Collapse of the Quantum Wavefunction

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We show using a realistic Hamiltonian-type model that definite outcomes of quantum measurements may emerge from quantum evolution of pure states, i.e. quantum dynamics provides a deterministic collapse of the wavefunction in a quantum measurement process. The relaxation of the wavefunction into a pointer state with classical properties is driven by the interaction with an environment. The destruction of superpositions, i.e. choosing a preferred attraction basin and thereby a preferred pointer state, is caused by a tiny nonlinearity in the macroscopic measurement apparatus. In more details, we numerically studied the many-body quantum dynamics of a closed Universe consisting of a system spin measured by a ferromagnet embedded in a spin-glass environment. The nonlinear term is the self-induced magnetic field of the ferromagnet. The statistics for the outcomes of this quantum measurement process depends on the size of the attraction basins in the measurement apparatus and are in accordance to Born’s rule.

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I. INTRODUCTION

Quantum mechanics [1–3] is perhaps the most successful theory ever developed and provides very accurate descriptions of Nature. For example the gyromagnetic ratio is correctly predicted by quantum electrodynamics to twelve significant figures of accuracy [4]. There are present no experimental findings that contradict the predictions made by the quantum theory! In spite of this enormous predicting power, the orthodox quantum theory [1–3] can only provide the probability for the outcomes of measurements. It can not describe how the complex quantum wavefunction describing the state of the system actually evolves into a single classical outcome, found in all measurement processes. To overcome this problem, the orthodox quantum theory postulates the discontinuous collapse of the wavefunction upon measurement, linking unitary quantum evolution to the classical perceived world of single well defined events. This probabilistic limitation of the quantum theory has, since its conception, troubled many physicists [5–13]. For example Einstein stated in a letter to Born in 1926: “I, at any rate, am convinced that He is not playing at dice.” [14].

Essential insight to the measurement problem has been gained through the decoherence program [15–17]. Most importantly, realistic macroscopic systems are never isolated, but interacting with their environments. The interaction with the environments forces the quantum wavefunction of macroscopic systems to evolve into classical-like pointer states [15–17]. Decoherence theory explains the emergence of classical properties of systems coupled to their surrounding environments. But the theory has nothing to say about single outcomes! Decoherence always refers to an ensemble average where the environmental degrees of freedom are traced out, resulting in a reduced density matrix stating the probabilities to find the system in different pointer states. There are no mechanism in the Decoherence theory that chooses a single outcome among all the possible pointer states. To many practitioners of quantum mechanics this is not a problem, since they interpret the wavefunction to only describe ensemble averages. However, we here choose to interpret the wavefunction as representing an individual system. With this interpretation a state in superposition of outcomes a and b cannot suddenly evolve to an outcome of a or b, even if decoherence is involved.

In this work, we show from the quantum evolution of pure states that single outcomes of measurement events are possible as a consequence of a phase transition in the measurement apparatus. The wavefunction of the measurement apparatus is forced into classical-like pointer states by the interaction with the environment. The destruction of superpositions and, thereby, selection of a preferred pointer state is caused by an infinitesimal non-linearity in the macroscopic measurement apparatus. Which pointer state the wavefunction actually collapses into depends on fine details of the initial conditions. The probability for the wavefunction to collapse into a particular pointer state is shown to be in close agreement with Born’s rule. The statistics for the outcomes depends on the size of the attraction basins in the measurement apparatus which again is affected by the initial state of the measured object.

To be more specific about our physical picture, consider first the quantum dynamics of a closed Universe consisting of a linear ferromagnet (A) embedded in a large environment (E). Assume that the ferromagnet has an easy z-axis. Let the initial state of the linear ferromagnet be a superposition between a state where nearly all spins are pointing up and a state where nearly all spins are pointing down, \( |A_0 \rangle = |A_\uparrow \rangle + |A_\downarrow \rangle \), and let the initial state of the environment be \( |E_0 \rangle \). When the ferromagnet interacts with the environment, the ferromagnet will transfer its entropy and energy into the environment and relax towards its two ground states, \( |A_\uparrow \rangle \) and \( |A_\downarrow \rangle \), which turn out to be pointer states. But since the time evolution is linear the dynamics of the distinct branches
in the superposition is completely independent of each other. The wavefunction of the whole system then becomes \(|\Psi\rangle = |A_1\rangle|E_i\rangle + |A_1\rangle|E_j\rangle\) after some transient relaxation time. Already here we see that interactions with an environment do not provide single outcomes. Quantum superpositions survive the interaction with environments. The closed Universe evolves into independent branches \([9]\), here, a superposition of two pointer states. There are no mechanisms that select a preferred evolution branch and the ferromagnet remains in a superposition with zero average magnetization. We clearly see the incapability of linear quantum mechanics to describe everyday observations. Macroscopic ferromagnets are not in superpositions, their magnetization have preferred directions. We show below that a small non-linear term in the ferromagnet may destroy the superpositions and select a preferred direction.

Nonlinearity has been argued to exist intrinsically in quantum mechanics \([18]\), for example at the Planck scale \([19]\). More importantly, macroscopic quantum coherent systems are often effectively described by nonlinear Schrödinger equations \([20]\). For example the dynamics of the quantum coherent superconducting state is governed by the nonlinear Ginzburg-Landau equation \([21]\). Another example is Bose-Einstein condensates where the mesoscopic quantum coherent wavefunction is governed by the nonlinear Gross-Pitaevskii equation \([22]\). The fractional quantum Hall state may also be described by an effective nonlinear equation \([23]\). From the “emergence” point of view, these kind of nonlinearity may naturally emerge as a consequence of the enormous number of interacting particles involved in the macroscopic coherent state which may have less symmetry compared to the underlying many-body Hamiltonian \([24]\). Finally, nonlinearity may also rise naturally as a consequence of the interaction between two fluctuating quantum fields in the mean-field limit, see Appendix A. In this paper we view nonlinearity as modeling physical reality, not only a convenient (and necessary) approximation.

