Few-particle vortex cluster equilibria in Bose–Einstein condensates: existence and stability

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Received 6 June 2013, in final form 4 September 2013
Published 18 October 2013
Online at stacks.iop.org/JPhysA/46/445001

Abstract
Motivated by recent experimental and theoretical studies of few-particle vortex clusters in Bose–Einstein condensates, we consider the ordinary differential equations of motion and systematically examine settings for up to \(N = 6\) vortices. We analyze the existence of corresponding stationary state configurations and also consider their spectral stability properties. We compare our particle model results with the predictions of the full partial differential equation system. Whenever possible, we propose generalizations of these results in the context of clusters of \(N\) vortices. Some of these we can theoretically establish, especially for the \(N\)-vortex polygons, while others we state as conjectures, e.g. for the \(N\)-vortex line equilibrium.

PACS numbers: 05.45.-a, 03.75.Lm

(Some figures may appear in colour only in the online journal)

1. Introduction

The realm of Bose–Einstein condensates (BECs) has in the past two decades offered a pristine setting for the exploration of numerous nonlinear wave structures and their interactions [1–3]. One of the most prominent examples of such states that has received considerable attention consists of matter wave vortices, which by now have been reviewed in numerous works [4–8]. Most of the relevant experimental studies have been concerned with various techniques of producing single charge vortices. Corresponding examples include, among others, phase imprinting methods between two hyperfine states [9], stirring the BEC above a critical rotation speed [10], supercritical dragging of defects through the BEC [11, 12], quenching through the condensation quantum phase transition [13, 14] or the effectively nonlinear interference of atomic BEC fragments [15]. In addition, experimental efforts were also focused on producing vortices of higher topological charge [16], as well as providing large amounts of angular momentum with the aim of generating robust triangular vortex lattices [17].
On the other hand, considerably less experimental effort was originally invested in the exploration of clusters of few vortices. It was realized early on that such clusters with the same circulation vortices could be created [10] and the expectation was that such states in the presence of angular momentum would shape up into canonical polygons [18] with or without a vortex located at the center. However, in more recent experimental efforts, states with two vortices in the form of a counter-circulating vortex dipole [12, 14, 19], as well as vortex tripole [21] (with two vortices of one sign, and one vortex of the opposite sign) have been produced and their dynamics monitored. A very recent work has also produced sets of 2-, 3-, 4-vortices exploring their dynamics in the absence of a rotational angular momentum induced term [22]. These experimental works have, in turn, either had as a preamble [23–31] or subsequently motivated [32–36] studies on the statics, stability and dynamics of such vortex clusters (predominantly, in fact, the vortex dipole).

Most of the above theoretical works on vortex clusters have stemmed from an improved understanding of the underlying partial differential equation (PDE) which describes the pancake-shaped (i.e., quasi-two-dimensional) condensates which contain such clusters. However, in parallel, a growing fraction of the literature has been advocating [19, 22, 31, 35, 36] the usefulness of a particle model corroborating the two principal features of the vortex motion. These are that each of the vortices has a precessional motion (dictated by its charge, distance from the origin of the parabolic trap confining the BEC and characteristics of the BEC, namely its background density at the center characterizing the so-called chemical potential $\mu$ and the trap confinement frequency $\Omega$) and also has a relative position dependent interaction with other vortices present in the BEC. In the limit of large chemical potentials (so-called Thomas–Fermi limit), where the size of the vortex core shrinks effectively to a point, hence the structure of the core plays no role in the dynamics, it is expected that incorporating these two principal features into particle-based ordinary differential equations (ODEs) should be sufficient to capture the possible vortex cluster equilibria and to assess their (spectral) stability. This approach can, in fact, also be justified rigorously; see for example the derivation of the relevant ODEs in [37].

The aim of the present paper is to carry out the above mentioned program to the extent that it is possible analytically and/or in a numerically assisted form. We focus especially on the context of configurations that arise as stationary states from the system with a few vortices i.e., $N = 3$ up to $N = 6$. We then attempt to generalize the conclusions drawn from these low dimensional systems to the extent possible for larger dimensional ones carrying, however, suitable symmetries. As principal examples, we present the case of $N$-gons where the vortices occupy the vertices of a canonical polygon, as well as that where the vortices are aligned along an axis of the BEC. In the former, we can prove some of the relevant stability conclusions (either analytically or in a numerically assisted form), while in the latter, we conjecture the general result based on our numerical observations, but leave the relevant proof as an open problem for future study.

Our presentation is structured as follows. In section 2, we briefly present the theoretical setup, equations of motion and associated conservation laws. In section 3, we focus on the realm of small vortex numbers $N = 3, \ldots, 6$, while in section 4, we attempt to generalize our conclusions to larger $N$, under suitable symmetry constraints. In section 5, we summarize our findings and present our conclusions, as well as identify a number of directions for future work.

2. Theoretical setup

At the PDE level, the system of interest can be described by a two-dimensional (2D) equation of the nonlinear Schrödinger (NLS) or Gross–Pitaevskii (GPE) type, where the trap strength is
parametrized by an effective frequency \( \Omega_{\text{eff}} = \alpha_1/\alpha_2 \) (i.e., the ratio between the in-plane and perpendicular to the plane trapping frequencies) and the density (at the center of the trap) by the chemical potential parameter \( \mu \), directly associated with the number of atoms in the BEC.

