Higher homotopy associativity in the Harris decomposition of Lie groups

Daisuke Kishimoto
Department of Mathematics, Kyoto University, Kyoto 606-8502, Japan
(kishi@math.kyoto-u.ac.jp)

Toshiyuki Miyauchi
Department of Applied Mathematics, Faculty of Science, Fukuoka University, Fukuoka 814-0180, Japan
(miyauchi@math.sci.fukuoka-u.ac.jp)

For certain pairs of Lie groups \((G, H)\) and primes \(p\), Harris showed a relation of the \(p\)-localized homotopy groups of \(G\) and \(H\). This is reinterpreted as a \(p\)-local homotopy equivalence \(G \simeq (p) H \times G/H\), and so there is a projection \(G(p) \to H(p)\). We show how much this projection preserves the higher homotopy associativity.

Keywords: Lie group; mod \(p\) decomposition; higher homotopy associativity; higher homotopy commutativity.

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1. Introduction

Lie groups decompose into products of small spaces when localized at a prime \(p\). This is called the mod \(p\) decomposition and is fundamental in the homotopy theory of Lie groups. Then it is important to study relations between the mod \(p\) decomposition and the group structures (or the loop space structures) of Lie groups, and there are several results on such relations \([2, 5–8, 12–14, 19, 23]\). In this paper, we study a relation between the group structures (or the loop structures) of Lie groups and maps between Lie groups arising from the classical mod \(p\) decomposition due to Harris \([3, 4]\).

Let \((G, H) = (SU(2n + 1), SO(2n + 1)), (SU(2n), Sp(n)), (SO(2n), SO(2n − 1)), (E_6, F_4), (Spin(8), G_2)\) and let \(p\) be any prime \(\geq 5\) for \((G, H) = (E_6, F_4)\), any prime \(\neq 3\) for \((G, H) = (Spin(8), G_2)\), and any odd prime otherwise. Then \(H\) is a subgroup of \(G\) in the obvious way so that there is a fibration \(H \to G \to G/H\). Harris \([3, 4]\) showed that the associated homotopy exact sequence splits after localizing at a prime \(p\) such that

\[
\pi_*(G)(p) \cong \pi_*(H)(p) \oplus \pi_*(G/H)(p). \tag{1.1}
\]
The proof of Harris actually implies a stronger result such that there is a $p$-local homotopy equivalence

$$G \simeq_p H \times G/H \quad (1.2)$$

which is one of the most classical mod $p$ decomposition of Lie groups. In particular, there is a projection $G(p) \to H(p)$ which has been treated only as a continuous map so far. However, we are interested in relations between the projection and the group structures (or the loop space structures) of $G$ and $H$, and so we naively ask how much this projection respects the group structures of $G$ and $H$. We make this naive question more precise. Stasheff [22] defined $A_k$-maps for $k \geq 2$ between loop spaces as $H$-maps preserving the $(k-2)^{th}$ higher homotopy associativity. Then $A_k$-maps form a gradation between continuous maps and loop maps, and so our question is made precise as follows.

**Question 1.1.** Given a prime $p$, for which $k$ is the projection $G(p) \to H(p)$ an $A_k$-map?

The aim of this paper is to answer this question. Since the case $(G, H) = (SU(2), Sp(1))$ is trivial, we will exclude it throughout.

**Theorem 1.2.** Let $(G, H), a_k$ and $p$ be as in the table 1 below. Then for $k \geq 2$ the following statements hold:

1. for $(G, H) \neq (SO(2n), SO(2n-1))$ the projection $G(p) \to H(p)$ is an $A_k$-map if and only if $p \geq a_k$;
2. for $(G, H) = (SO(2n), SO(2n-1))$
   (a) if $p \geq a_k$ then the projection $G(p) \to H(p)$ is an $A_k$-map;
   (b) if $p < a_k - n + 2$ then the projection $G(p) \to H(p)$ is not an $A_k$-map.

| $(G, H)$          | $(SU(2n+1), SO(2n+1))$ | $(SU(2n), Sp(n))$ | $(SO(2n), SO(2n-1))$ |
|-------------------|------------------------|-------------------|-----------------------|
| $a_k$             | $k(2n+1)$              | $2kn - 1$         | $2(k-1)(n-1) + n$     |
| $p$               | $p \geq 3$             | $p \geq 3$        | $p \geq 3$            |
| $(G, H)$          | $(E_6, F_4)$           | $(Spin(8), G_2)$  |
| $a_k$             | $12k - 5$              | $6k - 2$          |
| $p$               | $p \geq 5$             | $p \neq 3$        |

Harris [4] showed the decomposition (1.1) by constructing a specific map $G/H \to G$, which is a $p$-local section of the projection $G \to G/H$, from a finite order self-map of $G$. But if we use the mod $p$ decomposition of Mimura, Nishida, and Toda [18] and its naturality instead, then we get the decomposition (1.2) for more pairs of Lie groups, e.g. $(SU(n), SU(n-1))$ for $p \geq n$. Our method for showing the projection $G(p) \to H(p)$ is an $A_k$-map does not use a specific map $G/H \to G$, and so we get the following general result. We set notation to state it. Suppose that a connected
Lie group $G$ has no $p$-torsion in the integral homology for an odd prime $p$. Then its mod $p$ cohomology is an exterior algebra generated by odd degree elements. Suppose further that a pair of connected Lie groups $(G, H)$ admits the decomposition (1.2). Then the mod $p$ cohomology of $H$ and $G/H$ are also exterior algebras generated by odd degree elements. Let $2m - 1$ and $2l - 1$ be the largest dimensions of the mod $p$ cohomology generators of $H$ and $G/H$, respectively. We define

$$b_k = \max\{(k - 1)m + l, kl\}.$$  

Notice that if $(G, H)$ is as in theorem 1.2 then it satisfies the above conditions. The following table 2 gives a list of $(m, l)$ for $(G, H)$ in theorem 1.2.

| $(G, H)$ | $(SU(2n + 1), SO(2n + 1))$ | $(SU(2n), Sp(n))$ | $(SO(2n), SO(2n - 1))$ |
|----------|--------------------------|-----------------|----------------------|
| $(m, l)$ | $(2n + 1, 2n + 1)$       | $(2n, 2n - 1)$  | $(2n - 2, n)$        |
| $(G, H)$ | $(E_6, F_4)$             | $(Spin(8), G_2)$|                     |
| $(m, l)$ | $(12, 9)$                | $(6, 4)$        |                     |

It holds that $a_k = b_k$ for $(G, H)$ in theorem 1.2 except for $(G, H) = (E_6, F_4)$, in which case $a_k = b_k - 2$.

**Theorem 1.3.** Let $(G, H)$ be a pair of connected Lie groups, and let $p$ be an odd prime. Suppose $G$ has no $p$-torsion in the integral homology and the decomposition (1.2) holds. Then for $p \geq b_k$ the projection $G_{(p)} \rightarrow H_{(p)}$ is an $A_k$-map.

Theorem 1.3 is a consequence of the following stronger statement (proposition 2.7).

**Theorem 1.4.** Under the condition of theorem 1.3, there is an $A_k$-structure on $(G/H)_{(p)}$ such that the decomposition

$$G \simeq_{(p)} H \times G/H$$

is an $A_k$-equivalence.

The main technique for proving theorem 1.3 is a refinement of the reduction of the projective spaces of $p$-regular Lie groups established in the paper [8] on the higher homotopy commutativity of localized Lie groups. Then, indirectly though, our result is connected to higher homotopy commutativity, Sugawara and Williams $C_k$-spaces, where we refer to [8] for their definitions. For example, we have the following.

