Linear Nonbinary Covering Codes and Saturating Sets in Projective Spaces

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Abstract—Let $A_{R,q}$ denote a family of covering codes, in which the covering radius $R$ and the size $q$ of the underlying Galois field are fixed, while the code length tends to infinity. The construction of families with small asymptotic covering densities is a classical problem in the area of Covering Codes.

In this paper, infinite sets of families $A_{R,q}$, where $R$ is fixed but $q$ ranges over an infinite set of prime powers are considered, and the dependence on $q$ of the asymptotic covering densities of $A_{R,q}$ is investigated. It turns out that for the upper limit $\mu_q^*(R, A_{R,q})$ of the covering density of $A_{R,q}$, the best possibility is

$$\mu_q^*(R, A_{R,q}) = O(q).$$

The main achievement of the present paper is the construction of optimal infinite sets of families $A_{R,q}$, that is, sets of families such that (1) holds, for any covering radius $R \geq 2$.

We first showed that for a given $R$, to obtain optimal infinite sets of families it is enough to construct $R$ finite families $A_{R,q}^{(0)}, A_{R,q}^{(1)}, \ldots, A_{R,q}^{(R-1)}$ such that, for all $u \geq u_0$, the family $A_{R,q}^{(u)}$ contains codes of codimension $f_u(\gamma, (R, q)) = O(q^{\alpha_0})$ and $u_0$ is a constant. Then, we were able to construct the needed families $A_{R,q}^{(u)}$ for any covering radius $R \geq 2$, with $q$ ranging over the (infinite) set of $R$-th powers. A result of independent interest is that in each of these families $A_{R,q}^{(u)}$, the lower limit of the covering density is bounded from above by a constant independent of $q$.

The key tool in our investigation is the design of new small saturating sets in projective spaces over finite fields, which are used as the starting point for the $q^n$-concatenating constructions of covering codes. A new concept of $N$-fold strong blocking set is introduced. As a result of our investigation, many new asymptotic and finite upper bounds on the length function of covering codes and on the smallest sizes of saturating sets, are also obtained. Updated tables for these upper bounds are provided. An analysis and a survey of the known results are presented.

Index Terms—Linear covering codes, nonbinary codes, saturating sets in projective spaces, covering density

I. INTRODUCTION

Let $F_q$ be the Galois field with $q$ elements. Let $F_q^n$ be the $n$-dimensional vector space over $F_q$. Denote by $[n, n-r]_q$ a $q$-ary linear code of length $n$ and codimension (redundancy) $r$, that is, a subspace of $F_q^n$ of dimension $n-r$. For an introduction to coding theory, see [1], [2].

The Hamming distance $d(v, c)$ of vectors $v$ and $c$ in $F_q^n$ is the number of positions in which $v$ and $c$ differ. The smallest Hamming distance between distinct code vectors is called the minimum distance of the code. An $[n, n-r]_q$ code with minimum distance $\delta$ is denoted as an $[n, n-r, \delta]_q$ code. The sphere of radius $R$ with center $c$ in $F_q^n$ is the set $\{v : v \in F_q^n, d(v, c) \leq R\}$.

Definition 1.1: i) The covering radius of an $[n, n-r]_q$ code is the least integer $R$ such that the space $F_q^n$ is covered by spheres of radius $R$ centered on codewords.

ii) A linear $[n, n-r]_q$ code has covering radius $R$ if every column of $F_q^n$ is equal to a linear combination of $R$ columns of a parity check matrix of the code, and $R$ is the smallest value with such property.

Definition 1.1 makes sense for both linear and nonlinear codes. For linear codes Definitions 1.1 i) and ii) are equivalent. An $[n, n-r]_q R$ code $(n, n-r, d]_q R$ code, resp.) is an $[n, n-r]_q$ code $(n, n-r, d]_q R$ code, resp.) with covering radius $R$. For an introduction to coverings of vector Hamming spaces over finite fields, see [3]–[6].

The covering problem for codes is that of finding codes with small covering radius with respect to their lengths and dimensions. Codes investigated from the point of view of the covering problem are usually called covering codes (in contrast to error-correcting codes) [6].

Problems connected with covering codes are considered in numerous works, see e.g. [7] – [41] and the references therein, the references in [3]–[6], and the online bibliography of [42]. In this work we mainly give references to researches on nonbinary codes; some papers on binary codes are also mentioned as they contain useful general ideas. It should be noted that the monographs [3], [4] mostly deal with binary covering codes, and that no surveys of nonbinary covering codes have been recently published. In this work we try to make up for this deficiency for linear codes; in particular, for infinite linear code families. We obtain a number of new asymptotic optimal results, essentially improving the known estimates for both finite and infinitely growing code lengths. The description of new results is provided, along with a survey of the known ones and their updates.

Studying covering codes is a classical combinatorial task. Covering codes are connected with many areas of information theory and combinatorics, see, e.g., [3, Sec. 1.2] where problems of data compression, decoding errors and erasures, foot-
Clearly, by repeating the process it is possible to obtain an $[n + \delta, n + \delta - r]_{q}R$ code from $C$ for any integer $\delta \geq 1$. We will call such a code a $\delta$-extension of $C$.

For a given $R \geq 1$ and for a fixed prime power $q$, let $A_{R,q}$ denote an infinite sequence of $q$-ary linear $[n, n - r_{n}]_{q}R$ codes $C_{n}$, $n \geq R$, with fixed covering radius $R$. An infinite sequence $A_{R,q}$ of covering codes is called an infinite family of covering codes or an infinite code family, or simply infinite family.

For infinite families $A_{R,q}$ we consider asymptotic covering densities

$$
\Pi_{q}(R, A_{R,q}) = \liminf_{i \to \infty} \mu_{q}(n_{i}, R, C_{n_{i}}).
$$

(1.3)

$$
\mu_{q}^{*}(R, A_{R,q}) = \limsup_{i \to \infty} \mu_{q}(n_{i}, R, C_{n_{i}}).
$$

(1.4)

We will write $\Pi_{q}(R)\ (\mu_{q}^{*}(R)$ resp.) for $\Pi_{q}(R, A_{R,q})\ (\mu_{q}^{*}(R, A_{R,q})$ resp.) if the family $A_{R,q}$ is clear from the context.

For an infinite family $A_{R,q}$, the sequence of codimensions $r_{n}$ will be assumed to be non-decreasing. In fact, if $r_{n+1} < r_{n}$, for some $n$, then any $1$-extension $C^{*}$ of $C_{n}$ has a better covering density than $C_{n+1}$, and therefore it is convenient to replace $C_{n+1}$ with $C^{*}$.

A code $C_{n}$ will be called a supporting code of $A_{R,q}$ if $r_{n} > r_{n-1}$, a filling code otherwise. It is immediately seen that a filling code must have the same parameters of a $\delta$-extension of some supporting code, and this motivates our notation. The subsequence of supporting codes will be denoted as $C_{n_{i}}$.

Throughout the paper, constructing an infinite family, we will only describe supporting codes, whereas the filling codes will be assumed to be obtained via $\delta$-extension. The words “to construct a family” will mean “to construct the supporting codes of a family”.

In this work we will mainly deal with infinite families $A_{R,q}$ for which the lengths and the codimension of the supporting codes $C_{n_{i}}$ are linked by some function, namely $n_{i} = f_{q}(r_{i})$ where $f_{q}$ is an increasing function for a fixed $q$. In most cases, an explicit expression for the function $f_{q}$ will be given.

By (1.2), the covering density of an $[n + 1, n + 1 - r]_{q}R$ code is greater than that of an $[n, n - r]_{q}R$ code. Therefore,

$$
\Pi_{q}(R, A_{R,q}) = \liminf_{i \to \infty} \mu_{q}(n_{i}, R, C_{n_{i}}).
$$

(1.5)

$$
\mu_{q}^{*}(R, A_{R,q}) = \limsup_{i \to \infty} \mu_{q}(n_{i+1} - 1, R, C_{n_{i+1} - 1}).
$$

(1.6)

Note that by (1.5), (1.6), the lower limit of the asymptotic covering density depends only on the supporting codes, while the upper limit depends on filling codes.

The size $q$ of the base field $\mathbb{F}_{q}$ is fixed for a given family $A_{R,q}$. But, it is natural to consider an infinite set of families $A_{R,q}$ with fixed $R$ and infinitely growing $q$. In most constructions, $f_{q}(r)$ is an increasing function of $q$ for a fixed $r$. Therefore, a central problem for linear covering codes is the following:

For a fixed covering radius $R$, find a set of families $A_{R,q}$ of $q$-ary codes with $q$ running over an infinite set of prime power, such that the covering densities (1.3) and (1.4) are asymptotically as small as possible with respect to the size of the base field $q$.

This problem has distinct perspectives and solutions for lower and upper limits.
As to the lower limit (15), it can happen that the asymptotic covering density of a family $\mathcal{A}_{R,q}$ are bounded from above by a constant independent of $q$. In this case $\overline{\mu}(R, \mathcal{A}_{R,q}) = O(1)$ and the family $\mathcal{A}_{R,q}$ is said to be "good". Accordingly, an $[n, n-r]_q R$ covering code is called "short" if $n = O(q^{\frac{R}{R^2}})$.

By (1.2) and (1.3), a family $\mathcal{A}_{R,q}$ consisting of short codes is good. In this case, $f_q(r) = O(q^{\frac{R}{R^2}})$. A saturating set $K$ will be said to be "small" if the related covering code $\mathcal{C}_K$ is short.

A classical example is the direct sum [3] of $R$ copies of the $\left[\frac{q^j-1}{q-1}, \frac{q^j}{q-1}\right]$-1 perfect Hamming codes, which gives an infinite family $\mathcal{A}_{R,q}$ of $[n_i, n_i - r_i]_q R$ codes with parameters

$$A_{R,q} : n_i = R q^{i-1} - 1, r_i = R i, i = 1, 2, 3, \ldots.$$  

When the upper limit is considered, it is not possible to obtain an upper bound independent on $q$. This depends on the fact that

$$\mu_q(n_{i+1} - 1, R, C_{n_i+1}) = \frac{V_q(n_{i+1} - 1, R)}{q^{i+1}},$$

$$\mu_q(n_{i+1} - 1, R, C_{n_i+1}) = \frac{V_q(n_{i+1} - 1, R)}{q^{i+1}}.$$  

Since $r_{i+1} > r_i$, this implies that the optimal case is $\mu_q(R, A_{R,q}) = O(q)$. Then the following natural issue arises.

**Open Problem 1.** For any covering radius $R \geq 2$, construct an infinite code family $\mathcal{A}_{R,q}$ with $\mu_q(R, A_{R,q}) = O(q)$.

To solve Open Problem 1 it is convenient to proceed as follows. For any given integer $\gamma$ with $0 \leq \gamma \leq R - 1$, construct an infinite family $\mathcal{A}_{R,q}^{(\gamma)}$ such that its supporting codes are $[n_u, n_u - r_u]_q R$ codes with codimension $r_u = R u + \gamma$ and length $n_u = f_q^{(\gamma)}(r_u)$, where $u \geq u_0$ and a constant $u_0$ may depend on the family. Considering families of type $\mathcal{A}_{R,q}^{(\gamma)}$ is a standard method of investigation of linear covering codes, see [3], [17], [18], [24], [27–29], [35] and the references therein; families $\mathcal{A}_{R,q}^{(\gamma)}$ with distinct values of $\gamma$ often have distinct properties.

Assume that we have $R$ good infinite code families $\mathcal{A}_{R,q}^{(\gamma)}$, $\gamma = 0, 1, \ldots, R - 1$. Let us consider the infinite family $\hat{\mathcal{A}}_{R,q}$, whose supporting codes are the union of those of all the families $\mathcal{A}_{R,q}^{(\gamma)}$. The family $\hat{\mathcal{A}}_{R,q}$ contains an infinite sequence of $[n_j, n_j - j]_q R$ codes $\mathcal{C}_j$ with length $n_j = f_q^{(j)}(j)$, $\gamma_j = j \equiv j \pmod{R}$, where $j \geq j_0$ and $j_0$ is a constant depending on a constant $u_0$ of the starting families. Note that it may occur that $n_{v+1} \leq n_v$ for some $v$. In this case we replace the code $\mathcal{C}_v$ by an $[n_u + 1 - 1, n_u + 1 - v_0 q] R$ code that always can be obtained from $\mathcal{C}_{v+1}$ by removing a redundancy symbol and a suitable parity check. As a guaranty of correctness.

$$\mu_q(R, \hat{\mathcal{A}}_{R,q}) = \lim_{j \to \infty} \frac{V_q(n_{j+1} - 1, R)}{q^{j+1}} \frac{V_q(n_{j+1} - 1, R)}{q^{j+1}}.$$  

Since all families $\mathcal{A}_{R,q}^{(\gamma)}$ are good, we have

$$V_q(n_{j+1} - 1, R)/q^{j+1} = O(1).$$

Hence,

$$\hat{\mu}_q(R, \hat{\mathcal{A}}_{R,q}) = O(q).$$

So, to solve Open Problem 1 it is sufficient to find a solution to Open Problem 2.

