Invariant Killing spinors in 11D and type II supergravities

U Gran\textsuperscript{1}, J Gutowski\textsuperscript{2} and G Papadopoulos\textsuperscript{3}

\textsuperscript{1} Fundamental Physics, Chalmers University of Technology, SE-412 96 Göteborg, Sweden
\textsuperscript{2} DAMTP, Centre for Mathematical Sciences, University of Cambridge, Wilberforce Road, Cambridge, CB3 0WA, UK
\textsuperscript{3} Department of Mathematics, King's College London, Strand, London WC2R 2LS, UK

E-mail: ulf.gran@chalmers.se, J.B.Gutowski@damtp.cam.ac.uk and george.papadopoulos@kcl.ac.uk

Received 24 November 2008
Published 8 July 2009
Online at stacks.iop.org/CQG/26/155004

Abstract

We present all isotropy groups and associated $\Sigma$ groups, up to discrete identifications of the component connected to the identity, of spinors of 11-dimensional and type II supergravities. The $\Sigma$ groups are products of a Spin group and an $R$-symmetry group of a suitable lower dimensional supergravity theory. Using the case of $SU(4)$-invariant spinors as a paradigm, we demonstrate that the $\Sigma$ groups, and so the $R$-symmetry groups of lower dimensional supergravity theories arising from compactifications, have disconnected components. These lead us to discrete symmetry groups reminiscent of $R$-parity. We examine the role of disconnected components of the $\Sigma$ groups in the choice of Killing spinor representatives and in the context of compactifications.

PACS numbers: 04.65.+e, 11.30.Pb

1. Introduction

Supersymmetric supergravity backgrounds can be categorized into two classes. One class are those backgrounds for which the Killing spinors are invariant under some proper Lie subgroup of the gauge group of the associated theory, and another class are those backgrounds for which the isotropy group of the Killing spinors is the identity. Most of the known supergravity backgrounds belong to the former class, such as the vacua of string compactifications with or without fluxes, for a recent review see [1], and those backgrounds that have applications in gauge theory/duality correspondences; see e.g. [2–5]. A notable exception are the maximally and the near maximally supersymmetric backgrounds of ten- and 11-dimensional
supergravities which belong to the latter class and have been classified in [6] and [7–9], respectively.

The main aim of this paper is to initiate the classification of all supersymmetric 11-dimensional and type II backgrounds for which the Killing spinors are invariant under some proper Lie subgroup \( H \) of the gauge group. Some partial results are already known. In 11-dimensional supergravity, these include the \( N = 1 \) backgrounds with \( SU(5) \) and \( \text{Spin}(7) \times \mathbb{R}^9 \)-invariant Killing spinors [10]; the \( N = 2 \) backgrounds with \( SU(5) \)- and some \( N = 2 \) and \( N = 4 \) backgrounds with \( \text{Spin}(7) \)- and \( G_2 \times \mathbb{R}^9 \)-, and \( N = 4 \) backgrounds with \( G_2 \)- and \( SU(4) \)-invariant Killing spinors, and special cases of backgrounds with stability subgroups embeddable in \( \text{Spin}(7) \times \mathbb{R}^9 \) [13].

In IIB supergravity, the Killing spinor equations have been solved for \( N = 1 \) backgrounds in [14] and all backgrounds with the maximal number of \( H \)-invariant spinors have been classified in [15]. The method that we shall use to solve this problem is based on spinorial geometry [11] facilitated by the application of the \( \Sigma(P) \) groups defined in [16]. The \( \Sigma(P) \) groups are the subgroups of the gauge group of the supergravity theories that leave the plane \( P \) spanned by some spinors invariant. The importance of the \( \Sigma(P) \) groups has been demonstrated in the classification of all supersymmetric backgrounds of heterotic supergravity [16]. In particular, the \( \Sigma(P) \) groups have been used to find the solutions of the gaugino and dilatino Killing spinor equations given a solution of the gravitino Killing spinor equation.

To find the geometry of all 11-dimensional backgrounds that admit \( H \)-invariant Killing spinors, we shall first identify all the subgroups \( H \subset \text{Spin}(10, 1) \) that leave some spinors invariant. These subgroups will be given up to discrete identifications of the connected to the identity component and so they will be derived from a Lie algebra computation. Partial lists of such groups have appeared elsewhere [11, 17, 18]. Here we shall give the complete set relevant for 11-dimensional supergravity which is tabulated in Table 1. In addition, in Tables 2 and 3, we give explicitly the representatives of all the \( H \)-invariant spinors for all isotropy groups.

Table 1. The complete list of isotropy groups of spinors in 11-dimensional supergravity. \( H \) denotes the subgroups of \( \text{Spin}(10, 1) \), up to discrete identifications of the connected to the identity component, which leave some Majorana spinors invariant. \( N_H \) is the maximal number of \( H \)-invariant spinors.

| \( N_H \) | \( H \) |
|---|---|
| 1 | \( \text{Spin}(7) \times \mathbb{R}^9 \) |
| 2 | \( \text{Spin}(7), SU(5), SU(4) \times \mathbb{R}^9, G_2 \times \mathbb{R}^9 \) |
| 3 | \( Sp(2) \times \mathbb{R}^9 \) |
| 4 | \( SU(4), G_2, SU(2) \times SU(3), (SU(2)^2) \times \mathbb{R}^9, SU(3) \times \mathbb{R}^9 \) |
| 5 | \( SU(2) \times \mathbb{R}^9 \) |
| 6 | \( Sp(2), U(1) \times \mathbb{R}^9 \) |
| 8 | \( SU(3), SU(2)^2, SU(2) \times \mathbb{R}^9 \) |
| 10 | \( SU(2) \) |
| 12 | \( U(1) \) |
| 16 | \( SU(2), \mathbb{R}^9 \) |
| 32 | \{1\} |
is Lie algebraic. The list of all $\Sigma(\mathcal{P}_H)$ groups is presented in table 5. We find that all the $\Sigma(\mathcal{P}_H)$ groups are products $\text{Spin} \times R$, where $\text{Spin}$ and $R$ can be identified with the Spin and $R$-symmetry groups of a lower dimensional supergravity theory. Using the case of 11-dimensional backgrounds with $SU(4)$-invariant spinors as a paradigm, we demonstrate that the $\Sigma(\mathcal{P}_H)$ groups have disconnected components which are subgroups of the connected component $\text{Spin}^0(10, 1)$ of Spin(10, 1). In a compactification scenario on a manifold with an $H$-structure and with or without fluxes, this implies that the $R$-symmetry group of the associated lower dimensional supergravity theory is disconnected. In the $SU(4)$ case, representatives of the disconnected components act as a discrete symmetry via reflections on some components of the frame of the compact internal space. This action leaves the metric invariant but changes the fluxes, the fermions and the (almost) complex structure $I$ of the internal space to $-I$. Such discrete transformations are reminiscent\(^4\) of $R$-parity transformations imposed to suppress the rate of decay of the proton in supersymmetric theories; for a review see [20]. Since these discrete symmetries are remnants of the restricted Lorentz transformations of an 11-dimensional theory, it is natural to argue that they must be symmetries of the lower dimensional effective supergravity theories imposing restrictions on the couplings.

Furthermore, we explain how $\Sigma(\mathcal{P}_H)$ can be used in 11-dimensional supergravity to find the normal forms of $H$-invariant Killing spinors for backgrounds with $N < N_H$ number of supersymmetries. In particular, we shall demonstrate for $H = SU(4)$ that the group $\Sigma(\mathcal{P}_{SU(4)})$ can be used to find the Killing spinors for all backgrounds with $N < N_H = 4$. It will become apparent that the disconnected components of $\Sigma(\mathcal{P}_{SU(4)})$ can also be used to reduce the number of choices of Killing spinors.

The isotropy groups $H$ and the associated $\Sigma(\mathcal{P}_H)$ groups of IIA and IIB supergravities have been tabulated in tables 4 and 6, respectively. These groups for IIB supergravity can easily be read off from those of heterotic supergravity; see also [15]. For IIA supergravity, the isotropy groups $H$ and the associated $\Sigma(\mathcal{P}_H)$ groups have a close relationship to those of

---

\(^4\) The discrete symmetries have also a passing similarity with the Weyl subgroup [19] of the $U$-duality group though they cannot be directly identified because the former appears in $SU(4)$-structure compactifications while the latter appears in toric ones.
11-dimensional supergravity. However, there are isotropy and $\Sigma(\mathcal{P}_H)$ groups of the latter that do not have a IIA analogue.

This paper is organized as follows: In section 2, we give the isotropy groups and representatives of the invariant spinors of 11-dimensional and type II supergravities. In section 3, we give the $\Sigma$ groups of 11-dimensional and type II supergravity. In section 4, we use the $\Sigma(\mathcal{P}_{SU(4)})$ to give the normal forms of $SU(4)$-invariant Killing spinors, and emphasize the importance of the disconnected components of the group, and in section 5 we give our conclusions. In appendices A and B, we give details of the computation of the isotropy and $\Sigma$ groups, respectively.

