THRESHOLD DYNAMICS OF A REACTION-DIFFUSION EPIDEMIC MODEL WITH STAGE STRUCTURE

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ABSTRACT. A time-delayed reaction-diffusion epidemic model with stage structure and spatial heterogeneity is investigated, which describes the dynamics of disease spread only proceeding in the adult population. We establish the basic reproduction number $R_0$ for the model system, which gives the threshold dynamics in the sense that the disease will die out if $R_0 < 1$ and the disease will be uniformly persistent if $R_0 > 1$. Furthermore, it is shown that there is at least one positive steady state when $R_0 > 1$. Finally, in terms of general birth function for adult individuals, through introducing two numbers $\tilde{R}_0$ and $\hat{R}_0$, we establish sufficient conditions for the persistence and global extinction of the disease, respectively.

1. Introduction. As discussed in Thieme [30], when we describe the spread of infectious disease, all the interesting structures that could and should be considered. Spatial heterogeneity of the environment and spatial-temporal movement of individuals play an important role in the dynamics of infectious disease (see, e.g., [27, 40]). Assuming some types of host random movement, there are increasing interests to formulate and analyze infectious disease by reaction-diffusion equations (see, e.g., [1, 5, 10, 9]). Sometimes delay or non-local delay effects would be incorporated into reaction-diffusion epidemic models (see, e.g., [11, 18, 20, 26, 36] and the references therein). It is necessary to point that another common way to study the spread of disease in a heterogeneous population is to assume the immigration of infective individuals, which is described by patchy models (see, e.g., [17, 19, 34, 35]). In fact, a constant immigration term has a mildly stabilizing effect on the dynamics and tends to increase the minimum number of infective individuals in the models (see, e.g., [2]).

In nature, as usual, the individual members of population undergo life history through two stages, immature and mature. For vector-borne disease, Dengue fever is transmitted to humans by the mature female Aedes aegypti mosquito (see, e.g., [36]). For some disease, such as sexual disease, it is reasonable to consider the disease transmission in adult population and neglect transmission in juveniles (see, e.g., [35]). Sometimes it seems unreasonable for us to assume that all the individuals in a bounded habitat are commonly susceptible for the disease and have the ability to
transmit the disease. Therefore, it is important for us to incorporate stage structure of individual into epidemic model to understand the transmission dynamics of the infectious disease. This work intends to take stage structure, spatial heterogeneity and spatial-temporal movement of individuals into consideration of epidemic models. So in the following, we consider that the host population has two stages: juvenile stage and adult stage. For simplicity, we assume that (see, e.g., [35])

(A1): disease transmission occurs only in adult individuals, and juvenile individuals are immune to the disease;
(A2): juvenile individuals do not have the ability to reproduce, and adults are responsible for the reproduction of the population.

Let \( u_j(t, x) \) be the density of juvenile individuals at time \( t \) and location \( x \). Then

\[
u_j(t, x) = \int_0^\tau u(t, a, x) da,
\]

where \( u(t, a, x) \) is the density of individuals with age \( a \) at time \( t \) and location \( x \), and \( \tau \) the length of the juvenile period. Denote \( A(t, x) \) as the density of adult individuals at time \( t \) and location \( x \). Then \( u(t, a, x) \) and \( A(t, x) \) satisfy (see, e.g., Metz and Diekmann [23])

\[
\begin{aligned}
\partial_t u + \partial_a u &= d_j(a)\Delta u(t, a, x) - \mu_j(a)u(t, a, x), \quad 0 < a < \tau, x \in \Omega, \\
u(t, 0, x) &= f(x, A(t, x)), \quad t \geq -\tau, x \in \Omega, \\
\partial_t A &= d_m\Delta A(t, x) - \mu(x)A(t, x) + u(t, \tau, x), \quad t > 0, x \in \Omega,
\end{aligned}
\]

where \( f(x, A(t, x)) \) and \( \mu(x)A(t, x) \) is the birth and mortality function of adult individuals, respectively, and \( \mu_j(a) \) denotes the per capita mortality rate of juvenile at age \( a \), \( \Delta \) is the Laplacian operator on \( \mathbb{R}^N \), \( \Omega \) is a bounded and open subset of \( \mathbb{R}^N \) with a smooth boundary \( \partial\Omega \). The term \( u(t, \tau, x) \) of the third equation in (2) is the adults recruitment term, being those of maturation age \( \tau \). For simplicity, we assume \( d_j(a) = d_j, \mu_j(a) = \mu_j \), that is, diffusion rate and mortality rate of juvenile individuals are independent of age \( a \). Let \( v(r, a, x) = u(a+r, a, x) \) with \( r \geq 0 \). Then it follows that

\[
\begin{aligned}
\frac{\partial v(r, a, x)}{\partial a} &= \left[ \frac{\partial u(t, a, x)}{\partial t} + \frac{\partial u(t, a, x)}{\partial a} \right]_{t=a+r} \\
&= d_j\Delta u(a+r, a, x) - \mu_j u(a+r, a, x) \\
&= d_j\Delta v(r, a, x) - \mu_j v(r, a, x) \\
v(r, 0, x) &= f(x, A(r, x)).
\end{aligned}
\]

Regarding \( r \) as a parameter and integrating the last equation, we obtain

\[
v(r, a, x) = e^{-\mu_j r} \int_\Omega \Gamma(d_j a, x, y) f(x, A(r, y)) dy,
\]

where \( \Gamma \) is the Green function associated with \( \Delta \) and the Neumann boundary condition. Since \( u(t, \tau, x) = v(t-\tau, \tau, x), \forall t \geq \tau \), we have

\[
u(t, \tau, x) = e^{-\mu_j \tau} \int_\Omega \Gamma(d_j \tau, x, y) f(y, A(t-\tau, y)) dy.
\]

(3)

Differentiating (1) with respect to \( t \) and making use of (2) and (3), it then follows that \( u_j(t, x) \) and \( A(t, x) \) satisfy

\[
\begin{aligned}
\frac{\partial u_j}{\partial t} &= d_j\Delta u_j(t, x) + f(x, A(t, x)) - \mu_j u_j(t, x) - u(t, \tau, x), \\
\frac{\partial A}{\partial t} &= d_m\Delta A(t, x) - \mu(x)A(t, x) + u(t, \tau, x).
\end{aligned}
\]
We consider a disease transmission of SIS type with nonlinear incidence. According to the principle of mass action, bilinear incidence rate which reflects mechanism of disease transmission could be adopted in classical epidemic model. It has been shown that the disease transmission process may have a nonlinear incidence rate (see, e.g. [15, 14] and the reference therein). We employ saturating incidence to describe the transmission process of the disease. Let \( S = S(t, x), I = I(t, x) \) be sub-population of susceptible, infectious classes, respectively. Then \( A(t, x) = S(t, x) + I(t, x) \). Therefore, we obtain the following model:

\[
\begin{align*}
\frac{\partial S}{\partial t} &= d_S \Delta S(t, x) - \mu(x)S(t, x) - g(x, I(t, x))S(t, x) + \gamma(x)I(t, x), \\
\frac{\partial I}{\partial t} &= d_I \Delta I(t, x) - (\mu(x) + \gamma(x))I(t, x) + g(x, I(t, x))S(t, x),
\end{align*}
\]

where \( g(x, I(t, x)) = \frac{\beta(x)I(t, x)}{\gamma(x) + \beta(x)} \) is saturating incidence, both \( \alpha(x) \) and \( \beta(x) \) are positive Hölder continuous functions on \( \Omega \). Substituting (3) into the second equation of (4), and dropping the \( u_j(t, x) \) equation from (4) (since \( u(t, \tau, x) \) does not depend on the variables of juveniles) result in the following system containing \( S(t, x) \) and \( I(t, x) \) only:

\[
\begin{align*}
\frac{\partial S(t, x)}{\partial t} &= d_S \Delta S(t, x) - \mu(x)S(t, x) - g(x, I(t, x))S(t, x) + \gamma(x)I(t, x) \\
\frac{\partial I(t, x)}{\partial t} &= d_I \Delta I(t, x) - (\mu(x) + \gamma(x))I(t, x) + g(x, I(t, x))S(t, x).
\end{align*}
\]

For simplicity, letting \((u_1, u_2) = (S, I), (d_1, d_2) = (d_S, d_I)\), we investigate the following time-delayed and non-local reaction-diffusion system with Neumann boundary condition:

\[
\begin{align*}
\frac{\partial u_1(t, x)}{\partial t} &= d_1 \Delta u_1(t, x) - \mu(x)u_1(t, x) - g(x, u_2(t, x))u_1(t, x) + \gamma(x)u_2(t, x) \\
&+ e^{-\mu \tau} \int_{\Omega} \Gamma(d_1 \tau, x, y) f(y, A(t - \tau, y)) dy, \\
\frac{\partial u_2(t, x)}{\partial t} &= d_2 \Delta u_2(t, x) - (\mu(x) + \gamma(x))u_2(t, x) + g(x, u_2(t, x))u_1(t, x) \\
&+ e^{\mu \tau} \int_{\Omega} \Gamma(d_2 \tau, x, y) f(y, A(t - \tau, y)) dy, \\
\frac{\partial u_1(t, x)}{\partial n} &= \frac{\partial u_2(t, x)}{\partial n} = 0, \quad t > 0, x \in \partial \Omega.
\end{align*}
\]

We assume that

\((F)\): \( f(\cdot, 0) \equiv 0, \partial A f(x, 0) > 0, \forall x \in \Omega, f \) is bounded on \( \overline{\Omega} \times \mathbb{R}^+, \) and for each \( x \in \Omega, f(x, \cdot) : \mathbb{R}^+ \to \mathbb{R}^+ \) is strictly sub-homogeneous in the sense that \( f(x, \alpha A) > \alpha f(x, A), \forall \alpha \in (0, 1), A > 0. \)

The rest of this paper is organized as follows. In the next section, we study the well-posedness of model system (5) and introduce the basic reproduction number for model (6). In section 3, based on the monotonicity of the birth function on the density of adult individuals, we establish threshold dynamics in terms of the basic reproduction number. Section 4 is devoted to establish sufficient conditions for the persistence and global extinction of disease under the general birth function. Furthermore, a spatially homogeneous case of model (6) with the same diffusion rate of susceptible and infectious adult individuals is studied.

2. The basic reproduction number. Let \( X := C(\overline{\Omega}, \mathbb{R}^2) \) and \( Y := C(\overline{\Omega}, \mathbb{R}) \) be the Banach space of continuous functions with values in the real plane with the
strongly order spaces. Given a function \( \| \cdot \| \) by \( u \) or it can be rewritten as the following integral equation:

\[
\begin{align*}
\left( T_i(t) \varphi_i \right)(x) &= \int_{\Omega} \Gamma_i(t, x, y) \varphi_i(y) dy, \quad t \geq 0, \quad i = 1, 2,
\end{align*}
\]

where \( \Gamma_1 \) and \( \Gamma_2 \) are the Green functions associated with \( d_1 \Delta - \mu(\cdot)I \) and \( d_2 \Delta - (\mu(\cdot) + \gamma(\cdot))I \) subject to the Neumann boundary condition, respectively. It then follows that for each \( t > 0 \), \( T_i(t) : Y \rightarrow Y, i = 1, 2 \) is compact and strongly positive (see, e.g., [28, Section 7.1 and Corollary 7.2.3]). Moreover, \( T(t) := \text{diag}(T_1(t), T_2(t)) : X \rightarrow X, t \geq 0 \) is a \( C_0 \) semigroup. Let \( \mathcal{A}_i : \mathcal{D}(\mathcal{A}_i) \rightarrow Y \) be the generator of \( T_i(t), i = 1, 2 \). Then \( T(t) : X \rightarrow X \) is a semigroup generated by the operator \( \mathcal{A} = (\mathcal{A}_1, \mathcal{A}_2) \) defined on \( \mathcal{D}(\mathcal{A}) = \mathcal{D}(\mathcal{A}_1) \times \mathcal{D}(\mathcal{A}_2) \). Define \( F = (F_1, F_2) : C^+_\tau \rightarrow X \) by

\[
\begin{align*}
F_1(\phi)(x) := & -g(x, \varphi_2(0, x)) \varphi_1(0, x) + \gamma(x) \varphi_2(0, x) \\
& + e^{-\mu_1 \tau} \int_{\Omega} \Gamma(d_1 \tau, x, y) f(y, \varphi_1(-\tau, y) + \varphi_2(-\tau, y)) dy,
\end{align*}
\]

\[
F_2(\phi)(x) := g(x, \varphi_2(0, x)) \varphi_1(0, x)
\]

for \( x \in \Omega \) and \( \phi = (\varphi_1, \varphi_2) \in C^+_\tau \). Then [5] can be rewritten as the following abstract functional differential equation:

\[
\begin{align*}
\frac{d u}{d \tau} &= \mathcal{A} u + F(u), \\
u_0 &= \phi \in C^+_\tau,
\end{align*}
\]

or it can be rewritten as the following integral equation

\[
u(t) = T(t) \phi + \int_0^t T(t-s) F(u_s) ds,
\]

where \( u := (u_1, u_2) \). The following result asserts that solutions of system [5] exist globally on \([0, \infty)\).

