EXTENDED LETTERPLACE CORRESPONDENCE FOR NONGRADED NONCOMMUTATIVE IDEALS AND RELATED ALGORITHMS

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Abstract. Let $K\langle x_i \rangle$ be the free associative algebra generated by a finite or countable number of variables $x_i$. The notion of “letterplace correspondence” introduced in [18, 19] for the graded (two-sided) ideals of $K\langle x_i \rangle$ is extended in this paper also to the nongraded case. This amounts to the possibility of modelizing nongraded noncommutative presented algebras by means of a class of graded commutative algebras that are invariant under the action of the monoid $N$ of natural numbers. For such purpose we develop the notion of saturation for the graded ideals of $K\langle x_i, t \rangle$, where $t$ is an extra variable and for their letterplace analogues in the commutative polynomial algebra $K[x_{ij}, t_j]$, where $j$ ranges in $N$. In particular, one obtains an alternative algorithm for computing inhomogeneous noncommutative Gröbner bases using just homogeneous commutative polynomials. The feasibility of the proposed methods is shown by an experimental implementation developed in the computer algebra system Maple and by using standard routines for the Buchberger algorithm contained in Singular.

1. Introduction

Many structures and models in mathematics and physics are based on noncommutative associative algebras that are given by a presentation with a finite or countable number of generators. It is sufficient to mention the role of Hecke algebras or Temperley-Lieb ones in statistical mechanics and noncommutative geometry [8] [17], as well as the relevance of more classical enveloping algebras [10] or relatively free algebras defined for PI-algebras [12] [14]. A systematic way to control the consequences of the defining relations of a presented algebra consists in considering a well-ordering on the monomials of the free associative algebra (tensor algebra) that is compatible with multiplication and in computing what is modernly called a “Gröbner basis”. In fact, if it is possible to describe such a basis for the two-sided ideal of the relations satisfied by the generators of the associative algebra then a monomial linear basis is given for it that is one has some kind of generalization of the Poincaré-Birkhoff-Witt theorem.

Among the founding contributions to the theory of noncommutative Gröbner bases one has to mention [2] [15] [24] [27] [28] and of course [6] for the commutative case. Starting with the papers [18] [19], through a substantial development of the concept of letterplace embedding contained in [11], a new approach for the theory

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and computation of noncommutative Gröbner bases has been proposed. The basic idea is to define a bijective correspondence between all graded two-sided ideals of the free associative algebra and a class of multigraded invariant ideals of a commutative polynomial algebra in double-indexed (letter-place) variables where shift operators act over the place indices. Such bijection provides also a correspondence between the homogeneous Gröbner bases of these ideals. It follows that the notion of Gröbner basis in the commutative and noncommutative case and the related algorithms can be considered as special instances of a general theory of Gröbner bases for commutative ideals that are invariant under the action of suitable algebra endomorphisms \[5, 18, 19, 20\]. Since the endomorphisms acting on the letterplace algebra are just shift operators, note that these results contribute also to the theory of algebras of finite difference polynomials \[7, 21\].

The goal of the present paper is to complete the work initiated in \[18, 19\] by proposing an extension of the letterplace correspondence to the nongraded case. This is obtained by analyzing in detail the concept of saturation for nongraded ideals of the free associative algebra and for their letterplace analogues. Note that the homogenization and saturation processes for the noncommutative case were previously introduced in \[23, 24, 26\] (see also \[22\]). From the extended letterplace correspondence one obtains an alternative algorithm to compute inhomogeneous noncommutative Gröbner bases by using homogeneous polynomials in commutative variables. In fact, these methods can be easily implemented in any commutative computer algebra system. Then, one has that the theory and methods for commutative and noncommutative Gröbner bases are unified whenever they are homogeneous or not. The feasibility of the proposed algorithms is shown in practice by means of an experimental implementation and a test set consisting of relevant classes of noncommutative algebras.

