ON A CLASS OF PARAMETERIZED SOLUTIONS TO INTERVAL PARAMETRIC LINEAR SYSTEMS

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Abstract
Presented is a new method yielding parameterized solution to an interval parametric linear system. Some properties of this method are discussed. The solution enclosure it provides is compared to the enclosures by other methods. It is shown that an application, proposed by other authors, cannot be done in the general case.

Key words: Interval linear systems, dependent data, solution enclosure.

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1. Introduction. Parameterized solutions to interval parametric linear systems are linear functions of interval parameters that estimate the united solution set. Parameterized solutions present an alternative form of the traditional numerical interval vectors enclosing the solution set. They allow subsequent problems involving both the primary solution and the initial interval parameters to account better for the parameter dependencies, cf. [1, 2]. Methods for deriving parameterized solutions are developed in relation to many classical interval methods yielding interval vectors enclosing the united solution set, see [1–6] and the references therein mentioning most of the works on parameterized solutions. Recently, Kolev presented in [4] parameterized analogue of the generalized method of Neumaier and Pownuk [7, 8] and proposed an application of it. A methodological framework alternative to [4], [6] was proposed in [9] and its advantages for handling a class of interval parametric linear systems were demonstrated. A new parameterized solution, based on the methodology of [9] and different from the parameterized form (2) was proposed in [2] along with a new application direction.

Present work is motivated by [4]. While the parameterized solution in [4] is based on affine arithmetic, in Section 3 of this work we propose a parameterized solution in form (2) which is based on the numerical method in [9] and not using affine arithmetic. We show that the two forms of parameterized solutions related to the numerical method in [9] can be applied also to the numerical methods in [4, 6] without using affine arithmetic. The methodology in [9] has an expanded scope of applications and provides a sharper solution enclosure than most of the methods for a wide class of parametric systems involving rank one uncertainty structure. In this work we demonstrate that for the latter class of parametric systems, the proposed here parameterized solution provides sharper solution enclosure than a variety of parameterized solutions based on affine arithmetic and compared in [5]. In Section 4 we discuss in details and demonstrate by a numerical example that the application proposed in [4] cannot be done to arbitrary parametric linear systems with rank one uncertainty structure.

2. Preliminaries. Denote by $\mathbb{R}^{m \times n}$ the set of real $m \times n$ matrices. Vectors are considered as one-column matrices. A real compact interval is $\mathbf{a} = [a^-, a^+] := \{a \in \mathbb{R} | a^- \leq a \leq a^+\}$ and $\mathbb{IR}^{m \times n}$ denotes the set of interval $m \times n$ matrices. For $\mathbf{a} = [a^-, a^+]$, define its mid-point $\bar{a} := (a^- + a^+)/2$, the radius $\hat{a} := (a^+ - a^-)/2$ and the magnitude $|\mathbf{a}| := \max\{|a^-|, |a^+|\}$. These functions are applied to interval vectors and matrices.
componentwise. The inequalities are understood componentwise. The spectral radius of a matrix $A \in \mathbb{R}^{n \times n}$ is denoted by $\rho(A)$. The identity matrix of appropriate dimension is denoted by $I$. For $A_k \in \mathbb{R}^{n \times m}$, $1 \leq k \leq t$, $(A_1, \ldots, A_t) \in \mathbb{R}^{n \times tm}$ denotes the matrix obtained by stacking the columns of the matrices $A_k$. Denote the $i$-th column of $A \in \mathbb{R}^{n \times m}$ by $A_i$, and its $i$-th row by $A_i^\top$.

We consider systems of linear algebraic equations with linear uncertainty structure

$$A(p)x = a(p), \quad p \in \mathbb{R}^K,$$

(1)

$$A(p) := A_0 + \sum_{k=1}^K p_k A_k, \quad a(p) := a_0 + \sum_{k=1}^K p_k a_k,$$

where $A_k \in \mathbb{R}^{n \times n}$, $a_k \in \mathbb{R}^n$, $k = 0, \ldots, K$ and the parameters $p = (p_1, \ldots, p_K)^\top$ are considered to be uncertain and varying within given non-degenerate intervals $p = (p_1, \ldots, p_K)^\top$. Nonlinear dependencies between interval valued parameters in linear algebraic systems are usually linearized to the form (1) and methods for the latter are applied to bound the corresponding solution set. The so-called united parametric solution set of the system (1) is defined by

$$\Sigma_{uni}^p = \Sigma_{uni}(A(p), a(p), p) := \{ x \in \mathbb{R}^n \mid (\exists p \in \mathbb{R})(A(p)x = a(p)) \}. $$

