The complexity of cake cutting with unequal shares

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Abstract

An unceasing problem of our prevailing society is the fair division of goods. The problem of fair cake cutting is dividing a heterogeneous and divisible resource, the cake, among \( n \) players who value pieces according to their own measure function. The goal is to assign each player a not necessarily connected part of the cake that the player evaluates at least as much as her proportional share.

In this paper, we investigate the problem of proportional division with unequal shares, where each player is entitled to receive a predetermined portion of the cake. Our contribution is twofold. First we present a protocol that delivers a proportional solution in less queries than all known algorithms. We then show that our protocol is the fastest possible by giving an asymptotically matching lower bound. Both results remain valid in a highly general cake cutting model, which can be of independent interest.

1 Introduction

In cake cutting problems, the cake symbolizes a heterogeneous and divisible resource that is to be distributed among \( n \) players. Each player has her own measure function, which determines the value of any part of the cake for her. The aim of fair cake cutting is to allocate each player a piece that is worth at least as much as her proportional share, evaluated with her measure function \cite{17}.

The efficiency of fair division algorithm is measured by the number of queries. In the standard Robertson-Webb model \cite{15}, two kinds of queries are allowed. The first one is the cut query, in which a player is asked to mark the cake at a distance from a given starting point so that the piece between these two is worth a given value to her. The second operation is the eval query, in which a player is asked to evaluate a given piece according to her measure function.

If shares are meant to be equal for all players, then the proportional share is defined as \( \frac{1}{n} \) of the whole cake. In the unequal shares version of the problem (also called cake cutting with entitlements), proportional share is defined as a player-specific demand, summing up to the value of the cake over all players. The scope of this paper is to determine the query complexity of proportional cake cutting in the case of unequal shares. Robertson and Webb \cite{15} write 'Nothing approaching general theory of optimal number of cuts for unequal shares division has been given to date. This problem may prove to be very difficult.' We now settle the issue for the number of queries.

1.1 Related work

Equal shares Possibly the most famous cake cutting protocol belongs to the class of Divide and Conquer algorithms. Cut and Choose is a 2-player equal-shares method that guarantees proportional

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shares. It already appeared in the Old Testament, where Abraham divided the Canaan to two equally valuable parts and his brother Lot chose the one he valued more for himself. The first \( n \)-player variant of this algorithm is attributed to Banach and Knaster \[17\] and it requires \( \mathcal{O}(n^2) \) queries. Other methods include the continuous (but discretizable) Dubins-Spanier \[6\] protocol and the Even-Paz \[5\] protocol. Even and Paz show that their method requires \( \mathcal{O}(n \log n) \) queries at most. The complexity of proportional cake cutting in higher dimensions has been studied in several papers \[2, 3, 4, 9, 10, 16\], in which cuts are tailored to fit the shape of the cake.

**Unequal shares** The problem of proportional cake cutting with unequal shares is first mentioned by Steinhaus \[17\]. Motivated by dividing a leftover cake, Robertson and Webb \[15\] define the problem formally and offer a range of solutions for two players. Known methods for this case include cloning players \[5\], using Ramsey partitions \[12\] and most importantly, the Cut Near-Halves protocol \[15\]. The last method computes a fair solution for 2 players with demands \( d_1 \) and \( d_2 \) in \( 2[\lfloor \log_2(d_1 + d_2) \rfloor] \) queries. As Robertson and Webb’s Algorithm for More Than Two Players (Unequal Shares) \[15\] shows, any 2-player protocol can be generalized to \( n \) players in a recursive manner. Barbanel \[14\] studied the case of cutting the cake in an irrational ratio between two players and showed that there always exists a proportional division.

**Lower bounds** The drive towards establishing lower bounds on the complexity of cake cutting protocols is coeval to the cake cutting literature itself \[17\]. In the ’80s Even and Paz \[8\] conjectured that their protocol is the best possible, while Robertson and Webb explicitly write that “they would place their money against finding a substantial improvement on the \( n \log_2 n \) bound” for proportional cake cutting with equal shares. After circa 20 years of no breakthrough in the topic, Magdon-Ismail et al. \[11\] showed that any protocol must make \( \Omega(n \log_2 n) \) comparisons — but this was no bound on the number of queries. Essentially simultaneously, Woeginger and Sgall \[19\] came up with the lower bound \( \Omega(n \log_2 n) \) on the number of queries for the case where contiguous pieces are allocated to each player. Not much later, this condition was dropped by Edmonds and Pruhs \[7\] who completed the query complexity analysis of proportional cake cutting with equal shares by presenting a lower bound of \( \Omega(n \log_2 n) \). Brams et al. \[5\] study the minimum number of actual cuts in the case of unequal shares and prove that \( n - 1 \) cuts might not suffice — in other words, they show that there is no proportional allocation with contiguous pieces. However, no lower bound on the number of queries is known in the case of unequal shares.

