Optimization of Neutrino Oscillation Parameters Using Differential Evolution

Ghulam Mustafa,† Faisal Akram,‡ and Bilal Masud§

Centre for High Energy Physics, University of the Punjab, Lahore (54590), Pakistan

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Abstract We show how the traditional grid based method for finding neutrino oscillation parameters \( \Delta m^2 \) and \( \tan^2 \theta \) can be combined with an optimization technique, Differential Evolution (DE), to get a significant decrease in computer processing time required to obtain minimal chi-square \( \chi^2 \) in different regions of the parameter space. We demonstrate efficiency for the two-neutrinos case. For this, the \( \chi^2 \) function for neutrino oscillations is evaluated for grids with different density of points in standard allowed regions of the parameter space of \( \Delta m^2 \) and \( \tan^2 \theta \) using experimental and theoretical total event rates of chlorine (Homestake), Gallex+GNO, SAGE, Superkamiokande, and SNO detectors. We find that using DE in combination with the grid based method with small density of points can produce the results comparable with the one obtained using high density grid, in much lesser computation time.

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1 Introduction

The flux of solar neutrino was first measured by Raymond Davis Junior and John N. Bahcall at Homestake in late 1960s and a deficit was detected between theory i.e., Standard Solar Model (SSM) and experiment. This deficit is known as Solar Neutrino Problem.\(^1\) Several theoretical attempts have been made to explain this deficit.\(^2\)

One of these is neutrino oscillations, the change of electron neutrinos to another neutrino flavour during their travel from a source point in the sun to the detector at the earth surface.\(^3\) There was no experimental proof for the neutrino oscillations until 2002 when Sudbury Neutrino Observatory (SNO) provided strong evidence for neutrino oscillations.\(^4\) The exact amount of depletion, which may be caused by the neutrino oscillations, however, depends upon the neutrinos’ mass-squared difference \( \Delta m^2 \equiv m_3^2 - m_1^2 \) and mixing angle \( \theta \), in the interval \([0, \pi/2]\), defining the relation between flavor eigen-states and mass eigen-states of the neutrinos.

The data from different neutrino experiments have provided the base to explore the field of neutrino physics. In global analysis of solar neutrino data, theoretically expected event rates, with oscillations, at different detector locations are calculated and combined with experimental event rates statistically through the chi-square \( \chi^2 \) function, as defined below by Eq. (1), for a grid of values of the parameters \( \Delta m^2 \) and \( \tan^2 \theta \). The values of these parameters with minimum chi-square in different regions of the parameter space suggest different oscillation solutions. The names of these solutions, found in the literature, along with specification of the respective region in the parameter space are: Small Mixing Angle (SMA): \( 10^{-4} \leq \tan^2 \theta \leq 3 \times 10^{-2}, 3 \times 10^{-7} \text{eV}^2 \leq \Delta m^2 \leq 10^{-4} \text{eV}^2 \), Large Mixing Angle (LMA): \( 3 \times 10^{-2} \leq \tan^2 \theta \leq 2, 2 \times 10^{-6} \text{eV}^2 \leq \Delta m^2 \leq 10^{-3} \text{eV}^2 \), Low Probability Low Mass (LOW): \( 3 \times 10^{-2} \leq \tan^2 \theta \leq 2, 10^{-8} \text{eV}^2 \leq \Delta m^2 \leq 2 \times 10^{-6} \text{eV}^2 \) and Vacuum Oscillation (VO): \( 0.1 \leq \tan^2 \theta \leq 1, 10^{-11} \text{eV}^2 \leq \Delta m^2 \leq 10^{-8} \text{eV}^2 \).\(^5\)

Extensive work has been done on the global analysis of solar neutrino data\(^6\)\(^-\)^\(^16\) and now is the era of precision measurement of the neutrino oscillation parameters.\(^17\)\(^-\)^\(^18\)

