CORRESPONDENCES OF CATEGORIES FOR SUBREGULAR W-ALGEBRAS AND PRINCIPAL W-SUPERALGEBRAS

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ABSTRACT. Based on the Kazama–Suzuki type coset construction and its inverse coset between the subregular W-algebra for \( \mathfrak{sl}_n \) and the principal W-superalgebra for \( \mathfrak{sl}_{1|n} \), we prove a weight-wise linear equivalence of their representation categories. Our main results are then improvements of these correspondences incorporating the monoidal structures. First, in the rational case, we use a certain kernel VOA together with a relative semi-infinite cohomology functor to get functors from categories of modules for the subregular W-algebra for \( \mathfrak{sl}_n \) to categories of modules for the principal W-superalgebra for \( \mathfrak{sl}_{1|n} \) and vice versa. We study these functors and in particular prove isomorphisms between the superspaces of logarithmic intertwining operators. As a corollary, we obtain a correspondence of representation categories in the monoidal sense beyond the rational case as well.

1. Introduction

Let \( \mathfrak{g} \) be a simple Lie superalgebra, \( f \) an even nilpotent element in \( \mathfrak{g} \) and \( k \) a complex number. Then one associates via quantum Hamiltonian reduction to the affine vertex superalgebra of \( \mathfrak{g} \) at level \( k \), \( V^k(\mathfrak{g}, f) \), the W-superalgebra \( W^k(\mathfrak{g}, f) \) \[FF1\], \[KRW\], \[KW\]. Historically, principal W-algebras, that is \( f \) principal nilpotent and \( \mathfrak{g} \) a Lie algebra, have received most attention. However more general W-superalgebras and their cosets have recently received increased attention due to their relevance in quantum field theory \[GR\] and moduli spaces of instantons on surfaces \[RSYZ\]. These can be viewed as generalizations of the celebrated Alday–Gaiotto–Tachikawa correspondence \[AGT\]. One of the major predictions of this new development is that cosets of certain W-superalgebras obey isomorphisms, called triality \[GR\]. This has been proven by Andrew Linshaw and one of us whenever at least one of the three involved W-superalgebras is in fact a W-algebra \[CL2, CL3\]. We are interested in using these new relations of different W-algebras and superalgebras to get correspondences of their representation categories. Triality is a vast generalization of both Feigin–Frenkel duality and the coset realization of principal W-algebras. We now quickly review this principal case.

In the case of principal W-algebras one has Feigin–Frenkel duality \[FF2\], that is \( W^k(\mathfrak{g}, f_{\text{prin}}) \cong W^\ell(\mathfrak{Lg}, f_{\text{prin}}) \) with the levels being non-critical and related by

\[
r^\vee(k + h^\vee)(\ell + \mathcal{L}h^\vee) = 1 \quad (1.1)
\]

and \( r^\vee \) the lacety of \( \mathfrak{g} \), \( \mathfrak{Lg} \) the Lie algebra whose roots coincide with the coroots of \( \mathfrak{g} \) and \( h^\vee, \mathcal{L}h^\vee \) the dual Coxeter numbers. Moreover for simply-laced \( \mathfrak{g} \) one also has the coset realization of principal W-algebras \[ACL2\], that is for \( k + h^\vee \) not a non-positive rational number one has \( \text{Com}(V^k(\mathfrak{g}), V^{k-1}(\mathfrak{g}) \otimes L_1(\mathfrak{g})) \cong W^\ell(\mathfrak{g}, f_{\text{prin}}) \) with \( L_1(\mathfrak{g}) \) the simple quotient of \( V^1(\mathfrak{g}) \). In this instance \( k \) and \( \ell \) are related by

\[
\frac{1}{k + h^\vee} + \frac{1}{\ell + h^\vee} = 1. \quad (1.2)
\]
Let $P_+$ be the set of dominant weights of $\mathfrak{g}$ and for $\lambda \in P_+$ denote by $V^k(\lambda)$ the highest-weight module of $V^k(\mathfrak{g})$ whose top-level is the irreducible highest-weight module $E_\lambda$ of $\mathfrak{g}$ of highest-weight $\lambda$, see subsection 3.1 for the precise definition. Let $W^k(\lambda, f)$ be its image under quantum Hamiltonian reduction corresponding to the nilpotent element $f$. Then the coset Theorem can be restated as follows for generic $k$:

$$V^{k-1}(\mathfrak{g}) \otimes L_1(\mathfrak{g}) \cong \bigoplus_{\lambda \in P_+ \cap Q} V^k(\lambda) \otimes W^k(\lambda, f_{\text{prin}})$$  \hspace{1cm} (1.3)

with $Q$ the root lattice of $\mathfrak{g}$. We specialize to type $A$, that is $\mathfrak{g} = \mathfrak{sl}_n$. Denote by $\omega_i$ the $i$-th fundamental weight. Let $L = \sqrt{n} \mathbb{Z}$ and define the map $s : P_+ \rightarrow L'/L$ by $s(\lambda) = \frac{1}{\sqrt{n}} \ell + L$ if $\lambda = \sum_i \omega_i \mod Q$. With this notation the coset Theorem can be restated as a sum over all dominant weights, namely for generic $k$

$$V^{k-1}(\mathfrak{sl}_n) \otimes \mathcal{E} \otimes n \cong \bigoplus_{\lambda \in P_+} V^k(\lambda) \otimes W^k(\lambda, f_{\text{prin}}) \otimes V_{\ell}(\lambda)$$ \hspace{1cm} (1.4)

with $\mathcal{E}$ the vertex superalgebra of a pair of free fermions and $V_L$ the lattice vertex algebra of $L$. Decompositions of vertex algebras of this type together with the theory of vertex superalgebra extensions of [CKM1] can be very efficiently used to get nice functors between representation categories of different vertex algebras. Indeed there are fully faithful braided tensor functors from certain subcategories of the simple quotient $W_{\ell}(\lambda, f_{\text{prin}})$ of $W^\ell(\lambda, f_{\text{prin}})$ to certain categories of ordinary modules of $L_{k-1}(\mathfrak{g}) \otimes L_1(\mathfrak{g})$ if $k-1$ is an admissible level [CHY, C1] or $\mathfrak{g} = \mathfrak{sl}_2$ at generic level $k$ [CJORY]. Such correspondences of braided tensor categories have been conjectured in the context of the quantum geometric Langlands program [AFo]. In this work we will study correspondences between representation categories of subregular $W$-algebra of $\mathfrak{sl}_n$ and principal $W$-superalgebra of $\mathfrak{sl}_{n,1}$. Our first method will also use the theory of vertex superalgebra extensions. This method is very efficient, however it only applies if one can ensure the existence of vertex tensor category structure in the sense of [HLZ1]-[HLZ8]. This is usually a very difficult problem and hence in addition we seek a different method.

In order to motivate our second approach we need to discuss generalizations of Feigin–Frenkel duality. Both Feigin–Frenkel duality and the coset realization of principal $W$-algebras of type $A$ and $D$ are part of a large family of trialities relating cosets of $W$-(super)algebras [CL2, CL3]. We review the Feigin–Frenkel type duality in type $A$. Nilpotent elements of $\mathfrak{sl}_N$ are characterized by partitions of $N$. Let $N = n + m$ and $f_{n,m}$ the nilpotent element corresponding to the partition $(n, 1, \ldots, 1)$ of $N$. Let $\psi = k + n + m$ and for $n > 0$ one sets $W^\psi(n, m) := \mathcal{W}^k(\mathfrak{sl}_{n+m}, f_{n,m})$ and we only note that the $W^\psi(0, m)$-algebra is defined slightly differently. $W^\psi(n, m)$ has an affine subalgebra of $\mathfrak{gl}_m$ at level $k + n - 1$ and the coset is denoted by $C^\psi(n, m)$. Next, consider $\mathfrak{sl}_{n|m}$. In this case nilpotent elements are characterized by pairs of partitions of $n$ and $m$. Let $f_{n|m}$ be the nilpotent element corresponding to the partition $(n|1, \ldots, 1)$ of $(n|m)$ and set $\mathcal{V}^\psi(n, m) := \mathcal{W}^\psi(\mathfrak{sl}_{n|m}, f_{n|m})$ with $\phi = \ell + n - m$. Again the case $n = 0$ is slightly differently and in fact is precisely (1.4). The superalgebra $\mathcal{V}^\psi(n, m)$ has an affine subalgebra of $\mathfrak{gl}_m$ at level $-\ell - n + 1$ and the coset is denoted by $D^\psi(n, m)$ if $n \neq m$. We only note that the definition of $\mathcal{V}^\psi(n, n)$ and $D^\psi(n, n)$ are a little bit different since $\mathfrak{sl}_{n|n}$ is not a simple Lie superalgebra. The Feigin–Frenkel type duality is then [CL2]

$$C^\psi(n, m, m) \cong D^\psi(m, n, m)$$ \hspace{1cm} (1.5)

for $n \geq m$. The case $m = 0$ is precisely Feigin–Frenkel duality of type $A$. These results are a very good starting point for investigating the problem of finding a nice functor from certain representation categories of $W^\psi(n - m, m)$ to categories.
of $W^{-1}(n, m)$-modules. The idea is that one pairs $W^\psi(n - m, m)$ with a certain kernel vertex operator algebra, such that $W^{-1}(n, m)$ is realized as a cohomology. The candidate kernel vertex operator algebra is suggested from physics: $S$-duality for four-dimensional $N = 4$ supersymmetric GL-twisted gauge theory is a physics pendant of the quantum geometric Langlands program and it especially conjectures (and proves for $\mathfrak{sl}_2$) the existence of certain junction vertex operator algebras, also called quantum geometric Langlands kernel VOAs [CG], see also [CGL, FG] for subsequent studies. In the case of the central object is

$$A[\mathfrak{sl}_N, \psi] := \bigoplus_{\lambda \in P^+} V^k(\lambda) \otimes V^\ell(\lambda) \otimes V_{\lambda}(\lambda)$$

with $k, \ell$ related by (1.2) and $\psi = k + N$. This has conjecturally a simple vertex superalgebra structure for generic $\psi$. Comparing with (1.4), we see that the principal quantum Hamiltonian reduction on the second factor gives the decomposition of $V^{k-1}(\mathfrak{sl}_n) \otimes E^{\otimes n}$. In [CL2] it is proven that in fact all the $W^\psi(n, m)$ can be realized as a coset of a reduction of $A[\mathfrak{sl}_N, \psi]$ provided a certain existence and simplicity conjecture is true. There it is also conjectured that a certain relative semi-infinite Lie algebra cohomology of $W^\psi(n - m, m) \otimes A[\mathfrak{sl}_n, 1 - \psi] \otimes \pi^{k-\ell}$ with $\pi^{k-\ell}$ a rank one Heisenberg vertex algebra is isomorphic as a vertex superalgebra to $W^{-1}(n, m)$. This conjecture is also reduced to a certain existence and simplicity statement [CL2, Theorem 1.4]. In the case of $m = 1$, the kernel vertex algebra in this conjecture is replaced by $V_2 \otimes \pi$ with $\pi$ a rank one non-degenerate Heisenberg vertex algebra. One major result of this work is that we not only prove the Conjecture for $m = 1$, but especially understand how it provides a functor from $W^{\psi}(n - 1, 1)$-modules to $W^{-1}(n, 1)$-modules. Note that the $W^{\psi}(n - 1, 1)$-algebra is the subregular $W$-algebra of $\mathfrak{sl}_n$ and $W^{-1}(n, 1)$ is the principal $W$-superalgebra of $\mathfrak{sl}_{n+1}$ and in this case there is a very different proof of the duality that is rather useful for our understanding [CGN]. Let us now explain our main results and especially the very nice properties of the functor from $W^{\psi}(n - 1, 1)$-modules to $W^{-1}(n, 1)$-modules.

1.1. Results. In this paper we employ two different methods to establish a correspondence between the representation theory of the subregular $W$-algebra of $\mathfrak{sl}_n$ and of the principal $W$-superalgebra of $\mathfrak{sl}_{n+1} \simeq \mathfrak{sl}_n$. One method is a tensor categorical, which is based on the vertex tensor category theory [HLZ1]-[HLZ8] together with the vertex operator superalgebra extension theory [HKL, CKM1]. This method only applies for categories of modules that are vertex tensor categories, as e.g. for $C_2$-cofinite vertex operator algebras. Alternatively, the other method is of cohomological nature and it does not require the existence of vertex tensor categories. In order to state our main results on the correspondence of monoidal structures, we first briefly review correspondences of linear structures.

As mentioned earlier,

$$W^+ := W^{\psi}(n - 1, 1) = W^k(\mathfrak{sl}_n, f_{\text{sub}}), \quad W^- := W^{-\psi}(n, 1) \simeq W^\ell(\mathfrak{sl}_{n+1}, f_{\text{prim}})$$

contain affine $\mathfrak{gl}_1$ subalgebras, denoted by $\pi^+$ and $\pi^-$, respectively. In [CGN], the first three authors of this paper introduced two diagonal coset constructions

$$W^\pm \xrightarrow{\sim} \text{Com}(\Delta(\pi^\pm), W^\pm \otimes V_2), \quad W^+ \xrightarrow{\sim} \text{Com}(\Delta(\pi^-), W^\pm \otimes V_{\sqrt{\pi^+} \otimes \pi^2}).$$

Here $V_2$ stands for the lattice vertex superalgebra associated to an integral lattice $L$ and $\Delta(\pi^\pm)$ for certain diagonal Heisenberg subalgebras of $W^k \otimes V_{\sqrt{\pi^+} \otimes \pi^2}$. When $n = 2$, the former coincides with the Kazama–Suzuki coset construction firstly studied by [DVPYZ, KaSu] at unitary levels, and the latter coincides with its inverse coset construction firstly studied by [FST] at general levels (see also [Ad1, Ad2, HM, S1, CLRW]). Based on the same technique used in [S1] (see also [FST]),
we consider the categories of $\mathcal{W}^\pm$-modules on which $\pi^\pm$ acts semisimply with the gl$_3$-weight decomposition (see §5.1 for the definition)
\[
\mathcal{W}^\pm-\text{mod}_{\mathfrak{g}l_3} = \bigoplus_{|\lambda| \in \mathbb{C} / \mathbb{Z}} \mathcal{W}^\pm-\text{mod}_{\mathfrak{g}l_3}^{(\lambda)}
\]
and establish linear categorical equivalences
\[
\mathcal{W}^+\text{-mod}_{\mathfrak{g}l_3}^{(\lambda)} \cong \mathcal{W}^+\text{-mod}_{\mathfrak{g}l_3}^{(\lambda')}, \quad \mathcal{W}^-\text{-mod}_{\mathfrak{g}l_3}^{(\lambda)} \cong \mathcal{W}^-\text{-mod}_{\mathfrak{g}l_3}^{(\lambda')}
\]
for appropriate pairs $(\lambda, \lambda')$. In addition, these equivalences are also valid if we replace $\mathcal{W}^\pm$ with their simple quotients (see §5 for the precise statements). In the following, we lift this result to correspondences that include the monoidal structure of the categories.

1.1.1. Correspondences via categorical methods. The simple subregular W-algebra $\mathcal{W}_k(\mathfrak{sl}_n, \mathfrak{f}_{\text{sub}})$ at the levels
\[
k = -n + \frac{n + r}{n-1}, \quad (r \in \mathbb{Z}_{\geq 0}, \gcd(n + r, r - 1) = 1)
\]
is called exceptional. Its $C_2$-cofiniteness is established by Arakawa [Ar2] and rationality was first proven for $n = 3$ [Ar1] and $n = 4$ [CL1] and then in general by Arakawa and van Ekeren [AvE2]. In this case, the Heisenberg subalgebra $\pi^+$ inside the subregular W-algebra is extended to a lattice vertex algebra [Ma] and thus the representation theory at these levels can be studied by the theory of simple current extensions modulo the representation theory of the Heisenberg coset. Linshaw and one of the authors [CL2] identified the Heisenberg coset with a principal W-algebra
\[
\text{Com}(\pi^+, \mathcal{W}_{-n+\frac{n+r}{n-1}}(\mathfrak{sl}_n, \mathfrak{f}_{\text{sub}})) \cong \mathcal{W}_{-r+\frac{r+1}{r+1}}(\mathfrak{sl}_r, \mathfrak{f}_{\text{prin}}),
\]
which generalizes the level-rank duality for the sl$_2$-parafermion algebra when $n = 2$ [ALY] as well as the cases $n = 3$ [ACL1] and $n = 4$ [CL1]. Here the right-hand side in (1.12) for the cases $r = 0, 1$ is interpreted as $\mathbb{C}$, see Section 4.3. The principal $\mathcal{W}$-algebra at these levels are $C_2$-cofinite and rational [Ar2, Ar3] and the representation theory is completely understood [Ar3]: the complete set of representatives of simple modules is in one-to-one correspondence with that of the simple affine vertex algebra $L_n(\mathfrak{sl}_r)$ [Ar3, FKW] and both of their fusion rules are the same [AvE1, CL]. Hence it gives an isomorphism of their fusion rings
\[
\mathcal{X}(L_n(\mathfrak{sl}_r)) \cong \mathcal{X}(\mathcal{W}_{-n+\frac{n+r}{n-1}}(\mathfrak{sl}_n, \mathfrak{f}_{\text{sub}})), \quad L(\lambda) \mapsto L_{\mathcal{W}}(\lambda),
\]
where $L_n(\lambda)$ is the unique simple quotient of the previous $V^n(\lambda)$ and $\lambda$ runs through the set $P_+^n(r)$ of dominant weights of $\mathfrak{sl}_r$ at level $n$. Now, by [CL1, CL2], one can decompose the subregular W-algebra into
\[
\mathcal{W}_{n+\frac{n+r}{n-1}}(\mathfrak{sl}_n, \mathfrak{f}_{\text{sub}}) \simeq \bigoplus_{i \in \mathbb{Z}} L_{\mathcal{W}}(n \pi_i) \otimes V_{-r+r+i}^\pm + \sqrt{\pi_i} Z,
\]
where $\pi_i$ denotes the $i$-the fundamental weight, including the cases $r = 0, 1$ as $\mathbb{C} \otimes \mathbb{C}$ and $\mathbb{C} \otimes V_{n\pi_i}^\pm$, respectively (see Remark 4.2). Note that the Kazama–Suzuki coset construction (1.8) implies a similar description for the simple principal $\mathcal{W}$-superalgebra. Namely, we have by Theorem 4.7
\[
\mathcal{W}_{-(n-1)+\frac{n-1}{n+r}}(\mathfrak{sl}_1, \mathfrak{f}_{\text{prin}}) \simeq \bigoplus_{i \in \mathbb{Z}} L_{\mathcal{W}}(n \pi_i) \otimes V_{-r+r+i}^\pm + \sqrt{\pi_i} Z,
\]
including the cases $r = 1, 2$ as $\mathbb{C} \otimes \mathbb{C}$ and $\mathbb{C} \otimes V_{\sqrt{n+1} \pi_i}^\pm$, respectively (see Remark 4.8).

To deduce the classification of simple modules and the fusion ring of the simple subregular W-algebra and principal W-superalgebra, we view these (super)algebras
as extensions of the principal $W$-algebra tensored with lattice vertex (super)algebras. 

Based on earlier works [HKL, CKL, CKM1, KO] on simple current extensions, we consider the simple current extensions of the form

$$\mathcal{E} = \bigoplus_{a \in N/L} S_a \otimes V_{a+L},$$

where $N$ is an arbitrary lattice lying between a non-degenerate integral lattice $L$ and its dual lattice $L'$, and $\{V_{a+L} \mid a \in N/L\}$ (resp. $\{S_a \mid a \in N/L\}$) is a group of simple currents for the lattice vertex superalgebra $V_L$ associated to $L$ (resp. for a simple $C_2$-cofinite $1/2\mathbb{Z}$-graded vertex operator superalgebra $V$), satisfying some mild assumptions. Then the classification of simple modules and the description of fusion rings for the one are expressed by the other. In particular, the fusion rings are related in the following symmetric way:

$$\mathcal{K}(\mathcal{E}) \simeq \left(\mathcal{K}(V) \otimes S^N/L_{L'/L}\right)^{N/L},$$

$$\mathcal{K}(V) \simeq \left(\mathcal{K}(\mathcal{E}) \otimes S^{N'/L_{L'/L}}\right)^{N'/L},$$

(1.14)

see Theorem 2.3 and 2.5 for details. We note that this kind of duality in the rational setting between fusion rings for simple current extensions is essentially obtained by Yamada and Yamauchi [YY] and by Kanade, McRae and one of the authors [CKM1]. See also [ADJR] in the case of parafermion vertex algebras.

By applying this general theory of simple current extensions and the level-rank duality (B.2) of the fusion rings of $L_n(\mathfrak{sl}_1)$ and $L_\ell(\mathfrak{sl}_r)$ [Fr, OS], we obtain the following main theorem in the rational case:

**Main Theorem 1.** Let $r \in \mathbb{Z}_{\geq 0}$ such that $\gcd(n + r, r - 1) = 1$.

1. The complete set of simple modules for $W_k(\mathfrak{sl}_n, f_{\text{sub}})$ at $k = -n + \frac{n+r}{n-1}$ is in one-to-one correspondence with that of $L_\ell(\mathfrak{sl}_n)$, i.e., $P^r_\ell(n)$. Moreover, we have an isomorphism of fusion rings:

$$\mathcal{K}(W_k(\mathfrak{sl}_n, f_{\text{sub}})) \simeq \mathcal{K}(L_\ell(\mathfrak{sl}_n)).$$

2. The complete set of simple modules for $W_\ell(\mathfrak{sl}_1[n], f_{\text{prin}})$ at $\ell = -(n-1) + \frac{n-1}{n+r}$ is in one-to-one correspondence with

$$\{(\lambda, n) \in P^r_\ell(n) \times \mathbb{Z}_n(n+r) \mid \pi_P Q(\lambda) = a \in \mathbb{Z}_n\} / \mathbb{Z}_n.$$

Moreover, we have an isomorphism of fusion rings:

$$\mathcal{K}(W_\ell(\mathfrak{sl}_1[n], f_{\text{prin}})) \simeq \left(\mathcal{K}(L_\ell(\mathfrak{sl}_n)) \otimes S^{\mathbb{Z}_n(n+r)}\right)^{\mathbb{Z}_n}.$$

See Proposition 4.5 and Theorem 4.12, respectively. We note that for (1), the classification result of simple modules together with the description of the fusion ring when $n$ is even is already obtained in [AvE2]. For (2), the case of $n = 2$ corresponds to the $N = 2$ super Virasoro algebra and the classification result is obtained by Adamović [Ad1] and the description of fusion ring essentially coincides with the one in [Ad2]. We also note that the description of fusion ring in the case $n = 2$ as in (2) was inferred by Wakimoto [Wak] using the Verlinde formula. Since the coset constructions (1.8) in the rational case are indeed simple current extensions, the fusion rings for $W_k(\mathfrak{sl}_n, f_{\text{sub}})$ and $W_\ell(\mathfrak{sl}_1[n], f_{\text{prin}})$ in Main theorem 1 are actually related by the isomorphisms (1.14). This manifest the compatibility of the monoidal structure under the correspondence (1.8).

In subsection 4.5 we comment that this procedure only requires the existence of a vertex tensor category structure on the Heisenberg coset. This is however only...
known in two series of non-rational subregular \( W \)-algebras, namely for \( W_k(\mathfrak{g}_m, f_{sub}) \) for \( k = -n + \frac{m}{m+1} \) and \( k = -n + \frac{m}{n} \) by [ACGY, CL2]. The Heisenberg cosets are singlet algebras and certain vertex tensor subcategories are understood [CMY2]. We thus seek a second method that does not require the existence of vertex tensor category.

1.1.2. Correspondences via relative semi-infinite cohomology and a kernel VOA. One of the features of the coset construction (1.8) is that the difference between the two \( W \)-superalgebras \( W^k \) is the dressings by the Fock modules of the Heisenberg vertex algebra of rank one. In order to see this, let us decompose \( W^{\pm} \) as modules of tensor products of the common Heisenberg coset \( \mathcal{C}_0 \) i.e., (1.5) with \( m = 1 \) and the Heisenberg vertex algebra:

\[
W^+ \simeq \bigoplus_{a \in \mathbb{Z}} \mathcal{C}_a \otimes \pi^+_a, \quad W^- \simeq \bigoplus_{a \in \mathbb{Z}} \mathcal{C}_a \otimes \pi^-_a \tag{1.15}
\]

Here \( \mathcal{C}_a (a \in \mathbb{Z}) \) are \( \mathcal{C}_0 \)-modules appearing as the highest weight vectors for \( \pi^\pm \) of highest weight \( a \). We see that in order to go from \( W^+ \) to \( W^- \) we just need to replace \( \pi^+_a \) by \( \pi^-_a \). We are looking for functors that not only replace \( \pi^+_a \) by \( \pi^-_a \) and vice versa, but that also allow us to relate the monoidal structures of the representation categories of \( W^+ \) and \( W^- \) in an efficient way. Here comes the relative semi-infinite cohomology into play.

The relative semi-infinite cohomology was introduced by Feigin [Fe] and by I. Frenkel, Garland and Zuckerman [FGZ]. It is a vertex algebraic or infinite dimensional Lie algebraic analogue of the usual relative Lie algebra cohomology. Let \( \pi^\pm \) denote a Heisenberg vertex algebra generated by a field \( H^\pm(z) \) which satisfies the OPE equal to the minus of that of the generator \( H^{\pm}(z) \) of \( \pi^{\pm} \), i.e.,

\[
H^+(z)H^+(w) \sim - H^+(z)H^+(w).
\]

Then \( \pi^\pm \otimes \pi^\pm \) contains a commutative vertex algebra generated by a diagonal Heisenberg field \( A(z) = H^+_1(z) + H^-_1(z) \). We may consider the semi-infinite cohomology \( H^r_{\text{rel}}(\pi_a^\pm \otimes \pi^\pm_1) \) of \( \mathfrak{g}_{11}((z)) \) with coefficients in \( \pi^+_a \otimes \pi^+_1 \) relative to \( \mathfrak{g}_{11} \), which satisfies

\[
H^r_{\text{rel}}(\pi_a^\pm \otimes \pi^\pm_1) = \delta_{0,0}\delta_{a+b,0}\mathbb{C}.
\]

The non-zero cohomology class is spanned by the tensor product of the highest weight vectors. To obtain \( W^- \) from \( W^+ \) or \( W^+ \) from \( W^- \), we consider the direct sums:

\[
\bigoplus_{a \in \mathbb{Z}} \pi_{1,-a}^+ \otimes \pi_{-a}^- + \bigoplus_{a \in \mathbb{Z}} \pi_{1,-a}^- \otimes \pi_{-a}^+.
\]

By changing the bases of the Heisenberg fields inside \( \pi^\pm \otimes \pi^\pm_1 \), we find that they admit vertex superalgebra structure:

\[
V_{\mathbb{Z}} \otimes \pi^{\sqrt{-1}\mathbb{Z}}, \quad V_{\sqrt{-1}\mathbb{Z}} \otimes \pi^\mathbb{Z}. \tag{1.16}
\]

Indeed, \( V_{\mathbb{Z}} \) in the first is the kernel VOA (1.6) for \( m = 1 \). Then we may reconstruct \( W^+ \) from \( W^- \) and vice versa in the following way:

\[
W^- \simeq H^r_{\text{rel}}(W^+ \otimes V_{\mathbb{Z}} \otimes \pi^{\sqrt{-1}\mathbb{Z}}), \quad W^+ \simeq H^r_{\text{rel}}(W^- \otimes V_{\sqrt{-1}\mathbb{Z}} \otimes \pi^\mathbb{Z}),
\]

see Proposition 6.2. Note, that the first isomorphism in particular proves Conjecture 1.2 of [CL2] for the case \( m = 1 \), that is \( \mathfrak{g}_{11} \). We may generalize this construction for modules by replacing the second factor in (1.16) by their Fock modules with appropriate highest weights. This actually defines functors weight-wisely:

\[
H^r_{+,\lambda}: W^{+}\text{-mod}^\mathbb{Z}_{[\lambda]} \rightarrow W^-\text{-mod}^\mathbb{Z}_{[\lambda]}, \quad H^r_{-,\lambda}: W^-\text{-mod}^\mathbb{Z}_{[\lambda]} \rightarrow W^+\text{-mod}^\mathbb{Z}_{[\lambda]}, \tag{1.17}
\]
which turns out to be naturally isomorphic to the ones in (1.10), see Theorem 6.3. Therefore, \( H^\text{rel}_{\pm, \pm \lambda} \) and \( H^\text{rel}_{\pm, \lambda} \) themselves give an equivalence of categories as abelian categories. The upshot of this method is that the way of comparing the monoidal structure is clear: we may use the intertwining operators among Fock modules to compare the intertwining operators among \( W^\pm \)-modules related by the equivalence (1.17). The second main theorem of this paper is the following:

**Main Theorem 2.** For \( i = 1, 2, 3 \), let \( M^+_i \) be an object in \( W^+\text{-mod}_{[\lambda_i]} \) and \( M^-_i \) in \( W^-\text{-mod}_{[\lambda_i]} \).

