BAR COMPLEXES AND FORMALITY OF PULL-BACKS

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Abstract. We prove a result concerning formality of the pull-back of a fibration. Our approach is to use bar complexes in the category of commutative differential graded algebras.

1. Introduction

In this note, we show that the pull-back of a fibration by a formal map is formal. The fibration is required to be totally non-homologous to zero, and to be a formal map as well. We are here referring to the notion of formality in the setting of rational homotopy theory. This result extends a theorem of Vigué-Poirrier, [10], where it is proved that the fibre of such a fibration is a formal space. Our proof makes use of bar complexes, which, when we use a normalization due to Chen, become commutative differential graded algebras useful for rational homotopy theory. We conclude with an example which generalizes a result of Baum and Smith, [2].

2. Review of rational homotopy theory

In this section we briefly recall some notions from rational homotopy theory. References for this material are numerous and we mention [1], [3], [9].

We introduce the category of commutative differential graded algebras over a field $k$ of characteristic zero. We assume that all algebras are concentrated in non-negative degrees, have a differential which raises degree by one, and are augmented. Furthermore, we shall assume that $H^0(A) \approx k$ for all algebras $A$. We denote this category by $kCDGA$ and refer to objects in it as $kCDGA$’s. A morphism of $kCDGA$’s which induces an isomorphism on cohomology is called a quasi-isomorphism. There is a notion of homotopy between maps of $kCDGA$’s which becomes an equivalence relation on the set of maps from $A_1$ to $A_2$, when the source, $A_1$, is of a special type called a $KS$ complex, which is basically a free algebra whose differential respects an ordering on the generators.

Among the KS complexes in $kCDGA$ is an important class called minimal algebras, which are essentially characterized by being free with decomposable differential. For every $kCDGA A$, there exists a minimal $kCDGA \mathcal{M}$, and a quasi-isomorphism $\mathcal{M} \to A$.

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Such a minimal algebra is called a minimal model of $A$. It is unique up to isomorphism, and furthermore, a map of algebras $f : A_1 \to A_2$ determines a map $f : \mathcal{M}(A_1) \to \mathcal{M}(A_2)$ which is unique up to homotopy. An algebra $A$ is called formal if there are quasi-isomorphisms of $k$CDGA’s $A \leftarrow \mathcal{M}(A) \to H(A)$. This is equivalent to demanding that there be a sequence of $k$CDGA quasi-isomorphisms $A \leftarrow A_1 \leftarrow \cdots \leftarrow A_n \to H(A)$.

If $X$ is a path-connected topological space, then there is a functor which associates to $X$ a $k$CDGA $A(X)$, known as the Sullivan-de Rham algebra. For a large class of spaces, including simply-connected spaces of finite $\mathbb{Q}$-type (see definition below), the minimal model of $A(X)$ determines the rational homotopy of $X$, $\pi^*(X) \otimes \mathbb{Q}$. We say that a space $X$ is formal if $A(X)$ is formal. Then for these formal spaces, their rational homotopy is determined by their cohomology algebras.

3. Bar complexes and Eilenberg-Moore theory

In this section we shall discuss the theory of Eilenberg and Moore concerning pullbacks of fibrations. For references, see [7], [8], or [5]. For the rest of the paper, we shall assume that all spaces are path-connected and of finite $k$-type (meaning that $H^n(X; k)$ is finite-dimensional for all $n \geq 1$). We shall also assume that all fibrations are Serre fibrations.

Let us suppose that we have a fibration $F \to E \xrightarrow{p} B$ and a map $f : X \to B$, so that we obtain a pull-back diagram:

$$
\begin{array}{c}
E_f \\
\downarrow \phi \\
X \\
\downarrow f \\
B
\end{array}
\quad \quad \quad \quad \quad (1)
$$

Then the maps $f^*$ and $p^*$ make $A^*(X)$ and $A^*(E)$ (differential graded) modules over $A^*(B)$. Let us assume that $B$ is simply-connected. Then a theorem of Eilenberg and Moore asserts that there is an isomorphism

$$
\theta : Tor_{A^*(B)}(A^*(X), A^*(E)) \to H^*(E_f). \quad \quad \quad (2)
$$

We may use the bar resolution to obtain a resolution of, say, $A^*(X)$ by $A^*(B)$-modules. Since we are considering $A^*(-)$ to be the Sullivan-de Rham complex, we will be using Chen’s normalized bar resolution, see [4] or [6].

