Generalized Kähler Geometry in 
$(2,1)$ superspace

Chris Hull$^a$, Ulf Lindström$^b$, Martin Roček$^c$, 
Rikard von Unge$^d$ and Maxim Zabzine$^b$

$^a$ The Blackett Laboratory, Imperial College London 
Prince Consort Road, London SW7 2AZ, U.K.

$^b$Department of Physics and Astronomy Uppsala University, 
Box 516, SE-751 20 Uppsala, Sweden

$^c$C.N.Yang Institute for Theoretical Physics, Stony Brook University, 
Stony Brook, NY 11794-3840, USA

$^d$Institute for Theoretical Physics, Masaryk University, 
611 37 Brno, Czech Republic

Abstract

Two-dimensional $(2,2)$ supersymmetric nonlinear sigma models can be described in $(2,2)$, $(2,1)$ or $(1,1)$ superspaces. Each description emphasizes different aspects of generalized Kähler geometry. We investigate the reduction from $(2,2)$ to $(2,1)$ superspace. This has some interesting nontrivial features arising from the elimination of nondynamical fields. We compare quantization in the different superspace formulations.
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# 1 Introduction

Supersymmetric sigma models in two dimensions are useful for investigating complex geometry because of the constraints that supersymmetry imposes on the target space geometry. The geometry of models with (2,2) supersymmetry [2] has been explored in investigations of generalized complex geometry [1], in particular generalized Kähler geometry (GKG) [2]. A number of interesting mathematical structures have been revealed
this way. Examples include: the generalized Kähler potential underlying all geometrical objects [3]; a set of coordinates adapted to the full local description of GKG [3]; a new gerbe structure related to the global description of GKG [4]; and a generalization of the Monge-Ampère equation [5].

There are different ways of characterizing GKG, and sigma models with various numbers of manifest supersymmetries reflect these. The Lagrangian in (2, 2) superspace is the generalized Kähler potential, whereas the (1, 1) superspace Lagrangian is given directly in terms of the metric and $B$-field. Here we shall focus on the (2, 1) description where the geometry is governed locally by a one-form. This is closely related to the local symplectic description of GKG introduced in [4].

Our starting point will be the (2, 2) model which we reduce to (2, 1). Once we have the classical GKG model in (2, 1) superspace, it is natural to ask about quantum properties. Here there are also interesting differences between the (2, 2), (2, 1) and (1, 1) analyses and we investigate these.

The classical results are in sections 2-5, and the quantum aspects in section 6. After a brief introduction to (2, 2) superspace in section 3, we describe (2, 1) superspace and sigma models in section 4. The reduction from (2, 2) to (2, 1) as well as a discussion of the geometric significance is contained in section 5. Section 6 compares the structure of the one-loop counterterms in the various formulations and discusses the corresponding differences in the renormalization schemes.

Conventions and background material can be found in [4, 5].

2 Generalized Kähler Geometry

The bihermitean geometry discovered in [6] and reformulated as Generalized Kähler Geometry (GKG) in [2] is characterized by the data $(M, g, J_\pm)$ where $M$ is a manifold, $J_\pm$ are two complex structures and $g$ is a metric hermitean with respect to both of them. Moreover, the following integrability conditions are required

$$d_{\pm}^c \omega_\pm + d_{\mp}^c \omega_\mp = 0 , \quad dd_{\pm}^c \omega_\pm = 0 ,$$

where $\omega_\pm := gJ_\pm$ and $d_{\pm}^c$ is the $i(\bar{\partial} - \partial)$ operator for the corresponding complex structure. These conditions are equivalent to the existence of a closed three-form $H$ where

$$H := d_{\pm}^c \omega_\pm = -d_{\mp}^c \omega_\mp ,$$

(2.2)
which is used to construct connections $\Gamma^{(\pm)}$ with torsion $\pm \frac{1}{2} g^{-1} H$ that preserve $J_\pm$. Locally we can always find a potential for the torsion: $H = dB$. We refer to this potential as “the $B$-field”. Clearly, it is only defined up to a gauge transformation $\delta B = d\Lambda$, and it is often convenient to choose a particular gauge.

There is an alternative local description of GKG, derived in [4], which emphasizes the relation to symplectic geometry. We can choose a gauge $B = B_+$ for the $B$-field for which the $(1,1)$ part with respect to $J_+$ vanishes, or a gauge $B = B_-$ for the $B$-field for which the $(1,1)$ part with respect to $J_-$ vanishes. The GKG geometry can then be formulated in terms of

$$F_+ = \frac{1}{2} (B_+ - g) J_+ , \quad F_- = \frac{1}{2} (B_- + g) J_- .$$

(2.3)

(As explained in [4], a more global description can be given in terms of gerbe-connections.) Remarkably, $F_\pm$ are closed and non-degenerate, so that locally we may define one-forms $\lambda_\pm$:

$$F_\pm = d\lambda_\pm .$$

(2.4)

3 (2,2) superspace

Our point of departure is the full description of GKG in (2,2) superspace. As shown in [3], away from irregular points the geometry is completely specified in terms of a generalized Kähler potential $K$. If there are no irregular points, the action for the general (2,2) sigma model is

$$S = \int d^2 \xi \ d\theta^+ d\bar{\theta}^+ d\theta^- d\bar{\theta}^- \ K(\phi, \chi, X_L, X_R)$$

$$= \int d^2 \xi \ \bar{D}_+ \bar{D}_+ D_- D_- \ K(\phi, \chi, X_L, X_R) .$$

(3.5)

The (2,2) algebra is

$$\{\bar{D}_\pm, D_\pm\} = 2i \partial_\pm ,$$

(3.6)

and the arguments of $K$ are constrained (2,2) superfields satisfying

$$\text{(anti)chiral :} \quad \bar{D}_\pm \phi = 0 , \quad D_\pm \bar{\phi} = 0 ,$$

$$\text{twisted (anti)chiral :} \quad D_+ \chi = \bar{D}_- \chi = 0 , \quad D_+ \bar{\chi} = \bar{D}_- \bar{\chi} = 0 ,$$

$$\text{left semi(anti)chiral :} \quad \bar{D}_+ X_L = 0 , \quad D_+ \bar{X}_L = 0 ,$$

$$\text{right semi(anti)chiral :} \quad \bar{D}_- X_R = 0 , \quad D_- \bar{X}_R = 0 ,$$

(3.7)
where we adopt the following notation for superfields $\phi = (\phi, \bar{\phi})$, $\chi = (\chi, \bar{\chi})$, $X_L = (X_l, \bar{X}_l)$, $X_R = (X_r, \bar{X}_r)$.

