chemical potential shifts proportionally to the SDW order parameter preserving the original Fermi surface for one spin branch and gapping out the other spin branch. The uniaxial SDW state is therefore a ‘half-metal’ that allows for electrical control of spin currents. Such a state is highly desirable for nano-science applications.

The model: For definiteness we focus on doped graphene at 3/8 filling. Our point of departure is the tight binding model 12 with the nearest-neighbor dispersion $\varepsilon_k = \varepsilon_0 \sqrt{1 + 4 \cos \frac{k_x \sqrt{3}}{2} \cos \frac{3k_x}{2} + 4 \cos^2 \frac{k_x \sqrt{3}}{2} - \mu}$

where $\mu = -t_1$ at 3/8 filling. The FS then forms a perfect hexagon inscribed within a hexagonal BZ (Fig. 1).

FIG. 1: (Color online) a) The Fermi surface at the doping level of interest is a hexagon inscribed within a hexagonal Brillouin zone (BZ), for both honeycomb and triangular lattices. The FS has three inequivalent corners, which are saddle points of the dispersion, marked by a vanishing Fermi velocity and a divergent density of states. The three inequivalent saddle points $M_i$ are connected by three inequivalent nesting vectors $Q_i$, each of which is equal to half a reciprocal lattice vector, such that $Q_i = -Q_i$. b) Spin structure for the uniaxial SDW state. The SDW order quadruples the unit cell to a unit cell with eight sites (shaded). The enlarged unit cell has a large spin moment $3\Delta$ on two sites and a small spin moment $-\Delta$ on the other six. The total spin on each unit cell is zero.
The perfect nesting of the FS in doped graphene is quite robust – it is broken only by third and higher neighbor hoppings, which are generally quite small. The Fermi velocity vanishes near the hexagon corners $\mathbf{M}_1 = (2\pi/3, 0)$, $\mathbf{M}_2 = (\pi/3, \pi/\sqrt{3})$, $\mathbf{M}_3 = (\pi/3, -\pi/\sqrt{3})$, which are saddle points of the dispersion:

$$\varepsilon_{\mathbf{M}_i + \mathbf{k}} \approx \frac{3t_1}{4} (k_y^2 - 3k_x^2), \quad \varepsilon_{\mathbf{M}_2 + \mathbf{k}} \approx -\frac{3t_1}{4} 2k_y(k_y + \sqrt{3}k_x),$$

(2)

where each time $\mathbf{k}$ denotes the deviation from a saddle point. Saddle points give rise to a logarithmic singularity in the DOS and control the SDW instability at weak coupling. There are three inequivalent nesting vectors connecting inequivalent pairs of saddle points (see Fig. 1):

$$\mathbf{Q}_1 = (0, 2\pi/\sqrt{3}), \quad \mathbf{Q}_{2,3} = (\pm \pi/3, -\pi/\sqrt{3}).$$

(3)

Each $\mathbf{Q}_i$ is equivalent to $-\mathbf{Q}_i$ modulo a reciprocal lattice vector.

There are two electron-electron interactions that contribute to the SDW channel. One is a forward scattering interaction $|\mathbf{k}, \mathbf{k} + \mathbf{Q}_i\rangle \rightarrow |\mathbf{k}, \mathbf{k} + \mathbf{Q}_i\rangle$, while the other is an unklapp interaction, $|\mathbf{k}, \mathbf{k}'\rangle \rightarrow |\mathbf{k} + \mathbf{Q}_i, \mathbf{k}' + \mathbf{Q}_i\rangle$. We label these interactions $g_2$ and $g_3$, respectively, for consistency with the notation introduced in [3]. The partition function for $g_2 - g_3$ model can then be written as

$$Z = \int D[\psi^\dagger, \psi] \exp(-S[\psi^\dagger, \psi]),$$

where

$$S = \int_0^{1/T} \mathcal{L}(\mathbf{k}, \tau)$$

and

$$\mathcal{L} = \sum_{\alpha} \psi_{a,\alpha}^\dagger (\partial_\tau - \varepsilon_k + \mu) \psi_{a,\alpha}$$

$$- \sum_{a \neq \beta} g_3 \psi_{a,\alpha}^\dagger \psi_{a,\beta} \psi_{b,\beta} \psi_{b,\alpha} - g_2 \psi_{a,\alpha}^\dagger \psi_{b,\alpha} \psi_{b,\beta} \psi_{a,\beta},$$

(4)

where the action is written in terms of electron operators, $a, b$ are patch labels, and $\alpha$ and $\beta$ are spin components.

