Vanishing pressure limit for compressible Navier–Stokes equations with degenerate viscosities

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Received 14 January 2017, revised 27 July 2017
Accepted for publication 8 August 2017
Published 12 October 2017

Abstract

In this paper we study a vanishing pressure process for highly compressible Navier–Stokes equations as the Mach number tends to infinity. We first prove the global existence of weak solutions for the pressureless system in the framework (Li and Xin 2015 arXiv:1504.06826v2), where the weak solutions are established for compressible Navier–Stokes equations with degenerate viscous coefficients. Furthermore, a rate of convergence of the density in $L^\infty(0, T; L^2(\mathbb{R}^N))$ is obtained, in case when the velocity corresponds to the gradient of density at initial time.

Keywords: compressible Navier–Stokes, mach number limit, pressureless, global weak solutions, density-dependent viscosities

Mathematics Subject Classification numbers: 76N10, 35Q35

1. Introduction

The time evolution of a viscous compressible barotropic fluid occupying the whole space $\mathbb{R}^N (N = 2, 3)$ is governed by the equations

$$
\partial_t \rho_\varepsilon + \text{div}(\rho_\varepsilon u_\varepsilon) = 0,
$$

$$
\partial_t (\rho_\varepsilon u_\varepsilon) + \text{div}(\rho_\varepsilon u_\varepsilon \otimes u_\varepsilon) + \nabla P_\varepsilon - \text{div}S_\varepsilon = 0,
$$

where the unknown functions $\rho_\varepsilon$ and $u_\varepsilon$ are the density and the velocity. The pressure $P_\varepsilon = \varepsilon \rho_\varepsilon^\gamma$ with $\gamma > 1$ is given and $\varepsilon > 0$ is related to Mach number, and the stress tensor takes the form
in which $I$ is the identical matrix, $h$ and $g$ are functions of $\rho_\varepsilon$ satisfying the physical restrictions
\[ h(\rho_\varepsilon) > 0, \quad h(\rho_\varepsilon) + Ng(\rho_\varepsilon) \geq 0. \]  
For simplicity reasons, in this paper we assume
\[ h(\rho_\varepsilon) = \rho_\varepsilon^\gamma, \quad g(\rho_\varepsilon) = (\alpha - 1)\rho_\varepsilon^\gamma, \quad \text{for } \alpha > (N - 1)/N, \quad \varepsilon \in (0, 1). \]  
The initial functions are imposed as
\[ \rho_\varepsilon(x, t = 0) = \rho_0 \geq 0, \quad \rho_\varepsilon u_\varepsilon(x, t = 0) = m_0, \quad x \in \mathbb{R}^N. \]  
Lions [20] first proved the global existence of weak solutions of (1) if the adiabatic index $\gamma \geq 3N/(N + 2)$. Later, $\gamma$ was relaxed by Feireisl et al [7] to $\gamma > N/2$ and by Jiang and Zhang [13] to $\gamma > 1$ under some extra spherically symmetry assumptions. While for the case when viscosities are density dependent, Bresch and Desjardins [4, 5] developed a new entropy structure on condition that
\[ g(\rho_\varepsilon) = \rho_\varepsilon h'(\rho_\varepsilon) - h(\rho_\varepsilon). \]  
This gives an estimate on the gradient of density, and thereby, some further compactness information on density. Li et al [16] proposed the global entropy weak solution to system (1) in one-dimensional bounded interval and studied the vacuum vanishing phenomena in finite time span. Similar results in [16] were extended to the Cauchy problem in [14] by Jiu and Xin. Guo et al [10] obtained the global existence of weak solution to (1) if some spherically symmetric assumptions are made. However, the problem becomes much more difficult in general high dimension spaces. Mellet and Vasseur [21] provided a compactness framework which ensures the existence of weak solutions as a limit of approximation solutions, but leaves such approximations sequence open in [21]. Until recently, the problem was solved in two impressive papers by Vasseur and Yu [22] and Li and Xin [18], where they constructed separately appropriate approximations from different approaches. Vasseur and Yu [23] also considered the compressible quantum Navier–Stokes equations with damping, which helps to understand the existence of global weak solutions to the compressible Navier–Stokes equations.

Referring to [18, 21], we give the weak solution of system (1) in below

**Definition 1.1.** For fixed $\varepsilon > 0$, we call $(\rho_\varepsilon, u_\varepsilon)$ a weak solution to the problem (1)–(5), if

\[
\begin{align*}
0 \leq \rho_\varepsilon & \in L^\infty(0, T; L^1(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)) , \\
\nabla \rho_\varepsilon^{\alpha - 1/2} & , \sqrt{\rho_\varepsilon} u_\varepsilon \in L^\infty(0, T; L^2(\mathbb{R}^N)) , \\
\nabla \rho_\varepsilon^{(\alpha - 1)/2} & \in L^2(0, T; L^2(\mathbb{R}^N)) , \\
h(\rho_\varepsilon) \nabla u_\varepsilon & \in L^2(0, T; W^{-1,1}_{\text{loc}}(\mathbb{R}^N)) , \\
g(\rho_\varepsilon) \text{div} u_\varepsilon & \in L^2(0, T; W^{-1,1}_{\text{loc}}(\mathbb{R}^N)) ,
\end{align*}
\]
\((\sqrt{\rho}, u)\) satisfy (1)_1 in distribution sense, and the integral equality
\[
\int_{\mathbb{R}^N} m_0 \phi(x, 0) + \int_0^T \int_{\mathbb{R}^N} \sqrt{\rho} \partial_t \rho \phi + \sqrt{\rho} u \nabla \phi + \frac{2\alpha}{2\alpha - 1} \sqrt{\rho} u \partial_t \partial_t \phi
\]
holds true for any test functions \(\phi \in C^1_c(\mathbb{R}^N \times [0, T])\), where
\[
(h(\rho) \nabla u, \nabla \phi) + (g(\rho) \nabla u, \nabla \phi)
\]
and
\[
\frac{2\alpha}{2\alpha - 1} \sqrt{\rho} u \partial_t \partial_t \phi
\]

The following important existence results of weak solutions are obtained in [18] by Li and Xin.

**Proposition 1.1.** ([18]) Assume that the initial function in (5) satisfies
\[
\begin{align*}
0 \leq \rho_0 &\in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N), & \nabla \rho_0^{\alpha/2} &\in L^2(\mathbb{R}^N), \\
m_0 &\in L^{(\gamma+1)/(\gamma+1)}(\mathbb{R}^N), & \rho_0 &\neq 0, m_0 = 0 \text{ a.e. on } \{x \in \mathbb{R}^N | \rho_0 = 0\}, \\
\rho_0^{(1+\eta_0)} |m_0|^{2+\eta_0} &\in L^1(\mathbb{R}^N) \text{ for some } \eta_0 > 0.
\end{align*}
\]

Additionally, assume that for \(N = 2\)
\[
\alpha > 1/2, \quad \gamma > 1, \quad \gamma \geq 2\alpha - 1,
\]
and for \(N = 3\)
\[
\begin{align*}
\gamma &\in (1, 3), \\
\gamma &\in (1, 3\alpha - 3) \text{ if } \alpha \in [3/4, 1], \\
\gamma &\in [2\alpha - 1, 3\alpha - 1] \text{ and } \rho_0^{\alpha/2} |m_0|^{4} \in L^1(\mathbb{R}^3) \text{ if } \alpha \in (1, 2).
\end{align*}
\]

Then the problem (1)–(5) has global weak solutions \((\rho, u)\) in the sense of definition 1.1.

**Remark 1.1.** By (8), if we multiply (1)_2 by \(4|u|^2 u\) and compute directly, we infer
\[
\rho_c |u|^4 \in L^\infty(0, T; L^1(\mathbb{R}^3)).
\]

**Proof.** The rigorous proof is available in the appendix.

It seems rather natural to expect that, as \(\varepsilon \to 0^+\), the limit \((\rho, u)\) of \((\rho, u)\) satisfies the corresponding pressureless system
\[
\partial_t \rho + \text{div}(\rho u) = 0,
\]
\[
\partial_t (\rho u) + \text{div}(\rho u \otimes u) - \text{div}(h(\rho) \nabla u + g(\rho) \text{div} u) = 0.
\]

As in [11], we define the weak solution (called quasi-solution) to system (10).
Definition 1.2. The function \((\rho, u)\) is called a quasi-solution, if
\[
\begin{aligned}
0 \leq \rho &\in L^\infty\left(0, T; L^1(\mathbb{R}^N)\right), \\
\nabla \rho \rho^{-1/2} &\in L^\infty\left(0, T; L^2(\mathbb{R}^N)^N\right), \\
\n\sigma(h) \nabla u &\in L^2\left(0, T; \mathcal{W}_{\text{loc}}^{-1}(\mathbb{R}^N)^N\right), \\
\n\rho \nabla u &\in L^2\left(0, T; \mathcal{W}_{\text{loc}}^{-1}(\mathbb{R}^N)\right); \\
\end{aligned}
\]

in addition, \((\sqrt{\rho}, u)\) satisfy (10) in distribution sense, and the integral equality
\[
\int_{\mathbb{R}^N} m_0 \phi(x, 0) + \int_0^T \int_{\mathbb{R}^N} \sqrt{\rho}(\sqrt{\rho}u) \partial_t \phi + \sqrt{\rho} \otimes \sqrt{\rho} u : \nabla \phi
\]
holds true, where the quantities on the right side are defined as the same of \(\langle h(\rho) \nabla u, \nabla \phi \rangle\) and \(\langle g(\rho) \nabla u, \nabla \phi \rangle\).

