Research Article

New Bounds for the Randić Index of Graphs

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The Randić index of a graph $G$ is defined as the sum of weights $1/\sqrt{d_u d_v}$ over all edges $uv$ of $G$, where $d_u$ and $d_v$ are the degrees of the vertices $u$ and $v$ in $G$, respectively. In this paper, we will obtain lower and upper bounds for the Randić index in terms of size, maximum degree, and minimum degree. Moreover, we obtain a generally lower and a general upper bound for the Randić index.

1. Introduction

Let $G$ be a simple graph with a vertex set $V = V(G)$ and edge set $E(G)$. The integers $n = n(G) = |V(G)|$ and $m = m(G) = |E(G)|$ are the order and the size of the graph $G$, respectively. The open neighborhood of vertex $v$ is $N_G(v) = N(v) = \{u \in V(G) | uv \in E(G)\}$, and the degree of $v$ is $d_G(v) = d_v = |N(v)|$. One of the most active fields of research in contemporary chemical graph theory is the study of topological indices of graph invariants that can be used for describing and predicting physicochemical and pharmacologic properties of organic compounds. Topological indices have been used and have been shown to give a high degree of predictability of pharmaceutical properties. In 1947, Wiener [1] conceived the first molecular graph-based structure descriptor, eventually named the “Wiener index.” The information on the chemical constitution of the molecule is conventionally represented by a molecular graph. Graph theory was successfully provided by the chemist with a variety of very useful tools, namely, topological indices. Among the several hundred presently existing graph-based molecular structure descriptors [2], the Randić index $R(G)$ of a graph was introduced by the chemist Randić under the name of “branching index” in 1975 [3] as the sum of

$$R(G) = \sum_{uv \in E(G)} \frac{1}{\sqrt{d_u d_v}} \quad (1)$$

Also, it was designed in 1975 to measure the extent of branching of the carbon-atom skeleton of saturated hydrocarbons. It was demonstrated that the Randić index is well correlated with a variety of physicochemical properties of alkanes, such as boiling point, enthalpy of formation, surface area, and solubility in water.

The Randić index is certainly the most widely applied in chemistry and pharmacology, in particular for designing quantitative structure-property and structure-activity relations. Randić proposed this index to “quantitatively characterize the degree of molecular branching.” According to him, “the degree of branching of the molecular skeleton is a critical factor” for some molecular properties such as “boiling points of hydrocarbons and the retention volumes and the retention times obtained from chromatographic studies” (all citations are taken from [3]).

Zhou et al. [4] obtained lower and upper bounds for the general Randić index, and Du et al. [5] obtained new lower and
upper bounds for the Randić index in terms of other topology indices; for other bounds, see [6, 7]. Then, in this paper, we will obtain new lower and upper bounds for the Randić index.

2. Main Results

In this section, we present lower and upper bounds for the Randić index.

We make use of the following lemmas in this paper to obtain the results.

Lemma 1 (see [8]). Let $x_i$ and $y_i$, $i = 1, \ldots, n$, be real numbers such that $Xx_i \leq y_i \leq Yx_i$ for each $i = 1, \ldots, n$. Then,

$$\sum_{i=1}^{n} x_i y_i \geq \frac{1}{n} \sum_{i=1}^{n} x_i^2 + \frac{1}{n} \sum_{i=1}^{n} y_i^2,$$

with equality holding if and only if either $y_i = Xx_i$ or $y_i = Yx_i$ for each $i = 1, \ldots, n$.

Lemma 2 (see [9]). Let $s_1, s_2, \ldots, s_n$ be nonnegative real numbers with the property

$$s_1 + s_2 + \cdots + s_n = 1.$$

Further, let

$$b_1 \geq b_2 \geq \cdots \geq b_n,$$

be real numbers, and assume that there are $t, T \in R$ such that $0 < t \leq b_i \leq T < + \infty$ for each $i = 1, 2, \ldots, n$. Then,

$$\sum_{i=1}^{n} s_i b_i + t T \sum_{i=1}^{n} s_i \leq t + T.$$

Equality in equation (5) is obtained if and only if $T = b_1 = \cdots = b_k \geq b_{k+1} = \cdots = b_n = t$ for some $k$, $1 \leq k \leq n$.

Theorem 1. Let $G$ be a connected graph of size $m$, maximum degree $\Delta$, and minimum degree $\delta$. Then,

$$R(G) \geq \frac{\Delta^3 - \delta^3}{\Delta^3 + \delta^3} R_2(G) + M_2(G),$$

and the equality holds in (6) if and only if $G$ is a regular graph.

