Plane-wave model of neutrino oscillations revisited

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Abstract

The phenomenology of massive neutrinos – flavour mixing in the lepton sector causing oscillations between different neutrino-types along their propagation over macroscopic distances in vacuum – aims at relating observable quantities (oscillation frequency or, equivalently, oscillation length) to the neutrino properties: mixing angles $\theta_{ij}$ and mass-squared differences $\Delta m^2_{ij}$.

Calculation of the probabilities for a given neutrino-type either to survive or to mutate into another type, as functions of momentum $p$ and travelling distance $L$, are properly based on wave-packet models of varying complexity. Approximations neglecting subtle effects like decoherence result in the standard oscillation formulae with terms proportional to $\sin^2(\Delta m^2_{ij} L/4p)$.

The same result may also be derived by a simple plane-wave model as shown in most textbooks. However, those approaches rely on unphysical a-priory assumptions: either “equal energy” or “equal velocity” or “equal momentum” in the phases of different mass eigenstates – which are refuted elsewhere. In addition, some assume tacitly that interference occurs at time $t = L$.

This study re-examines the plane-wave model. No unphysical assumption is necessary for deriving the standard formulae: a heuristic approach relies only on carefully defining interference at time $t = L/\beta$, and is justified by coherence arguments based in a qualitative way on wave-packets.

1 Introduction

A minimal extension of the Standard Model for non-zero mass neutrinos requires flavour-mixing in the lepton sector, which is described (analogously to quark mixing) by the $3 \times 3$ unitary Pontecorvo-Maki-Nakagawa-Sakata (PMNS) matrix $U$:

$$ \begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = U \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix} = U^\dagger \begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} \quad (1) $$

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1 Present data do not exclude the possibility of the lowest mass being exactly zero.
where $|\nu_\ell\rangle$ ($\ell = e, \mu, \tau$) is the weak eigenstate of a neutrino created or absorbed in some charged-current weak interaction ("flavour eigenstate"), and $|\nu_i\rangle$ ($i = 1, 2, 3$) is the eigenstate of the free-particle Hamiltonian ("mass eigenstate") describing a neutrino’s kinematic behaviour – which is, however, not directly observable.

After applying the unitary constraints and removing unphysical phases, the matrix elements of $U \equiv (U_{\ell i})$ and of its inverse $U^{-1} = U^\dagger \equiv (U_{i \ell}^*)^T$ can be parametrized by three rotation angles ($\theta_{12}, \theta_{13}, \theta_{23}$) and one complex phase $\delta$, thus introducing the possibility of $CP$ violation in the lepton sector. Take care when a neutrino $|\nu_\ell\rangle$ couples as an adjoint spinor to the weak interaction vertex: its corresponding coupling factor (i.e. the matrix element $U_{\ell i}$ or $U_{i \ell}^*$) will appear as the complex conjugate $[2]$. A neutrino is produced weakly as a well-defined flavour state, but manifests itself as a coherent linear superposition of the three mass eigenstates:

$$\psi_\ell(0,0) \equiv |\nu_\ell\rangle = \sum_{i=1}^3 U_{\ell i}^* |\nu_i\rangle$$  \hspace{1cm} (2)

Each $|\nu_i\rangle$’s momentum $p_i$ and energy $E_i = \sqrt{m_i^2 + p_i^2}$ are separately determined by energy-momentum conservation in the production process. This fact is exploited by precision experiments for measuring the neutrino masses $[1, 4]$. An example is the 2-body decay $\pi^+ \to \nu_\mu \mu^+$ $[1, 4]$: if the $\pi$ mass, the $\mu$ mass and the $\mu$ momentum in the $\pi$ rest frame are known with sufficiently high precision, then the neutrino mass squared is kinematically determined. Such observation causes the superposition to collapse into one specific mass eigenstate $|\nu_\ell\rangle$ with probability $|U_{\mu \ell}|^2$; the measurements yield only an incoherently averaged muon-based effective mass squared $\sum_{i=1}^3 |U_{\mu i}|^2 m_i^2$.

## 2 Plane-wave model

In absence of such an observation, the wave function $\psi_\ell$ will propagate by evolving as coherently superposed plane-waves along e.g. the $x$-direction $[4]$:

$$\psi_\ell(t,x) = \sum_{i=1}^3 U_{\ell i}^* |\nu_i\rangle e^{-i\phi_i}, \text{ phase } \phi_i = E_i t - p_i x$$ \hspace{1cm} (3)

with different phases $\phi_i$ for each of its components $|\nu_\ell\rangle$. The interfering phases will steadily shift apart – this dispersion is the origin of the oscillation.

Note that for any plane-wave, the phase velocity $\equiv E_i/p_i = 1/\beta_i \geq 1$; the group velocity $\equiv dE_i/dp_i = p_i/E_i = \beta_i \leq 1$ is equal to the particle’s velocity in the lab frame. For a wave-packet, $\beta_i$ is the velocity of the packet’s centre.