In Section 2 we describe the bijective correspondence between all (two-sided) ideals of the free associative algebra \(F = K\langle X\rangle\) and the class of saturated graded ideals of the algebra \(\bar{F} = K\langle \bar{X}\rangle\), where \(\bar{X} = X \cup \{t\}\). If \(\mathbb{N} = \{n \in \mathbb{Z} \mid n \geq 0\}\) and \(\mathbb{N}^* = \mathbb{N} \setminus \{0\}\), for the letterplace algebras \(P = K[X \times \mathbb{N}^*]\) and \(\bar{P} = K[\bar{X} \times \mathbb{N}^*]\) we introduce the action of the monoid \((\mathbb{N}, +)\) on the place indices of the variables and also a multigrading based on such indices. Then, one obtains a bijection between all \(\mathbb{N}\)-invariant ideals of \(P\) and the class of saturated multigraded \(\mathbb{N}\)-ideals of \(\bar{P}\). In Section 3 we review some key results proved in \[18, 19\]. Precisely, the letterplace ideals of \(P\) are defined as \(\mathbb{N}\)-ideals generated by elements that are multilinear with respect to the place multigrading. Then, we introduce the letterplace correspondence as a bijection between all graded ideals of \(F\) and the class of letterplace ideals of \(P\). Note that under this correspondence a saturated ideal of \(\bar{F}\) does not map into a saturated ideal of \(\bar{P}\). It is necessary therefore to introduce the notion of \(L\)-saturation for letterplace ideals as a saturation property that involves only multilinear elements. By composing the above ideal correspondences, we finally obtain the extended letterplace correspondence which maps all ideals of \(F\) into the class of \(L\)-saturated letterplace ideals of \(\bar{P}\).

To develop effective methods for the \(L\)-saturation, in Section 4 we review the notion of monomial \(\mathbb{N}\)-ordering of \(P\) and the construction of an important class of such orderings that we call place \(\mathbb{N}\)-orderings. Then, we prove that they induce the graded right lexicographic ordering of the free associative algebra \(F\). We review finally the theory of Gröbner \(\mathbb{N}\)-bases for ideals of \(P\) that are invariant under shift
operators and the related letterplace algorithm that computes homogeneous non-commutative Gröbner bases by using just elements of the commutative algebra \( P \).

In Section 5 we solve the problem of computing \( L \)-saturations of letterplace ideals by using Gröbner \( L \)-bases that are Gröbner \( N \)-bases restricted to multilinear elements. The monomial orderings of \( P \) suitable for this task are place \( N \)-orderings which are of elimination for the extra variables \( t(j) \). As a byproduct one obtains finally a letterplace algorithm for computing inhomogeneous noncommutative Gröbner bases using homogeneous polynomials of \( \tilde{P} \). This method is illustrated in a detailed simple example in Section 6 and it is experimented in Section 7 for classes of presented associative algebras that are of interest in different areas of algebra. The experiments are performed by means of an implementation developed in the language of Maple and also by using standard routines for the Buchberger algorithm that are implemented in Singular \([9]\). Conclusions about the letterplace approach to non-commutative computations and further developments of it are finally discussed in Section 8.

2. Homogenized and saturated ideals

We start studying the notion of homogenization and saturation for ideals of the free associative algebra. These concepts have been introduced essentially in \([23, 24, 29]\) but we intend to clarify why commutators naturally arise in such constructions.

Denote by \( \bar{F} = K\langle X \rangle \) the free associative algebra freely generated by a finite or countable set \( X = \{x_1, x_2, \ldots \} \). Clearly, one has the algebra grading \( F = \bigoplus_{d \in \mathbb{N}} F_d \)
where \( F_d \) is the subspace of homogeneous polynomials of total degree \( d \). Let \( t \) be a new variable disjoint by \( X \). Define \( \bar{X} = X \cup \{t\}, \bar{F} = K\langle \bar{X} \rangle \). Consider the algebra endomorphism \( \varphi : \bar{F} \to \bar{F} \) such that \( x_i \mapsto x_i \) and \( t \mapsto 1 \) for all \( i \geq 1 \). Clearly \( \varphi^2 = \varphi \) and \( F = \varphi(F) \). Then, the map \( \varphi \) defines a bijective correspondence between all two-sided ideals of \( F \) and two-sided ideals of \( \bar{F} \) containing \( \ker \varphi = \langle t-1 \rangle \). In what follows, all the ideals of the algebras \( F, \bar{F} \) are assumed two-sided ones.

