Information and dimensionality of anisotropic random geometric graphs

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Abstract

This paper deals with the problem of detecting non-isotropic high-dimensional geometric structure in random graphs. Namely, we study a model of a random geometric graph in which vertices correspond to points generated randomly and independently from a non-isotropic $d$-dimensional Gaussian distribution, and two vertices are connected if the distance between them is smaller than some pre-specified threshold. We derive new notions of dimensionality which depend upon the eigenvalues of the covariance of the Gaussian distribution. If $\alpha$ denotes the vector of eigenvalues, and $n$ is the number of vertices, then the quantities $\left(\frac{\|\alpha\|_2}{\|\alpha\|_3}\right)^6/n^3$ and $\left(\frac{\|\alpha\|_2}{\|\alpha\|_4}\right)^4/n^3$ determine upper and lower bounds for the possibility of detection. This generalizes a recent result by Bubeck, Ding, Rácz and the first named author from [BDER15] which shows that the quantity $d/n^3$ determines the boundary of detection for isotropic geometry. Our methods involve Fourier analysis and the theory of characteristic functions to investigate the underlying probabilities of the model. The proof of the lower bound uses information theoretic tools, based on the method presented in [BG15].

1 Introduction

This study continues a line of work initiated by Bubeck, Ding, Rácz and the first named author [BDER15], in which the problem of detecting geometric structure in large graphs was studied. In other words, given a large graph one is interested in determining whether or not it was generated using a latent geometric structure. The main contribution of this study is a generalization of the results to the anisotropic case.

Extracting information from large graphs is an extensively studied statistical task. In many cases, a given network, or graph, reflects some underlying structure; for example, a biological neuronal network is likely to reflect certain characteristics of its functionality such as physical location and cell structure. The objective of this paper is thus the detection of such an underlying geometric structure.

As a motivating example, consider the graph representing a large social network. It may be assumed that each node (or user) is described by a set of numerical parameters representing its properties (such as geographical location, age, political association, interests etc). It is plausible to assume that two nodes are more likely to be connected when their two respective points in parameter space are more correlated. Adopting this assumption, the nodes of such a graph may
be thought of as points in a Euclidean space, with links appearing between two nodes when their distance is small enough. A natural question in this context would be: What can be said about the geometric structure by inspection of the graph itself? Specifically, can one distinguish between such a graph and a graph with no underlying geometric structure?

In Statistical terms, given a graph $G$ on $n$ vertices, our null hypothesis is that $G$ is an instance of the standard Erdős-Rényi random graph $G(n, p)$ [ER60], where the presence of each edge is determined independently, with probability $p$:

$$H_0 : G \sim G(n, p).$$

On the other hand, for the alternative, we consider the so-called random geometric graph. In this model each vertex is a point in some metric space and an edge is present between two points if the distance between them is smaller than some predefined threshold. Perhaps the most well-studied setting of this model is the isotropic Euclidean model, where the vertices are generated uniformly on the $d$-dimensional sphere or simply from the standard normal $d$-dimensional distribution. However, it seems that this model is too simplistic to reflect real world social networks. One particular problem, which we intend to tackle in this study, is the isotropicity assumption, which amounts to the fact that all of the properties associated with a node have the same significance in determining the network structure. It is clear that some parameters, such as geographic location, can be more significant than others. We therefore propose to extend this model to a non-isotropic setting. Roughly speaking, we replace the sphere with an ellipsoid; Instead of generating vertices from $N(0, I_n)$, they will be generated from $N(0, D_{\alpha})$ for some diagonal matrix $D_{\alpha}$ with non-negative entries. We denote the model by $G(n, p, \alpha)$ where $p$ is the probability of an edge appearing, and $D_{\alpha} = \text{diag}(\alpha) \in \mathbb{R}^d$. Formally, let $X_1, ..., X_n$ be i.i.d points generated from $N(0, D_{\alpha})$. In $G(n, p, \alpha)$ vertices correspond to $X_1, ..., X_n$ and two distinct vertices are joined by an edge if and only if $\langle X_i, X_j \rangle \geq t_{p, \alpha}$, where $t_{p, \alpha}$ is the unique number satisfying $\mathbb{P}(\langle X_1, X_2 \rangle \geq t_{p, \alpha}) = p$. Our alternative hypothesis is thus

$$H_1 : G \sim G(n, p, \alpha).$$

In this paper, we will focus on the high-dimensional regime of the problem. Namely, we assume that the dimension and covariance matrix can depend on $n$. This point of view becomes highly relevant when considering recent developments in data sciences, where big data and high-dimensional feature spaces are becoming more prevalent. We will focus on the dense regime, where $p$ is a constant independent of $n$ and $\alpha$.

### 1.1 Previous work

This paper can be seen a direct follow-up of [BDER15], which as noted above deals with the isotropic model of $G(n, p, d)$ in which $D_{\alpha} = I_d$. In the dense regime, it was shown that the total variation between the models depends asymptotically on the ratio $\frac{d}{n^3}$. The dependence is such that if $d >> n^3$, then $G(n, p, d)$ converges in total variation to $G(n, p)$. Conversely, on the other hand, if $d << n^3$ the total variation converges to 1.

Our starting point is thus the result of [BDER15] stated as follows:

**Theorem 1.** (a) Let $p \in (0, 1)$ be fixed and assume that $d/n^3 \to 0$. Then,

$$\text{TV}(G(n, p), G(n, p, d)) \to 1.$$
(b) Furthermore, if $d/n^3 \to \infty$ then

$$TV(G(n, p), G(n, p, d)) \to 0.$$ 

One of the fundamental differences between $G(n, p)$ and $G(n, p, d)$ is a consequence of the triangle inequality. That is, if two points $u$ and $v$ are both close to a point $w$, then $u$ and $v$ cannot be too far apart. This roughly means that if both $u$ and $v$ are connected to $w$, then there is an increased probability of $u$ being connected to $v$, unlike the case of the Erdős-Rényi graph where there is no dependence between the edges. Thus, counting the number of triangles in a graph seems to be a natural test to uncover geometric structure.

The idea of using triangles was extended in [BDER15] and a variant was proposed: the signed triangle. This statistic was successfully used to completely characterize the asymptotics of $TV(G(n, p), G(n, p, d))$ in the isotropic case. To understand the idea behind signed triangles, we first note that if $A$ is the adjacency matrix of $G$ then the number of triangles in $G$ is given by $\text{Tr}(A^3)$. The "number" of signed triangles is then given by $\text{Tr}((A - p\mathbf{1})^3)$ where $\mathbf{1}$ is the matrix whose entries are all equal to 1. It turns out that the variance of signed triangles is significantly smaller than the corresponding quantity for regular triangles.

The methods used in [BDER15] relied heavily on the symmetries of the sphere. As mentioned, our goal is to generalize this to the non-isotropic case, which requires us to apply different methods. The dimension $d$ of the isotropic space arises as a natural parameter when discussing the underlying probabilities of Theorem 1. Clearly, however, when different coordinates of the space have different scales, the dimension by itself has little meaning. For example, consider a $d$-dimensional ellipsoid with one axis being large and the rest being much smaller. This ellipsoid behaves more like a 1-dimensional sphere rather than a $d$-dimensional one, in the sense mentioned above. It would stand to reason the more anisotropic the ellipsoid is, the smaller its effective dimension would be.

### 1.2 Main results and ideas

In accordance to the above, our first task is to find a suitable notion of dimensionality for our model. For any $v \in \mathbb{R}^d$ and $q > 1$, denote the $q$-norm of $\mathbb{R}^d$ as $\|v\|_q = \left(\sum_{i=1}^d v_i^q\right)^{\frac{1}{q}}$. We derive the quantity $\left(\frac{\|\alpha\|_2}{\|\alpha\|_3}\right)^6$ as the new notion of the dimension, where $\alpha$ parametrizes the ellipsoid, and is considered as a $d$-dimensional vector. We note that, in the isotropic case, this quantity reduces to $d$ which also maximizes this expression.

This notion of dimension allows us to tackle the main objective of this paper. Studying the total variation, $TV(G(n, p), G(n, p, \alpha))$. Considering what we know about the isotropic case our question becomes: What conditions are required from $\alpha$, so that the total variation remains bounded away from 0? The following theorem provides a sufficient condition on $\alpha$ as well as a necessary one:

**Theorem 2.** (a) Let $p \in (0, 1)$ be fixed and assume that $\left(\frac{\|\alpha\|_2}{\|\alpha\|_3}\right)^6 / n^3 \to 0$. Then,

$$TV(G(n, p), G(n, p, \alpha)) \to 1.$$
Theorem 2. The main idea is to use Pinsker’s inequality to bound the total variation distance by the respective relative entropy. Thus we are interested in
\[
\text{Ent} [W(n, \alpha) | |M(n)|],
\]

Theorem 2(b) will then follow from the next result:

(b) Furthermore, if \( \left( \frac{\|\alpha\|_2}{\|\alpha\|_4} \right)^4 \rightarrow n^3 \rightarrow \infty \), then
\[
\text{TV}(G(n, p), G(n, p, \alpha)) \rightarrow 0.
\]

Note that there is a gap between the bounds 2(a) and 2(b) (for example, if \( \alpha_i \sim \frac{1}{\sqrt{n}} \), then \( \left( \frac{\|\alpha_i\|_2}{\|\alpha_i\|_4} \right)^6 \) is order of \( \frac{d}{n^2(d)} \), while \( \left( \frac{\|\alpha_i\|_2}{\|\alpha_i\|_4} \right)^4 \) is about \( d^2 \)). We conjecture that the bound 2(a) is tight:

Conjecture 1. Let \( p \in (0, 1) \) be fixed and assume that \( \left( \frac{\|\alpha\|_2}{\|\alpha\|_4} \right)^6 \rightarrow n^3 \rightarrow \infty \). Then
\[
\text{TV}(G(n, p), G(n, p, \alpha)) \rightarrow 0
\]

In the following we describe some of the ideas used to prove Theorem 2.

As discussed, the main idea underlying this work has to do with counting triangles. Given a graph \( G \) we denote by \( T(G) \) the number of triangles in the graph. It is easy to verify that \( \mathbb{E} T(G(n, p)) = (\begin{pmatrix} n \end{pmatrix}) p^3 \) and \( \text{Var}(T(G(n, p))) \) is of order \( n^4 \). In the isotropic case, standard calculations show that the expected number of triangles in \( G(n, p, d) \) is boosted by a factor proportional to \( 1 + \frac{1}{\sqrt{n}} \). The first difficulty that arises is to find a precise estimate for the probability increment in the non-isotropic case. In this case, we show that there is a constant \( \delta_p \) depending only on \( p \) such that \( \mathbb{E} T(G(n, p, \alpha)) \geq \left( \begin{pmatrix} n \end{pmatrix} \right) p^3 \left( 1 + \delta_p \left( \frac{\|\alpha\|_2}{\|\alpha\|_4} \right)^3 \right) \). This would imply a non-negligible total variation distance as long as \( \left( \begin{pmatrix} n \end{pmatrix} \right) \left( \frac{\|\alpha\|_4}{\|\alpha\|_2} \right)^3 \) is bigger than the standard deviation of \( T(G(n, p)) \). We incorporate the idea of using signed triangles which attain a similar difference between expected values but have a smaller variance. The number of signed triangles is defined as:
\[
\tau(G) = \sum_{\{i,j,k\} \in \begin{pmatrix} n \end{pmatrix}} (A_{i,j} - p)(A_{i,k} - p)(A_{j,k} - p),
\]
where \( A \) is the adjacency matrix of \( G \), which is proportional to \( \text{Tr}((A - p)^3) \). It was shown that \( \text{Var}(\tau(G(n, p))) \) is only of order \( n^3 \). Resolving the value of \( \text{Var}(\tau(G(n, p, \alpha))) \) leads to the following result (which implies Theorem 2(a)):

Theorem 3. Let \( p \in (0, 1) \) be fixed and assume that \( \left( \frac{\|\alpha\|_2}{\|\alpha\|_4} \right)^6 \rightarrow n^3 \rightarrow 0 \). Then
\[
\text{TV}(\tau(G(n, p)), \tau(G(n, p, \alpha))) \rightarrow 1
\]

To prove Theorem 2(b) we may view the random graph \( G(n, p, \alpha) \) as a measurable function of a random \( n \times n \) matrix \( W(n, \alpha) \) with entries proportional to \( \langle \gamma_i, \gamma_j \rangle \) where \( \gamma_i \) are drawn i.i.d from \( N(0, D_{\alpha}) \) and \( D_{\alpha} = \text{diag}(\alpha) \). Similarly, \( G(n, p) \) can be viewed as a function of an \( n \times n \) GOE random matrix denoted by \( M(n) \). In [BDER15] Theorem 1(b) was proven using direct calculations on the densities of the involved distributions. However, in our case, no simple formula exists, which makes their method inapplicable. The premise is instead proven using information theoretic tools, adopting ideas from [BG15]. The main idea is to use Pinsker’s inequality to bound the total variation distance by the respective relative entropy. Thus we are interested in
\[
\text{Ent} [W(n, \alpha) | |M(n)|].
\]
Theorem 4. Let \( p \in (0, 1) \) be fixed and assume that \( \left( \frac{\|\alpha\|_2}{\|\alpha\|_4} \right)^4 / n^3 \to \infty \). Then

\[
\text{Ent} \left[ W(n, \alpha) || M(n) \right] \to 0.
\]

We suspect, as stated in Conjecture 1, that Theorem 2(b) does not give a tight characterization of the lower bound. Indeed, in the dense regime of the isotropic case, signed triangles act as an optimal statistic. It would seem to reason that deforming the sphere shouldn’t affect the utility of such a local tool.

2 Preliminaries

We work in \( \mathbb{R}^n \), equipped with the standard Euclidean structure \( \langle \cdot, \cdot \rangle \). For \( q \geq 1 \), we denote \( \| \cdot \|_q \) the corresponding \( q \)-norm. That is, for \( (v_1, \ldots, v_n) = v \in \mathbb{R}^n, \| v \|_q = \left( \sum_{i=1}^{n} v_i^q \right)^{\frac{1}{q}} \). If \( \alpha = \{ \alpha_i \}_{i=1}^{d} \) is a multi-set with elements from \( \mathbb{R} \), we adopt the same notation for \( \| \alpha \|_q \). We abbreviate \( \| \cdot \| := \| \cdot \|_2 \), the usual Euclidean norm and denote by \( S_{n-1} \) the unit sphere under this norm. In our proofs, we will allow ourselves to use the letters \( c, C, c' , C' , c_1, C_1 \), etc. to denote absolute positive constants whose values may change between appearances. The letters \( x, y, z \) will usually denote spatial variables while \( a, b, c \) will denote the corresponding frequencies in the Fourier domain. The letters \( X, Y, Z \) will usually be used as random variables and vectors.