Details about the PDE level description and the reduction from the original three-dimensional (3D) system to the effective 2D one can be found in [33] for our setup. In some sense, our work will naturally complement the above manuscript, as the latter considers how vortex cluster states emerge from the linear limit of the quantum harmonic (2D) oscillator and bifurcations of nonlinear states from linear states thereof (i.e., regime of small chemical potential). Here, on the other hand, we will focus on the opposite limit of the large chemical potential and the particle system emerging when considering the vortices as isolated precessing and interacting entities characterized by their position within the 2D plane.

In this spirit, let \( x_j = (x_j, y_j) \) be a point vortex in the BEC system with charge \( S_j \) for \( j = 1, \ldots, N \) and consider the system

\[
\dot{x}_j = -S_j \Omega_1 y_j - \frac{b}{2} \sum_{k \neq j} S_k \frac{y_j - y_k}{|x_j - x_k|^2} \tag{1}
\]

\[
\dot{y}_j = S_j \Omega_1 x_j + \frac{b}{2} \sum_{k \neq j} S_k \frac{x_j - x_k}{|x_j - x_k|^2} \tag{2}
\]

where we fix \( b = 2 \) for simplicity.\(^3\) \( \Omega \) in equations (1)–(2) is the precession frequency of a single vortex in a trap, which is known to depend on the effective trap frequency \( \Omega_{\text{eff}} \) and the number of atoms in the BEC (as characterized by the so-called chemical potential) \([14, 31]\).

The above system is Hamiltonian with

\[
H(x_1, y_1, \ldots, x_N, y_N) = -\Omega N \sum_{j=1}^{N} S_j r_j^2 + \sum_{j<k}^{N} S_j \log (r_{jk}^2) \tag{3}
\]

where \( r_{jk} = |z_j - z_k| \), and \( z_j \) is the complex variable \( z_j = x_j + i y_j \). In terms of this variable, the equations reduce to

\[
i \dot{z}_j = -S_j \Omega_1 z_j + \sum_{k \neq j}^{N} \frac{S_k}{z_j - \bar{z}_k}. \tag{4}
\]

We also note in passing that the angular momentum \( L = \sum S_j r_j^2 \) is also a conserved quantity for the system of vortices. We use the term angular momentum for this conservation law in line with the tradition stemming from the literature of point vortices in fluid mechanics \([39, 40]\) (rather than the angular momentum of the full quantum mechanical problem). Finally, it will be useful to cast the system in polar coordinates \((r_j, \theta_j)\), in which case the equations of motion become:

\[
\dot{r}_j = \sum_{k \neq j} S_k r_k \sin(\theta_k - \theta_j) \frac{r_{jk}}{r_{jk}^2} \tag{5}
\]

\[
r_j \dot{\theta}_j = r_j S_j \Omega_1 + \sum_{k \neq j} S_k \left( r_j - r_k \cos(\theta_j - \theta_k) \right). \tag{6}
\]

\(^3\) Our considerations herein will not be significantly affected by the precise value of \( b \), as long as the latter assumes a constant value proximal to the one of the homogeneous limit that we assume here. We should note for completeness, however, that to improve the agreement between the ODE as regards the precise location of the fixed points, the work of [31] and others thereafter, considered an effective value of \( b = 1.35 \). This was intended to account for the density-induced ‘screening’ effect associated with the vortex interaction. A first-principles accounting of such screening would necessitate the study of integro-differential equations as analyzed in [38] (see equation (21) therein).
It should be noted here that in the present work the precession frequency of a single vortex will be assumed to be constant, an assumption that is fairly accurate between the center of the trap confining the BEC and half of its radial extent (the latter is often referred to as the Thomas–Fermi radius). More general position dependent precession frequency expressions can also be used in connection to experiments, such as most notably \( \tilde{\Omega}_j = \Omega_j (1 - r_j^2) \) [14, 19, 22]. However we have checked that these do not significantly modify the existence or stability conclusions for the solutions presented in this paper. Hence, for simplicity of the exposition, we restrict our presentation to the constant \( \Omega_j \) case hereafter.

3. Small \( N \)

We now focus more specifically on the case of small vortex clusters. It will be clear from what follows that the cases of interest to us will be those where the vortices do not all carry the same charge. In the latter case, the rotation imposed by the precession is only enhanced by the rotation induced by the interaction and hence the vortices cannot find themselves in a situation of genuine equilibrium. Rather, in the latter case, one can only talk about rigidly rotating states as was examined in the recent work of [22] for \( N = 2–4 \). For large \( N \), the latter setting has been examined too, with a recent example being the work of [41]; see also therein for relevant references. Here, on the other hand, we deal with genuine equilibria of the vortex cluster system and hence none of our configurations carry vortices of a single charge type.

3.1. \( N = 3 \)

The case \( N = 2 \) was examined in detail in [19, 34], in which the dipole was found to be the only fixed point and is linearly stable. Hence, we start by focusing our considerations to the case of \( N = 3 \).

