Error Analysis of Fully Discrete Scheme for the Cahn–Hilliard–Magneto-Hydrodynamics Problem

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Abstract
In this paper we analyze a fully discrete scheme for a general Cahn–Hilliard equation coupled with a nonsteady Magneto-hydrodynamics flow, which describes two immiscible, incompressible and electrically conducting fluids with different mobilities, fluid viscosities and magnetic diffusivities. A typical fully discrete scheme, which is comprised of conforming finite element method and the Euler semi-implicit discretization based on a convex splitting of the energy of the equation is considered in detail. We prove that our scheme is unconditionally energy stable and obtain some optimal error estimates for the concentration field, the chemical potential, the velocity field, the magnetic field and the pressure. The results of numerical tests are presented to validate the rates of convergence.

Keywords Nonstationary Magneto-hydrodynamics flow · Cahn–Hilliard equation · Fully discrete scheme · Conforming finite element · Energy stable · Optimal error estimates

Mathematics Subject Classification 65N30 · 76M10 · 76W05

1 Introduction

In the paper, we derive error estimates for a fully discrete, first order in time, finite element method for the Cahn–Hilliard–Magneto-hydrodynamics problem for two phase flow. Let \( \Omega \subset \mathbb{R}^d \) \((d = 2, 3)\) stands for an open convex polygonal or polyhedral domain with Lipschitz continuous boundary \( \partial \Omega \). For all \( \phi \in H^1(\Omega), u \in L^2(\Omega) \) and \( B \in L^2(\Omega) \), consider the following energy:

\[
E(\phi, u, B) = \int_{\Omega} \left( \frac{1}{2} |u|^2 + \frac{S_c}{2} |B|^2 + \frac{\lambda \epsilon}{2} |\nabla \phi|^2 + \frac{\lambda}{4\epsilon} (1 - \phi^2)^2 \right) dx,
\]

where \( \phi, u \) and \( B \) denote respectively the concentration field, the velocity field and the magnetic field, and the parameter \( \epsilon > 0 \) stands for the interfacial thickness between the
The Cahn–Hilliard-Magneto-hydrodynamics system is a gradient flows of this energy:

\[
\begin{align*}
\partial_t \phi - \epsilon \text{div}(\kappa(\phi)\nabla \mu) - \nabla \phi \cdot \mathbf{u} &= 0, \quad \text{in } \Omega_T := \Omega \times (0, T], \\
- \epsilon \Delta \phi + \epsilon^{-1}(\phi^3 - \phi) &= \mu, \quad \text{in } \Omega_T,
\end{align*}
\]

\[
\begin{align*}
\partial_t \mathbf{u} - \text{div}(2\nu(\phi)\mathbb{D}(\mathbf{u})) + (\mathbf{u} \cdot \nabla)\mathbf{u} + S_c \mathbf{B} \times \text{curl} \mathbf{B} + \nabla p &= \lambda \epsilon \kappa(\phi) \nabla \mu, \quad \text{in } \Omega_T,
\end{align*}
\]

\[
\begin{align*}
\partial_t \mathbf{B} + \text{curl}(\eta(\phi)\text{curl} \mathbf{B}) - \text{curl}(\mathbf{u} \times \mathbf{B}) &= 0, \quad \text{in } \Omega_T,
\end{align*}
\]

\[
\begin{align*}
\text{div} \mathbf{u} &= 0, \quad \text{in } \Omega_T,
\end{align*}
\]

\[
\begin{align*}
\text{div} \mathbf{B} &= 0, \quad \text{in } \Omega_T.
\end{align*}
\]

This problem is considered in conjunction with natural and no-flux/no-flow boundary conditions:

\[
\begin{align*}
\frac{\partial \phi}{\partial n} = \frac{\partial \phi}{\partial n} = 0, \quad &\text{on } \partial \Omega_T, \\
\mathbf{u} &= 0, \quad \text{on } \partial \Omega_T, \\
\mathbf{B} \cdot n &= 0, \quad \text{on } \partial \Omega_T, \\
\text{curl} \mathbf{B} \times n &= 0, \quad \text{on } \partial \Omega_T,
\end{align*}
\]

and initial conditions:

\[
\begin{align*}
\phi(x, 0) &= \phi_0(x), \quad \mathbf{u}(x, 0) = \mathbf{u}_0(x), \quad \mathbf{B}(x, 0) = \mathbf{B}_0(x), \quad \text{in } \Omega,
\end{align*}
\]

where \( T > 0 \) is time, \( \phi \approx \pm 1 \) represents two different fluids, \( \mu \) and \( p \) are respectively the chemical potential and the pressure. \( \mathbb{D}(\mathbf{u}) := \frac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^T) \) is the strain-rate tensor.

The given functions \( \kappa(\phi) > 0, \nu(\phi) > 0 \) and \( \eta(\phi) = \frac{1}{\eta_0 \sigma(\phi)} > 0 \) stand for respectively the mobility, the fluid viscous, the magnetic diffusivity with \( \sigma \) the electrical conductivity. \( \eta_0 \) is the magnetic permeability. \( S_c := \frac{1}{\eta_0 \rho_0} \) denotes the coupling coefficient, here \( \rho_0 \) is the reference density. \( \lambda \) denotes the mixing energy density.

The (2)–(4) system satisfies the following energy laws:

\[
E(\phi(t), \mathbf{u}(t), \mathbf{B}(t)) + \int_0^t \left( v(\phi) \| \nabla \mathbf{u} \|^2 + S_c \eta(\phi) \| \nabla \mathbf{B} \|^2 + \lambda \epsilon \kappa(\phi) \| \nabla \mu \|^2 \right) ds = E(\phi_0, \mathbf{u}_0, \mathbf{B}_0).
\]

In the last decades, phase field approaches for two phases incompressible flows have been widely developed to model and numerically solve the topological transitions of interfaces (cf. \cite{1, 6, 7, 11, 14, 19, 21, 32} and references therein). Recently, the research of interaction of electromagnetic fields with two immiscible, incompressible and electrically conducting fluids has become more and more important for the design and analysis of engineering field, such as fusion reactors, pump accelerators, metallurgical industry and Magneto-hydrodynamics generators \cite{4, 23, 24, 26}. Very recently, in the work \cite{33}, two phase Magneto-hydrodynamics problem about the diffuse interface between two different incompressible fluids is considered.

Though many error estimates are available for fully discrete scheme to the Magneto-hydrodynamics equations \cite{18, 22, 25, 30, 31} and for the Cahn–Hilliard/Navier–Stokes equations \cite{3, 5–10, 27}, it is not trivial to cope with the system which couple Magneto-hydrodynamics with Cahn–Hilliard, since the phase field dependent coefficients affects the whole system. The major difficulties cause from the phase field dependent coefficients and from the coupling nonlinear terms. To the best of our knowledge, error estimates for fully
discrete scheme of problem (2)–(4) are not yet set up [33]. However, the analysis in [3, 5, 7] cannot be easily developed to the fully discrete form of problem (2)–(4), as the time-space discretization raises another difficulty, particularly in deriving error estimates for the pressure.

In this paper, applying some useful techniques given in references [3, 5, 29], we analyze a fully discrete scheme for problem (2)–(4), which is comprised of conforming mixed finite element method in space and the Euler semi-implicit discretization with a convex splitting method in time. The main purpose of this paper is to derive some optimal error estimates for the concentration field, the chemical potential, the velocity field, the magnetic field and the pressure.

The highlight of this paper is to set up the error estimates for the fully discrete solution \((\phi^n_h, \mu^n_h, u^n_h, p^n_h, B^n_h)\) as follows:

\[
\max_{1 \leq n \leq N} \|\nabla (\phi(t_n) - \phi^n_h)\| + \left( \Delta t \sum_{n=1}^{N} \|\nabla (\mu(t_n) - \mu^n_h)\|^2 \right)^{1/2} \leq C \left( \Delta t + h^{k+1} \right),
\]

\[
\max_{1 \leq n \leq N} \|u(t_n) - u^n_h\| + \max_{1 \leq n \leq N} \|B(t_n) - B^n_h\| \leq C \left( \Delta t + h^{k+1} \right),
\]

\[
\left( \Delta t \sum_{n=1}^{N} \|p(t_n) - p^n_h\|^2 \right)^{1/2} \leq C \left( \Delta t + h^{k+1} \right).
\]

Numerical tests are given to confirm the theoretical rates of convergence.

The paper is organized as follows. In Sect. 2 we present some preliminary results and the well-posedness of weak solution of Cahn–Hilliard–Magneto-hydrodynamics system. In Sect. 3 we give fully discrete scheme and obtain its unconditionally energy stable. In Sect. 4 we prove some optimal error estimates for the concentration field, the chemical potential, the velocity field, the magnetic field and the pressure. In Sect. 5 we present some numerical tests to checked the theoretical results of the our scheme.

# 2 Continuous Problem

## 2.1 Preliminaries

In this subsection we give some standard notations. Let \(C^m(\Omega)\) \((m \in N)\) be the space of functions with up to \(m\) times continuously differentiable in \(\Omega\), and let \(C^{m,1}(\Omega)\) be the space of functions in \(C^m(\Omega)\) that are Lipschitz continuous in \(\Omega\). Let \((L^p(\Omega), \|\cdot\|_{L^p})\) and \((W^{k,p}(\Omega), \|\cdot\|_{k,p})\) denote respectively the Lebesgue spaces and Sobolev spaces. For simplicity, we denote \(\|\cdot\| := \|\cdot\|_{L^2}\), and denote by \(H^k(\Omega)\) the Sobolev space \(W^{k,2}(\Omega)\).

To define a weak formulation of problem (2)–(4), we introduce the following Sobolev spaces

\[
\mathcal{X} := H^0(\Omega)^d = \left\{ v \in H^1(\Omega)^d : v|_{\partial \Omega} = 0 \right\},
\]

\[
\mathcal{X}_0 := \left\{ v \in L^2(\Omega)^d : \nabla \cdot v = 0, v|_{\partial \Omega} = 0 \right\},
\]

\[
\mathcal{W} := H^1(\Omega)^d := \left\{ w \in H^1(\Omega)^d : w \cdot n|_{\partial \Omega} = 0 \right\}.
\]
\[ \begin{align*}
\mathcal{W}_0 & := \left\{ w \in H^1(\Omega)^d : \text{curl} \, w \times n|_{\partial\Omega} = 0 \right\}, \\
\mathcal{X}_0 & := \left\{ v \in \mathcal{X} : \text{div} v = 0 \right\}, \\
\mathcal{W}_n & := \left\{ w \in \mathcal{W} : \text{div} w = 0 \right\}, \\
Q & := \left\{ \varphi \in H^1(\Omega) : \frac{\partial \varphi}{\partial n}|_{\partial\Omega} = 0 \right\}, \\
M & := L^2_0(\Omega) = \left\{ q \in L^2(\Omega), \int_\Omega q \, dx = 0 \right\}.
\end{align*} \]

The following useful inequalities hold \[9, 12, 13, 15, 18, 28\]:

\[ \begin{align*}
\| v \| & \leq C_\Omega \| \nabla v \|, \quad \forall v \in \mathcal{X}, \\
\| v \|_{L^6} & \leq C_\Omega \| \nabla v \|, \quad \forall v \in \mathcal{X}, \\
\| \mathbb{D} v \| & \geq c_0 \| \nabla v \|, \quad \forall v \in \mathcal{X}, \\
\| v \|_{L^3} & \leq C_\Omega \| v \|^\frac{6-d}{3} \| \nabla v \|^\frac{d}{3}, \quad \forall v \in \mathcal{X}, \\
\| v \|_{L^4} & \leq C_\Omega \| v \|^\frac{4-d}{4} \| \nabla v \|^\frac{d}{4}, \quad \forall v \in \mathcal{X}, \\
\| v \|_{L^\infty} & \leq C_\Omega \| v \|^\frac{1}{2} \| \nabla v \|^\frac{1}{2}, \quad \forall v \in H^2(\Omega)^d, \\
c_1 \| \nabla B \|^2 & \leq \| \text{curl} \, B \|^2 + \| \text{div} B \|^2, \quad \forall B \in \mathcal{W}, \\
\| \text{curl} \, B \| & \leq \sqrt{2} \| \nabla B \|, \quad \| \text{div} B \| \leq \sqrt{d} \| \nabla B \|, \quad \forall B \in H^1(\Omega)^d, \\
\| \phi \|_{L^p} & \leq C_\Omega \| \phi \|_{H^1}, \quad (2 \leq p \leq 6) \quad \forall v \in \mathcal{Q}, \\
\| \phi \|_{L^3} & \leq C_\Omega \| \phi \|^\frac{6-d}{3} \| \nabla \phi \|^\frac{d}{3} + C_\Omega \| \phi \|, \quad \forall v \in \mathcal{Q}, \\
\| \phi \|_{L^\infty} & \leq C_\Omega \| \Delta \phi \|^\frac{3(d-1)}{3d} \| \phi \|^\frac{2d-3}{3} + C_\Omega \| \phi \|^\frac{3d}{3d-1}, \quad \forall v \in \mathcal{Q},
\end{align*} \]

where \( c_0, c_1 \) and \( C_\Omega \) are positive constants depending on \( \Omega \).