### 2. Isotropy groups and invariant spinors

#### 2.1. Isotropy groups

The gravitino and supersymmetry parameter of 11-dimensional supergravity are Majorana Spin(10, 1) spinors. To find the isotropy groups\(^5\) of these spinors, one begins with the results

\(^5\) These are determined up to discrete identifications of the connected to the identity component.
of [17, 21] which demonstrate that there are two kinds of orbits of Spin(10, 1) on the space of Majorana spinors, $\Delta_{32}$, with isotropy groups $SU(5)$ and Spin(7) $\times \mathbb{R}^9$. Then $\Delta_{32}$ is decomposed into irreducible representations of the above isotropy groups, and the procedure is repeated until all spinor singlets are found. This is essentially a Lie algebra computation, and a more detailed description is given in the appendices. The complete list of isotropy groups is given in table 1; for the spinor notation we use see [11, 12].

All non-compact isotropy groups are subgroups of Spin(7) $\times \mathbb{R}^9$. The same applies for the compact isotropy groups apart from $SU(2) \times SU(3)$ and $G_2$. The former is the subgroup only of $SU(5)$ while the latter is the subgroup only of Spin(7) $\times \mathbb{R}^9$. The rest of the compact isotropy groups are subgroups of both $SU(5)$ and Spin(7) $\times \mathbb{R}^9$. This easily follows from the analysis of the singlets in the appendices. Many of the groups in table 1 and the cases $H \subseteq \text{Spin}(7) \ltimes \mathbb{R}^8$ have previously appeared in [11, 17] and in [17, 18], respectively.

In the non-compact cases, the group is a semi-direct product of a compact group $K$ and $\mathbb{R}^9$. To specify the group, one has in addition to determine the representation of $K$ on $\mathbb{R}^9$. This is easily found from the results of table 3 which give explicitly the invariant spinors. In particular, one has (Spin(7) $\ltimes \mathbb{R}^8$) $\ltimes \mathbb{R}$, (SU(4) $\ltimes \mathbb{R}^8$) $\ltimes \mathbb{R}$, (SU(3) $\ltimes \mathbb{R}^9$) $\ltimes \mathbb{R}^9$, (SU(2) $\ltimes \mathbb{R}^9$) $\ltimes \mathbb{R}^9$, (SU(2) $\ltimes (\mathbb{C}^2 \times \mathbb{R}^4)$) $\ltimes \mathbb{R}$, (SU(2) $\ltimes \mathbb{R}^8$) $\ltimes (SU(2) \ltimes \mathbb{R}^4)$ $\ltimes \mathbb{R}$, (U(1) $\ltimes (\mathbb{C}^2 \times \mathbb{R}^4)$) $\ltimes \mathbb{R}$.

| $H \subseteq \text{Spin}(7) \ltimes \mathbb{R}^8$ | IIA/N$_H$ | IIB/N$_H$ | Singlets |
|---------------------------------|----------|-----------------|------------------------|
| Spin(7) $\ltimes \mathbb{R}^8$   | 1        | 1 + $e_{1234}$  |
| Spin(7)                          | 2        | $e_{1234}$, $e_5$ + $e_{1234}$  |
| $G_2 \ltimes \mathbb{R}^8$      | 2        | 1 + $e_{1234}$, $e_1$ + $e_{234}$ |
| $SU(4) \ltimes \mathbb{R}^8$    | 2        | 1 + $e_{1234}$, $e_1$ + $e_{234}$ |
| $Sp(2) \ltimes \mathbb{R}^8$    | 3        | 1, $i(e_{12} + e_{34})$ |
| SU(4)                            | 4        | 1, $e_5$ |
| $G_2$                            | 4        | 1 + $e_{1234}$, $e_5$ + $e_{1234}$, $e_1$ + $e_{234}$, $e_5$ + $e_{1234}$ |
| (SU(2))$^2 \ltimes \mathbb{R}^8$| 4        | $e_{12}$ |
| SU(3) $\ltimes \mathbb{R}^8$    | 4        | $e_{12}$, $e_{13}$ + $e_{24}$ |
| SU(2) $\ltimes \mathbb{R}^8$    | 5        | $e_{12}$, $e_{13}$ + $e_{24}$ |
| Sp(2)                            | 6        | 1, $i(e_{12} + e_{34}), e_5, i(e_{125} + e_{345})$ |
| $U(1) \ltimes \mathbb{R}^8$     | 6        | 1, $e_{12}$, $e_{13}$ |
| SU(2)$^2$                        | 8        | 1, $e_{12}$, $e_{13}$, $e_{15}$, $e_5$, $e_1$ |
| SU(3)$^2$                        | 8        | 1, $e_{15}$, $e_5$, $e_1$ |
| SU(2) $\ltimes \mathbb{R}^8$    | 8        | 1, $e_{12}$, $e_{13}$, $e_5$, $e_2$ |
| SU(2)                            | 10       | 1, $e_{12}$, $e_{13}$ + $e_{24}$, $e_5$, $e_{125}$, $e_{135}$ + $e_{245}$ |
| U(1)                             | 12       | 1, $e_{12}$, $e_{13}$, $e_5$, $e_{125}$, $e_{135}$ |
| $R^8$                            | 16       | 1, $e_{12}$, $e_{15}$, $e_{25}$, $e_1$, $e_2$, $e_5$, $e_{125}$ |
| $\{1\}$                         | 32       | $\Delta_{16}, \Delta_{16} \oplus \Delta_{16}$ |
and \( (SU(2) \ltimes \mathbb{R}^4) \times \mathbb{R}^5 \). The groups are stated in the order in which they are denoted in table 1.

2.2. Invariant spinors

2.2.1. Time-like basis. To solve the Killing spinor equations, it is useful to have an explicit basis in the space of the \( H \)-invariant spinors. The most straightforward way to give such a basis is in terms of the description of spinors in terms of forms. There are two ways to describe the Majorana spinors, in terms of forms, associated with the construction of \( \text{Clif}(\mathbb{R}^{10,1}) \) from either \( \text{Clif}(\mathbb{R}^{10}) \) or \( \text{Clif}(\mathbb{R}^{9,1}) \). These two constructions lead to different bases in \( \text{Clif}(\mathbb{R}^{10,1}) \), the ‘time-like’ and ‘null bases’, respectively. These bases have been constructed in [11, 12, 22], where one can also find the spinor conventions used in this paper. The ‘time-like’ and ‘null’ bases are suited to investigate backgrounds with \( H \subseteq SU(5) \)- and \( H \subseteq \text{Spin}(7) \ltimes \mathbb{R}^9 \)-invariant spinors, respectively. There are also several cases which can be investigated using both kinds of bases. For future use, we shall give the invariant spinors in both bases.

First, we consider the invariant spinors of \( H \subseteq SU(5) \) isotropy groups in the time-like basis [11, 12]. The results of the detailed analysis in appendix A are presented in table 2.

2.2.2. Null basis. The invariant spinors of \( H \subseteq \text{Spin}(7) \ltimes \mathbb{R}^9 \) isotropy groups in the null basis [12] are summarized in table 3. A detailed derivation of these singlets can be found in appendix A.2.

A consequence of the results summarized in tables 2 and 3 is that there are restrictions on the number of singlets that can occur. In particular, the isotropy group of more than 16 linearly independent spinors is the identity. In addition, there are no cases with \( N_H = 7, 11, 13, 14 \) and 15. Furthermore, for every non-compact isotropy group there is an associated compact one with twice the number of invariant Killing spinors.

2.3. Comparison with the isotropy groups of IIB and IIA supergravities

The relevant spinor representation of IIB supergravity is the complex chiral (Weyl) representation \( \Delta_{16}^c \) of \( \text{Spin}_c(9, 1) \). This is constructed from the associated Majorana–Weyl representation, \( \Delta_{16}^m \), associated with the heterotic string, by a straightforward complexification, \( \Delta_{16}^c = \Delta_{16}^m \otimes \mathbb{C} = \Delta_{16}^m \oplus i\Delta_{16}^m \). As a result the isotropy groups for spinors of IIB supergravity in \( \text{Spin}(9, 1) \) are precisely those found for the heterotic string. The associated invariant spinors are the complexification of those of heterotic supergravity. One consequence of this is that in IIB supergravity, there are always an even number of invariant spinors. For example, in heterotic supergravity the spinor \( 1 + e_{1234} \) is \( \text{Spin}(7) \ltimes \mathbb{R}^8 \)-invariant. The associated invariant spinors of IIB supergravity are \( 1 + e_{1234} \) and \( i(1 + e_{1234}) \).