### 1.2 Our contribution

We provide formal definitions in Section 2 and present the query analysis of the fastest known protocol for the \( n \)-player proportional cake cutting with demands \( D \) in total in Section 3. Then, in Section 4 we focus on our protocol for the problem. The main idea is that we recursively render the players in two batches so that these batches can simulate two players who aim to cut the cake into two approximately equal halves. Our method requires only \( 2(n - 1) \cdot \lfloor \log_2 D \rfloor \) queries, with this being the fastest procedure that derives a proportional division for the \( n \)-player cake cutting problem with unequal shares. Moreover, our protocol also works on a highly general cake (introduced in Section 5), extending the traditional notion of the cake to any finite dimension. We complement this result by showing a lower bound of \( \Omega(n \cdot \log_2 D) \) on the query complexity of the problem in Section 6. Our proof generalizes, but does not rely on the lower bound proof given by Edmonds and Pruhs \[7\] for the problem of proportional division with equal shares. Moreover, our lower bound remains valid in the generalized cake cutting and query model, allowing a more powerful notion of a query even on the usual cake.

### 2 Preliminaries

We begin with formally defining our input. Our setting includes a set of players of cardinality \( n \), denoted by \( \{P_1, P_2, \ldots, P_n\} \), and a heterogeneous and divisible good, which we refer to as the cake and project to the unit interval \([0, 1]\). Each player \( P_i \) has a non-negative, absolutely continuous measure function \( \mu_i \) that is defined on Lebesgue-measurable sets. Besides this, each player \( P_i \) has a demand \( d_i \in \mathbb{Z}^+ \), representing that \( P_i \) is entitled to receive \( \frac{d_i}{\sum_{j=1}^{n} d_j} \) part of \([0, 1]\) part of the whole cake. The value of the whole
cake is identical for all players, in particular it is the sum of all demands:

\[\forall 1 \leq i \leq n \quad \mu_i([0,1]) = D = \sum_{j=1}^{n} d_j.\]

The cake \([0,1]\) will be partitioned into subintervals in the form \([x, y]\), \(0 \leq x \leq y \leq 1\). A finite union of such subintervals forms a piece \(X_i\) allocated to player \(P_i\). We would like to stress that a piece is not necessarily connected.

**Definition 1.** A set \(\{X_i\}_{1 \leq i \leq n}\) of pieces is a division of the cake \([0,1]\) if \(\bigcup_{1 \leq i \leq n} X_i = [0,1]\) and \(X_i \cap X_j = \emptyset\) for all \(i \neq j\). We call division \(\{X_i\}_{1 \leq i \leq n}\) proportional if \(\mu_i(X_i) \geq d_i\) for all \(1 \leq i \leq n\).

In words, proportionality means that each player receives a piece with which her demand is satisfied. We do not consider alternative fairness notions such as envy-freeness or Pareto optimality in this paper.

We now turn to defining the measure of efficiency in cake cutting. We assume that \(1 \leq i \leq n\), \(x, y \in [0,1]\), and \(0 \leq \alpha \leq 1\). Oddly enough, the Robertson-Webb query model was not formalized explicitly by Robertson and Webb first, but by Wöeginger and Sgall [19], who attribute it to the earlier two. In their query model, a protocol can ask agents the following two types of queries.

- **Cut query** \((P_i, \alpha)\) returns the leftmost point \(x\) so that \(\mu_i([0,x]) = \alpha\). In this operation \(x\) becomes a so-called **cut point**.
- **Eval query** \((P_i, x)\) returns \(\mu_i([0,x])\). Here \(x\) must be a cut point.

Notice that this definition implies that choosing sides, sorting marks or calculating any other parameter than the value of a piece are not counted as queries and thus do not influence the efficiency of a protocol.

**Definition 2.** The number of queries in a protocol is the number of eval and cut queries until termination. We denote the number of queries for a \(n\)-player algorithm with total demand \(D\) by \(T(n, D)\).

The query definition of Wöeginger and Sgall is the strictest of the type Robertson-Webb. We now outline three options to extend the notion of a query, all of which have been used in earlier papers [7, 8, 15, 19].

1. **The query definition of Edmonds and Pruhs.** There is a slightly different and stronger formalization of the core idea, given by Edmonds and Pruhs [7] and also used by Procaccia [13, 14]. The crucial difference is that they allow both cut and eval queries to start from an arbitrary point in the cake.
   - **Cut query** \((P_i, x, \alpha)\) returns the leftmost point \(y\) so that \(\mu_i([x,y]) = \alpha\) or an error message if no such \(y\) exists.
   - **Eval query** \((P_i, x, y)\) returns \(\mu_i([x,y])\).

