Some \( q \)-analogues of supercongruences of
Rodriguez-Villegas

Victor J. W. Guo\(^1\) and Jiang Zeng\(^2\)

\(^1\)Department of Mathematics, Shanghai Key Laboratory of PMMP, East China Normal University,
500 Dongchuan Rd., Shanghai 200241, People’s Republic of China
jwguo@math.ecnu.edu.cn, \url{http://math.ecnu.edu.cn/~jwguo}

\(^2\)Université de Lyon; Université Lyon 1; Institut Camille Jordan, UMR 5208 du CNRS;
43, boulevard du 11 novembre 1918, F-69622 Villeurbanne Cedex, France
zeng@math.univ-lyon1.fr, \url{http://math.univ-lyon1.fr/~zeng}

Abstract. We study different \( q \)-analogues and generalizations of the ex-conjectures of Rodriguez-Villegas. For example, for any odd prime \( p \), we show that the known congruence

\[
\sum_{k=0}^{p-1} \binom{2k}{k}^2 \frac{1}{16^k} \equiv (-1)^{\frac{p-1}{2}} \pmod{p^2}
\]

has the following two nice \( q \)-analogues with \([p] = 1 + q + \cdots + q^{p-1}\):

\[
\sum_{k=0}^{p-1} \frac{q; q^2}{(q^2; q^2)_k} \frac{(1+\varepsilon)_k}{q^{(1+\varepsilon)_k}} \equiv (-1)^{\frac{p-1}{2}} q^{\frac{(p-1)\varepsilon}{4}} \pmod{[p]^2},
\]

where \((a; q)_0 = 1\), \((a; q)_n = (1 - a)(1 - aq) \cdots (1 - aq^{n-1})\) for \( n = 1, 2, \ldots \), and \( \varepsilon = \pm 1 \).
Several related conjectures are also proposed.

Keywords: congruences, least nonnegative residue, little \( q \)-Legendre polynomials, \( q \)-binomial theorem, \( q \)-Chu-Vandermonde formula

2000 Mathematics Subject Classifications: Primary 11B65, Secondary 05A10, 05A30

1. Introduction

Rodriguez-Villegas [14] discovered numerically some remarkable supercongruences between a truncated hypergeometric function associated to a Calabi-Yau manifold at a prime \( p \) and the number of its \( \mathbb{F}_p \)-points. In particular, Rodriguez-Villegas recorded four such supercongruences associated to elliptic curves. Following a strategy developed by Ahlgren and Ono [1], by using the Gross-Koblitz formula to write the Gaussian hypergeometric series in terms of the \( p \)-adic \( \Gamma \)-function, Mortenson [10,11] first proved the following four conjectured supercongruences of Rodriguez-Villegas [14, (36)].
Theorem 1.1 (Rodríguez-Villegas-Mortenson). Let $p \geq 5$ be a prime. Then

$$\sum_{k=0}^{p-1} \frac{(2k)^2}{16k} \equiv \left( \frac{-4}{p} \right) \pmod{p^2}, \quad (1.1)$$

$$\sum_{k=0}^{p-1} \frac{(3k)^2}{27k} \equiv \left( \frac{-3}{p} \right) \pmod{p^2}, \quad (1.2)$$

$$\sum_{k=0}^{p-1} \frac{(4k)^2}{64k} \equiv \left( \frac{-2}{p} \right) \pmod{p^2}, \quad (1.3)$$

$$\sum_{k=0}^{p-1} \frac{(6k)^2}{432k} \equiv \left( \frac{-1}{p} \right) \pmod{p^2}, \quad (1.4)$$

where $\left( \frac{\cdot}{p} \right)$ denotes the Legendre symbol modulo $p$.

Elementary proof of Theorem 1.1 has been given by Z.-H. Sun [16]. See also [15, 20] for two simple proofs of (1.1). Note that Van Hamme [23] and McCarthy and Osburn [9] have also studied some related interesting supercongruences. The starting point of this paper is the observation of the following striking $q$-analogue of Theorem 1.1.

Conjecture 1.2. Let $p \geq 5$ be a prime and let $\left( \frac{\cdot}{p} \right)$ be the Legendre symbol modulo $p$. Then

$$\sum_{k=0}^{p-1} \frac{(q; q^2)^2_k}{(q^2; q^2)^2_k} \equiv \left( \frac{-4}{p} \right) q^{\frac{1-q^2}{p}} \pmod{[p]^2},$$

$$\sum_{k=0}^{p-1} \frac{(q; q^3)_k(q^2; q^3)_k}{(q^3; q^3)^2_k} \equiv \left( \frac{-3}{p} \right) q^{\frac{1-q^2}{p}} \pmod{[p]^2},$$

$$\sum_{k=0}^{p-1} \frac{(q; q^4)_k(q^2; q^4)_k}{(q^4; q^4)^2_k} \equiv \left( \frac{-2}{p} \right) q^{\frac{3(1-p^2)}{8}} \pmod{[p]^2},$$

$$\sum_{k=0}^{p-1} \frac{(q^2; q^6)_k(q^6; q^6)_k}{(q^6; q^6)^2_k} \equiv \left( \frac{-1}{p} \right) q^{\frac{5(1-p^2)}{12}} \pmod{[p]^2},$$

where $[p] = 1 + q + \cdots + q^{p-1}$.

