LATTICE-ORDERED ABELIAN GROUPS FINITELY GENERATED AS SEMIRINGS

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Abstract. A lattice-ordered group (an $\ell$-group) $G(\oplus, \lor, \land)$ can be naturally viewed as a semiring $G(\lor, \oplus)$. We give a full classification of (abelian) $\ell$-groups which are finitely generated as semirings, by first showing that each such $\ell$-group has an order-unit so that we can use the results of Busaniche, Cabrer and Mundici [3]. Then we carefully analyze their construction in our setting to obtain the classification in terms of certain $\ell$-groups associated to rooted trees (Theorem 4.1).

This classification result has a number of important applications: for example it implies a classification of finitely generated ideal-simple (commutative) semirings $S(\oplus, \cdot)$ with idempotent addition and provides important information concerning the structure of general finitely generated ideal-simple (commutative) semirings, useful in obtaining further progress towards Conjecture 1.1 discussed in [2], [8].

1. Introduction

Lattice-ordered groups (or $\ell$-groups for short) have long played an important role in algebra and related areas of mathematics. Let us briefly mention the relation to functional analysis and logic via the correspondence with MV-algebras [1], [6], or the fact that the theory of factorization and divisibility on a Bézout domain yields an abelian $\ell$-group. For this and further applications see eg. [1] or [2].

Recently, there have been several interesting results concerning unital $\ell$-groups. For example, Busaniche, Cabrer and Mundici [3] classified finitely generated unital (abelian) $\ell$-groups $G$ using the combinatorial notion of a stellar sequence, which is a sequence $|\Delta_0| \supset |\Delta_1| \supset \ldots$ of certain simplicial complexes in $[0,1]^n$. The idea is that each such $G$ is of the form $G \simeq \mathcal{M}(\{0,1\}^n)/I$, where $\mathcal{M}(\{0,1\}^n)$ is the $\ell$-group of all piecewise linear functions $f : [0,1]^n \to \mathbb{R}$ and $I$ is the set of all functions $f$ such that $f(|\Delta_i|) = 0$ for some $i$.

The aim of this paper is to explore and use the connections between semirings and $\ell$-groups in the study of simple semirings. Namely, an $\ell$-group $G(\oplus, \lor, \land)$ is also a semiring $G(\lor, \oplus) = S(\oplus, \cdot)$ such that the semiring addition $\oplus$ is idempotent. By removing the idempotency condition, one obtains the notion of a parasemifield, i.e., a commutative semiring $S(\oplus, \cdot)$ such that its multiplicative structure forms a group. (See the beginnings of Sections 2 and 3 for precise definitions of the notions concerning $\ell$-groups and semirings, respectively.)

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In fact, it is not hard to observe that there is a one-to-one correspondence between lattice-ordered groups and additively idempotent parasemifields (i.e., satisfying $a + a = a$ for all $a$). This correspondence preserves finite generation in the sense that an $\ell$-group is finitely generated if and only if the corresponding parasemifield is finitely generated. However, these are not equivalent to the property of being finitely generated as a semiring, which is stronger.

We shall assume all $\ell$-groups and semirings to be automatically commutative, as we will be dealing only with these throughout the paper.

Our first result is Theorem 3.6 in which we show that every additively idempotent parasemifield, finitely generated as a semiring, is unital in the $\ell$-group sense. Hence it is natural to inquire whether we can identify the ones, which are finitely generated as semirings, among the unital $\ell$-groups from the classification. The answer is yes, although the proof is fairly involved and requires a careful discussion of the geometry of stellar sequences. The resulting Theorem 4.1 classifies all additively idempotent parasemifields which are finitely generated as semirings.

Given the basic and fundamental nature of the notion of a semiring, it is not surprising that there is a wide variety of applications of semirings and semifields, ranging from cryptography and other areas of computer science to dequantization, tropical mathematics and geometry – see for example, [7], [13], [14], and [20] for overviews of some of the applications and for further references.

Many parts of the structural theory of semirings and semifields mimic analogous results concerning rings and fields, see, e.g., [11]. However, much less is known overall: for instance, whereas simple commutative rings are just fields and are known very explicitly, the analogous results for semirings are more subtle. First of all, one has to distinguish between congruence-simple and ideal-simple semirings. Bashir, Hurt, Jančarík and Kepka [2] classified the congruence-simple ones and reduced the study of ideal-simple semirings to the study of parasemifields.

Together with their results, our Theorem 4.1 implies a full classification of additively idempotent finitely generated ideal-simple semirings. The structure of this classification follows Theorem 11.2 and Section 12 of [2], but it is fairly technical, so we don’t state the final result explicitly.

We have already mentioned that additively idempotent parasemifields are just $\ell$-groups; the present Theorem 4.1 classifies those which are finitely generated semirings. A natural question to ask then is what is the structure of such parasemifields without the idempotency assumption. Note that the corresponding result concerning rings is that if a field is finitely generated as a ring, then it is finite.

There are no finite parasemifields and in fact, we have the following conjecture:

**Conjecture 1.1** ([2], [8]). Every parasemifield which is finitely generated as a semiring is additively idempotent.

Ježek, Kala and Kepka [8] proved this in the case of at most two generators by studying the geometry of semigroups $C(S) \subset \mathbb{N}_0^2$ attached to parasemifields $S$. (For the definition and basic information on the semigroups $C(S)$, see Section 3.) Since each parasemifield $S$ has an additively idempotent factor $S/\sim$ such that the semigroup $C(S)$ is equal to $C(S/\sim)$, one can use Theorem 4.1 to obtain refined information on the structure of the semigroup $C(S) \subset \mathbb{N}_0^m$ in general.
In a work in progress, the author and Korbelář use this to prove Conjecture in the case of three generators. It seems quite possible that a similar approach will yield a proof of this conjecture in general. Our Theorem would then provide all parasemifields, finitely generated as a semiring and hence, again using the results of, imply a complete classification of finitely generated ideal-simple semirings (see for some details).

There are various natural ways of extending and generalizing the classification of finitely generated unital \( \ell \)-groups. Let us just mention the cases of \( \ell \)-groups which are not assumed to be unital, of finitely generated parasemifields, or even of non-commutative finitely generated parasemifields. To the author’s knowledge, not much is known about any of these interesting problems.