More specifically, the bar complex is

$$
B(A^*(X), A^*(B), A^*(E)) = \bigoplus_{i=0}^{\infty} A^*(X) \otimes_k (sA^*(B))^\otimes_i \otimes_k A^*(E) \quad \quad (3)
$$

where the tensor products are over the ground field $k$, and $s$ denotes the suspension functor on graded vector spaces which lowers degree by one. Hence the degree of an element $(\alpha, \omega_1, \ldots, \omega_k, \beta)$ is: $\deg(\alpha) + \sum_{i=1}^{k}(\deg(\omega_i) - 1) + \deg(\beta)$, where $\alpha \in A^*(X)$, $\omega_i \in A^*(B)$,
and $\beta \in A^\bullet(E)$. Actually, the bar complex is bigraded. We introduce the bar degree, denoted $B(A^\bullet(X), A^\bullet(B), A^\bullet(E))$. The bar degree of an element $(\alpha, \omega_1, \ldots, \omega_k, \beta)$ is defined to be $-k$. The other grading is the normal tensor product grading, the degree of an element $(\alpha, \omega_1, \ldots, \omega_k, \beta)$ being $\deg(\alpha) + \sum_{i=1}^k \deg(\omega_i) + \deg(\beta)$.

There are two differentials of total degree $+1$:

$$d(\alpha, \omega_1, \ldots, \omega_k, \beta) = (d\alpha, \omega_1, \ldots, \omega_k, \beta)$$

$$+ \sum_{i=1}^k (-1)^{i-1+1}(\alpha, \omega_1, \ldots, \omega_{i-1}, d\omega_i, \omega_{i+1}, \ldots, \omega_k, \beta)$$

$$+ (-1)^{i\bar{k}}(\alpha, \omega_1, \ldots, \omega_{k}, d\beta)$$

$$-\delta(\alpha, \omega_1, \ldots, \omega_k, n) = (-1)^{\bar{\varepsilon}_0}(\alpha \omega_1, \omega_2, \ldots, \omega_k, \beta)$$

$$+ \sum_{i=1}^{k-1} (-1)^{i}(\alpha, \omega_1, \ldots, \omega_{i-1}, \omega_i \omega_{i+1}, \omega_{i+2}, \ldots, \omega_k, \beta)$$

$$+ (-1)^{i-1+1}(\alpha, \omega_1, \ldots, \omega_{k-1}, \omega_k \beta)$$

where $\varepsilon_i = \deg(\alpha + \deg(\omega_1 + \cdots + \deg(\omega_i) - i$. The differential $\delta$ has degree $+1$ with respect to the bar grading, while the differential $d$ has degree $+1$ with respect to the tensor product grading. One may verify that $d\delta + \delta d = 0$, and we put $D \overset{\text{def}}{=} d + \delta$ to be the total differential. With the given bigrading, we get a double complex with the two differentials $d$ and $\delta$. If we filter the bar complex so that we take $d$-cohomology first in the associated spectral sequence, then we obtain the spectral sequence of Eilenberg and Moore.

Chen’s normalized version of this bar complex is the following. If $f \in A^0(B)$, let $S_i(f)$ be the operator on $B(A^\bullet(X), A^\bullet(B), A^\bullet(E))$ defined by

$$S_i(f)(\alpha, \omega_1, \ldots, \omega_k, \beta) = (\alpha, \omega_1, \ldots, \omega_{i-1}, f, \omega_i, \ldots, \omega_k, \beta)$$

for $1 \leq i \leq k+1$. Let $W$ be the subspace of $B(A^\bullet(X), A^\bullet(B), A^\bullet(E))$ generated by the images of $S_i(f)$ and $DS_i(f) - S_i(f)D$. Then define

$$\bar{B}(A^\bullet(X), A^\bullet(B), A^\bullet(E)) \overset{\text{def}}{=} B(A^\bullet(X), A^\bullet(B), A^\bullet(E))/W.$$ (7)

