Primal-Dual Method for Optimization Problems with Changing Constraints

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Abstract: We propose a modified primal-dual method for general convex optimization problems with changing constraints. We obtain properties of Lagrangian saddle points for these problems which enable us to establish convergence of the proposed method. We describe specializations of the proposed approach to multi-agent optimization problems under changing communication topology and to feasibility problems.

Key words: Convex optimization, changing constraints, primal-dual method, constrained multi-agent optimization, feasibility problem.

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1 Introduction

It is well known that the general optimization problem consists in finding the minimal value of some goal function \( \hat{f} \) on a feasible set \( \tilde{D} \). For brevity, we write this problem as

\[
\min_{v \in \tilde{D}} \to \hat{f}(v).
\]

In many cases, only some approximations are known instead of the exact values of the goal function and the feasible set. This situation is caused by various circumstances. On the one hand, this is due to inevitable calculation errors of values of cost and constraint functions. On the other hand, this is due to incompleteness of information about these functions since their parameters may be specialized during the computational process. Such problems are called non stationary; see e.g. [1] and [2, Chapter VI, §3]. Besides, some perturbations can be inserted for attaining better properties in comparison with the initial one as in various regularization methods; see e.g. [3]. In these problems,
only some sequences of approximations \{\tilde{D}_k\} and \{\tilde{f}_k\} are known, which however must converge in some sense to the exact values of \(\tilde{D}\) and \(\tilde{f}\). The case where the convergence is not obligatory seems more difficult, but it also appears in many applied problems. For instance, large-scale models may contain superfluous constraints and variables together with the necessary ones, but only some of them can be utilized at a given iterate. Various decentralized multi-agent optimization problems can serve as examples of such systems; see e.g. [4, 5] and the references therein.

In this paper we investigate just general convex optimization problems with changing constraints. First we obtain properties of Lagrangian saddle points for these problems. They enable us to propose a modification of the primal-dual method from [7] for finding their solutions. We establish different convergence properties of the proposed method under rather weak assumptions. We describe specializations of the proposed approach to multi-agent optimization problems under changing communication topology and to feasibility problems.

2 The general problem with changing constraints and its properties

Let us consider first a general optimization problem of the form

\[
\min_{x \in D} f(x)
\]

for some function \(f : \mathbb{E} \to \mathbb{R}\) and set \(D \subseteq \mathbb{E}\) in a finite-dimensional space \(\mathbb{E}\). The set of its solutions is denoted by \(D^*\), and the optimal function value by \(f^*\), i.e.

\[
f^* = \inf_{x \in D} f(x).
\]

It will be suitable for us to specialize this problem as follows. For each \(x \in \mathbb{E}\), let \(x = (x_i)_{i=1,...,m}\), i.e. \(x^\top = (x_i^\top, \ldots, x_m^\top)\), where \(x_i = (x_{i1}, \ldots, x_{in})^\top\) for \(i = 1, \ldots, m\), hence \(\mathbb{E} = \mathbb{R}^{mn}\). This means that each vector \(x\) is divided into \(m\) subvectors \(x_i \in \mathbb{R}^n\). In case \(n = 1\) we obtain the custom coordinates of \(x\). Next, we suppose that

\[
D = \{x \in X \mid Ax = b\},
\]

where \(X\) is a subset of \(\mathbb{R}^{mn}\), the matrix \(A\) has \(ln\) rows and \(mn\) columns, so that \(b = (b_i)_{i=1,...,l}\), \(b_i \in \mathbb{R}^n\) for \(i = 1, \ldots, m\), and \(b \in \mathbb{R}^ln\).

In what follows, we will use the following basic assumptions.

(A1) The set \(D^*\) is nonempty, \(X\) is a convex and closed set in \(\mathbb{R}^{mn}\).

(A2) \(f : \mathbb{R}^{mn} \to \mathbb{R}\) is a convex function.
For brevity, we set $M = \{1, \ldots, m\}$ and $L = \{1, \ldots, l\}$. It is clear that the matrix $A$ is represented as follows:

$$A = \begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_l \end{pmatrix},$$

where $A_i$ is the corresponding $n \times mn$ sub-matrix of $A$ for $i \in L$. We will write this briefly

$$A = \left( \{A_i^\top \}_{i \in L} \right)^\top.$$

Similarly, we can determine some other submatrices

$$A_I = \left( \{A_i^\top \}_{i \in I} \right)^\top$$

for any $I \subseteq L$, hence $A = A_L$. Setting

$$F_I = \{ x \in \mathbb{R}^{mn} \mid A_I x = b_I \} \text{ and } D_I = \{ x \in X \mid A_I x = b_I \} = X \cap F_I, \quad (3)$$