**Definition 2.1.** Denote by \( C \) the largest graded ideal contained in \( \ker \varphi \) that is the ideal generated by all homogeneous elements \( f \in \bar{F} \) such that \( \varphi(f) = 0 \).

**Proposition 2.2.** The ideal \( C \subset \bar{F} \) is generated by the commutators \( [x_i, t] = x_i t - t x_i \), for any \( i \geq 1 \).

**Proof.** Let \( f \in \bar{F} \) be a homogeneous element such that \( \varphi(f) = 0 \). Since the commutators \( [x_i, t] \) clearly belongs to \( C \), we have to prove that \( f \) is congruent to 0 modulo them. In fact, it is clear that \( f \) is congruent to a homogeneous element \( f' = t^{d'} \sum f_k t^{d'-k} \) where \( d' \geq 0 \) and \( f_k \in F \) is homogeneous of degree \( k \), for any \( k \). Then \( 0 = \varphi(f) = \varphi(f') = \sum f_k \) and hence \( f_k = 0 \) for all \( k \). We conclude that \( f' = 0 \). \( \square \)

We want now to define a bijective correspondence between all ideals of \( F \) and some class of graded ideals of \( \bar{F} \) containing \( C \).

**Definition 2.3.** Let \( I \) be any ideal of \( F \). We define \( I^* \subset \bar{F} \) the largest graded ideal contained in the preimage \( \varphi^{-1}(I) \) that is \( I^* \) is the ideal generated by all homogeneous elements in \( \varphi^{-1}(I) \). Clearly \( C = 0^* \subset I^* \). We call \( I^* \) the homogenization of the ideal \( I \).

**Definition 2.4.** Let \( f \in F, f \neq 0 \) and denote \( f = \sum f_k \) the decomposition of \( f \) in its homogeneous components. We denote \( \deg(f) = d = \max\{k\} \) and define \( f^* = \sum \)
\[ \sum_k f_k t^{d-k}. \] We call \( \deg(f) \) the top degree of \( f \) and \( f^* \) its homogenization. Clearly \( f^* \in \bar{F} \) is a homogeneous element such that \( \deg(f^*) = \deg(f) \) and \( \varphi(f^*) = f \).

**Proposition 2.5.** Let \( I \) be an ideal of \( F \). Then \( I^* = \langle f^* \mid f \in I, f \neq 0 \rangle + C \).

**Proof.** Denote \( J = \langle f^* \mid f \in I, f \neq 0 \rangle + C \). Clearly \( J \) is a graded ideal of \( \bar{F} \) such that \( \varphi(J) \subset I \) and hence \( J \subset I^* \). Let \( g \in I^* \) be a homogeneous element and define \( f = \varphi(g) \in I \). If \( f = 0 \) then \( g \in C \subset J \). Otherwise, denote \( d = \deg(f) \) and \( d' = \deg(g) \). Since clearly \( d' \geq d \) one has that \( g \) is congruent modulo \( C \) to the element \( t^{d-d'}f^* \) and hence \( g \in J \). \( \square \)

If \( I \subset F \) is an ideal one has clearly that \( \varphi(I^*) = I \). Moreover, if \( J \subset \bar{F} \) is a graded ideal containing \( C \) then in general \( J \subset \varphi(J)^* \).

**Definition 2.6.** Let \( C \subset J \subset \bar{F} \) be a graded ideal. Define \( \text{Sat}(J) = \varphi(J)^* = \langle \varphi(f)^* \mid f \in J, f \notin C, f \text{ homogeneous} \rangle + C \). Then \( J \subset \text{Sat}(J) \subset \bar{F} \) is a graded ideal that we call the saturation of \( J \).

**Definition 2.7.** Let \( J \subset \bar{F} \) be a graded ideal containing \( C \). We say that \( J \) is saturated if \( J \) coincides with its saturation \( \text{Sat}(J) \) that is for any homogeneous element \( f \in J, f \notin C \) one has that \( \varphi(f)^* \in J \). If \( I \) is an ideal of \( F \) then its homogenization \( I^* \) is clearly a saturated ideal.

Note that in \([23]\) an equivalent definition of saturated ideal is named \( \text{dh-closed} \). Then, a bijective correspondence is given between all ideals of \( F \) and the saturated graded ideals of \( \bar{F} \) containing \( C \). One can characterize such ideals in the following way.

