A Novel Solution to the Frenet-Serret Equations

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Abstract
A set of equations is developed to describe a curve in space given the curvature $\kappa$ and the angle of rotation $\theta$ of the osculating plane. The set of equations has a solution (in terms of $\kappa$ and $\theta$) that indirectly solves the Frenet-Serret equations, with a unique value of $\theta$ for each specified value of $\tau$. Explicit solutions can be generated for constant $\theta$. The equations break down when the tangent vector aligns to one of the unit coordinate vectors, requiring a reorientation of the local coordinate system.

1 Introduction
Given the curvature $\kappa$ and torsion $\tau$, the Frenet-Serret equations describe a curve in space parameterized by the arc length $s$:

$$\frac{dT}{ds} = \kappa N; \quad (1)$$

$$\frac{dN}{ds} = -\kappa T + \tau B; \quad (2)$$

$$\frac{dB}{ds} = -\tau N; \quad (3)$$

$$\frac{dR}{ds} = T. \quad (4)$$

Here $R$, $T$, $N$, and $B$ are the position, tangent, normal, and binormal vectors, respectively. These equations have no explicit solution (in terms of $\kappa$ and $\tau$) for the general case, although solutions for special cases exist.

It is shown here that a set of equations can be developed to describe a curve in space given the curvature $\kappa$ and the angle of rotation $\theta$ of the osculating plane. The set of equations has a solution (in terms of $\kappa$ and $\theta$) that indirectly solves the Frenet-Serret equations, and has a unique $\theta$ for every value of $\tau$. 

Many problems\textsuperscript{3–7} involve the use of the Frenet-Serret equations, requiring numerical approximations or the use of helical arc segments (each having constant $\tau$ and $\kappa$). Specifying $\kappa$ and $\theta$ to generate a solution may be useful if $\tau$ is not initially known. The torsion $\tau$ can then be determined from $\theta$.

2 Mathematical Development

A local coordinate system having the property $T = i'$ (figure 1) supports the definition of $N$:

$$N = j' \cos \theta + k' \sin \theta.$$  \hfill (5)

The curvature $\kappa$ and the angle of rotation $\theta$ of the osculating plane (containing $N$ and $T$) characterize the curve. When the plane containing $T$ and the global coordinate $j$ is normal to $k'$ (figure 2) then
Figure 2: Angular orientation of the local coordinate system with respect to the global coordinate system.

\[ k' = \frac{T \times j}{|T \times j|} = \frac{-i T_k + k T_i}{\sqrt{1 - T_j^2}}. \]  \hspace{1cm} (6)

Equation (6) breaks down when \( T = \pm j \), requiring an alternate expression for \( k' \) (developed in section 4). However, when \( T \neq \pm j \),

\[ j' = k' \times T = \frac{-i T_i T_j + j (1 - T_j^2) - k T_j T_k}{\sqrt{1 - T_j^2}}. \]  \hspace{1cm} (7)

Substituting (7) and (6) into (5):

\[ N_i = \frac{1}{\kappa} \frac{dT_i}{ds} = \frac{-T_k \sin \theta - T_i T_j \cos \theta}{\sqrt{1 - T_j^2}}; \]  \hspace{1cm} (8)

\[ N_j = \frac{1}{\kappa} \frac{dT_j}{ds} = \cos \theta \sqrt{1 - T_j^2}; \]  \hspace{1cm} (9)

\[ N_k = \frac{1}{\kappa} \frac{dT_k}{ds} = \frac{T_i \sin \theta - T_j T_k \cos \theta}{\sqrt{1 - T_j^2}}. \]  \hspace{1cm} (10)
Equation (9) can be integrated directly:

\[
\int_{T_{j0}}^{T_j} \frac{dT_j}{\sqrt{1 - T_j^2}} = \int_{s_0}^{s} \kappa \cos \theta d\sigma; \quad (11)
\]

leading to

\[
\sin^{-1} T_j = \sin^{-1} T_{j0} + \int_{s_0}^{s} \kappa \cos \theta d\sigma;
\]

\[
T_j = \sin \left[ \sin^{-1} T_{j0} + \int_{s_0}^{s} \kappa \cos \theta d\sigma \right] = \sin \delta;
\]

\[
T_j = T_{j0} \cos \int_{s_0}^{s} \kappa \cos \theta d\sigma + \sqrt{1 - T_{j0}^2} \sin \int_{s_0}^{s} \kappa \cos \theta d\sigma. \quad (12)
\]