**Proposition 3.1.** When \( N = 3 \), the only fixed point of the particle system is a collinear configuration.

**Proof.** Without loss of generality, fix \( y_1 = 0 \). Suppose \((x_1, 0, x_2, y_2, x_3, y_3)\) with charges \( S_1, S_2 \) and \( S_3 \) is a fixed point of the system. Fix \( S_1 = 1 \). Under these assumptions, the equations read

\[
\frac{S_2 y_2}{r_{12}^2} = -\frac{S_3 y_3}{r_{13}^2}, \tag{7}
\]

\[
\frac{S_3 (y_2 - y_3)}{r_{23}^2} = -S_2 \Omega y_2 - \frac{y_2}{r_{12}^2}, \tag{8}
\]

\[
\frac{S_2 (y_3 - y_2)}{r_{23}^2} = -S_3 \Omega y_3 - \frac{y_3}{r_{13}^2}. \tag{9}
\]

There are three possible cases to consider: \( S_2 = S_3 = \pm 1, S_2 = -S_3 = 1. \) In each case, we obtain the relation \( y_2 = -y_3. \) For instance, consider \( S_2 = -S_3 = 1. \) Then,

\[
\frac{y_2}{r_{12}^2} = \frac{y_3}{r_{13}^2},
\]

\[
\frac{y_3 - y_2}{r_{23}^2} = -\Omega y_2 - \frac{y_2}{r_{12}^2},
\]

\[
\frac{y_3 - y_2}{r_{23}^2} = \Omega y_3 - \frac{y_3}{r_{13}^2}.
\]

Thus inserting the first relation into the second equation and equating the second and third equations yields \( y_2 = -y_3. \)
Note that even though we have satisfied the equations for equilibrium in x, it is still possible that \( \dot{y} \neq 0 \). To see that a true collinear configuration exists, fix \( \Omega = 1 \) and set \( S_1 = 1, S_2 = S_3 = -1 \). One can check that the configuration \( x_1 = (0, 0), x_2 = (0, \sqrt{2}^2) \) and \( x_3 = (0, -\sqrt{2}^2) \) satisfies the fixed point equations.

One can then check by a direct calculation that the collinear fixed point is linearly unstable. The eigenvalues of the linearization matrix are \( \lambda_{1,2} = \pm i \sqrt{5}, \lambda_{3,4} = \pm i \sqrt{7} \) and \( \lambda_{5,6} = 0 \). In general we expect the collinear configuration to have \( N - 2 \) real directions of instability. This is consonant with the conclusions of [31, 33].

3.2. \( N = 4 \)

The case \( N = 4 \) is more subtle, as the system exhibits more than one fixed point, as we show analytically below. Unlike the classical point vortex problem, the system (1)–(2) does not exhibit translational symmetry and therefore the ‘center of vorticity’ is not conserved. In fact,

\[
\sum_{j=1}^{N} S_j \dot{x}_j = -\Omega \sum_{j=1}^{N} y_j \tag{10}
\]

\[
\sum_{j=1}^{N} S_j \dot{y}_j = \Omega \sum_{j=1}^{N} x_j \tag{11}
\]

which implies that the fixed points sought herein must have a center of mass located at \( (0, 0) \). Moreover, the system exhibits rotational symmetry which is clear by replacing \( \theta \mapsto \theta + \alpha t \).

We make the following conjecture.

**Conjecture 3.2.** All fixed points of (1)–(2) are symmetric about the origin.

Assuming the conjecture is true, one can prove

**Proposition 3.3.** For \( N = 4 \), the only fixed points of (1)–(2) are square or collinear (i.e., one in which all the vortices are located on a line going through the origin) configurations.

**Proof.** Choosing \( x_1 = (x_1, 0) \) implies that \( x_2 = (-x_1, 0) \). Then (10)–(11) imply \( x_3 = -x_4 \) and \( y_3 = -y_4 \). If \( y_3 = 0 \), the configuration is collinear, so assume \( y_3 \neq 0 \). Then the equation for \( \dot{x}_1 \) yields

\[
\frac{S_3}{r_{13}} = \frac{S_4}{r_{14}}.
\]

Thus \( r_{13} = r_{14} \), but since the configuration is symmetric about the origin this implies that it is a square (see figure 1).

Fixing \( x_1 = (1, 0), x_2 = (-1, 0), x_3 = (0, 1), x_4 = (0, -1) \), \( S_1 = S_2 = 1 \), and \( S_3 = S_4 = -1 \) gives a square fixed point with \( \Omega = 1/2 \). Linearizing about this equilibrium yields eigenvalues

\[
\lambda_{1,2} = \pm i \sqrt{2}, \lambda_{3,4} = \lambda_{5,6} = \pm \frac{i}{4}, \lambda_{7,8} = 0 \tag{12}
\]
Figure 1. Fixed points of the system in the case of $N = 4$ and with $\Omega = 1/2$. The square configuration has the vortices denoted by circles, while the collinear one has them denoted by stars. Notice that adjacent vortices have opposite sign (unit magnitude) charge.

with corresponding eigenvectors

$v_1 = \{1/3 i (i + 2 \sqrt{2}), 1/3 (1 - 2 i \sqrt{2}), 1/3 (1 - 2 i \sqrt{2}), 1/3 (i + 2 \sqrt{2}), -1, -1, 1, 1\} = \bar{v}_2$

$v_3 = \{0, i, 0, i, 0, 1, 0, 1\} = \bar{v}_4$

$v_5 = \{-i, 0, -i, 0, 1, 0, 1, 0\} = \bar{v}_6$

$v_7 = \{-1, -1, 1, 1, -1, -1, 1\} = \bar{v}_8$

$v_8 = 0$.

