GLANON GROUPOIDS

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Abstract. We introduce the notion of Glanon groupoids, which are Lie groupoids equipped with multiplicative generalized complex structures. It combines symplectic groupoids, holomorphic Lie groupoids and holomorphic Poisson groupoids into a unified framework. Their infinitesimal, Glanon Lie algebroids are studied. We prove that there is a bijection between Glanon Lie algebroids and source-simply connected and source-connected Glanon groupoids. As a consequence, we recover various integration theorems and obtain the integration theorem for holomorphic Poisson groupoids.

1. Introduction

In their study of quantization theory, Karasev [20], Weinstein [34], and Zakrzewski [40, 41] independently introduced the notion of symplectic groupoids. By a symplectic groupoid, we mean a Lie groupoid equipped with a multiplicative symplectic two-form on the space of morphisms. It is a classical theorem that the unit space of a symplectic groupoid is naturally a Poisson manifold [8]. Indeed the Lie algebroid of a symplectic groupoid \( \Gamma \Rightarrow P \) is naturally isomorphic to \( \left( T^*P \right)_\pi \), the canonical Lie algebroid associated to the Poisson manifold \( (P, \pi) \). In [29], Mackenzie and one of the authors proved that, for a given Poisson manifold \( (P, \pi) \), if the Lie algebroid \( \left( T^*P \right)_\pi \) integrates to a \( s \)-connected and \( s \)-simply connected Lie groupoid \( \Gamma \Rightarrow P \), then \( \Gamma \) is naturally a symplectic groupoid. The symplectic groupoid structure on \( \Gamma \) was also obtained by Cattaneo-Felder [7] using the Poisson sigma model. The general integration condition for symplectic groupoids was completely solved recently by Crainic-Fernandes [11].

Recently, there has been increasing interest in holomorphic Lie algebroids and holomorphic Lie groupoids. It is a very natural question to find the integrability condition for holomorphic Lie algebroids. In [24], together with Laurent-Gengoux, two of the authors studied holomorphic Lie algebroids and their relation with real Lie algebroids. In particular, they proved that associated to any holomorphic Lie algebroid \( A \), there is a canonical real Lie algebroid \( A_R \) such that the inclusion \( A \to A_\infty \) is a morphism of sheaves. Here \( A \) and \( A_\infty \) denote, respectively, the sheaf of holomorphic sections and the sheaf of smooth sections of \( A \). In other words, a holomorphic Lie algebroid can be considered as a holomorphic vector bundle \( A \to X \) whose underlying real vector bundle is endowed with a Lie algebroid structure such that, for any open subset \( U \subset X \), \( [\mathcal{A}(U), \mathcal{A}(U)] \subset \mathcal{A}(U) \) and the restriction of the Lie bracket \( [\cdot, \cdot] \) to \( \mathcal{A}(U) \) is \( \mathbb{C} \)-linear. In particular, they proved that a holomorphic Lie algebroid \( A \) is integrable as a holomorphic Lie algebroid if and only if its underlying real Lie algebroid \( A_R \) is integrable as a real Lie algebroid [25].

In this paper, we introduce the notion of Glanon groupoids, as a unification of symplectic groupoids and holomorphic groupoids. By a Glanon groupoid, we mean a Lie groupoid equipped with a generalized complex structure in the sense of Hitchin [16] which is multiplicative. Recall that a generalized complex structure on a manifold \( M \) is a smooth bundle...
map $J : TM \oplus T^*M \to TM \oplus T^*M$ such that $J^2 = -\text{Id}$, $JJ^* = \text{Id}$, and the $+i$-eigenbundle of $J$ is involutive with respect to the Courant bracket, or the Nijenhuis tensor of $J$ vanishes. A generalized complex structure $\mathcal{J}$ on a Lie groupoid $\Gamma \rightrightarrows M$ is said to be multiplicative if $\mathcal{J}$ is a Lie groupoid automorphism with respect to the Courant groupoid $TT \oplus T^*\Gamma \rightrightarrows TM \oplus A^*$ in the sense of Mehta [30]. When $\mathcal{J}$ is the generalized complex structure associated to a symplectic structure on $\Gamma$, it is clear that $\mathcal{J}$ is multiplicative if and only if the symplectic two-form is multiplicative. Thus its Glanon groupoid is equivalent to a symplectic groupoid. On the other hand, when $\mathcal{J}$ is the generalized complex structure associated to a complex structure on $\Gamma$, $\mathcal{J}$ is multiplicative if and only if the complex structure on $\Gamma$ is multiplicative. Thus one recovers a holomorphic Lie groupoid. On the infinitesimal level, a Glanon groupoid corresponds to a Glanon Lie algebroid, which is a Lie algebroid $A$ equipped with a compatible generalized complex structure, i.e. a generalized complex structure $\mathcal{J}_A : TA \oplus T^*A \to TA \oplus T^*A$ which is a Lie algebroid automorphism with respect to the Lie algebroid $TA \oplus T^*A \to TM \oplus A^*$. More precisely, we prove the following main result:

**Theorem A.** If $\Gamma$ is a s-connected and s-simply connected Lie groupoid with Lie algebroid $A$, then there is a one-one correspondence between Glanon groupoid structures on $\Gamma$ and Glanon Lie algebroid structures on $A$.

As a consequence, we recover the following standard results [29, 25]:

**Theorem B.** Let $(P, \pi)$ be a Poisson manifold. If $\Gamma$ is a s-connected and s-simply connected Lie groupoid integrating the Lie algebroid $(T^*P)_\pi$, then $\Gamma$ automatically admits a symplectic groupoid structure.

**Theorem C.** If $\Gamma$ is a s-connected and s-simply connected Lie groupoid integrating the underlying real Lie algebroid $A_R$ of a holomorphic Lie algebroid $A$, then $\Gamma$ is a holomorphic Lie groupoid.

Another important class of Glanon groupoids is when the generalized complex structure $\mathcal{J}$ on $\Gamma$ has the form

$$\mathcal{J} = \begin{pmatrix} N & \pi^* \\ 0 & -N^* \end{pmatrix}.$$ 

It is simple to see that in this case $(\Gamma \rightrightarrows M, \mathcal{J})$ being a Glanon groupoid is equivalent to $\Gamma$ being a holomorphic Poisson groupoid. On the other hand, on the infinitesimal level, one proves that their corresponding Glanon Lie algebroids are equivalent to holomorphic Lie bialgebroids. Thus as a consequence, we prove the following

**Theorem D.** Given a holomorphic Lie bialgebroid $(A, A^*)$, if the underlying real Lie algebroid $A_R$ integrates to a s-connected and s-simply connected Lie groupoid $\Gamma$, then $\Gamma$ is a holomorphic Poisson groupoid.

A special case of this theorem was proved in [25] using a different method, namely, when the holomorphic Lie bialgebroid is $((T^*X)_\pi, TX)$, the one corresponding to a holomorphic Poisson manifold $(X, \pi)$. In this case, it was proved that when the underlying real Lie algebroid of $(T^*X)_\pi$ integrates to a s-connected and s-simply connected Lie groupoid $\Gamma$, then $\Gamma$ is automatically a holomorphic symplectic groupoid. However, according to our knowledge, the integrating problem for general holomorphic Lie bialgebroids remain open. Indeed, solving this integration problem is one of the main motivation behind our study of Glanon groupoids.
It is known that the base manifold of a generalized complex manifold is automatically a Poisson manifold \[ \mathcal{L} \]. Thus it follows that a Glanon groupoid is automatically a (real) Poisson groupoid. On the other hand, a Glanon Lie algebroid \( A \) admits a Poisson structure, which can be shown a linear Poisson structure. Therefore \( A^* \) is a Lie algebroid. We prove that \( (A, A^*) \) is indeed a Lie bialgebroid, which is the infinitesimal of the associated Poisson groupoid of the Glanon groupoid.

Note that in this paper we confine ourselves to the standard Courant groupoid \( TT \oplus T^*\Gamma \). We can also consider the twisted Courant groupoid \( (TT \oplus T^*\Gamma)\# \), where \( H \) is a multiplicative closed 3-form. This will be discussed somewhere else. Also note that a multiplicative generalized complex structure on the groupoid \( \Gamma \) induces a pair of maps \( j_u : TM \oplus A^* \to TM \oplus A^* \) and \( j_A : A \oplus T^*M \to A \oplus T^*M \) on the units and on the core, respectively. Since a multiplicative generalized complex structure is the same as two complex conjugated multiplicative Dirac structures on the groupoid, we get following the results in \[ 19, 12 \] that a multiplicative generalized complex structure is the same as two complex conjugated Manin pairs over the set of units \( M \) (in the sense of \[ 6 \]). The detailed study of properties of the maps \( j_A, j_u \), the Manin pairs, and the associated Dorfman connections will be investigated in the spirit of \[ 24 \] in a future project.

**Acknowledgments.** We would like to thank Camille Laurent-Gengoux, Rajan Mehta, and Cristian Ortiz for useful discussions. We also would like to thank several institutions for their hospitality while work on this project was being done: Penn State University (Jotz), and IHES and Beijing International Center for Mathematical Research (Xu). Special thanks go to the Michea family and our many friends in Glanon, whose warmth provided us constant inspiration for our work in the past. Many of our friends from Glanon have left mathematics, but our memories of them serve as a reminder that a mathematician’s primary role should be to elucidate and enjoy the beauty of mathematics. We dedicate this paper to them.

1.1. Notation. In the following, \( \Gamma \Rightarrow M \) will always be a Lie groupoid with set of arrows \( \Gamma \), set of objects \( M \), source and target \( s, t : \Gamma \to M \) and object inclusion map \( \epsilon : M \to \Gamma \). The product of \( g, h \in \Gamma \) with \( s(g) = t(h) \) will be written \( m(g, h) = gh \) of simply \( gh \).

The Lie functor that sends a Lie groupoid to its Lie algebroid and Lie groupoid morphisms to Lie algebroid morphisms is \( A \). For simplicity, we will write \( A(\Gamma) = A \). The Lie algebroid \( q_A : A \to M \) is identified with \( T_T^A \Gamma, \) the bracket \( [\cdot, \cdot]_A \) is defined with the right invariant vector fields and the anchor \( \rho_A = \rho \) is the restriction of \( Tt \) to \( A \). Hence, as a manifold, \( A \) is embedded in \( TT \). The inclusion is \( \iota : A \to TT \). Note that the Lie algebroid of \( \Gamma \Rightarrow M \) can alternatively be defined using left-invariant sections. Given \( a \in \Gamma(A), \) the right-invariant section corresponding to \( a \) will simply be written \( a^r, \) i.e., \( a^r(g) = TR_g(a(t(g))) \) for all \( g \in \Gamma \). We will write \( a^l \) for the left-invariant vector field defined by \( a, \) i.e., \( a^l(g) = -T(L_g \circ \iota)(a(s(g))) \) for all \( g \in \Gamma \).

The projection map of a vector bundle \( A \to M \) will always be written \( q_A : A \to M, \) unless specified otherwise. For a smooth manifold \( M, \) we fix once and for all the notation \( p_M := qTM : TM \to M \) and \( c_M := qTM : T^*M \to M. \) We will write \( P_M \) for the direct sum \( TM \oplus T^*M \) of vector bundles over \( M, \) and \( pr_M \) for the canonical projection \( P_M \to M. \)

A bundle morphism \( P_M \to P_M, \) for a manifold \( M, \) will always be meant to be over the identity on \( M. \)

2. Preliminaries

2.1. Dirac structures. Let \( A \to M \) be a vector bundle with dual bundle \( A^* \to M. \) The natural pairing \( A \oplus A^* \to \mathbb{R}, (a_m, \xi_m) \mapsto \xi_m(a_m) \) will be written \( \langle \cdot, \cdot \rangle_A \) or \( \langle \cdot, \cdot \rangle_{q_A} \) if the
vector bundle structure needs to be specified. The direct sum $A \oplus A^*$ is endowed with a canonical fiberwise pairing $(\cdot, \cdot)_A$ given by
\[
((a_m, \xi_m), (b_m, \eta_m))_A = \langle b_m, \xi_m \rangle_A + \langle a_m, \eta_m \rangle_A
\] for all $m \in M$, $a_m, b_m \in A_m$ and $\xi_m, \eta_m \in A^*_m$.

In particular, the Pontryagin bundle $P_M := TM \oplus T^*M$ of a smooth manifold $M$ is endowed with the pairing $(\cdot, \cdot)_{TM}$, which will be written as usual $\langle \cdot, \cdot \rangle_M$.

The orthogonal space relative to the pairing $(\cdot, \cdot)_A$ of a subbundle $E \subseteq A \oplus A^*$ will be written $E^\perp$ in the following. An almost Dirac structure on $M$ is a Lagrangian vector subbundle $D \subset P_M$. That is, $D$ coincides with its orthogonal relative to $[10]$, $D = D^\perp$, so its fibers are necessarily dim $M$-dimensional.