Usually, the interval methods (providing interval enclosure of $\Sigma$) are considered to be uncertain and varying within given non-degenerate intervals $p$. A new type – parameterized solution – is proposed in [4]. This solution is in form of an affine-linear function of interval-valued parameters

$$x(p, r) = \hat{x} + V(\hat{p} - p) + r, \quad p \in \mathbb{R}, r \in \mathbb{R} = [-\hat{r}, \hat{r}],$$

(2)

where $\hat{x}, \hat{r} \in \mathbb{R}^n$, $V \in \mathbb{R}^{n \times K}$. Some representations move $x$ into the interval vector $r$ and consider the parameters $p, r$ varying independently within the interval $[-1, 1]$. The parameterized solution has the property $\Sigma_{uni}^p \subseteq x(p, r)$, where $x(p, r)$ is the interval hull of $x(p, r)$ over $p \in \mathbb{R}, r \in \mathbb{R}$. For a nonempty and bounded set $\Sigma \subset \mathbb{R}^n$, its interval hull is $\Sigma := \bigcap \{ x \in \mathbb{R}^n \mid \Sigma \subseteq x \}$.

In what follows we consider another form of the parametric system (1) and some numerical and parameterized solutions related to this form. Let $K = \{1, \ldots, K\}$ and $\pi', \pi''$ be two subsets of $K$ such that $\pi' \cap \pi'' = \emptyset$, $\pi' \cup \pi'' = K$, Card$(\pi') = K_1$. The permutation $\pi'$ denotes the indices of the parameters that appear in both the matrix and the right-hand side of the system, while $\pi''$ involves the indices of the parameters that appear only in $a(p)$ in (1). Denote $p_\pi = (p_{\pi_1}, \ldots, p_{\pi_K})$ and by $D_{p_\pi}$ a diagonal matrix with diagonal vector $p_\pi$. The system (1) has the following equivalent form

$$(A_0 + LD_{g(p_\pi)}^\gamma R)x = a_0 + LD_{g(p_\pi)}^\gamma t + F_{p_\pi}, \quad p \in \mathbb{R},$$

(3)

where $g(p_\pi) \in \mathbb{R}^\gamma$, $\gamma = \sum_{k=1}^{K_1} \gamma_k$, $g(p_\pi) = (g_1^\top(p_{\pi_1}), \ldots, g_{K_1}^\top(p_{\pi_K}))^\top$, $L = (L_1, \ldots, L_{K_1}) \in \mathbb{R}^{n \times \gamma}$, $R = (R_1^\gamma, \ldots, R_{K_1}^\gamma)^\top \in \mathbb{R}^{\gamma \times n}$ and for $1 \leq k \leq K_1$, $g_k(p_{\pi_k}) = (p_{\pi_1}, \ldots, p_{\pi_{k-1}})^\top \in \mathbb{R}^{\gamma_k}$, $p_{\pi_k} A_{\pi_k} = L_k D_{g_k(p_{\pi_k})} R_k$, and $\sum_{k \in \pi'} p_k a_k = LD_{g(p_\pi)}^\gamma t$. We assume that (3) provides an equivalent optimal rank one representation (cf. [4]) of either $A(p_{\pi'}) - A_0$, or of

\[\text{An interval } \mathbf{a} = [a^-, a^+] \text{ is degenerate if } a^- = a^+.\]
Every interval parametric linear system (1) has an equivalent, optimal, rank one representation (3) and there are various ways to obtain it, cf. [10], [9]. The following theorem presents a method for computing numerical interval enclosure of \( \Sigma_{\text{uni}}^p \).

**Theorem 1** ([9]). Let (3) be the equivalent optimal rank one representation of system (1) and let the matrix \( A(\hat{p}) \) be nonsingular. Denote \( C = A^{-1}(\hat{p}) \) and \( \hat{x} = Ca(\hat{p}) \). If

\[
\varrho \left( \left| (RCL)D_{g(\bar{p}_{\pi} - p_{\pi})} \right| \right) < 1,
\]

(i) \( \Sigma_{\text{uni}}(A(p), a(p), p) \) and the solution set \( \Sigma_{\text{uni}}(3) \) of system (3) are bounded

\[
(I - RCLD_{g(p_{\pi})}) y = R\hat{x} - RCFp_{\pi''} - RCLD_{g(p_{\pi})} t, \quad p \in [\bar{p}, \hat{p}],
\]

(ii) \( y \supseteq \Sigma_{\text{uni}}(5) \) is computable by methods that require (6) (cf. [9])

\[
\varrho \left( \sum_{i=1}^{K} \left| (A(\hat{p}))^{-1} A_i \right| \hat{p}_i \right) < 1,
\]

(iii) every \( x \in \Sigma_{\text{uni}}(A(p), a(p), p) \) satisfies

\[
x \in \hat{x} - (CF)(\bar{p}_{\pi''}, \hat{p}_{\pi''}) + (CL) \left( D_{g([\bar{p}_{\pi}, \hat{p}_{\pi}])} | y - t | \right) .
\]

The condition (6) is weaker and holds true when the condition (4) is satisfied, cf. [9]. The interval vector \( y \) in Theorem 1 (ii) can be obtained by a variety of numerical methods, many of them are discussed in [9].

