Renormalization of twist-three operators and integrable lattice models.

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Abstract

We address the problem of solution of the QCD three-particle evolution equations which govern the $Q^2$-dependence of the chiral-even quark-gluon-quark and three-gluon correlators contributing to a number of asymmetries at leading order and the transversely polarized structure function $g_2(x_{\text{Bj}})$. The quark-gluon-quark case is completely integrable in multicolour limit and corresponds to a spin chain with non-periodic boundary conditions, while the pure gluonic sector contains, apart from a piece in the Hamiltonian equivalent to XXX Heisenberg magnet of spin $s = -\frac{3}{2}$, a non-integrable addendum which can be treated perturbatively for a bulk of the spectrum except for a few lowest energy levels. We construct a quasiclassical expansion with respect to the total conformal spin of the system and describe fairly well the energy spectra of quark-gluon-quark and three-gluon systems.

Keywords: twist-three operators, evolution, three-particle problem, integrability, spectrum of eigenvalues

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1Alexander von Humboldt Fellow.
1 Twist-three effects in high energy scattering.

The naive parton model [1] provided a first successful microscopic description of several high energy inclusive processes and endowed them with intuitive probabilistic interpretation as the scattering of a probe on an incoherent bunch of free and collinear constituents (quark and gluons) in a hadron target by expressing the cross sections in terms of the densities of partons in a (incoming) hadron or hadrons in a (outgoing) parton. Later it was realized that this is only the first asymptotic term in the infinite series in inverse powers of a hard momentum scale $Q$ where all quantum mechanical interference effects enter the game as power corrections and, loosely speaking, can be interpreted as an overlap of hadron wave functions with different number of partons. For the deep inelastic scattering this series has a firm field theoretical ground in the form of the operator product expansion (OPE) [2, 3]. For other reactions which do not admit the OPE a set of factorization theorems [4] has been proven within the framework of QCD, however, not far beyond the first non-leading term [5] which goes under the name of higher twist. These are the latter terms which are of particular interest since being correlations of more than two field operators they can give new insights into the QCD dynamics in the non-perturbative domain which is still out of a systematic theoretical control.

Typical higher twists in hard reactions are usually associated with power suppressed contributions in a large momentum scale on the background of a dominating twist-two cross section and while being important cannot be studied experimentally with a high accuracy. However, it is not always the case as there exists a class of transverse spin dependent processes where the relevant observable is an asymmetry, $\Delta \sigma = \sigma(\vec{s}_\perp) - \sigma(-\vec{s}_\perp)$, (not a cross section $\sigma = \sigma(\vec{s}_\perp) + \sigma(-\vec{s}_\perp)$) and where the twist-three contributes as a leading effect not contaminated by a lower twist. The well-known twist-three structure function $g_2(x_{Bj})$ [6] measured in the polarized deep inelastic scattering [7] is their good representative. It acquires an operator definition via a Fourier transform of the matrix element of a non-local operator of the quark fields separated by a light-like distance\footnote{We have omitted here (and everywhere below) the path ordered exponential assuming the light-cone gauge $B_+ = 0$.}

$$g_T(x) \equiv g_1(x) + g_2(x) = \frac{1}{2} \int \frac{d\kappa}{2\pi} e^{ix\kappa} \langle h | \bar{\psi}(0) \gamma^\perp \gamma_5 \psi(\kappa n) | h \rangle. \quad (1)$$

Being of the form similar to the conventional unpolarized distributions this quantity is implicitly interaction dependent. This can be made manifest by virtue of QCD equations of motion and the Lorentz invariance. Its genuine twist-three part can be reduced in leading order in the QCD coupling constant to an integral of a quark-gluon-quark correlation function (whose precise definition
is given in Eq. (139) depending on two momentum fractions \( Y(x, x') \sim \langle \bar{\psi}G\psi \rangle \), namely \[ g_{2w-3}^{tw}(x) = \int_1^x \frac{dx'}{x'} \int \frac{dx''}{x'' - x'} \left[ \frac{\partial}{\partial x'} Y(x', x'') + \frac{\partial}{\partial x''} Y(x'', x') \right]. \] 

(2)

Evaluating the correction in \( \alpha_s \) to the structure function \( g_2 \) requires the introduction of three-gluon operators \( G\tilde{G}G \) (see Eq. (4)) which enter through the quark loop coupling\[4]\ and might be responsible for the small-\( x_{Bj} \) behaviour of \( g_2(x_{Bj}) \), — the issue which is under debate\[11].

Another set of observables is single transverse spin asymmetries in hadronic reactions where only one particle in the initial state is polarized (or only polarization of a single particle in the final state is tagged), i.e. the inclusive Compton scattering \( \gamma\vec{p} \to \gamma X \)\[12], the direct photon\[13, 14\], pion\[13\] or jet production in \( p\vec{p} \)-collisions, the Drell-Yan process with measured azimuthal angular dependence of a lepton\[15\] etc. They are of great interest since they involve almost the same quark-gluon-quark and three-gluon correlation functions discussed above but not folded with a coefficient function and thus depend on both (although equal) momentum fractions. This can be used for their extraction or at least for a direct confrontation with phenomenological models available.

Since the data are normally cannot be taken at the same values of the hard momentum variable \( Q \), the correlation functions which enter cross sections are measured with different resolutions and, thus, they differ for different scales. Therefore, the question naturally appears about their relation. Their dependence on \( Q \) is only logarithmic and within the context of QCD it results from the behaviour of the theory at the cut-off. This can be consistently described within the framework of the renormalization group for composite operators. In the language of the QCD improved parton model the task of its description is translated into the construction and solution of the evolution equation for the multi-parton correlation functions. At leading order in the coupling constant, to which we restrict ourselves, the former problem is relatively simple and can be resolved in a straightforward manner\[16, 17, 18\]. However, the second issue is by far more complicated since we face here a three-particle problem and a priori there is a little hope to find analytical solutions to it. The ultimate result of the scale dependence of a correlator is expected to be of the form

\[ F(x, x'|Q^2) = \sum_{\{\alpha\}} \Psi_{\{\alpha\}}(x, x') \left( \frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)} \right)^{f_{(c)p}(\alpha)/\beta_0} \langle \langle \mathcal{F}\{\alpha\}(Q_0^2) \rangle \rangle, \] 

(3)

where \( \mathcal{E}_{\{\alpha\}} \) are the eigenvalues of evolution kernels (and \( f_{(c)p} \) is an extracted colour factor) and \( \Psi_{\{\alpha\}}(x, x') \) are the corresponding eigenfunctions parametrized by a set quantum numbers \( \{\alpha\} \).

Note, that the results of this reference are not complete since the contribution from twist-three two-gluon operators which are related to the three-gluon ones\[10\] were not taken into account and must be included for complete treatment.
usual $\beta_0 = \frac{4}{3}T_F N_f - \frac{11}{3}C_A$ is the leading term of the QCD $\beta$-function and $\langle \langle F_{\alpha\beta}(Q^2_0) \rangle \rangle \rangle$ stand for reduced matrix elements of local operators at a low normalization point $Q_0$.

Very recently a guiding principle has been found which has allowed to cope the second problem and it is based on the integrability of spin lattice models to which the corresponding evolution equations can be reduced [19, 20, 21, 22]. These findings are analogous to a similar equivalence encountered in the problem of description of the Regge behaviour of QCD amplitudes [23, 24, 25, 26]. The studies have been undertaken for the three-quark [19, 20] and chiral-odd quark-gluon-quark [19, 21, 22] operators and allowed to find the anomalous dimensions, hereafter referred to as energies, of the eigenstates as well as their eigenfunctions in a WKB type manner where the rôle of the Planck constant has been played by the inverse total conformal spin of the system. These results compared quite well with an explicit numerical diagonalization of the mixing matrix and, therefore, they provided a reasonable approximation.

In the present paper we continue the study along this line and give almost complete description of the three-gluon and quark-gluon-quark chiral even sector. Our presentation will be organized as follows. Next section is devoted to the study of the purely gluonic evolution equation. This means that we discard completely from our consideration the mixing with operators containing quark fields. We construct the evolution equation for corresponding correlator and because of its complexity we are forced to find an effective approximation to it. By making use of the conformal symmetry the problem is reduced finally to a quantum mechanical problem for particles with a conformal invariant pair-wise interaction and it turns out that the corresponding total Hamiltonian is a sum of an integrable piece equivalent to the Heisenberg spin chain of spin $s = -\frac{3}{2}$ and an addendum which breaks the integrability but still can be treated as a small perturbation for a large part of the spectrum. However, it plays its crucial rôle for the generation of a fine structure of lowest energy levels. Next, we address the non-singlet quark-gluon-quark sector which in the limit of infinite number of colours is shown to be identical to an inhomogeneous spin chain with non-periodic boundary conditions. In both cases, gluon and quark-gluon problems, we are entirely interested in the polynomial solutions of the lattice models, which play an exceptional and distinguished rôle for physics being implied by the OPE, since each polynomial will correspond to a multiplicatively renormalizable local operator. However, non-polynomial solutions are equally interesting albeit their physical relevance within the present context remains obscure presently. Finally, we summarize.
2 Gluonic sector.

In this section we address the question of the diagonalization of the purely gluonic twist-three evolution equation for a correlation function which contributes to $g_2$. We will do it in parallel with an operator $\mathcal{T}$ which possesses exactly solvable interaction and thus is very insightful for the realistic case. Thus, we introduce

$$
\begin{align*}
\left\{ \begin{array}{l}
G_\alpha \\
T_{\alpha\beta\gamma}
\end{array} \right\}(\kappa_1, \kappa_2, \kappa_3) &= g \left\{ \begin{array}{l}
f_{abc} g_{\alpha\nu} g_{\mu\rho} \\
d_{abc} \tau_{\alpha\beta\gamma;\mu\nu\rho}
\end{array} \right\} G^{\alpha\perp}_{+\mu}(\kappa_3 n) G^{\beta\perp}_{+\nu}(\kappa_2 n) G^{\gamma\perp}_{+\rho}(\kappa_1 n),
\end{align*}
$$

where the totally symmetric, w.r.t. the independent permutation of $\{\alpha, \beta, \gamma\}$ and $\{\mu, \nu, \rho\}$ indices, tensor is $\tau_{\alpha\beta\gamma;\mu\nu\rho} \equiv \mathcal{P}(\alpha, \beta, \gamma) g_{\mu\alpha} (g_{\nu\beta} g_{\rho\gamma} + g_{\nu\gamma} g_{\rho\beta}) - \frac{1}{2} \mathcal{P}(\mu, \nu, \rho) g_{\mu\nu} (g_{\alpha\beta} g_{\rho\gamma} + g_{\alpha\gamma} g_{\rho\beta} + g_{\beta\gamma} g_{\rho\alpha})$. The momentum fraction space functions are

$$
F(x_1, x_3) = \int \frac{dk_1}{2\pi} \frac{dk_3}{2\pi} e^{ix_1 k_1 - ix_3 k_3} \langle h | F(k_1, 0, k_3) | h \rangle,
$$

with $\mathcal{F} = \mathcal{G}, \mathcal{T}$ and possess the symmetry properties $G(x_1, x_3) = -G(-x_3, -x_1), T(x_1, x_3) = T(-x_3, -x_1)$. They obey the evolution equation

$$
\frac{d}{d\ln Q^2} F(x_1, x_3) = -\frac{\alpha_s}{2\pi} \int \prod_{i=1}^{3} dx_i' \delta (x_1' - x_3 + x_2') K^F \left( \{x_i\}\{x_i'\} \right) F(x_1', x_3'),
$$

where $x_2 = x_3 - x_1$.

Since the existing studies \[8, 27\] did not provide an insight into the structure of the corresponding kernel $K^F$ we reanalyze the issue anew showing a simple structure of the result which allows for a reduction of the problem to a lattice model characterized by a Hamiltonian which consists of two parts: exactly integrable piece equivalent to the generalized Heisenberg spin chain and a one which violates this property and is responsible for the formation of a fine structure of low lying levels.

2.1 Three-gluon evolution equation.

To start with, note first of all that only ‘good’ transverse components of the fields enter the operators and, therefore, the latter belong to a special class of the so-called quasi-partonic operators. They are distinguished by the property that the renormalization does not move

\[4\] The $f_{abc}$-coupling in $\mathcal{T}$-operator gives identical zero provided the total derivatives are irrelevant like for the forward matrix elements considered presently. Obviously, due to negative $C$-parity the $d_{abc}$-coupling does not contribute to the deep inelastic scattering cross section with an electromagnetic probe.

\[5\] Here the action of the operator of cyclic permutation $\mathcal{P}$ is defined as $\mathcal{P}(1, 2, 3)f(1, 2, 3) \equiv f(1, 2, 3) + f(2, 3, 1) + f(3, 1, 2)$.
Figure 1: Diagrams (in the light-cone gauge) giving rise to the gluon-gluon two-particle evolution kernel. The graph with the crossed gluon line corresponds to the contact-type contribution arising from the use of the equation of motion.

them outside of the class and that at leading order the total evolution kernel is a linear combination of the twist-two non-forward kernels in subchannels.

The pair-wise gluon evolution kernel can be decomposed as follows into Lorentz tensors

\[
\begin{align*}
\mathcal{K}_{gg}^{a'b',\mu\nu}(x_1, x_2 | x'_1, x'_2) &= -\frac{\alpha_s}{2\pi} \left\{ \frac{1}{2} g_{\mu\nu} g_{a'b'} g_{\mu'\nu'} g_{gg}^{A} + \frac{1}{2} \epsilon_{\mu'\nu'} \epsilon_{\mu\nu} g_{gg}^{T} \right\}^{ab}_{\alpha\beta} (x_1, x_2 | x'_1, x'_2), \quad (7)
\end{align*}
\]

with \( g_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu - n^\ast_\mu n^\ast_\nu \), \( \epsilon_{\mu'\nu'} = \epsilon_{\mu'\nu'\rho\sigma} n_\rho n_\sigma \) and \( \tau_{\mu'\nu'\rho\sigma} = \frac{1}{2} \left( g_{\mu'\nu'\rho\sigma} + g_{\mu'\nu'\rho\sigma} - g_{\mu'\nu'\rho\sigma} \right) \); and colour structures

\[
\begin{align*}
\{ g_{gg}^{T} \}^{ab}_{a'b'} = \frac{1}{N_c^2 - 1} \delta_{ab} \delta_{a'b'} g_{gg}^{T(1)} + \frac{1}{N_c} f_{abc} f_{a'b'c} g_{gg}^{T(8)A} + \frac{N_c}{N_c^2 - 4} d_{abc} d_{a'b'c} g_{gg}^{T(8)A} + \cdots, \quad (8)
\end{align*}
\]

where the dots stand for the decuplet and 27-plet contributions which are irrelevant for our consequent discussion.

