ABSTRACT
We present a time-dependent model of the interaction between a neutron star magnetosphere and a thin (Shakura-Sunyaev) accretion disk, where the extent of the magnetosphere is determined by balancing outward diffusion and inward advection of the stellar magnetic field at the inner edge of the disc. The nature of the equilibria available to the system is governed by the magnetic Prandtl number $P_m$ and the ratio $\xi$ of the corotation radius to the Alfvén radius. For $\xi \gtrsim P_m^{0.3}$, the system can occupy one of two stable states where the torques are of opposite signs. If the star is spinning up initially, in the absence of extraneous perturbations, $\xi$ decreases until the spin-up equilibrium vanishes, the star subsequently spins down, and the torque asymptotes to zero. Vortex-in-cell simulations of the Kelvin-Helmholtz instability suggest that the transport speed across the mixing layer between the disc and magnetosphere is less than the shear speed when the layer is thin, unlike in previous models.

Key words: accretion, accretion discs – instabilities – MHD – pulsars: general – stars: neutron – X-rays: binaries

1 INTRODUCTION
A complete time dependent theory of mass and angular momentum transfer onto a neutron star accreting via a thin (Shakura-Sunyaev) disc needs to explain the stochastic fluctuations observed in the spin frequencies and X-ray luminosities of such systems (Bildsten et al. [1997]; Baykal & Oegelman [1993]), as well as regular phenomena like the negative (spin-down) torque on accreting millisecond pulsars (Galloway et al. [2002]; Chakrabarty et al. [2003]), quasi-periodic oscillations (QPOs) in low-mass X-ray binaries (LMXBs) (van der Klis [2001]), and torque reversals (Nelson et al. [1997]). Torque reversals, in which some X-ray pulsars alternate between sustained episodes of spin-up/down, present a particular challenge to theories of accretion because the transitions between spin-up/down episodes are sudden ($\sim 10\%$ of the duration of the episodes themselves).

In the standard picture of neutron star accretion, a thin disc is threaded by the stellar magnetic field. The torque on the star is the sum of the material torque $N_{\text{mat}}$ carried by material falling onto the star and the torque $N_B$ due to magnetic stresses. The material torque $N_{\text{mat}}$ always acts to spin up the star, while the magnetic torque $N_B$ may either spin up or spin down the star. The Ghosh-Lamb picture has been extended in several directions to accommodate time-dependent effects. Lovelace et al. [1993] proposed that the magnetosphere could alternate between configurations with and without outflows. Li & Wickramasinghe [1998] suggested that the maximum radius where the stellar magnetic field threads the disc is constrained to small and large values, with intermediate values excluded by an unstable feedback mechanism as the magnetospheric configuration alters the disc resistivity. Torkelsson [1998] considered the interaction between a disc dynamo and the stellar magnetic field. van Kerkwijk et al. [1998] considered the possibility that the inner disc flips over, due to warping, to form a retrograde disc. Yi & Vishniac [1999] suggested that the accretion flow changes abruptly from a geometrically thin and cool Keplerian flow to a geometrically thick and hot sub-Keplerian flow, accompanied by a change from spin-up to spin-down.

Spruit & Taam [1993] (ST93 hereafter) showed that the disc can become unstable, with mass accreted in bursts, based on the premise that accretion is enhanced by the Kelvin-Helmholtz instability (KHI) but inhibited when the magnetosphere rotates faster than the disc (Illarionov & Sunyaev [1975]). They compared the bursts predicted by their model to the type II bursts observed from the Rapid Burster MXB 1730-355.

In this paper, we generalize the accretion model of ST93 to include torque feedback onto the star and diffusive penetration of the stellar magnetic field into the inner edge of the accretion disc, balanced by inward advection. On the
basis of preliminary numerical simulations, we also neglect enhanced accretion by the KHI. We present the new model in Section 2. In Section 3, we show that the disc-magnetosphere system has two stable torque states, for certain combinations of the magnetic Prandtl number $Pm$ and the ratio $\xi$ of the corotation radius to the Alfén radius. If the star is initially spinning up, in the absence of extraneous perturbations, the spin-up equilibrium eventually vanishes and the star subsequently spins down. In Section 3, we show that the viscous torque in the mixing layer, omitted in the original ST93 model, is significant; when it is included, the region of parameter space for which the system has two stable torque states is reduced. In Section 4, we present the results of preliminary numerical simulations which indicate that the transport of material across the disc-magnetosphere mixing layer due to the KHI is suppressed when the mixing layer is thin. In Section 4, we discuss the implications of the generalized model for the torque reversals observed in accreting X-ray pulsars.

2 DISC-MAGNETOSPHERE COUPLING

2.1 Magnetospheric radius

We use cylindrical polar coordinates $(r, \phi, z)$ throughout and assume axisymmetry. The magnetospheric radius $r_m$ of the system is defined to include all material that corotates with the star (ST93). Such a radius may not exist if the magnetic field lines are swept back all the way to the stellar surface; Ghosh et al. (1977)]. Outside $r_m$, there is a thin, annular mixing layer $r_{m1} < r < r_{m2}$ within which the angular velocity of infalling material is brought from the Keplerian value $\Omega_k(r_{m2})$ to the stellar velocity $\Omega_*$ by magnetic torques. Once the material enters the mixing layer, it is transported on to the star on a time-scale much shorter than $r_{m1}/\dot{r}_{m1}$. The rate at which mass flows inward through a surface at a fixed radius $r$ in the disc is

$$\dot{M}(r, t) = -2\pi r v_r(r, t) \Sigma(r, t),$$

where $\Sigma$ is the surface mass density, and $v_r < 0$ is the radial drift velocity. In the steady state, $\dot{M}$ is constant, independent of $r$ (by continuity), and equals the accretion rate onto the star. Out of the steady state, the accretion rate onto the star is

$$\dot{M}'(t) = \dot{M}(r_{m2}, t) + 2\pi r v_r(r_{m2}, t) \Sigma(r_{m2}, t),$$

where the second term in (2) represents mass swept up as $r_{m2}$ changes.

We place an upper bound on the steady state value of $r_{m1}$ by noting that the torque required to maintain corotation in the region $r < r_{m1}$ cannot exceed that provided by the magnetic shear stress, yielding (ST93)

$$r_{m1,\text{max}} = \frac{\dot{M} \Omega_*}{\pi |S(r_{m1,\text{max}})|},$$

where $S(r)$, the magnetic stress at radius $r$, at the disc surface ($z = h$), is given by Shapiro & Teukolsky (1983)

$$|S(r)| = (2\pi)^{-1} |B_\phi(r) B_\phi(r)|_{z=h}.$$  \hspace{1cm} (4)

The toroidal field, generated by differential rotation, is limited to $B_\phi \lesssim B_z$, because as $B_\phi$ builds up, the magnetic pressure inflates the field lines outward until they open up, destroying the magnetic link between the star and the disc. Hence, we have

$$|S(r)| \approx \eta B_\phi^2(r) (4\pi)^{-1} \approx \eta \mu^2 r^{-6}(4\pi)^{-1},$$

where $B(r)$ is the strength of the dipole field in the absence of a disc, $\mu$ is the stellar dipole moment, and $\eta \sim 1$ encapsulates our ignorance of the true (time dependent) magnetic field modified by the disc.