**Theorem 2** ([2]). Let (3) be the equivalent, optimal rank one, representation of the system (1) and let the matrix \( A(\hat{p}) \) be nonsingular. Denote \( C = A^{-1}(\hat{p}) \) and \( \hat{x} = Ca(\hat{p}) \). If (4) holds true, then

i) there exists an united parameterized solution of the system (1), (3)

\[
x(p) = \hat{x} - (CF)(\bar{p}_{\pi''} - p_{\pi''}) + (CL) \left( D_{g([\bar{p}_{\pi}, \hat{p}_{\pi}])} | y - t | \right), \quad p \in p,
\]

where \( y \supseteq \Sigma_{\text{uni}}(5) \),

ii) with the same \( y \) used in (7) and in (8), interval evaluation \( x(p) \) of \( x(p) \) is equal to the interval vector \( x \) obtained by Theorem 1.

3. Another method for parameterized solution. In [4] Kolev proposes a parameterized solution based: (a) on a generalized method of Neumaier and Pownuk [8] (abbreviated here as iGNP), and (b) on affine arithmetic. It is reported in [4] that the implementation of the proposed there parameterized method is eight times slower than the interval method iGNP from [8]. We suppose that the considerable slow down is due to the affine arithmetic which is used in both the implementation of iGNP and the parameterized solution derivation. It is discussed in [4] that the proposed there interval method (Theorem 1, abbreviated as iGRank1, is applicable to the same expanded class of parametric systems as the method iGNP and provides interval solution enclosure of the same (sometimes better) quality while overcoming some specific features that have to
be considered in the implementation of iGNP. In what follows (Theorem \(3\)) we propose a new parameterized solution, abbreviated as pKRank1, which is based on Theorem \(1\) and does not require affine arithmetic. It will be shown (Corollary \(1\)) that the interval solution enclosures based on pKRank1 and pPRank1 (Theorem \(2\)) are the same in exact arithmetic. Also, the parameterized solutions pKRank1 and pPRank1 are applicable to the interval method iGNP with \(y\) obtained by the latter (Proposition \(1\)).

**Theorem 3.** Let \(A(\tilde{p})\) be the equivalent, optimal rank one, representation of \((\ref{eq:1})\) and let \(A(\tilde{p})\) be nonsingular. Denote \(C = A^{-1}(\tilde{p})\), \(\tilde{x} = C a(\tilde{p})\) and let \((\ref{eq:4})\) hold true.

(i) There exists a parameterized solution enclosure of \(\Sigma_{uni}(\ref{eq:1})\)

\[
x(p, r) = x - (CF) (\hat{p}_{\pi'\pi''} - p_{\pi''}) + (CLD_{\hat{y}-t}) g(\hat{p}_{\pi'} - p_{\pi'}) + r, \quad p \in p, r \in r = [-\hat{r}, \hat{r}],
\]

where \(\hat{y} = R\hat{x}, \hat{y}\) is that of Theorem \(1\) (ii), and \(\hat{r} = |CLD_{\hat{y}-\hat{y}}| g(\hat{p}_{\pi'})\).

(ii) The interval evaluation \(x(p, r)\) of the function in (i) is equal to the interval vector \(x\), obtained by Theorem \(1\) provided that both vectors are based on the same \(y\) of Theorem \(1\) (ii).

**Proof.** Since \(1\) holds true, Theorem \(1\) implies that every \(x \in \Sigma_{uni}(A(p), a(p), p)\) satisfies

\[
x(\hat{p}) = \tilde{x} - (CF) (\hat{p}_{\pi'\pi''} - p_{\pi''}) + (CLD_{\hat{y}-t}) g(\hat{p}_{\pi'} - p_{\pi'}) + r
\]

Consider the right-hand side in \(9\) as an interval function \(x(p, y)\) of \(p \in p, y \in y\) and rearrange it as follows.

\[
x(p, y) = \tilde{x} - (CF) (\hat{p}_{\pi'\pi''} - p_{\pi''}) + (CLD_{\hat{y}-t}) g(\hat{p}_{\pi'} - p_{\pi'}) + (CL) (D_{\hat{y}} g(\hat{p}_{\pi'} - p_{\pi'}))
\]

