Fine-grained complexity of graph homomorphism problem for bounded-treewidth graphs

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Abstract

For graphs $G$ and $H$, a homomorphism from $G$ to $H$ is an edge-preserving mapping from the vertex set of $G$ to the vertex set of $H$. For a fixed graph $H$, by $\text{Hom}(H)$ we denote the computational problem which asks whether a given graph $G$ admits a homomorphism to $H$. If $H$ is a complete graph with $k$ vertices, then $\text{Hom}(H)$ is equivalent to the $k$-COLORING problem, so graph homomorphisms can be seen as generalizations of colorings. It is known that $\text{Hom}(H)$ is polynomial-time solvable if $H$ is bipartite or has a vertex with a loop, and NP-complete otherwise [Hell and Nešetřil, JCTB 1990].

In this paper we are interested in the complexity of the problem, parameterized by the treewidth of the input graph $G$. If $G$ has $n$ vertices and is given along with its tree decomposition of width $\text{tw}(G)$, then the problem can be solved in time $|V(H)|^{\text{tw}(G)} \cdot n^{O(1)}$, using a straightforward dynamic programming. We explore whether this bound can be improved. We show that if $H$ is a projective core, then the existence of such a faster algorithm is unlikely: assuming the Strong Exponential Time Hypothesis (SETH), the $\text{Hom}(H)$ problem cannot be solved in time $(|V(H)| - \varepsilon)^{\text{tw}(G)} \cdot n^{O(1)}$, for any $\varepsilon > 0$. This result provides a full complexity characterization for a large class of graphs $H$, as almost all graphs are projective cores.

We also notice that the naive algorithm can be improved for some graphs $H$, and show a complexity classification for all graphs $H$, assuming two conjectures from algebraic graph theory. In particular, there are no known graphs $H$ which are not covered by our result.

In order to prove our results, we bring together some tools and techniques from algebra and from fine-grained complexity.

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1 Introduction

Many problems that are intractable for general graphs become significantly easier if the structure of the input instance is “simple”. One of the most successful measures of such a structural simplicity is the treewidth of a graph, whose notion was rediscovered by many authors in different contexts [3, 22, 39, 1]. Most classic NP-hard problems, including Independent Set, Dominating Set, Hamiltonian Cycle, or Coloring, can be solved in time \( f(\text{tw}(G)) \cdot n^{O(1)} \), where \( \text{tw}(G) \) is the treewidth of the input graph \( G \) and \( n \) is the number of its vertices [2, 7, 11, 13]. In other words, many problems become polynomially solvable for graphs with bounded treewidth.

In past few years the notion of fine-grained complexity gained popularity, and the researchers became interested in understanding what is the optimal dependence on the treewidth, i.e., the function \( f \) in the complexity of algorithms solving particular problems. This lead to many interesting algorithmic results and lower bounds [43, 6, 29, 33, 38, 13]. Note that the usual assumption that \( P \neq NP \) is not strong enough to obtain tight bounds for the running times of algorithms. In the negative results we usually assume the Exponential Time Hypothesis (ETH), or the Strong Exponential Time Hypothesis (SETH) [27, 28]. Informally speaking, the ETH asserts that 3-Sat with \( n \) variables and \( m \) clauses cannot be solved in time \( 2^{o(n+m)} \), while the SETH implies that CNF-SAT with \( n \) variables and \( m \) clauses cannot be solved in time \( (2 - \varepsilon)^n \cdot n^{O(1)} \), for any \( \varepsilon > 0 \).

For example, it is known that for every fixed \( k \), the \( k \)-COLORING problem can be solved in time \( k^{\text{tw}(G)} \cdot n^{O(1)} \), if a tree decomposition of \( G \) of width \( \text{tw}(G) \) is given. On the other hand, Lokshtanov, Marx, and Saurabh showed that this results is essentially optimal, assuming the SETH.

Theorem 1 (Lokshtanov, Marx, Saurabh [34]). Let \( k \geq 3 \) be a fixed integer. Assuming the SETH, the \( k \)-COLORING problem on a graph \( G \) with \( n \) vertices cannot be solved in time \( (k - \varepsilon)^{\text{tw}(G)} \cdot c \cdot n^d \) for any \( \varepsilon > 0 \) and any constants \( c, d \).

Homomorphisms. For two graphs \( G \) and \( H \), a homomorphism is an edge-preserving mapping from \( V(G) \) to \( V(H) \). The graph \( H \) is called the target of the homomorphism. The existence of a homomorphism from any graph \( G \) to the complete graph \( K_k \) is equivalent to the existence of a \( k \)-coloring of \( G \). Because of that we often refer to a homomorphism to \( H \) as an \( H \)-coloring and think of vertices of \( H \) as colors. We also say that a graph \( G \) is \( H \)-colorable if it admits a homomorphism to \( H \). For a fixed graph \( H \), by \( \text{Hom}(H) \) we denote the computational problem which asks whether a given instance graph \( G \) admits a homomorphism to \( H \). Clearly \( \text{Hom}(K_k) \) is equivalent to \( k \)-COLORING.

Since \( k \)-COLORING is one of the best studied computational problems, it is interesting to investigate how these results generalize to \( \text{Hom}(H) \) for non-complete targets \( H \). For example, it is known that \( k \)-COLORING is polynomial-time solvable for \( k \leq 2 \), and NP-complete otherwise. A celebrated result by Hell and Nešetřil [24] states that \( \text{Hom}(H) \) is polynomially solvable if \( H \) is bipartite or has a vertex with a loop, and otherwise is NP-complete. The polynomial part of the theorem is straightforward and the main contribution was to prove hardness for all non-bipartite graphs \( H \). The difficulty comes from the fact that the local structure of the graph \( H \) is not very helpful, but we need to consider \( H \) as a whole. This is the reason why the proof of Hell and Nešetřil uses a combination of combinatorial and algebraic arguments. Several alternative proofs of the result have appeared [10, 41], but none of them is purely combinatorial.

If it comes the to the running times of algorithms for \( k \)-COLORING, it is well-known that the trivial \( k^n \cdot n^{O(1)} \) algorithm for \( k \)-COLORING can be improved to \( c^n \cdot n^{O(1)} \) for a constant \( c \), which does not depend on \( k \) (currently best algorithm of this type has running time \( 2^n \cdot n^{O(1)} \) [4]). Analogously, we can ask whether the trivial \( |H|^n \cdot n^{O(1)} \) algorithm for \( \text{Hom}(H) \) can be improved. There were several algorithms
with running times $c(H)^n \cdot n^{O(1)}$, where $c(H)$ is some structural parameter of $H$, that could be much smaller than $|H|$ [20, 44, 40]. However, the question whether there exists an absolute constant $c$, such that for every $H$ the $\text{Hom}(H)$ problem can be solved in time $c^n \cdot n^{O(1)}$, remained open. Finally, it was answered in the negative by Cygan et al. [12], who proved that the $|H|^n \cdot n^{O(1)}$ algorithm is essentially optimal, assuming the ETH.

Using a standard dynamic programming approach, $\text{Hom}(H)$ can be solved in time $|H|^t \cdot n^{O(1)}$, if an input graph is given along with its tree decomposition of width $t$ [5, 13]. Theorem 1 asserts that this algorithm is optimal if $H$ is a complete graph with at least 3 vertices, unless the ETH fails. A natural extension of this result would be to provide analogous tight bounds for non-complete targets $H$.

Egri, Marx, and Rzążewski [15] considered this problem in the setting list homomorphisms. Let $H$ be a fixed graph. The input of the LHOM($H$) problem consists of a graph $G$, whose every vertex is equipped with a list of vertices of the target $H$. We ask if $G$ has a homomorphism to $H$, respecting the lists. Egri et al. provided a full complexity classification for the case if $H$ is reflexive, i.e., every vertex has a loop. It is perhaps worth mentioning that the P / NP-complete dichotomy for LHOM($H$) was first proved for reflexive graphs as well: If $H$ is a reflexive graph, then the LHOM($H$) problem is polynomial time-solvable if $H$ is an interval graph, and NP-complete otherwise [17]. Egri et al. defined a new graph invariant $i^*(H)$, based on incomparable sets of vertices and a new graph decomposition, and proved the following.

**Theorem 2 (Egri, Marx, Rzążewski [15]).** Let $H$ be a fixed non-interval reflexive graph with $i^*(H) = k$. Let $n$ and $t$ be, respectively, the number of vertices and the treewidth of an instance graph $G$.
(a) Assuming a tree decomposition of $G$ of width $t$ is given, the LHOM($H$) problem can be solved in time $k^t \cdot c \cdot n^d$, for some constants $c, d$.
(b) There is no algorithm solving LHOM($H$) in time $(k - \varepsilon)^t \cdot c \cdot n^d$ for any $\varepsilon > 0$, and any constants $c, d$, unless the SETH fails.