A tiny nonlinearity will, by been shown below, force a macroscopic ferromagnet embedded in an environment to choose a preferred direction, consistent with experimental observations. For ferromagnets, one may argue for a physically sound, nonlinear term by considering the self-induced magnetic field created by the magnetization. Assuming that the spins in a ferromagnet are carried by charged particles, then in connection to the spin each particle also possess a small magnetic moment, \(g \mu \beta \mathbf{S}\), where \(\mathbf{S}\) is a dimensionless spin operator, \(g\) is the gyro magnetic factor and \(\mu \beta\) is the Bohr magneton. The magnetic moments create a magnetic field which may be, crudely, approximated to \(\mathbf{B} = \mu_0 g \mu \beta (|\Psi\rangle \sum_{i \in A} \mathbf{S}_i |\Psi\rangle)\) where \(\mu_0\) is the magnetic constant and \(|\Psi\rangle\) is the conventional normalized wavefunction of the system. Here, we have completely neglected spatial variations, fluctuations and other internal degrees of freedom in the magnetic field. We believe that those additional degrees of freedom in \(\mathbf{B}\) does not fundamentally modify the physical picture presented below. It is important to note that we use the operator \(\langle \Psi | ... | \Psi \rangle\) to denote a (inner) scalar product, no ensemble average is included in the process. The self-induced magnetic field may again interact with the spins in the ferromagnet giving rise to a nonlinear term in the Hamiltonian which may, at the mean field level, be expressed as

\[
H_B = -\mu \langle \Psi | \sum_{i \in A} \mathbf{S}_i |\Psi\rangle \cdot \sum_{j \in A} \mathbf{S}_j,
\]

where the parameter \(\mu\) controls the nonlinearity and has the dimension of energy. This non-linear term will favor the pointer state where the spins are parallel to the magnetic field and disfavor all other pointer states. Here is the physical picture. Each measurement apparatus has a set of attraction basins, related to the set of eigenvalues. Each attraction basin has a stable fixed point that is the pointer state. Interactions with environments will force all the parts of a superposition that reside in an attraction basin to relax into the basin’s stable fixed point, i.e. the basin’s pointer state. A self-induced magnetic field, \(\mathbf{B}\), above a certain threshold will single out a preferred attraction basin and transform all other initially attraction basins into repulsive basins and their stable fixed points into unstable fixed points. Consequently, the measurement apparatus has now a single unique attraction basin with its corresponding pointer state. Hence, the ferromagnet will, in the presence of a self-induced magnetic field together with an environment, evolve into a single unique pointer state.

In the case of no coupling to an environment, there is no mechanism for the spins in the ferromagnet to relax and the ferromagnet is then forced to remain in a superposition regardless of nonlinearity. With an environment, however, non-zero fluctuations in the magnetization will create a magnetic field which, when strong enough, enhances parallel spins and thereby increase the magnetization along that direction. Thus, environment induced fluctuations may start a self-enhancing process which favors the pointer state with spins parallel to the direction of the initial fluctuation. This process forces the system to choose a preferred attraction basin. During this process, the interactions with the environment also cause the wavefunction to fall into the fixed point of the chosen attraction basin, i.e. its pointer state. The probability for the wavefunction to end up in a given pointer state is governed by the size of the attraction basins, which for a ferromagnet alone is of course 50\% for ending as \(|\Psi\rangle = |A_1\rangle|E_i\rangle\) and 50\% for ending as \(|\Psi\rangle = |A_1\rangle|E_j\rangle\). We will discuss the the size of the attraction basins in more details below. In summary, a non-linear term in the Hamiltonian as Eq. (1) may somewhere along the time evolution of the ferromagnet chose a preferred attraction basin. In the same time, interactions with the environment will force the wavefunction to fall into an unique classical-like pointer state which is the stable fixed point of the chosen attraction basin. Which attraction basin that is actually chosen depend on small details of the ini-
tial state of the ferromagnet and the environment. An infinitesimal small change in the initial state may force the wavefunction to collapse in a entirely different pointer state. However, the dynamics is completely deterministic. Given the same initial condition, the wavefunction will always choose the same attraction basin and collapse into the same final pointer state. In this sense, quantum mechanics is a deterministic theory.

Nonlinearity in the ferromagnet can in principle be infinitesimally small and still do its job in the selection of the preferred attraction basin. The reason is as follows. For simplicity, consider a ferromagnet in 2 dimensions with the Hamiltonian

\[ H_t = -J \sum_{\langle i,j \rangle} S_i^z S_j^z \]  

(2)

where \( \langle i, j \rangle \) denotes nearest neighboring spin-\( \frac{1}{2} \) spins. This ferromagnet has two degenerate ground states, \( | \uparrow \uparrow \ldots \uparrow \rangle \) and \( | \downarrow \downarrow \ldots \downarrow \rangle \) with the energies \( E_\uparrow = E_\downarrow = -J N_A / 2 \) where \( N_A \) is the number of spins in the ferromagnet. The ground states are separated by an energy barrier with height \( -J N_A / 2 \). A magnetic field given by Eq. (1) will lift the degeneracy between \( E_\uparrow \) and \( E_\downarrow \) by \( \Delta E = E_\uparrow - E_\downarrow = \mu B / 4 \). Note that this argument also applies for the case where the magnetic field decays a as a function of distance from its sources provided that the exchange interaction between the spins decays much faster. Thus for large \( N_A \), the energy from the initially fluctuating magnetic field will not only define a global minimum, but also eventually grow strong enough in order to completely remove the local minimum connected to the anti-parallel ground state forcing both branches of the wavefunction to fall into the same fixed point defined by the global minimum. Therefore, an infinitesimally small nonlinear parameter \( \mu \) is sufficient to define a global minimum and remove all other local minima for a macroscopic ferromagnet.

