Weil divisors on rational normal scrolls

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1 INTRODUCTION

Let $X \subset \mathbb{P}^n$ be a rational normal scroll of degree $f$ and dimension $r = n - f + 1$. $X$
is the image of a projective bundle $\tilde{X}$ over $\mathbb{P}^1$ of rank $r - 1$ through the birational
morphism $j$ defined by the tautological line bundle $\mathcal{O}_{\tilde{X}}(1)$. Depending on $X$, the
scroll $X$ may be smooth (in this case $j$ is an isomorphism) or singular. The aim of
this note is to study Weil divisors on a singular rational normal scroll. Let $V$ be
the vertex of $X$, we have that codim($V$, $X$)$\geq 2$. When codim($V$, $X$) = 2 the scroll
$X$ is a cone over a rational normal curve of degree $f$; in this case let $E = j^{-1}(V)$
be the exceptional divisor in $\tilde{X}$.

The paper is divided into four sections. In section 2 we set up the notation
and recall some standard facts about Weil divisors on normal varieties: the strict
image map $j_\# : \text{CaCl}(\tilde{X}) \to \text{Cl}(X)$ (defined in Prop 2.1), is used to describe the
group $\text{Cl}(X)$ of Weil divisors on $X$ modulo linear equivalence. It turns out that
there are two different cases: (i) when codim($V$, $X$) > 2, then $j_\#$ is an isomorphism;
(ii) when codim($V$, $X$) = 2, then $j_\#$ is surjective and ker($j_\#$) is generated by $E$.
Since the group CaCl($\tilde{X}$) of Cartier divisors on $\tilde{X}$ modulo linear equivalence is well
known, this provides an explicit description of Cl($X$) (Cor. 2.2). Furthermore we
introduce the basic theme of the next section: the relation between the strict image,
the scheme theoretic image and the direct image of the ideal sheaf of a divisor on $\tilde{X}$
(Lemma 2.5).

The goal of section 3 is to describe the sheaves corresponding to Weil divisors
on $X$. It is known (see Prop. 3.3) that the group Cl($X$) is in bijection with the set $\text{Div}(X)$ of divisorial sheaves on $X$ (i.e. coherent sheaves which are reflexive of
rank one), and induces on it a natural group structure. We describe explicitly the
group $\text{Div}(X)$ via the direct image morphism $j_* : \text{Pic}(\tilde{X}) \to \text{Coh}(X)$ of sheaves.
In particular, we will be able to say when the direct image $j_*F$ of an invertible
sheaf $F \in \text{Pic}(\tilde{X})$ is reflexive. The analysis naturally splits in the two cases above
mentioned. When codim($V$, $X$) > 2, we will prove with standard techniques that
$j_* \text{Pic}(\tilde{X}) = \text{Div}(X)$ (Cor. 3.10). In particular (Prop. 3.9) the divisorial sheaf $\mathcal{O}_X(D)$ of a Weil divisor $D$ on $X$ is

$$\mathcal{O}_X(D) \cong j_* \mathcal{O}_{\tilde{X}}(\tilde{D}),$$

where $\tilde{D}$ is the proper transform of $D$ in $\tilde{X}$. When codim($V, X$) = 2, this is no longer true. To overcome this problem we introduce the concept of integral total transform $D^* \subset \tilde{X}$ of a Weil divisor $D \subset X$ (see Def. 3.13). We prove (Th. 3.17) that $\text{Div}(X)$ consists of the direct images of those line bundles $\mathcal{F} \in \text{Pic}(\tilde{X})$ such that the degree $\deg(\mathcal{F}|_E)$ of $\mathcal{F}$ restricted to the exceptional divisor $E$ is $< f$, where $f$ is the degree of the scroll $X$. In particular we will prove the projection formula:

$$\mathcal{O}_X(D) \cong j_* (\mathcal{O}_{\tilde{X}}(D^*))$$

In section 4 we will study the intersection of Weil divisors on $X$ in the critical case, i.e. codim($V, X$) = 2. According to [6], Prop. 2.4. an effective Weil divisor $D$ on $X \subset \mathbb{P}^n$ is itself a closed subscheme of $X$ of pure codimension 1 with no embedded components. Given two effective divisors $D$ and $D'$ on $X$ with no common components, we can consider the scheme-theoretic intersection $Y = D \cap D' \subset \mathbb{P}^n$ and ask, for example, for its degree. Note that when codim($V, X$) $> 2$ one can compute the degree of a "complete intersection" of $l$ ($1 \leq l \leq r - 1$) divisors $D_1, \ldots, D_l$ using the natural intersection form on $\text{Cl}(X)$ inherited from $\text{Cl}(\tilde{X})$, via the isomorphism $j_\#$. When codim($V, X$) = 2, the unique linear theory of intersection which can be defined in $X$ is the generalization to higher dimension of the theory developed by Mumford in [7] for a normal surface. According to this theory, the intersection number of two divisors $D, D'$ on $X$ is a rational number defined as the intersection number of the corresponding rational total transforms in $\tilde{X}$ (see Def. 3.12), which is in general a rational number and does not represent the degree of the scheme theoretic intersection $Y = D \cap D'$. In Th. 4.6 we will use the integral total transform to find the minimal reflexive resolution of $\mathcal{O}_Y$ as an $\mathcal{O}_X$-module, which allows us to compute the degree of $Y$ (Prop. 4.11).

The last section is devoted to examples and applications. In particular in Ex. 5.1 we show that every divisor of degree $\geq n$ on a rational normal cone of degree $n - 1$ in $\mathbb{P}^n$ has maximal arithmetic genus. In Ex. 5.2 we show that, when codim($V, X$) = 2, every effective divisor $D \subset X$ and every scheme theoretic intersection $Y = D \cap D' \subset X$ of two effective divisors with no common components are arithmetically Cohen-Macaulay schemes in $\mathbb{P}^n$. In Ex. 5.4 we will compute the arithmetic genus of $Y$.

Much of this material was motivated by the subject of my doctoral thesis: Classification of curves of maximal genus in $\mathbb{P}^5$, since these curves lie on (possibly singular) rational normal three-folds. In that context it is necessary to compute the degree of the scheme theoretic intersection of two divisors on a rational normal cone $X$ over a twisted cubic in $\mathbb{P}^5$, and to develop some linkage techniques on $X$. Moreover, for the linkage problem, it is necessary to know the divisorial sheaves on $X$. Thanks to my advisor Ciro Ciliberto.

2 PRELIMINARIES

In this section we recall some basic facts about rational normal scrolls and we describe the group $\text{Cl}(X)$ in terms of $\text{CaCl}(\tilde{X})$. For more details about rational normal scrolls the reader may consult for example [8] or [3].
A rational normal scroll $X \subset \mathbb{P}^n$ is the image of a projective bundle $\pi : \mathbb{P}(\mathcal{E}) \to \mathbb{P}^1$ over $\mathbb{P}^1$ of rank $r - 1$ through the morphism $j$ defined by the tautological bundle $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$, where $\mathcal{E} \cong \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(a_r)$ with $0 \leq a_1 \leq \cdots \leq a_r$ and $\sum a_i = f$. If $a_1 = \cdots = a_l = 0$, $1 \leq l < r$, $X$ is singular and the vertex $V$ of $X$ has dimension $l - 1$. Let us denote $\mathbb{P}(\mathcal{E}) = \tilde{X}$. The morphism $j : \tilde{X} \to X$ is a rational resolution of singularities, i.e. $X$ is normal and arithmetically Cohen-Macaulay and $R^1 j_* \mathcal{O}_X = 0$ for $j > 0$. We will call $j : \tilde{X} \to X$ the canonical resolution of $X$.

