Estimate of Crossed-Boson-Exchange Contributions to the Binding Energy of Two-Body Systems

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January 13, 2022

Abstract

Binding energies calculated from using the Bethe-Salpeter equation in the simplest ladder approximation significantly differ from those obtained in the non-relativistic standard instantaneous approximation. While they should a priori be better, they turn out to be further away from an exact calculation in the case of scalar neutral particles or from experiment in the case of the Coulomb interaction. Part of the discrepancy is due to the omission in the interaction kernel of contributions corresponding to crossed-boson-exchange diagrams. The role of these contributions is examined numerically, using a simple approximation. The sensitivity to both the coupling constant and the mass of the exchanged boson is considered.

PACS numbers: 03.65.Ge, 11.10.St, 12.40.Qq
Keywords: crossed-boson exchange, effective interaction, binding energies

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1 Introduction

In calculating binding energies of a two-body system, it is a current practice to rely on a meson-exchange-mediated interaction in the instantaneous approximation. Apparent support for this approach in the non-relativistic domain comes from the success of the Coulomb interaction in explaining atomic spectra. In the case of a massive neutral scalar boson exchange, the standard Yukawa potential provides binding energies that are halfway from those given by a field-theory-motivated calculation [1]. Curiously, in both cases, calculations based on the Bethe-Salpeter equation [2] in the ladder approximation, which should represent an improvement over these non-relativistic calculations, are not doing so well. The binding energies so obtained, for a zero-mass boson case (Wick-Cutkosky model [3, 4]) or a massive one, are much lower [5, 6] and [1, 7] respectively.

The discrepancy has been examined in different papers [7, 8, 9] where it was shown that the standard instantaneous approximation, which leads to the Coulomb or Yukawa potentials, is too a drastic approximation. It neglects retardation effects in relation with the fact that, part of the time, the two constituents are accompanied by the bosons they are exchanging. Simplifying a little, one can say that, during this time, the constituents do not interact with each other as far as one relies on the ladder approximation. The force between the constituents is thus effectively reduced, hence smaller binding energies. However, if one imagines that these constituents are allowed to interact and that this interaction is the same as in absence of in-flight bosons, one can expect the effective interaction between the constituents to be transparent to these in-flight bosons, recovering in this case the standard instantaneous-approximation-potential [10, 11, 12, 13, 14, 15]. This supposes to incorporate in the interaction the contribution of crossed-boson exchanges. It also supposes that the exchanged bosons carry neither spin nor charge, otherwise the interaction between the constituents would differ, depending whether there are in-flight bosons or not.

There are rather few calculations incorporating the contribution of crossed-boson exchange diagrams. Moreover, they essentially involve two-boson exchange. This was considered in atomic physics [16] or for the nucleon-nucleon interaction [17, 18]. Within theoretical models, this was also considered for scalar particles where their effect was found to almost remove the above retardation effect on the strength of the interaction [19]. Later studies however revealed that the meaning of these results was not quite clear due to bad convergence properties with respect to the order of diagrams under consideration. Part of the problem arises from the fact that, for some diagram, it is possible to incorporate more or less of the higher order contributions (with respect to the coupling constant). Thus, an alternative approach [9] led to a significantly lower contribution from the two-crossed-boson exchange to the binding energy, leaving entire the problem of recovering at least the benchmark-binding-energies obtained in the instantaneous approximation. A similar conclusion could be drawn from results obtained by other approaches [20].

In the present work, we will consider the contribution of higher-order crossed-boson-exchange diagrams. As exact calculations are practically unfeasible\footnote{unless to use methods of ref. [1], but in that case no detailed information is available on the role of different orders for instance}, we will use some
approximation which, actually, underlies for a part the demonstrations tending to show that the crossed-boson-exchange allows one to recover the instantaneous potential in the case of spin-less and charge-less bosons [10, 11, 12, 13, 14, 15]. Taking for granted the above theoretical works, the main motivation of the present paper will be to illustrate numerically how the convergence to the expected result is achieved, depending on the coupling or on the mass of the exchanged boson. On the other hand, the above theoretical works are not so well known and we believe that it is not unnecessary to remember the role of the crossed-boson exchange in recovering the standard instantaneous-approximation-potential. This study is made at the lowest order in the inverse of the constituent mass.

The plan of the present paper is as follows. In the second part, we remind the expressions that we are using and their relation to a diagram expansion. The third part is devoted to the presentation of the results for the binding energies in terms of the coupling constant and the boson mass. This is done as a function of the order of the crossed diagrams that are taken into account. A short conclusion is given in the fourth section.