We define the following bilinear terms:

\[ a_\phi(\varphi; \phi, \theta) = \int_\Omega \kappa(\varphi) \nabla \phi \cdot \nabla \theta \, dx, \quad a_f(\varphi; u, v) = \int_\Omega 2\nu(\varphi) \mathbb{D}(u) : \mathbb{D}(v) \, dx, \]

\[ a_B(\varphi; B, H) = \int_\Omega \eta(\varphi) \text{curl} \, B \cdot \text{curl} \, H \, dx + \int_\Omega \eta(\varphi) \text{div} \, B \cdot \text{div} \, H \, dx, \]

\[ d(v, q) = \int_\Omega q \text{div} v \, dx, \]

and trilinear terms:

\[ b(w, u, v) = \frac{1}{2} \int_\Omega [(w \cdot \nabla) u] \cdot v - [(w \cdot \nabla)v] \cdot u \, dx = \int_\Omega [(w \cdot \nabla) u] \cdot v + \frac{1}{2} [(\nabla \cdot w) u] \cdot v \, dx, \]

\[ c_B(H, B, v) = \int_\Omega H \times \text{curl} \, B \cdot v \, dx, \quad c_B(u, B, H) = \int_\Omega (u \times B) \cdot \text{curl} \, H \, dx. \]

In addition, using the definition of \( b(\cdot, \cdot, \cdot) \), it follows that

\[ b(u, v, v) = 0, \quad u \in \mathcal{X}, v \in H^1(\Omega)^d. \]
Theorem 1

Suppose that the initial conditions were proved by the similar lines as in [2, 17, 20]. Thus, we skip the proofs of the following:

\[ c_B(B, B, u) - c_B(u, B, B) = 0, \quad u \in \mathcal{X}, B \in \mathcal{W}. \]  

(8)

The bilinear term \( d(\cdot, \cdot) \) satisfies the LBB condition [12, 28]:

\[ \sup_{v \in \mathcal{X}, v \neq 0} \frac{d(v, q)}{\|v\|_1} \geq \beta\|q\|, \quad \forall q \in \mathcal{M}, \]  

(9)

where \( \beta > 0 \) is constant depend on \( \Omega \).

A weak formulation for (2)–(4) may be written as follows: find \((\phi, \mu, u, p, B)\) such that

\[ \begin{aligned}
&\frac{\partial}{\partial t} \phi + \epsilon a_\phi(\phi; \mu, \psi) + (\nabla \phi \cdot u, \psi) = 0, \\
&\epsilon^{-1}(\phi^3 - \phi, \theta) + \epsilon (\nabla \phi, \nabla \theta) = (\mu, \theta), \\
&\frac{\partial}{\partial t} u + a_f(\phi; u, v) + b(u, u, v) + S c_B(B, B, v) \\
&- d(v, p) = \lambda(\mu \nabla \phi, v), \\
&d(u, q) = 0, \\
&\frac{\partial}{\partial t} B + a_B(\phi; B, H) - c_B(u, B, H) = 0,
\end{aligned} \]  

(10a–10e)

for all \((\psi, \theta, v, q, H) \in Q \times Q \times \mathcal{X} \times \mathcal{M} \times \mathcal{W} \).

2.2 Wellposedness of Solution

This subsection builds a well-posedness result of (10). For simplicity, we consider the following problem: find \((\phi, \mu, u, B) \in (Q, Q, \mathcal{X}_0, \mathcal{W})\), such that

\[ \begin{aligned}
&\frac{\partial}{\partial t} \phi + \epsilon a_\phi(\phi; \mu, \psi) + (\nabla \phi \cdot u, \psi) = 0, \\
&\epsilon^{-1}(\phi^3 - \phi, \theta) + \epsilon (\nabla \phi, \nabla \theta) = (\mu, \theta), \\
&\frac{\partial}{\partial t} u + a_f(\phi; u, v) + b(u, u, v) + S c_B(B, B, v) = \lambda(\mu \nabla \phi, v), \\
&\frac{\partial}{\partial t} B + a_B(\phi; B, H) - c_B(u, B, H) = 0,
\end{aligned} \]  

(11a–11d)

for all \((\psi, \theta, v, H) \in Q \times Q \times \mathcal{X}_0 \times \mathcal{W} \).

The next theorems give a well-posedness result for weak solution of problem (11). They were proved by the similar lines as in [2, 17, 20]. Thus, we skip the proofs of the following theorems.

**Theorem 1** Suppose that the initial conditions \( \phi_0, u_0, B_0 \) satisfy

\[ \phi_0 \in H^1(\Omega), \quad u_0 \in L^2(\Omega)^d, \quad B_0 \in L^2(\Omega)^d. \]  

(12)

Furthermore, assume that the given functions \( \kappa, \nu \) and \( \eta \) satisfy

\[ \kappa, \nu, \eta \in C(\overline{\Omega} \times [0, T] \times R; R^+). \]  

(13)

Then problem (11) has at least one solution \((\phi, \mu, u, B)\) such that

\[ \begin{aligned}
&\phi \in L^\infty(0, T; H^1(\Omega)) \cap L^2(0, T; L^2(\Omega)), \\
u \in L^\infty(0, T; L^2(\Omega)^d) \cap L^2(0, T; \mathcal{X}), \\
&B \in L^\infty(0, T; L^2(\Omega)^d) \cap L^2(0, T; \mathcal{W}) \\
&\mu \in L^2(0, T; Q).
\end{aligned} \]  

(14)
Theorem 2 Taking the place of (13) by \(v, \eta, \kappa \in C^{0,1}(\overline{\Omega} \times [0, T] \times R^+ \) and

\[
\begin{align*}
0 < \kappa_1 & \leq \kappa(x, t, \phi) \leq \kappa_2, \\
0 < v_1 & \leq v(x, t, \phi) \leq v_2, \\
0 < \eta_1 & \leq \eta(x, t, \phi) \leq \eta_2.
\end{align*}
\]

where \(\kappa_i, v_i\) and \(\eta_i\) \((i = 1, 2)\) are positive constants. Retaining the other assumptions in Theorem 1. In addition, we suppose \((\phi, \mu, u, B)\) satisfy

\[
\phi \in L^2(0, T, W^{1,\infty}(\Omega)), \quad u \in L^2(0, T, W^{1,\infty}(\Omega)^d), \quad B \in L^2(0, T, W^{1,\infty}(\Omega)^d). \quad (18)
\]

Then problem (11) has a unique solution \((\phi, \mu, u, B)\).

3 Fully Discrete Scheme

In this section, we give a fully discrete scheme based on applying conforming finite element method in space and Euler semi-implicit discretization with a convex splitting method in time for (10) and obtain some unconditionally energy stable.

Let \(K_h\) be a conforming, quasi-uniform family of triangulations of \(\Omega\) into triangles when \(d=2\) and tetrahedra when \(d = 3\), respectively. Furthermore, we introduce four finite element spaces \(\mathcal{X}_h, \mathcal{M}_h, \mathcal{W}_h, \mathcal{Q}_h\) with \(\mathcal{X}_h \subset \mathcal{X}, \mathcal{M}_h \subset \mathcal{M}, \mathcal{W}_h \subset \mathcal{W}, \mathcal{Q}_h \subset \mathcal{Q}\) as follows.

\[
\begin{align*}
\mathcal{X}_h &= \left\{ v \in C^0(\overline{\Omega})^d \cap \mathcal{X} : v|_K \in P_{r+1}(K)^d, \forall K \in K_h \right\}, \\
\mathcal{M}_h &= \left\{ q \in C^0(\overline{\Omega}) \cap \mathcal{M} : q|_K \in P_r(K), \forall K \in K_h \right\}, \\
\mathcal{W}_h &= \left\{ w \in C^0(\overline{\Omega})^d \cap \mathcal{W} : w|_K \in P_{r+1}(K)^d, \forall K \in K_h \right\}, \\
\mathcal{Q}_h &= \left\{ \omega \in C^0(\overline{\Omega}) \cap \mathcal{Q} : \omega|_K \in P_{r+1}(K), \forall K \in K_h \right\}, \\
\mathcal{X}_{0h} &= \left\{ v \in \mathcal{X}_h : d(v, w) = 0, \forall w \in \mathcal{M}_h \right\}.
\end{align*}
\]

As is noted that the \((\mathcal{X}_h, \mathcal{M}_h)\) is Taylor-Hood finite element pair. Therefore, the finite element pair \((\mathcal{X}_h, \mathcal{M}_h)\) satisfies the discrete LBB condition [12, 28]:

Assumption A1 The following discrete LBB condition holds:

\[
\exists \beta_0 > 0, \quad \sup_{v_h \in \mathcal{X}_h, v_h \neq 0} \frac{d(v_h, q_h)}{\|\nabla v_h\|} \geq \beta_0 \|q_h\|, \quad \forall q_h \in \mathcal{M}_h.
\]

Moreover, we suppose the finite element spaces satisfy following inverse inequality and finite element approximation properties:

Assumption A2 The following inverse inequality holds

\[
\|v_h\|_{m,q} \leq C h^{l-m+d(\frac{1}{q} - \frac{1}{p})} \|v_h\|_{l,p}, \quad \forall v_h \in \mathcal{X}_h, \mathcal{W}_h \text{ or } \mathcal{Q}_h,
\]

\[
0 \leq l \leq m \leq 1, \quad 1 \leq p \leq q \leq \infty.
\]
Assumption A3 There exists $k \geq 1$, such that for all $1 \leq l \leq k$,
\[
\inf_{v_{h}} \left[ \| v - v_{h} \| + h \| \nabla (v - v_{h}) \| \right] \leq C h^{l+1} \| v \|_{l+1}, \quad \forall v \in H^{l+1}(\Omega)^{d},
\]
\[
\inf_{B_{h}} \left[ \| B - B_{h} \| + h \| \nabla (B - B_{h}) \| \right] \leq C h^{l+1} \| B \|_{l+1}, \quad \forall B \in H^{l+1}(\Omega)^{d},
\]
\[
\inf_{\psi_{h}} \left[ \| \varphi - \varphi_{h} \| + h \| \nabla (\varphi - \varphi_{h}) \| \right] \leq C h^{l+1} \| \varphi \|_{l+1}, \quad \forall \varphi \in H^{l+1}(\Omega),
\]
\[
\inf_{q_{h} \in X_{h}} \| q - q_{h} \| \leq C h^{l} \| q \|_{l}, \quad \forall q \in H^{l}(\Omega).
\]

Let $N$ be a positive integer and $0 = t_{0} < t_{1} < \cdots < t_{N} = T$ be a uniform partition of $[0, T]$, with $\Delta t = t_{i} - t_{i-1}$ for $i = 1, \ldots, N$. Let us denote by $\phi^{n}$ the valve $\phi(t_{n})$ at the time $t_{n}$, and denote $\delta_{t}\phi := \frac{\phi^{n} - \phi^{n-1}}{\Delta t}$. Denote $\kappa^{n} := \kappa(x, t_{n}, \phi)$ $\nu^{n} := \nu(x, t_{n}, \phi)$ and $\eta^{n} := \eta(x, t_{n}, \phi)$.

With above preparation, we give the fully discrete scheme of problem (10).

Scheme 3.1 Given $(\phi_{h}^{0}, \mu_{h}^{0}, B_{h}^{0}) \in Q_{h} \times X_{h} \times W_{h}$, find $(\phi_{h}^{n}, \mu_{h}^{n}, u_{h}^{n}, p_{h}^{n}, B_{h}^{n}) \in Q_{h} \times Q_{h} \times X_{h} \times M_{h} \times W_{h}$, such that
\[
(\delta_{t}\phi_{h}^{n}, \psi_{h}) + \epsilon a_{\phi}(\phi_{h}^{n-1}, \mu_{h}^{n}, \psi_{h}) + (\nabla \phi_{h}^{n-1} \cdot u_{h}^{n}, \psi_{h}) = 0, \quad \text{(19a)}
\]
\[
e^{-1} \left( (\phi_{h}^{n})^{3} - \phi_{h}^{n-1} , \theta_{h} \right) + \epsilon (\nabla \phi_{h}^{n}, \nabla \theta_{h}) = (\mu_{h}^{n}, \theta_{h}). \quad \text{(19b)}
\]
\[
(\delta_{t} u_{h}^{n}, v_{h}) + a_{f}(v_{h}, \phi_{h}^{n-1}) + b_{f}(u_{h}^{n-1}, u_{h}^{n}, v_{h}) + S_{L}b_{\bar{B}}(B_{h}^{n-1}, B_{h}^{n}, v_{h}) - d(v_{h}, p_{h}^{n}) = \lambda(\mu_{h}^{n} \nabla \phi_{h}^{n-1}, v_{h}), \quad \text{(19c)}
\]
\[
d(u_{h}^{n}, q_{h}) = 0, \quad \text{(19d)}
\]
\[
(\delta_{t} B_{h}^{n}, H_{h}) + a_{B}(\eta_{h}(\phi_{h}^{n-1}), B_{h}^{n}, H_{h}) - c_{\bar{B}}(u_{h}^{n}, B_{h}^{n-1}, H_{h}) = 0, \quad \text{(19e)}
\]
for all $(\psi_{h}, \theta_{h}, v_{h}, q_{h}, H_{h}) \in Q_{h} \times Q_{h} \times X_{h} \times M_{h} \times W_{h}$.

