1 MATHEMATICAL BACKGROUND

In differentiable rendering, the image can be represented as an integral over an evolving manifold containing the path space, and scene derivatives can be evaluated using transport theorems. For completeness, we introduce here the related mathematical background.

We will follow existing terminology and definitions of fluid and continuum mechanics [Cermelli et al. 2005], mathematics [Seguin and Fried 2014], and their application to differentiable rendering [Zhang et al. 2020].

1.1 Mathematical Notions

We first add details of common concepts in mathematical analysis and differential geometry used in our work.

Open and closed sets. A set in the Euclidean space \( U \subset \mathbb{R}^n \) is called an open set if for any \( x \in U \) there exists \( \varepsilon > 0 \) s.t. \( \{y \in \mathbb{R}^n \mid ||y-x|| < \varepsilon\} \subset U \). A set \( U \subset \mathbb{R}^n \) is called an open set relative to \( X \) if for any \( x \in U \) there exists \( \varepsilon > 0 \) s.t. \( \{y \in X \mid ||y-x|| < \varepsilon\} \subset U \). A set \( A \subset \mathbb{R}^n \) (resp. \( A \subset \mathbb{R}^m \)) is called a closed set (resp. closed set relative to \( X \)) if \( \mathbb{R}^n - A \) (resp. \( X - A \)) is an open set (resp. open set relative to \( X \)).

Continuous functions. For sets \( X \) and \( Y \subset \mathbb{R}^n \), a function \( \varphi: X \rightarrow Y \) is called continuous if for any open set \( U \) relative to \( Y \), \( \varphi^{-1}(U) \) is open relative to \( X \). Note that this definition is equivalent to the epsilon-delta definition.

Manifolds. We define the halfspace as:

\[
\mathbb{R}_+^n := \left\{(x_1,...,x_n) \in \mathbb{R}^n \mid x_1 \geq 0\right\}.
\]

Then a set \( M \subset \mathbb{R}^n \) is called an \( m \)-dimensional manifold (or \( m \)-manifold) if for any \( x \in M \) there exists an open set \( U_x \) relative to \( M \) and a one-to-one function \( \phi_x: U_x \rightarrow \mathbb{R}^n \), called a chart, such that both \( \phi_x \) and \( \phi_x^{-1} \) are continuous. Note that if we can choose each chart \( \phi_x \) so that both \( \phi_x \) and \( \phi_x^{-1} \) are \( C^1 \)-differentiable then \( M \) is called a \( C^1 \)-differentiable manifold. The boundary (or boundary manifold) of \( M \) is defined as:

\[
\partial M := \{x \in M \mid \text{the first coordinate of } \phi_x(x) \text{ is zero}\}.
\]

Note that this definition is independent of the choice of a particular open set \( U_x \) and chart \( \phi_x \), and that \( \partial M \) is a \( (m-1) \)-manifold. The interior of \( M \) is defined as \( \text{Int}(M) := M - \partial M \).

We can intuitively understand an \( m \)-manifold \( M \) as a set of points of that need at least \( m \) real numbers to parameterize all points in \( M \). For instance, a surface mesh embedded in \( \mathbb{R}^3 \) is a 2-manifold, and the space of light paths \( \Omega_k \) with \( k+1 \) vertices is a \( 2(k+1) \)-manifold embedded in \( \mathbb{R}^{3(k+1)} \).

Tangent space. Suppose that \( M \subset \mathbb{R}^n \) is a \( C^1 \)-differentiable \( m \)-manifold. The tangent space of \( M \) on \( x \in M \) is then defined as:

\[
T_xM := \{\gamma \ (0) \mid \gamma: (-\epsilon, \epsilon) \rightarrow M \text{ is a differentiable curve,}
\]

\[
\text{and } \gamma(0) = x \subset \mathbb{R}^n,
\]

where the derivative (velocity w.r.t. parameterization) of the curve \( \gamma(0) \) can be evaluated as usual in \( \mathbb{R}^n \). Note that \( T_xM \) is an \( m \)-dimensional vector space; a vector in \( T_xM \) is called a tangent vector.

1.2 Evolving Manifolds

An evolving \( m \)-manifold \( M(\theta) \subset \mathbb{R}^n \) with respect to the scene parameter \( \theta \in \mathbb{R}^d \) can be considered as a function mapping each value of the parameter \( \theta \) to each \( m \)-manifold. Its trajectory is defined as \( \mathcal{J} := \{(x, \theta) \mid x \in M(\theta), \theta \in \mathbb{R}^d\} \subset \mathbb{R}^{n+d} \). When \( M(\theta) \) evolves continuously, we can assume that \( \mathcal{J} \) is an \( (m+d) \)-manifold.

In the following we fix the scene parameters vector \( \theta = (\theta_1,...,\theta_d) \) as a single scalar \( \theta \) for simplicity. Generalization to vector \( \theta \) will be introduced at the end of this section.

While we have the motion of the entire manifold \( M \) with respect to \( \theta \), describing the motion of a single point \( x \in M(\theta) \) cannot be defined in a trivial way. A local parameterization is defined as a one-to-one function \( \hat{x}: U \rightarrow \mathcal{J} \) such that \( U \) is open in \( \mathbb{R}^n \times \mathbb{R}^d \) and \( \hat{x}(p, \theta') \in M(\theta') \) for any \( (p, \theta') \in U \). Suppose a local parameterization for a given point \( x \in M(\theta) \), i.e., there exists \( (p_0, \theta) \in U \) s.t. \( \hat{x}(p_0, \theta) = x \).