In this paper we are interested in showing tight complexity bounds for the complexity of the non-list variant of the problem. Let us point out that despite the obvious similarity of Hom($H$) and LHOM($H$) problems, they behave very differently when it comes to showing hardness results. Note that if $H'$ is an induced subgraph of $H$, then any instance of LHOM($H'$) is also an instance of LHOM($H$), where the vertices of $V(H) \setminus V(H')$ do not appear in any list. Thus in order to prove hardness of LHOM($H$), it is sufficient to find a "hard part" $H'$ of $H$, and perform a reduction for the LHOM($H'$) problem. The complexity dichotomy for LHOM($H$) was proven exactly along these lines [17, 18, 19]. Also the proof of Theorem 2 (b) heavily uses the fact that we can work with some local subgraphs of $H$ and ignore the rest of vertices. In particular, all these proofs are purely combinatorial.

On the other hand, in Hom($H$) problem we need to capture the structure of the whole graph $H$, which is difficult using only combinatorial tools. This is why typical tools used in this area come from abstract algebra and algebraic graph theory.

For more information about graph homomorphisms we refer the reader to the comprehensive monograph by Hell and Nešetřil [26].

**Our contribution.** It is well known that in the study of graph homomorphisms the crucial role is played by the graphs that are cores, i.e., they do not have a homomorphism to any of its proper subgraphs. In particular, in order to provide a complete complexity classification of Hom($H$), it is sufficient to consider the case that $H$ is a core (we explain this in more detail in Section 2.1). Also, the complexity dichotomy by Hell and Nešetřil [24] implies that Hom($H$) is polynomial-time solvable if $H$ is a core on at most two
vertices. So from now on let us assume that $H$ is a fixed core, which is non-trivial, i.e., has at least three vertices.

We split the analysis into two cases, depending on the structure of $H$. First, in Section 4.1, we consider targets $H$, that are projective (the definition of this class is rather technical, so we postpone it to Section 2.2). We show that for projective cores the straightforward dynamic programming on a tree decomposition is optimal, assuming the SETH.

**Theorem 3.** Let $H$ be a fixed non-trivial projective core on $k$ vertices, and let $n$ and $t$ be, respectively, the number of vertices and the treewidth of an instance graph $G$.

(a) Assuming a tree decomposition of $G$ of width $t$ is given, the $\text{Hom}(H)$ problem can be solved in time $k^t \cdot c \cdot n^d$, for some constants $c, d$.

(b) There is no algorithm solving $\text{Hom}(H)$ in time $(k - \varepsilon)^t \cdot c \cdot n^d$ for any $\varepsilon > 0$, and any constants $c, d$, unless the SETH fails.

The proof brings together some tools and ideas from algebra and fine-grained complexity theory. The main technical ingredient is the construction of a so-called edge gadget, i.e., a graph $F$ with two specified vertices $u^*$ and $v^*$, such that:

(a) for any distinct vertices $x, y$ of $H$, there is a homomorphism from $F$ to $H$, which maps $u^*$ to $x$ and $v^*$ to $y$, and

(b) in any homomorphism from $F$ to $H$, the vertices $u^*$ and $v^*$ are mapped to distinct vertices of $H$.

Using this gadget, we can perform a simple and elegant reduction from $k$-COLORING. If $G$ is an instance of $k$-COLORING, we construct an instance $G^*$ of $\text{Hom}(H)$ by taking a copy of $G$ and replacing each edge $xy$ with a copy of the edge gadget, whose $u^*$-vertex is identified with $x$, and $v^*$-vertex is identified with $y$. By the properties of the edge gadget it is straightforward to observe that $G^*$ is $H$-colorable if and only if $G$ is $k$-colorable. Since the size of $F$ depends only on $H$, we observe that the treewidth of $G^*$ differs from the treewidth of $G$ by an additive constant, which is sufficient to obtain the desired lower bound.

Although the statement of Theorem 3 might seem quite specific, it actually covers a large class of graphs. Indeed, Hell and Nešetřil observed that almost all graphs are cores [25], see also [26, Corollary 3.28]. Moreover, Łuczak and Nešetřil proved that almost all graphs are projective [35]. From these two results, we can easily obtain that almost all graphs are projective cores. This, combined by Theorem 3, implies the following.

**Corollary 4.** For almost all graphs $H$, the $\text{Hom}(H)$ problem on instance graphs with $n$ vertices and treewidth $t$ cannot be solved in time $(|H| - \varepsilon)^t \cdot c \cdot n^d$ for any $\varepsilon > 0$, and any constants $c, d$, unless the SETH fails.

In Section 4.2 we consider the case that $H$ is a non-projective core. First, we show that the approach that we used for projective cores cannot work in this case: it appears that one can construct the edge gadget for a core $H$ with the properties listed above if and only if $H$ is projective. What makes studying non-projective cores difficult is that we do not understand their structure well. In particular, we know that a graph $H = H_1 \times H_2$, where $H_1$ and $H_2$ are non-trivial and $\times$ denotes the direct product of graphs (see Section 2.2 for a formal definition), is non-projective, and by choosing $H_1$ and $H_2$ appropriately, we can ensure that $H$ is a core. However, we do not know whether there are any non-projective non-trivial connected cores that are indecomposable, i.e., they cannot be constructed using direct products. This problem was studied in a slightly more general setting by Larose and Tardif [32, Problem 2], and it remains wide open. We restate it here, only for restricted case that $H$ is a core, which sufficient for our purpose.

**Conjecture 1.** Let $H$ be a connected non-trivial core. Then $H$ is projective if and only if it is indecomposable.
Since we do not know any counterexample to Conjecture 1, in the remainder we consider cores \( H \) that are built using the direct product. We show a lower complexity bound for \( \text{Hom}(H_1 \times \ldots \times H_m) \), where each \( H_i \) is indecomposable, under an additional assumption that one of the factors of \( H \) is truly projective.

The definition of truly projective graphs is rather technical and we present it in Section 4.2. Graphs with such a property (actually, a slightly more restrictive one) were studied by Larose [30, Problems 1b. and 1b’.] in connection with some problems related to unique colorings, considered by Greenwell and Lovász [21]. Larose [30, 31] defined and investigated even more restricted class of graphs, called strongly projective (see Section 5 for the definition). We know that every strongly projective graph is truly projective, and every truly projective graph is projective. Larose [30, 31] proved that all known projective graphs are in fact strongly projective. This raises a natural question whether projectivity and strong projectivity are in fact equivalent [30, 31]. Of course, an affirmative answer to this question would in particular mean that all projective cores are truly projective. Again, we state the problem in this weaker form, which is sufficient for our application.

**Conjecture 2.** Every projective core is truly projective.

Actually, it appears that if we assume both Conjecture 1 and Conjecture 2, we are able to provide a full complexity classification for the \( \text{Hom}(H) \) problem, parameterized by the treewidth of the input graph.

**Theorem 5.** Assume that Conjecture 1 and Conjecture 2 hold. Let \( H \) be a fixed non-trivial connected core with prime decomposition \( H_1 \times \ldots \times H_m \), and define \( k := \max_{i \in [m]} |H_i| \). Let \( n \) and \( t \) be, respectively, the number of vertices and the treewidth of an instance graph \( G \).

(a) Assuming a tree decomposition of \( G \) of width \( t \) is given, the \( \text{Hom}(H) \) problem can be solved in time \( k^t \cdot c \cdot n^d \), where \( c, d \) are constants.

(b) There is no algorithm solving \( \text{Hom}(H) \) in time \( (k - \varepsilon)^t \cdot c \cdot n^d \) for any \( \varepsilon > 0 \) and any constants \( c, d \), unless the SETH fails.

Let us point out that despite some work on both conjectures by members of graph homomorphisms community [32, 30, 31], we know no graph \( H \), for which the bounds from Theorem 5 do not hold.

## 2 Notation and preliminaries

For \( n \in \mathbb{N} \), by \( [n] \) we denote the set \( \{1, 2, \ldots, n\} \). All graphs considered in this paper are finite, undirected and with no multiple edges. For a graph \( G \), by \( V(G) \) and \( E(G) \) we denote the set of vertices and the set of edges of \( G \), respectively, and we write \( |G| \) for the number of vertices of \( G \). Let \( K^*_1 \) be the single-vertex graph with a loop. A graph is ramified if it has no two distinct vertices \( u \) and \( v \) such that the neighborhood of \( u \) is contained in the neighborhood of \( v \). For a graph \( G \), denote by \( \omega(G) \), \( \chi(G) \), and \( \text{og}(G) \), respectively, the size of the largest clique contained in \( G \), the chromatic number of \( G \), and the odd girth of \( G \).

