Learning Graphical Models Using Multiplicative Weights

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Abstract

We give a simple, multiplicative-weight update algorithm for learning undirected graphical models or Markov random fields (MRFs). The approach is new, and for the well-studied case of Ising models or Boltzmann machines, we obtain an algorithm that uses a nearly optimal number of samples and has running time \( \tilde{O}(n^2) \) (where \( n \) is the dimension), subsuming and improving on all prior work. Additionally, we give the first efficient algorithm for learning Ising models over non-binary alphabets.

Our main application is an algorithm for learning the structure of \( t \)-wise MRFs with nearly-optimal sample complexity (up to polynomial losses in necessary terms that depend on the weights) and running time that is \( n^{O(t)} \). In addition, given \( n^{O(t)} \) samples, we can also learn the parameters of the model and generate a hypothesis that is close in statistical distance to the true MRF. All prior work runs in time \( n^{O(d)} \) for graphs of bounded degree \( d \) and does not generate a hypothesis close in statistical distance even for \( t = 3 \). We observe that our runtime has the correct dependence on \( n \) and \( t \) assuming the hardness of learning sparse parities with noise.

Our algorithm— the Sparsitron— is easy to implement (has only one parameter) and holds in the on-line setting. Its analysis applies a regret bound from Freund and Schapire’s classic Hedge algorithm. It also gives the first solution to the problem of learning sparse Generalized Linear Models (GLMs).

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1 Introduction

Undirected graphical models or Markov random fields (MRFs) are one of the most well-studied and influential probabilistic models with applications to a wide range of scientific disciplines [KF09, Lau98, MRS13, HS86, KFL01, Sal09, Cli90, JEMF06]. Here we focus on binary undirected graphical models which are distributions \((Z_1, \ldots, Z_n)\) on \(\{1, -1\}^n\) with an associated undirected graph \(G\) - known as the dependency graph - on \(n\) vertices where each \(Z_i\) conditioned on the values of \((Z_j : j \text{ adjacent to } i \text{ in } G)\) is independent of the remaining variables.

Developing efficient algorithms for inferring the structure of the underlying graph \(G\) from random samples from \(D\) is a central problem in machine learning, statistics, physics, and computer science [AKN06, KS01, WRL06, BMS13, NBSS12, TR14] and has attracted considerable attention from researchers in these fields. A famous early example of such an algorithmic result is due to Chow and Liu from 1968 [CL68] who gave an efficient algorithm for learning graphical models where the underlying graph is a tree. Subsequent work considered generalizations of trees [ATHW11] and graphs under various strong assumptions (e.g., restricted strong convexity [NRWY10] or correlation decay [BMS13, RSS12]).

The current frontier of MRF learning has focused on the Ising model (also known as Boltzmann machines) on bounded-degree graphs, a special class of graphical models with only pairwise interactions and each vertex having degree at most \(d\) in the underlying dependency graph. We refer to [Bre15] for an extensive historical overview of the problem. Two important works of note are due to Bresler [Bre15] and [VMLC16] who learn Ising models on bounded degree graphs.

Bresler’s algorithm is a combinatorial (greedy) approach that runs in time \(\tilde{O}(n^2)\) but requires doubly exponential in \(d\) many samples from the distribution (only singly exponential is necessary). [VMLC16] use machinery from convex programming to achieve nearly optimal sample complexity for learning Ising models with zero external field and with running time \(\tilde{O}(n^4)\). Neither of these results are proved to hold over non-binary alphabets or for general MRFs.

1.1 Our Results

The main contribution of this paper is a simple, multiplicative-weight update algorithm for learning MRFs. Using our algorithm we obtain the following new results:

- An efficient online algorithm for learning Ising models on arbitrary graphs with nearly optimal sample complexity and running time \(\tilde{O}(n^2)\) per example (precise statements can be found in Section 5). In particular, for bounded degree graphs we achieve a run-time of \(\tilde{O}(n^2)\) with nearly optimal sample complexity. This subsumes and improves all prior work including the above mentioned results of Bresler [Bre15] and [VMLC16]. Our algorithm is the first that works even for unbounded-degree graphs as long as the \(\ell_1\) norm of the weight vector of each neighborhood is bounded, a condition necessary for efficiency (see discussion following Corollary 5.4).

- An algorithm for learning the dependency graph of binary \(t\)-wise Markov random fields with nearly optimal sample complexity and run-time \(n^{O(t)}\) (precise statements can be found in Section 7). Moreover, given access to roughly \(n^{O(t)}\) samples (pressuring necessary terms depending on the weights), we can also reconstruct the parameters of the model and output a \(t\)-wise MRF that gives a point-wise approximation to the original distribution.
As far as we are aware, these are the first efficient algorithms for learning higher-order MRFs. All previous work on learning general $t$-wise MRFs runs in time $n^{O(t)}$ (where $d$ is the underlying degree of the graph) and does not output a function $f$ that can generate an approximation to the distribution in statistical distance, even for the special case of $t = 3$. We give evidence that the $n^{O(t)}$ dependence in our running time is nearly optimal by applying a simple reduction from the problem of learning sparse parities with noise on $t$ variables to learning $t$-wise MRFs due to Bresler, Gamarnik, and Shah [BGS14] (learning sparse parities with noise is a notoriously difficult challenge in theoretical computer science). Bresler [Bre15] observed that even for the simplest possible Ising model where the graph has a single edge, beating $O(n^2)$ run-time corresponds to fast algorithms for the well-studied light bulb problem [Val88], for which the best known algorithm runs in time $O(n^{1.62})$ [Val15].

Moreover, our algorithm is easy to implement, has only one tunable parameter, and works in an on-line fashion. The algorithm—the Sparsitron—solves the problem of learning a sparse Generalized Linear Model. That is, given examples $(X,Y) \in [-1,1]^n \times [0,1]$ drawn from a distribution $D$ with the property that $E[Y|X = x] = \sigma(w \cdot x)$ for some monotonic, Lipschitz $\sigma$ and unknown $w$ with $\|w\|_1 \leq \lambda$, the Sparsitron efficiently outputs a $w'$ such that $\sigma(w' \cdot x)$ is close to $\sigma(w \cdot x)$ in squared-loss and has sample complexity $O(\lambda^2 \log n)$.

In an independent and concurrent work, Hamilton, Koehler, and Moitra [HKM17] generalized Bresler’s approach to hold for both higher-order MRFs as well as MRFs over general (non-binary) alphabets. For learning binary MRFs on bounded-degree—degree at most $d$—graphs, under the same non-degeneracy assumption taken by Hamilton et al.,¹ we obtain sample complexity that is singly exponential in $d^t$, whereas theirs is doubly exponential in $d^t$ (both of our papers obtain sample complexity that depends only logarithmically on $n$, the number of vertices).

### 1.2 Our Approach

For a graph $G = (V,E)$ on $n$ vertices, let $C_t(G)$ denote all cliques of size at most $t$ in $G$. We use the Hammersley-Clifford characterization of Markov random fields and define a binary $t$-wise Markov random field on $G$ to be a distribution $D$ on $\{-1,1\}^n$ where

$$Pr_{Z \sim D}[Z = z] \propto \exp \left( \sum_{I \in C_t(G)} \psi_I(z) \right),$$

and each $\psi_I : \mathbb{R}^n \to \mathbb{R}$ is a function that depends only on the variables in $I$.

For ease of exposition, we will continue with the case of $t = 2$, the Ising model, and subsequently describe the extension to larger values of $t$. Let $\sigma(z)$ denote the sigmoid function. That is $\sigma(z) = 1/1 + e^{-z}$. Since $t = 2$, we have

$$Pr[Z = z] \propto \exp \left( \sum_{i \neq j \in [n]} A_{ij} z_i z_j + \sum_i \theta_i z_i \right)$$

for a weight matrix $A \in \mathbb{R}^{n \times n}$ and $\theta \in \mathbb{R}^n$; here, a weight $A_{ij} \neq 0$ if and only if $\{i,j\}$ is an edge in the underlying dependency graph. For a node $Z_i$, it is easy to see that the probability

¹A previous version of this manuscript needed a slightly stronger non-degeneracy assumption.
In our setting will result in a Kanade, and Shamir [KKKS11] who introduced the “GLMtron” algorithm. As such, if we set \( X \equiv (Z_j : j \neq i) \) and \( Y = (1 - Z_i)/2 \), then the conditional expectation of \( Y \) given \( X \) is equal to a sigmoid with an unknown weight vector \( w \) and threshold \( \theta \). We can now rephrase our original unsupervised learning task as the following supervised learning problem: Given random examples \((X, Y)\) with conditional mean function \( \mathbb{E}[Y|X = x] = \sigma(w \cdot x + \theta) \), recover \( w \) and \( \theta \).

Learning a conditional mean function of the form \( u(w \cdot x) \) with a fixed, known transfer function \( u : \mathbb{R} \to \mathbb{R} \) is precisely the problem of learning a Generalized Linear Model or GLM and has been studied extensively in machine learning. The first provably efficient algorithm for learning GLMs \( \lambda^2 \) and depends on only one parameter achieves essentially optimal sample complexity. The Sparsitron is simple to describe (see Algorithm 

\[ \| w \|_2 \] is small, we still must prove that \( w' \) itself is close to \( w \) and not just that the \( \ell_2 \)-error as in Equation 1.1 is small. We address these two challenges next.

Our algorithm, the Sparsitron, uses a multiplicative-weight update rule for learning \( w \), as opposed to the GLMtron or Isotron, both of which use additive update rules. This enables us to achieve essentially optimal sample complexity. The Sparsitron is simple to describe (see Algorithm 2) and depends on only one parameter \( \lambda \), the upper bound on the \( \ell_1 \)-norm. Its analysis only uses a regret bound from the classic Hedge algorithm due to Freund and Schapire [FS97].

Although the Sparsitron algorithm finds a vector \( w' \in \mathbb{R}^n \) such that \( \mathbb{E}_X[(\sigma(w' \cdot X + \theta') - \sigma(w \cdot X + \theta)]^2 \) is small, we still must prove that \( w' \) is actually close to \( w \). Achieving such strong recovery guarantees for arbitrary distributions is typically a much harder problem (and can be provably hard in some cases for related problems [FGKP09, GR09]). In our case, we exploit the nature of MRFs by a clean property of such distributions: Call a distribution \( D \) on \( \{1, -1\}^n \) \( \delta \)-unbiased if each variable \( Z_i \) is 1 or \(-1\) with probability at least \( \delta \) conditioned on any setting of the other variables. It turns out that under conditions that are necessary for reconstruction, the distributions of MRFs are \( \delta \)-unbiased for a non-negligible \( \delta \). We show that for such \( \delta \)-unbiased distributions achieving reasonably small \( \ell_2 \)-error as in Equation 1.1 implies that the recovered coefficient \( w' \) is in fact close to \( w \).

To obtain our results for learning \( t \)-wise Markov random fields, we generalize the above approach to handle functions of the form \( \sigma(p(x)) \) where \( p \) is a degree \( t \) multilinear polynomial. Sparsitron can be straightforwardly extended to handle low-degree polynomials by linearizing such polynomials (i.e., working in the \((n^t)\)-dimensional space of coefficients). We then have to show that achieving small \( \ell_2 \)-error - \( \mathbb{E}_X[(\sigma(p(X)) - \sigma(q(X))]^2 \) \( \ll 1 \) - implies that the polynomials \( p, q \) are close. This

\[ Z_i = -1 \text{ conditioned on any setting of the remaining nodes to some value } x \in \{-1, 1\}^{[n]\setminus\{i\}} \] is equal to \( \sigma(w \cdot x + \theta) \) where \( w \in \mathbb{R}^{[n]\setminus\{i\}} \), \( w_j = -2A_{ij}, \theta = -\theta_i \).

