Further On the Fountain Effect in Superfluid Helium

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In the previous paper (Attard 2022d) on the fountain pressure in superfluid helium, it was shown that the experimentally confirmed expression of H. London (1939) was thermodynamically equivalent to equality of chemical potential. However this theoretical equivalence was not reflected in the experimental data. The problem has now been traced to errors in the enthalpy and entropy derived from the measured heat capacity by Donnelly and Barenghi (1998). In this paper the corrected thermodynamic data yields almost exact agreement between the two expressions and the measured fountain pressure. A physical explanation is given for energy minimization as the principle that drives the fountain effect and superfluid flow more generally.

I. INTRODUCTION

The fountain effect refers to the spurting of superfluid helium from a heated open chamber connected by a capillary to a chamber of liquid helium maintained below the condensation temperature (Allen 1938, Balibar 2017). Closing the heated chamber and measuring the fountain pressure is, in conjunction with an equation due to H. London (1939), a common experimental technique for obtaining the entropy (Donnelly and Barenghi 1998).

The accepted explanation for the fountain effect as an osmotic pressure (Tisza 1938, Balibar 2017) is quantitatively inaccurate and makes little sense (Attard 2022d). H. London’s (1939) expression for the fountain pressure—that the temperature derivative of the fountain pressure equals the entropy per unit volume—is known to be quantitatively accurate. I have criticized his derivation on thermodynamic grounds (Attard 2022d), which raises the question of the status of the result, and, importantly, its meaning. What does it tell us about superfluid flow, and how does it relate to the broader principles of thermodynamics? I showed that the expression was thermodynamically equivalent to equality of chemical potential in the two chambers at different temperatures (Attard 2022d). Although equality of chemical potential arises from energy minimization, it is not at all clear why this should be relevant in thermodynamics, or why it should be the principle that drives superfluid flow.

In contrast, for an equilibrium system the maximization of the total entropy would be the relevant principle, but this would lead to equality of chemical potential divided by temperature, which is not equivalent to the H. London (1939) expression. The present system is a non-equilibrium steady state system for which there is no principle of first entropy maximization (Attard 2012). And there is nothing specifically superfluid about entropy maximization.

This paper begins by recapitulating (Attard 2022d) (1) that equality of chemical potential arises from energy minimization, (2) that this is thermodynamically equivalent to the H. London (1939) expression for the fountain pressure, and (3) that equality of fugacity arises from entropy maximization. The new material shows (4) that 1 and 3 are borderline distinguishable by measurement, (5) that there are systematic thermodynamic errors in the enthalpy and entropy data reported by Donnelly and Barenghi (1998), (6) that the corrected thermodynamic data confirms the equivalence of 1 and 2 and their agreement with the measured fountain pressure (Hammel and Keller 1961), and (7) that the adiabatic nature of collisionless superfluid flow is the physical basis for the principle of energy minimization that underlies 1.

This paper is an excerpt from Ch. 9 on superfluidity in the forthcoming second edition of Attard (2023). It uses without justification some concepts from that chapter and the preceding Ch. 8 on Bose-Einstein condensation and the χ-transition. It is largely independent of the theory of high temperature superconductivity, Ch. 10.

II. THERMODYNAMIC ANALYSIS

A. Formal analysis

Following Attard (2022d), consider two closed chambers of helium, A and B, each in contact with its own thermal reservoir of temperature $T_A$ and $T_B$, and having pressure $p_A$ and $p_B$. The chambers are connected by a capillary through which fluid can flow. Chamber A in practice is at the lower temperature, and consists of saturated liquid and vapor. As H. London (1939) points out, in the optimum steady state the pressure of the second chamber is a function of its temperature and the pressure and temperature of the first chamber, $p_B = p(T_B; p_A, T_A)$.

The result given by H. London (1939) says that the derivative of the pressure of the second chamber with respect to its temperature for fixed first chamber equals the entropy density,

$$\frac{dp_B}{dT_B} = \rho_B s_B. \quad (2.1)$$

Here $\rho$ is the number density and $s$ is the entropy per particle.