4 (2, 1) superspace

4.1 Sigma models

A $(1, 1)$ sigma model $(i = 1, \ldots, 2n)$ is defined by

$$S = \frac{1}{2} \int d^2 \xi \ d^2 \theta \ D_+ \Phi^i(g_{ij} + B_{ij}) D_- \Phi^j,$$

where $\Phi^i$ are unconstrained $(1, 1)$ superfields and the $(1, 1)$ algebra is given by

$$\{D_\pm, D_\pm\} = 2i\partial_\pm. \quad (4.9)$$

In [7], a $(1, 1)$ sigma model $(i = 1, \ldots, 2n)$ with an extra (left) supersymmetry is lifted to $(2, 1)$ superspace as $(\alpha = 1, \ldots, n)$

$$S = -i \int d^2 \xi \ d\theta^+ d\bar{\theta}^+ d\theta^- (\lambda_\alpha D_+ \varphi^\alpha + \bar{\lambda}_\alpha D_- \varphi^{\bar{\alpha}}) \equiv -i \int d^2 \xi \ d\theta^+ d\bar{\theta}^+ d\theta^- (\lambda_i D_+ \varphi^i) \quad (4.10)$$

where $\varphi^i = (\varphi^\alpha, \varphi^{\bar{\alpha}})$ with $\mathbb{D}_+ \varphi = 0 = \mathbb{D}_- \varphi$. The $(2, 1)$ is given by

$$\{\mathbb{D}_+, \mathbb{D}_+\} = 2i\partial_\pm, \quad D_\pm^2 = i\partial_\pm. \quad (4.11)$$

The metric $g$ and $B$-field (in a suitable gauge) are given by

$$g_{\alpha\bar{\beta}} = i(\partial_\alpha \bar{\lambda}_{\bar{\beta}} - \partial_{\bar{\beta}} \lambda_\alpha),$$

$$B_{\alpha\bar{\beta}}^{(2,0)} = i(\partial_\alpha \lambda_{\bar{\beta}} - \partial_{\bar{\beta}} \lambda_\alpha), \quad \text{with}$$

$$B = B^{(2,0)} + B^{(0,2)}. \quad (4.12)$$

Alternatively, it is always possible to choose a gauge where

$$B_{\alpha\bar{\beta}}^{(1,1)} = i(\partial_\alpha \bar{\lambda}_{\bar{\beta}} + \partial_{\bar{\beta}} \lambda_\alpha). \quad (4.13)$$

The target space geometry of a $(2, 1)$ supersymmetric sigma model corresponds to a complex manifold $(M, J)$ with an hermitian metric $g$ such that

$$dd^c \omega = 0, \quad (4.14)$$

\footnote{Ref. [7] uses a one-form potential $k$ related to our $\lambda$ through $k_\alpha = i\lambda_\alpha$.}
where $\omega = gJ$. In the mathematics literature such geometry is called strong KT (strong Kähler with torsion). Here $H$ is defined by $H = d^c \omega$. Locally the geometry can be encoded in terms of the one-form potential $\lambda$ which appears in $(2,1)$ action (4.10). The geometry is invariant under the following symmetries:

$$\lambda_\alpha(\varphi, \bar{\varphi}) \rightarrow \lambda_\alpha(\varphi, \bar{\varphi}) + \partial_\alpha f(\varphi, \bar{\varphi}) + l_\alpha(\varphi). \quad (4.15)$$

These transformations change the lagrangian by terms which vanish when integrated over the full superspace. If $H$ represents an element of $H^3(M, \mathbb{Z})$ then the transformations of $\lambda$ are related to a gerbe with connection and the problem can be analyzed very much along the lines described in [4].

The $(2,1)$ superspace field equations follow from varying the action (4.10) with respect to $\varphi^i$; the chirality constraints imply that we may write the variation as

$$\delta \varphi = \bar{D}_+ \delta \psi_-, \quad \delta \bar{\varphi} = -D_+ \delta \bar{\psi}_- \quad (4.16)$$

for arbitrary unconstrained $(2,1)$ superfield variations $\psi = (\psi, \bar{\psi})$. After integration by parts, the variation of the action (4.10) may be written as:

$$\delta S = - \int d^2 \xi \ d\theta^+ d\bar{\theta}^+ d\theta^- g_{\alpha\beta} \left( \delta \psi_+^{\alpha} \nabla_+^{(-)} D_- \varphi_\beta - \delta \bar{\psi}_-^{\beta} \nabla_+^{(-)} D_- \varphi_\alpha \right), \quad (4.17)$$

where $\nabla_+^{(-)}$ is the pullback to $(2,1)$ superspace of the connection with torsion:

$$\nabla^{(-)} = \nabla^{(0)} - \frac{1}{2} g^{-1} H, \quad (4.18)$$

where $\nabla^{(0)}$ is the Levi-Civita connection.

### 4.2 Vector-fields and couplings

Assume that our matter action (4.10) is symmetric under

$$\varphi^I \rightarrow e^{i\gamma} \varphi^I, \quad \bar{\varphi}^I \rightarrow e^{-i\gamma} \bar{\varphi}^I \quad (4.19)$$

with $\gamma$ real. Geometrically it corresponds to having an isometry which preserves the strong KT geometry. We gauge this symmetry by turning the parameter $\gamma$ into a chiral superfield and introducing a $(2,1)$ vector multiplet $(v, A_-)$ which transforms according to

$$(v^v)' = e^{i\gamma} v^v e^{-i\gamma}, \quad A_-^v = e^{i\gamma} (A_- - iD_-) e^{-i\gamma} \quad (4.20)$$
Coupling to matter is achieved by introducing gauge covariantly chiral fields \( \varphi^1, \tilde{\varphi}^\dagger = \tilde{\varphi}^\dagger e^v \) which satisfy chirality constraints with respect to the gauge covariant derivatives \( D_+ = e^{-v} \tilde{D}_+ e^v \), \( \tilde{D}_+ = \tilde{D}_+ \), i.e. \( \tilde{D}_+ \varphi^1 = D_+ \tilde{\varphi}^\dagger = 0 \). The gauged action becomes

\[
S = -i \int d^2 \xi \ d \theta^+ d \bar{\theta}^- \left[ \lambda_1 (\varphi^1, \tilde{\varphi}^\dagger, ...) \nabla_- \varphi^1 + \lambda_1 (\varphi^1, \tilde{\varphi}^\dagger, ...) \nabla_- \tilde{\varphi}^\dagger + ... \right],
\]

where

\[
\nabla_- \varphi^1 = D_- \varphi^1 + i A_- \varphi^1
\]

and dots stand for the contribution of other fields (spectators) which remain unchanged compared to the action (4.10).