Traditionally, the whole parameter space \( (10^{-13} \text{eV}^2 \leq \Delta m^2 \leq 10^{-3} \text{eV}^2, 10^{-4} \leq \tan^2 \theta \leq 10) \) is divided into a grid of points by varying the logarithm of each parameter uniformly. The density of points in the grid is usually taken as \( 50 \times 50 \) point per decade (ppd), in logarithmic scale of the parameters \( \Delta m^2 \) and \( \tan^2 \theta \), which corresponds to \( 500 \times 250 \) points in the whole parameter space. The chi-square values are calculated for each point in the parameter space either by using \(^8\)B flux constrained by the SSM in our case or by using unconstrained \(^8\)B flux\(^10\) where it is varied about the value predicted by the Standard Solar Model. The global minimum chi-square value \( \chi^2_{\text{min}} \) is found and 100\% C.L. (Confidence Level) contours are drawn in the \( \tan^2 \theta - \Delta m^2 \) plane by joining points with \( \chi^2 = \chi^2_{\text{min}} + \Delta \chi^2 \) for different confidence levels. From the chi-square distribution one can easily find that \( \Delta \chi^2 = 2.28, 4.61, 5.99, 9.21, 11.83 \) for 68\%, 90\%, 95\%, 99\% and 99.73\% C.L. (with \( \beta = 0.68, 0.90, 0.95, 0.99 \) and 0.9973 respectively) for two degrees of freedom (d.o.f.). Minimum chi-square values are found in all the regions and the goodness-of-fit (g.o.f) corresponding to each of the minimum chi-square is calculated. To find each g.o.f. the chi-square distribution is used and 100(1 – \( \beta \))\% confidence level, corresponding to the minimum chi-square in the region and the d.o.f. of the analysis, is calculated.\(^5\)\(^,\)^\(^10\) It is

\(^*\)E-mail: g_mustafa61@yahoo.com
\(^\dagger\)E-mail: faisal.chep@pu.edu.pk
\(^\S\)E-mail: bilalmasud.chep@pu.edu.pk

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observed that the results of this analysis depend upon the choice of density of points in the parameter space. Increasing the grid density generally produces the results with lower chi-square and better g.o.f., but also requires high computing power. This limitation motivates us to explore other optimization techniques more efficient than simple grid based method. One such technique is the Differential Evolution (DE). It is a population based stochastic algorithm for optimization of real valued non-linear non-differentiable objective functions that has become very popular during the last decade. We combine DE technique with the grid based method for the optimization of $\Delta m^2$ and $\tan^2 \theta$. The result of this combined analysis is compared with the simple grid based method with different choices of grid density. We have used the terms “small density”, “large density” and “very large density” for different density grids. Our results show that DE in combination with grid based method with small density of points can produce the results of global analysis of solar neutrino data obtained with very large density of points. This saves a lot of computing time.

The paper is organized as follows: In Sec. 2, we define the chi-square ($\chi^2$) function for the solar neutrino oscillations that is used as objective function for DE algorithm as well as for the traditional method. In Sec. 3, we describe resulting values of the parameters with minimum chi-square along with respective g.o.f. in standard allowed regions for different density grids. In Sec. 4 we give our respective results obtained using DE. Some concluding remarks are made in the Sec. 5. Algorithm of DE is described in Appendix A.

2 Chi-Square ($\chi^2$) Function Definition

In our $\chi^2$ analysis, we used the updated data of total event rates of different solar neutrino experiments. We followed the $\chi^2$ definition of Ref. [19] and included chlorine (Homestake),[20] weighted average of Gallex and GNO[21] SAGE,[22] Superkamiokande,[23] SNO CC (Charged Current) and SNO NC (Neutral Current)[24] total rates. Expression for the $\chi^2$ function is given as:

$$\chi^2_{\text{Rates}} = \sum_{j_1,j_2=1,6} (R^\text{th}_{j_1,j_2} - R^\text{exp}_{j_1,j_2})^2 (\frac{1}{V_{j_1,j_2}})^{-2} (R^\text{th}_{j_2,j_2} - R^\text{exp}_{j_2,j_2})^2,$$

