On the existence of 3–way $k$–homogeneous Latin trades

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Abstract

A $\mu$–way Latin trade of volume $s$ is a collection of $\mu$ partial Latin squares $T_1, T_2, \ldots, T\mu$, containing exactly the same $s$ filled cells, such that if cell $(i, j)$ is filled, it contains a different entry in each of the $\mu$ partial Latin squares, and such that row $i$ in each of the $\mu$ partial Latin squares contains, set-wise, the same symbols and column $j$, likewise. It is called $\mu$–way $k$–homogeneous Latin trade, if in each row and each column $T_r$, for $1 \leq r \leq \mu$, contains exactly $k$ elements, and each element appears in $T_r$ exactly $k$ times. It is also denoted by $(\mu, k, m)$ Latin trade, where $m$ is the size of partial Latin squares.

We introduce some general constructions for $\mu$–way $k$–homogeneous Latin trades and specifically show that for all $k \leq m$, $6 \leq k \leq 13$ and $k = 15$, and for all $k \leq m$, $k = 4, 5$ (except for four specific values), a 3–way $k$–homogeneous Latin trade of volume $km$ exists. We also show that there are no $(3, 4, 6)$ Latin trade and $(3, 4, 7)$ Latin trade. Finally we present general results on the existence of 3–way $k$–homogeneous Latin trades for some modulo classes of $m$.

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1 Introduction

A Latin square $L$ of order $n$ is an $n \times n$ array usually on the set $N = \{1, \ldots, n\}$ where each element of $N$ appears exactly once in each row and exactly once in each column. We can represent each Latin square as a subset of $N \times N \times N$, 

$$L = \{ (i, j; k) \mid \text{element } k \text{ is located in position } (i, j) \}.$$ 

A partial Latin square $P$ of order $n$ is an $n \times n$ array of elements from the set $N$, where each element of $N$ appears at most once in each row and at most once in each column. The set $S_P = \{ (i, j) \mid (i, j; k) \in P \}$ of the partial Latin square $P$ is called the shape of $P$ and $|S_P|$ is called the volume of $P$. By $R_P^i$ and $C_P^j$ we mean the set of entries in row $i$ and column $j$, respectively of $P$. A $\mu$-way Latin trade, $(T_1, T_2, \ldots, T_\mu)$, of volume $s$ is a collection of $\mu$ partial Latin squares $T_1, T_2, \ldots, T_\mu$, containing exactly the same $s$ filled cells, such that if cell $(i, j)$ is filled, it contains a different entry in each of the $\mu$ partial Latin squares, and such that row $i$ in each of the $\mu$ partial Latin squares contains, set-wise, the same symbols and column $j$, likewise. If $\mu = 2$, $(T_1, T_2)$ is called a Latin bitrade. The study of Latin trades and combinatorial trades in general, has generated much interest in recent years. For a survey on the topic see [3], [9], and [6].

A $\mu$–way Latin trade which is obtained from another one by deleting its empty rows and empty columns, is called a $\mu$–way $k$–homogeneous Latin trade ($\mu \leq k$) or briefly a $(\mu, k, m)$ Latin trade, if it has $m$ rows and in each row and each column $T_r$, for $1 \leq r \leq \mu$, contains exactly $k$ elements, and each element appears in $T_r$ exactly $k$ times.

In Figure 1(a) a $(3, 5, 7)$ Latin trade is demonstrated. The elements of $T_2$ and $T_3$ are written as subscripts in the same array as $T_1$. ($\bullet$ means the cell is empty.)

![Figure 1: A (3, 5, 7) Latin trade and its base row](image-url)
A \((\mu, k, m)\) Latin trade \((T_1, T_2, \ldots, T_\mu)\) is called circulant, if it can be obtained from the elements of its first row, called the base row and denoted by \(\mu-B^k_m\), by permuting the coordinates cyclically along the diagonals. For example in Figure 1(b), a \(3-B^5_7\) base row, \((1,2,3)_{1}, (3,5,2)_{2}, (5,3,7)_{3}, (7,1,5)_{4}, (2,7,1)_{5}\), is shown. Actually if a base row \(B = \{(a_1, a_2, \ldots, a_\mu)_{c_l} | 1 \leq l \leq k\}\), where \(a_r \text{ and } c_l \in \{1,2,\ldots,m\}\), is given, we construct a set of \(\mu\) partial Latin squares as in the following manner:

\[
1 \leq r \leq \mu, \quad T_r = \{(1+i, c_l+i; a_r+i)(\text{mod} \ m)|0 \leq i \leq m-1, 1 \leq l \leq k\}.
\]

Algorithm 1 To check that \(B = \{(a_1, a_2, \ldots, a_\mu)_{c_l} | 1 \leq l \leq k\}\), where \(a_r \text{ and } c_l \in \{1,2,\ldots,m\}\), is a base row of a \((\mu, k, m)\) Latin trade: we note that for each \(r, 1 \leq r \leq \mu, \quad R^1_{T_r} = \{a_r | (a_1, a_2, \ldots, a_\mu)_{c_l} \in B \text{ and } 1 \leq l \leq k\}\) and \(C^m_{T_r} = \{a_r + m - c_l \equiv a_r - c_l(\text{mod} \ m) | (a_1, a_2, \ldots, a_\mu)_{c_l} \in B \text{ and } 1 \leq l \leq k\}\).