Equation (8) is solved by noting that

\[
T_k = \sqrt{1 - T_j^2 - T_i^2} = \sqrt{\cos^2 \delta - T_i^2}
\]

and introducing the variable \( \beta \) so that

\[
T_i = \cos \delta \cos \beta; \quad (13)
\]

\[
T_k = \cos \delta \sin \beta. \quad (14)
\]

Substituting into (8):

\[
\frac{dT_i}{ds} = -\kappa \cos \theta \sin \delta \cos \beta - \cos \delta \sin \beta \frac{d\beta}{ds} \]

\[
= -\kappa \sin \theta \cos \delta \sin \beta - \kappa \cos \theta \cos \delta \cos \beta \sin \delta \]

\[
= -\kappa \sin \theta \sin \beta - \kappa \cos \theta \cos \beta \sin \delta. \quad (15)
\]

Equation (15) simplifies to

\[
\frac{d\beta}{ds} = \frac{\kappa \sin \theta}{\cos \delta}; \quad (16)
\]

or

\[
\beta = \beta_0 + \int_{s_0}^{s} \frac{\kappa \sin \theta}{\cos \delta} d\sigma = \cos^{-1} \left( \frac{T_i}{\cos \delta} \right); \quad (17)
\]

so that

\[
T_i = T_i(0) \cos \delta \int_{s_0}^{s} \frac{\kappa \sin \theta}{\cos \delta} d\sigma - T_k(0) \cos \delta \int_{s_0}^{s} \frac{\kappa \sin \theta}{\cos \delta} d\sigma; \quad (18)
\]

where
\[
\cos \delta = \sqrt{1 - T^2_j} \cos \int_{s_0}^{s} \kappa \cos \theta d\sigma - T_{j0} \sin \int_{s_0}^{s} \kappa \cos \theta d\sigma. \quad (19)
\]

The solution for \( T_k \) follows from (14) and (16):
\[
T_k = T_{k0} \cos \delta \cos \delta_0 \cos \int_{s_0}^{s} \kappa \sin \theta \cos \theta d\sigma + T_{i0} \sin \int_{s_0}^{s} \kappa \sin \theta \cos \delta d\sigma. \quad (20)
\]

It can be easily verified that (12), (18), and (20) meet the requirement:
\[
\kappa = \left| \frac{dT}{ds} \right|. \quad (21)
\]

Generating an expression for the torsion \( \tau \) requires first computing \( N \) by substituting (12)-(14) into (8)-(10):
\[
N_i = - \cos \theta \sin \delta \cos \beta - \sin \beta \sin \theta; \quad (22)
N_j = \cos \theta \cos \delta; \quad (23)
N_k = - \cos \theta \sin \delta \sin \beta + \cos \beta \sin \theta. \quad (24)
\]

Next, \( B = T \times N \):
\[
B_i = \sin \delta \sin \theta \cos \beta - \cos \theta \sin \beta; \quad (25)
B_j = - \cos \delta \sin \theta; \quad (26)
B_k = \sin \delta \sin \theta \sin \beta + \cos \theta \cos \beta. \quad (27)
\]

Equation (28) expresses the torsion as a function of \( \theta \):
\[
\tau = \left| \frac{dB}{ds} \right| = \frac{d\theta}{ds} - \kappa \tan \delta \sin \theta. \quad (28)
\]

Equation (29) expresses \( \tau \) in terms of components of \( T \) and \( B \):
\[
\tau = \frac{d\theta}{ds} + \frac{\kappa T_j B_j}{1 - T^2_j}. \quad (29)
\]

Finally, (2) serves as a check on the solutions for \( T, N, B, \) and \( \tau \).

### 3 Discussion

Integrating (29) leads to the following expression for \( \theta \):
\[
\theta = \theta_0 + \int_{s_0}^{s} \left( \tau - \frac{\kappa T_j B_j}{1 - T^2_j} \right) d\sigma. \quad (30)
\]
Equation (30) indicates a unique value of $\theta$ for each specified value of $\tau$ when $T_j \neq \pm 1$. Thus, (12), (18), and (20) indirectly solve (1)-(3).

