SUMMAND ABSORBING SUBMODULES
OF A MODULE OVER A SEMIRING

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Abstract. An $R$-module $V$ over a semiring $R$ lacks zero sums (LZS) if $x + y = 0 \implies x = y = 0$. More generally, a submodule $W$ of $V$ is summand absorbing in $V$ if $\forall x, y \in V : x + y \in W \implies x \in W, y \in W$. These arise in tropical algebra and modules over idempotent semirings. We explore the lattice of summand absorbing submodules of a given LZS module, especially those that are finitely generated, in terms of the lattice-theoretic Krull dimension, and describe their explicit generation.

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1. Introduction

Throughout this paper, $R$ is a semiring [2], and $V$ is a (left) module (sometimes called “semimodule”) over $R$; i.e., $(V, +)$ is a semigroup satisfying the familiar module axioms as well as $r0_V = 0r_x = 0_V$ for all $r \in R, x \in V$. The zero submodule $\{0_V\}$ is usually written as $0$. We denote the set of all submodules of $V$ by $\text{Mod}(V)$. Since we cannot take quotient modules $V/W$ over semirings, we also write $\text{Mod}(V; W)$ for the submodules of $V$ containing $W$.

An $R$-module $V$ over a semiring $R$ lacks zero sums (where the term “zerosumfree” is used in [4]), abbreviated LZS, if

$$\forall x, y \in V : x + y = 0 \implies x = y = 0.$$  \hfill (LZS)

In this paper we continue the theory of LZS modules from [4], which applies immediately to tropical algebra and modules over idempotent semirings. Our results apply in particular

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to additive semigroups (with neutral element), which can be viewed as modules over the semiring \( \mathbb{N}_0 \) of natural numbers including zero.

The condition \([\text{LZS}]\) follows at once from the condition, called \textbf{upper bound} (ub), that \( a + b + c = a \) implies \( a + b = a \). (We take \( a = 0 \).) In \cite{4 §6} the obstruction in an \( R \)-module \( V \) to the ub condition was studied in terms of Green’s partial pre-order, a binary relation \( \preceq \) (more precisely denoted \( \preceq_V \)) given by

\[
    x \preceq y \iff \exists z \in V : x + z = y.
\]

This is a pre-order (also called a “quasi-ordering”) on \( V \); in other words, \( \preceq \) is reflexive \((x \preceq x)\) and transitive \((x \preceq y, y \preceq z \Rightarrow x \preceq z)\), but not necessarily antisymmetric. (Green’s partial preorder is generated by the relations \( 0_V \preceq z, x + z = y \) implies \( x = x + 0_V \preceq x + z = y \).)

To get a better understanding of this property, we introduce an equivalence relation \( \equiv \) on \( V \) as follows:

\[
    x \equiv y \iff x \preceq y, y \preceq x.
\]

As observed after \cite{4 Definition 6.1}, \( \equiv \) is a congruence, implying \( V := V/\equiv \) is again an \( R \)-module in the obvious way, with the induced operations \( \bar{x} + \bar{y} = \bar{x + y} \) and \( r \bar{x} = \bar{rx} \), where \( \bar{x} \) denotes the equivalence class of \( x \). We have the well-defined partial ordering \( \leq \) on the \( R \)-module \( \bar{V} \) induced by \( \preceq \):

\[
    \bar{x} \leq \bar{y} \iff x \preceq y
\]

for any \( x, y \in V \), which is compatible with addition and scalar multiplication.

**Proposition 1.1 \([4 \text{ Proposition 6.2}]\).**

(i) The monoid \( \bar{V} \) is upper bound.

(ii) \( V \) is upper bound if and only if the congruence \( \equiv \) is trivial.

For \( x \preceq y \) we want to examine those \( z \in V \) such that \( x \preceq z \preceq y \). For example, if \( \alpha + \beta = 1 \), then \( x \preceq \alpha x + \beta y \preceq y \), since \( x = \alpha x + \beta x \preceq \alpha x + \beta y \) and \( \alpha x + \beta y \preceq \alpha y + \beta y = y \).

**Definition 1.2.** A subset \( S \) of \( V \) is \textbf{convex} (in \( V \)) if for any \( x, y, z \in V \):

\[
    x \in S, y \in S, x \preceq z \preceq y \Rightarrow z \in S.
\]

It is easily seen that a set \( S \subset V \) is convex in \( V \) iff \( S \) partitions into full equivalence classes (of \( \equiv \)), and the image \( \bar{S} \) in the partially ordered set \( \bar{V} \) is convex.

1.1. \textbf{Summand absorbing submodules.}

We turn to the main notion of this paper, examining the LZS condition in terms of a related property which we call \( \text{SA} \).

**Definition 1.3.** A submodule \( W \) of \( V \) is \textbf{summand absorbing} (abbreviated \( \text{SA} \)) in \( V \) if

\[
    \forall x, y \in V : x + y \in W \Rightarrow x \in W, y \in W; \quad \text{(SA)}
\]

we then say that \( W \) is an \( \text{SA-submodule} \) \( V \). A submodule \( U \) of \( V \) is a \( \Sigma \text{SA-submodule} \) of \( V \) (or: an \( \text{SA-sum in} \ V \)) if \( U = \sum_{i \in I} W_i \) for some family \( (W_i \mid i \in I) \) of \( \text{SA-submodules} \).

**Remarks 1.4.**

a) The \( R \)-module \( V \) is \( \text{LZS} \) iff \( 0 \) is an \( \text{SA-submodule} \) of \( V \).

b) \( V \) is \text{ub} iff the semigroup \( V_y := \{ x \in V : x + y = y \} \) is \( \text{SA} \) (as an additive semigroup) for each \( y \in V \).

We also recall the notion of a weak complement of a submodule \( W \) of \( V \).
Definition 1.5 ([4, Definition 1.2]). A submodule $T$ of $V$ is a weak complement of $W$ (in $V$), denoted $V = W \oplus_T T$, if $V = W + T$ and for every $w \in V \setminus \{0\}$ the intersection $(w + T) \cap T$ is empty.

Thus $V = W \oplus T \implies V = W \oplus_w T$, although $V = W \oplus_w T$ does not necessarily imply that $V = T \oplus_w W$.

Lemma 1.6 ([4, Lemma 2.2]). Suppose that $W$ is a submodule of an LZS module $V$. Then $T$ is a weak complement of $W$, if and only if $T$ is SA with $T \cap W = 0$.

Thus the SA property is a natural continuation of the research in [4]. The following result, proved in [4, Lemma 2.3], leads to a theory of decompositions in tropical algebra and related structures, much stronger than the classical theory, since one gets unique decompositions.

Lemma 1.7. Suppose $V$ has an SA-submodule $T$. Then any decomposition of $V$ descends to a decomposition of $T$, in the sense that if $V = Y + Z$, then $T = (T \cap Y) + (T \cap Z)$.

Accordingly, we are led to study SA-submodules in their own right, particularly when they are finitely generated.

SA($V$) denotes the poset consisting of all SA-submodules of $V$, partially ordered by inclusion. We also write SA($V; W$) for the SA-submodules of $V$ containing $W$. SA$_f$ denotes the finitely generated SA-submodules. The set of all SA-sum submodules is denoted as $\Sigma$ SA($V$), regarded again as a poset by the inclusion relation (containing SA($V$) as a sub-poset). $\Sigma$ SA$_f$ is the set of all sums of finitely generated SA-submodules.

Proposition 1.8 ([4, Proposition 5.7]). A submodule $W$ of $V$ is in SA($V$) iff $W$ is a union of equivalence classes and $\overline{W}$ is SA in $\overline{V}$.

In other words, the elements of SA($V$) are just the convex submodules of $V$ under the relation $\leq$. We denote the convex hull conv$_V(W)$ of an $R$-submodule $W$ of $V$ more concisely as $\overline{W}$.

Any family $(W_i \mid i \in I)$ in the poset SA($V$) has the infimum $\bigcap_{i \in I} W_i$ and the supremum $(\sum_{i \in I} W_i)\wedge$ in SA($V$), and so SA($V$) is a complete lattice (in contrast to $\Sigma$ SA($V$), cf. Remark 2.10) below. Furthermore, we shall show in Proposition 3.1 that SA($V$) is a modular lattice. Accordingly many tools of classical module theory become available.

The first part of the paper ([2,3] covers the general theory of SA-submodules. In [2] we continue the theory of [4], and introduce decompositions of SA-modules, called “SA-decompositions,” proving the following results:

Theorem 2.2. Assume that $W$ and $T$ are submodules of $V$ with $W + T = V$, $W \cap T = 0$, and furthermore, that $T$ is an SA-submodule of $V$. Then $V = W \oplus_T T$.

Theorem 2.13. Any $R$-module $V$ has at most one SA-decomposition $(T_i \mid i \in I)$, where all $T_i$ are SA-indecomposable. This is the finest SA-decomposition of $V$.

Theorem 2.16. Assume that $W$ and $T$ are submodules of $V$ with $T \in$ SA($V$), $W + T = V$, $W \cap T = 0$, (whence $V = W \oplus_T T$ by Theorem 2.2). Let $(v_\lambda \mid 1 \leq \lambda \leq d)$ be a system of generators of $V$. Write $v_\lambda = w_\lambda + t_\lambda$ with $w_\lambda \in W$, $t_\lambda \in T$. Then $(t_\lambda \mid 1 \leq \lambda \leq d)$ is a system of generators of $T$.

Theorem 2.18. The SA-decompositions of $R$ correspond uniquely to the complete orthogonal systems of idempotents of $R$. 


In §3 we develop a structure theory of SA-submodules along the classical lines of the socle and various analogs of dimension theory, including SA-Kdim, which is Krull dimension in the sense of [3] (or, more precisely, [5]), but based on SA-submodules. The theory is made more explicit in §3.2 by use of SA-uniform submodules and the SA-uniformity dimension (Definition 3.28), described in Theorem 3.30.

In the second of the paper (§4–§6) we exhibit a reasonably broad class of $R$-modules $V$, over an arbitrary semiring $R$, called finitely SA-accessible, for which we can obtain some stronger results. We say that $V$ is SAF-hereditary if submodules of SAF-submodules are SAF-submodules. Those SAF-hereditary modules with SAF-Kdim, called SAF-accessible, are examined in §4, and put to use in §5 and §6, where the height of a module is defined in terms of the Krull dimension, and decisive results are obtained for modules in terms of their height.

In the third part of the paper (§7–§9) we study generation of SA-submodules. §7 brings in a somewhat technical condition, called spines on $R$, built from halos, which permit rather efficient generation of modules, cf. Theorem 7.10, and these are presented over $V$ in §8. One main result:

**Theorem 8.3.** Assume that $S$ is an additive spine of an $R$-module $V$. Then every SA-submodule $W$ of $V$ is generated by $W \cap S$, and moreover $W \cap S$ is an additive spine of $W$.

For rings, this specializes to:

**Theorem 7.8.** Assume that $S$ is a set of generators of a (left) $R$-module $V$, and $M$ is an additive spine of $R$. Then any SA-submodule $W$ of $V$ is generated by the set $W \cap (MS)$.

An application to matrices is given in Theorem 7.14, and more generally to monoid semirings in Theorem 7.17.

In §9 we obtain rather satisfactory results about the SA-submodules and the Σ SA-submodules of a finitely generated module $V$ over a semiring $R$ which has a finite additive spine. The main reason is that in this case all SA-submodules of $V$ are finitely generated.

## 2. Preliminary results

We make a fresh start, reworking easy facts from [4, §4] in a slightly different way which fits better into the present chain of arguments than a mere citation of parts of [4] would do. Our goal is to compare different “weak” decompositions of $V$ into SA-submodules.

### 2.1. Three basic principles.

For basic facts on SA-submodules we refer to [4], now being content to recall three general principles from that paper.

**A)** If $f : V \to V'$ is an $R$-linear map between $R$-modules and $W'$ is an SA-submodule of $V'$ then (clearly) $f^{-1}(W')$ is SA in $V$.

Using this principle we immediately see that if $V$ is LZS and $W$ is a direct summand of $V$, i.e., $V = W \oplus T$, then $W$ is in SA($V$), since $W$ is the kernel of a projection $p : V \to V$ with $p(V) = T$. (But usually an $R$-module $V$ lacking zero sums has many more SA-submodules than direct summands.) We note in passing that for this conclusion it suffices to assume that $T$ (instead of $V$) is LZS.

---

1In contrast to direct decompositions, a formal definition of such decompositions, named “SA-decompositions”, will be given only later (Definition 2.8).
B) If \((W_i \mid i \in I)\) is a family of SA-submodules of \(V\) then (evidently) \(\bigcap_{i \in I} W_i\) is SA in \(V\). If the family \((W_i)\) is upward directed, i.e., for \(i, j \in I\) there exists \(k \in I\) with \(W_i \subset W_k\), \(W_j \subset W_k\), then also \(\bigcup_{i \in I} W_i\) is in \(SA(V)\).

C) A subset \(S\) containing \(0_V\) is convex in \(V\) if and only if \(S\) is SA \([4, \text{Lemma } 6.6]\).

Our paper \([4]\) contains various facts about weak complements, but more can be done. Our results below are rooted in analyzing the situation where both \(V = W \oplus_w T\) and \(V = T \oplus_w W\) hold, cf. Definition \([1.5]\).

**Theorem 2.1** ([4] Lemmas 2.1 and 2.2). Assume that \(W\) is a submodule of \(V\) which is LZS, and that \(T\) is a weak complement of \(W\) in \(V\). Then the module \(T\) is in \(SA(V)\). If moreover \(T\) also is LZS, then \(V\) is LZS.

**Proof.** The first assertion is by Lemma \([1.6]\). Assume that \(v_1, v_2 \in V\) and \(v_1 + v_2 = t \in T\). Then \(v_i = w_i + t_i \in W\), \(t_i \in T\) \((i = 1, 2)\). By adding we obtain

\[(w_1 + w_2) + (t_1 + t_2) = t.\]

By Definition \([1.5]\) this implies \(w_1 + w_2 = 0\) and \(t_1 + t_2 = t\). Since \(W\) is LZS, it follows that \(w_1 = w_2 = 0\), and so \(v_i = t_i \in T\). Thus \(T\) is in \(SA(V)\).

If \(T\) is LZS and \(t = 0\), i.e. \(v_1 + v_2 = 0\), we obtain from \(t_1 + t_2 = 0\) that \(t_1 = t_2 = 0\), whence \(v_1 = v_2 = 0\). So \(V\) is LZS. \(\square\)

**Theorem 2.2.** Assume that \(W\) and \(T\) are submodules of \(V\) with \(W + T = V\), \(W \cap T = 0\), and furthermore, that \(T\) is an SA-submodule of \(V\). Then \(V = W \oplus_w T\).

**Proof.** Let \(w \in W \setminus \{0\}\), \(t \in T\), and suppose that \(w + t \in T\). Then \(w \in T\) since \(T\) is SA in \(V\). We conclude that \(w \in T \cap W = \{0\}\), a contradiction. Thus \((w + T) \cap T = \emptyset\). \(\square\)

**Lemma 2.3.** Assume that \(W, T, U\) are submodules of \(V\), such that \(V = W \oplus_w T\) and \(V = W + U\). Then \(T \subset U\).

**Proof.** The module \(T\) is in \(SA(V)\) (Theorem \([2.1]\)). Thus \(V = W + U\) implies that

\[T = (T \cap W) + (T \cap U)\]

But \(T \cap W = \{0\}\), whence \(T = T \cap U\), i.e., \(T \subset U\). \(\square\)

**Theorem 2.4** (Uniqueness of weak complements, cf. \([4]\ Corollary 2.5]). Assume that \(V = W \oplus_w T\) and \(V = W \oplus_w U\). Then \(T = U\).

