Spinning particle in an external linearized gravitational wave field

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Abstract

We study the interaction of a scalar and a spinning particle with a coherent linearized gravitational wave field treated as a classical spin two external field. The spin degrees of freedom of the spinning particle are described by skew-commuting variables. We derive the explicit expressions for the eigenfunctions and the Green’s functions of the theory. The discussion is exact within the approximation of neglecting radiative corrections and we prove that the result is completely determined by the semiclassical contribution.

1 Introduction

For many years Grassmann variables have been a common practice for giving a pseudoclassical framework to describe spin, [1]–[5], or other internal degrees of freedom of elementary particles, [6]. Of course the value of such models must be thought of in view of their quantization, that can be the canonical one or, in view of the inherent pseudoclassical description, the path-integral quantization. Although some technical caution had to be taken in extending the path-integral to skew-commuting variables (it is, for instance, relevant whether their number is even or odd), nevertheless the program was successfully carried out, see e.g. [7].

We have recently presented the quantization of the theory describing a spin $\frac{1}{2}$ particle in an external electromagnetic wave field by using the path-integral and external source method and we have produced the corresponding Feynman propagator, [8]. The electromagnetic interaction at the pseudoclassical level was introduced by the standard minimal coupling directly in the constraints describing the theory. The self-consistency of the theory automatically produced the non minimal electromagnetic coupling characteristic for the spin 1/2 particle (Pauli term). The calculations leading to the final result were straightforward, although somewhat complex, due to the above mentioned cautions. In this paper we consider the problem of describing the interaction of a scalar and a spinning particle at a first quantized level with an external gravitational field. Of course, if we don’t want to go
far beyond our possibilities of performing explicit calculations, the gravitational field must be taken in the linear approximation. Even with this simplifying assumption, however, we have to face new features with respect to \[8\], such as the treatment of covariant derivatives of Grassmann variables, responsible for the spin-gravitational field interaction and whose affine connections have to be substituted with their symmetric part, implying that the spinning particle cannot be directly coupled to the torsion, \[9\]. Following the approach started by \[10\] we are able to evaluate the eigenfunctions and the Feynman kernel if we assume that the linearized field is that of a gravitational wave. When restricting ourselves to the even sector, we recover the propagator of a scalar particle, calculated in \[11\], by using symmetry arguments and canonical transformations in the path-integral framework.

A brief summary of the paper is as follows. In Section 2 we give the main definitions of the physical quantities as well as the gauge conditions we will assume on the metric and on the affine connections. We then calculate the wave function for the scalar particle, whose Green’s function is discussed in Section 3 together with its semiclassical limit. The linearized theory for a spinning particle in the external gravitational wave is described in Section 4, while in the final Section 5 we consider the corresponding Green’s function and we show, for the spinning particles too, the semiclassical nature of the results.

2 Scalar particle in an external gravitational wave: linear theory

According to general relativity, the interaction of matter with a given gravitational field is introduced in a geometrical way by replacing the flat Minkowski metric \( \eta_{\mu\nu} = (+, -, -, -) \) by a tensor \( g_{\mu\nu}(x) \) depending upon the coordinates \( x^\mu = (x^0, \vec{x}) \equiv (t, \vec{x}) \) and by using the general covariance principle \[12\]. Therefore the action describing a massive scalar particle interacting with an external gravitational field can be written as:

\[
S = \int d^4x \sqrt{-g(x)} L(\phi(x), D_\mu \phi(x))
\]

(2.1)

where \( g(x) \) is the determinant of the metric and \( D_\mu \) the covariant derivative. Letting \( g^{\mu\nu}(x) \) be the inverse of the metric tensor, the Lagrangian density \( L \) has the form

\[
L = \frac{1}{2} g^{\mu\nu}(x) D_\mu \phi(x) D_\nu \phi(x) - \frac{1}{2} m^2 \phi^2(x)
\]

(2.2)

and produces the equation of motion

\[
\left( g^{\mu\nu}(x) D_\mu D_\nu + m^2 \right) \phi(x) = 0.
\]

(2.3)

The energy-momentum tensor of the matter field, in the absence of gravity, is obtained by the usual relation

\[
T_0^{\mu\nu}(x) = -\frac{2}{\sqrt{-g(x)}} \frac{\delta S}{\delta g_{\mu\nu}(x)} \bigg|_{g_{\mu\nu}(x) = \eta_{\mu\nu}} = -\eta^{\mu\nu} L_0 + \partial^\mu \phi(x) \partial^\nu \phi(x)
\]

