Bivariate Exponentiated Generalized Linear Exponential Distribution with Applications in Reliability Analysis

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Abstract

The aim of this paper, is to define a bivariate exponentiated generalized linear exponential distribution based on Marshall-Olkin shock model. Statistical and reliability properties of this distribution are discussed. This includes quantiles, moments, stress-strength reliability, joint reliability function, joint reversed (hazard) rates functions and joint mean waiting time function. Moreover, the hazard rate, the availability and the mean residual lifetime functions for a parallel system, are established. One data set is analyzed, and it is observed that, the proposed distribution provides a better fit than Marshall-Olkin bivariate exponential, bivariate generalized exponential and bivariate generalized linear failure rate distributions. Simulation studies are presented to estimate both the relative absolute bias, and the relative mean square error for the distribution parameters based on complete data.

Key words: Joint probability density function, Joint reversed (hazard) rates functions, Joint mean waiting time function, Simulation studies.

1 Introduction

Sarhan et al. (2013) introduced exponentiated generalized linear exponential distribution (EGLED), which generalized a lot of probability distributions such as exponential (E), generalized exponential (GE), linear exponential (LE), generalized linear exponential (GLE) distributions, among others. Furthermore, the EGLED provides more flexibility to analyze real data sets such as Leukemia data, drug data, among others.

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In many scientific practical situations, multivariate lifetime data arise frequently, so it is important to consider different multivariate models that could be used to model such multivariate lifetime data. Such these models are interesting in several applications, such as reliability engineering, industrial engineering and computer systems. So, the aim of this paper is to introduce a bivariate exponentiated generalized linear exponential distribution (BEGLED) based on Marshall-Olkin shock model (1967), whose marginal distributions are EGLED. In the mentioned applications of the bivariate distribution, could be the lifetimes of two components, the magnitudes of stress and strength components and drought intensities. A lot of bivariate distributions based on Marshall-Olkin model are studied by many authors, see Sarhan and Balakrishnan (2007), Al-Khedhairi and El-Gohary (2008), Kundu and Gupta (2009), Sarhan et al. (2011), Kundu and Gupta (2013), Balakrishna and Shiji (2014), El-Gohary et al. (2016), Rasool and Akbar (2016) and El-Bassiouny et al. (2016).

The random variable $X$ is said to have EGLED($a,b,\alpha,\theta$) if its CDF is

$$F_X(x) = \left(1 - e^{-\eta(x)}\right)^\theta; \quad x \geq 0,$$

where $\eta(x) = ax + \frac{b}{2}x^2$, the parameters $a,b \geq 0$ such that $a + b > 0$ and $\alpha,\theta > 0$.

The parameters $a$ and $b$ are scale parameters, while $\alpha$ and $\theta$ are shape parameters.

### 2 The BEGLED and Its Marginal Functions

Assume $U_i \sim \text{EGLED}(\alpha, a, b, \theta_i), \quad i = 1, 2, 3$ are three independent random variables. Define $X_k = \max\{U_k, U_3\}; k = 1, 2$. So, the bivariate vector $(X_1, X_2)$ has the BEGLED with parameters vector $\Phi = (\alpha, a, b, \theta_1, \theta_2, \theta_3)$. The joint CDF of $(X_1, X_2)$ is

$$F_{X_1,X_2}(x_1, x_2) = \left(1 - e^{-\eta(x_1)}\right)^{\theta_1} \left(1 - e^{-\eta(x_2)}\right)^{\theta_2} \left(1 - e^{-\eta(z)}\right)^{\theta_3}, \quad z = \min(x_1, x_2).$$

Also, we can get the joint PDF of $(X_1, X_2)$ as follows

$$f_{X_1,X_2}(x_1, x_2) = \begin{cases} f_1(x_1, x_2) & \text{if } 0 < x_1 < x_2 \\ f_2(x_1, x_2) & \text{if } 0 < x_2 < x_1 \\ f_3(x, x) & \text{if } x_1 = x_2 = x, \end{cases}$$

where

$$f_1(x_1, x_2) = \varphi_2 \left[\eta(x_1)\eta(x_2)\right]^{\alpha-1} e^{-\eta(x_1)+\eta(x_2)} \left(1 - e^{-\eta(x_2)}\right)^{\theta_2-1} \left(1 - e^{-\eta(x_1)}\right)^{\theta_1+\theta_3-1},$$

$$f_2(x_1, x_2) = \varphi_1 \left[\eta(x_1)\eta(x_2)\right]^{\alpha-1} e^{-\eta(x_1)+\eta(x_2)} \left(1 - e^{-\eta(x_1)}\right)^{\theta_1-1} \left(1 - e^{-\eta(x_2)}\right)^{\theta_2+\theta_3-1},$$

and

$$f_3(x, x) = \alpha \theta_3 (a + bx) \eta^{\alpha-1}(x) e^{-\eta(x)} \left(1 - e^{-\eta(x)}\right)^{\theta_1+\theta_2+\theta_3-1},$$

where
\[ \varphi_i = \alpha^2 \theta_i (\theta_3 - i + \theta_3)(a + bx_1)(a + bx_2), \quad i = 1, 2. \]

On the other hand, the marginal CDFs for the BEGLED can be represented as follows

\[ F_{X_i}(x_i) = \left( 1 - e^{-\eta^*(x_i)} \right)^{\theta_i + \theta_3}, \quad i = 1, 2. \]

Therefore, we can get the marginal PDFs for the BEGLED as follows

\[ f(x_i) = \alpha (\theta_i + \theta_3)(a + bx_i)\eta^{\alpha - 1}(x_i) e^{-\eta^*(x_i)} \left( 1 - e^{-\eta^*(x_i)} \right)^{\theta_i + \theta_3 - 1}, \quad i = 1, 2. \]

### 3 Statistical Properties

#### 3.1 The median of the BEGLED

Domma (2009) presented the median correlation coefficient \( M_{X_1,X_2} \) as a form

\[ M_{X_1,X_2} = 4F_{X_1,X_2}(M_{X_1}, M_{X_2}) - 1, \]

where \( M_{X_1} \) and \( M_{X_2} \) denote the median of \( X_1 \) and \( X_2 \) respectively.

If \( X_1 \sim EGED(\alpha, a, b, \theta_1 + \theta_3) \) and \( X_2 \sim EGED(\alpha, a, b, \theta_2 + \theta_3) \) then

\[ M_{X_i} = \frac{1}{b} \left( -a + \sqrt{a^2 + 2b \left[ -\ln \left( 1 - \left( \frac{1}{2} \right)^{\left( \frac{1}{\alpha + \theta_3} \right)} \right) \right]} \right), \quad i = 1, 2. \]

So, the coefficient of median correlation between \( X_1 \) and \( X_2 \) is

\[ M_{X_1,X_2} = \left\{ \begin{array}{ll} 4 \left( 1 - e^{-\eta^*(M_{X_2})} \right)^{\theta_2} \left( 1 - e^{-\eta^*(M_{X_1})} \right)^{\theta_1 + \theta_3} - 1 & \text{if } x_1 < x_2 \\ 4 \left( 1 - e^{-\eta^*(M_{X_1})} \right)^{\theta_1} \left( 1 - e^{-\eta^*(M_{X_2})} \right)^{\theta_2 + \theta_3} - 1 & \text{if } x_1 > x_2. \end{array} \right. \]

Equation (8) can be used to generate a bivariate data.