We can easily evaluate from diagrams in Fig. 1 the kernels for the parity even and transversity channels

\[
\begin{align*}
f_{(c)}^{-1} g_{gg}^{V} (x_1, x_2 | x'_1, x'_2) &= k(x_1, x_2 | x'_1, x'_2) + 2 \frac{x_1 x_2 + x'_1 x'_2}{x'_1 x'_2} \Theta_{111}^0 (x_1, -x_2, x_1 - x'_1) \\
&+ 2 \frac{x_1 x_2 x'_1 + x_2 x'_2}{x'_1 x'_2 (x_1 + x_2)^2} \Theta_{111}^0 (x_1, -x_2) \pm (x'_1 \leftrightarrow x'_2), \quad (9)
\end{align*}
\]

\[
\begin{align*}
f_{(c)}^{-1} g_{gg}^{T} (x_1, x_2 | x'_1, x'_2) &= k(x_1, x_2 | x'_1, x'_2) \pm (x'_1 \leftrightarrow x'_2), \quad (10)
\end{align*}
\]

with “+” sign for \( f_{(8)} = f_{(1)} / 2 = C_A / 4 \) and “−” sign for \( f_{(8)A} = C_A / 4 \). For the parity odd sector we have

\[
\begin{align*}
f_{(c)}^{-1} g_{gg}^{A} (x_1, x_2 | x'_1, x'_2) &= k(x_1, x_2 | x'_1, x'_2) + 2 \frac{x_1 x_2 + x'_1 x'_2}{x'_1 x'_2} \Theta_{111}^0 (x_1, -x_2, x_1 - x'_1) \\
&+ 2 \frac{x_1 x_2}{x'_1 x'_2} \Theta_{111}^0 (x_1, -x_2) \mp (x'_1 \leftrightarrow x'_2),
\end{align*}
\]
with “±” signs assigned vice versa to the previous case. The generalized step functions look as follows

$$\Theta_{11}(x_1, x_2) = \frac{\theta(x_1)}{x_1 - x_2} + \frac{\theta(x_2)}{x_2 - x_1},$$

$$\Theta_{111}(x_1, x_2, x_3) = \frac{x_1 \theta(x_1)}{(x_2 - x_1)(x_1 - x_3)} + \frac{x_2 \theta(x_2)}{(x_1 - x_2)(x_2 - x_3)} + \frac{x_3 \theta(x_3)}{(x_1 - x_3)(x_3 - x_2)}.$$

We have introduced the notation

$$k(x_1, x_2 | x_1', x_2') = \frac{x_1}{x_1'} \left[ \frac{x_1}{x_1 - x_1'} \Theta_{11}(x_1, x_1 - x_1') \right] + \frac{x_2}{x_2'} \left[ \frac{x_2}{x_2 - x_2'} \Theta_{11}(x_2, x_2 - x_2') \right] + \left( \frac{1}{2} \frac{\beta_0}{C_A} + 2 \right) \delta(x_1 - x_1'),$$

for the part which appears in all kernels.

Introducing the conventions

$$\tilde{K}(x_i, x_j | x_i', x_j') = K(x_i, x_j | x_i', x_j') \delta(x_i + x_j - x_i' - x_j'),$$

and $K^{i\pm j} \equiv \frac{1}{2} (K^i \pm K^j)$, we can contract the generalized kernels (7) with the tensor structure of the operators in question (4) and construct easily the total evolution equation for the three-gluon quasi-partonic correlator $G$ (see Fig. 2 for its pictorial representation)

$$K^G = 2^{gg} \tilde{K}_{(8A)}^{A+T}(x_1, x_2 | x_1', x_2') + 2^{gg} \tilde{K}_{(8A)}^{A+T}(x_2, -x_3 | x_2', -x_3')$$

$$+ \tilde{K}_{(8A)}^{V}(x_1, -x_3 | x_1', -x_3') - \tilde{K}_{(8A)}^{V-T}(x_1, x_2 | x_1', -x_3') - \tilde{K}_{(8A)}^{V-T}(x_2, -x_3 | x_1', -x_3')$$

$$- \frac{i}{2} \beta_0 \delta(x_1 - x_1') \delta(x_3 - x_3'),$$

and for the $T$

$$K^T = \tilde{K}_{(8S)}^{T}(x_1, x_2 | x_1', x_2') + \tilde{K}_{(8S)}^{T}(x_2, -x_3 | x_2', -x_3') + \tilde{K}_{(8S)}^{T}(x_1, -x_3 | x_1', -x_3')$$

$$- \frac{i}{2} \beta_0 \delta(x_1 - x_1') \delta(x_3 - x_3').$$

Figure 2: Structure of the total evolution kernel for the three-gluon correlation functions $G(x_1, x_3)$. The symbol m.c. stands for the mirror symmetrical contributions of the last two diagrams.
Defining the moments as
\[ F^j_l = \int dx_1 dx_3 x_1^l x_3^{l-j} F(x_1, x_3), \]
with the properties \( G^j_k = -(-1)^j G^{j-k}_{j-k} \), \( T^j_k = (-1)^j T^{j-k}_{j-k} \), we can write, given the results of Eqs. (13) and (14), the evolution equation for the moments, i.e. for the local operators \( F^j_l \propto g G^j_k (i\partial_+) \right)^j G^{j-k}_{j-k+1} \), in the form:
\[ \frac{d}{d \ln Q^2} F^j_l = -\frac{\alpha_s}{4\pi} \sum_{k=0}^{j} \Gamma_{jk}^F(J) F_k^j, \quad \Gamma_{jk}^F(J) = C_A \gamma_{jk}^F(J) + \beta_0 \delta_{jk}, \]
with the anomalous dimension matrices
\[ \gamma_{jk}^F(J) = \delta_{jk} \left\{ 3\psi(j+3) - 3\psi(1) + 2 \left( \frac{(-1)^{j-k}}{(j+1)^4} (j+1) C^{j-k+2}_{j-k+1} + 3(j+2) C^{j-k+1}_{j-k+1} \right) - \frac{1}{j+2} \right\} \]
\[ + \theta_{k,j+1} \left\{ 2(-1)^{k-j} \frac{C_{j-k+1}^{j+1}}{C_{j-k+1}^{j+1}} \left( \frac{k-j}{(j+1)(k+2)} + \frac{k-j}{(j+2)(j-k+1)} - \frac{(j+1)(j-k+1)}{(j+2)_j^3} \right) \right. \]
\[ - \left. \frac{(k+1)(j+1)}{(j+2)_j^3} - \frac{1}{j+2} \right\} - 2 \left( \frac{j-k+1}{(j+1)_j^3} \left( \frac{(-1)^{j-k} C^{j-k}_{j-k+2}}{(j+1)_j^4} \right) + 1 \right) \]
\[ + 2 \left( \frac{(-1)^{j-k}}{(j+1)_j^4} \left( \frac{1}{(j+1)_j^4} \right) \right) \left( C_{j-k+1}^{j-k+2} + (j+2)(3C_{j-k+1}^{j-k+1} - (j+4)C_{j-k+1}^{j-k+1}) \right) \]
\[ + 2 \left( \frac{(-1)^{j-k}}{(j+1)_j^4} \left( \frac{1}{(j+1)_j^4} \right) \right) \left( C_{j-k+1}^{j-k+2} + (j+2)(3C_{j-k+1}^{j-k+1} - (j+4)C_{j-k+1}^{j-k+1}) \right) \]
\[ - \frac{1}{k-j} \left( \frac{(-1)^{j-k} C_{j-k+1}^{j-k+2}}{C_{j-k+1}^{j-k+2}} + (j-k+1)_j^2 \right) \right\} + \left( j \rightarrow j-j, \quad k \rightarrow j-k \right), \]
and\( ^6 \)
\[ \gamma_{jk}^T(J) = \delta_{jk} \left\{ 3\psi(j+3) - 3\psi(1) \right\} \]
\[ - \theta_{k,j+1} \left( \frac{1}{k-j} \left\{ (-1)^{j-k} \frac{C^{j-k}_{j-k+2}}{C_{j-k+2}} + (j-k+1)_j^2 \right\} + \left( j \rightarrow j-j, \quad k \rightarrow j-k \right) \right), \]
where we introduced the Pochhammer symbol \( (j)_{\ell} = \frac{\Gamma(j+\ell)}{\Gamma(j)} \), the binomial coefficients \( C^j_k = \frac{\Gamma(j+1)}{\Gamma(j+1)\Gamma(j-k+1)} \) and the step functions \( \theta_{j,k} = \{ 1, \text{ if } j \geq k; 0, \text{ if } j < k \} \). Our mixing matrix \( ^{17} \) differs from the one obtained in Ref. \( ^{8} \) since the evolution equation was written there for a different quantity, i.e. a linear combination of \( G \)-functions considered here \( ^{4,5} \). But both matrices have indeed identical eigenvalues with different eigenvectors.

\(^6\)To get these anomalous dimensions we assumed the gluons being different so that there is no symmetrization in the pair-wise kernels. Identity of gluons will just pick up eigenstates with a definite symmetry. With the matrix \( ^{18} \) at hand we have the states with all possible symmetries.
2.2 Reduction to 1D lattice model.

The key observation is that not all of the parts of the anomalous dimension matrix (17) play an equally important role in the generation of the energy spectrum. Note first that due to the antisymmetry of the three-particle correlator \( G(x_1, x_3) \), w.r.t. the interchange of the momentum fractions of the first and last gluon, the leading rightmost \( \frac{1}{2j} \)-singularity in the \( j \)-plane of the vector kernel disappears. For the formation of the upper part of the spectrum \( j, k \sim J \) only \([1/(x_i - x_i')]_+\)-distributions are relevant while the fine structure of low levels is generated by the \( K^A + K^T \)-part of the total kernel. Making use of these observations we can deduce a simplified evolution kernel which reproduces with a good accuracy the exact one. Namely, the simplified anomalous dimension matrix reads

\[
\gamma_{jk}(J) = \delta_{jk} \left\{ 3\psi(j + 3) - 3\psi(1) - \frac{4}{(j + 2)^2} \right\}
- \theta_{j-1,k} \left\{ \frac{1}{k - j} (\frac{(-1)^{k-j} C_{j-k}^f}{C_{k+2}^f} + \frac{(J - k + 1)_2}{(J - j + 1)_2}) + 4 \frac{J - k + 1}{(J - j + 1)_3} \right\} + \left( j \to J - j \right), \tag{19} \]

and its eigenvalues coincide with the ones of Eq. (17) with a high precision, see Fig. 3. This anomalous dimension matrix corresponds to the kernel

\[
K = \frac{C_A}{2} \left\{ \hat{k}(x_1, x_2|x_1', x_2') + \hat{k}(x_2, -x_3|x_2', -x_3') + \hat{k}(x_1, -x_3|x_1', -x_3') \right.
+ \left. \hat{\Delta}(x_1, x_2|x_1', x_2') + \hat{\Delta}(x_2, -x_3|x_2', -x_3') \right\} - \frac{1}{2} \delta_0 \delta(x_1 - x_1') \delta(x_3 - x_3'), \tag{20} \]

with \( k \) defined in Eq. (14) and

\[
\Delta(x_1, x_2|x_1', x_2') = \frac{2 x_1 x_2 + x_1' x_2'}{x_1' x_2'} \Theta_{111}(x_1, -x_2, x_1 - x_1') + \frac{2 x_1 x_2}{x_1' x_2'} \Theta_{111}(x_1, -x_2). \tag{21} \]

As is well known the tree level conformal invariance of QCD [28] allows for the diagonalization of the pair-wise evolution kernels in subchannels by means of the Jacobi polynomials \( P_j^{(\nu_0, \nu_a)} \) [29-31] and express the two-particle Hamiltonians as a function of corresponding Casimir operators. Hence, we find

\[
\int dx_1 dx_2 P_j^{(2,2)} \left\{ \hat{k} \right\} (x_1, x_2|x_1, x_2') = \left\{ \begin{array}{ll}
2\psi(j + 3) - 2\psi(1) + \frac{\delta_0}{2 C_A} & \frac{4}{(j+2)(j+3)} \end{array} \right\} P_j^{(2,2)} \left( \begin{array}{c}
x_1 - x_2 \\
x_1' + x_2'
\end{array} \right), \tag{22} \]

where \( j \) is a conserved quantum number related to the conformal spin of the composite two-particle conformal operator \( O_{jk} = \phi_b (i \partial_a + i \partial_b) \delta^k P_j^{(\nu_0, \nu_a)} ((\partial_a - \partial_b)/(\partial_a + \partial_b)) \phi_a \), composed from elementary fields of conformal weight \( \nu = d + s - 1 = 2 \) for gluons. Namely, the eigenvalues of the pair-wise Casimir operator of the collinear conformal algebra \( su(1,1) \), \( J_{ab}^2 \), on these states are
Figure 3: The exact energy spectrum (17) versus the selected trajectories generated by the approximate formula (19).

\[ J_{ab}(J_{ab} - 1) \] where \( J_{ab} = j + \frac{1}{2}(\nu_a + \nu_b + 2) \). Therefore, we can substitute the original problem of the diagonalization of the kernel (20)

\[ K \equiv \frac{C_A}{2} H (\{x_i\}\{x'_i\}) + \frac{\beta_0}{2} \delta(x_1 - x'_1) \delta(x_3 - x'_3), \] (23)

by a quantum mechanical problem of the diagonalization of the Hamiltonian, \( H \rightarrow \mathcal{H} \),

\[ \mathcal{H} = \mathcal{H}_0 + \mathcal{V}, \] (24)

where

\[ \mathcal{H}_0 = h_{12} + h_{23} + h_{31}, \quad \mathcal{V} = v_{12} + v_{23}, \] (25)

with the pair-wise interaction

\[ h_{ab} = 2\psi(\hat{J}_{ab}) - 2\psi(1), \quad v_{ab} = -\frac{4}{\hat{J}_{ab}^2}. \] (26)

