QCD condensates from $\tau$-decay data: a functional approach

S. Ciulli a, C. Sebu a, K. Schilcher b, H. Spiesberger b

a Laboratoire de Physique-Mathématique et Théorique, Université de Montpellier II, F-34095 Montpellier, France
b Institut für Physik, Johannes-Gutenberg-Universität, Staudinger Weg 7, D-55099 Mainz, Germany

Received 26 January 2004; received in revised form 8 June 2004; accepted 11 June 2004
Available online 23 June 2004
Editor: P.V. Landshoff

Abstract

We study a functional method to extract the $V-A$ condensate of dimension 6 from a comparison of $\tau$-decay data with the asymptotic space-like QCD prediction. Our result is in agreement within errors with that from conventional analyses based on finite energy sum rules.

1. Introduction

Although QCD has been with us for three decades, the knowledge of the values of the various fundamental or effective parameters of the theory (with the possible exception of the coupling constant) such as quark masses and condensates is still astonishingly limited. The precise data on $\tau$-decay obtained by the ALEPH [1] and OPAL [2] Collaborations at CERN have offered an opportunity for new studies, which range from an extraction of the strange quark mass [3] to the determination of various condensate parameters. Of particular interest was the extraction of the dimension-6 condensate [4–8] which, in the chiral limit, determines, e.g., the $K \rightarrow \pi \pi$ matrix elements of the relevant electroweak penguin operators.

In this Letter we study a functional method [9,10] which allows us to extract within rather general assumptions the condensates from a comparison of the time-like experimental data with the asymptotic space-like results from theory. We will see that the price to be paid for the generality of assumptions are relatively large errors in the values of the extracted parameters.

We should add a few remarks concerning the distinction of our approach from the popular methods based on finite-energy sum rules (FESR). A first obvious remark is that it is not possible (not even in principle) to reconstruct the correlator in the space-like region from error afflicted time-like data, as this constitutes an analytic

---

* Supported by PROCOPE, EGIDE.
E-mail addresses: sebu@lpm.univ-montp2.fr (C. Sebu), lspi@thep.physik.uni-mainz.de (H. Spiesberger).

0370-2693 © 2004 Elsevier B.V. Open access under CC BY license.
doi:10.1016/j.physletb.2004.06.046
continuation from a finite domain. One has to stabilize this “ill-posed problem” by suitable additional assumptions. The simplest choice is probably to assume that the result of QCD and the operator product expansion (OPE) in the space-like region is simply a series in powers of $1/s$ times condensates (vacuum expectation values of operators) and that there are no truly non-perturbative terms like e.g. from instantons. This would be the ideal situation for the application of FESR: each moment would pick out a single operator. However, this is unfortunately not the case since logarithms arising within perturbation theory do not fall in this class of functions. The higher- and lower-dimensional condensates contribute to a given moment starting with the inclusion of corrections of order $O(\alpha^2_s)$ [11]. It is also known that the perturbation series starts to diverge at some, not very high, order. As higher orders of the relevant Wilson coefficients are not known, one can only hope that the extraction of the condensates is stable when higher orders are included. There is the additional problem of the contribution of the truly non-perturbative terms (non-OPE) to the integral on the circle in the complex plane, even if this uncertainty, which is expected to be most prominent near the physical cut, can be reduced by choosing suitable linear combinations of moments.

The assumptions of our approach are quite different and more general. Motivated by the amazing extent to which the (suitably modified) first and second Weinberg sum rules are satisfied precociously [7], we assume that the exact correlator in the space-like region roughly falls off like $1/\sqrt{s}$. By ‘roughly’ we mean, within some error band which must be supplied by hand. We parametrize the error band by a $1/s^4$ term with a scale of about $\Lambda_{\text{QCD}}$. In contrast to the case of FESR, these assumptions refer only to the negative axis in the $s$-plane. Furthermore our assumptions are quite independent of perturbation theory or indeed of QCD itself. Similarly to the Weinberg sum rules, they only depend on the pattern of chiral symmetry breaking. Of course, as QCD is well established, we like to discuss our results in the language of QCD and the OPE. For this reason we also include the $O(\alpha_s)$ correction to the $1/\sqrt{s}$ term and parametrize the unknown error corridor in the space-like QCD expression by an effective dimension-8 condensate. As will become clear below, one would not gain anything by taking into account a higher-dimension condensate in the characterization of the error corridor since this would only lead to a stronger suppression of the low-$s$ region.