where $b_I = (b_i)_{i \in I}$, we obtain a family of optimization problems

$$\min_{x \in D_I} \rightarrow f(x). \quad (4)$$

As above, we denote the solution set of problem (3)–(4) by $D^*_I$, and the optimal function value by $f^*_I$, so that $D^*_L = D^*$ and $f^*_L = f^*$. Clearly, if $I \subseteq J$, then $f^*_I \leq f^*_J$. We intend to establish some properties related to superfluous constraints. We will denote by $F^*$ the solution set of the optimization problem

$$\min_{x \in X} \rightarrow f(x),$$

and its optimal function value by $f^{**}$.

**Lemma 1** Suppose the set $F^* \cap F_I$ is nonempty for some $I \subseteq L$. Then $f^{**} = f^*_I$ and $F^* \cap F_I = D^*_I$.

**Proof.** If $x^* \in F^* \cap F_I$, then clearly $x^* \in D^*_I$, hence $f^{**} = f^*_I$. It follows that $F^* \cap F_I = D^*_I$. \hfill \Box

**Definition 1** We say that $I \subseteq J$ is a basic index set with respect to $J$ if

$$A_I x = b_I \implies A_J x = b_J.$$

We say that $I \subseteq L$ is a basic index set if it is a basic index set with respect to $L$.

From the definitions we obtain immediately the simple but useful properties.
Lemma 2

(i) If $I \subset J$ is a basic index set with respect to $J$, then $f_I^* = f_J^*$, $D_I = D_J$, and $D_I^* = D_J^*$.

(ii) If $I$ is a basic index set, then $f_I^* = f^*$, $D_I = D$, and $D_I^* = D^*$.

For each problem (3)–(4) associated with an index set $I \subseteq L$ we can define its Lagrange function

$$L_I(x, y) = f(x) + \langle y_I, A_I x - b_I \rangle$$

and the corresponding saddle point problem. It appears more suitable to utilize the general Lagrange function

$$L(x, y) = f(x) + \langle y, A x - b \rangle,$$

with the modified dual feasible set. Namely, we say that $w^* = (x^*, y^*) \in X \times Y_I$ is a saddle point for problem (3)–(4) if

$$\forall y \in Y_I, \quad L(x, y) \leq L(x^*, y^*) \leq L(x^*, y^*) \quad \forall x \in X,$$

where

$$Y_I = \{ y = (y_i)_{i \in L} \in \mathbb{R}^m \mid y_i = 0 \in \mathbb{R}^n \text{ for } i \notin I \}.$$

We denote by $W_I^* = D_I^* \times Y_I^*$ the set of saddle points in (5) since $D_I^*$ is precisely the solution set of problem (3)–(4), whereas $Y_I^*$ is the set of its Lagrange multipliers. Since $D_L^* = D^*$, we also set $Y^* = Y_L^*$, i.e. $W^* = D^* \times Y^*$ is the set of saddle points for the initial problem (1)–(2). Observe that (5) is rewritten equivalently as follows:

$$A_I x^* = b_I, \quad L(x^*, y^*) \leq L(x, y^*) \quad \forall x \in X.$$  (6)

Besides, if we take $I = \emptyset$, then $Y_I = \{0\}$, hence we can write $D_I^* = F^*$ and $Y_I^* = \{0\}$.

**Proposition 1** Suppose that assumptions (A1)–(A2) are fulfilled. If $I \subset J$ is a basic index set with respect to $J$, then $D_I^* = D_J^*$ and $Y_I^* \subseteq Y_J^*$.

**Proof.** The first equality follows from Lemma 2 (i). If $(x^*, y^*) \in D_I^* \times Y_I^*$, then (6) holds, which now implies (5) with $I = J$. Hence $y^* \in Y_J^*$.$\square$

**Corollary 1** Suppose that assumptions (A1)–(A2) are fulfilled. If $I$ is a basic index set, then $D_I^* = D^*$ and $Y_I^* \subseteq Y^*$.

We can establish similar relations for dual variables in case $F^* \cap F_I \neq \emptyset$.

**Proposition 2** Suppose that assumptions (A1)–(A2) are fulfilled, the set $F^* \cap F_I$ is nonempty for some $I \subseteq L$. Then $F^* \cap F_I = D_I^*$ and $0 \in Y_I^*$.