Fixing $\Omega = 1/2$ gives a collinear fixed point $x_1 = (c, 0) = -x_3$, $x_2 = (d, 0) = -x_4$ where

$c = \sqrt{1 + \sqrt{2(-1 + \sqrt{2})}}$

$d = \frac{1}{2} \left(10 \sqrt{1 + \sqrt{2(-1 + \sqrt{2})}} - 9(1 + \sqrt{2(-1 + \sqrt{2})})^{3/2} + 4(1 + \sqrt{2(-1 + \sqrt{2})})^{5/2} - (1 + \sqrt{2(-1 + \sqrt{2})})^{7/2}\right)$.

This configuration has eigenvalues $\lambda_{1,2} \approx \pm 2.15$, $\lambda_{3,4} \approx \pm 1.43 i$, $\lambda_{5,6} \approx \pm 0.914$, $\lambda_{7,8} = 0$, and hence possesses $N - 2 = 2$ directions of instability.

These results are consonant with the corresponding findings of the 2D PDE of the NLS/GPE type [31, 33]. For the square configuration the PDE predicts that such a state exists even at the linear limit and bifurcates therefrom as a linearly stable equilibrium (see also [42]) with the exception of a possibility for an oscillatory instability (Hamiltonian Hopf bifurcation) which can arise for an intermediate range of values of the chemical potential $\mu$; $\mu$ is the PDE parameter within the GPE that effectively controls the number of atoms in the BEC. Nevertheless, the state is linearly stable in the limit of large $\mu$, which is the regime of interest herein (see the relevant discussion in the introduction). It should be highlighted that
this state also has a neutral direction because of its radial symmetry within the isotropic 2D parabolic trap. Such a direction is also shared by the collinear configuration which also has a pair of zero eigenvalues due to its invariance under rotations of its linear axis. Moreover, the latter configuration has two directions of instability, in line with the expectation that each higher collinear configuration will have an additional real eigenvalue pair (than the previous one—starting with the dipole that has none, then the tripole has one, the aligned quadrupole two, etc). This expectation stems from the supercritical nature of the bifurcation of these collinear states from the dark soliton state as discussed in [31, 33].

3.3. $N = 5$

In the next section we prove that for general $N$ odd the $N$-gon (i.e., the canonical polygon with the vortices at its vertices) is not a fixed point. We thus hereafter investigate other configurations, confining our considerations to ones that are symmetric about the origin. Fix $(x_1, y_1) = (0, 0)$ and $(x_2, y_2) = (r_1, 0)$. Then by symmetry $(x_3, y_3) = (-r_1, 0)$. Set $(x_4, y_4) = (r_2 \cos \theta_2, r_2 \sin \theta_2)$, $0 \leq \theta_2 < \pi$, so that $(x_5, y_5) = (-r_2 \cos \theta_2, -r_2 \sin \theta_2)$. We compute

$$\lambda_1 = \frac{(S_3 - S_5) \sin \theta_2}{r_2}$$

and so either $\sin \theta_2 = 0$ or $S_2 = S_5$. Assuming the latter holds we find from the equation for $y_1$ that $S_2 = S_1$. Then from the equation for $x_2$ it follows that $\sin(2\theta_2) = 0$. If we choose the root $\theta_2 = \frac{\pi}{2}$, then the configuration must be a square or rhombus centered on the origin. Here, two cases are physically relevant: one where the vortices alternate in sign in the counterclockwise direction with the central vortex being of either sign, and another where the outer vortices have the same sign opposite to the sign of the vortex at the origin. In the first case, we find that the system has no fixed points. In the second, we find that if the central vortex has charge $M > 0$ and the outer vortices have charge $-1$, then the system has a fixed point if and only if $r_1 = r_2$. As an example, when the radius is one and $M = 2$ we compute the configuration to be unstable with eigenvalues $\lambda_{1,2} \approx 3.14i, \lambda_{3,4} = \sqrt{6}, \lambda_{5,6} \approx -1.15 \pm 1.07i, \lambda_{7,8} \approx 1.15 \pm 1.07i$ and $\lambda_{9,10} = 0$. More generally, considering the algebraic constraints, one finds that the configuration with a square surrounding the central vortex can be realized provided there are suitable constraints connecting the central vortex charge, the surrounding charges, the precession frequency and the square’s radius. The relevant condition reads

$$r_1^2 = \frac{2M - 3\tilde{M}}{2\Omega M}$$

(13)

where $-\tilde{M}$ are the surrounding charges to the central one. It is thus clear that such configurations will only exist if $\tilde{M} \times (2M - 3\tilde{M}) > 0$ and assuming $\tilde{M} > 0$ without loss of generality, this leads to $M > 3\tilde{M}/2$. In the case of $M = 1$, the lowest charge that will work is $M = 2$. This configuration can be found to be definitely unstable due to a real pair $\lambda = \pm 2\tilde{M}^{1/2}\Omega \sqrt{4M - 2\tilde{M} / (2M - 3\tilde{M})}$, while other eigenvalues have more complicated forms not provided here. As in the above numerical example, we find a quartet of complex eigenvalues, a real pair, an imaginary pair and a neutral pair associated with the rotational invariance of such a 5-vortex state. There are two observations to make here in connection to this. This configuration is the same as the one labeled ‘5x’ in the work of [33].4 The second is that in line with the numerical observations of [33], we find that this