Unfortunately, the sample complexity of the GLMtron depends on \( \|w\|_2 \), which results in suboptimal bounds on sample complexity for our setting \(^2\). We desire sample complexity dependent on \( \|w\|_1 \), essentially the sparsity of \( w \). In addition, we need an exact recovery algorithm. That is, we need to ensure that \( w' \) itself is close to \( w \) and not just that the \( \ell_2 \)-error as in Equation 1.1 is small.

Notice that \( \sigma(z) \) is both monotone and 1-Lipschitz. Therefore, directly applying the GLMtron in our setting would require \( \Omega(n) \) samples; we are aiming for an information-theoretically optimal logarithmic dependence in the dimension \( n \).

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\(^2\)GLMtron in our setting would require \( \Omega(n) \) samples; we are aiming for an information-theoretically optimal logarithmic dependence in the dimension \( n \).
presents several additional technical challenges; still, in a self-contained proof, we show this holds whenever the underlying distribution is $\delta$-unbiased as is the case for MRFs.

1.3 Best-Experts Interpretation of Our Algorithm

Our algorithm can be viewed as a surprisingly simple weighted voting scheme (a.k.a. “Best-Experts” strategy) to uncover the underlying graph structure $G = (\{v_1, \ldots, v_n\}, E)$ of a Markov random field. Consider an Ising model where for a fixed vertex $v_i$, we want to determine $v_i$’s neighborhood and edge weights. Let $Z = (Z_1, \ldots, Z_n)$ denote random draws from the Ising model.

- Initially, all vertices $v_j (j \neq i)$ could be neighbors. We create a vector of “candidate” neighbors of length $2n - 2$ with entries $(j, +)$ and $(j, -)$ for all $j \neq i$. Intuitively, since we do not know if node $v_j$ will be negatively or positively correlated with $v_i$, we include two candidate neighbors, $(j, +), (j, -)$ to cover the two cases.

- At the outset, every candidate is equally likely to be a neighbor of $v_i$ and so receives an initial weight of $1/(2n - 2)$. Now consider a random draw from the Ising model $Z = (Z_1, \ldots, Z_n)$. For each $j \neq i$ we view each $Z_j$ (and its negation $-Z_j$) as the vote of $(j, +)$ for the value $Z_i$ (respectively of $(j, -)$). The overall prediction $p$ of our candidates is equal to a weighted sum of their votes (we always assume the weights are non-negative and normalized appropriately).

- For a candidate neighbor $v_j$, let the penalty of the prediction $p$ (as motivated by the conditional mean function) be equal to $\ell_j = (\sigma(-2p) - (1 - Z_i)/2)Z_j$. Each candidate $v_j$’s weight is simply multiplied by $\beta^{\ell_j}$ (for some suitably chosen learning rate $\beta^3$). It is easy to see that candidates who predict $Z_i$ correctly will be penalized less than neighbors whose predictions are incorrect.

Remarkably, the weights of this algorithm will converge to the weights of the underlying Ising model, and the rate of this convergence is optimal. Weights of vertices that are not neighbors of $v_i$ will rapidly decay to zero.

For clarity, we present the updates for a single iteration of our Sparsitron algorithm applied to Ising model in Algorithm 1. The iterative nature of the algorithm is reminiscent of algorithms such as belief propagation and stochastic gradient descent that are commonly used in practice. Exploring connections with these algorithms (if any) is an intriguing question.

1.4 Organization

We begin by describing the Sparsitron algorithm for learning sparse generalized models and prove its correctness. We then show, given a hypothesis output by the Sparsitron, how to recover the underlying weight vector exactly under $\delta$-unbiased distributions. For ease of exposition, we begin by assuming that we are learning an Ising model.

We then describe how to handle the more general case of learning $t$-wise MRFs. This requires working with multilinear polynomials, and studying their behavior (especially, how small they can be) under $\delta$-unbiased distributions.

\footnote{For our analysis, the learning rate can be set using standard techniques, e.g., $\beta = 1 - \sqrt{\log n/T}$ when processing $T$ examples.}
Algorithm 1 Updates for Sparsitron applied to learning Ising models

Initialize $W_{ij}^+ = W_{ij}^- = 1/2(n-1)$ and $A_{ij} = 0$ for $i \neq j$.

PARAMETERS: Sparsity bound $\lambda$.

1: for each new example $(Z_1, \ldots, Z_n)$ do:
2: Compute the current predictions: $p_i = \sum_{j \neq i} A_{ij} Z_j$ for all $i$.
3: for each $i \neq j$ do
4: Compute the penalties: Set $\ell_{ij} = (\sigma(-2p_i) - (1 - Z_i)/2) \cdot Z_j$.
5: Update the weights: Set $W_{ij}^+ = W_{ij}^+ + \beta \ell_{ij}$; $W_{ij}^- = W_{ij}^- - \beta \ell_{ij}$.
6: for each $i \neq j$ do
7: Compute edge weights: $A_{ij} = \frac{\lambda}{\sum_{\ell \neq i}(W_{ij}^+ + W_{ij}^-)} \cdot \left( W_{ij}^+ - W_{ij}^- \right)$.

2 Preliminaries

We will use the following notations and conventions.

- For a vector $x \in \mathbb{R}^n$, $x_{-i} \in \mathbb{R}^{[n]\setminus\{i\}}$ denotes $(x_j : j \neq i)$.
- We write multilinear polynomials $p : \mathbb{R}^n \to \mathbb{R}$ as $p(x) = \sum \hat{p}(I) \prod_{i \in I} x_i$; in particular, $\hat{p}(I)$ denotes the coefficient of the monomial $\prod_{i \in I} x_i$ in the polynomial. Let $\|p\|_1 = \sum I |\hat{p}(I)|$.
- For a multilinear polynomial $p : \mathbb{R}^n \to \mathbb{R}$, we let $\partial_i p(x) = \sum_{J \cap \{i\} = \emptyset} \hat{p}(J \cup \{i\}) \prod_{j \in J} x_j$ denote the partial derivative of $p$ with respect to $x_i$. Similarly, for $I \subseteq [n]$, let $\partial_I p(x) = \sum_{J \subseteq [n], I \not\subseteq J} \hat{p}(J \cup I) \prod_{j \in J} x_j$ denote the partial derivative of $p$ with respect to the variables $(x_i : i \in I)$.
- For a multilinear polynomial $p : \mathbb{R}^n \to \mathbb{R}$, we say $I \subseteq [n]$ is a maximal monomial of $p$ if $\hat{p}(J) = 0$ for all $J \supsetneq I$ (i.e., there is no non-zero monomial that strictly contains $I$).

3 Learning Sparse Generalized Linear Models

We first describe our Sparsitron algorithm for learning sparse GLMs. In the next section we show how to learn MRFs using this algorithm. The main theorem of this section is the following:

**Theorem 3.1.** Let $\mathcal{D}$ be a distribution on $[n] \times \{0,1\}$ where for $(X,Y) \sim \mathcal{D}$, $E[Y|X = x] = u(x \cdot X)$ for a non-decreasing 1-Lipschitz function $u : \mathbb{R} \to [0,1]$. Suppose that $\|u\|_1 \leq \lambda$ for a known $\lambda \geq 0$. Then, there exists an algorithm that for all $\varepsilon, \delta \in [0,1]$ given $T = O(\lambda^2 (\ln(n/\delta\varepsilon))/\varepsilon^2)$ independent examples from $\mathcal{D}$, produces a vector $v \in \mathbb{R}^n$ such that with probability at least $1 - \delta$,

$$\mathbb{E}_{(X,Y) \sim \mathcal{D}}[(u(v \cdot X) - u(w \cdot X))^2] \leq \varepsilon. \quad (3.1)$$

The run-time of the algorithm is $O(nT)$. Moreover, the algorithm can be run in an online manner.

**Proof.** We assume without loss of generality that $w_i \geq 0$ for all $i$ and that $\|w\|_1 = \lambda$; if not, we can map examples $(x,y)$ to $((x,-x,0),y)$ and work in the new space. For any vector $v \in \mathbb{R}^n$, define the risk of $v$ $\varepsilon(v) = \mathbb{E}_{(X,Y) \sim \mathcal{D}}[(u(v \cdot X) - u(w \cdot X))^2]$. Let $1$ denote the all $1$’s vector.


Our approach is to use the regret bound for the Hedge algorithm of Freund and Schapire [FS97]. Let $T \geq 1$, $\beta \in [0,1]$ be parameters to be chosen later and $M = C''T \ln(1/\delta)/\varepsilon^2$ for a constant $C''$ to be chosen later. The algorithm is shown in Algorithm 2. The inputs to the algorithm are $T + M$ independent examples $(x^1, y^1), \ldots, (x^T, y^T)$ and $(a^1, b^1), \ldots, (a^M, b^M)$ drawn from $D$.

**Algorithm 2** Sparsitron

1: Initialize $w^0 = 1/n$.
2: for $t = 1, \ldots, T$ do
3: \hspace{1em} Let $p^t = w^{t-1}/\|w^{t-1}\|_1$.
4: \hspace{1em} Define $\ell^t \in \mathbb{R}^n$ by $\ell^t = (1/2)(1 + (\lambda p^t \cdot x^t) - y^t) x^t$.
5: \hspace{1em} Update the weight vectors $w^t$: for each $i \in [n]$, set $w^t_i = w^{t-1}_i \cdot \beta^t_i$.
6: for $t = 1, \ldots, T$ do
7: \hspace{1em} Compute the empirical risk
8: \hspace{1em} $\hat{\epsilon}(\lambda p^t) = (1/M) \sum_{j=1}^M (u(\lambda p^t \cdot a^j) - b^j)^2$.
9: \hspace{1em} Return $v = \lambda p^j$ for $j = \arg \min_{t \in [T]} \hat{\epsilon}(\lambda p^t)$.

We add the 1 in Step 4 of Algorithm 2 to be consistent with [FS97] who work with loss vectors in $[0,1]^n$.

We next analyze our algorithm and show that for suitable parameters $\beta, T, M$, it achieves the guarantees of the theorem. We first show that the sum of the risks $\varepsilon(\lambda p^1), \ldots, \varepsilon(\lambda p^T)$ is small with high probability over the examples; the claim then follows by a simple Chernoff bound to argue that for $M$ sufficiently big, the empirical estimates of the risk, $\hat{\epsilon}(\lambda p^1), \ldots, \hat{\epsilon}(\lambda p^T)$ are close to the true risks.

Observe that $\ell^t \in [0,1]^n$ and associate each $i = 1, \ldots, n$ with an expert and then apply the analysis of Freund and Schapire (c.f. [FS97], Theorem 5). In particular, setting $\beta = 1/(1 + \sqrt{\ln n}/T)$, we get that

$$\sum_{t=1}^T p^t \cdot \ell^t \leq \min_{i \in [n]} \sum_{t=1}^T \ell^t_i + O(\sqrt{T \ln n} + (\ln n)).$$

(3.2)

Let random variable $Q^t = p^t \cdot \ell^t - (w/\lambda) \cdot \ell^t$. Note that $Q^t \in [-1,1]$. Let

$$Z^t = Q^t - \mathbb{E}_{(x^t,y^t)}[Q^t | (x^1,y^1), \ldots, (x^{t-1},y^{t-1})].$$

Then, $Z^1, \ldots, Z^T$ form a martingale difference sequence with respect to the sequence $(x^1, y^1), \ldots, (x^T, y^T)$ and are bounded between $[-2,2]$. Therefore, by Azuma-Hoeffding inequality for bounded martingale difference sequences, with probability at least $1 - \delta$, we have $\left| \sum_{t=1}^T Z^t \right| \leq O(\sqrt{T \ln(1/\delta)})$.