Denote $\hat{S}_{h} = Q_{h} \cap M_{h}$. Let $\hat{\kappa} \in C^{0,1}(\overline{\Omega} ; R^{+})$ satisfy (15)–(17), we define the operator $T_{h} : \hat{S}_{h} \to \hat{S}_{h}$ by the following weak problem: given $\sigma \in \hat{S}_{h}$, find $T_{h}(\sigma) \in \hat{S}_{h}$ such that
\[
a_{\phi}(\hat{\kappa} ; T_{h}(\sigma), \varnothing) = (\sigma, \varnothing), \quad \forall \varnothing \in \hat{S}_{h}.
\]
(20)

As in the work [5], we have similar results.

Lemma 1 Let $\xi, \vartheta \in \hat{S}_{h}$ and denote
\[
(\xi, \vartheta)_{-1,h} := a_{\phi}(\hat{\kappa} ; T_{h}(\xi), T_{h}(\vartheta)) = (\xi, T_{h}(\vartheta)) = (T_{h}(\xi), \vartheta),
\]
and define the following negative norm:
\[
\| \xi \|_{-1,h} := \sqrt{(\xi, \xi)_{-1,h}} = \sup_{0 \neq \vartheta \in \hat{S}_{h}} \frac{(\xi, \vartheta)}{\| \nabla \vartheta \|}.
\]
Therefore, $\forall \vartheta \in X_{h}$ and $\forall \xi \in \hat{S}_{h}$, we have
\[
| (\xi, \vartheta) | \leq \| \xi \|_{-1,h} \| \nabla \vartheta \|.
\]
Then the following Poincaré inequality holds:
\[
\| \vartheta \|_{-1,h} \leq C \| \vartheta \|, \quad \forall \vartheta \in \hat{S}_{h}.
\]
We now are ready to state and prove unconditionally energy stable for the fully discrete scheme (19).

**Theorem 3** Suppose Assumptions A1–A3 and (15)–(17) hold. Then the finite element approximate solution \((\phi_h^n, \mu_h^n, u_h^n, p_h^n, B_h^n)\) of problem (19) satisfy the following discrete energy law:

\[
E(\phi_h^n, u_h^n, B_h^n) + \Delta t \sum_{m=1}^{n} \left[ \kappa_1 \varepsilon \|\nabla \mu_h^n\|^2 + \Delta t c_0 v_1 \|\nabla u_h^n\|^2 + S_c \Delta t c_1 \eta_1 \|\nabla B_h^n\|^2 \right] + \Delta t^2 \sum_{m=1}^{n} \left[ \frac{\epsilon}{2} \|\delta_t \phi_h^n\|^2 + \frac{1}{2\lambda} \|\delta_t \phi_h^n\|^2 + \frac{1}{2} \|\delta_t \phi_h^n\|^2 \right] \leq E(\phi_h^0, u_h^0, B_h^0).
\]

**Proof** Taking \(\psi_h = \mu_h^n\) in (19a), \(\theta_h = \delta_t \phi_h^n\) in (19b), \(v_h = u_h^n\) in (19c), \(q_h = p_h^n\) in (19d), \(H_h = B_h^n\) in (19e), one finds that

\[
(\delta_t \phi_h^n, \mu_h^n) + \epsilon a(\phi_h^{n-1}; \mu_h^n, \mu_h^n) + (\nabla \phi_h^{n-1}, \nabla \mu_h^n) = 0,
\]

\[
e^{-1} \left( (\phi_h^n)^3 - \phi_h^{n-1}, \delta_t \phi_h^n \right) + \epsilon (\nabla \phi_h^n, \nabla \delta_t \phi_h^n) = (\mu_h^n, \delta_t \phi_h^n),
\]

\[
(\delta_t \phi_h^n, u_h^n) = f_h^n(\phi_h^{n-1}; u_h^n, u_h^n) + d(u_h^n, p_h^n) = \lambda (\mu_h^n \nabla \phi_h^{n-1}, u_h^n),
\]

\[
d(u_h^n, p_h^n) = 0,
\]

\[
(\delta_t B_h^n, B_h^n) + a_B(\eta_h^n(\phi_h^{n-1}); B_h^n, B_h^n) - c_B(u_h^n, B_h^n - B_h^n = 0).
\]

Using the elementary identity

\[
2(a - b) = a^2 - b^2 + (a - b)^2,
\]

for all \(a, b \in \mathbb{R}^d\).

It follows that

\[
\epsilon (\nabla \phi_h^n, \nabla \delta_t \phi_h^n) = \frac{\epsilon}{2} \left[ \delta_t \|\nabla \phi_h^n\|^2 + \Delta t \|\nabla \delta_t \phi_h^n\|^2 \right],
\]

\[
e^{-1} \left( (\phi_h^n)^3 - \phi_h^{n-1}, \delta_t \phi_h^n \right) = \frac{e^{-1}}{4} \delta_t \|\phi_h^n\|^2 - 1^2
\]

\[
+ \frac{e^{-1} \Delta t}{4} \left[ \|\delta_t \phi_h^n\|^2 + 2 \|\phi_h^n \delta_t \phi_h^n\|^2 + 2 \|\delta_t \phi_h^n\|^2 \right],
\]

\[
(\delta_t \phi_h^n, u_h^n) = \frac{1}{2} \left[ \delta_t \|u_h^n\|^2 + \Delta t \|\delta_t \phi_h^n\|^2 \right],
\]

\[
(\delta_t B_h^n, B_h^n) = \frac{1}{2} \left[ \delta_t \|B_h^n\|^2 + \Delta t \|\delta_t B_h^n\|^2 \right].
\]

Then using the operator \(\Delta t \sum_{n=1}^{m} \) to the combined equations, the desired result is derived. The proof is completed. \(\square\)

The following Theorem give some bounds for the fully discrete solution. It is very important to derive optimal error estimates of problem (19).
Theorem 4 [5, 18, 33] Assumptions A1–A3 and (15)–(17) hold. Then the fully discrete solution \((\phi_h, \mu_h, u_h, p_h, B_h)\) of (19) satisfy the following bounds:

\[
\max_{0 \leq m \leq n} \left[ \| u_h^m \|^2 + \| B_h^m \|^2 + \| \nabla \phi_h^m \|^2 + \| (\phi_h^m)^2 - 1 \|^2 + \| \phi_h^m \|^4 
+ \| \phi_h^m \|^2 + \| \phi_h^m \|_{H^1} \right] \leq C,
\]

\[
\Delta t \sum_{m=1}^{n} \left[ \| \nabla u_h^m \|^2 + \| \nabla B_h^m \|^2 + \| \nabla \mu_h^m \|^2 \right] \leq C,
\]

\[
\sum_{m=1}^{n} \left[ \| \nabla (\phi_h^m - \phi_h^{m-1}) \|^2 + \| \phi_h^m - \phi_h^{m-1} \|^2 + \| \phi_h^m (\phi_h^m - \phi_h^{m-1}) \|^2 
+ \| (\phi_h^m)^2 - (\phi_h^{m-1})^2 \|^2 + \| \phi_h^m - \phi_h^{m-1} \|_{1,h}^2 
+ \| u_h^m - u_h^{m-1} \|^2 + \| B_h^m - B_h^{m-1} \|^2 \right] \leq C,
\]

\[
\Delta t \sum_{m=1}^{n} \left[ \| \delta_r \phi_h^m \|_{H^{-1}}^2 + \| \delta_r \phi_h^m \|_{-1,h}^2 + \| \nabla \phi_h^m \|^2 + \| \mu_h^m \|^2 
+ \| \mu_h^m \|_{L^\infty}^{4(d-1)} \right] \leq C,
\]

\[
\max_{0 \leq m \leq n} \left[ \| \mu_h^m \|^2 + \| \nabla \phi_h^m \|^2 + \| \mu_h^m \|_{L^\infty}^{4(d-1)} \right] \leq C,
\]

\[
\Delta t^2 \sum_{m=1}^{n} \left[ \| \delta_r u_h^m \|^2 + \| \delta_r B_h^m \|^2 \right] \leq C.
\]

4 Error Analysis

In this section we derive some optimal error estimates of the fully discrete scheme (19). Therefore, we assume that the solution \((\phi, \mu, u, p, B)\) has the following regularity.

Assumption A4 The weak solution \((\phi, \mu, u, p, B)\) of Cahn–Hilliard–Magneto-hydrodynamics (2)–(4) is sufficiently smooth such that

\[
\partial_t \phi \in L^\infty(0, T; H^1(\Omega)), \quad \phi, \partial_r \phi \in L^\infty(0, T; H^{k+2}(\Omega) \cap H^{1,\infty}(\Omega)),
\]

\[
\partial_t u \in L^\infty(0, T; H^1(\Omega)^d), \quad u, \partial_r u \in L^\infty(0, T; H^{k+2}(\Omega)^d \cap H^{1,\infty}(\Omega)^d),
\]

\[
\partial_t B \in L^\infty(0, T; H^1(\Omega)^d), \quad B, \partial_r B \in L^\infty(0, T; H^{k+2}(\Omega)^d \cap H^{1,\infty}(\Omega)^d),
\]

\[
\mu \in L^\infty(0, T; H^{k+2}(\Omega)), \quad p, \partial_r p \in L^\infty(0, T; H^{k+1}(\Omega)).
\]

We need the following technical lemmas to obtain the rate of convergence of the fully discrete scheme (19).

Lemma 2 [5] Assume \(\sigma \in H^1(\Omega)\), and \(v \in \mathcal{S}_h\). Then there exists \(C > 0\), independent of \(h\), such that

\[
|\langle \sigma, v \rangle| \leq C \| \nabla \sigma \|_1 \| v \|_{-1,h}.
\]
Lemma 3 [5] Assume that \((\phi, \mu, u, p, B), (\phi^n, \mu^n, u^n, p^n, B^n)\) are weak solution to problem (10) and (19), respectively. Then for any \(\Delta t > 0\), the following inequality holds
\[
\|\nabla (\phi^3 - (\phi^n)^3)\| \leq C \|\nabla (\phi - \phi^n)\|.
\] (28)

Let \(\hat{\nu}, \hat{\eta}, \hat{\kappa} \in C^{0,1}(\Omega; R^+)\) satisfy (15)–(17), we introduce the following four projections. Let \((u, p) \in \mathcal{X} \times \mathcal{M}, B \in \mathcal{W}, \theta \in Q\), we introduce Stokes projection \((Rh u, Jh p) \in \mathcal{X}_h \times \mathcal{M}_h\) as the solution of the weak problem as follows
\[
a_f (\hat{\nu}; u - Rh u, v) + d(v, p - Jh p) = 0, \quad \forall v \in \mathcal{X}_h,
\] (29)
\[
d(u - Rh u, q) = 0, \quad \forall q \in \mathcal{M}_h.
\] (30)

We have the following approximation result
\[
\|u - Rh u\| + h \left(\|\nabla (u - Rh u)\| + \|p - Jh p\|\right) \leq Ch^{k+2} \left(\|u\|_{k+2} + \|p\|_{k+1}\right).
\] (31)

Similarly, Maxwell projection \(Rmh B \in \mathcal{W}_h\) satisfying
\[
a_B (\hat{\eta}; B - Rmh B, H) = 0, \quad \forall H \in \mathcal{W}_h,
\] (32)
\[
\|B - Rmh B\| + h \|\nabla (B - Rmh B)\| \leq Ch^{k+2} \|B\|_{k+2}.
\] (33)

Ritz projections \(rh \theta \in Q_h\) and \(\tilde{\tau}_h \theta \in Q_h\) satisfying
\[
a_\phi (\hat{\kappa}; \theta - rh \theta, \psi) = 0, \quad \forall \psi \in Q_h,
\] (34)
\[
\|\theta - rh \theta\| + h \|\nabla (\theta - rh \theta)\| \leq Ch^{k+2} \|\theta\|_{k+2},
\] (35)
\[
(\nabla (\phi - \tilde{\tau}_h \phi), \nabla \psi) = 0, \quad \forall \psi \in Q_h,
\] (36)
\[
\|\phi - \tilde{\tau}_h \phi\| + h \|\nabla (\phi - \tilde{\tau}_h \phi)\| \leq Ch^{k+2} \|\phi\|_{k+2}.
\] (37)