The system does not necessarily adjust to $r_{m1} = r_{m1,\text{max}}$. The value of $r_{m1}$ depends on how rapidly the magnetic field (which induces corotation) penetrates the disc by Ohmic diffusion. Let $d_B = r_{m2} - r_{m1}$ be the depth to which the magnetic field penetrates the disc beyond $r_{m1}$, and let $\Delta j$ be the change in specific angular momentum across the mixing layer, given by

$$\Delta j = r_{m1}^2 \Omega_* - r_{m2}^2 \Omega_k(r_{m2}).$$  \hspace{1cm} (5)

Define $d_c$ to be the critical depth to which the field would need to penetrate in order to change the specific angular momentum of inflowing material by $\Delta j$, obtained by setting $M \Delta j = -2\pi r_{m1}^2 d_c S(r_{m1})$. One then finds

$$d_c = \frac{M(r_{m2}) \Omega_*}{2 \pi S(r_{m1})} \left( \frac{r_{m2}}{r_{m1}} \right)^2 \left( \frac{r_{m2}}{r_c} \right)^{-3/2} - 1,$$

where the corotation radius is given by $r_c = (\mathcal{G}M/\Omega_*)^{1/3}$, and our sign convention is such that a positive magnetic stress $S$ subtracts angular momentum from the disc, as in ST93. Since the magnetic stress acts in the direction to bring material crossing the mixing layer into corotation, one has $\text{sign}(S) = \text{sign}(r_c - r_{m1})$.

In equilibrium, one has $d_B = d_c$. If the equilibrium is perturbed by increasing $\dot{M}$, $d_c$ increases, inflowing material cannot be brought into corotation as it travels from $r_{m2}$ to $r_{m1}$, and $r_{m1}$ must decrease by an amount $\approx d_c - d_B$ on the local radial drift time $\approx |v_r(r_{m2})|/d_B$, giving

$$r_{m1} \approx \frac{d_B - d_c}{d_B} |v_r(r_{m2})|.$$

Similarly, if the equilibrium is perturbed by reducing $\dot{M}$, $d_c$ decreases, inflowing material is brought into corotation before travelling the full distance from $r_{m2}$ to $r_{m1}$, and $r_{m1}$ must increase by an amount $\approx |v_r(r_{m2})|/d_B$, as in (5).

2.2 Mixing layer thickness

The mixing layer thickness $d_l$ is determined by a competition between outward diffusion and inward advection. Let $v_B$ be the speed at which the magnetic field penetrates into the disc, as measured in a frame where $v_r = 0$. One finds $v_B \approx \nu_B / d_B$, as for any diffusive process, where $\nu_B$ is the magnetic diffusivity of the disc material. In the observers frame, where the magnetospheric radius is changing and matter flows radially inward, we have $v_B = \dot{r}_{m2} + |v_r(r_{m2})|$, and hence

$$\dot{r}_{m2} = \frac{\nu_B}{d_B} - |v_r(r_{m2})|.$$  \hspace{1cm} (6)

It might appear, from (9), that we are erring by modelling the transport of material across the disc-magnetosphere mixing layer, given by

$$\Delta j = r_{m1}^2 \Omega_* - r_{m2}^2 \Omega_k(r_{m2}).$$  \hspace{1cm} (6)

Define $d_c$ to be the critical depth to which the field would need to penetrate in order to change the specific angular momentum of inflowing material by $\Delta j$, obtained by setting $M \Delta j = -2\pi r_{m1}^2 d_c S(r_{m1})$. One then finds

$$d_c = \frac{M(r_{m2}) \Omega_*}{2 \pi S(r_{m1})} \left( \frac{r_{m2}}{r_{m1}} \right)^2 \left( \frac{r_{m2}}{r_c} \right)^{-3/2} - 1,$$

where the corotation radius is given by $r_c = (\mathcal{G}M/\Omega_*)^{1/3}$, and our sign convention is such that a positive magnetic stress $S$ subtracts angular momentum from the disc, as in ST93. Since the magnetic stress acts in the direction to bring material crossing the mixing layer into corotation, one has $\text{sign}(S) = \text{sign}(r_c - r_{m1})$.

In equilibrium, one has $d_B = d_c$. If the equilibrium is perturbed by increasing $\dot{M}$, $d_c$ increases, inflowing material cannot be brought into corotation as it travels from $r_{m2}$ to $r_{m1}$, and $r_{m1}$ must decrease by an amount $\approx d_c - d_B$ on the local radial drift time $\approx |v_r(r_{m2})|/d_B$, giving

$$r_{m1} \approx \frac{d_B - d_c}{d_B} |v_r(r_{m2})|.$$

Similarly, if the equilibrium is perturbed by reducing $\dot{M}$, $d_c$ decreases, inflowing material is brought into corotation before travelling the full distance from $r_{m2}$ to $r_{m1}$, and $r_{m1}$ must increase by an amount $\approx |v_r(r_{m2})|/d_B$, as in (5).

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reduces to \( \dot{d}_B = \nu_B/d_B \) and hence \( d_B(t = 0) = 0 \), as expected for diffusion.

Comparing our model with ST93, we note the following. ST93 set \( r_{m1} \) to the largest value for which the corotation of material within \( r_{m1} \) can be maintained by magnetic stresses. Subsequently, they calculate the mixing layer thickness by requiring that disc material be brought into corotation by magnetic stresses before reaching \( r_{m1} \). In our model, the mixing layer thickness is set by the advection-diffusion balance, and \( r_{m1} \) grows or shrinks depending on whether or not material can be brought into corotation as it crosses the mixing layer, as described in Section 4.4 ST93 allow for \( r_{m1} \) to decrease if the velocity shear across the mixing layer exceeds the radial drift velocity, arguing that in this case the KHI transports material across the boundary layer faster than it can be brought into corotation. In our model, material is transported across the mixing layer at the radial drift velocity. This is motivated by numerical simulations of the KHI (Section 4) showing that mass transport is inhibited in a thin mixing layer.