It is a general fact that Weil divisors on a normal scheme do not depend on closed subsets of codimension $\geq 2$. We refer to [3], Prop. II. 6.5. for this basic fact which we will continuously use in this note. In Th. 3.1 we will see the equivalent following way:

**PROPOSITION 2.1** Let $X$ be a singular rational normal scroll, let $V$ be its vertex and let $X_S$ be its smooth part. Let $j : \tilde{X} \to X$ be the canonical resolution. Then:

1. there is a surjective homomorphism $j_\#: \text{CaCl}(\tilde{X}) \to \text{Cl}(X)$ defined by $C = \sum n_i C_i \to \sum n_i j_!(C_i \cap j^{-1}X_S)$, where we ignore those $C_i \cap j^{-1}X_S$ which are empty;

2. if $\text{codim}(V, X) > 2$, then $j_\#: \text{Pic}(\tilde{X}) \to \text{Cl}(X)$ is an isomorphism;

3. if $\text{codim}(V, X) = 2$ and $E$ is the exceptional divisor of $j$, then there is an exact sequence: $0 \to \mathbb{Z} \to \text{CaCl}(\tilde{X}) \xrightarrow{j_\#} \text{Cl}(X) \to 0$ where the first map is defined by $1 \mapsto 1 \cdot E$.

Given a Cartier divisor $C$ on $\tilde{X}$, the Weil divisor $j_\#(C)$ on $X$ will be called the strict image of $C$ through $j$. It is well known that $\text{Pic}(\tilde{X}) = \mathbb{Z}[\tilde{H}] \oplus \mathbb{Z}[\tilde{R}]$, where $[\tilde{H}] = [\mathcal{O}_X(1)]$ is the hyperplane class and $[\tilde{R}] = [\pi^* \mathcal{O}_{\mathbb{P}^1}(1)]$ is the class of the fibre of the map $\pi : \tilde{X} \to \mathbb{P}^1$. Let $H = j_\# \tilde{H}$ and $R = j_\# \tilde{R}$ be the strict images of $\tilde{H}$ and $\tilde{R}$ respectively (i.e. respectively an hyperplane section and a divisor in the ruling of $X$). Then as a consequence of Prop. 2.1 we have the following:

**COROLLARY 2.2** Let $X \subset \mathbb{P}^n$ be a singular rational normal scroll of degree $f$ and let $j : \tilde{X} \to X$ be its canonical resolution. Then

1. if $\text{codim}(V, X) > 2$, $\text{Cl}(X) \cong \mathbb{Z}[H] \oplus \mathbb{Z}[R]$;

2. if $\text{codim}(V, X) = 2$, $E \sim \tilde{H} - f\tilde{R}$ and $\text{Cl}(X) \cong \mathbb{Z}[R]$.

**Proof.** 1) Follows immediately from Prop. 2.1 2). When $\text{codim}(V, X) = 2$ an hyperplane section $H$ passing trough $V$ splits in the union of $f$ fibers $R$, therefore we have $H \sim fR$, i.e. by Prop. 2.1 3) $E \sim \tilde{H} - f\tilde{R}$. □
LEMMA 2.5 Let $C$ be an effective Cartier divisor on $\tilde{X}$ and let $j_*(C)$ be its scheme-theoretic image in $X$. Then $j_*(\mathcal{I}_{C|\tilde{X}}) \cong \mathcal{I}_{j_*(C)|X}$.
Proof. Let us consider the following diagram of sheaves on $X$:

$$
0 \to \mathcal{I}_{j_*(C)}|_X \to \mathcal{O}_X \to \mathcal{O}_{j_*(C)} \to 0
$$

Since $j_*\mathcal{O}_X \cong \mathcal{O}_X$, the morphism $j^!: \mathcal{O}_X \to j_*\mathcal{O}_X$ is an isomorphism; therefore, by the Snake’s Lemma, we have to prove that the morphism $\mathcal{O}_{j_*(C)} \to j_*\mathcal{O}_C$ is injective. This follows because otherwise ker$(j^!|_{\mathcal{I}_{j_*(C)}}) \hookrightarrow \mathcal{O}_{j_*(C)}$ would define a subscheme $C'$ of $j_*(C)$ such that the morphism $\mathcal{O}_{j_*(C)} \to j_*\mathcal{O}_C$ factors in $\mathcal{O}_{j_*(C)} \to \mathcal{O}_C \to j_*\mathcal{O}_C$, but this cannot happen by universal property of $j_*(C)$. □

## 3 DIVISORIAL SHEAVES

We consider here the problem of describing the group $\text{Div}(X)$ of divisorial sheaves on a singular rational normal scroll $X$ in terms of the Picard group $\text{Pic}(\tilde{X})$ of the canonical resolution $\tilde{X}$. It is known (see Prop. 3.3 below) that $\text{Div}(X)$ is naturally isomorphic to the group $\text{Cl}(X)$ of Weil divisors modulo linear equivalence. The analysis naturally splits in two cases: $\text{codim}(V, X) > 2$ and $\text{codim}(V, X) = 2$. First we deal with the first case, which can be treated in a more or less standard way. The result (Cor. 3.10) is that the natural map $j_*: \text{Pic}(\tilde{X}) \to \text{Coh}(X)$ is in fact an isomorphism $j_*: \text{Pic}(\tilde{X}) \to \text{Div}(X)$. This is no longer true when $\text{codim}(V, X) = 2$.

To overcome this problem we will introduce the concept of integral total transform of a Weil divisor $D$ on $X$. But first let us use basic definitions and properties of divisorial sheaves on a normal scheme; for details and for a more general point of view the reader may consult [4], §2. Let $X$ be a normal scheme. We recall that a coherent sheaf $\mathcal{F}$ on $X$ is reflexive if the natural map $\mathcal{F} \to \mathcal{F}^{\vee\vee}$ is an isomorphism, where $\mathcal{F}^{\vee}$ denotes the dual sheaf $\text{Hom}(\mathcal{F}, \mathcal{O}_X)$. The following Theorem, which says that reflexive sheaves depend only on subsets of codimension 1, is a basic fact which we will use in this section.

**THEOREM 3.1** Let $X$ be a normal scheme and let $Y \subset X$ be a closed subset of codimension $\geq 2$. Then the restriction map induces an equivalence of categories from the category $\text{Ref}(X)$ of reflexive sheaves on $X$ to the category $\text{Ref}(X \setminus Y)$ of reflexive sheaves on $X \setminus Y$.