The interval evaluation \(x(p, y)\) of the last expression for \(x(p, y)\) is

\[
x(p, y) = \tilde{x} - (CF) (\hat{p}_{\pi'\pi''} - p_{\pi''}) + (CLD_{\hat{y}-t}) g(\hat{p}_{\pi'} - p_{\pi'}) + (CLD_{\hat{y}-\hat{y}}) g(\hat{p}_{\pi'} - p_{\pi'}),
\]

the latter implying the representation (i). In order to prove (ii) we need to prove that \(x(p, y)\) is \((\ref{eq:10})\). Since \(|\hat{y} - t - |y - \hat{y}|| + |y - \hat{y}|\).

\[
(CLD_{\hat{y}-\hat{y}}) g(\hat{p}_{\pi'} - p_{\pi'}) - (CLD_{\hat{y}-\hat{y}}) g(\hat{p}_{\pi'} - p_{\pi'}) \leq (CLD_{\hat{y}-t}) g(\hat{p}_{\pi'} - p_{\pi'}) \leq (CLD_{\hat{y}-\hat{y}}) g(\hat{p}_{\pi'} - p_{\pi'}).
\]

Since \(g(\hat{p}_{\pi'} - p_{\pi'})\) and \((CLD_{\hat{y}-\hat{y}}) g(\hat{p}_{\pi'} - p_{\pi'})\) are symmetric interval vectors, \((CLD_{\hat{y}-\hat{y}}) g(\hat{p}_{\pi'} - p_{\pi'}) = - (CLD_{\hat{y}-\hat{y}}) g(\hat{p}_{\pi'} - p_{\pi'})\), which implies the required assertion and (ii). \(\square\)

**Corollary 1.** Let \(A(\tilde{p})\) be the equivalent, optimal rank one, representation of \((\ref{eq:1})\) and let \(A(\tilde{p})\) be nonsingular. Denote \(C = A^{-1}(\tilde{p})\), \(\tilde{x} = C a(\tilde{p})\). If \(4\) holds true, then

\[
x(p, r) = x(p) = x,
\]

where \(x(p, r)\) is that of Theorem \(3\), \(x(p)\) is that of Theorem \(2\) and \(x\) is that of Theorem \(1\) provided that all computations are in exact arithmetic and both parameterized solutions use the same \(y\) of Theorem \(1\) (ii).
Proof. The proof is part of the proof of Theorem 3 since \( x(p) = \overline{10} = x. \)

**Proposition 1.** Let (3) be the equivalent, optimal rank one, representation of (1) and let \( A(p) \) be nonsingular. Denote \( C = A^{-1}(p) \) and let (3) holds true. If \( y \) is obtained by Theorem (3) and the implementation iteration thereafter, the interval vector

\[
x_{R1} = \hat{x} - (CF)g(p_{\pi^\nu} - p_{\pi'}) + (CL) (D_{g(p_{\pi^\nu} - p_{\pi'})}(y - t)),
\]

obtained by Theorem 3, and the interval vector

\[
x_{NP} = Ca_0 + (CF)p_{\pi''} + (CL) (D_{g(p_{\pi''})}t + D_{g(p_{\pi'})}(y - t)),
\]

obtained by the implementation of Theorem 4, are equal.

Proof. In the notation of (3), \( D_0 = \text{Diag}(g(p_{\pi'})) \) and \( D_0 \) and the implementation iteration thereafter, the interval vector

\[
\hat{x} = Ca(p) = Ca_0 + CF\tilde{p}_{\pi''} + CLD_{g(p_{\pi''})}t,
\]

we have the desired equality. □

Proposition 1 implies that the two kinds of parameterized solutions, obtained by Theorem 2 and Theorem 3, are applicable to the generalized method of Neumaier and Pownuk (iGNP) with \( y \) obtained by the latter method. Since (7) reports for better solution enclosures provided by Theorem 1 compared to iGNP for some problems, as well as for a better performance in a computing environment, it is expected that these advantages will be attributable to the above two kinds of parameterized solutions, obtained by Theorem 2 and Theorem 3. One advantage of the parameterized solutions involving the remainder term \( r \in [-\bar{r}, \bar{r}] \) is that they allow obtaining an inner estimate of the hull solution, presented in the next proposition.

