Elements of (super-)Hamiltonian formalism
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Abstract
In these lectures we discuss some basic aspects of Hamiltonian formalism, which usually do not appear in standard textbooks on classical mechanics for physicists. We pay special attention to the procedure of Hamiltonian reduction illustrating it by the examples related to Hopf maps. Then we briefly discuss the supergeneralisation(s) of the Hamiltonian formalism and present some simple models of supersymmetric mechanics on Kähler manifolds.

Introduction
The goal of these lectures is to convince the reader to construct the supersymmetric mechanics within the Hamiltonian framework, or, at least, to combine the superfield approach with the existing methods of Hamiltonian mechanics. The standard approach to construct the supersymmetric mechanics with more than two supercharges is the Lagrangian superfield approach. Surely, superfield formalism is a quite powerful method for the construction of supersymmetric theories. However, all superfield formalisms, being developed a priori for field theory, are convenient for the construction of the field-theoretical models, which are covariant with respect to space-time coordinate transformations. However, the supermultiplets (i.e. the basic ingredients of superfield formalisms) do not respect the transformations mixing field variables. On the other hand, in supersymmetric mechanics these variables appear as spatial coordinates. In other words, the superfield approach, being applied to supersymmetric mechanics, provides us with a local construction of mechanical models. Moreover, the obtained models need to be re-formulated in the Hamiltonian framework, for the subsequent quantization. In addition, many of the numerous methods and statements in the Hamiltonian formalism could be easily extended to supersymmetric systems and applied there. Independently from the specific preferences, the “Hamiltonian view” of the existing models of supersymmetric mechanics, which were built within the superfield approach, could establish unexpected links between different supermultiplets and models. Finally, the superfield methods seem to be too general in the context of simple mechanical systems.

For this reason, we tried to present some elements of Hamiltonian formalism, which do not usually appear in the standard textbooks on classical mechanics, but appear to be useful in the context of supersymmetric mechanics. We pay much attention to the procedure of Hamiltonian reduction, having in mind that it could be used for the construction of the lower-dimensional supersymmetric models from the existing higher-dimensional ones. Also, we devote a special attention to the Hopf maps and Kähler spaces, which are typical structures in supersymmetric systems. Indeed, to extend the number of supersymmetries (without extension of the fermionic degrees of freedom) we usually equip the configuration/phase space with complex structures and restrict them to be Kähler, hyper-Kähler, quaternionic and so on, often via a choice of the appropriate supermultiplets related to the real, complex, quaternionic structures. We illustrated these matters by examples of Hamiltonian reductions related with Hopf maps, having in mind that they could be straightforwardly applied to supersymmetric systems. Also, we included some less known material related with Hopf fibrations. It concerns the generalization of the oscillator to spheres, complex projective spaces, and quaternionic projective spaces, as well as the reduction of the oscillator systems to Coulomb ones.

Most of the presented constructions are developed only for the zero and first Hopf maps. We tried to present them in the way, which will clearly show, how to extend them to the second Hopf map and the quaternionic case.

The last two sections are devoted to the super-Hamiltonian formalism. We present the superextensions of the Hamiltonian constructions, underlying the specific “super”-properties, and present some examples. Then we provide the list of supersymmetric mechanics constructed within the Hamiltonian approach. Also in this case, we tried to arrange the material in such a way, as to make clear the relation of these constructions to complex structures and their possible extension to quaternionic ones.

The main references to the generic facts about Hamiltonian mechanics are the excellent textbooks [1, 2], and on the supergeometry there exist the monographs [3, 4]. There are numerous reviews on supersymmetric mechanics. In our opinion the best introduction to the subject is given in refs. [5, 6].
1 Hamiltonian formalism

In this Section we present some basic facts about the Hamiltonian formalism, which could be straightforwardly extended to the super-Hamiltonian systems.

We restrict ourselves to considering Hamiltonian systems with nondegenerate Poisson brackets. These brackets are defined, locally, by the expressions

\[ \{ f, g \} = \frac{\partial f}{\partial x^i} \omega^{ij} (x) \frac{\partial g}{\partial x^j}, \quad \det \omega^{ij} \neq 0, \]  

(1.1)

where

\[ \{ f, g \} = -\{ g, f \}, \quad \Leftrightarrow \quad \omega^{ij} = -\omega^{ji} \]  

(1.2)

\[ \{ \{ f, g \}, h \} + \text{cycl.perm}(f, g, h) = 0, \quad \Leftrightarrow \quad \omega^{ij}_{\alpha} \omega^{\alpha k} + \text{cycl.perm}(i, j, k) = 0. \]  

(1.3)

The Eq. (1.2) is known as a “antisymmetricity condition”, and the Eq. (1.3) is called Jacobi identity. Owing to the nondegeneracy of the matrix \( \omega^{ij} \), one can construct the nondegenerate two-form, which is closed due to Jacobi identity

\[ \omega = \frac{1}{2} \omega_{ij} dx^i \wedge dx^j : \quad d\omega = 0 \Leftrightarrow \omega_{ij,k} + \text{cycl.perm}(i, j, k) = 0. \]  

(1.4)

The manifold \( M \) equipped with such a form, is called symplectic manifold, and denoted by \( (M, \omega) \). It is clear that \( M \) is an even-dimensional manifold, \( \dim M = 2N \).

The Hamiltonian system is defined by the triple \( (M, \omega, H) \), where \( H(x) \) is a scalar function called Hamiltonian.

The Hamiltonian equations of motion yield the vector field preserving the symplectic form

\[ \frac{dx^i}{dt} = \{ H, x^i \} = V^i_H : \quad \mathcal{L}_V \omega = 0. \]  

(1.5)

Here \( \mathcal{L}_V \) denotes the Lie derivative along vector field \( V \).

Vice versa, any vector field, preserving the symplectic structure, is locally a Hamiltonian one. The easiest way to see it is to use homotopy formula

\[ \imath_V \omega + d\imath_V \omega = \mathcal{L}_V \omega \Rightarrow d\imath_V \omega = 0. \]  

(1.6)

Hence, \( \imath_V \omega \) is a closed one-form and could be locally presented as follows: \( \imath_V \omega = dH(x) \). The local function \( H(x) \) is precisely the Hamiltonian, generating the vector field \( V \). The transformations preserving the symplectic structure are called canonical transformations.

Any symplectic structure could be locally presented in the form (Darboux theorem)

\[ \omega_{\text{can}} = \sum_{a=1}^{N} dp_a \wedge dq^a, \]  

(1.7)

where \( (p_a, q^a) \) are the local coordinates of the symplectic manifold.

The vector field \( V \) defines a symmetry of the Hamiltonian system, if it preserves both the Hamiltonian \( H \) and the symplectic form \( \omega \): \( \mathcal{L}_V \omega = 0, \mathcal{L}_V H = 0 \). Hence,

\[ V = \{ \mathcal{J}, \_ \}, \quad \{ \mathcal{J}, H \} = 0. \]  

(1.8)

The 2N-dimensional Hamiltonian system is called an integrable system, when it has \( N \) functionally independent constants of motion being in involution (Liouville theorem),

\[ \{ \mathcal{J}_a, \mathcal{J}_b \} = 0, \quad \{ H, \mathcal{J}_b \} = 0, \quad H = \mathcal{J}_1, \quad a, b = 1, \ldots, N. \]  

(1.9)

When the constants of motion are noncommutative, the integrability of the system needs more than \( N \) constants of motion. If

\[ \{ \mathcal{J}_\mu, \mathcal{J}_\nu \} = f_{\mu\nu}(\mathcal{J}), \quad \text{corank} f_{\mu\nu} = K_0, \quad \mu, \nu = 1, \ldots, K \geq K_0, \]  

(1.10)

then the system is integrable, if \( 2N = K + K_0 \). The system with \( K + K_0 \geq 2N \) constants of motion is sometimes called a superintegrable system.
The cotangent bundle $T^*M_0$ of any manifold $M_0$ (parameterized by local coordinates $q^i$) could be equipped with the canonical symplectic structure $(1.7)$.

The dynamics of a free particle moving on $M_0$ is given by the Hamiltonian system

$$\left(T^*M_0, \omega_{can}, \mathcal{H}_0 = \frac{1}{2}g^{ab}(q)p_ap_b\right), \quad (1.11)$$

where $g^{ab}g_{bc} = \delta^a_c$, and $g_{ab}dq^a dq^b$ is a metric on $M_0$.

The interaction with a potential field could be incorporated in this system by the appropriate change of Hamiltonian,

$$\mathcal{H}_0 \rightarrow \mathcal{H} = \frac{1}{2}g^{ab}(q)p_ap_b + U(q), \quad (1.12)$$

where $U(q)$ is a scalar function called potential. Hence, the corresponding Hamiltonian system is given by the triplet $(T^*M_0, \omega_{can}, \mathcal{H})$.

In contrast to the potential field, the interaction with a magnetic field requires a change of symplectic structure. Instead of the canonical symplectic structure $\omega_{can}$, we have to choose

$$\omega_F = \omega_{can} + F, \quad F = \frac{1}{2}F_{ab}(q) dq^a \wedge dq^b, \quad dF = 0 \quad (1.13)$$

where $F_{ab}$ are components of the magnetic field strength.

Hence, the resulting system is given by the triplet $(T^*M_0, \omega_F, \mathcal{H})$. Indeed, taking into account that the two-form $F$ is locally exact, $F = dA$, $A = A_a(q) dq^a$, we could pass to the canonical coordinates $(\pi_a = p_a + A_a, q^a)$. In these coordinates the Hamiltonian system assumes the conventional form

$$\left(T^*M_0, \omega_{can} = d\pi_a \wedge dq^a, \mathcal{H} = \frac{1}{2}g^{ab}(\pi_a - A_a)(\pi_b - A_b) + U(q)\right).$$

Let us also remind, that in the three-dimensional case the magnetic field could be identified with vector, whereas in the two-dimensional case it could be identified with (pseudo)scalar.

The generic Hamiltonian system could be described by the following (phase space) action

$$S = \int dt \left(\mathcal{A}_i(x) \dot{x}^i - \mathcal{H}(x)\right), \quad (1.14)$$

where $\mathcal{A} = \mathcal{A}_i dx^i$ is a symplectic one-form: $d\mathcal{A} = \omega$. Indeed, varying the action, we get the equations

$$\delta S = 0, \quad \Rightarrow \quad \dot{x}^i \omega_{ij}(x) = \frac{\partial \mathcal{H}}{\partial x^j}, \quad \omega_{ij} = \frac{\partial A_i}{\partial x^j} - \frac{\partial A_j}{\partial x^i}. \quad (1.15)$$

Though $\mathcal{A}$ is defined up to closed (locally exact) one-form, $\mathcal{A} \rightarrow \mathcal{A} + df(x)$, this arbitrariness has no impact in the equations of motion. It change the Lagrangian on the total derivative $f, \dot{x}^i = df(x)/dt$.

As an example, let us consider the particle in a magnetic field. The symplectic one-form corresponding to the symplectic structure $(1.13)$, could be chosen in the form $\mathcal{A} = (p_a + A_a) dq^a$, $d\mathcal{A} = \omega_F$. Hence, the action (1.14) reads

$$S = \int dt \left((p_a + A_a)q^a - \frac{1}{2}g^{ab}(q)p_ap_b + U(q)\right). \quad (1.16)$$

Varying this action by $p$, we get, on the extrema, the conventional second-order action for the system in a magnetic field

$$S_0 = \int dt \left(\frac{1}{2}g_{ab}\dot{q}^a\dot{q}^b + A_a\dot{q}^a - U(q)\right). \quad (1.17)$$

The presented manipulations are nothing but the Legendre transformation from the Hamiltonian formalism to the Lagrangian one.
Particle in the Dirac monopole field

Let us consider the special case of a system on three-dimensional space moving in the magnetic field of a Dirac monopole. Its symplectic structure is given by the expression

$$\omega_D = dp_a \wedge dq^a + s \frac{q^a}{|q|^3} \epsilon_{abc} dq^b \wedge dq^c. \quad (1.18)$$

The corresponding Poisson brackets are given by the relations

$$\{p_a, q^b\} = \delta_a^b, \quad \{q^a, q^b\} = 0, \quad \{p_a, p_b\} = s \epsilon_{abc} \frac{q^c}{|q|^3}. \quad (1.19)$$

It is clear that the monopole field does not break the rotational invariance of the system. The vector fields generating $SO(3)$ rotations are given by the expressions

$$V_a = \epsilon_{abc} q^b \frac{\partial}{\partial q^c} - \epsilon_{abc} p_b \frac{\partial}{\partial p_c}, \quad \{V_a, V_b\} = \epsilon_{abc} V_c. \quad (1.20)$$

The corresponding Hamiltonian generators could be easily found as well

$$\iota_{V_a} \omega_D = dJ_a, \quad \{J_a, J_b\} = \epsilon_{abc} J_c, \quad \{J_0, H\} = 0, \quad (1.21)$$

where

$$J_a = \epsilon_{abc} q^b p_c + s \frac{q^a}{|q|}, \quad J_a q^a = s|q|. \quad (1.22)$$

Now, let us consider the system given by the symplectic structure (1.18), and by the $so(3)$-invariant Hamiltonian

$$H = \frac{p_a p_a}{2g} + U(|q|), \quad \{J_a, H\} = 0, \quad (1.23)$$

where $g(|q|) dq^a dq^a$ is $so(3)$-invariant metric on $M_0$. In order to find the trajectories of the system, it is convenient to direct the $q^3$ axis along the vector $J = (J_1, J_2, J_3)$, i.e. to assume that $J = J_3 \equiv J$. Upon this choice of the coordinate system one has

$$\frac{q^3}{|q|} = \frac{s}{J}. \quad (1.24)$$

Then, we introduce the angle

$$\phi = \arctan \frac{q^1}{q^2}, \quad \frac{d\phi}{dt} = \frac{J^2 - s^2}{J g |q|^2}, \quad (1.25)$$

and get, after obvious manipulations

$$E = \frac{J^2 - s^2}{2g |q|^2} + \left(\frac{J^2 - s^2}{2J g |q|^2}\right)^2 \left(\frac{d|q|}{d\phi}\right)^2 + U(|q|). \quad (1.26)$$

Here $E$ denotes the energy of the system.

From the expression (1.26) we find,

$$\phi = \left(\frac{J - s^2}{J}\right) \int \frac{d|q|}{\sqrt{2g |q|^2 (E - U) - J^2 + s^2}}. \quad (1.27)$$

It is seen that, upon the replacement

$$U(q) \rightarrow U(q) + \frac{s^2}{2g |q|^2}, \quad (1.28)$$

we shall eliminate in (1.27) the dependence on $s$, i.e. on a monopole field. The only impact of the monopole field on the trajectory will be the shift of the orbital plane given by (1.24).

Let us summarize our considerations. Let us consider the $so(3)$-invariant three-dimensional system

$$\omega_{can} = dp \wedge dq, \quad H = \frac{p^2}{2g} + U(|q|), \quad \{J_0, H\} = 0, \quad J_0 = p \times q. \quad (1.29)$$
The Poisson brackets associated with this symplectic structure read
\[ \{f, g\}_0 = i \frac{\partial f}{\partial z^a} g^{ab} \frac{\partial g}{\partial \bar{z}^b} - i \frac{\partial g}{\partial z^a} g^{ab} \frac{\partial f}{\partial \bar{z}^b}, \]
where \( g^{ab} g_{bc} = \delta^a_c. \)

From the closeness of (1.31) it immediately follows, that the Kähler metric can be locally represented in the form
\[ g_{ab} dz^a d\bar{z}^b = \frac{\partial^2 K}{\partial z^a \partial \bar{z}^b} dz^a d\bar{z}^b, \]
where \( K(z, \bar{z}) \) is some real function called the Kähler potential. The Kähler potential is defined modulo holomorphic and antiholomorphic functions
\[ K(z, \bar{z}) \rightarrow K(z, \bar{z}) + U(z) + \bar{U}(\bar{z}). \]

The local expressions for the differential-geometric objects on Kähler manifolds are also very simple. For example, the non-zero components of the metric connections (Christoffel symbols) look as follows:
\[ \Gamma^a_{bc} = g^{na} g_{b\bar{n}c}, \quad \Gamma^{\bar{a}}_{bc} = \Gamma^{\bar{a}}_{bc}, \]
while the non-zero components of the curvature tensor read
\[ R^a_{bcd} = -(\Gamma^a_{bc})_d, \quad R^{\bar{a}}_{bcd} = R^{\bar{a}}_{bcd}. \]

The isometries of Kähler manifolds are given by the holomorphic Hamiltonian vector fields
\[ V_{\mu} = V^a_{\mu} (z) \frac{\partial}{\partial z^a} + \bar{V}^a_{\bar{\mu}} (\bar{z}) \frac{\partial}{\partial \bar{z}^a}, \quad V_{\mu} = \{ h_{\mu}, \}_0, \]
where \( h_{\mu} \) is a real function, \( h_{\mu} = \bar{h}_{\bar{\nu}} \), called Killing potential. One has
\[ [V_{\mu}, V_{\nu}] = C^\lambda_{\mu\nu} V_\lambda, \quad \{ h_{\mu}, h_{\nu} \}_0 = C^\lambda_{\mu\nu} h_\lambda + \text{const}, \]
and
\[ \frac{\partial h_{\mu}}{\partial z^a} - \Gamma^a_{bc} \frac{\partial h_{\mu}}{\partial \bar{z}^c} = 0. \]

The dynamics of a particle moving on the Kähler manifold in the presence of a constant magnetic field is described by the Hamiltonian system
\[ \Omega_B = dz^a \wedge d\pi_a + d\bar{z}^a \wedge d\bar{\pi}_a + i B g_{ab} dz^a \wedge d\bar{z}^b, \quad \mathcal{H}_0 = g^{ab} \pi_a \bar{\pi}_b \]
for the Kähler structure define the Noether constants of motion
\[ \mathcal{J}_\mu = J_\mu + B h_\mu = V^a_{\mu} \pi_a + \bar{V}^a_{\bar{\mu}} \bar{\pi}_a + B h_\mu : \begin{cases} \{ \mathcal{H}_0, J_\mu \} = 0, \\ \{ J_\mu, J_{\nu} \} = C^\lambda_{\mu\nu} J_\lambda. \end{cases} \]
One can easily check that the vector fields generated by \( J_\mu \) are independent of \( B \)
\[ V = V^a(z) \frac{\partial}{\partial z^a} - \bar{V}^a_{\bar{\mu}} \frac{\partial}{\partial \bar{\pi}_a} + \bar{V}^a(\bar{z}) \frac{\partial}{\partial \bar{z}^a} - \bar{V}^a_{\bar{\mu}} \bar{\pi}_a \frac{\partial}{\partial \bar{\pi}_a}. \]
Hence, the inclusion of a constant magnetic field preserves the whole symmetry algebra of a free particle moving on a Kähler manifold.
Complex projective space

The most known nontrivial example of a Kähler manifold is the complex projective space $\mathbb{CP}^N$. It is defined as a space of complex lines in $\mathbb{C}^{N+1}$: $u^a \sim \lambda u^a$, where $u^a$, $\lambda \in \mathbb{C}$, $a = 0, \ldots, N$ are the Euclidean coordinates of $\mathbb{C}^{N+1}$, and $\lambda \in \mathbb{C} - \{0\}$. Equivalently, the complex projective space is the coset space $\mathbb{CP}^N = SU(N + 1)/U(N)$.