**Proof.** [4], Th. 1.12. We recall from this proof the way to extend a reflexive sheaf $\mathcal{G}$ on $X \setminus Y$ to a reflexive sheaf $\mathcal{F}$ on $X$. It consists in taking a coherent extension $\mathcal{F}_0$ of $\mathcal{G}$ in $X$ (which exists by a general result on extensions of coherent sheaves), and then to put $\mathcal{F} = \mathcal{F}_0^{\vee\vee}$. The double dual of any coherent sheaf is in fact always reflexive. □

**DEFINITION 3.2** Let $X$ be a normal scheme. Let $D$ be a Weil divisor on $X$. If $K(X)$ denotes the function field of $X$ ([4], pg. 91 and pg. 141), then the sheaf $\mathcal{O}_X(D)$ defined for every open set $U \subset X$ as

$$
\Gamma(U, \mathcal{O}_X(D)) = \{f \in K(X) | \text{div } f + D \geq 0 \text{ on } U\},
$$

is called the divisorial sheaf of $X$. 


The following Proposition (see [3], Prop. 2.8.) describes the equivalence between reflexive sheaves and divisorial sheaves. In point 4) is defined the group structure on $\text{Div}(X)$ induced by $\text{Cl}(X)$.

**Proposition 3.3** Let $X$ be a normal scheme.

1. For any Weil divisor $D$ the sheaf $\mathcal{O}_X(D)$ is reflexive and locally free of rank one at every generic point and at every point of codimension 1.

2. Conversely, every reflexive sheaf which is locally free of rank one at every generic point and at every point of codimension 1 is isomorphic to $\mathcal{O}_X(D)$ for some Weil divisor $D$.

3. If $D_1$ and $D_2$ are Weil divisors on $X$, $D_1 \sim D_2$ if and only if $\mathcal{O}_X(D_1) \cong \mathcal{O}_X(D_2)$ as $\mathcal{O}_X$-modules.

4. If $D$, $D_1$, $D_2$ are Weil divisors on $X$, $\mathcal{O}_X(-D) \cong \mathcal{O}_X(D)^\vee$ and $\mathcal{O}_X(D_1 + D_2) = (\mathcal{O}_X(D_1) \otimes \mathcal{O}_X(D_2))^{\vee \vee}$.

We come back now to the case of a singular rational normal scroll $X$. Our goal is to explicitly describe $\text{Div}(X)$ with its group structure and the idea is to compare $\text{Div}(X)$ with $\text{Pic}(\tilde{X})$, via the direct image map $j_*$ of sheaves. There is a natural surjective map, induced by the strict image map $j_* : \text{Pic}(\tilde{X}) \to \text{Cl}(X)$, which we call again $j_*$:

$$j_* : \text{Pic}(\tilde{X}) \to \text{Div}(X)$$

$$\mathcal{O}_{\tilde{X}}(C) \to j_*(\mathcal{O}_X(C))^{\vee \vee}$$

which is bijective when $\text{codim}(V, X) > 2$. The sheaf $j_*(\mathcal{O}_{\tilde{X}}(C))^{\vee \vee}$ must be the divisorial sheaf $\mathcal{O}_X(j_*(C))$ by Th. [3], since it is reflexive and isomorphic to $\mathcal{O}_X(j_*(C))$ outside a subset of codimension $\geq 2$ (in the open set $X_S$). We are then going to compare $j_* : \text{Pic}(\tilde{X}) \to \text{Div}(X)$ with $j_* : \text{Pic}(\tilde{X}) \to \text{Coh}(X)$ or, in other words, we are going to check when $j_* (\mathcal{O}_X(C))$ is a reflexive sheaf. Of course we will have, according to Prop. [2], two separate cases: a) $\text{codim}(V, X) > 2$ and b) $\text{codim}(V, X) = 2$.

First we briefly describe $j_*(\text{Pic}(\tilde{X}))$. We refer for details to [10] or [4]. Let us denote:

$$\mathcal{O}_{\tilde{X}}(a, b) := \mathcal{O}_{\tilde{X}}(a\tilde{H} + b\tilde{R})$$

the invertible sheaf associated to the class $[a\tilde{H} + b\tilde{R}]$ in $\text{Pic}(\tilde{X})$ and let us consider on $X$ their direct images:

$$\mathcal{O}_X(a, b) := j_* \mathcal{O}_{\tilde{X}}(a, b),$$

with $a, b \in \mathbb{Z}$. The cohomology of $\mathcal{O}_X(a, b)$ can be explicitly calculated using the Leray spectral sequence, which, since $R^i \pi_* \mathcal{O}_X(a, b) = 0$ for every $a, b \in \mathbb{Z}$ and $0 < i < r - 1$ (by Grauert’s Theorem), simplifies as follows:

$$\cdots \to H^i(\pi_* \mathcal{O}_X(a, b)) \to H^i(\mathcal{O}_X(a, b)) \to H^{i-r+1}(R^{r-1} \pi_* \mathcal{O}_X(a, b)) \to \cdots$$

$$\cdots \to H^{i+1}(\pi_* \mathcal{O}_X(a, b)) \to \cdots$$
Therefore for \( i < r - 1 \) and \( a \geq 0 \) we obtain:
\[
H^i(\tilde{X}, \mathcal{O}_{\tilde{X}}(a,b)) \cong H^i(\mathbb{P}^1, \pi_* \mathcal{O}_{\tilde{X}}(a,b)) \cong \sum_{|I|=a} H^i(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(b + \sum_{j \in I} a_j))
\] (3.4)
which are zero of course if \( a < 0 \) and for \( 1 < i < r - 1 \); while for \( j = r - 1, r \) we obtain:
\[
H^j(\tilde{X}, \mathcal{O}_{\tilde{X}}(a,b)) \cong H^{j-r+1}(\mathbb{P}^1, R^{r-1} \pi_* \mathcal{O}_{\tilde{X}}(a,b)),
\]
which can be computed also by Serre duality using (3.4). For \( a \geq 0 \) and \( b \geq -1 \) using (3.4) we compute:
\[
h^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(a,b)) = f \left( \frac{a + r - 1}{r} \right) + (b + 1) \left( \frac{a + r - 1}{r} - 1 \right).
\] (3.5)
For \( b < -1 \), the dimension \( h^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(a,b)) \) depends on the type of the scroll, i.e. on the integers \( a_1, \ldots, a_r \). We recall from [10] that we have the vanishing
\[
R^i j_* \mathcal{O}_{\tilde{X}}(a,b) = 0
\] (3.6)
for \( i > 0 \) and for all \( a \in \mathbb{Z} \) and \( b \geq -1 \), which implies, via the degenerate Leray spectral sequence associated to \( j \):
\[
h^i(O_X(a,b)) = h^i(O_{\tilde{X}}(a,b))
\] (3.7)
for \( i \geq 0 \). Moreover in [10] it is proved that the dualizing sheaf \( \omega_X \) of \( X \) is:
\[
\omega_X = j_* \mathcal{O}_{\tilde{X}}(K_{\tilde{X}}) = \mathcal{O}_X(-r, f - 2).
\] (3.8)
As we will see (Cor. 3.10 and Th. 3.17), \( \omega_X \) is always a divisorial sheaf, so it makes sense to talk about the canonical divisor \( K_X \sim -r H + (f - 2) R \) on \( X \).