(2.4)

where \( L_0 \) is the free Lagrangian density. When the gravitational field is weak and described by a small perturbation to the flat metric

\[
g_{\mu\nu}(x) \simeq \eta_{\mu\nu} + h_{\mu\nu}(x), \quad g^{\mu\nu}(x) \simeq \eta^{\mu\nu} - h^{\mu\nu}(x)
\]

(2.5)
we can consider the linearized theory with a Lagrangian density

\[ L = L_0 - \frac{1}{2} h_{\mu\nu}(x) \, T_0^{\mu\nu}(x) = (1 + \frac{1}{2} h(x)) L_0 - \frac{1}{2} h^{\mu\nu}(x) \partial_\mu \phi(x) \partial_\nu \phi(x) \] (2.6)

where \( h(x) = \eta_{\mu\nu} h^{\mu\nu}(x) = h_0^\mu(x) \). The Lagrange equations derived from (2.6) are thus

\[ \left( (1 + \frac{1}{2} h(x)) (\partial^\mu \partial_\mu + m^2) - h^{\mu\nu}(x) \partial_\mu \partial_\nu - (\partial_\mu h_0^\nu(x) - \frac{1}{2} \partial_\nu h(x)) \partial^\nu \right) \phi = 0 \] (2.7)

Upon multiplying on the left by the factor \((1 - h(x)/2)\), at this order in \( h_{\mu\nu}(x) \) equation (2.7) can be further simplified to

\[ \left( \partial^\mu \partial_\mu - h^{\mu\nu}(x) \partial_\mu \partial_\nu - (\partial_\mu h_0^\nu(x) - \frac{1}{2} \partial_\nu h(x)) \partial^\nu + m^2 \right) \phi = 0 \] (2.8)

From (2.8) it appears that we are considering a theory describing the interaction of scalar matter with a given gravitational field treated as a spin two external field in a flat Minkowski spacetime \[\text{[13]}\].

As is well known there are some conditions that can be imposed on the metric tensor, in analogy to the gauge choice for the vector potential \( A_\mu(x) \) in electrodynamics. A particularly convenient requirement is represented by the harmonic condition \[\text{[12]}\]

\[ g^{\mu\nu}(x) \Gamma^\lambda_{\mu\nu}(x) = 0 \] (2.9)

whose linearization reads

\[ \partial_\mu h_0^\nu(x) = \frac{1}{2} \partial_\nu h(x) \] (2.10)

To our purpose it turns out to be even more convenient to choose a restriction of the general harmonic gauge, that is

\[ \partial_\mu h_0^\nu(x) = 0, \quad h(x) = 0 \] (2.11)

This choice, in addition to some algebraic simplifications, also avoids the more serious ordering problem that would be present in some terms of equation (2.7). Finally we consider the linearized gravitational field as the one produced by a wave of arbitrary spectral composition and polarization properties, but propagating in a fixed direction:

\[ g_{\mu\nu}(x) \simeq \eta_{\mu\nu} + h_{\mu\nu}(\kappa_x), \quad h_{\mu\nu}(\kappa_x) = a_{\mu\nu} f(\kappa_x), \quad \kappa_x = k \cdot x. \] (2.12)

Here \( k^2 = 0 \) and \( a_{\mu\nu} = a_{\nu\mu} \) is the polarization tensor. Due to these choices the equation (2.11) then becomes

\[ k_\mu a_\nu^\mu = 0, \quad a_\mu^\mu = 0 \] (2.13)

and it is easy to cast the linearized Klein-Gordon wave equation (2.7) into the form

\[ \left( \partial^\mu \partial_\mu - h^{\mu\nu}(x) \partial_\mu \partial_\nu + m^2 \right) \phi(x) = 0 \] (2.14)

Remembering the derivation of the Volkov solution for the problem of spin \( \frac{1}{2} \) particle in a electromagnetic wave field \[\text{[3], [10]}\], we look for a solution of (2.14) in the form

\[ \phi_p(x) = \frac{2 p_0 V}{2} \exp \left( \frac{i p \cdot x - i F(\kappa_x)}{2} \right), \] (2.15)
where $p^\mu$ is a constant 4-vector and $F(\kappa_x) \to 0$ as $x^0 \to -\infty$. Moreover, since $F(\kappa_x)$ is still an undetermined function, by adding to $p^\mu$ an arbitrary 4-vector proportional to $k^\mu$ we obtain a wave function $\phi_p(x)$ of the same form. Therefore, without loss of generality we can impose on the 4-vector $p^\mu$ the mass-shell condition $p^2 = m^2$. After substituting the solution (2.15) into equation (2.14) and taking into account the mass-shell relation, the null character of $k^\mu$ and the harmonic gauge condition (2.11) we obtain for $F(\kappa_x)$ the following first-order differential equation:

$$2 k \cdot p F'(\kappa_x) - p_\mu p_\nu h^{\mu\nu}(\kappa_x) = 0$$  \hspace{1cm} (2.16)

where $F'(\kappa_x)$ is the derivative with respect to the argument. By integrating this first-order equation, from (2.16) we finally obtain for $\phi_p(x)$ the Volkov-like solution

$$\phi_p(x) = (2p_0 V)^{-\frac{3}{2}} \exp \left(-ip \cdot x - \frac{i}{2 k \cdot p} \int d\kappa h^{\mu\nu}(\kappa) p_\mu p_\nu \right).$$  \hspace{1cm} (2.17)