Remark 1.2. The quasi-solution in definition 1.2 was first proposed by Haspot to approximate in some sense the compressible Navier–Stokes equations [11].

In this paper, we choose \(\varepsilon = \eta^{-2} > 0\) with \(\eta\) being the Mach number. The readers can refer to [9, 11, 19, 20] for more information in this aspect. There are satisfactory results on the incompressible limit when \(\eta \to 0\), we refer readers to the pioneering works by Desjardins et al and Lions and Masmoudi [2, 3, 19] when the viscous coefficients are constant. Regrettably, results are rarely available up to publication when \(\eta \to \infty\). One major difficulty is the compactness lack of the density because its \(L^2\)-bound is no longer conserved for constant viscosities. However, the case of density-dependent viscosity is much different due to the new BD entropy inequality. Haspot [11] proved the highly compressible limit \((\varepsilon \to 0)\) in the sense of distribution in suitable Lebesgue spaces, and discussed the global existence of quasi-solutions as a convergence limit from approximation solutions of system (1), although such approximations are only a priori exist. It is worthy mentioning that in [11] the author constructed a family of explicit solutions \((\rho, u)\) with \(\rho\) satisfying the porous medium equation, the heat equation, or the fast diffusion equation, up to the choice of \(\alpha\). Haspot and Zatorska [12] consider the one-dimensional Cauchy problem and obtain a rate of convergence of \(\rho_e\) and other related properties.

We are interested in the limit procedure for the weak solutions \((\rho_e, u_e)\) of (1) as \(\varepsilon\) tends to zero, and then get some convergence rate the solutions. In particular, on the basis of existence results obtained in [18] by Li and Xin, we adopt some ideas in [11] and [21] and first show the quasi-solutions stability for the solutions \((\rho_e, u_e)\) of (1). Secondly, in the spirit of [12], we obtain a convergence rate of \(\rho - \rho_e\) in terms of \(\varepsilon\) in high dimensions by the argument of duality, as long as the initial velocity associated with the gradient of initial density.

Theorem 1.1. Let the conditions (6)–(8) in proposition 1.1 hold true. Then, for \(\alpha \geq 1\), the solution \((\rho_e, u_e)\) of (1) converges to a limit function \((\rho, u)\) which solves (10) in the sense of definition 1.2. Furthermore,
\[
\rho_e \to \rho \quad \text{in} \quad C\left([0, T]; L^1_{\text{loc}}(\mathbb{R}^N)\right), \tag{11}
\]
\[
\rho_e u_e \to \rho u \quad \text{in} \quad L^2\left(0, T; L^2_{\text{loc}}(\mathbb{R}^N)\right), \tag{12}
\]
where \(q_1 \in [1, \infty), \ q_2 \in [1, 2)\) if \(N = 2, q_1 \in [1, 6\alpha - 3), q_2 \in [1, \frac{12\alpha - 6}{6\alpha - 1})\) if \(N = 3\).
Remark 1.3. The assumptions in proposition 1.1 guarantee the existence of \((\rho_\varepsilon, u_\varepsilon)\) to (1), whose proof are available in [18]. We allow more general viscosities at the cost of stress tensor having the form (2), although it seems not appropriate from a physical point of view.

Remark 1.4. In case of \(\alpha = 1\), theorem 1.1 is valid for the symmetric viscous stress tensor \(S_\varepsilon = \text{div} \left( \rho_\varepsilon \frac{\nabla u_\varepsilon + (\nabla u_\varepsilon)^T}{2} \right)\), where the existence of \((\rho_\varepsilon, u_\varepsilon)\) are achieved in [18, 22]. Moreover, the case \(\alpha < 1\) can also be discussed by modifying slightly the argument in theorem 1.1.

Theorem 1.2. In addition to the assumptions made in theorem 1.1, let
\[
 u_0 + \alpha \rho_0^{\alpha - 2} \nabla \rho_0 = 0. \tag{13}
\]
Then there is a positive \(C\) which may depend on \(T\) such that for \(\alpha \geq 3/2\)
\[
 \sup_{0 \leq t \leq T} \| (\rho_\varepsilon - \rho)(\cdot, t) \|_{L^1(\mathbb{R}^N)} \leq C \varepsilon^\sigma,
\]
where \(\sigma < \frac{1}{2(2\alpha - 1)}\) if \(N = 2\), \(\sigma = \frac{4\alpha - 3}{4(2\alpha - 1)}\) if \(N = 3\).

Remark 1.5. For one-dimensional problem, Haspot and Zatorska [12] first obtained a rate of convergence of \(\rho_\varepsilon - \rho\) in suitable Sobolev spaces for \(1 < \alpha \leq 3/2\). We remark that the argument in [12] relies heavily on the upper bound of density.

In the rest of this paper, section 2 is for some useful lemmas and sections 3 and 4 are devoted to proving theorems 1.1 and 1.2 respectively.

2. Preliminaries

Lemma 2.1. (see [8, 15]) Let \(B_R = \{x \in \mathbb{R}^N : |x| < R\}\). For any \(v \in W^{1,q}(B_R) \cap L^r(B_R)\), it satisfies
\[
 \|v\|_{L^p(B_R)} \leq C_1 \|v\|_{L^r(B_R)} + C_2 \|\nabla v\|_{L^q(B_R)} \|v\|_{L^r(B_R)}^{1-\gamma}, \tag{14}
\]
where the constant \(C_i\) depends only on \(p, q, r, \gamma\); and the exponents \(0 \leq \gamma \leq 1\), \(1 \leq q, r \leq \infty\) satisfy \(\frac{1}{r} = \gamma \left(\frac{1}{q} - \frac{1}{N}\right) + (1 - \gamma) \frac{1}{p}\) and
\[
 \begin{align*}
 \min\{r, \frac{Nq}{N-q}\} & \leq p \leq \max\{r, \frac{Nq}{N-q}\}, & \text{if } q < N; \\
 r & \leq p < \infty, & \text{if } q = N; \\
 r & \leq p \leq \infty, & \text{if } q > N.
\end{align*}
\]

The following \(L^p\)-bound estimate is taken from [17, lemma 2.4], whose proof is available by adopting [6, lemma 12] and the elliptic theory due to Agmon et al [1].

Lemma 2.2. ([17, lemma 2.4]) Let \(p \in (1, +\infty)\) and \(k \in \mathbb{N}\). Then for all \(v \in W^{2+k,p}(B_R)\) with \(0\)-Dirichlet boundary condition, it holds that
\[
 \|\nabla^{2+k} v\|_{L^p(B_R)} \leq C \|\Delta v\|_{W^{k+2,p}(B_R)},
\]
where the \(C\) relies only on \(p\) and \(k\).

Lemma 2.3. Assume that \(f\) is increasing and convex in \(\mathbb{R}_+ = [0, +\infty)\) with \(f(0) = 0\). Then,
\[
 |x - y| f(|x - y|) \leq (x - y) (f(x) - f(y)), \quad \forall \ x, y \in \mathbb{R}_+.
\]
Proof. Define $F(x) = f(x) - f(y) - f(x - y)$. Since $f$ is convex, then $F'(x) \geq 0$ for $x \geq y \geq 0$. This and $F(0) = 0$ deduce $F(x) \geq 0$. Hence,
$$f(|x - y|) = f(x - y) - f(y) = |f(x) - f(y)|.$$ 

Repeating the argument when $y \geq x \geq 0$, we obtain
$$f(|x - y|) \leq |f(x) - f(y)|, \quad \forall \ x, y \in \mathbb{R}_+.$$ 

This, along with the monotonicity of $f$, leads to
$$|x - y|f(|x - y|) \leq |x - y||f(x) - f(y)| = (x - y)(f(x) - f(y)),$$
the required.  