**Proposition 2.** Let \( x(q) = \hat{x} + Uq + r, q \in [-\bar{q}, \bar{q}] = q, r \in [-\bar{r}, \bar{r}] = r, \) be the parameterized solution obtained by Theorem 3 where \( U = (-CF, CLD_{\bar{g}}), q = (p_{\pi''}^\top, g^\top(p_{\pi'}))^\top, \) \( \bar{q} = \tilde{p}_{\pi''}^\top, g^\top(p_{\pi'})^\top. \) Define

\[
v^{\text{Low}} := \hat{x} + (Uq)^- + r, \quad v^{\text{UP}} := \hat{x} + (Uq)^+ + r.
\]

With \( x_* = [x_*^-, x_*^+] = \square x(p)_0, (1) \),

\[
x_*^+ \in v^{\text{Low}}, \quad x_*^+ \in v^{\text{UP}}, \quad \text{that is} \quad x_* \subseteq v^{\text{Low}} \cup v^{\text{UP}} = x(q).
\]

Define \( x_{\text{in}}(q) := \hat{x} + Uq + r, \) where the interval evaluation is in Kaucher interval arithmetic (4) and \( r_- \) denotes dual(\( r \)). In classical interval arithmetic

\[
(x_{\text{in}}(q))^- = \hat{x} + U\bar{q} + \bar{r}, \quad (x_{\text{in}}(q))^+ = \hat{x} + U\bar{q} - \bar{r}.
\]

For every \( i, 1 \leq i \leq n, \) such that \( (x_{\text{in}}(q))^-(i) > (x_{\text{in}}(q))^+(i), \) substitute \( (x_{\text{in}}(q))_i = \emptyset. \) Then, it holds \( x_{\text{in}}(q) \subseteq \square x(p)_0 \subseteq x(q). \)

Proof. The proof can be based on the properties of Kaucher interval arithmetic (4), or to be done similarly to that of Theorem 1. □

The methodology in (3) has an expanded scope of applications for systems involving rank one uncertainty structure. Next example demonstrates the advantage of the proposed here parameterized solution (Theorem 3) to a variety of parameterized solutions based on affine arithmetic and compared in (3).
Example 1. Consider the parametric linear system

\[
\begin{pmatrix}
 p_1 + p_6 & -p_6 & 0 & 0 & 0 \\
 -p_6 & p_2 + p_6 + p_7 & -p_7 & 0 & 0 \\
 0 & -p_7 & p_3 + p_7 + p_8 & -p_8 & 0 \\
 0 & 0 & -p_8 & p_4 + p_8 + p_9 & -p_9 \\
 0 & 0 & 0 & -p_9 & p_5 + p_9
\end{pmatrix}
\begin{pmatrix}
 x_1 \\
 x_2 \\
 x_3 \\
 x_4 \\
 x_5
\end{pmatrix}
= \begin{pmatrix}
 10 \\
 0 \\
 10 \\
 0 \\
 0
\end{pmatrix}
\]

after [2], where the parameters vary within given intervals \( p_i \in [1 - \delta, 1 + \delta] \). This example is considered in [2] Example 5 and the outer solution enclosures obtained by six parameterized solutions based on affine arithmetic are compared to the direct parameterized method (abbreviated PDM) of [2]. Here we compare the parameterized inner and outer bounds for the solution set, obtained by the method of Theorem 3 and Proposition 2, and the corresponding bounds obtained by [3], thus comparing to the other six parameterized solutions considered in [2]. We present the results for the smallest uncertainty \( \delta = 0.01 \) and the largest uncertainty \( \delta = 0.25 \), considered in [3].

Table 1 presents inner and outer bounds obtained by us for \( \delta = 0.01 \). These bounds are much sharper than, and can be compared to, the bounds obtained by three other parameterized solutions reported in [2] Table 4. For the results in Table 1 Table 2 presents two measures of the quality of a solution enclosure: sharpness \( O_s \) of the solution enclosure \( x_{out} \) defined by \( Q_s(x_{in}, x_{out}) := \{ 0 \text{ if } x_{in} = \emptyset, \text{rad}(x_{in})/\text{rad}(x_{out}) \text{ otherwise} \} \), and percentage \( O_w \) by which an interval \( y \) overestimates the interval \( x, x \subseteq y \), defined by \( O_w(x, y) := (1 - \text{rad}(x)/\text{rad}(y)) \times 100 \). It is seen from Table 2 that the range of the sharpness measure is very close for the two methods pKR1 and PDM. On the other hand, the percentage by which PDM overestimates pKR1out is between 0.55% and 0.96%. Table 3 presents the two measures of the quality of a solution enclosure for the case of large parameter uncertainties \( \delta = 0.25 \) in Example 1. Although the percentage by which PDM overestimates pKR1out is more pronounced in this case, the ranges of sharpness is very close for these two methods and the methods compared in [2]. The first conclusion from Example 1 is that the methods based on condition 4 provide sharper solution enclosure than the methods based on condition 6 for systems with rank one uncertainty structure. The second important conclusion from this example is that the sharpness measure is not quite informative when comparing the solution enclosure of different methods in contrast to the percentage of overestimation.