**Proof.** By Lemma \([2.3]\) we have \(T \subset U\) and \(U \subset T\). \(\square\)

**Theorem 2.5.** Assume that \(W\) and \(T\) are submodules of \(V\) with \(W + T = V\) and \(W \cap T = 0\), and furthermore that both \(W\) and \(T\) are LZS. Then the following conditions are equivalent.

1. \(W, T\) are SA-submodules of \(V\).
2. \(V = W \oplus_w T = T \oplus_w W\).

If (1), (2) hold, then \(V\) is LZS.

**Proof.** (1) \(\Rightarrow\) (2) is clear by Theorem \([2.2]\) and (2) \(\Rightarrow\) (1) is clear by Theorem \([2.1]\). If (1), (2) hold, then it follows by the last sentence in Theorem \([2.1]\) that \(V\) is LZS. \(\square\)

The following lemma will be useful.

**Lemma 2.6.** Assume that \(W, T, Y, Z\) are submodules of \(V\) with

\[V = W \oplus_w T = Y + Z.\]

Assume also that \(W\) is LZS.
a) Then \( T = (T \cap Y) + (T \cap Z) \).

b) If moreover \( V = Y \oplus_w Z \), then \( T = (T \cap Y) \oplus_w (T \cap Z) \).

Proof. a): This is clear since \( T \) is in \( \text{SA}(V) \) (Theorem 2.4).

b): If \( y \in T \cap Y \) and \( y \neq 0 \) then \( (y + Z) \cap Z = \emptyset \), and so

\[
[y + (T \cap Z)] \cap T \cap Z = \emptyset. 
\]

\[\square\]

2.2. SA-decompositions.

We assume throughout that the \( R \)-module \( V \) is L\( Z\)S (and so all submodules of \( V \) also are L\( Z\)S), a natural hypothesis in view of the preceding Theorems 2.1 and 2.5. Given two decompositions

\[
V = W_1 \oplus_w W_1' = W_2 \oplus_w W_2',
\]

we know by Theorems 2.1 and 2.5 that all four submodules \( W_1, W_1', W_2, W_2' \) are in \( \text{SA}(V) \), and that

\[
V = W_1' \oplus_w W_1 = W_2' \oplus_w W_2.
\]

By Lemma 2.6 we have decompositions

\[
W_1 = W_1 \cap W_2 \oplus_w W_1' \cap W_2',
\]

\[
W_1' = W_1' \cap W_2 \oplus_w W_1' \cap W_2',
\]

and analogous decompositions of \( W_2, W_2' \). By adding we obtain

\[
V = [(W_1 \cap W_2) \oplus_w (W_1' \cap W_2')] \oplus_w [(W_1' \cap W_2) \oplus_w (W_1' \cap W_2')].
\]

Since \( W_1' \cap W_2' \) is in \( \text{SA}(V) \) we conclude by Theorem 2.2 that

\[
V = [(W_1 \cap W_2) + (W_1' \cap W_2') + (W_1' \cap W_2)] \oplus_w W_1' \cap W_2'.
\]

Furthermore, by adding the equalities

\[
W_1 = W_1 \cap W_2 + W_1 \cap W_2',
\]

\[
W_2 = W_1 \cap W_2 + W_1' \cap W_2,
\]

we obtain

\[
W_1 + W_2 = W_1 \cap W_2 + W_1 \cap W_2' + W_1' \cap W_2.
\]

Comparing (2.4) and (2.5) we learn the following.

**Proposition 2.7.** The present situation (2.1) implies that

\[
V = W_1 \oplus_w (W_1' \cap W_2').
\]

But we do not know whether \( W_1 + W_2 \) is in \( \text{SA}(V) \) or not. If \( W_1, W_2 \) are direct summands of \( V \) then we know from [4, Theorem 2.9] that \( W_1 + W_2 \) is a direct summand of \( V \), and so \( W_1 + W_2 \) is an SA-submodule of \( V \). This difficulty prompts the following somewhat unusual definition of “SA-decompositions” and “SA-summands.”

**Definition 2.8.**

a) We call a family \( (T_i \mid i \in I) \) in \( \text{SA}(V) \) **orthogonal**, if for any two indices \( i \neq j \), the intersection \( T_i \cap T_j \) equals 0.

b) An **SA-decomposition** of \( V \) is an orthogonal family \( (T_i \mid i \in I) \) in \( \text{SA}(V) \) with all \( T_i \neq 0 \), which spans \( V \), i.e., \( V = \sum_{i \in I} T_i \).

c) We call a submodule \( T \) of \( V \) an **SA-summand** of \( V \) if \( T \) is a member of an SA-decomposition \( (T_i \mid i \in I) \) of \( V \), i.e., \( T = T_i \) for some \( i \in I \).
d) We say that $V$ is SA-indecomposable, if $V$ does not have any SA-summand $T \neq V$, i.e., $\{V\}$ is the unique SA-decomposition of $V$.

**Remark 2.9.** If $(T_i \mid i \in I)$ is an orthogonal family in $\text{SA}(V)$ with all $T_i \neq 0$, then $(T_i \mid i \in I)$ is an SA-decomposition of $W = \sum_i T_i$, since every $T_i$ is also an SA-submodule of $W$.

Any family $(U_\lambda \mid \lambda \in \Lambda)$ in $\Sigma \text{SA}(V)$ has the supremum $\sum_{\lambda \in \Lambda} U_\lambda$ in $\Sigma \text{SA}(V)$, but even for two SA-sums $U_1, U_2$ no infimum in $\Sigma \text{SA}(V)$ is in sight. This changes if one of the modules $U_1, U_2$ is in $\text{SA}(V)$.

**Remark 2.10.** If $W \in \text{SA}(V)$, $U \in \Sigma \text{SA}(V)$ and $U = \sum_{i \in I} W_i$ with $W_i \in \text{SA}(V)$, then
\[ W \cap U = \sum_{i \in I} (W \cap W_i), \]  
and so $W \cap U$ is an SA-sum in $V$. The module $W \cap U$ is the infimum of $W$ and $U$ in $\Sigma \text{SA}(V)$. Furthermore $W + U$ is the supremum of $W$ and $U$ in $\Sigma \text{SA}(V)$.

We cannot build up finite SA-decompositions from binary SA-decompositions, as is common for finite direct decompositions, but nevertheless a finite SA-decomposition may be viewed as an iterated formation of weak complements of $\Sigma \text{SA}$-modules, due to the following fact.

**Proposition 2.11.** Assume that $(T_i \mid 1 \leq i \leq n)$ is a finite orthogonal family in $\text{SA}(V)$. Let $U := \sum_{i=1}^n T_i \in \Sigma \text{SA}(V)$. Then the chain in $\Sigma \text{SA}(V)$
\[ U_0 = \{0_v\} \subset U_1 \subset U_2 \subset \cdots \subset U_n = U \]  
with $U_r := \sum_{i=1}^r T_i$ ($1 \leq r \leq n$) has the property
\[ U_{r+1} = U_r \oplus w T_{r+1} \quad (0 \leq r \leq n). \]  
\[ \text{Proof.} \] This follows from Theorem 2.22 since $U_{r+1} = U_r + T_{r+1}$ and
\[ U_r \cap T_{r+1} = \sum_{i=1}^r (T_i \cap T_{r+1}) = 0. \]  
\[ \square \]

Note that conversely the chain $(\ast)$ in $\Sigma \text{SA}(V)$ determines the family $(T_i \mid 1 \leq i \leq n)$, due to the uniqueness of weak complements (Theorem 2.4). If an infinite orthogonal family $(T_i \mid i \in I)$ in $\text{SA}(V)$ is given, then we see in the same way that for any module $U_J := \sum_{i \in J} T_i$ and $k \in J$ we have
\[ U_J = U_J \setminus \{k\} \oplus_w T_k. \]  
\[ \text{Definition 2.12.} \] Given two SA-decompositions $(T_i \mid i \in I)$, $(S_j \mid j \in J)$ of $V$ we say that the second SA-decomposition refines the first one, if every module $S_j$ is contained in some module $T_i$.

If this happens then clearly every $S_j$ is contained in a unique module $T_i$, since different members of $(T_i \mid i \in I)$ have intersection zero. We thus have a unique map $\lambda : J \to I$ with
\( S_j \subset T_{\lambda(j)} \) for each \( j \in J \). This map \( \lambda \) is surjective, since otherwise \( (S_j \mid j \in J) \) would not span \( V \). It follows that for every \( i \in T \)

\[
T_i = \sum_{\lambda(j) = i} S_j \quad (2.10)
\]

and so \((S_j \mid \lambda(j) = i)\) is an SA-decomposition of \( T_i \). Now the following is obvious.

**Theorem 2.13.** Any \( R \)-module \( V \) has at most one SA-decomposition \((T_i \mid i \in I)\), where all \( T_i \) are SA-indecomposable. This is the finest SA-decomposition of \( V \).

**Proposition 2.14.** Any two SA-decompositions have a common refinement.

*Proof.* Assume that \((T_i \mid i \in I)\) and \((S_j \mid j \in J)\) are two decompositions of \( V \). We have

\[
V = \sum_{i \in I} T_i = \sum_{j \in J} S_j.
\]

Then, since \( T_i \in \text{SA}(V) \), we have

\[
T_i = \sum_{j \in J} T_i \cap S_j
\]

(cf. Lemma 2.6(a)), and so

\[
V = \sum_{(i,j) \in I \times J} T_i \cap S_j.
\]

Furthermore, \((T_i \cap S_j) \cap (T_k \cap S_\ell) = 0\) if \((i,j) \neq (k, \ell)\). Let \( K \) denote the subset of \( I \times J \) consisting of all \((i,j)\) with \( T_i \cap S_j \neq 0 \). Then \((T_i \cap S_j \mid (i,j) \in K)\) is a common refinement of the SA-decompositions \((T_i \mid i \in I)\) and \((S_j \mid j \in J)\). \(\Box\)

It is evident that the SA-decomposition just constructed is the coarsest common refinement of the SA-decompositions \((T_i \mid i \in I)\) and \((S_j \mid j \in J)\) of \( V \).

**Proposition 2.15.** If \( V \) is finitely generated, then every SA-decomposition \((T_i \mid i \in I)\) of \( V \) is finite (i.e., \( I \) is finite).

*Proof.* We pick a set of generators \( \{s_1, \ldots, s_r\} \) of \( V \). For every \( k \in \{1, \ldots, r\} \) there is a finite subset \( I_k \) of \( I \) such that \( s_k \in \sum_{i \in I_k} T_i \), whence \( R s_k \subset \sum_{i \in I_k} T_i \). Thus

\[
V = \sum_{k=1}^r R s_k \subset \sum_{i \in J} T_i
\]

with \( J := \bigcup_{k=1}^r I_k \) finite. Suppose that \( J \neq I \). Then choosing some \( \ell \in I \setminus J \) we have \( T_\ell \subset \sum_{i \in J} T_i \). But this is impossible since all intersections \( T_\ell \cap T_i \) with \( i \in J \) are zero, and so

\[
T_\ell = \bigcup_{i \in J} (T_\ell \cap T_i) = 0.
\]

Thus \( J = I \). \(\Box\)

**Theorem 2.16.** Assume that \( W \) and \( T \) are submodules of \( V \) with \( T \in \text{SA}(V) \), \( W + T = V \), \( W \cap T = 0 \), (whence \( V = W \oplus T \) by Theorem 2.23). Let \((v_\lambda \mid 1 \leq \lambda \leq d)\) be a system of generators of \( V \). Write \( v_\lambda = w_\lambda + t_\lambda \) with \( w_\lambda \in W, t_\lambda \in T \). Then \((t_\lambda \mid 1 \leq \lambda \leq d)\) is a system of generators of \( T \).
Proof. \( V = \sum_{\lambda \in \Lambda} R(w_\lambda + t_\lambda). \) It follows that

\[
V = \sum_{\lambda \in \Lambda} Rw_\lambda + \sum_{\lambda \in \Lambda} Rt_\lambda.
\]

Intersecting with \( T \) we obtain

\[
T = (\sum_{\lambda \in \Lambda} Rw_\lambda) \cap T + \left( \sum_{\lambda \in \Lambda} Rt_\lambda \right) \cap T = \sum_{\lambda \in \Lambda} Rt_\lambda.
\]

Corollary 2.17. Let \((T_i \mid i \in I)\) be an SA-decomposition of \( V \). Then \( V \) is finitely generated iff \( I \) is finite and each \( R \)-module \( T_i \) is finitely generated.

Proof. This is an immediate consequence of Proposition 2.15 and Theorem 2.16.

In the case that \( R \) is commutative and \( V = R \), considered as an \( R \)-module, we obtain the following explicit description of all SA-decompositions of \( R \). First note that by Proposition 2.15 all SA-decompositions of \( R \) are finite.

Theorem 2.18. Assume that the semiring \( R \) is commutative. Then the SA-decompositions \((T_i \mid 1 \leq i \leq n)\) of \( R \) are the families \((e_i R \mid 1 \leq i \leq n)\) given by the complete finite orthogonal systems \((e_i \mid 1 \leq i \leq n)\) of idempotents of \( R \) \((e_i e_j = \delta_{ij} e_i, \ \sum_{i=1}^n e_i = 1)\). In this way the SA-decompositions of \( R \) correspond uniquely to the complete orthogonal systems of idempotents of \( R \). Every SA-decomposition of \( R \) is a direct decomposition of \( R \).

Proof. If \((e_i \mid 1 \leq i \leq n)\) is a complete family of orthogonal idempotents of \( R \), then it is plain that

\[
R = \bigoplus_{i=1}^n e_i R,
\]

and so \((e_i R \mid 1 \leq i \leq n)\) is also an SA-decomposition of \( R \).

Conversely assume that \((T_i \mid 1 \leq i \leq n)\) is an SA-decomposition of \( R \). Then all \( T_i \) are ideals of \( R \), and so \( T_i T_j \subset T_i \cap T_j = 0 \) for \( i \neq j \). We pick elements \( e_1, \ldots, e_n \) of \( R \) with \( e_i \in T_i \) and

\[
1 = e_1 + \cdots + e_n.
\]

Multiplying by \( e_k \) for some \( k \in \{1, \ldots, n\} \) we obtain

\[
e_k = \sum_{i=1}^n e_k e_i.
\]

But for \( k \neq i \) we have \( e_k e_i \in T_k \cap T_i = 0 \), and conclude that \( e_k = e_k^2 \). Thus \( \{e_1, \ldots, e_n\} \) is a complete system of orthogonal idempotents of \( R \). If \( x \in T_i \) then

\[
x = \left( \sum_{j=1}^n e_j \right) x = \sum_{j=1}^n e_j x = e_i x,
\]

since \( e_j x \in T_j T_i = 0 \) for \( j \neq i \). Conversely if \( x \in R \) and \( x = e_i x \) then \( x \in T_i \), since \( e_i R \subset T_i \). This proves that \( T_i = e_i R \). The \( e_i \) are uniquely determined by the family of submodules \((T_i \mid 1 \leq i \leq n)\) of \( R \), since \( R = \bigoplus_{i=1}^n T_i \) and \( e_i x = x \) for \( x \in T_i \), while \( e_j x = 0 \) for \( x \in T_j \). \( \square \)
Remark 2.19. If \((T_i \mid i \leq i \leq n)\) is an SA-decomposition of a commutative semiring \(R\), viewed as an \(R\)-module, then the \(T_i\) are ideals of the semiring \(R\) with \(T_i \cap T_j = T_i T_j = 0\) for \(i \neq j\), and they can be viewed as semirings having as unit elements the idempotents \(e_i\) from above. Thus an SA-decomposition of \(R\) as an \(R\)-module is the same as a finite direct product decomposition

\[
R = T_1 \times \cdots \times T_n = \prod_{i=1}^{n} T_i
\]

of \(R\) as a semiring.