**Open Problem 2.** For any covering radius $R \geq 2$, construct $R$ infinite code families $\mathcal{A}_{R,q}^{(0)}, \mathcal{A}_{R,q}^{(1)}, \ldots, \mathcal{A}_{R,q}^{(R-1)}$ such that for each $\gamma = 0, 1, \ldots, R - 1$ the supporting codes of $\mathcal{A}_{R,q}^{(\gamma)}$ are $[n_u, n_u - r_u]_q R$ codes with codimension $r_u = R u + \gamma$ and length $n_u = f_q^{(\gamma)}(r_u)$ with $f_q^{(\gamma)}(r) = O(q^{\frac{R}{R^2}})$ and $u \geq u_0$ where a constant $u_0$ may depend on the family.

On one hand, infinite families $\mathcal{A}_{R,q}^{(0)}$ are provided by example (17): for $R = 2, 3$, families $\mathcal{A}_{R,q}^{(0)}$ with better parameters are obtained in [18], [24], [29]. On the other hand, for $\gamma \geq 1$, code families $\mathcal{A}_{R,q}^{(\gamma)}$ with density $\overline{\mu}_q(R, \mathcal{A}_{R,q}^{(\gamma)}) = O(1)$ are only known for $R = 2, \gamma = 1, q = (q')^2$ [24], and $R = 3, \gamma = 1, q = (q')^3$ [35].

In this paper, Open Problem 2 (and Open Problem 1) is solved for an arbitrary covering radius $R \geq 2$ and $q = (q')^R$ where $q'$ is a power of prime.

Our main tools are the $q^m$-concatenating constructions of covering codes, and the connection between covering codes and saturating sets in projective spaces.

The $q^m$-concatenating constructions are proposed in [10] and are developed in [15], [16, Supplement], [17–20], [24], [27–30], [37], see also [3, Sec. 5.4] and [4]. These constructions are the fundamental instrument for obtaining infinite families of covering codes with a fixed radius. Using a starting code as a "seed", the $q^m$-concatenating constructions yield an infinite family of new codes with the same covering radius and with almost the same covering density. If the starting code is short then the new infinite family is good.

Linear codes arising from small saturating sets are a convenient choice for the starting codes of the $q^m$-concatenating constructions [18], [24], [29], [35], [37].

The achievements of the present paper are mainly a consequence of new constructions of small saturating sets, some of which rely on the concept of a multifold strong blocking set that is introduced in this work. We have also thoroughly analyzed and collected the known results on the upper bounds on the length function, in particular for the cases $R = 2, 3$. We have updated tables about the upper bounds and formulas for infinite code families. As a result of our previously mentioned constructions, many new upper bounds on the length function are obtained.

The paper is organized as follows. In Section II the $q^m$-concatenating constructions, used in this work, are recalled. In Section III new constructions of small $\varphi$-saturating sets, including those relying on the new concept of strong blocking sets, are described. Section IV contains updated tables about the upper bounds on $f_q(r, R)$ for $R = 2, 3$, $r = 3, 4, 5$. In Sections V–VII we consider codes with covering radii $R = 2, 3, R \geq 4$. Section VIII provides results for nonprime covering radius.

Some of the results from this work were briefly presented without proofs in [64], [65].
II. \( q^m \)-Concatenating Constructions Lengthening Covering Codes

In this section we describe the common ideas and the popular versions of the \( q^m \)-concatenating constructions. Other versions can be found in [3, Sec. 5.4], [4], [10], [15], [16, Supplement], [17]–[20], [29], [30], [35], [37]. Specific constructions for \( R = 2 \) are given in detail in [24].

Using a starting \([n_0, n_0 - r_0]_q R\) code of length \( n_0 \), the \( q^m \)-concatenating constructions yield an infinite family of \([n, n - (r_0 + Rm)]_q R\) codes with the same covering radius \( R \) and length \( n = q^mn_0 + N_m \), where \( m \) ranges over an infinite set of integers. Here \( N_m \leq R\theta_{m,q} \), where

\[
\theta_{m,q} = \frac{q^m - 1}{q - 1}.
\]

It should be noted that all \( q^m \)-concatenating constructions have the contribution \( q^m n_0 \) into \( n \); two of them may differ by the value of \( N_m \).

Throughout this paper, all matrices and columns are \( q \)-ary. An element of \( F_q^m \) written in a \( q \)-ary matrix denotes an \( m \)-dimensional column containing its coordinates with respect to a fixed basis of \( F_q^m \) over \( F_q \); vice versa, an \( m \)-dimensional vector can be viewed as an element of \( F_q^m \).

A. \((R, \ell)\)-partitions and \((R, \ell)\)-objects

**Definition 2.1:** Let \( H \) be a parity-check matrix of an \([n, n - r]_q R\) code \( V \) and let \( 0 \leq \ell \leq R \).

1. A partition of the column set of the matrix \( H \) into nonempty subsets is called an \((R, \ell)\)-partition if every column of \( F_q^r \) (including the zero column) is equal to a linear combination with nonzero coefficients of at least \( \ell \) and at most \( R \) columns of \( H \) belonging to distinct subsets. For an \((R, 0)\)-partition we can formally treat the zero column as the linear combination of 0 columns.

An \( R \)-partition is an \((R, \ell)\)-partition for some \( \ell \geq 0 \).

2. If \( H \) admits a \((R, \ell)\)-partition, the code \( V \) is called an \((R, \ell)\)-object and is denoted as \([n, n - r]_q R, \ell\) code or as \([n, n - r, d]_q R, \ell\) code, where \( d \) is the minimum distance of \( V \).

Clearly, the trivial partition of a parity-check matrix of an \([n, n - r]_q R, \ell\) code into \( n \) one-element subsets is an \((R, \ell)\)-partition.

Note that in Definition 2.1 it is not necessary that \( \ell \) is the greatest value with the properties considered. Any \((R, \ell)\)-partition with \( \ell > 0 \) is also an \((R, \ell_1)\)-partition with \( \ell_1 = 1, 0, \ell - 1 \).

**Lemma 2.2:** [10], [17], [18] An \([n, n - r, d]_q R \) code is an \([n, n - r]_q R, \ell\) code with \( \ell \geq 1 \) if and only if \( d \leq R \). If \( d > R \) the maximum possible value of \( \ell \) is zero.

A spherical \((R, \ell)\)-capsule with center \( c \) in \( F_q^n \) is the set \( \{ v : v \in F_q^n, 0 \leq \ell \leq d(v, c) \leq R \} \) (see [10]). It is easy to see that spherical \((R, \ell)\)-capsules centered at vectors of an \((R, \ell)\)-object cover the space \( F_q^n \).

B. Basic \( q^m \)-Concatenating Constructions

We give a basic \( q^m \)-concatenating construction QM based on ideas in [10], [17], [18], [20].

**Basic Construction QM.** Let \( H_0 = [h_1, h_2, \ldots, h_{n_0}]_q \), with \( h_j \in F_q^r \), be a parity check matrix of an \([n_0, n_0 - n_0]_q R, \ell_0\) starting code \( V_0 \). Assume that \( H_0 \) has a starting \((R, \ell_0)\)-partition \( P_0 \) into \( p_0 \) subsets. Let \( m \geq 1 \) be an integer parameter depending on \( p_0 \) and \( n_0 \). To each column \( h_j \) we associate an element \( \beta_j \in F_q^m \cup \{\ast\} \) so that \( \beta_i \neq \beta_j \) if columns \( h_i \) and \( h_j \) belong to distinct subsets of \( P_0 \). If \( h_i \) and \( h_j \) belong to the same subset we are free to assign either \( \beta_i = \beta_j \) or \( \beta_i \neq \beta_j \). We call \( \beta_j \) an indicator of column \( h_j \). Let \( B = \{\beta_1, \beta_2, \ldots, \beta_n\} \) be an indicator set. It is necessary that \(|B| \geq p_0 \). Also, let \( C \) be an \((r_0 + Rm) \times N_m \) matrix with \( N_m \leq (R - \ell_0)\theta_{m,q} \).

Finally, define \( V \) as the \([n, n - (r_0 + Rm)]_q R V \) code with \( n = q^m n_0 + N_m \) and the parity-check matrix of the form

\[
H_V = [C \ B_1 \ B_2 \ \ldots \ B_{n_0}]_q, \quad (2.1)
\]

\[
B_j = \begin{bmatrix}
\beta_j \xi_1 & \beta_j \xi_2 & \ldots & \beta_j \xi_m \\
\xi_1 & \xi_2 & \ldots & \xi_m \\
\vdots & \vdots & \ddots & \vdots \\
\beta_j^{R-1} \xi_1 & \beta_j^{R-1} \xi_2 & \ldots & \beta_j^{R-1} \xi_m
\end{bmatrix}
\]

if \( \beta_j \in F_q^m \),

\[
B_j = \begin{bmatrix}
\beta_j \xi_1 & \beta_j \xi_2 & \ldots & \beta_j \xi_m \\
0 & 0 \ldots 0 \\
\xi_1 & \xi_2 & \ldots & \xi_m
\end{bmatrix}
\]

if \( \beta_j = \ast \)

where \( \{\xi_1, \xi_2, \ldots, \xi_m\} = F_q^m \), \( \xi_1 = 0 \), \( \xi_2 = 1 \). Note that the submatrix \( C \) is not needed if \( \ell_0 = R \).

If \( m \) and \( B \) are carefully chosen, then the covering radius \( R V \) of the new code \( V \) is equal to the covering radius \( R \) of the starting code \( V_0 \). Examples are shown in Constructions QM1 - QM4 below.

We use the following notations:

- \( W_m \) is a parity-check matrix of the \([\theta_{m,q}, \theta_{m,q} - m]_q \)1 Hamming code;
- \( A_{R, m} \) is a parity-check matrix of an \([n', n' - R'm]_q R' \) code \( V_{R', m} \) (in most cases we will assume that either \( n' = \ell_q(R'm, R') \) or \( n' = \ell_q(R'm, R') \));
- \( 0_k \) is the zero matrix with \( k \) rows (the number of columns will be clear by context);
- \( \Sigma_{R', m} \) is the “direct sum” of \( R' \) matrices \( W_m \), i.e. an \( R'' m \times R'' \theta_{m,q} \) matrix of the form

\[
\Sigma_{R', m} = \begin{bmatrix}
W_m & 0_m & \cdots & 0_m \\
0_m & W_m & \cdots & 0_m \\
& & \ddots & \ddots \\
& & & 0_m & 0_m & \cdots & W_m
\end{bmatrix},
\]

(2.2)

Note that \( \Sigma_{R', m} \) is a parity-check matrix of an \([R'' m, q, R'' \theta_{m,q} - R'' m]_q R'' \) code, see the direct sum construction in Section [\( \triangledown \)].
Construction QM$_1$. Here $R \geq 2$, $\ell_0 = 0$, $q^m + 1 \geq p_0$, $B \subseteq F_{q^m} \cup \{\ast\}$,
\[ C = \begin{bmatrix} 0_{r_0} & 0_{r_0} & \Sigma_{R_{r,m}} \end{bmatrix}, \quad n = q^mn_0 + R\theta_{m,q}. \quad (2.3) \]

Construction QM$_2$. Here $R \geq 2$, $1 \leq \ell_0 < R$, $q^m \geq p_0$, $B \subseteq F_{q^m}$,
\[ C = \begin{bmatrix} 0_{r_0 + t_{0,m}} & \Sigma_{R_{r,m}} \end{bmatrix}, \quad n = q^mn_0 + (R - \ell_0)\theta_{m,q}. \quad (2.4) \]

Construction QM$_3$. Here $R \geq 2$, $\ell_0 = R$, $q^m + 1 \geq p_0$, $B \subseteq F_{q^m} \cup \{\ast\}$,
\[ C \text{ is absent, } n = q^mn_0. \quad (2.5) \]

Construction QM$_4$. Here $R \geq 2$, $\ell_0 = 0$, $q^m - 1 \geq p_0$, $B \subseteq F_{q^m}$,
\[ C = \begin{bmatrix} 0_{r_0} & 0_{r_0} & \Sigma_{[R/2],m} & 0_{[R/2],m} \end{bmatrix}, \quad n \leq q^mn_0 + \lceil R/[R/2] \rceil \theta_{m,q} + q([R/2] m, [R/2]). \quad (2.6) \]

C. $q^m$-Concatenating Constructions with a Complete Set of Indicators (CSI)

In these versions of the basic Construction QM we must use all elements of $F_{q^m}$ or $F_{q^m} \cup \{\ast\}$ as indicators $\beta_j$. To this end, perhaps, we should assign distinct indicators to columns from the same subset of an $R$-partition. As a result the size of the submatrix $C$ is reduced.