### 4.3 Additional susy in (2, 1)

We may start from the (2, 1) sigma model and ask under which circumstances it has (2, 2) supersymmetry, i.e., what is the condition for an additional right supersymmetry. We thus consider the action (4.10) and make an ansatz for an additional supersymmetry:

\[
\delta \varphi^\alpha = \bar{D}_+ (\epsilon J_\alpha^i D_- \varphi^i) = \bar{\nabla}_+^{(-)} (\epsilon J_\alpha^i D_- \varphi^i), \quad i := (\alpha, \bar{\alpha}),
\]

where \( \bar{\nabla}_+^{(-)} \) is the pullback of the connection with the torsion given by \(-\frac{1}{2} g^{-1} H\), as discussed in section [2]. The superfields \( \{\varphi^i\} = \{\varphi^\alpha, \tilde{\varphi}^{\bar{\alpha}}\} \) are coordinates for the complex structure \( J_+ \), and are chiral (resp. antichiral): \( \bar{D}_+ \varphi^\alpha = \bar{D}_+ \tilde{\varphi}^{\bar{\alpha}} = 0 \). The second equality in (4.23) follows because \( \varphi^\beta \) is chiral (\( \bar{D}_+ \varphi^\beta = 0 \)) and \( \Gamma^{(+)} \) preserves \( J_+ \) holomorphic indices (\( \Gamma^{(+)}_{j\beta} = 0 \)):

\[
\bar{\nabla}_+^{(-)} \psi^\alpha - \bar{D}_+ \psi^\alpha \equiv \Gamma^{(+)}_{ij} \bar{D}_+ \psi^i \psi^j = \Gamma^{(+)}_{ij} \bar{D}_+ \tilde{\varphi}^{\bar{\alpha}} \psi^j = \Gamma^{(+)}_{j\beta} \bar{D}_+ \tilde{\varphi}^{\bar{\alpha}} \psi^j = 0.
\]

Complex conjugation gives:

\[
\delta \varphi^{\bar{\alpha}} = -D_+ (\epsilon J_\alpha^i D_- \varphi^i).
\]

The hermitean metric and \( B \)-field are given in terms of the vector potentials \( \lambda_\alpha \) as in (4.12). Matching with the (1, 1) reduction of the transformations (4.23), (4.25) implies that the transformation parameter \( \epsilon \) obeys

\[
(\bar{D}_+ - D_+) \epsilon = 0, \quad \partial_+ \epsilon = 0
\]

\[
(\bar{D}_+ + D_+) \epsilon := 2i \epsilon^-.
\]

6
We first consider invariance of the action: we vary (4.10) using (4.17) and (4.23):

\[
\delta S = -\int d^2 \xi \ d\theta^+ d\bar{\theta}^+ d\theta^- \ g_{\alpha \beta} \left( (\epsilon J^\alpha_i D_- \varphi^i) \nabla^{(-)}_+ D_- \bar{\varphi}^\beta + \text{c.c.} \right). \tag{4.27}
\]

Because the holomorphic superfields \( \varphi \) are chiral and the metric is hermitean with respect

to \( J_+ \), the variation of the action can be rewritten in terms of \( \omega_-=gJ_- \)

\[
\delta S = -\int d^2 \xi \ d\theta^+ d\bar{\theta}^+ d\theta^- \ \left( \epsilon \omega_{-ij} D_- \varphi^j \nabla^{(-)}_+ D_- \varphi^i \right) \tag{4.28}
\]

This cancels only if the symmetric part of \( \omega_- \) vanishes and hence the metric is hermitean

with respect to \( J_- \); then we have

\[
\delta S = -\int d\theta^+ d\bar{\theta}^+ d\theta^- \ \frac{1}{2} \left( -\epsilon \omega_{-ij} \nabla^{(-)}_+ (D_- \varphi^j D_- \varphi^i) + \bar{\epsilon} \omega_{-ij} \nabla^{(-)}_+ (D_- \varphi^i D_- \varphi^j) \right). \tag{4.29}
\]

Integrating by parts, we find that this vanishes when \( (4.26) \) is satisfied and when the

connection \( \nabla^{(-)} \) preserves \( \omega_- \), and hence

\[
\nabla^{(-)}_k J_{-j}^i = 0. \tag{4.30}
\]

Note that this allows us to rewrite the transformations \( (4.23) \) as

\[
\delta \varphi^\alpha = J^\alpha_i \nabla^{(-)}_+ (\epsilon D_- \varphi^i), \tag{4.31}
\]

which makes it clear that the \( \theta \) independent component of \( \epsilon \) generates central charge

transformations proportional to the field equations

\[
\nabla^{(-)}_+ D_- \varphi^\alpha = 0.
\]

These transformations are thus of interest only off-shell. The additional right supersymmetry has parameter \( \epsilon^- \).

Checking closure of the algebra generated by the transformations \( (4.23) \), we find that \( J_{-j}^i \) is an additional complex structure,

\[
J_{-j}^i = -\mathbb{I}, \quad \mathcal{N}(J_-) = 0, \tag{4.32}
\]

where \( \mathcal{N} \) is the Nijenhuis tensor. In addition, the commutator of the new right supersymmetry \( (4.23) \) with the existing left is proportional to field equations times \(-iJ^\alpha_{-j} \) and \( iJ^\alpha_{-\beta} \), which is just the commutator of the left and right complex structures (recall the \( J_+ \) has the canonical form \( \text{diag}(i, -i) \)).
From (2, 2) to (2, 1) superspace

In this section we discuss the reduction from (2, 2) to (2, 1) models. Here we adopt the following short-hand notations for the derivatives of $K$:

$$
K_C = \partial_\phi K = (K_c, K_c) = (\partial_\phi K, \partial_\phi K),
K_T = \partial_\chi K = (K_t, K_{\bar{t}}) = (\partial_\chi K, \partial_\chi K),
K_L = \partial_{X_L} K = (K_l, K_{\bar{l}}) = (\partial_{X_L} K, \partial_{X_L} K),
K_R = \partial_{X_R} K = (K_r, K_{\bar{r}}) = (\partial_{X_R} K, \partial_{X_R} K),
$$

(5.33)

where we suppress all coordinates indices. Analogously we define the matrices of double derivatives of $K$, e.g. $K_{\mu\nu}$ is our notation for the matrix of second derivatives $\partial_{X_i} \partial_{X_j} K$ etc. We use matrix and vector notation and suppress all indices. For further explanations of this notation the reader may consult [5].