Congruences modulo $[p]$ or $[p]^2$ (even $[p]^3$) have been studied by different authors (see [3, 12, 13, 19, 21]). Throughout the paper we will tacitly use the fact that when $p$ is a prime the $q$-integer $[p]$ is an irreducible polynomial in $\mathbb{Q}[q]$. Therefore $\mathbb{Q}[q]/[p]$ is a field. Furthermore, rational functions $r(q)/s(q)$ are well defined modulo $[p]$ or $[p]^r$ ($r \geq 1$) provided that $s(q)$ is relatively prime to $[p]$. For any two polynomials $A(x, q) = \sum_{k=0}^{n} a_k(q)x^k$ and $B(x, q) = \sum_{k=0}^{n} b_k(q)x^k$ in $x$ with coefficients being rational functions $r(q)/s(q)$ such that $s(q)$ is relatively prime to $[p]$, we use the convention that

$$A(x, q) \equiv B(x, q) \pmod{[p]^r} \iff a_k(q) \equiv b_k(q) \pmod{[p]^r} \quad \text{for} \quad k = 0, 1, \ldots, n.$$
There are several generalizations and variations of (1.1)–(1.4) in the literature, but no $q$-analogues seem to be investigated hitherto. Indeed, Tauraso [20] proved the following generalization of (1.1).

**Theorem 1.3** (Tauraso [20]). Let $p$ be an odd prime. Then

$$\sum_{k=0}^{p-1} \binom{2k}{k} \frac{2^{x^k}}{16^k} \equiv \sum_{k=0}^{p-1} \binom{p-1}{2\frac{k}{2}} \left(\frac{-x}{16}\right)^k (1-x)^{\frac{p-1}{2}-k} \pmod{p^2}. \quad (1.5)$$

Recently, Z.-H. Sun [16] introduced the generalized Legendre polynomials

$$P_n(a, x) = \sum_{k=0}^{n} \binom{a}{k} \left(\frac{1-a}{k}\right) \frac{(1-x)^k}{2^k} = \sum_{k=0}^{n} \binom{a}{k} \left(\frac{a+k}{k}\right) \frac{(x-1)^k}{2^k},$$

and proved many supercongruences related to $P_{p-1}(a, x)$. In particular, he obtained the following result.

**Theorem 1.4** (Z.-H. Sun [16]). Let $p$ be an odd prime and let $a$ be a $p$-adic integer. Then

$$P_{p-1}(a, x) \equiv (-1)^{(a)p} P_{p-1}(a, -x) \pmod{p^2}, \quad (1.6)$$

and so

$$\sum_{k=0}^{p-1} \binom{a}{k} \left(\frac{-1-a}{k}\right) (x^k - (-1)^{(a)p} (1-x)^k) \equiv 0 \pmod{p^2}. \quad (1.7)$$

where $(a)_p$ denotes the least nonnegative residue of $a$ modulo $p$.

It is easy to see that the Rodriguez-Villegas-Mortenson congruences (1.1)–(1.4) immediately follows from the congruence (1.7) by taking $x = 1$ and $a = -\frac{1}{2}, -\frac{1}{3}, -\frac{1}{4}, -\frac{1}{6}$. The aim of this paper is to give $q$-analogues of (1.5)–(1.7). It turns out that a complete $q$-analogue of (1.5) is easily given. However, for a general $p$-adic integer $a$, we can only give $q$-analogues of (1.6) and (1.7) in the modulus $p$ case. On the other hand, for $a = -\frac{1}{2}$, we can give complete $q$-analogues of them. Thus, the first congruence in Conjecture 1.2 is proved, while the other three congruences are still open. Some further related unsolved problems will also be presented in this paper.

### 2. Results, I: supercongruences modulo $[p]^2$

Recall that the $q$-shifted factorials are defined by $(a; q)_0 = 1$ and $(a; q)_n = (1 - a)(1 - aq) \cdots (1 - aq^{n-1})$ for $n = 1, 2, \ldots$. The $q$-binomial coefficients $\binom{[\alpha]}{[k]}_q$ are defined by

$$\binom{[\alpha]}{[k]}_q = \binom{[\alpha]}{[k]} = \begin{cases} \frac{(q^{\alpha-k+1}; q)_k}{(q; q)_k}, & \text{if } k \geq 0, \\ 0, & \text{if } k < 0. \end{cases}$$

We first give a $q$-analogue of Tauraso’s congruence (1.5).
Theorem 2.1. Let \( p \) be an odd prime. Then
\[
\sum_{k=0}^{p-1} \frac{(q; q^2)_k^2}{(q^2; q^2)_k^2} x^k \equiv \sum_{k=0}^{p-1} \frac{(-1)^{k} q^{1-x^2}}{q^2} \left( (-x)^x (x; q^2) \right) \left( \frac{1}{k} \right) \pmod{p^2}.
\] (2.1)

Since
\[
\lim_{q \to 1} \frac{(q; q^2)_k^2}{(q^2; q^2)_k^2} = \left( \prod_{j=1}^{k} \frac{2j-1}{2j} \right)^2 = \left( \frac{2k}{k} \right)^2 16^{-k},
\]
letting \( q \to 1 \) in (2.1), we obtain (1.5). Moreover, setting \( x = 1 \) in (2.1) yields the following \( q \)-analogue of (1.1).

Corollary 2.2. Let \( p \) be an odd prime. Then
\[
\sum_{k=0}^{p-1} \frac{(q; q^2)_k^2}{(q^2; q^2)_k^2} x^k \equiv \left( \frac{-1}{p} \right) q^{x^2-1} \pmod{p^2}.
\] (2.2)

Our second result is another generalization of (2.2).

Theorem 2.3. Let \( p \) be an odd prime. Then
\[
\sum_{k=0}^{p-1} \frac{(q; q^2)_k^2}{(q^2; q^2)_k^2} x^k \equiv (-1)^{x^2} q^{1-x^2} \sum_{k=0}^{p-1} \frac{(q; q^2)_k^2}{(q^2; q^2)_k^2} q^{2k} (x; q^2)_k \pmod{p^2}.
\] (2.3)

It is clear that, when \( x = 1 \), the congruence (2.3) reduces to (2.2). On the other hand, setting \( x = 0 \) in (2.3), we obtain the following dual form of (2.2).

Corollary 2.4. Let \( p \) be an odd prime. Then
\[
\sum_{k=0}^{p-1} \frac{(q; q^2)_k^2}{(q^2; q^2)_k^2} q^{2k} \equiv \left( \frac{-1}{p} \right) q^{x^2} \pmod{p^2}.
\]

Our third result is a \( q \)-analogue of the \( a = -\frac{1}{2} \) case of (1.6).