1. The functors \( \{ H^\text{rel}_{\pm, \pm \lambda_i} \} \) induce isomorphisms between the superspaces of logarithmic intertwining operators:

   \[
   H_{\pm} : I_{W^\pm} \left( \frac{M^\pm}{M^\pm_{1}} \right) \cong I_{W^\pm} \left( H^\text{rel}_{\pm, \pm \lambda_i} \left( M^\pm_{1} \right) \right).
   \]

2. Suppose that \( M^+_1 \) and \( M^+_2 \) admit a fusion product \( M^+_1 \otimes M^+_2 \). Then the corresponding \( W^\pm\)-module \( H^\text{rel}_{\pm, \pm \lambda_i} \left( M^+_1 \right) \otimes M^+_2 \) equipped with the image of the canonical intertwining operator of type \( \left( M^+_1 \otimes M^+_2 \right) \) gives a fusion product of \( H^\text{rel}_{\pm, \pm \lambda_i} \left( M^+_1 \right) \) and \( H^\text{rel}_{\pm, \pm \lambda_i} \left( M^+_2 \right) \). In particular, we have a natural isomorphism

   \[
   H^\text{rel}_{\pm, \pm \lambda_i} \left( M^+_1 \right) \otimes H^\text{rel}_{\pm, \pm \lambda_i} \left( M^+_2 \right) \simeq H^\text{rel}_{\pm, \pm \lambda_i} \left( M^+_1 \right) \otimes M^+_2 .
   \]

See Theorem 6.4 and Corollary 6.5 for more precise statements. See also Theorem 6.8 for an explicit description in the rational case.

1.2. Outlook. In [CL2, CL3] it has been conjectured that certain \( W \)-(super)algebras whose levels are related by a relation of type (1.1) are connected via tensoring with a certain kernel VOA and then taking a relative semi-infinite cohomology, see Sections 10 of [CL2] and Section 1.2 of [CL3]. We have proven that in the simplest case, that is when the kernel VOA is a lattice vertex algebra times a Heisenberg vertex algebra, this indeed works. Moreover we have seen that this approach is very efficient in connecting representation categories of the involved \( W \)-(super)algebras. It is thus a major future problem to lift our studies to higher rank cases, this means it is important to settle [CL2, Conjecture 1.1 and 1.2] and [CL3, Conjecture 1.2].

It is also interesting to study correspondences of correlation functions and conformal blocks and we expect that our relative semi-infinite cohomology approach will be quite useful. Recall that we benefitted from physics suggestions and so it is worth noting that physics suggests us an additional relation. The relation between correlation functions of the \( H^\pm_{\pm} \)-model (a non-compact version of the \( sl_2 \) WZW-theory) and \( N = 2 \) super Liouville theory (the underlying vertex algebra is the \( N = 2 \) super Virasoro algebra) is called the supersymmetric Fateev–Zamolodchikov–Zamolodchikov duality in physics. It is understood using mirror symmetry [HK] or the path integral [CHR]. The latter method actually relates correlation functions of both theories to correlation functions in Liouville theory (whose underlying vertex algebra is the Virasoro algebra) and uses Liouville theories self-duality, i.e. Feigin–Frenkel duality of the Virasoro algebra. This method generalizes to higher rank cases [CH] and that generalization is actually inspired from [CGN, CL2].

Also note that the connection between characters of modules is already quite interesting. For example in the simplest example of \( sl_2 \) at admissible level characters are formal distributions or expansions of meromorphic Jacobi forms [CR1, AdS], while characters of the \( N = 2 \) super Virasoro algebra are expressed in terms of Jacobi theta functions and mock Jacobi forms [STT]. This correspondence has been studied in [FSST, S3, KoSa, CLRW]. From our perspective the character
of a module that we obtain via $H^m_{+,\lambda}$ is the Euler-Poincaré supercharacter of the relative complex as higher cohomologies vanish.

It is usually a very difficult problem to establish the existence of a vertex tensor category structure on a category of modules of a vertex algebra $V$ that is not $C_2$-cofinite. However, Theorem 3.3.4 of [CY] tells us that if the category of ordinary modules of $V$ is of finite length and $C_1$-cofinite, then it is a vertex tensor category. This Theorem applies for example to $L_k(sl_2)$ at admissible level [CHY], but almost all modules in this case are not ordinary and in fact are often not even lower bounded. The dual $W$-superalgebra is the $N=2$ super Virasoro algebra, and modules are always ordinary [S1, CLRW]. It is a reasonable hope that they are in fact all $C_1$-cofinite and of finite length. Note that the duality between subregular $W$-algebras of $\mathfrak{sl}_n$ and the principal $W$-superalgebras of $\mathfrak{sl}_n|_1$ somehow extends to $n=1$, namely an analogous relation between the $\beta\gamma$-system and the affine vertex superalgebra of $\mathfrak{gl}_{1|1}$ at non-degenerate level. For the latter, it is indeed true that the category of ordinary modules is $C_1$-cofinite and of finite length [CMY3] and since the $\beta\gamma$-system times a pair of free fermions is a simple current extension of the affine vertex superalgebra of $\mathfrak{gl}_{1|1}$ at non degenerate level [CR2] one obtains the category of $\beta\gamma$-modules that includes relaxed-highest-weight modules. In fact, $\beta\gamma$ is the first example where one has a general vertex tensor category result including all relaxed-highest weight modules [AW], this result can thus be reproduced, but also $L_k(sl_{2|1})$ for $k=1$ and $k=-1/2$, see [CMY3, Section 5].

1.3. Organization. This paper is organized as follows. In Section 2 we first review and explore general results about the monoidal structure of categories of VOA modules and their super extensions, following [CKM1]. Based on purely categorical treatments of simple current extensions in Appendix A, we prove the ring isomorphisms (1.14) in Section 2.4. After we review of the Feigin–Semikhatov duality [CGN] in Section 3, we prove, in Section 4, Main Theorem 1 by using the ring isomorphisms (1.14) and the level-rank duality discussed in Appendix B. The weight-wise linear equivalences between module categories of the subregular $W$-algebra and the principal $W$-superalgebra are proved in Section 5. Lastly, in Section 6, we introduce the relative semi-infinite cohomology functor and prove Main Theorem 2. We note that the purely categorical perspective of simple current extensions in Appendix A is of independent interest.

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2. Fusion rules of lattice cosets

2.1. Vertex superalgebras and their modules. We recall basics on vertex superalgebras and their modules, following [HLZ1, CKM1]. Recall that a vector superspace $V$ (over $\mathbb{C}$) is a $\mathbb{Z}_2$-graded vector space $V = V_0 \oplus V_1$ where $\mathbb{Z}_2 = \{0, 1\}$ and that an element $a \in V_0$ (resp. $a \in V_1$) is all even (resp. odd) or of parity $\overline{a} = 0$ (resp. $\overline{a} = 1$). The $\mathbb{Z}_2$-grading on $V$ induces a vector superspace structure on the space of $\mathbb{C}$-linear endomorphisms $End V$ and thus $End V$ is naturally a Lie superalgebra by $[a, b] = ab - (-1)^{\overline{a}\overline{b}}ba$ for parity homogeneous elements $a, b \in End V$. 

A vertex superalgebra is a vector superspace $V$ equipped with a non-zero vector $1 \in V_0$, a linear map $\partial \in (\End V)_0$ and a parity-preserving linear map

$$Y(\cdot, z) : V \to \End V[z, z^{-1}], \quad a \mapsto Y(a, z) = a(z) = \sum_{n \in \mathbb{Z}} a_n z^{-n-1}$$

such that $Y(a, z)b \in V(\{z\})$ ($a, b \in V$) satisfying

(1) $Y(1, z)a = a$ and $Y(a, z)1 \in a + V[z]z$ for $a \in V$,
(2) $\partial 1 = 0$ and $[\partial, Y(a, z)] = \partial_z Y(a, z)$ for $a \in V$,
(3) for $a, b \in V$, there exists $m \in \mathbb{Z}_{\geq 0}$ such that $(z - w)^m [Y(a, z), Y(b, w)] = 0$.

It follows from the axioms that $[Y(a, z), Y(b, w)] = \sum_{n=0}^{\infty} Y(a_n b, w) \partial^n_z \delta(z - w)/n!$ where $\delta(z - w) = \sum_{n \in \mathbb{Z}} z^n w^{-n-1}$, which we write as

$$Y(a, z)Y(b, w) \sim \sum_{n=0}^{\infty} Y(a_n b, w) \frac{(z - w)^n}{n},$$

called the OPE of $Y(a, z)$ and $Y(b, w)$. If $V = V_0$, then it is called a vertex algebra. An even element $\omega \in V_0$ is called a conformal vector if the corresponding field $Y(\omega, z) = L(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}$ satisfies that $L_{-1} = \partial$, $L_0$ acts on $V$ semisimply, and

$$L(z) L(w) \sim \frac{\partial_w L(w)}{z - w} + \frac{2L(w)}{(z-w)^2} + \frac{c/2}{(z-w)^4}$$

for some $c \in \mathbb{C}$, called the central charge of $\omega$. The $L_0$-grading $V = \bigoplus_{\Delta \in \mathbb{C}} V_\Delta$ is called the conformal grading and in this paper we consider the case $V$ is at most $\frac{1}{2} \mathbb{Z}$-graded. A vertex operator superalgebra of CFT type is a pair $(V, \omega)$ such that $V$ is $\frac{1}{2} \mathbb{Z}_{\geq 0}$-graded by $L_0$ and $V_0 = \mathbb{C}1$ and $\dim V_{\Delta} < \infty$ for $\Delta \in \frac{1}{2} \mathbb{Z}_{>0}$.

In the following, we will consider a negative-definite lattice vertex superalgebra e.g. $V_{\sqrt{c} \mathbb{Z}}$, which does not satisfy this condition, see §2.3 below. This motivates us to consider an additional grading, following [HLZ1, Y]: given a finitely generated abelian group $A$, we consider an additional $A$-grading on a vertex superalgebra $V = \bigoplus_{a \in A} V^a$ as a vector superspace such that

$$Y(\cdot, z) : V^a \times V^b \to V^{a+b}(\{z\}), \quad (a, b \in A).$$

Then by a strongly $A$-graded vertex operator superalgebra of CFT type we mean a $\frac{1}{2} \mathbb{Z}$-graded vertex superalgebra $(V, \omega)$ with an additional $A$-grading

$$V = \bigoplus_{a \in A} V^a = \bigoplus_{\Delta \in \frac{1}{2} \mathbb{Z}, a \in A} V^a_{\Delta}, \quad V^a_{\Delta} = V^a \cap V_\Delta$$

as a vector superspace, satisfying

(1) $V^a_{\Delta}$ is finite dimensional for all $(\Delta, a)$ and $V^0_{0} = \mathbb{C}1$,
(2) $V^0_{\Delta} = 0$ for $\Delta < 0$ and $V^0_{\Delta + m} = 0$ for sufficiently negative $m$ for each $(\Delta, a)$.

Note that both of a lattice vertex superalgebra $V_L$ associated with a non-degenerate integral lattice $L$ with a natural strong $L$-grading and a $\frac{1}{2} \mathbb{Z}_{>0}$-graded vertex operator superalgebra with trivial strong $A$-grading satisfy this condition. In the following, we will consider modules for $V$ with a strong $A$-grading as above.

A weak $V$-module is a vector superspace $M$ equipped with a parity-preserving linear map

$$Y_M(\cdot, z) : V \to \End M[z, z^{-1}], \quad a \mapsto Y_M(a, z) = a^M(z) = \sum_{n \in \mathbb{Z}} a_n^M z^{-n-1}$$

such that $Y_M(\cdot, z) : V \times M \to M(\{z\})$ satisfying

(1) $Y_M(1, z)a = a$ for all $a \in M$,
(2) for all $a, b \in V$,

$$z_0^{-1} \delta \left( \frac{z_1 - z_2}{z_0} \right) a^M(z_1)b^M(z_2) - (-1)^{a \cdot b} z_0^{-1} \delta \left( \frac{z_2 - z_1}{z_0} \right) b^M(z_2)a^M(z_1) = z_2^{-1} \delta \left( \frac{z_1 - z_2}{z_0} \right) Y_M(a(z_0)b, z_1),$$

where $\delta(z) = \sum_{n \in \mathbb{Z}} z^n$. We note that the axioms imply $\partial_z Y_M(a, z) = Y_M(\partial a, z)$ for $a \in V$. We call a weak $V$-module $A$-gradable if it admits a grading as a vector superspace by an $A$-torsor $\tilde{A}$, denoted by $M = \bigoplus_{a \in \tilde{A}} M^a$, and $Y_M(\cdot, z)$ restricts to $V^a \times M^b \to M^{a+b}(\ell z)$. For $A$-gradable weak $V$-modules $M$ and $N$, a morphism from $M$ to $N$ is a linear map $f : M \to N$ such that

$$f \left( a^M(z)v \right) = a^N(z)f(v), \quad (a \in V, v \in M)$$

(2.1)

which induces a morphism of $A$-torsors on gradings of $M$ and $N$. In the following, the term $A$-gradable is dropped if $A$ is clear from context (especially, if $A$ is trivial).

Next, we recall several classes of modules. An $A$-gradable weak $V$-module $M$ is called an $(A$-gradable) generalized $V$-module if it admits a generalized $L^M_0$-eigenspace decomposition

$$M = \bigoplus_{\Delta \in \mathbb{C}} M_\Delta = \bigoplus_{\Delta \in \mathbb{C}, a \in \tilde{A}} M^a_\Delta, \quad M_\Delta = \{ v \in M \mid \exists N \geq 0, (L_0 - \Delta)^N v = 0 \}$$

(2.2)

A generalized $V$-module $M$ is called grading-restricted if dim $M^a_0 < \infty$ and $M^a_{\Delta+N} = 0$ for $a \in \tilde{A}$, $\Delta \in \mathbb{C}$ and sufficiently negative $N \in \mathbb{Z}$; ordinary if $M$ is grading-restricted and (2.2) is an $L^M_0$-eigenspace decomposition.

Let $V$-mod$_C$ be the category of grading-restricted generalized $V$-modules. It is naturally a supercategory [BrE], that is the Hom spaces are vector superspaces and the compositions of morphisms are compatible with the superstructure. (The symbol $\mathbb{C}$ indicated that the $L_0$-eigenvalues of objects are not restricted to $\mathbb{R} \subset \mathbb{C}$ whereas the later restriction is needed for monoidal structure as we will see in the next subsection.) Recall that the underlying category $V$-mod$_{\mathbb{R}}$ is the subcategory of $V$-mod$_C$ whose objects are the same as $V$-mod$_C$ but whose morphisms consist of even homomorphisms. Although $V$-mod$_{\mathbb{R}}$ is naturally an abelian category, $V$-mod is merely a $\mathbb{C}$-linear additive supercategory since the kernel and cokernel objects for parity-inhomogeneous morphisms usually do not exists. Then by a subquotient object in $V$-mod$_C$ we mean one in $V$-mod$_{\mathbb{R}}$, and by the notion of (semi)simplicity as well. Note that $V$-mod has an involutive autofunctor $\Pi$ which switches the superstructure of objects: $(HM)_i = M_{-i, i}$ ($i \in \mathbb{Z}_2$). Therefore, we may recover all the properties of parity-homogeneous morphisms by composing $\Pi$ if necessary.

### 2.2. Intertwining operators and fusion products

We recall the notion of (logarithmic) intertwining operators, $P(x)$-tensor products and a braided monoidal structure on module categories, following [HILZ], [HILZ8] and [CKM1]. We restrict the category $V$-mod$_C$ to the full subcategory $V$-mod consisting of objects $M$ such that the grading (2.2) for $M$ is supported on $\mathbb{R}$ and the size $N$ of Jordan block for $L_0$ is uniformly bounded with respect to $\Delta \in \mathbb{R}$, following [CKM1, Assumption 3.9]. We assume that the grading by an $A$-torsor for each object in $V$-mod all comes from a grading with respect to a common abelian group containing $A$, or otherwise we restrict to a full subcategory satisfying this property. For objects $M_i$ ($i = 1, 2, 3$) in $V$-mod, a parity-homogeneous (logarithmic) intertwining operator
of type \((M_1, M_2)\) is a parity-homogeneous bilinear map
\[
\mathcal{Y}(\cdot, z): M_1 \times M_2 \rightarrow M_3[z][\log z]
\]
\[
(m_1, m_2) \mapsto \mathcal{Y}(m_1, z)m_2 = \sum_{k=0}^{K} \sum_{n \in \mathbb{C}} (m_1)\mathcal{Y}(n, k)m_2z^{-n-1}(\log z)^k
\]
(2.3)
such that \(\mathcal{Y}(\cdot, z): M_1^a \times M_2^b \rightarrow M_3^{a+b}[\{z\}][\log z]\) satisfying
1. for all \(m_i \in M_i\) \((i = 1, 2)\), \((m_1)^\mathcal{Y}(n_i, N, k)m_2 = 0\) for sufficiently large \(N \in \mathbb{N}\),
2. \(\mathcal{Y}(L^M_{-1} m_1, z) = \partial_z \mathcal{Y}(m_1, z)\),
3. for all homogeneous \(a \in V\) and \(m_1 \in M_1\).
\[
(\mathcal{Y}(z_1 - z_2)) a^M_2(z_1) \mathcal{Y}(m_1, z_2) = z_2^{-1} \delta \left( \frac{z_1 - z_2}{z_2} \right) \mathcal{Y}(a^M_2(z_0) m_1, z_2).
\]
(2.4)
A general intertwining operator of type \((M_1, M_2)\) is a sum of parity-homogeneous ones and the vector superspace consisting of them is denoted by \(I_V\). In [HLZ1]-[HLZ8], quite a similar notion called a \(P(x)\)-intertwining map was introduced. Essentially, it is obtained from an intertwining operator by evaluation of \(z\) at a complex number in \(x \in C \setminus \{0\}\). To avoid the problem of convergence, in the definition of \(P(x)\)-intertwining map \(I_x(\cdot, \cdot)\), (2.3) is replaced by a formal one
\[
I_x(\cdot, \cdot): M_1 \times M_2 \rightarrow \overline{M}_3,
\]
where \(\overline{M}\) denote the algebraic completion \(\overline{M} = \overline{M_0} \oplus \overline{M_1}\) with \(\overline{M_i} = \prod_{\Delta \in \mathcal{C}} M_{\Delta,i}\). By [CKM1, Proposition 3.15], intertwining operators and \(P(x)\)-intertwining maps are in one-to-one correspondence by evaluation of \(z\) at \(x\) and extension from the point \(z = x\) by the differential equation (2) and we denote by \(I_x(M_1, M_2)\) the (super)space of \(P(x)\)-intertwining map of type \((M_1, M_2)\). For objects \(M_1\) and \(M_2\) in \(V\)-mod, if the superfunctor from \(V\)-mod to the supercategory of vector superspaces over \(C\) given by \(N \mapsto I_x(M_1, M_2)\) is representable, then the representing object, which is unique up to isomorphisms, is called the \(P(x)\)-tensor product and denoted by \(M_1 \boxtimes_P(x) M_2\). It admits a canonical \(P(x)\)-intertwining map \(I_x(M_1, M_2)(\cdot, \cdot): M_1 \boxtimes_P(x) M_2 \rightarrow \overline{M_1 \boxtimes_P(x) M_2}\) and thus a canonical intertwining operator
\[
\mathcal{Y}(M_1 \boxtimes_P(x) M_2)(\cdot, z): M_1 \times M_2 \rightarrow \overline{M_1 \boxtimes_P(x) M_2}[\{z\}][\log z].
\]
Note that the existence of \(P(x)\)-tensor product for some \(x\) implies that for all the other values in \(C \setminus \{0\}\) and they are all isomorphic \(M_1 \boxtimes_P(x) M_2 \cong M_1 \boxtimes_P(x') M_2\) by an even morphism [CKM1, Corollary 3.36].
Now suppose the existence of \(P(x)\)-tensor product for all objects in \(V\)-mod. It follows from the universality of representing objects that morphisms \(f_i: M_i \rightarrow N_i\) \((i = 1, 2)\) induce a unique morphism
\[
f_1 \boxtimes_P(x) f_2: M_1 \boxtimes_P(x) M_2 \rightarrow N_1 \boxtimes_P(x) N_2,
\]
which gives rise to a superfunctor \(\boxtimes_P(x): V\text{-mod} \times V\text{-mod} \rightarrow V\text{-mod}\). Moreover, the \(P(x)\)-intertwining map \(M_1 \boxtimes_P(x) M_2 \rightarrow M_1 \boxtimes_P(x) M_1\) defined by
\[
(m_1, m_2) \mapsto (-1)^{\overline{m}_1 \overline{m}_2} e^{-L(1)} \mathcal{Y}(M_1 \boxtimes_P(x) M_1)(m_1, z)m_2\]
induces a unique morphism \(\mathcal{R}^+_P(x, M_1, M_2): M_1 \boxtimes_P(x) M_2 \rightarrow M_2 \boxtimes_P(x) M_1\). Then under several assumptions on convergence properties for iterated \(P(x)\)-tensor products among several values \(x \in C \setminus \{0\}\), the category \(V\text{-mod}\) equipped with \(\boxtimes_P(x)\) and \(\mathcal{R}^+\) has the structure of vertex tensor supercategory. It is first established in
of simple currents (equivalently, simple invertible objects). The set $\text{Pic}(V)$ of such objects is an immediate consequence of Proposition 2.1, when $V = V_0$ and $\mathbb{Z}_{\geq 0}$-graded with trivial strong $A$-grading, and $V$-mod is restricted to the usual module category in this setting, that is, the full subcategory $V$-mod$_0 \subset V$-mod consisting of objects $M$ such that $M = M_0$. In particular, when $V = V_0$, the $\mathbb{C}_2$-cofiniteness condition, i.e., $\dim V/V(-2)V < \infty$ is sufficient by [Hun2, Proposition 4.1, Theorem 4.13]. In this paper, we consider the case for $V$-mod with $\frac{1}{2}\mathbb{Z}_{\geq 0}$-grading but trivial strong $A$-grading or with lattice vertex superalgebras associated with non-degenerate integral lattices. The first one is an immediate consequence of [CKM1, Theorem 3.65] and [Hun2, Proposition 4.1, Theorem 4.13]:

**Proposition 2.1.** Let $V$ be a $\mathbb{C}_2$-cofinite, $\frac{1}{2}\mathbb{Z}_{\geq 0}$-graded vertex operator superalgebra of CFT type. Then $V$-mod is a $\mathbb{C}$-linear additive braided monoidal supercategory satisfying the following property (P): all Hom spaces are finite dimensional, all object have finite length, and the fusion product $\boxtimes$ is right exact.

**Proof.** We start with the subalgebra $V_0^\mathbb{Z} = \bigoplus_{\Delta \in \mathbb{Z}_{\geq 0}} V_{\Delta,0}$, which is a vertex operator algebra of CFT type characterized as the fixed-point by finite order automorphisms $\theta_V = e^{2\pi i L_0}$ and $P_e = \text{id}_{V_0^\mathbb{Z}} - \text{id}_{V_0^\mathbb{Z}}$. Then $V_0^\mathbb{Z}$ is also $\mathbb{C}_2$-cofinite by [Mi] and thus $V_0^\mathbb{Z}$-mod$_0$ has a structure of $\mathbb{C}$-linear braided tensor category satisfying (P) by [Hun2, Theorem 3.24, 4.13] (moreover, it is finite as abelian category). Note that $V$ itself is an object in $V_0^\mathbb{Z}$-mod and that $V_0^\mathbb{Z}$-mod is equivalent to the superization of $V_0^\mathbb{Z}$-mod$_0$ as $\mathbb{C}$-linear additive braided monoidal supercategory, see Remark A.18 for the definition of superization. Since $V$ is an algebra object in $V_0^\mathbb{Z}$-mod, we may consider $V$-mod through the category $\text{Rep}^0(V)$ of local $V$-modules in $V_0^\mathbb{Z}$-mod, see §A.4 for a review on this subject. It follows from the construction that $\text{Rep}^0 V$ is a $\mathbb{C}$-linear additive braided monoidal supercategory [KO, CKM1] induced by $V_0^\mathbb{Z}$-mod satisfying the same property (P). Moreover, $\text{Rep}^0 V$ is equivalent to $V$-mod as $\mathbb{C}$-linear additive supercategories by [HKL, CKL] and in addition is the same as the one given in [HLZ1]-[HLZ8] by [CKM1, Theorem 3.65].

In the following, we denote by $\text{Irr}(V)$ the complete set of representatives of simple objects in $V$-mod and by $\text{Pic}(V) \subset \text{Irr}(V)$ the Picard group, i.e., the set consisting of simple currents (equivalently, simple invertible objects). The set $\text{Pic}(V)$ forms a group by tensor product, see Appendix A for their properties.

### 2.3. Lattice vertex superalgebras

Let us first fix some notation for Heisenberg vertex algebras and their Fock modules. Given a finite dimensional (even) vector space $a$ equipped with a bilinear form $\langle \cdot | \cdot \rangle$, we denote by $\pi^a$ the associated Heisenberg vertex algebra, which is generated by the fields $a(z)$ ($a \in a$) with OPE

$$a(z) b(w) \sim \frac{(a|b)}{(z-w)^2}.$$ 

When $\langle \cdot | \cdot \rangle$ is non-degenerate, $\pi^a$ has a conformal vector of central charge $\dim a$ by the Sugawara construction. For $\lambda \in a^*$, we denote by $\pi^a_\lambda$ the Fock module of $\pi^a$ of highest weight $\lambda$. It is generated by a highest weight vector which we denote by $|\lambda\rangle$.
Correspondences of categories for subregular $W$-algebras and principal $W$-superalgebras

2.4. Simple current extensions by lattices. Let $V$ be a simple, $C_2$-cofinite, $\frac{1}{2}\mathbb{Z}_{>0}$-graded vertex operator superalgebra of CFT type with trivial strong grading and $V_L$ be the lattice vertex superalgebra associated with $L$ be a non-degenerate integral lattice with strong $L$-grading. Here we consider the module category of simple current extensions of $V \otimes V_L$, following [CKM1] and Appendix A.