Proof. By the definition of the Randić index, we can write

$$R(G) = \frac{1}{\sum_{v \sim v, \delta(v) \leq \Delta} \sqrt{d_v d_w} \times \frac{1}{d_v d_w}}.$$

For each edge $v \sim v, \delta(v) \leq \Delta$, it holds that

$$\frac{\Delta^3}{d_v d_w} \leq \sqrt{d_v d_w} \leq \frac{\Delta^3}{d_v d_w},$$

where the left-hand side equality is attained if and only if $d_v = d_w = \Delta$ for $v \sim v, \delta(v) \leq \Delta$ and the right-hand side equality is attained if and only if $d_v = d_w = \Delta$ for $v \sim v, \delta(v) \leq \Delta$.

Setting $x_i = 1/d_v d_w, y_i = \sqrt{d_v d_w}, X = \Delta^3,$ and $Y = \Delta^3$ in Lemma 1 and Inequality (8), we have

$$R(G) = \sum_{v \sim v, \delta(v) \leq \Delta} \sqrt{d_v d_w} \times \frac{1}{d_v d_w}.$$

and the proof is completed.

Suppose that equality holds in (6). Then, the equality holds in (8). From the equality in (8), by Lemma 1, we obtain

$$\sqrt{d_v d_w} = \frac{\delta^3}{d_v d_w},$$

or

$$\sqrt{d_v d_w} = \frac{\Delta^3}{d_v d_w},$$

for each edge $v \sim v, \delta(v) \leq \Delta$. By the equality condition in (9), we must have $d_v = d_w = \Delta$ or $d_v = d_w = \Delta, \delta$ for each edge $v \sim v, \delta(v) \leq \Delta$. G, $G$ is regular as $G$ is connected.

Conversely, one can easily check that equality holds in (6) for regular graph.

For any nontrivial connected graph $G$, $R_2(G) \geq m/\Delta^4$ and $M_2(G) \geq m\delta^2$ with either equality if and only if $G$ is regular. From Theorem 1, it then follows immediately the following consequence.

Corollary 1. Let $G$ be a connected graph of size $m$, maximum degree $\Delta$, and minimum degree $\delta$. Then,

$$R(G) \geq \frac{m\delta^2}{\Delta^3 - \Delta \delta + \delta^3}.$$
Theorem 3. Let \( G \) be a graph of size \( m \), with maximum degree \( \Delta \) and minimum degree \( \delta \). Then, the following inequality holds:

\[
R(G) \geq \frac{m\delta}{\delta \Delta + \Delta^2} + \frac{m}{\Delta + \delta}
\]  

if and only if \( G \) is a regular graph.

Proof. Note that for each edge \( v_i v_j \in E(G) \), we have

\[
1/\Delta \leq 1/\sqrt{d_i d_j} \leq 1/\delta; \text{ hence, we can write that}
\]

\[
\left( \frac{1}{\sqrt{d_i d_j}} - \frac{1}{\Delta} \right) \left( \frac{1}{\sqrt{d_i d_j}} - \frac{1}{\delta} \right) \leq 0.
\]  

By Inequality (13), we have

\[
\frac{1}{d_i d_j} + \frac{\Delta}{\delta} \geq \frac{\Delta + \delta}{1/\Delta + 1/\delta},
\]

and this leads to the desired bound.

Suppose that equality holds in (13). Then, the equality holds in (14). From the equality in (14), we have

\[
\frac{1}{\sqrt{d_i d_j}} = \Delta,
\]

or

\[
\frac{1}{\sqrt{d_i d_j}} = \delta.
\]

Note that \( \Delta \) and \( \delta \) are natural numbers; hence, we have \( d_i = d_j = 1 \), for each edge \( v_i v_j \in E(G) \), i.e., \( G = K_2 \).

Conversely, one can easily see that equality holds in (13) for \( G = K_2 \).

Now, we present a lower bound for the Randić index in terms of size \( m \), maximum degree \( \Delta \), and minimum degree \( \delta \).

Theorem 4. Let \( G \) be a graph of size \( m \), with maximum degree \( \Delta \) and minimum degree \( \delta \). Then, the following equality holds:

\[
R(G) = \frac{m}{\sqrt{2\Delta^2 - \delta^2}}
\]

if and only if \( G \) is a regular graph.