### 3 Green’s function for the scalar particle

The standard way of obtaining the Green’s function for the wave equation is to solve the non homogeneous boundary value problem with an added delta-function source term (see e.g. [14]):

$$(\partial^\mu \partial_\mu - h^{\mu\nu}(x) \partial_\mu \partial_\nu + m^2) \Delta F(x,y) = -\delta^4(x-y).$$  \hspace{1cm} (3.1)

By extracting the singularity due to the mass shell condition, we can assume for the Green function the form

$$\Delta F(x,y) = \int \frac{d^4p}{(2\pi)^4} \frac{e^{-i \frac{p}{p} \cdot (x-y)}}{p^2 - m^2 + i\epsilon} e^{-i F_1(\kappa_x, \kappa_y)}$$  \hspace{1cm} (3.2)

where $\kappa_x = k \cdot x$, $\kappa_y = k \cdot y$. The complete Feynman’s propagator $\Delta F(x,y)$ is therefore a superposition of free propagators modulated by the phase factor $\exp \{ -i F_1(\kappa_x, \kappa_y) \}$. The equation for $F_1(\kappa_x, \kappa_y)$ is determined by (3.1-3.2) and reads

$$\int \frac{d^4p}{(2\pi)^4} e^{-i \frac{p}{p} \cdot (x-y) - i F_1(\kappa_x, \kappa_y)} \left[ 2 k \cdot p \frac{\partial F_1(\kappa_x, \kappa_y)}{\partial \kappa_x} - p_\mu p_\nu h^{\mu\nu}(\kappa_x) \right] =$$

$$\int \frac{d^4p}{(2\pi)^4} e^{-i \frac{p}{p} \cdot (x-y)} \left( 1 - e^{-i F_1(\kappa_x, \kappa_y)} \right)$$  \hspace{1cm} (3.3)

The term in square brackets on the left-hand side of (3.3) is just the equation (2.16). Therefore the left-hand side of (3.3) vanishes if we take

$$F_1(\kappa_x, \kappa_y) = \frac{1}{2 k \cdot p} \int d\kappa h^{\mu\nu}(\kappa)$$  \hspace{1cm} (3.4)

We now verify that the integral on the right-hand side of (3.3) vanishes too. We first separate the component of $p_\mu$ in the direction of $k_\mu$ [15]. $p^\mu = p'^\mu + \alpha k^\mu$ where $p'^\mu$ spans
a three-dimensional surface. Then the function $F_1(\kappa_x, \kappa_y)$ is independent of $\alpha$ and we can integrate over $\alpha$ getting

$$\int \frac{d\alpha}{2\pi} e^{-i\alpha(\kappa_x - \kappa_y)} \left( 1 - e^{-iF_1(\kappa_x, \kappa_y)} \right) = \delta(\kappa_x - \kappa_y) \left( 1 - e^{-iF_1(\kappa_x, \kappa_y)} \right). \quad (3.5)$$

The right hand side of (3.5) is obviously vanishing in view of (3.4). The Green’s function has therefore the form

$$\Delta F(x,y) = \int d^4p \left( \frac{2\pi}{4} \right) e^{-ip \cdot (x-y)} \exp \left( -\frac{i}{2k \cdot p} \int d\kappa h^{\mu\nu}(\kappa) p_\mu p_\nu \right) \quad (3.6)$$

This Feynman kernel was derived in paper [11] by looking at the problem in a semiclassical way and by evaluating the quantum mechanical path-integral in a closed form.