Theorem 2.5. Let \( p \) be an odd prime. Then
\[
\sum_{k=0}^{p-1} \frac{(q; q^2)_k^2 (x; q^2)_k q^{2k}}{(q^2; q^2)_k (1-q^2; q^2)_k} \equiv (-1)^{x^2} \sum_{k=0}^{p-1} \frac{(q; q^2)_k^2 (-x; q^2)_k q^{2k}}{(q^2; q^2)_k (-q^2; q^2)_k} \pmod{p^2}.
\] (2.4)

Letting \( x = -1 \) in (2.4), and noticing that \( \frac{(-1; q^2)_k}{(-q^2; q^2)_k} = \frac{2}{1+q^2} \), we are led to another \( q \)-analogue of (1.1).

Corollary 2.6. Let \( p \) be an odd prime. Then
\[
\sum_{k=0}^{p-1} \frac{2(q; q^2)_k^2 q^{2k}}{(q^2; q^2)_k (1+q^2)_k} \equiv \left( \frac{-1}{p} \right) \pmod{p^2}.
\] (2.5)
If \( \frac{p-1}{2} \equiv 1 \pmod{2} \), then from (2.4) we immediately deduce that \( P_{p-1,2,1}(q,0) \equiv 0 \pmod{[p]^2} \), which may be restated as follows.

**Corollary 2.7.** Let \( p \) be an odd prime and \( m, r \) two positive integers with \( p \nmid m \). Then

\[
\sum_{k=0}^{p-1} \frac{(q^r;q^m)_k(q^{m-r};q^m)_k x^k}{(q^m;q^m)_k^2} \equiv (-1)^{\left\lfloor \frac{r}{m} \right\rfloor} q^{\frac{r}{m}(1-q^m)} \sum_{k=0}^{p-1} \frac{(q^r;q^m)_k(q^{m-r};q^m)_k(x;q^m)_k q^m k^m}{(q^m;q^m)_k^2} \pmod{[p]}. \tag{3.1}
\]

In particular, if \( p \equiv \pm 1 \pmod{m} \), then

\[
\sum_{k=0}^{p-1} \frac{(q^r;q^m)_k(q^{m-r};q^m)_k x^k}{(q^m;q^m)_k^2} \equiv (-1)^{\left\lfloor \frac{r}{m} \right\rfloor} q^{\frac{r}{m}(1-q^m)} \sum_{k=0}^{p-1} \frac{(q^r;q^m)_k(q^{m-r};q^m)_k(x;q^m)_k q^m k^m}{(q^m;q^m)_k^2} \pmod{[p]}. \tag{3.2}
\]

Note that, for \( p \geq 5 \), we have

\[
(-1)^{\left\lfloor \frac{r}{m} \right\rfloor} = \begin{cases} 
(-1) \frac{p+1}{2} = 1, & \text{if } p \equiv 1 \pmod{3} \\
(-1) \frac{2p-1}{3} = -1, & \text{if } p \equiv 2 \pmod{3}
\end{cases}, \ 
(-1)^{\left\lfloor \frac{r}{m} \right\rfloor} = \begin{cases} 
(-1) \frac{p+1}{4} = -1, & \text{if } p \equiv 1 \pmod{4} \\
(-1) \frac{3p-1}{4} = 1, & \text{if } p \equiv 3 \pmod{4}
\end{cases}, \ 
(-1)^{\left\lfloor \frac{r}{m} \right\rfloor} = \begin{cases} 
(-1) \frac{p+1}{6} = -1, & \text{if } p \equiv 1 \pmod{6} \\
(-1) \frac{5p-1}{6} = 1, & \text{if } p \equiv 5 \pmod{6}
\end{cases}.
\]

Letting \( x = 1, r = 1, m = 3, 4, 6 \) in (3.2) with \( p \geq 5 \), we obtain the following result.

### 3. Results, II: congruences modulo \([p]\)

In this section, we first give \( q \)-analogues of (1.2)–(1.4). Actually we prove the following more general results.

**Theorem 3.1.** Let \( p \) be an odd prime of the form \( 4k + 3 \). Then

\[
\sum_{k=0}^{p-1} \frac{(q^r;q^m)_k(q^{m-r};q^m)_k x^k}{(q^m;q^m)_k^2} \equiv 0 \pmod{[p]^2}. \tag{2.6}
\]

Note that, when \( q = 1 \), the congruence (2.6) can be written as

\[
\sum_{k=0}^{p-1} \frac{(2k)^2}{32k} \equiv 0 \pmod{p^2} \quad \text{for } p \equiv 3 \pmod{4},
\]

which was conjecture by Z.-W. Sun [18] and proved by Tauraso [20] and Z.-H. Sun [15,16].
Corollary 3.2. Let \( p \geq 5 \) be a prime. Then

\[
\sum_{k=0}^{p-1} \frac{(q; q^3)_k(q^2; q^3)_k}{(q^3; q^3)_k} 
\equiv \left( -\frac{3}{p} \right) q^{\frac{p^2 - 1}{2}} \pmod{p},
\]

\[
\sum_{k=0}^{p-1} \frac{(q; q^4)_k(q^3; q^4)_k}{(q^4; q^4)_k} 
\equiv \left( -\frac{2}{p} \right) q^{\frac{3(p^2 - 1)}{8}} \pmod{p},
\]

\[
\sum_{k=0}^{p-1} \frac{(q; q^6)_k(q^5; q^6)_k}{(q^6; q^6)_k} 
\equiv \left( -\frac{1}{p} \right) q^{\frac{5(p^2 - 1)}{12}} \pmod{p}.
\]

In the same vein, letting \( x = 0, r = 1, \) and \( m = 3, 4, 6 \) in (3.1), we obtain