| \( x \) | pKR1 | outer | PDM | inner | PDM |
|---|---|---|---|---|---|
| \( x_1 \) | [7.01522, 7.16659] | [7.01480, 7.16702] | [7.01736, 7.16446] | [7.01777, 7.16405] |
| \( x_2 \) | [4.11780, 4.24583] | [4.11736, 4.24628] | [4.11987, 4.24377] | [4.12030, 4.24333] |
| \( x_3 \) | [5.39374, 5.51535] | [5.39331, 5.51578] | [5.39567, 5.51342] | [5.39609, 5.51300] |
| \( x_4 \) | [2.13805, 2.22558] | [2.13770, 2.22594] | [2.13962, 2.22401] | [2.13997, 2.22367] |
| \( x_5 \) | [1.06046, 1.12136] | [1.06017, 1.12165] | [1.06171, 1.12011] | [1.06200, 1.11982] |

Table 1: Bounds for \( [\Sigma] \) in Example 1 \( \delta = 0.01 \), obtained by pKR1 and PDM.

4. On an application of pKR1. In [2] Section 3 Kolev proposes to apply the parameterized solution of type Theorem 3 for determining \( [\Sigma]^p \) of parametric systems.
In order to simplify the presentation, we denote $\lambda$ in more details and demonstrate that this might be dangerous.

Now, in order to operate simultaneously with both $p$ and $\delta$ out of $A$ vectors $p \in \{ -s, s \}$, let (3) be the equivalent, optimal rank one, representation of (1), which involves only $\lambda \in \mathbb{P}^{n \times 1}$, and let (4) hold true. Let 

$$\lambda \in \mathbb{P}^{n \times 1},$$

for an $s_i = \lambda$ (4) be nonsingular. Denote $C = A^{-1}(\hat{y})$, $\hat{x} = C\hat{y}$, $\hat{y} = \tilde{R}\hat{x}$, and let (1) hold true. Let $i$ be arbitrary, $1 \leq i \leq n$, and let $(\Sigma_{uni}(1)) \in S$ be monotone with respect to each parameter $p_k$, $k \in \pi = (\pi^\top, (\pi^\top)^\top)^\top$, so that

$$\lambda \in \mathbb{P}^{n \times 1},$$

for an $s_i \in \{ -1, 1 \}^K (|s_i| = 1 \in \mathbb{R}^K)$, where $-1$ means decreasing and 1 = increasing. In order to simplify the notation, in what follows we will omit the subscript in $s_i$. Denote $s_i^\top = s_1^\top = (s_i, s_i)$. We consider $x$ in Theorem (1) as an interval evaluation of the function

$$x(p) = \hat{x} - (CF)(\hat{p}_{\pi''} - p_{\pi''}) + (CL)(D_{\hat{p}_{\pi''} - p_{\pi''}}(y(p) - t)),$$

where $y(p)$ is the solution of the system (3). Replacing in this function the two endpoint vectors $p^{-s}$, respectively $p^s$, we obtain

$$x_i(p^{-s}) = \hat{x}_i - C_iF(\hat{p}_{\pi''} - p_{\pi''}) + C_iLD_{\hat{y} - t}(\hat{p}_{\pi''} - p_{\pi''}) + C_iLD_{\hat{y} - t}(\hat{p}_{\pi''} - p_{\pi''})$$

$$x_i(p^s) = \hat{x}_i - C_iF(\hat{p}_{\pi''} - p_{\pi''}) + C_iLD_{\hat{y} - t}(\hat{p}_{\pi''} - p_{\pi''}) + C_iLD_{\hat{y} - t}(\hat{p}_{\pi''} - p_{\pi''}).$$

In order to simplify the presentation, we denote $\lambda'' = C_iF, \lambda' = C_iLD_{\hat{y} - t}$. For $p \in \mathbb{P}$ and $s \in \{ -1, 1 \}$, we have $p^{-s} = \hat{p} - sp, p^s = \hat{p} + sp$ and

$$\hat{p} - p^{-s} = \hat{p} - (\hat{p} - sp), p^s = \hat{p} - (\hat{p} - sp),$$

similarly, $\hat{p} - p^s = -sp$.