Construction QM$_5$. Here $R \geq 2$, $\ell_0 = 0$, $n_0 \geq q^m \geq p_0$, $B = F_{q^m}$,
\[ C = \begin{bmatrix} 0_{r_0 + t_{0,m}} & \Sigma_{R_{r,m}} \end{bmatrix}, \quad n = q^mn_0 + (R - 1)\theta_{m,q}. \quad (2.7) \]

Construction QM$_6$. Here $R \geq 3$, $\ell_0 = 0$, $n_0 \geq q^m + 1 \geq p_0$, $B = F_{q^m} \cup \{\ast\}$,
\[ C = \begin{bmatrix} 0_{r_0 + t_{0,m}} & \Sigma_{R_{r,m}} \end{bmatrix}, \quad n = q^mn_0 + (R - 1)\theta_{m,q}. \quad (2.8) \]

Construction QM$_7$. Here $R = 3$, $\ell_0 = 0$, $n_0 \geq q^m + 1 \geq p_0$, $B = F_{q^m} \cup \{\ast\}$, $q = 2^i$,
\[ C = \begin{bmatrix} 0_{r_0 + t_{0,m}} & W_m \end{bmatrix}, \quad n = q^mn_0 + \theta_{m,q}. \quad (2.9) \]

Construction QM$_8$. Here $R = 4$, $\ell_0 = 0$, $n_0 \geq q^m \geq p_0$, $B = F_{q^m}$; 3 does not divide $q^m - 1$,
\[ C = \begin{bmatrix} 0_{r_0 + t_{0,m}} & 0_{r_0 + t_{0,m}} & A_{2,m} & 0_{2,m} \\ W_m & 0_m \end{bmatrix}, \quad n \leq q^mn_0 + \ell_q(2m, 2) + \theta_{m,q}. \quad (2.10) \]

Other constructions CSI including these with $n_0 < q^m$ can be found in [18], [30].

D. Summary

Theorem 2.3: \cite{3}, \cite{10}, [18], [24], [35], [29] In all Constructions QM$_i$ the new code $V$ is an $[n, n-(r_0 + Rm), 3]_{qR, \ell}$ code with covering radius $R$ and $\ell \geq \ell_0$.

Corollary 2.4: It holds that
\[ \ell_q(r_0 + Rm, R) \leq q^m\ell_q(r_0, R) + \right\} \begin{array}{l} \ell_q(r_0 + Rm, R) \leq q^m\ell_q(r_0, R) + \ell_q(r_0, R) \\ \left[ R \right] \theta_{m,q} + \ell_q(\left[ R \right] m, \left[ R \right]) \end{array} \quad (2.11) \]

Proof: In Constructions QM$_1$ and QM$_2$ we put $n_0 = \ell_q(r_0, R)$ and then use the trivial $R$-partition. In the code $V_{R,m}$ with $R' = \left[ R \right]$ we put $n' = \ell_q(\left[ R \right] m, \left[ R \right])$. \hfill $\square$

Note that Constructions QM$_1$-QM$_4$ provide an infinite family of the new $[n, n-(r_0 + Rm)]_{qR}$ codes $V$ with growing codimension $r = r_0 + Rm$. In Constructions QM$_5$-QM$_8$ instead, the value of $m$ cannot assume arbitrarily large values. However, these construction can be used in an iterative process where the new codes are the starting ones for the following steps \cite{18}, [24]. As result we obtain an infinite code family, see, e.g., [24], Rem. 5. For this iterative process, it is important that in the new codes obtained by the $q^m$-concatenating constructions the value of $\ell$ is increasing and eventually reaches $R$, see \cite[Sec. IV]{18} and Examples in Section VI.

Remark 2.5: i) By (2.3)-(2.10), if the starting code $V_0$ is “good”, i.e. $n_0 = O(q^m)$, then, for the all new $[n, n-(r_0 + Rm)]_{qR}$ codes $V$, obtained by the $q^m$-concatenating constructions, the lower limit of the asymptotic covering density of the new family is somewhat greater than covering density of the starting code. However, it should be noted that the difference is not significant when the value of $R/q^m$ is small.

ii) By (2.3)-(2.10), if the starting code $V_0$ is “short”, i.e. $n_0 = O(q^m)$, then, for the all new $[n, n-(r_0 + Rm)]_{qR}$ codes $V$, obtained by the $q^m$-concatenating constructions, the lower limit of the asymptotic covering density of the new family is somewhat greater than covering density of the starting code. However, it should be noted that the difference is not significant when the value of $R/q^m$ is small.

ii) By (2.3)-(2.10), if the starting code $V_0$ is “short”, i.e. $n_0 = O(q^m)$, then, for the all new $[n, n-(r_0 + Rm)]_{qR}$ codes $V$, obtained by the $q^m$-concatenating constructions, the lower limit of the asymptotic covering density of the new family is somewhat greater than covering density of the starting code. However, it should be noted that the difference is not significant when the value of $R/q^m$ is small.

III. New “Small” Saturating Sets

A. Multifold Strong Blocking Sets

In a projective space a $t$-fold blocking set with respect to subspaces of some fixed dimension is a set of points that meets every such subspace in at least $t$ points. To describe new constructions of relatively small $\rho$ -saturating sets in spaces $PG(v, q)$ with $q = (q^p)^{+1}$ we introduce a new concept of $t$-fold strong blocking set.

Definition 3.1: Let $2 \leq t \leq v$. A pointset $B$ in a projective space $PG(v, q)$ is a $t$-fold strong blocking set if every $(t-1)$-dimensional subspace of $V$ is spanned by $t$ points in $B$.

Let $(x_0, x_1, \ldots, x_v)$, where $x_i \in F_q$, be homogenous coordinates for a point $P$ in $PG(v, q)$ and let $P_u = (x_0^u, x_1^u, \ldots, x_v^u)$. 


Theorem 3.2: Let $\rho$ be any positive integer. Let $q = (q')^{\rho+1}$. Let $v \geq \rho + 1$. Any $(\rho + 1)$-fold strong blocking set in a subgeometry $PG(v, q') \subset PG(v, q)$ is a $\rho$-saturating set in the space $PG(v, q)$.

Proof: Let $B$ be a $(\rho + 1)$-fold strong blocking set in $PG(v, q')$. Let $P$ be a point in $PG(v, q) \setminus B$. By definition of $(\rho + 1)$-fold strong blocking set we only need to show that there exists a $\rho$-dimensional subspace of $PG(v, q')$ passing through $P$. Consider the subspace $\Sigma(P)$ of $PG(v, q')$ generated by the point set $O(P) := \{P, P', P'q', P'q'^2, \ldots, P'q'^P\}$. As $\left(p'q'^v\right)' = p'^h = P$, the Frobenius collineation $X \rightarrow X'$ fixes $O(P)$. Therefore $\Sigma(P)$ is a subspace of $PG(v, q')$. Clearly $\Sigma(P)$ is contained in some $\rho$-dimensional subspace of $PG(v, q')$ (if the points in $O(P)$ are independent, then this subspace coincides with $\Sigma(P)$). As $P \in \Sigma(P)$, the assertion is proved.

B. Small $\rho$-Saturating Sets in Spaces $PG(\rho + 1, (q')^{\rho+1})$

Corollary 3.3: Let $q = (q')^2$. Any 2-fold blocking set in the subplane $PG(2, \sqrt{q}) \subset PG(2, q)$ is a 1-saturating set in the plane $PG(2, q)$.

Proof: As a line is spanned by any two its points, a 2-fold blocking set in a projective plane is always a 2-fold strong blocking set. Then we use Theorem 3.2.

Note that Corollary 3.3 is also given in [46].

Theorem 3.4: Let $q = (q')^4$. In $PG(2, q)$ there is a 1-saturating set of size $2\sqrt{q} + 2\sqrt{q} + 2$.

Proof: The union of two disjoint Baer subplanes in $PG(2, \sqrt{q})$ is a 2-fold blocking set [66]. Then we use Corollary 3.3.

In $PG(2, q)$, $q$ not a square, 2-fold blocking sets of size $b \leq 3q - 2$ are not known in the literature [66], [67]. We give here some results for $q = p^3$, $p$ prime.

Theorem 3.5: Let $q = p^3$, $p$ prime, $p \leq 73$. Then in $PG(2, q)$ there is a 2-fold blocking set of size $2\left(q + \sqrt{q^2} + \sqrt{q} + 1\right)$.

Proof: By [50, Lem. 13.8 (iii)], the point set

$B = \{(1, x, x^2) \mid x \in F_q \} \cup \{(0, 1, m) \mid m \in F_q, m^{p^3+1} = 1\}$

is a 1-blocking set in $PG(2, q)$ of size $q + p^3 + 1$. We are looking for a projectivity $\gamma$ for which $B \cap \gamma(B) = \emptyset$ holds. Then $B \cup \gamma(B)$ is a 2-fold blocking set in $PG(2, q)$.

Let $H$ be the multiplicative subgroup of $F_q^*$ consisting of the $(p+1)$th powers in $F_q$ (equivalently, $H = \{y \in F_q \mid y^{p+1} = 1\}$). For $a, b \in F_q^*$, $b \notin H$, we consider the projectivity $\gamma_{a,b}(r, s, t) = (r - t, a r, s b)$. Obviously, $\gamma_{a,b}(0, 1, m) = (0, 0, a) = (1, 0, m) \notin B$. Also, $\gamma_{a,b}(1, 1, 1) = (0, a b, a) = (0, b, 1) \notin B$ as $b^{p^3+1} = 1$.

Finally, for $x \neq 1$, $\gamma_{a,b}(1, x, x^2) = (1, a b/(x^p - 1), a x/(x^p - 1)) \in B$ if and only if $a b^{p-1} b^p = (x^p - 1)^{p-1} x$.

So, $B \cap \gamma_{a,b}(B) = \emptyset$ if and only if the equation $a b^{p-1} b^p = (x^p - 1)^{p-1} x$ has no solution in $F_q$.

Note, that any element $c \in F_q^* \setminus H$ can be expressed as a product $a b^{p-1} b^p$ with $a, b \in F_q^*$, $b \notin H$. In fact, $c$ belongs to some coset $d H$, $d \notin H$, and therefore $c = d a^{p-1}$ for some $a \in F_q^*$. Let $b = d b^p \notin H$, so that $b^p = d$.

Then the following claim is proved: if there is an element $c \in F_q^* \setminus H$ such that $c \neq (x^p - 1)^{p-1} x$ for any $x \in F_q$, then there exist $a, b$ such that $B \cap \gamma_{a,b}(B) = \emptyset$.

The existence of such element $c$ has been tested by computer for every prime $p \leq 73$.

Corollary 3.6: Let $q = (q')^6$, $q'$ prime, $q' \leq 73$. In $PG(2, q)$ there is a 1-saturating set of size $2\sqrt{q} + 2\sqrt{q} + 2\sqrt{q} + 2$.

Note that the smallest previously known 1-saturating sets in $PG(2, q)$, $q = (q')^2$, have size $3\sqrt{q} - 1$ [18, Th. 5.2], cf. Theorem 3.3 and Corollary 3.6.

Now we construct a 3-fold strong blocking set in $PG(3, q)$. Let $l_1, l_2, l_3$ be the lines with the following equations:

$l_1 : x_0 = x_2 = 0$; $l_2 : x_1 = x_3 = 0$; $l_3 : x_0 = x_3, \quad x_1 = x_2$.

These lines are pairwise skew, and are all contained in the hyperbolic quadric $Q : x_0 x_1 = x_2 x_3$. Let $g$ be any line disjoint from $Q$, and let

$B = l_1 \cup l_2 \cup l_3 \cup g$.

(3.1)

A possible choice for $g$ is the following:

$g : \begin{cases} x_0 = x_1, x_2 = k x_3, \quad k \text{ non-square in } F_q, & \text{if } q \text{ odd} \\ x_0 = x_1 + x_3, x_2 = k x_3,  \\ T^2 + T + k \text{ irreducible over } F_q, & \text{if } q \text{ even}. \end{cases}$

Theorem 3.7: The set $B$ of (3.1) has size $4q + 4$ and it is a 3-fold strong blocking set in $PG(3, q)$.