We illustrate Theorem 2.18 by some examples. Let \(X\) be a topological space, and, as common, let \(C(X)\) denote the ring of continuous \(\mathbb{R}\)-valued functions on \(X\). This ring is equipped with the “function ordering”, where \(f \leq g\) iff \(f(x) \leq g(x)\) for all \(x \in X\). Furthermore let \(C^+(X)\) denote the positive cone of this partial ordering on \(C(X)\), i.e.,

\[
C^+(X) := \{ f \in C(X) \mid f \geq 0 \},
\]

a semiring lacking zero sums. Our interest is in the SA-submodules of the semiring \(C^+(X)\), viewed as a \(C^+(X)\)-module. Note that \(C^+(X)\) is the set of all continuous functions on \(X\) with values in \(\mathbb{R}_{\geq 0} = [0, \infty]\). The restriction of the function ordering to \(C^+(X)\) coincides with the minimal ordering, since for \(f \leq g\) in \(C^+(X)\) we have

\[
g = f + (g - f),
\]

and \(g - f \in C^+(X)\).

It is plain that the function \(f \in C(X)\) is an idempotent of \(C(X)\) iff \(f\) has only values in \(\{0, 1\}\), and so \(f\) is the characteristic function \(\chi_U\) of a clopen (= closed and open) subset of \(X\); \(\chi_U(x) = 1\) if \(x \in U\), \(\chi_U(x) = 0\) if \(x \in X \setminus U\). All these idempotents lie in \(C^+(X)\). Thus a complete orthogonal system \((e_i \mid i \leq i \leq n)\) of \(C^+(X)\) corresponds uniquely to a finite disjoint decomposition \(X = \bigcup U_i\) of \(X\) into clopen subsets via \(e_i = \chi_{U_i}\). In particular \(C^+(X)\) itself is SA-indecomposable iff the topological space \(X\) is connected. In consequence of Theorem 2.18 we can describe all SA-decompositions of \(C^+(X)\) when the clopen subsets of \(X\) are known. We give three examples. Some more notation: \([\alpha, \beta]\) (resp. \([\alpha, \beta]\)) denotes the open (resp. closed) interval from \(\alpha\) to \(\beta\). Likewise for the half-closed intervals \([\alpha, \beta]\) and \([\alpha, \beta]\).

Examples 2.20. We fix a topological subspace \(X\) of the real line \(\mathbb{R}\). Let \(R := C^+(X)\).

a) If \(X\) is an interval of \(\mathbb{R}\) (open, half-open, closed), then \(R\) is SA-indecomposable.

b) Assume that \((x_n \mid n \in \mathbb{N})\) is a strictly increasing sequence in \(\mathbb{R}\) converging to \(x_\infty := \sup_{n \in \mathbb{N}} x_n \in \mathbb{R}\). Let \(X = \{ x_n \mid n \in \mathbb{N} \} \cup \{ x_\infty \}\). The primitive idempotents of \(R\) are precisely all elements

\[
e_n := \chi_{\{x_n\}} \quad (n \in \mathbb{N}),
\]

and so the SA-indecomposable summands of \(R\) are the ideals

\[
T_n := e_n R \quad (n \in \mathbb{N})
\]

of \(R\), consisting of the \(\mathbb{R}_+\)-valued functions \(f\) on \(X\) with \(f(x) = 0\) for \(x \in X \setminus \{x_n\}\). For every \(n \in \mathbb{N}\) we also have an idempotent \(g_n\) of \(R\) with

\[
e_1 + \cdots + e_n + g_n = 1
\]

and \(e_i g_n = 0\) for \(1 \leq i \leq n\), namely the characteristic function \(\chi_{Y_n}\) of

\[
Y_n := \{ x_i \mid i > n \} \cup \{ x_\infty \}.
\]
These clopen sets $Y_n$ constitute a fundamental system of neighborhoods of $x_\infty$ in $X$. The SA-decompositions $(T_1, \ldots, T_n, S_n)$ of $R$ which correspond to the orthogonal systems $(e_1, \ldots, e_n, g_n)$, i.e., $T_i = R e_i$, $S_n = R g_n$, are co-final in the set of all SA-decompositions of $R$ under refinement. Note that the “decomposition socle”

$$d \text{soc}(R) = \bigoplus_{i \in \mathbb{N}} R e_i$$

of $R$ (cf. [4, Definition 2.15]) is not an SA-summand of $R$, but is an SA-submodule of $R$. It is the set $\{ f \in R \mid f(x_\infty) = 0 \}$.

c) Let $X = \mathbb{Q} \subset \mathbb{R}$. The clopen subsets of $X$ are the disjoint unions of intervals

$$] \alpha, \beta[ := \{ x \in \mathbb{Q} \mid \alpha < x < \beta \}$$

with $\alpha, \beta \in (\mathbb{R} \cup \{-\infty, \infty\}) \setminus \mathbb{Q}$ and $\alpha < \beta$. Every such interval $] \alpha, \beta[$ provides an idempotent $e_{\alpha, \beta} := \chi_{] \alpha, \beta[}$ of $R = C^+(\mathbb{Q})$, and so an SA-submodule

$$T_{\alpha, \beta} := e_{\alpha, \beta}R$$

consisting of all $f \in R$ with $f(x) = 0$ for $x < \alpha$ or $x > \beta$. These submodules $T_{\alpha, \beta}$ are a co-final system of SA-summands of $R$ (with respect to reverse inclusion). If $\gamma \in ] \alpha, \beta[ \setminus \mathbb{Q}$, then

$$e_{\alpha, \beta} = e_{\alpha, \gamma} + e_{\gamma, \beta}, \quad e_{\alpha, \gamma} \cdot e_{\gamma, \beta} = 0.$$

Thus every module $T_{\alpha, \beta}$ is SA-decomposable. It follows that $R$ contains no SA-indecomposable SA-summands altogether.

Every finite sequence $\alpha_1 < \alpha_2 < \cdots < \alpha_n$ in $\mathbb{R} \setminus \mathbb{Q}$ gives a partition of $\mathbb{Q}$ into clopen intervals $]-\infty, \alpha_1[ \cup ]\alpha_1, \alpha_2[ \cup \cdots \cup ]\alpha_n, \infty[$, and thus a complete orthogonal system of idempotents $\{ e_{-\infty, \alpha_1}, e_{\alpha_1, \alpha_2}, \ldots, e_{\alpha_n, \infty} \}$ which corresponds to a direct sum decomposition

$$R = T_{-\infty, \alpha_1} \oplus T_{\alpha_1, \alpha_2} \oplus \cdots \oplus T_{\alpha_n, \infty}.$$

These decompositions are a co-final system in the set of all finite SA-decompositions of $R$ with respect to refinement.

3. The lattice $SA(V)$

**Proposition 3.1.** $SA(V)$ is a modular lattice. More precisely, if $W_i$ are submodules with $W_1 \subseteq W_2$ and $W_2$ is SA, then

$$W_1 + (W_2 \cap W_3) = W_2 \cap (W_1 + W_3).$$

**Proof.** ($\subseteq$) is immediate. To prove ($\supseteq$), suppose $x_2 = x_1 + x_3 \in W_2$ for $x_i \in W_i$. By hypothesis $x_1, x_3 \in W_2$, so $x_3 \in W_2 \cap W_3$ and $x_2 \in W_1 + (W_2 \cap W_3)$. $\square$

**Corollary 3.2.** Suppose $W_1 \subseteq W_2$, $W$ are SA submodules of $V$, with $W_1 \cap W = W_2 \cap W$ and $W_1 + W = W_2 + W$. If $W_1 + W$ is SA, then $W_1 = W_2$.

**Corollary 3.3.** Given a set $\{ W_i : i \in I \}$ of distinct submodules of an SA-module $V$, for $I$ infinite, and any submodule $W$, either $\{ W_i \cap W : i \in I \}$ or $\{ W_i + W : i \in I \}$ contains infinitely many distinct submodules.

**Definition 3.4.** Let $(T_i \mid i \in I)$ denote the set of all minimal non-zero SA-submodules of $V$. (It can happen that this set is empty.) We define the SA-socle of $V$ by

$$P := \text{soc}_{SA}(V) := \sum_{i \in I} T_i.$$
(read $P = 0$ if $I = \emptyset$).

Note that if $i, j \in I$ are different indices then $T_i \cap T_j = 0$, and that $T_i \in \text{SA}(P)$ for every $i \in I$. Thus $(T_i | i \in I)$ is an SA-decomposition of the SA-socle $P$.

3.1. Krull dimension (in the sense of Lemonnier-Gordon-Robson).

SA$(V)$, being a modular lattice, admits a satisfying dimension theory, denoted SA-Kdim, along the lines of Krull dimension, as defined and exposed elegantly in [3], which we use as our model. Since quotient modules do not play an effective role over semirings, we need to consider instead pairs $(V, W)$ where $V \nsubseteq W$. Fortunately, this theory already was developed at the level of lattices by Lemonnier [5], and is developed in this generality in [6], so all we need to do is put it in the present context.

**Definition 3.5.**

\begin{enumerate}
  \item The pair $(V, W)$ is **SA-artinian** if every descending chain
    \[ S_0 \supseteq S_1 \supseteq S_2 \supseteq \ldots \]
    of finitely generated non-zero SA-submodules of $V$ containing $W$ stops after finitely many steps. The $R$-module $V$ is **SA-artinian** if $(V, 0)$ is SA-artinian.
  \item The pair $(V, W)$ is **SA-noetherian** if every ascending chain
    \[ S_0 \subseteq S_1 \subseteq S_2 \subseteq \ldots \]
    of finitely generated non-zero SA-submodules of $V$ containing $W$ stops after finitely many steps. The $R$-module $V$ is **SA-noetherian** if $(V, 0)$ is SA-noetherian.
  \item The pair $(V, V)$ has SA-Kdim equal to $-1$. For $W \neq V$, SA-Kdim$(V, W) = 0$ if for every descending chain
    \[ W_0 \supseteq W_1 \supseteq W_2 \supseteq \ldots \]
    in SA$(V; W)$ stops after finitely many steps.
    In general, SA-Kdim$(V, W)$ (if it exists) is the smallest ordinal $\theta$ for which for every chain (3.1) one must have SA-Kdim$(W_i, W_{i+1}) < \theta$ for almost all $i$. Such a chain is called $\theta$-stable. SA-Kdim$(V)$ (if it exists) is SA-Kdim$(V, 0)$.
  \item A pair $(W, W')$ of SA-sumodules is called **SA-critical** if SA-Kdim$(W, W') = \theta$ but SA-Kdim$(W, W'') < \theta$ whenever $W' \supseteq W'' \supseteq W'$.
  \item The submodule $W$ is $\theta$-**SA-critical** if $(W, 0)$ is SA-critical.
\end{enumerate}

This leads to a natural generalization of the socle (cf. Definition 3.4). One can define the **SA-critical socle** [6, p. 146] to be the sum of all SA-critical submodules of $V$, of minimal SA-Kdim, but we do not go in that direction.

Unfortunately, SA-artinian $R$-modules seem to be not very frequent, but here is an instance.

**Definition 3.6.** A set of generators $T$ of $V$ is **SA-adapted** if every SA-submodule $W$ of $V$ is generated by the set $W \cap T$.

**Example 3.7.** If $V$ has an SA-adapted finite set of generators, then $V$ is certainly SA-artinian.

Hence, some of the results emerge more neatly for finitely generated SA-modules, and we need some more terminology.

**Notation 3.8.**
a) Given a module $V$ over a semiring $R$, we denote the set of all finitely generated SA-modules of $V$ by $\text{SA}_f(V)$. We furthermore denote the set of all sums $\sum W_i$ of families $(W_i \mid i \in I)$ in $\text{SA}_f(V)$ by $\Sigma \text{SA}_f(V)$ and the subset of all such sums with finite $I$ by $\Sigma_f \text{SA}_f(V)$.

b) Observe that the modules $U \in \Sigma_f \text{SA}_f(V)$ are again finitely generated. Moreover it is easily seen that $\Sigma_f \text{SA}_f(V)$ is the set of all finitely generated modules $U \in \Sigma \text{SA}_f(V)$.

c) We often call a module $W \in \text{SA}_f(V)$ an $\text{SA}_f$-submodule of $V$, and call a module $U \in \Sigma \text{SA}_f(V)$ a $\Sigma \text{SA}_f$-submodule of $V$, or a $\text{SA}_f$-sum in $V$, furthermore call a module $U \in \Sigma_f \text{SA}_f(V)$ a finite $\text{SA}_f$-sum in $V$.

d) The pair $(V, V)$ has $\text{SA}_f$-$\text{Kdim}$ equal to $-1$. For $W \neq V$, $\text{SA}_f$-$\text{Kdim}(V, W) = 0$ if for every descending chain

$$W_0 \supsetneq W_1 \supsetneq W_2 \supsetneq \cdots$$

in $\text{SA}_f(V; W)$ stops after finitely many steps. In general, $\text{SA}_f$-$\text{Kdim}(V, W)$ (if it exists) is the smallest ordinal $\theta$ for which for every chain (3.2) one must have $\text{SA}_f$-$\text{Kdim}(W_i, W_{i+1}) < \theta$ for almost all $i$. $\text{SA}_f$-$\text{Kdim}(V)$ (if it exists) is $\text{SA}_f$-$\text{Kdim}(V, 0)$.

The following results are really special cases of results in [5, 6] as indicated above.

**Proposition 3.9** ([6, 3.1.8]). Every $\text{SA}$-noetherian module has $\text{SA}$-$\text{Kdim}$.

**Proposition 3.10** ([6, 1.3.7]). Define the composition length $\ell(V, W)$ from $V$ to $W \subset V$ to be the length $m$ of a chain (if it exists)

$$V = W_0 \supsetneq W_1 \supsetneq \cdots \supsetneq W_m = W$$

for which, for each $i$, the chain $W_i \supsetneq W_{i+1}$ cannot be refined to $W_i \supsetneq W_i' \supsetneq W_{i+1}'$. Then $\ell(V, W)$ is well-defined (independent of the choice of chain (3.3)), and additive in the sense that

$$\ell(V, W) = \ell(V, W') + \ell(W', W), \quad \forall V \supsetneq W' \supsetneq W.$$

(This holds in either context, $\text{SA}(V)$ or $\text{SA}_f(V)$.) Analogously, we have

**Proposition 3.11.** $\text{SA}$-$\text{Kdim}(V) = \sup\{\text{SA}$-$\text{Kdim}(V, W), \text{SA}$-$\text{Kdim}(W)\}.$

3.2. $\text{SA}$-equivalence and $\text{SA}$-uniform modules.

A neuralgic point, for the sake of brevity often not adequately reflected in our terminology, is the fact that for $W \in \text{Mod}(V)$ the set $\text{SA}(W)$ is definitely bigger than $\text{Mod}(W) \cap \text{SA}(V)$ except in the case that $W \in \text{SA}(V)$.

We introduce on $\text{Mod}(V)$ an equivalence relation which plays a central role throughout the subsection. For the remainder of this section, the module $V$ is LZS.

**Definition 3.12.** Given $W_1, W_2 \in \text{Mod}(V)$, we say that $W_1$ and $W_2$ are $\text{SA}$-equivalent (in $V$) if for any $S \in \text{SA}(V)$ either $W_1 \cap S = W_2 \cap S = \{0\}$ or both $W_1 \cap S$ and $W_2 \cap S$ are nonzero (where “0” means the zero module $\{0_V\}$ ). We then write $W_1 \sim W_2$. In the rare case where a second module $V'$ is under consideration and $W_1, W_2$ are submodules of $V'$, we speak more precisely about the above equivalence relation as an $\text{SA}(V)$-equivalence, or specify “in $V'$”.

