A class of regular bouncing cosmologies

Milovan Vasilic

Institute of Physics, University of Belgrade,
P.O. Box 57, 11001 Belgrade, Serbia

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Abstract

In this paper, I construct a class of everywhere regular geometric sigma models that possess bouncing solutions. Precisely, I show that every bouncing metric can be made a solution of such a model. My previous attempt to do so by employing one scalar field has failed due to the appearance of harmful singularities near the bounce. In this work, I use four scalar fields to construct a class of geometric sigma models which are free of singularities. The models within the class are parametrized by their background geometries. I prove that, whatever background is chosen, the dynamics of its small perturbations is classically stable on the whole time axes. Contrary to what one expects from the structure of the initial Lagrangian, the physics of background fluctuations is found to carry 2 tensor, 2 vector and 2 scalar degrees of freedom. The graviton mass, that naturally appears in these models, is shown to be several orders of magnitude smaller than its experimental bound. I provide three simple examples to demonstrate how this is done in practice. In particular, I show that graviton mass can be made arbitrarily small.

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*Electronic address: mvasilic@ipb.ac.rs
I. INTRODUCTION

The motivation for this work comes from the need for systematization of the unpleasantly extensive number of cosmological models found in literature. Indeed, encouraged by the latest astronomical observations \[1-12\], the physical community made a serious effort to model the newly discovered accelerated expansion of the Universe. As a consequence, a variety of dark energy models appeared in literature \[13-31\]. In most of them, the authors search for the inflaton potential that makes the desired (observationally acceptable) background metric a solution to their equations.

In this paper, I demonstrate how a freely chosen bouncing metric can be made a stable solution of a simple geometric sigma model. Geometric sigma models differ from ordinary sigma models in two respects. First, all scalar fields can be gauged away, leaving us with a purely metric theory. Second, its construction follows a rather peculiar way. One first chooses the metric one would like to be the vacuum of the model, and then builds a theory that has this metric as its solution. These models have first been proposed in \[32\] in the context of fermionic excitations of flat geometry. Here, I use them for modeling dark energy dynamics of the Universe. In my previous paper \[33\], I considered the simplest geometric sigma models with one scalar field. Such geometric sigma models successfully generated various inflationary cosmologies, but their bouncing solutions were mostly unstable. In this paper, I shall consider geometric sigma models with 4 scalar fields. As will shortly be clear, such models are compatible with the existence of regular and stable bouncing solutions.

The results obtained in this paper are summarized as follows. First, a class of purely geometric dark energy models has been constructed. Every particular model is defined as a geometric sigma model of 4 scalar fields coupled to gravity. By construction, their background metrics are spatially flat, homogeneous and isotropic, while scalar fields are pure gauge. In particular, an arbitrarily chosen bouncing metric is made a solution of the properly defined geometric sigma model. Ultimately, one is provided with the class of dark energy models parametrized by their background geometries.

The second result concerns the absence of singularities in this class of models. It has been shown that the invertibility of the sigma model target metric is sufficient to ensure the absence of physical singularities. In particular, it has been demonstrated that small perturbations of the background regularly propagate through the bounce. This holds true
irrespectively of the fact that the chosen sigma model target metric is not positive definite. The third result establishes linear stability of the background fluctuations in the considered geometric sigma models. This is the main result of the paper. It has been proven true for all background geometries. In particular, whatever bouncing metric is chosen to be the background, there is a class of scalar field potentials that makes it classically stable on the whole time axes. This is a great achievement, as most of the bouncing models found in literature suffer from the appearance of instability at some moment in the history of the Universe. (For other bouncing scenarios, see the recent bouncing literature [34–41], and references therein.)

The fourth result concerns the particle spectrum of the considered geometric sigma models. As the initial action governs the dynamics of 4 scalar fields coupled to Einstein gravity, one expects to have one graviton and 4 scalar particles. Surprisingly, what one finds is the particle spectrum that consists of 2 tensor, 2 vector and 2 scalar degrees of freedom. The graviton mass, that appears as a byproduct, is shown to be several orders of magnitude smaller than its experimental bound.

Finally, I have repeated the analysis of Ref. [33] to show that inclusion of ordinary matter does not compromise the established background stability. In particular, the presence of a perfect fluid is shown to modify the background dynamics very much the same as ΛCDM model does.

The layout of the paper is as follows. In Sec. II the construction of geometric sigma models, as defined in [32], is recapitulated and subsequently applied to spatially flat, homogeneous and isotropic geometries. As a result, a class of action functionals of the Universe is obtained. Each of these action functionals has a nontrivial background solution that stands for the background geometry of the Universe. In Sec. III the dynamics of small perturbations of these nontrivial backgrounds is examined. Despite the fact that only scalar fields are coupled to the metric in the initial action, the particle spectrum is found to consist of 2 tensor, 2 vector and 2 scalar degrees of freedom. In Sec. IV the apparent singularities found in coefficients of the linearized field equations are shown to be unphysical. In Sec. V the background solutions are proven stable for all spatially flat, homogeneous and isotropic geometries. In Sec. VI the examples of three bouncing Universes are used to demonstrate how geometric sigma models are constructed in practice. Sec. VII is devoted to concluding remarks.
My conventions are as follows. The indexes \( \mu, \nu, \ldots \) and \( i, j, \ldots \) from the middle of the alphabet take values 0, 1, 2, 3. The indexes \( \alpha, \beta, \ldots \) and \( a, b, \ldots \) from the beginning of the alphabet take values 1, 2, 3. The spacetime coordinates are denoted by \( x^\mu \), the ordinary differentiation uses comma \( (X,_\mu \equiv \partial_\mu X) \), and the covariant differentiation uses semicolon \( (X;_\mu \equiv \nabla_\mu X) \). The repeated indexes denote summation: \( X_{\alpha\alpha} \equiv X_{11} + X_{25} + X_{33} \). The signature of the 4-metric \( g_{\mu\nu} \) is \(-, +, +, +\), and the curvature tensor is defined as \( R_{\mu\nu\lambda\rho} \equiv \partial_\lambda \Gamma^\mu_{\nu\rho} - \partial_\rho \Gamma^\mu_{\nu\lambda} + \Gamma^\mu_{\sigma\lambda} \Gamma^\sigma_{\nu\rho} - \Gamma^\mu_{\sigma\rho} \Gamma^\sigma_{\nu\lambda} \).

II. GEOMETRIC SIGMA MODELS

A. General considerations

Geometric sigma models are theories constructed out of the predefined spacetime metric \( g^{(o)}_{\mu\nu}(x) \). The metric \( g^{(o)}_{\mu\nu} \) is freely chosen, and the coordinates \( x^\mu \) are fully fixed. This way, the functional dependence on \( x \) in \( g^{(o)}_{\mu\nu}(x) \), and the corresponding Ricci tensor \( R^{(o)}_{\mu\nu}(x) \), is completely determined. I postulate the following Einstein like equations:

\[
R_{\mu\nu} = R^{(o)}_{\mu\nu}(x). \tag{1}
\]

The metric \( g^{(o)}_{\mu\nu} \) is a solution of the equation (1). In what follows, I shall call it vacuum.

The equation (1) obviously lacks general covariance. To covariantize it, I introduce a new set of coordinates \( \phi^i = \phi^i(x) \). In terms of these new coordinates, the equation (1) takes the form

\[
R_{\mu\nu} = H_{ij}(\phi) \phi^{i}_{,\mu} \phi^{j}_{,\nu}, \tag{2}
\]

where the functions \( H_{ij}(\phi) \) are defined through

\[
H_{ij}(\phi) \equiv R^{(o)}_{ij}(\phi). \tag{3}
\]

In other words, the ten functions \( H_{ij}(\phi) \) are obtained by replacing \( x \) with \( \phi \) in ten components of the Ricci tensor \( R^{(o)}_{\mu\nu}(x) \). The equation (2) is generally covariant once the new coordinates \( \phi^i \) are seen as scalar functions of the old coordinates \( x^\mu \). If the new coordinates are chosen to coincide with the old ones, \( \phi^i(x) \equiv x^i \), the covariant equation (2) is brought back to its non-covariant form (1).