The relevant spinor representation of IIA supergravity is the Majorana representation, \( \Delta_{32} \), of \( \text{Spin}(9, 1) \). This should be thought of as the direct sum of the positive and negative chirality Majorana–Weyl representations of \( \text{Spin}(9, 1) \), \( \Delta_{32} = \Delta_{16}^+ \oplus \Delta_{16}^- \). Since a single spinor in a Majorana–Weyl representation of \( \text{Spin}(9, 1) \) has isotropy group \( \text{Spin}(7) \ltimes \mathbb{R}^8 \), it is clear that all the isotropy groups of spinors of IIA supergravity must be subgroups of \( \text{Spin}(7) \ltimes \mathbb{R}^8 \). It is then straightforward to show that the isotropy groups are closely related to those of table 3 for 11-dimensional supergravity. If the isotropy group in table 3 is of type \( K \ltimes \mathbb{R}^9 \), then the associated isotropy group in IIA supergravity is \( K \ltimes \mathbb{R}^8 \). The additional generator is along the additional eleventh direction. Moreover, the invariant spinors are precisely as those given in
table 3 now interpreted as Spin(9, 1) spinors. This follows from the well-known fact that the Majorana representation of Spin(10, 1) decomposes under Spin(9, 1) as the sum of a positive and negative chirality Majorana–Weyl representation. The isotropy groups, up to discrete identifications of the connected to the identity component, of IIA and IIB supergravity spinors as well as the singlets in the null basis [22] are summarized in table 4.

It is clear that all the isotropy groups of IIB and IIA supergravities appear as subgroups of the isotropy groups of spinors of 11-dimensional supergravity but the converse is not true. In particular, there are isotropy groups of 11-dimensional supergravity that do not have a type II analogue. These are the SU(5) and SU(2) × SU(3) isotropy groups. Clearly, supersymmetric backgrounds with such invariant Killing spinors have a purely 11-dimensional origin.

3. The Σ(\(\mathcal{P}_H\)) groups

3.1. 11-dimensional supergravity

For generic backgrounds, the holonomy of the supercovariant connection of 11-dimensional supergravity is \(SL(32, \mathbb{R})\) [23–25]. A consequence of this, in the context of backgrounds with \(H\)-invariant Killing spinors, is that in most cases one expects that there are solutions with any number \(N\) of supersymmetries for \(N \leq N_H\), where \(N_H = \dim \Sigma(\mathcal{P}_H)\). Although this is the expectation, there are also exceptions [7, 8], and a conjecture for the fractions that can occur can be found in [26].

Assuming that the Killing spinors \(\epsilon\) of a supersymmetric background are \(H\)-invariant, \(N \leq N_H\), these can be written as a linear combination of a basis \((\eta_i), i = 1, \ldots, N_H\), in tables 2 or 3, i.e.

\[
\epsilon = \sum_{i=1}^{N_H} f_i \eta_i, 
\]

(1)

where \(f_i\) are spacetime functions. To solve the Killing spinor equation, it is convenient to bring the Killing spinors \(\epsilon\) to a normal form. To do this, one may consider the subgroup of Spin(10, 1) which leaves the plane \(\mathcal{P}_H\) of all \(H\)-invariant spinors invariant. This is clearly the remaining gauge group of the theory. Since the isotropy group \(H\) acts on \(\mathcal{P}_H\) with identity transformation, it is clear that we should consider those transformations of Spin(10, 1) which leave \(\mathcal{P}_H\) invariant up to transformations generated by \(H\). This is precisely the \(\Sigma(\mathcal{P}_H)\) group for the subspace \(\mathcal{P}_H\) using the definition of [16].

The computation of the \(\Sigma(\mathcal{P}_H)\) groups for each \(H\) can be done as in heterotic supergravity (type I) [16]. The details can be found in the appendices. The results are given in table 5.

It is clear from the results of table 5 that the \(\Sigma\) groups are a product, Spin × \(R\), where Spin and \(R\) are the Spin group and the \(R\)-symmetry group of a lower dimensional supergravity theory. This allows us to view the associated 11-dimensional supersymmetric backgrounds as being in the same universality class as those lifted from the lower dimensional supergravity theories constructed from compactification on a holonomy \(H\) manifold. However, this does not imply that the associated backgrounds have a lower dimensional origin. In two cases, those with isotropy groups \(SU(5)\) and \(SU(2) \times SU(3)\), the associated Spin group is the identity. This is because such 11-dimensional backgrounds are in the same universality class as those associated with compactifications of 11-dimensional supergravity on holonomy \(SU(5)\) and \(SU(2) \times SU(3)\) manifolds to one dimension, and the Spin group in one dimension, up to a discrete identification, is the identity.
Class. Quantum Grav. 26 (2009) 155004
U Gran et al

Table 5. The $\Sigma(\mathcal{P}_H)$ groups for each possible isotropy group of spinors in 11-dimensional supergravity. The explicit action of the generators of $\Sigma(\mathcal{P}_H)$ on $\mathcal{P}_H$ can be found in appendix B.

| $N_H$ | $H$ | $\Sigma(\mathcal{P}_H)$ |
|-------|-----|--------------------------|
| 1     | Spin(7) $\times \mathbb{R}^3$ | Spin(1, 1) |
| 2     | Spin(7) $\times \mathbb{R}^9$ | Spin(1, 1) $\times U(1)$ |
|       | SU(5) | $U(1)$ |
|       | SU(4) $\times \mathbb{R}^3$ | Spin(1, 1) $\times U(1)$ |
|       | $G_2$ $\times \mathbb{R}^9$ | Spin(1, 1) $\times U(1)$ |
| 3     | Sp(2) $\times \mathbb{R}^9$ | Spin(1, 1) $\times SU(2)$ |
| 4     | SU(4) | Spin(2, 1) $\times U(1)$ |
|       | $G_2$ | Spin(3, 1) |
|       | SU(2) $\times SU(3)$ | SU(2) $\times U(1)$ |
|       | $(SU(2))^3$ $\times \mathbb{R}^3$ | Spin(1, 1) $\times SU(2)^3$ |
|       | SU(3) $\times \mathbb{R}^9$ | Spin(1, 1) $\times SU(2)$ $\times U(1)$ |
| 5     | SU(2) $\times \mathbb{R}^9$ | Spin(1, 1) $\times Sp(2)$ |
| 6     | Sp(2) | Spin(2, 1) $\times SU(2)$ |
|       | U(1) $\times \mathbb{R}^9$ | Spin(1, 1) $\times SU(4)$ |
| 8     | SU(3) | Spin(4, 1) $\times U(1)$ |
|       | SU(2)$^2$ | Spin(2, 1) $\times SU(2)^2$ |
|       | SU(2) $\times \mathbb{R}^9$ | Spin(1, 1) $\times Sp(2)\times SU(2)$ |
| 10    | SU(2) | Spin(2, 1) $\times Sp(2)$ |
| 12    | U(1) | Spin(2, 1) $\times SU(4)$ |
| 16    | SU(2) | Spin(6, 1) $\times SU(2)$ |
|       | $\mathbb{R}^9$ | Spin(1, 1) $\times Spin(9)$ |
| 32    | $\{1\}$ | Spin(10, 1) |

It is also important to note that the $\Sigma(\mathcal{P}_H)$ groups do not capture the full expected $R$-symmetry groups of the associated lower dimensional supergravity theories. For example consider the $N_H = 8, H = SU(3)$ case in table 5. The associated lower dimensional theory is a five-dimensional supergravity with eight real supersymmetries, and the total Spin and $R$-symmetry group is expected to be Spin(4, 1) $\times SU(2)$. However, $\Sigma(\mathcal{P}_H) = Spin(4, 1) \times U(1) \subset Spin(4, 1) \times SU(2)$. This is because, unlike the $R$-symmetry groups of lower dimensional supergravities in general, the $R$-symmetry subgroups that appear in $\Sigma(\mathcal{P}_H)$ are required to be subgroups of Spin(10, 1). As a result, in some cases the $R$-symmetry group contained in $\Sigma(\mathcal{P}_H)$ is only a subgroup of the $R$-symmetry group of the lower dimensional supergravity theory.

3.2. IIA and IIB supergravities

For completeness, we also give the $\Sigma$ groups of the invariant spinors of type II supergravities. The $\Sigma$ groups of type IIB supergravity can easily be derived from those of type I supergravity. One difference is that there is an additional $U(1)$ generator because of the Spin, nature of IIB spinors. The $\Sigma$ groups of IIA supergravity can easily be derived from those of 11-dimensional supergravity. In particular, one simply excludes all the generators associated with the eleventh direction. The results are tabulated in table 6.
4. Disconnected components of $\Sigma$ groups

The Lie algebra computation that we have done, which is summarized in the appendices, identifies the component of each $\Sigma$ group that is connected to the identity. However, the $\Sigma$ groups may have disconnected components. Since these disconnected components of the $\Sigma$ groups were not essential for the solution of Killing spinor equations of heterotic supergravity, they were not stressed in that computation. This is no longer the case for type II and 11-dimensional supergravities.