Since $V \otimes V_L$ is a simple, strongly $L$-graded, $C_2$-cofinite, $\frac{1}{2}\mathbb{Z}_{>0}$-graded vertex operator superalgebra of CFT type, we have the $\mathbb{C}$-linear additive braided monoidal supercategory $V \otimes V_L$-mod by the previous two subsections. It is naturally isomorphic to the Deligne product $(V$-mod) $\otimes (V_L$-mod) by [CKM2, Theorem 5.5]:

$$V \otimes V_L$-mod \simeq (V$\otimes$-mod) $\otimes (V_L$-mod) $$

where $\Omega_a(M) := \{ m \in M \mid \forall \lambda \in L, \ n \geq 0, \ h_0 \lambda(m) = \delta_{n,0} \lambda(a)m, \}$. The equivalence implies the decomposition of the Picard groups:

$$\text{Pic}(V) \times \text{Pic}(V_L) \simeq \text{Pic}(V \otimes V_L), \quad (M, V_{a+L}) \mapsto M \otimes V_{a+L}. \quad (2.6)$$

Let $N$ be a sublattice of $L'$ such that $L \subset N \subset L'$. We consider an algebra object $\mathcal{E}$ in $V \otimes V_L$-mod of the form

$$\mathcal{E} = \bigoplus_{a \in N/L} E_a, \quad E_a = S_a \otimes V_{a+L}, \quad (2.7)$$

where $\{S_a\}_{a \in N/L}$ is a subgroup of $\text{Pic}(V)$ with $S_0 = V$. Then $\mathcal{E}$ is a categorical simple current extension, see §A.5. By Proposition 2.1 and 2.2, $V \otimes V_L$-mod satisfies Assumption A.2, A.4 and A.23. As in §A.5, we also suppose that $V \otimes V_L$-mod satisfies Assumption A.15 and that $\mathcal{E}$ satisfies (S1) in §A.5 and equations

$$\theta_{a+b} \cdot \theta_{E_a} \cdot \theta_{E_b} = \theta_{E_{a+b}}, \quad a, b \in N/L,$$
where \( \theta_M = e^{2\pi \sqrt{-1} \theta_M} \). Then (S2) in §A.5 is automatically satisfied since \( \text{Irr}(V_L) = \text{Pic}(V_L) \). Thus, by [CKM1, Theorem 3.42], \( \mathcal{E} \) is a simple, \( C_2 \)-cofinite, strongly \( L' \)-graded, \( \mathbb{Z}_{\geq 0} \)-graded vertex operator superalgebra of CFT type and we can use all the results in Appendix A. By Theorem A.12, we have the decomposition

\[
\text{V-mod} = \bigoplus_{\phi \in (N/L)^{\vee}} \text{V-mod}_{\phi},
\]

where \( (N/L)^{\vee} = \text{Hom}_{\text{Grp}}(N/L, \mathbb{C}^\times) \) and \( \text{V-mod}_{\phi} \) is the full subcategory of \( \text{V-mod} \) consisting of objects \( M \) such that \( \mathcal{M}_{S_n,M} = \phi(a) \text{id}_{S_n \otimes M} \) for \( a \in N/L \). We write \( \phi_M \) for \( \phi \) if \( M \in \text{V-mod}_{\phi} \). Similarly,

\[
\text{V}_{L'}-\text{mod} = \bigoplus_{\phi \in (N/L)^{\vee}} \text{V}_{L'}-\text{mod}_{\phi}.
\]

Since \( \text{V}_{L'}-\text{mod}_{\phi} \) is semisimple, it follows that

\[
\text{Irr}(\text{V}_{L'}-\text{mod}_{\phi}) = \left\{ V_{a+L} \mid b \in L'/L, \; e^{2\pi \sqrt{-1} \pi \beta(a)} = \phi(a) \text{ for all } b \in N/L \right\}.
\]

The group homomorphism

\[
\gamma: L' \to (N/L)^{\vee}, \quad a \mapsto (\gamma_a: N/L \ni b \mapsto \gamma_a(b) = e^{2\pi \sqrt{-1} \pi \beta(a)} \in \mathbb{C}^\times)
\]

induces an isomorphism \( L'/N' \simeq (N/L)^{\vee} \) and then \( \text{Irr}(\text{V}_{L'}-\text{mod}_{\phi}) = \{ V_{a+L} \mid a \in L'/L, \; \gamma_a = \phi \} \). Let \( (V \otimes V_L)^0 \) be the full subcategory of \( V \otimes V_{L'}-\text{mod} \) whose objects \( M \) satisfy that \( \mathcal{M}_{E,M} = \text{id}_{E \otimes M} \), or equivalently, that \( M \) is local for \( \mathcal{E} \). Then it follows from (2.5) that

\[
(V \otimes V_{L'}-\text{mod})^0 \simeq \bigoplus_{\phi \in (N/L)^{\vee}} (\text{V-mod})_{\phi} \otimes (\text{V}_{L'}-\text{mod})_{\phi^{-1}}.
\]

The following is a slight generalization to the irrational setting of [CKM1, Theorem 4.39, Theorem 4.41] and of [YY, Theorem 3.7, Theorem 3.13].

**Theorem 2.3** (cf. [CKM1, YY]). Let \( V, V_L \) and \( \mathcal{E} \) as above.

(i) The complete set of isomorphism classes of simple objects in \( \text{Rep}^0(\mathcal{E}) \simeq \mathcal{E}\text{-mod} \) is

\[
\text{Irr}(\mathcal{E}) \simeq \{ (M, a) \in \text{Irr}(V) \times (L'/L) \mid \phi_M \phi_{V_{a+L}} = 1 \}/(N/L)
\]

by \( (M, a) \mapsto \mathcal{F}(M \otimes V_{a+L}) = \mathcal{E} S_{V_{a+L}}(M \otimes V_{a+L}) \). In particular, we have \( |\text{Irr}(\mathcal{E})| = |\text{Irr}(V)| / |N'/L|/|N/L| \) and

\[
\text{Pic}(\mathcal{E}) \simeq \{ (M, a) \in \text{Pic}(V) \times (L'/L) \mid \phi_M \phi_{V_{a+L}} = 1 \}/(N/L).
\]

(ii) We have a ring isomorphism

\[
\mathcal{K}(\mathcal{E}) \simeq \left( \mathcal{K}(V) \otimes_{\mathbb{Z}[N/L]} \mathbb{Z}[L'/L] \right)^{N/L} \tag{2.9}
\]

where the tensor product over \( \mathbb{Z}[N/L] \) is given by \( [M \boxtimes S_a] \otimes b = [M] \otimes (-a + b) \) for \( a \in N/L \).

**Proof.** (i) is immediate from Corollary A.25. For (ii), note that \( \mathcal{K}(V_L) \simeq \mathbb{Z}[L'/L] \), \( (V_{a+L}) \mapsto a \). Since \( \mathcal{K}(\mathcal{E}) \) and \( \mathcal{K}(V_L) \) are \( (N/L)^{\vee} \)-graded \( \mathbb{Z}[N/L] \)-algebras by Theorem A.16, the fusion algebra \( \mathcal{K}((V \otimes V_L)^0) \) is a diagonal \( N/L \)-invariant subalgebra

\[
\mathcal{K}((V \otimes V_L)^0) \simeq \left( \mathcal{K}(V) \otimes \mathcal{K}(V_L) \right)^{N/L}
\]

\[
\simeq \left( \mathcal{K}(V) \otimes \mathbb{Z}[L'/L] \right)^{N/L}.
\]
Hence, by Corollary A.26 the induction functor $\mathcal{E} \otimes_{V \otimes V_L} \cdot$ induces an isomorphism

$$\mathcal{K}(\mathcal{E}) \simeq \mathcal{K}((V \otimes V_L)\text{-mod})^0/\mathcal{J}$$

where $\mathcal{J}$ is generated by $[M] \otimes b - [M \otimes S_a] \otimes (a + b)$ for an object $M$ in $V$-mod, $a \in N/L$ and $b \in L'/L$. This implies (2.9). □

Next, we consider an inverse statement of Theorem 2.3. Recall that $\text{Rep}(\mathcal{E})$ is a $\mathfrak{C}$-linear monoidal supercategory and the induction functor

$$\mathcal{F}(\cdot) = \mathcal{E} \otimes_{V \otimes V_L}(\cdot) : (V\text{-mod}) \otimes (V_L\text{-mod}) \to \text{Rep}(\mathcal{E})$$

is a $\mathfrak{C}$-linear monoidal superfunctor. By (S2), the functors

$$V\text{-mod} \to \text{Rep}(\mathcal{E}), \quad M \mapsto \mathcal{F}(M \otimes V_L),$$

$$V_L\text{-mod} \to \text{Rep}(\mathcal{E}), \quad M \mapsto \mathcal{F}(V \otimes M),$$

are embeddings so that we may consider $V$-mod and $V_L$-mod are $\mathfrak{C}$-linear monoidal subcategories of $\text{Rep}(\mathcal{E})$. From now on, we will write $V_{a+L}$ instead of $\mathcal{F}(V \otimes V_{a+L})$ as an object in $\text{Rep}(\mathcal{E})$ for $a \in L'/L$ by abuse of notation.

**Lemma 2.4.** Every simple object $M$ in $\text{Rep}(\mathcal{E})$ is isomorphic to $\widetilde{M} \otimes_{\mathcal{E}} V_{a+L}$ for some $\widetilde{M} \in \text{Irr}(\mathcal{E})$ and $a \in L'/L$. Moreover, $N'/L$ acts simply transitively on the set $\{(\widetilde{M}, a) \in \text{Irr}(\mathcal{E}) \times L'/L \mid \widetilde{M} \otimes_{\mathcal{E}} V_{a+L} \in \text{Irr}(\text{Rep}(\mathcal{E}))\}$ by

$$b \cdot (\widetilde{M}, a) = (\widetilde{M} \otimes_{\mathcal{E}} V_{b+L}, a - b), \quad b \in N'/L.$$ 

**Proof.** By Proposition A.24 (ii), we have $M \simeq \mathcal{F}(\widetilde{M} \otimes V_{a+L})$ for some $\widetilde{M} \in \text{Irr}(V)$ and $a \in L'/L$. Since $\mathcal{F}$ is a monoidal superfunctor, we may decompose

$$\mathcal{F}(\widetilde{M} \otimes V_{a+L}) \simeq \mathcal{F}(\widetilde{M} \otimes V_{b+L}) \otimes_{\mathcal{E}} V_{a-b+L} \quad (2.10)$$

for any $b \in L'/L$. By using the isomorphism $\gamma : L'/N' \simeq (N/L)'$, we may take $b \in L'/L$ such that $\widetilde{M} \otimes V_{b+L}$ is a local $V \otimes V_L$-module with respect to $\{S_e \otimes V_{b+L}\}_{e \in N/L}$. Hence, $\mathcal{F}(\widetilde{M} \otimes V_{b+L}) \in \text{Irr}(\mathcal{E})$. This proves the first part of the statement. The element $b \in L'/L$ in the decomposition (2.10) such that $\widetilde{M} \otimes V_{b+L}$ is local as above are uniquely determined up to $N'/L$. This implies the second part of the statement. □

Since objects of $\text{Rep}(\mathcal{E})$ consist of pairs $(M, \mu_M)$ with $M \in (V\text{-mod}) \otimes (V_L\text{-mod})$ and a morphism $\mu_M : \mathcal{E} \otimes_{\mathcal{V}} V_L M \to M$ in $(V\text{-mod}) \otimes (V_L\text{-mod})$, we have two families of monodromy actions

$$\mathcal{M}_N, \bullet = \mathcal{M}_{N \otimes V_L}, \bullet, \quad N \in V\text{-mod}, \quad \mathcal{M}_{V_{a+L}}, \bullet = \mathcal{M}_{V \otimes V_{a+L}}, \bullet, \quad a \in L'/L.$$ 

Clearly, any object $M$ in the subcategory $V$-mod of $\text{Rep}(\mathcal{E})$ has trivial monodromy with $V_{a+L}$, i.e., $\mathcal{M}_{V_{a+L}} \cdot (M \otimes V_L) = \text{id}$. Conversely, any simple object in $\text{Rep}(\mathcal{E})$ satisfying this monodromy-free property lies in $V$-mod. This implies the following slight generalization in the irrational setting of [CKM1, YY] as Theorem 2.3.

**Theorem 2.5** (cf. [CKM1, YY]). Let $V, V_L$ and $\mathcal{E}$ as above.

(i) The set $\text{Irr}(V)$ has one-to-one correspondence to

$$\{(M, a) \in \text{Irr}(\mathcal{E}) \times (L'/L) \mid \mathcal{M}_{M, \mathcal{E}} V_{a+L} = \text{id}, \quad (\forall b \in N'/L)\} / (N'/L)$$

by $(M, a) \mapsto \widetilde{M}$ such that $\mathcal{F}(\widetilde{M} \otimes V_L) \simeq M \otimes_{\mathcal{E}} V_{a+L}$. In particular, we have $|\text{Irr}(V)| = |\text{Irr}(\mathcal{E})| \cdot |N/L| / (N'/L)$ and $\text{Pic}(V)$ is naturally isomorphic to

$$\{(M, a) \in \text{Pic}(\mathcal{E}) \times (L'/L) \mid \mathcal{M}_{M, \mathcal{E}} V_{a+L} = \text{id}, \quad (\forall b \in N'/L)\} / (N'/L)$$

Correspondences of categories for subregular $W$-algebras and principal $W$-superalgebras
(ii) We have an isomorphism of rings

\[ \mathcal{K}(V) \cong \left( \mathcal{K}(E) \otimes_{\mathbb{Z}[N'/L]} \mathbb{Z}[L'/L] \right)^{N'/L} \]  

(2.11)

where the tensor product over \( \mathbb{Z}[N'/L] \) is given by \([M \otimes_k V_{a+L}] \otimes b = [M] \otimes (a + b)\) for \( a \in N'/L \).

Proof. (i) is immediate from Lemma 2.4. We show (ii). Since every object in \( V\text{-mod} \otimes V_L\text{-mod} \) has finite length, so does every object in \( \text{Rep}(\mathcal{E}) \) and \( \mathcal{E}\text{-mod} \). Thus, we may take bases of \( \mathcal{K}(\text{Rep}(\mathcal{E})) \) and \( \mathcal{K}(\mathcal{E}) \) by complete sets of simple objects. By Lemma 2.4, we have a natural isomorphism

\[ \mathcal{K}(\text{Rep}(\mathcal{E})) \cong \mathcal{K}(\mathcal{E}) \otimes_{\mathbb{Z}[L'/N]} \mathcal{K}(V_L) \cong \mathcal{K}(\mathcal{E}) \otimes_{\mathbb{Z}[L'/N]} \mathbb{Z}[L'/L]. \]

Note that \( \mathcal{K}(\text{Rep}(\mathcal{E})) \) is an \( (N'/L)\)-graded ring by the monodromy action of \( \{V_{a+L}\}_{a \in N'/L} \). Since the \( (N'/L)\)-invariant subring is spanned by \( \text{Irr}(V) \), we obtain the assertion. \( \square \)

3. Feigin–Semikhatov Duality

In this section we review the Feigin–Semikhatov duality [CGN, CL2] between the subregular \( W \)-algebra and the principal \( W \)-superalgebra conjectured by Feigin and Semikhatov in [FS].

3.1. Affine vertex superalgebras and \( W \)-superalgebras. Let \( \mathfrak{g} \) be a finite-dimensional Lie superalgebra equipped with a non-degenerate even supersymmetric invariant bilinear form \( (\cdot | \cdot) \) and \( \mathfrak{h} \) be its Cartan subalgebra. When \( \mathfrak{g} \) is simple, we always normalize the form \( (\cdot | \cdot) \) so that the highest even root of \( \mathfrak{g} \) has square length 2. Let \( \bar{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] \oplus CK \) be the affinization of \( \mathfrak{g} \) defined by

\[ [at^m, bt^n] = [a, b]t^{m+n} + m(a|b)\delta_{m+n, 0}K, \quad [K, at^m] = 0 \]

for \( a, b \in \mathfrak{g} \) and \( m, n \in \mathbb{Z} \). The induced \( \bar{\mathfrak{g}} \)-module

\[ V^k(\mathfrak{g}) = U(\bar{\mathfrak{g}}) \otimes_{U(\mathfrak{g}[t] \oplus CK)} C_k, \]

where \( C_k \) is the \((1|0)\)-dimensional \( \mathfrak{g}[t] \oplus CK \)-module by \( \mathfrak{g}[t] = 0 \) and \( K = k \text{id} \), has a vertex superalgebra structure and is called the universal affine vertex superalgebra associated to \( \mathfrak{g} \) at level \( k \). It is \( \mathbb{Z}_{\geq 0} \)-graded conformal if \( k + h^\vee \neq 0 \) by the Sugawara construction, where \( h^\vee \) is the dual Coxeter number of \( \mathfrak{g} \). Similarly, for a dominant weight \( \lambda \in P_+ \) of \( \mathfrak{g} \), let \( E_{\lambda, k} \) be the irreducible highest-weight representation of \( \mathfrak{g} \oplus CK \) of highest-weight \( \lambda \) on which \( K \) acts by \( k \text{id} \). This lifts to a \( \mathfrak{g}[t] \oplus CK \)-module by \( \mathfrak{g}[t] = 0 \) and we obtain the induced \( V^k(\mathfrak{g}) \)-module

\[ V^k(\lambda) = U(\bar{\mathfrak{g}}) \otimes_{U(\mathfrak{g}[t] \oplus CK)} E_{\lambda, k}. \]

Suppose that \( \mathfrak{g} \) is simple and take an even nilpotent element \( f \) in \( \mathfrak{g}_0 \) together with a good \( \frac{1}{2}\mathbb{Z} \)-grading \( \Gamma \) of \( \bar{\mathfrak{g}} \) adapted to \( f \) (see [KRW] for the definition). In [KRW], the universal \( W \)-algebra \( W^k(\mathfrak{g}, f; \Gamma) \) is defined by the generalized Drinfeld-Sokolov reduction of \( V^k(\mathfrak{g}) \) associated to \( (\mathfrak{g}, f, \Gamma) \). When \( k + h^\vee \neq 0 \), it has a standard conformal vector \( \omega_f \) and then is a \( \frac{1}{2}\mathbb{Z}_{\geq 0} \)-graded vertex operator superalgebra of CFT type. Once we fix \( \Gamma \), we abbreviate \( W^k(\mathfrak{g}, f; \Gamma) \) as \( W^k(\mathfrak{g}, f) \) and denote by \( W_k(\mathfrak{g}, f) \) its unique simple quotient. In the next two subsections, we consider two \( W \)-superalgebras which we consider in this paper.
3.2. Subregular $W$-algebras. Let $g = sl_n$, $f = f_{\text{sub}}$ be a subregular nilpotent element in $sl_n$ and $\Gamma$ the Dynkin grading corresponding to the weighted Dynkin diagram

for $m \in \mathbb{Z}_{\geq 1}$. Then the associated $W$-algebra $W^k(sl_n, f_{\text{sub}})$ is called the subregular $W$-algebra and $\mathbb{Z}_{\geq 0}$-graded (resp. $\frac{1}{2}\mathbb{Z}_{\geq 0}$-graded) with respect to $\omega_{\text{sub}} = \omega_\Gamma$ if $n = 2m$ (resp. $n = 2m + 1$). As in [CGN, §4.1] (see §3.5 for details), there exists a Heisenberg vertex subalgebra $\pi^H$ of $W^k(sl_n, f_{\text{sub}})$ such that $H^+_0$ acts diagonally with integer eigenvalues and

$$H^+(z)H^+(w) \sim \frac{\varepsilon_+}{(z-w)^2}, \quad \varepsilon_+ := \frac{n-1}{n(k+n)-1}. \quad (3.1)$$

The subregular $W$-algebra is strongly $\mathbb{Z}$-graded by $H^+_0$ and, accordingly, we work with the module category whose objects are strongly $\mathbb{C}$-graded by $H^+_0$.

3.3. Principal $W$-superalgebras. Let $g = sl_{1|n}$, $f = f_{\text{prin}}$ be a principal nilpotent element in $(sl_{1|n})_0 = sl_n$ and $\Gamma$ the $\mathbb{Z}$-grading of $sl_{1|n}$ in [CGN, §3.3]. By choosing a set $\{\beta_i\}_{i=0}^{n-1}$ of simple roots of $sl_{1|n}$ so that $(\beta_i|\beta_{i+1}) = -1$ and $\beta_0$ is a unique odd root, we can express the weighted Dynkin diagram for $\Gamma$ as

By [CGN, §4.1] (see §3.5 for details), there exists a Heisenberg vertex subalgebra $\pi^{H^+}$ of $W^k(sl_{1|n}, f_{\text{prin}})$ such that $H^-_0$ acts diagonally with integer eigenvalues and

$$H^-(z)H^-(w) \sim \frac{\varepsilon_-}{(z-w)^2}, \quad \varepsilon_- := \frac{n}{n-1}(k+n-1)+1. \quad (3.2)$$

In this paper we use $\omega_{\text{sup}} := \omega_\Gamma - \frac{1}{2}\partial H^-$ so that $W^k(sl_{1|n}, f_{\text{prin}})$ is $\frac{1}{2}\mathbb{Z}_{\geq 0}$-graded (resp. $\mathbb{Z}_{\geq 0}$-graded) if $n = 2m$ (resp. $n = 2m+1$). Similarly to the subregular case, the principal $W$-superalgebra is strongly $\mathbb{Z}$-graded by $H^-_0$ and we work with the module category whose objects are strongly $\mathbb{C}$-graded by $H^-_0$.

1In [CGN], another good grading $\Gamma_o$ corresponding to

is used. Since $W^k(sl_n, f_{\text{sub}}, \Gamma_o)$ is isomorphic to $W^k(sl_n, f_{\text{sub}}; \Gamma_o)$ as a vertex algebra (see [DSK] and [AKM, Theorem 3.2.6.1]), the element $H^+$ corresponds to $H_1$ in [CGN, §4.1].
3.4. Feigin–Semikhatov Duality. Let \( \kappa \in \mathbb{C} \setminus \{0, \frac{n}{n-1}\} \) and set

\[
W^+ := W^{-n}(\mathfrak{s}l_n, f_{\text{sub}}), \quad W^- := W^{+(n-1)}(\mathfrak{sl}_1, f_{\text{prim}}).
\]

By (3.1) and (3.2), the condition \( \kappa \neq \frac{n}{n-1} \) implies \( \varepsilon \neq 0 \). Then, by [KW, KS], the Heisenberg vertex subalgebra \( \pi^H \) acts semisimply on \( \mathcal{W} \) and thus we have a decomposition

\[
\mathcal{W} \simeq \bigoplus_{\lambda \in \mathbb{Z}} \Omega^{H \mathcal{W}}_{\lambda} (\mathcal{W}) \otimes \pi^H
\]

as a \( \text{Com}(\pi^H, \mathcal{W}) \otimes \pi^H \)-module, where

\[
\Omega^{H \mathcal{W}}_{\lambda} (\mathcal{W}) := \{ w \in \mathcal{W} \mid H^w_{(n)} w = \delta_{n,0} \lambda w \text{ for } n \geq 0 \}.
\]

By using lattice vertex superalgebras \( \mathcal{V}_+ = V_{\mathbb{Z}^n} \) with \( (\phi^\pm | \phi^\pm) = \pm 1 \) and certain free field realizations (see §3.5 below for details), we obtain the Kazama–Suzuki coset construction and its inverse [CGN, Corollary 5.15]:

\[
\mathcal{K}_\pm \colon \mathcal{W}_+ \cong \text{Com}(\pi^H_{\pm}, \mathcal{W} \otimes \mathcal{V}_\pm),
\]

where \( H^z \pm := \frac{1}{n} \sum_{i=1}^{n} (n-i) \alpha_i(z) - \phi_-(z) \). We have

\[
H^z \pm (w) H^z \mp (w) \sim \frac{\bar{\varepsilon}^2}{(z - w)^2}, \quad \bar{\varepsilon} := \varepsilon \pm 1 = \pm \left( \frac{n-1}{n} \kappa \right)^{\pm 1}.
\]

The restriction of (3.6) gives the Feigin–Semikhatov duality [CGN, CL2]:

\[
\text{Com}(\pi^H_+, \mathcal{W}_+) \cong \text{Com}(\pi^H_-, \mathcal{W}_-).
\]

3.5. Free field realization revisited. Here we briefly review the free field realization of \( \mathcal{W} \) used in [CGN] to construct the isomorphism (3.6) for later use.

Let \( \pi^h^+ \) denote the Heisenberg vertex algebra generated by fields \( \alpha_i(z) (i = 1, \ldots, n-1) \) corresponding to the simple roots of \( \mathfrak{sl}_n \), which satisfy the OPEs

\[
\alpha_i(z) \alpha_j(w) \sim \frac{\delta_{i,j}}{(z-w)^2}.
\]

Then the Miura map for \( \mathcal{W}^+ \) [KW] induces an embedding

\[
p_+ : \mathcal{W}^+ \hookrightarrow \pi^h^+ \otimes \mathcal{U}^+, \quad \mathcal{U}^+ = V_{\mathbb{Z}^n}^{\geq 0},
\]

which sends \( H^z(z) \) to \( \frac{1}{n} \sum_{i=1}^{n-1} (n-i) \alpha_i(z) - \phi_-(z) \). Here \( \mathcal{U}^+ \) denotes the vertex subalgebra of \( V_{\mathbb{Z}^n}^{\geq 0} \) generated by \( \{ \phi^+ \}(z), \phi^-(z), m(\phi^+ + \phi^-) \}(z) \mid m \in \mathbb{Z} \).

Similarly, let \( \pi^h^- \) denote the Heisenberg vertex algebra generated by fields \( \beta_i(z) (i = 0, \ldots, n-1) \) corresponding to the simple roots of \( \mathfrak{sl}_1 \), satisfying the OPEs

\[
\beta_i(z) \beta_j(w) \sim \frac{1}{(z-w)^2}.
\]

Then the Miura map for \( \mathcal{W}^- \) [KW] induces an embedding

\[
p_- : \mathcal{W}^- \hookrightarrow \pi^h^- \otimes \mathcal{U}^-,
\]

with \( \mathcal{U}^- = V_{\mathbb{Z}^n}^{> 0} \), which sends \( H^z(z) \) to \( -\frac{1}{n} \sum_{i=0}^{n} (n-i) \beta_i(z) + \phi^+(z) \).

We introduce vertex superalgebra embeddings \( \mathcal{K}_\pm \) by

\[
\beta_0(z) \rightarrow \frac{1}{n} \phi^+(z), \quad \beta_1(z) \rightarrow -\frac{1}{n} \alpha_1(z) + \phi^+(z), \quad \beta_i(z) \rightarrow -\frac{1}{n} \alpha_i(z), \quad (i = 2, \ldots, n-1), \quad \phi^+(z) \rightarrow \phi^+(z) + \phi^-(z), \quad |m\phi^+ \rangle \rightarrow |m(\phi^+ + \phi^-) \rangle \otimes |m\phi^+ \rangle,
\]

and use them to embed \( \mathcal{W} \) into \( \mathcal{W} \otimes \mathcal{V}_\pm \):
where \( \phi^+_L(z) \) (resp. \( \phi^+_R(z) \)) denotes the field \( \phi^+(z) \) in \( \mathcal{U}^+ \) (resp. \( \mathcal{V}^+ \)), and by

\[
\mathsf{KS}^- : \pi^{\mathcal{h}+} \otimes \mathcal{U}^+ \to \pi^{\mathcal{h}-} \otimes \mathcal{U}^- \otimes \mathcal{V}^-,
\]

\[
\alpha_i(z) \mapsto -\kappa(\beta_i(z) - \phi^+(z) - \phi^-(z)), \quad \alpha_i(z) \mapsto -\kappa \beta_i(z), \quad (i = 2, \ldots, n-1),
\]

\[
\phi^+(z) \mapsto \mp \kappa \beta_0(z) + \phi^+(z), \quad |m(\phi^+ + \phi^-)| \mapsto |m\phi^+| \otimes |m\phi^-|.
\]

Then, \( \mathsf{KS}_{\pm} \) is obtained as a restriction of \( \mathsf{KS}_{\pm} \) by \( p_{\pm} \) in the following way:

\[
\begin{array}{c c c c}
\mathcal{W}^- & \mathsf{KS}^+ & \mathcal{W}^+ \otimes \mathcal{V}^+ & \mathcal{W}^+ \otimes \mathcal{V}^- \\
\pi^{\mathcal{h}-} \otimes \mathcal{U}^- & \mathsf{KS}^+ & \pi^{\mathcal{h}+} \otimes \mathcal{U}^+ \otimes \mathcal{V}^+ & \pi^{\mathcal{h}+} \otimes \mathcal{U}^+ \otimes \mathcal{V}^-.
\end{array}
\]

From this explicit form of \( \mathsf{KS}_{\pm} \), the following lemma is obtained straightforwardly:

**Lemma 3.1.**

1. We have \( \psi^\pm(z) := \mathsf{KS}_{\pm}(H^\pm(z)) = \tilde{z}^\pm \mathcal{H}^\pm(z) \pm \phi^\pm(z) \).
2. The composition \( g_{\pm} := \mathsf{KS}_{\mp} \circ \mathsf{KS}_{\pm} : \mathcal{W}^\pm \hookrightarrow \mathcal{W}^\pm \otimes \mathcal{V}^+ \otimes \mathcal{V}^- \) satisfies

\[
g_{\pm}(w) = w \otimes |\lambda \phi^+| \otimes |\lambda \phi^-|, \quad (w \in \Omega^{H^\pm}_{\lambda}(\mathcal{W}^\pm), \lambda \in \mathbb{Z}).
\]

Note that Lemma 3.1 (1) implies

\[
g_{\pm}(H^\pm(z)) = H^\pm(z) + \varepsilon_{\pm}(\phi^+(z) + \phi^-(z)).
\]

Hence, if we write a general element \( v \in \Omega^{H^\pm}_{\lambda}(\mathcal{W}^\pm) \otimes \pi^{H^\pm} \) as \( v = F(H^\pm)\mathfrak{p} \) where \( F(H^\pm) \) is a polynomial in the variables \( \mathcal{H}^\pm(m) \) \( (m \in \mathbb{Z}_{>0}) \) and \( \mathfrak{p} \in \Omega^{H^\pm}_{\lambda}(\mathcal{W}^\pm) \), we have

\[
g_{\pm}(v) = F\left(g_{\pm}(H^\pm)\right)\mathfrak{p} \otimes |\lambda \phi^+| \otimes |\lambda \phi^-| \quad (3.9)
\]

by Lemma 3.1 (2). Now, the formal Taylor expansion formula

\[
F(x + y) = \exp \left( y \frac{d}{dx} \right) F(x)
\]

implies the following:

**Corollary 3.2.** Define an even linear operator \( \mathcal{H}_{\pm} \) on \( \mathcal{W}^\pm \otimes \mathcal{V}^+ \otimes \mathcal{V}^- \) by

\[
\mathcal{H}_{\pm} = \sum_{m=1}^{\infty} \frac{1}{m} \mathcal{H}^\pm_{(m)} \otimes \left( \phi^+_{(-m)} + \phi^-_{(-m)} \right).
\]

Then, we have \( g_{\pm}(v) = \exp(\mathcal{H}_{\pm})(v \otimes |\lambda \phi^+| \otimes |\lambda \phi^-|) \) for \( v \in \mathcal{W}^\pm(\mathcal{W}^\pm) \otimes \pi^{H^\pm}_{\lambda} \).