As for the contents of this paper, Section 2 reviews the definitions and basic facts on \( \ell \)-groups, including the statement of the classification of finitely generated unital ones (and the required notions concerning simplicial and abstract complexes). Then in Section 3 we briefly review some preliminaries on semirings and parasemifields and prove that if an additively idempotent parasemifield is finitely generated as a semiring, then it is unital. For the sake of completeness we outline the proofs of some classical results concerning semirings that we need. In Section 4 we then give the classification of additively idempotent parasemifields, finitely generated as semirings.

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2. \( \ell \)-Groups and Complexes

In this section we briefly review some basics about \( \ell \)-groups and simplicial complexes that we will need, including the classification of Busaniche, Cabrer and Mundici. Our treatment is quite terse, but we at least try to provide a brief (very) informal overview at the end of this section. For a more detailed treatment we refer the reader to the paper. Also see where rational polyhedra are used in the study of projective unital \( \ell \)-groups. For more general background information on \( \ell \)-groups see for example or.

A lattice-ordered abelian group (\( \ell \)-group for short) \((G, +, -, 0, \vee, \wedge)\) is an algebraic structure such that \((G, +, -, 0)\) is an abelian group, \((G, \vee, \wedge)\) is a lattice, and \(a + (b \vee c) = (a + b) \vee (a + c)\) for all \(a, b, c \in G\).

An order-unit \(u \in G\) is an element such that for each \(g \in G\) there exists \(n \in \mathbb{N}\) so that \(nu \geq g\) (i.e., \(nu \vee g = nu\)). A unital \( \ell \)-group \((G, u)\) is an \( \ell \)-group with an order-unit \(u\). A unital \( \ell \)-homomorphism is a homomorphism of \( \ell \)-groups which maps one order-unit to the other one. An \( \ell \)-ideal is the kernel of a unital \( \ell \)-homomorphism; any \( \ell \)-ideal \(I\) then determines the factor-homomorphism \(G \to G/I\).

Let us now review the classification of. Denote by \(\mathcal{M}([0,1]^n)\) the set of piecewise linear continuous functions \(f : [0,1]^n \to \mathbb{R}\) such that each piece has integral coefficients (and the number of pieces is finite). \(\mathcal{M}([0,1]^n)\) is a group under pointwise addition of functions and we can define \((f \vee g)(x) = \max(f(x), g(x))\) and
A deletion of a maximal set of \( \Sigma \) and \( \omega \) is an abstract simplicial complex and \( (0) \in \Delta \). Also, for \( D \subset \{0,1\}^n \) we define \( \mathcal{M}(D) \) as the \( \ell \)-group whose elements are restrictions \( f|D \in \mathcal{M}(\{0,1\}^n) \) to \( D \). \( \mathcal{M}(D) \) is thus a factor of \( \mathcal{M}(\{0,1\}^n) \).

The classification then says that each finitely generated unital \( \ell \)-group is of the form \( \mathcal{M}(\{0,1\}^n)/I \) for an explicitly defined \( \ell \)-ideal \( I \) (and provides a criterion for when two ideals give the same \( \ell \)-group). The ideal \( I \) comes from a stellar sequence \( \mathcal{W} \) of simplicial complexes as follows: from \( \mathcal{W} \) we construct a sequence \( \mathcal{P}_0 \supset \mathcal{P}_1 \supset \mathcal{P}_2 \supset \ldots \) of polyhedra in \([0,1]^n\) and define \( I = \{ f \in \mathcal{M}(\{0,1\}^n) | f(P_i) = 0 \text{ for some } i \} \). To give more details we first need to give some definitions concerning (abstract) simplicial complexes, following \([3]\).

We assume the reader is familiar with the usual notion of a (simplicial) complex in \( \mathbb{R}^n \). Let us just note that a simplex is a convex hull of a finite set of points, a \( k \)-simplex is a simplex of dimension \( k \), a complex \( \mathcal{K} \) is a finite set of simplexes such that if \( T_1, T_2 \) are simplexes with \( \dim T_1 = \dim T_2 - 1 \), \( T_1 \subset \partial T_2 \), and \( T_2 \in \mathcal{K} \), then also \( T_1 \in \mathcal{K} \) (where by \( \partial T \) we denote the boundary of \( T \) ). The support \( |\mathcal{K}| \) of a complex \( \mathcal{K} \) is the union of all simplexes in \( \mathcal{K} \). Throughout this paper we shall often identify a complex with its support. A simplex \( \text{conv}(v_0, \ldots, v_k) \) is rational if all the coordinates of all the vertices \( v_i \) are rational. A complex is rational if all its simplexes are rational. For more background information on simplicial complexes, see for example \([2]\).

**Definition 2.1** (\([3]\), page 262). A (finite) abstract simplicial complex is a pair \( H = (\mathcal{V}, \Sigma) \), where \( \mathcal{V} \) is a non-empty finite set of vertices of \( H \) and \( \Sigma \) is a collection of subsets of \( \mathcal{V} \) whose union is \( \mathcal{V} \) with the property that every subset of an element of \( \Sigma \) is again an element of \( \Sigma \). Given \( \{v, w\} \in \Sigma \) and \( a \notin \mathcal{V} \) we define the binary subdivision \( \langle\{v, w\}, a\rangle \) of \( H \) as the abstract simplicial complex \( \langle\{v, w\}, a\rangle \) obtained by adding \( a \) to the vertex set and replacing every set \( \{v, w, u_1, \ldots, u_k\} \in \Sigma \) by the two sets \( \{v, a, u_1, \ldots, u_k\} \) and \( \{a, w, u_1, \ldots, u_k\} \) and all their subsets.

A weighted abstract simplicial complex is a triple \( W = (\mathcal{V}, \Sigma, \omega) \) where \( (\mathcal{V}, \Sigma) \) is an abstract simplicial complex and \( \omega \) is a map of \( \mathcal{V} \) into \( \mathbb{N} \). For \( \{v, w\} \in \Sigma \) and \( a \notin \mathcal{V} \), the binary subdivision \( \langle\{v, w\}, a\rangle \mathcal{V}, \Sigma\rangle \) is the abstract simplicial complex \( \langle\{v, w\}, a\rangle \mathcal{V}, \Sigma\rangle \) equipped with the weight function \( \tilde{\omega} : \mathcal{V} \cup \{a\} \to \mathbb{N} \) given by \( \tilde{\omega}(a) = \tilde{\omega}(v) + \tilde{\omega}(w) \) and \( \tilde{\omega}(u) = \omega(u) \) for all \( u \in \mathcal{V} \).

