ON SUMMABILITY, MULTIPLIABILITY, PRODUCT INTEGRABILITY, AND PARALLEL TRANSLATION

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Dedicated to Professor Heikki Haahti on the occasion of his 85th birthday

Abstract

In this paper we provide necessary and sufficient conditions for the existence of the Kurzweil, McShane and Riemann product integrals of step mappings with well-ordered steps, and for right regulated mappings with values in Banach algebras. Our basic tools are the concepts of summability and multipliability of families in normed algebras indexed by well-ordered subsets of the real line. These concepts also lead to the generalization of some results from the usual theory of infinite series and products. Finally, we consider Stieltjes-type product integrals, Haahti products, and their relation to parallel translation operators.

Keywords: product integral, step mapping, right regulated mapping, well-ordered set, summability, multipliability, transfinite induction, parallel translation, Haahti product

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1 Introduction

The aim of this paper is to apply the concepts of summability and multipliability in order to generalize some results in the theory of infinite series and products, and also to derive criteria for product integrability of mappings which take values in Banach algebras. Product integrals and Haahti products are then used to define parallel translation operators and to study their properties.

The paper is organized as follows.

In Section 2, we begin by recalling the definition of summability introduced by S. Heikkilä in [7]. We consider sums of the form \( \sum_{\alpha \in \Lambda} x_\alpha \), where the index set \( \Lambda \) is a well-ordered subset of \( \mathbb{R} \cup \{ \infty \} \), and \( x_\alpha \) are elements of a normed vector space; sums of this type were used in [7] as a tool in the study of integrability and impulsive differential equations. Then we proceed to the related novel concept of multipliability and consider products of the form \( \prod_{\alpha \in \Lambda} x_\alpha \), where \( x_\alpha \) are elements of a normed algebra. In the case when \( \Lambda = \mathbb{N} \), our definitions and results correspond to the usual theory of infinite series and products in normed spaces and algebras.

In Section 3, we recall the general definition of the Kurzweil and McShane product integrals of the form \( \prod_{a}^{b} V(t, dt) \), which were studied in [8-13, 15-19], and which include the product integrals \( \prod_{a}^{b} (I + A(t) \ dt) \) and \( \prod_{a}^{b} (I + dA(t)) \) considered in the next sections. In infinite-dimensional Banach algebras, the Kurzweil and McShane product integrals lose some of their pleasant properties. To overcome this difficulty, we follow the
We define the sum $\sum_{\alpha \in \Lambda} x_\alpha$ of a family $(x_\alpha)_{\alpha \in \Lambda}$ of elements $x_\alpha \in E$ as the limit of the partial sums $S_N = \sum_{\alpha \in \Lambda, \alpha < N} x_\alpha$ if it exists. Obviously, for a fixed well-ordered set $\Lambda$, the set of all summable families $(x_\alpha)_{\alpha \in \Lambda}$ forms a linear space.

In Sections 4 and 5, we focus on the product integrals $\prod_{\alpha \in \Lambda} (I + A(t) dt)$ in the sense of Kurzweil, McShane and Riemann. We apply the results from Sections 2, 3 and from the papers [12, 17] to derive new sufficient and necessary conditions for product integrability of right-continuous step mappings having well-ordered steps, and then for right regulated mappings.

Section 6 is devoted to the Riemann-Stieltjes and Kurzweil-Stieltjes product integrals $\prod_{\alpha \in \Lambda} (I + dA(t))$. The main result here is concerned with Kurzweil-Stieltjes product integrability of right-continuous step mappings with well-ordered steps. The results from Sections 4, 5, 6 are illustrated on a number of examples.

In Section 7, we present an application of Stieltjes-type product integrals to differential geometry. In [6], H. Hájek and S. Heikkilä studied operators corresponding to parallel translation of vectors along paths on $C^0$-manifolds, and used product and Riemann-Stieltjes product integration techniques to establish the existence of these operators; their results are generalized in Section 7.

2 Summability, multipliability, and their properties

In this section, we generalize some results of the theory of infinite series and products. A nonempty subset $\Lambda$ of $\mathbb{R} \cup \{\infty\}$, ordered by the natural ordering $\prec$ of $\mathbb{R}$ together with the relation $t < \infty$ for every $t \in \mathbb{R}$, is well-ordered if every nonempty subset of $\Lambda$ has the smallest element. In particular, to every number $\beta$ of $\Lambda$, different from its possible maximum, there corresponds the smallest element in $\Lambda$ that is greater than $\beta$. It is called the successor of $\beta$ and is denoted by $S(\beta)$. There are no numbers of $\Lambda$ in the open interval $(\beta, S(\beta))$.

If an element $\gamma$ of $\Lambda$ is not a successor or the minimum of $\Lambda$, it is called a limit element. For every $\gamma \in \mathbb{R}$, we denote

$$\Lambda^{<\gamma} = \{\alpha \in \Lambda; \alpha < \gamma\},$$

$$\Lambda^{\leq \gamma} = \{\alpha \in \Lambda; \alpha \leq \gamma\}.$$

One of our basic tools in this paper is the following principle of transfinite induction:

If $\Lambda$ is well-ordered and $\mathcal{P}$ is a property such that if $\mathcal{P}(\beta)$ is true whenever $\mathcal{P}(\beta)$ is true for all $\beta \in \Lambda^{<\beta}$, then $\mathcal{P}(\gamma)$ is true of all $\gamma \in \Lambda$.

The following definition of summability is adopted from [7].

**Definition 2.1.** Let $E$ be a normed space, and let $\Lambda$ be a well-ordered subset of $\mathbb{R} \cup \{\infty\}$. Denote $a = \min \Lambda$, and $b = \sup \Lambda$. The family $(x_\alpha)_{\alpha \in \Lambda}$ with elements $x_\alpha \in E$ is called summable if for every $\gamma \in \Lambda \cup \{b\}$, there is an element $\sum_{\alpha \in \Lambda, \alpha < \gamma} x_\alpha$ of $E$, called the sum of the family $(x_\alpha)_{\alpha \in \Lambda^{<\gamma}}$, satisfying the following conditions:

(i) $\sum_{\alpha \in \Lambda^{<\beta}} x_\alpha = 0$, and if $\gamma = S(\beta)$ for some $\beta \in \Lambda$, then $\sum_{\alpha \in \Lambda^{<\gamma}} x_\alpha = x_\beta + \sum_{\alpha \in \Lambda^{<\beta}} x_\alpha$.

(ii) If $\gamma$ is a limit element, then for each $\varepsilon > 0$ there is a $\beta_\varepsilon \in \Lambda^{<\gamma}$ such that

$$\left\| \sum_{\alpha \in \Lambda^{<\beta}} x_\alpha - \sum_{\alpha \in \Lambda^{<\beta}} x_\alpha \right\| < \varepsilon, \quad \beta \in \Lambda \cap [\beta_\varepsilon, \gamma).$$

We define the sum $\sum_{\alpha \in \Lambda} x_\alpha$ of a summable family $(x_\alpha)_{\alpha \in \Lambda}$ as $\sum_{\alpha \in \Lambda^{<\beta}} x_\alpha$ if $b \notin \Lambda$, and $x_b + \sum_{\alpha \in \Lambda^{<b}} x_\alpha$ if $b \in \Lambda$. A family $(x_\alpha)_{\alpha \in \Lambda}$ is called absolutely summable if $(\|x_\alpha\|)_{\alpha \in \Lambda}$ is summable.

Obviously, for a fixed well-ordered set $\Lambda$, the set of all summable families $(x_\alpha)_{\alpha \in \Lambda}$ forms a linear space.
In a normed algebra with a unit element, we introduce the following related concept of multipliability. We point out that our concept of multipliable families is different from the definition of multipliable sequences given in the appendix of [1].

**Definition 2.2.** Let $E$ be a normed algebra with a unit element $I$, and let $\Lambda$ be a well-ordered subset of $\mathbb{R} \cup \{\infty\}$. Denote $a = \min \Lambda$, and $b = \sup \Lambda$. The family $(x_\alpha)_{\alpha \in \Lambda}$ with elements $x_\alpha \in E$ is called multipliable if for every $\gamma \in \Lambda \cup \{b\}$, there is an element $\prod_{\alpha \in \Lambda^{<\gamma}} x_\alpha$ of $E$, called the product of the family $(x_\alpha)_{\alpha \in \Lambda^{<\gamma}}$, satisfying the following conditions:

(i) $\prod_{\alpha \in \Lambda^{<\gamma}} x_\alpha = I$, and if $\gamma = S(\beta)$ for some $\beta \in \Lambda$, then $\prod_{\alpha \in \Lambda^{<\gamma}} x_\alpha = x_\beta \cdot \prod_{\alpha \in \Lambda^{<\beta}} x_\alpha$.

(ii) If $\gamma$ is a limit element, then for each $\epsilon > 0$ there is a $\beta_\epsilon \in \Lambda^{<\gamma}$ such that

$$\left\| \prod_{\alpha \in \Lambda^{<\beta}} x_\alpha - \prod_{\alpha \in \Lambda^{<\gamma}} x_\alpha \right\| < \epsilon, \quad \beta \in \Lambda \cap \beta_\epsilon, \gamma).$$

We define the product $\prod_{\alpha \in \Lambda} x_\alpha$ of a multipliable family $(x_\alpha)_{\alpha \in \Lambda}$ as $\prod_{\alpha \in \Lambda^{<\gamma}} x_\alpha$ if $b \not\in \Lambda$, and $x_b \cdot \prod_{\alpha \in \Lambda^{<b}} x_\alpha$ if $b \in \Lambda$.

**Remark 2.3.** In Definition 2.1, condition (ii) can be rephrased by saying that for each limit element $\gamma \in \Lambda$, we have

$$\lim_{\beta \to \gamma^-} \left( \sum_{\alpha \in \Lambda^{<\beta}} x_\alpha \right) = \sum_{\alpha \in \Lambda^{<\gamma}} x_\alpha.$$ Similarly, condition (ii) in Definition 2.2 says that for each limit element $\gamma \in \Lambda$, we have

$$\lim_{\beta \to \gamma^-} \left( \prod_{\alpha \in \Lambda^{<\beta}} x_\alpha \right) = \prod_{\alpha \in \Lambda^{<\gamma}} x_\alpha.$$

In the rest of this section, we assume that $\Lambda$ is a well-ordered set in $\mathbb{R} \cup \{\infty\}$ with $a = \min \Lambda$ and $b = \sup \Lambda$. Observe that if $(x_\alpha)_{\alpha \in \Lambda}$ is a summable family and $c \in (a, b)$, then $(x_\alpha)_{\alpha \in \Lambda \cap [c, b]}$ is summable, too; the corresponding partial sums from Definition 2.1 are simply $\sum_{\alpha \in \Lambda \cap [c, b]} x_\alpha = \sum_{\alpha \in \Lambda^{<\gamma}} x_\alpha - \sum_{\alpha \in \Lambda^{<c}} x_\alpha$.

On the other hand, multipliability of $(x_\alpha)_{\alpha \in \Lambda}$ does not necessarily imply the multipliability of $(x_\alpha)_{\alpha \in \Lambda \cap [c, b]}$. However, the statement becomes true if we assume that the elements of $(x_\alpha)_{\alpha \in \Lambda}$ and its product are invertible. In this case, the partial products from Definition 2.2 are given by

$$\prod_{\alpha \in \Lambda \cap [c, b]} x_\alpha = \left( \prod_{\alpha \in \Lambda^{<\gamma}} x_\alpha \right) \left( \prod_{\alpha \in \Lambda^{<c}} x_\alpha \right)^{-1},$$

where the invertibility of the last product is guaranteed by the next lemma.

**Lemma 2.4.** Suppose that $(x_\alpha)_{\alpha \in \Lambda^{<\gamma}}$ is a multipliable family in a normed algebra, and that its members and product are invertible. Then all products $\prod_{\alpha \in \Lambda^{<\gamma}} x_\alpha$, $\gamma \in \Lambda$, are invertible.

**Proof.** Assume there is a $\gamma_0 \in \Lambda$ such that $\prod_{\alpha \in \Lambda^{<\gamma_0}} x_\alpha$ is not invertible. We use transfinite induction to show that for every $\gamma \in \Lambda$ such that $\gamma \geq \gamma_0$, the product $\prod_{\alpha \in \Lambda^{<\gamma}} x_\alpha$ is not invertible; this will be in contradiction with the assumption that the product of the whole family is invertible.

We already know that $\prod_{\alpha \in \Lambda^{<\gamma_0}} x_\alpha$ is not invertible.

If $\gamma > \gamma_0$ and $\gamma = S(\beta)$, then

$$\prod_{\alpha \in \Lambda^{<\gamma}} x_\alpha = x_\beta \cdot \prod_{\alpha \in \Lambda^{<\beta}} x_\alpha$$

and $\prod_{\alpha \in \Lambda^{<\gamma}} x_\alpha$ cannot be invertible; otherwise,

$$\prod_{\alpha \in \Lambda^{<\beta}} x_\alpha = x_\beta^{-1} \cdot \prod_{\alpha \in \Lambda^{<\beta}} x_\alpha$$

would be invertible, too.
Finally, if \( \gamma > \gamma_0 \) and \( \gamma \) is a limit element, then
\[
\Pi_{\alpha \in \Lambda < \gamma} x_\alpha = \lim_{\beta \to \gamma^-} \left( \Pi_{\alpha \in \Lambda < \beta} x_\alpha \right)
\]
cannot be invertible, since the limit of noninvertible elements is always noninvertible (the set of all invertible elements is open).

The next lemma generalizes two well-known results from the theory of infinite series and products.

**Lemma 2.5.** (a) If \((x_\alpha)_{\alpha \in \Lambda}\) is a summable family in a normed space, then \(\lim_{\alpha \to \gamma^-} x_\alpha = 0\) for every limit element \(\gamma \in \Lambda\).

(b) If \((x_\alpha)_{\alpha \in \Lambda}\) is a multipliable family in a unital normed algebra, and if its elements as well as its product are invertible, then \(\lim_{\alpha \to \gamma^-} x_\alpha = I\) for every limit element \(\gamma \in \Lambda\).

**Proof.** We prove only the second statement; the proof of the first one is similar. Assume that \((x_\alpha)_{\alpha \in \Lambda}\) is multipliable and its elements as well as its product are invertible. Let \(\gamma \in \Lambda\) be an arbitrary limit element. Given \(\varepsilon > 0\), let \(\beta_\varepsilon\) be as in Definition 2.2 (ii). For every \(\beta \in \Lambda \cap [\beta_\varepsilon, \gamma)\), we have \(S(\beta) \in \Lambda \cap [\beta_\varepsilon, \gamma)\). Also, applying Definition 2.2 (ii) and the triangle inequality, we obtain
\[
\left\| \Pi_{\alpha \in \Lambda < S(\beta)} x_\alpha - \Pi_{\alpha \in \Lambda < \beta} x_\alpha \right\| < 2\varepsilon, \quad \beta \in \Lambda \cap [\beta_\varepsilon, \gamma).
\]
Consequently,
\[
\lim_{\beta \to \gamma^-} \left( \Pi_{\alpha \in \Lambda < S(\beta)} x_\alpha - \Pi_{\alpha \in \Lambda < \beta} x_\alpha \right) = 0.
\]
In view of this result, Lemma 2.4 and the continuity of \(x \mapsto x^{-1}\), we get
\[
\lim_{\beta \to \gamma^-} (x_\beta - I) = \lim_{\beta \to \gamma^-} \left( \left( \Pi_{\alpha \in \Lambda < S(\beta)} x_\alpha - \Pi_{\alpha \in \Lambda < \beta} x_\alpha \right)^{-1} \right) = 0; \quad \left( \Pi_{\alpha \in \Lambda < \gamma} x_\alpha \right)^{-1} = 0.
\]
The next lemma generalizes the well-known result that in a Banach space, every absolutely convergent series is convergent in the ordinary sense. The proof is based on the relation between summability and strong Henstock-Kurzweil integrability of vector-valued step mappings described in [7]. (For the definition of the strong Henstock-Kurzweil integral, see e.g. [16, 17]. In [7], this integral is referred to as the Henstock-Lebesgue integral.)

**Lemma 2.6.** Assume that \((x_\alpha)_{\alpha \in \Lambda^< b}\) is an absolutely summable family in a Banach space \(E\). Then \((x_\alpha)_{\alpha \in \Lambda^< b}\) is summable.

**Proof.** Without loss of generality, we can suppose that \(b = \sup \Lambda < \infty\). Otherwise, we can replace \(\Lambda\) by the well-ordered set \(\tilde{\Lambda} = \{1 - \exp(a - \alpha) / \alpha \in \Lambda\}\) with \(\min \tilde{\Lambda} = 0\) and \(\sup \tilde{\Lambda} = 1\); this transformation preserves (absolute) summability.

For every \(\alpha \in \Lambda^< b\), let \(z_\alpha = \frac{x_\alpha}{S(\alpha) - \alpha}\). Consider the mapping \(A : [a, b] \to E\) given by
\[
A(t) = \begin{cases} 
  z_\alpha, & t \in [\alpha, S(\alpha)), \ \alpha \in \Lambda^< b, \\
  0, & t = b.
\end{cases}
\]

By [7] Proposition 3.4, the absolute summability of \((x_\alpha)_{\alpha \in \Lambda^< b}\) implies that \(A\) is Bochner integrable. Consequently, \(A\) is strongly Henstock-Kurzweil integrable, which means by [7] Proposition 3.1] that \((S(\alpha) - \alpha)z_\alpha)_{\alpha \in \Lambda^< b} = (x_\alpha)_{\alpha \in \Lambda^< b}\) is summable. \(\square\)
Remark 2.7. In a unital Banach algebra $E$, we may introduce the exponential and logarithm function as follows:

$$\exp x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = I + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots, \quad x \in E,$$

$$\log x = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(x - I)^n}{n}, \quad \|x - I\| < 1.$$  

These functions have similar properties as in the familiar case when $E = \mathbb{R}^{n \times n}$, in particular:

1. The exponential and logarithm are continuous functions.
2. For every $x \in E$, $\exp x$ is an invertible element and its inverse is $\exp(-x)$.
3. If $x, y \in E$ are such that $xy = yx$, then $\exp(x + y) = \exp x \exp y$.
4. $\log(\exp x) = x$ if $\|x - I\| < \log 2$.
5. We have the estimates
   $$\|\exp x\| \leq \exp \|x\|, \quad x \in E,$$
   $$\|\exp x - I\| \leq \|x\| \exp \|x\|, \quad x \in E,$$
   which follow easily from the definition of the exponential function.

We now show that the formula $\exp(x) \exp(y) = \exp(x + y)$ can be generalized to families of commutative elements.

