Algebraic varieties in the Birkhoff strata of the Grassmannian $\text{Gr}^{(2)}$: Harrison cohomology and integrable systems

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Abstract

The local properties of the families of algebraic subsets $W_g$ in the Birkhoff strata $\Sigma_1^{2g}$ of $\text{Gr}^{(2)}$ containing the hyperelliptic curves of genus $g$ are studied. It is shown that the tangent spaces $T_g$ for $W_g$ are isomorphic to the linear spaces of 2-coboundaries. Particular subsets in $W_g$ are described by the integrable dispersionless coupled KdV systems of hydrodynamical type defining a special class of 2-cocycles and 2-coboundaries in $T_g$. It is demonstrated that the blows-ups of such 2-cocycles and 2-coboundaries and gradient catastrophes for associated integrable systems are interrelated.

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1. Introduction

In this paper, we continue the study of the structure and properties of algebraic curves and varieties in the Birkhoff strata $\Sigma_\ell$ of the Sato Grassmannian [1–3] within the approach proposed in [4]. It was shown in [4] that each stratum $\Sigma_\ell$ contains a subset $W$ closed with respect to pointwise multiplication which geometrically represents an infinite-dimensional algebraic variety defined by the relations

$$ p_j p_k = \sum_l C^l_{jk} p_l = 0, \quad (1) $$

$$ \sum_l \left( C^r_{jk} C^l_{m} - C^l_{ml} C^r_{ij} \right) = 0, \quad j, k, m, r = 0, 1, 2, 3, \ldots. \quad (2) $$

Algebraically relations (1) with the condition $C^l_{jk} = C^l_{kj}$ represent a table of multiplication of a commutative associative algebra $A$ in the basis $(p_0, p_1, p_2, \ldots)$, while (2) is the associativity condition of the structure constants $C^l_{jk}$. 

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In virtue of this algebro-geometrical duality, the subsets $W$ are of particular interest. Indeed, ‘the associativity relations are so natural that any information deduced from them should have some kind of meaning’ (see [5], chapter II, section 1.3). On the other hand, due to its relation to commutative algebras, the algebraic varieties defined by equations (1) and (2) are the natural objects for the Harrison [6] cohomology theory ‘which is particularly applicable to the coordinate ring of algebraic varieties’ [7].

Here we will study the local properties of the subsets $W$ in the Birkhoff strata of the Grassmannian $\text{Gr}(2)$. This Grassmannian is an important specialization of the universal Sato Grassmannian $\text{Gr}$ [1–3]. At the same time, all calculations are simplified drastically for $\text{Gr}(2)$, which allows us to perform a complete analysis.

In the previous work [8], we showed that the Birkhoff strata $\Sigma_1$ defined by (1) and (2) with $p_{2n} = z^{2n}$, $n = 0, 1, 2, 3, \ldots$ which are isomorphic to the infinite families of coordinate rings for rational normal (Veronese) curves of all odd orders ($g = 0$), for elliptic curves ($g = 1$) and hyperelliptic curves ($g > 1$),

$$p_{2k+1}^2 = \lambda_{2k+1} + \sum_{k=0}^{2g} u_k \lambda^k.$$  

Hyperelliptic curve (3) at fixed $u_k$ is contained in a point of the subset $W_g$. It was noted in [4] that tangent spaces for algebraic varieties in the Birkhoff strata of the Sato Grassmannian $\text{Gr}$ are isomorphic to the linear spaces of 2-coboundaries. Here we will show that the tangent spaces for the varieties $W_g$ are isomorphic to the linear spaces of 2-coboundaries of the special structure. In particular, the 2-cocycles $\psi_g(\alpha, \beta)$ and the 2-coboundaries $f_g(\gamma)$ for these varieties are related as

$$\psi_g(p_{2k+1}, p_{2k+1}) = 2p_{2k+1}f_g(p_{2k+1}), \quad k = g, g + 1, \ldots.$$  

We also present representations of the 2-cocycles and 2-coboundaries for $W_g$ in terms of the elements of the ideals $I(W_g)$ of these varieties.

We consider the special subvarieties $W^I_g$ defined only by equations (2) for which the differential one-forms

$$\omega_c = \sum_{k=0}^{g} p_{2(g+k)+1}(z) \, dx_{2(g+k)+1}$$  

are closed. It is shown that such varieties $W^I_g$ are characterized by the closedness of only $2g+1$ one-forms. Moreover, these subvarieties are described by $2g$ commuting $2g + 1$ component hydrodynamical type systems. For the subvarieties $W^I_g$ of the subsets $W_g$ for which the one-forms

$$\omega = \sum_{k=0}^{\infty} p_{2(g+k)+1}(z) \, dx_{2(g+k)+1}$$  

are closed, the associated systems form infinite hierarchies of integrable systems. For $g = 0$, it is the Burgers–Hopf hierarchy, while for $g \geq 1$, they are dispersionless coupled Korteweg–de Vries (dcKdV) hierarchies. Solutions of these integrable hierarchies provide us a special class of 2-cocycles and 2-coboundaries.

At $g = 1$, we also consider a particular class of the varieties $W_1$ for which $u_0 = 0$. It is shown that the corresponding reduced subset $W_1$ represents a two-dimensional family of coordinate rings for the elliptic curve with the fixed point at the origin $(p_3, \lambda) = (0, 0)$. The associated hierarchy is the dispersionless nonlinear Schrödinger (dNLS) equation or 1-layer...
that the dNLS equation is equivalent to the Hirota-type equation
\[
\det \begin{pmatrix}
\phi_{x,t} & \phi_{x,s} \\
\phi_{x,t} & \phi_{x,s}
\end{pmatrix} + (\phi_{x,t})^3 = 0.
\]  
(7)

Solutions of this equation provide the particular dNLS 2-cocycles and 2-coboundaries.

Interrelation between the properties of the dcKdV hierarchies describing the varieties \( W_g \) and the corresponding 2-cocycles and 2-coboundaries is discussed too. It is observed that the blow-ups of the 2-cocycles and 2-coboundaries (i.e. an unbounded increase of their values) and the gradient catastrophe for the above hydrodynamical type system happen on the same subvarieties of the finite codimension of the affine space of the variables \( x_k \).

The paper is organized as follows. Birkhoff strata in the Grassmannian and the results of [8] are briefly described in section 2. Harrison cohomology of the varieties \( W_g \) and \( W_{g} \) is discussed in section 3. The big cell in \( Gr^{(2)} \) and associated Burgers–Hopf hierarchy are considered in section 4. Section 5 is devoted to the stratum \( \Sigma_2 \) which contains a family of elliptic curves and corresponding integrable systems. Deformations of the moduli \( g_k \) and \( g_{\beta} \) of the elliptic curve and associated 2-cocycles are studied in section 6. Deformations of the elliptic curve with a fixed point in the origin described by the dispersionless NLS equation and associated Hirota-type equation are considered in section 7. Hyperelliptic curves in \( W_0 \) and corresponding dispersionless coupled KdV (dcKdV) hierarchies are discussed in section 8. Interpretation of the ideals of varieties \( W_{g}^{l} \) as the Poisson ideals is presented in section 9. Interrelation between cohomology blow-ups and gradient catastrophe for dcKdV hierarchies and comparison with Whitham theory are discussed briefly in section 10.

2. Birkhoff strata in the Grassmannian \( Gr^{(2)} \) and associated algebraic varieties

For completeness, we recall here briefly the basic facts about Birkhoff strata in \( Gr^{(2)} \) and the corresponding results of [8].

The Sato Grassmannian \( Gr \) can be viewed as the set of closed vector subspaces in the infinite-dimensional set of all formal Laurent series with coefficients in \( \mathbb{C} \) with certain special properties (see e.g. [2, 3]). Each subspace \( W \in Gr \) possesses an algebraic basis \((w_0(z), w_1(z), w_2(z), \ldots)\) with the basis elements
\[
w_n = \sum_{k=-\infty}^{n} d_k z^k
\]
(8)
of finite order \( n \). The Grassmannian \( Gr \) is a connected Banach manifold which exhibits a stratified structure [2, 3], i.e. \( Gr = \bigcup_{\Sigma} \Sigma \) where the stratum \( \Sigma \) is a subset in \( Gr \) formed by elements of the form (8) such that the possible values \( n \) are given by the infinite set \( S = \{s_0, s_1, s_2, \ldots\} \) of integers \( s_k \) with \( s_0 < s_1 < s_2 < \cdots \) and \( s_n = n \) for large \( n \). The big cell \( \Sigma_0 \) corresponds to \( S = \{0, 1, 2, \ldots\} \). Other strata are associated with the sets \( S \) different from \( S_0 \).

\( Gr^{(2)} \) is the subset of elements \( W \) of \( Gr \) obeying the condition \( z^2 \cdot W \subset W \) [2, 3]. This condition imposes strong constraints on the Laurent series and on the structure of the strata. Namely, the Birkhoff stratum \( \Sigma_0 \) in \( Gr^{(2)} \) corresponds to the sets \( S \) such that \( S + 2 \subset S \), i.e. all possible \( S \) having the form \([2, 3] \)

\[ S_m = \{-m, -m+2, -m+4, \ldots, m, m+1, m+2, \ldots\}, \]
(9)

with \( m = 0, 1, 2, \ldots \). The codimension of \( \Sigma_m \) is \( m(m+1)/2 \). One has \( Gr^{(2)} = \bigcup_{m\geq 0} \Sigma_m \).

In this paper, we consider only the strata \( \Sigma_{2g}, g = 0, 1, 2, \ldots \). The stratum \( \Sigma_{2g} \) with arbitrary \( g \) is characterized by \( S = \{-2g, -2g+2, -2g+4, \ldots, 0, 2, 4, \ldots, 2g, 2g+1, 2g+ \ldots \} \).
2, . . . }. So it does not contain, in particular, g elements of the orders 1, 3, 5, . . . , 2g − 1, and the positive-order elements of the canonical basis are given by

\[ p_0 = 1 + \sum_{k \geq 1} \frac{H_k^0}{z^k}, \]

\[ p_j = z^j + \sum_{k=0}^{j/2-1} H_{2k-1}^{j-2k+1} + \sum_{k \geq 1} \frac{H_k^j}{z^k}, \quad j = 2, 4, 6, . . . , 2g - 2, \]

\[ p_j = z^j + \sum_{k=0}^{g-1} H_{2k-1}^{j-2k+1} + \sum_{k \geq 1} \frac{H_k^j}{z^k}, \quad j = 2g, 2g + 1, 2g + 2, 2g + 3, . . . , \]

where \( H_k^j \) are arbitrary complex-valued parameters.

It was shown in [4] and [8] that

**Proposition 2.1.** The stratum \( \Sigma_{2g} \) for \( g = 1, 2, 3, . . . \) contains maximal subset \( W_g \) closed with respect to pointwise multiplication with elements of the form

\[ w = \sum_{k \geq 0} a_{2k}^j \lambda^k + \sum_{k \geq g} b_{2k+1}(\lambda) p_{2k+1}, \]

with the parameters \( H_k^j \) obeying the constraints

\[ H_{2k}^{2m} = 0, \quad m = 0, 1, 2, . . . , k = -2g + 2, -2g + 4, . . . , -2, 0, 1, 2, 3, . . ., \]

\[ H_{2k}^{2m+1} = 0, \quad m = 0, 1, 2, . . . , k = -g, -g + 1, -g + 2, . . ., \]

and

\[ H_{2k}^{2j+1} - H_{2k+1}^{2j+1} = 0, \]

\[ H_{2k}^{2j+1} + H_{2k+1}^{2j+1} + \sum_{s=0}^{l-g} \frac{H_{2s+1}^{2j+1} H_{2s+1}^{2j+1}}{H_{2s+1}^{2j+1}} = 0, \]

and \( p_{2m} = \lambda^m, m = 0, 1, 2, . . . , \lambda = z^2 \). One also has

\[ C_{2k+1} \equiv p_{2k+1}^2 = \left( \lambda^{2k+1} + \sum_{l=0}^{2g} u_l \lambda^l \right) = 0, \]

where \( u_k \) are certain polynomials in \( H_k^j \) and

\[ \lambda_{2m+1} \equiv p_{2m+1} - a_m(\lambda) p_{2g+1} = 0, \quad m = g + 1, g + 2, . . ., \]

where \( a_m(\lambda) \) are polynomials in \( \lambda \).

The subsets \( W_g \) have the following algebraic and geometrical interpretation.