Now if \(B\) satisfies the following conditions, then it will suffice to be a base row of a \((\mu, k, m)\) Latin trade.

(i) \(a_r\)'s are distinct, for each \((a_1, a_2, \ldots, a_\mu)_{c_l} \in B\).

(ii) \(c_l\)'s are distinct.

(iii) \(R^1_{T_1} = R^1_{T_2} = \cdots = R^1_{T_\mu}\).

(iv) \(C^m_{T_1} = C^m_{T_2} = \cdots = C^m_{T_\mu}\).

Lemma 1 For each \(k \geq \mu\), a \((\mu, k, k)\) Latin trade exists.

Proof. By taking a Latin square of order \(k\) and permuting its rows, cyclically, \(\mu\) times we obtain the desired Latin trade.

\[\quad \]

A \((\mu, \mu, \mu)\) Latin trade is called a \(\mu\)-intercalate.

\[
\begin{array}{ccc}
1 & 3 & 2 \\
3 & 2 & 1 \\
2 & 1 & 3 \\
\end{array}
\]

Figure 2: A 3–intercalate

The following question is of interest.
Question 1  For given $m$ and $k$, $m \geq k \geq \mu$, does there exist a $(\mu, k, m)$ Latin trade?

For Latin bitrades, Question 1 is discussed and is answered completely in [4], [5], [2], [1], and [7]. In this paper applying earlier results we introduce some general constructions for $(\mu, k, m)$ Latin trades and specifically concentrate on the case of $\mu = 3$. Our main result is stated in the following theorem.

Theorem 1  All $(3, k, m)$ Latin trades $(m \geq k \geq 3)$ exist, for

- $k = 4$, except for $m = 6$ and 7 and possibly for $m = 11$,
- $k = 5$, except possibly for $m = 6$,
- $6 \leq k \leq 13$,
- $k = 15$,
- $k \geq 4$ and $m \geq k^2$,
- $m$ a multiple of 5, except possibly for $m = 30$,
- $m$ a multiple of 7, except possibly for $m = 42$ and $(3, 4, 7)$ Latin trade.

2  General constructions

Theorem 2  If $l \neq 2, 6$ and for each $k \in \{k_1, \ldots, k_l\}$ there exists a $(\mu, k, p)$ Latin trade, then a $(\mu, k_1 + \cdots + k_l, lp)$ Latin trade exists. (Some $k_i$s can possibly be zero.)

Proof.  Since $l \neq 2, 6$, there exist two $l \times l$ orthogonal Latin squares. Denote these Latin squares by $L_1$ and $L_2$, with elements chosen from the sets $\{e_1, e_2, \ldots, e_l\}$ and $\{f_1, f_2, \ldots, f_l\}$, respectively. Assume that $L^*$ is a square that is formed by superposing $L_1$ and $L_2$. We replace each $(e_i, f_j)$ in $L^*$ with a $(\mu, k_j, p)$ Latin trade whose elements are from the set $\{(i - 1)p + 1, (i - 1)p + 2, \ldots, ip\}$. As a result we obtain a $(\mu, k_1 + \cdots + k_l, lp)$ Latin trade.

Theorem 3  If the number of mutually orthogonal Latin squares of order $k + 1$, MOLS($k+1$), is greater than or equal to $\mu + 1$, then there exists a $(\mu, k, k+1)$ Latin trade.

Proof.  By Exercise 5.2.11 of [10] page 103, there are $\mu$ idempotent MOLS($k+1$). If in each of those MOLS we delete the main diagonals, we obtain a $(\mu, k, k+1)$ Latin trade.
Actually by applying results of existence of idempotent MOLS$(n)$ ([4], Section 3.6, Table 3.83), we can improve Theorem 3 for the case $\mu = 3$ as follows.

**Theorem 4** If $k \geq 11$, then there exists a $(3, k, k + 1)$ Latin trade.

**Theorem 5** Any $(\mu, \mu, m)$ Latin trade, $T = (T_1, T_2, \ldots, T_\mu)$, can be partitioned into disjoint $\mu$–intercalates.

**Proof.** We prove this result by induction. Without loss of generality, let $(1,1; r) \in T_r$ for each $1 \leq r \leq \mu$. Therefore $\{1,2,\ldots, \mu\} \subset R^i_{T_r} \cap C^i_{T_r}$ for each $1 \leq i, r \leq \mu$. Since $|R^i_{T_r}| = |C^i_{T_r}| = \mu$ for each $1 \leq i, r \leq \mu$, $R^i_{T_r} = C^i_{T_r} = \{1,2,\ldots, \mu\}$ for each $1 \leq i, r \leq \mu$. Again without loss of generality, let $(i,1;i) \in T_1$ and $(1,j;j) \in T_1$ for $1 \leq i, j \leq \mu$. This implies that $\{(i,j) \mid 1 \leq i, j \leq \mu\}$ is a subset of shape of $T_1$. Therefore subarray $\{(i,j) \mid 1 \leq i, j \leq \mu\}$ with elements $\{1,2,\ldots, \mu\}$ is a $\mu$–intercalate. We can apply the same argument to the $(m-\mu) \times (m-\mu)$ subsquare obtained by removing rows $1,2,\ldots, \mu$ and columns $1,2,\ldots, \mu$. This completes the proof. ■

**Corollary 1** For every $m \geq 1$, there exists a $(\mu, k, m)$ Latin trade with $k = \mu$, if and only if $k|m$.

**Theorem 6** Assume that $m_i \geq k_i$, for $i = 1, 2$. If there exists a $(\mu_i, k_i, m_i)$ Latin trade for $i = 1, 2$, then there exists a $(\mu_1\mu_2, k_1k_2, m_1m_2)$ Latin trade.