where $R^\text{th}_{j_1,j_2}$ is the theoretically calculated event rate with oscillations at detector $j$ and $R^\text{exp}_{j_1,j_2}$ is the measured rate. For chlorine, Gallex+GNO and SAGE experiments $R^\text{th}_{j_1,j_2}$ and $R^\text{exp}_{j_1,j_2}$ are in the units of SNU (1 SNU=10$^{-36}$ captures/atom/sec) and for Superkamiokande, SNO CC and SNO NC, these are used as ratio to SSM Eq. (8) below. $V_{j_1,j_2}$ is the error matrix that contains experimental (systematic and statistical) errors and theoretical uncertainties that affect solar neutrino fluxes and interaction cross sections. For calculating the error matrix $V_{j_1,j_2}$ we followed Ref. [19] and for updated uncertainties we used Ref. [25]. For calculating the theoretical event rates, using Eqs. (4)–(7) below, we first found the time average survival probabilities, over the whole year, of electron neutrino $\langle P^k_{ee}(E_{\nu}) \rangle$ ($E_{\nu}$ is the neutrino energy in MeV) at the detector locations for the $k^{\text{th}}$ neutrino source and for a grid of values of $\Delta m^2/E_{\nu}$ and $\tan^2 \theta$ following the prescriptions described in Ref. [10]. For uniform grid interval distribution we used the parameters $\Delta m^2/E_{\nu}$ and $\tan^2 \theta$ as exponential functions of the variables $x_1$ and $x_2$ as:

$$\frac{\Delta m^2}{E_{\nu}} = 10^{0.1x_1 - 13},$$

$$\tan^2 \theta = 10^{-2(0.025x_2)},$$

so that discrete values of $x_1$ and $x_2$ from 0 to 100 cover the entire $\tan^2 \theta - \Delta m^2$ parameter space. We used the expression for the average expected event rate in the presence of oscillation in case of Chlorine and Gallium detectors given as:

$$R^\text{th}_{j} = \sum_{k=1}^{8} \phi_k \int_{E^\text{th}_k}^{E^\max} dE_{\nu} \lambda_k(E_{\nu}) \{\sigma_{e,j}(E_{\nu})(P^k_{ee}(E_{\nu}))\}.$$

Here $E^\text{th}_k$ is the process threshold for the $j$-th detector ($j=1,2,3$ for Homestake, Gallex+GNO and SAGE respectively). The values of energy threshold $E^\text{th}_j$ for Cl, Ga detectors are 0.814, 0.233 MeV respectively,[26] $\phi_k$ are the total neutrino fluxes taken from BS05(OP).[27] Different sources of solar neutrino flux are: pp, pep, hep, $^7$Be, $^8$B, $^{13}$N, $^{15}$O and $^{17}$F. For Gallium detector all fluxes contribute whereas for Chlorine detector all fluxes except pp flux contribute. $\lambda_k(E_{\nu})$ are normalized solar neutrino energy spectra for different neutrino sources from the sun, taken from Refs. [28–29], and $\sigma_{e,j}$ is the interaction cross section for $\nu_e$ in the $j$-th detector. Numerical data of energy dependent neutrino cross sections for chlorine and gallium experiments is available from Ref. [28].

Superkamiokande and SNO detectors are sensitive for higher energies, so $\phi_{e}$ are the total $^8$B and hep fluxes for these detectors respectively. Expression of the average expected event rate with oscillations for elastic scattering at SK detector is as below:

$$N^\text{th}_{\text{SK}} = \sum_{k=1}^{8} \phi_k \int_{0}^{E^\max} dE_{\nu} \lambda_k(E_{\nu}) \{\sigma_{e}(E_{\nu})(P^k_{ee}(E_{\nu})) + \sigma_{\mu}(E_{\nu})(1 - \langle P^k_{ee}(E_{\nu})\rangle)\}.$$

Here $\sigma_{e}$ and $\sigma_{\mu}$ are elastic scattering cross sections for electron and muon neutrinos that we took from Ref. [30].