The set of sections $\Gamma(P_M)$ of the Pontryagin bundle of $M$ is endowed with the Courant bracket, given by
\[
[X + \alpha, Y + \beta] = [X, Y] + \left( \mathcal{L}_X \beta - \mathcal{L}_Y \alpha + \frac{1}{2} d(\alpha(Y) - \beta(X)) \right)
\] for all $X + \alpha, Y + \beta \in \Gamma(P_M)$.

An almost Dirac structure $D$ on a manifold $M$ is a Dirac structure if its set of sections is closed under this bracket, i.e., $[\Gamma(D), \Gamma(D)] \subset \Gamma(D)$. Since $\langle X + \alpha, Y + \beta \rangle_M = 0$ if $(X, \alpha), (Y, \beta) \in \Gamma(D)$, integrability of the Dirac structure is expressed relative to a non-skew-symmetric bracket that differs from (2) by eliminating in the second line the third term of the second component. This truncated expression is called the Dorfman bracket in the literature:
\[
[X + \alpha, Y + \beta] = [X, Y] + (\mathcal{L}_X \beta - i_Y d\alpha)
\] for all $X + \alpha, Y + \beta \in \Gamma(D)$.

### 2.2. Generalized complex structures

Let $V$ be a vector space. Consider a linear endomorphism $J$ of $V \oplus V^*$ such that $J^2 = -\text{Id}_V$ and $J$ is orthogonal with respect to the inner product
\[
\langle X + \xi, Y + \eta \rangle_V = \xi(Y) + \eta(X), \quad \forall X, Y \in V, \ \xi, \eta \in V^*.
\]

Such a linear map is called a linear generalized complex structure by Hitchin [16]. The complexified vector space $(V \oplus V^*) \otimes \mathbb{C}$ decomposes as the direct sum
\[
(V \oplus V^*) \otimes \mathbb{C} = E_+ \oplus E_-
\]
of the eigenbundles of $J$ corresponding to the eigenvalues $\pm i$ respectively, i.e.,
\[
E_\pm = \{(X + \xi) \mp iJ(X + \xi) \mid X + \xi \in V \oplus V^*\}.
\]

Both eigenspaces are maximal isotropic with respect to $\langle \cdot, \cdot \rangle$ and they are complex conjugate to each other.

The following lemma is obvious.

**Lemma 2.1.** The linear generalized complex structures are in 1-1 correspondence with the splittings $(V \oplus V^*) \otimes \mathbb{C} = E_+ \oplus E_-$ with $E_\pm$ maximal isotropic and $E_- = \overline{E_+}$.

Now, let $M$ be a manifold and $J$ a bundle endomorphism of $P_M = TM \oplus T^*M$ such that $J^2 = -\text{Id}_{P_M}$, and $J$ is orthogonal with respect to $\langle \cdot, \cdot \rangle_M$. Then $J$ is a generalized almost complex structure. In the associated eigenbundle decomposition
\[
T_{\mathbb{C}}M \oplus T^*_{\mathbb{C}}M = E_+ \oplus E_-,
\]
if $\Gamma(E_+)$ is closed under the (complexified) Courant bracket, then $E_+$ is a (complex) Dirac structure on $M$ and one says that $\mathcal{J}$ is a generalized complex structure [10] [15]. In this case, $E_-$ must also be a Dirac structure since $E_- = \mathcal{E}_+$. Indeed $(E_+, E_-)$ is a complex Lie bialgebroid in the sense of Mackenzie-Xu [28], in which $E_+$ and $E_-$ are complex conjugate to each other.

**Definition 2.2.** Let $\mathcal{J} : P_M \rightarrow P_M$ be a vector bundle morphism. Then the generalized Nijenhuis tensor associated to $\mathcal{J}$ is the map

$$N_\mathcal{J} : P_M \times_M P_M \rightarrow P_M$$

defined by

$$N_\mathcal{J}(\xi, \eta) = [[\mathcal{J}\xi, \mathcal{J}\eta]] + \mathcal{J}^2[[\xi, \eta]] - \mathcal{J}([[\mathcal{J}\xi, \eta]] + [[\xi, \mathcal{J}\eta]])$$

for all $\xi, \eta \in \Gamma(P_M)$, where the bracket is the Courant-Dorfman bracket.

The following proposition gives two equivalent definitions of a generalized complex structure.

**Proposition 2.3.** A generalized complex structure is equivalent to any of the following:

(a) A bundle endomorphism $\mathcal{J}$ of $P_M$ such that $\mathcal{J}$ is orthogonal with respect to $\langle \cdot, \cdot \rangle_M$, $\mathcal{J}^2 = -\text{Id}$ and $N_\mathcal{J} = 0$.

(b) A complex Lie bialgebroid $(E_+, E_-)$ whose double is the standard Courant algebra $T\mathbb{C}M \oplus T^*\mathbb{C}M$, and $E_+$ and $E_-$ are complex conjugate to each other.

For a two-form $\omega$ on $M$ we denote by $\omega^\flat : TM \rightarrow T^*M$ the bundle map $X \mapsto i_X \omega$, while for a bivector $\pi$ on $M$ we denote by $\pi^\flat : T^*M \rightarrow TM$ the contraction with $\pi$. Also, we denote by $[\cdot, \cdot]_\pi$ the bracket defined on the space of 1-forms on $M$ by

$$[\xi, \eta]_\pi = L_{\pi^\flat} \xi \eta - L_{\pi^\flat} \eta \xi - d\pi(\xi, \eta)$$

for all $\xi, \eta \in \Omega^1(M)$.

A generalized complex structure $\mathcal{J} : P_M \rightarrow P_M$ can be written

$$\mathcal{J} = \begin{pmatrix} N & \pi^\flat \\ \omega^\flat & -N^* \end{pmatrix}$$

(5)

where $\pi$ is a bivector field on $M$, $\omega$ is a two-form on $M$, and $N : TM \rightarrow TM$ is a bundle map, satisfying together a list of identities (see [10]). In particular $\pi$ is a Poisson bivector field.

Let $\mathcal{I} : P_M \rightarrow P_M$ be the endomorphism

$$\mathcal{I} = \begin{pmatrix} \text{Id}_{TM} & 0 \\ 0 & -\text{Id}_{T^*M} \end{pmatrix}.$$ 

Then we have

$$\langle \mathcal{I}(\cdot), \mathcal{I}(\cdot) \rangle_M = -\langle \cdot, \cdot \rangle_M,$$

$$[[\mathcal{I}(\cdot), \mathcal{I}(\cdot)] = \mathcal{I}[\cdot, \cdot]$$

and the following proposition follows.

**Proposition 2.4.** If $\mathcal{J}$ is an almost generalized complex structure on $M$, then

$$\mathcal{\bar{J}} := \mathcal{I} \circ \mathcal{J} \circ \mathcal{I}$$

is an almost generalized complex structure. Furthermore,

$$N_{\mathcal{\bar{J}}} = \mathcal{I} \circ N_\mathcal{J} \circ (\mathcal{I}, \mathcal{I}).$$

Hence, $\mathcal{\bar{J}}$ is a generalized complex structure if and only if $\mathcal{J}$ is a generalized complex structure.
The following are two standard examples [16].

**Examples 2.5.**

(a) Let $J$ be an almost complex structure on $M$. Then

$$
J = \begin{pmatrix}
J & 0 \\
0 & -J^*
\end{pmatrix}
$$

is $\langle \cdot, \cdot \rangle_M$-orthogonal and satisfies $J^2 = -\text{Id}$. $J$ is a generalized complex structure if and only if $J$ is integrable.

(b) Let $\omega$ be a nondegenerate 2-form on $M$. Then

$$
J = \begin{pmatrix}
0 & -(\omega^*)^{-1} \\
\omega^* & 0
\end{pmatrix}
$$

is a generalized complex structure if and only if $d\omega = 0$, i.e., $\omega$ is a symplectic 2-form.

### 2.3. Pontryagin bundle over a Lie groupoid.

**The tangent prolongation of a Lie groupoid.** Let $\Gamma \rightrightarrows M$ be a Lie groupoid. Applying the tangent functor to each of the maps defining $\Gamma$ yields a Lie groupoid structure on $T\Gamma$ with base $TM$, source $Ts$, target $Tt$ and multiplication $Tm : T(\Gamma \times_M \Gamma) \to T\Gamma$. The identity at $v_p \in T_pM$ is $1_{v_p} = T_p\epsilon_{v_p}$. This defines the tangent prolongation $T\Gamma \rightrightarrows TM$ of $\Gamma \rightrightarrows M$ or the tangent groupoid associated to $\Gamma \rightrightarrows M$.

**The cotangent Lie groupoid defined by a Lie groupoid.** If $\Gamma \rightrightarrows M$ is a Lie groupoid with Lie algebroid $\Lambda \Gamma \to M$, then there is also an induced Lie groupoid structure on $T^*\Gamma \rightrightarrows \Lambda^*\Gamma = (TM)^\circ$. The source map $\hat{s} : T^*\Gamma \to \Lambda^*\Gamma$ is defined by

$$
\hat{s}(\alpha_g) \in A^*_s(g), \quad \hat{s}(\alpha_g)(a(s(g))) = \alpha_g(a^l(g))
$$

for all $a \in \Gamma(A)$, and the target map $\hat{t} : T^*\Gamma \to \Lambda^*\Gamma$ is given by

$$
\hat{t}(\alpha_g) \in A^*_t(g), \quad \hat{t}(\alpha_g)(a(t(g))) = \alpha_g(a^r(g))
$$

for all $a \in \Gamma(A)$. If $\hat{s}(\alpha_g) = \hat{t}(\alpha_h)$, then the product $\alpha_g \star \alpha_h$ is defined by

$$
(\alpha_g \star \alpha_h)(v_g \star v_h) = \alpha_g(v_g) + \alpha_h(v_h)
$$

for all composable pairs $(v_g, v_h) \in T_{(g,h)}(\Gamma \times_M \Gamma)$.

This Lie groupoid structure was introduced in [8] and is explained for instance in [8] [92] [27]. Note that the original definition was the following: let $\Lambda$ be the graph of the partial multiplication $m$ in $\Gamma$, i.e.,

$$
\Lambda = \{ (g, h, g \star h) \mid g, h \in \Gamma, s(g) = t(h) \}.
$$

The isomorphism $\psi : (T^*\Gamma)^3 \to (T^*\Gamma)^3$, $\psi(\alpha, \beta, \gamma) = (\alpha, \beta, -\gamma)$ sends the conormal space $(T\Lambda_G)^\circ \subseteq (T^*\Gamma)^3|_{\Lambda}$ to a submanifold $\Lambda_s$ of $(T^*\Gamma)^3$. It is shown in [8] that $\Lambda_s$ is the graph of a groupoid multiplication on $T^*\Gamma$, which is exactly the multiplication defined above.

**The “Pontryagin groupoid” of a Lie groupoid.** If $\Gamma \rightrightarrows M$ is a Lie groupoid with Lie algebroid $A \to M$, according to [30], there is hence an induced VB-Lie groupoid structure on $\mathcal{P}_\Gamma = T\Gamma \oplus T^*\Gamma$ over $TM \oplus A^*$, namely, the product groupoid, where $\Gamma \times \Gamma \rightrightarrows \Gamma$ and $TM \oplus A^*$ are identified with the fiber products $T\Gamma \times T\Gamma$ and $T\Gamma \times_M A^*$, respectively. It is called a Courant groupoid by Mehta [30].

**Proposition 2.6.** Let $\Gamma \rightrightarrows M$ be a Lie groupoid with Lie algebroid $A \to M$. Then the Pontryagin bundle $\mathcal{P}_\Gamma = T\Gamma \oplus T^*\Gamma$ is a Lie groupoid over $TM \oplus A^*$, and the canonical projection $\mathcal{P}_\Gamma \to \Gamma$ is a Lie groupoid morphism.
We will write $\mathcal{Tt}$ for the target map

$$\mathcal{Tt} : \mathcal{P}_\Gamma \to TM \oplus A^*$$

$$(v_g, \alpha_g) \mapsto (\mathcal{Tt}(v_g), \mathcal{t}(\alpha_g))$$

$\mathcal{Ts}$ for the source map

$$\mathcal{Ts} : \mathcal{P}_\Gamma \to TM \oplus A^*$$

and $\mathcal{Tt}$, $\mathcal{Tt}$, $\mathcal{Tr}$ for the embedding of the units, the inversion map and the multiplication of this Lie groupoid, i.e. for instance

$$\mathcal{Ts}(e) = (s_s, \hat{s}_s)(e) \in T^s_g M \times A^*_{s(g)}$$

for all $e \in \mathcal{P}_\Gamma(g)$.