Here the operator \( \hat{J}_{ab} \) is defined as a formal solution of the equation \( \hat{J}_{ab}^2 = \hat{J}_{ab}(\hat{J}_{ab} - 1) \) and possesses the eigenvalues \( J_{ab} \). Obviously, \([\mathcal{H}, \hat{J}_{ab}^2]_-=0\) where \( \hat{J}^2 = \hat{J}_{12}^2 + \hat{J}_{23}^2 + \hat{J}_{31}^2 - 9/4 \) is the total Casimir operator of the gluonic system and, therefore, it is the integral of motion. The spectrum of \( \mathcal{H}_0 \) part corresponds to the anomalous dimensions of the operator \( \mathcal{T} \).
The explicit form of the operators \( \hat{J}_{ab}^2 \) depends on a basis where their action is defined. For our purposes we find it convenient to use the space spanned by the elements \( \theta^k \equiv \frac{\partial^k \phi}{\Gamma(k+\nu+1)} \) \[21, 22\]. Then, it is easy to read off their form from the action of the generators of the collinear conformal group on the elementary field operators. Finally, we have for the step-up, the step-down and the grade operators the representation

\[
\hat{J}^+ = (\nu + 1) \theta + \theta^2 \frac{\partial}{\partial \theta}, \quad \hat{J}^- = \frac{\partial}{\partial \theta}, \quad \hat{J}^3 = \frac{1}{2} (\nu + 1) + \theta \frac{\partial}{\partial \theta}. \tag{27}
\]

The quadratic Casimir operator is given by \( \hat{J}^2 \) and equals \( \hat{J}^3 \hat{J}^3 - 1 \) for a one-particle state. The two-particle Casimir which enters Eq. (26) is

\[
\hat{J}_{ab}^2 = -\theta_{ab}^{-1}(\nu_a + \nu_b)/2 \partial_a \partial_b \theta_{ab}^{1+(\nu_a + \nu_b)/2} + \frac{1}{2} (\nu_b - \nu_a) \theta_{ab} (\partial_a + \partial_b). \tag{28}
\]

### 2.3 Isotropic Heisenberg magnet.

In this section we show that the Hamiltonian \( H_0 \) possesses, apart from \( \hat{J}^2 \), an extra integral of motion which makes the former exactly integrable. We will find that it corresponds to the generalized homogeneous XXX chain \[33\] with sites of equal spin \( s = -\frac{\nu}{2} = -\frac{3}{2} \). To demonstrate this we use the formalism of the Quantum Inverse Scattering Method \[36, 37, 38, 39\] and consider the lattice with equal conformal weights on each site \( \nu_\ell = \nu \). To prove the integrability of the Schrödinger equation

\[
H_0 \Psi_0 = E_0 \Psi_0, \tag{29}
\]

let us define a matrix \( R_{a,b} \) acting on the product of the vector space of the site \( V_a \) and an auxiliary space \( V_b \). It satisfies the Yang-Baxter equation \[37, 38, 39, 40\]

\[
R_{a,b}(\lambda - \mu) R_{c,a}(\lambda) R_{c,b}(\mu) = R_{b,c}(\mu) R_{a,c}(\lambda) R_{a,b}(\lambda - \mu). \tag{30}
\]

The solution to this equation for the case when the dimensions of the quantum and auxiliary spaces coincide and which is of the most interest for us is given by \[41, 42, 43\]

\[
R_{a,b}(\lambda) = f(\nu, \lambda) \frac{\Gamma(\hat{J}_{ab} + \lambda) \Gamma(\nu + 1 - \lambda)}{\Gamma(\hat{J}_{ab} - \lambda) \Gamma(\nu + 1 + \lambda)}. \tag{31}
\]

It is defined up to an arbitrary c-number function \( f(\nu, \lambda) \). In the case when the auxiliary space is two-dimensional the \( R \)-matrix gives the Lax operator

\[
L_a(\lambda) \equiv R_{a,b} \left( \lambda - \frac{1}{2} \right) = \lambda \mathbb{I} + \sigma^3 \hat{J}^3, \tag{32}
\]

where \( \hat{J} = (\hat{J}^1, \hat{J}^2, \hat{J}^3) \) with \( \hat{J}^1 = \frac{1}{2} (\hat{J}^- - \hat{J}^+) \) and \( \hat{J}^2 = \frac{1}{2} (\hat{J}^- + \hat{J}^+) \).
Following a standard procedure we define the auxiliary and the fundamental monodromy matrices

\[ T_2(\lambda) = L_{a_1}(\lambda)L_{a_2}(\lambda)L_{a_3}(\lambda), \quad T_b(\lambda) = R_{a_{1,b}}(\lambda)R_{a_{2,b}}(\lambda)R_{a_{3,b}}(\lambda), \]

satisfying the \(RTT\)-equation which is Eq. (30) but with two \(R\)-matrices replaced by the monodromies. It means that \(R(\lambda-\mu)\) intertwines the two possible co-multiplications of the \(T\)-matrices. This immediately leads to the commutation relations

\[
[t_{b_1}(\lambda_1), t_{b_2}(\lambda_2)]_- = 0
\]

between the transfer matrices,

\[ t_b(\lambda) = \text{tr}_b T_b(\lambda), \]

which act on the total space of the spin chain \(\bigotimes_{\ell=1}^3 V_{a_{\ell}}\).

The fundamental transfer matrix gives us the total Hamiltonian of the chain \([40, 41, 42]\)

\[
\mathcal{H}_0 = \left. \frac{d}{d\lambda} \right|_{\lambda=0} \ln t_b(\lambda) = h_{a_1,a_2} + h_{a_2,a_3} + h_{a_3,a_1},
\]

with the two-site Hamiltonians

\[ h_{a,b} = R_{a,b}(0)R'_{a,b}(0) = 2\psi \left( \hat{\mathcal{J}}_{a,b} \right) - 2\psi(1), \]

coinciding with Eq. (26) for the one-particle conformal weight \(\nu = 2\) and provided \(f(\nu, \lambda) = \frac{\Gamma(1-\lambda) \Gamma(\nu+1+\lambda)}{\Gamma(1+\lambda) \Gamma(\nu+1-\lambda)}\). The expansion of the auxiliary transfer matrix in the rapidity \(\lambda\) gives

\[
t_{\frac{1}{2}}(\lambda) = 2\lambda^3 + \lambda \left( \hat{\mathcal{J}}^2 - 3\hat{\mathcal{J}}^2 \right) + iQ,
\]

where the total conformal spin and the ‘hidden’ charge are

\[
\hat{\mathcal{J}}^2 = (\hat{\mathcal{J}}_1 + \hat{\mathcal{J}}_2 + \hat{\mathcal{J}}_3)^2, \quad Q = 2\epsilon_{ijk}\hat{\mathcal{J}}_i^2\hat{\mathcal{J}}_j^2\hat{\mathcal{J}}_k^2,
\]

respectively. Note that \(Q\) can be represented as a commutator of the two-particle Casimir operators \([\hat{\mathcal{J}}_2^2, \hat{\mathcal{J}}_3^2]_- = [\hat{\mathcal{J}}^2_{12}, \hat{\mathcal{J}}^2_{23}]_- = [\hat{\mathcal{J}}^2_{13}, \hat{\mathcal{J}}^2_{12}]_- = [\hat{\mathcal{J}}^2_{23}, \hat{\mathcal{J}}^2_{13}]_-\). For the later convenience we introduce a convention for the eigenvalues of the quadratic Casimir operator of the chain, namely,

\[
\eta^2 \equiv \left( J + 3/2(\nu + 1) \right) \left( J + 3/2(\nu + 1) - 1 \right).
\]

For gluons at large \(J\) we have \(\eta = J + 4 + O(J^{-1})\).
2.4 Conformal basis and recursion relation.

The eigenfunction of the three-particle system can be found making use of the existence of the additional non-trivial integral of motion

\[ Q\Psi_0 = q\Psi_0, \]  

where \( q \) are the eigenvalues of \( Q \).

The operator product expansion suggests that we must be primarily interested in polynomial solutions of the above equation. We can make further profit of conformal invariance of the system by looking the solution to this equation in the form of expansion w.r.t. a three-particle basis:

\[ \Psi_0 = \sum_{j=0}^J \Psi_j \mathcal{P}_{J,j}(\theta_1, \theta_2), \]  

where the basis vectors

\[ \mathcal{P}_{J,j}(\theta_1, \theta_2) = \frac{(2j + 5)}{(j + 3)^{j+3}} \frac{\Gamma(J + j + 6)}{\Gamma^{1/2}(2J + 8)} \theta_1^{J} \theta_2^{-J} F_1 \left( \frac{j - J, j + 3}{2j + 6}; \theta \right), \quad \text{with} \quad \theta \equiv \frac{\theta_{13}}{\theta_{23}}, \]  

diagonalize simultaneously \([20, 21, 22]\) the total \( \hat{J}^2 \) and \( \hat{J}_{13}^2 \) Casimir operators

\[ \hat{J}^2 \mathcal{P}_{J,j}(\theta_1, \theta_2) = \left( J + \frac{9}{2} \right) \left( J + \frac{7}{2} \right) \mathcal{P}_{J,j}(\theta_1, \theta_2), \]  
\[ \hat{J}_{13}^2 \mathcal{P}_{J,j}(\theta_1, \theta_2) = (j + 2) (j + 3) \mathcal{P}_{J,j}(\theta_1, \theta_2), \]  
and are normalized to unity \( \langle \mathcal{P}_{J',j'}(\theta_1,\theta_2|\theta_3)\mathcal{P}_{J,j}(\theta_1,\theta_2|\theta_3) \rangle = \delta_{J',J}\delta_{j',j} \) w.r.t. \( SU(1,1) \) invariant scalar product

\[ \langle \chi_2(\theta_1, \theta_2, \theta_3) | \chi_1(\theta_1, \theta_2, \theta_3) \rangle = \int \prod_{|\theta_i| \leq 1} d\theta_i d\bar{\theta}_i (1 - \theta_i \bar{\theta}_i)^{\nu - 1} \chi_2(\bar{\theta}_1, \bar{\theta}_2, \bar{\theta}_3) \chi_1(\theta_1, \theta_2, \theta_3). \]  

The Hamiltonian \( \mathcal{H}_0 \) of the system exhibits cyclic permutation symmetry of the sites of the chain. Therefore, its eigenfunctions \( \Psi_0 \) could change at most their phase under this transformation

\[ P\Psi_0(\theta_1, \theta_3|\theta_2) \equiv \Psi_0(\theta_2, \theta_1|\theta_3) = e^{i\varphi}\Psi_0(\theta_1, \theta_3|\theta_2). \]  

Then

\[ \theta \xrightarrow{P} 1 - \frac{1}{\theta} \xrightarrow{P} \frac{1}{1 - \theta}, \]  

for the “anharmonic” ratio of the conformal basis. Since the triple action of \( P \) leads to the system at its initial state, the eigenvalues of the operator are cubic roots of unity

\[ P^3 = 1, \quad \varphi = 0, \frac{2}{3}\pi, \frac{4}{3}\pi. \]  

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Since this phase appears as a result of the cyclic shift of the chain sites it has an obvious interpretation of the momentum of the lattice. Obviously, it is the function of the conserved charges and is expressed through the equality

$$\varphi = \arg \frac{\sum_{j=0}^J (-i)^j (2j+5)\Upsilon_j}{\sum_{j=0}^J i^j (2j+5)\Upsilon_j},$$

with $\Upsilon_j$ related to the expansion coefficients $\Psi_j$ via Eq. (52). It follows from this definition that $\varphi(-q) = -\varphi(q)$.

The cyclic permutation transforms different three-particle bases and since each of them corresponds just to a different quantum mechanical addition of three spins they are related by the conventional $6j$-symbols as

$$P_{J;j}^3(\theta_2, \theta_1|\theta_3) = \sum_{k=0}^J W_{jk}(J)P_{J;j}(\theta_1, \theta_3|\theta_2).$$

The Racah coefficients fulfill the following properties: $\sum_{\ell=0}^J W_{j\ell}W_{k\ell} = \delta_{jk}$ which is a consequence of the completeness condition $\sum_{j=0}^J |P_{J;j}\rangle\langle P_{J;j}| = 1$; $\sum_{\ell,m=0}^J W_{j\ell}W_{\ell m}W_{mk} = \delta_{jk}$ coming from $P^3 = 1$; and $W_{jk} = (-1)^{j+k}W_{kj}$ reflecting the relation $P^2 = P_{13}PP_{13}$.

Introducing new conventions

$$\Psi_j = (-i)^j \varrho_j \Upsilon_j,$$

with

$$\varrho_j \equiv \left[ \frac{(j+1)4(j+j+6)2(j-j+1)2}{2j+5} \right]^{-1/2},$$

we can deduce from Eq. (51) the recursion relation for the expansion coefficients $\Upsilon_j$

$$q\Upsilon_j = a_j\Upsilon_{j+1} + b_j\Upsilon_{j-1},$$

where

$$a_j = \frac{(j+1)(j+3)}{2(2j+5)}(J+j+6)(J-j+2), \quad b_j = \frac{(j+2)(j+4)}{2(2j+5)}(J+j+7)(J-j+1),$$

and satisfying the boundary conditions

$$\Upsilon_{-1} = \Upsilon_{J+1} = 0.$$ 

Obviously, the coefficients $\Upsilon_j$ are related for positive and negative values of the eigenvalues of integral of motion due to $P_{13}QP_{13}^{-1} = -Q$ as

$$\Upsilon_j(-q) = (-1)^j\Upsilon_j(q).$$

And since $P_{13}P_{J;j}(\theta_1, \theta_3|\theta_2) = (-1)^jP_{J;j}(\theta_1, \theta_3|\theta_2)$ we conclude from here that $P_{13}\Psi_0(q) = \Psi_0(-q)$. 