4.1. Discrete subgroups

We shall not attempt to compute the disconnected components of all $\Sigma$ groups. Instead we shall focus on the case of $SU(4)$ invariant spinors. Before we proceed with this, let us establish some notation. It is known that field theories with chiral couplings may violate parity invariance. So the minimal requirement imposed on a relativistic theory is that it should be covariant under restricted Lorentz transformations, i.e. the transformations of the connected component of the Lorentz group. These transformations can also be characterized as proper and orthochronous, i.e. those that preserve both the orientation of spacetime and the direction of time. The spin group associated with the restricted Lorentz transformations

\begin{table}
\centering
\begin{tabular}{lll}
$H$ & $\Sigma(\mathcal{P}_{IIA}(N_H))$ & $\Sigma(\mathcal{P}_{IIB}(N_H))$
\hline
$Spin(7) \times \mathbb{R}^8$ & $Spin(1, 1)(1)$ & $Spin(1, 1)(2)$
$Spin(7)$ & $Spin(1, 1)(2)$ & $
$SU(4) $\times \mathbb{R}^8$ & $Spin(1, 1) \times U(1)(2)$ & $Spin(1, 1) \times U(1)(4)$
$G_2 \times \mathbb{R}^8$ & $Spin(1, 1)(2)$ & $
$Sp(2) $\times \mathbb{R}^8$ & $Spin(1, 1) \times SU(2)(3)$ & $Spin(1, 1) \times SU(2)(6)$
$SU(4)$ & $Spin(1, 1) \times U(1)(4)$ & $
$G_2 & $Spin(2, 1)(4)$ & $Spin(2, 1)(4)$
$(SU(2)^2) \times \mathbb{R}^8$ & $Spin(1, 1) \times SU(2)^2(4)$ & $Spin(1, 1) \times SU(2)^2(8)$
$SU(3) \times \mathbb{R}^8$ & $Spin(1, 1) \times U(1)^2(4)$ & $
$SU(2) $\times \mathbb{R}^8$ & $Spin(1, 1) \times Sp(2)(5)$ & $Spin(1, 1) \times Sp(2)(10)$
$Sp(2)$ & $Spin(1, 1) \times SU(2)(6)$ & $
$U(1) $\times \mathbb{R}^8$ & $Spin(1, 1) \times SU(4)(6)$ & $Spin(1, 1) \times SU(4)(12)$
$SU(3)$ & $Spin(3, 1) \times U(1)(8)$ & $Spin(3, 1) \times U(1)(8)$
SU(2)$^2$ & $Spin(1, 1) \times SU(2)^2(8)$ & $
$SU(2) $\times \mathbb{R}^8$ & $Spin(1, 1) \times Sp(4) \times SU(2)(8)$ & $
$SU(2) & $Spin(1, 1) \times Sp(2)(10)$ & $
$U(1) & $Spin(1, 1) \times SU(4)(12)$ & $
$SU(2) & $Spin(5, 1) \times SU(2)(16)$ & $Spin(5, 1) \times SU(2)(16)$
$\mathbb{R}^8$ & $Spin(1, 1) \times Spin(8)(16)$ & $Spin(1, 1) \times Spin(8)(16)$
\{1\} & $Spin(9, 1)(32)$ & $Spin(9, 1)(32)$
\end{tabular}
\caption{The $\Sigma(\mathcal{P}_H)$ groups for each possible isotropy group of spinors in IIA and IIB supergravity. The first column contains the isotropy groups of spinors of type II supergravities. The second column denotes the $\Sigma$ groups of IIA supergravity. The number in ( ) denotes the real dimension of $\mathcal{P}_H$. The third column contains the $\Sigma$ groups of IIB supergravity and — denotes the IIB cases that do not occur.}
\end{table}
is the connected component Spin\(^0\) of Spin; Spin is the double cover of the proper Lorentz transformations.

Suppose that we consider those \(\Sigma\) groups that are constructed from Spin\(^0\)(\(n, 1\)) transformations. It is expected that in such a case \(\Sigma = \text{Spin}^0(d, 1) \times R\), where \(d < n\). Although Spin\(^0\)(\(d, 1\)) is connected, we shall see that \(R\) can be disconnected. For this, let us consider the case of 11-dimensional supergravity backgrounds with SU(4)-invariant spinors. In this case, the connected component \(\Sigma^0\) of the \(\Sigma\) group, up to discrete identifications that preserve component connected to the identity, is \(\Sigma^0(\mathcal{P}_{SU(4)}) = \text{Spin}^0(2, 1) \times U(1)\); see appendix B. These are not the only transformations of Spin\(^0\)(\(10, 1\)) that preserve \(\mathcal{P}_{SU(4)}\). It is easy to see that the discrete Spin\(^0\)(\(10, 1\)) transformations,

\[
\begin{align*}
\Gamma_{1234}, &\quad \Gamma_{6234}, &\quad \Gamma_{1734}, &\quad \Gamma_{1284}, &\quad \Gamma_{1239}, &\quad \Gamma_{6734}, &\quad \Gamma_{6284}, \\
\Gamma_{6239}, &\quad \Gamma_{1784}, &\quad \Gamma_{1739}, &\quad \Gamma_{1289}, &\quad \Gamma_{1789}, &\quad \Gamma_{6289}, &\quad \Gamma_{6739}, \\
\Gamma_{6784}, &\quad \Gamma_{6789},
\end{align*}
\]

(2)

also leave \(\mathcal{P}_{SU(4)}\) invariant and are not in Spin\(^0\)(\(2, 1\) \times U(1)).

4.2. Killing spinors

To illustrate the importance of the additional transformations in \(\Sigma(\mathcal{P}_{SU(4)})\), let us investigate the orbits of Spin\(^0\)(\(2, 1\) \times U(1)) on \(\mathcal{P}_{SU(4)}\). It turns out that there are three types of orbits with representatives

\[
1 + e_{12345}, \quad e_5 + e_{1234}, \quad 1 + e_{12345} + e_5 + e_{1234}.
\]

(3)

The first two represent orbits with compact isotropy groups and the last represents an orbit with a non-compact isotropy group. This means that there are three cases that should be investigated for backgrounds with \(N = 1\) supersymmetry. This appears to be a contradiction because it is known that there are only two types of orbits of Spin\(^0\)(\(10, 1\)) on the space of Majorana spinors with isotropy groups SU(5) and Spin(7) \(\cong \mathbb{R}^9\) \cite{17, 21}. Of course, this may imply that the transformation of Spin\(^0\)(\(10, 1\)) which relates the first two orbits does not preserve \(\mathcal{P}_{SU(4)}\). However, this is not the case. Observe that the first element in (2) transforms the representative of the first orbit into the second. This in turn gives only two distinct cases with \(N = 1\) supersymmetry. This may not seem to be advantageous since for selecting the first Killing spinor the whole Spin\(^0\)(\(10, 1\)) could be used. However, it has an effect when investigating cases with \(N = 3\) supersymmetry where only the \(\Sigma(\mathcal{P}_{SU(4)})\) group can be used. So there are two cases with \(N = 3\) supersymmetry and SU(4)-invariant spinors that should be investigated instead of the three which arise from considering only the connected component of the \(\mathcal{P}_{SU(4)}\) group.

Of course, the representatives \(1 + e_{12345}\) and \(e_5 + e_{1234}\) of the orbits can also be related by the \(\Gamma_{01}, \Sigma = 10\), transformation written in the time-like basis, which is in Spin(2, 1). However \(\Gamma_{01}\) is not in either Spin\(^0\)(\(10, 1\)) or Spin\(^0\)(\(2, 1\)) because it induces non-orthochronous Lorentz transformations. So if the 11-dimensional theory is only assumed to be invariant under restricted Lorentz transformations, then the only way to relate the two compact orbits is with a discrete \(R\) transformation in (2). The analysis presented above can easily be modified if one begins with Spin(10, 1) rather than Spin\(^0\)(\(10, 1\)).

4.3. Compactifications

To investigate the role of the discrete transformations in the context of compactifications, consider an \(\mathbb{R}^{2,1} \times X\) compactification with four SU(4)-invariant Killing spinors, where
$X$ is an $SU(4)$-structure manifold. Again we assume that the 11-dimensional theory we compactify has local $Spin^0(10, 1)$ invariance. The vacuum configuration breaks this into the $\Sigma(\mathcal{P}_{SU(4)}) = Spin^0(2, 1) \times R$ group.

To separate the $Spin^0(2, 1)$ transformations from those of $R$, it suffices to see how they act on the frame of $\mathbb{R}^{2,1} \times X$. The frame of the internal space $X$ is $(e^a, e^{5\alpha})$, $\alpha = 1, 2, 3, 4$, and the remaining three directions are those of $\mathbb{R}^{2,1}$. It is easy to see that $Spin^0(2, 1)$ acts with restricted Lorentz transformations on the frame of $\mathbb{R}^{2,1}$ and leaves invariant the frame of the internal space, while the $R$ transformations leave the frame of $\mathbb{R}^{2,1}$ invariant while transforming the frame of the internal space.