Although we do not claim that our method is superior to other approaches, we hope that our results lend additional confidence to the numerical results obtained with the help of FESR.

2. QCD condensates

We consider the polarization operator of hadronic vector and axial-vector charged currents, $J_\mu = V_\mu = \bar{u}_d \gamma_\mu d$ and $J_\mu = A_\mu = \bar{u}_d \gamma_\mu \gamma^5 d$,

$$\Pi^{(j)}_{\mu\nu} = i \int dx e^{iqx} \langle T J_\mu(x) J_\nu(0) \rangle = (\gamma_{\mu\nu} q^2 + q_\mu q_\nu) \Pi^{(1)}_J(q^2) + q_\mu q_\nu \Pi^{(0)}_J(q^2).$$

The conservation of the vector current implies $\Pi^{(0)}_V = 0$. The spectral functions are related to the absorptive part of the correlators

$$v_j(s) = 4\pi \text{Im} \Pi^{(j)}_V(s), \quad a_j(s) = 4\pi \text{Im} \Pi^{(j)}_A(s)$$

and can be measured in hadronic $\tau$-decays. We consider specifically the $V - A$ component which is related to the branching ratios of $\tau$ decays through

$$R_{\tau, V - A} = \frac{B(\tau \rightarrow \nu_\tau + \text{hadrons}, V - A)}{B(\tau \rightarrow \nu_\tau + e + \bar{\nu}_\tau)} = \left(1 - \frac{s}{m_\tau^2}\right)^2 \left[1 + \frac{s}{m_\tau^2} \left(v_1 - a_1 - a_0\right) + \frac{2}{m_\tau^2} s a_0\right].$$
Here, $V_{ud}$ is the weak mixing CKM-matrix element, $|V_{ud}|^2 = 0.9752 \pm 0.0007$, the $\tau$ mass is denoted $m_\tau = 1.777$ GeV and $S_{\text{EW}} = 1.0194 \pm 0.0040$ accounts for electroweak radiative corrections [12]. The spin-0 axial-vector contribution $a_0(s)$ is dominated by the one-pion state, $a_0(s) = 2\pi^2 f_\pi^2 \delta(s - m_\pi^2)$, with the $\pi$-decay constant $f_\pi = 0.1307$ GeV. Its contribution in the last term of (3) is tiny

$$\Delta R_{\tau, V-A}|_{a_0} \simeq 24\pi^2 \frac{f_\pi^2 m_\pi^2}{m_\tau^4} \simeq 0.0074 \tag{4}$$

and will be neglected. The contribution of the pion pole to the first term in Eq. (3) is well identified in the data and concentrated at low $s \simeq m_\pi^2$; thus it can be removed from the data and taken into account explicitly without introducing sizable additional uncertainties. The experimental data [1,2] are given by binned and normalized event numbers related to the differential distribution $dR_{\tau, V-A}/ds$ and can therefore be viewed as a measurement of the function

$$\omega_{V-A}(s) = v_1(s) - a_1(s) - a_0(s). \tag{5}$$

The $(V - A)$ correlator is special since it vanishes identically in the chiral limit ($m_q = 0$) to all orders in QCD perturbation theory. Renormalon ambiguities are thus avoided. Non-perturbative terms can be calculated for large $|s|$ by making use of the operator product expansion (OPE) of QCD

$$\Pi_{V-A}^{(0+1)}(s) = \sum_{D \geq 4} O_{D}^{V-A} \left( \frac{\alpha_s}{\pi} \right) \left( 1 + c_D \frac{\alpha_s}{\pi} \right) \tag{6}$$

where $O_{D}^{V-A}$ are vacuum matrix elements of local operators of dimension $D$ (so-called condensates). Their contribution is known up to dimension 8 and read, at leading order,

$$O_4^{V-A} = (m_u + m_d) \langle \bar{q} q \rangle = -f_\pi^2 m_\pi^2, \tag{7}$$

$$O_6^{V-A} = -\frac{32\pi}{9} \alpha_s \langle \bar{q} q \rangle^2, \tag{8}$$

$$O_8^{V-A} = 4\pi \alpha_s \langle \bar{q} G_{\mu\nu} G^{\mu\nu} q \rangle. \tag{9}$$