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Figure 4: The log of the absolute value of the numerically diagonalized conserved charge $q$ with $q = 0$ trajectory being removed. The two possibilities of the description are clearly seen: from the top by the trajectories which behave at large $J$ as $q \propto J^3$; and from the bottom with the behaviour $q \propto J^2$.

2.5 Conserved charge.

The exact analytical solution based on the Eq. (54) is difficult. We, therefore, perform our analysis for large conformal spin $J$.

There exists, however, the exceptional point $q = 0$ when the exact solution can be easily found since Eq. (54) is reduced to a two-term relation. Due to Eq. (57) we have $\Upsilon_{2k+1}(0) \equiv 0$ with $k = 0, 1, \ldots, J/2$ and from the condition of fulfilling the boundary condition $\Upsilon_{J+1} = 0$ we conclude that this solution exists only for even $J$. For even $j \equiv 2k$ we get

$$\Upsilon_{2k}(0) = (-1)^k \Upsilon_0(0) \prod_{\ell=1}^k \frac{b_{2\ell-1}}{a_{2\ell-1}} = \frac{(-1)^k (2k + 1)!!(2k + 3)!!}{3 \cdot 4^k \cdot k!(k + 1)!} \left( \frac{J+2}{2} - k \right)_k \left( \frac{J+8}{2} \right)_k \Upsilon_0(0). \tag{58}$$

Now we proceed with the rest of the spectrum. From the condition $q^2 \leq 4a_jb_j$ of the existence of solutions to the recursion relation (54) we can easily deduce the critical values of the conserved charge for asymptotical conformal spins $J \to \infty$ to be

$$-\frac{J^3}{3\sqrt{3}} \leq q \leq \frac{J^3}{3\sqrt{3}}. \tag{59}$$
This equation gives the upper boundary for the quantized values of integral of motion shown in Fig. 4. This picture demonstrates as well that there exist two possibilities of parametrization of the spectrum by different sets of trajectories.

The trajectories following the $q = 0$ behave at $J \to \infty$ as $q \propto J^2$. Introducing the convention $q^* \equiv q/\eta^2$ and the continuum variable $\tau = j/J$ the recursion relation can be reduced in leading-$J$ approximation to a simple first order differential equation the solution to which reads

$$\Upsilon_j = J i^j \phi(\tau), \quad \phi(\tau) = e^{-i\phi} |\Gamma(3/2 + iq^*)|^{-2} \left[\tau^2(1 - \tau^2)\right]^{1/2} \left(\frac{1 - \tau^2}{\tau^2}\right)^{iq^*}. \quad (60)$$

It is valid for whole interval of $j$ namely for $j \gg 1$, $J - j \gg 1$ except of the vicinities of the reflection points $j_{\text{end}} \sim 1, J$ where oscillating WKB solution is not applicable and we have to solve Eq. (54) close to the end points exactly. We have for $J \gg J - j \sim 1$

$$2q^* \Upsilon_j = (J - j + 2) \Upsilon_{j+1} + (J - j + 1) \Upsilon_{j-1}, \quad (61)$$

while for $J \gg j \sim 1$

$$2q^* \Upsilon_j = \frac{(j + 1)(j + 3)}{2j + 5} \Upsilon_{j+1} + \frac{(j + 2)(j + 4)}{2j + 5} \Upsilon_{j-1}. \quad (62)$$

To solve the first limiting recursion relation we make the following ansatz

$$\Upsilon_j = i^{J-j} \int_0^1 d\lambda \lambda^{iq^*-1}(1 - \lambda)^{-iq^*-1} v_{J-j}(\lambda). \quad (63)$$

Substituting it into Eq. (61) we get the equation

$$2\lambda(1 - \lambda) \frac{d}{d\lambda} v_\ell(\lambda) = (\ell + 1)v_{\ell+1}(\lambda) - (\ell + 2)v_{\ell-1}(\lambda), \quad (64)$$

with the solution

$$v_\ell(\lambda) = C [\lambda(1 - \lambda)]^{-1/2}(1 - 2\lambda)^{\ell+2}. \quad (65)$$

The solution to the second recurrent relation can be found by exploiting the cyclic symmetry of the problem and the known WKB solution (60). Using the Racah coefficients, $W_{jk}$, which relate the different three-particle bases (51), we can write

$$\Psi_j = e^{i\phi} \sum_{k=0}^J W_{jk}(J) \Psi_k. \quad (66)$$

From the recursion relation which is obeyed by the coefficients $W$ (see e.g. Refs. [20, 22]) we can obtain the formula for the limit $\tau \equiv \frac{k}{J} = \text{fixed}$, $j = \text{fixed}$, $J \to \infty$, namely,

$$W_{jk} = \frac{1}{\sqrt{J}} (-1)^j w_j(\tau), \quad \text{with} \quad w_j(\tau) = \sqrt{2} \left[\frac{\tau^5(1 - \tau^2)^2}{N_j}\right]^{1/2} C_j^{5/2}(2\tau^2 - 1), \quad (67)$$
with the normalization coefficient $N_j^{-1} = (3 \cdot 4)^{2j+5}/(j+1)_4$. Then the solution for $j \sim 1$ region is given by

$$\Upsilon_j = i^j (-1)^j e^{i\varphi} \sqrt{2} \left[ \frac{(j+1)_4}{2j+5} \right]^{1/2} \int_0^1 d\tau \frac{\phi(\tau) w_j(\tau)}{[\tau^3(1-\tau^2)]^{1/2}}. \quad (68)$$

A simple calculation gives us finally the result

$$\Upsilon_j = (-1)^{j-j} \frac{(j+1)_4}{4} p_j \left( q^* \bigg| \frac{3}{2}, \frac{3}{2} \right) \quad (69)$$

in terms of Askey-Wilson polynomials.

$$p_j (x|\alpha, \beta) = i^j {}_3 F_2 \left( \begin{array}{c} -j, \ j+2\alpha+2\beta-1, \ \alpha-ix \\ \alpha+\beta, \ 2\alpha \end{array} \right| 1), \quad (70)$$

which are orthogonal on the interval $-\infty < x < \infty$

$$\int_{-\infty}^{\infty} dx \, p_j (x|\alpha, \beta) p_k (x|\alpha, \beta) |\Gamma(\alpha + ix)\Gamma(\beta + ix)|^2 = 0, \quad j \neq k. \quad (71)$$

Matching the WKB with the end-point solutions in the region of their overlap we get the quantization conditions for the charge

$$q^* \ln \eta = \arg \Gamma \left( \frac{3}{2} + iq^* \right) + \frac{\pi}{6} (2m + \sigma_J), \quad (72)$$

where $\sigma_J = [1 - (-1)^J]/2$. The first iteration at large $\eta$ is

$$q^* = \frac{\pi}{6 \ln \eta - \psi(3/2)} \quad (73)$$

with accuracy $O(\ln^{-3} \eta)$.

Let us turn to the second way of describing the spectrum of $q$. The maximum of the eigenfunctions describing the trajectories which behave as $q \to J^3/\sqrt{27}$ is achieved for $j_{\text{max}} = J/\sqrt{3}$ and in the vicinity of this point the system behaves as a classical one. To find the quasiclassical corrections we expand the recursion relation around this point $j = \frac{J}{\sqrt{3}} (J + \lambda \sqrt{J})$, and look for the solution to Eq. (54) in the form of the series w.r.t. the inverse powers of $J$

$$\Phi(\lambda) = \sum_{\ell=0}^{\infty} \Phi(\ell)(\lambda) J^{-\ell/2}, \quad \text{and} \quad q(J, n) = \frac{J^3}{\sqrt{3}} \sum_{\ell=0}^{\infty} q^{(\ell)}(n) J^{-\ell}, \quad (74)$$

where $\Upsilon_j \equiv \Phi(\lambda)$. This leads to the infinite sequence of differential equations

$$D_{(1)} \Phi(0)(\lambda) = 0, \quad D_{(1)} \Phi(1)(\lambda) + D_{(2)} \Phi(0)(\lambda) = 0, \quad D_{(1)} \Phi(2)(\lambda) + D_{(2)} \Phi(1)(\lambda) + D_{(3)} \Phi(0)(\lambda) = 0, \quad \ldots, \quad (75)$$

\footnote{See also Refs. \cite{24,25} in connection to solution of the two-reggeon Baxter equation.}
supplied with the boundary conditions \( \Phi(\pm \infty) = 0 \) and the differential operators given by the formulae
\[
D_{(1)} = \frac{d^2}{d\lambda^2} + \left( 8 - 2q^{(1)} - \lambda^2 \right),
\]
\[
D_{(2)} = -\frac{d}{d\lambda} + \lambda \left( 8 - 5\sqrt{3} - \frac{1}{3} \lambda^2 \right),
\]
\[
D_{(3)} = \frac{1}{4} \frac{d^4}{d\lambda^4} + \frac{3}{2} \left( 8 - \lambda^2 \right) \frac{d^2}{d\lambda^2} + 4\lambda \frac{d}{d\lambda} - \frac{1}{2} \left( 11 - 40\sqrt{3} + 4q^{(2)} + 5\sqrt{3}\lambda^2 \right),
\]
\[\ldots\]
The solutions \( \Phi_{(\ell)}(\lambda) = \phi_{(\ell)}(\lambda) \exp(-\lambda^2/2) \) to these equations are expressed by a linear combination of the Hermite polynomials
\[
\phi_{(0)}(\lambda) = H_n(\lambda),
\]
\[
\phi_{(1)}(\lambda) = \frac{1}{18} n(n-1)(n-2)H_{n-3}(\lambda) + \frac{n}{4} (n-14 + 10\sqrt{3}) H_{n-1}(\lambda)
- \frac{1}{8} (n-17 + 10\sqrt{3}) H_{n+1}(\lambda) - \frac{1}{144} H_{n+3}(\lambda),
\]
\[\ldots\]
where \( n = 0, 1, \ldots \) and gives the number of nodes of the solutions in the classically allowed region. Deriving these equations we have implicitly assumed \( n \ll J \) in order the expansion (74) for the charge to be meaningful.

The quantization for the charge \( q \) immediately follows from boundary conditions. In this way we derive a few next-to-leading corrections to the charge
\[
q^{(0)}(n) = \frac{1}{3},
\]
\[
q^{(1)}(n) = \frac{7}{2} - n,
\]
\[
q^{(2)}(n) = \frac{85}{9} - \frac{22}{3} n + \frac{2}{3} n^2,
\]
\[\ldots\]
which compare quite well with the eigenvalues deduced from the numerical diagonalization of Eq. (74), see Fig. [3].

2.6 Eigenvalues of Hamiltonian \( \mathcal{H}_0 \).

Having found the expansion coefficients \( \Upsilon_j \), which completely specify the eigenfunction \( \Psi_0 \), we can express the three-particle energy \( \mathcal{E}_0 \) in their terms by exploiting the permutation symmetry of the system
\[
\mathcal{E}_0(J, q) = 2Re \frac{\sum_{j=0}^{J} (-i)^j (2j + 5) \epsilon(j) \Upsilon_j}{\sum_{j=0}^{J} (-i)^j (2j + 5) \Upsilon_j} + 3,
\]
\[\text{(79)}\]
Figure 5: The numerical quantized values of the charge (crosses) and WKB approximation (74,78).

with the two-particle energy
\[ \epsilon(j) = 2\psi(j + 3) - 2\psi(1). \]  
(80)

Conventionally, taking the average of the Hamiltonian \( H_0 \) w.r.t. the eigenfunctions \( \Psi_0 \) we find an equivalent expression for the energy
\[ E_0(J, q) = 3 \frac{\sum_{j=0}^{J} \epsilon(j) |\Psi_j|^2}{\sum_{j=0}^{J} |\Psi_j|^2}. \]  
(81)

These results provide the exact energy of the three-gluon system with the same helicity of all particles.

Substituting the eigenfunction (58) corresponding to the special case when \( q = 0 \) we easily obtain the lowest energy trajectory for even conformal spin \( J \)
\[ E_0(J, 0) = 2\psi \left( \frac{J}{2} + 3 \right) + 2\psi \left( \frac{J}{2} + 2 \right) - 4\psi(1) + 4. \]  
(82)

Next, using the WKB solutions found in the preceding section we can get the explicit result for the set of trajectories describing the spectrum from below using Eq. (50) as
\[ E_0(J, q) = 4 \ln \eta - 6\psi(1) + 2\text{Re} \psi \left( \frac{3}{2} - iq^* \right), \]  
(83)
Figure 6: The numerical eigenvalues of the Hamiltonian $\mathcal{H}_0$ versus the analytical energy given in Eq. (84) for $m = 0, 1$ (odd $J$) and Eq. (83) for higher trajectories $m = 3, 5$ (even $J$), $m = 7, 13$ (odd $J$).

where $q^* \equiv q/\eta^2$ and is valid with $O(\eta^{-1})$ accuracy. For the few lowest trajectories following the $q = 0$ one we can write from here the approximate formula

$$E_0(J, q) = 4 \ln \frac{\eta}{2} - 4\psi(1) + 4 - \psi''(3/2) \frac{\pi^2}{36} \left( \frac{2m + \sigma_J}{\ln \eta - \psi(3/2)} \right)^2,$$

(84)

where we have used the explicit form of the quantized $q$ from Eq. (73). For comparison with the numerical diagonalization see Fig. 6.

For the trajectories starting from the top of the energy spectrum we get using Eqs. (76,77) and (81)

$$E(J, q) = 2 \ln q - 6\psi(1) + O(J^{-2}),$$

(85)

where we have to substitute the explicit WKB expansion (78) for the integral of motion. Evaluation of the higher WKB corrections to this equation is extremely complicated task because of the lengthy form of the WKB functions $\phi(\ell)$ for $\ell \geq 2$ so that even $O(J^{-2})$ correction to energy becomes intractable.
2.7 Bethe ansatz and Baxter equation.