These queries can be simulated as trivial concatenations of the queries defined by Wöeginger and Sgall. To pin down the starting point \(x\) of a cut query \((P_i, x, \alpha)\) we introduce the cut point \(x\) with the help of a dummy players Lebesgue-measure, ask \(P_i\) to evaluate the piece \([0,x]\) and then we eval query with value \(\alpha' = \alpha + \mu_i([0,x])\). Similarly, to generate an eval query \((P_i, x, y)\) one only needs to artificially generate the two cut points \(x\) and \(y\) and then ask two eval queries of the Wöeginger-Sgall model, \((P_i, x)\) and \((P_i, y)\). We remark that such a concatenation of Wöeginger-Sgall queries reveals more information than the single query in the model of Edmonds and Pruhs.

2. **Proportional cut query.** The term proportional cut query stands for extended cut queries of the sort \(P_i\) cuts the piece \([x,y]\) in ratio \(a:b\), where \(a, b\) are integers. As Wöeginger and Sgall also note it, two eval queries and one cut query with ratio \(\alpha = \frac{a}{a+b}, \mu_i([x,y])\) are sufficient to execute such an operation if \(x, y\) are cut points, otherwise five queries suffice. Notice that the eval queries are only used by \(P_i\) when she calculates \(\alpha\), and their output does not need to be revealed to any other player or even to the protocol.

3. **Reindexing.** It is especially useful with recursive algorithms if one is allowed to reindex any piece \([x,y]\) so that it represents the interval \([0,1]\) for \(P_i\). Any further cut and eval query on \([x,y]\) can also be substituted by at most five queries on the whole cake. Similarly as above, there is no need to reveal the result of the necessary eval queries addressed to a player.
These workaround ensure that protocols require asymptotically the same number of queries in both model formulations, even if reindexing and proportional queries are allowed. We opted for utilizing all three extensions of the Woeginger-Sgall query model in our upper bound proofs, because the least restrictive model allows the clearest proofs. As of regarding our lower bound proof, it holds even if we allow a highly general query model including all of the above extensions, which we define in Section 3.

3 Known protocols

To provide a base for comparison, we sketch the known protocols for proportional cake cutting with unequal shares and bound their query complexity.

The most naive approach to the case of unequal shares is the cloning technique, where each player $P_i$ with demand $d_i$ is substituted by $d_i$ players with unit demands. In this way a $D$-player equal shares cake cutting problem is generated, which can be solved in $O(D \log_2 D)$ queries.

As Robertson and Webb [15] point out, any 2-player protocol can be generalized to an $n$-player protocol. They list two 2-player protocols, Cut Near-Halves and their Ramsey Partition Algorithm and also remark that for 2 players, Cut Near-Halves is always at least as efficient as Ramsey Partition Algorithm. Therefore, we restrict ourselves to analyzing the complexity of the generalized Cut Near-Halves protocol.

Cut Near-Halves is a simple procedure, in which the cake of value $D$ is repeatedly cut in approximately half by players $P_1$ and $P_2$ with demands $d_1 \leq d_2$ as follows. $P_1$ cuts the cake into two near-halves, more precisely, in ratio $[\frac{D}{2}] : [\frac{D}{2}]$. Then, $P_2$ picks a piece that she values at least as much as $P_1$. This piece is awarded to $P_2$ and her claim is reduced accordingly, by the respective near-half value of the cake. In the next round, the same is repeated on the remaining part of the cake, and so on, until $d_1$ or $d_2$ is reduced to zero. Notice that the person of the cutter is always the player with the lesser current demand, and thus it might change from round to round.

The recursive protocol of Robertson and Webb runs as follows. We assume that $k - 1 < n$ players $P_1, P_2, \ldots, P_{k-1}$ have already divided the whole cake of value $D = d_1 + d_2 + \ldots + d_n$. The next player $P_k$ then challenges each of the first $k - 1$ players separately to redistribute the piece already assigned to them. In these rounds, $P_k$ claims $\frac{d_1 + d_2 + \ldots + d_n}{d_k}$ part of each piece. That is, $P_k$ claims $d_k$ in total and the other players altogether claim $d_1 + \ldots + d_{n-1}$. This generates $k - 1$ rounds of the Cut Near-Halves protocol, each with 2 players. Notice that this protocol tends to assign a highly fractured piece of cake to every player.

The following theorem summarizes the results known about the complexity of the 2-player and $n$-player versions of the Cut Near-Halves protocol.

**Theorem 1** (Robertson and Webb [15]). The 2-player Cut Near-Halves protocol with demands $d_1, d_2$ requires $T(2) = 2 \log_2(d_1 + d_2)$ queries at most. The recursive $n$-player version is a finite.