2 Expressions used for the contribution of crossed diagrams to the interaction

We first consider in this section the expression of the effective potential introduced to account for some retardation effects, going therefore beyond the standard instantaneous approximation [8, 9]. In the following part, we start from this effective interaction to introduce a contribution that could globally account for crossed diagrams. How this is realized in practice is detailed.

2.1 Effective interaction in the ladder approximation

Our starting point for our study is a single-boson-exchange contribution, field-theory motivated interaction, represented by the time-ordered diagrams represented in Fig. 1.

Figure 1: Time-ordered single-boson-exchange contributions to the two-body interaction with indication of the kinematics in the center of mass.

Its expression is obtained from second-order perturbation theory. In the center of mass, the two terms representing the contributions of the diagrams displayed in Fig. 1 are equal,
providing some simplification and allowing one to write:

\[ V_E(\vec{p}, \vec{p}') = \frac{g^2}{\omega_k} \left( \frac{1}{E - \omega_k - e - e'} \right), \]

with \( \omega_k = \sqrt{\mu^2 + k^2}, \quad e = \frac{\vec{p}^2}{2m}, \quad e' = \frac{\vec{p}'^2}{2m} \) \hfill (1)

\( \vec{k}, \vec{p}, \vec{p}' \) representing respectively the momentum of the exchanged boson and of the constituents in the initial and final states. Anticipating on the non-relativistic character of the present work, the different energies appearing in the above interaction, Eq. (1), have been replaced by their non-relativistic counter-part. The energy dependence appearing at the denominator of the meson propagator, which stems from the field-theory character of the approach, is nevertheless fully kept. As shown in [7, 8, 9], it is the key ingredient that allows one to approximate results from the Bethe-Salpeter equation in the ladder approximation while providing departure to the instantaneous-approximation results. These last ones are recovered when the term at the denominator, \( E - e - e' \), is neglected.

Consistently with neglecting relativistic corrections, the above interaction is used with the following non-relativistic equation:

\[ \left( \frac{\vec{p}^2}{m} - E \right) \psi(\vec{p}) = - \int \frac{d\vec{p}'}{(2\pi)^3} \left( V_E(\vec{p}, \vec{p}') + \ldots \right) \psi(\vec{p}'), \] \hfill (2)

where dots stand for higher order contributions. The use of an energy-dependent-interaction raises some problem such as non-orthogonality of the solutions. This energy dependence can be removed by a transformation quite similar in its spirit to the Foldy-Wouthuysen one, at the price of dressing the original degrees of freedom to get effective ones (see also refs. [21, 22]). It also results an effective interaction. At the lowest order \( \left( \frac{1}{m} \right)^0 \), this interaction can be obtained by replacing the energy-dependent term at the denominator in Eq. (1), \( E - e - e' \), by the interaction itself, obtaining in this way a self-consistency equation for the effective potential. In configuration space, this equation reads:

\[ V_{\text{eff,sc}}(r) = -g^2 \frac{1}{2\pi^2} \int \frac{dk k^2 j_0(kr)}{\omega_k (\omega_k - V_{\text{eff,sc}}(r))}. \] \hfill (3)

This potential may also be used in a non-relativistic equation:

\[ \left( V_{\text{eff,sc}}(r) + \frac{\vec{p}^2}{m} - E \right) \phi(r) = 0, \] \hfill (4)

which is a typical Schrödinger one. Its solutions, which are now orthogonal, will provide the binding energies of interest here. It has been shown that it retains most of the genuine character related to the energy dependence of the original interaction, Eq. (1) [8, 9]. The change from the function, \( \psi(\vec{p}) \), in Eq. (2) to the function, \( \phi(r) \), in Eq. (4) mainly reminds that they correspond to different formalisms (energy-dependent and energy-independent), the change from momentum to configuration space being unimportant here. In the limit where one could neglect the interaction term in the denominator at the r.h.s. of Eq. (3), the effective interaction would identify to the standard instantaneous interaction:

\[ V_0(r) = -g^2 \frac{e^{-\mu r}}{4\pi r}. \] \hfill (5)
2.2 Beyond the ladder approximation

In deriving the interaction, Eq. (1), a standard approach has been used. This one does not incorporate possible interaction between the two constituents while a boson is exchanged. To get some guess on its effect, a first approach would consist in adding this interaction to the denominator of the meson propagator in this equation, as part of the energy in the intermediate state, or, better, in Eq. (3). From this one, the following expression is immediately obtained:

\[
V_{\text{eff,sc}}(r) = -g^2 \frac{1}{2\pi^2} \int \frac{dk \, k^2 j_0 (kr)}{\omega_k \left( \omega_k - V_{\text{eff,sc}}(r) + V_{\text{eff,sc}}(r) \right)} = V_0(r). \tag{6}
\]

