Critical behaviour for scalar nonlinear waves

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In the long-wave regime, nonlinear waves may undergo a phase transition from a smooth to a fast oscillatory behaviour. We show that this phenomenon, commonly known as dispersive shock, shares many features of the tri-critical point in statistical systems and we build a dictionary between nonlinear waves and statistical mechanics. We provide a classification of Universality classes and the explicit description of the transition by means of special functions, extending Dubrovin’s universality conjecture [1, 2] to a wider class of equations.

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Dispersive shock waves in 1+1 dimensions are a class of shock waves recently observed in a variety of physical situations in which the media are dispersive or not-strictly diffusive. Examples include plasma physics [3], Bose-Einstein condensates [4], nonlinear optics [5, 6] and hydrodynamics [7, 8]. Through the shock, the waves experience an abrupt phase transition from a regular to a rapidly-oscillatory behavior, and the transition is conjectured to be universal [1]. Evidently, this terminology comes from statistical mechanics but while the connection between dissipative shocks and statistical models is known (see [11, 12] and references therein), an analogue description of the dispersive case is still missing.

We consider a very general model equation for 1-dimensional scalar unidirectional waves in fluid of the form

\[ u_t + a(u)u_x + N[u] = 0, \tag{1} \]

where \( a(u) \) is a non constant function (in most relevant cases \( a(u) = u \)) and \( N \) is a nonlinear - and possibly non-local - differential operator, which models the phenomena into examination by taking into account the relevant physical effects like dispersion, dissipation, pressure, the interfacial interaction between two different fluids, etc. The critical behaviour arises when we consider solutions that at time \( t = 0 \) vary on a large scale (compared to the natural scale of the system), say \( 1/\varepsilon \) with \( \varepsilon \) small, and study whether at a later time fluctuations on a smaller scale arise. We assume that the nonlinear operator admits a long-wave expansion

\[ N[u(x/\varepsilon)] = \varepsilon^{\beta+1} N[u(x)] + o(\varepsilon^{\beta+1}), \quad \beta > 0, \]

so that in the long-wave regime, after the change of variables \( x \to x/\varepsilon, t \to t/\varepsilon \), the wave satisfies the rescaled equation

\[ u_t + a(u)u_x + \varepsilon^\beta N[u(x)] = 0, \quad \varepsilon \to 0, \]

with initial data independent on \( \varepsilon \). If the latter equation is well-posed, then in this regime (1) is a small perturbation of the scalar conservation law (or Hopf equation)

\[ u_t + a(u)u_x = 0, \quad \text{at least as long as the wave remains smooth}. \]

The Hopf equation describes a wave where every particle on the profile travels with constant velocity \( a(u) \), i.e. the solution is constant along the characteristic lines

\[ x(t; x_0) = x_0 - a(\varphi(x_0))t, \quad u(x(t; x_0)) = \varphi(x_0). \tag{2} \]

Up to the critical time \( t_c \) all lines are distinct and the solution is uniquely determined, while after it the lines start to intersect and the wave develops a shock - a point with vertical derivative. Close to and after the shock, the term \( N[u] \) is not anymore negligible and it makes the wave fluctuate on a smaller scale, provided the perturbation is not strictly dissipative. To understand this transition, we look at the well understood case of the dispersionless (or semiclassical) limit of Korteweg-de Vries (KdV) [13, 14, 15]

\[ u_t + uu_x - \varepsilon^2 u_{xxx} = 0, \quad \varepsilon \to 0; \quad u(x, t = 0, \varepsilon) = \varphi(x). \]

We assume the initial data to be smooth, positive, fast decaying and with a single hump. The \( (x, t) \)-plane is divided in two zones (see Figure 1(a)): the semiclassical zone, which is the union of the points belonging to a single characteristic line, and its complement, the Whitham zone. In the semiclassical zone the limit \( \lim_{\varepsilon \to 0} u(x(t, \varepsilon)) = u(x, t) \) exists and corresponds to the

| Wave equation in 1+1 dimensions | Statistical Model |
|------------------------------|------------------|
| Long-wave limit | Thermo-dynamic Limit |
| Critical Point of Gradient Catastrophe | Tricritical Point |
| Wave Amplitude in the Whitham Zone | Order Parameter |
| Unfolding of Cubic Singularity | Mean Field of \( a^\beta \) model |
| Monocline at the critical point | Renormalization group flow |
| Scaling Linear perturbations of Hopf | Fixed point of renormalization group |
solution of the Hopf equation. In the Whitham zone, \( u(x,t,\varepsilon) \) develops oscillations of vanishing wavelength \( O(\varepsilon) \) (see Figure 1(b)) and the limit exists only in a weak sense: there exists a function \( \tilde{u}(x,t) \) (see Figure 1(c)) which averages the oscillations, uniquely defined by the weak limit

\[
\lim_{\varepsilon \to 0} \int \psi(x)u(x,t,\varepsilon)dx = \int \psi(x)\tilde{u}(x,t)dx,
\]

for any test function \( \psi(x) \).