Corollary 3.3. Let \( p \geq 5 \) be a prime. Then

\[
\sum_{k=0}^{p-1} \frac{(q; q^3)_k(q^2; q^3)_k}{(q^3; q^3)_k^2} q^{3k} 
\equiv \left( -\frac{3}{p} \right) q^{\frac{p^2 - 1}{2}} \pmod{p},
\]

\[
\sum_{k=0}^{p-1} \frac{(q; q^4)_k(q^3; q^4)_k}{(q^4; q^4)_k^2} q^{4k} 
\equiv \left( -\frac{2}{p} \right) q^{\frac{3(p^2 - 1)}{8}} \pmod{p},
\]

\[
\sum_{k=0}^{p-1} \frac{(q; q^6)_k(q^5; q^6)_k}{(q^6; q^6)_k^2} q^{6k} 
\equiv \left( -\frac{1}{p} \right) q^{\frac{5(p^2 - 1)}{12}} \pmod{p}.
\]

Theorem 3.4. Let \( p \) be an odd prime and \( m, r \) two positive integers with \( p \nmid m \). Then we have the following congruence modulo \([p]\):

\[
\sum_{k=0}^{p-1} \frac{(q^r; q^m)_k(q^{m-r}; q^m)_k x^k}{(q^m; q^m)_k^2} 
\equiv \sum_{k=0}^{\frac{p-1}{2}} \left[ \left( -\frac{r}{m} \right) \right]_p^2 q^{\frac{mk(k-1)}{2}} (-x)^k (x; q^m)(-\frac{r}{m})_{p-k}.
\]

Next, we give \( q \)-analogues of (1.6)–(1.7) in the modulus \( p \) case. For this end, we introduce the following polynomials

\[
P_{n,m,r}(q, x) = \sum_{k=0}^{n} \frac{(q^r; q^m)_k(q^{m-r}; q^m)_k x^k}{(q^m; q^m)_k^2(-q^m; q^m)_k}.
\]

Note that \( P_{n,m,r}(1, x) \) is the generalized Legendre polynomial \( P_n(a, x) \) with \( a = -\frac{r}{m} \) in [16].

Theorem 3.5. Let \( p \) be an odd prime and \( m, r \) two positive integers with \( p \nmid m \). Then

\[
P_{p-1,m,r}(q, x) \equiv (-1)^{\frac{p-1}{2}} P_{p-1,m,r}(q, -x) \pmod{p}.
\]  \( (3.3) \)

Letting \( x = 0 \) in (3.3), we obtain
Corollary 3.6. Let \( p \) be an odd prime and \( m, r \) two integers with \( p \nmid m \) and \( \langle r/m \rangle_p \equiv 0 \pmod{2} \). Then
\[
\sum_{k=0}^{p-1} \frac{(q^r; q^m)_k (q^{m-r}; q^m)_k q^{nk}}{(q^m; q^m)_k^2 (-q^m; q^m)_k} \equiv 0 \pmod{p}.
\] (3.4)

Taking \((m, r) = (3, 1), (4, 1), (6, 1)\) in (3.4), we get the following congruences.

Corollary 3.7. Let \( p \) be an odd prime. Then
\[
\sum_{k=0}^{p-1} \frac{(q; q^3)_k (q^2; q^3)_k q^{3k}}{(q^3; q^3)_k^2 (-q^3; q^3)_k} \equiv 0 \pmod{p}, \quad \text{for } p \equiv 2 \pmod{3},
\] (3.5)
\[
\sum_{k=0}^{p-1} \frac{(q; q^4)_k (q^3; q^4)_k q^{4k}}{(q^4; q^4)_k^2 (-q^4; q^4)_k} \equiv 0 \pmod{p}, \quad \text{for } p \equiv 5, 7 \pmod{8},
\] (3.6)
\[
\sum_{k=0}^{p-1} \frac{(q; q^6)_k (q^5; q^6)_k q^{6k}}{(q^6; q^6)_k^2 (-q^6; q^6)_k} \equiv 0 \pmod{p}, \quad \text{for } p \equiv 3 \pmod{4}.
\] (3.7)

Letting \( x = -1 \) in (3.3), we obtain
\[
P_{p-1, m, r}(q, -1) \equiv (-1)^{\langle -x \rangle_p} \pmod{p}.
\] (3.8)

Taking \((m, r) = (3, 1), (4, 1), (6, 1)\), we get the following result.

Corollary 3.8. Let \( p \geq 5 \) be a prime. Then
\[
\sum_{k=0}^{p-1} \frac{2(q; q^3)_k (q^2; q^3)_k q^{3k}}{(q^3; q^3)_k^2 (1 + q^{3k})} \equiv \left( -\frac{3}{p} \right) \pmod{p},
\] (3.9)
\[
\sum_{k=0}^{p-1} \frac{2(q; q^4)_k (q^3; q^4)_k q^{4k}}{(q^4; q^4)_k^2 (1 + q^{4k})} \equiv \left( -\frac{2}{p} \right) \pmod{p},
\] (3.10)
\[
\sum_{k=0}^{p-1} \frac{2(q; q^6)_k (q^5; q^6)_k q^{6k}}{(q^6; q^6)_k^2 (1 + q^{6k})} \equiv \left( -\frac{1}{p} \right) \pmod{p}.
\] (3.11)

4. Proofs of Theorems 2.1 and 2.3

Recall that the little \( q \)-Legendre polynomials are defined by
\[
P_n(x|q) = \sum_{k=0}^n \binom{n}{k} \binom{n+k}{k} q^{\frac{k(k+1)}{2}} x^n (-x)^k.
\] (4.1)
They can also be written as (see [22])

\[ P_n(x|q) = (-1)^n q^{-\frac{n(n+1)}{2}} \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} (-1)^k q^{\frac{k(k+1)}{2}}(-x)^k(xq)_k. \]  

(4.2)

We now give a new expansion for the little \( q \)-Legendre polynomials.