**Corollary 1.5.** Let $p$ be an odd prime. The following are equivalent:

1. the projection $SU(2n + 1)_{(p)} \rightarrow SO(2n + 1)_{(p)}$ is an $A_k$-map;
2. $SU(2n + 1)_{(p)}$ is a Sugawara $C_k$-space;
3. $SU(2n + 1)_{(p)}$ is a Williams $C_k$-space.
It would be interesting to find a direct connection between an $A_k$-structure of the projection $G(p) \to H(p)$ and the higher homotopy commutativity of $G(p)$ and $H(p)$.

2. Projective spaces for products

Throughout this section, we assume that spaces are path-connected. This section recalls the reduction of the projective space of a product of $A_n$-spaces established in [8] and shows its property that we will use. We refer to [8, 10, 22] for the basics of $A_n$-spaces, their projective spaces and $A_n$-maps. Let $X$ be an $A_n$-space and $P^kX$ be the $k$th projective space of $X$ for $k \leq n$. If $X$ is an $A_\infty$-space then we write the canonical map $P^kX \to BX$ by $j_k$. For a map $f : X \to Y$ where $Y$ is a topological monoid, let $\tilde{f} : \Sigma X \to BY$ denote the adjoint of $f$. The following is shown in [10] (cf. [8]).

**Lemma 2.1.** Let $X$ be an $A_n$-space and $Y$ be a topological monoid. A map $f : X \to Y$ is an $A_n$-map if and only if there is a map $P^n X \to BY$ satisfying a homotopy commutative diagram

$$
\begin{array}{ccc}
\Sigma X & \to & BY \\
\downarrow & & \downarrow \\
P^n X & \to & BY.
\end{array}
$$

Let $X_1, \ldots, X_l$ be $A_n$-spaces and let

$$
\tilde{P}^n(X_1, \ldots, X_l) = \bigcup_{i_1 + \cdots + i_l = n} P^{i_1} X_1 \times \cdots \times P^{i_l} X_l.
$$

We regard $X_1 \times \cdots \times X_l$ as an $A_n$-space by the product of multiplications of $X_1, \ldots, X_l$. The following is proved in [8].

**Lemma 2.2.** Let $X_i$ be $A_n$-spaces for $i = 1, 2$. There are maps $\tilde{P}^i(X_1, X_2) \to P^i(X_1 \times X_2)$ and $P^i(X_1 \times X_2) \to \tilde{P}^i(X_1, X_2)$ for $i = 1, 2, \ldots, n$ satisfying a homotopy commutative diagram

$$
\begin{array}{ccccccc}
\Sigma X_1 \vee \Sigma X_2 & \to & \tilde{P}^2(X_1, X_2) & \to & \cdots & \to & \tilde{P}^n(X_1, X_2) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Sigma(X_1 \times X_2) & \to & P^2(X_1 \times X_2) & \to & \cdots & \to & P^n(X_1 \times X_2) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Sigma p_1 + \Sigma p_2 & \to & \Sigma X_1 \vee \Sigma X_2 & \to & \tilde{P}^2(X_1, X_2) & \to & \cdots & \to & \tilde{P}^n(X_1, X_2)
\end{array}
$$
where the upper left arrow is the inclusion and $p_i: X_1 \times X_2 \to X_i$ is the $i^{th}$ projection for $i = 1, 2$.

In order to apply lemma 2.2 to our case, we need the following simple lemma. Let $X$ be an H-space. Then the projective space $P^2X$ is the cofibre of the Hopf construction $H: \Sigma X \wedge X \to \Sigma X$, where we write the inclusion $\Sigma X \to P^2X$ by $j$.

**Lemma 2.3.** Let $X_i$ be H-spaces for $i = 1, 2$. Then there is a homotopy commutative diagram

\[
\begin{array}{ccc}
\Sigma(X_1 \times X_2) & \xrightarrow{j} & P^2(X_1 \times X_2) \\
\Sigma(p_1 + p_2) & \downarrow & \\
\Sigma(X_1 \vee \Sigma X_2) & \xrightarrow{j} & P^2(X_1 \times X_2)
\end{array}
\]

where $p_i: X_1 \times X_2 \to X_i$ is the $i^{th}$ projection for $i = 1, 2$ and $j$ is the restriction of $j$.

**Proof.** Let $h$ be the composite of the inclusion $\Sigma(X_1 \wedge X_2) \to \Sigma(X_1 \times X_2) \wedge (X_1 \times X_2)$ and the Hopf construction $H: \Sigma(X_1 \times X_2) \wedge (X_1 \times X_2) \to \Sigma(X_1 \times X_2)$. By the definition of the Hopf construction, $h$ is the right homotopy inverse of the projection $\Sigma(X_1 \times X_2) \to \Sigma(X_1 \wedge X_2)$, and so the map

\[
\Sigma i_1 \vee \Sigma i_2 \vee h: \Sigma X_1 \vee \Sigma X_2 \vee \Sigma(X_1 \wedge X_2) \to \Sigma(X_1 \times X_2)
\]

is a homotopy equivalence, where $i_k: X_k \to X_1 \times X_2$ is the inclusion for $k = 1, 2$. Let $r'$ be the composite of the homotopy inverse $(\Sigma i_1 \vee \Sigma i_2 \vee h)^{-1}$ and the projection $r: \Sigma X_1 \vee \Sigma X_2 \vee \Sigma(X_1 \wedge X_2) \to \Sigma X_1 \vee \Sigma X_2$. Then there is a homotopy cofibration

\[
\Sigma(X_1 \wedge X_2) \xrightarrow{h} \Sigma(X_1 \times X_2) \xrightarrow{r'} \Sigma X_1 \vee \Sigma X_2
\]

and so one gets a homotopy commutative diagram

\[
\begin{array}{ccc}
\Sigma(X_1 \times X_2) & \xrightarrow{j} & P^2(X_1 \times X_2) \\
r' & \downarrow & \\
\Sigma X_1 \vee \Sigma X_2 & \xrightarrow{j} & P^2(X_1 \times X_2)
\end{array}
\]

It remains to show $j \circ r' \simeq j \circ (\Sigma p_1 + \Sigma p_2)$. Since $\Sigma X_1 \times \Sigma X_2 \subset \tilde{P}^2(X_1 \times X_2)$, it follows from lemma 2.2 that $j$ factors through the inclusion $\Sigma i_1 \vee \Sigma i_2: \Sigma X_1 \vee \Sigma X_2 \to \Sigma X_1 \vee \Sigma X_2$.

\[
\begin{array}{ccc}
\Sigma i_1 \vee \Sigma i_2 \vee h: \Sigma X_1 \vee \Sigma X_2 \vee \Sigma(X_1 \wedge X_2) & \to & \Sigma X_1 \vee \Sigma X_2
\end{array}
\]
Then it suffices to show \((\Sigma i_1 \lor \Sigma i_2) \circ r' \simeq (\Sigma i_1 \lor \Sigma i_2) \circ (\Sigma p_1 + \Sigma p_2)\), or equivalently, \((\Sigma i_1 \lor \Sigma i_2) \circ (\Sigma p_1 + \Sigma p_2) \circ h \simeq *\). Now \((\Sigma i_1 \lor \Sigma i_2) \circ (\Sigma p_1 + \Sigma p_2) \circ h = (\Sigma(i_1 \circ p_1) \circ h + \Sigma(i_2 \circ p_2) \circ h)\). On the other hand, \(\Sigma p_k \circ h \simeq *\) for \(k = 1, 2\) by the definition of the Hopf construction. Thus \((\Sigma i_1 \lor \Sigma i_2) \circ (\Sigma p_1 + \Sigma p_2) \circ h = \Sigma (i_1 \circ p_1) \circ h + \Sigma (i_2 \circ p_2) \circ h\).