Thus, with probability at least $1 - \delta$,

$$\sum_{t=1}^T \mathbb{E}_{(x^t,y^t)}[Q^t | (x^1,y^1), \ldots, (x^{t-1},y^{t-1})] \leq \sum_{t=1}^T Q^t + O(\sqrt{T \ln(1/\delta)}).$$

(3.3)
Now, for a fixed \((x^t, y^t)\)\(,\ldots, (x^{t-1}, y^{t-1})\), taking expectation with respect to \((x^t, y^t)\), we have
\[
\mathbb{E}_{(x^t, y^t)}[Q^t | (x^1, y^1), \ldots, (x^{t-1}, y^{t-1})] = \mathbb{E}_{(x^t, y^t)}[(p^t - (1/\lambda)w) \cdot \ell^t] = (1/2) \mathbb{E}_x[(p^t - (1/\lambda)w) \cdot (u(\lambda p^t \cdot x^t) - y^t)x^t] = (1/2\lambda) \mathbb{E}_x[(\lambda p^t \cdot x^t - w \cdot x^t)(u(\lambda p^t \cdot x^t) - u(w \cdot x^t))] \geq (1/2\lambda) \mathbb{E}_x[(u(\lambda p^t \cdot x^t) - u(w \cdot x^t))^2].
\]
For all \(a, b \in \mathbb{R}\), \((a - b)(u(a) - u(b)) \geq (u(a) - u(b))^2\).
\[
= (1/2\lambda) \cdot \varepsilon(\lambda p^t).
\]
Therefore, for a fixed \((x^1, y^1), \ldots, (x^{t-1}, y^{t-1})\), we have
\[
(1/2\lambda)\varepsilon(\lambda p^t) \leq \mathbb{E}_{(x^t, y^t)}[Q^t | (x^1, y^1), \ldots, (x^{t-1}, y^{t-1})].
\]
Combining the above with Equations 3.2, 3.3, we get that with probability at least \(1 - \delta\),
\[
(1/2\lambda)\sum_{t=1}^T \varepsilon(\lambda p^t) \leq \sum_{t=1}^T Q^t + O(\sqrt{T \ln(1/\delta)}) \leq \min \sum_{t=1}^T \ell_t^t - \sum_{t=1}^T (w/\lambda) \cdot \ell^t + O(\sqrt{T \ln n + (\ln n)}) + O(\sqrt{T \ln(1/\delta)}).
\]
Now, let \(L = \sum_{t=1}^T \ell^t\). Then,
\[
\min \sum_{t=1}^T \ell_t^t - \sum_{t=1}^T (1/\lambda)w \cdot \ell^t = \min \sum_{t=1}^T \ell_t^t - \sum_{t=1}^T (w/\lambda) \cdot \ell^t = \min L_t - (w/\lambda) \cdot L \leq 0,
\]
where the last inequality follows as \(||w||_1 = \lambda\). Therefore, with probability at least \(1 - \delta\),
\[
(1/2\lambda)\sum_{t=1}^T \varepsilon(\lambda p^t) = O(\sqrt{T \ln(1/\delta)}) + O(\sqrt{T \ln n + (\ln n)}).
\]
In particular, for \(T > C''n^2(\ln(n/\delta))/\varepsilon^2\) for a sufficiently big constant \(C''\), with probability at least \(1 - \delta\),
\[
\min_{t \in [T]} \varepsilon(\lambda p^t) \leq O(\lambda) \cdot \frac{\sqrt{T \ln(1/\delta)} + \sqrt{T \ln n + \ln n}}{T} \leq \varepsilon/2.
\]
Now set \(M = C'''\ln(T/\delta)/\varepsilon^2\) so that by a Chernoff-Hoeffding bound as in Fact 3.2, with probability at least \(1 - \delta\), for every \(t \in [T]\), \(|\varepsilon(\lambda p^t) - \bar{\varepsilon}(\lambda p^t)| \leq \varepsilon/4\). Therefore, with probability at least \(1 - 2\delta\), \(\varepsilon(v) \leq \varepsilon/4 + \bar{\varepsilon}(v) \leq \varepsilon\). Note that the number of samples needed is \(T + M = O(\lambda^2 \ln(n/\varepsilon \delta)/\varepsilon^2)\). The theorem follows.
\[
\square
\]
**Fact 3.2.** There exists a constant \(C > 0\) such that the following holds. Let \(v \in \mathbb{R}^n\) and let \((a^1, b^1), \ldots, (a^M, b^M)\) be independent examples from \(\mathcal{D}\). Then, for all \(\rho, \gamma \geq 0\), and \(M \geq C\ln(1/\rho)/\gamma^2\),
\[
\Pr \left( \left[ \frac{1}{M} \left( \sum_{j=1}^M (u(v \cdot a^j) - b^j)^2 \right) - \varepsilon(v) \right] \geq \gamma \right) \leq \rho.
\]
4 Recovering affine functions from $\ell_2$ minimization

In this section we show that running the Sparsitron algorithm with sufficiently low error parameter $\varepsilon$ will result in an $\ell_\infty$ approximation to the unknown weight vector. We will use this strong approximation to reconstruct the dependency graphs of Ising models as well as the edge weights.

Our analysis relies on the following important definition:

**Definition 4.1.** A distribution $\mathcal{D}$ on $\{1, -1\}^n$ is $\delta$-unbiased if for $X \sim \mathcal{D}$, $i \in [n]$, and any partial assignment $x$ to $(X_j : j \neq i)$,

$$\min(\Pr[X_i = 1|X_{-i} = x], \Pr[X_i = -1|X_{-i} = x]) \geq \delta.$$  

We will use the following elementary property of sigmoid.

**Claim 4.2.** For $a, b \in R$,

$$|\sigma(a) - \sigma(b)| \geq e^{-|a| - 3} \cdot \min(1, |a - b|).$$

**Proof.** Fix $a \in R$ and let $\gamma = \min(1, |a - b|)$. Then, since $\sigma$ is monotonic

$$|\sigma(a) - \sigma(b)| \geq \min(\sigma(a + \gamma) - \sigma(a), \sigma(a) - \sigma(a - \gamma)).$$  

Now, it is easy to check by a case-analysis that for all $a, a' \in \mathbb{R}$,

$$|\sigma(a) - \sigma(a')| \geq \min(\sigma'(a), \sigma'(a')) \cdot |a - a'|.$$  

Further, for any $t$, $\sigma'(t) = 1/(2 + e^t + e^{-t}) \geq e^{-|t|}/4$. Combining the above two, we get that

$$\sigma(a + \gamma) - \sigma(a) \geq (1/4) \min(e^{-|a+\gamma|}, e^{-|a|}) \cdot \gamma \geq (1/4)e^{(-|a| - \gamma)}.\gamma.$$  

Similarly, we get

$$\sigma(a) - \sigma(a - \gamma) \geq 4 \min(e^{-|a-\gamma|}, e^{-|a|}) \cdot \gamma \geq (1/4)e^{(-|a| - \gamma)\gamma}.$$  

The claim now follows by substituting $\gamma = \min(1, |a - b|)$ (and noting that $1/4 \geq e^{-2}$). \qed

**Lemma 4.3.** Let $D$ be a $\delta$-unbiased distribution on $\{1, -1\}^n$. Suppose that for two vectors $v, w \in \mathbb{R}^n$ and $\alpha, \beta \in \mathbb{R}$, $\mathbb{E}_{X \sim D}[(\sigma(w \cdot X + \alpha) - \sigma(v \cdot X + \beta))^2] \leq \varepsilon$ where $\varepsilon < \delta \cdot \exp(-2\|w\|_1 - 2|\alpha| - 6)$. 

Then,

$$\|v - w\|_\infty \leq O(1) \cdot e^{\|w\|_1 + |\alpha|} \cdot \sqrt{\varepsilon/\delta}.$$  

**Proof.** For brevity, let $p(x) = w \cdot x + \alpha$, and $q(x) = v \cdot x + \beta$. Fix an index $i \in [n]$ and let $X \sim D$. 

Now, for any $x \in \{1, -1\}^n$, by Claim 4.2,

$$|\sigma(p(x)) - \sigma(q(x))| \geq e^{-\|w\|_1 - |\alpha| - 3} \cdot \min(1, |p(x) - q(x)|).$$  

Let $x_i^+ \in \{1, -1\}^n$ (respectively $x_i^-$) denote the vector obtained from $x$ by setting $x_i = 1$ (respectively $x_i = -1$). Note that $p(x_i^+) - p(x_i^-) = 2w_i$ and $q(x_i^+) - q(x_i^-) = 2v_i$. Therefore,

$$p(x_i^+) - q(x_i^+) - (p(x_i^-) - q(x_i^-)) = 2(w_i - v_i).$$
Thus, 
\[ \max \left( |p(x^{i,+}) - q(x^{i,+})|, |p(x^{i,-}) - q(x^{i,-})| \right) \geq |w_i - v_i|. \]

Therefore, for any fixing of \(X_{-i}\), as \(X\) is \(\delta\)-unbiased,
\[ \Pr_{X_i|X_{-i}} \left[ |p(X) - q(X)| \geq |w_i - v_i| \right] \geq \delta. \]

Hence, combining the above inequalities,
\[ \varepsilon \geq \mathbb{E}_X \left[ (\sigma(p(X)) - \sigma(q(X)))^2 \right] \geq e^{-2\|w\|_1 - 2|\alpha|-6} \cdot \delta \cdot \min \left( 1, |w_i - v_i|^2 \right). \]

As \( \varepsilon < e^{-2\|w\|_1 - 2|\alpha|-6}\delta \), the above inequality can only hold if \( |w_i - v_i| < 1 \) so that
\[ |w_i - v_i| < e^{\|w\|_1 + |\alpha|+3} \cdot \sqrt{\varepsilon/\delta}. \]

The claim now follows. \( \square \)

5 Learning Ising Models

**Definition 5.1.** Let \( A \in \mathbb{R}^{n \times n} \) be a weight matrix and \( \theta \in \mathbb{R}^n \) be a mean-field vector. The associated \( n\)-variable Ising model is a distribution \( \mathcal{D}(A, \theta) \) on \( \{1, -1\}^n \) given by the condition
\[ \Pr_{Z \leftarrow \mathcal{D}(A, \theta)} [Z = z] \propto \exp \left( \sum_{i \neq j \in [n]} A_{ij} z_i z_j + \sum_i \theta_i z_i \right). \]

The dependency graph of \( \mathcal{D}(A, \theta) \) is the graph \( G \) formed by all pairs \( \{i, j\} \) with \( |A_{ij}| \neq 0 \). We define \( \lambda(A, \theta) = \max_i (\sum_j |A_{ij}| + |\theta_i|) \) to be the width of the model.

We give a simple, sample-efficient, and online algorithm for recovering the parameters of an Ising model.

**Theorem 5.2.** Let \( \mathcal{D}(A, \theta) \) be an \( n\)-variable Ising model with width \( \lambda(A, \theta) \leq \lambda \). There exists an algorithm that given \( \lambda, \varepsilon, \rho \in (0, 1) \), and \( N = O(\lambda^2 \exp(\Theta(\lambda))/\varepsilon^4) \cdot (\log(n/\rho\varepsilon)) \) independent samples \( Z^1, \ldots, Z^N \leftarrow \mathcal{D}(A, \theta) \) produces \( A \) such that with probability at least 1 - \( \rho \),
\[ \|A - \hat{A}\|_\infty \leq \varepsilon. \]

The run-time of the algorithm is \( O(n^2 N) \). Moreover, the algorithm can be run in an online manner.