**Proposition 2.8.** Let \( C \subset J \subset \bar{F} \) be a graded ideal. Then \( J \) is saturated if and only if \( tf \in J \) with \( f \in \bar{F} \) implies that \( f \in J \).

**Proof.** Suppose that \( J \) is saturated and let \( tg \in J \) with \( g \in \bar{F} \). Since \( J \) is graded, we can assume that \( g \) is homogeneous. Put \( f = \varphi(g) = \varphi(tg) \). If \( f = 0 \) then \( g \in C \subset J \). Otherwise, since \( J \) is saturated and \( tg \in J \) we obtain that \( f^* \in J \). Moreover, one has clearly that \( g \) is congruent modulo \( C \) to an element \( t^df^* \in J \) for some \( d \geq 0 \) and hence \( g \in J \). Suppose now that \( tg \in J \) implies \( g \in J \) and let \( g \in J, g \notin C \) be a homogeneous element. If \( f = \varphi(g) \) then \( g \) is congruent modulo \( C \subset J \) to an element \( t^df^* \). We conclude that \( t^df^* \in J \) and therefore \( f^* \in J \). \( \square \)

**Proposition 2.9.** Let \( J \subset \bar{F} \) be a graded ideal containing \( C \). Then, we have \( \text{Sat}(J) = \{ f \in F \mid t^i f \in J, \text{ for some } i \geq 0 \} \).

**Proof.** Denote \( J' = \{ f \mid t^if \in J, \text{ for some } i \} \). Let \( g \in \bar{F} \) and \( f \in J' \) that is \( t^if \in J \), for some \( i \). Clearly \( gt^if \in J \) and also \( t^igf \in J \) since \( C \subset J \). We conclude that \( gf \in J' \). With similar arguments one proves that \( J' \) is a graded ideal of \( \bar{F} \) containing \( J \). Moreover, by Proposition 2.8 it follows immediately that \( J' \) is a saturated ideal. Finally, we have clearly that \( \varphi(J') = \varphi(J) \) and hence \( J' = \varphi(J')^* = \varphi(J)^* = \text{Sat}(J) \). \( \square \)

We start now considering commutative polynomial algebras with the purpose of defining analogues of the above noncommutative constructions. Denote \( \mathbb{N}^* = \mathbb{N} \setminus \{0\} \) and consider the product set \( X(\mathbb{N}^*) = X \times \mathbb{N}^* \). For the elements of this set we make use of the notation \( x_i(j) = (x_i, j) \), for all \( i, j \geq 1 \). Define \( P = K[X(\mathbb{N}^*)] \) the polynomial algebra in all commuting variables \( x_i(j) \). The algebra \( P \) is called the
letterplace algebra \[\text{[14]}\]. It is clear that the monoid \(\mathbb{N}\) acts (faithfully) by algebra monomorphisms on \(P\) by putting \(k\cdot x_i(j) = x_i(k \cdot j)\), for all \(i, j, k\). Precisely, one has that \(P\) is a free commutative \(\mathbb{N}\)-algebra generated by the set \(X(1) = \{x_i(1) \mid i \geq 1\}\).

An ideal \(I \subset P\) is said a \(\mathbb{N}\)-invariant ideal or a \(\mathbb{N}\)-ideal if \(\mathbb{N} \cdot I \subset I\). Clearly, we have the algebra grading \(P = \bigoplus_{d \in \mathbb{N}} P_d\) where \(P_d\) is the subspace of homogeneous polynomials of total degree \(d\). The algebra \(P\) has another natural multigrading defined as follows. If \(m = x_i(1) \cdots x_d(j_d) \in \text{Mon}(P)\) then we denote \(\partial(m) = \mu = (\mu_k)_{k \in \mathbb{N}}\), where \(\mu_k = \#\{\alpha \mid j_\alpha = k\}\). If \(P_d \subset P\) is the subspace spanned by all monomials of multidegree \(\mu\) then \(P = \bigoplus_{\mu} P_\mu\) is clearly a multigrading. Note that the multidegrees \(\mu = (\mu_k)_{k \in \mathbb{N}}\) have finite support and one can define \(|\mu| = \sum_k \mu_k\). Then, one has clearly \(P_d = \bigoplus_{|\mu|=d} P_\mu\) that is multihomogeneous elements are also homogeneous ones. Note that the multigrading is compatible with the \(\mathbb{N}\)-algebra structure on \(P\). Precisely, if \(\mu = (\mu_k)\) is a multidegree then we denote \(i \cdot \mu = (\mu_{k-i})_{k \in \mathbb{N}}\), where we put \(\mu_{k-i} = 0\) when \(k-i < 0\). Then, for all \(i \geq 0\) and for any multidegree \(\mu\) one has that \(i \cdot P_\mu \subset P_i \cdot \mu\).

