Super Liouville action for Regge surfaces \(^1\).  

Pietro Menotti  
*Dipartimento di Fisica dell’Università, Pisa 56100, Italy and*  
*INFN, Sezione di Pisa*  
and  
Giuseppe Policastro  
*Scuola Normale Superiore, Pisa 56100, Italy and*  
*INFN, Sezione di Pisa*  

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Abstract

We compute the super Liouville action for a two dimensional Regge surface by exploiting the invariance of the theory under the superconformal group for sphere topology and under the supermodular group for torus topology. For sphere topology and torus topology with even spin structures, the action is completely fixed up to a term which in the continuum limit goes over to a topological invariant, while the overall normalization of the action can be taken from perturbation theory. For the odd spin structure on the torus, due to the presence of the fermionic supermodulus, the action is fixed up to a modular invariant quadratic polynomial in the fermionic zero modes.

1 Introduction

Discretized models of field theory serve the purpose of reducing the infinite number of degrees of freedom to a finite one. Here we shall be concerned with a discrete approach to two dimensional supergravity.

In two dimensions on the continuum, the reduction of the functional integral to the superconformal gauge gives rise to the super Liouville action [1, 2]. In theories related to gravity several discretization schemes have been proposed [3] one of which is to approximate a smooth manifold by one which is everywhere flat except for a finite number of $D - 2$ dimensional simplices i.e. the Regge model [4].

In principle one can think of other schemes of reducing the number of degrees of freedom to a finite one; e.g. in two dimensional gravity one could expand the conformal factor, in the case of spherical topology, in spherical harmonics on the surface of the sphere and keep only a finite number of modes.

However the Regge scheme has the remarkable advantage that the family of Regge conformal factors is closed under the invariance groups of the theory which are $SL(2, C)$ for spherical topology and modular group for torus topology. This would not occur e.g. by expanding in spherical harmonics and keeping a finite number of modes because under a $SL(2, C)$ transformation they would mix with an infinite number of modes.
In ref. [5] the ordinary non supersymmetric case was considered; the Liouville action for a Regge surface and also the measure for the conformal factor was derived and shown that the resulting theory is exactly invariant under $SL(2, C)$ for sphere topology. For torus topology the procedure provided also a non formal explicit proof of the modular invariance of the theory. Moreover the derived measure is Weyl invariant [6, 7] thus providing a Weyl invariant discretization scheme.

The procedure for computing the Liouville action was to exploit the heat kernel technique similarly to what is done on the continuum. It was later realized [8] that the same result (except for a function which in the continuum limit contributes to a topological term) can be more easily obtained by exploiting the invariance of the action under the $SL(2, C)$ group, for sphere topology and under the modular group for torus topology. In this paper we extend such a treatment to the supersymmetric case. Here the role is played by the superconformal and by the supermodular groups acting on the superconformal factor.

Obviously one could also follow the standard heat kernel procedure as it was done in [5] for the non supersymmetric case i.e. by computing the short time behavior of the heat kernel on a singular Riemann surface and by choosing the correct selfadjoint extension of the Lichnerowicz-De Rahm operator by exploiting the Riemann-Roch theorem. The procedure in the supersymmetric case is further complicated by the fact that the superlaplacian is not a definite positive operator and thus in order to apply the heat kernel procedure one has to compute the heat kernel of the square of the operator and then take the square root of the result.

Here we shall follow the simpler procedure of ref. [8]. We shall see that for sphere topology and for the even spin structures for torus topology the action is fixed up to a function of the conical defects which in the continuum limit goes over to a topological invariant. The overall normalization of the action can be taken from perturbation theory. For the odd spin structure on the torus, due to the presence of the fermionic supermodulus and the associated fermionic zero modes, group theory determines the action up to a polynomial quadratic in the amplitudes of the two fermionic zero modes. Modular invariance
imposes certain restrictions on the coefficients of such polynomial which however are not sufficient to determine them completely. Thus it appears that for the odd spin structure case a closer appeal the structure of the the heat kernel derivation is necessary to fix the action completely.

The computation of the integration measure for the superconformal factor which is the remaining ingredient in the discretized functional integral, is left for an other paper.

The two dimensional Regge surface in [5] was described by a conformal factor given in terms of the positions and the strengths of the singularities to which, for torus topology the Teichmüller parameters have to be added. We recall that such a description is completely equivalent to the usual one in terms of triangulations but in two dimensions the use of complex coordinates appears more powerful. Similarly in the supersymmetric case, exploiting a well known result by Howe [9], we shall describe the supergeometry in terms of a superconformal factor supplemented in the case of torus topology, by the supermoduli. The idea of using the complex plane and supercomplex plane to describe a Regge geometry is due to Foerster [10] where he also makes an heuristic guess of the action and of the integrations measure; such guesses however do not agree with the exact results of [5].

The supersymmetric approach that we shall describe in the following allows also to introduce spinning particle in a natural way on a Regge surface.