**Proof.** The first equality follows from Lemma 1. Take any $x^* \in F^* \cap F_I$, then $x^* \in D_I^*$ and (6) holds with $y^* = 0$. Therefore, $0 \in Y_I^*$.$\square$
3 Primal-dual method for the family of saddle point problems

We intend to find saddle points in (5) by a modification of the primal-dual method that was proposed in [7]. First we note that the set of saddle points for the initial problem (1)–(2) is nonempty under the assumptions in (A1)–(A2); see e.g. [8, Corollary 28.2.2]. Therefore, this is the case for each saddle point problem in (5) associated with a basic index set \( I \). Denote by \( \pi_U(u) \) the projection of \( u \) onto \( U \). Also, for simplicity we will write \( Y(k) = Y_{I_k} \), \( Y^* = Y^*_{I_k} \), etc. Then the method is described as follows.

Method (PDM). Step 0: Choose an index set \( I_0 \subseteq L \), a point \( w_0 = (x_0, y_0) \in X \times Y(0) \). Set \( k = 1 \).

Step 1: Choose an index set \( I_k \subseteq L \) and a number \( \lambda_k > 0 \).

Step 2: Take \( p^k = \pi_{Y(k)}[y^{k-1} + \lambda_k(Ax^{k-1} - b)] \).

Step 3: Take \( x^k = \text{argmin}\{f(x) + \langle p^k, Ax - b \rangle + 0.5\lambda_k^{-1}\|x - x^{k-1}\|^2 \mid x \in X\} \).

Step 4: Take \( y^k = \pi_{Y(k)}[y^{k-1} + \lambda_k(Ax^k - b)] \). Set \( k = k + 1 \) and go to Step 1.

First we observe that

\[
p^k = \text{argmin}\{-L(x^{k-1}, p) + 0.5\lambda_k^{-1}\|p - y^{k-1}\|^2 \mid p \in Y(k)\}
\]

and

\[
y^k = \text{argmin}\{-L(x^k, y) + 0.5\lambda_k^{-1}\|y - y^{k-1}\|^2 \mid y \in Y(k)\}.
\]

Therefore, each iteration of (PDM) involves two projection (proximal) steps in the dual variable \( y \) and one proximal step in the primal variable \( x \). The point \( w^k = (x^k, y^k) \) belongs to \( X \times Y(k) \). The next two properties follow the usual substantiation schemes for this method; see [7] and also [9].

Lemma 3 Suppose \( U \) is a closed convex set in a finite-dimensional space \( E \), \( \varphi : E \to \mathbb{R} \) is a convex function, \( u \) is a point in \( E \). If

\[\mu(z) = \varphi(z) + 0.5\lambda^{-1}\|z - u\|^2, \quad \lambda > 0,\]

and

\[v = \text{argmin}\{\mu(z) \mid z \in U\},\]

then

\[2\lambda\{\varphi(v) - \varphi(z)\} \leq \|z - u\|^2 - \|z - v\|^2 - \|v - u\|^2 \quad \forall z \in U. \tag{7}\]

Proof. Since the function \( \mu \) is strongly convex with constant \( \lambda^{-1} \), we have

\[\mu(z) - \mu(v) \geq 0.5\lambda^{-1}\|z - v\|^2 \quad \forall z \in U.\]

This inequality gives \( \Box \).
Proposition 3 Suppose that assumptions (A1)–(A2) are fulfilled. For any pair \( w^* = (x^*, y^*) \in D^*_k \times Y^*_k \) we have

\[
\|w^k - w^*\|^2 \leq \|w^{k-1} - w^*\|^2 - \|p^k - y^k\|^2 - \|p^k - y^{k-1}\|^2 - \|x^k - x^{k-1}\|^2 \\
+ 2\lambda_k \langle y^k - p^k, A(x^k - x^{k-1}) \rangle \\
= \|w^{k-1} - w^*\|^2 - \|p^k - y^k\|^2 - \|p^k - y^{k-1}\|^2 - \|x^k - x^{k-1}\|^2 \\
+ 2\lambda_k^2\|A(k)(x^k - x^{k-1})\|^2.
\] (8)

Proof. Choose any \( w^* = (x^*, y^*) \in D^*_k \times Y^*_k \). Setting \( \varphi(z) = L(z, p^k), \lambda = \lambda_k, U = X, u = x^{k-1}, v = x^k, \) and \( z = x^* \) in (7) gives

\[
2\lambda_k \{L(x^k, p^k) - L(x^*, p^k)\} \leq \|x^* - x^{k-1}\|^2 - \|x^* - x^k\|^2 - \|x^k - x^{k-1}\|^2.
\]

Also, using (5) with \( I = I_k, x = x^k, \) and \( y = p^k \) gives

\[
2\lambda_k \{L(x^*, p^k) - L(x^k, y^*)\} \leq 0.
\]