As we have seen above the evaluation of the higher WKB correction to the energy \((85)\) is a rather non-trivial task within the formalism used above as it requires cumbersome calculations due to a badly organized expansion in \(J\): the actual expansion parameter for the eigenfunctions was \(\sqrt{J}\). In this section we will address this issue starting from the Baxter equation for the \(Q\)-operator. For the polynomial solutions the latter is equivalent to the usual algebraic Bethe ansatz \([36, 37, 38, 39, 43]\).

To construct the diagonalized auxiliary transfer matrix \(t_\pm(\lambda)\) one defines a local vacuum state on each site \(|\omega_\ell\rangle\) so that

\[
\hat{J}^-_\ell |\omega_\ell\rangle = 0, \quad \hat{J}^3_\ell |\omega_\ell\rangle = \frac{\nu + 1}{2} |\omega_\ell\rangle,
\]

where from the definitions of the \(su(1,1)\) generators we conclude that \(|\omega_\ell\rangle = \text{const}\) but it is not annihilated by the generator of the conformal spin \(\hat{J}\) and thus is a legitimate choice. Then the vacuum of the chain is \(|\Omega\rangle = \otimes_{\ell=1}^3 |\omega_\ell\rangle\) and the auxiliary monodromy matrix acts as

\[
T_\pm(\lambda) |\Omega\rangle \equiv \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix} |\Omega\rangle = \begin{pmatrix} (\lambda + \frac{\nu + 1}{2})^3 & 0 \\ \ast & (\lambda - \frac{\nu + 1}{2})^3 \end{pmatrix} |\Omega\rangle,
\]

where the \(\ast\) stands for a term whose explicit form does not matter for us. Hence, as we see the element \(B(\lambda)\) is the annihilation operator while \(C(\lambda)\) can treated as the creation one. Therefore, we can construct a special set of the states of the chain by acting with \(C(\lambda)\) on the vacuum \(|\Omega\rangle\), the so-called Bethe states

\[
|\Psi_J\rangle = \prod_{\ell=1}^J C(\lambda_\ell) |\Omega\rangle,
\]

whose spin is \(\sum_{\ell=1}^3 \hat{J}^3_\ell |\Psi_J\rangle = (J + \frac{3}{2}(\nu + 1)) |\Psi_J\rangle\) \([43]\). Making use of the exchange relations stemming from the \(RTT\) fundamental commutation relation one can easily deduce that these states are the eigenvectors of the auxiliary transfer matrix \(t_\pm(\lambda) = A(\lambda) + D(\lambda)\) with the eigenvalues \([36, 38, 39, 43]\)

\[
t_\pm(\lambda) = \left(\lambda + \frac{\nu + 1}{2}\right) \prod_{\ell=1}^J \frac{\lambda - \lambda_\ell + 1}{\lambda - \lambda_\ell - 1} + \left(\lambda - \frac{\nu + 1}{2}\right) \prod_{\ell=1}^J \frac{\lambda - \lambda_\ell - 1}{\lambda - \lambda_\ell},
\]

provided the Bethe roots \(\{\lambda_\ell | \ell = 1, \ldots, J\}\) satisfy the Bethe Ansatz equation

\[
(\lambda_\ell + \frac{\nu + 1}{2}) \prod_{\ell=1, \ell \neq \ell'}^J (\lambda_\ell - \lambda_{\ell'}) = (\lambda_\ell - \frac{\nu + 1}{2}) \prod_{\ell=1, \ell \neq \ell'}^J (\lambda_\ell - \lambda_{\ell'} - 1).
\]

\(^8\)Note that we can equally treat \(\hat{J}^-_\ell\) as the creation and \(\hat{J}^+_\ell\) as the annihilation operators, respectively, as is done conventionally \([43]\). This corresponds to the second choice of the pseudovacuum. In this case the local vacuum is defined as \(|\omega_\ell\rangle = \theta_\ell^{-\nu - 1}\) and the eigenvalue of \(\hat{J}^3_\ell\) just changes the sign as compared to the Eqs. \((86)\): \(\hat{J}^3_\ell |\omega_\ell\rangle = -\frac{\nu + 1}{2} |\omega_\ell\rangle\). In this case the Bethe states are \(|\Psi_J\rangle = \prod_{\ell=1}^J B(\lambda_\ell) |\Omega\rangle\) and the eigenvalues of the auxiliary transfer matrix as well as the Bethe Ansatz equation remains intact.
On this condition the Bethe states become degree-$J$ homogeneous translation invariant polynomials in $\theta_\ell$.

By virtue of Eq. (34) one can assume that the Bethe states are also the eigenstates of the fundamental transfer matrix. It is really the case and for the Hamiltonian of the lattice we have the expression in terms of Bethe roots \cite{42, 43}

$$E_0 = \sum_{\ell=1}^{J} \frac{d}{d\lambda} \ln \frac{\lambda_\ell + \frac{\nu+1}{2}}{\lambda_\ell - \frac{\nu+1}{2}} + 3 \frac{d}{d\lambda} \bigg|_{\lambda=0} f(\nu, \lambda). \tag{91}$$

Introducing finally the function

$$Q(\lambda) = \prod_{\ell=1}^{J} (\lambda - \lambda_\ell), \tag{92}$$

we have from Eq. (89) the Baxter equation \cite{13, 16, 47}

$$t_{\frac{1}{2}}(\lambda) Q((\lambda + \frac{\nu+1}{2})^3 Q(\lambda + 1) + (\lambda - \frac{\nu+1}{2})^3 Q(\lambda - 1) \tag{93}$$

for the eigenvalues of the $Q$-operator where the $t_{\frac{1}{2}}(\lambda)$ stands for the eigenvalues of the transfer matrix, rather then the operator, and reads

$$t_{\frac{1}{2}}(\lambda) = 2\lambda^3 + \left(\eta^2 - 3\tilde{f}^2\right) \lambda + iq. \tag{94}$$

The usefulness of the Baxter operator is that the energy of the system can be rewritten concisely in its terms making use of the known Bethe ansatz representation \cite{24}

$$E_0 = \frac{Q'\left(\frac{\nu+1}{2}\right)}{Q\left(\frac{\nu+1}{2}\right)} - \frac{Q'\left(-\frac{\nu+1}{2}\right)}{Q\left(-\frac{\nu+1}{2}\right)} + 6\psi(\nu + 1) - 6\psi(1), \tag{95}$$

provided $f(\nu, \lambda)$ is given by the expression after Eq. (37). Thus, the knowledge of the Baxter operator as a function of the conserved charges allows to find immediately the energy as a function of the latter. The Eq. (93) will be the main object of our analysis in this section.

### 2.8 Quasiclassical expansion.

Let us analyze the Baxter equation for large values of the total conformal spin, $J$, of the three-particle system as was done before when we have dealt with the recursion relation \cite{24}. In these case the system behaves as a quasiclassical one, and we can apply appropriate methods used previously for the analysis of the Toda \cite{18} and $s = -1$ Heisenberg spin chain \cite{26}.

To get rid of the large factor $J$ in the transfer matrix let us rescale the spectral parameter as $\lambda \to J\lambda$ and introduce instead of $Q$ and $t_{\frac{1}{2}}$ the functions

$$\xi(\lambda) = Q(J\lambda), \quad T(\lambda) = (J\lambda)^{-3} t_{\frac{1}{2}}(J\lambda). \tag{96}$$
Then Eq. (93) takes the form

\[
\left( \lambda + \frac{\nu + 1}{2} J^{-1} \right)^3 \xi(\lambda + J^{-1}) + \left( \lambda - \frac{\nu + 1}{2} J^{-1} \right)^3 \xi(\lambda - J^{-1}) = \lambda^3 T(\lambda) \xi(\lambda),
\]

where \(T(i\lambda)\) is a real function. Moreover we introduce conventionally the WKB function as

\[
\xi(\lambda) = e^{JS(\lambda)},
\]

so that the energy is expressed now as

\[
\mathcal{E}_0 = S'(\frac{\nu + 1}{2} J^{-1}) - S'(\frac{-\nu + 1}{2} J^{-1}) + 6\psi(\nu + 1) - 6\psi(1),
\]

and, thus, for the limit \(J \to \infty\) we can primarily be interested in the small \(\lambda\) asymptotics of Eq. (97). We look the solution to this equation in terms of series w.r.t. inverse powers of the conformal spin

\[
S(\lambda) = \sum_{\ell=0}^{\infty} S(\ell)(\lambda) J^{-\ell} \quad \text{with} \quad T(\lambda) = \sum_{\ell=0}^{\infty} T(\ell)(\lambda) J^{-\ell},
\]

where

\[
T(0)(\lambda) = 2 + \lambda^{-2} + \frac{i}{\sqrt{3}} q^{(0)} \lambda^{-3}, \quad T(1)(\lambda) = 8\lambda^{-2} + \frac{i}{\sqrt{3}} q^{(1)} \lambda^{-3},
\]

\[
T(2)(\lambda) = \frac{27}{2} \lambda^{-2} + \frac{i}{\sqrt{3}} q^{(2)} \lambda^{-3}, \quad T(\ell)(\lambda) = \frac{i}{\sqrt{3}} q^{(\ell)} \lambda^{-3}, \quad \text{for} \quad \ell \geq 3.
\]

In leading order approximation we have the equation for \(S'(0)\)

\[
2 \cosh S'(0)(\lambda) = T(0)(\lambda),
\]

with the solution

\[
S'(0)(\lambda) = 2 \ln \left\{ \sqrt{\frac{1}{4} T(0)(\lambda) + \frac{1}{2}} + \sqrt{\frac{1}{4} T(0)(\lambda) - \frac{1}{2}} \right\},
\]

which has a correct asymptotics for \(\lambda \to \infty\) since \(S(\lambda) \sim \ln \lambda\) as follows from Eq. (102) because \(Q(\lambda) \sim \lambda^J\). In the small \(\lambda\) region we get from Eq. (103)

\[
S'(0)(\lambda \to \pm 0) = \pm \ln \frac{i}{\sqrt{3}} q^{(0)} \lambda^{-3}.
\]

From here we can immediately find that for large conformal spin \(\mathcal{E}_0(J \to \infty) = 6 \ln J\) and coincides with the result (93). Unfortunately, due to the fact that for the calculation of the energy from Eq. (99) the function \(S\) enters in the point \(\sim J^{-1}\) in order to estimate a non-leading correction we have to solve an infinite series of differential equations stemming from Eq. (97). This is, of course, not feasible. Therefore, we have to perform an effective resummation of these series.
To do this we notice that from Eq. (104) it follows that for Re\(\lambda > 0\) we have
\[
\exp\left( S'(0) \lambda \right) \gg \exp\left( -S'(0) \lambda \right)
\]
and one can easily convince oneself [26] that up to \(O(\lambda^6)\) the small \(\lambda\) expansion of Eq. (97), i.e. Eqs. (114), can be substituted by the simplified equation
\[
\frac{\xi(\lambda + J^{-1})}{\xi(\lambda)} = \lambda^3 \left( \lambda + \frac{\nu + 1}{2} J^{-1} \right)^{-3} T(\lambda).
\] (105)
Similarly for Re\(\lambda < 0\) we can omit the first term on the l.h.s. of Eq. (97). Now these simplified equation can be solved exactly as
\[
S'(\lambda) = 3\psi(J\lambda) + 3\psi\left( J\lambda + \frac{\nu + 1}{2} \right) + \frac{J^{-1}\partial}{\exp(J^{-1}\partial) - 1} \ln T(\lambda), \quad \text{for} \quad \text{Re}\lambda > 0.
\] (106)
And analogously for the Re\(\lambda < 0\) obtained from this equation by \(J \rightarrow -J\) and changing the sign of the third term. One can readily recognize the differential operator as the generating function
\[
t/(e^t - 1) = \sum_{\ell=0}^{\infty} B\ell t^\ell/\ell! \quad \text{for the Bernoulli numbers of the first order [50].}
\]
Finally, we have for the energy
\[
E_0 = 6\psi\left( \frac{\nu + 1}{2} \right) - 6\psi(1) + 2\text{Re} \left. \frac{J^{-1}\partial}{\exp(J^{-1}\partial) - 1} \ln T(\lambda) \right|_{\lambda=\nu+1/2} + \mathcal{O}(J^{-6}).
\] (107)
Making use of the representation of the auxiliary transfer matrix in terms of its zeros \(r_\ell\)
\[
t_{1/2}(\lambda) = 2 \prod_{\ell=1}^{3} (\lambda - r_\ell),
\] (108)
one can easily obtain a concise expression of the energy (cf. [26])
\[
E_0 = 2 \ln 2 + 2\text{Re} \sum_{\ell=1}^{3} \left( \psi\left( \frac{\nu + 1}{2} - r_\ell \right) - \psi(1) \right) + \mathcal{O}(J^{-6}).
\] (109)

2.9 Quantization of \(q\) and energy.

Expressing the energy directly in terms of the integrals of motion we have to find their quantized values. Although it was done in section 2.4 from the study of the recursion relation we address this question here from the point of view of the Baxter equation. From Eq. (92) we get for \(S'(\lambda)\)
\[
S'(\lambda) = \sum_{\ell=1}^{J} (J\lambda - \lambda_\ell)^{-1},
\] (110)
and thus
\[
\oint_C \frac{d\lambda}{2\pi i} S'(\lambda) = 1,
\] (111)
for a contour in the complex \(\lambda\)-plane which encircles all the roots \(\lambda_\ell/J\). From the leading WKB solution we conclude that the zeros of \(Q(\lambda)\) are accumulated on the interval \(|T(\lambda)| \geq 2\) on the
imaginary axis. For the case at hand we have two intervals of instability for the classical motion and they are determined by the system of cuts \([\lambda_1, \lambda_2] \cup [\lambda_3, \lambda_4]\). Therefore, we have \(n\) zeros on \([\lambda_1, \lambda_2]\) and \(J-n\) on \([\lambda_3, \lambda_4]\) and, thus,

\[
\oint_{C_1} \frac{d\lambda}{2\pi i} S'(\lambda) = \frac{n}{J},
\]  

(112)

with the contour \(C_1\) which encircles the interval \([\lambda_1, \lambda_2]\) (see Fig. 7). It is obvious that \(0 \leq n \leq J\). Let us assume for the following that \(n \sim O(J^0)\) and substitute the expansion (100) into quantization condition (112) to get the sequence

\[
\oint_{C_1} \frac{d\lambda}{2\pi i} S'_0(\lambda) = 0, \quad \oint_{C_1} \frac{d\lambda}{2\pi i} S'_{(1)}(\lambda) = n, \quad \oint_{C_1} \frac{d\lambda}{2\pi i} S'_{(\ell)}(\lambda) = 0, \quad \text{for } \ell \geq 2.
\]  

(113)

The functions \(S'_{(\ell)}\) can be easily deduced from the Baxter equation (97) and read

\[
S'_{(1)} = \frac{T_{(1)}}{2 \sinh S'_0} - \frac{1}{2} S''_0 \coth S'_0 - \frac{9}{2} \lambda^{-1},
\]

\[
S'_{(2)} = \frac{T_{(2)}}{2 \sinh S'_0} - \left\{ \frac{1}{2} S''_0 + \frac{1}{8} \left( S''_0 \right)^2 + \frac{1}{2} \left( S'_{(1)} \right)^2 + \frac{9}{2} \lambda^{-1} S'_{(1)} + \frac{27}{4} \lambda^{-2} \right\} \coth S'_0 \]

\[- \frac{1}{6} S''_0 - \frac{1}{2} S'_{(1)} S''_0 - \frac{9}{4} \lambda^{-1} S''_0,
\]  

(114)

and \(S'_0\) has been found before in Eq. (103). From the quantization condition for \(S'_0\) and Fig. 6 it is seen that the function \(S'_0\) should not have singularities on the interval \([\lambda_1, \lambda_2]\) which is the case provided the latter shrinks into the point \(\lambda_{\text{crit}}\). These critical values are defined by the equations \(T_{(0)}(\lambda_{\text{crit}}) = 0\) and \(T_{(0)}(\lambda_{\text{crit}}) = -2\) with the solution

\[
q_{\text{crit}} \equiv q^{(0)} = \frac{1}{3}, \quad \lambda_{\text{crit}} = -\frac{i}{2\sqrt{3}},
\]  

(115)

where we have restricted ourselves to \(q^{(0)} > 0\).