2 Sphere topology

As usual we shall describe the supersphere by a single chart given by the complex superplane $z = (z, \theta)$ with $z = x + iy$, completed by the point at infinity. It was proven in [5] that any two dimensional supergeometry is locally superconformally flat. As for the sphere topology there are no Teichmüller parameters, the supergeometry of the supersphere can be given in terms of a super Weyl transformation applied to the flat background described
by the superzweibein

\[ E_M^A = \left( \begin{array}{ccc} \delta_m^a & 0 \\ \frac{1}{2}(\gamma^a)_{\mu \beta} & \delta_\mu^\alpha \end{array} \right) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \theta & 0 & 1 \\ 0 & -\tilde{\theta} & 0 & 1 \end{pmatrix} \] (1)

and zero connection. The indices \( M = (m, \mu) \) are Einstein indices; \( m \) runs over the values \( z, \bar{z} \) while \( \mu \) runs over \( \theta, \bar{\theta} \). The indices \( A = (a, \alpha) \) are Lorentz internal indices; \( a \) runs over the values \( u, \bar{u} \) and \( \alpha \) over the values +,−. The new zweibein resulting from the super Weyl transformations is given by \[ E_M^a = e^{\Sigma} \hat{E}_M^a; \quad E_M^\alpha = e^{\Sigma/2} \hat{E}_M^\alpha - 2 \hat{E}_M^b(\gamma_b)^{\alpha \beta} \hat{D}_\beta e^{\Sigma/2} \] (2)

being

\[ \hat{D}_+ = \hat{E}_+^M \partial_M = \frac{\partial}{\partial \theta} - \theta \frac{\partial}{\partial z}; \quad \hat{D}_- = \hat{E}_-^M \partial_M = \frac{\partial}{\partial \theta} + \bar{\theta} \frac{\partial}{\partial \bar{z}} \] (3)

where \( E_A^M \) denotes the inverse superzweibein. The superzweibein (2) satisfies the torsion constraints \[ \[4, 9, 11, 12\] \] for any \( \Sigma \).

This description of the supergeometry is invariant under the superconformal group given by \( \[12\] \)

\[ z' = \frac{az + b + \alpha \theta}{cz + d + \beta \theta}; \quad \theta' = \frac{\gamma z + \delta + A \theta}{cz + d + \beta \theta} \] (4)

with the restrictions

\[ ad - bc + \gamma \delta = 1; \quad a\beta - c\alpha + A\gamma = 0 \]
\[ b\beta - d\alpha + A\delta = 0; \quad A^2 - 2\alpha \beta = 1 \] (5)

On the continuum the super Liouville action is given by \( \[1, 2, 12\] \)

\[ S_{sL}(\Sigma) = -\frac{10}{8\pi} \int d^2z \tilde{E}(\tilde{D}_+ \Sigma \tilde{D}_- \Sigma + \tilde{R}_{+-} \Sigma) \] (6)

where \( \tilde{E} \) is the superdeterminant of the background superzweibein. Taking into account that \( \[12\] \)

\[ \mathcal{R}_{+-} = e^{-\Sigma}(\tilde{R}_{+-} - 2\tilde{D}_+ \tilde{D}_- \Sigma) \equiv e^{-\Sigma}(\tilde{R}_{+-} - 2\tilde{\Box}^{-} \Sigma) \] (7)
being $\mathcal{R}_{+-}$ the supercurvature, with an integration by parts eq. (8) can be reduced to Polyakov’s non local covariant form

\[ S_{sL} = \frac{10}{32\pi} \left[ \int d^2z(E\mathcal{R}_{+-})(z)G(z,z')d^2z'(E\mathcal{R}_{+-})(z') \right. \\
\left. -4\ln\left(\frac{A}{A_0}\right)\int d^2z' E\mathcal{R}_{+-} \right] \] (8)

where $A$ is the area and $A_0$ is the reference area. The origin of the last term is due to the contribution of the zero modes as stressed in [13]. $G$ is the Green function of the operator $\Box(-)$ i.e.

\[ \Box(-)G(z,z') = \delta^2(z - z')(\bar{\theta} - \bar{\theta}')(\theta - \theta')\frac{1}{E(z')} \] (9)

As $\Box(-) = e^{-\Sigma}\Box(-)$ we have that $G(z,z') = \hat{G}(z,z')$ defined by

\[ \hat{\Box}(-)\hat{G}(z,z') = \delta^2(z - z')(\bar{\theta} - \bar{\theta}')(\theta - \theta') \] (10)

The solution of eq. (10) is [12]

\[ \hat{G}(z,z') = \frac{1}{\pi}\ln[(z - z' + \theta\bar{\theta})(\bar{z} - z' - \bar{\theta}\theta) + \varepsilon^2] \] (11)

being $z - z' + \theta\bar{\theta}$ the superinvariant displacement. In the following the $\varepsilon^2$ will be understood. It follows that the superconformal factor $\Sigma$ describing a super Regge two dimensional surface with the topology of the sphere is given by

\[ \Sigma(z) = \sum_i(\alpha_i - 1)\ln[(z - z_i + \theta\bar{\theta})(\bar{z} - z_i - \bar{\theta}\theta_i)] + \lambda_0 \] (12)

being $1 - \alpha_i = \delta_i$ the conical defects at the points $(z_i, \theta_i)$. The $\alpha_i$ are constrained by the Gauss-Bonnet theorem

\[ \int d^2zd\theta d\bar{\theta} \mathcal{R}_{+-} = \pi\chi(M), \] (13)