Then we can define the local velocity of \( x \) as:

\[
\nu(x, \theta) := \frac{\partial}{\partial \theta'} \hat{x}(p_0, \theta') \bigg|_{\theta'=\theta} \subset \mathbb{R}^n.
\]

The local velocity depends on the choice of local parameterization. Unlike fluid or continuum mechanics, we should eliminate this dependency to get well-defined formulations on evolving manifolds. When the codimension of \( M \) is one, i.e., \( n = m+1 \), the scalar normal velocity \( V_M(x, \theta) \) and local tangential velocity \( v_{\text{tan}}(x, \theta) \) of a given point \( x \in M(\theta) \) are defined as:

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\[ \mathcal{V}_M(\mathbf{x}, \theta) := v(\mathbf{x}, \theta) - n(\mathbf{x}, \theta), \quad (5) \]
\[ v_{\text{tan}}(\mathbf{x}, \theta) := v(\mathbf{x}, \theta) - \mathcal{V}_M(\mathbf{x}, \theta) n(\mathbf{x}, \theta), \quad (6) \]

where \( n(\mathbf{x}, \theta) \) denotes the unit normal vector of \( M(\theta) \) at \( x \). Then the local tangential velocity \( v_{\text{tan}} \) still depends on the local parameterization, but the scalar normal velocity \( \mathcal{V}_M(\mathbf{x}, \theta) \) is independent of such local parameterization. However, while the surface geometry embedded in \( \mathbb{R}^3 \) has codimension one, the space of light paths \( \mathcal{L}_k \) has a higher codimension \( (k + 1) \), so the normal vector \( n \) cannot be defined. Thus, more generally, the local tangential velocity and the vector normal velocity \( \mathcal{V}_M(\mathbf{x}, \theta) \) are defined as:

\[ v_{\text{tan}}(\mathbf{x}, \theta) := \Pi_{\mathcal{T}_M}(v(\mathbf{x}, \theta)), \quad (7) \]
\[ \mathcal{V}_M(\mathbf{x}, \theta) := v(\mathbf{x}, \theta) - v_{\text{tan}}(\mathbf{x}, \theta), \quad (8) \]

where \( \Pi_{\mathcal{T}_M} : \mathbb{R}^n \to \mathcal{T}_M \) denotes the canonical projection (orthogonal projection) from a vector space onto a subspace. Note the definitions of \( v_{\text{tan}} \) in Equations (6) and (7) are equivalent, while \( \mathcal{V}_M \) in Equations (5) and \( \mathcal{V}_M \) in Equation (8) are related as \( \mathcal{V}_M = \mathcal{V}_M \mathcal{B} \) when \( n = m + 1 \). Steady-state path-space differentiable rendering [Zhang et al. 2020] did not use the vector normal velocity since the authors relied on the Reynolds transport relation on a 2D manifold embedded in \( \mathbb{R}^3 \). This is no longer possible in transient state, so that the need to apply the generalized transport theorem on the path space \( \mathcal{L}_k \subset \mathbb{R}^{n(k+1)} \).

The boundary of the evolving manifold, \( \partial M(\theta) \), is also an evolving manifold. We can define a local velocity \( v_{\partial M}(\mathbf{x}, \theta) \), a local tangential velocity \( v_{\text{tan}, \partial M}(\mathbf{x}, \theta) \), and a vector normal velocity \( \mathcal{V}_{\partial M}(\mathbf{x}, \theta) \) on \( \partial M(\theta) \) in the same fashion, but the scalar normal velocity \( \mathcal{V}_{\partial M}(\mathbf{x}, \theta) \) at \( x \in \partial M(\theta) \) is defined differently as:

\[ v_{\text{tan}, \partial M}(\mathbf{x}, \theta) := \Pi_{\mathcal{T}_{\partial M}}(v_{\partial M}(\mathbf{x}, \theta)), \quad (9) \]
\[ \mathcal{V}_{\partial M}(\mathbf{x}, \theta) := v_{\partial M}(\mathbf{x}, \theta) - v_{\text{tan}, \partial M}(\mathbf{x}, \theta), \quad (10) \]
\[ v_{\partial M}(\mathbf{x}, \theta) := \mathcal{V}_{\partial M}(\mathbf{x}, \theta) \cdot n_{\partial M}(\mathbf{x}, \theta), \quad (11) \]

where \( n_{\partial M}(\mathbf{x}, \theta) \in \mathcal{T}_{\partial M}(\theta) \). Note that when restricting the direction of the unit normal vector \( n_{\partial M}(\mathbf{x}, \theta) \) into \( \mathcal{T}_{\partial M}(\theta) \) (instead of into \( \mathbb{R}^n \)) the outgoing normal direction \( n_{\partial M} \) from \( M(\theta) \) is determined uniquely, so that we can use the scalar normal velocity of \( \partial M(\theta) \) under the general dimensionality of \( M(\theta) \) and \( \mathbb{R}^n \).

