Newton’s Method with Deflation for Isolated Singularities of Polynomial Systems

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Abstract
We present a modification of Newton’s method to restore quadratic convergence for isolated singular solutions of polynomial systems. Our method is symbolic-numeric: we produce a new polynomial system which has the original multiple solution as a regular root. Using standard bases, a tool for the symbolic computation of multiplicities, we show that the number of deflation stages is bounded by the multiplicity of the isolated root. Our implementation performs well on a large class of applications.

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1 Introduction

Let $F(x) = 0$ be a polynomial system of $N$ equations in $n$ unknowns $x \in \mathbb{C}^n$. We are interested in $x^∗$, an isolated solution of $F(x) = 0$:

$$\{ y \in \mathbb{C}^n : ||y - x^∗|| < \epsilon \} \cap F^{-1}(0) = \{ x^∗ \}.$$ (1)

Denote by $A(x)$ the Jacobian matrix of the system $F(x) = 0$. We call $x^∗$ a singular solution of $F(x) = 0$ $\iff$ $\text{rank}(A(x^∗)) < n$. Let $m$ be the multiplicity of the isolated solution $x^*$ of $F(x) = 0$.
Newton’s method (also called the method of Gauss-Newton when \( N > n \)) generates a sequence of approximations \( x_k \) for \( x^* \). If \( x^* \) is nonsingular, then the sequence converges quadratically (i.e.: \( \| x_k - x_{k+1} \| = O((\| x_{k-1} - x_k \|)^2) \)) to \( x^* \), which justifies its widespread usage. But otherwise, if \( x^* \) is singular, the convergence slows down and gets lost when \( x_k \approx x^* \).

The brutal force approach is to use a working precision of \( m \times D \) decimal places to achieve \( D \) correct decimal places in the final approximation. Even as multiprecision arithmetic is widely available and nowadays less expensive to use, this approach can only work if all coefficients in the system \( F \) have their first \( m \times D \) decimal places correct. Our goal is to restore the quadratic convergence of a sequence converging to an isolated singular root without imposing extra requirements of precision on \( F \).

Newton’s method for singular solutions has been extensively researched. The research up to the mid eighties is surveyed in [18]. We classify research related to our work in two domains:

1. **Detection and treatment of bifurcation points.** When following a solution path of a system defined by a parameter, the solution path may turn back or bifurcate for increasing values of the parameter. Techniques to detect and compute such bifurcation points are generally done via Liapunov-Schmidt reduction [1] [15]; see also [29, 30], [10], [23], and the references cited therein.

2. **Deflation method for polynomial systems.** A symbolic deflation method was presented in [36], and further developed in [33], [34], and [35]. We discovered this approach from the reference list of [25].

A theoretical framework to study the complexity and numerical stability of Newton’s method was developed by Shub and Smale, see [4], and was generalized to overdetermined systems in [9]. See [13, 14] for recent generalizations of this \( \alpha \)-theory to multiple roots.

The contribution of this paper is twofold. First – as announced in [40] – we provide a numerically stable implementation of a modified symbolic deflation method. Second, using standard bases [16], we show that the number of deflations needed to restore the quadratic convergence of Newton’s method is bounded by the multiplicity. In the next section we describe our method, followed by an introduction to standard bases and our proof in the third section. Our symbolic-numeric implementations and numerical results are described in sections four and five.

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2 A Modified Deflation Method

A singular root $x^*$ of a square (i.e., $N = n$) system $F(x) = 0$ satisfies

\[
\begin{align*}
F(x) &= 0 \\
\det(A(x)) &= 0.
\end{align*}
\]

The augmented system (2) forms the basic idea for deflation. If $x^*$ is isolated and $\text{corank}(A(x^*)) = 1$, then $x^*$ as root of (2) has a lower multiplicity.

We find deflation used repeatedly first in [36], and later modified in [33] and applied in [34, 35].

In theory, $\det(A(x)) = 0$ (or maximal minors) could be used to form new equations. But this is neither good symbolically because the determinant is usually of high degree and leads to expression swell, nor numerically, as the evaluation of the determinant is numerically unstable: $\|\det(A(x)) - \det(A(x^*))\| \gg \|A(x) - A(x^*)\|$. Instead of using the determinant, on a system $F$ of $N$ equations in $n$ variables, we proceed along the following three steps to form new equations:

1. Let $r = \text{rank}(A(x_0))$ for $x_0 \approx x^*$. For numerical stability, we compute the rank via a Singular Value Decomposition (SVD) of the matrix $A = A(x_0)$.