### 2.4. Pontryagin bundle over a Lie algebroid.

Given any vector bundle $q_A : A \to M$, the map $Tq_A : TA \to TM$ has a vector bundle structure obtained by applying the tangent functor to the operations in $A \to M$. The operations in $TA \to TM$ are consequently vector bundle morphisms with respect to the tangent bundle structures in $TA \to A$ and $TM \to M$ and $TA$ with these two structures is therefore a double vector bundle which we call the tangent double vector bundle of $A \to M$ (see [26] and references given there).

If $(q_A : A \to M, [\cdot, \cdot]_A, \rho_A)$ is a Lie algebroid, then there is a Lie algebroid structure on $Tq_A : TA \to TM$ defined in [28] with respect to which $p_A : TA \to A$ is a Lie algebroid morphism over $p_M : TM \to M$; we now call this the tangent prolongation of $A \to M$.

For a general vector bundle $q : A \to M$, there is also a double vector bundle

$$T^*A \xrightarrow{r_A} A^* \quad \xrightarrow{q_A} M$$

Here the map $r_A$ is the composition of the Legendre transformation $T^*A \to T^*A^*$ with the projection $T^*A^* \to A^*$ [28].

Elements of $T^*A$ can be represented locally as $(\omega, a, \phi)$ where $\omega \in T^*_m M$, $a \in A_m$, $\phi \in A^*_m$ for some $m \in M$. In these terms the Legendre transformation

$$R : T^*A \to T^*A^*$$

can be defined by $R(\omega, a, \phi) = (-\omega, a, \phi)$; for an intrinsic definition see [28]. This $R$ is an isomorphism of double vector bundles preserving the side bundles; that is to say, it is a vector bundle morphism over both $A$ and $A^*$. Since $A$ is a Lie algebroid, its dual $A^*$ has a linear Poisson structure, and the cotangent space $T^*A^*$ has a Lie algebroid structure over $A^*$. Hence, there is a unique Lie algebroid structure on $r_A : T^*A \to A^*$ with respect to which the projection $T^*A \to A$ is a Lie algebroid morphism over $A^* \to M$. As a consequence, one can form the product Lie algebroid $TA \times_A T^*A \to TM \times M A^*$. Thus we have the following

**Proposition 2.7.** Let $A \to M$ be a Lie algebroid. Then the Pontryagin bundle $P_A$ is naturally a Lie algebroid. Moreover, the canonical projection $P_A \to A$ is a Lie algebroid morphism.

The double vector bundle $(P_A, TM \oplus A^*, A, M)$ is a VB-Lie algebroid in the sense of Gracia-Saz and Mehta [13].
2.5. **Canonical identifications.** The canonical pairing \( \langle \cdot, \cdot \rangle_A : A \oplus A^* \to \mathbb{R} \) induces a nondegenerate pairing \( \langle \langle \cdot, \cdot \rangle \rangle_A = \text{pr}_2 \circ T \langle \cdot, \cdot \rangle_A \) on \( TA \times_{TM} TA^* \), where \( \text{pr}_2 : T\mathbb{R} = \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) (see [27]):

\[
\begin{array}{c}
A \oplus A^* \\
\langle \cdot, \cdot \rangle_A \\
\text{pr} \downarrow \\
\mathbb{R}
\end{array}
\quad 
\begin{array}{c}
TA \times_{TM} TA^* \\
\langle \langle \cdot, \cdot \rangle \rangle_A \\
\text{pr}_2 \downarrow \\
\mathbb{R}
\end{array}
\]

That is, if \( \chi = \frac{d}{dt} |_{t=0} \varphi(t) \in TA^* \) and \( \xi = \frac{d}{dt} |_{t=0} a(t) \in TA \) are such that \( Tq_A(\xi) = Tc_A(\chi) \), then \( \langle \langle \chi, \xi \rangle \rangle_A = \frac{d}{dt} |_{t=0} \langle a(t), \varphi(t) \rangle_T \). For instance, if \( X \in \Gamma(A) \) and \( \xi \in \Gamma(A^*) \), then \( TX \in \Gamma_{TM}(TA) \) and \( T\xi \in \Gamma_{TM}(TA^*) \) are such that \( Tq_A(TX) = \text{Id}_{TM} = Tc_A(T\xi) \) and we have for all \( v_p = \dot{c}(0) \in T_p M \):

\[
\langle \langle TX(v_p), T\xi(v_p) \rangle \rangle_A = \frac{d}{dt} |_{t=0} \langle X, \xi \rangle_A (c(t)) = \text{pr}_2(T_p(\xi(X))(v_p)).
\]

(8)

If \( (Tq_A)^\vee : (TA)^\vee \to TM \) is the vector bundle that is dual to the vector bundle \( Tq_A : TA \to TM \), there is an induced isomorphism \( I \)

\[
\begin{array}{ccc}
TA^* & \overset{I}{\longrightarrow} & (TA)^\vee \\
Tc_A \downarrow & & (Tq_A)^\vee \downarrow \\
TM & \longrightarrow & TM
\end{array}
\]

that is defined by

\[
\langle \xi, I(\chi) \rangle_{TA \to TM} = \langle \langle \xi, \chi \rangle \rangle_A
\]

(9)

for all \( \chi \in TA^* \) and \( \xi \in TA \) such that \( Tc_A(\chi) = Tq_A(\xi) \). That is, the following diagram commutes:

\[
\begin{array}{ccc}
TA \times_{TM} TA^* & \overset{T\langle \cdot, \cdot \rangle_A}{\longrightarrow} & T\mathbb{R} \\
(\text{Id}, I) \downarrow & & \langle \langle \cdot, \cdot \rangle \rangle_A \downarrow \\
TA \times_{TM} (TA)^\vee & \overset{T\langle \cdot, \cdot \rangle_{TA}}{\longrightarrow} & \mathbb{R}
\end{array}
\]

Applying this to the case \( q_A = p_{TM} \), we get an isomorphism

\[
\begin{array}{ccc}
T(T^*M) & \overset{I}{\longrightarrow} & (TTM)^\vee \\
T_{TM} \downarrow & & (Tp_{TM})^\vee \downarrow \\
TM & \longrightarrow & TM
\end{array}
\]

We have also the canonical involution

\[
\begin{array}{ccc}
TTM & \overset{\sigma}{\longrightarrow} & TTM \\
Tp_{TM} \downarrow & & p_{TM} \downarrow \\
TM & \longrightarrow & TM
\end{array}
\]

Recall that for \( V \in \mathfrak{X}(M) \) the map \( TV : TM \to TTM \) is a section of \( TP_{TM} : TTM \to TM \) and \( \sigma(TV) \) is a section of \( p_{TM} : TTM \to TM \), i.e. a vector field on \( TM \).
We get an isomorphism \( \varsigma := \sigma^* \circ I : T(T^*M) \to T^*(TM) \)
\[
\begin{array}{ccc}
T(T^*M) & \xrightarrow{\varsigma} & T^*(TM) \\
\downarrow Tc_M & & \downarrow c_{TM} \\
TM & \xrightarrow{\varsigma} & TM
\end{array}
\]

**Proposition 2.8.** The map \( \Sigma := (\sigma, \varsigma) : TP_M \to P_{TM} \)
\[
\begin{array}{ccc}
TP_M & \xrightarrow{\Sigma} & P_{TM} \\
\downarrow Tpr_M & & \downarrow pr_{TM} \\
TM & \xrightarrow{\Sigma} & TM
\end{array}\]

where \( pr_M : P_M \to M \) is the projection, establishes an isomorphism of vector bundles.

### 2.6. Lie functor from \( P_\Gamma \) to \( P_A \).

**Proposition 2.9.** Let \( \Gamma \Rightarrow M \) be a Lie groupoid with Lie algebroid \( A \). Then the Lie algebroid of \( P_\Gamma \) is canonically isomorphic to \( P_A \).

Moreover, the pairing \( \langle \cdot, \cdot \rangle_\Gamma \) is a groupoid morphism: \( P_\Gamma \times_\Gamma P_\Gamma \to \mathbb{R} \). Its corresponding Lie algebroid morphism coincides with the pairing \( \langle \cdot, \cdot \rangle_A : P_A \times_A P_A \to \mathbb{R} \), under the canonical isomorphism \( A(P_\Gamma) \cong P_A \).

This is a standard result. Below we recall its proof which will be useful in the following.

For any Lie groupoid \( \Gamma \Rightarrow M \) with Lie algebroid \( A \to M \), the tangent bundle projection \( p_\Gamma : TT \to \Gamma \) is a groupoid morphism over \( p_M : TM \to M \) and applying the Lie functor gives a canonical morphism \( A(p_\Gamma) : A(TT) \to A \). This acquires a vector bundle structure by applying \( A(\cdot) \) to the operations in \( TT \to \Gamma \). This yields a system of vector bundles

\[
\begin{array}{ccc}
A(TT) & \xrightarrow{q_{A(TT)}} & TM \\
\downarrow A(p_\Gamma) & & \downarrow p_M \\
A & \xrightarrow{q_A} & M
\end{array}
\]

in which \( A(TT) \) has two vector bundle structures, the maps defining each being morphisms with respect to the other; that is to say, \( A(TT) \) is a double vector bundle.

Associated with the vector bundle \( q_A : A \to M \) is the tangent double vector bundle

\[
\begin{array}{ccc}
TA & \xrightarrow{Tq_A} & TM \\
\downarrow p_A & & \downarrow p_M \\
A & \xrightarrow{q_A} & M
\end{array}
\]

It is shown in [28] that the canonical involution \( \sigma : T(TT) \to T(TT) \) restricts to a canonical map

\( \sigma_T : A(TT) \to TA \)

which is an isomorphism of double vector bundles preserving the side bundles. Note that here, \( A \) is seen as a submanifold of \( TT \).

Similarly, the cotangent groupoid structure \( T^*\Gamma \Rightarrow A^* \) is defined by maps which are vector bundle morphisms and, reciprocally, the operations in the vector bundle \( c_T : T^*\Gamma \to \Gamma \)
are groupoid morphisms. Taking the Lie algebroid of $T^*\Gamma \to A^*$ we get a double vector bundle

$$A(T^*\Gamma) \xrightarrow{\sigma_{T^*\Gamma}} A^*$$  \hspace{1cm} \text{(11)} \]

where the vector bundle operations in $A(T^*\Gamma) \to A$ are obtained by applying the Lie functor to those in $T^*\Gamma \to \Gamma$.

It follows from the definitions of the operations in $T^*\Gamma \Rightarrow A^*$ that the canonical pairing $\langle \cdot, \cdot \rangle_{TT} : TT \times \Gamma T^*\Gamma \to \mathbb{R}$ is a groupoid morphism into the additive group(oid) $\mathbb{R}$. Hence $\langle \cdot, \cdot \rangle_{TT}$ induces a Lie algebroid morphism $A(\langle \cdot, \cdot \rangle_{TT}) : A(TT) \times_A A(T^*\Gamma) \to A(\mathbb{R}) = \mathbb{R}$. Note that $A(\langle \cdot, \cdot \rangle_{TT})$ is the restriction to $A(TT) \times_A A(T^*\Gamma)$ of $(\langle \cdot, \cdot \rangle_{TT}) : T(TT) \times \Gamma T(T^*\Gamma) \to \mathbb{R}$.

As noted in [28], $A(\langle \cdot, \cdot \rangle_{TT})$ is nondegenerate, and therefore induces an isomorphism of double vector bundles $I_{\Gamma} : A(T^*\Gamma) \to A(TT)^\vee$, where $A(TT)^\vee$ is the dual of $A(TT) \to A$. Now dualizing $\sigma_{T^{-1}} : TA \to A(TT)$ over $A$, we define

$$\varsigma_{\Gamma} = (\sigma_{T^{-1}})^* \circ I_{\Gamma} : A(T^*\Gamma) \to T^*A;$$

this is an isomorphism of double vector bundles preserving the side bundles. The Lie algebroids $T^*A \to A^*$ and $A(T^*\Gamma) \to A^*$ are isomorphic via $\varsigma_{\Gamma}$.

The Lie algebroid of the direct sum $P_{\Gamma} = TT \oplus T^*\Gamma$ is equal to

$$A(P_{\Gamma}) = T^*_U P_{\Gamma},$$

where we write $U$ for the unit space of $P_{\Gamma}$; i.e., $U := TM \oplus A^*$. By the considerations above, we have a Lie algebroid morphism $\Sigma_{\Gamma} = \Sigma|_{A(P_{\Gamma})} = (\sigma_{\Gamma}, \varsigma_{\Gamma})$:

$$\Sigma_{\Gamma} : \Sigma_{\Gamma} = (\sigma_{\Gamma}, \varsigma_{\Gamma}) : P_{\Gamma} = TA \oplus T^*A$$

preserving the side bundles $A$ and $TM \oplus A^*$.