**Proposition 2.2.** [8] Each point of the subset \( W_g \) (\( g = 0, 1, 2, . . . \)) corresponding to the fixed values of \( H_k^j \) which obey conditions (12) and (13) is an infinite-dimensional commutative algebra \( A_g \) isomorphic to \( C(\lambda, p_{2g+1})/C_{2g+1} \). The subset \( W_g \) is isomorphic to an infinite-dimensional algebraic variety defined by the equations

\[ f_{jk} \equiv p_j p_k - \sum_i C_{ijk}^l p_l = 0 \]
and by relations (12) and (13) in the affine space with local coordinates
\[ \lambda, \; p_{2(g+m)+1}, H_{-(2g-k)-1}, \; k, \; m = 1, 2, 3, \ldots. \] (17)

Its ideal is
\[ I_g = \{ C_{2g+1}, C_{2g+3}, \ldots \} \] (18)

where \( I_{2m+1}^{(q)} \) are given by (15).

With small abuse of notation, we will refer to this infinite-dimensional algebraic variety
\[ W_g \] as the infinite family of the coordinate ring for the family of elliptic curves (22).

Relations (12) and (13) are equivalent to the associativity conditions
\[ \sum_s (C_{jk}^t C_{ts}^r - C_{kt}^s C_{js}^r) = 0 \] (19)

for the structure constants \( C_{jk}^t \) given by
\[ C_{2j,2k}^{2l} = \delta_{j+k}^l, \]
\[ C_{2j+1,2l}^{2l+1} = \delta_{j+k}^l + H_{2(k-j)-1}^{2j+1}, \]
\[ C_{2j,2k}^{2l} = \delta_{j+k}^l + H_{2(j-l)+1}^{2j+1} + H_{2(k-j)+1}^{2k+1} + \sum_{s=-g-1}^{-1} \sum_{r=-g-1}^{1} H_{2s+1}^{2j+1} H_{2r+1}^{2k+1} \delta_{j+k}^{s-r}, \] (20)

The variety \( W_g \) is the intersection of quadrics (16) and (19).

Sections of these varieties \( W_g \) by the planes with all \( H_s^t = \text{const} \) represent well-known algebraic curves. At \( g = 0 \), relations (14) and (15) are equivalent to the following:
\[ \lambda = p_1^2 - 2H_1^1, \]
\[ p_3 = p_1^3 - 3H_1^1 p_1, \]
\[ p_5 = p_1^5 - 5H_1^1 p_1^3 + \frac{15}{2} H_1^1 p_1^3, \] (21)

Formulae (21) represent the well-known parameterization of the rational normal (Veronese) curves of odd order (see e.g. [9]). So, for the big cell, the subset \( W_0 \) is the infinite-dimensional family of the infinite tower of the Veronese curves of all odd orders.

For \( g = 1 \), relations (14) and (15) take the form
\[ C_3 = p_3^2 - (\lambda^3 + u_2 \lambda^2 + u_1 \lambda + u_0), \] (22)

\[ I_{2m+1}^{(1)} = p_{2m+1} = \left( \lambda^{m-1} - \sum_{k=0}^{m-2} H_{-1}^{2(m-k)-1} \lambda^k \right) p_3, \; \; m = 2, 3, \ldots. \] (23)

Thus, the subset \( W_1 \) is the infinite family of the coordinate ring for the family of elliptic curves (22).

At \( g > 1 \), one has a hyperelliptic curve of genus \( g \) defined by equation (14) and an infinite family of its coordinate rings.

We would like to note that one can view these subsets \( W_g \) in different ways. First one can interpret them as the infinite families of the deformed basic curves (rational normal curves...
two-sided refer to such subvarieties as obeying constraints (14) and (15) or (12) and (13). Second, since these subsets \( W_g \) are defined by the quadratic equations (16) and (19), they are isomorphic to infinite-dimensional algebraic varieties in the affine spaces with the local coordinates \( \lambda, p_{2g+1}, p_{2g+3}, \ldots, H^g_2 \). Finally for all \( p_j = \text{const} \) and \( \lambda = \text{const} \), the subsets \( W_g \) coincide with the varieties of structure constants \( C^l_{jk} \) of commutative associative algebras which are of their own interest (see e.g. [5]). We will refer to such subvarieties as \( W_g \). All these aspects of the subsets \( W_g \) are equally important and arise in various applications.

3. Harrison cohomology

Study of the properties of the varieties \( W_g \) described in the previous section begins, as usual, with the analysis of their local structure. A standard way to do this is to consider a tangent bundle \( T_{W_g} \) for \( W_g \). In virtue of equations (1) and (2), \( T_{W_g} \) is defined by the following systems of linear equations:

\[
\pi_j p_k + p_j \pi_k - \sum_l \Delta^l_{jk} p_l - \sum_l C^l_{jk} \pi_l = 0, \tag{24}
\]

\[
\sum_l \left( \Delta^l_{jk} p^m + C^l_{jk} \Delta^p_{lm} - \Delta^l_{mk} C^p_{lj} - C^l_{mk} \Delta^p_{lj} \right) = 0, \tag{25}
\]

where \( \pi_j, \Delta^l_{jk} \in T_{W_g} \). Subsystem (25) defines a tangent bundle for the subvarieties \( W_g \). The linear system

\[
\pi^*_j p_k + p_j \pi^*_k - \sum_l \Delta^l^*_{jk} p_l - \sum_l C^l^*_{jk} \pi^*_l = 0, \tag{26}
\]

\[
\sum_l \left( \Delta^l^*_{jk} p^m + C^l^*_{jk} \Delta^p^*_{lm} - \Delta^l^*_{mk} C^p^*_{lj} - C^l^*_{mk} \Delta^p^*_{lj} \right) = 0 \tag{27}
\]

defines a cotangent bundle \( T^*_{W_g} \) for the variety \( W_g \). The elements of \( T_{W_g} \) and \( T^*_{W_g} \) can be realized, as usual (see e.g. [5, 9, 10]), as

\[
\pi_j = X(p_j), \quad \Delta^l_{jk} = X(C^l_{jk}), \tag{28}
\]

\[
\pi^*_j = dp_j, \quad \Delta^l^*_{jk} = dc^l_{jk}, \tag{29}
\]

where \( \pi_1 \) is a vector field on \( W_g \), and \( dp_j \) and \( dc^l_{jk} \) are differentials of \( p_j \) and \( C^l_{jk} \), respectively. In particular, the system (25) for \( dc^l_{jk} \) defines a module of differentials for the variety \( W_g \) of the structure constants \( C^l_{jk} \).

Cohomology theory of commutative associative algebras proposed by Harrison in [6] is the most appropriate to analyze the local properties of the varieties \( W_g \).

Harrison’s cohomology is a commutative version of the classical Hochschild cohomology for associative algebras [11, 12]. We review it briefly to make this paper self-consistent. Let \( K \) be a field and \( A \) be an associative algebra (possibly infinite dimensional) over \( K \). Let \( \Lambda \) be a two-sided \( A \)-module. The set of all \( A \)-valued \( n \)-cochains (or \( (n, \Lambda) \)-cochains) of \( A \) is denoted by \( C^n(A, \Lambda) \). The coboundary operator \( \delta \) maps each \( C^n(A, \Lambda) \) onto \( C^{n+1}(A, \Lambda) \) according to the following rule: for \( f \in C^n(A, \Lambda) \) and \( a_1, \ldots, a_n \in A \),

\[
(\delta f)(a_1, a_2, \ldots, a_{n+1}) = a_1 \cdot f(a_2, a_3, \ldots, a_{n+1}) + \sum_{k=1}^{n} (-1)^{k} f(a_1, \ldots, \hat{a}_k, \ldots, a_{n+1}) + (-1)^{n+1} f(a_1, \ldots, a_n) \cdot a_{n+1}, \tag{30}
\]
where in the first and the last terms, the dot denotes the action of $A$ on the module $\Lambda$, and in the middle term, it denotes the product in $A$. For any cochain $f$, one has $\delta(\delta f) = 0$. An element $f \in C^n(A, \Lambda)$ is called an $n$-dimensional cocycle if $\delta f = 0$. An element of the form $\delta g$ with $g \in C^{n-1}(A, \Lambda)$ is called an $n$-dimensional coboundary. For instance, 2-cocycle is a bilinear map $f$ from $A \times A$ to $\Lambda$ such that

$$a_1 \cdot f(a_2, a_3) - f(a_1 \cdot a_2, a_3) + f(a_1, a_2 \cdot a_3) - f(a_1, a_2) \cdot a_3 = 0,$$

(31)

while the 2-coboundary $\phi$ is defined by

$$\phi(a_1, a_2) = a_1 \cdot g(a_2) - g(a_1 \cdot a_2) + g(a_1) \cdot a_2,$$

(32)

where $g$ is a linear mapping.

The coboundaries form a subgroup $B^n(A, A)$ of the group $Z^n(A, \Lambda)$ of $n$-dimensional cocycles. The $n$-dimensional cohomology group $H^n(A, \Lambda)$ is defined as the quotient $Z^n(A, \Lambda)/B^n(A, \Lambda)$ \cite{13, 14}.

In the case of commutative associative algebras considered by Harrison, one has some new properties, but on the other hand, one can define only the first three cohomology modules $H^1, H^2, H^3$ \cite{6}. The main advantage of the Harrison cohomology theory is that it is quite suitable for an analysis of the local properties of algebraic varieties \cite{6, 7, 13–17}.

Particular examples of algebraic varieties defined by the associativity condition for structure constants of commutative associative algebras in a fixed basis $\{p_{ij}\}$ are of great general interest (see \cite{5}, chapter II, section 1.3). Tangent spaces for such algebras are defined by equations (24) and (25). Introducing the bilinear mapping $\psi_{g}(\alpha, \beta)$ to a tangent space of $A_{g}$ defined by (see \cite{5}, p 91)

$$\psi_{g}(p, p) = \sum_{l} \Delta_{jk}^{l} p_{l},$$

(33)

one observes that equations (25) are equivalent to

$$p_{j} \psi_{g}(p, p) - \psi_{g}(p, p) + \psi_{g}(p, p) - p_{k} \psi_{g}(p, p) = 0, \quad j, k, l = 2g + l, \ldots,$$

(34)

or to

$$\alpha \psi_{g}(\beta, \gamma) - \psi_{g}(\alpha \beta, \gamma) + \psi_{g}(\alpha, \beta \gamma) - \gamma \psi_{g}(\alpha, \beta) = 0,$$

(35)

where $\alpha, \beta, \gamma \in W_{g}$. So comparing (35) and (31), one concludes that the tangent spaces at a point of the varieties $W_{g}$ of the structure constants are isomorphic to the linear spaces of 2-cocycles on $W_{g}$ \cite{5}.

These 2-cocycles exhibit a more specific property being considered in larger varieties $W_{k}$. Indeed, one observes that the system (24) is equivalent to the equation

$$\psi_{g}(\alpha, \beta) = \alpha f_{g}(\beta) + \beta f_{g}(\alpha) - f_{g}(\alpha \beta),$$

(36)

where $f_{g}(\alpha)$ denotes a linear map defined by the relations $f_{g}(p_{k}) = \pi_{k}$. So

$$\psi_{g}(\alpha, \beta) = (\delta f_{g})(\alpha, \beta).$$

(37)

Thus, due to definition (32), $\psi_{g}(\alpha, \beta)$ is a 2-coboundary and one has the following.

**Proposition 3.1.** Tangent spaces of the varieties $W_{k}$ are isomorphic to the linear spaces of 2-coboundaries (see also \cite{44}).
Consequently, the Harrison cohomology moduli $H^2(W_g)$ and $H^1(W_g)$ vanish.

In more general setting, the system (24) and (25) defines any $W_g$-module $E$. So, for the same reason as above, a $W_g$-module $E$ and, in particular, the cotangent spaces of $W_g$ are isomorphic to linear spaces of 2-coboundaries.