**Lemma 4.1.** Let \( n \) be a nonnegative integer. Then

\[ P_n(x|q) = \sum_{k=0}^{n} \binom{n}{k}^2 q^{\frac{k(k+1)}{2}}(-nk)(-x)^k(xq)_{n-k}. \]  

(4.3)

**Proof.** By the \( q \)-binomial theorem (see [2, p. 36, Theorem 3.3]):

\[ (x; q)_N = \sum_{k=0}^{N} \binom{N}{k} (-x)^k q^{\frac{k(k+1)}{2}}, \]  

(4.4)

one sees that, for \( 0 \leq m \leq n \), the coefficient of \( x^m \) in the right-hand side of (4.3) is given by

\[
\sum_{k=0}^{m} \binom{n}{k}^2 q^{\frac{k(k+1)}{2}}(-nk)(-1)^k \binom{n-k}{m-k}(-1)^{m-k}q^{\frac{(m-k)(m-k+1)}{2}}
\]

\[
= (-1)^m \binom{n}{m} \sum_{k=0}^{m} \binom{m}{k} \binom{n}{k} q^{(m-k)(n-k)-mn+\frac{m(m+1)}{2}}
\]

\[
= (-1)^m \binom{n}{m} \binom{n+m}{m} q^{-mn+\frac{m(m+1)}{2}},
\]

where the last step follows from the \( q \)-Chu-Vandermonde formula (see [2, p. 37, Theorem 3.4]). This completes the proof. \( \square \)

We also need the following result.

**Lemma 4.2.** Let \( p \) be an odd prime and \( 0 \leq k \leq p-1 \). Then

\[
\frac{(q; q^2)^k}{(q^2; q^2)^k} \equiv (-1)^k \left(\frac{p-1}{2}\right)^k \left(\frac{p-1}{2}+k\right) q^{k^2-kp} \pmod{p^2}.
\]  

(4.5)

**Proof.** Observing that

\[
(1 - q^{2j-1})^2 + (1 - q^{p-2j+1})(1 - q^{p+2j-1})q^{2j-1-p} = (1 - q^p)^2 q^{2j-1-p},
\]

we have

\[
(1 - q^{2j-1})^2 \equiv -(1 - q^{p-2j+1})(1 - q^{p+2j-1})q^{2j-1-p} \pmod{p^2}.
\]
It follows that
\[
\frac{(q; q^2)_k^2}{(q^2; q^2)_k^2} = \prod_{j=1}^{k} \frac{(1 - q^{2j-1})^2}{(1 - q^2j)^2} \equiv (-1)^k \prod_{j=1}^{k} \frac{(1 - q^{p-2j+1})(1 - q^{p+2j-1})q^{2j-1-p}}{(1 - q^{2j})^2} \\
\equiv (-1)^k \left[ \frac{p-1}{2} \right]_q \left[ \frac{p-1}{2} + k \right]_q q^{k^2-kp} \pmod{[p]^2},
\]
as desired. □

*Proof of Theorem 2.1.* Letting \( n = \frac{p-1}{2} \), replacing \( q \) and \( x \) by \( q^2 \) and \( xq^{-2} \) respectively in (4.1) and (4.3), and then applying (4.5), we obtain (2.1). □

*Proof of Theorem 2.3.* Similarly to the proof of Theorem 2.1, just compare (4.1) and (4.2). □

## 5. Proof of Theorem 2.5

### Lemma 5.1.
Let \( n \) be a positive integer and \( 0 \leq j \leq n \). Then
\[
\sum_{k=j}^{n} (-1)^k \left[ \begin{array}{c} n + k \cr k \end{array} \right] \left[ \begin{array}{c} n - j \cr k - j \end{array} \right] \frac{q^{k(k+1)/2-nk}}{(-q; q)_k} \frac{(q^{n+1}; q^2)_j \frac{(q^{n+2j-1} - j(j-1)}{4}}{(-q; q)_n(q^{n+j+1}; q^2)_j}, \quad \text{if } n \equiv j \equiv 0 \pmod{2},
\]
\[
\left[ \begin{array}{c} n - 1 \cr n - k \end{array} \right] \frac{(q^{n+2}; q^2)_i \frac{(q^{n+2j-1} - j(j-1)}{4}}{(-q; q)_{n-1}(q^{n+j+1}; q^2)_i}, \quad \text{if } n \equiv j \equiv 1 \pmod{2},
\]
\[
0, \quad \text{otherwise.}
\]

*Proof.* Replacing \( k \) by \( k + j \), we can write the left-hand side of (5.1) as
\[
\sum_{k=0}^{n-j} (-1)^{k+j} \left[ \begin{array}{c} n + k + j \cr n \end{array} \right] \left[ \begin{array}{c} n - j \cr k \end{array} \right] \frac{q^{(k+j)(k+j+1)/2-n(k+j)}}{(-q; q)_{k+j}} \]
\[
= \sum_{k=0}^{n-j} (-1)^{k+j} \frac{(q; q)_{n+k+j}(q; q)_{n-j}q^{(k+j)(k+j+1)/2-n(k+j)}}{(q; q)_{n}(q; q)_{k+j}(q; q)_{k}(q; q)_{n-j-k}(-q; q)_{k+j}} \\
= \sum_{k=0}^{n-j} (-1)^{j} \frac{(q; q)_{n+j}(q^{n+j+1}; q)_{k}(q^{-n+j}; q)_{k}q^{j(j+1)/2-nj}}{(q; q)_{n+j}(q; q)_{j}(q^{j+1}; q)_{k}(q; q)_{k}(q^{-j+1}; q)_{k}} \\
= (-1)^{j} \left[ \begin{array}{c} n + j \cr j \end{array} \right] \frac{q^{j(j+1)/2-nj}}{(-q; q)_{j}} \sum_{k=0}^{n-j} \frac{(q^{n+j+1}; q)_{k}(q^{-n+j}; q)_{k}q^{k}}{(-q; q)_{k}(q^{2j+2}; q^{2})_{k}}, \quad (5.2)
\]
where we have used the relation
\[
\frac{(q; q)_{n-j}}{(q; q)_{n-j-k}} = (-1)^j (q^{n+j}; q)_k q^{k(2n-2j-k+1)/2}.
\]