\[\text{Proposition 2.4. } \text{Let } X_1, \ldots, X_l \text{ be } A_n\text{-spaces, } Y \text{ be a topological monoid, and } f : X_1 \times \cdots \times X_l \to Y \text{ be a map such that the restriction } f|_{X_i} \text{ is an } A_n\text{-map for each } i. \text{ Then } f \text{ is an } A_n\text{-map if and only if there is a map } \hat{P}^n(X_1, \ldots, X_l) \to BY \text{ satisfying a homotopy commutative diagram}
\]

\[
\begin{array}{ccc}
\Sigma X_1 \lor \cdots \lor \Sigma X_l & \xrightarrow{j} & BY \\
\downarrow & & \downarrow \\
\hat{P}^n(X_1, \ldots, X_l) & \xrightarrow{} & BY
\end{array}
\]

where \(\hat{f}\) is the restriction of \(\bar{f}\).

\[\text{Proof. } \text{We first consider the case } n = 2. \text{ The map } f \text{ is an } A_2\text{-map, that is, an } H\text{-map if and only if the Samelson products } \langle f|_{X_i}, f|_{X_j} \rangle \text{ are trivial for all } i \neq j. \text{ By the adjointness of Samelson products and Whitehead products, this is equivalent to that } \hat{f} \text{ extends to } \hat{P}^2(X_1, \ldots, X_l) \to BY.\]

Thus since each \(f|_{X_i} \text{ is an } H\text{-map, such an extension exists if and only if } \hat{f} \text{ extends to } \hat{P}^2(X_1, \ldots, X_l) \to BY.\)

We next consider the case \(n \geq 3. \text{ By the } n = 2 \text{ case, we may assume that } f \text{ is an } H\text{-map. Then by lemma 2.3, there is a homotopy commutative diagram}
\]

\[
\begin{array}{ccc}
\Sigma(X_1 \times \cdots \times X_l) & \xrightarrow{j} & P^2(X_1 \times \cdots \times X_l) \\
\downarrow & & \downarrow \\
\Sigma p_1 + \cdots + \Sigma p_l & \xrightarrow{} & P^2Y \\
\downarrow & & \downarrow \\
\Sigma X_1 \lor \cdots \lor \Sigma X_l & \xrightarrow{j} & P^2(X_1 \times \cdots \times X_l) \\
\downarrow & & \downarrow \\
\Sigma X_1 \cdots \lor \Sigma X_l & \xrightarrow{j} & P^2Y \\
\downarrow & & \downarrow \\
BY & \xrightarrow{j^2} & BY
\end{array} \tag{2.1}
\]

where the composite of the upper row is \(\bar{f}\) and the composite of the lower row is \(\bar{f}\). Now suppose that there is a map \(\hat{P}^n(X_1, \ldots, X_l) \to BY \text{ extending } \bar{f}. \text{ Then it follows from lemmas 2.2 and 2.3 together with (2.1) that there is a homotopy commutative diagram}
and thus by lemma 2.1 \( f \) is an \( A_n \)-map.

If \( f \) is an \( A_n \)-map then by lemma 2.1 there is a map \( P^n(X_1, \ldots, X_l) \to BY \) extending \( \bar{f} \). Thus by lemma 2.2, one gets the desired map \( \hat{P}(X_1, \ldots, X_l) \to BY \).

Therefore, the proof is complete. \( \square \)

Hereafter, let \( p \) be an odd prime and we localize at \( p \). Let \( G \) be a connected Lie group. By the classical result of Hopf, the rational cohomology of \( G \) is an exterior algebra generated by odd degree elements. If generators are in dimensions \( 2n_1-1, \ldots, 2n_r-1 \) for \( n_1 \leq \cdots \leq n_r \) then we say that the type of \( G \) is \((n_1, \ldots, n_r)\).

Recall that \( G \) is called \( p \)-regular if it is homotopy equivalent to a product of spheres such that

\[
G \simeq S^{2n_1-1} \times \cdots \times S^{2n_r-1}.
\]

It is well known that \( G \) is \( p \)-regular if \( p > n_r \). On the other hand, it is also well known that any odd sphere is an \( A_{p-1} \)-space. A cell decomposition of the projective spaces of odd spheres is given in [8] as follows.

**Lemma 2.5.** For \( l \leq p-1 \) there is a \( p \)-local cell decomposition

\[
P^l S^{2n-1} \simeq S^{2n} \cup e^{4n} \cup \cdots \cup e^{2ln}.
\]

The following proposition is proved in [8], which shows an intrinsic feature of the decomposition (2.2) with respect to higher homotopy associativity.

**Proposition 2.6.** Let \( G \) be a connected Lie group of type \((n_1, \ldots, n_r)\). If \( p > kn_r \), implying \( G \) is \( p \)-regular, then the product \( A_k \)-structure on \( S^{2n_1-1} \times \cdots \times S^{2n_r-1} \) and the standard \( A_k \)-structure on \( G \) are equivalent.

We slightly improve this proposition in our setting. Let \((G, H)\) and \( p \) be as in theorem 1.3 and suppose \( G \) is \( p \)-regular. Then \( G/H \) is a product of odd spheres,
and if
\[ G/H \simeq S^{2l_1-1} \times \cdots \times S^{2l_t-1} \]
for \( l_1 \leq \cdots \leq l_t \) then we say that \( G/H \) has type \((l_1, \ldots, l_t)\). Hereafter, let \((n_1, \ldots, n_r), (m_1, \ldots, m_s)\) and \((l_1, \ldots, l_t)\) be the types of \( G, H \) and \( G/H \), respectively. Then \((m_1, \ldots, m_s)\) and \((l_1, \ldots, l_t)\) are subsequences of \((n_1, \ldots, n_r)\).

**Proposition 2.7.** Let \((G, H)\) and \(p\) be as in theorem 1.3. If \( p \geq b_k \) with \( k \geq 2 \) then the product \( A_k\)-structure on \( H \times S^{2l_1-1} \times \cdots \times S^{2l_t-1} \) and the standard \( A_k\)-structure on \( G \) are equivalent.

**Proof.** Note that \( G \) is \( p\)-regular for \( p \geq b_k \). By lemma 2.1 and proposition 2.4, it suffices to show that the natural map \( \Sigma(H \vee S^{2l_1-1} \vee \cdots \vee S^{2l_t-1}) \to BG \) extends to a map \( \hat{P}^k(H, S^{2l_1-1}, \ldots, S^{2l_t-1}) \to BG \). Let
\[ X = \bigcup_{i+j=k, i \neq k} \hat{P}^i(S^{2m_1-1}, \ldots, S^{2m_s-1}) \times \hat{P}^j(S^{2l_1-1}, \ldots, S^{2l_t-1}). \]
We first construct an extension \( X \to BG \) for \( p \geq b_k \). Let \( b_k = \max\{(k-1)m_s + l_t, kl_t\} \). By lemma 2.5, \( X \) consists of even dimensional cells and \( \dim X \leq 2b_k \). Since \( G \) is \( p\)-regular, its homotopy groups can be calculated from those of spheres in [24]. In particular, \( \pi_{2i-1}(BG) = 0 \) for \( i \leq p \), and so we get an extension \( X \to BG \) for \( p \geq b_k \).

