Some Notes on Complex Symmetric Operators

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Abstract

In this paper we show that every conjugation $C$ on the Hardy-Hilbert space $H^2$ is of type $C = T^*C_1 T$, where $T$ is an unitary operator and $C_1 f(z) = \overline{f(z)}$, with $f \in H^2$. In the sequence, we extend this result for all separable Hilbert space $H$ and we prove some properties of complex symmetry on $H$. Finally, we prove some relations of complex symmetry between the operators $T$ and $|T|$, where $T = U|T|$ is the polar decomposition of bounded operator $T \in \mathcal{L}(H)$ on the separable Hilbert space $H$.

1 Introduction

Let $\mathcal{L}(H)$ be the space of bounded linear operators on a separable Hilbert space $H$. A conjugation $C$ on $H$ is an antilinear operator $C : H \rightarrow H$ such that $C^2 = I$ and $\langle Cf, Cg \rangle = \langle g, f \rangle$, for all $f, g \in H$. An operator $T \in \mathcal{L}(H)$ is said to be complex symmetric if there exists a conjugation $C$ on $H$ such that $CT = T^*C$ (we will often say that $T$ is $C$-symmetric). Complex symmetric operators generalize the concept of symmetric matrices of linear algebra. Indeed, it is well known (\cite[Lemma 1]{5}) that given a conjugation $C$, there exists an orthonormal basis $\{f_n\}_{n=0}^{\infty}$ for $H$ such that $Cf_n = f_n$. Hence, if $T$ is $C$-symmetric then

$$\langle Tf_n, f_m \rangle = \langle Cf_m, CTf_n \rangle = \langle f_m, T^*Cf_n \rangle = \langle Tf_m, f_n \rangle,$$

(1)

that is, $T$ has a symmetric matrix representation. The reciprocal of this fact is also true. That is, if there is an orthonormal basis such that $T$ has a symmetric matrix representation, then $T$ is complex symmetric.

The complex symmetric operators class was initially addressed by Garcia and Putinar \cite{5} \cite{6} and includes the normal operators, Hankel operators and Volterra integration operators.

Now, let $L^2$ be the Hilbert space on the unit circle $\mathbb{T}$ and let $L^\infty$ be the Banach space of all essentially bounded functions on $\mathbb{T}$. It is known that $\{e_n(e^it) := e^{int} : n \in \mathbb{Z}\}$ is

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an orthonormal basis for $L^2$. The *Hardy-Hilbert space*, denoted by $H^2$, consists of all analytic functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ on the unit disk $\mathbb{D}$ such that $\sum_{n=0}^{\infty} |a_n|^2 < \infty$. It is clear that $\mathcal{B} := \{e_n(z) = z^n : n = 0, 1, 2, \ldots \}$ is an orthonormal basis for $H^2$.

For each $\phi \in L^\infty$, the *Toeplitz operator* $T_\phi : H^2 \to H^2$ is defined by
\[
T_\phi f = P(\phi f),
\]
for each $f \in H^2$, where $P : L^2 \to H^2$ is the orthogonal projection. The concept of Toeplitz operators was initiated by Brown and Halmos [1] and generalizes the concept of Toeplitz matrices.

In [7], Guo and Zhu raised the question of characterizing complex symmetric Toeplitz operators on $H^2$ in the unit disk. In order to obtain such characterization, Ko and Lee [8] introduced the family of conjugations $C_\lambda : H^2 \to H^2$, given by
\[
C_\lambda f(z) = \overline{f(\lambda \bar{z})}
\]
with $\lambda \in \mathbb{T}$ and proved the following result:

**Theorem 1.1.** Let $\phi(z) = \sum_{n=-\infty}^{\infty} \hat{\phi}(n) z^n \in L^\infty$. Then $T_\phi$ is $C_\lambda$-symmetric if, and only if, $\hat{\phi}(-n) = \lambda^n \hat{\phi}(n)$, for all $n \in \mathbb{Z}$.

### 2 Canonical Conjugations

Our first objective in this paper is to study relations between an arbitrary conjugation $C$ on $H^2$ and the conjugation $C_1 f(z) = \overline{f(\bar{z})}$. Once the conjugation $C_1$ is a kind of canonical conjugation on $H^2$, we observe a close relationship between conjugations of $H^2$ and conjugation $C_1$, namely:

**Theorem 2.1.** If $C$ is a conjugation on $H^2$, then exists an unitary operator $T : H^2 \to H^2$ such that $TC = C_1 T$.