Adding these inequalities, we obtain

\[
\|x^k - x^*\|^2 \leq \|x^{k-1} - x^*\|^2 - \|x^k - x^{k-1}\|^2 + 2\lambda_k \langle p^k - y^*, Ax^k - b \rangle.
\] (9)

On the other hand, setting \( \varphi(z) = -L(x^{k-1}, z), \lambda = \lambda_k, U = Y_k, u = y^{k-1}, v = p^k, \) and \( z = y^k \) in (7) gives

\[
2\lambda_k \{L(x^{k-1}, y^k) - L(x^{k-1}, p^k)\} \leq \|y^k - y^{k-1}\|^2 - \|p^k - y^k\|^2 - \|p^k - y^{k-1}\|^2.
\]

Next, setting \( \varphi(z) = -L(x^k, z), \lambda = \lambda_k, U = Y_k, u = y^{k-1}, v = y^k, \) and \( z = y^* \) in (7) gives

\[
2\lambda_k \{L(x^{k-1}, y^*) - L(x^k, y^k)\} \leq \|y^* - y^{k-1}\|^2 - \|y^* - y^k\|^2 - \|y^k - y^{k-1}\|^2.
\]

Adding these inequalities, we obtain

\[
\|y^k - y^*\|^2 \leq \|y^{k-1} - y^*\|^2 - \|p^k - y^k\|^2 - \|p^k - y^{k-1}\|^2 \\
- 2\lambda_k \{\langle y^* - y^k, Ax^{k-1} - b \rangle + \langle y^k - p^k, Ax^k - b \rangle\}.
\] (10)

Now adding (9) and (10) gives the first inequality in (8). Since

\[
\langle y^k - p^k, A(x^k - x^{k-1}) \rangle = \lambda_k \|A(k)(x^k - x^{k-1})\|^2,
\]

we conclude also that the second relation in (8) holds true. \( \square \)

Now we can indicate conditions that provide basic convergence properties.
**Theorem 1** Suppose that assumptions (A1)–(A2) are fulfilled,

\[
\bigcap_{k=1}^\infty W^*_k \neq \emptyset,
\]  

(11)

the sequence \(\{\lambda_k\}\) satisfies the condition:

\[
\lambda_k \in \left[\tau, \frac{\sqrt{(1-\tau)}}{\sqrt{2\|A(k)\|}}\right]
\]  

(12)

for some \(\tau \in (0, 1)\). Then:

(i) the sequence \(\{w^k\}\) has limit points,

(ii) each of these limit points is a solution of problem (5) for some \(I \subseteq L\),

(iii) for any limit point \(\bar{w}\) of \(\{w^k\}\) such that

\[
\bar{w} \in \bigcap_{k=1}^\infty W^*_k
\]  

it holds that

\[
\lim_{k \to \infty} w^k = \bar{w}.
\]  

(13)

**Proof.** Take any point

\[
w^* \in \bigcap_{k=1}^\infty W^*_k.
\]

Then from (8) and (12) we have

\[
\|w^k - w^*\|^2 \leq \|w^{k-1} - w^*\|^2 - \|p^k - y^k\|^2 - \|p^k - y^{k-1}\|^2 - \tau\|x^k - x^{k-1}\|^2
\]  

(14)

for \(k = 1, 2, \ldots\) Hence, the sequence \(\{w^k\}\) is bounded and has limit points, i.e. part (i) is true. Besides, (14) gives

\[
\lim_{k \to \infty} \|w^k - w^*\| = \sigma \geq 0
\]  

(15)

and

\[
\lim_{k \to \infty} \|p^k - y^k\| = \lim_{k \to \infty} \|p^k - y^{k-1}\| = \lim_{k \to \infty} \|x^k - x^{k-1}\| = 0,
\]  

(16)

hence

\[
\lim_{k \to \infty} \|y^k - y^{k-1}\| = 0.
\]  

(17)

Let \(\bar{w} = (\bar{x}, \bar{y})\) be an arbitrary limit point of \(\{w^k\}\), i.e.

\[
\bar{w} = \lim_{s \to \infty} w^{k_s}.
\]
Then there exists $J \subseteq L$ such that $J = I_{k_s}$ for infinitely many times. Without loss of generality we can suppose that $J = I_{k_s}$ for any $s$. Then $w^{k_s} = (x^{k_s}, y^{k_s}) \in X \times Y_J$ for any $s$, hence $\bar{w} = (\bar{x}, \bar{y}) \in X \times Y_J$. Setting $\varphi(z) = \mathcal{L}(z, p^k)$, $\lambda = \lambda_k$, $U = X$, $u = x^{k-1}$, $v = x^k$, and $z = x \in X$ in (7) gives

$$2\lambda_k \{\mathcal{L}(x^k, p^k) - \mathcal{L}(x, p^k)\} \leq \|x - x^{k-1}\|^2 - \|x - x^k\|^2 - \|x^k - x^{k-1}\|^2.$$