For convenience, let us define, for every \(n \geq 0\),
\[
e^\phi_n := \phi^n - \phi_h^n, \quad e^\mu_n := \mu^n - \mu_h^n,
\]
\[
e^u_n := u^n - u_h^n, \quad e^p_n := p^n - p_h^n,
\]
\[
e^B_n := B^n - B_h^n,
\]
which gives
\[
e^\phi_n = \eta^\phi_n - e^\phi_n, \quad e^\mu_n = \eta^\mu_n - e^\mu_n,
\]
\[
e^u_n = \eta^u_n - e^u_n, \quad e^p_n = \eta^p_n - e^p_n,
\]
\[
e^B_n = \eta^B_n - e^B_n.
\]

with
\[
\eta^\phi_n := \phi^n - rh \phi^n \in Q, \quad e^\phi_n := \phi^n - rh \phi^n \in Q_h,
\]
\[
\eta^\mu_n := \mu^n - \tilde{\tau}_h \mu^n \in Q, \quad e^\mu_n := \mu^n - \tilde{\tau}_h \mu^n \in Q_h,
\]
\[
\eta^u_n := u^n - Rh u^n \in \mathcal{X}, \quad e^u_n := u^n - Rh u^n \in \mathcal{X}_h,
\]
\[
\eta^p_n := p^n - Jh p^n \in \mathcal{M}, \quad e^p_n := p^n - Jh p^n \in \mathcal{M}_h,
\]
\[
\eta^B_n := B^n - Rmh B^n \in \mathcal{W}, \quad e^B_n := B^n - Rmh B^n \in \mathcal{W}_h.
\]

We are now in a position to give and derive the first main theorem of this section for the concentration field, the chemical potential, the velocity field and the magnetic field.
\textbf{Theorem 5} Suppose that Assumptions A1–A4 and (15)–(17) hold with two positive constants $h_0$, $\Delta t_0$, and suppose the scheme (19) is initialized such that

\[ \| \phi^0 - \phi_h^0 \| + \| u^0 - u_h^0 \| + \| B^0 - B_h^0 \| \leq C \left( \Delta t + h^{k+1} \right). \]  

(38)

For $h \in (0, h_0]$ and $\Delta t \in (0, \Delta t_0]$, the finite element approximate solution $(\phi_h^n, \mu_h^n, u_h^n, p_h^n, B_h^n)$ of (19) satisfy the following error equations:

\begin{align*}
\max_{1 \leq n \leq N} \| \nabla (\phi(t_n) - \phi_h^n) \|^2 + \Delta t \sum_{n=1}^{N} \| \nabla (\mu(t_n) - \mu_h^n) \|^2 & \leq C \left( \Delta t^2 + h^{2k+2} \right), \\
\max_{1 \leq n \leq N} \| u(t_n) - u_h^n \|^2 + \Delta t \sum_{n=1}^{N} \| \nabla (u(t_n) - u_h^n) \|^2 & \leq C \left( \Delta t^2 + h^{2k+2} \right), \\
\max_{1 \leq n \leq N} \| B(t_n) - B_h^n \|^2 + \Delta t \sum_{n=1}^{N} \| \nabla (B(t_n) - B_h^n) \|^2 & \leq C \left( \Delta t^2 + h^{2k+2} \right)
\end{align*}

Proof Applying (10), (19), (29)–(30), (32), (34) and (36), we can obtain the following error estimates:

\begin{align*}
(\delta_t \epsilon^n, \psi_h) + \epsilon a_v(\kappa^n(\phi_h^n - 1); \epsilon^n, \psi_h) = (A^n_h, \psi_h) + (R^n_\phi, \psi_h), \\
(\delta_t \epsilon^n_v, \psi_h) + \epsilon a_v(\kappa^n(\phi_h^n - 1); \epsilon^n_v, \psi_h) - d(\psi_h, \epsilon^n_v) = (\Phi^n_h, \psi_h) + (R^n_v, \psi_h), \\
(\delta_t \epsilon^n, \psi_h) + S_c (\hat{\epsilon} \psi_h) + \epsilon a_v(\kappa^n(\phi_h^n - 1); \epsilon^n, \psi_h) = (\hat{\Phi}^n_h, \psi_h) + S_c (R^n_b, \psi_h).
\end{align*}

(39a) \hspace{1cm} (39b) \hspace{1cm} (39c) \hspace{1cm} (39d)

where $(R^n_\phi, \psi_h), (R^n_v, \psi_h), (R^n_B, \psi_h), (A^n_h, \psi_h), (\hat{A}^n_h, \theta_h), (\Phi^n_h, \psi_h)$ and $(\hat{\Phi}^n_h, \psi_h)$ are denoted by

\begin{align*}
(R^n_\phi, \psi_h) & := \left( \partial_t \phi^n - \delta_t r h \phi^n, \psi_h \right), \\
(R^n_v, \psi_h) & := \left( \partial_t u^n - \delta_t R h u^n, \psi_h \right), \\
(R^n_B, \psi_h) & := \left( \partial_t B^n - \delta_t R h B^n, \psi_h \right), \\
(A^n_h, \psi_h) & := a_v(\kappa^n(\phi^n); \mu^n, \psi_h) - a_v(\kappa^n(\phi_h^n - 1); \mu^n, \psi_h) \\
& \quad + (u^n \cdot \nabla \phi^n, \psi_h) - (u^n_h \cdot \nabla \phi_h^n - 1, \psi_h), \\
(\hat{A}^n_h, \theta_h) & := \left( \eta^n_h, \theta_h \right) - \frac{\Delta t}{\epsilon} (\delta_t \phi^n, \theta_h) \\
& \quad - \epsilon^{-1} (\phi^n - 1, \phi_h^n - 1, \theta_h) + \epsilon^{-1} \left( (\phi^n)^3 - (\phi_h^n)^3 \right), \\
(\Phi^n_h, \psi_h) & := a_f(\kappa^n(\phi^n); u^n, \psi_h) - a_f(\kappa^n(\phi_h^n - 1); u^n, \psi_h) \\
& \quad + b(u^n, u^n_h, \psi_h) - b(u^n_h - 1, u^n, \psi_h) \\
& \quad + S_c \hat{\epsilon} (B^n, B^n_h, \psi_h) - S_c \hat{\epsilon} (B^n_h - 1, B^n_h, \psi_h) \\
& \quad + \lambda (\mu_h^n \nabla \phi_h^n - 1, \psi_h) - \lambda (\mu^n \nabla \phi^n, \psi_h).
\end{align*}

(40) \hspace{1cm} (41) \hspace{1cm} (42) \hspace{1cm} (43) \hspace{1cm} (44) \hspace{1cm} (45)
\[
(\tilde{\Phi}_h^n, H_h) := S_c a_B(\eta^n(\phi^n); B^n, H_h) - S_c a_B(\eta^n(\phi_h^{n-1}); B^n, H_h) + S_c c_B(u_h^n, H_h^{-1}, B^n, H_h) - S_c c_B(u^n, B^n, H_h).
\]  
(46)

Setting \(\psi_h = \epsilon^n \in Q_h, \theta_h = \kappa_1 \epsilon^n_h \in Q_h\) in (39a)–(39b), respectively, we have

\[
\begin{align}
(\delta_t \epsilon^n, \epsilon^n) + \epsilon \kappa_1 (\nabla \epsilon^n, \nabla \epsilon^n) &\leq (\Lambda^n_h, \epsilon^n) + (R^n_h, \epsilon^n), \\
-\kappa_1 \epsilon (\nabla \epsilon^n, \nabla \epsilon^n) + \kappa_1 (\epsilon^n, \epsilon^n) &\leq -\kappa_1 (\Lambda^n_h, \epsilon^n).
\end{align}
\]  
(47a)

(47b)

Setting \(\psi_h = \lambda \epsilon^n_h \in Q_h, \theta_h = \lambda \delta_h \epsilon^n_h \in Q_h, v_h = \epsilon^n_u \in \chi_h, H_h = \epsilon^n_B \in \mathcal{W}_h\) in (39), respectively, it follows that

\[
\begin{align}
\lambda (\delta_t \epsilon^n_h, \epsilon^n_h) + \lambda \epsilon \kappa_1 (\nabla \epsilon^n_h, \nabla \epsilon^n_h) &\leq \lambda (\Lambda^n_h, \epsilon^n_h) + \lambda (R^n_h, \epsilon^n_h), \\
\epsilon \lambda (\nabla \epsilon^n_h, \nabla \delta_h \epsilon^n_h) &\leq \lambda (\Lambda^n_h, \delta_h \epsilon^n_h), \\
(\delta_t \epsilon^n_u, \epsilon^n_u) + c_0 v_1 (\nabla \epsilon^n_u, \nabla \epsilon^n_u) &\leq (\Phi^n_h, \epsilon^n_u) + (R^n_u, \epsilon^n_u), \\
S_c (\delta_t \epsilon^n_B, \epsilon^n_B) &\leq (\Phi^n_h, \epsilon^n_B) + S_c (R^n_B, \epsilon^n_B).
\end{align}
\]  
(48a)

(48b)

(48c)

(48d)

Combining (47) and (48), we obtain

\[
\begin{align}
\frac{1}{2} \left( \| \epsilon^n_h^2 - \| \epsilon_{n-1}^2 + \| \epsilon^n_h - \epsilon_{n-1}^2 \right)
+ \frac{\epsilon \lambda}{2} \left( \| \nabla \epsilon^n_h^2 - \| \nabla \epsilon_{n-1}^2 + \| \nabla \epsilon^n_h - \nabla \epsilon_{n-1}^2 \right)
+ \frac{1}{2} \left( \| \epsilon^n_u^2 - \| \epsilon_{n-1}^2 + \| \epsilon^n_u - \epsilon_{n-1}^2 \right)
+ \frac{S_c}{2} \left( \| \epsilon^n_B^2 - \| \epsilon_{n-1}^2 + \| \epsilon^n_B - \epsilon_{n-1}^2 \right) + \Delta t \lambda \kappa_1 \| \nabla \epsilon^n_h^2
+ \Delta t \lambda \kappa_1 \| \nabla \epsilon^n_h^2 + \Delta t c_0 v_1 (\nabla \epsilon^n_u, \nabla \epsilon^n_u) + \Delta t c_1 \| \nabla \epsilon^n_B^2
\leq \Delta t \lambda (R^n_h, \epsilon^n_h) + \Delta t (R^n_h, \epsilon^n_h) + \Delta t (R^n_u, \epsilon^n_u)
+ \Delta t S_c (R^n_B, \epsilon^n_B) + \Delta t \lambda (\Lambda^n_h, \epsilon^n_h) + \Delta t (\Lambda^n_h, \epsilon^n_h)
+ \Delta t (\Lambda^n_h, \delta_h \epsilon^n_h) - \Delta t \kappa_1 (\Lambda^n_h, \epsilon^n_h)
+ \Delta t (\Phi^n_h, \epsilon^n_u) + \Delta t (\tilde{\Phi}_h^n, \epsilon^n_B)
= \sum_{i=1}^{10} \gamma_i.
\end{align}
\]  
(49)

We now bound the terms on the RHS of (49). Define time dependent spatial mass average as follows

\[
\tilde{\epsilon}_{\lambda}^n := \| \Omega \|^{-1} (\epsilon_{\lambda}^n, 1), \quad 1 \leq n \leq N.
\]

For terms \(\gamma_1 - \gamma_4\), using (6), Taylor’s theorem, Assumption 4 and Young’s inequality, one finds that

\[
\begin{align}
\gamma_1 &= \Delta t \lambda (R^n_h, \epsilon^n_h - \tilde{\epsilon}_h^n)
\leq \Delta t \lambda \| \partial_t \phi^n - \partial_t r_h \phi_r \| \| \nabla \epsilon_h^n \|
\leq C \Delta t \left( \Delta t \int_{t-\Delta t}^{t} \| \partial_s^2 \phi(s) \|^2 \| F \| ds \right)
\end{align}
\]
\[ + \frac{h^{k+1}}{\sqrt{\Delta t}} \int_{t-\Delta t}^{t} \| \partial_s \phi(s) \|^2_{H^{k+1}} \, ds \right] \frac{1}{2} \| \nabla \varepsilon^n \| \\
\leq C \Delta t \left( \Delta t^2 + h^{2k+2} \right) + \kappa_1 \lambda_{\Delta t} \frac{1}{8} \| \nabla \varepsilon^n \|^2, \quad (50) \]

\[ \gamma_2 \leq C \Delta t \left\{ \Delta t \int_{t-\Delta t}^{t} \| \partial_{ss}^2 \phi(s) \|^2_{L^2} \, ds \right. \\
+ \frac{h^{k+1}}{\sqrt{\Delta t}} \int_{t-\Delta t}^{t} \| \partial_s \phi(s) \|^2_{H^{k+1}} \, ds \right\} \frac{1}{2} \varepsilon^n \| \\
\leq C \Delta t \left( \Delta t^2 + h^{2k+2} \right) + C \Delta t \| \varepsilon^n \|^2. \quad (51) \]