To proceed further we consider an infinitesimal interval around \(\lambda_{\text{crit}}\)

\[
\lambda_2 - \lambda_1 = i\varepsilon, \quad \lambda_1 = \lambda_{\text{crit}} - i\varepsilon/2, \quad \lambda_2 = \lambda_{\text{crit}} + i\varepsilon/2,
\]  

(116)

to ensure the presence of singularities of \(S'_{(1)}\) on it and to fulfill the second quantization condition (113). Here \(T_{(0)}(\lambda_1) = T_{(0)}(\lambda_2) = -2\) as is seen from the Fig. 6. Expansion of the rescaled transfer matrix around the critical point \(\lambda = \lambda_{\text{crit}} + \frac{i}{2}\varepsilon(2x-1)\) gives

\[
T_{(0)} = -2 + 144x(x-1)\varepsilon^2 + \mathcal{O}(\varepsilon^3).
\]  

(117)
Inserting it into Eqs. (103, 114) we get

\[ S'(0) = i\pi + 12i\sqrt{x(x-1)e} + \mathcal{O}(e^2), \]

\[ S'(1) = \frac{i}{24e} \frac{T(1)(\lambda_{\text{crit}})}{\sqrt{x(x-1)}} + \frac{i}{4e} \frac{2x-1}{x(x-1)} + \mathcal{O}(e^0). \]  

(118)

Substituting the last equation into the quantization condition \( i\varepsilon \oint_{C_1} dx \frac{d}{dx} S'(1)(\lambda) = n \) we can deform the integration contour away from the cut \([\lambda_1, \lambda_2]\) reverting its direction and take the residue at infinity \( x = \infty \) to get

\[ -i\varepsilon \text{Res}_{x=\infty} \frac{i}{xe} \left( \frac{1}{24} T(1)(\lambda_{\text{crit}}) + \frac{1}{2} \right) = n. \]  

(119)

Using explicit form of \( T(1) \) from Eq. (101) one can immediately convince oneself that this is exactly \( q(1) \) in Eq. (78).

Since the rescaled transfer matrix possesses the singularities only on the interval \( x \in [0, 1] \) and infinity, we can always deform the integration contour and evaluate the residue at \( x = \infty \). Thus, for evaluation of the quantized \( q \) one has to keep only \((xe)^{-1}\)-terms in the expansions. This simplifies considerably the task of calculation of the higher WKB corrections (and is easy to formalize) as compared to the method based on the solution of the recursion relation in section 2.4. For instance, to find \( q(2) \) we have just to keep more terms in the expansion in Eqs. (117, 118):

\[ \frac{1}{4} T(0) = -\frac{1}{2} + 36(xe)^2 + 192\sqrt{3}(xe)^3 + 2160(xe)^4 + \ldots, \]
\[
\frac{1}{24} T_1 = (q^{(1)} - 4) + 2\sqrt{3}(3q^{(1)} - 8)(x\varepsilon) + 72(q^{(1)} - 2)(x\varepsilon)^2 + \ldots,
\]

(120)

which was done having in mind that we will be interested only in the residue at \( x = \infty, x\varepsilon = \text{fixed}. \)

And, thus, we have

\[
S_1'(0) = i \left\{ \pi + 12(x\varepsilon) + 32\sqrt{3}(x\varepsilon)^2 + 304(x\varepsilon)^3 + \ldots \right\},
\]

\[
S_1'(1) = i \left\{ \frac{2q^{(1)} - 7}{2(x\varepsilon)} + \frac{10q^{(1)} - 39}{\sqrt{3}} + \frac{2}{3} \left( 66q^{(1)} - 239 \right)(x\varepsilon) + \ldots \right\}
\]

(121)

to the same accuracy. Therefore, we have finally the necessary terms

\[
S_2'(2) = i \frac{1}{x\varepsilon} \left\{ \frac{1}{24} T_2(\lambda_{\text{crit}}) - \frac{2}{3} \left( q^{(1)} \right)^2 - \frac{8}{3} q^{(1)} + \frac{533}{36} \right\} + \ldots
\]

(122)

From the requirement that this residue must equal zero we get the result in Eq. (78).

One can generalize in a straightforward way this consideration for \( q^{(\ell)} \) with \( \ell \geq 3 \), however, we will not pursue this goal here since in the consequent consideration of the perturbed Hamiltonian (24) we will be able to calculate the energy only up to \( O(J^{-3}) \) accuracy. The main motivation to study the Baxter equation was to express the energy directly via the integral of motion which allows to find the former in a very economic way. Applying Eq. (107) we easily get

\[
\mathcal{E}_0 = 2 \ln \frac{q^{(0)} J^3}{\sqrt{3}} + 2 \frac{q^{(1)}}{J q^{(0)}} + \frac{2}{J^2} \left( \frac{q^{(2)}}{q^{(0)}} - \frac{4}{8} \frac{\left( q^{(1)} \right)^2 - 11}{(q^{(0)})^2} \right) - 6\psi(1) + O(J^{-3}),
\]

(123)

where we have to plug in the explicit expressions for the expansion coefficients of the charge (78).

### 2.10 Perturbed Hamiltonian.

Let us turn to the perturbed Hamiltonian (24) whose eigenvalues are responsible for the scale dependence of the three-gluon correlator which contributes to the transverse spin structure function \( g_2 \). Of course, the problem now is more complicated since the interaction is not integrable. But it is easy to figure out that the non-integrable addendum \( V \) is small for a bulk of the spectrum and this allows to consider it as a perturbation.

To start with, let us introduce the more general Hamiltonian

\[
\mathcal{H}_\alpha = \mathcal{H} + \alpha V
\]

(124)

specified by a coupling constant \( \alpha \). Obviously \( \mathcal{H}_\alpha = 0 = \mathcal{H} \) and \( \mathcal{H}_\alpha = \mathcal{H} \). Considering the flow of the energy levels given in Fig. 8 we conclude that the upper part of the spectrum is not affected significantly by the perturbation \( V \) and it represents a small correction. At the same time it modifies in a sizable way the level spacing for the lowest trajectories.
The present problem does not manifest the cyclic symmetry but only w.r.t. the permutation of the first and third gluons. Therefore, due to antisymmetry of the three-gluon operator we will be interested in the eigenfunctions, $\Psi^{(-)}$, of the Hamiltonian (24)

$$\mathcal{H}\Psi^{(-)} = \mathcal{E}\Psi^{(-)},$$

which are odd functions w.r.t. the permutation of the end gluons

$$P_{13}\Psi^{(-)} = -\Psi^{(-)}.$$  

For the top of the spectrum we have $|\mathcal{V}| \ll |\mathcal{H}_0|$ for large $J$ and, therefore, as a first approximation we take

$$\Psi^{(-)} = \Psi^{(-)}_0, \quad \text{with} \quad \Psi^{(-)}_0(q) = \frac{1}{\sqrt{2}} \{ \Psi_0(q) - \Psi_0(-q) \}.$$  

Then the matrix element of $\mathcal{V}$ in this basis reads

$$\mathcal{V}_{q',q} \equiv \left[ \langle \Psi^{(-)}_0(q')|\Psi^{(-)}_0(q')\rangle \langle \Psi^{(-)}_0(q)|\Psi^{(-)}_0(q)\rangle \right]^{-1/2} \langle \Psi^{(-)}_0(q')|\mathcal{V}|\Psi^{(-)}_0(q)\rangle$$

with the explicit eigenfunctions being plugged in there, it gives

$$\mathcal{V}_{q',q} = -8 \left( \sum_{j=0}^{J} |\Psi_j(q')|^2 \sum_{j=0}^{J} |\Psi_j(q)|^2 \right)^{-1/2}$$
\[ \times \sum_{j=0}^{J} \frac{\Psi_j^*(q')\Psi_j(q)}{(j+2)(j+3)} \left\{ \cos(\varphi(q') - \varphi(q)) - (-1)^j \cos(\varphi(q') + \varphi(q)) \right\}. \] (129)

It defines the correction to the eigenfunctions

\[ \Psi^{(-)}(n) = \Psi_0^{(-)}(q) + \sum_{q' \neq q} \frac{\mathcal{V}_{q',q}}{E_0(q) - E_0(q')} \Psi_0^{(-)}(q'), \]

where \( q = q(n) \) and \( q' = q(n') \), and correction to the energy which can be easily evaluated making use of Eqs. (76,77) with the result

\[ \mathcal{V}_{q,q} = -\frac{24}{J^2} - \frac{48}{J^3}(2n - 3) + \mathcal{O}(J^{-4}). \] (130)

Therefore, it affects the spacing of integrable Hamiltonian \( \mathcal{H}_0 \) only at order \( \mathcal{O}(J^{-2}) \) while the net spacing is \( \delta E_0(q \rightarrow J^2/\sqrt{27}) \propto \mathcal{O}(J^{-1}) \) and the former is a small effect for large \( J \).

As is seen from the picture of the flow of the energy eigenvalues of the Hamiltonian \( \mathcal{H}(\alpha) \) the dependence on \( \alpha \) is linear. Thus, going to the limit \( \alpha \rightarrow \infty \), we can consider the integrable Hamiltonian as a correction on the background of \( \mathcal{V} \). Calculating its matrix elements w.r.t. eigenfunctions of \( \mathcal{V} \) and extrapolating the result to the point \( \alpha = 1 \) we can estimate the spacing for the low trajectories of the \( \mathcal{H} \).

The eigenfunctions of the \( \mathcal{V} \) kernel which diagonalize it in the limit of large \( J \) are

\[ \Psi_{J,m}^{(-)} = \frac{1}{\sqrt{2}} \left\{ \mathcal{P}_{J,m}(\theta_1, \theta_2|\theta_3) - \mathcal{P}_{J,m}(\theta_3, \theta_2|\theta_1) \right\}, \] (131)

with required antisymmetry w.r.t. the \( P_{13} \) permutation. They are normalized as \( \langle \Psi_{J,m}^{(-)}|\Psi_{J,m}^{(-)} \rangle = \delta_{m,m'} + \mathcal{O}(J^{-2}) \). We have the eigenvalues

\[ \mathcal{V}(m) \equiv \langle \Psi_{J,m}^{(-)}|\mathcal{V}|\Psi_{J,m}^{(-)} \rangle/\langle \Psi_{J,m}^{(-)}|\Psi_{J,m}^{(-)} \rangle = -\frac{4}{(m+2)(m+3)} + \mathcal{O}(J^{-2}). \] (132)

The \( \mathcal{O}(J^{-2}) \) terms come form \( \delta \mathcal{V}(m) = \mathcal{V}(m) + \frac{4}{(m+2)(m+3)} = \frac{4(-1)^mW_{m,0}}{(m+2)(m+3)} - 4\sum_{\ell=0}^{J} \frac{W_{m,\ell}W_{m,\ell}}{\ell(\ell+1)(\ell+2)} \) which can be estimated as follows. In the sum the main contributions come for the regions \( m, \ell \sim 1 \) and \( 0 \leq \tau \leq 1 \) with \( \ell = \tau J \). For the latter we can replace the sum by the integral, \( \sum \rightarrow J \int \frac{1}{\ell} \) and use \( W \) from Eq. (67) so that one can immediately conclude that sum \( \propto J^{-2} \). For the small fixed \( m \) and \( \ell \), and asymptotical \( J \), as can be deduced from the recursion relation obeyed by \( W \), the Racah coefficients are given by

\[ W_{jk} = \frac{(-1)^j}{j^2}(2k+5)(j+2)(j+3) + \mathcal{O}(J^{-4}). \] (133)

and, therefore, we have finally for the matrix element \( \delta \mathcal{V}(m) \propto \mathcal{O}(J^{-2}) \). This estimation is an example of the asymptotic addition law for anomalous dimensions of scalar multi-particles composite operators [31], according to which, for the present case, the limiting point of the spectrum
at large $J$ is given by the naive sum of the anomalous dimension of a field and of a two-particle composite operator. One can immediately recognize in Eq. (132) the anomalous dimensions of the two-particle local composite operators in $\phi^3_6$-theory.