**Definition 2.2** (\([3]\), page 264). Let \( W = (\mathcal{V}, \Sigma, \omega) \) and \( W' \) be two weighted abstract simplicial complexes. A map \( b : W \to W' \) is a stellar transformation if it is either a deletion of a maximal set of \( \Sigma \) or a binary subdivision or the identity map.

A sequence \( \mathcal{W} = (W_0, W_1, W_2, \ldots) \) of weighted abstract simplicial complexes is stellar if \( W_{i+1} \) is obtained from \( W_i \) by a stellar transformation.

**Definition 2.3** (\([3]\), page 263). Let now \( W = (\mathcal{V}, \Sigma, \omega) \) be an abstract simplicial complex with the set of vertices \( \mathcal{V} = \{v_1, \ldots, v_n\} \). Choose the standard basis \( e_1, \ldots, e_n \) of \( \mathbb{R}^n \) and let \( \Delta_W \) be the complex whose vertices are \( v'_0 = e_1/\omega(v_1) \), \( v'_1 = e_1/\omega(v_2) \), and \( v'_n = e_1/\omega(v_n) \) and whose \( k \)-dimensional simplexes are given by \( \text{conv}(v'_0, \ldots, v'_{i(k)}) \in \Delta_W \) if and only if \( \{v_{i(0)}, \ldots, v_{i(k)}\} \in \Sigma \).
Then $\Delta_W$ is a complex, $|\Delta_W| \subset [0,1]^n$ and we have a map $i : V \to |\Delta_W|$ given by $i(v) = v'$, the so called canonical realization of $W$.

Definition 2.4 ([3], pages 256-257). Let $K$ be a complex and $p \in |K| \subset \mathbb{R}^n$ a point in $K$. Then blow-up $K(p)$ of $K$ at $p$ is the complex obtained by replacing each simplex $T \in K$ that contains $p$ by the set of all simplexes of the form $\text{conv}(F \cup \{p\})$, where $F$ is any face of $T$ not containing $p$.

For a rational 1-simplex $E = \text{conv}(v,w) \in \mathbb{R}^n$ we define the Farey mediant of $E$ as the rational point $u = \frac{\text{den}(v)+\text{den}(w)}{\text{den}(v)+\text{den}(w)} \in E$ (where $\text{den}(v)$ denotes the least common denominator of the coordinates of a vector $v$).

If $E$ belongs to a rational complex $K$ and $v$ is the Farey mediant of $E$, the (binary) Farey blow-up is the blow-up $K(v)$.

Remark 2.5 ([3], Lemma 4.4). Note that if $W = (W_0, W_1, W_2, \ldots)$ is a stellar sequence of weighted abstract simplicial complexes and $i_0 : V_0 \to |\Delta_0|$ the canonical realization, we can naturally extend this to attach a complex $\Delta_i = \Delta_{W_i}$ to each $W_i$:

Let $b_0 : W_0 \to W_1$ be the given stellar transformation. We define $\Delta_1$ as follows:

If $b_1$ deletes a maximal set $M \in \Sigma$, we delete the corresponding maximal simplex from $\Delta_0$. If $b_1$ is a binary subdivision $\{\{a,b\}, c\}W_0$ at some $E = \{a,b\} \in \Sigma$, let $e$ be the Farey mediant of the 1-simplex $\text{conv}(i_0(E))$. Then $\Delta_1$ is the Farey blow-up of $\Delta_0$ at $e$. If $b_1$ does not do anything, we also keep $\Delta_0$ unchanged.

In all cases we accordingly modify $i_0$ to obtain a realization $i_1 : V_1 \to |\Delta_1|$. Then we can continue by considering $b_1 : W_1 \to W_2$, and so on.

Eventually we get a sequence of complexes corresponding to $[0,1]^n \supset |\Delta_0| \supset |\Delta_1| \supset \ldots$.

Definition 2.6 ([3], Lemma 2.3). Given a sequence $P = (P_1 \supset P_2 \supset \ldots)$ of subsets of $[0,1]^n$, define an $\ell$-ideal $I$ of $\mathcal{M}([0,1]^n)$ by $I = \{ f \in \mathcal{M}([0,1]^n) | f(P_i) = 0 \text{ for some } i \}$. This gives an $\ell$-group $\mathcal{M}([0,1]^n)/I$.

Theorem 2.7 ([3], Theorem 5.1). For every finitely generated unital $\ell$-group $(G, u)$ there is a stellar sequence $W = (W_0, W_1, W_2, \ldots)$ such that $(G, u) \simeq \mathcal{G}(W)$, where $\mathcal{G}(W) = \mathcal{M}([0,1]^n)/I$ for $I = \text{the ideal corresponding to the sequence } [0,1]^n \supset |\Delta_0| \supset |\Delta_1| \supset \ldots$ defined using $W$ as in 2.6.

All this is not nearly as complicated as it sounds: we start with suitable complex $\Delta_0$ and then modify it in infinitely many steps. In each step we either

- delete a maximal simplex from the previous complex, or
- suitably divide a 1-dimensional simplex $E$ into two (and then we have to correspondingly divide all the simplexes containing $E$), or
- don’t do anything.

This produces a sequence $[0,1]^n \supset |\Delta_0| \supset |\Delta_1| \supset \ldots$ and we define $G = \mathcal{M}([0,1]^n)/I$, where $I$ is the set of all functions $f$ such that $f(|\Delta_i|) = 0$ for some $i$. Every finitely generated unital $\ell$-group is obtained in this way.

3. Existence of order-unit

Let us now review the connection between $\ell$-groups and semirings.

By a (commutative) semiring we shall mean a non-empty set $S$ equipped with two associative and commutative operations (addition and multiplication) where the multiplication distributes over the addition from both sides. We shall be dealing with commutative semirings only, so we usually just call them semirings.
A non-trivial semiring $S$ is a parasemifield if the multiplication defines a non-trivial group. A non-trivial semiring $S$ is a semifield if there is an element $0 \in S$ such that $0 \cdot S = 0$ and such that the set $S \setminus \{0\}$ is a group (for the semiring multiplication).

A semiring is additively idempotent if $x + x = x$ for all $x \in S$.

**Proposition 3.1** ([13], [18]). There is a one-to-one correspondence between additively idempotent parasemifields (i.e., $a + a = a$ for all $a \in S$) and $\ell$-groups.

