Scalable programmable quantum gates and a new aspect of the additivity problem for the classical capacity of quantum channels

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(Dated: October 22, 2001)

We consider two apparently separated problems: in the first part of the paper we study the concept of a scalable (approximate) programmable quantum gate (SPQG). These are special (approximate) programmable quantum gates, with nice properties that could have implications on the theory of universal computation. Unfortunately, as we prove, such objects do not exist in the domain of usual quantum theory.

In the second part the problem of noisy dense coding (and generalizations) is addressed. We observe that the additivity problem for the classical capacity obtained is of apparently greater generality than for the usual quantum channel (completely positive maps): i.e., the latter occurs as a special case of the former, but, as we shall argue with the help of the non-existence result of the first part, the former cannot be reduced to an instance of the latter.

We conclude by suggesting that the additivity problem for the classical capacity of quantum channels, as posed until now, may conceptually not be in its appropriate generality.

PACS numbers: 03.65.Ta, 03.67.Hk
Keywords: Quantum gate, scaling, channel capacity, additivity.

I. INTRODUCTION

The present paper brings together two subjects in the realm of quantum information theory that might at first glance seem far apart: the theory of universal computation in a quantum computer, and noise resistant coding of classical information in quantum channels.

The former deals with implementing arbitrary transformation of the (quantum) data in the memory of a computer by a sequence of commands (a program) that are themselves presented to the machine as data. From the first days of the theory of quantum computation this issue was of central importance, as a tool to show that there is essentially only one quantum Turing machine, and to parallel Turing’s insight of the existence of universal classical machines: see Deutsch 11, and Bernstein and Vazirani 6. A great deal of work has been invested into finding small universal sets of “quantum gates”, acting on only few qubits at a time, so that by concatenation any multi–qubit unitary can be approximated arbitrarily (Deutsch 11, DiVincenzo 13, Barenco 2, Deutsch et al. 12, and Barenco et al. 3). This concatenation (represented as a certain directed graph with labelled nodes) can be given to a machine as classical data, which then interprets it as a series of controlled actions on the quantum data.

The universality problem was studied abstractly by Nielsen and Chuang 24, in the notion of programmable quantum gate (PQG), where one allows arbitrary quantum data for a program, their results being further developed by Vidal, Masanes, and Cirac 31, 32. We review these studies, as far as they are relevant for the present purpose, in section I. Then, in section II, the notion of scalability is introduced, which captures the idea that a sufficiently powerful universal programmable quantum gate might give a universal gate if tensored with itself and fed with entangled programs. Unfortunately, it turns out that such objects do not exist, and we point out some implications for the general theory of universal computation.

Then, we switch to the apparently completely distinct problem of quantum channel coding of classical data: a quantum channel usually is modelled by a completely positive, trace preserving, linear map ϕ : B(H1) → B(H2). We may use this channel to communicate by choosing states σ on H1 at the sender’s side, the receiver getting ϕ(σi). By the result of Holevo 18, and Schumacher and Westmoreland 28 the maximum rate at which classical information can be transmitted asymptotically reliably (the capacity) is given by

C(ϕ) = \max_{\{σ, p\}} I(p; ϕ(σ)),

(1)

with the Holevo mutual information

I(p; ϕ(σ)) = H \left( \sum_i p_i ϕ(σ_i) \right) − \sum_i p_i H(ϕ(σ_i)),

H being the von Neumann entropy on density operators. This holds when in the block coding (implicit in the statement) for ϕ⊗n one is restricted to using product states σi ⊗ ... ⊗ σin. Strictly speaking, we should write “sup” over all probability distributions p on states, and integrals instead of finite sums. However, we restrict

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our attention to finite dimensional spaces, and there it is possible to show that the supremum is achieved by a finitely supported measure \( p \) (see \[3\]).

Unfortunately it is unknown whether it is sufficient to restrict coding to product states. It would be if the additivity conjecture

\[
C(\varphi \otimes \vartheta) = C(\varphi) + C(\vartheta)
\]

is true. To show this one would need to consider input state ensembles with entangled states, and prove that the corresponding Holevo information can be achieved by an ensemble without entangled states, or more directly, that a code using entangled states can be modified to an equally good code (in terms of error probability and rate) without entangled states. Neither of these has been achieved in generality so far, though there have been advances recently: see \[1\] and \[22\].

In section \[11\] we present an example of a special classical–quantum channel as a case study: dense coding in the presence of noisy entanglement, and by use of a general quantum channel, in particular a noiseless one. Here, coding is done by selecting not a state of a system, to be sent down the channel, but by selecting an operation on a given state. This is a more general concept of coding, as we demonstrate in section \[11\]. It appears that the coding of such a channel can be approximated by programmable quantum gates (in this sense the new model is a special case of the old one), but that the parallel use of these systems cannot: there will always be actions on the combined space that cannot be mimicked by entangled inputs to the PQG-augmented channel.

We conclude with the suggestion that the additivity problem for classical capacities of quantum channels has not been posed until now in its appropriate generality.