**Proof.** Since $C$ is a conjugation, there exists an orthonormal basis $\mathcal{B}' = \{f_n\}_{n=0}^{\infty}$ of $H^2$ such that $C f_n = f_n$. Now, let $\mathcal{B} = \{e_n\}_{n=0}^{\infty}$ the standard orthonormal basis of $H^2$ and the linear isomorphism $T : H^2 \to H^2$ given by
\[
T \left( \sum_{n=0}^{\infty} a_n f_n \right) = \sum_{n=0}^{\infty} a_n e_n.
\]

Note that $T f_n = e_n$, for all $n \geq 0$, and therefore $T$ is unitary. Now, for
\[ f(z) = \sum_{n=0}^{\infty} a_n e_n \in H^2, \] we get

\[
C_1 f(z) = \sum_{n=0}^{\infty} a_n e_n \\
= \sum_{n=0}^{\infty} a_n T(f_n) \\
= T\left(\sum_{n=0}^{\infty} a_n C f_n\right) \\
= (TC) \left(\sum_{n=0}^{\infty} a_n f_n\right) \\
= (TC) \left(\sum_{n=0}^{\infty} a_n T^{-1}(e_n)\right) \\
= (TCT^{-1}) f(z),
\]

whence \( C_1 T = TC. \) \( \square \)

The previous theorem says that all complex symmetric Toeplitz operator is unitarily equivalent to a \( C_1 \)-symmetric operator. Indeed:

**Remark 2.2.** Let \( T_\phi : H^2 \to H^2 \) an Toeplitz operator. Observe that, if \( T_\phi \) is \( C \)-symmetric, since the operator \( T \) of previous theorem is unitary, we have

\[ C_1 = TCT^*, \]

therefore the operator \( T_2 := TT_\phi T^* \) is \( C_1 \)-symmetric (see \[5\] p. 1291). This shows that \( T_\phi \) and \( T_2 \) are unitarily equivalent operators. Moreover, is obvious that, if \( T \) commutes with \( C_1 \) or \( C \), then \( C = C_1 \).

**Corollary 2.3.** Let \( A \in \mathcal{L}(H^2) \). Then \( A \) is \( C_1 \)-symmetric if, and only if, the matrix of \( A \) with respect the canonical basis of \( H^2 \) is symmetric.

**Proof.** If \( A \) is \( C_1 \)-symmetric, then \( C_1 A = A^* C_1 \). Moreover, by previous theorem there exists an isomorphism \( T \) on \( H^2 \) such that \( TC_1 = C_1 T \). Consider \( B = \{ e_n \}_{n=0}^{\infty} \) and \( B' = \{ f_n \}_{n=0}^{\infty} \) orthonormal basis of \( H^2 \) such that

\[ T f_n = e_n \text{ and } C_1 f_n = f_n. \]

Thus, we must

\[ C_1 e_n = C_1 (T f_n) = TC_1 (f_n) = T f_n = e_n, \]

that is \( C_1 e_n = e_n, \forall n \geq 0. \) Therefore, by \([\Pi]\), follows that \([A]_B = [A]_B^t\).

Reciprocally, suppose that \( A \) is \( C \)-symmetric such that \( C e_n = e_n \). By previous theorem, \( TC = C_1 T \) and \( T e_n = e_n \). Hence, \( T \) is the identity operator and so \( C = C_1 \). \( \square \)
In fact, the reciprocal of the Theorem 2.1 is true:

**Proposition 2.4.** If \( T : H^2 \to H^2 \) is an unitary operator, then \( C := T^{-1}C_1T \) is an conjugation on \( H^2 \).

**Proof.** It is easy to see that \( C \) is an antilinear operator. Now, since \( T \) is an unitary operator, considering \( B = \{e_n\}_{n=0}^{\infty} \) the orthonormal basis of \( H^2 \), we have

\[
\langle Ce_n, Ce_m \rangle = \langle T^*C_1Te_n, T^*C_1Te_m \rangle = \langle TT^*C_1Te_n, C_1Te_m \rangle = \langle Te_n, Te_m \rangle = \langle e_m, T^*Te_n \rangle = \langle e_m, e_n \rangle.
\]

By other hand, once \( C^2 = (T^{-1}C_1T)(T^{-1}C_1T) = I \), follow the desired. \( \Box \)

In short, the Theorem 2.1 and the Proposition 2.4 tell us that:

**Corollary 2.5.** If \( T : H^2 \to H^2 \) an linear isomorphism and \( C := T^{-1}C_1T \), then \( T \) is unitary if, and only if, \( C \) is a conjugation on \( H^2 \).