We will from now on speak of the ferromagnet \( \mathcal{A} \) as an measurement apparatus, referring to the system as our measurement object. Including a measurement object, that in our case is a system spin \( S_{sys} \) interacting with the ferromagnet, will alter the size of the attraction basins of the ferromagnet. Here we provide an estimate for the probabilities showing that this change of size provides statistics in accordance with Born’s rule. First, let us exclusively focus on the case of a ferromagnet embedded in an environment with no measurement object. For clarity we again restrict the discussion to an Ising ferromagnet with spin-\( \frac{1}{2} \) spins. Define the dimensionless magnetic field for the ferromagnet as \( \tilde{B}(t) = \langle \Psi(t) | \sum_{i \in A} S_i^z | \Psi(t) \rangle \). Let the initial magnetization of the ferromagnet be zero, \( \tilde{B}(t=0) = 0 \). Interactions with environments force \( \tilde{B}(t) \) to deterministically fluctuate back and forth around zero. Assume that when \( \tilde{B} \) reaches a threshold, \( \pm \tilde{B}_c \), the nonlinear term dominates over the fluctuations and starts a self enhancing process which ends up in a macroscopic stable magnetization, see Fig. 1. A measurement apparatus that can measure the direction of a single spin must be sensitive to a flip of a single spin. Thus, \( \tilde{B}_c \approx \pm \frac{\beta}{2} \). In other words, when the environment has flipped one spin in the ferromagnet, nonlinearity will set in and force all other spins to align to that spin. Note that this value of \( B_c \) applies only for apparatus that are able to detect the direction of a single spin-\( \frac{1}{2} \) spin. If the initial states of the ferromagnet and the environment are symmetric with respect to the up and down pointer states, \( \tilde{B}(t) \) will fluctuate symmetrically around zero. Hence, the probability for that \( \tilde{B}(t) \) reaches \( -\tilde{B}_c \) and the ferromagnet will eventually choose the up state is 0.5. Similarly, the probability for that \( \tilde{B}(t) \) reaches \( -\tilde{B}_c \) and the ferromagnet will eventually end up in the down state is 0.5, see Fig. 1. Note that the dynamic from a state with zero magnetization into a state with macroscopic magnetization closely resembles the dynamics of a phase transition. The state of the system evolves from a state with the same symmetry as the underlying Hamiltonian into a state with less (broken) symmetry [24].

Now, add a system spin to be measured by a ferromagnet embedded in an environment. Let the system spin be in the initial state

\[ |S_{sys}(t=0)\rangle = \alpha | \uparrow \rangle + \beta | \downarrow \rangle \]  

(3)

where \( \alpha \) and \( \beta \) are complex scalars. Let for simplicity the initial state of the ferromagnet be in an antiferromagnetic configuration

\[ |A_0\rangle = \downarrow \downarrow \uparrow \uparrow \downarrow \downarrow \uparrow \downarrow \downarrow \ldots \downarrow \downarrow \ldots \uparrow \downarrow \]  

(4)

Of course, other states including those with superpositions which has zero magnetization will do the job. After a short time the system spin has interacted with the ferromagnet and flipped its spin leading to a combined state where one spin in the ferromagnet is flipped, e.g.

\[ \Psi = | \uparrow \rangle | \downarrow \downarrow \uparrow \uparrow \ldots \downarrow \downarrow \ldots \downarrow \rangle \]  

(5)

The dimensionless magnetic field is now \( \tilde{B} = \frac{1}{2} (\alpha^2 - \beta^2) \) so the starting point of \( \tilde{B}(t) \) is no longer at zero but at \( \frac{1}{2} (\alpha^2 - \beta^2) \), see Fig. 1. We see also that the size of the “up” attraction basin is \( \frac{1}{2} + \frac{1}{2} (\alpha^2 - \beta^2) \) while the size of the “down” attraction basin is \( \frac{1}{2} - \frac{1}{2} (\alpha^2 - \beta^2) \). The probability ratio for \( \tilde{B} \) to reach \( \tilde{B}_c \) first versus \( -\tilde{B}_c \) is therefore, see Fig. 1,

\[ \frac{P_\uparrow}{P_\downarrow} = \frac{\frac{1}{2} + \frac{1}{2} (\alpha^2 - \beta^2)}{\frac{1}{2} - \frac{1}{2} (\alpha^2 - \beta^2)} = \frac{\alpha^2}{\beta^2} \]  

(6)

In the last equality we have used the normalization condition \( \alpha^2 + \beta^2 = 1 \). Thus, the probability for the ferromagnet to end with a stable up state is \( P_\uparrow = \alpha^2 \). And, the probability for the ferromagnet to end with a stable down state is \( P_\downarrow = \beta^2 \), which are in accordance with Born’s rule.

Our approach should not be confused with that of the dynamical reduction program [12, 25], where the Schrödinger equation is altered by inclusion of stochastic
nonlinear terms in order to achieve dynamical or spontaneous localisation of wave packets. The basic idea behind the dynamical reduction program is that the wavefunction randomly collapses all the time, where the rate of collapse is related to the mass of the system. This applies to all particles whether isolated or interacting [12, 25]. In our approach, an isolated system remains unperturbed until the different systems start to interact. It is the asymmetry between the size of the environment and the rest that causes a classical appearance of the state of the measurement apparatus and the measured object upon measurements.

Ref. [26] considered a self-collapse mechanism where the quantum wavefunction automatically exhibit dynamical collapse, without any measurement apparatus nor environment, due to nonlinearity originating from non-Abelian gauge fields. Our description of a measurement process is completely different from Ref. [26]. We find that the dynamics of the wavefunction strongly depend on the existence of a macroscopic measurement apparatus embedded in an environment that is able to absorb entropy and energy from the measurement apparatus.