II. PROGRAMMABLE QUANTUM GATES

In classical computers there is no fundamental distinction in a universal machine’s memory between data and program. In fact a program may modify itself during the computation (a feature considered essential by von Neumann when he designed his computer model). To which extent can a quantum computer memory be used to modify other parts of the memory in a program–like fashion? More precisely (following \[24\]): assume that a unitary process \( G \) acts on \( \mathcal{H}_D \otimes \mathcal{H}_P \), with the data register \( \mathcal{H}_D \) and the program register \( \mathcal{H}_P \):

\[
|\zeta\rangle \otimes |\psi\rangle \mapsto G(|\zeta\rangle \otimes |\psi\rangle).
\]

We call \( |\psi\rangle \) a program if it has the property that

\[
\forall|\zeta\rangle \quad G(|\zeta\rangle \otimes |\psi\rangle) = U_{\psi}|\zeta\rangle \otimes |\psi'\rangle.
\]

Note that — though a priori \( |\psi'\rangle \) could also depend on \( |\zeta\rangle \) — for \( |\zeta_1\rangle, |\zeta_2\rangle \in \mathcal{H}_D \) the corresponding \( |\psi_1'\rangle, |\psi_2'\rangle \) are linearly dependent:

\[
G((\alpha|\zeta_1\rangle + \beta|\zeta_2\rangle) \otimes |\psi\rangle) = \alpha U_{\psi}|\zeta_1\rangle \otimes |\psi'\rangle + \beta U_{\psi}|\zeta_2\rangle \otimes |\psi'\rangle,
\]

which generally is entangled unless \( |\psi'\rangle \in \mathbb{C}|\psi_{2}'\rangle \).

(We thus can have a global phase — which we shall systematically ignore). Henceforth we assume that \( |\psi'\rangle \) is independent of \( |\zeta\rangle \), just as equation \[6\] suggests. It follows that \( U_{\psi} \) is unitary, which is encoded (via \( G \)) in the program \( |\psi\rangle \). How many unitaries can be implemented in this way?

**Theorem 1 (Nielsen, Chuang \[24\])** If \( U_{\psi_1} \neq \gamma U_{\psi_2} \) for all \( \gamma \in \mathbb{C} \), then \( |\psi_1\rangle \perp |\psi_2\rangle \).

**Proof.** Let

\[
G(|\zeta\rangle \otimes |\psi_1\rangle) = U_{\psi_1}|\zeta\rangle \otimes |\psi_{1}'\rangle,
\]

\[
G(|\zeta\rangle \otimes |\psi_2\rangle) = U_{\psi_2}|\zeta\rangle \otimes |\psi_{2}'\rangle.
\]

Hence

\[
(|\psi_{1}'\rangle \otimes |\psi_{1}\rangle) G^*G(|\zeta\rangle \otimes |\psi_2\rangle) = \langle |\psi_{1}'\rangle |\zeta\rangle |U_{\psi_{1}}^* U_{\psi_{2}} |\zeta\rangle.
\]

If \( \langle |\psi_{1}'\rangle |\psi_{2}\rangle = 0 \), also \( \langle |\psi_1\rangle |\psi_2\rangle = 0 \), and we are done. Else \( \langle |\zeta\rangle |U_{\psi_{1}}^* U_{\psi_{2}} |\zeta\rangle \) is a constant, independent of \( |\zeta\rangle \), hence \( U_{\psi_{1}}^* U_{\psi_{2}} = \gamma \mathbb{1} \), contradicting the assumption. \( \square \)

As a consequence we have only at most \( \log \dim \mathcal{H}_P \) many essentially different programs. There is no way to encode all possible unitaries on \( \mathcal{H}_D \) by “quantum code” unless we allow for an infinite–dimensional \( \mathcal{H}_P \).

We have already in the introduction pointed out that it is well possible to implement arbitrarily good approximations to all unitaries (at the cost of ever increasing \( \dim \mathcal{H}_P \)). In \[24\], however, there was proposed a more interesting solution: a probabilistic programmable quantum gate, i.e. an encoding of unitaries in a state, and a process that performs the encoded unitary with some probability, and otherwise fails (does something else): the process is able to report which of the two events happened. This result was refined in subsequent work of Vidal, Masanes, and Cirac \[21\] \[22\], but we will not follow this line of research here.

To fix notions, let us define our concept of approximation: a (unitary) gate \( G \) on \( \mathcal{H}_D \otimes \mathcal{H}_P \) is said to be \( \epsilon \)--approximating if for every unitary \( U \) on \( \mathcal{H}_D \) there is a state vector \( |\psi\rangle \in \mathcal{H}_P \) (it is easily seen that pure state program register contents suffice) such that

\[
\forall|\zeta\rangle \quad \| U|\zeta\rangle \langle \zeta | U^* - \text{Tr}_{\mathcal{H}_P} (G(|\zeta\rangle \langle \zeta | \otimes |\psi\rangle \langle \psi |) G^*) \|_1 \leq \epsilon.
\]

Of course there are \( \epsilon \)--approximating gates such that the approximating induced maps

\[
\Gamma_\psi(\sigma) = \text{Tr}_{\mathcal{H}_P} (G(\sigma \otimes |\psi\rangle \langle \psi |) G^*)
\]
in the above equation all may be chosen unitary, but the present formulation has the appropriate generality for the nonexistence theorem of the following section.

