Quasinormal modes of charged squashed Kaluza-Klein black holes in the Gödel Universe

Xi He and Bin Wang
Department of Physics, Fudan University, 200433 Shanghai, China

Songbai Chen
Institute of Physics and Department of Physics, Hunan Normal University, Changsha, 410081 Hunan, China

Abstract

We study the quasinormal modes of scalar perturbation in the background of five-dimensional charged Kaluza-Klein black holes with squashed horizons immersed in the Gödel universe. Besides the influence due to the compactness of the extra dimension, we disclose the cosmological rotational effect in the wave dynamics. The wave behavior affected by the Gödel parameter provides an interesting insight into the Gödel universe.

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**I. INTRODUCTION**

Quasinormal mode (QNM) plays an essential role in the study of black hole physics. It is known as a unique fingerprint of the black hole existence, since quasinormal frequencies only depend on black hole parameters. It is expected that QNM can be detected in the gravitational wave observations to be realized in the near future [1]. Besides its potential observational interest in astronomy to detect the black hole existence, theoretically QNM is believed as a testing ground of fundamental physics. It is widely believed that the study of QNM can help us get deeper insights into the AdS/CFT [2] and dS/CFT [3] correspondences (for a review on this topic and more complete list of references can be found in [4] [5]). Recently further motivation of studying the QNMs has been pointed out in [6, 7, 8] by arguing that QNMs can provide a phenomenological signature of black hole thermal phase transition. Further evidences of the non-trivial relation between the dynamical and thermodynamical properties of black holes were provided in [9, 10]. Furthermore string theory has the radial prediction that spacetime has extra dimensions and gravity propagates in higher dimensions, QNM has recently been argued as a useful tool to disclose the existence of the extra dimensions [11, 12, 13]. Most studies of the QNM are concentrated on black holes immersed in the rather idealized isotropic homogeneous universe.

It was argued that it is more reasonable to consider the universe background as homogeneous but with global rotation [14]. An original exact solution of Einstein equations with pressureless matter and negative cosmological constant for the rotating universe was found by Gödel [15] in four-dimensional spacetime, which exhibits close timelike curves through every point. Supersymmetric generalizations of the Gödel universe in five dimensions have been found in [16]. Exact solutions describing various black holes embedded in a Gödel universe have been presented in [17, 18]. Applying the squashing transformation, the new squashed Kaluza-Klein (KK) black hole solutions in the rotating universe have been obtained in [19, 20]. In [21, 22], the possible influence of the rotation of the universe on the QNM of the Schwarzschild-Gödel black hole and the Hawking radiation in the rotating Gödel black hole have been investigated respectively. It was found that the quasinormal spectrum and Hawking radiation are considerably affected by the rotating cosmological background. In this work we are going to explore specifically the QNM of the new solution obtained by applying the squashing transformation to the general charged black hole embedded in the Gödel universe [19]. Besides the influence due to the compactness of the extra dimension on the QNM [10, 23], here we will also disclose the cosmological rotation effect. As done in [21, 22] we will limit our attention to the case of slow
rotation of the cosmological background. This is mainly because that phenomenologically the small rotation of the universe is the most reasonable situation. In addition, the small rotation can help to separate the variables in perturbation equations.

The plan of the paper is as follows. In Sec. II we first go over the charged squashed KK Gödel black hole background and then derive the master equation of scalar perturbation in the limit of small Gödel parameter \( j \). In Sec. III, we discuss our numerical result on the quasinormal frequencies for the scalar field perturbations. In the end we present our conclusions and further discussions.