To see the transformations induced by the discrete transformations (2), observe that $\Gamma_{1234}$ acts on $e^a$ as

$$e^a \rightarrow -e^a, \quad a = 1, 2, 3, 4,$$

leaving the rest of the frame directions invariant. It is straightforward to compute the action of the rest of the elements in (2). Such transformations leave the metric of the internal space invariant but act on the fluxes. Moreover they act on the spinors of the theory and change the (almost) complex structure $I$ of the internal space to $-I$.

The action on the fluxes can be identified by observing that $\Gamma_{1234}$ changes the $(p, q)$ tensors, with respect to $I$, to $(q, p)$ ones. The action on the fermions is straightforward since $\Gamma_{1234}$ is an element of the $Spin^0(10, 1)$ group. Since from the perspective of the lower dimensional theory (4) is a remnant of the restricted Lorentz group in eleven dimensions, it is natural to argue that it must remain a symmetry after compactification. It is clear that such an assertion will put restrictions on the couplings of the lower dimensional effective theory.

The action of $\Gamma_{1234}$ on the fluxes and spinors is a $\mathbb{Z}_2$ action. As such it resembles an $R$-parity transformation [20]. Though in our case the $\Sigma$ group has many disconnected components, so the representatives (2) form a larger group of reflections.

5. Concluding remarks

We have given the isotropy groups of spinors of 11-dimensional and type II supergravities as well as representatives of the singlets. Using these, we have computed the $\Sigma(\mathcal{P}_H)$ group of $\mathcal{P}_H$, where $\mathcal{P}_H$ is the plane spanned by all $H$-invariant spinors. These are the subgroups of the gauge groups of ten- and 11-dimensional supergravities which preserve $\mathcal{P}_H$. We have found that the $\Sigma(\mathcal{P}_H)$ groups are of type Spin $\times R$, i.e. they are the product of a Spin group and an $R$-symmetry group of a lower dimensional supergravity. In some cases though the $R$-symmetry group contained in $\Sigma(\mathcal{P}_H)$ is a subgroup of the $R$-symmetry group of the associated lower dimensional supergravity. This is because not all $R$-symmetry groups of lower dimensional supergravities are subgroups of either Spin($9, 1$) or Spin($10, 1$).

The solution of the Killing spinor equations of backgrounds with $H$-invariant spinors can proceed as described in [11] and [7]. In particular, if $N$ is small, then $\Sigma(\mathcal{P}_H)$ can be used to find simplified canonical forms for the Killing spinors. On the other hand, if $N$ is near $N_H$, then $\Sigma(\mathcal{P}_H)$ can be used to find the canonical form of the normals to the Killing spinors in $\mathcal{P}_H$ or its dual. In addition, we have emphasized the role of the disconnected components of the $\Sigma(\mathcal{P}_H)$ groups in choosing representatives for the Killing spinors. It is expected that in this way one can solve the Killing spinor equations of 11-dimensional and type II supergravities for all $H$-invariant Killing spinors.

As we have mentioned the disconnected components of the $\Sigma(\mathcal{P}_H)$ groups in the context of compactifications should lead to discrete symmetries in the lower dimensional supergravity effective theories. These are reminiscent of $R$-parity type of transformations; see e.g. [20].
In the $SU(4)$ case that we have investigated in some detail, the discrete group has several generators which act like reflections on the frame, on the fluxes and on the fermions of the theory. In a consistently constructed effective theory for a compactification, the invariance under such discrete $R$-symmetries may be manifest. This is because they are remnants of the Lorentz symmetry in higher dimensions. Nevertheless in the absence of a constructive method for specifying an effective theory, they may provide additional symmetry information which may lead to the suppression of some couplings.

Acknowledgments

The authors would like to thank Daniel Persson, Axel Kleinschmidt and Antoine Van Proeyen for discussions. The work of UG is funded by the Swedish Research Council.

Appendix A. Invariant spinors

A.1. The $SU(5)$ series

There are two orbits of Spin$(10, 1)$ on the Majorana representation $\Delta_{32}$ of Spin$(10, 1)$ with isotropy groups $SU(5)$ and Spin$(7) \ltimes \mathbb{R}^9$. First consider the $SU(5)$ case. To simplify notation, let us use standard notation and denote the representations by the dimension. The 32-dimensional Majorana Spin$(10, 1)$ representation decomposes under $SU(5)$ as

$$32 = 1 + \bar{1} + 5 + \bar{5} + 10 + \bar{10}.$$  \hfill (A.1)

The singlets can be arranged in the directions

$$1 + e_{12345}, \quad i(1 - e_{12345}).$$  \hfill (A.2)

To find all the singlets and isotropy groups, we shall do the computations in several steps.

**Step 1:** An additional singlet can be either in the $5$ or in $10$ representations. If it is in the $5$ representation the isotropy group is $SU(4)$. Moreover under $SU(4)$, one has that

$$32 = 1 + 1 + 5 + \bar{5} + 10 + \bar{10}.$$  \hfill (A.3)

The additional $SU(4)$-invariant spinors are

$$e_5 + e_{1234}, \quad i(e_5 - e_{1234}).$$  \hfill (A.4)

On the other hand if the additional singlet is in $10$, there are three possibilities because there are three kinds of orbits [11]. One is the generic orbit with isotropy group $SU(2)^2$, and two special orbits with stability subgroups $Sp(2)$ and $SU(2) \times SU(3)$, respectively. Under $SU(2)^2$, we have the decomposition

$$32 = +^21 + 1 + ^2\bar{1} + ^24 + 4 + ^2\bar{4} + 6 + \bar{6}.$$  \hfill (A.5)

The additional singlets to those of (A.2) are

$$e_5 + e_{1234}, \quad i(e_5 - e_{1234}), \quad e_{12} - e_{345}.$$  \hfill (A.6)

Decomposing the spinors under $Sp(2)$, we have

$$32 = +^21 + ^21 + ^2\bar{1} + ^25 + 5 + 6.$$  \hfill (A.7)

The additional singlets to those of (A.2) are

$$e_5 + e_{1234}, \quad i(e_5 - e_{1234}), \quad e_{12} + e_{345} - e_{125}, \quad i(e_{12} + e_{345} + e_{125}).$$  \hfill (A.8)
Considering the isotropy group $SU(2) \times SU(3)$, the decomposition reads

$$32 = +^1 +^1 \bar{1} + (2, 3) + (\bar{2}, \bar{3}) + 2 \otimes 3 + 2 \otimes \bar{3} + (1, 3) + (1, \bar{3}).$$ \hspace{1cm} (A.9)

The additional singlets to those of (A.2) are

$$e_{12} - e_{345}, \quad i(e_{12} + e_{345}).$$ \hspace{1cm} (A.10)

**Step 2-1:** Let us begin with (A.3). The singlets can be either in the 4 or in the 6 representation. If the additional singlets are in 4, then the isotropy group is $SU(3)$. In addition, the Majorana representation decomposes under $SU(3)$ as

$$32 = +^4 +^4 \bar{1} +^4 3 +^4 \bar{3}.$$ \hspace{1cm} (A.11)

The additional singlets to those of (A.4) can be chosen as

$$e_4 - e_{1235}, \quad i(e_4 + e_{123}), \quad e_{45} - e_{123}, \quad i(e_{45} + e_{123}).$$ \hspace{1cm} (A.12)

Now if the additional singlet is in the 6 representation, there are different isotropy groups that can occur depending on the choice of orbit. These are two different orbits with isotropy groups $SU(2)^2$ and $Sp(2)$, respectively. The decomposition according to the former is as in (A.5) while for the latter is as in (A.7).

**Step 2-2:** The additional singlet in (A.5) can be either in the $(2, 2)$ or in the $2 \otimes 2$ representations. In the former case, the isotropy group is $SU(2)$ and the decomposition is

$$32 = +^8 +^4 1 +^4 3 +^4 \bar{3}.$$ \hspace{1cm} (A.13)

This is the standard $SU(2)$ case which preserves 16 supersymmetries. The additional singlets to those of (A.2) and (A.4) are

$$e_4 - e_{1235}, \quad i(e_4 + e_{123}), \quad e_{45} - e_{123}, \quad i(e_{45} + e_{123}).$$ \hspace{1cm} (A.14)

If on the other hand the singlet is in $2 \otimes 2$, there are two orbits to consider one with isotropy group $SU(2)$ and the other with isotropy group $U(1)$. In the $SU(2)$ case, the decomposition is

$$32 \rightarrow +^5 +^5 \bar{1} +^4 3 +^4 2 +^4 2.$$ \hspace{1cm} (A.15)

The additional singlets to those of (A.2) and (A.6) are

$$e_{13} + e_{24} + e_{245} + e_{135}, \quad i(e_{13} + e_{24} - e_{245} - e_{135}).$$ \hspace{1cm} (A.16)

For the $U(1)$ orbit, the decomposition is

$$32 = +^6 +^6 \bar{1} +^{10} 2.$$ \hspace{1cm} (A.17)

The additional singlets to those of (A.2) and (A.6) are

$$e_{13} + e_{245}, \quad i(e_{13} - e_{245}), \quad e_{24} + e_{135}, \quad i(e_{24} - e_{135}).$$ \hspace{1cm} (A.18)
Step 2-3: Next consider the Sp(2) case. The additional singlets can be either in the 5 representation or in the 8 representation. In the former case, it reduces to the SU(2)\(^2\) case (A.5), and in the latter it reduces to SU(2) (A.13).