**Proof.** The starting point for our algorithm is the following observation. Let \( Z \leftarrow \mathcal{D}(A, \theta) \). Then, for any \( i \in [n] \) and any \( x \in \{1, -1\}^{[n] \setminus \{i\}} \),
\[ \Pr[Z_i = -1|Z_{-i} = x] = \frac{1}{1 + \exp(2 \sum_{j \neq i} A_{ij} x_j + \theta_i)} = \sigma(w(i) \cdot x + \theta_i), \quad (5.1) \]

where we define \( w(i) \in \mathbb{R}^{[n] \setminus \{i\}} \) with \( w(i)_j = -2A_{ij} \) for \( j \neq i \). This allows us to use our Sparsitron algorithm for learning GLMs.
Lemma 4.3 and Equation 5.2, we get

For Fact 5.3.

We now apply the Sparsitron algorithm to compute a vector \( \sigma \) further, \( \sigma \) is a monotone 1-Lipschitz function. Let \( \gamma \in (0,1) \) be a parameter to be chosen later. We set \( \hat{A}_{nj} = - (v(n)_{j})/2 \) for \( j < n \). We next argue that Equation 5.2 in fact implies \( \| w(n) - v(n) \|_{\infty} < 1 \). To this end, we will use the following easy fact (see e.g. Bresler [Bre15]):

**Fact 5.3.** For \( Z \sim \mathcal{D}(A, \theta) \), \( i \in [n] \), and any partial assignment \( x \) to \( Z_{-i} \),

\[
\min \left( \Pr[Z_i = -1 | Z_{-i} = x], \Pr[Z_i = 1 | Z_{-i} = x] \right) \geq (1/2)e^{-2\lambda(A, \theta)} \geq (1/2)e^{-2\Lambda}.
\]

That is, the distribution \( Z \) is \( \delta \)-unbiased for \( \delta = (1/2)e^{-2\lambda} \). Note that \( w(n) \cdot X = \sum_{j < n} w(n)_j Z_j + w(n)_n \) and \( v(n) \cdot X = \sum_{j < n} v(n)_j Z_j + v(n)_n \). Therefore, as \( (Z_1, \ldots, Z_{n-1}) \) is \( \delta \)-unbiased, by Lemma 4.3 and Equation 5.2, we get

\[
\max_{j < n} |v(n)_j - w(n)_j| \leq O(1) \exp(2\lambda) \cdot \sqrt{\gamma}/\delta,
\]

if \( \gamma \leq c\delta \cdot \exp(-4\lambda) \leq c \exp(-5\lambda) \) for a sufficiently small \( c \). Thus, if we set \( \gamma = c' \exp(-5\lambda)\varepsilon^2 \) for a sufficiently small constant \( c' \), then we get

\[
\max_{j < n} |A_{nj} - \hat{A}_{nj}| = (1/2)\|v(n) - w(n)\|_{\infty} \leq \varepsilon.
\]

By a similar argument for \( i = 1, \ldots, n-1 \) and taking a union bound, we get estimates \( \hat{A}_{ij} \) for all \( i \neq j \) so that with probability at least \( 1 - \rho \),

\[
\max_{i \neq j} |A_{ij} - \hat{A}_{ij}| \leq \varepsilon.
\]

Note that by Theorem 3.1, the number of samples needed to satisfy Equation 5.2 is

\[
O((\lambda/\gamma)^2 \cdot (\log(n/\rho\gamma))) = O(\lambda^2 \exp(10\lambda) / \varepsilon^4) \cdot (\log(n/\rho\varepsilon)).
\]

This proves the theorem.

The above theorem immediately implies an algorithm for recovering the dependency graph of an Ising model with nearly optimal sample complexity.

**Corollary 5.4.** Let \( \mathcal{D}(A, \theta) \) be an \( n \)-variable Ising model with width \( \lambda(A, \theta) \leq \lambda \) and each non-zero entry of \( A \) at least \( \eta > 0 \) in absolute value. There exists an algorithm that given \( \lambda, \eta, \rho \in (0,1) \), and \( N = O(\exp(\Omega(\lambda)) / \eta^4) \cdot (\log(n/\eta\varepsilon)) \) independent samples \( Z^1, \ldots, Z^N \sim \mathcal{D}(A, \theta) \) recovers the underlying dependency graph of \( \mathcal{D}(A, \theta) \) with probability at least \( 1 - \rho \). The run-time of the algorithm is \( O(n^2 N) \). Moreover, the algorithm can be run in an online manner.
Proof. The claim follows immediately from Theorem 5.2 by setting \( \varepsilon = \eta/2 \) to compute \( \hat{A} \) and taking the edges \( E \) to be \( \{i, j\} : |\hat{A}_{ij}| \geq \eta/2 \). \( \square \)

It is instructive to compare the upper bounds from Corollary 5.4 with known unconditional lower bounds on the sample complexity of learning Ising models with \( n \) vertices due to Santhanam and Wainwright [SW12]. They prove that, even if the weights of the underlying graph are known, any algorithm for learning the graph structure must use \( \Omega\left(\frac{2^{\log n}}{n^2}\right) \) samples. Hence, the sample complexity of our algorithm is near the best-known information-theoretic lower bound.

6 Recovering polynomials from \( \ell_2 \) minimization

In order to obtain results for learning general Markov Random Fields, we need to extend our learning results from previous sections to the case of sigmoids of low-degree polynomials. In this section, we prove that for any polynomial \( p : \mathbb{R}^n \to \mathbb{R} \), minimizing the \( \ell_2 \)-loss with respect to a sigmoid under a \( \delta \)-unbiased distribution \( D \) also implies closeness as a polynomial. That is, for two polynomials \( p, q : \mathbb{R}^n \to \mathbb{R} \) if \( E_{X \sim D}[(\sigma(p(X)) - \sigma(q(X)))^2] \) is sufficiently small, then \( \|p - q\|_1 \ll 1 \) (Lemma 6.4) and that the coefficients of maximal monomials of \( p \) can be inferred from \( q \) (Lemma 6.2). These results will allow us to recover the structure and parameters of MRFs when combined with Sparsitron.

The exact statements and arguments here are similar in spirit to Lemma 4.3 and its proof but are more subtle. To start with, we need the following property of \( \delta \)-unbiased distributions.\[ \]

**Lemma 6.1.** There is a constant \( c > 0 \) such that the following holds. Let \( D \) be a \( \delta \)-unbiased distribution on \( \{1, -1\}^n \). Then, for any multilinear polynomial \( s : \mathbb{R}^n \to \mathbb{R} \), and any maximal monomial \( I \neq \emptyset \subseteq [n] \) in \( s \),

\[
\Pr_{X \sim D}[|s(X)| \geq |\hat{s}(I)|] \geq \delta^{|I|}.
\]

**Proof.** We prove the claim by induction on \( |I| \). For an \( i \in [n] \), let \( x_i^+ \in \{1, -1\}^n \) (respectively \( x_i^- \)) denote the vector obtained from \( x \) by setting \( x_i = 1 \) (respectively \( x_i = -1 \)). Note that \( x_i^+, x_i^- \) only depend on \( x_{-i} \). Let \( X \sim D \).

Suppose \( I = \{i\} \) so that \( s(x) = \hat{s}(\{i\})x_i + s'(x_{-i}) \) for some polynomial \( s' \) that only depends on \( x_{-i} \). Note that \( \max(|s(x_i^+)|, |s(x_i^-)|) \geq |\hat{s}(\{i\})| \). Therefore, for any fixing of \( X_{-i} \), as \( X \) is \( \delta \)-unbiased,

\[
\Pr_{X_i|X_{-i}}[|s(X)| \geq |\hat{s}(\{i\})|] \geq \delta.
\]

Now, suppose \( |I| = \ell \geq 2 \) and that the claim is true for all polynomials and all monomials of size at most \( \ell - 1 \). Let \( i \in I \). Then, \( s(x) = x_i \cdot \partial_i(s(x_{-i})) + s'(x_{-i}) \) for some polynomial \( s' \) that only depends on \( x_{-i} \). Thus, \( \max(|s(x_i^+)|, |s(x_i^-)|) \geq \partial_is(x_{-i}) \). Therefore, for any fixing of \( X_{-i} \), as \( X \) is \( \delta \)-unbiased,

\[
\Pr_{X_i|X_{-i}}[|s(X)| \geq |\partial_is(X_{-i})|] \geq \delta.
\]

Now, let \( J = I \setminus \{i\} \) and observe that \( J \) is a maximal monomial in \( r(x_{-i}) \equiv \partial_is(x_{-i}) \) with \( \hat{r}(J) = \hat{s}(I) \). Therefore, by the induction hypothesis,

\[
\Pr_{X_{-i}}[|\partial_is(X_{-i})| \geq |\hat{s}(I)|] \geq \delta^{\ell-1}.
\]
Combining the last two inequalities, we get that $\Pr[|s(X)| \geq \tilde{s}(I)] \geq \delta^\ell$. The claim now follows by induction. 

The next lemma shows that for unbiased distributions $\mathcal{D}$, and two low-degree polynomials $p, q : \mathbb{R}^n \to \mathbb{R}$, if $\mathbb{E}_{X \sim \mathcal{D}}[(\sigma(p(x)) - \sigma(q(x))^2] \leq \varepsilon$, then one can infer the coefficients of the maximal monomials of $p$ from $q$.

**Lemma 6.2.** Let $\mathcal{D}$ be a $\varepsilon$-unbiased distribution on $\{1, -1\}^n$. Let $p, q$ be two multilinear polynomials $p, q : \mathbb{R}^n \to \mathbb{R}$ such that $\mathbb{E}_{X \sim \mathcal{D}}[(\sigma(p(x)) - \sigma(q(x))^2] \leq \varepsilon$. Then, for every maximal monomial $I \subseteq [n]$ of $p$, and any $\rho > 0$,

$$\Pr_{X \sim \mathcal{D}}[|\tilde{p}(I) - \partial_I q(X)| > \rho] \leq \frac{e^{2\|p\|_1 + 6\varepsilon}}{\rho^2 \delta|I|}.$$

**Proof.** Let $X \sim \mathcal{D}$ and fix a maximal monomial $I \subseteq [n]$ in $p$. Now, for any $x \in \{1, -1\}^n$, by Claim 4.2,

$$|\sigma(p(x)) - \sigma(q(x))| \geq e^{-\|p\|_1 - 3} \cdot \min(1, |p(x) - q(x)|).$$

Therefore,

$$\mathbb{E} \left[ \min \left(1, |p(X) - q(X)|^2 \right) \right] \leq e^{2\|p\|_1 + 6\varepsilon}.$$

Hence, for every $\rho \in (0, 1)$,

$$\Pr_X[|p(X) - q(X)| > \rho] \leq e^{2\|p\|_1 + 6\varepsilon}/\rho^2.$$

Now consider a fixing of all variables not in $I$ to $z \in \{1, -1\}^{[n] \setminus I}$ and let $r_z(x_I)$ be the polynomial obtained by the resulting fixing. Now,

$$\Pr_X[|p(X) - q(X)| > \rho] = \sum_{z \in \{1, -1\}^{[n] \setminus I}} \Pr[X_{[n] \setminus I} = z] \cdot \Pr[|r_z(X_I)| > \rho \mid X_{[n] \setminus I} = z].$$

Further, $\tilde{p}(I) = \tilde{p}(I) - \partial_I q(z)$ as $I$ is maximal in $p$.

Conditioned on the event that $|\tilde{p}(I)| > \rho$, for a random choice of $X_{[n] \setminus I}$, we have from Lemma 6.1 that $\Pr_{X_I}[|r_z(X_I)| > \rho] \geq \delta^{|I|}$. Thus we have

$$\Pr_X[|p(X) - q(X)| > \rho] \geq \delta^{|I|} \cdot \Pr_{X_{[n] \setminus I}}[|\tilde{p}(I) - \partial_I q(X_{[n] \setminus I})| > \rho].$$

Combining the above equations we get that

$$\Pr_X[|\tilde{p}(I) - \partial_I q(X)| > \rho] \leq \frac{e^{2\|p\|_1 + 6\varepsilon}}{\rho^2 \delta|I|}.$$

The next claim shows that under the assumptions of Lemma 6.2, the highest degree monomials of $p, q$ are close to each other.