Proof. Since for any \( x, y \in \mathbb{R}^* \), we have

\[
x^2 - xy + y^2 \geq xy.
\]

By the definition of the Randić index and setting \( x = d_i \) and \( y = d_j \) in Inequality (26), we can write that
\[ R(G) = \frac{1}{\sqrt{d_i d_j}} \geq \frac{1}{\Delta^2 - d_i d_j + \Delta^2} \]

\[ \geq \frac{1}{\Delta^2 - d_i d_j + \Delta^2} \]

\[ = \frac{m}{\sqrt{2\Delta^2 - \delta^2}} \]

as desired.

Suppose that equality holds in (25). Then, the equality holds in (26). From the equality in (26), we have

\[ x^2 - xy + y^2 = xy, \]

or

\[ x^2 + y^2 = 2xy \implies (x - y)^2 = 0. \]

Note that \( x, y > 0 \); hence, we get \( x = y \); therefore, \( d_i = d_j \) for each edge \( v_i v_j \in E(G) \).

Also, suppose that equality holds in (25). Then, the equality holds in (27). From the equality in (27), we have

\[ \frac{1}{\sqrt{d_i d_j}} = \frac{1}{\sqrt{d_i^2 - d_i d_j + d_j^2}}. \]

or

\[ \sqrt{d_i d_j} = \sqrt{d_i^2 - d_i d_j + d_j^2}. \]

By equality (31),

\[ 2d_i d_j = d_i^2 + d_j^2. \]

Therefore, \( d_i = d_j \) for each edge \( v_i v_j \in E(G) \).

Conversely, one can easily see that equality holds in (25) for regular graphs.

For any real number \( \alpha \), the general Randić index, \( R_\alpha \), is defined in [10] as

\[ M_\alpha^2 = R_\alpha = R_\alpha(G) = \sum_{uv \in E(G)} (d_u d_v)^\alpha. \]

The concept of the first general Zagreb index introduced by Li et al. [11] is defined as

\[ M_\alpha(G) = \sum_{u \in V(G)} d_\alpha^G(u) = \sum_{uv \in E(G)} \left( (d_{G}^{-1}(u) + d_{G}^{-1}(v)) \right), \]

where \( \alpha \in \mathbb{R} \).

In [12], the following upper bounds for \( ID \) were also established:

\[ \text{ID}(G) \leq \frac{(\Delta^\alpha + \delta^\alpha)^2 n^2}{4M_1^\alpha(G)\Delta^\alpha}, \quad \text{if } \alpha \geq 1, \quad (35) \]

\[ \text{ID}(G) \leq \frac{(\Delta^\alpha + \delta^\alpha)^2 n^2}{4M_1^\alpha(G)\Delta^\alpha\delta}, \quad \text{if } \alpha \leq 1, \quad (36) \]

\[ \text{ID}(G) \leq \frac{M_1^\alpha(G)}{\Delta^{\alpha+1}}, \quad \text{if } \alpha \leq -1, \quad (37) \]

\[ \text{ID}(G) \leq \frac{M_1^\alpha(G)}{\delta^{\alpha+1}}, \quad \text{if } \alpha \geq -1, \quad (38) \]

\[ \text{ID}(G) \leq \frac{M_1^\alpha(G) + m\Delta^{2\alpha}}{\Delta^{2\alpha+2}}, \quad \text{if } \alpha \leq -2, \quad (39) \]

\[ \text{ID}(G) \leq \frac{M_1^\alpha(G) + m\delta^{2\alpha}}{\delta^{2\alpha+2}}, \quad \text{if } \alpha \geq -2. \quad (40) \]

We need the following lemma to prove the next theorem.

**Lemma 3** (see [13]). Let \( G \) be an undirected, simple graph of order \( n \geq 2 \) with no isolated vertices. Then,

\[ R_{-1}(G) \leq \frac{1}{2} \sum_{i=1}^{n} \frac{1}{d_i}. \]

Equality holds if and only if \( G \) is a \( k \)-regular graph, \( 1 \leq k \leq n - 1 \).

Applying Inequalities (35)–(40), we establish upper bounds for the Randić index in terms of the general first and second Zagreb indices maximum degree \( \Delta \) and minimum degree \( \delta \).