Each nesting vector $\mathbf{Q}_i$ has associated with it an SDW order parameter $\Delta_i = \Delta_{a,\alpha} = \frac{2\pi}{\sqrt{3}} g_1 \sum_{k} \langle \psi_{a,\alpha}^\dagger \sigma_{a,\beta} \psi_{a,\beta} \rangle$. The condition for the emergence of each $\Delta_i$ is the same: $(g_2 + g_3)/t_1 \log^2 t_1/T_N = O(1)$ [3], leaving a large number of SDW states as potential candidates. We study the selection of the SDW order within Ginzburg-Landau theory and by comparing different SDW solutions in the mean-field approximation for Eq. [4] at arbitrary $T < T_N$.

**Ginzburg-Landau theory:** To construct the Ginzburg-Landau theory, we decouple the quartic interaction terms by restricting the interaction to the spin channel and performing a Hubbard Stratonovich transformation to introduce the order parameters $\Delta_i$. We integrate out the fermions in the Matsubara frequency representation and obtain an action in terms of the order parameter fields $\mathbf{\Delta}_i$, which takes the form

$$\mathcal{L} = T \sum_{n=-\infty}^{\infty} \int \frac{d^2 k}{(2\pi)^2} \left[ \frac{2}{g_2 + g_3} \sum_i (\Delta_i)^2 \right] + \text{Tr} \ln \left( i\omega_n - \varepsilon_k - \sum_i \mathbf{\Delta}_i \cdot \mathbf{\sigma} \right).$$

(5)

For $T \approx T_N$, we can expand (5) in small $\Delta_i/T_N$. It is useful to define the expansion coefficients

$$Z_i = T \sum_{\omega_n} \int \frac{d^2 k}{(2\pi)^2} \xi_i$$

(6)

where the integrands $\xi_i$ are expressed in terms of fermionic Green functions $G = (i\omega_n - \varepsilon_k - \mu)^{-1}$, $G_i = (i\omega_n - \varepsilon_k + \mathbf{Q}_i - \mu)^{-1}$, and $G_{i+j} = (i\omega_n - \varepsilon_k + \mathbf{Q}_i + \mathbf{Q}_j - \mu)^{-1}$ as

$$\xi_1 = G^2 G_3^2,$$

$$\xi_2 = G^2 G_3 G_1,$$

$$\xi_3 = G G_3 G_1 G_{1+3},$$

$$\xi_4 = G^2 G_3^2 G_4.$$  

(7)

Diagrammatically, $Z_1 - Z_4$ are given by ‘square’ diagrams with four fermionic propagators and $\mathbf{\sigma}_{a,\beta}$ in the vertices, and $Z_4$ is given by a ‘hegagonal’ diagram with six fermionic propagators, (see Fig. 2). The free energy evaluated at $T \approx T_N$ can be expressed in terms of these coefficients as

$$\mathcal{L} \propto \alpha(T - T_N) \sum_i \Delta_i^2 + Z_1(\Delta_1^2 + \Delta_2^2 + \Delta_3^2)^2 + 2(Z_2 - Z_1 - Z_3)(\Delta_1^2 \Delta_2^2 + \Delta_2^2 \Delta_3^2 + \Delta_3^2 \Delta_1^2)$$

$$+ 4Z_3((\Delta_1 \cdot \Delta_2)^2 + (\Delta_2 \cdot \Delta_3)^2 + (\Delta_3 \cdot \Delta_1)^2) - 4Z_4(\Delta_1 \cdot \Delta_2 \times \Delta_3)^2 + \cdots$$

(8)

where $\alpha$ is an inessential positive constant.