Taking \( a = q^{n+j+1} \) and \( b = q^{-n+j} \) in Andrews’ \( q \)-analogue of Gauss’ \( 2F_1(-1) \) sum (see \([4, 5]\) or \([7, \text{Appendix (II.11)}]\)):
\[
\sum_{k=0}^{\infty} \frac{(a; q)_k (b; q)_k q^{k(k+1)/2}}{(q; q)_k (abq; q^2)_k} = \frac{(aq; q^2)_\infty (bq; q^2)_\infty}{(q; q^2)_\infty (abq; q^2)_\infty},
\]
where \((x; q)_\infty = \lim_{n \to \infty} (x; q)_n\), we have
\[
\sum_{k=0}^{n-j} \frac{(q^{n+j+1}; q)_k (q^{-n+j}; q)_k q^{k(k+1)/2}}{(q; q)_k (q^{2j+2}; q^2)_k} = \frac{(q^{n+j+2}; q^2)_\infty (q^{-n+j+1}; q^2)_\infty}{(q; q^2)_\infty (q^{2j+2}; q^2)_\infty} \cdot \frac{(q^{-n+j+1}; q^2)_{n-j}}{(q^{2j+2}; q^2)_{n-j}},
\]
\[
= \begin{cases} 
\frac{(q^{-n+j+1}; q^2)_{n-j}}{(q^{2j+2}; q^2)_{n-j}}, & \text{if } n \equiv j \pmod{2}, \\
0, & \text{otherwise}.
\end{cases}
\]
(5.4)

Replacing \( q \) by \( q^{-1} \) in (5.4) and noticing that \((q^{-m}; q^{-1})_k = (-1)^k q^{-mk-\frac{k(k+1)}{2}} (q^m; q)_k\), we get
\[
\sum_{k=0}^{n-j} \frac{(q^{n+j+1}; q)_k (q^{-n+j}; q)_k q^{k(k+1)/2}}{(q; q)_k (q^{2j+2}; q^2)_k} = \begin{cases} 
\frac{(q^{n+j+2}; q^2)_\infty (q^{-n+j+1}; q^2)_\infty}{(q; q^2)_\infty (q^{2j+2}; q^2)_\infty} \cdot \frac{(q^{-n+j+1}; q^2)_{n-j}}{(q^{2j+2}; q^2)_{n-j}}, & \text{if } n \equiv j \pmod{2}, \\
0, & \text{otherwise}.
\end{cases}
\]
(5.5)

Substituting (5.5) into (5.2) and making some simplifications, we obtain the desired identity (5.1). \(\square\)

**Lemma 5.2.** Let \( n \) be a positive integer and
\[
F_n(x, q) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \left[ \frac{n+k}{k} \right] \frac{(x; q)_k q^{k(k+1)/2-nk}}{(-q; q)_k}.
\]
(5.6)

Then
\[
F_n(x, q) = (-1)^n F_n(-x, q).
\]
(5.7)

**Proof.** By the \( q \)-binomial theorem (4.4), the coefficient of \( x^j \) \((0 \leq j \leq n)\) in the right-hand side of (5.6) is given by
\[
q^{\frac{j(j-1)}{2}} \sum_{k=j}^{n} (-1)^{k-j} \binom{n}{k} \left[ \frac{n+k}{k} \right] \frac{q^{k(k+1)/2-nk}}{(-q; q)_k}
= q^{\frac{j(j-1)}{2}} \sum_{k=j}^{n} (-1)^{k-j} \binom{n}{k-j} \left[ \frac{n-j}{k-j} \right] \frac{q^{k(k+1)/2-nk}}{(-q; q)_k},
\]
\[
= q^{\frac{j(j-1)}{2}} \sum_{k=j}^{n} (-1)^{k-j} \binom{n}{k-j} \left[ \frac{n-j}{k-j} \right] \frac{q^{k(k+1)/2-nk}}{(-q; q)_k}.
\]

which, by Lemma 5.1, is equal to 0 if \( n - j \equiv 1 \pmod{2} \). This proves (5.7).

**Proof of Theorem 2.5.** Note that, for \( \frac{p-1}{2} < k < p \), there holds \((q; q^2)^2_k \equiv 0 \pmod{[p]^2}\).

By Lemma 4.2, we have

\[
\sum_{k=0}^{p-1} \frac{(q; q^2)^2_k (x; q^2)^2_k}{(q; q^2)^2_k (-q^2; q^2)_k} k^{2k} \equiv \sum_{k=0}^{p-1} (-1)^k \left[ \frac{p-1}{2} \right] q^{k^2 + 2k - kp} \mod{[p]^2}.
\]

The theorem then follows from Lemma 5.2. \( \square \)

**Remark.** Another application of Andrews’s \(q\)-analogue of Gauss’s \( _2F_1(-1) \) sum (5.3) to \(q\)-congruences can be found in [8].

### 6. Proofs of Theorems 3.1, 3.4 and 3.5

**Proof of Theorems 3.1.** First of all, if the theorem holds for \((r, m)\), then replacing \(q\) by \(q^d\) \((d \geq 1)\) means that it also holds for \((rd, md)\). Without loss of generality, we may assume that \(\gcd(m, r) = 1\). Secondly, since \(q^r \equiv q^{r-p} \pmod{[p]}\), we may further assume that \(1 \leq r < p\). When \(m = 1\), we have

\[
\sum_{k=0}^{p-1} \frac{(q^r; q)_k (q^{1-r}; q)_k x^k}{(q^2; q^2)_k} \equiv \sum_{k=0}^{r-1} \left[ \frac{r-1}{k} \right] \left[ \frac{r-1+k}{k} \right] (-x) k q^{(k+1)/2} - k(r-1),
\]

and

\[
\sum_{k=0}^{p-1} \frac{(q^r; q)_k (q^{1-r}; q)_k (x; q)_k q^k}{(q^2; q^2)_k} \equiv \sum_{k=0}^{r-1} \left[ \frac{r-1}{k} \right] \left[ \frac{r-1+k}{k} \right] (-1) k (x; q)_k q^{(k+1)/2} - k(r-1).
\]

The proof then follows from (4.1) and (4.2) with \(P_n(x|q)\) replaced by \(P_r(xq^{-1}|q)\).