5.1 Sigma Models

Let us consider the reduction to (2, 1) superspace of a sigma model in (2, 2) superspace with action:

$$
S = \int d^2 \xi \ d\theta^+ d\bar{\theta}^+ d\theta^- d\bar{\theta}^- K(\phi, \chi, X_L, X_R).
$$

(5.1)

Of the (2, 2) superspace derivatives, we keep $D_+, \bar{D}_+$ and write

$$
D_- = \frac{1}{\sqrt{2}} (D_- - iQ_-),
\bar{D}_- = \frac{1}{\sqrt{2}} (D_- + iQ_-),
$$

(5.2)

where $Q_-$ is defined such that it anticommutes with $D_-$ and moreover these expressions are compatible with the definitions (3.6) and (4.9). We further separate $\theta^-$ into its real and imaginary parts, and reduce to (2, 1) by dropping the dependence on the imaginary part (denoted by a vertical bar). We find

$$
S = i \int d^2 \xi \ D_+ \bar{D}_+ D_- Q_- K(\phi, \chi, X_L, X_R) | = i \int d^2 \xi \ D_+ \bar{D}_+ D_- (K_L \psi_L - K_R JD_- x_R + K_C JD_- z - K_T JD_- w),
$$

(5.3)
where $J$ is the canonical complex structure $diag(i, -i)$ and where $x_L, \psi_L, x_R, z, w$ are $(2, 1)$ superfields defined according to

$$
x_L := X_L \mid , \quad \psi_L = Q_X L \mid , \quad x_R := X_R \mid , \quad z := \phi \mid , \quad w := \chi \mid .
$$

The $(2, 1)$ superfields $x_L = (x_l, \bar{x}_l)$, $\psi_L = (\psi_l, \bar{\psi}_l)$, $z = (z, \bar{z})$ and $w = (w, \bar{w})$ are (anti)chiral $(2, 1)$ superfields, respectively:

$$
\begin{align*}
\bar{D}^+ x_l &= 0 = D^+ \bar{x}_l , \\
\bar{D}^+ \psi_l &= 0 = D^+ \bar{\psi}_l , \\
\bar{D}^+ z &= 0 = D^+ \bar{z} , \\
\bar{D}^+ w &= 0 = D^+ \bar{w} ,
\end{align*}
$$

and $x_R = (x_r, \bar{x}_r)$ is an unconstrained $(2, 1)$ superfield. In $(2, 1)$ superspace $K$ is a function of these $(2, 1)$ superfields. To find the usual $(2, 1)$ superspace action (4.10), we impose the $\psi_L$-field equation;

$$
\bar{D}^+ K_l = D^+ K_{\bar{l}} = 0 ,
$$

which we can solve for $x_R$ by introducing chiral superfields $y_l = (y_l, \bar{y}_l)$:

$$
K_l = y_l , \quad K_{\bar{l}} = \bar{y}_{\bar{l}}
$$

(5.7)

to find $x_R(x_L, y_L, z, w)$. As $\psi_L$ is chiral, (5.6) implies that the $\psi_l$ term does not enter in the final $(2, 1)$ superspace Lagrangian. We are left with

$$
i \int d^2 \xi \ \bar{D}_+ D_+ D_+ D_- \left( K_R JD_- x_R(x_L, y_L, z, w) + K_C JD_- z - K_T JD_- w \right).
$$

(5.8)

We also need to check that the field equations of $x_R$ impose no new constraints – they yield the consistent equation $\psi_{-L} = D_- x_L$.

Introducing the notation

$$
\varphi^\alpha = \begin{pmatrix} x_l \\ y_l \\ z \\ w \end{pmatrix},
$$

we can further reduce to $(1, 1)$ superspace by removing the dependence on the imaginary part of $\theta^+$. We then find spinor superfields $\psi_+$ together with the reduction of $\psi_-$; these are auxiliary as $(1, 1)$ superfields [8].
the action may be written as in (4.10) with
\[ \lambda_{\alpha} = -\begin{pmatrix}
-\mathcal{K}_R J K_{LR}^{-1} K_{LI} \\
\mathcal{K}_R J K_{IR}^{-1} \\
iK_c - \mathcal{K}_R J K_{LR}^{-1} K_{LC} \\
iK_t - \mathcal{K}_R J K_{LR}^{-1} K_{LT}
\end{pmatrix}. \] (5.10)

Considering the one-form \( \lambda := \lambda_{\alpha} d\phi^\alpha + \bar{\lambda}_\bar{\alpha} d\bar{\phi}^{\bar{\alpha}} \) and comparing to the expressions in [4] shows that we have recovered the one-form \( \lambda^{(+)} \) from (2.4), which in (2,2) coordinates \( X_R, X_L, \phi, \chi \) reads:
\[ \lambda^{(+)} = -\mathcal{K}_R J dX_R - \mathcal{K}_C J d\phi + \mathcal{K}_T J d\chi . \] (5.11)

Because all (2,1) fields are chiral it is straightforward to check that reducing to (1,1) superspace produces an action
\[ -\int D_+ D_- \left[ D_+ \varphi^i(J_+), k(d\lambda^{(+)})_{kl} D_- \varphi^l \right] \] (5.12)
which, using (2.3) and (2.4), is clearly the sigma model (4.8) written in coordinates adapted to \( J_+ \).

### 5.2 Vector multiplets

In (2,2) superspace there are different vector multiplets that are used to gauge various types of isometries [9, 10, 11]. The Kähler vector (or twisted Kähler) multiplets that gauges isometries in the chiral (or twisted chiral) sector reduce straightforwardly to the (2,1) form described in section (4.2). The basic Yang-Mills multiplet in (2,2) supersymmetry consists of a real unconstrained superfield \( V \) transforming as
\[ e^V \rightarrow e^{i\Lambda} e^V e^{-i\Lambda} , \] (5.13)
where \( \Lambda \) is a (2,2) chiral superfield.

Reducing to (2,1) superspace we define the components
\[ e^V| = e^v , \]
\[ e^{-V} Q_- e^V| = -2A_- - ie^{-v} D_- e^v , \] (5.14)
with gauge transformations
\[ e^v \rightarrow e^{i\tilde{\alpha}} e^v e^{-i\Lambda} , \]
\[ A_- \rightarrow e^{i\tilde{\alpha}} (A_- - i D_-) e^{-i\Lambda} , \]
\[ \tilde{A}_- \rightarrow e^{i\tilde{\alpha}} (\tilde{A}_- - i D_-) e^{-i\tilde{\alpha}} , \] (5.15)
where $\lambda$ is a $(2,1)$ chiral superfield and there is a reality constraint
\[
\bar{A}_- = e^\bar{v} (A_- - iD_-) e^{-\bar{v}}.
\] (5.16)

The covariant derivative is $\nabla_- \phi = D_- \phi + iA_- \cdot \xi$ where $\xi$ is the Killing vector.