We next construct an extension \( \hat{P}^k(H, S^{2l_1-1}, \ldots, S^{2l_t-1}) \to BG \) from an extension \( X \to BG \). Let
\[ Y = \hat{P}^{k-1}(S^{2m_1-1}, \ldots, S^{2m_s-1}) \subset X. \]
By construction, we may assume that the restriction of an extension \( X \to BG \) to \( Y \) decomposes as \( Y \to BH \to BG \), where the second map is the induced map from the inclusion \( H \to G \). By proposition 2.6, there is a map \( g: P^{k-1}H \to Y \) such that the composite \( f \circ g: P^{k-1}H \to BH \) restricts to the canonical map \( \Sigma H \to BH \). Then \( f \circ g \) gives the standard \( A_{k-1}\)-structure of the identity map of \( H \), and so by [25] there is an \( A_{k-1}\)-map \( h: H \to H \) such that \( f \circ g \circ P^{k-1}h \) is homotopic to the canonical map \( j_{k-1}: P^{k-1}H \to BH \), where \( P^{k-1}h: P^{k-1}H \to P^{k-1}H \) is the induced map from \( h \). Then the composite
\[ \bigcup_{i+j=k, i \neq k} P^iH \times \hat{P}^j(S^{2l_1-1}, \ldots, S^{2l_t-1}) \xrightarrow{g \circ P^{k-1}h \times 1} X \to BG \]
restricts to the canonical map \( j_{k-1}: P^{k-1}H \to BG \). Since the canonical map \( j_{k-1} \) extends to \( j_k: P^kH \to BG \), we finally obtain the desired extension \( \hat{P}^k(H, S^{2l_1-1}, \ldots, S^{2l_t-1}) \to BG \), completing the proof. \( \square \)

Now we prove theorem 1.3.

**Proof of theorem 1.2.** By lemma 2.1 and proposition 2.7, it suffices to show that the adjoint \( \tilde{q}: \Sigma G \to BH \) of the projection \( q: G \to H \) extends to
\( \hat{P}^k(H, S^{2l_1-1}, \ldots, S^{2l_t-1}) \to BH \). Note that \( q: G \to H \) is identified with the projection
\[ H \times S^{2l_1-1} \times \cdots \times S^{2l_t-1} \to H. \]
Then there is a homotopy commutative diagram
\[
\begin{array}{ccc}
\Sigma G & \xrightarrow{\Sigma q} & \Sigma H \\
\downarrow & & \downarrow \\
\hat{P}^k(H, S^{2l_1-1}, \ldots, S^{2l_t-1}) & \longrightarrow & \hat{P}^kH
\end{array}
\]
where the bottom map is the projection. Thus since the composite \( \Sigma G \xrightarrow{\Sigma q} \Sigma H \to BH \) is \( \bar{q} \), the composite
\[ \hat{P}^k(H, S^{2l_1-1}, \ldots, S^{2l_t-1}) \to \hat{P}^kH \xrightarrow{j_k} BH \]
is the desired extension of \( \bar{q} \). □

3. Cohomology calculation

For the rest of the paper, cohomology is assumed to be with mod \( p \) coefficients for an odd prime \( p \). This section calculates \( P^1x \) for some cohomology classes \( x \) of the classifying spaces of Lie groups, which we are going to use. We first consider \( SU(n) \).

The cohomology of \( BSU(n) \) is given by
\[ H^*(BSU(n)) = \mathbb{Z}/p[c_2, \ldots, c_n] \] (3.1)
where \( c_i \) is the Chern class. In [21], \( P^1c_k \) is determined, and in particular,
\[
P^1c_k = \sum_{2i_2+3i_3+\cdots+ni_n = k+p-1} (-1)^{i_2+\cdots+i_n-1}(i_2+\cdots+i_n-1)! \frac{i_2! \cdots i_n!}{i_2! \cdots i_n!} \times \left( k-1 - \frac{\sum_{j=2}^{k-1}(k+p-1-j)i_j}{i_2+\cdots+i_n-1} \right)i_2^2 \cdots i_n^n. \] (3.2)

If a polynomial \( P \) includes a monomial \( M \) then we write \( P \geq M \). By (3.2) one gets:

**Lemma 3.1.** Let \( k \geq 2 \).

1. In \( H^*(BSU(2n+1)) \),
\[
P^1c_2 \geq \begin{cases} 
(-1)^{k-1}c_{2n+1}^{k-1} & (p = k(2n+1) - 1) \\
-3c_{2n+1}c_3 & (p = 2n+1) 
\end{cases}
\]
\[
P^1c_4 \geq -3c_{2n+1}c_3 \quad (p = 2n+1). 
\]
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2. In $H^*(BSU(2n))$ with $k,n \geq 2$,

\[
\mathcal{P}^1c_2 \geq (-1)^{k-1}(k-1)c_{2n-2}c_{2n-1}c_{p-2(k-1)n+2} \quad (2(k-1)n \leq p < 2kn-1)
\]

\[
\mathcal{P}^1c_4 \geq (-1)^{k-1}3(k-1)c_{2n-2}c_{2n-1}c_3 \quad (p = 2(k-1)n - 1).
\]

We next consider $SO(2n+1)$. The cohomology of $BSO(2n+1)$ is given by

\[
H^*(BSO(2n+1)) = \mathbb{Z}/p[p_1, \ldots, p_n]
\]  

(3.3)

where $p_i$ is the Pontrjagin class and $e_i$ is the Euler class. Then the inclusion $c: SO(2n+1) \to SU(2n+1)$ satisfies $c^*(c_{2k}) = (-1)^k p_k$ and $c^*(c_{2k+1}) = 0$, and so by (3.2),

\[
\mathcal{P}^1p_k = \sum_{i_1 + 2i_2 + \cdots + ni_n = k + \frac{p-1}{2}} (-1)^{i_1+i_2+\cdots+i_n+i_1+i_2+\cdots+i_n} (i_1+i_2+\cdots+i_n-1)!
\]

\[
\times \left(2k-1 - \frac{\sum_{j=1}^{k-1} (2k+p-1-2j)i_j}{i_1+i_2+\cdots+i_n-1}\right) p_1^{i_1} \cdots p_n^{i_n}.
\]  

(3.4)

Thus by focusing on $BSO(15)$, one gets the following.

**Lemma 3.2.** Let $k \geq 2$. In $H^*(BSO(15))$ the following hold:

\[
\mathcal{P}^1p_1 \geq \begin{cases} 
(-1)^{k+\frac{p-1}{2}} \frac{5}{12} p_r p_6^{k-2} p_2 & (p = 12k-7) \\
(-1)^{k+\frac{p-1}{2}} p_r p_6^{k-2} & (p = 12k-11)
\end{cases}
\]

\[
\mathcal{P}^1p_2 \geq (-1)^{k+\frac{p-1}{2}} 3p_r p_6^{k-2} \quad (p = 12k-13)
\]

\[
\mathcal{P}^1p_4 \geq \begin{cases} 
(-1)^{k+\frac{p-1}{2}} \frac{29}{12} p_r p_6^{k-2} p_2 & (p = 12k-13) \\
(-1)^{k+\frac{p-1}{2}} p_r p_6^{k-2} & (p = 12k-17)
\end{cases}
\]

The cohomology of $BSO(2n)$ is given by

\[
H^*(BSO(2n)) = \mathbb{Z}/p[p_1, \ldots, p_{n-1}, e_n]
\]

where $p_i$ is the Pontrjagin class and $e_n$ is the Euler class. Then the inclusion $j: SO(2n) \to SO(2n+1)$ satisfies $j^*(p_i) = p_i$ for $i = 1, \ldots, n-1$ and $j^*(p_n) = e_n^2$. Thus by (3.4) one gets:

**Lemma 3.3.** In $H^*(BSO(2n))$, for $2(k-2)(n-1) + 2 \leq p < 2(k-1)(n-1) + 2$ with $k \geq 3$

\[
\mathcal{P}^1p_1 \geq (-1)^{k+\frac{p-1}{2}} (k-2)p_{n-1}^{k-3} p_{\frac{p-1}{2}-(k-2)(n-1)} e_n^2
\]

and for $p \leq 2n-1$

\[
\mathcal{P}^1p_{n-\frac{p-1}{2}} \geq (-1)^{\frac{p-1}{2}} 2n e_n^2.
\]
Lemma 3.4. Let $k \geq 2$. In $H^*(BSO(8))$ the following hold:

$$\mathcal{P}^1p_1 \geq (-1)^{k+\frac{p-k}{2}} \frac{1}{12} p_3^{k-2} p_2^2 \quad (p = 6k - 5)$$

$$\mathcal{P}^1p_2 \geq (-1)^{k+\frac{p-k}{2}} \frac{1}{4} p_3^{k-2} p_2^2 \quad (p = 6k - 7).$$