In both our model and ST93, magnetic stresses act only on the mixing layer, so the magnitude of the magnetic torque on the disc is smaller than in models of star-disc interactions where magnetic stresses act on the entire disc, such as Rappaport et al. (2004) (RFS04 henceforth). For an order-of-magnitude comparison, one can take the representative values \( d_B = 0.01r_c \), \( r_{m1} = 1.1r_c \), and integrate our assumed form of the magnetic stress \( |S(r)| = \mu^2 r^{-4}(4\pi)^{-1} \) over the mixing layer, to determine the approximate magnitude of the magnetic torque on the mixing layer in our model. One can similarly calculate the magnetic torque on the disc in the RFS04 model, by integrating the assumed form of the magnetic stress in RFS04, \( |S(r)| = \mu^2 r^{-4}(2\pi)^{-1}[1 - \Omega_K(r)/\Omega_*] \), over the entire disc outside \( r_c \). We find that the magnetic torque on the mixing layer in our model is of order 0.03 times the magnetic torque on the disc in RFS04, assuming identical stellar field strengths and spin frequencies in the two models. For an alternative perspective, one can take the assumed form of the magnetic stress in RFS04 and compute the ratio of the magnetic torque integrated over the entire disc outside \( r_c \) to the torque integrated only over the region \( 1.1r_c < r < 1.11r_c \). We find that these torques are in the approximate ratio 1:0.008.

### 2.3 Surface density of the disc

The surface density of the disc outside \( r_{m2} \) evolves according to the thin disc equation

\[
\frac{\partial \Sigma}{\partial t} = \frac{3}{r} \frac{\partial}{\partial r} \left[ r^{1/2} \frac{\partial}{\partial r} \left( \nu \Sigma r^{1/2} \right) \right],
\]

where \( \nu \) is the kinematic viscosity. We neglect the possibility that the magnetic pressure in the mixing layer causes the disc to thicken Wang 1987. The outer boundary condition is

\[
\lim_{r \to \infty} \dot{M}(r,t) = \dot{M}_\infty,
\]

where \( \dot{M}_\infty \) is some value set by the details of Roche lobe overflow. At the inner boundary, if \( r_{m2} > r_c \), the magnetosphere presents a centrifugal barrier and one expects \( \Sigma(r_{m2}) \) to be larger than if \( r_{m2} < r_c \) ST93; see also RFS04, so we choose a boundary condition of the form

\[
\nu \Sigma(r_{m2}) \propto \exp[a(r_{m2}/r_c - 1)],
\]

with \( a > 0 \). We fix the constant of proportionality by considering steady state solutions, which have the form

\[
\nu \Sigma = \frac{M}{3\pi} \left[ 1 - \beta \left( \frac{r_{m2}}{r_c} \right)^{1/2} \right],
\]

(12)

with \( \beta < 1 \) Frank et al. 1992. If \( r_{m2} = r_c \), the magnetosphere corotates with the mixing layer, so the disc structure does not depend on the location of \( r_{m2} \) (ST93) and (12) implies \( \beta = 0 \) and \( \nu \Sigma = M/(3\pi r_c) \). We therefore choose the boundary condition at the inner edge to be

\[
\nu \Sigma(r_{m2}) = \frac{M}{3\pi} \exp[a(r_{m2}/r_c - 1)].
\]

(13)

Comparison of (12) and (13) shows that

\[
\beta = 1 - \exp[a(r_{m2}/r_c - 1)]
\]

(14)

in steady state. We set \( a = 1 \) in this paper, except where specified otherwise. We show in Section 4 and Section 5 that our results remain qualitatively unchanged for \( 0 \lesssim a \lesssim 2 \).

The boundary condition (13) differs slightly from that in ST93, where \( \nu \Sigma(r_{m2}) \propto \exp[a((r_{m2}/r_c)^{1/2} - 1)] \) and there is also a power-law dependence of \( \nu \Sigma(r_{m2}) \) on the mass accretion rate.

The viscosity \( \nu \) in ST93 is constant in time and decreases as a power law in radius. In our model, we assume that \( \nu \) is constant in time and independent of radius for simplicity. Later (Section 3.5), we show that including a power law dependence of viscosity on radius leaves our results qualitatively unchanged.

### 2.4 Stellar spin frequency

The stellar spin frequency, \( \Omega_* \), evolves according to the equation of motion

\[
\dot{\Omega}_* = N,
\]

(15)

where \( I \) is the star’s moment of inertia and

\[
N = \dot{M}' \Omega_K(r_{m2}) \frac{r_{m2}^2}{\nu \Sigma(r_{m2})} + N_v(r_{m2})
\]

(16)

is the torque on the star, set equal to the rate of change of angular momentum inside the surface \( r = r_{m2} \). There is no magnetic term in (16) since, by assumption, there is no magnetic stress on the disc for \( r > r_{m2} \). The first term in (16) is the material torque and the second term is the viscous torque, given by

\[
N_v = 2\pi r \nu \Sigma r^2 \frac{\partial \Omega}{\partial r}
\]

(17)

In the steady state, (16) reduces to

\[
N = \beta \dot{M} \Omega_K(r_{m2}) \frac{r_{m2}^2}{\nu \Sigma(r_{m2})}
\]

(18)

From (16) and (18), the steady state torque on the star is positive (negative) when \( r_{m2} \) is less than (greater than) \( r_c \) Away from the steady state, this is no longer necessarily true, since (16) and (18) hold true only in the steady state. [As matter accretes onto the star, the star’s moment of inertia changes slightly, but this is a second order effect; Ghosh et al. 1977] We show that long-term changes in \( \Omega \) alter the equilibria of the star-disc system (Section 3.4); in contrast, ST93 held \( \Omega_* \) constant.

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2.5 Order-of-magnitude estimates of physical quantities

In this section, we estimate the key physical parameters in the problem for typical systems. Let \( M_{1.4}, I_{45}, P_{10}, \) and \( M_{-9} \) denote the mass, moment of inertia, spin period and accretion rate of a neutron star, normalised by \( M = 1.4 \, M_\odot, I = 10^{45} \, \text{g cm}^2, P_{\text{spin}} = 10 \, \text{s}, \) and \( M = 10^{-9} \, M_\odot \, \text{yr}^{-1} \) respectively. The corotation radius, defined by \( \Omega_r = \Omega_K(r_c) \), is given by

\[
r_c = 7.78 \times 10^8 M_{1.4}^{1/3} P_{10}^{2/3} \text{ cm}.
\]  

(19)

The characteristic torque \( N_{\text{char}} \) at the corotation radius is

\[
N_{\text{char}} = \dot{M} (GMr_c)^{1/2} = 2.40 \times 10^{-4} M_{-9} M_{1.4}^{3/2} P_{10}^{1/3} \, \text{g cm}^{-2} \, \text{s}^{-2}.
\]  

(20)

The corresponding rate of change of the spin frequency is

\[
\dot{\Omega} = \Omega r_c/(2\pi) = 38.2 \times 10^{-13} I_{45} M_{-9} M_{1.4}^{3/2} P_{10}^{1/3} \, \text{Hz s}^{-1}.
\]  

(22)

Pulsars exhibiting torque reversals spin up (or down) at a comparable rate, with \( |\dot{\Omega}|/(2\pi) = 7-60 \, \text{Hz s}^{-1} \) (Nelson et al 1997). In Section 2.4, we show that our model predicts spin up/down rates of this order of magnitude.