The equation (2) has the form of the Einstein’s equation in which four scalar fields \( \phi^i(x) \) of some nonlinear sigma model are coupled to gravity. The "matter field equations"
are obtained from the Bianchi identities \(2R^\mu_{\nu;\nu} = g^{\mu\nu}R_{\nu\nu}\). If the condition \(\det \phi^i,_{\mu} \neq 0\) is fulfilled, one obtains
\[
H_{ij} \nabla^2 \phi^i = \frac{1}{2} \left( \frac{\partial H_{ik}}{\partial \phi^i} - \frac{\partial H_{ki}}{\partial \phi^i} - \frac{\partial H_{ij}}{\partial \phi^k} \right) \phi^j \phi^k. \tag{4}
\]
The equation (4) is not an independent equation, as it follows from (2) and the Bianchi identities. It is straightforward to verify that the equations (2) and (4) follow from the action functional
\[
I_g = \frac{1}{2\kappa} \int d^4x \sqrt{-g} \left[ R - H_{ij}(\phi)\phi^i_{,\mu} \phi^j{,\mu} \right]. \tag{5}
\]
The target metric \(H_{ij}(\phi)\) is constructed out of the background metric \(g_{\mu\nu}^{(o)}\), through its defining relation (3). This way, an action functional is associated with every freely chosen background metric. This action functional describes a nonlinear sigma model coupled to gravity, and has the nontrivial vacuum solution
\[
\phi^i = x^i, \quad g_{\mu\nu} = g_{\mu\nu}^{(o)}. \tag{6}
\]
The physics of small perturbations of the vacuum (6) does not violate the condition \(\det \phi^i,_{\mu} \neq 0\), which enables one to fix the gauge \(\phi^i(x) = x^i\). This gauge brings us back to the non-covariant geometric equation (1).

Equation (11) is not the unique geometric equation that allows the solution \(g_{\mu\nu} = g_{\mu\nu}^{(o)}\). A simple generalization of this equation is obtained by adding terms proportional to \(g_{\mu\nu} - g_{\mu\nu}^{(o)}\).

The simplest choice is the equation
\[
R_{\mu\nu} = R_{\mu\nu}^{(o)}(x) + \frac{1}{2} V(x) \left( g_{\mu\nu} - g_{\mu\nu}^{(o)} \right). \tag{7}
\]
It defines a class of geometric theories parametrized by metrics \(g_{\mu\nu}^{(o)}\), and potentials \(V\). The covariantization of the non-covariant equation (7) ultimately leads to the action functional
\[
I_g = \frac{1}{2\kappa} \int d^4x \sqrt{-g} \left[ R - F_{ij}(\phi)\phi^i_{,\mu} \phi^j{,\mu} - V(\phi) \right], \tag{8}
\]
where the target metric \(F_{ij}(\phi)\) is defined by
\[
F_{ij}(x) \equiv R_{ij}^{(o)}(x) - \frac{1}{2} V(x) g_{ij}^{(o)}(x). \tag{9}
\]
The class of theories defined by (8) possesses the vacuum solution (6) for any choice of the potential \(V(\phi)\). The physics of small perturbations of this vacuum allows the gauge condition \(\phi^i = x^i\), which brings us back to the geometric equation (7).
B. Cosmology

In what follows, I shall construct a class of geometric sigma models based on a spatially flat, homogeneous and isotropic metric $g^{(o)}_{\mu\nu}$, defined by

$$ds^2 = -dt^2 + a^2(t) (dx^2 + dy^2 + dz^2).$$  

(10)

The nonzero components of the corresponding Ricci tensor take the form

$$R^{(o)}_{00} = -3 \frac{\ddot{a}}{a}, \quad R^{(o)}_{\alpha\beta} = \left( a \ddot{a} + 2 \dot{a}^2 \right) \delta_{\alpha\beta},$$

where "dot" denotes time derivative. It is seen that both, $g^{(o)}_{\mu\nu}$ and $R^{(o)}_{\mu\nu}$, are functions of time only. If the potential $V(x)$ is also chosen to be independent of spatial coordinates, so will be the target metric $F_{ij}$. Indeed, the nonzero components of $F_{ij}$ are found to be

$$F_{00} = W - 2 \dot{H}, \quad F_{ab} = -a^2 W \delta_{ab},$$

(11)

where $H \equiv \dot{a}/a$ is the Hubble parameter, and $W$ is defined by

$$V \equiv 2 \left( W + \dot{H} + 3H^2 \right).$$

(12)

The target metric $F_{ij}(\phi)$, and the potential $V(\phi)$ are obtained by the substitution $x^i \rightarrow \phi^i$ in $F_{ij}(x)$ and $V(x)$. Owing to their independence of spatial coordinates, this leads to the target metric and the potential that depend on $\phi^0$ only. The corresponding action is that of (8), with $F_{ij}$ and $V$ defined by (11) and (12). It governs the dynamics of gravity coupled to 4 scalar fields, and has the vacuum solution (6). The precise form of the target metric $F_{ij}(\phi^0)$, and the potential $V(\phi^0)$ is determined once the functions $a(t)$ and $W(t)$ are specified. The class of action functionals (8) represents a collection of dark energy models parametrized by $a(t)$ and $W(t)$.

Before I go on, let me mention that the authors of Refs. [42–44] developed another way to parametrize cosmologies by scale factors. Their procedure was named "cosmological reconstruction", and was primarily intended for the construction of $f(R)$ cosmological models. In particular, it was shown that any bouncing metric could be obtained by a proper choice of the function $f(R)$. However, the problem with $f(R)$ bounces is that they are typically unstable. Indeed, it is well known that every $f(R)$ theory can be rewritten as a scalar-tensor theory with a single scalar field [42–44]. These single-scalar theories are known to be plagued
with instabilities for most of their bouncing solutions. In fact, this is exactly what I demonstrated in my previous paper [33], and what lead me to consider multiple scalars in this paper. It will become clear later that the multi-scalar concept developed in this paper leads to an everywhere stable dynamics, irrespectively of the type of bounce considered. Shortly, my class of geometric sigma models allows any regular bounce to be the background solution whose small perturbations have stable propagation on the whole time axes. This is certainly not the case with F(R) bounces.

Standard physical requirements that ensure the absence of ghosts and tachyons restrain the target metric $F_{ij}(\phi)$ to be positively definite, and the potential $V(\phi)$ to be bounded from below. Unfortunately, this is not the case with our class of sigma models. Indeed, the potential $W$ must be everywhere negative to ensure $F_{ab} > 0$. But then, the Hubble parameter $H$ must monotonously decrease to prevent $F_{00}$ from becoming negative. As a result, the bouncing solutions are excluded. In what follows, I shall consider everywhere negative $W$. Aware of the fact that $F_{00}$ cannot be everywhere positive, I shall demand that $F_{00}$ is everywhere negative. This way, the target metric $F_{ij}$ becomes an everywhere invertible matrix ($\det F_{ij} \neq 0$). The conditions

$$W < 0, \quad F_{00} < 0$$

will be justified later when I demonstrate their necessity for proving regularity and stability. (The possibility that scalar fields with the wrong sign in the kinetic term may be observationally allowed has been considered before [45–50].)

In the next section, I shall examine the dynamics of small perturbations of the vacuum (6). It will be shown that the class of geometric sigma models (8) supports regular and stable bouncing backgrounds, irrespectively of the violation of the standard physical requirements.

III. DYNAMICS OF SMALL PERTURBATIONS

A. Preliminaries

In this section, I shall examine dynamics of small perturbations of the vacuum (6), as governed by the action functional (8). The variables of the theory are the scalar perturbation $\varphi^i$, and the metric perturbation $h_{\mu\nu}$. They are defined by

$$\phi^i = x^i + \varphi^i, \quad g_{\mu\nu} = g_{\mu\nu}^{(o)} + h_{\mu\nu}.$$  

(14)
The infinitesimal change of coordinates $x^\mu \rightarrow x^\mu + \xi^\mu(x)$ leaves the action invariant, and allows for a gauge fixing. In the gauge $\phi^i = x^i$, the matter field equations are identically satisfied, and we are left with the gravitational field equation (7). In this analysis, however, I shall use the gauge condition

$$\phi^0 = t, \quad g^{0\alpha} = 0.$$  \hfill (15)

Then, the residual diffeomorphisms are defined by the constraints

$$\xi^0 = 0, \quad \dot{\xi}^\alpha = 0.$$  \hfill (16)

With respect to the residual diffeomorphisms, the variables of the gauge fixed theory transform as

$$\delta_0 \varphi^\alpha = -\xi^\alpha + O_2,$$

$$\delta_0 h_{\alpha\beta} = -a^2 (\xi_{\alpha,\beta} + \xi_{\beta,\alpha}) + O_2,$$

$$\delta_0 h_{00} = O_2,$$  \hfill (17)

where $\delta_0$ is the form variation, and $O_2$ denotes higher order terms. Here, and in what follows, I adopt the convention to lower spatial indexes by the Kronecker delta. Thus,

$$\xi_\alpha \equiv \delta_{\alpha\beta} \xi^\beta, \quad \varphi_\alpha \equiv \delta_{ab} \varphi^b, \ldots$$

The field equations obtained by varying the action (8) read

$$R_{\mu\nu} - \frac{1}{2} V g_{\mu\nu} = F_{ij} \dot{\phi}^i_{,\mu} \dot{\phi}^j_{,\nu},$$

$$2F_{ki} \Box \phi^i + (2F_{kj} - F_{ij,k}) \dot{\phi}^i_{,\mu} \dot{\phi}^{j\mu} - V_k = 0.$$  \hfill (18)

They govern the dynamics of the metric $g_{\mu\nu}$, and the scalar fields $\phi^i$. To examine the dynamics of their small perturbations, it suffices to consider only linear terms.