Recall that we have also a map

$$\Sigma = (\sigma, \varsigma) : TP_{\Gamma} \to P_{TT}.$$

**Lemma 2.10.** Let $\Gamma \Rightarrow M$ be a Lie groupoid and $u$ an element of $A(P_{\Gamma}) \subseteq TP_{\Gamma}$ projection to $a_m \in A$ and $(v_m, \alpha_m) \in T_m M \times A^*_m$. Then, if $\Sigma_{\Gamma}(u) = (v_{am}, \alpha_{am}) \in P_{\Gamma}(a_m)$ and $\Sigma(u) = (v_m, \alpha_m) \in P_{TT}(a_m)$, we have

$$\iota_m v_{am} = \tilde{v}_{am} \quad \text{and} \quad \alpha_{am} = \tilde{\alpha}_{am}|_{T_m A}.$$
Proposition 3.2. In particular, we have the following
are all Lie groupoid morphisms.

Note also that $J\phi$ is a generalized complex structure on $\Gamma$ under the canonical isomorphism
This is equivalent to $D\pi\circ T\langle \cdot, \cdot \rangle$.

Proof. The first equality follows immediately from the definition of $\sigma_T$.

Choose $T^s\alpha, A \ni w_{am} = \sigma_T(y)$ for some $y \in A(T\Gamma) \subseteq T(T\Gamma)$ and $\tilde{w}_{am} := t_\ast w_{am}$. Write also $U = (x, \xi)$ with $x \in A(p_T)$ and $\xi \in A(T^s\Gamma) \subseteq T(T^s\Gamma)$, i.e. $\alpha_{am} = \sigma_T(\xi)$. Then we have $A(p_T)(y) = A(c_T)(\xi)$ and we can compute

$$\langle w_{am}, \alpha_{am} \rangle_{TA} = \langle y, I_T(\xi) \rangle_{A(T\Gamma)}$$
$$= \langle y, I_T(\xi) \rangle_{T(T\Gamma)}$$
$$= \langle \sigma(y), \xi \rangle_{T(T\Gamma)} = \langle \tilde{w}_{am}, \tilde{\alpha}_{am} \rangle_{T(T\Gamma)} = \langle w_{am}, \xi^\ast \tilde{\alpha}_{am} \rangle_{TA}.$$

The pairing $\langle \cdot, \cdot \rangle_T$ is a groupoid morphism $p_T \times_T p_T \rightarrow \mathbb{R}$. Hence, we can consider the Lie algebroid morphism

$$A(\langle \cdot, \cdot \rangle_T) : A(p_T) \times_A A(p_T) \rightarrow A(\mathbb{R}) = \mathbb{R}.$$ 

We have

$$A(\langle \cdot, \cdot \rangle_T) = (pr_2 \circ T\langle \cdot, \cdot \rangle) |_{A(p_T) \times_A A(p_T)}.$$ 

(12)

We can see from the proof of the last lemma that $A(\langle \cdot, \cdot \rangle_{T(T\Gamma)})$ coincides with $\langle \cdot, \cdot \rangle_{TA}$ under the isomorphism $\Sigma_T$. Hence, $A(\langle \cdot, \cdot \rangle_T)$ coincides with the pairing $\langle \cdot, \cdot \rangle : P_A \times_A P_A \rightarrow \mathbb{R}$, under the canonical isomorphism $A(p_T) \cong P_A$.

3. MULTIPlicative GENERALIZED COMPLEX GEOMETRY

3.1. Glanon groupoids.

Definition 3.1. Let $\Gamma \Rightarrow M$ be a Lie groupoid with Lie algebroid $A \rightarrow M$. A multiplicative generalized complex structure on $\Gamma$ is a Lie groupoid morphism

such that $\mathcal{J}$ is a generalized complex structure.

The pair $(\Gamma \Rightarrow M, \mathcal{J})$ is then called a Glanon groupoid.

This is equivalent to $D_{\mathcal{J}}$ and $\overline{D}_{\mathcal{J}}$, the eigenspaces of $\mathcal{J} : (TT \oplus T^s\Gamma) \otimes \mathbb{C} \rightarrow (TT \oplus T^s\Gamma) \otimes \mathbb{C}$ to the eigenvalues $i$ and $-i$ being multiplicative Dirac structures on $\Gamma \Rightarrow M$ [31] [58].

Note also that $\mathcal{J}$ is a Lie groupoid morphism if and only if the maps $N : TT \rightarrow TT$, $\phi^* : T^s\Gamma \rightarrow TT$, $N^* : T^s\Gamma \rightarrow T^s\Gamma$, and $\omega^* : TT \rightarrow T^s\Gamma$ such that

$$\mathcal{J} = \begin{pmatrix} N & \pi^* \\ \omega^* & -N^* \end{pmatrix}$$

are all Lie groupoid morphisms.

In particular, we have the following

Proposition 3.2. If $\Gamma$ is a Glanon groupoid, then $\Gamma$ is naturally a Poisson groupoid.
Example 3.3 (Glanon groups). Let $G \cong \{*\}$ be a Glanon Lie group. Since any multiplicative two-form must vanish (see for instance [18]), the underlying generalized complex structure on $G$ is equivalent to a multiplicative holomorphic Poisson structure. Therefore, Glanon Lie groups are in one-one correspondence with complex Poisson Lie groups.

Example 3.4 (Symplectic groupoids). Consider a Lie groupoid $\Gamma \rightrightarrows M$ equipped with a non-degenerate two-form $\omega$. Then the map

$$J_\omega = \begin{pmatrix} 0 & -\omega^1 \omega^2^{-1} \\ \omega^1 & 0 \end{pmatrix},$$

where $\omega^1 : T\Gamma \to T^*\Gamma$ is the bundle map $X \mapsto i_X\omega$, defines a Glanon groupoid structure on $\Gamma \rightrightarrows M$ if and only if $(\Gamma, \omega)$ is a symplectic groupoid.

Example 3.5 (Holomorphic Lie groupoids). Let $\Gamma$ be a holomorphic Lie groupoid with $J_\Gamma : T\Gamma \to T^*\Gamma$ being its almost complex structure. Then the map

$$J = \begin{pmatrix} J_\Gamma & 0 \\ 0 & -J_\Gamma^* \end{pmatrix},$$

defines a Glanon groupoid structure on $\Gamma$.

3.2. Glanon Lie algebroids. Let $A$ be a Lie algebroid over $M$. Recall that $P_A = TA \oplus T^*A$ has the structure of a Lie algebroid over $TM \oplus A^*$.

Definition 3.6. A Glanon Lie algebroid is a Lie algebroid $A$ endowed with a generalized complex structure $J_A : P_A \to P_A$ that is a Lie algebroid morphism.

Proposition 3.7. If $A$ is a Glanon Lie algebroid, then $(A, A^*)$ is a Lie bialgebroid.

Example 3.8 (Glanon Lie algebra). Let $g$ be a Lie algebra. Then $P_g = g \times (g \oplus g^*)$ is a Lie algebroid over $g^*$. Hence, a map

$$J : P_g \to P_g,$$

$$J(x, y, \xi) = (x, J_{x, g}(y, \xi), J_{x, g^*}(y, \xi))$$

can only be a Lie algebroid morphism if $J_{x, g}(y, \xi) = J_{x', g}(y', \xi)$ for all $x, x', y \in g$. In particular, $J_{x, g^*}(y, \xi) = J_{x, g^*}(y', \xi)$ for all $x, y, y' \in g$ and the map $J_x : \{x\} \times g \times g^* \to \{x\} \times g \times g^*$ has the matrix

$$\begin{pmatrix} n_x & \pi_x^1 \\ 0 & -n_x^* \end{pmatrix}.$$ 

It thus follows that it must be equivalent to a complex Lie bialgebra.

Example 3.9 (Symplectic Lie algebroid). Let $(M, \pi)$ be a Poisson manifold. Let $A$ be the cotangent Lie algebroid $A = (T^*M)_\pi$. Then the map

$$\begin{pmatrix} 0 & (\omega_A^1)^{-1} \\ \omega_A^2 & 0 \end{pmatrix}$$

where $\omega_A$ is the canonical cotangent symplectic structure on $A$, defines a Glanon Lie algebroid structure on $A$.

Example 3.10 (Holomorphic Lie algebroids). Let $A$ be a holomorphic Lie algebroid. Let $A_R$ be its underlying real Lie algebroid and $j : TA_R \to TA_R$ the corresponding almost complex structure. Then the map

$$\begin{pmatrix} j & 0 \\ 0 & -j^* \end{pmatrix}$$

defines a Glanon Lie algebroid structure on $A_R$.  

3.3. Integration theorem. Now we are ready to state the main theorem of this paper.

**Theorem 3.11.** If $\Gamma$ is a Glanon groupoid with Lie algebroid $A$, then $A$ is a Glanon Lie algebroid.

Conversely, given a Glanon Lie algebroid $A$, if $\Gamma$ is a $s$-connected and $s$-simply connected Lie groupoid integrating $A$, then $\Gamma$ is a Glanon groupoid.

**Remark 3.12.** Ortiz shows in his thesis [31] that multiplicative Dirac structures on a Lie groupoid $\Gamma \Rightarrow M$ are in one-one correspondence with morphic Dirac structures on its Lie algebroid, i.e. Dirac structures $D_A \subseteq P_A$ such that $D_A$ is a subalgebroid of $P_A \to TM \oplus A^*$ over a set $U \subseteq TM \oplus A^*$.

By extending this result to complex Dirac structures and using the fact that a multiplicative generalized complex structure on $\Gamma \Rightarrow M$ is the same as a pair of transversal, complex conjugated, multiplicative Dirac structures in the complexified $P_\Gamma$, one finds an alternative method for the proof of our main theorem.

Applying this theorem to Example 3.4 and Example 3.9, we obtain immediately the following

**Theorem 3.13.** Let $(P, \pi)$ be a Poisson manifold. If $\Gamma$ is a $s$-connected and $s$-simply connected Lie groupoid integrating the Lie algebroid $(T^*P)_\pi$, then $\Gamma$ automatically admits a symplectic groupoid structure.

Similarly, applying Theorem 3.11 to Examples 3.5 and 3.10, we obtain immediately the following

**Theorem 3.14.** If $\Gamma$ is a $s$-connected and $s$-simply connected Lie groupoid integrating the underlying real Lie algebroid $A_R$ of a holomorphic Lie algebroid $A$, then $\Gamma$ is a holomorphic Lie groupoid.

A Glanon groupoid is automatically a Poisson groupoid, while a Glanon Lie algebroid must be a Lie bialgebroid. The following result reveals their connection

**Theorem 3.15.** Let $\Gamma$ be a Glanon groupoid with its Glanon Lie algebroid $A$, $(\Gamma, \pi)$ and $(A, A^*)$ their induced Poisson groupoid and Lie bialgebroid respectively. Then the corresponding Lie bialgebroid of $(\Gamma, \pi)$ is isomorphic to $(A, A^*)$.

3.4. Tangent Courant algebroid. In [3], Boumaiza-Zaalani proved that the tangent bundle of a Courant algebroid is naturally a Courant algebroid. In this section, we study the Courant algebroid structure on $P_{TM}$ in terms of the isomorphism $\Sigma : TP_M \to P_{TM}$ defined in [10].

We need to introduce first some notations.

**Definition 3.16.** (a) The map $T : \mathfrak{X}(M) \to \mathfrak{X}(TM)$ sends $V \in \mathfrak{X}(M)$ to $TV = \sigma(TV) \in \mathfrak{X}(TM)$.

(b) The map $T : \Omega^1(M) \to \Omega^1(TM)$ sends $\alpha \in \Omega^1(M)$ to $T\alpha = \zeta(T\alpha) \in \Omega^1(TM)$.

(c) The map $T : C^\infty(M) \to C^\infty(TM)$ is given by

$$Tf := pr_2 \circ Tf \in C^\infty(TM, \mathbb{R}),$$

for all functions $f \in C^\infty(M)$.

Note that if $v_m = \dot{c}(0) \in T_m M$, then

$$Tf(v_m) = Tf(\dot{c}(0)) = \left. \frac{d}{dt} \right|_{t=0} (f \circ c(t)), \quad (13)$$

13
that is,
\[ Tf = df : TM \to \mathbb{R}. \]

Now introduce the map \( \mathbb{T} : \Gamma(P_M) \to \Gamma(P_{TM}) \) given by
\[ \mathbb{T}(V, \alpha) = \Sigma(\mathbb{T}V, \mathbb{T}\alpha) = (\mathbb{T}V, \mathbb{T}\alpha) \]
for all \((V, \alpha) \in \Gamma(P_M)\).