To better understand the transition we introduce an order parameter \( W(x,t) \) which measures the amplitude of the oscillations in the Whitham zone

\[
W(x,t) = \lim_{\varepsilon \to 0} \sup \{u(x,t,\varepsilon) - \varpi(x,t)\}.
\]

In the KdV case the function \( W \), shown in Figure 1(d), can be computed exactly [16]. Let us fix \( t \) at a value bigger than \( t_c \). The Whitham zone then is an interval \( (x_-(t),x_+(t)) \) of the real line. The order parameter \( W(x,t) \) is zero outside that interval, it behaves like \( W(x,t) \sim 1/\log(x_+(t) - x) \) close to the right boundary, while it is discontinuous at the left boundary: \( \lim_{x \to x_-} W(x,t) > 0 \). Therefore, the solution undergoes a second order phase transition at \( x = x_+ \) – the order parameter is continuous but not differentiable – and a first order phase transition at \( x = x_- \) – the order parameter is discontinuous. The boundary of the Whitham zone, given parametrically by \( (x(z),t(z)) = (z - \varphi(z),\frac{1}{\varphi(z)}) \), is made of a curve of second order phase transitions and a curve of first order phase transitions, which meet at a point \( (x_c,t_c) \), which is therefore a tricritical point [17] (see Figure 1(a)).

In what follows we investigate the local behaviour of solutions close to the tricritical point for a general PDE [1], we argue that it is universal and we characterize the universality classes (to avoid a cumbersome notation and since the final result is independent on \( a \), we stick with \( a = u \). The few modifications needed in formulas below can be found in [1]. For the Hopf equation, the critical point \( x_c,t_c \) is the point where the wave breaks and the solution becomes multivalued, a singular behavior known as gradient catastrophe. It is well known that the generic singularity is a cubic one. Indeed, it follows from [2] that solutions can locally be expressed by the implicit formula

\[
u(x - \varphi(x)t) = \varphi(x) \quad \text{or} \quad x - ut = f(u), \quad f = \varphi^{-1};
\]

if we let \( u_c = u(x_c,t_c) \) and suppose \( f''(u_c) \neq 0 \), then we can introduce the scale variables

\[
X = \frac{x - x_c - u_c(t - t_c)}{\lambda}, \quad U = \left(\frac{\gamma}{6}\right)^{1/3} \frac{u - u_c}{\lambda^2},
\]

\[
T = \left(\frac{6}{\gamma}\right)^{1/3} \frac{(t - t_c)}{\lambda^2},
\]

with \( \gamma = -f''(u_c) > 0 \) and \( \lambda \) a small parameter, to get for small \( \lambda \to 0^+ \)

\[
X - UT + U^3 = 0,
\]

which is the miniversal unfolding of the cubic singularity. A good picture of the transition can be derived from [3]. If \( \beta_1 \geq \beta_2 \geq \beta_3 \) are the three roots of \( U = 0 \) for \( T > 0 \), then \( W_{MF}(x,t) = 2(\beta_1(x,t) - \beta_2(x,t)) \) measures the envelope of the solution. This has the same phase diagram and qualitative behaviour of the exact order parameter KdV, but a different exponent \( W_{MF} \sim (x_+(t) - x)^{1/2} \). This Mean Field description of the phase transition gives the correct behavior but discards the true nature of the oscillations. It shows the same structure of the \( \varphi^0 \) mean field theory [17].