**4. Fusion rules of \( \mathcal{W} \)-superalgebras: Rational cases**

Here we describe the fusion rings of \( \mathcal{W}_e(\mathfrak{sl}_n, f_{\text{sub}}) \) and \( \mathcal{W}_r(\mathfrak{sl}_{|n|}', f_{\text{prin}}) \) in the rational case, that is,

\[
\mathcal{W}_{s_{\text{sub}}}(n, r) := \mathcal{W}_{-n-1} \otimes_x (\mathfrak{sl}_n, f_{\text{sub}}), \quad \mathcal{W}_{s_{\text{prin}}}(n, r) := \mathcal{W}_{-(n-1)+} \otimes_x (\mathfrak{sl}_n, f_{\text{prin}}) \quad (4.1)
\]

for \( n, r \in \mathbb{Z}_{\geq 2} \) with \( \gcd(n-1, r+1) = 1 \) in terms of that of the simple affine vertex algebra of type \( A \).
4.1. Fusion rules of affine vertex algebras. Here we recall the fusion ring of
the simple affine vertex algebra \( L_n(sl_r) \). Let \( \mathfrak{h} \) denote the Cartan subalgebra of \( sl_r \),
\( Q \) the root lattice, \( P \) the weight lattice and \( P_+ \) the set of dominant integral weights.
The quotient \( P/Q \) is represented by the set of fundamental weights \( \{ \varpi_i \}_{i=1}^{r+1} \)
and is isomorphic to \( \mathbb{Z}_r \) as groups by \( \varpi_i \mapsto i \), where \( \varpi_0 = 0 \). The sum \( \varpi = \sum_{i=1}^{r+1} \varpi_i \)
is called the Weyl vector.

Let \( \mathfrak{h}_{aff} \) denote the Cartan subalgebra of the affinzation \( \widehat{sl}_r = sl_r[t, t^{-1}] \oplus \mathbb{C}K \),
which is \( \mathfrak{h}_{aff} = \mathfrak{h} \oplus \mathbb{C}K \), and \( \{ \Lambda_i \}_{i=0}^{r+1} \) the set of affine fundamental weights,
\( \widehat{P} \) the affine weight lattice, \( \widehat{P}_+ \) the set of dominant integral affine weights, and \( \rho = \sum_{i=0}^{r+1} \Lambda_i \) the affine Weyl vector. We identify \( h_{aff}^* \) with \( h^* \oplus \mathbb{C} \Lambda_0 \) and write \( h_{aff}^* \ni \lambda \mapsto \lambda + \Xi \in h^* \) for the natural projection. Note that we have a group homomorphism
\[
\pi_{P/Q} : \widehat{P} \rightarrow P \rightarrow P/Q \simeq \mathbb{Z}_r, \quad \Lambda_i \mapsto i.
\] (4.2)

By [FZ], the set of simple \( L_n(sl_r) \)-modules \( \text{Irr}(L_n(sl_r)) \) is identified as
\[
\widehat{P}_+^n(r) := \{ \lambda \in \widehat{P}_+ \mid \lambda(K) = n \} \xrightarrow{\sim} \text{Irr}(L_n(sl_r)), \quad \lambda \mapsto L(\lambda).
\] (4.3)
We denote by \( N_{\lambda,\mu}^\nu \) the fusion rule
\[
L(\lambda) \boxtimes L(\mu) \simeq \bigoplus_{\nu \in \widehat{P}_+^n(r)} N_{\lambda,\mu}^\nu L(\nu),
\] (4.4)
which is calculated by the Kac–Walten formula, see e.g. [DFMS, §16.2]. By [Fu],
the group of simple currents is \( \text{Pic}(L_n(sl_r)) = \{ L(n\Lambda_i) \}_{i \in \mathbb{Z}_r} \) and is isomorphic to \( \mathbb{Z}_r \) by \( L(n\Lambda_i) \mapsto i \). Thus we have a \( \mathbb{Z}_r \)-action on \( \text{Irr}(L_n(sl_r)) \) by fusion product
\[
L(n\Lambda_i) \boxtimes L(\mu) \simeq L(\sigma^*(\mu))
\] (4.5)
for \( i \in \mathbb{Z}_r \) and \( \mu \in \widehat{P}_+^n(r) \), where \( \sigma \) is the cyclic permutation \( \sigma(\Lambda_i) = \Lambda_{i+1} \).

4.2. Fusion rules of principal \( W \)-algebras. The principal \( W \)-algebra \( W^k(sl_r) := W^k(sl_r, j_{\text{prim}}) \) is the \( W \)-algebra associated with \( sl_r \), the principal nilpotent element \( j_{\text{prim}} \) and the principal \( \mathbb{Z} \)-grading on \( sl_r \). The simple quotient \( W_k(sl_r) \) at the level
\[
k = -r + \frac{r + n}{r + 1}, \quad (n \in \mathbb{Z}_r, \gcd(n - 1, r + 1) = 1),
\]
which we denote by \( W_{pr}(r, n) \), is \( C_2 \)-cofinite [Ar2] and rational [Ar3]. In addition, we have \( \text{Irr}(W_{pr}(r, n)) = \{ L_W(\lambda) \mid \lambda \in \widehat{P}_+^n(r) \} \) where
\[
L_W(\lambda) = H_0^\lambda \left( L(\lambda - (k + r)h + r\Lambda_0) \right), \quad \left( k = -r + \frac{r + n}{r + 1}, \quad \lambda \in \widehat{P}_+^n(r) \right), \quad (4.6)
\]
and \( H_0^\lambda(?) \) denotes the \( \sim \)-"-\-) reduction functor introduced in [FKW]. Note that \( L_W(\lambda) \) has the lowest conformal dimension
\[
h_W^\lambda := \frac{(\lambda|\lambda + 2\rho)}{2(k + r)} - (\lambda|\rho).
\] (4.7)
It follows from [FKW, C1, AvE1] that the fusion rules of \( W_{pr}(r, n) \) are given by
\[
L_W(\lambda) \boxtimes L_W(\mu) \simeq \bigoplus_{\nu \in \widehat{P}_+^n(r)} N_{\lambda,\mu}^\nu L_W(\nu),
\]
where \( N_{\lambda,\mu}^\nu \) is determined by (4.4). Thus we have an isomorphism of fusion rings
\[
\mathcal{K}(L_n(sl_r)) \xrightarrow{\sim} \mathcal{K}(W_{pr}(r, n)), \quad L(\lambda) \mapsto L_W(\lambda).
\] (4.8)
In particular, we have an isomorphism of groups
\[
\text{Pic}(W_{pr}(r, n)) = \{ L_W(n\Lambda_i) \}_{i \in \mathbb{Z}_r} \simeq \mathbb{Z}_r, \quad L_W(n\Lambda_i) \mapsto i.
\]
4.3. Fusion rules of subregular $W$-algebras. We consider the subregular $W$-algebra $W_{sb}(n, r)$ as in (4.1). The norm of the Heisenberg field (3.1) is $\varepsilon_+ = \frac{n}{2}$.

**Lemma 4.1** ([CL2]). Let $n, r \in \mathbb{Z}_{\geq 2}$ such that $\gcd(n - 1, r + 1) = 1$. Then

$$\text{Com} \left( \pi^{H^+}, W_{sb}(n, r) \right) \simeq W_{pr}(r, n). \tag{4.9}$$

**Proof.** Since (4.9) is proven for $r \geq 3$ [CL2, Corollary 6.15], we show the case $r = 2$. We first prove that $C_0 := \text{Com} \left( \pi^{H^+}, W_{sb}(n, 2) \right)$ is a conformal extension of $W_{pr}(2, n)$. By [AvEM, Lemma 2.8], it suffices to show that the asymptotic growth of $C_0$ is less than 1. Since $W_{sb}(n, 2)$ is of CFT type and $\pi^{H^+}$ acts semisimply on it, both $C_0$ and $\text{Com} \left( C_0, W_{sb}(n, 2) \right)$ are simple and the latter is isomorphic to a positive-definite lattice vertex algebra $V_L$ (cf. [LX]). Then $C_0$ is rational by [CKLR, Theorem 4.12] and its asymptotic growth coincides with the effective central charge $c_{\text{eff}}(C_0)$ by [AvEM, Proposition 2.4]. Since the effective central charge of a simple $W$-algebra at admissible level is given by

$$c_{\text{eff}} \left( W_{-h^+} \left( \mathfrak{g}, f \right) \right) = \dim(\mathfrak{g}) - \frac{h^+ \dim(\mathfrak{g})}{pq},$$

and the effective central charge of a vertex algebra as e.g. a lattice vertex algebra of a positive definite lattice coincides with the central charge, we obtain

$$c_{\text{eff}}(W_{sb}(n, 2)) - c_{\text{eff}}(V_L) = \frac{n}{n + 2} \geq c_{\text{eff}}(C_0).$$

The last inequality follows from the fact that every simple $W_{sb}(n, 2)$-module decomposes into a direct sum of simple $C_0 \otimes V_L$-modules. Hence the asymptotic growth of $C_0$ is less than 1.

Next we decompose $W_{sb}(n, 2)$ as a $W_{spr}(n, 2) \otimes \pi^{H^+}$-module. Since $W_{sb}(n, 2)$ is completely reducible, we have

$$W_{sb}(n, 2) = \bigoplus_{m \in \mathbb{Z}} C_m \otimes \pi^H_m,$$

where $C_m$ is a certain simple $C_0$-module. By [CL2, Theorem 9.1], $W_{sb}(n, 2)$ is weakly generated by $W_{spr}(n, 2) \otimes \pi^{H^+}$ and the fields $G^+(z)$, which are the highest weight vectors in $C_{\pm 1} \otimes \pi^H_{\pm 1}$. By the equality of conformal weight of $G^+(z)$

$$\frac{n}{2} = \Delta W_{w}(n, 2) \left( G^+(z) \right) = \Delta W_{w}(n, 2) \left( G^+(z) \right) + \Delta \pi^{H^+} \left( G^+(z) \right)$$

and $\Delta \pi^{H^+}(G^+(z)) = n/4$, we have $\Delta W_{w}(n, 2)(G^+(z)) = n/4$, which is maximal among the conformal weights for the $(n + 2, 3)$-minimal series representations. Indeed it coincides uniquely with the conformal weight of the simple current $L_{\omega}(n\Lambda_1)$ of $W_{spr}(n, 2)$. Recalling that $G^+(z)$ generates the lattice $\sqrt{2n}\mathbb{Z}$, we have a conformal embedding

$$W_{spr}(n, 2) \otimes V_{\sqrt{2n}\mathbb{Z}} \oplus L_{\omega}(n\Lambda_1) \otimes V_{\sqrt{2n} + \sqrt{2n}\mathbb{Z}} \hookrightarrow W_{sb}(n, 2),$$

which completes the weak generators. Therefore, the above embedding is surjective. This completes the proof. \hfill $\Box$

**Remark 4.2.** It is straightforward to show that we have $W_{sb}(n, 0) \simeq \mathbb{C}$ and $W_{sb}(2m, 1) \simeq V_{\sqrt{m}\mathbb{Z}}$ for $n \in \mathbb{Z}_{\geq 2}$ and $m \in \mathbb{Z}_{\geq 1}$.

It follows from [CL2, Theorem 9.4] that

$$\text{Com} \left( W_{pr}(r, n), W_{sb}(n, r) \right) \simeq V_{\sqrt{m}\mathbb{Z}}, \tag{4.10}$$

$$W_{sb}(n, r) \simeq \bigoplus_{i \in \mathbb{Z}} L_{\omega}(n\Lambda_i) \otimes V_{\sqrt{m} + \sqrt{m}\mathbb{Z}}. \tag{4.11}$$
In particular, \( W_{ab}(n, r) \) is a simple current extension of \( W_{pr}(r, n) \otimes V_{\sqrt{n}rZ} \) and thus is \( C_2 \)-cofinite and rational (see [CGN, Corollary 5.19 (1)]). Then, by using \( \pi_{P/Q} \) given in (4.2) and Theorem 2.3, we obtain the following theorem.

**Theorem 4.3.** There exists a one-to-one correspondence

\[
\text{Irr}(W_{ab}(n, r)) \simeq \{ (\lambda, a) \in \hat{P}^n_{+}(r) \times \mathbb{Z}_{nr} \mid \pi_{P/Q}(\lambda) = a \in \mathbb{Z}_r \} / \mathbb{Z}_r,
\]

where the \( \mathbb{Z}_r \)-action on \( \hat{P}^n_{+}(r) \times \mathbb{Z}_{nr} \) is defined by \( m \cdot (\lambda, a) = (\sigma^m(\lambda), a + mn) \) for \( m \in \mathbb{Z}_r \). Moreover, we have a \( W_{pr}(r, n) \otimes V_{\sqrt{n}rZ} \)-module decomposition

\[
L_{ab}(\lambda, a) \simeq \bigoplus_{i \in \mathbb{Z}_r} L_W(\sigma^i(\lambda)) \otimes V_{\sqrt{i+1}r + \sqrt{nr}Z} \quad (4.12)
\]

and the fusion product formula

\[
L_{ab}(\lambda, a) \boxtimes L_{ab}(\mu, b) \simeq \bigoplus_{\nu \in \hat{P}^n_{+}(r)} N^\nu_{\lambda\mu} L_{ab}(\nu, a + b),
\]

where \( L_{ab}(\lambda, a) \) denotes the simple \( W_{ab}(n, r) \)-module corresponding to \((\lambda, a)\) and \( N^\nu_{\lambda\mu} \) is the affine fusion rule given by (4.4).

**Proof.** Using (4.7), it follows that \( h^W_{n\lambda_i} = in(r - i)/2r \) so that the monodromy operator \( M_{L_W(n\lambda_i), L_W(\lambda)} \) is

\[
M_{L_W(n\lambda_i), L_W(\lambda)} = e^{-i\pi_{P/Q}(\lambda)} \cdot \zeta_r = e^{2\pi i/r}.
\]

Thus the simple module \( L_W(\lambda) \otimes V_{\sqrt{nr} + \sqrt{nr}Z} \) is local with respect to the simple currents \( L_W(n\lambda_i) \otimes V_{\sqrt{nr} + \sqrt{nr}Z} \) if and only if

\[
\pi_{P/Q}(\lambda) = a \in \mathbb{Z}_r.
\]

Here \( a \in \mathbb{Z}_{nr} \) is regarded as an element of \( \mathbb{Z}_r \) by the natural projection \( \mathbb{Z}_{nr} \rightarrow \mathbb{Z}_r \). Thus the assertion follows from Corollary A.25 and A.26.

**Corollary 4.4.** The modular \( S \)-matrix for \( W_{ab}(n, r) \) is given by

\[
S_{(\lambda, a), (\mu, b)}^\text{ab} = e^{\frac{2\pi i n}{nr}} \sqrt{\frac{r}{n}} S^W_{\lambda, \mu},
\]

where \( S^W_{\lambda, \mu} \) is the modular \( S \)-matrix for \( W_{pr}(r, n) \).

Theorem 4.3 together with (B.2) implies the following isomorphism.

**Proposition 4.5.** The one-to-one correspondence

\[
\text{Irr}(L_r(\mathfrak{s}_n)) \xrightarrow{\sim} \text{Irr}(W_{ab}(n, r)); \ L(\lambda) \mapsto L_{ab}(\lambda^\ell, \ell(\lambda)) \quad (\lambda \in P^r_+ (n)),
\]

gives an isomorphism of fusion rings

\[
K(L_r(\mathfrak{s}_n)) \simeq K(W_{ab}(n, r)). \quad (4.13)
\]

**Remark 4.6.** In [AvE2], T. Arakawa and J. van Ekeren construct another ring isomorphism \( K(L_r(\mathfrak{s}_n)) \simeq K(W_{ab}(n, r)) \) for even \( n \). Let \( v \) be coprime to \( n + r \) and \( \ell = -n + \frac{n}{2r} \), then the fusion rings of \( K(L_r(\mathfrak{s}_n)) \) and \( K(L_\ell(\mathfrak{s}_n)) \) coincide and in fact for \( v = n - 1 \) there is an equivalence of braided tensor categories between the category of ordinary modules of \( L_\ell(\mathfrak{s}_n) \) and those of \( W_{ab}(n, r) \) [ACF, Theorem 10.4]. We expect that these two parameterization of simple modules coincide.
4.4. Fusion rules of principal $W$-superalgebras. We consider the principal $W$-superalgebra $\mathcal{W}_{sp}(n,r)$ as in (4.1). The norm of the Heisenberg field (3.2) is $\varepsilon_+ = \frac{r}{n+r}$.

Theorem 4.7. There exists an isomorphism of vertex superalgebras

$$\text{Com}(\mathcal{W}_{pr}(r,n), \mathcal{W}_{sp}(n,r)) \simeq V_{\sqrt{(n+r)r}}.$$  

Moreover, we have

$$\mathcal{W}_{sp}(n,r) \simeq \bigoplus_{\lambda \in \mathbb{Z}} L_W(n\Lambda_i) \otimes V_{\sqrt{(n+r)r}+\sqrt{(n+r)r}}$$  

as $\mathcal{W}_{pr}(r,n) \otimes V_{\sqrt{(n+r)r}r}$-modules. In particular, $\mathcal{W}_{sp}(n,r)$ is a simple current extension of $\mathcal{W}_{pr}(r,n) \otimes V_{\sqrt{(n+r)r}r}$ and thus is $C_2$-cofinite and rational.

Proof. First of all, the generator of $\sqrt{nr}Z$ in (4.11) is given by $nH^+$ and we have

$$\mathcal{W}_{sb}(n,r) = \bigoplus_{i \in \mathbb{Z}_r, \lambda \in \mathbb{Z}} L_W(n\Lambda_i) \otimes \pi_{r\lambda+1}^{H^+}.$$  

Then, since $(n+r)\phi^+ = nH^+ + r\phi^+$ by Lemma 3.1 (1), we have

$$\mathcal{W}_{sb}(n,r) \otimes V_Z = \bigoplus_{i \in \mathbb{Z}_r, \lambda \in \mathbb{Z}} L_W(n\Lambda_i) \otimes \pi_{r\lambda+1}^{H^+} \otimes \pi_{r\lambda+1}^{H^+}$$  

$$\simeq \bigoplus_{i \in \mathbb{Z}_r, \lambda \in \mathbb{Z}} L_W(n\Lambda_i) \otimes \pi_{r\lambda+1}^{H^+} \otimes \pi_{r\lambda+1}^{(n+r)\phi^+}$$  

as $\mathcal{W}_{pr}(r,n) \otimes \pi^{H^+} \otimes \pi^{(n+r)\phi^+}$-modules. Thus, by [CGN, Corollary 5.16], we obtain

$$\mathcal{W}_{sp}(n,r) \simeq \text{Com} \left( \pi^{H^+}, \mathcal{W}_{sb}(n,r) \otimes V_Z \right)$$  

$$\simeq \bigoplus_{i \in \mathbb{Z}_r, \lambda \in \mathbb{Z}} L_W(n\Lambda_i) \otimes \pi_{(n+r)\phi^+}^{(n+r)\phi^+}.$$  

This implies the assertion since we have

$$\text{Com}(\mathcal{W}_{pr}(r,n), \mathcal{W}_{sp}(n,r)) \simeq \bigoplus_{\lambda \in \mathbb{Z}} \pi_{(n+r)\phi^+}^{(n+r)\phi^+} \simeq V_{\sqrt{(n+r)r}}$$  

by the characterization of lattice vertex superalgebras.

Remark 4.8. As a counterpart to Remark 4.2, we have $\mathcal{W}_{sp}(n,0) \simeq \mathbb{C}$ and $\mathcal{W}_{sp}(2m,1) \simeq V_{\sqrt{2m+1}r}$ for $n \in \mathbb{Z}_{\geq 2}$ and $m \in \mathbb{Z}_{\geq 1}$.

By the same argument as in the proof of Theorem 4.3, the simple current extension (4.14) implies the following.

Theorem 4.9. There exists a one-to-one correspondence

$$\text{Irr}(\mathcal{W}_{sp}(n,r)) \simeq \{(\alpha, \ell) \in \hat{P}_n^+(r) \times Z_{(n+r)r} \mid \sigma_{P_{\mathcal{Q}}} (\lambda) = a \in \mathbb{Z}_r\}/\mathbb{Z}_r,$$

where the $\mathbb{Z}_r$-action on $\hat{P}_n^+(r) \times Z_{(n+r)r}$ is defined by $m \cdot (\alpha, \lambda) = (\alpha, a + m(n+r))$ for $m \in \mathbb{Z}_r$. Moreover, we have a $\mathcal{W}_{pr}(r,n) \otimes V_{\sqrt{(n+r)r}r}$-module decomposition

$$L_{sp}(\lambda, a) \simeq \bigoplus_{i \in \mathbb{Z}_r} L_W(\sigma^i(\lambda)) \otimes V_{\sqrt{(n+r)r}r}$$  

and the fusion product is given by

$$L_{sp}(\lambda, a) \boxtimes L_{sp}(\mu, b) \simeq \bigoplus_{\nu \in \hat{P}_n^+(r)} N_{\nu}^{\mu} L_{sp}(\nu, a + b),$$

where $L_{sp}(\lambda, a)$ denotes the simple $\mathcal{W}_{sp}(n,r)$-module corresponding to $(\lambda, a)$ and $N_{\nu}^{\mu}$ is the affine fusion rule given by (4.4).
Corollary 4.10. We obtain the following.

1. We have a ring isomorphism
   \[
   \mathcal{K}(\mathcal{W}_{\text{spr}}(n, r)) \simeq \left( \mathcal{K}(L_n(\mathfrak{sl}_r)) \otimes \mathbb{Z}[\mathbb{Z}_{(n+r)}] \right)^{\mathbb{Z}_r}.
   \]

2. The number of inequivalent simple \(\mathcal{W}_{\text{spr}}(n, r)\)-modules is \(\binom{n+r}{n}\).

3. The group of simple currents of \(\mathcal{W}_{\text{spr}}(n, r)\)-modules is
   \[
   \text{Pic}(\mathcal{W}_{\text{spr}}(n, r)) = \{ L_{\text{spr}}(n\Lambda_0, ar) \}_{a \in \mathbb{Z}_{n+r}} \simeq \mathbb{Z}_{n+r},
   \]
   where the isomorphism is given by \( L_{\text{spr}}(n\Lambda_0, ar) \mapsto a \).

Corollary 4.11. The formal character
\[
\text{ch}(L_{\text{spr}}(\lambda, a))(q, z) = \text{tr}_{L_{\text{spr}}(\lambda, a)}(q^{L_0 - z/24} z^{H_0})
\]
is given by
\[
\frac{1}{\eta(q)} \sum_{\mu \in \mathbb{Z}_r} \sum_{w \in W} \varepsilon(w) q^{\frac{(\mu + 1/2)(\lambda + 1/2) - (n+r)(\lambda + 1/2)}{2(n+r)(1+r)}} \sum_{\mu \in (n+r)(1+r)} q^{\frac{(\mu + 1/2)^2}{2(n+r)}} \mathfrak{s}^{w}_{\mu}
\]
and the modular \(S\)-matrix for \(\mathcal{W}_{\text{spr}}(n, r)\) is given by
\[
S_{\text{spr}}^{(\lambda, a), (\mu, b)} = e^{\frac{2\pi i}{(n+r)+1}} \sqrt{\frac{r}{n+r}} S_{\lambda, \mu}^{W}.
\]

Finally, we give another description of \(\text{Irr}(\mathcal{W}_{\text{spr}}(n, r))\) by the Kazama–Suzuki coset construction. It follows from (4.14) and (4.15) that
\[
\text{Com}(\mathcal{W}_{\text{spr}}(n, r), \mathcal{W}_{\text{ab}}(n, r) \otimes V_Z) \simeq \bigoplus_{a \in \mathbb{Z}} \pi_{\text{spr}}^H_{(n+r), a} \simeq V_{\sqrt{n+r}} Z
\]
as vertex superalgebras. Hence, \(\mathcal{W}_{\text{ab}}(n, r) \otimes V_Z\) decomposes into
\[
\bigoplus_{i \in \mathbb{Z}_r} \bigoplus_{\mu \in \mathbb{Z}_{n+r}} L_Y(n\Lambda_i) \otimes V_{\sqrt{n+r}} Z_{\sqrt{n+r}} \otimes \mathcal{W}_{\text{spr}}(n\Lambda_0, \mu r) \otimes V_{\sqrt{n+r}} Z_{\sqrt{n+r}} \simeq \bigoplus_{\mu \in \mathbb{Z}_{n+r}} L_{\text{spr}}(n\Lambda_0, \mu r) \otimes V_{\sqrt{n+r}} Z_{\sqrt{n+r}}
\]
as a \(\mathcal{W}_{\text{spr}}(n, r) \otimes V_{\sqrt{n+r}} Z_{\sqrt{n+r}}\)-module. Thus, \(\mathcal{W}_{\text{ab}}(n, r) \otimes V_Z\) is an order \((n+r)\) simple current extension of \(\mathcal{W}_{\text{spr}}(n, r) \otimes V_{\sqrt{n+r}} Z_{\sqrt{n+r}}\). Since \(V_Z\) is a holomorphic vertex operator superalgebra,
\[
\mathcal{W}_{\text{ab}}(n, r) \otimes V_Z \simeq \mathcal{W}_{\text{spr}}(n, r) \otimes V_{\sqrt{n+r}} Z_{\sqrt{n+r}}
\]
is an isomorphism of braided tensor supercategories. Hence Theorem 2.5 together with the isomorphism (4.13) imply the following.

Theorem 4.12. There exists a one-to-one correspondence
\[
\text{Irr}(\mathcal{W}_{\text{spr}}(n, r)) \simeq \left\{ (\lambda, a) \in \hat{P}_+^r(n) \times \mathbb{Z}_{(n+r)n} \mid \pi_{P/Q}(\lambda) = -a \in \mathbb{Z}_n \right\} / \mathbb{Z}_n
\]
where \(\mathbb{Z}_n\) acts on \(\hat{P}_+^r(n) \times \mathbb{Z}_{(n+r)n}\) by \(m \cdot (\lambda, a) = (\sigma^m(\lambda), a - m(n+r))\), \((m \in \mathbb{Z}_n)\). Let \(L_{\text{spr}}(\lambda, a)\) denote the simple \(\mathcal{W}_{\text{spr}}(n, r)\)-module under this parameterization. Then we have an isomorphism of fusion rings
\[
\mathcal{K}(\mathcal{W}_{\text{spr}}(n, r)) \simeq \left( \mathcal{K}(L_r(\mathfrak{sl}_n)) \otimes \mathbb{Z}[\mathbb{Z}_{(n+r)}] \right)^{\mathbb{Z}_n} ; L_{\text{spr}}(\lambda, a) \mapsto L(\lambda) \otimes [a].
\]
4.5. On more correspondences. In this section, we used that the Heisenberg coset of our subregular $W_{1b}(n, r)$-algebra and our principal $W_{op}(n, r)$-superalgebra are rational and $C_2$-cofinite. In particular the representation categories of the Heisenberg cosets are vertex tensor categories. This is in fact all we need in order to apply the simple current extension procedure. Note, that the theory of vertex tensor categories applies also to infinite order extensions [CMY1] and the existence of vertex tensor category on Fock modules for the Heisenberg vertex algebra is known as long as the Heisenberg weight is real [CKLR]. There are only a few cases of subregular $W$-algebras where it is known that its Heisenberg coset has categories of modules that are vertex tensor categories. In all these case the singlet algebra $M(p)$ [Ad4, AM] appears as a coset.

The first example are the $B_p$-algebras for $p \in \mathbb{Z}_{\geq 2}$ of [CRW]. The $B_3$-algebra is just the vertex algebra of $\beta\gamma$-ghosts. This is not a subregular $W$-algebra, but $B_3$ is $L_{-4/3}(\mathfrak{sl}_2)$ [Ad3] and in general $B_p$ is $W_k(\mathfrak{sl}_{p-1}, f_{sub})$ for $k = -(p - 1)^2/p$ [ACGY]. The $B_p$-algebras have received attention since they are the chiral algebras of certain quantum field theories, called Argyres–Douglas theories [BN, C2], proven in [ACGY, ACKR]. The second example is the $\mathbb{Z}_n$-orbifold of $\beta\gamma$-ghosts, which is $W_k(\mathfrak{sl}_n, f_{sub})$ for $k = -n + \frac{2p + n}{n}$ [CL2, Theorem 9.4]. The Heisenberg coset of the $B_p$-algebra is the singlet algebra $M(p)$ and the Heisenberg coset of $\beta\gamma$-ghosts and hence their $\mathbb{Z}_n$-orbifolds is the singlet algebra $M(2)$. Existence of vertex tensor category is not completely understood, but at least it is established for all modules that appear as summands inside triplet algebra modules [CMY2]. The $B_p$-algebra as a simple current extension of a Heisenberg vertex algebra times the singlet algebra has already been investigated in [ACKR].