The main result of this section is the following:

**Proposition 3.17.** For any \( e_1, e_2 \in \Gamma(P_M) \), we have
\[
\begin{align*}
[\mathbb{T}e_1, \mathbb{T}e_2] &= \mathbb{T}[e_1, e_2] \\
(\mathbb{T}e_1, \mathbb{T}e_2)_{TM} &= \mathbb{T}\langle e_1, e_2 \rangle_M.
\end{align*}
\]

The following results show that \( \mathbb{T}f \in C^\infty(TM) \), \( \mathbb{T}V \in \mathfrak{X}(TM) \) and \( \mathbb{T}\alpha \in \Omega^1(TM) \) are the complete lifts of \( f \in C^\infty(M) \), \( V \in \mathfrak{X}(M) \) and \( \alpha \in \Omega^1(M) \) in the sense of [39].

**Lemma 3.18.** For all \( f \in C^\infty(M) \) and \( V \in \mathfrak{X}(M) \), we have
\[ \mathbb{T}V(\mathbb{T}f) = \mathbb{T}(V(f)). \]

**Proof.** Let \( \phi \) be the flow of \( V \). For any \( v_m = \dot{c}(0) \in T_mM \), we have
\[
(\mathbb{T}V(\mathbb{T}f))(v_m) = (\mathbb{T}V)(v_m)(\mathbb{T}f) = \sigma \left( \frac{d}{dt} \bigg|_{t=0} \frac{d}{ds} \bigg|_{s=0} \phi_s(c(t)) \right) \mathbb{T}f
\]
\[
= \left( \frac{d}{dt} \bigg|_{s=0} \frac{d}{ds} \bigg|_{t=0} \phi_s(c(t)) \right) (\mathbb{T}f)
\]
\[
= \left( \frac{d}{ds} \bigg|_{s=0} \mathbb{T}f \left( \frac{d}{dt} \bigg|_{t=0} \phi_s(c(t)) \right) \right)
\]
\[
= \left( \frac{d}{dt} \bigg|_{t=0} \frac{d}{ds} \bigg|_{s=0} (f \circ \phi_s)(c(t)) \right)
\]
\[
= \left( \frac{d}{dt} \bigg|_{t=0} V(f)(c(t)) \right) \mathbb{T}(V(f))(v_m).
\]

\( \Box \)

The following lemma characterizes the sections \( \mathbb{T} \alpha \) of \( \Omega^1(TM) \).

**Lemma 3.19.** (a) For all \( \alpha \in \Omega^1(M) \) and \( V \in \mathfrak{X}(M) \), we have
\[ \mathbb{T}[TV, \mathbb{T}\alpha]_{TM} = \mathbb{T}\langle [V, \alpha]_{TM} \rangle. \]

(b) \( \xi \in \Omega^1(TM) \) satisfies
\[ \mathbb{T}[TV, \xi]_{TM} = 0 \]
for all \( V \in \mathfrak{X}(M) \) if and only if \( \xi = 0 \).

**Proof.** (a) Choose \( V \in \mathfrak{X}(M) \). Then, using \( \sigma^2 = \text{Id}_{TM} \):
\[
\begin{align*}
\mathbb{T}[TV, \mathbb{T}\alpha]_{prTM} &= \mathbb{T}[\sigma(TV), (\sigma^* \circ f)(T\alpha)]_{prTM} \\
&= \mathbb{T}[TV, I(T\alpha)]_{TM} \\
&= \mathbb{T}\langle [TV, T\alpha] \rangle_{TM}.
\end{align*}
\]
(b) Let $\xi \in \Omega^1(TM)$ be such that $\xi(T V) = 0$ for all $V \in \mathfrak{X}(M)$. For any $u \in TM$ with $u \neq 0$ and $v \in T_u(TM)$, there exists a vector field $V \in \mathfrak{X}(M)$ such that $TV(u) = v$. This yields $\xi(v) = 0$. Therefore, $\xi$ vanishes at all points of $TM$ except for the zero section of $TM$. By continuity, we get $\xi = 0$.  

□

Using this, we can show the following formulas.

Lemma 3.20.  
(a) For all $V, W \in \mathfrak{X}(M)$, we have  
$$[TV, TW] = T[V, W].$$

(b) For any $\alpha, \beta \in \Omega^1(M)$ and $V, W \in \mathfrak{X}(M)$, we have  
$$L_{TV}T\beta - i_{TW}dT\alpha = T(L_V\beta - i_W\alpha).$$

Proof.  
(a) This is an easy computation, using the fact that if $\phi$ is the flow of the vector field $V$, then $T\phi$ is the flow of $TV$ (alternatively, see [27]).

(b) For any $U \in \mathfrak{X}(M)$, we can compute  
$$[[TU, L_{TV}T\beta - i_{TW}dT\alpha]]_{PTM} = TV(T\beta, TU) - \langle T\beta, T[V, U] \rangle - TW(T\alpha, TU)$$
$$+ TU\langle T\alpha, TW \rangle + \langle T\alpha, T[W, U] \rangle$$
$$= TV(T\beta, U) - \langle T\beta, [V, U] \rangle - TW(T\alpha, U)$$
$$+ TU\langle T\alpha, W \rangle + \langle T\alpha, [W, U] \rangle$$
$$= T(\langle V\langle \beta, U \rangle \rangle - \langle T\beta, [V, U] \rangle - \langle W\langle \alpha, U \rangle \rangle$$
$$+ U\langle \alpha, W \rangle + \langle \alpha, [W, U] \rangle$$
$$= \langle TU, L_V\beta - i_W d\alpha \rangle_{PTM}$$
$$= \langle TU, (L_V\beta - i_W d\alpha) \rangle_{PTM}.$$  

We get  
$$[[TU, T(L_V\beta - i_W d\alpha) - (L_{TV}T\beta - i_{TW}dT\alpha)]_{PTM} = 0$$

for all $U \in \mathfrak{X}(M)$ and we can conclude using Lemma 3.19.  

□

Proof of Proposition 3.17.  
Equation (15) follows immediately from (17).  
Formula (14) for the Dorfman bracket on sections of $P_{TM} = T(TM) \oplus T^*(TM)$ follows from Lemma 3.20 (Note that we can show in the same manner that the Courant bracket is compatible with $\mathbb{T}$.)

3.5. Nijenhuis tensor. Now let $J : P_M \to P_M$ be a vector bundle morphism over the identity. Consider the map  
$$TJ : TP_M \to TP_M,$$

and the map $TJ$ defined by the commutative diagram  
$$\begin{array}{ccc}
TP_M & \xrightarrow{TJ} & TP_M \\
\Sigma & \downarrow & \Sigma \\
P_{TM} & \xrightarrow{TJ} & P_{TM}
\end{array}$$

i.e.,  
$$TJ = \Sigma \circ T J \circ \Sigma^{-1}.$$  

Then, by definition, we get for all $e \in \Gamma(P_M)$:  
$$TJ(Te) = (\Sigma \circ TJ)(Te) = \Sigma(T(J(e))) = T(J(e)).$$  

(18)
The following lemma is immediate.

**Lemma 3.21.**  
(a) \( T(\text{Id}_M) = \text{Id}_{T^*M} \).  
(b) Let \( J : P_M \to P_M \) be a vector bundle morphism over the identity. Then  
\[ T(J^2) = (T(J))^2. \]

Given a vector bundle morphism \( J : P_M \to P_M \), we can consider as above  
\[ TN_J : TP_M \times_{TP} TP_M \to TP_M. \]

Define \( TN_J : P_{TM} \to P_{TM} \) by the following commutative diagram:

\[
\begin{array}{ccc}
TP_M \times_{TP} TP_M & \xrightarrow{T(N_J)} & TP_M \\
\Sigma \times \Sigma & \downarrow & \downarrow \Sigma \\
P_{TM} \times_{TP} P_{TM} & \xrightarrow{T(N_J)} & P_{TM}
\end{array}
\]

An easy computation using (18) and (13) yields  
\[ TN_J(Te_1, Te_2) = N_{T(J)}(Te_1, Te_2) \]
for all \( e_1, e_2 \in \Gamma(P_M) \). As in the proof of Lemma 3.19, this implies that \( TN_J \) and \( N_{T(J)} \) coincide at all points of \( TM \) except for the zero section of \( TM \). By continuity, we obtain the following theorem.

**Theorem 3.22.** Let \( J : P_M \to P_M \) be a vector bundle morphism. Then  
\[ TN_J = N_{T(J)}. \]

### 3.6. Multiplicative Nijenhuis tensor

Let \( \Gamma \rightrightarrows M \) and \( \Gamma' \rightrightarrows M' \) be Lie groupoids. A map \( \Phi : \Gamma \to \Gamma' \) is a groupoid morphism if and only if the map \( \Phi \times \Phi \times \Phi \) restricts to a map  
\[ \Lambda_{\Gamma} \to \Lambda_{\Gamma'} \]
where \( \Lambda_{\Gamma} \) and \( \Lambda_{\Gamma'} \) are the graphs of the multiplications in \( \Gamma \rightrightarrows M \) and respectively \( \Gamma' \rightrightarrows M' \).

Consider the graphs of the multiplications on \( TT \) and \( T^*\Gamma \):

\[ \Lambda_{TT} = \{(v_g, v_h, v_g \ast v_h) \mid v_g, v_h \in TT, Tt(v_h) = Ts(v_g)\} = T\Lambda_{\Gamma} \]
and
\[ \Lambda_{T^*\Gamma} = \{((\alpha_g, \alpha_h, \alpha_g \ast \alpha_h) \mid \alpha_g, \alpha_h \in T^*\Gamma, \hat{t}(\alpha_h) = \hat{s}(\alpha_g)\}. \]

Recall that if  
\[ (\Lambda_{T^*\Gamma})^{\text{op}} = \{(\alpha_g, \alpha_h, -(\alpha_g \ast \alpha_h)) \mid \alpha_g, \alpha_h \in T^*\Gamma, \hat{t}(\alpha_h) = \hat{s}(\alpha_g)\}, \]
i.e.
\[ \Lambda_{TT} \oplus_{\Lambda_{\Gamma}} (\Lambda_{T^*\Gamma})^{\text{op}} = (\text{Id} \times \text{Id} \times \mathcal{I}) (\Lambda_{TT} \oplus_{\Lambda_{\Gamma}} \Lambda_{T^*\Gamma}), \]
then  
\[ (\Lambda_{T^*\Gamma})^{\text{op}} = (T\Lambda_{\Gamma})^{\circ}. \]

A map \( J : P_{\Gamma} \to P_{\Gamma} \) is a groupoid morphism if and only if \( \Lambda_{TT} \oplus_{\Lambda_{\Gamma}} \Lambda_{T^*\Gamma} \) is stable under the map \( J \times J \times J \). This yields:

**Lemma 3.23.** The map \( J \) is a groupoid morphism if and only if \( \Lambda_{TT} \oplus_{\Lambda_{\Gamma}} (\Lambda_{T^*\Gamma})^{\text{op}} \) is stable under the map \( J \times J \times J \), where \( J \) is defined by (9).
Since $\Lambda TT = T\Lambda_\Gamma$ and $(\Lambda T\Gamma)^{op} \cong (T\Lambda_\Gamma)^{\circ}$, we get that $J$ is multiplicative if and only if $T\Lambda_\Gamma \oplus (T\Lambda_\Gamma)^{\circ}$ is stable under $J \times J \times \bar{J}$.

Similarly, the map $N_J : P_\Gamma \times_\Gamma P_\Gamma \rightarrow P_\Gamma$ is multiplicative if and only if $N_J \times N_J \times N_J$ restricts to a map

$$(\Lambda TT \oplus_{\Lambda_\Gamma} \Lambda T\Gamma) \times_{\Lambda_\Gamma} (\Lambda TT \oplus_{\Lambda_\Gamma} \Lambda T\Gamma) \rightarrow \Lambda TT \oplus_{\Lambda_\Gamma} \Lambda T\Gamma.$$  

**Lemma 3.24.** The map $N_J$ is multiplicative if and only if

$$(T\Lambda_\Gamma \oplus_{\Lambda_\Gamma} \Lambda T\Gamma \oplus_{\Lambda_\Gamma} (T\Lambda_\Gamma)^{\circ}) \rightarrow T\Lambda_\Gamma \oplus_{\Lambda_\Gamma} (T\Lambda_\Gamma)^{\circ}.$$ 

The following lemma is easy to prove.

**Lemma 3.25.** If $M$ is a smooth manifold and $N$ a submanifold of $M$, then the Courant-Dorfman bracket on $P_M$ restricts to sections of $TN \oplus T(N)^{\circ}$.

Note that $TN \oplus (TN)^{\circ}$ is a generalized Dirac structure in the sense of [1]. We get the following theorem.