The twisted Kähler multiplet transforms with twisted chiral parameters $\tilde{\Lambda}$
\[
e^{\tilde{V}} \to e^{i\tilde{\Lambda}} e^{\tilde{V}} e^{-i\tilde{V}}.
\] (5.17)

The $(2,1)$ components are defined as
\[
\begin{align*}
e^{\tilde{V}} &= e^{\bar{v}}, \\
e^{-\tilde{V}} Q_- e^{\tilde{V}} &= ie^{-\bar{v}} D_- e^{\bar{v}} + 2\bar{A}_-,
\end{align*}
\] (5.18)

where $e^{\bar{v}}$ and $\bar{A}_-$ transforms exactly as $e^v$ and $A_-$ and they satisfy the same reality constraint. That is, both $V$ and $\tilde{V}$ reduce to the same multiplet in $(2,1)$ superspace.

In addition there are the Large Vector Multiplet gauges isometries that act on both chiral and twisted chiral coordinates and the semichiral vector multiplet that gauges isometries among the semichiral coordinates. These multiplets introduce novel features when reduced and will be treated elsewhere.

### 5.3 Comment on generalized Kähler geometry and superspace

The superfields of $(2,1)$ superspace are necessarily complex: they are chiral and their complex conjugates are antichiral. Geometrically this means that they are holomorphic (resp. antiholomorphic) coordinates that put $J_+$ into its canonical form. All the holomorphic coordinates are on an equal footing, but we have singled out one of the two complex structures $J_+$ for preferential treatment. Similarly, in $(1,2)$ superspace, $J_-$ is diagonalized.

In contrast, in $(2,2)$ superspace, we must choose a *polarization* for the semichiral superfields – the coordinates along the symplectic leaves on which $[J_+, J_-]$ is invertible. Along these leaves, we choose half of the $J_+$-holomorphic coordinates, and half of the $J_-$-holomorphic coordinates to write the generalized Kähler potential that is the $(2,2)$ superspace Lagrange density. Different choices of polarization give rise to different generalized Kähler potentials; of course, they all give rise to the same $(2,1)$ Lagrange density up to holomorphic coordinate reparameterizations. The $(2,2)$ superspace description, while requiring a choice of polarization, treats the two complex structures on equal footing.
6 Superspace Counterterms and Renormalization

6.1 Quantization in (1, 1) Superspace

The (2, 2) sigma model can be formulated in (1, 1), (2, 1) or (2, 2) superspaces and in each case can be quantized using the corresponding superspace Feynman rules to obtain superspace counterterms. These are of interest due to their relation to the field equations governing string backgrounds. The one-loop counterterm in (1, 1) superspace is given in terms of the Ricci curvature with torsion, and comparing this with the counterterms in (2, 1) and (2, 2) superspace gives interesting expressions for the Ricci curvature in terms of potentials. The aim of this section is to explore and exploit these relations.

The (1, 1) superspace action is

\[ S = \int d^2\xi \ d\theta^+ d\theta^- E_{ij} D_+ \Phi^i D_- \Phi^j, \]

where \( E = g + B \). The one-loop counterterm is proportional to

\[ \Delta = \int d^2\xi \ d\theta^+ d\theta^- \left( R_{ij}^{(+)} + \partial_{[i} \alpha_{j]} \right) D_+ \Phi^i D_- \Phi^j - U^i \frac{\delta S}{\delta \Phi^i} \].

Here \( R_{ij}^{(+)} \) is the Ricci tensor with torsion and the term involving \( \alpha \) is a total derivative. The term proportional to \( U \) vanishes when the classical field equation \( \delta S / \delta \Phi^i = 0 \) is imposed, and off-shell can be absorbed into a field redefinition\(^3\) of \( \Phi: \Phi \to \Phi + U(\Phi) \).

Integrating by parts (and shifting \( \alpha \)), this can be rewritten as

\[ \Delta = \int d^2\xi \ d\theta^+ d\theta^- \left( R_{ij}^{(+)} + 2 \nabla_{(i} U_{j)} + H_{ijk} U^k + \partial_{[i} \alpha_{j]} \right) D_+ \Phi^i D_- \Phi^j, \]

or as

\[ \Delta = \int d^2\xi \ d\theta^+ d\theta^- \left( R_{ij}^{(+)} + \mathcal{L}_U E_{ij} + \partial_{[i} \alpha_{j]} \right) D_+ \Phi^i D_- \Phi^j \]

after a further shift of \( \alpha \), where \( \mathcal{L}_U \) is the Lie derivative with respect to \( U \). For a given geometry, one-loop finiteness requires that there is a choice of vector \( U \) and 1-form \( \alpha \) such that\(^12\)

\[ R_{ij}^{(+)} + 2 \nabla_{(i} U_{j)} + H_{ijk} U^k + \partial_{[i} \alpha_{j]} = 0. \]

\(^3\)There are more general field redefinitions which may involve the derivatives \( D_\pm \) and dimensionful parameters. These field redefinitions are not relevant at the given order. A similar comment is applicable to (2, 1) and (2, 2) superspace.
For geometries without torsion, this gives the condition for finiteness

$$R_{ij} + 2\nabla_i(U_j) = 0.$$ \hspace{1cm} (6.6)

For a Kähler manifold, this becomes

$$R_{\alpha\bar{\beta}} + 2\nabla_\alpha(U_{\bar{\beta}}) = 0, \quad \nabla_\alpha(U_{\bar{\beta}}) = \nabla_{\bar{\alpha}}(U_{\beta}) = 0.$$ \hspace{1cm} (6.7)

### 6.2 The (2, 2) model without Torsion

Before turning to the (2, 1) and (2, 2) supersymmetric cases with torsion, it will be useful to first review the case of (2, 2) sigma models without torsion, with Kähler target space.

The general one-loop counterterm in this case is

$$\Delta_{(2,2)} = \int d^2\xi d^4\theta \left[ \frac{1}{2} \ln(\det(K_{\alpha\bar{\alpha}})) + Z^\alpha(\phi)K_{\alpha} + \bar{Z}^{\bar{\alpha}}(\bar{\phi})K_{\bar{\alpha}} \right],$$ \hspace{1cm} (6.8)

where $Z^\alpha(\phi)$ is an arbitrary holomorphic vector field and corresponds to the field redefinitions

$$\phi^\alpha \rightarrow \phi^\alpha + Z^\alpha(\phi).$$ \hspace{1cm} (6.9)

The two last terms in (6.8) are proportional to the equations of motion. The requirement for finiteness thus becomes

$$\frac{1}{2} \ln(\det(K_{\alpha\bar{\alpha}})) + Z^\alpha(\phi)K_{\alpha} + \bar{Z}^{\bar{\alpha}}(\bar{\phi})K_{\bar{\alpha}} = f(\phi) + \bar{f}(\bar{\phi}).$$ \hspace{1cm} (6.10)