2. Let $h \in \mathbb{C}^{r+1}$ be a random vector and $B \in \mathbb{C}^{n \times (r+1)}$ be a random matrix.

   For numerical stability, we generate random numbers on the unit circle.

3. Let $C(x) = A(x)B$, notice that $C = [c_1, c_2, \ldots, c_{r+1}]$ is an $N \times (r+1)$ matrix with polynomial entries.

With probability one (exceptional pairs of vectors $h$ and matrices $B$ belong to a proper algebraic subset of $\mathbb{C}^{r+1} \times \mathbb{C}^{n \times (r+1)}$) we have

$$\text{rank}(A(x^*)) = r \iff \text{corank}(C(x^*)) = 1$$

$$\iff \exists! \lambda = \left( \begin{array}{c} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_{r+1} \end{array} \right): G(x^*, \lambda) = \left( \begin{array}{c} \sum_{i=1}^{r+1} \lambda_i c_i(x^*) \\ \sum_{i=1}^{r+1} h_i \lambda_i - 1 \end{array} \right) = 0.$$

The random $h$ and $B$ guarantee the existence and uniqueness of the solution $\lambda$ to $G(x, \lambda)$ when $x = x^*$.

In one deflation step, we add the equations of $G(x, \lambda)$ instead of $\det(A(x)) = 0$ to the system $F(x) = 0$, adding $r + 1$ extra variables $\lambda_1, \lambda_2, \ldots, \lambda_{r+1}$.

For $r = n - 1$, adding $G(x, \lambda)$ to $F(x) = 0$ is equivalent to the Liapunov-Schmidt reduction [1] used to compute bifurcation points, see e.g. [10].
Input: $F(x) = 0$ polynomial system; $x_0$ initial approximation for $x^*$.

\[ A^+ := \text{SVD}(A(x_k)); \]
\[ x_{k+1} := x_k - A^+ F(x_k); \]

Quadratic convergence?  

Yes: Output $F; x_{k+1}$. 

No: 

\[ r := \text{rank}(A^+); \]
\[ F := \begin{cases} F(x) = 0 \\ G(x, \lambda) = 0 \end{cases}; \]
\[ \lambda := \text{LeastSquares}(G(x_{k+1}, \lambda)); \]
\[ k := k + 1; \quad x_k := (x_k, \lambda); \]

Figure 1: Flowchart for a modified deflation method. In practice, the test for quadratic convergence is implemented by a test whether the numerical rank of $A(x_k)$ equals the number of columns of $A(x_k)$.

3 A Bound on the Number of Deflations

The termination of our algorithm in Figure 1 depends on the following theorem.

**Theorem 3.1** The number of deflations needed to restore the quadratic convergence of Newton’s method converging to an isolated solution is strictly less than the multiplicity of the isolated solution.

The answer to the question “How much less?” can be understood by looking at a standard basis for the ideal generated by the given polynomials in the system. We use standard bases to prove the termination of our algorithm, as explained in the next two subsections.
3.1 Standard Bases for Local Orderings

Let \( R = k[x_1, ..., x_n] \) be the ring of polynomials in \( n \) variables with coefficients in the field \( k \). We use the following multi-degree notation: \( x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} \), where \( \alpha = (\alpha_1, ..., \alpha_n) \) is a vector of nonnegative integers.

A multiplicative ordering \( \leq \) on the monoid \( \{ x^\alpha \mid \alpha \in \mathbb{Z}_{\geq 0}^n \} \) is a local ordering if \( 1 > x^\alpha \) for all \( \alpha \neq (0,0, \ldots, 0) \).

To any weight vector \( \omega \in \mathbb{R}_{\geq 0}^n \) we may associate the weight ordering \( \leq_\omega \) by setting

\[
x^\alpha \leq_\omega x^\beta \iff \langle \alpha, \omega \rangle \leq \langle \beta, \omega \rangle \text{ or } [ \langle \alpha, \omega \rangle = \langle \beta, \omega \rangle \text{ and } x^\alpha \leq_{\text{lex}} x^\beta ],
\]

where \( \langle \cdot, \cdot \rangle \) is the usual inner product and the lexicographic ordering, \( \leq_{\text{lex}} \), is used to break ties.