Step 2-4: The additional singlets can either be in the (2, 3) or in the 2 \(\oplus\) 3 representations. The former case reduces to either (A.11) or (A.5) cases. In the latter case, it reduces to the (A.5) case and its descendants\(^6\).

Step 3-1: Consider first the SU(3) case. The additional singlet can be in 3. In this case, it reduces to the SU(2) case and the singlets are given in (A.2), (A.4), (A.12) and (A.14).

Step 3-2: If there is an additional singlet in the decomposition (A.13), then the stability subgroup is \{1\}, i.e. all spinors are singlets.

Step 3-3: Additional singlets in the decomposition (A.15) can either be in the 2 or in 3 representations. If the singlet is in 2, then the isotropy group is \{1\} and all spinors are invariant. If the singlet is in 3, the isotropy group is U(1) and it reduces to the (A.17) case above.

The existence of any additional singlets in the decompositions described above has isotropy group \{1\}. As a result, all spinors will be invariant. The results have been summarized in table 2.

A.2. The Spin(7) \(\ltimes\mathbb{R}^9\) series

Next, let us consider the case of the Spin(7) \(\ltimes\mathbb{R}^9\) orbit. In this case, it is best to view the 32 representation of Spin(10, 1) as the sum of the Majorana–Weyl 16\(^+\) and anti-Majorana–Weyl 16\(^-\) representations of Spin(9, 1). Under Spin(7) \(\subset\) Spin(9, 1) the 16\(^+\) and 16\(^-\) decompose as

\[
16^+ = 1 \oplus 7 \oplus 8, \quad 16^- = 1 \oplus 7 \oplus 8. \quad (A.19)
\]

Using the null spinor basis, one can easily see that the isotropy group of either singlet is Spin(7) \(\ltimes\mathbb{R}^9\), while the isotropy group of both is Spin(7). To continue one can proceed as in the heterotic case, which involves the identification of singlets in 16\(^+\). Then one also decomposes the 16\(^-\) under the maximal compact subgroup \(K\) of the singlets of 16\(^-\). This will give all the singlets of \(K\). It turns out that in all cases half of the total number of singlets of \(K\) that lie either in 16\(^+\) or in 16\(^-\) have an enhanced isotropy group of type \(K \ltimes \mathbb{R}^9\). So the isotropy groups of all the spinors in 32 are of type \(K\) and \(K \ltimes \mathbb{R}^9\), where \(K\) is the maximal compact subgroup of the isotropy groups that appear in the heterotic supergravity. The results are summarized in table 3.

The isotropy groups Sp(2) (6), SU(2)\(^2\) (8), SU(3) (8), SU(2) (10), U(1) (12), SU(2) (16) can occur as subgroups of SU(5) and Spin(7) \(\ltimes\mathbb{R}^9\), where \((\cdot)\) denotes \(N_H\). However, SU(2) \(\times\) SU(3) appears as a subgroup only of SU(5), while \(G_2\) appears only as a subgroup of Spin(7) \(\ltimes\mathbb{R}^9\).

It is worth remarking that the lists above include the isotropy groups up to discrete identifications. There are more possibilities if disconnected groups are also allowed. For example, consider the subgroup \(\mathbb{Z}_2\) generated by \(\Gamma_\gamma\). Clearly \(\Gamma_\gamma \in\) Spin(10, 1). The invariant spinors are 16\(^-\). One can also consider \(\mathbb{Z}_2 \times\) \(\mathbb{R}^9\), then the invariant spinors are those in 16\(^-\) which do not contain the basis form \(e_8\). There are many more possibilities.

\(^6\) Observe that SU(2) \(\times\) SU(3) has two kinds of orbits on 2 \(\times\) 3 represented by the rank 1 and rank 2 2 \(\times\) 3 matrices. The latter case reduces to (A.5) and the former to the U(1) case.
Appendix B. $\Sigma$ groups

The computation of the $\Sigma$ groups in 11-dimensional supergravity is similar to that in the heterotic case [16]. We shall not give the details of how this Lie algebraic computation is done. Instead we shall state explicitly, the generators of the $\Sigma$ groups up to generators of the isotropy group $H$. The generators of the $\Sigma$ groups are given in the time-like basis for the $SU(5)$-series and in the null basis for the $\text{Spin}(7) \times \mathbb{R}^5$ series [12]. For notation and conventions see [11, 12].

B.1. The $SU(5)$ series

B.1.1. $SU(5), N_H = 2$. In this case we find $\Sigma(\mathcal{P}) = U(1)$, where the generator of $U(1)$ can be chosen as $i\Gamma^{11}$.

B.1.2. $SU(4), N_H = 4$. Here we get $\Sigma(\mathcal{P}) = \text{Spin}(2, 1) \times U(1)$, where $\text{spin}(2, 1)$ is generated by the real span of $\Gamma^{05}, \Gamma^{0\bar{5}}$ and $i\Gamma^{5\bar{5}}$, and the generator of $u(1)$ can be chosen as $i\Gamma^{11}$.

B.1.3. $SU(2) \times SU(3), N_H = 4$. For this case we find $\Sigma(\mathcal{P}) = SU(2) \times U(1)$, where the generator of $u(1)$ can be chosen as $i\Gamma^{33}$ and, in the basis given above, $su(2) = \mathbb{R}(\Gamma^{12}, \Gamma^{1\bar{2}}, \frac{1}{2}(\Gamma^{11} + \Gamma^{22}))$.

B.1.4. $Sp(2), N_H = 6$. In this case we get $\Sigma(\mathcal{P}) = \text{Spin}(2, 1) \times SU(2)$, where $\text{spin}(2, 1) = so(2, \mathbb{R})$ is generated by the real span of $\Gamma^{05}, \Gamma^{0\bar{5}}$ and $i\Gamma^{5\bar{5}}$ and $SU(2)$ acts on $\mathcal{P}_H$ with the three-dimensional representation. In particular, in the basis given above $su(2)$ is spanned by $\Gamma^{12} + \Gamma^{34}, \Gamma^{1\bar{2}} + \Gamma^{3\bar{4}}, \frac{1}{2}(\Gamma^{11} + \Gamma^{22} + \Gamma^{33} + \Gamma^{44})$.

B.1.5. $SU(3), N_H = 8$. Here we get $\Sigma(\mathcal{P}) = \text{Spin}(4, 1) \times U(1)$, where $\text{spin}(4, 1)$ is generated by real span of $\Gamma^i$, with $i = 0, 4, 4, 5, \bar{5}$, and the generator of $u(1)$ can be chosen as $i\Gamma^{11}$.

B.1.6. $SU(2)^2, N_H = 8$. For this case we find $\Sigma(\mathcal{P}) = \text{Spin}(2, 1) \times SU(2)^2$, where $\text{spin}(2, 1)$ is generated by the real span of $\Gamma^{05}, \Gamma^{0\bar{5}}$ and $i\Gamma^{5\bar{5}}$, and the two factors of $SU(2)$ act on $\mathcal{P}_H$ with the three-dimensional representation. In particular, in the basis given above the two factors of $SU(2)$ are generated by $su(2) = \mathbb{R}(\Gamma^{12}, \Gamma^{1\bar{2}}, \frac{1}{2}(\Gamma^{11} + \Gamma^{22}))$ and $su(2) = \mathbb{R}(\Gamma^{34}, \Gamma^{3\bar{4}}, \frac{1}{2}(\Gamma^{33} + \Gamma^{44}))$, respectively.

B.1.7. $SU(2), N_H = 10$. In this case we get $\Sigma(\mathcal{P}) = \text{Spin}(2, 1) \times Sp(2)$, where $\text{spin}(2, 1)$ is generated by the real span of $\Gamma^{05}, \Gamma^{0\bar{5}}$ and $i\Gamma^{5\bar{5}}$, and $Sp(2)$ acts on $\mathcal{P}_H$ with the five-dimensional vector representation, $Sp(2) = \text{Spin}(5)$. This can be verified by a direct computation.

B.1.8. $U(1), N_H = 12$. Here we get $\Sigma(\mathcal{P}) = \text{Spin}(2, 1) \times SU(4)$, where $\text{spin}(2, 1)$ is generated by the real span of $\Gamma^{05}, \Gamma^{0\bar{5}}$ and $i\Gamma^{5\bar{5}}$, and $SU(4)$ acts on $\mathcal{P}$ with the real six-dimensional vector representation, $SU(4) = \text{Spin}(6)$. This works exactly as in the corresponding type I case [16] and can easily be seen from previous results by a direct computation.
B.2. The Spin(7) × ℝ⁹ series

The generators of the Σ groups for this series are given in the null basis.