Due to the constraint $z^3W_g \subset W_g$, i.e. $p_{2j} = z^{2j}$, the two cocycles $\psi_g(\alpha, \beta)$ and linear maps $f_g(\alpha)$ have certain specific properties. Indeed, taking into account the explicit form of equations (1) for the varieties $W_g$ [8], i.e. $(p_{2j} = z^{2j})$

\[ p_{2j}p_{2k} = p_{2(j+k)}, \]
\[ p_{2j}p_{2k+1} = p_{2(j+k+1)} + \sum_{m=-g}^{k-1} H_{2m+1}^{2j+1} p_{2(k-s)-1}, \]
\[ p_{2j+1}p_{2k+1} = p_{2(j+k+1)} + \sum_{m=-g}^{j} H_{2m+1}^{2j+1} p_{2(j-s)} + \sum_{m=-g}^{j} H_{2m+1}^{2k+1} p_{2(k-s)} \]

\[ + \sum_{m=-g}^{j-1} \sum_{m=-g}^{z-m} H_{2m+1}^{2j+1} H_{2r+1}^{2k+1} p_{2(s+r)} + \sum_{m=-g}^{j} \sum_{m=-g}^{z-m} H_{2m+1}^{2j+1} H_{2r+1}^{2k+1} p_{2(s+r)} \]

\[ + \sum_{r=-g}^{z-r-1} \sum_{m=-g}^{j} H_{2m+1}^{2j+1} H_{2r+1}^{2k+1} p_{2(s+r)} \]

and formulae (24) and (25), one finds that

\[ \psi_g(p_0, p_k) = 0, \quad k = 2g + 1, 2g + 3, \ldots, \]

and

\[ \psi_g(p_{2n}, p_{2n}) = 0, \quad f_g(p_{2n}) = 0, \quad n, m = 0, 1, 2, \ldots. \]

Relation (41) between 2-cocycles and 2-coboundaries allows us to find the explicit form of the map $f_g(\alpha)$. Using (38), one first finds that

\[ p_{2k+1}^2 = \lambda_{2k+1}^{2k+1} + \sum_{m=0}^{2k} v_m k^m, \]

where $v_k$ are certain polynomials of $H_1^{2k}$. Hence,

\[ \psi_g(p_{2k+1}, p_{2k+1}) = \sum_{m=0}^{2k} \Delta_m z^{2m} \]

is an element of $T_{W_g}$. So one finds that

\[ f_g(p_{2k+1}) = \frac{\sum_{m=0}^{2k} \Delta_m z^{2m}}{2p_{2k+1}}, \quad k = g, g + 1, g + 2, \ldots. \]

Since

\[ p_{2k+1} = z^{2k+1} + \sum_{m=-g}^{\infty} \frac{H_{2m+1}^{2k+1} z^{2m+1}}{z^{2m+1}} \]
and $H^j_k$ obey the associativity conditions (13), $f(p_{2k+1})$ have the form

$$f_k(p_{2k+1}) = \sum_{m=-g}^{\infty} A_{2m+1}^{2k+1}.$$

Then a straightforward calculation shows that

$$A_{2m+1}^{2k+1} = A_{2m+1}^{2k+1},$$

where $A_{2m+1}^{2k+1} \in T_{W_k}$.

Thus, one has

$$f_k(p_{2k+1}) = \sum_{m=-g}^{\infty} \Delta_{2m}^{2k+1} = \Delta(p_{2k+1}),$$

which is quite a natural result if one treats $p_{2k+1}$ as a Laurent series (45). Instead relation (41) seems to be far from trivial.

Finally we note that considering a vector field $X_c$ acting in the variety $W_{ gc}$, one can obtain a simple realization of the mapping $\psi_g(\alpha, \beta)$ and $f_k(\alpha)$, namely

$$f(p_j) = X_c(p_j), \quad \psi_k(p_l, p_l) = -X_c(f_{kl}).$$

4. Integrable subvariety of $W_0$ in the big cell and Burgers–Hopf hierarchy

A way to select particular classes of the varieties $W_k$ or their subvarieties is to require that their tangent or cotangent bundles have some additional properties. The cotangent bundle $\Omega_{W_g}$ for the variety $W_g$ is isomorphic to the module $\Omega_{W_g}$ of differentials over $W_g$ defined by relations (26) and (27). Having in mind the concept of Lagrangian submanifolds of symplectic manifolds (see e.g. [10]), it is quite natural for this purpose to consider special subsets in $W_g$ for which certain one-forms are closed.

Let us begin with the simplest case of the big cell. For $W_0$, one has [8]

$$C_{2l,2m}^l = \delta_{m+n},$$

$$C_{2l+1,2m+1}^l = \delta_{m+n} + H_{2m+1,2l(n-1)+1},$$

$$C_{2l,2m+1,2l+1} = \delta_{m+n+1} + H_{2(m-1)+1}^l + H_{2l+1}^l,$$

and the subset $W_{0,0}$ is defined by the equations

$$H_{2k+1}^{2m+1} - H_{2k+1}^{2(m+n)+1} + \sum_{j=0}^{n-1} H_{2k+1}^{2j+1} H_{2k+1}^{2(n-j)-1} = 0,$$

$$H_{2k+1}^{2m+1} + H_{2k+1}^{2(m+n)+1} + \sum_{j=0}^{k-1} H_{2l+1}^{2(l+1)} H_{2(k-l)-1}^{2n+1} = 0.$$

Relations (51) imply that

$$kH_j^l = jH_k^l, \quad j, k = 1, 2, 3, \ldots.$$
The lowest of relations (51) and (52) are

\[
(H_1^1)^2 + 2H_1^1 = 0, \\
H_3^1 + H_1^1 H_1^1 = 0, \\
H_3^3 - 3H_1^1 - 3H_1^1 H_1^1 - (H_1^1)^3 = 0, \\
2H_1^1 + 2H_1^1 H_1^1 + H_3^2 = 0,
\]

\ldots

and

\[
3H_1^1 = H_3^3, \\
5H_1^1 = H_5^3, \\
7H_1^1 = H_7^3, \ldots.
\]

These equations are equivalent to

\[
H_3^3 + \frac{3}{2} (H_1^1)^2 = 0, \\
H_5^3 - \frac{5}{2} (H_1^1)^3 = 0, \\
H_3^3 - (H_1^1)^3 = 0, \\
H_7^3 + \frac{35}{8} (H_1^1)^4 = 0,
\]

\ldots

In general, one has

\[
H_{2k-1}^{2k-1} - 2^k (2k - 1) \left(\frac{1}{2^k}\right) (H_1^1)^k = 0, \quad k = 1, 2, 3, \ldots
\]

and

\[
H_{2k+1}^{2k+1} + \alpha_{jk} (H_1^1)^{j+k+1} = 0, \quad k = 1, 2, 3, \ldots,
\]

where \(\alpha_{jk}\) are some constants. So all \(H_j^k\) are parameterized by the single variable \(H_1^1\). Hence, \(W_0\) is the one-dimensional variety and \(W_0\) is a one-parametric family of the towers of rational normal curves of all odd orders.

Thus, the situation in the Grassmannian \(Gr^{(2)}\) is drastically degenerate with respect to the general Sato Grassmannian where in the variety \(W_0\) in the big cell, one has the infinite-parametric family of rational normal curves [4]. Having in mind this degeneration, we will proceed in a way which will be applicable to other strata in \(Gr^{(2)}\) and in the general Grassmannian.

Thus, let \(x_1\) and \(x_3\) be local coordinates in the variety \(W_0\). Let us consider a subvariety \(W_{0c}\) for which the one-form

\[
\omega_{31} = H_1^1 \, dx_3 + H_1^1 \, dx_1
\]

is closed. So in \(W_{0c}\), one has

\[
\frac{\partial H_3^1}{\partial x_1} - \frac{\partial H_1^1}{\partial x_3} = 0.
\]

Combining equations (55) and (59), one obtains

\[
\frac{\partial H_1^1}{\partial x_3} + 3H_1^1 \frac{\partial H_1^1}{\partial x_1} = 0,
\]
which is the well-known Burgers–Hopf (BH) equation. On the other hand, condition (59) locally implies the existence of a function $S_1$ such that

$$H_3^1 = \frac{\partial S_1}{\partial x_1}, \quad H_1^1 = \frac{\partial S_1}{\partial x_3}$$

and

$$\omega_{31} = dS_1.$$  \hspace{1cm} (62)

Here and in the rest of the paper, our consideration is pure local. Substitution of the second of equations (55) in the first one gives

$$\frac{\partial S_1}{\partial x_3} + \frac{3}{2} \left( \frac{\partial S_1}{\partial x_1} \right)^2 = 0,$$  \hspace{1cm} (63)

which is the potential form of the BH equation (60).

Using relations (55), one obtains the following.

**Lemma 4.1.** Equation (59) implies that

$$\frac{\partial H_3^k}{\partial x_1} - \frac{\partial H_1^k}{\partial x_3} = 0$$  \hspace{1cm} (64)

for all $k = 1, 3, 5, 7, \ldots$ and, hence,

$$\frac{\partial p_3(z)}{\partial x_1} - \frac{\partial p_1(z)}{\partial x_3} = 0.$$  \hspace{1cm} (65)

Thus, the closedness of the form (58) implies the closedness of the infinite set of one-forms given by

$$\omega_{31}(z) = p_3(z) \, dx_3 + p_1(z) \, dx_1 = z^3 \, dx_3 + z \, dx_1 + \sum_{k \geq 1} \frac{1}{2k+1} (H_{2k+1}^3 \, dx_3 + H_{2k+1}^1 \, dx_1).$$  \hspace{1cm} (66)

One can easily repeat such a construction starting with any closed one-form

$$\omega_{jk} = H_j^1 \, dx_j + H_k^1 \, dx_k,$$  \hspace{1cm} (67)

instead of the form (58) and obtaining the infinite set of the forms

$$\omega_{jk}(z) = p_j(z) \, dx_j + p_k(z) \, dx_k,$$  \hspace{1cm} (68)

where $j$ and $k$ are any two distinct indices of the set $\{1, 3, 5, \ldots\}$. Combining these results, one obtains the following.

**Proposition 4.2.** Existence of the closed one-forms

$$\omega_{jk} = H_j^1 \, dx_j + H_k^1 \, dx_k,$$  \hspace{1cm} (69)

for any two distinct indices $j, k$ of the set $\{1, 3, 5, \ldots\}$, is a necessary and sufficient condition for the closedness of the one-form

$$\omega = \sum_{j=1}^{\infty} p_j(z) \, dx_j.$$  \hspace{1cm} (70)

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The closedness of the form (70) locally implies the existence of the formal Laurent series (action)

\[ S(z, x) = \sum_{k=0}^{\infty} z^{2k+1} x_{2k+1} + \sum_{m=0}^{\infty} \frac{S_{2m+1}}{z^{2m+1}} \]  

(71)
such that

\[ p_j = \frac{\partial S}{\partial x_j}, \quad j = 1, 3, 5, \ldots. \]  

(72)

In particular,

\[ H'_k = \frac{\partial S_k}{\partial x_j}. \]  

(73)

Substitution of expressions (73) into algebraic relations (55) gives an infinite set of differential equations which is equivalent to the following:

\[ \frac{\partial H'_1}{\partial x_{2k-1}} - 2^k k (2k - 1) \left( \frac{1/2}{k} \right) (H'_1)^{k-1} \frac{\partial H'_1}{\partial x_1} = 0, \quad k = 1, 2, 3, \ldots. \]  

(74)

This is a standard form of the BH hierarchy. Another form of the BH hierarchy is given by the infinite system of the Hamilton–Jacobi-type equations

\[ \frac{\partial S}{\partial x_j} \frac{\partial S}{\partial x_k} - \sum_{l=1,3,5,\ldots} C'_{jk} \frac{\partial S}{\partial x_l} = 0, \]  

(75)

with \( S \) given by (71) and \( C'_{jk} \) given by (50).

The closed one-forms of type (70) and action \( S(x, z) \) (71) are basis objects in various approaches to the dispersionless integrable hierarchies [18–20]. In the usual construction (see e.g. [18–20]), one starts with the action \( S(x, z) \) of the form (71) obeying the Hamilton–Jacobi-type equations. Proposition 4.2 shows that for validness of such a scheme, it is sufficient to require the closedness only of the forms (69).

Finally we note that formulae (52) and (73) imply the existence of the function \( F \) such that (e.g. [35])

\[ H'_k = -\frac{1}{k} \frac{\partial^2 F}{\partial x_j \partial x_k}, \quad j, k = 1, 3, 5, \ldots. \]  

(76)

Consequently the algebraic relations (55) become differential equations for \( F \) which are equivalent to the well-known Hirota equations (see e.g. [18–20])

\[ \frac{\partial^2 F}{\partial x_1 \partial x_3} - \frac{3}{2} \left( \frac{\partial F}{\partial x_1^2} \right)^2 = 0, \]

\[ \frac{\partial^2 F}{\partial x_1 \partial x_5} + \frac{5}{2} \left( \frac{\partial F}{\partial x_1^2} \right)^3 = 0, \]

(77)

and so on for the BH hierarchy.
Solutions of the BH hierarchy or the BH’s Hirota equations provide us with the particular class of 2-cocycles (33) given by

$$
\psi_0(p_{2j+1}, p_{2k+1}) = - \sum_{l=1} \left( \frac{1}{2(j-l) + 1} \frac{\partial^2 \Delta F}{\partial x_{2j+1} \partial x_{2(j-l)+1}} + \frac{1}{2(k-l) + 1} \frac{\partial^2 \Delta F}{\partial x_{2j+1} \partial x_{2(k-l)+1}} \right) z^{2l},
$$

(78)

$$
\psi_0(p_{2j+1}, p_{2k+1}) = - \sum_{l=1} \left( \frac{1}{2(j-l) - 1} \frac{\partial^2 \Delta F}{\partial x_{2j+1} \partial x_{2(j-l)-1}} \right) p_{2l+1},
$$

where \( \Delta F \) denotes a variation of the function \( F \), for example, \( \Delta F = \sum_{m=1}^\infty \alpha_m \frac{\partial F}{\partial x_{2m+1}} \) where \( \alpha_m \) are arbitrary constants. For the 2-coboundary \( f_0(p_{2k+1}) \), one has

$$
f_0(p_{2j+1}) = D(z) \Delta F = \sum_{m=1}^\infty \frac{1}{2^{2m+1}} \frac{\partial \Delta F}{\partial x_{2m+1}},
$$

(79)

where \( D(z) \) is the standard vertex operator.