**Theorem 2.1.** Let (F) holds. For any \( \phi \in C^+_\tau \), system [5] has a unique solution \( u(t, \cdot; \phi) \) on \([0, \infty)\) with \( u_0 = \phi \). Moreover, the solution semiflow \( \Phi(t) = u_t(\cdot) : C^+_\tau \rightarrow C^+_\tau, t \geq 0, \) has a global compact attractor.

**Proof.** Firstly, we show the global existence of the unique solution. According to (F) and the boundedness of \( \Gamma(d_1 \tau, x, y) \), there exists a positive number \( \tilde{H} \) such that

\[
e^{-\mu_1 \tau} \int_{\Omega} \Gamma(d_1 \tau, x, y) f(y, \psi(t - \tau, y)) dy \leq \tilde{H}, \quad \forall t \geq 0, \quad \psi \in C^+_\tau.
\]
For any \( \varphi = (\varphi_1, \varphi_2) \in C^+_r \), consider the following linear cooperative reaction-diffusion system

\[
\begin{cases}
\frac{\partial v_1(t,x)}{\partial t} = d_1 \Delta v_1(t,x) - \mu(x)v_1(t,x) + \gamma(x)v_2(t,x) + \tilde{H}, & t > 0, \ x \in \Omega, \\
\frac{\partial v_2(t,x)}{\partial t} = d_2 \Delta v_2(t,x) - (\mu(x) + \gamma(x))v_2(t,x) + \frac{\beta(x)}{\alpha(x)}v_1(t,x), & t > 0, \ x \in \Omega, \\
\frac{\partial v_1(t,x)}{\partial n} = 0, & t > 0, \ x \in \partial \Omega, \\
v_1(0,x) = \varphi_1(0,x), v_2(0,x) = \varphi_2(0,x), & x \in \Omega.
\end{cases}
\]

(6)

By [8, Theorem 4.2], we conclude that the unique of the system (6) exists globally on \([0, \infty)\). For any \( \varphi \in C^+_r \), let \( v^+(t,x,\varphi(0,x)) \) be the solution of (6) with initial value \( v^+(0,x) = \varphi(0,x) \). Define

\[v^+_1(t,x,\varphi) = \begin{cases} v^+(t + \theta, x, \varphi(0)), & t + \theta > 0, t > 0, \theta \in [-\tau, 0], \\
\varphi(t + \theta, x), & t + \theta \leq 0, t > 0, \theta \in [-\tau, 0]. \end{cases}\]

For any \( \varphi \in C^+_r \), set

\[
B^+(\varphi)(x) = \begin{pmatrix} B^+_1(\varphi)(x) \\ B^+_2(\varphi)(x) \end{pmatrix} = \begin{pmatrix} \gamma(x)\varphi_2(0,x) + \tilde{H} \\ \frac{\beta(x)}{\alpha(x)}\varphi_1(0,x) \end{pmatrix},
\]

and

\[
B^-(\varphi)(x) = \begin{pmatrix} B^-_1(\varphi)(x) \\ B^-_2(\varphi)(x) \end{pmatrix} = \begin{pmatrix} \frac{\beta(x)}{\alpha(x)}\varphi_1(0,x) \\ 0 \end{pmatrix}.
\]

Then it is easy to see that the functions \( v^+(t,x) \) and \( v^-(t,x) \equiv 0 \) on \((t,x) \in [-\tau, \infty) \times \Omega \) satisfy

\[
v^+(t) = T(t-s)v^+(s) + \int_s^t T(t-r)B^+(v^+_1)dr
\]

and

\[
v^-(t) = T(t-s)v^-(s) + \int_s^t T(t-r)B^-(v^-_1)dr.
\]

Take \( B(\varphi) = F(\varphi) \) for any \( \varphi \in C^+_r \). Then \( B \) is Lipschitz continuous on \( C^+_r \). For any \( \varphi \in C^+_r \) with \( v^+(t + \theta) \leq \varphi \leq v^+(t,x) \) for \( t \in [0, \infty) \) and \( \theta \in [-\tau, 0] \), since

\[
v^+(t) - \varphi(0) + h \left[ B^+(v^+_1) - B(\varphi) \right] \geq 0 \text{ in } X^+
\]

and

\[
\varphi(0) - v^+(t) + h \left[ B(\varphi) - B^-(v^-_1) \right] \geq 0 \text{ in } X^+,
\]

we have

\[
\lim_{h \to 0^+} \frac{1}{h} \text{dist } (v^+(t) - \varphi(0) + h \left[ B^+(v^+_1) - B(\varphi) \right], X^+) = 0
\]

and

\[
\lim_{h \to 0^+} \frac{1}{h} \text{dist } (\varphi(0) - v^-(t) + h \left[ B(\varphi) - B^-(v^-_1) \right], X^+) = 0
\]

for any \( t \geq 0 \). It then follows from \[22, Proposition 3\] with \( S^+(t,s) = T(t-s), S(t,s) = T(t-s) \) and \( T(t,s) = T(t-s) \) that there exists a unique solution \( u(t, x, \phi) \) of \([5]\) on \([0, \infty)\) such that \( v^-(t,x) \leq u(t,x,\phi) \leq v^+(t,x) \) for any \( \phi \in C^+_r \). Thus, the system \([5]\) generates a semiflow \( \Phi(t) = u_t(\cdot) : C^+_r \to C^+_r \) by \( (\Phi(t)\phi)(\theta,x) = u(t + \theta, x, \phi) \) for \( t \geq 0, \theta \in [-\tau, 0] \) and \( x \in \Omega \).

In the following we prove the point dissipativeness of the solution semiflow \( \Phi(t) \). Given \( \phi \in C^+_r \), let \( (u_1(t,x), u_2(t,x)) = (u_1(t, \phi)(x), u_2(t, \phi)(x)) \) denote the solution
of system (5). Set \( \bar{\mu} = \min_{x \in \Omega} \mu(x) \) and \( \bar{A}(t) = \int_{\Omega} [u_1(t,x) + u_2(t,x)] dx \). Since \( A(t,x) = u_1(t,x) + u_2(t,x) \), by (5) and Green’s formula, it follows that

\[
\frac{d\bar{A}(t)}{dt} \leq -\bar{\mu} \bar{A}(t) + e^{-\mu_1 t} \int_{\Omega} \int_{\Omega} \Gamma(d_j \tau, x, y) f(y, A(t-\tau, y)) dy dx, \ t > \tau.
\]

By (F) and the boundedness of \( \Gamma(d_j \tau, x, y) \), there exists a positive number \( k_1 \) independent of \( \phi \), such that

\[
\frac{d\bar{A}(t)}{dt} \leq -\bar{\mu} \bar{A}(t) + k_1, \ t \geq 0,
\]

that is,

\[
\frac{d(e^{\bar{\mu} t} \bar{A}(t))}{dt} \leq k_1 e^{\bar{\mu} t}, \ t \geq 0.
\]

Thus, there exists a \( t_1 = t_1(\phi) \) \( \geq \tau \) such that \( \bar{A}(t) \leq \frac{k_1}{\bar{\mu}} + 1 \) for all \( t > t_1 \), which implies that

\[
\|u_i(t, \cdot)\|_{L^1(\Omega)} \leq \frac{k_1}{\bar{\mu}} + 1, \ \forall t > t_1, \ i = 1, 2.
\]

Consequently, with the aid of [24, Lemma 3.1] (see also, [16, Theorem 1 and Corollary 1]), we conclude that there exists a positive constant \( K \) independent of \( \phi \) such that

\[
\|u_i(t, \cdot)\|_{L^\infty(\Omega)} \leq K, \ \forall t > t_1 = t_1(\phi), \ i = 1, 2,
\]

which implies that the solutions of system (5) are ultimately bounded, and hence, \( \Phi(t) : C_t^+ \rightarrow C_t^+ \) is point dissipative. Moreover, \( \Phi(t) : C_t^+ \rightarrow C_t^+ \) is compact for each \( t > \tau \) (see, [39, Theorem 2.2.6]). It then follows from [12, Theorem 3.4.8] that \( \Phi(t) : C_t^+ \rightarrow C_t^+, \forall t \geq 0, \) has a global compact attractor.

Consider the following time-delayed reaction-diffusion equation

\[
\begin{cases}
\frac{\partial A(t,x)}{\partial t} = d \Delta A(t,x) - \mu(x) A(t,x) \\
+ e^{-\mu_1 t} \int_{\Omega} \Gamma(d_j \tau, x, y) f(y, A(t-\tau, y)) dy, \ t > 0, \ x \in \Omega, \\
\frac{\partial A(t,x)}{\partial n} = 0, \ t > 0, \ x \in \partial \Omega, \\
A(s, x) = \psi(s, x), \ s \in [-\tau, 0], \ x \in \Omega,
\end{cases}
\]

(7)

where \( d > 0, \mu(x) \) is a positive Hölder continuous function on \( \Omega \). Let

\[
Z = C([-\tau, 0], Y) \quad \text{and} \quad Z^+ = C([-\tau, 0], Y^+),
\]

where \( Y^+ := C(\Omega, \mathbb{R}_+) \). Define \( \| \cdot \|_{Z} := \max_{\theta \in [-\tau, 0]} \| \psi(\theta) \|_Y \). Then, both \( (Y, Y^+) \) and \( (C, C^+) \) are strongly ordered spaces. In view of the proof of [41, Theorem 3.1], we have the following result.

**Lemma 2.2.** Assume (F) holds. For each \( \psi \in Z^+ \), problem (7) admits a unique mild solution \( A(t, \cdot; \psi) \) on \([0, \infty)\) with \( A_0 = \psi, \) and \( A(t, \cdot; \psi), \forall t > \tau, \) is a classical solution of (7). Moreover, the solution semiflow \( Q(t) = A_t(\cdot) : Z^+ \rightarrow Z^+, t \geq 0, \) has a global compact attractor.

Using arguments similar to those in [28, Theorem 7.6.1] (see also [32, Theorem 2.2]), it is shown that the following nonlocal elliptic eigenvalue problem

\[
\begin{cases}
\lambda \bar{\psi}(x) = d \Delta \bar{w} - \mu(x) \bar{w} + e^{-\mu_1 \tau} \int_{\Omega} \Gamma(d_j \tau, x, y) \partial_A f(y, 0) \bar{w}(y) dy, \ x \in \Omega, \\
\frac{\partial \bar{w}(x)}{\partial n} = 0, \ x \in \partial \Omega
\end{cases}
\]

is equivalent to

\[
\lambda \bar{\psi}(x) = d \Delta \bar{w} - \mu(x) \bar{w} - e^{-\mu_1 \tau} \int_{\Omega} \Gamma(d_j \tau, x, y) \partial_A f(y, 0) \bar{w}(y) dy, \ x \in \Omega, \\
\frac{\partial \bar{w}(x)}{\partial n} = 0, \ x \in \partial \Omega.
\]
has a principal eigenvalue denoted by $\lambda_0(d, \tau, \partial_A f(\cdot, 0))$. By [32] Theorem 2.2, the following nonlocal elliptic eigenvalue problem

$$
\begin{align*}
\lambda \tilde{w}(x) &= d \Delta \tilde{w} - \mu(x) \tilde{w} + e^{-\mu \tau} e^{-\lambda \tau} \int_\Omega \Gamma(d_j \tau, x, y) \partial_A f(y, 0) \tilde{w}(y) dy, x \in \Omega, \\
\frac{\partial \tilde{w}(x)}{\partial n} &= 0, \quad x \in \partial \Omega,
\end{align*}
$$

has a principal eigenvalue denoted by $\tilde{\lambda}_0(d, \tau, \partial_A f(\cdot, 0))$, and $\tilde{\lambda}_0(d, \tau, \partial_A f(\cdot, 0))$ has the same sign as $\lambda_0(d, \tau, \partial_A f(\cdot, 0))$.

For any $\psi \in Z^+$, let $A(t; \psi)(\cdot) = A(t; \cdot; \psi)$ denote the solution of (7). At what follows, we establish the threshold dynamics for system (7).

**Lemma 2.3.** Let (F) hold and let $\psi^* \in \text{int}(Y^+)$ be fixed.

(i): If $\lambda_0(d, \tau, \partial_A f(\cdot, 0)) < 0$, then the solution $A(t; \cdot; \psi)$ of (7) satisfies $\lim_{t \to \infty} A(t, x; \psi) = 0$ uniformly for $x \in \Omega$.