The last two results are based on the vacuum dominance or factorization approximation which holds, e.g., in the large-$N$ limit. It would be desirable to avoid this assumption, given the fact that the complete expression for $O_6$, which involves two operators, is known to $O(\alpha_s)$ [13]. Our method, however, is limited in practice to determine a single constant and we therefore have to adhere to the factorization assumption. The argument can be inverted: if we find consistency between theory and data, then this fact lends additional support to the factorization assumption.

The numerical value of $O_4^{V-A}$ is very small and this condensate can be neglected in our analysis. For $O_6^{V-A}$ we use the expression in Eq. (8) and include the corresponding next-to-leading-order corrections which have been calculated in [14,15]. They depend on the regularization scheme implying that the value of the condensate itself is a scheme-dependent quantity. Explicitly,

$$O_6^{V-A} = -\frac{32\pi}{9} \alpha_s \langle \bar{q} q \rangle^2 \left( 1 + \frac{\alpha_s(\mu^2)}{4\pi} \left( c_6 + \ln \left( \frac{\mu^2}{s} \right) \right) \right), \tag{10}$$

where

$$c_6 = \begin{cases} 
\frac{247}{48} & \text{BM-scheme [14]}, \\
\frac{89}{38} & \text{anticommuting } \gamma_5 \text{ [15].} 
\end{cases} \tag{11}$$

1 We use the ALEPH data [1] because of their smaller experimental errors.
The renormalization scale $\mu^2$ is conveniently chosen to be $-s$.

The result for $O_8^{V-A}$, Eq. (9), is taken from [6]. It involves a quark–gluon condensate for which various estimates exist. The typical scales determining the condensates are around 300 MeV, e.g., $\langle \bar{q}q \rangle \simeq (250 \text{ MeV})^3$, $(\alpha_s/\pi)|G^2| \simeq (300 \text{ MeV})^5$. Assuming a similar scale for the condensate entering $O_8^{V-A}$, we expect $O_8^{V-A}$ to be of order $10^{-3}$ GeV$^8$. This is small enough so that the OPE makes sense. If $O_8^{V-A}$ would be much larger, radiative corrections to higher-dimension condensates would mix significantly with the lower-dimension condensates through their imaginary parts. There exist a number of QCD sum rule extractions of the value of $\langle \bar{q}q \rangle$ comparable to $(\alpha_s/\pi)\langle \bar{q}q \rangle$.

3. An $L^2$ norm approach

We consider a set of functions $F(s)$ (where $F(s)$ relates to $\Pi_{V-A}^{(0+1)}(s)$) expressed in terms of some squared energy variable $s$ which are admissible as a representation of the true amplitude if

(i) $F(s)$ is a real analytic function in the complex $s$-plane cut along the time-like interval $\Gamma_R = [s_0, \infty)$. The value of the threshold $s_0$ depends on the specific physical application ($s_0 = \left(2m_\pi\right)^2$ for $\Pi_V$, $s_0 = \frac{1}{4}m_\pi^2$ for $\Pi_A$).

(ii) The asymptotic behavior of $F(s)$ is restricted by fixing the number of subtractions in the dispersion relation

\begin{equation}
F(s) = \frac{1}{\pi} \int_{s_0}^{\infty} \frac{f(x)}{x-s} dx. \tag{12}
\end{equation}

We have two sources of information which will be used to determine $F(s)$ and $f(s)$. First, there are experimental data in a time-like interval $\Gamma_{\text{exp}} = [s_0, s_{\text{max}}]$ with $s_0 > 0$ for the imaginary part of the amplitude. Although these data are given on a sequence of adjacent bins, we describe them by a function $f_{\text{exp}}(s)$. We assert that $f_{\text{exp}}$ is a real, not necessarily continuous function. The experimental precision of the data is described by a covariance matrix $\Sigma(s, s')$.