The complex projective space $\mathbb{CP}^N$ could be covered by $N + 1$ charts marked by the indices $\hat{a} = 0, a$. The zero chart could be parameterized by the functions (coordinates) $z^\hat{a}_0 = u^\hat{a}/u^0$, $a = 1, \ldots N$; the first chart by $z^{\hat{a}}_1 = z^\hat{a}/z^1$, $a = 0, 2, 3, \ldots, N$, and so on.

Hence, the transition function from the $\hat{b}$-th chart to the $\hat{c}$-th one has the form

$$z^{\hat{c}}_a = \frac{z^{\hat{b}}_\hat{a}}{z^{\hat{b}}_\hat{c}}, \quad \text{where} \quad z^{\hat{c}}_a = 1.$$  

One can equip the $\mathbb{CP}^N$ by the Kähler metric, which is known under the name of Fubini-Study metric

$$g_{\hat{a}\hat{b}}dz^a dz^b = \frac{dz d\bar{z}}{1 + z\bar{z}} - \frac{(z d\bar{z})(z d\bar{z})}{(1 + z\bar{z})}. \quad (1.42)$$

Its Kähler potential is given by the expression

$$K = \log(1 + z\bar{z}). \quad (1.43)$$

Indeed, it is seen that upon transformation from one chart to the other, given by (1.41), this potential changes by holomorphic and anti-holomorphic functions, i.e. the Fubini-Study metric is globally defined on $\mathbb{CP}^N$.

The Poisson brackets on $\mathbb{CP}^N$ are defined by the following relations:

$$\{ z^a, z^b \} = (1 + z\bar{z})(\delta^{ab} + z^a z^b), \quad \{ z^a, \bar{z}^b \} = \{ \bar{z}^a, z^b \} = 0.$$  

It is easy to see that $\mathbb{CP}^N$ is a constant curvature space, with the symmetry algebra $su(N + 1)$. This algebra is defined by the Killing potentials

$$h_{ab} = \frac{z^a z^b - N \delta_{ab}}{1 + z\bar{z}}, \quad h^a_a = \frac{z^a}{1 + z\bar{z}}, \quad h^\bar{a}_a = \frac{\bar{z}^a}{1 + z\bar{z}}. \quad (1.45)$$

The manifold $\mathbb{CP}^1$ (complex projective plane) is isomorphic to the two-dimensional sphere $S^2$. Indeed, it is covered by the two charts, with the transition function $z \rightarrow 1/z$. The symmetry algebra of $\mathbb{CP}^1$ is $su(2) = so(3)$

$$\{ x^i, x^j \} = \epsilon^{ijk} x^k, \quad i, j, k = 1, 2, 3 \quad (1.46)$$

where the Killing potentials $x^i$ look as follows:

$$x^1 + ix^2 = \frac{2z}{1 + z\bar{z}}, \quad x^3 = \frac{1 - z\bar{z}}{1 + z\bar{z}} \quad (1.47)$$

It is seen that these Killing potentials satisfy the condition

$$x^i x^i = 1,$$

i.e. $x^i$ defines the sphere $S^2$ in the three-dimensional ambient space $\mathbb{R}^3$. It is straightforwardly checked that $z$ are the coordinates of the sphere in the stereographic projection on $\mathbb{R}^2 = \mathbb{C}$. The real part of the Fubini-Study structure gives the linear element of $S^2$, and the imaginary part coincides with the volume element of $S^2$.

On the other hand, these expressions give the embedding of the $S^2$ in $S^3$ (with ambient coordinates $u^1, u^2$) defining the so-called first Hopf map $S^3/S^1 = S^2$. Below we shall describe this map in more detail.

**Hopf maps**

The Hopf maps (or Hopf fibrations) are the fibrations of the sphere over a sphere,

$$S^{2p-1}/S^{p-1} = S^p, \quad p = 1, 2, 4, 8.$$  

(1.48)
These fibrations reflect the existence of real \((p = 1)\), complex \((p = 2)\), quaternionic \((p = 4)\) and octonionic \((p = 8)\) numbers.

We are interested in the so-called zero-th, first and second Hopf maps:
\[
\begin{align*}
S^1/S^0 &= S^1 \quad \text{(zero Hopf map)} \\
S^3/S^1 &= S^2 \quad \text{(first Hopf map)} \\
S^7/S^3 &= S^4 \quad \text{(second Hopf map)}
\end{align*}
\]  
(1.49)

Let us describe the Hopf maps in explicit terms. For this purpose, we consider the functions \(x(u, \bar{u}), x_0(u, \bar{u})\)
\[
x = 2u_1\bar{u}_2, \quad x_{p+1} = u_1\bar{u}_1 - u_2\bar{u}_2,
\]  
(1.50)

where \(u_1, u_2\), could be real, complex or quaternionic numbers. So, one can consider them as a coordinates of the \(2p\)-dimensional space \(\mathbb{R}^{2p}\), where \(p = 1\) when \(u_{1,2}\) are real numbers; \(p = 2\) when \(u_{1,2}\) are complex numbers; \(p = 4\) when \(u_{1,2}\) are quaternionic numbers; \(p = 8\) when \(u_{1,2}\) are octonionic ones.

In all cases \(x_{p+1}\) is a real number, while \(x\) is, respectively, a real number \((p = 1)\), complex number \((p = 2)\), quaternion\((p = 4)\), or octonion \((p = 8)\). Hence, \((x_0, x)\) parameterize the \((p + 1)\)-dimensional space \(\mathbb{R}^{p+1}\).

The functions \(x, x_{p+1}\) remain invariant under transformations
\[
u_a \rightarrow g u_a, \quad \text{where} \quad \bar{g}g = 1.
\]  
(1.51)

Hence
\[
g = \pm 1 \quad \text{for} \quad p = 1
\]  
(1.52)
\[
g = \lambda_1 + i\lambda_2, \quad \lambda_1^2 + \lambda_2^2 = 1 \quad \text{for} \quad p = 2
\]  
(1.53)
\[
g = \lambda_1 + i\lambda_2 + j\lambda_3 + k\lambda_4, \quad \lambda_1^2 + \ldots + \lambda_4^2 = 1 \quad \text{for} \quad p = 4
\]  
(1.54)

and similarly for the octonionic case \(p = 8\).

So, \(g\) parameterizes the spheres \(S^{p-1}\) of unit radius. Notice that \(S^1, S^3, S^7\) are the only parallelizable spheres. We shall also use the following isomorphisms between these spheres and groups: \(S^0 = \mathbb{Z}_2, S^1 = U(1), S^3 = SU(2)\).

We get that (1.50) defines the fibrations
\[
\mathbb{R}^2/S^0 = \mathbb{R}^2, \quad \mathbb{R}^4/S^1 = \mathbb{R}^3, \quad \mathbb{R}^8/S^3 = \mathbb{R}^5, \quad \mathbb{R}^{16}/S^7 = \mathbb{R}^9.
\]  
(1.55)

One could immediately check that the following equation holds:
\[
xx + x_{p+1}^2 = (u_1\bar{u}_1 + u_2\bar{u}_2)^2.
\]  
(1.56)

Thus, defining the \((2p - 1)\)-dimensional sphere in \(\mathbb{R}^{2p}\) of the radius \(r_0\): \(u_a\bar{u}_a = r_0^2\), we will get the \(p\)-dimensional sphere in \(\mathbb{R}^{p+1}\) with radius \(R_0 = r_0^2\)
\[
1 u_1\bar{u}_1 + u_2\bar{u}_2 = r_0^2 \quad \Rightarrow \quad xx + x_0^2 = r_0^4.
\]  
(1.57)

So, we arrive at the Hopf maps given by (1.49). The last, fourth Hopf map, \(S^{15}/S^7 = S^8\), corresponding to \(p = 8\), is related to octonions in the same manner.

For our purposes it is convenient to describe the the expressions (1.50) in a less unified way. For the zero Hopf map it is convenient to consider the initial and resulting ambient space \(\mathbb{R}^2\) as complex spaces \(\mathbb{C}\), parameterized by the single complex coordinates \(w\) and \(z\). In this case the map (1.50) could be represented in the form
\[
w = z^2,
\]  
(1.58)

which is known as a Bohlin (or Levi-Civita) transformation relating the Kepler problem with the circular oscillator.

For the first and second Hopf maps it is convenient to represent the transformation (1.50) in the following form:
\[
x = u\gamma\bar{u}.
\]  
(1.59)

Here, for the first Hopf map \(x = (x^1, x^2, x^3)\) parameterizes \(\mathbb{R}^3\), and \(u_1, u_2\) parameterize \(\mathbb{C}^2\), and \(\gamma = (\sigma^1, \sigma^2, \sigma^3)\) are Pauli matrices. This transformation is also known under the name of Kustaanheimo-Stiefel transformation. For the second Hopf map \(x = (x^1, \ldots, x^5)\) parameterizes \(\mathbb{R}^5\), and \(u_1, \ldots, u_4\) parameterize \(\mathbb{C}^4 = \mathbb{H}^2\), and \(\gamma = (\gamma^1, \ldots, \gamma^4)\), \(\gamma^5 = \gamma^1\gamma^2\gamma^3\gamma^4\), where \(\gamma^1, \ldots, \gamma^4\) are Euclidean four-dimensional gamma-matrices. The latter transformation is sometimes called Hurwitz transformation, or “generalized Kustaanheimo-Stiefel” transformation.
2 Hamiltonian reduction

A Hamiltonian system which has a constant(s) of motion, can be reduced to a lower-dimensional one. The corresponding procedure is called Hamiltonian reduction. Let us explain the meaning of this procedure in the simplest case of the Hamiltonian reduction by a single constant of motion.

Let \((\omega, H)\) be a given 2\(N\)-dimensional Hamiltonian system, with the phase space (local) coordinates \(x^A\), and let \(\mathcal{J}\) be its constant of motion, \(\{H, \mathcal{J}\} = 0\). We go from the local coordinates \(x^A\) to another set of coordinates, \((\tilde{H}, y^i, u)\), where \(y^i = y^i(x)\) are 2\(N \to 2\) independent functions, which commute with \(\mathcal{J}\),

\[
\{y^i, \mathcal{J}\} = 0, \quad i = 1, \ldots, 2N - 2.
\]  

(2.60)

In this case the latter coordinate, \(u = u(x)\), necessarily has a non-zero Poisson bracket with \(\mathcal{J}\) (because the Poisson brackets are nondegenerate):

\[
\{u(x), \mathcal{J}\} \neq 0.
\]  

(2.61)

Then, we immediately get that in these coordinates the Hamiltonian is independent of \(u\)

\[
\{\mathcal{J}(H, y, u), H\} = \frac{\partial H}{\partial u} \cdot \{u, \mathcal{J}\} \neq 0, \Rightarrow H = H(\mathcal{J}, y).
\]  

(2.62)

On the other hand, from the Jacobi identity we get

\[
\{\{y^i, y^j\}, \mathcal{J}\} = \frac{\partial^2 \{y^i, y^j\}}{\partial u} \{u, \mathcal{J}\} = 0 \Rightarrow \{y^i, y^j\} = \omega^{ij}(y, \mathcal{J}).
\]  

(2.63)

Since \(\mathcal{J}\) is a constant of motion, we can fix its value

\[
\mathcal{J} = c,
\]  

(2.64)

and describe the system in terms of the local coordinates \(y^i\) only

\[
(\omega(x), H(x)) \rightarrow (\omega_{\text{red}}(y, c) = \omega_{ij}(y, c)dy^i \wedge dy^j, H_{\text{red}} = H(y, c)).
\]  

(2.65)

Hence, we reduced the initial 2\(N\)-dimensional Hamiltonian system to a \((2N - 2)\)-dimensional one.

Geometrically, the Hamiltonian reduction by \(\mathcal{J}\) means that we fix the \((2N - 1)\)-dimensional level surface \(M_c\) by the Eq.(2.64), and then factorize it by the action of a vector field \(\{\mathcal{J}, \}\), which is tangent to \(M_c\). The resulting space \(\mathcal{M}_0 = M_c/\{\mathcal{J}, \}\) is a phase space of the reduced system.

The Hamiltonian reduction by the \(K\) commuting constants of motion \(\mathcal{J}, \{\mathcal{J}_\alpha, \mathcal{J}_\beta\} = 0\) is completely similar to the above procedure. It reduces the \(2N\) dimensional Hamiltonian system to a \(2(N - K)\) dimensional one.

When the constants of motion do not commute with each other, the reduction procedure is a bit more complicated.

Let the initial Hamiltonian system have \(K\) constants of motion,

\[
\{\mathcal{J}_\alpha, H\} = 0, \quad \{\mathcal{J}_\alpha, \mathcal{J}_\beta\} = \omega_{\alpha\beta}(\mathcal{J}), \quad \text{corank} \ \omega_{\alpha\beta}|_{\mathcal{J}_\alpha = c_\alpha} = K_0.
\]  

(2.66)

Hence, one could choose the \(K_0\) functions, which commute with the whole set of the constants of motion

\[
\tilde{\mathcal{J}}_\tilde{\alpha}(\mathcal{J}) : \{\tilde{\mathcal{J}}_\tilde{\alpha}, \mathcal{J}_\beta\}|_{\mathcal{J}_\alpha = c_\alpha} = 0, \quad \tilde{\alpha} = 1, \ldots K_0.
\]  

(2.67)

The vector fields \(\{\tilde{\mathcal{J}}_\tilde{\alpha}, \}\) are tangent to the level surface

\[
M_c : \mathcal{J}_\alpha = c_\alpha \quad \text{dim}M_c = 2N - K.
\]  

(2.68)

Factorizing \(M_c\) by the action of the commuting vector fields \(\{\tilde{\mathcal{J}}_\tilde{\alpha}, \}\), we arrive at the phase space of the reduced system, \(\mathcal{M}_0 = M_c/\{\mathcal{J}, \}\), whose dimension is given by the expression

\[
\text{dim}\mathcal{M}_0 = 2N - K - K_0.
\]  

(2.69)

In contrast to the commuting case, the reduced system could depend on the parameters \(c_\alpha\) only.

Notice that the Hamiltonian system could also possess a discrete symmetry. In this case the reduced system has the same dimension as the previous one. To be more precise, the reduction by the discrete symmetry group could be described by a local canonical transformation. However, the quantum mechanical counterpart of this canonical transformation could yield a system with non-trivial physical properties.

Below, we shall illustrate the procedure of (Hamiltonian) reduction by discrete, commutative, and noncommutative symmetry generators on examples related to Hopf maps.
Zero Hopf map. Magnetic flux tube

The transformation of the Hamiltonian system associated with the zero Hopf map corresponds to the reduction of the system by the discrete group $Z_2$. It is a (local) canonical transformation. As a consequence, the resulting system has the same dimension as the initial one.

Let us consider the Hamiltonian system with four-dimensional phase space, parameterized by the pair of canonically conjugated complex coordinates, $(\omega = d\pi \wedge dz + d\bar{\pi} \wedge d\bar{z}, H)$, which is invariant under the following action of $Z_2$ group:

$$H(z, \bar{z}, \pi, \bar{\pi}) = H(-z, -\bar{z}, -\pi, -\bar{\pi}), \quad \omega(\pi, \bar{\pi}, z, \bar{z}) = \omega(-\pi, -\bar{\pi}, -z, -\bar{z}).$$

We can pass now to the coordinates, which are invariant under this transformation (clearly, it is associated with the zero Hopf map)

$$w = z^2, \quad p = \pi/2z$$
$$\omega = d\pi \wedge dz + d\bar{\pi} \wedge d\bar{z} = dp \wedge dw + d\bar{p} \wedge d\bar{w}.$$  \hspace{1cm} (2.70)

(2.71)

However, one can see that the angular momentum of the initial systems looks as a doubled angular momentum of the transformed one

$$J = i(z\pi - \bar{z}\bar{\pi}) = 2i(wp - \bar{w}\bar{p}).$$  \hspace{1cm} (2.72)

This indicates that the global properties of these two systems could be essentially different. This difference has to be reflected in the respective quantum-mechanical systems.

Let us consider the Schrödinger equation

$$H(\pi, \bar{\pi}, z, \bar{z})\Psi(z, \bar{z}) = E\Psi(z, \bar{z}), \quad \pi = -i\partial_z, \quad \bar{\pi} = -i\partial_{\bar{z}},$$

(2.73)

with the wavefunction which obeys the condition

$$\Psi(|z|, \arg z + 2\pi) = \Psi(|z|, \arg z).$$  \hspace{1cm} (2.74)

Let us reduce it by the action of $Z_2$ group, restricting ourselves to even ($\sigma = 0$) or odd ($\sigma = \frac{1}{2}$) solutions of Eq. (2.73)

$$\Psi_\sigma(z, \bar{z}) = \psi_\sigma(z^2, \bar{z}^2)e^{2i\sigma \arg z}, \quad \sigma = 0, 1/2,$$

(2.75)

and then perform the Bohlin transformation (2.70). According to Eq.(2.75), the wave functions $\psi_\sigma$ satisfy the condition

$$\psi_\sigma(|w|, \arg w + 2\pi) = \psi_\sigma(|w|, \arg w),$$  \hspace{1cm} (2.76)

which implies that the range of definition $\arg w \in [0, 4\pi)$ can be restricted, without loss of generality, to $\arg w \in [0, 2\pi)$. In terms of $\psi_\sigma$ the Schrödinger equation (2.73) reads

$$H(\hat{p}_\sigma, \hat{p}^+_\sigma, w, \bar{w})\psi_\sigma(w, \bar{w}) = E\psi_\sigma, \quad \hat{p}_\sigma = -i\partial_w - \frac{i\sigma}{w}.$$  \hspace{1cm} (2.77)

Equation (2.77) can be interpreted as the Schrödinger equation of a particle with electric charge $e$ in the static magnetic field given by the potential $A_w = \frac{ie}{\omega w}, \sigma = 0, 1/2$. It is a potential of an infinitely thin solenoid—magnetic flux tube (or magnetic vortex, in the two-dimensional interpretation): it has zero strength of the magnetic field $B = \text{rot}A_w = 0$ ($w \in \mathbb{C}$) and nonzero magnetic flux $2\pi\sigma/e$.