The linearized field equations are obtained by rewriting (18) in terms of $h_{\alpha\beta}$ and $\varphi_\alpha$, and subsequently neglecting higher order terms. The straightforward calculation leads to quite cumbersome expressions, which I choose not to display here. Instead, I shall first simplify them by decomposing $h_{\alpha\beta}$ and $\varphi_\alpha$ to their irreducible components with respect to the rotational group. In the first step, divergences are subtracted in the decomposition

$$h_{\alpha\beta} = \tilde{h}_{\alpha\beta} + \bar{h}_{\alpha,\beta} + \bar{h}_{\beta,\alpha} + h_{,\alpha\beta},$$

$$\varphi_\alpha = \tilde{\varphi}_\alpha + \varphi_{,\alpha}.$$  \hfill (19a)

Here, the new variables $\tilde{h}_{\alpha\beta}$, $\bar{h}_\alpha$ and $\tilde{\varphi}_\alpha$ are constrained by

$$\tilde{h}_{\alpha\beta,\beta} = \tilde{h}_{\alpha,\alpha} = \tilde{\varphi}_{,\alpha,\alpha} = 0.$$
(The variable $\varphi^i$ is treated as a vector despite the scalar nature of the original $\phi^i$. This will be clarified at the end of this section.) In the second step, the trace is subtracted from $\tilde{h}_{\alpha\beta}$. The traceless part of $\tilde{h}_{\alpha\beta}$ is defined by

$$\hat{h}_{\alpha\beta} \equiv \tilde{h}_{\alpha\beta} - \frac{1}{2} \tilde{h}_{\gamma\gamma} \delta_{\alpha\beta} + \frac{1}{2} \partial_\alpha \partial_\beta \left( \Delta^{-1} \tilde{h}_{\gamma\gamma} \right),$$

where $\Delta^{-1}$ stands for the inverse of the Laplacian $\Delta \equiv \delta_{\alpha\beta} \partial_\alpha \partial_\beta$. In what follows, I shall simplify the analysis by the assumption that metric perturbations are spatially localized. This means that the perturbations $h_{\mu\nu}$ and $\varphi^i$ are assumed to decrease sufficiently fast in spatial infinity. With this assumption, Laplacian $\Delta$ becomes an invertible operator, and the equations like $\partial_\alpha X = 0$, or $\Delta X = 0$ have the unique solution $X = 0$. The new variable $\hat{h}_{\alpha\beta}$ is constrained by

$$\hat{h}_{\alpha\alpha} = \hat{h}_{\alpha\beta,\beta} = 0.$$ 

**B. Field equations**

With these preliminaries, the linearized field equations are given as follows. First, the equations

$$\left[ \frac{1}{a^2} \hat{h}_{\alpha\alpha} \right]_0 - 2W a^2 \dot{\varphi} + 2H h_{00} = 0, \quad (20a)$$

$$\left[ \frac{1}{a^2} \tilde{h}_\alpha \right]_{00} + 3H \left[ \frac{1}{a^2} \tilde{h}_\alpha \right]_0 + 2W \left( \varphi - \frac{1}{2a^2} \hat{h}_\alpha \right) = 0, \quad (20b)$$

$$\frac{1}{2H} \left[ \frac{1}{a^2} \tilde{h}_{\alpha\alpha} \right]_{,0} + \left[ \frac{1}{a^2} \tilde{h}_{\alpha\alpha} \right]_0 - 2a^2 \left( \varphi + \left( 5H + \frac{\dot{W}}{W} + \frac{W}{2H} \right) \dot{\varphi} \right) + \Delta \left( 2\varphi - \frac{1}{a^2} \hat{h} \right) = 0 \quad (20c)$$

are used to solve for $h_{00}$, $\tilde{h}_\alpha$, and $h$. Thus, these variables carry no degrees of freedom. The remaining six variables are dynamical. They consist of two tensor modes $\hat{h}_{\alpha\beta}$, two vector modes $\tilde{h}_\alpha$, and two scalars $\tilde{h}_{\alpha\alpha}$ and $\varphi$. Their equations are as follows.

The tensor modes satisfy the equation

$$\left[ \frac{1}{a^2} \hat{h}_{\alpha\beta} \right]_{00} + 3H \left[ \frac{1}{a^2} \hat{h}_{\alpha\beta} \right]_0 - \frac{1}{2a^2} \Delta \left[ \frac{1}{a^2} \hat{h}_{\alpha\beta} \right]_0 - 2W \left[ \frac{1}{a^2} \hat{h}_{\alpha\beta} \right] = 0. \quad (21)$$

Being subject to the constraints $\hat{h}_{\alpha\alpha} = \hat{h}_{\alpha\beta,\beta} = 0$, the variable $\hat{h}_{\alpha\beta}$ carries two physical degrees of freedom. Note, however, the presence of the mass term in (21). It will be demonstrated later that the corresponding graviton mass $m_g = \sqrt{-2W}$ can be made ten
orders of magnitude smaller than its experimental bound. In some cases, it is possible to make it arbitrarily small.

The vector modes are governed by the equation

\[
\dddot{\tilde{\chi}}_\alpha - \left(3H + \frac{\dot{W}}{W}\right)\ddot{\tilde{\chi}}_\alpha - \frac{1}{a^2} \Delta \tilde{\chi}_\alpha - 2W\tilde{\chi}_\alpha = 0 ,
\]

where \(\tilde{\chi}_\alpha\) stands for

\[
\tilde{\chi}_\alpha \equiv a^3 \left[\frac{1}{a^2} \tilde{h}_\alpha\right]_\alpha .
\]

Although the equation (22) is a second order differential equation with respect to \(\tilde{\chi}_\alpha\) it is of third order with respect to the original variable \(\tilde{h}_\alpha\). It seems as if \(\tilde{h}_\alpha\) carried more than two degrees of freedom. This is, however, not the case. Indeed, after the gauge fixing (15), we are left with the residual gauge symmetry which can further be fixed. (The residual parameters \(\xi_\alpha\) are arbitrary functions of spatial coordinates, alone). It is straightforward to verify that \(\tilde{h}_\alpha/a^2\) transforms as

\[
\delta_0 \left[\frac{1}{a^2} \tilde{h}_\alpha\right] = -\tilde{\xi}_\alpha ,
\]

where \(\tilde{\xi}_\alpha\) is the divergence free part in the decomposition

\[
\xi_\alpha = \tilde{\xi}_\alpha + \xi,\alpha.
\]

The restriction \(\dot{\xi}_\alpha = 0\) then implies that \(\tilde{\chi}_\alpha\) is gauge invariant. Let me now integrate (23). One finds

\[
\frac{1}{a^2} \tilde{h}_\alpha = \int_0^t \tilde{\chi}_\alpha a^3 dt + \tilde{c}_\alpha ,
\]

where \(\tilde{c}_\alpha\) is a divergent free, but otherwise arbitrary function of \(\vec{x}\). The transformation law (24) then tells us that

\[
\delta_0 \tilde{c}_\alpha = \delta_0 \left[\frac{1}{a^2} \tilde{h}_\alpha\right] = -\tilde{\xi}_\alpha ,
\]

Both, \(\tilde{c}_\alpha\) and \(\tilde{\xi}_\alpha\), are divergence free functions of spatial coordinates, only. This enables one to impose the gauge condition

\[
\tilde{c}_\alpha = 0 ,
\]

thereby establishing 1−1 correspondence between \(\tilde{\chi}_\alpha\) and \(\tilde{h}_\alpha\). As a consequence, the equation (22) carries exactly two degrees of freedom. After this gauge fixing, the residual gauge symmetry is defined by

\[
\xi_\alpha = \xi,\alpha , \quad \dot{\xi} = 0 .
\]
The scalar field equations are most complicated. They govern the dynamics of \( \tilde{h}_{\alpha\alpha} \) and \( \varphi \), which I decide to display in matrix form. First, I define the matrix variable

\[
\Psi \equiv \begin{pmatrix} \psi' & \psi'' \end{pmatrix} \equiv \begin{pmatrix} \frac{1}{a^2} \tilde{h}_{\alpha\alpha} \\ -4W a^2 \dot{\varphi} \end{pmatrix}.
\]