Lemma 2.8. Let $(x_{\alpha})_{\alpha \in \Lambda^{<b}}$ be a summable family in a unital Banach algebra. If $x_{\alpha}x_{\beta} = x_{\beta}x_{\alpha}$ whenever $\alpha, \beta \in \Lambda$, then the family $(\exp x_{\alpha})_{\alpha \in \Lambda^{<b}}$ is multipliable, and

$$\prod_{\alpha \in \Lambda^{<b}} \exp x_{\alpha} = \exp \left( \sum_{\alpha \in \Lambda^{<b}} x_{\alpha} \right). \quad (2.1)$$

Proof. Using the assumption that $x_{\alpha}x_{\beta} = x_{\beta}x_{\alpha}$ for all $\alpha, \beta \in \Lambda$, it follows by transfinite induction with respect to $\beta$ that

$$x_{\beta} \left( \sum_{\alpha \in \Lambda^{<\beta}} x_{\alpha} \right) = \left( \sum_{\alpha \in \Lambda^{<\beta}} x_{\alpha} \right) x_{\beta}, \quad \beta \in \Lambda. \quad (2.2)$$

To prove that the family $(\exp x_{\alpha})_{\alpha \in \Lambda^{<b}}$ is multipliable and (2.1) holds, it is enough to check that the conditions from Definition 2.2 are satisfied with

$$\prod_{\alpha \in \Lambda^{<\gamma}} \exp x_{\alpha} = \exp \left( \sum_{\alpha \in \Lambda^{<\gamma}} x_{\alpha} \right), \quad \gamma \in \Lambda \cup \{b\}. \quad (2.3)$$

Clearly,

$$\prod_{\alpha \in \Lambda^{<\gamma}} \exp x_{\alpha} = \exp \left( \sum_{\alpha \in \Lambda^{<\gamma}} x_{\alpha} \right) = \exp 0 = I.$$  

If $\gamma = S(\beta)$ for some $\beta \in \Lambda$, it follows that

$$\prod_{\alpha \in \Lambda^{<\gamma}} \exp x_{\alpha} = \exp \left( \sum_{\alpha \in \Lambda^{<\gamma}} x_{\alpha} \right) = \exp \left( x_{\beta} + \sum_{\alpha \in \Lambda^{<\beta}} x_{\alpha} \right) = \exp x_{\beta} \cdot \prod_{\alpha \in \Lambda^{<\beta}} \exp x_{\alpha},$$

where the third equality is a consequence of (2.2) and the third property mentioned in Remark 2.7. Thus condition (i) of Definition 2.2 is satisfied.
Assume next that \( \gamma \) is a limit element of \( \Lambda \cup \{ b \} \), and let \( \varepsilon > 0 \) be given. Since the exponential is a continuous function, it is possible to find \( \delta > 0 \) such that

\[
\left\| \exp x - \exp \left( \sum_{\alpha \in \Lambda^{< \gamma}} x_\alpha \right) \right\| < \varepsilon
\]

for all \( x \in E \) such that \( \left\| x - \sum_{\alpha \in \Lambda^{< \gamma}} x_\alpha \right\| < \delta \). Because the family \( (x_\alpha)_{\alpha \in \Lambda^{< b}} \) is summable, there exists a \( \beta_\delta \in \Lambda^{< \gamma} \) such that

\[
\left\| \sum_{\alpha \in \Lambda^{< \beta}} x_\alpha - \sum_{\alpha \in \Lambda^{< \gamma}} x_\alpha \right\| < \delta, \quad \beta \in \Lambda \cap [\beta_\delta, \gamma).
\]

It then follows that

\[
\left\| \prod_{\alpha \in \Lambda^{< \beta}} \exp x_\alpha - \prod_{\alpha \in \Lambda^{< \gamma}} \exp x_\alpha \right\| = \left\| \exp \left( \sum_{\alpha \in \Lambda^{< \beta}} x_\alpha \right) - \exp \left( \sum_{\alpha \in \Lambda^{< \gamma}} x_\alpha \right) \right\| < \varepsilon, \quad \beta \in \Lambda \cap [\beta_\delta, \gamma).
\]

This proves that condition (ii) of Definition 2.2 is satisfied. To conclude the proof, we substitute \( \gamma = b \) in \( \text{Lemma 2.8} \) to get \( \text{Lemma 2.10} \).

**Lemma 2.9.** Let \( (p_\alpha)_{\alpha \in \Lambda^{< b}} \) be a family of real numbers. If \( (\exp p_\alpha)_{\alpha \in \Lambda^{< b}} \) is multipliable and its product is nonzero, then \( (p_\alpha)_{\alpha \in \Lambda^{< b}} \) is summable, and

\[
\sum_{\alpha \in \Lambda^{< b}} p_\alpha = \log \left( \prod_{\alpha \in \Lambda^{< b}} \exp p_\alpha \right).
\]

*Proof.* Assume for contradiction that \( (p_\alpha)_{\alpha \in \Lambda^{< b}} \) is not summable. Then there is a limit element \( \gamma \in \Lambda \cup \{ b \} \) such that \( (p_\alpha)_{\alpha \in \Lambda^{< b}} \) is summable for every \( \beta \in \Lambda^{< \gamma} \), but \( (p_\alpha)_{\alpha \in \Lambda^{< b}} \) is not summable. Lemma 2.8 implies that

\[
\prod_{\alpha \in \Lambda^{< \beta}} \exp p_\alpha = \exp \left( \sum_{\alpha \in \Lambda^{< \beta}} p_\alpha \right), \quad \beta \in \Lambda^{< \gamma}.
\]

Since all partial products of \( (\exp p_\alpha)_{\alpha \in \Lambda^{< b}} \) are nonzero by Lemma 2.4 we get

\[
\sum_{\alpha \in \Lambda^{< \beta}} p_\alpha = \log \left( \prod_{\alpha \in \Lambda^{< \gamma}} \exp p_\alpha \right), \quad \beta \in \Lambda^{< \gamma}.
\]

Using the continuity of the logarithm function, we obtain

\[
\lim_{\beta \to \gamma^{-}} \left( \sum_{\alpha \in \Lambda^{< \beta}} p_\alpha \right) = \log \left( \prod_{\alpha \in \Lambda^{< \gamma}} \exp p_\alpha \right), \quad \beta \in \Lambda^{< \gamma},
\]

which contradicts the fact that \( (p_\alpha)_{\alpha \in \Lambda^{< b}} \) is not summable.

The following consequence of Lemmas 2.8 and 2.9 shows that absolute summability of \( (x_\alpha)_{\alpha \in \Lambda^{< b}} \) and multipliability of \( (\exp \|x_\alpha\|)_{\alpha \in \Lambda^{< b}} \) are equivalent.

**Lemma 2.10.** A family \( (x_\alpha)_{\alpha \in \Lambda^{< b}} \) in a normed space is absolutely summable if and only if the family \( (\exp \|x_\alpha\|)_{\alpha \in \Lambda^{< b}} \) is multipliable. In this case,

\[
\sum_{\alpha \in \Lambda^{< b}} \|x_\alpha\| = \log \left( \prod_{\alpha \in \Lambda^{< b}} \exp \|x_\alpha\| \right).
\]

*Proof.* By Lemma 2.8 summability of \( (\|x_\alpha\|)_{\alpha \in \Lambda^{< b}} \) implies multipliability of \( (\exp \|x_\alpha\|)_{\alpha \in \Lambda^{< b}} \) and the relation \( \text{2.4} \). Conversely, if \( (\exp \|x_\alpha\|)_{\alpha \in \Lambda^{< b}} \) is multipliable, then it is obvious that \( \prod_{\alpha \in \Lambda^{< b}} \exp \|x_\alpha\| \geq 1 \). It follows from Lemma 2.9 with \( p_\alpha = \|x_\alpha\| \), that \( (\|x_\alpha\|)_{\alpha \in \Lambda^{< b}} \) is summable.
The next two results generalize the well-known relations between infinite series and products.

**Lemma 2.11.** A family \((x_\alpha)_{\alpha \in \Lambda^{<\gamma}}\) in a normed space is absolutely summable if and only if the family of real numbers \((1 + \|x_\alpha\|)_{\alpha \in \Lambda^{<\gamma}}\) is multipliable.

**Proof.** Assume first that \((x_\alpha)_{\alpha \in \Lambda^{<\gamma}}\) is absolutely summable. We use transfinite recursion to define the partial products \(\Pi_{\alpha \in \Lambda^{<\beta}} (1 + \|x_\alpha\|)\), \(\gamma \in \Lambda \cup \{b\}\), so that the conditions of Definition 2.2 will be satisfied.

First, let \(\Pi_{\alpha \in \Lambda^{<\gamma}} (1 + \|x_\alpha\|) = 1\). Next, assume that \(\Pi_{\alpha \in \Lambda^{<\beta}} (1 + \|x_\alpha\|)\) is defined for each \(\beta \in \Lambda^{<\gamma}\), where \(\gamma \in (\Lambda \cup \{b\}) \setminus \{a\}\). If \(\gamma = S(\beta)\), we let

\[
\Pi_{\alpha \in \Lambda^{<\gamma}} (1 + \|x_\beta\|) = (1 + \|x_\beta\|) \cdot \Pi_{\alpha \in \Lambda^{<\beta}} (1 + \|x_\alpha\|),
\]

which ensures that condition (i) of Definition 2.2 is satisfied.

Finally, assume that \(\gamma \) is a limit element of \(\Lambda \cup \{b\}\). By Lemma 2.1, the family \((\exp \|x_\alpha\|)_{\alpha \in \Lambda^{<\gamma}}\) is multipliable. Moreover, it is easy to show by transfinite induction that

\[
\Pi_{\alpha \in \Lambda^{<\beta}} (1 + \|x_\alpha\|) \leq \Pi_{\alpha \in \Lambda^{<\gamma}} \exp \|x_\alpha\|, \quad \beta \in \Lambda^{<\gamma}.
\]

Thus,

\[
s = \sup_{\beta \in \Lambda^{<\gamma}} \left( \Pi_{\alpha \in \Lambda^{<\beta}} (1 + \|x_\alpha\|) \right)
\]

is finite. Given \(\varepsilon > 0\), there exists a \(\beta_\varepsilon \in \Lambda^{<\gamma}\) such that

\[
\left| \Pi_{\alpha \in \Lambda^{<\beta_\varepsilon}} (1 + \|x_\alpha\|) - s \right| < \varepsilon.
\]

However, since \(\left( \Pi_{\alpha \in \Lambda^{<\beta}} (1 + \|x_\alpha\|) \right)_{\beta \in \Lambda^{<\gamma}}\) is a nondecreasing transfinite sequence, it follows that

\[
\left| \Pi_{\alpha \in \Lambda^{<\beta}} (1 + \|x_\alpha\|) - s \right| < \varepsilon, \quad \beta \in [\beta_\varepsilon, \gamma) \cap \Lambda.
\]

Thus, condition (ii) of Definition 2.2 will be satisfied if we define \(\Pi_{\alpha \in \Lambda^{<\gamma}} (1 + \|x_\alpha\|) = s\).

The above reasoning implies that \(\Pi_{\alpha \in \Lambda^{<\gamma}} (1 + \|x_\alpha\|)\) is defined for each \(\gamma \in \Lambda \cup \{b\}\), whence the family \((1 + \|x_\alpha\|)_{\alpha \in \Lambda^{<\gamma}}\) is multipliable. Assume conversely that \((1 + \|x_\alpha\|)_{\alpha \in \Lambda^{<\gamma}}\) is multipliable. We use transfinite recursion to define the partial sums \(\sum_{\alpha \in \Lambda^{<\gamma}} \|x_\alpha\|, \gamma \in \Lambda \cup \{b\}\), so that the conditions of Definition 2.1 will be satisfied. At the same time, we are going to prove that

\[
\sum_{\alpha \in \Lambda^{<\beta}} \|x_\alpha\| \leq \Pi_{\alpha \in \Lambda^{<\beta}} (1 + \|x_\alpha\|)
\]

for all \(\beta \in \Lambda \cup \{b\}\). First, let \(\sum_{\alpha \in \Lambda^{<\beta}} \|x_\alpha\| = 0\) and note that \(\sum_{\alpha \in \Lambda^{<\gamma}} \|x_\alpha\| = 0 < 1 = \Pi_{\alpha \in \Lambda^{<\gamma}} (1 + \|x_\alpha\|)\). Next, assume that \(\sum_{\alpha \in \Lambda^{<\beta}} \|x_\alpha\|\) is defined for each \(\beta \in \Lambda^{<\gamma}\), where \(\gamma \in (\Lambda \cup \{b\}) \setminus \{a\}\), and that (2.5) holds for all \(\beta \in \Lambda^{<\gamma}\). If \(\gamma = S(\beta)\), we let

\[
\sum_{\alpha \in \Lambda^{<\gamma}} \|x_\alpha\| = \|x_\beta\| + \sum_{\alpha \in \Lambda^{<\beta}} \|x_\alpha\|,
\]

which ensures that condition (i) of Definition 2.1 is satisfied. Also, note that

\[
\sum_{\alpha \in \Lambda^{<\gamma}} \|x_\alpha\| = \|x_\beta\| + \sum_{\alpha \in \Lambda^{<\beta}} \|x_\alpha\| \leq \|x_\beta\| + \Pi_{\alpha \in \Lambda^{<\beta}} (1 + \|x_\alpha\|)
\]

\[
\leq \|x_\beta\| \Pi_{\alpha \in \Lambda^{<\beta}} (1 + \|x_\alpha\|) + \Pi_{\alpha \in \Lambda^{<\beta}} (1 + \|x_\alpha\|)
\]

\[
= (1 + \|x_\beta\|) \Pi_{\alpha \in \Lambda^{<\beta}} (1 + \|x_\alpha\|) = \Pi_{\alpha \in \Lambda^{<\gamma}} (1 + \|x_\alpha\|),
\]
i.e., (2.5) holds when \( \beta = \gamma \).

Finally, assume that \( \gamma \) is a limit element of \( \Lambda \cup \{ b \} \). We know from (2.5) that

\[
\sum_{\alpha \in \Lambda^< \beta} \|x_\alpha\| \leq \Pi_{\alpha \in \Lambda^< \gamma} (1 + \|x_\alpha\|), \quad \beta \in \Lambda^< \gamma.
\]

Thus,

\[
s' = \sup_{\beta \in \Lambda^< \gamma} \left( \sum_{\alpha \in \Lambda^< \beta} \|x_\alpha\| \right)
\]

is finite. Given \( \varepsilon > 0 \), there exists a \( \beta_\varepsilon \in \Lambda^< \gamma \) such that

\[
\left| \sum_{\alpha \in \Lambda^< \beta} \|x_\alpha\| - s' \right| < \varepsilon.
\]

However, since \( \left( \sum_{\alpha \in \Lambda^< \beta} \|x_\alpha\| \right)_{\beta \in \Lambda^< \gamma} \) is a nondecreasing transfinite sequence, it follows that

\[
\left| \sum_{\alpha \in \Lambda^< \beta} \|x_\alpha\| - s' \right| < \varepsilon, \quad \beta \in [\beta_\varepsilon, \gamma) \cap \Lambda
\]

Thus, condition (ii) of Definition 2.11 will be satisfied if we define \( \sum_{\alpha \in \Lambda^< \gamma} \|x_\alpha\| = s' \). Also, we have

\[
\sum_{\alpha \in \Lambda^< \gamma} \|x_\alpha\| = \sup_{\beta \in \Lambda^< \gamma} \left( \sum_{\alpha \in \Lambda^< \beta} \|x_\alpha\| \right) \leq \sup_{\beta \in \Lambda^< \gamma} \left( \Pi_{\alpha \in \Lambda^< \beta} (1 + \|x_\alpha\|) \right) = \Pi_{\alpha \in \Lambda^< \gamma} (1 + \|x_\alpha\|),
\]

i.e., (2.5) holds when \( \beta = \gamma \).

The above reasoning implies that \( \sum_{\alpha \in \Lambda^< \gamma} \|x_\alpha\| \) is defined for each \( \gamma \in \Lambda \cup \{ b \} \), whence the family \( \{ \|x_\alpha\| \}_{\alpha \in \Lambda^< b} \) is summable.

**Lemma 2.12.** Let \( (x_\alpha)_{\alpha \in \Lambda^< b} \) be a family in a normed space. Assume that \( 0 \leq \|x_\alpha\| < 1 \) for all \( \alpha \in \Lambda^< b \). Then \( (x_\alpha)_{\alpha \in \Lambda^< b} \) is absolutely summable if and only if the product of the family \( (1 - \|x_\alpha\|)_{\alpha \in \Lambda^< b} \) is positive.

**Proof.** Since \( 0 \leq \|x_\alpha\| < 1 \) for all \( \alpha \in \Lambda^< b \), it can be shown by transfinite induction that the family \( (1 - \|x_\alpha\|)_{\alpha \in \Lambda^< b} \) is multiplicable, and that the products \( \Pi_{\alpha \in \Lambda^< b} (1 - \|x_\alpha\|) \) form a decreasing transfinite sequence with values in [0, 1].

Suppose that \( (x_\alpha)_{\alpha \in \Lambda^< b} \) is absolutely summable. Assume for contradiction that \( \Pi_{\alpha \in \Lambda^< b} (1 - \|x_\alpha\|) = 0 \) for some \( \gamma \in \Lambda \cup \{ b \} \). Because \( \Lambda \cup \{ b \} \) is well-ordered, there is the smallest element \( \gamma \in \Lambda \cup \{ b \} \) with that property. It is a limit element of \( \Lambda \cup \{ b \} \) since \( 1 - \|x_\alpha\| > 0 \) for each \( \alpha \in \Lambda^< b \). The assumption that \( (x_\alpha)_{\alpha \in \Lambda^< b} \) is summable implies by Lemma 2.11 the existence of a \( \beta \in \Lambda^< b \) such that \( \|x_\alpha\| \leq \frac{1}{2} \) when \( \alpha \in [\beta, \gamma) \cap \Lambda \).

Thus \( 1 - \|x_\alpha\| \geq \exp(-2\|x_\alpha\|) \) when \( \alpha \in [\beta, \gamma) \cap \Lambda \), so that

\[
\Pi_{\alpha \in [\beta, \gamma) \cap \Lambda} (1 - \|x_\alpha\|) \geq \Pi_{\alpha \in [\beta, \gamma) \cap \Lambda} \exp(-2\|x_\alpha\|) = \exp \left( -2 \sum_{\alpha \in [\beta, \gamma) \cap \Lambda} \|x_\alpha\| \right) > 0.
\]

Consequently,

\[
\Pi_{\alpha \in \Lambda^< \gamma} (1 - \|x_\alpha\|) \geq \exp \left( -2 \sum_{\alpha \in [\beta, \gamma) \cap \Lambda} \|x_\alpha\| \right) \Pi_{\alpha \in \Lambda^< \beta} (1 - \|x_\alpha\|) > 0,
\]

a contradiction. Thus \( \Pi_{\alpha \in \Lambda^< \gamma} (1 - \|x_\alpha\|) > 0 \) for every \( \gamma \in \Lambda \cup \{ b \} \), and hence also when \( \gamma = b \).

Assume conversely that \( \Pi_{\alpha \in \Lambda^< \gamma} (1 - \|x_\alpha\|) > 0 \). Since \( 0 \leq \|x_\alpha\| < 1 \), we have \( 1 - \|x_\alpha\| \leq \exp(-\|x_\alpha\|) \leq 1 \) for all \( \alpha \in \Lambda^< b \). Using transfinite induction, we conclude that the family \( \{ \exp(-\|x_\alpha\|) \}_{\alpha \in \Lambda^< b} \) is multiplicable and

\[
\Pi_{\alpha \in \Lambda^< b} \exp(-\|x_\alpha\|) \geq \Pi_{\alpha \in \Lambda^< b} (1 - \|x_\alpha\|) > 0.
\]
By Lemma 2.8 with \( p_\alpha = -\|x_\alpha\| \), the family \((-\|x_\alpha\|)_{\alpha \in \Lambda^{<b}}\) is summable. Consequently, the family \((x_\alpha)_{\alpha \in \Lambda^{<b}}\) is absolutely summable. 