A sequence \((G^{(n)})_{n \in \mathbb{N}}\) of programmable quantum gates \(G^{(n)}\) on \(\mathcal{H}_P \otimes \mathcal{H}_D\) is called approximating for \(\mathcal{H}_D\) if each \(G^{(n)}\) is \(\epsilon_n\)-approximating, with \(\epsilon_n \to 0\) for \(n \to \infty\).

III. SCALABLE PROGRAMMABLE QUANTUM GATES

Given \(\epsilon > 0\) we can devise \(\epsilon\)-approximating quantum gates \(G_1\) and \(G_2\) for given data registers \(\mathcal{H}_{D_1}\) and \(\mathcal{H}_{D_2}\), respectively, by allowing for sufficiently large program registers.

Programming, however, is about making act together data in a potentially unlimited number of registers. In general, to approximately perform an arbitrary unitary on \(\mathcal{H}_{D_1} \otimes \mathcal{H}_{D_2}\) it is necessary to define a new quantum gate \(G\).

This motivates us to the following definition: we say that two sequences \((G_1^{(n)})_{n \in \mathbb{N}}\) and \((G_2^{(n)})_{n \in \mathbb{N}}\) of programmable quantum gates that are approximating for \(\mathcal{H}_{D_1}\) and \(\mathcal{H}_{D_2}\), respectively, are scalable, if the sequence \((G_1^{(n)} \otimes G_2^{(n)})_{n \in \mathbb{N}}\) is approximating for \(\mathcal{H}_{D_1} \otimes \mathcal{H}_{D_2}\).

Such approximating gate sequences thus spare us the task to find an implement new programmable quantum gates when we scale up our computing system.

Unfortunately, nature does not supply us with such objects:

**Theorem 2** Let \((G_1^{(n)})_{n \in \mathbb{N}}\) and \((G_2^{(n)})_{n \in \mathbb{N}}\) be sequences of programmable quantum gates with fixed data registers \(\mathcal{H}_{D_1}\) and \(\mathcal{H}_{D_2}\), respectively. Assume that the unitary \(U\) on \(\mathcal{H}_{D_1} \otimes \mathcal{H}_{D_2}\) is approximated arbitrarily close by programs \(\psi^{(n)} \in \mathcal{H}_{P_1} \otimes \mathcal{H}_{P_2}\), i.e.

\[
\text{Tr}_{\mathcal{H}_{P_1} \otimes \mathcal{H}_{P_2}} \left[ G_1^{(n)} \otimes G_2^{(n)} \left( |\zeta\rangle \langle \zeta | \otimes |\psi\rangle \langle \psi | \right) G_1^{(n)^*} \otimes G_2^{(n)^*} \right] 
\to U|\zeta\rangle \langle \zeta | U^*
\]

as \(n \to \infty\). Then \(U\) is not entangling, i.e. it is of the form \(U = U_1 \otimes U_2\).

*Proof.* Consider the expressions of eq. (3) for data of the form \(|\zeta\rangle = |\zeta_1\rangle \otimes |\zeta_2\rangle\). The first claim is that the reduced state of the left hand side of \(\mathcal{H}_{D_1}\) is independent of \(\zeta_2\): this becomes clear by first tracing out \(\mathcal{H}_{D_2} \otimes \mathcal{H}_{P_2}\) and then \(\mathcal{H}_{P_2}\). Then the same applies to the limit at the right hand side.

So, for fixed \(|\zeta_1\rangle\) we have

\[
\text{Tr}_{\mathcal{H}_{D_2}} (|\zeta_1\rangle \langle \zeta_1 | \otimes |\zeta_2\rangle \langle \zeta_2 |) U^* = \rho_0 = \sum_i \lambda_i |e_i\rangle \langle e_i |,
\]

with a constant state \(\rho_0\) (that we wrote in diagonalized form), regardless of \(|\zeta_2\rangle\).

Now assume that \(U\) is entangling, and choose \(|\zeta_1\rangle\) such there exists \(|\zeta_2\rangle\) so that \(U|\zeta_1\rangle \otimes |\zeta_2\rangle\) is entangled. Then \(\rho_0\) is mixed, and its diagonalization contains at least two terms. We shall derive a contradiction from this: first observe that for arbitrary \(|\zeta_2\rangle\) the state \(U|\zeta_1\rangle \otimes |\zeta_2\rangle\) is a purification of \(\rho_0\), hence, by eq. (4) there exists an orthonormal basis \(\{|f_i\rangle\}\) of \(\mathcal{H}_{D_2}\) such that

\[
U|\zeta_1\rangle \otimes |\zeta_2\rangle = \sum_i \sqrt{\lambda_i} |e_i\rangle \otimes |f_i\rangle.
\]

For \(|\zeta_2\rangle\) orthogonal to \(|\zeta_2\rangle\) there is another such basis \(\{|f'_i\rangle\}\) with

\[
U|\zeta_1\rangle \otimes |\zeta_2\rangle = \sum_i \sqrt{\lambda_i} |e_i\rangle \otimes |f'_i\rangle.
\]

By linearity we get thus

\[
U|\zeta_1\rangle \otimes (\alpha|\zeta_2\rangle + \beta|\zeta_2\rangle) = \sum_i \sqrt{\lambda_i} |e_i\rangle \otimes (\alpha|f_i\rangle + \beta|f'_i\rangle),
\]

for \(|\alpha|^2 + |\beta|^2 = 1\). This again must be a purification of \(\rho_0\), so the resulting \(\{|f_i\rangle + \beta|f'_i\rangle\}\) must form an orthonormal basis: this leads quickly to the condition (for all \(i, j\))

\[
\pi \beta (f_i|f_j\rangle + \alpha \beta (f'_i|f_j\rangle = 0,
\]

implying \(f_i|f'_j\rangle = (f'_i|f_j\rangle = 0\), otherwise \(z\) and \(\pi\) would be linearly dependent over the complex field.