### 1.3 Generalized Transport Theorem for Evolving Manifolds

Suppose that there is a scalar field \( \varphi : \mathcal{F} \to \mathbb{R} \) defined on an evolving manifold. Generally, the field \( \varphi \) could not be defined on the entire space \( \mathbb{R}^{n+1} \), so the partial derivative of the field \( \varphi \) with respect to \( \theta \) cannot be defined in a trivial way. In other words, when trying to evaluate \( \lim_{\varepsilon \to 0} \frac{\varphi(x(\mathbf{x}, \theta + \varepsilon)) - \varphi(x(\mathbf{x}, \theta))}{\varepsilon} \), the numerator cannot be defined unless \( x \) lies on both \( M(\theta) \) and \( M(\theta + \varepsilon) \). Therefore, we first define a derivative which depends on the choice of local parameterization, then we can define a parameterization-independent derivative from the dependent one. The derivative \( \psi(\mathbf{x}, \theta) \) w.r.t. \( \theta \), which depends on choice of local parameterization, and the normal derivative \( \hat{\psi}(\mathbf{x}, \theta) \) w.r.t. \( \theta \), which is independent of local parameterization, are defined as follows:

\[ \psi(\mathbf{x}, \theta) := \left. \frac{\partial}{\partial \theta} \varphi \right|_{\theta=0} \left( \hat{x}(\{p_0, \theta'\}, \theta') \right) |_{\theta'=0}, \quad (12) \]
\[ \hat{\psi}(\mathbf{x}, \theta) := \psi(\mathbf{x}, \theta) - v_{\text{tan}}(\mathbf{x}, \theta) \cdot \nabla M(\varphi)(\mathbf{x}, \theta). \quad (13) \]

Now we investigate the derivative of the integration over evolving manifolds. Seguin et al. [2014] showed that the derivative of the integral of \( \varphi \) over the evolving manifold \( M(\theta) \) can be represented as the transport theorem for evolving manifolds:

\[ \frac{d}{d\theta} \int_{M(\theta)} \varphi d\mu_M = \int_{M(\theta)} \left\{ \frac{\partial}{\partial \theta} - v_{\text{tan}}(\mathbf{x}, \theta) \cdot \nabla M(\varphi)(\mathbf{x}, \theta) \right\} d\mu_M + \int_{\partial M(\theta)} \varphi \mathcal{V}_M d\mu_{\partial M}. \quad (14) \]

where \( \hat{x} \) is the total curvature vector, \( m \) times the mean curvature vector on the \( m \)-manifold \( M(\theta) \), and \( \mu_M \) and \( \mu_{\partial M} \) are the measures on \( M(\theta) \) and \( \partial M(\theta) \), respectively. The mean curvature vector on a \( m \)-manifold embedded in \( \mathbb{R}^n \) has been defined in differential geometry [Carmo 1992; Chen 1975]. According to their definition, the mean curvature vectors can be well defined for arbitrary codimensions, i.e., even if \( n > m + 1 \), so that the unit normal vector on the manifold is not uniquely defined.

Note that this transport theorem has been given different names depending on the dimension \( m \) and the codimension \( n \). For the simplest case, \( m = n - 1 \), the theorem is called the Leibniz integral rule, and for the case of \( m = n = 3 \) (or any case of \( m = n \)) the theorem is usually called the Reynolds transport theorem. Zhang et al. [2020] used this transport theorem for the particular case of \( m = 2 \) and \( n = 3 \), and applied it iteratively. Our work generalizes this for any dimension and codimension.

### 1.4 Generalized Transport Theorem with Discontinuity

In this section we will treat the case when the scalar field \( \varphi \) contains discontinuities. For simplicity, our notations follow previous work [Zhang et al. 2020]. The discontinuity set (discontinuity submanifold) of \( M(\theta) \) with respect to \( \varphi \) is defined as:

\[ \Delta M(\varphi)(\theta) := \{ x \in M(\theta) | \varphi(\cdot, \theta) \text{ is discontinuous at } x \}. \quad (15) \]

We assume that \( \Delta M(\varphi)(\theta) \) can be represented as a finite union of \( (m - 1) \)-submanifolds of \( M(\theta) \), and that \( \Delta M(\varphi)(\theta) \) itself evolves continuously. Then the continuous interior \( \text{Int}^{c}(M)(\varphi)(\theta) \) and the extended boundary \( \partial \overline{M}(\varphi)(\theta) \) of \( M \) with respect to \( \varphi \) are defined as follows:

\[ \text{Int}^{c}(M)(\varphi)(\theta) := \text{Int}(M)(\varphi)(\theta) - \Delta M(\varphi)(\theta) \quad (16) \]
\[ \partial \overline{M}(\varphi)(\theta) := \partial M(\varphi)(\theta) \cup \Delta M(\varphi)(\theta). \quad (17) \]

Note that when \( M \) is a 2D manifold (surface), \( \Delta M \) can also be called a discontinuity curve, as in Zhang et al. [2020]. We will often omit the dependency \( \{\varphi \} \) or \( \{\theta\} \) for simplicity.