**Proof.** Let $S$ be an additively idempotent parasemifield and define $a \lor b = a + b$, $a \land b = (a^{-1} + b^{-1})^{-1}$. Then $(S, \cdot, -1, 1, \lor, \land)$ is an $\ell$-group. Conversely, if $(S, \cdot, -1, 1, \lor, \land)$ is an $\ell$-group (written multiplicatively), then $(S, +, \cdot)$ is an additively idempotent parasemifield, where $a + b = a \lor b$. □

We define an ordering $\leq$ on a semiring $S$ by $a \leq b$ if and only if $a = b$ or there exists $c \in S$ such that $a + c = b$. Note that it is preserved by addition and multiplication in $S$. Also, this is the same ordering as the one on the corresponding $\ell$-group.

Note that if $S$ is a parasemifield, then the ordering $\leq$ on $S$ is antisymmetric:

**Proposition 3.2** ([7], Proposition 20.37). Let $S$ be a parasemifield. For all $a, b, c \in S$ we have:

a) If $a + b + c = a$, then $a + b = a$.

b) If $a \leq b \leq a$, then $a = b$.

**Proof.** a) Let $a + b + c = a$. Multiply both sides by $a^{-2}b$ and then add $a^{-1}c$. We get $a^{-1}b + a^{-2}b^2 + a^{-2}bc + a^{-1}c = a^{-1}b + a^{-1}c$, and so $(a^{-1}b + a^{-1}c)(a^{-1}b + 1) = (a^{-1}b + a^{-1}c)$. Dividing by $a^{-1}b + a^{-1}c$ we get $a^{-1}b + 1 = 1$ as needed.

b) Write $b = a + x$ and $a = a + x + y$. By part a), $a = a + x = b$. □

**Definition 3.3.** An additively idempotent parasemifield $S$ is order-unital if there exists an element $u \in S$ such that for each $s \in S$ there is $n \in \mathbb{N}$ so that $u^n s + 1 = 1$.

Note that this definition is equivalent to the corresponding definition of a unital $\ell$-group. For if $v \in S$ is an order-unit in the $\ell$-group sense, we have that for each $s \in S$ there is some $n \in \mathbb{N}$ so that $v^n \geq s$. Choose now $u = v^{-1}$. Then $1 \geq u^n s$, and so $1 = u^n s + t$ for some $t \in S$. Now $1 + u^n s = (u^n s + t) + u^n s = u^n s + t = 1$. Conversely, if $u$ is an element from the definition then $v = u^{-1}$ will be an order-unit in the $\ell$-group sense.

For more background information on semirings see for example [7].

As usual, $\mathbb{N}$ and $\mathbb{Q}^+$ denote the semirings of positive integers and rational numbers, respectively; $\mathbb{N}_0$ is the semiring of non-negative integers.

We will need some basic properties of (finitely generated) parasemifields.

In the rest of this section, let $S$ be a parasemifield $m$-generated as a semiring. That means that there is a surjective semiring homomorphism $\varphi : \mathbb{N}[x_1, \ldots, x_m] \to S$ (where $x_i$ are indeterminates). For $a = (a_1, \ldots, a_m) \in \mathbb{N}_0^m$ we use the notation $x^a = x_1^{a_1} \cdots x_m^{a_m}$.

Let $A$ be the prime subparasemifield of $S$, i.e., the smallest (possibly trivial) parasemifield contained in $S$.

Let $Q$ be the subsemiring of elements which are smaller than some element of $A$, i.e., $Q = \{ s \in S | \exists q \in A : s \leq q \}$. Let $\mathcal{C} = \mathcal{C}(S) = \{ a \in \mathbb{N}_0^m | \varphi(x^a) \in Q \}$ be the corresponding semigroup (or a cone) in $\mathbb{N}_0^m$. 


The structure of $Q$ and $C$ carries a lot of information about $S$. For example, in 
\cite{8} it was used to show that every parasemifield, two generated as a semiring, is
additively idempotent. In general, Scholle has studied these semigroups $C$ in quite
some detail in his Master’s Thesis \cite{17}.

**Proposition 3.4.** a) If $S$ is additively idempotent, then $A = \{1\}$. Otherwise
$A \cong Q^+.

b) For $q_1, q_2 \in S$ we have $q_1 + q_2 \in Q$ if and only if $q_1, q_2 \in Q$. For $q \in S$, $n \in \mathbb{N}$
we have $q^n \in Q$ if and only if $q \in Q$.

c) $C$ is a pure subsemigroup of $\mathbb{N}_0^m$, i.e., it is closed under addition and for $n \in \mathbb{N}$
and $a \in \mathbb{N}_0^m$ we have $na \in C$ if and only if $a \in C$.

**Proof.** This is a summary of various statements in \cite{8} and \cite{11}. We just sketch the
proofs.

a) $Q^+$ is the free 0-generated parasemifield. Therefore $A$ is a factor of $Q^+$. Now
it suffices only to note that $Q^+$ is congruence simple.

b) $q_1 + q_2 \in Q$ means that $q_1 + q_2 \leq s$ for some $s \in A$. Therefore $q_i \leq q_1 + q_2 \leq s$
and $q_i \in Q$ ($i = 1, 2$).

c) Follows directly from b). \hfill \Box

We shall use the structure of $C$ to show Theorem \ref{thm:3.6}. In particular, we will need
the following proposition which essentially says that there is an element $c$ which is
“inside” the cone $C$.

**Proposition 3.5.** There exists $c \in C$ such that:

a) $c + e_i \in C$ for each $i = 1, \ldots, m$, where $e_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{N}_0^m$ is the
vector having 1 at the $i$-th position and 0 elsewhere.

b) $nc + a \in C$ for each $a = (a_i) \in \mathbb{N}_0^m$, where $n = a_1 + \cdots + a_m$.

**Proof.** a) Take $f = 1 + x_1 + \cdots + x_m \in \mathbb{N}[x_1, \ldots, x_m]$. Since $S$ is a parasemifield,
there is $g = \sum_j a_j x^{c(j)}$ (where $a_j \in \mathbb{N}$) such that $\varphi(g)$ is the inverse of $\varphi(f)$ in $S$,
i.e., $\varphi(fg) = 1$. Thus $\varphi(fg) \in Q$, and since $fg = \sum_j a_j (x^{c(j)} + x^{e_1 + \cdots + x^{e_m}})$,
by \ref{prop:3.3} b), each of the monomials $x^{c(j)} + x^{e_1 + \cdots + x^{e_m}}$ lies in $Q$,
and so $c^{(j)} + e_1, \ldots, c^{(j)} + e_m$ all lie in $C$. Hence we can just choose $c = c^{(j)}$ for
any $j$.

b) $a = a_1 e_1 + \cdots + a_m e_m$, and so $nc + a = a_1 (c + e_1) + \cdots + a_m (c + e_m) \in C$. \hfill \Box

We are now ready to prove the main result of this section:

**Theorem 3.6.** Let $S$ be an additively idempotent parasemifield, finitely generated
as a semiring. Then $S$ is order-unital.