**Theorem 5.** Let \( G \) be a graph of order \( n \), size \( m \), maximum degree \( \Delta \), minimal degree \( \delta \), and with no isolated vertices. Then,

\[ R(G) \leq \frac{(\Delta^\alpha + \delta^\alpha)^2 n^2}{8M_1^\alpha(G)\Delta^{\alpha-1}}, \quad \text{if } \alpha \geq 1, \quad (42) \]

\[ R(G) \leq \frac{(\Delta^\alpha + \delta^\alpha)^2 n^2}{8M_1^\alpha(G)\Delta^{\alpha-1}\delta}, \quad \text{if } \alpha \leq 1, \quad (43) \]

\[ R(G) \leq \frac{M_1^\alpha(G)}{2\Delta}, \quad \text{if } \alpha \leq -1, \quad (44) \]

\[ R(G) \leq \frac{\Delta M_1^\alpha(G)}{2\delta^{\alpha+1}}, \quad \text{if } \alpha \geq -1, \quad (45) \]

\[ R(G) \leq \frac{M_1^\alpha(G) + m\Delta^{2\alpha}}{2\Delta^{2\alpha+2}}, \quad \text{if } \alpha \leq -2, \quad (46) \]

\[ R(G) \leq \frac{M_1^\alpha(G) + m\delta^{2\alpha}}{2\delta^{2\alpha+2}}, \quad \text{if } \alpha \geq -2. \quad (47) \]
Proof. By the definition of the Randić index, we can write
\[
R(G) = \sum_{v, v' \in E(G)} \frac{1}{\sqrt{d_v d_{v'}}} = \sum_{v, v' \in E(G)} \sqrt{d_v d_{v'}} \times \frac{1}{d_v d_{v'}} 
\]
\[
\leq \Delta \sum_{v, v' \in E(G)} \frac{1}{d_v d_{v'}} 
= \Delta R(G).
\]
By Lemma 3, we have
\[
R(G) \leq \frac{\Delta}{2} n \sum_{i=1}^{n} \frac{1}{d_i},
\]
and from (35)–(40) and (49), we arrive at (42)–(47) directly.
Here, we obtain the upper bound for the inverse degree index that helps us to obtain the next result. □

Lemma 4. Let \( G \) be a simple graph of order \( n \geq 2 \), with \( m \) edges and with no isolated vertices. Then, the following equality holds:
\[
\sum_{i=1}^{n} \frac{1}{d_i} \leq \frac{n(\Delta + \delta) - 2m}{\Delta \delta},
\]
if and only if \( G \) is a \( k \)-regular graph, \( 1 \leq k \leq n - 1 \).

Proof. Setting \( s_i = 1/n \) and \( b_i = d_i \) for \( i = 1, 2, \ldots, n \), \( t = \delta \) and \( T = \Delta \), Inequality (5) becomes
\[
\frac{1}{n} \sum_{i=1}^{n} d_i + \frac{\Delta \delta}{n} \sum_{i=1}^{n} \frac{1}{d_i} \leq \Delta + \delta. 
\]
Note that
\[
\sum_{i=1}^{n} d_i = 2m.
\]
Therefore, we have
\[
\sum_{i=1}^{n} \frac{1}{d_i} \leq \frac{n(\Delta + \delta) - 2m}{\Delta \delta}.
\]
Suppose that equality holds in (50). Then, the equality holds in (51). Equality in (51) holds if and only if \( d_1 = \cdots = d_k \) and \( d_{k+1} = \cdots = d_n \).
By Inequality (49) and Lemma 3, we get the next result. □

Corollary 2. Let \( G \) be a graph of order \( n \), size \( m \), maximum degree \( \Delta \), and minimal degree \( \delta \). Then,
\[
R(G) \leq \frac{n^2(\Delta^2 + \delta^2)}{16m\Delta^2 \delta^2},
\]
where
\[
d = 2m^2 - (n - 1)m\Delta 
+ \frac{\Delta - 1}{2} \left[ 2m\left(\frac{2m}{n}\right) + 1 - 2m\left(1 + \frac{2m}{n}\right)n \right].
\]
By using Inequality (48) and Inequalities (55)–(57), we get the next results.

Corollary 3. Let \( G \) be an undirected, simple graph of order \( n \), size \( m \), maximum degree \( \Delta \), and minimal degree \( \delta \). Then,
\[
R(G) \leq \frac{n^2(\Delta^2 + \delta^2)}{16m\Delta^2 \delta^2},
\]
where
\[
c = 2m^2 - (n - 1)m\Delta 
+ \frac{\Delta - 1}{2} \left[ \Delta^2 + \delta^2 + \frac{(2m - \Delta - \delta)^2}{n - 2} \right].
\]
Denote by \( A \) the adjacency matrix of the graph \( G \) and by \( D \) the diagonal matrix of its vertex degrees. The eigenvalues of \( A \) is
\[
\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_w.
\]
The energy of graph \( G \) is defined as
\[
E_A(G) = \sum_{i=1}^{w} |\lambda_i|.
\]
This concept was introduced by Gutman and is intensively studied in chemistry since it can be used to approximate the total \( \pi \)-electron energy of a molecule (see, e.g., [14, 15]).