Taking the limit $k = k_s \to \infty$ due to (16)–(17) gives

$$\mathcal{L}(\bar{x}, \bar{y}) - \mathcal{L}(x, \bar{y}) \leq 0. \tag{18}$$

Also, setting $\varphi(z) = -\mathcal{L}(x^k, z)$, $\lambda = \lambda_k$, $U = Y_J$, $u = y^{k-1}$, $v = y^k$, and $z = y \in Y_J$ in (7) gives

$$2\lambda_k \{\mathcal{L}(x^k, y) - \mathcal{L}(x^k, p^k)\} \leq \|y^{k-1} - y\|^2 - \|y^k - y\|^2 - \|y^k - y^{k-1}\|^2.$$

Taking the limit $k = k_s \to \infty$ due to (16)–(17) gives

$$\mathcal{L}(\bar{x}, y) - \mathcal{L}(\bar{x}, \bar{y}) \leq 0. \tag{19}$$

It follows from (18) and (19) that $\bar{w} = (\bar{x}, \bar{y}) \in W_J^* = D_J^* \times Y_J^*$. Hence, part (ii) is also true.

Next, if

$$\bar{w} \in \bigcap_{k=1}^{\infty} W_{(k)}^*,$$

we can set $w^* = \bar{w}$ in (15). However, now $\sigma = 0$, which gives (13) and part (iii) is true. $\Box$

These properties enable us to establish convergence to a solution under suitable conditions.

**Theorem 2.** Suppose that assumptions (A1)–(A2) are fulfilled, the sequence $\{\lambda_k\}$ satisfies condition (12) for some $\tau \in (0, 1)$.

(i) If there exists a nonempty basic index set $I \subseteq L$ such that $I \subseteq I_k$, then the sequence $\{w^k\}$ has limit points and each of these limit points belongs to $W^*$.

(ii) If in addition $I = I_{k_s}$ for some infinite subsequence $\{w^{k_s}\}$, then

$$\lim_{k \to \infty} w^k = w^* \in W^*. \tag{20}$$

**Proof.** By definition, the sets $W_I^*$ and $W^*$ are now nonempty. Due to Proposition $\blacksquare$, $W_I^* \subseteq W_{(k)}^*$, hence condition (11) holds. Then the sequence $\{w^k\}$ has limit points due to Theorem $\blacksquare$ (i). Also, there exists $J \subseteq L$ such that $I \subseteq J = I_{k_s}$ for infinitely many times. But now $J$ is a nonempty basic index set, hence $W_J^* \subseteq W^*$. Following
the lines of part (ii) of Theorem 1 we obtain that any limit point of \( \{w^{k_s}\} \) will belong to \( W^*_j \subseteq W^* \). Therefore, part (i) is true. In case (ii) we have similarly that any limit point \( w^* \) of \( \{w^{k_s}\} \) will belong to \( W^*_I \), but

\[
 w^* \in \bigcap_{k=1}^{\infty} W^*_k^{(k)}.
\]

The result now follows from Theorem 1 (iii).

**Theorem 3** Suppose that assumptions (A1)–(A2) are fulfilled, \( F^* \cap F_L \neq \emptyset \), the sequence \( \{\lambda_k\} \) satisfies condition (12) for some \( \tau \in (0,1) \). Then:

(i) the sequence \( \{w^k\} \) has limit points,

(ii) if each \( I_k \) is a basic index set, all the limit points of \( \{w^k\} \) belong to \( W^* \),

(iii) if there exists a nonempty basic index set \( I \subseteq L \) such that \( I \subseteq I_k \) and \( I = I_k \) for some infinite subsequence \( \{w^{k_s}\} \), the sequence \( \{w^k\} \) converges to a point of \( W^* \).

**Proof.** Due to Proposition 2, we now have \( F^* \cap F_L = D^*_I \), \( D^*_I \neq \emptyset \), and \( 0 \in Y_i^* \) for any \( I \subseteq L \). It follows that

\[
 \left\{ F^* \cap F_L \right\} \times \{0\} \subseteq \bigcap_{k=1}^{\infty} W^*_k^{(k)}.
\]

Therefore, (11) holds and assertion (i) follows from Theorem 1 (i). Following the lines of part (ii) of Theorem 1 we obtain that any limit point of \( \{w^{k_s}\} \) will belong to \( W^*_j \subseteq W^* \) where \( J \) is a nonempty basic index set. Therefore, assertion (ii) is also true. Assertion (iii) clearly follows from Theorem 2.