---

$^4$Note that under the hypothesis of the lemma, the coefficients of $p$ and $q$ can nevertheless be far.
Lemma 6.3. Let $\mathcal{D}$ be a $\delta$-unbiased distribution on $\{1, -1\}^n$. Let $p, q$ be two multilinear polynomials $p, q : \mathbb{R}^n \rightarrow \mathbb{R}$ such that $E_{X \sim \mathcal{D}}[(\sigma(p(x)) - \sigma(q(x))^2] \leq \varepsilon$ where $\varepsilon < e^{-2\|p\|_1 \delta^4}$. Then, for every maximal monomial $I \subseteq [n]$ of $(p - q)$,

$$|\hat{p}(I) - \hat{q}(I)| \leq e^{\|p\|_1 + 3} \cdot \sqrt{\varepsilon / |I|}.$$  

Proof. Fix a maximal monomial $I \subseteq [n]$ in $(p - q)$. Now, for any $X$, by Claim 4.2,

$$|\sigma(p(X)) - \sigma(q(X))| \geq e^{-2\|p\|_1 - 3} \cdot \min(1, |p(X) - q(X)|).$$

On the other hand, as $X$ is $\delta$-unbiased, by Lemma 6.1, with probability at least $\delta |I|$, $|p(X) - q(X)| \geq |\hat{p}(I) - \hat{q}(I)|$. Therefore,

$$\varepsilon \geq E_X \left[(\sigma(p(X)) - \sigma(q(X))^2] \geq e^{-2\|p\|_1 - 6} \cdot \delta |I| \cdot \min(1, |\hat{p}(I) - \hat{q}(I)|^2).$$

As $\varepsilon < e^{-2\|p\|_1 \delta^4}$, the above inequality can only hold if $|\hat{p}(I) - \hat{q}(I)| < 1$ so that

$$|\hat{p}(I) - \hat{q}(I)| < e^{\|p\|_1 + 3} \sqrt{\varepsilon / |I|}.$$  

The claim follows. \qed

We next show that if $E_{X \sim \mathcal{D}}[(\sigma(p(x)) - \sigma(q(x))^2] \ll n^{-\ell}$ is sufficiently small, then $\|p - q\|_1 \ll 1$.

Lemma 6.4. Let $\mathcal{D}$ be a $\delta$-unbiased distribution on $\{1, -1\}^n$. Let $p, q$ be two multilinear polynomials $p, q : \mathbb{R}^n \rightarrow \mathbb{R}$ of degree $t$ such that $E_{X \sim \mathcal{D}}[(\sigma(p(x)) - \sigma(q(x))^2] \leq \varepsilon$ where $\varepsilon < e^{-2\|p\|_1 \delta^4}$. Then,

$$\|p - q\|_1 = O(1) \cdot (2t)^t e^{\|p\|_1} \cdot \sqrt{\varepsilon / \delta^4} \cdot \left(\frac{n}{t}\right).$$

Proof. For a polynomial $s : \mathbb{R}^n \rightarrow \mathbb{R}$ of degree at most $t$, and $\ell \leq t$, let $s_{\leq \ell}$ denote the polynomial obtained from $s$ by only taking monomials of degree at most $\ell$ and let $s_{= \ell}$ denote the polynomial obtained from $s$ by only taking monomials of degree exactly $\ell$.

For brevity, let $r = p - q$, and for $\ell \leq t$, let $\rho_\ell = \|r_{= \ell}\|_1 = \|p_{= \ell} - q_{= \ell}\|_1$. We will inductively bound $\rho_t, \rho_{t-1}, \ldots, \rho_1$.

From Lemma 6.3 applied to the polynomials $p, q$, we immediately get that

$$\rho_\ell = \|r_{= \ell}\|_1 \leq e^{\|p\|_1 + 3} \cdot \sqrt{\varepsilon / \delta^4} \cdot \left(\frac{n}{t}\right) \equiv \varepsilon_0. \quad (6.1)$$

Now consider $I \subseteq [n]$ with $|I| = \ell$. Then, by an averaging argument, there is some fixing of the variables not in $X_I$ so that for the polynomials $p_I, q_I$ obtained by this fixing, and for the resulting distribution $\mathcal{D}_I$ on $\{1, -1\}^I$,

$$E_{Y \sim \mathcal{D}_I}[(\sigma(p_I(Y)) - \sigma(q_I(Y))^2] \leq \varepsilon.$$  

Note that $\mathcal{D}_I$ is also $\delta$-unbiased. Therefore, by Lemma 6.3 applied to the polynomials $p, q$, letting $r_I = p_I - q_I$, we get that

$$|\hat{r}_I(I)| = |\hat{p}_I(I) - \hat{q}_I(I)| \leq e^{\|p\|_1 + 3} \cdot \sqrt{\varepsilon / \delta^4}.$$
We next relate the coefficients of $r_I$ to that of $r$. As the polynomial $r_I$ is obtained from $r$ by fixing the variables not in $I$ to some values in $\{1, -1\}$, 

$$|\hat{r}_I(I)| \geq |\hat{r}(I)| - \sum_{J: J \supset I} |\hat{r}(J)|.$$ 

Combining the above two inequalities, we get that 

$$|\hat{r}(I)| \leq e\|p\|_{1+3} \cdot \sqrt{\varepsilon / \delta^t} + \sum_{J \supset I} |\hat{r}(J)|.$$ 

Summing the above equation over all $I$ of size exactly $\ell$, we get 

$$\|r=\ell\|_1 = \sum_{I: |I|=\ell} |\hat{r}(I)| \leq e\|p\|_{1+3} \cdot \sqrt{\varepsilon / \delta^t} \cdot \left(\frac{n}{\ell}\right) + \sum_{I: |I|=\ell} \left(\sum_{J \supset I} |\hat{r}(J)|\right)$$ 

$$= \varepsilon_0 + \sum_{j=\ell+1}^t \binom{j}{\ell} \left(\sum_{|J|=j} |\hat{r}(J)|\right) = \varepsilon_0 + \sum_{j=\ell+1}^t \binom{j}{\ell} \|r=\ell\|_1.$$ 

Therefore, we get the recurrence, 

$$\rho_\ell \leq \varepsilon_0 + \sum_{j=\ell+1}^t \binom{j}{\ell} \rho_j. \quad (6.2)$$

We can solve the above recurrence by induction on $\ell$. Specifically, we claim that the above implies $\rho_j \leq (2t)^{t-j} \cdot \varepsilon_0$. For $j = t$, the claim follows from Equation 6.1. Now, suppose the inequality holds for all $j > \ell$. Then, by Equation 6.2, as $\binom{j}{\ell} \leq j^{j-\ell}$, 

$$\rho_\ell \leq \varepsilon_0 + \sum_{j=\ell+1}^t j^{j-\ell}(2t)^{t-j} \leq \varepsilon_0 + \sum_{j=\ell+1}^t t^{j-\ell}(2t)^{t-j} \leq \varepsilon_0 + \sum_{j=\ell+1}^t 2^{t-j} \leq t^{t-\ell} \cdot \varepsilon_0 \cdot 2^{t-\ell}.$$ 

Therefore, 

$$\|r\|_1 = \sum_{\ell=0}^t \|r=\ell\|_1 \leq \sum_{\ell=0}^t (2t)^{t-\ell} \varepsilon_0 \leq \varepsilon_0 \cdot 2^{t+1} t.$$ 

The lemma now follows by plugging in the value of $\varepsilon_0$. 

7 Learning Markov Random Fields 

We now describe how to apply the Sparsitron algorithm to recover the structure as well as parameters of binary $t$-wise MRFs.
We will use the characterization of MRFs via the Hammersley-Clifford theorem. Given a graph \( G = (V,E) \) on \( n \) vertices, let \( C_t(G) \) denote all cliques of size at most \( t \) in \( G \). A binary \( t \)-wise MRF with dependency graph \( G \) is a distribution \( D \) on \( \{1,-1\}^n \) where the probability density function of \( D \) can be written as

\[
\Pr_{Z \sim D}[Z = x] \propto \exp \left( \sum_{I \in S} \psi_I(x) \right),
\]

where \( S \subseteq C_t(G) \) and each \( \psi_I : \mathbb{R}^n \to \mathbb{R} \) is a function that depends only on the variables in \( I \). Note that if \( t = 2 \), this corresponds exactly to the Ising model. We call \( \psi(x) = \sum_{I \in S} \psi_I(x) \) the factorization polynomial of the MRF and \( G \) the dependency graph of the MRF.

Note that the factorization polynomial is a polynomial of degree at most \( t \). However, different graphs and factorizations (i.e., functions \( \{\psi_I\} \)) could potentially lead to the same polynomial. To get around this we enforce the following non-degeneracy condition:

**Definition 7.1.** For a \( t \)-wise MRF \( D \) on \( \{1,-1\}^n \) we say an associated dependency graph \( G \) and factorization

\[
\Pr_{Z \sim D}[Z = x] \propto \exp \left( \sum_{I \in S} \psi_I(x) \right),
\]

for \( S \subseteq C_t(G) \) is \( \eta \)-identifiable if for every maximal monomial \( J \) in \( \psi(x) = \sum_{I \in S} \psi_I(x) \), \( \left| \hat{\psi}(J) \right| \geq \eta \) and every edge in \( G \) is covered by a non-zero monomial of \( \psi \).

We now state our main theorems for learning MRFs. Our first result is about structure learning, i.e., recovering the underlying dependency graph of a MRF. Roughly speaking, using \( N = 2^{O(\lambda t)} \log(n/\eta)/\eta^4 \) samples we can recover the underlying dependency graph of a \( \eta \)-identifiable MRF where \( \lambda \) is the maximum \( \ell_1 \)-norm of the derivatives of the factorization polynomial. The run-time of the algorithm is \( O(M \cdot n^t) \). Note that \( \max_i \| \partial_i \psi \|_1 \) is analogous to the notion of width for Ising models (as in Corollary 5.4). Thus, exponential dependence on it is necessary as in the Ising model and our sample complexity is in fact nearly optimal in all parameters.

**Theorem 7.2.** Let \( D \) be a \( t \)-wise MRF on \( \{1,-1\}^n \) with underlying dependency graph \( G \) and factorization polynomial \( p(x) = \sum_{I \in C_t(G)} p_I(x) \) with \( \max_i \| \partial_i p \|_1 \leq \lambda \). Suppose that \( D \) is \( \eta \)-identifiable. Then, there exists an algorithm that given \( \lambda, \eta, \rho \in (0,1/2) \), and

\[
N = \frac{e^{O(t)} e^{O(\lambda t)}}{\eta^4} \cdot (\log(n/\rho \eta))
\]

independent samples from \( D \), recovers the underlying dependency graph \( G \) with probability at least \( 1 - \rho \). The run-time of the algorithm is \( O(N \cdot n^t) \). Moreover, the algorithm can be run in an online manner.