It is in general a very difficult problem to get the existence of vertex tensor categories and so we will now study a procedure that does not require this.

5. Beyond rational case: coset functors

In this section, we establish a linear equivalence of weight module categories for the subregular $W$-algebra $W^+$ and the principal $W$-superalgebra $W^-$ at arbitrary levels under the duality relation (3.3). This generalizes the known results for the $N = 2$ super Virasoro algebra and the affine vertex algebra associated to $\mathfrak{sl}_2$ in [FST, S1], which corresponds to the case $n = 2$.

5.1. Coset functors. Recall that we have the category $W^{\pm{\text{-mod}}}$ of strongly graded grading-restricted generalized $W^\pm$-modules. Here we introduce a full subcategory $W^{\pm{\text{-mod}}}^{\text{wt}}$ of $W^{\pm{\text{-mod}}}$ whose objects are semisimple as $\pi^{H^{\pm}}$-modules. For an object $M \in \text{Ob}(W^{\pm{\text{-mod}}}^{\text{wt}})$, let $\text{Supp}(M)$ denote the set of $H^{0}_{(0)}$-eigenvalues of $M$. Since $\text{Supp}(W^\pm) = \mathbb{Z}$ (see §3.2-3.3), $\text{Supp}(M)$ defines naturally a subset of $\mathbb{C}/\mathbb{Z}$. We may decompose $M$ into

$$M = \bigoplus_{\lambda \in \text{Supp}(M)} M_{\lambda} \simeq \bigoplus_{\lambda \in \text{Supp}(M)} \Omega_{\lambda}^{H^{\pm}}(M) \otimes \pi_{\lambda}^{H^{\pm}}$$  \hspace{1cm} \text{(5.1)}$$

as $\text{Com}(\pi^{H^{\pm}}, \mathbb{W}^{\pm}) \otimes \pi^{H^{\pm}}$-modules where

$$\Omega_{\lambda}^{H^{\pm}}(M) := \{w \in M \mid \forall m \geq 0, H^{\pm}_{(m)} w = \delta_{m,0} \lambda \}.$$  

Note that for each $\lambda \in \mathbb{C}$, $M_{[\lambda]} := \bigoplus_{m \in \mathbb{Z}} M_{\lambda}^{m} \subset M$ is a submodule by (3.4). Thus we have a decomposition $M = \bigoplus_{[\lambda] \in \mathbb{C}/\mathbb{Z}} M_{[\lambda]}$ as a $W^\pm$-module. Motivated by this, we introduce full subcategories $W^{\pm{\text{-mod}}}^{\text{wt}}_{[\lambda]}$ consisting of objects $M$ such that $\text{Supp}(M) \subset \lambda + \mathbb{Z}$. Then we have a decomposition

$$W^{\pm{\text{-mod}}}^{\text{wt}} = \bigoplus_{[\lambda] \in \mathbb{C}/\mathbb{Z}} W^{\pm{\text{-mod}}}^{\text{wt}}_{[\lambda]}.$$
For an object $M$ in $W^\pm$-mod$^\text{wt}$, we have a decomposition
\[ M \otimes V_+ \simeq \bigoplus_{\xi \in \mathbb{C}} \Omega_{\xi}^\pm (M \otimes V_+) \otimes \pi_{\xi}^\pm \]
as a Com($\pi^\pm$, $W^\pm \otimes V_+$) module. Then the Com($\pi^\pm$, $W^\pm \otimes V_+$)-module $\Omega_{\xi}^\pm (M \otimes V_+)$ has a $W^\pm$-module structure through the isomorphism $\text{KS}_\pm$ in (3.6), which we denote by $\Omega_{\xi}^\pm (M)$. For $M \in \text{Ob}(W^\pm$-mod$^\text{wt}_{[\lambda]}$), it follows from
\[ M \otimes V_+ \simeq \bigoplus_{\mu \in \lambda + Z} \Omega_{\mu}^\pm (M) \otimes \pi_{\mu}^\pm \]
as follows lemma will play a key role in the proof:
\[ (\Omega_{\mu}^\pm (M) \otimes \pi_{\mu}^\pm) \simeq \bigoplus_{\nu \in Z} \Omega_{\nu + \xi}^\pm (M) \otimes \pi_{\nu + \xi}^\pm \ 	ext{for } \xi \in \pm\lambda + Z \]
that $\Omega_{\xi}^\pm (M)$ is non-zero only if $\xi \in \mp\lambda + Z$ and in this case
\[ \Omega_{\xi}^\pm (M) \simeq \bigoplus_{\nu \in Z} \Omega_{\nu - \xi}^\pm (M) \otimes \pi_{\nu - \xi}^\pm \]
Now, it is obvious that the assignments $\Omega_{\xi}^\pm$ induce $\mathbb{C}$-linear superfunctors
\[ \Omega_{\pm}^\pm : W^+\text{-mod}^\text{wt}_{[\lambda]} \rightarrow W^-\text{-mod}^\text{wt}_{[\mp\lambda]}, \Omega_{\pm}^- \rightarrow W^+\text{-mod}^\text{wt}_{[\pm\lambda]} \]
which are non-zero only if $\xi \in \mp\lambda + Z$, respectively. We call $\Omega_{\xi}^\pm$ coset functors.

5.2. Equivalence of categories. Here we establish an equivalence of categories $W^\pm$-mod weight-wisely:

**Theorem 5.1.** For $\lambda \in \mathbb{C}$, the two functors
\[ \Omega_{-\lambda}^+ : W^+\text{-mod}^\text{wt}_{[\lambda]} \rightarrow W^-\text{-mod}^\text{wt}_{[-\lambda]}, \Omega_{-\lambda}^- : W^-\text{-mod}^\text{wt}_{[-\lambda]} \rightarrow W^+\text{-mod}^\text{wt}_{[\lambda]} \]
are mutually quasi-inverse to each other and give an equivalence of categories.

The following lemma will play a key role in the proof:

**Lemma 5.2.** For an object $M$ in $W^\pm\text{-mod}^\text{wt}_{[\lambda]}$, the linear map $g_{\pm,\lambda}^M : M \rightarrow M \otimes V_+ \otimes V_-$ defined by
\[ g_{\pm,\lambda}^M (w) = \exp (\mathcal{H}_{\pm} (w \otimes |\mu\phi^+\rangle \otimes |\mu\phi^-\rangle)) \]
is an embedding of $W^\pm$-modules.

**Proof.** Since $\exp (\mathcal{H}_{\pm})$ is invertible, $g_{\pm,\lambda}^M$ is injective. By Corollary 3.2, $g_{\pm,\lambda}^M$ is already a $\pi^\pm$-homomorphism. Therefore, it remains to show
\[ g_{\pm}(a)(z)g_{\pm,\lambda}^M (w) = g_{\pm,\lambda}^M (a(z)w) \]
for $\pi^\pm$-singular vectors $a \in \Omega_{\mu}^\pm (W^\pm)$ and $w \in \Omega_{\lambda + \mu}^\pm (M)$. For this, we decompose the structure map of $M$
\[ Y_M (\cdot, z) : \Omega_{\nu}^\pm (W^\pm) \otimes \pi_{\nu}^\pm \rightarrow \Omega_{\lambda + \mu}^\pm (M) \otimes \pi_{\lambda + \mu}^\pm \]
to a Com($\pi^\pm$, $W^\pm$)--module (5.4) intertwining operator $Y (\cdot, z) = I_1 (\cdot, z) \otimes I_2 (\cdot, z)$ so that
\[ I_2 ([|\nu\rangle \otimes |\lambda + \mu\rangle |\lambda + \mu + \nu\rangle) = z^{\nu(\lambda + \mu) / \pm} E \left( -\frac{\nu}{\pm} H^\pm, z \right) \]
where we denote
\[ E (h, z) = \exp \left( \sum_{m=1}^{\infty} \frac{h (-m)}{m} z^m \right) \]
for a Heisenberg field \( h(z) = \sum_{m \in \mathbb{Z}} h(m) z^{-n-1} \). Then, by Lemma 3.1, we have
\[
g_M^\dagger(a)(z)g_M^\dagger(w) = Y_M(a \otimes |\phi^+ \rangle \otimes |\phi^- \rangle, z)(w \otimes |\phi^+ \rangle \otimes |\phi^- \rangle)
\]
\[
= I_1(a, z)w \otimes I_2([\nu], z)(\lambda + \mu) \otimes Y_{\nu z}([\nu \phi^+ \rangle, z)(\mu \phi^- \rangle) \otimes Y_{\nu z}(\nu \phi^+, z) |\mu \phi^- \rangle
\]
\[
= I_1(a, z)w \otimes z^{\nu(\lambda+\mu)/\varepsilon} E\left(-\frac{\nu}{\varepsilon \tilde{H}^\pm}, z\right) |\lambda + \mu + \nu\rangle
\]
\[
\otimes E(\nu \phi^+, z)(|\mu + \nu \rangle \phi^+ \rangle \otimes E(\nu \phi^-, z)(|\mu + \nu \rangle \phi^- \rangle
\]
\[
= I_1(a, z)w \otimes z^{\nu(\lambda+\mu)/\varepsilon} E\left(-\frac{\nu}{\varepsilon g_M(H^\pm)}, z\right) |\lambda + \mu + \nu\rangle \otimes (|\mu + \nu \rangle \phi^+ \rangle \otimes (|\mu + \nu \rangle \phi^- \rangle
\]
\[
= g_M^\dagger(a(z)w).
\]
This completes the proof. \( \square \)

**Proof of Theorem 5.1.** We show \( \Omega^-_{-\varepsilon, \lambda} \circ \Omega^+_{-\lambda} \). For an object \( M \) in \( \mathcal{W}^+ - \text{mod}_{[\lambda]} \), it follows from (5.2) that the \( \mathcal{W}^+ \)-module \( \Omega^-_{-\varepsilon, \lambda} \circ \Omega^+_{-\lambda}(M) \) is realized as the subspace of
\[
\bigoplus_{\mu \in \mathbb{Z}} M_{\lambda+\mu} \otimes \pi_{\mu^+} \otimes \pi_{\mu^-} \subset M \otimes \mathcal{V}^+ \otimes \mathcal{V}^-,
\]
consisting of highest weight vectors of \( \pi^{\mathcal{H}^+} \otimes \pi^{\mathcal{H}^*} \), i.e., elements \( w \) satisfying
\[
\tilde{H}^+(m)w = -\lambda \delta_{m,0} w, \quad \tilde{H}^-(m)w = -\tilde{\varepsilon} \lambda \delta_{m,0} w, \quad (m \geq 0).
\]
Hence, the \( \mathcal{W}^+ \)-module \( \Omega^-_{-\varepsilon, \lambda} \circ \Omega^+_{-\lambda}(M) \) coincides with the image of \( g_M^\dagger \) in Lemma 5.2. Therefore, the family of \( \mathcal{W}^+ \)-homomorphisms
\[
\left\{ g_M^\dagger: M \to \Omega^-_{-\varepsilon, \lambda} \circ \Omega^+_{-\lambda}(M) \mid M \in \text{Ob}(\mathcal{W}^+ - \text{mod}_{[\lambda]}^w) \right\}
\]
gives a desired natural isomorphism. Similarly, one can show that
\[
\left\{ g_M^-: M \to \Omega^+_{-\varepsilon, \lambda} \circ \Omega^-_{-\lambda}(M) \mid M \in \text{Ob}(\mathcal{W}^- - \text{mod}_{[\lambda]}^w) \right\}
\]
gives a natural isomorphism \( \text{id} \simeq \Omega^+_{-\varepsilon, \lambda} \circ \Omega^-_{-\lambda} \). This completes the proof. \( \square \)

6. Relative semi-infinite cohomology functor

Here we reinterpret the coset functors in terms of the relative semi-infinite cohomology and then study its compatibility with monoidal structure on the module categories of \( \mathcal{W}^\pm \).

6.1. Semi-infinite Cohomology of \( \tilde{g}\mathfrak{sl}_1 \) relative to \( \mathfrak{sl}_1 \). Let \( \pi^A \) be a degenerate (i.e., commutative) Heisenberg vertex algebra generated by a field \( A(z) \) and \( E \) be the \( bc \)-system vertex superalgebra generated by (odd) fields \( \varphi(z), \varphi^*(z) \) satisfying the OPEs
\[
\varphi(z)\varphi^*(w) \sim \frac{1}{z - w}, \quad \varphi(z)\varphi(w) \sim 0 \sim \varphi^*(z)\varphi^*(w).
\]
We introduce a cohomological grading \( \mathcal{E}^\bullet = \bigoplus_{\ell \in \mathbb{Z}} \mathcal{E}^\ell \) on \( \mathcal{E} \) as a strong \( \mathbb{Z} \)-grading given by \( \varphi \in \mathcal{E}^{-1} \) and \( \varphi^* \in \mathcal{E}^1 \). For a \( \pi^A \)-module \( M \), the \( \pi^A \otimes \mathcal{E} \)-module \( C^\bullet_{\infty}(M) := M \otimes \mathcal{E}^\bullet \) forms a cochain complex with differential \( d_{(0)} \) where \( d(z) = A(z) \otimes \varphi^*(z) \).

The cohomology
\[
H^\bullet_{\infty}(M) = H(C^\bullet_{\infty}(M), d_{(0)})
\]
is called the semi-infinite cohomology of $\hat{\mathfrak{g}}_1$ with coefficients in $M$ [Fe, FGZ]. In the following, we will use cohomologies obtained from a subcomplex

$$ C^{rel, \bullet}_\infty (M) := \{ u \in C^\bullet_\infty (M) \mid A_{(0)} u = \varphi_{(0)} u = 0 \}, $$

which coincides with $M^{A_{(0)}} \otimes \mathcal{S}_F$. Here $M^{A_{(0)}}$ is the space of $A_{(0)}$-invariants in $M$ and $\mathcal{S}_F$ is a subalgebra of $\mathcal{E}$ generated by $\varphi(z)$ and $\partial \varphi^*(z)$. The cohomology

$$ H^{rel, \bullet}_\infty (M) = H( C^{rel, \bullet}_\infty (M), d_{(0)} ) $$

is called the relative semi-infinite cohomology of $\hat{\mathfrak{g}}_1$ with coefficients in $M$ [Fe, FGZ]. Note that if $V$ is a vertex superalgebra which contains $\pi^A$, then $H^{rel, 0}_\infty (V)$ is naturally a vertex superalgebra and if $M$ is a $V$-module, then $H^{rel, p}_\infty (M)$ is naturally a $H^{rel, 0}_\infty (V)$-module for each $p \in \mathbb{Z}$.

Consider the following spacial case: let $\pi^{B_\pm}$ be Heisenberg vertex algebras generated by fields $B_\pm(z)$ satisfying the OPEs

$$ B^\pm(z)B^\pm(w) \sim \frac{\pm b}{(z-w)^2} $$

for some scalar $b \neq 0$. Then the Fock module $\pi^{B_\pm} \otimes \pi^{B_\pm}$ is a module over the degenerate Heisenberg vertex algebra generated by $A(z) = B^+(z) + B^-(z)$. We will use the following fundamental result, which is a special case of [FGZ, CFL].

**Proposition 6.1** ([FGZ, CFL]). For $p \in \mathbb{Z}$,

$$ H^{rel, p}_\infty (\pi^{B_\pm} \otimes \pi^{B_\pm}) = \delta_{p, 0} \delta_{\lambda + \mu, 0} C[[\lambda]] \otimes [\mu] \}. $$

**Proof.** We give a proof for the completeness of the paper. Since $C^{rel, \bullet}_\infty (\pi^{B_\pm} \otimes \pi^{B_\pm})$ is 0 if $\lambda + \mu \neq 0$, we may assume $\mu = -\lambda$. Then the complex is isomorphic to the tensor product $\otimes_{n \geq 1} \mathbb{C}[n]$ where each $\mathbb{C}[n]$ is isomorphic to the same complex $\mathcal{D}^\bullet = \mathbb{C}[x, y] \otimes \bigwedge_p \psi^\bullet$, given by

$$ 0 \to \mathbb{C}[x, y] \psi \to \mathbb{C}[x, y] \otimes \mathbb{C}[x, y] \psi \psi^* \to \mathbb{C}[x, y] \psi^* \to 0 $$

with differential $D = \psi^* \partial / \partial y + x \partial / \partial \psi$ by change of variables

$$ x \mapsto \frac{1}{2\imath \hbar} \left( B^{(-n)}_{-\lambda} - B^{(-n)}_{-\mu} \right), \quad y \mapsto B^{(-n)}_{-\lambda} + B^{(-n)}_{-\mu}, \quad \psi \mapsto \varphi_{(-n)}, \quad \psi^* \mapsto \varphi^*_{(-n)}. $$

It is straightforward to show $H^p(\mathcal{D}^\bullet) = \delta_{p, 0} C$ and then the assertion follows from the Künneth formula. \hfill \Box

### 6.2. Coset as relative semi-infinite cohomology

We reinterpret the coset functors in §5.2 in terms of relative semi-infinite cohomology. For this, we begin with the $W$-superalgebras $W^\pm$.

Fix a scalar $\epsilon$ satisfying $\epsilon^2 = \epsilon / \epsilon^+$. By using $\epsilon$, we introduce new sets of generating fields $\{ s^\pm (z), t^\pm (z) \}$ of $\pi^{\phi^\pm} \otimes \pi^{\phi^\pm}$ by

$$ \begin{pmatrix} s^+ \\ t^+ \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ \epsilon & \epsilon \end{pmatrix} \begin{pmatrix} \phi^+ \\ \phi^\pm \end{pmatrix}, \quad \begin{pmatrix} s^- \\ t^- \end{pmatrix} = \begin{pmatrix} -\epsilon & -1 \\ \epsilon & -1 \end{pmatrix} \begin{pmatrix} \phi^- \\ \phi^- \end{pmatrix}, $$

which satisfy the OPEs

$$ s^\pm(z)s^\pm(w) \sim \frac{-\epsilon^\pm}{(z-w)^2}, \quad t^\pm(z)t^\pm(w) \sim \frac{\epsilon^\pm}{(z-w)^2}, \quad s^\pm(z)t^\pm(w) \sim 0. $$

Then the vertex superalgebra $\hat{\mathcal{V}}^\pm = V_{\lambda + \mu} \otimes \pi^{\phi^\pm}$ decomposes into

$$ \hat{\mathcal{V}}^\pm \simeq \bigoplus_{\mu \in \mathbb{Z}} \pi^\pm_{\lambda + \mu} \otimes \pi^\pm_{\mu} $$
as $\pi^+ \otimes \pi^-$-modules. The tensor product $W^+ \otimes \widetilde{V}^\pm$ has a degenerate Heisenberg vertex subalgebra generated by a field $A^\pm(z) = H^\pm(z) + s^\pm(z)$. It follows from (3.4) and Proposition 6.1 that

$$H^{rel,0}_\infty(W^+ \otimes \widetilde{V}^\pm) \simeq \bigoplus_{\mu \in \mathbb{Z}} \Omega^{H^\pm}_\mu(W^\pm) \otimes \pi^\pm_\mu$$

(6.2)
as $\text{Com}(\pi^{H^\pm}, W^\pm) \otimes \pi^\pm$-modules. Since $\pi^\pm \simeq \pi^{H^\pm}$ by $t^\pm(z) \mapsto H^\pm(z)$, we have $H^{rel,0}_\infty(W^+ \otimes \widetilde{V}^\pm) \simeq W^\mp$ as $\text{Com}(\pi^{H^\pm}, W^\mp) \otimes \pi^{H^\pm}$-modules by (3.8). Indeed, this is an isomorphism of vertex superalgebra through (3.6) as follows:

**Proposition 6.2.** The natural embedding $W^\pm \otimes V^\pm \hookrightarrow C^\infty(W^\pm \otimes \widetilde{V}^\pm)$ induces an isomorphism of vertex superalgebras

$$\phi_\pm: \text{Com}(\pi^{\widetilde{H}^\pm}, W^\pm \otimes V^\pm) \xrightarrow{\simeq} H^{rel,0}_\infty(W^\pm \otimes \widetilde{V}^\pm).$$

**Proof.** We show the case $\eta_+. It follows from

$$W^+ \otimes V^\pm \subset W^+ \otimes V^+ \otimes \phi^- \otimes \mathcal{E} = C^\infty_\mathbb{Z}(W^+ \otimes \widetilde{V}^+),$$
as $A^+ = -(\widetilde{H}^\pm(z) + \epsilon^{-1}\phi^-)(z)$ that the embedding is restricted to

$$\text{Com}(\pi^{\widetilde{H}^+}, W^+ \otimes V^+) \hookrightarrow C^\infty(W^+ \otimes \widetilde{V}^+),$$

and that $\text{Com}(\pi^{\widetilde{H}^+}, W^+ \otimes V^+)$ is annihilated by the differential $d_{(0)}$. Thus, the embedding induces a homomorphism of vertex superalgebras

$$\eta_+: \text{Com}(\pi^{\widetilde{H}^+}, W^+ \otimes V^+) \rightarrow H^{rel,0}_\infty(W^+ \otimes \widetilde{V}^+).$$

Under the isomorphism (6.2), $\eta_+$ is the identity on $\Omega^{H^\pm}_\mu(W^\pm)$ for each $\mu \in \mathbb{Z}$ and sends $H^-$ to $t^\pm(z)$. Hence $\eta_+$ is an isomorphism. The case $\eta_-$ can be shown in the same way. This completes the proof. \square

Next, we consider the relative semi-infinite cohomology for arbitrary $W^\pm$-modules in $W^\pm$-$\text{mod}^{wt}$. For this purpose, we replace the vertex superalgebras $\widetilde{V}^\pm$ with their modules:

$$\widetilde{V}^\pm_\lambda = W^\pm \otimes \pi^\pm_\lambda, \quad \widetilde{V}^\pm_\lambda = W^\pm \otimes \pi^\pm_\lambda, \quad (\lambda \in \mathbb{C}).$$

As a $\pi^\pm \otimes \pi^\pm$-module, $\widetilde{V}^\pm_\lambda$ decomposes into

$$\widetilde{V}^+_\lambda \simeq \bigoplus_{\mu \in \mathbb{Z}} \pi^{\pm}_{-\mu - \epsilon - 1 \lambda} \otimes \pi^{\pm}_{\mu + \lambda}, \quad \widetilde{V}^-_\lambda \simeq \bigoplus_{\mu \in \mathbb{Z}} \pi^{\pm}_{-\mu - \epsilon \lambda} \otimes \pi^{\pm}_{\mu + \epsilon - 1 \lambda},$$

respectively. Then for a $W^\pm$-module $M$ in $W^\pm$-$\text{mod}^{wt}$, the tensor product $M \otimes \widetilde{V}^\pm_\lambda$ is a module of a degenerate Heisenberg vertex algebra $\pi^{H^\pm}$ generated by $A^\pm(z) = H^\pm(z) + s^\pm(z)$. The relative semi-infinite cohomology $H^{rel,\mu}(M \otimes \widetilde{V}^\pm_\lambda)$ is naturally a $W^\mp$-module by the isomorphism

$$\gamma_\pm := \eta_+ \circ \text{KS}_\pm: W^\mp \xrightarrow{\simeq} H^{rel,0}_\infty(W^\pm \otimes \widetilde{V}^\pm_\lambda)$$

by (3.6) and Proposition 6.2. non-zero only for $p = 0$ by Proposition 6.1. Therefore, this assignment induce exact $\mathbb{C}$-linear superfunctors

$$H^{rel}_\pm: W^\pm$-$\text{mod}^{wt} \rightarrow W^-$-$\text{mod}^{wt}, \quad \text{H}^{rel}_\mp: W^-$-$\text{mod}^{wt} \rightarrow W^+$-$\text{mod}^{wt},$$

respectively. More precisely, for objects $M_+$ in $W^+$-$\text{mod}^{wt}_{[\lambda]}$ and $M_-$ in $W^-$-$\text{mod}^{wt}_{[\xi,\lambda]}$ and $\xi \in \lambda + \mathbb{Z}$, we have

$$\text{H}^{rel}_{\pm, \xi}(M_+) \simeq \bigoplus_{\mu \in \mathbb{Z}} \Omega^{H^+}_{\xi + \mu}(M_+) \otimes \pi^{H^+_\mu}_{\xi + \mu}, \quad \text{H}^{rel}_{\pm, \xi}(M_-) \simeq \bigoplus_{\mu \in \mathbb{Z}} \Omega^{H^-}_{\xi + \mu}(M_-) \otimes \pi^{H^-}_{\xi + \mu}$$
as $\text{Com}(\pi_{1,\infty}^+ \mathcal{W}^\infty) \otimes \pi_{1,\infty}^H - \text{modules}$ by Proposition 6.1. It follows from (5.3) that these decomposition coincide with $\Omega^+_{i\xi}(\mathcal{M}_+)$ and $\Omega^-_{i\xi}(\mathcal{M}_-)$, respectively. The following can be proven in the same way as Proposition 6.2.

**Theorem 6.3.** For objects $M_\pm$ in $\mathcal{W}^1 - \text{mod}_{\mathcal{V}_{\lambda}}^{\text{rel}}$ and $M_- \in \mathcal{W}^2 - \text{mod}_{\mathcal{V}_{\lambda}}^{\text{rel}}$, the natural embeddings $M_\pm \otimes \mathcal{V}^\infty \hookrightarrow C_{\infty}^\bullet(M \otimes \mathcal{V}^\infty_{e\lambda})$ induce isomorphisms between the relative semi-infinite cohomology functors $\eta_{+,\lambda}^M : \Omega^+_{+,\lambda}(M) \xrightarrow{\approx} H^\text{rel}_{+,e\lambda}(M)$, $\eta_{-,\lambda}^M : \Omega^-_{-,\lambda}(M) \xrightarrow{\approx} H^\text{rel}_{-,e\lambda}(M)$ (6.3)
of $\mathcal{W}^\infty$-modules, respectively. Moreover, the family $\{\eta_{\pm,\lambda}^M\}$ gives natural isomorphisms of superfunctors $\Omega^+_{-,\lambda} \xrightarrow{\approx} H^\text{rel}_{+,e\lambda} : \mathcal{W}^+ - \text{mod}_{\mathcal{V}_{\lambda}}^{\text{rel}} \rightarrow \mathcal{W}^- - \text{mod}_{\mathcal{V}_{\lambda}}^{\text{rel}}$, $\Omega^-_{+,\lambda} \xrightarrow{\approx} H^\text{rel}_{-,e\lambda} : \mathcal{W}^- - \text{mod}_{\mathcal{V}_{\lambda}}^{\text{rel}} \rightarrow \mathcal{W}^+ - \text{mod}_{\mathcal{V}_{\lambda}}^{\text{rel}}$.

By composing the natural isomorphisms in (5.5) and Theorem 6.3, we obtain natural isomorphisms $\Upsilon_{\pm,\lambda}^M : M \xrightarrow{\approx} H^\text{rel}_{+,e\lambda} \circ H^\text{rel}_{-,e\lambda}(M)$ (6.4) for each object $M$ in $\mathcal{W}^\pm - \text{mod}_{\mathcal{V}_{\lambda}}^{\text{rel}}$. More explicitly, first note that $H^\text{rel}_{+,e\lambda} \circ H^\text{rel}_{-,e\lambda}(M)$ is a subquotient of the total complex:

$$(M \otimes V_\mathcal{Z} \otimes \pi_{\lambda}^{\phi^-} \otimes \mathcal{E}) \otimes V_{\sqrt{\mathcal{Z}}} \otimes \pi_{\lambda}^{\phi^+} \otimes \mathcal{E} \simeq (M \otimes V_\mathcal{Z} \otimes V_{\sqrt{\mathcal{Z}}}) \otimes \pi_{\lambda}^{\phi^+} \otimes \pi_{\lambda}^{\phi^-} \otimes \mathcal{E}^{\otimes 2}.$$ (6.5)

Then $\Upsilon_{\pm,\lambda}^M$ is given by

$$\Upsilon_{\pm,\lambda}^M(w) = [g_{\pm,\lambda}^M(w) \otimes |\epsilon \lambda \phi^+| \otimes |\lambda \epsilon \phi^-| \otimes 1]
= [(\exp(H_{\pm,\lambda})(w) \otimes |\mu \phi^\pm_L| \otimes |\mu \phi^\pm_R|) \otimes |\epsilon \lambda \phi^+| \otimes |\lambda \epsilon \phi^-| \otimes 1]$$ (6.6)

for $w \in M_{\lambda+\mu}$ ($\mu \in \mathbb{Z}$). Here we have replaced the notation $\phi^\pm_L$ (reps. $\phi^\pm_R$) standing for the elements in $V_\mathcal{Z}$ or $V_{\sqrt{\mathcal{Z}}}$ (resp. $\pi_{\lambda}^{\phi^+}$ or $\pi_{\lambda}^{\phi^-}$) for clarity.