On the other hand, we have a theoretical model, in fact QCD. From perturbative QCD we can obtain a prediction for the amplitude in a space-like interval $\Gamma_L = [s_2, s_1]$. This model amplitude $F_{\text{QCD}}(s)$ is a continuous function of real type, but does not necessarily conform to the analyticity property (i). Since perturbative QCD is expected to be reliable for large energies, we expect that there is also useful information about the imaginary part of the amplitude provided that $|s|$ is large, i.e., we can also use $f_{\text{QCD}}(s) = \text{Im} F_{\text{QCD}}(s + i0)|_{s \in (s_{\text{max}}, \infty)}$. In order to compare the true amplitude with theory, we can therefore split the integral in the dispersion relation (12),

\begin{equation}
F(s) = \frac{1}{\pi} \int_{s_{\text{max}}}^{\infty} \frac{f(x)}{x-s} dx = \frac{1}{\pi} \int_{s_0}^{s_{\text{max}}} \frac{f(x)}{x-s} dx, \tag{13}
\end{equation}

We do not exclude the case $s_2 \rightarrow -\infty$.\footnote{We do not exclude the case $s_2 \rightarrow -\infty$.}
and test the hypothesis whether the left-hand side can be described by QCD.

We also need an a priori estimate of the accuracy of the QCD predictions. This will be described by a continuous, strictly positive function \( \sigma_L(s) \) for \( s \in \Gamma_L \) which should describe errors due to the truncation of the perturbative series and the operator product expansion and is expected to decrease as \(|s| \to \infty\) and diverge for \( s \to 0\). In the case of \( \Pi_{V-A} \) which does not have perturbative contributions, we will take the contribution of the dimension \( D = 8 \) operator as an error and use \( \sigma_L(s) = O_8/s^4 \) with \( O_8 \) in the order of \( 10^{-3} \text{GeV}^8 \). If the perturbative part dominates, as is the case for the individual vector or axial-vector correlators, the last known term of the perturbation series could be used as a sensible estimate of the error corridor.

The goal is to check whether there exists any function \( F(s) \) with the above analyticity properties, the true amplitude, which is in accord with both the data in \( \Gamma_{\text{exp}} \) and the QCD model in \( \Gamma_L \). In order to quantify the agreement we will define functionals \( \chi^2_{\text{L}}[f] \) and \( \chi^2_{\text{R}}[f] \) using an \( L^2 \) norm. For the time-like interval we simply compare the true amplitude \( f(s) \) with the data and use the covariance matrix of the experimental data as a weight function:

\[
\chi^2_{\text{R}}[f] = \int_{s_{\text{max}}}^{s_0} dx \int_{s_{\text{max}}}^{s_0} dx' V^{-1}(x,x')(f(x) - f_{\text{exp}}(x))(f(x') - f_{\text{exp}}(x')).
\]

Experimental data correspond to cross sections measured in bins of \( s \), so that we can calculate this integral in terms of a sum over data points. The ALEPH data which we use are given for 65 equal-sized bins of width \( \Delta s = 0.05 \text{GeV}^2 \) between 0 and 3.25 GeV\(^2\). \( \chi^2_{\text{R}} \) given in (14) is in fact the conventional definition of a \( \chi^2 \) and has a probabilistic interpretation: for uncorrelated data obeying a Gaussian distribution we would expect to obtain \( \chi^2_{\text{R}} = N \), where \( N \) is the number of data points. Since experimental data at different energies are correlated, we instead expect

\[
\chi^2_{\text{exp}} = \sum_{i,j} \sqrt{V(s_i,s_i)V(s_j,s_j)} V^{-1}(s_i,s_j).
\]

In order to define a measure for the agreement of the true function \( f(s) \) with theory, we use the left-hand side of (13) which is well-defined and expected to be a reliable prediction of QCD in the space-like interval for not too small \(|s|\). This expression can be compared with the corresponding integral over the true function. Thus we define

\[
\chi^2_{\text{L}}[f] = \frac{1}{|\Gamma_L|} \int_{\Gamma_L} w_L(x) \left( F_{\text{QCD}}(x) - \frac{1}{\pi} \int_{s_{\text{max}}}^{\infty} \frac{f_{\text{QCD}}(x')}{x' - x} dx' - \frac{1}{\pi} \int_{s_0}^{s_{\text{max}}} \frac{f(x')}{x' - x} dx' \right)^2 dx,
\]

where \( w_L \) is the weight function for the space-like interval and identified with \( 1/\sigma^2_L(s) \). The integral is normalized to unity for the case where the difference within parentheses saturates the error \( \sigma_L \).