II. SCALAR FIELD PERTURBATION OF THE CHARGED SQUASHED KK BLACK HOLE IN GÖDEL UNIVERSE

The line element of the general charged rotating squashed KK Gödel black hole is

\[
    ds^2 = -f(r)dt^2 + \frac{k(r)^2}{V(r)} dr^2 + 2g(r)\sigma_3 dt + h(r)\sigma_3^2 + \frac{j^2}{4} \left[ k(r)(\sigma_1^2 + \sigma_2^2) + \sigma_3^2 \right],
\]

where

\[
    \sigma_1 = \cos \psi d\theta + \sin \psi \sin \theta d\phi,
\]

\[
    \sigma_2 = -\sin \psi d\theta + \cos \psi \sin \theta d\phi,
\]

\[
    \sigma_3 = d\psi + \cos \theta d\phi,
\]

with coordinates \( \theta \in [0, \pi), \phi \in [0, 2\pi) \) and \( \psi \in [0, 4\pi) \), and \( r \) runs in the range \( (0, r_\infty) \). The metric functions read

\[
    f(r) = 1 - \frac{2M}{r^2} + \frac{q^2}{r^4}, \\
    g(r) = jr^2 + 3jq + \frac{(2M-q)\bar{a}}{2r^2} - \frac{q^2\bar{a}^2}{2r^4}, \\
    h(r) = -j^2r^2(r^2 + 2M + 6q) + 3jq\bar{a} + \frac{(M-q)\bar{a}^2}{2r^2} - \frac{q^2\bar{a}^2}{4r^4}, \\
    V(r) = 1 - \frac{2M}{r^2} + \frac{8j(M+q)[\bar{a} + 2j(M+2q)]}{r^2} + \frac{2(M-q)\bar{a}^2 + q^2[1 - 16j\bar{a} - 8j^2(M+3q)]}{r^4}, \\
    k(r) = \frac{V(r_\infty)r_\infty^4}{(r^2 - r_\infty^2)^2}.
\]

Constant \( M, q, \bar{a} \) represent the mass, charge and rotation of the black hole, and \( j \) denotes the scale of the Gödel background. We will concentrate on the black hole without rotation by setting \( \bar{a} = 0 \). When \( j = 0, \bar{a} = 0 \), Eq. (1) reduces to the five-dimensional charged KK black hole with squashed horizon\[24\].
\( r_\infty \to \infty \) \((k(r) \to 1)\), the squashing effect disappears and one recovers the five-dimensional charged black hole in the Gödel universe.

Considering the reasonable phenomenology that our universe must rotate slowly even if there is global rotation in the cosmological background, we will expand Eq. (1) in the small \( j \) limit by discarding terms over the order \( O(j^2) \). In [21, 22], it was argued that small \( j \) limit is required to separate variables in perturbation equations. Using coordinate transformation \( \rho = \rho_0 r_\infty^2 r \) and \( \tau = \sqrt{V(r_\infty)} t \), Eq. (1) can be rewritten in the form

\[
ds^2 = -F(\rho)d\tau^2 + \frac{K^2(\rho)}{F(\rho)}d\rho^2 + \rho^2 K^2(\rho)(d\theta^2 + \sin^2 \theta d\phi^2) + \frac{r^2_\infty}{4K^2(\rho)} \sigma^2_3 - 2H(\rho)\sigma_3 d\tau, \tag{4}\]

with

\[
F(\rho) = \left(1 - \frac{\rho_+}{\rho}\right)\left(1 - \frac{\rho_-}{\rho}\right), \tag{5}
\]

\[
K(\rho)^2 = 1 + 1, \tag{6}
\]

\[
H(\rho) = \frac{r^3_\infty}{2\rho_0 K^2 j} + \frac{3r^2_\infty}{\rho_0} \sqrt{\rho_+ \rho_-}, \tag{7}
\]

where \( \rho_0^2 = \frac{r^4_\infty}{V(r_\infty)} \), so that \( r^2_\infty = 4(\rho_+ + \rho_0)(\rho_- + \rho_0) \). \( \rho_+ \), \( \rho_- \) denote the outer and inner horizons of the black hole in the new coordinate. For simplicity we will follow [10] to take \( \rho_+ = 1 - \frac{a^2}{2} \) and \( \rho_- = \frac{a^2}{2} \) and adjust the parameter \( a \) to indicate the extent on how far the outer horizon is away from the inner horizon.