Thus, we have

$$x_i(p^{-s}) = \hat{x}_i - \lambda''(s')\hat{p}_{\pi'} + \lambda'(s')\hat{p}_{\pi'} + C_iLD_{\hat{y} - t}(\hat{p}_{\pi'} - p_{\pi'}),$$

$$x_i(p^s) = \hat{x}_i - \lambda''(-s')\hat{p}_{\pi'} + \lambda'(-s')\hat{p}_{\pi'} + C_iLD_{\hat{y} - t}(-\hat{p}_{\pi'} - p_{\pi'}).$$

Now, in order to operate simultaneously with both $p^{-s}, p^s$, as well as simultaneously with both $x_i(p^{-s}), x_i(p^s)$, we use Kaucher interval arithmetic and the relations between
proper and improper intervals. Consider the following interval expression in \( \mathbb{K} \mathbb{R} \):

\[
(11) \quad \hat{x}_i - \lambda''(p''_\pi)_{-s''} + \lambda'(p'_\pi)_{-s'} + [r^-_{s,i}, r^+_{s,i}],
\]

where \([r^-_{s,i}, r^+_{s,i}] = [C_{i\pi} LD_{y(s \tilde{p}_\pi)} -g (s' \hat{p}_\pi), C_{i\pi} LD_{y(-s \tilde{p}_\pi)} -g (-s' \hat{p}_\pi)] \) and \( p'_\pi = [-\hat{p}_\pi, \tilde{p}_\pi] \).

(11) is equivalent to

\[
(12) \quad \hat{x}_i - [\lambda''(p''_\pi)_{-s''} \lambda'(p'_ \pi)_{-s'}] + [r^-_{s,i}, r^+_{s,i}].
\]

If

\[
(13) \quad s_{\lambda''} := \text{sign}(\lambda'') = -s'' \quad \text{and} \quad s_{\lambda'} := \text{sign}(\lambda') = -s',
\]

then

\[
-\lambda''(p''_\pi)_{-s''} = -s_{\lambda''} \lambda''(p''_\pi)_{-s''} = -s_{\lambda''} \lambda''(p''_\pi)_{-s''} s_{\lambda''} \lambda''(p''_\pi) = -\lambda''(p''_{\tilde{\pi}^\prime})
\]

\[
= -[-s'' \lambda'' \tilde{p}_{\pi}^\prime, \lambda'' \tilde{p}_{\pi}^\prime] \quad (13) = [s'' \lambda'' \tilde{p}_{\pi}^\prime, -s'' \lambda'' \tilde{p}_{\pi}^\prime],
\]

similarly \( \lambda'(p'_\pi)_{-s'} = [\lambda'(s' \tilde{p}_{\pi}^\prime), -\lambda'(s' \tilde{p}_{\pi}^\prime)] \). Thus, (11), (12), become equivalently

\[
(14) \quad \hat{x}_i - [\lambda''(s'' \tilde{p}_{\pi}^\prime), \lambda'(s' \tilde{p}_{\pi}^\prime)] + [\lambda'(s' \tilde{p}_{\pi}^\prime), -\lambda'(s' \tilde{p}_{\pi}^\prime)] + [r^-_{s,i}, r^+_{s,i}] = \left[ x_i(p_{-s_i}), x_i(p_{s_i}) \right].
\]

Thus, by (13), (12) is equivalent to \( \hat{x}_i - \lambda''(p''_\pi) + \lambda'(p'_\pi) + [r^-_{s,i}, r^+_{s,i}] \).

Now, we compare (12) to \( x_i(p, r) \), where \( x(p, r) \) is the parameterized solution from Theorem 3. The coefficients \( \lambda'', \lambda' \) are the same in both expressions. Consider three cases.

- Obviously, under (13), the first three terms in the two expressions are equivalent.
- If for some \( k \in \pi, \lambda_k = 0 \), then the equality relation (14) is preserved and the equivalence between the first three terms in (12) and \( x_i(p, r) \) is also preserved. However, \( s_k \neq s_{\lambda_k} \) and \( s_k \) cannot be inferred from \( \lambda_k \).
- If for some \( k \in \pi, 0 \neq s_{\lambda_k} \neq s_k \), the equality relation (14) turns into inclusion (due to \( |\lambda_k||p'_k| \subseteq |\lambda_k||p''_k| \)), which contradicts to the initial assumption. In this case, the first three terms in (12) and \( x_i(p, r) \) are equivalent but \( s_k \) also cannot be inferred from \( \lambda_k \).

Thus, we have proven the following theorem.

**Theorem 4.** Let (3) be the equivalent, optimal rank one, representation of the system (1), which involves only rank one interval parameters. Let the matrix \( A(p) \) be nonsingular. Denote \( C = A^{-1}(\tilde{p}) \), \( z = C a(\tilde{p}) \), \( y = R \tilde{x} \), and let the condition (4) hold true. If for any \( i, 1 \leq i \leq n, (\Sigma_{\text{uni}}(1))_i \) is monotone with respect to each parameter \( p_k, k \in \pi = ((\pi'')^\top, (\pi')^\top)^\top \), with type of monotonicity specified by the sign vector \( s_i \), and if sign \((CF, CLD_{\tilde{g}^{-1}}) = s_i \), then the parameterized solution defined in Theorem 3 can be used for determining \( \hat{x}_i(\Sigma_{\text{uni}}(1))_i \).
It follows from (14) that with given \( s_i \), \( \Box(\Sigma_{\text{uni}}(1)) \), can be obtained by solving (1) for \( p^{-s_i} \), respectively, for \( p^{s_i} \), \( p \in \mathbb{R}^R \), or by solving the equivalent centered system