#### 3.2 The mathematical expectation

We can derive the marginal expectation (\( r \)th moment) of \( X_i \) when \( X_i \sim EGED(\alpha, a, b, \theta_i + \theta_3) \) such that \( i = 1, 2 \) as follows

\[ E(X_i^r) = \int_0^\infty x_i^r f_{X_i}(x_i) dx_i, \]

by using Equation (5), Maclaurin expansion, binomial expansion and gamma function, we get

\[ E(X_i^r) = \sum_{j=0}^\infty \sum_{k=0}^\infty \sum_{l=0}^\infty \zeta_j^l(a + b\Theta) \Gamma(\Theta), \]

where
\[
\zeta_{j,k}^{(t)} = \alpha (\theta_i + \theta_3) \left( \frac{(-1)^j e^{kt}}{t!} \right)^{\alpha-k-1} \left( \frac{\alpha - 1}{k} \right) \left( \frac{\theta_i + \theta_3 - 1}{j} \right),
\]

and

\[
\Theta = r + l + 2\alpha - k - 1,
\]

and

\[
U_j^{(t)} = \frac{d}{dx_i} \left( \frac{\exp(-(j+1)\eta^{\alpha}(x_i) + x_i))}{x_i=0}. \right.
\]

### 3.3 The conditional probability density functions

The conditional probability density function of \(X_i\) given \(X_j = x_j\), \((i, j = 1, 2, i \neq j)\) is given by

\[
f_{X_i|X_j}(x_i | x_j) = \begin{cases} f_{X_i,X_j}^{(1)}(x_i | x_j) & \text{if } x_i > x_j > 0 \\ f_{X_i,X_j}^{(2)}(x_i | x_j) & \text{if } x_j > x_i > 0 \\ f_{X_i,X_j}^{(3)}(x_i | x_j) & \text{if } x_i = x_j > 0, \end{cases}
\]

where

\[
f_{X_i|X_j}^{(1)}(x_i | x_j) = \alpha \theta_i (a + bx_i) \eta^{\alpha-1}(x_i) e^{-\eta^{\alpha}(x_i)} \left( 1 - e^{-\eta^{\alpha}(x_i)} \right)^{\theta_i-1},
\]

\[
f_{X_i|X_j}^{(2)}(x_i | x_j) = \frac{\alpha \theta_j (\theta_i + \theta_3) (a + bx_i) \eta^{\alpha-1}(x_i) e^{-\eta^{\alpha}(x_i)} (1 - e^{-\eta^{\alpha}(x_i)})^{\theta_i + \theta_3 - 1}}{(\theta_j + \theta_3) (1 - e^{-\eta^{\alpha}(x_i)})^{\theta_3}},
\]

and

\[
f_{X_i|X_j}^{(3)}(x_i | x_j) = \frac{\theta_3}{\theta_j + \theta_3} \left[ 1 - e^{-\eta^{\alpha}(x_i)} \right] \theta_i.
\]

Equation (11) can be getting by substituting from Equations (3) and (5) in the following relation

\[
f_{X_i|X_j}(x_i | x_j) = \frac{f_{X_i,X_j}(x_i, x_j)}{f_{X_j}(x_j)}, \quad (i \neq j = 1, 2). \tag{12}
\]

### 3.4 The distributions of \(T = \max(X_1, X_2)\) and \(S = \min(X_1, X_2)\)

In the mentioned applications \(X_1\) and \(X_2\) could be exchange rates in two time periods. So, it is important to get the distributions of \(T\) and \(S\). If the bivariate vector \((X_1, X_2)\) has the BEGLED then

\[
F_T(t) = P(\max(X_1, X_2) \leq t) = P(\max(U_1, U_3) \leq t, \max(U_2, U_3) \leq t) = F_{EGLD}(\alpha, a, b, \theta_1 + \theta_2 + \theta_3). \tag{13}
\]

Also, we can get the distribution of \(S\) as follows

\[
F_S(t) = P(\min(X_1, X_2) \leq t) = P(X_1 < t) + P(X_2 < t) - P(X_1 < t, X_2 < t) = F_{EGLD}(\alpha, a, b, \theta_1 + \theta_2 + \theta_3) + F_{EGLD}(\alpha, a, b, \theta_2 + \theta_3) - F_{EGLD}(\alpha, a, b, \theta_1 + \theta_2 + \theta_3). \tag{14}
\]
4 Reliability Properties

In this section, we present the stress-strength reliability, the joint reliability function, the joint reversed (hazard) functions and the joint of mean waiting time function. Also, we present the hazard rate, the availability and the mean residual lifetime functions for a parallel system with two components.

4.1 Stress-strength reliability

Let $X_1$ is a random variable represents stress, and $X_2$ is a random variable represents strength, and the random vector $(X_1, X_2)$ has the BEGLED then, the reliability function $R$ is

$$R = P[X_1 < X_2] = P(U_1 < U_3 < U_2) + P(U_3 < U_1 < U_2) = \frac{\theta_2 + \theta_3}{\theta_1 + \theta_2 + 2\theta_3}. \quad (15)$$

4.2 The joint reliability function

Assume $(X_1, X_2)$ be two dimensional random variable with CDF $F_{X_1, X_2}(x_1, x_2)$, and the marginal functions are $F_{X_1}(x_1)$ and $F_{X_2}(x_2)$ then, the joint reliability function $R_{X_1, X_2}(x_1, x_2)$ is

$$R_{X_1, X_2}(x_1, x_2) = 1 - F_{X_1}(x_1) - F_{X_2}(x_2) + F_{X_1, X_2}(x_1, x_2). \quad (16)$$

Assume the random vector $(X_1, X_2)$ has the BEGLED then, the joint reliability function of $(X_1, X_2)$ is given by

$$R_{X_1, X_2}(x_1, x_2) = \begin{cases} 
R_1(x_1, x_2) & \text{if } 0 < x_1 < x_2 \\
R_2(x_1, x_2) & \text{if } 0 < x_2 < x_1 \\
R_3(x, x) & \text{if } x_1 = x_2 = x, \quad (17)
\end{cases}$$

where

$$R_1(x_1, x_2) = 1 - \left(1 - e^{-\eta^\alpha(x_1)}\right)^{\theta_1 + \theta_3} - \left(1 - e^{-\eta^\alpha(x_2)}\right)^{\theta_2 + \theta_3} + \left(1 - e^{-\eta^\alpha(x_2)}\right)^{\theta_2} \left(1 - e^{-\eta^\alpha(x_1)}\right)^{\theta_1 + \theta_3},$$

$$R_2(x_1, x_2) = 1 - \left(1 - e^{-\eta^\alpha(x_1)}\right)^{\theta_1 + \theta_3} - \left(1 - e^{-\eta^\alpha(x_2)}\right)^{\theta_2 + \theta_3} + \left(1 - e^{-\eta^\alpha(x_1)}\right)^{\theta_1} \left(1 - e^{-\eta^\alpha(x_2)}\right)^{\theta_2 + \theta_3},$$

$$R_3(x, x) = 1 - \left(1 - e^{-\eta^\alpha(x)}\right)^{\theta_1 + \theta_3} - \left(1 - e^{-\eta^\alpha(x)}\right)^{\theta_2 + \theta_3} + \left(1 - e^{-\eta^\alpha(x)}\right)^{\theta_1 + \theta_2 + \theta_3}. $$
4.3 The joint reversed (hazard) rate functions

4.3.1 The joint hazard rate function and its marginal functions

Assume \((X_1, X_2)\) be two dimensional random variable with PDF \(f_{X_1,X_2}(x_1,x_2)\), and reliability function \(R_{X_1,X_2}(x_1,x_2)\). Basu (1971) defined the bivariate hazard rate function as

\[
h(x_1,x_2) = \frac{f_{X_1,X_2}(x_1,x_2)}{R_{X_1,X_2}(x_1,x_2)}. \tag{18}\]