As a consequence, to each orthonormal system of \(|\zeta_2\rangle\)'s of \(\mathcal{H}_{D_2}\) we would get an orthonormal system of \(|f_i\rangle\)'s of at least double size, contradicting the finite dimension of \(\mathcal{H}_{D_2}\). Thus \(U\) cannot be entangling, forcing \(U = U_1 \otimes U_2\). To see this either consult eq. (4) or follow this simple argument: since

\[
\sigma_{12} \mapsto U \sigma_{12} U^*
\]

maps product states to product states, the map

\[
T_1 : \sigma \mapsto \text{Tr}_2 U (\sigma \otimes |\zeta_2\rangle \langle \zeta_2 |) U^*
\]

maps pure states to pure states and is completely positive, entailing that it has to be of the form \(T_1(\sigma) = U_1 \sigma U_1^*\) (this may be viewed as an easy kind of Wigner-theorem). Here \(U_1\) is a unitary which cannot — except for a global phase — depend of \(|\zeta_2\rangle\), or else there would be entangled states \(U \sigma_1 \otimes \sigma_2) U^*\). The same applies to the second factor, yielding a unitary \(U_2\). In total we have that the unitary \(U_1 \otimes U_2\) coincides with \(U\) on te pure states, hence \(U = U_1 \otimes U_2\) (again except for an unimportant global phase). \(\square\)

Observe the following peculiarity of the argument: it is not true that the reduced state at the left hand side of eq. (3) is always a product (if it is, our proof is simplified drastically). For example \(G_1\) and \(G_2\) may be swapping operations, so their product may be used to swap in any entangled state! What is true however is, that entangled
states cannot occur as a result of a unitary action on the 
data registers.

This nonexistence should not be mixed up with the 
existence of the beautiful model of one-way quantum computer 
by Raussendorf and Briegel [27]: there, too, a single state is prepared 
and acted on locally (even only by measurements), to produce any given effect on the data 
register. There is no contradiction, however, to our result, 
as there is implied classical communication between the sites 
of these quantum operations, which we had to exclude.

In a sense, the result had to be expected: it reproduces 
on a somewhat different level the insight in universal 
computation that single qubit actions are not sufficient 
for universality, but one needs interacting gates like the C–NOT gate.

We shall show in the following, however, that this 
nonexistence result has some bearing on quantum channel coding.

IV. NOISY DENSE CODING CAPACITY

Consider the following communication scenario: a sender A and a receiver B share a state $\rho$ on the $d_A \times d_B$–system $\mathcal{H}_A \otimes \mathcal{H}_B$, i.e. $\dim \mathcal{H}_A = d_A$, $\dim \mathcal{H}_B = d_B$. They have at their disposal a quantum channel from A to B 
that allows noiseless transmission of an arbitrary quantum state in $\mathcal{H} \simeq \mathbb{C}^d$. They want to use this channel to 
communicate classical information, taking advantage of the correlation (or even entanglement) of $\rho$. The most 
general thing possible for A to do is to subject her share 
of the state to an operation, and send the result through the channel. It is well known that, if $\rho$ supplies only 
classical correlation (for instance, if

$$\rho = \sum_{i=0}^{d_A-1} \sum_{j=0}^{d_B-1} p_{ij} |i\rangle \langle i| \otimes |j\rangle \langle j|,$$

for orthogonal bases $\{|i\rangle : i = 0, \ldots, d_A - 1\}$ and $\{|j\rangle : j = 0, \ldots, d_B - 1\}$ of $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively), then 
this is of no help at all, and the capacity is just that of the noiseless channel: $\log d$ (in this paper log and exp are to basis 2).

However, for entangled $\rho$ the phenomenon of dense coding arises, which was first described in [26]: there $d_A = d_B = d = 2$ was considered, with the joint singlet state

$$\rho = |\Psi^+\rangle \langle \Psi^-| = 1/2(|01\rangle - |10\rangle)(|01\rangle - |10\rangle).$$

It was demonstrated that by applying one of the three Pauli unitaries $\sigma_x, \sigma_y, \sigma_z$, or the identity $\mathbb{1}$, A can drive 
the state to any of the four Bell states, hence can encode 2 bits. It is quite clear that by starting with any maximally entangled state, e.g.

$$\rho = |\Phi\rangle \langle \Phi|, \text{ with } |\Phi\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |i\rangle \otimes |i\rangle$$

on the system $\mathbb{C}^d \otimes \mathbb{C}^d$, i.e. $d_A = d_B = d$, one can devise a scheme to transmit $2 \log d = \log d^2$ bits (see [26] for a detailed discussion).