being $\chi(M)$ the Euler characteristic of the manifold. Substituting eq. (12) into eq. (7) one obtains

\[ 2\pi = -2\pi \sum_i \int d^2z\delta(z - z_i)(\alpha_i - 1) = 2\pi \sum_i \delta_i. \] (14)
We note that this is at variance with the ordinary non supersymmetric case where \( \sum_i \delta_i = 2 \); it is due to the fact that \( \Sigma \) is related to the superzweibein instead of to the metric. We recall that under a superconformal transformation the displacement \( z_{ij} = z_i - z_j + \theta_i \theta_j \) behave as

\[
z'_{ij} = z'_i - z'_j + \theta'_i \theta'_j = \frac{z_{ij}}{(cz_i + d + \beta \theta_i)(cz_j + d + \beta \theta_j)}. \tag{15}
\]

The corresponding transformation of the superconformal factor is as follows

\[
\Sigma'(z', \theta'; z_i, \theta_i, \alpha_i, \lambda_0) = \Sigma(z(z', \theta'), \theta(z', \theta'); z_i, \theta_i, \alpha_i, \lambda_0) + \\
+ \ln \left| \text{sdet} \partial(z, \bar{z}, \theta, \bar{\theta}) \partial(z', \bar{z}', \theta', \bar{\theta}') \right| = \\
\sum_i (\alpha_i - 1) \ln (z'_i - z'_j + \theta'_i \theta'_j)(z'_i - z'_j + \bar{\theta}'_i \bar{\theta}'_j) + \lambda_0 \\
+ \sum_i (\alpha_i - 1) \ln (cz_i + d + \beta \theta_i)(cz_i + d + \beta \theta_i) \
\tag{16}
\]

where we have taken into account the constraint \( \sum_i (1 - \alpha_i) = 1 \). Thus we have

\[
\Sigma'(z', \theta'; z_i, \theta_i, \alpha_i, \lambda_0) = \Sigma(z', \theta'; z'_i, \theta'_i, \alpha'_i, \lambda'_0) \tag{17}
\]

with

\[
z'_i = \frac{az_i + b + \alpha \theta_i}{cz_i + d + \beta \theta_i}; \quad \theta'_i = \frac{\gamma z_i + \delta + A \theta_i}{cz_i + d + \beta \theta_i} \tag{18}
\]

\[
\lambda'_0 = \lambda_0 + \sum_i (\alpha_i - 1) \ln (cz_i + d + \beta \theta_i)(cz_i + d + \beta \theta_i); \\
\alpha'_i = \alpha_i. \tag{19}
\]

The condition \( \sum_i \delta_i = 1 \) plays a crucial role in the above transformation which shows that the family of Regge superconformal factors is closed under the superconformal group.

The discrete transcription of the nonlocal action (8) is

\[
S_{sL} = \frac{1}{2} \sum_{i,j} K_{ij}[\alpha] \ln [(z_i - z_j + \theta_i \theta_j)(\bar{z}_i - \bar{z}_j - \bar{\theta}_i \bar{\theta}_j)] + B(\lambda_0, \alpha) \tag{20}
\]

with \( K_{ij} = K_{ji} \) and \( K_{ii} = 0 \). Under a superconformal transformation the action goes over to

\[
S_{sL} \rightarrow S_{sL} - \sum_{ij} K_{ij}[\alpha] \ln (cz_i + d + \beta \theta_i)
\]
\[- \sum_{ij} K_{ij}[\alpha] \ln(\bar{c} \bar{z}_i + \bar{d} - \bar{\beta}\theta_i) + B(\lambda'_0, \alpha) - B(\lambda_0, \alpha) \] (21)

and thus we must have

\[- \sum_{ij} K_{ij}[\alpha] \ln(c z_i + d + \beta \theta_i) - \sum_{ij} K_{ij}[\alpha] \ln(\bar{c} \bar{z}_j + \bar{d} - \bar{\beta}\bar{\theta}_j) + B(\lambda'_0, \alpha) = B(\lambda_0, \alpha). \] (22)

In order to find the structure of \(B\) let us consider first the transformation eq. (4)

\[a = 1; \quad d = k; \quad A = 1\] (23)

and all other parameters equal to zero. Then we have

\[2 \ln k \sum_{ij} K_{ij}[\alpha] + B(\lambda'_0, \alpha) = B(\lambda_0, \alpha)\] (24)

with \(\lambda'_0 = \lambda_0 + 2 \ln k\), which tells us that \(B\) is a linear function of \(\lambda_0\) i.e.

\[B(\lambda_0, \alpha) = -\lambda_0 \sum_{ij} K_{ij}[\alpha] + F[\alpha]\] (25)

so reaching the structure

\[S_{sL} = \frac{1}{2} \sum_{ij} K_{ij}[\alpha] \ln(z_{ij}\bar{z}_{ij}) - \lambda_0 \sum_{ij} K_{ij}[\alpha] + F[\alpha].\] (26)