The neutrino will eventually be detected at a distance $x = L$ by some charged-current weak interaction, and its absorbed flavour $\ell'$ can be identified. Therefore, the mass eigenstates $|\nu_i\rangle$ of eq. (3) have to be re-expressed in terms of flavour eigenstates $|\nu_\ell\rangle$ while keeping into account the evolved individual phases $\phi_i$:

$$\psi_\ell(t,x) = \sum_{\ell' = e, \mu, \tau} \left( \sum_{i=1}^3 U_{\ell i}^* U_{\ell' i} e^{-i\phi_i} \right) |\nu_\ell\rangle$$ \hspace{1cm} (4)

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2. So far, only upper limits can be derived from the effective masses squared $[4]$.  
3. Notwithstanding any uncertainties at production, free-particle propagation is always on-shell $[7]$.  
Assuming vacuum only, no matter effects like MSW need to be taken into account.
At this point the detection causes the wave function \( \psi_L \) to collapse into a specific flavour state \( |\nu_\ell\rangle \), and the probability of the original flavour \( \ell \) to be observed in the detector as flavour \( \ell^* \) (including survival if \( \ell^* = \ell \)) is
\[
P(\nu_\ell \rightarrow \nu_\ell) = |\langle \nu_\ell | \psi_L(t, L) \rangle|^2
\] (5)

Evaluating that in terms of PMNS matrix elements \( U_{\ell i} \) and phases \( \phi_i \) is straightforward, albeit involving some calculations. The results can be found in textbooks, e.g. [2]. As an example, the survival probability is given by
\[
P(\nu_\ell \rightarrow \nu_\ell) = 1 - \sum_{i=1}^{3} 4 |U_{\ell i}|^2 |U_{\ell j}|^2 \sin^2 \left( \frac{\phi_i - \phi_j}{2} \right), \quad \text{with } j = (i \text{ mod } 3) + 1
\] (6)

### 2.1 Two-flavour mixing

The principal consequences of mixing (except \( CP \)-violating effects) are manifest by simply regarding only two neutrino flavours (say, \( \nu_e \) and \( \nu_\mu \)) together with two mass eigenstates \((\nu_1 \text{ and } \nu_2)\)\(^4\). In this scenario, the PMNS matrix is reduced to a \( 2 \times 2 \) orthogonal matrix with only one real parameter, the rotation angle \( \theta = \theta_{12} \):
\[
U = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \quad \text{and} \quad U^{-1} = U^T = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}
\] (7)

The survival probability of \( |\nu_e\rangle \) follows from eqs. [6, 7] with \( i = 1 \), and the mutation probability \( |\nu_e\rangle \rightarrow |\nu_\mu\rangle \) as 1-complement of that value:
\[
P(\nu_e \rightarrow \nu_e) = 1 - \sin^2(2 \theta) \sin^2 \left( \frac{\phi_1 - \phi_2}{2} \right)
\]
\[
P(\nu_e \rightarrow \nu_\mu) = 1 - P(\nu_e \rightarrow \nu_e) = \sin^2(2 \theta) \sin^2 \left( \frac{\phi_1 - \phi_2}{2} \right)
\] (8)

Defining the mean energy and mean momentum \((E \text{ and } p, \text{ respectively})\), the energy and momentum differences \((\Delta E \text{ and } \Delta p, \text{ respectively})\), and the velocity \( \beta \) of a fictive particle moving with mean 4-momentum \((E, p, 0, 0)\):
\[
E = (E_1 + E_2)/2 \quad \Delta E = E_1 - E_2 \neq 0
\]
\[
p = (p_1 + p_2)/2 \quad \Delta p = p_1 - p_2 \neq 0
\]
\[
\beta = p/E \neq (\beta_1 + \beta_2)/2
\] (9)

hence, the difference between two phases in \( \psi_e(t, x) \) is
\[
\Delta \phi = \phi_1 - \phi_2 = \Delta E t - \Delta p x
\] (10)

The probabilities of eqs. [6, 7] oscillate both in space \((k \sim \Delta p)\) and in time \((\omega \sim \Delta E)\), with an amplitude depending on the mixing angle \( \theta \):
\[
P(\nu_e \rightarrow \nu_\mu) = 1 - P(\nu_e \rightarrow \nu_e) = \sin^2(2 \theta) \sin^2 \left( \frac{\Delta E t - \Delta p x}{2} \right),
\] (11)

but the oscillation in time is artificial, caused by the infinite plane-waves instead of more adequate wave-packets. The oscillation in space is genuine and experimentally proven [2, 3]. The task is how to relate \( \Delta E \) and \( \Delta p \) to the neutrinos’ kinematic attributes, in particular to their masses \( m_i \) and momenta \( p_i \) (or energies \( E_i \)).