The kinematic viscosity in a disc is conventionally parametrized by the \( \alpha \) parameter (Shakura & Sunyaev 1973). From the standard unmagnetised \( \alpha \)-disc solution (Frank et al 1992),

\[
\nu = 4.0 \times 10^{-10} \alpha^4/5 M_{1.4}^{1/10} M_{-9}^{1/4} (r/10^9 \, \text{cm}) \, \text{cm}^2 \, \text{s}^{-1},
\]  

(23)

taking \( r \approx r_c \) and \( 0.001 < \alpha < 1.0 \), we conclude that \( \tau_c \) lies in the range \( 10^{10} \to 10^{13} \, \text{cm}^2 \, \text{s}^{-1} \). It is convenient to define the viscous time-scale corresponding to the length-scale \( r_c \),

\[
\tau \nu = 2r^2/(3\nu) = 4.06 \times 10^4 M_{1.4}^{2/3} P_{10}^{1/3} \nu_{13} \, \text{s},
\]  

(25)

with \( \nu_{13} = \nu/(10^{13} \, \text{cm}^2 \, \text{s}^{-1}) \). Using the above estimates for \( \nu \) and \( \tau_c \), we find \( \tau_{\nu} \) in the range 1 to 10^3 day; this is the time-scale for torque transitions in our model (Section 2.4). By comparison, Ferdkszian (1998) found \( \tau_{\nu} \) to be typically less than 0.5 day.

If the magnetic diffusivity is due to the same turbulence that is responsible for the kinematic viscosity (e.g., the magneto-rotational instability [Balbus & Hawley 1991]), one expects the magnetic Prandtl number to satisfy \( P_m = \nu_B/\nu \sim 1 \). By contrast, in the literature dealing with the diffusion of magnetic fields through discs, \( P_m \) is frequently considered as a free parameter in the range \( 10^{-2} \leq P_m \lesssim 10^{4} \) (Reves-Ruzi & Stepinsk 1996, Lubow et al 1994). The work in this paper is valid for \( P_m \ll 1 \), since for higher \( P_m \) the mixing layer is not thin (Section 8).

3 TORQUE BISTABILITY

In this section, we show that the model of the disc-magnetosphere interaction presented in Section 2 possesses two stable equilibria, with opposite signs of torque, under certain conditions. In searching for equilibrium, we consider that the time-scale \( \tau_c/\Omega_c \sim 10^{11} \) s is longer than the time-scale \( \tau_{\nu} \) for \( r_{m1} \) and \( d_B \) to change. Referring to (3) and (9), we observe that the condition for equilibrium is

\[
d_B = \nu_B/|\nu_{\nu}(r_{m2})| = \Omega_c.
\]  

(26)

First, we obtain dimensionless expressions for \( d_B, \Omega_c \) and \( r_{m1,max} \) in equilibrium. From (26), (15), (1), and the definition of the Prandtl number \( P_m = \nu_B/\nu \), we find

\[
d_B/r_c = 2 r_{m2}/3 P_m \exp[a(r_{m2}/r_c - 1)].
\]  

(27)

A thin mixing layer (\( d_B \ll r_c \)) therefore requires a small Prandtl number (Section 2.6). From (17) and (19), we find

\[
d_c/r_c = \xi^{7/2} \sqrt{2} (r_{m1}/r_c) \Omega_{\nu} \Omega_c/r_c^2 - 3/2 \Omega_{\nu} \Omega_c/r_c^2 - 1,
\]  

(28)

where

\[
\xi = \eta^{2/7} r_c/r_A
\]  

(29)

to find the characteristic Alfvén radius for spherical accretion (Elsner & Lamb 1977). From (30), (20) and (26), we find

\[
r_{m1,max}/r_c = 2^{-3/10} \xi^{-7/10}.
\]  

(31)

Anticipating that \( r_{m1} \sim r_c \), we initially restrict our investigation to the regime \( \xi \leq 2^{-3/10} r_{m1,max} \gg r_c \), so that the constraint \( r_{m1} \leq r_{m1,max} \) is satisfied.

We now find equilibrium by plotting \( d_B/r_c \) and \( d_c/r_c \) against \( r_{m2}/r_c \) and solving (26) and (27) graphically. Equations (26) and (27) always admit one solution for \( r_{m2} > r_c \), which we denote \( A \) (see Fig. 3). This is a spin-down equilibrium, since the torque is negative whenever \( r_{m2} > r_c \) (Section 2.4).

3.1 Multiple equilibria

Under certain conditions, two other equilibria become available to the system. The top panel of Fig. 3 shows that for \( \xi = 0.35 \), \( P_m = 0.01 \), the curves of \( d_B/r_c \) and \( d_c/r_c \) as functions of \( r_{m2}/r_c \) have only a single intersection, so the disc-magnetosphere system has only a single, spin-down, equilibrium, denoted \( A \). However, setting \( \xi < 0.40 \), \( P_m = 0.01 \), as in the bottom panel of Fig. 3, we find one of the curves of \( d_B/r_c \) and \( d_c/r_c \) has three intersections, and hence the disc-magnetosphere system has three equilibria, denoted \( A, B, C \). These equilibria have \( r_{m2}(A) = 1.08 r_c, r_{m2}(B) = 0.90 r_c \), and \( r_{m2}(C) = 0.56 r_c \), so that \( A \) is a spin-down equilibrium while \( B \) and \( C \) are spin-up equilibria. Combining (26) and (27), one obtains

\[
\dot{\Omega}_c = (r_{m2}/r_c)^{1/2} \dot{\Omega}_{\nu,\text{char}},
\]  

(32)

giving \( \dot{\Omega}_{\nu}(A) = -0.082 \dot{\Omega}_{\nu,\text{char}}, \dot{\Omega}_{\nu}(B) = 0.089 \dot{\Omega}_{\nu,\text{char}}, \) and \( \dot{\Omega}_{\nu}(C) = 0.27 \dot{\Omega}_{\nu,\text{char}} \). In Section 6, we show that equilibria \( A \) and \( C \) are stable to perturbations in \( r_{m1} \) and \( r_{m2} \), while equilibrium \( B \) is unstable. The system is thus bistable, with the two stable equilibria having opposite signs of torque. A single equilibrium bifurcates into three equilibria when the curves of \( d_B/r_c \) and \( d_c/r_c \) in Fig. 3 are tangential at some \( r_{m2} \ll r_c \). Solving for this condition numerically, we obtain Fig. 3, which shows the combinations of \( P_m \) and \( \xi \) for which multiple equilibria exist. For \( a = 1 \), the critical
value $\xi_{\text{crit}}$ separating the single and multiple equilibria is well approximated by

$$\xi_{\text{crit}} = 1.36Pm^{0.287},$$  \hspace{1cm} (33)$$

with a fractional error of $<1\%$ in the range $0 < Pm < 0.05$.