Let us consider the case when \( \text{codim}(V, X) > 2 \).

**PROPOSITION 3.9** Let \( \text{codim}(V, X) > 2 \), let \( D \sim a H + b R \) be a Weil divisor on \( X \) and let \( \tilde{D} \sim a H + b \tilde{R} \) be its proper transform on \( \tilde{X} \). Then
\[
j_* \mathcal{O}_{\tilde{X}}(\tilde{D}) = \mathcal{O}_X(D) = \mathcal{O}_X(a, b).
\]

**Proof.** It is sufficient to consider the local situation. Let \( U \) be an open set containing the vertex \( V \) and let \( U' = j^{-1}(U) \). Call \( V' = j^{-1}(V) \). Since \( \text{codim}(V', \tilde{X}) \geq 2 \), then \( H^0(U', \mathcal{O}_{\tilde{X}}(\tilde{D})) \cong H^0(U' \setminus V', \mathcal{O}_{\tilde{X}}(D)) \); moreover \( H^0(U' \setminus V', \mathcal{O}_{\tilde{X}}(\tilde{D})) \cong H^0(U \setminus V, \mathcal{O}_X(D)) \cong H^0(U, \mathcal{O}_X(D)) \) where the last isomorphism follows from Th. 3.3. \( \square \)

**COROLLARY 3.10** Let \( \text{codim}(V, X) > 2 \), then \( \mathcal{O}_X(a, b) \) is a reflexive sheaf for every \( a, b \in \mathbb{Z} \) and
\[
\text{Div}(X) = \{ \mathcal{O}_X(a, b) \mid a, b \in \mathbb{Z} \}.
\]
Moreover the natural group structure on \( \text{Div}(X) \) inherited from \( \text{Cl}(X) \) is given by
\[
< \mathcal{O}_X(a, b), \mathcal{O}_X(a', b') > \mapsto \mathcal{O}_X((a + a'), (b + b')) \text{ and } \mathcal{O}_X(a, b)^\vee \cong \mathcal{O}_X(-a, -b).
Proof. The first assertions follows directly from Prop. 3.3. The description of the group structure on \( \text{Div}(X) \) follows from Prop. 3.9 and part 4) of Prop. 3.3. \qed

**REMARK 3.11** Note that \( O_X((a + a'), (b + b')) \cong (O_X(a, b) \otimes O_X(a', b'))^{\vee \vee} \), by part 4) of Prop. 3.3. In general \( O_X(a, b) \otimes O_X(a', b') \) is not reflexive. For example \( O_X(0, 1) \otimes O_X(0, -1) \) is not reflexive. Otherwise we would have \( O_X(0, 1) \otimes O_X(0, -1) \cong (O_X(0, 1) \otimes O_X(0, -1))^{\vee \vee} \cong O_X \). i.e. \( O_X(0, 1) \) would be invertible, which is a contradiction since \( R \) is not Cartier (see Remark 2.3).

When the codimension of \( V \) is 2 we know (Remarks 2.4 3)) that the map \( j\# : \text{Pic}(\tilde{X}) \to \text{Cl}(X) \) is not injective and therefore also the map \( j\# : \text{Pic}(\tilde{X}) \to \text{Div}(X) \) is not injective. It turns out that the study of \( \text{Div}(X) \) becomes very simple if we introduce, for any Weil divisor \( D \) on \( X \), a Cartier divisor \( D^* \) on \( \tilde{X} \), which we will call the integral total transform of \( D \), which plays, in some sense, the role of the proper transform in case \( \text{codim}(V, X) \geq 2 \).

The problem of defining, for a Weil divisor \( D \) trough \( V \) on \( X \), the total transform \( \tilde{X} \) has been considered and solved by Mumford in 6) on a normal surface, with the goal of developing the bilinear intersection theory on normal surfaces (see also 8)). Mumford’s theory can be generalized on rational normal cones by defining the total transform \( j^*D \) of a divisor \( D \) on \( X \) as:

**DEFINITION 3.12** Let \( \text{codim}(V, X) = 2 \) and let \( D \) be a Weil divisor on \( X \). Then the (rational) total transform of \( D \) in \( \tilde{X} \) is:

\[
j^*D = \tilde{D} + qE,
\]

where \( E \) is the exceptional divisor on \( \tilde{X} \) and \( q \) is a rational number uniquely determined by the equation: \( (\tilde{D} + qE) \cdot E \cdot \tilde{H}^{r-2} = 0 \).

If \( \tilde{D} \sim a\tilde{H} + b\tilde{R} \), then we find that \( q = \frac{a}{2} \). If \( D \) is effective, since \( \tilde{D} \) does not contain \( E \), we have that \( b = \tilde{D} \cdot E \cdot \tilde{H}^{r-2} \geq 0 \), i.e. \( q \geq 0 \). Mumford’s total transform \( j^*D \) is in general a \( \mathbb{Q}\)-divisor and it is integral if and only if \( D \) is Cartier. As we will see, it is more convenient for our purposes to consider, if \( D \) is effective, the round-up \([j^*D]\) of \( j^*D \), i.e. the smallest integral divisor on \( \tilde{X} \) containing \( j^*D \). So let us give the following definition:

**DEFINITION 3.13** Let \( \text{codim}(V, X) = 2 \). Let \( D \) be an effective Weil divisor on \( X \), we define the integral total transform \( D^* \) of \( D \) as:

\[
D^* = [j^*D] = \tilde{D} + [q]E
\]

where \( \tilde{D} \sim a\tilde{H} + b\tilde{R} \) is the proper transform of \( D \), \( q = \frac{b}{2} \) is the same number appearing in Def. 3.12 and \([q]\) is the round-up of \( q \), i.e. the smallest integer \( \geq q \). We define the total transform of \(-D\) as \((-D)^* = -D^* \).

**NOTE 3.14** It is a simple computation to get the following equivalent expression of \( D^* \). Let \( D \sim dR \) be effective, i.e. \( d \geq 0 \), and divide \( d - 1 = kf + h \) (\( k \geq -1 \) and \( 0 \leq h < f \)). Then:

\[
D^* \sim (k + 1)\tilde{H} - (f - h - 1)\tilde{R}. \tag{3.15}
\]
The relations between these coefficients and the ones in Def. 3.13 are \( k + 1 = a + [q] \) and \( f - h - 1 = f'[q] - b \). From formula (3.15) it is clear that \( D^* \) is uniquely determined by the class of linear equivalence of \( D \sim dR \), i.e. by the degree \( d \).