It is less clear what happens if the state is not maximally entangled, or even mixed: however, since the protocol A and B have to follow depends even in the maximally mixed case on the actual state, we allow them to use the protocol optimally adapted to $\rho$. Formally:

A chooses an operation (i.e. a completely positive, trace preserving linear map)

$$T : \mathcal{L}(\mathcal{H}_A) \rightarrow \mathcal{L}(\mathcal{H}),$$

and applies it to her part of $\rho$, after which she sends the resulting state to B. He thus receives the signal state

$$\rho^T := (T \otimes \text{id})\rho.$$

We here assume that one copy of $\rho$ is available per use of the noiseless channel. Below we will discuss the case of more or unlimited many copies per round.

Then we can compute the mutual information

$$I(\mu; \rho) := H(\int \text{d}\mu(T)\rho^T) - \int \text{d}\mu(T)H(\rho^T),$$

with respect to a probability measure $\mu$ on the space $\text{CP}(\mathcal{H}_A, \mathcal{H})$ of quantum operations (i.e., completely positive, trace preserving, linear maps) from $B(\mathcal{H}_A)$ to $B(\mathcal{H})$.

By the quantum channel coding theorem, eq. (1), of [18], and [28], the dense coding capacity

$$DC(d, \rho) := \sup_{\mu} I(\mu; \rho)$$

is the classical capacity of the channel with signal states $\rho^T$, when block coding using product states

$$\rho^{T_1 \otimes \cdots \otimes T_n} = ((T_1 \otimes \text{id})\rho) \otimes \cdots \otimes ((T_n \otimes \text{id})\rho)$$

is allowed. If we impose no restriction on the block coding, i.e. all states

$$(T \otimes \text{id}^\otimes n)\rho^\otimes n,$$

with $T \in \text{CP}(\mathcal{H}_A^\otimes n, \mathcal{H}_B^\otimes n)$ are admissible, we get the ultimate dense coding capacity

$$\overline{DC}(d, \rho) = \lim_{n \to \infty} \frac{1}{n} DC(\rho^\otimes n).$$

Note that the limit exists by the trivial superadditivity of $DC$:

$$DC(d_1 d_2, \rho \otimes \sigma) \geq DC(d_1, \rho) + DC(d_2, \sigma).$$
Our first task is the evaluation of $DC(d, \rho)$:
Assume any probability distribution $\mu$ on $\text{CP}(\mathcal{H}_A, \mathcal{H})$, and denote the Haar measure on the group $\mathcal{U}(d)$ of unitaries on $\mathcal{H}$ as $d\mu$. Then for every unitary $U$ we have (by unitary invariance of entropy)
\[
H \left( \int d\mu(T) \rho^T \right) = H \left( \int d\mu(T) (U \otimes \mathbb{1}) \rho^T (U \otimes \mathbb{1})^* \right),
\]
\[
H (\rho^T) = H ((U \otimes \mathbb{1}) \rho^T (U \otimes \mathbb{1})^*).
\]
I.e. $I(\mu; \rho) = I(\mu^U; \rho)$, with the translated measure $\mu^U(F) = \mu(U^* FU)$, for measureable $F \subset \text{CP}(\mathcal{H}_A, \mathcal{H})$.

With concavity of $H$ we find
\[
I(\mu; \rho) = \int_{\mathcal{U}(d)} d\mu \left[ H \left( \int d\mu(T) (U \otimes \mathbb{1}) \rho^T (U \otimes \mathbb{1})^* \right) - \int d\mu(T) H \left( (U \otimes \mathbb{1}) \rho^T (U \otimes \mathbb{1})^* \right) \right] 
\leq H \left( \int d\mu(T) \int dU (U \otimes \mathbb{1}) \rho^T (U \otimes \mathbb{1})^* \right) - \int d\mu(T) H (\rho^T).
\]
The latter quantity is exactly $I(\rho, \rho)$, with $\rho = \int dU \mu^U$.

Now it is straightforward to prove (essentially by Schur’s lemma) that
\[
\int dU (U \otimes \mathbb{1}) \rho^T (U \otimes \mathbb{1})^* = \frac{1}{d} \mathbb{1} \otimes \rho_B,
\]
where we observed that by definition
\[
\text{Tr}_A \rho^T = \text{Tr}_A ((T \otimes \mathbb{1}) \rho) = \text{Tr}_A \rho = \rho_B.
\]

Hence maximization yields
\[
DC(d, \rho) = \log d + H(\rho_B) - \inf_{\mu} \int d\mu(T) H (\rho^T).
\]

This infimum in turn is achieved at the point mass on a $T$ minimizing $H (\rho^T)$.

Hence we arrive at the result:

**Theorem 3** The dense coding capacity of the state $\rho$ and a $d$–level noiseless transmission system, using one copy of $\rho$ per round and product states for coding, is given by
\[
DC(d, \rho) = \log d + H(\rho_B) - \min_T H ((T \otimes \mathbb{1}) \rho),
\]
where the minimization is over all quantum operations $T : \mathcal{B}(\mathcal{H}_A) \to \mathcal{B}(\mathcal{H})$.