Now the correction to the spectrum (132) is given by the $\alpha^{-1}\langle \Psi^{(-)}_{J,m} | H_0 | \Psi^{(-)}_{J,m} \rangle$ and reads

$$\langle \Psi^{(-)}_{J,m} | H_0 | \Psi^{(-)}_{J,m} \rangle / \langle \Psi^{(-)}_{J,m} | \Psi^{(-)}_{J,m} \rangle = \alpha(m) (1 - (-1)^m W_{mm}) + \sum_{\ell=0}^{J} (2 - (-1)\ell) \epsilon(\ell) W_{m\ell} W_{m\ell}. \quad (134)$$

Making use of the explicit asymptotics of the Racah coefficients we get finally (up to $O(\eta^{-1})$)

$$\mathcal{E}_\alpha(m) = \alpha \mathcal{V}(m) + 4 \ln \eta + 4\psi(m+3) + 2\psi(m+5) - 2\psi(2m+5) - 2\psi(2m+6) - 6\psi(1). \quad (135)$$

Since the dependence on $\alpha$ observed for the energy flow is linear we can extrapolate this result to the point $\alpha = 1$ and estimate the spacing for the lowest trajectories of $\mathcal{E}$. For instance for $m = 0$ we get $\mathcal{E}_1(0) - \mathcal{E}_0(J, 0) = -0.46$ which is rather close to the numerical value found from the explicit diagonalization.

Making use of these results we design the following formulae which describe well (see Figs. 10, 11) the spectrum of the perturbed Hamiltonian

$$\mathcal{E}(J, m) = \mathcal{E}_0(J, 0) - \Delta(m), \quad \text{for} \quad m = 0, 1; \quad (136)$$

$$\mathcal{E}(J, m) = \mathcal{E}_0(J, q(m)) - \Delta(m), \quad \text{for} \quad m \geq 2. \quad (137)$$

Here $\mathcal{E}_0(J, 0)$ is the ground state energy (84) and $\mathcal{E}_0(J, q(m))$ is Eq. (83) with the $q(m)$ trajectories deduced from the quantization condition (72). The function $\Delta$ is

$$\Delta(0) = 0.54, \quad \Delta(1) = 0.08, \quad \Delta(m \geq 2) = (\delta_0 - \delta(m - 2)) \theta(\delta_0 - \delta(m - 2)), \quad (138)$$

with $\delta_0 = 0.15$ and $\delta = 0.01$. It serves to generate a shift of the non-perturbed energy trajectories. The top of the spectrum coincides with the $\mathcal{E}_0$ trajectories.

3 Quark-gluon sector.

Although we have started with the pure gluonic sector, chronologically, it is the non-singlet quark-gluon-quark sector which has been first addressed within the context of the transversely polarized structure function $g_2(x_B)$. The first study which correctly identified a complete basis of operators was by Shuryak and Vainshtein and they have calculated a lowest anomalous dimension $[52]$. The evolution equation for the correlation function $Y$ in Eq. (2) which is defined as the Fourier transform

$$Y(x_1, x_3) = \int \frac{d\kappa_1}{2\pi} \frac{d\kappa_3}{2\pi} e^{ix_1\kappa_1 - ix_3\kappa_3} \langle h | Y(\kappa_1, 0, \kappa_3) | h \rangle \quad (139)$$
Figure 9: The exact energy eigenvalues versus the approximate formulae (136).

Figure 10: Same as in Fig. 9 but for large $J$. The spectrum is cut at the top where it coincides with the eigenvalues of the integrable Hamiltonian $\mathcal{H}_0$. 
of the \( C \)-even combination
\[
\mathcal{Y}(\kappa_1, 0, \kappa_3) = \frac{1}{2} \left\{ \pm \mathcal{S}(\kappa_1, 0, \kappa_3) + \mp \mathcal{S}(-\kappa_3, 0, -\kappa_1) \right\},
\]
(140)
of the three-particle quark-gluon-quark operators
\[
\pm \mathcal{S}(\kappa_1, \kappa_2, \kappa_3) = \frac{1}{2} \dot{\psi}(\kappa_3 n) i g \gamma_+ \left[ i \dot{G}_{\sigma^+}^\perp(\kappa_2 n) \pm \gamma_5 G_{\sigma^+}^\perp(\kappa_2 n) \right] \psi(\kappa_1 n),
\]
(141)
has been derived in Refs. \([8, 53]\) as well as the anomalous dimensions of the local operators were calculated. Consequently this was redone in a number of studies \([27, 54]\) with almost the same results.

3.1 Evolution equation.

In spite of the available results \([53]\) let us discuss the evolution equation from our present point of view. The function \( Y \) can be decomposed according to this discussion as \( Y = \dot{\psi}(\kappa_3 n) \gamma_+ \left[ i \dot{G}_{\sigma^+}^\perp(\kappa_2 n) \pm \gamma_5 G_{\sigma^+}^\perp(\kappa_2 n) \right] \psi(\kappa_1 n) \),
\[
(142)
\]
and their explicit form can be found in Ref. \([21]\). Analogous expression holds for \( \mp \mathcal{S} \) with \( T \) and \( V \) subscripts being interchanged in the quark-gluon kernels. Since these kernels are equivalent we consider in the following only \( \dot{\psi}(\kappa_3 n) \gamma_+ \left[ i \dot{G}_{\sigma^+}^\perp(\kappa_2 n) \pm \gamma_5 G_{\sigma^+}^\perp(\kappa_2 n) \right] \psi(\kappa_1 n) \),
\[
(143)
\]
with the large-\( N_c \) anomalous dimension matrix \([53, 54]\)
\[
\gamma_{jk}^S = \delta_{jk} \left( \psi(j + 1) + \psi(j + 4) + \psi(J - j + 2) + \psi(J - j + 3) - 4 \psi(1) - \frac{3}{2} \right)
- \theta_{j,k+1}^+(J-k+1) \frac{(J-k+1)_2}{(j-k)(j+2)_2} - \theta_{k,j+1}^+(J-j+1) \frac{(J-j+1)_2}{(k-j)(J-j+1)_2},
\]
(144)
For large \( N_c \) we have \( \hat{q}_K^{(V)}(\mathbf{x}_1,\mathbf{x}_2) \propto 1/N_c \) and the eigenvalues of the other kernels read, up to terms suppressed in \( N_c \),

\[
\int dx_1 dx_2 P_{j}^{(1,2)} \left( \frac{x_1 - x_2}{x_1 + x_2} \right) \left\{ \frac{\hat{q}_K^{(V)}}{\hat{q}_K^{(3)}} \right\} (x_1, x_2 | x_1', x_2') = \frac{N_c}{2} \left\{ \psi(j + 1) + \psi(j + 4) - 2\psi(1) - \frac{5}{3} \right\} P_{j}^{(1,2)} \left( \frac{x_1' - x_2'}{x_1' + x_2'} \right).
\]

Therefore, similarly to the previous study we can replace the original problem of the diagonalization of the three-particle kernel \( K^{+S} = \frac{N_c}{2} H \rightarrow \frac{N_c}{2} \mathcal{H} \) by the solution of the Schrödinger equation

\[
\mathcal{H}\psi = E\psi,
\]

with the Hamiltonian

\[
\mathcal{H} = h_{12} + h_{23} - \frac{3}{2},
\]

where the pair-wise Hamiltonians

\[
h_{12} = \psi \left( \mathcal{J}_{12} + \frac{3}{2} \right) + \psi \left( \mathcal{J}_{12} - \frac{3}{2} \right) - 2\psi(1), \quad h_{23} = \psi \left( \mathcal{J}_{23} + \frac{1}{2} \right) + \psi \left( \mathcal{J}_{23} - \frac{1}{2} \right) - 2\psi(1),
\]

can be easily read from Eq. (145).

### 3.2 Inhomogeneous open spin chain.

It turns out that the problem (146-148) is exactly solvable. Let us consider the inhomogeneous spin chain which is characterized by ‘inhomogeneities’ \( \delta_\ell \). The monodromy matrix is defined as before but with the shifted spectral parameters

\[
T_b(\lambda) = R_{a_1,b}(\lambda - \delta_1)R_{a_2,b}(\lambda - \delta_2)R_{a_3,b}(\lambda - \delta_3).
\]

The generating function of the conserved integrals of motion, i.e. the transfer matrix, reads [57]

\[
t_b(\lambda) = \text{tr}_b K^+(\lambda)T_b(\lambda)K^-(\lambda)T_b^{-1}(-\lambda),
\]

with boundary reflection matrices \( K^\pm \) which we set equal to unit matrix in order to ensure the conformal invariance of \( t_b \). The matrices (150) obey the commutation relation (34). The shifted and renormalized auxiliary transfer matrix \( \prod_{\ell=1}^{3} \left\{ \lambda - \delta_\ell \right\} \mathcal{T}_{\frac{1}{2}}(\lambda - \frac{1}{2}) \rightarrow \mathcal{T}_{\frac{1}{2}}(\lambda) [57]

\[
\mathcal{T}_{\frac{1}{2}}(\lambda) = \text{tr}_{\frac{1}{2}} L_{a_1}(\lambda - \delta_1)L_{a_2}(\lambda - \delta_2)L_{a_3}(\lambda - \delta_3)\sigma_2 L_{a_1}^1(-\lambda - \delta_3)L_{a_2}^1(-\lambda - \delta_2)L_{a_3}^1(-\lambda - \delta_1)\sigma_2,
\]

See Ref. [53] for spin models appeared in the solution of the quark-gluon reggeon interaction.
which is an even function in λ, has the following expansion in the rapidity\[10\]
\[ t_\frac{1}{2}(\lambda) = \Omega(\lambda) - (4\lambda^2 - 1) \left(\lambda^2 - \delta_2^2 - \delta_2^2\right) \hat{J}^2 - \frac{1}{2}(4\lambda^2 - 1)Q_\delta, \]  
(152)
where \(\Omega(\lambda)\) is the c-number function
\[ \Omega(\lambda) = -2 \prod_{\ell=1}^3 \left(\lambda^2 + \delta_\ell^2 - \delta_\ell^2\right) + (4\lambda^2 - 1) \sum_{\ell=1}^3 \delta_\ell^2 \left(\lambda^2 + \delta_\ell^2 + \delta_\ell^2 - \delta_\ell^2\right) + (4\lambda^2 - 1)\delta_2^2\delta_3^2, \]
\(\hat{J}^2\) is the total Casimir operator and \(Q_\delta\) is the non-trivial integral of motion
\[ Q_\delta = [\hat{J}^2_{12}, \hat{J}^2_{23}]_+ - 2(\delta_1^2 - \delta_2^2)\hat{J}^2_{23} - 2(\delta_2^2 - \delta_3^2)\hat{J}^2_{23} + 8i\delta_2\epsilon_{ijk}\hat{j}_{1}^i\hat{j}_{2}^j\hat{j}_{3}^k. \]  
(153)
As will be clear later on, only \(\delta_2 = 0\) spin chains appear in QCD within the twist-three context. Therefore, we set in what follows \(\delta_2 = 0\).

The Hamiltonian which commutes with the charges found above is deduced from the fundamental transfer matrix and reads \[57\] \(H_\delta = h_{12} + h_{23}\), with the two-site Hamiltonians expressed in terms of the fundamental \(R\)-matrices \(h_{a_1,a_2} = R_{a_1,a_2}(-\delta)R_{a_1,a_2}'(-\delta)\) so that explicitly
\[ h_{12} = \psi(\hat{J}_{12} + \delta_1) + \psi(\hat{J}_{12} - \delta_1) - 2\psi(1), \quad h_{23} = \psi(\hat{J}_{23} + \delta_3) + \psi(\hat{J}_{23} - \delta_3) - 2\psi(1), \]  
(154)
provided we set in the bundle \(R_{ab}(\lambda)\) \[54\] the function \(f(\nu, \lambda)\) as was done after Eq. (37). The commutation rule for the \(h_{a_1,a_2}\) with \(Q_\delta\) is \([Q_\delta, h_{12}]_+ = 2[\hat{J}^2_{23}, \hat{J}^2_{12}]_-\) and \([Q_\delta, h_{23}]_+ = 2[\hat{J}^2_{12}, \hat{J}^2_{23}]_-\). So that the property \([Q_\delta, H]_+ = 0\) is obvious.

Setting \(\delta_1 = 3/2, \delta_3 = 1/2\) we get the QCD result \(148\) and \(Q_S\) charge of Ref. \[19\]. The conformal weights of the end quarks are \(\nu_1 = \nu_3 = 1\) and \(\nu_2 = 2\) for the gluon.

### 3.3 Master equation and integral of motion.

As before we can solve now the simplified problem
\[ Q_S\Psi = q_S\Psi, \]  
(155)
instead of Eq. (146), where \(Q_S\) is Eq. (153) with \(\delta\)’s set earlier. The main steps of the solution of this problem are the same as we have pursued in the preceding section and in Ref. \[22\] so that we just outline them briefly.