B.2.1. Spin(7) × ℝ⁹, \(N_H = 1\). For this case, \(\Sigma(\mathcal{P}) = \text{Spin}(1, 1)\), where \text{spin}(1, 1) is generated by \(\Gamma^\pm\).

B.2.2. Spin(7), \(N_H = 2\). For this case, \(\Sigma(\mathcal{P}) = \text{Spin}(2, 1)\), where \text{spin}(2, 1) is generated by \(\Gamma^\pm\), \(\Gamma^\mp\).

B.2.3. \(G_2 \times ℝ⁹\), \(N_H = 2\). For this case \(\Sigma(\mathcal{P}) = \text{Spin}(1, 1) \times U(1)\), where \text{spin}(1, 1) is generated by \(\Gamma^\pm\) and \(u(1)\) has generator \((\Gamma^1 + \Gamma^1)\Gamma^2\).

B.2.4. \(SU(4) \times ℝ⁹\), \(N_H = 2\). For this case \(\Sigma(\mathcal{P}) = \text{Spin}(1, 1) \times U(1)\), where \text{spin}(1, 1) is generated by \(\Gamma^\pm\) and \(u(1)\) is generated by \(i(\Gamma^{11} + \Gamma^{33} + \Gamma^{34} + \Gamma^{44})\).

B.2.5. \(Sp(2) \times ℝ⁹\), \(N_H = 3\). For this case \(\Sigma(\mathcal{P}) = \text{Spin}(1, 1) \times SU(2)\), where \text{spin}(1, 1) is generated by \(\Gamma^\pm\) and \(u(2)\) is generated by \(\tilde{\Gamma}^{12} + \Gamma^{12} + \Gamma^{34} + \Gamma^{33}, i(\Gamma^{12} - \Gamma^{12} + \Gamma^{34} - \Gamma^{33}), i(\Gamma^{11} + \Gamma^{22} + \Gamma^{33} + \Gamma^{44})\).

B.2.6. \(SU(4), \ N_H = 4\). For this case \(\Sigma(\mathcal{P}) = \text{Spin}(2, 1) \times U(1)\), where \text{spin}(2, 1) has generators \(\Gamma^\pm, \Gamma^\mp, \Gamma^\mp(\Gamma^1 + \Gamma^1), \Gamma^\mp(\Gamma^1 - \Gamma^1), \Gamma^\mp(\Gamma^1 + \Gamma^1)\).

B.2.7. \(G_2, N_H = 4\). For this case \(\Sigma(\mathcal{P}) = \text{Spin}(3, 1)\), where \text{spin}(3, 1) is generated by \(\Gamma^\pm, \Gamma^\mp, \Gamma^\mp(\Gamma^1 + \Gamma^1), \Gamma^\mp(\Gamma^1 - \Gamma^1), \Gamma^\mp(\Gamma^1 + \Gamma^1)\).

B.2.8. \((SU(2) \times SU(2)) \times ℝ⁹\), \(N_H = 4\). For this case \(\Sigma(\mathcal{P}) = \text{Spin}(1, 1) \times SU(2) \times SU(2)\), where \text{spin}(1, 1) is generated by \(\Gamma^\pm\) and the two \(su(2)\) factors are generated by \(\Gamma^{12} + \Gamma^{12}, i(\Gamma^{12} - \Gamma^{12}, i(\Gamma^{11} + \Gamma^{22})\) and \(\Gamma^{34} + \Gamma^{34}, i(\Gamma^{34} - \Gamma^{34}), i(\Gamma^{33} + \Gamma^{44})\), respectively.

B.2.9. \(SU(3) \times ℝ⁹\), \(N_H = 4\). For this case \(\Sigma(\mathcal{P}) = \text{Spin}(1, 1) \times SU(3) \times U(1)\), where \text{spin}(1, 1) is generated by \(\Gamma^\pm\), \(\text{spin}(3)\) is generated by all \(\Gamma^{AB}\) for \(A, B = 1, \bar{1}, \bar{1}\) and \(u(1)\) is generated by \(i(\Gamma^{22} + \Gamma^{33} + \Gamma^{44})\).

B.2.10. \(SU(2) \times ℝ⁹\), \(N_H = 5\). For this case \(\Sigma(\mathcal{P}) = \text{Spin}(1, 1) \times Spin(5)\), where \text{spin}(1, 1) is generated by \(\Gamma^\pm\) and \text{spin}(5) is generated by the real span of \(i(\Gamma^{11} + \Gamma^{22}), \Gamma^{12} + \Gamma^{12}, i(\Gamma^{12} - \Gamma^{12}), i(\Gamma^{11} + \Gamma^{22}), \Gamma^{34} + \Gamma^{34}, i(\Gamma^{34} - \Gamma^{34}), \Gamma^{33} + \Gamma^{33} + \Gamma^{24} + \Gamma^{24}, i(\Gamma^{33} - \Gamma^{33} + \Gamma^{24} - \Gamma^{24}), \Gamma^{14} + \Gamma^{14} + \Gamma^{32} + \Gamma^{32}, i(\Gamma^{14} - \Gamma^{14} - \Gamma^{32} + \Gamma^{32})\).

B.2.11. \(Sp(2), \ N_H = 6\) For this case \(\Sigma(\mathcal{P}) = \text{Spin}(2, 1) \times SU(2)\), where \text{spin}(2, 1) is generated by \(\Gamma^\pm, \Gamma^\mp, \Gamma^\mp\) and \text{spin}(2) is generated by \(\Gamma^{12} + \Gamma^{12} + \Gamma^{34} + \Gamma^{34}, i(\Gamma^{12} - \Gamma^{12} + \Gamma^{34} - \Gamma^{34}), i(\Gamma^{11} + \Gamma^{22} + \Gamma^{33} + \Gamma^{44})\).
2.2.12. \( U(1) \times \mathbb{R}^9, N_H = 6 \) For this case \( \Sigma(\mathcal{P}) = \text{Spin}(1, 1) \times \text{Spin}(6) \), where \( \text{spin}(1, 1) \) is generated by \( \Gamma^{\pm}, \Gamma^{-}, \Gamma^{+} \) and the two su(2) factors are generated by \( i(\Gamma^{11} + \Gamma^{22}), \Gamma^{12} + \Gamma^{12}, i(\Gamma^{12} - \Gamma^{12}), i(\Gamma^{12} + \Gamma^{12}), \Gamma^{34} + \Gamma^{34}, i(\Gamma^{34} - \Gamma^{34}), i(\Gamma^{11} + \Gamma^{11}), \Gamma^{13} + \Gamma^{13}, i(\Gamma^{13} - \Gamma^{13}), \Gamma^{24} + \Gamma^{24}, i(\Gamma^{24} - \Gamma^{24}), \Gamma^{23} - \Gamma^{23}, i(\Gamma^{23} + \Gamma^{23}), \Gamma^{41} - \Gamma^{41}, i(\Gamma^{41} + \Gamma^{41}) \).

2.2.13. \( SU(2) \times SU(2), N_H = 8 \). For this case \( \Sigma(\mathcal{P}) = \text{Spin}(2, 1) \times \text{Spin}(5) \times SU(2), \) where \( \text{spin}(2, 1) \) is generated by \( \Gamma^{\pm}, \Gamma^{-}, \Gamma^{+} \) and the two su(2) factors are generated by \( i(\Gamma^{11} + \Gamma^{22}), \Gamma^{12} + \Gamma^{12}, i(\Gamma^{12} - \Gamma^{12}) \) and \( i(\Gamma^{33} + \Gamma^{44}), \Gamma^{34} + \Gamma^{34}, i(\Gamma^{34} - \Gamma^{34}) \), respectively.

2.2.14. \( SU(3), N_H = 8 \). For this case \( \Sigma(\mathcal{P}) = \text{Spin}(4, 1) \times U(1) \), where \( \text{spin}(4, 1) \) is generated by the real span of \( \Gamma^{\pm}, \Gamma^{-}, \Gamma^{+}, (\Gamma^{11} + \Gamma^{11}), i(\Gamma^{11} + \Gamma^{11}) \) and \( i(\Gamma^{11} - \Gamma^{11}) \) and \( u(1) \) is generated by \( i\Gamma^{22} \).

2.2.15. \( SU(2) \times \mathbb{R}^9, N_H = 8 \). For this case \( \Sigma(\mathcal{P}) = \text{Spin}(1, 1) \times \text{Spin}(5) \times SU(2), \) where \( \text{spin}(1, 1) \) is generated by \( \Gamma^{\pm}, \Gamma^{-}, \Gamma^{+} \) and \( \text{spin}(5) \) is generated by \( \Gamma^{13} + \Gamma^{13} + \Gamma^{24} + \Gamma^{24}, i(\Gamma^{12} + \Gamma^{12}), i(\Gamma^{12} - \Gamma^{12}), i(\Gamma^{33} + \Gamma^{44}), \Gamma^{34} + \Gamma^{34}, i(\Gamma^{34} - \Gamma^{34}) \).