Imposing invariance under the general superconformal transformation we have

\[0 = - \sum_{ij} K_{ij}[\alpha] \ln(c z_i + d + \beta \theta_i)(\bar{c} \bar{z}_i + \bar{d} - \bar{\beta}\bar{\theta}_i) - \sum_i (\alpha_i - 1) \ln(c z_i + d + \beta \theta_i)(\bar{c} \bar{z}_i + \bar{d} - \bar{\beta}\bar{\theta}_i) \sum mn K_{mn}[\alpha]\] (27)

which gives, due to the arbitrariness of \(z_i\) and \(\theta_i\),

\[0 = \sum_j K_{ij}[\alpha] + (\alpha_i - 1) \sum mn K_{mn}[\alpha].\] (28)

Defining \(K_{ij}[\alpha] = (1 - \alpha_i)(1 - \alpha_j)h_{ij}[\alpha]\) the above equation becomes

\[\sum_{j \neq i} (1 - \alpha_j) h_{ij}[\alpha] = \sum_{m,n \neq m} (1 - \alpha_m)(1 - \alpha_n) h_{mn}[\alpha].\] (29)
We assume that \( h_{mn}[^\alpha] \) will depend only on \( \alpha_n, \alpha_m \), i.e.

\[
    h_{mn}[^\alpha] = h(\alpha_m, \alpha_n). \tag{30}
\]

Let us choose the \( \alpha_i \ (i = 1, N) \) as follows: \( \alpha_1 \) and \( \alpha_2 \) free and for \( i \geq 3 \) \( \alpha_i = \bar{\alpha} \) with

\[
    1 - \bar{\alpha} = \frac{\alpha_1 + \alpha_2 - 1}{N - 2}. \tag{31}
\]

Substituting into (29) we obtain with \( i = 1 \)

\[
    (1 - \alpha_2)(2\alpha_1 - 1)h(\alpha_1, \alpha_2) + (\alpha_1 + \alpha_2 - 1)(2\alpha_1 - 1)h(\alpha_1, \bar{\alpha}) = \\
    2(1 - \alpha_2)(\alpha_1 + \alpha_2 - 1)h(\alpha_2, \bar{\alpha}) + \frac{(N - 2)(N - 3)}{(N - 2)^2}(\alpha_1 + \alpha_2 - 1)^2 h(\bar{\alpha}, \bar{\alpha}). \tag{32}
\]

The above equation has to hold for any \( N \), and in the limit \( N \to \infty \) it reduces to

\[
    (2\alpha_1 - 1)(1 - \alpha_2)h(\alpha_1, \alpha_2) + (\alpha_1 + \alpha_2 - 1)(2\alpha_1 - 1)h(\alpha_1, 1) = \\
    2(1 - \alpha_2)(\alpha_1 + \alpha_2 - 1)h(\alpha_2, 1) + (\alpha_1 + \alpha_2 - 1)^2 h(1, 1). \tag{33}
\]

Setting \( \alpha_2 = 1 \) we obtain

\[
    \alpha_1(2\alpha_1 - 1)h(\alpha_1, 1) = \alpha_1^2 h(1, 1) \quad \text{i.e.} \\
    h(\alpha_1, 1) = \frac{\alpha_1}{2\alpha_1 - 1} h(1, 1) \tag{34}
\]

and substituting into (33) we finally obtain

\[
    h(\alpha_1, \alpha_2) = h(1, 1) \frac{1}{2} \left( \frac{1}{2\alpha_1 - 1} + \frac{1}{2\alpha_2 - 1} \right). \tag{35}
\]

We notice that the derived \( h(\alpha_1, \alpha_2) \) satisfy exactly eqn. (32) for any \( N \) and also eq. (29).

In conclusion the action takes the form

\[
    S_{sL} = \text{const} \left[ \sum_{i,j \neq i} \frac{(1 - \alpha_i)(1 - \alpha_j)}{2\alpha_i - 1} \ln(z_{ij} \bar{z}_{ij}) + \frac{\lambda_0}{2} \sum_i \left( 2\alpha_i - 1 - \frac{1}{2\alpha_i - 1} \right) + F[^\alpha] \right]. \tag{36}
\]

Group theory alone cannot fix the proportionality constant and the function \( F[^\alpha] \). The latter, as happens in the usual non supersymmetric case, represents the analogue of the
renormalized electrostatic self energies of the point charges (singular curvatures) and
from the structure of the heat kernel derivation of eq. [2] we know it will have the form
\[ F[\alpha] = \sum_i f(\alpha_i) \]. The explicit form of \( f(\alpha) \) can be obtained only from the full heat kernel
derivation [2], as in the usual case. However, in the continuum limit it becomes
\[ \sum_i f(\alpha_i) = \sum_i f(1) - f'(1) \sum_i \delta_i = N f(1) - f'(1) \] (37)
which, due to (14) is a constant term of topological nature. The proportionality constant
can be borrowed from perturbation theory and has the value 5/4. Summing up, for sphere
topology we have
\[ S_{sL} = \frac{5}{4} \left( \sum_{i,j \neq i} \frac{(1 - \alpha_i)(1 - \alpha_j)}{2\alpha_i - 1} \ln(z_{ij} \bar{z}_{ij}) + \frac{\lambda_0}{2} \sum_i \left( 2\alpha_i - 1 - \frac{1}{2\alpha_i - 1} \right) + \sum_i f(\alpha_i) \right) \] (38)
This expression, for \( \alpha_i \approx 1 \) and for a dense set of \( z_i \) goes over to the continuum result (8).