We would now like to comment on the classical nature of the results so far obtained for the wave function (2.17) and the Green’s function (3.6). Indeed the classical action for a relativistic scalar particle in an external gravitational field,

$$S = -m \int_{\tau_f}^{\tau_i} \left( g^{\mu\nu}(x) \dot{x}_\mu \dot{x}_\nu \right)^{1/2} d\tau \quad (3.7)$$

is obtained by integrating a singular Lagrangian, giving rise to the first class constraint $\chi = g^{\mu\nu}(x) p_\mu p_\nu - m^2$ and to a vanishing canonical Hamiltonian. An extended Hamiltonian proportional to the constraint, $\mathcal{H}_E(x,p) = \alpha_1 \chi$ is then defined. For a linearized gravitational wave field as (2.12) and in the harmonic gauge (2.11) we have

$$\mathcal{H}_E(x,p) = \alpha_1 \left( g^{\mu\nu}(x) p_\mu p_\nu - h^{\mu\nu}(x) p_\mu p_\nu - m^2 \right) \quad (3.8)$$

Writing the corresponding covariant Hamilton-Jacobi equation

$$\eta^{\mu\nu} \frac{\partial S(x,p)}{\partial x^\mu} \frac{\partial S(x,p)}{\partial x^\nu} - h^{\mu\nu}(x) \frac{\partial S(x,p)}{\partial x^\mu} \frac{\partial S(x,p)}{\partial x^\nu} - m^2 = 0 \quad (3.9)$$

we shall look for a solution of the characteristic Hamilton function $S(x,p)$, expressed as a function of the coordinates $x^\mu$ and the initial momentum $p^\mu$, of the general form

$$S(x,p) = -p \cdot x - S_1(\kappa_x). \quad (3.10)$$

From (3.9) and (2.13) we get the differential equation

$$2k \cdot p S_1'(\kappa_x) - h^{\mu\nu}(\kappa_x) p_\mu p_\nu = 0, \quad (3.11)$$

identical to equation (2.16) for $F(\kappa_x)$. Hence

$$S(x,p) = -p \cdot x - \frac{1}{2k \cdot p} \int_{-\infty}^{k_x} d\kappa h^{\mu\nu}(\kappa) p_\mu p_\nu \quad (3.12)$$

so that the Volkov-like solution (2.14) turns out to be

$$\phi_p(x) = (2p_0V)^{-\frac{1}{2}} e^{iS(x,p)} \quad (3.13)$$

The WKB method applied to (2.14) therefore gives exact result in the first order approximation, i.e. we have found an exact classical solution of equation (2.14).
4 Dirac particle in an external gravitational wave: linear theory

We now study the same problem for a spin $\frac{1}{2}$ particle, considering the gravitational wave as an external field defined in a flat Minkowski background. (For the connection with the canonical intrinsic approach using Dirac equation in curved space-time, see \cite{13}). Consequently the minimal linearized Lagrangian describing the interaction of fermionic matter with a given gravitational field can also be considered as obtained from the Lagrangian density established in \cite{12}.

The Lagrangian density \( L \) then becomes

\[
L = L_0 - \frac{1}{2} h_{\mu\nu}(x) T_0^{\mu\nu}(x),
\]

in complete analogy with the scalar case described by eq. (2.4). The Lagrangian density of the free fermionic matter \( L_0 \) and the corresponding symmetrized energy-momentum tensor \( T_0^{\mu\nu}(x) \) are given by the equations

\[
L_0 = \frac{1}{2} \bar{\psi}(x) (i\gamma^\mu \partial_\mu - m) \psi(x) + \frac{1}{2} \bar{\psi}(x) (-i\gamma^\mu \partial_\mu - m) \psi(x)
\]

\[
T_0^{\mu\nu}(x) = -\eta^{\mu\nu} L_0 + \frac{i}{4} \left( \bar{\psi}(x) (\gamma^\mu \partial^\nu - \gamma^\nu \partial^\mu) \psi(x) + \bar{\psi}(x) (\gamma^\nu \partial^\mu - \gamma^\mu \partial^\nu) \psi(x) \right)
\]

The Lagrangian density \( L \) then becomes

\[
L = (1 + \frac{1}{2} h(x)) L_0 - \frac{i}{4} h^{\mu\nu}(x) \bar{\psi}(x) (\gamma_\mu \partial_\nu - \gamma_\nu \partial_\mu) \psi(x)
\]

and it can also be considered as obtained from the Lagrangian density established in general relativity for the interaction of the Dirac field with a prescribed gravitational field in the linear approximation \cite{12}. The Dirac equation we derive from \cite{14} turns out to be \cite{13}

\[
(1 + \frac{1}{2} h(x)) (i\gamma^\mu \partial_\mu - m) - \frac{i}{2} h^{\mu\nu}(x) \gamma_\mu \partial_\nu - \frac{i}{4} \partial_\nu h^{\mu\nu}(x) \gamma_\mu + \frac{i}{4} \partial_\mu h(x) \gamma^\mu \psi(x) = 0
\]

or, multiplying it again by \( (1 - \frac{1}{2} h(x)) \) and taking the linear terms in \( h_{\mu\nu}(x) \),

\[
(\gamma^\mu \partial_\mu - m) - \frac{i}{2} h^{\mu\nu}(x) \gamma^\mu \partial_\nu - \frac{i}{4} \partial_\nu h^{\mu\nu}(x) \gamma_\mu + \frac{i}{4} \partial_\mu h(x) \gamma^\mu \psi(x) = 0
\]