The scalar field perturbations in charged squashed KK Gödel black hole background are governed by the Klein-Gordon equation \( \Box \Phi = 0 \). We can separate the variables of the field equation by adopting the limit of small \( j \) and representing the wave function as \( \Phi(\tau, \rho, \theta, \phi, \psi) = e^{i\omega \tau + im\phi - i\lambda \psi} R(\rho)S(\theta) \). Then the perturbation equations reduce to

\[
\frac{F}{\rho^2 K^2} \frac{\partial}{\partial \rho} \left[ \rho^2 F \frac{\partial R}{\partial \rho} \right] + \left[ \omega^2 - \frac{8\omega \lambda K^2}{r^2_\infty} - \frac{4F\lambda^2 K^2}{r^2_\infty} - \frac{F}{\rho^2 K^2} E_{lm\lambda} \right] R = 0, \tag{6}
\]

\[
\frac{d^2 S}{d\theta^2} + \cot \theta \frac{dS}{d\theta} + \left[ E_{lm\lambda} - (m - \lambda \cos \theta)^2 \csc^2 \theta \right] S = 0, \tag{7}
\]

where \( E_{lm\lambda} = l(l + 1) - \lambda^2 \) is the eigenvalue of the spin-weighted spherical function (7), \( \lambda \) is the separation constant of the fifth dimension.

Boundary conditions on the wave function \( R(\rho) \) at the outer horizon and the spatial infinity can be expressed as

\[
R(\rho) \sim \begin{cases} 
(\rho - \rho_+)^\alpha, & \rho \to \rho_+; \\
\rho^2 e^{i\lambda \rho}, & \rho \to \infty. 
\end{cases} \tag{8}
\]
A solution of equation (6) that satisfies the above boundary condition can be written as

\[ R(\rho) = e^{i(\omega - \rho -)} \chi (\rho - \rho_+)^{\alpha} (\rho - \rho_-)^{\beta + \gamma} \sum_{m=0}^{\infty} a_m \left( \frac{\rho - \rho_+}{\rho - \rho_-} \right)^m, \tag{9} \]

where constants \( \alpha, \beta, \gamma \) and \( \chi \) are

\[ \alpha = \frac{1}{\sqrt{\rho_0}(\rho_- - \rho_+)} \left\{ -\rho_+^2 (\rho_0 + \rho_+) \omega \right. \]
\[ \left. - 8 j \rho_+ \left( 3 \sqrt{\rho_0 - \rho_+} + \frac{r_\infty}{2} \right) \lambda + \rho_0 \left( \rho_+ \omega - 24 j \sqrt{\rho_0 - \rho_+} \lambda \right) \right\}^{\frac{1}{2}}, \tag{10} \]

\[ \beta = \frac{1}{\sqrt{\rho_0}(\rho_+ - \rho_-)} \left\{ -\rho_-^2 (\rho_0 + \rho_-) \omega \right. \]
\[ \left. - 8 j \rho_- \left( 3 \sqrt{\rho_0 - \rho_-} + \frac{r_\infty}{2} \right) \lambda - 8 j \rho_+ \left( 3 \sqrt{\rho_0 - \rho_-} + \frac{r_\infty}{2} \right) \lambda \right\}^{\frac{1}{2}}, \tag{11} \]

\[ \gamma = \frac{-i}{\rho_0 r_\infty^2} \left( \frac{\omega^2 - \frac{3 \lambda^2}{r_\infty^2}}{\omega_0} - \frac{8 j}{\rho_0} \left( 3 \sqrt{\rho_0 - \rho_+} + \frac{r_\infty}{2} \right) \omega \lambda \right) \left\{ \rho_0 (2 \rho_0 + \rho_- + \rho_+) \lambda^2 \right. \]
\[ + 2 j r_\infty^2 \left[ 2 (\rho_+ + \rho_-) \left( 3 \sqrt{\rho_0 - \rho_+} + \frac{r_\infty}{2} \right) + \rho_0 \left( 6 \sqrt{\rho_0 - \rho_+} + \frac{r_\infty}{2} \right) \right] \omega \lambda \]
\[ - \rho_0 r_\infty^2 \left[ (\rho_0 + 2 \rho_+ + 2 \rho_-) \omega^2 + 2 i \sqrt{\omega^2 - \frac{4 \lambda^2}{r_\infty^2}} - \frac{8 j}{\rho_0} \left( 3 \sqrt{\rho_0 - \rho_-} + \frac{r_\infty}{2} \right) \omega \lambda \right]\}, \tag{12} \]