Interference of phases \( \phi_1 \) and \( \phi_2 \) must occur at the same space-time point \((t, x, 0, 0)\). Neglecting the sizes of production and detection regions w.r.t. their distance \( L \), the space point is at \( x = L \) and is fixed. Defining the corresponding time \( t \) is a more subtle question, the answer of which requires wave-packet arguments.

\(^4\) These consequences hold, of course, also for \( \nu_e \leftrightarrow \nu_\tau \) and \( \nu_\mu \leftrightarrow \nu_\tau \) mixing.
2.2 Heuristic approach

Fixing the space point in eq. (10) at \( x = L \), the oscillation frequency \( \Delta \phi \) of eqs. (8) can easily be calculated without unphysical assumptions:

\[
\Delta p = p_1 - p_2 = \frac{p_1^2 - p_2^2}{2p} = \frac{E_1^2 - E_2^2 - (m_1^2 - m_2^2)}{2p} = \frac{E \Delta E}{p} - \frac{m_1^2 - m_2^2}{2p} = \frac{\Delta E}{\beta} - \frac{\Delta m^2}{2p},
\]

(12)

\[
\Delta \phi = \Delta E t - \Delta p L = \Delta E \left( t - \frac{L}{\beta} \right) + \frac{\Delta m^2}{2p} L, \quad \text{with } \Delta m^2 = m_1^2 - m_2^2
\]

Each neutrino mass eigenstate \( |\nu_i\rangle \), or equivalently the centre of its wave-packet, arrives at a different time \( t_i = L/\gamma_i \) at \( x = L \). Defining \( T = L/\gamma \) being the arrival time of the fictive particle of mean 4-momentum in eq. (9), and assuming w.l.o.g. masses \( m_1 < m_2 \), then velocities \( \beta_1 > \beta > \beta_2 \) and arrival times \( t_1 < T < t_2 \).

Interference \( |\nu_1\rangle \leftrightarrow |\nu_2\rangle \) requires the phase difference \( \phi_1 - \phi_2 \) to be observed at a common time \( t \) at which a “snapshot” of the interference pattern can be taken. Therefore \( |\nu_1\rangle \) and \( |\nu_2\rangle \) must be described as wave-packets of finite size in \( x \) which partly overlap at \( x = L \). Since \( t_1 \) and \( t_2 \) are the arrival times of the packets’ centres, it is reasonable to fix interference at time \( t = T = L/\gamma \) [1]. With eqs. (11, 12)

\[
t - \frac{L}{\beta} = 0 \implies \Delta \phi = \frac{\Delta m^2}{2p} L
\]

(13)

\[
\mathcal{P}(\nu_e \rightarrow \nu_\mu) = 1 - \mathcal{P}(\nu_e \rightarrow \nu_e) = \sin^2(2\theta) \sin^2 \left( \frac{\Delta m^2}{4p} L \right)
\]

(14)

The oscillation length \( L_{osc} \) is found by equating the argument to \( \pm \pi \) (conversion to conventional units done with \( h\gamma = 0.19733 \text{ GeV·fm} \)):

\[
L_{osc} = 4\pi \frac{p}{|\Delta m^2|} \approx 4\pi \frac{E}{|\Delta m^2|} = 2.48 \cdot \frac{E/\text{GeV}}{|\Delta m^2|/\text{eV}^2} \text{ km}
\]

(15)

Eqs. (14) and (15) agree with the standard oscillation formulae.

2.3 Wave-packet aspects

In sec. 2.2 fixing interference at \( t = T = L/\gamma \) may look arbitrary. In fact, any time \( t \) within the coherence time interval \( \Delta t \) at which the wave-packets overlap at \( x = L \) could be chosen as well. Which are the consequences if \( t \neq T \)?

Wave-packets are characterized by their finite size \( \sigma_x \) in \( x \). The tail of the faster \( |\nu_1\rangle \) arrives at \( t_{1T} = t_1 + \sigma_x/2\gamma_1 \), and the front of the slower \( |\nu_2\rangle \) arrives at \( t_{2F} = t_2 - \sigma_x/2\gamma_2 \) (remember \( t_1 < t_2 \)). In order to be able to interfere, the wave-packets must overlap [6,8]. Hence, the coherence time interval is defined by

\[
\Delta t = t_{1T} - t_{2F} \approx \frac{\sigma_x}{\beta} - (t_2 - t_1) > 0
\]

(16)

and shrinks from originally \( \sigma_x/\beta \) (full overlap at production) as the wave-packets get increasingly staggered in the course of propagation.