In accordance with (2.72), the angular momentum transforms as follows:

$$J \rightarrow 2J_\sigma, \quad J_\sigma = \frac{i}{\hbar}(w\hat{p}_\sigma - \bar{w}\hat{p}^+_\sigma),$$  \hspace{1cm} (2.78)

where $J_\sigma$ is the angular momentum operator of the reduced system. Hence, the eigenvalues of the angular momenta of the reduced and initial systems, $m_\sigma$ and $M$, are related by the expression $M = 2m_\sigma$, from which it follows that

$$m_\sigma = \pm\sigma, \pm(1 + \sigma), \pm(2 + \sigma), \ldots.$$  \hspace{1cm} (2.79)

Hence, the $Z_2$-reduction related to zero Hopf map transforms the even states of the initial system to the complete basis of the resulting one. The odd states of the initial system yield the wave functions of the resulting system in the presence of magnetic flux generating spin 1/2. Similarly to the above consideration, one can show that the reduction of the two-dimensional system by the $Z_N$ group yields the $N$ systems with the fractional spin $\sigma = 0, 1/N, 2/N, \ldots, (N-1)/N$ (see [9]).
1st Hopf map. Dirac monopole

Now we consider the Hamiltonian reduction by the action of the $U(1)$ group, which is associated with the first Hopf map. It is known under the name of Kustaanheimo-Stiefel transformation.

Let us consider the Hamiltonian system on the four-dimensional Hermitean space $(M_0, g_{ab} dz^a d\zbar^b)$, dim $M_0 = 2$,

$$T^* M_0, \quad \omega = dz^a \wedge d\pi_a + d\zbar^a \wedge d\bar{\pi}_a, \quad \mathcal{H} = g^{ab} \pi_a \bar{\pi}_b + V(z, \zbar).$$  \hfill (2.80)

We define, on the $T^* M_0$ space, the Hamiltonian action of the $U(2)$ group given by the generators

$$J = iz \sigma \pi - i\zbar \bar{\pi} \zbar, \quad J_0 = iz \pi - i\zbar \bar{\pi} \zbar : \quad \{J_0, J_k\} = 0, \quad \{J_k, J_l\} = 2\epsilon_{klm} J_m,$$  \hfill (2.81)

where $\sigma$ are Pauli matrices.

Let us consider the Hamiltonian reduction of the phase space $(T^* M_0, \omega)$ by the (Hamiltonian) action of the $U(1) = S^1$ group given by the generator $J_0$. Since $J_0$ commutes with $J_i$, the latter will generate the Hamiltonian action of the $su(2) = so(3)$ algebra on the reduced space as well.

In order to perform the Hamiltonian reduction, we have to fix the level surface

$$J_0 = 2s,$$  \hfill (2.82)

and then factorize it by the action of the vector field $\{J_0, \}$.

The resulting six-dimensional phase space $T^* M_0^{\text{red}}$ could be parameterized by the following $U(1)$-invariant functions:

$$y = z \sigma \zbar, \quad \pi = \frac{z \sigma \pi + \zbar \bar{\pi} \zbar}{2z \zbar} : \quad \{y, J_0\} = \{\pi, J_0\} = 0.$$  \hfill (2.84)

In these coordinates the reduced symplectic structure and the generators of the angular momentum are given by the expressions (compare with (1.18),(1.22))

$$\Omega_{\text{red}} = d\pi \wedge dy + s \frac{y \cdot (dy \times dy)}{2|y|^3}, \quad J_{\text{red}} = J/2 = \pi \times y + s \frac{y}{|y|}.$$  \hfill (2.85)

Hence, we get the phase space of the Hamiltonian system describing the motion of a nonrelativistic scalar particle in the magnetic field of the Dirac monopole.

Let $M_0$ be a $U(2)$-invariant Kähler space with a metric generated by the Kähler potential $K(z, \zbar)$ [10]

$$g_{ab} = \frac{\partial^2 K(z, \zbar)}{\partial z^a \partial \zbar^b} = a(z, \zbar) \delta_{ab} + a'(z, \zbar) \zbar^a z^b,$$  \hfill (2.86)

where

$$a(y) = \frac{dK(y)}{dy}, \quad a'(y) = \frac{d^2 K(y)}{dy^2}. \hfill (2.87)$$

Let the potential be also $U(2)$-invariant, $V = V(z, \zbar)$, so that $U(2)$ is a symmetry of the Hamiltonian: $\{J_0, \mathcal{H}\} = \{J_i, \mathcal{H}\} = 0$.

Hence, the Hamiltonian could also be restricted to the reduced six-dimensional phase space. The reduced Hamiltonian looks as follows:

$$\mathcal{H}_{\text{red}} = \frac{1}{a} \left[y\pi^2 - b(y\pi)^2\right] + s^2 \frac{1 - by}{ay} + V(y).$$  \hfill (2.88)

where

$$y \equiv |y|, \quad b = \frac{a'(y)}{a + ya'(y)}.$$  \hfill (2.89)

Let us perform the canonical transformation $(y, \pi) \rightarrow (x, p)$ to the conformal-flat metric

$$x = f(y)y, \quad \pi = fp + \frac{df(y)p}{dy}y,$$

where

$$\left(1 + \frac{yf'(y)}{f}\right)^2 = 1 + \frac{ya'(y)}{a} \Rightarrow \left(\frac{d\log x}{dy}\right)^2 = \frac{d\log ya(y)}{ydy}, \quad x < 1.$$
In the new coordinates the Hamiltonian takes the form
\[ H_{\text{red}} = \frac{x^2(y)}{ya(y)}p^2 + \frac{s^2}{y(a + ya'(y))} + V(y(x)). \]

In order to express the \( y, a(y), a'(y) \) via \( x \), it is convenient to introduce the function
\[ \tilde{A}(y) = \int (a + ya'(y))yf(y)dy \]
and consider its Legendre transform \( A(x) \),
\[ A(x) = A(x,y)|_{\frac{\partial A(x,y)}{\partial y}}, \quad A(x,y) = xa(y)y - \tilde{A}(y). \]

Then, we immediately get
\[ \frac{dA(x)}{dx} = a(y)y, \quad x\frac{d^2A}{dx^2} = y\sqrt{a(a + ya'(y))}. \]

By the use of these expressions, we can represent the reduced Hamiltonian as follows:
\[ H_{\text{red}} = \frac{x^2}{N^2}p^2 + \frac{s^2}{(2xN'(x))^2} + V(y(x)), \quad N^2(x) = \frac{dA}{dx}, \]
(2.91)

The Kähler potential of the initial system is connected with \( N \) via the equations
\[ \frac{dK}{dx} = \frac{N^3(x)}{2x^2N'(x)}, \quad \frac{d\log y}{dx} = \frac{N}{2x^2N'(x)}. \]
(2.92)

Hence, for \( s = 0 \) we shall get the system (1.29). However, when \( s \neq 0 \), by comparing the reduced system with (1.30), we conclude that the only Kähler space which yields a “well-defined system with monopole” is flat space.

\( \mathbb{C}^{N+1} \rightarrow \mathbb{C}P^N \) and \( T^*\mathbb{C}^{N+1} \rightarrow T^*\mathbb{C}P^N \)

Now, we consider the Hamiltonian reduction of the the space \( (\mathbb{C}^{N+1}, \omega = du^0d\bar{u}^0 + du^a d\bar{u}^a) \), to the complex projective space \( \mathbb{C}P^N \).

The \( U(N + 1) = U(1) \times SU(N) \) isometries of this space are defined by the following Killing potentials:
\[ J_0 = u\bar{u}, \quad J_{su(N+1)} = u\bar{T}u, \quad \{J_0, J_{su(N+1)}\} = 0, \]
(2.93)

where \( T = T^i, TrT = 0 \) are \( (N + 1) \times (N + 1) \) dimensional traceless matrices defining the \( su(N + 1) \) algebra. The Poisson brackets, corresponding to the Kähler structure, are defined by the relations \( \{u^0, \bar{u}^0\} = i, \{u, \bar{u}^b\} = i\delta^{ab} \).

Let us perform the Hamiltonian reduction by the action of \( J_0 \). The reduced phase space is a 2\( N \) dimensional one. Let us choose for this space the following local complex coordinates:
\[ z^a = \frac{u^a}{u^0}; \quad \{z^a, J_0\} = 0, \quad a = 1, \ldots, N \]
(2.94)

and fix the level surface
\[ J_0 = r_0^2 \Rightarrow |u^0|^2 = \frac{r_0^2}{1 + \sum z\bar{z}}. \]
(2.95)

Then, we immediately get the Poisson brackets for the reduced space
\[ \{z^a, \bar{z}^b\} = \frac{i}{r_0^2} (1 + \sum z\bar{z}) (\delta^{ab} + z^a\bar{z}^b), \quad \{z^a, z^b\} = \{\bar{z}^a, \bar{z}^b\} = 0. \]
(2.96)

Hence, the reduced Poisson bracket are associated with the Kähler structure. It could be easily seen, that this Kähler structure is given by the Fubini-Study metric (1.42) multiplied on \( r_0^2 \). The restriction of the generators \( J_{su(N+1)} \) on the level surface (2.95) yields the expressions (1.45).
In the above example $\mathcal{C}^{N+1}$ and $\mathbb{C}P^N$ appeared as the phase spaces. Now, let us show, how to reduce the $T^*\mathcal{C}^{N+1}$ to $T^*\mathbb{C}P^N$, i.e. let us consider the case when $\mathcal{C}^{N+1}$ and $\mathbb{C}P^N$ play the role of the configuration spaces of the mechanical systems. Since the dimension of $T^*\mathcal{C}^{N+1}$ is $4(N + 1)$, and the dimension of $T^*\mathbb{C}P^N$ is $4N$, the reduction has to be performed by two commuting generators.

Let us equip the initial space with the canonical symplectic structure (2.80), and perform the reduction of this phase space by the action of the generators

$$J_0 = i\pi u - \bar{\pi} \bar{u}, \quad h_0 = u \bar{u}: \quad \{J_0, h_0\} = 0. \quad (2.97)$$

We choose the following local coordinates of the reduced space:

$$z^a = \frac{u^a}{\bar{u}^0}, \quad p_a = g_{ab} (z, \bar{z}) \left( \frac{\bar{u}^a}{\bar{u}^0} - \bar{z}^a \frac{\bar{u}^0}{\bar{u}^0} \right): \quad \{z^a, J_0\} = \{z^a, h_0\} = \{p_a, J_0\} = \{p_a, h_0\} = 0, \quad (2.98)$$

where $g_{ab}$ is defined by the expression (1.42). Then, calculating the Poisson brackets between these functions, and fixing the value of the generators $J_0, h_0$,

$$h_0 = r_0^2, \quad J_0 = 2s, \quad (2.99)$$

we get

$$\{p_a, z^b\} = \delta^b_a, \quad \{p_a, \bar{p}_b\} = \frac{s}{r_0^2} g_{ab} (z, \bar{z}). \quad (2.100)$$

Hence, we arrive at the phase space structure of the particle moving on $\mathbb{C}P^N$ in the presence of a constant magnetic field with $B_0 = s/r_0^2$ strength.

### 2nd Hopf map. $SU(2)$ instanton

In the above examples we have shown that the zero Hopf map is related to the canonical transformation corresponding to the reduction of the two-dimensional system by the discrete group $Z_2 = S^0$, and transforms the system with two-dimensional configuration space to the system of the same dimension, which has a spin $\sigma = 0, 1/2$. The first Hopf map corresponds to the reduction of the system with four-dimensional configuration space by the Hamiltonian action of $U(1) = S^1$ group, and yields the system moving on the three-dimensional space in the presence of the magnetic field of the Dirac monopole. Similarly, with the second Hopf map one can relate the Hamiltonian reduction of the cotangent bundle of eight-dimensional space (say, $T^*\mathcal{C}^4 = T^*\mathbb{H}P^2$) by the action of $SU(2) = S^3$ group. When the $SU(2)$ generators $I_i$ have non-zero values, $I_i = c_i = 1$, $|c_i| \neq 0$, the reduced space is a $(2 \cdot 8 - 3 - 1 =) 12$-dimensional one, $T^*\mathbb{H}^3 \times S^3$. It is the phase space of a coloured particle moving on $\mathbb{H}^3$ in the presence of the $SU(2)$ Yang monopole [11] (here $S^2$ appears as a isospin space).

When $c_1 = c_2 = c_3 = 0$, the $J_i$ generators commute with each other, and the reduced space is a $(2 \cdot 8 - 2 \cdot 3 =) 10$-dimensional one, $T^*\mathbb{H}^3$. Such a reduction is also known under the name of Hurwitz transformation relating the eight-dimensional oscillator with the five-dimensional Coulomb problem.

We shall describe a little bit different reduction, associated with the fibration $\mathbb{C}P^3/\mathbb{C}P^1 = S^4$ [12]. This fibration could be immediately obtained by factorization of the second Hopf map $S^7/S^3 = S^4$ by $U(1)$. Indeed, the second Hopf map is described by the formulae (1.50), (1.51), where $S^7$ is embedded in the two-dimensional quaternionic space $\mathbb{H}^3 = \mathbb{C}^4$, parameterized by four complex (two quaternionic) Euclidean coordinates

$$u_i = v_i + j v_{i+1}, \quad i = 1, 2, \quad u_1, u_2 \in \mathbb{H}; \quad v_1, v_2, v_3, v_4 \in \mathbb{C}. \quad (2.101)$$

Here $S^4$ is embedded in $\mathbb{R}^5$ parameterized by the Euclidean coordinates $(x, x_5)$ given by (1.50). This embedding is invariant under the right action of a $SU(2)$ group given by (1.51), so that $g$ defines a three-sphere (1.54). The complex projective space $\mathbb{C}P^3$ is defined as $S^7/U(1)$, while the inhomogeneous coordinates $z_a$ appearing in the Fubini-Study metric of $\mathbb{C}P^3$, are related to the coordinates of $\mathbb{C}^4$ as follows: $z_a = v_a/v_4$, $a = 1, 2, 3$. The expressions (1.50) defining $S^4$ are invariant under $U(1)$-factorization, while $S^3/U(1) = S^2$. Thus, we arrive to the conclusion that $\mathbb{C}P^3$ is the $S^2$-fibration over $S^4 = \mathbb{H}P^1$. The expressions for $z_a$ yield the following definition of the coordinates of $S^4$:

$$w_1 = \frac{z_2 + z_1 z_4}{1 + z_3 z_3}, \quad w_2 = \frac{z_2 z_3 - z_1}{1 + z_3 z_3}. \quad (2.102)$$
Choosing \( z_3 \) as a local coordinate of \( S^2 = \mathbb{CP}^1 \),

\[
    u = z_3 ,
\]

we get the expressions

\[
    z_1 = w_1 u - \bar{w}_2, \quad z_2 = w_2 u + \bar{w}_1, \quad z_3 = u.
\]

(2.103)

In these coordinates the Fubini-Study metric on \( \mathbb{CP}^3 \) looks as follows:

\[
    g_{\bar{a}b} dz_a d\bar{z}_b = \frac{dz d\bar{z}}{1 + z \bar{z}} - \frac{(\bar{z} dz)(dz \bar{z})}{(1 + z \bar{z})^2} = \frac{dw_i d\bar{w}_i}{(1 + w \bar{w})^2} + \frac{(du + A)(du + \bar{A})}{(1 + u \bar{u})^2},
\]

(2.104)

where

\[
    A = \frac{(\bar{w}_1 + w_2 u)(udw_1 - d\bar{w}_2) + (\bar{w}_2 - w_1 u)(udw_2 + d\bar{w}_1)}{1 + w \bar{w}}.
\]

(2.105)

Hence, \( w_1, w_2 \) and \( u \) are the conformal-flat complex coordinates of \( S^4 = \mathbb{HP}^1 \) and \( S^2 = \mathbb{CP}^1 \), while the connection \( A \) defines the \( SU(2) \) gauge field.

Now, let us consider the Hamiltonian system describing the motion of a free particle on \( \mathbb{CP}^3 \)

\[
    \mathcal{H}_{\mathbb{CP}^3} = g^{\bar{a}b} \pi_a \bar{\pi}_b, \quad \{z_a, \pi_b\} = i\delta_{ab}
\]

(2.106)

Let us extend the coordinate transformation (2.104) to the \( T^* \mathbb{CP}^3 \), by the following transformation of momenta:

\[
    \pi_1 = \bar{\bar{p}}_1 \bar{p}_2 \frac{1}{1 + w \bar{u}}, \quad \pi_2 = \bar{\bar{p}}_2 p_1 \frac{1}{1 + w \bar{u}},
\]

\[
    \pi_3 = p_u + \bar{\bar{p}}_2 w_1 - \bar{\bar{p}}_1 w_2 - \bar{u}(w_1 p_1 + w_2 p_2) \frac{1}{1 + u \bar{u}}.
\]

(2.107)

This extended transformation is a canonical transformation,

\[
    \{w_i, p_j\} = \delta_{ij}, \quad \{u, p_u\} = 1.
\]

(2.108)

In the new coordinates the Hamiltonian reads

\[
    \mathcal{H}_{\mathbb{CP}^3} = (1 + w \bar{w})^2 P_1 \bar{P}_1 + (1 + u \bar{u})^2 p_u \bar{p}_u.
\]

(2.109)

Here we introduced the covariant momenta

\[
    P_1 = p_1 - i \frac{\bar{w}_1}{1 + w \bar{w}} I_1 - \frac{w_2}{1 + w \bar{w}} I_+, \quad P_2 = p_2 - i \frac{\bar{w}_2}{1 + w \bar{w}} I_1 + \frac{w_1}{1 + w \bar{w}} I_+,
\]

(2.110)

and the \( su(2) \) generators \( I_+, I_- \) defining the isometries of \( S^2 \)

\[
    I_1 = -i(p_u \bar{u} - \bar{p}_u u), \quad I_- = p_u + \bar{u}^2 \bar{p}_u, \quad I_+ = \bar{p}_u + u^2 p_u
\]

(2.111)

\[
    \{I_+, I_1\} = \mp i I_+ \bar{\bar{p}}_u, \quad \{I_+, I_-\} = 2i I_1.
\]

(2.112)

The nonvanishing Poisson brackets between \( P_i, w_i \) are given by the following relations (and their complex conjugates):

\[
    \{w_i, P_j\} = \delta_{ij}, \quad \{P_1, P_2\} = - \frac{2 I_+}{(1 + w \bar{w})^2}, \quad \{P_1, \bar{P}_1\} = - \frac{2 I_-}{(1 + w \bar{w})^2}.
\]

(2.113)

The expressions in the r.h.s. define the strength of a homogeneous \( SU(2) \) instanton (the “angular part” of the \( SU(2) \) Yang monopole), written in terms of conformal-flat coordinates of \( S^4 = \mathbb{HP}^1 \). Hence, the first part of the Hamiltonian, i.e. \( \mathcal{D}_4 = (1 + w \bar{w})^2 P_1 \bar{P}_1 \), describes a particle on the four-dimensional sphere in the field of a \( SU(2) \) instanton.