Let \( X \) be a real valued random variable. The characteristic function of \( X \) is a function \( \varphi : \mathbb{R} \to \mathbb{R} \), given by

\[
\varphi_X(t) = \mathbb{E}[e^{itX}].
\]

More generally, if \( X \) is an \( n \)-dimensional random vector, then the characteristic function of \( X \) is a function \( \varphi : \mathbb{R}^n \to \mathbb{R} \) given by

\[
\varphi_X(t) = \mathbb{E}[e^{i\langle t, X \rangle}].
\]

By elementary Fourier analysis, one can use the characteristic function to recover the distribution, whenever the random vector is integrable. We will be interested in the specific case where the dimension of \( X \) is 3. Assume \( X = (X_1, X_2, X_3) \) has a density, denoted by \( f \), a characteristic function, denoted by \( \varphi \) and cumulative distribution function

\[
F(t_1, t_2, t_3) = \mathbb{P}(X_1 > t_1, X_2 > t_2, X_3 > t_3),
\]

with marginals onto the first 1 or 2 coordinates denoted as \( F(t_1, t_2) \) and \( F(t_1) \) respectively. Then e.g., [She91, Theorem 5] states that

\[
\frac{1}{\pi^3} \int_{\mathbb{R}^4} \varphi(a,b,c)e^{-i(ata_1+bt_2+ct_3)} \frac{dadbdc}{abc} = \left( 1 \right)
\]

\[
8F(t_1, t_2, t_3) - 4(F(t_1, t_2) + F(t_2, t_3) + F(t_1, t_3)) + 2(F(t_1) + F(t_2) + F(t_3)) - 1,
\]

where the integral is taken as a Cauchy principal value; In \( \mathbb{R}^3 \), the Cauchy principal value of a function \( g \), which we henceforth denote by \( \int_{\mathbb{R}^3} g \), is defined as

\[
\int_{0}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} \Delta_a \Delta_b \Delta_c g(a,b,c) dadbdc,
\]

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where $\Delta_ag(a, b, c) := g(a, b, c) + g(-a, b, c)$ and likewise for $b, c$. In the following, for multivariate functions, we interpret the definition of an odd (resp. even) function in the following sense: $g$ is odd (resp. even) if it is antisymmetric (resp. symmetric) under change of sign of any coordinate, while keeping the values of the rest of the coordinates intact. We note that the principal value of an odd function vanishes, and if $g$ is integrable then $\int g = \int g$. Furthermore, by denoting
\[
\text{sgn}_{(t_1,t_2,t_3)}(x, y, z) = \text{sgn}(x-t_1)\text{sgn}(y-t_2)\text{sgn}(z-t_3),
\]
a simple calculation shows the following equality:
\[
\int_{\mathbb{R}^3} f(x, y, z) \cdot \text{sgn}_{(t_1,t_2,t_3)}(x, y, z) dx dy dz =
\]
\[
8F(t_1, t_2, t_3) - 4(F(t_1, t_2) + F(t_2, t_3) + F(t_1, t_3)) + 2(F(t_1) + F(t_2) + F(t_3)) - 1.
\]
Since the Fourier transform is an isometry we have that
\[
\int_{\mathbb{R}^3} f \cdot \text{sgn}_{(t_1,t_2,t_3)} = \frac{1}{\pi^3} \int_{\mathbb{R}^3} \varphi \cdot \hat{\text{sgn}}_{(t_1,t_2,t_3)},
\]
(2)
where $\hat{\text{sgn}}_{(t_1,t_2,t_3)}$ is the Fourier transform of $\text{sgn}_{(t_1,t_2,t_3)}$, when considered as a tempered distribution. Putting all of the above together yields
\[
\text{sgn}_{(t_1,t_2,t_3)}(a, b, c) = e^{-i(ata_1 + bt_2 + ct_3)} ab^c.
\]
(3)
For a positive semi-definite $n \times n$ matrix $\Sigma$, we denote by $\mathcal{N}(0, \Sigma)$ the law of the centered Gaussian distribution with covariance $\Sigma$. If $X \sim \mathcal{N}(0, \Sigma)$ then $X^T X$ has the law $\mathcal{W}_n(\Sigma, 1)$ of the Wishart distribution with 1 degree of freedom. The characteristic function of $X^T X$ is known (see [Eat07]) and given by
\[
\Theta \rightarrow \text{det}(I - 2i\Theta \Sigma)^{-\frac{1}{2}}.
\]
(4)
If $Z$ is distributed as a standard Gaussian, then $Z^2$ has the $\chi^2$ distribution with 1 degree of freedom. For such a distribution, we have $\mathbb{E}[\chi^2] = 1$ and $\text{Var}(\chi^2) = 2$. The $\chi^2$ distribution has a sub-exponential tail which may be bounded using a Bernstein’s type inequality (Ver12), in the following way. If $\{\chi^2\}_{i=1}^n$ are independent $\chi^2$ random variables, then for every $(\upsilon_1, ..., \upsilon_n) = v \in \mathbb{R}^n$ and every $t > 0$
\[
\mathbb{P}\left( \left| \sum \chi_i^2 - \sum \upsilon_i \right| \geq t \right) \leq 2 \exp \left( -\frac{t}{2 \|v\|_\infty} \right).
\]
(5)
Let $X_1, ..., X_n$ be independent random variables with 0 mean and variance $\mathbb{E}[X_i^2] = \sigma_i^2$. Define
\[
\sigma_n^2 = \sum_{i=1}^n \sigma_i^2 \quad \text{and} \quad S_n = \sum_{i=1}^n \frac{X_i}{\sqrt{s_n^2}}.
\]
Under appropriate regularity conditions the central limit theorem states that $S_n$ converges in distribution to $\mathcal{N}(0, 1)$, the standard normal distribution.

Berry-ESseen’s inequality [Pet95] quantifies this convergence. Suppose that the absolute third moments of $X_i$ exist and $\mathbb{E}[|X_i|^3] = \rho_i$. If we denote by $Z$ a standard Gaussian and define $S_n$ as above then, for every $x \in \mathbb{R}$,
\[
|\mathbb{P}(S_n < x) - \mathbb{P}(Z < x)| \leq \frac{\sum_i \rho_i}{S_n^3}.
\]
(6)
This can be generalized to higher dimensions, as found in [Ben05, Theorem 1.1]. In that case assume $X_1, \ldots, X_n$ are independent random vectors in $\mathbb{R}^d$ and $S_n = \sum_{i=1}^n X_i$ has covariance $\Sigma^2$. Assume that $\Sigma$ is invertible and denote $\mathbb{E}[\|\Sigma^{-1}X_i\|^3] = \rho_i$. If $Z_d$ is a $d$-dimensional standard Gaussian vector, then there exists a universal constant $C_{be} > 0$, such that for any convex set $A$:

$$|\mathbb{P}(\Sigma^{-1}S_n \in A) - \mathbb{P}(Z_n \in A)| \leq C_{be}d^4 \sum_i \rho_i.$$  \hfill (7)

For a random vector $X$ on $\mathbb{R}^n$ with density $f$, the differential entropy of $X$ is defined

$$\text{Ent}[X] = -\int_{\mathbb{R}^n} f(x) \ln(f(x)) dx.$$ 

If $Y$ is another random vector with density $g$, the relative entropy of $X$ with respect to $Y$ is

$$\text{Ent}[X||Y] = \int_{\mathbb{R}^n} f(x) \ln\left(\frac{f(x)}{g(x)}\right) dx.$$ 

Pinsker’s inequality connects between the relative entropy and the total variation distance,

$$\text{TV}(X, Y) \leq \sqrt{\frac{1}{2} \text{Ent}[X||Y]}.$$  \hfill (8)

The chain rule for relative entropy states that for any random vectors $X_1, X_2, Y_1, Y_2$,

$$\text{Ent}[(X_1, X_2)|| (Y_1, Y_2)] = \text{Ent}[X_1||Y_1] + \mathbb{E}_{x \sim \lambda_1} \text{Ent}[X_2|X_1 = x||Y_2|Y_1 = x],$$  \hfill (9)

where $\lambda_1$ is the marginal of $X_1$, and $X_2|X_1 = x$ is the distribution of $X_2$ conditioned on the event $X_1 = x$ (similarly for $Y_2|Y_1 = x$).

### 3 Estimates for a triangle in a random geometric graph

In this section we derive a lower bound for the probability that an induced subgraph, of size 3, of a random geometric graph forms a triangle. This calculation is instrumental for the derivation of Theorem 2(a). Using the notation of the introduction, let $X_1, X_2, X_3 \sim \mathcal{N}(0, D_\alpha)$ be independent normal random vectors with coordinates $X_1^i, X_2^i, X_3^i$ for $1 \leq i \leq d$. We denote by $f$ the joint density of $(\langle X_1, X_2 \rangle, \langle X_1, X_3 \rangle, \langle X_2, X_3 \rangle)$. Consider the event

$$E_p = \{\langle X_1, X_2 \rangle \geq t_{p,\alpha}, \langle X_1, X_3 \rangle \geq t_{p,\alpha}, \langle X_2, X_3 \rangle \geq t_{p,\alpha}\},$$

that the corresponding vertices form a triangle in $G(n, p, \alpha)$. The main result of this section is the following theorem.

**Theorem 5.** Let $p \in (0, 1)$ and assume $\|\alpha\|_\infty = 1$. One has

$$p^3 + \Delta \left(\frac{\|\alpha\|_3}{\|\alpha\|_2}\right)^3 \geq \mathbb{P}(E_p) \geq p^3 + \delta_p \left(\frac{\|\alpha\|_3}{\|\alpha\|_2}\right)^3$$

whenever $\|\alpha\|_2 > c_p$, for constants $\Delta, \delta_p, c_p > 0$ which depend only on $p$. 

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3.1 Lower bound; the case $p = \frac{1}{2}$

It will be instructive to begin the discussion with the (easier) case $p = \frac{1}{2}$, in which $t_{p,\alpha} = 0$. We are thus interested in the probability that $\langle X_1, X_2 \rangle, \langle X_1, X_3 \rangle, \langle X_2, X_3 \rangle > 0$. Note that the triplet $(\langle X_1, X_2 \rangle, \langle X_1, X_3 \rangle, \langle X_2, X_3 \rangle)$ can be realized as a linear combination of upper off-diagonal elements taken from $d$ independent 3-dimensional Wishart random matrices (see below for an elaborated explanation). Unfortunately, there is no known closed expression for the density of such a distribution. The following lemma utilizes the characteristic function of the joint distribution to derive a closed expression for the desired probability.

**Lemma 1.**

$$
P(E_{\frac{1}{2}}) = \frac{1}{8} + \int_{\mathbb{R}^3} \frac{i}{8abc\pi^3} \left( \prod_i (1 + \alpha_i^2(a^2 + b^2 + c^2) + 2\alpha_i abc)^{-\frac{1}{2}} \right) dadbdc. \tag{10}
$$

**Proof.** Consider the event $\{\langle X_1, X_2 \rangle > 0, \langle X_1, X_3 \rangle < 0, \langle X_2, X_3 \rangle < 0\}$. The map $(x, y, z) \mapsto (x, -y, -z)$ is measure preserving by the symmetry of $X_3$. Thus,

$$
P(\{\langle X_1, X_2 \rangle > 0, \langle X_1, X_3 \rangle < 0, \langle X_2, X_3 \rangle < 0\})
= P(\{\langle X_1, X_2 \rangle > 0, \langle X_1, X_3 \rangle > 0, \langle X_2, X_3 \rangle > 0\}).
$$

By the same argument,

$$
P(\{\langle X_1, X_2 \rangle > 0, \langle X_1, X_3 \rangle > 0, \langle X_2, X_3 \rangle < 0\})
= P(\{\langle X_1, X_2 \rangle < 0, \langle X_1, X_3 \rangle < 0, \langle X_2, X_3 \rangle < 0\}).
$$

We denote the event on the right side by $P(I_{\frac{1}{2}})$, the probability of an induced independent set on 3 vertices.

From the above observation, it is clear that $4 \left( P(E_{\frac{1}{2}}) + P(I_{\frac{1}{2}}) \right) = 1$. Also, we may note that $\int_{\mathbb{R}^3} \text{sgn}(xyz) \cdot f(x, y, z) \, dx \, dy \, dz = 4 \left( P(E_{\frac{1}{2}}) - P(I_{\frac{1}{2}}) \right)$. Combining the two equalities yields $P(E_{\frac{1}{2}}) = \frac{1}{8} + \frac{1}{8} \int_{\mathbb{R}^3} \text{sgn}(xyz) \cdot f(x, y, z) \, dx \, dy \, dz$. As noted, no closed expression for $f$ is known, so the calculation of the above integral cannot be carried out in a straightforward manner. Instead, (2) allows us to rewrite the integral as

$$
\int_{\mathbb{R}^3} \text{sgn}(xyz) \cdot f(x, y, z) \, dx \, dy \, dz = \frac{1}{8} \int_{\mathbb{R}^3} \widehat{\text{sgn}}(abc) \cdot \varphi(a, b, c) \, dadbdc,
$$

where $\varphi$ is the characteristic function of $f$, and $\widehat{\text{sgn}}$ is the Fourier transform of $\text{sgn}$ as in (3).

Thus, we are required to calculate $\varphi(a, b, c)$. Consider three independent normal random variables, $X, Y, Z$, with mean 0 and variance $\sigma^2$, the characteristic function of $(XY, XZ, YZ)$ is defined by $(a, b, c) \rightarrow E[\exp(i(a \cdot XY + b \cdot XZ + c \cdot YZ))]$. We have that

$$
a \cdot XY + b \cdot XZ + c \cdot YZ = \text{Tr} \left( \begin{bmatrix} 0 & a & b \\ a & 0 & c \\ b & c & 0 \end{bmatrix} \cdot \begin{bmatrix} X^2 & XY & XZ \\ XY & Y^2 & YZ \\ XZ & YZ & Z^2 \end{bmatrix} \right).$$
If we consider the Wishart distribution $\mathcal{W}_3(\Sigma, 1)$, where $\Sigma$ is a $\sigma^2$ scalar matrix, we note that the above function equals the characteristic function of $\mathcal{W}_3(\Sigma, 1)$ on the matrix

$$
\begin{bmatrix}
0 & \frac{a}{i} & \frac{b}{i} \\
\frac{a}{i} & 0 & \frac{c}{i} \\
\frac{b}{i} & \frac{c}{i} & 0
\end{bmatrix}.
$$

Using the formula (4), this equals $\det \left( \begin{bmatrix}
1 & -i\sigma^2a & -i\sigma^2b \\
-i\sigma^2a & 1 & -i\sigma^2c \\
-i\sigma^2b & -i\sigma^2c & 1
\end{bmatrix} \right)^{-\frac{1}{2}}$, which may be written otherwise as $(1 + (\sigma^2)^2(a^2 + b^2 + c^2) + 2(\sigma^2)^3 abc i)^{-\frac{1}{2}}$.

By the convolution-multiplication theorem [Dur10, Theorem 3.3.2], the characteristic function of a sum of independent variables is the multiplication of their characteristic functions, it then follows that:

$$
\varphi(a, b, c) = \prod_{i=1}^{d} (1 + \alpha_i^2(a^2 + b^2 + c^2) + 2\alpha_i^3 abc i)^{-\frac{1}{2}},
$$

which results in:

$$
\int_{\mathbb{R}^3} \text{sgn} \cdot \varphi(a, b, c) \, dadbdc = \int_{\mathbb{R}^3} \frac{i}{abc} \prod_{i} (1 + \alpha_i^2(a^2 + b^2 + c^2) + 2\alpha_i^3 abc i)^{-\frac{1}{2}} \, dadbdc.
$$

This concludes the proof.

In view of the above, it suffices to estimate the integral in (10). The next result will be useful in the coming calculations.

**Lemma 2.** Let $n \geq 3$ and $\gamma = \{\gamma_i\}_{i=1}^{d}$, suppose that $\gamma_i \in [0, 1]$ for $1 \leq i \leq d$. Define

$$
I(T) = \int_{T}^{\infty} \frac{r^2 \, dr}{\sqrt{\prod_{i} (1 + \gamma_i^2 r^2)}}, \quad \forall T \geq 1,
$$

and denote $\|\gamma\|_2^2 = \sum_{i} \gamma_i^2$, then there exists constants $c_n, C_n > 0$, depending only on $n$, such that whenever $\|\gamma\|_2^2 > c_n$ we have that $I(T) \leq C_n \left( \frac{1}{\|\gamma\|_2} \right)^{\frac{1}{2}} \frac{1}{T^{n-3}}$.