We finally consider the exceptional Lie groups. Let $p$ be a prime $> 5$. The cohomology of $BE_8$ is given by

$$H^*(BE_8) = \mathbb{Z}/p[x_4, x_{16}, x_{24}, x_{28}, x_{36}, x_{40}, x_{48}, x_{60}], \quad |x_i| = i.$$  

Let $j_2 : Spin(15) \to E_8$ be the canonical inclusion. It is shown in [6] that the generators $x_i$ can be chosen such that

\[
j_2^*(x_4) = p_1 \\
j_2^*(x_{16}) = 12p_4 - \frac{18}{5} p_3 p_1 + p_2^2 + \frac{1}{10} p_2 p_1^2 \\
j_2^*(x_{24}) = 60p_6 - 5p_5 p_1 - 5p_4 p_2 + 3p_3^2 - p_3 p_2 p_1 + \frac{5}{36} p_2^3 \\
j_2^*(x_{28}) = 480p_7 + 40p_5 p_2 - 12p_4 p_3 - 3p_3 p_2^2 - 3p_4 p_2 p_1 + \frac{24}{5} p_3^2 p_1 + \frac{11}{36} p_2^3 p_1 \\
j_2^*(x_{36}) = 480p_7 p_2 + 72p_6 p_3 - 30p_5 p_4 - \frac{25}{2} p_5 p_2^2 + 9p_4 p_3 p_2 - 18 \frac{p_3^3}{5} - \frac{1}{4} p_3 p_2^3 \mod (p_1)
\]

where $BSpin(15) \simeq BSO(15)$ since we are localizing at an odd prime. Then in particular, $x_{60}$ can be chosen such that $j_2^*(x_{60})$ does not include the monomial $a p_7 p_6 p_2$ for $a \in \mathbb{Z}/p$. Thus by lemma 3.2 and a degree reason, one gets:

Lemma 3.5. Let $p$ be a prime $> 5$ and $k \geq 2$. In $H^*(BE_8)$,

$$\mathcal{P}^1x_4 \geq \begin{cases} (-1)^{k+\frac{p-k}{2}} \frac{5}{96-60k^2} x_{36} x_{24}^{k-2} & (p = 12k - 7) \\ (-1)^{k+\frac{p-k}{2}} \frac{35}{8-60k^2} x_{28} x_{24}^{k-2} & (p = 12k - 11) \end{cases}$$

$$\mathcal{P}^1x_{16} \geq \begin{cases} (-1)^{k+\frac{p-k}{2}} \frac{7}{8-60k^2} x_{28} x_{24}^{k-2} & (p = 12k - 17) \end{cases}$$

There is a commutative square of inclusions

$$\begin{array}{ccc}
Spin(10) & \xrightarrow{i_1} & Spin(15) \\
\downarrow j_1 & & \downarrow j_2 \\
E_6 & \xrightarrow{i_2} & E_8.
\end{array}$$

(3.5)
Let $p$ be a prime $> 5$. The cohomology of $BE_6$ and $BSO(2m)$ are given by

$$H^*(BE_6) = \mathbb{Z}/p[x_4, x_{10}, x_{12}, x_{16}, x_{18}, x_{24}], \quad |x_i| = i. \quad (3.6)$$

It is shown in [6] that $x_{10}$ and $x_{18}$ are chosen as

$$j_1^*(x_{10}) = e_5, \quad j_1^*(x_{12}) = -6p_3 + p_2p_1, \quad j_1^*(x_{18}) = p_2e_5$$

and that $i_2^*(x_i) = x_i$ for $i = 4, 12, 16, 24$. Then by the choice of the generators $x_i$ of $H^*(BE_8)$ and the commutative diagram (3.5),

$$i_2^*(x_{28}) = 40x_{18}x_{10} + \frac{1}{6}x_{16}x_{12}, \quad i_2^*(x_{36}) = -10x_{18}^2 - \frac{5}{2}x_{16}x_{10}^2.$$ 

Thus by lemma 3.5 one finally obtains the following.

**Corollary 3.6.** Let $p$ be a prime $> 5$ and $k \geq 2$. In $H^*(BE_6)$,

$$\mathcal{P}^1_{x_4} \geq \begin{cases} (-1)^{k+\frac{r+1}{2}} \frac{25}{48} x_{24}^{-2} x_{18}^2 & (p = 12k - 7) \\ (-1)^{k+\frac{r+1}{2}} \frac{5}{69} x_{24}^{-2} x_{18} & (p = 12k - 11) \end{cases}$$

$$\mathcal{P}^1_{x_{16}} \geq \begin{cases} (-1)^{k+\frac{r+1}{2}} \frac{175}{48} x_{24}^{-2} x_{18}^2 & (p = 12k - 13) \\ (-1)^{k+\frac{r+1}{2}} \frac{35}{69} x_{24}^{-2} x_{18} x_{10} & (p = 12k - 17) \end{cases}.$$ 

**4. Proof of the main theorem**

Let $\langle a, b \rangle$ denote the Samelson product of $a, b \in \pi_*(X)$ for an H-group $X$. We will need the following lemma.

**Lemma 4.1.** Let $f : X \to Y$ be a map between H-groups. Suppose there are $a, b \in \pi_*(X)$ such that $f_*(a) = 0$ and $f_*(\langle a, b \rangle) \neq 0$. Then $f$ is not an H-map.

**Proof.** If $f$ is an H-map then by the naturality of Samelson products

$$0 \neq f_*(\langle a, b \rangle) = \langle f_*(a), f_*(b) \rangle = 0,$$

which is a contradiction. Thus $f$ is not an H-map.

All Samelson products that we need to apply lemma 4.1 are calculated in [1, 5] except for the case $(G, H) = (Spin(8), G_2)$ and $p = 2$. Then we calculate a certain Samelson product in $Spin(8)$ for $p = 2$. Recall that the fibration $Spin(7) \to Spin(8) \to S^7$ is trivial such that $Spin(8) = Spin(7) \times S^7$. Let $\tilde{\partial} : \pi_{s+1}(S^8) \to \pi_{s}(Spin(8))$ be the connecting map of the homotopy exact sequence of a fibration $Spin(8) \to Spin(9) \to S^8$. We will freely use the notation of the homotopy groups of spheres in [24].

**Lemma 4.2.** Let $p = 2$ and $\langle \iota_7 \rangle : S^7 \to Spin(7) \times S^7 = Spin(8)$ be the inclusion. Then

$$\langle \iota_7, \iota_7 \rangle = \pm \tilde{\partial}(2\sigma_8 - \Sigma\sigma').$$
Proof. Consider a commutative diagram with fibration rows and columns

\[
\begin{array}{ccc}
\text{Spin}(7) & \longrightarrow & \text{Spin}(7) \\
\downarrow & & \downarrow \\
\text{Spin}(8) & \longrightarrow & \text{Spin}(9) \\
\downarrow & & \downarrow \\
S^7 & \longrightarrow & S^{15} \\
\end{array}
\]

Then in the homotopy exact sequence of the middle row

\[
\cdots \rightarrow \pi_{s+1}(S^8) \xrightarrow{\partial} \pi_s(\text{Spin}(8)) \rightarrow \pi_s(\text{Spin}(9)) \rightarrow \pi_s(S^8) \rightarrow \cdots
\]

one has \(\partial(\iota_8) = [\iota_7]\), where \(\iota_8\) is the identity map of \(S^8\). Thus by the adjointness of Whitehead products and Samelson products, \(\partial([\iota_8, \iota_8]) = \langle [\iota_7], [\iota_7] \rangle\). On the other hand, it is shown in [24, p. 50] that \(\pi_{15}(\text{Spin}(7)) = Z/8\{\nu_6 + \epsilon_6\} \oplus Z/2\{\nu_5\nu_8^2\} \oplus Z/8\{\eta_5\nu_6\}\), where for a cyclic group \(A\), \(A\{x\}\) is a cyclic group generated by \(x\) which is isomorphic to \(A\). Thus one gets the desired equality. \(\Box\)