This condition must be compatible with the condition (6.7) obtained earlier from the (1, 1) analysis. Integrating over two of the fermionic coordinates to obtain a (1, 1) superspace form of this counterterm must then give a Ricci tensor term plus terms involving vector fields, so this implies that there must be an identity involving a relation between the Ricci tensor and an expression with two derivatives acting on $\ln(\det(K_{\alpha\bar{\alpha}}))$. There is indeed such an expression, the well-known identity for Kähler manifolds:

$$R_{\alpha\bar{\alpha}} = \partial_\alpha\bar{\partial}_{\bar{\alpha}}\ln(\det(K_{\beta\bar{\beta}})).$$ \hspace{1cm} (6.11)

However, if this had not been known, the superspace arguments would have led us to discover it. This kind of argument will lead to interesting identities in the cases with torsion. Then acting on (6.10) with $\partial_\alpha\bar{\partial}_{\bar{\alpha}}$ and using the expression (6.11) yields the condition for finiteness

$$R_{\alpha\bar{\alpha}} + 2\nabla_\alpha(Z_{\bar{\alpha}}) = 0.$$ \hspace{1cm} (6.12)
This is of the same form as the condition \((6.7)\) found above with \(U = Z\), but has the extra restriction that the vector \(Z\) is required to be holomorphic \((\bar{\partial}_\alpha Z^\alpha = 0)\). If equation \((6.7)\) is satisfied with a nonholomorphic \(U\), the theory is one-loop finite when regarded as a \((1, 1)\) sigma model but the wave function renormalisation required for off-shell finiteness does not respect the full \((2, 2)\) supersymmetry.

### 6.3 The \((2, 1)\) Sigma Model

Next we turn to the \((2, 1)\) sigma-model \((4.10)\). The one-loop counterterm was given in \([13]\). The one-loop renormalization of \(\lambda_i\) is proportional to \(\Gamma_j^{(+)}\) where \(\Gamma^{(+)}\) is the \(U(1)\) part of the connection with torsion:

\[
\Gamma_i^{(+)} = J_{+k}^{i} \Gamma_j^{(+)}^{k} . \tag{6.13}
\]

It can be written \([13]\) in terms of a one form \(v^{(+)}\) (known as the Lee form for \(\omega^{(+)}\) in Hermitian geometry) defined by

\[
v_i^{(+)} = J_{+j}^{i} \nabla_k J_{+j}^{k} = -\frac{1}{2} J_{+j}^{i} H_{jk}^{l} J_{+l}^{k} . \tag{6.14}
\]

(which vanishes if the torsion vanishes) and the determinant of the metric. In complex coordinates adapted to \(J_+\),

\[
v_\alpha^{(+)} = -g^{\beta\gamma} H_{\alpha\beta\gamma} = g^{\beta\gamma} (g_{\alpha^\gamma,\beta} - g_{\beta^\gamma,\alpha}) \tag{6.15}
\]

and

\[
\Gamma_\alpha^{(+)} = i \left( 2v_\alpha^{(+)} + \partial_\alpha \ln \det g_{\beta\gamma} \right) . \tag{6.16}
\]

The curvature of the \(U(1)\) part of the connection is

\[
C_{ij}^{(+)} = \partial_i \Gamma_j^{(+)} - \partial_j \Gamma_i^{(+)} . \tag{6.17}
\]

Allowing for terms that vanish on-shell and total derivatives, the general form of the one-loop counterterm is

\[
S = -\frac{i}{2} \int d^2 \xi \ d\theta^+ d\bar{\theta}^+ d\theta^- (\Gamma_i^{(+)} + \mathcal{L}_{\mathcal{V}^{(+)}} \lambda_i^{(+)} + \partial_i \rho^{(+)} ) D_\varphi \varphi^i . \tag{6.18}
\]

Here \(\mathcal{L}_{\mathcal{V}^{(+)}}\) denotes the Lie derivative with respect to a vector field \((\mathcal{V}^{(+)})^i = ((\mathcal{V}^{(+)})^\alpha, (\bar{\mathcal{V}}^{(+)})^{\bar{\alpha}})\), with \((\mathcal{V}^{(+)})^\alpha(\varphi)\) a holomorphic vector field. The term involving \(\rho_i^{(+)}\) is a total derivative,
included for generality. The term involving \( V(\cdot) \) again vanishes on-shell (up to a surface term). The condition for one-loop finiteness is then

\[
\Gamma_i^{(\cdot)} + \mathcal{L}_{V(\cdot)} \lambda_i^{(\cdot)} + \partial_i \rho^{(\cdot)} = f_i(\varphi) + \bar{f}_i(\bar{\varphi}).
\] (6.19)

The results from (2, 1) superspace must be compatible with those from (1, 1) superspace. In particular, the (2, 1) counterterm \( 6.18 \) with \( V^{(\cdot)} = 0 \) and \( \rho^{(\cdot)} = 0 \) gives a (1, 1) superspace counterterm involving derivatives of \( \Gamma_i^{(\cdot)} \). This must agree with the counterterm \( 6.1 \) for some choice of \( U, \alpha \), and for this to be the case, there must be some identities relating \( R_{ij}^{(\cdot)} \) to derivatives of \( \Gamma_i^{(\cdot)} \). This is indeed the case, and leads to the remarkable identities \([13],[14],[15]\)

\[
R_{\alpha\beta}^{(\cdot)} = \nabla_\alpha v_\beta^{(\cdot)},
\] (6.20)

\[
R_{a\bar{b}}^{(\cdot)} = \nabla_a v_{\bar{b}}^{(\cdot)} - \frac{i}{2} C_{a\bar{b}}^{(\cdot)} - (\partial_a v_{\bar{b}}^{(\cdot)} - \partial_{\bar{b}} v_a^{(\cdot)}),
\] (6.21)

which may be written covariantly as

\[
R_{ik}^{(\cdot)} = \nabla_i v_k^{(\cdot)} - \frac{1}{2} (J_k^{(\cdot)} C^{(\cdot)})_{ik} - (dv^{(\cdot)})_{ik}
\] (6.22)

which imply consistency between \( 6.21 \) and \( 6.5 \) with

\[
U_i = -\frac{1}{2} v_i^{(\cdot)}, \quad \alpha_i = v_i^{(\cdot)}.
\] (6.23)