**Theorem 3.26.** Let $(\Gamma \Rightarrow M, J)$ be an almost generalized complex groupoid. Then the Nijenhuis tensor $N_J$ is a Lie groupoid morphism

$$N_J : P_\Gamma \times_\Gamma P_\Gamma \rightarrow P_\Gamma.$$ 

**Proof.** Choose sections $\xi_1, \xi_2, \xi_3, \eta_1, \eta_2, \eta_3 \in \Gamma(P_\Gamma)$ such that

$$(\xi_1, \xi_2, \xi_3)|_{\Lambda_\Gamma}, (\eta_1, \eta_2, \eta_3)|_{\Lambda_\Gamma} \in \Gamma(T\Lambda_\Gamma \oplus_{\Lambda_\Gamma} (T\Lambda_\Gamma)^{\circ}).$$

Then we have

$$(J\xi_1, J\xi_2, J\xi_3)|_{\Lambda_\Gamma}, (J\eta_1, J\eta_2, J\eta_3)|_{\Lambda_\Gamma} \in \Gamma(T\Lambda_\Gamma \oplus_{\Lambda_\Gamma} (T\Lambda_\Gamma)^{\circ}).$$

From Lemma 3.25, it follows that

$$(N_J \times N_J \times N_J)((\xi_1, \xi_2, \xi_3), (\eta_1, \eta_2, \eta_3))$$

takes values in

$$T\Lambda_\Gamma \oplus_{\Lambda_\Gamma} (T\Lambda_\Gamma)^{\circ}$$
on $\Lambda_\Gamma$. By Lemma 3.24 we are done. \[\square\]

### 3.7. Infinitesimal multiplicative Nijenhuis tensor.

**Definition 3.27.** Let $\mathcal{J} : P_\Gamma \rightarrow P_\Gamma$ be a Lie groupoid morphism. The map

$$A(\mathcal{J}) : P_A \rightarrow P_A$$
is defined by the commutative diagram

$$\begin{array}{ccc}
P_A & \overset{\Sigma_\Gamma}{\longrightarrow} & P_A \\
A(\mathcal{J}) \downarrow & & \downarrow A(\mathcal{J}) \\
A(P_\Gamma) & \overset{\Sigma_\Gamma}{\longrightarrow} & P_A
\end{array}$$

The following lemma can be found in [5].
Lemma 3.28. Let $J : P_G \to P_G$ be a multiplicative map. Then

$$\mathcal{A}(J) : P_A \to P_A$$

is a vector bundle morphism if and only if

$$J : P_G \to P_G$$

is a vector bundle morphism.

Since the map $N_J : P_G \times_G P_G \to P_G$ is then also a Lie groupoid morphism by Theorem 3.20, we can also consider

$$\mathcal{A}(N_J) : P_A \times_A P_A \to P_A$$

defined by

$$\begin{array}{ccc}
A(P_G) \times_A A(P_G) & \xrightarrow{\Sigma^2} & P_A \times_A P_A \\
\downarrow \mathcal{A}(N_J) & & \downarrow \mathcal{A}(N_J) \\
A(P_G) & \xrightarrow{\Sigma} & P_A
\end{array}$$

The main result of this section is the following.

Theorem 3.29. Let $J : P_G \to P_G$ be a vector bundle homomorphism and a Lie groupoid morphism. Then

(a) The map $\mathcal{A}(J)$ is $(\cdot, \cdot)_A$-orthogonal if and only if $J$ is $(\cdot, \cdot)_G$-orthogonal and, in that case,

$$\mathcal{A}(N_J) = N_{\mathcal{A}(J)}.$$  \hspace{2cm} (20)

For the proof, we need a couple of lemmas.

Definition 3.30. Let $M$ be a smooth manifold and $\iota : N \hookrightarrow M$ a submanifold of $M$.

(a) A section $e_N = : X_N + \alpha_N$ is $\iota$-related to $e_M = X_M + \alpha_M$ if $\iota_* X_N = X_M |_N$ and $\alpha_N = \iota^* \alpha_M$. We write then $e_N \sim_\iota e_M$.

(b) Two vector bundle morphisms $J_N : P_N \to P_N$ and $J_M : P_M \to P_M$ are said to be $\iota$-related if for each section $e_N$ of $P_N$, there exists a section $e_M \in P_M$ such that $e_N \sim_\iota e_M$ and $J_N(e_N) \sim_\iota J_M(e_M)$.

(c) Two vector bundle morphisms $N_N : P_N \times_N P_N \to P_N$ and $N_M : P_M \times_M P_M \to P_M$ are $\iota$-related if for each pair of sections $e_N, f_N \in \Gamma(P_N)$, there exist sections $e_M, f_M \in \Gamma(P_M)$ such that $e_N \sim_\iota e_M$, $f_N \sim_\iota f_M$ and $N_N(e_N, f_N) \sim_\iota N_M(e_M, f_M)$.

Lemma 3.31. Let $M$ be a manifold and $N \subseteq M$ a submanifold. If $e_N, f_N \in \Gamma(P_N)$ are $\iota$-related to $e_M, f_M \in \Gamma(P_M)$, then $[e_N, f_N] \sim_\iota [e_M, f_M]$, for the Dorfman and the Courant bracket.

Proof. This is an easy computation, see also [33].

Lemma 3.32. If $M$ is a manifold, $N \subseteq M$ a submanifold and $J_N : P_N \to P_N$ and $J_M : P_M \to P_M$ two $\iota$-related vector bundle morphisms. Then the generalized Nijenhuis tensors $N_J$ and $N_J$ are $\iota$-related.

Proof. This follows immediately from Lemma 3.31 and the definition.
Recall that the Lie algebroid $A$ of a Lie groupoid $\mathcal{G} \rightrightarrows M$ is an embedded submanifold of $TT$, $\iota : A \hookrightarrow TT$.

**Remark 3.33.** Note that Lemma 2.10 states that if $u \in \Gamma_A(A(P_{\mathcal{G}}))$ and $\tilde{u} \in \Gamma_{TT}(TP_{\mathcal{G}})$ is an extension of $u$, then
\[ \Sigma_{\mathcal{G}} \circ u \sim \Sigma \circ \tilde{u}. \]

**Lemma 3.34.** Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid and $J : P_{\mathcal{G}} \rightarrow P_{\mathcal{G}}$ a vector bundle morphism. If $J$ is multiplicative, then
\[ (a) \quad \mathfrak{A}(J) : P_A \rightarrow P_A \text{ and } \mathcal{T}J : P_{TT} \rightarrow P_{TT} \text{ are } \iota\text{-related, and} \]
\[ (b) \quad \mathfrak{A}(\mathcal{N}_J) \text{ and } \mathcal{T}\mathcal{N}_J \text{ are } \iota\text{-related.} \]

**Proof.** (a) Choose a section $e_A$ of $P_A$. Then we have $\Sigma_{\mathcal{G}}^{-1}(e_A) =: u \in \Gamma(A(P_{\mathcal{G}}))$ and since $A(P_{\mathcal{G}}) \subseteq TP_{\mathcal{G}}$, we find a section $\tilde{u}$ of $TP_{\mathcal{G}}$ such that $\tilde{u}$ restricts to $u$. Set $e_{TT} := \Sigma \circ \tilde{u}$. By Lemma 2.10 we have then $e_A \sim e_{TT}$. Furthermore, by construction of $A(J)$, we know that $A(J) \circ u = (TJ) \circ \tilde{u}$ on $TM \oplus A^* = TM \oplus TM^* \subseteq P_{\mathcal{G}}$. We have then
\[ \mathfrak{A}(J)(e_A) = \Sigma \circ A(J) \circ u \sim \Sigma \circ TJ \circ \tilde{u} = \mathcal{T}J(e_{TT}). \]

(b) By definition of $\mathfrak{A}(\mathcal{N}_J)$, this can be shown in the same manner. \hfill $\square$

**Proof of Theorem 3.29** (a) The map $\mathfrak{A}(J)$ is $\langle \cdot, \cdot \rangle_A$-orthogonal if and only if
\[ \langle \cdot, \cdot \rangle_A \circ (\mathfrak{A}(J), \mathfrak{A}(J)) = \langle \cdot, \cdot \rangle_A. \]

We have
\[ \langle \cdot, \cdot \rangle_A = \mathfrak{A}((\cdot, \cdot)_{\Gamma}) = A((\cdot, \cdot)_{\Gamma}) \circ (\Sigma_{\mathcal{G}}^{-1}, \Sigma_{\mathcal{G}}^{-1}) \]
by definition and
\[ \langle \cdot, \cdot \rangle_A \circ (\mathfrak{A}(J), \mathfrak{A}(J)) = A((\cdot, \cdot)_{\Gamma}) \circ (\Sigma_{\mathcal{G}}^{-1}, \Sigma_{\mathcal{G}}^{-1}) \circ (\Sigma_{\mathcal{G}}, \Sigma_{\mathcal{G}}) \circ (A(J), A(J)) \circ (\Sigma_{\mathcal{G}}^{-1}, \Sigma_{\mathcal{G}}^{-1}) \]
\[ = A((\cdot, \cdot)_{\Gamma} \circ (J, J)) \circ (\Sigma_{\mathcal{G}}^{-1}, \Sigma_{\mathcal{G}}^{-1}). \]

Since $\Sigma_{\mathcal{G}} : A(P_{\mathcal{G}}) \rightarrow P_A$ is an isomorphism, we get that $\mathfrak{A}(J)$ is $\langle \cdot, \cdot \rangle_A$-orthogonal if and only if
\[ A((\cdot, \cdot)_{\Gamma}) = A((\cdot, \cdot)_{\Gamma} \circ (J, J)) \]
and we can conclude.

(b) Choose sections $e_A, f_A \in \Gamma(P_A)$ and $u, v \in \Gamma_A(A(P_{\mathcal{G}}))$ such that $e_A = \Sigma \circ u$, $f_A = \Sigma \circ v$. Choose as in the proof of Lemma 3.34 two extensions $\tilde{u}$ and $\tilde{v} \in \Gamma_{TT}(TP_{\mathcal{G}})$ of $u$ and $v$ and set $e_{TT} := \Sigma \circ \tilde{u}$ and $f_{TT} := \Sigma \circ \tilde{v} \in \Gamma(P_{TT})$. Then we have
\[ e_A \sim e_{TT} \quad f_A \sim f_{TT}, \]
\[ \mathfrak{A}(J)(e_A) \sim \mathcal{T}J(e_{TT}) \quad \mathfrak{A}(J)(f_A) \sim \mathcal{T}J(f_{TT}) \]
and
\[ \mathfrak{A}(\mathcal{N}_J)(e_A, f_A) \sim \mathcal{T}\mathcal{N}_J(e_{TT}, f_{TT}). \]

But by Lemma 3.32 (21) yields also
\[ \mathfrak{N}_{\mathcal{N}_J}(e_A, f_A) \sim \mathcal{T}\mathcal{N}_J(e_{TT}, f_{TT}). \]

Since $\mathcal{N}_{\mathcal{T}J} = \mathcal{T}\mathcal{N}_J$ by Theorem 3.22 we get that
\[ \mathfrak{A}(\mathcal{N}_J)(e_A, f_A) \sim \mathcal{N}_{\mathcal{T}J}(e_{TT}, f_{TT}) \quad \text{and} \quad \mathfrak{N}_{\mathcal{N}_J}(e_A, f_A) \sim \mathcal{N}_{\mathcal{T}J}(e_{TT}, f_{TT}). \]

This yields
\[ \mathfrak{N}_{\mathcal{N}_J}(e_A, f_A) = \mathcal{N}_{\mathcal{N}_J}(e_A, f_A) \]
and the proof is complete. \hfill $\square$
The following corollary is immediate.

**Corollary 3.35.** Let \( \mathcal{J} : P_\Gamma \to P_\Gamma \) be a vector bundle morphism and a Lie groupoid morphism. Then

\[
\mathcal{N}_{\mathcal{A}(\mathcal{J})} = 0 \quad \text{if and only if} \quad \mathcal{N}_\mathcal{J} = 0.
\]

### 3.8. Proof of the integration theorem. Proof of Theorem 3.11

By Lemma 3.28 the map \( \mathcal{A}(\mathcal{J}) : P_A \to P_A \) is a vector bundle morphism. Since \( \mathcal{J}^2 = -\text{Id}_{P_\Gamma} \), we have using Lemma 3.21

\[
(\mathcal{A}(\mathcal{J}))^2 = -\text{Id}_{P_A}.
\]

By Corollary 3.35 and Theorem 3.29 we get

\[
\mathcal{N}_{\mathcal{A}(\mathcal{J})} = 0
\]

and

\( \mathcal{A}(\mathcal{J}) \) is \( \langle \cdot , \cdot \rangle_A \)-orthogonal.