Proof: We need to show that any plane $\pi$ of $PG(3, q)$ meets $B$ in three non collinear points. If one of lines of $B$ lies on $\pi$, then the assertion is trivial. Let $l_1 = \pi \cap l_i$. Assume that $P_1, P_2, P_3$ are collinear. Then the line $l$ through $P_1, P_2$ and $P_3$ is contained in $Q$, by the “three then all” principle for quadrics in projective spaces. As $R = \pi \cap Q \notin Q$, we have that $R$ is not collinear with $P_1$ and $P_2$.

Remark 3.8: Any 3-fold strong blocking set $B$ in $PG(3, q)$ has at least $3q + 3$ points. Let $l$ be any line such that $l \cap B = 0$. Then each of the $q + 1$ planes in the pencil through $l$ must contain three points of $B$.

Corollary 3.9: Let $q = (q')^3$. In $PG(3, q)$ there is a 2-saturating set of size $4q^3 + 4$ consisting of four pairwise skew lines of $PG(3, q') \subset PG(3, q)$.

Proof: We use Theorems 3.2 and 3.7.

We now give an inductive construction of $\nu$-fold strong blocking sets in $PG(v, q)$.

Construction A. Let $H \cong PG(v, q)$ be a hyperplane in $PG(v + 1, q)$, and let $B \subset H$ be a $\nu$-fold strong blocking set in $H$. Let $P_1, P_2, \ldots, P_{v+1}$ be $v + 1$ independent points in $H$, and let $l_1, \ldots, l_{v+1}$ be concurrent lines in $PG(v, q)$ such that $l_i \cap H = P_i$ for each $i$. Let

$B' = B \cup \bigcup_{i=1}^{v+1} (l_i \setminus \{P_i\})$.

(3.2)

Theorem 3.10: Let $B$ be a $\nu$-fold strong blocking set in $PG(v, q)$ of size $k$. Then the set $B'$ of Construction A is a $(\nu + 1)$-fold strong blocking set in $PG(v + 1, q)$ of size $k + 1 + (v + 1)(q - 1)$. 
Proof: Let $H$ be the hyperplane in $\text{PG}(v, q)$ as in Construction A. Let $H_1$ be any hyperplane in $\text{PG}(v, q)$. We need to show that $H_1$ is generated by $v + 1$ points in $B^*$. When $H = H_1$, this follows from the fact that $B$ must contain $v + 1$ independent points. Assume then that $H \neq H_1$, and let $\Sigma = H \cap H_1$. As $\Sigma$ is a hyperplane in $H$, there exist points $Q_1, \ldots, Q_\rho$ in $B$ which generate $\Sigma$. Note that $\Sigma$ does not pass through a point $P_i$ for some $i \in \{1, \ldots, v + 1\}$, as otherwise $\Sigma$ would coincide with $H$. Let $Q = H_1 \cap \sum_i$. As $Q \notin \Sigma$, and as $\Sigma$ is a hyperplane of $H_1$, we have that $H_1 = \langle \Sigma, Q \rangle = \langle Q_1, \ldots, Q_\rho, Q \rangle = \times 2$. This proves that $B^*$ is a $(v + 1)$-fold blocking set. The size of $B^*$ can be easily calculated from (3.2).

Corollary 3.11: In $\text{PG}(v, q)$, $v \geq 3$, there exists a $v$-fold strong blocking set of size
\[(q - 1)\left(\frac{v(v + 1)}{2} - 2\right) + v + 5.\] (3.3)

Proof: By Theorem 3.7 in $\text{PG}(3, q)$ there exists a 3-fold strong blocking set of size $4q + 4$. Then the assertion follows by Theorem 3.10 taking into account that $4q + 4 + 1 + 4(q - 1) + 1 + 5(q - 1) + \ldots + 1 + v(q - 1) = 4q + 4 + (v - 3) + (q - 1)(v + 1)/2 - 6$.

From Theorem 3.2 we deduce the following result.

Corollary 3.12: Let $q = (q')^{\rho+1}$, $\rho \geq 2$. Then there exists a $\rho$-saturating set in $\text{PG}(\rho + 1, q)$ of size
\[(\rho + 1)(\rho + 2) - 2\] (3.4)

Note that the smallest previously known $\rho$-saturating sets in $\text{PG}(\rho + 1, (q')^{\rho+1})$, $\rho \geq 2$, have size $n = \frac{1}{q}((\rho + 1)(\rho + 2) + \rho + 2)$ [58, Th. 6], e.g. $n = 6q' - 2$ for $\rho = 2$ and $n = 10q' - 5$ for $\rho = 3$; from (3.4) we obtain sizes $4q' + 4$ and $8q' + 1$, respectively.

Remark 3.13: The codes associated to the saturating sets of Corollaries 3.9 and 3.12 will be used as starting codes for $q^\ell$-concatenating constructions, see Sections VI and VII. Therefore, we need to treat such codes as $(R, \ell)$-objects with $\ell > 0$ and to obtain the corresponding $(R, \ell)$-partitions, see Definition 2.1. To this end, it is useful to represent some point $P_i$ of a line $l$ in $\text{PG}(v, q)$ as a linear combination with nonzero coefficients of $u$ other points of $l$. We compute some of the admissible values of $u$. Let $l = \{P_0, P_1, \ldots, P_{q-1}\}$. Without loss of generality we identify $l$ with the projective line $P(1, q)$, and assume that $P_0 = (0, 1), P_1 = (1, 0), P_i = (1, b_i)$, $i \geq 2$, where $b_2, \ldots, b_{q-1} = F_q$.

1. Clearly, for each $i = 0, 1, \ldots, \lfloor(q - 2)/2\rfloor$, the point $P_i$ can be written as $P_i = c_{2i+1}P_{2i+1} + c_{2i+2}P_{2i+2}$ and $c_{2i+1}, c_{2i+2} \in F_q$. So, $P_0 = c_1P_1 + c_2P_2 = c_1c_2P_3 + c_1c_2P_4 + c_2P_5 + c_2P_6$. Then $c_1 = c_2 = 0$.

2. Note that $P_1 = (1, 0) = -\sum_{i=2}^{q-1}(b_i)$, $\sum_{i=2}^{q-1}(b_i)$ is admissible.

3. Let $q \geq u \geq 3, q \geq 4$. Then, for any $d_i \in F_q^*$, one can always choose $a_0$ and $a_1$ in $F_q$ so that $a_0(0, 1) + a_1(1, 0) + \sum_{i=2}^{q-1}d_i(1, b_i) = a_2(2, b_2)$ with some $a_2 \in F_q^*$.

C. Small $\rho$-Saturating Sets in Spaces $\text{PG}(v, (q')^{\rho+1}), v = \rho + 2, \rho + 3, \ldots, 2\rho - 1$

Lemma 3.14: Fix $1 \leq k \leq v - 1$. Let $B_k$ be the subset of $\text{PG}(v, q)$ consisting of the points whose Hamming weight is at most $v - k + 1$, i.e., $B_k$ is the union of the $(v + 1)$-subspaces of equation $x_i = \ldots = x_k = 0$, where $0 \leq i_1 < i_2 < \ldots < i_k \leq v$. Then $B_k$ is a $(k + 1)$-strong blocking set.

Proof: Let $W$ be any $k$-dimensional subspace of $\text{PG}(v, q)$. Let $w_1, \ldots, w_{k+1}$ be $k + 1$ independent points of $W$. Consider the matrix
\[
A_W = \begin{bmatrix}
\vdots \\
- & w_1 & - & \cdots & - \\
\vdots \\
- & w_{k+1} & - & \cdots & - 
\end{bmatrix}
\]
whose rows are homogenous coordinates of points $w_1, \ldots, w_{k+1}$. As the rank of $A_W$ is equal to $k + 1$, there exists a non singular $(k + 1) \times (k + 1)$ matrix $M = (m_{ij})$ such that $MA_W$ contains a matrix $I_{k+1}$. Note that the rows of $MA_W$ are the coordinates of $(k + 1)$ points of $W$: more precisely the $i$th-row of $MA_W$ is $m_{i1}w_1 + m_{i2}w_2 + \ldots + m_{i(k+1)}w_{k+1}$. Clearly these points are independent, and they are contained in $B_k$ as $I_{k+1}$ is a submatrix of $MA_W$.

Note that in the previous lemma
\[
|B_k| = \frac{1}{q - 1} \sum_{i=1}^{v-k+1} (q - 1)^i (\frac{v + 1}{i}) = V_q(v + 1, v - k + 1) - 1, \quad (3.5)
\]
see (1.1). Therefore the order of magnitude of the size of $B_k$ is $(q^{\frac{v-k}{k}})^{\rho}$.

Theorem 3.15: Let $\rho$ be a positive integer. Let $q = (q')^{\rho+1}$ and $v > \rho + 1$. Then in $\text{PG}(v, q)$ there exists a $\rho$-saturating set of size
\[
\frac{V_q(v + 1, v - \rho + 1) - 1}{q^{\rho + 1}} \sim \left(\frac{v + 1}{\rho}ight)^{\frac{\rho}{q^{\rho+1}}} \quad (3.6)
\]

Proof: By Theorem 3.2, the set $B_\rho \subseteq \text{PG}(v, q')$, where $B_\rho$ is defined as in Lemma 3.14 is the desired $\rho$-saturating set.

For some values of $v$ and $\rho$, the coefficient $(\rho + 1)$ can be improved. We show that this is possible for $v = 4, \rho = 2$. Let $q = (q')^3$. Let $E_0 = (0, 0, 0, 0), E_1 = (0, 1, 0, 0), E_2 = (0, 0, 1, 0), E_3 = (0, 0, 0, 1)$ be points in $\text{PG}(4, q)$. For $i, j \in \{0, 1, 2, 3, 4\}$, $i < j$, let $\pi_{k, i,j}$ be the plane in $\text{PG}(4, q)$ generated by $E_k, E_i$ and $E_j$. Let $\pi_1 = \pi_{0, 1, 0}, \pi_2 = \pi_{0, 1, 3}, \pi_3 = \pi_{0, 1, 3}, \pi_4 = \pi_{0, 1, 4}, \pi_5 = \pi_{0, 1, 4}, \pi_6 = \pi_{1, 2, 3}, \pi_7 = \pi_{1, 2, 4}, \pi_8 = \pi_{1, 3, 4}, \pi_9 = \pi_{2, 3, 4}$. Let
\[
S = \left(\bigcup_{k=1}^{9} \pi_k\right) \cap \text{PG}(4, q') \quad (3.7)
\]
The union $S$ of the nine planes $\pi_k$ consists of all points of $\text{PG}(4, q')$, apart from those belonging to the following three disjoint classes: points with all non-zero coordinates; points
with precisely one zero coordinate; points \((x, 0, y, z, 0)\) with \(xyz \neq 0\). Therefore,

\[|S| = \theta_{b,q} - (q'-1)^2 - 5q' - 1)^2 - 9q' + 4. \]

**Theorem 3.16:** Let \(q = (q')^3\). The set \(S\) as in (3.7) has size \(9q'^2 - 8 \sqrt{q'} + 4\), and it is a 2-saturating set in \(PG(4, q).\)

**Proof:** Let \(P\) be a point in \(PG(4, q).\) Let \(\pi\) be any plane of \(PG(4, q')\) containing the subspace generated by \(P, P', P'(q')^2\). Clearly \(\pi\) passes through \(P\). Assume that \(\pi\) does not pass through \(E_0\). Then among the points in \(\{\pi \cap \pi_s \mid s = 1, \ldots, 5\}\) there are at least three non-collinear points of \(S\). Assume that \(\pi\) passes through \(E_0\). Let \(H_0\) be the hyperplane generated by \(E_1, \ldots, E_4\). Then \(\pi \cap H_0\) consists of a line \(\ell\). Obviously, \(\ell\) meets \(\cup_{i=0}^3 \pi_i\) in at least two non-collinear points. Then \(\pi\) passes through 3 non-collinear points in \(S\).

**Open problem:** Reduce the coefficient \((v+1)_p\) in (3.6), for generic \(v\) and \(p\).

### IV. TABLES OF UPPER BOUNDS ON THE LENGTH FUNCTION \(\ell_q(r, R)\) FOR SMALL \(r\) AND \(R\)

We give tables of the values of \(\tilde{\ell}_q(r, R)\), i.e., the smallest known lengths of a \(q\)-ary linear code with codimension \(r\) and covering radius \(R\). Obviously, \(\ell_q(r, R) \leq \tilde{\ell}_q(r, R)\) holds. The dot “.” appears in a table when \(\ell_q(r, R) = \tilde{\ell}_q(r, R)\) holds. Subscripts indicate the minimum distance \(d\) of the \(\tilde{\ell}_q(r, R) = \ell_q(r, R) - d_{\text{PG}} R\) codes. Multiple subscripts mean that the value of \(\tilde{\ell}_q(r, R)\) is provided by codes with distinct distances.