4 However, note that the latter work inadvertently mentioned a quadrupole as surrounding the central vortex in p 1453; the correct statement is that four same-charge vortices, opposite in sign to the doubly charged central one are surrounding it.
configuration at large $\mu$ contains a complex eigenfrequency quartet, as well as a real eigenvalue pair as manifestations of its instability (see accordingly the third row, right panel of figure 4 in p 1453 of [33]).

We now turn to the case when $\theta_2 = 0$; here, we have a collinear fixed point which only exists when the vortices have alternating charges. This fixed point can be described by the configuration $(0, 0), (\pm r_1, 0), (\pm r_2, 0)$ where

$$r_1 = \sqrt{\frac{2 - \sqrt{3}}{2\Omega}}, \quad r_2 = \sqrt{\frac{\sqrt{3}}{2\Omega}}$$

This configuration was computed to be unstable with eigenvalues $\lambda_{1,2} \approx \pm 6.95\Omega, \lambda_{3,4} \approx \pm 6.69\Omega, \lambda_{5,6} \approx \pm 3.64\Omega, \lambda_{7,8} \approx \pm 2.8\Omega, \lambda_{9,10} = 0$ and so again we see that there are $N - 2 = 3$ directions of real instability, as well as a neutral direction. This is in line with the expectations of the earlier works of [31, 33], examining the PDE limit of such 5-vortex collinear configurations. Both $N = 5$ configurations are depicted in figure 2.

3.4. $N = 6$

We are aware of two fixed points when $N = 6$, shown in figure 3: the hexagon and the collinear configuration. Both of these are configurations of alternating charges (either along the ring or along a line, respectively). The linearization about the hexagon when all vortices are on the unit circle and $\Omega = 1/2$ yields the following eigenvalues

$$\lambda_{1,2} = \pm \frac{3i}{\sqrt{2}}, \quad \lambda_{3,4} = \lambda_{5,6} = \pm i, \quad \lambda_{7,8} = \lambda_{9,10} = \pm \frac{1}{\sqrt{2}}, \quad \lambda_{11,12} = 0$$

and so, unlike the square, the hexagon is unstable. This is in line with earlier observations at the PDE level for this vortex ring; see e.g. the discussion of [33] (top right of figure 5 in p 1454.
and the associated discussion). There, it is inferred (coming from the opposite limit of small chemical potential) that the hexagon supercritically bifurcates from the already unstable (even off of and near the linear limit) state of the dark soliton ring [43], inheriting its instability.

The collinear configuration can be represented by \((\pm a, 0), (\pm b, 0), (\pm c, 0)\) where \(a \approx 0.23, b \approx 0.67\) and \(c \approx 1.58\) were computed numerically. The eigenvalues of the linearization about this fixed point were computed to be \(\lambda_1, \lambda_2 \approx \pm 4.88, \lambda_3, \lambda_4 \approx \pm 4.86, \lambda_5, \lambda_6 \approx \pm 2.65, \lambda_7, \lambda_8 \approx \pm 2.21, \lambda_9, \lambda_{10} \approx \pm 0.96,\) and \(\lambda_{11,12} = 0\) and so this configuration also has \(N - 2 = 4\) directions of instability. We note in this case too that in order to generalize the relevant results in the case of arbitrary \(\Omega_1\), one has to scale the fixed point positions by \(1/\sqrt{\Omega}\) and the corresponding eigenvalues linearly by \(\Omega\).

4. Large \(N\)

4.1. The \(N\)-gon fixed point

Up to now, we have provided detailed (existence and stability) features of setups containing only small numbers of vortices. We now generalize our conclusions by offering some proofs, as well as some conjectures for more general vortex configurations featuring large numbers of vortices \(N\). As we will see, there are some significant differences between the cases of an even number of vortices and those of an odd number of vortices, both at the level of existence, as well as at that of stability. We will assume hereafter that the vortices are of alternating charge, namely \(S_j = (-1)^{j+1}, j = 1, 2, \ldots, N.\)

Proposition 4.1. The polygonal \(N\)-vortex (hereafter, sometimes referred to as ‘\(N\)-gon’) configuration described above is a fixed point of the point vortex system if and only if \(N\) is even. Moreover, the configuration is linearly stable for \(N = 4\) and linearly unstable otherwise.

First, if the \(N\)-gon is an equilibrium any radial scaling is also an equilibrium for some different choice of \(\Omega\), and so we may assume that each vortex lies on the unit circle. This leaves a single free real parameter \(\Omega\), and \(z_1 = e^{2\pi i/N}, z_2 = e^{4\pi i/N}, \ldots, z_N = 1\). We have examined the \(N = 4\) and \(N = 6\) (i.e., the marginally stable and the first unstable) cases in detail above. Notice that by abusing notation, we can also consider the dipole as an example of this type.
with \( N = 2 \), which is stable at the particle level in accordance with the earlier analysis e.g. of [34].