Since

\[
\begin{array}{ccc}
A(P_\Gamma) & \xrightarrow{A(\mathcal{J})} & A(P_\Gamma) \\
\downarrow & & \downarrow \\
TM \oplus A^* & \longrightarrow & TM \oplus A^*
\end{array}
\]

is a Lie algebroid morphism and

\[
\begin{array}{ccc}
A(P_\Gamma) & \xrightarrow{\Sigma|A(P_\Gamma)} & P_A \\
\downarrow & & \downarrow \\
TM \oplus A^* & \longrightarrow & TM \oplus A^*
\end{array}
\]

is a Lie algebroid isomorphism over the identity, the map

\[
\mathcal{A}(\mathcal{J}) = \Sigma|A(P_\Gamma) \circ A(\mathcal{J}) \circ (\Sigma|A(P_\Gamma))^{-1} = \Sigma|A(P_\Gamma) \circ A(\mathcal{J}) \circ \Sigma^{-1}|_{P_A},
\]

\[
\begin{array}{ccc}
P_A & \xrightarrow{\mathcal{A}(\mathcal{J})} & P_A \\
\downarrow & & \downarrow \\
TM \oplus A^* & \longrightarrow & TM \oplus A^*
\end{array}
\]

is a Lie algebroid morphism.

For the second part, consider the map

\[
A_\mathcal{J} := \Sigma^{-1}|_{P_A} \circ \mathcal{J}_A \circ \Sigma|_{A(P_\Gamma)} : A(P_\Gamma) \to A(P_\Gamma).
\]

Since \( \mathcal{J}_A : P_A \to P_A \) is a Lie algebroid morphism, \( A_\mathcal{J} \) is a Lie algebroid morphism and there is a unique Lie groupoid morphism \( \mathcal{J} : P_\Gamma \to P_\Gamma \) such that \( A_\mathcal{J} = A(\mathcal{J}) \). By Lemma 3.28 \( \mathcal{J} \) is a morphism of vector bundles.

We get then immediately \( \mathcal{J}_A = \mathcal{A}(\mathcal{J}) \). Since \( \mathcal{J}_A^2 = -\text{Id}_A \), we get

\[
\mathcal{A}(\mathcal{J}^2) = -\text{Id}_A = \mathcal{A}(-\text{Id}_{P_\Gamma})
\]

and we can conclude by Theorem 3.29.

This concludes the proof of Theorem 3.11.
4. Application

4.1. Holomorphic Lie bialgebroids. Holomorphic Lie algebroids were studied for various purposes in the literature. See [4, 13, 24, 17, 36] and references cited there for details.

By definition, a holomorphic Lie algebroid is a holomorphic vector bundle $A \to X$, equipped with a holomorphic bundle map $A \xrightarrow{\rho} TX$, called the anchor map, and a structure of sheaf of complex Lie algebras on $A$, such that

(a) the anchor map $\rho$ induces a homomorphism of sheaves of complex Lie algebras from $A$ to $\Theta_X$;

(b) and the Leibniz identity

$$[V, fW] = (\rho(V)f)W + f[V, W]$$

holds for all $V, W \in A(U)$, $f \in \mathcal{O}_X(U)$ and all open subsets $U$ of $X$.

Here $A$ is the sheaf of holomorphic sections of $A \to X$ and $\Theta_X$ denotes the sheaf of holomorphic vector fields on $X$.

By forgetting the complex structure, a holomorphic vector bundle $A \to X$ becomes a real (smooth) vector bundle, and a holomorphic vector bundle map $\rho : A \to TX$ becomes a real (smooth) vector bundle map. Assume that $A \to X$ is a holomorphic vector bundle whose underlying real vector bundle is endowed with a Lie algebroid structure $(A, \rho, [\cdot, \cdot])$ such that, for any open subset $U \subset X$,

(a) $[A(U), A(U)] \subset A(U)$

(b) and the restriction of the Lie bracket $[\cdot, \cdot]$ to $A(U)$ is $\mathbb{C}$-linear.

Then the restriction of $[\cdot, \cdot]$ and $\rho$ from $A_\infty$ to $A$ makes $A$ into a holomorphic Lie algebroid, where $A_\infty$ denotes the sheaf of smooth sections of $A \to X$.

The following proposition states that any holomorphic Lie algebroid can be obtained of such a real Lie algebroid in a unique way.

**Proposition 4.1 ([24]).** Given a structure of holomorphic Lie algebroid on the holomorphic vector bundle $A \to X$ with anchor map $A \xrightarrow{\rho} TX$, there exists a unique structure of real smooth Lie algebroid on the vector bundle $A \to X$ with respect to the same anchor map $\rho$ such that the inclusion of sheaves $A \subset A_\infty$ is a morphism of sheaves of Lie algebras.

By $A_R$, we denote the underlying real Lie algebroid of a holomorphic Lie algebroid $A$. In the sequel, by saying that a real Lie algebroid is a holomorphic Lie algebroid, we mean that it is a holomorphic vector bundle and its Lie bracket on smooth sections induces a $\mathbb{C}$-linear bracket on $A(U)$, for all open subset $U \subset X$.

Assume that $(A \to X, \rho, [\cdot, \cdot])$ is a holomorphic Lie algebroid. Consider the bundle map $j : A \to A$ defining the fiberwise complex structure on $A$. It is simple to see that the Nijenhuis torsion of $j$ vanishes [24]. Hence one can define a new (real) Lie algebroid structure on $A$, denoted by $(A \to X, \rho_j, [\cdot, \cdot)_j)$, where the anchor $\rho_j$ is $\rho \circ j$ and the bracket on $\Gamma(A)$ is given by [22]

$$[V, W]_j = [jV, W] + [V, jW] - j[V, W] = -j[V, jW], \quad \forall V, W \in \Gamma(A). \quad (22)$$

In the sequel, $(A \to X, \rho_j, [\cdot, \cdot]_j)$ will be called the underlying imaginary Lie algebroid and denoted by $A_I$. It is known that

$$j : A_I \to A_R \quad (23)$$

is a Lie algebroid isomorphism [22].
Definition 4.2. A holomorphic Lie bialgebroid is a pair of holomorphic Lie algebroids $(\mathcal{A}, \mathcal{A}^\ast)$ over the base $M$ such that for any open subset $U$ of $M$, the following compatibility holds

$$d_\ast [X, Y] = [d_\ast X, Y] + [X, d_\ast Y].$$

(24)

for all $X, Y \in \mathcal{A}(U)$.

Given a holomorphic vector bundle $A \to M$, by $\mathcal{A}^k$ we denote the sheaf of holomorphic sections of $\wedge^k A \to M$. Then, similar to the smooth case, $A$ is a holomorphic Lie algebroid if and only if there is a sheaf of Gerstenhaber algebras $(\mathcal{A}^\ast, \wedge, [\cdot, \cdot])$ over $M$ [23].

Proposition 4.3. Let $A \to M$ be a holomorphic vector bundle. Then $(\mathcal{A}, \mathcal{A}^\ast)$ is a holomorphic Lie bialgebroid if and only if there is a sheaf of differential Gerstenhaber algebras $(\mathcal{A}^\ast, \wedge, [\cdot, \cdot], d_\ast)$ over $M$.

Proof. The proof is exactly the same as in the smooth case. See [38, 23, 21]. For the completeness, we sketch a proof below.

We note that, if $(\mathcal{A}, \mathcal{A}^\ast)$ is a holomorphic Lie bialgebroid, the holomorphic Lie algebroid structure on $\mathcal{A}$ induces a complex of sheaves $d_\ast : \mathcal{A}^k \to \mathcal{A}^{k+1}$ over $M$. From the compatibility condition (24) and the Leibniz rule, one can prove as in [21], that

$$d_\ast [X, f] = [d_\ast X, f] + [X, d_\ast f].$$

(25)

for all $X \in \mathcal{A}(U)$ and $f \in \mathcal{O}_P(U)$. Therefore by Leibniz rule, one proves that

$$d_\ast [X, Y] = [d_\ast X, Y] + [X, d_\ast Y]$$

(26)

for all $X, Y \in \mathcal{A}^\ast(U)$. Thus $(\mathcal{A}^\ast, \wedge, [\cdot, \cdot], d_\ast)$ is a sheaf of differential Gerstenhaber algebras over $M$. The converse is obvious. □

Let $(\mathcal{A}, \mathcal{A}^\ast)$ be a holomorphic Lie bialgebroid with anchor $\rho$ and $\rho_\ast$, respectively. The following proposition shows that the holomorphic bundle map

$$\pi^\#_M = (\rho \circ \rho_\ast^\#) : T^* M \to TM.$$  

(27)

is skew-symmetric and defines a holomorphic Poisson structure on $M$.

Proposition 4.4. Let $(\mathcal{A}, \mathcal{A}^\ast)$ be a holomorphic Lie bialgebroid with anchor $\rho$ and $\rho_\ast$, respectively. Then

(a) $L_\rho f X = -d_\ast f, X$ for any $f \in \mathcal{O}_P(U)$ and $X \in \mathcal{A}(U)$;
(b) $[d_\ast f, d_\ast g] = d_\ast \{f, g\}$, $\forall f, g \in \mathcal{O}_P(U)$;
(c) $\rho \circ \rho_\ast^\# = -\rho_\ast \circ \rho^\#$ and $\pi_M$ defined by (27) is a holomorphic Poisson structure on $M$.

Proof. The proof is similar to the proofs of Proposition 3.4, Corollary 3.5 and Proposition 3.6 in [28], and is left to the reader. □

4.2. Associated real Lie bialgebroids. By $A_R$ and $A_I$ we denote the underlying real and imaginary Lie algebroids of $A$, respectively. We write $A^*_R := (A^*)_R$ and $A^*_I := (A^*)_I$ for the underlying real and imaginary Lie algebroids of the complex dual $A^\ast$, respectively.

Lemma 4.5. Let $A$ be a holomorphic Lie algebroid. For any $\alpha \in \Gamma(\mathcal{A}^\ast)$ and $f \in \mathcal{C}^\infty(M)$, we have

$$d^I \alpha = -(j^* dR \circ j^*) \alpha,$$

$$d^I f = (j^* dR) f.$$
Proposition 4.8. Let \((A, A^*)\) be a pair of holomorphic Lie algebroids. The following are equivalent:

(a) \((A, A^*)\) is a holomorphic Lie bialgebroid;
(b) \((A_R, A^*_R)\) is a Lie bialgebroid;
(c) \((A_R, A^*_I)\) is a Lie bialgebroid.

Proof. (b)⇒(a) It is clear that if \((A_R, A^*_R)\) is a real Lie bialgebroid then the compatibility condition for \((A, A^*)\) being a holomorphic Lie bialgebroid is automatically satisfied.

(a)⇒(b) Assume that \(U \subset M\) is any open subset, and \(X \in \mathcal{A}(U)\) any holomorphic section. Consider the operator \(C^\infty(U, \mathbb{C}) \to \Gamma(A_R|U \otimes \mathbb{C})\) defined by

\[
\mathcal{L}_X f = d^R_X [X, f] - [d^R_X X, f] - [X, d^R_X f]
\]

for all \(f \in C^\infty(U, \mathbb{C})\). Here \(d^R : \Gamma(A^* \otimes \mathbb{C}) \to \Gamma(A^* \otimes \mathbb{C})\) is the Lie algebroid cohomology differential with the trivial coefficients \(\mathbb{C}\) of the Lie algebroid \(A_R\), and \(X\) is considered as a section of \(A_R|U\). It is simple to check that \(\mathcal{L}_X\) is a derivation, i.e.,

\[
\mathcal{L}_X(fg) = f\mathcal{L}_X g + g\mathcal{L}_X f.
\]

Since \((A, A^*)\) is a holomorphic Lie bialgebroid, it follows that \(\mathcal{L}_X f = 0, \forall f \in \mathcal{O}_P(U)\) according to \([23]\). Here we use the fact that \(d^*_X f = d^R_X f\) for any \(f \in \mathcal{O}_P(U)\). On the other hand, we also have \(\mathcal{L}_X f = 0\) since each term in \([30]\) vanishes \([23]\).

Hence it follows that \(\mathcal{L}_X f = 0\) for all \(f \in C^\infty(U, \mathbb{C})\). Finally, when \(U\) is a contractible open subset of \(M\), \(A|U\) is a trivial bundle, \(\Gamma(A|U \otimes \mathbb{C})\) is spanned by \(\mathcal{A}(U)\) over \(C^\infty(U, \mathbb{C})\). It thus follows that \(d^*_X [X, Y] = [d^*_X X, Y] + [X, d^*_X Y]\) for any \(X, Y \in \Gamma(A|U \otimes \mathbb{C})\). Hence \((A_R, A^*_I)\) is indeed a real Lie bialgebroid.

(a)⇔(c): The equivalence between (a) and (c) can be proved similarly, using the equality \(d^*_X f = i \cdot d^*_X f\) for all \(f \in \mathcal{A}(U)\).

It is well known that Lie bialgebroids are symmetric. Hence \((A_R, A^*_R)\) is a Lie bialgebroid if and only if \((A^*_R, A_R)\) is a Lie bialgebroid. As a consequence, we have

Corollary 4.7. \((A, A^*)\) is a holomorphic Lie bialgebroid iff \((A^*, A)\) is a holomorphic Lie bialgebroid.