(ii): If $\lambda_0(d, \tau, \partial_A f(\cdot, 0)) > 0$, then (7) admits at least one positive steady state $A^*(\cdot)$, and there exists a $\varsigma > 0$ such that for every $\psi \in C^+$ with $\psi(0, \cdot) \not\equiv 0$, there is a $t_0 = t_0(\psi) > 0$ such that $A(t, x; \psi) \geq \varsigma \psi^*(x)$ for all $t \geq t_0, x \in \Omega$.

(iii): If $\lambda_0(d, \tau, \partial_A f(\cdot, 0)) > 0$, and $\partial_A f(x, A) \geq 0$ for all $x \in \Omega$ and $A \in (0, \infty)$. Then (7) admits a unique positive steady state $A^*$, which satisfies $\lim_{t \to \infty} A(t, x; \psi) = A^*(\cdot)$ for every $\psi \in Z^+$ with $\psi(0, \cdot) \not\equiv 0$.

The proofs of Lemma 2.3 (i) and (ii) are completely similar to those in [41] Theorem 3.1 and the proof of Lemma 2.3 (iii) is also similar to that in [41] Theorem 3.2 (1)], so we omit the details of the proofs of Lemma 2.3. In order to find the disease-free equilibrium (infection-free steady state), we set $u_2 = 0$ in system (5), leading to the following equation for the density of susceptible host population:

$$
\begin{align*}
\frac{\partial u_1(t, x)}{\partial t} &= d_1 \Delta u_1(t, x) - \mu(x) u_1(t, x) + e^{-\mu \tau} \int_\Omega \Gamma(d_j \tau, x, y) f(y, u_1(t - \tau, y)) dy, \quad t > 0, \quad x \in \Omega, \\
\frac{\partial u_1(t, x)}{\partial n} &= 0, \quad t > 0, \quad x \in \partial \Omega.
\end{align*}
$$

As in Lemma 2.3, the nonlocal elliptic eigenvalue problem

$$
\begin{align*}
\lambda \hat{w}(x) &= d_1 \Delta \hat{w}(x) - \mu(x) \hat{w}(x) + e^{-\mu \tau} \int_\Omega \Gamma(d_j \tau, x, y) \partial_{u_1} f(y, 0) \hat{w}(y) dy, \quad x \in \Omega, \\
\frac{\partial \hat{w}(x)}{\partial n} &= 0, \quad x \in \partial \Omega
\end{align*}
$$

has a principal eigenvalue, which is denoted by $\lambda_0(d_1, \tau, \partial_{u_1} f(y, 0))$. We further make the following assumption:

(F1): $\lambda_0(d_1, \tau, \partial_{u_1} f(\cdot, 0)) > 0$, and $\partial_A f(x, A) \geq 0$ for all $x \in \Omega$ and $A \in (0, \infty)$.

Then by Lemma 2.3 (iii), we have the following result on the uniqueness of disease-free equilibrium.

**Theorem 2.4.** Let (F) and (F1) hold. Then equation (8) admits a positive steady state $u_1^*(x)$ which is globally attractive in $Y^+ \setminus \{0\}$, and hence, system (5) admits a unique disease-free equilibrium $(u_1^*(x), 0)$.

Linearizing system (5) at the disease-free equilibrium $(u_1^*(x), 0)$, we get the following system for infectious component $u_2$:

$$
\begin{align*}
\frac{\partial u_2}{\partial t} &= d_2 \Delta u_2(t, x) - (\mu(x) + \gamma(x)) u_2(t, x) + \beta(x) u_1^*(x) u_2(t, x), \quad t > 0, \quad x \in \Omega, \\
\frac{\partial u_2}{\partial n} &= 0, \quad t > 0, \quad x \in \partial \Omega.
\end{align*}
$$
Substituting $u_2(t, x) = e^{\lambda t} \varphi(x)$ into (9), we obtain the following eigenvalue problem:

\[
\begin{cases}
\lambda \varphi(x) = d_2 \Delta \varphi(x) - (\mu(x) + \gamma(x))\varphi(x) + \beta(x)u_1^*(x)\varphi(x), & x \in \Omega, \\
\frac{\partial \varphi(x)}{\partial n} = 0, & x \in \partial \Omega.
\end{cases}
\]

(10)

It then follows from [28, Theorem 7.6.1] that (10) has a principal eigenvalue denoted by $\lambda(d_2, u_1^*)$ with a positive eigenfunction.

In the following, we introduce the basic reproduction number for system (5).

Suppose that host population is near the disease-free equilibrium. We introduce the next generation operator:

\[
L(\varphi)(x) := \int_0^\infty \beta(x)u_1^*(x)T_2(t)\varphi(x)\,dt, \quad x \in \Omega.
\]

Motivated by [7, 33, 31, 36, 37], we define the spectral radius of $L$ as the basic reproduction number for model (5), that is,

\[
R_0 := r(L).
\]

By [37, Theorem 3.1] with diffusion rate independent on spatial variable $x$, we have the following observation.

**Lemma 2.5.** $R_0 - 1$ and $\lambda(d_2, u_1^*)$ have the same sign.

3. **Threshold dynamics.** In this section, we establish the threshold dynamics of the system (5) in terms of the basic reproduction number $R_0$.

Before we show the main results of this section, we will propose the following results which play an important role in establishing persistence of (5).

**Lemma 3.1.** Let $u(t, x; \phi)$ be the solution of system (5) with $u_0(\phi) = \phi \in C^+$. If there is a $T > 0$ such that $u_2(T, \cdot; \phi) \neq 0$, then $u_2(t, \cdot; \phi) > 0$ and $u_1(t, \cdot; \phi) > 0, \forall t > T$.

**Proof.** From system (5), it is easy to see that $u_2(t, x; \phi)$ satisfies

\[
\begin{cases}
\frac{\partial u_2(t, x)}{\partial t} \geq d_2 \Delta u_2(t, x) - (\mu(x) + \gamma(x))u_2(t, x), & t > 0, \ x \in \Omega, \\
\frac{\partial u_2(t, x)}{\partial n} = 0, & t > 0, \ x \in \partial \Omega.
\end{cases}
\]

If $u_2(T, \cdot, \phi) \neq 0$ for some $T \geq 0$, it then follows from parabolic maximum principle (see, e.g., [25, Theorem 4]) that $u_2(t, x; \phi) > 0$, for $t > T$ and $x \in \Omega$.

For $u_1$, we have that

\[
\begin{cases}
\frac{\partial u_1(t, x)}{\partial t} \geq d_1 \Delta u_1(t, x) - \mu(x)u_1(t, x) - \frac{\beta(x)}{\alpha(x)}u_1(t, x) + \gamma(x)u_2(t, x), & t > 0, \ x \in \Omega, \\
\frac{\partial u_1(t, x)}{\partial n} = 0, & t > 0, \ x \in \partial \Omega.
\end{cases}
\]

(11)
Let $U(t) : Y \rightarrow Y, i = 1, 2$ are the $C_0$ semigroups associated with $d_1 \Delta - (\mu(\cdot) + \frac{\beta(\cdot)}{\alpha(\cdot)})I$ subject to the Neumann boundary condition. Thus, the equation
\[
\begin{aligned}
\frac{\partial u(t,x)}{\partial t} &= d_1 \Delta u(t,x) - \left( \mu(x) + \frac{\beta(x)}{\alpha(x)} \right) u(t,x) + \gamma(x)u_2(t,x), \quad t > T, \ x \in \Omega, \\
\frac{\partial u(t,x)}{\partial n} &= 0, \quad t > T, \ x \in \partial \Omega, \\
w(T,x) &= \psi(x) \in Y, \ x \in \Omega
\end{aligned}
\]
(12)
can be rewritten as
\[
w(t,x; \psi) = U(t)(\psi)(x) + \int_0^t U(t-s)(\gamma(x)u_2(s,x)) \, ds, \quad t \geq T, \ x \in \Omega, \psi \in Y. \quad (13)
\]
Since $u_2(t,x; \phi) > 0$ for all $t > T$ and $x \in \Omega$, it follows from (13) and the property of $C_0$ semigroup $U(T)$ that for any $\psi \geq 0$ with $\psi \neq 0$, the solution of (12) satisfies $w(t,x; \psi) > 0$ for all $t > T$ and $x \in \Omega$. Hence, by (11) and the comparison principle, we have $u_1(t,x; \phi) > 0$ for all $t > T$ and $x \in \Omega$.

The following conclusion indicates that $R_0$ is a threshold index for disease extinction or persistence.

**Theorem 3.2.** Let (F) and (F1) hold, and $u_1^*(x)$ be the unique positive steady state of (8). Then the following two statements valid:

1. If $R_0 < 1$, then for any $H > 0$, there exists $\delta = \delta(H) > 0$ such that for any $\phi \in C^+_\infty$ with $0 \leq \phi_1(0, \cdot) \leq H$ and $0 \leq \phi_2(0, \cdot) \leq \delta$, there holds
   \[
   \lim_{t \to \infty} (u_1(t,x; \phi), u_2(t,x; \phi)) = (u_1^*(x), 0),
   \]
   uniformly for $x \in \Omega$.

2. If $R_0 > 1$, then system (5) admits at least one positive steady state $\hat{u}(x)$, and there exists an $\iota > 0$ such that for any $\phi \in C^+_\infty$ with $\phi_2(0, \cdot) \neq 0$, we have
   \[
   \lim_{t \to \infty} u_i(t,x; \phi) \geq \iota, \quad \forall i = 1, 2
   \]
   uniformly for all $x \in \Omega$.

**Proof.** (1). In the case where $R_0 < 1$, it follows from Lemma 2.3 that $\lambda(d_2, u_1^*(\cdot)) < 0$. For $\epsilon > 0$, consider the following reaction-diffusion equation:
\[
\begin{aligned}
\frac{\partial \hat{u}_1}{\partial t} &= d_1 \Delta \hat{u}_1(t,x) - \mu(x)\hat{u}_1(t,x) + \gamma(x)\epsilon + e^{-\mu \tau} \int_\Omega \Gamma(dy, y)f(y, \hat{u}_1(t - \tau, y) + \epsilon)dy, \ t > 0, \ x \in \Omega, \\
\frac{\partial \hat{u}_1}{\partial n} &= 0, \quad t > 0, \ x \in \partial \Omega,
\end{aligned}
\]
(14)
It then follows from Lemma 2.3 that (14) admits a globally attractive and positive steady state $u_1^*(x, \epsilon)$. Clearly, there holds
\[
\lim_{\epsilon \to 0} u_1^*(x, \epsilon) = u_1^*(x) \quad \text{uniformly in} \ x \in \Omega.
\]

In addition, since
\[
\lim_{\epsilon \to 0^+} \lambda(d_2, u_1^*(\cdot, \epsilon) + \epsilon) = \lambda(d_2, u_1^*(\cdot)) < 0,
\]
there exists an $\epsilon > 0$ such that $\lambda(d_2, u_1^*(\cdot, \epsilon) + \epsilon) < 0$, where $\lambda(d_2, u_1^*(\cdot, \epsilon) + \epsilon)$ is the principle eigenvalue of (10) with replacing $u_1^*(\cdot)$ by $u_1^*(\cdot, \epsilon) + \epsilon$.