A tree decomposition of a graph \( G \) is a pair \( \left( \mathcal{T}, \{X_a\}_{a \in V(\mathcal{T})} \right) \), in which \( \mathcal{T} \) is a tree, whose vertices are called nodes and \( \{X_a\}_{a \in V(\mathcal{T})} \) is the family of subsets (called bags) of \( V(G) \), such that

1. every \( v \in V(G) \) belongs to at least one bag \( X_a \),
2. for every \( uv \in E(G) \) there is at least one bag \( X_a \) such that \( u, v \in X_a \),
3. for every \( v \in V(G) \) the set \( T_v := \{a \in V(\mathcal{T}) \mid v \in X_a\} \) induces a connected subgraph of \( \mathcal{T} \).

The width of a tree decomposition \( \left( \mathcal{T}, \{X_a\}_{a \in V(\mathcal{T})} \right) \) is the number \( \max_{a \in V(\mathcal{T})} |X_a| - 1 \). The minimum possible width of a tree decomposition of \( G \) is called the treewidth of \( G \) and denoted by \( \text{tw}(G) \).
2.1 Graph homomorphisms and cores

For graphs $G$ and $H$, a function $f : V(G) \to V(H)$ is a homomorphism, if it preserves edges, i.e., for every $uv \in E(G)$ it holds that $f(u)f(v) \in E(H)$ (see Figure 1). If $G$ admits a homomorphism to $H$, we denote this fact by $G \to H$ and we write $f : G \to H$ if $f$ is a homomorphism from $G$ to $H$. If there is no homomorphism from $G$ to $H$, we write $G \not\to H$. Graphs $G$ and $H$ are homomorphically equivalent if $G \to H$ and $H \to G$, and incomparable if $G \not\to H$ and $H \not\to G$. Observe that homomorphic equivalence is an equivalence relation on the class of all graphs. An endomorphism of $G$ is any homomorphism from $f : G \to G$.

A graph $G$ is a core if $G \not\to H$ for every proper subgraph $H$ of $G$. Equivalently, we can say $G$ is a core if and only if every endomorphism of $G$ is an automorphism. Note that a core is always ramified. If $H$ is a subgraph of $G$ such that $G \to H$ and $H$ is a core, we say that $H$ is a core of $G$. Notice that if $H$ is a subgraph of $G$, then it always holds that $H \to G$, so every graph is homomorphically equivalent to its core. Moreover, if $H$ is a core of $G$, then $H$ is always an induced subgraph of $G$, because every endomorphism $f : G \to H$ restricted to $H$ must be an automorphism. It was observed by Hell and Nešetřil that every graph has a unique core (up to an isomorphism) [25]. Note that if $f : G \to H$ is a homomorphism from $G$ to its core $H$, then it must be surjective.

We say that a core is trivial if it is isomorphic to $K_1$, $K^*_1$, or $K_2$. It is easy to observe that these three graphs are the only cores with fewer than 3 vertices. In general, finding a core of a given graph is computationally hard; in particular, deciding if a graph is a core is coNP-complete [25]. However, the graphs whose cores are trivial are simple to describe.

**Observation 6.** Let $G$ be a graph, whose core $H$ is trivial.
(a) $H \simeq K_1$ if and only if $\chi(G) = 1$, i.e., $G$ has no edges,
(b) $H \simeq K_2$ if and only if $\chi(G) = 2$, i.e., $G$ is bipartite and has at least one edge,
(c) $H \simeq K^*_1$ if and only if $G$ has vertex with a loop. □

In particular, there are no non-trivial cores with loops. The following conditions are necessary for $G$ to have a homomorphism into $H$.

**Observation 7 ([26]).** Assume that $G \to H$ and $G$ and $H$ have no loops. Then $\omega(G) \leq \omega(H)$, $\chi(G) \leq \chi(H)$, and $\text{og}(G) \geq \text{og}(H)$. □

We denote by $H_1 + \ldots + H_m$ a disconnected graph with connected components $H_1, \ldots, H_m$. Observe that if $f$ is a homomorphism from $G = G_1 + \ldots + G_\ell$ to $H = H_1 + \ldots + H_m$, then it maps every connected component of $G$ into some connected component of $H$. Also note that a graph does not have to
be connected to be a core, in particular the following characterization follows directly from the definition of a core.

**Observation 8.** A disconnected graph \( H \) is a core if and only if its connected components are pairwise incomparable cores.

An example of a pair of incomparable cores is shown in Figure 2: it is the Grötzsch graph, denoted by \( G_G \), and the clique \( K_3 \). Clearly, \( \text{og}(G_G) > \text{og}(K_3) \) and \( \chi(G_G) > \chi(K_3) \), so by Observation 7, they are incomparable. Therefore, by Observation 8, the graph \( G_G + K_3 \) is a core.

![Figure 2: An example of incomparable cores, the Grötzsch graph (left) and \( K_3 \) (right).](image)

Finally, let us observe that we can construct arbitrarily large families of pairwise incomparable cores. Let us start the construction with an arbitrary non-trivial core \( H_0 \). Now suppose we have constructed pairwise incomparable cores \( H_0, H_1, \ldots, H_k \) and we want to construct \( H_{k+1} \). Let \( \ell = \max_{i \in \{0, \ldots, k\}} \text{og}(H_i) \) and \( r = \max_{i \in \{0, \ldots, k\}} \chi(H_i) \). By the classic result of Erdős [16], there is a graph \( H \) with \( \text{og}(H) > \ell \) and \( \chi(H) > r \). We set \( H_{k+1} \) to be the core of \( H \). Observe that \( \text{og}(H_{k+1}) = \text{og}(H) > \ell \) and \( \chi(H_{k+1}) = \chi(H) > r \), so, by Observation 7, we have that for every \( i \in \{0, \ldots, k\} \) the core \( H_{k+1} \) is incomparable with \( H_i \).

### 2.2 Graph products

Define the **direct product** of graphs \( H_1 \) and \( H_2 \), denoted by \( H_1 \times H_2 \), as follows: \( V(H_1 \times H_2) = \{(x, y) \mid x \in V(H_1) \text{ and } y \in V(H_2)\} \) and \( E(H_1 \times H_2) = \{(x_1, y_1)(x_2, y_2) \mid x_1x_2 \in E(H_1) \text{ and } y_1y_2 \in E(H_2)\} \). If \( H = H_1 \times H_2 \), then \( H_1 \times H_2 \) is a factorization of \( H \), and \( H_1 \) and \( H_2 \) are its factors. Note that \( H_1 \times H_2 \simeq H \) if and only if \( H_1 \simeq K_1^r \) or \( H_2 \simeq K_1^s \). Clearly, the binary operation \( \times \) is commutative, so will identify \( H_1 \times H_2 \) and \( H_2 \times H_1 \). Since \( \times \) is also associative, we can extend the definition for more than two factors:

\[
H_1 \times \cdots \times H_{m-1} \times H_m := (H_1 \times \cdots \times H_{m-1}) \times H_m.
\]

Moreover, in the next sections, we will sometimes consider products of graphs, that are products themselves. Formally, the vertices of such graphs are tuples of tuples. If it does not lead to confusion, for \( \bar{x} := (x_1, \ldots, x_{k_1}) \) and \( \bar{y} := (y_1, \ldots, y_{k_2}) \), we will treat tuples \((\bar{x}; \bar{y}), (x_1, \ldots, x_{k_1}, y_1, \ldots, y_{k_2}), (\bar{x}, y_1, \ldots, y_{k_2}), (x_1, \ldots, x_{k_1}, \bar{y})\) as equivalent. This notation is generalized to more factors in a natural way. We denote by \( H^m \) the product of \( m \) copies of \( H \).

The direct product appears in the literature under different names: **tensor product, cardinal product, Kronecker product, relational product**. It is also called **categorical product**, because it is the product in the category of graphs (see [23, 37] for details).
We say that a graph $H$ is directly indecomposable (or indecomposable for short) if the fact that $H = H_1 \times H_2$ implies that either $H_1 \cong K^+_1$ or $H_2 \cong K^+_2$. A graph that is not indecomposable, is decomposable. A factorization, where each factor is directly indecomposable and not isomorphic to $K^+_1$, is called a prime factorization. Clearly, $K^+_1$ does not have a prime factorization.