If a family of nonnegative real numbers \((p_\alpha)_{\alpha \in \Lambda^{<b}}\) is not absolutely summable, we write \( \sum_{\alpha \in \Lambda^{<b}} p_\alpha = \infty \). Then we get the following consequence of Lemma 2.12 which generalizes [3, Lemma 8.3.3].

**Corollary 2.13.** Let \((p_\alpha)_{\alpha \in \Lambda^{<b}}\) be a family of real numbers. Assume that \( 0 \leq p_\alpha < 1 \) for all \( \alpha \in \Lambda^{<b} \). Then \( \prod_{\alpha \in \Lambda^{<b}} (1 - p_\alpha) = 0 \) if and only if \( \sum_{\alpha \in \Lambda^{<b}} p_\alpha = \infty \).

**Example 2.14.** The increasing sequence formed by the numbers

\[
b - 2^{-n}(b - a), \quad n \in \mathbb{N}_0,
\]

is a well-ordered subset of the interval \([a, b) \subseteq \mathbb{R}\). The smallest number of this sequence is \(a\) and its supremum is \(b\). When \(a = 0\) and \(b = 1\), the numbers in (2.6) form the increasing sequence

\[
\Lambda_0 = \{\alpha(n_0) = 1 - 2^{-n_0}; n_0 \in \mathbb{N}_0\}.
\]

Clearly, \(\Lambda_0\) is a well-ordered subset of \([0, 1)\). The points of \(\Lambda_0\) divide the interval \([0, 1)\) into disjoint subintervals \([1 - 2^{-n_0}, 1 - 2^{-n_0+1})\), \(n_0 \in \mathbb{N}_0\). Choosing \(a = 1 - 2^{-n_0}, b = 1 - 2^{-n_0+1}\) in (2.6) and renaming \(n\) to \(n_1\), we obtain in each of these subintervals increasing sequences, which together form the well-ordered set

\[
\Lambda_1 = \{\alpha(n_0, n_1) = 1 - 2^{-n_0+1} - 2^{-n_0-n_1+1}; n_0, n_1 \in \mathbb{N}_0\}.
\]

All numbers of \(\Lambda_0 \setminus \{0\}\) are limit elements of \(\Lambda_1\).

If the above process is repeated, one can obtain additional examples of well-ordered sets \(\Lambda_m, m \in \mathbb{N},\) with a more complicated structure; see [7, Example 2.1].

We now construct a family \((x_\alpha)_{\alpha \in \Lambda_1}\) in the following way: Choose a vector \(z \neq 0\) of \(E\), and let

\[
x_{\alpha(n_0, n_1)} = \frac{(-1)^{n_0+n_1}}{(n_0+1)(n_1+1)}z, \quad n_0, n_1 \in \mathbb{N}_0.
\]

The family \((x_\alpha)_{\alpha \in \Lambda_1}\) is summable, and its sum can be evaluated as the double sum (cf. [7, page 4])

\[
\sum_{\alpha \in \Lambda_1} x_\alpha = \sum_{n_0=0}^{\infty} \sum_{n_1=0}^{n_0} \frac{(-1)^{n_0+n_1}}{(n_0+1)(n_1+1)}z = \sum_{n_0=0}^{\infty} \frac{(-1)^{n_0}}{n_0+1} \sum_{n_1=0}^{\infty} \frac{(-1)^{n_1}}{n_1+1} z = (\log 2)^2 z.
\]

Clearly, \(x_\alpha x_\beta = x_\beta x_\alpha\) whenever \(\alpha, \beta \in \Lambda_1\). It then follows from Lemma 2.8 that the family \((\exp x_\alpha)_{\alpha \in \Lambda_1}\) is multiplicable, and its product is

\[
\prod_{\alpha \in \Lambda_1} \exp x_\alpha = \exp \left( \sum_{n_0=0}^{\infty} \sum_{n_1=0}^{\infty} \frac{(-1)^{n_0+n_1}}{(n_0+1)(n_1+1)}z \right) = \exp \left( (\log 2)^2 z \right).
\]

Note that \((x_\alpha)_{\alpha \in \Lambda_1}\) is not absolutely summable. Thus neither \(\exp \|x_\alpha\|)_{\alpha \in \Lambda_1}\) nor \((1 + \|x_\alpha\|)_{\alpha \in \Lambda_1}\) is multiplicable, and the product of \((1 + \|x_\alpha\|)_{\alpha \in \Lambda_1}\) is zero.

### 3 Product integrals and their properties

The concept of product integration was originally introduced by V. Volterra (see e.g. [18, 21]): Given a continuous matrix-valued function \(A : [a, b] \to \mathbb{R}^{n \times n}\), he considered products of the form

\[
(I + A(\xi_m)(t_m - t_{m-1}))(I + A(\xi_{m-1})(t_{m-1} - t_{m-2})) \cdots (I + A(\xi_1)(t_1 - t_0)),
\]

(3.1)
where \( a = t_0 < t_1 < \ldots < t_m = b \) and \( \xi_i \in [t_{i-1}, t_i] \), \( i \in \{1, \ldots, m\} \). The product integral \( \prod_{\xi}^t (I + A(t) \, dt) \) is then defined as the limit of the product (3.1) when the lengths of all subintervals \([t_{i-1}, t_i]\) approach zero. The motivation for introducing this concept stems from the fact that the indefinite product integral \( t \mapsto \prod_{\xi}^t (I + A(s) \, ds) \), \( t \in [a, b] \), corresponds to the fundamental matrix of a system of \( n \) homogeneous linear ordinary differential equations \( x'(t) = A(t)x(t) \). In [10], P. R. Masani generalized this concept to mappings \( A : [a, b] \to E \), where \( E \) is a unital normed algebra, and \( A \) is Riemann integrable. Other authors have considerably extended the class of product integrable mappings by introducing new definitions of product integrals in the spirit of Lebesgue, Bochner, Kurzweil, or McShane; see [2, 3, 5, 13, 17, 18, 19].

If the products (3.1) are replaced by

\[
(I + A(t_m) - A(t_{m-1}))(I + A(t_{m-1}) - A(t_{m-2})) \cdots (I + A(t_1) - A(t_0)),
\]

we obtain the Stieltjes-type product integral \( \prod_{\xi}^t (I + dA(t)) \). The basic references on this topic are the second part of the book [2] by R. M. Dudley and R. Norvaiša, and the paper [5] by R. D. Gill and S. Johansen, who also provide a detailed overview of applications to survival analysis and Markov processes. Another motivation for considering Stieltjes-type product integrals comes from the theory of integral equations (also known as generalized linear differential equations; see [14, 15, 12]) of the form

\[
x(t) = x(a) + \int_a^t d[A(s)]x(s), \quad t \in [a, b],
\]

(3.2)

where \( A : [a, b] \to \mathbb{R}^{n \times n} \), the unknown function \( x \) takes values in \( \mathbb{R}^n \), and the integral on the right-hand side is the Kurzweil-Stieltjes integral. Equations of this form encompass other types of equations, such as ordinary differential equations with impulses, dynamic equations on time scales, or functional differential equations (cf. [11, 12, 14, 20]). It turns out that under certain assumptions on \( A \), the indefinite Stieltjes product integral \( t \mapsto \prod_{\xi}^t (I + dA(s)) \), \( t \in [a, b] \), corresponds to the fundamental matrix of Eq. (3.2).

We now summarize some basic facts about product integration that will be needed later, including several new results about strong Kurzweil product integrals. Throughout the rest of the paper, we assume that \( E \) is a unital Banach algebra.

A tagged partition of an interval \([a, b]\) is a collection of point-interval pairs \( D = (\xi_i, [t_{i-1}, t_i])_{i=1}^m \), where \( a = t_0 < t_1 < \cdots < t_m = b \) and \( \xi_i \in [t_{i-1}, t_i] \) for every \( i \in \{1, \ldots, m\} \). If we relax the assumption \( \xi_i \in [t_{i-1}, t_i] \) and replace it by \( \xi_i \in [a, b] \), then the collection \( D \) is called a free tagged partition. (Note that each tagged partition is also a free tagged partition.)

Given a function \( \delta : [a, b] \to \mathbb{R}^+ \) (called a gauge on \([a, b]\)), a free tagged partition is called \( \delta \)-fine if

\[
[t_{i-1}, t_i] \subset (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i)), \quad i \in \{1, \ldots, m\}.
\]

Let \( \mathcal{I} \) be the set of all compact subintervals of \([a, b]\). Assume that a point-interval function \( V : [a, b] \times \mathcal{I} \to E \) is given. For an arbitrary free tagged partition \( D = (\xi_i, [t_{i-1}, t_i])_{i=1}^m \) of the interval \([a, b]\), we denote

\[
P(V, D) = \prod_{i=1}^m V(\xi_i, [t_{i-1}, t_i]) = V(\xi_m, [t_{m-1}, t_m])V(\xi_{m-1}, [t_{m-2}, t_{m-1}]) \cdots V(\xi_1, [t_0, t_1]).
\]

**Definition 3.1.** A function \( V : [a, b] \times \mathcal{I} \to E \) is called Kurzweil product integrable, if there exists an invertible element \( P_V \in E \) with the following property: For each \( \varepsilon > 0 \), there exists a gauge \( \delta : [a, b] \to \mathbb{R}^+ \) such that

\[
\| P(V, D) - P_V \| < \varepsilon \quad (3.3)
\]

for all \( \delta \)-fine tagged partitions of \([a, b]\). In this case, \( P_V \) is called the Kurzweil product integral of \( V \) and will be denoted by \( \prod_{\xi}^t V(t, dt) \).
If (3.3) holds for all $\delta$-fine free tagged partitions of $[a, b]$, then $V$ is called McShane product integrable over $[a, b]$. The McShane product integral $\prod_V$ will again be denoted by $\prod_{a}^{b} V(t, dt)$.

The definition of Riemann product integrability is obtained from the definition of Kurzweil product integrability if the gauge $\delta$ is assumed to be constant on $[a, b]$. In this case, the integral $\prod_{a}^{b} V(t, dt)$ is called the Riemann product integral.

It follows from the definition that Riemann or McShane product integrability implies Kurzweil product integrability.

In practice, the most common types of product integrals are obtained by taking a function $V : [a, b] \to E$ and defining $V : [a, b] \times I \to E$ as follows:

- For $V(t, [x, y]) = I + A(t)(y - x)$, the corresponding product integrals $\prod_{a}^{b} V(t, dt)$ are simply referred to as the product integrals of $A$ and are usually denoted by $\prod_{a}^{b} (I + A(t) dt)$; see Definition 3.8.

- For $V(t, [x, y]) = \exp(A(t)(y - x))$, the corresponding product integrals $\prod_{a}^{b} V(t, dt)$ are called the exponential product integrals of $A$ and are denoted by $\prod_{a}^{b} \exp(A(t) dt)$; see [15] Definition 3.7.

- For $V(t, [x, y]) = I + A(y) - A(x)$, the corresponding product integrals $\prod_{a}^{b} V(t, dt)$ are called the Stieltjes product integrals of $A$ and are denoted by $\prod_{a}^{b} (I + dA(t))$; see Definition 6.1.

To obtain a reasonable theory of product integrals, we need to impose certain additional assumptions on the function $V : [a, b] \times I \to E$. The following conditions are taken over from [15], where they are collectively referred to as the condition $C$:

(V1) $V(t, [t, t]) = I$ for every $t \in [a, b]$.

(V2) For every $t \in [a, b]$ and $\epsilon > 0$ there is a $\sigma > 0$ such that

$$\|V(t, [x, y]) - V(t, [t, y])V(t, [x, t])\| < \epsilon$$

for all $x, y \in [a, b]$, $t - \sigma < x \leq t \leq y < t + \sigma$.

(V3) For every $t \in [a, b]$, there exists an invertible element $V_{+}(t) \in E$ such that $\lim_{y \to t_{+}} V(t, [t, y]) = V_{+}(t)$.

(V4) For every $t \in [a, b]$, there exists an invertible element $V_{-}(t) \in E$ such that $\lim_{x \to t_{-}} V(t, [x, t]) = V_{-}(t)$.

The next statement from [15] Theorem 1.7] summarizes some basic properties of the Kurzweil/McShane product integrals. (In [15], the statement is formulated for $E = \mathbb{R}^{n}$, but the proof remains valid in every unital Banach algebra; see also [15] Remark 1.17.)

**Theorem 3.2.** Assume that $V : [a, b] \times I \to E$ satisfies conditions (V1)–(V4) and the Kurzweil/McShane product integral $\prod_{a}^{b} V(t, dt)$ exists. Then for every $c \in (a, b)$, the Kurzweil/McShane product integrals $\prod_{a}^{c} V(t, dt)$ and $\prod_{c}^{b} V(t, dt)$ exist and the equality

$$\prod_{a}^{b} \prod_{c}^{b} V(t, dt) = \prod_{a}^{b} V(t, dt). \quad (3.4)$$

holds. Moreover, the functions $s \mapsto \prod_{a}^{s} V(t, dt)$ and $s \mapsto (\prod_{a}^{s} V(t, dt))^{-1}$ are bounded on $[a, b]$.

According to the next proposition from [15] Lemma 1.11], conditions (V1)–(V4) imply that the indefinite Kurzweil product integral is a regulated function.
Theorem 3.3. Assume that \( V : [a, b] \times \mathcal{I} \to E \) satisfies conditions (V1)-(V4) and the Kurzweil product integral \( \prod_{a}^{b} V(t, dt) \) exists. Then

\[
\begin{align*}
\lim_{\beta \to s-} \prod_{a}^{b} V(t, dt) &= V_-(s)^{-1} \cdot \prod_{a}^{s} V(t, dt), & s \in (a, b], \\
\lim_{\beta \to s+} \prod_{a}^{b} V(t, dt) &= V_+(s) \cdot \prod_{a}^{s} V(t, dt), & s \in [a, b].
\end{align*}
\]

We now define the concept of the strong product integral \( \prod_{a}^{b} V(t, dt) \), which generalizes the definitions of the strong product integrals \( \prod_{a}^{b} (I + A(t) dt) \) and \( \prod_{a}^{b} \exp(A(t) dt) \) from [17] Definitions 3.4 and 3.8. The motivation for introducing strong product integrals is explained in [17] Section 3; the main reason is that in infinite dimension, ordinary product integrals no longer possess the same pleasant properties as their finite-dimensional counterparts, while the theory of strong product integrals closely parallels the finite-dimensional case.

Definition 3.4. A function \( V : [a, b] \times \mathcal{I} \to E \) is called strongly Kurzweil product integrable if there is a function \( W : [a, b] \to E \) such that \( W(t)^{-1} \) exists for all \( t \in [a, b] \), both \( W \) and \( W^{-1} \) are bounded, and for every \( \varepsilon > 0 \), there is a gauge \( \delta : [a, b] \to \mathbb{R}^+ \) such that

\[
\sum_{i=1}^{m} \| V(\xi_i, [t_{i-1}, t_i]) - W(t_i)W(t_{i-1})^{-1} \| < \varepsilon \tag{3.5}
\]

for every \( \delta \)-fine tagged partition of \([a, b] \). In this case, we define the strong Kurzweil product integral as

\[
\prod_{a}^{b} V(t, dt) = W(b)W(a)^{-1}.
\]

If (3.5) holds for all \( \delta \)-fine free tagged partitions of \([a, b] \), then \( A \) is called strongly McShane product integrable over \([a, b] \). The strong McShane product integral is again defined as \( \prod_{a}^{b} V(t, dt) = W(b)W(a)^{-1} \).

The next statement is a generalization of [17] Theorem 3.5.

Theorem 3.5. If \( V : [a, b] \times \mathcal{I} \to E \) is strongly Kurzweil/McShane product integrable, then it is also Kurzweil/McShane product integrable and the values of the product integrals coincide.

Proof. Let us prove the statement concerning Kurzweil product integrals; the proof of the McShane counterpart is a straightforward modification. Consider the function \( W \) from Definition 3.3. There exists a constant \( M > 0 \) such that \( \| W(t) \| \leq M \) and \( \| W(t)^{-1} \| \leq M \) for all \( t \in [a, b] \). Take an arbitrary \( \varepsilon \in (0, \frac{1}{M^2}) \). There exists a gauge \( \delta : [a, b] \to \mathbb{R}_+ \) such that

\[
\sum_{i=1}^{m} \| V(\xi_i, [t_{i-1}, t_i]) - W(t_i)W(t_{i-1})^{-1} \| < \varepsilon
\]

for every \( \delta \)-fine tagged partition of \([a, b] \). Consequently,

\[
\sum_{i=1}^{m} \| W(t_i)^{-1}V(\xi_i, [t_{i-1}, t_i])W(t_{i-1}) - I \| < M^2\varepsilon < 1.
\]

We need the following estimate, which follows from [3] Lemma 2.1: If \( y_1, \ldots, y_m \in E \) are such that \( \sum_{i=1}^{m} \| y_i \| \leq 1 \), then

\[
\|(I + y_1) \cdots (I + y_m) - I\| \leq \sum_{i=1}^{m} \| y_i \| + \left( \sum_{i=1}^{m} \| y_i \| \right)^2.
\]
By letting \( y_i = W(t_i)^{-1}V(\xi_i, [t_{i-1}, t_i])W(t_{i-1}) - I, \ i \in \{1, \ldots, m\} \), we get
\[
\left\| W(t_m)^{-1} \left( \prod_{i=m}^{1} V(\xi_i, [t_{i-1}, t_i]) \right) W(t_0) - I \right\| = \left\| \prod_{i=m}^{1} W(t_i)^{-1}V(\xi_i, [t_{i-1}, t_i])W(t_{i-1}) - I \right\|
\]
\[
= \left\| (I + y_m) \cdots (I + y_1) - I \right\| \leq \sum_{i=1}^{m} \| y_i \| + \left( \sum_{i=1}^{m} \| y_i \| \right)^2 < M^2 \varepsilon + M^4 \varepsilon^2.
\]
It follows that
\[
\left\| \prod_{i=m}^{1} V(\xi_i, [t_{i-1}, t_i]) - W(b)W(a)^{-1} \right\| = \left\| \prod_{i=m}^{1} V(\xi_i, [t_{i-1}, t_i]) - W(t_m)W(t_0)^{-1} \right\| < M^4 \varepsilon + M^6 \varepsilon^2
\]
for every \( \delta \)-fine tagged partition of \([a, b]\), which proves that the Kurzweil product integral \( \prod_a^b V(t, dt) \) exists and equals \( W(b)W(a)^{-1} \). ∎

It is straightforward to see that strong Kurzweil product integrability on \([a, b]\) implies integrability on every subinterval of \([a, b]\). In the next theorem, we show that strong integrability on two adjacent intervals \([a, c]\) and \([c, b]\) implies strong integrability on \([a, b]\).

**Theorem 3.6.** Assume that \( V : [a, b] \times \mathcal{I} \rightarrow E \) satisfies conditions (V1)–(V4). Moreover, suppose that for a certain \( c \in [a, b] \), the strong Kurzweil product integrals \( \prod_a^c V(t, dt) \) and \( \prod_c^b V(t, dt) \) exist. Then the strong Kurzweil product integral \( \prod_a^b V(t, dt) \) exists as well.