So, the bivariate hazard rate function for the random vector \((X_1, X_2)\) which has the BEGLED is

\[
h_{X_1,X_2}(x_1,x_2) = \begin{cases} h_1(x_1,x_2) & \text{if } 0 < x_1 < x_2 \\ h_2(x_1,x_2) & \text{if } 0 < x_2 < x_1 \\ h_3(x,x) & \text{if } x_1 = x_2 = x, \end{cases} \tag{19}\]

where

\[
h_1(x_1,x_2) = \frac{\phi_2 \left[ \eta(x_1)\eta(x_2) \right]^{\alpha-1} e^{-\eta(x_1)\eta(x_2)} (\Psi(x_2))^{\theta_2-1} (\Psi(x_1))^{\theta_1+\theta_3-1}}{1 - (\Psi(x_1))^{\theta_1+\theta_3} - (\Psi(x_2))^{\theta_2+\theta_3} + (\Psi(x_1))^{\theta_1} (\Psi(x_2))^{\theta_2}},
\]

\[
h_2(x_1,x_2) = \frac{\phi_1 \left[ \eta(x_1)\eta(x_2) \right]^{\alpha-1} e^{-\eta(x_1)\eta(x_2)} (\Psi(x_1))^{\theta_1-1} (\Psi(x_2))^{\theta_2+\theta_3-1}}{1 - (\Psi(x_1))^{\theta_1+\theta_3} - (\Psi(x_2))^{\theta_2+\theta_3} + (\Psi(x_1))^{\theta_1} (\Psi(x_2))^{\theta_2}},
\]

and

\[
h_3(x,x) = \frac{\alpha \theta_3(a+bx)\eta^{\alpha-1}(x) e^{-\eta(x)} (\Psi(x))^{\theta_1+\theta_2+\theta_3-1}}{1 - (\Psi(x))^{\theta_1+\theta_3} - (\Psi(x))^{\theta_2+\theta_3} + (\Psi(x))^{\theta_1} (\Psi(x))^{\theta_2} (\Psi(x))^{\theta_3}},
\]

where \(\Psi(.) = 1 - e^{-\eta(.)}\).

Also, the marginal hazard rate functions \(h_i(x_i), i = 1, 2\) of the BEGLED are

\[
h_i(x_i) = \frac{\alpha (\theta_i + \theta_3) (a+bx_i) \eta^{\alpha-1}(x_i) e^{-\eta(x_i)} (\Psi(x_i))^{\theta_i+\theta_3-1}}{1 - (\Psi(x_i))^{\theta_i+\theta_3}}, \quad i = 1, 2. \tag{20}\]

4.3.2 The joint reversed hazard rate function and its marginal functions

Assume \((X_1, X_2)\) be two dimensional random variable with CDF \(F_{X_1,X_2}(x_1,x_2)\), the joint reversed hazard rate function is

\[
r(x_1,x_2) = \frac{f_{X_1,X_2}(x_1,x_2)}{F_{X_1,X_2}(x_1,x_2)}. \tag{21}\]

So, the joint reversed hazard rate function for the random vector \((X_1, X_2)\) which has the BEGLED is

\[
r_{X_1,X_2}(x_1,x_2) = \begin{cases} r_1(x_1,x_2) & \text{if } 0 < x_1 < x_2 \\ r_2(x_1,x_2) & \text{if } 0 < x_2 < x_1 \\ r_3(x,x) & \text{if } x_1 = x_2 = x, \end{cases} \tag{22}\]

where
Also, the marginal reversed hazard rate functions mean waiting time functions $m_i(x)$ of the waiting time is to describe different maintenance strategies to any system. The joint mean waiting time function $M$ can be used for prediction the distribution function. So, one of the most important applications of the waiting time is to describe different maintenance strategies to any system. The joint mean waiting time function $M(t_1, t_2)$ is defined as follows

$$M_w(t_1, t_2) = \frac{1}{F(t_1, t_2)} \int_0^{t_1} \int_0^{t_2} F(x_1, x_2) \, dx_2 \, dx_1.$$  

Assume the random vector $(X_1, X_2)$ has the BEGLED. Using Maclaurin and binomial expansions then, the joint mean waiting time function $M_w(t_1, t_2)$ is

$$M_w(t_1, t_2) = \left\{ \begin{array}{ll}
M_{w1}(t_1, t_2) & \text{if } t_1 > t_2 > 0 \\
M_{w2}(t_1, t_2) & \text{if } 0 < t_1 < t_2 \\
M_{w3}(t, t) & \text{if } t_1 = t_2 = t,
\end{array} \right.$$  

where

$$M_{w1}(t_1, t_2) = \frac{1}{F(t_1, t_2)} \sum_{j,k=0}^{\infty} \frac{g_j^{(k)} \psi_j^{(k)}}{(k+1)!} \left( \frac{\theta_1}{\theta_3} \right)^j \left( \frac{\theta_3 + \theta_3}{\theta_3} \right)^j (t_1 t_2)^{k+1}; \quad i = 1, 2,$$

$$M_{w2}(t, t) = \frac{1}{F(t, t)} \sum_{j,k=0}^{\infty} \frac{(-1)^j Q_j^{(k)}(t_1 + \theta_2 + \theta_3)}{(k+1)!} \left( \frac{\theta_1 + \theta_2 + \theta_3}{\theta_3} \right)^j t^{k+1},$$

and

$$g_j^{(k)} = \frac{d^k}{dx_1^k} \left( e^{-j \eta^a(x_1)} \right)_{x_1=0}, \quad \psi_j^{(k)} = \frac{d^k}{dx_2^k} \left( e^{-j \eta^a(x_2)} \right)_{x_2=0}, \quad Q_j^{(k)} = \frac{d^k}{dx^k} \left( e^{-j \eta^a(x)} \right)_{x=0}.$$  

Also, the marginal mean waiting time functions $m_{w_i}(t)$ for $X_1$ and $X_2$ can be written as:

$$m_{w_i}(t) = \frac{1}{F_X_i(t)} \int_0^t F_X_i(x_i) \, dx_i$$

$$= \frac{1}{F_X_i(t)} \sum_{j,k=0}^{\infty} \frac{(-1)^j g_j^{(k)}}{(k+1)!} \left( \frac{\theta_1 + \theta_3}{\theta_3} \right)^j t^{k+1}; \quad i = 1, 2.$$

4.4 The joint mean waiting time and its marginal functions

The waiting time is closely related to important random variable reversed hazard rate function, which the failure occurs in the interval $[0, t]$. The observations of waiting times can be used for prediction the distribution function. So, one of the most important applications of the waiting time is to describe different maintenance strategies to any system. The joint mean waiting time function $M_w(t_1, t_2)$ is defined as follows
where

\[ g_{x_j}^{(k)} = \frac{d^k}{dx^k} \left( e^{-j\eta^*(x_j)} \right) \bigg|_{x=0}. \]

### 4.5 The hazard rate, the availability and the mean residual lifetime functions for a parallel system

Cox (1972) defined the joint hazard rate function as a vector, which is useful to calculate the total life span of a two component parallel system \((2-out-of-2 : F)\) as follows

\[ h(x^*) = (h_X(x), h_{12}(x_1|x_2), h_{21}(x_2|x_1)), \tag{27} \]

where the first element \(h(x)\) in the vector \(h(x^*)\), gives the hazard function of the system using the information that both the component has survived beyond \(x\), where \(X = \min(X_1, X_2)\). The second element \(h_{12}(x_1|x_2)\), gives the hazard function span of the first component given that it has survived to an age \(x_1\), and the other has failed at \(x_2\). Similar argument holds for the third element \(h_{21}(x_2|x_1)\).