3. Proof of theorem 1.1

In what follows, the operations are based on hypotheses imposed in proposition 1.1, and the generic constant $C > 0$ is $\epsilon$ independent.

Firstly, for all existing time $t \geq 0$, we have
$$\|\rho_\epsilon(\cdot, t)\|_{L^1(\mathbb{R}^n)} = \|\rho_0\|_{L^1(\mathbb{R}^n)}$$
(15)

and
$$\sup_{0 \leq t \leq T} \int_{\mathbb{R}^n} (\rho_\epsilon |u_\epsilon|^2 + \epsilon \rho_\epsilon^2) + \int_0^T \int_{\mathbb{R}^n} \rho_\epsilon^2 \rho^2 |\nabla u_\epsilon|^2 + (\alpha - 1) \rho_\epsilon^2 (\text{div} u_\epsilon)^2$$
$$\leq \int_{\mathbb{R}^n} \frac{|m_\epsilon|^2}{\rho_0} + \epsilon \int_{\mathbb{R}^n} \rho_0^2.$$ 
(16)

Following in [21], a straight calculation shows
$$\sup_{0 \leq t \leq T} \int_{\mathbb{R}^n} (\rho_\epsilon |u_\epsilon + \alpha \rho_\epsilon^{2-\gamma} \nabla \rho_\epsilon|^2 + \epsilon \rho_\epsilon^2) + \alpha \gamma \epsilon \int_0^T \int_{\mathbb{R}^n} \rho_\epsilon^{2+\gamma-3} |\nabla \rho_\epsilon|^2$$
$$\leq \int_{\mathbb{R}^n} \rho_0 |u_0 + \alpha \rho_0^{2-\gamma} \nabla \rho_0|^2 + \epsilon \int_{\mathbb{R}^n} \rho_0^2.$$ 
(17)

The initial condition (6) and (15)–(17) guarantee
$$\sup_{0 \leq t \leq T} \int_{\mathbb{R}^n} \left( \rho_\epsilon + \left| \nabla \rho_\epsilon^{\alpha-1/2} \right|^2 \right) dx \leq C.$$ 
(18)

We claim that
$$\rho_\epsilon \in L^\infty \left( 0, T; L^1(\mathbb{R}^n) \cap L^q(\mathbb{R}^n) \right), \quad \nabla \rho_\epsilon^{\alpha-1/2} \in L^\infty \left( 0, T; L^2(\mathbb{R}^n) \right),$$
(19)
where $q < \infty$ if $N = 2$, and $q = 6\alpha - 3$ if $N = 3$.

Proof. [Proof of (19)]. If $\alpha \leq 3/2$, by (14) we have
$$\|\rho_\epsilon^{\alpha-1/2}\|_{L^1(\mathbb{R}^n)} \leq C \left( \|\rho_\epsilon^{\alpha-1/2}\|_{L^{(\alpha-1)/2-1}(\mathbb{R}^n)} + \|\nabla \rho_\epsilon^{\alpha-1/2}\|_{L^2(\mathbb{R}^n)} \right)$$
$$\leq C \left( \|\rho_\epsilon\|_{L^1(\mathbb{R}^n)} + \|\nabla \rho_\epsilon^{\alpha-1/2}\|_{L^2(\mathbb{R}^n)} \right),$$
(20)
where \( p \geq (\alpha - 1/2)^{-1} \) for \( N = 2 \) and \( p = 6 \) for \( N = 3 \). While for \( \alpha > 3/2 \), by (14) and interpolation theorem, one has
\[
\| \rho_{\varepsilon}^{\alpha-1/2} \|_{L^p(\mathbb{R}^N)} \leq C \left( \| \rho_{\varepsilon}^{\alpha-1/2} \|_{L^1(\mathbb{R}^N)} + C \| \nabla \rho_{\varepsilon}^{\alpha-1/2} \|_{L^2(\mathbb{R}^N)} \right) \\
\leq C \left( \| \rho_{\varepsilon}^{(1-\theta)(\alpha-1/2)} \|_{L^p(\mathbb{R}^N)} + \| \nabla \rho_{\varepsilon}^{\alpha-1/2} \|_{L^2(\mathbb{R}^N)} \right),
\tag{21}
\]
where \( \theta = \frac{p(\alpha-1/2)}{p-(\alpha-1/2)} > 1 \) if \( N = 2 \) and \( p = 6 \) if \( N = 3 \). The (19) thus follows from (18), (20) and (21).

The key issue in proving theorem 1.1 is to get the \( \varepsilon \)-independent estimates and take \( \varepsilon \)-limit in definition 1.1. In terms of (8) and (19), one has
\[
\varepsilon \int_0^T \int_{\mathbb{R}^N} \rho_{\varepsilon}^2 \text{div} \phi \to 0 \quad \text{as} \quad \varepsilon \to 0.
\]
Besides, we also need to justify (11) and (12) and the strong convergence of \( \sqrt{\rho_{\varepsilon}} u_{\varepsilon} \). For that purpose it suffices to prove the lemmas 3.1–3.3 below.

**Lemma 3.1.** Upon to some subsequence, it satisfies
\[
\rho_{\varepsilon} \rightharpoonup \rho \quad \text{in} \quad C \left( [0,T]; L_{\text{loc}}^{q_1}(\mathbb{R}^N) \right),
\tag{22}
\]
where \( q_1 \in [1, \infty) \) if \( N = 2 \) and \( q_1 \in [1, 6\alpha - 3) \) if \( N = 3 \).

**Proof.** By (19) and Hölder inequality, we have
\[
\| \nabla \rho_{\varepsilon}^\alpha \|_{L^1(\mathbb{R}^N)} \leq C \| \rho_{\varepsilon}^{1/2} \|_{L^1(\mathbb{R}^N)} \| \nabla \rho_{\varepsilon}^{\alpha-1/2} \|_{L^2(\mathbb{R}^N)} \leq C.
\tag{23}
\]
Since \( 1 \leq \alpha \leq 2\alpha - 1 < 6\alpha - 3 \), from (19) and (16) we deduce
\[
\begin{align*}
\| \rho_{\varepsilon}^\alpha u_{\varepsilon} \|_{L^1(\mathbb{R}^N)} + \int_0^T \| \rho_{\varepsilon}^\alpha \text{div} u_{\varepsilon} \|_{L^1(\mathbb{R}^N)} \\
\leq \| \rho_{\varepsilon}^{\alpha-1/2} \|_{L^2(\mathbb{R}^N)} \| \sqrt{\rho_{\varepsilon}} u_{\varepsilon} \|_{L^2(\mathbb{R}^N)} \\
+ \sup_{0 \leq t \leq T} \| \rho_{\varepsilon}^{\alpha/2} \|_{L^2(\mathbb{R}^N)} \int_0^T \| \rho_{\varepsilon}^{\alpha/2} \text{div} u_{\varepsilon} \|_{L^2(\mathbb{R}^N)} \leq C.
\end{align*}
\]
This, along with
\[
\partial_t \rho_{\varepsilon}^\alpha = (1-\alpha) \rho_{\varepsilon}^\alpha \text{div} u_{\varepsilon} - \text{div}(\rho_{\varepsilon}^\alpha u_{\varepsilon}),
\tag{24}
\]
ensures that \( \partial_t \rho_{\varepsilon}^\alpha \in L^2 \left( 0,T; W_{\text{loc}}^{1,1}(\mathbb{R}^N) \right) \). By the Aubin–Lions lemma, we get
\[
\rho_{\varepsilon}^\alpha \rightharpoonup \rho^\alpha \quad \text{in} \quad C \left( [0,T]; L_{\text{loc}}^\beta(\mathbb{R}^N) \right) \quad \text{for} \quad \beta \in [1,3/2).
\]
Therefore, up to some subsequence,
\[
\rho_{\varepsilon}^\alpha \to \rho^\alpha, \quad \text{almost everywhere.}
\tag{25}
\]
So, (19) and (25) guarantee the strong convergence of $\rho_\varepsilon$ to $\rho$ in $L^\infty \left(0, T; L^2_{\text{loc}}(\mathbb{R}^N) \right)$ with $q \in [1, \infty)$ if $N = 2$ and $q \in [1, 6\alpha - 3]$ if $N = 3$. Choosing $\alpha = 1$ implies $\partial_t \rho_\varepsilon \in L^2 \left(0, T; W^{1,1}_{\text{loc}}(\mathbb{R}^N) \right)$, we conclude (22) by the Aubin–Lions lemma.