\[ \chi = \sqrt{\frac{\omega^2 - \frac{8 j \lambda^2}{\rho_0} \left( 3 \sqrt{\rho_0 - \rho_+} + \frac{r_\infty}{2} \right) \lambda - \frac{4 \lambda^2}{r_\infty^2}}}, \tag{13} \]

We will employ the continued fraction method to find the accurate quasinormal frequencies of the charged squashed KK black hole in Gödel universe. The radial equation (6) for the perturbations can be solved using a series expansion around some irregular points, such as the event horizon. The coefficients \( a_m \) of the expansion are then determined by a recursion relation starting from \( a_0 = 1 \) with the form

\[ \alpha_0 a_1 + \beta_0 a_0 = 0, \tag{14} \]

\[ \alpha_m a_{m+1} + \beta_m a_m + \gamma_m a_{m-1} = 0, \quad m = 1, 2, \ldots \]

where the recursion coefficients \( \alpha_m, \beta_m, \gamma_m \) are given by

\[ \alpha_m = m^2 + (C_0 + 1)m + C_0, \tag{15} \]

\[ \beta_m = -2m^2 + (C_1 + 2)m + C_3, \]

\[ \gamma_m = m^2 + (C_2 - 3)m + C_4 - C_2 + 2. \]

\( C_m \)'s are functions of the frequency \( \omega \) and when \( j = 0 \), they reduce to (23)-(26) in [10]. For conciseness, here we will not write out their tedious expressions when \( j \neq 0 \). The boundary conditions are satisfied when the continued fraction condition on the recursion coefficients holds. The series in [14] converge for the given \( l \).
TABLE I: Quasnormal frequencies of scalar perturbations of charged squashed Kaluza-Klein black holes in the Gödel universe with the change of $j$ for chosen $\rho_0$, for $l = 1$, $n = 0$ and $a = 0.3$.

| $j$ | $\rho_0 = 30$ | $\rho_0 = 60$ | $\rho_0 = 80$ |
|-----|---------------|---------------|---------------|
| 0.001 | 0.148491 - 0.0310149i | 0.107443 - 0.0222711i | 0.094073 - 0.0193677i |
| 0.002 | 0.155686 - 0.031394i | 0.114733 - 0.0223486i | 0.101447 - 0.0194209i |
| 0.003 | 0.163248 - 0.031927i | 0.122536 - 0.0223911i | 0.109415 - 0.0193456i |
| 0.004 | 0.171186 - 0.031677i | 0.130872 - 0.0221666i | 0.118004 - 0.0191933i |
| 0.005 | 0.179511 - 0.031563i | 0.139760 - 0.0218721i | 0.127240 - 0.0182739i |

The frequency $\omega$ is a root of the continued fraction equation

$$
\left[ \beta_m - \frac{\alpha_m \gamma_m - \cdots \alpha_0 \gamma_0}{\beta_{m-1} - \beta_{m-2} - \cdots - \beta_0} \right] = \left[ \frac{\alpha_m \gamma_{m+1} - \cdots \alpha_0 \gamma_{m+1}}{\beta_{m+1} - \beta_{m+2} - \cdots} \right], \quad (m = 1, 2, \ldots). \tag{16}
$$

Solving the above continued fraction equation numerically, we can obtain the QNMs for the charged squashed KK black hole in Gödel universe.