Generalizing a procedure described in [1], we are able to give for quite a general perturbation \( N \) a much finer description of the tri-critical phase transition, which takes into account the precise nature of the oscillations and which is remarkably universal. In fact, the local behaviour of \( u \) around the tricritical point is uniquely characterized by the linearization of \( N \) at the constant function \( u = u_c \):

\[
\mathcal{N}[u_c + \delta u] = T_{u_c}[\delta u] + O((\delta u)^2).
\]

First of all, let us change variables as in [4] but with the scaling parameter \( \lambda = \varepsilon^{\frac{1}{3}} \) depending on \( \varepsilon \), then the wave equation 1 reduces to

\[
U_T + UU_X + \varepsilon^{\frac{\alpha - 1}{3}} L_{u_c}[U] + \text{higher order terms} = 0.
\]

The balance of the intrinsic and extrinsic scales \( \varepsilon, \lambda \) is achieved when \( \alpha = 1 + \frac{1}{33} \), or equivalently if \( u \) admits

\[
\frac{\alpha}{3}.
\]
Critical universality classes are thus characterized by

\[ u(x, t, \varepsilon) \simeq u_c + \varepsilon^{1/\alpha} U \left( \frac{x - x_c - u_c(t - t_c)}{\varepsilon^{\beta/\alpha}}, t - t_c \right) + O(\varepsilon^{2\beta/\alpha}), \]

and the leading term \( U \) of \( \tilde{U} \) is a solution of the linearized perturbation

\[ U_T + UU_X + \tilde{L}_{u_c}[U] = 0. \tag{7} \]

The local behavior close to the tricritical point emerges on a meso-scale \( \varepsilon^{1/\alpha} \), lying between the microscopic \( O(\varepsilon) \) scale and the macroscopic one \( O(\varepsilon^0) \), and on this scale just universal corrections matter. This is the exact analog of the renormalization group in statistical mechanics, where universality arises by magnifying the theory at the meso-scale where block spin or phase-space renormalization is performed \[15\].

Before describing the distinguished solution \( U(X, T) \) of \eqref{7} which gives the universal correction at the tricritical point, we consider the classification of Universality Classes. Since any positive constant in front of \( L_c \) can be factored out trivially, we say that two nonlinear PDEs \( N, N' \) belong to the same universality class if \( \tilde{L}_{u_c} = \tilde{L}_{u_c} \) up to a (positive) scalar multiple. The Classification of Universality Classes follows easily from the observation that, since \( N[u(x/\varepsilon)] = \varepsilon^{\beta+1} \tilde{N} + o(\varepsilon^{\beta+1}) \), then for any \( u_c \)

\[ \tilde{L}_{u_c}[u(x/\varepsilon)] = \varepsilon^{\beta+1} \tilde{L}_{u_c}[u(x)]. \]

If we assume that \( \tilde{N} \) is translationally invariant, then classifying universality classes is tantamount to classify translational invariant scaling pseudo-differential operators. The general operator of this class can be written as \( L[U](x) := \int_{-\infty}^{+\infty} e^{ipx} m(p) \hat{U}(p) dp \), where \( \hat{U} \) is the Fourier transform of \( U \) and the function \( m \) is the symbol (or Fourier multiplier) of the operator. In order for the operator to define a meaningful evolution, it must map real functions into real functions and it must be dissipative (the dissipation possibly vanishing, i.e. \( \int U(X)L[U(X)]dX \geq 0 \)); the two conditions read \( m(-p) = m^*(p), Re(m(p)) \geq 0 \). The scaling property required above further constrains the symbol to be of the form \( m(p) = \kappa|p|^{\beta+1+i\theta} \), for some \( (\kappa, \theta) \in \mathbb{R}^2 \setminus \{0\} \), \( \kappa > 0 \), and \( \beta > 0 \). Explicitly, we have:

\[ U_T + UU_X + \int_{-\infty}^{+\infty} e^{ipX} (\kappa + i\theta \operatorname{sign}(p)) |p|^\beta \hat{U}(p) dp = 0. \tag{8} \]

Critical universality classes are thus characterized by a pair of parameters \( (\alpha, \beta) \). Since the transformation \( U(X, T) \rightarrow U(-X, T) \) sends \( \theta \) to \(-\theta \), we can assume \( \theta \geq 0 \). Notice that if \( \theta = 0 \) then the perturbation is purely dissipative while if \( \kappa = 0 \) it is dispersive and possesses the Hamiltonian \( H[U] = \int_{-\infty}^{+\infty} U^3 - \theta U K[U]dX \), where \( K[U] = \int e^{ipX}|p|^{\beta} \hat{U}(p) dp \).