Consequently, the (19) and (25) implies that

$$\sqrt{\rho_\varepsilon} \to \sqrt{\rho} \text{ in } L^2 \left(0, T; L^2_{\text{loc}}(\mathbb{R}^N) \right), \quad \rho_\varepsilon^{\alpha - 1/2} \to \rho^{\alpha - 1/2} \text{ in } L^2 \left(0, T; H^1_{\text{loc}}(\mathbb{R}^N) \right).$$

**Lemma 3.2.** Upon to some subsequence, it satisfies

$$\sqrt{\rho_\varepsilon} u_\varepsilon \to \sqrt{\rho} u \text{ in } L^2 \left(0, T; L^2_{\text{loc}}(\mathbb{R}^N) \right). \quad (26)$$

**Proof.** The process is divided into several steps.

**Step 1.** Define $m_\varepsilon = \left(\chi(\rho_\varepsilon)\rho_\varepsilon^\alpha + (1 - \chi(\rho_\varepsilon))\rho_\varepsilon^{(1+\alpha)/2}\right) u_\varepsilon$ with $\chi(x)$ being smooth and satisfying $\chi(x) = 1$ if $|x| \leq 1$ and $\chi(x) = 0$ if $|x| \geq 2$.

We claim that, for $p \in [1, 3/2)$,

$$m_\varepsilon \to \left(\chi(\rho)\rho^\alpha + (1 - \chi(\rho))\rho^{(1+\alpha)/2}\right) u \text{ in } L^2 \left(0, T; L^2_{\text{loc}}(\mathbb{R}^N) \right).$$

Consequently,

$$m_\varepsilon \to \left(\chi(\rho)\rho^\alpha + (1 - \chi(\rho))\rho^{(1+\alpha)/2}\right) u, \text{ almost everywhere}. \quad (27)$$

In fact, we deduce from (19) and (16) that

$$\int_0^T \|\nabla(\chi(\rho_\varepsilon)\rho_\varepsilon^\alpha u_\varepsilon)\|^2_{L^2(\mathbb{R}^N)} dt \leq C \int_0^T \left(\|\rho_\varepsilon^\alpha |u_\varepsilon||\nabla \rho_\varepsilon^{(1+\alpha)/2}\|^2_{L^2(\{1 \leq \rho_\varepsilon \leq 2\})} + \|\chi \sqrt{\rho_\varepsilon} |u_\varepsilon||\nabla \rho_\varepsilon^{1/2}\|^2_{L^2(\mathbb{R}^N)} + \chi \rho_\varepsilon^\alpha |\nabla u_\varepsilon|^2_{L^2(\mathbb{R}^N)}\right) dt \leq C \int_0^T \left(\|\sqrt{\rho_\varepsilon} u_\varepsilon\|^2_{L^2(\mathbb{R}^N)} \|\nabla \rho_\varepsilon^{1/2}\|^2_{L^2(\mathbb{R}^N)} + \|\rho_\varepsilon^{3/2} \nabla u_\varepsilon\|^2_{L^2(\mathbb{R}^N)} + \|\rho_\varepsilon^\alpha |u_\varepsilon||\nabla \rho_\varepsilon^{1/2}\|^2_{L^2(\mathbb{R}^N)} + \chi \rho_\varepsilon^\alpha |\nabla u_\varepsilon|^2_{L^2(\mathbb{R}^N)}\right) dt \leq C,$$
where we have used $\mathbf{1}_{(\rho_{c}, \varepsilon)}(\rho_{c}^{2(\alpha-\beta)}/2) \leq \mathbf{1}_{(\rho_{c}, \varepsilon)}(\rho_{c}^{1/2})$ since $\alpha \geq 1$. The last two inequalities guarantees

$$\nabla m_{\varepsilon} \in L^{2}(0, T; L^{2}(\mathbb{R}^{N})) .$$

Furthermore, if

$$\partial_{t} \rho_{c} \in L^{2}(0, T; W_{\text{loc}}^{-1, 1}(\mathbb{R}^{N})) .$$

(29)

Then, the Aubin–Lions lemma shows there is a $m \in L^{2}(0, T; L^{3/2}(\mathbb{R}^{N}))$ such that

$$m_{\varepsilon} \rightarrow m,$$

almost everywhere.

This combining with (25) and $\sqrt{\rho_{c}} u_{\varepsilon} \in L^{\infty}(0, T; L^{2}(\mathbb{R}^{N}))$ provides

$$\int_{\rho_{c} \leq 1} \frac{m_{\varepsilon}^{2}}{\rho_{c}^{2\alpha-1}} = \int_{\rho_{c} \leq 1} \lim_{\varepsilon \rightarrow 0} \frac{(\rho_{c}^{\alpha} u_{\varepsilon})^{2}}{\rho_{c}^{2\alpha-1}} \leq \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^{N}} \rho_{c} |u_{\varepsilon}|^{2} \leq C .$$

(30)

So, $m = 0$ on vacuum sets. We define $u = m (\chi(\rho) \rho^{\alpha} + (1 - \chi(\rho)) \rho^{1+\alpha/2})^{-1}$ if $\rho > 0$ and $u = 0$ if $\rho = 0$. The proof is thus completed.

We need to check (29). Let us first prove

$$\partial_{t} \left( (1 - \chi(\rho_{c})) \rho_{c}^{(1+\alpha)/2} u_{\varepsilon} \right) \in L^{2}(0, T; W^{-1, 1}(\mathbb{R}^{N})) .$$

(31)

By (1) and (24), a careful calculation shows

$$\partial_{t} \left( (1 - \chi(\rho_{c})) \rho_{c}^{(1+\alpha)/2} u_{\varepsilon} \right)
\begin{align*}
= & - \frac{2}{\alpha + 1} \chi' \rho_{c} u_{\varepsilon} \partial_{t} \rho_{c}^{(\alpha+1)/2} + (1 - \chi(\rho_{c})) \partial_{t} \rho_{c}^{(1+\alpha)/2} u_{\varepsilon} \\
= & \frac{2}{\alpha + 1} \chi' \rho_{c} u_{\varepsilon} \left( \frac{\alpha - 1}{2} \rho_{c}^{(\alpha+1)/2} \text{div} u_{\varepsilon} + \text{div} \rho_{c}^{(1+\alpha)/2} u_{\varepsilon} \right) \\
& + (1 - \chi(\rho_{c})) \rho_{c} u_{\varepsilon} \left( \frac{3 - \alpha}{2} \rho_{c}^{\alpha-1/2} \text{div} u_{\varepsilon} - \text{div} \rho_{c}^{(\alpha-1)/2} u_{\varepsilon} \right) \\
& + (1 - \chi(\rho_{c})) \rho_{c}^{(\alpha-1)/2} \left[ \text{div} \rho_{c}^{\alpha/2} \nabla u_{\varepsilon} + (\alpha - 1) \rho_{c}^{\alpha} \text{div} u_{\varepsilon} \right] - \varepsilon \psi_{\varepsilon} - \varepsilon \text{div} \rho_{c} u_{\varepsilon} \cdot u_{\varepsilon} .
\end{align*}
$$

(32)

The terms in (32) are dealt with as follows: firstly,

$$\chi' \rho_{c} u_{\varepsilon} \left( \frac{\alpha - 1}{2} \rho_{c}^{(\alpha+1)/2} \text{div} u_{\varepsilon} + \text{div} \rho_{c}^{(1+\alpha)/2} u_{\varepsilon} \right)
\begin{align*}
= & \frac{\alpha + 1}{\alpha + 3} \chi' \rho_{c}^{(\alpha+3)/2} u_{\varepsilon} \text{div} u_{\varepsilon} + \frac{\alpha + 1}{\alpha + 3} u_{\varepsilon} \cdot \nabla \chi(\rho_{c}^{(\alpha+3)/2}) \\
& - \frac{\alpha + 1}{\alpha + 3} \partial_{t} \rho_{c}^{(1 - \chi(\rho_{c}^{(\alpha+3)/2})) (\partial_{t} u_{\varepsilon} \cdot u_{\varepsilon} + u_{\varepsilon} \cdot u_{\varepsilon})} \\
\in & L^{2}(0, T; W^{-1, 1}(\mathbb{R}^{N})) ,
\end{align*}
$$

where we have used
\[
\| (1 - \chi(\rho_e^{(\alpha+3)/2})) u^e_\varepsilon u^e_\varepsilon \|_{L^1(\mathbb{R}^n)}^2 \\
+ \int_0^T \| \rho_e^{(\alpha+3)/2} \chi'(\rho_e) u_\varepsilon \div u_\varepsilon + (1 - \chi(\rho_e^{(\alpha+3)/2}))(u_\varepsilon \cdot \nabla u_\varepsilon + u_\varepsilon \div u_\varepsilon) \|_{L^1(\mathbb{R}^n)}^2 \\
\leq \| \| u_\varepsilon^2 \|_{L^2((1 \leq \rho_e^{(\alpha+3)/2}))}^2 \\
+ \int_0^T \left( \| u_\varepsilon \|_{\div u_\varepsilon} \|_{L^2((1 \leq \rho_e^{(\alpha+3)/2}))}^2 + \| u_\varepsilon \|_{\| \nabla u_\varepsilon \|_{L^1((1 \leq \rho_e^{(\alpha+3)/2)})}}^2 \right) \\
\leq C \| \rho_e \|_{L^2(\mathbb{R}^n)}^2 + \int_0^T \| \sqrt{\rho_e} u_\varepsilon \|_{L^2(\mathbb{R}^n)}^2 \| \rho_e^{\alpha/2} \nabla u_\varepsilon \|_{L^2(\mathbb{R}^n)}^2 \\
\leq C,
\]

due to (16).