In presence of a monomial ordering \( \leq_\omega \), a polynomial

\[
f(x) = \sum_{\alpha \in \mathbb{Z}_{\geq 0}^n} c_\alpha x^\alpha \in R, \text{ where supp}(f) = \{ \alpha \mid c_\alpha \neq 0 \} \text{ is finite},
\]

has the following attributes associated with it:

\[
\begin{align*}
\text{le}(f) &= \text{the leading exponent} = \max_{\leq_\omega} \text{supp}(f) \\
\text{lm}(f) &= \text{the leading monomial} = x^{\text{le}(f)} \\
\text{lc}(f) &= \text{the leading coefficient} = c_{\text{le}(f)} \\
\text{lt}(f) &= \text{the leading term} = \text{lc}(f) \text{lm}(f)
\end{align*}
\]

Let \( I \subset R \) be an ideal. We call a set of polynomials \( S \subset I \) a standard basis of \( I \) if for any \( f \in I \) there is \( g \in S \) such that \( \text{lm}(g) \mid \text{lm}(f) \). Alternatively, \( S \) is a standard basis iff the initial ideal \( \text{in}(I) = \langle \{ \text{lm}(f) \mid f \in I \} \rangle \) is generated by the leading monomials \( \text{in}(S) = \{ \text{lm}(g) \mid g \in S \} \).

The monomials that do not belong to the initial ideal \( \text{in}(I) \) are called standard monomials. The minimal generators of \( \text{in}(I) \) shall be called the corners of the staircase, the corners of the form \( x_i^a \) for some \( i \) and \( a \) are called the endpoints of the staircase.

A standard basis \( S \) is called reduced if the leading monomials of its elements form a minimal generating set for the initial ideal \( \text{in}(I) \) and the tail \( g - \text{lt}(g) \) contains only standard monomials.

Graphically, any monomial ideal can be represented by a staircase in the nonnegative integer lattice \( \mathbb{Z}_{\geq 0}^n \). For example, let \( I \) be the ideal of \( R = k[x_1, x_2] \) generated by

\[
\begin{align*}
f_1 &= x_1^3 + x_1 x_2^2; \\
f_2 &= x_1 x_2^2 + x_3^3; \\
f_3 &= x_1^2 x_2 + x_1 x_2^2.
\end{align*}
\]
The initial ideal depends on the ordering chosen: the staircase at the left in Figure 2 represents $\mathfrak{in}_w(I)$, where $\omega = (-1, -2)$. The staircase at the right in Figure 2 represents $\mathfrak{in}_w(I)$, where $\omega = (-2, -1)$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{staircases.png}
\caption{Two different staircases of the standard basis of $I$ with respect to different local orderings $\leq_{(-1,-2)}$ (at the left) and $\leq_{(-2,-1)}$ (at the right). Monomials generating $\mathfrak{in}(I)$ are represented by black disks, while the standard monomials are shown as empty circles.}
\end{figure}

Observe in Figure 2 that the number of standard monomials is the same for both orderings. This is always so, for any local ordering, as the standard monomials form a basis of the $k$-linear space $R_{(x_1,...,x_n)}/R_{(x_1,...,x_n)}I$, where $R_{(x_1,...,x_n)}$ is the localization of the polynomial ring $R$ at the origin and $R_{(x_1,...,x_n)}I$ is the extension of the ideal $I$ in this localized ring (see [5] for details). This linear space is of finite dimension if the origin is an isolated solution; its dimension, which is the multiplicity of the origin, then equals the number of the standard monomials for any local ordering.