First it is convenient to take the polynomials, which diagonalize simultaneously \(\hat{J}^2\) and \(\hat{J}^2_{12},\)
\[ P_{J,j}(\theta_1, \theta_2 | \theta_3) = \frac{\theta_j^{-1}}{\sqrt{2}(j+1)(j+3)} \frac{\Gamma(j+3) \Gamma(J+j+5)}{\Gamma(2j+4) \Gamma^3(2J+6)} \theta_{J,12}^j \theta_{J,2}^j F_1 \left( \begin{array}{c} j-J, j+2 \noindent \hfill 2 \noindent j+5 \noindent \hfill \theta \end{array} \right), \]  
(156)
\[10\] We would like to thank G. Korchemsky for a discussion on this point (see also recent Ref. \[56\]).
with \(\theta \equiv \frac{\theta_{32}}{\theta_{12}}\), as basis vectors since the off-diagonal elements of the \(Q_S\) in this basis will be the same as for \(Q_T\) in [22] so that only diagonal elements will be slightly modified. Expanding the eigenfunctions \(\Psi\) in the way it was done in (42)

\[
\Psi = \sum_{j=0}^{J} \Psi_j P_{J,j}(\theta_1, \theta_2 | \theta_3),
\]

it is easy to show that Eq. (155) is equivalent to the three-term recursion relation

\[
(2j + 3)\Upsilon_{j+1} + (2j + 5)\Upsilon_{j-1} + g_j^2 (2j + 3)(2j + 5) \left([Q_S(\frac{j}{2}, \frac{j}{2})]_{j,j} - q_S\right)\Upsilon_j = 0,
\]

with boundary conditions \(\Upsilon_{-1} = \Upsilon_{J+1} = 0\), and where

\[
[Q_S(\delta_1, \delta_3)]_{j,j} = -2\delta_3^2 \left( j + \frac{3}{2} \right) \left( j + \frac{5}{2} \right) + \left\{ \left( j + \frac{3}{2} \right) \left( j + \frac{5}{2} \right) - \delta_1^2 \right\} \\
\times \left\{ \frac{3}{4} - \left( j + \frac{3}{2} \right) \left( j + \frac{5}{2} \right) + \left( J + \frac{7}{2} \right) \left( J + \frac{5}{2} \right) + \frac{3}{4} \left( J + \frac{5}{2} \right) \left( J + \frac{7}{2} \right) \right\}.
\]

Here the new expansion coefficients are introduced as follows

\[
\Psi_j \equiv \varrho_j \Upsilon_j, \quad \text{with} \quad \varrho_j = \left[ \frac{(j + 1)^3(j + 3)^3}{(j + 2)^3} (J - j + 1)(J + j + 5) \right]^{-1/2}.
\]

However, due to the loss of any symmetry of the integral of motion w.r.t. the permutation of the sites it is instructive to study the recurrence relation based on the expansion of the eigenfunctions in the basis \(P_{J,j}(\theta_3, \theta_2 | \theta_1)\)

\[
\Psi = \sum_{j=0}^{J} \Phi_j P_{J,j}(\theta_3, \theta_2 | \theta_1),
\]

where

\[
P_{J,j}(\theta_3, \theta_2 | \theta_1) = \sqrt{2} \varsigma_j^{-1} \frac{\Gamma(j + 3) \Gamma(J + j + 5)}{\Gamma(2j + 5) \Gamma^{1/2}(2J + 1)} \frac{\theta_{32}^j \theta_{12}^{J-j}}{2} \left( j - J, j + 2 \right)_{2j+5} F_1 \left( j - J, j + 2 \right)_{2j+5} \left( \theta \right), \quad \text{with} \quad \theta \equiv \frac{\theta_{32}}{\theta_{12}}.
\]

Introducing

\[
\Phi_j \equiv \varsigma_j \Xi_j, \quad \text{with} \quad \varsigma_j = [(j + 1)(j + 2)(j + 3)(J - j + 1)(J + j + 5)]^{-1/2},
\]

we have the recursion relation for \(\Xi_j\)

\[
(2j + 3)\Xi_{j+1} + (2j + 5)\Xi_{j-1} + \varsigma_j^2 (2j + 3)(2j + 5) \left([Q_S(\frac{j}{2}, \frac{j}{2})]_{j,j} - q_S\right)\Xi_j = 0.
\]

From the condition of existence of a solution to the recursion relations which satisfies the boundary conditions we find that the allowed values of \(q_S\) asymptotically lie in the band \(0 \leq q_S/J^4 \leq \frac{1}{2}\). The upper boundary is achieved at \(j_{\text{max}} = \frac{1}{\sqrt{2}}J\). In complete analogy with \(Q_T\) [22],
the spectrum of $Q_S$ can be described by means of two different sets of trajectories. The first ones which behave as $J^2$ and another ones as $J^4$, at large $J$.

Let us turn to the first possibility. One can immediately find from Eq. (158) the exact lowest trajectory for the charge

$$q_{S}^{\text{exact}}(J) = \left( J + \frac{5}{2} \right) \left( J + \frac{7}{2} \right) + \frac{5}{8}. \tag{165}$$

For other levels we construct an effective WKB approximation. For the case at hand the classical motion is allowed for the whole interval of $j$ except of the vicinities of the reflection points $j_{\text{end}} \sim 1, J$, which will be parametrized by the continuous parameter $\tau \equiv j/J$. Introducing the new function as $Y_j = J(-1)^j v(\tau)$ and $q_{S}^{*} = q_{S}/J^2$, we have from (158) the differential equation

$$\tau^2 (1 - \tau^2) v''(\tau) - \tau (1 - \tau^2) v'(\tau) + 2(q_{S}^{*} - 1)v(\tau) = 0, \tag{166}$$

the solution to which reads

$$v(\tau) = C^{(+)} \tau_{1+2i\eta S} F_1\left( \frac{1}{2} + i\eta S, -\frac{1}{2} + i\eta S \bigg| \tau^2 \right) + C^{(-)} \tau_{1+2i\eta S} F_1\left( \frac{1}{2} - i\eta S, -\frac{1}{2} - i\eta S \bigg| \tau^2 \right), \tag{167}$$

where $\eta S = \frac{1}{2}\sqrt{2q_{S}^{*} - 3}$. Analogously for $\Xi_j = J(-1)^j \xi(\tau)$ we have the equation

$$\tau^2 (1 - \tau^2) \xi''(\tau) - \tau (1 - \tau^2) \xi'(\tau) + 2(q_{S}^{*} - 1 + 4\tau^2)\xi(\tau) = 0, \tag{168}$$

with the solution

$$\xi(\tau) = \tilde{C}^{(+)} \tau_{1+2i\eta S} F_1\left( \frac{3}{2} + i\eta S, -\frac{3}{2} + i\eta S \bigg| \tau^2 \right) + \tilde{C}^{(-)} \tau_{1+2i\eta S} F_1\left( \frac{3}{2} - i\eta S, -\frac{3}{2} - i\eta S \bigg| \tau^2 \right). \tag{169}$$

Here the arbitrary complex constants $C^{(\pm)}$ and $\tilde{C}^{(\pm)}$ have to be fixed from a sewing procedure with the solutions $\Psi_j$ and $\Phi_j$ close to the boundaries.

For $J \gg J - j \sim 1$ we can easily solve (158)

$$Y_j = \Xi_j = (-1)^j (J - j + 1). \tag{170}$$

To solve the recursion relation for $J \gg j \sim 1$ is more difficult but we accept the same strategy as for Eq. (152). However, since the permutation symmetry is lost we have to modify the procedure accordingly. Namely, the expansion coefficients in Eqs. (157,161) are related to each other by

$$\Psi_j = \sum_{k=0}^{J} W_{kj} \Phi_k, \quad \Phi_j = \sum_{k=0}^{J} W_{jk} \Psi_k, \tag{171}$$

with the Racah coefficients $W_{jk}$, $P_{J,j}(\theta_3,\theta_2|\theta_1) = \sum_{k=0}^{J} W_{jk}(J)P_{J,k}(\theta_1,\theta_2|\theta_3)$, which can be read off from the general result of Ref. 22. Then, for the region in question the solutions can be written as

$$\begin{cases} \begin{aligned} Y_j \\ \Xi_j \end{aligned} = (-1)^j (j + 3) \int_0^1 d\tau P_j^{(1,2)}(2\tau - 1) \left\{ \frac{(j+1)(j+3)}{(j+2)} \xi(\tau) \right\}. \tag{172} \end{cases}$$
Evaluating the integrals we obtain (see Ref. [22])

\[
\gamma_j = (-1)^j \frac{(j+1)(j+3)}{j+2} \left\{ \tilde{C}^{(+)} \left[ {}_2F_1 \left( \frac{3}{2} + \frac{i\eta_s}{1+2i\eta_s} \mid 1 \right) \right] + (-1)^j \frac{\Gamma \left( \frac{3}{2} + i\eta_s \right) \Gamma \left( \frac{5}{2} + j - i\eta_s \right)}{\Gamma \left( \frac{3}{2} - i\eta_s \right) \Gamma \left( \frac{5}{2} + j + i\eta_s \right)} \times {}_4F_3 \left( \begin{array}{c} -\frac{3}{2} + i\eta_s, -\frac{1}{2} + i\eta_s, \frac{3}{2} + i\eta_s, \frac{5}{2} + j + i\eta_s, 1 \end{array} \right) \right\},
\]

and

\[
\xi_j = (-1)^j (j+2) \left\{ C^{(+)} \left[ {}_2F_1 \left( \frac{3}{2} + i\eta_s, -\frac{1}{2} + i\eta_s \mid 1 \right) \right] + (-1)^j \frac{\Gamma \left( \frac{3}{2} + i\eta_s \right) \Gamma \left( \frac{5}{2} + j - i\eta_s \right)}{\Gamma \left( \frac{3}{2} - i\eta_s \right) \Gamma \left( \frac{5}{2} + j + i\eta_s \right)} \times {}_4F_3 \left( \begin{array}{c} \frac{1}{2} + i\eta_s, -\frac{1}{2} + i\eta_s, \frac{1}{2} + i\eta_s, \frac{3}{2} + i\eta_s, 1 \end{array} \right) \right\},
\]

(173)

Comparing Eqs. (167) and (170,173,174) in the region of their overlap we find

\[
\frac{C^{(+)} \left[ {}_2F_1 \left( \frac{3}{2} + i\eta_s, -\frac{1}{2} + i\eta_s \mid 1 \right) \right]}{C^{(-) \left[ {}_2F_1 \left( \frac{3}{2} + i\eta_s, -\frac{1}{2} + i\eta_s \mid 1 \right) \right]}} = -\frac{\Gamma \left( \frac{3}{2} - i\eta_s \right) \Gamma \left( \frac{5}{2} + j + i\eta_s \right)}{\Gamma \left( \frac{3}{2} + i\eta_s \right) \Gamma \left( \frac{5}{2} + j - i\eta_s \right)} \times {}_4F_3 \left( \begin{array}{c} -\frac{3}{2} + i\eta_s, -\frac{3}{2} - j + i\eta_s, \frac{5}{2} + j + i\eta_s, 1 \end{array} \right),
\]

(174)

and the quantization condition for the eigenvalues \( q_s \) (cf. [19])

\[
4\eta_s \ln J - \arg \frac{\tilde{C}^{(-)} \left[ {}_2F_1 \left( \frac{3}{2} + i\eta_s \mid 1 \right) \right]}{C^{(+) \left[ {}_2F_1 \left( \frac{3}{2} + i\eta_s, -\frac{1}{2} + i\eta_s \mid 1 \right) \right]}} \Gamma^2 \left( \frac{3}{2} + i\eta_s \right) = 2\pi m,
\]

(176)

where \( m \in \mathbb{Z}_+ \).

For the upper part of the spectrum \( q_s \propto J^4 \), the classical motion is concentrated in the vicinity of the point \( j_{\text{max}} \). Looking the solution to the recursion relation in the form of the series \([74]\) we find immediately that the differential equations resulted from these are almost the same, to the required accuracy, as for the chiral-odd sector so that we can immediately adapt the results of Ref. [22]. Therefore, the WKB expansion of the charge reads

\[
q_s(J) = \sum_{\ell=0}^{\infty} \frac{q_s^{(\ell)}(J)}{J^\ell},
\]

(177)

with the expansion coefficients

\[
q_s^{(0)}(n) = \frac{1}{2},
\]

\[
q_s^{(1)}(n) = 6 - \frac{1}{\sqrt{2}} - \sqrt{2}n,
\]

\[
q_s^{(2)}(n) = \frac{1}{16} \left( 403 - 72\sqrt{2} \right) + \frac{1}{8} \left( 11 - 72\sqrt{2} \right)n + \frac{11}{8}n^2,
\]

\[
\ldots,
\]

(178)
with \( n = 0, 1, \ldots \). It differs from the \( q_T \) expression in Ref. [22] only at the level of \( O(J^{-2}) \) corrections.

### 3.4 Energy spectrum.

The knowledge of eigenfunctions allows to find the energy of the quark-gluon-quark system. For the trajectories starting from the top of the spectrum we have

\[
E(J, q_S) = \ln q_S/2 - 4\psi(1) - \frac{3}{2} + O(J^{-1}),
\]

(179)

with \( q_S \) taken from Eq. (177).

For the levels behaving as \( E \propto 2 \ln J \) we can find the exact lowest trajectory [58] corresponding to the charge (165)

\[
E(J) = \psi(J + 3) + \psi(J + 4) - 2\psi(1) - \frac{1}{2}.
\]

(180)

The remainder of the spectrum is described by the formula [14, 22]

\[
E(J, q_S) = 2 \ln J - 4\psi(1) + 2 \text{Re} \psi \left( \frac{3}{2} + i\eta_S \right) - \frac{3}{2},
\]

(181)

and it compares quite well with the explicit numerical diagonalization of Eq. (144) as shown in Figs. [11, 12].

### 4 Conclusion.

In this paper we have demonstrated that the one-loop QCD evolution equations for the three-gluon and quark-gluon-quark correlation functions can be reduced to (almost) integrable one-dimensional lattice models. The appearance of the integrable interaction deduced from the Yang-Baxter bundle (31), within the present context, is a consequence of the logarithmic behaviour of QCD in collinear regime since the Mellin transform w.r.t. momentum of the emitted gluon from a particle diverges as \( \ln J \) for large \( J \), i.e. for the kinematics with negligible momentum transfer to a gluon. For finite \( J \) it stems from the \( \psi(J) \)-function, i.e. the one which appears in local Hamiltonians. It is well known that in the region of momenta mentioned above, QCD can be replaces by the theory of eikonalized fields which should possesses, hence, more symmetries than the original QCD Lagrangian.

Integrability allows to disentangle the energy spectra of the three-particle systems and an additional hidden charge serves to count the trajectories giving rise to different scale dependence of the corresponding multiplicatively renormalizable components of the correlators. The explicit analytical solutions have been found in the form of WKB type expansions w.r.t. the total conformal
Figure 11: The analytical trajectories from the Eqs. (180,181) and the numerically diagonalized anomalous dimension matrix (144).

Figure 12: Same as in Fig. 11 but for large $J$. The spectrum is cut from the top where it is almost continuous.
spin $J$ of the system. A few first terms in the series gave a good accuracy as compared to the numerical results obtained by a brute force diagonalization of the anomalous dimension matrices. Moreover, our analytical expressions are valid up to a very low $J$ and, therefore, this allows to apply the solutions for realistic calculations of the effects of scaling violations in physical observables.

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