It would be of interest to extend the above formulas to the case of the KP hierarchy using the dispersive version of formula (76) given, for instance, in [36].

5. Stratum \( \Sigma_2 \): elliptic curve and associated integrable equations

For the stratum \( \Sigma_2 \), the system (1) is equivalent to the system (14) and (15) with \( g = 1 \) and \( p_{2n} = z^{2n} = \lambda^n \), i.e.

$$
p_3^2 = \lambda^3 + u_2 \lambda^2 + u_1 \lambda + u_0,
$$

(80)

and

$$
p_{2m+1} = \left( \lambda^{m-1} + \sum_{k=0}^{m-2} H_{2(m-k)-1} \lambda^k \right) p_3,
$$

(81)

where

$$
u_2 = 2H_{2}^3, \quad u_1 = 2H_{1}^3 + (H_{2}^3)^2, \quad u_0 = 2H_{1}^3 + 2H_{3}^3 H_{3}^-.
$$

(82)

The associativity condition (2) is reduced to the infinite system of equations for \( H_{l}^{j+1} \), the first of which are given by

$$
H_{l}^5 = H_{l}^1 \cdot (H_{l}^{-1} H_{l}^-)^2,
$$

$$
H_{l}^3 = -H_{l}^1 H_{l}^- + H_{l}^3
$$

$$
H_{l}^3 = -\frac{1}{2} (H_{l}^3)^2 - 2H_{3} H_{3}^-,
$$

(83)

and

$$
H_{l}^7 = -2H_{l}^1 H_{l}^- + H_{l}^3 + (H_{l}^{-1} H_{l}^-)^3,
$$

$$
H_{l}^7 = -\frac{3}{2} (H_{l}^3)^2 + H_{3} H_{3}^- - 2H_{3}^3 H_{3}^-,
$$

$$
H_{l}^7 = H_{l}^1 (H_{l}^3)^2 + 3H_{3}^3 (H_{l}^{-1} H_{l}^-)^2 - 2H_{3}^3 H_{3}^-,
$$

(84)

\[ \ldots \]
and
\[
5H_3^5 - 3H_3^3 = -(H_1^3)^2 + H_1^3 H_{-1}^3,
\]
\[
7H_2^3 - 3H_3^2 = \frac{1}{2} H_1^3 (H_1^3)^2 - 2H_1^3 (H_{-1}^3)_1^2 - H_1^3 H_1^3,
\]
\[
9H_2^4 - 3H_3^2 = 3H_1^3 (H_{-1}^3)^3 + \frac{3}{2} (H_1^3)^3,
\]
\[
7H_3^5 - 5H_2^2 = H_{-1}^3 (H_{-1}^3)^3 - \frac{3}{2} (H_1^3)^3 + H_1^3 H_1^3 H_{-1}^3 + \frac{1}{2} H_{-1}^3 (H_1^3)^2 - (H_1^3)^2,
\]

These and other such relations show that there are only three independent elements among \(H_2^{2+1}\). It is convenient to choose \(H_{-1}^3, H_1^3\) and \(H_2^3\) as the independent ones since these variables first define an elliptic curve \((80)\) and second they provide a simple parameterization of the algebraic variety \(W_{1\epsilon}\), defined by equations \((83)-(85)\) and so on. One readily obtains the following.

**Proposition 5.1.** The variety \(W_{1\epsilon}\) generically is a three-dimensional one and the variety \(W_1\) is the three parametric family of the coordinate rings for elliptic curves.

In particular, relations \((83)\) define a three-dimensional subvariety immersed into the six-dimensional Euclidean space with the coordinates \(H_{-1}^3, H_1^3, H_2^3, H_3^2, H_{-1}^3\) and \(H_2^3\). The induced metric on this subvariety is
\[
\begin{align*}
\mathrm{d}s^2 &= (1 + 4y_1^2 + y_2^2 + 4y_3^2) \, \mathrm{d}y_1^2 + (2 + y_1^2 + y_2^2) \, \mathrm{d}y_2^2 + (2 + 4y_1^2) \, \mathrm{d}y_3^2 \\
&\quad + 2(-2y_1 + y_1y_2 + 2y_2y_3) \, \mathrm{d}y_1 \, \mathrm{d}y_2 + 2(-y_2 + 4y_1y_3) \, \mathrm{d}y_1 \, \mathrm{d}y_3 \\
&\quad + 2(-y_1 + 2y_1y_2) \, \mathrm{d}y_2 \, \mathrm{d}y_3,
\end{align*}
\]

where \(H_{-1}^3 = y_1, H_1^3 = y_2\) and \(H_2^3 = y_3\) are chosen as the local coordinates. The Riemannian curvature tensor has the following nonzero components:

\[
R_{1212} = \frac{-2 - y_2^2 - 16y_1y_3 + 4y_2 - 8y_1^2 - 12y_1y_2y_3 - 8y_3^2 - 8y_1y_2^2 - 16y_1^3 - 8y_1^3y_3 + 2y_2^3}{D},
\]

\[
R_{1213} = \frac{2y_1y_2 - 8y_1 + 8y_2y_3 + 8y_1y_2^2 - 16y_1^3 + 8y_1^3y_2}{D},
\]

\[
R_{1223} = \frac{-8 - 18y_1^2 - 8y_1^4 - 4y_2^2 - 8y_1^2y_2}{D},
\]

\[
R_{1313} = \frac{-16 - 8y_2^2 - 36y_1^2 - 16y_1^3y_2 - 16y_1^4}{D},
\]

where
\[
D = 4 + 4y_2^2 + 17y_1^2 - 4y_1y_2y_3 + 32y_1y_2y_3 + 16y_3^2 + y_2^4 + 24y_1y_2^2 + 8y_1^2y_2 + 32y_1^2y_1^6 + 4y_3^2y_1^2 + 24y_1^4 + 16y_1^3y_3.
\]
and
\[
\Delta^7_{-1} = -2 \Delta^3_{-1} H^3_1 - 2 H^3_{-1} \Delta^3_1 + \Delta^3_1 + 3 (H^3_{-1})^2 \Delta^3_{-1},
\]
\[
\Delta^7_i = -3 H^3_{-1} \Delta^3_1 + \Delta^3_1 H^3_{-1} + 2 H^3_1 \Delta^3_{-1} - 2 \Delta^3_1 H^3_{1} - 2 H^3_1 \Delta^3_{-1},
\]
\[
\Delta^7_3 = \Delta^3_{-1} (H^3_1)^2 + 2 H^3_{-1} H^3_1 \Delta^3_1 + 3 \Delta^3_1 (H^3_{-1})^2 + 6 H^3_1 H^3_{-1} \Delta^3_{-1} - 2 \Delta^3_1 H^3_{3} - 2 H^3_1 \Delta^3_{3},
\]
(90)

and so on.

The variety $W_{1c}$ is a regular one and, hence, the bundles $T_{W_{1c}}$ and $T^*_{W_{1c}}$ are the three-dimensional ones.

Now, following the general idea described in the previous section, we will consider a particular subvariety $W^I_{1c}$ for which the three one-forms
\[
\omega_i = H^3_{1} dx_7 + H^3_{5} dx_5 + H^3_{3} dx_3, \quad i = -1, 1, 3,
\]
(91)

where $x_3$, $x_5$, and $x_7$ are some local coordinates in $W_{1c}$, are closed. Equivalently $W^I_{1c}$ is a subvariety of $W_{1c}$ such that
\[
\left. d\omega_i \right|_{W^I_{1c}} = 0.
\]
(92)
The conditions $d\omega_i = 0$ locally imply that
\[
H^3_i = \frac{\partial S_i}{\partial x_3}, \quad H^3_{i} = \frac{\partial S_i}{\partial x_5}, \quad H^3_{i} = \frac{\partial S_i}{\partial x_7}, \quad i = -1, 1, 3,
\]
(93)

where $S_i$ ($i = -1, 1, 3$) are three functions such that $\omega_i = dS_i$. Substitution of these expressions into (83) and (84) gives rise to the partial differential equations
\[
\frac{\partial S_{-1}}{\partial x_5} = \frac{\partial S_1}{\partial x_3} - \left( \frac{\partial S_{-1}}{\partial x_3} \right)^2,
\]
\[
\frac{\partial S_1}{\partial x_5} = -\frac{\partial S_{-1}}{\partial x_3} \frac{\partial S_1}{\partial x_3} + \frac{\partial S_3}{\partial x_3},
\]
\[
\frac{\partial S_3}{\partial x_5} = -\frac{1}{2} \left( \frac{\partial S_1}{\partial x_3} \right)^2 - 2 \frac{\partial S_3}{\partial x_3} \frac{\partial S_{-1}}{\partial x_3},
\]
(94)

and
\[
\frac{\partial S_{-1}}{\partial x_7} = -2 \frac{\partial S_{-1}}{\partial x_3} \frac{\partial S_1}{\partial x_3} + \frac{\partial S_3}{\partial x_3} + \left( \frac{\partial S_{-1}}{\partial x_3} \right)^3,
\]
\[
\frac{\partial S_1}{\partial x_7} = -\frac{3}{2} \left( \frac{\partial S_1}{\partial x_3} \right)^2 + \frac{\partial S_1}{\partial x_3} \frac{\partial S_{-1}}{\partial x_3} + 2 \frac{\partial S_3}{\partial x_3} \frac{\partial S_{-1}}{\partial x_3},
\]
\[
\frac{\partial S_3}{\partial x_7} = \frac{\partial S_{-1}}{\partial x_3} \left( \frac{\partial S_1}{\partial x_3} \right)^2 + 3 \frac{\partial S_3}{\partial x_3} \frac{\partial S_{-1}}{\partial x_3} - 2 \frac{\partial S_3}{\partial x_3} \frac{\partial S_{-1}}{\partial x_3},
\]
(95)

while the comparison with equations (89) and (90) shows that
\[
\Delta^k_i = \frac{\partial \Delta S_i}{\partial x_k}, \quad i = -1, 1, 3, \quad k = 3, 5, 7, \ldots.
\]
(96)
Lemma 5.2. The closedness of the forms (91) implies that all one-forms
\[ \omega_i = H_i^7 \, dx_7 + H_i^2 \, dx_5 + H_i^3 \, dx_3, \]
for all \( i = -1, 1, 3, 5, 7, \ldots \), or equivalently the one-form
\[ \omega_i = p_i(z) \, dx_7 + p_2(z) \, dx_5 + p_3(z) \, dx_3, \]
is closed.

Proof is by direct calculation with the use of the algebraic relations (83)–(85) and others. Thus locally,
\[ \omega(z) = dS(z), \quad p_k(z) = \frac{\partial S}{\partial x_k}, \quad k = 3, 5, 7, \]
where
\[ S(z) = x_7 z_7^7 + x_5 z_5^3 + x_3 z_3^3 + \sum_{k=0}^{\infty} \frac{S_{2k+1}}{2k+1}. \]
Moreover, one can show that the differential consequences of all other equations defining the subvariety \( W_1 \) are satisfied due to equations (94) and (95) and flows given by these equations commute.

So one has the following.

Proposition 5.3. The subvariety \( W_1 \) for which one-forms (93) are closed is characterized (locally) by the compatible system of PDEs (94), (95).