Thanks to Mora [32], there is an algorithm for computing standard bases, generalized by Greuel and Pfister [16], and implemented in the computer algebra system Singular [17]. We will not use standard bases explicitly except for theoretical purposes but note an analytic interpretation of the local ordering $\leq_w$: as we approach the origin along a smooth curve

$$c : \mathbb{C} \rightarrow \mathbb{C}^n \text{ such that } c(t) = \begin{cases} b_1 t^{-\omega_1} (1 + O(t)) \\ \vdots \\ b_n t^{-\omega_n} (1 + O(t)) \end{cases}$$

(6)

with $\omega \in \mathbb{Z}_{\geq 0}^n$ and $(b_1, ..., b_n) \in \mathbb{C}^n \setminus \{0\}$, for every $f \in I$ the leading term $\text{lt}_w(f)$ becomes dominant, i.e.:

$$f(c(t)) = \text{lt}_w(f)(c(t)) + O(t^{-\langle \omega, \text{le}(f) \rangle}).$$

(7)
3.2 Understanding the Deflation Method

First of all let us formulate the goal of what we would call the *symbolic deflation* process: Given a system of polynomial equations $f_i = 0$, $i = 1, 2, \ldots, N$ with the point $x^* \in \mathbb{C}^n$ as an isolated solution of multiplicity $m > 1$, find a system $g_i = 0$, $i = 1, 2, \ldots, N'$ such that $x^*$ is still an isolated solution of multiplicity less than $m$.

The best deflation one can cook up is the one that corresponds to the maximal ideal annihilating $x_i^*$, i.e.: $g_i = x_i - x_i^*$, $i = 1, 2, \ldots, n$. However, from a practical angle of numerical methods what we actually need is an algorithm that would relate the deflated system to the original one in a numerically stable way and taking into account the fact that the isolated solution $x^*$ may be known only approximately.

### 3.2.1 A Symbolic Deflation Method

Here we assume that everything is exact and, therefore, w.l.o.g. we may assume that the isolated solution $x^*$ is the origin.

Consider the ideal $I$ generated by the polynomials $f_i$ of the original system. We call an ideal $I'$ a *deflation* of $I$ if $I' \supset I$, $I' \neq R$, and the multiplicity of the origin for $I'$ is lower than that for the original ideal $I$.

If the multiplicity $m > 1$, it means that the initial ideal $\text{in}(I)$ does not contain $x_i$ for some $i$.

**Proposition 3.2** Suppose $m > 1$ and let $g$ be an element of a reduced standard basis of $I$ with respect to a local monomial ordering $\preceq$, such that $\text{lm}(g) = x_i^d$, for some $i \in \{1, \ldots, n\}$ and $d > 1$. Then the ideal $I' = I + \langle \partial g/\partial x_i \rangle$ is a deflation of $I$.

**Proof.** The derivative $\partial g/\partial x_i$ can not contain monomials $> x_i^{d-1}$. Therefore, $I'$ contains $I$ properly, since $\text{lm}(\partial g/\partial x_i) = x_i^{d-1}$ is a standard monomial for $I$. The appended generator $\partial g/\partial x_i$ still vanishes at the origin, hence, $I' \neq R$. □

Life is easy if one can compute a standard basis. However, this procedure is expensive symbolically and unstable numerically. Can we find $x_i$ and $g$ in the proposition in a less straightforward way? The next lemma gives a positive answer.

A linear coordinate change $T : \mathbb{C}^n \to \mathbb{C}^n$ induces an automorphism of the polynomial ring $R = \mathbb{C}[x_1, \ldots, x_n]$, which we call $T$ as well: $T(f)(x) = f(T(x))$. The ideal $T(I) = \{T(f) \mid f \in I\} = \langle T(f_1), \ldots, T(f_N) \rangle$ represents the system after the change of coordinates.

Let $A(x)$ be the Jacobian matrix of the system $F(x) = 0$, i.e.: an $N$-by-$n$ matrix with polynomial entries $A_{ij}(x) = \partial f_i/\partial x_j$. The origin is singular iff $c = \text{corank}(A(0)) > 0$. Since the Jacobian matrix is rank deficient, the kernel of $A(0)$ is nonzero.
Lemma 3.3  Take a nonzero vector $\lambda \in \ker A(0) \subset \mathbb{C}^n$ and let $T : \mathbb{C}^n \rightarrow \mathbb{C}^n$ be a linear coordinate transformation such that:

$$T_i(x) = \lambda_i x_1 + \sum_{j=2}^{n} \mu_{ij} x_j, \quad \text{for } i = 1, 2, \ldots, n, \quad (8)$$

where $[\lambda, \mu_2, \ldots, \mu_n]$ is a nonsingular matrix.