**Proof.** We construct a $(\mu_1\mu_2, k_1k_2, m_1m_2)$ Latin trade in the following way:

Suppose $(T_1, T_2, \ldots, T_{\mu_1})$ is a $(\mu_1, k_1, m_1)$ Latin trade and $U = (U_1, U_2, \ldots, U_{\mu_2})$ is a $(\mu_2, k_2, m_2)$ Latin trade. For each entry $i$ in $T_1, T_2, \ldots, T_{\mu_1}$, we replace $i$ with a copy of $U$ where elements are chosen from the set $\{(i-1)m_2 + 1, (i-1)m_2 + 2, \ldots, im_2\}$; replace the empty cells in $T_1, T_2, \ldots, T_{\mu_1}$ with an empty $m_2 \times m_2$ array. As a result we obtain a $(\mu_1\mu_2, k_1k_2, m_1m_2)$ Latin trade. ■

**Corollary 2** Suppose $k = k_1k_2$ and $m = m_1m_2$ where $m_i \geq k_i \geq 2$, for $i = 1, 2$. Then there exists a $(4, k, m)$ Latin trade, provided that if $k_j = 2$, for some $j$, then $m_j$ must be assumed to be even.

**Proof.** It is shown that Latin homogeneous bitrades (i.e $(2, k, m)$ Latin trade) exist for all $m \geq k \geq 3$ and for all even $m$, when $k = 2$. (See [1], [2], [3], and [7].) ■

**Theorem 7** For every $k$, if there exists a $(\mu, k, m)$ Latin trade and a $(\mu, k, n)$ Latin trade, then there exists a $(\mu, k, m+n)$ Latin trade.
Proof. Let $T_1$ be a $(\mu, k, m)$ Latin trade and $T_2$ be a $(\mu, k, n)$ Latin trade such that the elements of $T_1$ are in the set $\{1, \ldots, m\}$ and the elements of $T_2$ are chosen from the set $\{m+1, \ldots, m+n\}$. Therefore, the following Latin trade is a $(\mu, k, m+n)$ Latin trade.

\[
\begin{array}{c|c}
T_1 & \hline
\end{array}
\begin{array}{c}
T_2
\end{array}
\]

Corollary 3 If the number of MOLS$(k+1) \geq \mu+1$, then for each $m$ where $m \geq k^2$, there exists a $(\mu, k, m)$ Latin trade.

Proof. If $m \geq k^2$, then we can write $m$ as $m = rk + s(k+1)$, where $r, s \geq 0$. Theorem 7 and Theorem 3 lead us to a conclusion.

By Theorems 7 and 4 we have:

Corollary 4 If $k \geq 11$, then for each $m$ where $m \geq k^2$, there exists a $(3, k, m)$ Latin trade.

Theorem 8 Consider an arbitrary natural number $k$. If for every $k+1 \leq l \leq 2k-1$ there exists a $(\mu, k, l)$ Latin trade, then for any $m \geq k$ there exists a $(\mu, k, m)$ Latin trade.

Proof. For every $m \geq 2k$, we can write $m = rk + sl$, where $r, s \geq 0$ and $k+1 \leq l \leq 2k-1$. Since there exist a $(\mu, k, k)$ Latin trade and a $(\mu, k, l)$ Latin trade, by Theorem 7 we conclude that there exists a $(\mu, k, m)$ Latin trade.

3 $\mu = 3$

In this section we apply the above constructions to establish the existence of 3–way $k$–homogeneous Latin trades for specific values of $k$, and when $m$ is a multiple of 5 or 7. We also show that there is no $(3, 4, 6)$ Latin trade.

3.1 Small even $k$

Proposition 1 There exists a $(3, 4, m)$ Latin trade for every $m \geq 4$, except possibly for $m = 6, 7$ and 11.
Proof. By Lemma 1 and Theorem 3 there exist a (3, 4) Latin trade and a (3, 4, 5) Latin trade, respectively. Since $8 = 2 \times 4$, $9 = 4 + 5$, $10 = 2 \times 5$, $12 = 3 \times 4$, $13 = 2 \times 4 + 5$, $14 = 4 + 2 \times 5$, and $15 = 3 \times 5$; Theorem 7 results that there exist (3, 4, m) Latin trades for $m = 8, 9, 10, 12, 13, 14$, and 15. Since the number $\text{MOLS}(5) = 4$, then by Corollary 3 there exists a (3, 4, m) Latin trade, for every $m \geq 16$.

Proposition 2 There is no (3, 4, 6) Latin trade.

Proof. By contradiction. Suppose $T = (T_1, T_2, T_3)$ is a (3, 4, 6) Latin trade. By applying some permutations on rows and columns, if necessary, we may assume that all cells containing the element 1 form a $4 \times 4$ array minus a transversal $\tau$, which will be labeled $L$. For example in Figure 3 one of the possible positions of 1 is shown. Note that there are 12 cells in $L$ each of which has a 1 in one of the $T_i$’s. In what follows the argument is based only on the assumption that in each of those cells there exists one 1 from one of the $T_i$’s. (● means the cell is empty.)