Table [I] gives values of \(\tilde{\ell}_q(3, 2)\). We used [60, Tabs 2.4], [35, Tab I], [68, Tab 3]. Theorem 3.3 and Corollary 3.6 give the relation \(\ell_q(r, R) \leq 3q' - 1\) [18, Th. 5.2], and computer search made in this work. Note that the distance \(d = 4\) occurs when the code arises from a complete arc in the plane \(PG(2, q)\).

From Table I the following result is obtained.

**Theorem 4.1:** For the length function \(\ell_q(3, 2)\),

\[\ell_q(3, 2) \leq a_q \sqrt{q}, \quad \text{with } a_q < 3 \text{ if } q \leq 109, \]

\[a_q < 3.5 \text{ if } q \leq 349, \quad a_q < 4 \text{ if } q \leq 1217. \quad (4.1)\]

In Table II we give a number of concrete sizes of 1-saturating sets and complete caps in \(PG(2, q)\), \(q = p^{2k+1}\), taken from [63, Tab 2], [69, Ap., Lem. 4.3], and [70, Tab 1]. These sizes are the values of \(\tilde{\ell}_q(3, 2)\).

Using [58, Tab 1], [35, Tabs II, III], [37, Tabs III-V], Theorem 3.7 and Corollary 3.12 we obtained Table III where values of \(\tilde{\ell}_q(4, 3)\) are listed. The distances \(d = 4\) and \(d = 5\) occur, respectively, when the code arises from an incomplete cap and a complete arc in \(PG(3, q)\) [35, 37].

From Table III we obtain the following theorem.

**Theorem 4.2:** For the length function \(\ell_q(4, 3)\),

\[\ell_q(4, 3) \leq b_q \sqrt{q}, \quad \text{with } b_q < 4 \text{ if } q \leq 83, \]

\[b_q < 4.5 \text{ if } q \leq 343, \quad b_q < 5 \text{ if } q \leq 563. \quad (4.2)\]

In Table IV the values of \(\tilde{\ell}_q(5, 3)\) are given. We use [58, Tab 1], [37, Tabs III-IV], for \(q \leq 7\) and the computer search made in this work for \(8 \leq q \leq 32\). For \(37 \leq q \leq 43\), we apply the direct sum (see Section V) of the \(\tilde{\ell}_q(3, 2), \tilde{\ell}_q(3, 2) - 3q^2\) code of Table I and the \([q + 1, q - 1]_q\) Hamming code. The distances \(d = 4\) and \(d = 6\) occur, respectively, when the code arises from an incomplete cap and an arc in \(PG(4, q)\).

From Table IV we obtain the following theorem.

**Theorem 4.3:** For the length function \(\ell_q(5, 3)\),

\[\ell_q(5, 3) \leq c_q \sqrt{q}, \quad \text{with } c_q < 4 \text{ if } q \leq 27, \]

\[c_q < 4.2 \text{ if } q \leq 32, \quad c_q < 5 \text{ if } q \leq 43. \]

### V. CODES WITH COVERING RADIUS \(R = 2\)

**A. Direct sum and doubling constructions**

The direct sum construction (DS) forms an \([n_1 + n_2, n_1 + n_2 - (r_1 + r_2)]_{q_{1}}R\) code \(V\) with \(R = R_1 + R_2\) from two codes: an \([n_1, n_1 - r_1]_{q_{1}}R\) code \(V_1\) and an \([n_2, n_2 - r_2]_{q_{2}}R\) code \(V_2\) [6], [3], [4]. The parity-check matrix \(H\) of the new code \(V\) has the form

\[H = \begin{bmatrix} H_1 & 0_{r_1} & H_2 \end{bmatrix}, \quad q = 3, \quad (5.2)\]

where \(H_1\) and \(H_2\) are parity-check matrices of the starting codes \(V_1\) and \(V_2\), respectively. Construction DS is denoted by \(\oplus\), i.e., \(V_1 \oplus V_2 = V\) or

\[[n_1, n_1 - r_1]_{q_{1}}R_1 \oplus [n_2, n_2 - r_2]_{q_{2}}R_2 = [n_1 + n_2, n_1 + n_2 - (r_1 + r_2)]_{q_{1}}R_1 + R_2. \quad (5.1)\]

DS construction yields that

\[\ell_q(r_1 + r_2, R_1 + R_2) \leq \ell_q(r_1, R_1) + \ell_q(r_2, R_2).\]

In [19] Construction CP1 (“codimension plus one”) is proposed. The construction is similar to the construction in [13]. From an \([n, n-r]_q\) code \(V_1\) Construction CP1 forms an \([f_q(n), f_q(n) - (r + 1)]_q\) code \(V\) where \(f_q(n) = 2n, f_q(n) = 3n - 1, f_q(n) = 3n\). For \(q = 3\) Construction CP1 is a doubling construction. In this case the parity-check matrix \(H\) of the new code \(V\) has the form

\[H = \begin{bmatrix} 0 & 1 & H_1 \end{bmatrix}, \quad q = 3, \quad (5.2)\]

where \(0\) and \(1\) is the row of all zeroes and units, respectively, and \(H_1\) is a parity-check matrix of the starting code \(V_1\). By \(5.2\), see also [23],

\[\ell_3(r + 1, 2) \leq 2\ell_3(r, 2). \quad (5.3)\]

**B. Infinite Code Families of Even Codimension \(r = 2t\)**

Let \(q = 3\). By applying the doubling construction of (5.2) to the codes of [21, Th. 1], [27, Th. 4] and by using the codes of [27, Th. 11] we obtain an infinite family of \([n, n - r]_q\) codes with the following parameters

\[A_{2,3}^{(0)}: R = 2, \quad r = 2t \geq 4, \quad q = 3, \quad r \neq 8, \quad \pi_3(2) \approx \frac{25}{18}, \]

\[n = \begin{cases} \frac{r}{2} \cdot 3^{r-1} - 1, & \text{if } r = 4c + 2 \\frac{r}{2} \cdot 3^{r-1} - \frac{1}{2}, & \text{if } r = 8c + 4. \end{cases} \quad (5.4)\]

For \(r = 4\), from (5.4) we obtain an \([8, 4]_q\) code. Note that by [22, Tab II], \(\ell_3(4, 2) = 8\) holds.
### TABLE I
Upper Bounds $\ell_q = \ell_q(3, 2)$ on the Length Function $\ell_q(3, 2)$

| $q$ | $\ell_q$ | $q$ | $\ell_q$ | $q$ | $\ell_q$ | $q$ | $\ell_q$ | $q$ | $\ell_q$ | $q$ | $\ell_q$ | $q$ | $\ell_q$ |
|-----|----------|-----|----------|-----|----------|-----|----------|-----|----------|-----|----------|-----|----------|
| 3   | 4k       | 4   | 3k        | 5   | 2k+1     | 6   | 5k+1     | 7   | 7k+2     | 8   | 7k+2     | 9   | 7k+2     |
| 3k  | 4k       | 4k  | 3k        | 5k  | 2k+1     | 6k  | 5k+1     | 7k  | 7k+2     | 8k  | 7k+2     | 9k  | 7k+2     |

### TABLE II
Upper Bounds $\ell_q = \ell_q(3, 2)$ on the Length Function $\ell_q(3, 2)$ for $q = \rho^{2k+1}$

| $q$ | $\ell_q$ | $q$ | $\ell_q$ | $q$ | $\ell_q$ | $q$ | $\ell_q$ | $q$ | $\ell_q$ | $q$ | $\ell_q$ |
|-----|----------|-----|----------|-----|----------|-----|----------|-----|----------|-----|----------|
| 2k  | 2k+1     | 3   | 3k+1     | 5   | 2k+1     | 6   | 5k+1     | 7   | 7k+2     | 8   | 7k+2     | 9   | 7k+2     |
| 2k+1| 2k+1     | 3k  | 3k+1     | 5k  | 2k+1     | 6k  | 5k+1     | 7k  | 7k+2     | 8k  | 7k+2     | 9k  | 7k+2     |

### TABLE III
Upper Bounds $\ell_q = \ell_q(4, 3)$ on the Length Function $\ell_q(4, 3)$

| $q$ | $\ell_q$ | $q$ | $\ell_q$ | $q$ | $\ell_q$ | $q$ | $\ell_q$ | $q$ | $\ell_q$ | $q$ | $\ell_q$ |
|-----|----------|-----|----------|-----|----------|-----|----------|-----|----------|-----|----------|
| 2   | 3k+1     | 3   | 5k+1     | 5   | 7k+2     | 7   | 7k+2     | 8   | 7k+2     | 9   | 7k+2     |
| 3   | 5k+1     | 3k  | 5k+1     | 5k  | 7k+2     | 7k  | 7k+2     | 8k  | 7k+2     | 9k  | 7k+2     |

### TABLE IV
Upper Bounds $\ell_q = \ell_q(5, 3)$ on the Length Function $\ell_q(5, 3)$

| $q$ | $\ell_q$ | $q$ | $\ell_q$ | $q$ | $\ell_q$ | $q$ | $\ell_q$ |
|-----|----------|-----|----------|-----|----------|-----|----------|
| 2   | 6k+1     | 3   | 8k+1     | 5   | 10k+2    | 7   | 12k+3    |
| 3   | 7k+2     | 5k  | 10k+2    | 7k  | 12k+3    | 9k  | 14k+4    |
Let $q \geq 4$. The geometrical constructions (named “oval plus line”) give $[2q + 1, 2q - 3]_2$ codes, see [8, p. 104] for even $q$ and [15, Th. 3.1], [18, Th. 5.1] for arbitrary $q$. By computer, using the back-tracking algorithms [59], [68], we have proved the following proposition.

**Proposition 5.1:** $\ell_4(4, 2) = 9$.

No examples of $[n, n - 4]_2$ codes with $n < 2q + 1$, seem to be known.

**Open problem.** To prove that $\ell_4(q, 2) = 2q + 1$ for $q \geq 5$.

In [29] the parity-check matrices of the codes of [8], [18] are modified and used as starting $(R, \ell)$-objects in $q^2$-concatenations. As a result, an infinite family of $[n, n - r]_q$ codes is obtained with the following parameters [29, Th. 9]:

$$A_{2,q}^{(0)} : R = 2, r = 2t \geq 4, q \geq 7, q \neq 9, r \neq 8, 12, 20, n = 2q^{-\frac{r-2}{r}} + \frac{q^{-2}}{r}.$$

Also, in [29] codes with $r = 8, 12, n = 2q^{-\frac{r-2}{r}} + \frac{q^{-2}}{r} + \frac{q^{-\frac{r-2}{r}}}{q^{-\frac{r-2}{r}}} + \frac{q^{-\frac{r-2}{r}}}{q^{-\frac{r-2}{r}}}$, $q \geq 7, q \neq 9$, are given.

For $q = 4, 5, 9$ in [24, Ex. 5] an infinite family of $[n, n - r]_q$ codes is obtained with:

$$A_{2,q}^{(0)} : R = 2, r = 2t \geq 4, q = 4, 5, 9, n = 2q^{-\frac{r-2}{r}} + \frac{q^{-2}}{r} + \frac{q^{-\frac{r-2}{r}}}{q^{-\frac{r-2}{r}}}, \frac{q^{-2}}{r} - \frac{2}{q} - \frac{1}{2q^2} + \frac{2}{q^2} + \frac{2}{q^4},$$

with $r \neq 8, 12$ if $q = 5, 9$, $\overline{r}(2) < 2 - \frac{2}{q} - \frac{1}{2q^2} - \frac{2}{q^2} + \frac{2}{q^4}$.

Also, codes with $q = 4, 5, 9, n = 2q^3 + q^2 + 2q + 2, r = 8, n = q^5 + \theta_b, r = 12, 14, 20$.

**C. More on 1-Saturating Sets in Projective Planes $PG(2, q)$**

We recall here some of the known results on small 1-saturating sets in $PG(2, q)$. (For the new 1-saturating sets obtained in this paper we refer to Section III and Tables III, II of Section IV.)

For large $q$ the existence of 1-saturating sets in $PG(2, q)$ of size at most $5\sqrt{q \log q}$ was shown by means of probabilistic methods in [45], [61].

The following results are given by explicit constructions.

In $PG(2, q)$, $q = (q')^2$, a 1-saturating set of size $3\sqrt{q} - 1$ is obtained in [18, Th. 5.2].

In the plane $PG(2, q)$, $q = (q')^m$, $m \geq 2$, projectively non-equivalent 1-saturating sets of size $2q_{m-1} + \sqrt{q}$ are obtained in [58, Th. 2], [63, Th. 3.2].