Then $\partial_1(T(I)) = \{ \frac{\partial}{\partial x_1} f \mid f \in T(I) \}$ is a deflation of $T(I)$.

Proof. For all $i = 1, 2, \ldots, N$,

$$\frac{\partial}{\partial x_1}(f_i(T(x))) = \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j}(T(x)) \frac{\partial T_j}{\partial x_1}(x) = \sum_{j=1}^{n} \left( \frac{\partial f_i}{\partial x_j}(T(x)) \right) \lambda_j. \quad (9)$$

The last expression is equal to 0 when $x = 0$, since $\lambda \in \ker A(0)$.

Take any $g = b_1 T(f_1) + \cdots + b_N T(f_N) \in T(I)$, where $b_i \in R$ for all $i$. Then

$$\frac{\partial g}{\partial x_1} = b_1 \frac{\partial (T(f_1))}{\partial x_1} + \cdots + b_N \frac{\partial (T(f_N))}{\partial x_1} \quad + \quad T(f_1) \frac{\partial b_1}{\partial x_1} + \cdots + T(f_N) \frac{\partial b_N}{\partial x_1}.$$

In view of (9), the last expression evaluates to 0 at $x = 0$. Therefore, $\partial_1(T(I))$ is a proper ideal annihilating the origin. On the other hand, there is an element $g$ of a reduced standard basis of $T(I)$ with respect to a local ordering such that $\text{in}(g) = x_1^d$ with $d > 1$. According to the Proposition 3.2 the ideal $I' = T(I) + \langle \partial g/\partial x_1 \rangle$ is a deflation of $T(I)$. So is $\partial_1(T(I))$, for it contains $I'$. \hfill $\square$

Lemma 3.3 leads to Algorithm 1.

Algorithm 1  \textbf{G = Symbolic\_Deflation}(F)

\textbf{Require:} $F$, a finite set of polynomials in $R$, such that the ideal $\langle F \rangle$ has multiplicity $m > 1$ at the origin.

\textbf{Ensure:} $G$, a finite set of polynomials in $R$, such that the ideal $\langle G \rangle + \langle F \rangle$ is a deflation of $\langle F \rangle$.

\begin{itemize}
  \item Compute the Jacobian $A$ of $F$ at the origin;
  \item Pick a nonzero vector $\lambda \in \ker A(0)$;
  \item $G := \left\{ \sum_{i=1}^{n} \lambda_i \frac{\partial f}{\partial x_i} \mid f \in F \right\}$.
\end{itemize}

\textbf{3.2.2  A Numerical Deflation Method}

This method (see Algorithm 2) does boil down to the Algorithm 1 and here is why.
Algorithm 2 $G = \text{Numeric\_Deflation}(F; x_0)$

**Require:** $F = \{f_1, \ldots, f_N\}$, a finite set of polynomials in $R$, such that the ideal $\langle F \rangle$ has multiplicity $m > 1$ at the point $x^* \approx x_0$.

**Ensure:** $G$, a finite set of polynomials in $R' = R[\lambda_1, \ldots, \lambda_{r+1}]$, where $r = \text{rank}A(x_0)$, such that
- the ideal $\langle G \rangle \subset R'$ has an isolated solution at the point $P = (x^*, \lambda^*) \in \mathbb{C}^{n,r+1}$;
- the vector $\lambda^*$ is determined uniquely;
- the multiplicity of $P$ is less than $m$.

1. Compute the Jacobian matrix $A(x)$ of $F$;
2. $r := \text{rank}A(x_0)$;
3. (R1) Generate a random matrix $B \in \mathbb{C}^{n \times (r+1)}$;
4. $C(x) := A(x)B$ ($N \times (r + 1)$ matrix with polynomial entries);
5. Let $\lambda = (\lambda_1, \ldots, \lambda_{r+1})^T$ be a vector of indeterminates;
6. Consider $N$ new polynomials $g_i(x, \lambda) = (C(x)\lambda)_i \in R'$;
7. (R2) $h(\lambda) := h_1\lambda_1 + \cdots + h_{r+1}\lambda_{r+1} - 1$, where the $h_i$ are random numbers in $\mathbb{C}$;
8. $G := F \cup \{g_1, \ldots, g_N\} \cup \{h\}$.