When \(m \geq 2\), let

\[
s = \frac{m \left( -\frac{r}{m} \right) p + r}{p}.
\]

Then \(s\) is a positive integer, \(m|ps - r\), and so

\[
\frac{(q^r; q)_k (q^{m-r}; q^m)_k}{(q^m; q^m)_k} \equiv \prod_{j=1}^{k} \frac{(1 - q^{mj-r})(1 - q^{mj+r-m})}{(1 - q^{mj})^2} \equiv (-1)^k \prod_{j=1}^{k} \frac{(1 - q^{ps+ mj-j})(1 - q^{ps - mj - r + m})}{(1 - q^{2j})^2} q^{m k (k-1)/2} + kr
\]

\[
= (-1)^k \left[ \left( \frac{ps-r}{m} \right) K \left[ \frac{ps-r}{m} + k \right] \right] q^m \left[ \left( \frac{-r}{m} \right) K + k \right] q^{m k (k-1)/2} - k(ps-r) \pmod{[p]}.
\]

(6.1)
It follows that
\[
\sum_{k=0}^{p-1} \frac{(q^r; q^m)_k(q^{m-r}; q^m)_k x^k}{(q^m; q^m)_k^2} = \sum_{k=0}^{\langle -\frac{r}{m} \rangle_p} \left( -\frac{r}{m} \right)_k^p \sum_{k'}^{\langle -\frac{r}{m} \rangle_q + k} \frac{(-x)^k q^{mk(k+1)} - mk(-\frac{r}{m})^p}{q^{mk} - mk(-\frac{r}{m})^p} \pmod{p},
\]
and
\[
\sum_{k=0}^{p-1} \frac{(q^r; q^m)_k(q^{m-r}; q^m)_k(x; q^m)_k q^{mk}}{(q^m; q^m)_k^2} = \sum_{k=0}^{\langle -\frac{r}{m} \rangle_p} \left( -\frac{r}{m} \right)_k^p \sum_{k'}^{\langle -\frac{r}{m} \rangle_q + k} \frac{(-1)^k(x; q^m)_k q^{mk} - mk(-\frac{r}{m})^p}{q^{mk} - mk(-\frac{r}{m})^p} \pmod{p}.
\]

The proof of (3.1) then follows from the two expressions (4.1) and (4.2) for \( P_{(-\frac{r}{m})_p}(xq^{-m}|q^m) \). Moreover, if \( p \equiv \pm 1 \pmod{m} \), then \( \frac{r(m-r)(1-p^2)}{2m} \) is an integer and
\[
\frac{-m\langle -\frac{r}{m} \rangle_p(-\frac{r}{m})_p + 1}{2} \equiv \frac{r(m-r)(1-p^2)}{2m} \pmod{p}.
\]
This proves (3.2).

Proof of Theorems 3.4. Apply (4.3) to \( P_{(-\frac{r}{m})_p}(xq^{-m}; q^m) \).

Proof of Theorems 3.5. Similarly as before, we have
\[
P_{p-1,m,r}(q, x) = \sum_{k=0}^{p-1} \frac{(q^r; q^m)_k(q^{m-r}; q^m)_k(x; q^m)_k q^{mk}}{(q^m; q^m)_k^2} = \sum_{k=0}^{\langle -\frac{r}{m} \rangle_p} (-1)^k \langle -\frac{r}{m} \rangle_q \frac{(-1)^k(x; q^m)_k q^{mk} - mk(-\frac{r}{m})^p}{q^{mk} - mk(-\frac{r}{m})^p}.
\]
The proof then follows directly from Lemma 5.2.

7. Concluding remarks and open problems

We have the following two stronger conjectural results for Theorems 3.1.

Conjecture 7.1. Let \( p \) be an odd prime and \( m, r \) two positive integers with \( p \nmid m \). Then there exists a unique integer \( f_{p,m,r} \) such that
\[
\sum_{k=0}^{p-1} \frac{(q^r; q^m)_k(q^{m-r}; q^m)_k x^k}{(q^m; q^m)_k^2} \equiv (-1)^{\langle -\frac{r}{m} \rangle_p} q^{f_{p,m,r}} \sum_{k=0}^{p-1} \frac{(q^r; q^m)_k(q^{m-r}; q^m)_k(x; q^m)_k q^{mk}}{(q^m; q^m)_k^2} \pmod{[p]^2}.
\]
Furthermore, the numbers \( f_{p,m,r} \) satisfy the following recurrence relation:

\[
f_{p,m,m+r} = \begin{cases} 
-f_{p,m,r}, & \text{if } r \equiv 0 \pmod{p}, \\
f_{p,m,r} - r, & \text{otherwise}.
\end{cases}
\]