Ref [27] studied a collapse of the quantum wavefunction due to a spontaneous symmetry breaking process in the measurement apparatus, an anti-ferromagnet in the thermodynamical limit. To induce the collapse Ref [27] used a time dependent, fluctuating, symmetry breaking, non-unitary, staggered magnetic field with an undetermined origin. A major difference between our model and the model in Ref [27] is the importance of the environment in the collapse of the wavefunction. We studied the evolution of a pure wavefunction describing a closed Universe.

Note that the results of this work by no means contradict the predictions of quantum mechanics. Merely, the results shows that quantum mechanics may be regarded as a complete theory which can describe measurement processes and predicts single outcomes of classical alike states without the collapse postulate. In other words, the results show that quantum mechanics is a deterministic theory. The price to pay is a tiny non-linearity effectively only in systems of macroscopic sizes.

II. MODEL

To support the ideas discussed above in the introduction, we perform numerical modelling of an idealized, but realistic, model of a quantum measurement process. Our model is composed of a system spin, $S_{sys}$, being measured by a ferromagnet ($A$) embedded in a spin-glass environment ($E$). The system spin is a spin-$\frac{1}{2}$ spin. The measurement apparatus is a ferromagnet consisting of a number $N_{A} = 4$ or $N_{A} = 8$ spin-$\frac{1}{2}$ spins. In addition to the traditional spin-spin exchange couplings, the spins in the ferromagnet also interact with its self-induced magnetic field, giving rise to a weak non-linear term. Finally, the system spin and the ferromagnet are embedded in an environment consist of a $N_{E} = 15$ spin-$\frac{1}{2}$ spins with random, frustrated, spin-glass interactions.

A. Hamiltonian

Our model Hamiltonian may be written as

$$H = H_A + H_E + H_{AE} + H_{SA} + H_{SE} + H_B \quad (7)$$

where

- $H_A = -\sum_{i,j \in A} \sum_{\alpha} J_{ij}^{\alpha} S_{i}^{\alpha} S_{j}^{\alpha}$
- $H_{AE} = \sum_{i \in A} \sum_{j \in E} \sum_{\alpha} \Delta_{ij}^{\alpha} S_{i}^{\alpha} I_{j}^{\alpha}$
- $H_E = \sum_{i,j \in E} \sum_{\alpha} \Theta_{ij}^{\alpha} I_{i}^{\alpha} I_{j}^{\alpha}$
- $H_{SE} = \sum_{i \in E} \sum_{\alpha} \Omega_{i}^{\alpha} S_{sys}^{\alpha} I_{i}^{\alpha}$
- $H_{SA} = \sum_{i \in A} \sum_{\alpha} \Gamma_{i} S_{sys}^{\alpha} S_{i}^{\alpha}$
- $H_B = -\mu \langle \Psi | \sum_{i \in A} S_{i}^{z} | \Psi \rangle \sum_{j \in A} S_{j}^{z}$

The exchange couplings $J_{ij}^{\alpha}$, $\Theta_{ij}^{\alpha}$ and $\Delta_{ij}^{\alpha}$ control the interaction between spins in the ferromagnet, the spin-glass environment and between the spins in the ferromagnet and the environment, respectively. Furthermore, $\Gamma_{i}$ controls the interaction between the system spin and the apparatus while $\Omega_{i}^{\alpha}$ controls the interaction between the system spin and the apparatus.
spin and the environment. In more physical terms, the measurement apparatus is modelled by a ferromagnet, $H_A$. A spin glass, $H_E$, serves the role of an environment which absorbs the energy and entropy of the ferromagnet allowing it to relax towards its ferromagnetic ground states. The system to be measured is a single spin-$\frac{1}{2}$ spin, $S_{sys}$, which role is to tilt the up-down symmetry of the ferromagnet. The interaction between the ferromagnet and the spin glass is modelled by $H_{AE}$. While $H_{SA}$ describes the interaction between the system spin and the ferromagnet. To make the measurement process more realistic, we have also a coupling between the system spin and the environment, $H_{SE}$.

The Hamiltonian given by Eq. (7) without the non-linear term, $H_B$, is linear and will not be able to conduct any measurement process as discussed in the introduction and shown in more details below. This linear part of the Hamiltonian can not choose a preferred pointer states among the possible pointer states for the ferromagnet. To model a ferromagnet that shows the behavior found in experiments, we introduce a non-linear term, $H_B$, which describes the interaction between the self-induced magnetic field and the spins in the ferromagnet. Physically, the self-induced magnetic field ($B$) originates from the magnetization of the ferromagnet, $B \propto M \propto \sum_{i \in \Lambda} \langle \Psi | S_i^z | \Psi \rangle$. Again, $| \Psi \rangle$ is the normalized wavefunction for the whole system, i.e. closed Universe. We use the parameter $\mu$ to control the non-linear self coupling between the magnetic field and the spins $S_i$ in the ferromagnet. In the macroscopic limit, an infinitesimal small $\mu$ will force the ferromagnet to choose a direction. Note that these kind of one-particle non-linear interaction can also be argued to originate from linear terms like $S_i \cdot S_j \rightarrow \langle S_i \rangle \cdot S_j$. This kind of mean field approximation has been used with tremendous success in other macroscopic many-body quantum systems as superconductivity [21], superfluidity [22] and fractional quantum hall states [28–30]. There, the mean field solution accurately describes the properties of complicated many-body quantum wavefunctions in the macroscopic limit. Another source for the nonlinear term is the interaction between $\Psi$ and a magnetic field treated in the macroscopic limit, an infinitesimal small $\mu$ will force the ferromagnet to choose a direction. Note that these kind of one-particle non-linear interaction can also be argued to originate from linear terms like $S_i \cdot S_j \rightarrow \langle S_i \rangle \cdot S_j$. This kind of mean field approximation has been used with tremendous success in other macroscopic many-body quantum systems as superconductivity [21], superfluidity [22] and fractional quantum hall states [28–30].