**Proof.** Indeed, assume $\|\gamma\|_2^2 > n$. Note that necessarily $d \geq n$ in this case. Thus we can give a non trivial lower bound of $\prod_{i} (1 + \gamma_i^2 r^2)$ by considering the sum of all products of $n$ different elements of $\gamma$. That is

$$
\prod_{i} (1 + \gamma_i^2 r^2) \geq \left( \sum_{S \subset \gamma} \prod_{\gamma_j \in S} \gamma_j^2 \right) r^{2n}.
$$

We claim now that:

$$
\sum_{S \subset \gamma} \prod_{\gamma_j \in S} \gamma_j^2 \geq \frac{1}{n!} \prod_{k=0}^{n-1} (\|\gamma\|_2^2 - k) .
$$

(12)
To see that, we may rewrite
\[
\sum_{S \subset \gamma} \prod_{i \in S} \gamma_i^{2} = \frac{1}{n} \sum_{i} \gamma_i^{2} \sum_{S \subset \gamma \setminus \{\gamma_i\}} \prod_{j \in S} \gamma_j^{2},
\]
where we have counted each \( S \subset \gamma \) \( n \) times. But, \( \gamma_i \leq 1 \) for every \( 1 \leq i \leq d \), and so \( \|\gamma \|_2^2 > \|\gamma\|_2^2 - 1 \). (12) now follows by induction, since
\[
\frac{1}{n} \sum_{i} \gamma_i^{2} \sum_{S \subset \gamma \setminus \{\gamma_i\}} \prod_{j \in S} \gamma_j^{2} \geq \frac{1}{n} \sum_{i} \gamma_i^{2} \prod_{k=0}^{n-2} (\|\gamma\|_2^2 - k) = \frac{1}{n} \prod_{k=0}^{n-1} (\|\gamma\|_2^2 - k)
\]
If we further assume that \( \|\gamma\|_2^2 \geq 2n \), then \( \|\gamma\|_2^2 - k > \frac{1}{2} \|\gamma\|_2^2 \), for every \( 0 \leq k \leq n - 1 \). Plugging this into (12) produces
\[
\prod_{i} (1 + \gamma_i^{2} r^{2}) \geq \left( \frac{\|\gamma\|_2^2}{n!^2} \right)^n r^{2n},
\]
which implies
\[
I \leq \left( \frac{n!^2}{\|\gamma\|_2^2} \right)^{\frac{n}{2}} \int_{T} dr \frac{dr}{r^{n-2}} = \left( \frac{n!^2}{n - 3} \right)^{\frac{n}{2}} \frac{1}{T^{n-3}},
\]
as desired.

\[\square\]

Remark: The constants obtained in the above proof are far from optimal, but will suffice for our needs.

We will use the above result in order bound from below the integral in formula (10). For this, we will assume \( W.L.O.G. \) that the variances are normalized in the following way:
\[
\alpha_1 = 1 \text{ and } \alpha_i \in [0,1] \text{ for } 1 \leq i \leq d.
\]
(13)

We note that this normalization yields the following properties for \( n, m \in \mathbb{N} \), which we shall use freely:

- For every \( k > 0 \), \( \|\alpha\|_k^k \geq 1 \) and thus \( \left( \|\alpha\|_k^n \right)^n \leq \left( \|\alpha\|_k^m \right)^m \) when \( n \leq m \).
- \( \alpha_i^n \geq \alpha_i^m \) and \( \|\alpha\|_n^n \geq \|\alpha\|_m^m \) when \( n \leq m \).
- For any \( n > 2 \) and \( \varepsilon > 0 \) there exists \( c > 0 \) such that whenever \( \|\alpha\|_2^2 > c \) we have \( \left( \frac{\|\alpha\|_2^3}{\|\alpha\|_2} \right)^n < \varepsilon \).

Lemma 3. There exists a constant \( c_{1/2} > 0 \) such that whenever \( \|\alpha\|_2^2 > c_{1/2} \) then
\[
\int_{\mathbb{R}^3} \prod_{i} \left( 1 + \alpha_i^2 (a^2 + b^2 + c^2) + 2\alpha_i^3 abc \right)^{-\frac{1}{2}} \text{ dabc} \geq \frac{1}{8} \left( \frac{\|\alpha\|_3^3}{\|\alpha\|_2^3} \right).\]
Proof. First, we have the privilege of knowing the integral evaluates to some probability. Therefore, the principal value of its imaginary part must vanish. This becomes evident by noting that the imaginary part is an odd function. Thus, we are interested in:

\[
\text{Re} \left( \int_{\mathbb{R}^3} \frac{1}{abc} \prod_i \left( 1 + \alpha_i^2(a^2 + b^2 + c^2) + 2\alpha_i^3 abc \right)^{-\frac{1}{2}} \, dadbdc \right)
\]

First, we will prove that the following holds:

\[
\sin \left( \frac{1}{2} \sum_i \arctan \left( \frac{2\alpha_i^3 abc}{1 + \alpha_i^2(a^2 + b^2 + c^2)} \right) \right) \geq \sum_i \frac{\alpha_i^3 abc}{1 + \alpha_i^2(a^2 + b^2 + c^2)} - 3 \|\alpha\|_3^2 (abc)^2.
\]

Indeed, since \(\sin(x) \geq x - x^2\) we have that

\[
\sin \left( \frac{1}{2} \sum_i \arctan \left( \frac{2\alpha_i^3 abc}{1 + \alpha_i^2(a^2 + b^2 + c^2)} \right) \right) \\
\geq \frac{1}{2} \sum_i \arctan \left( \frac{2\alpha_i^3 abc}{1 + \alpha_i^2(a^2 + b^2 + c^2)} \right) - \frac{1}{4} \left( \sum_i \arctan \left( \frac{2\alpha_i^3 abc}{1 + \alpha_i^2(a^2 + b^2 + c^2)} \right) \right)^2 \\
\geq \frac{1}{2} \sum_i \arctan \left( \frac{2\alpha_i^3 abc}{1 + \alpha_i^2(a^2 + b^2 + c^2)} \right) - \left( \sum_i \alpha_i^3 \right)^2 (abc)^2.
\]
With the last inequality following from the fact that $\arctan^2(x) \leq x^2$. Now, using the inequality $\arctan(x) \geq x - x^2$ yields

\[
\frac{1}{2} \sum_i \arctan \left( \frac{2\alpha_i^3 abc}{1 + \alpha_i^2(a^2 + b^2 + c^2)} \right) \geq \sum_i \frac{\alpha_i^3 abc}{1 + \alpha_i^2(a^2 + b^2 + c^2)} - 2 \left( \sum_i \alpha_i^6 \right) (abc)^2 - \left( \sum_i \alpha_i^3 \right)^2 (abc)^2 
\]

When $(a, b, c) \in B_1$, then $\alpha_i^2(a^2 + b^2 + c^2) \leq \frac{\alpha_i^2 \parallel \alpha \parallel^2}{2} \leq 1$ and we have

\[
\sum_i \frac{\alpha_i^3 abc}{1 + \alpha_i^2(a^2 + b^2 + c^2)} - 3 \parallel \alpha \parallel^6 (abc)^2 \geq \frac{1}{2} \parallel \alpha \parallel^3 abc - 3 \parallel \alpha \parallel^6 (abc)^2. \tag{16}
\]

Next, we note that for $(a, b, c) \in B_1$:

\[
1 \geq \frac{1}{\prod_i \left[ \left( 1 + \alpha_i^2(a^2 + b^2 + c^2) \right)^2 + 4\alpha_i^6 a^2 b^2 c^2 \right]^{\frac{1}{4}}} \geq \frac{1}{\prod_i \left[ \left( 1 + \frac{\alpha_i^2}{\parallel \alpha \parallel^2} \right)^2 + \frac{4\alpha_i^6}{\parallel \alpha \parallel^4} \right]^{\frac{1}{4}}}
\]

Since, in (13), we’ve assumed that $\alpha_i \leq 1$ for each $i$ while $\sum_i \alpha_i^2 \geq 1$, we may now lower bound the above by $\frac{1}{\prod_i \left( 1 + \frac{\alpha_i^2}{\parallel \alpha \parallel^2} \right)^{\frac{1}{4}}}$, and since $\ln \left( \prod_i \left( 1 + \frac{7\alpha_i^2}{\parallel \alpha \parallel^2} \right) \right) \leq \frac{\pi}{\parallel \alpha \parallel^2} \sum_i \alpha_i^2 = 7$, we have

\[
\frac{1}{\prod_i \left( 1 + \frac{7\alpha_i^2}{\parallel \alpha \parallel^2} \right)^{\frac{1}{4}}} \geq e^{-2}. \tag{17}
\]

By combining (16) and (17) into (11) we may see for $(a, b, c) \in B_1$ the following holds:

\[
\text{Im} (\varphi(a, b, c)) \geq \left( \frac{1}{2} \parallel \alpha \parallel^3 abc - 3 \parallel \alpha \parallel^6 (abc)^2 \right) e^{-2} \text{ when } abc > 0.
\]

Also, it is not hard to see that $\text{Im}(\varphi)$ is an odd function, which makes $\frac{\text{Im}(\varphi(a, b, c))}{abc}$ even. Hence, if $H = \{(a, b, c) \in \mathbb{R}^3 \mid abc > 0\}$, then

\[
\int_{B_1} \frac{\text{Im}(\varphi(a, b, c))}{abc} \, dadbdc = 2 \int_{B_1 \cap H} \frac{\text{Im}(\varphi(a, b, c))}{abc} \, dadbdc.
\]

Finally, since the volume of $B_1$ is $\frac{4\pi}{3\parallel \alpha \parallel^2}$, and as long as $\parallel \alpha \parallel^2$ is large enough:

\[
\int_{B_1 \cap H} \frac{\text{Im}(\varphi(a, b, c))}{abc} \, dadbdc \geq \frac{1}{e^2} \int_{B_1 \cap H} \left( \frac{1}{2} \parallel \alpha \parallel^3 - 3 \parallel \alpha \parallel^6 abc \right) \, dadbdc 
\]

\[
\geq \frac{\pi}{3e^2} \left( \parallel \alpha \parallel^3 \right)^3 - \frac{3 \parallel \alpha \parallel^6}{e^2} \int_{B_1} |abc| \, dadbdc \geq \frac{\pi}{3e^2} \left( \frac{\parallel \alpha \parallel^3}{\parallel \alpha \parallel^2} \right)^3 - \frac{3}{e^2} \left( \frac{\parallel \alpha \parallel^3}{\parallel \alpha \parallel^2} \right)^6,
\]

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where the last inequality uses the fact
\[
\int_{B_1} |abc| \, dadbdc \leq \frac{1}{\|\alpha\|_2^6}.
\]
That is, by using the properties of the normalization (13), there is a constant \(c_1 > 0\) such that whenever \(\|\alpha\|_2^2 > c_1\) then
\[
\int_{B_1} \frac{\text{Im}(\phi(a, b, c))}{abc} \, dadbdc > \frac{1}{4} \left( \frac{\|\alpha\|_3}{\|\alpha\|_2} \right)^3.
\]

**Step 2 - The integrand is positive on** \(B_2 = \left\{ x \in \mathbb{R}^3 : \|x\|^2 \leq \frac{1}{\|\alpha\|_2^{11/12}} \right\}\), **the ball of radius** \(\frac{1}{\|\alpha\|_2^{11/12}}\).

We first note that when \(\sum_i \arctan\left(\frac{2\alpha_i^3 abc}{1+\alpha_i^2(a^2+b^2+c^2)}\right) < \pi\), the sign of \(\sin\left(\prod_i \left[1 + \alpha_i^2(a^2 + b^2 + c^2) + 2\alpha_i^3 abc\right]\right)\) is the same as that of \(abc\), which in turn implies that \(\frac{\text{Im}(\phi(a, b, c))}{abc} > 0\). Thus, it will be enough to show that whenever \((a, b, c) \in B_2\) and \(abc > 0\), we have that \(\sum_i \arctan\left(\frac{2\alpha_i^3 abc}{1+\alpha_i^2(a^2+b^2+c^2)}\right) < \pi\).

Indeed, for \((a, b, c) \in B_2\), \(abc < \left(\frac{\|\alpha\|_2^{-11/12}}{3}\right)^3 \leq \frac{1}{\|\alpha\|_2^3}\) which, under the assumption \(abc > 0\), results in
\[
\sum_i \arctan\left(\frac{2\alpha_i^3 abc}{1+\alpha_i^2(a^2+b^2+c^2)}\right) \leq \sum_i \frac{2\alpha_i^3 abc}{1+\alpha_i^2(a^2+b^2+c^2)} \leq \frac{2\|\alpha\|_3^3}{\|\alpha\|_2^2} < 2 < \pi,
\]
as desired.

**Step 3 - The absolute value of the integrand is negligible on the spherical shell** \(B \setminus B_2\) **where** \(B\) **is the unit ball in** \(\mathbb{R}^3\).

Observe that,
\[
\left|\frac{\sin\left(\frac{1}{2} \sum_i \arctan\left(\frac{2\alpha_i^3 abc}{1+\alpha_i^2(a^2+b^2+c^2)}\right)\right)}{abc}\right| \leq \frac{1}{2} \sum_i \frac{2\alpha_i^3 |abc|}{1+\alpha_i^2(a^2+b^2+c^2)} \leq \|\alpha\|_3^3.
\]

On the other hand, for \((a, b, c) \notin B_2\) we have that:
\[
\prod_i \left[1 + \alpha_i^2(a^2 + b^2 + c^2)\right]^{1/4} \leq \prod_i \left[1 + \alpha_i^2(a^2 + b^2 + c^2)\right]^{1/2} \leq \prod_i \left(1 + \frac{\alpha_i^2}{\|\alpha\|_2^{22/12}}\right)^{-1/2}.
\]
Using the elementary inequality $\ln(1 + x) \geq x - \frac{x^2}{2}$ for $x > 0$ yields:

$$\ln \left( \prod_{i} \left( 1 + \frac{\alpha_i^2}{\|\alpha\|_2^{22/12}} \right) \right) = \sum_{i} \ln \left( 1 + \frac{\alpha_i^2}{\|\alpha\|_2^{22/12}} \right) \geq \|\alpha\|_2^{2/12} - \|\alpha\|_4^{4/12} \geq \|\alpha\|_2^{2/12} - 1$$

where the last inequality follows from the fact that $\|\alpha\|_4^4 \leq \|\alpha\|_2^2$. In turn, this implies

$$\prod_{i} \left( 1 + \frac{\alpha_i^2}{\|\alpha\|_2^{22/12}} \right)^{-\frac{1}{2}} \leq e^{-\frac{1}{2}\|\alpha\|_2^{2/12} - 1}.$$

Finally, since the volume of the unit ball is $\frac{4\pi}{3}$, this gives that

$$\int_{B \setminus B_2} \left| \frac{\text{Im}(\varphi(a, b, c))}{abc} \right| \ dadbdc < \frac{4\pi}{3} \|\alpha\|_3^3 e^{-\|\alpha\|_2^{2/12} - 1}. \quad (19)$$

Consequently, there is a constant $c_2$ such that whenever $\|\alpha\|_2^2 > c_2$ then

$$\int_{B \setminus B_2} \left| \frac{\text{Im}(\varphi(a, b, c))}{abc} \right| \ dadbdc \leq \frac{1}{16} \left( \frac{\|\alpha\|_3}{\|\alpha\|_2} \right)^3.$$

**Step 4 - The integral is negligible outside of $B$.**

For $(a, b, c) \notin B$ we use (18) to achieve

$$\sin \left( \frac{1}{2} \sum_{i} \arctan \left( \frac{2\alpha_i^3 abc}{1 + \alpha_i^2(a^2 + b^2 + c^2)} \right) \right) < \frac{\|\alpha\|_3^3}{\prod_{i} \left( 1 + \alpha_i^2(a^2 + b^2 + c^2) \right)^{1/2}}.$$  

By passing to spherical coordinates we obtain:

$$\int_{\mathbb{R}^3 \setminus B} \frac{1}{\prod_{i} \left( 1 + \alpha_i^2(a^2 + b^2 + c^2) \right)^{1/2}} \ dadbdc = 4\pi \int_{1}^{\infty} \frac{r^2 \ dr}{\prod_{i} \left( 1 + \alpha_i^2 r^2 \right)^{1/2}}.$$

Applying Lemma 2 with $n = 4$ and $T = 1$, shows the existence of constants $C, c'_3 > 0$ such that whenever $\|\alpha\|_2^2 > c'_3$,

$$\int_{1}^{\infty} \frac{r^2 \ dr}{\prod_{i} \left( 1 + \alpha_i^2 r^2 \right)^{1/2}} \leq C \left( \frac{1}{\|\alpha\|_2^2} \right)^2 = C \frac{1}{\|\alpha\|_2^2}.$$

Thus, there exists a constant $c_3 = \max(c'_3, (16C)^2)$ such that whenever $\|\alpha\|_2^2 > c_3$ then

$$\int_{\mathbb{R}^3 \setminus B} \left| \frac{\text{Im}(\varphi(a, b, c))}{abc} \right| \ dadbdc \leq \frac{1}{16} \left( \frac{\|\alpha\|_3}{\|\alpha\|_2} \right)^3.$$
Final Step - \[ \int_{\mathbb{R}^3} \text{Im}(\varphi(a,b,c)) \, dadbdc \geq \frac{1}{8} \left( \frac{\|\alpha\|_3}{\|\alpha\|_2} \right)^3 \]

We may now decompose the integral

\[ \int_{\mathbb{R}^3} \text{Im}(\varphi(a,b,c)) \, dadbdc = \int_{B_2} \text{Im}(\varphi(a,b,c)) \, dadbdc + \int_{\mathbb{R}^3 \setminus B_2} \text{Im}(\varphi(a,b,c)) \, dadbdc \]

Letting \( \|\alpha\|_2^2 > \max(c_1, c_2, c_3) \) steps 1 and 2 show that

\[ \int_{B_2} \text{Im}(\varphi(a,b,c)) \, dadbdc \geq \frac{1}{4} \left( \frac{\|\alpha\|_3}{\|\alpha\|_2} \right)^3, \] (20)

while steps 2 and 3 show

\[ \int_{\mathbb{R}^3 \setminus B_2} \left| \text{Im}(\varphi(a,b,c)) \right| \, dadbdc \leq \frac{1}{8} \left( \frac{\|\alpha\|_3}{\|\alpha\|_2} \right)^3. \]

The required bound then follows by combining the above two estimates.

\[ \square \]

3.2 Arbitrary \( 0 < p < 1 \)

We now consider the case for arbitrary \( p \). First, we would like to derive bounds on the behavior of \( t_{p,\alpha} \), which constitute the following lemma.