If \((X_1, X_2)\) is a BEGLE random vector, then the joint hazard rate function \(h(x^*)\) is

\[ h_X(x) = \frac{\alpha(\theta_1 + \theta_2 + \theta_3)(a + bx)\eta_1(x)e^{-\eta_1(x)}(1 - e^{-\eta_1(x)})^{-1}}{\left(1 - e^{-\eta_1(x)}\right)^{\theta_1+\theta_2+\theta_3-1} - 1}, \tag{28} \]

\[ h_{12}(x_1|x_2) = \frac{f_{X_1}(x_1)(1 - e^{-\eta_1(x_1)})^{\theta_3}}{1 - (1 - e^{-\eta_1(x_1)})^{\theta_1}}, \tag{29} \]

and

\[ h_{21}(x_2|x_1) = \frac{f_{X_2}(x_2)(1 - e^{-\eta_2(x_2)})^{\theta_3}}{1 - (1 - e^{-\eta_2(x_2)})^{\theta_2}}. \tag{30} \]

Also, the joint availability function can be defined as a vector, which is useful to calculate the expected lifetime of a parallel system with two component as follows

\[ V(x^*) = (v_X(x), v_{12}(x_1|x_2), v_{21}(x_2|x_1)), \tag{31} \]

where the first element \(v(x)\) in the vector \(V(x^*)\), gives the expected lifetime of the system using the information that both the component has survived beyond \(x\), where \(X = \min(X_1, X_2)\). The second element \(v_{12}(x_1|x_2)\), gives the expected lifetime span of the first component given that it has survived to an age \(x_1\), and the other has failed at \(x_2\). Similar argument holds for the third element \(v_{21}(x_2|x_1)\).

If \((X_1, X_2)\) is a BEGLE random vector, then the joint availability function \(V(x^*)\) is

\[ V(x) = \frac{1}{A} \int_x^\infty y f_X(y) \, dy; \quad A = \int_x^\infty f_X(y) \, dy, \tag{32} \]

using Maclaurin expansion, binomial expansion and upper incomplete gamma function, we get

\[ V(x) = \frac{1}{1 - (\Psi(x))^{\theta_1+\theta_2+\theta_3}} \sum_{i=0}^{\alpha-1} \sum_{j=0}^{\alpha-1} \sum_{k=0}^{\infty} \phi_{i,j}^{(k)} (b\Gamma(\alpha^*, x) + b(\Gamma(\alpha^* + 1, x))). \tag{33} \]
Similarly,

\[ V_{12}(x_1|x_2) = \frac{1}{B} \int_{x_1}^{\infty} y f(y, x_2) \, dy; \quad B = \int_{x_1}^{\infty} f(y, x_2) \, dy, \quad x_1 > x_2, \]

\[ = \frac{1}{1 - (\Psi(x_1))^\alpha} \sum_{j=0}^{\alpha-1} \sum_{i,k=0}^\infty \xi_{i,j}^{(k)} (a \Gamma(\alpha^*, x_1) + b(\Gamma(\alpha^* + 1, x_1))), \quad (34) \]

and

\[ V_{21}(x_2|x_1) = \frac{1}{C} \int_{x_2}^{\infty} y f(x_1, y) \, dy; \quad C = \int_{x_2}^{\infty} f(x_1, y) \, dy, \quad x_1 < x_2, \]

\[ = \frac{1}{1 - (\Psi(x_2))^\alpha} \sum_{j=0}^{\alpha-1} \sum_{i,k=0}^\infty \Omega_{i,j}^{(k)} (a \Gamma(\alpha^*, x_1) + b(\Gamma(\alpha^* + 1, x_1))), \quad (35) \]

where

\[ \theta_{i,j}^{(k)} = \alpha (\theta_1 + \theta_2 + \theta_3) \frac{(-1)^j a^{\alpha-j-1} Q_{*i}^{(k)}}{k!} \left( \frac{b}{2} \right)^j \frac{(\alpha - 1)}{j} \left( \frac{\theta_1 + \theta_2 + \theta_3 - 1}{i} \right), \]

\[ \xi_{i,j}^{(k)} = \alpha \theta_1 \frac{(-1)^j a^{\alpha-j-1} Q_{*i}^{(k)}}{k!} \left( \frac{b}{2} \right)^j \frac{(\alpha - 1)}{j} \left( \frac{\theta_1 - 1}{i} \right), \]

\[ \Omega_{i,j}^{(k)} = \alpha \theta_2 \frac{(-1)^j a^{\alpha-j-1} Q_{*i}^{(k)}}{k!} \left( \frac{b}{2} \right)^j \frac{(\alpha - 1)}{j} \left( \frac{\theta_2 - 1}{i} \right), \]

\[ \alpha^* = \alpha + j + k + 1, \]

and

\[ Q_{*i}^{(k)} = \frac{d^k}{dy^k} (\exp(-(1+i)\eta^\alpha (y) + y))|_{y=0}. \]

On the other hand, Asha and Jagathnath (2008) defined the joint mean residual lifetime \( m(x^*) \), which is useful to compute the mean residual lifetime (MRL) to two component in a parallel system, as follows

\[ m(x^*) = (m_X(x), m_{12}(x_1|x_2), m_{21}(x_2|x_1)), \quad (36) \]

where the first element \( m(x) \) in the vector \( m(x^*) \), gives the MRL of the system using the information that both the component has survived beyond \( x \), where \( X = \min(X_1, X_2) \). The second element \( m_{12}(x_1|x_2) \), gives the MRL span of the first component given that it has survived to an age \( x_1 \), and the other has failed at \( x_2 \). Similar argument holds for the third element \( m_{21}(x_2|x_1) \). The joint MRL function related to the joint vitality function by the relationships

\[ m_X(x) = v_X(x) - x, \quad x > 0. \quad (37) \]

\[ m_{12}(x_1|x_2) = v_{12}(x_1|x_2) - x_1, \quad x_1 > x_2. \quad (38) \]

\[ m_{21}(x_2|x_1) = v_{21}(x_2|x_1) - x_2, \quad x_1 < x_2. \quad (39) \]

So, if \( (X_1, X_2) \) is a BEGLE random vector, then it is easy to get the vector \( m(x^*) \).
5 Maximum Likelihood Estimation (MLE)

In this section, we want to estimate the unknown parameters of the BEGLED. We will use the maximum likelihood method. Suppose that \((x_{11}, x_{21}), (x_{12}, x_{22}), \ldots, (x_{1n}, x_{2n})\) is a sample of size \(n\), from the BEGLED. We use the following notation \(I_1 = \{x_{1i} < x_{2i}\}\), \(I_2 = \{x_{1i} > x_{2i}\}\), \(I_3 = \{x_{1i} = x_{2i} = x_i\}\), \(I = I_1 \cup I_2 \cup I_3\), \(|I_1| = n_1\), \(|I_2| = n_2\), \(|I_3| = n_3\), and \(|I| = n_1 + n_2 + n_3 = n\). Based on the observations, the likelihood function \(L(\Phi)\) of this sample is

\[
l(\Phi) = \prod_{i=1}^{n_1} f_1(x_{1i}, x_{2i}) \prod_{i=1}^{n_2} f_2(x_{1i}, x_{2i}) \prod_{i=1}^{n_3} f_3(x_i).
\]

Substituting from Equation (3) into Equation (40), the log-likelihood function \(L(\Phi)\) can be written as

\[
L(\Phi) = n_1 \ln (\alpha^2 \theta_2 (\theta_1 + \theta_3)) + \sum_{i=1}^{n_1} \ln(a + bx_{1i}) + (\alpha - 1) \sum_{i=1}^{n_1} \ln(\eta(x_{1i}))
\]

\[
+ (\theta_1 + \theta_3 - 1) \sum_{i=1}^{n_1} \ln \left(1 - e^{-\eta^\alpha(x_{1i})}\right) - \sum_{i=1}^{n_1} \eta^\alpha(x_{1i}) + \sum_{i=1}^{n_1} \ln(a + bx_{2i})
\]

\[
+ (\alpha - 1) \sum_{i=1}^{n_1} \ln(\eta(x_{1i})) + (\theta_2 - 1) \ln \left(1 - e^{-\eta^\alpha(x_{2i})}\right) - \sum_{i=1}^{n_1} \eta^\alpha(x_{2i})
\]

\[
+ n_2 \ln(\alpha^2 \theta_1 (\theta_2 + \theta_3)) + \sum_{i=1}^{n_2} \ln(a + bx_{1i}) + (\alpha - 1) \sum_{i=1}^{n_2} \ln(\eta(x_{1i}))
\]

\[
+ (\theta_1 - 1) \sum_{i=1}^{n_2} \ln \left(1 - e^{-\eta^\alpha(x_{1i})}\right) - \sum_{i=1}^{n_2} \eta^\alpha(x_{1i}) + \sum_{i=1}^{n_2} \ln(a + bx_{2i})
\]

\[
+ (\alpha - 1) \sum_{i=1}^{n_2} \ln(\eta(x_{1i})) + (\theta_2 + \theta_3 - 1) \sum_{i=1}^{n_2} \ln \left(1 - e^{-\eta^\alpha(x_{2i})}\right)
\]

\[
- \sum_{i=1}^{n_2} \eta^\alpha(x_{2i}) + n_3 \ln(\alpha \theta_3) + \sum_{i=1}^{n_3} \ln(a + bx_i) + (\alpha - 1) \sum_{i=1}^{n_3} \ln(\eta(x_i))
\]

\[
+ (\theta_1 + \theta_2 + \theta_3 - 1) \sum_{i=1}^{n_3} \ln \left(1 - e^{-\eta^\alpha(x_i)}\right) - \sum_{i=1}^{n_3} \eta^\alpha(x_i).
\]