In [56], [54], [62], [45] 1-saturating sets in $PG(2, q)$ of size approximately $c_0 q^{-p}$ with a constant $c_0$ independent of $q$ are constructed.

In [63] constructions of 1-saturating $n$-sets in $PG(2, q)$ of size $n$ about $3q^2$ are proposed. In particular the following upper bounds on $n$ are obtained for $p$ prime:

$$n \leq \begin{cases} 2q^p + (p^2 - 1)p + 1, & q = p^m, m \geq 2t; \\
\frac{2}{p} \sqrt{\frac{np}{p^2 - 1}} + \frac{1}{\sqrt{q}} p^{m} + 1, & q = p^{m-1}; \\
\min_{v=1, \ldots, 2t + 1} \left\{ (v + 1)p^{t+1} + \frac{(p^2 - 1)p^v}{p^{t+1} + 1} + 2 \right\}, & q = p^{2t+1}. \end{cases}$$

Several triples $(t, p, v)$ such that $n < 5\sqrt{q \log q}$ are obtained in [63].

**D. Infinite Code Families of Odd Codimension $r = 2t + 1$**

Let $q = 3$. By [21, Th. 1], [27, Ths 4 and 9], there exists an infinite family of $[n, \omega - r]_3$ codes with the following parameters:

$$A_{2,3}^{(1)} : R = 2, r = 2t + 1 \geq 5, q = 3, q \neq 7, \overline{r}(2) \leq \frac{25}{24},$$

$$n = \frac{5}{4} \sqrt{3} \cdot \frac{3^{r-2}}{2} - \frac{1}{4} + \left\{ \begin{array}{ll} 0 & \text{if } r = 4e + 1 \\
\frac{3}{4} \cdot \frac{3^{r-2}}{2} - \frac{1}{4} & \text{if } r = 8c + 3 \end{array} \right. (5.8)$$

Let $q = 4$. In [28] an infinite family of $[n, \omega - r]_4$ codes is obtained with parameters

$$A_{2,4}^{(1)} : R = 2, r = 2t + 1 \geq 5, q = 4, q \neq 7, 11, 13, 19,

n = 2 \cdot \frac{3^{r-2}}{2} + \frac{3}{2} \cdot \frac{4^{r-2}}{2}, \overline{r}(4) \approx 1.587. \quad (5.9)$$

Let $q = 5$. In [27, Ths 5.10] an infinite family of $[n, \omega - r]_5$ codes is obtained with

$$A_{2,5}^{(1)} : R = 2, r = 2t + 1 \geq 7, q = 5, q \neq 9, \overline{r}(5) \approx \frac{8}{5},$$

$$n = \sqrt{5} \cdot \frac{5^{r-2}}{2} + \left\{ \begin{array}{ll} \ell_5(\frac{r-2}{2}, 2) & \text{if } r = 4c + 3 \\
(\ell_5(\frac{r-2}{2} + \frac{1}{2}), 2) & \text{if } r = 8c + 5. \end{array} \right. \quad (5.10)$$

Now we construct infinite code families by using the $q^m$-concatenations in [24]. Terminology and notation of [24] will be used; in particular, we are going to consider $E^r$-partitions, $2^r$-partitions, and their cardinalities $h^r(H)$ and $h^r(H)$, see [24, Def. 1, Rem. 1]. The starting codes will be the codes associated to the 1-saturating sets described in the part C of this section.

In [24, Ex. 6, form. (33)] an infinite family of $[n, \omega - r]_2$ codes is constructed with

$$A_{2,2}^{(4)} : R = 2, r = 2t + 1 \geq 3, q = (q')^2 \geq 16,$$

$$n = \left( 3 - \frac{2}{\sqrt{q}} \right) q^{r-2} + [q^{-\frac{r-2}{2}}];$$

$$\overline{r}(2) < 4.5 - \frac{3}{2} \sqrt{q} + \frac{17}{2q} + \frac{9}{q \sqrt{q}} + \frac{5}{2q^2}.$$ \quad (5.11)

The starting code $V_0$ (denoted as $W$) is based on the previously mentioned 1-saturating $(3\sqrt{q} - 1)$-set. In [24] it is noted that $h^r(H_{V_0}) \leq 4$ and that this inequality allows us to obtain an effective iterative code chain. A similar situation arises if one takes as $V_0$ the $[n_0 = 2\sqrt{q} + 2\sqrt{q} + 2, m_0 = 3]_2$ code based on Theorem 3.4. We partition the column set of the parity-check matrix into subsets $T_1, \ldots, T_3$ so that $|T_1| = |T_2| = 2, T_1 \cup T_2 = \pi_1, T_3 \cup T_4 = \pi_2, \pi_1, \pi_2$ are disjoint Baer subplanes in $PG(2, \sqrt{q})$. An arbitrary point of $PG(2, q) \setminus \{ \pi_1 \cup \pi_2 \}$ lies on a line through two points belonging to the distinct subplanes. So, we obtain a 2-partition, see [24, Def. 1] and Definition 2.1. Moreover, as every point of a subplane $\pi$ is a linear combination of two other points of $\pi$, this 2-partition...
is a $2^E$-partition \cite[Rem. 1]{24} and $h^E(H_{W_1}) \leq 4$. Now, by changing $3\sqrt{q} - 1$ by $2\sqrt{q} + 2\sqrt{q} + 2$ in (5.11), we obtain the following theorem.

**Theorem 5.2:** For $q = (q')^4$ there is an infinite family of $[n, n - r]_q 2$ codes with

$$A_{2, q}^{(1)} : R = 2, \ r = 2t + 1 \geq 3, \ q = (q')^4, \ n = \left(2 + \frac{2}{\sqrt{q}} + \frac{2}{\sqrt{q}}\right) q^{r-2} + \left[q^{r-2}\right], \ \overline{\mu}_q(2) < 2 + \frac{2}{\sqrt{q}} + \frac{6}{\sqrt{q}} + \frac{4}{\sqrt{q^3}} - \frac{2}{q - 8\sqrt{q}}. \tag{5.12}\$$

**Theorem 5.3:** Let $q \geq 7$. Assume that there exists an $[n_q, n_q - 3]_q 2$ code $V_0$ with $n_q < q$. Then there exists an infinite family of $[n, n - r]_q 2$ codes with

$$A_{2, q}^{(1)} : R = 2, \ r = 2t + 1 \geq 3, \ q \geq 7, \ r \neq 9, 13, \ a_q = \frac{n_q}{\sqrt{q}}, \ n = a_q \cdot q^{r-2} + 2\left[q^{r-2}\right] + \left\{\begin{array}{ll}
0 & \text{if } 2p_0 \leq q + 1 \\
\lfloor q^{r-2} \rfloor & \text{if } 2p_0 > q + 1
\end{array}\right., \ \overline{\mu}_q(2) \approx \frac{a_q^2}{2} - \frac{a_q^2}{q} + \frac{2a_q}{q\sqrt{q}}. \tag{5.13}\$$

For $r = 9, 13, \ a_q \cdot q^{r-2} + 2\sqrt{q} + q^{r-2} + q^{r-2}$ holds.

**Proof:** Take $V_0$ as the starting code for the constructions of [24]. Then, changing $n_q$ by $p_0$, we use the same argument of [24, Ex. 6] on partition cardinalities $h^+(H_{W_1}), h^E(H_{W_1})$. As a result, (5.13) is obtained, cf. [24, form. (32)]. □

**Theorem 5.3** is the main tool to obtain infinite code families with growing odd codimension.

**Theorem 5.4:** For $q = (q')^6$ there is an infinite family of $[n, n - r]_q 2$ codes with

$$A_{2, q}^{(1)} : R = 2, \ r = 2t + 1 \geq 3, \ r \neq 9, 13, \ q \neq 9, (q')^6, \ n = \left(2 + \frac{2}{\sqrt{q}} + \frac{2}{\sqrt{q}}\right) q^{r-2} + 2\left[q^{r-2}\right], \ \overline{\mu}_q(2) < 2 + \frac{4}{\sqrt{q}} + \frac{6}{\sqrt{q}} + \frac{8}{\sqrt{q^3}} + \frac{5}{\sqrt{q^5}}. \tag{5.14}\$$

**Proof:** The assertion follows from Theorem 5.3 and Corollary 3.6. □

**Lemma 5.5:** For an $[n_q, n_q - 3]_q 2$ code $V_0$ we have $p_0 \leq n_q - 1$.

**Proof:** In a parity-check matrix $H$ of $V_0$ there are three linear dependent columns. Let two of these columns form one subset of a partition $P_0$ of $H$, while the other subsets of $P_0$ contain precisely one column. By Definition 2.1, $P_0$ is a 2-partition. □

**Theorem 5.6:** For any $q \leq 1217$, there exists an infinite family of $[n, n - r]_q 2$ codes with

$$A_{2, q}^{(1)} : R = 2, \ r = 2t + 1 \geq 3, \ q \leq 1217, \ n = a_q \cdot q^{r-2} + 2\left[q^{r-2}\right] + \left\{\begin{array}{ll}
0 & \text{if } 16 \leq q \leq 1217 \\
\lfloor q^{r-2} \rfloor & \text{if } 7 \leq q \leq 13
\end{array}\right., \ \overline{\mu}_q(2) \leq 5. \tag{5.15}\$$

**Table VI:**

| $q$ | $\gamma = 0$ | $\gamma = 1$ |
|-----|---------------|---------------|
| 3   | 1.369         | 1.042         |
| 4   | 1.504         | 1.587         |
| 5   | 1.606         | 1.600         |
| 6   | 1.729         | 1.880         |
| 7   | 1.872         | 1.707         |
| 8   | 1.943         | 2.183         |

For each of the infinite families (5.8) - (5.13) the covering density is bounded from above by a constant. If in (5.13) we take as $V_0$ a code with length $n_q \sim f(q)\sqrt{q}$, where $f(q)$ is some increasing function of $q$, such as in (5.7), then the asymptotic covering density increases like $f^2(q)$. However for concrete $q$ new code families can be supportable, e.g. Table 1.[1]

We end this section with Tables VI and VII which have been obtained from [5.1], [5.3], [5.6], [5.3], [5.10], [5.15]. Table 1 Proposition 5.1 and [22, Tab. II], [27, Tab. 1], [28, Tab. I], [29, Tab. 1].

**VI. CODES WITH COVERING RADIUS $R = 3$**

### A. Infinite Code Families of Codimension $r = 3t$

Let $q \geq 4$. The geometrical construction (named “two ovals plus line”) \cite[Th. 7]{58} gives a $[3q + 1, 3q - 5]_{q^3} 3$ code. So,

$$\ell_q(6, 3) \leq 3q + 1 \text{ if } q \geq 4. \tag{6.1}\$$

To our knowledge, no examples of $[n, n - 6]_{q^3} 3$ code with $n < 3q + 1$ are known.

**Open problem.** To prove that $\ell_q(6, 3) = 3q + 1$ for $q \geq 4$.

The parity-check matrix of the code of [58, Th. 7] is modified in [29, Th.6] and then it is used as the starting point in $q^{m}$-concatenating constructions. As a result, an infinite family of $[n, n - r]_{q^3} 3$ codes is obtained with the following parameters

$$A_{3, q}^{(0)} : R = 3, \ r = 3t \geq 6, \ q \geq 5, \ n = 3q^{r-2} + q^{r-2}, \ r \neq 9, \ \overline{\mu}_q(3) \leq \frac{9}{2q} + \frac{3}{2q^2} + \frac{14}{3q^3} - \frac{1}{2q^4}. \tag{6.2}\$$

Also, in [29] it is shown that codes with parameters as in (6.2) exist for $r = 9$ if $q = 16$ or $q \geq 23$. For $q = 7, 8, 11, 13, 17, 19, 21$ the codes of (5.5) and the $[\theta_{3, q}, \theta_{3, q} - 3]_{q^3} 1$ Hamming codes gives $[3q^2 + 2q + 1, n - 9]_{q^3} 3$ codes. For $q = 5, 9$ and $r = 9$, by (5.6), codes with length $n = 3q^2 + 2q + 2$ are obtained.