$\text{SA}$-equivalence is closely related to a notion of “$\text{SA}$-essential extension” of $R$-modules, to be defined now, which vaguely resembles the all-important notion of “essential extension” in the theory of modules over rings.
Definition 3.13. If \( W, W' \in \text{Mod}(V) \), we say that \( W' \) is an **SA-essential extension** of \( W \) (in \( V \)), and write \( W \subseteq_e W' \), if \( W \subseteq W' \) and for \( S \cap W \neq 0 \) every \( S \in \text{SA}(V) \) with \( S \cap W' \neq 0 \).

Note in particular that \( W \).

Remark 3.14. If \( W, W' \) are submodules of \( V \) with \( W \subseteq W' \), then \( W \subseteq_e W' \) means the same as \( W \sim_e W' \).

We list easy facts about SA-equivalence and SA-extensions.

Lemma 3.15. Assume that \( W_1, W_2, X \) are submodules of \( V \) with \( W_1 \subseteq X \subseteq W_2 \). Then the following are equivalent.

1. \( W_1 \subseteq_e W_2 \).
2. \( W_1 \subseteq_e X \) and \( X \subseteq_e W_2 \).

*Proof.* (1) \( \Rightarrow \) (2): If \( T \in \text{SA}(V) \) with \( T \cap X \neq 0 \), then \( T \cap W_2 \neq 0 \), whence \( T \cap W_1 \neq 0 \). This proves that \( W_1 \subseteq_e X \). We have \( X \sim_e W_1 \sim_e W_2 \), and so \( X \sim_e W_2 \), whence \( X \subseteq_e W_2 \).

(2) \( \Rightarrow \) (1): We have \( W_1 \sim_e X \sim_e W_2 \), whence \( W_1 \sim_e W_2 \), and so \( W_1 \subseteq_e W_2 \).

\( \square \)

Remark 3.16. Assume that \( (W_i \mid i \in I) \) and \( (W'_i \mid i \in I) \) are families in \( \text{Mod}(V) \) with \( W_i \sim_e W'_i \) for every \( i \in I \). Then

\[ \sum_{i \in I} W_i \sim_e \sum_{i \in I} W'_i. \]  

(3.4)

*Proof.* Let \( S \in \text{SA}(V) \). Then

\[ S \cap \left( \sum_i W_i \right) = \sum_i S \cap W_i, \quad S \cap \left( \sum_i W'_i \right) = \sum_i S \cap W'_i, \]

and so

\[ S \cap \sum_i W_i \neq 0 \ \Leftrightarrow \ \exists_i S \cap W_i \neq 0 \ \Leftrightarrow \ \exists_i S \cap W'_i \neq 0 \ \Leftrightarrow \ S \cap \sum_i W'_i \neq 0. \]

Note in particular that

\[ W \sim_e W' \ \Rightarrow \ W + X \sim_e W' + X \]  

(3.5)

for any \( X \in \text{Mod}(V) \).

Remark 3.17. Let \( W_1, W_2 \in \text{Mod}(V) \) and \( S \in \text{SA}(V) \) be given. Then

\[ W_1 \sim_e W_2 \ \Rightarrow \ S \cap W_1 \sim_e S \cap W_2. \]

*Proof.* If \( T \in \text{SA}(V) \) and \( W_1 \sim_e W_2 \), then \( S \cap T \in \text{SA}(V) \), and so

\[ (W_1 \cap S) \cap T \neq 0 \ \Leftrightarrow \ (W_2 \cap S) \cap T \neq 0. \]

\( \square \)

Proposition 3.18. Assume that \( W_1 \) and \( W_2 \) are SA-submodules of \( V \). Then

\[ W_1 \sim_e W_2 \ \Leftrightarrow \ W_1 \cap W_2 \subseteq_e W_1 \quad \text{and} \quad W_1 \cap W_2 \subseteq_e W_2. \]

\( \text{It may seem appropriate to reserve the letter } ^e \text{ for a straight generalization of “essential extensions” to modules over semirings, as defined in } [2] \text{ p.95} \) (there called “essential module-monomorphisms”), and to label SA-equivalences and SA-extensions by “sae” instead of “e”. But in the present paper the true essential extensions do not show up, and so we feel free to use the simpler label “e”.

2
Theorem 3.23. Let

\[ W_1 \cap W_2 \subset_e W_1, \ W_1 \cap W_2 \subset_e W_2. \]

Proof. \((\Leftarrow)\): By use of Remark 3.14 we see that \(W_1 \cap W_2 \sim_e W_1\) and \(W_1 \cap W_2 \sim_e W_2\), whence \(W_1 \sim_e W_2\).

\((\Rightarrow)\): Using Remark 3.17 we see that \(W_1 \cap W_2 \sim_e W_2 \cap W_2 = W_2\) and in the same way that \(W_1 \cap W_2 \sim_e W_1\). Thus, again by Remark 3.14,

\[ W_1 \cap W_2 \subset_e W_1, \ W_1 \cap W_2 \subset_e W_2. \]

\[ \square \]

Proposition 3.19. Assume that \((V_i \mid i \in I)\) is an orthogonal family in \(SA(V)\), and furthermore that \((W_i \mid i \in I)\) is a family in \(\text{Mod}(V)\) with \(W_i \subset V_i\) for each \(i \in I\). Then

\[ \sum_i W_i \subset_e \sum_i V_i \iff \forall i \in I: W_i \subset_e V_i. \]

Proof. \((\Rightarrow)\): Pick some \(k \in I\). Then

\[ V_k \cap \left( \sum_i W_i \right) = \sum_i V_k \cap W_i = V_k \cap W_k = W_k, \]

and \(V_k \cap \left( \sum_i V_i \right) = V_k\). We conclude by means of Remarks 3.14 and 3.17 that \(W_k \subset_e V_k\).

\((\Leftarrow)\): We have \(W_i \sim_e V_i\) for each \(i \in I\) and conclude by Remark 3.10 that \(\sum_i W_i \sim_e \sum_i V_i\), whence by Remark 3.14 that \(\sum_i W_i \subset_e \sum_i V_i\). (Here we did not need the orthogonality assumption on \((V_i \mid i \in I)\).) \[ \square \]

Lemma 3.20. If \(V\) is \(SA\)-artinian, Then \(\text{soc}_{SA}(V) \subset_e V\).

Proof. It is immediate that every non-zero \(S \in SA(V)\) contains a minimal non-zero \(SA\)-module \(T_i\). Thus \(S \cap \text{soc}_{SA}(V) \neq 0\). \[ \square \]

Definition 3.21. We call a submodule \(W\) of \(V\) \(SA\)-uniform (in \(V\)), if for every \(S \in SA(V)\) with \(S \cap W \neq 0\) the extension \(S \cap W \subset W\) is \(SA\)-essential (in \(V\)). We denote the set of all these submodules \(W\) of \(V\) by \(\text{Mod}_u(V)\) and its subset \(SA(V) \cap \text{Mod}_u(V)\) by \(SA_u(V)\). \[ \square \]

Note that the zero submodule 0 is \(SA\)-uniform in \(V\), and furthermore, that \(V \in \text{Mod}_u(V)\) iff \(S \cap T \neq 0\) for any two non-zero \(SA\)-submodules \(S, T\) of \(V\).

Theorem 3.22. \(\text{Mod}_u(V)\) is a union of full \(SA\)-equivalence classes in \(\text{Mod}(V)\). In other words, if \(W, W' \in \text{Mod}(V)\) are \(SA\)-equivalent (in \(V\)) and \(W\) is \(SA\)-uniform, then \(W'\) is \(SA\)-uniform.

Proof. Let \(S \in SA(V)\) be given with \(S \cap W' \neq 0\). From \(W \sim_e W'\) we conclude by Remark 3.17 that \(W \cap S \sim_e W' \cap S\), whence \(W \cap S \neq 0\), and so \(W \cap S \sim_e W\). Since also \(W \sim_e W'\) we conclude that \(W' \cap S \sim_e W'\). \[ \square \]

Theorem 3.23. Let \(\xi\) be an \(SA\)-equivalence class in \(\text{Mod}_u(V)\). Then there exists a unique member \(M(\xi)\) of \(\xi\) such that

\[ \xi = \{ W \in \text{Mod}(V) \mid W \subset_e M(\xi) \}. \tag{3.6} \]

\[ ^3\text{These notions can be viewed in terms of the general theory from [6]. By [6, 3.2.4] any pair \((M, N)\) with \(SA\)-Kdim has \(SA\)-critical submodules (cf. Definition 3.5). By [6, 3.2.6] every \(SA\)-critical module is \(SA\)-uniform.} \]
Proof. We choose a labeling of all elements of $\xi$, $\xi = (W_i \mid i \in I)$, and fix an index $0 \in I$. We then define $M(\xi) = \sum_{i \in I} W_i$. Since $W_i \sim_e W_0$ for every $i \in I$, we conclude by Remark 3.16 that 

$$M(\xi) \sim_e \sum_{i \in I} W_0 = W_0.$$ 

Thus $M(\xi) \in \xi$, and more precisely $M(\xi)$ is the unique maximal element of the poset $\xi$. If $W$ is a submodule of $M(\xi)$ then $W \in \xi$ iff $W \sim_e M(\xi)$ iff $W \subset_e M(\xi)$. □

By this theorem the $M(\xi)$ are precisely all maximal SA-uniform submodules of $V$.

We know nearly nothing about the equivalence classes $\xi$ in $\text{Mod}_u(V)$ with $\xi \cap \text{SA}(V) = \emptyset$, but when $\xi$ contains SA-submodules of $V$ we get more insight about $\xi$ (than provided by Theorems 3.22 and 3.23) by SA-restricting $\xi$ to $\Sigma \text{SA}(V) \cap \text{Mod}_u(V)$, as we explain now. We first give a description of the SA-uniform modules in $\Sigma \text{SA}(V)$.

**Proposition 3.24.** Let $(W_i \mid i \in I)$ be a family of non-zero SA-submodules of $V$. The following are equivalent.

1. $U := \sum_{i \in I} W_i$ is SA-uniform in $V$.
2. All $W_i \in \text{SA}_u(V)$, and $W_i \sim_e W_j$ for $i \neq j$.
3. All $W_i \in \text{SA}_u(V)$, and $W_i \cap W_j \neq \emptyset$ for $i \neq j$.

Proof. (1) $\Rightarrow$ (2): Since $W_i$ is in $\text{SA}(V)$ and $W_i \neq 0$, we have $W_i = W_i \cap U \subset_e U$, whence $W_i \sim_e U$ and $W_i$ is SA-uniform (cf. Theorem 3.22). It follows that $W_i \sim_e W_j$ for $i \neq j$.

(2) $\Leftrightarrow$ (3): Evident.

(2) $\Rightarrow$ (1): Let $S \in \text{SA}(V)$ and $S \cap U \neq \emptyset$. We have $S \cap U = \sum_{i \in I} S \cap W_i$, and so there exists $i \in I$ with $S \cap W_i \neq 0$, implying $S \cap W_i \subset_e W_i$. Fixing an index $0 \in I$ we have $U = \sum_{i \in I} W_i \sim_e \sum_{i \in I} W_0 = W_0$ and conclude by Theorem 3.22 that $U$ is SA-uniform. □

Assume now that $\xi$ is an SA-equivalence class in $\text{Mod}_u(V) \setminus \{0\}$ with $\xi \cap \text{SA}(V) \neq \emptyset$. We write $\xi \cap \text{SA}(V) = \xi \cap \text{SA}_u(V) = \{W_i \mid i \in J\}$ and define

$$U(\xi) := \sum_{i \in J} W_i \in \Sigma \text{SA}(V).$$

(3.7)

Choosing an index $0 \in J$ we have

$$U(\xi) \sim_e \sum_{i \in J} W_0 = W_0.$$ 

(3.8)

Thus $U(\xi)$ is the unique biggest module in the set $\xi \cap \Sigma \text{SA}(V)$.

**Proposition 3.25.** Assume again that $\xi$ contains a non-zero SA-submodule of $V$. Then

$$\xi \cap \text{SA}(V) = \{W \in \text{SA}_u(V) \mid W \neq 0, W \subset U(\xi)\},$$

(3.9)

$$\xi \cap \Sigma \text{SA}(V) = \{U \in \Sigma \text{SA}(V) \mid U \subset_e U(\xi)\}.$$ 

(3.10)

If in addition $V$ is SA-artinian, then $\xi \cap \Sigma \text{SA}(V)$ contains a smallest module $P(\xi)$, and

$$\xi \cap \Sigma \text{SA}(V) = \{U \in \Sigma \text{SA}(V) \mid P(\xi) \subset_e U \subset U(\xi)\}.$$ 

(3.11)

Proof. In (3.9) the inclusion ”$\subset$” is obvious, while “$\subset$” follows from (3.8). If $U \in \Sigma \text{SA}(V)$ and $U \in \xi$ then $U \subset U(\xi)$ and $U \sim_e U(\xi)$, whence $U \subset_e U(\xi)$. Thus (3.10) is evident. If $V$ is SA-artinian, then the set $\xi \cap \Sigma \text{SA}(V)$ contains a smallest module $P(\xi)$ and so $\xi \cap \text{SA}(V)$
is the set of all $U \in \Sigma \text{SA}(V)$ with $P(\xi) \subset_e U \subset_e U(\xi)$. Since we know that $P(\xi) \subset_e U(\xi)$ the “e” in these inclusions can be omitted.

**Lemma 3.26.** Assume that $S$ and $T$ are non-zero SA-uniform SA-submodules of $V$. Then $S \sim_e T$ iff $S \cap T \neq 0$.

**Proof.** If $S \sim_e T$, then $S \cap T \subset_e S$ and thus certainly $S \cap T \neq 0$. Conversely, if $S \cap T \neq 0$ then, due to the SA-uniformity of $S$ and $T$, we have $S \cap T \subset_e S$ and $S \cap T \subset_e T$, and so $S \sim_e T$. □

**Theorem 3.27.**

a) Assume that $(T_i \mid i \in I)$ is a maximal orthogonal family of non-zero SA-uniform submodules of $V$. Then $(T_i \mid i \in I)$ is a system of representatives of all SA-equivalence classes in $\text{SA}_u(V) \setminus \{0\}$.

b) If $(S_j \mid j \in J)$ is a second such family, then there is a bijection $\lambda: I \rightarrow J$ with $T_i \sim_e S_{\lambda(i)}$ for all $i \in I$, and $\sum_{i \in I} T_i \sim_e \sum_{j \in J} S_j$.

**Proof.** This is an immediate consequence of the preceding Lemma 3.26. □

We are ready to define an invariant for $R$-modules lacking zero sums.

**Definition 3.28.** The **SA-uniformity dimension** $\dim_{\text{sau}}(V)$ of $V$ is the cardinality of the set of all SA-equivalence classes of nonzero SA-uniform submodules of $V$. In other terms, $1 + \dim_{\text{sau}}(V) = \text{card}(\text{SA}_u(V)/\sim_e)$.

(In particular, $\dim_{\text{sau}}(V) = 0$ iff $V$ does not contain any non-zero SA-uniform submodule.)

Theorem 3.27 provides the following more elementary description of this invariant.

**Corollary 3.29.** $\dim_{\text{sau}}(V)$ is the cardinality $|I|$ of any maximal orthogonal family $(T_i \mid i \in I)$ of non-zero SA-uniform submodules of $V$.