The quadratic term and the first quartic term in (8) set the overall magnitude of $\Delta^2 = \sum_i \Delta_i^2$, but do not differentiate between different SDW states. The second quartic term in (8) determines whether SDW order develops only at one nesting vector, or at all three (depending on the sign of $Z_2 - Z_1 - Z_3$). Finally, the third quartic term and sixth order term control the relative orientation of the vector order parameters, if SDW order develops at multiple wavevectors. Close to $T_N$ the expansion to or-
FIG. 2: (Color online) The terms quartic in $\Delta$ are produced by processes represented diagrammatically by square diagrams. The diagrams for $Z_2$ and $Z_3$ correspond to patterns $\Delta_2, \Delta_1, \Delta_1$ and $\Delta_3, \Delta_1, \Delta_1$, respectively. The sixth order chirality sensitive term is produced by 'hexagonal diagrams.' Sample square and hexagonal diagrams are shown above. The integrals are dominated by momenta that bring all the fermion propagators to the vicinity of one of the saddle points of the dispersion.

der $\Delta^4$ is generally sufficient, but we include the sixth order term because $Z_3$ is suppressed by an extra factor of $T_N/t_1$, which is exponentially small in the weak coupling limit. The relative smallness of $Z_3$ arises because in the integrals for $Z_1, Z_2$, and $Z_4$, all fermions can be simultaneously brought to the saddle points, whereas in the integral for $Z_3$, three fermions can be brought simultaneously to saddle points, but the remaining fermion stays far away from not only the saddle point but also the FS.

We evaluate the coefficients $Z_1 - Z_4$ to leading order in small $T_N/t_1$ and obtain

$$Z_1 = \frac{0.20 \log \frac{T_N}{t_1}}{\pi^4 T_N^2 t_1}, \quad Z_2 = \frac{0.58}{\pi^4 T_N^2 t_1},$$

$$Z_3 = -\frac{0.08}{\pi^2 T_N^2 t_1}, \quad Z_4 = -\frac{0.1}{T_N^4 t_1}$$

The positivity of $Z_1$ guarantees a second order phase transition, with the type of SDW order depending on the signs and relative magnitudes of $Z_2, Z_3,$ and $Z_4$. Since $Z_3$ is smaller by $T_N/t_1$ than $Z_{1,2}$, and $Z_2$ is smaller by $\log \frac{T_N}{t_1}$ than $Z_1$, it follows that $Z_2 - Z_1 - Z_3 < 0$, so the system forms SDW order simultaneously at all three nesting vectors (the $3Q$ state). Meanwhile, the relative orientation of the three SDW order parameters is controlled by the sign of $Z_3$ at the smallest $\Delta_i$, and by the sign of $Z_4$ at somewhat larger $\Delta_i$. Both $Z_3$ and $Z_4$ are negative and favor the non-chiral SDW order with the three $\Delta_i$ all aligned along the same axis.

An order parameter of the form $\Delta(e^{iQ_2 \cdot r} + e^{iQ_1 \cdot r} \pm e^{iQ_3 \cdot r})$ leads to spin moments on the lattice of the form shown in Fig. 11. A quarter of lattice sites have spin moment $3\Delta$, the other three quarters have moment $-\Delta$. Such an order cannot be obtained from any spin Hamiltonian for local moments of constant magnitude on every site. Our result differs from earlier mean-field analysis [11] which found non-coplanar insulating SDW order at weak coupling. We note, however, the $3Q$ state that we found, with non-equal spin length on different sites, was not considered in that work and other earlier considerations of the type of SDW order. We found analogous results for fermions on a triangular lattice at Van Hove filling. This system is identical to graphene, except that the nesting is less robust and is spoiled already by second neighbor hopping.

Properties of a uniaxial SDW: Is the uniaxial SDW state a metal or an insulator? To address this issue we need to compute the fermionic spectrum. Without loss of generality, we take the SDW to be uniaxial along the $z$ axis, so that $S^z$ is a good quantum number, and spin-up and spin-down fermions decouple. Consider the state with $\Delta_1 = \Delta_2 = \Delta_3 = \Delta z \sigma_3$. The up spins near the three Van Hove points are described by a simple $3 \times 3$ Hamiltonian

$$H = \begin{pmatrix} \varepsilon_{1,k} - \delta \mu & \Delta & \Delta \\ \Delta & \varepsilon_{2,k} - \delta \mu & \Delta \\ \Delta & \Delta & \varepsilon_{3,k} - \delta \mu \end{pmatrix}$$