As a consequence we get:

**Theorem 4** Without the restriction on product state encoding, but still using one copy of $\rho$ per round, the capacity is
\[
DC(d, \rho) = \log d + H(\rho_B) - \lim_{n \to \infty} \frac{1}{n} \min_T H ((T \otimes \mathbb{1}^\otimes n) \rho^\otimes n),
\]
where the minimization is over all quantum operations $T : \mathcal{B}(\mathcal{H}_A^\otimes n) \to \mathcal{B}(\mathcal{H}^\otimes n)$.

Note that the argument describes at the same time a distribution on $\text{CP}(\mathcal{H}_A, \mathcal{H})$ that achieves the capacity: $A$ should apply a fixed minimizing $T$, followed by uniformly distributed unitary rotations. The effect of the latter can be achieved equally by a uniform distribution on an orthogonal basis of unitaries (with respect to the Hilbert–Schmidt inner product $(A, B) = \text{Tr} A^* B$ on operators), see [23].

As applications of the theorem we can see immediately that for pure states $|\psi\rangle$
\[
DC(d, |\psi\rangle\langle\psi|) = \log d + E(\psi) = \log d + H(\text{Tr}_B |\psi\rangle\langle\psi|),
\]
a result already reported in [1] and [13], and that $DC(d, \rho) = \log d$ if $\rho$ is separable (below, theorem 6 we will see that this holds true even for non–distillable $\rho$): in the first case the optimizing $T$ is any unitary map, in the second case it is the projection onto any pure state (note that $DC(d, \rho) \leq \log d$ follows from the inequality $H(\sigma_B) - H(\sigma) \leq 0$ for separable $\sigma$). This latter choice shows that always $DC(d, \rho) \geq \log d$ (it amounts to ignoring the correlation provided by $\rho$).

In general, however, the minimization required by the theorem seems not an easy task itself.

**Remark 5** The quantity $H(\sigma_B) - H(\sigma)$, $\sigma = (T \otimes \mathbb{1}) \rho$, from theorem 3 has appeared in another context before: it is the coherent information of Schumacher [27].

**Remark 6** Until now we stuck to using one copy of $\rho$ per use of the noiseless channel. In recent work by Horodecki et al. [22] this restriction was lifted: unlimited many copies of $\rho$ were assumed to be available. Of course, the theorem can be used to obtain a formula for that case too, which we give, because it interestingly differs from that than the one in [24] (though of course the numbers coincide):

Assume $k$ copies of $\rho$ may be used per round. Obviously the resulting dense coding capacity is
\[
DC^{(k)}(d, \rho) = DC(d, \rho^\otimes k),
\]
and for unlimited use of $\rho$ we get
\[
DC^{(\infty)}(d, \rho) = \lim_{k \to \infty} DC(d, \rho^\otimes k).
\]

Similarly for the ultimate dense coding capacity with $k$ copies of $\rho$ per round:
\[
DC^{(k)}(d, \rho) = DC(d, \rho^\otimes k) = \lim_{n \to \infty} \frac{1}{n} DC(d^n, \rho^\otimes k^n),
\]
and with unlimited use of $\rho$:

$$\DC^{(\infty)}(d, \rho) = \lim_{k \to \infty} \DC(k)(d, \rho)$$

$$= \lim_{k \to \infty} \lim_{n \to \infty} \frac{1}{n} \DC(d^n, \rho^\otimes kn)$$

$$= \lim_{n \to \infty} \lim_{k \to \infty} \frac{1}{n} \DC(d^n, \rho^\otimes kn)$$

$$= \lim_{n \to \infty} \frac{1}{n} \DC^{(\infty)}(d, \rho)$$

$$= \DC^{(\infty)}(d, \rho).$$

(The limits are exchangeable because the double lim is actually a joint sup over $n$ and $k$, because of monotonicity.)

In [21] the differently looking expression (for the case $d = 2$)

$$\DC(\rho) = \sup_{\mathbf{T}} \left\{ 1 + \frac{n H(\rho_B) - H\left( (T \otimes \text{id}^{\otimes n}) (\rho_B^{\otimes n}) \right)}{H\left( (T \rho_A^{\otimes n}) \right)} \right\},$$

was given, the sup being over all quantum operations $\mathbf{T}$ defined on $\mathcal{B}(\mathcal{H}^{\otimes n}_A)$. However, the derivation in that work is sufficiently close to ours so as see identity of the results.

Let us comment here a bit on other related work, and the relation of $\DC(d, \rho)$ to entanglement:

In the works [8] and [9] the relation of the dense coding capacity to entanglement measures was stressed. With our results, it is easy to reproduce the observations of these papers, and go even a little further.

We use the following inequality from [22]: for a (two-way) non–distillable state $\sigma$

$$H(\rho_B) - H(\rho) \leq \DC(\rho||\sigma).$$

Applying $T \otimes \text{id}$ to both $\rho$ and $\sigma$, and invoking the monotonicity of relative entropy under completely positive maps, we find

$$H(\rho_B) - H((T \otimes \text{id}) \rho) \leq \DC((T \otimes \text{id}) \rho || (T \otimes \text{id}) \sigma) \leq \DC(\rho || \sigma).$$

Now minimize over $T$ and non–distillable $\sigma$: this proves

**Theorem 7** For all states $\rho$ one has

$$\DC(d, \rho) \leq \log d + E_{\text{ve}}(\rho),$$

where $E_{\text{ve}}(\rho) = \inf_{\sigma \in \mathcal{D}} \DC(\rho || \sigma)$ is the relative entropy of entanglement with respect to the set $\mathcal{D}$ of non–distillable states.