**Proof.** By the assumption, we have a pair of functions \( W_1 : [a, c] \rightarrow E \), \( W_2 : [c, b] \rightarrow E \) with the properties specified in Definition 3.3. Without loss of generality, assume that \( W_1(c) = W_2(c) \); otherwise, we can replace \( W_2 \) by the function \( W_2 \) given by \( W_2(t) = W_2(t)W_2(c)^{-1}W_1(c) \).

Let \( M > 0 \) be such that \( \|W_i(t)\| \leq M \) and \( \|W_i(t)^{-1}\| \leq M \) for all \( t \) and \( i \in \{1, 2\} \).

For an arbitrary \( \varepsilon > 0 \), we have a pair of gauges \( \delta_1 : [a, c] \rightarrow \mathbb{R}^+ \), \( \delta_2 : [c, b] \rightarrow \mathbb{R}^+ \) having the properties specified in Definition 3.3. Also, thanks to the conditions (V2) and (V4), there exists a \( \delta_c > 0 \) such that
\[
\| V(c, [x, y]) - V(c, [c, y])V(c, [x, c]) \| < \varepsilon \quad \text{and} \quad \| V(c, [x, c]) - V_c((c, c)) \| < \varepsilon \tag{3.6}
\]
for all \( x, y \in [a, c] \) with \( c - \delta_c < x \leq c \leq y < c + \delta_c \). Let \( \delta : [a, b] \rightarrow \mathbb{R}^+ \) be given by
\[
\delta(t) = \begin{cases} 
\min(\delta_1(t), c-t), & t \in [a, c), \\
\delta_c, & t = c, \\
\min(\delta_2(t), t-c), & t \in (c, b]. 
\end{cases}
\]

Consider an arbitrary \( \delta \)-fine partition \( (\xi_i, [t_{i-1}, t_i])_{i=1}^{m} \) of \([a, b]\). Our choice of \( \delta \) implies the existence of a unique index \( j \in \{1, \ldots, m\} \) such that \( t_{j-1} \leq \xi_j = c \leq t_j \). Obviously, we have
\[
\sum_{i=1}^{j-1} \| V(\xi_i, [t_{i-1}, t_i]) - W(t_i)W(t_{i-1})^{-1} \| = \sum_{i=1}^{j-1} \| V(\xi_i, [t_{i-1}, t_i]) - W(t_i)W(t_{i-1})^{-1} \| < \varepsilon,
\]
\[
\sum_{i=j+1}^{m} \| V(\xi_i, [t_{i-1}, t_i]) - W(t_i)W(t_{i-1})^{-1} \| = \sum_{i=1}^{j-1} \| V(\xi_i, [t_{i-1}, t_i]) - W(t_i)W(t_{i-1})^{-1} \| < \varepsilon.
\]
Moreover, using (3.6), we get
\[
\| V(\xi_j, [t_{j-1}, t_j]) - W(t_j)W(t_{j-1})^{-1} \| = \| V(c, [t_{j-1}, t_j]) - V(c, [c, t_j])V(c, [t_{j-1}, c]) \|< \varepsilon.
\]
\[ + \|V(c, [t_j])V(c, [t_{j-1}, c]) - W(t_j)W(t_{j-1})^{-1}\| < \varepsilon + \|V(c, [t_j])V(c, [t_{j-1}, c]) - W(t_j)W(c)^{-1}V(c, [t_{j-1}, c])\| \\
\]

\[ + \|W(t_j)W(c)^{-1}V(c, [t_{j-1}, c]) - W(t_j)W(c)^{-1}W(c)W(t_{j-1})^{-1}\| \leq \varepsilon + \|V(c, [t_j]) - W(t_j)W(c)^{-1}\| \cdot \|V(c, [t_{j-1}, c]) - W(c)W(t_{j-1})^{-1}\| \\
\]

\[ \leq \varepsilon + \varepsilon \cdot (\varepsilon + \|V(c)\|) + M^2 \varepsilon = \varepsilon \cdot (1 + \varepsilon + \|V(c)\| + M^2). \]

Consequently,
\[
\sum_{i=1}^{m} \|V(\xi_i, [t_{i-1}, t_i]) - W(t_i)W(t_{i-1})^{-1}\| < \varepsilon \cdot (3 + \varepsilon + \|V(c)\| + M^2),
\]

which proves that \( V \) is strongly Kurzweil product integrable on \([a, b]\). \( \square \)

For strong Kurzweil product integrals, we have the following Hake-type theorem (the corresponding statement for ordinary Kurzweil product integrals can be found in \[15, \text{Theorem 1.13}\]; the proof still works in unital Banach algebras).

**Theorem 3.7.** Assume that \( V : [a, b] \times \mathcal{I} \to E \) satisfies conditions \((V1)-(V4)\) and that for every \( c \in [a, b] \), the strong Kurzweil product integral \( \prod_{a}^{c} V(t, dt) \) exists. Suppose also that
\[
\lim_{c \to b^-} V(b, [c, b]) \prod_{a}^{c} V(t, dt) = L, \tag{3.7}
\]

where \( L \in E \) is invertible. Then the strong Kurzweil product integral \( \prod_{a}^{b} V(t, dt) \) exists and equals \( L \).

**Proof.** Let \( W(t) = \prod_{a}^{t} V(s, ds), t \in [a, b), \) and \( W(b) = L \). Eq. (3.7) together with condition \((V4)\) imply that \( \lim_{c \to b^-} W(c) = V_\sim(b)^{-1}W(b) \), and therefore
\[
\lim_{c \to b^-} W(c)^{-1} = W(b)^{-1}V_\sim(b). \tag{3.8}
\]

Let \( M > 0 \) be such that \( \|W(t)\| \leq M \) for all \( t \in [a, b) \).

Consider an arbitrary \( \varepsilon > 0 \). Let \( \{b_n\}_{n=1}^{\infty} \) be an increasing sequence in \([a, b)\) with \( \lim_{n \to \infty} b_n = b \). For every \( n \in \mathbb{N} \), there exists a gauge \( \delta_n : [a, b_n) \to \mathbb{R}^+ \) such that the inequality
\[
\sum_{i=1}^{m} \|V(\xi_i, [t_{i-1}, t_i]) - W(t_i)W(t_{i-1})^{-1}\| < \varepsilon \cdot \frac{1}{2^{n+1}}
\]

holds for each \( \delta \)-fine partition \( (\xi_i, [t_{i-1}, t_i])_{i=1}^{m} \) of \([a, b_n]\).

For an arbitrary \( t \in [a, b) \), there is an \( n \in \mathbb{N} \) such that \( t \in [a, b_n) \). Let \( \delta(t) > 0 \) be an arbitrary number satisfying \( \delta(t) < \min(\delta_n(t), b_n - t) \). Also, thanks to condition \((V4)\) and Eq. (3.8), there is a \( \delta(b) > 0 \) such that \( \|V(b, [b, t]) - V_\sim(b)\| < \varepsilon \) and \( \|W(t)^{-1} - W(b)^{-1}V_\sim(b)\| < \varepsilon \) whenever \( t \in (b - \delta(b), b) \). We have now defined a gauge \( \delta : [a, b) \to \mathbb{R}^+ \). Consider an arbitrary \( \delta \)-fine partition \( (\xi_i, [t_{i-1}, t_i])_{i=1}^{m} \) of \([a, b] \). Our choice of \( \delta \) guarantees that \( \xi_i < b \) for \( i \in \{1, \ldots, m-1\} \) and \( \xi_m = b \). Moreover, if \( \xi_i \in [a, b_n] \), then \( [t_{i-1}, t_i] \subset [a, b_n] \).

Consequently,
\[
\sum_{i=1}^{m} \|V(\xi_i, [t_{i-1}, t_i]) - W(t_i)W(t_{i-1})^{-1}\| = \sum_{i=1}^{m} \|V(\xi_i, [t_{i-1}, t_i]) - W(t_i)W(t_{i-1})^{-1}\|
\]
\[+ \|W(b, [t_{m-1}, b]) - W(b)W(t_{m-1})^{-1}\| \leq \sum_{n=1}^{\infty} \sum_{i; \xi_i \in [a, b_n]} \|V(\xi_i, [t_{i-1}, t_i]) - W(t_i)W(t_{i-1})^{-1}\|
\]
\[+ \|V(b, [t_{m-1}, b]) - V_\sim(b)\| + \|W(b)W(b)^{-1}V_\sim(b) - W(b)W(t_{m-1})^{-1}\| \leq \sum_{n=1}^{\infty} \varepsilon \cdot \frac{1}{2^{n+1}} + \varepsilon + M \varepsilon = \varepsilon (2 + M),
\]

which proves that \( V \) is strongly Kurzweil product integrable on \([a, b]\). \( \square \)
We conclude our overview of product integration theory with some information about the product integrals of the form $\prod_{a}^{b}(I + A(t)\,dt)$, which are defined as follows.

**Definition 3.8.** A mapping $A : [a, b] \rightarrow E$ is called Kurzweil/McShane/Riemann product integrable if the function $V : [a, b] \times I \rightarrow E$ given by $V(t, [x, y]) = I + A(t)(y - x)$ is Kurzweil/McShane/Riemann product integrable in the sense of Definition 3.3. In this case, the product integral of $A$ is defined as $\prod_{a}^{b}(I + A(t)\,dt) = \prod_{a}^{b}V(t, dt)$.

A is called strongly Kurzweil/McShane product integrable if $V$ is strongly Kurzweil/McShane product integrable in the sense of Definition 3.3.

**Remark 3.9.** For an arbitrary $A : [a, b] \rightarrow E$, consider the function $V : [a, b] \times I \rightarrow E$ given by $V(t, [x, y]) = I + A(t)(y - x)$. Then the condition (V1) is obviously satisfied, and (V3), (V4) hold with $V_+(t) = V_-(t) = I$. Finally, if $t \in [a, b]$ and $\varepsilon > 0$, take an arbitrary $\sigma > 0$ such that $\|A(t)\|^2\sigma^2 < \varepsilon$. Then, if $x, y \in [a, b]$ and $t - \sigma < x \leq t \leq y < t + \sigma$, we have

$$\|V(t, [x, y]) - V(t, [t, y])V(t, [x, t])\| = \|A(t)^2(y - t)(t - x)\| < \|A(t)\|^2\sigma^2 < \varepsilon,$$

which shows that (V2) is satisfied.

According to Theorem 3.3, the indefinite Kurzweil product integral $s \mapsto \prod_{a}^{s}(I + A(t)\,dt)$ is continuous.

The next theorem provides a simple criterion for the existence of the Riemann product integral; the proof can be found in [10, Section 5] or [18, Section 5.5].

**Theorem 3.10.** A function $A : [a, b] \rightarrow E$ is Riemann product integrable if and only if it is Riemann integrable.

Next, let us recall the so-called strong Luzin condition.

**Definition 3.11.** A mapping $W : [a, b] \rightarrow E$ is said to satisfy the strong Luzin condition on $[a, b]$ if for every $\varepsilon > 0$ and $Z \subset [a, b]$ of measure zero, there exists a function $\delta : Z \rightarrow \mathbb{R}^+$ such that

$$\sum_{j=1}^{m} \|W(v_j) - W(u_j)\| < \varepsilon$$

for every collection of point-interval pairs $(\tau_j, [u_j, v_j])_{j=1}^{m}$ with $[u_j, v_j] \subset [a, b]$, $\tau_j \in Z$, and $\tau_j - \delta(\tau_j) < u_j \leq v_j < \tau_j + \delta(\tau_j)$ for all $j \in \{1, 2, \ldots, m\}$.

It is easily verified that every mapping which satisfies the strong Luzin condition is necessarily continuous, and that a product of two mappings satisfying the strong Luzin condition again satisfies the same condition. The strong Luzin condition appears in the following characterization of strongly Kurzweil product integrable mappings from [17, Corollary 4.8].

**Theorem 3.12.** For every mapping $A : [a, b] \rightarrow E$, the following conditions are equivalent:

1. $A$ is strongly Kurzweil product integrable.
2. There is a mapping $W : [a, b] \rightarrow E$ which satisfies the strong Luzin condition, $W(t)^{-1}$ exists for all $t \in [a, b]$, and $W'(t) = A(t)W(t)$ for almost all $t \in [a, b]$.

**Remark 3.13.** We point out that if $A$ is strongly Kurzweil product integrable, then the mapping $W$ from the second condition of Theorem 3.12 can be chosen as $W(t) = \prod_{a}^{t}(I + A(s)\,ds)$, $t \in [a, b]$; this follows from [17, Theorems 4.2 and 4.6].
The next theorem is concerned with the question whether the sum \( A_1 + A_2 \) of two strongly Kurzweil product integrable mappings \( A_1, A_2 \) is again strongly Kurzweil product integrable. Although we do not know the answer in general, the next result, which is sufficient for our purposes and will be needed in Section 5, provides an affirmative answer in the simpler case when one of the mappings is Bochner integrable. Recall that by [17, Theorem 4.14], Bochner integrability is equivalent to strong McShane product integrability, which in turn implies strong Kurzweil product integrability.

**Lemma 3.14.** If \( A_1, A_2 : [a, b] \to E \) are such that \( A_1 \) is strongly Kurzweil product integrable and \( A_2 \) is Bochner integrable, then \( A_1 + A_2 \) is strongly Kurzweil product integrable.

**Proof.** According to Remark 3.13, the indefinite product integrals

\[
W_i(t) = \prod_a^t (I + A_i(s)) \, ds, \quad t \in [a, b], \quad i \in \{1, 2\},
\]

satisfy the strong Luzin condition, \( W_i(t)^{-1} \) exists for every \( t \in [a, b] \), and

\[
W_i'(t)W_i(t)^{-1} = A_i(t) \quad \text{for almost all } t \in [a, b].
\]

Next, observe that \( W_1^{-1}A_2W_1 \) is the product of two continuous mappings and one Bochner integrable mapping, and is therefore Bochner integrable. Let

\[
V(t) = \prod_a^t (I + W_1(s)^{-1}A_2(s))W_1(s) \, ds, \quad t \in [a, b],
\]

\[
U(t) = W_1(t)V(t), \quad t \in [a, b].
\]

By Remark 3.13, \( V(t)^{-1} \) exists for all \( t \in [a, b] \), and we have

\[
V'(t)V(t)^{-1} = W_1(t)^{-1}A_2(t)W_1(t)
\]

for almost all \( t \in [a, b] \). Consequently,

\[
U'(t)U(t)^{-1} = (W_1'(t)V(t) + W_1(t)V'(t))V(t)^{-1}W_1(t)^{-1}
= A_1(t)W_1(t)V(t)V(t)^{-1}W_1(t)^{-1} + W_1(t)W_1(t)^{-1}A_2(t)W_1(t)W_1(t)^{-1} = A_1(t) + A_2(t)
\]

for almost all \( t \in [a, b] \). Since \( W_1 \) and \( V \) satisfy the strong Luzin condition, it follows that \( U \) satisfies the same condition. By Theorem 3.12, the existence of a mapping \( U \) with the properties described above implies that \( A_1 + A_2 \) is strongly Kurzweil product integrable. \( \Box \)

### 4 Product integrability of step mappings

In the present section, we focus on the existence of the product integral \( \prod_a^b (I + A(t)) \, dt \) corresponding to a mapping \( A : [a, b] \to E \), where \( E \) is a unital Banach algebra (see Definition 3.8).

For step mappings with finitely many steps, the Riemann, strong Kurzweil and strong McShane product integrals always exist and are easy to calculate: if there is a partition \( a = t_0 < t_1 < \cdots < t_m = b \) and \( A(t) = A_i \in E \) for all \( t \in (t_{i-1}, t_i) \), then it was shown in [17, Example 4.15] that

\[
\prod_a^b (I + A(t)) \, dt = \prod_{i=0}^m \prod_{t_{i-1}}^{t_i} (I + A(t)) \, dt = \prod_{i=0}^m \exp(A_i(t_i - t_{i-1})).
\]
In this section, we study the existence of the product integral \( \prod_a^b (I + A(t) \, dt) \) in the case when \( A \) is a step mapping with well-ordered steps. More precisely, we assume the existence of a well-ordered subset \( \Lambda \) of \([a, b]\) such that \( \min \Lambda = a \) and \( \max \Lambda = b \), and a family \((z_\alpha)_{\alpha \in \Lambda}\) of \( E \) such that

\[
A(t) = \begin{cases} 
  z_\alpha, & t \in [\alpha, S(\alpha)), \, \alpha \in \Lambda^{<b}, \\
  z_b, & t = b.
\end{cases}
\]

(4.1)

Because \([a, b]\) is a countable union of disjoint intervals \([\alpha, S(\alpha)), \alpha \in \Lambda\), \( A \) is well-defined on \([a, b]\) by (4.1).

Each mapping of this form has at most countably many discontinuities. Hence, the following result from [17, Theorem 5.3] is applicable in our situation.

**Theorem 4.1.** If \( A : [a, b] \to E \) has countably many discontinuities, then the following conditions are equivalent:

1. \( A \) is strongly Kurzweil product integrable.
2. \( A \) is Kurzweil product integrable.
3. There is a continuous mapping \( W : [a, b] \to E \) such that \( W(t) = A(t)W(t) \) for all \( t \in [a, b] \setminus Z \), where \( Z \) is countable.

We now show that Kurzweil product integrability of step mappings is closely related to the concept of multipliability introduced in Section 2. The proof is inspired by the proof of [7, Proposition 3.1].

**Theorem 4.2.** Let \( A : [a, b] \to E \) be a step mapping with representation (4.1). Then the following conditions are equivalent:

1. \( A \) is strongly Kurzweil product integrable.
2. The family \((\exp((S(\alpha) - \alpha)z_\alpha))_{\alpha \in \Lambda^{<b}}\) is multipliable and its product is invertible.

If any of these conditions is satisfied, we have

\[
\prod_a^b (I + A(t) \, dt) = \prod_{\alpha \in \Lambda^{<b}} \exp((S(\alpha) - \alpha)z_\alpha).
\]

In particular, the product on the right-hand side is an invertible element of \( E \).

**Proof.** We begin by proving the implication 1 \( \Rightarrow \) 2. Denote \( x_\alpha = \exp((S(\alpha) - \alpha)z_\alpha), \alpha \in \Lambda^{<b} \). To prove that the family \((x_\alpha)_{\alpha \in \Lambda^{<b}}\) is multipliable, it suffices to show that conditions (i) and (ii) of Definition 2.2 are satisfied with

\[
\prod_{\alpha \in \Lambda} x_\alpha = \prod_{\alpha \in \Lambda} (I + A(t) \, dt), \quad \beta \in \Lambda.
\]

Clearly, \( \prod_{\alpha \in \Lambda} x_\alpha = \prod_a^b (I + A(t) \, dt) = I \). Assume next that \( \gamma \in \Lambda \) is a successor, i.e., \( \gamma = S(\beta) \) for some \( \beta \in \Lambda \). Then [17, Example 5.11] implies that \( \prod_{\beta} (I + A(t) \, dt) = \exp((S(\beta) - \beta)z_\beta) = x_\beta \). Thus

\[
x_\beta \cdot \prod_{\alpha \in \Lambda^{<\beta}} x_\alpha = \prod_{\beta} (I + A(t) \, dt) \cdot \prod_{\alpha \in \Lambda^{<\gamma}} (I + A(t) \, dt) = \prod_{\alpha \in \Lambda^{<\gamma}} (I + A(t) \, dt) \cdot \prod_{\alpha \in \Lambda^{<\beta}} x_\alpha
\]

and condition (i) of Definition 2.2 is satisfied.
Assume finally that $\gamma$ is a limit element, and let $\varepsilon > 0$ be given. Since $t \mapsto \int_a^t (I + A(s)) \, ds$ is continuous at $t = \gamma$ and $\gamma$ is a limit element, there exists a $\beta_\varepsilon \in \Lambda^{<\gamma}$ such that

\[
\left| \prod_{\alpha \in \Lambda^{<\beta}} (I + A(t)) \, dt - \prod_{\alpha \in \Lambda^{<\gamma}} (I + A(t)) \, dt \right| < \varepsilon, \quad \beta \in \Lambda \cap [\beta_\varepsilon, \gamma).
\]

Consequently,

\[
\left| \prod_{\alpha \in \Lambda^{<\beta}} x_\alpha - \prod_{\alpha \in \Lambda^{<\gamma}} x_\alpha \right| < \varepsilon, \quad \beta \in \Lambda \cap [\beta_\varepsilon, \gamma)
\]

and condition (ii) of Definition 2.2 is also satisfied.