**Theorem 3.30.** Assume that $(V_\lambda \mid \lambda \in \Lambda)$ is a family of SA-submodules of $V$ with $\sum_{\lambda \in \Lambda} V_\lambda = V$.

a) Then $\dim_{\text{sau}}(V) \leq \sum_{\lambda \in \Lambda} \dim_{\text{sau}}(V_\lambda)$. \hspace{1cm} (3.12)

b) If in addition the family $(V_\lambda)$ is orthogonal (i.e., $V_\lambda \cap V_\mu = 0$ for $\lambda \neq \mu$), then we have equality, $\dim_{\text{sau}}(V) = \sum_{\lambda \in \Lambda} \dim_{\text{sau}}(V_\lambda)$. \hspace{1cm} (3.13)

**Proof.** a): Let a non-zero SA-uniform module $S$ be given. Then $S = \sum_{\lambda \in \Lambda} S \cap V_\lambda$, and thus $S \cap V_\lambda \neq 0$ for at least one index $\lambda$. This implies that $S \cap V_\lambda \in \text{SA}_u(V_\lambda)$ and $S \sim_e S \cap V_\lambda$. Thus the natural map $\bigcup_{\lambda \in \Lambda} (\text{SA}_u(V_\lambda) \setminus \{0\})/\sim_e \rightarrow (\text{SA}_u(V) \setminus \{0\})/\sim_e$ is surjective. Comparing cardinalities gives the first claim (3.12).
b): If $V_\lambda \cap V_\mu = 0$ for $\lambda \neq \mu$, then any two non-zero modules $S \in \text{SA}(V_\lambda)$, $T \in \text{SA}(V_\mu)$ have intersection zero and thus certainly are not SA-equivalent. Thus now the map (*) is also injective, and in (3.12) holds equality. □

**Remark 3.31.** It is clear that for every SA-submodule $V'$ of $V$ we have

$$\dim_{\text{sau}}(V') \leq \dim_{\text{sau}}(V).$$

Thus we can complement (3.12) by the inequality

$$\sup_{\lambda \in \Lambda} \dim_{\text{sau}}(V_\lambda) \leq \dim_{\text{sau}}(V). \quad (3.14)$$

If $\dim_{\text{sau}}(V)$ is infinite it follows from (3.12), (3.14) that

$$\dim_{\text{sau}}(V) = \max_{\lambda \in \Lambda} \dim_{\text{sau}}(V_\lambda) = \sum_{\lambda \in \Lambda} \dim_{\text{sau}}(V_\lambda).$$

In a similar vein we see, in the case that the set $\Lambda$ is finite, that $\dim_{\text{sau}}(V)$ is finite iff $\dim_{\text{sau}}(V_\lambda)$ is finite for each $\lambda$. Then (3.13) holds iff the family $(V_\lambda \mid \lambda \in \Lambda)$ is orthogonal.

4. SA-f-hereditary modules with SA-Kdim

By working only with finitely generated SA-submodules of $V$, we obtain results on a wide class of submodules $U$ of $V$, to be put to use in §3 and §6. Throughout §4–§6 we assume for simplicity that the $R$-module $V$ is Lzs.

**Definition 4.1.**

a) We say that the $R$-module $V$ is SA-f-hereditary if for any submodule $W \subset V_f(V)$, $W' \subset \text{SA}_f(V)$ for all $W' \subset \text{SA}(V)$ with $W' \subset W$.

b) We say that $V$ is finitely SA-accessible (= SAF-accessible for short) if $V$ is both SA-f-hereditary and SA-Kdim(V) exists.

Note that if $V$ is finitely generated and SA-f-hereditary, then $\text{SA}(V) = \text{SA}_f(V)$. We present some ways to obtain new SAF-accessible modules from old ones.

**Proposition 4.2.** Assume that $V$ is an $R$-module and $(V_i \mid i \in I)$ is a family of submodules of $V$ with $V = \bigcup_{i \in I} V_i$. Assume that this family is upwardly directed, i.e., for every $i, j \in I$ there is some $k \in I$ with $V_i \subset V_k$, $V_j \subset V_k$. Then, if every $V_i$ is SA-f-hereditary, $V$ also is SA-f-hereditary; and if every $\text{SA}_f$-$\text{Kdim}(V_i) \leq \theta$, then $V$ also has $\text{SA}_f$-$\text{Kdim} \leq \theta$.

**Proof.** a): Assume that the $V_i$ are SA-f-hereditary. Let $W \subset \text{SA}_f(V)$, $W' \subset \text{SA}(V)$ and $W' \subset W$. Since $W$ is finitely generated, there exists some $i \in I$ with $W \subset V_i$. Both $W$ and $W'$ are in $\text{SA}(V)$, and so are $W_i$. Because $V_i$ is SA-f-hereditary and $W$ is finitely generated, $W_i$ also is finitely generated.

b): Assume now that $V$ has $\text{SA}_f$-$\text{Kdim} \leq \theta$. Given a descending chain $(W_i \mid i \in I)$ in $\text{SA}_f(V)$ then for any $W_0$ there exist some $i \in I$ with $W_0 \subset V_i$. By the same argument as above all $W_i \in \text{SA}_f(V_i)$ for all $i > i_0$. Since $\text{SA}_f$-$\text{Kdim}(V_i) \leq \theta$, the chain is $\theta$-stable. □

For later reference we also quote an obvious fact.

**Lemma 4.3.** Assume again that $V$ is the union of an upward directed family $(V_i \mid i \in I)$ of submodules. If $U$ is a finitely generated submodule of $V$ then $U \subset V_i$ for some $i \in I$.

**Proposition 4.4.** Assume that a direct decomposition $V = \bigoplus_{i \in I} V_i$ of an $R$-module $V$ is given. If each $V_i$ is SA-f-hereditary, then $V$ is SA-f-hereditary. If each $\text{SA}_f$-$\text{Kdim}(V_i) \leq \theta$, then $V$ has $\text{SA}_f$-$\text{Kdim} \leq \theta$. 
Proof. a) It is immediate from the definition of the summand absorbing property (cf. \((\text{SA})\)) that the SA-submodules of \(V\) are the direct sums \(\bigoplus_{i \in I} W_i\) with each \(W_i\) an SA-submodule of \(V_i\). It follows by use of Lemma 4.3 that the finitely generated SA-submodules of \(V\) are the direct sums \(\bigoplus_{i \in J} W_i\) with \(W_i \in \text{SA}_f(V_i)\) and \(J \subset I\) finite.

b) If the modules \(V_i\) are \(\text{SA}_f\)-hereditary and \(W = \bigoplus_{i \in J} W_i \in \text{SA}_f(V)\), then every SA-submodule \(W'\) of \(W\) has the form \(W' = \bigoplus_{i \in I} W'_i\) with \(W'_i \subset W_i\), and so \(W'_i \in \text{SA}_f(V_i)\), whence \(W' \in \text{SA}_f(V)\). Thus \(V\) is \(\text{SA}_f\)-hereditary.

c) Assume now that each \(\text{SA}_f\)-Kdim \((V_i) \leq \theta\) and \(I\) is finite. Let \((W'_k \mid k \in \mathbb{N}_0)\) be a decreasing chain in \(\text{SA}_f(V)\). We want to verify that this chain is \(\theta\)-stable. We have \(W'_k = \bigoplus_{i \in I} W'_{i,k}\) with \(W'_{i,k} \in \text{SA}_f(V_i)\). Since \(\text{SA}_f\)-Kdim \((V_i) \leq \theta\), each chain \((W'_{i,k} \mid k \in \mathbb{N}_0)\) is \(\theta\)-stable.

It follows that the chain \((W'_k)\) is \(\theta\)-stable. This proves that \(V\) has \(\text{SA}_f\)-Kdim \(\leq \theta\) for \(I\) finite. If \(I\) is infinite, and all \(\text{SA}_f\)-Kdim \((V_i) \leq \theta\), then for every finite \(J \subset I\) the submodule \(V_J := \bigoplus_{i \in J} V_i\) has \(\text{SA}_f\)-Kdim \(\leq \theta\), as proved. Invoking Proposition 4.2 we see that \(V\) has \(\text{SA}_f\)-Kdim \(\leq \theta\). \(\square\)

**Theorem 4.5.** Assume that \(U\) is an \(\Sigma \text{SA}_f\)-submodule of an \(R\)-module \(V\).

a) If \(V\) is \(\text{SA}_f\)-hereditary, then \(U\) is \(\text{SA}_f\)-hereditary, and

\[
\Sigma \text{SA}_f(U) = \{X \in \Sigma \text{SA}_f(V) \mid X \subset U\}. \tag{4.1}
\]

b) If \(V\) is \(\text{SA}_f\)-accessible then \(U\) is \(\text{SA}_f\)-accessible.

Proof. We choose a family \((W_i \mid i \in I)\) in \(\text{SA}_f(V)\) with \(U = \sum_{i \in I} W_i\).

a): Let \(W \in \text{SA}_f(U)\), \(W' \in \text{SA}(U)\) and \(W' \subset W\). Then

\[
W = \sum_{i \in I} W \cap W_i, \quad W' = \sum_{i \in I} W' \cap W_i.
\]

Every \(W_i\) is in \(\text{SA}(U)\) and so \(W \cap W_i\) and \(W' \cap W_i\) are in \(\text{SA}(U)\). These modules are contained in the \(\text{SA}\)-submodule \(W_i\) of \(U\) and so are \(\text{SA}\) in \(W_i\). Since \(W_i\) is \(\text{SA}\) in \(V\), they are \(\text{SA}\) in \(V\). Moreover, since \(V\) is \(\text{SA}_f\)-hereditary, all the modules \(W \cap W_i\), \(W' \cap W_i\) are finitely generated. Since \(W = \sum_{i \in I} W \cap W_i\) this proves that \(W\) is an \(\text{SA}_f\)-sum in \(V\), whence \(\text{SA}_f(U) \subset \Sigma \text{SA}_f(V)\), and thus

\[
\Sigma \text{SA}_f(U) \subset \Sigma \text{SA}_f(V).
\]

On the other hand, every \(\text{SA}_f\)-submodule \(X\) of \(V\) which is contained in \(U\) is a \(\text{SA}_f\)-sum in \(U\). This proves assertion \((4.1)\).

Moreover, if \(I\) is finite, we conclude from \(W' = \sum_{i \in I} W' \cap W_i\) that \(W' \in \text{SA}_f(U)\), since all \(W' \cap W_i \in \text{SA}_f(U)\). This proves for \(I\) finite that \(U\) is \(\text{SA}_f\)-hereditary. If \(I\) is infinite, then the \(R\)-module \(U_J := \sum_{i \in J} W_i\) is \(\text{SA}_f\)-hereditary for every finite \(J \subset I\). Invoking Proposition 4.2 we see that \(U\) is \(\text{SA}_f\)-hereditary.

b): Assume now that \(V\) is \(\text{SA}_f\)-accessible. We first consider the case that \(I\) is finite. We proceed as in the last part of the proof of Proposition 4.4. Assume that \((W'_k \mid k \in \mathbb{N}_0)\) is a decreasing chain in \(\text{SA}_f(U)\) with \(W'_0 = W\). For every \(i \in I\) the modules \(W_i \cap W'_k\) are \(\text{SA}_f\)-submodules of \(V_i\), as just proved, and so \((W_i \cap W'_k \mid k \in \mathbb{N}_0)\) is a decreasing chain in \(\text{SA}_f(V)\). Since \(V\) has \(\text{SA}_f\)-Kdim \(\leq \theta\), all these chains \((W'_i \cap W'_k \mid k \in \mathbb{N})\) with \(i\) running through the finite set \(I\), are \(\theta\)-stable. Since \(W'_k = \sum_{i \in I} W_i \cap W'_k\) it follows that the chain \((W'_k \mid k \in \mathbb{N})\) is \(\theta\)-stable. This proves that \(U\) has \(\text{SA}_f\)-Kdim \(\leq \theta\) and so is \(\text{SA}_f\)-accessible.

If \(I\) is infinite, then for any finite \(J \subset I\) the module \(U_J := \sum_{i \in J} W_i\) is \(\text{SA}_f\)-accessible, and so \(U\) is \(\text{SA}_f\)-accessible by Proposition 4.2. \(\square\)
Given a submodule $U$ and an $SA$-submodule $W$ of $V$, we say that $U$ dominates $W$, if $W$ is contained in the convex hull $U$ of $U$ in $V$, i.e., in the smallest $SA$-submodule of $V$ containing $U$, cf. Proposition [1.8]

Let $On$ denote the set of ordinal numbers of cardinality $\leq 2^{\aleph_0}$. In [6] we will gain some insight in the dominance relation for submodules $W, U \in \Sigma SA_f(V)$ (cf. Notations [3.8]) by use of a "height function"

$$h : \Sigma SA_f(V) \to On,$$

to be established now. The modules $V$, in which this works well, are the SAF-accessible modules defined in [4].

We first construct a family $(V_t^0 \mid t \leq \omega)$ and a strictly increasing chain $V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_\omega$ in $\Sigma SA_f(V)$, indexed by ordinal numbers. We proceed by transfinite induction. We do not assume anything about the $R$-module $V$, except that $V$ is LZS, as always, but it seems that the construction is really useful only if $V$ has some SA-Kdim.

**Construction 5.2.** Let

$$V_0 := V_0^0 = \{0\},$$

$$V_1^0 := \text{the sum of all minimal } W \neq 0 \text{ in } SA_f(V),$$

$$V_1 := V_0 + V_1^0 = V_1^0.$$

Assume that $V_s^0$ and $V_s$ are already defined for all $s < t \in On$.

A) Assume that $t$ is not a limit ordinal, so $t = \tau + 1$ for a unique $\tau \in On$.

Case I: There exists no $SA$-critical $W \in SA_f(V)$ with $W \not\subseteq V_\tau$. The construction stops with $\omega := \tau$.

Case II: Otherwise. We define $V_t^0 = V_{\tau+1}^0$ as the sum of all $W \in SA_f(V)$ which are $SA$-critical and not contained in $V_\tau$, and define $V_t := V_t + V_t^0$.

B) Assume that $t$ is a limit ordinal. We put $V_t^0 = \{0\}$, $V_t = \bigcup_{s < t} V_s$.

Note that $V_s \subsetneq V_t$ for all $s < t \leq \omega$, and that all modules $V_t^0$ and $V_t$ are elements of $\Sigma SA_f(V)$. The strictly ascending chain $(V_t)$ stops with a module $V_\omega, \omega \in On$, which may or may not be a limit ordinal.

**Definition 5.3.**

a) The height $h_V(U)$ of a submodule $U$ of $V$ with $U \subseteq V_\omega$ is the minimum of all ordinals $t \leq \omega$ with $U \subset V_t$. This minimum exists, since the set $\{t \in On \mid t \leq \omega\}$ is well ordered.

b) Clearly, if $U \subseteq V_\omega$ and $t \in O_n, t \leq \omega$, then

$$h_V(U) \leq t \iff U \subset V_t.$$ (5.1)

We call the family $(V_t \mid t \leq \omega)$ the height filtration in $V$ (or: of $V_\omega$).

Given any module $V$ over a semiring $R$ we denote by $\bar{V}$ the sum of all $W \in SA_f(V)$. This is the top element of the poset $\Sigma SA_f(V)$. Clearly $\bar{V}$ is also the union of all $U \in \Sigma_f SA_f(V)$ (cf. Notations [3.8]). If $V$ is finitely generated then, of course, $\bar{V} = V$.  

**5. The height filtration**
Theorem 5.4. Assume that $\text{SA-Kdim}(V) \leq \theta$. Then $V_\omega$ coincides with the maximal $\text{SA}_f$-sum $\tilde{V}$ in $V$.

**Proof.** Clearly $V_\omega \subset \tilde{V}$. Suppose that $V_\omega \neq \tilde{V}$. Then there exists some $W \in \text{SA}_f(V)$ with $W \not\subset V_\omega$. Since $\text{SA-Kdim}(V) \leq \theta$, there exists some $\text{SA}$-critical $W' \in \text{SA}_f(V)$ with $W' \subset W$, with $W' \not\subset V_\omega$. But this contradicts the definition of $V_\omega$. Thus $V_\omega = \tilde{V}$. \hfill $\square$

In the following we usually write $h(U)$ instead of $h_V(U)$, whenever it is clear from the context, which $R$-module $V$ is under consideration. We concentrate on a study of the heights of the $\text{SA}_f$-sums in $V$. We will assume almost everywhere that $\text{SA-Kdim}(V) \leq \theta$, so that we catch all $\text{SA}_f$-sums in the height filtration due to Theorem 5.4.