![Figure 3: Positions of 1 in $T = (T_1, T_2, T_3)$](image)

In the first stage we show that the cells of $\tau$ in $T$ are empty. Suppose without loss of generality the cell $T_{14}$ in $\tau$ is not empty. Then $T_{54}$ and $T_{64}$ must be empty. Thus at least 4 cells of {$T_{51}, T_{52}, T_{53}, T_{61}, T_{62}, T_{63}$} must be filled. Then by pigeonhole principal there exists a column in $T$ with at least 5 filled cells, a contradiction. So all cells of $\tau$ are empty. Therefore exactly 4 cells of {$T_{51}, T_{52}, T_{53}, T_{54}, T_{61}, T_{62}, T_{63}, T_{64}$} are filled, and from $T$ being 4-homogeneous all the cells: {$T_{55}, T_{56}, T_{65}, T_{66}$} are filled.

In the second stage we show that no element, other than 1, appears more than two times in any row or in any column of $L$. For example let us denote by $\{1, x, y, z\}$, the elements which appear in the first row and without loss of generality $T_{15}$ is another filled cell of that row. In contrary, assume that $x$ appears three times in the first row of $L$, i.e. in the cells $T_{11}, T_{12}$, and $T_{13}$. This leaves only two elements $y$ and $z$ to appear in $T_{15}$, which is a contradiction for $T$ being a 3-way Latin trade. So each of the elements other than 1, either does not appear in a row of $L$ or it appears exactly two times in a row of $L$. Now each element other than 1 if it appears in $L$, it occupies 4, 6, or 8 cells.
In the third stage we show that no element occupies 6 or 8 cells of $L$. If an element, say $u \neq 1$ appears 8 times in $L$, then since $u$ appears 2 times in each row and in each column of $L$, so it appears once in each row of the $[1, \ldots, 4] \times [5, 6]$ block. This means that $u$ appears at least 16 times in $T$, which is a contradiction. If $u \neq 1$ appears 6 times in $L$ then three rows and three columns of $L$ each contains $u$ twice. So without loss of generality one of the following cases happens.

![Figure 4: Positions of u in the fifth and sixth columns of T](image)

In case (a) the fifth column has at least 5 filled cells which is a contradiction. In case (b) there are five columns of $T$ which have $u$ and since each column containing an $u$ will contain 3 of them, so there are at least 15 cells containing $u$ in $T$, which is a contradiction.

Now we have shown that each $u \neq 1$ if it appears in $L$, it appears exactly 4 times. The array $L$ has exactly $36 - 12 = 24$ places for elements different from 1 to occupy while the 5 other elements can fill at most $5 \times 4 = 20$ places, which is a contradiction.

**Proposition 3** There is no $(3, 4, 7)$ Latin trade.

**Proof.** By contradiction. Suppose $T = (T_1, T_2, T_3)$ is a $(3, 4, 7)$ Latin trade. By applying some permutations on rows and columns, if necessary, we may assume that all cells containing the element 1 form a $4 \times 4$ array minus a transversal $\tau$, which will be labeled $L$. For example in Figure 5 one of the possible positions of 1 is shown. Note that there are 12 cells in $L$ each of which has a 1 in one of the $T_i$’s. In what follows the argument is based only on the assumption that in each of those cells there exists one 1 from one of the $T_i$’s. (• means the cell is empty.)
Figure 5: Positions of 1 in $T = (T_1, T_2, T_3)$

If we focus on the placement of the remaining filled cells in $T$, we see that rows 1 to 4 of $T$ each have one additional filled cell in one of columns 5, 6 or 7. Likewise for columns 1 to 4 of rows 5, 6 or 7. Further, the subsquare defined by the intersection of rows 5, 6, and 7 with columns 5, 6, and 7, can have at most three filled cells in any row or column. Hence it follows that without loss of generality columns 5 has two filled cell in rows 1 to 4 (similarly row 5 has two filled cells in columns 1 to 4) and columns 6 and 7 have one filled cell in rows 1 to 4 (similarly rows 6 and 7 have one filled cell in columns 1 to 4). Thus we may assume cell $(5, 5)$ is empty and one possible distribution of empty cells (one out of 36) is:

![Diagram]

We can assume that the cell $T_{15}$ contains symbols 2, 3, 4. Then the first row must contain only symbols 1, 2, 3, 4, and these are distributed among the four filled cells (in the first row) according to one of three possible ways:

- $123, 124, 134, 234$ (or $123, 134, 124, 234$)
- $124, 134, 123, 234$ (or $124, 123, 134, 234$)
- $134, 124, 123, 234$ (or $134, 123, 124, 234$)

The idea is to label the filled columns with one of these configurations, to label the first row 1234, and then attempt to complete the labeling of the rows and columns as follows:

- each row and column is labeled by 4 elements from \(\{1, \ldots, 7\}\),
- the first 4 rows and first 4 columns contain 1 in its label,
- first row is labeled \(\{1, 2, 3, 4\}\),
- columns with filled cells in the first row are filled as above,
- for any \(i\), the number \(i\) appears in precisely 4 row labels and in precisely 4 column labels,
• if the cell $T_{ij}$ is filled, $A$ is the label of row $i$ and $B$ is the label of row $j$, then $|A \cup B| \leq 5$ (because the cell $T_{ij}$ contains three elements of $A \cap B$).