The harmonic condition \( (2.11) \) further simplifies the wave equation \( (4.5) \) giving the final form

\[
(i\gamma^\mu \partial_\mu - \frac{i}{2} h^{\mu\nu}(x) \gamma_\mu \partial_\nu - m) \psi(x) = 0
\]

Moreover, since the external field represents a linearized gravitational wave, we assume again for \( h_{\mu\nu}(x) \) the expression \( h_{\mu\nu}(k_x) \) given in \cite{12}. Instead of solving the first order Dirac equation \( (4.7) \), in analogy to what is done for the electromagnetic interactions \cite{10}, we find it easier to consider the second order equation obtained by applying to \( (4.7) \) the operator \( (i\gamma^\mu \partial_\mu - \frac{i}{2} h^{\mu\nu}(k_x) \gamma_\mu \partial_\nu + m) \). By using the supplementary gauge condition \( (2.11) \) we finally get

\[
(\partial^\mu \partial_\mu - h^{\mu\nu}(k_x) \partial_\mu \partial_\nu + \frac{i}{2} \sigma_{\mu\nu} (\partial^\mu h^{\nu\rho}(k_x)) \partial_\rho + m^2) \psi(x) = 0
\]
If we compare (4.8) with the scalar analogous equation (2.14), we can see that they are identical but for a term representing the spin-gravitational field interaction, as should be expected. Hence we shall assume for the solutions of (4.8) a general form

$$\psi_{p,s}(x) = (2p_0 V)^{-\frac{1}{2}} e^{-ip \cdot x} G(\kappa_x) u(p,s)$$  \hspace{1cm} (4.9)$$

with the initial condition

$$\psi_{p,s}(x) \xrightarrow{x^0 \to -\infty} (2p_0 V)^{-\frac{1}{2}} e^{-ip \cdot x} u(p,s)$$  \hspace{1cm} (4.10)$$

where $u(p,s)$ is a constant spinor which is a solution of the corresponding free Dirac equation

$$\left(\gamma^\mu p_\mu - m\right) u(p,s) = 0.$$  \hspace{1cm} (2.14)$$

Using the relations $k^2 = k^\mu h^\mu_{\nu} (\kappa_x) = h^\mu_{\nu}(\kappa_x) = 0$ and $p^2 = m^2$, from (4.9) and (4.8) we get the first order differential equation

$$2 k \cdot p G'_{\kappa_x} + i \left( h^{\mu\nu}(\kappa_x) p_\mu p_\nu - \frac{1}{2} k^\mu p_\nu \frac{dh^{\nu\rho}(\kappa_x)}{d\kappa} \sigma_{\rho\mu} \right) G(\kappa_x) = 0$$  \hspace{1cm} (4.11)$$

with solution

$$G(\kappa_x) = \exp\left(-\frac{i}{2k \cdot p} \int_{-\infty}^{\kappa_x} d\kappa \left( h^{\mu\nu}(\kappa) p_\mu p_\nu - \frac{1}{2} k^\mu p_\nu \frac{dh^{\nu\rho}(\kappa)}{d\kappa} \sigma_{\rho\mu} \right) \right).$$  \hspace{1cm} (4.12)$$

Taking into account that $(k^\mu p_\nu a^{\nu\rho} \sigma_{\rho\mu})^n = 0$ for $n > 1$, after some straightforward simplifications the Volkov-like solution (4.9) can finally be put into the form

$$\psi(x) = (2p_0 V)^{-\frac{1}{2}} \left( 1 + \frac{i}{4} k^\mu p_\nu h^{\nu\rho}(\kappa_x) \sigma_{\rho\mu} \right) \exp\left(-ip \cdot x - \frac{i}{2k \cdot p} \int_{-\infty}^{\kappa_x} d\kappa h^{\mu\nu}(\kappa) p_\mu p_\nu \right) u(p,s)$$  \hspace{1cm} (4.13)$$

where, as usual, $\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu]$.  \hspace{1cm} \hspace{1cm} 