### 6.4 The (1, 2) Sigma model

For completeness we give the corresponding formulas for the (1, 2) sigma model. The \( U(1) \) part of the connection is

\[
\Gamma_i^{(-)} = J^j_{-i} \Gamma_j^{(-)}
\] (6.24)

which can be written in terms of the Lee-form \( v^{(-)} \)

\[
v_i^{(-)} = J^j_{-i} \nabla_j v^k = \frac{1}{2} J^l_{-i} H_{jk}^l J_{-l},
\] (6.25)

and the determinant of the metric. In complex coordinates adapted to \( J_- \) we have

\[
v_\alpha^{(-)} = g^{\beta\gamma} H_{\alpha\beta\gamma} = -g^{\beta\gamma}(g_{\alpha\beta\gamma} - g_{\beta\gamma\alpha}),
\] (6.26)

and

\[
\Gamma_\alpha^{(-)} = i(2v_\alpha^{(-)} + \partial_\alpha \ln \det g_{\beta\gamma})
\] (6.27)
The curvature of the $U(1)$ part of the connection is
\[ C_{ij}^{(-)} = \partial_i \Gamma_j^{(-)} - \partial_j \Gamma_i^{(-)} \tag{6.28} \]

The one-loop counterterm is given by
\[ S = -\frac{i}{2} \int d^2\xi d\theta^- d\theta^+ (\Gamma^{(-)} + \mathcal{L}_{V^{(-)}} \lambda^{(-)} + \partial_i \rho^{(-)} D_+ \phi^i) \tag{6.29} \]

The relation between the $U(1)$ curvature and the Ricci tensor is
\[ R_{ik}^{(-)} = \nabla_i^{(+)} v_k^{(-)} - \frac{1}{2} (J_- C^{(-)})_{ik} - (d v^{(-)})_{ik} \tag{6.30} \]

6.5 The $(2,2)$ model with Torsion

We turn now to the $(2,2)$ case. The one-loop counterterm is proportional to
\[ \int d^2\xi \ d^4\theta \ [K^1 + \mathcal{L}_W K] \ , \tag{6.31} \]

where $K^1$ is the one-loop counterterm calculated in \[10\], given by
\[ K^1 = \ln \left( \frac{A}{B} \right) \ , \tag{6.32} \]

where $A$ and $B$ are given in (6.61) below.

The results from $(2,2)$ superspace must be compatible with the results in $(2,1)$ and $(1,2)$ as well as the result in $(1,1)$ superspace. In particular, the $(2,2)$ counterterm \[6.31\] with $W = 0$ reduces to the counterterms \[6.18\] and \[6.29\] with in general nonzero $V^{(\pm)}$ and $\rho^{(\pm)}$.

We can investigate in which cases we get nonzero $V^{(\pm)}$ by reducing the two counterterms further to $(1,1)$ superspace. If we had $V^{(\pm)} = 0$ we would get the $(1,1)$ counterterms
\[ -\frac{1}{2} \int d^2\xi d^2\theta D_+ \phi^i (J_+ C^{(+)})_{ik} D_- \phi^k \tag{6.33} \]

and
\[ \frac{1}{2} \int d^2\xi d^2\theta D_+ \phi^i (C^{(-)} J_-)_{ik} D_- \phi^k \tag{6.34} \]

Since they both come from the same $(2,2)$ lagrangian $K^1$ we know that they should differ by a closed two-form
\[ J_+ C^{(+) + } C^{(-)} J_- = d(K^1_T d\chi - K^1_C d\phi) \tag{6.35} \]
But from (6.22) and (6.30) and the fact that $R_{ik}^{(+)} = R_{ki}^{(-)}$ we know that

$$J_+ C^{(+)} + C^{(-)} J_- = dX + \mathcal{L}_Y g_{ik} + H_{ik}^l Y_l$$  \hspace{1cm} (6.36)

where

$$X = v^{(+)} + v^{(-)}$$  \hspace{1cm} (6.37)
$$Y = v^{(+)} - v^{(-)}$$  \hspace{1cm} (6.38)

which proves that $V^{(\pm)} \neq 0$ when $Y \neq 0$. As we will see in the next section, in the case of commuting complex structures, $v^{(+)} = v^{(-)}$.

### 6.5.1 $[J_+, J_-] = 0$

When $[J_+, J_-] = 0$, there are only chiral and twisted chiral superfields and the $(2, 2)$ superspace Lagrangian is given by a potential

$$K = K(\phi, \bar{\phi}, \chi, \bar{\chi}) .$$  \hspace{1cm} (6.39)

On converting to $(2, 1)$ superspace, this gives a potential

$$\chi^{(+)} = -J_- dK = d_\bar{c} K .$$  \hspace{1cm} (6.40)

The $(2, 2)$ one-loop counterterm (6.31) where

$$A = \det(-K_{\bar{c}i}), \quad B = \det(K_{c\bar{c}})$$  \hspace{1cm} (6.41)

and the vector $W$ is holomorphic with respect to both $J_\pm$:

$$W = (W^c(\phi), W^{\bar{c}}(\bar{\phi}), W^i(\chi), W^{\bar{i}}(\bar{\chi})) .$$  \hspace{1cm} (6.42)

gives a $(2, 1)$ one-loop counterterm with

$$\chi^{+1} = -J_- dK^1 .$$  \hspace{1cm} (6.43)

Note that

$$\sqrt{\det(g_{ij})} = AB$$  \hspace{1cm} (6.44)

and $v^{(+)}$ is given by

$$v^{(+)}_{i} = -\partial_i \ln A , \quad v^{(+)}_{\bar{c}} = -\partial_{\bar{c}} \ln B ,$$  \hspace{1cm} (6.45)
so that using (6.16), $\Gamma^{(+)}_a$ is given by

$$\Gamma^{(+)}_c = i (-2 \partial_c \ln A + \partial_c \ln (AB)) = -i \partial_c \ln (A/B) \quad (6.46)$$

and

$$\Gamma^{(+)}_t = i (-2 \partial_t \ln B + \partial_t \ln (AB)) = i \partial_t \ln (A/B) \quad . \quad (6.47)$$

As a result

$$\Gamma^{(+)} = \lambda^+ = -J_- dK^1 \quad (6.48)$$

and the $(2, 2)$ counterterm (6.31) with $W = 0$ gives the $(2, 1)$ counterterm (6.18) with $V^{(+)} = 0, \rho^{(+)} = 0$.