The existence of multiple equilibria can be understood as follows. One can think of $d_c$ as measuring how ‘difficult’ it is for magnetic stresses to change the angular velocity of infalling material from $\Omega_K(r_{m2})$ to $\Omega_c$ as it crosses the mixing layer. At $r_{m2} \approx r_c$, there is no velocity shear across the mixing layer, so $d_c \approx 0$. As $r_{m2}$ is displaced from $r_c$, the velocity shear increases and hence $d_c$ increases. For small enough $r_{m2}$, $d_c$ decreases again because the magnetic stress is proportional to $r_{m1}^{-6}$ so it becomes ‘easy’ for the magnetic field to bring material crossing the mixing layer into corotation. The shape of $d_c$ as a function of $r_{m1}$ suggests that even if $d_B$ was determined differently than in Section 2.2 (e.g. with a more sophisticated model of diffusion, or modified by the action of the KHI), $d_B(r_{m2})$ and $d_c(r_{m2})$ might still have multiple intersections and bistability might still occur.

### 3.2 Stability

Suppose that an equilibrium is perturbed by slightly increasing $r_{m1}$. From (33), if $d_B - d_c > 0$, $r_{m1}$ will continue to increase, whereas if $d_B - d_c < 0$, $r_{m1}$ will tend back to the equilibrium value. This suggests that those equilibria for which $d_B - d_c$ is a decreasing (increasing) function of $r_{m2}$ are stable (unstable). Thus we expect equilibria $A$ and $C$ to be stable, and equilibrium $B$ to be unstable.

Elaborating on the above argument, we calculate the evolution of a perturbation about each equilibrium. As a (rough) first approximation to the equations of motion, we assume that changes in the surface density profile of the disc occur rapidly compared to changes in $r_{m1}$, so that $\Sigma(r,t)$ can be regarded as passing through a succession of equilibrium states given by (22). In reality, $r_{m1}$ and $\Sigma$ change on similar time-scales; $\Sigma$ adjusts over a length-scale $r_{m1}$ in a time $\sim r_{m1}^2/\nu$ (Frank et al. 1992, p.70), while the radial drift velocity in a thin disc

$$v_r = \frac{3}{\Sigma r^{1/2}} \frac{\partial}{\partial r} \left( \nu \Sigma^{1/2} \right),$$  \hspace{1cm} (34)$$

together with (33), implies $r_{m1}/r_{m2} \sim r_{m1}^2/\nu$. None the less, in our approximation, we find the following evolution equations

$$\tau_0 \frac{d}{dt} \left( \frac{r_{m1}}{r_c} \right) = \frac{d_B - d_c}{r_{m2}} \frac{r_c}{r_{m1}} \exp[a(1 - r_{m2}/r_c)]$$  \hspace{1cm} (35)

$$\equiv f_1,$$

$$\tau_0 \frac{d}{dt} \left( \frac{r_{m2}}{r_c} \right) = \frac{2}{3} Pm \frac{r_c}{r_{m2}} \frac{r_c}{r_{m2}} \exp[a(1 - r_{m2}/r_c)]$$  \hspace{1cm} (36)

where the instantaneous value of $d_c$ is given by (33), $\tau_0$ is given by (24), and $d_B = r_{m2} - r_{m1}$.

Linearising about an equilibrium $\{r_{m1}^{(eq)}, r_{m2}^{(eq)}\}$, we write the equations of motion in the form $\tau_0 \dot{x}_i = A_{ij} x_j$, with $x = [(r_{m1} - r_{m1}^{(eq)})/r_c, (r_{m2} - r_{m2}^{(eq)})/r_c]$ and $A_{ij} = \partial f_i/\partial x_j$. The eigenvalues of $A_{ij}$ give the growth rate of perturbations. Obtaining analytic expressions for the growth rates is not tractable, so we calculate them numerically. For equilibria $A$, $B$ and $C$ of Fig. 1, we find that the dominant modes of a perturbation have growth rates of $-7.1$, $3.8$, and $-4.7$ respectively, in units of $\tau_0^{-1}$, indicating that $A$ and $C$ are stable, while $B$ is unstable.

We emphasise that the above analysis provides an indication of stability only, since it is based on the assumption that changes in the surface density profile of the disc are
rapid compared to changes in the position of the magnetosphere. To prove stability rigorously, one must to solve the disc PDEs subject to our chosen boundary conditions (e.g. ST93). However, this lies beyond the scope of this paper, especially in view of the uncertainties surrounding the boundary conditions themselves (e.g. the true, time dependent nature of the magnetic field in the vicinity of the mixing layer is unknown).

3.3 Externally driven evolution

How does the bistable system in Section 3.1 switch from one equilibrium to another? One possibility is that an extraneous perturbation, such as a change in $M_{\infty}$, pushes the system from $A$, through $B$ (which is unstable), to $C$, in Fig. 1. To explore this possibility, we integrate the equations of motion (35) and (36), starting near the unstable equilibrium $B$. Typical results are shown in Fig. 2. One finds that the system tends to either of the stable equilibria, depending on the initial conditions. In each case, we set the initial value of $d_B$ to the value corresponding to equilibrium $B$, but we set the initial value of $r_{m2}$ to be slightly larger (smaller) than $r_{m2}^{(B)}$, whereupon the system tends to $A$ ($C$). In each case, the transition between equilibria occurs on a time-scale $\sim \tau_V$.

3.4 Long-term evolution due to torque feedback

The system can also switch from one equilibrium to another as $\Omega_*$ changes secularly in response to the disc torque. To explore this possibility, let us assume that the system is initially in a state of spin-up (equilibrium $C$). From (24), $\xi \propto \delta \propto \Omega_*^{-2/3}$, so as the star spins up, $\xi$ decreases, the system moves from the upper to the lower region of parameter space in Fig. 1, and equilibria $B$ and $C$ vanish. The system then occupies the remaining equilibrium, $A$, and spins down. From (31), $r_{m1,\max}/r_c \propto \xi^{-7/10} \propto \Omega_*^{7/15}$, so as the star spins down, $r_{m1,\max}/r_c$ decreases, until the state $r_{m1} = r_{m1,\max}$ is reached. Ultimately, $r_{m1,\max}$ approaches $r_c$, the torque tends to zero, and the system stops spinning down.

Due to the dependences of $\xi$ and $r_{m1,\max}/r_c$ on $\Omega_*$, $\Omega_*$ undergoes fractional changes of order unity in this evolutionary scenario. For example, suppose $Pm = 1$, $\xi = 0.4$, and initially $\Omega_* = 1$ in arbitrary units. Then the star spins up to $\Omega_* = 1.16$, and subsequently spins down, with $\Omega_* \to 0.40$ as $t \to \infty$.