H. London (1939) purported to derive this result using a work-heat flow cycle, which derivation is dubious (At-
tard 2022d appendix A). (See also section [II C] Possibly H. London (1939) simply guessed this result and worked backwards: the left hand side has units of Boltzmann’s constant per unit volume, and the only thermodynamic quantity with those units is the entropy density.

To derive this from a general axiom, one should focus on variational principles for extensive quantities, since these supply the foundations for thermodynamics (Attard 2002, 2012, 2022d section IIA). Consider therefore the total entropy, which is equivalent to the negative of the free energy divided by temperature (Attard 2002). Hence maximization of the former gives the same result as minimization of the latter. The first axiom that might determine the fountain pressure is that the total entropy of the total system is a maximum. This is of course just the Second Law of Thermodynamics, albeit applied to a non-equilibrium steady state system. Usually maximization of the first entropy gives the same result as minimization of the latter. The first axiom that might determine the fountain pressure is that the total entropy is a minimum. The energy of each chamber is a function of its entropy, volume, and number,

\[ S_{\text{tot}} = \frac{-F(N_A, V_A, T_A)}{T_A} - \frac{F(N_B, V_B, T_B)}{T_B}, \quad (2.2) \]

where \( N \) is the number, \( V \) is the volume, and \( F \) is the Helmholz free energy. With the total number of helium atoms fixed, \( N = N_A + N_B \), its derivative is

\[ \frac{\partial S_{\text{tot}}}{\partial N_A} = -\frac{\mu_A}{T_A} + \frac{\mu_B}{T_B}, \quad (2.3) \]

where \( \mu \) is the chemical potential. The maximum total entropy occurs when this is zero, which gives the condition for the optimum steady state as

\[ \frac{\mu_A}{T_A} = \frac{\mu_B}{T_B}. \quad (2.4) \]

Since the fugacity is \( z = e^{\beta \mu} \), where \( \beta = 1/k_B T \), this is equivalent to \( z_A = z_B \) and it can be called the constant fugacity condition.

Measurements of the fountain pressure involve two closed chambers held at different temperatures by a heater and a cooler. Hence it is a non-equilibrium steady state system. Usually maximization of the first entropy plays no direct role in determining the optimum state of such systems (Attard 2012, 2023 chapter 3). Also, there is nothing specifically fluid about this result.

The energy is also an extensive variable, and the second possible axiom is that the total energy is a minimum. The energy of each chamber is a function of its entropy, volume, and number, \( E_{\text{tot}} = E(S_A, V_A, N_A) + E(S_B, V_B, N_B) \) (see sections 2.2.4 and 2.3.4 of Attard (2023)). In this case the derivative at fixed total \( N \) is

\[ \frac{\partial E_{\text{tot}}}{\partial N_A} = \mu_A - \mu_B. \quad (2.5) \]

The energy is minimized (Attard 2023 section 9.4.5) when

\[ \mu_A = \mu_B. \quad (2.6) \]

For the physical interpretation of this result, it is important to note that the derivative is at constant entropy. There is currently no principle of energy minimization in thermodynamics or statistical mechanics. In mechanics, the force points toward the potential energy minimum (Newton 1687), but even in this case the total energy is constant on a trajectory. Further, mechanical laws on their own have no direct application to thermodynamic or statistical systems.

This same result can be obtained by minimizing the simple sum of free energies, Helmholtz or Gibbs. But it is difficult to justify simply adding them together without dividing each by its respective chamber temperature.

The chemical potential is the Gibbs free energy per particle, \( \mu = G(N, p, T)/N \) (Attard 2002). The derivative of equation (2.6) with respect to \( T_B \) at constant pressure and temperature of the first chamber, and number of the second, is (Attard 2002, 2023 section 2.3.1)

\[ 0 = \frac{d(G_B/N_B)}{dT_B} = \frac{\partial g_B}{\partial T_B} + \frac{\partial g_B}{\partial p_B} \frac{dp_B}{dT_B} = -s_B + v_B \frac{dp_B}{dT_B}, \quad (2.7) \]

where \( g \), \( s \), and \( v \) are the Gibbs free energy, entropy, and volume per particle, respectively. This is the same as H. London’s expression, equation (2.1). (Since \( T_B = T_A \) implies \( \mu_B = \mu_A \), the constant of integration must be zero.) Therefore equality of chemical potential, equation (2.6), is thermodynamically equivalent to H. London’s expression for the fountain pressure, equation (2.1).  