In particular, when $\rho$ is non–distillable, $\DC(d, \rho) = \log d$ (see also [22] for this observation). One may wonder, whether the inverse is true, too: when $\rho$ is distillable, does $\DC(d, \rho) > \log d$ follow?

To compare this result to the statements in [8, 12], and the result of the recently published [22] we have to note that in these works only unitary encodings were considered. Hence our $\DC(d, \rho)$ is typically a strict upper bound to the capacity in the cited works. Still, questions raised in [8, 12] receive answers: the conjectured capacity formulas and inequalities from these works follow immediately, by the same method of Haar averaging we employed above (see also [22]).

To get a bound in the other direction is not so easy. We might try to go further on the road of entanglement, and find an entanglement measure lower bound. For example, if we could prove that

$$f(\rho) = \DC^{(\infty)}(d, \rho) - \log d$$

is an entanglement measure itself, we would find the lower bound

$$\DC^{(\infty)}(d, \rho) \geq \log d + E_D(\rho),$$

with the distillable entanglement $E_D(\rho)$: this follows from general inequalities in [22]. We leave this question, however, to another occasion.

We would like now to discuss the additivity of $D$, i.e. whether for states $\rho$, $\sigma$

$$\DC(d_1, d_2, \rho \otimes \sigma) = \DC(d_1, \rho) + \DC(d_2, \sigma).$$

Note that if this is true for $\rho$ and all $\sigma = \rho^{\otimes n}$ (e.g., inductively), it immediately follows that $\DC^{(\infty)}(d, \rho) = \DC(d, \rho)$. In particular, all ultimate capacities in remark 6 are identical to their “un–barred” versions. The capacity with unlimited use of $\rho$ from [22] would then read

$$\DC^{(\infty)}(d, \rho) = \DC^{(\infty)}(d, \rho) = \log d + \inf_k \min_T H((T \otimes \text{id}^{\otimes k}) \rho_B^{\otimes k}),$$

where the minimization is over all quantum operations $T : \mathcal{B}(\mathcal{H}^{\otimes k}_A) \to \mathcal{B}(\mathcal{H})$.

By theorem 3 the statement of eq. (3) is equivalent to asking, if

$$\min_{T_{12}} H((T_{12} \otimes \text{id}^{\otimes 2}) \rho \otimes \sigma)) = \min_{T_1} H((T_1 \otimes \text{id}) \rho) + \min_{T_2} H((T_2 \otimes \text{id}) \sigma).$$

Obviously, and fitting with the superadditivity of $D$, “$\leq$” (subadditivity) is trivial, and the question is if “$<$” can occur. Note that in this generality it is quite easy to come up with states that violate the additivity property, see the discussion below. The problem is rather to find conditions where additivity holds.

Generalizing, one may assume not a noiseless, but a noisy channel $\varphi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ between $A$ and $B$, and consider the dense coding capacities

$$\DC(\varphi, \rho), \quad \DC^{(\infty)}(\varphi, \rho), \quad \DC^{(k)}(\varphi, \rho), \quad \DC^{(\infty)}(\varphi, \rho).$$
For example, we can define

$$DC(\varphi, \rho) = \sup_{\mu} I(\mu; \varphi \circ \rho),$$

over all probability distributions $\mu$ on $\text{CP}(\mathcal{H}_A, \mathcal{H})$, with

$$I(\mu; \varphi \circ \rho) := H \left( \int d\mu(T) \rho^{\varphi \circ T} \right) - \int d\mu(T) H \left( \rho^{\varphi \circ T} \right).$$

Observe that our previous $DC(d, \rho)$ is reproduced in the new definition as $DC(\text{id}_d, \rho)$. Further, observe that for a pure entangled state $\rho$ the definition relates to the entanglement assisted classical capacity $\text{ECC}$ of the quantum channel $\varphi$: in fact, $\text{DC}^{(\infty)}(\varphi, \rho)$ is this latter quantity.

Again, the superadditivity

$$DC(\varphi \otimes \vartheta, \rho \otimes \sigma) \geq DC(\varphi, \rho) + DC(\vartheta, \sigma) \quad (6)$$

trivially holds, and we may study conditions for equality in eq. (4), i.e. additivity.

Note that it is fairly easy to come up with situations $(\varphi, \vartheta, \rho, \sigma)$ where strict superadditivity holds. In fact one can even have either $\varphi = \vartheta$ or $\rho = \sigma$: e.g. consider

$$\varphi = \vartheta = \text{id}_{\mathcal{S}(C^2)}, \quad \left\{ \begin{array}{l} \rho = |00\rangle \langle 00| \quad \text{(unentangled)} \ , \\ \sigma = |\Psi^-\rangle \langle \Psi^-| \otimes \sigma, \end{array} \right.$$  

or alternatively

$$\varphi = \text{id}_{\mathcal{S}(C^2)}, \quad \vartheta = \frac{1}{2} \mathbb{1} \quad \text{(constant map)}, \quad \left\{ \begin{array}{l} \rho = \sigma = |\Psi^-\rangle \langle \Psi^-|. \end{array} \right.$$  

But with both these conditions simultaneously it seems not so easy. It may even be that weak additivity holds, i.e.

$$DC(\varphi^{\otimes n}, \rho^{\otimes n}) = n \ DC(\varphi, \rho),$$

for all channels $\varphi$ and joint states $\rho$, but we could not reach a conclusive result on this question.