Define \(\bar{P} = K[X(\mathbb{N}^*)]\) and consider the \(\mathbb{N}\)-algebra endomorphism \(\psi: \bar{P} \rightarrow \bar{P}\) such that \(x_i(1) \mapsto x_i(1)\) and \(t(1) \mapsto 1\) for all \(i \geq 1\). Clearly, the map \(\psi\) is idempotent and \(P = \psi(\bar{P})\). Moreover, one has that the \(\mathbb{N}\)-ideal \(\ker(\psi) = (t(1) - 1)\bar{P}\) does not contain any multihomogeneous element different from zero. We define now a bijective correspondence between all \(\mathbb{N}\)-ideals of \(P\) and some class of multigraded \(\mathbb{N}\)-ideals of \(\bar{P}\).

**Definition 2.10.** Let \(I\) be any \(\mathbb{N}\)-ideal of \(P\). We define \(I^* \subset \bar{P}\) the largest multigraded \(\mathbb{N}\)-ideal contained in the preimage \(\psi^{-1}(I)\) that is \(I^*\) is the ideal generated by all multihomogeneous elements in \(\psi^{-1}(I)\). We call \(I^*\) the multihomogenization of the ideal \(I\). Note that \(0^* = 0\).

**Definition 2.11.** Let \(f \in P, f \neq 0\) and denote \(f = \sum_{\mu} f_{\mu}\) the decomposition of \(f\) in its multihomogeneous components. We denote \(\partial(f) = \nu = (\max_{\mu} \{\mu_k\})_{k \in \mathbb{N}}\) and define \(f^* = \sum_{\mu} f_{\mu} \prod_{k} t(k)^{\nu_{k} - \mu_{k}}\). We call \(\partial(f)\) the top multidegree of \(f\) and \(f^*\) its multihomogenization. Clearly \(f^* \in \bar{P}\) is a multihomogeneous element such that \(\partial(f^*) = \partial(f)\) and \(\psi(f^*) = f\). Moreover, one has that \((i \cdot f)^* = i \cdot f^*\), for all \(i \geq 0\).

**Proposition 2.12.** Let \(I\) be a \(\mathbb{N}\)-ideal of \(P\). Then \(I^* = \langle f^* \mid f \in I, f \neq 0 \rangle\).

**Proof.** Denote \(J = \langle f^* \mid f \in I, f \neq 0 \rangle\). Clearly \(J\) is a multigraded \(\mathbb{N}\)-ideal of \(\bar{P}\) such that \(\psi(J) \subset I\) and hence \(J \subset I^*\). Let \(g \in I^*\) be a multihomogeneous element and define \(f = \psi(g) \in I\). Denote \(\mu = \partial(f)\) and \(\nu = \partial(g)\). Since clearly \(\nu_k \geq \mu_k\) for all \(k\), one has that \(g = \prod_k t(k)^{\nu_k - \mu_k} f^*\) and hence \(g \in J\).

If \(I \subset P\) is a \(\mathbb{N}\)-ideal one has clearly that \(\psi(I^*) = I\). Moreover, if \(J \subset \bar{P}\) is a multigraded \(\mathbb{N}\)-ideal then in general \(J \subset \psi(J)^*\).

**Definition 2.13.** Let \(J \subset \bar{P}\) be a multigraded \(\mathbb{N}\)-ideal. Define \(\text{Sat}(J) = \psi(J)^* = \langle \psi(f)^* \mid f \in J, f \text{ multihomogeneous} \rangle\). Then \(J \subset \text{Sat}(J) \subset \bar{P}\) is a multigraded \(\mathbb{N}\)-ideal that we call the saturation of \(J\).