The Poisson brackets between \( P_i \) and \( u, \bar{u}, p_u, \bar{p}_u \) are defined by the following nonzero relations and their complex conjugates:

\[
    \{P_i, p_u\} = - \frac{\bar{w}_i + 2 \epsilon_{ij} w_j u}{1 + w \bar{w}} p_u, \quad \{P_i, \bar{p}_u\} = \frac{\bar{w}_i \bar{p}_u}{1 + w \bar{w}},
\]

(2.114)

\[
    \{P_i, u\} = \frac{\bar{w}_i + \epsilon_{ij} w_j u}{1 + w \bar{w}}, \quad \{\bar{P}_i, u\} = \frac{\epsilon_{ij} \bar{w}_j - \bar{w}_i u}{1 + w \bar{w}}.
\]

(2.115)
The second part of the Hamiltonian defines the motion of a free particle on the two-sphere. It could be represented as a Casimir of \( SU(2) \)

\[
D_{S^2} = (1 + u\bar{u})^2 p_\alpha \bar{p}_\alpha = I_+ I_- + I_z^2 = I^2.
\]

It commutes with the Hamiltonian \( D_0 \), as well as with \( I_1, I_\pm \) and \( P_1, w_1 \)

\[
\{ D_{\mathbb{R}^3}, I^2 \} = \{ P_1, I^2 \}_B = \{ w_1, I^2 \}_B = \{ I_1, I^2 \}_B = \{ I_\pm, I^2 \}_B = 0.
\]

Hence, we can perform a Hamiltonian reduction by the action of the generator \( D_2 \), which reduces the initial twelve-dimensional phase space \( T^* \mathbb{S}^3 = T^* (S^4 \times S^2) \) to a ten-dimensional one. The relations (2.117) allow us to parameterize the reduced ten-dimensional phase space in terms of the coordinates \( P_1, w_1, I_\pm, I_1 \), where the latter obey the relation

\[
I_+ I_- + I_1^2 \equiv I^2 = \text{const}.
\]

Thus, the reduced phase space is nothing but \( T^* S^4 \times S^2 \), where \( S^2 \) is the internal space of the instanton.

Let us collect the whole set of non-zero expressions defining the Poisson brackets on \( T^* S^4 \times S^2 \)

\[
\{ w_1, P_j \} = \delta_{ij},
\]

\[
\{ P_1, P_2 \} = -\frac{2I_+}{(1 + w\bar{w})^2},
\]

\[
\{ P_1, \bar{P}_j \} = -i \frac{2I_1 \delta_{ij}}{1 + w\bar{w}}^2,
\]

\[
\{ P_1, I_1 \} = i w_1 I_+ w_1 \frac{I_+}{1 + w\bar{w}},
\]

\[
\{ P_1, I_\pm \} = i w_1 I_\mp \frac{I_\pm}{1 + w\bar{w}},
\]

\[
\{ I_+, I_- \} = i w_1 I_\mp \frac{I_\pm}{1 + w\bar{w}}.
\]

The reduced Hamiltonian is \( H_{\mathbb{R}^3}^{red} = (1 + w\bar{w})^2 P^2 + I^2 \). So, the Hamiltonian of the coloured particle on \( S^4 \) interacting with the \( SU(2) \) instanton is connected with the Hamiltonian of a particle on \( \mathbb{R}^3 \) as follows:

\[
D_{S^4}^{red} = D_{\mathbb{R}^3}^{red} - I^2 \quad (> 0).
\]

This yields an intuitive explanation of the degeneracy in the ground state in the corresponding quantum system on \( S^4 \). Indeed, since the l.h.s. is positive, the ground state of the quantum system on \( S^4 \) corresponds to the excited state of a particle on \( \mathbb{R}^3 \), which is a degenerate one. On the other hand, the ground state of a particle on \( \mathbb{R}^3 \) can be reduced to the free particle on \( S^4 \), when \( I = 0 \).

Now, let us consider a similar reduction for the particle on \( \mathbb{R}^3 \), in the presence of constant magnetic field \( (1.38) \).

Passing to the coordinates (2.104) and momenta (2.111) we get the Poisson brackets defined by the nonzero relations given by (2.115) and

\[
\{ p_\alpha, \bar{p}_\beta \} = \frac{iB}{1 + u\bar{u}}^2.
\]

\[
\{ w_1, P_j \} = \delta_{ij}, \quad \{ P_1, P_2 \} = -\frac{2I_+}{1 + w\bar{w}},
\]

\[
\{ P_1, \bar{P}_j \} = -i \frac{2I_1 \delta_{ij}}{1 + w\bar{w}},
\]

where \( I_\pm, I_1 \) are defined by the expressions

\[
I_1 = I_1 + B \frac{1 - u\bar{u}}{2(1 + u\bar{u})}, \quad I_- = I_- + B \frac{i\bar{u}}{1 + u\bar{u}}, \quad I_+ = I_+ + B \frac{iu}{1 + u\bar{u}}.
\]
Notice that the expressions (2.123) are similar to (2.113) and the generators (2.124) form, with respect to the new Poisson brackets, the $su(2)$ algebra

$$\{T_\pm, T_1\}_B = \mp i T_\pm, \quad \{T_+, T_-\} = 2i T_1. \quad (2.125)$$

It is clear that these generators define the isometries of the “internal” two-dimensional sphere with a magnetic monopole located at the center.

Once again, as in the absence of a magnetic field, we can reduce the initial system by the Casimir of the $SU(2)$ group

$$I^2 \equiv I_+^2 + I_-^2 = D_{S^2} + B^2/4, \quad I \geq B/2. \quad (2.126)$$

In order to perform the Hamiltonian reduction, we have to fix the value of $I^2$, and then factorize by the action of the vector field $\{T^2, \}_B$.

The coordinates (2.102), (2.111) commute with the Casimir (2.126),

$$\{P_i, T^2\}_B = \{w_i, T^2\}_B = \{I_1, T^2\}_B = \{I_\pm, T^2\}_B = 0. \quad (2.127)$$

Hence, as we did above, we can choose $P_i$, $w_i$, and $I_\pm$ as the coordinates of the reduced, ten-dimensional phase space.

The coordinates $I_\pm$, $I_1$ obey the condition

$$I_1^2 + I_+ I_- = I^2 = \text{const.} \quad (2.128)$$

The resulting Poisson brackets are defined by the expressions (2.119), with $I_1, I_\pm$ replaced by $I_\pm, I_1$.

Hence, the particle on $\mathbb{CP}^3$ moving in the presence of a constant magnetic field reduces to a coloured particle on $S^4$ interacting with the instanton field. The Hamiltonians of these two systems are related as follows:

$$D_{S^4} = D_{\mathbb{CP}^3} - I^2 + B^2/4, \quad I \geq B/2 \quad (2.129)$$

Notice that, upon quantization, we must replace $I^2$ by $I(I+1)$ and require that both $I$ and $B$ take (half) integer values (since we assume unit radii for the spheres, this means that the “monopole number” obeys a Dirac quantization rule). The extension of this reduction to quantum mechanics relates the theories of the quantum Hall effect on $S^4$ [13] and on $\mathbb{CP}^3$ [14].

Notice that the third Hopf map could also be related with the generalized quantum Hall effect theory [15].

### 3 Generalized oscillators

Among the integrable systems with hidden symmetries the oscillator is the simplest one. In contrast to other systems with hidden symmetries (e.g. Coulomb systems), its symmetries form a Lie algebra. The $N$-dimensional oscillator on $T^N \mathbb{R}^N$,

$$\mathcal{H} = \frac{1}{2} \left( p_a p_a + \alpha^2 q^a q^a \right), \quad \omega_{can} = dp_a \wedge dq^a, \quad a = 1, \ldots, N \quad (3.130)$$

besides the rotational symmetry $so(N)$, has also hidden ones, so that the whole symmetry algebra is $su(N)$.

The symmetries of the oscillator are given by the generators

$$J_{ab} = p_a q^b - p_b q^a, \quad I_{ab} = p_a p_b + \alpha^2 q^a q^b. \quad (3.131)$$

The huge number of hidden symmetries allows us to construct generalizations of the oscillator on curved spaces, which inherit many properties of the initial system.

The generalization of the oscillator on the sphere was suggested by Higgs [16]. It is given by the following Hamiltonian system:

$$\mathcal{H} = \frac{1}{2} g^{ab} p_a p_a + \alpha^2 q^a q^a, \quad \omega = dp_a \wedge dq^a, \quad q^a = \frac{x_a}{x_0}, \quad (3.132)$$

where $x^a, x_0$ are the Euclidean coordinates of the ambient space $\mathbb{R}^{N+1}$: $x^2 + x^a x^a = 1$, and $g_{ab} dq^a dq^b$ is the metric on $S^N$. This system inherits the rotational symmetries of the flat oscillator given by (3.131), and possesses the hidden symmetries given by the following constants of motion (compare with (3.131)):

$$I_{ab} = J_a J_b + \alpha^2 q^a q^b, \quad (3.133)$$
where \( J_a \) are the translation generators on \( S^N \).

In contrast to the flat oscillator, whose symmetry algebra is \( su(N) \), the spherical (Higgs) oscillator has a nonlinear symmetry algebra.

This construction has been extended to the complex projective spaces in Ref. [17], where the oscillator on \( \mathbb{C}P^N \) was defined by the Hamiltonian

\[
\mathcal{H} = g^{ab} \pi_a \pi_b + \alpha^2 z \bar{z},
\]

with \( z^a = u^a / u^0 \) denoting inhomogeneous coordinates of \( \mathbb{C}P^N \) and \( g_{\bar{a}b} dz^a d\bar{z}^b \) being Fubini-Study metric (1.42).

It is easy to see that this system has constants of motion given by the expressions

\[
J_{ab} = i(\bar{z}^b \pi_a - \bar{\pi}_b z^a), \quad J_{\bar{a}b} = J_{a\bar{b}}^+ + \omega^2 z^a z^b, \quad (3.135)
\]

where \( J_{a+} = \pi_a + (\bar{z} \bar{\pi}) z^a, \quad J_{a-} = J_{a+}^0 \) are the translation generators on \( \mathbb{C}P^N \). The generators \( J_{\bar{a}b} \) define the kinematical symmetries of the system and form a \( su(N) \) algebra. When \( N > 1 \), the generators \( I_{\bar{a}b} \) are functionally independent of \( \mathcal{H}, \ J_{\bar{a}b} \) and define hidden symmetries. As in the spherical case, their algebra is a nonlinear one

\[
\{ J_{\bar{a}b}, J_{\bar{c}d} \} = i \delta_{\bar{a}d} J_{\bar{b}c} - i \delta_{\bar{b}d} J_{\bar{a}c}, \quad \{ J_{\bar{a}b}, J_{\bar{c}d} \} = i \delta_{\bar{a}d} J_{\bar{b}c} - i \delta_{\bar{b}d} J_{\bar{a}c},
\]

\[
\{ I_{\bar{a}b}, I_{\bar{c}d} \} = i \alpha^2 \delta_{\bar{a}d} J_{\bar{b}c} - i \alpha^2 \delta_{\bar{b}d} J_{\bar{a}c} + i I_{\bar{c}d}(J_{\bar{a}b} + J_{\bar{b}a}) - i I_{\bar{a}d}(J_{\bar{b}c} + J_{\bar{c}b}) \quad (3.136)
\]

Hence, it is seen that for \( N = 1 \), i.e. in the case of the two-dimensional sphere \( S^2 = \mathbb{C}P^1 \), the suggested system has no hidden symmetries, as opposed to the Higgs oscillator on \( S^2 \). Nevertheless, this model is exactly solvable both for \( N = 1 \) and \( N > 1 \) [18]. Moreover, it remains exactly solvable, even after inclusion of a constant magnetic field, for any \( N \) (including \( N = 1 \), when it has no hidden symmetries). The magnetic field does not break the symmetry algebra of the system! As opposed to the described model, the constant magnetic field breaks the hidden symmetries, as well as the exact solvability, of the Higgs oscillator on \( S^2 = \mathbb{C}P^1 \).

**Remark.** The Hamiltonian (3.134) could be represented as follows:

\[
\mathcal{H} = g^{a\bar{b}} (\pi_a \bar{\pi}_b + \alpha^2 \partial_a K \partial_b K), \quad (3.137)
\]

where \( K(z, \bar{z}) = \log(1 + z \bar{z}) \) is the Kähler potential of the Fubini-Study metric.

Although this potential is not uniquely defined, it provides the system with some properties, which are general for the few oscillator models on Kähler spaces. By this reason we postulate it as an oscillator potential on arbitrary Kähler manifolds.

Now, let us compare these systems with the sequence which we like: real, complex, quaternionic numbers (and zeroth, first, second Hopf map). Let us observe, that the \( S^N \)-oscillator potential is defined, in terms of the ambient space \( \mathbb{R}^{N+1} \), in complete similarity to the \( \mathbb{C}P^N \)-oscillator potential in terms of the “ambient” space \( \mathbb{C}P^{N+1} \). The latter system preserves its exact solvability in the presence of a constant magnetic (\( U(1) \) gauge) field.

Hence, continuing this sequence, one can define on the quaternionic projective spaces \( \mathbb{H}P^N \) the oscillator-like system given by the potential

\[
V_{\mathbb{H}P^N} = \alpha^2 w^a \bar{w}^a = \alpha^2 \frac{u^a_1 \bar{u}^a_1 + u^a_2 \bar{u}^a_2}{u^1_1 \bar{u}^1_1 + u^1_2 \bar{u}^1_2}, \quad (3.138)
\]

where

\[
w^a = \frac{u^a_1 + j u^a_2}{u^1_1 + j u^1_2}, \quad \bar{w}^a = \frac{u^a_1 \bar{u}^a_1 + u^a_2 \bar{u}^a_2 + u^1_1 \bar{u}^1_1 + u^2_2 \bar{u}^2_2}{u^1_1 \bar{u}^1_1 + u^2_2 \bar{u}^2_2} = 1.
\]

Here \( w^a \) are inhomogeneous (quaternionic) coordinates of the quaternionic projective space \( \mathbb{H}P^N \), and \( u^a_1 + j u^a_2, \ u^0_1 + j u^0_2 \) are the Euclidean coordinates of the “ambient” quaternionic space \( \mathbb{H}P^{N+1} = \mathbb{C}P^{2N+2} \).

One can expect that this system will be a superintegrable one and will be exactly solvable also in the presence of a \( SU(2) \) instanton field.

In the simplest case of \( \mathbb{H}P^1 = S^4 \) we shall get the alternative (with respect to the Higgs) model of the oscillator on the four-dimensional sphere. In terms of the ambient space \( \mathbb{R}^5 \), its potential will be given by the expression

\[
V_{S^4} = \alpha^2 \frac{1 - x^0 / x}{1 + x^0 / x} = \alpha^2 \frac{1 - \cos \theta}{1 + \cos \theta} \quad (3.139)
\]
Checking this system for this simplest case, we found, that it is indeed exactly solvable in the presence of the instanton field [19].

Let us mention that the Higgs (spherical) oscillator could be straightforwardly extended to (one- and two-sheet) hyperboloids, and the $\mathbb{CP}^N$-oscillator to the Lobachevsky spaces $\mathcal{L}_N = SU(N+1)/U(N)$. In both cases these systems have hidden symmetries.

Notice also that, on the spheres $S^N$, there exists the analog of the Coulomb system suggested by Schrödinger [20]. It is given by the potential

$$V_{Coulomb} = -\frac{\gamma}{r_0} \frac{y^{N+1}}{|y|}, \quad y^{N+1} + |y|^2 = r_0^2. \quad (3.140)$$

This system inherits the hidden symmetry of the conventional Coulomb system on $\mathbb{R}^N$.

Probably, as in the case of the oscillator, one can define superintegrable analogs of the Coulomb system on the complex projective spaces $\mathbb{CP}^N$ and on the quaternionic projective spaces $\mathbb{HP}^N$. However, up to now, this question has not been analyzed.

**Relation of the (pseudo)spherical oscillator and Coulomb systems**

The oscillator and Coulomb systems, being the best known among the superintegrable mechanical systems, possess many similarities both at the classical and quantum mechanical levels. Writing down these systems in spherical coordinates, one can observe that the radial Schrödinger equation of the $(p+1)$-dimensional Coulomb system could be transformed in the Shrödinger equation of the $2p$-dimensional oscillator by the transformation (see, e.g. [21])

$$r = R^2,$$

where $r$ and $R$ are the radial coordinates of the Coulomb and oscillator systems, respectively.

Due to the existence of the Hopf maps, in the cases of $p = 1, 2, 4$ one can establish a complete correspondence between these systems. Indeed, their angular parts are, respectively, $p$- and $(2p-1)$-dimensional spheres, while the above relation follows immediately from (1.56). Considering the Hamiltonian reductions related to the Hopf maps (as it was done in the previous section), one can deduce, that the $(p+1)$-dimensional Coulomb systems could be obtained from the $2p$-dimensional oscillator, by a reduction under the $G = S^{(p-1)}$ group. Moreover, for non-zero values of those generators we shall get generalizations of the Coulomb systems, specified by the presence of a magnetic flux ($p = 1$), a Dirac monopole ($p = 2$), a Yang monopole ($p = 4$) [9, 22, 23]. However, this procedure assumes a change in the roles of the coupling constants and the energy. To be more precise, these reductions convert the energy surface of the oscillator in the energy surface of the Coulomb-like system, while there is no one-to-one correspondence between their Hamiltonians.

As we have seen above, there exists well-defined generalizations of the oscillator systems on the spheres, hyperboloids, complex projective spaces and Lobachevsky spaces. The Coulomb system could also be generalized on the spheres and hyperboloids. Hence, the following natural question arises. Is it possible to relate the oscillator and Coulomb systems on the spheres and hyperboloids, similarly to those in the flat cases? The answer is positive, but it is rather strange. The oscillators on the 2$p$-dimensional sphere and two-sheet hyperboloid (pseudosphere) result in the Coulomb-like systems on the $(p+1)$-dimensional pseudosphere, for $p = 1, 2, 4$ [24].

Below, following [24], we shall show how to relate the oscillator and Coulomb systems on the spheres and two-sheet hyperboloids. In the planar limit this relation results in the standard correspondence between the conventional (flat) oscillator and the Coulomb-like system. We shall discuss mainly the $p = 1$ case, since the treatment could be straightforwardly extended to the $p = 2, 4$ cases.