B. Relationship between the two principles

Differentiating the condition of constant fugacity, equation (2.3), gives

\[ 0 = \frac{1}{T_B} \frac{d\mu_B}{dT_B} - \frac{\mu_B}{T_B^2}. \quad (2.8) \]

On the saturation curve the chemical potential divided by temperature must be small and negative because the liquid is in equilibrium with a gas. Using measured values for the enthalpy and the entropy for \(^4\text{He} \) (Donnelly and Barenghi 1998), the value at \( T = 1 \text{ K} \) is \( \beta u_{\text{sat}} = -1.26 \times 10^{-3} \), and at \( T = 2.15 \text{ K} \) it is \( \beta u_{\text{sat}} = -1.05 \times 10^{-1} \). The fugacity for bosons must be bounded above by unity, \( z < 1 \), otherwise the denominator of the momentum state distribution would pass through zero. (For the case of ideal bosons, F. London (1938) showed that \( z^{\text{id}} \rightarrow 1 \) below the \( \lambda \)-transition (Attard 2022a, 2023 section 8.1).) Since the compressibility is positive, and since the fountain pressure is greater than the saturation pressure, on a fountain path one must have
\[ \mu_B^{\text{sat}} \leq \mu_B < 0, \text{ or} \]
\[ -1 \ll \beta_B \mu_B < 0. \quad (2.9) \]

This result may be confirmed using measured fountain pressures (Hammel and Keller 1961) and saturation data (Donnelly and Barenghi 1998).

This reduces the derivative above to
\[ \frac{d\mu_B}{dT_B} = \mathcal{O}(10^{-3})k_B. \quad (2.10) \]

This says that the condition of constant fugacity is equivalent to the condition of constant chemical potential to within about one part in one thousand. Closer to the \( \lambda \)-transition the difference is about one part in ten. One can see from this that there is some question whether current measurements have the accuracy to distinguish the two principles.

In the present case it may be said that the condition of chemical potential equality is a rigorous mathematical consequence of energy minimization at constant entropy. But not all mathematical results have physical relevance. Given the fact that energy minimization plays no role in the usual thermodynamic systems, if it is indeed the underlying principle for superfluid flow then a physical argument or explanation is required (see the conclusion).

Equally, it must be conceded that there is no reason to believe that maximizing the entropy is a principle relevant to the present non-equilibrium steady state system (Attard 2012, 2023 chapter 3).

The tests against measured data (see section III), and the arguments that can be made (see the conclusion), both favor the principle of energy minimization. Perhaps in time it will become axiomatic that this is the principle that drives superfluid flow.

C. Inconsistency in the derivation of the H. London expression

In addition to the previous criticisms of H. London’s (1939) derivation (Attard 2022d appendix A), here is a new argument that it is unsound. The point is not whether his expression is correct, but whether it has been mathematically proved to be an exact result.

An essential step in H. London’s (1939) derivation is where he invokes the Nernst heat theorem to conclude that the enthalpy divided by temperature and the entropy both go to zero at absolute zero. These mean that the chemical potential must go to zero at absolute zero. H. London (1939) also assumes that there exists a continuous fountain path to absolute zero from an arbitrary thermodynamic point.

Since the H. London (1939) expression implies that the chemical potential is constant on the fountain path, equation (2.6), these assumptions imply that it is zero everywhere on the fountain path, which is demonstrably false for saturated \(^4\)He. It follows therefore that if the H. London (1939) expression is exact, then either the Nernst heat theorem is invalid, or else that there does not exist a fountain path to absolute zero, or both. In any case, the failure of either of these assumptions renders the derivation of H. London (1939) invalid.

Without assessing the Nernst heat theorem in detail, I note that if H. London (1939) is correct in that it implies that the chemical potential divided by temperature vanishes at absolute zero, then it would imply that the fugacity equaled unity at absolute zero. But the fugacity is bound to be strictly less than unity, at least for a system with single particle energy states (F. London 1938, Attard 2022a, 2023 section 8.1). Contrariwise, the number of bosons would be infinite. Since \(^4\)He is dominated by ideal statistics deep below the \( \lambda \)-transition, (see sections 9.1 and 9.2 of Attard (2023)), a finite-sized system must arguably violate the Nernst heat theorem.