The normalized Laplacian matrix of a graph \( G \) is denoted by \( L \) and \( L = I - D^{-1/2}AD^{-1/2} \). Its eigenvalues
\[
\rho_1 \geq \rho_2 \geq \cdots \geq \rho_n = 0,
\]
are normalized Laplacian eigenvalues of graph \( G \). The normalized Laplacian energy (or L-energy) of a graph \( G \) is
In [16], the following upper bounds for \( R_{-1} \) were also established:

\[
R_{-1}(G) \leq \frac{E_L(G)}{2}, \quad (64)
\]

\[
E_L(G) \leq \frac{E_A(G)}{\delta}. \quad (65)
\]

Applying Inequalities (48), (64), and (65), we establish upper bounds for the Randić index in terms of the energy and the normalized Laplacian energy.

**Corollary 4.** Let \( G \) be a graph with maximum degree \( \Delta \) and minimum degree \( \delta \). Then,

\[
R(G) \leq \frac{\Delta E_L(G)}{2}, \quad (66)
\]

\[
R(G) \leq \frac{\Delta E_A(G)}{2\delta}. \quad (67)
\]

In [17], the following lower bounds for \( R_{-1} \) were also established:

\[
R_{-1}(G) \geq \frac{n}{2(n - 1)} + \frac{1}{4}(\rho_1 - \rho_{n-1})^2. \quad (68)
\]

Equality holds if and only if \( G \cong K_n \).

**Proof.** By the definition of the Randić index and by Inequality (67), we can write

\[
R(G) = \sum_{v, v \in E(G)} \frac{1}{\sqrt{d_i d_j}}
\]

\[
= \sum_{v, v \in E(G)} \sqrt{d_i d_j} \times \frac{1}{d_i d_j}
\]

\[
\geq \delta \sum_{v, v \in E(G)} \frac{1}{d_i d_j}
\]

\[
= \delta R_{-1}(G) \geq \frac{\delta n}{2(n - 1)} + \frac{\delta}{4}(\rho_1 - \rho_{n-1})^2, \quad (69)
\]

and the proof is completed.

Suppose that equality holds in (68). Then, the equality holds in (69) and (70). From the equality in (69), we have \( d_i = d_j = \delta \) for each edge \( v, v \in E(G) \). From the equality in (70), we have \( G \cong K_n \).

Conversely, one can easily see that equality holds in (68) for graph \( K_n \).

Denote

\[
\Lambda(G) = \{T(G) \mid \text{is a topology index of graph } G\},
\]

\[
\Omega(G) = \{T(G) \mid T \text{ is a topology index of graph } G \text{ and } T(G) \geq R(G)\},
\]

where \( R(G) \) is the Randić index.

Next, we present a general upper bound for the Randić index in terms of other topological indices. \( \Box \)

**Theorem 6.** Let \( G \) be graph of order \( n \) and minimal degree \( \delta \). Then,

\[
R(G) \geq \frac{\delta n}{2(n - 1)} + \frac{\delta}{4}(\rho_1 - \rho_{n-1})^2. \quad (71)
\]

Equality holds if and only if \( G \cong K_n \).

**Proof.** For \( x, y, z \geq 0 \) and \( x \leq z \), we have

\[
(x + y)(z - x) \geq 0,
\]

and by solving this inequality, we get

\[
x \leq \frac{\sqrt{4yz + (z - y)^2} + z - y}{2} \quad \text{or}
\]

\[
x \geq \frac{-\sqrt{4yz + (z - y)^2} + z - y}{2}
\]

By setting \( x = R(G), y = \Lambda(G) \), and \( z = \Omega(G) \) in Inequality (75), we obtain

\[
R(G) \leq \frac{4\Lambda(G)\Omega(G) + (\Omega(G) - \Lambda(G))^2 + \Omega(G) - \Lambda(G)}{2}. \quad (72)
\]

Suppose that equality holds in (73). Then, the equality holds in (74); hence, we have

\[
(x + y)(z - x) = 0.
\]

By the equality condition in (77), we obtain

\[
(x + y) = 0 \text{ or } (z - x) = 0.
\]

Therefore, we have two following cases:

**Case 1.** If \( (x + y) = 0 \), note that \( x, y \geq 0 \); hence, we obtain \( x = y = 0 \).

**Case 2.** If \( (z - x) = 0 \), then we obtain \( x = z \); by this fact and by Inequality (75), we get \( x = z = 0 \). Since \( x = R(G), y = \Lambda(G) \), and \( z = \Omega(G) \), therefore, we have \( G \cong K_n \).