3 Torus topology

The supertorus can be defined [14] as the quotient of the complex superplane \( \mathbb{C}^{(1,1)} \) with
respect to an abelian group \( G \) (the fundamental group of the torus) with two generators
\( g_1, g_2 \) which act on \( \mathbb{C}^{(1,1)} \) in a properly discontinuous manner, leaving invariant the metric
element \( (dz + d\theta \theta)(d\bar{z} - d\bar{\theta} \bar{\theta}) \). We will denote \( g \) by \( (a, b, \alpha) \), i.e. \( (a, b, \alpha)(z, \theta) = (z + ab + a\alpha, b\theta + a\alpha) \). The conditions imposed on \( g_1 \) and \( g_2 \) imply that \( a_1 \) and \( a_2 \) are equal to \( \pm 1 \).

We must distinguish between two cases
1. Even spin structures

These are given by \( a_1 \) and \( a_2 \) not both equal to 1. It is known [14] that by means of
a conjugation with an element of \( G \) and with a uniform rescaling \( z \rightarrow k^2 z, \ \theta \rightarrow k\theta \), the
generators can be reduced to the canonical form
\[ g_1 = (a_1, 1, 0) \]
\[ g_2 = (a_2, \tau, 0) \]
where $\tau$ is a bosonic modulus.

2. **Odd spin structure**

This is given by $a_1 = a_2 = 1$. By means of a conjugation with a uniform rescaling we can reduce the generators to the form

$g_1 = (1, 1, \alpha)$

$g_2 = (1, \tau, \alpha \tau)$.

If furthermore one conjugates with respect to the element of the superconformal group $f(z, \theta) = (z(1 + \alpha \theta), \theta + \alpha z)$ one obtains

$g_1 = (1, 1, 0)$

$g_2 = (1, \tau, \chi)$

where $\chi = (c - \tau)\alpha$ is the fermionic supermodulus which is absent in the even spin structures. The superzweibein is not left unchanged under such a transformation, instead it goes over to

$$
\begin{pmatrix}
1 + 2\alpha\theta' & \alpha \\
\theta' & 1
\end{pmatrix}
$$

(39)

for the $(z, \theta)$ sector, and similarly for the $(\bar{z}, \bar{\theta})$ sector. However, such a zweibein can be obtained from $\hat{E}$ by means of the super Weyl transformation generated by $e^{\Sigma} = (1 + \alpha \theta')(1 - \bar{\alpha} \bar{\theta}')$, supplemented by the internal $U(1)$ transformation given by the rotation $\frac{1 + \bar{\alpha} \bar{\theta}'}{1 - \alpha \theta'}$. As the superdeterminant of the last transformation is 1, everything reduces to a change in the superconformal factor.

In case 1, i.e. even spin structures, the modular group is given by the usual modular transformations

$$
\begin{align*}
z' &= \frac{z}{c\tau + d} & \theta' &= \frac{\theta}{(c\tau + d)^{1/2}} \\
\tau' &= \frac{a\tau + b}{c\tau + d} & \text{with } ad - bc &= 1, \; a, b, c, d \in \mathbb{Z}
\end{align*}
$$

(40)

which are generated by the two transformations

$$
\tau' = \tau + 1 \quad \text{and} \quad \tau' = -\frac{1}{\tau}.
$$

(41)

Writing $z = x + \tau y$, $\bar{z} = x + \bar{\tau} y$ the fundamental region in $(x, y)$ is $[0, 1] \times [0, 1]$. 

10
In case 2, i.e. odd spin structure, we have the supermodular transformations given by

\[
\begin{align*}
    z' &= \frac{z}{ct + d + c\chi \theta}; \\
    \theta' &= \frac{\theta}{(ct + d)^{1/2}} - \frac{c\chi z}{(ct + d)^{3/2}}; \\
    \tau' &= \frac{a\tau + b}{ct + d}; \\
    \chi' &= \frac{\chi}{(ct + d)^{3/2}};
\end{align*}
\]

whose generators are

\[
\begin{align*}
    (\tau', \chi') &= (\tau + 1, \chi) \\
    (\tau', \chi') &= (-\frac{1}{\tau}, \frac{\chi}{\tau^{3/2}}).
\end{align*}
\]

We recall moreover that the two elements \( g = (1, \tau, \chi) \) and \( (1, \tau, e^{i\phi} \chi) \) with \( \phi = n \pi \) are equivalent under conjugation \( [14] \).

On a torus equipped with an even spin structure \((a, b)\) with \(a_1 = (-1)^a\), \(a_2 = (-1)^b\) the flat superzweibein is still given by eq. (1). The Green function \( \hat{G}_{ab}(z, z'; \tau) \) satisfies

\[
\int d^2z' \hat{\Box}(-) \hat{G}_{ab}(z, z'; \tau) f_{ab}(z') = f_{ab}(z)
\]

for any \( f_{ab}(z) \) belonging to the \((a, b)\) spin structure and orthogonal to the zero modes of \( \hat{\Box}(-) \) i.e. such that \( \int dz d\bar{z} d\theta d\bar{\theta} f_{ab} = 0 \); in fact even if the supermetric \( (f_{ab}, f_{ab}) = \int d^2z\hat{E} f_{ab} f_{ab} \) is not definite positive, one can explicitly check that Range(\( \hat{\Box}(-) \)) = (Ker(\( \hat{\Box}(-) \)))\perp.