The following property will be very useful (see also Theorem 8.17 in [23]).

**Theorem 9 (McKenzie [36]).** Any connected non-bipartite graph with more than one vertex has a unique prime factorization into directly indecomposable factors (with possible loops).

Let $i \in [m]$ and let $H_1 \times \ldots \times H_m$ be some factorization of $H$ (not necessary prime). A function $\pi_i : V(H) \to V(H_i)$ such that for every $(x_1, \ldots, x_m) \in V(H)$ it holds that $\pi_i(x_1, \ldots, x_m) = x_i$ is a projection on the $i$-th coordinate. It follows from the definition of the direct product that every projection $\pi_i$ is a homomorphism from $H$ to $H_i$.

Below we summarize some basic properties of direct products.

**Observation 10.** Let $H$ be a graph on $k$ vertices. Then

(a) $H \times K_1$ consists of $k$ isolated vertices, in particular its core is $K_1$,

(b) if $H$ has at least one edge, then the core of $H \times K_2$ is $K_2$,

(c) the graph $H^m$ contains a subgraph isomorphic to $H$, induced by the set $\{(x, \ldots, x) \mid x \in V(H)\}$; in particular, if $m \geq 2$, then $H^m$ is never a core,

(d) if $H = H_1 \times \ldots \times H_m$, then for every $G$ it holds that $G \to H$ if and only if $G \to H_i$ for all $i \in [m]$.

**Proof.** Items (a), (b), (c) are straightforward to observe. To prove (d), consider a homomorphism $f : G \to H$. Clearly, $H \to H_i$ for every $i \in [m]$ because each projection $\pi_i : H \to H_i$ is a homomorphism. So $\pi_i \circ f$ is a homomorphism from $G$ to $H_i$. On the other hand, if we have some $f_i : G \to H_i$ for every $i \in [m]$, then we can define a homomorphism $f : G \to H$ by $f(x) := (f_1(x), \ldots, f_m(x))$. \hfill \Box

A homomorphism $f : H^m \to H$ is idempotent, if for every $x \in V(H)$ it holds that $f(x, x, \ldots, x) = x$. One of the main characters of the paper is the class of projective graphs, considered e.g. in [30, 31, 32]. A graph $H$ is projective (or idempotent trivial), if for every $m \geq 2$, every idempotent homomorphism from $H^m$ to $H$ is a projection.

**Observation 11.** If $H$ is a projective core and $f : H^m \to H$ is a homomorphism, then $f \equiv g \circ \pi_i$ for some $i \in [m]$ and some automorphism $g$ of $H$.

**Proof.** If $f$ is idempotent, then it is a projection and we are done. Assume $f$ is not idempotent and define $g : V(H) \to V(H)$ by $g(x) = f(x, \ldots, x)$. The function $g$ is an endomorphism of $H$ and $H$ is a core, so $g$ is in fact an automorphism of $H$. Observe that $g^{-1} \circ f$ is an idempotent homomorphism, so it is equal to $\pi_i$ for some $i \in [m]$, because $H$ is projective. From this we get that $f \equiv g \circ \pi_i$. \hfill \Box

It is known that projective graphs are always connected [32]. Observe that the definition of projective graphs does not imply that their recognition is decidable. However, an algorithm to recognize these graph follows from the following, useful characterization.

**Theorem 12 (Larose, Tardif [32]).** A connected graph $H$ with at least three vertices is projective if and only if every idempotent homomorphism from $H^2$ to $H$ is a projection.
Recall from the introduction that almost all graphs are projective cores \([26, 35]\). It appears that the properties of projectivity and being a core are independent. In particular, the graph in Figure 3 is not a core, as it can be mapped to a triangle. However, Larose \([30]\) proved that all non-bipartite, connected, ramified graphs which do not contain \(C_4\) as a (non-necessarily induced) subgraph, are projective (this will be discussed in more detail in Section 5, see Theorem 25). On the other hand, there are also non-projective cores, an example is \(G \times K_3\), see Figure 2. We discuss such graphs in detail in Section 4.2.

![Figure 3: An example of a projective graph which is not a core.](image)

### 3 Complexity of finding graph homomorphisms

Note that if two graphs \(H_1\) and \(H_2\) are homomorphically equivalent, then the \(\text{Hom}(H_1)\) and \(\text{Hom}(H_2)\) problems are also equivalent. So in particular, because every graph is homomorphically equivalent to its core, we may restrict our attention to graphs \(H\) which are cores. Also, recall from Observation 6 that \(\text{Hom}(H)\) can be solved in polynomial time if \(H\) is isomorphic to \(K_1^*, K_1,\) or \(K_2\). So we will be interested only in non-trivial cores \(H\). In particular, we will assume that \(H\) is non-bipartite and has no loops.

We are interested in understanding the complexity bound of the \(\text{Hom}(H)\) problem, parameterized by the treewidth of the input graph. The standard dynamic programming approach (see for example Cygan et al. \([13]\)) gives us the following upper bound.

**Theorem 13 (Folklore).** Let \(H\) be a fixed graph on \(k\) vertices. Assuming a tree decomposition of width \(t\) of the instance graph on \(n\) vertices is given, the \(\text{Hom}(H)\) problem can be solved in time \(k^t \cdot c \cdot n^d\) for some constants \(c, d\).

By Theorem 1, this bound is tight if \(H\) is a complete graph with at least three vertices, unless the SETH fails. We are interested in extending this result for other graphs \(H\).

First, let us observe that there are cores, for which the bound from Theorem 13 can be improved. Indeed, let \(H\) be a decomposable core, isomorphic to \(H_1 \times \ldots \times H_m\) (see discussion in Section 4.2 for more about cores that are products.). Recall from Observation 10 (d) that for every graph \(G\) it holds that

\[
G \rightarrow H \text{ if and only if } G \rightarrow H_i \text{ for every } i \in [m].
\]

So, given an instance \(G\) of \(\text{Hom}(H)\), we can call the algorithm from Theorem 13 to solve \(\text{Hom}(H_i)\) for each \(i \in [m]\) and return a positive answer if and only if we get a positive answer in each of the calls. This way we obtain the following result.

**Theorem 14.** Let \(H\) be a fixed core with prime factorization \(H_1 \times \ldots \times H_m\). Assuming a tree decomposition of width \(t\) of the instance graph with \(n\) vertices is given, the \(\text{Hom}(H)\) problem can be solved in time \(\max_{j \in [m]} |H_j|^t \cdot c \cdot n^d\) for some constants \(c, d\). \(\square\)

Let us conclude this section with a simple observation about the complexity of \(\text{Hom}(H)\) for disconnected cores \(H\).
Theorem 15. Consider a fixed disconnected core \( H = H_1 + \ldots + H_m \). Consider an instance \( G \) with \( n \) vertices, given along with its tree decomposition of width \( t \).

(a) Assume that for every \( i \in [m] \) the \( \text{Hom}(H_i) \) problem can be solved in time \( \alpha^t \cdot c \cdot n^d \), where \( \alpha, c, d \) are constants. Then \( \text{Hom}(H) \) can be solved in time \( \alpha^t \cdot c' \cdot n^d \) for some constant \( c' \).

(b) Assume that \( \text{Hom}(H) \) can be solved in time \( \alpha^t \cdot c \cdot n^d \), where \( \alpha, c, d \) are constants. Then for every \( i \in [m] \) the \( \text{Hom}(H_i) \) problem can be solved in time \( \alpha^t \cdot c' \cdot n^d \) for some constant \( c', d \).

Proof. First, observe that if \( G \) is disconnected, say \( G = G_1 + \ldots + G_t \), then \( G \rightarrow H \) if and only if \( G_i \rightarrow H \) for every \( i \in [t] \). Also, \( \text{tw}(G) = \max_{i \in [t]} \text{tw}(G_i) \). It means that if the instance graph is disconnected, we can just consider the problem separately for each of its connected components.

So we assume that \( G \) is connected. First, recall from Observation 8 that \( G \rightarrow H \) if and only if \( G \rightarrow H_i \) for some \( i \in [m] \). Again, we can solve \( \text{Hom}(H_i) \) for each \( i \in [m] \) (for the same instance \( G \)) and return a positive answer for \( \text{Hom}(H) \) if and only if we get a positive answer for at least one \( i \in [m] \). The total complexity of this algorithm is \( m \cdot \alpha^t \cdot c \cdot n^d = \alpha^t \cdot c' \cdot n^d \) for \( c' = c \cdot m \). This proves (a).