Then $W$ is closed under $D$ and when $H^0(B) = k$ (B is connected), then $W$ is acyclic so that $\bar{B}(A^\bullet(X), A^\bullet(B), A^\bullet(E))$ is quasi-isomorphic to $B(A^\bullet(X), A^\bullet(B), A^\bullet(E))$. Notice that in the normalized bar complex, there are no elements of negative degree, and with our assumption that $B$ is simply-connected, we are assured convergence of the associated Eilenberg-Moore spectral sequence. The map $\theta$ mentioned above is induced by the map

$$\theta : B(A^\bullet(X), A^\bullet(B), A^\bullet(E)) \to A^\bullet(E_f)$$

(8)
which sends all tensor products to zero except for $A^\bullet(X) \otimes_k A^\bullet(E)$, where the map is: 
\[ \alpha \otimes \beta \mapsto \bar{\mu}^* \alpha \wedge \bar{\mu}^* \beta. \]
Note that $\theta(W) = 0$, so that we get an induced map
\[ \theta : B(A^\bullet(X), A^\bullet(B), A^\bullet(E)) \to A^\bullet(E_f). \] (9)
The bar complex computes Tor, and the theorem of Eilenberg and Moore states that this map $\theta$ is a quasi-isomorphism.

The normalized bar complex may also be augmented. The augmentation, $\varepsilon$, maps all elements of positive total degree to zero. The elements of degree zero have the form $\theta f, g \in \epsilon_0$ so that the pull-back diagram above preserves all base-points, then on the bar complex via the shuffle product.

To the bar complex, we also obtain a commutative differential graded algebra structure which preserves the order of the $a_i < j$'s as well as the order of the $x_i$'s. That is, we demand that if $i < j$, then $\sigma(a_i) < \sigma(a_j)$ and $\sigma(b_i) < \sigma(b_j)$.

We obtain a product on $\bar{B}(A^\bullet(X), A^\bullet(B), A^\bullet(E))$ by first taking the normal tensor product on the $A^\bullet(X) \otimes A^\bullet(E)$ factors, then taking the tensor product of this product with the shuffle product on the $A^\bullet(B)^{\otimes i}$ factors. As usual, we introduce a sign $(-1)^{|\alpha||\beta|}$ whenever $\alpha$ is moved past $\beta$. The actual formula for the product is the following. Let $a, x \in A^\bullet(X)$, $b_i, y_i \in A^\bullet(B)$, and $c, z \in A^\bullet(E)$.

\[ (a, b_1, \ldots, b_k, c) \bullet (x, y_1, \ldots, y_l, z) = \sum_{\sigma} (-1)^{n + n_{\sigma}} (ax, \sigma(b_1, \ldots, b_k; y_1, \ldots, y_l), cz) \] (10)
where $n = |c||x| + |x|\{|b_1| + \ldots + |b_k| - k\} + |c|\{|y_1| + \ldots + |y_l| - l\}$, the sum is over all shuffles $\sigma$ of the set $(b_1, \ldots, b_k)$ with the set $(y_1, \ldots, y_l)$, and
\[ n_{\sigma} = \sum_{(i,j)}(|b_i| - 1)(|y_j| - 1) \]
where the sum is over all pairs $(i,j)$ such that $b_i$ is moved past $y_j$ in the shuffle $\sigma$. One may check that this product is associative, graded commutative, and that the differential $D$ is a derivation with respect to this product.

**Lemma 3.1.** The product defined above induces a product on Chen’s normalized bar complex.

**Proof.** We have that $\bar{B}(A^\bullet(X), A^\bullet(B), A^\bullet(E)) = B(A^\bullet(X), A^\bullet(B), A^\bullet(E))/W$. We will show that $W$ is an ideal. Now $W$ is generated by the images of $S_i(f)$ and $DS_i(f) - S_i(f)D$. It is obvious that $\alpha \bullet S_i(f)\beta \in W$ for any $\alpha, \beta$. Moreover, $\alpha \bullet (DS_i(f)\beta -$
\[ S_i(f)D\beta = \alpha \cdot DS_i(f)\beta - \alpha \cdot S_i(f)D\beta. \] Now \( \alpha \cdot S_i(f)D\beta \in W \) as we just noted. Moreover, since \( D \) is a derivation, we have

\[ D(\alpha \cdot S_i(f)\beta) = D\alpha \cdot S_i(f)\beta + (-1)^{[\alpha]} \alpha \cdot DS_i(f)\beta \quad (11) \]