The above result supposes that the interaction with and without in-flight bosons are the same, which implies that the exchanged bosons couple to the constituents in a scalar way (no spin, no charge). The effect of the extra interaction between the constituents is depicted on the left diagram in Fig. 2. Replacing the bubble representing this interaction by a single-boson-exchange immediately suggests that this effect actually involves crossed-boson-exchange diagrams as can be seen on the r.h.s. of this figure.

\[
\begin{align*}
\text{Figure 2: Schematic representation of the interaction between the constituents while a boson is exchanged (left diagram) and when this one is represented by crossed diagrams with two and three bosons exchanged.}
\end{align*}
\]

This feature can be checked by expanding the denominator in Eq. (6) with respect to the effective interaction that has just been introduced:

\[
V_{\text{eff,sc}}(r) = -g^2 \frac{1}{2\pi^2} \int \frac{dk \, k^2 j_0 (kr)}{\omega_k \left( \omega_k - V_{\text{eff,sc}}(r) \right)} \left( 1 + \frac{-V_{\text{eff,sc}}(r)}{\omega_k - V_{\text{eff,sc}}(r)} + \left( \frac{-V_{\text{eff,sc}}(r)}{\omega_k - V_{\text{eff,sc}}(r)} \right)^2 + \ldots \right). \tag{7}
\]

It is easy to see that, apart from the first term representing a single-boson exchange, there is a one to one correspondance between this expansion and diagrams shown in Fig. 2. Symbolically, the expression of the nth order contribution reads:

\[
\Delta V^{(n)} = \frac{-g^2}{E - H_0} \left( V \frac{1}{E - H_0} \right)^n, \tag{8}
\]

where the quantity, \( E - H_0 \), can be written as \( E - e - e' - \omega_k \). The quantity \( V \) represents the (effective) interaction between the two constituents. Assuming that \( E - e - e' = V_{\text{eff,sc}}(r) \),
as we assumed for the derivation of the effective interaction in the ladder approximation [9], one gets:

\[ \Delta V^{(n)}(r) = -g^2 \frac{1}{2\pi^2} \int \frac{dk}{\omega_k} \frac{k^2 j_0(kr)}{(\omega_k - V_{eff,sc}(r)) (\omega_k - V_{eff,sc}(r))} . \]  (9)

This expression is identical to the nth order term in the expansion given in Eq. (7). We now comment on what the above expansion accounts for. At first sight, it seems to only involve part of the crossed-boson-exchange diagrams.

\[ \[
\]

Figure 3: Actual time-ordered contributions (r.h.s.) approximately accounted for by the first-order crossed diagram (l.h.s.). Notice that the very right diagram, which has a non-crossed character, is required to factorize the expression of an instantaneous interaction when added to the other ones.

First of all, it is noticed that expansion whose general term is given by Eq. (8) contains the most singular terms. In the zero-mass boson case and in absence of the effective interaction at the denominator, the nth order contribution contains a factor \( \omega_k^{-n} \) and is therefore infra-red diverging. This divergence is hopefully regularized by the effective interaction. Secondly, the interaction \( V \) at the numerator accounts for many more diagrams than inferred from Fig. 2, including some non-crossed diagrams that have a small contribution. This can be checked in the small coupling limit and has been done explicitly in ref. [8] for two- and three-boson exchange. Diagrams that are actually included in the two-boson-exchange case are shown in Fig. 3. The contribution of each diagram in this figure is rather complicated but great simplification occurs when they are summed up, with the result that propagators relative to each meson factorize. Lastly, we stress that the developments presented here suppose that the effective interaction can be approximated by a local one. This makes it possible to derive relatively simple expressions and facilitates calculations. Going beyond would require tremendous work. Moreover, as the corrections are of order \( \frac{1}{m} \) in the minimal case, it is likely that one should also consider for consistency relativistic corrections of same order. This involves \( \sqrt{m/e} \) factors in the interaction, Eq. (1), and many boson-exchange contributions involving Z-type diagrams.

3 Results

In calculating the contribution of each crossed diagram, a first possibility is to calculate the contribution of each term in the expansion of the effective potential, Eq. (7), assuming that the problem is solved and that, according to Eq. (6), this self-consistent effective
potential, $V_{\text{eff,sc}}(r)$, is equal to the instantaneous one, $V_0(r)$. This procedure however provides results whose interpretation is somewhat ambiguous. While the contribution of a given crossed diagram is obtained, the contribution due to the kinetic energy contains contributions at all orders when calculated with wave functions that are solutions of the problem. As a result, the sum of the kinetic energy and the first order term, for instance, may not evidence any binding. We will therefore proceed differently.