The first partial derivatives of Equation (41) with respect to \(\alpha, a, b, \theta_1, \theta_2\) and \(\theta_3\) are
\[ \frac{\partial L}{\partial \alpha} = \frac{2n_1}{\alpha} + \sum_{i=1}^{n_1} \ln(\eta(x_{1i})) + (\theta_1 + \theta_3 - 1) \sum_{i=1}^{n_1} \frac{\eta^\alpha(x_{1i}) \ln(\eta(x_{1i}))}{e^{\eta^\alpha(x_{1i})} - 1} \\
- \sum_{i=1}^{n_1} \eta^\alpha(x_{1i}) \ln(\eta(x_{1i})) + \sum_{i=1}^{n_1} \ln(\eta(x_{2i})) - \sum_{i=1}^{n_1} \eta^\alpha(x_{2i}) \ln(\eta(x_{2i})) \\
+ (\theta_2 - 1) \sum_{i=1}^{n_2} \frac{\eta^\alpha(x_{2i}) \ln(\eta(x_{2i}))}{e^{\eta^\alpha(x_{2i})} - 1} + \frac{2n_2}{\alpha} + \sum_{i=1}^{n_2} \ln(\eta(x_{1i})) \\
+ (\theta_1 - 1) \sum_{i=1}^{n_2} \frac{\eta^\alpha(x_{1i}) \ln(\eta(x_{1i}))}{e^{\eta^\alpha(x_{1i})} - 1} - \sum_{i=1}^{n_2} \eta^\alpha(x_{1i}) \ln(\eta(x_{1i})) \\
+ \sum_{i=1}^{n_2} \ln(\eta(x_{1i})) + (\theta_2 + \theta_3 - 1) \sum_{i=1}^{n_2} \frac{\eta^\alpha(x_{2i}) \ln(\eta(x_{2i}))}{e^{\eta^\alpha(x_{2i})} - 1} \\
- \sum_{i=1}^{n_2} \eta^\alpha(x_{2i}) \ln(\eta(x_{2i})) + \sum_{i=1}^{n_3} \ln(\eta(x_i)) - \sum_{i=1}^{n_3} \eta^\alpha(x_i) \ln(\eta(x_i)) \\
+ \frac{n_3}{\alpha} + (\theta_1 + \theta_2 + \theta_3 - 1) \sum_{i=1}^{n_3} \frac{\eta^\alpha(x_i) \ln(\eta(x_i))}{e^{\eta^\alpha(x_i)} - 1}, \quad (42) \]

\[ \frac{\partial L}{\partial a} = \sum_{i=1}^{n_1} \frac{1}{a + bx_{1i}} + (\alpha - 1) \sum_{i=1}^{n_1} \frac{x_{1i}}{\eta(x_{1i})} + (\theta_1 + \theta_3 - 1) \sum_{i=1}^{n_1} \frac{ax_{1i} \eta^{\alpha-1}(x_{1i})}{e^{\eta^\alpha(x_{1i})} - 1} \\
- \alpha \sum_{i=1}^{n_1} x_{1i} \eta^{\alpha-1}(x_{1i}) + \sum_{i=1}^{n_1} \frac{1}{a + bx_{2i}} + (\alpha - 1) \sum_{i=1}^{n_2} \frac{x_{2i}}{\eta(x_{2i})} \\
+ (\theta_2 - 1) \sum_{i=1}^{n_2} \frac{ax_{2i} \eta^{\alpha-1}(x_{2i})}{e^{\eta^\alpha(x_{2i})} - 1} - \alpha \sum_{i=1}^{n_2} x_{2i} \eta^{\alpha-1}(x_{2i}) + \sum_{i=1}^{n_2} \frac{1}{a + bx_{2i}} \\
+ (\alpha - 1) \sum_{i=1}^{n_2} \frac{x_{1i}}{\eta(x_{1i})} - \alpha \sum_{i=1}^{n_2} x_{1i} \eta^{\alpha-1}(x_{1i}) + (\theta_1 - 1) \sum_{i=1}^{n_2} \frac{ax_{1i} \eta^{\alpha-1}(x_{1i})}{e^{\eta^\alpha(x_{1i})} - 1} \\
+ \sum_{i=1}^{n_2} \frac{1}{a + bx_{2i}} + (\alpha - 1) \sum_{i=1}^{n_2} \frac{x_{2i}}{\eta(x_{2i})} + (\theta_2 + \theta_3 - 1) \sum_{i=1}^{n_2} \frac{ax_{2i} \eta^{\alpha-1}(x_{2i})}{e^{\eta^\alpha(x_{2i})} - 1} \\
- \alpha \sum_{i=1}^{n_2} x_{2i} \eta^{\alpha-1}(x_{2i}) + \sum_{i=1}^{n_3} \frac{1}{a + bx_{i}} + (\theta_1 + \theta_2 + \theta_3 - 1) \sum_{i=1}^{n_3} \frac{ax_{i} \eta^{\alpha-1}(x_{i})}{e^{\eta^\alpha(x_{i})} - 1} \\
+ (\alpha - 1) \sum_{i=1}^{n_3} \frac{x_{i}}{\eta(x_{i})} - \alpha \sum_{i=1}^{n_3} x_{i} \eta^{\alpha-1}(x_{i}), \quad (43) \]
the solution has to be obtained numerically. By Equating the Equations (42-47) by zeros, we get the non-linear normal Equations. So,

\[ \frac{\partial L}{\partial b} = \sum_{i=1}^{n_1} \frac{x_{1i}}{\alpha + bx_{1i}} + \frac{\alpha - 1}{2} \sum_{i=1}^{n_1} (x_{1i})^2 \frac{\theta_1 + \theta_3}{\eta(x_{1i})} + \frac{\theta_1 + \theta_3 - 1}{2} \sum_{i=1}^{n_1} \alpha(x_{1i})^2 \eta^{\alpha - 1}(x_{1i}) - 1 \]

\[ + \sum_{i=1}^{n_1} \frac{x_{2i}}{\alpha + bx_{2i}} - \frac{\alpha}{2} \sum_{i=1}^{n_1} (x_{2i})^2 \eta^{\alpha - 1}(x_{2i}) + \frac{\alpha - 1}{2} \sum_{i=1}^{n_2} \frac{x_{1i}}{\alpha + bx_{1i}} \]

\[ + \frac{\alpha - 1}{2} \sum_{i=1}^{n_2} (x_{1i})^2 \frac{\theta_1 - 1}{\eta(x_{1i})} + \frac{\alpha - 1}{2} \sum_{i=1}^{n_2} \alpha(x_{1i})^2 \eta^{\alpha - 1}(x_{1i}) - 1 \]