Along with learning the dependency graph, given more samples, we can also approximately learn the parameters of the MRF: i.e., compute a \( t \)-wise MRF whose distribution is close as a pointwise-approximation to the original probability density function.
Theorem 7.3. Let $D$ be a $t$-wise MRF on $\{1, -1\}^n$ with underlying dependency graph $G$ and factorization polynomial $\psi(x) = \sum_{i \in C_t(G)} \rho_i(x)$ with $\max_i \|\partial_i \psi\|_1 \leq \lambda$. There exists an algorithm that given $\lambda$, and $\varepsilon, \rho \in (0, 1/2)$, and

$$N = \frac{(2t)^O(t) \cdot e^{O(\lambda t)}}{\varepsilon^4} \cdot n^4 \cdot (\log(n/\rho\varepsilon))$$

independent samples $Z^1, \ldots, Z^N \leftarrow D$ produces a $t$-wise MRF $D'$ with dependency graph $H$ and a factorization polynomial $\varphi(x) = \sum_{I \in C_t(H)} \rho_I(x)$ such that with probability at least $1 - \rho$:

$$\forall x, \Pr_{Z \sim D'} [Z = x] = (1 \pm \varepsilon) \cdot \Pr_{Z \sim D'} [Z = x].$$

The algorithm runs in time $O(Nn^4)$ and can be run in an online manner.

We in fact show how to recover the parameters of a log-polynomial density defined as follows:

Definition 7.4. A distribution $D$ on $\{1, -1\}^n$ is said to be a log-polynomial distribution of degree $t$ if for some multilinear polynomial $p : \mathbb{R}^n \to \mathbb{R}$ of degree $t$,

$$\Pr_{X \sim D} [X = x] \propto \exp(p(x)).$$

Theorem 7.5. Let $D$ be a log-polynomial distribution of degree at most $t$ on $\{1, -1\}^n$ with the associated polynomial $p : \mathbb{R}^n \to \mathbb{R}$ such that $\max_i \|\partial_i p\|_1 \leq \lambda$. There exists an algorithm that given $\lambda$, and $\varepsilon, \rho \in (0, 1)$ and

$$N = \frac{(2t)^O(t) \cdot e^{O(\lambda t)}}{\varepsilon^4} \cdot (\log(n/\rho\varepsilon)),$$

independent samples $Z^1, \ldots, Z^N \leftarrow D$, finds a multilinear polynomial $q : \mathbb{R}^n \to \mathbb{R}$ such that with probability at least $1 - \rho$

$$\|p - q\|_1 \leq \varepsilon \cdot \left(\frac{n}{t}\right).$$

Moreover, we can also find coefficients $(\hat{s}(I) : I \subseteq [n], |I| \leq t)$ such that with probability at least $1 - \rho$, for every maximal monomial $I$ of $p$, we have $|\hat{p}(I) - \hat{s}(I)| < \varepsilon$. The run-time of the algorithm is $O(N \cdot n^4)$ and the algorithm can be run in an online manner.

7.1 Learning the structure of MRFs

The following elementary properties of MRFs play a critical role in our analysis.

Lemma 7.6. Let $D$ be a $t$-wise MRF on $\{1, -1\}^n$ with underlying dependency graph $G$ and factorization polynomial $p(x) = \sum_{I \in C_t(G)} \rho_I(x)$ with $\max_i \|\partial_i p\|_1 \leq \lambda$. Then, the following hold for $Z \leftarrow D$:

- For any $i$, and a partial assignment $x \in \{1, -1\}^{[n] \setminus \{i\}}$, $\Pr[Z_i = -1 | Z_{-i} = x] = \sigma(-2\partial_i p(x)).$
- $D$ is $(e^{-2\lambda}/2)$-unbiased.
Proof. For any $x \in \{1, -1\}^{[n] \setminus \{i\}}$,
\[
\frac{\Pr[Z_i = 1 | Z_{-i} = x]}{\Pr[Z_i = -1 | Z_{-i} = x]} = \exp(2\partial_ip(x)).
\]
Thus,
\[
\Pr[Z_i = -1 | Z_{-i} = x] = \sigma(-2\partial_ip(x)).
\]
Next, for each $i$, and any partial assignment $x$ to $Z_{-i}$,
\[
\min(\Pr[Z_i = -1 | Z_{-i} = x], \Pr[Z_i = 1 | Z_{-i} = x]) = \min(\sigma(-2\partial_ip(x)), 1 - \sigma(-2\partial_ip(x))) \geq (1/2)e^{-2||\partial_ip||_1} \geq (1/2)e^{-2\lambda}.
\]
\[
\square
\]
We also need the following elementary fact about median:

**Claim 7.7.** Let $X$ be a real-valued random variable such that for some $\alpha, \gamma \in \mathbb{R}$, $\Pr[|X - \alpha| > \gamma] < 1/4$. Then, for $K$ independent copies of $X, X_1, X_2, \ldots, X_K$,
\[
\Pr[|\text{median}(X_1, \ldots, X_K) - \alpha| > \gamma] \leq 2\exp(-\Omega(K)).
\]

**Proof of Theorem 7.2.** We will show how to recover neighbors of the vertex $n$ (for ease of notation). By repeating the argument for all $i \in [n]$, we will get the graph $G$.

The starting point for our algorithm is Lemma 7.6 that allow us to use Sparsitron algorithm via feature expansion and the properties of $\delta$-unbiased distributions developed in Section 6.

Concretely, let $p' = -2\partial_n p$ and $p' = (\hat{p}'(I) : I \subseteq [n - 1], |I| \leq t - 1)$. Similarly, for $x \in \{1, -1\}^{n-1}$, let $v(x) = (\prod_{i \in I} x_i : I \subseteq [n - 1], |I| \leq t - 1)$. Let $Z \sim \mathcal{D}$ and $X$ be the distribution of $v(Z_{-n})$ and let $Y = (1 - Z_n)/2$. Then, by Lemma 7.6, we have
\[
\mathbb{E}[Y|X] = \sigma(p' \cdot X).
\]

Let $\delta = e^{-2\lambda}/2$, and let $\epsilon \in (0, 1), K \geq 1$ be parameters to be chosen later. Our algorithm is shown in Figure 3. The intuition is as follows: We first apply Sparsitron to recover a polynomial $q$ that approximates $\partial_n p$ in the sense that
\[
\mathbb{E}_Z[|\sigma(-2\partial_n p(Z)) - \sigma(-2q(Z))|^2] < \epsilon.
\]

However, the above does not guarantee that the coefficients of $q$ are close to those of $\partial_n p$. To overcome this, we exploit Lemma 6.2 that guarantees that for any maximal monomial $I$ in $\partial_n p$, $\partial_I q(Z)$ is close to $\partial_i p(I)$ with high probability for $Z \sim \mathcal{D}$; concretely, in steps (4), (5), (6), we draw fresh samples from $\mathcal{D}$ and use the median evaluation of $\partial_I q(\ )$ as our estimate for $\partial_i p(I)$.

We next argue that for a suitable choice of $\epsilon, K$, with probability at least $1 - \rho/n$, the graph $H$ contains all edges of $G$ adjacent to vertex $n$.

Observe that by our definitions of $p', q, X$
\[
\mathbb{E}_Z[|\sigma(-2\partial_n p(Z)) - \sigma(-2q(Z))|^2] = \mathbb{E}_Z[(\sigma(p' \cdot X) - \sigma(q \cdot X))^2] \leq \epsilon.
\]
Algorithm 3 MRF Recovery

1: Initialize $H = \emptyset$ to be the empty graph.
2: Apply the Sparsitron algorithm as in Theorem 3.1 to compute a vector $q$ such that with probability at least $1 - \rho/2n^2$,

$$\mathbb{E}[(\sigma(p' \cdot X) - \sigma(q \cdot X))^2] \leq \varepsilon.$$ 

3: Define a polynomial $q : \mathbb{R}^{n-1} \rightarrow \mathbb{R}$ by setting $\widehat{q}(I) = (-1/2)q_I$ for all $I \subseteq [n-1]$.
4: Let $Z^1, \ldots, Z^K$ be additional independent samples from $\mathcal{D}$.
5: for each $I \subseteq [n-1], |I| \leq t - 1$ do
6: If $|\text{median} (\partial_I q(Z^1), \ldots, \partial_I q(Z^K))| > \eta/2$, then add the complete graph on $\{n\} \cup I$ to $H$.

(Here, we abuse notation and write $q(Z) = q(Z_1, \ldots, Z_{n-1})$ as the latter does not depend on $Z_n$.)

Further, as $Z$ is $\delta$-unbiased by Lemma 7.6, by Lemma 6.2 for any maximal monomial $I \subseteq [n-1]$ of $\partial_n p$, we have

$$\Pr \left[ |\widehat{\text{median}}(\partial_I q(Z^1), \partial_I q(Z^2), \ldots, \partial_I q(Z^K)) - \widehat{\partial_n p}(I)| > \eta/4 \right] < 16\frac{e^{2|q||1+6\varepsilon}}{\eta^2 \delta^4 |I|}.$$ 

Let $\varepsilon = e^{-2\lambda - 6\eta^2 \delta^4} / 64$ so that

$$\Pr \left[ |\widehat{\text{median}}(\partial_I q(Z^1), \partial_I q(Z^2), \ldots, \partial_I q(Z^K)) - \widehat{\partial_n p}(I)| > \eta/4 \right] < 1/4.$$ 

Therefore, by Claim 7.7,

$$\Pr \left[ |\text{median}(\partial_I q(Z^1), \partial_I q(Z^2), \ldots, \partial_I q(Z^K)) - \text{median}(\partial_I q(Z^1), \ldots, \partial_I q(Z^K)) - \widehat{\partial_n p}(I)| > \eta/4 \right] < 2 \exp(-\Omega(K)).$$ 

Taking $K = C \log(n^t / \rho)$ for a sufficiently big constant $C$, we get that with probability at least $1 - \rho/n$, for all maximal monomials $I$ of $\partial_n p$, $|\text{median}(\partial_I q(Z^1), \partial_I q(Z^2), \ldots, \partial_I q(Z^K)) - \widehat{\partial_n p}(I)| < \eta/4$.

Now, whenever the above happens, as the coefficients of maximal monomials of $p$ are at least $\eta$ in magnitude (by $\eta$-identifiability), our algorithm will add the complete graph on the variables of all maximal monomials of $p$ involving vertex $n$ to $H$.

Thus, the algorithm recognizes the neighbors of vertex $n$ exactly with probability at least $1 - \rho/n$. Repeating the argument for each vertex $i \in [n]$ and taking a union bound over all vertices gives us the recovery guarantee of the theorem. It remains to bound the sample-complexity.

Note that $\|p'\|_1 = 2\|\partial_n p\|_1 \leq 2\lambda$. Therefore, by Theorem 3.1, the number of samples needed for the call to Sparsitron in Step (2) of Algorithm 3 is

$$O(\lambda^2 \cdot \ln(n^t / \rho \varepsilon) / \varepsilon^2) = e^{O(t)} \cdot e^{O(\lambda \cdot t)} \cdot \ln(n / \rho \eta) \cdot (1/\eta^4).$$

As $K = C t \ln(n / \rho)$, the above bound dominates the number of samples proving the theorem. \qed

7.2 Learning log-polynomial densities and parameters of MRFs

We first observe that Theorem 7.5 implies Theorem 7.3
Similarly, for feature expansion GLMs via

The theorem now follows. \( \square \)

We next prove Theorem 7.5. The proof is similar to that of Theorem 5.2 and Theorem 7.2.

**Proof of Theorem 7.5.** For each \( i \), we will show how to recover a polynomial \( q_i \) such that \( \| \partial_i p - q_i \|_1 < \epsilon \cdot \binom{n}{t-1} \). We can then combine these polynomials to obtain a polynomial \( q \). One way to do so is as follows: For each \( I \subseteq [n] \), let \( i = \arg \min (I) \), and define \( \tilde{q}(I) = \tilde{q}_i (I \setminus \{ i \}) \). Then,

\[
\| p - q \|_1 = \sum_{I} | \tilde{p}(I) - \tilde{q}(I) | = \sum_{i=1}^{n} \sum_{I: \arg \min (I) = i} | \tilde{p}(I) - \tilde{q}(I) | \\
\leq \sum_{i=1}^{n} \| \partial_i p - q_i \|_1 \leq \epsilon \cdot n \cdot \left( \frac{n}{t-1} \right).
\]

Here we show how to find a polynomial \( q_n \) such that with probability at least \( 1 - \rho/n \),

\[
\| \partial_n p - q_n \|_1 < \epsilon \cdot \left( \frac{n}{t-1} \right). \tag{7.1}
\]

The other cases can be handled similarly and the theorem then follows from the above argument.