Now, \( \alpha \cdot S_i(f)\beta \in W \), and \( W \) is closed under \( D \), so the left-hand side of (11) is in \( W \). Also, \( D\alpha \cdot S_i(f)\beta \in W \). Hence, \( \alpha \cdot DS_i(f)\beta \in W \) as well.

We have arrived at the following lemma.

**Lemma 3.2.** Assume that we have the pull-back diagram 1, where \( p \) is a fibration and \( B \) is simply connected. Then the normalized bar complex

\[ \overline{B}(A^\bullet(X), A^\bullet(B), A^\bullet(E)) \]

is a \( kCDGA \). Moreover,

\[ \theta : \overline{B}(A^\bullet(X), A^\bullet(B), A^\bullet(E)) \rightarrow A^\bullet(E_f) \]

is a quasi-isomorphism of \( kCDGA \)'s.

**Remark 3.3.** We note that Chen’s normalization is functorial. That is, if we have a commutative diagram of \( kCDGA \)’s

\[ \begin{array}{ccc}
A_2 & \leftarrow & B_2 \\
\uparrow & & \uparrow \\
A_1 & \leftarrow & B_1
\end{array} \quad \begin{array}{c}
\longrightarrow \quad \longrightarrow \\
\uparrow & & \uparrow \\
C_2 & \rightarrow & C_1
\end{array} \quad (12) \]

then we get a map of \( kCDGA \)’s \( \overline{B}(A_1, B_1, C_1) \rightarrow \overline{B}(A_2, B_2, C_2) \).

The next lemma concerns quasi-isomorphisms of bar complexes. The main idea of the proof may be found in [10], Lemme 4.3.3, and so we omit the proof here.

**Lemma 3.4.** Suppose that \( A_1 \leftarrow B_1 \rightarrow C_1 \) and \( A_2 \leftarrow B_2 \rightarrow C_2 \) are two sequences of maps of \( kCDGA \)’s with \( H^1(B_1) = 0 = H^1(B_2) \). Suppose further that \( B_1 \) is a KS-complex, and that we have a homotopy commutative diagram of the form

\[ \begin{array}{ccc}
A_2 & \leftarrow & B_2 \\
\uparrow & & \uparrow \\
A_1 & \leftarrow & B_1
\end{array} \quad \begin{array}{c}
\longrightarrow \quad \longrightarrow \\
\uparrow & & \uparrow \\
C_2 & \rightarrow & C_1
\end{array} \quad (13) \]

where the vertical arrows are all \( kCDGA \) quasi-isomorphisms. Then \( \overline{B}(A_1, B_1, C_1) \) is quasi-isomorphic to \( \overline{B}(A_2, B_2, C_2) \) in \( kCDGA \) (via a sequence of \( kCDGA \) quasi-isomorphisms).
4. Formality of pull-backs

We can use the normalized bar complex to extend a result of Vigué-Poirrier concerning formality of the fiber of a fibration, [10], Théorème 4.4.4. Our proof is also shorter and more direct than in [10], which deals with more general considerations.

**Definition 4.1.** Suppose that \( A \xleftarrow{f} B \xrightarrow{g} C \) are morphisms of \( k \)CDGA’s. Then we shall say that \( f \) and \( g \) are *compatibly formal* if there exists a homotopy commutative diagram

\[
\begin{array}{ccc}
A & \xleftarrow{f} & B \xrightarrow{g} C \\
\uparrow & & \uparrow & & \uparrow \\
\mathcal{M}(A) & \xleftarrow{f} & \mathcal{M}(B) \longrightarrow \mathcal{M}(C) \\
\downarrow & & \downarrow & & \downarrow \\
H(A) & \xleftarrow{f} & H(B) \xrightarrow{g} H(C)
\end{array}
\] (14)

where the middle row are minimal models for \( A, B, \) and \( C \), and the vertical arrows are quasi-isomorphisms. We shall say that maps, \( f, g \), of spaces \( X \xrightarrow{f} Y \xleftarrow{g} Z \) are *compatibly formal* if the the corresponding maps \( A(X) \xleftarrow{f} A(Y) \xrightarrow{g} A(Z) \) are compatibly formal.