III. THE BEHAVIOR OF QNMS FOR THE CHARGED SQUASHED KK BLACK HOLE IN GÖDEL UNIVERSE

![Graphs showing the behavior of Real(\(\omega\)) and Imag(\(\omega\)) with change in \(j\) for chosen \(\rho_0\). The solid line is for \(\rho_0 = 1\), the dotted line for \(\rho_0 = 5\) and the dashed line for \(\rho_0 = 10\). In plotting the figure, we have chosen \(\lambda = 1\), \((l, n) = (1, 0)\).]

We now report our numerical results of QNMs for the charged squashed KK black hole in Gödel universe. The fundamental QNMs will be our primary interest, since fundamental modes dominate in the late time
oscillations. We generalized our previous numerical program used in [10] by including small Gödel parameter $j$. When $j = 0$, it was shown in [10] that our numerical calculation is reliable and precise by comparing with results in [23].

In Fig.(1) we display the dependence of quasinormal frequencies of scalar perturbations on the Gödel parameter $j$ in the regime when $j$ is small. We found that for chosen values of $(l, n)$, $\lambda = 1$, $a$ and $\rho_0$, the real part of quasinormal frequency increases with the increase of $j$. Our numerical results are for small values of $\rho_0$, which means that the squashed effect is strong in our result. The behavior of the oscillation frequency is different from that found in the Schwarzschild black hole in the Gödel universe [21], where it was observed that the oscillation frequency linearly decreases with $j$. Considering that the quasinormal spectrum is sensitive to boundary conditions at the event horizon and at spatial infinity and in our case the asymptotic structures near the event horizon and the spatial infinity are different from those in the background of [21], it is not strange to find different behaviors of the real part frequencies in these two different backgrounds. The spectrum we observed is considerably affected by the squashed effect.

The behavior of the imaginary part of the quasinormal frequency on the Gödel scale parameter $j$ has also been shown in Fig. (1). For not very small $\rho_0$, we observed that the absolute values of the imaginary frequency monotonically decrease with the increase of $j$. This tells us that when the universe has more rotation, it is more difficult for the perturbation around the black hole to calm down. The scalar field damps more slowly for bigger values of $j$. The dependence of damping rate on the Gödel parameter $j$ agrees with that found in [21]. In the limit of small $j$ the QNMs govern the decay of the scalar field at late times. For severely squashed case with very small $\rho_0$, we found that with the increase of $j$, the absolute value of the imaginary frequency first increases and then decreases. More detailed numerical analysis on this phenomenon has been carried out and the results are shown in table I for $a = 0.3$. It is clear to see that with the increase of $\rho_0$, the maximum value of the absolute imaginary frequency appears for smaller $j$ and when $\rho_0 \geq 300$, the imaginary frequency changes monotonically with the increase of $j$. For bigger $a$, the critical value to obtain the monotonic behavior of the imaginary frequency becomes smaller. This shows that when the squashed effect is considerable, the perturbation around the black hole will even decay faster in the slower rotating universe. This can be attributed to the fact that near the event horizon the black hole is severely squashed and the asymptotic structure there has been drastically changed. However with the increase of $j$, the squashed effect seems diluted and the damping time increases again with $j$.

In Fig. (2) we show the dependence of quasinormal frequencies of scalar perturbations around the charged
FIG. 2: The behaviors of $Re(\omega)$ and $Im(\omega)$ with the change of $\rho_0$ for chosen $j$. The solid line is for $j = 0.01$, the dashed line for $j = 0.02$, dashed-dotted line for $j = 0.03$, short dotted line for $j = 0.04$ and the left line is for $j = 0.05$.

squashed KK black holes in G"odel universe on the parameter $\rho_0$ for different chosen values of $j$. For fixed parameters of $(l, n), \lambda, a$ and chosen $j$, we see that the real part of the quasinormal frequency decreases with the increase of $\rho_0$, which is consistent with the result in the non-rotating squashed KK black holes when $j = 0$ observed in [10, 23].