Fixed $\epsilon > 0$ with $\lambda(d_2, u_1^*(\cdot, \epsilon) + \epsilon) < 0$. Let the positive function $w(x)$ be the eigenfunction corresponding to the principle eigenvalue $\lambda(d_2, u_1^*(\cdot, \epsilon) + \epsilon)$. Let $\xi > 0$
satisfy $\xi e^{-(d_2, u_1^*(\cdot, \cdot) + \epsilon)r} w(x) < \epsilon$ for any $x \in \overline{\Omega}$. For any given $H > 0$, it follows from the proof of Theorem 2.1 that there exists $T_1 = T_1(H) > \tau$ such that, for any $\phi \in C^+_T$ with $0 \leq \phi_1(\cdot, \cdot) \leq H$,

$$u_i(t, x; \phi) \leq K, \quad t \geq T_1 - \tau, \; x \in \overline{\Omega}, \; i = 1, 2,$$

where $K > 0$ is a positive constant independent of $H$ and $\phi$. Let $\tilde{u}_1^*(t, x; K)$ be the solution of (14) with $\tilde{u}_1^*(s, x; K) = K$ for $(s, x) \in [-\tau, 0] \times \Omega$. By the global attractivity of $u_1^*(\cdot, \epsilon)$, we have that there exists $T_2 > \tau$ such that $\tilde{u}_1^*(t, x; K) < u_1^*(x, \epsilon) + \epsilon$ for any $t \geq T_2 - \tau$ and $x \in \overline{\Omega}$. Let

$$\Lambda(H) := \max \left\{ K, \sup_{t \in [-\tau, T_1]} \sup_{x \in \overline{\Omega}, 0 \leq \phi_i \leq H} u_1(t, x; \phi) \right\}.$$

Then $u_2(t, x; \phi)$ satisfies that

$$\frac{\partial u_2}{\partial t} \leq d_2 \Delta u_2(t, x) - (\mu(x) + \gamma(x)) u_2(t, x) + \Lambda(H) \beta(x) u_2(t, x), \quad 0 \leq t \leq T_1 + T_2.$$ 

Let \( \hat{u}_2(t, x; \delta) \) be the solution of the following equation

$$\begin{cases}
\frac{\partial \hat{u}_2}{\partial t} = d_2 \Delta \hat{u}_2(t, x) - (\gamma(x) + \mu(x)) \hat{u}_2(t, x) + \Lambda(H) \beta(x) \hat{u}_2(t, x), & 0 \leq t \leq T_0, \; x \in \Omega, \\
\frac{\partial \hat{u}_2}{\partial n} = 0, & 0 \leq t \leq T_0, \; x \in \partial \Omega
\end{cases}$$

with $\hat{u}_2(0, \cdot) \equiv \delta$, where $T_0 = T_1 + T_2$. Take $\delta = \delta(H) > 0$ with $\delta < H$ small enough such that

$$\hat{u}_2(t, x; \delta) < \epsilon \quad \text{for} \; t \in [0, T_0] \; \text{and} \; x \in \overline{\Omega}$$

and

$$\hat{u}_2(T_0 + s, x; \delta) < \xi e^{t \lambda d_2, u_1^*(\cdot, \cdot) + \epsilon}s w(x), \quad \text{for} \; s \in [-\tau, 0] \; \text{and} \; x \in \overline{\Omega}.$$ 

Applying the comparison principle yields that

$$u_2(t, x; \phi) \leq \hat{u}_2(t, x; \delta) < \epsilon \quad \text{for} \; t \in [0, T_0] \; \text{and} \; x \in \overline{\Omega}$$

and

$$u_2(T_0 + s, x; \phi) \leq \hat{u}_2(T_0 + s, x; \delta) < \xi e^{t \lambda d_2, u_1^*(\cdot, \cdot) + \epsilon}s w(x) \quad \text{for} \; (s, x) \in [-\tau, 0] \times \overline{\Omega} \quad (15)$$

provided that $\phi_2(0, x) \leq \delta$ for any $x \in \overline{\Omega}$. On the other hand, since $u_1(t, x; \phi) \leq K$ for $(t, x) \in [T_1 - \tau, T_1] \times \Omega$ and $u_2(t, x; \phi) \leq K$ for $(t, x) \in [T_1 - \tau, T_0] \times \Omega$, the comparison principle implies that

$$u_1(t, x; \phi) \leq \hat{u}_1^*(t - T_1, x; K) \quad \forall (t, x) \in [T_1, T_0] \times \overline{\Omega}.$$ 

In particular, $u_1(T_0 + s, x; \phi) \leq \hat{u}_1^*(T_2 + s, x; K) < u_1^*(x, \epsilon) + \epsilon$ for $(s, x) \in [-\tau, 0] \times \overline{\Omega}$.

Let $\overline{u}_2(t, x) = \xi e^{t \lambda d_2, u_1^*(\cdot, \cdot) + \epsilon} w(x)$. It is clear that $\overline{u}_2(t, x)$ is a solution of the following reaction-diffusion equation:

$$\begin{cases}
\frac{\partial \overline{u}_2}{\partial t} = d_2 \Delta \overline{u}_2(t, x) + \beta(x)(u_1^*(x, \epsilon) + \epsilon) \overline{u}_2(t, x)

- (\gamma(x) + \mu(x)) \overline{u}_2(t, x), & t > 0, \; x \in \Omega, \\
\frac{\partial \overline{u}_2}{\partial n} = 0, & t > 0, \; x \in \partial \Omega.
\end{cases}$$

Let $(u_1(t, x), u_2(t, x)) = (u_1(t, x; \phi), u_2(t, x; \phi))$ be the nonnegative solution of system (3) with $\phi \in C^+_T$ satisfying $0 \leq \phi_1(\cdot, \cdot) \leq H$ and $0 \leq \phi_2(0, \cdot) \leq \delta$. We claim that

$$u_2(t, x) \leq \xi e^{t \lambda d_2, u_1^*(\cdot, \cdot) + \epsilon}(t - T_0) w(x), \quad \forall t \geq T_0, \; x \in \overline{\Omega}.$$
On the contrary we suppose that the claim does not holds. Then in view of (15) and comparison principle, there exist $T_3 > T_0$ and $x_0 \in \Omega$ such that

$$u_2(t, x) - \xi e^{\lambda d_2 u_1^*(x, \epsilon) + \epsilon (T - T_0)} w(x) < \epsilon, \quad T_0 - \tau \leq t < T_3, \quad x \in \Omega,$$

$$u_2(T_3, x_0) = \xi e^{\lambda d_2 u_1^*(x, \epsilon)} (T_3 - T_0) w(x_0).$$

(17)

It is easy to see that

$$\frac{\partial u_1}{\partial t} \leq d_1 \Delta u_1(t, x) - \mu(x) u_1(t, x) + \gamma(x) \epsilon$$

$$+ e^{-\mu_{3\tau}} \int_{\Omega} \Gamma(d_{3\tau}, x, y) f(y, u_1(t - \tau, y) + \epsilon) dy$$

(18)

holds for $T_0 \leq t \leq T_3$. By standard comparison argument, we obtain

$$u_1(t, x) \leq \bar{u}_1^*(t - T_1, x; K), \quad \forall (t, x) \in [T_1, T_3] \times \Omega.$$

Therefore,

$$u_1(t, x) \leq u_1^*(x, \epsilon) + \epsilon, \quad \forall (t, x) \in [T_0, T_3] \times \Omega.$$

Consequently, we have

$$\frac{\partial u_2}{\partial t} \leq d_2 \Delta u_2(t, x) + \beta(x)(u_1^*(x, \epsilon) + \epsilon) u_2(t, x) - (\mu(x) + \gamma(x)) u_2(t, x)$$

for any $(t, x) \in [T_0, T_3] \times \Omega$. Since $u_2(T_0, x) \leq \xi w(x)$ for any $x \in \Omega$, it follows from the parabolic strong maximum principle that

$$u_2(t, x) < \xi e^{\lambda d_2 u_1^*(x, \epsilon) + \epsilon (t - T_0)} w(x), \quad \forall (t, x) \in [T_0, T_3] \times \Omega,$$

which contradicts (17). Therefore, the claim holds.

By the above claim, we have

$$u_2(t, x) < \xi e^{\lambda d_2 u_1^*(x, \epsilon) + \epsilon (t - T_0)} w(x), \quad \forall t > T_0, \quad x \in \Omega,$$

which implies that $u_2(t, x) \to 0$ as $t \to \infty$ uniformly for $x \in \Omega$. In addition, we have

$$u_1(t, x) \leq u_1^*(x, \epsilon) + \epsilon, \quad \forall t \geq T_0 - \tau, \quad x \in \Omega.$$

Denote the omega limit set $\varpi = \omega(\phi)$ of $\phi$ for $\Phi_1$ as

$$\varpi = \{ \phi^* = (\phi^*_1, \phi^*_2) \in C^+_T : \exists \{ k_n \} \to \infty \text{ such that } \lim_{n \to \infty} \| \Phi_1^n(\phi) - \phi_1^* \| = 0 \}.$$

It is clear that $\Phi_1^n(\phi) = \Phi_n(\phi)$. Since $\lim_{t \to \infty} u_2(t, \cdot) = 0$, we have $\varpi = \omega_1 \times \{ 0 \}$, where $0$ denotes the constant function identically zero in $Z$. We now prove $\omega_1 \neq \{ 0 \}$.

Suppose not, it follows that $\lim_{n \to \infty} \| \Phi_n(\phi) \| = 0$. Following from the continuous dependence of solutions on initial values, we have that $\lim_{t \to \infty} u_1(t, \cdot) = 0$, which contradicts the condition (F1). Clearly, $\Phi_1(\psi_1, 0) = (S_1(\psi_1), 0)$ for any $\psi_1 \in C^+$, where $S_1(t \geq 0)$ is the solution semiflow associated with $\Phi_1$. By [43] Lemma 1.2.1, $\varpi$ is an internally chain transitive set for $\Phi_1$, and hence $\omega_1$ is an internally chain transitive set for $S_1$. It follows from Theorem 2.4 that $u_1^*(x)$ is globally attractive for $S_1$ in $Z^+ \setminus \{ 0 \}$. Since $\varpi = \omega_1(\varpi) = (S_1(\omega_1), 0)$ (hence, $\omega_1 = S_1^n(\omega_1) = S_n(\omega_1)$), we have $\omega_1 \cap W^s(u_1^*) \neq \emptyset$, where $W^s(u_1^*)$ is the stable set of $u_1^*$ on the map $S_1$. By [43] Theorem 1.2.1, we obtain $\omega_1 = \{ u_1^* \}$, and hence $\varpi = \{ (u_1^*, 0) \}$, that is

$$\lim_{n \to \infty} \| \Phi_n(\phi) - (u_1^*, 0) \| = 0.$$

By virtue of the continuous dependence of solutions on initial values, we then have

$$\lim_{t \to \infty} \{ u_1(t, x; \phi), u_2(t, x; \phi) \} = (u_1^*(x), 0).$$
uniformly for \( x \in \overline{\Omega} \).

(2). In the case where \( R_0 > 1 \), we can conclude from Lemma 2.5 that \( \lambda(d_2, u^*_1) > 0 \).

For any \( \phi \in C_{r}^+ \) with \( \phi_2(0, \cdot) \neq 0 \), let \((u_1(t, x; \phi), u_2(t, x; \phi))\) be the solution of system \((\overline{5})\). Define

\[
\mathcal{W}_0 := \{ \phi \in C_{r}^+ : \phi_2(0, \cdot) \neq 0 \},
\]

\[
\partial \mathcal{W}_0 = C_{r}^\dagger \setminus \mathcal{W}_0 = \{ \phi \in C_{r}^+ : \phi_2(0, \cdot) \equiv 0 \}.
\]

Note that for every \( \phi \in \mathcal{W}_0 \), Lemma 3.1 implies that \( u_2(t, x; \phi) > 0 \), \( \forall t > 0, x \in \overline{\Omega} \), in other words, \( \Phi(t)\mathcal{W}_0 \subseteq \mathcal{W}_0 \) and \( \Phi(t)(\partial \mathcal{W}_0) \subseteq \partial \mathcal{W}_0 \). \( \forall t \geq 0 \). Let

\[
M_0 := \{ \phi \in \partial \mathcal{W}_0 : \Phi(t)\phi \in \partial \mathcal{W}_0, \forall t \geq 0 \}.
\]

Let \( M_1 = \{(0, 0)\} \) and \( M_2 = \{(u_1^*, 0)\} \). Furthermore, we have the following three claims.

**Claim 1.** \( \varpi(\psi) = \cup_{\psi \in M_0} M_1 \cup M_2 \).

Since \( \psi \in M_0 \), we have \( \Phi(t)\psi \in M_0 \), \( \forall t \geq 0 \). Thus \( u_2(t, \cdot; \psi) \equiv 0 \), \( \forall t \geq 0 \). Then \( u_1 \) satisfies the time-delayed reaction-diffusion equation \((\overline{8})\), and hence, either \( \lim_{t \to \infty} u_1(t, x) = 0 \) or \( \lim_{t \to \infty} u_1(t, x) = u_1^*(x) \) uniformly for \( x \in \overline{\Omega} \). Consequently, we have \( \varpi(\psi) = \cup_{\psi \in M_0} M_1 \cup M_2 \).