It remains to prove the implication $2 \Rightarrow 1$. Assume that the family $(\exp((S(\alpha) - \alpha)z_\alpha))_{\alpha \in \Lambda^{<\gamma}}$ is multipliable and its product is invertible. Consider the mapping $W : [a, b] \to E$ given by

\[
W(t) = \exp((t - \gamma)z_\gamma) \left( \prod_{\alpha \in \Lambda^{<\gamma}} \exp((S(\alpha) - \alpha)z_\alpha) \right), \quad t \in [\gamma, S(\gamma)), \quad \gamma \in \Lambda^{<b},
\]

\[
W(b) = \prod_{\alpha \in \Lambda^{<\gamma}} \exp((S(\alpha) - \alpha)z_\alpha).
\]

To finish the proof, it is enough to verify that $W$ satisfies condition 3 of Theorem 4.1. Note that $(t - \gamma)^{-1}$ exists for every $t \in [a, b]$, and

\[
W'(t) = z_\gamma \exp((t - \gamma)z_\gamma) \left( \prod_{\alpha \in \Lambda^{<\gamma}} \exp((S(\alpha) - \alpha)z_\alpha) \right) = A(t)W(t), \quad t \in (\gamma, S(\gamma)), \quad \gamma \in \Lambda^{<b},
\]

i.e., $W'(t) = A(t)W(t)$ for every $t \in (a, b) \setminus \Lambda^{<b}$. In particular, $W$ is continuous at every point $t \in (a, b) \setminus \Lambda^{<b}$.

Let us show that $W$ is in fact continuous on the whole interval $[a, b]$. By definition, $W$ is right-continuous at every point $t \in [a, b]$. We need to show that $W$ is left-continuous at every point $\gamma \in \Lambda$.

If $\gamma = S(\beta)$ for some $\beta \in \Lambda$, then

\[
\lim_{t \to \gamma^-} W(t) = \exp((\gamma - \beta)z_\beta) \left( \prod_{\alpha \in \Lambda^{<\gamma}} \exp((S(\alpha) - \alpha)z_\alpha) \right) = \left( \prod_{\alpha \in \Lambda^{<\gamma}} \exp((S(\alpha) - \alpha)z_\alpha) \right) = W(\gamma).
\]

If $\gamma$ is a limit element, we know that

\[
\lim_{\beta \to \gamma^- - \alpha \in \Lambda^{<\gamma}} \exp((S(\alpha) - \alpha)z_\alpha) = \prod_{\alpha \in \Lambda^{<\gamma}} \exp((S(\alpha) - \alpha)z_\alpha)
\]

(4.2)

Also, the second part of Lemma 2.5 implies $\lim_{\beta \to \gamma^-} \exp((S(\beta) - \beta)z_\beta) = I$; using the continuity of the logarithm function, we get

\[
\lim_{\beta \to \gamma^-} (S(\beta) - \beta)z_\beta = \lim_{\beta \to \gamma^-} \log(\exp((S(\beta) - \beta)z_\beta)) = \log I = 0.
\]

(4.3)

Now, for an arbitrary $t \in [a, \gamma)$, there exists a $\beta \in \Lambda \cap [a, \gamma)$ such that $t \in [\beta, S(\beta))$. Note that

\[
\|W(t) - W(\beta)\| = \left\| \left( \exp((t - \beta)z_\beta) - I \right) \prod_{\alpha \in \Lambda^{<\beta}} \exp((S(\alpha) - \alpha)z_\alpha) \right\|
\]

\[
\leq \|(t - \beta)z_\beta\| \exp(\|t - \beta\|z_\beta) \left\| \prod_{\alpha \in \Lambda^{<\beta}} \exp((S(\alpha) - \alpha)z_\alpha) \right\|
\]

\[
\leq \|(S(\beta) - \beta)z_\beta\| \exp(\|(S(\beta) - \beta)z_\beta\|) \left\| \prod_{\alpha \in \Lambda^{<\beta}} \exp((S(\alpha) - \alpha)z_\alpha) \right\|.
\]

(4.4)
For $t \to \gamma^-$, we have $\beta \to \gamma^-$ and the expression in (4.4) tends to 0 because of (4.2) and (4.3). Hence,
\[
\lim_{t \to \gamma^-} W(t) = \lim_{t \to \gamma^-} W(\beta) + \lim_{t \to \gamma^-} (W(t) - W(\beta)) = \lim_{\beta \to \gamma^-} W(\beta) = W(\gamma),
\]
where the last equality follows from (4.2). This proves that $W$ is left-continuous at every point $\gamma \in \Lambda$.

In the commutative case, we obtain the following criterion.

**Theorem 4.3.** Let $A : [a, b] \to E$ be a step mapping with representation (4.1). Assume that $z_\alpha z_\beta = z_\beta z_\alpha$ whenever $\alpha, \beta \in \Lambda^{<b}$. Then the following conditions are equivalent:

1. $A$ is strongly Kurzweil product integrable.
2. $A$ is strongly Henstock-Kurzweil integrable.
3. The family $((S(\alpha) - \alpha)z_\alpha)_{\alpha \in \Lambda^{<b}}$ is summable.
4. The family $(\exp((S(\alpha) - \alpha)z_\alpha))_{\alpha \in \Lambda^{<b}}$ is multipliable and its product is invertible.

If any of these conditions is satisfied, we have
\[
\prod_{a}^{b} (I + A(t) \, dt) = \exp \left( \int_{a}^{b} A(t) \, dt \right) = \prod_{\alpha \in \Lambda^{<b}} \exp((S(\alpha) - \alpha)z_\alpha) = \exp \left( \sum_{\alpha \in \Lambda^{<b}} (S(\alpha) - \alpha)z_\alpha \right). \tag{4.5}
\]

**Proof.** Conditions 1 and 4 are equivalent by Theorem 4.2, and conditions 2 and 3 are equivalent by [7, Proposition 3.1]. The commutativity assumption and [17, Theorems 3.12, 3.13] imply that conditions 1 and 2 are equivalent and the first equality in (4.5) holds. The second equality follows from Theorem 4.2, and the third one from Lemma 2.8.

For the Riemann product integral, we have an even simpler condition.

**Theorem 4.4.** Let $A : [a, b] \to E$ be a step mapping with representation (4.1). Then the following conditions are equivalent:

1. $A$ is Riemann product integrable.
2. The family $(z_\alpha)_{\alpha \in \Lambda}$ is bounded.

**Proof.** Recall that $A$ is Riemann product integrable if and only if it is Riemann integrable. By [7, Proposition 3.5], this happens if and only if the family $(z_\alpha)_{\alpha \in \Lambda}$ is bounded.

Finally, we have the following characterization of strong McShane/Bochner product integrability.

**Theorem 4.5.** Let $A : [a, b] \to E$ be a step mapping with representation (4.1). Then the following conditions are equivalent:

1. $A$ is strongly McShane product integrable.
2. $A$ is Bochner integrable.
3. The family $((S(\alpha) - \alpha)z_\alpha)_{\alpha \in \Lambda^{<b}}$ is absolutely summable.
4. The family $(\exp((S(\alpha) - \alpha)\|z_\alpha\|))_{\alpha \in \Lambda^{<b}}$ is multipliable.
5. The family $(1 + (S(\alpha) - \alpha)\|z_\alpha\|)_{\alpha \in \Lambda^{<b}}$ is multipliable.
Choose a vector \( z \),

Example 4.8. As noticed in Example 2.14, the set

Using a similar approach, we get the next statement, which generalizes one part of Lemma 2.10.

Lemma 4.6. Let \( \Lambda \) be a well-ordered set in \( \mathbb{R} \cup \{\infty\} \) with \( a = \min \Lambda \) and \( b = \sup \Lambda \). Assume that \((x_{\alpha})_{\alpha \in \Lambda \times b}\) is a family in a unital Banach algebra \( E \) such that \( x_{\alpha}x_{\beta} = x_{\beta}x_{\alpha} \) whenever \( \alpha, \beta \in \Lambda \). If the family \((\exp x_{\alpha})_{\alpha \in \Lambda \times b}\) is multipliable and its product is invertible, then \((x_{\alpha})_{\alpha \in \Lambda \times b}\) is summable.

Proof. As in the proof of Lemma 2.8, we can suppose that \( b < \infty \). For every \( \alpha \in \Lambda \times b \), let \( z_{\alpha} = \frac{z_{\alpha}}{x_{\alpha}S(\alpha)\rightarrow 0} \) and consider the mapping \( A : [a, b] \rightarrow E \) given by

\[
A(t) = \begin{cases} 
  z_{\alpha}, & t \in [\alpha, S(\alpha)), \ \alpha \in \Lambda \times b, \\
  0, & t = b. 
\end{cases}
\]

We know that \((\exp x_{\alpha})_{\alpha \in \Lambda \times b}\) is multipliable and has its product is invertible. By Theorem 4.3, the family \((S(\alpha) - \alpha)z_{\alpha})_{\alpha \in \Lambda \times b}\) is summable.

Using a similar approach, we get the next statement, which generalizes one part of Lemma 2.10.

Lemma 4.7. Let \( \Lambda \) be a well-ordered set in \( \mathbb{R} \cup \{\infty\} \) with \( a = \min \Lambda \) and \( b = \sup \Lambda \). Assume that \((x_{\alpha})_{\alpha \in \Lambda \times b}\) is an absolutely summable family in a unital Banach algebra \( E \). Then \((\exp x_{\alpha})_{\alpha \in \Lambda \times b}\) is multipliable and has an invertible product.

Proof. As in the proof of Lemma 2.6 we can suppose that \( b < \infty \). Let \((z_{\alpha})_{\alpha \in \Lambda \times b}\) and \( A : [a, b] \rightarrow E \) be the same meaning as in the proof of Lemma 4.6. By Theorem 4.3, the absolute summability of \((x_{\alpha})_{\alpha \in \Lambda \times b} = ((S(\alpha) - \alpha)z_{\alpha})_{\alpha \in \Lambda \times b}\) implies that \( A \) is strongly McShane product integrable, and therefore also strongly Kurzweil product integrable. By Theorem 4.3, the family \((\exp x_{\alpha})_{\alpha \in \Lambda \times b}\) is multipliable and its product is invertible.

Example 4.8. As noticed in Example 2.14, the set

\[
\Lambda_1 = \{\alpha(n_0, n_1) = 1 - 2^{-n_0 - 1} - 2^{-n_0 - n_1 - 1}; \ n_0, n_1 \in \mathbb{N}_0\}
\]

is a well-ordered subset of \([0, 1]\). Routine calculations show that for every \( \alpha = \alpha(n_0, n_1) = 1 \in \Lambda_1, \) we have

\[
S(\alpha) - \alpha = \alpha(n_0, n_1 + 1) - \alpha(n_0, n_1) = 2^{-n_0 - n_1 - 2}.
\]

Choose a vector \( z \neq 0 \) of \( E \), and let \( A : [0, 1] \rightarrow E \) have the representation \((\frac{(-1)^{n_0 + n_1}}{(n_0 + 1)(n_1 + 1)})\), where \( \Lambda = \Lambda_1 \cup \{1\} \) and

\[
z_{\alpha} = z_{\alpha(n_0, n_1)} = \frac{(-2)^{n_0 + n_1 + 2}}{(n_0 + 1)(n_1 + 1)} z, \ \alpha = \alpha(n_0, n_1) \in \Lambda_1, \ \ z_1 = 0.
\]

Note that

\[
(S(\alpha) - \alpha)z_{\alpha} = \frac{(-1)^{n_0 + n_1}}{(n_0 + 1)(n_1 + 1)} z.
\]

Hence, the family \((S(\alpha) - \alpha)z_{\alpha})_{\alpha \in \Lambda \times b}\) is equal to the family \((x_{\alpha})_{\alpha \in \Lambda_1}\) considered in Example 2.14 and

\[
\sum_{\alpha \in \Lambda \times b} (S(\alpha) - \alpha)z_{\alpha} = \sum_{\alpha \in \Lambda_1} x_{\alpha} = \sum_{n_0 = 0}^{\infty} \sum_{n_1 = 0}^{\infty} \frac{(-1)^{n_0 + n_1}}{(n_0 + 1)(n_1 + 1)} z = (\log 2)^2 z.
\]
Since \( z_\alpha z_\beta = z_\beta z_\alpha \) whenever \( \alpha, \beta \in \Lambda \), it follows from Theorem 11.3 that \( A \) is strongly Kurzweil product integrable and
\[
\prod_0^1 (I + A(t) \, dt) = \exp \left( \int_0^1 A(t) \, dt \right) = \exp \left( \sum_{\alpha \in \Lambda < 1} (S(\alpha) - \alpha)z_\alpha \right) = \exp \left( (\log 2)^2 z \right).
\]

On the other hand, since \( (x_\alpha)_{\alpha \in \Lambda_1} \) is neither bounded nor absolutely summable, Theorems 11.3 and 11.5 imply that \( A \) is neither Riemann product integrable nor strongly McShane product integrable.

**Example 4.9.** Let \( \Lambda_1 \) be as in Example 4.8. Choose a vector \( z \neq 0 \) of \( E \), and let \( A : [0, 1] \rightarrow E \) have the representation (4.1), where \( \Lambda = \Lambda_1 \cup \{ 1 \} \) and
\[
z_\alpha = z_{\alpha(n_0,n_1)} = \frac{2^{n_0+n_1+2}}{(n_0+1)^2(n_1+1)^2}z, \quad \alpha = \alpha(n_0,n_1) \in \Lambda_1, \quad z_1 = 0.
\]
In this case, the family \( \{ (S(\alpha) - \alpha)z_\alpha \}_{\alpha \in \Lambda < 1} \) is absolutely summable, and
\[
\sum_{\alpha \in \Lambda < 1} (S(\alpha) - \alpha)z_\alpha = \sum_{n_0=0}^{\infty} \sum_{n_1=0}^{\infty} \frac{1}{(n_0+1)^2(n_1+1)^2}z = \left( \frac{\pi^2}{6} \right)^2 z.
\]
It follows from Theorem 11.3 that \( A \) is strongly McShane product integrable. By Theorem 11.5, we get
\[
\prod_0^1 (I + A(t) \, dt) = \exp \left( \int_0^1 A(t) \, dt \right) = \exp \left( \sum_{\alpha \in \Lambda < 1} (S(\alpha) - \alpha)z_\alpha \right) = \exp \left( \left( \frac{\pi^2}{6} \right)^2 z \right).
\]
On the other hand, \( A \) is not Riemann product integrable because the family \( (z_\alpha)_{\alpha \in \Lambda_1} \) is unbounded.

### 5 Product integrability of right regulated mappings

In this section we study product integrability of mappings \( A \) from \([a, b] \) to a unital Banach algebra \( E \) which are right regulated, i.e., which have right limits at all points of \([a, b]\). The main difference between right regulated mappings and regulated mappings, which have also left limits at every point of \([a, b]\), is that the former ones may have discontinuities of the second kind, while regulated mappings can have only discontinuities of the first kind. Another difference is that regulated mappings are always Riemann product integrable, whereas right regulated mappings need not be even Kurzweil product integrable.

By [7] Lemma 2.6, every right regulated mapping is strongly measurable and has at most countably many discontinuities. Thus, Theorem 11.1 is applicable. In this section we provide additional necessary and sufficient conditions for Kurzweil product integrability of right regulated mappings. Our basic tool is the following lemma; it is a consequence of [7] Lemma 2.5 and its proof, which is based on a generalized iteration method presented in [8].

**Lemma 5.1.** Let \( A : [a, b] \rightarrow E \) be right regulated. Then for every \( \epsilon > 0 \), there is a well-ordered set \( \Lambda_\epsilon \subset [a, b] \) such that \([a, b]\) is a disjoint union of the intervals \([\beta, S(\beta)], \) \( \beta \in \Lambda_\epsilon^b \), and \( \| A(s) - A(t) \| \leq \epsilon \) whenever \( s, t \in (\beta, S(\beta)) \) and \( \beta \in \Lambda_\epsilon^b \).

\( \Lambda_\epsilon \) is determined by the following properties:
\[
a = \min \Lambda_\epsilon, \quad a < \gamma \in \Lambda_\epsilon \quad \text{if and only if} \quad \gamma = \sup \{ G_\epsilon(x); x \in \Lambda_\epsilon^\gamma \}, \quad (5.1)
\]
where \( G_\epsilon : [a, b] \rightarrow [a, b] \) is defined by
\[
G_\epsilon(x) = \sup \{ y \in (x, b]; \| A(s) - A(t) \| \leq \epsilon \text{ for all } s, t \in (x, y) \}, \quad x \in [a, b], \quad G_\epsilon(b) = b. \quad (5.2)
\]

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For a right regulated mapping $A : [a, b] \to E$ and an arbitrary $\varepsilon > 0$, we introduce the step mapping $A_\varepsilon : [a, b] \to E$ given by
\[
A_\varepsilon(t) = A(\beta+) - \varepsilon, \quad t \in [\beta, S(\beta)), \quad \beta \in \Lambda^b_{\varepsilon}, \quad A_\varepsilon(b) = A(b).
\] (5.3)

Note that $\|A_\varepsilon(t) - A(t)\| \leq \varepsilon$ for all $t \in (\beta, S(\beta))$ and $\beta \in \Lambda^b_{\varepsilon}$, i.e., for all $t \in [a, b]$ with countably many exceptions. In this way, we can approximate right regulated mappings by step mappings. Moreover, the following results show that this approximation preserves the existence or nonexistence of product integrals. Hence, we can use criteria from Section 4 to study product integrability of right regulated mappings.

Our first result provides necessary and sufficient conditions for strong Kurzweil product integrability of right regulated mappings. The proof is inspired by the proof of [7, Proposition 4.1]; note however that it relies on Lemma [3.14]whose statement is far from obvious.

**Theorem 5.2.** Let $A : [a, b] \to E$ be a right regulated mapping. Given an arbitrary $\varepsilon > 0$, let $\Lambda_\varepsilon$ be the well-ordered subset from Lemma [5.1]. Then the following properties are equivalent:

1. $A$ is strongly Kurzweil product integrable.
2. The step mapping $A_\varepsilon : [a, b] \to E$ given by (5.3) is strongly Kurzweil product integrable.
3. The family $(\exp((S(\beta) - \beta)A(\beta+)))_{\beta \in \Lambda^b_{\varepsilon}}$ is multipliable and has an invertible product.