where $\varepsilon_{1,2,3}$ are the dispersions near the Van Hove points, Eq. (2), and $\delta \mu$ is the SDW-induced shift of the chemical potential. The $3 \times 3$ Hamiltonian describing the spin down branch is obtained by taking $\Delta \rightarrow -\Delta$. At $k = 0$ (i.e., at Van Hove points) the energies of spin-up excitations $E_k - \delta \mu$ are $-\Delta, -\Delta$, and $2\Delta$, and the energies of spin-down excitations are $\Delta, \Delta$, and $-2\Delta$. In conventional SDW states (e.g., SDW on a 2D square lattice) $\delta \mu/\Delta \propto T_N/E_F$ is negligibly small and can be safely neglected. We find, however, that in our case $\delta \mu = -\Delta$, so that gapless excitations arise in the spin-down spectrum.

To see the unexpected shift of the chemical potential, we diagonalize Eq. (10) and the corresponding equation for down spins and inspect six branches of excitations. We find that fixing $\delta \mu = -\Delta$ ensures that both in the paramagnetic and in the $3Q$ uniaxial SDW state there are four bands with $E_k \leq \mu$ and two bands with $E_k \geq \mu$ for all momenta in the reduced BZ (see Fig. 3). Since the chemical potential is fixed by the constraint that the total number of electrons (equal to the number of states below the chemical potential) must not change between $\Delta = 0$ and $\Delta \neq 0$ [13], it follows that we must set $\delta \mu = -\Delta$. For verification, we computed the thermodynamic potential $\Omega(\Delta, \mu)$ from (5), numerically solved the simultaneous equations $\partial \Omega/\partial \Delta = 0$ and $\partial \Omega/\partial \mu = -N$, and confirmed that $\delta \mu = -\Delta$ to a high accuracy.

Having determined that $\delta \mu = -\Delta$, we find from (10) that gapless excitations emerge when $\varepsilon_{1,k} \varepsilon_{2,k} \varepsilon_{3,k} = 0$, which has solutions along three lines passing through each Van Hove point. Two of then coincide with the original FS, the third is directed towards the center of the BZ. The $3Q$ uniaxial SDW state is then obviously a
electrical control of spin currents, which may be beneficial for nanoscience applications particularly because charge currents will necessarily also be spin currents, which allows for electrical control of the latter.

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SUPPLEMENT

In this supplement we present the calculations that were quoted in the main text.

Calculation of $Z_1$

We wish to evaluate

$$Z_1 = T \sum_{\omega_n} \int \frac{d^2k}{(2\pi)^2} G^2(k, \omega_n) G^2(k + Q_3, \omega_n)$$  \hspace{1cm} (11)

The integral over the Brillouin zone is dominated by those values of $k$ where both Green functions correspond to states near a saddle point. Expanding the energy about the saddle points, we rewrite the integral as

$$Z_1 \approx T \sum_{\omega_n} \int \frac{d^2k}{(2\pi)^2} \frac{1}{(i\omega_n - \frac{3}{4} (3k_y^2 - k_y^2))^2 (i\omega_n - \frac{3}{4} 2k_y (k_y - \sqrt{3}k_x))^2}$$  \hspace{1cm} (12)

Where the integral is understood to have a UV cutoff for $k$ with logarithmic accuracy. Performing the integral over $a$, we rewrite the integral as

$$Z_1 = T \sum_{\omega_n} \frac{2}{3\sqrt{3}t_1} \int_{-\sqrt{t_1}a}^{\sqrt{t_1}a} \frac{da}{2\pi |a|} \int_{-\sqrt{t_1}a}^{\sqrt{t_1}a} \frac{dx}{2\pi} 1 \left( \frac{1}{(i\omega_n - a(a + b))^2} \right)$$  \hspace{1cm} (13)

We now define $x = ab$ and rewrite the integral as

$$Z_1 = T \sum_{\omega_n} \frac{2}{3\sqrt{3}t_1} \int_{-\sqrt{t_1}a}^{\sqrt{t_1}a} \frac{da}{2\pi |a|} \int_{-\sqrt{t_1}a}^{\sqrt{t_1}a} \frac{dx}{2\pi} 1 \left( \frac{1}{(i\omega_n - x)^2} \right)$$  \hspace{1cm} (14)