**PROPOSITION 3.16** Let \( \operatorname{codim}(V, X) = 2 \). Let \( D \) be an effective Weil divisor on \( X \). Then the integral total transform \( D^* = D + [q]E \) is the biggest Cartier divisor \( C \) on \( X \) such that \( j_*(C) = j_#(C) = D \). More precisely:

\[
j_*(\tilde{D} + \alpha E) = j_#(\tilde{D} + \beta E) = D
\]

if and only if \( 0 \leq \alpha \leq [q] \); in this case \( j_!\mathcal{I}_{D + \alpha E} = \mathcal{I}_D \).

Proof. In Remarks 2.4 we have seen that \( j_#(\tilde{D}) = j_!(\tilde{D}) = D \) and \( j_#(\tilde{D} + mE) = D \) for every \( m \in \mathbb{Z} \). Therefore it is enough to prove that \( j_!(D^*) = D \) and that \( D \) is a proper subscheme of \( j_!(D^* + mE) \) for every integer \( m > 0 \). Proving the first equality is equivalent to prove that \( j_!(D^*) \) does not have embedded components. Let us fix a divisor \( D' \sim (f - h - 1)R \) on \( X \); then \( F = D + D' \sim (k + 1)H \) is a Cartier divisor on \( X \) cut out by a hypersurface \( F \) of degree \( k + 1 \). Let us consider on \( \tilde{X} \) the divisor \( j^*F \sim (k + 1)\tilde{H} \). We have that \( D^* + \tilde{D}' = j^*F \), in fact they are linearly equivalent and coincide outside \( E \). Therefore the scheme-theoretic union \( \mathcal{F} \) of the scheme-theoretic images \( j_*\mathcal{D}' \) and \( j_*\mathcal{D}^* \) is contained in \( j_!(j^*F) = F \), which is equal to the scheme theoretic union of the strict images \( j_!\mathcal{D}' = D' \) and \( j_!\mathcal{D}^* = D \). Therefore we must have \( \mathcal{F} = F \). This implies that \( \mathcal{F} \) cannot have embedded components, since it is cut out by a hypersurface on the arithmetically Cohen-Macaulay scheme \( X \). Since the ideal sheaf of \( \mathcal{F} \) is given by the product of the ideal sheaves of \( j_!(\mathcal{D}^*) \) and \( \mathcal{D}' \), an embedded components of \( j_!(\mathcal{D}^*) \) would be an embedded components of \( \mathcal{F} \), but this is not possible. On the other side \( j_!(\mathcal{D}^* + mE) \) for \( m > 0 \) can not be equal to \( D \); in fact \( D^* + mE \sim (k + 1 + m)\tilde{H} - (m + 1)f - h - 1)\tilde{R} \) is not contained in \( j^*F \) since \( j^*F - D^* - mE - (f - h - 1)\tilde{R} \) is not effective. The isomorphism between the ideals follows from Lemma 2.3.

**THEOREM 3.17** Let \( \operatorname{codim}(V, X) = 2 \). A sheaf \( \mathcal{O}_X(a, b) \) with \( a, b \in \mathbb{Z} \) is reflexive if and only if \( b = \deg \mathcal{O}_X(a, b) |_E \leq f \).

Proof. The divisors on \( \tilde{X} \) of type \( d\tilde{R} \) with \( d \geq 0 \) are proper transforms of divisors on \( X \), therefore the sheaves \( \mathcal{O}_X(0, -d) \) are ideal sheaves on \( X \) by Prop. 3.10, and therefore they are reflexive. The divisors \( \tilde{H} - (f - h - 1)\tilde{R} \) with \( 0 \leq h < f \) are total transforms of divisors \( \sim (h + 1)R \) on \( X \); by Prop. 3.10 \( \mathcal{O}_X(-1, f - h - 1) \) with \( 0 \leq h < f \) are ideal sheaves on \( X \), therefore they are reflexive by Th. 3.1. By the projection formula we have that \( \mathcal{O}_X(a + a', b) = \mathcal{O}_X(a, b) \otimes \mathcal{O}_X(a', 0) \) for every \( a, a', b \in \mathbb{Z} \). Since the tensor product of a reflexive sheaf with an invertible sheaf is reflexive, we obtain that the sheaves \( \mathcal{O}_X(a, b) \) are reflexive for every \( a \in \mathbb{Z} \) and \( b < f \). It remains to prove that the sheaves \( \mathcal{O}_X(0, d) \) are not reflexive for \( d \geq f \). Let us suppose they are reflexive and divide \( d - 1 = kf + h + 1 \) as usual, with \( k \geq -1 \) and \( 0 \leq h < f \); then the sheaves \( \mathcal{O}_X(k, h + 1) \) and \( \mathcal{O}_X(0, d) \) with \( h < f - 1 \) (or the sheaves \( \mathcal{O}_X(k + 1, 0) \) and \( \mathcal{O}_X(0, (k + 1)f) \) if \( h = f - 1 \) are both reflexive and isomorphic on \( X_S \), by Th. 3.1 they should be isomorphic on \( X \), but this is
not possible. In fact the dimension of the respective zero cohomology groups are different, as one may compute using (3.7) and (3.4). □

As a consequence of Th. 3.17 we have that we can write the divisorial sheaf associated to a Weil divisor \( D \) in \( X \) in more than one form. To be more precise, let \( d > 0 \) and divide \( d = kf + h + 1 \geq 0 \), with \( k \geq -1 \) and \( 0 \leq h < f \). Let \( D \sim dR \) and let us suppose that \( D \) is not Cartier, i.e. \( d \not\equiv 0 \mod f \) (otherwise the representation is unique and it is \( \mathcal{O}_X(D) \cong \mathcal{O}_X(k + 1, 0) \)), then:

\[
\mathcal{O}_X(D) \cong \mathcal{O}_X((k + 1), -f + h + 1) \cong \mathcal{O}_X(k, h + 1).
\]

Let \( D \sim -dR \) (now \( D \) may be Cartier), then

\[
\mathcal{O}_X(D) \cong \mathcal{O}_X(-(k + 1), f - h - 1) \cong \mathcal{O}_X(-k, -(h + 1)) \cong \cdots \cong \mathcal{O}_X(0, -d).
\]

In the next Corollary we explicitly describe the group \( \text{Div}(X) \), fixing a particular form in which we write a divisorial sheaf. In this way it is evident the bijection between \( \text{Div}(X) \) and \( \text{Cl}(X) \cong \mathbb{Z} \).

**COROLLARY 3.18** Let \( \text{codim}(V, X) = 2 \). Then:

\[
\text{Div}(X) = \{ \mathcal{O}_X(a, b) \mid a, b \in \mathbb{Z}, 0 \leq b < f \}.
\]

The natural group structure on \( \text{Div}(X) \) is given by: \(< \mathcal{O}_X(a, b), \mathcal{O}_X(a', b') >\mapsto \mathcal{O}_X(a + a' + \lfloor \frac{b + b'}{f} \rfloor, b + b' \mod f) \) and \( \mathcal{O}_X(a, b) \cong \mathcal{O}_X(-a, -b) \), where \( \lfloor \cdot \rfloor \) denotes the integral part. Moreover for a fixed \( b : 0 \leq b < f \), the set \( \text{Div}_b \{ \mathcal{O}_X(a, b) \mid a \in \mathbb{Z} \} \) is the set of divisorial sheaves of Weil divisors \( D \sim dR \) with \( d = b \mod f \).