To see (b), consider an instance \( G \) of \( \text{Hom}(H_i) \) on \( n \) vertices and treewidth \( t \). Let \( V(H_i) = \{ z_1, \ldots, z_k \} \) and let \( u \) be some fixed vertex of \( G \). We construct an instance \( G^* \) of \( \text{Hom}(H) \) as follows. We take a copy \( G' \) of \( G \) and a copy \( H_k \) of \( H_i \), and identify the vertex corresponding to \( u \) in \( G' \) with the vertex corresponding to \( (z_1, \ldots, z_k) \) in \( H_k \). Denote this vertex of \( G^* \) by \( \bar{z} \).

We claim that \( G \rightarrow H_i \) if and only if \( G^* \rightarrow H \). Indeed, if \( f : G \rightarrow H_i \), then there exists \( j \in [k] \) such that \( f(z_j) = z_j \), so we can define a homomorphism \( g : G^* \rightarrow H_i \) (which is also a homomorphism from \( G^* \) to \( H \)) by

\[
g(x) = \begin{cases} f(x) & \text{if } x \in G', \\ \pi_j(x) & \text{otherwise.} \end{cases}
\]

Clearly, both \( f \) and \( \pi_j \) are homomorphisms and \( \bar{z} \) is a cutvertex in \( G^* \) for which \( f(\bar{z}) = \pi_j(\bar{z}) \), so \( g \) is a homomorphism from \( G^* \) to \( H_i \).

Conversely, if we have \( g : G^* \rightarrow H_i \), we know that \( g \) maps \( G^* \) to a connected component \( H_j \) of \( H_i \), for some \( j \in [m] \), because \( G^* \) is connected. But \( G^* \) contains an induced copy \( H_k \) of \( H_i \), so also an induced copy of \( H_i \), say \( \bar{H}_i \) (recall Observation 10 (c)). So \( g|_{V(H_i)} \) is in fact a homomorphism from \( H_i \) to \( H_j \). Recall from Observation 7 that since \( H_1 + \ldots + H_m \) is a core, its connected components are pairwise incomparable cores — so \( j \) must be equal to \( i \). It means that \( g|_{V(G')} \) is a homomorphism from \( G' \) to \( H_i \), so we conclude that \( G \rightarrow H_i \).

Note that the number of vertices of \( G^* \) is \( n + |H_k| - 1 \leq |H_k| \cdot n \). Now let \( (T, \{ X_a \}_{a \in V(T)} \) be a tree decomposition of \( G \) of width \( t \), and let \( b \) be a node of \( T \), such that \( u \in X_b \). Define \( X'_b := X_b \cup V(H_k) \) and let \( V(T^*) = V(T) \cup \{ b' \} \) and \( E(T^*) = E(T) \cup \{ bb' \} \). Clearly, \( (T^*, \{ X_a \}_{a \in V(T^*)} \) is a tree decomposition of \( G^* \). This means that \( \text{tw}(G^*) \leq t + |H_k| \). The graph \( H_i \) is fixed, so the number of vertices of \( H_k \) is a constant. By our assumption we can decide if \( G^* \rightarrow H \) in time \( \alpha^{\text{tw}(G^*)} \cdot c \cdot |G^*|^d \), so we can decide if \( G \rightarrow H_i \) in time \( \alpha^{\text{tw}(G^*)} \cdot c \cdot (|H_k| \cdot n)^d \leq \alpha^{t \cdot |H_k|} \cdot c \cdot |H_k|^d \cdot n^d = \alpha^t \cdot c' \cdot n^d \), where \( c' = c \cdot \alpha |H_k| \cdot |H_k|^d \).

Theorem 15 implies that for our purpose it is sufficient to consider connected cores.

### 4 Lower bounds

In this section we will investigate the lower bounds for the complexity of \( \text{Hom}(H) \). The section is split into two main parts. In Section 4.1 we consider projective cores. Then, in Section 4.2, we consider non-projective cores.
4.1 Projective cores

The main result of this section is Theorem 3.

**Theorem 3.** Let $H$ be a fixed non-trivial projective core on $k$ vertices, and let $n$ and $t$ be, respectively, the number of vertices and the treewidth of an instance graph $G$.

(a) Assuming a tree decomposition of $G$ of width $t$ is given, the $\text{Hom}(H)$ problem can be solved in time $k^t \cdot c \cdot n^d$, for some constants $c, d$.

(b) There is no algorithm solving $\text{Hom}(H)$ in time $(k - \varepsilon)^t \cdot c \cdot n^d$ for any $\varepsilon > 0$, and any constants $c, d$, unless the SETH fails.

Observe that Theorem 3 (a) follows from Theorem 13, so we need to show the hardness counterpart, i.e., the statement (b). A crucial building block in our reduction will be the graph called the edge gadget, whose construction is described in the following lemma.

**Lemma 16.** For every non-trivial projective core $H$, there exists a graph $F$ with two specified vertices $u^*$ and $v^*$, satisfying the following:

(a) for every $x, y \in V(H)$ such that $x \neq y$, there exists a homomorphism $f : F \to H$ such that $f(u^*) = x$ and $f(v^*) = y$,

(b) for every $f : F \to H$ it holds that $f(u^*) \neq f(v^*)$.

**Proof.** Let $V(H) = \{z_1, \ldots, z_k\}$. For $i \in [k]$ denote by $z_i^{k-1}$ the $(k - 1)$-tuple $(z_1, \ldots, z_i)$ and by $\overline{z_i}$ the $(k - 1)$-tuple $(z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_k)$. We claim that $F := H^{(k-1)k}$ and vertices

$$u^* := (z_1^{k-1}, \ldots, z_k^{k-1}) \quad \text{and} \quad v^* := (\overline{z_1}, \ldots, \overline{z_k})$$

satisfy the statement of the lemma. Note that the vertices of $F$ are $k(k - 1)$-tuples.

To see that (a) holds, observe that if $x$ and $y$ are distinct vertices from $V(H)$, then there always exists $i \in [k(k - 1)]$ such that $\pi_i(u^*) = x$ and $\pi_i(v^*) = y$. This means that $\pi_i$ is a homomorphism from $F = H^{k(k-1)}$ to $H$ satisfying $\pi_i(u^*) = x$ and $\pi_i(v^*) = y$.

To prove (b), recall that since $H$ is projective, by Observation 11, the homomorphism $f$ is a composition of some automorphism $g$ of $H$ and $\pi_i$ for some $i \in [k(k - 1)]$. Observe that $u^*$ and $v^*$ are defined in a way such that $\pi_j(u^*) \neq \pi_j(v^*)$ for every $j \in [k(k - 1)]$. As $g$ is an automorphism, it is injective, which gives us $f(u^*) = g(\pi_i(u^*)) \neq g(\pi_i(v^*)) = f(v^*)$.

Finally, we are ready to prove Theorem 3 (b).

**Proof of Theorem 3 (b).** Note that since $H$ is non-trivial, we have $k \geq 3$. Since $H$ is projective, it is also connected. We reduce from $k$-COLORING, let $G$ be an instance with $n$ vertices and treewidth $t$. We construct an instance $G^*$ of $\text{Hom}(H)$ as follows. First, for every $z \in V(G)$ we introduce a vertex $z'$ of $V(G^*)$. Let $V'$ denote the set of these vertices. Now, for every edge $xy$ of $G$, we introduce to $G^*$ a copy of the edge gadget, constructed in Lemma 16, and denote it by $F_{xy}$. We identify the vertices $u^*$ and $v^*$ of $F_{xy}$ with vertices $x'$ and $y'$, respectively. This completes the construction of $G^*$.

We claim that $G$ is $k$-colorable if and only if $G^* \to H$. Indeed, let $\varphi$ be a $k$-coloring of $G$. For simplicity of notation, we label the colors used by $\varphi$ in the same way as the vertices of $H$, i.e., $z_1, z_2, \ldots, z_k$. Define $g : V(V') \to V(H)$ by setting $g(v') := \varphi(v')$ Now consider an edge $xy$ of $G$ and the edge gadget $F_{xy}$. Since $c$ is a proper coloring, we have $g(x') \neq g(y')$. So by Lemma 16 (a), we can find a homomorphism $f_{xy} : F_{xy} \to H$, such that $f_{xy}(x') = g(x')$ and $f_{xy}(y') = g(y')$. Repeating this for every edge gadget, we can extend $g$ to a homomorphism from $G^*$ to $H$.
Conversely, from Lemma 16 (b), we know that for any \( f : G^* \rightarrow H \) and every edge \( xy \) of \( G \) it holds that \( f(x') \neq f(y') \), so any homomorphism from \( G^* \) induces a \( k \)-coloring of \( G \).