3.5 Radial viscosity gradient

Thus far we have assumed that the viscosity is constant in time and independent of radius, whereas in ST93 $\nu$ was allowed to vary as a power law in radius:

$$\nu(r) = \nu(r_c) \left( r/r_c \right)^n.$$  \hfill (37)

Adopting this prescription in our model, we find that the locations of the equilibria presented in Section 3.1 are unchanged, provided that we take $Pm$ to be uniform (for simplicity). This is because the derivations of (27) and (28) do not rely upon the assumption of uniform $\nu$. With regard to the stability analysis in Section 3.2, one finds that the evolution equations (35) and (36) are each modified by a factor of $(r_{m2}/r_c)^n$ on the right hand sides, and the $\nu$ appearing in (28) is given by $\nu(r_c)$. Numerical evaluation of the dominant linear growth rate (maximum eigenvalue of $A_{ij}$), for a range of $n$, reveals that the growth rate changes in magnitude but not in sign (Figure 3), so the stability of each equilibrium is qualitatively unchanged from the case of uniform viscosity.

More realistically, one might expect the viscosity $\nu$ to vary as a function of the disc temperature and density, and therefore $Pm$ may not be uniform. The detailed study of this scenario is beyond the scope of this paper.

3.6 Wind torque

For the low magnetic Prandtl numbers adopted in this study ($Pm \approx 10^{-2}$), significant bending of the magnetic field takes place, which might lead to the formation of a centrifugal wind (e.g. Labow et al. 1993; Reeves-Ruiz & Stepinski 1996), accompanied by a magnetic torque that removes angular momentum from the disc.

We can investigate the effect of the wind torque on our calculations with a toy model, in which we assume that a fraction $\delta$ of the field lines threading the mixing layer are open and form a wind, and the magnetic stresses associated with these field lines remove angular momentum from the mixing layer (wind torque). The remaining fraction $(1-\delta)$ of field lines threading the mixing layer are closed (connected...
to the star) and can either add or subtract angular momentum from the mixing layer, as in Section 4.1 depending on the relative velocities of the star and disc. For simplicity, we assume that the toroidal field associated with the open field lines at the disc surface is equal in magnitude to the toroidal field associated with the closed field lines, though its direction may be different, giving $|S_{\text{wind}}| \approx m^2 r^{-6} (4\pi)^{-1}$.

In this crude picture, the wind torque modifies the equation for $d_c$, since $d_c$ is calculated by equating the magnetic torque on the mixing layer to the rate of change of angular momentum of material crossing the mixing layer. In the case $r_{m2} < r_c$, both the open and closed field lines subtract angular momentum from the mixing layer, and one finds that equation (35) for $d_c$ is unchanged. In the case $r_{m1} > r_c$, the closed field lines add angular momentum to the mixing layer while the open field lines subtract angular momentum from the mixing layer, and one finds that equation (35) is modified by a factor of $1/(1-2\delta)$ on the right hand side. For $0 < \delta < 1/2$, this has the effect of elevating the section of the curve of $d_c$ to the right of $c_2$ in Figure 4, so that equilibrium $A$ occurs closer to corotation. For $\delta > 1/2$, one has $d_c < 0$ for $r_{m1} > r_c$, so equilibrium $A$ vanishes, and it is possible that there are no solutions for equilibria; the toy model breaks down.

In our calculations, with or without the toy model for the wind, we have neglected magnetic torques on the disc outside the mixing layer. It is possible that there is a wind and an associated wind torque in this region. The wind torque might even dominate the viscous torque (see for example Pelletier 1992), and the omission of this torque is a limitation of our model.

4 VISCOS TORQUE AT THE MIXING LAYER

The net torque exerted on the mixing layer induces the corotation of inflowing material and hence governs the evolution of the magnetosphere. It is the sum of magnetic and viscous components. The magnetic component results from the magnetic stresses integrated over the top and bottom surfaces of the mixing layer (at $|z| = h$), while the viscous component results from the mechanical shear stress integrated over the surface $r = r_{m2}$. In this section, we show that the viscous torque, neglected by ST93, is actually $\propto r_{m2}/d_B$ times larger than the magnetic torque. When the viscous torque is included in our model, only one stable equilibrium is available to the disc-magnetosphere system under a wider range of conditions than otherwise, although this depends sensitively on the velocity profile near $r_{m2}$.

4.1 Ratio of viscous and magnetic torques

Material within $r_{m1}$ corotates with the star, so the viscous stress vanishes at the inner edge of the mixing layer. By assumption, the angular velocity profile of the disc is approximately Keplerian beyond $r_{m2}$. The viscous torque $N_v$ acting on the mixing layer is therefore

$$N_v = 2\pi r_{m2} \Sigma_r^2 \left( \frac{\partial \Omega_r}{\partial r} \right)_{r=r_{m2}}$$

$$= -3\pi \Sigma (r_{m2}) \Omega_k (r_{m2}) r_{m2}^2,$$

The negative sign indicates that the viscous torque subtracts angular momentum from material in the mixing layer. In the steady state, at $r_{m2}$, the surface density is given by $\nu \Sigma (r_{m2}) = M (1-\beta)/(3\pi)$, implying

$$|N_v| = \frac{M}{\nu} \Omega_k (r_{m2}) r_{m2}^2 |1-\beta|.$$

Note that, for fixed $M$, $|N_v|$ does not tend to zero in the limit $\nu \to 0$, since (40) is independent of $\nu$. In this limit, from (40), $v_r \propto \nu$ tends to zero, and $\Sigma \propto M/\nu \propto \nu^{-1}$ grows without bound, in such a way that $N_v \propto \nu \Sigma$ remains fixed.

The magnetic torque $N_B$ on the mixing layer is

$$N_B = -2\pi r_{m1} S(r_{m1}) r_{m1}.$$  

From (3), with $r_{m1} \approx r_{m1,\text{max}}$ for order-of-magnitude purposes, we obtain $M \Omega_k \approx \pi r_{m1} S(r_{m1})$, implying

$$|N_B| \approx 2\pi r_{m1} d_B M \Omega_k.$$

Dividing (41) by (42), we find

$$\frac{|N_v|}{N_B} \approx \frac{1}{2} \frac{\Omega_k (r_{m2})}{\Omega_k (r_{m1})} \frac{r_{m2}^2}{r_{m1} d_B} |1-\beta| \approx \frac{d_B}{2 d_B} |1-\beta|,$$

where we have used $\Omega_k (r_{m2}) \approx \Omega_*$ (close to corotation) and $r_{m1} \approx r_{m2}$ (thin mixing layer). The expression (43) shows that the viscous torque is much greater than the magnetic torque for $d_B \ll r_{m2}$, Neglecting the viscous torque is not permissible, except for $\beta$ close to unity.