In order to find the true function \( f(s) \), we can combine the information contained in \( \chi^2_{\text{R}} \), (14), and \( \chi^2_{\text{L}} \), (16) in the following way [9,10]. We fix

\[
\chi^2_{\text{R}}[f] = \chi^2_{\text{exp}},
\]

and minimize \( \chi^2_{\text{L}} \):

\[
\chi^2_{\text{L}}[f] \to \text{least} \quad (\equiv \chi^2_{\text{L,min}}).
\]

These conditions are equivalent to finding the unrestricted minimum of the functional

\[
\mathcal{F}[f] = \chi^2_{\text{L}}[f] + \mu \chi^2_{\text{R}}[f].
\]
where $\mu$ is the Lagrange multiplier, which will be found later. The solution of the condition $\chi^2_R[f] = \chi^2_0$ will be denoted by $f(x; \mu)$:

$$\int_{s_0}^{s_{\text{max}}} dx \int_{x'}^{s_{\text{max}}} d'x' V^{-1}(x,x') [f(x; \mu) - f_{\text{exp}}(x)][f(x'; \mu) - f_{\text{exp}}(x')] = \chi^2_0.$$ 

To this end we require the Fréchet derivative of $F$ to be zero

$$\frac{\partial F[f,Y]}{\partial \alpha} \equiv \lim_{\alpha \to 0} \frac{\partial F[f + \alpha Y]}{\partial \alpha} = 0,$$

for any function $Y$. This leads to the following integral equation for the imaginary part $f(x; \mu)$:

$$f(x; \mu) = f_{\text{exp}}(x) + \frac{1}{\pi} \int_{s_0}^{s_{\text{max}}} dx' V(x,x') \int_{\Gamma_L} dx'' w_L(x'') F_{\text{QCD}}(x'') \frac{1}{x' - x''}$$

$$- \frac{\lambda}{\pi} \int_{s_0}^{s_{\text{max}}} dx' V(x,x') \int_{s_{\text{max}}}^{\infty} dx'' \frac{f_{\text{QCD}}(x'')}{(x' - y)(x'' - y)}$$

$$+ \lambda \int_{s_0}^{s_{\text{max}}} dx' K(x,x') f(x'; \mu),$$

(19)

where $\lambda = 1/\mu$ and

$$K(x,x') = -\frac{1}{\pi^2} \int_{s_0}^{s_{\text{max}}} dx'' V(x,x'') \int_{\Gamma_L} dy \frac{w_L(y)}{(x' - y)(x'' - y)}.$$

Eq. (19) is a Fredholm equation of the second kind which is stable against variations of its input. At this stage we should notice that if one had claimed that in the space-like region the function $F(s)$ was given by some analytic expression (e.g., by some few QCD terms), this would be equivalent to saying that $\chi^2_L$ vanished identically. But then, from the definition of the functional $F$ and the vanishing of its Fréchet derivative, it follows that $\mu = 1/\lambda$ is zero which will turn the integral equation (19) into a Fredholm equation of the first kind which is known to be unstable.

The integral equation will be solved numerically by expanding $f(s)$ in terms of Legendre polynomials. The algorithm to determine an acceptable value for the condensate is then the following:

(i) For a fixed value of $\chi^2_0 = \chi^2_{\text{exp}}$ we determine the solution (19) and calculate the corresponding value of $\chi^2_L[f]$ as a function of the condensate $\alpha_s \langle \bar{q}q \rangle^2$. The Lagrange multiplier $\mu$ is determined by iteration such that the condition $\chi^2_R[f] = \chi^2_0$ is fulfilled.

(ii) We minimize this $\chi^2_L[f]$ with respect to $\alpha_s \langle \bar{q}q \rangle^2$ and call the minimal value $\chi^2_{L_{\text{min}}}$ and the corresponding $\alpha_s \langle \bar{q}q \rangle^2$ is the value for the condensate we are looking for.