**6.3. Relation to fusion product.** Here we prove that the superfunctors $H^\text{rel}_{+,\lambda}$ respect the fusion products of the module categories $\mathcal{W}^\pm - \text{mod}_{\mathcal{V}_{\lambda}}^{\text{rel}}$. More strongly, the relative semi-infinite cohomology functors induce isomorphisms between the spaces of (logarithmic) intertwining operators.

Let $\lambda_i \in \mathbb{C}$ ($i = 1, 2, 3$) with $\lambda_3 = \lambda_1 + \lambda_2$ and $Y^\pm(\cdot, z)$ denote the $\pi_{\lambda_3}^{\phi^\pm}$-intertwining operator $Y^\pm(\cdot, z) : \pi_{\lambda_1}^{\phi^+} \otimes \pi_{\lambda_2}^{\phi^+} \rightarrow \pi_{\lambda_3}^{\phi^+}(z)$ satisfying

$$Y^\pm ([\pm \epsilon\lambda_1 \phi^\pm], z) = \pm \epsilon\lambda_2 E(\pm \epsilon\lambda_1 \phi^\pm, z) = \pm \epsilon\lambda_3 \phi^\pm,$$

see (5.4) for the definition of $E(\cdot, z)$. Take a $\mathcal{W}^+\mathcal{M}_i$ in $\mathcal{W}^+ - \text{mod}_{\mathcal{V}_{\lambda}}^{\text{rel}}$ for each $i = 1, 2, 3$ and a $\mathcal{W}^+\mathcal{M}_i$-intertwining operator $Y(\cdot, z)$ of type $(\mathcal{M}_i \setminus \mathcal{M}_{3})$. Then the product $Y(\cdot, z) : Y_\mathcal{Z}(\cdot, z) \otimes Y^\pm(\cdot, z) \otimes Y^\pm(\cdot, z)$ is a $C^\infty_\mathcal{Z}(\mathcal{W}^+ \otimes V^+)$-intertwining operator $C^\infty_\mathcal{Z}(\mathcal{M}_1 \otimes \hat{V}_{\lambda_1}) \otimes C^\infty_\mathcal{Z}(\mathcal{M}_2 \otimes \hat{V}_{\lambda_2}) \rightarrow C^\infty_\mathcal{Z}(\mathcal{M}_3 \otimes \hat{V}_{\lambda_3})(z)[\log z]$.

Recall that by (2.4) an intertwining operator $Y$ satisfies the derivation property:

$$Y(a(0)m, z) = [a(0), Y(m, z)].$$
It follows that $\mathcal{Y}^\text{rel}(\cdot, z)$ can be restricted to
\[
C_\infty^\text{rel}(M_1 \otimes \tilde{V}_{\epsilon, \lambda_2}^+) \otimes C_\infty^\text{rel}(M_2 \otimes \tilde{V}_{\epsilon, \lambda_2}^+) \to C_\infty^\text{rel}(M_3 \otimes \tilde{V}_{\epsilon, \lambda_2}^+) \{z\}[\log z]
\]
and induces a $W^-$-intertwining operator
\[
H_+(\mathcal{Y})(\cdot, z) : H^\text{rel}_{+, \epsilon, \lambda_1}(M_1) \otimes H^\text{rel}_{+, \epsilon, \lambda_2}(M_2) \to H^\text{rel}_{+, \epsilon, \lambda_3}(M_3) \{z\}[\log z].
\]
Similarly, by using $Y^+(\cdot, z)$, we can construct from a $W^+$-intertwining operator $\mathcal{Y}(\cdot, z)$, a $W^+$-intertwining operator
\[
H_-(\mathcal{Y})(\cdot, z) : H^\text{rel}_{-, \epsilon, \lambda_1}(M_1) \otimes H^\text{rel}_{-, \epsilon, \lambda_2}(M_2) \to H^\text{rel}_{-, \epsilon, \lambda_3}(M_3) \{z\}[\log z],
\]
where $M_i (i = 1, 2, 3)$ denotes a $W^-$-module in $W^-\text{-mod}_{\pi_{\lambda_i}}$ by abuse of notation.

**Theorem 6.4.** The linear maps
\[
H_{\pm} : I_{W^\pm}(M_3/M_1 M_2) \to I_{W^\pm}(M_3/M_1 M_2)
\]
are isomorphisms of vector superspaces.

**Proof.** Let us abbreviate the compositions $H_{\pm \circ s} = H_{\pm} \circ H_{s}$ and $H_{\pm \circ \sigma} = H_{\pm} \circ H_{\sigma}$. Then to show that $H_{\pm}$ are isomorphisms, it suffices to show that the compositions
\[
I_{W^\pm}(M_3/M_1 M_2) \xrightarrow{H_{\pm \circ s}} I_{W^\pm}(H_{\pm \circ s}(M_1) H_{\pm \circ s}(M_2)) \xrightarrow{\simeq} I_{W^\pm}(M_3/M_1 M_2)
\]
are the identity, where the last isomorphism is induced by (6.4). We show the claim for $H_+$. By taking a $W^+$-intertwining operator $\mathcal{Y}(\cdot, z)$ of type $(\pi^{M_3}_{\lambda_3}, \pi)$, we need to show that the diagram
\[
\begin{array}{ccc}
M_1 \otimes M_2 & \xrightarrow{\mathcal{Y}(\cdot, z)} & M_3 \{z\}[\log z] \\
\bigwedge_{s}^{M_3} \otimes \bigwedge_{s}^{M_2} & \downarrow \bigwedge_{s}^{M_3} \otimes \bigwedge_{s}^{M_2} & \\
H_{\circ \sigma}^\text{rel}(M_1) \otimes H_{\circ \sigma}^\text{rel}(M_2) & \xrightarrow{H_{\circ \sigma}^\text{rel}(\mathcal{Y})(\cdot, z)} & H_{\circ \sigma}^\text{rel}(M_3) \{z\}[\log z]
\end{array}
\]
commutes. By using the decomposition (5.1) of modules $M_i (i = 1, 2)$, we may take $w_1 \in \Omega^{H_{\pm \circ \sigma \sigma}}_{\lambda_1 + \mu_i}(M_1) \otimes \pi^{H_{\pm \circ \sigma \sigma}}_{\lambda_1 + \mu_i}$ for some $\mu_i \in \mathbb{Z}$ without loss of generality. It follows from (6.6) that
\[
H_{\circ \sigma}^\text{rel}(\mathcal{Y})(\bigwedge_{s}^{M_1 \lambda_1}(w_1), \bigwedge_{s}^{M_2}(w_2)) = H_{\circ \sigma}^\text{rel}(\mathcal{Y})(\bigwedge_{s}^{M_1 \lambda_1}(w_1), \bigwedge_{s}^{M_2}(w_2))
\]
where $\mu_3 = \mu_1 + \mu_2$. By Proposition 6.1, the Heisenberg field
\[
t^+(z) + s^-(z) = (\phi_L^+ + \phi_L^-)(z) - \epsilon(\phi_R^+ + \phi_R^-)(z)
\]
in (6.5) is a coboundary. Hence, the right-hand side in the last equality is equal to

\[
\left[ y(w_1, z) w_2 \otimes E \left( (\lambda_1 + \mu_1)(\phi_L^+ + \phi^+_L), z \right) |\mu_3\phi^+_L) \otimes |\epsilon\lambda_3\phi^+_R) \otimes | - \epsilon\lambda_3\phi^-_R) \right].
\]

Then by decomposing the intertwining operator \( y(\cdot, z) \) into a \( \text{Com}(\pi^H, W^+) \otimes \pi^H \)-intertwining operator, the argument in the proof of Lemma 5.2 implies that the last cohomology class is equal to

\[
\left[ (\exp(\mathcal{G}_*) y(w_1, z) w_2) \otimes |\mu_3\phi^+_L) \otimes |\epsilon\lambda_3\phi^+_R) \otimes | - \epsilon\lambda_3\phi^-_R) \right]
\]

\[= \mathcal{Y}^{M_3}_{\lambda_3} (y(w_1, z) w_2). \]

Therefore, we obtain the commutativity of the diagram (6.7). The claim for \( H_{+\pm} \) can be shown in the same way. This completes the proof. \( \Box \)

Suppose that there exist full subcategories

\[ S^\pm = \bigoplus_{|\lambda| \in C/\mathbb{Z}} S^\pm_{|\lambda|} \subset \mathcal{W}^\pm_{\text{mod wt}}, \]

which have the vertex tensor supercategory structure [HLZ1, CKM1] and that the relative semi-infinite cohomology functors give an equivalence of categories

\[ H_{+\pm, \lambda}^\text{rel}: \mathcal{S}^\pm_{|\lambda|} \xrightarrow{\sim} \mathcal{S}^\pm_{|\lambda'|}, \quad H_{+\pm, \lambda}^\text{rel}: \mathcal{S}^\pm_{|\lambda|} \xrightarrow{\sim} \mathcal{S}^\pm_{|\lambda'|}, \quad (\lambda \in \mathbb{C}). \]

**Corollary 6.5.** For objects \( M^\pm_i (i = 1, 2) \) in \( W^+_{\text{mod wt}}[|\lambda|] \) (resp. \( W^-_{\text{mod wt}}[|\lambda|] \)), the pair \( (H^\text{rel}_{\pm, \epsilon(\lambda_1 + \lambda_2)}(M^\pm_1 \otimes M^\pm_2), H_{\pm}(y_{M^\pm_1 \otimes M^\pm_2})) \) induced by the canonical intertwining operator \( y_{M^\pm_1 \otimes M^\pm_2} (\cdot, z) \) of \( M^\pm_1 \) and \( M^\pm_2 \), is the fusion product of \( H^\text{rel}_{\pm, \epsilon\lambda_1}(M^\pm_1) \) and \( H^\text{rel}_{\pm, \epsilon\lambda_2}(M^\pm_2) \). In particular, we have an isomorphism

\[ \psi_{M_1, M_2}: H^\text{rel}_{\pm, \epsilon\lambda_1}(M^\pm_1) \boxtimes H^\text{rel}_{\pm, \epsilon\lambda_2}(M^\pm_2) \cong H^\text{rel}_{\pm, \epsilon(\lambda_1 + \lambda_2)}(M^\pm_1 \otimes M^\pm_2). \]

**Proof.** Since the proofs for \( (M^\pm_1, M^\pm_2) \) are quite similar, we only show the case \( (M^\pm_1, M^\pm_2) \). By the definition of fusion product, it suffices to show that given an arbitrary object \( N \) in \( S^- \) and an intertwining operator

\[ I(\cdot, z): H^\text{rel}_{\pm, \epsilon\lambda_1}(M^\pm_1) \otimes H^\text{rel}_{\pm, \epsilon\lambda_2}(M^\pm_2) \rightarrow N[log z], \]

there exists a unique \( W^- \)-homomorphism \( I^*: H^\text{rel}_{\pm, \epsilon\lambda_1}(M^\pm_1) \boxtimes H^\text{rel}_{\pm, \epsilon\lambda_2}(M^\pm_2) \rightarrow N \) such that \( I(\cdot, z) = I^* \circ H_+ (y_{M^\pm_1 \otimes M^\pm_2}) \). For this, we may assume \( N \) is an object in \( S^\pm_{|\lambda|} \) with \( \lambda_3 = \lambda_1 + \lambda_2 \). Then by (6.8), there exists an object \( \tilde{N} \) in \( S^\pm_{|\lambda|} \) such that \( H^\text{rel}_{\pm, \epsilon\lambda_3}(\tilde{N}) \cong N \). Then by Theorem 6.4, there exists a unique intertwining operator

\[ \tilde{I}(\cdot, z): M^\pm_1 \otimes M^\pm_2 \rightarrow \tilde{N}[log z] \]

corresponding to (6.10). Hence there exists a unique \( W^+ \)-homomorphism \( \tilde{I}^*: M^\pm_1 \boxtimes M^\pm_2 \rightarrow \tilde{N} \) such that \( \tilde{I}(\cdot, z) = \tilde{I}^* \circ y_{M^\pm_1 \otimes M^\pm_2} (\cdot, z) \). Therefore, \( \tilde{I}^* := H_+ (\tilde{I}^*) \) is the unique \( W^- \)-homomorphism satisfying \( I(\cdot, z) = I^* \circ H_+ (y_{M^\pm_1 \otimes M^\pm_2}) (\cdot, z) \) by (6.8). \( \Box \)

**Remark 6.6.** By using the associators of vertex tensor supercategories, one can prove that the triple of families of isomorphisms

- \( H^\text{rel}_{\pm, \epsilon\lambda_1}: S^\pm_{|\lambda|} \xrightarrow{\sim} S^\pm_{|\lambda'|}, \)
- \( \psi_{M, N}: H^\text{rel}_{\pm, \epsilon\lambda}(M) \boxtimes H^\text{rel}_{\pm, \epsilon\mu}(N) \cong H^\text{rel}_{\pm, \epsilon(\lambda + \mu)}(M \boxtimes N) \)
- \( \mathcal{Y}: W^- \xrightarrow{\sim} H^\text{rel}_{\pm, \epsilon}(W^+) \)
Correspondences of categories for subregular $W$-algebras and principal $W$-superalgebras

satisfy the same axiom of tensor superfunctors, that is, the following commutative diagrams:

\[
\begin{align*}
\big(H_{+,\lambda}^{\text{rel}}(M)\boxtimes H_{+,\mu}^{\text{rel}}(N)\big)\boxtimes H_{+,\nu}^{\text{rel}}(L) & \xrightarrow{A^2} H_{+,\lambda\mu\nu}^{\text{rel}}(N)\boxtimes H_{+,\mu\nu}^{\text{rel}}(L) \\
\Psi_{M\boxtimes N}^+ \boxtimes \text{id} & \quad \text{id} \boxtimes \Psi_{N,L}^+
\end{align*}
\]

where $A^\pm$ is the associativity isomorphism, $\ell^\pm$ (resp. $r^\pm$) is the left (resp. right) unit isomorphism on $S^\pm$. (See also Appendix A.1). Therefore, $(H_{+,\lambda}^{\text{rel}}, \Psi_{+,\lambda}^+, Y^+)$ is a lax tensor superfunctor. In this point of view, the decomposition $S^\pm = \bigoplus_{\lambda} S^\pm_{\lambda}$, which a priori is defined by the eigenvalues for $H^+_\lambda$, should be interpreted as the decomposition in terms of the tensor autofunctor $S = \exp(2\pi \sqrt{-1} H^+_\lambda)$. Similarly, $(H_{-,\lambda}^{\text{rel}}, \Psi_{-,\lambda}^- Y^-)$ forms a lax tensor superfunctor in the same sense and these two functors are weight-wise quasi-inverse to each other. However, these two functors do not preserve the braided structure since we use Fock modules $\pi_{\lambda}^\pm$ to define $H_{+,\lambda}^{\text{rel}}$, which give a nontrivial additional braiding themselves.

6.4. Rational case. We present Corollary 6.5 more explicitly in the rational cases treated in §4. Recalling the notation $W_{sb}(n, r)$ and $W_{sup}(n, r)$, we have

\[
\epsilon_+ = \frac{r}{n}, \quad \epsilon_- = \frac{r}{n+r}, \quad \epsilon = \sqrt{\frac{\epsilon_+}{\epsilon_-}} = \sqrt{\frac{n}{n+r}}
\]

and obtain the following proposition by comparing the branching rule.

Proposition 6.7. Let $\xi \in \mathbb{C}$ and $\lambda \in \mathbb{P}^n_+(r)$.

1. For $a \in \mathbb{Z}_{nr}$, we have

\[
H_{+,\lambda}^{\text{rel}}(\lambda, a) \simeq \begin{cases}
L_{\text{sup}}(\lambda, \frac{n}{n+r}) & \text{if } \xi \equiv a \mod n\mathbb{Z}, \\
0 & \text{otherwise}
\end{cases}
\]

2. For $a \in \mathbb{Z}_{(n+r)r}$, we have

\[
H_{-,\lambda}^{\text{rel}}(\lambda, a) \simeq \begin{cases}
L_{\text{sup}}(\lambda, \frac{na+r\xi}{n+r}) & \text{if } \xi \equiv a \mod (n + r)\mathbb{Z}, \\
0 & \text{otherwise}
\end{cases}
\]
Proof. Since the proofs for (1) and (2) are similar, we only prove (i). Since $L_{ab}(\lambda, a) \otimes \bigodot_{\xi}^{\frac{n}{m(n+r)}}$ decomposes into the direct sum
\[
\bigoplus_{i \in \mathbb{Z}_r} L_W(\sigma^i(\lambda)) \otimes \bigoplus_{m \in \mathbb{Z}} \pi^H_{\frac{n}{m} + i + r m} \otimes \bigoplus_{\mu \in \mathbb{Z}} \pi^F_{\frac{n}{m} - \mu} \otimes \pi^F_{\frac{1}{m} + \mu', \mu},
\]
we obtain by Proposition 6.1
\[
H^\text{rel}_{\frac{n}{m(n+r)}}(L_{ab}(\lambda, a)) \simeq \begin{cases} \bigoplus_{i \in \mathbb{Z}_r} L_W(\sigma^i(\lambda)) \otimes \bigoplus_{m \in \mathbb{Z}} \pi^H_{\frac{n}{m} + i + r m} & \text{if } \xi \equiv a \mod n\mathbb{Z}, \\
0 & \text{otherwise}. \end{cases}
\]
as $\mathcal{W}_{pr}(r, n) \otimes \pi^H$-modules. This decomposition and Theorem 4.9 imply (1). \[\Box\]

The following is Corollary 6.5 in the rational case, which can be deduced directly from the above proposition and the fusion rules obtained in §4.

Theorem 6.8. Let $\xi_1, \xi_2 \in \mathbb{C}$ and $\lambda_1, \lambda_2 \in \tilde{L}_{pr}^\infty(r)$.

(1) For $a_1, a_2 \in \mathbb{Z}_{n+r}$, we have
\[
H^\text{rel}_{\frac{n}{m(n+r)}}(L_{ab}(\lambda_1, a_1)) \boxtimes H^\text{rel}_{\frac{n}{m(n+r)}}(L_{ab}(\lambda_2, a_2)) \simeq \begin{cases} H^\text{rel}_{\frac{n}{m(n+r)}}(L_{ab}(\lambda_1, a_1)) \boxtimes L_{ab}(\lambda_2, a_2)) & \text{if } \xi_i \equiv a_i \mod n\mathbb{Z} \text{ for } i = 1, 2, \\
0 & \text{otherwise}. \end{cases}
\]

(2) For $a_1, a_2 \in \mathbb{Z}_{(n+r)r}$, we have
\[
H^\text{rel}_{\frac{n}{m(n+r)}}(L_{apr}(\lambda_1, a_1)) \boxtimes H^\text{rel}_{\frac{n}{m(n+r)}}(L_{apr}(\lambda_2, a_2)) \simeq \begin{cases} H^\text{rel}_{\frac{n}{m(n+r)}}(L_{apr}(\lambda_1, a_1)) \boxtimes L_{apr}(\lambda_2, a_2)) & \text{if } \xi_i \equiv a_i \mod (n+r)\mathbb{Z} \text{ for } i = 1, 2, \\
0 & \text{otherwise}. \end{cases}
\]

Appendix A. Categorical aspects of simple currents

We consider the theory of simple currents of vertex operator algebras purely in the categorical manner, following [CKL, CKM1]. We refer to a category enriched by the category of $\mathbb{Z}_2$-graded sets as a supercategory. For a supercategory $\mathcal{C}$, we write $\mathcal{C}_\text{ab}$ for its underlying category, namely, the objects in $\mathcal{C}_\text{ab}$ are the same as $\mathcal{C}$ and the morphisms in $\mathcal{C}_\text{ab}$ are the even morphisms of $\mathcal{C}$. For an additive supercategory $\mathcal{C}$ such that $\mathcal{C}_\text{ab}$ is an abelian category, by a subquotient object in $\mathcal{C}$ we mean one in the underlying category $\mathcal{C}_\text{ab}$. We use the notion of simplicity as well. We stress that we use this terminology even if $\mathcal{C}$ is an abelian supercategory.

A.1. Preliminaries. Let $\mathcal{C}$ be an essentially small, $\mathbb{C}$-linear, monoidal supercategory whose underlying category is abelian. We write
\[
\boxtimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}, \quad (M, N) \mapsto M \boxtimes N
\]
for the monoidal product superfunctor with unit object $1_{\mathcal{C}}$, by
\[
l_\bullet : 1_{\mathcal{C}} \boxtimes \bullet \xrightarrow{\cong} \bullet, \quad (l_M : 1_{\mathcal{C}} \boxtimes M \xrightarrow{\cong} M),
\]
\[
r_\bullet : \bullet \boxtimes 1_{\mathcal{C}} \xrightarrow{\cong} \bullet, \quad (r_M : M \boxtimes 1_{\mathcal{C}} \xrightarrow{\cong} M),
\]
\[
A_{\bullet, \bullet, \bullet} : \bullet \boxtimes (\bullet \boxtimes \bullet) \xrightarrow{\cong} (\bullet \boxtimes \bullet) \boxtimes \bullet, \quad \left(A_{M, N, L} : M \boxtimes (N \boxtimes L) \xrightarrow{\cong} (M \boxtimes N) \boxtimes L \right),
\]
the structural natural isomorphisms of superfunctors satisfying the pentagon and triangle axioms, see [BaK, EGNO]. The last one is called the associativity isomorphism. Here we use the convention that the parity of a parity-homogeneous
morphism $f$ is denoted by $\bar{f}$ and the composition of two parity-homogeneous morphisms $(f_1, f_2)$, $(g_1, g_2)$ in $\mathfrak{C} \times \mathfrak{C}$ is given by $(-1)^{f_2g_1}(f_1g_1, f_2g_2)$.

For an object $M \in Ob(\mathfrak{C})$, a right dual of $M$ is a triple $(M^*, e^R_M, i^R_M)$ of $M^* \in Ob(\mathfrak{C})$ and even morphisms

$$e^R_M: M^* \boxtimes M \to 1_\mathfrak{C}, \quad i^R_M: 1_\mathfrak{C} \to M^* \boxtimes M^*,$$

(A.1)
satisfying the rigidity axioms. A left dual of $M$ is similarly defined to be a triple $(^*M, e^L_M, i^L_M)$ consisting of $^*M \in Ob(\mathfrak{C})$ and even morphisms

$$e^L_M: M \boxtimes M^* \to 1_\mathfrak{C}, \quad i^L_M: 1_\mathfrak{C} \to M \boxtimes M.$$

(A.2)

By [EGNO, Proposition 2.10.15], right (left) duals are unique up to even isomorphisms if they exist. The supercategory $\mathfrak{C}$ is called rigid if every object in $\mathfrak{C}$ has a right and left dual. We call an object $M$ of $\mathfrak{C}$ an invertible object if it admits a left and right dual such that the morphisms in (A.1) and (A.2) are isomorphisms.

**Lemma A.1.** For an invertible object $S$, the superfunctors

$$S \boxtimes •: \mathfrak{C} \to \mathfrak{C}, \quad M \mapsto S \boxtimes M,$$

$$• \boxtimes S: \mathfrak{C} \to \mathfrak{C}, \quad M \mapsto M \boxtimes S,$$

are exact.

**Proof.** Since the proofs for $S \boxtimes •$ and $• \boxtimes S$ are similar, we prove only for $S \boxtimes •$. Note that any object lies in the image of $S \boxtimes •$ since for any object $M \in Ob(\mathfrak{C})$,

$$S \boxtimes (S^* \boxtimes M) \simeq (S \boxtimes S^*) \boxtimes M \simeq 1_\mathfrak{C} \boxtimes M \simeq M.$$  

Now, take a short exact sequence in $\mathfrak{C}$

$$0 \to M \to N \to L \to 0.$$  

We show that the complex

$$0 \to S \boxtimes M \to S \boxtimes N \to S \boxtimes L \to 0$$  

(A.3)
is exact. For the left exactness of (A.3), it suffices to show that for any object $A \in Ob(\mathfrak{C})$, the induced complex

$$0 \to \text{Hom}_\mathfrak{C}(A, S \boxtimes M) \to \text{Hom}_\mathfrak{C}(A, S \boxtimes N) \to \text{Hom}_\mathfrak{C}(A, S \boxtimes L)$$  

(A.4)
is exact. Indeed, by the left exactness of $\text{Hom}_\mathfrak{C}(A, •)$, we have the exact sequence

$$0 \to \text{Hom}_\mathfrak{C}(A, M) \to \text{Hom}_\mathfrak{C}(A, N) \to \text{Hom}_\mathfrak{C}(A, L).$$  

(A.5)

Then, by the functoriality of the following isomorphisms

$$\text{Hom}_\mathfrak{C}(A, M) \simeq \text{Hom}_\mathfrak{C}(A, 1_\mathfrak{C} \boxtimes M) \simeq \text{Hom}_\mathfrak{C}(A, (S \boxtimes S) \boxtimes M)$$

$$\simeq \text{Hom}_\mathfrak{C}(A, S \boxtimes (S \boxtimes M)) \simeq \text{Hom}_\mathfrak{C}(S \boxtimes A, S \boxtimes M),$$

(see [EGNO, Proposition 2.10.8]), the exactness of (A.5) implies that of

$$0 \to \text{Hom}_\mathfrak{C}(S \boxtimes A, S \boxtimes M) \to \text{Hom}_\mathfrak{C}(S \boxtimes A, S \boxtimes N) \to \text{Hom}_\mathfrak{C}(S \boxtimes A, S \boxtimes L).$$

Replacing $A$ by $S^* \boxtimes A$ in $S \boxtimes A$, we conclude that (A.4) is exact. We can prove the right exactness of (A.3) in a similar way by showing the exactness of

$$0 \to \text{Hom}_\mathfrak{C}(S \boxtimes L, A) \to \text{Hom}_\mathfrak{C}(S \boxtimes N, A) \to \text{Hom}_\mathfrak{C}(S \boxtimes M, A)$$

for any object $A \in Ob(\mathfrak{C})$ and thus we omit it. This completes the proof.

We call a simple invertible object a *simple current* following the terminology of the theory of vertex algebra. By Lemma A.1, a simple current exists if and only if the unit object $1_\mathfrak{C}$ is simple.

**Assumption A.2.** The unit object $1_\mathfrak{C}$ is simple.
Definition A.3. We call the group of isomorphism classes of simple currents in \( \mathcal{C} \), denoted by \( \text{Pic}(\mathcal{C}) \), the Picard group of \( \mathcal{C} \).

A braided monoidal supercategory is a monoidal supercategory \( \mathcal{C} \) equipped with a natural isomorphism of superfunctors, called the braiding,

\[
\mathcal{R} \circlearrowleft : (\bullet \boxtimes \bullet) \xrightarrow{\sim} (\bullet \boxtimes \bullet) \circ \sigma, \quad (\mathcal{R}_{M,N} : M \boxtimes N \xrightarrow{\sim} N \boxtimes M),
\]

where \( \sigma \) is the superfunctor

\[
\sigma : \mathcal{C} \times \mathcal{C} \to \mathcal{C} \times \mathcal{C}, \quad (M,N) \mapsto (N,M),
\]

under which every parity-homogeneous morphism \((f,g)\) maps to \((-1)^{fg}(g,f)\). The natural isomorphism \( \mathcal{R} \) is required to satisfy the hexagon identity. The natural isomorphism

\[
\mathcal{M} \circlearrowleft : (\bullet \boxtimes \bullet) \xrightarrow{\sim} (\bullet \boxtimes \bullet), \quad (\mathcal{M}_{M,N} : M \boxtimes N \xrightarrow{\sim} M \boxtimes N)
\]

is called the monodromy. It is straightforward to check that, in a braided monoidal supercategory, the existence of a right dual implies that of a left dual and vice versa. Indeed, given a right dual \((M^*,e^R_M,i^R_M)\) of an object \( M \), the triple \((M^n,e^n_M \circ \mathcal{R}^{-1}_{M,M} \circ i^R_M)\) defines a left dual of \( M \) and conversely, given a left dual \((^*M,e^L_M,i^L_M)\), a triple \((^*M,e^L_M \circ \mathcal{R}_{M,M}^{-1} \circ i^L_M)\) defines a right dual.