By applying a depth-first search, we found no solutions (Indeed, we tried all 36 distributions of filled cells and all three configurations in the first row). The search takes a minute with no optimization. So, it is already impossible to distribute elements in rows and columns according to the restrictions of the $(3, 4, 7)$ Latin trade disregarding how the cell symbols are distributed among the three components of the purported Latin trade. Therefore, there is no $(3, 4, 7)$ Latin trade.

At this point we will show the existence of some $(3, k, m)$ Latin trades. For this purpose we will need some small cases. We have found base rows of those Latin trades computationally, sometimes by trial and errors. But we have checked all of them by Algorithm [1].

**Theorem 9** If $k = 6, 8, 10$ and $12$ then there exists a $(3, k, m)$ Latin trade for every $m \geq k$.

**Proof.** We will show for the given $k$, there exist $(\mu, k, l)$ Latin trades for $l$, where $k + 1 \leq l \leq 2k - 1$. Then by Theorem [3] we will get all $m \geq k$ where $k = 6, 8, 10$, and 12.

• $k = 6$.

If $8 \leq m = 2l \leq 10$, by Corollary [2] a $(3, 6, m)$ Latin trade exists.

And the following are the base rows of a $(3, 6, m)$ Latin trade for $m = 7, 9, 11$:

$3-B_6^7 = \{(1, 5, 4)_1, (3, 4, 2)_2, (5, 3, 1)_3, (7, 2, 5)_4, (2, 1, 7)_5, (4, 7, 3)_6\},$
$3-B_6^9 = \{(1, 8, 3)_1, (3, 2, 1)_2, (2, 5, 6)_3, (6, 3, 2)_4, (8, 6, 5)_5, (5, 1, 8)_7\},$
$3-B_{11}^6 = \{(1, 6, 3)_1, (3, 2, 7)_2, (6, 4, 1)_3, (2, 7, 4)_4, (7, 3, 6)_5, (4, 1, 2)_10\}.$

• $k = 8$.

If $10 \leq m = 2l \leq 14$, by Corollary [2] a $(3, 8, m)$ Latin trade exists.

And the following are the base rows of a $(3, 8, m)$ Latin trade for $m = 9, 11, 13, 15$:

$3-B_8^9 = \{(1, 8, 7)_1, (3, 2, 9)_2, (2, 4, 3)_3, (7, 1, 6)_4, (9, 7, 4)_5, (8, 9, 1)_6, (4, 6, 8)_7, (6, 3, 2)_8\},$
$3-B_{11}^8 = \{(1, 5, 4)_1, (3, 2, 9)_2, (2, 4, 3)_3, (7, 1, 6)_4, (9, 7, 4)_5, (8, 9, 1)_6, (4, 6, 8)_7, (6, 3, 2)_8\},$
$3-B_{13}^8 = \{(1, 5, 3)_1, (3, 1, 5)_2, (2, 6, 11)_3, (6, 4, 2)_4, (8, 3, 4)_5, (4, 8, 6)_6, (11, 2, 8)_7, (5, 11, 1)_10\},$
$3-B_{15}^8 = \{(1, 11, 4)_1, (3, 2, 6)_2, (2, 4, 3)_3, (6, 7, 2)_4, (8, 3, 7)_5, (4, 8, 1)_6, (11, 6, 8)_7, (7, 1, 11)_12\}.$
• $k = 10.$

If $12 \leq m = 2l \leq 18$, by Corollary 2 a $(3, 10, m)$ Latin trade exists.

And the following are the base rows of a $(3, 10, m)$ Latin trade for $m = 13, 15, 17, 19$:

$3-B_{13}^{10} = \{(1, 11, 6)_{1}, (3, 2, 13)_{2}, (2, 4, 3)_{3}, (6, 8, 7)_{4}, (8, 7, 4)_{5}, (4, 5, 2)_{6}, (11, 3, 8)_{7}, (13, 6, 5)_{8}, (5, 1, 11)_{9}, (7, 13, 1)_{10}\},$

$3-B_{15}^{10} = \{(1, 6, 5)_{1}, (3, 2, 4)_{2}, (2, 4, 14)_{3}, (6, 8, 3)_{4}, (8, 1, 2)_{5}, (4, 3, 6)_{6}, (11, 5, 8)_{7}, (5, 7, 11)_{8}, (14, 11, 7)_{9}, (7, 14, 1)_{11}\},$

$3-B_{17}^{10} = \{(1, 6, 4)_{1}, (3, 2, 6)_{2}, (2, 7, 14)_{3}, (6, 1, 2)_{4}, (8, 4, 5)_{5}, (4, 8, 3)_{6}, (11, 5, 8)_{7}, (5, 11, 7)_{8}, (14, 3, 11)_{9}, (7, 14, 1)_{13}\},$

$3-B_{19}^{10} = \{(1, 6, 2)_{1}, (3, 2, 6)_{2}, (2, 4, 14)_{3}, (6, 8, 7)_{4}, (8, 7, 3)_{5}, (4, 3, 5)_{6}, (11, 5, 4)_{7}, (5, 11, 8)_{8}, (14, 1, 11)_{9}, (7, 14, 1)_{15}\}.$

• $k = 12.$

If $14 \leq m = 2l \leq 22$ or $m = 15, 21$, by Corollary 2 a $(3, 12, m)$ Latin trade exists.