Secondly, by virtue of (16) and (19),
\[
(1 - \chi(\rho_e)) \rho_e^{(\alpha-1)/2} \div (\rho_e^{\alpha} \nabla u_\varepsilon) \\
= \div (1 - \chi(\rho_e)) \rho_e^{(\alpha-1)/2} \rho_e^{\alpha/2} \nabla u_\varepsilon - \frac{\alpha - 1}{2} (1 - \chi(\rho_e)) \rho_e^{\alpha/2} \nabla u_\varepsilon \rho_e^{\alpha-1/2} \\
+ \chi'(\rho_e^{(\alpha-1)/2}) \rho_e^{\alpha-1/2} \nabla \rho_e \cdot \nabla u_\varepsilon \\
\in L^2(0, T; W^{-1, 1}(\mathbb{R}^n)).
\]

By similar argument, we receive
\[
(1 - \chi(\rho_e)) \rho_e^{(\alpha-1)/2} \div (\rho_e^{\alpha} \nabla u_\varepsilon) \in L^2(0, T; W^{-1, 1}(\mathbb{R}^n)).
\]

Next,
\[
(1 - \chi(\rho_e)) \rho_e u_\varepsilon \left( \frac{3}{2} \alpha \rho_e^{(\alpha-1)/2} \div u_\varepsilon - \div (\rho_e^{(\alpha-1)/2} u_\varepsilon) \right) \\
- (1 - \chi(\rho_e)) \rho_e^{(\alpha-1)/2} \div (\rho_e u_\varepsilon \otimes u_\varepsilon) \\
= \frac{1 - \alpha}{2} (1 - \chi(\rho_e)) \rho_e^{(\alpha+1)/2} u_\varepsilon \div u_\varepsilon - \frac{2 \alpha + 3}{2 \alpha} (1 - \chi(\rho_e^{(\alpha+3)/2}))(u_\varepsilon \div u_\varepsilon + u_\varepsilon \cdot \nabla u_\varepsilon) \\
+ \partial_j \left( \frac{2 \alpha + 3}{(2 \alpha + 3)(1 - \chi(\rho_e^{(\alpha+3)/2}))} u^e_k u^e_l - (1 - \chi(\rho_e)) u^e_k u^e_l \rho_e^{(\alpha+1)/2} \right) \\
\in L^2(0, T; W^{-1, 1}(\mathbb{R}^n)),
\]

where the following inequality has been used
\[
\int_0^T \| (1 - \chi(\rho_e)) \rho_e^{(\alpha+1)/2} |u_\varepsilon|^2 \|_{L^1(\mathbb{R}^n)}^2 \\
\leq C \left\{ \int_0^T \| \nabla ((1 - \chi(\rho_e)) \rho_e^{(\alpha+1)/2} |u_\varepsilon|) \|_{L^2(\mathbb{R}^n)}^2 + \| u_\varepsilon \|_{L^2((\rho_e \geq 1)})^2 \right\}, \quad N = 2 \\
\leq C \left\{ \int_0^T \| \rho_e^{\alpha/2} \|_{L^2}^2 + \| \rho_e^{\alpha/2} \|_{L^2}^2 \right\}, \quad N = 3
\]

due to (9), (19) and (28).
Finally, since (19) and \((2\gamma + \alpha - 1)/2 \in (1, 6\alpha - 3)\), it has
\[
(1 - \chi(\rho \varepsilon)) \rho \varepsilon^{(\alpha - 1)/2} \nabla \rho \varepsilon \gamma
\]
\[
= \frac{2\gamma}{2\gamma + \alpha - 1} \left( \nabla ((1 - \chi(\rho \varepsilon)) \rho \varepsilon^{(2\gamma + \alpha - 1)/2} + \rho \varepsilon^{(2\gamma + \alpha - 1)/2} \chi \nabla \rho \varepsilon) \right)
\]
\[
in L^\infty(0, T; W^{-1,1}(\mathbb{R}^N)).
\]
A similar argument yields
\[
\partial_t (\chi(\rho \varepsilon) \rho \varepsilon^2) \in L^2(0, T; W^{-1,1}(\mathbb{R}^N)),
\]
which combining with (31) gives the desired (29).

Step 2. It satisfies
\[
\sqrt{\rho \varepsilon} \ln^{1/2}(e + |u|^2) \in L^\infty(0, T; L^2(\mathbb{R}^N)).
\] (34)
To this end, let us first check
\[
\sqrt{\rho \varepsilon} u \ln^{1/2}(e + |u|^2) \in L^\infty(0, T; L^2(\mathbb{R}^N)).
\] (35)
Clearly, the (35) follows directly from (9) and (16) in case of \(N = 3\). Now let us pay attention to \(N = 2\). Following in [11, 21], we have for any \(\delta \in (0, 2)\)
\[
\frac{d}{dt} \int_{\mathbb{R}^2} \rho \varepsilon (1 + |u|^2) \ln(1 + |u|^2) + \int_{\mathbb{R}^2} \rho \varepsilon (1 + \ln(1 + |u|^2)) |\nabla u|^2
\]
\[
\leq C \int_{\mathbb{R}^2} \rho \varepsilon^2 |\nabla u|^2 + C \varepsilon^2 \left( \int_{\mathbb{R}^2} \rho \varepsilon^{(\delta - 2\alpha - \delta)/2} \right)^{-\frac{2}{\delta - 2}} \left( \int_{\mathbb{R}^2} \rho \varepsilon (1 + |u|^2) \right)^{\delta/2}.
\] (36)
Thus, using (15) and (16), integration of (36) in time conclude the (35), so long as
\[
\varepsilon^2 \int_0^T \left( \int_{\mathbb{R}^2} \rho \varepsilon^{(\delta - 2\alpha - \delta)/2} \right)^{-\frac{2}{\delta - 2}} \leq C,
\]
which is fulfilled because of (19). Making use of (25), (27), (35), the Fatou lemma, we get (34).

Step 3. Given constant \(M > 1\), the (25) and (27) ensure that \(\sqrt{\rho \varepsilon} \ln^{1/2}(e + |u|^2) \to \sqrt{\rho \varepsilon} |u| \leq M\) almost everywhere when \(\rho > 0\). If we also define \(\sqrt{\rho \varepsilon} |u| \leq M\) on sets \(\{ \rho = 0 \}\), then
\[
\sqrt{\rho \varepsilon} |u| \leq M \sqrt{\rho \varepsilon} \to 0 = \sqrt{\rho \varepsilon} |u| \leq M \quad \text{for} \quad \rho = 0.
\]
Recalling (19), it satisfies for \(q > 2\)
\[
\sqrt{\rho \varepsilon} u \ln^{1/2}(e + |u|^2) \in L^\infty(0, T; L^q),
\]
and therefore,
\[
\int_0^T \int_{\mathbb{R}^2} |\sqrt{\rho \varepsilon} u | u | \leq M - \sqrt{\rho \varepsilon} |u| \leq M |^2 \to 0 \quad \text{as} \quad \varepsilon \to 0.
\]
On the other hand, it follows from (35) and (34) that
\[
\int_0^T \int_{\mathbb{R}^2} \left( \sqrt{\rho \varepsilon} u | u | \leq M + \sqrt{\rho \varepsilon} |u| \leq M \right)^2
\]
\[
\leq C \int_0^T \int_{\mathbb{R}^2} \left( \rho \varepsilon |u|^2 \ln(1 + |u|^2) + \rho |u|^2 \ln(1 + |u|^2) \right)
\]
\[
\to 0 \quad \text{as} \quad M \to \infty.
\]
In conclusion, sending $\varepsilon \to 0$ first and then $M \to \infty$ yields
\[
\int_0^T \int_{\mathbb{R}^N} \left| \sqrt{\rho_z u_z} - \sqrt{\rho u} \right|^2 \leq 2 \int_0^T \int_{\mathbb{R}^N} \left| \sqrt{\rho_z u_z} |u_z| \right| \leq M - \sqrt{\rho u} |u| \leq M \right|^2 + 2 \int_0^T \int_{\mathbb{R}^N} \left( \left| \sqrt{\rho_z u_z} |u_z| > M \right|^2 + \left| \sqrt{\rho u} |u| > M \right|^2 \right) \to 0.
\]

**Lemma 3.3.** It satisfies
\[
\rho_z u_z \to \rho u \quad \text{in} \quad L^2 \left( 0, T ; L^{q_2}(\mathbb{R}^N) \right),
\]
where $q_2 \in [1, 2)$ if $N = 2$ and $q_2 \in [1, \frac{2\alpha - 6}{6\alpha - 1})$ if $N = 3$.