Here we give an estimate for the number of queries of the recursive protocol.

**Theorem 2.** The number of cuts in the recursive $n$-player Cut Near-Halves protocol is at most

$$T(n, D) = \sum_{i=1}^{n-1} 2i \cdot \left\lfloor \log_2 \left( \sum_{j=1}^{i+1} d_j \right) \right\rfloor \leq n(n-1) \cdot \lceil \log D \rceil.$$ 

**Proof.** The first round consists of players $P_1$ and $P_2$ sharing the cake using $2 \lceil \log_2 (d_1 + d_2) \rceil$ queries. The second round then has two 2-player runs, each of them requiring $2 \lceil \log_2 (d_1 + d_2 + d_3) \rceil$ queries. In general, the $i$th round terminates after $i \cdot 2 \lceil \log_2 \left( \sum_{j=1}^{i+1} d_j \right) \rceil$ queries at most. The number of rounds is $n - 1$. Now we add up the total number of queries.
The following example proves that the calculated bound can indeed be reached asymptotically in instances with an arbitrary number of players.

**Example 1.** The estimation for the query number is asymptotically sharp if

\[ \lceil \log_2 \left( \sum_{j=1}^{n} d_j \right) \rceil = \lceil \log_2 D \rceil \]

holds for at least a fixed portion of all \( 1 \leq i \leq n-1 \), say, for the third of them. This is easy to reach if \( n \) is a sufficiently large power of 2 and all but one players have demand 1, while there is another player with demand 2. Notice that this holds for every order for the agents. If one sticks to a decreasing order of demand when indexing the players, then not only asymptotic, but also strict equality can be achieved by setting \( d_1 \) much larger than all other demands.

### 4 Our protocol

In this section, we present a simple and elegant protocol that beats all three above mentioned methods in query number. The main idea is that we recursively render the players in two batches so that these batches can simulate two players who aim to cut the cake into two approximately equal halves. For now we work with the standard cake and query model defined in Section 2. Later, in Section 5 we will show how our protocol can be extended to a more general cake. We remind the reader that cutting near-halves means to cut in ratio \( \left\lfloor \frac{D}{2} \right\rfloor : \left\lceil \frac{D}{2} \right\rceil \).

To ease the notation we assume that the players are indexed so that when they mark the near-half of the cake, the marks appear in an increasing order from 1 to \( n \). In the subsequent rounds, we reindex the players to keep this property intact. Based on these marks, we choose ‘the middle player’, this being the player whose demand reaches the near-half of the cake when summing up the demands in the order of marks. This player cuts the cake and each player is ordered to the piece her mark falls to. The middle player is cloned if necessary so that she can play on both pieces. The protocol is then repeated on both generated subinstances.

#### Proportional division with unequal shares

Each player marks the near-half of the cake. Sort the players according to their marks.

Calculate the smallest index \( j \) such that \( \left\lfloor \frac{D}{2} \right\rfloor \leq \sum_{i=1}^{j} d_i =: a \).

Cut the cake in two along \( P_j \)’s mark.

Define two instances of the same problem and solve them recursively.

1. Players \( P_1, P_2, \ldots, P_j \) share the slice on the left in ratio \( d_1 : d_2 : \cdots : d_{j-1} : d_j - a + \left\lfloor \frac{D}{2} \right\rfloor \).

2. Players \( P_j, P_{j+1}, \ldots, P_n \) share the slice on the right in ratio \( a - \left\lfloor \frac{D}{2} \right\rfloor : d_{j+1} : d_{j+2} : \cdots : d_n \).

**Example 2.** We present our protocol on an example with \( n = 3 \). Every step of the protocol is depicted in Figure 7. Let \( d_1 = 1, d_2 = 3, d_3 = 1 \). Since \( D = 5 \) is odd, all players mark the near-half of the cake in ratio 2:3. The cake is then cut at \( P_2 \)’s mark, since \( d_1 < \left\lfloor \frac{D}{2} \right\rfloor \), but \( d_1 + d_2 \geq \left\lceil \frac{D}{2} \right\rceil \). The first subinstance will consist of players \( P_1 \) and \( P_2 \) with demands 1 and 1, respectively, whereas the second subinstance will...
have the second copy of player $P_2$ alongside $P_3$ with demands 2 and 1, respectively. In the first instance, both players mark half of the cake and the one who marked it closer to the left will receive the leftmost piece, while the other player is allocated the remaining 1 or 2 pieces. The players in the second instance mark the cake in ratio 1 : 2. Suppose that the player demanding more marks it closer to 0. The leftmost piece is then allocated to her and the same two players share the remaining piece in ratio 1 : 1. The player with the mark on the left will be allocated the piece on the left, while the other players takes the remainder of the piece. These rounds require $3 + 2 + 2 + 2 = 9$ proportional cut queries and no eval query.