For the SNO CC reaction, $\nu_e d \rightarrow e^- pp$, we calculated event rate using the expression:

$$N^\text{th}_{\text{CC}} = \sum_{k=1}^{8} \phi_k \int_{0}^{E^\max} dE_{\nu} \lambda_k(E_{\nu}) \sigma_{\text{CC}}(E_{\nu}) \langle P^k_{ee}(E_{\nu}) \rangle.$$

Here $\sigma_{\text{CC}}$ is $\nu d$ CC cross section of which calculational method and updated numerical results are given in Refs. [31] and [32] respectively.

Expression for the SNO NC reaction, $\nu_e d \rightarrow \nu_e pn$ ($x = e, \mu, \tau$), event rate is given as:

$$N^\text{th}_{\text{NC}} = \sum_{k=1,2} \phi_k \int_{0}^{E^\max} dE_{\nu} \lambda_k(E_{\nu}) \times \sigma_{\text{NC}}(E_{\nu}) \langle (P^k_{ee}(E_{\nu})) + (P^k_{ee}(E_{\nu}))\rangle.$$
Here $\sigma_{NC}$ is $\nu d$ NC cross section and $\langle P^k_{ee}(E_\nu) \rangle$ is the time average probability of oscillation of $\nu_e$ into any other active neutrino. We used updated version of CC and NC cross section data from the website given in Ref. [32]. In case of oscillation of the $\nu_e$ into active neutrino only, $\langle P^k_{ee}(E_\nu) \rangle + \langle P^k_{ee}(E_\nu) \rangle = 1$ and $N^\text{th}_{NC}$ is a constant.

For superkamiokande$^{[23]}$ and SNO$^{[24]}$ experiments, the event rates come in the unit of $10^6 \text{ cm}^{-2} \cdot \text{s}^{-1}$. We converted these rates into ratios to SSM predicted rate. We also calculated theoretical event rates as ratios to SSM predicted rate in order to cancel out all energy independent efficiencies and normalizations.$^{[9]}$

$$R^\text{th}_{ij} = \frac{N^\text{th}_{j}}{N^\text{SSM}_j}. \quad (8)$$

Here $N^\text{SSM}_j$ ($j = 4, 5, 6$ for SK, SNO CC, and SNO NC respectively) is the predicted number of events assuming no oscillations. We used the SSM BS05(OP)$^{[27]}$ in our calculations. Theoretical event rates, so calculated, were used in Eq. (1) to calculate the chi-square function for different points in the $\tan^2 \theta - \Delta m^2$ parameter space.

3 Analysis from Different Density Grids

Table 1 shows oscillation parameters with minimum $\chi^2$ and respective g.o.f. values, in standard allowed regions described in Sec. 1, calculated for different density grids in the parameter space. The density of points in grid-based searches is usually denoted by the number of points per decade (ppd), i.e., the interval between $x$ and $10x$ in logarithmic scale, for each parameter. Different density grids we selected are: $10 \times 20$ ppd (small density), $50 \times 50$ ppd (large density) and $100 \times 100$ ppd (very large density). Calculation of g.o.f. is described in the introduction section. We used chi-square function definition of Sec. 2. Small density grid for $\Delta m^2$ and $\tan^2 \theta$ was obtained by simply varying $x_1$ and $x_2$ in Eqs. (2) and (3) with step size equal to 1. Other grids given in the Table 1 were obtained by varying $x_1$ and $x_2$ with appropriate step size.