**Example 1** Conservation laws \( u_t + \partial_x f(u, u_x, \ldots) = 0 \), with \( f \) some smooth function, admit a long-wave regime with \( \tilde{N}[u(x)] = \partial_x(n(u)u_x) \), where \( n(u) = \frac{\partial}{\partial u} u_x(x_0(u), x_1(u), \ldots) \). Provided \( n(u_c) \neq 0 \), then \( \beta = 1 \) and \( L_n = n(u_c)u_x \). The universal model for these equations is thus the Burgers equation:

\[ U_t + U U_x + n(u_c) U_{xx} = 0. \]

The critical behaviour for these conservation laws is typical of dissipative shocks and it has been considered in \[2, 17\].

**Example 2** Local Hamiltonian PDEs are equations in the form \( u_t = \frac{1}{2\pi} \int h(u, u_x, \ldots)dx \), for a smooth function \( h \) s.t. \( h(0, 0, \ldots) = 0 \). They admit a long-wave expansions with \( \tilde{N}[u] = \partial_x(b(u)u_x^2 + 2b(u)u_{xx}) \), \( b(u) = \frac{\partial}{\partial u} u_x(x_0(u), x_1(u), \ldots) \). Provided \( b(u_c) \neq 0 \), then \( \beta = 2 \) and \( L_n = b(u_c)u_{xxx} \). Therefore the universal model for these equations is KdV:

\[ U_t + U U_x + b(u_c) U_{xxx} = 0. \]

The critical behaviour for this class has been considered in \[7\].

**Example 3** The Benjamin-Ono (B-O) equation \[20, 21\]

\[ u_t + u u_x - \mathcal{H}[u_{xx}] = 0, \]

where \( \mathcal{H} \) is the Hilbert transform: \( \mathcal{H}[u](x) = \frac{1}{\pi} \int \frac{n(u)}{x-y}dy \) is an integrable Hamiltonian equation as KdV, but it is non-local. It is already in the long-wave form \[8\], with \( \beta = 1, \kappa = 0 \) and \( \theta = 1 \). Thus, \( \tilde{N} = L = \mathcal{H}[u_{xx}] \). It has the same exponent as Burgers but being Hamiltonian like KdV its solutions undergo a dispersive shock \[22\]. It corresponds therefore to a novel universality class, which we name Benjamin-Ono universality class. All equations of B-O hierarchy (see \[23\] for the definition) belong to the B-O universality class.

**Example 4** The Camassa-Holm equation \[24\]:

\[ u_t - u_{xxx} + u u_x = 2 \frac{3}{2} u_x u_{xx} + \frac{1}{3} u u_{xxx}, \]

and the Benjamin-Ono-Mahony equation \[25\]

\[ u_t - u_{xxx} + u u_x = 0 \]

can be written in the standard form \[1\] by inverting \( 1 - \partial_x^2 \). Provided \( u_c \neq 0 \), they also belong to the KdV universality class \[1\].

To compute the universal correction \( U(X, T) \), we generalize an argument by Dubrovin \[1\] and developed mathematically by the authors \[26\] (see also \[19, 27\]). We argue that \( U(X, T) \) satisfies a (possibly infinite) deformation of the cubic equation \[3\], known as string equation. First note that the rescaled function \( U^\mu(X, T) = \frac{U(X, T)}{N[X]} \)
\(\mu^{3+1} U(X/\mu^{9+3}, T/\mu^{6+4})\) satisfies \(U_T = U U_X + \mu L[U]\), and \(U^0\) is the solution of the cubic equation \([5]\). Therefore we can expect the cubic equation to be just the first term of an expansion in power of \(\mu\)

\[X - U^\mu T + (U^\mu)^3 = O(\mu)\,.
\]

Moreover, note that the unique real solution \(U^0(X,T)\) (for \(T < 0\)) of the cubic equation is the unique stationary point, vanishing at \(X = T = 0\), of the flow

\[U_S = \partial_X (X - UT + U^3)\,.
\]

which commutes with the Hopf equation: the above flow is a symmetry of Hopf. To find higher order corrections of the cubic equation we take advantage of the fact \([28]\) that in the small \(\mu\) regime, symmetries of Hopf can be deformed to symmetries of the full equation \(U_T + U U_X + \mu L[U(X)] = 0\); we look for the relevant symmetry as a (unique) power series \(\mu\) such that

\[U_S = \partial_X (X - UT + U^3 + \mu \alpha_1[U] + \mu^2 \alpha_2[U] + \ldots)\,.
\]

commutes order by order with the above equation. By definition, the string equation is the equation for the vanishing of the right hand-side of \([10]\). The function \(U(X,T)\) is uniquely characterized to be the solution of the string equation with the boundary behaviour

\[U(X,T) \sim - \text{sign}(X) |X|^{\frac{1}{2}} \quad \text{as} \quad |X| \to \infty, \quad \forall T,
\]

which assures the correct \(\mu \to 0\) limit.