The number of vertices of \( G^* \) is at most \( |F| n^2 \). Now let \( \{T, \{X_a\}_{a \in V(T)}\} \) be a tree decomposition of \( G \) of width \( t \). Let us extend it to a tree decomposition of \( G^* \). For each edge \( xy \) of \( G \) there exists a bag \( X_b \) such that \( x, y \in X_b \). We add to \( T \) the node \( b' \) with \( X_{b'} := X_b \cup V(F_{xy}) \), and an edge \( bb' \). It is straightforward to observe that by repeating this step for every edge of \( G \), we obtain a tree decomposition of \( G^* \) of width at most \( t + |F| \). Recall that \( H \) is fixed, so \( |F| \) is a constant. So if we could decide if \( G^* \rightarrow H \) in time \((k - \epsilon)^{tw(G^*)} \cdot c \cdot |G^*|^d \leq (k - \epsilon)^{t+|F|} \cdot c \cdot |F|^d \cdot n^{2d} \), then we would be able to decide if \( G \) is \( k \)-colorable in time \((k - \epsilon)^t \cdot c' \cdot n^{d'} \) for constants \( c' = c \cdot (k - \epsilon)^{|F|} \cdot |F|^d \) and \( d' = 2d \). By Theorem 1 this contradicts the SETH. \( \square \)

4.2 Non-projective cores

Now we will focus on non-trivial connected cores, which are additionally non-projective, i.e., they do not satisfy the assumptions of Theorem 3. First, let us argue that the approach from Section 4.1 cannot work in this case. In particular, we will show that an edge gadget with properties listed in Lemma 16 cannot be constructed for non-projective graphs \( H \).

We will need the definition of constructible sets, see Larose and Tardif [32]. For a graph \( H \), a set \( C \subseteq V(H) \) is constructible if there exists a graph \( K \), vertices \( x_0, \ldots, x_\ell \in V(F) \) and \( y_1, \ldots, y_\ell \in V(H) \) such that

\[
\{ f(x_0) \in V(H) \mid f : K \rightarrow H \text{ such that } f(x_i) = y_i \text{ for every } i \in [\ell] \} = C.
\]

We might think of \( C \) as the set of colors that might appear on the vertex \( x_0 \), when we precolor each \( x_i \) with the color \( y_i \) and try to extend this partial mapping to a homomorphism to \( H \). The tuple \((K, x_0, \ldots, x_\ell, y_1, \ldots, y_\ell)\) is called a construction of \( C \).

It appears that the notion of constructible sets is closely related to projectivity.

**Theorem 17 (Larose, Tardif [32]).** A graph \( H \) on at least three vertices is projective if and only if every subset of its vertices is constructible.

Now we show that Lemma 16 cannot work for non-projective graphs \( H \).

**Proposition 18.** Let \( H \) be a fixed non-trivial connected core. Then an edge gadget \( F \) with properties listed in Lemma 16 exists if and only if \( H \) is projective.

**Proof.** The ‘if’ statement follows from Lemma 16. Let \( k := |H| \) and suppose that there exists a graph \( F \) with properties given in Lemma 16. Consider a set \( C \subseteq V(H) \) and define \( \ell := |C| \). Let \( \{y_1, \ldots, y_{k-\ell}\} \) be the complement of \( C \) in \( V(H) \). Take \( k - \ell \) copies of \( F \), say \( F_1, \ldots, F_{k-\ell} \) and denote the vertices \( u^* \) and \( v^* \) of \( i \)-th copy \( F_i \) by \( u_i^* \) and \( v_i^* \), respectively. Identify the vertices \( u_i^* \) of all these copies, denote the obtained vertex by \( u^* \), and the obtained graph by \( K \). Now set \( x_0 := u^* \) and for each \( i \in [k - \ell] \) set \( x_i := v_i^* \).

It is easy to verify that this is a construction of the set \( C \). Indeed, observe that if \( x \in C \), then, from Lemma 16 (a), for each copy \( F_i \) there exists a homomorphism \( f_i : F_i \rightarrow H \) such that \( f_i(v_i^*) = f_i(x_i) = y_i \) and \( f_i(u_i^*) = f_i(x_0) = x \). Combining these homomorphisms yields a homomorphism \( f : K \rightarrow H \). On the other hand, if \( x \notin C \), then \( x = y_i \) for some \( i \in [k - \ell] \). But from Lemma 16 (b) we know that for every homomorphism \( f : F \rightarrow H \) it holds that \( x = y_i = f(v_i^*) \neq f(u^*) = f(x_0) \), so \( x_0 \) cannot be mapped to \( x \) by any homomorphism from \( K \) to \( H \). \( \square \)
Observe that if \( H \) is projective, then it must be indecomposable. Indeed, assume that for some non-trivial \( H \) it holds that \( H = H_1 \times H_2, H \not\cong K_1^* \) and \( H_2 \not\cong K_1^* \). Consider a homomorphism \( f : (H_1 \times H_2)^2 \rightarrow H_1 \times H_2 \), defined as \( f((x, y), (x', y')) = (x, y') \). Note that it is idempotent, but not a projection, so \( H \) is not projective.

In the light of the observation above, it is natural to ask whether indecomposability implies projectivity. This problem was already stated e.g. by Larose and Tardif [32, Problem 2] and, to the best of our knowledge, no significant progress in this direction was made. Let us recall it here.

**Conjecture 1.** Let \( H \) be a connected non-trivial core. Then \( H \) is projective if and only if it is indecomposable.

Since we know no connected non-trivial non-projective cores that are indecomposable, in the remainder of the section we will assume that \( H \) is a decomposable, non-trivial connected core. By Theorem 9 we know that \( H \) has a unique prime factorization \( H_1 \times \ldots \times H_m \) for some \( m \geq 2 \). To simplify the notation, for any given homomorphism \( f : G \rightarrow H_1 \times \ldots \times H_m \) and \( i \in [m] \), we define \( f_i \equiv \pi_i \circ f \). Then for each vertex \( x \) of \( G \) it holds that \( f(x) = (f_1(x), \ldots, f_m(x)) \), and \( f_i \) is a homomorphism from \( G \) to \( H_i \).

The following observation follows from Observation 10.

**Observation 19.** Let \( H \) be a connected, non-trivial core with factorization \( H = H_1 \times \ldots \times H_m \), such that \( H_i \not\cong K_1^* \) for all \( i \in [m] \). Then for \( i \in [m] \) the graph \( H_i \) is a connected non-trivial core, incomparable with \( H_j \) for \( j \in [m] \setminus \{i\} \).

Now let us consider the complexity of \( \text{Hom}(H) \), where \( H \) has a prime factorization \( H_1 \times H_2 \times \ldots H_m \) for \( m \geq 2 \). By Theorem 14, the problem can be solved in time \( \left( \max_{i \in [m]} |H_i| \right)^t \cdot c \cdot n^d \), where \( n \) and \( t \) are the number of vertices and the treewidth of the input graph, respectively, and \( c, d \) are constants. We believe that this bound is actually tight, and prove a matching lower bound under some additional assumption.

We say that a graph \( H \) is **truly projective** if it has at least three vertices and for every \( s \geq 2 \) and every connected core \( W \) incomparable with \( H \), it holds that the only homomorphisms \( f : H^s \times W \rightarrow H \) which satisfy \( f(x, x, \ldots, x) = x \) for any \( x \in V(H) \), \( y \in V(W) \), are projections. It is easy to verify that truly projective graphs are projective. Indeed, by Theorem 12, we need to show that any idempotent homomorphism \( g : H^2 \rightarrow H \) is a projection. Consider a core \( W \), which is incomparable with \( H \), and a homomorphism \( f : H^2 \times W \rightarrow H \), defined by \( f(x_1, x_2, y) := g(x_1, x_2) \). Since \( H \) is truly projective, \( f \) is a projection, and so is \( g \).

We show the following lower bound.

**Theorem 20.** Let \( H \) be a fixed non-trivial connected core, with prime factorization \( H_1 \times \ldots \times H_m \). Assume there exists \( i \in [m] \) such that \( H_i \) is truly projective. Unless the SETH fails, there is no algorithm solving \( \text{Hom}(H) \) in time \( (|H_i| - \varepsilon)^t \cdot c \cdot n^d \), for any \( \varepsilon > 0 \) and any constants \( c, d \), where \( n \) and \( t \) are, respectively, the number vertices and the treewidth of an instance graph.