We see from Table 1 that increasing density of the grid gives the parameters with smaller $\chi^2$ and better goodness-of-fit. The effect is more prominent in LOW and SMA regions. Selecting a finer grid with larger number of points in the parameter space, of course, will give better results but limitations of the computation time restrict us$^{[5]}$ to a grid with limited density of points. But we point out that even without increasing the number of points in the grid we can get lower $\chi^2$ and better g.o.f. by fine tuning of the oscillation parameters using DE. We describe what we mean by fine tuning and report our results obtained this way in the next section.

4 Optimization of the Chi-square Function Using DE

The steps of the DE algorithm we used, for the optimization of chi-square function, are described in detail in Appendix A. We wrote the subroutine of the chi-square function, denoted by $\chi^2$, following its definition described in Sec. 2, that depends on $x_1$ and $x_2$ and used it as objective function of the DE algorithm. We used the survival probabilities $\langle P^k_{ee}(E_\nu) \rangle$ already calculated for the small density grid and interpolated them to the continuous values of $x_1$ and $x_2$ to calculate event rates and chi-square function in DE algorithm. We used cubic polynomial fit for the interpolation purpose to fit the data. The results of the interpolated probabilities were verified to be equivalent to the originally calculated probabilities with reasonable accuracy. For each of the four regions we used the point with minimum chi-square of the small density grid, given in Table 1, as the starting point and explored the space around it for the fine tuning using DE. That is, in each region we searched for points with smaller $\chi^2$ values and better g.o.f. of the oscillation parameters.

### Table 1

$\chi^2_{\text{min}}$ values for 4 d.o.f. (6(rates)-2(parameters: $\tan^2 \theta, \Delta m^2$)) and respective goodness-of-fit (g.o.f.) values of the oscillation parameters $\Delta m^2$ and $\tan^2 \theta$, in standard allowed regions, for different grid sizes in points per decade (ppd) in logarithmic scale of the parameters.

| Solution | $\Delta m^2$ (points per decade) | $\tan^2 \theta$ | $\Delta m^2$/eV$^2$ | $\tan^2 \theta$ | $\chi^2_{\text{min}}$ | g.o.f. |
|----------|---------------------------------|----------------|------------------|----------------|-----------------|-------|
|          | $\Delta m^2$                   |                |                  |                |                 |       |
|          | $\tan^2 \theta$                |                |                  |                |                 |       |
| LMA      | 10                              | 20             | 2.512 \cdot 10^{-5} | 3.981 \cdot 10^{-1} | 0.808           | 93.74% |
|          | 50                              | 50             | 2.512 \cdot 10^{-5} | 3.981 \cdot 10^{-1} | 0.808           | 93.74% |
|          | 100                             | 100            | 2.455 \cdot 10^{-5} | 3.981 \cdot 10^{-1} | 0.805           | 93.78% |
| VAC      | 10                              | 20             | 6.31 \cdot 10^{-11} | 1.00 \cdot 10^0 | 1.268           | 86.68% |
|          | 50                              | 50             | 6.61 \cdot 10^{-11} | 1.00 \cdot 10^0 | 1.259           | 86.82% |
|          | 100                             | 100            | 6.61 \cdot 10^{-11} | 1.00 \cdot 10^0 | 1.259           | 86.82% |
| LOW      | 10                              | 20             | 1.00 \cdot 10^{-8} | 1.122 \cdot 10^0 | 4.09            | 39.46% |
|          | 50                              | 50             | 3.02 \cdot 10^{-8} | 1.059 \cdot 10^0 | 4.04            | 40.12% |
|          | 100                             | 100            | 3.09 \cdot 10^{-8} | 1.029 \cdot 10^0 | 3.968           | 41.04% |
| SMA      | 10                              | 20             | 6.31 \cdot 10^{-6} | 1.585 \cdot 10^{-3} | 7.78            | 10.01% |
|          | 50                              | 50             | 5.74 \cdot 10^{-6} | 1.679 \cdot 10^{-3} | 1.949           | 74.52% |
|          | 100                             | 100            | 5.46 \cdot 10^{-6} | 1.728 \cdot 10^{-3} | 1.86            | 76.24% |
Fig. 1 Track of the DE algorithm for optima in different regions using the strategy DE/best/2/bin are shown by the vector arrows. The symbols □, • and ◇ show the best points of different density grids in units of points per decade (ppd), in logarithmic scale of the parameters $\Delta m^2$ and $\tan^2 \theta$, shown in Table 1 and triangle symbol △ shows the best point after fine tuning using DE.