Then, the two scalar equations take the matrix form

\[
\ddot{\Psi} - \frac{1}{a^2} \Delta \Psi + \mathcal{F} \dot{\Psi} + \mathcal{M} \Psi = 0,
\]

where \( \mathcal{F} \) and \( \mathcal{M} \) are matrix valued coefficients. The friction coefficient \( \mathcal{F} \), and the mass matrix \( \mathcal{M} \) have the form

\[
\mathcal{F} \equiv \begin{pmatrix} 3H - 2 \frac{\dot{H}}{H} + \frac{\dot{F}_{00}}{F_{00}} - \frac{W}{F_{00}} L & -\frac{H}{F_{00}} L \\
2\frac{W}{F_{00}} \left( 2\dot{H} + \frac{\dot{\varphi}}{H} \right) & 5H + \left( \frac{W}{F_{00}} - 1 \right) L \end{pmatrix},
\]

\[
\mathcal{M} \equiv \begin{pmatrix} -2W \left( 1 + \frac{2H}{F_{00}} L \right) & -\frac{H}{F_{00}} K \\
4W \frac{W}{F_{00}} L & 3H \left( \frac{\dot{H}}{H} - \frac{\dot{W}}{W} \right) - \frac{W}{H} \left( 3H + \frac{\dot{H}}{H} \right) + \frac{W}{F_{00}} K \end{pmatrix},
\]

where the shorthand notation

\[
L \equiv 2H + \frac{\dot{W}}{W},
\]

\[
K \equiv H \left[ 3L - \left( \frac{1}{H} \right)_{,00} \right] + W \left[ 1 - \left( \frac{1}{H} \right)_{,0} \right]
\]

is introduced for convenience. The field equations (26) carry two physical degrees of freedom, despite the fact that they are third order differential equations with respect to the original variables. To verify this, I shall make use of the residual gauge symmetry (25) to show that all but two scalar degrees of freedom are nonphysical. First, I integrate \( \dot{\varphi} \) to obtain

\[
\varphi = \int_0^t \dot{\varphi} dt + c,
\]

where \( c \) is arbitrary function of spatial coordinates, only. Then, the transformation law \( \delta_0 \varphi = -\xi \) implies that \( \dot{\varphi} \) is gauge invariant, and consequently, \( \delta_0 c = -\xi \). This allows the final gauge fixing

\[
c = 0
\]
which leaves us with no free gauge parameters. The gauge fixing $c = 0$ establishes $1 - 1$ correspondence between $\varphi$ and $\dot{\varphi}$, and consequently, between $\Psi$ and $\{\varphi, \tilde{h}_{\alpha\alpha}\}$. Thus, the equation (26) has two degrees of freedom.

C. Particle spectrum

The theory considered in this paper is defined by the action (8), which governs the dynamics of four scalar fields coupled to gravity. One would expect the particle content of the theory to be 2 tensor + 4 scalar degrees of freedom. However, the preceding results suggest the structure 2 tensor + 2 vector + 2 scalar physical fields. Let me clarify this situation. The nature of physical fields is determined by their transformation properties with respect to symmetries of the vacuum. Usually, it is the $SO(3,1)$ invariant Minkowski vacuum that leads to the well known classification of elementary particles. In cosmology, however, the vacuum is $SO(3)$ invariant. Indeed, the cosmological background is only spatially homogeneous and isotropic, while its time dependence is nontrivial. Thus, in what follows, I shall consider the rotational group, defined by

$$\xi^\alpha = \omega_{\alpha\beta} x^\beta. \quad (27)$$

The parameters $\omega_{\alpha\beta} = -\omega_{\beta\alpha}$ are constant and antisymmetric. With respect to diffeomorphisms $x^\mu \to x^\mu + \xi^\mu$, the scalars $\phi^i$ and the metric $g_{\mu\nu}$ transform as

$$\delta_0 \phi^i = -\xi^\rho \phi^i_{,\rho},$$
$$\delta_0 g_{\mu\nu} = -\xi^\rho_{,\mu} g_{\rho\nu} - \xi^\rho_{,\nu} g_{\rho\mu} + \xi^\rho g_{\mu\nu,\rho}.$$

Their small perturbations, however, transform differently. Indeed, in the gauge (15), their transformation law with respect to rotations (27) becomes

$$\delta_0 \varphi^\alpha = -\xi^\alpha - \xi^\beta \varphi_{,\beta}^\alpha,$$
$$\delta_0 h_{00} = -\xi^\alpha h_{00,\alpha},$$
$$\delta_0 h_{\alpha\beta} = -\xi^\gamma_{,\alpha} h_{\gamma\beta} - \xi^\gamma_{,\beta} h_{\gamma\alpha} - \xi^\gamma h_{\alpha\beta,\gamma}.$$

It is seen that $h_{\alpha\beta}$ transforms as a tensor, while $h_{00}$ and $\varphi_{,\alpha}^\alpha$ are scalars. The variable $\varphi^\alpha$, on the other hand, is neither a vector nor a collection of three scalars. This state of affairs can change once we realize that the action (8) has an extra global symmetry. Indeed, it is easily shown that the action (8) is invariant with respect to

$$\delta_1 \phi^a = \epsilon^a_b \phi^b, \quad \delta_1 \phi^0 = \delta_1 g_{\mu\nu} = 0,$$
where $\epsilon_{ab} = -\epsilon_{ba}$ are constant parameters independent of $\omega_{ab}$. The full global symmetry is then defined by the total variation $\delta_0 + \delta_1$. Let us consider the subgroup defined by

$$\epsilon_{ab} = \omega_{ab}.$$ 

Its action on the variables $\varphi^\alpha$ and $h_{\alpha\beta}$ is given by

$$\left[\delta_0(\omega) + \delta_1(\omega)\right] \varphi^\alpha = \xi^\alpha_{\alpha} \varphi^\beta - \xi^\beta \varphi^\alpha_{,\beta},$$

$$\left[\delta_0(\omega) + \delta_1(\omega)\right] h_{\alpha\beta} = -\xi^\gamma_{,\alpha} h_{\gamma\beta} - \xi^\gamma_{,\beta} h_{\gamma\alpha} - \xi^\gamma h_{\alpha\beta,\gamma}$$

where $\xi^\alpha \equiv \omega^\alpha_{,\beta} x^\beta$. It is seen that $\varphi^\alpha$ transforms as a vector, and $h_{\alpha\beta}$ as a tensor with respect to global rotations. Therefore, the terminology "irreducible representations of the rotational group", used for the description of decomposition (19), is justified. As a consequence, the particle spectrum has the structure

$$2 \text{ tensor} + 2 \text{ vector} + 2 \text{ scalar}$$

degrees of freedom, in contrast with what one might expect from the action (8).

IV. SINGULARITIES

In the preceding section, we have seen that the evolution of small physical perturbations is governed by the equations (21), (22) and (26). Let me examine their regularity.

In the first step, one examines if the coefficients of these equations are everywhere regular. It is straightforward to verify that the conditions (13) ensure the full regularity of the tensor and vector equations, (21) and (22). This leaves us with the scalar equations (26), whose coefficients become singular in $H = 0$. This is an improvement with respect to Ref. [33] because the real harmful singularities $F_{00} = 0$ do not appear in the present approach. In fact, it has been suggested in [33] that singularity of the bounce $(H = 0)$ is not physical. In what follows, I shall demonstrate that the propagation of scalar modes in the vicinity of the bounce is regular, irrespectively of the apparent singularity of the $F$ and $M$ coefficients.

Let me first choose the moment of the bounce as a natural origin of time coordinate. Then, the singular point $H = 0$ is identified with $t = 0$. Without loss of generality, I shall consider the scale factor $a(t)$ that behaves as

$$a(t) = a_0 + a_2 t^2 + O(t^4)$$
in the vicinity of $t = 0$. (The only reason for dropping term proportional to $t^3$ is to simplify cumbersome expressions.) A straightforward calculation then shows that $\mathcal{F}$ and $\mathcal{M}$ take the form

$$\mathcal{F} = -\frac{2}{t} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \mathcal{O}_0, \quad \mathcal{M} = -\frac{1}{t} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \mathcal{O}_0,$$

where $\mathcal{O}_0$ stands for regular terms. Let me now rewrite the equation (26) in terms of the new variable $\Psi_1$, defined as

$$\Psi_1 = U^{-1} \Psi = \begin{pmatrix} 1 & t \\ 2 & 1 \end{pmatrix} \Psi.$$

With this, the equation (26) takes the form

$$\ddot{\Psi}_1 - \frac{1}{a^2} \Delta \Psi_1 + \mathcal{F}_1 \dot{\Psi}_1 + \mathcal{M}_1 \Psi_1 = 0,$$

(29)

where $\mathcal{F}_1$ and $\mathcal{M}_1$ are given by

$$\mathcal{F}_1 = U^{-1} \left( \mathcal{F} U + 2 \dot{U} \right),$$

$$\mathcal{M}_1 = U^{-1} \left( \mathcal{M} U + \mathcal{F} \dot{U} + \ddot{U} \right)$$

(30)

for every nonsingular $U$. In the case under consideration, one finds

$$\mathcal{F}_1 = \mathcal{F} + \mathcal{O}_0, \quad \mathcal{M}_1 = \begin{pmatrix} -2W & 0 \\ 4W^2 & 3\dot{H} - 2W \end{pmatrix} + \mathcal{O}_1.$$