**Proof.** The equivalence 2 $\iff$ 3 follows immediately from Theorem [4.2] it remains to prove the equivalence 1 $\iff$ 2. Both $A_\varepsilon$ and $A$ are strongly measurable. We know that $\|A_\varepsilon(t) - A(t)\| \leq \varepsilon$ for all $t \in (\beta, S(\beta))$ and $\beta \in \Lambda^b_{\varepsilon}$. Consequently, the inequality $\|A_\varepsilon(t) - A(t)\| \leq \varepsilon$ holds almost everywhere on $[a, b]$, and $\|A_\varepsilon - A\|$ is Lebesgue integrable. This means that both $A_\varepsilon - A$ and $A - A_\varepsilon$ are Bochner integrable. According to Lemma [3.14] if $A$ is strongly Kurzweil product integrable, then $A_\varepsilon = A + (A_\varepsilon - A)$ is strongly Kurzweil product integrable; conversely, if $A_\varepsilon$ is strongly Kurzweil product integrable, then $A = A_\varepsilon + (A - A_\varepsilon)$ is strongly Kurzweil product integrable. 

The next theorem provides additional criteria applicable in the commutative case.

**Theorem 5.3.** Let $A : [a, b] \to E$ be a right regulated mapping such that $A(t_1)A(t_2) = A(t_2)A(t_1)$ for all $t_1, t_2 \in [a, b]$. Given an arbitrary $\varepsilon > 0$, let $\Lambda_\varepsilon$ be the well-ordered subset from Lemma [5.1]. Then the following properties are equivalent:

1. $A$ is strongly Kurzweil product integrable.
2. $A$ is strongly Henstock-Kurzweil integrable.
3. The step mapping $A_\varepsilon : [a, b] \to E$ given by (5.3) is strongly Henstock-Kurzweil integrable.
4. The family $(\exp((S(\beta) - \beta)A(\beta+)))_{\beta \in \Lambda^b_{\varepsilon}}$ is summable.

**Proof.** The conditions 1 and 2 are equivalent by [17] Theorems 3.12 and 3.13, while conditions 2, 3 and 4 are equivalent by [7] Proposition 4.1. 

The next result is concerned with strong McShane product integrability and extends [7] Proposition 4.3.

**Theorem 5.4.** Let $A : [a, b] \to E$ be a right regulated mapping. Given an arbitrary $\varepsilon > 0$, let $\Lambda_\varepsilon$ be the well-ordered subset from Lemma [5.1]. Then the following properties are equivalent:

1. $A$ is strongly McShane product integrable.
2. $A$ is Bochner integrable.

3. The step mapping $A_\varepsilon : [a, b] \rightarrow E$ given by (5.3) is Bochner integrable.

4. The family $((S(\beta) - \beta)A(\beta^+))_{\Lambda_0^+}$ is absolutely summable.

5. The family $(\exp((S(\beta) - \beta)\|A(\beta^+)\|))_{\Lambda_0^+}$ is multipliable.

6. The family $(1 + (S(\beta) - \beta)\|A(\beta^+)\|)_{\Lambda_0^+}$ is multipliable.

Proof. The conditions 1 and 2 are equivalent by [7, Theorem 4.14], conditions 2 and 3 are equivalent by [7, Proposition 4.3], and conditions 3, 4, 5, 6 are equivalent by Theorem 5.5.

Remark 5.5. The integrability results derived above have analogous counterparts for left regulated mappings, i.e., for mappings which have left limits at every point of $(a,b]$. We now present examples of product integrable right regulated mappings $A : [0,1] \rightarrow l^\infty$, which have discontinuities of the second kind at every rational point of $(0,1]$. The examples are adopted from [7, Examples 4.1, 4.2 and 4.3], only the space $c_0$ is replaced by the space $l^\infty$ of all bounded real sequences, which is a unital commutative Banach algebra with respect to componentwise addition, multiplication, scalar multiplication, and the supremum norm. We use the symbol $\lceil x \rceil$ to denote the largest integer not greater than $x$.

Example 5.6. Define a mapping $A_0 : [0,1] \rightarrow l^\infty$ by $A_0(0) = \left( \frac{1}{i} \right)_{i=1}^\infty$, and

$$A_0(t) = \left( \sum_{i=1}^{\infty} \frac{1}{i^2} \left( 2(nt - \lfloor nt \rfloor) \cos \left( \frac{\pi}{2(nt - \lfloor nt \rfloor)} \right) + \frac{\pi}{2} \sin \left( \frac{\pi}{2(nt - \lfloor nt \rfloor)} \right) \right) \right)_{i=1}^\infty, \quad t \in (0,1].$$

As noted in [7, Example 4.1], $A_0$ is right regulated, the set of all rational numbers of $[0,1]$ is the set of discontinuity points of $A_0$, and all positive discontinuities are of the second kind. Because $A_0$ is bounded, it is Riemann integrable, and hence also Riemann product integrable.

Example 5.7. Let $A_0$ be as in Example 5.6. For every $m \in \mathbb{N}$, define $A_m : [0,1] \rightarrow l^\infty$ by $A_m(0) = \left( \frac{1}{i} \right)_{i=1}^\infty$, and

$$A_m(t) = A_0(t) + \left( \frac{1}{i} \sum_{n=1}^{\min(i,m)} \left( \cos \left( \frac{\pi}{2(nt - \lfloor nt \rfloor)} \right) + \frac{\pi}{2} \sin \left( \frac{\pi}{2(nt - \lfloor nt \rfloor)} \right) \right) \right)_{i=1}^\infty, \quad t \in (0,1].$$

As noted in [7, Example 4.2], each $A_m$ is right regulated, the rational numbers of $[0,1]$ form the set of discontinuity points of $A_m$, it is strongly Henstock-Kurzweil integrable, but neither Bochner nor Riemann integrable. It then follows from Theorems 5.3 and 5.4 that each $A_m$ is strongly Kurzweil product integrable, but neither strongly McShane product integrable nor Riemann product integrable.

Example 5.8. Let $A_0$ be as in Example 5.6. For every $m \in \mathbb{N}$, define $A^m : [0,1] \rightarrow l^\infty$ by $A^m(0) = \left( \frac{1}{i} \right)_{i=1}^\infty$, and

$$A^m(t) = A_0(t) + \left( \frac{1}{i} \sum_{n=1}^{\min(i,m)} \frac{1}{2\sqrt{nt - nt}} \right)_{i=1}^\infty, \quad t \in (0,1].$$

As noted in [7, Example 4.3], each $A^m$ is right regulated, the rational numbers of $[0,1]$ form its set of discontinuity points, $A^m$ is unbounded and Bochner integrable. Thus, it is strongly McShane product integrable by Theorem 5.4 but not Riemann product integrable.
6 Stieltjes product integrability

This section is devoted to Stieltjes-type product integrals of the form \( \prod_a^b (I + dA(t)) \), which are defined as follows.

**Definition 6.1.** A mapping \( A : [a, b] \to E \) is called Kurzweil-Stieltjes/Riemann-Stieltjes product integrable if the function \( V : [a, b] \times I \to E \) given by \( V(t, [x, y]) = I + A(y) - A(x) \) is Kurzweil/Riemann product integrable in the sense of Definition [3.1]. In this case, the Kurzweil-Stieltjes/Riemann-Stieltjes product integral of \( A \) is defined as \( \prod_a^b (I + dA(t)) = \prod_a^b V(t, dt) \).

\( A \) is called strongly Kurzweil-Stieltjes product integrable if \( V \) is strongly Kurzweil product integrable in the sense of Definition [3.1].

We begin by recalling an elegant criterion for Riemann-Stieltjes product integrability, which was derived by R. M. Dudley and R. Norvaisa and is based on the notion of \( p \)-variation.

Given a mapping \( A : [a, b] \to E \) and a number \( p > 0 \), the \( p \)-variation of \( A \) is defined as

\[
\sup \left\{ \left( \sum_{i=1}^m \|A(t_i) - A(t_{i-1})\|^p \right)^{1/p} : \{t_i\}_{i=0}^m \text{ is a partition of } [a, b] \right\}.
\]

It is known that each mapping with finite \( p \)-variation, for some \( p \in (0, \infty) \), is regulated (see [3, Lemma 2.4]). We use the notation \( \Delta^+ A(t) = A(t+) - A(t) \) for \( t \in [a, b] \), and \( \Delta^- A(t) = A(t) - A(t-) \), \( t \in (a, b] \).

The next theorem combines [3, Theorem 4.26] and [3, Proposition 4.30].

**Theorem 6.2.** Assume that \( A : [a, b] \to E \) satisfies the following conditions:

1. \( A \) has a finite \( p \)-variation for a certain \( p \in (0, 2) \).

2. \( A \) is left-continuous or right-continuous at each point of \( (a, b) \).

3. \( I + \Delta^+ A(t) \) is invertible for all \( t \in [a, b] \), and \( I + \Delta^- A(t) \) is invertible for all \( t \in (a, b] \).

Then \( A \) is Riemann-Stieltjes product integrable.

We now turn our attention to Kurzweil-Stieltjes product integrals and start with a simple example.

**Example 6.3.** Let \( z_a, z_b \in E \) be arbitrary and consider the function \( A : [a, b] \to E \) given by \( A(t) = z_a \) for \( t \in [a, b] \), and \( A(b) = z_b \). Then the Riemann-Stieltjes and Kurzweil-Stieltjes product integrals \( \prod_a^b (I + dA(t)) \) exist if and only if \( z_b - z_a \) is invertible; this is an easy consequence of the fact that for an arbitrary partition \( a = t_0 < t_1 < \cdots < t_m = b \), we have \( \prod_{i=m}^0 (I + A(t_i) - A(t_{i-1})) = I + z_b - z_a \).

If \( I + z_b - z_a \) is invertible, then \( A \) is also strongly Kurzweil-Stieltjes product integrable. Define \( W : [a, b] \to E \) by \( W(t) = I \) for \( t \in [a, b] \) and \( W(b) = I + z_b - z_a \). For an arbitrary partition \( a = t_0 < t_1 < \cdots < t_m = b \), we have

\[
\sum_{i=1}^m \|I + A(t_i) - A(t_{i-1}) - W(t_i)W(t_{i-1})^{-1}\| = \|I + A(t_m) - A(t_{m-1}) - W(t_m)W(t_{m-1})^{-1}\| = 0,
\]

i.e., the strong Kurzweil-Stieltjes integral \( \prod_a^b (I + dA(t)) \) exists and equals \( I + z_b - z_a \).

Next, we focus on more complicated step mappings having the form [1.1].

**Theorem 6.4.** Let \( A : [a, b] \to E \) be a step mapping with representation [1.1]. Assume that for each limit element \( \gamma \in A \), \( \lim_{\beta \to \gamma} (I + z_\gamma - z_\beta) \) exists and is invertible. Then the following conditions are equivalent:
1. A is strongly Kurzweil-Stieltjes product integrable.

2. A is Kurzweil-Stieltjes product integrable.

3. The family \((x_\alpha)_{\alpha \in \Lambda}\) given by

\[
 x_\alpha = \begin{cases} 
 I & \text{if } \alpha = a, \\
 I + z_\alpha - z_\beta & \text{if } \alpha = S(\beta), \\
 \lim_{\beta \to \alpha^-} (I + z_\alpha - z_\beta) & \text{if } \alpha \text{ is a limit element}
\end{cases}
\] (6.1)

is multipliable, and its elements as well as its product are invertible.

If any of these conditions is satisfied, we have

\[
 \prod_{\alpha}^{b} (I + dA(t)) = \prod_{\alpha \in \Lambda} x_\alpha.
\]

Proof. Let us start by checking whether the function \(V : [a, b] \times \mathcal{I} \to E\) given by \(V(t, [x, y]) = I + A(y) - A(x)\) satisfies conditions (V1)–(V4) from Section 3.

The statement of (V1) is obviously true. To prove that condition (V2) holds, assume first that \(t \in [a, b]\). Because \(A\) has the representation (4.1), then \(t \in [\alpha, S(\alpha)]\) for some \(\alpha \in \Lambda\). Choosing \(\sigma = S(\alpha) - t\), then 

\[
 A(y) = A(t) = z_\alpha \quad \text{when } y < t < t + \sigma, \quad \text{whence}
\]

\[
 \|I + A(y) - A(x) - (I + A(y) - A(t))(I + A(t) - A(x))\| = 0 \quad (6.2)
\]

for all \(x, y \in [a, b]\) such that \(t - \sigma < x \leq t \leq y < t + \sigma\). Eq. (6.2) also holds when \(t = b\), since then \(y = t\) and \(A(y) = A(t) = z_b\). This proves that (V2) is satisfied.

Since \(A\) is right-continuous, it follows immediately that condition (V3) holds with \(V_+(t) = I\). To prove (V4), assume first that \(t \in [a, b] \setminus \Lambda\). Then \(t \in (\beta, S(\beta))\) for some \(\beta \in \Lambda\), and

\[
 V_+(t) = \lim_{x \to \beta} (I + A(t) - A(x)) = I + z_\beta - z_\beta = I
\]

is invertible. Next, if \(t = \gamma\) for some limit element \(\gamma \in \Lambda\), then

\[
 V_+(t) = \lim_{x \to \gamma^-} (I + A(t) - A(x)) = \lim_{\beta \to \gamma^-} (I + z_\gamma - z_\beta),
\]

and the last limit exists and is invertible. Assume finally that \(t = \gamma \in \Lambda\) is a successor, say \(\gamma = S(\beta), \beta \in \Lambda\). Then

\[
 V_+(t) = \lim_{x \to t^-} (I + A(t) - A(x)) = I + z_\gamma - z_\beta.
\]

We do not apriori know whether the last element is invertible. However, if condition 1 or 2 is satisfied, then the product integral \(\prod_\beta (I + dA(t))\) exists, and it follows from Example 5.3 that \(I + z_\gamma - z_\beta\) has to be invertible. Also, if condition 3 is satisfied, then \(I + z_\gamma - z_\beta\) is obviously invertible. This shows that condition (V4) is satisfied if at least one of the conditions 1, 2, and 3 holds.

Now, let us show that conditions 1, 2, and 3 are equivalent.

We begin with the implication 3 \(\Rightarrow\) 1. We use transfinite induction to prove that for every \(\gamma \in \Lambda\), the strong product integral \(\prod_\alpha^{\gamma} (I + dA(t))\) exists and

\[
 \prod_{\alpha}^{\gamma} (I + dA(t)) = \prod_{\alpha \in \Lambda \leq \gamma} x_\alpha.
\]
The statement is obvious for \( \gamma = a \). Next, we make an induction hypothesis: Suppose that \( \gamma \in \Lambda \setminus \{a\} \) and
\[
\prod_{a}^{\beta}(I + dA(t)) = \prod_{a} \gamma x_{a} \text{ for every } \beta \in \Lambda^{< \gamma}.
\]
Assume first that \( \gamma \) is a successor, i.e., \( \gamma = S(\beta) \) for a certain \( \beta \in \Lambda \). By Example 6.3, the strong Kurzweil-Stieltjes product integral \( \prod_{a}^{\beta}(I + dA(t)) \) exists and equals \( I + z_{\gamma} - z_{\beta} \). Consequently,
\[
\prod_{a}^{\gamma}(I + dA(t)) = \prod_{\beta}^{\gamma}(I + dA(t)) \prod_{a}^{\beta}(I + dA(t)) = (I + z_{\gamma} - z_{\beta}) \prod_{a}^{\beta} x_{a} = x_{\gamma} \prod_{a}^{\beta} x_{a} = \prod_{a}^{\gamma} x_{a}.
\]
Assume next that \( \gamma \in \Lambda \) is a limit element. Then
\[
\prod_{a}^{\gamma} x_{a} = \lim_{\beta \to \gamma^{-}} \left( \prod_{a}^{\beta} x_{a} \right).
\]
For an arbitrary \( s \in [a, \gamma) \), there is a \( \beta \in \Lambda \) such that \( s \in [\beta, S(\beta)) \). Since \( A \) is constant on \( [\beta, S(\beta)) \), we have \( \prod_{a}^{s}(I + dA(t)) = I \), and
\[
\prod_{a}^{s}(I + dA(t)) = \prod_{\beta}^{s}(I + dA(t)) \prod_{a}^{\beta}(I + dA(t)) = \prod_{a}^{\beta}(I + dA(t)) = \prod_{a}^{\gamma} x_{a}.
\]
Consequently,
\[
\lim_{s \to \gamma^{-}} \left( I + A(\gamma) - A(s) \right) \prod_{a}^{s}(I + dA(t)) = \left( \lim_{\beta \to \gamma^{-}} (I + z_{\gamma} - z_{\beta}) \right) \prod_{a}^{\beta} x_{a} = x_{\gamma} \prod_{a}^{\beta} x_{a}, \tag{6.3}
\]
where the third equality follows from Lemma 2.3. Note that by Lemma 2.1 the product on the right-hand side of (6.3) is invertible. These facts imply that
\[
\lim_{s \to \gamma^{-}} \prod_{a}^{s}(I + dA(t)) = \prod_{a}^{\gamma} x_{a} = x_{\gamma} \prod_{a}^{\beta} x_{a}, \tag{6.4}
\]
and the right-hand side is invertible. According to Theorem 3.7, the strong Kurzweil-Stieltjes product integral \( \prod_{a}^{\beta}(I + dA(t)) \) exists and equals \( x_{\gamma} \prod_{a}^{\beta} x_{a} = \prod_{a}^{\gamma} x_{a} \), which completes the proof by transfinite induction.

The implication 1 \( \Rightarrow \) 2 follows immediately from Theorem 3.7.

It remains to verify the implication 2 \( \Rightarrow \) 3. We use transfinite induction to prove that for every \( \gamma \in \Lambda \), the family \( (x_{a})_{a \in \Lambda^{< \gamma}} \) is multiplicable, and
\[
\prod_{a \in \Lambda^{< \gamma}} x_{a} = \prod_{a}^{\gamma} x_{a} \tag{6.5}
\]
The statement is obvious for \( \gamma = a \). Next, we make an induction hypothesis: Suppose that \( \gamma \in \Lambda \setminus \{a\} \) and
\[
\prod_{a}^{\beta}(I + dA(t)) = \prod_{a} \gamma x_{a} \text{ for every } \beta \in \Lambda^{< \gamma}.
\]
Assume first that \( \gamma \) is a successor, i.e., \( \gamma = S(\beta) \) for a certain \( \beta \in \Lambda \). Then \( (x_{a})_{a \in \Lambda^{< \gamma}} \) is obviously multiplicable. By Example 6.3, we have \( \prod_{a}^{\beta}(I + dA(t)) = I + z_{\gamma} - z_{\beta} \). Therefore, \( x_{\gamma} = I + z_{\gamma} - z_{\beta} \) has to be invertible, and
\[
\prod_{a \in \Lambda^{< \gamma}} x_{a} = x_{\gamma} \prod_{a \in \Lambda^{< \beta}} x_{a} = (I + z_{\gamma} - z_{\beta}) \prod_{a \in \Lambda^{< \gamma}} x_{a} = \prod_{a}^{\gamma}(I + dA(t)) \prod_{a}^{\beta}(I + dA(t)) = \prod_{a}^{\gamma}(I + dA(t)).
\]
Assume next that \( \gamma \in \Lambda \) is a limit element. Then we claim that the family \( (x_{a})_{a \in \Lambda^{< \gamma}} \) is multiplicable with its product being equal to
\[
\prod_{a \in \Lambda^{< \gamma}} x_{a} = \lim_{\beta \to \gamma^{-}} \prod_{a}^{\beta}(I + dA(t))
\]
(the existence of the limit is guaranteed by Theorem 3.3). Indeed, the previous equality together with the induction hypothesis imply that

$$\Pi_{\alpha \in \Lambda \leq \gamma} x_\alpha = \lim_{\beta \to \gamma^-} \left( \prod_{\alpha \in \Lambda \leq \beta} x_\alpha \right) = \lim_{\beta \to \gamma^-} \left( \Pi_{\alpha \in \Lambda < \beta} x_\alpha \right),$$

and therefore condition (ii) of Definition 2.2 is satisfied. Consequently, the family $\langle x_\alpha \rangle_{\alpha \in \Lambda \leq \gamma}$ is multiplicable as well, and we get

$$\Pi_{\alpha \in \Lambda \leq \gamma} x_\alpha = x_\gamma \Pi_{\alpha \in \Lambda < \gamma} x_\alpha = \lim_{\beta \to \gamma^-} (I + z_\gamma - z_\beta) \lim_{\beta \to \gamma^-} \prod_a (I + dA(t)) = V^- (\gamma) \lim_{\beta \to \gamma^-} \prod_a (I + dA(t)) = \prod_a (I + dA(t)),$$

where the last equality follows from Theorem 3.3; this completes the proof by transfinite induction.