(iii) We determine the error on $\alpha_s \langle \bar{q}q \rangle^2$ by solving $\chi^2_L(\alpha_s \langle \bar{q}q \rangle^2) = \chi^2_{L_{\text{min}}} + 1$.

4. Numerical results and discussion

A typical situation resulting from this algorithm is shown in Fig. 1. The left part of this figure shows $\chi^2_L$ which has the expected quadratic dependence of $\alpha_s \langle \bar{q}q \rangle^2$. The values of $\alpha_s \langle \bar{q}q \rangle^2$ corresponding to the minimum of $\chi^2_L$
Fig. 1. A typical result for $\chi^2_L$ as a function of $\alpha_s(\bar{q}q)^2$ (left) and the regularized function compared with data [1] (right). We have chosen $O_8 = 10^{-3}$ GeV$^8$ and $c_6 = 89/48$.

Table 1
Results of the determination of $\alpha_s(\bar{q}q)^2$ (in units of $10^{-4}$ GeV$^6$) for the two choices of $c_6$ in (11) and with different values for $O_8$ to fix the error channel in the space-like interval

| $O_8$           | $\alpha_s(\bar{q}q)^2$ for $c_6 = \frac{89}{48}$ | $\alpha_s(\bar{q}q)^2$ for $c_6 = \frac{247}{48}$ |
|----------------|---------------------------------|---------------------------------|
| $1.0 \times 10^{-3}$ GeV$^8$ | $1.6 \pm 1.0$                | $1.1 \pm 0.6$                |
| $1.25 \times 10^{-3}$ GeV$^8$  | $1.6 \pm 1.1$                | $1.1 \pm 0.8$                |
| $1.5 \times 10^{-3}$ GeV$^8$  | $1.6 \pm 1.2$                | $1.1 \pm 0.9$                |

are listed in Table 1 for various choices of $O_8$ as discussed above. The regularized function shown in the right part of Fig. 1 follows nicely the data points, except at large $s$. Here the experimental errors are large and hence, as it should happen, the regularizing effect by means of the functional (14) is not as effective. The data points at the upper end of the spectrum do not contribute significantly to $\chi^2_L$ and our results do not change when we discard the last ten points.

For the evaluation of $\chi^2_L$ we have restricted the range of integration within limits $s_2 \leq s \leq s_1 < 0$. We checked that our result is insensitive to changes of $s_2$ as soon as its absolute value is chosen larger than $O(100)$ GeV$^2$. Since the error channel defined by $O_8/s^4$ diverges for $s \to 0$, one could, in principle, choose the upper limit $s_1 = 0$. Numerical instabilities require a non-zero value. We observe a well-defined plateau for the result for $\alpha_s(\bar{q}q)^2$ as a function of $s_1$ between $-1.0$ and $-0.5$ GeV$^2$ and quote the values for $s_1 = -0.7$ GeV$^2$.

We stress that $O_8$ is only needed to define an error channel in the space-like region. We therefore expect that the resulting central value for $O_8$ should not depend strongly on $O_8$, which is indeed reflected by the numerical results shown in the table. However, increasing $O_8$, i.e., opening the error channel, leads to larger uncertainties for $O_8$. The fact that we observe independence of $\alpha_s(\bar{q}q)^2$ on $s_1$ in the range between $-1.0$ and $-0.5$ GeV$^2$ shows that there is a negligible contribution to $\chi^2_L$ for $|s_1| < 1$ GeV$^2$. This means that increasing the error channel in this region of small $|s_1|$ even further, for example by using a combination of $O_8$ and $O_{10}$, would not change anything.
In the numerical evaluation we have used the NLO expression for $\alpha_s(s)$ with $\Lambda_{Nf}^{MS} = 3$ GeV. The result for $\alpha_s(\bar{q}q)^2$ is not sensitive to changing $\Lambda_{Nf}^{MS}$ within the present experimental error $\pm 0.030$ GeV. Moreover, the results for the two different values of the NLO coefficients $c_6$ agree within errors and we could have chosen to determine $\alpha_s(\bar{q}q)^2$ directly without taking into account its $s$-dependent NLO correction. We repeat that we have used the NLO QCD expression of Eq. (10) in order to compare our results with those of other approaches within the framework of QCD. However, our method would work for any $s$-dependent ansatz for $\alpha_s(\bar{q}q)^2$ as well.