\[
\left( A(\hat{p}) - LD_{p'} \right) x = a(\hat{p}) - Fp' - LD_{p'} t, \quad p' \in [-\hat{p}, \hat{p}]
\]

for \( p' = s_i \hat{p} \), respectively, for \( p' = -s_i \hat{p} \), that is \((p')^{s_i} \), resp., \((p')^{-s_i} \).

Note, that the matrix \((CF, CLD_{\tilde{g}-t})\) is different from the matrix \((-CF, CLD_{\tilde{g}-t})\) in the parameterized solution of Theorem 3. Note also, that the interval \([r_{i-1}, r_{i+1}] \in \mathbb{R}\) is not symmetric in general and differs from the symmetric interval \([-r, r]\) in Theorem 3.

By Proposition 1 it follows that (for rank one uncertainty structure of the system) the parameterized solution obtained by the method of [4] (based on affine arithmetic) will have the same signs of the parameter coefficients as the parameterized solution of Theorem 3. The example, considered in [4], illustrates the first case \( (s_k = s_i) \) in the proof of Theorem 3. By the following example we illustrate the last case \( (0 \neq s_k \neq -s_k) \) in the proof of Theorem 4 which implies that the parameterized solution of type Theorem 4 cannot be used in general for determining the hull solution to interval parametric linear systems involving rank one parameters.

**Example 2.** Consider the parametric linear system

\[
\begin{pmatrix}
\frac{1}{4} + p_1 & \frac{1}{4} + p_{12} & p_{22} \\
p_2 & \frac{1}{4} + p_{12} & \frac{1}{3} + p_1 \\
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
\end{pmatrix}
= 
\begin{pmatrix}
\frac{s_1}{4} - p_1 \\
\frac{s_2}{4} + \frac{p_{12}}{4} \\
\end{pmatrix}
\]

\( p_1, p_{12} \in [-\frac{3}{4}, \frac{3}{4}], \quad p_2, p_{22} \in [-\frac{1}{4}, \frac{1}{4}], \quad p_3 \in [-\frac{1}{2}, \frac{1}{2}]. \)

The coefficient matrix for everyone of the parameters has rank one. Therefore, by Corollary 3, the parameteric united solution set has linear boundary and its interval hull is obtained for particular endpoints of the parameter intervals. Table 4 (right) presents global monotonicity (single entry) or local monotonicity (two entries respectively for the lower and the upper bounds) type of the parametric solution set with respect to interval parameters. An equivalent representation of the system is defined by

\[
L = \begin{pmatrix}
0 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 \\
\end{pmatrix}, \quad R = \begin{pmatrix}
1 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1 \\
\end{pmatrix}, \quad g(p) = (p_1, p_{12}, p_2, p_{22})^T, \quad F = (-1, \frac{1}{2}, \frac{1}{2})^T, \quad t = 0.
\]

Applying Theorem 5 we obtain a parameterized solution with reminder term \( x(p, r) = \hat{x} + U(p_3, p_1, p_{12}, p_2, p_{22})^T + r \), where \( r \in [-\hat{r}, \hat{r}] \) and

\[
U = \begin{pmatrix}
1.07065 & 0.502836 & 1.89414 & -0.0321066 & -0.930657 \\
-0.282609 & -2.01134 & -0.31569 & 0.128426 & 0.117557 \\
-0.143116 & 0.167612 & 0.63138 & -0.995304 & -0.00979639 \\
\end{pmatrix},
\]

\[
\hat{x} = (-2.9538, 1.81522, -0.901268)^T, \quad \hat{r} = (52.7807, 39.2595, 22.8547)^T.
\]

The sign of matrix \( U' = (CF, CLD_{\tilde{g}-t}) \) is presented in Table 4 (left). Comparing the left and right tables of monotonicity, it is clear that the sign of matrix \( U' \) does not represent the true monotonic dependence. Furthermore, no one of the elements of \( U, U' \) is zero. Applying the monotonicity defined by \( \text{sign}(U') \) we obtain an interval vector, which is contained in \( \Box \Sigma^p \). Therefore, using \( \text{sign}(U') \) is dangerous.
Table 4: Monotonic dependence of the solution components on the interval parameters. Left: $\text{sign}(U)$ in the parameterized solution; Right: true dependence.

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