As far as the variety \( W_1 \) is concerned, the system (94) and (95) defines a special family of the coordinate rings for the elliptic curves (80) or, equivalently, a special family of the deformed elliptic curves parameterized by the solutions of the system (94) and (95). In terms of the coefficients \( u_0, u_1, u_2 \) (see (82)) defining an elliptic curve, equations (94) and (95) are given by
\[
\frac{\partial u_2}{\partial x_5} = -\frac{3}{2} \left( \frac{\partial}{\partial x_3} u_2 \right) u_2 + \frac{\partial}{\partial x_3} u_1, \\
\frac{\partial u_1}{\partial x_5} = \frac{\partial}{\partial x_3} u_0 - \frac{1}{2} \left( \frac{\partial}{\partial x_3} u_1 \right) u_2 - \left( \frac{\partial}{\partial x_3} u_2 \right) u_2, \\
\frac{\partial u_0}{\partial x_5} = -\frac{1}{2} \left( \frac{\partial}{\partial x_3} u_0 \right) u_2 - \left( \frac{\partial}{\partial x_3} u_2 \right) u_0, 
\]
and
\[
\frac{\partial}{\partial x_7} u_2 = -\frac{3}{2} \left( \frac{\partial}{\partial x_3} u_1 \right) u_2 + \frac{15}{8} \left( \frac{\partial}{\partial x_3} u_2 \right) (u_2)^2 - \frac{3}{2} \left( \frac{\partial}{\partial x_3} u_2 \right) u_1 + \frac{\partial}{\partial x_3} u_0, \\
\frac{\partial}{\partial x_7} u_1 = -\frac{1}{2} \left( \frac{\partial}{\partial x_3} u_0 \right) u_2 - \frac{3}{2} \left( \frac{\partial}{\partial x_3} u_1 \right) u_1 + \frac{3}{2} \left( \frac{\partial}{\partial x_3} u_2 \right) u_1 u_2 \\
- \left( \frac{\partial}{\partial x_3} u_2 \right) u_0 + \frac{3}{8} \left( \frac{\partial}{\partial x_3} u_1 \right) u_2^2, \\
\frac{\partial}{\partial x_7} u_0 = -u_0 \frac{\partial}{\partial x_3} u_1 - \frac{1}{2} \left( \frac{\partial}{\partial x_3} u_0 \right) u_1 + \frac{3}{8} \left( \frac{\partial}{\partial x_3} u_2 \right) u_2^2 + \frac{3}{2} \left( \frac{\partial}{\partial x_3} u_2 \right) u_0 u_2. \]
The system (101) is the well-known and well-studied dispersionless coupled KdV system (see e.g. [21–23]), while (102) gives its first higher order symmetry. The systems (101) and (102) are integrable hydrodynamical type systems with a number of remarkable properties (see e.g. [21–23]): they have infinite set of symmetries and conservation laws, they belong to the infinite
hierarchy, etc. Within a different approach [22], they arose in the Birkhoff stratum $\Sigma_2$ of $Gr^{(2)}$ as the hidden BH equations. We would like to emphasize that in the present context, they have a meaning of equations describing a special class of algebraic varieties $W^c_\ell$.

Passing to the infinite-dimensional variety $W_1$, one has an infinite-dimensional cotangent bundle $T^*_W$. Hence, one can consider special varieties $W^c_\ell$ for which the one-forms

$$\omega_i = \sum_{k=1}^{\infty} H^{2k+1}_i \, dx_{2k+1}, \quad i = -1, 1, 3,$$

(103)

are closed or, equivalently, the one-form

$$\omega(z) = \sum_{k=1}^{\infty} p_{2k+1}(z) \, dx_{2k+1},$$

(104)

where $x_{2k+1}, k = 1, 2, 3, \ldots$, are the local coordinates in $W_1$, is closed. In this case, $\omega(z) = dS(z)$,

(105)

where

$$S(z) = \sum_{m=1}^{\infty} z^{2m+1} x_{2m+1} + z S_{-1}(x) + \sum_{k=0}^{\infty} S_{2k+1} x_{2k+1}$$

(106)

and $p_j(z) = \frac{\delta S}{\delta x_j}, j = 3, 5, 7, \ldots$. The action $S(z)$ (106) is of the form found by a different method in [22] for the hidden BH hierarchy.

6. Deformations of moduli for elliptic curves and 2-cocycles

The systems (101) and (102) are of particular interest for the theory of deformations of elliptic curves and corresponding Riemann surfaces. Each solution of this system provides us with a nontrivial deformation of the curve (80).

In terms of the moduli $g_2$ and $g_3$ of an elliptic curve, i.e. in terms of (see e.g. [24])

$$g_2 = u_1 - \frac{1}{3} u_2^2 = 2 H^3 - \frac{1}{3} (H^{-1})^2,$$

$$g_3 = u_0 + \frac{2}{7} u_2^2 - \frac{1}{6} u_1 u_2 = \frac{2}{7} H^6 - \frac{1}{6} (H^{-1})^3 + 2 H^3,$$

(107)

equations (94), (95) or (101), (102) are of the form (see also [25])

$$\frac{\partial g_2}{\partial x_5} = \frac{\partial g_3}{\partial x_3} = \frac{5}{6} \frac{\partial g_2}{\partial x_2} - \frac{2}{3} \frac{\partial u_2}{\partial x_3} g_2,$$

$$\frac{\partial g_2}{\partial x_5} = \frac{5}{6} \frac{\partial g_3}{\partial x_2} u_2 - \frac{1}{6} \frac{\partial u_2}{\partial x_3} g_2 - \frac{\partial u_2}{\partial x_3} g_3,$$

$$\frac{\partial u_2}{\partial x_5} = \frac{5}{6} \frac{\partial g_2}{\partial x_3} - \frac{\partial u_2}{\partial x_3} u_2,$$

and

$$\frac{\partial g_2}{\partial x_7} = -\frac{7}{6} \frac{\partial}{\partial x_3} g_3 + \frac{7}{9} \frac{\partial u_2}{\partial x_3} \left( \frac{\partial}{\partial x_3} u_2 \right) g_2 + \frac{35}{12} \frac{\partial u_2}{\partial x_3} g_2 + \frac{5}{6} \frac{\partial}{\partial x_3} g_2 = \frac{3}{2} \frac{\partial g_2}{\partial x_3} g_2 - \left( \frac{\partial}{\partial x_3} u_2 \right) g_3,$$

$$\frac{\partial g_3}{\partial x_7} = \frac{7}{6} \frac{\partial}{\partial x_3} g_3 u_2 - g_3 \frac{\partial}{\partial x_3} g_2 - \frac{5}{6} \left( \frac{\partial}{\partial x_3} g_3 \right) g_2 + \frac{35}{72} \left( \frac{\partial}{\partial x_3} g_3 \right) u_2^2$$

$$+ \frac{2}{9} \left( \frac{\partial}{\partial x_3} u_2 \right) g_2^2 + \frac{7}{18} u_2 g_2 \frac{\partial}{\partial x_3} g_2,$$

$$\frac{\partial u_2}{\partial x_7} = -\frac{7}{6} \frac{\partial}{\partial x_3} g_2 + \frac{35}{72} \left( \frac{\partial}{\partial x_3} u_2 \right) u_2^2 - \frac{7}{6} \left( \frac{\partial}{\partial x_3} u_2 \right) g_2 + \frac{\partial}{\partial x_3} g_3$$

(109)
or equivalently
\[
\frac{\partial g_2}{\partial x_5} = \frac{\partial g_3}{\partial x_3} - \frac{5}{3} \frac{\partial g_2}{\partial x_3} \frac{\partial S_{-1}}{\partial x_3} - \frac{4}{3} \frac{\partial^2 S_{-1}}{\partial x_3^2} g_2,
\]
\[
\frac{\partial g_3}{\partial x_5} = \frac{5}{3} \frac{\partial g_3}{\partial x_3} \frac{\partial S_{-1}}{\partial x_3} - \frac{1}{3} \frac{\partial g_2}{\partial x_3} g_2 - \frac{2}{3} \frac{\partial^2 S_{-1}}{\partial x_3^2} g_3,
\]
\[
\frac{\partial S_{-1}}{\partial x_5} = \frac{1}{2} g_2 - \frac{5}{6} \left( \frac{\partial S_{-1}}{\partial x_3} \right)^2,
\]
and
\[
\frac{\partial g_2}{\partial x_7} = -\frac{7}{3} \frac{\partial S_{-1}}{\partial x_3} \frac{\partial}{\partial x_3} g_3 + \frac{28}{9} \frac{\partial S_{-1}}{\partial x_3} \left( \frac{\partial^2}{\partial x_3^2} S_{-1} \right) g_2 + \frac{35}{18} \left( \frac{\partial S_{-1}}{\partial x_3} \right)^2 \frac{\partial}{\partial x_3} g_2 - \frac{3}{2} g_2 \frac{\partial}{\partial x_3} g_2,
\]
\[
\frac{\partial g_3}{\partial x_7} = \frac{7}{3} g_3 \left( \frac{\partial}{\partial x_3} u_2 \right) \frac{\partial S_{-1}}{\partial x_3} - \frac{6}{9} g_3^2 \frac{\partial}{\partial x_3} g_2 - \frac{5}{6} \left( \frac{\partial}{\partial x_3} g_3 \right) g_2 + \frac{35}{18} \left( \frac{\partial}{\partial x_3} g_3 \right) \left( \frac{\partial S_{-1}}{\partial x_3} \right)^2
\]
\[
+ \frac{4}{9} \left( \frac{\partial^2}{\partial x_3^2} S_{-1} \right) g_2^2 + \frac{7}{9} g_2^2 \frac{\partial S_{-1}}{\partial x_3} \frac{\partial}{\partial x_3} g_3,
\]
\[
\frac{\partial S_{-1}}{\partial x_7} = \frac{7}{6} g_2^2 \frac{\partial}{\partial x_3} S_{-1} + \frac{35}{54} \left( \frac{\partial}{\partial x_3} S_{-1} \right)^3 + \frac{1}{2} g_3^3.
\]
Deformations of the discriminant \( \Delta = -16 \left( 2g_3^3 + 27g_3^2 \right) \) of an elliptic curve (80) are defined by the equation
\[
\frac{\partial \Delta}{\partial x_5} = -192 \left( g_2 \right)^2 \frac{\partial}{\partial x_3} g_3 + 128 \left( g_2 \right)^3 \frac{\partial}{\partial x_3} u_2 + 160 \left( g_2 \right)^2 u_2 \frac{\partial}{\partial x_3} u_2 + 720 g_3 u_2 \frac{\partial}{\partial x_3} g_3 + 864 \left( \frac{\partial}{\partial x_3} u_2 \right) \left( g_3 \right)^2 + 288 g_3^2 \frac{\partial}{\partial x_3} g_2^2.
\]
and

$$f_1(p_{2j+1}) = \frac{\partial \Delta S_{-1}}{\partial x_{2j+1}} + \sum_{m=0}^{\infty} \frac{1}{z^{2m+1}} \frac{\partial \Delta S_{2m+1}}{\partial x_{2j+1}},$$

(114)

where $\Delta S_{2j+1}$ denote variations of $S_{2j+1}$. As a special case, one has $\Delta S_{2j+1} = \sum_{m=1}^{\infty} \frac{\partial \Delta S_{2m+1}}{\partial x_{2j+1}}$, where $d_m$ are arbitrary constants.

These formulae are quite similar to those for the big cell written in terms of the corresponding $S_{2k+1}$. In the big cell, due to relations (52), one can go further and express all $S_{2k+1}$ as the derivatives of a single function $F$ and obtain Hirota equations (77). Such a property is not valid, in general, for the variety $W_1$ in the stratum $\Sigma_2$. Indeed one has relations (85) instead of (52). So, even all $H_{2k+1}^{2j+1} = \frac{\partial \Delta S_{2j+1}}{\partial x_{2j+1}}$, the relations (85), in contrast to relations (52) apparently, do not imply the existence of a single function $F$ such that $H_{2k+1}^{2j+1} = \frac{\partial \Delta S_{2j+1}}{\partial x_{2j+1}}$. This fact supports the observation made in [22].

7. Deformations of elliptic curves with a fixed point and dispersionless NLS equation

Particular subvarieties in $W_1$ and $W_{1c}$ and corresponding reductions of the system (101) (102) are of interest too. The simplest corresponds to the constraint $u_0 = 0$ or

$$H_1^3 + H_1^3 = 0.$$  

(115)

The elliptic curve (80) in this case assumes the form

$$p_2^2 = k^3 + u_2 k^2 + u_1 k,$$

(116)

which corresponds to the vanishing of one of roots for the cubic polynomial on the rhs of (80). Under constraint (115), the subvariety $W_{1c}$ becomes two dimensional, and $H_1^3$ can be chosen as the local coordinates on it. Consequently, a natural analog of the closedness condition discussed in the previous section is given by

$$d\omega_i = 0, \quad \omega_i = H_i^3 dx_3 + H_i^3 dx_3, \quad i = -1, 1, 3, \quad (117)$$

under constraint (115). Similar to lemma 5.2, one can show that condition (117) implies the closedness of the form

$$\omega(z) = p_2(z) dx_3 + p_3(z) dx_3.$$  

(118)

and, then, validity of formulae (99) and (100) with $x_7 = 0$.