\[ - \frac{\alpha}{2} \sum_{i=1}^{n_2} (x_{1i})^2 \eta^{\alpha - 1}(x_{1i}) + \sum_{i=1}^{n_2} \frac{x_{2i}}{\alpha + bx_{2i}} + \frac{\alpha - 1}{2} \sum_{i=1}^{n_2} (x_{2i})^2 \eta^{\alpha - 1}(x_{2i}) \]

\[ + \frac{\theta_2 + \theta_3 - 1}{2} \sum_{i=1}^{n_2} \alpha(x_{2i})^2 \eta^{\alpha - 1}(x_{2i}) - 1 \]

\[ + \frac{\theta_2 + \theta_3 - 1}{2} \sum_{i=1}^{n_2} \alpha(x_{2i})^2 \eta^{\alpha - 1}(x_{2i}) - 1 \]

\[ - \frac{\alpha}{2} \sum_{i=1}^{n_3} (x_{i})^2 \eta^{\alpha - 1}(x_{i}), \quad (44) \]

\[ \frac{\partial L}{\partial \theta_1} = \frac{n_1}{\theta_1 + \theta_3} + \sum_{i=1}^{n_3} \ln \left(1 - e^{-\eta^{\alpha}(x_{i})}\right) + \frac{n_2}{\theta_1} + \sum_{i=1}^{n_1} \ln \left(1 - e^{-\eta^{\alpha}(x_{1i})}\right) \]

\[ + \sum_{i=1}^{n_3} \ln \left(1 - e^{-\eta^{\alpha}(x_{i})}\right), \quad (45) \]

\[ \frac{\partial L}{\partial \theta_2} = \frac{n_1}{\theta_2} + \sum_{i=1}^{n_1} \ln \left(1 - e^{-\eta^{\alpha}(x_{1i})}\right) + \frac{n_2}{\theta_2 + \theta_3} + \sum_{i=1}^{n_2} \ln \left(1 - e^{-\eta^{\alpha}(x_{2i})}\right) \]

\[ + \sum_{i=1}^{n_3} \ln \left(1 - e^{-\eta^{\alpha}(x_{i})}\right), \quad (46) \]

and

\[ \frac{\partial L}{\partial \theta_3} = \frac{n_1}{\theta_1 + \theta_3} + \sum_{i=1}^{n_3} \ln \left(1 - e^{-\eta^{\alpha}(x_{i})}\right) + \frac{n_2}{\theta_2 + \theta_3} + \sum_{i=1}^{n_2} \ln \left(1 - e^{-\eta^{\alpha}(x_{2i})}\right) + \frac{n_3}{\theta_3} \]

\[ + \frac{n_2}{\theta_2} + \sum_{i=1}^{n_3} \ln \left(1 - e^{-\eta^{\alpha}(x_{i})}\right). \quad (47) \]

By Equating the Equations (42-47) by zeros, we get the non-linear normal Equations. So, the solution has to be obtained numerically.
6 Data Analysis

In this section, we have analyzed one bivariate real data set to explicate that the BEGLED can be a good lifetime model, comparing with Marshall-Olkin bivariate exponential distribution (MOBED), bivariate generalized exponential distribution (BVGED) and bivariate generalized linear failure rate distribution (BGLFRD). To make this comparison, we will use the log-likelihood values (L), Akaike information criterion (AIC), correct Akaike information criterion (CAIC), Hannan-Quinn information criterion (HQIC) and the likelihood ratio test (Λ).

The data set in Table 1 has been obtained from Meintanis (2007). This data represents football (soccer) data of the UEFA Champion’s League data for the year 2004 : 2005 and 2005 : 2006. This data describes the games which satisfy the following two conditions:

1. At least one kick goal scored by any team have been considered.
2. The home team must be scored at least one goal.

Note that, the kick goal is the goal which scored directly from foul kick, penalty kick or any other direct free kick. Here the variables $X_1$ and $X_2$ are as follows:

$X_1$ : represents the time in minutes of the first kick goal scored by any team.
$X_2$ : represents the first goal of any type scored by the home team.

|                  | 2005 : 2006 | 2004 : 2005 |
|------------------|-------------|-------------|
| Lyon : Real Madrid | 26 20       | Internazionale : Bremen 34 34 |
| Milan : Fenerbahce | 63 18       | Real Madrid : Roma 53 39 |
| Chelsea : Anderlecht | 19 19     | Man. United : Fenerbahce 54 7 |
| Club Brugge : Juventus | 66 85     | Bayern : Ajax 51 28 |
| Fenerbahce : PSV | 40 40       | Moscow : PSG 76 64 |
| Internazionale : Rangers | 49 49    | Barcelona : Shakhtar 64 15 |
| Panathinaikos : Bremen | 8 8       | Leverkusen : Roma 26 48 |
| Ajax : Arsenal | 69 71       | Arsenal : Panathinaikos 16 16 |
| Man. United : Benfica | 39 39    | Dynamo Kyiv : Real Madrid 44 13 |
| Real Madrid : Rosenborg | 82 48    | Man. United : Sparta 25 14 |
| Villarreal : Benfica | 72 72      | Bayern : M. TelAviv 55 11 |
| Juventus : Bayern | 66 62       | Bremen : Internazionale 49 49 |
| Club Brugge : Rapid | 25 9        | Anderlecht : Valencia 24 24 |
| Olympiacos : Lyon | 41 3        | Panathinaikos : PSV 44 30 |
| Internazionale : Rapid | 16 75     | Arsenal : Rosenborg 42 3 |
| Schalke : PSV | 18 18       | Liverpool : Olympiacos 27 47 |
| Barcelona : Bremen | 22 14       | M. Tel-Aviv : Juventus 28 28 |
| Milan : Schalke | 42 42       | Bremen : Panathinaikos 2 2 |
| Rapid : Juventus | 36 52       | |

To analyze this data by the BEGLED, we fit at first the marginals $X_1$ and $X_2$ of the BEGLED separately one by one on this data. The following Tables obtain the MLEs, L, Anderson-Darling ($A^*$) and Cramér-Von Mises ($W^*$) values for the marginals $X_1$ and $X_2$. 

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respectively for each model.

**Table 2.** The MLE(s), $L$, $A^*$, $W^*$ values for $X_1$.

| Model | $\hat{a}$ | $\hat{b}$ | $\hat{\theta}$ | $\hat{\alpha}$ | -L  | $A^*$ | $W^*$ |
|-------|----------|----------|----------------|----------------|-----|-------|-------|
| E     | 0.0245   | -        | -              | -              | 174.30 | 0.5202 | 0.0686 |
| GE    | 0.0449   | -        | 3.119          | -              | 165.82 | 0.6171 | 0.0826 |
| GLFR  | 0.0052   | 0.0009   | 1.302          | -              | 162.68 | 0.2637 | 0.0399 |
| EGLE  | 0.0022   | 0.0006   | 0.492          | 1.897          | 161.89 | 0.2530 | 0.0396 |

**Table 3.** The MLE(s), $L$, $A^*$, $W^*$ values for $X_2$.

| Model | $\hat{a}$ | $\hat{b}$ | $\hat{\theta}$ | $\hat{\alpha}$ | -L  | $A^*$ | $W^*$ |
|-------|----------|----------|----------------|----------------|-----|-------|-------|
| E     | 0.0304   | -        | -              | -              | 166.219 | 0.3651 | 0.0549 |
| GE    | 0.0413   | -        | 1.678          | -              | 163.937 | 0.3859 | 0.0576 |
| GLFR  | 0.0192   | $6 \times 10^{-4}$ | 1.14          | -              | 162.938 | 0.2713 | 0.04478 |
| EGLE  | 0.0172   | $2 \times 10^{-4}$ | 0.622         | 1.705          | 162.672 | 0.2640 | 0.0436 |

We can conclude that, the EGLE distribution fits the data better than E, GE and GLFR distributions for the marginals, because it has the smallest value among $-L$, $A^*$ and $W^*$.