Such a Green function is given by \( [12, 15, 16] \)

\[
G_{ab}(z, z', \tau) = G_0(z - z', \tau) + [\theta' S_{ab}(z - z') + c.c.] \quad (\text{for } z \neq z')
\]

where \( [15] \)

\[
G_0(z - z', \tau) = \frac{1}{\pi} \ln \left| \frac{\vartheta_{11}(z - z', \tau)}{\eta(\tau)} \right|^2 + \frac{i(z - \bar{z} - z' + \bar{z}')^2}{\tau - \bar{\tau}}
\]

and

\[
S_{ab}(z - z') = \frac{1}{\pi} \frac{\vartheta'_{11}(0, \tau)}{\vartheta_{ab}(0, \tau)} \frac{\vartheta_{ab}(z - z', \tau)}{\vartheta_{11}(z - z', \tau)}.
\]

The transformation properties of the Green functions under modular transformations are the following...
for \((z, \theta) \to (z, \theta)\) and \(\tau \to \tau + 1\), \(G_{ab} \to G_{a,a+b+1}\);

for \((z, \theta) \to \left(\frac{z}{\tau}, \frac{\theta}{\tau}\right)\) and \(\tau \to -\frac{1}{\tau}\), \(G_{ab} \to G_{ba}\);

(with \(a\) and \(b\) defined modulo \(2\)) that is, the even spin structures \((0, 0), (0, 1)\) and \((1, 0)\) undergo a permutation.

We come now to the odd spin structure. For clearness sake we shall denote by \(\hat{z}, \hat{\theta}\) the original variables \(z, \theta\) which appear in eq.(42,43) and are associated to the zweibein \(\hat{E}\) of eq.(1). One can perform the change of variables

\[
\hat{z} = z + \zeta(z - \bar{z}) \quad \hat{\theta} = \theta + \zeta(z - \bar{z})
\]

where \(\zeta = \frac{\chi}{\tau - \bar{\tau}}\); the fundamental region in \(z, \theta\) takes a particularly simple form, i.e. the product of \(\theta\) by an ordinary torus of modulus \(\tau\). The superzweibein \(\hat{E}\) is transformed into

\[
\tilde{E}_M^A = \begin{pmatrix}
1 + 2\zeta \theta & 2\zeta \bar{\theta} & \zeta & -\bar{\zeta} \\
-2\zeta \theta & 1 - 2\zeta \bar{\theta} & -\zeta & \bar{\zeta} \\
\theta & 0 & 1 & 0 \\
0 & -\bar{\theta} & 0 & 1
\end{pmatrix}
\]

\(\tilde{E}_M^A\) is invariant under the usual translations and the two super Killing vectors

\[
z' = z + \varepsilon \zeta \theta; \quad \theta' = \theta + \varepsilon \zeta \\
z' = z - \varepsilon \bar{\zeta} \bar{\theta}; \quad \bar{\theta'} = \bar{\theta} + \varepsilon \bar{\zeta}.
\]

The superinvariant derivatives are

\[
\tilde{D}_+ = (1 + \theta \zeta + \theta \bar{\theta} \zeta \bar{\zeta}) \partial_\theta - \theta \bar{\zeta} \partial_{\bar{\theta}} - \theta \partial_z - \theta \bar{\bar{\theta}} \zeta \partial_{\bar{z}}; \\
\tilde{D}_- = (1 - \bar{\zeta} + \theta \bar{\theta} \zeta \bar{\zeta}) \partial_{\bar{\theta}} + \theta \zeta \partial_\theta + \theta \bar{\bar{\theta}} \zeta \partial_z + \theta \partial_{\bar{z}}.
\]

The Green function of \(\tilde{\Omega}^{(-)}\) for the odd spin structure \(G_{++}(z, z'; \tau, \chi)\) satisfies

\[
\int d^2z' \tilde{E}' \tilde{\Omega}^{(-)} G_{++}(z, z'; \tau, \chi) f_{++}(z') = f_{++}(z)
\]

for any \(f_{++} = f_0 + \theta \phi - \bar{\theta} \bar{\phi} + \theta \bar{\bar{\theta}} f_A\) belonging to the odd spin structure with

\[
\int d^2z(\phi - \zeta f_0) = \text{const} \zeta \bar{\zeta}; \\
\int d^2z(\bar{\phi} - \bar{\zeta} \bar{f}_0) = \text{const} \zeta \bar{\zeta}; \\
\int d^2z(f_A - 2\zeta \bar{\zeta} f_0) = 0,
\]
i.e. orthogonal to the kernel of $\tilde{\Gamma}^{(-)}$. In fact, as it happens for the even spin structures, one can explicitly check that $\text{Range}(\tilde{\Gamma}^{(-)}) = (\text{Ker}(\tilde{\Gamma}^{(-)}))^\perp$. $G_{++}$ is given by a supersymmetric generalization of the function $G$ of the usual case, i.e.\cite{12}

$$
G_{++}(z, z'; \tau, \chi) = \frac{1}{\pi} \left[ \ln \left| \frac{\varphi_{11}(\hat{z} - \hat{z}' + \hat{\theta} \hat{\theta}', \tilde{\tau})}{\eta(\tilde{\tau})} \right|^2 + \frac{i\pi}{\tilde{\tau} - \tilde{\tau}'}(\hat{z} - \hat{z}' + \hat{\theta} \hat{\theta}' - \overline{\hat{z}} + \overline{\hat{z}'} + \overline{\hat{\theta}} + \overline{\hat{\theta}'})^2 \right] 
$$