4.2 Equilibrium states

In this section, we examine the modified equilibria of the disc-magnetosphere system after taking into account the viscous torque. Following a similar analysis to that presented in Section 3.1, one finds that equation (29) for $d_B/r_c$ is still valid, but (29) needs to be modified as follows:

$$d_c/r_c = \frac{\xi^2/2}{\sqrt{2} (r_{m1}/r_c)^6 \text{sgn}(r_c - r_{m1})} \times \left[ \beta \left( \frac{r_{m2}}{r_{m1}} \right)^2 \left( \frac{r_{m2}}{r_c} \right)^{-3/2} - 1 \right],$$

with $\beta$ given by (40).

Examples of graphical solutions for equilibria, with the viscous torque included, are shown in Fig. 4. The top panel uses the same values of dimensionless parameters ($P_m = 0.01$, $\xi = 0.4$) as the bottom panel of Fig. 1, but clearly the shape of $d_c$ as a function of $r_{m2}$ is very different, and there is only one equilibrium. The bottom panel, with $P_m = 0.001$, $\xi = 0.4$, shows a system with three equilibria; $A$ and $C$ are stable, with zero and spin-up torque respectively, while equilibrium $B$ is unstable. In both panels, the discontinuity in $d_c$ arises because the magnetic torque changes sign at $r_{m2} = r_c$, while the viscous torque does not; in reality, the change in the magnetic torque occurs gradually over the domain $r_c - d_B < r_{m2} < r_c + d_B$. Fig. 6 shows the regions in $P_m - \xi$ parameter space for which there are multiple equilibria. Note that, for a given $\xi$, the range of $P_m$ giving rise to multiple equilibria is smaller compared to the case where the viscous torque is neglected (Fig. 2).

The viscous torque on the mixing layer depends on
and spin-up respectively, and one unstable, spin-up equilibrium, marked $A$. Including the viscous torque at the mixing layer. The top panel shows $d_B/r_c$ (solid curve) and $d_c/r_c$ (dashed curve) as functions of $r_{m2}/r_c$, for $P_m = 0.01$, $\xi = 0.4$. There is only one equilibrium, marked $A$, with zero net torque. The bottom panel shows the same information but for $P_m = 0.001$, $\xi = 0.4$. There are two stable equilibria, marked $A$ and $C$, corresponding to zero torque and spin-up respectively, and one unstable, spin-up equilibrium, marked $B$.

Figure 5. Equilibrium states for the disc-magnetosphere system including the viscous torque at the mixing layer. The top panel shows $d_B/r_c$ (solid curve) and $d_c/r_c$ (dashed curve) as functions of $r_{m2}/r_c$, for $P_m = 0.01$, $\xi = 0.4$. There is only one equilibrium, marked $A$, with zero net torque. The bottom panel shows the same information but for $P_m = 0.001$, $\xi = 0.4$. There are two stable equilibria, marked $A$ and $C$, corresponding to zero torque and spin-up respectively, and one unstable, spin-up equilibrium, marked $B$.

Figure 6. Regions in $P_m - \xi$ parameter space for which the system has a single equilibrium or multiple equilibria, with the viscous torque included.

$\partial \Omega/\partial r$ at $r_{m2}$ through $\xi$. The derivation of $\xi$ assumes that the disc velocity profile is Keplerian beyond $r_{m2}$, so that the viscous torque is given by $\xi$. This assumption is questionable, because viscous stresses may result in a non-Keplerian velocity profile for a short distance beyond $r_{m2}$. Fig. 6 sketches two hypothetical velocity profiles in the vicinity of the mixing layer. In one case, the velocity profile is Keplerian outside $r_{m2}$, and the viscous torque on the mixing layer is negative, while in the other case the velocity profile

Figure 7. Schematic diagram of possible angular velocity profiles near the mixing layer (not to scale). Profile $A$ is Keplerian beyond $r_{m2}$, and corresponds to a $-\text{ve}$ torque on the boundary layer ($\partial \Omega/\partial r < 0$ at $r_{m2}$). Profile $B$ is non-Keplerian for a short distance beyond $r_{m2}$, and corresponds to a $+\text{ve}$ torque ($\partial \Omega/\partial r > 0$ at $r_{m2}$).

is non-Keplerian for a short distance beyond $r_{m2}$, and the viscous torque on the mixing layer is positive. Consequently, the system is very sensitive to the velocity profile near the mixing layer. ST93 did not address this issue, because the viscous torque on the mixing layer was neglected in their model. In the special case $\partial \Omega/\partial r = 0$ at $r = r_{m2}$, the viscous torque on the mixing layer vanishes, and the analysis in Section 4 and ST93 is valid.

5 TURBULENCE IN THE MIXING LAYER

The interaction between the accretion disc and the stellar magnetic field at the disc-magnetosphere boundary is not well understood. The boundary is subject to magnetohydrodynamic (MHD) instabilities, e.g. the magnetic interchange instability (MII) and the Kelvin-Helmholtz instability (KHI), which control the turbulent transport of material through the mixing layer, into the magnetosphere, and onto the star (Ghosh & Lamb 1978, 1979; Scharlemann 1973; Spruit & Taam 1993). Such instabilities may also explain the quasi-periodic oscillations in X-ray sources (Baan 1979; Li & Narayan 2004).

In this section, we assess the validity of the claim that the KHI transports material across the mixing layer with a speed $(ST93)$

$$v_{\text{KH}} = r_{m1} |\Omega_k(r_{m1}) - \Omega_s|.$$  \hspace{1cm} (45)

To this end, we wrote a vortex-in-cell (VIC) code to perform two dimensional (2D), inviscid, incompressible, hydrodynamic simulations of the KHI (with $B = 0$ and effective gravity $g_{\text{eff}} = 0$) in an elongated domain, representing a thin mixing layer $(ST93)$. We emphasise that the simulations give some insight into the behaviour of the KHI but they do not faithfully simulate the global physics of the mixing layer. Ideally, in a full treatment, one would perform simulations in an unbounded domain and see how thick the mixing layer grows of its own accord, rather than enforcing a thin layer and testing what sort of flow is consistent with this assumption, as we do here.
5.1 Vortex-in-cell code

The VIC algorithm (Liu & Dooley 2006) discretizes the vorticity field $\omega = \nabla \times \mathbf{v}$ into point vortices, or vortons. The vortons are tracked as particles, while the velocity field $\mathbf{v} = \nabla \times \mathbf{A}$ is obtained by solving Poisson’s equation,

$$\nabla^2 \mathbf{A} = -\omega,$$  

(46)
on a grid, where $\mathbf{A}$ is the vector potential for the velocity field. Vortons are advected by the velocity field; in turn, one can reconstruct the velocity field from the vorton distribution.