Consider the following eigenvalue problem:

\[
\begin{align*}
\lambda \hat{w}(x) &= d_1 \Delta \hat{w}(x) - \left( \mu(x) + \frac{\beta(x)\eta}{1+2\eta} \right) \hat{w}(x) \\
&\quad + e^{-\nu_0 \tau} \int_{\Omega} \Gamma(d, \tau, t, x, y) \partial d_{1, f}(y)(0, \hat{w}(y))dy, \quad x \in \Omega, \\
\frac{\partial \hat{w}(x)}{\partial n} &= 0, \quad x \in \partial \Omega.
\end{align*}
\]

Since \( \lambda_0(d_1, \tau, \partial d_1 f(\cdot, 0)) > 0 \), there is a sufficiently small \( \eta_0 > 0 \) such that \((\overline{19})\) admits a principal eigenvalue \( \lambda_0^*(d_1, \tau, \partial d_{1, f}(\cdot, 0)) > 0 \) for any \( \eta \in [0, \eta_0] \). By a similar argument to \((\overline{8})\), we can conclude that the following time-delayed reaction-diffusion equation

\[
\begin{align*}
\frac{\partial \hat{u}_1(t, x, \eta)}{\partial t} &= d_1 \Delta \hat{u}_1(t, x, \eta) - \mu(x) \hat{u}_1(t, x, \eta) - \frac{\beta(x)\eta}{1+2\eta} \hat{u}_1(t, x, \eta) \\
&\quad + e^{-\nu_0 \tau} \int_{\Omega} \Gamma(d, \tau, t, x, y, \eta) f(y, \hat{u}_1(t - \tau, y, \eta))dy, \quad t > 0, \quad x \in \Omega, \\
\frac{\partial \hat{u}_1(t, x, \eta)}{\partial n} &= 0, \quad t > 0, \quad x \in \partial \Omega.
\end{align*}
\]

has a unique positive steady state \( \hat{u}_1^*(x, \eta) \) for any \( \eta \in [0, \eta_0] \), which is globally attractive in \( Z^+ \setminus \{0\} \). Thus, there is a \( \epsilon > 0, \zeta_0 > 0 \) and \( t_1 > 0 \) such that any solution \( \hat{u}_1(t, x, \eta) \) with \( \phi_1(0, \cdot) \neq 0 \) satisfies

\[
\hat{u}_1(t, x, \eta) > \hat{u}_1^*(x, \eta) - \epsilon \geq \zeta_0, \quad \forall \eta \in [0, \eta_0], \quad t > t_1, \quad x \in \overline{\Omega}.
\]

Let \( \eta_1 < \min\{\zeta_0, \eta_0\} \), then we show the second claim.

**Claim 2.** \( M_1 \) is a uniform weak repeller for \( \mathcal{W}_0 \) in the sense that

\[
\limsup_{t \to \infty} \|\Phi(t)(\phi)\| \geq \eta_1, \quad \forall \phi \in \mathcal{W}_0.
\]

Suppose, by contradiction, there exists \( \hat{\phi} \in \mathcal{W}_0 \) such that

\[
\limsup_{t \to \infty} \|\Phi(t)(\hat{\phi})\| < \eta_1.
\]
Then, there exists $t_2 > \tau$ such that $u_i(t, x, \hat{\phi}) < \eta_1$, $i = 1, 2$, $\forall t \geq t_2$, $x \in \overline{\Omega}$. In addition, Lemma 3.1 implies that $u_i(t, x; \hat{\phi}) > 0$. Thus, according to the monotonicity of $f$, $u_1(t; x; \hat{\phi})$ satisfies

$$
\begin{cases}
\frac{\partial u_1(t, x)}{\partial t} > d_1 \Delta u_1(t, x) - \mu(x) u_1(t, x) - \beta(x) \eta_0 u_1(t, x) \\
+ e^{-\eta_1 \tau} \int_{\Omega} \Gamma(d, \tau, x, y) f(y, u_1(t - \tau, y)) dy, \quad t \geq t_2, \; x \in \overline{\Omega}, \\
\frac{\partial u_1(t, x)}{\partial n} = 0, \quad t \geq t_2, \; x \in \partial \Omega.
\end{cases}
$$

It then follows from (20) and the standard comparison principle that for $t > t_2 + 1$, $u_1(t, x; \hat{\phi}) > \bar{u}_1(t, x, \eta_1) > \zeta_0$, which contradicts with $u_1(t, x; \hat{\phi}) < \eta_1 < \zeta_0$ as $t$ is large enough. This contradiction proves Claim 2.

Since $\lim_{t_2 \to 0} \lambda \left( d_2, \frac{\bar{u}_1 - \eta_2}{1 + \sigma_{\eta_2}} \right) = \lambda(d_2, u_1^*) > 0$, there is an $\eta_2 > 0$ such that $\lambda \left( d_2, \frac{\bar{u}_1 - \eta_2}{1 + \sigma_{\eta_2}} \right) > 0$. We now prove the third claim.

**Claim 3.** $M_2$ is a uniform weak repeller for $\mathbb{W}_0$ in the sense that

$$
\limsup_{t \to \infty} \| \Phi(t)(\phi) - (u_1^*, 0) \| \geq \eta_2, \; \forall \phi \in \mathbb{W}_0.
$$

Suppose, by contradiction, there exists $\phi_0 \in \mathbb{W}_0$ such that

$$
\limsup_{t \to \infty} \| \Phi(t)(\phi_0) - (u_1^*, 0) \|_{C_\tau} < \eta_2.
$$

Then there exists $t_0 > \tau$ such that $u_1(t, x; \phi_0) > u_1^*(x) - \eta_2$, $0 < u_2(t, x; \phi_0) < \eta_2$, $\forall t > t_0$, $x \in \overline{\Omega}$. Thus $u_2(t, x; \phi_0)$ satisfies

$$
\begin{cases}
\frac{\partial u_2(t, x)}{\partial t} > d_2 \Delta u_2(t, x) - (\mu(x) + \gamma(x)) u_2(t, x) \\
+ \frac{\beta(x)}{1 + \sigma_{\eta_2}} (u_1^*(x) - \eta_2) u_2(t, x), \quad t > t_0, \; x \in \overline{\Omega}, \\
\frac{\partial u_2(t, x)}{\partial n} = 0, \quad t > t_0, \; x \in \partial \Omega.
\end{cases}
$$

Let $\tilde{\psi}$ be the strongly positive eigenfunction corresponding to $\lambda \left( d_2, \frac{\bar{u}_1 - \eta_2}{1 + \sigma_{\eta_2}} \right)$. Note that $e^{\lambda \left( d_2, \frac{\bar{u}_1 - \eta_2}{1 + \sigma_{\eta_2}} \right) t} \tilde{\psi}$ is a solution of the following linear system:

$$
\begin{cases}
\frac{\partial v(t, x)}{\partial t} = d_2 \Delta v(t, x) - (\mu(x) + \gamma(x)) v(t, x) \\
+ \frac{\beta(x)}{1 + \sigma_{\eta_2}} (u_1^*(x) - \eta_2) v(t, x), \quad t > 0, \; x \in \overline{\Omega}, \\
\frac{\partial v(t, x)}{\partial n} = 0, \quad t > 0, \; x \in \partial \Omega.
\end{cases}
$$

In view of Lemma 3.1, there exists $\varepsilon_0 > 0$ such that $u_2(t, x; \phi_0) \geq \varepsilon_0 e^{\lambda \left( d_2, \frac{\bar{u}_1 - \eta_2}{1 + \sigma_{\eta_2}} \right) t} \tilde{\psi}$ for all $x \in \overline{\Omega}$. By the standard comparison principle, we have

$$
u_2(t, x; \phi_0) \geq \varepsilon_0 e^{\lambda \left( d_2, \frac{\bar{u}_1 - \eta_2}{1 + \sigma_{\eta_2}} \right) t} \tilde{\psi}, \; \forall t \geq t_0, \; x \in \overline{\Omega},
$$

which implies $u_2(t, x; \phi_0)$ is unbounded, a contradiction.

Define a continuous function $p : C_\tau^+ \to [0, \infty)$ by

$$
p(\phi) := \min_{x \in \overline{\Omega}} p(0, x), \; \forall \phi \in C_\tau^+.
$$

It is obvious that $p^{-1}(0, \infty) \subseteq \mathbb{W}_0$. By Lemma 3.1, $p$ has the property that if $p(\phi) > 0$ or $\phi \in \mathbb{W}_0$ with $p(\phi) = 0$, then $p(\Phi(t)\phi) > 0$, $\forall t > 0$. That is, $p$ is a generalized distance function for the semiflow $\Phi(t) : C_\tau^+ \to C_\tau^+$ (see, e.g., [29]).

From the above claims, it follows that any forward orbit of $\Phi(t)$ in $M_0$ converges to $M_1$ or $M_2$. In view of Claim 2 and Claim 3, we conclude that $M_1$ and $M_2$ are two
isolated invariant sets in $C_t^+$, and that $W^*(M_i) \cap \mathbb{W}_0 = \emptyset$, $i = 1, 2$, where $W^*(M_i)$ is the stable set of $M_i$. It is clearly that no subset of $\{M_1, M_2\}$ forms a cycle in $\partial \mathbb{W}_0$. It then follows from [29, Theorem 3] that there exists an $\hat{\eta} > 0$ such that

$$\min_{\psi \in \omega(\phi)} p(\psi) > \hat{\eta}, \; \forall \phi \in \mathbb{W}_0.$$ 

Hence, $\liminf_{t \to \infty} u_2(t, :, \phi) \geq \hat{\eta}, \; \forall \phi \in \mathbb{W}_0$. On the other hand, according to Theorem [31] and Lemma [3.1], there exists a $\varrho > 0$ such that $0 < u_2(t, :, \phi) \leq \varrho, \forall t \geq t'_2 = t'_2(\varrho), x \in \Omega$. Consequently, for large enough $t$, $u_1(t, x)$ satisfies that

$$\begin{align*}
\partial u_1(t, x) / \partial t &\geq \alpha_1 \Delta u_1(t, x) - \mu(x)u_1(t, x) - \beta(x)u_1(t, x) + \gamma(x)u_2(t, x), \; t > 0, x \in \Omega, \\
\partial u_1(t, x) / \partial n &\leq 0, \; t > 0, x \in \partial \Omega.
\end{align*}$$

By using similar arguments to [29, Lemma 1], it follows that the following reaction-diffusion equation

$$\begin{align*}
\partial w(t, x) / \partial t &= d_1 \Delta w(t, x) - \mu(x)w(t, x) - \beta(x)w(t, x) + \gamma(x)w_2(t, x), \; t > 0, x \in \Omega, \\
\partial w(t, x) / \partial n &= 0, \; t > 0, x \in \partial \Omega.
\end{align*}$$

admits a unique positive steady state $w_*(\cdot)$ which is globally attractive in $X_1$. With the aid of the standard parabolic comparison principle, we obtain that

$$\liminf_{t \to \infty} u_1(t, :, \phi) \geq w_*(\cdot).$$

Therefore, there exists an $\alpha$ with $0 < \alpha \leq \hat{\eta}$ such that

$$\liminf_{t \to \infty} u_i(t, :, \phi) \geq \alpha, \; \forall \phi \in \mathbb{W}_0, \; i = 1, 2.$$

Hence, the uniform persistence stated in the conclusion are valid.

It remains to prove the existence of a positive steady state. Let $\Phi : X^+ \to X^+, t \geq 0$ be the solution semiflow for the following non-local reaction-diffusion system

$$\begin{align*}
\partial u_1(t, x) / \partial t &= d_1 \Delta u_1(t, x) - \mu(x)u_1(t, x) - g(x, u_2(t, x))u_1(t, x) + \gamma(x)u_2(t, x) \\
&\quad + e^{-\mu t} \int \Omega \Gamma(d_2 r, x, y)f(y, u_1(t, y) + u_2(t, y))dy, \; t > 0, x \in \Omega, \\
\partial u_2(t, x) / \partial t &= d_2 \Delta u_2(t, x) - (\mu(x) + \gamma(x))u_2(t, x) + g(x, u_2(t, x))u_1(t, x), \; t > 0, x \in \Omega, \\
\partial u_1(t, x) / \partial n &= \partial u_2(t, x) / \partial n = 0, \; t > 0, x \in \partial \Omega.
\end{align*}$$

Define $\mathbb{W}_0 := \{(\phi_1, \phi_2) \in X^+ : \phi_2(\cdot) \neq 0\}, \partial \mathbb{W}_0 := X^+ \setminus \mathbb{W}_0$. By the same arguments as $\Phi(t) : C^+_t \to C^+_t$, we can obtain that $\Phi(t) : X^+ \to X^+$ is point dissipative, compact for each $t > 0$ and uniformly persistent with respect to $(\mathbb{W}_0, \partial \mathbb{W}_0)$. It then follows from [33, Theorem 1.3.7] that $\Phi(t)$ admits an equilibrium $\bar{u} \in \mathbb{W}_0$, i.e., $\Phi(t)\bar{u} = \bar{u}, \forall t \geq 0$. Fix a $t > 0$, we then get $\Phi(t)\bar{u} \in \text{int}(X^+)$. It is obvious that $\bar{u}(\cdot) = (\bar{u}_1(\cdot), \bar{u}_2(\cdot))$ is a positive steady state of (5).