Proof. By Th. 3.17, for every \( d \in \mathbb{Z} \), we can write the divisorial sheaf associated to \( D \sim dR \) in the form \( \mathcal{O}_X(a, b) \) with \( a \in \mathbb{Z} \) and \( 0 \leq b < f \). □

This particular choice is convenient if we want to compute \( h^0(\mathcal{O}_X(D)) \), in fact by (3.7) we know how to compute it.

**COROLLARY 3.19** Let \( \text{codim}(V, X) = 2 \). Let \( D \sim dR \) be an effective divisor; divide \( d = kf + h + 1 \) with \( k \geq -1 \) and \( 0 \leq h < f \). Let \( |D| \) be the complete linear system of \( D \). Then:

\[
\dim |D| = f \left( \begin{array}{c} k + r - 1 \\ r \end{array} \right) + (h + 2) \left( \begin{array}{c} k + r - 1 \\ r - 1 \end{array} \right) - 1,
\]

if \( h \neq f - 1 \) (i.e. \( D \) is not Cartier), or

\[
\dim |D| = f \left( \begin{array}{c} k + r \\ r \end{array} \right),
\]

if \( D \) is Cartier.

Proof. Cor. 3.18, (3.7), (3.3). □

We conclude with the projection formula:

**COROLLARY 3.20** (Projection formula) Let \( \text{codim}(V, X) = 2 \). Let \( D \) be a Weil divisor on \( X \) and let \( D^* \) be its integral total transform. Then:

\[
j_* \mathcal{O}_X(D^*) = \mathcal{O}_X(D)
\]

Proof. By Th. 3.17 \( j_* \mathcal{O}_X(D^*) \) is a reflexive sheaf; since it is the divisorial sheaf associated to \( D \) in the open set \( X_S \), then by Th. 3.1 it is the divisorial sheaf of \( D \) an all \( X \). □
4 INTERSECTION OF WEIL DIVISORS

As we have already noted, an effective Weil divisor $D$ on $X \subset \mathbb{P}^n$ is a closed subscheme of $X$ of pure codimension 1 with no embedded components ([1], Prop. 2.4.). For this reason we can regard $D \subset \mathbb{P}^n$ as a projective scheme. Given two effective divisors $D$ and $D'$ on $X$ with no common components, we can consider the scheme-theoretic intersection $D \cap D' \subset \mathbb{P}^n$ and ask for its degree. For degree of a scheme $Y \subset \mathbb{P}^n$ we mean the length $h^0(\mathcal{O}_Y)$ of the zero-dimensional scheme $Y$, which represents a generic (codim $Y$)-dimensional linear section of $Y$. As we will see, this problem has an immediate solution when codim($V, X$) > 2, via the isomorphism Cl($X$) $\cong$ CaCl($X$). In this section we show how to use the integral total transform to compute this degree in case codim($V, X$) = 2.

If codim($V, X$) > 2, by Cor. 2.2 we can define an intersection form on $\text{Div}(X)$:

$$I : \text{Div}(X)^r \to \mathbb{Z}$$

$$< D_1, D_2, \ldots, D_r > \mapsto \tilde{D}_1 \cdot \tilde{D}_2 \cdots \tilde{D}_r$$

exactly as in $\tilde{X}$; i.e. $I$ is determined by the rule:

$$H' = f \quad H'^{-1} \cdot R = 1 \quad H'^{-2} \cdot R^2 = 0.$$  

The intersection form $I$ determines the degree of the intersection scheme $Y = D_1 \cap \cdots \cap D_l \subset \mathbb{P}^n$ of $l (l \leq r)$ effective divisors which intersect properly.

If codim($V, X$) = 2, the linear theory of intersection developed by Mumford in [6] in the case of a normal surface can be generalized to our case by

$$D_1 \cdots D_r = j^* D_1 \cdots j^* D_r$$

where $j^* D_i$ is the Mumford’s total transform of $D_i$. Given two effective Weil divisors $D$ and $D'$ with no common components, the intersection number $D \cdot D' \cdot H'^{-2}$ does not represent in this case the degree of the intersection scheme $Y = D \cap D' \subset \mathbb{P}^n$. To compute this degree we will find the minimal resolution of $\mathcal{O}_Y$ as $\mathcal{O}_X$-module (Th. [7]); other applications of this resolution are described in the next section. First we need to prove some properties of the integral total transform.

LEMMA 4.1 Let codim($V, X$) = 2 and let $D$ be an effective Weil divisor on $X$. Then:

$$p_a(D) = p_a(D^*) = (4.2).$$

Proof. Let $D \sim dR$ and divide $d - 1 = kf + h$ with $k \geq -1$ and $0 \leq h < f$; since $D^* \sim (k + 1)H - (f - h - 1)\tilde{R}$ with $f - h - 1 \geq 0$, from the exact sequence

$$0 \to \mathcal{I}_{D^*|X} \to \mathcal{O}_X \to \mathcal{O}_{D^*} \to 0$$

we obtain

$$0 \to j_* \mathcal{I}_{D^*|X} \to j_* \mathcal{O}_X \to j_* \mathcal{O}_{D^*} \to 0.$$  

Since $j_* \mathcal{I}_{D^*|X} = \mathcal{I}_D|X$ by Prop. [5.16] and $j_* \mathcal{O}_X = \mathcal{O}_X$, we get $j_* \mathcal{O}_{D^*} = \mathcal{O}_D$. By [6.7] we obtain

$$p_a(D) = 1 - \chi(\mathcal{O}_D) = 1 - \chi(\mathcal{O}_{D^*}) = p_a(D^*).$$
DEFINITION 4.3 With the notation of Def. 3.13 define:
\[ \epsilon := [q] - q. \]

By definition \( \epsilon \) is a rational number in the interval \([0, 1]\) of the kind \( \epsilon = \frac{l-1}{f} \) with \( l = 1, 2, \ldots, f. \) With the notation of 3.13 we have:
\[ \epsilon = \frac{f - h - 1}{f}. \]

We note that given \( D \sim dR \) on \( X, \epsilon \) is uniquely determined by the class of linear equivalence of \( D \), i.e. by \( d. \)

LEMMA 4.4 Let \( \text{codim}(V, X) = 2 \) and let \( D_1 \sim d_1R \) and \( D_2 \sim d_2R \) be two effective divisor on \( X. \) Then:
\[ (D_1 + D_2)^* = \begin{cases} D_1^* + D_2^* & \text{if } [\epsilon_1 + \epsilon_2] = 0 \\ D_1^* + D_2^* - E & \text{if } [\epsilon_1 + \epsilon_2] = 1 \end{cases} \]
where \([ \cdot ] \) denotes the integral part.