A similar Hamiltonian without the system spin and the non-linear self induced magnetic field $B$ has been studied in Ref. [31]. There, they study, numerically, how the spin glass (E) relaxes the ferromagnet (A) [31–33]. It was found that frustrated spin-glasses where very effective with respect to relaxation and decoherence of the central system, even for small numbers of spins ($N_E \approx 10$) in the environments. Spin glasses are therefore ideal as environments for small spin systems.

B. Preparation and numerical method

This section describes the preparation of the initial state and the procedure for the numerical time integration of the Schrödinger equation.

First, we prepare an initial state

$$| \Psi(t=0) \rangle = | S_{sys}(t=0) \rangle \otimes | A_0 \rangle \otimes | E_0 \rangle.$$  \hspace{1cm} (8)

The system spin to be measured is prepared in a general superposition

$$| S_{sys}(t=0) \rangle = | \uparrow \rangle \cos \theta + | \downarrow \rangle \sin \theta e^{i \phi},$$  \hspace{1cm} (9)

where the phase angle $\phi \in [0, 2\pi]$ is not so important for the physical picture and henceforth set to 0. The ferromagnet is for simplicity prepared in an antiferromagnetic state $| A_0 \rangle = | \uparrow \downarrow \ldots \uparrow \downarrow \rangle$. Of course, other states with zero magnetization would do the same job. The environment is prepared in a state close to the spin-glass ground state $| E_0 \rangle$ computed by the Lanczos method [34].

Next, we iteratively time integrate the Schrödinger equation $i \frac{\partial \Psi}{\partial t} = H \Psi$ using the finite difference method,

$$\Psi(t + dt) = \Psi(t) - iH \Psi dt,$$  \hspace{1cm} (10)

starting from the initial state, Eq. (8). The Hamiltonian $H$ is given by Eq. (7). For clarity, we have set $\hbar = 1$ in Eq (10) and throughout the article. Each iterative step in Eq. (10) is found numerically by the Chebyshev expansion method [35–37]. The method is based on a polynomial expansion of the propagator $U(t, t_0) = e^{i(t-t_0)H} \approx J_0(t) + 2 \sum_{k=1}^{K} J_k(t) T_k(H)$, where $T_k(x)$ is the Chebyshev polynomial of the first kind and $J_k(x)$ is the Bessel function of integer order $k$. The Chebyshev moments $T_k(H)$ is computed by the recursive application of the Hamiltonian operator $H$ on the wavefunction $| \Psi \rangle$. We have verified that the numerical method preserves the norm of the wavefunction, $\langle \Psi(t) | \Psi(t) \rangle$ for all $t$. Furthermore, the conserved energy of the closed universe

$$E_U = \langle \Psi | H - \frac{1}{2} H_B | \Psi \rangle$$  \hspace{1cm} (11)
has also been verified to be conserved in the numerical calculations. A short derivation of this form of the conserved energy for our model $E_U$ is given in Appendix A.

The Hilbert space of the composite system $H = H_S \otimes H_A \otimes H_E$ is of dimension $2^L$, where $L$ is the total number of spins, system spin plus ferromagnet plus spin-glass. Present computing power limits us to model systems where $L \leq 24$.

The following parameters are fixed in all simulations: $\Delta = 0.3$, $\Omega = 0.8$, $\Theta = 0.5$, $N_E = 15$. The remaining parameters: $N_A = 4 \rightarrow \mu = 12.0$ or $N_A = 8 \rightarrow \mu = 6.0$. The nonlinear parameter $\mu$ is varied such that the effect of maximum magnetization on each spin in the ferromagnet is constant, $\mu N_A = 48.0$. This allows direct comparisons between the measurement results from ferromagnets of different sizes. Since our closed Universe is very small, the non-linear parameter had been set relatively large to optimize the measurement properties of our tiny measurement apparatus. In the macroscopic limit the non-linear parameter may be infinitesimal small and still do the same job, as argued for in the introduction.

III. RESULTS

In this section we present the numerical results for the model given by Eq. (7). The results support our claim that the wavefunction of the measurement apparatus, in the presence of a small nonlinear term, is forced to choose one of the two classical pointer states $| A_I \rangle$ or $| A_J \rangle$. Without the non-linear term, the ferromagnet remains in the superposition regardless of the interaction with the environment. We also present statistics for measurements with different initial configurations for various $\theta$, and show that the presence of a measurement object will alter the size of the attraction basins in the measurement apparatus in a way such that the outcome is consistent with Born’s rule to the numerical precision.

A. Time evolution

It is difficult to extract useful information directly from the complex many-body quantum state of the whole system which is, here, a $2^{20}$ or $2^{24}$ dimensional complex vector. In order to characterize the ferromagnet, we use the dimensionless magnetization

$$M(t) = \sum_{i \in A} \langle \Psi(t)| S^z_i | \Psi(t) \rangle$$

(12)

which determines the degree of ferromagnetic order in the measurement apparatus. In addition, we use the exchange energy

$$E(t) = -\sum_{i,j \in A} J_{ij} \langle \Psi(t)| S_i \cdot S_j | \Psi(t) \rangle$$

(13)

to characterize the relaxation of the ferromagnet. To characterize the state of the system spin we use the expectation value

$$\langle S^z_{sys}(t) \rangle = \langle \Psi(t)| S^z_{sys} | \Psi(t) \rangle.$$  

(14)

Again, note that we use the operator $\langle \Psi| \cdots | \Psi \rangle$ to denote a (inner) scalar product, no ensemble average is performed.

