Matrix nil-clean factorizations over abelian rings

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Abstract

A ring $R$ is nil-clean if every element in $R$ is the sum of an idempotent and a nilpotent. A ring $R$ is abelian if every idempotent is central. We prove that if $R$ is abelian then $M_n(R)$ is nil-clean if and only if $R/J(R)$ is Boolean and $M_n(J(R))$ is nil. This extend the main results of Breaz et al. [2] and that of Koşan et al. [4].

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1 Introduction

Let $R$ be a ring with an identity. An element $a \in R$ is called nil-clean if there exists an idempotent $e \in R$ such that $a - e \in R$ is a nilpotent. A ring $R$ is nil-clean provided that every element in $R$ is nil-clean. In [1, Question 3], Diesl asked: Let $R$ be a nil clean ring, and let $n$ be a positive integer. Is $M_n(R)$ nil clean? In [2, Theorem 3], Breaz et al. proved that their main theorem: for a field $K$, $M_n(K)$ is nil-clean if and only if $K \cong \mathbb{Z}_2$. They also asked if this result could be extended to division rings. As a main result in [4], Koşan et al. gave a positive answer to this problem. They showed that the preceding equivalence holds for any division ring.

A ring $R$ is abelian if every idempotent in $R$ is central. Clearly, every division ring is abelian. We extend, in this article, the main results of Breaz et al. [2, Theorem 3] and that of Koşan et al. [4, Theorem 3]. We shall prove that for an abelian ring $R$, $M_n(R)$ is nil-clean if and only if $R/J(R)$ is Boolean and $M_n(J(R))$ is nil. As a corollary, we also prove that the converse of a result of Koşan et al.’s is true.

Throughout, all rings are associative with an identity. $M_n(R)$ will denote the ring of all $n \times n$ full matrices over $R$ with an identity $I_n$. $GL_n(R)$ stands for the $n$-dimensional general linear group of $R$.

2 The main result

We begin with several lemmas which will be needed in our proof of the main result.

Lemma 2.1 [7, Proposition 3.15] Let $I$ be a nil ideal of a ring $R$. Then $R$ is nil-clean if and only if $R/I$ is nil-clean.

Lemma 2.2 [4, Theorem 3] Let $R$ be a division ring. Then the following are equivalent:

(1) $R \cong \mathbb{Z}_2$.
(2) $M_n(R)$ is nil-clean for all $n \in \mathbb{N}$.

(3) $M_n(R)$ is nil-clean for some $n \in \mathbb{N}$.

Recall that a ring $R$ is an exchange ring if for every $a \in R$ there exists an idempotent $e \in aR$ such that $1 - e \in (1 - a)R$. Clearly, every nil-clean ring is an exchange ring.

**Lemma 2.3** Let $R$ be an abelian exchange ring, and let $x \in R$. Then $RxR = R$ if and only if $x \in U(R)$.

**Proof** If $x \in U(R)$, then $RxR = R$. Conversely, assume that $RxR = R$. As in the proof of [3, Proposition 17.1.9], there exists an idempotent $e \in R$ such that $e \in xR$ such that $ReR = R$. This implies that $e = 1$. Write $xy = 1$. Then $yx = y(xy)x = (yx)^2$. Hence, $yx = y(yx)x$. Therefore $1 = x(yx)y = xy(xy)xy = yx$, and so $x \in U(R)$. This completes the proof. □

Set

$$J^*(R) = \bigcap \{P \mid P \text{ is a maximal ideal of } R\}.$$  

We will see that $J(R) \subseteq J^*(R)$. In general, they are not the same. For instance, $J(R) = 0$ and $J^*(R) = \{x \in R \mid \dim_F(xV) < \infty\}$, where $R = \text{End}_F(V)$ and $V$ is an infinite-dimensional vector space over a field $F$.

**Lemma 2.4** Let $R$ be an abelian exchange ring. Then $J^*(R) = J(R)$.

**Proof** Let $M$ be a maximal ideal of $R$. If $J(R) \nsubseteq M$, then $J(R) + M = R$. Write $x + y = 1$ with $x \in J(R), y \in M$. Then $y = 1 - x \in U(R)$, an absurd. Hence, $J(R) \subseteq M$. This implies that $J(R) \subseteq J^*(R)$. Let $x \in J^*(R)$, and let $r \in R$. If $R(1 - xr)R \neq R$, then we can find a maximal ideal $M$ of $R$ such that $R(1 - xr)R \subseteq M$, and so $1 - xr \in M$. It follows that $1 = xr + (1 - xr) \in M$, which is imposable. Therefore $R(1 - xr)R = R$. In light of Lemma 2.3, $1 - xr \in U(R)$, and then $x \in J(R)$. This completes the proof. □

A ring $R$ is local if $R$ has only maximal right ideal. As is well know, a ring $R$ is local if and only if for every $a \in R$, either $a$ or $1 - a$ is invertible if and only $R/J(R)$ is a division ring.