There is a commutative diagram of inclusions

\[
\begin{array}{ccc}
\text{SU}(3) & \longrightarrow & G_2 \\
\downarrow & & \downarrow i \\
\text{SU}(3) & \longrightarrow & \text{Spin}(6) \\
\downarrow j' & & \downarrow j \\
\text{Spin}(7) & \longrightarrow & \text{Spin}(8). \\
\end{array}
\]

It is shown in [11] that

\[
\pi_{14}(\text{Spin}(7)) = Z/8\{\nu_6 + \epsilon_6\} \oplus Z/2\{\nu_5\nu_8^2\} \oplus Z/8\{\eta_5\nu_6\}. \]

Then since \(\text{Spin}(8) = \text{Spin}(7) \times S^7\),

\[
\pi_{14}(\text{Spin}(8)) = Z/8\{j''_6([\nu_6 + \epsilon_6])\} \oplus Z/2\{j''_5([\nu_5]\nu_8^2)\}
\]

\[
\oplus Z/8\{j''_6([\eta_5\nu_6])\} \oplus Z/8\{[\iota_7]\sigma'\},
\]

where \(\pi_{14}(S^7) = Z/8\{\sigma'\}\). The following is proved in [11].

Lemma 4.3. For an odd integer \(u_1\) and an integer \(u_2\),

\[
\begin{align*}
\partial(\sigma_8) & \equiv [\iota_7]\sigma' + u_1j''_6([\nu_6 + \epsilon_6]) + u_2j''_5([\nu_5\nu_8^2]) \mod j''_5([\nu_5]\nu_8^2) \\
\partial(\Sigma\sigma') & \equiv 2[\iota_7]\sigma' + 4j''_6([\nu_6 + \epsilon_6]) - j''_5([\eta_5\nu_6]) \mod j''_5([\nu_5]\nu_8^2).
\end{align*}
\]
Lemma 4.5.\[ \square \]

As desired.

Proof. Consider a commutative diagram with fibration rows

\[
\begin{array}{ccc}
SU(2) & \longrightarrow & SU(3) \\
\downarrow & & \downarrow \pi' \\
Spin(5) & \longrightarrow & Spin(6) \\
\end{array}
\]

\[
\begin{array}{ccc}
& & S^5 \\
\pi & \longrightarrow & S^5.
\end{array}
\]

As in [16], there is \( x \in \pi_8(Spin(6)) \) such that \( \pi'_6(x) = \nu_5 \), and \( [\nu_5]_7 \in \pi_3(Spin(7)) \) is chosen as \( [\nu_5]_7 = j'_*(x) \), where \( \pi_{n+3}(S^n) = \mathbb{Z}/8\{\nu_n\} \) for \( n \geq 5 \). On the other hand, in [16], \( [\nu^2_5] \in \pi_{11}(SU(3)) \) is chosen to be any \( y \in \pi_{11}(SU(3)) \) satisfying \( \bar{\pi}_*(y) = \nu^2_5 \).

Since \( Spin(6)/SU(3) = S^7 \) and \( \pi_k(S^7) = 0 \) for \( k = 11, 12 \), \( i' \) is an isomorphism in \( \pi_{11} \). Then since \( \pi'_*(xv_8) = \nu^2_5 \), we may put \( [\nu^2_5] = (i'_*)^{-1}(xv_8) \). Thus

\[
j_*(i_*([\nu^2_5]_7)) = j'_*(xv^2_5) = [\nu_5]_7v^2_5
\]

as desired. \( \square \)

Lemma 4.4. We may choose \( [\nu_5]_7 \in \pi_3(Spin(7)) \) and \( [\nu^2_5] \in \pi_{11}(SU(3)) \) such that

\[
[\nu_5]_7v^2_5 = j_*(i_*([\nu^2_5]_7)).
\]

Proof. Consider a commutative diagram with fibration rows

\[
\begin{array}{ccc}
SU(2) & \longrightarrow & SU(3) \\
\downarrow & & \downarrow \pi' \\
Spin(5) & \longrightarrow & Spin(6) \\
\end{array}
\]

\[
\begin{array}{ccc}
& & S^5 \\
\pi & \longrightarrow & S^5.
\end{array}
\]

As in [11], there is \( x \in \pi_8(Spin(6)) \) such that \( \pi'_6(x) = \nu_5 \), and \( [\nu_5]_7 \in \pi_3(Spin(7)) \) is chosen as \( [\nu_5]_7 = j'_*(x) \), where \( \pi_{n+3}(S^n) = \mathbb{Z}/8\{\nu_n\} \) for \( n \geq 5 \). On the other hand, in [16], \( [\nu^2_5] \in \pi_{11}(SU(3)) \) is chosen to be any \( y \in \pi_{11}(SU(3)) \) satisfying \( \bar{\pi}_*(y) = \nu^2_5 \).

Since \( Spin(6)/SU(3) = S^7 \) and \( \pi_k(S^7) = 0 \) for \( k = 11, 12 \), \( i' \) is an isomorphism in \( \pi_{11} \). Then since \( \pi'_*(xv_8) = \nu^2_5 \), we may put \( [\nu^2_5] = (i'_*)^{-1}(xv_8) \).

Thus

\[
j_*(i_*([\nu^2_5]_7)) = j'_*(xv^2_5) = [\nu_5]_7v^2_5
\]

as desired. \( \square \)

Lemma 4.5. Let \( q: Spin(8) \to G_2 \) be the projection

\[
Spin(8) = Spin(7) \times S^7 \simeq G_2 \times S^7 \times S^7 \to G_2.
\]

Then \( q_*([\nu_7], [\nu_7]) \neq 0 \).
Proof. Consider a commutative diagram with fibration rows

$$
\begin{array}{ccc}
SU(3) & \xrightarrow{i} & G_2 \\
\downarrow & & \downarrow \pi \\
Spin(6) & \xrightarrow{j'} & Spin(7)
\end{array}
\quad (4.2)
$$

In [11, 16], $([\bar{\nu}_6 + \epsilon_6])$ and $[\bar{\nu}_6 + \epsilon_6]$ are chosen such that $\pi_4([\bar{\nu}_6 + \epsilon_6]) = \pi_4([\bar{\nu}_6 + \epsilon_6])$. Let $(\bar{\nu}_6 + \epsilon_6) = \bar{\nu}_6 + \epsilon_6$, where $\pi_{14}(S^6) = Z/8\{\bar{\nu}_6\} \oplus Z/2\{\epsilon_6\}$. Then by lemma 4.4,

$$
j_*(([\bar{\nu}_6 + \epsilon_6])) = [\bar{\nu}_6 + \epsilon_6] + t_1j_*(i_*(\bar{\nu}_6^2[\nu]_{11})) + t_2[\eta_5\epsilon_6]_7
$$

for some integers $t_1, t_2$. Thus by (4.1) and lemmas 4.2, 4.3 and 4.4,

$$
\langle [\nu], [\nu] \rangle \equiv \pm(2u_1 - 4 + s_1 - 2(u_1t_2 - t_2 - u_2)s_1)j''(j_*(\bar{\nu}_6 + \epsilon_6)) \mod (j''(j_*(\bar{\nu}_6^2[\nu]_{11}))), j''([\nu])').
$$

Since $q \circ j'' \circ j \simeq 1$ and $q_*(j''([\nu]')) = 0$,

$$
q_*(([\nu], [\nu])) \equiv \pm(2u_1 - 4 + s_1 - 2(u_1t_2 - t_2 - u_2)s_1)\bar{\nu}_6 + \epsilon_6 \mod i_*(\bar{\nu}_6^2[\nu]_{11})
$$

and so it suffices to show $s_1 \equiv 4 \mod 8$.