**Remark 6.5.** In Theorem 6.4, we encountered the assumptions that $I + z_\alpha - z_\beta$ is invertible whenever $\alpha = S(\beta)$, and that $\lim_{\beta \to \gamma^-} (I + z_\alpha - z_\beta)$ exists and is invertible whenever $\alpha$ is a limit element. In terms of the step mapping $A$, one can equivalently say that $I + \Delta^- A(t)$ exists and is invertible for all $t \in (a, b]$; note that the symmetric expression $I + \Delta^+ A(t)$ is always invertible since $A$ is right-continuous.

The following consequence of Theorem 6.4 is useful in applications.

**Corollary 6.6.** Let $A : [a, b] \to E$ be a step mapping with representation (4.4). Assume that for each limit element $\gamma \in \Lambda$, we have $\lim_{\beta \to \gamma^-} z_\beta = z_\gamma$. Then the following conditions are equivalent:

1. $A$ is strongly Kurzweil-Stieltjes product integrable.
2. $A$ is Kurzweil-Stieltjes product integrable.
3. The family $\langle x_\alpha \rangle_{\alpha \in \Lambda}$ given by

$$\begin{cases} x_\alpha = I & \text{if } \alpha = a, \text{ or if } \alpha \text{ is a limit element,} \\ x_{S(\beta)} = I + z_{S(\beta)} - z_\beta & \text{if } \beta \in \Lambda < b \end{cases}$$

is multiplicable, and its elements as well as its product are invertible.

If any of these conditions is satisfied, we have

$$\prod_a^b (I + dA(t)) = \Pi_{\alpha \in \Lambda} x_\alpha.$$

In the following examples we make the convention that $\sum_{n=1}^0 x_n = 0$, and $\prod_{n=1}^b x_n = I$.

**Example 6.7.** Let $q \in (0, 2)$, $C > \frac{1}{3} \left( \frac{2}{5} \right)^{1/q}$, and an interval $[a, b] \subset \mathbb{R}$ be given. Define

$$\Lambda = \left\{ \alpha(n) = b - 2^{-n} (b - a); \ n \in \mathbb{N}_0 \right\} \cup \{b\}, \ \text{and}$$

$$z_\alpha = z_{\alpha(n)} = \sum_{k=1}^n \frac{(-1)^{k+1}}{C \left( k + \frac{(-1)^{k+1}}{2} \right)^{1/q}} + \frac{(-1)^k}{2} \ 	ext{if } \alpha = \alpha(n) \in \Lambda < b.$$

The fractions in the last sum make sense because the choice of $C$ ensures that the denominator of every fraction is positive.
The only limit element of $\Lambda$ is $b$. Let

$$z_b = \lim_{\alpha \to b} z_\alpha = \lim_{n \to \infty} z_{\alpha(n)} = \sum_{k=1}^{\infty} \frac{(-1)^k}{C \left( k + \frac{(-1)^{k+1}}{2} \right)^{1/q} + \frac{(-1)^k}{2}} I.$$

We claim that the last series is convergent. Indeed, the terms of this series approach zero as $k \to \infty$, and by summing pairs of consecutive terms corresponding to $k = 2n - 1$ and $k = 2n$, $n \in \mathbb{N}$, we get

$$z_b = \sum_{n=1}^{\infty} \frac{1}{C^2 \left( 2n - \frac{1}{2} \right)^{2/q} - \frac{1}{4}} I,$$

which is finite because $q \in (0, 2)$.

The mapping $A : [a, b] \to E$ defined by

$$A(t) = z_{\alpha(n)}, \quad b - 2^{-n}(b - a) \leq t < b - 2^{-n-1}(b - a), \quad n \in \mathbb{N}, \quad A(b) = z_b,$$

has the representation (4.1), and the hypothesis of Corollary 6.6 is satisfied. We are now going to show that condition 3 of this corollary holds. Indeed, consider the family $(x_\alpha)_{\alpha \in \Lambda}$ given by (6.5). Because $S(\alpha(n)) = \alpha(n+1)$, $n \in \mathbb{N}$, it follows that

$$x_{S(\alpha(n))} = I + z_{S(\alpha(n))} - z_{\alpha(n)} = \left( 1 + \frac{(-1)^n}{C \left( n + 1 + \frac{(-1)^n}{2} \right)^{1/q} + \frac{(-1)^{n+1}}{2}} \right) I, \quad n \in \mathbb{N}. \quad (6.6)$$

The assumption $C > \frac{1}{2} \left( \frac{q}{2} \right)^{1/q}$ guarantees that the last element is a nonzero multiple of $I$, i.e., it is invertible. Next, observe that

$$\prod_{n=0}^{\infty} \left( 1 + \frac{(-1)^n}{C \left( n + 1 + \frac{(-1)^n}{2} \right)^{1/q} + \frac{(-1)^{n+1}}{2}} \right) \left( 1 + \frac{1}{C \left( \frac{q}{2} \right)^{1/q} - \frac{1}{2}} \right) \left( 1 + \frac{1}{C \left( \frac{q}{2} \right)^{1/q} + \frac{1}{2}} \right) \cdots = 1, \quad (6.7)$$

because the terms of this product approach 1 as $n \to \infty$, and the products of consecutive pairs of terms are equal to 1. Thus it follows from (6.6) and (6.7) that $(x_\alpha)_{\alpha \in \Lambda < b}$ is multipliable and

$$\prod_{\alpha \in \Lambda < b} x_\alpha = I.$$

By Corollary 6.6 $A$ is strongly Kurzweil-Stieltjes product integrable, and

$$\prod_{\alpha}^{b} (I + dA(t)) = \prod_{\alpha \in \Lambda \leq b} x_\alpha = \prod_{\alpha \in \Lambda < b} x_\alpha = I.$$

Since

$$\|z_{S(\alpha(n))} - z_{\alpha(n)}\| = \frac{1}{C \left( n + 1 + \frac{(-1)^n}{2} \right)^{1/q} + \frac{(-1)^{n+1}}{2}}, \quad n \in \mathbb{N},$$

it is not difficult to see that the $p$-variation of $A$ is finite when $0 < p \leq q$ and infinite when $p > q$. Hence, by Theorem 6.2 $A$ is also Riemann-Stieltjes product integrable.

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Example 6.8. Consider the well-ordered sets

$$\Lambda_0 = \{ \alpha(n_0) = 1 - 2^{-n_0}; n_0 \in \mathbb{N}_0 \}, \quad \Lambda_1 = \{ \alpha(n_0, n_1) = 1 - 2^{-n_0 - 1} - 2^{-n_0 - n_1 - 1}; n_0, n_1 \in \mathbb{N}_0 \}$$

from Example 2.14 and let $\Lambda = \Lambda_1 \cup \{ 1 \}$. Denote

$$q(n_0) = 2 - 2^{-n_0}, \quad n_0 \in \mathbb{N}_0.$$ 

It is not difficult to show that for every fixed $n_0 \in \mathbb{N}_0$, the function $x \mapsto \sum_{n=1}^{\infty} \frac{1}{x(2n - \frac{1}{2})^{1/q(n_0)} - \frac{1}{4}}$ is defined on $\left( \frac{1}{2} \left( \frac{2}{3} \right)^{1/q(n_0)}, \infty \right)$, it is continuous and strictly decreasing on this interval, and its range is $(0, \infty)$. Thus there exists a $C(n_0) > 0$ so that

$$\sum_{n=1}^{\infty} C(n_0)^2 \left(2n - \frac{1}{2}\right)^{2/q(n_0)} - \frac{1}{4} = \exp(-n_0), \quad n_0 \in \mathbb{N}_0. \quad (6.8)$$

Because the series in (6.8) has positive terms, then its first term, and also its lower bound $\frac{1}{2C(n_0)^2}$, is less than $\exp(-n_0)$. Since $\exp(-n_0) \to 0$ as $n \to \infty$, it follows that

$$\lim_{n_0 \to \infty} C(n_0) = \infty. \quad (6.9)$$

Consider the family $(z_\alpha)_{\alpha \in \Lambda}$ given by

$$z_\alpha = \sum_{k=1}^{n_1} \frac{(-1)^{k+1}}{C(n_0 + 1) \left( k + \frac{(-1)^{k+1}}{2} \right)^{1/q(n_0+1)} + \frac{(-1)^k}{2}}, \quad \alpha = \alpha(n_0, n_1) \in \Lambda_1, \quad z_1 = 0, \quad (6.10)$$

and the corresponding mapping $A : [0, 1] \to E$ defined by

$$A(t) = z_{\alpha(n_0, n_1)}, \quad t \in [\alpha(n_0, n_1), S(\alpha(n_0, n_1))], \quad \alpha(n_0, n_1) \in \Lambda_1, \quad A(1) = z_1.$$ 

Let us verify that $A$ satisfies the assumption of Theorem 6.4, i.e., $\lim_{\beta \to \alpha_-} (I + z_\alpha - z_\beta)$ exists and is invertible for each limit element $\alpha \in \Lambda$. The limit elements of $\Lambda$ are all numbers of $(\Lambda_0 \setminus \{ 0 \}) \cup \{ 1 \}$.

Every $\alpha \in \Lambda_0 \setminus \{ 0 \}$ has the form $\alpha(n_0 + 1, 0)$ for some $n_0 \in \mathbb{N}_0$; in this case, we obtain

$$\lim_{\beta \to \alpha_-} z_\beta = \lim_{n_1 \to \infty} z_{\alpha(n_0, n_1)} = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{C(n_0 + 1) \left( k + \frac{(-1)^{k+1}}{2} \right)^{1/q(n_0+1)} + \frac{(-1)^k}{2}} I = \exp(-n_0 + 1)I.$$

Because $z_\alpha = z_{\alpha(n_0+1, 0)} = 0$ by (6.10) and our convention, it follows that

$$\lim_{\beta \to \alpha_-} (I + z_\alpha - z_\beta) = I + 0 - \exp(-(n_0 + 1))I = (1 - \exp(-(n_0 + 1)))I,$$

which is an invertible element. Next, we consider the limit $\lim_{\beta \to 1-} z_\beta$. Each $\beta \in \Lambda_1$ has the form $\alpha(n_0, n_1)$ for some $n_0, n_1 \in \mathbb{N}_0$. If $n_1$ is even, then

$$\|z_\beta\| = \sum_{n=1}^{n_1/2} \frac{1}{C(n_0 + 1)^2 \left(2n - \frac{1}{2}\right)^{2/q(n_0+1)} - \frac{1}{4}} < \exp(-n_0 + 1),$$
and the right-hand side tends to zero as \( n_0 \to \infty \). If \( n_1 \) is odd, then

\[
\|z_\beta\| \leq \sum_{n=1}^{(n_3-1)/2} \frac{1}{C(n_0 + 1)^2 (2n - \frac{1}{2})^{2/q(n_0+1)} - \frac{1}{2}} + \frac{1}{C(n_0 + 1) (n_1 + \frac{1}{2})^{1/q(n_0+1)} - \frac{1}{2}} < \exp(-(n_0 + 1)) + \frac{1}{C(n_0 + 1) - \frac{1}{2}},
\]

and the right-hand side again tends to zero as \( n_0 \to \infty \) because of (6.9).

It follows that \( \lim_{\beta \to 1-} z_\beta = 0 \), and therefore \( \lim_{\beta \to 1-} (I + z_1 - z_\beta) = I \), which confirms that \( A \) satisfies the assumption of Theorem 6.4. We now verify that condition 3 of this theorem holds.

The family \( (x_\alpha)_{\alpha \in \Lambda} \) given by (6.1) is

\[
x_\alpha = \begin{cases} 
I & \text{if } \alpha = 0 \text{ or } \alpha = 1, \\
I + z_\alpha(n_0,n_1+1) - z_\alpha(n_0,n_1) & \text{if } \alpha = \alpha(n_0,n_1) \text{ for some } n_0,n_1 \in \mathbb{N}, \\
(1 - \exp(-(n_0 + 1)))I & \text{if } \alpha = \alpha(n_0 + 1,0) \text{ for some } n_0 \in \mathbb{N},
\end{cases} \tag{6.11}
\]

It follows from (6.10) that

\[
x_\alpha(n_0,n_1+1) = I + z_\alpha(n_0,n_1+1) - z_\alpha(n_0,n_1) = I + \frac{(-1)^{n_1}}{C(n_0 + 1) (n_1 + 1 + (-1)^{n_1} 2^{1/q(n_0+1)} + (-1)^{n_1+1})} I.
\]

Hence, all elements of \( (x_\alpha)_{\alpha \in \Lambda} \) are invertible. It remains to show that \( (x_\alpha)_{\alpha \in \Lambda} \) is multipliable and its product is invertible. Take an arbitrary \( n_0 \in \mathbb{N} \). As in Example 6.7 with \( a = \alpha(n_0) \) and \( b = \alpha(n_0 + 1) \), we have

\[
\prod_{\alpha \in \Lambda \cap \{\alpha(n_0),\alpha(n_0+1)\}} x_\alpha = \prod_{n_1=0}^{\infty} \left( I + \frac{(-1)^{n_1}}{C(n_0 + 1) (n_1 + 1 + (-1)^{n_1} 2^{1/q(n_0+1)} + (-1)^{n_1+1})} I \right) = I,
\]

because the product of each pair of consecutive terms is \( I \). Therefore

\[
\prod_{\alpha \in \Lambda \cap \{\alpha(n_0),\alpha(n_0+1)\}} x_\alpha = x_\alpha(n_0+1) \left( \prod_{\alpha \in \Lambda \cap \{\alpha(n_0),\alpha(n_0+1)\}} x_\alpha \right) = (1 - \exp(-(n_0 + 1)))I,
\]

and consequently

\[
\prod_{\alpha \in \Lambda \cap \{\alpha(n_0),\alpha(n_0+1)\}} x_\alpha = \prod_{n=1}^{n_0} (1 - \exp(-n))I, \quad n_0 \in \mathbb{N}.
\]

For an arbitrary pair \( n_0,n_1 \in \mathbb{N} \), we have

\[
\prod_{\alpha \in \Lambda \cap \{0,\alpha(n_0,n_1)\}} x_\alpha = \left( \prod_{\alpha \in \Lambda \cap \{\alpha(n_0,0),\alpha(n_0,n_1)\}} x_\alpha \right) \left( \prod_{\alpha \in \Lambda \cap \{\alpha(n_0,0),\alpha(n_0+1)\}} x_\alpha \right) = \prod_{k=1}^{n_0} x_{\alpha(n_0,k)} \prod_{n=1}^{n_0} (1 - \exp(-n))I.
\]

If \( n_1 \) is even, we get

\[
\prod_{\alpha \in \Lambda \cap \{0,\alpha(n_0,n_1)\}} x_\alpha = \prod_{n=1}^{n_0} (1 - \exp(-n))I,
\]

and if \( n_1 \) is odd, we obtain

\[
\prod_{\alpha \in \Lambda \cap \{0,\alpha(n_0,n_1)\}} x_\alpha = x_{\alpha(n_0,n_1)} \prod_{n=1}^{n_0} (1 - \exp(-n))I = \left( 1 + \frac{1}{C(n_0 + 1) (n_1 + \frac{1}{2})^{1/q(n_0+1)} - \frac{1}{2}} \right) \prod_{n=1}^{n_0} (1 - \exp(-n))I.
\]
Using (6.9), we conclude that \( \lim_{\beta \to 1} \left( \prod_{\alpha \in \Lambda^<} x_\alpha \right) = \prod_{n=1}^\infty (1 - \exp(-n))I \). Hence, the family \((x_\alpha)_{\alpha \in \Lambda}\) is multiplicable and its product is \( \prod_{n=1}^\infty (1 - \exp(-n))I \). This product is invertible by Corollary 2.13 since the family \((\exp(-n))_{n \in \mathbb{N}}\) is absolutely summable.

By Theorem 6.4, \( A \) is Kurzweil-Stieltjes product integrable and \( \prod_{t=0}^1(I + dA(t)) = \prod_{n=1}^\infty (1 - \exp(-n))I \). Note that the \( p \)-variation of \( A \) is infinite for every \( p \in (0, 2) \). Thus the assumptions of Theorem 6.2 are not satisfied, and the question of Riemann-Stieltjes product integrability of \( A \) is left open.

The next theorem will be needed in the following section and deals with Kurzweil-Stieltjes product integrability of step mappings with idempotent values.

**Theorem 6.9.** Let \( A : [a, b] \to E \) be a step mapping with representation (4.1). Assume that \( A \) satisfies the following assumptions:

1. \( z_\alpha \cdot z_\alpha = z_\alpha \) for all \( \alpha \in \Lambda \).
2. For each limit element \( \gamma \in \Lambda \), we have \( \lim_{\beta \to \gamma} z_\beta = z_\gamma \).
3. The family \((x_\alpha)_{\alpha \in \Lambda}\) given by

\[
\begin{align*}
x_\alpha &= I \\
x_{S(\beta)} &= I + z_{S(\beta)} - z_\beta \\
&\quad \text{if } \beta \in \Lambda^<b
\end{align*}
\]

is multiplicable, and its elements as well as its product are invertible.

Then \( A \) is Kurzweil-Stieltjes integrable, the family \((z_\alpha)_{\alpha \in \Lambda}\) is multiplicable, and

\[
\prod_{\alpha=a}^{b} (I + dA(t)) \cdot A(a) = \left( \prod_{\alpha \in \Lambda} x_\alpha \right) z_a = \prod_{\alpha \in \Lambda} z_\alpha. \tag{6.12}
\]

**Proof.** The Kurzweil-Stieltjes integrability of \( A \) is guaranteed by Corollary 6.6. The first equality of (6.12) follows also from Corollary 6.6 and the fact that \( A(a) = z_a \). It remains to prove the second equality of (6.12). We assume that \( z_a \neq 0 \) (the equality is obvious if \( z_a = 0 \)).