In our analysis, the values of DE control variables $F$ and CR were taken as 0.4 and 0.9 respectively for the LMA and VAC regions. For the LOW and SMA regions $F$ and CR were both taken as 0.3 for better convergence. Maximum number of iterations were taken to be 50 for all regions. We took the best point in the small density grid in Table 1 as the first member of the population in the first iteration, as described above, and used the strategy DE/best/2/bin for DE mutation in all the remaining iterations or generations. The steps of DE algorithm, described in Appendix A, were repeated for the number of iterations specified.

Table 2 and Fig. 1 show the results in different regions of parameter space during and after optimization of the parameters using DE. The values of chi-square function after each accepted mutation are given in column 5, whereas, columns 3 and 4 contain the corresponding values of $\Delta m^2$ and $\tan^2 \theta$. Table 2 also shows that DE causes a small decrease in the value of $\chi^2_{\text{min}}$ and accordingly a small change in the values of $\Delta m^2$ and $\tan^2 \theta$ for LMA and VAC solutions. This result is consistent with the earlier observation, in grid based method as shown in Table 1, that LMA and VAC solutions are not significantly affected by the increase in the density of points. It is interesting to note that in case of LMA solution a slight decrease in $\chi^2_{\text{min}}$ occurs only when the density of points is taken to be $100 \times 100$ ppd, whereas DE obtains the same result within only 50 iterations, with 20 points randomly selected in each iteration. Though the improvement in this case is very modest but it requires much lesser computing time when DE is used instead of the grid based method alone with very large density of points. In case of LOW solution, though DE produces a small decrease in $\chi^2_{\text{min}}$, a significant change in $\Delta m^2$ can be noted merely after a few iterations. This result is again consistent with the variations in these parameters when density of point is increased in the grid based method. In case of the SMA solution, we note a large decrease in $\chi^2_{\text{min}}$ and accordingly some change in
\( \Delta m^2 \) and \( \tan^2 \theta \). Table 2 shows that after merely 7 iterations DE produces the value of \( \Delta m^2 \) which we obtained with large density grid and after 50 iteration the result of DE were comparable with those obtained using very large density grid. This comparison shows that combining the DE with the grid based method using a small density grid can produce the results of global analysis of solar neutrino data obtained by using a large or even very large density grid in much lesser computing time. The same results are also shown by Fig. 1 in which progress in values of \( \Delta m^2 \) and \( \tan^2 \theta \) due to DE is represented by vectors. For comparison, the results of grid based method using different density of points are also shown in the same figure by different symbols.