Consider again the pull-back diagram 1 where \( p \) is a fibration with fiber \( F \), and \( B \) is simply-connected.

**Theorem 4.2.** Assume that \( p \) and \( f \) are compatibly formal maps, and suppose that the Serre spectral sequence for the fibration \( p \) degenerates at the \( E_2 \) term. Then the pull-back \( E_f \) is formal.

**Proof.** By 3.2 we have a quasi-isomorphism of \( k \)CDGA’s

\[
\theta : \bar{B}(A^\bullet(X), A^\bullet(B), A^\bullet(E)) \rightarrow A^\bullet(E_f)
\] (15)

By the assumption of compatible formality, we have a homotopy commutative diagram whose vertical arrows are quasi-isomorphisms

\[
\begin{array}{ccc}
A^\bullet(X) & \xleftarrow{f} & A^\bullet(B) \longrightarrow A^\bullet(E) \\
\uparrow & & \uparrow & & \uparrow \\
\mathcal{M}(X) & \xleftarrow{f} & \mathcal{M}(B) \longrightarrow \mathcal{M}(E) \\
\downarrow & & \downarrow & & \downarrow \\
H^\bullet(X) & \xleftarrow{f} & H^\bullet(B) \longrightarrow H^\bullet(E)
\end{array}
\] (16)
Then we obtain a sequence of $k$CDGA quasi-isomorphisms

$$\bar{B}(A^*(X), A^*(B), A^*(E)) \leftarrow \cdots \bar{B}(H^*(X), H^*(B), H^*(E))$$

by 3.4. Now the bar complex

$$\bar{B}(H^*(X), H^*(B), H^*(E))$$

has only a single differential, $\delta$. Since we have assumed the Serre spectral sequence for the fibration $p$ to degenerate at the $E_2$ term, it follows that $H^*(E)$ is a free $H^*(B)$-module. Thus if we grade according to the bar degree for $\delta$, we find that

1. $H_+(\bar{B}(H^*(X), H^*(B), H^*(E))_*) = 0$
2. $H_0(\bar{B}(H^*(X), H^*(B), H^*(E))_*) \approx H_*(\bar{B}(H^*(X), H^*(B), H^*(E))_*)$

Hence the projection to cohomology

$$\bar{B}(H^*(X), H^*(B), H^*(E))_* \rightarrow \bar{B}(H^*(X), H^*(B), H^*(E))_0$$

$$\rightarrow H_*(\bar{B}(H^*(X), H^*(B), H^*(E))_*)$$

$$\approx H^*(E_f)$$

is a $k$CDGA quasi-isomorphism and $E_f$ is consequently formal.

5. An Example

Suppose that $B$ is a simply-connected space with the property that the cohomology of $B$ is a free $k$CDGA. Then $B$ is a formal space, and the cohomology, $H^*(B)$, is a minimal model for $A^*(B)$. Let $E \overset{p}{\rightarrow} B$ be a fibration with $E$ a formal space, and let $X$ be a formal space with a map to $B$, $f : X \rightarrow B$. Then it is easy to see that $f$ and $p$ are compatibly formal.

Let $E_f$ be the pull-back of the fibration $p$ by the map $f$, as in diagram 1. Then 3.2 and 3.4 imply that $H^*(E_f)$ and $\text{Tot}_{H^*(B)}(H^*(X), H^*(E))$ are isomorphic as algebras. This extends a result of Baum and Smith, [2], where this was proven by other means for $X$ a compact globally symmetric space, and $E = BH$, $B = BG$, for $G$ a compact, connected Lie group, and $H \subset G$ a closed, connected subgroup. Moreover, if the Serre spectral sequence for the fibration $E \overset{p}{\rightarrow} B$ degenerates at the $E_2$ term, then by 4.2, we have that $E$ is a formal space.

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