The values for $\alpha_s(\bar{q}q)^2$ given in the table translate into values for the condensate $O_{V-A}^6$ according to Eq. (10).

In order to compare with other results from the literature we use $\alpha_s(s) = 0.6$ at the scale $s = 1$ GeV$^2$. For the dimension-8 condensate which we use to parametrize the unknown error corridor, $\sigma_L$, in the space-like QCD expression we take $O_8 = 1.0 \times 10^{-3}$ GeV$^8$. Then we obtain

$$O_{V-A}^6 = \begin{cases} (-0.0020 \pm 0.0014) \text{ GeV}^6 & \text{for } c_6 = \frac{89}{18}, \\ (-0.0015 \pm 0.0009) \text{ GeV}^6 & \text{for } c_6 = \frac{247}{98}. \end{cases}$$

These results can be compared with the lowest-order vacuum saturation expression

$$O_{V-A}^6|_{VS} = -\frac{32\pi}{9} \alpha_s(\bar{q}q)^2 \simeq -0.0013 \text{ GeV}^6,$$

where we used $\langle \bar{q}q \rangle = -0.014$ GeV$^3$. On the other hand, analyses based on finite-energy sum rules [4–8] typically find results

$$O_{V-A}^6 = (-0.004 \pm 0.001) \text{ GeV}^6,$$

which are not inconsistent with our number. The fact that we find agreement within errors is not trivial. We do not mean to imply that our method is, from a practical point of view, superior to the FESR; our numerical answers are actually less accurate. But because we use more general assumptions, our approach gives additional confidence to the numerical results obtained with the help of QCD sum rules.

Acknowledgement

We are grateful to A. Höcker for providing us with the ALEPH data and for helpful discussions.

References

[1] R. Barate, et al., ALEPH Collaboration, Z. Phys. C 76 (1997) 15;
R. Barate, et al., ALEPH Collaboration, Eur. Phys. J. C 4 (1998) 409.
[2] K. Ackerstaff, et al., OPAL Collaboration, Eur. Phys. J. C 7 (1999) 571.
[3] For a recent review, see R. Gupta, hep-ph/0311033.
[4] M. Davier, L. Girlanda, A. Höcker, J. Stern, Phys. Rev. D 58 (1998) 096014.
[5] J. Bijnens, E. Gamiz, J. Prades, JHEP 0110 (2001) 009.
[6] B.L. Ioffe, K.N. Zyablyuk, Nucl. Phys. A 687 (2001) 437.
[7] C.A. Dominguez, K. Schilcher, hep-ph/0309285.
[8] V. Cirigliano, E. Golowich, K. Maltman, Phys. Rev. D 68 (2003) 054013.
[9] G. Auberson, G. Mennessier, Commun. Math. Phys. 121 (1989) 49.
[10] G. Auberson, M.B. Causse, G. Mennessier, in: S. Ciulli, F. Scheck, W. Thirring (Eds.), Rigorous Methods in Particle Physics, in: Springer Tracts in Modern Physics, vol. 119, Springer, New York, 1990;
M.B. Causse, G. Mennessier, Z. Phys. C 47 (1990) 611.
[11] G. Launer, Z. Phys. C 32 (1986) 557.
[12] W. Marciano, A. Sirlin, Phys. Rev. Lett. 61 (1988) 1815.
[13] V. Cirigliano, J.F. Donoghue, E. Golowich, K. Maltman, Phys. Lett. B 522 (2001) 245;
        V. Cirigliano, J.F. Donoghue, E. Golowich, K. Maltman, Phys. Lett. B 555 (2003) 71.
[14] K.G. Chetyrkin, V.P. Spiridonov, S.G. Gorishnii, Phys. Lett. B 160 (1985) 149;
        L.V. Lanin, V.P. Spiridonov, K.G. Chetyrkin, Yad. Fiz. 44 (1986) 1372.
[15] L.-E. Adam, K.G. Chetyrkin, Phys. Lett. B 329 (1994) 129.