To get a better insight on the role of the different crossed diagrams, we will truncate the expansion of the effective potential, Eq. (6), at some order, determine the associated effective potential, insert this potential in the Schrödinger equation, Eq. (4), and look for the corresponding binding energy. In this way, the kinetic energy is calculated consistently with the employed potential. We thus obtain a series of binding energies which ultimately should converge to the binding energy obtained with the standard instantaneous potential. Our interest is in the convergence of this series to the expected value, depending on the strength of the coupling constant, or on the mass of the exchanged boson.

For the coupling constant, denoted $\alpha = \frac{g^2}{4\pi}$ in analogy with the QED one, we consider two values, $\alpha = 0.5$ and $\alpha = 1.5$. In each case, two boson masses are considered, so that the sensitivity to this mass can be studied. In the first case, we included the boson masses, $\mu = 0$ and $\mu = 0.15 m$, the first of them being appropriate to the study of a Coulomb-like interaction. In the other case, we included the masses, $0.15 m$ and $0.5 m$, that have been used in various works dealing with the scalar-particle model. The first of them is identical to one of the values introduced for the coupling, $\alpha = 0.5$, so that to also provide insight on the sensitivity of results to this coupling. The results are presented in two figures, Fig. 4 for $\alpha = 0.5$ and Fig. 5 for $\alpha = 1.5$.

From examination of the two parts of Fig. 4, it is seen that the convergence to the asymptotic value is slower when the boson mass decreases. The same observation holds for results presented in Fig. 5. This feature, which is perhaps counter-intuitive, can be explained by the fact that, with a lower boson mass, the force extends to larger distances, making the effect bigger. There is an evident relationship with the increasing singular
character of the terms of the expansion of the effective interaction, Eq. \( \text{} \), when the effective interaction at the meson propagator is neglected and the boson mass goes to zero.

The comparison of results of Figs. 4 and 5 for the same boson mass, \( \mu = 0.15 \, m \), shows that the convergence is slower when the coupling strength increases. This result is much less surprising as the role of higher order diagrams in the interaction is expected to increase with the strength of the coupling. Thus, altogether, it turns out that the more or less rapid character of the convergence to the asymptotic value of the binding energy is governed by the global interaction. This one combines both the intensity of the coupling and the range of the force, the interaction being bigger when it extends to larger distances. Roughly, the slowness of the convergence increases with the binding energy.

While the first-order crossed diagram provides more than 2/3 of the missing binding energy in the less bound case \( (E = -0.013 \, m, \text{ right part of Fig. 4}) \), in the other cases, it hardly provides a third of it and one has to include the exchange of three crossed bosons to reach a half. In these cases, the interaction of crossed diagrams up to order 7 does not allow one to approach the asymptotic value at better than 7-8%.

The results obtained here are in complete agreement with those obtained in refs. [9, 20]. They do not support however those, much larger, of ref. [19], based on a dispersion calculation of the crossed-box diagram. This discrepancy is interesting. It mainly reveals that individual contributions can be quite large, but these ones may be dramatically reduced by higher order effects arising from the appearance of the effective interaction at the denominator of the boson propagator, \( (\omega_k - V_{eff,sc}(r))^{-1} \). This has the advantage to stabilize the corresponding contribution of each diagram in such a way that they always have the same sign, making the expansion safer. In the other approach [19], it is likely that higher order terms in the interaction will appear with different signs, making difficult to make an accurate prediction.
4 Conclusion

It is not so well known that the validity of the standard instantaneous interaction largely relies on the cancellation of two effects and essentially supposes a scalar coupling of spin- and charge-less bosons to constituents. The first effect is a renormalization of the instantaneous interaction, which gets effectively reduced due to the absence of interaction between the two constituents while they are exchanging bosons. This effect is in agreement with that one found with the Bethe-Salpeter equation in the ladder approximation or an energy-dependent interaction, which, both, represent an improvement upon the standard instantaneous interaction. The second effect is mainly due to the contribution of crossed diagrams, which precisely account for the interaction between these constituents while bosons are exchanged. Our study was aimed to study the role of these higher order contributions in calculating binding energies.

Technically, we assumed that the above cancellation holds exactly. We could thus look at the convergence to the expected binding energies, depending on the mass of the exchanged boson or on the size of the coupling constant. Except for very low binding energies (1% of the constituent mass), we found that the convergence is rather slow. This feature is enhanced when the strength of the coupling increases or when the boson mass decreases. For usual coupling strengths employed in the domain of strongly interacting systems, retaining crossed diagrams up to order 6 does not allow one to approach the expected result with an accuracy much better than 10%.