Since, the E, GE and GLFR distributions are special cases from the EGLE distribution, we perform the following three testing of hypotheses for $X_1$ and $X_2$ separately:

Test 1: $H_{01}: \alpha = 1, b = 0, \theta = 1$ (ED) against $H_{11}: \alpha \neq 1, b > 0, \theta \neq 1$ (EGLED).

Test 2: $H_{02}: \alpha = 1, b = 0$ (GED) against $H_{12}: \alpha \neq 1, b > 0$ (EGLED).

Test 3: $H_{03}: \alpha = 1$ (GLFRD) against $H_{13}: \alpha \neq 1$ (EGLED).

The likelihood ratio test statistics ($\Lambda$), the degree of freedom (d.f) and the corresponding p-values for the three tests of hypotheses in case of $X_1$ and $X_2$ are presented in Tables 4 and 5 respectively.

**Table 4.** The likelihood ratio test statistics, d.f and p-values for $X_1$.

| Model | $H_o$ | $\Lambda$ | d.f | p-values |
|-------|-------|----------|-----|----------|
| E     | $\alpha = 1, b = 0, \theta = 1$ | 24.824 | 3   | 0.00001681 |
| GE    | $\alpha = 1, b = 0$ | 7.846 | 2   | 0.01978166 |
| GLFR  | $\alpha = 1$ | 1.576 | 1   | 0.20933780 |

**Table 5.** The likelihood ratio test statistics, d.f and p-values for $X_2$.

| Model | $H_o$ | $\Lambda$ | d.f | p-values |
|-------|-------|----------|-----|----------|
| E     | $\alpha = 1, b = 0, \theta = 1$ | 7.094 | 3   | 0.06896126 |
| GE    | $\alpha = 1, b = 0$ | 2.53 | 2   | 0.2822393 |
| GLFR  | $\alpha = 1$ | 0.532 | 1   | 0.46576723 |

When the level of significance $\delta$ equals 0.05, it is clear that:
(a) The EGLED provides a significantly better fit in case of $X_1$ and $X_2$ compared to the ED.

(b) The EGLED provides a significantly better fit in case of $X_1$ compared to the GED.

(c) The EGLED provides a better fit for $X_2$ compared to the GED.

(d) The EGLED provides a better fit in case of $X_1$ and $X_2$ compared to the GLFRD.

On the other hand, after studying the marginals $X_1$ and $X_2$, we fit the BEGLED on the UEFA Champion’s League data. The following tables obtain the MLEs, $L$, AIC, CAIC and HQIC values.

**Table 6.** The MLEs and $L$ values

| Model | $\hat{\alpha}$ | $\hat{a}$ | $\hat{b}$ | $\hat{\theta}_1$ | $\hat{\theta}_2$ | $\hat{\theta}_3$ | $-L$ |
|-------|----------------|----------|----------|-----------------|-----------------|----------------|------|
| MOBE  | -              | -        | 0.012    | 0.014           | 0.022           | 339.0          |
| BVGE  | -              | 0.039    | -        | 1.351           | 0.465           | 1.153          | 296.9|
| BGLFR | -              | 0.0002   | 0.0008   | 0.492           | 0.411           | 0.411          | 293.4|
| BEGLE | 0.089          | 0.0107   | 2.711    | 0.00017         | 0.249           | 0.220          | 291.7|

**Table 7.** The AIC, CAIC and HQIC values.

| Model | AIC   | CAIC  | HQIC  |
|-------|-------|-------|-------|
| MOBE  | 684.0 | 684.7 | 685.8 |
| BVGE  | 601.9 | 603.1 | 604.1 |
| BGLFR | 596.8 | 598.7 | 599.6 |
| BEGLE | 595.4 | 598.2 | 598.8 |

It is clear that, the BEGLED provides a better fit than MOBE, BVGE and BGLFR distributions because it has the smallest value among $-L$, AIC, CAIC and HQIC. Since, the BVGE and BGLFR distributions are special cases from the BEGLED, then we perform the following two testing of hypotheses:

Test 1: $H_{02}: \alpha = 1, b = 0$ (BVGED) against $H_{12}: \alpha \neq 1, b > 0$ (BEGLED).

Test 2: $H_{03}: \alpha = 1$ (BVGLFRD) against $H_{13}: \alpha \neq 1$ (BEGLED).

The likelihood ratio test statistics, d.f and p-values for the BVGE and the BGLFR distributions are given in the following Table.

**Table 8.** The likelihood ratio test statistics, d.f and p-values.

| Model      | $H_0$ | $\lambda$ | d.f | p-values |
|------------|-------|-----------|-----|----------|
| BVGE       | $\alpha = 1, b = 0$ | 10.466 | 2   | 0.00533749 |
| BGLFR      | $\alpha = 1$       | 3.354  | 1   | 0.06704192 |

We note that the p-value is not large. So, we prefer the BEGLED for analyzing this data.
7 Simulation Study

In this section, the MLE method is used to estimate the parameters $\alpha, a, b, \theta_1, \theta_2$ and $\theta_3$ of the BEGLED. The population parameters are generated using software "Mathcad prime 3" package. The sampling distributions are obtained for different sample sizes $n = [30, 50, 100, 200]$ from $N = 1000$ replications. This study presents an assessment of the properties of the MLE for the parameters in terms of bias, variance (Var), mean square error (MSE) and 95% confidence intervals (C.I), which be obtained in the following Tables.

Table 9. The MLEs, Bias, Var, MSE and C.I values

| n   | a   | b   | $\theta_1$ | $\theta_2$ | $\theta_3$ | Estimate | Bias     | Var       | MSE      | C.I       |
|-----|-----|-----|------------|------------|------------|----------|----------|-----------|----------|-----------|
| 30  | 1.5 | 0.7 | 0.8        | 1.2        | 1.3        | 1.529447 | 0.029447 | 0.0422195 | 0.0430866 | (1.2060, 2.0056) |
| 50  | 0.5 | 0.7 | 0.8        | 1.2        | 1.3        | 0.505473 | 0.005473 | 0.0050818 | 0.0051117 | (0.3717, 0.6442) |
| 100 | 1.5 | 0.7 | 0.8        | 1.2        | 1.3        | 1.30607  | 0.10007  | 0.3690383 | 0.37916   | (0.5189, 2.7853) |
| 200 | 0.5 | 0.7 | 0.8        | 1.2        | 1.3        | 0.499603 | -0.006396| 0.0031267 | 0.0031268 | (0.3929, 0.6071) |
|     | 1.5 | 0.7 | 0.8        | 1.2        | 1.3        | 0.913251 | 0.113251 | 0.2811745 | 0.2940002 | (0.2862, 2.2053) |
|     | 1.5 | 0.7 | 0.8        | 1.2        | 1.3        | 1.5092043 | 0.0092043 | 0.1491417 | 0.1524966 | (0.7602, 2.2375) |
|     | 0.5 | 0.7 | 0.8        | 1.2        | 1.3        | 0.499514 | -0.000486 | 0.0007961 | 0.0007961 | (0.4468, 0.5564) |
|     | 1.5 | 0.7 | 0.8        | 1.2        | 1.3        | 0.7016369 | 0.0016369 | 0.0059615 | 0.0059642 | (0.5624, 0.8559) |
|     | 1.5 | 0.7 | 0.8        | 1.2        | 1.3        | 0.901861 | 0.101861 | 0.1651336 | 0.1755093 | (0.3843, 1.9003) |
|     | 1.5 | 0.7 | 0.8        | 1.2        | 1.3        | 1.2920736 | 0.0920736 | 0.2386965 | 0.247174 | (0.5865, 2.4482) |
|     | 1.5 | 0.7 | 0.8        | 1.2        | 1.3        | 1.3579218 | 0.057218 | 0.1491417 | 0.1524966 | (0.7602, 2.2375) |
|     | 0.5 | 0.7 | 0.8        | 1.2        | 1.3        | 0.901861 | 0.101861 | 0.1651336 | 0.1755093 | (0.3843, 1.9003) |
|     | 1.5 | 0.7 | 0.8        | 1.2        | 1.3        | 1.2920736 | 0.0920736 | 0.2386965 | 0.247174 | (0.5865, 2.4482) |
|     | 1.5 | 0.7 | 0.8        | 1.2        | 1.3        | 1.3579218 | 0.057218 | 0.1491417 | 0.1524966 | (0.7602, 2.2375) |