![Figure 2: Typical time evolution of the magnetization $M$ (solid) and the exchange energy $E$ (dashed) for a linear ferromagnet and the expectation value of the system spin (dashed dot). The initial state of the system spin is $| S_{sys}(t=0) \rangle = \frac{1}{\sqrt{2}}(| \uparrow \rangle + | \downarrow \rangle)$. The number of spins in the ferromagnet is $N_A = 4$.](image)

Fig. 2 shows a typical time evolution of the magnetization $M$ and the exchange energy $E$ characterizing the state of a linear ferromagnet, i.e. $\mu = 0$. The system spin is initially prepared in the state $| S_{sys}(t=0) \rangle = \frac{1}{\sqrt{2}}(| \uparrow \rangle + | \downarrow \rangle)$. We see that the exchange energy of the ferromagnet decreases rapidly, as the ferromagnet relaxes from the anti-ferromagnetic state towards its ferromagnetic ground states, $| A_I \rangle$ and $| A_J \rangle$. Note that due to the limited number of spins used in the ferromagnet, $N_A = 4$, the interaction with the large environment prevents the ferromagnet to relax all the way down to the ferromagnetic ground states where $E = -1.35$. We also see in Fig. 2 that the magnetization $M$ shows small fluctuations, but no sustained net magnetization characterizing the expected classical ground state. Fig. 2 also shows that the expectation value of the measured system spin is zero, indicating that $| S_{sys} \rangle$ remains in a superposition between $| \uparrow \rangle$ and $| \downarrow \rangle$. A more careful examination of the stationary state shows that the composite wavefunction of our closed universe is still in a superposition. Thus, the magnetization fluctuates around zero and the ferromagnet does not choose any preferred pointer state. From Fig. 2, one may conclude that linear ferromagnets are not able to conduct any measurement. The superposition persists regardless of the interaction with the environment. The time evolution

$$\langle (| \uparrow \rangle + | \downarrow \rangle)| A_0 \rangle | E_0 \rangle \rightarrow | \uparrow \rangle| A_1 \rangle | E_1 \rangle + | \downarrow \rangle| A_1 \rangle | E_1 \rangle,$$

(15)

describes the dynamics in the linear case. The initial state of the system becomes entangled with the apparatus and the environment. The two branches of Eq. (15)
evolve completely independent of each other, and the initial superposition has through interaction with its environment extended to our entire closed universe. Thus, decoherence, in the sense of relaxation to the pointer states, alone is not sufficient to remove the initial superposition. It should be noted that the persistence of the superposition in this case is entirely due to the exact linearity of the dynamical equation, and has nothing to do with the size of the apparatus. It is straightforward to show that superpositions survive any linear finite sized ferromagnet, see Appendix B.

![Figure 3](image-url) Figure 3: A typical time evolution of the magnetization $M$ (solid) and the exchange energy $E$ (dashed) of a non-linear ferromagnet, and the expectation value $\langle S_{sys} \rangle$ (dashed dot). The system spin is initially prepared in the state $|S_{sys}(t = 0)\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle)$. The number of spins in the ferromagnet is $N_A = 4$.

![Figure 4](image-url) Figure 4: Same as in Fig. 3. The only difference is the initial state of the environment.

Typical time evolutions of the magnetization $M$ of a non-linear ferromagnet is shown in Figs. 3 and 4. After initial fluctuations due to the environmental interaction, one of the two possible magnetization directions is enhanced due to the feedback provided by interaction with the self-induced magnetic field $B$. A “phase transition” in the ferromagnet takes place, where the ferromagnet relaxes into the pointer state of the attraction basin selected by the emerged magnetic field. We have quoted the term “phase transition” here since our tiny ferromagnets consist of only 4 spins and phase transitions happen in principle only in the thermodynamical limit. Rather, finite size effects [38] are expected to dominate the behavior of our small ferromagnets. This can be seen in the actual values for $M(t)$, $E(t)$ and $S_{sys}(t)$. They are all significantly less than the values expected in the thermodynamical limit which are: $M = 0.5 + N_A = 2.0$, $E = -1.35$ and $S_{sys} = 0.5$. However, the stationary values of $M$, $E$ and $S_{sys}$ are stabilized well beyond the noise and one may clearly conclude that a measurement has been conducted by an admittedly rather poor measurement apparatus.

In more physical terms, when the magnetization reaches a critical value, the non-linear term sets in and turns all attraction basins, except the chosen one, into repulsive basins. From this point there is no turning back, all superpositions in the wavefunction of the closed Universe are forced to evolve such that the state of the ferromagnet becomes the unique pointer state chosen by the magnetic field. Thus, as a result of the measurement dynamics the spin expectation value of our system $\langle S_{sys} \rangle$ will follow the direction of the established magnetization $M$. The state of the measurement object has been driven into the state corresponding to the final outcome of the measurement apparatus. The final result is in correspondence to the result obtained by the collapse postulate of orthodox quantum theory where an initial state ($|\uparrow\rangle + |\downarrow\rangle$) $\otimes$ $|A_0\rangle$ is instantaneously and discontinuously collapsed upon measurement to either $|\uparrow\rangle \otimes |A_1\rangle$ or $|\downarrow\rangle \otimes |A_1\rangle$. We might therefore conclude that a quantum measurement of a single event has been accomplished. The total wavefunction is

$$|\Psi\rangle = |\uparrow\rangle|A_1\rangle|\tilde{E}_1\rangle$$ (16)

after a measurement where the up pointer state is chosen. Note that the final state of the environment $|\tilde{E}_1\rangle$ still contains superpositions inherited from the initial superposition of the measured spin.

The only difference in the initial state $|\Psi\rangle$ of the composite system between Fig. 3 and Fig. 4 is the different initial state of the environment. Note that the different initial states of the environment are completely equivalent, they all belong to the set of degenerate ground states of the spin glass. We see that a small difference in the initial state of the environment does in this case alter the dynamics of the measurement process completely. In this example the slight altering of the initial state resulted in a different final pointer state of the measurement apparatus. It is this sensitivity to the initial state that gives rise to the indoctrinated randomness in orthodox quantum mechanics, and to Born’s rule.