Condition (115) gives rise to the equations

$$\frac{\partial u_2}{\partial x_3} = - \frac{3}{2} \left( \frac{\partial}{\partial x_3} u_2 \right)^2 u_2 + \frac{\partial}{\partial x_3} u_1,$$

$$\frac{\partial u_1}{\partial x_3} = \frac{1}{2} \left( \frac{\partial}{\partial x_3} u_1 \right)^2 u_2 - \left( \frac{\partial}{\partial x_3} u_2 \right) u_1.$$  

(119)

This system describes deformations of the elliptic curve (116) for which the origin $(p_3 = x = 0)$ is the fixed point.

At $u_0 = 0$, the definition (107) implies that

$$\left( \frac{u_2}{3} \right)^3 + \frac{u_2}{3} g_2 + g_3 = 0.$$  

(120)

Hence, equations (119) or (108) for the moduli $g_2$ and $g_3$ become

$$\frac{\partial g_2}{\partial x_3} = \frac{5}{6} \frac{\partial g_2}{\partial x_3} u_2 - \frac{2}{3} \frac{\partial u_2}{\partial x_3} g_2,$$

$$\frac{\partial g_3}{\partial x_3} = \frac{5}{6} \frac{\partial g_2}{\partial x_3} u_2 - \frac{1}{3} \frac{\partial u_2}{\partial x_3} g_2.$$  

(121)
where $u_2$ is a root of the cubic equation (120). The original system (108) is compatible with constraint (120), which is equivalent to $u_0 = 0$. Another form of the system (108) under constraint (120) is given by the system

\[
\begin{align*}
\frac{\partial g_2}{\partial x_5} &= -\frac{1}{9} u_2^5 \frac{\partial u_2}{\partial x_3} - \frac{7}{6} \frac{\partial g_2}{\partial x_3} - \frac{\partial u_2}{\partial x_3}, \\
\frac{\partial u_2}{\partial x_5} &= \frac{\partial g_2}{\partial x_3} - \frac{5}{6} \frac{\partial u_2}{\partial x_3}.
\end{align*}
\]

(122)

Solving this system, one reconstructs $g_3 = -(\frac{u_2}{5})^3 - \frac{u_2}{5} g_2$.

For the discriminant $\Delta = 16u_1^2(u_2^2 - 4u_1)$, one has

\[
\begin{align*}
\frac{\partial \Delta}{\partial x_5} &= \frac{592}{3} (u_2 g_2)^2 \left( \frac{\partial}{\partial x_3} u_2 \right) + \frac{192}{3} (g_2)^3 u_2 + 128 (g_2)^2 u_2 \frac{\partial}{\partial x_3} g_2 + \frac{112}{27} (u_2)^5 \frac{\partial}{\partial x_3} u_2 \\
&\quad + \frac{512}{9} (u_2)^4 \left( \frac{\partial}{\partial x_3} u_2 \right)^2 g_2 + \frac{80}{9} (u_2)^5 \frac{\partial}{\partial x_3} g_2 + \frac{208}{3} (u_2)^3 g_2 \frac{\partial g_2}{\partial x_3}.
\end{align*}
\]

(123)

We emphasize that the elliptic curve (116) generically is not singular and it remains almost everywhere regular under deformations given by equations (119)–(123). There are two obvious constraints, namely $u_1 = 0$ and $u_2 = 4u_1$ under which curve (116) becomes singular ($\Delta = 0$). Under both these constraints, the system (119) is reduced to the BH equation. So the BH equation describes deformations of the degenerate plane cubic in agreement with the observation made in [4]. Thus, systems (119) or (121) are of importance for the theory of elliptic curves.

In fact system (119) is a well-known one in the theory of dispersionless integrable systems. It is the so-called dispersionless Jaulent–Miodek system (see e.g. [23]). Under the change of the dependent variables

\[
u = -u_2, \quad v = -u_1 + \frac{1}{4} u_2^2
\]

(124)

and $x_3 = x$, $x_5 = -t$, it becomes the 1-layer Benney system

\[
\begin{align*}
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial x} &= 0, \\
\frac{\partial v}{\partial t} + \frac{\partial}{\partial x} (uv) &= 0,
\end{align*}
\]

(125)

which describes long waves on the shallow water [27]. Moreover, the system (125) is the quasiclassical limit [28] of the famous nonlinear Schrödinger (NLS) equation

\[
i \epsilon \frac{\partial}{\partial t} \psi + \frac{\epsilon^2}{2} \frac{\partial^2}{\partial x^2} \psi + |\psi|^2 \psi = 0
\]

(126)

as

\[
\psi = A e^{i \phi}, \quad u = S_x, \quad v = -A^2
\]

(127)

and $\epsilon \to 0$.

The NLS equation (126) and its quasiclassical limit arise in the numerous nonlinear phenomena in physics and problems in mathematics (see e.g. [29, 30]). However, its relevance to the deformation theory for elliptic curves seems to have not been mentioned before.

In addition to a number of remarkable properties typical for integrable hydrodynamical type equations, the system (125) implies the existence of a single function $\phi$ such that it is equivalent to the single Hirota-type equation. To demonstrate this it is convenient to use the system (94) under constraint (115), i.e.

\[
\frac{\partial S_3}{\partial x_3} + \frac{\partial S_{-1}}{\partial x_3} \frac{\partial S_1}{\partial x_3} = 0,
\]

(128)
that is
\[
\frac{\partial S_{-1}}{\partial x_5} = \frac{\partial S_1}{\partial x_3} - \left( \frac{\partial S_{-1}}{\partial x_3} \right)^2,
\]
(129)
\[
\frac{\partial S_1}{\partial x_5} = -2 \frac{\partial S_1}{\partial x_3} \frac{\partial S_{-1}}{\partial x_3}.
\]
Differentiating the first equation (129) with respect to \(x_3\) and expressing \(\frac{\partial S_{-1}}{\partial x_3}\) in terms of \(S_1\), using the second equation of (129), one obtains
\[
\frac{\partial^2 S_1}{\partial x_2^2} = \frac{\partial}{\partial x_3} \left( -\left( \frac{\partial S_1}{\partial x_3} \right)^2 + \left( \frac{\partial S_{-1}}{\partial x_3} \right)^2 \right).
\]
(130)
This equation implies the existence of the function \(\phi\) such that
\[
S_1 = \frac{\partial \phi}{\partial x_3}.
\]
(131)
In terms of \(\phi\), equation (130) (choosing vanishing integration constants) is
\[
\phi_{x_3x_3} \phi_{x_3x_3} - \left( \phi_{x_3x_3} \right)^2 + \left( \phi_{x_3x_3} \right)^3 = 0,
\]
(132)
where \(\phi_{x_3x_3} = \frac{\partial^2 \phi}{\partial x_3^2}\).

Thus, one can construct solutions of the system (129) solving Hirota- (and Hessian-) type equation (132) and using formula (131) and \(\frac{\partial S_{-1}}{\partial x_3} = -\frac{1}{2} \phi_{x_3x_3}\). Solutions of system (119) are given by
\[
u_2 = \frac{-\phi_{x_3x_3}}{\phi_{x_3x_3}}, \quad \nu_1 = 2\phi_{x_3x_3} + \frac{1}{4} \phi_{x_3x_3}^2,
\]
(133)
and for the moduli \(g_2\) and \(g_3\), one has
\[
g_2 = 2\phi_{x_3x_3} - \frac{1}{12} \phi_{x_3x_3}^2, \quad g_3 = \frac{2}{3} \phi_{x_3x_3} - \frac{1}{108} \phi_{x_3x_3}^3,
\]
(134)
while the solution of the 1-layer Benney system (125) is \((t = -x_3)\)
\[
u = \frac{\phi_{x_3x_3}}{\phi_{x_3x_3}}, \quad \nu = -2\phi_{x_3x_3}.
\]
(135)
One can obtain equation (132) directly from the 1-layer Benney system (125) too. In fact, the second equation (125) implies the existence of the function \(\tilde{\phi}\) such that \(v = 2\tilde{\phi}\), and, hence, \(u = -\tilde{\phi}\). Substituting these expressions into the first equations (125), one obtains
\[
\tilde{\phi}_{tt} = \left( \frac{\tilde{\phi}_t^2}{\tilde{\phi}_x} + \tilde{\phi}_x^2 \right) x.
\]
(136)
This implies that \(\tilde{\phi} = \tilde{\phi}_x\), and equation (136) becomes
\[
\tilde{\phi}_{tt} = \frac{\tilde{\phi}_t^2}{\tilde{\phi}_x} + \tilde{\phi}_x^2
\]
(137)
that coincides with (132) modulo substitution \(x = -x_3, \phi = -\tilde{\phi}\).
Thus, the Hirota equation (132) governs deformations of the elliptic curve (116). It should be relevant also to the study of the quasiclassical limit of the NLS equation. Equation (132) has several interesting properties. For example, it is invariant under the scale transformations

\[ x_3 \rightarrow \rho x_3, \quad x_5 \rightarrow \rho^2 x_5, \quad \phi \rightarrow \phi. \]  

(138)

Hence, it admits the self-similar solutions

\[ \phi = f \left( \frac{x_3^2}{x_5} \right) \]  

(139)

for which it is reduced to the following ODE:

\[ y^2 \varphi \varphi' + 4 (\varphi + 2y \varphi')^3 = 0, \]  

(140)

where \( y = \frac{x_3^2}{x_5} \) and \( \varphi = \frac{df}{dy} \). One can show that the only monomial solution of the equation (140) is \( \varphi = -\frac{1}{108} y \). This implies that \( H_{3}^{1} = -\frac{1}{108} \frac{x_3^2}{x_5} \) and \( H_{3}^{2} = \frac{1}{54} \frac{x_3^2}{x_5} \) and the corresponding family of elliptic curves is degenerate and it is given by

\[ P_s^2 = \lambda^3 + \frac{2x_3^2}{3x_5} \lambda^2. \]  

(141)

Solutions of equations (132) provide us with a particular class of 2-cocycles \( \psi_{1}^{\text{dNLS}} \) and 2-cocoboundaries \( f_{1}^{\text{dNLS}} \) defined by formulae (113) and (114) under reduction (115), for example,

\[ \psi_{1}^{\text{dNLS}}(p_3, p_1) = (-\Delta u)\lambda^2 + \left( -\Delta v + \frac{1}{2} u \Delta u \right) \lambda = \left( \frac{\Delta \phi}{\phi_{x_3 x_5}} + \frac{\phi_{x_3 x_5}}{\phi_{x_5}^2} (\Delta \phi)_{x_3 x_5} \right) \lambda^2 \]

\[ + \left( 2 (\Delta \phi)_{x_3 x_5} \frac{1}{2} \phi_{x_5}^2 (\Delta \phi)_{x_3 x_5} - \frac{1}{2} \phi_{x_3 x_5}^2 (\Delta \phi)_{x_3 x_5} \right) \lambda. \]  

(142)

One can refer to such 2-cocycles as dNLS 2-cocycles.