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Table 10. The MLEs, Bias, Var, MSE and C.I values

| n  | parameter | Estimate | Bias    | Var   | MSE    | C.I                        |
|----|-----------|----------|---------|-------|--------|----------------------------|
| 30 | $\alpha = 2$ | 1.8540053 | -0.1459947 | 0.0861492 | 0.1074636 | (1.3579, 2.5402) |
|    | $\alpha = 0.2$ | 0.2043051 | 0.0043051 | 0.0047227 | 0.0047412 | (0.0793, 0.3422) |
|    | $\theta_1 = 1$ | 1.5124662 | 0.0124662 | 0.0168988 | 0.0170542 | (1.2850, 1.7882) |
|    | $\theta_1 = 0.5$ | 0.5857591 | 0.0057591 | 0.0092056 | 0.0094106 | (0.2091, 1.3758) |
|    | $\theta_2 = 0.6$ | 0.6827111 | 0.0827111 | 0.108878 | 0.115192 | (0.2713, 1.4742) |
|    | $\theta_3 = 0.9$ | 0.9424471 | 0.0424471 | 0.085705 | 0.0873723 | (0.4933, 1.6149) |
| 50 | $\alpha = 2$ | 1.8900924 | -0.1099076 | 0.054245 | 0.0663247 | (1.4403, 2.4056) |
|    | $\alpha = 0.2$ | 0.2016888 | 0.0016888 | 0.0029133 | 0.0029162 | (0.0983, 0.3127) |
|    | $\theta_1 = 1$ | 1.5054807 | 0.0054807 | 0.0103271 | 0.0103572 | (1.3217, 1.7219) |
|    | $\theta_1 = 0.5$ | 0.5604422 | 0.0604422 | 0.0566666 | 0.0593188 | (0.2489, 1.1597) |
|    | $\theta_2 = 0.6$ | 0.657993 | 0.057993 | 0.063326 | 0.066958 | (0.3180, 1.3028) |
|    | $\theta_3 = 0.9$ | 0.9271563 | 0.0271563 | 0.053646 | 0.054384 | (0.5486, 1.4589) |
| 100| $\alpha = 2$ | 1.9448174 | -0.0551826 | 0.0271835 | 0.0302286 | (1.6074, 2.2876) |
|    | $\alpha = 0.2$ | 0.200691 | 0.000691 | 0.0014431 | 0.0014436 | (0.1273, 0.2765) |
|    | $\theta_1 = 1$ | 1.5025237 | 0.0025237 | 0.0050839 | 0.005093 | (1.3686, 1.6470) |
|    | $\theta_1 = 0.5$ | 0.5351945 | 0.0351945 | 0.0261092 | 0.0273479 | (0.2977, 0.9242) |
|    | $\theta_2 = 0.6$ | 0.6309327 | 0.0309327 | 0.0311299 | 0.0320867 | (0.3745, 1.0547) |
|    | $\theta_3 = 0.9$ | 0.9152326 | 0.0152326 | 0.0263485 | 0.026505 | (0.6396, 1.2798) |
| 200| $\alpha = 2$ | 1.977254 | -0.022746 | 0.0112714 | 0.0117888 | (1.7652, 2.1946) |
|    | $\alpha = 0.2$ | 0.2002004 | 0.0002004 | 0.0006825 | 0.0006825 | (0.1481, 0.2525) |
|    | $\theta_1 = 1$ | 1.5009306 | 0.0009306 | 0.0023884 | 0.0023893 | (1.4047, 1.6006) |
|    | $\theta_1 = 0.5$ | 0.5172642 | 0.0172642 | 0.0120296 | 0.0123276 | (0.3456, 0.7748) |
|    | $\theta_2 = 0.6$ | 0.6161236 | 0.0161236 | 0.014622 | 0.014882 | (0.4252, 0.9035) |
|    | $\theta_3 = 0.9$ | 0.9085774 | 0.0085774 | 0.0124857 | 0.0125593 | (0.7099, 1.1482) |

From Tables 9 and 10, we note that the bias is reduced as the sample size is increased.

8 Conclusions

In this paper, we have proposed a bivariate exponentiated generalized linear exponential distribution (BEGLED), whose marginals are exponentiated generalized linear exponential distributions. We discussed some statistical and reliability properties of the new distribution. Since the joint CDF and the joint PDF are in a closed form, therefore the BEGLED can be used in practice for non-negative and positively correlated random variables. The maximum likelihood estimates (MLE) of the six parameters index to the BEGLED are discussed. Moreover, a real data set is analyzed to show the usefulness of the proposed distribution. Also, the bias of the parameters is calculated using simulation studies. We hope our new distribution (BEGLED) might attract wider sets of applications in reliability analysis.

References

[1] Al-Khedhairi, A. and El-Gohary, A., (2008). A new class of bivariate Gompertz distributions and its mixture. International journal of mathematical analysis, 2(5), 235-253.
[2] Asha, G. and Jagathnath, K. M., (2008). Modeling and characterizations of a bivariate Pareto distribution. Stat and appli, 7(4), 435-452.

[3] Balakrishna, N. and Shiji, K., (2014). On a class of bivariate exponential distributions. Statistics and probability letters, 85,153-160.

[4] Basu, A. P., (1971). Bivariate failure rate. American statistics association, 66, 103-104.

[5] Cox, D. R.,(1972). Regression models and life tables. Royal statistics society, 34, 187-220.

[6] Domma, F., (2009). Some properties of the bivariate Burr type III distribution. Statistics. DOI: 10.1080/02331880902986547.

[7] El-Bassiouny, A. H., EL-Damcese, M., Abdelfattah, M. and Eliwa, M. S., (2016). Bivariate exponentiated generalized Weibull-Gompertz distribution. Journal of applied probability and statistics, 11(1), 25-46.

[8] El-Gohary, A., El-Bassiouny, A. H. and El-Morshedy, M., (2016). Bivariate exponentiated modified Weibull extension distribution. J. stat. appl. pro, 5(1 ), 67-78.

[9] Kundu, D. and Gupta, R. D., (2009). Bivariate generalized exponential distribution. Journal of multivariate analysis,100, 581-593.

[10] Kundu, D. and Gupta, K., (2013). Bayes estimation for the Marshall-Olkin bivariate Weibull distribution. Journal of computational statistics and data analysis, 57(1), 271-281.

[11] Marshall, A. W. and Olkin, I., (1967). A multivariate exponential model. Journal of the American statistical association, 62, 30-44.

[12] Meintanis, S. G., (2007). Test of fit for Marshall-Olkin distributions with applications. Journal of statistical planning and inference, 137, 3954-3963.

[13] Rasool, R. and Akbar, A. J., (2016). On bivariate exponentiated extended Weibull family of distributions. Ciências Natura, Santa Maria, 38(2), 564-576.

[14] Sarhan, A. and Balakrishnan, N., (2007). A new class of bivariate distributions and its mixture. Journal of multivariate analysis, 98,1508-1527.

[15] Sarhan, A. M., Hamilton, D. C., Smith, B., and Kundu, D., (2011). The bivariate generalized linear failure rate distribution and its multivariate extension. Computational statistics and data analysis, 55(1),644-654.

[16] Sarhan, A., Abd EL-Baset, A. A., Ibtesam, A. A., (2013). Exponentiated generalized linear exponential distribution. Applied mathematical modelling, 37, 2838-2849.