And the following are the base rows of a $(3, 12, m)$ Latin trade for $m = 17, 19, 23$:

$3-B_{17}^{12} = \{(1, 16, 4)_{1}, (3, 7, 2)_{2}, (2, 4, 14)_{3}, (6, 8, 3)_{4}, (8, 5, 11)_{5}, (4, 3, 10)_{6}, (11, 1, 8)_{7}, (5, 14, 6)_{8}, (14, 11, 5)_{9}, (16, 6, 7)_{10}, (7, 10, 16)_{11}, (10, 2, 1)_{16}\},$

$3-B_{19}^{12} = \{(1, 16, 7)_{1}, (3, 2, 6)_{2}, (2, 4, 3)_{3}, (6, 9, 1)_{4}, (8, 7, 4)_{5}, (4, 3, 11)_{6}, (11, 5, 2)_{7}, (5, 11, 9)_{8}, (14, 8, 5)_{9}, (16, 14, 8)_{10}, (7, 6, 14)_{11}, (9, 1, 16)_{14}\},$

$3-B_{23}^{12} = \{(1, 7, 5)_{1}, (3, 2, 8)_{2}, (2, 4, 1)_{3}, (6, 9, 3)_{4}, (8, 1, 7)_{5}, (4, 3, 9)_{6}, (11, 5, 2)_{7}, (5, 11, 4)_{8}, (14, 8, 6)_{9}, (16, 14, 11)_{10}, (7, 6, 16)_{11}, (9, 16, 14)_{14}\}.$

3.2 Small odd $k$

**Proposition 4** There exists a $(3, 5, m)$ Latin trade for every $m \geq 5$, except possibly $m = 6$.

**Proof.** By Lemma 1 there exists a $(3, 5, 5)$ Latin trade. The following are the base rows of a $(3, 5, m)$ Latin trade for $m = 7, 8, 9, 11$:

$3-B_{7}^{5} = \{(1, 3, 2)_{1}, (3, 2, 5)_{2}, (5, 7, 3)_{3}, (7, 5, 1)_{4}, (2, 1, 7)_{5}\},$

$3-B_{8}^{5} = \{(1, 6, 2)_{1}, (3, 2, 4)_{2}, (2, 4, 3)_{3}, (6, 3, 1)_{4}, (4, 1, 6)_{7}\},$

$3-B_{9}^{5} = \{(1, 4, 3)_{1}, (4, 3, 8)_{2}, (7, 1, 4)_{3}, (3, 8, 7)_{6}, (8, 7, 1)_{7}\},$

$3-B_{11}^{5} = \{(1, 6, 9)_{1}, (9, 2, 11)_{5}, (11, 1, 6)_{6}, (2, 11, 1)_{7}, (6, 9, 2)_{9}\}.$

By Theorem 7 a $(3, 5, 10)$ Latin trade exists. So a $(3, 5, m)$ Latin trade exists for 5 consecutive values $m \in \{7, 8, \ldots, 11\}$. Thus a $(3, 5, m)$ Latin trade exists for all $m \geq 7$ by Theorem 7.
**Theorem 10** If $k = 7, 9, 11$ and 13 then there exists a $(3, k, m)$ Latin trade for every $m \geq k$.

**Proof.** We introduce the following base rows:

- $k = 7$.
  \[
  m \geq 8: \quad 3-B^7_m = \{(1, 4, 2)_1, (3, 1, 4)_2, (2, 3, 6)_3, (6, 5, 1)_4, (8, 2, 3)_5, (4, 8, 5)_6, (5, 6, 8)_8\}.
  \]
- $k = 9$.
  \[
  m = 10: \quad 3-B^9_10 = \{(1, 7, 9)_1, (3, 2, 8)_2, (2, 4, 5)_3, (7, 6, 4)_4, (9, 3, 2)_5, (8, 9, 7)_6, (4, 1, 6)_7, (6, 5, 1)_8, (5, 8, 3)_9\}.
  \]
  \[
  m \geq 11: \quad 3-B^9_m = \{(1, 5, 4)_1, (3, 4, 6)_2, (2, 3, 1)_3, (6, 2, 5)_4, (8, 1, 2)_5, (4, 7, 8)_6, (11, 6, 3)_7, (5, 11, 7)_8, (7, 8, 11)_11\}.
  \]
- $k = 11$.
  \[
  m \geq 11: \quad 3-B^{11}_m = \{(6, 1, 2)_1, (1, 7, 4)_2, (7, 2, 1)_3, (2, 8, 7)_4, (8, 3, 10)_5, (3, 9, 5)_6, (9, 4, 11)_7, (4, 10, 3)_8, (10, 5, 9)_9, (5, 11, 6)_10, (11, 6, 8)_11\}.
  \]
- $k = 13$.
  \[
  m \geq 13: \quad 3-B^{13}_m = \{(7, 1, 2)_1, (1, 8, 4)_2, (8, 2, 1)_3, (2, 9, 3)_4, (9, 3, 8)_5, (3, 10, 11)_6, (10, 4, 13)_7, (4, 11, 12)_8, (11, 5, 6)_9, (5, 12, 10)_10, (12, 6, 5)_11, (6, 13, 7)_12, (13, 7, 9)_13\}.
  \]