The scalar field \( \varphi \) is continuous on each connected component of the continuous interior \( \text{Int}^{c}(M) \), so the integral over \( M \) can be represented as the sum of integrals over each connected component of \( \text{Int}^{c}(M) \). Then we can apply the transport theorem (14) for each connected component, and finally obtain the transport theorem with discontinuities:
\[ \frac{d}{d\theta} \int_{M(\theta)} \varphi d\mu_M = \int_R \left( \varphi - \varphi \cdot \hat{V}_M \right) d\mu_M + \int_{\partial M} \Delta \varphi V_{\partial M} d\mu_M, \]

where \( \Delta \varphi (x, \theta) \) on \( \partial M \) is defined as:

\[ \Delta \varphi (x, \theta) := \begin{cases} \varphi (x, \theta), & \text{if } x \in \partial M \\ \varphi^-(x, \theta) - \varphi^+(x, \theta), & \text{if } x \in \partial M \setminus \{ \partial M \} \end{cases}. \]

Here, \( \varphi^-(x, \theta) \) and \( \varphi^+(x, \theta) \) are defined as the limits of \( \varphi(x, \theta) \) when approaching \( x \) from \( -\Delta_M \) and \( \Delta_M \), respectively. Note that the normal vectors \( n_{\partial M} \) and \( n_{\partial M} \) on \( \partial M \), the scalar normal velocities \( V_{\partial M} \) and \( V_{\partial M} \), and measures \( \mu_{\partial M} \) and \( \mu_{\partial M} \) can be defined in similar ways to \( \partial M \).

1.5 Generalization to Vector Parameters

Taking multiple parameters \( \theta = (\theta_1, \ldots, \theta_d) \) into account is easily achievable by repeating the formulations for each parameter \( \theta_i \). Then \( \mathbb{R}^n \) vector velocity terms \( \varphi \) and \( \hat{V} \) change to \( \mathbb{R}^{n \times d} \) Jacobians, and scalar velocity and derivative terms \( V, \psi, \phi \) change to \( \mathbb{R}^{1 \times d} \) gradients. Then changing the inner product term \( \hat{V}_M \) into a matrix product \( \hat{V}_M \) from Equations (14) and (18) generalizes the transport theorem for multiple parameters. However, since the transport theorem for multiple parameters is equivalent to enumerating the theorem for each parameter, we keep writing formulations for a single parameter \( \theta \) for the sake of simplicity.

2 DERIVATION DETAILS ON DIFFERENTIAL TRANSIENT PATH INTEGRAL

2.1 Formal Definition of Scene Geometry

In physically-based rendering, the scene geometry is usually represented as a 2D manifold, which is unfortunately riddled with discontinuities at the edges of the polygons. To clarify how to treat this discontinuity, we first define piece-wise differentiable manifolds in this section.

The scene geometry \( M \) is a piece-wise differentiable 2D manifold, which satisfies that:

- \( M \) is represented as a finite union of differentiable 2D manifolds, i.e., \( M = \bigcup_{i=1}^{p} M_{[i]} \).
- For any two distinct pieces \( M_{[i]} \) and \( M_{[j]} \), if they intersect \( (M_{[i]} \cap M_{[j]} \neq \emptyset) \) then the intersection \( M_{[i]} \cap M_{[j]} \) is a differentiable 1D manifold (curve).

Then the boundary of the piece-wise differentiable 2D manifold \( M \) is a piece-wise differentiable 1D manifold, defined as \( \partial M := \bigcup_{i=1}^{p} \partial M_{[i]} \), and the interior of the piece-wise differentiable 2D manifold \( M \) is defined as \( \text{Int} (M) := M - \partial M = \bigcup_{i=1}^{k} \text{Int} (M_{[i]}) \). Note that in usual polygonal representations, each differentiable piece \( M_{[i]} \) corresponds to each planar polygon (usually triangles), and the boundary of the scene geometry \( \partial M \) becomes the union of all edges in all polygon meshes. We do not consider self-intersection of scene geometry.

When a point \( x \) is in the boundary of the scene \( \partial M \), we observe that \( x \) belongs to one of two cases: it is either contained in exactly one differentiable piece \( \partial M_{[i]} \), or in the intersection of two differentiable pieces \( \partial M_{[i]} \cap \partial M_{[j]} \). We will call the first case the boundary edges, and the second case the sharp edges [Zhang et al. 2020, 2019]. Then the boundary of the scene geometry \( \partial M \) becomes the union of the boundary edges and the sharp edges.

Generally, we can define piece-wise differentiable \( m \)-manifolds in a similar way, and extend the transport theorem in Equation (18) to piece-wise differentiable manifolds by taking the summation of the equation for each differentiable piece. Then we can use the same equation defining the transport theorem on a piece-wise differentiable 2 \((k + 1)\)-manifold \( \Omega_k = M^{k+1} \).

2.2 Product Space Rules

When an evolving manifold \( \mathcal{N} (\theta) \) is formed as the product of two other evolving manifolds, \( \mathcal{N} (\theta) = M_1 (\theta) \times M_2 (\theta) \), and there is a scalar function \( \varphi (\cdot, \theta) : \mathcal{N} (\theta) \rightarrow \mathbb{R} \) which is the product of scalar functions \( \varphi_1 (\cdot, \theta) : M_1 (\theta) \rightarrow \mathbb{R} \) and \( \varphi_2 (\cdot, \theta) : M_2 (\theta) \rightarrow \mathbb{R} \), i.e., \( \varphi (x_1, x_2, \theta) = \varphi_1 (x_1, \theta) \varphi_2 (x_2, \theta) \), we can use the same transport theorem described in Equation (18) by substituting \( M \) by \( \mathcal{N} \). Then Equation (18) can be evaluated in terms of \( M_1 \) and \( M_2 \) as follows:

\[ \frac{d}{d\theta} \varphi (x_1, x_2, \theta) = \varphi_1 (x_1, \theta) \varphi_2 (x_2, \theta) = \frac{\partial M_1}{\partial \theta} (\mathcal{N}, \theta) \times M_2 (\theta) + M_1 (\theta) \times \frac{\partial M_2}{\partial \theta} (\mathcal{N}, \theta), \]

\[ \frac{\partial \mathcal{N} (\mathcal{N})}{\partial \theta} (\mathcal{N}, \theta) = \frac{\partial M_1 (\mathcal{N})}{\partial \theta} (\mathcal{N}, \theta) \times \frac{\partial M_2 (\mathcal{N})}{\partial \theta} (\mathcal{N}, \theta), \]

\[ \frac{\partial \mathcal{N} (\mathcal{N})}{\partial \theta} (\mathcal{N}, \theta) = \frac{\partial \mathcal{N}_1 (\mathcal{N})}{\partial \theta} (\mathcal{N}, \theta) \times \frac{\partial \mathcal{N}_2 (\mathcal{N})}{\partial \theta} (\mathcal{N}, \theta). \]

These rules can be extend in a similar way to an arbitrary number of products, such as the order-\( k \) path space \( \Omega_k = M^{k+1} \).

2.3 Terms in the Path Integral

Recall the transient path integral, the path throughput, and the correlated importance described in the main paper, respectively, for negligible scattering delays surface path vertices:

\[ l = \int_{\Omega} \mathcal{F} (\mathcal{X}) \ d\mu (\mathcal{X}), \]

\[ \mathcal{F} (\mathcal{X}) := \int_{\Omega} \mathcal{S}_\mathcal{E} (\mathcal{X}) \ d\mu (\mathcal{X}), \]

\[ \mathcal{S}_\mathcal{E} (\mathcal{X}) := \int_{0}^{\infty} L_{\mathcal{E}} (x_0, x_1) \ W_{\mathcal{E}} (x_0, x_1) \ d\mu (\mathcal{X}), \]

\[ \mathcal{S}_\mathcal{E} (\mathcal{X}) := \int_{0}^{\infty} L_{\mathcal{E}} (x_0, x_1) \ W_{\mathcal{E}} (x_0, x_1) \ d\mu (\mathcal{X}), \]

\[ \mathcal{S}_\mathcal{E} (\mathcal{X}) := \int_{0}^{\infty} L_{\mathcal{E}} (x_0, x_1) \ W_{\mathcal{E}} (x_0, x_1) \ d\mu (\mathcal{X}). \]

For a mathematically rigorous derivation, we define each term in the path integral: \( L_{\mathcal{E}}, \rho, W_{\mathcal{E}}, G, \) and \( V \).
We will denote this function as $G(x, y) = \begin{cases} \frac{\omega_0 \omega_y}{||x-y||^2} & x \neq y \\ 0 & x = y \end{cases}$.

Equation (25)

Also, when the domain is restricted to differentiable pieces of the scene geometry, denoted by $\mathcal{M}_{ij} \times \mathcal{M}_{ij}$, which contains their boundaries $\partial \mathcal{M}_{ij}$ and $\partial \mathcal{M}_{ij}$, we will denote this function as $G(\mathcal{M}_{ij} \times \mathcal{M}_{ij}) = \mathbb{R}$, where $G(\mathcal{M}_{ij} \times \mathcal{M}_{ij})$ is a continuous function. Note that the entire geometric function $G$ satisfies $\Delta (M^2) [G] = \delta (M^2)$.

Definition 2.2. For given scene geometry $M$, the visibility function $V: M \times M \rightarrow \mathbb{R}$ is defined as:

$$V(x, y) = \begin{cases} 1 & \text{openlineseg}(x, y) \cap M = \emptyset \\ 0 & \text{otherwise} \end{cases} \tag{26}$$

where $\text{openlineseg} := \{(x + (1 - \lambda)y) | 0 < \lambda < 1\}$ means the open line segment between two given points. Unlike the geometric function $G$ and the visibility function $V$, the light source emission function $L_e$, surface scattering function $(BSDF)$, and the sensor sensitivity function $W_e$ vary depending on the scene. We can introduce the following conditions which those terms should satisfy in practice:

A.1 For any fixed $x_0, x_1 \in M$ and $L_e$ ($x_0 \rightarrow x_1$) as a function of $t$ contains a finite number of joint discontinuities at $t_{i1} \ldots t_{ir}$ and a finite number of Dirac delta distributions at $t_{i1} \ldots t_{ir}$. Except for $t_{i1} \ldots t_{ir}$ and $t_{i1} \ldots t_{ir}$, $L_e(x_0 \rightarrow x_1) = \emptyset$ is continuous. Also, $t_{i1} \ldots t_{ir}$ and $t_{i1} \ldots t_{ir}$ vary continuously when $x_0$ and $x_1$ vary continuously.

A.2 For any fixed $x_{n1}, x_n \in M$, $W_e(x_{n1} \rightarrow x_n)$ as a function of $t$ contains a finite number of jump discontinuities at $t_{w1} \ldots t_{ws}$. Except for $t_{w1} \ldots t_{ws}$, $W_e(x_0 \rightarrow x_1) = \emptyset$ is continuous. Also, $t_{w1} \ldots t_{ws}$ vary continuously when $x_0$ and $x_1$ vary continuously.