Proposition 5.5. Assume that $\text{SA-Kdim}(V) \leq \theta$. Assume furthermore that $(U_\lambda \mid \lambda \in \Lambda)$ is a family in $\Sigma \text{SA}_f(V)$ and $U := \sum_{\lambda \in \Lambda} U_\lambda$. Then

$$h(U) = \sup_{\lambda \in \Lambda} h(U_\lambda). \quad (5.2)$$

**Proof.** Let $t := h(U)$. Of course $h(U_\lambda) \leq t$ for all $\lambda \in \Lambda$. Suppose there exists an ordinal number $\tau < t$ with $h(U_\lambda) \leq \tau$ for all $\lambda \in \Lambda$. Then $U_\lambda \subset V_\tau$ for all $\lambda$, and so $U \subset V_\tau$. But this means that $h(U) \leq \tau$, a contradiction. Thus $t$ is the least upper bound of $(h(U_\lambda) \mid \lambda \in \Lambda)$. \hfill $\square$

Corollary 5.6. Assume again that $\text{SA-Kdim}(V) \leq \theta$ and $(U_\lambda \mid \lambda \in \Lambda)$ is a family in $\Sigma \text{SA}_f(V)$. Assume furthermore that the height of $U := \sum_{\lambda \in \Lambda} U_\lambda$ is not a limit ordinal. Then

$$h(U) = \max_{\lambda \in \Lambda} h(U_\lambda). \quad (5.3)$$

**Proof.** Let $t := h(U)$, $t_\lambda := h(U_\lambda)$ for $\lambda \in \Lambda$. Then $\sup_{\lambda \in \Lambda} t_\lambda = r$, as we have seen. If $t$ is not a limit ordinal, this implies that there exists $\lambda \in \Lambda$ with $t_\lambda = t$. \hfill $\square$

In the following proposition we do not need the assumption that $\text{SA-Kdim}(V) \leq \theta$.

Proposition 5.7. Assume that $U$ is a sum of finitely many finitely generated $\text{SA}$-submodules of $V$ (i.e., $U \in \Sigma_f \text{SA}_f(V)$, cf. Definition 3.8). Then $h(U)$ is not a limit ordinal.

**Proof.** $U$ has a finite system $S$ of generators, $S = \{s_1, \ldots, s_m\}$. For each $i \in \{1, \ldots, m\}$ there is a smallest ordinal $t_i$ with $s_i \in V_{t_i}$, and clearly $t_i$ is not a limit ordinal. Let $t_k$ denote the largest of the $t_i$. This is the smallest ordinal $\tau$ of $\leq \omega$ with $S \subset V_\tau$, whence $U \subset V_\tau$. Thus $h(U) = t_k$. \hfill $\square$

Definition 5.8.

a) Assume that $t$ is an ordinal number with $t \leq \omega$, and that $t$ is not a limit ordinal. As common we denote the ordinal number $\tau$ with $\tau + 1 = t$ by $t - 1$. We call a module $W \in \text{SA}_f(V)$ **$t$-critical**, if $W$ is $\text{SA}$-critical with $W \not\subset V_{t-1}$. We denote the set of all $t$-critical $\text{SA}_f$-modules in $V$ by $\text{SA}_t(V)$.

b) If $\tau \leq \omega$ is a limit ordinal we put $\text{SA}_\tau(V) := \emptyset$. We furthermore define

$$\text{SA}_{\min}(V) := \bigcup_{t \leq \omega} \text{SA}_t(V),$$

and we call the elements of this set the **height-critical $\text{SA}_f$-submodules** of $V$. 


Theorem 5.9. Assume that \( V \) is SAF-accessible, and that \( U \) is an SA-submodule of \( V \) of height \( h(U) = t \). Then for any \( \tau \leq t \) the following holds:

\[
\text{SA}_\tau(U) = \{ W \in \text{SA}_\tau(V) \mid W \subset U \}, \tag{5.4}
\]

\[
U_\tau = U \cap V_\tau. \tag{5.5}
\]

Proof. We verify this by induction on \( \tau \). For \( \tau = 0 \) both assertions are obvious. Let \( \tau > 0 \), and assume first that \( \tau \) is not a limit ordinal and [5.4], [5.5] are true for \( \tau - 1 \). If \( W \) is an SA-submodule of \( U \), then

\[
W \in \text{SA}_f(V) \iff W \in \text{SA}_f(U),
\]

since \( U \) is in \( \text{SA}(V) \). We conclude from \( U_{\tau-1} = U \cap V_{\tau-1} \) that \( W \in \text{SA}_\tau(U) \) iff \( W \not\subset U_{\tau-1} \) iff \( W \not\subset V_{\tau-1} \) iff \( W \in \text{SA}_\tau(V) \). This proves [5.4] for the ordinal \( \tau \). Let \( \{W_i \mid i \in I\} \) denote the set of all \( \tau \)-critical submodules of \( U \) and \( \{W_k \mid k \in K\} \) denote the set of \( \tau \)-critical submodules of \( V \) not contained in \( U \). Thus

\[
U_\tau = U_{\tau-1} + \sum_{i \in I} W_i,
\]

\[
V_\tau = V_{\tau-1} + \sum_{i \in I} W_i + \sum_{k \in K} W_k',
\]

whence

\[
U \cap V_\tau = U \cap V_{\tau-1} + \sum_{i \in I} W_i + \sum_{k \in K} U \cap W_k'
\]

\[
= U_{\tau-1} + \sum_{i \in I} W_i + \sum_{k \in K} U \cap W_k'
\]

\[
= U_\tau + \sum_{k \in K} U \cap W_k'.
\]

Now \( U \cap W_k' \in \text{SA}_f(V) \) since \( V \) is SA\(_f\)-hereditary and \( U \cap W_k' \not\subset W_k' \). Due to the \( \tau \)-criticality of \( W_k' \) it follows that \( U \cap W_k' \subset V_{\tau-1} \), and thus \( U \cap W_k' \subset U \cap V_{\tau-1} = U_{\tau-1} \), so that altogether we obtain that \( U \cap V_\tau = U_\tau \).

Assume finally that \( \tau \) is a limit ordinal. Then \( \text{SA}_\tau(U) = \text{SA}_\tau(V) = \emptyset \), and so [5.4] holds trivially. By induction hypothesis \( U \cap V_\sigma = U_\sigma \) for \( \sigma < \tau \). Thus

\[
U \cap V_\tau = U \cap \left( \sum_{\sigma < \tau} V_\sigma \right) = \sum_{\sigma < \tau} U \cap V_\sigma = \sum_{\sigma < \tau} U_\sigma = U_\tau,
\]

which proves [5.5].

Corollary 5.10. Assume again that \( V \) is SAF-accessible. Let \( U \in \Sigma \text{SA}_f(V) \). Recall from Theorem 4.5.(b) that \( U \) is SAF-accessible. Let \( t := h_V(U) \).

a) If \( U' \in \Sigma \text{SA}_f(U) \) then \( U' \in \Sigma \text{SA}_f(V) \) and \( h_U(U') = h_V(U') \). In particular \( t = h_U(U) \).

b) \( U \) is the sum of all modules \( W \in \text{SA}_\tau(V) \) with \( W \subset U \), \( \tau \leq t \).

Proof. a): The height \( h_U(U') \) is the minimal ordinal \( \tau \) such that \( U' \subset U_\tau \). Since \( U_\tau = U \cap V_\tau \) (cf. [5.5]), this is also the minimal ordinal \( \tau \) with \( U'' \subset V_\tau \), and so \( h_U(U') = h_V(U') \).

b): We have \( t = h_U(U) \). Now \( U \) is the sum of all \( W \in \text{SA}_\tau(U) \) with \( \tau \leq t \), as is clear by Construction [5.2] and Theorem 5.4. By [5.4] these are the \( W \in \text{SA}_\tau(V) \) with \( W \subset U \) and \( \tau \leq t \).
6. Primitive $SA_f$-modules

**Definition 6.1.** Assume that $SA$-Kdim($V$) ≤ θ.

a) We call a module $W \in SA_f(V)$ **primitive** in $V$ if $W$ is $\tau$-critical for some $\tau \leq \omega$ (and so $W \subset V^0_\tau$, $h(W) = \tau$), but $W \not\subset \hat{V}_{\tau-1}$ (i.e., $W$ is not dominated by $V_{\tau-1}$, cf. Definition 6.4). We define
\[
SA_{\tau,\text{prim}}(V) := \text{set of all primitive } W \in SA_f(V) \text{ of height } \tau.
\]
(6.1)

\[
SA_{\text{prim}}(V) := \bigcup_{\tau \leq \omega} SA_{\tau,\text{prim}}(V).
\]
(6.2)

b) If $T \in \Sigma SA_f(V)$, $h(T) = r$, we set for $\tau \leq t$
\[
SA_{\text{prim}}(T, V) = \{W \in SA_{\text{prim}}(V) \mid W \subset T\}
\]
and for $\tau \leq t$
\[
SA_{\tau,\text{prim}}(T, V) = \{W \in SA_{\tau,\text{prim}}(V) \mid W \subset T\}.
\]
(6.4)

**Theorem 6.2.** Assume that $V$ is SAF-accessible. Let $T, U \in \Sigma SA_f(V)$. Assume furthermore that all primitive $SA_f$-submodules of $V$, which are contained in $T$, are also contained in $U$. Then $T \subset \hat{U}$.

**Proof.** We know by Corollary 5.10(b) that $T$ is the sum of all $W \in SA_r(T)$ with $\tau \leq t := h(T)$. Furthermore, it is clear by Definition 7.1, that every $W \in SA_r(T)$ is dominated by the sum $X_{\tau}$ of all $W' \in SA_{\text{prim}}(T, V)$ with $h(W') \leq \tau$. Since we assume that every $W' \in SA_{\text{prim}}(T, V)$ is contained in $U$, it follows that $T \subset (\sum_{r \leq t} X_{\tau})^\wedge \subset \hat{U}$.

**Theorem 6.3.** Assume conversely that $T \subset \hat{U}$. Then all primitive $SA_f$-submodules of $V$ which are contained in $T$ are contained in $U$.

**Proof.** Let $W \in SA_{\tau,\text{prim}}(T)$ be given, i.e. $W \in SA_f(V)$, $h(W) = \tau$, $W \subset T$, $W$ primitive in $V$. We have $W \subset V^0_\tau$, $W \not\subset \hat{V}_{\tau-1}$, but $W \subset \hat{U}$. This is only possible if $W \subset U$ (and so $W \in SA_{\tau,\text{prim}}(U)$).

**Definition 6.4.** The **primitivity socle** $\text{prsoc}(T)$ of a module $T \in \Sigma SA_f(V)$ is the sum of all primitive $SA_f$-submodules $W$ of $V$ contained in $T$.

We state an immediate consequence of Theorems 6.2 and 6.3.

**Corollary 6.5.** Assume that $V$ is SAF-accessible. For modules $T, U \in \Sigma SA_f(V)$ the following are equivalent:

1) $\hat{T} \subset \hat{U}$,
2) $T \subset \hat{U}$,
3) $\text{prsoc}(T) \subset \text{prsoc}(U)$.

**Proof.** (1) ⇔ (2): Obvious, cf. Proposition 1.8.
(2) ⇔ (3): Clear by Theorems 6.2 and 6.3.

**Proposition 6.6.** Assume again that $V$ is SAF-accessible. Let $T \in \Sigma SA_f(V)$. The **primitivity socle** $\text{prsoc}(T)$ is the smallest module $U \in \Sigma SA_f(V)$ contained in $T$ which dominates $T$.

**Proof.** Let $T_0 := \text{prsoc}(T)$. By definition of the primitivity socle it is evident that $\text{prsoc}(T_0) = T_0$, and thus $\text{prsoc}(T) = \text{prsoc}(T_0)$. It follows by Corollary 6.2 that $\hat{T} = \hat{T}_0$, and so $T_0$ dominates $T$. If $U \in \Sigma SA_f(V)$ and $U \subset T \subset \hat{U}$, then $\hat{U} \subset \hat{T} \subset \hat{U}$, and so $\hat{U} = \hat{T}$. Again by Corollary 6.1 we conclude that $\text{prsoc}(U) = \text{prsoc}(T) = T_0$. Thus certainly $T_0 \subset U$.
7. Generating SA-submodules by use of additive spines

Given an $R$-module $V$ and a set $S$ of generators of $V$ we want to establish a new set $T$ of generators of $V$, which is “small” in some sense if $S$ is “small”, and gives us sets of generators of all SA-submodules $W$ of $V$ in a coherent way. Recall SA-adapted from Definition 3.6.

We will obtain a reasonable SA-adapted set of generators $T$ from a given set of generators $S$ by employing the so-called additive spine $M$ of a module (Definition 8.1) the semiring $R$ (Definition 7.2). In the special case that both $M$ and $S$ are finite it will turn out that also $T$ is finite, and so all SA-submodules $W$ of $V$ are generated by $|T|$ elements.

We first define additive spines of $R$, state basic facts about them, and give first examples.

Notation 7.1. Given (nonempty) subsets $A$, $B$ of $R$, we denote the set of products $ab$ with $a \in A$, $b \in B$ (or $A \cdot B$). Similarly, if $A \subset R$, $X \subset V$ then $AX$ denotes the set of products $ax$ with $a \in A$, $x \in X$. Furthermore $\sum A$ and $\sum X$ denote the set of all finite sums of elements of $A$ in $R$ and of $X$ in $V$ respectively. Admitting also the empty sum of elements of $A$ or $X$, we always have $0_R \in \sum A$, $0_V \in \sum X$. If necessary we write more precisely $\sum_R A$ and $\sum_V X$ instead of $\sum A$ and $\sum X$.

In this notation a set $S \subset V$ generates the $R$-module $V$ if $V = \sum RS$.

Definition 7.2. Given a subset $M$ of $R$,

a) We define the set

$$\tilde{M} := \{x \in R \mid \exists y, z \in R : yx \in M, zyx = x\},$$

which we call the halo of $M$ in $R$.

b) If the halo $\tilde{M}$ generates $R$ additively, i.e., $R = \sum \tilde{M}$, we call $M$ an additive spine of $R$.

We state some facts about halos which are immediate consequences of Definition 7.2.a.

Remarks 7.3.

i) $M \subset \tilde{M}$ for any set $M \subset R$.

ii) If $M \subset N \subset R$ then $\tilde{M} \subset \tilde{N}$.

iii) If $(M_i \mid i \in I)$ is a family of subsets of $R$, then

$$\left(\bigcup_{i \in I} M_i\right)\sim = \bigcup_{i \in I} \tilde{M}_i.$$

iv) $\{0\}\sim = \{0\}$ and $(M \setminus \{0\})\sim = \tilde{M}$.

Due to the last remark we may assume in any study of halos that $0 \in M$ or $0 \not\in M$, whatever is more convenient.

Here are the perhaps most basic examples of halos deserving interest.

Example 7.4. Let $M = \{1_R\}$. Then $\tilde{M}$ is the set of left invertible elements of $R$. Indeed, if $x \in \tilde{M}$, then there exists $y \in R$ with $yx = 1$. Conversely, if $x$ is left-invertible there exists $y \in R$ with $yx = 1$, and so $xyx = x$, which proves that $x \in \tilde{M}$. 
Example 7.5. Let $M = \{e\}$ with $e$ an idempotent of $R$. If $x \in \tilde{M}$, then there exist $y, z \in R$ with $yx = e$, $ze = x$. It follows that $xe = x$, yielding the von Neumann condition $xyx = x$. Conversely, if $yx = e$ and $xyx = x$, then clearly $x \in M$. This proves that
\[
\{e\}^\sim = \{x \in R \mid \exists y \in R : yx = e, xyx = x\}.
\]

Let $\text{Id}(R)$ denote the set of all idempotents of $R$. Starting from Example 7.5, we obtain the following fact.