Consider a point $P = (x, \lambda) \in \mathbb{C}^{n,r+1}$ and let $x = x_0$. When this specialization is performed, the values for $\lambda$ are determined by the following system of $N + 1$ linear equations:

$$\begin{cases} A(x_0)\lambda = 0 \\ \langle h, \lambda \rangle = 1 \end{cases}$$

where $h = (h_1, \ldots, h_{r+1})$ is a vector of random complex numbers.

Observe how $C$ is created: the randomization step R1 insures that the $r + 1$ columns of $C(x^*)$ are random combinations of the columns of $A(x^*)$ and, therefore, corank$C(x^*) = 1$ with probability one. Then $\lambda$ is bound to live in the one-dimensional ker $C(x^*)$. The randomization step R2 makes sure one nonzero vector is picked out from the kernel. This proves the uniqueness of $\lambda^*$.

Since $C(x^*)\lambda = A(x^*)B\lambda = 0$, the vector $B\lambda$ corresponds to $\lambda$ in Algorithm 1, provided $x^*$ is the origin. Therefore, the multiplicity drops by Lemma 3.3.

## 4 A Symbolic-Numeric Implementation

The method was tested and developed in Maple 9. It is implemented in PHCpack [39], publicly available in release 2.3. While floating-point arithmetic is used, the result is symbolic, in the form of a new polynomial system with a well conditioned root.

In our implementation we computed the full SVD, although we only need to know the rank of the Jacobian matrix at the current approximation for the
root. To speed up the algorithm, we refer to the recent techniques presented in [31].

Another issue is the exploitation of the structure in the deflated systems. This issue is related to automatic differentiation [19, 37]. See also [24].

The deflation of the system \( F(x) = 0 \) with Jacobian matrix \( A(x) \) leads to a system \( F^{(1)}(x) = 0 \) whose Jacobian matrix \( A^{(1)}(x, \lambda) \) has an obvious block structure. At the right of formula (11), the matrix \( A^{(1)}(x, \lambda) \) has two columns: the first with derivatives with respect to \( x \) and the second with derivatives with respect to \( \lambda \).

\[
F^{(1)}(x, \lambda) = \begin{cases} 
F(x) = 0 \\
A(x)B\lambda = 0 \\
h\lambda = 1 
\end{cases} 
A^{(1)}(x, \lambda) = \begin{bmatrix} 
A(x) & 0 \\
(\frac{\partial}{\partial x}A(x))B\lambda & A(x)B \\
0 & h 
\end{bmatrix}
\]

where the derivative of a matrix of polynomials \( A(x) \) with respect to a vector of unknowns \( x = (x_1, x_2, \ldots, x_n) \) gives a sequence of matrices:

\[
\frac{\partial}{\partial x}A(x) = \begin{bmatrix} 
\frac{\partial A(x)}{\partial x_1} & \frac{\partial A(x)}{\partial x_2} & \cdots & \frac{\partial A(x)}{\partial x_n} 
\end{bmatrix}.
\]

(12)

To evaluate \( A^{(1)}(x, \lambda) \) efficiently, we must first evaluate \( A(x) \) before we multiply with \( B \). Otherwise, multiplying first the polynomials in \( A(x) \) with the matrix \( B \) will give rise to multiple occurrences of the same polynomials in \( A(x) \). Expanding \( A(x)B \) as polynomials in \( x \) makes matters even worse. Likewise, we first evaluate \( \frac{\partial}{\partial x}A(x) \) before multiplying with the vector \( B\lambda \).

With the exploitation of structure, the system and its Jacobian matrix can be evaluated about twice as fast, as our experiences on the cyclic 9-roots problem suggest. With 7 multipliers added to the original 9 variables, the time for 1000 evaluations of \( F^{(1)}(x, \lambda) \) at a random point drops from 1.75 to 0.94 cpu seconds\(^1\), and the time to evaluate \( A^{(1)}(x, \lambda) \) 1000 times drops from 11.0 to 5.3 cpu seconds. With 8 multipliers, the drop is even more significant: from 2.07 to 0.92 cpu seconds for 1000 evaluations of \( F^{(1)}(x, \lambda) \), and from 12.5 to 5.1 cpu seconds for 1000 evaluations of \( A^{(1)}(x, \lambda) \). When the rank deficiency is modest, the number of multipliers is close to the original number of variables and we can evaluate about twice as fast.