The angle $\theta$ can also be expressed in terms of components of $T$, $N$, $B$:

$$
\theta = -\sin^{-1}\frac{B_j}{\sqrt{1 - T_j^2}} = \cos^{-1}\frac{N_j}{\sqrt{1 - T_j^2}} = -\tan^{-1}\frac{B_j}{N_j},
$$  

(31)

3.1 Constant $\theta$

An explicit solution often results when $\theta$ is constant. Setting $T_{i0} = 1$, so that $\beta_0 = \delta_0 = 0$ (and setting $s_0 = 0$) leads to

$$
\delta = \int_0^s \kappa(\sigma) \cos \theta_0 d\sigma;
$$  

(32)

$$
\beta = 2 \tan \theta_0 \tanh^{-1}(\tan \delta/2);
$$  

(33)

so that

$$
T_i = \cos\left[2 \tan \theta_0 \tanh^{-1}(\tan \delta/2)\right] \cos \int_0^s \kappa(\sigma) \cos \theta_0 d\sigma;
$$  

(34)

$$
T_j = \sin \int_0^s \kappa(\sigma) \cos \theta_0 d\sigma;
$$  

(35)

$$
T_k = \sin\left[2 \tan \theta_0 \tanh^{-1}(\tan \delta/2)\right] \cos \int_0^s \kappa(\sigma) \cos \theta_0 d\sigma.
$$  

(36)

The torsion becomes

$$
\tau(s) = -\kappa(s) \sin \theta_0 \tan \int_0^s \kappa(\sigma) \cos \theta_0 d\sigma. 
$$  

(37)

As an example, when

$$
\kappa = \kappa_0 e^{-s^2},
$$  

(38)

$$
T_i = \cos\left[\frac{\kappa_0 \sqrt{\pi}}{2} \text{erf}(s) \cos \theta_0\right] \cos \left[2 \tan \theta_0 \tanh^{-1}\left(\tan\left[\frac{\kappa_0 \sqrt{\pi}}{4} \text{erf}(s) \cos \theta_0\right]\right)\right];
$$  

(39)

$$
T_j = \sin\left[\frac{\kappa_0 \sqrt{\pi}}{2} \text{erf}(s) \cos \theta_0\right];
$$  

(40)

$$
T_k = \cos\left[\frac{\kappa_0 \sqrt{\pi}}{2} \text{erf}(s) \cos \theta_0\right] \sin \left[2 \tan \theta_0 \tanh^{-1}\left(\tan\left[\frac{\kappa_0 \sqrt{\pi}}{4} \text{erf}(s) \cos \theta_0\right]\right)\right];
$$  

(41)

$$
\tau(s) = -\kappa_0 e^{-s^2} \sin \theta_0 \tan\left[\frac{\kappa_0 \sqrt{\pi}}{2} \text{erf}(s) \cos \theta_0\right].
$$  

(42)
3.2 Constant $\kappa$

When $\kappa = \kappa_0$ but $\theta \neq \theta_0$, the solution will typically involve undetermined integrals. For example, when $\kappa = \kappa_0$ and $\theta = \kappa_0 s$,

$$T_i = \cos (\sin \kappa_0 s) \cos \int_0^s \frac{\kappa_0 \sin \kappa_0 \sigma}{\cos (\sin \kappa_0 \sigma)} d\sigma; \quad (43)$$

$$T_j = \sin (\sin \kappa_0 s); \quad (44)$$

$$T_k = \cos (\sin \kappa_0 s) \sin \int_0^s \frac{\kappa_0 \sin \kappa_0 \sigma}{\cos (\sin \kappa_0 \sigma)} d\sigma; \quad (45)$$

and

$$\tau(s) = \kappa_0 - \kappa_0 \tan (\sin \kappa_0 s) \sin \kappa_0 s. \quad (46)$$

3.3 Constant $\kappa$ and $\theta$

When $\kappa = \kappa_0$ and $\theta = \theta_0$, (34)-(37) become:

$$T_i = \cos (\kappa_0 s \cos \theta_0) \cos \left[2 \tan \theta_0 \tanh^{-1} \left(\tan \left[\frac{\kappa_0 s}{2} \cos \theta_0 \right]\right)\right]; \quad (47)$$

$$T_j = \sin (\kappa_0 s \cos \theta_0); \quad (48)$$

$$T_k = \cos (\kappa_0 s \cos \theta_0) \sin \left[2 \tan \theta_0 \tanh^{-1} \left(\tan \left[\frac{\kappa_0 s}{2} \cos \theta_0 \right]\right)\right]; \quad (49)$$

$$\tau(s) = -\kappa_0 \sin \theta_0 \tan (\kappa_0 s \cos \theta_0). \quad (50)$$

When $\theta_0 = \pi/2$, $\tau(s) = 0$, confining $T$ and $N$ to a plane. When $T$ aligns with $j$, $\tau \rightarrow \infty$ in (50), and the equations break down.