Proof. By definition 3.13 of integral total transform we have:
\[ (D_1 + D_2)^* = D_1 + D_2 + [q_1 + q_2]E, \]

since the proper transform of \( D_1 + D_2 \) is \( \tilde{D}_1 + \tilde{D}_2. \) We find the two cases \([q_1 + q_2] = [q_1] + [q_2] \) if \( 0 \leq \epsilon_1 + \epsilon_2 < 1 \) and \([q_1 + q_2] = [q_1] + [q_2] - 1 \) if \( 1 \leq \epsilon_1 + \epsilon_2 < 2. \)

THEOREM 4.6 Let \( \text{codim}(V, X) = 2 \) and let \( Y \subset X \subset \mathbb{P}^n \) be a "complete intersection" of two effective divisors \( D_1 \) and \( D_2 \) on \( X. \) The following sequence is exact:
\[ 0 \to j_* \mathcal{O}_X(-(D_1 + D_2)^*) \to j_* \mathcal{O}_X(-D_1^*) \oplus j_* \mathcal{O}_X(-D_2^*) \to \mathcal{I}_{Y|X} \to 0 \]
and therefore it is a "reflexive resolution" of \( \mathcal{O}_Y \) as a \( \mathcal{O}_X \)-module.

Proof. First let \( f_i : \mathcal{O}_{D_i + D_2} \rightarrow \mathcal{O}_{D_i} \) for \( i = 1, 2 \) be the projection and let \( g_i : \mathcal{O}_{D_i} \rightarrow \mathcal{O}_{D_i \cap D_2} \). Then there is an exact sequence of sheaves of \( \mathcal{O}_X \)-modules:
\[ 0 \to \mathcal{O}_{D_1 + D_2} \xrightarrow{(f_1+f_2)} \mathcal{O}_{D_1} \oplus \mathcal{O}_{D_2} \xrightarrow{(g_1-g_2)} \mathcal{O}_{D_1 \cap D_2} \to 0. \]

For any effective Weil divisor \( D \) on \( X \) we have already seen the resolution of \( \mathcal{O}_D: \)
\[ 0 \to j_* \mathcal{O}_X(-D^*) \to \mathcal{O}_X \to \mathcal{O}_D \to 0 \]
in the proof of Lemma 3.1. The mapping cone (see [1] pg. 432, pg. 657) between the resolution (of type 4.3) of \( \mathcal{O}_{D_1 + D_2} \) and of \( \mathcal{O}_{D_1} \oplus \mathcal{O}_{D_2} \) gives a resolution of \( \mathcal{O}_{D_1 \cap D_2}: \)
\[ j_* \mathcal{O}_X(-(D_1 + D_2)^*) \hookrightarrow \mathcal{O}_X \oplus j_* \mathcal{O}_X(-D_1^*) \oplus j_* \mathcal{O}_X(-D_2^*) \to \mathcal{O}_X \oplus \mathcal{O}_X. \]

In this resolution we can suppress redundant terms and obtain the required resolution:
\[ j_* \mathcal{O}_X(-(D_1 + D_2)^*) \to j_* \mathcal{O}_X(-D_1^*) \oplus j_* \mathcal{O}_X(-D_2^*) \to \mathcal{O}_X. \]
\( \square \)
NOTE 4.10 The resolution \( L \) allows us to find a resolution, in general not minimal, of \( \mathcal{O}_Y \) as \( \mathcal{O}_{\mathbb{P}^n} \)-module. Indeed, as shown in \( L \), a minimal resolution of \( \mathcal{O}_X(a, b) \) as \( \mathcal{O}_{\mathbb{P}^n} \)-module is given by the Eagon-Northcott type complex \( C^b(a) \) for \( b \geq -1 \). Since each of the terms in \( L \) is a sheaf \( \mathcal{O}_X(a, b) \) with \( b \geq 0 \), then a suitable mapping cone between the complexes \( C^b(a) \)'s gives us the required resolution in \( \mathbb{P}^n \).

PROPOSITION 4.11 Let \( \text{codim}(V, X) = 2 \) and let \( D \) and \( D' \) be two effective divisors on \( X \) with no common components. Then the degree of the "complete intersection" scheme \( Y = D \cap D' \) is given by:

\[
\deg (D \cap D') = \begin{cases} 
D^* \cdot D'^* \cdot \tilde{H}^{-2} & \text{if } |\epsilon + \epsilon'| = 0 \\
D^* \cdot D'^* \cdot \tilde{H}^{-2} + f(\epsilon + \epsilon' - 1) + 1 & \text{if } |\epsilon + \epsilon'| = 1
\end{cases} 
\tag{4.12}
\]

Proof. Let us call \( X_L, Y_L, D_L \) and \( D'_L \) general \( (r - 2) \)-dimensional linear sections of \( X, Y, D \) and \( D' \) respectively, and let us call \( \tilde{X}_L \) the canonical resolution of the rational normal surface \( X_L \). The resolution of \( \mathcal{O}_{Y_L} \) as an \( \mathcal{O}_{X_L} \)-module is by Th. 4.3

\[
0 \to j_* \mathcal{O}_{\tilde{X}_L}(- (D_L + D'_L)^*) \to j_* \mathcal{O}_{\tilde{X}_L}(- D_L^*) \oplus j_* \mathcal{O}_{\tilde{X}_L}(- D'_L^*) \to \mathcal{O}_{X_L}.
\]

By the proof of Lemma 4.1 we have that \( \chi(j_* \mathcal{O}_{\tilde{X}_L}(- D^*)) = p_a(D^*) \) for every effective divisor \( D \) on \( X_L \). Let \( |\epsilon + \epsilon'| = 0 \); by Lemma 4.3 \( (D_L + D'_L)^* = D_L^* + D'_L^* \).

Looking at the resolution of \( \mathcal{O}_{Y_L} \), by adjunction formula we compute:

\[
h^0(\mathcal{O}_{Y_L}) = \frac{1}{2} (D_L^* + D'_L^* + K_{\tilde{X}_L}) \cdot (D_L^* + D'_L^*) - \frac{1}{2} (D_L^* + K_{\tilde{X}_L}) \cdot D_L^* - \frac{1}{2} (D'_L^* + K_{\tilde{X}_L}) \cdot D'_L^* = D_L^* \cdot D'_L^* = D^* \cdot D'^* \cdot \tilde{H}^{-2},
\]

where \( K_{\tilde{X}_L} \sim -2\tilde{H} + (f - 2)\tilde{R} \) is the canonical divisor of \( \tilde{X}_L \). Let \( |\epsilon + \epsilon'| = 1 \); by Lemma 4.4 we have that \( (D_L + D'_L)^* = D_L^* + D'_L^* - E \). By an analogous computation we find:

\[
h^0(\mathcal{O}_{Y_L}) = D_L^* \cdot D'_L^* - (D_L^* + D'_L^*) \cdot E - \frac{1}{2} (K_{\tilde{X}_L} - E) \cdot E = D_L^* \cdot D'_L^* + (2f - h_1 - h_2 - 2) - f + 1 = D^* \cdot D'^* \cdot \tilde{H}^{-2} + f(\epsilon + \epsilon' - 1) + 1
\]

\[\Box\]