Similarly we have

\[ \gamma_3 \leq C \Delta t \| \nabla \varepsilon^n \| \| \partial_t u^n - \delta_t R_h u^n \| \\
\leq C \Delta t \left\{ \Delta t \int_{t-\Delta t}^{t} \| \partial_{ss}^2 u(s) \|^2_{L^2} \, ds \right. \\
+ \frac{h^{k+1}}{\sqrt{\Delta t}} \int_{t-\Delta t}^{t} \| (\partial_s u(s), \partial_s p(s)) \|^2_{H^{k+1} \times H^{k+1}} \, ds \right\} \frac{1}{2} \| \nabla \varepsilon^n \|^2 \\
\leq C \Delta t \left( \Delta t^2 + h^{2k+2} \right) + \frac{\varepsilon_0 u_1 \Delta t}{8} \| \nabla \varepsilon^n \|^2, \quad (52) \]

and

\[ \gamma_4 \leq C \Delta t \| \nabla \varepsilon^n_B \| \| \partial_t B^n - \delta_t R_{mh} B^n \| \\
\leq C \Delta t \left\{ \Delta t \int_{t-\Delta t}^{t} \| \partial_{ss}^2 B(s) \|^2_{L^2} \, ds \right. \\
+ \frac{h^{k+1}}{\sqrt{\Delta t}} \int_{t-\Delta t}^{t} \| \partial_s B \|^2_{H^{k+1}} \, ds \right\} \frac{1}{2} \| \nabla \varepsilon^n_B \| \\
\leq C \Delta t \left( \Delta t^2 + h^{2k+2} \right) + \frac{\Delta t c_1 \eta_1 S_c}{4} \| \nabla \varepsilon^n_B \|^2. \quad (53) \]

For nonlinear term \( \gamma_5 \), adding and subtracting some terms, we can rewrite

\[ \gamma_5 = \Delta t \lambda \left\{ \varepsilon a_{\phi} (k^n (\phi^n) - \kappa^n (\phi^{n-1})); \mu^n, \varepsilon^n_{\mu} \right. \\
+ \varepsilon a_{\phi} (k^n (\phi^{n-1}) - \kappa^n (\phi^{n-1})); \mu^n, \varepsilon^n_{\mu} + (u^n \cdot \nabla (\phi^n - \phi^{n-1}), \varepsilon^n) \\
+ \left( (u^n - R_h u^n) \cdot \nabla \phi^{n-1}, \varepsilon^n_{\mu} \right) + \left( R_h u^n \cdot \nabla (\phi^{n-1} - R_h \phi^{n-1}), \varepsilon^n_{\mu} \right) \\
+ \left( R_h u^n \cdot \nabla \varepsilon^{n-1}_{\phi}, \varepsilon^n_{\mu} \right) + \left( e^n u \cdot \nabla \phi^{n-1}, \varepsilon^n_{\mu} \right) \right\} \\
= \sum_{i=1}^{6} A_i^n + \Delta t \lambda (\varepsilon^n u \cdot \nabla \phi^{n-1}, \varepsilon^n_{\mu}). \quad (54) \]
For terms $A_1^n - A_2^n$, thanks to Hölder inequality, Taylor’s theorem and (35), one finds that

$$A_1^n + A_2^n \leq C \Delta t |k|_{C^{0,1}(\Omega \times R; R)} \| \nabla \mu^n \|_{L^\infty} \left\{ \| \Delta t \delta_t \phi^n \| + \| \phi^{n-1} - \hat{r}_h \phi_h^{n-1} \| + \| \hat{r}_h \phi_h^{n-1} - \phi_h^{n-1} \| \right\} \| \nabla \varepsilon^n \|$$

$$\leq C \Delta t \left\{ (\Delta t \int_{t-\Delta t}^t \| \partial_s \phi(s) \|_{L^2}^2 ds) \right\}^{\frac{1}{2}} + h^{k+1} \| \partial_t \phi \|_{H^{k+1}}$$

$$+ \| \varepsilon_{\phi}^{n-1} \| \| \nabla \varepsilon^n \|,$$

where

$$|k|_{C^{0,1}(\Omega \times R; R)} := \sup \left\{ \left| k(x, \phi) - v(y, \varphi) \right| \left| (x, \phi) - (y, \varphi) \right| : (x, \phi), (y, \varphi) \in \Omega \times \mathbb{R} \right\}.$$ 

Using (6), (31) and (35), the terms $A_3^n - A_6^n$ can be bounded by

$$A_3^n \leq C \Delta t \| u^n \|_{L^4} \| \Delta t \nabla \delta_t \phi^n \| \| \varepsilon^n \|_{L^4}$$

$$\leq C \Delta t \left( \Delta t \int_{t-\Delta t}^t \| \partial_s \phi(s) \|_{L^2}^2 ds \right) \| \nabla \varepsilon^n \|_{L^2},$$

$$A_4^n \leq C \Delta t \| u^n - R_h u^n \|_{L^4} \| \nabla \phi^{n-1} \|_{L^4} \| \varepsilon^n \|_{L^4}$$

$$\leq C \Delta t h^{k+1} \| (u, p) \|_{H^k} \| \nabla \varepsilon^n \|,$$

$$A_5^n \leq C \Delta t \| R_h u^n \|_{L^4} \| \nabla (\phi^{n-1} - r_h \phi_h^{n-1}) \| \| \varepsilon^n \|_{L^4}$$

$$\leq C \Delta t h^{k+1} \| \phi \|_{H^{k+2}} \| \nabla \varepsilon^n \|,$$

$$A_6^n \leq C \Delta t \| R_h u^n \|_{L^4} \| \nabla \phi^{n-1} \|_{L^4} \| \varepsilon^n \|_{L^4}$$

$$\leq C \Delta t \| \nabla \phi^{n-1} \| \| \nabla \varepsilon^n \|.$$

Combining (54) with above estimates, it follows that

$$\mathcal{Y}_5 \leq C \Delta t \left( h^{2k+2} + \Delta t^2 \right) + \Delta t \| \varepsilon_{\phi}^{n-1} \|^2$$

$$+ \frac{\Delta t \lambda k_1}{8} \| \nabla \varepsilon^n \|^2 + \Delta t \lambda (\varepsilon_u \cdot \nabla \phi^{n-1}, \varepsilon^n).$$

(55)

Similarly, for nonlinear term $\mathcal{Y}_6$, we rewrite

$$\mathcal{Y}_6 = \Delta t \left\{ \varepsilon a_\phi (\kappa^n (\phi^n) - \kappa^n (\phi^{n-1})); \mu^n, \varepsilon^n \right\}$$

$$+ \varepsilon a_\phi (\kappa^n (\phi^{n-1}) - \kappa^n (\phi^{n-1})); \mu^n, \varepsilon^n) + (u^n \cdot \nabla (\phi^n - \phi^{n-1}), \varepsilon^n)$$

$$+ \left( (u^n - R_h u^n) \cdot \nabla \phi^{n-1}, \varepsilon^n \right) + \left( R_h u^n \cdot \nabla (\phi^{n-1} - r_h \phi_h^{n-1}), \varepsilon^n \right)$$

$$+ (R_h u^n \cdot \nabla \phi^{n-1}, \varepsilon^n) + (u^n \cdot \nabla \phi^{n-1}, \varepsilon^n) \right\}$$

$$= \sum_{i=7}^{13} A_i^n.$$

(56)
Applying (6), (31) and (35), we bound the terms $\Lambda_7^n - \Lambda_{13}^n$ as follows

$$\Lambda_7^n + \Lambda_8^n \leq C \Delta t \left( \left( \Delta t \int_{t-\Delta t}^t \| \partial_s \phi(s) \|_{L^2}^2 ds \right)^{\frac{1}{2}} + h^{k+1} + \| \epsilon^{n-1}_\phi \| \right) \| \nabla \epsilon^n_\phi \|,$$

$$\Lambda_9^n \leq C \Delta t \| u^n \|_{L^\infty} \| \nabla (\phi^n - \phi^{n-1}) \| \| \epsilon^n_\phi \|,$$

$$\Lambda_{10}^n \leq C \Delta t \left( \Delta t \int_{t-\Delta t}^t \| \partial_s \nabla \phi(s) \|_{L^2}^2 ds \right)^{\frac{1}{2}} \| \epsilon^n_\phi \|,$$

$$\Lambda_{11}^n \leq C \Delta t \left( \Delta t \int_{t-\Delta t}^t \| \partial_s \nabla \phi(s) \|_{L^2}^2 ds \right)^{\frac{1}{2}} \| \epsilon^n_\phi \|,$$

$$\Lambda_{12}^n \leq C \Delta t \| R_h u^n \|_{L^\infty} \| \nabla (\phi^{n-1} - r_h \phi^{n-1}) \| \| \epsilon^n_\phi \|,$$

$$\Lambda_{13}^n \leq C \Delta t \| u^n \|_{L^6} \| \nabla \phi_h^{n-1} \|_{L^3} \| \epsilon^n_\phi \| \leq C \Delta t \| \nabla \epsilon^n_\phi \| \| \epsilon^n_\phi \|.$$

Combining (56) with above estimates, one finds that

$$\gamma_6 \leq C \Delta t \left( h^{2k+2} + \Delta t^2 \right) + C \Delta t \| \epsilon^{n-1}_\phi \|_1^2 + C \Delta t \| \nabla \epsilon^{n-1}_\phi \|^2$$

$$+ C \Delta t \| \nabla \epsilon^n_\phi \|^2 + \frac{\Delta t c_{\nu_1}}{8} \| \nabla \epsilon^n_\phi \|^2 + C \Delta t \| \epsilon^n_\phi \|^2. \quad (57)$$

For nonlinear term $\gamma_7$, we can rewrite

$$\gamma_7 : = \Delta t \left\{ \lambda (\eta^n_{\mu}, \delta_t \epsilon^n_\phi) - \frac{\Delta t \lambda}{\epsilon} (\delta_t \phi^n, \delta_t \epsilon^n_\phi) - \lambda \epsilon^{-1} (\epsilon^{n-1}_\phi, \delta_t \epsilon^n_\phi) + \lambda \epsilon^{-1} \left( (\phi^n)^3 - (\phi_h^n)^3, \delta_t \epsilon^n_\phi \right) \right\}$$

$$= \Delta t \sum_{i=1}^{4} \tilde{\Lambda}_i^n. \quad (58)$$

Using (6), (37) and Lemmas 2–3, we bound the terms $\tilde{\Lambda}_1^n - \tilde{\Lambda}_4^n$ as follows

$$\tilde{\Lambda}_1^n \leq C \| \nabla \eta^n_{\mu} \| \| \delta_t \epsilon^n_\phi \|_{-1,h} \leq C h^{k+1} \| \mu \|_{k+2} \| \delta_t \epsilon^n_\phi \|_{-1,h},$$

$$\tilde{\Lambda}_2^n \leq \frac{1}{\epsilon} \| \Delta t \nabla \phi^n \| \| \delta_t \epsilon^n_\phi \|_{-1,h} \leq C \left( \Delta t \int_{t-\Delta t}^t \| \partial_s \nabla \phi(s) \|_{L^2}^2 ds \right)^{\frac{1}{2}} \| \delta_t \epsilon^n_\phi \|_{-1,h},$$

$$\tilde{\Lambda}_3^n \leq C \| \epsilon^{n-1}_\phi \| \| \delta_t \epsilon^n_\phi \|_{-1,h} \leq C \| \epsilon^{n-1}_\phi \| \| \delta_t \epsilon^n_\phi \|_{-1,h} + C \| \eta^{n-1}_\phi \| \| \delta_t \epsilon^n_\phi \|_{-1,h},$$

$$\tilde{\Lambda}_4^n \leq C \| \nabla (\phi^n)^3 - (\phi_h^n)^3 \| \| \delta_t \epsilon^n_\phi \|_{-1,h} \leq C \| \nabla (\phi^n - \phi_h^n) \| \| \delta_t \epsilon^n_\phi \|_{-1,h}$$

$$\leq C \| \nabla \epsilon^n_\phi \| \| \delta_t \epsilon^n_\phi \|_{-1,h} + C \| \nabla \eta^n_{\phi} \| \| \delta_t \epsilon^n_\phi \|_{-1,h}. \quad \blacksquare$$
Combining (58) with above estimates, it follows that
\[
\mathcal{Y}_7 \leq C \Delta t \left( h^{2k+2} + \Delta t^2 \right) + C \Delta t \| e^n_{\theta} - e^{n-1}_{\phi} \|^2 \\
+ C \Delta t \| \nabla e^n_{\phi} \|^2 + \alpha \Delta t \| \delta_t e^n_{\phi} \|^2_{-1,h}. \tag{59}
\]