The first proposition relies on the following two lemmas.

**Lemma 4.2.** If \( N \) is odd, the \( N \)-gon is not a fixed point. If \( N \) is even, the \( N \)-gon is a fixed point.

**Proof.** This is easiest to see in polar coordinates. First assume that the point vortices lie on the unit circle. When \( N \) is even, note that by symmetry \( \dot{r}_j = 0 \) for each \( j \). For \( r = 1 \), one finds

\[
\dot{\theta}_j = (-1)^{j+1} \Omega + \frac{1}{2} \sum_{k \neq j} (-1)^{k+1} \notag
\]

\[
= \Omega - \frac{1}{2}, \quad \text{if } j \text{ is odd} \notag
\]

\[
= -\Omega + \frac{1}{2}, \quad \text{if } j \text{ is even} \notag
\]

and so the \( N \)-gon is a fixed point when \( \Omega = 1/2 \). On the other hand, when \( N \) is odd we have

\[
\dot{\theta}_j = (-1)^{j+1} \Omega + \frac{1}{2} \sum_{k \neq j} (-1)^{k+1} \notag
\]

\[
= \Omega, \quad \text{if } j \text{ is odd} \notag
\]

\[
= -\Omega + 1, \quad \text{if } j \text{ is even} \notag
\]

and so for no value of \( \Omega \) is the odd \( N \)-gon a fixed point. \( \square \)

**Lemma 4.3.** When the \( N \)-gon is a fixed point, it is unstable for \( N \geq 6 \).

**Proof.** This result will require a detailed examination of the corresponding stability matrix. We again consider the system in polar coordinates. The resulting matrix, denoted by \( M \), of the linearization is a \( 2N \times 2N \) block matrix made up of \( N \times N \) blocks

\[
M := \begin{pmatrix} A & B \\ C & D \end{pmatrix}. \notag
\]

We compute each of these blocks explicitly. Let \( a_{j,k} \) denote the entry of \( A \) in the \( j \)th row and \( k \)th column, and similarly define \( b_{j,k} \), \( c_{j,k} \), and \( d_{j,k} \). Let \( \theta_{jk} = \theta_j - \theta_k \) and \( r_{jk} = |z_j - z_k| \).

Then

\[
a_{j,k} = S_j \left( r_k^2 - r_j^2 \right) \cos(\theta_{jk}) \notag \quad \text{for } j \neq k
\]

\[
a_{j,j} = \sum_{k \neq j} 2r_kS_k(r_j - r_k \cos(\theta_{jk})) \sin(\theta_{jk}). \notag
\]

In the case of the \( N \)-gon where \( S_k = (-1)^{k+1} \) and \( r_j = 1 \) for all \( j \) we see that the above entries are identically zero, hence \( A = 0 \), the zero matrix. Next,

\[
b_{j,k} = \frac{r_jS_j(r_j^2 + r_k^2) \cos(\theta_{jk}) - 2r_j r_k}{r_{jk}^4}, \quad j \neq k \notag
\]

\[
b_{j,j} = \sum_{k \neq j} \frac{r_jS_k(2r_k r_j - (r_j^2 + r_k^2) \cos(\theta_{jk}))}{r_{jk}^4}. \notag
\]
In this case, when all radii are set to 1 we find
\[ b_{j,k} = \frac{S_j}{r_{jk}} \]
\[ b_{j,j} = \sum_{k \neq j} \frac{S_k}{r_{jk}}. \]

The matrix \( C \) is
\[ c_{j,k} = \frac{S_k (r_j^2 + r_k^2) \cos(\theta_{jk}) - 2r_j r_k}{r_k r_j^4}, \quad j \neq k \]
\[ c_{j,j} = -\sum_{k \neq j} S_k (r_j^2 - 2r_j r_k \cos(\theta_{jk}) + r_k^2 \cos(2\theta_{jk})) - \sum_{k \neq j} S_k (r_j - r_k \cos(\theta_{jk})). \]

Again setting \( r = 1 \) one obtains after some simplification
\[ c_{j,k} = -\frac{S_j}{r_{jk}} \]
\[ c_{j,j} = \sum_{k \neq j} \frac{S_k}{r_{jk}} - \sum_{k \neq j} S_k. \]

Here we have used that \( r_{jk}^2 = 2 - 2 \cos(\theta_{jk}) \) when \( r = 1 \). Finally, we have
\[ d_{j,k} = \frac{S_j r_j (r_j^2 - r_k^2) \sin(\theta_{jk})}{r_k r_j^4}, \quad j \neq k \]
\[ d_{j,j} = \sum_{k \neq j} S_k r_k (r_j^2 - r_k^2) \sin(\theta_{jk}) \]

which is the zero matrix when \( r = 1 \). We are interested in the determinant of the matrix
\[ \begin{pmatrix} -\lambda I & B \\ C & -\lambda I \end{pmatrix}. \]

Since the two matrices in the bottom row of \( M \) commute, one has the formula
\[ \det(M) = \det(AD - BC) \]
and
\[ \det(M - \lambda I_{2N \times 2N}) = \det(\lambda^2 I_{2N} - BC). \]