Similarly, on a fountain path to absolute zero, situating the high temperature chamber on the saturation curve would require a negative pressure in the other chamber at absolute zero. (The saturated vapor pressure of the high temperature chamber is relatively low, but it must still be much greater than the pressure in the chamber at absolute zero.) Conversely, if one insists upon a stable thermodynamic state at absolute zero, then this places a lower bound on the pressure in the high temperature chamber that would exceed the saturation pressure. In other words, not all thermodynamic state points lie on a fountain path to a stable point at absolute zero.

III. COMPARISON WITH EXPERIMENT

A. Three expressions and dubious data

In practice in experimental application the H. London (1939) expression for the derivative of the fountain pressure, equation (2.1), is integrated along the saturation curve (Hammel and Keller 1961),

\[ p_B - p_A = \int_{T_A}^{T_B} dT' \rho^{\text{sat}}(T')s^{\text{sat}}(T'). \quad (3.1) \]

Strictly speaking, the integral for the fountain pressure should be evaluated on the fountain path rather than the saturation path. But corrections for this effect involve the thermal expansivity, \( \alpha \sim \mathcal{O}(10^{-3}) \), and amount to only about \(-0.5\%\) at the highest fountain pressures (Attard 2022d).

The second equation for the fountain pressure comes from the equality of chemical potential, equation (2.6). This can be used to obtain the fountain pressure by writing \( \mu_B = \mu_B^{\text{sat}} + (p_B - p_B^{\text{sat}})\rho_B^{\text{sat}} \), which holds for an incompressible liquid. Rearranging gives

\[ p_B - p_A \approx p_B^{\text{sat}} - p_A + p_B^{\text{sat}}(\mu_B - \mu_B^{\text{sat}}) \]
\[ = p_B^{\text{sat}} - p_A + p_B^{\text{sat}}(\mu_A - \mu_B^{\text{sat}}). \quad (3.2) \]
Invariably the experimental measurements are performed at saturation of chamber A, and so all quantities on the right hand side, including \( \mu_s = \Delta h_{sat} - T_s \Delta s_{sat} \), can be obtained from standard tables such as those given by Donnelly and Barenghi (1998). It is emphasized that any difference between the fountain pressure given by equation (3.1) and that given by equation (3.2) must be due to experimental error, and the difference between them gives a guide to the reliability of the measurements.

The third equation for the fountain pressure comes from the equality of fugacity, equation (2.4). Again using the incompressible liquid expression for the departure from the equality of fugacity, equation (2.4). The calculated curves use the enthalpy (table 7.6) and entropy (table 8.5) obtained from measured heat capacity data by Donnelly and Barenghi (1998).

Figure 1 compares the three expressions for the fountain pressure (Attard 2012, 2023 section 2.4.2)

\[
C_p = -\frac{1}{T^2} \frac{\partial^2 (\overline{G}(N, p, T))/T}{\partial (1/T)^2} = \frac{\partial\overline{H}(N, p, T)}{\partial T} = T \frac{\partial \overline{S}(N, p, T)}{\partial T}. \tag{3.4}
\]

The second equality can be seen by dividing both sides of the definition of the unconstrained Gibbs free energy, \( G(E, V|N, p, T) = E + pV - TS(E, V, N) \), by \( T \) and differentiating with respect to \( 1/T \), holding as usual \( E \) and \( V \) fixed. The third equality follows by expressing the first equality as a derivative with respect to \( T \).

In practice measurements are made along the saturation curve, \( p_{sat}(T) \). I believe that the change in enthalpy, \( \Delta H = H(N, p_{sat}(T_2), T_2) - H(N, p_{sat}(T_1), T_1) \), is measured as the energy input. At constant number,

\[
C_{sat} = \frac{\partial \overline{H}(N, p, T)}{\partial T} \bigg|_{N} = \frac{\partial \overline{H}(N, p, T)}{\partial T} + \frac{\partial \overline{H}(N, p, T)}{\partial p} \frac{\partial p_{sat}(T)}{dT} = C_p + T \frac{\partial p_{sat}(T)}{dT}. \tag{3.5}
\]

This heat capacity at constant saturation is larger than that at constant pressure. I believe this to be the quantity \( C_s \) in table 7.4 of Donnelly and Barenghi (1998).