A.3 The source emission $L_e$ does not have non-zero energy on spatially zero-measure sets.

A.4 The scattering function $\rho$ does not contain a Dirac delta, i.e., there is no ideal specular reflection, and is continuous except when the incoming or outgoing directions are perpendicular to the surface normal.

A.1 and A.2 are our novel assumptions for transient rendering, while A.3 and A.4 are common assumptions used in physically-based differentiable rendering [Bangaru et al. 2020; Li et al. 2018; Loubet et al. 2019; Zhang et al. 2020, 2019]. Note that these assumptions cover most of practical cases.

2.4 Differential Transient Path Integral

To differentiate Equation (21) using the transport theorem in Equation (18), we will first evaluate the boundary path space $\partial \Omega_k [f_T] (\Omega)$. For simplicity, we first fix $k$ and evaluate the order-$k$ boundary path space $\partial \Omega_k$. By definition of extended boundary (Equation (17)), $\partial \Omega_k$ consists of the (geometric) boundary $\partial \Omega_k$ and the discontinuity set $\Delta \Omega_k [f_T]$. Since the total throughput $f_T$ is the product of partial terms $S_e, \rho, V, \text{and } G$, the entire discontinuity set can be represented as the union of discontinuity sets caused by each of those partial terms as follows:

$$\partial \Omega_k [f_T] = \partial \Omega_k \cup \Delta \Omega_k [f_T]$$

$$= \partial \Omega_k \cup \Delta \Omega_k [S_e] \cup \Delta \Omega_k [G_1, \ldots, G_k]$$

$$\cup \Delta \Omega_k [V_1 \ldots V_k] \cup \Delta \Omega_k [\rho_1, \ldots, \rho_{k-1}] \tag{27}$$

Note that both boundary edges and sharp edges are contained in $\partial \Omega_k$ as mentioned in §2.1.

Applying the product space rule Eq. (20) to Eq. (27), $\Delta \Omega_k [G_1, \ldots, G_k]$, $\Delta \Omega_k [V_1 \ldots V_k]$, and $\Delta \Omega_k [\rho_1, \ldots, \rho_{k-1}]$ can be rewritten as follows:

$$\Delta \Omega_k [G_1, \ldots, G_k] = \bigcup_{j=1}^{k} M_0 \times \ldots \times M_{i-2} \times \Delta M^2 [G] \times M_{i+1} \times \ldots \times M_k$$

$$\Delta \Omega_k [V_1 \ldots V_k] = \bigcup_{j=1}^{k} M_0 \times \ldots \times M_{i-2} \times \Delta M^2 [V] \times M_{i+1} \times \ldots \times M_k$$

$$\Delta \Omega_k [\rho_1, \ldots, \rho_{k-1}] = \bigcup_{j=1}^{k} M_0 \times \ldots \times M_{i-2} \times \Delta M^3 [\rho] \times M_{i+1} \times \ldots \times M_k \tag{28}$$

Vanishing and overlapping discontinuities. Some discontinuity sets of partial terms, $\Delta \Omega_k [S_e], \Delta M^2 [G], \Delta M^2 [V]$ and $\Delta M^3 [\rho]$ may not contribute to the total discontinuity set. As discussed in previous work [Zhang et al. 2020], $\Delta M^3 [\rho]$ is caused at the horizontal ray direction, but in that case the geometric term $G$ is continuously zero. For a formal mathematical analysis we rely on the following proposition:

Proposition 2.3. Given a set $X \in \mathbb{R}^n$ and a subset $A \subset X$, suppose that a function $\varphi_1: X \rightarrow \mathbb{R}$ is continuously zero on $A$ and a function $\varphi_2: X \rightarrow \mathbb{R}$ is a bounded function. Then even if $\varphi_2$ is discontinuous in $A$, $A$ does not contribute to the discontinuity set of $\varphi_1 \varphi_2$, i.e., $A \cap \Delta X [\varphi_1 \varphi_2] = \emptyset$.

Thus, we can omit $\Delta \Omega_k [\rho_1, \ldots, \rho_{k-1}]$ from Equation (27). The visibility function $V$ also presents a discontinuity at the horizontal ray direction, which vanished by the geometric function $G$. Then Equation (27) can be rewritten in the following simpler form:

$$\Delta \Omega_k [f_T] = \partial \Omega_k \cup \Delta \Omega_k [S_e] \cup \Delta \Omega_k [G_1, V_1 \ldots V_k] - \partial \Omega_k \tag{29}$$

Note that in the last term of Equation (29) we will evaluate the discontinuity set of the product term $G V$ directly, without considering discontinuities on $\partial \Omega_k$, already accounted for in the first term of Equation (29).