We now assume $T_N \ll t_1$ (which should certainly be the case for weak/moderate coupling). In this limit, we can perform the integral over $x$ approximately, using the Cauchy integral formula, to get

$$Z_1 = T \sum_{\omega_n} \frac{2}{3\sqrt{3}t_1} \int_{-\sqrt{t_1}a}^{\sqrt{t_1}a} \frac{da}{2\pi |a|} \frac{2i\text{sign}\omega_n}{(a^2 - 2i\omega_n)^3} = T \sum_{\omega_n} \frac{4}{3\sqrt{3}t_1} \int_{-\sqrt{t_1}a}^{\sqrt{t_1}a} \frac{da}{2\pi |a|} \frac{1}{(a^2 + 4\omega_n^2)^3}$$  \hspace{1cm} (15)

The imaginary part of the above integral is odd in $\omega$ and hence vanishes upon performing the Matsubara sum to leave an integral that is purely real

$$Z_1 = T \sum_{\omega_n} \frac{8|\omega_n|}{3\sqrt{3}t_1} \int_{-\sqrt{t_1}a}^{\sqrt{t_1}a} \frac{da}{2\pi |a|} \frac{4a^2 - 3a^4}{(a^4 + 4\omega_n^2)^3} \approx T \sum_{\omega_n} \frac{8|\omega_n|}{3\sqrt{3}t_1} \int_{-\sqrt{t_1}a}^{\sqrt{t_1}a} \frac{da}{2\pi |a|} \frac{4a^2}{(a^4 + 4\omega_n^2)^3}$$  \hspace{1cm} (16)

with logarithmic accuracy. Performing the integral over $a$ (again with logarithmic accuracy) gives

$$Z_1 \approx T \sum_{\omega_n} \frac{1}{12\pi \sqrt{3}t_1 |\omega_n|^3} \ln \frac{t_1}{\omega_n} = \frac{1}{48\pi^4 \sqrt{3}T_N^2 t_1} (16.8 \ln \frac{t_1}{2\pi T} + 10.5) \approx \frac{16.8 \ln \frac{t_1}{2\pi T}}{48\pi^4 \sqrt{3}T_N^2 t_1}$$  \hspace{1cm} (17)

Where we take $\omega_n = 2\pi(n + 1/2)T_N$, $T = T_N$ and perform the discrete sum on mathematica.

Calculation of $Z_2$

This time we want to evaluate

$$Z_2 = T \sum_{\omega_n} \int \frac{d^2k}{(2\pi)^2} G^2(k, \omega_n) G(k + Q_3, \omega_n) G(k + Q_4, \omega_n)$$  \hspace{1cm} (18)
Again, we anticipate this integral will be dominated by regions of the Brillouin zone where all three Green functions correspond to states near saddle points. Expanding the dispersion about the saddle points, we obtain

\[ Z_2 \approx T \sum_{\omega_n} \int \frac{d^2k}{(2\pi)^2} \frac{1}{(i\omega_n - \frac{3\pi^2}{4}(3k_x^2 - k_y^2))(i\omega_n - \frac{3\pi^2}{4}2k_y(k_y - \sqrt{3}k_x))(i\omega_n - \frac{3\pi^2}{4}2k_y(k_y + \sqrt{3}k_x))} \]

(19)

Making the same coordinate substitutions as in the preceding section, we recast this as

\[ Z_2 = T \sum_{\omega_n} \frac{2}{3\sqrt{3}t_1} \int_{-\sqrt{\pi}t_1}^{\sqrt{\pi}t_1} dadb \frac{1}{(i\omega_n + ab)^2(i\omega_n - a(a + b))(i\omega_n - b(a + b))} \]

(20)

After scaling out \( \omega_m \), we can rewrite it as

\[ Z_2 = T \sum_{\omega_n} \frac{2}{3\sqrt{3}t_1|\omega_n|^3} \int_{-\sqrt{\pi}t_1}^{\sqrt{\pi}t_1} dadb \frac{1}{(i + ab)^2(i - a(a + b))(i - b(a + b))} = T \sum_{\omega_n} \frac{2}{12\pi^2\sqrt{3}t_1|\omega_n|^3}^{2.9} \]