### B. Infinite Code Families of Codimension $r = 3t + 1$

Let $q = 3$. DS of the codes of (5.8) and the $[\theta_{3, q}, \theta_{3, q} - t]_{q^3} 1$ Hamming codes forms an infinite family of $[n, n - r]_{q^3} 3$ codes
TABLE V

| \( r \) | \( \ell_3(r, 2) \) | \( \ell_4(r, 2) \) | \( \ell_5(r, 2) \) | \( \ell_7(r, 2) \) | \( r \) | \( \ell_3(r, 2) \) | \( \ell_4(r, 2) \) | \( \ell_5(r, 2) \) | \( \ell_7(r, 2) \) |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 3   | 4   | 5   | 6   | 6   | 14  | 1822 | 9522 | 35000 | 292105 |
| 4   | 8   | 11  | 15  | 15  | 15  | 2915 | 19456 | 78256 | 741909 |
| 5   | 11  | 19  | 28  | 44  | 16  | 5588 | 37888 | 175000 | 1764735 |
| 6   | 22  | 37  | 50  | 105 | 17  | 8201 | 77824 | 410937 | 5193363 |
| 7   | 40  | 85  | 131 | 309 | 18  | 16402 | 151552 | 875000 | 12353145 |
| 8   | 76  | 154 | 281 | 743 | 19  | 24785 | 316672 | 1953828 | 36353541 |
| 9   | 101 | 304 | 703 | 2164 | 20  | 49328 | 611328 | 4375000 | 86472015 |
| 10  | 202 | 592 | 1400 | 5145 | 21  | 73811 | 1245184 | 9853906 | 25474787 |
| 11  | 223 | 1227 | 3153 | 15141 | 22  | 147622 | 2424892 | 21875000 | 605034105 |
| 12  | 620 | 2389 | 7031 | 36407 | 23  | 223073 | 4980736 | 48831278 | 1781323509 |
| 13  | 911 | 4948 | 16406 | 106636 | 24  | 443960 | 9699328 | 109375000 | 4237128735 |

Also, [431, 415]_3 codes and [3887, 3865]_3 codes are given in [18, Tab. I, form.(37)], and a [14, 7]_3 code is obtained in [22, Tab. II].

**Theorem 6.1:** Denote by \( Q_3 \) the set of values of \( q \) for which there is an \( ([\ell_4(4,3), \ell_4(4,3) - 4, 3])_3 \) code with minimum distance \( d = 3 \). Then, for \( 7 \leq q \leq 563 \), there is an infinite family of \([n, n - r]_3\) codes with

\[
A_{3, q}^{(1)} : R = 3, \quad r = 3t + 1, \quad 7 \leq q \leq 563, \quad \ell(q, 3) \approx 2.382, \quad n = \frac{7}{4} \cdot 3^\frac{2}{3} \cdot 3^\frac{-r}{3} + \begin{cases} \frac{3}{2} \sqrt[3]{3} \frac{3^{r+2}}{2} - \frac{3}{2} & \text{if } r = 6c + 1 \\ \frac{3}{2} \sqrt[3]{3} \frac{3^{r+2}}{2} - \frac{3}{2} & \text{if } r = 12c + 4 \\ \frac{3}{2} \sqrt[3]{3} \frac{3^{r+2}}{2} - \frac{3}{2} & \text{if } r = 12c + 10 \end{cases},
\]

\[
\begin{align*}
&b_q = \frac{\ell(q, 4, 3) - 3}{\sqrt{q}}, \quad \overline{p}_q(3) < \frac{b_q^2}{6}, \\
n = b_q \cdot q^{\frac{-r-1}{q-1}} + q^{\frac{-r-1}{q-1}} - 1 + \begin{cases} \frac{3^{r+1}}{q-1} & \text{if } q \in Q_3 \\
\ell_q(2, \frac{r+2}{q-1}, 2) & \text{if } q \notin Q_3 \end{cases}, \\
\overline{p}_q(3) < 10.7 & \text{ if } q \leq 83, \quad \overline{p}_q(2) < 15.2 & \text{ if } q \leq 343, \\
\overline{p}_q(2) < 20.9 & \text{ if } q \leq 563.
\end{align*}
\]

**Proof:** By Lemma 2.2 for \( q \in Q_3 \) we have \( \ell_q(4,3), \ell_q(4,3) - 4, 3) \) \( \ell_4 \) codes with \( \ell_0 \geq 1 \). By Table III for \( q \geq 7 \) we have that \( \ell_q(4,3) \leq q \) if \( q \in Q_3 \) and \( \ell_q(4,3) \leq q - 2 \) if \( q \notin Q_3 \). We take the \( ([\ell_q(4,3), \ell_q(4,3) - 4])_3 \) codes of Table III as the codes \( V_0 \) for Constructions QM_2 (if \( q \in Q_3 \)) and QM_4 (if \( q \notin Q_3 \)), using the trivial partition and letting \( m \geq 1 \). Now the assertion follows from (2.4) and (2.6).

We denote by \( p^{(e)}(V) \) the upper bound of the minimal possible cardinality of an \((R, \ell)\)-partition for a parity-check matrix of an \([n, n - r]_q R, \ell \) code \( V \).

**Theorem 6.2:** For \( q = (q')^3 \geq 64 \) there exists an infinite family of \([n, n - r]_3\) codes with

\[
A_{3,q}^{(1)} : R = 3, \quad r = 3t + 1, \quad 7 \leq q \leq 64, \quad (q')^{3} \geq 64, \quad \ell(q, 3) \approx 2.382, \quad n = \left( 4 + \frac{4}{\sqrt{q}} \right) q^{\frac{-r}{q-1}} + \overline{p}_q(3) < \frac{32}{3} + \frac{32}{\sqrt{q}} + \frac{32}{\sqrt{q^2}} - \frac{64}{3q},
\]

\[
\overline{p}_q(3) < 10.7 & \text{ if } q \leq 27, \quad \overline{p}_q(2) < 12.4 & \text{ if } q \leq 32, \\
\overline{p}_q(2) < 20.9 & \text{ if } q \leq 43.
\]

**Proof:** By Lemma 2.2 and Table IV for \( q \neq 2, 5, 19 \), we have \( ([\ell_q(5,3), \ell_q(5,3) - 5, 3])_3 \) \( \ell_4 \) codes with \( \ell_0 \geq 1 \). By Table IV for \( q \geq 3 \) we have that \( \ell_q(5,3) \leq q^2 \). We take the \( ([\ell_q(5,3), \ell_q(5,3) - 5])_3 \) codes of Table IV as the codes \( V_0 \) for Constructions QM_2 (if \( q \neq 2, 5, 19 \)) and QM_4 (otherwise) using the trivial partition and letting \( m \geq 2 \). Now the assertion follows from (2.4) and (2.6).
Theorem 6.4: For $q = (q')^3 \geq 27$ there exists an infinite family of $[n, n-r]_q$ codes with
\[ A_{3,q}^{(2)} : R = 3, \quad r = 3t + 2 \geq 8, \quad n = (q')^3 \geq 27, (6.8) \]
\[ n = \left( 9 - \frac{8}{\sqrt{q}} + \frac{4}{\sqrt{q^2}} \right) q^{-\frac{r}{2}} \left( \frac{1}{\sqrt{q}} + \frac{7}{\sqrt{q}} \right). \]

Proof: Let $S$ be as (3.7). For any plane of $S$, let \( \{P_1, P_2, \{P_3, P_4\}, \{P_5, \ldots, P_{q^3+q^2+1}\} \) be a partition of the set of its points such that $P_1, P_2 \notin l_{3,4}$ and $P_3, P_4 \notin l_{1,2}$, where $l_{i,j}$ is the line through points $P_i, P_j$. It can be easily shown that if $u \in \{2, 3\}$, then every point of the plane is equal to a linear combination with nonzero coefficients of $u$ other points belonging to distinct subsets of the partition. The corresponding partition of the columns of the parity-check matrix of the related code $C_S$ is a (3,3)-partition with $p^{(3)}(C_S) = 3 \cdot 9 = 27 \leq q$. Therefore we may take $C_S$ as the starting $[n_0 = 9 \sqrt{q^2 - 8 \sqrt{q} + 4}, n_0 - 4]_q$ code $V_0$ for Construction QM$_3$ with $m \geq 1$. \hfill \Box

VII. CODES WITH COVERING RADIUS $R \geq 4$

A. Infinite Code Families of Codimension $r = Rt$ and Arbitrary $q$

In this Section we obtain a code $V$ of covering radius $R \geq 4$ and codimension $Rt$ from DS of $g_2$ codes $V_2$ with radius two and $g_3$ codes $V_3$ with radius three. More precisely, let
\[ V = V_2 \oplus \ldots \oplus V_2 \oplus V_3 \oplus \ldots \oplus V_3 \]
where $V$ is an $[n, n-Rt]_q$ code, $V_2$ is an $[n_2, n_2-2t]_q$ code, $V_3$ is an $[n_3, n_3-3t]_q$ code, $n = g_2n_2 + g_3n_3$, $n_2 + g_2 + 3g_3 = R$, and
\[ g_2 = \begin{cases} 0 & \text{if } R \equiv 0 \pmod{3} \\ 1 & \text{if } R \equiv 2 \pmod{3}, g_3 = \left[ \frac{R}{3} \right] - g_2. \end{cases} \]

Theorem 7.1: Let $R \geq 4$ and let $q \geq 4$. Then there exists an $[n = R + [R/3], n - 2R, [R/3]]_q$ code with $\ell \geq 1$.

Proof: Geometrical constructions of a $[2q + 1, 2q - 3]_q$ code $V_2$ ("oval plus line") and of a $[3q + 1, 3q - 5]_q$ code $V_3$ ("two ovals plus line") are given in [8, p. 104], [18, Th. 5.1], [58, Th. 7]. Using these codes in (7.1) and (7.2) with $t = 2$, we obtain an $[n = Rq + [R/3], n - 2R]_q$ code $V$. Minimum distance $d = 3$ follows from the fact that the point sets associated to $V_2$ and $V_3$ contain triples of collinear points. The value $\ell \geq 1$ follows from Lemma 2.2. \hfill \Box

Open problem: To obtain $[n, n-2R]_q$ codes with $R \geq 4$, $q \geq 4$, $n < Rq + [R/3]$. In particular, for $R \geq 4$, to generalize the geometrical constructions "oval plus line" and "two ovals plus line".

Theorem 7.2: There exist infinite families of $[n, n-r]_q$ codes with the parameters
\[ i) A_{4,q}^{(0)} : R \geq 4, \quad r = R \geq 5R, \quad q \geq 7, \quad n \neq 9, \]
\[ n = Rq - R + 2R \quad \text{and} \quad [R/3], n^{q-2R} \quad \text{if} \quad R \neq 6R. \]  \hspace{1cm} (7.3)
\[ ii) A_{4,q}^{(0)} : R \geq 4, \quad r = R \geq 2R, \quad q = 5, 9, \quad R \neq 3R, 4R, 6R, \]
\[ n = Rq - R + 2R \quad \text{and} \quad [R/3] + g_2 \cdot q^{-1} \] \hspace{1cm} (7.4)

\[ n^{q-2R} \quad \text{if} \quad R \neq 6R. \]

Proof: We use the construction of (7.1) with the codes $V_2$ and $V_3$ taken from (5.3), (6.2) and (5.6), (6.2) for the cases i) and ii), respectively.

It should be noted that the main term of the asymptotic covering density $\overline{p}_3(R, A_{3,q})$ for the family of (7.3) is $\frac{R^R}{n^R}$; it does not depend on $q$.

By the results on cases $r = 8, 12$ and $r = 9$ reported after (5.5), (5.6), and (6.2), one can easily fill up gaps in (7.3) for codes with $r = 3R, 4R$, and $6R$.

B. Infinite Code Families of Codimension $r = Rt + 1$, $q = (q')^R$

Theorem 7.3: Let $q = (q')^R$. Then there exists an infinite family of $[n, n-r]_q$ codes with
\[ A_{R,q}^{(1)} : R \geq 4, \quad r = R + 1, \quad q = (q')^R, \]
\[ t = 1 \quad \text{and} \quad t \geq t_0, \quad q^{t-1} \geq n_{R,q}, \]
\[ n_{R,q}^{(1)} = (\sqrt{q} - 1) \left( \frac{R(R + 1)}{2} - 2 \right) + R + 5, \]
\[ n = n_{R,q}^{(1)} \cdot q^{\frac{R(R + 1)}{2}} \quad \text{if} \quad q' \geq 4, \]
\[ n = \begin{cases} 0 & \text{if} \quad q' < 4 \quad \text{and} \quad w \in \{0, 1\}, \end{cases} \]
\[ w = \frac{\frac{R(R + 1)}{2} - 5}{q - 1}, \quad w \in \{0, 1\}, \quad \text{if} \quad q' = 3. \]

Proof: As the starting code $V_0$ for Constructions QM$_2$, QM$_3$ we take an $[n_{R,q}^{(1)}, n_{R,q}^{(1)} - (R + 1), 3R]_q$ code $C_K$ related to the $(R - 1)$-saturating set $K \in PG(R, q') \subset PG(R, q)$ described in Corollary 3.12 see also (3.2), Construction A and Corollary 3.11. Note that $K$ contains four pairwise skew lines of $PG(R, q')$, whereas for other $\frac{R(R + 1)}{2} - 6 \geq 2R - 4$ all but one point belong to $K$. These latter lines are partitioned into $R - 3$ sets of concurrent lines. By Definition 2.1 and Remark 3.13 the code $C_K$ is an $(R, t_0)$-object with $t_0 = 5$ if $q \geq 4$ and $t_0 \geq R - 1$ if $q = 3$. The trivial partition of its parity-check matrix is an $(R, t_0)$-partition into $n_{R,q}^{(1)} \leq q^{t_0} - 1$ subsets. Finally, we use (2.4) and (2.5) to get the assertion.