2.4.1 Discontinuity of $G V$ terms. The term $\Delta \Omega_k [G_1 V_1 \ldots V_k] - \partial \Omega_k$ caused by silhouette edges can be represented by the product rule (Equation (20)) as:

$$\Delta \Omega_k [G_1 V_1 \ldots V_k] - \partial \Omega_k = \bigcup_{j=1}^{k} M_0 \times \ldots \times M_{i-2} \times \Delta M^2 [G V] \times M_{i+1} \times \ldots \times M_k \tag{30}$$
where the term $\{\Delta M^2 [GV] - \partial M^2\}$ is obtained by the following lemma:

**Lemma 2.4.** The discontinuity submanifold of GV can be specified as:

$$\Delta M^2 [GV] - \partial M^2 = \Delta \left( \int \left( M^2 \right) \right) [GV]$$

$$= \{(x, y) \in \int \left( M^2 \right) | \text{line segment between } x, y \text{ only intersects silhouette edges of } M\}. \tag{31}$$

A line segment between $(x, y)$ intersects a silhouette edge of $M$ if and only if:

$$\exists z \in \text{open lineseg}(x, y) \cap M \text{ s.t. one of the three cases holds:}$$

1. $z \in \int(M)$ and $n(z) \cdot (y - x) = 0$,
2. $z \in \partial M\{i\} \text{ (boundary edge)}$,
3. $\{z \in \partial M\{i\} \cap \partial M\{j\} \text{ with } i \neq j \text{ (sharp edge)}\}$ and
   $$\begin{cases} (n_i(z) \cdot y - x) & (n_j(z) \cdot (y - x)) \leq 0. \end{cases} \tag{32}$$

To prove the lemma, we first show that $V^{-1}(\{1\}) \cap \int(M^2) \subset \int(\rho)(M^2) [GV]$, i.e. for any pair of mutually visible points $(x, y) \in \int(M^2)$ with $V(x, y) = 1$, GV is continuous on $(x, y)$. If $y - x$ is orthogonal to $n(x)$ or $n(y)$, the claim holds since $G$ is continuously zero on $(x, y)$. Otherwise, we can easily observe that the visible pair of points are still visible when they move within a sufficiently small distance.

Second, we observe that if $(x, y) \in V^{-1}(\{0\}) \cap \int(M^2)$ and the line segment between them intersects a point $z \in M$ which is not on a silhouette edge, then $V$ is continuous on $(x, y)$. Note that by definition of silhouette described in Lemma 2.4, $(x, y) \in \int(M^2)$ with $V(x, y) = 1$ is orthogonal to $n(x)$ or $n(y)$, the claim holds since $G$ is continuously zero on $(x, y)$. Otherwise, we can easily observe that the visible pair of points are still visible when they move within a sufficiently small distance, the line segment between them still penetrates the open disk on $(x, y)$.

**2.4.2 Discontinuities of the $S_e$ term.** Recall the assumptions A.1 and A.2 for source emission $L_e$ and sensor sensitivity $W_e$. Given $x_0$ and $x_1$, $L_e(x_0 \rightarrow x_1, \cdot)$ has Dirac delta mass at $t = t_{1j} \cdots t_{lj}$, and given $x_{k-1}$ and $x_k$, $W_e(x_{k-1} \rightarrow x_k, \cdot)$ has discontinuity at $t = t_{W1} \cdots t_{WK}$. By the following lemma, we can obtain the discontinuity of $S_e$.

**Lemma 2.5.** With assumptions A.1 and A.2, the discontinuity of $S_e$ can be determined as:

$$\Delta \Omega_{\kappa} \left[ S_e \right] = \left\{ x \in \Omega_{\kappa} | \text{tof}(x) = t_{Wi} - t_j \right\}$$

$$\text{for some } 1 \leq i \leq s, 1 \leq j \leq r. \tag{33}$$

Note that this representation does not include a vanishing part for $G \equiv 0$. To prove this lemma, first let $u_{t_0}(t)$ denote the unit step function, which is defined as $u_{t_0}(t) = 1$ for $t > t_0$ and $u_{t_0}(t) = 0$ otherwise. Also, $\delta_{t_0}(t)$ denotes the Dirac delta function centered at $t_0$ so that $\delta_{t_0}(t) = 1(t - t_0)$. Important facts to prove Lemma 2.5 is that $u_{t_0} \star u_{t_1}$ is a continuous function and $\delta_{t_0} \star u_{t_1} = u_{t_0} \star \delta_{t_0}$ where $\star$ denotes the cross-correlation. Since $S_e(x_0, x_1, x_{k-1}, x_k, t = \text{tof}(x))$ is the correlation of $L_e$ and $W_e$ in the temporal domain, the only discontinuity occurs when tof $(x)$ lies on the discontinuous point of $\delta_{t_0} \star u_{t_1}$.

For example, suppose that the source is $L_e(x_0, x_1, t) = L_e(x_0, x_1) \delta(t)$ and the sensor is $W_e(x_{N-1}, x_N, t) = W_e(x_{N-1}, x_N) \text{box}(t; t_{\text{start}}, t_{\text{end}})$ where box $(t; t_{\text{start}}, t_{\text{end}})$ refers to the unit box function starting from $t_{\text{start}}$ and ending at $t_{\text{end}}$. In this case the discontinuity of $S_e$, $\Delta \Omega_{\kappa} \left[ S_e \right]$, is the set of paths with travel time $t_{\text{start}}$ or $t_{\text{end}}$. In practice, if the light source and the sensor are not Dirac deltas then $\Delta \Omega_{\kappa} \left[ S_e \right] = \emptyset$.