One of the simplest examples of braided monoidal supercategory is the supercategory \( \text{SVect}_\mathbb{C} \) of vector superspaces over \( \mathbb{C} \) of at most countable dimension equipped with the tensor product \( M \boxtimes N = M \otimes_{\mathcal{C}} N \) and the braiding

\[
R_{M,N} : M \otimes_{\mathcal{C}} N \to N \otimes_{\mathcal{C}} M, \quad m \otimes n \mapsto (-1)^{mn}n \otimes m.
\]

Obviously, the supercategory \( \mathcal{C} \) is \( \mathbb{C} \)-linear and its underlying category \( \underline{\mathcal{C}} \) is abelian. Note that the supercategory \( \text{SVect}_\mathcal{C} \) has admits a natural parity reversing endofunctor \( \Pi \), which exchanges the parity of objects.

In this paper, we make the following assumption on our monoidal supercategory \( \mathcal{C} \) in consideration:

Assumption A.4. Every object in \( \mathcal{C} \) has a structure of a \( \mathbb{C} \)-vector superspace of at most countable dimension, the forgetful functor \( \mathcal{C} \to \text{SVect}_\mathbb{C} \) is a \( \mathbb{C} \)-linear exact faithful superfunctor, and there exists an involutive auto functor \( \Pi_{\mathcal{C}} \) of \( \mathcal{C} \) which coincides with the parity reversing auto functor \( \Pi \) of \( \text{SVect}_\mathbb{C} \) through the forgetful functor. In addition, each \( \mathbb{C} \)-vector superspace of morphisms in \( \mathcal{C} \) has a finite dimension.

Such a situation naturally appears when we consider module categories of suitable superalgebras over \( \mathbb{C} \) like vertex operator superalgebras or Hopf superalgebras consisting of modules of at most countable dimension. The existence of the forgetful functor implies Diximier’s lemma (or Schur’s lemma) for \( \mathcal{C} \).

Lemma A.5. A simple object \( M \in \text{Ob}(\mathcal{C}) \) satisfies \( \text{End}_{\mathcal{C}}(M)_0 = \mathbb{C} \text{id}_M \) and \( \text{End}_{\mathcal{C}}(M)_1 = 0 \) or \( \mathbb{C} \Pi \) where \( \Pi \) is a certain linear map satisfying \( \Pi^2 = \text{id}_M \).

Proof. The first assertion follows from the argument in the purely even case, see e.g. [Wal, §0.5]. For the second one, we suppose \( \text{End}_{\mathcal{C}}(M)_1 \neq 0 \) and take a nonzero morphism \( f \in \text{End}_{\mathcal{C}}(M)_1 \). We prove \( \text{End}_{\mathcal{C}}(M)_1 = \mathbb{C} f \). Since \( f^2 \) is an even isomorphism, we may assume that \( f^2 = \text{id}_C \) by rescaling. Then it suffices to show \( g = \pm f \) for \( g \in \text{End}_{\mathcal{C}}(M)_1 \) such that \( g^2 = \text{id}_C \). Let \( \alpha \in \mathbb{C} \) be the scalar such that \( fg = \alpha \text{id}_C \). Since \( \text{id}_C = f(fg)g = \alpha fg = \alpha^2 \text{id}_C \), we obtain \( \alpha = \pm 1 \). Then, by \( fg = \pm \text{id}_C \) and \( (fg)(gf) = \text{id}_C \), we have \( fg = gf \). Now, we have \((f + g)(f - g) = f^2 - g^2 = 0 \), which implies that \( f + g = 0 \) or \( f - g = 0 \). This completes the proof. \( \square \)
In practice, the case $\text{End}_C(M)_1 = \mathbb{C}I$ occurs when $C$ is a module category of a $\mathbb{C}$-superalgebra $A = A_0 \oplus A_1$ and an object $M = M_0 \oplus M_1$ satisfies $f: M_0 \cong M_1$ as $A_0$-modules. In this case, an isomorphism $(f, f^{-1}): M = M_0 \oplus M_1 \cong M_1 \oplus M_0 = M$ as $A_0$-modules has odd parity and it might define an isomorphism as $A$-modules, see [CW, §3.1].

A.2. Block decomposition and Monodromy filtration by Simple currents. Let $C$ be a monoidal supercategory as in §2.1 equipped with a braided monoidal supercategory structure. Here we study decompositions and filtrations on $A$. Block decomposition and Monodromy filtration by Simple currents.

Proof. We show the assertion for $C$ is an isomorphism of $A$-superalgebras. Then by $(\text{A.7})$, we have

$$\text{End}_C(M) \cong \text{End}_C(S^n \otimes M), \quad (M \in \text{Ob}(\mathbb{C})).$$

Lemma A.6. The map

$$\text{End}_C(M) \rightarrow \text{End}_C(S^n \otimes M), \quad f \mapsto \text{id}_{S^n} \otimes f$$

is an isomorphism of $\mathbb{C}$-superalgebras.

Proof. It is straightforward to show that the map

$$\text{End}_C(S^n \otimes M) \rightarrow \text{End}_C(S^{-n} \otimes (S^n \otimes M)), \quad f \mapsto \text{id}_{S^{-n}} \otimes (\text{id}_{S^n} \otimes f)$$

is an inverse of (A.7) under the natural isomorphisms

$$\text{End}_C(S^{-n} \otimes (S^n \otimes M)) \cong \text{End}_C((S^{-n} \otimes S^n) \otimes M) \cong \text{End}_C(1 \otimes M) \cong \text{End}_C(M).$$

By Lemma A.6, the monodromy

$$M_{S^n,M} = R_{M,S^n} \circ R_{S^n,M} \in \text{End}_C(S^n \otimes M)$$

defines a unique element $m_S(n) \in \text{End}_C(M)$ satisfying

$$M_{S^n,M} = \text{id}_{S^n} \otimes m_S(n).$$

Proposition A.7. The endomorphism $m_S(n)$ is invertible and moreover

$$m_S(n) = m_S(1)^n \quad (n \in \mathbb{Z}).$$

Proof. We show the assertion for $n \geq 1$ by induction on $n$. By the hexagon identity, the following diagram commutes.

$$\begin{array}{cccc}
(S \otimes S^{-1}) \otimes M & \xrightarrow{\mathbb{R}_{S^n,M}} & M \otimes (S \otimes S^{-1}) & \xrightarrow{\mathbb{R}_{M,S^n}} & (S \otimes S^{-1}) \otimes M \\
S \otimes (S^{-1} \otimes M) & \cong & S \otimes (S^{-1} \otimes M) & \cong & S \otimes (S^{-1} \otimes M) \\
\text{id} \otimes \otimes & \cong & \text{id} \otimes \otimes & \cong & \text{id} \otimes \otimes \\
S \otimes (M \otimes S^{-1}) & \cong & S \otimes (M \otimes S^{-1}) & \cong & S \otimes (M \otimes S^{-1}) \\
(S \otimes M) \otimes S^{-1} & \xrightarrow{\mathbb{id} \otimes \mathbb{R}} & (M \otimes S) \otimes S^{-1} & \xrightarrow{\mathbb{id} \otimes \mathbb{R}} & (S \otimes M) \otimes S^{-1}.
\end{array}$$
Here all the isomorphisms without symbols are associativity isomorphisms. Then the morphisms in the bottom are composed as
\[
(R_{M,S} \boxtimes \text{id}_{S^{-1}}) \circ (R_{S,M} \boxtimes \text{id}_{S^{-1}}) = M_{S,M} \boxtimes \text{id}_{S^{-1}} = (\text{id}_S \boxtimes m_S(1)) \boxtimes \text{id}_{S^{-1}}.
\]
Now, we may use the naturality of the associativity \(\mathcal{A}_{,\bullet,\bullet}\) and the braiding \(\mathcal{R}_{,\bullet}\) to conclude \(M_{S^n,M} = \text{id}_{S^n} \boxtimes m_S(1)^n\) since
\[
M_{S^n,M} = R_{M,S^n} \circ R_{S^n,M}
= A_{S,S^{-1},M} \circ (\text{id}_S \boxtimes R_{S^{-1},M}) \circ A_{S,M,S^{-1}}^{-1}
= (\text{id}_{S^n} \boxtimes m_S(1)) \circ A_{S,S^{-1},M} \circ (\text{id}_S \boxtimes R_{S^{-1},M}) \circ A_{S,M,S^{-1}}^{-1}
= (\text{id}_{S^n} \boxtimes m_S(1)) \circ A_{S,S^{-1},M} \circ (\text{id}_S \boxtimes M_{S^{-1},M}) \circ A_{S,S^{-1},M}^{-1}
= (\text{id}_{S^n} \boxtimes m_S(1)) \circ A_{S,S^{-1},M} \circ (\text{id}_S \boxtimes m_S(1)) \circ m_S(1)^{n-1}) \circ A_{S,S^{-1},M}^{-1}
= \text{id}_{S^n} \boxtimes m_S(1)^n.
\]
Similarly, we can show \(m_S(-n) = m_S(-1)^n\) for \(n \geq 1\). Thus it remains to prove
\[
m_S(1)m_S(-1) = \text{id}_M.
\]
For this, note that by using the same argument as above, we obtain that
\[
M_{S^n,M} = \text{id}_{S^n} \boxtimes (m_S(1) \text{id}_M(-1)).
\]
Then \(S \boxtimes S^* \simeq 1_{\mathcal{C}}\) and \(M_{1e,M} = \text{id}_{1e} \boxtimes M\) (see [Kas, Proposition XIII. 1.2]) implies the assertion. This completes the proof.

**Proposition A.8.** An object \(M \in \text{Ob}(\mathcal{C})\) admits the generalized eigenspace decomposition of \(m_S(1)\)
\[
M = \bigoplus_{\alpha \in \mathbb{C}^\times} M_\alpha,
\]
where
\[
M_\alpha = \bigcup_{n \in \mathbb{Z}_{\geq 0}} M_\alpha[n], \quad M_\alpha[n] := \text{Ker}(m_S(1) - \alpha)^n.
\]

**Proof.** It is immediate that we have
\[
M \supset \sum_{\alpha \in \mathbb{C}^\times} M_\alpha = \bigoplus_{\alpha \in \mathbb{C}^\times} M_\alpha \tag{A.8}
\]
and that \(\{M_\alpha[n]\}_{n \in \mathbb{Z}_{\geq 0}}\) defines a filtration \(M_\alpha[p] \subset M_\alpha[q], (p < q)\). Thus it remains to prove the equality of (A.8). For this, we use the multiplication of \(m_S(1)\) on \(\text{End}_\mathcal{C}(M)\). Since the \(\mathbb{C}\)-vector superspace \(\text{End}_\mathcal{C}(M)\) is finite dimensional by Assumption A.4, the multiplication of \(m_S(1)\) gives a generalized eigenspace decomposition
\[
\text{End}_\mathcal{C}(M) = \bigoplus_{\alpha \in \mathbb{C}^\times} \text{End}_\mathcal{C}(M)_\alpha,
\]
\[
\text{End}_\mathcal{C}(M)_\alpha := \{f \in \text{End}_\mathcal{C}(M)_\alpha \mid (m_S(1) - \alpha)^n f = 0, \quad (\forall n \gg 0)\}.
\]
This gives the decomposition \(\text{id}_M = \sum_{\alpha} \pi_\alpha\). Thus we obtain
\[
M = \text{id}_M M = \sum_{\alpha \in \mathbb{C}^\times} \pi_\alpha M \subset \sum_{\alpha \in \mathbb{C}^\times} M_\alpha.
\]
This completes the proof.
We introduce full subcategories \( \mathcal{C}_\alpha[n] \subset \mathcal{C} \), \((\alpha \in \mathbb{C}^\times, n \in \mathbb{Z}_{\geq 0})\) defined by
\[
\mathcal{C}_\alpha[n] := \{ M \in \text{Ob}(\mathcal{C}) \mid (m_S(1) - \alpha)^{n+1} \text{id}_M = 0 \}.
\]
For a fixed \( \alpha \in \mathbb{C}^\times \), they give a filtration
\[
\mathcal{C}_\alpha[0] \subset \mathcal{C}_\alpha[1] \subset \cdots \subset \mathcal{C}_\alpha := \bigcup_{n \geq 0} \mathcal{C}_\alpha[n]. \tag{A.9}
\]
Then by Proposition A.8, we have the following block decomposition of \( \mathcal{C} \):
\[
\mathcal{C} = \bigoplus_{\alpha \in \mathbb{C}^\times} \mathcal{C}_\alpha. \tag{A.10}
\]
We call it the monodromy decomposition of \( \mathcal{C} \) by \( S \) and (A.9) the monodromy filtration of \( \mathcal{C} \) by \( S \).

**Proposition A.9.** (i) Any object \( M \) in \( \mathcal{C}_\alpha[m] \) is an extension
\[
0 \to M_1 \to M \to M_2 \to 0
\]
for some \( M_1 \in \text{Ob}(\mathcal{C}_\alpha[0]) \) and \( M_2 \in \text{Ob}(\mathcal{C}_\alpha[m-1]) \).
(ii) We have
\[
\boxtimes : \mathcal{C}_\alpha[m] \times \mathcal{C}_\beta[n] \to \mathcal{C}_{\alpha\beta}[m+n].
\]
**Proof.** (i) is immediate from the definition of \( \mathcal{C}_\alpha[m] \). We prove (ii). Take \( M \in \text{Ob}(\mathcal{C}_\alpha[m]) \) and \( N \in \text{Ob}(\mathcal{C}_\beta[n]) \). Let us write the monodromy operator \( m_S(1) \) for \( M \) as \( m_{S,M}(1) \) for clarity. By using the same diagram in the proof of Proposition A.7, we obtain
\[
M_{S,M\otimes N} = \text{id}_S \boxtimes (m_{S,M}(1) \boxtimes m_{S,N}(1)),
\]
and thus \( m_{S,M\otimes N}(1) = m_{S,M}(1) \boxtimes m_{S,N}(1) \). Now the assertion holds since
\[
(m_{S,M\otimes N}(1) - \alpha \beta)^{m+n+1} = (m_{S,M}(1) \boxtimes m_{S,N}(1) - \alpha \beta)^{m+n+1}
\]
\[
= (m_{S,M}(1) - \alpha) \boxtimes m_{S,N}(1) + \alpha \text{id}_M \boxtimes (m_{S,N}(1) - \beta))^{m+n+1}
\]
\[
= \sum_{k=0}^{m+n+1} \binom{m+n+1}{k} \alpha^k (m_{S,M}(1) - \alpha)^{m+n+1-k} \boxtimes m_{S,N}(1)^{m+n+1-k} (m_{S,N}(1) - \beta)^k
\]
\[
= 0.
\]
**Corollary A.10.** The full subcategory
\[
\mathcal{C}[0] := \bigoplus_{\alpha \in \mathbb{C}^\times} \mathcal{C}_\alpha[0]
\]
is a braided monoidal supercategory.

**Remark A.11.** If the simple current \( S \) is of finite order \( S^n \simeq 1_\mathcal{C} \), then the decomposition (A.10) holds without Assumption A.4. Indeed, by the complete reducibility of representations of finite abelian groups, \( m_S(1)^n = 1 \) implies the block decomposition
\[
\mathcal{C} = \mathcal{C}[0] = \bigoplus_{\alpha \in \mathbb{Z}/n \mathbb{Z}} \mathcal{C}_\alpha[0],
\]
where \( \mathbb{Z}_n \hookrightarrow \mathbb{C}^\times \), \([1] \mapsto e^{2\pi i/n}\) (cf. [CKL, Lemma 3.17]).

Next, we generalize (A.10) to a simultaneous decomposition by simple currents \( \{S_g\}_{g \in G} \) parametrized by a group \( G \), i.e., \( S_g \boxtimes S_h \simeq S_{gh}, \ (g,h \in G) \). Since \( \mathcal{C} \) is braided, we have
\[
S_{gh} \simeq S_g \boxtimes S_h \xrightarrow{S_g \boxtimes S_h} S_h \boxtimes S_g \simeq S_{hg},
\]
and $G$ is necessarily abelian. As the functoriality of $M_{\bullet, \bullet}$ implies that each monodromy operator $m_{S_\phi}(1)$ on $M \in Ob(\mathcal{C})$ lies in the center of $\text{End}_{\mathcal{C}}(M)$, the abelian group $G$ acts on $M$ by $g \mapsto m_{S_\phi}(1) (g \in G)$. We write $G' := \text{Hom}_{\text{Grp}}(G, \mathbb{C}^\times)$ for the dual of $G$ and introduce full subcategories $\mathcal{C}_\phi \subset \mathcal{C}$, $(\phi \in G')$, by

$$\mathcal{C}_\phi := \{ M \in Ob(M) \mid \forall g \in G, (m_{S_\phi}(1) - \phi(g))^N = 0, (\forall N \gg 0) \}.$$ 

Then Proposition A.8 and the proof of Proposition A.9 implies the following immediately.

**Theorem A.12.** The supercategory $\mathcal{C}$ admits a decomposition

$$\mathcal{C} = \bigoplus_{\phi \in G'} \mathcal{C}_\phi$$

as additive supercategories and the monoidal product respects the decomposition, i.e., $\boxtimes: \mathcal{C}_\phi \times \mathcal{C}_\psi \to \mathcal{C}_{\phi \psi}$.

Finally, we consider the case that $G$ is finitely generated. In this case, by the fundamental theorem of finitely generated abelian group, we have $G \cong G_{\text{fin}} \times \mathbb{Z}^n$ for some finite abelian group $G_{\text{fin}}$ and non-negative integer $n \geq 0$. Then the dual group $G'$ is $G' \cong G_{\text{fin}}' \times (\mathbb{C}^\times)^n$. Let $\mathcal{C}_{\phi, \alpha}[p] \subset \mathcal{C}$, $(\phi \in G_{\text{fin}}', \alpha \in (\mathbb{C}^\times)^n, p \in \mathbb{Z}_0^n)$, denote the full subcategory whose objects consist of $M \in Ob(\mathcal{C})$ such that

$$m_{S_\phi}(1) = \phi(g), \ (\forall g \in G_{\text{fin}}), \ (m_{S, 1, i} - \alpha)^p | M = 0, \ (1 \leq i \leq n),$$

where $1, (1 \leq i \leq n)$, denotes the generator of the $i$-th component of $\mathbb{Z}^n \subset G$. For each $(\phi, \alpha) \in G'$, we have $\mathcal{C}_{\phi, \alpha}[p] \subset \mathcal{C}_{\phi, \alpha}[q]$ if $q - p \in \mathbb{Z}_0^n$ and define

$$\mathcal{C}_{\phi, \alpha} := \bigcup_{p \in \mathbb{Z}_0^n} \mathcal{C}_{\phi, \alpha}[p].$$

Then we have the following:

**Theorem A.13.** Assume that the group $G$ is finitely generated. Then,

(i) the supercategory $\mathcal{C}$ admits a decomposition

$$\mathcal{C} = \bigoplus_{(\phi, \alpha) \in G'} \mathcal{C}_{\phi, \alpha},$$

and the monoidal product respects the filtration, i.e.,

$$\boxtimes: \mathcal{C}_{\phi, \alpha}[p] \times \mathcal{C}_{\phi, \beta}[q] \to \mathcal{C}_{\phi, \alpha \beta}[p + q],$$

(ii) the full subcategory

$$\mathcal{C}[0] := \bigoplus_{(\phi, \alpha) \in G'} \mathcal{C}_{\phi, \alpha}[0]$$

is naturally a braided monoidal subsupercategory,

(iii) every object in $\mathcal{C}_{\phi, \alpha}[p]$ is expressed as an extension of certain objects in $\mathcal{C}_{\phi, \alpha}[0]$ and objects in $\mathcal{C}_{\phi, \alpha}[\{p_1, \cdots, p_j - 1, \cdots, p_n\}]$ for some $j$.

We call the decomposition (A.11) the monodromy decompositions by $G$ and the filtration $\{ \mathcal{C}_{\phi, \alpha}[p] \}$ the monodromy filtration by $G$.

**Remark A.14.** By Theorem A.13 (iii), every simple object lies in $\mathcal{C}[0]$. Thus $\mathcal{C} = \mathcal{C}[0]$ holds if $\mathcal{C}$ is semisimple. Equivalently, $\mathcal{C} \neq \mathcal{C}[0]$ implies that the supercategory $\mathcal{C}$ is not semisimple.
A.3. Fusion rings. Let \( \mathcal{C} \) be a braided monoidal supercategory as in §2.2 with Assumption A.2 and A.4. Let \( \mathcal{K}(\mathcal{C}) \) denote its Grothendieck group, which is generated over \( \mathbb{Z} \) by isomorphism classes \([M]\) of objects \( M \) in \( \mathcal{C} \). Note that, if every object in \( \mathcal{C} \) of finite length, then the set \( \text{Irr} \mathcal{C} \) of simple objects of \( \mathcal{C} \) gives a \( \mathbb{Z} \)-basis of \( \mathcal{K}(\mathcal{C}) \). From now on, we assume the following condition:

**Assumption A.15.** The bifunctor \( \boxtimes: \mathcal{C} \times \mathcal{C} \to \mathcal{C} \) is biexact.

Then \( \mathcal{K}(\mathcal{C}) \) is a commutative ring by

\[
\mathcal{K}(\mathcal{C}) \times \mathcal{K}(\mathcal{C}) \to \mathcal{K}(\mathcal{C}), \quad ([M], [N]) \mapsto [M \boxtimes N]
\]

and is called the fusion ring of \( \mathcal{C} \). The set of simple currents in \( \mathcal{C} \), denoted by \( \text{Pic} \mathcal{C} \), is naturally an abelian group by \( \boxtimes \) and we regard the group ring \( \mathbb{Z}[\text{Pic} \mathcal{C}] \) as a subring of \( \mathcal{K}(\mathcal{C}) \).

Suppose that we have a set of simple currents \( \{S_g\}_{g \in G} \) parametrized by a finitely generated abelian group \( G \) as in §A.2. By Theorem A.13 (i), (ii), the fusion ring is \( G' \)-graded

\[
\mathcal{K}(\mathcal{C}) = \bigoplus_{\xi \in G'} \mathcal{K}(\mathcal{C}_\xi).
\]

By Theorem A.13 (iii), the embedding \( \mathcal{C}[0] \subset \mathcal{C} \) induces an isomorphism \( \mathcal{K}(\mathcal{C}[0]) \simeq \mathcal{K}(\mathcal{C}) \). Finally, we note that \( \mathcal{K}(\mathcal{C}) \) is a \( \mathbb{Z}[G] \)-algebra, where \( \mathbb{Z}[G] \) denotes the group ring of \( G \), by

\[
\mathbb{Z}[G] \times \mathcal{K}(\mathcal{C}) \to \mathcal{K}(\mathcal{C}), \quad (g, [M]) \mapsto [S_g \boxtimes M].
\]

Before we remark a criterion for the \( \mathbb{Z}[G] \)-freeness of \( \mathcal{K}(\mathcal{C}) \), we recall that Lemma A.1 implies that \( G \) acts on \( \text{Irr} \mathcal{C} \) by

\[
G \times \text{Irr} \mathcal{C} \to \text{Irr} \mathcal{C}, \quad (g, E) \to S_g \boxtimes E.
\]

Then it is clear that if every object in \( \mathcal{C} \) is of finite length, then \( \mathcal{K}(\mathcal{C}) \) is free over \( \mathbb{Z}[G] \) if and only if the \( G \)-action on \( \text{Irr} \mathcal{C} \) is free. Thus we have proved the following proposition:

**Proposition A.16.** For a set of simple currents \( \{S_g\}_{g \in G} \) in \( \mathcal{C} \) parametrized a finitely generated abelian group \( G \), the fusion rings \( \mathcal{K}(\mathcal{C}[0]) \) and \( \mathcal{K}(\mathcal{C}) \) are \( G' \)-graded \( \mathbb{Z}[G] \)-algebras. Moreover, the embedding \( \mathcal{C}[0] \subset \mathcal{C} \) gives an isomorphism \( \mathcal{K}(\mathcal{C}[0]) \simeq \mathcal{K}(\mathcal{C}) \) as \( G' \)-graded \( \mathbb{Z}[G] \)-algebras. If every object in \( \mathcal{C} \) is of finite length, then \( \mathcal{K}(\mathcal{C}) \) is free over \( \mathbb{Z}[G] \) if and only if the \( G \)-action on \( \text{Irr} \mathcal{C} \) is free.

A.4. Algebra objects and Induction functor. Following [CKM1], we review the notion of (unital, associative, commutative) algebra objects and their module objects in a braided monoidal supercategory, originally introduced by [KO] in the purely even case. We note that the authors of [CKM1] deal with superizations of non-super categories (see Remark A.18), but the proofs apply to our setting.

In this subsection, \( \mathcal{C} \) denotes an essentially small, \( \mathbb{C} \)-linear, monoidal supercategory whose underlying category is abelian, satisfying the following assumption which is a weaker version of Assumption A.15:

**Remark A.17.** In this subsection, we may replace Assumption A.15 by a weaker one, that is, the bifunctor \( \boxtimes \) is right exact.

An algebra object in \( \mathcal{C} \) is a triple \((\mathcal{E}, \mu_{\mathcal{E}}, \iota)\), (or \( \mathcal{E} \) for simplicity), consisting of \( \mathcal{E} \in \text{Ob}(\mathcal{C}) \) and even morphisms \( \mu_{\mathcal{E}} \in \text{Hom}_\mathcal{C}(\mathcal{E}, \mathcal{E})_0 \) and \( \iota \in \text{Hom}_\mathcal{C}(1_\mathcal{C}, \mathcal{E})_0 \) satisfying

- The morphism \( \iota: 1_\mathcal{C} \to \mathcal{E} \) is injective

and the following commutative diagrams:
Correspondences of categories for subregular $W$-algebras and principal $W$-superalgebras

(A1) Unity
\[
1_C \boxtimes C \xrightarrow{\overline{id}_C} C \boxtimes C \xrightarrow{\mu_C} \mathcal{E},
\]

(A2) Associativity
\[
\begin{array}{c}
\mathcal{E} \boxtimes (\mathcal{E} \boxtimes \mathcal{E}) \\
\xrightarrow{\mu_C} \mathcal{E} \boxtimes \mathcal{E} \\
\xrightarrow{\mu_C} \mathcal{E}
\end{array}
\]

(A3) Commutativity
\[
\begin{array}{c}
\mathcal{E} \boxtimes \mathcal{E} \\
\xrightarrow{\mu_C} \mathcal{E} \boxtimes \mathcal{E} \\
\xrightarrow{\mu_C} \mathcal{E}
\end{array}
\]

Remark A.18.
(1) A typical example of our braided monoidal supercategory is the superization $SC$ of a braided abelian monoidal category $C$, whose objects are pairs $M = (\bar{M}_0, M_1)$ of $M_i \in \text{Ob}(C)$. This induces a $\mathbb{Z}_2$-graded structure on the set of morphisms. In this case, an algebra object in $SC$ is called a superalgebra object in $C$, see [CKM1].

(2) For an application to extensions of vertex superalgebras, the condition
\[
\text{Hom}_C(1_C, \mathcal{E}) \simeq \text{End}_C(1_C)
\]
is often assumed so that the extended vertex superalgebra is of CFT type.

An $\mathcal{E}$-module is a pair $(M, \mu_M)$, (or $M$ for simplicity), consisting of $M \in \text{Ob}(\mathcal{E})$ and an even morphism $\mu_M \in \text{Hom}_C(\mathcal{E} \boxtimes M, M)$ satisfying the following commutative diagrams:

(M1) Unity
\[
\begin{array}{c}
1_C \boxtimes M \\
\xrightarrow{\overline{id}_M} \mathcal{E} \boxtimes M \\
\xrightarrow{\mu_M} M.
\end{array}
\]

(M2) Associativity
\[
\begin{array}{c}
\mathcal{E} \boxtimes (\mathcal{E} \boxtimes M) \\
\xrightarrow{\mu_M} \mathcal{E} \boxtimes M \\
\xrightarrow{\mu_M} M
\end{array}
\]

An $\mathcal{E}$-module $(M, \mu_M)$ is called local if it further satisfies the following commutative diagram:

(M3) Locality
\[
\begin{array}{c}
\mathcal{E} \boxtimes M \\
\xrightarrow{\mu_M} \mathcal{E} \boxtimes M \\
\xrightarrow{\mu_M} M.
\end{array}
\]
A morphism of \( \mathcal{E} \)-modules from \((M, \mu_M)\) to \((N, \mu_N)\) is a morphism \( f \in \text{Hom}_\mathcal{C}(M, N)\) satisfying the following commutative diagram:

\[
\begin{array}{ccc}
\mathcal{E} \boxtimes M & \xrightarrow{\mu_M} & \mathcal{E} \boxtimes N \\
\mu_M \downarrow & & \downarrow \mu_N \\
M & \xrightarrow{f} & N.
\end{array}
\]

Let \( \text{Rep}(\mathcal{E}) \) denote the supercategory of \( \mathcal{E} \)-modules with morphisms of \( \mathcal{E} \)-modules, and \( \text{Rep}^0(\mathcal{E}) \) the full subcategory of \( \text{Rep}(\mathcal{E}) \) consisting of local \( \mathcal{E} \)-modules.