**Lemma 2.5** Let $R$ be a ring with no non-trivial idempotents, and let $n \in \mathbb{N}$. Then the following are equivalent:

(1) $M_n(R)$ is nil-clean.

(2) $R/J(R) \cong \mathbb{Z}_2$ and $M_n(J(R))$ is nil.

**Proof** (1) $\Rightarrow$ (2) In view of [1, Proposition 3.16], $J(M_n(R))$ is nil, and then so is $M_n(J(R))$.

Let $a \in R$. By hypothesis, $M_n(R)$ is nil-clean. If $n = 1$, then $R$ is nil-clean. Then $a \in N(R)$ or $a - 1 \in N(R)$. This shows that $a \in U(R)$ or $1 - a \in U(R)$, and so $R$ is local. That is, $R/J(R)$ is a division ring. As $R/J(R)$ is nil-clean, it follows from Lemma 2.3 that $R/J(R) \cong \mathbb{Z}_2$. We now assume that $n \geq 2$. Then there exists an idempotent $E \in M_n(R)$ and a nilpotent $W \in GL_n(R)$ such that
\[
I_n + \begin{pmatrix}
a & 0 & \cdots & 0 \\
0 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & 0
\end{pmatrix} = E + W. \text{ Set } U = -I_n + W. \text{ Then } U \in GL_n(R). \text{ Hence, }
\]
\[
U^{-1} \begin{pmatrix}
a & 0 & \cdots & 0 \\
0 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & 0
\end{pmatrix} = U^{-1}E + I_n = (U^{-1}EU)U^{-1} + I_n.
\]

Set \( F = U^{-1}EU \). Then \( F = F^2 \in M_n(R) \), and that
\[
(I_n - F)U^{-1} \begin{pmatrix}
a & 0 & \cdots & 0 \\
0 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & 0
\end{pmatrix} = I_n - F.
\]

Write \( I_n - F = \begin{pmatrix} e & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 0 \end{pmatrix} \). As \( R \) possesses no non-trivial idempotents, \( e = 0 \) or \( 1 \). If \( e = 0 \), then \( I_n - F = 0 \), and so \( E = I_n \). This shows that \( \begin{pmatrix} a & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 0 \end{pmatrix} = W \) is nilpotent; hence that \( a \in R \) is nilpotent. Thus, \( 1 - a \in U(R) \).

If \( e = 1 \), then \( F = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix} \). Write \( U^{-1} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \), where \( \alpha \in R, \beta \in M_{1 \times (n-1)}(R), \gamma \in M_{(n-1) \times 1}(R) \) and \( \delta \in M_{(n-1) \times (n-1)}(R) \). Then
\[
\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} a & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & I_{n-1} \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} + I_n,
\]

where \( x \in M_{(n-1) \times 1}(R) \). Thus, we get
\[
\alpha a = 1, \gamma a = x\alpha + \gamma, 0 = x\beta + \delta + I_{n-1}.
\]

One easily checks that
\[
\begin{pmatrix} 1 & \beta \\ 0 & I_{n-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ x & I_{n-1} \end{pmatrix} U^{-1} \begin{pmatrix} 1 & 0 \\ \gamma a & I_{n-1} \end{pmatrix} = \begin{pmatrix} \alpha + \beta \gamma a & 0 \\ 0 & -I_{n-1} \end{pmatrix}.
\]

This implies that \( u := \alpha + \beta \gamma a \in U(R) \). Hence, \( \alpha = u - \beta \gamma a \). It follows from \( \alpha a = 1 \) that \( (u - \beta \gamma a)a = 1 \). As \( R \) is connected, we see that \( a(u - \beta \gamma a) = 1 \), and so \( a \in U(R) \). This shows that \( a \in U(R) \) or
1 − a ∈ U(R). Therefore R is local, and then R/J(R) is a division ring. Since M_n(R) is nil-clean, we see that so is M_n(R/J(R)). In light of Lemma 2.2 R/J(R) ∼= Z_2, as desired.

(2) ⇒ (1) In view of Lemma 2.1, M_n(R/J(R)) is nil-clean. Since M_n(R)/J(M_n(R)) ∼= M_n(R/J(R)) and J(M_n(R)) = M_n(J(R)) is nil, it follows from Lemma 2.2 that M_n(R) is nil-clean, as asserted.