We note that when \( |\epsilon + \epsilon'| = 1 \), the quantity \( f(\epsilon + \epsilon' - 1) \) is bigger than or equal to zero, therefore in this case \( \deg(D \cap D') \) is strictly bigger than the number \( D^* \cdot D'^* \cdot \tilde{H}^{-2} \). The scheme theoretic intersection \( Y \) contains the vertex \( V \) as a component with a certain multiplicity \( \geq 0 \), we call this number the integral intersection multiplicity \( m(D, D'; V) \) of \( D \) and \( D' \) in \( V \), i.e.

\[m(D, D'; V) = \deg(D \cap D') - D \cdot D' \cdot \tilde{H}^{-2}.
\]
If \( \tilde{D} \sim a\tilde{H} + b\tilde{R} \) and \( \tilde{D}' \sim a'\tilde{H} + b'\tilde{R} \) are the proper transforms of \( D \) and \( D' \), then by Prop. 4.11 we explicitly compute:

\[
m(D, D'; V) = \begin{cases} \frac{bb'}{bb'} - f(\epsilon \cdot \epsilon') & \text{if } [\epsilon + \epsilon'] = 0 \\ \frac{bb'}{bb'} - f(\epsilon \cdot \epsilon') + f(\epsilon + \epsilon' - 1) + 1 & \text{if } [\epsilon + \epsilon'] = 1. \end{cases}
\]

**NOTE 4.13** The intersection multiplicity of two effective divisors \( D \) and \( D' \) with no common components through the singular locus \( V \) on a normal surface \( X \), is defined in the linear intersection theory of Mumford as the rational number: \( i(D, D'; V) = j^*D \cdot j^*D' - D \cdot \tilde{D}' \). If \( X \) is a rational normal cone, using the same notations as above, we find that the linear intersection multiplicity \( i(D, D'; V) \) is exactly \( \frac{bb'}{f^2} \).

## 5 EXAMPLES AND APPLICATIONS

In this section we show some applications of the previous results. In particular in Ex. 5.1 we use Th. 4.6 to compute the arithmetic genus of the scheme theoretic intersection \( Y \) of two effective divisors on a rational normal cone.

**EXAMPLE 5.1** In this Example we show that every effective non degenerate divisor on a rational normal cone \( X \) of degree \( n - 1 \) in \( \mathbb{P}^n \) is a curve of maximal arithmetic genus \( p_a(C) = G(n, d) \).

Let \( C \sim dR \) with \( d > n - 1 \), let us divide \( d - 1 = m(n - 1) + \delta \) with \( m \geq 1 \), \( 0 \leq \delta \leq n - 2 \); then \( G(n, d) = \binom{n}{2}(n - 1) + m\delta \). By Lemma 4.1 we know that \( p_a(C) = p_a(C^*) \), where \( C^* \sim (m + 1)\tilde{H} - (n - 2 - \delta)\tilde{R} \). By adjunction formula on \( X \) we then compute \( p_a(C^*) = G(n, d) \).

**EXAMPLE 5.2** Let \( \text{codim}(V, X) = 2 \), then every effective divisor \( D \) and every "complete intersection" \( Y \) of two divisors \( D, D' \) on \( X \subset \mathbb{P}^n \) is arithmetically Cohen-Macaulay.

In the case of one divisor we know by (4.9), (3.7) and (3.4) that

\[
h^i(\mathcal{I}_{D/X}(k)) = h^i(\mathcal{I}_{D'/X}(k)) = 0
\]

for \( 1 \leq i \leq r - 1 \) and every \( k \). Since \( X \) is arithmetically Cohen-Macaulay from the exact sequence \( 0 \rightarrow \mathcal{I}_X|\mathbb{P}^n \rightarrow \mathcal{I}_{D|\mathbb{P}^n} \rightarrow \mathcal{I}_{D/X} \rightarrow 0 \) we conclude \( h^i(\mathcal{I}_{D|\mathbb{P}^n}(k)) = 0 \) for \( 1 \leq i \leq r - 1 \) and every \( k \). Looking at the resolution (4.7) and using (3.7) and (3.4) we compute: \( h^i(\mathcal{I}_{Y|X}(k)) = 0 \) for \( 1 \leq i \leq r - 2 \) and every \( k \); as in the previous case, since \( X \) is arithmetically Cohen-Macaulay we conclude \( h^i(\mathcal{I}_{Y|\mathbb{P}^n}(k)) = 0 \) for \( 1 \leq i \leq r - 2 \) and every \( k \).

**EXAMPLE 5.3** If \( \text{codim}(V, X) > 2 \) and \( Y \) is a "complete intersection" of \( l \) \( (1 \leq l \leq r - 1) \) divisors \( D_i \sim a_iH - b_iR \) with \( b_i \geq 0 \), then the resolution of \( \mathcal{O}_Y \) as an \( \mathcal{O}_X \)-module, is a Koszul complex (see (10), Ex. 3.6.):

\[
0 \rightarrow \mathcal{O}_X(-(a_1 + \cdots + a_i), b_1 + \cdots + b_l) \rightarrow \cdots \\
\cdots \rightarrow \sum_{i_1 < i_2} \mathcal{O}_X(-(a_{i_1} + a_{i_2}), b_{i_1} + b_{i_2}) \rightarrow \sum_i \mathcal{O}_X(-a_i, b_i) \rightarrow \mathcal{O}_X.
\]

From this resolution using (3.7) and (3.4) we conclude that \( Y \) is arithmetically Cohen-Macaulay iff \( b_1 + \cdots + b_l < f \).


EXAMPLE 5.4 Let $\text{codim}(V, X) = 2$ and $r \geq 3$. Resolution 4.7 can be used to compute the arithmetic genus of the intersection scheme $Y$ of two effective divisors $D$ and $D'$ with no common components.

Let us suppose that $Y$ is non degenerate, i.e. $d, d' > f$, then using (3.7), (3.4) and (3.5) we compute:

$$p_a(Y) = h^0(\mathcal{O}_{\tilde{X}}(K_{\tilde{X}} + (D + D')^*)) - h^0(\mathcal{O}_{\tilde{X}}(K_{\tilde{X}} + D^*)) - h^0(\mathcal{O}_{\tilde{X}}(K_{\tilde{X}} + D'^*))$$

$$= \sum_{i=-1}^{1} (-1)^i \left( \binom{\alpha_i - 1}{r} + (f - 1 - \beta_i) \binom{\alpha_i - 1}{r - 1} \right)$$

where $\alpha_{-1} = \left[ \frac{d'}{f} \right]$ and $\beta_{-1} = f \left[ \frac{d'}{f} \right] - d'$; $\alpha_1 = \left[ \frac{d}{f} \right]$ and $\beta_1 = f \left[ \frac{d}{f} \right] - d$; $\alpha_0 = \left[ \frac{d + d'}{f} \right]$ and $\beta_0 = f \left[ \frac{d + d'}{f} \right] - (d' + d)$. With these notations the degree of $Y$ is:

$$\text{deg}(Y) = \sum_{i=-1}^{1} (-1)^i \left( \binom{\alpha_i - 1}{r} + (f - 1 - \beta_i) \binom{\alpha_i - 1}{r - 2} \right).$$

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