| Solution | Iterations | \( \Delta m^2 /\text{eV}^2 \) | \( \tan^2 \theta \) | \( \chi^2_{\text{min}} \) | \( \chi^2_{\text{best}} \) | g.o.f. |
|----------|------------|----------------------------|-----------------|----------------|----------------|------|
| 1-7      | 2.511 81 \cdot 10^{-5} | 3.981 07 \cdot 10^{-1} | 0.808 314 | 0.807 316 | 93.77% |
| 10-13    | 2.499 27 \cdot 10^{-5} | 3.976 84 \cdot 10^{-1} | 0.804 711 | 0.804 564 | 86.82% |
| LMA      | 2.493 07 \cdot 10^{-5} | 3.970 28 \cdot 10^{-1} | 0.804 289 | 0.804 424 | 86.82% |
| 22-29    | 2.432 64 \cdot 10^{-5} | 3.980 97 \cdot 10^{-1} | 0.803 192 | 0.803 192 | 93.77% |
| 30-50    | 2.450 84 \cdot 10^{-5} | 3.975 1 \cdot 10^{-1} | 0.802 953 | 0.802 953 | 86.82% |
| SMA      | 6.309 57 \cdot 10^{-11} | 1.0   | 1.267 79 | 1.260 239 | 1.259 39 | 86.82% |
| 6-7      | 5.897 98 \cdot 10^{-6} | 1.899 28 \cdot 10^{-3} | 1.295 77 | 1.295 77 | 1.295 77 | 1.295 77 | 86.82% |
| 8-15     | 5.762 76 \cdot 10^{-6} | 1.906 13 \cdot 10^{-3} | 2.803 98 | 2.803 98 | 2.803 98 | 2.803 98 | 86.82% |
| 16-50    | 5.455 14 \cdot 10^{-6} | 1.7452 \cdot 10^{-3} | 1.821 26 | 1.821 26 | 1.821 26 | 1.821 26 | 76.89% |
| LOW      | 1.0 \cdot 10^{-8} | 1.122 98 | 4.188 97 | 4.188 97 | 4.188 97 | 4.188 97 | 39.46% |
| 2-4      | 2.378 07 \cdot 10^{-8} | 1.031 98 | 3.983 39 | 3.983 39 | 3.983 39 | 3.983 39 | 39.46% |
| 10       | 3.3042 \cdot 10^{-8} | 1.030 69 | 3.976 05 | 3.976 05 | 3.976 05 | 3.976 05 | 39.46% |
| 11       | 2.807 96 \cdot 10^{-8} | 1.027 41 | 3.972 87 | 3.972 87 | 3.972 87 | 3.972 87 | 39.46% |
| 12-20    | 3.173 57 \cdot 10^{-8} | 1.027 41 | 3.962 67 | 3.962 67 | 3.962 67 | 3.962 67 | 39.46% |
| 21-50    | 3.145 43 \cdot 10^{-8} | 1.027 23 | 3.961 25 | 3.961 25 | 3.961 25 | 3.961 25 | 41.12% |

5 Conclusions

Fine tuning of the neutrino oscillation parameters using Differential Evolution has been introduced as a solution to the impasse faced due to computation limitations of the larger density grid alternative. We find that the results of combining DE with the grid based method using small density grid, described in Sec. 4, are comparable with those obtained through grid based method alone with large and very large density of points, after 10 and 50 iterations of DE algorithm, with 20 points selected in each iteration respectively. Thus DE approaches to the optima in much lesser computing time as compared to the grid based method alone. So we conclude that combining DE with grid based method, a small density grid can produce the results of the analysis of solar neutrino data with very large density of points in the parameter space. As we go from large to very large density grid, the \( \chi^2 \) value changes in some cases, indicating that the saturation is not achieved. The implied need for going beyond even the presently accessible very large density can be satisfied if we combine the very large density with DE while using only a manageable computing time. In the present work we have not included the energy bin data of SK and SNO experiments and the reactor neutrino data for the sake of simplicity to check the implementation and efficiency of the DE technique. However, this study shows that the
introduced technique represents a useful methodological advancement in analysis of solar neutrino data that could be applied to more up-to-date data.

Appendix

Algorithm of Differential Evolution

The general algorithm of DE[33] for minimizing an objective function carries out a number of steps. Here we summarize the steps we carried out for minimizing the $\chi^2$ function defined in Sec. 2. We did optimization of the $\chi^2$ function individually for different regions of the parameter space to do one fine tuning in each region.