It is seen that the mass matrix $\mathcal{M}_1$ is everywhere regular, so that $\mathcal{F}_1$ remains the only coefficient with singular behavior. Now, I am ready to solve the equation (29) in the vicinity of $t = 0$. First, I use the Fourier decomposition

$$\Psi_1 = \text{Re} \int d^3k Q(k,t) e^{ik \cdot x}$$

to rewrite the equation (29) in terms of its Fourier modes:

$$\dot{Q} + \mathcal{F}_1 \dot{Q} + \left( \mathcal{M}_1 + \frac{k^2}{a^2} \right) Q = 0.$$

(31)

In the vicinity of $t = 0$, the coefficients $\mathcal{F}_1$ and $\mathcal{M}_1$ are well approximated by

$$\mathcal{F}_1 \approx -\frac{2}{t} I, \quad \mathcal{M}_1 \approx J,$$
where constant matrices $I$ and $J$ are defined by

$$ I \equiv \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad J \equiv \left( M_1 + \frac{k^2}{a^2} \right)_{t=0}. $$

The matrix elements of the matrix valued coefficients $I$ and $J$ are all finite. Using this approximation, and multiplying (31) by $t$, one finally arrives at

$$ t\ddot{Q} - 2I\dot{Q} + tJQ = 0. \quad (32) $$

The solution of the equation (32) is searched for in the form of the power series

$$ Q = \sum_{n=0}^{\infty} q_n t^n. $$

One straightforwardly finds

$$ Iq_1 = 0, \quad Jq_1 + \begin{pmatrix} 0 & 0 \\ 0 & 6 \end{pmatrix} q_3 = 0, \quad (33a) $$

and for $n \neq 1$,

$$ q_{n+2} = -\frac{1}{n+2} \begin{pmatrix} 1 \\ n-1 \\ 0 \\ 0 \\ 1 \\ n+1 \end{pmatrix} Jq_n. \quad (33b) $$

The equations (33b) tell us that all the coefficients $q_n$ are determined in terms of $q_0$, $q_1$ and $q_3$. The coefficients $q_0$, $q_1$ and $q_3$, on the other hand, are constrained by (33a), but are not completely determined. Let us see how many degrees of freedom we are left with. To this end, I shall use the notation

$$ q_n \equiv \begin{pmatrix} q'_n \\ q''_n \end{pmatrix} $$

to rewrite the equations (33a) in the component form. Thus, one finds

$$ q'_1 = 0, \quad 6 q'''_3 + J_{22} q''_1 = 0. \quad (34) $$

Now, we clearly see that the components $q'_0$, $q'_0$, $q'''_1$ and $q'_3$ remain undetermined. Thus, there exists a class of regular solutions to the singular equation (31), parametrized by four free parameters

$$ q'_0, \quad q''_0, \quad q''_1 \quad \text{and} \quad q'_3. $$

These four parameters stand for 2 physical degrees of freedom. As a consequence,
the class of cosmological models considered in this paper is everywhere regular.

Let me emphasize that it is very important to have the full number of degrees of freedom in \( t = 0 \). If it is not the case, only very special initial conditions in the past lead to trajectories that regularly pass the bounce. My result is that solutions regular in \( t = 0 \) carry the maximal number of degrees of freedom allowed by the model. Only then every perturbation formed in the past regularly passes the bounce. I have verified this result by numerically solving differential equations with this kind of singularity. The conditions (34) have also been checked. In particular, I have shown that \( Q'(t) \), formed in the past, always approaches \( t = 0 \) as a constant (\( q'_1 = 0 \)). Owing to the condition \( q'_1 = 0 \), the nonphysical variables \( h_{00}, h \) and \( \varphi_\alpha \), obtained by solving equations (20), are also everywhere regular.

Finally, I want to draw your attention to the structure of initial conditions that can be chosen in \( t = 0 \). As opposed to \( t \neq 0 \), the initial conditions in \( t = 0 \) cannot be the value of the field and of its first time derivative. Indeed, while initial conditions for the scalar \( Q''(t) \) can be chosen in the standard way \( (q'_0, q''_1) \), the scalar \( Q'(t) \) is determined by its value \( q'_0 \), and the value of its third time derivative \( q'_3 \).

V. STABILITY ANALYSIS

A. Tensor and vector modes

In this section, I shall examine stability of the vacuum against its small perturbations, as governed by the equations \( (21), (22) \) and \( (26) \). It is immediately seen that both, tensor and vector equations, have stable dynamics for any \( a(t) \), and any \( W \) that respects the conditions (13). Indeed, the two mass terms are everywhere positive, leading to an oscillatory behavior of small vacuum perturbations. What one might see as a problem is that tensor modes, which are usually identified with the graviton, have nonzero mass

\[
m_g = \sqrt{-2W}.
\]  

This is not what one would like to have. Notice, however, that the term \( \sqrt{-2W} \) is time dependent, and therefore, cannot stand for the conventional mass on the whole time axis. Instead, one should examine the behavior of \( \sqrt{-2W} \) at the present epoch. At the present epoch, the spacetime is close to being flat, and has rather slow expansion rate. In a situation like this, the term \( \sqrt{-2W} \) can well be considered the graviton mass.
The estimation of the present value of $m_g$ can be done by making use of the restrictions \cite{13}. One finds

$$m_g^2 > 4 (q + 1) H^2,$$

where the Hubble parameter $H$, and deceleration parameter $q$ are defined by

$$H \equiv \frac{\dot{a}}{a}, \quad q \equiv -\frac{\ddot{a}}{aH^2}. \quad (36)$$

The observed values of the Hubble and deceleration parameters are

$$H_0 \approx 1.6 \cdot 10^{-33} \text{eV}, \quad q_0 \approx -0.5,$$

so that the present graviton mass obeys

$$m_g > 2.2 \cdot 10^{-33} \text{eV}.$$

This is more than ten orders of magnitude smaller than the upper bound reported by the LIGO experiment \cite{51}:

$$m_g < 1.2 \cdot 10^{-22} \text{eV}.$$

Thus, there is plenty of room for choosing a plausible cosmological model from the class of models described in this paper. What one should do is to make a proper choice of the potential $W$. I shall demonstrate this in examples of the next section.

Let me mention one more thing about the estimation of graviton mass. It is seen that $m_g$ is sensitive to the actual value of the deceleration parameter $q$. In particular, the graviton mass can be made arbitrarily close to zero if $q < -1$. There is a good reason why one might believe in such a scenario. Namely, the realistic Universe, where all the measurements are done, is filled with ordinary matter, too. The presence of ordinary matter increases the measured value of $q$. Thus, it may happen that the vacuum value of $q$ is either equal to $-1$ (ΛCDM model), or even smaller than $-1$. In the examples of the next section, I shall demonstrate how the presence of matter increases the vacuum value $q < -1$ to the measured value $q = -1/2$.

**B. Scalar modes**

The stability of the scalar equation \cite{26} is examined by solving the eigenvalue problem of the mass matrix $\mathcal{M}$. What one hopes to find is that both eigenvalues of $\mathcal{M}$ are real and positive. Only then the vacuum solution of \cite{26} is stable against its small perturbations.

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Skipping unnecessary details, the conditions that ensure reality and positivity of the mass eigenvalues have the form

\[ \det \mathcal{M} \geq 0, \quad \mathcal{M}_{11} + \mathcal{M}_{22} - 2\sqrt{\det \mathcal{M}} \geq 0. \] (38)

The conditions (38) are not satisfied for every \( W \) and \( a \). However, it can be shown that, for every \( a(t) \), there exists a class of potentials \( W(t) \) such that (38) holds true. As an example, let me consider \( W \) defined by

\[ W = -\frac{\omega^2}{a^2}, \] (39)

where \( \omega \) is constant with the dimension of mass. Naturally, this choice of the potential must preserve the regularity conditions (13). If \( \omega \) is chosen large enough, the conditions (13) are satisfied for every \( a(t) \) with linear or slower growth in the asymptotic region. For scale factors that grow faster, one may be led to make a different choice of the potential. However, the choice (39) always respects (13) in a finite, arbitrarily chosen time interval \(-T < t < T\). This allows one to keep most of the potential (39), and just modify its asymptotic behavior.