When $\lambda \neq 0$, we see from Fig. (2) that the absolute imaginary part of the quasinormal frequency first increases and then decreases with the increase of $\rho_0$. This result holds for all selected small $j$ and consistent with the findings in the gravitational QNMs in [23] when $K = 1, 2$ and $j = 0$ there. For bigger values of $j$, we observed deeper U-turn in the absolute imaginary frequency and it is easier for $|\omega_I| \to 0$ if the black hole is not severely squashed. This is consistent with that shown in Fig. (1), telling us that the damping time of the perturbation is longer if the universe rotates more. We have also investigated the imaginary part of the quasinormal frequency when $\lambda = 0$, the result is shown in Fig. (3). Comparing with the case when $\lambda \neq 0$, we found that there is no turning point in $|\omega_I|$ and $|\omega_I|$ decreases monotonically with the increase of $\rho_0$. This result holds for all our selected values of $j$ and when $j = 0$ it is consistent with the gravitational perturbation result when $K = 0$ in [23]. Remembering that $\lambda$ is the separation constant due to the fifth dimension, we learnt that the turning point in $|\omega_I|$ is brought by the extra dimensional effect. When $j = 0$, $|\omega_I|$ approaches to a constant nonzero value which reduces to the result obtained in [10, 23].

In addition to the fundamental modes, we have also calculated the contribution of the overtones. The numerical behavior of the first few overtones of the charged squashed KK black hole in G"odel universe is
FIG. 3: Graphs of $Im(\omega)$ with the change of $\rho_0$ for different $\lambda$. The solid line is for $\lambda = 0$, the dashed line is for $\lambda = 0.5$ and the dotted line for $\lambda = 1$.

shown in Fig. 4. The trajectories described by the modes in the complex-$\omega$ plan exhibit the spiral-like behavior when $\rho_+$ approaches to $\rho_-$. For $j = 0$, this behavior was observed in asymptotically flat black holes in $[0, 25]$ and also in the squashed KK black holes in $[10]$. We observed that such a spiralling behavior sets in for larger overtone number $n_c$ as the decrease of $\rho_0$ for selected $j$. For $j = 0$, this behavior was obtained in $[10]$. For fixed $\rho_0$, we found that with the increase of $j$, the spiralling behavior sets in for smaller values of the overtone $n_c$.

IV. CONCLUSION AND DISCUSSIONS

We have investigated the scalar perturbation in the background of charged squashed KK black holes immersed in a rotating cosmological background. In the limit of small Gödel parameter $j$, the quasinormal frequencies have been calculated. It was found that due to the squashed effect, the real part of QNM frequency increases when the cosmological background rotates. For the imaginary part of the quasinormal frequency, the strong squashed effect will cause $|\omega_I|$ to increase with $j$ when the universe does not rotate so fast, however when $j$ gets bigger, $|\omega_I|$ will finally decrease with $j$. In the small Gödel parameter limit, all found modes are damping which shows the stability of the charged squashed black hole in the Gödel universe against scalar perturbation. However we saw the tendency that with the increase of $j$, especially for bigger $\rho_0$, the $|\omega_I|$
tends zero, which tells us that the damping time becomes longer and longer. This could imply that there is a danger to have unstable charged squashed KK black hole in the Gödel cosmological background for big Gödel parameter \( j \). This needs careful examination in the future. For big \( j \) we have problems in separating radial equations. On the other hand, remembering that the stability of the spacetime is mainly determined by the gravitational perturbation, it is of great interest to generalize the study here to the gravitational perturbations in the future.

In the homogeneous isotropic universe with \( j = 0 \), it was observed that the complex \( \omega \) plan starts to exhibit the spiral-like shape when the black hole parameter reaches a critical value and the nontrivial relation between the dynamical and thermodynamical properties of black holes were claimed existing in the loop behavior in the complex \( \omega \) plan[8, 9]. In the Gödel universe with small \( j \), although the spiral-like behavior in the complex \( \omega \) has also been observed, it is not clear at this moment whether it can tell us some relation between the thermodynamical and dynamical properties. For bigger \( j \) in our limit, the computation time becomes longer and especially when two black hole horizons come closer. Careful examinations of the dynamical properties through the QNMs and comparisons with the thermodynamical properties in [20] are called for for the Gödel universe.

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