(21)

Where the rescaled integral is fully convergent, and can be done numerically on mathematica. The sum over Matsubara frequencies can also be done on mathematica, and yields the answer

\[ Z_2 = T \sum_{\omega_n} \frac{2.9 \times 16.8}{48\pi^3\sqrt{3}t_1T_N^2} \]

(22)

Comparing with the previous expression for \( Z_1 \), we see that \( Z_2 \approx Z_1 \times 2.9/\ln(T/t_1) \). Thus, \( Z_2 \ll Z_1 \) provided the log is large. (If the log is not large then the evaluation of \( Z_1 \) with logarithmic accuracy does not suffice, and sub-logarithmic contributions to \( Z_1 \) must also be taken into account.)

**Calculation of \( Z_3 \)**

We want to evaluate

\[ Z_3 = T \sum_{\omega_n} \int \frac{d^2k}{(2\pi)^2} G(k, \omega_n)G(k + Q_3, \omega_n)G(k + Q_1, \omega_n)G(k + Q_1 + Q_3, \omega_n) \]

(23)

This time it is not possible to place all the Green functions at the saddle points. In fact, we cannot even place all the Green functions at the Fermi surface - the best that can be done is to place three of the Green functions near a saddle point, but the fourth has to be off Fermi surface. Thus, we obtain,

\[ Z_3 \approx T \sum_{\omega_n} \int \frac{d^2k}{(2\pi)^2} \frac{1}{(i\omega_n - \frac{3\pi^2}{4}(3k_x^2 - k_y^2))(i\omega_n - \frac{3\pi^2}{4}2k_y(k_y - \sqrt{3}k_x))(i\omega_n - \frac{3\pi^2}{4}2k_y(k_y + \sqrt{3}k_x))(i\omega - 2t_1)} \]

(24)

Making the usual substitutions, and assuming \( t_1 \gg T_N \), we obtain

\[ Z_3 \approx T \sum_{\omega_n} \frac{T_N}{12\sqrt{3}\pi^2t_1^2\omega_n^2} \int_{-t_1}^{t_1} \frac{dadb}{(i\omega_n + ab)(i\omega_n - a(a + b))(i\omega_n - b(a + b))} \]

(25)

\[ \approx T \sum_{\omega_n} \frac{T_N}{12\sqrt{3}\pi^2t_1^2\omega_n^2} \int_{-t_1/\omega_n}^{t_1/\omega_n} \frac{dadb}{(i + ab)(i - a(a + b))(i - b(a + b))} \]

(26)

The integral is convergent. As usual, the imaginary part is odd in \( \omega \) and vanishes and we care only about the real part. Performing the integral on mathematica and taking the real part, we obtain

\[ Z_3 \approx T \sum_{\omega_n} \frac{6.5T_N}{12\sqrt{3}\pi^2t_1^2\omega_n^2} \approx \frac{6.5}{48\sqrt{3}\pi^2t_1^2T_N} \]

(27)

Which is parametrically smaller than \( Z_1 \) and \( Z_2 \) by \( T_N/t_1 \).
When after some analysis of diagrams we find that, narrower region of stability, and the transition into the chiral SDW phase happens quite close to $T = T_0$. Nevertheless, if we apply our analysis because fermions can no longer be approximated as free particles (the self-energy corrections to fermionic lines and coupling $\Sigma$ are generally of order one).

The above calculations were performed at weak/moderate coupling, assuming $T_N/t_1 \ll 1$. However, the integrals can be evaluated at arbitrary $T_N/t_1$. To this end, it is useful to define the scaling functions $f_i(T_N/t_1) = Z_i(T_N/t_1)/Z_i(0)$. These scaling functions are evaluated numerically and shown in Fig. 4.