In what follows, I shall use the variable \( \Psi_1 \), defined in (28). This simplifies the form of the mass matrix. Using (39), the mass matrix \( \mathcal{M}_1 \) becomes

\[
\mathcal{M}_1 = \frac{1}{2} \begin{pmatrix}
-4W & -3H + 3t \left( \dot{H} + 2H^2 \right) \\
0 & -4W + 6 \left( \dot{H} + 2H^2 \right)
\end{pmatrix}.
\]

It is now easy to check the conditions (38). Indeed, one straightforwardly finds

\[
(\mathcal{M}_1)_{11} = -2W > 0,
\]

\[
(\mathcal{M}_1)_{22} = \frac{1}{2} \left( 12H^2 - 3F_{00} - W \right) > 0
\]

whenever the conditions (13) are fulfilled. As a consequence, the determinant is also positive,

\[
\det \mathcal{M}_1 = (\mathcal{M}_1)_{11} (\mathcal{M}_1)_{22} > 0,
\]

and so is the second of the conditions (38):

\[
(\mathcal{M}_1)_{11} + (\mathcal{M}_1)_{22} - 2\sqrt{\det \mathcal{M}_1} = \left[ \sqrt{(\mathcal{M}_1)_{11}} - \sqrt{(\mathcal{M}_1)_{22}} \right]^2 > 0.
\]

Thus, we have proved that the mass matrix \( \mathcal{M}_1 \) has two real positive eigenvalues whenever the potential \( W \) has the form (39). This certainly holds true in any finite time interval, and
for any choice of the scale factor $a(t)$. As for the asymptotic region, it is not difficult to show that $W$ can always be chosen to ensure the asymptotic stability of the scalar modes. For example, let us consider the power law behavior of $a(t)$ and $W(t)$. Starting with

$$a \sim t^\alpha, \quad W \sim t^\beta,$$

one straightforwardly finds that

$$(2\alpha + \beta)(\alpha - 1) > 0$$

ensures the validity of (38). In particular, for all $a(t)$ that grow faster than $t$ in the asymptotic region, it is enough to choose $W \sim 1/t^2$. The faster the expansion of the Universe is, the smaller values of $W$ are allowed. In some cases, it is possible to define $W$ with arbitrarily fast approach to zero. To conclude, I have proven in this section that

- for every scale factor $a(t)$, there is a class of potentials $W(t)$ that makes the dynamics linearly stable.

C. Matter fields

So far, I have proved regularity and stability of the class of dark energy models in which ordinary matter has been neglected. One may wonder if the presence of ordinary matter might spoil the nice results obtained so far. A similar problem has already been studied in [33], with the result that matter fields do not compromise earlier results. There, only one scalar field has been coupled to gravity. Nevertheless, the present problem turns out to be technically identical to that of Ref. [33]. For that reason, I shall present a short version of the full analysis.

Let me consider the action

$$I = I_g + I_m,$$  \hspace{1cm} (40)

where $I_g$ is the geometric action (8), and $I_m$ stands for the action of matter fields. Usually, the matter Lagrangian is taken to be that the standard model of elementary particles, minimally coupled to gravity. Matter fields are collectively denoted by $\Omega$. Owing to the minimal coupling to the metric, the matter field equations

$$\frac{\delta I_m}{\delta \Omega} = 0$$
are trivially satisfied by
\[ \Omega = \Omega_0, \]
where \( \Omega_0 \) stands for the well known vacuum of the standard model of elementary particles. The vacuum value of the stress-energy tensor \( T_{\mu \nu}^m \) is also zero. Formally,
\[ \Omega = \Omega_0 \implies T_{\mu \nu}^m = 0. \]

With this, the inflaton and Einstein’s equations reduce to those considered in the preceding sections. Indeed, owing to the absence of the direct matter-inflaton couplings, the inflaton equations take the form
\[ \frac{\delta I}{\delta \phi^i} = \frac{\delta I_g}{\delta \phi^i} = 0, \]
while Einstein’s equations become
\[ R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} R = T_{\phi}^{\mu \nu}, \]
whenever \( \Omega = \Omega_0 \). As a consequence, the model (40) has the vacuum solution
\[ g_{\mu \nu} = g_{\mu \nu}^{(o)}, \quad \phi^i = x^i, \quad \Omega = \Omega_0. \quad (41) \]
Thus, the presence of matter fields does not compromise the sigma model vacuum of the preceding sections.

The linear stability of the vacuum (41) is examined by inspecting the linearized field equations of the action (40). It is immediately seen that, after linearization, the inflaton and Einstein’s equations reduce to those of the geometric sigma model of the preceding sections. Indeed, the stress-energy tensor \( T_{\mu \nu}^m \), being at least quadratic in perturbations of matter fields, does not appear on the r.h.s. of the linearized Einstein’s equations. At the same time, the inflaton does not couple to matter fields, at all. Hence, the linearized inflaton and metric equations of motion remain unchanged by the inclusion of matter. They are diffeomorphism invariant, so that the complete gauge fixing procedure of Sec. III is still valid. In this gauge, the tensor, vector and scalar equations obtained from \( I \) coincide with those obtained from the geometric action \( I_g \). Their linear stability has already been proven. As for the stability of matter itself, it is enough to recall that matter field equations reduce to the standard model of elementary particles, in the inertial reference frame. To summarize, the vacuum (41) is linearly stable against small perturbations governed by the action (40). The dynamics of its geometric part remains the same as found in Sec. III. In conclusion,
the presence of matter fields does not violate the established linear stability.

As a final step, one might consider the inclusion of direct matter–inflaton couplings. Skipping unnecessary details, I shall only emphasize that typical matter–inflaton couplings preserve the regularity and stability arguments given earlier. In particular, the interaction terms which are at least quadratic in matter fields do not compromise the previous analysis. More details can be found in Ref. [33].

VI. EXAMPLES

In this section, I shall analyze three simple models of the Universe. In the first, a toy bouncing model is used for the demonstration of how geometric sigma models are constructed in practice. In the second, I consider a slowly contracting Universe with an exponential expansion after the bounce. The influence of dust matter is studied as an example of how ordinary matter modifies the background dynamics. The third example demonstrates how graviton mass can be made arbitrarily small.

A. Toy model

Here, I shall study a homogeneous, isotropic and spatially flat geometry with the scale factor of the form

\[ a(t) = \sqrt{1 + \omega^2 t^2} , \]  

and the potential defined by

\[ W(t) = -2 \frac{\omega^2}{a^2} . \]  

Its graph is displayed in Fig. 1. It describes a linearly contracting Universe with the bounce at \( t = 0 \), and linear expansion afterwards. The constant \( \omega \) is a free parameter with the dimension of mass. The scale factor (42a) is a solution of the sigma model \( (8) \) in which the target metric \( F_{ij}(\phi) \) and the potential \( V(\phi) \) are defined by the replacement \( x^i \rightarrow \phi^i \) in the expressions \( (11) \) and \( (12) \). As neither \( F_{ij}(x) \) nor \( V(x) \) depend on spatial coordinates, the resulting target metric

\[ F_{00} = -\frac{4 \omega^2}{(1 + \omega^2 \phi_0^2)^2} , \quad F_{ab} = 2 \omega^2 \delta_{ab} , \]  

(43a)
and the potential
\[ V = -\frac{2 \omega^2}{(1 + \omega^2 \phi_0^2)^2} \]  
depend on \( \phi_0 \), only. The equations (42b) and (43a) show that regularity conditions (13) are satisfied on the whole time axis. Then, the general arguments of the preceding sections ensure that small perturbations of the vacuum have nonsingular, and everywhere stable dynamics.

The free parameter \( \omega \) is determined from the observed values of the Hubble and deceleration parameters, as given by (37). First, we use (42) to obtain
\[ \omega = \frac{1 - q_0}{\sqrt{-q_0 H_0}}, \quad t_0 = \frac{1}{(1 - q_0)H_0}, \]
where \( t_0 \) stands for the present time. Then, the substitution of (37) yields
\[ \omega \approx 3.4 \cdot 10^{-33} \text{ eV}, \quad t_0 \approx 8.9 \text{ Gyr}. \]
With these values of \( \omega \) and \( t_0 \), the contemporary value of the graviton mass \( m_g = \sqrt{-2W} \) becomes
\[ m_g \approx 3.6 \cdot 10^{-33} \text{ eV}, \]
which is more than ten orders of magnitude smaller than the upper bound reported by the LIGO experiment \[51\]. In what follows, I shall demonstrate how this value of \( m_g \) can be made even smaller.

**B. Simple bouncing Universe**

In the second example, I shall examine the scale factor
\[ a(t) = \sqrt[3]{e^{\omega t} - \omega t} \]  
(44a)
with the potential
\[ W = -\frac{\omega^2}{a^2}. \]

The corresponding background dynamics is shown in Fig. 2. It is a slowly contracting Universe with an exponential expansion after the bounce. In this respect, its late time behavior resembles that of the ΛCDM model. The target metric and the potential, needed for the construction of the action (8), are obtained by the replacement \( x^i \rightarrow \phi^i \) in the expressions (11) and (12). Skipping the details of the calculation, I shall only emphasize that \( F_{00} \) is easily checked to be strictly negative. Thus, the reality conditions (13) are everywhere satisfied. As a consequence, the dynamics of small perturbations of the background (44a) is regular and stable at all times.