It is easy to see that our protocol returns a proportional division. In a generated subinstance, each player is entitled to demand a piece (or two pieces, in the case of $d_j$) of total value at least as much as her demand in the previous round. This holds because their demand remains the same $d_i$, the total demand of the players in the same subinstance is near-half the previous demand $D$, and they play on a cake that is worth at least near-half of the previous cake to every player ordered into this subinstance.

Having shown its correctness, we now present our estimation for the number of queries our protocol needs.

**Theorem 3.** If $2 \leq n$ and $n < D$, then the number of queries in our $n$-player protocol on a cake of total value $D$ can be estimated as

$$T(n, D) \leq 2(n - 1) \cdot \lceil \log_2 D \rceil.$$  

**Proof.** If $n = 2$, then our algorithm simulates the Cut Near-Halves algorithm – except that it uses cut queries exclusively – and according to Theorem 1 it requires $2 \lceil \log_2 D \rceil$ queries at most. This matches the formula stated in Theorem 3. For $n > 2$, the following recursion formula corresponds to our rules.

$$T(n, D) = n + \max_{1 \leq i \leq n} \left\{ T(i, \floor{\frac{D}{2}}) + T(n - i + 1, \lceil \frac{D}{2} \rceil) \right\}$$

We now substitute our formula into the right side of this expression.

$$n + \max_{1 \leq i \leq n} \left\{ T(i, \floor{\frac{D}{2}}) + T(n - i + 1, \lceil \frac{D}{2} \rceil) \right\} =$$

$$n + \max_{1 \leq i \leq n} \left\{ 2(i - 1)\lceil \log_2 \frac{D}{2} \rceil + 2(n - i)\lceil \log_2 \frac{D}{2} \rceil \right\} \leq (*)$$

$$n + \max_{1 \leq i \leq n} \left\{ 2(i - 1)(\lceil \log_2 D \rceil - 1) + 2(n - i)(\lceil \log_2 D \rceil - 1) \right\} =$$

$$n + 2(n - 1)(\lceil \log_2 D \rceil - 1) =$$

$$-n + 2\lceil \log_2 D \rceil \leq 2(n - 1) \cdot \lceil \log_2 D \rceil = T(n, D)$$

The inequality marked by (*) is trivially correct if $D$ is even. For odd $D$, we rely on the fact that $\log_2 D$ cannot be an integer.
\[ \log_2 \frac{D}{2} \leq \log_2 \left( \frac{D + 1}{2} \right) = \log_2 \left( \frac{D + 1}{2} - \log_2 2 \right) = \log_2 (D + 1) - 1 = \log_2 D - 1. \]

We remark that our protocol does not exactly reduce to Cut Near-Halves or Cut and Choose in the case of 2 players. In those protocols, only one player marks the cake and the other player uses an eval query to choose a side. We first make both players mark half of the cake and then allocate one piece to a player. This does not induce any difference in the number of queries, but it matters if one considers the type of the query. Namely, our protocol uses proportional cut queries exclusively. Eval queries are only utilized as technical workarounds to determine the value of the piece the player is restricted to and their result is never revealed to any other player or even the protocol itself. This can be seen as an advantage if one considers the perception of fairness from the point of view of a player. It is arguably more fair if all players need to answer the exact same queries as the other players in each round – our protocol thus guarantees anonymity [18]. Moreover, no information about the preferences of one player is passed on to other players, unlike in Cut and Choose and Cut Near-Halves, where the player who is asked to evaluate a piece might easily speculate that she was offered the piece because the other player cut it off the cake.

5 Generalizations

In this section we introduce a far generalization of cake cutting, where the cake is a measurable set in arbitrary finite dimension and cuts are defined by a monotone function. At the end of the section we prove that even in the generalized setting, \( \mathcal{O}(n \log D) \) queries suffice to construct a proportional division.

5.1 A general cake definition

Our players remain \( \{P_1, P_2, \ldots, P_n\} \) with demands \( d_i \in \mathbb{Z}^+ \), but the cake is now a Lebesgue-measurable subset \( X \) of \( \mathbb{R}^k \) such that \( 0 < \lambda(X) < \infty \). Each player \( P_i \) has a non-negative, absolutely continuous measure function \( \mu_i \) defined on the Lebesgue-measurable subsets of \( X \). An important consequence of this property is that for every \( Z \subseteq X \), \( \mu_i(Z) = 0 \) if and only if \( \lambda(Z) = 0 \). The value of the whole cake is identical for all players, in particular it is the sum of all demands:

\[ \forall 1 \leq i \leq n \quad \mu_i(X) = \sum_{j=1}^{n} d_j. \]

A measurable subset \( Y \) of the cake \( X \) is called a piece. The volume of a piece \( Y \) is the value \( \lambda(Y) \) taken by the Lebesgue-measure on \( Y \). We say that \( Y' \) is a quasisubset of \( Y \) if \( \lambda(Y' \setminus Y) = 0 \). Notice that in this case, \( \lambda(Y') \leq \lambda(Y) \). The cake \( X \) will be partitioned into pieces \( X_1, \ldots, X_n \).