In general, we expect the symmetry to be an infinite (possibly not converging) power series in \(\mu\); in that case the string equation will be valid only asymptotically for \(X \gg 0\), or equivalently for \(\mu\) small. However, if the string equation truncates, we can safely put \(\mu = 1\) to get the exact form of \(U\). This happens for at least three universality classes: Burgers, Benjamin-Ono and KdV. We point out that the method of the string equation is a valid alternative to the classical approach to shock by means of step-function initial data \([13]\), for it contains all universal information.

**Burgers Universality Class** The symmetry is \(U_S = \partial_X (X - UT + U^3 - 6U U_X + 4 U_{XX})\) (see \([19]\)) and string equation is therefore

\[X - UT + U^3 - 6U U_X + 4 U_{XX} = 0.
\]

The unique solution satisfying \([11]\) can be explicitly written in terms of the Pearcey integral \([2, 19]\), and it is plotted in Fig \([\text{h}]\). It is proved \([29]\) that sufficiently regular solutions of the Burgers equation admits an expansion \([\text{clusters}]\) where \(U(X, T)\) is exactly the solution of \([12]\).

**Korteweg-de Vries Universality Class** The equation \(U_S = \partial_X (X - UT + U^3 - 3 U^2_X - 6U U_X X + \frac{18}{5} U_{XXX})\) is a symmetry of the KdV equation \([1]\). The string equation satisfied by \(U(X, T)\) is

\[X - UT + U^3 - 3 U^2_X - 6U U_X X + \frac{18}{5} U_{XXX} = 0.
\]

The unique solution satisfying the boundary condition \([11]\) is plotted in Fig \([\text{g}]\). This ODE is known in literature as the second equation of the Painlevé I hierarchy and appeared in the context of random matrix theory \([31, 32]\). It is known that Painlevé equations can be linearized by means of an isomonodromic system \([33]\). It was proved in \([34, 35]\) that sufficiently regular solutions of any equation of the KdV hierarchy admit an expansion \([\text{clusters}]\) where \(U(X, T)\) is exactly the solution of \([13]\). The extension of this result to other local Hamiltonian PDEs – yet to be proved – goes under the name of Dubrovin’s universality conjecture \([1]\).

**Benjamin-Ono Universality Class: a new Painlevé equation?** The Benjamin-Ono String equation for \(U\) is finite and given by the formula

\[X - UT + U^3 - 3 U H[U_X] - 3 H[U U_X] - 4 U_{XX} = 0,
\]

as both \(\partial_X (X - UT)\) and \(\partial_X (U^3 - 3 U H[U_X] - 3 H[U U_X] - 4 U_{XX})\) are symmetries of B-O \([23]\). This is the first time that a nonlocal ODE resembling a Painlevé equation appears in the literature. We investigated equation \([14]\) numerically using a spectral method where the Hilbert transform is computed following \([36]\). According to our numerical results -see Fig \([\text{h}]\)-, for any real \(T\) equation \([14]\) admits a unique solution satisfying \([11]\) and such \(U\) solves Benjamin-Ono.

We point out that equation \([14]\) is a candidate for a new class of Painlevé equations. In fact, both \([12]\) and \([13]\) can be linearized and satisfy the Painlevé property \([32]\), for any solution extends to a meromorphic function in the complex plane. Here various important questions arise: does the unique solution of \([14]\) satisfying \([11]\) expand to a meromorphic function? Does \([14]\) admit a linearization by means of an isomonodromic system?
we showed that it corresponds to a tri-critical point. Generalizing an argument by Dubrovin [1], we then refined the coarse description of the transition using the amplitude of oscillations in the Whitham zone and we unveiled the precise local behaviour of the wave close the shock by means of the string equation, which encodes all universal features of the transition. Albeit our interest is primarily the dispersive-shock, our classification of universality classes does comprehend also dissipative and dispersive-dissipative equations alike, modeling media where dispersion and diffusion are balanced. Interestingly, their universal classes are PDEs with non-local interactions, similar to those used in experimental realization of dispersive shocks [6]. We are currently building the necessary mathematical technology to explicitly compute the string equation for these more general cases.

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[37] It is interesting to compare our Table with Table 1 of [12] concerning the dissipative shock.