**Proof.** Choose $u = \varphi(x^c)$ with $c \in C$ chosen by Proposition \ref{prop:3.4}. We want to show
that $u^n s + 1 = 1$ for each $s \in S$ and some $n \in \mathbb{N}$. Clearly it suffices to show it for
$s = \varphi(x^a), a \in \mathbb{N}_0^m$ (each element of $S$ is a finite sum of elements of this form).

By \ref{prop:3.3} b), we can choose $n$ large enough so that $nc + a \in C$. Thus $u^n s \in Q$. Since
$S$ is additively idempotent, $A = \{1\}$, and so this means that $u^n s \leq 1$. Therefore
$1 \leq u^n s + 1 \leq 1$, and so by Proposition \ref{prop:3.4} $u^n s + 1 = 1$ and $S$ is unital. \hfill \Box
4. The classification

By Theorem 3.6 we know that every additively idempotent parasemifield, finitely generated as a semiring, is order-unital. Hence it comes from the construction of \([3]\).

In this section we use this construction to classify all such parasemifields, namely, we show the following theorem:

**Theorem 4.1.** Let \( S \) be an additively idempotent parasemifield, finitely generated as a semiring. Then \( S \) is a (finite) product of parasemifields of the form \( G(T_i, v_i) \), where \( (T_i, v_i) \) are rooted trees and \( G(T_i, v_i) \) are associated additively idempotent parasemifields (or equivalently \( \ell \)-groups), defined in Definition 4.2.

Two such products \( \prod_{i=1}^{k} G(T_i, v_i) \) and \( \prod_{j=1}^{k'} G(T'_j, v'_j) \) are isomorphic parasemifields if and only if \( k = k' \) and there is some permutation \( \sigma \) of \( \{1, \ldots, k\} \) such that for all \( i \) we have \( (T_i, v_i) \cong (T'_{\sigma(i)}, v'_{\sigma(i)}) \) as rooted trees.

It feels slightly more natural to define \( G(T, v) \) in the language of \( \ell \)-groups (see Proposition 3.1 for the easy correspondence with parasemifields). First let us briefly introduce some notions related to rooted trees.

Note that a rooted tree \( (T, v) \) is a (finite, non-oriented) connected graph \( T \) containing no cycles and having a specified vertex, the root \( v \). By an initial segment \( T' \) of a rooted tree \( (T, v) \) we shall mean a (possibly empty) subtree such that if \( w \in T' \), then all the vertices on the (unique) path in \( T \) from \( v \) to \( w \) lie in \( T' \). If \( T' \) is a non-empty initial segment of a rooted tree \( (T, v) \), the set of next vertices \( N(T') \) is the set of all vertices \( w \in T \setminus T' \) such that there is \( t \in T' \) and an edge \((w, t)\) in \( T \). If \( T' \) is empty, we set \( N(T') = \{v\} \). For a vertex \( w \) define a tree \( T_w \subset T \) consisting of exactly all the vertices \( u \in T \) such that the (unique) path from \( u \) to the root \( v \) passes through \( w \).

We are now ready to define \( G(T, v) \).

**Definition 4.2.** Let \( T \) be a tree with root \( v \). Define an \( \ell \)-group \( G(T, v) \) as follows: First attach a copy of the group of integers \( \mathbb{Z} = \mathbb{Z}_w \) to each vertex \( w \) of \( T \). Then \( G(T, v) \) as an additive group is just the direct product of these groups \( \mathbb{Z}_w \). We shall denote elements of \( G(T, v) \) as tuples \( (g_w) \) with \( g_w \in \mathbb{Z}_w \).

Now take tuples \( (g_w) \) and \( (h_w) \) and define \( (g_w) \lor (h_w) = (k_w) \) and \( (g_w) \land (h_w) = (m_w) \) as follows: Let \( T' \) be the largest initial segment of \( T \) such that \( g_w = h_w \) for all \( w \in T' \). For \( w \in T' \), set \( k_w = m_w = g_w (= h_w) \). Take now \( w \in N(T') \). Then \( g_w \neq h_w \), without loss of generality assume that \( g_w > h_w \). Then define \( k_u = g_u \) and \( m_u = h_u \) for all \( u \in T_w \).

It is straightforward to check that \( G(T, v) \) is indeed an abelian lattice ordered group; note that the lattice operations come essentially from some lexicographical ordering on \( G(T, v) \) with respect to the structure of the tree.

We will need a few properties of the construction of \([3]\) and of piecewise linear convex functions, especially in relation to being finitely generated as semiring. They are collected in the following three lemmas.

**Lemma 4.3.** Let \( \mathcal{W} = (W_0, W_1, \ldots) \) be the stellar sequence corresponding to the \( \ell \)-group \( G = \mathcal{M}([0, 1]^n) / I \). Let \( [0, 1]^n \supset D_0 \supset D_1 \supset D_2 \supset \ldots \) be the corresponding sequence of complexes, let \( D = \bigcap D_i \) and consider the \( \ell \)-group \( \mathcal{M}(D) \) of restrictions of functions in \( \mathcal{M}([0, 1]^n) / I \) to \( D \).

Then there is a surjection \( G = \mathcal{M}([0, 1]^n) / I \rightarrow \mathcal{M}(D) \).
Proof. Let \( \text{res} : \mathcal{M}([0,1]^n) \to \mathcal{M}(D) \) be the restriction map and let \( \pi : \mathcal{M}([0,1]^n) \to \mathcal{M}([0,1]^n)/I \) be the projection. By the definition of \( I \), if \( \pi(f) = \pi(g) \) then \( \text{res}(f) = \text{res}(g) \), and so \( \text{res} \) factors through \( \pi \), i.e., \( \text{res} : \mathcal{M}([0,1]^n) \to \mathcal{M}([0,1]^n)/I \to \mathcal{M}(D) \). Let \( r : \mathcal{M}([0,1]^n)/I \to \mathcal{M}(D) \) be the corresponding map. Since \( \text{res} \) is a surjective homomorphism by definition, \( r \) is surjective as well. \( \square \)

**Lemma 4.4.** Let \( A \subset [0,1]^n \) be a simplex and \( f, g \in W(A) \) convex functions. Then \( \text{max}(f,g) \) and \( f + g \) are also convex.