The conditions of part (ii) of Theorem 2 are satisfied if for instance we take the rule \( I_k \subseteq I_{k+1} \) or \( I_{k+1} \subseteq I_k \) for index sets. These rules can be also applied in part (iii) of Theorem 3. In all the above theorems we utilized some conditions that must hold for each iterate \( k \). Obviously, all the assertions of the theorems will be true if we require for the same conditions to hold only for \( k \geq k' \) where \( k' \) is some fixed number.

4 **Primal-dual method for multi-agent optimization problems**

We now describe a specialization of the proposed approach to the multi-agent optimization problem

\[
 \min \rightarrow \left\{ \sum_{i=1}^{m} f_i(v) \left| \bigcap_{i=1}^{m} X_i \right. \right\},
\]

(21)
where $m$ is the number of agents (units) in the system. That is, the information about the function $f_i$ and set $X_i$ is known only to the $i$-th agent and may be unknown even to its neighbours. Besides, it is usually supposed that the agents are joined by some transmission links for information exchange so that the system is usually a connected network, whose topology may vary from time to time. This decentralized system has to find a concordant solution defined by (21).

For this reason, we replace (21) with the family of optimization problems of the form

$$\min_{x \in D_I} f(x) = \sum_{i=1}^{m} f_i(x_i),$$

(22)

where $x = (x_i)_{i=1,...,m} \in \mathbb{R}^{mn}$, i.e. $x^\top = (x_1^\top, \ldots, x_m^\top)$, $x_i = (x_{i1}, \ldots, x_{in})^\top$ for $i = 1, \ldots, m$.

$$D_I = X \bigcap F_I, \quad X = X_1 \times \ldots \times X_m = \prod_{i=1}^{m} X_i, \quad X_i \subseteq \mathbb{R}^n, \quad i = 1, \ldots, m;$$

(23)

the set $F_I$ describes the information exchange scheme within the current topology of the communication network, and $I$ is the index set of arcs of the corresponding oriented graph. More precisely, the maximal (full) communication network with non-oriented edges denoted by $\mathcal{F}$ corresponds to the set

$$\tilde{F} = \{ x \in \mathbb{R}^{mn} \mid x_s = x_t, \quad s, t = 1, \ldots, m, \quad s \neq t \},$$

i.e. each edge is associated with two directions or equations ($x_s = x_t$ and $x_t = x_s$). However, this definition of topology is superfluous. It seems more suitable to introduce some other graph topology for writing the multi-agent optimization problem in addition to the graph $\mathcal{F}$. For this reason, we associate each pair of vertices (agents) $(s, t)$ to one oriented arc $i$, so that $L = \{1, \ldots, l\}$ is the index set of all these arcs, hence $l = m(m - 1)/2$. That is, each arc $(s, t)$ is in fact used in both the directions in the communication network $\mathcal{F}$, but we fix only one direction for definition of the multi-agent optimization problem and obtain the graph $\mathcal{G}$. Taking subsets $I \subseteq L$, we obtain various constraint sets

$$F_I = \{ x \in \mathbb{R}^{mn} \mid x_s - x_t = 0, \quad i = (s, t) \in I \},$$

(24)

corresponding to the oriented graphs $\mathcal{G}_I$ in the the multi-agent optimization problem formulation. Replacing the arcs in $\mathcal{G}_I$ with non-oriented edges, we obtain the corresponding communication network $\mathcal{F}_I$ of the system. It follows that $\mathcal{F} = \mathcal{F}_L$, $\mathcal{G} = \mathcal{G}_L$, and $F = F_L$. Next, for each arc $i = (s, t)$ we can define the $n \times mn$ sub-matrix

$$A_i = (A_{i1} \cdots A_{im}),$$

where

$$A_{ij} = \begin{cases} E, & \text{if } j = s, \\ -E, & \text{if } j = t, \\ \Theta, & \text{otherwise,} \end{cases}$$
$E$ is the $n \times n$ unit matrix, $\Theta$ is the $n \times n$ zero matrix. Then clearly

$$F_I = \{ x \in \mathbb{R}^{mn} \mid A_I x = 0 \},$$

where

$$A_I = \left( \{ A_i^T \}_{i \in I} \right)^T,$$

which corresponds to the definition in (3) for $b_I = 0$ and any $I \subseteq L$, hence we can set $A = A_L$. Therefore, our problem (22)–(24) corresponds to (3)–(4).