The first result of above theorem shows that if there is a small invasion, then the disease-free equilibrium $(u_1^*, 0)$ is globally attractive. At what follows, a special case is considered where the diffusion rates of susceptible and infective individuals are equal, then we have the following result on global attractivity of the disease-free equilibrium without small invasion condition.
Theorem 3.3. Let (F) and (F1) hold. Assume that $R_0 < 1$. If $d_1 = d_2 = d$, then for any $\phi \in C_+^\infty$ with $\phi_1(0, \cdot) \neq 0$, the solution $(u_1(t, x; \phi), u_2(t, x; \phi))$ of (5) satisfies

$$\lim_{t \to \infty} (u_1(t, x; \phi), u_2(t, x; \phi)) = (u_1^*(x), 0)$$

uniformly for $x \in \overline{\Omega}$.

Proof. When $R_0 < 1$, it follows from Lemma 2.3 that $\lambda(d_2, u_1^*) < 0$. Since $d_1 = d_2 = d$, system (5) satisfies

$$\begin{align*}
\frac{\partial u_1(t, x)}{\partial t} &= d \Delta u_1(t, x) - \mu(x) u_1(t, x) - g(x, u_2(t, x)) u_1(t, x) + \gamma(x) u_2(t, x) \\
\frac{\partial u_2(t, x)}{\partial t} &= d \Delta u_2(t, x) - \mu(x) u_2(t, x) + g(x, u_2(t, x)) u_1(t, x)
\end{align*}$$

and hence, it follows from $A(t, x) = u_1(t, x) + u_2(t, x)$ that

$$\begin{align*}
\frac{\partial A(t, x)}{\partial t} &= d \Delta A(t, x) - \mu(x) A(t, x) + e^{-\mu t} \int_0^t \Gamma(d, \tau, x, y) f(y, A(t-\tau, y)) dy, \quad t > 0, x \in \Omega, \\
\frac{\partial A(t, x)}{\partial n} &= 0, \quad t > 0, x \in \partial \Omega.
\end{align*}$$

(22)

From (F1) and Lemma 2.3, equation (22) has a positive steady state $u_1^*(x)$ which is globally attractive in $Z \setminus \{0\}$. Then for any $\rho > 0$ and sufficiently large $t$, it follows that $A(t, x) = u_1(t, x) + u_2(t, x) < u_1^*(x) + \rho$ uniformly for $x \in \overline{\Omega}$. Thus, there exists $t_1$ large enough such that

$$\begin{align*}
\frac{\partial u_2(t, x)}{\partial t} < d \Delta u_2(t, x) - (\mu(x) + \gamma(x)) u_2(t, x) \\
+ \beta(x) (u_1^*(x) + \rho) u_2(t, x), \quad t > t_1, x \in \Omega,
\end{align*}$$

$$\frac{\partial u_2(t, x)}{\partial n} = 0, \quad t > t_1, x \in \partial \Omega.$$

We consider the following reaction-diffusion equation

$$\begin{align*}
\frac{\partial w(t, x)}{\partial t} &= d \Delta w(t, x) - (\mu(x) + \gamma(x)) w(t, x) \\
+ \beta(x) (u_1^*(x) + \rho) w(t, x), \quad t > 0, x \in \Omega,
\end{align*}$$

$$\frac{\partial w(t, x)}{\partial n} = 0, \quad t > 0, x \in \partial \Omega.$$

Since $\lim_{\rho \to 0} \lambda(d, u_1^*(\cdot) + \rho) = \lambda(d, u_1^*(\cdot)) < 0$, there exists an $\rho > 0$ such that $\lambda(d, u_1^*(\cdot) + \rho) < 0$. Let $\hat{\psi}$ be the strongly positive eigenfunction corresponding to $\lambda(d, u_1^*(x) + \rho)$. For $u_2(t_1, x)$, there exists some $\zeta > 0$ such that

$$u_2(t_1, x) \leq \zeta \hat{\psi}(x), \forall x \in \overline{\Omega}.$$

The standard comparison principle implies that

$$u_2(t, x) \leq \zeta e^{\lambda u_1^*(\cdot) + \rho} t - t_1 \hat{\psi}(x), \quad t > t_1, \ x \in \overline{\Omega},$$

which implies $\lim_{t \to \infty} u_2(t, \cdot) = 0$. For any $\phi \in C_+^\infty$ with $\phi_1(0, \cdot) \neq 0$, we have $\phi_1(0, \cdot) + \phi_2(0, \cdot) \neq 0$, and hence,

$$\lim_{t \to \infty} (u_1(t, \cdot; \phi) - u_1^*(\cdot)) = \lim_{t \to \infty} (A(t, \cdot; \phi) - u_2(t, \cdot; \phi) - u_1^*(\cdot)) = 0.$$

Thus, the global attractivity stated in conclusion holds. \qed
4. General birth function. In the previous section, under the condition (\(F1\)), we obtained some conclusions on the persistence and extinction of disease. It should be noted that in \((F1)\), the birth function \(f\) is monotone on the density of adult individuals \(A \in (0, \infty)\). In this section, for more general birth function \(f\), we intend to investigate the threshold dynamics for model (5) satisfying \(d_1 = d_2 = d\), that is,

\[
\begin{aligned}
\frac{\partial u_1(t,x)}{\partial t} &= d \Delta u_1(t,x) - \mu(x)u_1(t,x) - g(x,u_2(t,x))u_1(t,x) + \gamma(x)u_2(t,x) \\
&+ e^{-\mu \tau} \int_{\Omega} \Gamma(d_\tau, x, y)f(y, A(t-\tau, y))dy, \quad t > 0, x \in \Omega, \\
\frac{\partial u_2(t,x)}{\partial t} &= d \Delta u_2(t,x) - (\mu(x) + \gamma(x))u_2(t,x) \\
&+ g(x,u_2(t,x))u_1(t,x), \quad t > 0, x \in \Omega, \\
\frac{\partial u_1(t,x)}{\partial n} &= \frac{\partial u_2(t,x)}{\partial n} = 0, \quad t > 0, x \in \partial \Omega.
\end{aligned}
\]  

(23)

Then \(A(t,x) = u_1(t,x) + u_2(t,x)\) satisfies

\[
\begin{aligned}
\frac{\partial A(t,x)}{\partial t} &= d \Delta A(t,x) - \mu(x)A(t,x) \\
&+ e^{-\mu \tau} \int_{\Omega} \Gamma(d_\tau, x, y)f(y, A(t-\tau, y))dy, \quad t > 0, x \in \Omega, \\
\frac{\partial A(t,x)}{\partial n} &= 0, \quad t > 0, x \in \partial \Omega.
\end{aligned}
\]  

(24)

Setting \(u_2 = 0\) in system (23), we have the following equation for the density of susceptible host population:

\[
\begin{aligned}
\frac{\partial u_1(t,x)}{\partial t} &= d \Delta u_1(t,x) - \mu(x)u_1(t,x) \\
&+ e^{-\mu \tau} \int_{\Omega} \Gamma(d_\tau, x, y)f(y, u_1(t-\tau, y))dy, \quad t > 0, x \in \Omega, \\
\frac{\partial u_1(t,x)}{\partial n} &= 0, \quad t > 0, x \in \partial \Omega.
\end{aligned}
\]  

(25)

Assume that \((F)\) holds. It then follows from Lemma 2.2 that \((25)\) generates a semiflow \(Q(t) = u_{1t}(\cdot) : C^+ \to C^+, t \geq 0\), (in this case of \(d_1 = d\), we see that \(Y = X_1\)), which admits a global compact attractor \(B_0\).

As in Lemma 2.3 the following nonlocal elliptic eigenvalue problem

\[
\begin{aligned}
\lambda \hat{w}(x) &= d \Delta \hat{w}(x) - \mu(x)\hat{w}(x) + e^{-\mu \tau} \int_{\Omega} \Gamma(d_\tau, x, y)\partial_{u_1}f(y, 0)\hat{w}(y)dy, \quad x \in \Omega, \\
\frac{\partial \hat{w}(x)}{\partial n} &= 0, \quad x \in \partial \Omega
\end{aligned}
\]

has a principal eigenvalue, which is denoted by \(\lambda_0(d, \tau, \partial_{u_1}f(y,0))\). We further make the following assumption:

\((F2)\): \(\lambda_0(d, \tau, \partial_{u_1}f(y,0)) > 0\).

Assume that \((F)\) and \((F2)\) hold. Then by Lemma 2.3 ii), we have that for \(\phi_1(0, \cdot) \neq 0\),

\[
\begin{aligned}
u_1(t, x; \phi_1) &\geq \zeta u_1(t, x; \phi_1) \geq \zeta u_1(t, x; \phi_1) \\
&\geq \zeta u_1(t, x; \phi_1) \geq \zeta u_1(t, x; \phi_1) \geq \zeta u_1(t, x; \phi_1) \geq \zeta u_1(t, x; \phi_1)
\end{aligned}
\]  

(26)

where \(\zeta > 0\) and \(t_0\) are determined by Lemma 2.3 ii) with \(\psi(s, x) = \phi_1(s, x)\) for all \(s \in [-\tau, 0]\) and \(x \in \Omega\). Hence, the solution semiflow of \((25)\) defined by \((\hat{Q}(t)\phi_1)(s, x) := u_1(t + s, x; \phi_1)\) for \(s \in [-\tau, 0]\) and \(x \in \Omega\) admits a positive global compact attractor \(B_0 \subset \text{int}(C^+)\). We mention that under the conditions \((F)\) and \((F1)\), compact attractor \(B_0\) degenerates into a singleton set (see Theorem 2.4). As such, in Section 2, with the aid of the unique disease-free equilibrium \((u_1^*, 0)\), we can define the basic reproduction number \(R_0\) via the next generation operator and get the threshold result, see Theorems 3.2 and 3.3. However, in the present section, due to the non-monotonicity of birth function, the compact attractor \(B_0\) may not be a singleton set. Therefore, it is impossible to get a threshold dynamics by defining a unique number \(\mathcal{R}_0\) as that in Section 2. To establish the similar threshold results,
in the following we introduce two $\mathcal{R}_0$-like numbers $\mathcal{R}_0$ and $\bar{\mathcal{R}}_0$ by virtue of the lower and upper bounds of $\mathcal{B}_0$ respectively and then establish the dynamics of system (23).

For $u^*_1 \in \mathcal{B}_0$, in view of (26) and the properties of $\mathcal{B}_0$, there exists a $\vartheta > 0$ such that

$$\hat{u}^*_1(x) := \vartheta w^*(x) \leq u^*_1(s, x) \leq \varsigma w^*(x) =: \check{u}^*_1(x), \quad \forall s \in [-\tau, 0], \ x \in \Omega,$$  
(27)

where $w^* \in \text{int}(X^+_1)$ is fixed. In terms of $\hat{u}^*_1(x)$ and $\check{u}^*_1(x)$, we introduce the following operators:

$$\bar{L}(\varphi)(x) := \int_0^\infty \beta(x)\hat{u}^*_1(x)T_2(t)\varphi dt = \beta(x)\hat{u}^*_1(x)\int_0^\infty T_2(t)\varphi dt$$
and

$$\check{L}(\varphi)(x) := \int_0^\infty \beta(x)\check{u}^*_1(x)T_2(t)\varphi dt = \beta(x)\check{u}^*_1(x)\int_0^\infty T_2(t)\varphi dt,$$
respectively. Clearly, $\check{L}(\varphi) \leq \check{L}(\varphi)$ for any $\varphi \in \mathcal{D}(A_2)$.

It then follows from [3, Theorem 1.1] that

$$\bar{\mathcal{R}}_0 := r(\bar{L}) \leq r(\check{L}) =: \bar{\mathcal{R}}_0,$$

where $r(\bar{L})$ and $r(\check{L})$ are the spectral radius of $\bar{L}$ and $\check{L}$, respectively.

As in Section 2, for $m(\cdot) \in X^+_1$ with $m(x) > 0, x \in \Omega$, we consider the following elliptic eigenvalue problem:

$$\begin{aligned}
\lambda \varphi(x) &= d\Delta \varphi(x) - (\mu(x) + \gamma(x))\varphi(x) + \beta(x)m(x)\varphi(x), \quad x \in \Omega, \\
\frac{\partial \varphi(x)}{\partial n} &= 0, \quad x \in \partial \Omega.
\end{aligned}$$

(28)

It then follows from [28, Theorem 7.6.1] that [28] has a principal eigenvalue denoted by $\lambda(d, \hat{u}^*_1)$ with a positive eigenfunction. By [27, Theorem 3.1] with diffusion rate independent on spatial variable $x$, we have the following observation.

**Lemma 4.1.** $\bar{\mathcal{R}}_0 - 1$ and $\bar{\mathcal{R}}_0 - 1$ have the same sign as $\lambda(d, \hat{u}^*_1)$ and $\lambda(d, \check{u}^*_1)$, respectively.