Similarly, going to $(1, 2)$ superspace we have

$$\lambda^{(-)} = J_+ dK = -d_+ K \quad (6.49)$$

and the one loop counterterm becomes

$$\lambda^{-1} = J_+ dK^1 \quad . \quad (6.50)$$

In this case we have

$$v^{(-)} = v^{(+)} \quad (6.51)$$

$$\Gamma^{(-)}_c = -i \partial_c \ln (A/B) \quad (6.52)$$

$$\Gamma^{(-)}_t = -i \partial_t \ln (A/B) \quad . \quad (6.53)$$

As a result

$$\Gamma^{(-)} = -\lambda^{-1} = -J_+ dK^1 \quad (6.54)$$

Note that if the generalized Monge-Ampère equation [5]

$$A = B \quad (6.55)$$

is satisfied, then $K^1 = 0$ and

$$v_{i}^{(\pm)} = -2 \partial_i \Phi \quad , \quad (6.56)$$

where

$$\Phi = -2 \ln A \quad (6.57)$$

is the dilation (not to be confused with a $(1, 1)$ superfield).
6.5.2 General \([J_+, J_-] \neq 0\)

In the general case, the (2,2) superspace Lagrangian is given by the potential

\[
K = K(\phi, \bar{\phi}, \chi, \bar{\chi}, X_l, \bar{X}_l, X_r, \bar{X}_r) .
\]  

(6.58)

The one-loop counterterm is then proportional to

\[
\int d^2 \xi d^4 \theta \left[ K^1 + W^c(\phi)K_c + \bar{W}^t(\chi)K_t + W^l(\phi, \chi, X_l)K_l + W^r(\phi, \bar{\chi}, X_r)K_r + c.c. \right] ,
\]  

(6.59)

where \(K^1\) is the one-loop counterterm calculated in [17]. It was shown in [5] that \(K^1\) can be rewritten as

\[
K^1 = \ln \left( \frac{A}{B} \right) ,
\]  

(6.60)

where

\[
A = \det \begin{pmatrix} -K_{l\bar{t}} & -K_{t\bar{l}} & -K_{t\bar{t}} \\ -K_{\bar{l}l} & -K_{\bar{r}r} & -K_{\bar{r}l} \\ -K_{\bar{l}t} & -K_{l\bar{r}} & -K_{l\bar{t}} \end{pmatrix} ,
\]

\[
B = \det \begin{pmatrix} K_{l\bar{r}} & K_{l\bar{l}} & K_{t\bar{c}} \\ K_{r\bar{r}} & K_{r\bar{l}} & K_{r\bar{c}} \\ K_{c\bar{r}} & K_{c\bar{l}} & K_{c\bar{c}} \end{pmatrix} .
\]  

(6.61)

The condition for one-loop finiteness is then

\[
K^1 + W^c(\phi)K_c + \bar{W}^t(\chi)K_t + W^l(\phi, \chi, X_l)K_l + W^r(\phi, \bar{\chi}, X_r)K_r + c.c. \\
= f^+(\phi, \chi, X_l) + \bar{f}^+(\phi, \bar{\chi}, \bar{X}_l) + f^-(\phi, \bar{\chi}, X_r) + \bar{f}^-(\phi, \chi, \bar{X}_r) .
\]  

(6.62)

Note that the determinant of the metric in these coordinates is [5]

\[
\sqrt{\det g_{\mu\nu}} = \frac{(-1)^{d_1 d_3}}{\det K_{L,R}} AB
\]  

(6.63)

or in coordinates adapted to either of the complex structures is

\[
\det g_{a\bar{b}} = \frac{1}{(\det K_{L,R})^2 AB} .
\]  

(6.64)

The counterterm [6.59] for any given choice of the vector field \(W\) (including \(W = 0\)) must give a counterterm in (1,1) superspace of the form [6.1] for some definite \(U, \alpha\) (which will depend on the choice of \(W\)).
6.6 Renormalization

We can also compare with the \((2, 1)\) superspace counterterm \([6.18]\). Note that there is a subtlety here: Quantization in \((2, 1)\) superspace preserves \(J_+\) and the chiral constraints, so that the ambiguity involves a holomorphic vector field \(V^{(+)}\). In \((2, 2)\) superspace, the complex structures are implicitly defined in terms of the holomorphic coordinates, for \(J_+\) they are \(\phi; X; Y = \partial K/\partial X_i\). Quantization in \((2, 2)\) superspace preserves the structure leading to the superfield constraints (the semichirality constraints on \(X_L\) and \(X_R\)) but does not preserve \(J_+\). This means that for models with semichiral superfields \(X\) in \((2, 2)\) superspace, the counterterm in \((2, 1)\) superspace arising from the \((2, 2)\) counterterm differs from the the \((2, 1)\) counterterm \([6.18]\) by a choice of \(V^{(+)}\) that is incompatible with holomorphy, and the two results can be reconciled only by descending all the way to \((1, 1)\) superspace.

The one-loop quantum theory in \((2, 2)\) superspace is given in terms of a potential

\[
\hat{K} = K + \hbar tK^1 + O(\hbar^2),
\]

where \(K\) is the classical potential, \(\hbar\) is the loop-counting parameter and \(t\) is a scale-dependent term. (Here \(t\) is proportional to \(\log(\mu^2/m^2)\) where \(\mu\) is the ultraviolet regularisation mass scale and \(m\) is the infrared regularisation mass scale.) To find the \((2, 1)\) superspace form, we use the results from sections 3 and 4 and use \(\hat{K}\) instead of \(K\) in equations \((5.3),(5.6),(5.7),(5.8),(5.10),(5.11)\). For the one-loop corrections, we seek the terms linear in \(\hbar\). The change \(K^1\) to the potential does not change the definition of the fields \(X_{L,R}, \phi, \chi;\) however, it does change the \(J_+, J_-\) holomorphic fields \(Y_L, Y_R\), respectively, and hence it renormalizes the complex structures \(J_{\pm}\) (when the model has semichiral fields, \(i.e.,\) when \([J_+, J_-] \neq 0\)). The reason why the holomorphic fields \(Y\) change is that they are defined through derivatives of \(K\) which changes. For instance

\[
\hat{Y}_L = \frac{\partial K}{\partial X_L} = Y_L + \hbar tY_L^{(1)} + O(\hbar^2),
\]

where

\[
Y_L^{(1)} = K_L^1.
\]

\[\text{4}\]This may seem surprising, but there is an analog in the usual \((2, 2)\) chiral superfield description of hyperkähler manifolds: the \((2, 2)\) Lagrangian is the Kähler potential with respect to a given complex structure; changing the complex structure leads to a different Kähler potential that cannot be related to the original by any holomorphic coordinate redefinition, but descending to \((1, 1)\) superspace, one may find a real coordinate redefinition that shows the equivalence of the apparently different \((2, 2)\) models.
7 Conclusions

We have investigated the role of (2, 1) superspace in generalized Kähler geometry. In particular we have discussed the treatment of the one-loop quantum corrections in (2, 1) superspace. The main advantage of (2, 1) superspace is that all fields are chiral in contrast to the multitude of superfields necessary to describe the most general sigma-model in (2, 2) superspace. It is interesting to notice that the one-loop counterterm is proportional to the $U(1)$ connection of the target space geometry. We have shown how the counterterms reduces when one integrates out part of the superspace coordinates showing the necessity of nontrivial wave function renormalization to reconcile the results. An open question is how to express the $U(1)$ connection in terms of the generalized Kähler potential.

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