The parameter \( \omega \) is determined from the measured values of the Hubble and deceleration parameters. To avoid cumbersome expressions, I shall do this by a graphical method. The graphs of \( H(t) \) and \( q(t) \) are displayed in Fig. 3. As one can see, for positive values of \( H \), the deceleration parameter \( q \) can never get close to the measured value \( q = -0.5 \). There are three possibilities of how to handle this situation. The first is to accept the closest allowed

![FIG. 2: Simple bouncing Universe.](image)

![FIG. 3: Hubble and deceleration parameters.](image)
value of \( q \), which is, in this case, \( q \approx -0.82 \). However, this is not well justified, as there is no reason to have such a confidence in a model that is still under construction. Instead, one can reformulate the model to allow for the needed value of \( q \). This sounds better, but it has its drawbacks. For one thing, ordinary matter has to be included. As ordinary matter increases the value of \( q \), it will spoil the chosen vacuum value \( q = -1/2 \). In fact, one should start with the smaller vacuum value to reach the needed \( q = -1/2 \) after the inclusion of matter. But we already have that smaller value, namely \( q \approx -0.82 \). Therefore, we should check how it is modified by the inclusion of matter.

Let me consider the simple case of a perfect fluid. Its stress-energy tensor has the form

\[
T^m_{\mu\nu} = (\rho + p) u_\mu u_\nu + pg_{\mu\nu},
\]

where \( \rho \) and \( p \) stand for the fluid energy density and pressure, and \( u^\mu \) is the fluid 4-velocity. In the case of spatially homogeneous and isotropic spaces, \( u^\alpha = 0 \) and \( u^0 = 1 \). In this example, I shall work in the gauge \( \phi^i = x^i \). In this gauge, the field equations take the form

\[
R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa T_{\mu\nu},
\]

where \( \kappa \equiv 8\pi G \) is the gravitational constant,

\[
T_{\mu\nu} \equiv T^m_{\mu\nu} + \frac{1}{\kappa} T^\phi_{\mu\nu},
\]

and

\[
T^\phi_{\mu\nu} \equiv F_{\mu\nu} - \frac{1}{2} (V + g^{\rho\sigma} F_{\rho\sigma}) g_{\mu\nu}.
\]

In the absence of matter, these field equations reduce to the geometric equations (7).

In what follows, I shall make use of the fact that \( T_{\mu\nu} \) can be rewritten as

\[
T_{\mu\nu} = (\bar{\rho} + \bar{p}) u_\mu u_\nu + \bar{p} g_{\mu\nu},
\]

where the effective energy density and pressure read

\[
\bar{\rho} \equiv \rho + \frac{1}{2\kappa} \left( V + F_{00} - 3W \frac{a^2}{\dot{a}^2} \right),
\]

\[
\bar{p} \equiv p - \frac{1}{2\kappa} \left( V - F_{00} - W \frac{a^2}{\dot{a}^2} \right).
\]

Here, the 4-velocity \( u_\mu \) equals \( \delta^0_\mu \) in accordance with the assumed spatial isotropy, and \( \dot{a} \) stands for the scale factor of the metric \( g_{\mu\nu} \). This is because the solution for \( g_{\mu\nu} \) is searched for in the form

\[
ds^2 = -dt^2 + \dot{a}^2(t) \left( dx^2 + dy^2 + dz^2 \right).
\]
(The scale factor \(a(t)\) determines the vacuum metric \(g^{(\omega)}_{\mu\nu}\), as defined by (10).) Now, the Einstein’s equations reduce to the familiar Friedman, and continuity equations

\[
\bar{H} = \frac{\kappa}{3} \bar{\rho}, \quad \dot{\bar{\rho}} + 3\bar{H} (\bar{\rho} + \bar{p}) = 0,
\]

where \(\bar{H} \equiv \dot{\bar{a}}/\bar{a}\). The matter field equations are represented by the equation of state

\[
p = w\rho .
\]

Now, one can use (45) and (46) to derive the differential equation for the scale factor \(\bar{a}\). As it turns out, it has the form

\[
2\bar{a} \ddot{\bar{a}} + (3w + 1) \dot{\bar{a}}^2 + \alpha \bar{a}^2 + (3w + 1) \beta = 0,
\]

where the coefficients \(\alpha\) and \(\beta\) are defined by

\[
\alpha \equiv -\frac{1}{2} \left[(w + 1) V + (w - 1) F_{00}\right], \quad \beta \equiv \frac{1}{2} W a^2.
\]

The fluid energy density \(\rho\) becomes

\[
\rho = \frac{3}{2\kappa} \left[2 (\bar{H}^2 - H^2) + W a^2 \left(\frac{1}{\bar{a}^2} - \frac{1}{a^2}\right)\right].
\]

The nonlinear differential equation (47) is too complicated to be solved analytically. For this reason, I solved it numerically for a number of initial conditions. The simplest solution is obtained if we choose \(\bar{a}(0) = 1, \dot{\bar{a}}(0) = 0\), which yields

\[
\bar{a}(t) = a(t), \quad \rho = 0
\]

for every \(w\). This simple solution can easily be verified analytically. It states that the absence of ordinary matter \((\rho = 0)\) brings us back to the geometric vacuum (10). The nontrivial solutions are obtained for other choices of initial conditions. But first, let me point out that a straightforward analysis of (47) shows that regular solutions with everywhere positive \(\rho\) do not exist. The regular solutions for \(\bar{a}\) turn out to oscillate around \(a\), thereby causing \(\rho\) to oscillate around \(\rho = 0\). The situation is similar to that of the ΛCDM model. To avoid negative \(\rho\), one must reconcile with the existence of singularities. As an example, I shall consider dust matter \((w = 0)\), and the initial conditions \(\bar{a}(0) \ll 1, \dot{\bar{a}}(0) \gg \omega\). These initial conditions ensure that singularity resides at \(t \approx 0\), and determine the amount of matter that
FIG. 4: Scale factor modified by dust matter.

FIG. 5: Hubble and deceleration parameters in the presence of dust matter.

agrees with observations. For \( \bar{a}(0) = 10^{-9}, \bar{a}(0) = 10^4 \omega \), the graph of the scale factor \( \bar{a}(t) \) is displayed in Fig. 4. The present time \( t_0 \), and the value of the parameter \( \omega \) are determined from the observed values of the Hubble and deceleration parameters. The graphs of \( \bar{H}(t) \) and \( \bar{q}(t) \) are displayed in Fig. 5. Unlike the matter-free values of \( H(t) \) and \( q(t) \), displayed in Fig. 3, the modified Hubble and deceleration parameters \( \bar{H}(t) \) and \( \bar{q}(t) \) have the solution \( t = t_0 \) for which \( \bar{q}(t_0) = -1/2, \bar{H}(t_0) > 0 \). One straightforwardly reads the values

\[
\omega t_0 = 2.70, \quad \bar{H}_0 = 0.43 \omega,
\]

which lead to

\[
\omega = 3.7 \times 10^{-33} \text{ eV}, \quad t_0 = 15.4 \text{ Gyr}.
\]

Now, it is straightforward to determine the graviton mass at the present time. One finds

\[
m_g = 2.3 \times 10^{-33} \text{ eV},
\]

which is only a fraction larger than its lower bound established in Sec. V A. In the next section, I shall provide an example of the Universe whose background value of \( q_0 \) is smaller than \(-1\). With the proper choice of the potential, the graviton mass will go far below the bound of Sec. V A.
Finally, let me calculate the present value of the energy density $\rho$, as given by the equation \[(48)\]. The corresponding density parameter $\Omega_m$ is defined by $\Omega_m = \rho/\rho_c$, where $\rho_c \equiv 3H^2/\kappa$ is the critical density at time $t$. The graph of the function $\Omega_m(t)$ is shown in Fig. 6. Given the present time $\omega t_0 = 2.7$, the current value of $\Omega_m$ is straightforwardly read to be

$$\Omega_m(t_0) \approx 0.3.$$ 

One should have in mind, however, that the value of $\Omega_m$ is very sensitive to the initial conditions chosen to solve the equation \[(47)\]. The value $\Omega_m = 0.3$ is obtained with the choice $\ddot{a}(0) = 10^{-9}$, $\dot{a}(0) = 10^4 \omega$. For different choices of initial conditions, one can obtain both, smaller and larger values of $\Omega_m$.