Let us introduce the complex coordinate $z$ parameterizing the sphere by the complex projective plane $\mathbb{CP}^1$ and the two-sheeted hyperboloid by the Poincaré disk (Lobachevsky plane, pseudosphere) $\mathcal{L}$

$$x \equiv x_1 + ix_2 = R_0 \frac{2z}{1 + \epsilon \bar{z} z}, \quad x_3 = R_0 \frac{1 - \epsilon \bar{z} z}{1 + \epsilon \bar{z} z}. \quad (3.141)$$

In these coordinates the metric becomes conformally-flat

$$ds^2 = R_0^2 \frac{4dz d\bar{z}}{(1 + \epsilon \bar{z} z)^2}. \quad (3.142)$$

Here $\epsilon = 1$ corresponds to the system on the sphere, and $\epsilon = -1$ to that on the pseudosphere. The lower hemisphere and the lower sheet of the hyperboloid are parameterized by the unit disk $|z| < 1$, while the upper
hemisphere and the upper sheet of the hyperboloid are specified by \(|z| < 1\), and transform one into another by the inversion \(z \to 1/z\). In the limit \(R_0 \to \infty\) the lower hemisphere (the lower sheet of the hyperboloid) turns into the whole two-dimensional plane. In these terms the oscillator and Coulomb potentials read

\[
V_{\text{osc}} = \frac{2\alpha^2 R_0^2 z \bar{z}}{(1 - \epsilon z \bar{z})^2}, \quad V_C = -\frac{\gamma}{R_0} \frac{1 - \epsilon z \bar{z}}{2|z|},
\]

(3.143)

Let us equip the oscillator phase space \(T^*\mathbb{C}P^1\) (\(T^*\mathcal{L}\)) with the symplectic structure

\[
\omega = d\pi \wedge dz + d\bar{\pi} \wedge d\bar{z}
\]

(3.144)

and introduce the rotation generators defining the \(su(2)\) algebra for \(\epsilon = 1\) and the \(su(1,1)\) algebra for \(\epsilon = -1\)

\[
J \equiv \frac{iJ_1 - J_2}{2} = \pi + \epsilon \bar{z} \bar{\pi}, \quad J \equiv \frac{\epsilon J_1}{2} = i(\pi - \bar{\pi}).
\]

(3.145)

These generators, together with \(x/R_0, x_3/R_0\), define the algebra of motion of the (pseudo)sphere via the following non-vanishing Poisson brackets:

\[
\{J, x\} = 2x_3, \quad \{J, x_3\} = -\epsilon x, \quad \{J, x\} = i\epsilon x, \quad \{J, J\} = -2i\epsilon J, \quad \{J, J\} = iJ.
\]

(3.146)

In these terms, the Hamiltonian of a free particle on the (pseudo)sphere reads

\[
H_0 = \frac{J^2 + \epsilon J^2}{2R_0^2} = \frac{(1 + \epsilon z \bar{z})^2 \pi \bar{\pi}}{2R_0^2}.
\]

(3.147)

whereas the oscillator Hamiltonian is given by the expression

\[
H_{\text{osc}}^{\epsilon}(\alpha, R_0|\pi, \bar{\pi}, z, \bar{z}) = \left(\frac{1 + \epsilon z \bar{z})^2 \pi \bar{\pi}}{2R_0^2} + \frac{2\alpha^2 R_0^2 z \bar{z}}{(1 - \epsilon z \bar{z})^2}\right).
\]

(3.148)

It can be easily verified that the latter system possesses the hidden symmetry given by the complex (or vectorial) constant of motion [16]

\[
I = I_1 + iI_2 = \frac{J^2}{2R_0^2} + \frac{\alpha^2 R_0^2 \bar{x} \bar{z}^2}{x_3^2},
\]

(3.149)

which defines, together with \(J\) and \(H_{\text{osc}}\), the cubic algebra

\[
\{I, J\} = 2iI, \quad \{\bar{I}, I\} = 4i \left(\alpha^2 J + \frac{\epsilon J H_{\text{osc}}}{R_0^2} - \frac{J^3}{2R_0^2}\right).
\]

(3.150)

The energy surface of the oscillator on the (pseudo)sphere \(H_{\text{osc}}^{\epsilon} = E\) reads

\[
\frac{(1 - (z \bar{z})^2)^2 \pi \bar{\pi}^2}{2R_0^2} + 2 \left(\alpha^2 + \epsilon \frac{E}{R_0^2}\right) z \bar{z} = \frac{E}{R_0^2} \left(1 + (z \bar{z})^2\right).
\]

(3.151)

Now, performing the canonical Bohlin transformation (2.70) one can rewrite the expression (3.151) as follows:

\[
\frac{(1 - w \bar{w})^2 p \bar{p}}{2R_0^2} - \frac{\gamma}{R_0} \frac{1 + w \bar{w}}{2|w|} = \mathcal{E}_C,
\]

(3.152)

where we introduced the notation

\[
r_0 = R_0^2, \quad \gamma = \frac{E}{2}, \quad -2\mathcal{E}_C = \alpha^2 + \epsilon \frac{E}{r_0}.
\]

(3.153)

Comparing the l.h.s. of (3.152) with the expressions (3.143), (3.147) we conclude that (3.152) defines the energy surface of the Coulomb system on the pseudosphere with “radius” \(r_0\), where \(w, p\) denote the complex stereographic coordinate and its conjugated momentum, respectively. In the above, \(r_0\) is the “radius” of the pseudosphere, while \(\mathcal{E}_C\) is the energy of the system. Hence, we related classical isotropic oscillators on the sphere and pseudosphere with the classical Coulomb problem on the pseudosphere.
The constants of motion of the oscillators, $J$ and $I$ (which coincide on the energy surfaces (3.151)) are converted, respectively, into the doubled angular momentum and the doubled Runge-Lenz vector of the Coulomb system

$$J \to 2J_C, \quad I \to 2A, \quad A = -\frac{iJ_CJ_C}{r_0} + \gamma \frac{\vec{x}_C}{|x_C|},$$

(3.154)

where $J_C, J_C, x_C$ denote the rotation generators and the pseudo-Euclidean coordinates of the Coulomb system.

We have shown above that, for establishing the quantum-mechanical correspondence, we have to supplement the quantum-mechanical Bohlin transformation with the reduction by the $Z_2$ group action, choosing either even ($\sigma = 0$) or odd ($\sigma = 1/2$) wave functions (2.75). The resulting Coulomb system is spinless for $\sigma = 0$, and it possesses spin $1/2$ for $\sigma = 1/2$.

The presented construction could be straightforwardly extended to higher dimensions, concerning the $2p$-dimensional oscillator on the (pseudo)sphere and the $(p + 1)$-dimensional Coulomb-like systems, $p = 2, 4$. It is clear, that the $p = 2$ case corresponds to the Hamiltonian reduction, associated with the first Hopf map, and the $p = 4$ case is related to the second Hopf map. Indeed, the oscillator on the $2p$-dimensional (pseudo)sphere is also described by the Hamiltonian (3.148), where the following replacement is performed: $(z, \pi) \to (z^n, \pi_a), a = 1, \ldots, p$, with the summation over these indices understood. Consequently, the oscillator energy surfaces are again given by Eq. (3.151). Then, performing the Hamiltonian reduction, associated with the $p$-th Hopf maps (see the previous Section) we shall get the Coulomb-like system on the $(p + 1)$-dimensional pseudosphere.

For example, if $p = 2$, we reduce the system under consideration by the Hamiltonian action of the $U(1)$ group given by the generator $J = i(z\pi - x\tilde{z})$. This reduction was described in detail in Section 2. For this purpose, we have to fix the level surface $J = 2s$ and choose the $U(1)$-invariant stereographic coordinates in the form of the conventional Kustaanheimo-Stiefel transformation (2.84). The resulting symplectic structure takes the form (1.18). The oscillator energy surface reads

$$\frac{(1 - q^2)^2}{8r_0^2}(p^2 + s^2q^2) - \gamma \frac{1 + q^2}{2|q|} = \mathcal{E}_C,$$

(3.155)

where $r_0, \gamma, \mathcal{E}_C$ are defined by the expressions (3.153).

Interpreting $q$ as the (real) stereographic coordinates of the three-dimensional pseudosphere

$$x = r_0 \frac{2q}{1 - q^2}, \quad x_4 = r_0 \frac{1 + q^2}{1 - q^2},$$

(3.156)

we conclude that (3.155) defines the energy surface of the pseudospherical analog of a Coulomb-like system proposed in Ref. [7], which is also known under the name of “MIC-Kepler” system.

In the $p = 4$ case, we have to reduce the system by the action of the $SU(2)$ group and choose the $SU(2)$-invariant stereographic coordinates and momenta in the form corresponding to the standard Hurwitz transformation, which yields a pseudospherical analog of the so-called $SU(2)$-Kepler (or Yang-Coulomb) system [23]. The potential term of the resulting system will be given by the expression

$$V_{SU(2)-Kepler} = \frac{I^2}{r_0^2} \left( \frac{x_5^2}{2x^2 - 2} \right) - \frac{\gamma x_5}{2r_0 |x|},$$

(3.157)

where $(x, x_5)$ are the (pseudo)Euclidean coordinates of the ambient space $\mathbb{R}^{1,5}$ of the five-dimensional hyperboloid, $|x|^2 - x_5^2$; $I^2$ is the value of the generator $J_a^2$, under which the $SU(2)$ reduction has been performed. The constants $r_0, \gamma$ are defined by the expressions (3.153).

It is interesting to clarify, which systems will the $\mathbb{C}P^N$-oscillators, after similar reductions, result in. We have checked it only for the first Hopf map, corresponding to the case $p = 2$ [25, 17]. To our surprise, we found that the oscillators on $\mathbb{C}P^2$ and $L_2$ also resulted, after reduction, in the pseudospherical MIC-Kepler system!

4 Supersymplectic structures

In the previous Sections we presented some elements of Hamiltonian formalism which, in our belief, could be useful in the study of supersymmetric mechanics.
In the present Section we shall briefly discuss the Hamiltonian formalism on superspaces (super-Hamiltonian formalism). The super-Hamiltonian formalism, in its main lines, is a straightforward extension of the ordinary Hamiltonian formalism to superspace, with a more or less obvious placement of sign factors. Probably, from the supergeometrical viewpoint, the only qualitative difference appears in the existence of the odd Poisson brackets (antibrackets), which have no analogs in ordinary spaces, and in the respect of the differential forms to integration. Fortunately, these aspects are inessential for our purposes.

The Poisson brackets of the functions \( f(x) \) and \( g(x) \) on superspaces are defined by the expression

\[
\{ f, g \}_\kappa = \frac{\partial f}{\partial x^A} \Omega^{AB}_\kappa \frac{\partial g}{\partial x^B}, \quad \kappa = 0, 1. \tag{4.158}
\]

They obey the conditions

\[
p(\{ f, g \}_\kappa) = p(f) + p(g) + \kappa \quad \text{(grading)},
\]

\[
\{ f, g \}_\kappa = -(-1)^{(p(f)+\kappa)(p(g)+\kappa)} \{ g, f \}_\kappa \quad \text{("antisymmetricity")},
\]

\[
(-1)^{(p(f)+1)(p(h)+\kappa)} \{ f, \{ g, h \}_\kappa \}_\kappa + \text{cycl.perm.}(f, g, h) = 0 \quad \text{(Jacobi id.)}. \tag{4.159}
\]

Here \( x^A \) are local coordinates of superspace, while \( \frac{\partial}{\partial x^A} \) and \( \frac{\partial}{\partial \theta^A} \) denote right and left derivatives, respectively.

It is seen that the nondegenerate odd Poisson brackets can be defined on the \((2N.N)\)-dimensional superspace, and the nondegenerate even Poisson brackets could be defined on the \((2N.M)\)-dimensional ones. In this case the Poisson brackets are associated with the supersymplectic structure

\[
\Omega_{\kappa} = dz^A \Omega_{(\kappa)AB} dz^B, \quad d\Omega_{\kappa} = 0 \tag{4.161}
\]

where \( \Omega_{(\kappa)AB} \Omega^{BC} = \delta^C_A \).

The generalization of the Darboux theorem states that locally, the nondegenerate Poisson brackets could be transformed to the canonical form. The canonical odd Poisson brackets look as follows:

\[
\{ f, g \}_\kappa^{\text{can}} = \sum_{i=1}^N \left( \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial \theta^i} - \frac{\partial f}{\partial \theta^i} \frac{\partial g}{\partial x^i} \right), \tag{4.162}
\]

where \( p(\theta_i) = p(x^i) + 1 = 1 \). The canonical even Poisson brackets read

\[
\{ f, g \}_0 = \sum_{i=1}^N \left( \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x^i + N} - \frac{\partial f}{\partial x^i + N} \frac{\partial g}{\partial x^i} \right) + \sum_{\alpha=1}^M \epsilon_{\alpha} \frac{\partial f}{\partial \theta^\alpha} \frac{\partial g}{\partial \theta^\alpha}, \quad \epsilon_{\alpha} = \pm 1. \tag{4.163}
\]

Here \( x^i, x^{i+N} \) denote even coordinates, \( p(x) = 0 \), and \( \theta^\alpha \) are the odd ones \( p(\theta) = 1 \).

In a completely similar way to the ordinary (non-"super") space, one can show that the vector field preserving the supersymplectic structure is a locally Hamiltonian one. Hence, both types of supersymplectic structures can be related with the Hamiltonian systems, which have the following equations of motion:

\[
\frac{dx^A}{dt} = \{ \mathcal{H}_\kappa, x^A \}_\kappa, \quad p(\mathcal{H}_\kappa) = \kappa. \tag{4.164}
\]

Any supermanifold \( \mathcal{M} \) underlyed by the bosonic manifold \( M_0 \) can be associated with some vector bundle \( VM_0 \) of \( M_0 \) [3], in the following sense. One can choose on \( \mathcal{M} \) local coordinates \((x^i, \theta^\mu)\), such that the transition functions from one chart (parameterized by \((x^i, \theta^\mu)\)) to the other chart (parameterized by \((\tilde{x}^i, \tilde{\theta}^\mu)\)) look as follows:

\[
\tilde{x}^i = x^i(x), \quad \tilde{\theta}^\mu = A_{\alpha}^\mu(x) \theta^\alpha. \tag{4.165}
\]

Changing the parity of \( \theta \): \( p(\theta^\mu) = 1 \rightarrow p(\theta^\mu) = 0 \), we shall get the vector bundle \( VM_0 \) of \( M_0 \).

Any supermanifold equipped with the odd symplectic structure, is associated with the cotangent bundle of \( M_0 \) [26], so that the odd symplectic structure could be globally transformed to the canonical form, with the odd Poisson bracket given by the expression (4.162). Hence, the functions on the odd symplectic manifold could be interpreted as contravariant antisymmetric tensors on \( M_0 \).
The structure of the even symplectic manifold is not so rigid: there is a variety of ways to extend the
given symplectic manifold \( M_0, \omega \) to the supersymplectic ones, associated with the vector bundle \( VM_0 \). On these
supermanifolds one can (globally) define the even symplectic structure

\[
\Omega = \omega + d(\theta^\mu g_{\mu\nu}(x) D\theta^\nu) = \omega + \frac{1}{2} R_{\nu\rho k} \theta^\rho \theta^\mu dx^k + g_{\mu\nu} D\theta^\nu \wedge D\theta^\mu, \tag{4.166}
\]

Here \( x^i \) are local coordinates of \( M_0 \) and \( \theta^\mu \) are the (odd) coordinates in the bundle; \( g_{\mu\nu} = g_{\mu\nu}(x) \) are
the components of the metrics in the bundle, while \( D\theta^\mu = d\theta^\mu + \Gamma^\mu_{\nu i} \theta^\nu dx^i \), where \( \Gamma^\mu_{\nu i} \) are the connection
components respecting the metric in the bundle

\[
g_{\mu\nu;k} = g_{\mu\nu,k} - g_{\nu\alpha} \Gamma^\alpha_{k\nu} - g_{\alpha\nu} \Gamma^\alpha_{k\mu} = 0. \tag{4.167}
\]

We used the following notation as well: \( R_{\mu\nu k i} = g_{\mu\nu} R^\alpha_{\nu k i}, \) where \( R^\alpha_{\nu k i} \) are the components of connection’s
curvature

\[
R^\nu_{\alpha k i} = -\Gamma^\nu_{\kappa a,i} + \Gamma^\nu_{\kappa a,k} \Gamma^\kappa_{\alpha i} - \Gamma^\nu_{\kappa b} \Gamma^\kappa_{\beta a}; \quad R^\nu_{\alpha k i} = -R^\nu_{\alpha k i}.
\]

Let us consider the coordinate transformation (4.165). With respect to this transformation, the connection
components transform as follows:

\[
\Gamma^\mu_{\nu i} = A^\mu_{\lambda} \Gamma^\lambda_{\kappa a} \partial_\kappa x^k B^\alpha_{\nu \nu} - A^\mu_{\kappa a} B^\alpha_{\nu \nu} \partial_\kappa x^k, \quad A^\mu_{\nu} B^\lambda_{\nu} = \delta^\lambda_{\mu}. \tag{4.168}
\]

Since \( D\theta^\mu \) transforms homogeneously under (4.165), \( D\tilde{\theta}^\nu = D\theta^\mu A^\nu_{\mu}(x) \), we conclude that the supersymplectic
structure (4.166) is covariant under (4.165) as well.

The corresponding Poisson brackets look as follows:

\[
\{ f, g \} = (\nabla_i f) \tilde{\omega}^{ij} (\nabla_j g) + \alpha \frac{\partial f}{\partial \theta^\mu} g^{\mu\nu} \frac{\partial g}{\partial \theta^\nu}; \tag{4.169}
\]

where

\[
\tilde{\omega}^{im} (\omega_{mj} + \frac{1}{2} R_{\nu mj \nu} \theta^\nu \theta^\mu) = \delta^i_j, \quad \nabla_i = \frac{\partial}{\partial x^i} - \Gamma^k_{ij} (x) \theta^\nu \frac{\partial}{\partial \theta^\nu}.
\]

On the supermanifolds one can define also the analog of the Kähler structures. We shall call the complex
symplectic supermanifold an even (odd) Kähler one, when the even (odd) symplectic structure is defined by the
expression

\[
\Omega_{\kappa} = i(-1)^{p_A(p_B + k+1)} g_{(\kappa)\bar{A}B} dz^A \wedge d\bar{z}^B, \tag{4.170}
\]

where

\[
g_{(\kappa)\bar{A}B} = (-1)^{(p_A + k+1)(p_B + k+1)+k+1} \frac{y(g_{(\kappa)BA})}{y(g_{(\kappa)\bar{A}B})} p(g_{(\kappa)\bar{A}B}) = p_A + p_B + k.
\]

Here and in the following, the index \( \kappa = 0(1) \) denotes the even(odd) case.