**Definition 3.** A set \( \{X_i\}_{1 \leq i \leq n} \) of pieces is a division of \( X \) if \( \bigcup_{1 \leq i \leq n} X_i = X \) and \( X_i \cap X_j = \emptyset \) for all \( i \neq j \). We call division \( \{X_i\}_{1 \leq i \leq n} \) proportional if \( \mu_i(X_i) \geq d_i \) holds for all \( 1 \leq i \leq n \).

We will show in Section 5.3 that a proportional division always exists.

5.2 A stronger query definition

The more general cake clearly requires a more powerful query notion. Cut and eval queries are defined on an arbitrary measurable subset \( I \subseteq X \). Beyond this, each cut query specifies a value \( \alpha \in \mathbb{R}^+ \) and a monotone mapping \( f : [0, \lambda(I)] \to 2^I \) (representing a moving knife) such that \( f(x) \subseteq f(y) \) and \( \lambda(f(x)) = x \) holds for every \( 0 \leq x \leq y \leq \lambda(I) \).

- **Eval query** \( (P_i, I) \) returns \( \mu_i(I) \).
- **Cut query** \( (P_i, I, f, \alpha) \) returns an \( x \leq \lambda(I) \) with \( \mu_i(f(x)) = \alpha \) or an error message if such an \( x \) does not exist.
As queries involve an arbitrary measurable subset $I$ of $X$, our extended queries automatically cover the generalization of the previously discussed Edmonds-Pruhs queries, proportional queries and reindexing. If we restrict our attention to the usual unit interval cake $[0, 1]$, generalized queries open up a number of new possibilities for a query, as the following example shows.

**Example 3.** Defined on the unit interval cake $[0, 1]$, the following rules qualify as generalized queries.

- Evaluate an arbitrary measurable set.
- Cut a piece of value $\alpha$ surrounding a point $x$ so that $x$ is the midpoint of the cut piece.
- For disjoint finite sets $A$ and $B$, cut a piece $Z$ of value $\alpha$ such that $Z$ contains the $\varepsilon$-neighborhood of $A$ and avoids the $\varepsilon$-neighborhood of $B$ for a maximum $\varepsilon$.
- Determine $x$ such that the union of intervals $[0, x], [\frac{1}{n}, \frac{1}{n} + x], \ldots, [\frac{n-1}{n}, \frac{n-1}{n} + x]$ is of value $\alpha$.

The new notions also allow us to define cuts on a cake in higher dimensions.

**Example 4.** Defined on the generalized cake $X \subseteq \mathbb{R}^k$, the following rules qualify as generalized queries.

- Evaluate an arbitrary measurable set.
- Cut a piece of value $\alpha$ of piece $I$ so that the cut is parallel to a given hyperplane.
- Two queries on the same piece: one player always cuts piece $I \subset \mathbb{R}^2$ along a horizontal line, the other player cuts the same piece along a vertical line.

### 5.3 The existence of a proportional division

Our algorithm ‘Proportional division with unequal shares’ in Section 4 extends to the above described general setting and hence proves that a proportional division always exists.

**Theorem 4.** If $2 \leq n$ and $n < D$, then the number of generalized queries in our $n$-player protocol on the generalized cake of total value $D$ can be estimated as

$$T(n, D) \leq 2(n-1) \cdot \lfloor \log_2 D \rfloor.$$ 

**Proof.** The proof of Theorem 4 carries over without essential changes, thus we only discuss the differences here.

First we observe that proportional queries in ratio $a : b$ can still be substituted by a constant number of eval and cut queries. In the generalized model, proportional query $(P_i, I, f, a, b)$ returns $x \leq \lambda(I)$ such that $b \cdot \mu_i(f(x)) = a \cdot \mu_i(I \setminus f(x))$. Similarly as before, $P_i$ first measures $I$ by a single eval query and then uses the cut query $(P_i, I, f, a, \alpha)$ with $\alpha = \frac{b \cdot \mu_i(I)}{a \cdot \mu_i(I)}$. In the first round of our generalized algorithm, all players are asked to cut the cake $X$ in near-halves using the same $f$ function. Then $P_j$ is calculated, just as in the simpler version and we cut $X$ into the two near-halves according to $P_j$’s $f$-cut and copy $P_j$ if necessary. Due to the monotonicity of $f$, this sorts each player to a piece she values at least as much as the full demand on all players sorted to that piece. The next round is played in the same manner and so on.