**C. Bouncing Universe with negligible graviton mass**

In this subsection, I shall examine the bouncing Universe defined by the scale factor

$$a = e^{\omega t} + \ln \left[1 + e^{-(\omega t + 9)}\right],$$ \[(49a)\]

and the potential

$$W = -2\omega^2 \exp \left[-e^{2(\omega t + 5)}\right].$$ \[(49b)\]

It is straightforward to verify the regularity conditions \[(13)\], but it is not so easy to prove stability. Indeed, the potential \[(49b)\] is not of the general form \[(39)\], for which the general stability arguments of Sec. \[36\] hold. This leads us to check the stability conditions \[(38)\] by direct calculation. First, I change the variables from $\Psi$ to $\Psi_1$, as defined in \[(28)\]. The transformed mass matrix $M_1$ is calculated from \[(30)\]. Once the matrix elements of $M_1$...
are known, one can determine the expressions $\det \mathcal{M}_1$ and $(\mathcal{M}_1)_{11} + (\mathcal{M}_1)_{22} - 2\sqrt{\det \mathcal{M}_1}$ to see if they are everywhere non-negative, as required by (38). As opposed to the case $W = -\omega^2/a^2$, these expressions are very complicated, and practically unusable for analytic studies. Instead, a computing program is used for drawing their graphs, and establishing their positivity. The graphs are displayed in Fig. 7. It is seen that the conditions (38) are everywhere satisfied, and therefore, the stability of the model is verified.

The model under consideration shares many features with the model of the preceding subsection. For this reason, I shall restrain from drawing graphs, and elaborating details. Let me just describe the basic features of the model. For one thing, it is a bouncing Universe with a linear contraction, and exponential expansion. The bounce is located at $\omega t \approx -4.5$. Both, the Hubble parameter $H$, and the deceleration parameter $q$ are monotonously increasing functions after the bounce. In particular, $q(t) < -1$ during the whole expanding phase. As in the preceding example, the inclusion of dust matter increases $q(t)$ to take values well above $q = -1/2$. The differential equation (47) has been solved numerically. With the initial conditions $\bar{a}(0) = 10^{-3}$ and $\dot{\bar{a}}(0) = 10^3 \omega$, one finds

$$\omega = 1.3 \cdot 10^{-33} \text{ eV}, \quad t_0 = 12.4 \text{ Gyr}.$$ 

With these data, one can readily calculate the present energy density $\rho_0$, and the present value of the graviton mass. The former is found to satisfy $\kappa \rho_0 = 1.5 \omega^2$, wherefrom one finds

$$\Omega_m \approx 0.3.$$ 

The latter is calculated by evaluating the potential $W$ in $t = t_0$. Thus, one obtains

$$m_g = 1.8 \cdot 10^{-22084} \text{ eV}.$$ 

FIG. 7: Graphs of $\det \mathcal{M}_1$ (black), and $(\mathcal{M}_1)_{11} + (\mathcal{M}_1)_{22} - 2\sqrt{\det \mathcal{M}_1}$ (gray).
This result shows that graviton mass can indeed be arbitrarily small. Precisely, the class of geometric sigma models considered in this paper contains a subclass characterized by the arbitrarily small graviton mass.

Before I close this subsection, let me comment on other masses that appear in the particle spectrum. The vector modes, as governed by the equation (22), have the same mass as tensor modes. Thus, their masses are as negligible as the graviton mass. However, their friction term is significantly different. Indeed, when calculated in \( t = t_0 \), it takes the value \( 2 \cdot 10^5 H_0 \), which is five orders of magnitude larger than that of the graviton. With such a big friction term, vector particles have probably decayed long time ago. As for the scalar modes, their mass squares are the eigenvalues \( \lambda_{\pm} \) of the mass matrix \( M_1 \). They are defined by

\[
\lambda_{\pm} = \frac{1}{2} \left\{ \begin{array}{c}
(M_1)_{11} + (M_1)_{22} \\
\pm \sqrt{[(M_1)_{11} + (M_1)_{22}]^2 - 4 \det M_1}
\end{array} \right\}.
\]

The present time values of the masses are found by calculating \( \lambda_{\pm} \) at \( t = t_0 \). One obtains

\[
m' \approx 10^{-30} \text{ eV}, \quad m'' \approx 0.8 \cdot 10^{-22079} \text{ eV}.
\]

At the same time, the respective friction coefficients \( f' \) and \( f'' \) are obtained by calculating the eigenvalues of the matrix \( F_1 \). They take the values

\[
f' \approx 2 \cdot 10^5 H_0, \quad f'' \approx H_0.
\]

It is seen that the heavier scalar mode quickly decays, and we are left with one scalar mode of negligible mass. The effective particle spectrum thus consists of one scalar and two tensor massless modes.

At the end, let me note that the above result is not unexpected. Indeed, the potential \( W \), as defined by (49b), has an extremely fast approach to zero as \( t \to \infty \). The zero value of \( W \), on the other hand, makes the target metric \( F_{ij} \) degenerate, as seen from the definition (11). The sigma model (8) then reduces to the sigma model with only one scalar field. As a consequence, the resulting dynamics at late times is expected to carry three effective degrees of freedom. This is an example of practical impossibility to distinguish between truly massless fields and those with extremely small masses.
VII. RECAPITULATION

I have shown in this paper that every bouncing metric can be a stable solution of a simple geometric sigma model. In many respects, these models look like ordinary sigma models that govern dynamics of four scalar fields minimally coupled to gravity. What makes them different is the way they are constructed. Namely, one first chooses the metric one would like to be the vacuum of the model, and then builds up the theory such that this metric becomes one of its solutions. The procedure that associates an action functional with every homogeneous, isotropic and spatially flat geometry is explained in Sec. II. It is seen that all four scalar fields of the model can be gauged away, leaving us with a purely metric theory. This is the reason these models are called geometric.

The construction scheme of Sec. II does not guarantee that small perturbations of the vacuum have stable dynamics. In fact, the resulting indefiniteness of the sigma model target metric suggests the opposite. This led me to perform a separate stability analysis. First, in Sec. IV I have proven that singularities that appear in the field equations are nonphysical. In particular, I have shown that all perturbations formed in the past regularly pass the bounce. In Sec. V their stability is proven. Specifically, for every background value of the scale factor $a(t)$ there is a class of potentials $W(t)$ that makes the dynamics of small perturbations stable. Vacuum stability against matter fluctuations is only shortly mentioned, because it is technically identical to that of Ref. [33]. The important conclusion is that ordinary matter does not compromise the stability of geometric sigma models.

In Sec. VI three simple examples are considered. The first has only been used for the demonstration of how the procedure described in Sec. II works in practice. The second example is about slowly contracting Universe that has exponential expansion after the bounce. The measured values of the Hubble and deceleration parameters are obtained after ordinary dust matter has been included. In this respect, this example resembles the ΛCDM model. The graviton mass is calculated to be more than ten orders of magnitude smaller than its upper bound reported by the LIGO experiment [51]. The third example has been included to show that, in some cases, the graviton mass can be made arbitrarily small. In this particular example, an extremely high friction coefficient makes the two vector and one scalar mode decay quickly after the bounce. Only two tensor modes, and one scalar mode survive. As it turns out, their masses are of the order $10^{-22000}$ eV. Moreover, with an appropriate
choice of the potential $W$, one can make these masses arbitrarily small.

Before I close this section, let me say something about physical consequences of the considered multi-scalar cosmological models. Specifically, a curious reader might be interested in what kind of experiment is needed to justify (or rule out) the suggested class of models. There are several observational possibilities to distinguish my multi-scalar theory from single-scalar theories commonly discussed in literature. First, as contrasted with the typical single-scalar models, the graviton in my model is necessarily massive. As the present time value of the graviton mass goes far beyond its observational bound, the only way to measure modifications caused by the graviton mass is the detection of possible anomalies in the behavior of large cosmic structures. We already know that massive graviton can only weaken the gravitational force at large distances, which is quite the opposite of what one needs to explain the anomalous galactic curves. This leaves us with clusters of galaxies, or voids, as possible candidates for anomalous behavior due to graviton mass. Second, as opposed to massive graviton, the scalars of the theory typically enhance gravitational force. Thus, one may hope that a properly defined cosmological scalar-tensor theory may explain the anomalous galactic curves. Obviously, multi-scalar theory is expected to be more effective than a single-scalar theory, when it comes to modification of the gravitational force. Additional calculations along these lines are needed for the full comparison of the models. The third possibility is the comparison of calculated cosmological parameters. The problem with this is that the considered class of geometric sigma models has a subclass which is arbitrarily close to the class of single-scalar models. As shown in the example [VIC], the two vector modes, and one scalar mode of this subclass have extremely high friction terms, so that they quickly decay after the bounce. The remaining 2 tensor and 1 scalar mode have negligible masses, which makes them practically indistinguishable from typical single-scalar degrees of freedom. As a consequence, if cosmological parameters of a single-scalar theory agree with observations, the difference between the two models cannot be established by mere comparison of calculated parameters. Otherwise, the observed cosmological parameters might clearly distinguish between the two theories. To examine this possibility, additional calculations are needed. Finally, one can try to detect the predicted scalar and vector particles in collisions of high energy particles. Owing to the extreme weakness of their coupling to ordinary matter, this seems unlikely to happen in the near future.

To summarize, I have shown in this paper that any bouncing metric can be a stable
solution of a simple model. One hopes that the class of bouncing cosmologies thus obtained could accommodate a viable cosmological model one searches for. The realization of this program is expected from future investigations along these lines.

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