### B. Statistics

According to Born’s rule [39], in an ideal quantum mechanical measurement the probability for our measure-
moment apparatus to end in the state of positive magnetization $|A\uparrow\rangle$, when the object is prepared in the state $|S_{sys}(t=0)\rangle$ of Eq. (9) is

$$P_1(\theta) = \cos^2 \theta.$$ (17)

Correspondingly, the probability for ending in the negative magnetization state is

$$P_1(\theta) = \sin^2 \theta.$$ (18)

To obtain measurement statistics, we run a number of 96 independent simulations for each choice of $\theta$. For each simulation, a different spin glass ground state is used as the initial state for the environment.

The probability for the measurement apparatus and the system spin to end up as a function of $\theta$ for a ferromagnet consist of $N_A = 4$ spins is shown in Fig. 5. Here, the non-linear parameter is $\mu = 12.0$. We have verified that $P_1(\theta) + P_1(\theta) = 1$. We see in Fig. 5 that the probabilities of obtaining the pointer state $|\uparrow\rangle|A\uparrow\rangle$ as the outcome of the measurement process resemble Born’s rule. The discrepancies at small and large $\theta$ are due to the huge finite size effect in the tiny measurement apparatus. The subsystem of system spin plus apparatus are so small, that the interaction with the large environment affects the measurement result. Needless to say, a 4-spins ferromagnet is far from a perfect measurement apparatus. It is, for us, surprising that this tiny ferromagnet actually provides results so close to Born’s rule. One reason may be due to the up-down symmetry of the ferromagnet and of the environment. Since the subsystem of system spin plus ferromagnet is so small, the “noise” from the environment dominates and the probability for ending up is close to 50% regardless of the initial state of the system spin. Thus, $P_1(\theta = 45) = 0.5$ is guaranteed by symmetry.

To show that larger ferromagnets will actually provide better measurement results, we have also carried out simulations with a system with a $N_A = 8$ spins ferromagnet. Here, the non-linear parameter is $\mu = 6.0$. Recall that, this value of $\mu$ is chosen to conserve the effect of maximum magnetization on each spin in the ferromagnet, $\mu N_A = 48$, such that a direct comparison with the $N_A = 4$ case is possible. All other parameters are identical as above. As seen in Fig. 6, the probability for the ferromagnet to end in an up state follows Born’s rule more closely compared to the $N_A = 4$ spins ferromagnet. This is expected, since the finite size effect is smaller here. By comparing the cases $N_A = 4 \rightarrow \mu = 12.0$ and $N_A = 8 \rightarrow \mu = 6.0$, we see that the nonlinear parameter $\mu$ decreases for increasing size of the ferromagnet. Thus, for a macroscopic system the non-linear parameter $\mu$ may be infinitesimally small and still do its job: selecting a preferred attraction basin and consequently a pointer state.

Obviously the obtained statistics is highly dependent on the choice of parameters, for example on the strength of the nonlinear interaction $\mu$. We have chosen the parameters of our model in such a way that our measurement apparatus is optimized for measuring the $z$-component of a single spin.

![Figure 5](image5.png)

Figure 5: Probability for a ferromagnet of 4 spins to end in an up pointer state as a function of the angle $\theta$ of the initial system spin state $|S_{sys}(t=0)\rangle = |\uparrow\rangle \cos \theta + |\downarrow\rangle \sin \theta$. Solid line shows ideal statistics according to Born’s rule.

![Figure 6](image6.png)

Figure 6: Probability for a ferromagnet of 8 spins to end in an up pointer state as a function of the angle $\theta$ of the initial system spin state $|S_{sys}(t=0)\rangle = |\uparrow\rangle \cos \theta + |\downarrow\rangle \sin \theta$. Solid line shows ideal statistics according to Born’s rule.

IV. CONCLUSION

We show from a model study of a simple closed Universe that definite outcomes of quantum measurements can emerge continuously from pure quantum evolution. Each measurement apparatus has a set of attraction basins corresponding to the set of eigenvalues. The selection of a preferred attraction basin in a measurement process is caused by an infinitesimally small nonlinearity in the measurement apparatus. Interaction with the environment then forces all superpositions of the measured object and the measurement apparatus to fall into the same fixed point (pointer state) of the preferred attraction basin. The dynamics of this measurement process which strongly resembles the dynamics through phase
transitions is entirely deterministic! Given the same initial condition the wavefunction will always fall into the same pointer state. However, small changes in the initial state, for example in the environment, may cause the wavefunction to fall into a completely different pointer state. Thus, the uncontrollable degrees of freedom in the environments account for the intrinsic statistical behavior of the quantum measurement process. The probability for falling into a certain pointer state is governed by the size of the attraction basins of the measurement apparatus and is shown to be in close agreement with Born’s rule.

How the dynamics of non-local experiments, e.g. the EPR experiment [40], will be affected by non-linear measurement apparatus is an interesting future study.

To conclude, we have presented a model where the probabilistic behaviour of Quantum mechanics has exactly the same origin as the random outcomes of a rolling dice, i.e. its sensitive behaviour of uncontrollable initial conditions. Thus, even though neither He nor Quantum Mechanics are playing at dice, we physicists still have to.

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Appendix A: CONSERVATION OF ENERGY

In this appendix, we derive the expression for the conserved energy for our model, Eq. (11). The non-linear term is shown to introduce an extra term in the expression for the conserved energy in addition to the standard average of the Hamiltonian. The appendix also shows how a non-linear term as the one used in our model may naturally arise from the interaction between two different fields.