Here are some values of \( f_{p,m,r} \):

\[
\begin{align*}
    f_{7,2,1} &= -12, \quad f_{7,2,3} = -13, \quad f_{7,2,5} = -16, \quad f_{7,2,7} = -21, \quad f_{7,2,9} = 21, \quad f_{7,2,11} = 12, \quad f_{7,2,13} = 1, \\
f_{7,2,15} &= -12, \quad f_{7,2,17} = -27, \quad f_{7,2,19} = -44, \quad f_{7,2,21} = -63, \quad f_{7,2,23} = 63, \quad f_{7,2,25} = 40, \\
f_{3,5,1} &= -5, \quad f_{3,5,2} = -3, \quad f_{3,5,6} = -6, \quad f_{3,5,7} = -5, \quad f_{3,5,8} = 3, \quad f_{3,5,9} = -9, \\
f_{7,5,1} &= -29, \quad f_{7,5,2} = -19, \quad f_{7,5,6} = -30, \quad f_{7,5,7} = -21, \quad f_{7,5,8} = 22, \quad f_{7,5,9} = -33, \\
f_{11,7,1} &= -86, \quad f_{11,7,2} = -103, \quad f_{11,7,3} = -51, \quad f_{11,7,8} = -87, \quad f_{11,7,9} = -105, \quad f_{11,7,10} = -54.
\end{align*}
\]

**Conjecture 7.2.** Let \( p \) be an odd prime and \( m, r \) two positive integers with \( r < m \) and \( p \equiv \pm 1 \pmod{m} \). Then

\[
f_{p,m,r} = \frac{r(m-r)(1-p^2)}{2m}.
\]

In particular, the congruences in Corollaries 3.2 and 3.3 modulo \([p]^2\) are still true.

Note that Conjecture 7.1 is a complete \( q \)-analogue of (1.7) and Theorem 3.5 is a weak \( q \)-analogue of (1.6), of which we speculate the following true \( q \)-analogue.

**Conjecture 7.3.** Let \( p \) be an odd prime and \( m, r \) two positive integers with \( p \nmid m \). Then

\[
P_{p-1,m,r}(q,x) \equiv (-1)^{\frac{r}{2}}(-\frac{r}{m})_p P_{p-1,m,r}(q,-x) \pmod{[p]^2}.
\]

There are some similar congruences in the literature. For example, Van Hamme [23] proved the following conjecture of Beukers [6]:

\[
\sum_{k=0}^{p-1} \left( \frac{2k}{k} \right)^3 \equiv 0 \pmod{p^2}, \quad \text{for } p \equiv 3 \pmod{4}. \quad (7.1)
\]

In the light of our results, a nice \( q \)-analogue of (7.1) (at least in the modulus \( p \) case) would be desirable.

**References**

[1] S. Ahlgren and K. Ono, A Gaussian hypergeometric series evaluation and Apéry number congruences, J. Rine Angew. Math. 518 (2000), 187–212.

[2] G.E. Andrews, The Theory of Partitions, Cambridge University Press, Cambridge, 1998.

[3] G.E. Andrews, \( q \)-Analogs of the binomial coefficient congruences of Babbage, Wolstenholme and Glaisher, Discrete Math. 204 (1999), 15–25.
[4] G.E. Andrews, On the $q$-analog of Kummer's theorem and applications, Duke Math. J. 40 (1973) 525–528.
[5] G.E. Andrews, Applications of basic hypergeometric functions, SIAM Rev. 16 (1974) 441–484.
[6] F. Beukers, Another congruence for the Apéry numbers, J. Number Theory 25 (1987), 201–210.
[7] G. Gasper and M. Rahman, Basic Hypergeometric Series, Second Edition, Encyclopedia of Mathematics and Its Applications, Vol. 96, Cambridge University Press, Cambridge, 2004.
[8] V.J.W. Guo and J. Zeng, Some congruences involving central $q$-binomial coefficients, Adv. Appl. Math. 45 (2010), 303-316.
[9] D. McCarthy and R. Osburn, A $p$-adic analogue of a formula of Ramanujan, Arch. Math. 91 (2008), 492–504.
[10] E. Mortenson, A supercongruence conjecture of Rodriguez-Villegas for a certain truncated hypergeometric function, J. Number Theory 99 (2003), 139–147.
[11] E. Mortenson, Supercongruences between truncated $_2F_1$ hypergeometric functions and their Gaussian analogs, Trans. Amer. Math. Soc. 355 (2003), 987–1007.
[12] H. Pan, A $q$-analogue of Lehmer’s congruence, Acta Arith. 128 (2007), 303–318.
[13] L.-L. Shi and H. Pan, A $q$-analogue of Wolstenholme’s harmonic series congruence, Amer. Math. Monthly 114 (2007), 529–531.
[14] F. Rodriguez-Villegas, Hypergeometric families of Calabi-Yau manifolds, in: Calabi-Yau Varieties and Mirror Symmetry (Toronto, ON, 2001), Fields Inst. Commun., 38, Amer. Math. Soc., Providence, RI, 2003, pp. 223–231.
[15] Z.-H. Sun, Congruences concerning Legendre polynomials, Proc. Amer. Math. Soc. 139 (2011), 1915–1929.
[16] Z.-H. Sun, Generalized Legendre polynomials and related congruences modulo $p^2$, preprint, 2012, arXiv:1101.5386.
[17] Z.-W. Sun, Super congruences and Euler numbers, Sci. China Math. 54 (2011), 2509–2535.
[18] Z.-W. Sun, On sums involving products of three binomial coefficients, Acta Arith. 156 (2012) 123–141.
[19] A. Straub, A $q$-analog of Ljunggren’s binomial congruence, in: 23rd International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2011), Discrete Math. Theor. Comput. Sci. Proc., AO, Assoc. Discrete Math. Theor. Comput. Sci., Nancy, 2011, pp. 897–902.
[20] R. Tauraso, An elementary proof of a Rodriguez-Villegas supercongruence, preprint, 2009, arXiv:0911.4261v1.
[21] R. Tauraso, Some $q$-analogs of congruences for central binomial sums, preprint, 2012, arXiv:1201.6152.
[22] W. Van Assche, Little $q$-Legendre polynomials and irrationality of certain Lambert series, Ramanujan J. 5 (2001), 295–310.
[23] L. Van Hamme, Some conjectures concerning partial sums of generalized hypergeometric series, in: $p$-adic Functional Analysis (Nijmegen, 1996), Lecture Notes in Pure and Appl. Math., Vol. 192, Dekker, 1997, pp. 223–236.