Step 1 An array of vectors was initialized to define a population of size $NP = 20$ with $D = 2$ parameters as

$$x_i = x_{j,i} \quad \text{where} \quad i = 1, 2, \ldots, NP \quad \text{and} \quad j = 1, \ldots, D. \ (A1)$$

The parameters, involved here, are $x_1$ and $x_2$ of Eqs. (2) and (3) on which $\Delta m^2/E_\nu$ and $\tan^2 \theta$ depend. Upper and lower bounds ($b_{j,U}$ and $b_{j,L}$), individually for different regions of the parameter space described in the introduction section, for the $x$ values were specified and each vector $i$ was assigned a value according to

$$x_{j,i} = \text{rand}_j(0, 1) \cdot (b_{j,U} - b_{j,L}) + b_{j,L} \ , \quad \text{(A2)}$$

where $\text{rand}_j \in [0, 1]$ is $j$-th evaluation of a uniform random number generator. The $\chi^2$ function was calculated for each vector of the population and the vector with least $\chi^2$ value was selected as base vector $x_{r_0}$.

Step 2 Weighted difference of two randomly selected vectors from the population was added to the base vector $x_{r_0}$ to produce a mutant vector population $v_i$ of NP trial vectors. The process is known as mutation.

$$v_i = x_{r_0} + F \cdot (x_{r_1} - x_{r_2}) \ , \quad \text{(A3)}$$

Here the scale factor $F \in [0, 2]$ is a real number that controls the amplification of the differential variation. The indices $r_1, r_2 \in [1, NP]$ are randomly chosen integers and are different from $r_0$.

Different variants of DE mutation are denoted by the notation “DE/x/y/z”, where $x$ specifies the vector to be mutated which can be “rand” (a randomly chosen vector) or “best” (the vector of the lowest $\chi^2$ value from the current population), $y$ is the number of difference vectors used and $z$ is the crossover scheme. The above mentioned variant Eq. (11) is DE/best/1/bin, where the best member of the current population is perturbed with $y = 1$ and the scheme bin indicates that the crossover is controlled by a series of independent binomial experiments. The two variants, reported in Refs. [34–35], very useful for their good convergence properties, are DE/rand/1/bin

$$v_i = x_{r_0} + F \cdot (x_{r_2} - x_{r_3}) \ , \quad \text{(A4)}$$

and DE/best/2/bin

$$v_i = x_{r_0} + F \cdot (x_{r_1} + x_{r_2} - x_{r_3} - x_{r_4}) \ . \quad \text{(A5)}$$

For our problem, we used the variant DE/best/2/bin Eq. (13) for DE mutation, where 2 difference vectors $(x_{r_1} - x_{r_3}$ and $x_{r_2} - x_{r_4}$) were added to the base vector. The values of $F$ we used are reported in Sec. 4.

Step 3 The parameters of mutant vector population Eq. (13) were mixed with the parameters of target vectors Eq. (9) in a process called uniform crossover or discrete recombination. After the cross over the trial vector became:

$$u_i = u_{j,i} = \begin{cases} v_{j,i} & \text{if} \ (\text{rand}_j(0, 1) \leq \text{Cr} \text{ or } j = j_{\text{rand}}), \\ x_{j,i} & \text{otherwise}. \end{cases} \quad \text{(A6)}$$

Here $\text{Cr} \in [0, 1]$ is the cross over probability that controls fraction of the parameters inherited from the mutant population (the values of $\text{Cr}$ we used are given in Sec. 4), $\text{rand}_j \in [0, 1]$ is a output of a random number generator and $j_{\text{rand}} \in [1, 2]$ is a randomly chosen index.

Step 4 The $\chi^2$ function was evaluated for each of the trial vector $u_i$ obtained from Eq. (14). If the trial vector resulted in lower objective function than that of the target vector $x_i$, it replaced the target vector in the following generation. Otherwise the target vector was retained. (This operation is called selection.) Thus the target vector for the next generation became:

$$x_i' = \begin{cases} u_i & \text{if} \ \chi^2(u_i) \leq \chi^2(x_i), \\ x_i & \text{otherwise}. \end{cases} \quad \text{(A7)}$$

The processes of mutation, crossover and selection were repeated until the optimum was achieved or the number of iterations (generations) specified in Sec. 4 were completed.

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