**Lemma 4.** Let \( p \in (0, 1) \) and denote by \( \Phi \) the cumulative distribution function of the standard Gaussian. If \( t_p = \Phi^{-1}(p) \) then \( \|\alpha\|_2^2 t_p - k_p \leq t_{p,\alpha} \leq \|\alpha\|_2^2 t_p + k_p \), for a constant \( k_p \), depending only on \( p \). Furthermore, if \( p' := \Phi^{-1} \left( \frac{t_{p,\alpha}}{\|\alpha\|_2^2} \right) \) then \( |p - p'| \leq 3 \left( \frac{\|\alpha\|_3}{\|\alpha\|_2} \right)^3 \).

**Proof.** Let \( W = \frac{\langle X_1, X_2 \rangle}{\|\alpha\|_2^2} \) where \( X_1, X_2 \) are defined as in the beginning of the section. We may consider \( \langle X_1, X_2 \rangle \) as sum of independent random variables \( X_i \cdot X_i' \), where for each \( 1 \leq i \leq d \), \( X_i \) and \( X_i' \) are independently distributed as \( \mathcal{N}(0, \alpha_i) \). It then holds that \( \mathbb{E}[X_i \cdot X_i'^2] = \alpha_i^2 \). The absolute third moments are given as a product of absolute third moments of Gaussians. That is, \( \mathbb{E}[|X_i \cdot X_i'|^3] = \frac{8\alpha_i^3}{\pi} < 3\alpha_i^3 \).

Let \( t > 0 \) be such that \( p = \mathbb{P}(W \geq t) \), in which case we also have \( t_{p,\alpha} = t \|\alpha\|_2^2 \). Note that

\[ \frac{\sum_i \mathbb{E}[|X_i \cdot X_i'|^3]}{\|\alpha\|_2^3} \leq \frac{3\|\alpha\|_3^3}{\|\alpha\|_2^3}. \]

Thus, if we denote by \( Z \) a \( d \)-dimensional standard Gaussian vector, Berry-Esseen’s inequality, (6), yields for every \( s \in \mathbb{R} \):

\[ \left| \mathbb{P}(W > s) - \mathbb{P}(Z > s) \right| \leq \frac{3\|\alpha\|_3^3}{\|\alpha\|_2^3}. \]

If \( t_p = \Phi^{-1}(p) \) then \( \mathbb{P}(Z > t_p) = p \) and

\[ |\Phi(t_p) - \Phi(t)| = |\mathbb{P}(Z > t_p) - \mathbb{P}(Z > t)| = |\mathbb{P}(W > t) - \mathbb{P}(Z > t)| \leq \frac{3\|\alpha\|_3^3}{\|\alpha\|_2^3}. \]
Since $|p - p'| = |\Phi(t_p) - \Phi(t)|$, this shows the second part of the statement. To finish the proof, denote $m = \inf_{s \in [t_p, t]} (\Phi'(s))$. By Lagrange’s theorem

$$m|t_p - t| \leq |\Phi(t_p) - \Phi(t)| \leq 3 \frac{\|\alpha\|_2^3}{\|\alpha\|_2} \leq \frac{3}{m},$$

which shows $t_{p, \alpha} \in \|\alpha\|_2 t_p \pm \frac{3}{m}$. \hfill \Box

Before proceeding, we need some further definitions. Let $X_1', X_2', X_3'$ be independent copies of $X_1, X_2, X_3$ and consider the joint distribution $(\langle X_1, X_2 \rangle, \langle X_1', X_3 \rangle, \langle X_2', X_3' \rangle)$. This distribution has independent coordinates. Denote its density by $g$ and corresponding characteristic function by $\psi$. If $N_1, N_2$ are two independent standard Gaussians then the characteristic function of their product can be derived from (4) as $\mathbb{E} e^{i X_1 N_2} = (1 + t^2)^{-\frac{3}{2}}$. From this, it follows that the characteristic function of $(X_1, X_2)$ is $\mathbb{E} e^{i Y_1 Y_2} = \prod_i (1 + \alpha_i^2 t^2)^{-\frac{3}{2}}$, and we have, by independence

$$\psi(a, b, c) = \prod_i \left((1 + \alpha_i^2 a^2)(1 + \alpha_i^2 b^2)(1 + \alpha_i^2 c^2)\right)^{-\frac{3}{2}}. \quad (21)$$

We denote by $\psi_1 (a', b', c') = \psi\left(\frac{a'}{\|\alpha\|_2}, \frac{b'}{\|\alpha\|_2}, \frac{c'}{\|\alpha\|_2}\right)$ and $\varphi_1 (a', b', c') = \varphi\left(\frac{a'}{\|\alpha\|_2}, \frac{b'}{\|\alpha\|_2}, \frac{c'}{\|\alpha\|_2}\right)$ for the characteristic function $\varphi$, (11). The following result will help us relate the independent version of the distribution and the original one.

**Lemma 5.** There exist absolute constants $c, C, \varepsilon > 0$ such that whenever $\|\alpha\|_2^2 > c$ then

$$\int_{\mathbb{R}^3} |\text{Re}(\varphi_1) - \psi_1| da'db'dc' \leq C \left(\frac{\|\alpha\|_2}{\|\alpha\|_3}\right)^{3+\varepsilon}.$$

**Proof.** Note that since $\psi_1$ and $\text{Re}(\varphi_1)$ are characteristic functions, then $|\psi_1|, |\text{Re}(\varphi_1)| \leq 1$, and so $|\psi_1 - \text{Re}(\varphi_1)| \leq |\ln(\psi_1) - \ln(\text{Re}(\varphi_1))|$. Now, let

$$B_{0.01} = \left\{ x \in \mathbb{R}^3 : |x|^2 \leq \left(\frac{\|\alpha\|_2}{\|\alpha\|_3}\right)^{0.01} \right\}.$$

Clearly, $|\text{Re}(\varphi_1)| \leq |\varphi_1| = \prod_i \left(1 + \frac{\alpha_i^2}{\|\alpha\|_2^2} (a^2 + b^2 + c^2)^2 + 4 \frac{\alpha_i}{\|\alpha\|_2} a^2 b^2 c^2\right)^{-\frac{3}{4}}$, and since

$$|a'b'c'| \leq (a^2 + b^2 + c^2)^{\frac{3}{4}} \leq \left(\frac{\|\alpha\|_2}{\|\alpha\|_3}\right)^{0.015} \text{ for } (a', b', c') \in B_{0.01},$$

we have

$$|\arg \left(1 + \frac{\alpha_i^2}{\|\alpha\|_2^2} (a^2 + b^2 + c^2)^2 + 2 \frac{\alpha_i^2}{\|\alpha\|_2^3} a'b'c'\right)| \leq 2 \frac{\alpha_i^2}{\|\alpha\|_2^3} \left(\frac{\|\alpha\|_2}{\|\alpha\|_3}\right)^{0.015}.$$

By using the inequality $\cos(x) \geq 1 - x^2$, we achieve

$$\text{Re}(\varphi_1) \geq \cos \left(2 \frac{\|\alpha\|_3^2}{\|\alpha\|_2^3} \left(\frac{\|\alpha\|_2}{\|\alpha\|_3}\right)^{0.015}\right) |\varphi_1| \geq \left(1 - 4 \frac{\|\alpha\|_6^2}{\|\alpha\|_2^6} \left(\frac{\|\alpha\|_2}{\|\alpha\|_3}\right)^{0.03}\right) |\varphi_1|.$$
Using the above, together with the triangle inequality gives

\[ |\ln(\psi_1) - \ln(\text{Re}(\varphi_1))| \leq |\ln(\psi_1) - \ln(|\varphi_1|)| + \left| \ln \left( 1 - 4 \frac{\|\alpha\|_2^6}{\|\alpha\|_2^2} \left( \frac{\|\alpha\|_2}{\|\alpha\|_3} \right)^{0.03} \right) \right|. \]  (22)

For \( x \in (0, \frac{1}{2}) \) we have the inequality \( |\ln(1 - x)| \leq 2x \), thus, as long as \( \|\alpha\|_2^2 \) is large enough

\[ \left| \ln \left( 1 - 4 \frac{\|\alpha\|_2^6}{\|\alpha\|_2^2} \left( \frac{\|\alpha\|_2}{\|\alpha\|_3} \right)^{0.03} \right) \right| \leq 8 \frac{\|\alpha\|_2^6}{\|\alpha\|_2^2} \left( \frac{\|\alpha\|_2}{\|\alpha\|_3} \right)^{0.03}, \]

and

\[ 8 \int_{B_{0.01}} \frac{\|\alpha\|_2^6}{\|\alpha\|_2^2} \left( \frac{\|\alpha\|_2}{\|\alpha\|_3} \right)^{0.03} da'b'dc' \leq 32\pi \frac{\|\alpha\|_2^6}{\|\alpha\|_2^2} \left( \frac{\|\alpha\|_2}{\|\alpha\|_3} \right)^{0.045} = 32\pi \left( \frac{\|\alpha\|_2}{\|\alpha\|_3} \right)^{5.955}. \]  (23)

By using the inequality \( |\ln(1 + x) - x| \leq x^2 \) for \( x > 0 \) we bound \( \ln(\psi_1) \) with

\[ \ln(\psi_1(a', b', c')) = -\frac{1}{2} \sum \ln \left( 1 + \frac{\alpha_1^2a_1^2}{\|\alpha\|_2^2} \right) + \ln \left( 1 + \frac{\alpha_2b_2^2}{\|\alpha\|_2^2} \right) + \ln \left( 1 + \frac{\alpha_3c_3^2}{\|\alpha\|_2^2} \right) \]

\[ = -\frac{1}{2} \left( \alpha_1^2 + b_2^2 + c_3^2 \right) + O \left( \frac{\|\alpha\|_2^4}{\|\alpha\|_2^2} \right) \left( \alpha_1^4 + b_4^4 + c_4^4 \right). \]

Similar considerations show

\[ \ln(|\varphi_1|) = -\frac{1}{4} \sum \ln \left( 1 + \frac{2\alpha_1^2}{\|\alpha\|_2^2} (\alpha_1^2 + b_2^2 + c_3^2) + \frac{\alpha_1^4}{4} (\alpha_1^2 + b_2^2 + c_3^2)^2 + \frac{\alpha_1^6}{6} a_1^2b_2^2c_3^2 \right) \]

\[ = -\frac{1}{2} \left( \alpha_1^2 + b_2^2 + c_3^2 \right) - \frac{\|\alpha\|_2^4}{4} (\alpha_1^2 + b_2^2 + c_3^2)^2 - \frac{\|\alpha\|_2^6}{6} a_1^2b_2^2c_3^2 \]

\[ + O \left( \frac{\|\alpha\|_2^4}{\|\alpha\|_2^2} \right) \left( (\alpha_1^2 + b_2^2 + c_3^2)^2 + (\alpha_1^2 + b_2^2 + c_3^2)^4 + a_1^4b_4^4c_4^4 \right) \]

\[ = -\frac{1}{2} \left( \alpha_1^2 + b_2^2 + c_3^2 \right) + O \left( \frac{\|\alpha\|_2^4}{\|\alpha\|_2^2} \right) \left( 1 + (\alpha_1^2 + b_2^2 + c_3^2)^2 \right). \]  (24)

The above shows the existence of a constant \( C > 0 \) such that

\[ \int_{B_{0.01}} \left| \ln(\psi_1) - \ln(|\varphi_1|) \right| \leq C \left( \frac{\|\alpha\|_2^4}{\|\alpha\|_2^2} \right) \int_{B_{0.01}} (\alpha_1^2 + b_2^2 + c_3^2)^6 da'b'dc' \]

\[ = 4\pi C \left( \frac{\|\alpha\|_2^4}{\|\alpha\|_2^2} \right) \left( \frac{\|\alpha\|_2^6}{\|\alpha\|_3^2} \right)^{0.075} \leq 4\pi C \left( \frac{\|\alpha\|_3^4}{\|\alpha\|_2^2} \right) \left( \frac{\|\alpha\|_2^6}{\|\alpha\|_3^2} \right)^{0.075} = 4\pi C \left( \frac{\|\alpha\|_3^4}{\|\alpha\|_2^2} \right)^{3.925}. \]  (25)

By combining (23),(25) and (22), we obtain

\[ \int_{B_{0.01}} |\psi_1 - \text{Re}(\varphi_1)| da'b'dc' \leq \pi (4C + 32) \left( \frac{\|\alpha\|_3^4}{\|\alpha\|_2^2} \right)^{3.925}. \]
To bound the integral in \( \mathbb{R}^3 \setminus B_{0.01} \) we proceed in similar fashion to step 3 in Lemma 3. First, note that
\[
|\varphi_1|, |\psi_1| \leq \frac{1}{\prod_i \left( 1 + \frac{\alpha_i^2}{\|\alpha\|_2^2} \right)}.
\]
Denoting \( r = \sqrt{a^2 + b^2 + c^2} \), \( T = \left( \frac{\|\alpha\|_2}{\|\alpha\|_3} \right)^{0.005} \) and passing to spherical coordinates yields
\[
\int_{\mathbb{R}^3 \setminus B_{0.01}} |\text{Re}(\varphi_1) - \psi_1| da'db'dc' \leq \int_{\mathbb{R}^3 \setminus B_{0.01}} |\text{Re}(\varphi_1)| + |\psi_1| da'db'dc' \leq 8\pi \int_T^\infty \frac{r^2 \, dr}{\prod_i \left( 1 + \frac{\alpha_i^2}{\|\alpha\|_2^2} \right)^{2}}
\]
Invoking Lemma 2 with \( n > 606 \) shows the existence of constants \( C, c > 0 \) such that
\[
\int_T^\infty \frac{r^2 \, dr}{\prod_i \left( 1 + \frac{\alpha_i^2}{\|\alpha\|_2^2} \right)^{2}} \leq CT^{-603} = C \left( \frac{\|\alpha\|_3}{\|\alpha\|_2} \right)^{3.015},
\]
whenever \( \|\alpha\|_2^2 > c \). This concludes the proof when we take \( \varepsilon = 0.015 \).