Our simulations are performed in a 2D box of dimensions $0 < x < W = 1$, $-L/2 < y < L/2$. The velocity field is tracked on a regular cartesian grid $(x, y)$ tiled by $n_x \times n_y$ cells. The domain is periodic in one direction,

$$\mathbf{v}(x = 0, y, t) = \mathbf{v}(x = 1, y, t),$$  

(47)consistent with the cylindrical geometry of the annular mixing layer. In the other direction, representing the inner and outer edges of the annulus, we set

$$v_y(x, -L/2, t) = v_y(x, L/2, t) = 0,$$  

(48)$$\omega(x, -L/2, t) = \omega(x, L/2, t) = 0.$$  

(49)
To be absolutely consistent with ST93, one would prefer to specify $v_y$ at $|y| = L/2$. Doing so, however, requires the creation of new vortons at these boundaries, a feature that is hard to implement in our code. Initially, the fluids in the inner and outer halves of the annulus are approximately counter-streaming, with $v(x, y, 0) = 0.5\hat{x}$ for $y < 0$ and $v(x, y, 0) = -0.5\hat{x}$ for $y > 0$. Thus, one unit of time in our simulations is equal to the length of the simulation domain divided by the shear speed; physically, this is approximately the spin period of the neutron star. To provide an initial perturbation from which the KHI can grow, each vortex is given a random initial $y$ coordinate in the range $-0.005 < y < 0.005$, as in the top panel of Fig. 8.

5.2 Transport speed

The simulations track the root-mean-square and the maximum of the $y$ component of velocity, $v_{y,\text{rms}}$ and $v_{y,\text{max}}$, from which we can estimate the average and maximum rates at which material is transported across the mixing layer. We performed simulations for $L = 1, 1/2, 1/4, 1/8, 1/16, 1/32$, with $n_y = 64$, $n_x = 64/L$, and $5 \times 10^3$ vortons. Fig. 8 shows that the first vortical structures form on the smallest scale $\lambda$, consistent with the linear KHI growth rate $5\gamma_{KHI} \propto \lambda^{-1}$. These structures merge hierarchically until a single vortex occupies the simulation domain; energy cascades from small to large scales in 2D (Batchelor 1953). The top panel of Fig. 8 shows that $v_{y,\text{max}}$ rises initially, then decays within $\sim 20$ crossing times, asymptoting to a value that depends upon $W/L$. The top panel of Fig. 9 shows that higher aspect ratios $W/L$ give a lower saturated value of $v_{y,\text{max}}$. The behaviour of $v_{y,\text{rms}}$ as a function of $W/L$ is similar to $v_{y,\text{max}}$ with $v_{y,\text{rms}} \approx 0.2v_{y,\text{max}}$ (bottom panels of Fig. 8 and Fig. 9).

Our results imply that when the KHI is confined to an elongated domain, the maximum transport speed across the mixing layer is significantly less than the velocity shear.

However, these results must be interpreted with caution. Our simulations are 2D; they neglect the Coriolis force, gravity, magnetic fields and compressibility; and the vorticity vanishes artificially at the boundaries. One would expect the inclusion of three-dimensionality, magnetic fields and compressibility to decrease $v_{y,\text{max}}$ further, since energy cascades from large to small scales in 3D (Batchelor 1953). compressibility tends to suppress the KHI (Wang & Robertson 1984), and magnetic fields tend to disrupt large eddies (Malagoli et al. 1995). In contrast, it is hard to predict the effect of different boundary conditions, the Coriolis force, and gravity ($g_{\text{eff}} \neq 0$).

6 DISCUSSION AND CONCLUSIONS

In this paper, we present a generalized model of the disc-magnetosphere interaction, which incorporates diffusion of the stellar magnetic field into the disc. We show that, for $\xi \gtrsim Pm^{0.3}$, the system possesses two stable equilibria, corresponding to spin-up and spin-down. (Our calculations indicate rather than prove stability, since they are based on the assumption that changes in the surface density profile of the disc are rapid compared to changes in the position of the magnetosphere.) We suggest that transitions between stable equilibria can be induced, on the viscous time-scale $\tau_{\nu}$, by changes in the mass accretion rate at the outer edge of the disc (extraneous) or changes in the spin frequency of the star (torque feedback). We also show that the viscous torque $N_{\nu}$ is generally much larger than the magnetic torque $NB$; when it is included, the dynamics of the disc-magnetosphere system are very sensitive to the velocity profile at $r_{mz}$, but typically there are fewer combinations of $Pm$ and $\xi$ for which the system is bistable. Preliminary numerical simulations of the KHI indicate that the transport speed across the mixing layer between the disc and magnetosphere is less than the shear speed when the layer is thin. In our model, material is transported across the mixing layer at the radial drift velocity.

Given the presence of two stable equilibria in our model,
it is interesting to ask whether we can draw any parallels with the phenomenon of torque reversals (Nelson et al. 1997). The X-ray pulsars GX 1+4, 4U-1626-67, Cen X-3 and OAO 1657-415 have all been observed to alternate between episodes of spin-up and spin-down, such that the time-scale for transitions between episodes (days) is much shorter than the the duration of individual torque episodes (weeks or years). A successful model for torque reversals should explain the observed properties: (i) Transitions never occur between torques of the same sign. (ii) The magnitude of the torque is $\sim 0.1 N_{\text{char}}$. (iii) Transitions between equilibria occur over $\sim 1$ day ($\tau_\nu$) for large values of viscosity ($\alpha \sim 1$), as observed, but are slow ($\sim 50$ day) for $\alpha \sim 0.01$ (iv) The anti-correlated torque and X-ray luminosity in GX 1+4 is not explained by our model. Also, our model predicts that, in the absence of extraneous perturbations, a star that is spinning up initially eventually transitions to spin down, then evolves toward a state of zero torque, i.e. the model does not exhibit repeated torque reversals in its current, simple form.

The model we have proposed, based on ST93, omits several effects which may be important. First, the disc may generate its own magnetic field via an MHD dynamo, driven by the shear in the mixing layer (as at the solar tachocline). Torkelsson (1998) proposes dynamo action as a mechanism for torque reversals. Second, material may be
transported off the disc to form an outflow or be funneled onto the stellar poles. For the low magnetic Prandtl numbers adopted in our study ($P_m \approx 10^{-2}$), significant bending of the field lines takes place which might lead to a centrifugal wind (Reyes-Ruiz & Stepinski 1996; Lubow et al. 1994). In the disc outside the mixing layer, the magnetic torque due to such a wind might dominate the viscous torque (Blandford & Payne 1982; Pelletier & Pudritz 1992), whereas our model neglects magnetic torques outside of the mixing layer. Third, the mixing layer may not be thin in the radial direction. For example, even if the magnetic field does not penetrate far into the disc, the KHI may cause the mixing layer to thicken. Fourth, we assume the neutron star to be an aligned rotator, whereas the disc-magnetosphere system behaves differently for an oblique rotator (Wang 1997). Finally, our model does not solve for the disc structure in detail. Ideally, one would combine a detailed solution for the disk structure (e.g. Rappaport et al. 2004) with a detailed model for the location of the inner disk radius (which includes magnetic diffusion among other effects).

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