8. Hyperelliptic curves in \( W_g \) and dispersionless coupled KdV equations

For the strata \( \Sigma_{2g} (g > 1) \), the variety \( W_g \) is defined by relations (14) and (15) and associativity conditions [8]

\[ H_{2m}^{2m} = 0, \quad m = 0, 1, 2, \ldots, \quad k = -2g + 2, -2g + 4, \ldots, -2, 0, 1, 2, 3, \ldots, \]  

(143)

\[ H_{2k}^{2m+1} = 0, \quad m = 0, 1, 2, \ldots, \quad k = -g, -g + 1, -g + 2, \ldots, \]

and

\[ H_{2k+1}^{2j+1} + H_{2k+1}^{2(j+1)} + \sum_{l=0}^{j} H_{2r+1}^{2(j+1)} H_{2(l-r)-1} = 0, \]

\[ H_{2k+1}^{2j+1} + H_{2k+1}^{2(j+1)} + \sum_{l=0}^{j} H_{2r+1}^{2(j+1)} H_{2(l-r)-1} = 0, \]

(144)

and

\[ p_{2k+1}^2 = \lambda^{2g+1} + \sum_{k=0}^{2g} u_k \lambda^k. \]  

(145)
where the coefficients \( u_k \) in (14) can be obtained from

\[
p_{2g+1}^2 = \lambda^{2g+1} + 2 \sum_{j=0}^{2g+1} H_{2g+1}^{2j+1} \lambda^j + \sum_{k=-g}^{g+1} \sum_{s=0}^{f-k-1} H_{2s+1}^{2k+1} H_{2s+1}^{2k+1} \lambda^s.
\]  

(146)

**Lemma 8.1.** The subvariety \( W_{gc} \) of the coefficients \( H_j \) has the dimension \( 2g + 1 \).

**Proof.** We first observe that a hyperelliptic curve (14) is parameterized by \( 2g + 1 \) variables \( H_{2g+1}^{-2g-1}, H_{2g+1}^{-2g-3}, \ldots, H_{2g+1}^{-1}, H_{2g+1}^1, \ldots, H_{2g+1}^{2g+1} \). Then, evaluating coefficients in front of \( \frac{1}{\lambda^k} \), \( k = 0, 1, 2, \ldots, \) in equation (14), one concludes that all of them are certain polynomials of these \( 2g + 1 \) variables. For instance,

\[
H_{2g+1}^{2g+1} = -\frac{1}{2} (H_{2g+1}^{2g+1})^2 - \sum_{j=1}^g H_{2j+1}^{2g+1} H_{1-2j}^{2g+1},
\]  

(147)

Further, the part of conditions (143) and (144) encoded in relations (15), or equivalently in the relations

\[
p^k_{2g+1} = p_{2g+1}^{2g+1} + H_{-2g-1}^{2g+1} p_{2g+1} = 0, \quad k = g, g+1, g+2, \ldots,
\]  

(148)

allows us to express recursively all the variables \( H_{2g+1}^{2k+1} \), \( k = g, g+1, g+2, \ldots \), in terms of \( H_1^{2g+1} \) and, hence, in terms of \( H_j^{2g+1} \) with \( j = -(2g-1), -(2g-3), \ldots, 1, \ldots, 2g+1 \). In particular, one has

\[
H_{2g+1}^{2g+1} - H_{2g+1}^{2g+1} + H_{2g+1}^{2g+1} H_{2g+1}^{2g+1} = 0,
\]  

(149)

and

\[
H_{-2g-1}^{2g+1} = p^k_n (H_{2g+1}^{2g+1}),
\]  

(150)

where \( p^k_n \) are certain polynomials of \( 2g + 1 \) variables \( \{H_{-2g-1}^{2g+1}, H_{3-2g}^{2g+1}, H_{5-2g}^{2g+1}, \ldots, H_{1+2g}^{2g+1}\} \). \( \square \)

For example, at \( g = 2 \) one has the set \( \{H^5\} = \{H_{-3}^5, H_{-1}^5, H_1^5, H_3^5, H_5^5\} \) and the first of relations (149) and (150) are

\[
H_3^7 = H_{-3}^5 - H_{-5}^5, \\
H_{1}^7 = H_3^5 - H_{-3}^5 H_1^5, \\
H_{3}^7 = H_5^5 - H_{-3}^3 H_3^5, \\
H_{3}^7 = -H_5^5 H_1^5 - \frac{1}{2} H_1^2 - 2 H_3^5 H_5^5, \\
H_{3}^9 = H_5^7 - 2 H_3^5 H_1^5 + H_{-5}^5, \\
H_{1}^9 = -H_{3}^7 + H_{-3}^5 H_1^5 + H_1^5 - H_{-3}^5 H_1^5, \\
H_{1}^9 = H_{-3}^5 + H_{-3}^3 H_3^5 - H_3^5 H_1^5 + H_1^5 H_{-3}^5, \\
H_{3}^9 = -H_{-3}^5 H_3^5 + H_{-3}^3 H_3^5 - \frac{1}{2} H_1^2 - 2 H_3^5 H_5^5, \\
H_{3}^9 = -H_{-3}^5 H_5^5 + 3 H_{-3}^3 H_3^5 + 2 H_{-3}^5 H_{-3}^5 H_3^5 + H_{-3}^3 H_1^2 - H_{-3}^3 H_5^5.
\]  

(152)
Proposition 8.2. The variety $W_g$ represents a $(2g + 1)$-dimensional family of the coordinate rings of the deformed hyperelliptic curves (14) of genus $g$.

The tangent and cotangent spaces of $W_g$ are also $(2g + 1)$ dimensional. They are defined by the linearized versions of relations (143), (144) or (149), (150). For example, the part corresponding to relations (151) at $g = 2$ is given by

\begin{align*}
\Delta_{-1}^7 &= \Delta_{-1}^5 - 2\Delta_{-3}^5 \Delta_{-3}^3, \\
\Delta_{-1}^7 &= \Delta_{-1}^5 - \Delta_{-3}^5 \Delta_{-3}^3 - H_{-3}^5 \Delta_{-3}^3, \\
\Delta_1^7 &= \Delta_1^5 - \Delta_{-3}^5 \Delta_1^5 - H_{-3}^5 \Delta_{-3}^5, \\
\Delta_3^7 &= \Delta_3^5 - \Delta_{-3}^5 \Delta_3^3 - H_{-3}^5 \Delta_{-3}^3, \\
\Delta_5^7 &= -\Delta_{-1}^5 H_5^5 - H_{-3}^5 \Delta_5^3 - 2 \Delta_{-3}^5 H_{-3}^5 - 2 H_{-3}^5 \Delta_{-3}^3.
\end{align*}

(153)

Definition 8.3. The variety $W_{g}^{1}$ is the subvariety of $W_{g}$ for which $2g + 1$ one-forms

\[ \omega_i = H_i^{2g+1} \partial x_{2g+1} + H_i^{2g+3} \partial x_{2g+3} + \cdots + \partial H_i^{6g+1} \partial x_{6g+1}, \]

(154)

where $x_{2g+1}, x_{2g+3}, \ldots, x_{6g+1}$ are local coordinates in $W_{g}$, are closed.

Lemma 8.4. The closedness of the forms (154) implies the closedness of all form of type (154) with all $i = 2g + 3, 2g + 5, \ldots$.

Proof is by direct but rather cumbersome calculation. Thus, one has the following.

Proposition 8.5. The closedness of $2g + 1$ one-forms (154) in $W_{g}$ is the necessary and sufficient condition for the closedness of the one-form

\[ \omega(z) = \sum_{k=1}^{2g} z^{-k} \omega_k = \sum_{i=0}^{2g} \frac{p_{2(g+i)+1}(z)}{z^{2(g+i)+1}} \partial x_{2(g+i)+1}. \]

(155)

Corollary 8.6. For the variety $W_{g}^{1}$, one has locally

\[ \omega(z) = dS(z, x), \]

(156)

where

\[ S(x, z) = \sum_{k=g}^{3g} \frac{x_{2k+1}^{2k+1}}{z^{2k+1}} + \sum_{l=1}^{1} \frac{z^{2l-1} S_{1-2l} + \sum_{m=0}^{\infty} S_{2m+1}}{z^{2m+1}} \]

(157)

and

\[ p_{2j+1}(z) = \frac{dS(x, z)}{\partial x_{2j+1}}, \quad j = -g, \ldots, 3g \]

\[ H_k^{2j+1} = \frac{dS(x, z)}{\partial x_{2j+1}}, \quad k = 1 - 2g, 3 - 2g, 5 - 2g, \ldots \]

(158)
In virtue of (158), the algebraic relations (149) and (150) become systems of $2g + 1$ PDEs of Hamilton–Jacobi type for $2g + 1$ unknown $S_{1-2g}, S_{3-2g}, \ldots, S_1, \ldots, S_{2g+1}$. In particular, the system (149) takes the form

$$
\frac{\partial S_{1-2k}}{\partial x_{2g+1}} = -\frac{\partial S_{3-2k}}{\partial x_{2g+1}} + \frac{\partial S_{2g-1}}{\partial x_{3g+1}} \frac{\partial S_{2k-1}}{\partial x_{2g+1}} = 0, \quad k = 1 - g, 2 - g, \ldots, g
$$

(159)

while relations (150) give

$$
\frac{\partial S_{1+2k}}{\partial x_{2g+1}} = \frac{\partial S_{2g+1}}{\partial x_{3g+1}} \left( \frac{\partial S_{1-2l}}{\partial x_{2g+1}} \right)_{1=g, g, k} + \frac{\partial S_{2g-1}}{\partial x_{2g+1}} \frac{\partial S_{2g+1}}{\partial x_{2g+1}} = 0,
$$

(159)

where $F_{n_k}$ are certain polynomials on $S_{1-2k}$.

It is a straightforward check that the systems of PDEs (159) and (160) commute between themselves. So one can state the following.

**Proposition 8.7.** The variety $W^I_k$ is a family of deformations of the coordinate rings of the hyperelliptic curves (14) governed by the $2g$ commuting $2g + 1$ component systems of PDEs (159) and (160).

As the concrete example, we take relations (151) for $g = 2$. The corresponding system of PDEs is

$$
\frac{\partial S_{-3}}{\partial x_1} = \frac{\partial S_{-1}}{\partial x_1} - \left( \frac{\partial S_{-3}}{\partial x_1} \right)^2,
$$

$$
\frac{\partial S_{-1}}{\partial x_1} = \frac{\partial S_{1}}{\partial x_1} - \frac{\partial S_{-3}}{\partial x_5} \frac{\partial S_{-1}}{\partial x_5},
$$

$$
\frac{\partial S_{1}}{\partial x_1} = \frac{\partial S_{3}}{\partial x_5} - \frac{\partial S_{-3}}{\partial x_5} \frac{\partial S_{1}}{\partial x_5},
$$

$$
\frac{\partial S_{3}}{\partial x_1} = \frac{\partial S_{5}}{\partial x_5} - \frac{\partial S_{-3}}{\partial x_5} \frac{\partial S_{3}}{\partial x_5},
$$

$$
\frac{\partial S_{5}}{\partial x_7} = -\frac{\partial S_{-1}}{\partial x_5} \frac{\partial S_{3}}{\partial x_5} - \frac{1}{2} \left( \frac{\partial S_{1}}{\partial x_5} \right)^2 - 2 \frac{\partial S_{-3}}{\partial x_5} \frac{\partial S_{5}}{\partial x_5}.
$$

(161)

Relation (152) and those for $H^{11}, H^{13}$ give rise to the three other systems of PDEs of Hamilton–Jacobi type. These four five-component systems of PDEs describe a special class of deformations of the $g = 2$ curve (14) parameterized by five variables.

The systems (159) and (160) have various equivalent forms. For example, introducing $v_j = \frac{\partial S_j}{\partial x_{2g+1}}, j = 1 - 2g, \ldots, 1 + 2g$, one rewrites the system (149) and (150) as the set of $2g$ systems of conservation laws

$$
\frac{\partial v_{2j+1}}{\partial x_{2(g+1)+1}} = \frac{\partial}{\partial x_{2g+1}} F_{2j+1,k}(v), \quad j = -g, \ldots, g, k = 1, 2, \ldots, 2g.
$$

(162)

In terms of the coefficients $u_j$ of the hyperelliptic curve (14), the system (149) and (150) becomes the set of systems of hydrodynamical type

$$
\frac{\partial u_{2j+1}}{\partial x_{2(g+1)+1}} = V^{(k)}_{j}(u) \frac{\partial u_{j}}{\partial x_{2g+1}}, \quad j = 0, 1, 2, \ldots, 2g.
$$

(163)

where $V^{(k)}_{j}(u)$ are certain polynomials on $u_j$. 25
The systems (159), (160), (162) and (163) represent three different forms of the same system and bi-Hamiltonian structures (see e.g. [21–23]). In [22], they arose as the hidden BH hierarchies have all properties typical for integrable systems: infinite sets of symmetries, conservation laws, integrable hydrodynamical type systems, called the dispersionless coupled KdV systems. They are different from the previous one: they describe the local properties of a special class of algebraic representation found in that paper.

For the whole variety $W^g$, a special subvariety $W^g_I$ is defined by the requirement that $2g + 1$ one-forms

$$\omega_i = \sum_{k=0}^{\infty} H_{2^{i-1}+1} \, dz_{2^{i-1}+1}, \quad i = g, \ldots, g,$$

(166)

where $z_{2^{i-1}+1}$, $k = 0, 2, \ldots$, are local coordinates in $W^g$, are closed. This condition is equivalent to the condition of closedness of the one-form

$$\omega(z) = \sum_{k=0}^{\infty} \rho_{2^{i-1}+1}(z) \, dz_{2^{i-1}+1}.$$

(167)

The system (165) and the corresponding system for $g > 2$ are the well-known examples of the integrable hydrodynamical type systems, called the dispersionless coupled KdV systems. They have all properties typical for integrable systems: infinite sets of symmetries, conservation laws, bi-Hamiltonian structures (see e.g. [21–23]). In [22], they arose as the hidden BH hierarchies in the Birkhoff strata, and the compact form of such hierarchies has been found too. Also the fact that they can be written in the form (162) of conservation laws also follows from their representation found in that paper.