Setting \( \lambda^{(1)} = \lambda, B^{(1)} = B, \) and \( h^{(1)} = h \), it is straightforward to formally write down the \( k \)th deflated system:

\[
F^{(k)}(x, \lambda^{(1)}, \ldots, \lambda^{(k)}) = \begin{cases} 
F^{(k-1)}(x, \lambda^{(1)}, \ldots, \lambda^{(k-1)}) = 0 \\
A^{(k-1)}(x, \lambda^{(1)}, \ldots, \lambda^{(k-1)})B^{(k)}\lambda^{(k)} = 0 \\
h^{(k)}\lambda^{(k)} = 1
\end{cases}
\]

(13)

The benefit of exploiting the structure becomes more significant, as illustrated by the evaluation of \( F^{(2)} \), continuing the experiment on cyclic 9-roots.

\(^1\) Execution done on a 2.4GHz Linux machine with full optimization of the code.
After adding 8 multipliers (corank 2), in the first deflation, we have 17 variables. As the corank cannot decrease, a second deflation will need 16 multipliers. The time to evaluate $F^{(2)}(x, \lambda^{(1)}, \lambda^{(2)})$ for 1000 random values of $x$, $\lambda^{(1)}$, and $\lambda^{(2)}$ dropped from 45.1 to 5.66 cpu seconds.

The columns of the Jacobian matrix of $F^{(k)}$ are organized in $k + 1$ blocks. For example, for $k = 2$, $A^{(2)}(x, \lambda^{(1)}, \lambda^{(2)})$ has in its three block columns the derivatives with respect to $x$, $\lambda^{(1)}$, and $\lambda^{(2)}$ respectively:

$$
\begin{bmatrix}
    A & 0 & 0 \\
    \left(\frac{\partial A}{\partial x}\right) & B^{(1)}\lambda^{(1)} & AB^{(1)} \\
    0 & h^{(1)} & 0 \\
    \left(\frac{\partial A^{(1)}}{\partial \lambda}\right) & B^{(2)}\lambda^{(2)} & \left(\frac{\partial A^{(1)}}{\partial \lambda}\right) B^{(2)}\lambda^{(2)} \\
    0 & 0 & h^{(2)}
\end{bmatrix},
$$

(14)

where $A = A(x)$, $A^{(1)} = A^{(1)}(x, \lambda^{(1)})$, and $\frac{\partial}{\partial \lambda^{(1)}}$ means the same as in (12). Notice that the first three block rows and first two block columns of $A^{(2)}(x, \lambda^{(1)}, \lambda^{(2)})$ equal $A^{(1)}(x, \lambda^{(1)})$.

5 Applications and Numerical Results

The implementation has been tested on seventeen examples, available at http://www.math.uic.edu/~jan/demo.html. The initial approximations for Newton’s method were taken from the end points of solution paths defined by a polynomial homotopy to find all isolated solutions (see [26, 27] for a recent surveys). The numerical results reported in the table below are obtained with standard machine arithmetic.

Observe the improved numerical conditioning in Table 1. This observation justifies the naming2 of our method as a “re-conditioning” method.

One of the interesting examples is taken from [36] and listed as “ojika3” in Table 1. This system has two isolated roots: one with multiplicity two, and the other one has multiplicity four. Both roots need only one deflation, but at the double root, the rank of the Jacobian matrix is two, while the rank is one at the other 4-tuple root. The program produces two different deflated systems: one with three multipliers (for the double root) and the other with two multipliers (for the 4-tuple root).

The so-called cyclic 9-roots problem is the last and largest example in Table 1. This system is a widely used benchmark in the field of polynomial system solving, e.g.: [2, 3], [11], [12], [20], [28], with theoretical results in [21]. There are 5,594 (333 orbits of size 18) isolated regular cyclic 9-roots, in addition to the 162 isolated solutions of multiplicity four. One deflation suffices to restore quadratic convergence on all 162 quadruple roots of this large application.