**Boundary Contribution $\Delta f_r$ and Normal Velocities $V_{\partial \Omega_{\kappa}}$.** When $\Delta \Omega_{\kappa} \left[ S_e \right]$ becomes an empty set, the boundary path space $\partial \Omega_{\kappa}$ consists of $\partial \Omega_{\kappa} \left[ G_1 \cdot V_1 \cdots G_k \cdot V_k \right] = \partial \Omega_{\kappa}$. Then the discontinuity comes from the visibility, so $\Delta f_r$ becomes the same as $f_r$ and the normal velocity $V_{\partial \Omega_{\kappa}}$ converges to its steady-state counterpart.

3 VALIDATION

We validate our results against the baseline transient renderer by Jarabo and colleagues [2014], using a short light pulse of 0.01 fs.

![Fig. 1. Validation of our differentiable transient rendering against the transient path integral method of Jarabo et al. [2014]. (a) The transient sequences show how the pulse of light illuminates the floor first at 14.68 ns, then the Bunny at 27.02 ns, and finally the red wall at 43.11 ns. At 55.04 ns there is almost no signal. The last row shows the absolute difference between the frames, amplified by a factor of 50. (b) Radiance and absolute difference plot. Dotted vertical lines indicate the exact frames shown above. Our framework matches the baseline closely.](image)

4 ALGORITHM TABLES

We append Algorithms 1 and 2 for estimating the interior and boundary integrals, respectively. Variables $d$ and $\delta$ store the optical path
**Algorithm 1:** Estimating the interior integral

**Data:** scene, pixel index \((i, j)\), max bounce \(k\)

**Result:** Rendered temporal histogram of the \((i, j)\)-th pixel \(I[i, j, \cdot]\) and its scene derivative \(I[i, j, \cdot, \cdot]\)

\[
x[k + 2], y[k + 2] \leftarrow \text{New arrays of 3D positions on the scene geometry};
\]

\[
d_x[k + 2], d_y[k + 2] \leftarrow \text{New arrays of floating numbers (path distance)};
\]

\[
f_k[k + 2], f_y[k + 2] \leftarrow \text{New arrays of floating numbers (path distance)};
\]

\[
f_y[k + 1], f_y[k + 1] \leftarrow \text{New arrays of floating numbers (path distance)};
\]

\[
x[0] \leftarrow \text{camera position}, (f_x[0], f_y[0]) \leftarrow (1, 0);
\]

**for** \(1 \leq i < k + 2\) **do**

\[
\text{if } i = 1 \text{ then}
\]

\[
\text{Sample } \tilde{\omega}_0, p \sim \mathbb{P}_{\text{cameraPrimaryRay}, i, j};
\]

\[
\text{else}
\]

\[
\text{Sample } \omega[p, p] \sim \mathbb{P}_{\text{bidir}}(x[i - 1], \dot{\omega}_i, \cdot);
\]

\[
x_{\text{ci}} \leftarrow \text{rayTrace}(x[i - 1], \dot{\omega}_i);
\]

**if** \(x_{\text{ci}}\) **is valid** **then**

\[
x[i] \leftarrow x_{\text{ci}};
\]

\[
(\alpha, \delta) \leftarrow \text{The value and scene derivative of:}
\]

\[
\rho(x[i - 2] \rightarrow x[i - 1] \rightarrow y[i]) G(x[i - 1], x[i]) f(x[i]);
\]

\[
p \leftarrow p[i\alpha(x[i] - \tilde{\omega}_i)] / \|x[i] - x[i - 1]\|;\
\]

\[
f_x[i] \leftarrow f_x[i - 1] + \tilde{\omega}_i;\
\]

\[
f_y[i] \leftarrow f_y[i - 1] + \tilde{\omega}_i;\
\]

\[
\delta, \delta \leftarrow \text{The value and scene derivative of:}
\]

\[
\eta \|x[i] - y[i - 1];
\]

\[
d_x[i] \leftarrow d_x[i] + d_y[i - 1] + \delta;\
\]

\[
d_y[i] \leftarrow d_y[i - 1] + \delta;\
\]

\[
\tilde{\omega}_i \leftarrow -\tilde{\omega}_i;
\]

**else**

\[
\text{break;}
\]

\[
\text{Sample } y[0] \sim \mathbb{P}_{\text{emitter}}, f_y[0] \sim \mathbb{P}_{\text{emitter}}(y[0]);
\]

Similarly construct the light subpath \(y[s], dy[s], dy[s], f_y[s], f_y[s];\)

**for** \(0 \leq s \leq k - 1\) **do**

\[
(f, f) \leftarrow \text{combineSubpaths}(x[0 + : s + 1], y[0 : k - s]);
\]

\[
(\delta, \delta) \leftarrow \text{The value and scene derivative of:}
\]

\[
\eta \|x[s] - y[k - s - 1];
\]

\[
d_x[s] \leftarrow d_x[s] + d_y[k - s - 1] + \delta;\
\]

\[
d_y[s] \leftarrow d_y[s +] + d_y[k - s] + \delta;\
\]

\[
w \leftarrow \text{CombinationStrategy}; \quad \text{// Use Chapter 9 in [Veatch 1997]}
\]

**for** \(1 \leq s \leq \text{Scene range}(d)\) **do**

\[
s \leftarrow S_n[l] (y[0], y[1], x[1], x[0], d/c);
\]

\[
\delta \leftarrow S_n[l] (y[0], y[1], x[1], x[0], d/c);
\]

\[
l[i, j, l] \leftarrow l[i, j, l] + wfs;
\]

\[
l[i, j, l] \leftarrow l[i, j, l] + wfs + wfs;\]

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