**Proof.** Making use of lemma 3.2, (19) and (34), and the inequality
\[
\left| \sqrt{x} - \sqrt{y} \right| \leq \sqrt{|x - y|}, \quad \forall \ x \geq 0, \ y \geq 0,
\]
we conclude the (37) from the following
\[
\left\| \rho_z u_z - \rho u \right\|_{L^2(\mathbb{R}^N)} \leq \left\| \sqrt{\rho_z} \left( \sqrt{\rho_z u_z} - \sqrt{\rho u} \right) \right\|_{L^2(\mathbb{R}^N)} + \left\| \sqrt{\rho_z} - \sqrt{\rho} \right\|_{L^2(\mathbb{R}^N)} \leq C \left( \left\| \left( \sqrt{\rho_z} u_z - \sqrt{\rho u} \right) \right\|_{L^2(\mathbb{R}^N)} + \left\| \rho_z - \rho \right\|^2_{L^2(\mathbb{R}^N)} \right),
\]
where $q_2 = (1/q + 1/2)^{1}$ with $q \geq 2$ if $N = 2$ and $q \in [2, 6\alpha - 3]$ if $N = 3$. \hfill \Box

4. **Proof of theorem 1.2**

Utilizing (4), we deduce from (10) that
\[
\left\| \rho(\cdot, t) \right\|_{L^1(\mathbb{R}^N)} = \left\| \rho_z(\cdot, t) \right\|_{L^1(\mathbb{R}^N)} = \left\| \rho_0 \right\|_{L^1(\mathbb{R}^N)},
\]
\[
\sup_{t \in [0, T]} \int_{\mathbb{R}^N} \rho u_t^2 + \int_0^T \int_{\mathbb{R}^N} \left( \rho^\alpha |\nabla u|^2 + (\alpha - 1) \rho^\alpha |\text{div} u|^2 \right) \leq \int_{\mathbb{R}^N} \frac{|\rho_0|^2}{\rho_0}
\]
and
\[
\sup_{t \in [0, T]} \int_{\mathbb{R}^N} \rho |u + \alpha \rho^{\alpha - 2} \nabla \rho| \leq \int_{\mathbb{R}^N} \rho_0 |u_0 + \alpha \rho_0^{\alpha - 2} \nabla \rho_0|^2.
\]

In addition, the same method as (19) runs
\[
\rho \in L^\infty \left( 0, T ; L^1(\mathbb{R}^N) \cap L^q(\mathbb{R}^N) \right), \quad \nabla \rho^{\alpha - 1/2} \in L^\infty \left( 0, T ; L^2(\mathbb{R}^N) \right),
\]
where $q < \infty$ if $N = 2$ and $q = 6\alpha - 3$ if $N = 3$.

Set $z = \rho_z - \rho$. Subtracting (10) from (1) receives
\[
\left\{ \begin{array}{l}
\partial_t z = \Delta (\rho_z^\alpha - \rho^\alpha) - \text{div} \left[ \rho_z (u_z + \alpha \rho_z^{\alpha - 2} \nabla z) \right] - (u + \alpha \rho^{\alpha - 2} \nabla \rho), \\
z(x, t = 0) = 0.
\end{array} \right.
\]
Multiplying the above by \( \varphi(x,t) \in C^\infty_c(\mathbb{R}^N \times [0,+\infty)) \) and integrating the expression by parts give rise to

\[
\int_{\mathbb{R}^N} z \varphi(x,t) = \int_0^t \int_{\mathbb{R}^N} z (\varphi_t + a \Delta \varphi)
+ \int_0^t \int_{\mathbb{R}^N} \left[ \rho_\varepsilon (u_\varepsilon + \alpha \rho_\varepsilon^{\alpha-2} \nabla \rho_\varepsilon) - \rho (u + \alpha \rho^{\alpha-2} \nabla \rho) \right] \cdot \nabla \varphi,
\]  
(43)

where \( a = (\rho_\varepsilon^\alpha - \rho^\alpha)/\varepsilon \) if \( z \neq 0 \) and \( a = 0 \) if \( z = 0 \).

To be continued, consider the following backward parabolic equation

\[
\begin{align*}
\partial_t \varphi_R + a_n \Delta \varphi_R &= 0, & x \in B_R, \; 0 \leq s < t, \\
\varphi_R &= 0, & x \in \partial B_R, \; 0 \leq s < t, \\
\varphi_R(x,t) &= \theta(x) \in H^1_0(B_R), & x \in B_R,
\end{align*}
\]  
(44)

where \( a_n = \eta_{1/n} * a_{K,\varepsilon} \in [2^{-1},2K] \) and \( \eta_{1/n} \) being the standard Friedrichs’ mollifier such that as \( n \to \infty \),

\[
a_n \to a_{K,\varepsilon} = \begin{cases} K, & a > K, \\ a, & \varepsilon \leq a \leq K, \\ \varepsilon, & a < \varepsilon. \end{cases}
\]  
(45)

The classical linear parabolic theory (see [15]) ensures that (44) has a unique solution \( \varphi_R \in L^\infty (0,t; H^1_0) \cap L^2(0,t; H^2) \). If we multiply (44) by \( \Delta \varphi_R \), we infer for \( \tau \in [0,t] \)

\[
\frac{1}{2} \int_{B_R} |\nabla \varphi_R|^2(x,\tau) - \frac{1}{2} \int_{B_R} |\nabla \varphi_R|^2(x,t) + \int_{\tau}^t \int_{B_R} a_n |\Delta \varphi_R|^2 = 0,
\]

and thus,

\[
\int_{B_R} |\nabla \varphi_R|^2(x,\tau) + \int_{\tau}^t \int_{B_R} a_n |\Delta \varphi_R|^2 \leq \int_{B_R} |\nabla \theta|^2.
\]  
(46)

Define a smooth cut-off function \( \xi_R \) satisfying

\[
\xi_R = 1 \text{ in } B_{R/2}, \quad \xi_R = 0 \text{ in } \mathbb{R}^N \setminus B_R, \quad |\nabla^k \xi_R| \leq C R^{-k}, \; (k = 1, 2).
\]  
(47)

If we extend \( \varphi_R \) to \( \mathbb{R}^N \) by zero and replace \( \varphi_R \) in (43) with \( \varphi = \xi_R \varphi_R \), the first term on the right-hand side of (43) satisfies

\[
\int_0^t \int_{\mathbb{R}^N} z (\varphi_s + a \Delta \varphi)
\]

\[
= \int_0^t \int_{\mathbb{R}^N} z \xi_R (\partial_t \varphi_R + a \Delta \varphi_R) + \int_0^t \int_{\mathbb{R}^N} (\rho_\varepsilon^\alpha - \rho^\alpha) (2 \nabla \xi_R \nabla \varphi_R + \varphi_R \Delta \xi_R)
\]

\[
= \int_0^t \int_{\mathbb{R}^N} z \xi_R (a - a_n) \Delta \varphi_R + \int_0^t \int_{\mathbb{R}^N} (\rho_\varepsilon^\alpha - \rho^\alpha) (2 \nabla \xi_R \nabla \varphi_R + \varphi_R \Delta \xi_R)
\triangleq I_1 + I_2,
\]  
(48)