Although \( \text{Rep}(\mathcal{E}) \) is just a \( \mathbb{C} \)-linear additive supercategory, the underlying category \( \text{Rep}(\mathcal{E}) \) is an abelian category. Furthermore, by the existence of the involutive autofunctor \( \Pi_E \), every parity-homogeneous morphism in \( \text{Rep}(\mathcal{E}) \) admits kernel and cokernel objects. The same is true for \( \text{Rep}^0(\mathcal{E}) \). Each \( \text{Rep}(\mathcal{E}) \) and \( \text{Rep}^0(\mathcal{E}) \) admits a natural monoidal structure in the following way [CKM, §2]. Consider the two compositions

\[
\begin{align*}
\xi_1 & : \mathcal{E} \boxtimes (M \otimes N) \xrightarrow{A_{E, E}} (\mathcal{E} \boxtimes M) \otimes N \xrightarrow{\mu_M \otimes \text{id}_N} M \boxtimes N, \\
\xi_2 & : \mathcal{E} \boxtimes (M \otimes N) \xrightarrow{A_{E, E}} (\mathcal{E} \boxtimes M) \otimes N \xrightarrow{\mu_M \otimes \text{id}_N} (M \otimes \mathcal{E}) \otimes N \xrightarrow{\text{id}_M \otimes \xi_E} M \boxtimes \mathcal{E} \otimes N.
\end{align*}
\]

Then \( M \boxtimes_E N \) is defined by \( M \boxtimes_E N := \text{Coker}(\xi_1 - \xi_2) \), which is an object of \( \mathcal{E} \). Let \( \eta_{M,N} \) denote the canonical surjection \( M \boxtimes N \rightarrow M \boxtimes_E N \). The \( \mathcal{E} \)-module structure \( \mu_{M \boxtimes_E N} : \mathcal{E} \boxtimes (M \boxtimes_E N) \rightarrow M \boxtimes_E N \) is the unique even morphism, which makes the diagram

\[
\begin{array}{ccc}
\mathcal{E} \boxtimes (M \otimes N) & \xrightarrow{\xi_i} & M \boxtimes N \\
\mu_M \otimes \text{id}_N \downarrow & & \downarrow \eta_{M,N} \\
\mathcal{E} \boxtimes (M \boxtimes_E N) & \xrightarrow{\mu_{M \boxtimes_E N}} & M \boxtimes_E N.
\end{array}
\]

commutes for \( i = 1, 2 \). The associativity \( A_{E, E} \circ \bullet \boxtimes_E (\bullet \boxtimes_E \bullet) \simeq (\bullet \boxtimes_E \bullet) \boxtimes_E \bullet \) is given by the family of unique even morphisms \( A_{E, E}^M : M \boxtimes_E (N \boxtimes_E L) \simeq (M \boxtimes_E N) \boxtimes_E L \) for \( M, N, L \in \text{Ob}(\text{Rep}(\mathcal{E})) \), which follows the following diagram commutes:

\[
\begin{array}{ccc}
M \boxtimes (N \otimes L) & \xrightarrow{A_{M,N,L}} & (M \boxtimes N) \otimes L \\
\mu_M \otimes \text{id}_L \downarrow & & \downarrow \eta_{M,N,L} \\
M \boxtimes (N \boxtimes_E L) & \xrightarrow{\mu_{M \boxtimes_E N,L}} & (M \boxtimes_E N) \boxtimes_E L.
\end{array}
\]

The unit object of \( \text{Rep}(\mathcal{E}) \) and \( \text{Rep}^0(\mathcal{E}) \) is \( \mathcal{E} \) equipped with even natural morphisms

\[
l^E_\mathcal{E} : \mathcal{E} \boxtimes \mathcal{E} \simeq \bullet, \quad r^E_\mathcal{E} : \bullet \boxtimes \mathcal{E} \simeq \bullet
\]

given by the family of unique even morphisms \( l^E_M : \mathcal{E} \boxtimes \mathcal{E} M \simeq M \) and \( r^E_M : M \boxtimes \mathcal{E} \simeq M \), which make the following diagrams commute:

\[
\begin{array}{ccc}
\mathcal{E} \boxtimes M & \xrightarrow{\mu_M} & M \\
\eta_{E,N} \downarrow & & \downarrow \eta_{M,N} \\
\mathcal{E} \boxtimes_E M & \xrightarrow{l^E_M} & M
\end{array}
\quad
\begin{array}{ccc}
M \boxtimes \mathcal{E} & \xrightarrow{\mu_M} & M \\
\eta_{M,E} \downarrow & & \downarrow \eta_{M,E} \\
M \boxtimes \mathcal{E} & \xrightarrow{r^E_M} & M
\end{array}
\quad
\begin{array}{ccc}
\mathcal{E} \boxtimes M & \xrightarrow{\mu_M} & M \\
\eta_{E,N} \downarrow & & \downarrow \eta_{M,N} \\
\mathcal{E} \boxtimes \mathcal{E} M & \xrightarrow{l^E_M} & M
\end{array}
\quad
\begin{array}{ccc}
M \boxtimes \mathcal{E} & \xrightarrow{\mu_M} & M \\
\eta_{M,E} \downarrow & & \downarrow \eta_{M,E} \\
M \boxtimes \mathcal{E} & \xrightarrow{r^E_M} & M
\end{array}
\quad
\begin{array}{ccc}
\mathcal{E} \boxtimes M & \xrightarrow{\mu_M} & M \\
\eta_{E,N} \downarrow & & \downarrow \eta_{M,N} \\
\mathcal{E} \boxtimes \mathcal{E} M & \xrightarrow{l^E_M} & M
\end{array}
\quad
\begin{array}{ccc}
M \boxtimes \mathcal{E} & \xrightarrow{\mu_M} & M \\
\eta_{M,E} \downarrow & & \downarrow \eta_{M,E} \\
M \boxtimes \mathcal{E} & \xrightarrow{r^E_M} & M
\end{array}
\quad
\begin{array}{ccc}
\mathcal{E} \boxtimes M & \xrightarrow{\mu_M} & M \\
\eta_{E,N} \downarrow & & \downarrow \eta_{M,N} \\
\mathcal{E} \boxtimes \mathcal{E} M & \xrightarrow{l^E_M} & M
\end{array}.\]
The braiding \( \mathcal{R}_\bullet \) on \( \mathcal{E} \) induces a braiding \( \mathcal{R}_{\mathcal{E}} \) on \( \text{Rep}^0(\mathcal{E}) \), which is a family of unique even morphisms \( \mathcal{R}_{M,N}^E : M \boxtimes E N \simeq N \boxtimes E M \), which make the diagram

\[
\begin{array}{c}
M \boxtimes E N \xrightarrow{\mathcal{R}_{M,N}^E} N \boxtimes E M \\
\Downarrow \eta_{M,N} \quad \quad \Downarrow \eta_{N,M}
\end{array}
\]

commute. To summarize, we obtain the following.

**Theorem A.19** ([CKM1, Theorem 2.53]). *The supercategory \( \text{Rep}(\mathcal{E}) \) (resp. \( \text{Rep}^0(\mathcal{E}) \)) is naturally a \( \mathbb{C} \)-linear additive (resp. braided) monoidal supercategory such that the underlying category \( \text{Rep}(\mathcal{E}) \), (resp. \( \text{Rep}^0(\mathcal{E}) \)) is an abelian category.*

For \( M \in \text{Ob}(\mathcal{E}) \), we define an \( \mathcal{E} \)-module \( (\mathcal{F}(M), \mu_{\mathcal{F}(M)}) \) by

\[
\mathcal{F}(M) := \mathcal{E} \boxtimes M, \\
\mu_{\mathcal{F}(M)} : \mathcal{E} \boxtimes (\mathcal{E} \boxtimes M) \simeq (\mathcal{E} \boxtimes \mathcal{E}) \boxtimes M \xrightarrow{\mu_{\boxtimes \mathcal{E}}} \mathcal{E} \boxtimes M.
\]

We also define a superfunctor \( \mathcal{F} : \mathcal{E} \to \text{Rep}(\mathcal{E}) \) by \( M \mapsto (\mathcal{F}(M), \mu_{\mathcal{F}(M)}) \) and \( \mathcal{F}(f) := \text{id}_\mathcal{E} \boxtimes f \in \text{Hom}_{\text{Rep}(\mathcal{E})}(\mathcal{F}(M), \mathcal{F}(N)) \) for \( f \in \text{Hom}_{\mathcal{E}}(M, N) \). The superfunctor \( \mathcal{F} \) is called the induction functor and enjoys the following property.

**Theorem A.20** ([CKM1, Theorem 2.59]). *The induction functor \( \mathcal{F} : \mathcal{E} \to \text{Rep}(\mathcal{E}) \) is a \( \mathbb{C} \)-linear, additive, strong monoidal superfunctor.*

Let \( \mathcal{E}^\Theta \) denote the full subcategory of \( \mathcal{E} \) consisting of objects \( M \in \text{Ob}(\mathcal{E}) \) such that \( M_{\mathcal{E},\mathcal{E}} = \text{id}_{\mathcal{E},\mathcal{E}} \). We call \( \mathcal{E}^\Theta \) the category of \( \mathcal{E} \)-local objects.

**Theorem A.21** ([CKM1]). *We have the following.*

1. *The supercategory \( \mathcal{E}^\Theta \) is a braided monoidal subsupercategory of \( \mathcal{E} \).*
2. *For \( M \in \text{Ob}(\mathcal{E}) \), the object \( \mathcal{F}(M) \) lies in \( \text{Rep}^0(\mathcal{E}) \) if and only if \( M \in \text{Ob}(\mathcal{E}^\Theta) \).*
3. *The restriction of \( \mathcal{F} \) to \( \mathcal{E}^\Theta \) gives a braided monoidal superfunctor \( \mathcal{F} : \mathcal{E}^\Theta \to \text{Rep}^0(\mathcal{E}) \).*

**Proof.** By [CKM1, Theorem 2.67], \( \mathcal{E}^\Theta \) is a \( \mathbb{C} \)-linear additive braided monoidal supercategory, Thus to show (1), it remains to show that \( \mathcal{E}^\Theta \) is closed under kernel and cokernel, which immediately follows from the exactness of \( \mathcal{E} \boxtimes \bullet \) in Assumption A.15. (2) is [CKM1, Proposition 2.65] and (3) is [CKM1, Theorem 2.67]. \( \square \)

At last, the induction functor \( \mathcal{F} \) is related to the forgetful functor

\[
\mathcal{G} : \text{Rep}(\mathcal{E}) \to \mathcal{E}, \quad (M, \mu_M) \mapsto M.
\]

by the Frobenius reciprocity:

**Proposition A.22** ([CKM1, Lemma 2.61]). *The superfunctor \( \mathcal{G} : \text{Rep}(\mathcal{E}) \to \mathcal{E} \) is right adjoint to the superfunctor \( \mathcal{F} : \mathcal{E} \to \text{Rep}(\mathcal{E}) \), that is, we have a natural isomorphism

\[
\text{Hom}_{\text{Rep}(\mathcal{E})}(\mathcal{F}(N), M) \simeq \text{Hom}_{\mathcal{E}}(N, \mathcal{G}(M)),
\]

for \( M \in \text{Ob}(\text{Rep}(\mathcal{E})) \) and \( N \in \text{Ob}(\mathcal{E}) \). More explicitly, for \( f \in \text{Hom}_{\mathcal{E}}(N, \mathcal{G}(M)) \), the corresponding morphism of \( \mathcal{E} \)-modules is given by

\[
\mathcal{F}(N) = \mathcal{E} \boxtimes N \to M, \quad a \boxtimes m \mapsto \mu_M(a \boxtimes f(m)).
\]
A.5. Categorical simple current extensions. Let $\mathcal{C}$ be a braided monoidal supercategory as in §A.2, satisfying Assumption A.2–A.15 and the following assumption.

Assumption A.23. Every object in $\mathcal{C}$ has finite length.

Let $\mathcal{E}$ be an algebra object in $\mathcal{C}$ of the form

$$\mathcal{E} = \bigoplus_{g \in G} S_g,$$

where $\{S_g\}_{g \in G} \subset \text{Pic} \mathcal{C}$ is a set of simple currents parametrized by a finite abelian group $G$ with $S_e = 1_{\mathcal{C}}$. Here $e \in G$ denotes the unit of $G$. We call $\mathcal{E}$ a categorical simple current extension of $1_{\mathcal{C}}$. In the rest of this subsection, we assume

(S1) the product $\mu_\mathcal{E}$ restricts to a non-zero morphism $S_g \boxtimes S_h \to S_{gh}$ for $g, h \in G$,

(S2) the action of $G$ on $\text{Irr} \mathcal{C}$ is fixed-point free.

Note that these assumptions imply $S_g \simeq S_h$ if and only if $g = h$.

Proposition A.24. Suppose (S1) and (S2).

1. If $M$ is a simple object in $\mathcal{E}$, so is $\mathcal{F}(M)$ in $\text{Rep}(\mathcal{E})$.
2. For a simple object $M$ in $\text{Rep}(\mathcal{E})$, there exists a simple object $N$ in $\mathcal{C}$ such that $M \simeq \mathcal{F}(N)$ in $\text{Rep}(\mathcal{E})$.
3. For simple objects $M$ and $N$ in $\mathcal{C}$, we have $\mathcal{F}(M) \simeq \mathcal{F}(N)$ in $\text{Rep}(\mathcal{E})$ if and only if we have $M \simeq S_g \boxtimes N$ for some $g \in G$.
4. The supercategory $\mathcal{C}$ is semisimple if and only if $\text{Rep}(\mathcal{E})$ is semisimple. In this case, $\mathcal{E}^I \subset \mathcal{C}$ and $\text{Rep}^I(\mathcal{E}) \subset \text{Rep}(\mathcal{E})$ are also semisimple.

Proof. Although these statements are well-known in the theory of vertex algebras, (see e.g., [CKM1, Proposition 4.5]), we include a proof for the completeness of the paper. First we prove (1). Let $M$ be a simple object in $\mathcal{C}$. Then $\mathcal{F}(M) = \bigoplus_{g \in G} S_g \boxtimes M$ and all summands are pairwise non-isomorphic by (S2). Let $N$ be a nonzero subobject of $\mathcal{F}(M)$ in $\text{Rep}(\mathcal{E})$. Since $N$ is a semisimple object in $\mathcal{C}$, it has a simple subobject which is isomorphic to $S_g \boxtimes M$ for some $g \in G$. By (S1), the structure morphism $\mu_{\mathcal{F}(M)}$ restricts to an isomorphism $S_g \boxtimes (S_g \boxtimes M) \xrightarrow{\sim} S_{gh} \boxtimes M$ for any $h \in G$. Since $N$ is closed under the $\mathcal{E}$-action, it contains $\bigoplus_{h \in G} S_{gh} \boxtimes M = \mathcal{F}(M)$. This proves (1). Then (2) and (3) follow from (1) and Proposition A.22. Finally, we show (4). Assume that $\text{Rep}(\mathcal{E})$ is semisimple and take $N \in \text{Ob}(\mathcal{C})$. Since $\mathcal{F}(N)$ is semisimple, we have

$$\mathcal{F}(N) \simeq \bigoplus_{i \in I} N_i$$

for some simple objects $N_i$ in $\text{Rep}(\mathcal{E})$ indexed by a finite set $I$. Then by (2), we may replace $N_i$ by $\mathcal{F}(N_i)$ for some simple objects $N'_i$ in $\mathcal{C}$. Thus,

$$M \subset \mathcal{F}(N) \simeq \bigoplus_{i \in I} \mathcal{F}(N'_i) = \bigoplus_{g \in G} S_g \boxtimes N'_i.$$

Since $S_g \boxtimes N'_i$ are all simple objects in $\mathcal{C}$, $N$ is semisimple. To prove the converse, assume that $\mathcal{C}$ is semisimple. Since every object in $\mathcal{C}$ has finite length, so does every object in $\text{Rep}(\mathcal{E})$. Thus to show that $\text{Rep}(\mathcal{E})$ is semisimple, it suffices to show the splitting of any short exact sequence

$$0 \to N_1 \to N \to N_2 \to 0 \quad (A.12)$$

in $\text{Rep}(\mathcal{E})$ where $N_1, N_2$ are simple. By (2), we may assume $N_i = \mathcal{F}(N_i)$ for some simple object $N_i \in \text{Ob}(\mathcal{E})$. If $\mathcal{F}(N_1) \ncong \mathcal{F}(N_2)$, then $S_g \boxtimes N_1 \ncong S_h \boxtimes N_2$ for all $g, h \in G$ in $\mathcal{C}$. This implies the splitting of (A.12) in $\text{Rep}(\mathcal{E})$. Thus we may assume
\( \mathcal{F}(N_1) \cong \mathcal{F}(N_2) \) and moreover \( N := N_1 = N_2 \) from the beginning. Since \( G \) is a finite abelian group, it is isomorphic to some direct product of cyclic groups \( \prod_n \mathbb{Z}_{n_i} \). Then it suffices to show the splitting in the case of \( G = \mathbb{Z}_n \) for some \( n \in \mathbb{Z}_{\geq 0} \). Let \( S_p \) denote the simple current corresponding to \( p \in \mathbb{Z}_n \). Since the space of intertwining operators \( I(S_{p_1}) \otimes \mathcal{F}(N) \) is one dimensional, we may take its basis by the intertwining operator

\[
S_1 \otimes (S_p \otimes N) \cong (S_1 \otimes S_p) \otimes N \cong S_{p+1} \otimes N
\]

used in the definition of \( \mathcal{F}(N) \). On the other hand, the restriction of the \( \mathcal{E} \)-module structure of \( N \) gives an intertwining operator \( S_1 \otimes N \rightarrow N \). Along with the decomposition \( N \cong \mathcal{F}(N) \oplus \mathcal{F}(N) \) in \( \mathcal{E} \), the intertwining operator is expressed by the matrix:

\[
K := \begin{pmatrix} E & A \\ 0 & E \end{pmatrix}
\]

where \( E = \sum_{i \in \mathbb{Z}_n} E_{i+1,i} \) and \( A = \sum_{i \in \mathbb{Z}_n} a_i E_{i+1,i} \) for some \( a_i \in \mathbb{C} \). Since \( S_p \cong 1_{\mathcal{E}} \), we have \( \sum_i a_i = 0 \). Then it is straightforward to check that \( K \) is conjugate to the matrix \( K \) with \( a_i = 0 \) for all \( i \). This implies that we may take a decomposition \( N = \mathcal{F}(N) \oplus \mathcal{F}(N) \) in \( \mathcal{E} \) which is preserved by the action of \( S_1 \subset \mathcal{E} \). Since the other action of \( S_p \subset \mathcal{E} \) is obtained from the action of \( S_1 \) via iteration, the above decomposition of \( N \) is actually the decomposition as an \( \mathcal{E} \)-module. The remaining statements in (4) are now obvious. This completes the proof. \( \square \)

In particular, we have the following.

**Corollary A.25.** The induction functor \( \mathcal{F}: \mathcal{E} \rightarrow \text{Rep}(\mathcal{E}) \) induces the following natural identifications:

1. \( \text{Irr}(\text{Rep}(\mathcal{E})) \cong \text{Irr}(\mathcal{E})/G \) and \( \text{Pic}(\text{Rep}(\mathcal{E})) \cong \text{Pic}(\mathcal{E})/G \);
2. \( \text{Irr}(\text{Rep}^0(\mathcal{E})) \cong \text{Irr}(\mathcal{E})/G \) and \( \text{Pic}(\text{Rep}^0(\mathcal{E})) \cong \text{Pic}(\mathcal{E})/G \).

Since the induction functor is a monoidal superfunctor, we may write the fusion ring of \( \text{Rep}(\mathcal{E}) \) in terms of \( \mathcal{E} \).

**Corollary A.26.** Suppose (S1) and (S2). If the superfunctor \( \boxtimes_{\mathcal{E}} \) is bi-exact, then the induction functor \( \mathcal{F}: \mathcal{E} \rightarrow \text{Rep}(\mathcal{E}) \) induces the following isomorphisms of rings:

1. \( \mathcal{K}(\text{Rep}(\mathcal{E})) \cong \mathcal{K}(\mathcal{E})/\mathcal{J} \) where \( \mathcal{J} = \langle [M] - [S_p \boxtimes M] \mid g \in G, M \in \text{Ob}(\mathcal{E}) \rangle \);
2. \( \mathcal{K}(\text{Rep}^0(\mathcal{E})) \cong \mathcal{K}(\mathcal{E})/\mathcal{J}' \) where \( \mathcal{J}' = \langle [M] - [S_p \boxtimes M] \mid g \in G, M \in \text{Ob}(\mathcal{E}) \rangle \).

**Proof.** (1) follow from Proposition A.24. (2) follows from (1) and Theorem A.21. \( \square \)

**Appendix B. Proof of Proposition 4.5**

We recall the level-rank duality between \( L_m(\mathfrak{sl}_n) \) and \( L_n(\mathfrak{sl}_m) \) for \( n,m \geq 2 \). \([\text{Fr}, \text{OS}]\). The isomorphism \( \mathbb{C}^{nm} \cong \mathbb{C}^n \otimes \mathbb{C}^m \) induces an embedding of Lie algebras \( \mathfrak{sl}_n \otimes \mathfrak{sl}_m \hookrightarrow \mathfrak{sl}_{nm} \) and thus \( L_m(\mathfrak{sl}_n) \otimes L_n(\mathfrak{sl}_m) \hookrightarrow L_1(\mathfrak{sl}_{nm}) \). It is a conformal embedding and gives a finite decomposition

\[
L_1(\mathfrak{sl}_{nm}) \cong \bigoplus_{\lambda \in \hat{P}^m_n(n)} L_m(\lambda) \otimes L_n(\lambda^t)
\]

as \( L_m(\mathfrak{sl}_n) \otimes L_n(\mathfrak{sl}_m) \)-modules where \( \lambda \mapsto \lambda^t \) denotes the transpose. More precisely, let \( C_{n,m} \) denote the set of Young diagrams lying in the \( n \times m \) rectangle. Then we have an embedding \( \hat{P}^m_n(n) \hookrightarrow C_{n,m} \). (\( \lambda = \sum_{i \in \mathbb{Z}_n} a_i \Lambda_i \mapsto \sum_{i=1}^n a_i R_i \)) where \( R_i \)
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denote the column of boxes of height \( i \). \( (R_n) \) is identified with the empty set.) Then the transpose \( \lambda \mapsto \lambda^t \) is just the transpose of Young diagrams, e.g.,

\[
\tilde{P}_+(3) \ni \lambda_0 + \lambda_1 + 3\lambda_2 = \begin{array}{ccc}
\boxed{} & \boxed{} & \boxed{} \\
\boxed{} & \boxed{} & \boxed{} \\
\boxed{} & \boxed{} & \boxed{}
\end{array} \quad \overset{\lambda}{\rightarrow} \quad \begin{array}{ccc}
\boxed{} & \boxed{} & \boxed{} \\
\boxed{} & \boxed{} & \boxed{} \\
\boxed{} & \boxed{} & \boxed{}
\end{array} = \lambda_0 + \lambda_3 + \lambda_4 \in \tilde{P}_+(5).
\]

Note that \( \pi_{P/Q}(\lambda) = \ell(\lambda) \in \mathbb{Z} \) where \( \ell(\lambda) \) denotes the number of boxes of the Young diagram of \( \lambda \). By the Frenkel–Kac construction, the natural embedding \( Q(\mathfrak{sl}_n) \hookrightarrow \mathbb{Z}^m \) gives rise to a vertex algebra embedding \( L_1(\mathfrak{sl}_n) \simeq V_{Q(\mathfrak{sl}_n)} \hookrightarrow V_{\mathbb{Z}^m} \). Then \( \text{Com}(L_1(\mathfrak{sl}_n), V_{\mathbb{Z}^m}) \simeq V_{\mathbb{Z}^{nm}} \) with \( \sqrt{nm} \mapsto \mathbb{Z}^{nm} \); \( a\sqrt{nm} \mapsto (a, a, \ldots, a) \) and we have

\[
V_{\mathbb{Z}^m} \simeq \bigoplus_{\lambda \in \tilde{P}_+(n)} L_m(\lambda) \otimes \bigoplus_{\lambda \in \tilde{P}_+(n)} L_n(\sigma^{-\ell(\lambda)}(\lambda^t)) \otimes V_{\mathbb{Z}^{nm}} + \sqrt{nm}Z.
\]

as \( L_m(\mathfrak{sl}_n) \otimes V_{\mathbb{Z}^{nm}} \)-modules. Now, by the branching law of \( L_1(\mathfrak{sl}_n) \) as an \( L_m(\mathfrak{sl}_n) \otimes L_n(\mathfrak{sl}_n) \)-module [OS, Theorem 4.1], we obtain

\[
V_{\mathbb{Z}^m} \simeq \bigoplus_{\lambda \in \tilde{P}_+(n)} L_m(\lambda) \otimes \bigoplus_{\lambda \in \tilde{P}_+(n)} L_n(\sigma^{-\ell(\lambda)}(\lambda^t)) \otimes V_{\mathbb{Z}^{nm}} + \sqrt{nm}Z.
\]

as \( L_m(\mathfrak{sl}_n) \otimes L_n(\mathfrak{sl}_n) \otimes V_{\mathbb{Z}^{nm}} \)-modules, where we identify \( a \in \mathbb{Z}^m \) with its image in \( \mathbb{Z}_m \) under the natural projection \( \mathbb{Z}^m \rightarrow \mathbb{Z}_m \). Thus, \( \mathcal{E}_{m,n} := \text{Com}(L_m(\mathfrak{sl}_n), V_{\mathbb{Z}^m}) \) is an order \( m \) simple current extension of \( L_n(\mathfrak{sl}_n) \otimes V_{\mathbb{Z}^{nm}} \) and we have

\[
\text{Com}(L_m(\mathfrak{sl}_n), V_{\mathbb{Z}^m}) \simeq \bigoplus_{a \in \mathbb{Z}_m} L_n(n\lambda_a) \otimes V_{\mathbb{Z}^{nm}} + \sqrt{nm}Z.
\]

Therefore, by Theorem 2.3, we have

\[
\text{Irr}((\mathcal{E}_{m,n})_0) = \left\{ (\lambda, a) \in \tilde{P}_+(m) \times \mathbb{Z}^m | a \in \mathbb{Z}_m \right\},
\]

where \( \mathbb{Z}_m \) acts on \( \tilde{P}_+(m) \times \mathbb{Z}_m \) by \( r \cdot (\lambda, a) = (\sigma^r(\lambda), a + r n) \), \( (r \in \mathbb{Z}_m) \), and

\[
\mathcal{X}(\mathcal{E}_{m,n}) \simeq (\mathcal{X}(L_n(\mathfrak{sl}_n)) \otimes \mathbb{Z}[\mathbb{Z}_m]) \mathbb{Z}_m, \quad (\lambda, a) \mapsto L_n(\lambda) \otimes \mathbb{Z}_m.
\]

Now, (B.1) implies the decomposition

\[
V_{\mathbb{Z}^m} \simeq \bigoplus_{\lambda \in \tilde{P}_+(n)} L_m(\lambda) \otimes \mathbb{M}(\lambda^t, \ell(\lambda))
\]

as \( L_m(\mathfrak{sl}_n) \otimes \mathcal{E}_{m,n} \)-modules. This gives an one-to-one correspondence

\[
\text{Irr}(L_m(\mathfrak{sl}_n)) \rightarrow \text{Irr}(\mathcal{E}_{m,n}), \quad L_m(\lambda) \mapsto \mathbb{M}(\lambda^t, \ell(\lambda))
\]

which implies a braided-reverse equivalence of braided tensor categories between \( L_m(\mathfrak{sl}_n)\text{-mod} \) and \( \mathcal{E}_{m,n} \text{-mod} \) by [CKM2] and, in particular, an isomorphism

\[
\mathcal{X}(L_m(\mathfrak{sl}_n)) \simeq \left( \mathcal{X}(L_n(\mathfrak{sl}_n)) \otimes \mathbb{Z}[\mathbb{Z}_m] \right) \mathbb{Z}_m, \quad L_m(\lambda) \mapsto L_n(\lambda^t) \otimes \mathbb{Z}_m.
\]
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