**Proof.** A function \( h \) is convex if its graph is convex (in \( A \times \mathbb{R} \)), i.e., if the line segment between any two points in \( G(h) \) lies in \( G(h) \). Let \( X, Y \in G(\text{max}(f,g)) \) and denote \( XY \) the line segment between these points. Since \( f \) and \( g \) are both convex, \( XY \in G(f) \) and \( XY \in G(g) \). But then \( XY \in G(f) \cap G(g) = G(\text{max}(f,g)) \).

For \( f + g \), choose \( X = (x_1, x_2), Y = (y_1, y_2) \in G(f + g) \) \( (x_1, y_1) \) are \( n \)-tuples in \( A \) and \( x_2, y_2 \in \mathbb{R} \). Then there are \( X' = (x_1, x'), Y' = (y_1, y') \in G(f) \) and \( X'' = (x_1, x''), Y'' = (y_1, y'') \in G(g) \) such that \( x_2 = x' + x'' \) and \( y_2 = y' + y'' \). If we now take points \( X_0 = (a, b) \in G(f) \) and \( Y_0 = (a, c) \in G(g) \) on the line segments \( X'Y' \) and \( X''Y'' \), respectively, then the point \( (a, b + c) \) is a point on the line segment \( XY \) and lies in \( G(f + g) \) (and each point of the line segment \( XY \) is of this form). \( \square \)

**Lemma 4.5.** Let \( a_1, a_2, \ldots \) be a sequence of points in \( D \) such that \( \lim a_i = a \in [0,1]^n \). Then \( \mathcal{M}\{a_1, a_2, \ldots\} \) and \( \mathcal{M}(D) \) are not finitely generated semirings.

**Proof.** Assume that there are \( f_1, \ldots, f_k \in \mathcal{M}([0,1]^n) \) whose restrictions generate \( \mathcal{M}\{a_1, a_2, \ldots\} \) as a semiring. Since each \( f_i \) is piecewise linear, we can find a simplex \( A \) such that each \( f_i \) is linear on \( A \) and infinitely many of the \( a_i \) lie in \( A \). Denote \( B \) the set of all such \( a_i \). Using the (surjective) restriction map \( \mathcal{M}\{a_1, a_2, \ldots\} \to \mathcal{M}(B) \), we see that the functions \( f_1, \ldots, f_k \) generate \( \mathcal{M}(B) \) as well.

Now consider the subset \( M \) of \( \mathcal{M}(A) \) semiring-generated by \( f_1, \ldots, f_k \). Since each linear function is convex, each function in \( M \) is convex by Lemma 4.4. But there are clearly functions in \( \mathcal{M}(B) \) which are not restrictions of convex functions on \( A \), a contradiction.

The restriction map is a surjection from \( \mathcal{M}(D) \) onto \( \mathcal{M}\{a_1, a_2, \ldots\} \). Thus neither \( \mathcal{M}(D) \) is finitely generated. \( \square \)

Note that the same proof shows the following corollary:

**Corollary 4.6.** Let \( A \subset [0,1]^n \) be a simplex of dimension \( \geq 1 \). Then \( W(A) \) is not finitely generated semiring.

We are now ready to start discussing the structure of additively idempotent parasemifields. We will first show that our parasemifield \( S \) is a direct product of finitely many parasemifields corresponding to germs of functions at certain points.

**Definition 4.7.** Let \( p \) be a point in \([0,1]^n\) and let \( \mathcal{P} = ([0,1]^n \supset P_0 \supset P_1 \supset \ldots) \) be a sequence of complexes such that \( \bigcap P_i = \{p\} \). Then we define the \( \mathcal{P} \)-germ of functions at \( p \) as \( \mathcal{M}_\mathcal{P}(p) = \mathcal{M}([0,1]^n)/I \), where \( I \) is the ideal corresponding to the sequence \( \mathcal{P} \), i.e., \( I \) consists of functions \( f \in \mathcal{M}([0,1]^n) \) such that \( f(P_i) = 0 \) for some \( i \).

The germ of functions at a point \( p \) is exactly what it should intuitively be: it is the set of all functions viewed locally at \( p \) “in the directions given by \( \mathcal{P} \).”
Proposition 4.8. Let $S$ be an additively idempotent parasemifield, finitely generated as semiring. View $S$ as a (unital) $\ell$-group and let $W = (W_0, W_1, \ldots)$ be the corresponding stellar sequence, $D = ([0,1]^n \supset D_0 \supset D_1 \supset \ldots)$ the corresponding sequence of complexes, and $I$ the defining ideal. Then $D = D(W) = \bigcap D_i = \{d_1, \ldots, d_k\}$ is finite and $S = M(\{0,1\}^n)/I$ is isomorphic to the direct product of $S_i = M_D(d_i)$, where $D_i = ([0,1]^n \supset D_0 \supset D_1 \supset \ldots)$ with $D_i := D_j \cap C^j$ for some fixed simplex $C^j$ containing an open neighborhood of the given point $d_i$.

Remark 4.9. The formulation of Proposition 4.8 is fairly technical, but the idea is simple. The intersection $D$ is finite and the parasemifield $S$ will decompose as a direct product of parasemifields $S_i$, each of which corresponds to a germ of functions at a point $d_i \in D$.

Note that strictly speaking, the local sequences of complexes $D^i$ we are using do not come from a stellar sequence. This is just a technicality, though: we can modify the stellar sequence $W$ by first deleting all the simplexes outside of $C^i$ (using suitable subdivisions) and only then continuing with the stellar transformations which created $W$. This produces a stellar sequence $W'$ whose corresponding sequence of complexes is $D'^i = ([0,1]^n \supset D'_0 \supset D'_1 \supset \ldots \supset D'_k \supset D'_1 \supset D'_2 \supset \ldots)$, which differs from $D^i$ only in finitely many complexes, and so produces the same germ of functions.

Proof. (of Proposition 4.8) Assume that $D$ is not finite. Since $D \subset [0,1]^n$, we see that $D$ has a cumulation point. Thus by Lemma 4.3 it follows that $M(D)$ is not finitely generated. By Lemma 4.3, $M(\{0,1\}^n)/I = S$ surjects onto $M(D)$, and so neither $S$ is a finitely generated semiring.