5 The importance of the “same space-time point” condition for interference can best be illustrated by violating it, e.g. by taking the phases \( \phi_i \) at the different arrival times \( t_i \): \( E, t_i = E_i L/\beta_i = E_i^2 L/p_i = \left( m_1^2 + p_i^2 \right) L/p_i \Rightarrow \phi_i = E_i t_i - p_i L = m_1^2 L/p_i \Rightarrow \Delta \phi = \phi_1 - \phi_2 = \left( m_1^2/p_1 - m_2^2/p_2 \right) L \approx \Delta m^2 L/p \), which is obviously wrong – it disagrees by the factor 2 w.r.t. the correct value of eq. (13).

6 Decoherence by complete separation \( (t_2 - t_1 > \sigma_x/\beta) \) may occur at cosmic distances.
For $t$ within this interval, and re-writing eq. (12):

$$|t - T| < \Delta t \leq \frac{\alpha_E}{\beta}$$

$$\Delta \phi = \frac{\Delta m^2}{2p} L + \Delta \phi', \quad \text{with} \quad |\Delta \phi'| = |\Delta E| \cdot |t - T| < \frac{|\Delta E|}{\beta} \sigma_x$$

(17)

being an additional phase shift caused by the finite packet size.

Further analyses by a realistic wave-packet model [7] identify coherence conditions for possible interference[7] which subsequently constrain $\sigma_x$ to be

$$\sigma_x \ll \frac{\beta}{|\Delta E|}$$

(18)

Thus, the term $\Delta \phi'$ can be neglected, and eq. (17) reproduces eq. (13). This is, in retrospect, a justification for the heuristic ansatz of sec. 2.2.

### 2.4 Textbook approaches

The conventional plane-wave approaches are based on unphysical assumptions: they manifestly violate energy-momentum conservation[7] and also violate Lorentz invariance [5]. They are outlined below for stimulating criticism.

- **Equal energy** ($E_1 = E_2$)
  
  Starting from eq. (12) and setting $\Delta E = 0$ trivially yields $\Delta \phi$ of eq. (13).

- **Equal velocity** ($\beta_1 = \beta_2$)

  Hidden as an exercise in [2]: starting from eq. (12), setting $\beta_1 = \beta_2 = \beta$ and fixing $t = L/\beta$ yields $\Delta \phi$ of eq. (13). This approach misses the fact that assuming $\beta_1 = \beta_2$ is not necessary: it is sufficient to properly define $\beta$ as in sec. 2.2 above.

- **Equal momentum** ($p_1 = p_2$)

  A somewhat confused derivation, using the approximation $m_i \ll p_i \approx p$:

  $$E_i = \sqrt{p_i^2 + m_i^2} \approx p_i + \frac{m_i^2}{2p_i} \quad \Rightarrow \quad \Delta E \approx \Delta p + \frac{\Delta m^2}{2p}$$

  $$\Delta \phi = \Delta E t - \Delta p L \approx \Delta p (t - L) + \frac{\Delta m^2}{2p} t$$

  Setting $\Delta p = 0$ yields $\Delta \phi \approx \frac{\Delta m^2}{2p} t$, thereafter tacitly fixing $t = T_0 = L$ yields $\Delta \phi$ of eq. (13). But assuming $\Delta p = 0$ was not necessary; fixing $t$ would suffice.

  A closer look shows that $T_0$ is the time of arrival at $x = L$ of a fictive zero-mass particle, and $T_0 < t_1 < t_2$. Interference requires the slower wave-packet $|\nu_2\rangle$ to have a big enough size for its front having arrived at $L$ already at time $t = T_0$.

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[7] The coherence condition $|\Delta E| \ll \sigma_E$ [7], together with Heisenberg’s approximate uncertainty relation $\sigma_x \cdot \sigma_p \approx 1$ and the dispersion relation $\sigma_E/\sigma_p \approx dE/dp = p/E = \beta$, yield eq. (18).

[8] Energy-momentum conservation is essential for the measurement of neutrino masses.
3 Summary

A thorough review of neutrino oscillations, covering both theoretical and experimental aspects, is given in [3] with exhaustive references. There is general consent that a proper treatment can only be based on sophisticated wave-packet models. Many such models exist, but are too complex for usual textbooks.

Resorting to the simpler plane-wave model, conventional approaches, however, rely on unphysical a-priory assumptions (outlined in sec. 2.4) which are vigorously refuted in several theoretical papers, e.g. [6–8]. But they still dominate most textbooks, sometimes with caveats and a reference to wave-packets [2].

Sec. 2.2 presents a non-conventional heuristic approach, based on the same plane-wave model, albeit without relying on any of those unphysical assumptions. The key is to pay attention how to define a time at which interference takes place [1]. The results agree with the standard formulae shown in the textbooks.

Sec. 2.3 gives further justification for this heuristic approach, using coherence arguments based qualitatively on wave-packets [7].

Hopefully this study will contribute to a better understanding of the plane-wave model, and may inspire the authors of future textbooks.

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