The Kähler potential on the supermanifold is a local real even (odd) function \( K_{\kappa}(z, \bar{z}) \) defining the Kähler
structure

\[
g_{(\kappa)\bar{A}B} = \frac{\partial \theta^A}{\partial z^A} \frac{\partial \theta^B}{\partial \bar{z}^B} K_{\kappa}(z, \bar{z}). \tag{4.171}
\]

As in the usual case, \( K_{\kappa} \) is defined up to arbitrary holomorphic and antiholomorphic functions.

With the even (odd) form \( \Omega_{\kappa} \) one can associate the even (odd) Poisson bracket

\[
\{ f, g \}_{\kappa} = i \left( \frac{\partial f}{\partial z^A} g_{(\kappa)\bar{A}B} \frac{\partial g}{\partial \bar{z}^B} - (-1)^{(p_A + k)(p_B + k)} \frac{\partial f}{\partial z^A} g_{(\kappa)\bar{A}B} \frac{\partial g}{\partial \bar{z}^B} \right), \tag{4.172}
\]

where

\[
g_{(\kappa)\bar{A}B} g_{(\kappa)BA} = \delta^A_C, \quad g_{(\kappa)\bar{A}B} = (-1)^{(p_A + k)(p_B + k)} g_{(\kappa)\bar{B}A}.
\]

Example. Let us consider the supermanifold \( \Lambda M \) associated with the tangent bundle of the Kähler manifold
\( M_0 \). On this supermanifold one can define the even and odd Kähler potentials[27]

\[
K_0 = K(z, \bar{z}) + F(i g_{a\bar{a}} \sigma^a \bar{\sigma}^\bar{a}), \quad K_1 = \frac{\partial K(z, \bar{z})}{\partial \bar{z}^a} \sigma^a + \frac{\partial K(z, \bar{z})}{\partial z^a} \bar{\sigma}^\bar{a}, \tag{4.173}
\]
where $K(z,\bar{z})$ is a Kähler potential on $M_0$, $g_{\alpha\beta} = \partial^2 K/\partial z^\alpha \partial \bar{z}^\beta$, and $F(x)$ is a real function which obeys the condition $F'(0) \neq 0$. It is clear that these functions define even and odd Kähler structures on $\Lambda M_0$, respectively.

Finally, let us notice that the analog of the Liouville measure for the even supersymplectic symplectic structure $\Omega_0$ reads
\[ \rho = \sqrt{\text{Ber}\Omega_{(0)AB}}, \] (4.174)
while the odd symplectic structure has no similar invariant [28]. Indeed, one can verify that the even super-Hamiltonian vector field is always divergenceless, $\text{str}\{H,\}_{0} = 0$ (similarly to the non-superHamiltonian vector field), while in the case of the odd super-Hamiltonian vector field this property of the Hamiltonian vector field fails. As a consequence, in the latter case the so-called $\Delta$-operator can be defined [30], which plays a crucial role in the Batalin-Vilkovisky formalism (Lagrangian BRST quantization formalism) [29].

**Odd super-Hamiltonian mechanics**

Let us consider the supermanifold $\Lambda M$, associated with the tangent bundle of the symplectic manifold $(M, \omega)$, i.e. the external algebra of $(M, \omega)$. In other words, the odd coordinates $\theta^i$ transform from one chart to another like $dx^i$, and they can be interpreted as the basis of the 1-forms on $M$. By the use of the $\omega$ we can equip $\Lambda M$ with the odd symplectic structure
\[ \Omega_1 = d (\omega_{ij} \theta^i dx^j) = \omega_{ij} d x^i \wedge d \theta^j + \frac{1}{2} \omega_{kij} \theta^i dx^k \wedge dx^i. \] (4.175)
The corresponding odd Poisson brackets are defined by the following relations:
\[ \{x^i, x^j\}_1 = 0, \quad \{x^i, \theta^j\}_1 = \omega^{ij}, \quad \{\theta^i, \theta^j\}_1 = \frac{\partial \omega^{ij}}{\partial x^k} \theta^k, \] (4.176)
where $\omega^{ij} \omega_{jk} = \delta^i_k$.

Let us define, on $\Lambda M$, the even function
\[ F = -\frac{1}{2} \theta^j \omega_{ij} \theta^i, : \quad \{F, F\}_1 = 0, \] (4.177)
where the latter equation holds due to the closeness of $\omega$. By making use of this function, one can define the map of any function on $M$ in the odd function on $\Lambda M$
\[ f(x) \rightarrow Q_f(x, \theta) = \{f(x), F(x, \theta)\}_1, \] (4.178)
which possesses the following important property:
\[ \{f(x), g(x)\} = \{f(x), Q_g(x, \theta)\}_1 \text{ for any } f(x), g(x). \] (4.179)
In particular, (4.178) maps the Hamiltonian mechanics $(M, \omega, H(x))$ in the following super-Hamiltonian one: $(\Lambda M, \Omega_1, Q_H = \{H, F\}_1)$, where $Q_H$ plays the role of the odd Hamiltonian on $\Lambda M$.

The functions $H, F, Q_H$ form the superalgebra
\[ \{H \pm F, H \pm F\}_1 = \pm 2Q_H, \]
\[ \{H + F, H - F\}_1 = \{H \pm F, Q_H\}_1 = \{Q_H, Q_H\}_1 = 0, \] (4.180)
i.e. the resulting mechanics possesses the supersymmetry transformation defined by the “supercharge” $H + F$. This superalgebra has a transparent interpretation in terms of base manifold $(M, \omega)$
\[ \{H, \}_{1} = \xi^i_H \frac{\partial}{\partial \theta^i} \rightarrow \dot{i}_H - \text{contraction with } \xi_H, \]
\[ \{F, \}_{1} = \theta^i \frac{\partial}{\partial x^i} \rightarrow \dot{d} - \text{exterior differential}, \]
\[ \{Q, \}_{1} = \xi^i_H \frac{\partial}{\partial x^i} + \xi^i_H \theta^k \frac{\partial}{\partial \theta^i} \rightarrow \dot{\mathcal{L}}_H - \text{Lie derivative along } \xi_H, \]
while, using the Jacobi identity (4.160), we get
\[ \{ H, F \}_1 = Q_H \to \hat{d}i_H + i_H \hat{d} = \hat{\mathcal{L}}_H \to \text{homotopy formula.} \] (4.181)

Hence, the above dynamics could be useful for the description of the differential calculus on the symplectic (and Poisson) manifolds. Particularly, it has a nice application in equivariant cohomology and related localization formulæ (see [31] and refs therein).

However, the presented supersymmetric model has no deep dynamical meaning, since the odd Poisson brackets do not admit any consistent quantization scheme. Naively, this is reflected in the fact that conjugated operators should have opposite Grassmann grading, so that the Planck constant must be a Grassmann-odd number.

Moreover, the presented supersymmetric mechanics is not interesting even from the classical viewpoint. Its equations of motion read
\[ \frac{d\xi^i}{dt} = \{ x^i, Q_H \}_1 = \xi_{1H}, \quad \frac{d\theta^i}{dt} = \{ \theta^i, Q_H \}_1 = \frac{\partial \xi_{1H}}{\partial x^j} \theta^j, \]
i.e. the “fermionic” degrees of freedom have no impact in the dynamics of the “bosonic” degrees of freedom.

Nevertheless, the odd Poisson brackets are widely known, since 1981, in the theoretical physics community under the name of “antibrackets”. That was the year, when Batalin and Vilkovisky suggested their Covariant Lagrangian BRST quantization formalism (which is known presently as the Batalin-Vilkovisky formalism) [29], where the antibrackets (odd Poisson brackets) play the key role. However, only decades after, this elegant formalism was understood in terms of conventional supergeometrical constructions [30, 26]. It seems that the Batalin-Vilkovisky formalism could also be useful for the geometrical (covariant) formulation of the superfield approach to the construction of supersymmetric Lagrangian field-theoretical and mechanical models [32].

We shall not touch upon these aspects of super-Hamiltonian systems, and will restrict ourselves to the consideration of supersymmetric Hamiltonian systems with even symplectic structure.

**Hamiltonian reduction:** \( \mathbb{C}^{N+1,M} \to \mathbb{C}P^{N,M}, \Lambda \mathbb{C}^{N+1} \to \Lambda \mathbb{C}P^N \)

The procedure of super-Hamiltonian reduction is very similar to the Hamiltonian one. The main difference is in the counting of the dimensionality of the phase superspace. Namely, we should separately count the number of “fermionic” and “bosonic” degrees of freedom, which were eliminated during the reduction.

Instead of describing the extension of the Hamiltonian reduction to the supercase, we shall illustrate it by considering superextensions of the reduction \( \mathbb{C}^{N+1} \to \mathbb{C}P^N \) presented in Third Section. These examples were considered in details in Ref. [33].

Let us consider the complex superspace \( \mathbb{C}^{N+1,M} \) parameterized by the complex coordinates \( (u^\tilde{a}, \eta^n) \), \( \tilde{a} = 0, 1, \ldots, N \), \( n = 1, \ldots, M \). Let us equip it with the canonical symplectic structure
\[ \Omega^0 = i(du^\tilde{a} \wedge d\tilde{u}^\tilde{a} - id\eta^n \wedge d\bar{\eta}^n) \]
and with the corresponding even Poisson bracket
\[ \{ f, g \}_0 = i \left( \frac{\partial f}{\partial u^\tilde{a}} \frac{\partial g}{\partial \tilde{u}^\tilde{a}} - \frac{\partial f}{\partial \tilde{u}^\tilde{a}} \frac{\partial g}{\partial u^\tilde{a}} \right) + \frac{\partial f}{\partial \eta^n} \frac{\partial g}{\partial \bar{\eta}^n} + \frac{\partial f}{\partial \bar{\eta}^n} \frac{\partial g}{\partial \eta^n}. \] (4.182)

The (super-)Hamiltonian action of the \( U(1) \) group is given, on this space, by the generator
\[ \mathcal{J}_0 = u^\tilde{a} \tilde{u}^\tilde{a} - i\eta^n \bar{\eta}^n. \] (4.183)

For the reduction of \( \mathbb{C}^{N+1,M} \) by this generator, we have to factorize the \( (2N + 1.2M)_{\mathbb{R}} \)-dimensional level supersurface
\[ \mathcal{J}_0 = r_0^2 \] (4.184)
by the even super-Hamiltonian vector field \( \{ \mathcal{J}_0, \} \) (which is tangent to that surface). Hence, the resulting phase superspace is a \( (2N;2M)_{\mathbb{R}} \)-dimensional one.

Hence, for the role of local coordinates of the reduced phase space, we have to choose the \( N \) even and \( M \) odd complex functions commuting with \( \mathcal{J}_0 \). On the chart \( u^\tilde{a} \neq 0 \), appropriate functions are the following ones:
\[ z^A_{(\tilde{a})} = \left( \begin{array}{c} z^a_{(\tilde{a})} = \frac{u^a}{u^\tilde{a}}, \quad \theta^\tilde{a}_{(\tilde{a})} = \frac{\eta^k}{u^a}, \quad a \neq \tilde{a} \end{array} \right) : \{ z^A_{(\tilde{a})}, \mathcal{J}_0 \}_0 = 0. \] (4.185)
The reduced Poisson brackets could be defined by the expression \( \{ f, g \}_{0}^{\text{red}} = \{ f, g \}_{0} |_{\gamma_0 = r_0^2} \), where \( f, g \) are functions depending on the coordinates \( z^{A}_{(a)}, \tilde{z}^{A}_{(b)} \). Straightforward calculations yield the result

\[
\{ z^{A}, \tilde{z}^{B} \}_{0}^{\text{red}} = \{ \tilde{z}^{A}, w^{B} \}_{0}^{\text{red}} = 0,
\]

\[
\{ z^{A}, z^{B} \}_{0}^{\text{red}} = (i)^{pA_pB+1} \frac{1 + (-i)^{pC} z^{C} z^{C}}{r_0^2} (\delta^{AB} + (-i)^{pA_pB} z^{A} \tilde{w}^{B}).
\]

It is seen that these Poisson brackets are associated with a Kähler structure. This Kähler structure is defined by the potential

\[
K = r_0^2 \log(1 + (-i)^{pC} z^{C} z^{C}). \tag{4.186}
\]

The transition functions from the \( \tilde{a} \)-th chart to the \( \tilde{b} \)-th one look as follows:

\[
z^{\tilde{c}}_{(a)} = \frac{z^{\tilde{c}}_{(b)}}{z^{\tilde{a}}_{(b)}}, \quad \Theta^b_{(a)} = \frac{\theta^b_{(a)}}{z^{\tilde{a}}_{(b)}}, \quad \text{where} \quad z^{\tilde{a}}_{(b)} = \left( w^a_{(b)}, w^0_{(b)} = 1 \right). \tag{4.187}
\]

Upon these transformations the Kähler potential changes on the holomorphic and anti-holomorphic functions, i.e. the reduced phase space is indeed a Kähler supermanifold. We shall refer to it as \( \mathbb{C}P^{N,M} \). The quantization of this supermanifold is considered in [34].

Now, let us consider the Hamiltonian reduction of the superspace \( \mathbb{C}P^{N+1,N+1} \) by the action of the \( N = 2 \) superalgebra, given by the generators

\[
\mathcal{J}_0 = u^a \tilde{u}^\tilde{a} - i \eta^a \eta^\tilde{a}, \quad \Theta^+ = u^a \tilde{\eta}^\tilde{a}, \quad \Theta^- = \tilde{u}^\tilde{a} \eta^a ; \quad \{ \Theta^+, \Theta^- \} = \{ \Theta^\pm, \mathcal{J}_0 \} = 0. \tag{4.188}
\]

The equations

\[
J_0 = r_0^2, \quad \Theta^\pm = 0 \tag{4.189}
\]

define the \( (2N + 1.2N) \)-dimensional level surface \( M_{0,0,0} \). The reduced phase superspace can be defined by the factorization of \( M_{0,0,0} \) by the action of the tangent vector field \( \{ \mathcal{J} \} \). Hence, the reduced phase superspace is a \( (2N.2N) \)-dimensional one. The conventional local coordinates of the reduced phase superspace could be chosen as follows (on the chart \( u^b \neq 0 \)):

\[
\sigma^a = -i \{ z^a, \Theta^+ \} = \theta^a - \theta^0 z^a, \quad w^a = z^a + i \frac{\Theta^-}{\mathcal{J}_0} \sigma^a, \tag{4.190}
\]

where \( z^a, \theta^a, \sigma^a \) are defined by (4.185). The reduced Poisson brackets are defined as follows:

\[
\{ f, g \}_{0}^{\text{red}} = \{ f, g \}_{0} |_{\mathcal{J} = r_0^2, \Theta^\pm = 0}, \tag{4.191}
\]

where \( f, g \) are the functions on \( (w^a, \sigma^a) \). Straightforward calculations result in the following relations:

\[
\{ w^A, w^B \}_{0}^{\text{red}} = \{ \tilde{w}^A, w^B \}_{0}^{\text{red}} = 0, \quad \text{where} \quad w^A = (w^a, \sigma^a)
\]

\[
\{ w^a, \tilde{w}^b \}_{0}^{\text{red}} = i \frac{A}{r_0^2} (\delta^{ab} + w^a \tilde{w}^b) - \frac{\sigma^a \tilde{\sigma}^b}{r_0^2}.
\]

\[
\{ w^a, \tilde{\sigma}^b \}_{0}^{\text{red}} = i \frac{A}{r_0^2} (w^a \tilde{\sigma}^b + \mu (\delta^{ab} + w^a \tilde{w}^b)) \tag{4.191}
\]

\[
\{ \sigma^a, \tilde{\sigma}^b \}_{0}^{\text{red}} = A \frac{A}{r_0^2} \left( (1 + i \mu) \delta^{ab} + w^a \tilde{w}^b + i (\sigma^a + \mu w^a) (\tilde{\sigma}^b + \tilde{\mu} \tilde{w}^b) \right),
\]

and

\[
A = 1 + w^a \tilde{w}^a - i \sigma^a \tilde{\sigma}^a + i \sigma^a \tilde{w}^a \tilde{\sigma}^b w^b 1 + w^a \tilde{w}^a, \quad \mu = \frac{\tilde{w}^a \sigma^a}{1 + w^a \tilde{w}^a}.
\]

These Poisson brackets are associated with the Kähler structure defined by the potential

\[
K = r_0^2 \log A(w, \tilde{w}, \sigma, \tilde{\sigma}) = r_0^2 \log(1 + w^a \tilde{w}^a) + r_0^2 \log(1 - i g_{ab} \sigma^a \tilde{\sigma}^b). \tag{4.192}
\]

where \( g_{ab}(w, \tilde{w}) \) is the Fubini-Study metric on \( \mathbb{C}P^N \).
The transition functions from the $\tilde{a}$-th chart to the $\tilde{b}$-th one reads
\[ w^\tilde{b}(\tilde{a}) = \frac{w^\tilde{c}(\tilde{a})}{w^\tilde{b}(\tilde{a})}, \quad \sigma^\tilde{b}(\tilde{a}) = \frac{\sigma^\tilde{c}(\tilde{a}) x^\tilde{b}(\tilde{a})}{(w^\tilde{b}(\tilde{a}))^2}, \]
where $(w^\tilde{a}) = 1, \sigma^\tilde{b}(\tilde{a}) = 0)$. Hence, $\sigma^a$ transforms like $du^a$, i.e. the reduced phase superspace is $\Lambda\mathbb{CP}^N$, the external algebra of the the complex projective space $\mathbb{CP}^N$.

Remark 1. On $\mathbb{C}^{N+1, N+1}$ one can define the odd Kähler structure as well, $\Omega^1 = du^n \wedge d\bar{\eta}^n + d\bar{a}^n \wedge d\eta^n$. It could be reduced to the odd Kähler structure on $\Lambda\mathbb{CP}^N$ by the action of the generators
\[ J_0 = z \bar{z}, \quad Q = z \bar{\eta} + \bar{z} \eta. \]

Remark 2. The generalization of the reduction $T^*\mathbb{C}^2 \to T^*\mathbb{R}^3$, where the latter is specified by the presence of a Dirac monopole, is also straightforward. One should consider the $(4,M)$-dimensional superspace equipped with the canonical even symplectic structure $\Omega_0 = dx \wedge dz + d\bar{z} \wedge d\bar{z} + d\eta \wedge d\bar{\eta}$, and reduce it by the Hamiltonian action of the $U(1)$ group given by the generator $J = i\pi z - i\bar{\pi} \bar{z} - i\eta \bar{\eta}$. The resulting space is a $(6.2M)\mathbb{R}$-dimensional one. Its even local coordinates could be defined by the same expressions, as in the bosonic case, Eq.(2.84), while the odd coordinates could be chosen as follows: $\theta^m = f(z\bar{z})\bar{z}0\eta^m$.