In what follows, we will use the following basic assumptions.

**(B1)** For each $i = 1, \ldots, m$, $X_i$ is a convex and closed set in $\mathbb{R}^n$, $f_i : \mathbb{R}^n \to \mathbb{R}$ is a convex function.

**(B2)** The set $D^* = D_L^*$ is nonempty.

These assumptions imply (A1)–(A2). If the graph $\mathcal{F}_I$ for some $I \subseteq L$ is connected, then $I$ a basic index set. Now we present an implementation of Method (PDM) for the multi-agent optimization problem (22)–(24), where each agent (or unit) receives information only from its neighbours. Given an oriented graph $\mathcal{G}_I$ and an agent $j$, we denote by $\mathcal{N}_I^+(j)$ and $\mathcal{N}_I^-(j)$ the sets of incoming and outgoing arcs at $j$. Since many oriented graphs $\mathcal{G}_I$ are associated with the same graph $\mathcal{F}_I$, we suppose that agent $j$ is responsible for calculation of the current values of the primal variable $x_j$ and all the dual variables $y_i$ and $p_i$ such that $i \in \mathcal{N}_I^-(j)$. That is, we will fix the oriented graph $\mathcal{G}$ and its subgraphs $\mathcal{G}_I$ such that agent $j$ is associated with all the outgoing arcs for vertex $j$. The general Lagrange function for problems (22)–(24) is written as follows:

$$L(x, y) = f(x) + \langle y, Ax \rangle = \sum_{j \in M} f_j(x_j) + \sum_{i \in L} \langle y_i, A_ix \rangle$$

$$= \sum_{j \in M} \left\{ f_j(x_j) + \sum_{i \in \mathcal{N}_I^-(j)} \langle y_i, x_j \rangle - \sum_{i \in \mathcal{N}_I^+(j)} \langle y_i, x_j \rangle \right\}.$$  

The saddle point problems are defined in (5). As in Section 3 for simplicity we will write $Y_{(k)} = Y_{I_k}$, $Y^*_{(k)} = Y^*_{I_k}$, etc.

**Method (PDMI).** At the beginning, the agents choose the communication topology by choosing the active arc index set $I_0 \subseteq L$. Next, each $s$-th agent chooses $x^0_i$ and $y^0_i$ for $i \in \mathcal{N}_0^-(s)$ and reports these values to its neighbours. This means that $y^0_i = 0$ for $i \notin I_0$.

At the $k$-th iteration, $k = 1, 2, \ldots$, each $s$-th agent has the values $x^{k-1}_s$ and $y^{k-1}_i$, $i \in \mathcal{N}_{(k-1)}^-(s)$, and the same values of its neighbours. The agents choose the current communication topology by choosing the active arc index set $I_k \subseteq L$ and determine the stepsize $\lambda_k$. This means that they set $y^k_i = 0$ for $i \notin I_k$. 

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Step 1: Each $s$-th agent sets
\[ p_i^k = y_i^{k-1} + \lambda_k(x_s^k - x_t^k) \quad \forall i = (s, t), \; i \in \mathcal{N}_i^-(s). \] (26)

Then each $s$-th agent reports these values to its neighbours.

Step 2: Each $s$-th agent calculates
\[ v_s^k = \sum_{i \in \mathcal{N}_i^-(s)} p_i^k - \sum_{i \in \mathcal{N}_i^+(s)} p_i^k \]
and
\[ x_s^k = \arg \min_{x_s \in X_s} \left\{ f_s(x_s) + \langle v_s^k, x_s \rangle + 0.5\lambda_k^{-1} \| x_s - x_s^{k-1} \|_2 \right\} \] (27)
and reports this value to its neighbours.

Step 3: Each $s$-th agent sets
\[ y_i^k = y_i^{k-1} + \lambda_k(x_s^k - x_t^k) \quad \forall i = (s, t), \; i \in \mathcal{N}_i^-(s). \] (28)

Then each $s$-th agent reports these values to its neighbours. The $k$-th iteration is complete.

We observe that the agents do not store the dual variables related to the inactive arcs, i.e. $y_i^k = 0$ for $i \notin I_k$. If some arc $i = (s, t) \notin I_{k-1}$ becomes active at the $k$-th iteration, i.e. $i \in I_k$, then agent $s$ simply sets $y_i^{k-1} = 0$.

Due to (25), relations (26)–(28) correspond to Steps 2–4 of (PDMI), respectively. Hence, the convergence properties of (PDMI) will follow directly from Theorems \ref{thm:corollary2} and \ref{thm:corollary3}.