It should be noted that the main term of the asymptotic covering density $\overline{p}_3(R, A_{3,q}^{(1)})$ for the family of (7.5) is $(\frac{R^R}{2^n})$; it does not depend on $q$.

C. Infinite Code Families of Codimension $r = R + 2, \ldots, R(t+1) - 1$, $q = (q')^R$

To our knowledge, for $R \geq 4, r = R + 2, \ldots, R(t+1) - 1$, no infinite families with density asymptotically independent on $q$ are known.

Theorem 7.4: Let $q = (q')^R$. Fix $\gamma \in \{2, 3, \ldots, R\}$. Then there exists an infinite family of $[n, n-r]_q$ codes with
\[ A_{R,q}^{(\gamma)} : R \geq 4, \quad r = R + \gamma, \quad q = (q')^R, \]
\[ \gamma = 2, 3, \ldots, R - 1, \quad t = 1 \quad \text{and} \quad t \geq t_0, \quad q^{t-1} \geq n_{R,q}, \]
\[ n_{R,q}^{(\gamma)} = \frac{\sum_{i=1}^{(R+1)} (q - 1) \left( \frac{R+1}{q} \right) }{R - 1} \sim (R + \gamma) \frac{n_{R,q}}{q^n}, \]
\[ n = n_{R,q}^{(\gamma)} \cdot q^{\frac{R(R + 1)}{2}} + w \frac{R(R + 1) - 5}{q - 1}, \quad 0 \leq w \leq R - 3. \]
VIII. CODES WITH NONPRIME COVERING RADIUS

We consider the case when covering radius \( R \) is nonprime, i.e., \( R = sR' \) with integer \( s \) and \( R' \).

\textbf{Lemma 8.1:} Let \( R = sR' \). Assume that there exists an \([n', n'] - (R't + t')qR'\) code \( C_R \) with \( R' > t' \). Then there exists an \([n'R, n'R] - (Rt + 
R't)qR\) code \( C \). Moreover, if the starting code \( C_R \) is short the new code \( C \) is short too.

\textbf{Proof:} We apply Construction \( D \) to \( s \) copies of \( C_R \). If the code \( C_R \) is short then \( n' = O(q^{(R't+t'-R')/R'}) \) or, in other words, \( n' = cq^{(R't+t'-R')/R'} \) where \( c \) is a constant independent of \( q \). Also, \( R'(t'+t'-R')/R'R' = (Rt+s't'-R)/R \). Therefore \( n'R = c\frac{R'q}{R}(Rt+\frac{s't'}{R}/R) \). Then the assertion is proved. □

\textbf{Corollary 8.2:} For even \( R \geq 4 \) there exist infinite families \( A_{R,q}^{(R/2)} \) of \([n,n-r]qR \) codes with codimension \( r = Rt + \frac{R}{2} \) and the following parameters:

\begin{enumerate}
  \item \( q = (q')^2 \), \( t \geq 1 \),
  \[
  n = \frac{R}{2} \left( 3 - \frac{1}{\sqrt{q}} \right) q^{-R} + \frac{R}{2} \left[ \frac{1}{\sqrt{q}} q^{-2R} \right] .
  \]
  \item \( q = (q')^4 \), \( t \geq 1 \),
  \[
  n = R \left[ 1 + \frac{1}{\sqrt{q}} + \frac{1}{\sqrt{q}} \right] q^{-R} + \frac{R}{2} \left[ \frac{1}{\sqrt{q}} q^{-2R} \right] .
  \]
  \item \( q = (q')^6 \), \( q' \) prime, \( q' \leq 73 \), \( t \geq 1, t \neq 4, 6 \),
  \[
  n = R \left[ 1 + \frac{1}{\sqrt{q}} + \frac{1}{\sqrt{q}} \right] q^{-R} + R \left[ \frac{1}{\sqrt{q}} q^{-2R} \right] .
  \]
\end{enumerate}

\textbf{Proof:} Put \( R' = 2 \) and use the codes of \( 5.11 \). \( 5.12 \), and \( 5.14 \) as the code \( C \) of Lemma \( 8.1 \). □

\textbf{Corollary 8.3:} Let \( q = (q')^3 \) and assume that \( 3 \) divides \( R \). Then there exist infinite families of \([n,n-r]qR \) codes with

\begin{enumerate}
  \item \( A_{R,q}^{(3)} \): \( R = 3s \), \( r = Rt + \frac{R}{3} \), \( q = (q')^3 \geq 64 \),
    \[
    t \geq 1, \quad n = \frac{4R}{3} \left( 1 + \frac{1}{\sqrt{q}} \right) q^{-R} .
    \]
  \item \( A_{R,q}^{(2)} \): \( R = 3s \), \( r = Rt + \frac{2R}{3} \), \( q = (q')^3 \geq 27 \),
    \[
    t \geq 1, \quad n = \frac{R}{3} \left( 9 - \frac{8}{\sqrt{q}} + \frac{4}{\sqrt{q}} \right) q^{-R} .
    \]
\end{enumerate}

\textbf{Proof:} Put \( R' = 3 \) and use the codes of \( 6.5 \) and \( 6.8 \) as the code \( C \) of Lemma \( 8.1 \). □

\textbf{Corollary 8.4:} Let \( R = sR' \). Let \( q = (q')^R \). Then there exist an infinite family of \([n,n-r]qR \) codes with

\[
A_{R,q}^{(s)} : R = sR', \quad R' \geq 4, \quad r = Rt + \frac{R}{R'} q = (q')^{R'},
\]

\[
t = 1 \quad \text{and \quad} t \geq t_0, \quad q^{t_0-1} \geq n_{R',q}^{(1)},
\]

\[
n_{R',q}^{(1)} = \left( \frac{R' + 1}{2} \right) - 2 + R' + 5,
\]

\[
n = R^{(R' + 1)} R' \left( \frac{R - (R + 1)}{q - 1} \right) \left( R + \frac{R}{R'} \right) R',
\]

\[
\left\{ \begin{array}{l}
0, \quad \text{if} \ \ q' \geq 4 \\\n\frac{R + R'}{R'} R, \frac{R'+R}{q-1}, \ w \in \{0,1\}, \text{if} \ \ q' = 3
\end{array} \right.
\]

\textbf{Proof:} We use the codes of \( 7.5 \) as the code \( C \) of Lemma \( 8.1 \). □

\textbf{Corollary 8.5:} Let \( R = sR' \). Let \( q = (q')^{R'} \). Fix \( \gamma \in \{2,3,\ldots,R-1\} \). Then there exists an infinite family of \([n,n-r]qR \) codes with

\[
A_{R,q}^{(s)} : R = sR', \quad R' \geq 4, \quad r = Rt + \frac{R}{R'} q = (q')^{R'},
\]

\[
\gamma = 2, 3, \ldots, R-1, \quad t = 1 \quad \text{and} \quad t \geq t_0, \quad q^{t_0-1} \geq n_{R',q}^{(1)},
\]

\[
n_{R',q}^{(1)} = \sum_{i=1}^{\gamma+1} \left( \sqrt[\gamma]{q} - 1 \right) i \left( \sqrt[\gamma]{q} \right) \sim \left( \sqrt[\gamma]{q} \right) R' - 1,
\]

\[
n = R^{(R' + 1)} R' \left( \frac{R - (R + 1)}{q - 1} \right) R, \quad R' \geq 3, \quad \frac{R + R'}{R'} R, \frac{R'+R}{q-1}, \ w \in \{0,1\}, \text{if} \ \ q' = 3
\]

\[
0 \leq w \leq R - 3.
\]

\textbf{Proof:} We use the codes of \( 7.6 \) as the code \( C \) of Lemma \( 8.1 \). □

It should be emphasized that for the infinite families \( 5.1-5.7 \), the main term of the lower limit of covering density \( \Omega_{R}(R, A_{R,q}^{(s)}) \) is, respectively, \( \frac{R^n}{R} \left( \frac{1}{4} \right) \), \( \frac{R^n}{R} \left( \frac{1}{4} \right) \), \( \frac{R^n}{R} \left( \frac{1}{4} \right) \), \( \frac{R^n}{R} \left( \frac{1}{4} \right) \), \( \frac{R^n}{R} \left( \frac{1}{4} \right) \). All these terms do not depend on \( q \).

\textbf{Remark 8.6:} It should be emphasized that codes of Corollaries \( 8.2, 8.3, 8.5 \) are “short” for \( R = sR' \) though as a rule in these codes \( q \neq (q')^{R} \). Usually we have this property when \( q = (q')^{R} \).

IX. CONCLUSION

We considered infinite sequences \( A_{R,q} \) of linear nonbinary covering codes \( C_{n} \) of type \([n,n-r_{n}]qR \). Without loss of generality, we assumed that the sequence of codimension \( r_{n} \) is not decreasing. For a given family \( A_{R,q} \), the covering radius \( R \) and the size \( q \) of the underlying Galois field are fixed. We considered also infinite sets of the families \( A_{R,q} \), where \( R \) is fixed but \( q \) ranges over an infinite set of prime powers.

Each infinite family \( A_{R,q} \) consists of supporting and filling codes. The supporting codes are the codes \( C_{n} \) such that \( r_{n} > r_{n+1} \). Non-supporting codes are called filling codes. This terminology is motivated by the fact that the parameters of the codes in a family are completely determined by those
of its supporting codes. However, considering filling codes is necessary to investigate not only the lower limit (lim inf) of the covering densities of a family, but also its upper limit (lim sup).

Such lower and upper limits (denoted by $\overline{\mu}_q(R, A_{R,q})$ and $\underline{\mu}_q(R, A_{R,q})$ respectively) are the most considerable asymptotic features of families $A_{R,q}$. It is also relevant how these limits depend on $q$ in infinite sets of families $A_{R,q}$ with fixed $R$. We showed that for the upper limit the best possibility is $\mu_q(R, A_{R,q}) = O(q)$. The problem of constructing infinite sets of families $A_{R,q}$ with $\mu_q(R, A_{R,q}) = O(q)$ is open in the general case. We call it Open Problem 1. In the literature, a solution to Open Problem 1 was known only for $R = 2, q$ square.

We first showed in Introduction that Open Problem 1 for covering radius $R$ is solved provided that a solution to the following Open Problem 2 is achieved: construct $R$ infinite code families $A_{R,q}$, $\gamma = 0, \ldots , R - 1$, such that $\overline{\mu}_q(R, A_{R,q}) = O(1)$ holds. Here $A_{R,q}$ is an infinite family such that its supporting codes are a sequence of $[n_u, n_u - r_u]_q$ codes with codimension $r_u = R_u + \gamma$ and length $n_u = f_q^\gamma(r_u)$, where $u \geq n_0, f_q^\gamma$ is an increasing function for a fixed $q$.

The main achievement of the paper is a solution to Open Problem 2 (and, thereby, to Open Problem 1) for an arbitrary covering radius $R \geq 2$. This solution consists of infinite sets of families $A_{R,q}$ where $q = (q')^R$, $q'$ is power of prime.

The main tool was using codes related to saturating sets in projective spaces as starting points for $q^m$-concatenating constructions of covering codes. Combining $q^m$-concatenating constructions and the saturating sets turned out to be very effective.

In addition, the methods used for solving Open Problems 1 and 2 allowed us to obtain a number of results on covering codes of independent interest. In particular, we obtained many new upper bounds on the asymptotic covering density $\overline{\mu}_q(R, A_{R,q})$ for distinct $R$ and $\gamma$. We obtained also several new asymptotic and finite upper bounds on the length function.

It was natural to analyze and survey the previously known results, as well as presenting the new ones. In particular, this was done for covering radius $R = 2, 3$. A survey of the most used $q^m$-concatenating constructions is also given. It should be noted that no surveys of nonbinary linear covering codes have been recently published.

We also point out that new upper bounds on the length function are also new upper bounds on the smallest possible sizes of saturating sets. More generally, the new results and methods concerning small saturating sets in projective spaces over finite fields that have been given in this paper, such as the new concept of multifold strong blocking sets, seem to be of independent interest.

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