The proof of Theorem 20 is similar to the proof of Theorem 3 (b). We start with constructing an appropriate edge gadget. We will use the following result (to avoid introducing new definitions, we state the theorem in a slightly weaker form, using the terminology used in this paper, see also [23, Theorem 8.18]).

**Theorem 21 (Dörrfler, [14]).** Let \( \varphi \) be an automorphism of a connected, non-bipartite, ramified graph \( H \), with the prime factorization \( H_1 \times \ldots \times H_m \). Then for each \( i \in [m] \) there exists an automorphism \( \varphi_i^{(i)} \) of \( H_i \) such that \( \varphi_i(t_1, \ldots, t_m) \equiv \varphi_i^{(i)}(t_i) \).
In particular, it implies the following.

**Corollary 22.** Let $\mu$ be an automorphism of a connected, non-trivial core $H = H_1 \times R$, where $H_1$ is indecomposable and $R \not\cong K_1^r$. Then there exist automorphisms $\mu(1) : H_1 \to H_1$ and $\mu(2) : R \to R$ such that

$$
\mu(t, t') \equiv (\mu(1)(t), \mu(2)(t')).
$$

**Proof.** By Observation 19, $R$ is a non-trivial core, so it admits a unique prime factorization, say $R = H_2 \times \cdots \times H_m$. Therefore $H_1 \times H_2 \times \cdots \times H_m$ is the unique prime factorization of $H$. From Theorem 21 we know that for each $i \in [m]$ there exists an automorphism $\varphi_i$ of $H_i$ such that $\mu_i(t_1, \ldots, t_m) \equiv (\varphi_i(t_1), \ldots, \varphi_i(m)(t_m))$. Define $\mu(1)$ by setting $\mu(1)(t) := \varphi(1)(t)$ for every vertex $t \in V(H_1)$. Analogously, we define $\mu(2)$ by setting $\mu(2)(t_2, \ldots, t_m) := (\varphi(2)(t_2), \ldots, \varphi(m)(t_m))$ for every vertex $(t_2, \ldots, t_m)$ of $R$ (for each $i \in [m] \setminus \{1\}$ we have $t_i \in V(H_i)$). It is straightforward to verify that $\mu(1)$ and $\mu(2)$ satisfy the statement of the corollary. \(\square\)

In the following lemma we construct an edge gadget, that will be used in the hardness reduction. The construction is similar to the one in Lemma 16, but more technically complicated.

**Lemma 23.** Let $H = H_1 \times R$ be a connected, non-trivial core, such that $H_1$ is truly projective and $R \not\cong K_1^r$. Let $w$ be a fixed vertex of $R$. Then there exists a graph $F$ and vertices $u^*, v^*$ of $F$, satisfying the following conditions:

(a) for every $xy \in E(H_1)$ there exists $f : F \to H$ such that $f(u^*) = (x, w)$ and $f(v^*) = (y, w)$,

(b) for any $f : F \to H$ it holds that $f_1(u^*)f_1(v^*) \in E(H_1)$.

**Proof.** Let $E(H_1) = \{e_1, \ldots, e_s\}$ and let $e_i = u_iv_i$ for every $i \in [s]$ (clearly, one vertex can appear many times as some $u_i$ or $v_i$). Consider the vertices

\[
\begin{align*}
  u & := (u_1, \ldots, u_s, v_1, \ldots, v_s) \\
  v & := (v_1, \ldots, v_s, u_1, \ldots, u_s)
\end{align*}
\]

of $H_1^{2s}$. Let $F := H_1^{2s} \times R$, and let $u^* := (u, w)$ and $v^* := (v, w)$. We will treat vertices $u$ and $v$ as $2s$-tuples, and vertices $u^*$ and $v^*$ as $(2s + 1)$-tuples.

Observe that, if $xy \in E(H_1)$, then, by the definition of $u^*$ and $v^*$, there exists $i \in [2s]$ such that $x = \pi_i(u)$ and $y = \pi_i(v)$. Define a function $f : V(F) \to V(H)$ as $f(x_1, \ldots, x_{2s}, w) := (\pi_1(x_1, \ldots, x_{2s}), w)$. Observe that this is a homomorphism, for which $f(u^*) = f(u, w) = (x, w)$ and $f(v^*) = f(v, w) = (y, w)$, which is exactly the condition (a) in the statement of Lemma 23.

We prove (b) in two steps. First, we observe the following.

**Claim.** Let $\varphi : F \to H$. If for every $z \in V(H_1)$ and $r \in V(R)$ it holds that $\varphi_1(z, \ldots, z, r) = z$ then $\varphi_1(u^*)\varphi_1(v^*) \in E(H_1)$.

**Proof of Claim.** Recall that $R$ is a connected core incomparable with $H_1$, and $H_1$ is truly projective. It means that if $\varphi_1 : H_1^s \times R \to H_1$ satisfies the assumption of the claim, then it is equal to $\pi_i$ for some $i \in [2s]$. From the definition of $u^*$ and $v^*$ we have that $\pi_i(u^*)\varphi_1(v^*) \in E(H_1)$. \(\blacksquare\)

Note that the set $\{(z, \ldots, z, r) \in F \mid z \in V(H_1), r \in V(R)\}$ induces in $F$ a subgraph isomorphic to $H$, let us call it $\tilde{H}$. Let $\sigma$ be an isomorphism from $\tilde{H}$ to $H$ defined as $\sigma(z, \ldots, z, r) := (z, r)$.

Consider any homomorphism $f : F \to H$. We observe that $f|_{V(\tilde{H})}$ is an isomorphism from $\tilde{H}$ to $H$, because $H$ is a core. If $f|_{V(\tilde{H})} \equiv \sigma$ then for every $z \in V(H)$ and $r \in V(R)$ it holds that $f_1(z, \ldots, z, r) = \ldots$
\[ \sigma_1(z, \ldots, z, r) = z, \] so, by the Claim above, we are done. If not, observe that there exists the inverse isomorphism \( g : H \to \tilde{H} \) such that \( g \circ f \mid V(\tilde{H}) \) is the identity function on \( V(\tilde{H}) \). Define \( \mu := \sigma \circ g \).

Observe that \( \mu \) is an endomorphism of \( H_1 \times \tilde{R} \), so an automorphism, since \( H_1 \times \tilde{R} \) is a core. Also note that \( (\mu \circ f) : F \to H_1 \times \tilde{R} \) is a homomorphism such that for every \((z, \ldots, z, r) \in V(\tilde{H})\) it holds that
\[
(\mu \circ f)(z, \ldots, z, r) = (\sigma \circ g \circ f)(z, \ldots, z, r) = (\sigma \circ \text{id})(z, \ldots, z, r) = \sigma(z, \ldots, z, r) = (z, r),
\]
so \((\mu \circ f)_1(z, \ldots, z, z') = z\). This means that \( \mu \circ f \) satisfies the assumption of the Claim, so
\[
(\mu \circ f)_1(u^*)(\mu \circ f)_1(v^*) \in E(H_1). \tag{1}
\]

Clearly, for every vertex \( \bar{z} \) of \( F \) it holds that
\[
(\mu \circ f)(\bar{z}) = \mu(f_1(\bar{z}), f_2(\bar{z})) = \left( \mu_1(f_1(\bar{z}), f_2(\bar{z})), \mu_2(f_1(\bar{z}), f_2(\bar{z})) \right). \tag{2}
\]

Note that Corollary 22 implies that there exist automorphisms \( \mu^{(1)} \) and \( \mu^{(2)} \) of \( H_1 \) and \( R \), respectively, such that for every \( \bar{z} \in V(F) \) it holds that
\[
\begin{align*}
\mu_1(f_1(\bar{z}), f_2(\bar{z})) &= \mu^{(1)}(f_1(\bar{z})) \\
\mu_2(f_1(\bar{z}), f_2(\bar{z})) &= \mu^{(2)}(f_2(\bar{z})),
\end{align*} \tag{3}
\]

In particular, (2) and (3) imply that \((\mu \circ f)_1 = \mu^{(1)} \circ f_1\). Combining this with (1) we get that
\[
\left( \mu^{(1)} \circ f_1 \right)(u^*) \left( \mu^{(1)} \circ f_1 \right)(v^*) \in E(H_1). \tag{4}
\]

Since \( \mu^{(1)} \) is the automorphism of \( H_1 \), there exists the inverse automorphism \( \left( \mu^{(1)} \right)^{-1} \) of \( H_1 \). Because \( \left( \mu^{(1)} \right)^{-1} \) is an automorphism, (4) implies that \( f_1(u^*)f_1(v^*) \in E(H_1) \), which completes the proof.

Now we can proceed to the proof of Theorem 20.