where in the second equality we used (44).
By (45) and (46), one has
\[
|J_1| \leq \left( \int_0^T \int_{B_R} \frac{\varepsilon^2 (a - a_n)^2}{a_n} \right)^{1/2} \left( \int_0^T \int_{B_R} \left|a_n \Delta \varphi_R \right|^2 \right)^{1/2}
\]
\[
\leq \|\nabla \varphi\|_{L^2(B_R)} \left( \int_0^T \int_{B_R} \frac{\varepsilon^2 (a - a_n)^2}{a_n} \right)^{1/2}
\]
\[
\leq \sqrt{2} \varepsilon^{-1} \|\nabla \varphi\|_{L^2(B_R)} \left( \int_0^T \int_{B_R} \varepsilon^2 (a - a_{K,\varepsilon})^2 + \varepsilon^2 (a_{K,\varepsilon} - a_n)^2 \right)^{1/2}
\]
\[
\leq C \varepsilon^{1/2} \|\nabla \varphi\|_{L^2(B_R)},
\]
where the last inequality owes to (19) and (42), and the following two inequalities:
\[
\int_0^T \int_{B_R} \varepsilon^2 (a - a_{K,\varepsilon})^2 \leq C \int_0^T \|z\|^2_{L^\infty} \|z\|^2_{L^\infty} \left|a_{K,\varepsilon} - a_n\right| \rightarrow 0 \quad (n \rightarrow \infty)
\]
and
\[
\int_0^T \int_{B_R} \varepsilon^2 (a - a_{K,\varepsilon})^2 \leq C \varepsilon^2 \sup_{0 \leq t \leq T} \|z\|^2_{L^2} + \int_0^T \int_{B_R \cap \{z \in \mathbb{R}^n, a > K\}} (\rho^\alpha - \rho^\alpha)^2 \\
\leq C \varepsilon^2 \quad (K \rightarrow \infty).
\]
Therefore, (48) is estimated as
\[
\int_0^T \int_{\mathbb{R}^n} \varepsilon (\varphi_t + a \Delta \varphi) \leq C \varepsilon^{1/2} \|\nabla \varphi\|_{L^2(B_R)} + I_2. \tag{49}
\]

The second term on the right-hand side of (43) satisfies for \( \varphi = \xi_R \varphi_R \)
\[
\int_0^T \int_{\mathbb{R}^n} \left[ \rho \left( u \rho^\alpha - \alpha \rho^\alpha \right) - \rho (u + \alpha \rho^\alpha \nabla \rho) \right] \cdot \nabla \xi_R \varphi_R \\
\leq C \int_0^T \int_{B_R} \left( \rho |u| \rho^\alpha + \alpha \rho^\alpha \nabla \rho \right) |\nabla \xi_R| |\varphi_R| \\
+ \int_0^T \int_{B_R} \left( \rho |u| \rho^\alpha + \alpha \rho^\alpha \nabla \rho \right) |\nabla \xi_R| |\varphi_R| \\
\triangleq J_1 + J_2. \tag{50}
\]

Owing to (13) and (17),
\[
\|\sqrt{\rho} |u| + \alpha \rho^\alpha \nabla \rho \|_{L^2(\mathbb{R}^n)} \leq \epsilon \|\rho_0\|_{L^\gamma(\mathbb{R}^n)} \leq C \epsilon. \tag{51}
\]
This together with (13) and (41) shows for \( \bar{q} > 2 \)
\[
J_1 \leq C \int_0^T \|\sqrt{\rho} |u| + \alpha \rho^\alpha \nabla \rho \|_{L^2(B_R)} \|\nabla \varphi_R\|_{L^2(B_R)} \\
\leq C \varepsilon^{1/2} \int_0^T \|\sqrt{\rho} \|_{L^{1/2 - 1/q}(B_R)} \|\nabla \varphi_R\|_{L^2(B_R)}. \tag{52}
\]

We discuss \( J_1 \) in two cases.
• Let \( \bar{q} = 2 + \delta \) with \( \delta > 0 \) small in case of \( N = 2 \).

By (19) and (14),

\[
J_1 \leq C \varepsilon^{1/2} \int_0^T \| \sqrt{\rho_e} \|_{L^{(1/2 - 1/(2 + \delta))^{-1}}(B_\varepsilon)} \| \nabla \varphi_R \|_{L^{2+\delta}(B_\varepsilon)}
\]

\[
\leq C \varepsilon^{1/2} \int_0^T \| \nabla \varphi_R \|_{L^{2+\delta}(B_\varepsilon)}
\]

\[
\leq C \varepsilon^{1/2} \int_0^T \left( \| \nabla \varphi_R \|_{L^2(B_\varepsilon)} + \varepsilon^{1/2} \| \nabla \varphi_R \|_{L^2(B_\varepsilon)} \| \sqrt{\alpha} \Delta \varphi_R \|_{L^2(B_\varepsilon)} \right)
\]

\[
\leq C \varepsilon^{1/2} \left( \sup_{t \in [0,T]} \| \nabla \varphi_R \|_{L^2(B_\varepsilon)} + \left( \int_0^T \| \sqrt{\alpha} \Delta \varphi_R \|_{L^2(B_\varepsilon)}^2 \right)^{1/2} \right)
\]

\[
\leq C \varepsilon^{1/2} \| \nabla \varphi \|_{L^2(B_{\bar{q}})},
\]

(53)

where in the third inequality we have used

\[
\| \nabla^2 \varphi_R \|_{L^2(B_{\bar{q}})} \leq C \| \Delta \varphi_R \|_{L^2(B_{\bar{q}})} \leq \varepsilon^{-1/2} \| \sqrt{\alpha} \Delta \varphi_R \|_{L^2(B_{\bar{q}})},
\]

(54)

owing to (45) and lemma 2.2.

• Let \( \bar{q} = \frac{6a - 3}{3a - 2} \) in case of \( N = 3 \).

Similar to (53), we deduce

\[
J_1 \leq C \varepsilon^{1/2} \int_0^T \| \sqrt{\rho_e} \|_{L^{(2a - 4/(2a - 3))^{-1}}(B_{\varepsilon})} \| \nabla \varphi_R \|_{L^{\frac{3a - 3}{2 - a}}(B_{\varepsilon})}
\]

\[
\leq C \varepsilon^{1/2} \int_0^T \left( (1 + \varepsilon^{2a - 3}) \| \nabla \varphi_R \|_{L^2(B_{\varepsilon})} + \varepsilon^{2a - 3} \| \sqrt{\alpha} \Delta \varphi_R \|_{L^2(B_{\varepsilon})} \right)
\]

\[
\leq C \varepsilon^{\frac{4a - 3}{2a - 3}} \| \nabla \varphi \|_{L^2(B_{\bar{q}})}.
\]

(55)

With the aid of (52), (53) and (55), the (50) satisfies

\[
\int_0^T \int_{\mathbb{R}^n} \left( \rho_z u_z + \alpha \rho_z^{2 - 2} \nabla \rho_z + \rho(u + \alpha \rho^{2 - 2} \nabla \rho) \right) \cdot \nabla (\xi_R \varphi_R)
\]

\[
\leq J_2 + C \| \nabla \theta \|_{L^2(B_{\bar{q}})} \begin{cases} \varepsilon^{1/2}, & N = 2, \\ \varepsilon^{\frac{4a - 3}{2a - 3}}, & N = 3. \end{cases}
\]

which, along with (43) and (49), implies

\[
\int_{\mathbb{R}^v} z \xi_R \varphi_R(x, t) \leq I_2 + J_2 + C \| \nabla \theta \|_{L^2(B_{\bar{q}})} \begin{cases} \varepsilon^{1/2}, & N = 2, \\ \varepsilon^{\frac{4a - 3}{2a - 3}}, & N = 3. \end{cases}
\]

(56)

Next, by (19), (42), (47), the Poincaré inequality, we deduce from (48) that

\[
I_2 = \int_0^T \int_{\mathbb{R}^v} \left( \rho_{\varepsilon_{\text{avg}}} - \rho^\alpha \right) (2 \nabla \xi_R \nabla \varphi_R + \varphi_R \Delta \xi_R)
\]

\[
\leq C \varepsilon^{-1} \int_0^T \| \rho_{\varepsilon_{\text{avg}}} - \rho^\alpha \|_{L^2(B_{\varepsilon_{\text{avg}}})} \| \nabla \varphi_R \|_{L^2(B_{\varepsilon_{\text{avg}}})}
\]

\[
\leq C \varepsilon^{-1} \| \nabla \theta \|_{L^2(B_{\bar{q}})} \rightarrow 0 \quad (R \to \infty).
\]

(57)
By (13), (41), (19), (46), (47), (51) and (54), the Poincaré inequality, we deduce from (50) that

\[ J_2 \leq C \int_0^T \int_{B_N \setminus B_{N/2}} \left( \rho \epsilon |u_\epsilon + \alpha \rho \epsilon^{\alpha-2} \nabla \rho \epsilon | + \rho |u + \alpha \rho \epsilon^{\alpha-2} \nabla \rho| \right) |