From this one sees that the difference in enthalpy on the saturation curve is

\[
H(N, p_{sat}, T) - H(N, p_{0sat}, T_0) = \int_{T_0}^{T} dT' C_{sat}(T'). \tag{3.6}
\]

This contradicts the expression in note 11 to section 7 of Donnelly and Barenghi (1998), who appear to have mixed up \( C_p \) and \( C_{sat} \). The results they present for the enthalpy in their table 7.6 have a relative systematic error of \( O(10^{-2}) \). Although this is comparable to the random measurement and fitting error, because it is a systematic error, and because the chemical potential is the difference

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**Figure 1:** Measured and calculated fountain pressure for \( T_A = 1.502 \) K (left), 1.724 K (middle), and 1.875 K (right). The symbols are measured data (Hammel and Keller 1961). The short dashed curve is the saturation line integral form of the H. London (1939) expression, equation (3.1), the dotted curve is for fixed chemical potential, equation (3.2), and the long dashed curve is for fixed fugacity, equation (3.3). The calculated curves use the enthalpy (table 7.6) and entropy (table 8.5) obtained from measured heat capacity data by Donnelly and Barenghi (1998).
between two comparable quantities, it leads to errors on the order of 5% in the fountain pressure (figure 1).

The temperature derivative of the entropy along the saturation curve at constant number is

\[
\frac{dS(N,p,T)}{dT} = \frac{\partial S(N,p,T)}{\partial T} + \frac{\partial S(E,V,N)}{\partial p} \frac{dp^{\text{sat}}(T)}{dT} \\
= \frac{1}{T} C_p - \frac{\partial V(N,p,T)}{\partial T} \frac{dp^{\text{sat}}(T)}{dT} \\
= \frac{1}{T} C_p - \frac{\alpha(V(N,p,T))}{\partial T} \frac{dp^{\text{sat}}(T)}{dT} \\
= \frac{1}{T} C^{\text{sat}} - N \rho^{-1} \left[ \frac{1}{T} + \alpha \right] \frac{dp^{\text{sat}}(T)}{dT}.
\]

(3.7)

Accordingly the difference in entropy is

\[
S(N,p^{\text{sat}}(T),T) - S(N,p^{\text{sat}}(T_0),T_0)
= \int_{T_0}^{T} \frac{1}{T'} \left\{ C^{\text{sat}}(T') - \frac{N}{\rho'} \left[ 1 + \alpha'T' \right] \frac{dp^{\text{sat}}(T')}{dT'} \right\} dt.
\]

This contradicts the expression in note 8 to section 11 of Donnelly and Barenghi (1998), which neglects the second term in the braces. Compared to the present expression, the results for the calorimetric entropy in their table 8.5 have a relative systematic error of \(\mathcal{O}(10^{-2})\), with a similar error for the fountain pressure when used in the integral form of the H. London (1939) expression, equation (3.1) (figure 1).

Figure 2 tests the various equations for the fountain pressure against the measured values (Hammel and Keller 1961). The calculations use the measured data for the heat capacity at constant saturation, \(C^{\text{sat}}\) (Donnelly and Barenghi 1998 table 7.4), from which the enthalpy, equation (3.6), and the entropy, equation (3.8), and the chemical potential, \(\mu = h - Ts\), are obtained. This entropy was also used for the integrated H. London (1939) expression, equation (3.1).

It can be seen the fountain pressure predicted by the H. London expression evaluated as an integral along the fountain curve, equation (3.1), and that evaluated by equal chemical potential, equation (3.2), are virtually indistinguishable. This confirms the thermodynamic equivalence of the two, the validity of the thermodynamic analysis that corrects the results of Donnelly and Barenghi (1998), and the reliability of the experimental data when analyzed correctly.

Hammel and Keller (1961) estimated the error in their fossil pressure measurements as about 2% and found that the predicted fountain pressure, equation (3.1), using calorimetric entropy values (Kramers et al. 1951, Hill and Lounasmaa 1957) agreed within the error.