We apply transfinite induction to prove that for every \( \gamma \in \Lambda \), the product \( \prod_{\alpha \in \Lambda^<\gamma} z_\alpha \) is well defined and

\[
\prod_{\alpha \in \Lambda^<\gamma} z_\alpha = \left( \prod_{\alpha \in \Lambda^<\gamma} x_\alpha \right) \cdot z_a.
\]

Notice first that \((z_\alpha)_{\alpha \in \Lambda^<\gamma}\) is obviously multiplicable, and

\[
\prod_{\alpha \in \Lambda^<\gamma} z_\alpha = \prod_{\alpha \in \Lambda^<\gamma} x_\alpha \cdot z_a = \left( \prod_{\alpha \in \Lambda^<\gamma} x_\alpha \right) \cdot z_a. \tag{6.13}
\]

Next, make an induction hypothesis: assume that for every \( \beta \in \Lambda^<\gamma \), \((z_\alpha)_{\alpha \in \Lambda^<\beta}\) is multiplicable, and its product is \( \left( \prod_{\alpha \in \Lambda^<\beta} x_\alpha \right) \cdot z_a \).

If \( \gamma = S(\beta) \) for some \( \beta \in \Lambda^<\gamma \), then \((z_\alpha)_{\alpha \in \Lambda^<\gamma}\) is obviously multiplicable, \( x_\gamma = I + z_{S(\beta)} - z_\beta \), and

\[
\left( \prod_{\alpha \in \Lambda^<\gamma} x_\alpha \right) \cdot z_a = x_\gamma \cdot \left( \prod_{\alpha \in \Lambda^<\beta} x_\alpha \right) \cdot z_a = (I + z_{S(\beta)} - z_\beta) \left( \prod_{\alpha \in \Lambda^<\beta} z_\alpha \right) = (I + z_{S(\beta)} - z_\beta) \cdot z_\beta \cdot \left( \prod_{\alpha \in \Lambda^<\beta} z_\alpha \right).
\]

Because \( z_\beta z_\beta = z_\beta \), we get

\[
(I + z_{S(\beta)} - z_\beta) z_\beta = z_\beta + z_{S(\beta)} z_\beta - z_\beta z_\beta = z_\beta + z_{S(\beta)} z_\beta - z_\beta = z_{S(\beta)} z_\beta,
\]
and therefore
\[
\left( \prod_{\alpha \in \Lambda \leq \gamma} x_{\alpha} \right) \cdot z_{\alpha} = z_{S(\beta)} z_{\beta} \cdot \left( \prod_{\alpha \in \Lambda < \beta} z_{\alpha} \right) = \prod_{\alpha \in \Lambda \leq \gamma} z_{\alpha}.
\]

Assume next that \( \gamma \) is a limit element of \( \Lambda \), and let \( \varepsilon > 0 \) be given. Since the family \( (x_{\alpha})_{\alpha \in \Lambda} \) is multipliable, there is by Definition 2.2 (ii) a \( \beta_\varepsilon \in \Lambda^{< \gamma} \) such that
\[
\left\| \prod_{\alpha \in \Lambda \leq \beta} x_{\alpha} - \prod_{\alpha \in \Lambda < \gamma} x_{\alpha} \right\| < \frac{\varepsilon}{\|z_{\alpha}\|}, \quad \beta \in \Lambda \cap [\beta_\varepsilon, \gamma).
\]

Then
\[
\left\| \prod_{\alpha \in \Lambda \leq \beta} x_{\alpha} \cdot z_{\alpha} - \prod_{\alpha \in \Lambda < \gamma} x_{\alpha} \cdot z_{\alpha} \right\| \leq \|z_{\alpha}\| \left\| \prod_{\alpha \in \Lambda \leq \beta} x_{\alpha} - \prod_{\alpha \in \Lambda < \gamma} x_{\alpha} \right\| < \varepsilon, \quad \beta \in \Lambda \cap [\beta_\varepsilon, \gamma).
\]

In view of the above result and the induction hypothesis we get
\[
\left\| \prod_{\alpha \in \Lambda \leq \beta} z_{\alpha} - \prod_{\alpha \in \Lambda < \gamma} x_{\alpha} \cdot z_{\alpha} \right\| < \varepsilon, \quad \beta \in \Lambda \cap [\beta_\varepsilon, \gamma).
\]

This result, condition (ii) of Definition 2.2 and the fact that \( x_\gamma = I \) imply that
\[
\prod_{\alpha \in \Lambda \leq \gamma} z_{\alpha} = \prod_{\alpha \in \Lambda < \gamma} x_{\alpha} \cdot z_{\alpha} = \prod_{\alpha \in \Lambda \leq \gamma} x_{\alpha} \cdot z_{\alpha}.
\]

This completes the proof by transfinite induction. \( \square \)

### 7 Parallel translation

Parallel translation (also known as parallel transport) is one of the basic concepts of differential geometry and surface theory, and has important applications in physics. In the present section we apply Stieltjes-type product integrals and Haathi products to define parallel translation operators, and study their existence and properties. The ideas presented here go back to the paper [6] by H. Haathi and S. Heikkilä, who considered translation of vectors on Banach manifolds and employed the Riemann-Stieltjes product integral although they did not call it by that name. We start by recalling an elementary description of parallel translation from the beginning of [6].

Consider a polyhedron \( M \) in \( \mathbb{R}^3 \) and an oriented path \( \ell \) on \( M \) which does not cross any vertex of \( M \). We would like to define a parallel translation \( T_\ell \) of tangent vectors of \( M \) along \( \ell \). Assume that \( \ell \) can be decomposed into a finite union of subpaths \( \ell_0 \cup \cdots \cup \ell_m \), where for every \( i \in \{1, \ldots, m - 1\} \), the endpoints of \( \ell_i \) are the only points of \( \ell_i \) lying on the edges of \( M \). For every \( i \in \{0, \ldots, m\} \), let \( H_i \) be the face of \( M \) that contains \( \ell_i \). Also, let \( M_i \) denote the tangent space of \( H_i \), i.e., the 2-dimensional subspace of \( \mathbb{R}^3 \) parallel to \( H_i \).

If \( m = 0 \), i.e., the whole path is contained within the single face \( H_0 \), then the parallel translation \( T_\ell \) of tangent vectors along \( \ell \) is just the Euclidean parallel translation. Thus \( T_\ell \) is the identity operator on the tangent space \( M_0 \). Alternatively, \( T_\ell \) can be interpreted as the restriction to \( M_0 \) of the orthogonal projection operator \( P_0 \) which maps \( \mathbb{R}^3 \) onto \( M_0 \); we write \( T_\ell = P_0 \).

If \( m = 1 \), the tangent vectors are first translated in the Euclidean sense along \( \ell_0 \). At the terminal point of \( \ell_0 \), which is in \( H_0 \cap H_1 \), they are projected by the orthogonal projection operator \( P_1 \) from \( \mathbb{R}^3 \) onto the tangent space \( M_1 \), and finally translated in the Euclidean sense along \( \ell_1 \). This yields the translation operator \( T_\ell = P_1 P_0 \).

Continuing in this way, we conclude that in the general case when \( M \) has \( m + 1 \) faces, we get the translation operator \( T_\ell = P_m \cdots P_0 \), where \( P_m \) is the orthogonal projection from \( \mathbb{R}^3 \) onto \( M_m \). Notice that \( T_\ell \) does not depend on the exact shape of \( \ell \); only the sequence of faces crossed by \( \ell \) is important.
Next, consider the more complicated situation when \( M \) is a smooth surface in \( \mathbb{R}^3 \), and \( \ell : [a, b] \to M \) is a path of finite length with the initial point \( x = \ell(a) \) and terminal point \( y = \ell(b) \). To obtain the translation operator \( T_\ell \), it is natural to approximate \( M \) along \( \ell \) by a sequence of tangent planes, i.e., by a polyhedral surface. Choose \( m + 1 \) successive points \( x_i = \ell(t_i), i \in \{0, \ldots, m\} \), corresponding to a partition \( D : a = t_0 < t_1 < \cdots < t_m = b \). For every \( i \in \{0, \ldots, m\} \), let \( H_i \) be the tangent plane of \( M \) at \( x_i \), and \( M_i \) the tangent space of \( H_i \). Assume that if the partition \( D \) is fine enough, then each two successive tangent planes \( H_i \) and \( H_{i+1} \) have an intersection. (For example, this assumption is true if \( \ell \) is continuously differentiable.) Let \( \ell_D \) be a path on \( H_0 \cup \cdots \cup H_m \), starting from \( x \), passing through \( x_1, \ldots, x_{m-1} \), and terminating at \( y \). We already know that the parallel translation operator corresponding to translation along \( \ell_D \) is \( T_{\ell_D} = P(x_m)P(x_{m-1}) \cdots P(x_0) \), where \( P(x) \) is the orthogonal projection from \( \mathbb{R}^3 \) onto \( M_i \).

If the limit of \( T_{\ell_D} \) exists when the norm of the partition \( D \) tends to zero, it is denoted by \( T_\ell \) and called the parallel translation operator along \( \ell \). Thus \( T_\ell = \lim_{|D| \to 0} \prod_{i=m}^0 P(\ell(t_i)), \) where \( |D| = \max_{1 \leq i \leq m} (t_i - t_{i-1}) \). If we denote \( A = P \circ \ell \), then \( T_\ell \) is the Riemann-Haahti product of \( A \) in the sense of the following definition.

**Definition 7.1.** Consider a mapping \( A : [a, b] \to E \), where \( E \) is a Banach algebra. Assume there exists an element \( P_A \in E \) with the following property: For each \( \varepsilon > 0 \), there exists a gauge \( \delta : [a, b] \to \mathbb{R}^+ \) such that if \( ([t_i], [t_{i-1}], t_i))_{i=1}^m \) is a \( \delta \)-fine tagged partition of \([a, b]\), then \( \left\| P_A - \prod_{i=m}^0 A(t_i) \right\| < \varepsilon \). In this case, \( P_A \) is called the Kurzweil-Haahti product of \( A \), and will be denoted by \( \prod_{a} A(t) \).

If \( \delta \) is assumed to be constant on \([a, b]\), then \( P_A = \prod_{a} A(t) \) is called the Riemann-Haahti product of \( A \).

Note that the definition of \( \prod_{a} A(t) \) is very similar to the definition of the product integral \( \prod_{t} V(t, dt) \), where \( A(t, [x, y]) = A(y) \); the only difference is that we don’t require the element \( P_A \) to be invertible.

The next theorem shows that the Haahti product \( \prod_{a} A(t) \) exists if \( A \) is an idempotent mapping, i.e., if \( A(t) \cdot A(t) = A(t) \) for all \( t \in [a, b] \), and if \( A \) is Stieltjes product integrable. The idea of the proof is borrowed from [R] Lemma 2.1.

**Theorem 7.2.** Assume that \( E \) is a unital Banach algebra, and that \( A : [a, b] \to E \) is an idempotent mapping. If the Kurzweil-Stieltjes or the Riemann-Stieltjes product integral \( \prod_{I} (I + dA(t)) \) exists, then the Kurzweil-Haahti or Riemann-Haahti product \( \prod_{a} A(t) \) exists as well, and \( \prod_{a} A(t) = \left( \prod_{I} (I + dA(t)) \right) \cdot A(a) \).

**Proof.** Consider an arbitrary partition \( a = t_0 < t_1 < \cdots < t_m = b \). For every \( i \in \{1, \ldots, m\} \), we have

\[
A(t_i)A(t_{i-1}) = (A(t_{i-1}) + (A(t_i) - A(t_{i-1})))A(t_{i-1}) = A(t_{i-1})A(t_{i-1}) + (A(t_i) - A(t_{i-1}))A(t_{i-1})
\]

\[
= A(t_{i-1}) + (A(t_i) - A(t_{i-1}))A(t_{i-1}) = (I + (A(t_i) - A(t_{i-1})))A(t_{i-1}).
\]

Using this result repeatedly for \( k = m, m - 1, \ldots, 1 \), we obtain

\[
\prod_{i=m}^0 A(t_i) = \prod_{i=m}^1 (I + A(t_i) - A(t_{i-1}))A(a),
\]

which implies the statement of the theorem.

The following example shows that the Riemann-Haahti product can exist although neither the Riemann-Stieltjes nor the Kurzweil-Stieltjes product integral exist.

**Example 7.3.** Choose \( E = l^\infty \), and for every \( n \in \mathbb{N} \), let \( e^n \in l^\infty \) be the sequence \((e^n_i)_{i=1}^\infty \) given by

\[
e^n_i = \begin{cases} 1, & n = i, \\ 0, & n \neq i. \end{cases}
\]
Let $A : [0, 1] \to l^\infty$ be defined by

$$A(t) = \begin{cases} e^1 + e^{n+2}, & 1 - 2^n < t \leq 1 - 2^{-(n+1)}, \ n \in \mathbb{N}_0, \\ e^1, & t \in [0, 1]. \end{cases}$$

Clearly, $A$ is an idempotent mapping. For every partition $0 = t_0 < t_1 < \cdots < t_m = 1$, we have $\prod_{i=0}^{m-1} A(t_i) = e^1$. Hence, the Riemann-Haahti product of $A$ exists and $\prod_{i=0}^{m-1} A(t) = e^1$.

On the other hand, $A$ is neither Riemann-Stieltjes nor Kurzweil-Stieltjes product integrable. To see this, consider again a partition $D : 0 = t_0 < t_1 < \cdots < t_m = 1$. For every $i \in \{1, \ldots, m-1\}$, we have $A(t_i) = e^1 + e^{n_i}$ for a certain $n_i \geq 2$. Hence,

$$P(D) = \prod_{i=m}^{1} (I + A(t_i) - A(t_{i-1})) = (I - e^{n_{m-1}})(I + e^{-n_{m-2}})\cdots(I + e^{n_2} - e^{n_1})(I + e^{n_1}).$$

The right-hand side represents an element of $l^\infty$ whose components at positions $n_1, \ldots, n_{m-1}$ are zero, and all other components are equal to 1. Thus, no matter how we choose a gauge $\delta : [0, 1] \to \mathbb{R}^+$, we can always find two $\delta$-fine partitions $D_1, D_2$ of $[0, 1]$ such that $\|P(D_1) - P(D_2)\| = 1$. It follows that $A$ cannot be Kurzweil-Stieltjes product integrable.

**Remark 7.4.** The Banach algebra $E$ in Definition 7.1 need not be unital. In this case, neither the Kurzweil-Stieltjes nor the Riemann-Stieltjes product integral is defined. On the other hand, the Riemann-Haahti or Kurzweil-Haahti product may exist. For instance, it is enough to replace $l^\infty$ in Example 7.3 by $c_0$, which is not unital.

In a more abstract setting, the problem of parallel translation described in the introduction corresponds to the following situation: $E = L(X)$ is the Banach algebra of all bounded linear operators on a certain Banach space $X$, $\mathcal{M}$ is a Hausdorff topological space, and $A = P \circ \ell$, where $\ell : [a, b] \to \mathcal{M}$ is a continuous path and $P : \mathcal{M} \to L(X)$ is a projection, i.e., $P^2 = P$. In [6], the authors were dealing with the case where $\mathcal{M}$ is a $C^0$-manifold modelled on Banach spaces. The next result is an immediate consequence of Theorems 6.2 and 7.2.

**Theorem 7.5.** Let $\mathcal{M}$ be a Hausdorff topological space, $X$ a Banach space, $P : \mathcal{M} \to L(X)$ a projection, and $\ell : [a, b] \to \mathcal{M}$ a continuous path in $\mathcal{M}$. Assume that $A = P \circ \ell$ is right- or left-continuous at each point of $(a, b)$, has a finite $p$-variation for a certain $p \in (0, 2)$, $I + \Delta^+ A(t)$ is invertible for all $t \in [a, b]$, and $I + \Delta^- A(t)$ is invertible for all $t \in (a, b)$. Then both the Riemann-Stieltjes product integral and the Riemann-Haahti product of $P \circ \ell$ exist, and

$$\prod_{a}^{b} P(\ell(t)) = \prod_{a}^{b} (I + dP(\ell(t))) \cdot P(\ell(a)).$$

In [6], the Riemann-Stieltjes product integral $\prod_{a}^{b} (I + dP(\ell(t)))$ and the Riemann-Haahti product $\prod_{a}^{b} P(\ell(t))$ are referred to as the parallel translation operators and are denoted by $B\ell, T\ell$. A sufficient condition for the existence of these operators was presented in [6] Theorem 3.1], where it was assumed that $A = P \circ \ell$ has bounded variation, is right- or left-continuous at each point of $(a, b)$, and has only a finite number of discontinuity points. Theorem 7.5 replaces the assumption of bounded variation by the finiteness of $p$-variation for some $p \in (0, 2)$. Also, $A$ can have up to countably many discontinuities (recall that a mapping with finite $p$-variation is necessarily regulated, and therefore has at most countably many discontinuities). The extra assumptions concerning the invertibility of $I + \Delta^+ A(t)$ and $I + \Delta^- A(t)$ guarantee that the parallel
integral and the Kurzweil-Haahti product of
and Kurzweil products. The next theorem provides new existence results for the Kurzweil-Stieltjes product
The definitions of parallel translation operators can also be based on Kurzweil-Stieltjes product integrals

tangent vectors remains constant.
connexions generalizes the result that in the classical Levi-Civita pa rallelism, the scalar product of any two

\[ B(t) = I + \int_a^t d(P(\ell(s)))B(s), \quad \text{and} \quad T(t) = P(\ell(a)) + \int_a^t d(P(\ell(s)))T(s), \quad t \in [a, b] \]

have unique solutions \( B, T : [a, b] \to L(X) \), and \( B_t = B(b), T_t = T(b) \). If \( M \) is a \( C^1 \)-manifold and the path \( \ell \) is smooth, the two integral equations reduce to the initial value problems

\[ B'(t) = P'(\ell(t))\ell'(t)B(t), \quad B(a) = I, \quad \text{and} \quad T'(t) = P'(\ell(t))\ell'(t)T(t), \quad T(a) = P(\ell(a)). \]

When \( M \) is a \( C^0 \)-manifold and \( P : M \to L(X) \) is a continuous projection, the mappings \( T = \ell \mapsto T_\ell \) and \( B = \ell \mapsto B_\ell \), defined for those \( \ell \) for which \( P \circ \ell \) has bounded variation, are called in [6] \( P \)-connexions.
A result on the invariance of a scalar product, defined by a bounded bilinear function of \( X \), under these connexions generalizes the result that in the classical Levi-Civita parallelism, the scalar product of any two
tangent vectors remains constant.

The definitions of parallel translation operators can also be based on Kurzweil-Stieltjes product integrals
and Kurzweil products. The next theorem provides new existence results for the Kurzweil-Stieltjes product
integral and the Kurzweil-Haahti product of \( P \circ \ell \). It is a straightforward consequence of Corollary 6.6
Theorems 6.9 and 7.2.

**Theorem 7.6.** Let \( M \) be a Hausdorff topological space, \( X \) a Banach space, \( P : M \to L(X) \) a projection, and
\( \ell : [a, b] \to M \) a continuous path in \( M \). Assume \( A = P \circ \ell \) has representation \([4.1]\), where \( \lim_{\beta \to \gamma} z_\beta = z_\gamma \)
for each limit element \( \gamma \in \Lambda \), the family \( (z_\alpha)_{\alpha \in \Lambda} \) given by \([6.3]\) is multiplyable, and its elements as well as
its product are invertible. Then both the Kurzweil-Stieltjes product integral and the Kurzweil-Haahti product
of \( P \circ \ell \) exist, and

\[ \prod_a^b P(\ell(t)) = \prod_a^b (I + dP(\ell(t))) \cdot P(\ell(a)) = \prod_{\alpha \in \Lambda} z_\alpha. \]

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