5 Supersymmetric mechanics

In the previous Sections we presented some basic elements of the Hamiltonian and super-Hamiltonian formalism. We paid special attention to the examples, related with Kähler geometry, keeping in mind that the latter is of a special importance in supersymmetric mechanics. Indeed, the incorporation of the Kähler structure(s) is one of the standard ways to increase the number of supersymmetries of the system.

Our goal is to construct the supersymmetric mechanics with $\mathcal{N} \geq 2$ supersymmetries. This means that, on the given phase superspace equipped with even symplectic structure, we should construct the Hamiltonian $\mathcal{H}$ which has $\mathcal{N} = N$ odd constants of motion $Q_i$ forming the superalgebra
\[ \{Q_i, Q_j\} = 2\delta_{ij}\mathcal{H}, \quad \{Q_i, \mathcal{H}\} = 0. \quad (5.193) \]
This kind of mechanics is referred to as “$\mathcal{N} = N$ supersymmetric mechanics”.

It is very easy to construct the $\mathcal{N} = 1$ supersymmetric mechanics with single supercharges: we should simply take the square (under a given nondegenerate even Poisson bracket) of the arbitrary odd function $Q_1$, and consider the resulting even function as the Hamiltonian
\[ \{Q_1, Q_1\} \equiv 2\mathcal{H}_{SUSY} \Rightarrow \{Q_1, \mathcal{H}_{SUSY}\} = 0. \quad (5.194) \]
However, the case of $\mathcal{N} = 1$ supersymmetric mechanics is not an interesting system, both from the dynamical and field-theoretical viewpoints.

If we want to construct the $\mathcal{N} > 1$ supersymmetric mechanics, we must specify both the underlying system and the structure of phase superspace.

Let us illustrate it on the simplest examples of $\mathcal{N} = 2$ supersymmetric mechanics. For this purpose, it is convenient to present the $\mathcal{N} = 2$ superalgebra as follows:
\[ \{Q^+, Q^-\} = \mathcal{H}, \quad \{Q^\pm, Q^\mp\} = 0, \quad (5.195) \]
where $Q^\pm = (Q_1 \pm iQ_2)/\sqrt{2}$. Hence, we have to find the odd complex function, which is nilpotent with respect to the given nondegenerate Poisson bracket, in order to construct the appropriate system.

Let us consider a particular example, when the underlying system is defined on the cotangent bundle $T^*M_0$, and it is given by (1.12).

In order to supersymmetrize this system, we extend the canonical symplectic structure as follows:
\[ \Omega = dp_a \wedge dx^a + \frac{1}{2} R_{abcd} \theta^a \theta^b dx^c \wedge dx^d + g_{ab} D\theta^a \wedge D\theta^b, \quad (5.196) \]
where $D\theta^a = d\theta^a + \Gamma^a_{bc} \theta^b dx^c$, and $\Gamma^a_{bc}, R_{abcd}$ are the components of the connection and curvature of the metrics $g_{ab} dx^a dx^b$ on $M_0$.  

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We choose the following candidate for a complex supercharge:

\[ Q_+ = (p_a + iW_a)\theta^a_+ : \{Q_+, Q_+\} = 0. \]  
(5.197)

Hence, the supersymmetric Hamiltonian could be constructed by the calculation of the Poisson brackets of these supercharges.

\[ \mathcal{H} = \{Q_+, Q_-\} = \frac{1}{2}\theta^a_+ (p_a + W_a) + W_{ab}\theta^a_+ \theta^b_- + R_{abcd}\theta^a_- \theta^b_- \theta^c_- \theta^d_- . \]  
(5.198)

The "minimal" coupling of the magnetic field, \( \Omega \to \Omega + F_{ab}dx^a \wedge dx^b \), breaks the \( \mathcal{N} = 2 \) supersymmetry of the system

\[ \{Q_+, Q_-\} = F_{ab}\theta^a_+ \theta^b_-, \quad \{Q_+, Q_-\} = J + iF_{ab}\theta^a_+ \theta^b_- . \]

Notice that the Higgs oscillator on the sphere \( S^N \), considered in Section 3, could be supersymmetrised in this way, choosing \( W = \frac{q}{2} \log \frac{2+z^2}{2-z^2} \), with \( q \) being the conformal coordinates of the sphere.

One of the ways to extend this construction to \( \mathcal{N} = 4 \) supersymmetric mechanics is the doubling of the number of odd degrees of freedom. It was considered, within the (Lagrangian) superfield approach in Ref.[35]. In this paper the authors considered the \( (2N,2N)_{\text{rel}} \)-dimensional superspace and the supercharges containing term cubic on odd variables. Calculating the Poisson brackets, the authors found that the admissible metrics of the configuration space of that system should have the following local form:

\[ g_{ab} = \frac{\partial^2 A(z)}{\partial x^a \partial x^b} . \]  
(5.199)

The admissible set of potentials looks, in this local coordinates, as follows: \( V = g_{ab}c^{ab} + g^{ab}d_{af} \), where \( c^{ab} \) and \( d_{ab} \) are constant matrices.

So, considering the Hamiltonian system with generic phase spaces, we found that without any efforts it could be extended to \( \mathcal{N} = 1 \) supersymmetric mechanics. For the construction of \( \mathcal{N} = 2 \) supersymmetric mechanics we were forced to restrict ourselves to systems on the cotangent bundle of Riemann manifolds. Even after this strong restriction, we found that the inclusion of a magnetic field breaks the supersymmetry of the system. On the other hand, in trying to construct \( \mathcal{N} = 4 \) supersymmetric mechanics, we found that in this case even the metric of the configuration space and the admissible set of potentials are strongly restricted.

In further examples we shall show that the transition to Kähler geometry makes these restrictions much weaker.

**\( \mathcal{N} = 2 \) supersymmetric mechanics with Kähler phase space**

Let us consider a supersymmetric mechanics whose phase superspace is the external algebra of the Kähler manifold \( M, g_{ab}(z, \bar{z})dz^a d\bar{z}^b \) is the phase space of the underlying Hamiltonian mechanics [36]. The phase superspace is \( (D|D)_{\text{rel}} \)-dimensional supermanifold equipped with the Kähler structure

\[ \Omega = i\partial \bar{\partial} \left( K - ig_{ab}\theta^a \theta^b \right) = i \left( g_{ab} + iR_{abcd}\theta^d \theta^a \right)dz^a \wedge d\bar{z}^b + g_{ab}D\theta^a \wedge D\bar{\theta}^b , \]

where \( D\theta = d\theta + \Gamma^a_{bc} \theta^c dz^b \), and \( \Gamma^a_{bc} \), \( R_{abcd} \) are the Cristoffel symbols and curvature tensor of the underlying Kähler metrics \( g_{ab} = \partial_a \partial_b \bar{K}(z, \bar{z}) \), respectively.

The corresponding Poisson bracket can be presented in the form

\[ \{ , \} = ig^{ab}\nabla_a \wedge \nabla_b + g^{ab} \frac{\partial}{\partial \theta^a} \wedge \frac{\partial}{\partial \theta^b} \]  
(5.200)

where

\[ \nabla_a = \frac{\partial}{\partial z^a} - \Gamma^c_{ab} \theta^b \frac{\partial}{\partial \theta^c} \], \quad \tilde{g}^{-1} = (g_{ab} + iR_{abcd}\theta^d \theta^a) . \]

On this phase superspace one can immediately construct \( \mathcal{N} = 2 \) supersymmetric mechanics, defined by the supercharges

\[ Q_+^0 = \partial_a K(z, \bar{z}) \theta^a, \quad Q_-^0 = \partial_a K(z, \bar{z}) \bar{\theta}^a \]  
(5.201)

where \( K(z, \bar{z}) \) is the Kähler potential of \( M \), defined up to holomorphic and anti-holomorphic functions, \( K(z, \bar{z}) \rightarrow K(z, \bar{z}) + U(z) + \bar{U}(\bar{z}) \).
Let us show that the Hamiltonian mechanics (1.12) could be easily extended to the \( N = 4 \) manifolds and the mechanics, when the configuration space \( M \) and the potential term has the form of supercharges and get the underlying phase space to be the cotangent bundle of the Kähler manifold, we will double the number of supercharges and get the N = 4 supersymmetric mechanics. On the other hand, the hyper-Kähler manifolds are the cotangent bundle of the Kähler manifolds equipped with Ricci-flat metrics.

Another example of \( N = 2 \) supersymmetric mechanics is defined by the supercharges

\[
Q^c_+ = \partial_a G(z, \bar{z}) \theta^a, \quad Q^c_- = \partial_a G(z, \bar{z}) \bar{\theta}^a,
\]

where the real function \( G(z, \bar{z}) \) is the Killing potential of the underlying Kähler structure

\[
\partial_a \partial_b G - \Gamma^c_{ab} \partial_c G = 0, \quad G^a(z) = g^{ab} \partial_b G(z, \bar{z}).
\]

In this case the Hamiltonian of system reads

\[
\mathcal{H}^c = g_{ab} G^a G^b + i \bar{\theta}^d G_{c\bar{d}} \bar{g}^{\bar{d}b} G_{c\bar{b}} \theta^c,
\]

where \( G_{ab} = \partial_a \partial_b G(z, \bar{z}) \).

The commutators of the supercharges in these particular examples read

\[
\{Q^c_+, Q^0_\pm\} = \mathcal{R}_\pm, \quad \{Q^c_+, Q^0_\mp\} = \mathcal{Z},
\]

where

\[
\mathcal{Z} \equiv G(z, \bar{z}) + i G_{ab}(z, \bar{z}) \theta^a \bar{\theta}^b, \quad \mathcal{R}_+ = i \theta^{c\bar{d}} K_{c\bar{d}} \bar{g}^{\bar{d}b} G_{b\bar{c}} \theta^c, \quad \mathcal{R}_- = \bar{\mathcal{R}}_+.
\]

Hence, introducing the supercharges

\[
\Theta_\pm = Q^0_\pm \pm i Q^0_\mp
\]

we can define \( N = 2 \) SUSY mechanics specified by the presence of the central charge \( \mathcal{Z} \)

\[
\{\Theta_+, \Theta_-\} = \mathcal{H}, \quad \{\Theta_+, \Theta_\pm\} = \pm i \mathcal{Z} \quad \{\mathcal{Z}, \Theta_\pm\} = 0, \quad -\{\mathcal{H}, \Theta_\mp\} = 0, \quad \{\mathcal{Z}, \mathcal{H}\} = 0.
\]

The Hamiltonian of this generalized mechanics is defined by the expression

\[
\tilde{\mathcal{H}} = \mathcal{H}_0 + \mathcal{H}_c + i \mathcal{R}_+ - i \mathcal{R}_-.
\]

A “fermionic number” is of the form

\[
\tilde{\mathcal{F}} = ig_{ab} \theta^a \bar{\theta}^b : \{\tilde{\mathcal{F}}, \Theta_\pm\} = \pm i \Theta_\pm.
\]

It seems that, on the external algebra of the hyper-Kähler manifold, in the same manner one could construct \( N = 4 \) supersymmetric mechanics. On the other hand, the hyper-Kähler manifolds are the cotangent bundle of the Kähler manifolds equipped with Ricci-flat metrics.

We shall demonstrate, in the next examples, that these restrictions can be too strong. Namely, choosing the underlying phase space to be the cotangent bundle of the Kähler manifold, we will double the number of supercharges and get the \( N = 4 \) supersymmetric mechanics on the cotangent bundles of generic Kähler manifolds and the \( N = 8 \) ones on the cotangent bundles of the special Kähler manifolds.

### \( N = 4 \) supersymmetric mechanics

Let us show that the Hamiltonian mechanics (1.12) could be easily extended to the \( N = 4 \) supersymmetric mechanics, when the configuration space \( M_0 \) is the Kähler manifold \( (M_0, g_{ab} dz^a d\bar{z}^b) \), \( g_{ab} = \partial^2 K(z, \bar{z}) / \partial z^a \partial \bar{z}^b \), and the potential term has the form

\[
V(z, \bar{z}) = \frac{\partial U(z)}{\partial z^a} g^{\bar{a}b} \frac{\partial U(z)}{\partial \bar{z}^b}.
\]

For this purpose, let us define the supersymplectic structure

\[
\Omega = \omega_0 - i \theta \bar{\theta} g = d\pi_\alpha \wedge dz^\alpha + d\bar{\pi}_a \wedge d\bar{z}^a + R_{ab\bar{d}} \bar{\eta}_b d\pi_\alpha \wedge d\bar{z}^a + g_{ab} D\eta^a \wedge D\bar{\eta}_b.
\]
where

\[ g = ig_{ab} \eta^a \sigma_0 \bar{\eta}^b. \]
\[ D \eta^a_i = d \eta^a_i + \Gamma^a_{bc} \eta^b_i dz^c, \quad i = 1, 2 \]
\[ \Gamma^a_{bc}, R_{abcd} \] are the connection and curvature of the Kähler structure, respectively, and the odd coordinates \( \eta^a_i \) belong to the external algebra \( \Lambda M_0 \), i.e. they transform as \( dz^a \). This symplectic structure becomes canonical in the coordinates \((p_a, \chi^m)\)

\[
p_a = p_a - i \partial_a g, \quad \chi^m = e^m_b \eta^b_i = e^m_b \eta^b, \quad \Omega = dp_a \wedge dz^a + d \bar{p}_a \wedge d \bar{z}^a + dx^m \wedge d \chi^m,
\]

where \( e^m_a \) are the einbeins of the Kähler structure: \( e^m_a \delta_{mn} \bar{e}^n_b = g_{ab} \). The corresponding Poisson brackets are defined by the following non-zero relations (and their complex-conjugates):

\[
\{ \pi_a, \chi^b \} = \delta^b_a, \quad \{ \pi_a, \eta^b_i \} = -\Gamma^b_{ac} \eta^c_i, \quad \{ \pi_a, \bar{\eta}^b_i \} = -R_{abcd} \bar{\eta}^c_k \bar{\eta}^d_l, \quad \{ \eta^b_i, \bar{\eta}^b_j \} = g^{ab} \delta_{ij},
\]

Let us represent the \( \mathcal{N} = 4 \) supersymmetry algebra as follows:

\[
\{ Q^+_i, Q^-_j \} = \delta_{ij} \mathcal{H}, \quad \{ Q^\pm_i, Q^\pm_j \} = \{ Q^\pm_i, \mathcal{H} \} = 0, \quad i, j = 1, 2,
\]

and choose the supercharges given by the functions

\[
Q^+_1 = \pi a \eta^a + i U_a \bar{\eta}^a, \quad Q^+_2 = \pi a \eta^a - i U_a \bar{\eta}^a.
\]

Then, calculating the commutators (Poisson brackets) of these functions, we get that the supercharges (5.215) belong to the superalgebra (5.214), when the functions \( U_a, \bar{U}_a \) are of the form

\[
U_a(z) = \frac{\partial U(z)}{\partial z^a}, \quad \bar{U}_a(z) = \frac{\partial \bar{U}(\bar{z})}{\partial \bar{z}^a},
\]

while the Hamiltonian reads

\[
\mathcal{H} = g^{ab}(\pi a \bar{\eta}^a + U_a \bar{U}^b) - i U_a \delta \bar{\eta}^a + i \bar{U}_a \bar{\eta}^a \eta^b - R_{abcd} \bar{\eta}^c_k \bar{\eta}^d_l \eta^b k.\]

(5.217)

where \( U_{a,b} \equiv \partial_a \partial_b U - \Gamma^c_{ab} \partial_c U \).

Now, following [37], let us extend this system to \( \mathcal{N} = 4 \) supersymmetric mechanics with central charge

\[
\{ \Theta^+_i, \Theta^-_j \} = \delta_{ij} \mathcal{H} + \mathcal{Z} \sigma^3_{ij}, \quad \{ \Theta^\pm_i, \Theta^\pm_j \} = 0, \quad \{ \mathcal{Z}, \mathcal{H} \} = \{ \mathcal{Z}, \Theta^\mp_k \} = 0.
\]

(5.218)

For this purpose one introduces the supercharges

\[
\Theta^+_1 = (\pi a + i G_a (z, \bar{z})) \eta^a + i \bar{U}_a (z) \bar{\eta}^a, \quad \Theta^+_2 = (\pi a - i G_a (z, \bar{z})) \eta^a - i \bar{U}_a (z) \bar{\eta}^a.
\]

(5.219)

where the real function \( G(z, \bar{z}) \) obeys the conditions (5.204) and \( \partial_b G g^{ab} U_b = 0 \). So, \( G \) is a Killing potential defining the isometry of the underlying Kähler manifold (given by the vector \( \mathbf{G} = G^a (z) \partial_a + G^\alpha (z) \partial_\alpha \), \( G^a = ig^{ab} \partial_b G \)) which leaves the holomorphic function \( U(z) \) invariant

\[
\mathcal{L}_G U = 0 \Rightarrow G^a(z) U_a(z) = 0.
\]

Calculating the Poisson brackets of these supercharges, we get explicit expressions for the Hamiltonian

\[
\mathcal{H} \equiv g^{ab}(\pi a \bar{\eta}^a + G_a G_b + U_a \bar{U}^b) - i U_a \delta \bar{\eta}^a + i \bar{U}_a \bar{\eta}^a \eta^b + \frac{1}{2} G_{ab}(\bar{\eta}^c_k \bar{\eta}^d_l) - R_{abcd} \bar{\eta}^c_k \bar{\eta}^d_l \eta^b k.
\]

(5.220)

and for the central charge

\[
\mathcal{Z} = i(G^a \pi a + G^\alpha \pi_\alpha) + i \frac{1}{2} \partial_a \partial_b G (\bar{\eta}^a \sigma_3 \bar{\eta}^b).
\]

(5.221)

It can be checked by a straightforward calculation that the function \( \mathcal{Z} \) indeed belongs to the center of the superalgebra (5.218). The scalar part of each phase with standard \( \mathcal{N} = 2 \) supersymmetry can be interpreted as a particle moving on the Kähler manifold in the presence of an external magnetic field, with strength \( F = i G_{ab} dz^a \wedge d \bar{z}^b \), and in the potential field \( U_a(z) g^{ab} \bar{U}_b(z) \).