(55)

where $\tilde{\tau} = \tau + \chi(\hat{\theta} + \hat{\theta}')$ and $\hat{z}, \hat{\theta}$ are given by eq. (49). It is easily checked that $G_{++}$, in addition of possessing the two ordinary Killing vectors $\hat{z} \to \hat{z} + \epsilon$ and the two super Killing vectors $\hat{z} \to \hat{z} + \epsilon \hat{\theta}$, $\hat{\theta} \to \hat{\theta} + \epsilon$, is invariant under the supermodular transformations eq.(42,43) in the variables $\hat{z}, \hat{\theta}$.

For even spin structures the Regge superconformal factor is given by

$$
\Sigma(z, z_i, \alpha_i, \lambda_0, \tau) = \sum_i (\alpha_i - 1) \pi G_{ab}(z, z_i, \tau) + \lambda_0
$$

(56)

where $\lambda_0$ is the unique zero mode of $\tilde{\Gamma}^{(-)}$ in the space of even spin structure functions. The Gauss-Bonnet theorem now imposes $\sum_i (1 - \alpha_i) = 0$. The discrete transcription of the action $\Sigma$ is given by

$$
S_{sL} = \text{const} \left[ \frac{1}{2} \sum_{ij} K_{ij}[\alpha] \pi G_{ab}(z_i, z_j; \tau) + B(\lambda_0, \alpha, \tau) \right].
$$

(57)

We shall now exploit the fact, explicitly verified in the non supersymmetric case and which follows from the nature of the heat kernel derivation, that the singular behavior of the action at short distances is independent of the topology. Thus we find

$$
K_{ij}[\alpha] = (1 - \alpha_i)(1 - \alpha_j)(\frac{1}{2\alpha_i - 1} + \frac{1}{2\alpha_j - 1}).
$$

(58)

Next we impose the invariance of the action under modular transformations. The superconformal factor transforms as

$$
\Sigma'(z') = \Sigma(z) + \ln |\tau c + d|
$$

(59)
\[ \Sigma'(z', \theta', z_i, \theta_i, \alpha_i, \lambda_0, \tau) = \Sigma(z', \theta', z'_i, \theta'_i, \alpha'_i, \lambda'_0, \tau') \]  
(60) 

with
\[ z'_i = \frac{z_i}{c\tau + d}; \quad \theta'_i = \frac{\theta_i}{(c\tau + d)^{1/2}}; \quad \alpha'_i = \alpha_i; \]
\[ \lambda'_0 = \lambda_0 + \ln |c\tau + d|; \quad \tau' = \frac{a\tau + b}{c\tau + d}. \]  
(61)

Thus \( e^{\lambda_0} \) transforms as the modulus of a modular form of weight 1. Posing
\[ B(\lambda_0, \alpha, \tau) = C(\lambda_0 - \ln |2\pi \eta^2(\tau)|, \alpha, \tau) \]  
(62)

we have that
\[ C(x, \alpha, \tau) = C(x, \alpha, \frac{a\tau + b}{c\tau + d}). \]  
(63)

To further specialize the structure of \( C \) we proceed as in ref.\[8\] considering the transformation \( \lambda_0 \rightarrow \lambda_0 - \ln k^2, \ z_i \rightarrow k^2 z_i, \ \theta_i \rightarrow k\theta_i \). Taking into account that \( \sum_i (1 - \alpha_i) = 0 \) we find
\[ B = -2(\lambda_0 - \ln |2\pi \eta^2(\tau)|) \sum_i \frac{(1 - \alpha_i)^2}{2\alpha_i - 1} + F[\alpha, \tau]. \]  
(64)

The only requirement on \( F[\alpha, \tau] \) imposed by group theoretical considerations is to be a modular invariant function of \( \tau \). However, if we limit ourselves, as in ref. \[8\] , to the realm of modular functions, the only choice which is free of singularities in the upper half plane, infinity included, is to take \( F \) independent of \( \tau \) \[17\]. This also follows more directly from the nature of the heat kernel derivation according to which \( F[\alpha] \) has the structure \( \sum_i f(\alpha_i) \). In fact from inspection one sees that such a term arises form the “direct” terms \[18\] in the variation of the logarithm of the determinant, which depend only on the short time behavior of the heat kernel in the proximity of \( z_i \) and thus only on the \( \alpha_i \) and are independent of the topology and the moduli of the surface. Summing up, for the even spin structures we obtain the action
\[ S_{sL,ab} = \frac{5}{4} \left( \sum_{i,j \neq i} \frac{(1 - \alpha_i)(1 - \alpha_j)}{2\alpha_i - 1} \pi G_{ab}(z_i, z_j, \tau) - 2(\lambda_0 - \ln |2\pi \eta^2(\tau)|) \sum_i \frac{(1 - \alpha_i)^2}{2\alpha_i - 1} + \sum_i f(\alpha_i) \right). \]  
(65)
For the odd spin structure the Regge superconformal factor takes the form

$$
\sum_i (\alpha_i - 1) \pi G_{++}(z, z_i, \tau, \chi)
$$

(66)

with $G_{++}$ given by eq. (53), to which the zero modes of $\Box^{(-)}$ (see eq. (52)) have to be added. These are given by

$$
\lambda_0 + (c_1 \chi + \bar{c}_2 \bar{\chi}) \theta - (\bar{c}_1 \bar{\chi} + c_2 \chi) \bar{\theta}
$$