For the heat capacity \(C_s = C^{\text{sat}}\), Donnelly and Barenghi (1998 table 7.4) give the measurement error as 1–3% and the fitting error as 1–2%.

The values of the fountain pressure predicted by equality of fugacity, equation (3.6), lie systematically below the measured values. The difference is on the order of 3–5%, which appears significant compared to the various measurement errors. On the basis of the results in figure 2 one can tentatively conclude that the measured data favor the principle of energy minimization at constant entropy for superfluid flow, and that they likely rule out the principle of entropy maximization.

C. Convective flow

In the fountain effect with closed chambers a non-equilibrium steady state exists with viscous flow of He I from the high to the low temperature chamber and superfluid flow of He II in the other direction to maintain mass balance. These flows occur simultaneously in the same capillary. There is also net energy flow from the high temperature chamber to the low. The flow of He I is viscous Poiseuille flow driven by the pressure difference (but see next) and it carries the energy convectively (F. London and Zilsel 1948, Keller and Hammel 1960). The superfluid flow of He II arriving in the high temperature chamber is in total equal and opposite to the viscous flow of He I leaving it. A similar balance but in the opposite direction occurs for the low temperature chamber. In the steady state the total number of \(^4\text{He}\) atoms in each of the two closed chambers is conserved. There is of course a net energy flux between the two chambers, with energy supplied by a heater in one and removed by a refrigerator in the other.

In normal convective flow the two species are hot and
cold particles and their spatially distinct flows are driven by respective entropy gradients. One would expect similar gradients in the present fountain system. The evidence is that \( \mu_A = \mu_B < 0 \) (figure 2). Since \( T_B > T_A \), this means that \((-\mu_A/T_A) > (-\mu_B/T_B)\), which means that there is an entropy gradient that drives number from \( B \) to \( A \). (Recall \( \partial S(E,V,N)/\partial N = -\mu/T \).) This is what really drives the viscous Poiseuille flux of normal He, \( J_v \), from \( B \) to \( A \).

But what drives the steady flow of condensed He from \( A \) to \( B \)? Recall that when \( \mu_A = \mu_B \) the energy is minimized and there is no driving force. One concludes that there must strictly be a gradient with \( \mu_B < \mu_A \), and, to linear order, \( J_0 = c_2(\mu_A - \mu_B) \). In this case, then strictly speaking the measured fountain pressure should lie between that predicted by constant chemical potential and that predicted by constant fugacity. The fact that the measured fountain pressure lies so close to the prediction from \( \mu_A = \mu_B \) indicates that superfluid flow is extremely efficient at eliminating gradients, \( c_2 \gg \beta_A J_v \).

It seems likely that the thinner the capillary, and the lower the temperature difference, the closer to equality would be the chemical potentials (because the Poiseuille flow is reduced, and a smaller balancing superfluid flow requires a smaller energy gradient). It would appear that one needs a wide slit and better than the 2% precision of current measurements to confirm or refute the hypothesis \( \mu_B < \mu_A \) (equivalently, \( p_B^{\text{meas}} < p_B^{\text{H.London}} \)).

In the experiments of Keller and Hammel (1960), the mean velocity of the viscous flow of He I in the case of the greatest fountain pressure is on the order of 60 times the critical velocity for superfluid flow predicted by the momentum gap for the slit by the formula given in Attard (2023 section 9.3.4). Assuming a comparable speed for the condensed bosons, this suggests that collisions are strong enough to convert a proportion of the back flow of superfluid He II to viscous He I. But in the case of the fountain effect, this does not block the capillary because the fountain pressure is so large that substantial Poiseuille flow continues.

**IV. CONCLUSION**

Experimental data is no substitute for a mathematic derivation when it comes to proving that a result is exact. But experimental data has historically been used to formulate general scientific principles that can then be used axiomatically to derive exact and approximate expressions to describe that and other data. It is usually the case that such principles gain acceptance over time when no contradictory evidence emerges, when they explain a range of physical phenomena, and when scientists become familiar with them. A rationale for the principle of energy minimization at constant entropy for superfluid flow is now offered.