Taking \( T_h(\delta_t e^n_{\phi}) \) in (39a), and using (6), Lemmas 1–2, we derive
\[
\| \delta_t e^n_{\phi} \|^2_{-1,h} = -\epsilon a_0 k(\phi^n) (\phi^n_{-1}); e^n_{\mu}, T_h(\delta_t e^n_{\phi}) + (R^n_{\phi}, T_h(\delta_t e^n_{\phi})) + (\Lambda^n_{\phi}, T_h(\delta_t e^n_{\phi})) \\
\leq \epsilon C \| \nabla e^n_{\mu} \| \| \delta_t e^n_{\phi} \|_{-1,h} + \| \delta_t \phi^n - \delta_t r_h \phi^n \| \| T_h(\delta_t e^n_{\phi}) \| \\
+ \| T_h(\delta_t e^n_{\phi}) \|_{L^6} + \| \eta^n_{\phi} \| \| \nabla \phi^n \| \| T_h(\delta_t e^n_{\phi}) \|_{L^6} \\
+ \| R_h \| \| \nabla \phi^n \| \| T_h(\delta_t e^n_{\phi}) \|_{L^6} + C \| \nabla e^n_{\mu} \|^2 + C \| \nabla e^n_{\phi} \|^2 \\
+ C \| \nabla e^n_{\phi} \|^2 + C \left( h^{2k+2} + \Delta t^2 \right). \tag{60}
\]

Then we obtain
\[
\| \delta_t e^n_{\phi} \|^2_{-1,h} \leq 2C_1 \| \nabla e^n_{\mu} \|^2 + 2C_2 \| \nabla e^n_{\phi} \|^2 + C \| \nabla e^n_{\phi} \|^2 + C \left( h^{2k+2} + \Delta t^2 \right). \tag{61}
\]

From (59) and (61), and taking \( \alpha = \min \left\{ \frac{\lambda_k}{8C_1}, \frac{c_0 v_1}{8C_2} \right\} \) in (59), one finds that
\[
\mathcal{Y}_7 \leq C \Delta t \left( h^{2k+2} + \Delta t^2 \right) + \frac{\Delta t c_0 v_1}{8} \| \nabla e^n_{\mu} \|^2 + \frac{\Delta t \lambda_k}{8} \| \nabla e^n_{\phi} \|^2 \\
+ C \Delta t \| \nabla e^n_{\phi} \|^2 + C \Delta t \| \nabla e^n_{\phi} \|^2. \tag{62}
\]

Similarly, for nonlinear term \( \mathcal{Y}_8 \), we rewrite
\[
\mathcal{Y}_8 := \Delta t \left\{ (\eta^n_{\phi}, e^n_{\mu}) + \frac{\Delta t}{\epsilon} (\delta_t \phi^n, e^n_{\mu}) \\
+ \epsilon^{-1} (e^n_{\phi} - e^n_{\phi}, e^n_{\mu}) + \epsilon^{-1} \left( (\phi^n)^3 - (\phi^n)^3, e^n_{\mu} \right) \right\} \\
= \sum_{i=5}^8 \hat{A}_i^n. \tag{63}
\]

Using (6), (35), (37) and Lemma 3, we bound the terms \( \hat{A}^n - \hat{A}^n \) as follows
\[
\hat{A}^n \leq C \Delta t \| \eta^n_{\phi} \| \| e^n_{\mu} \| \leq C h^{k+1} \| \mu_{k+1} \| e^n_{\mu}, \\
\hat{A}^n \leq C \Delta t \| \nabla \delta_t \phi^n \| \| e^n_{\mu} \| \leq C \Delta t \left( \int_{t-\Delta t}^{t} \| \delta_s \nabla \phi(s) \|^2_{L^2} ds \right) \| e^n_{\mu} \|, \\
\hat{A}^n \leq C \Delta t \| e^n_{\phi} - e^n_{\phi} \| \| e^n_{\mu} \| \leq C \Delta t \| e^n_{\phi} - e^n_{\phi} \| \| e^n_{\mu} \| + C \Delta t \| \nabla \eta^n_{\phi} \| \| e^n_{\mu} \| \\
\hat{A}^n \leq C \Delta t \| (\phi^n)^3 - (\phi^n)^3 \| \| e^n_{\mu} \| \leq C \Delta t \| (\phi^n - \phi^n)^3 \| \| e^n_{\mu} \| \\
\leq C \Delta t \| e^n_{\phi} \| \| e^n_{\mu} \| + C \Delta t \| \nabla \eta^n_{\phi} \| \| e^n_{\mu} \|. 
\]
Combining (63) with above inequalities, we have

$$
\gamma_8 \leq C \Delta t \left( h^{2k+2} + \Delta t^2 \right) + \frac{\Delta t \kappa_1}{8} \| \epsilon_{\mu}^n \|^2 \\
+ C \Delta t \| \epsilon_{\phi}^n \|^2 + C \Delta t \| \epsilon_{\phi}^{n-1} \|^2.
$$

(64)

For nonlinear term $\gamma_0$, by adding and subtracting some terms, we rewrite

$$
\gamma_0 = \Delta t \left\{ a f \left( v^n (\phi^n) - v^n (\phi^{n-1}), u^n, \epsilon_u^n \right) \\
+ a f \left( v^n (\phi^{n-1}) - v^n (\phi^n), u^n, \epsilon_u^n \right) + b(\eta u^n - R_h u^{n-1}, u^n, \epsilon_u^n) + b(\eta u^{n-1}, \eta u^n, \epsilon_u^n) \\
+ b(u_h^{n-1}, \eta u^n, \epsilon_u^n) + b(u_h^{n-1}, \eta u^n, \epsilon_u^n) + S_c c_B (R_{mh} B^n - R_{mh} B^{n-1}, B^n, \epsilon_u^n) + S_c c_B (B^{n-1}, B^n, \epsilon_u^n) \\
- \lambda (\mu^n \nabla \phi^n, \epsilon_u^n) - \lambda (\mu^n \nabla (r_h \phi^n - r_h \phi^{n-1}), \epsilon_u^n) \\
- \lambda (\mu^n \nabla \epsilon_{\phi}^{n-1}, \epsilon_u^n) - \lambda (\mu^n \nabla \phi^{n-1}, \epsilon_u^n) - \lambda (\epsilon_{\mu}^{n} \nabla \phi^{n-1}, \epsilon_u^n) \\
= \Delta t \left\{ \sum_{i=1}^{14} \phi_{i}^{n} + b(u_h^{n-1}, \epsilon_u^n, \epsilon_u^n) + S_c c_B (B^{n-1}, \epsilon_u^n, \epsilon_u^n) \\
- \lambda (\epsilon_{\mu}^{n} \nabla \phi^{n-1}, \epsilon_u^n) \right\}.
$$

(65)

For the terms $\Phi_1^n - \Phi_2^n$, using Hölder inequality, (37) and Taylor’s theorem, we have

$$
\Phi_1^n + \Phi_2^n \leq C |v|_{C^{0,1}(\Omega \times \mathcal{R}; \mathcal{R})} \| (u^n) \|_{L^\infty} \left\{ \| \phi^n - \phi^{n-1} \| \\
+ \| \phi^{n-1} - \tilde{r}_h \phi^{n-1} \| + \| \tilde{r}_h \phi^{n-1} - \phi^{n-1} \| \right\} \| \nabla \epsilon_u^n \| \\
\leq C \left\{ \left( \Delta t \int_{t-\Delta t}^{t} \| \partial_s \phi(s) \|_{L^2}^2 ds \right)^{\frac{1}{2}} h^{k+1} \\
+ \| \epsilon_{\phi}^{n-1} \| \right\} \| \nabla \epsilon_u^n \|.
$$

Making use of (6) and (31), we estimate the terms $\Phi_3^n - \Phi_6^n$ as follows

$$
\Phi_3^n \leq C \| \nabla \eta_u^n \| \| \nabla u^n \| \| \nabla \epsilon_u^n \| \\
\leq C h^{k+1} \| (u^n, p) \|_{H^{k+2} \times H^{k+1}} \| \nabla \epsilon_u^n \|,
$$

$$
\Phi_4^n \leq C \| \Delta t R_{mh} \delta_i B^n \| \| \nabla B^n \|_{L^3} \| \nabla \epsilon_u^n \| \\
\leq C \left( \Delta t \int_{t-\Delta t}^{t} \| \partial_s (B(s)) \|_{L^2}^2 ds \right)^{\frac{1}{2}} \| \nabla \epsilon_u^n \|,
$$

$$
\Phi_5^n \leq C \| \epsilon_{\mu}^{n-1} \| \| \nabla u^n \|_{L^3} \| \nabla \epsilon_u^n \|.
$$
Thanks to (6) and (33), the terms $\Phi^n_{11} - \Phi^n_{14}$ can be bounded by

$$\Phi^n_{11} \leq C \| \mu^n \|_{L^3} \| \Delta t \| \| \phi^n \| \| \phi^n \|_{L^5}$$

$$\leq C h^{k+1} \| \phi \|_{H^{k+2}} \| \nabla \phi^n \|,$$

$$\Phi^n_{12} \leq C \| \mu^n \|_{L^3} \| \Delta t \| \| \nabla \phi^n \| \| \phi^n \|_{L^5}$$

$$\leq C \left( \int_{t-\Delta t}^{t} \| \phi_s(t) \|_{L^2} ds \right)^{\frac{1}{2}} \| \nabla \phi^n \|,$$

$$\Phi^n_{13} \leq C \| \mu^n \|_{L^3} \| \nabla \phi^n \| \| \phi^n \| \| \phi^n \|_{L^5}$$

$$\leq C \| \nabla \phi^n \| \| \phi^n \|_{L^5},$$

$$\Phi^n_{14} \leq C \| \nabla \phi^n \| \| \phi^n \| \| \phi^n \|_{L^5}$$

$$\leq C h^{k+1} \| \mu \|_{H^{k+1}} \| \nabla \phi^n \|.$$

Applying Hölder inequality, (6), (35) and (37), the terms $\Phi^n_{11} - \Phi^n_{14}$ can be estimated as follows

$$\Phi^n_{11} \leq C \| \mu^n \|_{L^3} \| \nabla \phi^n \| \| \phi^n \|_{L^5}$$

$$\leq C h^{k+1} \| \phi \|_{H^{k+2}} \| \nabla \phi^n \|,$$

$$\Phi^n_{12} \leq C \| \mu^n \|_{L^3} \| \Delta t \| \| \nabla \phi^n \| \| \phi^n \|_{L^5}$$

$$\leq C \left( \int_{t-\Delta t}^{t} \| \phi_s(t) \|_{L^2} ds \right)^{\frac{1}{2}} \| \nabla \phi^n \|,$$

$$\Phi^n_{13} \leq C \| \mu^n \|_{L^3} \| \nabla \phi^n \| \| \phi^n \|_{L^5}$$

$$\leq C \| \nabla \phi^n \| \| \phi^n \|_{L^5},$$

$$\Phi^n_{14} \leq C \| \nabla \phi^n \| \| \phi^n \| \| \phi^n \|_{L^5}$$

$$\leq C h^{k+1} \| \mu \|_{H^{k+1}} \| \nabla \phi^n \|.$$

Combining (65) with above bounds, one finds that

$$\Upsilon_0 \leq C \Delta t \left( h^{k+1} + \Delta t \| \epsilon^{n-1} \| + \| \epsilon^{n-1} \| + \| \epsilon^{n-1} \| \right) \| \nabla \epsilon^n \|$$

$$+ \Delta t b(u_h^{n-1}, \epsilon^n \phi^n \phi^{n-1}) - \Delta t s_{c_B} \left( B_h^{n-1}, \epsilon^n \phi^n \phi^{n-1} \right) - \Delta t \lambda(\epsilon^{n-1} \nabla \phi^{n-1} \epsilon^n \phi^n \phi^{n-1})$$

$$\leq C \left( \Delta t \left( h^{2k+2} + \Delta t \| \epsilon^{n-1} \| + \Delta t \| \epsilon^{n-1} \| + \Delta t \| \epsilon^{n-1} \| + \Delta t \| \epsilon^{n-1} \| \right) \right) + \frac{\Delta t c_0 v_1}{2} \| \nabla \epsilon^n \|$$

$$+ \Delta t b(u_h^{n-1}, \epsilon^n \phi^n \phi^{n-1}) - \Delta t s_{c_B} \left( B_h^{n-1}, \epsilon^n \phi^n \phi^{n-1} \right) - \Delta t \lambda(\epsilon^{n-1} \nabla \phi^{n-1} \epsilon^n \phi^n \phi^{n-1}).$$

(66)

For nonlinear term $\Upsilon_{10}$, by adding and subtracting some terms, we can rewrite

$$\Upsilon_{10} = \Delta t \left( s_c a_B(\eta^n (\phi^n) - \eta^n (\phi^{n-1}), \phi^n \phi^n \phi^{n-1}) \right)$$

$$+ s_c a_B(\eta^n (\phi^{n-1}) - \eta^n (\phi^{n-1}), \phi^n \phi^n \phi^{n-1}) - s_c a_B(\eta^n \phi^n \phi^{n-1}, \phi^n \phi^n \phi^{n-1})$$

$$- s_c a_B(\eta^n, R_{mh} B^{n-1}, \phi^n \phi^n \phi^{n-1}) - s_c a_B(\eta^n, \phi^n \phi^n \phi^{n-1}, \phi^n \phi^n \phi^{n-1}).$$
- $S_c c_B(\eta^n, B_h^{n-1}, \epsilon_B^n) - S_c c_B(\epsilon^n, B_h^{n-1}, \epsilon_B^n) \bigg) \\
= \Delta t \left\{ \sum_{i=1}^{6} \hat{\Phi}_i^n - S_c c_B(\epsilon^n, B_h^{n-1}, \epsilon_B^n) \right\}.