We are now ready to bound from below the probability of an induced triangle occurring in the general setting. Set \( p \in (0, 1) \) and \( t := t_{p, \alpha} \). We are interested in the event
\[
\left\{ \min \left( \langle X_1, X_2 \rangle, \langle X_1, X_3 \rangle, \langle X_2, X_3 \rangle \right) > t \right\}.
\]
As before, let \( f \) be the joint density of \( \langle X_1, X_2 \rangle, \langle X_1, X_3 \rangle, \langle X_2, X_3 \rangle \) and consider the integral:
\[
I_p := \int_{\mathbb{R}^3} f(x, y, z) \text{sgn}(x - t) \text{sgn}(y - t) \text{sgn}(z - t) \, dx dy dz.
\]
Note that, in the above formula, replacing \( f \) with \( g \), the density of the coordinate-independent version, as defined above, would yield \( I_p = p^3 + 3(1 - p)^2 p - 3(1 - p)^2 - (1 - p)^3 = (2p - 1)^3 \). The following lemma shows that the dependency between the coordinates induces an increased probability for triangles and induced edges.

**Lemma 6.** Fix \( p \in (0, 1) \). There exist constants \( \delta_p', c_p > 0 \) depending only on \( p \) such that whenever \( \|\alpha\|_2^2 > c_p \) then \( I_p \geq (2p - 1)^3 + \delta_p' \left( \frac{\|\alpha\|_3}{\|\alpha\|_2} \right)^3 \).

**Proof.** As in (1), we may write the Fourier transform of \( \text{sgn}(x - t) \text{sgn}(y - t) \text{sgn}(z - t) \) as \( \widehat{\text{sgn}}(a, b, c)e^{-2\pi it(a+b+c)} \). Thus, by (2), we have the equality
\[
I_p = \frac{1}{\pi^3} \int_{\mathbb{R}^3} \varphi(a, b, c) \text{sgn}(a, b, c)e^{-2\pi it(a+b+c)} \, dadbdc,
\]
where \( \varphi \), as in (11), is the characteristic function of \( f \). Since \( I_p \) represents a real number, we only need to consider the real part of the integral:

\[
I_p = \frac{1}{\pi^3} \int \text{Re} \left( \varphi(a, b, c) \text{sgn}(a, b, c) \cos(2\pi t(a + b + c)) \right) |_{\alpha} \, dadbdc
\]

\[
-\frac{1}{\pi^3} \int \text{Im} \left( \varphi(a, b, c) \text{sgn}(a, b, c) \sin(2\pi t(a + b + c)) \right) |_{\alpha} \, dadbdc
\]

\[
= \frac{1}{\pi^3} \int \frac{\text{Im} \left( \varphi(a, b, c) \right)}{|\alpha|} \cos(2\pi t(a + b + c)) \, dadbdc
\]

\[
+ \frac{1}{\pi^3} \int \frac{\text{Re} \left( \varphi(a, b, c) \right)}{|\alpha|} \sin(2\pi t(a + b + c)) \, dadbdc.
\]

We denote

\[
I_p' = \frac{1}{\pi^3} \int \frac{\text{Re} \left( \varphi(a, b, c) \right)}{abc} \sin(2\pi t(a + b + c)) \, dadbdc
\]

and

\[
I_p'' = \frac{1}{\pi^3} \int \frac{\text{Im} \left( \varphi(a, b, c) \right)}{abc} \cos(2\pi t(a + b + c)) \, dadbdc.
\]

We begin by showing that \( I_p'' > 2\beta_p' \left( \left\| \alpha \right\|_{\ell^2} \right)^3 \). First, it is not hard to see that the integrand in \( I_p'' \) is continuous, up to a removable discontinuity, and we may pass to standard integration. Let \( R \) be an arbitrary orthogonal transformation which takes \((1, 0, 0)\) to \(\frac{1}{\sqrt{3}}(1, 1, 1)\). Consider the set

\[
K = R \left( \left[ -\frac{1}{\left\| \alpha \right\|_{\ell^2}^{11/12}}, \frac{1}{\left\| \alpha \right\|_{\ell^2}^{11/12}} \right] \times \left[ -\frac{1}{\left\| \alpha \right\|_{\ell^2}^{11/12}}, \frac{1}{\left\| \alpha \right\|_{\ell^2}^{11/12}} \right] \times \left[ -\frac{1}{\left\| \alpha \right\|_{\ell^2}^{11/12}}, \frac{1}{\left\| \alpha \right\|_{\ell^2}^{11/12}} \right] \right).
\]

Note that if \( B_2 = \{ x \in \mathbb{R}^3 | \| x \| \leq \frac{1}{\left\| \alpha \right\|_{\ell^2}^{1/12}} \} \) and \( B_2' = \{ x \in \mathbb{R}^3 | \| x \| \leq \frac{4}{\left\| \alpha \right\|_{\ell^2}^{1/12}} \} \) then,

\[
B_2 \subseteq K \subseteq B_2'.
\]

Now, recall from (14) that,

\[
\frac{\text{Im} \left( \varphi(a, b, c) \right)}{abc} = \frac{\sin \left( \frac{1}{2} \sum_i \arctan \left( \frac{2\alpha_i^3 abc}{1 + \alpha_i^2 (a^2 + b^2 + c^2)} \right) \right)}{abc \prod_i \left( 1 + \alpha_i^2 (a^2 + b^2 + c^2) \right)^2}.
\]

From (18) and (15), we have

\[
\left\| \alpha \right\|_{\ell^3}^3 \geq \left| \frac{\sin \left( \frac{1}{2} \sum_i \arctan \left( \frac{2\alpha_i^3 abc}{1 + \alpha_i^2 (a^2 + b^2 + c^2)} \right) \right)}{abc} \right| \geq \sum_i \frac{\alpha_i^3}{1 + \alpha_i^2 (a^2 + b^2 + c^2)} - 3 \left\| \alpha \right\|_{\ell^3}^3 \left| abc \right|.
\]

Along with the inequality \( \frac{\alpha_i^3}{1 + \alpha_i^2 (a^2 + b^2 + c^2)} \geq \alpha_i^3 (1 - \alpha_i^2 (a^2 + b^2 + c^2)) \), the above yields

\[
\left| \sin \left( \frac{1}{2} \sum_i \arctan \left( \frac{2\alpha_i^3 abc}{1 + \alpha_i^2 (a^2 + b^2 + c^2)} \right) \right) \right| = \left\| \alpha \right\|_{\ell^5}^5 \left( a^2 + b^2 + c^2 \right) - 3 \left\| \alpha \right\|_{\ell^3}^3 \left| abc \right|.
\]
Therefore
\[
\int_K \frac{\text{Im} (\varphi(a, b, c))}{\alpha} \cos(2\pi t(a + b + c)) \operatorname{dadbdc}
\]
\[
\geq \|\alpha\|_3^3 \int_K \frac{\cos(2\pi t(a + b + c)) \operatorname{dadbdc}}{\prod_i (1 + \alpha_i^4 (a^2 + b^2 + c^2)^2 + 4\alpha_i^6 a^2 b^2 c^2)^2}
\]
\[
-3 \|\alpha\|_3^6 \int_K \left| abc \right| \operatorname{dadbdc} - \|\alpha\|_5^5 \int_K (a^2 + b^2 + c^2) \operatorname{dadbdc},
\]
with
\[
3 \|\alpha\|_3^6 \int_K \left| abc \right| \operatorname{dadbdc} \leq C_1 \frac{\|\alpha\|_3^6}{\|\alpha\|_2^2} = C_1 \left( \frac{\|\alpha\|_3}{\|\alpha\|_2} \right)^3 \frac{1}{\|\alpha\|_2} \leq C_1 \left( \frac{\|\alpha\|_3}{\|\alpha\|_2} \right)^3 \frac{1}{\|\alpha\|_2} ^{0.5},
\]
\[
\|\alpha\|_5^5 \int_K (a^2 + b^2 + c^2) \operatorname{dadbdc} \leq C_1 \frac{\|\alpha\|_5^5}{\|\alpha\|_2^{5/12}} \leq C_1 \left( \frac{\|\alpha\|_5}{\|\alpha\|_2} \right)^3 \frac{1}{\|\alpha\|_2} ^{0.5},
\]
for an absolute constant \(C_1 > 0\). Recalling that
\[
|\varphi(a, b, c)| = \prod_i \left( (1 + \alpha_i^4 (a^2 + b^2 + c^2)^2 + 4\alpha_i^6 a^2 b^2 c^2)^2 \right)^{-\frac{1}{4}},
\]
we would like approximate \(|\varphi(a, b, c)|\) by \(e^{-\frac{\|\alpha\|_2^2}{2}(a^2 + b^2 + c^2)}\). For that, we note that
\[
\left| |\varphi(a, b, c)| - e^{-\frac{\|\alpha\|_2^2}{2}(a^2 + b^2 + c^2)} \right| \leq \left| \ln(|\varphi(a, b, c)|) - \ln \left( e^{-\frac{\|\alpha\|_2^2}{2}(a^2 + b^2 + c^2)} \right) \right|.
\]
Since \(|\ln(x + 1) - x| \leq x^2\), similar considerations as in (24), show for \((a, b, c) \in K:\)
\[
\ln(|\varphi|) = -\frac{1}{4} \sum_i \ln \left( 1 + 2\alpha_i^2 (a^2 + b^2 + c^2) + \alpha_i^4 (a^2 + b^2 + c^2)^2 + 4\alpha_i^6 a^2 b^2 c^2 \right)
\]
\[
= -\frac{\|\alpha\|_2^2}{4} \left( a^2 + b^2 + c^2 \right) - \frac{\|\alpha\|_2^4}{4} \left( a^2 + b^2 + c^2 \right)^2 - \|\alpha\|_6^6 a^2 b^2 c^2
\]
\[
+ O \left( \|\alpha\|_4^4 \left( a^2 + b^2 + c^2 \right)^2 + (a^2 + b^2 + c^2) + a^2 b^2 c^2 \right).
\]
This shows the existence of an absolute constant \(C_2 > 0\) such that for \((a, b, c) \in K:\)
\[
\left| |\varphi(a, b, c)| - e^{-\frac{\|\alpha\|_2^2}{2}(a^2 + b^2 + c^2)} \right| \leq C_2 \|\alpha\|_4^4 \left( a^2 + b^2 + c^2 \right)^2.
\]
Hence
\[
\int_K |\varphi(a, b, c)| \cos(2\pi t(a + b + c)) \operatorname{dadbdc}
\]
\[
\geq \int_K e^{-\frac{\|\alpha\|_2^2}{2}(a^2 + b^2 + c^2)} \cos(2\pi t(a + b + c)) \operatorname{dadbdc} - C_2 \|\alpha\|_4^4 \int_K (a^2 + b^2 + c^2)^2 \operatorname{dadbdc},
\]
(27)

\text{20}
and 
\[ C_2 \| \alpha \|_2^4 \int_{K} (a^2 + b^2 + c^2)^{\frac{3}{2}} \, dabdc \leq C_3 \left( \frac{\| \alpha \|_2^4}{\| \alpha \|_2^{7/12}} \right) \leq C_3 \left( \frac{1}{\| \alpha \|_2^{\frac{3}{2}}}, \right) \]

for an absolute constant \( C_3 > 0 \). By rotational invariance of \( e^{-\frac{3}{2} (a^2 + b^2 + c^2)} \), we may apply \( R \) as a unitary coordinate change, which shows

\[ \int_{K} e^{-\frac{3}{2} \| \alpha \|_2^2} (a^2 + b^2 + c^2) \cos(2 \pi t(a + b + c)) \, dabdc = \int_{R^{-1} K} e^{-\frac{3}{2} \| \alpha \|_2^2} (a^2 + b^2 + c^2) \cos(2 \sqrt{3} \pi t) \, dabdc \]

\[ = \int_{\| \alpha \|_2^{1/12}} \int_{\| \alpha \|_2^{1/12}} \int_{\| \alpha \|_2^{1/12}} e^{-\frac{3}{2} \| \alpha \|_2^2} \, \frac{1}{\| \alpha \|_2^{1/12}} \, \frac{1}{\| \alpha \|_2^{1/12}} \, \frac{1}{\| \alpha \|_2^{1/12}} \, \cos(\sqrt{12} \pi t) \, da 
\]

where the last equality is a result of a second coordinate change. By Lemma 4, we know that

\[ |t_p| - \frac{k_p}{\| \alpha \|_2} \leq \left| \frac{t}{\| \alpha \|_2} \right| \leq |t_p| + \frac{k_p}{\| \alpha \|_2} \]

for constants \( k_p, t_p \) depending on \( p \). Also, a well known calculation shows that

\[ \int_{-\infty}^{\infty} e^{-\frac{3}{2} t^2} \cos\left(\sqrt{12} \pi \frac{t}{\| \alpha \|_2} a \right) \, da = \sqrt{2\pi e^{-\frac{6\pi^2}{\| \alpha \|_2^2}}}. \]

Thus, since the above integral is convergent, whenever \( \| \alpha \|_2^{1/12} \) is larger than some constant, which depends only on \( t \), we have

\[ \int_{-\| \alpha \|_2^{1/12}}^{\| \alpha \|_2^{1/12}} e^{-\frac{3}{2} t^2} \cos\left(\sqrt{12} \pi \frac{t}{\| \alpha \|_2} a \right) \, da \geq \frac{1}{2} \sqrt{2\pi e^{-\frac{6\pi^2}{\| \alpha \|_2^2}}}. \]

Together with the observation \( \int_{-1}^{1} e^{-\frac{2}{x^2}} \, dx > 1 \), this shows that the expression (28) is lower bounded by \( \frac{1}{2} \sqrt{2\pi e^{-\frac{6\pi^2}{\| \alpha \|_2^2}}} \). Combining the above, along with (26) and (27) shows

\[ \int_{K} \frac{\text{Im} (\varphi(a, b, c))}{abc} \cos(2 \pi t(a + b + c)) \, dabdc \]

\[ \geq \| \alpha \|_3^3 \int_{K} \cos(2 \pi t(a + b + c)) \| \varphi(a, b, c) \| \, dabdc - 2C_1 \left( \frac{\| \alpha \|_3^3}{\| \alpha \|_2} \right)^3 \frac{1}{\| \alpha \|_2^{0.5}} \]

\[ \geq \| \alpha \|_3^3 \int_{K} e^{-\frac{3}{2} \| \alpha \|_2^2} \cos(2 \pi t(a + b + c)) \, dabdc - C_3 \left( \frac{\| \alpha \|_3^3}{\| \alpha \|_2} \right)^6 - 2C_1 \left( \frac{\| \alpha \|_3^3}{\| \alpha \|_2} \right)^3 \frac{1}{\| \alpha \|_2^{0.5}} \]

\[ \geq \frac{1}{2} \sqrt{2\pi e^{-\frac{6\pi^2}{\| \alpha \|_2^2}}} \left( \frac{\| \alpha \|_3^3}{\| \alpha \|_2} \right)^3 - C_3 \left( \frac{\| \alpha \|_3^3}{\| \alpha \|_2} \right)^6 - 2C_1 \left( \frac{\| \alpha \|_3^3}{\| \alpha \|_2} \right)^3 \frac{1}{\| \alpha \|_2^{0.5}} \geq 4\delta_p \left( \frac{\| \alpha \|_3^3}{\| \alpha \|_2} \right)^3. \]
whenever $\|\alpha\|^2 > c''_p$, for $c''_p, \delta'_p$ constants, depending only on $p$. From (19), we can choose a constant $c'_p > c''_p > 0$ such that

$$\int_{\mathbb{R}^3 \setminus B_2} |\text{Re} (\varphi(a, b, c) \text{sgn}(a, b, c))| \, dadbdc < 2\delta'_p \left(\frac{\|\alpha\|_3}{\|\alpha\|_2}\right)^3,$$

whenever $\|\alpha\|^2 > c'_p$. Thus

$$I''_p > \int_K \frac{\text{Im} (\varphi(a, b, c))}{abc} \, dadbdc - \int_{\mathbb{R}^3 \setminus B_2} \left|\frac{\text{Im} (\varphi(a, b, c))}{abc}\right| \, dadbdc \geq 2\delta'_p \left(\frac{\|\alpha\|_3}{\|\alpha\|_2}\right)^3.$$