Assuming that \( (M_0, g_{ab} dz^a \wedge d \bar{z}^b) \) is the hyper-Kähler metric, \( U(z) + \bar{U}(\bar{z}) \) is a tri-holomorphic function and \( G(z, \bar{z}) \) defines a tri-holomorphic Killing vector, one should get \( \mathcal{N} = 8 \) supersymmetric mechanics. In this case, instead of the phase with standard \( \mathcal{N} = 2 \) supersymmetry arising in the Kähler case, we shall get the phase with standard \( \mathcal{N} = 4 \) supersymmetry. This system could be straightforwardly constructed by the dimensional reduction of the \( \mathcal{N} = 2 \) supersymmetric \((1+1)\) dimensional sigma-model by Alvarez-Gaumé and Freedman [40].
$N = 8$ mechanics

We have seen that the transition from the generic Riemann space to the generic Kähler space allows one to double the number of supersymmetries from $N = 2$ to $N = 4$, with the appropriate restriction of the admissible set of potentials.

On the other hand, we mentioned that the doubling of the number of odd variables and the restriction the Riemann metric allow one to construct the $N = 4$ supersymmetric mechanics [35]. Now, following the paper [38], we shall show that a similar procedure, applied to the systems on Kähler manifolds, permits to construct the $N = 8$ supersymmetric mechanics, with the supersymmetry algebra

$$\{Q_{iα}, Q_{jβ}\} = \{\overline{Q}_{iα}, \overline{Q}_{jβ}\} = 0,$$

$$\{Q_{iα}, \overline{Q}_{jβ}\} = \epsilon_{αβ}\epsilon_{ij}\mathcal{H}_{\text{SUSY}}, \quad (5.222)$$

where $i, j = 1, 2$, $α, β = 1, 2$.

We present the results for the mechanics without (bosonic) potential term. The respective systems with potential terms are constructed in [41].

In order to construct the $N = 8$ supersymmetric mechanics, let us define the $(2d.4d)$-dimensional symplectic structure

$$Ω = dA = dπ_α ∧ dz^α + dπ_α ∧ dz^β − R_{abc}η_α^c dη^b_α dz^a ∧ dz^b + g_{ab}Dη_α^a ∧ Dη^{b|α},$$

where

$$A = π_α dz^α + π_α dz^β + \frac{1}{2}η_α^a g_{ab} Dη^{b|α} + \frac{1}{2}η_α^b g_{ab} Dη^{a|α}, \quad (5.223)$$

and $Dη_α^a = dq_α^a + Γ_α^a b_α^b dz^c$. The corresponding Poisson brackets are given by the following non-zero relations (and their complex-conjugates):

$$\{π_α, z^β\} = δ^β_α, \quad \{π_α, η^a_α\} = −Γ_α^{ab} η^b_α, \quad \{π_α, \overline{π}_β\} = R_{abc} η^c_α η^{d|α}, \quad \{η^a_α, η^{b|α}\} = g^{ab} δ^β_α δ^β_β.$$

(5.224)

Let us search the supercharges among the functions

$$Q_{iα} = π_α η^α_α + \frac{1}{3} f_{abc} T_{iα}^abc, \quad \overline{Q}_{iα} = \overline{π}_α \overline{η}_α + \frac{1}{3} \overline{f}_{abc} T_{iα}^{abc} \quad (5.225)$$

where $T_{iα}^{abc} \equiv η^a_β η^{b|β} η^c_α$.

Calculating the mutual Poisson brackets of $Q_{iα}, \overline{Q}_{iα}$ one can get, that they obey the $N = 8$ supersymmetry algebra, provided the following relations hold:

$$\frac{∂}{∂z^d} f_{abc} = 0, \quad R_{abcd} = −f_{ace} g^{ce} \overline{f}_{ebd}. \quad (5.226)$$

The above equations guarantee, respectively, that the first and second equations in (5.222) are fulfilled. Then we could immediately get the $N = 8$ supersymmetric Hamiltonian

$$\mathcal{H}_{\text{SUSY}} = π_α g^{ab} \overline{η}_b + \frac{1}{3} f_{abc} \Lambda_{iα}^{abc} + \frac{1}{3} \overline{f}_{abc} \overline{Λ}_{iα}^{abc} + f_{abc} g^{ce} \overline{f}_{cde} \Lambda_{0}^{abc}, \quad (5.227)$$

where

$$\Lambda_{iα}^{abc} = −\frac{1}{4} η^a_α η^{b|β} η^c_β δ^d_α, \quad \overline{Λ}_{iα}^{abc} = \frac{1}{2} (η^a_α η^{b|α} η^{c|β} η^d_β + η^a_β η^{b|β} η^{c|α} η^d_α),$$

and $f_{abc} = f_{abc} − Γ_{da} f_{ebc} − Γ_{db} f_{aec} − Γ_{dc} f_{abe}$ is the covariant derivative of the third rank covariant symmetric tensor.

The equations (5.226) precisely mean that the configuration space $M_0$ is a special Kähler manifold of the rigid type [42]. Taking into account the symmetrizing of $f_{abc}$ over spatial indices and the explicit expression of $R_{abcd}$ in terms of the metric $g_{ab}$, we can immediately find the local solution for equations (5.226)

$$f_{abc} = \frac{∂^3 f(z)}{∂z^a∂z^b∂z^c}, \quad g_{ab} = e^{iν} \frac{∂^2 f(z)}{∂z^a∂z^b} + e^{-iν} \frac{∂^2 \overline{f}(z)}{∂z^a∂z^b}, \quad (5.228)$$

where $ν = \text{const} \in \mathbb{R}$.

Redefining the local function, $f \rightarrow i e^{-iν} f$, we shall get the $ν$-parametric family of supersymmetric mechanics,

---

1 We use the following convention: $ε_{ij} ε^{jk} = δ^k_i$, $ε_{12} = ε^{21} = 1$. 

---
whose metric is defined by the Kähler potential of a special Kähler manifold of the rigid type. Surely, this local solution is not covariant under arbitrary holomorphic transformation, and it assumes the choice of a distinguished coordinate frame.

The special Kähler manifolds of the rigid type became widely known during last decade due to the so-called “T-duality symmetry”: in the context of $\mathcal{N} = 2, d = 4$ super Yang-Mills theory, it connects the UV and IR limit of the theory[43]. The “T-duality symmetry” is expressed in the line below

\[
(z^a, f(z)) \Rightarrow \left( u_a = \frac{\partial f(z)}{\partial z^a}, \tilde{f}(u) \right), \quad \text{where} \quad \frac{\partial^2 \tilde{f}(u)}{\partial u_a \partial u_c} \frac{\partial f}{\partial z^a \partial z^b} = -\delta^a_b . \tag{5.229}
\]

It is clear that the symplectic structure is covariant under the following holomorphic transformations:

\[
\tilde{z}^a = \tilde{z}^a(z), \quad \tilde{\eta}^a_\beta = \frac{\partial \tilde{z}^a(z)}{\partial z^\beta}, \quad \tilde{\pi}_a = \frac{\partial \tilde{z}^b}{\partial z^a} \pi_b , \tag{5.230}
\]

By the use of (5.230), we can extend the duality transformation (5.229) to the whole phase superspace $(\pi_a, z^a, \eta^a_\beta) \rightarrow (p^a, u_a, \psi_{\alpha|\bar{\beta}})\) to the presymplectic one-form (5.223), we can easily perform the Legendre transformation of the Hamiltonian to the (second-order) Lagrangian

\[
\mathcal{L} = \mathcal{A}(d/dt) - \mathcal{H}_{\text{SUSY}}|_{\pi_a = g_{ab} \tilde{z}^b} =
\]

\[
= g_{ab} \tilde{z}^a \tilde{z}^b + \frac{1}{2} \eta^a_\beta g_{ab} \frac{D\tilde{\eta}^b_\alpha}{dt} + \frac{1}{2} \eta^a_\beta g_{ab} \frac{D\eta^b_\alpha}{dt} - \frac{1}{2} f_{abcd} D^b_{\alpha \beta} - \frac{1}{2} \tilde{f}_{abcd} D_{\alpha \beta} - f_{abc} g^{de} \tilde{f}_{cde} \Lambda_{0}^{abcd} . \tag{5.232}
\]

Here we denoted $d/dt = \tilde{z}^a \partial/\partial z^a + \eta^a_\beta \partial/\partial \eta^a_\beta + c.\ c.\ c.\ c.$ Clearly, the Lagrangian (5.232) is covariant under holomorphic transformations (5.230), and the duality transformation as well. The prepotential $\tilde{f}(u)$ is connected with $f(z)$ by the Legendre transformation

\[
\tilde{f}(u) = \tilde{f}(u, z)|_{u_a = \partial_a f(z)}; \quad \tilde{f}(u, z) = u_a z^a - f(z).
\]

**Supersymmetric Kähler Oscillator**

So far, the Kähler structure allowed us to double the number of supersymmetries in the system. One can hope that in some cases this could be preserved after inclusion of constant magnetic field, since this field usually respects the Kähler structure. We shall show, on the example of the Kähler oscillator (3.137), that it is indeed a case.

Let us consider, following [17, 39], the supersymmetrization of a specific model of Hamiltonian mechanics on the Kähler manifold $(M_0, g_{ab} dz^a dz^b)$ interacting with the constant magnetic field $B$, viz

\[
\mathcal{H} = g^{ab} (\pi_a \bar{\pi}_b + \alpha^2 \partial_a K \bar{\partial}_b K), \quad \Omega_0 = d\bar{\pi}_a \wedge dz^a + d\pi_a \wedge d\bar{z}^a + i B g_{ab} dz^a \wedge d\bar{z}^b , \tag{5.233}
\]

where $K(z, \bar{z})$ is a Kähler potential of configuration space.

Remind, that the Kähler potential is defined up to holomorphic and antiholomorphic terms, $K \rightarrow K + U(z) + \bar{U}(\bar{z})$. Hence, in the limit $\omega \rightarrow 0$ the above Hamiltonian takes the form

\[
\mathcal{H} = g^{ab} (\pi_a \bar{\pi}_b + \partial_a U(z) \bar{\partial}_b \bar{U}(\bar{z})) , \tag{5.234}
\]

i.e. it admits, in the absence of magnetic field, a $\mathcal{N} = 4$ superextension.

Notice, also, that in the “large mass limit”, $\pi_a \rightarrow 0$, this system results in the following one:

\[
\mathcal{H}_0 = \omega^2 g^{ab} \partial_a K \bar{\partial}_b K, \quad \Omega_0 = i B g_{ab} dz^a \wedge d\bar{z}^b ,
\]

which could be easily extended to $\mathcal{N} = 2$ supersymmetric mechanics.
We shall show that, although the system under consideration does not possess a standard $\mathcal{N} = 4$ superextension, it admits a superextension in terms of a nonstandard superalgebra with four fermionic generators, including, as subalgebras, two copies of the $\mathcal{N} = 2$ superalgebra. This nonstandard superextension respects the inclusion of a constant magnetic field.

We use the following strategy. At first, we extend the initial phase space to a $(2N,2N)_{\text{d}}$-dimensional superspace equipped with the symplectic structure

$$
\Omega = \Omega_B - i R_{abcd} \eta^a_i \bar{\eta}^d_j dz^a \wedge d\bar{z}^b + g_{ab} D \eta^a_i \wedge D \bar{\eta}^b_j,
$$

(5.235)

where $\Omega_B$ is given by (1.38). The corresponding Poisson brackets are defined by the following non-zero relations (and their complex-conjugates):

$$
\{\pi_a, \pi_b\} = \delta_a^b, \quad \{\pi_a, \eta^b_i\} = -\Gamma^b_{ac} \eta^c_i, \quad \{\eta^a_i, \eta^b_j\} = \delta^{ab} \delta_{ij},
$$

(5.236)

Then, in order to construct the system with the exact $\mathcal{N} = 2$ supersymmetry (5.195), we shall search for the odd functions $Q^\pm$, which obey the equations $\{Q^+, Q^\pm\} = 0$ (we restrict ourselves to the supersymmetric mechanics whose supercharges are linear in the Grassmann variables $\eta^a_i$, $\bar{\eta}^b_j$).

Let us search for the realization of supercharges among the functions

$$
Q^\pm = \cos \lambda \Theta^\pm_1 + \sin \lambda \Theta^\pm_2,
$$

(5.237)

where

$$
\Theta^+_1 = \pi_a \eta^a_1 + i \bar{\partial}_a W \bar{\eta}^a_2, \quad \Theta^+_2 = \bar{\pi}_a \bar{\eta}^a_2 + i \partial_a W \theta^a_1, \quad \Theta^-_{1,2} = \bar{\Theta}^+_1, \quad \Theta^-_{1,2}
$$

(5.238)

and $\lambda$ is some parameter.

Calculating the Poisson brackets of the functions, we get

$$
\{Q^+, Q^\pm\} = i (B \sin 2\lambda + 2\alpha \cos 2\lambda) \mathcal{F}_\pm,
$$

(5.239)

$$
\{Q^+, Q^-\} = \mathcal{H}^0_{\text{SUSY}} + (B \cos 2\lambda - 2\alpha \sin 2\lambda) \mathcal{F}_3/2.
$$

(5.240)

Here and further, we use the notation

$$
\mathcal{H}^0_{\text{SUSY}} = \mathcal{H} - R_{abcd} \eta^a_1 \bar{\eta}^b_2 \bar{\eta}^c_2 \bar{\eta}^d_1 - i W_{a,b} \eta^a_1 \eta^b_2 + i W_{a,b} \bar{\eta}^a_1 \bar{\eta}^b_2 + B \frac{i g_{ab} \eta^a_1 \bar{\eta}^b_1}{2},
$$

(5.241)

where $\mathcal{H}$ denotes the oscillator Hamiltonian (3.137), and

$$
\mathcal{F} = \frac{i}{2} g_{ab} \eta^a_1 \bar{\eta}^b_2 \sigma_{ij}, \quad \mathcal{F}_\pm = F_1 \pm F_2.
$$

(5.242)

One has, then

$$
\{Q^\pm, Q^\pm\} = 0 \Leftrightarrow B \sin 2\lambda + 2\alpha \cos 2\lambda = 0,
$$

(5.243)

so that $\lambda = \lambda_0 + (i - 1)\pi/2, \quad i = 1,2$.

Here the parameter $\lambda_0$ is defined by the expressions

$$
\cos 2\lambda_0 = \frac{B/2}{\sqrt{\alpha^2 + (B/2)^2}}, \quad \sin 2\lambda_0 = -\frac{\alpha}{\sqrt{\alpha^2 + (B/2)^2}}.
$$

(5.244)

Hence, we get the following supercharges:

$$
Q^\pm_\nu = \cos \lambda_0 \Theta^\pm_1 + (-1)^\nu \sin \lambda_0 \Theta^\pm_2,
$$

(5.245)

and the pair of $\mathcal{N} = 2$ supersymmetric Hamiltonians

$$
\mathcal{H}^0_{\text{SUSY}} = \{Q^+_\nu, Q^-_\nu\} = \mathcal{H}^0_{\text{SUSY}} - (-1)^i \sqrt{\alpha^2 + (B/2)^2} \mathcal{F}_3.
$$

(5.246)

Notice that the supersymmetry invariance is preserved in the presence of a constant magnetic field.

Calculating the commutators of $Q^\pm_1$ and $Q^\pm_2$, we get

$$
\{Q^\pm_1, Q^\pm_2\} = 2 \sqrt{\alpha^2 + (B/2)^2} \mathcal{F}_\pm, \quad \{Q^+_1, Q^-_2\} = 0.
$$

(5.247)
The Poisson brackets between \( \mathcal{F}_\pm \) and \( Q^\pm_\nu \) look as follows:

\[
\{ Q^\pm_i, \mathcal{F}_\pm \} = 0, \quad \{ Q^\pm_i, \mathcal{F}_\mp \} = \pm \epsilon_{ij} Q^\pm_j, \quad \{ Q^\pm_i, \mathcal{F}_3 \} = \pm i Q^\pm_i. \tag{5.248}
\]

In the notation \( S^\pm_1 \equiv Q^\pm_1, \quad S^\pm_2 \equiv Q^\pm_2 \) the whole superalgebra reads

\[
\{ S^\pm_i, S^\mp_j \} = \delta_{ij} \mathcal{H}^0_{SU(2)} + \Lambda \sigma^0_{ij} \mathcal{F}_\mu, \\
\{ S^\pm_i, \mathcal{F}_\mu \} = \pm i \sigma^0_{ij} \mathcal{F}_j, \quad \{ \mathcal{F}_\mu, \mathcal{F}_\nu \} = \epsilon_{\mu\nu\rho} \mathcal{F}_\rho, \tag{5.249}
\]

where

\[
\Lambda = \sqrt{\omega^2 + (B/2)^2}. \tag{5.250}
\]

This is precisely the weak supersymmetry algebra considered by A. Smilga [44]. In the particular case \( \omega = 0 \) it yields the \( N = 4 \) supersymmetric mechanics broken by the presence of a constant magnetic field.

Let us notice the \( \alpha \) and \( B \) appear in this superalgebra in a symmetric way, via the factor \( \sqrt{\alpha^2 + (B/2)^2} \).

**Remark** In the case of the oscillator on \( \mathbb{C}^N \) we can smoothly relate the above supersymmetric oscillator with a \( N = 4 \) oscillator, provided we choose

\[
K = \cos \gamma \, z\bar{z} + \sin \gamma \, (z^2 + \bar{z}^2)/2, \quad \gamma \in [0, \pi/2].
\]

Hence,

\[
\mathcal{H} = \pi\bar{\pi} + \alpha_0^2 z\bar{z} + \sin 2\gamma \, \alpha_0^2 (z^2 + \bar{z}^2)/2,
\]

i.e. for \( \gamma = 0, \pi/2 \) we have a standard harmonic oscillator, while for \( \gamma \neq 0, \pi/2 \) we get the anisotropic one, which is equivalent to two sets of \( N \) one-dimensional oscillators with frequencies \( \alpha_0 \sqrt{1 \pm \sin 2\gamma} \). The frequency \( \alpha \) appearing in the superalgebra, is of the form: \( \alpha = \alpha_0 \cos \gamma \).

**Conclusion**

We presented some constructions of the Hamiltonian formalism related with Hopf maps and Kähler geometry, and a few models of supersymmetric mechanics on Kähler manifolds. One can hope that the former constructions could be useful in supersymmetric mechanics along the following lines. Firstly, one could try to extend the number of supersymmetries, passing from the Kähler manifolds to quaternionic ones. The model suggested in [45] indicates that this could indeed work. One could also expect that the latter system will respect the inclusion of an instanton field. Secondly, one can try to construct the supersymmetric mechanics, performing the Hamiltonian reduction of the existing systems, related with the Hopf maps. In this way one could get new supersymmetric models, specified by the presence of Dirac and Yang monopoles, as well as with constant magnetic and instanton fields.

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