As in Theorem 7.2, we exploit Lemma 7.6 to employ our Sparsitron algorithm for learning GLMs via feature expansion. Concretely, let \( p' = -2 \partial_n p \) and \( p' = (\tilde{p}(I) : I \subseteq [n-1], |I| \leq t-1) \).

Similarly, for \( x \in \{1, -1\}^{n-1} \), let \( \psi(x) = (\prod_{i=1}^{n} x_i : I \subseteq [n-1], |I| \leq t-1) \). Let \( Z \sim D \) and \( X \) be the distribution of \( \psi(x) \) and let \( Y = (1 - Z_n)/2 \). Then, from the above arguments, we have

\[
\mathbb{E}[Y|X] = \sigma(p' \cdot X).
\]

Note that \( \| p' \|_1 = 2 \| \partial_n p \|_1 \leq 2\lambda \). Let \( \gamma \in (0, 1) \) be a parameter to be chosen later. We now apply the Sparsitron algorithm as in Theorem 3.1 to compute a vector \( q' \in \mathbb{R}^n \) such that with probability at least \( 1 - \rho/n \),

\[
\mathbb{E}[\sigma(p' \cdot X) - \sigma(q' \cdot X)]^2 \leq \gamma.
\]

We define polynomial \( q_n \) by setting \( \tilde{q}_n(I) = (-1/2) \cdot q'_I \) for all \( I \subseteq [n-1] \). Then, the above implies that

\[
\mathbb{E}_Z \left[ \sigma(-2 \partial_n p(Z)) - \sigma(-2q_n(Z)) \right]^2 \leq \gamma. \tag{7.2}
\]

Now, an argument similar to that of Lemma 7.6 shows that \( Z \) is \( \delta \)-unbiased for \( \delta = e^{-2\lambda}/2 \). Therefore, by Equation 7.2, and Lemma 6.4, for \( \gamma < c \exp(-4\lambda) \cdot \delta^{-t} \) for a sufficiently small constant \( c \), we get

\[
\| \partial_n p - q_n \|_1 \leq O(1)(2t)^t \cdot e^{2\lambda} \cdot \sqrt{\gamma/\delta^t} \cdot \left( \frac{n}{t-1} \right) \leq \epsilon \cdot \left( \frac{n}{t-1} \right),
\]

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where \( \gamma = \varepsilon^2 \exp(-C\lambda t)/C(2t)^2t \) for a sufficiently large constant \( C > 0 \). Note that by Theorem 3.1, the number of samples needed to satisfy Equation 7.2 is

\[
O((\lambda/\gamma)^2 \cdot (\log(n/\rho\gamma))) = \frac{(2t)^O(t) \cdot \epsilon^{O(\lambda t)}}{\varepsilon^4} \cdot (\log(n/\rho\varepsilon)).
\]

This proves Equation 7.1 and hence the main part of the theorem. The moreover part of the statement follows from an argument nearly identical to that of Theorem 7.2 and is omitted here. □

8 Extension of the Ising Model to general alphabet

Here we extend our results for the Ising model from Section 5 to work over general alphabet.

**Definition 8.1.** Let \( W = (W_{ij} \in \mathbb{R}^{k \times k} : i \neq j \in [n]) \) be a collection of matrices and \( \theta \in \mathbb{R}^{[n] \times [k]} \).

Then, the non-binary Ising model \( D = D(W, \theta) \) is the distribution on \([k]^n\) where

\[
\Pr_{X \sim D} [X = x] \propto \exp \left( \sum_{i \neq j} W_{ij}(x_i, x_j) + \sum_{i=1}^{n} \theta_i(x_i) \right).
\]

The dependency graph \( G \) of \( D \) is the graph formed by pairs \( \{i, j\} \) such that \( W_{ij} \neq 0 \). The width of \( D \) is \( \lambda(D) = \max_{i,a} \left( \sum_{j \neq i} \max_{b \in [k]} |W_{ij}(a, b)| + \theta_i(a) \right) \).

It is easy to see that for \( k = 2 \), the above description corresponds exactly to our discussion of Ising models in Section 5. Our goal here is to learn the structure and parameters of a general alphabet model as above given samples from it. We will show that arguments from Section 5 can be extended to case of general alphabet as well leading to algorithms to learn the structure and parameters of distributions \( D(W, \theta) \) with nearly optimal sample complexity. First, we address some necessary non-degeneracy conditions.

Note that as described above, different parameter settings can lead to the same probability density function. To get around this, throughout this section we assume without loss of generality that for every \( \{i, j\} \), the rows and columns of the matrices \( W_{ij} \in \mathbb{R}^{k \times k} \) are centered, i.e., sum to zero. We can do so because of the following elementary claim:

**Fact 8.2.** Given \( D(W, \theta) \) as above, we may assume without loss of generality that for every \( i, j \), the rows and columns of the matrix \( W_{ij} \) are centered, i.e., have mean zero.

**Proof.** For simplicity fix a particular \( x = x_1, \ldots, x_n \). The above pdf simplifies to

\[
\Pr_{X \sim D} [X = x] \propto \exp \left( \sum_{i \neq j} W_{ij}(x_i, x_j) + \sum_{i} \theta_i(x_i) \right).
\]

Now define \( W'_{ij}(x_i, x_j) \) equal to \( W_{ij}(x_i, x_j) - (1/k) \sum_a W_{ij}(a, x_j) \) and define \( \theta'_j(x_j) \) equal to \( \theta_j(x_j) + \sum_{i < j} (1/k) \sum_a W_{ij}(a, x_j) \). Then by inspection the column sums of \( W'_{ij} \) equal zero and \( \sum_{ij} W_{ij}(x_i, x_j) + \sum_j \theta_j(x_j) = \sum_{ij} W'_{ij}(x_i, x_j) + \sum_j \theta'(x_j) \). Repeating the argument for rows gives us the claim. □

Next, we make the following identifiability assumption analogous to our assumptions for the Ising model.
**Definition 8.3.** \( \mathcal{D}(W, \theta) \) is \( \eta \)-identifiable if for every edge in the dependency graph of \( \mathcal{D} \), \( \|W_{ij}\|_\infty = \max_{a,b}(|W_{ij}(a,b)|) \geq \eta \) (assuming the matrices \( W_{ij} \) are centered).

We prove the following extension of **Theorem 5.2** to non-binary Ising models:

**Theorem 8.4.** Let \( \mathcal{D}(W, \theta) \) be a \( n \)-variable non-binary Ising model with alphabet size \( k \), width \( \lambda \) that is \( \eta \)-identifiable. There exists an algorithm that given \( \lambda, \eta, \rho \in (0,1) \), and \( N = O(\exp(O(\lambda))/\eta^4) \cdot (\log(n/\rho \eta)) \cdot k^3 \) independent samples \( Z^1, \ldots, Z^N \leftarrow \mathcal{D}(A, \theta) \) recovers the underlying dependency graph of \( \mathcal{D}(W, \theta) \) with probability at least \( 1 - \rho \). The run-time of the algorithm is \( O(k^2n^2N) \). Moreover, the algorithm can be run in an online manner.

At a high-level, the proof proceeds as follows. Let \( Z \sim \mathcal{D}(W, \theta) \). For each \( i \in [n] \), a partial assignment \( x \in [k]^{[n]\setminus \{i\}} \), and for each \( \alpha \neq \beta \in [k] \),

\[
\Pr[Z_i = \beta \mid Z_i \in \{\alpha, \beta\} \land Z_{-i} = x] = \sigma \left( \sum_{j \neq i} W_{ij}(\alpha, x_j) - W_{ij}(\beta, x_j) \right) + \theta_i(\alpha) - \theta_i(\beta).
\]

This is analogous to **Equation 5.1** and allows us to use an argument similar to that used in **Theorem 5.2**. To this end, we first show an analogue of unbiasedness for non-binary Ising models. We say a distribution \( \mathcal{D} \) on \([k]^n\) is \( \delta \)-unbiased if for any \( i \in [n] \), and any partial assignment \( x \) to \((X_j : j \neq i)\), \( \min_a(\Pr[X_i = a | X_{-i} = x]) \geq \delta \). Just as for Ising models, non-binary Ising models \( \mathcal{D}(W, \theta) \) also turn out to be \( \delta \)-unbiased for \( \delta = (e^{-2\lambda}/k) \) depending on the width of the distribution.

**Lemma 8.5.** A non-binary Ising distribution \( \mathcal{D}(W, \theta) \) on \([k]^n\) is \( \delta \)-unbiased for \( \delta = (e^{-2\lambda}/k) \) where \( \lambda \) is the width of \( \mathcal{D}(W, \theta) \).

**Proof.** Let \( X \sim \mathcal{D}(W, \theta) \). Note that for any \( \alpha, \beta \in [k], i \in [n] \), and a partial assignment \( x \in [k]^{[n]\setminus \{i\}} \),

\[
\frac{\Pr[X_i = \alpha | X_{-i} = x]}{\Pr[X_i = \beta | X_{-i} = x]} = \frac{\exp \left( \sum_{j \neq i} W_{ij}(\alpha, x_j) + \theta_i(\alpha) \right)}{\exp \left( \sum_{j \neq i} W_{ij}(\beta, x_j) + \theta_i(\beta) \right)} \leq \exp(-2\lambda).
\]

Therefore, \( \min_a \Pr[X_i = \alpha | X_{-i} = x] \geq \exp(-2\lambda)/k \). \( \square \)

We are now ready to prove **Theorem 8.4**.

**Proof of Theorem 8.4.** Just as in the proof of **Theorem 5.2**, the starting point is the following connection to learning GLMs. Let \( Z \sim \mathcal{D}(W, \theta) \). Then, for any \( \alpha, \beta \in [k], i \in [n] \), and a partial assignment \( x \in [k]^{[n]\setminus \{i\}} \),

\[
\frac{\Pr[Z_i = \alpha \mid Z_{-i} = x]}{\Pr[Z_i = \beta \mid Z_{-i} = x]} = \frac{\exp \left( \sum_{j \neq i} W_{ij}(\alpha, x_j) + \theta_i(\alpha) \right)}{\exp \left( \sum_{j \neq i} W_{ij}(\beta, x_j) + \theta_i(\beta) \right)}.
\]

Thus,

\[
\frac{\Pr[Z_i = \beta \mid Z_{-i} = x]}{\Pr[Z_i = \alpha \mid Z_{-i} = x] + \Pr[Z_i = \beta \mid Z_{-i} = x]} = \sigma \left( \sum_{j \neq i} W_{ij}(\alpha, x_j) - W_{ij}(\beta, x_j) \right) + \theta_i(\alpha) - \theta_i(\beta).
\]
In the following we show how to recover the edges of the dependency graph involving vertex $n$; we can repeat the argument for each of the other vertices to infer the graph.

Fix $\alpha, \beta \in [k]$. We will show how to approximate $W_{ni}(\alpha, b) - W_{ni}(\beta, b)$ for all $i \in [n - 1]$ and $b \in [k]$. Once we have such an approximation, we can recover the values $W_{ni}(\alpha, b)$ as

$$W_{ni}(\alpha, b) = \sum_{\beta} (W_{ni}(\alpha, b) - W_{ni}(\beta, b)).$$

To this end, let $Z' \equiv Z \mid Z_n \in \{\alpha, \beta\}$. Note that we can obtain independent draws from $Z'$ by restricting to examples where $Z_n$ takes only values in $\{\alpha, \beta\}$. This will not cost us too much in sample complexity as by Lemma 8.5 $\Pr[Z_n \in \{\alpha, \beta\}] \geq 2 \exp(-2\lambda)/k$.