(67)

For the terms $\hat{\Phi}_1^n - \hat{\Phi}_2^n$, thanks to Hölder inequality, (37) and Taylor’s theorem, we get

$\hat{\Phi}_1^n + \hat{\Phi}_2^n \leq C|\eta|_{C^0([0,T \times R; R])} \| \nabla B^n \|_{L^\infty} \left\{ \| \Delta t \phi^n \| + \| \phi^n - \hat{\phi} h \phi^{n-1} \| + \| \hat{\phi} h \phi^{n-1} - \phi_h^{n-1} \| \right\} \| \nabla \epsilon_B^n \|

\leq C \left\{ \left( \Delta t \int_{t-\Delta t}^{t} \| \partial_t \phi(s) \|_{L^2}^2 \, ds \right)^{\frac{1}{2}} + h^{k+1} \right\} \| \nabla \epsilon_B^n \|

\leq C \left\{ \left( \Delta t \int_{t-\Delta t}^{t} \| \partial_t \phi(s) \|_{L^2}^2 \, ds \right)^{\frac{1}{2}} + h^{k+1} \right\} \| \nabla \epsilon_B^n \|

By (6), (31) and (33), the terms $\hat{\Phi}_3^n - \hat{\Phi}_6^n$ can be bounded by

$\hat{\Phi}_3^n \leq C \| u^n \|_{L^\infty} \| \eta_B^n \| \| \text{curl} \, \epsilon_B^n \|

\leq C h^{k+1} \| B \|_{H^{k+1}} \| \nabla \epsilon_B^n \|

\hat{\Phi}_4^n \leq C \| u^n \|_{L^\infty} \| \Delta t R_m h \delta B^n \| \| \text{curl} \, \epsilon_B^n \|

\leq C \left( \Delta t \int_{t-\Delta t}^{t} \| \partial_t \phi(s) \|_{L^2}^2 \, ds \right)^{\frac{1}{2}} \| \nabla \epsilon_B^n \|

\hat{\Phi}_5^n \leq C \| u^n \|_{L^\infty} \| \epsilon_B^{n-1} \| \| \text{curl} \, \epsilon_B^n \|

\leq C \| \epsilon_B^{n-1} \| \| \nabla \epsilon_B^n \|

\hat{\Phi}_6^n \leq C \| \eta_B^n \|_{L^6} \| B_h^{n-1} \| \| \text{curl} \, \epsilon_B^n \|

\leq C h^{k+1} \| (u, p) \|_{H^{k+2} \times H^{k+1}} \| B_h^{n-1} \| \| \nabla \epsilon_B^n \|.

Combining (67) with above inequalities, it follows that

$\gamma_{10} \leq C \Delta t \left\{ h^{k+1} + \Delta t t + \| \epsilon_B^{n-1} \| + \| \phi^{n-1} \| \right\} \| \nabla \epsilon_B^n \|

- \Delta t S_c c_B(\epsilon^n, B_h^{n-1}, \epsilon_B^n) \leq C \Delta t \left( h^{k+2} + \Delta t t^2 \right) + \Delta t \| \epsilon_B^{n-1} \|^2 + \Delta t \| \phi^{n-1} \|^2

+ \frac{c_1 \eta_1 S_c \Delta t}{4} \| \nabla \epsilon_B^n \|^2 - \Delta t S_c c_B(\epsilon^n, B_h^{n-1}, \epsilon_B^n).

(68)

Combining (49) with (7)–(8), (50)–(53), (55), (57), (62), (64), (66) and (68), we have

$\frac{1}{2} \left( \| \phi^{n-1} \|^2 - \| \phi^{n-1} \|^2 + \| \phi^{n-1} - \phi^{n-1} \|^2 \right)

+ \frac{\epsilon \lambda}{2} \left( \| \nabla \phi^{n-1} \|^2 - \| \nabla \phi^{n-1} \|^2 + \| \nabla \phi^{n-1} - \nabla \phi^{n-1} \|^2 \right)$
\[
\sum_{n=1}^{m} \left( \frac{1}{2} \left( \|e_u^n\|^2 - \|e_u^{n-1}\|^2 + \|e_u^{n-1} - e_u^n\|^2 \right) + \frac{S_c}{2} \left( \|e_B^n\|^2 - \|e_B^{n-1}\|^2 + \|e_B^{n-1} - e_B^n\|^2 \right) + \Delta t \lambda K_1 \|\nabla e]\mu^n\|^2 \\
+ \Delta t K_1 \|e_\mu^n\|^2 + \Delta t c_0 v_1 \|\nabla e_u^n\|^2 + \Delta t c_1 \eta_1 S_c \sum_{n=1}^{m} \|\nabla e_B^n\|^2 \right) \\
\leq C \Delta t \left( h^{2k+2} + \Delta t^2 \right) + C \Delta t \left( \|e_\phi^n\|^2 + \|\nabla e_\phi^n\|^2 \right) \\
+ C \Delta t \left( \|e_\phi^{n-1}\|^2 + \|e_\phi^{n-1} - e_\phi^n\|^2 + \|e_\phi^{n-1} - e_\phi^n\|^2 \right) \right). 
\]

(69)

Summing (69) from \( n = 1 \) to \( m \) and using (38), we obtain

\[
\|e_\phi^m\|^2 + \epsilon \lambda \|\nabla e_\phi^m\|^2 + \|e_u^m\|^2 + \|e_B^m\|^2 + \Delta t \lambda K_1 \sum_{n=1}^{m} \|\nabla e_\mu^n\|^2 \\
+ \Delta t K_1 \sum_{n=1}^{m} \|e_\mu^n\|^2 + \Delta t c_0 v_1 \sum_{n=1}^{m} \|\nabla e_u^n\|^2 + \Delta t c_1 \eta_1 S_c \sum_{n=1}^{m} \|\nabla e_B^n\|^2 \\
\leq C \left( h^{2k+2} + \Delta t^2 \right) + C \Delta t \sum_{n=1}^{m} \|e_\phi^n\|^2 + C \Delta t \sum_{n=1}^{m} \|\nabla e_\phi^n\|^2 \\
+ C \Delta t \sum_{n=1}^{m} \|e_\phi^{n-1}\|^2 + C \Delta t \sum_{n=1}^{m} \|\nabla e_\phi^{n-1}\|^2. 
\]

(70)

If \( 0 < \Delta t \leq \Delta t_0 := \frac{1}{2 \max\{C_3, C_4 \epsilon^{-1}\}} \leq \frac{1}{\max\{C_3, C_4 \epsilon^{-1}\}} \), since \( 1 \leq \frac{1}{1 - \max\{C_3, C_4 \epsilon^{-1}\}} \leq 2 \), it follows from (70) that

\[
\|e_\phi^m\|^2 + \epsilon \lambda \|\nabla e_\phi^m\|^2 + \|e_u^m\|^2 + \|e_B^m\|^2 + \Delta t \lambda K_1 \sum_{n=1}^{m} \|\nabla e_\mu^n\|^2 \\
+ \Delta t K_1 \sum_{n=1}^{m} \|e_\mu^n\|^2 + \Delta t c_0 v_1 \sum_{n=1}^{m} \|\nabla e_u^n\|^2 + \Delta t c_1 \eta_1 S_c \sum_{n=1}^{m} \|\nabla e_B^n\|^2 \\
\leq C \left( h^{2k+2} + \Delta t^2 \right) + \frac{(C_3 + C_7) \Delta t}{1 - \max\{C_3, C_4 \epsilon^{-1}\}} \sum_{n=1}^{m} \|e_\phi^{n-1}\|^2 \\
+ \frac{(C_4 + C_8) \Delta t}{1 - \max\{C_3, C_4 \epsilon^{-1}\}} \sum_{n=1}^{m} \|\nabla e_\phi^{n-1}\|^2 + \frac{C \Delta t}{1 - \max\{C_3, C_4 \epsilon^{-1}\}} \sum_{n=1}^{m} \|e_u^{n-1}\|^2 \\
+ \frac{C \Delta t}{1 - \max\{C_3, C_4 \epsilon^{-1}\}} \sum_{n=1}^{m} \|e_B^{n-1}\|^2. 
\]

(71)

By using the discrete Grönwall inequality, one finds that

\[
\|e_\phi^m\|^2 + \epsilon \lambda \|\nabla e_\phi^m\|^2 + \|e_u^m\|^2 + \|e_B^m\|^2 + \Delta t \lambda K_1 \sum_{n=1}^{m} \|\nabla e_\mu^n\|^2 
\]
\[
+ \Delta t \kappa_1 \sum_{n=1}^{m} \| e^n_{\mu} \|^2 + \Delta t c_0 v_1 \sum_{n=1}^{m} \| \nabla e^n_{u} \|^2 + \Delta t c_1 \eta_1 \sum_{n=1}^{m} \| \nabla e^n_{B} \|^2 \\
\leq C \left( h^{2k+2} + \Delta t^2 \right).
\] (72)

The desired result follow from (31), (33), (35), (37) and the triangle inequality. The proof is completed. \[ \square \]

In order to derive optimal error estimates on pressure, we define the seminorm for \( v \in H^{-1}(\Omega)^d \):
\[
\| v \|_{+,h} = \sup_{v_h \in \mathcal{X}_h} \frac{(v, v_h)}{\| \nabla v_h \|}.
\]

And we define the following inverse Stokes operator: \( S : H^{-1}(\Omega)^d \to \mathcal{X} \) and discrete inverse Stokes operator: \( S_h : H^{-1}(\Omega)^d \to \mathcal{X}_h \). Let \( \hat{v} \in C^{0,1}(\overline{\Omega}; R^+) \) satisfy (15)–(17), for all \( v \in H^{-1}(\Omega)^d \), \( (S(v), r) \in \mathcal{X} \times \mathcal{M} \) is the weak solution of the following problem:
\[
a_f(\hat{v}; S(v), v) + d(v, r) = (v, v), \quad \forall v \in \mathcal{X},
\]
\[
d(S(v), w) = 0, \quad \forall w \in \mathcal{M}.
\]

We have \( H^2 \) regularity results [28]:
\[
\| S(v) \|_2 + \| \nabla r \| \leq C \| v \|, \forall v \in L^2(\Omega)^d.
\]

And for all \( v \in H^{-1}(\Omega)^d \), \( (S_h(v), r_h) \in \mathcal{X}_h \times \mathcal{M}_h \) is the weak solution of the following discrete problem:
\[
a_f(\hat{v}; S_h(v), v_h) + d(v, r_h) = (v, v_h), \quad \forall v_h \in \mathcal{X}_h,
\]
\[
d(S_h(v), w_h) = 0, \quad \forall w_h \in \mathcal{M}_h.
\]

Lemma 4 For \( v \in \mathcal{X}_{0h} \), there exists constant \( C > 0 \) independent of \( h \) such that
\[
\| v \|_{+,h} \leq C \| \nabla S_h(v) \|.
\] (73)

Proof The proof follows the similar lines as in [3]. Here we skip it. \[ \square \]

Theorem 6 Suppose that assumptions of Theorem 5 hold. We have the following estimates
\[
\| \nabla S_h(e^n_u - e^{n-1}_u) \| \leq C \sqrt{\Delta t} (\Delta t + h^{k+1}),
\]
\[
\left( \Delta t \sum_{n=1}^{N} \| \nabla S_h(e^n_u - e^{n-1}_u) \|^2 \right)^{1/2} \leq C \Delta t (\Delta t + h^{k+1}).
\]

Proof Using Assumption A1 and (39a), we obtain
\[
(\delta_t e^n_u, v_h) + a_f(v^n(\phi^n_{h^{-1}}); e^n_u, v_h) - d(v_h, e^n_p) = (\Phi^n, v_h) + (R^n, v_h), \quad (74)
\]
\[
d(e^n_u, q_h) = 0. \quad (75)
\]

Setting \( v_h = S_h(e^n_u - e^{n-1}_u) \in \mathcal{X}_h \) in (74), noting that \( d\left( S_h(e^n_u) - S_h(e^{n-1}_u), q_h \right) = 0 \), one finds that
\[
\frac{\| \nabla S_h(e^n_u - e^{n-1}_u) \|^2}{\Delta t} = -a_f\left( v^n(\theta^n_{h^{-1}}); e^n_u, S_h(e^n_u - e^{n-1}_u) \right).
\]
Applying the same arguments as those in the proof of Theorem 5, the RHS of (76) can be estimated as

\[-a_f(v^n(\theta^n_h); e^n_u, S_h(e^n_u - e^{n-1}_u)) \leq \frac{1}{6\Delta t} \| \nabla S_h(e^n_u - e^{n-1}_u) \|^2 + C \Delta t \| \nabla e^n_u \|^2,
\]

\[
\left( R^n_u, S_h(e^n_u - e^{n-1}_u) \right) \leq \frac{1}{6\Delta t} \| \nabla S_h(e^n_u - e^{n-1}_u) \|^2 + C \Delta t (\Delta t^2 + h^{2k+2}),
\]

\[
(\Phi^n_h, S_h(e^n_u - e^{n-1}_u)) \leq \frac{1}{6\Delta t} \| \nabla S_h(e^n_u - e^{n-1}_u) \|^2 + C \Delta t (\Delta t^2 + h^{2k+2})
\]

\[
+ C \Delta t \left( \| \nabla e^n_\mu \|^2 + \| \nabla e^n_u \|^2 \right)
\]

\[
+ C \Delta t \left( \| \nabla e^{n-1}_\mu \|^2 + \| \nabla e^{n-1}_u \|^2 \right)
\].