It now remains to show that $I'_p$ is small, compared to $I''_p$. Let $g$ be the density of the coordinate free version of $f$, as in Lemma 5, and let $\psi$ be its characteristic function (21). Evidently, we have the equality:

$$\frac{1}{\pi^3} \int_{\mathbb{R}^3} \psi(a, b, c) \text{sgn}(a, b, c) e^{-2\pi it(a+b+c)} \, dadbdc = (2p - 1)^3.$$

Thus, by rewriting $I'_p$ as

$$\frac{1}{\pi^3} \int_{\mathbb{R}^3} (\text{Re}(\varphi(a, b, c)) + \psi(a, b, c) - \psi(a, b, c)) \frac{\sin(2\pi t(a + b + c))}{abc} \, dadbdc,$$

we obtain

$$I'_p = (2p - 1)^3 + \frac{1}{\pi^3} \int_{\mathbb{R}^3} (\text{Re}(\varphi(a, b, c)) - \psi(a, b, c)) \frac{\sin(2\pi t(a + b + c))}{abc} \, dadbdc.$$

Next, we rewrite $\sin(2\pi t(a + b + c))$ as:

$$\sin(2\pi ta) \sin(2\pi tb) \sin(2\pi tc) + \cos(2\pi ta) \cos(2\pi tb) \cos(2\pi tc) + \sin(2\pi ta) \cos(2\pi tb) \cos(2\pi tc).$$

One may now verify that $\text{Re}(\varphi(a, b, c) - \psi(a, b, c)) \frac{1}{abc}$ is an odd function. Thus, when taken as a principal value, we see that:

$$\int_{\mathbb{R}^3} \text{Re}(\varphi(a, b, c)) - \psi(a, b, c) \frac{\cos(2\pi ta) \cos(2\pi tb) \sin(2\pi tc)}{abc} = 0,$$

and the same can be said for the other similar terms. We are then left to consider an integrable function:

$$I'_p - (2p - 1)^3 = \int_{\mathbb{R}^3} \frac{\sin(2\pi ta) \sin(2\pi tb) \sin(2\pi tc)}{abc} (\text{Re}(\varphi(a, b, c) - \psi(a, b, c)) \, dadbdc.$$

By making the substitution $a' = \|\alpha\|_2 a, b' = \|\alpha\|_2 b, c' = \|\alpha\|_2 c$, and denoting $t' = \frac{t}{\|\alpha\|_2}$ the above equals

$$\int_{\mathbb{R}^3} \frac{\sin(2\pi t'a') \sin(2\pi t'b') \sin(2\pi t'c')}{a'b'c'} \left(\text{Re}(\varphi_1(a', b', c') - \psi_1(a', b', c'))\right) \, da'db'dc'.$$
where $\varphi_1$ and $\psi_1$ are as defined before. By Lemma 4, we know that $|t'| < |t_p| + \frac{k_p}{\|\alpha\|^2}$. Thus

$$\sup_{(a',b',c') \in \mathbb{R}^3} \left| \frac{\sin(2\pi t'a') \sin(2\pi t'b') \sin(2\pi t'c')}{a'b'c'} \right| \leq \left(2\pi \left(|t_p| + \frac{k_p}{\|\alpha\|^2}\right)\right)^3.$$ 

And so

$$|I_p' - (2p - 1)^3| \leq \left(2\pi \left(|t_p| + \frac{k_p}{\|\alpha\|^2}\right)\right)^3 \int_{\mathbb{R}^3} |\text{Re}(\varphi_1(a', b', c')) - \psi_1(a', b', c')| da' db' dc'. $$

Lemma 5 asserts that $\int_{\mathbb{R}^3} |\text{Re}(\varphi_1) - \psi_1| \leq C \left(\frac{\|\alpha\|^3}{\|\alpha\|^2}\right)^{3+\epsilon}$ for large enough $\|\alpha\|^2$. Thus,

$$|I_p' - (2p - 1)^3| \leq \left(2\pi \left(|t_p| + \frac{k_p}{\|\alpha\|^2}\right)\right)^3 C \left(\frac{\|\alpha\|^3}{\|\alpha\|^2}\right)^{3+\epsilon}.$$ 

Since we’ve assumed $\alpha$ to be normalized as in (13), $\frac{\|\alpha\|^3}{\|\alpha\|^2}$ can be made as small as needed. The proof concludes by choosing $c_p > c_p'$ to be such that

$$\left(2\pi \left(|t_p| + \frac{k_p}{\|\alpha\|^2}\right)\right)^3 C \left(\frac{\|\alpha\|^3}{\|\alpha\|^2}\right)^{3+\epsilon} < \delta_p' \left(\frac{\|\alpha\|^3}{\|\alpha\|^2}\right)^3$$ whenever $\|\alpha\|^2 > c_p$.

$$\square$$

Now, by definition $\mathbb{P}(\langle X_1, X_2 \rangle > t_{p,\alpha}) = p$ and $\mathbb{P}(\langle X_1, X_2 \rangle > t_{p,\alpha}, \langle X_1, X_3 \rangle > t_{p,\alpha}) = p^2$. We note that Lemma 6, along with (1) produces:

$$(2p - 1)^3 + \delta_p^2 \left(\frac{\|\alpha\|^3}{\|\alpha\|^2}\right)^3 \leq 8\mathbb{P}(E_p) - 12p^2 + 6p - 1.$$ 

This establishes the lower bound of Theorem 5

$$p^3 + \frac{\delta_p^2}{8} \left(\frac{\|\alpha\|^3}{\|\alpha\|^2}\right)^3 \leq \mathbb{P}(E_p).$$

### 3.3 Upper bound

To finish the proof of Theorem 5 it remains to prove the upper bound. This is done in the following lemma.

**Lemma 7.** Let $p \in (0, 1)$, $\mathbb{P}(E_p) - p^3 \leq \Delta \left(\frac{\|\alpha\|^3}{\|\alpha\|^2}\right)^3$, for a universal constant $\Delta > 0$.

**Proof.** The proof of this lemma will use the higher dimensional analogue of the Berry-Esseen’s inequality.

Define the random vector $V = (\langle X_1, X_2 \rangle, \langle X_1, X_3 \rangle, \langle X_2, X_3 \rangle)$. It is straightforward to check
that the covariance matrix of $V$ is $\|\alpha\|_2^2 I_3$ where $I_3$ is the identity matrix. We decompose $V$ into $V_i = (X_i X_i^2, X_i X_i^3, X_i^2 X_i^3)$. Clearly $V = \sum_{i=1}^d V_i$ and, since $X_i, X_i^2, X_i^3$ are i.i.d. Gaussians,

$$\mathbb{E} \|V\|^3 \leq \sqrt{3} \mathbb{E} \left[ \left( (X_i^4 X_i^2)^2 + (X_i^2 X_i^3)^2 + (X_i^2 X_i^3)^2 \right)^3 \right]$$

$$= \sqrt{3} \mathbb{E}[(X_i^4 X_i^2)^6] + 18 \mathbb{E}[(X_i^4 X_i^2)^4(X_i^2)^2] + 6 \mathbb{E}[(X_i^4 X_i^2)^4(X_i^2)^4] \leq 50\sqrt{\alpha_i^6} = 50\alpha_i^3.$$  

Thus, if $Z_3$ a 3-dimensional standard Gaussian random vector, by (7) there is a constant $C_{be}$ such that for any convex set $K \subset \mathbb{R}^3$ we have that

$$|\mathbb{P}(V/\|\alpha\|_2 \in K) - \mathbb{P}(Z_3 \in K)| \leq 100 C_{be} \left( \frac{\|\alpha\|_3}{\|\alpha\|_2} \right)^3.$$

In particular, this holds for the convex set

$$E_p = \left\{ \frac{\langle X_1, X_2 \rangle}{\|\alpha\|_2}, \frac{\langle X_1, X_3 \rangle}{\|\alpha\|_2}, \frac{\langle X_2, X_3 \rangle}{\|\alpha\|_2} > t_{p,\alpha} \right\}.$$

If we denote $p' = \Phi^{-1} \left( \frac{t_{p,\alpha}}{\|\alpha\|_2} \right)$, the above shows

$$|\mathbb{P}(E_p) - p^{\alpha}| \leq 100 C_{be} \left( \frac{\|\alpha\|_3}{\|\alpha\|_2} \right)^3.$$

By Lemma 4, $|p - p'| \leq 3 \left( \frac{\|\alpha\|_3}{\|\alpha\|_2} \right)^3$. Also

$$|p^3 - p'^3| = |p - p'| (p^2 + pp' + p'^2) \leq 9 \left( \frac{\|\alpha\|_3}{\|\alpha\|_2} \right)^3.$$

We then have

$$|\mathbb{P}(E_p) - p^3| \leq |\mathbb{P}(E_p) - p'^3| + |p^3 - p'^3| \leq (9 + 100 C_{be}) \left( \frac{\|\alpha\|_3}{\|\alpha\|_2} \right)^3$$

as desired.

\[ \square \]

## 4 Proof of Theorem 3

Recall from the introduction that $\tau(G)$ denotes the number of signed triangles of a graph $G$. If $A$ is the adjacency matrix of $G$ with entries $A_{i,j}$ we denote the centered adjacency matrix of $G$ as $\bar{A}$ with entries $\bar{A}_{i,j} := A_{i,j} - \mathbb{E}[A_{i,j}]$. Given three distinct vertices $i,j,k$ and $K$ the signed triangle induced by those 3 vertices is $\tau_G(i,j,k) := \bar{A}_{i,j} \bar{A}_{i,k} \bar{A}_{j,k}$. It then holds that for a graph $G = (V,E)$ the number of signed triangles is given by:

$$\tau(G) := \sum_{\{i,j,k\} \in \binom{V}{3}} \tau_G(i,j,k).$$

Analysis of $\tau(G(n,p))$ was done in [BDER15], where it was shown that $\mathbb{E}\tau(G(n,p)) = 0$ while $\text{Var}(\tau(G(n,p))) \leq n^3$.  

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To prove Theorem 3 it will suffice to show that $\mathbb{E} \tau(G(n, p, \alpha))$ is asymptotically bigger than both the standard deviation of $\tau(G(n, p))$ and of $\tau(G(n, p, \alpha))$, provided that $\left( \frac{||\alpha||_3}{||\alpha||_2} \right)^6 \ll n^3$.

To estimate $\mathbb{E} \tau(G(n, p, \alpha))$, we note that since $\mathbb{E} \tau(G(n, p, \alpha)) = \binom{n}{3} \mathbb{E} \tau(G(n, p, \alpha))(1, 2, 3)$ it is enough to estimate $\mathbb{E} \tau_G(n, p, \alpha)(1, 2, 3)$,

$$
\mathbb{E} \tau_G(n, p, \alpha)(1, 2, 3) = \mathbb{E} A_{1,2} A_{1,3} A_{2,3} = \mathbb{E}(A_{1,2} - p)(A_{1,3} - p)(A_{2,3} - p)
$$

$$
= \mathbb{E} A_{1,2} A_{1,3} A_{2,3} - p \left( \mathbb{E} A_{1,2} A_{2,3} + \mathbb{E} A_{1,3} A_{1,2} + \mathbb{E} A_{1,3} A_{2,3} \right)
$$

$$
+ p^2 (\mathbb{E} A_{1,2} + \mathbb{E} A_{1,3} + \mathbb{E} A_{2,3}) - p^3
$$

$$
= \mathbb{E} A_{1,2} A_{1,3} A_{2,3} - p^3,
$$

where the last equality follows from the fact that $\mathbb{E} A_{i,j} = p$ and $\mathbb{E} A_{i,j} A_{i,k} = p^2$ for all triples $\{i, j, k\} \in \binom{V}{3}$. The lower bound of Theorem 5 then yields

$$
\mathbb{E} \tau_G(n, p, \alpha)(1, 2, 3) \geq \delta_p \left( \frac{||\alpha||_3}{||\alpha||_2} \right)^3
$$

for a constant $\delta_p$, which shows

$$
\mathbb{E} \tau(G(n, p, \alpha)) \geq \delta_p \left( \frac{n}{3} \right) \left( \frac{||\alpha||_3}{||\alpha||_2} \right)^3.
$$

The upper bound of $\text{Var}(\tau(G(n, p, \alpha)))$ follows from the following lemma.

**Lemma 8.** Let $p \in (0, 1)$, then there exists a constant $M_p > 0$, depending only on $p$, such that

$$
\mathbb{E}[\tau_G(n, p, \alpha)(1, 2, 3) \tau_G(n, p, \alpha)(1, 2, 4)] \leq M_p \left( \frac{||\alpha||_3}{||\alpha||_2} \right)^6.
$$

**Proof.** The main observation utilized here is that conditioned on $A_{1,2}$, the random variables $\tau_G(n, p, \alpha)(1, 2, 3)$ and $\tau_G(n, p, \alpha)(1, 2, 4)$ are independent. Thus, by the law of total expectation

$$
\mathbb{E}[\tau_G(n, p, \alpha)(1, 2, 3) \tau_G(n, p, \alpha)(1, 2, 4)]
$$

$$
= \mathbb{E}[\tau_G(n, p, \alpha)(1, 2, 3) \tau_G(n, p, \alpha)(1, 2, 4) | \{A_{1,2} = 1\}] p
$$

$$
+ \mathbb{E}[\tau_G(n, p, \alpha)(1, 2, 3) \tau_G(n, p, \alpha)(1, 2, 4) | \{A_{1,2} = 0\}] (1 - p)
$$

$$
= \mathbb{E}[\tau_G(n, p, \alpha)(1, 2, 3) | \{A_{1,2} = 1\}]^2 p + \mathbb{E}[\tau_G(n, p, \alpha)(1, 2, 3) | \{A_{1,2} = 0\}]^2 (1 - p)
$$

$$
= \frac{1}{p} \mathbb{E}[\tau_G(n, p, \alpha)(1, 2, 3) 1\{A_{1,2} = 1\}]^2 + \frac{1}{1 - p} \mathbb{E}[\tau_G(n, p, \alpha)(1, 2, 3) 1\{A_{1,2} = 0\}]^2.
$$

Now, using the identities $1\{A_{1,2} = 0\} = 1 - A_{1,2}$ and $(1 - A_{1,2}) A_{1,2} = 0$ and following a similar calculation to the one in (29), we get

$$
\mathbb{E}[\tau_G(n, p, \alpha)(1, 2, 3) 1\{A_{1,2} = 0\}] = \mathbb{E}[(A_{1,2} - p)(A_{1,3} - p)(A_{2,3} - p)(1 - A_{1,2})]
$$

$$
= -p \mathbb{E}[A_{1,3} A_{2,3} (1 - A_{1,2})] + p^2 (\mathbb{E} A_{2,3} (1 - A_{1,2}) + \mathbb{E} A_{1,3} (1 - A_{1,2})) - p^3 \mathbb{E} [1 - A_{1,2}]
$$

$$
= -p^3 + p \mathbb{E} A_{1,2} A_{2,3} A_{1,3} + 2p^2 (p - p^2) - p^3 (1 - p) = p (\mathbb{E} A_{1,2} A_{2,3} A_{1,3} - p^3)
$$

$$
=p (\mathbb{E} [\tau_G(n, p, \alpha)(1, 2, 3)].
$$
Together with the fact that
\[
E[\tau_{G(n,p,\alpha)}(1, 2, 3)1\{A_{1,2} = 1\}] + E[\tau_{G(n,p,\alpha)}(1, 2, 3)1\{A_{1,2} = 0\}] = E[\tau_{G(n,p,\alpha)}(1, 2, 3)],
\]
the above yields
\[
E[\tau_{G(n,p,\alpha)}(1, 2, 3)1\{A_{1,2} = 1\}] = (1 - p)E[\tau_{G(n,p,\alpha)}(1, 2, 3)]
\]
and
\[
E[\tau_{G(n,p,\alpha)}(1, 2, 3)1\{A_{1,2} = 0\}] = pE[\tau_{G(n,p,\alpha)}(1, 2, 3)].
\]
By plugging this into (30) and using (29) it follows that
\[
E[\tau_{G(n,p,\alpha)}(1, 2, 3)\tau_{G(n,p,\alpha)}(1, 2, 4)] = \left(\frac{(1 - p)^2}{p} + \frac{p^2}{1 - p}\right) (E_{1,2}A_{1,3}A_{2,3} - p^6)^2.
\]
By Lemma 7, there exists a constant \(\Delta > 0\) such that
\[
(E_{1,2}A_{1,3}A_{2,3} - p^6)^2 \leq \Delta^2 \left(\frac{\|\alpha\|_3}{\|\alpha\|_2}\right)^6.
\]