Corollary 2 Suppose that assumptions (B1)–(B2) are fulfilled, the sequence $\{\lambda_k\}$ satisfies condition (12) for some $\tau \in (0, 1)$.

(i) If there exists a nonempty basic index set $I \subseteq L$ such that $I \subseteq I_k$, then the sequence $\{w^k\}$, $w^k = (x^k, y^k)$, generated by (PDMI) has limit points and each of these limit points belongs to $W^*$.

(ii) If in addition $I = I_k$, for some infinite subsequence $\{w^{k_s}\}$, then (20) holds.

Corollary 3 Suppose that assumptions (B1)–(B2) are fulfilled, the sequence $\{\lambda_k\}$ satisfies condition (12) for some $\tau \in (0, 1)$, $F^* \cap F_L \neq \emptyset$. Then:

(i) the sequence $\{w^k\}$, $w^k = (x^k, y^k)$, generated by (PDMI) has limit points,

(ii) if each $I_k$ is a basic index set, all the limit points of $\{w^k\}$ belong to $W^*$,

(iii) if there exists a nonempty basic index set $I \subseteq L$ such that $I \subseteq I_k$ and $I = I_{k_s}$ for some infinite subsequence $\{w^{k_s}\}$, the sequence $\{w^k\}$ converges to a point of $W^*$. 

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Convergence of (PDMI) requires for all the agents to choose the stepsize $\lambda_k$ in accordance with (12), hence they have to evaluate the norm $\|A_{(k)}\|$ at the $k$-th iteration. Fix some $I \subseteq L$, then
\[ A_I^\top A_I = H_I \otimes E, \]
where $H_I$ is the Kirchhoff matrix of the graph $F_I$, $\otimes$ denotes the Kronecker product of matrices. Application of the Gershgorin theorem (see Theorem 5 in [10, Chapter XIV]) gives
\[ \|A_I\| = \sqrt{\|H_I\|} \leq \sqrt{2d(F_I)}, \]
where $d(F_I)$ is the maximal vertex degree of the graph $F_I$. There exist more precise estimates for some special classes of graphs; see e.g. [11, 12]. Together with (12) we obtain the bound
\[ \lambda_k \in \left[ \tau, 0.5 \sqrt{(1 - \tau) / d(F_{(k)})} \right] \quad (29) \]
for some $\tau \in (0, 1)$. In case of varying topology the separate agents may meet difficulties in evaluation of $d(F_{(k)})$ since the graph then may be non-regular. The concordant value of $\lambda = \lambda_k$ satisfying (29) can be obtained by determining some upper bound for $d(F_{(k)})$.

It seems suitable to apply the following strategy. First we choose the fixed topology that corresponds to an arc index set $J \subset L$ so that it gives the connected graph $F_J$ and $J \subseteq I_k$ for any $k$. This means that all the arcs in $J$ remain always active. The status of the other arcs may vary, but the maximal vertex degree of the graph $F_{(k)}$ can not exceed some fixed number $v$. Then each agent can take $\lambda = 0.5 \sqrt{(1 - \tau) / v}$ and the assumptions of Corollary 2 (i) and Corollary 3 (i)–(ii) on the choice of parameters hold.

We now give a natural example of problem (22)–(24) such that $F^* \cap F_L \neq \emptyset$. Namely, set $X_i = \mathbb{R}^n$, $f_i(v) = (1/p)(\max\{h_i(v), 0\})^p$, $p \geq 1$ for $i = 1, \ldots, m$. Then (22)–(24) corresponds to a penalized problem for finding a point of the set
\[ \tilde{V} = \{ u \in \mathbb{R}^n \mid h_i(u) \leq 0, \, i = 1, \ldots, m \}. \]
If $\tilde{V} \neq \emptyset$, then clearly $F^* \cap F_L \neq \emptyset$, which gives stronger convergence properties.

It should be noticed that primal-dual methods are usually applied to large-scale convex optimization problems with binding constraints in order to keep the decomposability properties. However, the streamlined primal-dual gradient projection method requires strengthened assumptions. Utilization of extrapolation steps enables one to attain convergence under custom convex-concavity; see [13]. These methods admit a fixed positive stepsize that yields a linear rate of convergence; see e.g. [14, Chapter VI] and the references therein. However, replacing projections with proximal steps also will enhance convergence, besides the method becomes applicable to non-smooth problems. This primal-dual method with proximal steps was proposed in [7]. Similar methods were described in [9, 15]. It should be also noticed that known iterative methods for multi-agent optimization problems with changing communication topology are based on different conditions; see e.g. [16, 17].
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