Now, we are in a position to prove the main results of this section.

**Theorem 4.2.** Let (F) and (F2) hold, and $u^*, \check{u}^* \in \text{int}(X^+_1)$ be fixed. Then the following two statements valid:

**(1):** If $\bar{\mathcal{R}}_0 > 1$, then system [28] admits at least one positive steady state $\hat{u}(x)$, and there exists an $i > 0$ such that for any $\phi \in C^+_\tau$ with $\phi_2(0, \cdot) \neq 0$, we have

$$\liminf_{t \to \infty} u_i(t, x; \phi) \geq \iota, \quad \forall \ i = 1, 2$$

uniformly for all $x \in \Omega$.

**(2):** If $\bar{\mathcal{R}}_0 < 1$, then there exists an $\iota' > 0$ such that, for any $\phi \in C^+_\tau$ with $\phi_1(0, \cdot) \neq 0$, the solution $(u_1(t, x; \phi), u_2(t, x; \phi))$ of [23] satisfies

$$\liminf_{t \to \infty} u_1(t, \cdot; \phi) \geq \iota'$$

and

$$\lim_{t \to \infty} \|u_2(t, \cdot; \phi)\|_{X_2} = 0.$$
Proof. In the following, we shall use the previous analysis of this section and similar arguments to those in [32, Theorem 3.3] to prove the conclusion stated in (1).

In view of (F2) and (20), we have
\[ u^*_t(s, x) \geq c v^*(x), \forall u^*_t \in B_0, s \in [-\tau, 0), x \in \overline{\Omega}. \]
For any \( u^*_t \in B_0 \), it is obvious that the solution \( u_1(t, x; \phi_1) \) of equation (25) is well defined for \( t \in \mathbb{R} \) and \( (u_1(t, \cdot; u^*_t), 0) \) is also a solution of equation (23) defined for all \( t \in \mathbb{R} \). Define
\[ B_0 = \{ \tilde{u}^*_1(\cdot, \cdot) \in Y : \tilde{u}^*_1(\cdot, \cdot) := u_1(t + \cdot, \cdot; u^*_t), \forall t \in \mathbb{R}, u^*_t \in B_0 \}. \]
For any \( \phi \in C^+_\Omega \) with \( \phi_2(0, \cdot) \neq 0 \), let \( (u_1(t, x; \phi), u_2(t, x; \phi)) \) be the solution of system (23). Define
\[ W_0 := \{ \phi \in C^+_\Omega : \phi_2(0, \cdot) \neq 0 \}, \]
\[ \partial W_0 = C^+_\Omega \setminus W_0 = \{ \phi \in C^+_\Omega : \phi_2(0, \cdot) \equiv 0 \}. \]
For any \( \phi \in W_0 \), it follows from Lemma 3.1 that \( u_2(t, x; \phi) > 0, \forall t > 0, x \in \overline{\Omega} \), that is, \( \Phi(t)W_0 \subseteq W_0 \), \( \forall t \geq 0 \). Let
\[ M_0 := \{ \phi \in \partial W_0 : \Phi(t) \phi \in \partial W_0, \forall t \geq 0 \}. \]
Let \( M_1 := \{ (0, 0) \}, M_2 := \{ \tilde{u}^*_1, 0 \} : \tilde{u}^*_1 \in B_0 \} \) and \( \varpi(\psi) \) be the omega limit set of the orbit \( \gamma^+(\phi) := \{ \Phi(t) \phi : \forall t \geq 0 \} \) for \( \phi \in C^+_\Omega \), where \( \omega \) denote the constant function identically zero in \( Y \) and \( Y' \). Next, we prove the following three claims.

**Claim 1.** \( U_{\psi \in M_0} \varpi(\psi) = M_1 \cup M_2, \forall \psi \in M_0. \)

Since \( \psi \in M_0 \), we have \( \Phi(t) \psi \in M_0, \forall t \geq 0 \). Then \( u_2(t, \cdot; \psi) \equiv 0, \forall t \geq 0 \). Thus, \( u_1 \) satisfies the time-delayed reaction-diffusion equation (25), and hence, \( U_{\psi \in M_0} \varpi(\psi) = M_1 \cup M_2 \), which implies Claim 1.

For any \( \phi = (\phi_1, \phi_2) \in W_0 \), we have \( \phi_1(0, \cdot) + \phi_2(0, \cdot) \neq 0 \), and hence, it follows from Lemma 3(ii) that there exist \( \eta_1 > 0 \) and \( t_1 = t_1(\phi_1 + \phi_2) > 0 \) such that \( A(t, x; \phi_1 + \phi_2) > 2k_\beta \eta_1, \forall t \geq t_1, x \in \overline{\Omega} \).

Then we show the second claim.

**Claim 2.** \( M_1 \) is a uniform weak repeller for \( W_0 \) in the sense that
\[ \lim_{t \to \infty} \sup_{\phi} \| \Phi(t) \phi \|_{C^r} \geq \eta_1, \forall \phi \in W_0. \]

Let \( \eta_1 = \eta_1'/(2k_\beta) \). Suppose, by contradiction, there exists \( \tilde{\phi} \in W_0 \) such that
\[ \lim_{t \to \infty} \sup_{\phi} \| \Phi(t) \phi \|_{C^r} < \eta_1. \]
Then, there exists \( t_2 > \tau \) such that
\[ \left\| u_i(t, \cdot; \tilde{\phi}) \right\|_{\infty} \leq k_\beta \left\| u_i(t, \cdot; \tilde{\phi}) \right\|_{X_i} < \eta_1'/2, \ i = 1, 2, \]
for all \( t > t_2 \), which yield that \( u_i(t, x, \tilde{\phi}) < \eta_1'/2, \ i = 1, 2, \ \forall t \geq t_2, x \in \overline{\Omega} \). Thus, \( A(t, x; \phi_1 + \phi_2) = u_1(t, x; \phi) + u_2(t, x; \phi) < \eta_1' = 2k_\beta \eta_1, \forall t \geq t_1 + t_2, x \in \overline{\Omega} \), which is a contradiction. This completes the proof of Claim 2.

In the case where \( R_0 > 1 \), in view of (27) and Lemma 4.1 we have \( \lambda(d, \tilde{u}^*_1) > 0 \).

**Claim 3.** \( M_2 \) is a uniform weak repeller for \( W_0 \) in the sense that there exists \( \eta_2 > 0 \) such that
\[ \lim_{t \to \infty} \inf_{\xi \in M_2} \| \Phi(t) \phi - \xi \|_{C^r} \geq \eta_2, \forall \phi \in W_0. \]
Since \( \lambda(d, \bar{u}_1(x)) > 0 \), there exists a sufficiently small positive number \( \eta'_2 > 0 \) such that \( \lambda \left(d, \frac{\bar{u}_1(x) - \eta'_2}{1 + \eta'_2} \right) > 0 \). Let \( \eta_2 = \eta'_2/(2k\beta) \). Suppose, by contradiction, that for some \( \phi_0 \in \mathbb{W}_0 \),

\[
\limsup_{t \to \infty} \inf_{\xi \in \mathbb{M}_2} \| \Phi(t)(\phi_0) - \xi \|_{C_r} \geq \eta_2.
\]

Then there exists \( t'_1 > \tau \) such that

\[
\inf_{\xi \in \mathbb{M}_2} \| \Phi(t)(\phi_0) - \xi \|_{C_r} < \frac{3}{2} \eta_2, \quad \forall t > t'_1.
\]

Due to the compactness of \( \mathbb{M}_2 \), it follows that there exists a \( \xi^* \in \mathbb{M}_2 \) such that

\[
\| \Phi(t)(\phi_0) - \xi^* \|_{C_r} \leq \frac{3}{2} \eta_2, \quad \forall t > t'_1.
\]

As a consequence, we have

\[
\| u_1(t + s, \cdot; \phi_0) - \xi^*(\cdot, \cdot) \|_{C_r} \leq k_{\beta} \| u_1(t + s, \cdot; \phi_0) - \xi^*(\cdot, \cdot) \|_{C_r} \leq \eta'_2
\]

for all \( t > t'_1 \) and \( s \in [-\tau, 0] \), which leads to \( u_1(t, x; \phi_0) > \bar{u}_1^*(x) - \eta'_2 \) for all \( t > t'_1 \) and \( x \in \Omega \). In addition, by means of \( \xi^*_2 = 0 \), we have

\[
\limsup_{t \to \infty} \| u_2(t, \cdot; \phi_0) \|_{C_r} \leq \limsup_{t \to \infty} k_{\beta} \| u_2(t, \cdot; \phi_0) \|_{C_r} \leq \eta'_2
\]

for all \( t > t'_1 \) and \( s \in [-\tau, 0] \), which yields \( u_2(t, x; \phi_0) < \eta'_2 \) for all \( t > t'_1 \) and \( x \in \Omega \). Thus \( u_2(t, x; \phi_0) \) satisfies

\[
\begin{cases}
\frac{\partial u_2(t, x)}{\partial t} > d_2 \Delta u_2(t, x) - (\mu(x) + \gamma(x))u_2(t, x) + \frac{\beta(x)}{1 + \eta'_2} \bar{u}_1^*(x) - \eta'_2, & t > t'_1, \quad x \in \Omega, \\
\frac{\partial u_2(t, x)}{\partial n} = 0, & t > t'_1, \quad x \in \partial \Omega.
\end{cases}
\]

Let \( \psi^\eta_2 \in \text{int}(X_2) \) be the positive eigenfunction associated with \( \lambda \left(d, \frac{\bar{u}_1 - \eta'_2}{1 + \eta'_2} \right) > 0 \).

Note that the linear equation

\[
\begin{cases}
\frac{\partial v_2(t, x)}{\partial t} = d \Delta v_2(t, x) - (\mu(x) + \gamma(x))v_2(t, x) + \frac{\beta(x)}{1 + \eta'_2} \bar{u}_1^*(x) - \eta'_2, & t > 0, \quad x \in \Omega, \\
\frac{\partial v_2(t, x)}{\partial n} = 0, & t > 0, \quad x \in \partial \Omega
\end{cases}
\]

admits a solution \( v_2(t, x) = e^{-t \left(d, \frac{\bar{u}_1 - \eta'_2}{1 + \eta'_2} \right)} \psi^\eta_2(x) \). Since \( u_2(t, \cdot; \phi_0) \in \text{int}(X_2) \) for all \( t > \tau \), there exists \( \rho > 0 \) such that \( u_2(t'_1, \cdot; \phi_0) \geq \rho \psi_2(t'_1, \cdot) \). By the standard parabolic comparison principle, we have

\[
u_2(t, x; \phi_0) \geq e^{t \left(d, \frac{\bar{u}_1 - \eta'_2}{1 + \eta'_2} \right)} \psi^\eta_2(x), \quad \forall t > t'_1, \quad x \in \Omega,
\]

which implies that \( u_2(t, x; \phi_0) \) is unbounded, a contradiction. Thus, we complete the proof of Claim 3.

Define a continuous function \( p : C^+_\tau \to [0, \infty) \) by

\[
p(\phi) := \min_{x \in \Omega} \phi(0, x), \quad \forall \phi \in C^+_\tau.
\]

It is obvious that \( p^{-1}(0, \infty) \subseteq \mathbb{W}_0 \). By Lemma 3.1, \( p \) has the property that if \( p(\phi) > 0 \) or \( \phi \in \mathbb{W}_0 \) with \( p(\phi) = 0 \), then \( p(\Phi(t)(\phi)) > 0 \), \( \forall t > 0 \). That is, \( p \) is a generalized distance function for the semiflow \( \Phi(t) : C^+_\tau \to C^+_\tau \) (see, e.g., [20]).

From the above claims, it follows that any forward orbit of \( \Phi(t) \) in \( \mathbb{M}_0 \) converges to \( \mathbb{M}_1 \) or \( \mathbb{M}_2 \). In view of Claim 2 and Claim 3, we conclude that \( \mathbb{M}_1 \) and \( \mathbb{M}_2 \) are two
isolated invariant sets in $C^+_t$, and that $W^s(M_i) \cap \mathbb{W}_0 = \emptyset$, $i = 1, 2$, where $W^s(M_i)$ is the stable set of $M_i$. It is obvious that no subset of $\{M_1, M_2\}$ forms a cycle in $\partial \mathbb{W}_0$. It then follows from [29, Theorem 3] that there exists an $\eta > 0$ such that

$$\min_{\psi \in \omega(\phi)} p(\psi) > \eta, \quad \forall \phi \in \mathbb{W}_0.$$ 

Hence, $\liminf_{t \to \infty} u_2(t, \cdot; \phi) \geq \eta$, $\forall \phi \in \mathbb{W}_0$. By similar argument to the proof in Theorem 3.2, there exists an $\iota$ with $0 < \iota \leq \eta$ such that

$$\liminf_{t \to \infty} u_i(t, \cdot; \phi) \geq \iota, \quad \forall \phi \in \mathbb{W}_0, \ i = 1, 2.$$ 

By means of [21, Theorem 3.7 and Remark 3.10], we assert that $\Phi(t) : \mathbb{W}_0 \to \mathbb{W}_0$ has a global attractor $\mathfrak{A}_0$. It then follows from [21, Theorem 4.7] that $\Phi(t)$ has an equilibrium $(\tilde{u}_1(\cdot), \tilde{u}_2(\cdot)) \in \mathbb{W}_0$. It is not difficult to see that $(\tilde{u}_1(\cdot), \tilde{u}_2(\cdot))$ is a positive steady state of system (23).