Thus $D = \{d_1, \ldots, d_k\}$ is finite and we can find suitable disjoint simplexes $C_i$ containing open neighborhoods of the points $d_i$ and define $D^i$ and $S_i = M_D(d_i)$ as in the statement of the proposition. Then the restriction map gives a surjection $r : S \to \prod S_i$ similarly as in Lemma 4.3.

To show that $r$ is injective, assume that $r(f) = 0$ for some $f \in M([0,1]^n)$, i.e., there is $j$ such that $f(D_j^i) = 0$ for all $i$. We want to show that $\pi(f) = 0$. Since an open neighborhood of $D = \bigcap D_i$ is contained in $\bigcup_i D_j^i$, we see that there is $k$ such that $D_k \supset \bigcup_i D_j^i$. Thus $f(D_k) = 0$, which means that $f \in I$ and $\pi(f) = 0$. \hfill \Box

Therefore to finish the classification we just need to describe the structure of the germs $M_D(d)$. This will be given in terms of $\ell$-groups $G(T, v)$ associated to rooted trees, defined in Definition 4.2.

Proposition 4.10. Let $W = (W_0, W_1, \ldots)$ be a stellar sequence, $D = ([0,1]^n \supset D_0 \supset D_1 \supset D_2 \supset \ldots)$ the corresponding sequence of complexes and $D = \bigcap D_i$. Assume that $D = \{d\}$ has one element. Then the corresponding $\ell$-group of germs $G = M([0,1]^n)/I = M_D(d)$ is either not finitely generated as a semiring or is isomorphic to an $\ell$-group $G(T, v)$ associated to a (finite) rooted tree $(T, v)$.

Proof. Assume that $M_D(d)$ is finitely generated as a semiring. To prove the proposition we shall modify the sequence $D$ in several steps while preserving the $\ell$-group $M_D(d)$. The fairly long proof is divided into 5 steps:

1. Simplexes containing $d$

First form a new sequence of complexes $D^1 = (D_0^1 \supset D_1^1 \supset \ldots)$, where $D_1^1$ is obtained from $D_1$ by recursively removing all maximal simplexes not containing $d$. 

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Note that the simplexes not containing \( d \) play no role in determining the germ of local functions, and so \( \mathcal{M}_D(d) = \mathcal{M}_D(d) \). Also note that \( D^1 \) is still obtained from a stellar sequence (taking into account the potential need for making modifications as in Remark 4.9; we shall not mention this in the future).

2. Stable subspaces

By a stable line in \( D^1 \) we shall mean a line \( \ell \) passing through \( d \) such that \( \ell \cap D^1_i \) is a line segment (and not just the point \( d \)) for each \( i \). This means that while the stellar transformations which give \( D^1 \) may (and will) subdivide the 1-dimensional simplex which gives a line segment lying on \( \ell \), they will never delete this simplex.

Let us point out that stable lines give non-trivial elements in \( \mathcal{M}_D(d) \): the germ of linear functions on \( \ell \) will lie in \( \mathcal{M}_D(d) \). A linear function on a line is determined by its slope (and value at the point \( d \)) and since the functions we are considering are restrictions of linear functions with integral coefficients, the set of possible slopes is \( \mathbb{Z} \). Thus to each stable line \( \ell \) corresponds a copy of \( \mathbb{Z} \subset \mathcal{M}_D(d) \).

Similarly for \( k \geq 1 \) we can define a stable \( k \)-subspace in \( D^1 \) as a \( k \)-dimensional (affine) space \( L \) containing \( d \) such that \( L \cap D^1_i \) has dimension \( k \) for each \( i \). (A stable 1-subspace is just a stable line.)

By the definition of stable subspaces it follows that if a simplex in \( D^1_i \) intersects every stable line only in the point \( d \) (and thus the same is true for the intersection with any stable subspace), then it does not contribute to \( \mathcal{M}_D(d) \). Therefore we can form \( D^2 \) and \( D^2 \) by omitting all such simplexes with no non-trivial intersection with a stable line. Then \( \mathcal{M}_D(d) = \mathcal{M}_D(d) \).

3. Simplexes defined using the generators

By an open simplex we shall mean a point, or the interior of a \( k \)-simplex for \( k \geq 1 \).

Denote the (semiring) generators of \( S = \mathcal{M}_D(d) \) by \( f_1, \ldots, f_k \) (as usual, we identify a function \( f \in \mathcal{M}([0,1]^n) \) with its image in \( \mathcal{M}_D(d) \)). Each of these functions is piecewise linear, and so there is a finite set \( \mathcal{P}_0 \) of open simplexes which cover \( [0,1]^n \) and such that the restriction of each \( f_j \) to any \( P \in \mathcal{P}_0 \) is linear. In fact, we can modify \( \mathcal{P}_0 \) to get the following lemma:

**Lemma 4.11.** There is a finite set \( \mathcal{P} \) of open simplexes such that

(i) elements of \( \mathcal{P} \) are pairwise disjoint,

(ii) the restriction \( f_j|P \) is linear for all \( j \) and all \( P \in \mathcal{P} \),

(iii) \( \dim P \cap D^1_i = \dim P \) for all \( i \) and all \( P \in \mathcal{P} \),

(iv) \( P \cap \ell = \emptyset \) for all stable lines \( \ell \) and all \( P \in \mathcal{P} \) with \( \dim P > 1 \),

(v) for each \( e \) and each \( P \in \mathcal{P} \) with \( \dim P > e \) there is exactly one \( Q \in \mathcal{P} \) such that \( \dim Q = e \) and \( Q \subset \bar{P} \) (\( \bar{P} \) denotes the closure in \( \mathbb{R}^n \)),

(vi) \( \mathcal{M}_{D^3}(d) = \mathcal{M}_D(d) \), where \( D^3 = D^2 \cap U = (D^2_3 \cap U \cup D^2_2 \cap U \cup \ldots) \) and \( U = \bigcup_{P \in \mathcal{P}} P \).

Note that the simplexes \( P \in \mathcal{P} \) from the lemma can be viewed as a refinement of the notion of stable subspaces.

**Proof.** We shall modify \( \mathcal{P}_0 \) recursively in several steps while making sure that \( \mathcal{M}_{D^2 \cap U}(d) \) remains unchanged and equal to \( S = \mathcal{M}_D(d) \) (this is clearly true at the beginning as \( \bigcup_{P \in \mathcal{P}_0} P = [0,1]^n \)).