Conversely, one can easily see that equality holds in (73) for \( G \cong K_n \).
Denote $\Psi (G) = \{ T (G) \mid T \text{ is a topology index of graph } G \}$ and $T (G) \leq R (G)$, where $R (G)$ is the Randić index.

For $x, y, z \geq 0$ and $x \geq z$, we have

$$(x + y) (z - x) \leq 0.$$  \hfill (79)

Next, we present a general lower bound for the Randić index in terms of other topological indices.

By Inequality (79) and similarly with proof of Theorem 7, we get the next result.

**Theorem 8.** Let $G$ be a graph with the Randić index $R (G)$. Then,

$$R (G) \geq \frac{\sqrt{4\Lambda (G)\Psi (G) + (\Psi (G) - \Lambda (G))^2 + \Psi (G) - \Lambda (G)}}{2}.$$  \hfill (80)

Here, we recall some of the topological indices that will need to explain one of the applications of Theorems 7 and 8.

The first and second Zagrebi indices are vertex-degree-based graph invariants defined as

$$M_1 = M_1 (G) = \sum_{uv \in E (G)} (d_u + d_v),$$

$$M_2 = M_2 (G) = \sum_{uv \in E (G)} d_u d_v.$$  \hfill (81)

The quantity $M_1$ was first considered in 1972 [18], whereas $M_2$ in 1975 [19].

The forgotten topological index has been introduced by Furtula and Gutman [20] as

$$F (G) = \sum_{uv \in E (G)} (d_u^2 + d_v^2).$$  \hfill (82)

The harmonic index, denoted by $H (G)$, was defined in [21] as

$$H (G) = \sum_{uv \in E (G)} \frac{2}{d_u + d_v}.$$  \hfill (83)

The (first) geometric-arithmetic index of a graph was defined in [22] as

$$GA (G) = \sum_{uv \in E (G)} \frac{2 \sqrt{d_u d_v}}{d_u + d_v}.$$  \hfill (84)

### 3. Applications of Theorems 7 and 8

Using Theorems 7 and 8, we can obtain many upper and lower bounds for the Randić index in terms of other topological indices. For example, by Theorem 7, we can obtain following upper bounds.

Note that for $x, y \geq 0$, we have

$$\frac{1}{\sqrt{xy}} \leq \sqrt{xy},$$  \hfill (85)

$$\frac{1}{\sqrt{xy}} \leq xy,$$  \hfill (86)

$$\frac{1}{\sqrt{xy}} \leq \frac{x + y}{2}.$$  \hfill (87)

By Inequalities (85)–(87) and definitions of topological indices, we have

$$R (G) \leq \frac{M_1 (G)}{2},$$

and hence \{ $R_1 (G), M_1 (G), M_2 (G)$ \} $\in \Omega (G)$. Here, we let \{ $F (G), GA (G), H (G)$ \} $\in \Lambda (G)$; therefore, by Theorem 7, we get the following results:

$$R (G) \leq \frac{\sqrt{4R_1 (G)F (G) + (R_1 (G) - F (G))^2 + R_1 (G) - F (G)}}{2},$$

$$R (G) \leq \frac{\sqrt{4M_2 (G)GA (G) + (M_2 (G) - GA (G))^2 + M_2 (G) - GA (G)}}{2},$$

$$R (G) \leq \frac{\sqrt{4H (G)M_1 (G) + (M_1 (G) - H (G))^2 + M_1 (G) - H (G)}}{2},$$

$$R (G) \leq \frac{\sqrt{4M_2 (G)F (G) + (M_2 (G) - F (G))^2 + M_2 (G) - F (G)}}{2},$$

$$R (G) \leq \frac{\sqrt{4M_1 (G)GA (G) + (M_1 (G) - GA (G))^2 + M_1 (G) - GA (G)}}{2}.$$
Analogously above, we can obtain lower bounds for the Randić index by using Theorem 8.

**Data Availability**

The data involved in the examples of our manuscript are included within the article.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.

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