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2We are grateful to Erich Kaltofen for this naming at the ISSAC’04 poster session.
Table 1: Numerical Results, $D$ is the number of deflations needed to restore quadratic convergence. The fifth column shows the decrease in the corank of the Jacobian matrix for all stages in the deflation. In the second to last column we see the change in the estimate for the inverse condition number of $A(x)$ at the start of the deflation to the end of the deflation for $x \approx x^*$. The last column lists the increase in the number of correct digits from the initial guess to the final approximation.

| System  | $n$ | $m$ | $D$ | corank($A(x)$) | Inverse Condition# | #Digits |
|---------|----|----|-----|----------------|-------------------|--------|
| eg1     | 2  | 4  | 1   | $2 \rightarrow 0$ | 8.3e-09 $\rightarrow$ 5.0e-01 | 9 $\rightarrow$ 25 |
| eg2     | 2  | 2  | 1   | $1 \rightarrow 0$ | 1.2e-08 $\rightarrow$ 1.8e-01 | 9 $\rightarrow$ 24 |
| eg3     | 2  | 2  | 1   | $1 \rightarrow 0$ | 5.6e-09 $\rightarrow$ 1.2e-01 | 9 $\rightarrow$ 25 |
| eg4     | 2  | 3  | 2   | $1 \rightarrow 1 \rightarrow 0$ | 3.0e-10 $\rightarrow$ 6.4e-02 | 5 $\rightarrow$ 15 |
| eg5     | 2  | 4  | 3   | $1 \rightarrow 1 \rightarrow 1 \rightarrow 0$ | 6.4e-11 $\rightarrow$ 5.7e-03 | 4 $\rightarrow$ 23 |
| baker[22]| 2  | 2  | 1   | $1 \rightarrow 0$ | 1.7e-08 $\rightarrow$ 3.8e-01 | 9 $\rightarrow$ 24 |
| cbms1[38]| 3  | 11 | 1   | $3 \rightarrow 0$ | 4.2e-05 $\rightarrow$ 5.0e-01 | 5 $\rightarrow$ 20 |
| cbms2[38]| 3  | 8  | 1   | $3 \rightarrow 0$ | 1.2e-08 $\rightarrow$ 5.0e-01 | 8 $\rightarrow$ 18 |
| mth191  | 3  | 4  | 1   | $2 \rightarrow 0$ | 1.3e-08 $\rightarrow$ 3.5e-02 | 7 $\rightarrow$ 13 |
| decker1[8]| 2  | 3  | 2   | $1 \rightarrow 1 \rightarrow 0$ | 3.4e-10 $\rightarrow$ 2.6e-02 | 6 $\rightarrow$ 11 |
| decker2[6]| 2  | 4  | 3   | $1 \rightarrow 1 \rightarrow 1 \rightarrow 0$ | 4.5e-13 $\rightarrow$ 6.9e-03 | 5 $\rightarrow$ 16 |
| decker3[7]| 2  | 2  | 1   | $1 \rightarrow 0$ | 4.6e-08 $\rightarrow$ 2.5e-02 | 8 $\rightarrow$ 17 |
| ojika1[33]| 2  | 3  | 2   | $1 \rightarrow 1 \rightarrow 0$ | 9.3e-12 $\rightarrow$ 4.3e-02 | 5 $\rightarrow$ 12 |
| ojika2[33]| 3  | 2  | 1   | $1 \rightarrow 0$ | 3.3e-08 $\rightarrow$ 7.4e-02 | 6 $\rightarrow$ 14 |
| ojika3[36]| 3  | 2  | 1   | $1 \rightarrow 0$ | 1.7e-08 $\rightarrow$ 9.2e-03 | 7 $\rightarrow$ 15 |
| ojika4[35]| 4  | 1  | 2   | $2 \rightarrow 0$ | 6.5e-08 $\rightarrow$ 8.0e-02 | 6 $\rightarrow$ 13 |
| cyclic9 | 9  | 4  | 1   | $2 \rightarrow 0$ | 5.6e-10 $\rightarrow$ 1.8e-03 | 5 $\rightarrow$ 15 |

6 Conclusion

Our modified deflation method works in general, is numerically stable, relatively simple to implement; and perhaps most importantly, a preliminary implementation on a wide class of examples performs quite well.

However, the doubling of the number of equations by deflation quickly leads to huge systems, we will search for ways to limit the number of deflations.

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