\varepsilon \nabla \xi | \varphiR | \, dx \, dt \]

\[ \leq CR^{-1} \int_0^T \int_{B_N \setminus B_{N/2}} \| \sqrt{\rho \epsilon} |u_\epsilon + \alpha \rho \epsilon^{\alpha-2} \nabla \rho \epsilon | \| \| \varphiR \| L^1(B_N) \]

\[ \leq C \epsilon^{1/2} \sup_{t \in [0,T]} \| \rho \epsilon \|_{L^1(B_N \setminus B_{N/2})} \int_0^T \| \nabla \varphiR \| L^1(B_N) \]

\[ \leq C \sup_{t \in [0,T]} \| \rho \epsilon \|_{L^1(B_N \setminus B_{N/2})} \int_0^T \| \nabla \varphiR \| L^1(B_N) + \| \sqrt{\alpha \Delta \varphiR} \| L^2(B_N) \]

\[ \leq C \sup_{t \in [0,T]} \| \rho \epsilon \|_{L^1(B_N \setminus B_{N/2})} \| \nabla \| L^1(B_N) \rightarrow 0 \quad (R \rightarrow \infty). \]  

(58)

Particularly, if we replace the function \( \varphiR(x,t) \) in (56) with \( \varphiR(x,t) = \xi_R \left( \rho \epsilon^{\alpha-1} - \rho \epsilon^{\alpha-1} \right)(x,t) \),  
we conclude from (56)–(58) that by sending \( R \rightarrow \infty \),

\[ \int_{R^N} (\rho_\epsilon - \rho)(\rho \epsilon^{\alpha-1} - \rho \epsilon^{\alpha-1}) = C \left\{ \begin{array}{ll}
\frac{1}{\alpha + \frac{1}{2}}, & N = 2, \\
\frac{1}{\alpha + \frac{1}{2}}, & N = 3.
\end{array} \right. \]

In terms of lemma 2.3, it satisfies for \( \alpha \geq \frac{3}{2} \)

\[ \int_{R^N} |\rho_\epsilon - \rho|^{\alpha+\frac{1}{2}} \leq C \left\{ \begin{array}{ll}
\frac{1}{\alpha + \frac{1}{2}}, & N = 2, \\
\frac{1}{\alpha + \frac{1}{2}}, & N = 3.
\end{array} \right. \]

The proof of theorem 1.2 is complete by exploiting (19) and (42), and interpolation inequalities.

Acknowledgments

The author is supported by NNSFC Grant No. 11301422. The author is grateful for the anonymous reviewers’ helpful comments and suggestions which improved both the mathematical results and the way to present them.

Appendix. Proof of (9)

Multiplying equations (1) by \( 4|u_\epsilon|^2 u_\epsilon \) yields

\[ \frac{d}{dt} \int_{R^3} \rho_\epsilon |u_\epsilon|^2 + 4 \int_{R^3} \rho_\epsilon |u_\epsilon|^2 |\nabla u_\epsilon|^2 \]

\[ + 8 \int_{R^3} \rho_\epsilon |u_\epsilon|^2 |\nabla u_\epsilon|^2 + 4(\alpha - 1) \int_{R^3} \rho_\epsilon^2 \nabla u_\epsilon \nabla u_\epsilon \cdot (\nabla u_\epsilon |u_\epsilon| + 2u_\epsilon \cdot \nabla |u_\epsilon|) \]

\[ = 4\epsilon \int_{R^3} \nabla \rho_\epsilon \cdot \nabla (|u_\epsilon|^2 u_\epsilon). \]

By Young’s inequality, it satisfies for all \( \alpha > 1/2 \)
\[
8 \int_{\mathbb{R}^{3}} \rho_{\varepsilon}^{4} |u_{\varepsilon}|^{2} |\nabla u_{\varepsilon}|^{2} + 4(\alpha - 1) \int_{\mathbb{R}^{3}} \rho_{\varepsilon}^{2} \text{div} u_{\varepsilon} |u_{\varepsilon}| (\text{div} u_{\varepsilon} |u_{\varepsilon}| + 2u_{\varepsilon} \cdot \nabla |u_{\varepsilon}|) \\
\geq \frac{5}{2} \int_{\mathbb{R}^{3}} \rho_{\varepsilon}^{2} (\text{div} u_{\varepsilon})^{2} |u_{\varepsilon}|^{2} \geq \frac{5}{2} \int_{\mathbb{R}^{3}} \rho_{\varepsilon}^{4} |\nabla u_{\varepsilon}|^{2} |u_{\varepsilon}|^{2}.
\]

Next,
\[
\varepsilon \int_{\mathbb{R}^{3}} \nabla \rho_{\varepsilon}^{2} \text{div} (|u_{\varepsilon}|^{2} u_{\varepsilon}) \\
\leq \varepsilon \int_{\mathbb{R}^{3}} \rho_{\varepsilon}^{2} |u_{\varepsilon}|^{2} |\nabla u_{\varepsilon}|^{2} + C \varepsilon \int_{\mathbb{R}^{3}} \rho_{\varepsilon}^{2-\gamma-\alpha} |u_{\varepsilon}|^{2} \\
\leq \varepsilon \int_{\mathbb{R}^{3}} \rho_{\varepsilon}^{2} |u_{\varepsilon}|^{2} |\nabla u_{\varepsilon}|^{2} + C \epsilon \|\rho_{\varepsilon}^{2-\gamma-\alpha-1/2} \|_{L^{2}(\mathbb{R}^{3})} \left(1 + \|\rho_{\varepsilon}^{4} |u_{\varepsilon}|^{2} \|_{L^{4}(\mathbb{R}^{3})}^{2}\right) .
\]

For small \( \varepsilon \leq 1/2 \), the above three inequalities ensure that
\[
\frac{d}{dt} \int_{\mathbb{R}^{3}} \rho_{\varepsilon}^{2} |u_{\varepsilon}|^{4} + \int_{\mathbb{R}^{3}} \rho_{\varepsilon}^{2} |u_{\varepsilon}|^{2} |\nabla u_{\varepsilon}|^{2} \\
\leq C \epsilon \|\rho_{\varepsilon}^{2-\gamma-\alpha-1/2} \|_{L^{2}(\mathbb{R}^{3})} \left(1 + \|\rho_{\varepsilon}^{4} |u_{\varepsilon}|^{2} \|_{L^{4}(\mathbb{R}^{3})}^{2}\right) .
\]

The proof can be done by means of the Gronwall inequality, provided
\[
\varepsilon \int_{0}^{T} \|\rho_{\varepsilon}^{2-\gamma-\alpha-1/2} \|_{L^{2}(\mathbb{R}^{3})} \leq C .
\]

In fact, since (7) and (8) implies \( 1 \leq 4 \gamma - 2 \alpha - 1 \), it has
\[
\|\rho_{\varepsilon}^{2-\gamma-\alpha-1/2} \|_{L^{2}(\mathbb{R}^{3})}^{2} = \left(\int_{\{\rho_{\varepsilon} \leq 1\}} + \int_{\{\rho_{\varepsilon} \geq 1\}}\right) \rho_{\varepsilon}^{4 \gamma - 2 \alpha - 1} \\
\leq \int_{\{\rho_{\varepsilon} \leq 1\}} \rho_{\varepsilon} + \int_{\{\rho_{\varepsilon} \geq 1\}} \rho_{\varepsilon}^{2 \gamma + 4 \alpha - 3} \leq C + C \|\nabla \rho_{\varepsilon}^{(\gamma + \alpha - 1)/2} \|_{L^{4}(\mathbb{R}^{3})}^{4} ,
\]

where the last inequality owes to (19), Sobolev inequality and the following
\[
\int_{\mathbb{R}^{3}} \rho_{\varepsilon}^{2 \gamma + 4 \alpha - 3} \leq C \|\rho_{\varepsilon}^{(\gamma + \alpha - 1)/2} \|_{L^{4}(\mathbb{R}^{3})}^{4} ,
\]

Therefore,
\[
\varepsilon \int_{0}^{T} \|\rho_{\varepsilon}^{2-\gamma-\alpha-1/2} \|_{L^{2}(\mathbb{R}^{3})} \leq C \varepsilon + \varepsilon \int_{0}^{T} \|\nabla \rho_{\varepsilon}^{(\gamma + \alpha - 1)/2} \|_{L^{2}(\mathbb{R}^{3})}^{2} \leq C ,
\]

where the last inequality owes to (17) and the \( C \) is independent of \( \varepsilon \).

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