**Proof of Theorem 20.** Since \( \times \) is commutative, without loss of generality we can assume that \( H_1 \) is truly projective. Define \( R := H_2 \times \ldots \times H_m \), so \( H = H_1 \times R \). Since \( H_1 \) is truly projective, it is projective, so Theorem 3 can be applied here. Hence we know that assuming the SETH, there is no algorithm which solves instances of \( \text{Hom}(H_1) \) with \( n \) vertices and treewidth \( t \) in time \((|H_1| - \varepsilon)^t \cdot c' \cdot n^{d'} \), for any \( \varepsilon > 0 \) and constants \( c', d' \).

Let \( G \) be an instance of \( \text{Hom}(H_1) \) with \( n \) vertices and treewidth \( t \). The construction of the instance \( G^* \) of \( \text{Hom}(H) \) is analogous as in the proof of Theorem 3 (b). Let \( w \) be a fixed vertex of \( R \) and let \( F \) be a graph obtained by calling Lemma 23 for \( H \) and \( w \). For every vertex \( z \) of \( G \), we introduce to \( G^* \) a vertex \( z' \). Then we add a copy \( F_{xy} \) of \( F \) for every pair of vertices \( x', y' \), which corresponds to an edge \( xy \) in \( G \), and identify vertices \( x' \) and \( y' \) with vertices \( u^* \) and \( v^* \) of \( F_{xy} \), respectively.

As in the proof of Theorem 3, we observe that \( G^* \) is a yes-instance of \( \text{Hom}(H) \) if and only if \( G \) is a yes-instance of \( \text{Hom}(H_1) \). Moreover, \(|G^*| \leq |F| \cdot n^2 \) and \( \text{tw}(G^*) \leq t + |F| \). Thus, if we could decide if \( G^* \to H \) in time \((|H_1| - \varepsilon)^{tw(G^*)} \cdot |G^*|^{d} \cdot c \), then we would be able to decide if \( G \to H_1 \) in time \((|H_1| - \varepsilon)^{d} \cdot n^{d'} \cdot c' \) for constants \( c', d' \). By Theorem 3 (b), such an algorithm contradicts the SETH.
Note that combining the results from Theorem 14 and Theorem 20 we obtain a tight complexity bound for graphs $H$, whose largest factor is truly projective.

**Corollary 24.** Let $H$ be a non-trivial, connected core with prime factorization $H_1 \times \ldots \times H_m$ and let $H_i$ be the factor with the largest number of the vertices. Assume that $H_i$ is truly projective. Let $n$ and $t$ be, respectively, the number of vertices and the treewidth of an instance graph $G$.
(a) If a tree decomposition of $G$ of width $t$ is given, the $\text{Hom}(H)$ problem can be solved in time $|H_i|^t \cdot c \cdot n^d$, for some constants $c, d > 0$.
(b) There is no algorithm solving $\text{Hom}(H)$ in time $(|H_i| - \varepsilon)^t \cdot c \cdot n^d$ for any $\varepsilon > 0$ and any constants $c, d$, unless the SETH fails. \hfill \Box

5 Conclusion

Recall that in Theorem 20 and Corollary 24 we presented lower complexity bounds for $\text{Hom}(H)$ in the case that one of factors of $H$ is truly projective. In the light of Conjecture 1, we would like to weaken this assumption by substituting "truly projective" with "projective". Let us discuss the possibility of obtaining such a result.

As mentioned in the introduction, a class of graphs very close to truly projective graphs was considered by Larose [30]. In the same paper, he defined and studied the so-called strongly projective graphs. A graph $H$ on at least three vertices is strongly projective, if for every connected graph $W$ on at least two vertices and every $s \geq 2$, the only homomorphisms $f : H^s \times W \to H$ satisfying $f(x, \ldots, x, y) = x$ for all $x \in V(H)$ and $y \in V(W)$, are projections. Note that this definition is very similar, but more restrictive than the definition of truly projective graphs. Indeed, for truly projective graphs $H$ we restricted the homomorphisms from $H^s \times W$ to $H$ only for connected cores $W$, that are incomparable with $H$. Thus it is clear that every strongly projective graph is truly projective, and, as observed before, every truly projective graph is projective. Among other properties of strongly projective graphs, Larose [30, 31] shows that their recognition is decidable – note that this does not follow directly from the definition.

Let us recall some results on strongly projective graphs, as they show that many natural graphs satisfy the assumptions of Theorem 20 and Corollary 24. We say that graph is square-free if it does not contain a copy of $C_4$ as a (not necessarily induced) subgraph. Larose proved the following.

**Theorem 25 (Larose [30]).** If $H$ is a square-free, connected, non-bipartite core, then it is strongly projective.

**Example.** Consider the graph $G_B$ on 21 vertices, shown on Figure 4 (left), it is called the Brinkmann graph [9]. It is connected, its chromatic number of 4 and its girth is 5. In particular, it is square-free. Thus by Theorem 25 we know that $G_B$ is strongly projective. By exhaustive computer search we verified that $K_3 \times G_B$ is a core. Let us consider the complexity of $\text{Hom}(K_3 \times G_B)$ for input graphs with $n$ vertices and treewidth $t$. The straightforward dynamic programming approach from Theorem 13 results in the running time $63^t \cdot c \cdot n^d$, where $c$ and $d$ are constants. However, Theorem 14 gives us a faster algorithm, whose running time is $21^t \cdot c \cdot n^d$. Moreover, by Corollary 24 we know that this algorithm is likely to be asymptotically optimal, i.e., there is no algorithm with running time $(21 - \varepsilon)^t \cdot c \cdot n^d$ for any $\varepsilon > 0$ and any constants $c, d$, unless the SETH fails.

A graph is said to be primitive if there is no non-trivial partition of its vertices which is invariant under all automorphisms of this graph (see e.g. [42]).

**Theorem 26 (Larose [30]).** If $H$ is a directly indecomposable primitive core, then it is strongly projective.
In particular, Theorem 26 implies that Kneser graphs are strongly projective. Note that Kneser graphs might have 4-cycles, so this statement does not follow from Theorem 25.

Interestingly, Larose [30, 31] proved that all known projective graphs are in fact strongly projective (and thus of course truly projective). He also asked whether the same holds for all projective graphs. We recall this problem in a weaker form, which would be sufficient in our setting.

**Conjecture 2.** Every projective core is truly projective.

Clearly, if both Conjecture 1 and Conjecture 2 are true, there is another characterization of non-trivial indecomposable connected cores.

**Observation 27.** Assume that Conjecture 1 and Conjecture 2 hold. Let \( H \) be a connected non-bipartite core. Then \( H \) is indecomposable if and only if it is truly projective. \( \square \)

Note that Theorem 3, Corollary 24, and Observation 27 imply the following result.

**Theorem 5.** Assume that Conjecture 1 and Conjecture 2 hold. Let \( H \) be a fixed non-trivial connected core with prime decomposition \( H_1 \times \ldots \times H_m \), and define \( k := \max_{i \in [m]} |H_i| \). Let \( n \) and \( t \) be, respectively, the number of vertices and the treewidth of an instance graph \( G \).

(a) Assuming a tree decomposition of \( G \) of width \( t \) is given, the \( \text{HOM}(H) \) problem can be solved in time \( k^t \cdot c \cdot n^d \), where \( c, d \) are constants.

(b) There is no algorithm solving \( \text{HOM}(H) \) in time \( (k - \varepsilon)^t \cdot c \cdot n^d \) for any \( \varepsilon > 0 \) and any constants \( c, d \), unless the SETH fails.

We conjecture that the bounds from Theorem 5 are tight for all connected cores.

Finally, let us point out one more problem, related to the ones discussed in this paper. Recall that if \( H = H_1 \times H_2 \) is a connected, non-trivial core and \( H_1 \not\cong K^*_1, H_2 \not\cong K^*_1 \), then \( H_1 \) and \( H_2 \) must incomparable cores. We believe it would be interesting to know if the opposite implication holds as well. To motivate the study on this problem, we state the following conjecture.

**Conjecture 3.** Let \( H_1 \) and \( H_2 \) be connected, indecomposable, incomparable cores. Then \( H_1 \times H_2 \) is a core.
We confirmed this conjecture by exhaustive computer search for some small graphs. In particular, the conjecture is true for graphs $K_3 \times H$, where $H$ is any 4-vertex-critical, triangle-free graph with at most 14 vertices [8], the Grötzsch graph (see Figure 2), the Brinkmann graph (see Figure 4 (left)), or the Chvátal graph (see Figure 4 (right)).

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