Proposition 4.8. Let \((A, A^*)\) be a pair of holomorphic Lie algebroids.

(a) \((A_R, A^*_R)\) is a Lie bialgebroid, if and only if, \((A_I, A^*_I)\) is a Lie bialgebroid,
(b) \((A_R, A^*_I)\) is a Lie bialgebroid, if and only if, \((A_I, A^*_R)\) is a Lie bialgebroid.
Proof. Recall from (22) that the Lie bracket on $A_I$ is given by $[V, W]_j = -j[V, jW]$ for all $V, W \in \Gamma(A_I)$.

(a) Assume that $(A_R, A^\ast_R)$ is a Lie bialgebroid. Then we have for all $V, W \in \Gamma(A_R)$:
\[
d^I_d[V, W]_j = (j \circ d^R_{\ast})((-j[V, jW])) = (j \circ d^R_{\ast})([jV, jW])
\]
\[
= j \left( [d^R(jV), jW] + [jV, d^R(jW)] \right)
\]
\[
= -j \left( [j(d^R_V), jW] + [jV, j(d^R_W)] \right)
\]
\[
= [d^I_dV, W]_j + [V, d^I_dW]_j,
\]
which shows that $(A_I, A^\ast_I)$ is a Lie bialgebroid. The converse can be shown in the same manner.

(b) Assume that $(A_I, A^\ast_R)$ is a Lie bialgebroid. Then, for all $V, W \in \Gamma(A_R)$, we get
\[
d^I_d[V, W] = -(j \circ d^R_{\ast})([V, W]) = -(j \circ d^R_{\ast})(-j[V, jW])
\]
\[
= -j(d^R_V)(jW)_j - j[V, d^R(jW)]_j
\]
\[
= -j[V, j(d^R_W)]_j + [V, j(d^R_W)]_j
\]
\[
= [d^I_dV, W] + [V, d^I_dW].
\]
Hence $(A_R, A^\ast_I)$ is a Lie bialgebroid. The converse can be proved similarly.

**Proposition 4.9.** Let $(A, A^\ast)$ be a pair of holomorphic Lie algebroids. Then $(A_R, A^\ast_I)$ is a Lie bialgebroid if and only if $A_R$ endowed with the map $J_{A_R} : P_{A_R} \to P_{A_R}$,
\[
J_{A_R} = \begin{pmatrix}
J_{A_R} & \pi^I_{A^\ast_I} \\
0 & -J^R_{A_R}
\end{pmatrix}
\]
(31)
is a Glanon Lie algebroid, where $\pi^I_{A^\ast_I}$ is the Poisson structure on $A_R$ that is induced by the Lie algebroid structure on $A^\ast_I$.

Proof. Assume first that $(A_R, J_{A_R})$ is a Glanon Lie algebroid. Then the map $\pi^I_{A^\ast_I}$ is a morphism of Lie algebroids $T^*A \to TA$, and it follows that $(A_R, A^\ast_I)$ is a Lie bialgebroid.

Conversely, if $(A_R, A^\ast_I)$ is a Lie bialgebroid, the map $\pi^I_{A^\ast_I}$ is a morphism of Lie algebroids $T^*A \to TA$. According to Proposition 3.12 in [25] up to a scalar, $\pi_A(\cdot, \cdot) = \pi_{A^\ast_I}(\cdot, \cdot) + i\pi_{A^\ast_I}(J^R_{A_R}, \cdot)$ is the holomorphic Lie Poisson structure on $A$ induced by the holomorphic Lie algebroid $A^\ast$. By Theorem 2.7 in [24], $J_{A_R}$ is hence a generalized complex structure. Since $A$ is a holomorphic Lie algebroid, the map $J_{A_R} : TA_R \to TA_R$ is a morphism of Lie algebroids according to [25].

We summarize our results in the following:

**Theorem 4.10.** Let $(A, A^\ast)$ be a pair of holomorphic Lie algebroids. The following assertions are all equivalent:

(a) $(A, A^\ast)$ is a holomorphic Lie bialgebroid;
(b) $(A_R, A^\ast_R)$ is a Lie bialgebroid;
(c) $(A_R, A^\ast_I)$ is a Lie bialgebroid;
(d) $(A_I, A^\ast_R)$ is a Lie bialgebroid;
(e) $(A_I, A^\ast_I)$ is a Lie bialgebroid;
(f) the Lie algebroid $A_R$ endowed with the map $J_{A_R} : P_{A_R} \to P_{A_R}$,

$$J_{A_R} = \begin{pmatrix} J_{A_R} & \pi_I^R \\ 0 & -J_{A_R} \end{pmatrix}$$

is a Glanon Lie algebroid.

Example 4.11. Consider a holomorphic Poisson manifold $(X, \pi)$, where $\pi = \pi_R + i\pi_I \in \Gamma(\wedge^2 T^{1,0} X)$ is a holomorphic Poisson tensor. Let $A = TX$ and $A' = (T^* X)_\pi$, the cotangent Lie algebroid associated to $\pi$. Then $(A, A')$ is a holomorphic Lie bialgebroid. In this case, $A_R = TX$, $A_I = (TX)_J$, $A_R^* = (TX)^*_4 \pi_R$ and $A_I^* = (TX)^*_4 \pi_I$.

4.3. Holomorphic Poisson groupoids.

Definition 4.12. A holomorphic Poisson groupoid is a holomorphic Lie groupoid equipped with a holomorphic Poisson structure $\pi_T \in \Gamma(\wedge^2 T^{1,0} \Gamma)$ such that the graph of the multiplication $\Lambda \subset \Gamma \times \Gamma \times \Gamma$ is a cosotropic submanifold, where $\Lambda$ stands for $\Gamma$ equipped with the opposite Poisson structure.

Many properties of (smooth) Poisson groupoids have a straightforward generalization in this holomorphic setting. We refer to [35, 28, 37] for details. In particular, if $\Gamma \equiv M$ is a holomorphic Poisson groupoid, then $M$ is naturally a holomorphic Poisson manifold. More precisely, there exists a unique holomorphic Poisson structure on $M$ such that the source map $s : \Gamma \to M$ is a holomorphic Poisson map, while the target map is then an anti-Poisson map.

As a consequence, $M$ naturally inherits a holomorphic Poisson structure.

Theorem 4.13 ([37, 23]). Let $\Gamma \equiv M$ be a holomorphic Lie groupoid, and $\pi_T$ a holomorphic Poisson tensor on $\Gamma$. Then $(\Gamma, \pi_T)$ is a holomorphic Poisson groupoid if and only if $\pi^\#: T^* \Gamma \to T \Gamma$ is a morphism of holomorphic Lie groupoids over some map $\alpha_* : A^* \to TM$ (which is then the anchor of the Lie algebroid $A^*$).

For any open subset $U \subset M$ and $X \in \mathcal{A}^k(U)$, it follows as in [37, Theorem 3.1] that $[X^*, \pi_T]$ is a right-invariant holomorphic $(k + 1)$-vector field on $\Gamma_U$. Hence it defines an element, denoted $d_{X^*}X$, in $\mathcal{A}^{k+1}(U)$, i.e.

$$(d_{X^*}X) = [X^*, \pi_T].$$

As in [38], one proves that $(\mathcal{A}^*, \wedge, [\cdot, \cdot], d_{X^*})$ is a sheaf of differential Gerstenhaber algebras over $M$. This proves the following proposition.

Proposition 4.14. A holomorphic Poisson groupoid naturally induces a holomorphic Lie bialgebroid.

Proposition 4.15. Let $(\Gamma \equiv M, \pi)$ be a holomorphic Poisson groupoid with holomorphic Lie bialgebroid $(A, A^*)$, where $\pi_T = \pi_R + i\pi_I \in \Gamma(\wedge^2 T^{1,0} \Gamma)$. Then

(a) both $(\Gamma \equiv M, \pi_R)$ and $(\Gamma \equiv M, \pi_I)$ are Poisson groupoids;

(b) the Lie bialgebroids of $(\Gamma \equiv M, \pi_R)$ and $(\Gamma \equiv M, \pi_I)$ are $(A_R, A^*_1/4\pi_R)$ and $(A_R, A^*_1/4\pi_I)$ respectively.

Proof. (a) If $\pi$ is a multiplicative holomorphic Poisson structure on $\Gamma \equiv M$, then its real and imaginary parts are also multiplicative. It is shown in [24] that both $\pi_R$ and $\pi_I$ are Poisson bivector fields.

We write $A^*_1/4\pi_R$ (respectively $A^*_1/4\pi_I$) for the Lie algebroid $(A^*_R, 1/4[\cdot, \cdot], \rho_{A^*_R})$ (respectively $(A^*_I, 1/4[\cdot, \cdot], \rho_{A^*_I})$).

---

\footnote{We write $A^*_1/4\pi_R$ (respectively $A^*_1/4\pi_I$) for the Lie algebroid $(A^*_R, 1/4[\cdot, \cdot], \rho_{A^*_R})$ (respectively $(A^*_I, 1/4[\cdot, \cdot], \rho_{A^*_I})$).}
Proof. (a) By Theorem 2.7 in [23], \(J_{\pi}(T^*\Gamma)\pi_\pi = (T^*\Gamma)_{A\pi I}\) and \(\text{Re}((T^*\Gamma)_{A\pi I}) = (T^*\Gamma)_{A\pi I}\). The Lie algebra structure on \((T^*\Gamma)_{A\pi I}\) restricts to the holomorphic Lie algebra structure on \(A^*\) as follows; the map \(\rho : A^* \to TM\) is just the restriction of \(\pi^A\) to \(A^* = TM^0\) seen as a subbundle of \(T^*_M\Gamma\), and the bracket on \(T^*\Gamma\) restricts to a bracket on \(A^*\). In the same manner, the Lie groupoid \((\Gamma\to M, \pi^A_{\pi I})\) induces a Lie algebra structure on \(A_R^*\) that is the restriction of the Lie algebra structure on \((T^*\Gamma)_{\pi R}\). Hence, we can conclude easily.

\[\square\]

**Proposition 4.16.** Let \((\Gamma\to M, \pi)\) be a holomorphic Poisson groupoid, with the almost complex structure \(J_{\pi} : \pi \to \pi\) given by the matrix
\[
\begin{pmatrix}
J_{\pi} & \pi^\ast_{\pi I} \\
0 & -J^\ast_{\pi I}
\end{pmatrix}
\]
defines a Glanon groupoid structure on \(\Gamma\).

(a) The Lie algebroid morphism \(\mathfrak{X}(J_{\pi}) : \mathfrak{P}_A \to \mathfrak{P}_A\) is given by
\[
\begin{pmatrix}
J^\ast_{\pi A} & \pi^\ast_{A^*_I/A} \\
0 & -J^\ast_{\pi A}
\end{pmatrix}
\]
where \(\pi^\ast_{A^*_I/A}\) is the linear Poisson structure defined on \(A_R^*\) by the Lie algebroid \(A^*_I/A\).

Proof. (a) By Theorem 2.7 in [23], \(J_{\pi}\) is a generalized complex structure. Since \((\Gamma\to M, J_{\pi})\) is a holomorphic Lie groupoid and \((\Gamma\to M, \pi)\) is a Poisson groupoid, the maps \(J_{\pi} : \pi \to \pi\), its dual \(J_{\pi}^* : T^*\Gamma \to T^*\Gamma\) and \(\pi_{\pi I}^* : T^*\Gamma \to \pi \to \pi\) are all multiplicative.

(b) It is shown in [25] that \(\sigma_{\pi} \circ A(J_{\pi}) \circ \sigma_{\pi}^{-1} = J^\ast_{\pi A}\), and in [29] that \(\sigma_{\pi} \circ A(\pi^\ast_{\pi I}) \circ \sigma_{\pi}^{-1} = \pi^\ast_{\pi A^*_I/A}\), since \((A_R, A^*_I/A)\) is the Lie bialgebroid of \((\Gamma\to M, \pi)\).

\[\square\]

A holomorphic Lie bialgebroid \((A, A^*)\) is said to be integrable if it is isomorphic to the induced holomorphic Lie bialgebroid of a holomorphic Poisson groupoid.

As a consequence of Theorem 3.11 and Proposition 4.16, we obtain the following main result of this section.

**Theorem 4.17.** Given a holomorphic Lie bialgebroid \((A, A^*)\), if the underlying real Lie algebroid \(A_R\) integrates to a \(s\)-connected and \(s\)-simply connected Lie groupoid \(\Gamma\), then \(\Gamma\) is a holomorphic Poisson groupoid.

**Remark 4.18.** This is proved in [25] in the special case where \((A, A^*)\) is the holomorphic Lie bialgebroid \(((T^*M)_{\pi}, TM)\) induced from a holomorphic Poisson manifold \((M, \pi)\).

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