The query number for $n = 2$ follows from the fact that each of the two players are asked a proportional cut query in every round until recursively halving $\lfloor \frac{D}{2} \rfloor$ reaches 1, which means $\lfloor \log_2 D \rfloor$ queries in total. The recursion formula remains intact in the generalized model, and thus the query number too.

### 6 The lower bound

In this section, we prove our lower bound on the number of queries any deterministic protocol needs to make when solving the proportional cake cutting problem with unequal shares. This result is valid in two relevant settings: 1. on the $[0, 1]$ cake with Robertson-Webb or with extended queries, 2. on the general cake and queries introduced in Section 4.

The lower bound proof is presented in two steps. In Section 6.1 we define a single-player cake-cutting game where the goal is to identify a piece of small volume and positive value for the sole player. For this game, we design an adversary strategy and specify the minimum volume of the identified piece as a function of the number of queries asked. Then, in Section 6.2 we turn to the problem of proportional cake cutting.
We define our single-player game on a generalized cake of value $D$ that is equipped with an unknown measure function $\mu$. The aim of the sole player is to identify a piece of positive value according to $\mu$ by asking queries from the adversary. We would like to point out that the single-player thin-rich game of Edmonds and Pruhs [7] defined on the unit interval cake has a different goal. There, the player needs to receive a piece that has value not less than 1 and width at most 2. Moreover, their proof is restricted to instances with $n = 2 \cdot 3^\ell, \ell \in \mathbb{Z}^+$, whereas ours is valid for any $n \in \mathbb{Z}^+$.

In our single-player game, a set of queries reveals information on the value of some pieces of the cake. Each extended eval query $(P, I)$ partitions the cake into two pieces; $I$ and $X \setminus I$. An executed cut query $(P, I, x)$ partitions the cake into three; $f(x), I \setminus f(x)$ and $X \setminus I$. To each step of a protocol we define the currently smallest building blocks of the cake, which we call crumbles. Two points of $X$ belong to the same crumble if and only if they are in the same partition in all queries asked so far. At start, the only crumble is the cake itself and every new query can break an existing crumble into more crumbles. More precisely, $q$ queries can generate $3^q$ crumbles at most. Crumbles are disjoint and their union is the cake. The exact value of a crumble is not necessarily known and no real subset of a crumble can have a revealed value. As a matter of fact, the exact same information are known about the value of any subset of a crumble.

**Example 5.** In Figure 2 we illustrate an example for crumbles on the unit interval cake and two queries. The upper picture depicts a cut query defined on the green set $I$. It generates a piece of value $\alpha$ so that it contains the $\varepsilon$-neighborhood of points $A_1, A_2, A_3$ for maximum $\varepsilon$. This piece is marked red in the figure and it is a crumble. The second crumble at this point is the remainder of $I$ (marked in green only), while the third crumble is the set of points in black. These three crumbles are illustrated in the second picture.

The second query evaluates the blue piece in the third picture. It cuts the existing crumbles into 6 crumbles in total, as depicted in the bottom picture, where different colors mark different crumbles.

The main importance of crumbles lies in the very last stage of the protocol, when a piece of the cake is allocated to the player, as the following lemma shows. We remind the reader that a quasisubset of a piece $Y$ is a set $Y'$ with $\lambda(Y' \setminus Y) = 0$.

**Lemma 1.** The adversary can achieve that the value of a piece $Z$ is the total value of those crumbles that are quasisubsets of $Z$.

**Proof.** Clearly, if a crumble $C$ is a quasisubset of $Z$ then the full value of $C$ contributes to the value of $Z$. However, if some crumble $C'$ is not a quasisubset of $Z$ then the adversary can say that $C' \setminus Z$ carries the entire value of the $C'$, hence ensuring the statement of the lemma.
Lemma 1 shows that it is sufficient to restrict our attention to allocating entire crumbles to the player. We now proceed to construct an adversary strategy that bounds the volume of any crumble with positive value. Our adversary can actually reveal more information than asked; we allow her to disclose the value of each crumble in the cake. She answers the queries in accordance to the following rules.

- **eval query**

  When the player is asked to evaluate a piece, the piece cuts all existing crumbles into at most two disjoint pieces. For each crumble, the smaller piece gets value 0, while the other one inherits the full value of the crumble. If both pieces are of equal volume, we choose arbitrarily. With this, the value of every new crumble is specified. The answer to the eval query is the sum of the values of crumbles it contains. Notice that after the query, a crumble is by definition fully inside or fully outside the queried piece.