(67)

with $\lambda_0$ real and $c_1, c_2$ complex bosonic variables. Performing a modular transformation the zweibein $\tilde{E}_A^M$ goes over to a new zweibein given by

$$
\tilde{E}'_M^A = \frac{\partial z^N}{\partial z'^M} \tilde{E}_N^A.
$$

(68)

$\tilde{E}'_M^A$ is not of our canonical form (50) but can be reduced to it (obviously with $\zeta$ and $\theta$ replaced by $\zeta'$ and $\theta'$ ) through a super Weyl transformation and a $U(1)$ internal rotation. The new superconformal factor $\Sigma'(z')$ is given by

$$
\Sigma'(z') = \Sigma(\hat{z}) + \ln |c \tau + d| + \frac{c \chi \theta}{c \tau + d} - \frac{c \bar{\chi} \bar{\theta}}{c \bar{\tau} + d}
$$

(69)

Such a transformation is obtained by taking into account that the supermodular transformations are

$$
z' = \frac{z}{c \tau + d}; \quad \theta' = \theta (c \tau + d)^{1/2}
$$

(70)

as follows from eqs. (42) and (43) for the $\hat{z}, \hat{\theta}$ variables, and using $\sdet(\tilde{E}_M^A) = 1 + \zeta \theta - \bar{\zeta} \bar{\theta}$, which transforms according to (70) and (43). The modular invariance of $G_{++}$ implies the following transformations

$$
\lambda'_0 = \lambda_0 + \ln |\tau c + d|
$$

$$
c'_1 = c_1 (c \tau + d)^2 + c (c \tau + d)
$$

$$
c'_2 = c_2 (c \bar{\tau} + d)^{3/2} (c \tau + d)^{1/2}
$$

(71)
By exploiting the independence from the topology of the singular behaviour of the action at short distances we obtain the structure

\[ S_{sL} = \text{const} \left[ \sum_{i,j \neq i} \frac{(1 - \alpha_i)(1 - \alpha_j)}{2\alpha_i - 1} \pi G_{++}(z_i, z_j, \tau, \chi) + B(\alpha, \lambda_0, \tau, \chi, c_1\chi, c_2\chi) \right] . \]  

(72)

In order to obtain a more explicit form of \( B \) we consider first the case \( \chi = 0 \). \( B \) now depends only on \( \alpha, \lambda_0, \tau \) and using the same argument as for the even spin structures we obtain

\[ B(\alpha, \lambda_0, \tau, 0, 0, 0) = -2(\lambda_0 - \ln |2\tau\eta|^2) \sum_i \frac{(1 - \alpha_i)^2}{2\alpha_i - 1} + \sum_i f(\alpha_i) . \]  

(73)

Because of the nihilpotency of \( \chi \) and the bosonic nature of the action we now have

\[ B(\alpha, \lambda_0, \tau, \chi, c_1\chi, c_2\chi) = B(\alpha, \lambda_0, \tau, 0, 0, 0) + \chi\bar{\chi} \left[ F_0 + \sum_{m,n=1}^{2} F_{mn}c_m\bar{c}_n + (\sum_{m=1}^{2} F_m c_m + c.c.) \right] . \]  

(74)

where the \( F \)'s are functions of the modular invariant \( \alpha_i \) and of \( \tau \).

Invariance of the action under modular transformations (47) and (53) gives a set of restrictions on the \( F \)'s, which however are not sufficient to determine them completely. Probably for the odd spin structure case a more profound appeal to the structure of the heat kernel derivation is needed to pinpoint completely the unknown \( F \)'s. Summing up, the result we have achieved for the odd spin structure is the following

\[ S_{sL,++} = \frac{5}{4} \left( \sum_{i,j \neq i} \frac{(1 - \alpha_i)(1 - \alpha_j)}{2\alpha_i - 1} \pi G_{++}(z_i, z_j, \tau, \chi) - 2(\lambda_0 - \ln |2\tau\eta|^2) \right) \sum_i \frac{(1 - \alpha_i)^2}{2\alpha_i - 1} \]

\[ + \sum_i f(\alpha_i) + \chi\bar{\chi} \left[ F_0(\alpha, \tau) + \sum_{m,n=1}^{2} F_{mn}(\alpha, \tau)c_m\bar{c}_n + (\sum_{m=1}^{2} F_m(\alpha, \tau)c_m + c.c.) \right] . \]  

(75)

For amplitudes to which both even and odd spin structures contribute the relative weight is fixed by the factorization property for multiloop amplitudes [19].

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