The above results have been obtained by making some approximations, neglecting in particular non-locality effects and/or relativistic corrections. They are expected to hold at the order \( \left( \frac{1}{m} \right)^0 \). Despite these drawbacks, they roughly agree with some of the more elaborate calculations for the first-order crossed diagram [9, 20]. They however disagree with results of ref. [19]. With this respect, we notice that the present calculation of each diagram includes some higher order effect through the appearance of the effective interaction in its expression. Therefore, the expansion we used for the binding energy in terms of crossed diagrams is not equivalent to an expansion in terms of the coupling constant, which for a part underlies the results of the above work.

In view of the above remarks and in absence of other calculations, necessarily more complicated, we believe that the present results can give insight on the role of higher order terms in the interaction due to crossed diagrams. We nevertheless expect some change in the detail, arising from various relativistic corrections in particular or other \( \frac{1}{m} \) corrections. To get information on this, the summation of a subset of crossed diagrams should be performed, what supposes to solve a three-body problem in the minimal case.

Taking into account the present results as well as those obtained elsewhere in the ladder approximation, it appears that the instantaneous approximation employed for the usual derivation of interactions is highly misleading. The fact that it could effectively work in some cases hides cancellations that are in no way small. Moreover, these cases exclude two important physical examples: the nucleon-nucleon interaction where the exchanged single pion both carries charge and couples to the nucleon spin, and the quark-quark interaction where the exchanged gluon carries spin and color charge. Actually, for practical applications, it is essential to include the single-boson exchange contribution which is essential to describe the long-range part of the force. At shorter distances, where the
above corrections should show up, the interaction can be most usefully fitted to a few experimental data (cross-sections, binding energies). As our study shows, describing the interaction by single-boson exchange in this range, as done in the nucleon-nucleon interaction case, is largely elusive. As a support to this practice however, we notice that interaction models that completely lack of a theoretical description in this range, like Argonne V18 [23], do quite well.

Acknowledgments
We are very grateful to A. Amghar for an important observation concerning the role of the boson mass in our calculations.

References

[1] Taco Nieuwenhuis and J.A. Tjon: Phys. Rev. Lett. 77, (1996) 814.
[2] E.E. Salpeter and H.A. Bethe: Phys. Rev. 84, (1951) 1232.
[3] G.C. Wick: Phys. Rev. 96, (1954) 1124.
[4] R.E. Cutkosky: Phys. Rev. 96, (1954) 1135.
[5] B. Silvestre-Brac et al.: Phys. Rev. D29, (1984) 2275.
[6] A. Bilal and P. Schuck: Phys. Rev. D31, (1985) 2045.
[7] D.R. Phillips and S.J. Wallace: Phys. Rev. C54 (1996) 507.
[8] A. Amghar, B. Desplanques, Few-Body Systems 28 (2000) 65.
[9] A. Amghar, B. Desplanques, and L. Theußl, Nucl. Phys. A694 (2001) 439.
[10] I.T. Todorov: Phys. Rev. D3, (1971) 2351.
[11] E. Brezin, C. Itzykson and J. Zinn-Justin: Phys. Rev. D1, (1970) 2349.
[12] J.L. Friar: Phys. Rev. C22, (1980) 796.
[13] F. Gross: Phys.Rev. C26, (1982) 2203.
[14] A.R. Neghabian and W. Glöckle: Can. J. Phys. 61, (1983) 85.
[15] C. Itzykson and J.B. Zuber (eds.): Quantum Field Theory. McGraw-Hill International Editions (1985).
[16] G.P. Lepage, nucl-th/9706029.
[17] M. Lacombe et al., Phys. Rev. C21 (1980) 861.
[18] R. Machleidt, K. Holinde and Ch. Elster: Phys. Rep. 149, (1987) 1.
[19] L. Theußl and B. Desplanques: Few-Body Systems 30 (2001) 5.
[20] J. Tjon, private communication; I.R. Afnan and D.R. Phillips, private communication.

[21] N. Fukuda K. Sawada M. and Takekami: Prog. Theor. Phys. 12, (1954) 156.

[22] S. Okubo: Prog. Theor. Phys. 12, (1954) 603;
M. Sugawara and S. Okubo: Phys. Rev. 117, (1960) 605.

[23] R.B. Wiringa, V.G.J. Stoks and R. Schiavella: Phys. Rev. C51 (1995) 38.