**Theorem 11** If $k = 15$ and $m \geq 15$ then there exists a $(3, 15, m)$ Latin trade.

**Proof.** By Lemma 1 Theorem 3 and Corollary 2 we have a $(3, 15, m)$ Latin trade for $m = 15, 16, 18$ and 20. The following is a base row of a $(3, 15, m)$ Latin trade for $m \geq 21$:

\[
3-B^{15}_m = \{(1, 5, 4)_1, (3, 1, 2)_2, (2, 9, 11)_3, (6, 11, 3)_4, (8, 6, 7)_5, (4, 14, 10)_6, (11, 4, 8)_7, (5, 3, 6)_8, (14, 7, 5)_9, (16, 10, 1)_10, (7, 2, 16)_11, (19, 8, 9)_12, (21, 16, 19)_13, (9, 19, 21)_14, (10, 21, 14)_19\}
\]

The following are the base rows of a $(3, 15, m)$ Latin trade for $m = 17, 19$:

\[
3-B^{15}_{17} = \{(5, 2, 12)_3, (7, 15, 11)_4, (9, 17, 4)_5, (11, 13, 14)_6, (13, 16, 5)_7, (15, 11, 13)_8, (17, 14, 6)_9, (2, 12, 16)_10, (4, 9, 7)_11, (6, 8, 15)_12, (8, 10, 17)_13, (10, 7, 8)_14, (12, 6, 10)_15, (14, 5, 9)_16, (16, 4, 2)_17\},
\]

\[
3-B^{15}_{19} = \{(1, 2, 11)_1, (3, 4, 2)_2, (5, 17, 4)_3, (7, 10, 9)_4, (9, 15, 14)_5, (11, 9, 13)_6, (13, 19, 10)_7, (15, 13, 8)_8, (17, 6, 1)_9, (19, 14, 3)_10, (2, 11, 15)_11, (4, 1, 7)_12, (6, 3, 19)_13, (10, 7, 17)_15, (14, 5, 6)_17\}.\]
3.3 General cases

**Theorem 12** Let \( m \equiv 1 \pmod{6} \) and \( m \geq 7 \). Then there exists a \((3, m - 2, m)\) Latin trade.

**Proof.** The following is a base row of a \((2, m - 2, m)\) Latin trade:

\[
2 - B_{m-2}^m = \bigcup_{i=0}^{(m-13)/6} \{(6i + 2, 6i + 3)_{3i+1}, (6i + 4, 6i + 2)_{3i+2}, (6i + 3, 6i + 4)_{3i+3},
\]

\[
(6i+5, 6i+6)_{(m+3)/2+3i+1}, (6i+7, 6i+5)_{(m+3)/2+3i+2}, (6i+6, 6i+7)_{(m+3)/2+3i+3}\}
\]

\[
\bigcup\{(m - 5, m - 4)_{(m-7)/2+1}, (m - 2, m - 5)_{(m-7)/2+2}, (m - 4, m)_{(m-7)/2+3},
\]

\[
(1, m - 2)_{(m-7)/2+4}, (m, 1)_{(m-7)/2+5}\}.
\]

Now, for \( 1 \leq i \leq m - 2 \) we put \( 2i - 1 \pmod{m} \) in \( i \)-th cell of \( 2 - B_{m-2}^m \), as a result we obtain a base row of a \((3, m - 2, m)\) Latin trade.

**Example 1** As an example of the previous theorem, the following is a base row of a \((3, 11, 13)\) Latin trade:

\[
3 - B_{13}^{11} = \{(1, 2, 3)_1, (3, 4, 2)_2, (5, 3, 4)_3, (7, 8, 9)_4, (9, 11, 8)_5, (11, 9, 13)_6, (13, 1, 11)_7,
\]

\[
(2, 13, 1)_8, (4, 5, 6)_9, (6, 7, 5)_{10}, (8, 6, 7)_{11}\}.
\]

**Theorem 13** For every \( m = 5l \) and \( 4 \leq k \leq m, l \neq 6 \), there exists a \((3, k, m)\) Latin trade.

**Proof.** The theorem trivially holds for \( l = 1 \). If \( l = 2 \), then by Theorem 3, Theorem 7, Theorem 10 and Theorem 9, we can construct a \((3, k, 10)\) Latin trade for every \( 4 \leq k \leq 10 \). By Theorems 9 and 10 there exists a \((3, k, m)\) Latin trade for \( k = 6, 7 \) and 11, so suppose that \( k \neq 6, 7 \) and 11.

We may also assume that \( m > k \).

We have the following cases to consider, each case follows from Theorem 2:

- \( k = 5l' \).
  
  We set \( k_i = 5 \) for \( 1 \leq i \leq l' \) and \( k_i = 0 \) for \( l' + 1 \leq i \leq l \) and \( p = 5 \).

- \( k = 5l' + 1 \).
  
  We set \( k_i = 5 \) for \( 1 \leq i \leq l' - 3 \) and \( k_i = 4 \) for \( l' - 2 \leq i \leq l' + 1 \) and \( k_i = 0 \) for \( l' + 2 \leq i \leq l \) and \( p = 5 \).