Combining (76) with above bounds, it follows that

\[
\frac{\| \nabla S_h(e^n_u - e^{n-1}_u) \|^2}{2\Delta t} \leq C \left\{ \Delta t (\Delta t^2 + h^{2k+2}) + C \Delta t \left( \| \nabla e^n_u \|^2 + \| \nabla e^n_\mu \|^2 \right)
\]

\[
+ \| \nabla e^{n-1}_\mu \|^2 + \| \nabla e^{n-1}_u \|^2 \right\}.
\]

Applying Theorems 5 and the triangle inequality, the desired result is obtained. The proof is finished. \(\square\)

Then we give and prove the second main result of this section for the pressure.

**Theorem 7** Suppose that assumptions of Theorem 5 and \(\Delta t \leq C h\) hold. Then the finite element approximate \(p^n_h\) in (19) satisfy following bound:

\[
\left( \Delta t \sum_{n=1}^{N} \| p(t_n) - p^n_h \|^2 \right)^{1/2} \leq C \left( \Delta t + h^{k+1} \right).
\]

**Proof** From Assumption A and (39a), one finds that

\[
\beta_0 \| e^n_p \| \leq \sup_{v_h \in X_h, v_h \neq 0} \frac{d(e^n_p, v_h)}{\| \nabla v_h \|}
\]

\[
\leq \sup_{v_h \in X_h, v_h \neq 0} \left\{ \frac{(\delta_t e^n_u, v_h) + a_f(v^n(\phi^n_h - 1); e^n_u, v_h)}{\| \nabla v_h \|} \right\}
\]

\[
\leq \sup_{v_h \in X_h, v_h \neq 0} \left\{ \frac{-(\Phi^n_h, v_h) - (\delta_t u^n - R_h u^n, v_h)}{\| \nabla v_h \|} \right\}
\]

\[
\leq \sup_{v_h \in X_h, v_h \neq 0} \frac{1}{\| \nabla v_h \|} \left\{ B(u^{n-1}_h, e^n_u, v_h) + S_c B(b^{n-1}_h, e^n_u, v_h)
\]

\[
- \lambda(e^n_\mu \nabla \phi^n_h - 1, v_h) \right\} + C \left\{ \| \delta_t e^n_u \|_{\mu, h} + \| \nabla e^n_u \| \right\}
\]

\[
+ C \left( \Delta t + h^{k+1} \right) + \left( \| e^{n-1}_u \| + \| e^{n-1}_\mu \| + \| e^{n-1}_\phi \| \right)
\].
\[
\leq \sup_{v_h \in V_h, v_h \neq 0} \frac{1}{\|\nabla v_h\|} \left\{ b(u_h^{n-1} - u^{n-1}, e^n_u, v_h) + b(u^{n-1}, e^n_u, v_h) + \sum c(c_B^n - B^{n-1}, e^n_B, v_h) + \sum c(c_B^n - B^{n-1}, e^n_B, v_h) - \lambda \left( \epsilon^{n} \nabla (\phi^{n-1} - \phi^{n-1}), v_h \right) - \lambda \left( \epsilon^{n} \nabla (\phi^{n-1} - \phi^{n-1}), v_h \right) \right\} + C \left\{ \|\delta^t e^n_u\|_{*h} + \|\nabla e^n_u\| + C \left( \Delta t + h^{k+1} \right) + \left( \|e^n_{B_u}\| + \|e_B^n\| + \|e^n_\phi\| \right) \right\}.
\] (77)

By Assumption A2 and Theorem 5, it follows that

\[
\|\nabla e^n_{u_h}\| \leq C \min \left\{ h^{-1} \|e^n_{u_h}\|, \|\nabla e^n_{u_h}\| \right\} 
\leq C \min \left\{ h^{-1} (\Delta t + h^{k+1}), \Delta t^{-1/2} (\Delta t + h^{k+1}) \right\} \leq C,
\]

\[
\|\nabla e^n_{B_h}\| \leq C \min \left\{ h^{-1} \|e^n_{B_h}\|, \|\nabla e^n_{B_h}\| \right\} 
\leq C \min \left\{ h^{-1} (\Delta t + h^{k+1}), \Delta t^{-1/2} (\Delta t + h^{k+1}) \right\} \leq C,
\]

\[
\|\nabla e^n_{\phi_h}\| \leq C \min \left\{ h^{-1} \|e^n_{\phi_h}\|, \|\nabla e^n_{\phi_h}\| \right\} 
\leq C \min \left\{ h^{-1} (\Delta t + h^{k+1}), \Delta t^{-1/2} (\Delta t + h^{k+1}) \right\} \leq C.
\]

Using Hölder inequality and (6), we get

\[
b(u_h^{n-1} - u^{n-1}, e^n_u, v_h) \leq C \|\nabla e^n_{u_h}\| \|\nabla v_h\| \leq C \|\nabla e^n_{u_h}\| \|\nabla v_h\|,
\]

\[
b(u^{n-1}, e^n_u, v_h) \leq C \|\nabla u^{n-1}\| \|\nabla e^n_u\| \|\nabla v_h\| \leq C \|\nabla e^n_u\| \|\nabla v_h\|,
\]

\[
\sum c(c_B^n - B^{n-1}, e^n_B, v_h) \leq C \|\nabla B^n\| \|\nabla e^n_B\| \|\nabla v_h\| \leq C \|\nabla e^n_B\| \|\nabla v_h\|,
\]

\[
\sum c(c_B^n - B^{n-1}, e^n_B, v_h) \leq C \|\nabla B^n\| \|\nabla e^n_B\| \|\nabla v_h\| \leq C \|\nabla e^n_B\| \|\nabla v_h\|,
\]

\[
-\lambda \left( \epsilon^{n} \nabla (\phi^{n-1} - \phi^{n-1}), v_h \right) - \lambda \left( \epsilon^{n} \nabla (\phi^{n-1} - \phi^{n-1}), v_h \right) \leq C \|e^n_{\phi}\|_{L^2} \|\nabla e^n_{\phi_h}\| \|\nabla v_h\|_{L^4} \leq C \|\nabla e^n_{\phi}\| \|\nabla v_h\|.
\]

Combining (77) with above inequalities, we derive

\[
\|e^n_{\phi}\| \leq C \left\{ \|\delta^t e^n_u\|_{*h} + \|\nabla e^n_{u}\| + \|\nabla e^n_{B}\| + \|\nabla e^n_{\phi}\| + \left( \Delta t + h^{k+1} \right) \right\} + \left( \|e^n_{u}\| + \|e^n_{B}\| + \|e^n_{\phi}\| \right).
\]

Due to

\[
\|\delta^t e^n_u\|_{*h} \leq C \left( \frac{1}{\Delta t} \|\nabla S_h (e^n_u - e^{n-1}_u)\| \right) \leq C \sqrt{\frac{\Delta t}{\Delta t}} (\Delta t + h^{k+1}),
\]
\[
\left( \Delta t \sum_{n=1}^{N} \| \delta_t e_u^n \|_{h, \pi}^2 \right)^{1/2} \leq C \left( \Delta t + h^{k+1} \right).
\]

Using Theorems 5, 6 and the triangle inequality, we obtain the desired result. The proof is finished. \(\square\)

5 Numerical Results

In this section, two numerical tests are given to confirm the theoretical convergence rates and energy stable of the fully discrete scheme (19). The experiments have been finished with applying the finite element libraries from the Fenics Project [16]. The following functions \(\kappa, \nu, \mu\) are given

\[
\kappa(\phi) = e^{\phi}, \quad \nu(\phi) = e^{-\phi}, \quad \eta(\phi) = e^{\phi}.
\]

The finite element pair \(P_2 - P_2 - P_2 - P_1\) for the concentration field, the chemical potential, the velocity field, the magnetic field and the pressure is considered.

Note that the fully discrete scheme (19) is a nonlinear problem, thus we solve the problem (19) by a Picard type iteration. Namely, we fix the velocity field \(u_h\), the magnetic field \(B_h\) and the pressure \(p_h\) at a given time step, then compute for the phase field \(\phi_h\) and the chemical potential \(\mu_h\). Then we compute the velocity field \(u_h\), the magnetic field \(B_h\) and the pressure \(p_h\) with these updated.

5.1 Test 1

In the test, we consider a square domain \(\Omega = [0, 1]^2\), and the following functions are given

\[
\begin{align*}
\phi(x, y, t) &= 2 + \sin(t) \cos(\pi x) \cos(\pi y), \\
u_1(x, y, t) &= \pi \sin(2\pi y) \sin^2(\pi x) \sin(t), \\
u_2(x, y, t) &= -\pi \sin(2\pi x) \sin^2(\pi y) \sin(t), \\
p(x, y, t) &= \cos(\pi x) \sin(\pi y) \sin(t), \\
B_1(x, y, t) &= \sin(\pi x) \cos(\pi y) \sin(t), \\
B_2(x, y, t) &= -\sin(\pi y) \cos(\pi x) \sin(t),
\end{align*}
\]

as the exact solutions, and some source terms are taken such that the exact solutions satisfy (2)–(4).

The parameters are set to \(S_c = 1, \epsilon = 0.05, \lambda = 1\) and \(\Delta t = 0.1 h^2\), and the uniform triangles meshes are employed. We plot the error estimates of the phase field, the velocity field, the magnetic field and the pressure between the numerical solution and the exact solution at \(t = 0.5\) with different space sizes in Fig. 1. We observe that the rates of convergence the fully discrete numerical scheme (19) are second order accurate for all variables.
Fig. 1 Convergence rates: $H^1$ errors of the concentration field $\phi$, the velocity field $u$, the chemical potential $\mu$, the magnetic field $B$ and $L^2$ the pressure as the space mesh size $h$.

![Graph](image)

(a)

![Graph](image)

(b)

5.2 Test 2

In this test, we present the unconditionally energy stable of our proposed scheme 3.1. To this goal, we take the following initial conditions:

$$
\begin{align*}
\phi(x, y, 0) &= \frac{1}{2}(1 - \cos(4\pi x))(1 - \cos(2\pi y)) - 1, \\
u_1(x, y, 0) &= \sin(2\pi y) \sin^2(\pi x), \\
u_2(x, y, 0) &= -\sin(2\pi x) \sin^2(\pi y), \\
B_1(x, y, 0) &= \sin(\pi x) \cos(\pi y), \\
B_2(x, y, 0) &= -\sin(\pi y) \cos(\pi x).
\end{align*}
$$

We consider computing domain to be $\Omega = [0, 1]^2$. The parameters are set to $\epsilon = 0.05$, $\lambda = 1$ and $S_c = 1$. In this test, we take the terminal $T = 15$ and mesh $h = \frac{1}{24}$. In Fig. 2, we give the time evolution of the energy $E(t)$ for time step size $\Delta t = 0.01$ until $T = 15$. It is easy to see that our scheme is unconditionally energy stable.
Conclusions

In this paper, we have analyzed a fully discrete scheme for computing Cahn–Hilliard–Magneto-hydrodynamics system. The scheme is based on using conforming finite element method in space and Euler semi-implicit discretization with convex splitting in time. We have prove our scheme is unconditionally energy stable and obtain optimal error estimates for the concentration field, the chemical potential, the velocity field, the magnetic field and the pressure. Numerical tests are shown to confirm the theoretical rates of the our scheme.

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Data Availability Enquiries about data availability should be directed to the author.

Declarations

Conflict of interest The author declare no competing interests.

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