V. REDUCTIONS AMONG ADDITIVITY QUESTIONS

We have encountered two paradigms of coding in quantum channels, the first in the established discussion (a good overview is in [19], and some recent developments are reviewed in [21]), the second in the previous section:

1. **State preparation:** The encoder may prepare any state on the input system space $\mathcal{H}_1$ for the quantum channel $\varphi : \mathcal{B}(\mathcal{H}_1) \rightarrow \mathcal{B}(\mathcal{H}_2)$.

2. **Action on given state:** On the input system a state is given in advance (possibly entangled with the receiver), and the encoder may act on it in an arbitrary way, and the result is sent down the channel $\varphi$.

It is quite obvious that method 1 can be reduced to method 2: the previously given state is just any state not entangled with the receiver (say, a pure state). Then by executing an appropriate operation the encoder can drive the input into any desired state.

Less obvious, but still quite canonical, is the converse reduction: any operation $T : \mathcal{B}(\mathcal{H}_A) \rightarrow \mathcal{B}(\mathcal{H}_1)$ can be implemented as a unitary

$$U : \mathcal{H}_A \otimes \mathcal{H}' \rightarrow \mathcal{H}_1 \otimes \mathcal{H}'',$$

followed by a partial trace over $\mathcal{H}''$, the system $\mathcal{H}'$ being prepared initially in a null state $\sigma_0$. This is a formulation of the Stinespring dilation theorem [22], and it is quite easy to see that $\dim \mathcal{H}'$ can be chosen fixed and finite for all possible $T$. Now pick an $\epsilon$-approximating quantum gate $G_\epsilon$ for $\mathcal{H}_A \otimes \mathcal{H}'$, with program register $\mathcal{H}_P$: by choosing $\psi$ in the program register appropriately one obtains (using monotonicity of the trace norm under partial trace), for all states $\sigma$ on $\mathcal{H}_A$,

$$\|T(\sigma) - \text{Tr}_{\mathcal{H}' \otimes \mathcal{H}_P} (G_\epsilon(\sigma \otimes |0\rangle \langle 0) \otimes |\psi\rangle \langle \psi|)G_\epsilon^*)\|_1 \leq \epsilon. \quad (7)$$

Thus every coding process by acting on the input system can be arbitrarily well approximated by coding via choice of $|\psi\rangle \in \mathcal{H}_P$.

These two reductions, however, are of a very different nature, as we can see by considering their behaviour under tensor products of channels: while the reduction $1 \rightarrow 2$ scales alright (any entangled input state can be obtained by a suitable entangling operation on the product of the initial states), the reduction $2 \rightarrow 1$ that we proposed does not. In fact, as we have seen in theorem 2 on a product $\mathcal{H}_{A1} \otimes \mathcal{H}_{A2}$ of two input systems we can never implement an entangling operation, once we have chosen approximating quantum gates for each of them individually according to eq. (6), and tensor them.

We have seen that there are channels where classical information is encoded after method 1 (these are just the operations $\varphi$), and that there are channels where it is encoded after method 2 (the generalized noisy dense coding channels). The above reductions show that the two approaches are equivalent in the sense that a channel of the one kind can be simulated to arbitrary accuracy by one of the other kind.

However, for the additivity question of channel capacity one has to look at higher tensor products of the channel at hand. By the above argument the reduction $1 \rightarrow 2$ provides a reduction of the additivity question for channels of the first type to those of the second type. It is unknown to us if the additivity question can be reduced in the other direction: the construction above, summarized in eq. (6), at least does not provide this, as we have seen. On the other hand, it appears to be most natural: it seems the most reasonable thing to do to associate a channel of the first type to the given channel of the second type that has the same properties with respect to classical information transmission, by simply enabling to emulate the effect of any encoding transformation $T$ by a suitable input state.
VI. CONCLUSION

By studying entanglement assisted classical communication via quantum channels, attention was drawn towards channels which require actions for signalling rather than state preparations like the usual quantum dynamics, represented by completely positive maps. An attempted reduction of the more general scenario to the usual one was shown to fail, because no scalable programmable quantum gates exist. This was taken to indicate that the new concept is strictly more general, which leads us to conjecture that the additivity question for quantum channel capacity really is not about “whether entangled inputs help”, but rather “whether entangling inputs help”. It must be stressed that in the more general vista we presented, additivity is not a mere matter of “right” or “false”. Rather it becomes (as we demonstrated by examples) a question of characterization of the situations where it holds. Note finally that the very occurrence of the above mentioned distinction in coding concepts is a purely quantum phenomenon.

Acknowledgments

Part of this work was carried out during the author’s visits to Universität Bielefeld (July 2001), and the JST ERATO project “Quantum Computation and Information”, Tokyo (August/September 2001). The hospitality of these institutions is gratefully acknowledged.

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