4 Alternate Set of Equations

The equations break down when $T_j \rightarrow \pm 1$, requiring a different orientation for the local coordinate system. The angle of rotation of the osculating plane is designated $\phi$ here. In general, $\phi \neq \theta$, reflecting differences in angular orientation between the local and global coordinate systems for the two cases. Defining $k'$ as the normal to the plane containing $T$ and $i$, i.e.,

$$k' = \frac{i \times T}{|i \times T|} = \frac{-j T_k + k T_j}{\sqrt{1 - T_i^2}}. \quad (51)$$

The $j'$ unit vector becomes:

$$j' = k' \times T = \frac{-i (1 - T_i^2) + j T_i T_j + k T_i T_k}{\sqrt{1 - T_i^2}}. \quad (52)$$
Substituting into the expression for $N$:

\[ N_i = \frac{1}{\kappa} \frac{dT_i}{ds} = -\cos \phi \sqrt{1 - T_i^2}; \]  
\[ N_j = \frac{1}{\kappa} \frac{dT_j}{ds} = \frac{-T_k \sin \phi + T_i T_j \cos \phi}{\sqrt{1 - T_i^2}}; \]  
\[ N_k = \frac{1}{\kappa} \frac{dT_k}{ds} = \frac{T_j \sin \phi + T_i T_k \cos \phi}{\sqrt{1 - T_i^2}}. \]  

Equations (53)-(55) have the following solution:

\[ T_i = \sin \gamma; \]  
\[ T_j = \cos \gamma \cos \alpha; \]  
\[ T_k = \cos \gamma \sin \alpha; \]  
\[ N_i = -\cos \gamma \cos \phi; \]  
\[ N_j = \sin \gamma \cos \alpha \cos \phi - \sin \alpha \sin \phi; \]  
\[ N_k = \sin \gamma \sin \alpha \cos \phi + \cos \alpha \sin \phi; \]  
\[ B_i = \cos \gamma \sin \phi; \]  
\[ B_j = \sin \gamma \cos \alpha \sin \phi + \sin \alpha \cos \phi; \]  
\[ B_k = -\sin \gamma \sin \alpha \sin \phi + \cos \alpha \cos \phi; \]  
\[ \tau = \frac{d\phi}{ds} - \kappa \tan \gamma \sin \phi = \frac{d\phi}{ds} - \frac{\kappa B_i T_i}{1 - T_i^2}. \]  

Here

\[ \gamma = \sin^{-1} T_{i0} - \int_{s_0}^{s} \kappa \cos \phi d\sigma; \]  
\[ \alpha = \cos^{-1} \left( \frac{T_j}{\cos \gamma} \right) = \alpha_0 + \int_{s_0}^{s} \frac{\kappa \sin \phi}{\cos \gamma} d\sigma; \]  
\[ T_j = T_{j0} \frac{\cos \gamma}{\cos \gamma_0} \cos \int_{s_0}^{s} \frac{\kappa \sin \phi}{\cos \gamma} d\sigma - T_{k0} \frac{\cos \gamma}{\cos \gamma_0} \sin \int_{s_0}^{s} \frac{\kappa \sin \phi}{\cos \gamma} d\sigma; \]  
\[ \phi = -\tan^{-1} \frac{B_i}{N_i}. \]  

Even though $\theta$ and $\phi$ both represent the angle of rotation of the osculating plane, (31) and (69) differ because of differences in angular orientation of the local coordinate system.

When $T_j \rightarrow \pm 1$ or $T_i \rightarrow \pm 1$, switching from one set of equations to another avoids numerical difficulties.
5 Concluding Remarks

Unlike the Frenet-Serret equations, (8)-(10) are nonlinear, and do not involve $N$, $B$, or $\tau$. The solution (in terms of $\kappa$ and $\theta$) indirectly solves the Frenet-Serret equations, and leads to a precise definition of $\tau$ as a function of $\kappa$ and $\theta$. A unique value of $\theta$ can be obtained for each specified value of $\tau$ through a first order ordinary differential equation. The equations break down when $T \to \pm j$, requiring an alternative set of equations that break down when $T \to \pm i$. The expressions for the angle of the osculating plane in the two approaches differ because of differences in the angular orientation of the local coordinate system.

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