(2). When $\mathcal{R}_0 < 1$, we can conclude from (27) and Lemma 4.1 that $\lambda(d, \tilde{u}_1^+) < 0$.

From (F2) and Lemma 2.3(ii), it follows that the solution semiflow of (21) admits the global compact attractor $\mathfrak{B}_0$, and hence, there exist $\kappa > 0$ such that

$$A(t, x) = u_1(t, x) + u_2(t, x) \leq \kappa \epsilon^*(x) + \epsilon_0 = \tilde{u}_1^+(x) + \epsilon_0, \quad \forall t \geq t_3, \ x \in \Omega.$$ 

It then follows that $u_1(t, x) \leq \kappa \epsilon^*(x) + \epsilon_0 = \tilde{u}_1^+(x) + \epsilon_0, \ \forall t \geq t_3, \ x \in \Omega$. Consequently, by similar arguments to those in Theorem 3.3 we can conclude that

$$\limsup_{t \to \infty} \|u_2(t, \cdot; \phi)\|_{X_2} = 0.$$ 

Give $\phi = (\phi_1, \phi_2) \in C^+_t$ with $\phi_1(0, \cdot) \neq 0$. Regarding $u_2(t, x)$ as a fixed function on $\mathbb{R}^+ \times \Omega$, it follows that $u_1(t, x)$ satisfies the following non-autonomous time-delayed and non-local reaction-diffusion equation:

$$\begin{cases} \frac{\partial u_1(t, x)}{\partial t} = d \Delta u_1(t, x) - \mu(x) u_1(t, x) - g(x, u_2(t, x))u_1(t, x) + \gamma(x) u_2(t, x) \\
+ e^{-\nu t} \int_0^t \Gamma(d_j, \tau, x, y) f(y, A(t - \tau, y)) dy, \quad t > 0, \ x \in \Omega, \\
\frac{\partial u_1(t, x)}{\partial \nu} = 0, \quad t > 0, \ x \in \partial \Omega. \quad (29) \end{cases}$$ 

Since $\lim_{t \to \infty} \|u_2(t, \cdot; \phi)\|_{X_2} = 0$ and hence $\lim_{t \to \infty} \|u_2(t, \cdot; \phi)\|_{X_\infty} = 0$, equation (29) is asymptotic to equation (25). In view of (F) and (F2), it is easy to see from Lemma 2.3(ii) that for any $\phi \in C^+_t$ with $\phi_1(0, \cdot) \neq 0$, there exists an $\iota' > 0$ such that

$$\liminf_{t \to \infty} u_1(t, \cdot; \phi) \geq \iota'.$$ 

This completes the proof. \hfill \Box

**Remark 1.** For system (23), based on the conditions (F) and (F2), if we further provide the monotonicity of $f$, then $\mathcal{R}_0 = \mathcal{R}_0 = \mathcal{R}_0$. This implies that the Theorem 3.3 is a special case of Theorem 4.2.

In the rest of this section, we consider a special case of model system (23), where all the coefficients in system (23) are independent of the spatial variable $x$ and birth function is given. Namely, we study the following spatially homogeneous non-local and time-delayed reaction-diffusion model:
where $\mu, \gamma, \beta, \alpha, p$ and $q$ are positive constants. For the birth function $f$, we choose the Richer type function $f(A) = pAe^{-qA}, p > 0, q > 0$. Then $A(t, x) = u_1(t, x) + u_2(t, x)$ satisfies

\[
\left\{ \begin{array}{l}
\frac{\partial u_1(t,x)}{\partial t} = d\Delta u_1(t,x) - \mu u_1(t,x) - \frac{\beta u_1(t,x)u_2(t,x)}{1+\alpha u_2(t,x)} + \gamma u_2(t,x) \\
\frac{\partial u_2(t,x)}{\partial t} = d\Delta u_2(t,x) - (\mu(x) + \gamma(x))u_2(t,x) + \frac{\beta u_1(t,x)u_2(t,x)}{1+\alpha u_2(t,x)}, \ t > 0, x \in \Omega, \\
\frac{\partial u_1(t,x)}{\partial n} = 0, \ t > 0, x \in \partial \Omega, \\
\frac{\partial u_2(t,x)}{\partial n} = 0, \ t > 0, x \in \partial \Omega,
\end{array} \right.
\]

(30)

where $\mu, \gamma, \beta, \alpha, p$ and $q$ are positive constants. For the birth function $f$, we choose the Richer type function $f(A) = pAe^{-qA}, p > 0, q > 0$. Then $A(t, x) = u_1(t, x) + u_2(t, x)$ satisfies

\[
\left\{ \begin{array}{l}
\frac{\partial A(t,x)}{\partial t} = d\Delta A(t,x) - \mu A(t,x) \\
+e^{-\nu_{i}\tau}\int_{\Omega} \Gamma(d_j\tau, x, y) \left(pA(t-x, y)e^{-qA(t-x, y)}\right) dy, \ t > 0, x \in \Omega, \\
\frac{\partial A(t,x)}{\partial n} = 0, \ t > 0, x \in \partial \Omega.
\end{array} \right.
\]

(31)

Setting $u_2 = 0$ in system (30), we obtain the following equation for the density of susceptible host population:

\[
\left\{ \begin{array}{l}
\frac{\partial u_1(t,x)}{\partial t} = d\Delta u_1(t,x) - \mu u_1(t,x) \\
+e^{-\nu_{i}\tau}\int_{\Omega} \Gamma(d_j\tau, x, y) \left(pu_1(t-x, y)e^{-q_{A1}(t-x, y)}\right) dy, \ t > 0, x \in \Omega, \\
\frac{\partial u_1(t,x)}{\partial n} = 0, \ t > 0, x \in \partial \Omega.
\end{array} \right.
\]

(32)

With the aid of [42] Lemma 5.1 (see also [44]), we have the following observation.

**Lemma 4.3.** Assume $p/\mu e^{\mu \nu_{i}\tau} \in (1, e^{\nu_{i}^{2}}]$. Then for any $\phi_1 \in C^{+}$ with $\phi_1(0, \cdot) \neq 0$, the equation (32) admits a unique positive constant steady state $u_1^* = \frac{1}{q} \ln(p/\mu e^{\mu \nu_{i}\tau})$, and the solution $u_1(t, x; \phi_1)$ satisfies

\[\lim_{t \to \infty} \|u_1(t, x; \phi_1) - u_1^*\|_{X_1} = 0\]

uniformly for $x \in \Omega$.

By means of Lemma 4.3, it follows that system (30) admits a unique disease-free equilibrium $(u_1^*, 0)$. We assume that the state variable is near the disease-free equilibrium $(u_1^*, 0)$ and introduce the distribution of initial infectious individuals $\varphi(x)$ at time $t = 0$. Then the next generation operator

\[L(\varphi) = \int_{0}^{\infty} \beta u_1^* T_2(t)\varphi dt = \beta u_1^* \int_{0}^{\infty} T_2(t)\varphi dt.\]

Taking $\varphi \equiv 1$, it is easy to obtain that

\[R_0 = r(L) = \frac{\beta}{(\mu + \gamma)q} \ln(p/\mu e^{\mu \nu_{i}\tau}).\]

On the other hand, in view of (28) with $m(\cdot) = u_1^*$, it follows that $\lambda(d, u_1^*) = \lambda(d, \frac{1}{q} \ln(p/\mu e^{\mu \nu_{i}\tau}))$ satisfies

\[
\left\{ \begin{array}{l}
\lambda \varphi(x) = d\Delta \varphi(x) - (\mu + \gamma)\varphi(x) + \beta u_1^* \varphi(x), \ x \in \Omega, \\
\frac{\partial \varphi(x)}{\partial n} = 0, \ x \in \partial \Omega.
\end{array} \right.
\]

(33)

Substituting $\varphi(x) \equiv 1$ into (33), we have the principal eigenvalue of (33)

\[\lambda \left( \frac{1}{q} \ln(p/\mu e^{\mu \nu_{i}\tau}) \right) = -(\mu + \gamma) + \frac{\beta}{q} \ln(p/\mu e^{\mu \nu_{i}\tau}).\]

(34)
Thus, the following results show that the threshold dynamics of the model (5) can be determined by parameters.

**Theorem 4.4.** Assume \( p/\mu^{p/\tau} \in (1, e^2] \). The following statements are valid:

(i): if \( q(\mu + \gamma) > \beta \ln(p/\mu^{p/\tau}) \), for any \( \phi \in C^\pm_x \) with \( \phi_1(0, \cdot) \neq 0 \), then the solution \((u_1(t, x; \phi), u_2(t, x; \phi))\) of (30) satisfies
\[
\lim_{t \to \infty} \left\| (u_1(t, \cdot; \phi), u_2(t, x; \phi)) - \left( \frac{1}{q} \ln(p/\mu^{p/\tau}), 0 \right) \right\|_X = 0.
\]

(ii): if \( q(\mu + \gamma) < \beta \ln(p/\mu^{p/\tau}) \), the system (5) admits at least one positive steady state, and there exists an \( \eta > 0 \) such that for any \( \phi \in C^\pm_x \) with \( \phi_2(0, \cdot) \neq 0 \), we have
\[
\liminf_{t \to \infty} u_i(t, x; \phi) \geq \eta, \quad \forall \ i = 1, 2
\]
uniformly for all \( x \in \Omega \).

Based on the discussions in [42, Section 5] and [42, Lemma 5.2] (see also [38]), we have the following lemma.

**Lemma 4.5.** Assume \( pe^{-\mu/\tau}/\omega_1 \in (e^2, \infty) \). Then for any \( \phi \in C^\pm_x \) with \( \phi_1(0, \cdot) \neq 0 \), the solution \( u_1(t, x; \phi_1) \) of equation (32) satisfies
\[
w^*_{\text{max}} \leq \liminf_{t \to \infty} \inf_{x \in \Omega} u_1(t, x; \phi_1) \leq \limsup_{t \to \infty} \sup_{x \in \Omega} u_1(t, x; \phi_1) \leq \frac{pe^{-\mu/\tau}}{qa\mu},
\]
where \( w^*_{\text{max}} = \frac{pe^{-\mu/\tau} + pe^{-\mu/\tau}}{qa\mu} \exp\left(-\frac{pe^{-\mu/\tau} + qa\mu}{\epsilon\mu}\right) \).

By similar arguments to the previous analysis in this section, it follows from Lemma 4.5 and some simple calculation that
\[
\mathcal{R}_0 = \frac{\beta w^*_{\text{max}}}{\mu + \gamma} \quad \text{and} \quad \bar{\mathcal{R}}_0 = \frac{\beta pe^{-\mu/\tau}}{qa\mu(\mu + \gamma)}.
\]

Now, by means of Theorem 4.2, we are in a position to show the following threshold dynamics in terms of the model parameters.

**Theorem 4.6.** Assume \( pe^{-\mu/\tau}/\omega_1 \in (e^2, \infty) \). Then the following two statements valid:

(1): If \( \beta w^*_{\text{max}} > \mu + \gamma \), then there exists an \( \iota > 0 \) such that for any \( \phi \in C^\pm_x \) with \( \phi_2(0, \cdot) \neq 0 \), we have
\[
\liminf_{t \to \infty} u_i(t, x; \phi) \geq \iota, \quad \forall \ i = 1, 2
\]
uniformly for all \( x \in \Omega \).

(2): If \( \beta pe^{-\mu/\tau} < qa\mu(\mu + \gamma) \), then there exists an \( \iota' > 0 \) such that, for any \( \phi \in C^\pm_x \) with \( \phi_1(0, \cdot) \neq 0 \), the solution \((u_1(t, x; \phi), u_2(t, x; \phi))\) of (23) satisfies
\[
\liminf_{t \to \infty} u_i(t, \cdot; \phi) \geq \iota'
\]
and
\[
\lim_{t \to \infty} \| u_2(t, \cdot; \phi) \|_{X_2} = 0.
\]

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