5 Green’s function for the Dirac particle

We present here a direct calculation of the electron Green’s function in the presence of an external gravitational plane wave. The Green’s function satisfies the equation

$$\left( i\gamma^\mu \partial_\mu - \frac{i}{2} h^{\mu\nu}(\kappa_x) \gamma_\mu \partial_\nu - m \right) S_F(x,y) = \delta^4(x - y)$$  \hspace{1cm} (5.1)$$

with the usual associated boundary conditions. As in Section 4 we prefer working with a second order equation. In fact, if we let

$$S_F(x,y) = \left( i\gamma^\mu \partial_\mu - \frac{i}{2} h^{\mu\nu}(\kappa_x) \gamma_\mu \partial_\nu + m \right) \Delta_F(x,y)$$  \hspace{1cm} (5.2)$$

then, using again the gauge condition (2.11), we find that $\Delta_F(x,y)$ satisfies

$$\left( \partial^\mu \partial_\mu - h^{\mu\nu}(\kappa_x) \partial_\mu \partial_\nu + \frac{i}{2} \sigma_{\mu\nu} (\partial^\mu h^{\nu\rho}(\kappa_x)) \partial_\rho + m^2 \right) \Delta_F(x,y) = -\delta^4(x - y)$$  \hspace{1cm} (5.3)$$
Equation (5.3) is identical to the analogous equation (3.1) for the scalar particles but for the spin term \( (i/2) \sigma_{\mu\nu} (\partial^\mu h^{\rho\nu}(\kappa_x)) \partial_\rho \). It is therefore natural to look for a solution of the form

\[
\Delta_F(x, y) = \int \frac{d^4p}{(2\pi)^4} \frac{e^{-i p \cdot (x - y)}}{p^2 - m^2 + i\epsilon} e^{-i F_1(\kappa_x, \kappa_y)} N(\kappa_x, \kappa_y) \tag{5.4}
\]

where \( F_1(\kappa_x, \kappa_y) \) is the same as in the scalar case (see eq. (5.3)) and \( N(\kappa_x, \kappa_y) \) is an unknown function to be determined. From (5.4) and (5.3) the equation for \( N(\kappa_x, \kappa_y) \) becomes

\[
\int \frac{d^4p}{(2\pi)^4} \frac{e^{-i p \cdot (x - y) - i F_1(\kappa_x, \kappa_y)}}{p^2 - m^2 + i\epsilon} \left( \frac{1}{2} k^\mu p_\nu \frac{dh^{\nu\rho}(\kappa_x)}{d\kappa_x} \sigma_{\rho\mu} \right) N(\kappa_x, \kappa_y) + 2ik \cdot p \frac{\partial N(\kappa_x, \kappa_y)}{\partial \kappa_x} = \int \frac{d^4p}{(2\pi)^4} e^{-i p \cdot (x - y)} \left( 1 - N(\kappa_x, \kappa_y) e^{-i F_1(\kappa_x, \kappa_y)} \right) \tag{5.5}
\]

The left-hand side of eq. (5.5) vanishes by choosing

\[
N(\kappa_x, \kappa_y) = \exp \left( \frac{i}{4k \cdot p} \int_{-\infty}^{\kappa_x} dk \ k^\mu p_\nu \frac{dh^{\nu\rho}(\kappa)}{d\kappa} \sigma_{\rho\mu} \right) \tag{5.6}
\]

and, as in the scalar case, we easily verify that the right-hand side vanishes too. Thus

\[
\Delta_F(x, y) = \int \frac{d^4p}{(2\pi)^4} \left( 1 + \frac{i}{4k \cdot p} k^\mu p_\nu h^{\nu\rho}(\kappa_x) \sigma_{\rho\mu} \right) e^{-i p \cdot x - \frac{i}{2k \cdot p} \int_{-\infty}^{\kappa_x} dk \ h^{\mu\nu}(\kappa) p_\mu p_\nu} \int_{-\infty}^{k_x} dk \ h^{\mu\nu}(\kappa) p_\mu p_\nu \tag{5.7}
\]

is a solution of the inhomogeneous second order Dirac equation (5.3) and from equation (5.4) we finally get for the Feynman propagator \( S_F(x, y) \) the expression

\[
S_F(x, y) = \left( i\gamma^\mu \partial_\mu - \frac{i}{2} h^{\mu\nu}(\kappa_x) \gamma_\nu \partial_\nu + m \right) \Delta_F(x, y)
\]

\[
= \int \frac{d^4p}{(2\pi)^4} \left( 1 + \frac{i}{4k \cdot p} k^\mu p_\nu h^{\nu\rho}(\kappa_x) \sigma_{\rho\mu} \right) \frac{\gamma^\mu p^\mu + m}{p^2 - m^2 + i\epsilon} \left( \frac{1}{4k \cdot p} k^\mu p_\nu h^{\nu\rho}(\kappa_y) \sigma_{\rho\mu} \right) \exp \left( -i p \cdot (x - y) - \frac{i}{2k \cdot p} \int_{-\infty}^{\kappa_x} dk \ h^{\mu\nu}(\kappa) p_\mu p_\nu \right) \tag{5.8}
\]