2.2.16. \( U(1), N_H = 12 \). For this case \( \Sigma(\mathcal{P}) = \text{Spin}(2, 1) \times \text{Spin}(6), \) where \( \text{spin}(2, 1) \) is generated by \( \Gamma^{\pm}, \Gamma^{-}, \Gamma^{+} \) and \( \text{spin}(6) \) is generated by \( \Gamma^{12} + \Gamma^{12}, i(\Gamma^{12} + \Gamma^{12}), i(\Gamma^{12} - \Gamma^{12}), \Gamma^{34} + \Gamma^{34}, i(\Gamma^{34} - \Gamma^{34}), \Gamma^{24} + \Gamma^{24}, i(\Gamma^{24} - \Gamma^{24}), \Gamma^{13} + \Gamma^{13}, i(\Gamma^{13} - \Gamma^{13}), \Gamma^{14} - \Gamma^{14}, i(\Gamma^{14} + \Gamma^{14}), \Gamma^{23} - \Gamma^{23}, i(\Gamma^{23} - \Gamma^{23}), \Gamma^{41} + \Gamma^{41}, i(\Gamma^{41} + \Gamma^{41}), \Gamma^{23} - \Gamma^{23}, i(\Gamma^{23} + \Gamma^{23}), \Gamma^{41} - \Gamma^{41}, i(\Gamma^{41} - \Gamma^{41}) \).

2.2.17. \( SU(2), N_H = 16 \). For this case \( \Sigma(\mathcal{P}) = \text{Spin}(6, 1) \times SU(2), \) where \( \text{su}(2) \) is generated by \( i(\Gamma^{33} + \Gamma^{44}), \Gamma^{34} + \Gamma^{34}, i(\Gamma^{34} - \Gamma^{34}) \), and \( \text{spin}(6, 1) \) is generated by the real span of \( \Gamma^{AB} \) for \( A, B = +, -, \bar{z}, 1, \bar{1}, 2, \bar{2} \).

2.2.18. \( \mathbb{R}^9, N_H = 16 \). For this case \( \Sigma(\mathcal{P}) = \text{Spin}(1, 1) \times \text{Spin}(9), \) where \( \text{spin}(1, 1) \) is generated by \( \Gamma^{\pm} \) and \( \text{spin}(9) \) is generated by the real span of \( \Gamma^{AB} \) for \( A, B = 1, 2, 3, 4, 1, 2, 3, 4, \bar{z} \).

References

[1] Grana M 2006 Flux compactifications in string theory: a comprehensive review Phys. Rep. 423 91 (arXiv:hep-th/0509003)
[2] Klebanov I R and Strassler M J 2000 Supergravity and a confining gauge theory: duality cascades and chSB-resolution of naked singularities J. High Energy Phys. JHEP08(2000)052 (arXiv:hep-th/0007191)
[3] Chamseuddine A H and Volkov M S 1997 Non-Abelian BPS monopoles in N = 4 gauged supergravity Phys. Rev. Lett. 79 3343 (arXiv:hep-th/9707116)
[4] Maldacena J M and Nunez C 2001 Towards the large N limit of pure N = 1 super Yang-Mills Phys. Rev. Lett. 86 588 (arXiv:hep-th/0008001)
[5] Papadopoulos G and Tseytlin A A 2001 Complex geometry of conifolds and 5-brane wrapped on 2-sphere Class. Quantum Grav. 18 1333 (arXiv:hep-th/0012034)
[6] Figueroa-O’Farrill J and Papadopoulos G 2003 Maximally supersymmetric solutions of ten- and 11-dimensional supergravities J. High Energy Phys. JHEP03(2003)048 (arXiv:hep-th/0211089)

Figueroa-O’Farrill J and Papadopoulos G 2003 Pluecker-type relations for orthogonal planes (arXiv:math.ag/0211170)

[7] Gran U, Gutowski J, Papadopoulos G and Roest D 2007 N = 31 is not IIB J. High Energy Phys. JHEP02(2007)044 (arXiv:hep-th/0606049)

Gran U, Gutowski J, Papadopoulos G and Roest D 2007 N = 31, D = 11 J. High Energy Phys. JHEP02(2007)043 (arXiv:hep-th/0611031)

Gran U, Gutowski J, Papadopoulos G and Roest D 2007 IIB solutions with N > 28 Killing spinors are maximally supersymmetric J. High Energy Phys. JHEP12(2007)070 (arXiv:hep-th/0710.1829)

[8] Bandos I A, de Azcarraga J A and Varela O 2006 On the absence of BPS preonic solutions in IIA and IIB supergravities J. High Energy Phys. JHEP09(2006)009 (arXiv:hep-th/0607060)

[9] Figueroa-O’Farrill J and Gadhiia S 2007 M-theory preons cannot arise by quotients J. High Energy Phys. JHEP06(2007)043 (arXiv:hep-th/0702055)

Gauntlett J P and Pakis S 2003 The geometry of D = 11 Killing spinors J. High Energy Phys. JHEP04(2003)039 (arXiv:hep-th/0212008)

Gauntlett J P, Gutowski J B and Pakis S 2003 The geometry of D = 11 null Killing spinors J. High Energy Phys. JHEP12(2003)049 (arXiv:hep-th/0311112)

[11] Gillard J, Gran U and Papadopoulos G 2005 The spinorial geometry of supersymmetric backgrounds Class. Quantum Grav. 22 1033 (arXiv:hep-th/0410155)

Gran U, Papadopoulos G and Roest D 2005 Systematics of M-theory spinorial geometry Class. Quantum Grav. 22 2701 (arXiv:hep-th/0503046)

Cariglia M and Mac Conamhna O A P 2005 Classification of supersymmetric spacetimes in eleven dimensions Phys. Rev. Lett. 94 161601 (arXiv:hep-th/0412116)

Mac Conamhna O A P 2005 Eight-manifolds with G-structure in eleven-dimensional supergravity Phys. Rev. D 72 086007 (arXiv:hep-th/0504028)

Mac Conamhna O A P 2006 The geometry of extended null supersymmetry in M-theory Phys. Rev. D 73 045012 (arXiv:hep-th/0505230)

[14] Gran U, Gutowski J and Papadopoulos G 2005 The spinorial geometry of supersymmetric IIB backgrounds Class. Quantum Grav. 22 2453 (arXiv:hep-th/0501177)

Gran U, Gutowski J and Papadopoulos G 2006 The G(2) spinorial geometry of supersymmetric IIB backgrounds Class. Quantum Grav. 23 143 (arXiv:hep-th/0505074)

[15] Gran U, Gutowski J, Papadopoulos G and Roest D 2006 Maximally supersymmetric G-backgrounds of IIB supergravity Nucl. Phys. B 753 118 (arXiv:hep-th/0604079)

Gran U, Papadopoulos G, Roest D and Sloane P 2007 Geometry of all supersymmetric type I backgrounds J. High Energy Phys. JHEP08(2007)047 (arXiv:hep-th/0703143)

Gran U, Papadopoulos G and Roest D 2007 Supersymmetric heterotic string backgrounds Phys. Lett. B 656 119 (arXiv:hep-th/0706.4407)

[17] Figueroa-O’Farrill J M 2000 Breaking the M-waves Class. Quantum Grav. 17 2925 (arXiv:hep-th/9904124)

Cariglia M and Mac Conamhna O A P 2006 Null structure groups in eleven dimensions Phys. Rev. D 73 045011 (arXiv:hep-th/0411079)

[19] Lu H, Pope C N, Szczepaniak P S 1996 Dilatonic p-brane solitons Phys. Lett. B 371 46 (arXiv:hep-th/9511203)

Barbier R et al 2005 R-parity violating supersymmetry Phys. Rep. 420 1 (arXiv:hep-ph/0406039)

Bryant R L 2000 Pseudo-Riemannian metrics with parallel spinor fields and vanishing Ricci tensor Séminaires Congrès 45 53 (arXiv:math.DG/0004073)

[21] Gran U, Lohrmann P and Papadopoulos G 2006 The spinorial geometry of supersymmetric heterotic string backgrounds J. High Energy Phys. JHEP02(2006)063 (arXiv:hep-th/0511076)

[22] Hull C 2003 Holonomy and symmetry in M-theory arXiv:hep-th/0305039

Duff M J and Liu J T 2003 Hidden spacetime symmetries and generalized holonomy in M-theory Nucl. Phys. B 674 517 (arXiv:hep-th/0303140)

Papadopoulos G and Tsamparlis D 2003 The holonomy of the supercovariant connection and Killing spinors J. High Energy Phys. JHEP07(2003)018 (arXiv:hep-th/0306117)

Duff M J 2002 M-theory on manifolds of G(2) holonomy: the first twenty years arXiv:hep-th/0201062