Now, once every separable Hilbert space has an orthonormal basis, follows that the Corollary 2.5 is true for any separable Hilbert space \( \mathcal{H} \). In fact, if \( \mathcal{B} = \{f_n\} \) is an orthonormal basis on \( \mathcal{H} \), then \( J : \mathcal{H} \to \mathcal{H} \) given by

\[
J \left( \sum_{n=0}^{\infty} \lambda_n f_n \right) = \sum_{n=0}^{\infty} \overline{\lambda_n} f_n.
\]  \hspace{1cm} (2)

is a conjugation on \( \mathcal{H} \). Thus, we have:

**Theorem 2.6.** If \( T : \mathcal{H} \to \mathcal{H} \) an linear isomorphism and \( C := T^{-1}JT \), then \( T \) is unitary if, and only if, \( C \) is a conjugation on \( \mathcal{H} \).

**Proof.** Analogous to Theorem 2.1 and Proposition 2.4 \( \Box \)

**Remark 2.7.** Note that in the Hardy-Hilbert space \( H^2 \), we have \( J = C_1 \).

We already know that every normal operator is complex symmetric and that the reciprocal in general is not true. However, for Toeplitz operators, Theorem 1.1 gives us:

**Fact 2.8.** If \( T_\phi \) is \( J \)-symmetric, then \( T_\phi \) is normal.

Now note that if \( T_\phi \) is normal not necessarily \( T_\phi \) is \( J \)-symmetric. In fact, if \( \phi(z) = -\overline{z} + z \) then \( T_\phi \) is normal, however is not \( J \)-symmetric.
3 Properties of Complex Symmetry

In the following, we present some properties of complex symmetry in Hilbert spaces. The first result gives us a way to get complex symmetric operators from another complex symmetric operator. First, we need some lemmas:

**Lemma 3.1.** ([6, Lemma 1]) If $C$ and $J$ are conjugations on a Hilbert space $\mathcal{H}$, then $U = CJ$ is a unitary operator. Moreover, $U$ is both $C$-symmetric and $J$-symmetric.

**Lemma 3.2.** ([3, Lemma 2.2]) If $U : \mathcal{H} \rightarrow \mathcal{H}$ is a unitary and complex symmetric operator with conjugation $C$, then $UC$ is a conjugation.

**Proposition 3.3.** Let $T : \mathcal{H} \rightarrow \mathcal{H}$ an operator and $C$ and $J$ conjugations on $\mathcal{H}$. Then $T$ is $C$-symmetric if, and only if, $UT$ is $UC$-symmetric, where $U = CJ$.

**Proof.** We already know that $U$ is unitary and $C$ and $J$-symmetric and that $UC = CJC$ is a conjugation, by Lemmas 3.1 and 3.2. Now since $U^* = U^{-1} = JC$ and $T$ is $C$-symmetric, we have

$$UT(UC) = UTCU^* = UCT^*U^* = UC(UT)^*.$$  

Reciprocally, suppose that $UC(UT)^* = UT(UC)$. Thus

$$CT^*U^* = C(UT)^* = U^*UC(UT)^* = U^*UTUC = TUC = TCU^*,$$

whence $CT^* = TC$.  

**Lemma 3.4.** If $T : \mathcal{H} \rightarrow \mathcal{H}$ is both $C$-symmetric and $J$-symmetric, then $T$ is both $CJC$-symmetric and $JCJ$-symmetric.

**Proof.** By Lemma 3.1 we have that $U := CJ$ is unitary and $C$ and $J$-symmetric. Hence, by Lemma 3.2 $UC = CJC$ is a conjugation on $\mathcal{H}$. Thus, since $CT = T^*C$ and $JT = T^*J$ we get

$$(CJC)T = C(TJ)C = T^*(CJC),$$

and so $T$ is $CJC$-symmetric. Analogous, we prove that $T$ is $JCJ$-symmetric.

**Proposition 3.5.** If $T : \mathcal{H} \rightarrow \mathcal{H}$ is both $C$ and $J$-symmetric, then $TU$ is $C$-symmetric, where $U = CJ$.

**Proof.** In fact, once $T$ is both $C$-symmetric and $J$-symmetric, we have by Lemma 3.4 that $T$ is $CJC$-symmetric and so

$$(TU)C = T(CJC) = CU^*T^* = C(TU)^*.$$
Proposition 3.6. Let $A : \mathcal{H} \to \mathcal{H}$ an invertible operator and $C$-symmetric. If $T$ is an operator on $\mathcal{H}$ such that $TA = AT$, then $T$ is $C$-symmetric if, and only if, $TA$ is $C$-symmetric.