For clarity, we consider a simple quantum field model having the basic properties of our model, Eq. (7). The Lagrangian of our quantum field model with Einstein sum-convention over repeated indices assumed can be written as

$$L = i \overline{\Psi}_a \dot{\Psi}_a - \Psi^*_a H_{ab} \Psi_b - \Phi_\alpha \Psi_a^* \Gamma_{ab} \Psi_b + \frac{1}{2} \overline{\Phi}_\alpha \dot{\Phi}_\alpha - V(\Phi).$$

(A1)

Here, $\Psi$ is a complex field representing the wavefunction and $H_{ab}$ is the Hamiltonian for $\Psi$ where $a$ and $b$ are multi-indices which may include spin number, spin direction and also continuous variables like space coordinated etc. In Eq. (A1), $\Phi$ is a scalar field which may be associated with for example a harmonic oscillator with mass $\epsilon$ or in certain limit the electromagnetic field. The multi-index $\alpha$ denoting all the variables necessary to describe the field $\Phi$ which lives in a potential $V(\Phi)$.

From the Euler-Lagrange equation

$$\partial_t \left( \frac{\partial L}{\partial (\dot{\Psi}_a)} \right) - \frac{\partial L}{\partial \Psi_a} = 0 \quad (A2)$$

$$\partial_t \left( \frac{\partial L}{\partial (\dot{\Phi}_\alpha)} \right) - \frac{\partial L}{\partial \Phi_\alpha} = 0, \quad (A3)$$

we deduce the dynamical equations

$$i \dot{\Psi}_a = H_{ab} \Psi_b + \Phi_\alpha \Gamma_{ab}^\alpha \Psi_b \quad (A4)$$

$$\epsilon \dot{\Phi}_\alpha + \frac{\partial}{\partial \Phi_\alpha} V(\Phi) = -\Psi^*_a \Gamma_{ab}^\alpha \Psi_b. \quad (A5)$$

Without the coupling between the fields when $\Gamma_{ab}^\alpha = 0$, Eq. (A4) becomes the conventional Schrödinger equation for $\Psi$ in the first quantized language. Furthermore, Eq. (A5) take the form of Newton’s second law for the field $\Phi$.

To deduce the conserved energy we now assume that the field $\Phi$ is, similar to the electromagnetic field, mass-less i.e $\epsilon = 0$ and that the potential is of the form

$$V(\Phi) = \frac{1}{2} \Phi_\alpha \Phi_\alpha \quad (A6)$$

corresponding to the potential energy for the electromagnetic field $\frac{1}{2} B^2$. Equation (A5) is then simplified to the constraint equation

$$\Phi_\alpha = -\Psi^*_a \Gamma_{ab}^\alpha \Psi_b, \quad (A7)$$

which inserted into the Lagrangian Eq. (A1) give the effective Lagrangian for the field $\Psi$

$$L' = i \Psi^*_a \dot{\Psi}_a - \Psi^*_a H_{ab} \Psi_b - \frac{1}{2} \Phi_\alpha \Psi^*_a \Gamma_{ab}^\alpha \Psi_b. \quad (A8)$$

We can now deduce the conserved energy

$$E = \Psi^*_a H_{ab} \Psi_b + \frac{1}{2} \Phi_\alpha \Psi^*_a \Gamma_{ab}^\alpha \Psi_b. \quad (A9)$$

The conserved energy in our model, Eq. (7), is then

$$E_U = \langle \Psi | H - \frac{1}{2} H_B | \Psi \rangle \quad (A10)$$

if one relate the interaction term $-\Phi_\alpha \Psi^*_a \Gamma_{ab}^\alpha \Psi_b$ in Eq. (A1) with our non-linear term $\frac{1}{2} H_B = -\mu \left( \sum_{i \in A} \langle S_i^z \rangle \right) \sum_{j \in A} S_j^z$ and $\Phi$ with $\sum_{i \in A} \langle S_i^z \rangle$. Thus, we see that the nonlinear term $H_B$ gives rise to an additional term $-\frac{1}{2} H_B$ in the expression for the conserved energy compared to the conventional $E = \langle H \rangle$.

More importantly, one see that the interaction between the fields $\Phi$ and $\Psi$ give rise to a nonlinear term for the dynamics of $\Psi$ in the mean field limit, i.e when $\Phi \to \langle \Phi \rangle$. 

Appendix B: LINEAR TIME EVOLUTION

We will in this section show that superpositions survive interaction with arbitrary macroscopic measurement apparatus as long as the time evolution is linear. Consider a system to be measured, initially prepared in the superposition $|\uparrow\rangle + |\downarrow\rangle$. The system is being measured by a measurement apparatus $A$, interacting with an environment $E$. We can write the initial state of the composite system as

$$\Psi = (|\uparrow\rangle + |\downarrow\rangle) \otimes |A_0\rangle \otimes |E_0\rangle \quad \text{(B1)}$$

According to linear quantum mechanics, time evolution is governed by the time evolution operator $U(t, t_0)$, acting linearly on any quantum state

$$U(t, t_0)(\Psi_A + \Psi_B) = U(t, t_0)\Psi_A + U(t, t_0)\Psi_B. \quad \text{(B2)}$$

Thus the time evolution of branch $A$ of the wavefunction is completely independent of branch $B$.

When applied to a quantum measurement linear quantum evolution lead to the von Neumann chain of infinite regress [3]

$$\left(|\uparrow\rangle + |\downarrow\rangle\right) |A_0\rangle |E_0\rangle \xrightarrow{t} \left(|\uparrow\rangle |A_1\rangle |E_1\rangle + |\downarrow\rangle |A_1\rangle |E_1\rangle \right). \quad \text{(B3)}$$

Since the environment ($E$) can in principle denote every degree of freedom in the universe, including observers, we see that linear time evolution result in the initial superposition of the system being measured eventually extending to the entire universe. This time evolution (i.e linear quantum mechanics without the collapse postulate), naturally lead to the Everett relative state interpretation [9]. Where the universe continuously split in distinct independent branches.