Let $e : [k] \mapsto \{0, 1\}^k$ be a function that maps an alphabet symbol $a$ to a string of length $k$ that is all zeros except for a 1 in position $a$. Then for a string $x \in [k]^{n-1}$, let $\bar{x}$ be a string of length $k(n - 1)$ equal to $(e(x_1), \ldots, e(x_{n-1}))$. Let $w$ be a string broken into $n$ blocks each of size $k$ where the $b$'th value of block $i$ denoted $w_i(b)$ is $W_{ni}(\alpha, b) - W_{ni}(\beta, b)$. Then, from the above arguments, we have

$$\Pr[Z'_n = \beta \mid Z_{-n} = x] = \sigma \left( \sum_{i<n} W_{ni}(\alpha, x_i) - W_{ni}(\beta, x_i) + \theta_i(\alpha) - \theta_i(\beta) \right) = \sigma(w \cdot \bar{\alpha} + \theta'),$$

where $\theta' = \theta_i(\alpha) - \theta_i(\beta)$.

Note that as the matrices $W_{ij}$ are centered, we have $\sum_{a \in [k]} w_j(a) = 0$ for all $j < n$. Let $X$ be the distribution of $\bar{z}_{-n}$ for $z$ drawn from $Z'$ and let $Y = 1$ if $Z_n = \beta$ and 0 if $Z_n = \alpha$. Then,

$$\mathbb{E}[Y|X] = \sigma(w \cdot X + \theta').$$

Now, for some parameter $\gamma, \rho' \in (0, 1)$ to be chosen later, assume we ran our Sparsitron algorithm as in Theorem 3.1 to obtain a vector $u \in \mathbb{R}^{(n-1)k}$ and $\theta'' \in \mathbb{R}$ such that with probability at least $1 - \rho'$,

$$\mathbb{E} \left[ (\sigma(w \cdot X + \theta') - \sigma(u \cdot X + \theta''))^2 \right] \leq \gamma.$$

Here, we also assume that for each $i \in [n - 1]$, $\sum_{a \in [k]} u_i(a) = 0$; for if not, we can set $u'_i(a) = u_i(a) - (1/k) \sum_b u_i(b)$ and $\theta''' = \theta'' + (1/k) \sum_i \sum_b u_i(b)$. Next, analogous to Lemma 4.3, we show that the above condition implies that $\|w - u\|_\infty \ll 1$.

**Claim 8.6.** Let $Y$ be a $\delta$-unbiased distribution on $[k]^{n-1}$ and let $w = (w_i(a) : i < n, a \in [k]), u = (u_i(a) : i < n, a \in [k])$ be centered (i.e., for every $i$, $\sum_a w_i(a) = \sum_a u_i(a) = 0$) and for some $\theta, \theta' \in \mathbb{R}$, $\gamma < e^{-2\|w\|_1 - 2|\theta'|-\delta}$,

$$\mathbb{E} \left[ (\sigma(w \cdot \bar{Y} + \theta') - \sigma(u \cdot \bar{Y} + \theta''))^2 \right] \leq \gamma.$$

Then, $\|w - u\|_\infty \leq O(1) \cdot e^{\|w\|_1 + |\theta'|} \sqrt{\gamma/\delta}$.

Indeed, from the above claim and by Lemma 8.5, we get that for every $i < n, a \in [k]$,

$$|W_{ni}(\alpha, a) - W_{ni}(\beta, a) - u_i(a)| \leq O(1) e^{O(\lambda)} \sqrt{k \gamma}.$$
The above argument was for a fixed $\alpha, \beta \in [k]$. Repeating the argument for all $\alpha, \beta \in [k]$, we find estimates $U_i^{\alpha, \beta}(a)$ for $i < n$ and $a \in [k]$ such that

$$|W_{ni}(\alpha, a) - W_{ni}(\beta, a) - U_i^{\alpha, \beta}(a)| < O(1)e^{O(\lambda)}\sqrt{k\gamma}.$$

Now, set $U_{ni}(\alpha, a) = (1/k)\sum_{\beta}U_i^{\alpha, \beta}(a)$. Then, by the above inequality and the fact that $\sum_{\beta}W_{ni}(\beta, a) = 0$, we get that for every $i < n$ and $\alpha, a \in [k]$,

$$|W_{ni}(\alpha, a) - U_{ni}(\alpha, a)| < O(1)e^{O(\lambda)}\sqrt{k\gamma} < \eta/2$$

if we set $\gamma = \eta^2e^{-C\lambda}/Ck$ for a sufficiently big constant $C$. Now, given the above, we can identify the neighbors of vertex $n$ as follows: For each $i < n$ if $\max_{\alpha, a}|U_{ni}(\alpha, a)| > \eta/2$, $i$ is a neighbor of $n$. This works as $D(W, \theta)$ is $\eta$-identifiable.

It remains to bound the number of samples needed. As per Theorem 3.1, the number of samples of $Z'$ needed is

$$O(\lambda^2 \log(n/\rho^2)/\gamma^2) = O(1)k^2 \cdot e^{O(\lambda)}\log(n/\rho^2) = N'\text{.}$$

Finally, as $Z$ is $\delta$-unbiased for $\delta = e^{-2\lambda}/k$, to get the above number of samples of $Z'$ with probability $1 - \rho/n^2k^2$, it suffices to take

$$O(N'/\delta) = O(1)k^3 e^{O(\lambda)}\log(n \cdot n^2k^2/\rho\eta)$$

samples. Finally, we can take a union bound over vertices $i \in [n]$ and $\alpha, \beta \in [k]$. This completes the proof of the theorem.

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**Proof of Claim 8.6.** For $y \in [k]^{n-1}$ define $p(y) = \langle w, \bar{y} \rangle + \theta'$ and $q(y) = \langle u, \bar{y} \rangle + \theta''$. Let $y^{i:a}$ be the string obtained from $y$ by setting the $i$th coordinate of $y$ to $a$. Fix an index $i < n$. Then, as in the binary case, by Claim 4.2 and the fact that $Y$ is drawn from a $\delta$-unbiased distribution, we have

$$\mathbb{E}_{Y_i|Y_{-i}}\left[ (\sigma(p(Y)) - \sigma(q(Y)))^2 \right] \geq \delta \cdot e^{-\|w\|_1 - |\theta'| - \beta} \cdot \min_{a} \left( 1, \max_{a} |p(Y^{i:a}) - q(Y^{i:a})|^2 \right). \quad (8.1)$$

Since we assumed that the sum of the values in each block of $w$ are zero, we have that

$$\sum_{b} \left( p(y^{i:a}) - p(y^{i:b}) \right) = \sum_{b} (w_i(a) - w_i(b)) = k w_i(a). \quad (8.2)$$

and similarly

$$\sum_{b} \left( q(y^{i:a}) - q(y^{i:b}) \right) = \sum_{b} (u_i(a) - u_i(b)) = k u_i(a). \quad (8.3)$$

Subtracting equation (8.3) from equation (8.2) and applying the triangle inequality we obtain

$$k \cdot |(w_i(a) - u_i(a))| \leq \sum_{b} |p(y^{i:a}) - q(y^{i:a})| + |p(y^{i:b}) - q(y^{i:b})| \quad (8.4)$$

$$\leq 2k \cdot \max_{b \in [k]} |p(y^{i:b}) - q(y^{i:b})|. \quad (8.5)$$

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Thus we have $\max_{b \in \mathbb{K}} |p(y^{i,b}) - q(y^{i,b})| \geq 1/2 \cdot \max_{a \in \mathbb{K}} |w_i(a) - u_i(a)|$. We also have, that $\mathbb{E}_Y \left( (\sigma(p(Y)) - \sigma(q(Y)))^2 \right) \leq \gamma$. We therefore must have

$$\max_{a \in \mathbb{K}} |w_i(a) - u_i(a)| \leq O(1) \cdot \sqrt{\gamma/\delta} \cdot e^{\lVert w \rVert_1 + |\theta|}.$$

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A Hardness of Learning t-wise Markov random fields

Bresler, Gamarnik, and Shah [BGS14] showed how to embed parity learning as a Markov random field and showed that a restricted class of algorithms must take time \( n^{\Omega(d)} \) to learn degree MRFs defined on degree \( d \) graphs. Here we use the same construction, and for completeness we give a proof. The conclusion is that, assuming the hardness of learning sparse parities with noise, for degree \( d \) graphs, learning \( t \)-wise MRFs \( (t < d) \) will require time \( n^{\Omega(t)} \). Our positive results match this conditional lower bound.

- Let \( \chi_S : \{-1,1\}^n \rightarrow \{-1,1\}^n \) be an unknown parity function on a subset \( S \), \( |S| \leq k \), of \( n \) inputs bits (i.e., \( f(x) = \prod_{i \in S} x_i \)). Let \( \mathcal{C}_k \) be the concept class of all parity functions on subsets \( S \) of size at most \( k \). Let \( \mathcal{D} \) be the uniform distribution on \( \{-1,1\}^n \).

- Fix an unknown \( c \in \mathcal{C}_k \) and consider the following random experiment: \( x \) is drawn according to \( \mathcal{D} \) and with probability \( 1/2 + \eta \) (for some constant \( \eta \)), the tuple \( (x, c(x)) \) is output. With probability \( 1/2 - \eta \), the tuple \( (x, c(x') ) \) is output where \( c'(x) \) is the complement of \( c(x) \).

- The \( k \)-LSPN problem is as follows: Given i.i.d. such tuples as described above, find \( h \) such that \( \Pr_x[h(x) \neq c(x)] \leq \varepsilon \).

Now we reduce \( k \)-LSPN to learning the graph structure of a \((k + 1)\)-wise Markov random field. Let \( S \) denote the \( k \) indices of the unknown parity function. Let \( G \) be a graph on \( n + 1 \) vertices.
\(X_1, \ldots, X_n, Y\) equal to a clique on the set of vertices corresponding to set \(S\) and vertex \(Y\). Consider the probability distribution

\[
\Pr[Z = (x_1, \ldots, x_n, y)] \propto \exp(\gamma \chi_S(x)y)
\]

for some constant \(\gamma\). A case analysis shows that \(p_1 = \Pr[Y = \chi_S(X)] \propto e^\gamma\) and \(p_2 = \Pr[Y \neq \chi_S(X)] \propto e^{-\gamma}\). Hence the ratio \(p_1/p_2\) is approximately \(1 + 2\gamma\). Since \(p_1 + p_2 = 1\), by choosing \(\gamma\) to be a sufficiently small, we will have \(p_1 \geq 1/2 + \eta\) and \(p_2 \leq 1/2 - \eta\) for some small (but constant) \(\eta\).

Further, it is easy to see that the parity of any subset of \(X_i\)'s is unbiased. By the Vazirani XOR lemma [Gol95], this implies that the \(X_i\)s are uniformly distributed. Therefore, the distribution encoded by this \((k+1)\)-wise MRF is precisely the distribution described in the \(k\)-LSPN problem. If we could discover the underlying clique in the Markov random field, we would be able to learn the underlying sparse parity. Hence, learning \(k+1\)-MRFs is harder than \(k\)-LSPN.

The current best algorithm for learning \(k\)-LSPN is due to Valiant [Val15] and runs in time \(n^{\Omega(0.8k)}\). Any algorithm running in time \(n^{o(k)}\) would be a major breakthrough in theoretical computer science.