Proposition 3.7. Let $U : \mathcal{H} \to \mathcal{H}$ an unitary operator $J$-symmetric. If $T$ is an operator such that $UT^* = TU$ (that is, $T$ and $T^*$ are unitarily equivalents), then:
(i) $JT^* = T^*J \Leftrightarrow T$ is $UJ$-symmetric.
(ii) $UJT = TJU^* \Leftrightarrow T$ is $J$-symmetric.

Proposition 3.8. An operator $T : \mathcal{H} \to \mathcal{H}$ is $C$-symmetric if, and only if, $JT^*C = (CJ)^*T$.

Proof. We already know that $U = CJ$ is unitary and both $C$ and $J$-symmetric. Now, note that
$$JT^*C = (CJ)^*T \Leftrightarrow UT^*C = CU^*T.$$ First see that if $T$ is $C$-symmetric, then $UT^*C = U(CT) = (CU^*)T$. Reciprocally, we have
$$CT^* = CU^*(UT^*C)C = CU^*(CU^*TC) = (UCCU^*)TC = TC.$$

Proposition 3.9. Let $T : \mathcal{H} \to \mathcal{H}$ an operator and $C$ a conjugation on $\mathcal{H}$. If $TC = CT$, then $T$ is $C$-symmetric if, and only if, $T$ is self-adjoint.

4 Complex Symmetry of Aluthge and Duggal Transforms

Recall that the polar decomposition of an operator $T : \mathcal{H} \to \mathcal{H}$ is uniquely expressed by $T = U|T|$, where $|T| = \sqrt{T^*T}$ is a positive operator and $U$ is a partial isometry such that $\text{Ker}(U) = \text{Ker}|U|$ and $U$ maps $\text{cl}(\text{Ran}|T|)$ onto $\text{cl}(\text{Ran}(T))$. In this case, the Aluthge and Duggal Transforms are given, respectively, by $\widetilde{T} = |T|^\frac{1}{2}U|T|^\frac{1}{2}$ and $\hat{T} = |T|U$.

We already known that the Aluthge transform of a complex symmetric operator is also complex symmetric (see [1] Theorem 1]). In this section we study relations between complex symmetry of $T$ and $|T|$ with relation the conjugations $C$ and $J$, as well as the operators $\widetilde{T}$ and $\hat{T}$.

Proposition 4.1. If $T$ is complex symmetric, then $|T|$ is also complex symmetric.
Proof. If $CT = T^*C$, we have by Remark of [3, Lemma 1] that $T = CJ\hspace{1mm}|T|$, where $J$ commutes with $|T|$. Thus, once that $CJ$ is a unitary operator, follows that

\[ J\hspace{1mm}|T| = C(CJ\hspace{1mm}|T|) = |T|^*(CJ)^*C = |T|^*J. \]

\[ \square \]

**Corollary 4.2.** If $T$ is complex symmetric, then $|T|$ is self-adjoint.

**Proposition 4.3.** Let $C$ and $J$ conjugations on $\mathcal{H}$ such that $T = CJ\hspace{1mm}|T|$. If $|T|$ is $C$-symmetric, then $T$ is also $C$-symmetric.

Proof. First, let’s show that $|T|$ is $J$-symmetric. In fact, see that

\[ J(JC\hspace{1mm}|T|) = C\hspace{1mm}|T| = |T|^*C = (|T|^*JC)J, \]

and so $JC\hspace{1mm}|T|$ is $J$-symmetric. Thus, by Proposition 3.3, $|T|$ is $J$-symmetric. Therefore, it is enough to see that:

\[
\begin{align*}
CT &= C(CJ\hspace{1mm}|T|) \\
   &= |T|^*J \\
   &= (|T|^*JC)C \\
   &= (CJ\hspace{1mm}|T|)^*C \\
   &= T^*C.
\end{align*}
\]

\[ \square \]

**Corollary 4.4.** Let $T = CJ\hspace{1mm}|T|$. If $|T|$ is $C$-symmetric, then $\hat{T} = T$.

**Corollary 4.5.** Let $T = CJ\hspace{1mm}|T|$. Then $|T|$ is $C$-symmetric if, and only if, $\hat{T}$ is $J$-symmetric.

**Proposition 4.6.** Let $T = CJ\hspace{1mm}|T|$. If $C\hspace{1mm}|T| = |T|^*C$ and $CJ = JC$, then $T$ is $J$-symmetric.

Proof. In fact, we have that

\[
\begin{align*}
JT &= J(CJ\hspace{1mm}|T|) \\
   &= C\hspace{1mm}|T| \\
   &= |T|^*JJC \\
   &= |T|^*JCJ \\
   &= (CJ\hspace{1mm}|T|)^*J \\
   &= T^*J.
\end{align*}
\]

\[ \square \]
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