By the definition of $[\eta_5\epsilon_6]_7$ in [11], $\pi_4([\eta_5\epsilon_6]_7) = 0$, and so $\pi_4([\bar{\nu}_6 + \epsilon_6] + s_3\pi_4\langle[\nu]_{11}\rangle \equiv 0$. Consider a homotopy exact sequence of the lower fibration of (4.2). Since $Spin(6) = SU(4)$ and $\pi_6(SU(4)) \cong \pi_6(SU(\infty)) = 0$, one gets $\pi_6(Spin(6)) = \pi_6(SU(4)) = 0$, and so $\pi''_4: \pi_7(Spin(7)) \to \pi_7(S^6)$ is surjective. By [24] and [16], $\pi_7(S^6) = Z/2\{\eta_6\}$ and $\pi_7(Spin(7)) = Z\{\nu'_{11}\}$, implying $\pi''_4\langle[\nu]_{11}\rangle = \eta_6$. Thus $s_1(\bar{\nu}_6 + \epsilon_6) + s_3\eta_6\epsilon_6' = 0$. Now by [24, p. 64], $\eta_6\epsilon_6' = 4\bar{\nu}_6$, and so $(s_1 + 4s_3)\bar{\nu}_6 + s_1\epsilon_6 = 0$. Then $s_1$ is even and $s_1 + 4s_3 \equiv 0 \mod 8$. On the other hand, since $[\eta_5\epsilon_6]_7$ has order 8, either $s_1$ or $s_3$ must be odd. Then $s_3$ is odd, implying $s_1 \equiv 4 \mod 8$ as desired. 

PROPOSITION 4.6. Let $(G, H), a_k$ and $p$ be as in theorem 1.2. If $p < a_1$ then the projection $q: G \to H$ is not an $H$-map.

Proof. Let $(G, H) = (SU(2n + 1), SO(2n + 1))$. Then $p < 2n + 1$. Since $SO(2n + 1)$ is a direct summand of $SU(2n + 1)$, $\pi_4(SO(2n + 1))$ is a direct summand of $\pi_4(SU(2n + 1))$ as well. As in [17] $\pi_{4n+2}(SU(2n + 1))$ is a cyclic group. Since $SO(2n + 1) \cong Sp(n)$ where we are localizing at an odd prime $p$, $\pi_{4n+2}(SO(2n + 1)) \neq 0$ as in [17]. Then the projection $q$ induces an isomorphism in $\pi_{4n+2}$. Let $\epsilon_i$ be a generator of $\pi_{2i+1}(SU(2n + 1)) \cong Z(p)$ for $i = 1, \ldots, 2n - 1$. By [1] that the Samelson product $\langle \epsilon_{2n-p+1}, \epsilon_{p-1} \rangle$ in $SU(2n + 1)$ is non-trivial. Then in particular, since $q$ is an isomorphism in $\pi_{4n+2}$, $q_\ast(\langle \epsilon_{2n-p+1}, \epsilon_{p-1} \rangle) \neq 0$. On the other hand, since $\pi_{2p-1}(SO(2n + 1)) = 0$ for $p < 2n + 1$, $q_\ast(\epsilon_{p-1}) = 0$. Thus the proof is done by lemma 4.1.

The case $(G, H) = (SU(2n), Sp(n))$ follows from the same argument using $\langle \epsilon_{2n-p+1}, \epsilon_{p-1} \rangle$ in $SU(2n)$, where $p < 2n - 1$. There is nothing to do for $(G, H) = (SO(2n), SO(2n - 1))$ since $a_1 = 1$. 


Let \((G, H) = (E_6, F_4)\). Then \(p = 5\), and so
\[
E_6 \simeq F_4 \times B(9, 17)
\]
as in [17] such that the projection \(q: E_6 \rightarrow F_4\) is identified with the projection \(F_4 \times B(9, 17) \rightarrow F_4\), where \(B(9, 17)\) is a certain \(S^9\)-bundle over \(S^{17}\). Let \(\epsilon: B(9, 17) \rightarrow E_6\) be the inclusion. In [5] it is shown that the Samelson product \(\langle \epsilon, \epsilon \rangle\) is non-trivial, and its proof actually shows that \(\langle \epsilon|_{S^9}, \epsilon|_{S^9} \rangle\) is non-trivial, where \(\epsilon|_{S^9}\) is the restriction of \(\epsilon\) to the bottom cell \(S^9 \subset B(9, 17)\). The homotopy groups of \(B(9, 17)\) are calculated in [13] such that \(\pi_{18}(B(9, 17)) = 0\). Then \(q_*\langle \langle \epsilon|_{S^9}, \epsilon|_{S^9} \rangle \rangle \neq 0\). On the other hand, \(q_*\langle \epsilon|_{S^9} \rangle = 0\), and so by lemma 4.1 \(q\) is not an H-map.

Let \((G, H) = (\text{Spin}(8), G_2)\). Then \(p = 2\). By [16], the inclusion \(G_2 \rightarrow \text{Spin}(8)\) is injective in \(\pi_{14}\). Then by lemmas 4.1 and 4.5, \(q\) is not an H-map. Thus the proof is complete. \(\square\)

We set notation on cohomology. Let \(G\) be a connected Lie group whose integral homology has no \(p\)-torsion. Recall from [9] that there is an isomorphism of unstable algebras
\[
H^*(P^k G) \cong H^*(BG)/\tilde{H}^*(BG)^{k+1} \oplus S
\]
for some unstable algebra \(S\) depending on \(G\) and \(k\), where the natural map \(j_k: P^k G \rightarrow BG\) induces the obvious projection in cohomology. Let \((G, H)\) and \(p\) be in theorem 1.2. The cohomology of \(BG\) and \(BH\) are given by
\[
H^*(BG) = \mathbb{Z}/p[x_{n_1}, \ldots, x_{n_r}] \quad \text{and} \quad H^*(BH) = \mathbb{Z}/p[y_{m_1}, \ldots, y_{m_s}] \quad (4.4)
\]
where \(|x_i| = |y_i| = 2i\).

If \(m_i\) in \((m_1, \ldots, m_s)\) and \(n_j\) in \((n_1, \ldots, n_r)\) are equal as a member of the sequence \((n_1, \ldots, n_r)\), then we say that \(m_i = n_j\) in the type of \(G\), where \((m_1, \ldots, m_s)\) is a subsequence of \((n_1, \ldots, n_r)\). Now we state a cohomological criterion for the projection \(G \rightarrow H\) not being an \(A_1\)-map.

**Lemma 4.7.** Let \((G, H)\) and \(p\) be as in theorem 1.2, and let \(j: H \rightarrow G\) be the inclusion. Suppose that presentations (4.4) satisfy
\[
j^*(x_{n_i}) = \begin{cases} 
x_{m_j} & \text{if } n_i = m_j \text{ in the type of } G \\
0 & \text{otherwise.}
\end{cases}
\]
Suppose also that there are an integer \(m_k\) in the type of \(H\) and a monomial \(M = ax_{n_1} \cdots x_{n_{i_1}} \in H^*(BG)\) with \(a \neq 0\) and \(n_{i_1} \leq \cdots \leq n_{i_t}\) satisfying the following conditions:

1. \(n_{i_j} > m_k\) for \(j \geq 2\) and \(m_k - n_{i_1}\) is not a sum of integers in the type of \(G\);
2. for any map \(f: H^*(BH) \rightarrow H^*(BG)\) satisfying \(f(y_{m_i}) \equiv \pm x_{m_i} \mod H^*(BG)^2\) for all \(i\), there is no polynomial \(P \in \text{Im } f\) such that
\[
P \equiv M \mod I
\]
where \(m_k = n_{i_h}\) in the type of \(G\) and \(I = (x_{n_1}, \ldots, x_{n_{i_1}}, \ldots, x_{n_{i_t}})\).
Then the projection $q: G \to H$ is not an $A_l$-map.