When \( \Omega = 1/2, BC \) is a circulant matrix and hence its eigenvalues are real. Therefore, \( \lambda \) must be real or purely imaginary. To determine the eigenvalues of a circulant matrix, it is sufficient to know the first row of the matrix. Let \( \tilde{B} = BC \). Then
\[ \tilde{b}_{1,1} = \left( \sum_{k \neq 1} \frac{S_k}{r_{1k}} \right)^2 - \sum_{k \neq 1} S_k \sum_{k \neq 1} \frac{S_k}{r_{1k}} + \sum_{k \neq 1} \frac{S_k S_k}{r_{1k}} \]
\[ \tilde{b}_{1,2} = \left( \sum_{k \neq 1} \frac{S_k}{r_{1k}} \right)^2 - \sum_{k \neq 1} S_k \sum_{k \neq 2} \frac{S_k}{r_{2k}} + \sum_{k \neq 1} \frac{S_k S_k}{r_{1k}} \]
\[ \tilde{b}_{1,3} = \left( \sum_{k \neq 1} \frac{S_k}{r_{1k}} \right)^2 - \sum_{k \neq 1} S_k \sum_{k \neq 3} \frac{S_k}{r_{3k}} + \sum_{k \neq 1} \frac{S_k S_k}{r_{1k}} \]
and in general for \( m \neq 1 \)

\[
\tilde{b}_{1,m} = (-1)^{m+1} \frac{S_1}{r_{1m}} \left[ \sum_{k \neq 1} \frac{S_k}{r_{1k}} + \sum_{k \neq m} \frac{S_k}{r_{mk}} - \sum_{k \neq m} S_k \right] + \sum_{k \neq 1, m} \frac{S_k}{r_{1k}^m}.
\]  

(14)

When \( S_k = (-1)^k + 1 \), we have

\[
\tilde{b}_{1,1} = \left( \sum_{k \neq 1} \frac{(-1)^{k+1}}{r_{1k}^2} \right)^2 + \sum_{k \neq 1} \frac{(-1)^{k+1}}{r_{1k}^2} + \sum_{k \neq m} \frac{(-1)^{k+1}}{r_{mk}^4}.
\]

\[
\tilde{b}_{1,m} = (-1)^{m+1} \frac{1}{r_{1m}^2} \left[ \sum_{k \neq 1} \frac{(-1)^{k+1}}{r_{1k}^2} + \sum_{k \neq m} \frac{(-1)^{k+1}}{r_{mk}^2} + (-1)^{m+1} \right] + \sum_{k \neq 1, m} \frac{(-1)^{k+1}}{r_{1k}^m}.
\]

The eigenvalues of \( \tilde{B} \) are given by

\[
\gamma_j = \sum_{k=1}^{N} \tilde{b}_{1,k} \omega_j^{k-1}, \quad j = 1, \ldots, N
\]

and the eigenvectors are

\[
v_j = \{1, \omega_j, \omega_j^2, \ldots, \omega_j^{N-1}\}
\]

where \( \omega_j = e^{2\pi i / N} \) is the \( j \)th root of unity.

In particular, we find that for chosen even \( N \) we can explicitly compute any eigenvalue although a simplified expression for the eigenvalues was not found. For \( j = N - 2, \gamma_j \), the eigenvalues of \( \tilde{B} \) will be nonnegative when \( N \geq 4 \). Using Mathematica, we explicitly compute \( \gamma_{N-2} = \left( \frac{\sqrt{2}}{2} N - \sqrt{2} \right)^2 \geq 0 \). Thus \( \lambda_j = \pm \sqrt{\gamma_j} \) is real and nonzero when \( N > 4 \), and the corresponding \( N \)-gon is unstable. By the same token, the \( N \)-gon with \( N = 4 \) will be marginally stable. \( \square \)

We point out here that these results, as well as the numerical results both at the level of the particle equations and at that of the underlying PDE [33] suggest the following.

**Conjecture 4.4.** For \( N \geq 6 \), the \( N \)-gon has \( \frac{N}{2} - 2 \) distinct pairs of real eigenvalues.

At the ODE level, this stems from direct observations for \( N = 2, N = 4, N = 6 \) and \( N = 8 \), while at the PDE, the relevant observation is that each higher order vortex polygon (for even \( N \geq 6 \)) emerges from subsequent supercritical pitchfork bifurcations of the unstable ring dark soliton [43] and hence each additional destabilization adds a real eigenvalue pair to the linear spectrum [33] (see figure 5, top right and bottom left, p 1454). The \( N = 4 \) state on the other hand emerges as a spectrally stable one from the linear limit. Notice, however, that the stability identified here arises at the large \( \mu \) limit. For intermediate values of \( \mu \) (i.e., for an intermediate interval thereof) instabilities of oscillatory type may arise due to the PDE nature of the system, i.e., due to collisions of some of the internal modes of the vortex particle system with those of its host background.

### 4.2. The collinear fixed point

We now briefly discuss the generalization of the features of the collinear fixed point that we have numerically observed for larger numbers of vortices.
Conjecture 4.5. For all \( N \), there exists a symmetric, collinear fixed point of the equations of motion for alternating sign vortex configurations. For even \( N \) such configurations are symmetrically placed around the origin i.e., \((\pm a, 0), (\pm b, 0), (\pm c, 0)\) ... while for odd \( N \) they have the same structure plus an additional vortex placed at the origin.