It was shown in section II that H. London’s (1939) expression for the temperature derivative of the fountain pressure corresponds to chemical potential equality of the two chambers. It was also shown that equal chemical potential minimizes the energy at constant entropy.

Equivalence of chemical potential offers a thermodynamic mechanism for how the fountain pressure arises. Imagine that initially the high temperature chamber is at saturation, \( \mu_B^{\text{ini}} = \mu_B^{\text{sat}}(T_B) \). The low temperature chamber is always at saturation, \( \mu_A = \mu_A^{\text{sat}}(T_A) \). For \(^4\text{He}\), on the saturation curve the chemical potential from the measured enthalpy and entropy (Donnelly and Barenghi 1998) decreases with increasing temperature, \( d\mu^{\text{sat}}/dT < 0 \). Hence \( \mu_B^{\text{sat}} < \mu_A^{\text{sat}} \). Since \( \partial\mu/\partial p = v > 0 \), the only way that the chemical potential of the high temperature chamber can be increased to achieve equality is by increasing the pressure beyond its saturation value. This occurs as more \(^4\text{He}\) arrives in the second chamber because each atom occupies a certain impenetrable volume (i.e. the compressibility is positive). Given the goal of equalizing the chemical potentials, these are the reasons why condensed \(^4\text{He}\) initially flows down the chemical potential gradient from the low temperature chamber to the high temperature chamber, and why the high temperature chamber subsequently settles at a higher pressure.

The H. London (1939) expression, and the thermodynamically equivalent equality of chemical potential, agree with the measured values for the fountain pressure within experimental error, figure 2. This strongly suggests that energy minimization is the principle that determines superfluid flow. This raises \( O(10^3) \) questions: Why is the principle of entropy maximization inapplicable? Why is energy minimization the operative principle? And why is it at constant entropy?

In sections 9.1 and 9.2 of Attard (2023), it is shown that the far side of the \( \lambda \)-transition is dominated by the non-local permutation entropy of bosons in multiply-occupied momentum states. This entropy provides a barrier that suppresses momentum-changing collisions (Attard 2022a). Conversely, flow without such collisions conserves the permutation entropy. These results show that superfluid flow is flow at constant permutation entropy. And in so far as permutation entropy dominates the entropy of condensed \(^4\text{He}\), we may say that it is flow at constant entropy. That superfluid flow does not change entropy explains why the experimental data rules out conventional entropy maximization as the underlying superfluid principle, figure 2.

Now consider a condensed boson (i.e. one in a multiply-occupied momentum state) traversing the capillary from \( A \) to \( B \). It must do so adiabatically (i.e. with fixed total energy), since it experiences no momentum-changing collisions. Since it is condensed, this boson carries only mechanical energy \( \mu_A \), which is given by the derivative at constant entropy, \( \partial E(S,V,N)/\partial N = \mu \). In contrast, \( \partial E(N,V,T)/\partial N = \mu - T\partial\mu/\partial T \), which has an entropic component. On arriving in \( B \), our boson equilibrates via collisions to the temperature \( T_B \), the spacing of transverse momentum states being much smaller in the chamber than in the capillary (Attard 2022d). The change
in entropy of chamber $B$ and its thermal reservoir is the sum of that due to the change in energy, $\mu_A/T_B$, and that due to the change in number, $-\mu_B/T_B$. The change in entropy of chamber $A$ and its thermal reservoir is the sum of that due to the change in energy, $-\mu_A/T_A$, and that due to the change in number, $\mu_A/T_A$, which cancel. Hence the total change in total entropy upon transfer of a condensed boson from $A$ to $B$ is $(\mu_A - \mu_B)/T_B$. In the initial transient phase of the fountain effect, $\mu_A > \mu_B$ (see above), and so the change in the entropy of the universe is positive. One concludes that the ultimate driving force for the fountain effect, and superfluid flow more generally, is entropy creation. Who knew?!

In summary, superfluid flow transports energy adiabatically from regions of high to low specific mechanical energy, dissipating there the excess and creating entropy. The flow is adiabatic because superfluid flow is collisionless (Attard 2022a, 2023 chapter 9). This explains the principle of energy minimization as the immediate driving force for superfluid flow.

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