- \( k = 5l' + 2 \).
  
  We set \( k_i = 5 \) for \( 1 \leq i \leq l' - 2 \) and \( k_i = 4 \) for \( l' - 1 \leq i \leq l' + 1 \) and \( k_i = 0 \) for \( l' + 2 \leq i \leq l \) and \( p = 5 \).

- \( k = 5l' + 3 \).
  
  We set \( k_i = 5 \) for \( 1 \leq i \leq l' - 1 \), \( k_{l' - 1} = k_{l' + 1} = 4 \), and \( k_i = 0 \) for \( l' + 2 \leq i \leq l \), and \( p = 5 \).
• $k = 5l' + 4$.
  We set $k_i = 5$ for $1 \leq i \leq l'$, $k_{l'+1} = 4$, and $k_i = 0$ for $l' + 2 \leq i \leq l$, and $p = 5$.

**Theorem 14**  For every $m = 7l$ and $5 \leq k \leq m$, $l \neq 6$, there exists a $(3, k, m)$ Latin trade.

**Proof.** The theorem trivially holds for $l = 1$. If $l = 2$, then by Theorem 7, Theorem 10 and Theorem 9, we can construct a $(3, k, 14)$ Latin trade for every $5 \leq k \leq 14$. For $l \neq 2, 6$ by Theorems 9 and 10 there exists a $(3, k, m)$ Latin trade for $k = 8, 9$, so suppose that $k \neq 8, 9$.

We may also assume that $m > k$.

We have the following cases to consider, each case follows from Theorem 2:

• $k = 7l'$.
  We set $k_i = 7$ for $1 \leq i \leq l'$ and $k_i = 0$ for $l' + 1 \leq i \leq l$ and $p = 7$.

• $k = 7l' + 1$.
  We set $k_i = 7$ for $1 \leq i \leq l' - 2$ and $k_i = 5$ for $l' - 1 \leq i \leq l' + 1$ and $k_i = 0$ for $l' + 2 \leq i \leq l$ and $p = 7$.

• $k = 7l' + 2$.
  We set $k_i = 7$ for $1 \leq i \leq l' - 2$ and $k_{l'-1} = k_{l'} = 5$, $k_{l'+1} = 6$ and $k_i = 0$ for $l' + 2 \leq i \leq l$ and $p = 7$.

• $k = 7l' + 3$.
  We set $k_i = 7$ for $1 \leq i \leq l' - 1$, $k_{l'} = k_{l'+1} = 5$, and $k_i = 0$ for $l' + 2 \leq i \leq l$, and $p = 7$.

• $k = 7l' + 4$.
  We set $k_i = 7$ for $1 \leq i \leq l' - 1$, $k_{l'} = 6$, $k_{l'+1} = 5$ and $k_i = 0$ for $l' + 2 \leq i \leq l$, and $p = 7$.

• $k = 7l' + 5$.
  We set $k_i = 7$ for $1 \leq i \leq l'$, $k_{l'+1} = 5$, and $k_i = 0$ for $l' + 2 \leq i \leq l$, and $p = 7$.

• $k = 7l' + 6$.
  We set $k_i = 7$ for $1 \leq i \leq l'$, $k_{l'+1} = 6$, and $k_i = 0$ for $l' + 2 \leq i \leq l$, and $p = 7$.

Now by the results given above we have proved Theorem 1, given at the end of the Introduction.

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References

[1] Behrooz Bagheri Gh. and E. S. Mahmoodian. On the existence of \( k \)-homogeneous Latin bitrades. *Util. Math.*, 85:333–345, 2011.

[2] Richard Bean, Hoda Bidkhori, Maryam Khosravi, and E. S. Mahmoodian. \( k \)-homogeneous Latin trades. *Bayreuth. Math. Schr.*, 74:7–18, 2005.

[3] Elizabeth J. Billington. Combinatorial trades: a survey of recent results. In *Designs, 2002*, volume 563 of *Math. Appl.*, pages 47–67. Kluwer Acad. Publ., Boston, MA, 2003.

[4] Nicholas Cavenagh, Diane Donovan, and Aleš Drápal. 3-homogeneous Latin trades. *Discrete Math.*, 300(1-3):57–70, 2005.

[5] Nicholas Cavenagh, Diane Donovan, and Aleš Drápal. 4-homogeneous Latin trades. *Australas. J. Combin.*, 32:285–303, 2005.

[6] Nicholas J. Cavenagh. The theory and application of Latin bitrades: a survey. *Math. Slovaca*, 58(6):691–718, 2008.

[7] Nicholas J. Cavenagh and Ian M. Wanless. On the number of transversals in Cayley tables of cyclic groups. *Discrete Appl. Math.*, 158(2):136–146, 2010.

[8] Charles J. Colbourn and Jeffrey H. Dinitz, editors. *Handbook of combinatorial designs*. Discrete Mathematics and its Applications (Boca Raton). Chapman & Hall/CRC, Boca Raton, FL, second edition, 2007.

[9] A. D. Keedwell. Critical sets in Latin squares and related matters: an update. *Util. Math.*, 65:97–131, 2004.

[10] C. C. Lindner and C. A. Rodger. *Design theory*. CRC Press LLC, 1997.