- **cut query**

  The adversary uses a moving-knife protocol that increases the value of \( x \) in \( f(x) \), starting from \( f(0) \). Notice that we do not need to bother about discretizing this moving knife protocol, since it is part of an adversary strategy. Some crumbles cut into \( I \). For every such crumble \( C \), the adversary determines \( \mu(C \cap I) \) according to the following rule. If \( \lambda(C \cap I) \leq \frac{2}{3} \cdot \lambda(C) \), then \( \mu(C \cap I) = 0 \) and \( \mu(C \setminus I) = \mu(C) \). Otherwise, \( \mu(C \cap I) = \mu(C) \) and \( \mu(C \setminus I) = 0 \). In this way, the old crumbles are broken into two new crumbles, one in \( X \setminus I \) and one in \( I \). We now proceed to break the crumbles in \( I \) to further crumbles, if necessary.

  From time to time, the moving knife \( f(x) \) ‘swallows’ a crumble \( C' \subseteq I \) in the following manner. When \( \lambda(C' \cap f(x)) \geq \frac{1}{2} \cdot \lambda(C') \), then the adversary sets \( \mu(C' \cap f(x)) = \mu(C') \) – provided that \( \mu(C') \leq \alpha \). In this case, \( \alpha \) is decreased by \( \mu(C') \) and the knife moves further. We continue this until \( \alpha = 0 \) (when we stop) or we stumble upon a crumble whose value is larger than the remainder of \( \alpha \). In the latter case the cut is made at the exact half of the crumble and both new crumbles carry a positive value; the crumble in the cut will get the remainder of \( \alpha \), while the crumble outside the cut will carry the rest of the value of the original crumble. If more than one crumble reaches \( \lambda(C' \cap f(x)) \geq \frac{1}{2} \cdot \lambda(C') \) at the exact same \( x \), then we process them in arbitrary order, updating \( \alpha \) after each crumble.

**Lemma 2.** After \( q \) queries in the single-player game, the volume of any crumble with positive value is at least \( \frac{D}{2} \).

**Proof.** The statement is true at the very beginning of the protocol. Eval queries assign positive value to crumbs that are at least as large as half of the previous crumble they belonged to prior to the query. Cut queries assign positive value either to at least one third of a crumble \( (C \setminus I) \) or to at least halves of at least two-thirds of a crumble \( (C \cap I) \), yielding at least \( \frac{1}{3} \) of the volume of the previous crumble.

We remark that our adversary strategy keeps the number of crumbles with positive value low. More precisely, eval queries do not change the number of crumbs with positive value, while each cut query increases it by one at most.

### 6.2 The \( n \)-player game

We now place our single-player game into the framework of the original problem. For arbitrary \( D \geq n \geq 2 \), the constructed instance consists of players \( P_1, P_2, \ldots, P_n \) with demands \( d_1 = d_2 = \ldots = d_{n-1} = 1 \) and \( d_n = D - (n - 1) \), respectively. We fix \( \mu_n \) to be the Lebesgue-measure and do not reveal any information on \( \mu_1, \mu_2, \ldots, \mu_{n-1} \) unless a query is asked. The next lemma shows that there is an adversary strategy with which one needs \( \Omega(n[\log D]) \) queries to derive a proportional division.

**Lemma 3.** At least \( \frac{2(n-1)}{3} \) players must receive at least \( [\log_3 D] \) queries in order to ensure their proportional share.

**Proof.** First we observe that queries addressed to a player do not reveal any information on the value function of other players. Since \( d_n = D - (n - 1) \) and \( \mu_n \) is the Lebesgue-measure, in any proportional division, players \( P_1, P_2, \ldots, P_{n-1} \) share a piece of volume at most \( n - 1 \). Due to Lemma 1 some crumble found in the single-player game must be a quasisubset of this piece. We have presented an adversary strategy with which it takes \( [\log_3 D] \) queries to identify a piece of positive value and volume at most 3
for any player in $P_1, P_2, \ldots, P_{n-1}$, as Lemma \ref{lem:proportional} states. In order to stay in a piece of volume $n-1$ at most, at least two-third of the players $P_1, P_2, \ldots, P_{n-1}$ must be allocated a piece of volume at most 3. Consequently, at least $\frac{2(n-1)}{3}$ single-player games must contain at least $\lceil \log_3 D \rceil$ queries.

We now can conclude our theorem on the lower bound.

**Theorem 5.** To construct a proportional division in an $n$-player unequal shares cake cutting problem with demands summing up to $D$ one needs $\Omega(n \lceil \log D \rceil)$ queries.

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