The existence of such a configuration for any \( N \) suggests that there are always real numbers \( x_1, x_2, \ldots, x_N \) such that for every \( j = 1, \ldots, N \), \( \sum_{k=1}^{N} (-1)^{k-j}/(x_k - x_j) = 1 \) where in the summation, it is implied that \( k \neq j \). In the case of \( N \) even, the statement is precisely that (and the solution, as our numerical results indicate, has the vortices symmetrically placed around 0, i.e., \(-x_N/2, -x_{(N-2)/2}, \ldots, -x_1, x_1, \ldots, x_{(N-2)/2}, x_N/2\)). In the case of \( N \) odd, one of the vortices is always placed at 0, so one can rephrase the statement as \( \sum_{k=1}^{N} (-1)^{k-j}/(x_k - x_j) = x_j \).

The remaining vortices are placed symmetrically around 0 as above. It is clear that this is essentially an algebraic/number-theoretic problem. Effectively, this can be rewritten as a set of \( N \) polynomial equations in \( N \) unknowns. As such, it defines a variety, and the question is whether this variety always has a real point \((x_1, \ldots, x_N)\).\(^5\)

In addition, we can generalize our conclusions for the stability of the collinear vortex state as follows.

Conjecture 4.6. The collinear fixed point is unstable for \( N \geq 3 \) with \( N - 2 \) real directions of instability.

5 We thank Farshid Hajir for this very interesting observation.

5. Conclusions and future challenges

In the present work, we revisited the topic of few-vortex crystal configurations in two-dimensional Bose–Einstein condensates. While earlier studies focused on the PDE approach attempting to infer conclusions for the vortex dynamics from the vicinity of the linear limit [23–31, 33], our approach here has taken a complementary view whereby the vortices have been examined as interacting particle systems (as was done earlier chiefly for the dipole [34–36], but also for larger vortex numbers in the case where rotation is present [18] or absent [22] for co-circulating vortices). We have shown this approach to be fairly informative towards an understanding of the configurations that may arise for small vortex numbers \( N \) and the
identification of their stability characteristics. Moreover, the systematic progression towards higher vortex numbers $N$ has enabled us to extract generalizations of the conclusions obtained for lower vortex numbers. In some cases (e.g. for the non-existence of $N$-gons for $N$ odd, or for the stability characteristics of $N$-gons with $N$ even), we have been able to prove relevant conclusions in their full generality. In other cases (as e.g. for the collinear configurations), we have formulated general conjectures that may, in turn, stimulate non-trivial connections with other areas of mathematics such as algebra/number theory. These concern, for instance, the existence of solutions mapping into the existence of a real point of a certain variety and the analysis of the properties of the corresponding near-circulant stability matrices. A deeper cross-pollinating view that may address such open questions would certainly be a welcome addition to the literature in the near future.

However, there are additional extensions or generalizations of the questions posed herein that merit future investigation in their own right. On the one hand, here we have restricted our attention (for the reasons explained) to the case of counter-rotating vortices that may produce fixed point configurations. However, as recent experiments have naturally argued, it is of particular interest to also explore co-circulating vortex states and especially rigidly rotating examples thereof (where all the vortices rotate with the same angular momentum), rendering the co-rotating frame of reference the right one for seeking stationary states of the system. In the latter context, and in the absence of precession a classical result is that of Havelock proving the instability of $N$-gon configurations with $N \geq 8$ [44]. It would be relevant to explore whether this conclusion persists in the presence of precessional terms. Furthermore, it should be pointed out that cases of $N$-gons with same (as rotating configurations) and opposite charges (as fixed points) will have definite differences between their respective stability properties for the same $N$. As an immediate example, we mention the hexagon of $N = 6$, which may be stable (for suitable values of the angular momentum) when all six vortices have $S = 1$, but was always found to be unstable in our considerations herein with three vortices of $S = 1$ and three of $S = -1$.

On the other hand, a natural generalization of the present considerations is that of exploring the dynamics of vortex rings in three-dimensional BECs; see e.g. [3, 45] for relevant reviews. In this context, it is also possible to write ordinary differential equations (ODEs) characterizing the interaction of the rings and their intrinsic translational dynamics as e.g. in [46]. However, we have not been able to identify simple ODEs that would describe the motion of such rings in a three-dimensional parabolic trap—a key ingredient for the system of ODEs, as we saw above for the case of vortices. The exploration of such vortex rings as interacting particle systems is emerging as an extremely interesting topic for future work and will be deferred for future publications.

Acknowledgments

We thank C E Wayne for discussions at the early stage of this work and R Carretero-González for numerous iterations (including on the numerical results of [33]). We also thank F Hajir for bringing to our attention algebraic connections of the existence problems formulated herein. PGK is grateful to the IMA for its hospitality during the completion of this work and also acknowledges support from NSF-DMS-0806762, NSF-CMMI-1000337 and the US AFOSR via award FA9550-12-1-0332, as well as the Binational Science Foundation under grant BSF-2010239.
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