Using Lemma 8 we may now upper bound the variance of \(\tau(G(n, p, \alpha))\). Repeating the calculations done in [BDER15] and using the observation that \(\tau_G(i, j, k)\) is independent from \(\tau_G(i', j', k')\) whenever \(|\{i, j, k\} \cap \{i', j', k'\}| \leq 1\) shows
\[
\text{Var}(\tau(G(n, p, \alpha))) = \sum_{i,j,k} \sum_{i',j',k'} E[\tau_{G(n,p,\alpha)}(i, j, k)\tau_{G(n,p,\alpha)}(i', j', k')] - E[\tau_{G(n,p,\alpha)}(i, j, k)] E[\tau_{G(n,p,\alpha)}(i', j', k')]
\]
\[
\leq \sum_{i,j,k} E[\tau_{G(n,p,\alpha)}(i, j, k)\tau_{G(n,p,\alpha)}(i, j, k)] + \sum_{i,j,k,l} \sum_{i',j',k'} E[\tau_{G(n,p,\alpha)}(i, j, k)\tau_{G(n,p,\alpha)}(i, j, l)]
\]
\[
= \binom{n}{3} E[\tau_{G(n,p,\alpha)}(1, 2, 3)\tau_{G(n,p,\alpha)}(1, 2, 3)] + \binom{n}{4} \binom{4}{2} E[\tau_{G(n,p,\alpha)}(1, 2, 3)\tau_{G(n,p,\alpha)}(1, 2, 4)].
\]
Noting that \(E[\tau_{G(n,p,\alpha)}(1, 2, 3)\tau_{G(n,p,\alpha)}(1, 2, 3)] \leq 1\), in conjunction with Lemma 8 yields
\[
\text{Var}(\tau(G(n, p, \alpha))) \leq n^3 + M_p n^4 \left(\frac{\|\alpha\|_3}{\|\alpha\|_2}\right)^6.
\]
Combining all of the above
\[
E[\tau(G(n, p))] = 0, \quad E[\tau(G(n, p, \alpha))] \geq \delta_p \binom{n}{3} \left(\frac{\|\alpha\|_3}{\|\alpha\|_2}\right)^3,
\]
and
\[
\max\{\text{Var}(\tau(G(n, p, \alpha))), \text{Var}(G(n, p))\} \leq n^3 + M_p n^4 \left(\frac{\|\alpha\|_3}{\|\alpha\|_2}\right)^6.
\]
Using Chebyshev’s inequality implies that
\[
P\left(\tau(G(n, p, \alpha)) \leq \frac{1}{2} E[\tau(G(n, p, \alpha))]\right) \leq 200 \left(\frac{\|\alpha\|_3}{\|\alpha\|_2}\right)^6 \frac{n^3 + M_p n^4}{\delta_p^2 n^6}.
\]

Putting the two above expressions together we thus have:

$$TV(\tau(G(n,p,\alpha)), \tau(G(n,p ))) \geq 1 - C \left( \frac{\|\alpha\|_2}{\|\alpha\|_3} \right)^6 n^3 - C \frac{1}{n^2},$$

for a constant $C$ depending only on $p$. This concludes the proof of Theorem 3.

5 Proof of the lower bound

As stated in the introduction, we can view $G(n, p, \alpha)$ as a function of an appropriate random matrix, as follows. Let $Y$ be a random $n \times d$ matrix with rows sampled i.i.d from $N(0, D_\alpha)$. Define $W = W(n, \alpha) = YYYY^T/\|\alpha\|_2 - \text{diag}(YYYY^T/\|\alpha\|_2)$. Note that for $i \neq j$, $W_{ij} = \langle \gamma_i, \gamma_j \rangle/\|\alpha\|_2$, where $\gamma_i, \gamma_j$ are the rows of $Y$. Thus the $n \times n$ matrix $A$ defined as

$$A_{i,j} = \begin{cases} 1 & \text{if } W_{ij} \geq t_{p,\alpha}/\|\alpha\|_2 \text{ and } i \neq j \\ 0 & \text{otherwise} \end{cases}$$

has the same law as the adjacency matrix of $G(n, p, \alpha)$. Denote the map that takes $W$ to $A$ by $H_{p,\alpha}$, i.e., $A = H_{p,\alpha}(W)$.

Similarly, we may view $G(n, p)$ as function of an $n \times n$ matrix with independent Gaussian entries. Let $M(n)$ be a symmetric $n \times n$ random matrix with 0 entries in the diagonal, and whose entries above the diagonal are i.i.d. standard normal random variables. If $\Phi$ is the cumulative distribution function of the standard Gaussian, then the $n \times n$ matrix $B$, defined as

$$B_{i,j} = \begin{cases} 1 & \text{if } M(n)_{ij} \geq \Phi^{-1}(p) \text{ and } i \neq j \\ 0 & \text{otherwise} \end{cases}$$

has the same law as the adjacency matrix of $G(n, p)$. Denote the map that takes $M(n)$ to $B$ by $K_p$, i.e., $B = K_p(M(n))$.

Using the triangle inequality and by the previous two paragraphs, we have that for any $p \in (0, 1)$

$$TV(G(n, p), G(n, p, \alpha)) = TV(K_p(M(n)), H_{p,\alpha}(W(n, \alpha)))$$

$$\leq TV(H_{p,\alpha}(M(n)), H_{p,\alpha}(W(n, \alpha))) + TV(K_p(M(n)), H_{p,\alpha}(M(n)))$$

$$\leq TV(M(n), W(n, \alpha)) + TV(K_p(M(n)), H_{p,\alpha}(M(n))).$$

The second term is of lower order and will dealt with later. The first term is bounded using Pinsker’s inequality , (8), yielding

$$TV(M(n), W(n, \alpha)) \leq \sqrt{\frac{1}{2} \text{Ent}(M(n)||W(n, \alpha))}. $$

We’ll use a similar argument to the one presented in [BG15] which follows an inductive proof using the chain rule for relative entropy. We observe that a sample of $W(n + 1, \alpha)$ may be
constructed from $W(n, \alpha)$ by adjoining the column vector (and symmetrically the row vector) $\mathbb{Y}Y/\|\alpha\|_2$ where $Y \sim \mathcal{N}(0, D_\alpha)$ is independent of $\mathbb{Y}$. Thus, using the notation, $Z_n$ for a standard Gaussian in $\mathbb{R}^n$, by (9), we obtain

$$\operatorname{Ent}[W(n+1, \alpha) | M(n+1)] = \operatorname{Ent}[W(n, \alpha) | M(n)] + \mathbb{E}_Y \operatorname{Ent}[\mathbb{Y}Y/\|\alpha\|_2 | W(n, \alpha) | Z_n].$$

Since $W(n, \alpha)$ is a function of $\mathbb{Y}$, standard properties of relative entropy (see [CT12], chapter 2) show

$$\mathbb{E}_\mathbb{Y} \operatorname{Ent}[\mathbb{Y}Y/\|\alpha\|_2 | W(n, \alpha) | Z_n] = \mathbb{E}_\mathbb{Y} \operatorname{Ent}[\mathbb{Y}Y/\|\alpha\|_2 | \mathbb{Y}^T/\|\alpha\|_2 | Z_n] \leq \mathbb{E}_\mathbb{Y} \operatorname{Ent}[\mathbb{Y}Y/\|\alpha\|_2 | \mathbb{Y} | Z_n].$$

Note that $\mathbb{Y}Y/\|\alpha\|_2 | \mathbb{Y}$ is distributed as $\mathcal{N}(0, \frac{1}{\|\alpha\|_2^2} \mathbb{Y}D_\alpha \mathbb{Y}^T)$. The relative entropy between two $n$-dimensional Gaussians, (see [Duc07]) $\mathcal{N}_1 \sim \mathcal{N}(0, \Sigma_1), \mathcal{N}_2 \sim \mathcal{N}(0, \Sigma_2)$ is given by

$$\operatorname{Ent}[\mathcal{N}_1 | \mathcal{N}_2] = \frac{1}{2} \left( \operatorname{tr} \left( \Sigma_2^{-1} \Sigma_1 \right) + \ln \left( \frac{\det \Sigma_2}{\det \Sigma_1} \right) - n \right).$$

In our case $\Sigma_2 = I_n$ and $\mathbb{E}_\mathbb{Y} \operatorname{tr}(\mathbb{Y}D_\alpha \mathbb{Y}^T) = n \|\alpha\|_2^2$. Thus the following holds:

$$\mathbb{E}_\mathbb{Y} \operatorname{Ent} \left[ \frac{1}{\|\alpha\|_2} \mathbb{Y}Y | \mathbb{Y} | Z_n \right] = -\frac{1}{2} \left( \mathbb{E}_\mathbb{Y} \ln \det \left( \frac{1}{\|\alpha\|_2^2} \mathbb{Y}D_\alpha \mathbb{Y}^T \right) \right).$$

Theorem 4 is then implied by the following lemma.

**Lemma 9.** $-\mathbb{E}_\mathbb{Y} \ln \det \left( \frac{1}{\|\alpha\|_2^2} \mathbb{Y}D_\alpha \mathbb{Y}^T \right) \leq C \left( n^2 \left( \frac{\|\alpha\|_4}{\|\alpha\|_2} \right)^4 + \sqrt{n} \left( \frac{\|\alpha\|_4}{\|\alpha\|_2} \right)^4 \right)$ for a universal constant $C > 0$.

**Proof.** We follow similar lines as Lemma 2 in [BG15]. We decompose the expectation on the event that the smallest eigenvalue of $\frac{1}{\|\alpha\|_2^2} \mathbb{Y}D_\alpha \mathbb{Y}^T$, denoted by $\lambda_{\min}$, is larger than $\frac{1}{2}$. We first use the inequality $-\ln(x) \leq 1 - x + (1 - x)^2$ for $x \geq \frac{1}{2}$:

$$-\mathbb{E}_\mathbb{Y} \left( \ln \det \left( \frac{\mathbb{Y}D_\alpha \mathbb{Y}^T}{\|\alpha\|_2^2} \right) 1 \left\{ \lambda_{\min} \geq \frac{1}{2} \right\} \right) \leq \mathbb{E}_\mathbb{Y} \left( \left| \operatorname{tr} \left( I_n - \frac{\mathbb{Y}D_\alpha \mathbb{Y}^T}{\|\alpha\|_2^2} \right) \right| + \left| I_n - \frac{\mathbb{Y}D_\alpha \mathbb{Y}^T}{\|\alpha\|_2^2} \right|_{HS}^2 \right),$$

where $\|\cdot\|_{HS}$ denotes the Hilbert-Schmidt norm. Before proceeding, we first calculate several quantities. For $1 \leq j \leq n$ denote by $A_j$ the $j^{th}$ row of $\sqrt{\mathbb{D}_\alpha}$ with entries $\{\sqrt{\alpha_i} y_{i,j}, i = 1 \}$.

1. The expected squared norm of $A_j$ is given by $\mathbb{E} \|A_j\|^2 = \sum_i \mathbb{E} \alpha_i y_{j,i}^2 = \sum_i \alpha_i^2 = \|\alpha\|^2_2$.

Since $y_{j,i}$ is a centered Gaussian with variance $\alpha_i$.

2. When $j \neq k$, $A_j$ and $A_k$ are independent, and so $\mathbb{E} \|A_j\|^2 \|A_k\|^2 = \left( \sum_i \alpha_i^2 \right)^2 = \|\alpha\|^4_2$.  

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Similarly, we may deal with the second term:

\[ \mathbb{E}(A_j, A_k)^2 = \mathbb{E} \left( \sum_{i=1}^{d} \alpha_i y_{j,i} y_{k,i} \right)^2 \]

\[ = \sum_{i=1}^{d} \alpha_i^2 \mathbb{E} y_{j,i}^2 y_{k,i}^2 + \sum_{i_1 \neq i_2} \alpha_i \alpha_{i_2} \mathbb{E} y_{j,i_1} y_{k,i_2} y_{j,i_1} y_{k,i_2} = \sum_{i=1}^{d} \alpha_i^4 = \| \alpha \|^4_4. \]

4. The expected 4th power of the norm is given by

\[ \mathbb{E} \| A_j \|^4 = \mathbb{E} \left( \sum_i \alpha_i y_{j,i}^2 \right)^2 \]

\[ = \sum_i \alpha_i^2 \mathbb{E} y_{j,i}^4 + \sum_{i \neq k} \alpha_i \alpha_k \mathbb{E} y_{j,i}^2 y_{j,k}^2 \]

\[ \leq 3 \sum_i \alpha_i^4 + \left( \sum_i \alpha_i^2 \right)^2 = 3 \| \alpha \|^4_4 + \| \alpha \|^4_2, \]

when we remember that the 4th moment of a centered Gaussian with variance \( \alpha_i \) is \( 3 \alpha_i^2 \).

We turn to bound each term of the sum (31):

\[ \mathbb{E}_Y \left| \text{tr} \left( I_n - \frac{1}{\| \alpha \|^2_2} \mathbb{Y} D_\alpha \mathbb{Y}^T \right) \right| \]

\[ \leq \mathbb{E}_Y \left( \text{tr}^2 \left( I_n - \frac{1}{\| \alpha \|^2_2} \mathbb{Y} D_\alpha \mathbb{Y}^T \right) \right) = \mathbb{E}_Y \left( \sum_{j=1}^{n} \left( 1 - \frac{\| A_j \|^2_2}{\| \alpha \|^2_2} \right) \right)^2 \]

\[ = \mathbb{E}_Y \left( n^2 - \frac{2n}{\| \alpha \|^2_2} \sum_{j=1}^{n} \| A_j \|^2_2 + \frac{1}{\| \alpha \|^2_2} \sum_{j \neq k} \| A_j \|^2_2 \| A_k \|^2_2 + \frac{1}{\| \alpha \|^2_2} \sum_{j=1}^{n} \| A_j \|^4_4 \right) \]

\[ \leq \sqrt{n^2 - 2n^2 + 2 \left( \frac{n}{2} \right) + \frac{n}{\| \alpha \|^4_2} \left( 3 \| \alpha \|^4_4 + \| \alpha \|^4_2 \right)} = \sqrt{3n \frac{\| \alpha \|^4_4}{\| \alpha \|^4_2}}. \]

Similarly, we may deal with the second term:

\[ \mathbb{E}_Y \left| I_n - \frac{1}{\| \alpha \|^2_2} \mathbb{Y} D_\alpha \mathbb{Y}^T \right|_{HS}^2 = \left( \sum_{k,j} \frac{1}{\| \alpha \|^2_2} \mathbb{E}_Y \langle A_j, A_k \rangle \right)^2 - n = \]

\[ \frac{1}{\| \alpha \|^4_2} \sum_{j=1}^{n} \mathbb{E}_Y \| A_j \|^4_4 + \frac{1}{\| \alpha \|^4_2} \sum_{j \neq k} \langle A_j, A_k \rangle^2 - n \leq \]

\[ \frac{n}{\| \alpha \|^4_2} \left( 3 \| \alpha \|^4_4 + \| \alpha \|^4_2 \right) + \frac{2}{\| \alpha \|^2_2} \left( \frac{n}{2} \right) \| \alpha \|^4_4 - n = \]

\[ 3n \frac{\| \alpha \|^4_4}{\| \alpha \|^4_2} + \left( n^2 - n \right) \frac{\| \alpha \|^4_4}{\| \alpha \|^2_2} \leq 3n^2 \frac{\| \alpha \|^4_4}{\| \alpha \|^2_4}. \]

Combining (31) with the last two displays gives

\[ \mathbb{E}_Y \left( \ln \det \left( \frac{1}{\| \alpha \|^2_2} \mathbb{Y} D_\alpha \mathbb{Y}^T \right) \right) \left\{ \lambda_{\min} \geq \frac{1}{2} \right\} \leq 3 \left( n^2 \left( \frac{\| \alpha \|^4_4}{\| \alpha \|^4_2} \right)^4 + \sqrt{n \left( \frac{\| \alpha \|^4_4}{\| \alpha \|^2_2} \right)^4} \right). \]
To bound the integral on the event \( \{ \lambda_{\min} < \frac{1}{2} \} \) we observe that for any \( \xi \in (0, \frac{1}{2}) \):

\[
-\mathbb{E}_Y \left( \ln \det \left( \frac{YD_\alpha Y^T}{\|\alpha\|_2^2} \right) \mathbb{1} \left\{ \lambda_{\min} < 1/2 \right\} \right) \leq n \mathbb{E} \left( -\log(\lambda_{\min}) \mathbb{1} \left\{ \lambda_{\min} < 1/2 \right\} \right) = n \int_0^\infty \mathbb{P}(\lambda_{\min} > \theta \log(2)) d\theta
\]

\[
= n \int_0^\infty \mathbb{P}(\lambda_{\min} < s) ds = \frac{n}{\xi} \mathbb{P}(\lambda_{\min} < 1/2) + n \int_0^\xi \frac{1}{s} \mathbb{P}(\lambda_{\min} < s) ds.
\]

(32)

By allowing \( \xi \) to be some small constant, we’ll need to bound \( \mathbb{P}(\lambda_{\min} < 1/2) \) and \( \mathbb{P}(\lambda_{\min} < s) \) for small \( s \).