Proof. We assume $q$ is an $A_l$-map and show a contradiction. By lemma 2.1 the adjoint $\overline{q}: \Sigma G \to BH$ extends to $\overline{q}: P^l G \to BH$. Let $S$ be as in (4.3). By the condition (1.1),

$$\overline{q}^*(y_{m_1}) \equiv \pm x_{m_1} \mod I^2 + S.$$  

(4.5)

Then by the Cartan formula,

$$\overline{q}^*(P^1 y_{m_k}) \equiv P^1 \overline{q}^*(y_{m_k}) \equiv P^1 x_{m_k} \geq M \neq 0 \mod I + S.$$  

(4.6)

Let $f: H^*(BH) \to H^*(BG)$ be the composite of a lift of $\overline{q}^*: H^*(BH) \to H^*(P^l G)$ through the projection $H^*(BG) \oplus S \to H^*(P^l G)$ under the isomorphism (4.3) and the projection $H^*(BG) \oplus S \to H^*(BG)$. Then by (4.5), $f(y_{m_1}) \equiv \pm x_{m_1} \mod H^*(BG)$, and so by condition (1.2) there is no $P \in \text{Im} f$ such that $P \geq M \mod I$. Thus there is no $Q \in \text{Im} \overline{q}^*$ such that $Q \geq M \mod I + S$, which contradicts (4.6). Therefore the proof is complete. \hfill \Box

**Proposition 4.8.** Let $(G, H), a_k$ and $p$ be as in theorem 1.2, and let $k \geq 2$. Then the following statements hold:

1. if $a_k - 1 < p < a_k$ then the projection $G \to H$ is not an $A_k$-map except for $(G, H) = (SO(2n), SO(2n - 1))$;

2. if $a_k - 1 - n + 2 < p < a_k - n + 2$ then the projection $q: SO(2n) \to SO(2n - 1)$ is not an $A_k$-map.

Proof. (1) Let $(G, H) = (SU(2n + 1), SO(2n + 1))$. The inclusion $c: SO(2n + 1) \to SU(2n + 1)$ satisfies $c^*(e_{2i}) = (-1)^i p_i$ and $c^*(e_{2i+1}) = 0$. Then the presentations (3.1) and (3.3) satisfy the condition in lemma 4.7. By a dimensional consideration, we see that the monomial $(-1)^{k-1}e_{2n+1}^{k-1}c_{(k-1)(2n+1)+1}$ satisfies the condition (2) of lemma 4.7. Thus by lemmas 3.1 and 4.7, the projection $q$ is not an $A_k$-map for $a_k - 1 < p < a_k$.

Let $(G, H) = (SU(2n), Sp(n))$. The cohomology of $BSp(n)$ is given by

$$H^*(BSp(n)) = \mathbb{Z}/p[q_1, \ldots, q_n]$$

where $q_i$ is the symplectic Pontrjagin class. Then the inclusion $c': Sp(n) \to SU(2n)$ satisfies $(c')^*(e_{2i}) = (-1)^i q_i$ and $(c')^*(e_{2i+1}) = 0$. Thus by lemmas 3.1 and 4.7, $q$ is not an $A_k$-map for $a_k - 1 < p < a_k$.

Let $(G, H) = (E_6, F_4)$. The cohomology of $BF_4$ is given by

$$H^*(BF_4) = \mathbb{Z}/p[x_4, x_{12}, x_{16}, x_{24}], \quad |x_i| = i.$$  

In [6], it is shown that we may choose generators of $H^*(BF_4)$ such that the inclusion $j: F_4 \to E_6$ satisfies $j^*(x_i) = x_i$ for $i = 4, 12, 16, 24$ and $j^*(x_i) = 0$ for $i = 10, 18$. Then by corollary 3.6 and lemma 4.7, $q$ is not an $A_k$-map for $a_k - 1 < p < a_k$ except
for $p = 7$ with $k = 2$. Let $p = 7$. We aim to show that $q$ is not an H-map. As in [17], there is a homotopy equivalence

$$E_6 \simeq F_4 \times S^9 \times S^{17}.$$  

Let $\epsilon : S^{17} \to E_6$ be the inclusion. Then by [5] the Samelson product $\langle \epsilon, \epsilon \rangle$ is non-trivial. By [24], $\pi_34(S^9) = \pi_34(S^{17}) = 0$, and so $q_*(\langle \epsilon, \epsilon \rangle) \neq 0$. On the other hand, $q_*(\epsilon) = 0$. Thus by lemma 4.1, $q$ is not an H-map.

Let $(G, H) = (Spin(8), G_2)$ and $j : G_2 \to Spin(8)$ be the inclusion. Since $BSO(8) \simeq BSpin(8)$ at the odd prime $p$,

$$H^*(BSpin(8)) = \mathbb{Z}/p[p_1, p_2, p_3, e_4].$$

The cohomology of $BG_2$ is given by

$$H^*(BG_2) = \mathbb{Z}/p[x_4, x_{12}], \quad |x_i| = i$$

and so by a degree reason, we may assume

$$j^*(p_1) = x_4, \quad j^*(p_2) = 0, \quad j^*(p_3) = x_{12}, \quad j^*(e_4) = 0.$$  

Then by lemmas 3.4 and 4.7, $q$ is not an $A_k$-map for $a_{k-1} \leq p < a_k$.

(2) Let $(G, H) = (SO(2n), SO(2n-1))$. The inclusion $j : SO(2n-1) \to SO(2n)$ satisfies $j^*(p_i) = p_i$ and $j^*(e_n) = 0$. Then by lemmas 3.3 and 4.7, one gets that $q$ is not an $A_k$-map for $a_{k-1} - n + 2 \leq p < a_k - n + 2$ unless $p = n$ for $k = 2$. Let $p = n$ and suppose $q$ is an H-map. Let $\theta : S^{2n-1} \to S^{2n-1} \times SO(2n-1) \simeq SO(2n)$ be the inclusion. In [15] it is shown that the Samelson product $\langle \theta, \theta \rangle$ in $SO(2n)$ is non-trivial. Then since $\pi_{4n-2}(S^{2n-1}) = 0$ as in [24], $q_*(\langle \theta, \theta \rangle)$ is non-trivial. But since $q_*(\theta) = 0$, we obtain a contradiction by lemma 4.1. Thus $q$ is not an H-map. □

**Proof of theorem 1.2.** The proof is done by theorem 1.3 and propositions (4.6) and (4.8) except that $q : E_6 \to F_4$ is an $A_k$-map for $p = 12k - 5$. Let $p = 12k - 5$. From the proofs of theorem 1.3 and proposition 2.7, one can see that it suffices to show that the adjoint of the identity map $\Sigma E_6 \to BE_6$ extends to $X \to BE_6$, where $X$ is as in the proof of proposition 2.7. Note that $X = X^{(24k-12)} \cup e^{24k-6}$. Then since $\pi_{2i-1}(BE_6) = 0$ for $i \leq 12k - 6$ and $12k - 3$ by [24], there is an extension $X \to BE_6$ as desired. Thus the proof is complete. □

**Proof of corollary 1.5.** The equivalence of (2) and (3) is proved in [8], and Saumell [20] proved that $SU(2n+1)$ is a Williams $C_k$-space if and only if $p > k(2n+1)$. Thus the result follows from theorem 1.2. □

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