Proposition 7.6. If $R$ is any semiring then
\[
\text{Id}(R)^\sim = \{x \in R \mid \exists y \in R : xyx = x\}.
\]

Proof. $\text{Id}(R)^\sim$ is the union of the sets $\{e\}^\sim$ with $e$ an idempotent of $R$ (cf. Remark 7.3.iii). Thus it is clear from Example 2.6 that for every $x \in \text{Id}(R)^\sim$ there exists some $y \in R$ with $xyx = x$.

Conversely, if $xyx = x$, then $yx \cdot yx = yx$, and so $e := yx$ is an idempotent of $R$. Moreover $xe = x$, and so $x \in \{e\}^\sim$. □

We state an immediate consequence of this proposition.

Corollary 7.7. For any subset $M$ of $R$ we have
\[
[M \cap \text{Id}(R)]^\sim = \{x \in \tilde{M} \mid \exists y \in R : xyx = x\},
\]
and $\tilde{M}$ is the disjoint union of this set and $[M \setminus \text{Id}(R)]^\sim$.

The set $[M \cap \text{Id}(R)]^\sim$ may be regarded as the “easy part” of the halo $\tilde{M}$.

We are ready for a central result.

Theorem 7.8. Assume that $S$ is a set of generators of a (left) $R$-module $V$, and $M$ is an additive spine of $R$. Then any SA-submodule $W$ of $V$ is generated by the set $W \cap (MS)$.

Proof. Since $V = \bigoplus R S$ and $R = \bigoplus \tilde{M}$, we have $V = \bigoplus \tilde{M} S$.

Let $w \in W$, $w \neq 0$, be given. Then
\[
w = \sum_{i=1}^{n} x_i s_i
\]
with $n \in \mathbb{N}$, $s_i \in S$, $x_i \in \tilde{M}$. Since $W$ is in $\text{SA}(V)$, it follows that $x_i s_i \in W$ for $1 \leq i \leq n$.

Now choose $y_i, z_i \in R$ such that $m_i := y_i x_i \in M$ and $x_i = z_i m_i$. Then
\[
y_i (x_i s_i) = m_i s_i \in W \cap (MS)
\]
and
\[
z_i m_i s_i = z_i y_i x_i s_i = x_i s_i.
\]

From (A) we obtain that
\[
w = \sum_{i=1}^{n} z_i (m_i s_i).
\]

We conclude from (B) and (C) that $W \cap (MS)$ generates $W$. □

Corollary 7.9. Assume that $R$ has a finite additive spine $M$ and $V$ has a finite set of generators $S$. Then every SA-submodule $W$ of $V$ is finitely generated, more precisely, generated by at most $|M| \cdot |S|$ elements (independent of the choice of $W$!!).
**Theorem 7.10.** Assume that $V$ is a module over a semiring $R$ which is additively generated by the set of its left invertible elements. Then every set of generators $S$ of $V$ is SA-adapted.

**Proof.** We read off from Example 7.4 that $\{1_R\}$ is an additive spine of $R$. So by Theorem 7.8 every SA-submodule $W$ of $V$ is generated by $W \cap S = W \cap (1_R S)$.

We take a look at additive spines of matrix semirings.

**Example 7.11.** Assume that $C$ is a semiring which is additively generated by $\{1_C\}$, 

$$C = \sum_{i=1}^{\infty} \{1_C\}.$$

In other terms, the unique homomorphism $\varphi : \mathbb{N}_0 \rightarrow C$ with $\varphi(1) = 1_C$ is surjective. Then the semiring 

$$R = M_n(C) = \sum_{i,j=1}^{n} C e_{ij}$$

of $(n \times n)$-matrices with entries in $C$, and $e_{ij}$ the usual matrix units, has the additive spine 

$$D := \{e_{11}, e_{22}, \ldots, e_{nn}\}.$$

Indeed, for every $j \in \{1, \ldots, n\}$

$$\{e_{jj}\}^\sim \supset \{e_{ij} \mid 1 \leq i \leq n\},$$

since $e_{jj} e_{ij} = e_{jj}, e_{ij} e_{jj} = e_{ij},$ and so $\tilde{D} = \bigcup_j \{e_{jj}\}^\sim$ contains the set $E := \{e_{ij} \mid 1 \leq i, j \leq n\}$ of all matrix units, which by the nature of $C$ generates $M_n(C)$ additively.

This example can be amplified to a theorem about additive spines in arbitrary matrix rings $M_n(A)$ by use of a general principle to “multiply” additive spines, which runs as follows:

**Proposition 7.12.** Assume that $R_1$ and $R_2$ are subsemirings of a semiring $R$, such that $R$ is additively generated by $R_1 R_2$, i.e., $R = \sum R_1 R_2$. Assume furthermore that the elements of $R_1$ commute with those of $R_2$. Assume finally that $M_i$ is an additive spine of $R_i$. Let $\tilde{M}_i$ denote the halo of $M_i$ in $R_i (i = 1, 2)$. Then $\tilde{M}_1 \tilde{M}_2$ is contained in the halo $(M_1 M_2)^\sim$ of $M_1 M_2$ in $R$, and $M_1 M_2$ is an additive spine of $R$.

**Proof.** Let $x_i \in \tilde{M}_i (i = 1, 2)$ be given. We have elements $y_i, z_i$ of $R_i$ with $m_i := y_i x_i \in M_i$ and $z_i m_i = x_i$. Now 

$$(y_1 y_2)(x_1 x_2) = (y_1 x_1)(y_2 x_2) = m_1 m_2$$

and 

$$(z_1 z_2)(m_1 m_2) = (z_1 m_1)(z_2 m_2) = x_1 x_2.$$ 

This proves that $x_1 x_2 \in (M_1 M_2)^\sim$. It follows that 

$$(\sum \tilde{M}_1) \cdot (\sum \tilde{M}_2) = R_1 R_2$$

and then that 

$$R = \sum \tilde{M}_1 \tilde{M}_2 = \sum \tilde{M}_1 \tilde{M}_2.$$
Theorem 7.13. Assume that $R$ is the semiring of $(n \times n)$-matrices over any semiring $A$, so

$$R := M_n(A) = \sum_{i,j=1}^{n} A e_{ij}$$

with the usual matrix units $e_{ij}$. Let $N$ be an additive spine of $A$. Then the set $M := \bigcup_{i=1}^{n} Ne_{ii}$, consisting of the diagonal matrices with entries in $N$, is an additive spine of $R$.

Proof. Let $C$ denote the smallest subsemiring of $A$, $C = \{ n \cdot 1_A \mid n \in \mathbb{N} \}$. We have seen that $R_1 := M_n(C)$ has the additive spine $D := \{ e_{ii} \mid 1 \leq i \leq n \}$ (Example 7.11). Let $R_2 := A \cdot 1_R$. This is the subsemiring of $R$ consisting of all matrices $aI$ with $a \in A$, where $I$ is the identity matrix. It has the additive spine $N \cdot 1_{R_2}$. Now $R = \sum R_1 R_2$, and the elements of $R_1$ commute with those of $R_2$. Thus, by Proposition 7.12 $R$ has the additive spine $D \cdot (N1_{R_2}) = \bigcup_{i=1}^{n} Ne_{ii}$. □

Recalling Theorem 7.8 we obtain

Theorem 7.14. Assume that $V$ is an $M_n(A)$-module, $A$ any semiring, and $S$ a system of generators of $V$. Assume furthermore that $N$ is an additive spine of $A$. Then any $SA$-submodule $W$ of $M_n(A)$ is generated by the set

$$W \cap \left( \bigcup_{i=1}^{n} Ne_{ii} \right) = \bigcup_{i=1}^{n} W \cap (Ne_{ii}).$$

If $N$ is finite then $W$ can be generated by at most $n \cdot |N|$ elements.

The proof of Theorem 2.14 can be seen in a much wider context, as we explain now.

Definition 7.15. Let $S = (S, \cdot)$ be a monoid, in multiplicative notation. We call a subset $T$ of $S$ a spine of $S$ (= monoid spine), if for any $s \in S$ there exist $s_1, s_2 \in S$ such that $t := s_1s \in T$ and $s_2t = s$.

Given any semiring $A$ and monoid $S = (S, \cdot)$ we denote, as common, the monoid-semiring of $S$ over $A$ by $A[S]$.

In the case that the monoid $S$ is without zero, i.e., $S$ does not contain an absorbing element $0$, $(0 : S = S \cdot 0 = 0$ for all $s \in S)$, the elements $x$ of $R := A[S]$ are the formal sums

$$x = \sum_{s \in S} a_s s,$$

with coefficients $a_s \in A$ uniquely determined by $x$, only finitely many non-zero. The multiplication is determined by the rule $(as) \cdot (bt) = (ab)(st)$ for $a, b \in A$, $s, t \in S$. Identifying $a = a \cdot 1_S$, $s = 1_A \cdot s$, we regard $A$ as a subsemiring of $R$ and $S$ as a submonoid of $(R, \cdot)$.

If the monoid $S$ has a zero $0 = 0_S$, we take for $R = A[S]$ the free $A$-module with base $S \setminus \{0\}$ and multiplication rule $(as) \cdot (bt) = (ab)(st)$ if $st \neq 0_S$, $(as)(bt) = 0$ otherwise. Now the nonzero elements of $R = A[S]$ are formal sums $\sum_{s \neq 0} a_s s$. We identify again $a = a \cdot 1_S$, $s = 1_A \cdot s$ for $s \in S \setminus \{0\}$, and now also $0_S = 0_A$. Then again $A$ becomes a subsemiring of $R$ and $S$ a submonoid of $(R, \cdot)$. We have $R = \sum AS$ in both cases.

Example 7.16. The matrix semiring $M_n(A)$ coincides with $A[S]$, where $S$ is the monoid $\{ e_{ij} \mid 1 \leq i, j \leq n \} \cup \{0\}$ with multiplication rule $e_{ij} e_{kl} = \delta_{jk} e_{il}$. Note that $S$ has the monoid spine $\{ e_{11}, \ldots, e_{nn} \} \cup \{0\}$.
Theorem 7.17. Assume that $S$ is a multiplicative monoid (with zero or without zero) and $T$ is a spine of $S$. Assume furthermore that $A$ is a semiring and $N$ is an additive spine of $A$. Then $N \cdot T$ is an additive spine of $A[S]$.

Proof. Let $R := A[S]$ and $R_1 := C[S] \subset R$, with $C$ the image of the (unique) homomorphism $\mathbb{N}_0 \to A$. It is obvious that $R_1 = \sum S$ and that $S$ is contained in the halo $\tilde{T}$ of $T$ in $R_1$. Thus $T$ is a spine of $R_1$. (In fact it can be verified that $\tilde{T} = \tilde{S} = S$.) Let $R_2 := A \subset R$. Then $R = \sum R_1 \cdot R_2$ and the elements of $R_1$ commute with those of $R_2$. The assertion follows from Proposition 7.12. \hfill $\Box$

8. Halos and additive spines in $R$-modules

Halos and additive spines can be defined and studied on any $R$-module instead of the semiring $R$ itself. Although at present perhaps of limited practical value, this will make the theory of generators of $SA$-submodules more transparent.

Definition 8.1. Assume that $S$ is a subset of $V$.

a) The Halo $\tilde{S}$ of $S$ in $V$ is the set of all $v \in V$ such that there exist $\lambda, \mu \in R$ with $\lambda v \in S$ and $\mu \lambda v = v$.

b) $S$ is called an additive spine of the $R$-module $V$ if $V$ is additively generated by $\tilde{S}$, $V = \sum \tilde{S}$.

Thus the additive spines on $R \cdot R$, i.e., of $R$ considered as left $R$-module, are the same objects as the additive spines on $R$ as defined in §2.

Example 8.2. If $S$ is a set of generators of the $R$-module $V$ and $M$ is an additive spine of $R$, then we know by Theorem 7.8 that $MS$ is an additive spine of $V$.

Theorem 7.8 generalizes as follows:

Theorem 8.3. Assume that $S$ is an additive spine of an $R$-module $V$. Then every $SA$-submodule $W$ of $V$ is generated by $W \cap S$, and moreover $W \cap S$ is an additive spine of $W$.

Proof. a) We first verify that $V$ itself is generated by $S$. Since $V$ is additively generated by $\tilde{S}$, for given nonzero $v \in V$ we have

$$v = \sum_{i=1}^{n} v_i, \quad \text{(A)}$$

with $n \in \mathbb{N}$, $v_i \in \tilde{S}$. There exist $\lambda_i, \mu_i \in R$ such that

$$s_i := \lambda_i v_i \in S, \quad \text{(B)}$$

$$v_i = \mu_i s_i, \quad \text{(C)}$$

and so by (A)

$$v = \sum_{i=1}^{n} \mu_i s_i,$$

and we are done.

b) If now $W$ is an $SA$-submodule of $V$, and the above element $v$ lies in $W$, then in (A) all summands $v_i$ are in $W$, and so the $s_i$ from (B) are in $W \cap S$. We conclude from (B) and (C) that all $s_i$ are in the halo $(W \cap S)\sim$ of $W \cap S$ in $W$, and we infer from (A) that $W$ is additively generated by $(W \cap S)\sim$, i.e., $W \cap S$ is an additive spine of $W$. As proved in a) the set $W \cap S$ generates the $R$-module $W$. \hfill $\Box$
We write down a chain of propositions which turn out to be useful in working with halos and additive spines. For clarity we sometimes denote the halo of a set $S$ in an $V$ more elaborately by $\text{hal}_{V}(S)$ instead of $\tilde{S}$.

**Proposition 8.4.** If $S$ is a subset of an $R$-module $V$ and $W$ a submodule of $V$, then

\[ W \cap \text{hal}_{V}(S) = \text{hal}_{W}(W \cap S) = \text{hal}_{V}(W \cap S). \]

**Proof.** Let $v \in \text{hal}_{V}(S)$ be given. We choose $\lambda, \mu \in R$ with $\lambda v = s \in S$ and $\mu s = v$. If now $v \in W$ then $\lambda v = s \in W \cap S$, and so $v \in \text{hal}_{W}(W \cap S)$. This proves that

\[ W \cap \text{hal}_{V}(S) \subset \text{hal}_{W}(W \cap S). \] (A)

Trivially

\[ \text{hal}_{W}(W \cap S) \subset \text{hal}_{V}(W \cap S). \] (B)

If $v \in \text{hal}_{V}(W \cap S)$, then there exist $\lambda, \mu \in R$ with $\lambda v = s \in W \cap S$ and $\mu s = v$. It follows that $v \in W \cap \text{hal}_{V}(S)$. This proves

\[ \text{hal}_{V}(W \cap S) \subset W \cap \text{hal}_{V}(S). \] (C)

(A)–(C) together imply the assertion of the proposition. □

In case $S \subset W$ the proposition reads as follows:

**Corollary 8.5.** Let $S \subset V$. Then the halo of $S$ in any submodule $W \supset S$ of $V$ coincides with the halo of $S$ in $V$.

Thus in practice the notation $\text{hal}_{V}(S)$ instead of $\tilde{S}$ is rarely needed.

**Proposition 8.6.** Let $(V_i \mid i \in I)$ be a family of submodules of the $R$-module $V$ and assume that for every $i \in I$ there is given a set $S_i \subset V_i$.

a) Then

\[ \bigcup_{i \in I} \text{hal}_{V_i}(S_i) = \text{hal}_{V} \left( \bigcup_{i \in I} S_i \right). \]

b) If $\sum V_i = V$ and each $S_i$ is an additive spine of $V_i$, then $\bigcup S_i$ is an additive spine of $V$.