We would like to emphasize that in our approach they arise in a manner which is completely different from the previous one: they describe the local properties of a special class of algebraic varieties in $W^g_I$ in the Birkhoff strata $\text{Gr}(2)$. 
One more feature of the approach presented here is that it reveals a close interrelation between the special algebraic varieties of the type (143), (144) and integrable hydrodynamical type systems (163).

Similar connection in different setting, namely between hyperbolic systems of conservation laws and congruences of lines in projective spaces, has been noted and studied in [31–33]. Comparison of formulae (149), (150) and (162) with the first formulae from the papers [31–33] indicates that these two approaches could be connected. However, one can show that, for instance, the system (101) does not belong to the Temple class studied in [34].

9. Ideals of varieties $W^I_g$ as Poisson ideals

Any cotangent bundle carries a natural symplectic structure (see e.g. [5, 10]). Formulae (155) and (156) indicate that the symplectic structure on $T^*_{W^I_g}$ should be

$$\Omega^{gc} = \sum_{k=0}^{2g} dp_{2(g+k)+1} \wedge dx_{2(g+k)+1}. \quad (168)$$

In virtue of (156) for the subvariety $W^I_g$, one has

$$\Omega^{gc}|_{W^I_g} = 0, \quad (169)$$

i.e. the subvariety $W^I_g$ is the Lagrangian subvariety in $W_g$ if one treats $W_g$ as a symplectic variety equipped with the symplectic two-form (168) and $p_j$ and $x_j$ being the classical Darboux coordinates.

Passing then to the infinite-dimensional varieties $W_g$, i.e. the $2g$-parametric families of the coordinate rings for the hyperelliptic curve, one should consider an infinite-dimensional symplectic variety equipped with 2-form,

$$\Omega_g = \sum_{k=0}^{\infty} dp_{2(g+k)+1} \wedge dx_{2(g+k)+1}. \quad (170)$$

Within such a symplectic interpretation, it is quite natural to require that the ideal $I(W_g)$ of the variety $W_g$ has some properties typical for symplectic or Poisson varieties. One of the most natural requirements is that the ideal $I(W_g)$ is a Poisson ideal, i.e.

$$\{I(W_g), I(W_g)\} \subset I(W_g), \quad (171)$$

where $\{,\}$ is the Poisson bracket. Condition (171) of the closedness of the ideal $I$ has been used in [25] to define the so-called coisotropic deformations of algebraic varieties. Earlier this idea has been proposed in [35] in the context of coisotropic deformations of commutative associative algebras.

The ideal of the variety $W_g$ is given by (18) with $C_{2g+1}$ and $I^{(g)}_{2g+1}$, defined by (14) and (15) or, equivalently,

$$I(W_g) = \langle C_{2g+1}, \{M_k\}_g \rangle, \quad (172)$$

where $M_k$ is given by the lhs of (148), i.e.

$$M_k = \frac{aH^{2(k+1)+1}}{\partial x_{2g+1}} C_{2g+1} \quad (173)$$

In the basis of the ideal $I(W_g)$ composed by $C_{2g+1}$ and $M_k$, the closedness condition (171) with canonical Poisson bracket for the Darboux coordinates $x_j$, $p_j$ is equivalent to the following:

$$\{C_{2g+1}, M_k\} = \frac{aH^{2(k+1)+1}}{\partial x_{2g+1}} C_{2g+1}, \quad (174)$$

$$\{M_l, M_k\} = 0.$$
while $H_1^i$ and $u_m$ should obey the differential equations
\begin{align}
\frac{\partial H^{2(g+k)+1+1-2r}}{\partial x_{2(g+k)+1}} &= \frac{\partial H^{2(g+k)+1+1-2r}}{\partial x_{2(g+k)+1}}, \\
\frac{\partial H^{2(g+k)+1+1-2r}}{\partial x_{2(g+k)+1}} + H^{2(g+k)+1} \frac{\partial H^{2(g+k)+1+1-2r}}{\partial x_{2g+1}} - H^{2(g+k)+1} \frac{\partial H^{2(g+k)+1+1-2r}}{\partial x_{2g+1}} &= 0,
\end{align}
(175)
\begin{align}
\frac{\partial u_m}{\partial x_{2(g+k)+3}} - (1 - \delta_{m,0}) \frac{\partial u_{m-1}}{\partial x_{2(g+k)+1}} - H^{2(g+k)+1} \frac{\partial u_m}{\partial x_{2g+1}} - 2u_m \frac{\partial H^{2(g+k)+1+1-2r}}{\partial x_{2g+1}} &= 0,
\end{align}
(176)
k, l = 0, 1, 2, \ldots, m = 0, 1, 2, \ldots, 2g,

where $\delta_{m,0}$ is the Kronecker symbol. This is an infinite hierarchy of equations for $2g + 1$ unknowns $H_{1^2-2g}, H_{3^2-2g}, \ldots, H_{2g+1}, H_{2^2-2g+1}$, since all $H^{2(g+k)+1}$ are polynomials of these $2g + 1$ variables.

It is a straightforward check that the system (175)–(177) is equivalent to the hierarchy of the systems associated with the system (163). In other words, the hydrodynamical type systems discussed in the previous sections represent coisotropic deformations of curves (14).

We note that in our approach these systems arise within the study of the local properties of the special subvariety $W_{2g}$ carried a set of $2g + 1$ closed 1-forms.

Within the symplectic interpretation of the varieties $W_{g}$, one has simple realizations of the 2-cocycles, namely
\begin{align}
\psi_{g}(p_j, p_k) = [\alpha, f_{jk}]|_{W_{g}},
\end{align}
(178)
where $\alpha = \sum_{i \geq 2g+1} \alpha_i p_i$, $\alpha_i$ are arbitrary constants.

10. Discussion: cohomology blow-ups and comparison with Whitham theory

In the previous sections, it was shown that dispersionless coupled KdV (dcKdV) hierarchies (BH hierarchy for $g = 0$) provide us with a special class of 2-cocycles and 2-coboundaries which are well defined for regular solutions of the dcKdV hierarchies. Singular solutions of dcKdV of these hierarchies give rise to singular behavior of 2-cocycles and 2-coboundaries.

It is well known (see e.g. [56]) that the singular sector for the BH equation and BH hierarchy is composed by solutions which exhibit gradient catastrophe when derivatives of their solutions blow-up (become unbounded). A similar situation takes place also for dcKdV hierarchies [22]. Formulae (78), (79), (113) and (178) readily show that the blow-ups of $\frac{\partial u_m}{\partial x_l}$ lead to the blow-up of the corresponding 2-cocycles and 2-coboundaries (unboundedness of their values). Thus, gradient catastrophe for BH and dcKdV hierarchies and blow-ups for Harrison cohomology of the subvarieties $W_{g}$ are intimately connected. The analysis of the singular sectors of the BH hierarchy and dcKdV hierarchies performed in [22, 23] shows that the gradient catastrophe for these hierarchies happens on the subvarieties in the space of independent variables $x_j$ of finite codimensions and it is associated with the transition from one Birkhoff stratum to another. So such a transition is accompanied also by the blow-up of 2-cocycles and 2-coboundaries. Whether or not the blow-ups of Harrison cohomology happen on the subspaces of finite codimension (i.e. of zero measure) and are associated with certain transition between different strata is an open problem.

To compare our results with those of Whitham theory, we would like to note the following. Theory of finite-gap solutions and the theory of Whitham equations are two of the best
known chapters in the story of interconnection between algebraic curves and integrable nonlinear PDEs. Theory of periodic (or finite-gap) solutions for the KdV equation was first discussed in [37–41] and then extended to many other integrable nonlinear PDEs (see e.g. [42–48], review [49] and references therein). In this theory, an algebraic curve (a hyperelliptic curve in most cases) is fixed by initial data and, hence, remains unchanged during the evolution. Choosing different curves, e.g. hyperelliptic curves of different genus, one constructs classes of exact solutions of the same integrable PDE [37, 49].

The situation is quite different in Whitham theory. Whitham was the first to observe [50] (see also [51]) that the description of slow modulations of the simple phase traveling wave solutions of the KdV equation requires to consider an elliptic curve with varying parameters (moduli). Within the averaging method developed in [50, 51], the variations of wave solutions of the KdV equation requires the introduction of a family of Riemann surfaces varying gradually with X and T during the evolution. Choosing different curves, e.g. hyperelliptic curves, one constructs classes of exact solutions of the same integrable PDE [37, 49].

Reformulation of Whitham equations (180) and (181) in terms of Abelian differentials on a Riemann surface and generalization to modulated N-phase KdV waves associated with hyperelliptic curves given in [52, 53] clearly demonstrated that ‘... the description of the modulations of finite-gap solutions of KdV requires the introduction of a family of Riemann surfaces varying gradually with X and T. The mathematical problem is therefore one of deformation of hyperelliptic curves’ ([52], p 740). Further works on Whitham equations (see e.g. [54–60]) provide us with a wide class of deformations of algebraic curves, in particular, of hyperelliptic curves given in [52, 53] clearly demonstrated that ‘... the description of slow modulations of the simple phase traveling wave solutions of the KdV equation requires to consider an elliptic curve with varying parameters (moduli). Within the averaging method developed in [50, 51], the variations of wave solutions of the KdV equation requires the introduction of a family of Riemann surfaces varying gradually with X and T during the evolution. Choosing different curves, e.g. hyperelliptic curves of different genus, one constructs classes of exact solutions of the same integrable PDE [37, 49].

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The formulation of deformations of hyperelliptic curves considered in this paper looks, at first sight, similar to that from Whitham theory, particularly to that of Krichever’s universal Whitham hierarchy [57]. In fact, some objects like the action \( S(z) \) ((71), (157)) and closed one-forms ((70), (167)) are common in both approaches. However, the comparison of the Whitham deformations of hyperelliptic curves and those given in sections 5–8 as well as in [21–23, 25] clearly shows their difference. Let us consider the simplest case of an elliptic curve. Whitham deformations of the curve (179) are described by the system of equations (see e.g. [29])

\[
\frac{1}{4} \frac{\partial \beta_i}{\partial T} + v_i(\beta_1, \beta_2, \beta_3) \frac{\partial \beta_i}{\partial X} = 0, \quad i = 1, 2, 3, \tag{180}
\]

where \( X \) and \( T \) are slow variables and the characteristic velocities \( v_i \) are

\[
v_1 = \frac{1}{2}(\beta_1 + \beta_2 + \beta_3) + (\beta_1 - \beta_2) \frac{K(s)}{E(s)},
\]

\[
v_2 = \frac{1}{2}(\beta_1 + \beta_2 + \beta_3) + (\beta_2 - \beta_1) \frac{sK(s)}{E(s) - (1 - s)K(s)}, \tag{181}
\]

\[
v_3 = \frac{1}{2}(\beta_1 + \beta_2 + \beta_3) + (\beta_2 - \beta_3) \frac{K(s)}{E(s) - K(s)}.
\]

Here \( s = \frac{\beta_3 - \beta_1}{\beta_3 - \beta_2} \) and \( K(s) \) and \( E(s) \) are complete elliptic integrals of the first and second kind.

Rewritten in terms of the Riemann invariants \( \gamma_1, \gamma_2, \gamma_3 \) are of the form [21]

\[
\frac{\partial \gamma_i}{\partial x} = \frac{1}{2}(\gamma_1 + \gamma_2 + \gamma_3 + 2\gamma_i) \frac{\partial \gamma_i}{\partial x_j}, \quad i = 1, 2, 3. \tag{183}
\]
The systems (180), (181) and (183) are apparently quite different. Moreover, it is easy to see that there is no birational transformation \((p, \lambda) \rightarrow (q, \mu)\) between the elliptic curves (179) and (182) which simultaneously converts the system (183) into the system (180) and (181).

We would like to emphasize that in our approach, algebraic curves are completely defined by the structures of the Birkhoff strata of the Grassmannian \(\text{Gr}(2)^{1/2}\) and integrable systems of hydrodynamical type arise and are associated with the special subvarieties in Birkhoff strata. Possible interrelation between Whitham deformations and coisotropic deformations of hyperelliptic curves and other algebraic curves is an intriguing open problem. We hope to clarify it in future publications.

The cohomological structure of algebraic varieties and integrable equations in various settings have been discussed earlier, e.g. in [61–64]. Harrison cohomology for the families of the Veronese, elliptic and hyperelliptic curves studied in this paper seems to be quite different from those considered before. Possible interconnection between all these cohomological constructions will be considered elsewhere.

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