Recall that for any \( s, \lambda_{\min} < s \) implies the existence of \( \theta \in S^{n-1} \) such that

\[
\theta^T \frac{YD_\alpha Y^T}{\|\alpha\|_2^2} \theta < s, \text{ or equivalently } \left\| \sqrt{D_\alpha Y^T} \theta \right\|^2 < s \|\alpha\|_2^2.
\]

Also, if \( \theta \) is such that \( \left\| \sqrt{D_\alpha Y^T} \theta \right\| < \sqrt{s} \), then for any \( \theta' \in S^{n-1} \),

\[
\left\| \sqrt{D_\alpha Y^T} \theta' \right\| < \sqrt{s} + \sqrt{\lambda_{\max}} \|\theta - \theta'\|,
\]

where \( \lambda_{\max} \) is the largest eigenvalue of \( \frac{YD_\alpha Y^T}{\|\alpha\|_2^2} \).

We will first bound \( \mathbb{P}(\lambda_{\min} < 1/2) \), using an \( \varepsilon \)-net argument. Note that for each \( \theta, \sqrt{D_\alpha Y^T} \theta \) is distributed as \( \mathcal{N}(0, D_\alpha^2) \). Consider the Euclidean metric on \( S^{n-1} \) and let \( 0 < \varepsilon < 1 \). We may cover \( S^{n-1} \) with \( \left( \frac{1}{\varepsilon} \right)^n \) balls of radius \( \varepsilon \) (see Lemma 2.3.4 in [Tao12], for example) to achieve

\[
\mathbb{P}(\lambda_{\min} < 1/2) \leq \left( \frac{3}{\varepsilon} \right)^n \mathbb{P}\left( \|\mathcal{N}(0, D_\alpha^2)\| < \sqrt{\frac{1.1}{2} \|\alpha\|_2^2} \right) + \mathbb{P}\left( \sqrt{\lambda_{\max}} > \frac{0.1}{\sqrt{2\varepsilon}} \right). \tag{33}
\]

To bound \( \mathbb{P}\left( \sqrt{\lambda_{\max}} > \frac{0.1}{\sqrt{2\varepsilon}} \right) \) we will use another \( \varepsilon \)-net argument with \( \varepsilon = \frac{1}{2} \). Along with the fact that \( \|\theta - \theta'\| = \frac{1}{2} \) implies \( \left\| \sqrt{D_\alpha Y^T} (\theta - \theta') \right\| \leq \lambda_{\max} \), we may see that

\[
\mathbb{P}\left( \sqrt{\lambda_{\max}} > \frac{0.1}{\sqrt{2\varepsilon}} \right) \leq 6^n \mathbb{P} \left( \left\| \sqrt{D_\alpha Y^T} \theta \right\|^2 > \frac{0.01 \|\alpha\|_2^2}{4\varepsilon^2} \right)
\]

\[
= 6^n \mathbb{P} \left( \|\mathcal{N}(0, D_\alpha^2)\| > \sqrt{\frac{0.01 \|\alpha\|_2^2}{4\varepsilon^2}} \right). \tag{34}
\]
But, for any $x > 0$:

$$
P\left(\left\|\mathcal{N}(0, D_\alpha^2)\right\| > \sqrt{x \|\alpha\|^2_2}\right) = P\left(\sum_i \alpha_i^2 \chi_i^2 > x \|\alpha\|^2_2\right),
$$

where the $\chi_i^2$ are i.i.d. Chi-squared random variables with 1 degree of freedom. Observe that $\mathbb{E}[\alpha_i^2 \chi_i^2] = \alpha_i^2$.

We may now utilize the sub-exponential tail of the $\chi^2$ distribution and apply (5) with $\nu_i = \alpha_i^2$, noting that, by the normalization, (13), $\|\alpha\|_\infty = 1$. Thus, provided that $x > 3$:

$$
P\left(\sum_i \alpha_i^2 \chi_i^2 > x \|\alpha\|^2_2\right) \\
\leq P\left(\left|\sum_i \alpha_i^2 \chi_i^2 - \|\alpha\|^2_2\right| > (x - 1) \|\alpha\|^2_2\right) \\
\leq 2 \exp\left(-\frac{x - 1}{2} \|\alpha\|^2_2\right) \leq 2 \exp\left(-\|\alpha\|^2_2\right). \quad (35)
$$

Substituting $x$ for $\frac{0.04}{\varepsilon^2}$ in (34) shows that when $\frac{0.04}{\varepsilon^2} > 3$ then

$$
P\left(\sqrt{\lambda_{\max}} > \frac{0.1}{\sqrt{2\varepsilon}}\right) \leq 6^n \exp(-\|\alpha\|^2_2).
$$

The exact same considerations as in (35) also show that

$$
P\left(\left\|\mathcal{N}(0, D_\alpha^2)\right\| < \frac{1.1}{\sqrt{2}} \|\alpha\|^2_2\right) \\
\leq P\left(\left|\sum_i \alpha_i^2 \chi_i^2 - \|\alpha\|^2_2\right| > \frac{0.9}{2} \|\alpha\|^2_2\right) \\
\leq 2 \exp\left(-\frac{0.9}{4} \|\alpha\|^2_2\right) \leq 2 \exp\left(-\frac{\|\alpha\|^2_2}{8}\right).
$$

Plugging the above two displays into (33), when $\varepsilon$ is small enough, yields

$$
P(\lambda_{\min} < 1/2) \leq 2 \left(\frac{3}{\varepsilon}\right)^n e^{-\frac{\|\alpha\|^2_2}{8}} + 2 \cdot 6^n e^{-\|\alpha\|^2_2} \leq 4 \exp\left(\frac{3n}{\varepsilon} - \frac{\|\alpha\|^2_2}{8}\right). \quad (36)
$$

For general $0 < s < 1/2$, in a similar fashion to (33), using an $s$-net gives the bound

$$
P(\lambda_{\min} < s) \leq \left(\frac{3}{s}\right)^n P\left(\left\|\mathcal{N}(0, D_\alpha^2)\right\| < \sqrt{1.1s \|\alpha\|^2_2}\right) + P\left(\sqrt{\lambda_{\max}} > 0.1/\sqrt{s}\right). \quad (37)
$$

Now, $\mathcal{N}(0, D_\alpha^2)$ can be written as $D_\alpha Z_d$ where $Z_d$ is a standard Gaussian $d$-dimensional vector. In [LMOTJ07], Proposition 2.6, it was shown that there exists universal constants $C_L, C' > 0$ such that for any $t < C'$:

$$
P\left(\|D_\alpha Z\| < t \|D_\alpha\|^2_{HS}\right) \leq \exp\left(C_L \ln(t) \left(\frac{\|D_\alpha\|_{HS}}{\|D_\alpha\|_{op}}\right)^2\right) = \exp\left(C_L \ln(t) \|\alpha\|^2_2\right) = t^{C_L\|\alpha\|^2_2},
$$

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with equality stemming from the facts that \( \|D_\alpha\|_{HS} = \|\alpha\|_2 \) and \( \|D_\alpha\|_{op} = \|\alpha\|_\infty = 1 \). Thus
\[
P \left( \|N(0, D_\alpha^2)\| < \sqrt{1.1s \|\alpha\|_2^2} \right) \leq 2s^{\frac{C_2}{2}} \|\alpha\|_2^2.
\] (38)

By revisiting (34) and replacing \( \sqrt{2\varepsilon} \) with \( \sqrt{s} \) we note that for small \( s \)
\[
P \left( \sqrt{\lambda_{\text{max}}} > 0.1/\sqrt{s} \right) \leq 6^n \exp \left( -\frac{1}{2s} \left( \frac{0.01}{2} - s \right) \|\alpha\|_2^2 \right) \leq 6^n e^{-\frac{0.01\|\alpha\|_2^2}{4s}}.
\] (39)

And, provided that \( s \leq \frac{0.01}{4} \), (35) shows
\[
P(\lambda_{\text{min}} < s) \leq 2 \left( \frac{3}{s} \right)^n s^{\frac{C_2}{2} \|\alpha\|_2^2} + \exp \left( 2n - \frac{0.01\|\alpha\|_2^2}{4s} \right), \forall s \leq \frac{0.01}{4}.
\]

We have thus shown, by combining (36), together with the last inequality into (32) and choosing \( \xi \) to be a small enough constant:
\[
\frac{n}{\xi} P(\lambda_{\text{min}} < 1/2) + n \int_0^\xi \frac{1}{s} P(\lambda_{\text{min}} < s) ds \leq
\]
\[
\frac{n}{\xi} 12 \exp \left( \frac{3n}{\varepsilon} - \frac{\|\alpha\|_2^2}{8} \right) + n \int_0^\xi 3^n s^{\frac{C_2}{2} (\|\alpha\|_2^2 - n - 1)} + \frac{1}{s} e^{\left( 2n - \frac{0.01\|\alpha\|_2^2}{4s} \right)} ds.
\]

Assuming that \( \xi \leq \frac{1}{\varepsilon} \) and that \( \|\alpha\|_2^2 > n + 1 \),
\[
n \int_0^\xi 3^n s^{\frac{C_2}{2} (\|\alpha\|_2^2 - n)} ds \leq n 3^n \xi^{\frac{C_2}{2} (\|\alpha\|_2^2 - n)} \leq ne^{\frac{C_2}{2} (n - \|\alpha\|_2^2 + 2n)},
\]
\[
n \int_0^\xi \frac{1}{s} e^{\left( 2n - \frac{0.01\|\alpha\|_2^2}{4s} \right)} ds \leq ne^{2n} \int_0^\xi e^{-\frac{0.01\|\alpha\|_2^2}{8s}} ds \leq ne^{2n} \xi e^{-\frac{0.01\|\alpha\|_2^2}{8\xi}}.
\]

To obtain the desired we observe that if \( n^3 \left( \frac{\|\alpha\|_2^2}{\|\alpha\|_4^2} \right)^4 \rightarrow 0 \) then \( \left( \frac{\|\alpha\|_2^2}{\|\alpha\|_4^2} \right)^4 \gg n^3 \), the inequality
\[
\|\alpha\|_2^2 = \left( \frac{\|\alpha\|_4}{\|\alpha\|_2} \right)^{4/3} \|\alpha\|_2^2 \gg n \|
\]
implies \( \|\alpha\|_2^2 \gg n \), which shows the existence of a constant \( C' > 0 \) for which
\[
E_Y \left( \ln \det \left( \frac{1}{\|\alpha\|_2^2} \mathbf{Y}D_\alpha \mathbf{Y}^T \right) \mathbb{I}_{\{\lambda_{\text{min}} < 1/2\}} \right) < n \exp(-C' \|\alpha\|_2^2).
\]
To finish the prove of Theorem 2(b) we must now deal with $TV(K_p(M(n)), H_{p,\alpha}(M(n)))$.

**Lemma 10.** Assume $n^3 \left( \frac{\|\alpha\|_4}{\|\alpha\|_2} \right)^4 \to 0$, then $TV(K_p(M(n)), H_{p,\alpha}(M(n))) \to 0$.

**Proof.** First, we again pass to relative entropy using (8), Pinsker’s inequality:

$$TV(K_p(M(n)), H_{p,\alpha}(M(n))) \leq \sqrt{\text{Ent}[K_p(M(n)) || H_{p,\alpha}(M(n))]}.$$  

We note that both $K_p(M(n))$ and $H_{p,\alpha}(M(n))$ are simply Bernoulli matrices. The entries of $K_p(M(n))$ are i.i.d. Bernoulli$(p)$, while the entries of $H_{p,\alpha}(M(n))$ are i.i.d. Bernoulli$(p')$ where $p' = \Phi^{-1}(\frac{\|\alpha\|_2^2}{\|\alpha\|_4^2})$. Defining $\text{Ent}[p||p'] := \text{Ent}[\text{Bernoulli}(p)||\text{Bernoulli}(p')]$ and using the chain rule (9) for relative entropy yields

$$\text{Ent}[K_p(M(n)) || H_{p,\alpha}(M(n))] \leq n^2 \text{Ent}[p||p'].$$

One may verify that

$$\lim_{p' \to p'} \frac{\text{Ent}[p||p']}{(p-p')^2} = \lim_{p' \to p} \frac{p \ln(\frac{p}{p'}) + (1-p) \ln(\frac{1-p}{1-p'})}{(p-p')^2} = \frac{1}{2p-2p^2}.$$

So, $\frac{\text{Ent}[p||p']}{(p-p')^2}$ is a continuous function on $(0, 1) \times (0, 1)$ and is bounded on every compact subset of its domain. Thus, there exists a constant $C_p$, depending on $p$ such that

$$\text{Ent}[p||p'] \leq C_p (p-p')^2.$$

By Lemma 4, $|p-p'| \leq 3 \left( \frac{\|\alpha\|_3}{\|\alpha\|_2} \right)^3$, which affords the bound

$$\text{Ent}[p||p'] \leq 9C_p \left( \frac{\|\alpha\|_3}{\|\alpha\|_2} \right)^6.$$

But now, by Cauchy-Schwartz’s inequality, $\|\alpha\|_3^3 = \sum_i \alpha_i \alpha_i^2 \leq \sqrt{\|\alpha\|_2^2 \|\alpha\|_4^2}$. Combining all of the above

$$TV(K_p(M(n)), H_{p,\alpha}(M(n)))^2 \leq \text{Ent}[K_p(M(n)) || H_{p,\alpha}(M(n))]$$

$$\leq n^2 \text{Ent}[p||p'] \leq 9C_p n^2 \left( \frac{\|\alpha\|_3}{\|\alpha\|_2} \right)^6 \leq n^2 \frac{\|\alpha\|_4^4 \|\alpha\|_2^2}{\|\alpha\|_2^6} < n^3 \left( \frac{\|\alpha\|_4}{\|\alpha\|_2} \right)^4.$$

\[
\square
\]

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