Example 2.6 Let K be a field, and let R = K[x, y]/(x, y)^2. Then M_n(R) is nil-clean if and only if K ∼= Z_2. Clearly, J(R) = (x, y)/(x, y)^2, and so R/J(R) ∼= K. Thus, R is a local ring with a nilpotent Jacobson radical. Hence, R has no non-trivial idempotents. Thus, we are done by Lemma 2.2.

We come now to our main result.

Theorem 2.7 Let R be abelian, and let n ∈ N. Then the following are equivalent:

(1) M_n(R) is nil-clean.

(2) R/J(R) is Boolean and M_n(J(R)) is nil.

Proof (1) ⇒ (2) Clearly, M_n(J(R)) is nil. Let M be a maximal ideal of R, and let ϕ_M : R → R/M. Since M_n(R) is nil-clean, then so is M_n(R/M). Hence, R/M is an exchange ring with all idempotents central. In view of [3 Lemma 17.2.5], R/M is local, and so R/M is connected. In view of Lemma 2.2, R/M (R/M) ∼= Z_2. Write J(R/M) = K/M. Then K is a maximal ideal of R, and that M ⊆ K. This implies that M = K; hence, R/M ∼= Z_2. Construct a map ϕ_M : R/J*(R) → R/M, r + J*(R) ↦ r + M. Then ∩ M Kerϕ_M = ∩{r + J*(R) | r ∈ M} = 0. Therefore R/J*(R) is isomorphic to a subdirect product of some Z_2. Hence, R/J*(R) is Boolean. In light of Lemma 2.4, R/J(R) is Boolean, as desired.

(2) ⇒ (1) Since R/J(R) is Boolean, it follows by [2] Corollary 6] that M_n(R/J(R)) is nil-clean. That is, M_n(R)/J(M_n(R)) is nil-clean. But J(M_n(R)) = M_n(J(R)) is nil. Therefore we complete the proof, by Lemma 2.1.

We note that the "(2) ⇒ (1)" in Theorem 2.7 always holds, but "abelian" condition is necessary in "(1) ⇒ (2)". Let R = M_n(Z_2) (n ≥ 2). Then R is nil-clean. But R/J(R) is not Boolean. Here, R is not abelian.

Corollary 2.8 Let R be commutative, and let n ∈ N. Then the following are equivalent:

(1) M_n(R) is nil-clean.

(2) R/J(R) is Boolean and J(R) is nil.

(3) For any a ∈ R, a − a^2 ∈ R is nilpotent.

Proof (1) ⇒ (3) Let a ∈ R. In view of Theorem 2.7, a − a^2 ∈ J(R). Since R is commutative, we see that J(R) is nil if and only if J(M_n(R)) is nil. Therefore a − a^2 ∈ R is nilpotent.

(3) ⇒ (2) Clearly, R/J(R) is Boolean. For any a ∈ J(R), we have (a − a^2)^n = 0 for some n ≥ 1. Hence, a^n(1 − a)^n = 0, and so a^n = 0. This implies that J(R) is nil.

(2) ⇒ (1) As R is commutative, we see that M_n(J(R)) is nil. This completes the proof, by Theorem 2.7.

Furthermore, we observe that the converse of [2 Corollary 7] is true as the following shows.

Corollary 2.9 A commutative ring R is nil-clean if and only if M_n(R) is nil-clean.
**Proof**  One direction is obvious by \[2\] Corollary 7. Suppose that \(M_n(R)\) is nil-clean. In view of Corollary \[2.8\] that \(R/J(R) \cong \mathbb{Z}_2\) is nil-clean, and that \(J(R)\) is nil. Therefore \(R\) is nil-clean, by Lemma \[2.1\].

**Example 2.10**  Let \(m, n \in \mathbb{N}\). Then \(M_n(\mathbb{Z}_m)\) is nil-clean if and only if \(m = 2^r\) for some \(r \in \mathbb{N}\). Write \(m = p_1^{r_1} \cdots p_s^{r_s}\) (\(p_1, \cdots, p_s\) are distinct primes, \(r_1, \cdots, r_s \in \mathbb{N}\)). Then \(\mathbb{Z}_m \cong \mathbb{Z}_{p_1^{r_1}} \oplus \cdots \oplus \mathbb{Z}_{p_s^{r_s}}\). In light of Corollary \[2.8\] \(M_n(\mathbb{Z}_m)\) is nil-clean if and only if \(s = 1\) and \(\mathbb{Z}_{p_1^{r_1}}\) is nil-clean. Therefore we are done by Lemma \[2.3\].

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