As we have already discussed for the scalar particle in Sect.3, it would be interesting to show the pseudoclassical nature of the results obtained for the Dirac particle too. We start with the pseudoclassical Lagrangian of the results obtained for the Dirac particle too. We start with the pseudoclassical Lagrangian for a spin-\( \frac{1}{2} \) particle in an arbitrary external gravitational field [9]

\[
L = -\frac{i}{2} g_{\mu\nu}(x) \zeta^\mu (\dot{\zeta}^\nu + \Gamma_\nu^\rho(x) \dot{x}^\rho \dot{\zeta}^\lambda) - \frac{i}{2} \zeta_5 \dot{\zeta}_5 - m \left( g_{\mu\nu}(x) (\dot{x}^\mu - \frac{i}{m} \zeta^\nu \dot{\zeta}_5) (\dot{x}^\nu - \frac{i}{m} \zeta^\mu \dot{\zeta}_5) \right)^{\frac{1}{2}} \tag{5.9}
\]
where $\Gamma_{\nu \lambda \rho}^\mu(x)$ are the usual Christoffel symbols and the Grassmann variables $\zeta^\mu$ are quite naturally introduced, since in quantization they correspond to the use of the spin matrices conform to metric $\gamma^\mu$ (see for instance [17] and references therein.) By introducing the vierbein field $G_{\mu}^A(x)$ and its inverse $H_{\mu}^B(x)$

$$g_{\mu \nu}^\mu(x) = \eta_{AB} G_{\mu}^A(x) G_{\nu}^B(x), \quad g^{\mu \nu}(x) = \eta^{AB}(x) H_{A}^\mu(x) H_{B}^\nu(x) \quad (5.10)$$

$$H_{A}^\mu(x) G_{\nu}^B(x) = \delta_{\nu}^{\nu}, \quad H_{A}^\mu(x) G_{\mu}^B(x) = \delta_{B}^{B} \quad (5.11)$$

we can transform a world four-vector $\zeta^\mu$ into a “local” one and we can also introduce Grassmann variables $\xi^A$ corresponding, after quantization, to the usual Dirac matrices (flat space spin matrices)

$$\xi^A = G_{\mu}^A(x) \zeta^\mu, \quad \zeta^\mu = H_{\mu}^A(x) \xi^A, \quad \xi_5 = \zeta_5. \quad (5.12)$$

The singular Lagrangian (5.9) gives rise to the two first class constraints $\chi_D = \mathcal{P}_\mu \zeta^\mu - \text{im}(\pi_5 - (i/2) \zeta_5)$ and $\chi = g^{\mu \nu}(x) \mathcal{P}_\mu \mathcal{P}_\nu - m^2$, where $\mathcal{P}_\mu$ is the canonical momentum and $\mathcal{P}_\mu = P_\mu - (i/2) g_{\mu \nu}(x) \zeta^\rho \Gamma_{\nu \lambda}^\rho(x) \zeta^\lambda$ is the mechanical one. We also have a second class constraint

$$\pi_\mu = \frac{i}{2} g_{\mu \nu}(x) \zeta^\nu \quad (5.13)$$

and, in addition, we can further require [3, 9]

$$\pi_5 = -\frac{i}{2} \zeta_5. \quad (5.14)$$

The relations (5.13-5.14) form a set of second-class constraints whose relevant Dirac brackets are

$$\{\zeta_5, \zeta_5\} = -i, \quad \{\zeta^\mu, \zeta^\nu\} = ig^{\mu \nu}(x), \quad \{\xi^A, \xi^B\} = i\eta^{AB}. \quad (5.15)$$

Some remarks on the quantization are now in order. The Dirac brackets (5.15) give the anticommutation rules

$$[\hat{\xi}_5, \hat{\xi}_5]_+ = 1, \quad [\hat{\xi}^A, \hat{\xi}^B]_+ = -\eta^{AB}, \quad [\hat{\zeta}^\mu, \hat{\zeta}^\nu]_+ = -g^{\mu \nu}(x). \quad (5.16)$$

The first and the second relation can be satisfied by putting

$$\hat{\xi}_5 = 2^{-\frac{1}{2}} \gamma_5, \quad \hat{\xi}^A = 2^{-\frac{1}{2}} \gamma_5 \gamma^A \quad (5.17)$$

where $\gamma_5$ and $\gamma^A$ are the usual flat space Dirac matrices. We must now choose a representation for the algebra of the operators $\hat{\zeta}^\mu$. A possible choice which will correspond to the use of the spin matrices conform to metric is

$$\hat{\zeta}^\mu = H_{\mu}^A(x) \hat{\xi}^A = 2^{-\frac{1}{2}} \gamma_5 \gamma^\mu \quad (5.18)$$

where

$$[\gamma^\mu, \gamma^\nu]_+ = 2g^{\mu \nu}(x) \quad (5.19)$$
As usual, the extended Hamiltonian $H_E$ is written as a linear combination of the first-class constraints with arbitrary coefficients and the simplest choice appears to be

$$H_E = \alpha_1 \left( g^{\mu\nu}(x) \mathcal{P}_\mu \mathcal{P}_\nu - m^2 \right).$$