**Proof.** Let $S := \bigcup_{i \in I} S_i$.

a): We have $\text{hal}_{V}(S) = \bigcup_{i \in I} \text{hal}_{V_i}(S_i)$ in complete analogy to Remark 7.3.iii. Furthermore $\text{hal}_{V_i}(S_i) = \text{hal}_{V_i}(S_i)$ by Corollary 8.5.

b): Let $\tilde{S}_i := \text{hal}_{V_i}(S_i)$. Then $\bigcup \tilde{S}_i = \tilde{S}$, $\sum \tilde{S}_i = V$, and so

\[ \sum \tilde{S} = \sum \left( \sum \tilde{S}_i \right) = \sum V_i = V. \]

We now have a good hold on all additive spines of a free $R$-module as follows:

**Proposition 8.7.** Assume that $V$ is a free $R$-module with base $(v_i \mid i \in I)$. Then every additive spine $S$ of $V$ has the shape

\[ S = \bigcup_{i \in I} M_i v_i \]

with every $M_i$ an additive spine of $R$, as defined in 4.
Proof. We have \( V = \bigoplus_{i \in I} V_i \) with \( V_i = Rv_i \cong _RR \). The claim follows from Proposition \ref{8.6} \( \square \)

Proposition \ref{8.8} (Functoriality of halos and additive spines). Let \( \varphi : V \to V' \) be an \( R \)-linear map between \( R \)-modules.

a) If \( S \) is a subset of \( V \) then
\[
\varphi(\tilde{S}) \subset \varphi(S)^\sim.
\]

b) If \( S \) is an additive spine of \( V \), then the \( R \)-module \( \varphi(V) \) is additively generated by \( \varphi(\tilde{S}) \), and so \( \varphi(S) \) is an additive spine of \( \varphi(V) \).

Proof. a): Let \( x \in \tilde{S} \). We have \( \lambda, \mu \in R \) with \( \lambda x = s \in S \), \( \mu s = x \). It follows that \( \lambda \varphi(x) = \varphi(s) \), \( \mu \varphi(s) = \varphi(x) \), whence \( \varphi(x) \in \varphi(S)^\sim \).

b): By Corollary \ref{8.5} we may replace \( V \) by \( \varphi(V) \), and so assume that \( \varphi \) is surjective. We have \( \sum \tilde{S} = V \). Applying \( \varphi \) we obtain
\[
\sum \varphi(\tilde{S}) = \varphi(V).
\]

It follows by a) that \( \sum \varphi(S)^\sim = \varphi(V) \). \( \square \)

Corollary \ref{8.9}. Assume that \( R, T \) are semirings and \( V \) is an \((R,T)\)-bimodule, i.e., \( V \) is a left \( R \)-module, a right \( T \)-module, and
\[
\forall \lambda \in R, \mu \in T, v \in V : (\lambda v)\mu = \lambda(v\mu).
\]

Let \( S \) be a subset of \( V \). As before let \( \tilde{S} \) denote the halo of \( S \) in \( R \), \( V \) as a left \( R \)-module). Then, for any \( t \in T \)
\[
\tilde{St} \subset (St)^\sim.
\]

If \( S \) is an additive spine of \( V \) then \( \tilde{St} \) generates the left \( R \)-module \( Vt \) additively, and so \( St \) is an additive spine of \( Vt \).

Proof. Apply Proposition \ref{8.8} to the endomorphism \( v \mapsto vt \) of \( RV \). \( \square \)

Corollary \ref{8.10}. If again \( V \) is an \((R,T)\)-bimodule and \( t \) is a unit of \( T \), then \( \tilde{St} = (St)^\sim \), and \( S \) is an additive spine of \( V \) iff \( \tilde{St} \) is an additive spine of \( V \).

Proof. Let \( u := t^{-1} \). Then by Corollary \ref{8.9} \( (\tilde{St})u \subset (St)^\sim u \subset (Stu)^\sim = \tilde{S} \). Multiplying by \( t \), we obtain \( \tilde{St} \subset (St)^\sim \subset \tilde{St} \), whence \( \tilde{St} = (St)^\sim \), and then
\[
\sum (St)^\sim = \left( \sum \tilde{S} \right) t.
\]

Example \ref{8.11}. \( R \) is an \((R,R)\)-bimodule in the obvious way. Thus, if \( M \) is an additive spine of \( R \) (as defined already in \[\square\]), and if \( u \) is a unit of \( R \), then \( Mu \) is again an additive spine of \( R \).

Example \ref{8.12}. Assume that \( C \) is a semiring which is a homomorphic image of \( \mathbb{N}_0 \), and \( R := M_n(C) \). We have seen in Example 2.12 that \( \{e_{11}, \ldots, e_{nm}\} \) is an additive spine of \( R \). Let \( \sigma \in \Gamma_n \). Then \( u := \sum e_{i,\sigma(i)} \) is a unit of \( R \), namely \( u \) is the permutation matrix of \( \sigma^{-1} \).

We have \( e_{ii}u = e_{i,\sigma(i)} \), and conclude that \( \{e_{1,\sigma(1)}, \ldots, e_{n,\sigma(n)}\} \) is an additive spine of \( M_n(C) \).

We can generalize Proposition \[\square\] as follows:
Proposition 8.13. Assume that $R_1, R_2$ are subsemirings of a semiring $R$ with $R = \bigoplus R_1 R_2$, and that $V_1, V_2$ are left modules over $R_1$ and $R_2$ respectively. Assume furthermore that there is given a composition $V_1 \times V_2 \xrightarrow{\sim} V$ such that

$$(\lambda_1 \lambda_2)(v_1 \cdot v_2) = (\lambda_1 v_1) \cdot (\lambda_2 v_2)$$

for any $\lambda_i \in R, v_i \in V_i (i = 1, 2)$. Assume finally that $V = \bigoplus V_i \cdot V_2$. Then, given subsets $S_i \subset V_i$ with halos $\tilde{S}_i$ in the $R_i$-module $V_i (i = 1, 2)$, the following holds.

a) $\tilde{S}_1 \cdot \tilde{S}_2$ is contained in the halo $(S_1 \cdot S_2)^\sim$ of $S_1 \cdot S_2$ in $V$.

b) If $S_i$ is an additive spine of $V_i (i = 1, 2)$ then

$$V = \bigoplus \tilde{S}_1 \cdot \tilde{S}_2$$

and $S_1 \cdot S_2$ is an additive spine of $V$.

Proof. Let $v_i \in \tilde{S}_i (i = 1, 2)$. We have $\lambda_i, \mu_i \in R_i$ with $\lambda_i v_i = s_i \in S_i, \mu_i s_i = v_i$. Now

$$(\lambda_1 \lambda_2)(v_1 \cdot v_2) = (\lambda_1 v_1) \cdot (\lambda_2 v_2) = s_1 \cdot s_2$$

and $(\mu_1 \mu_2)(s_1 \cdot s_2) = (\mu_1 s_1) \cdot (\mu_2 s_2) = v_1 \cdot v_2$. This proves that $\tilde{S}_1 \cdot \tilde{S}_2 \subset (S_1 \cdot S_2)^\sim$. If now $\sum \tilde{S}_i = V_i (i = 1, 2)$, then

$$\sum (\tilde{S}_1 \cdot \tilde{S}_2) \supset \left( \sum \tilde{S}_1 \right) \cdot \left( \sum \tilde{S}_2 \right) = V_1 \cdot V_2,$$

and so $\sum (\tilde{S}_1 \cdot \tilde{S}_2) \supset \sum V_1 \cdot V_2 = V$, whence $\sum \tilde{S}_1 \cdot \tilde{S}_2 = V$. A fortiori $\sum (S_1 \cdot S_2)^\sim = V$. \qed

Note that Proposition 8.12 is indeed a special case of this proposition: Given an $R$-module $V$, take $R_1 = R_2 = R, V_1 = R, V_2 = V$ and the scalar product $R \times V \rightarrow V$.

9. The posets $\text{SA}(V), \Sigma \text{SA}(V)$ and $\Sigma_f \text{SA}_f$ in good cases

Assume now that $R$ has a finite additive spine $M$ consisting of $m := |M|$ elements. We have seen in [7] that, when $S$ is a set of generators of $V$, then every $W \in \text{SA}(V)$ is generated by the set $W \cap (MS)$. Thus, if $s := |S|$ is finite, we see that the lattice $\text{SA}(V)$ is finite, consisting of at most $2^{m|S|}$ elements. More generally we have the following fact.

Theorem 9.1. Assume that $V_0$ is a submodule of an $R$-module $V$ and $S$ is a subset of $V$, such that $V$ is generated over $V_0$ by $S$, i.e.,

$$V = V_0 + \sum R S.$$  \hfill (9.1)

Let $W_0 \in \text{SA}(V)$ be given with $W_0 \subset V_0$, and consider the set

$$\text{SA}(V; W_0, V_0) = \{ W \in \text{SA}(V) \mid W \cap V_0 = W_0 \}.$$  \hfill (9.2)

Then if $s := |S|$ is finite, this set $\text{SA}(V; W_0, V_0)$ consists of at most $2^{ms}$ elements. Furthermore, any chain $W_0 \subsetneq W_1 \subsetneq \cdots \subsetneq W_r$ in $\text{SA}(V; W_0, V_0)$ has length $r \leq ms$.

Proof. Let $U$ denote the submodule of $V$ generated by $S$. We have $V = V_0 + U$. If $W \in \text{SA}(V; W_0, V_0)$ then by (1.1)

$$W = W \cap V_0 + W \cap U = W_0 + W \cap U,$$  \hfill (9.3)
and, of course, \( W \cap U \in \text{SA}(U) \). Since \( |\text{SA}(U)| \leq 2^{ms} \), as stated above, we infer that \( |\text{SA}(V; W_0, V_0)| \leq 2^{ms} \). Also, if \( W_0 \not\subseteq W_1 \not\subseteq \cdots \not\subseteq W_r \) is a chain in \( \text{SA}(V; W_0, V_0) \), we conclude from (\ref{eq:1}) for \( U_i := W_i \cap U \) that
\[
U_0 \not\subseteq U_1 \not\subseteq \cdots \not\subseteq U_r.
\]
Every \( U_i \) is generated by the set \( U_i \cap (MS) \) and so
\[
U_0 \cap (MS) \not\subseteq U_1 \cap (MS) \not\subseteq \cdots \not\subseteq U_r \cap (MS).
\]
This implies that \( r \leq |MS| = ms \).

We return to an arbitrary semiring \( R \).

**Theorem 9.2.** Assume that \( T \) is an additive spine of the \( R \)-module \( V \) (cf. Def. \ref{def:spine}).

a) Then any \( U \in \Sigma \text{SA}(V) \) is generated by the set \( U \cap T \).

b) If \( T \) is finite, \( |T| = t \), then \( |\Sigma \text{SA}(V)| \leq 2^t \), and any chain
\[
U_0 \not\subseteq U_1 \not\subseteq \cdots \not\subseteq U_r
\]
in \( \Sigma \text{SA}(V) \) has length \( r \leq t \).

**Proof.** a): Write \( U = \bigcup_{i \in I} W_i \) with \( W_i \in \text{SA}(V) \). We know by Theorem \ref{thm:finite} that every \( W_i \) is generated by \( W_i \cap S \). Thus \( U \) is generated by the set
\[
\bigcup_{i \in I}(W_i \cap S) = \left( \bigcup_{i \in I} W_i \right) \cap S.
\]
A fortiori \( U \) is generated by \( U \cap S \).

b): Every \( U \in \Sigma \text{SA}(V) \) is generated by the set \( U \cap T \subset T \). We have at most \( 2^t \) possibilities for this set, and so \( |\Sigma \text{SA}(V)| \leq 2^t \). Furthermore, if \( U_0 \not\subseteq \cdots \not\subseteq U_r \) is a chain in \( \Sigma \text{SA}(V) \), then
\[
U_0 \cap T \not\subseteq U_1 \cap T \not\subseteq \cdots \not\subseteq U_r \cap T,
\]
since each \( U_i \) generated by \( U_i \cap T \), and so \( r \leq t \).

By a variation of our previous arguments we obtain

**Theorem 9.3.** Assume that \( R \) has a finite additive spine \( M \), furthermore that \( U \in \Sigma_f \text{SA}_f(V) \). Let \( S \) be a finite set of generators of \( U \). Then every \( W \in \text{SA}_f(U) \) is generated by the finite set \( W \cap (MS) \) and every chain
\[
W \not\subseteq W_1 \not\subseteq W_2 \not\subseteq \cdots \not\subseteq W_r
\]
in \( \text{SA}(U) \), hence in \( \text{SA}_f(U) \), has length \( r \leq |M| \cdot |S| \). A fortiori this holds if \( W \) and all \( W_i \) are in \( \text{SA}_f(V) \).

**Proof.** Every \( W \in \text{SA}_f(U) \) is generated by the finite set \( W \cap (MS) \), cf. Theorem \ref{thm:finite}.

Furthermore by the same theorem every \( W_i \) is generated by the subset \( W_i \cap (MS) \) of \( W \cap (MS) \). It follows that
\[
W \cap (MS) \not\subseteq W_1 \cap (MS) \not\subseteq \cdots \not\subseteq W_r \cap (MS)
\]
and so \( r \leq |W \cap (MS)| \leq |M| \cdot |S| \). It is obvious that every \( \text{SA} \)-submodule of \( V \) contained in \( U \) is \( \text{SA} \) in \( U \).

**Example 9.4.** We read off from Theorem \ref{thm:finite} that, if \( R \) has a finite additive spine, then every module \( U \in \Sigma_f \text{SA}_f(V) \) is \( \text{SAF} \)-accessible.
Our final result in this section refers to modules with additive spines which are not necessarily finite.

**Theorem 9.5.** Assume that \( T \subset V \) is an additive spine of the \( R \)-module \( V \), and \( U \in \Sigma \text{SA}(V) \).

1. Then \( U \) is generated by the set \( U \cap T \).
2. If \( U \) is an \( \text{SA}_f \)-sum in \( V \), and \( (W_i \mid i \in I) \) is a family of finitely generated \( \text{SA}_f \)-submodules of \( V \) with \( U = \sum_{i \in I} W_i \), then every \( W_i \) is generated by a finite subset \( T_i \) of \( W_i \cap T \), and so \( U \) is generated by the subset \( \bigcup_{i \in I} T_i = T' \) of \( T \). This subset \( T' \) is an additive spine of \( U \).
3. If \( U \in \Sigma_f \text{SA}_f(V) \) then \( U \) is generated by a finite subset of \( U \cap T \), and this is an additive spine of \( U \).

**Proof.** We choose a family \( (W_i \mid i \in I) \) in \( \text{SA}_f(V) \) with \( U = \sum_{i \in I} W_i \).

1. a): Done before (Theorem 8.2).
2. b): We assume now that all \( W_i \) are finitely generated. Every \( W_i \) is generated by \( W_i \cap T \) (Theorem 8.3). It follows that \( W_i \) is generated by a finite subset \( T_i \) of \( W_i \cap T \). Indeed, given generators \( s_1, \ldots, s_r \) of \( W_i \) for \( i \) fixed, write every \( s_j \) as a linear combination of a finite subset \( T_{ij} \) of \( W_i \cap T \). Then \( T_i := \bigcup_{j=1}^r T_{ij} \) does it. It follows by Theorem 8.3 that \( T_i \) is an additive spine of \( W_i \). It now is clear that \( T' := \bigcup_{i \in I} T_i \) generates \( U = \sum_{i \in I} W_i \), and it follows by Proposition 8.6 that \( T' \) is an additive spine of \( U \).
3. c): Now evident, since the index set \( I \) can be assumed to be finite, and so \( T' = \bigcup_{i \in I} T_i \) is a finite additive spine of \( U \).

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