**Calculation of $Z_4$**

We now calculate the coefficient of the sixth order chirality sensitive term in the free energy, $v(\Delta_1 \cdot (\Delta_2 \times \Delta_3))^2$. After some analysis of diagrams we find that,

$$Z_4 = T \sum_{\omega_n} \int_{\omega_n}^{t_1/\omega_n} \frac{d^2 k}{(2\pi)^2} G^2(k, \omega_n)G^2(k + \mathbf{Q}_3, \omega_n)G^2(k + \mathbf{Q}_1, \omega_n)$$  \hspace{1cm} (28)

We can now place all the Green functions on the near a saddle point. Making the usual substitutions, we obtain

$$Z_4 = \sum_{\omega_n} \frac{T_N}{12\sqrt{3\pi^2}t_1|\omega_n|^5}\int_{-t_1/\omega_n}^{t_1/\omega_n} \frac{dadb}{(i + ab)^2(i - a(a + b))^2(i - b(a + b))^2}$$  \hspace{1cm} (29)

$$= \sum_{\omega_n} \frac{T_N}{12\sqrt{3\pi^2}t_1|\omega_n|^5}\int_{-t_1/T_N}^{t_1/T_N} \frac{dadb}{(i + ab)^2(i - a(a + b))^2(i - b(a + b))^2}$$  \hspace{1cm} (30)

where in the second line we have assumed that the Matsubara sum is controlled by the first few Matsubara frequencies. The integral can be done numerically, and it is negative for small $T_N/t_1$. It then follows that at weak/moderate coupling $Z_4 < 0$, so that the Free energy at sixth order also disfavors chirality.

**Scaling functions**

The above calculations were performed at weak/moderate coupling, assuming $T_N/t_1 \ll 1$. However, the integrals can be evaluated at arbitrary $T_N/t_1$. To this end, it is useful to define the scaling functions $f_i(T_N/t_1) = Z_i(T_N/t_1)/Z_i(0)$. These scaling functions are evaluated numerically and shown in Fig. 4.

**PHASE DIAGRAM AT STRONG COUPLING**

At strong coupling, when $g_{2,3} \geq t_1$ and $T_N \sim t_1$, our analysis based on Ginzburg-Landau expansion is less accurate because fermions can no longer be approximated as free particles (the self-energy corrections to fermionic lines and vertex corrections to square and hexagonal diagrams are generally of order one). Nevertheless, if we apply our analysis to $T_N \sim t_1$, we find that $Z_3$, $Z_4$ and $Z_2 - Z_1 - Z_3$ all change signs at some $T_N/t_1$ (see Fig. 4).

The first sign change occurs in the sixth order chirality sensitive term $Z_4$, which becomes positive for $T_N/t > 0.1$. When $Z_4$ is positive, the chiral SDW state [3] is energetically favored, provided we are sufficiently far below $T_N$ for the sixth order term to dominate over the quartic term $Z_3$. Thus, at large $T_N/t_1$, the uniaxial SDW phase has a much narrower region of stability, and the transition into the chiral SDW phase happens quite close to $T_N$.

The term $Z_4$ is next to change sign, becoming negative for $T_N/T > 0.35$. Once this term becomes negative, the system prefers instead a 1Q collinear state, of the form discussed in [11], wherein SDW order develops only at a single nesting vector. The subsequent sign change of $Z_3$ at $T_N/t \approx 0.55$ has no physical consequences.
The $1Q$ collinear SDW state that forms at $T_N/t > 0.35$ is a (full) metal because the entire FS is not gapped out. The competition between a metallic collinear state and non-coplanar insulating state has been detected numerically in the mean-field analysis at strong coupling [10], and our results for the strong coupling case are in line with this earlier study. Our strong coupling results are also consistent with the studies that found a non-coplanar, chiral SDW order in the models of spins of the same fixed length at every lattice site [3, 10]. However, in the weak/moderate coupling limit, our results indicate that the preferred state is a uniaxial $3Q$ state of a sort not considered before, which can only be realized in a model starting from itinerant fermions.

We also analyzed the evolution of $Z_i$ with $T_N/t_1$ for fermions on a triangular lattice. We found similar trends, e.g., sign change of $Z_4$. However, for a triangular lattice, the first sign change (in $Z_4$) occurs at a much larger $T_N/t_1 \sim 0.5$, when the itinerant approach is very questionable.

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[15] The details are relegated to the online supplementary material.