(5.20)

The constraint $\chi_D$ becomes $\chi_D = \mathcal{P}_\mu \zeta^\mu - m \zeta_5$, once eq. (5.14) is accounted for. It could be used, after quantization, for constructing the spinorial states. Considering again the restriction to a linearized gravitational wave field in the harmonic gauge, (2.12-2.13) and taking into account the linearized form $H^\mu_A(\kappa_x) = \delta^\mu_A - (1/2) h^A_\mu(\kappa_x)$, $G^A_\mu(\kappa_x) = \delta^A_\mu + (1/2) h^A_\mu(\kappa_x)$ of the vierbein fields we finally get

$$H_E = \alpha_1 \left( \eta^{\mu\nu} \mathcal{P}_\mu \mathcal{P}_\nu - h^{\mu\nu}(\kappa_x) \mathcal{P}_\mu \mathcal{P}_\nu + i \mathcal{P}_\rho \frac{\partial h^{\mu\nu}(\kappa_x)}{\partial \mathcal{X}_\mu} \mathcal{X}_\mu \mathcal{X}_\nu - m^2 \right)$$

(5.21)

Therefore the variables $\zeta^\mu$ satisfy the linearized relations

$$\zeta^\mu = \xi^\mu - \frac{1}{2} h^A_\mu(\kappa_x) \mathcal{X}^A$$

(5.22)

and we finally obtain the following equation for the Hamilton’s characteristic function $S(x)$:

$$\eta^{\mu\nu} \frac{\partial S(x)}{\partial x^\mu} \frac{\partial S(x)}{\partial x^\nu} - h^{\mu\nu}(x) \frac{\partial S(x)}{\partial x^\mu} \frac{\partial S(x)}{\partial x^\nu} - i \mathcal{P}_\rho \frac{\partial h^{\mu\nu}(x)}{\partial \mathcal{X}_\mu} \mathcal{X}_\mu \mathcal{X}_\nu - m^2 = 0.$$ 

(5.23)

The solution (3.12) obtained in the scalar case suggests, for the pseudoclassical Hamilton characteristic function, the ansatz

$$S(x) = -p \cdot x - \frac{1}{2} \frac{k \cdot p}{k_0} \int_{-\infty}^{\kappa_x} d\kappa \mathcal{K}_\mu \mathcal{K}_\nu h^{\mu\nu}(\kappa) + S_D(\kappa_x).$$

(5.24)

The unknown function $S_D$ will satisfy the first-order differential equation

$$2 k \cdot p \frac{dS_D(\kappa_x)}{d\kappa_x} + i k^{\rho\mu} \mathcal{K}_\rho \frac{dh^{\mu\nu}(\kappa_x)}{d\kappa_x} \mathcal{K}_\nu = 0.$$ 

(5.25)

with solution

$$S_D(\kappa_x) = -\frac{i}{2} \frac{k \cdot p}{k_0} \int_{-\infty}^{\kappa_x} d\kappa k^{\mu\nu} \frac{dh^{\mu\nu}(\kappa)}{d\kappa} \mathcal{X}_\mu \mathcal{X}_\nu.$$ 

(5.26)

It can easily be seen that the operator $\hat{S}$ obtained upon quantization – through the use of eqs. (5.17) – of the pseudoclassical $S(x)$ given in (5.24) can be represented as

$$\hat{S} = -p \cdot x - \frac{1}{2} \frac{k \cdot p}{k_0} \int_{-\infty}^{\kappa_x} d\kappa \left( h^{\mu\nu}(\kappa) \mathcal{K}_\mu \mathcal{K}_\nu - \frac{1}{2} k^{\mu\nu} \frac{dh^{\mu\nu}(\kappa)}{d\kappa} \mathcal{K}_\mu \mathcal{K}_\nu \right)$$

(5.27)

Therefore by applying the operator $(2p_0V)^{-\frac{1}{2}} e^{i\hat{S}}$ to a free constant spinor $u(p, s)$ we see that the Volkov-like solution (4.13) will be written as follows

$$\psi(x) = (2p_0V)^{-\frac{1}{2}} e^{i\hat{S}} u(p, s)$$

(5.28)

completely showing the semiclassical nature of the solution.
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