To start, let \( \mathcal{P} = \mathcal{P}_0 \). Now recursively repeat the following set of modifications:
1. For determining $S$ are relevant only those simplexes $P \in \mathcal{P}$ which have non-empty intersection with infinitely many (and hence all) of the $D_i^2$. Hence we can delete all other $P$ from $\mathcal{P}$. Continue to Step 2.

2. If there is $P \in \mathcal{P}$ and finitely many open simplexes $S_1, \ldots, S_n \subset P$ such that $\dim S_i < \dim P$ and $P \setminus (S_1 \cup \cdots \cup S_n)$ has non-empty intersection with all $D_i^2$, then replace $P$ by $S_1, \ldots, S_n$ in $\mathcal{P}$. Return to Step 1 if $\mathcal{P}$ has been modified, else continue to Step 3.

Note that $\mathcal{P}$ has finitely many elements at any time and dimensions of elements of $\mathcal{P}$ are decreasing, so this step will happen only finitely many times. Also note that after being done with steps 1 and 2, $\mathcal{P}$ contains only open simplexes $P$ with $\dim(P \cap D_i^2) = \dim P$ for all $i$.

3. Assume that $d \neq P \in \mathcal{P}$ has non-empty intersection with infinitely many stable lines. Arguing similarly as in the proof of Lemma 4.5 we see that $S$ is then not finitely generated as a semiring, a contradiction: Namely, any function $f$ which is linear along finitely many of these lines (and suitably defined at the other lines) will be non-trivial in $S$. By considering the set of slopes of $f$ along these lines, it’s easy to construct a function $f \in S$ which will not be convex on $P$. But this contradicts Lemma 4.4 as all the semiring generators are linear on $P$. Thus every $d \neq P \in \mathcal{P}$ has non-empty intersection with only finitely many stable lines.

Suppose that $\dim P > 1$, $\ell$ is a stable line and $\ell \cap P \neq \emptyset$. Choose then a $(\dim P - 1)$-dimensional hypersurface $H$ containing $\ell$ and subdivide $P$ along this hypersurface, i.e., $P = (P \cap H) \cup (P \setminus H)$ and $P \setminus H$ has two connected components, $P_1$ and $P_2$. We can choose $H$ so that $P \cap H, P_1$, and $P_2$ are all open simplexes; in $\mathcal{P}$ then replace $P$ by $P \cap H, P_1$ and $P_2$.

After doing this finitely many times (because for each $P$ there are only finitely many stable lines), we arrive at $\mathcal{P}$ satisfying property (iv). Return to Step 1 if $\mathcal{P}$ has been modified, else continue to Step 4.

4. Assume that there are $P, Q, R \in \mathcal{P}$ such that $Q, R \subset \bar{P}$ and $Q \not\subset R$ and $R \not\subset Q$. Take such a $P$ of the smallest dimension. Since $Q$ and $R$ are disjoint, we can again subdivide $P$ by a $(\dim P - 1)$-dimensional hypersurface $H$ as above so that $(Q \subset P_1$ and $R \subset P_2$) or $(Q \subset P_1$ and $R \subset P \cap H$) or $(R \subset P_1$ and $Q \subset P \cap H$) (and replace $P$ in $\mathcal{P}$ by $P \cap H, P_1$ and $P_2$).

After doing this finitely many times (because for each $P$ such a situation can occur only finitely many times), we arrive at $\mathcal{P}$ satisfying property (v): We have just ensured the uniqueness of such $Q$, its existence easily follows from the fact that $\dim P \cap D_i^2 = \dim P$.

Return to Step 1 if $\mathcal{P}$ has been modified, else we are done.

Note that the whole algorithm terminates after finitely many steps and that (i) – (vi) are satisfied at the end. □

4. Construction of the tree $T$

Now we can easily construct a rooted tree $(T, v)$ attached to the sequence $\mathcal{M}_{D_i}(d)$ obtained using Lemma 4.11. Associate a vertex $v_P$ to each $P \in \mathcal{P}$, there will be an edge connecting vertices $v_P$ and $v_Q$ if and only if $(P \subset Q$ and $\dim P = \dim Q - 1)$ or $(Q \subset P$ and $\dim Q = \dim P - 1)$. The vertex $v_d$ is the root $v$.

By Lemma 4.11 we see that $(T, v)$ is a (connected) rooted tree.
5. Description of $M_D(d)$

The germ of a function $f$ in $M_{D^3}(d) = M_D(d)$ can have any value at $d$, which gives the $Z_v$ at the root $v$ of the tree $T$.

Given $f \in M_{D^3}(d)$, choose a small ball $B$ containing $d$ so that the restriction of $f$ to $B \cap r$ is linear for all rays $r \ni d$. Since we are considering only the germ of functions at $d$, $f$ is uniquely determined by $f|B$ as an element of $M_{D^3}(d) = M_D(d)$.

Take a 1-simplex $P \in \mathcal{P}$. The value of $f$ at the endpoint $d$ of $\bar{P}$ has already been selected, and so the restriction $f|(P \cap B)$ (which is linear by Lemma 4.11 (ii)) is uniquely determined by its value at any point $p \in P \cap B$. The choice of this value gives the $Z_{vp}$ at the vertex $v_P$ of the tree $T$.

After having dealt with all the 1-simplexes, take a 2-simplex $P \in \mathcal{P}$. There is a unique 1-simplex $Q \in \mathcal{P}, Q \subset \bar{P}$; $f|(Q \cap B)$ has already been determined, and so the restriction $f|(P \cap B)$ is uniquely determined by its value at any point $p \in P$. The choice of this value gives the $Z_{vp}$ at the vertex $v_P$ of the tree $T$.

We can continue in this way, successively dealing with simplexes of larger and larger dimensions, until we have covered the whole tree $T$ and uniquely determined the function $f|B$ as an element of $M_{D^3}(d) = M_D(d)$.

Now it is straightforward to check that the $\ell$-group $M_D(d)$ is exactly $G(T, v)$. □

Together with Proposition 4.8, this finishes the proof of Theorem 4.1. The uniqueness statement follows from the proof and from the uniqueness statement of [3], Corollary 5.4. □

Let us note that as a group, each $G(T_i, v_i)$ is just $\mathbb{Z}^{n_i}$ for some $n_i$, and so we obtain the following corollary to Theorem 4.1.

**Corollary 4.12.** If an additively idempotent paraseifield is finitely generated as a semiring $S(+, \cdot)$, then it is finitely generated as a group $S(\cdot) \simeq \mathbb{Z}^n$.

We are not aware of any more direct or elementary proof of this surprising fact. It would certainly be very interesting to obtain one.

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