**Definition 2.14.** Let \(J \subset \bar{P}\) be a multigraded \(\mathbb{N}\)-ideal. We say that \(J\) is saturated if \(J\) coincides with its saturation \(\text{Sat}(J)\) that is if \(f \in J\) is a multihomogeneous element then \(\psi(f)^* \in J\). If \(I\) is a \(\mathbb{N}\)-ideal of \(P\) then its multihomogenization \(I^*\) is clearly a saturated ideal.
Then, a bijective correspondence is given between all \(\mathbb{N}\)-ideals of \(P\) and the saturated multigraded \(\mathbb{N}\)-ideals of \(\bar{P}\). One can characterize such ideals in the following way.

**Proposition 2.15.** Let \(J \subset \bar{P}\) be a multigraded \(\mathbb{N}\)-ideal. Then \(J\) is saturated if and only if \(t(j)f \in J\) with \(f \in \bar{P}\), \(j \geq 1\) implies that \(f \in J\).

*Proof.* Suppose that \(J\) is saturated and let \(t(j)g \in J\) with \(g \in \bar{F}, \ j \geq 1\). Since \(J\) is multigraded, we can assume that \(g\) is multihomogeneous. Put \(f = \psi(g) = \psi(t(j)g)\).

Since \(J\) is saturated and \(t(j)g \in J\) we obtain that \(f^* \in J\). Moreover, one has that \(g = \prod_k t(k)^{\mu_k}f^* \in J\) for some multidegree \(\mu\) and hence \(g \in J\). Suppose now that \(t(j)g \in J\) implies \(g \in J\) and let \(g \in J\) be a multihomogeneous element. If \(f = \psi(g)\) then clearly \(\prod_k t(k)^{\mu_k}f^* = g \in J\) for some \(\mu\). We conclude that \(f^* \in J\). \(\Box\)

**Proposition 2.16.** Let \(J \subset \bar{P}\) be a multigraded \(\mathbb{N}\)-ideal. Then, we have that \(\text{Sat}(J) = \{f \in \bar{P} \mid \prod_k t(k)^{\mu_k}f \in J, \text{ for some multidegree } \mu\}\).

*Proof.* Put \(J' = \{f \mid \prod_k t(k)^{\mu_k}f \in J, \text{ for some } \mu\}\). Let \(i \geq 0\) and \(f \in J'\) that is \(m \cdot f \in J\), for some \(m = \prod_k t(k)^{\mu_k}\). Since \(J\) is a \(\mathbb{N}\)-ideal, we have that \((i \cdot m)(i \cdot f) = i \cdot (m \cdot f) \in J\) where \((i \cdot m) = \prod_k t(i + k)^{\mu_k}\). We conclude that \(i \cdot f \in J'\). With similar arguments one proves that \(J'\) is a multigraded \(\mathbb{N}\)-ideal of \(\bar{P}\) containing \(J\).

By Proposition 2.15 we obtain also that \(J'\) is a saturated ideal. Finally, we have clearly that \(\psi(J') = \psi(J)\) and hence \(J' = \psi(J)^* = \psi(J)^* = \text{Sat}(J)\). \(\Box\)

3. Letterplace Correspondence and \(L\)-Saturation

Consider the \(K\)-linear embedding \(\iota : F \to P\) such that \(\iota(m) = x_{i_1}(1) \cdots x_{i_d}(d)\) for all monomials \(m = x_{i_1} \cdots x_{i_d} \in \text{Mon}(F)\). This mapping was introduced in [11].

Note that the map \(\iota\) preserves the total degree. Then, define \(V = \bigoplus_d V_d\) the graded subspace of \(P\) which is the image of map \(\iota\). For all \(d \geq 0\), denote \(1^d\) the multidegree \(\mu = (\mu_k)\) such that \(\mu_k = 1\) for \(k \leq d\) and \(\mu_k = 0\) otherwise.

Clearly one has that \(V_d = P_{1^d}\).

**Definition 3.1.** Denote \(L = \bigcup_d V_d\) the set of multihomogeneous elements of \(V\).

We call such elements the letterlinear elements of \(P\).

There is a bijective correspondence between all graded ideals of \(F\) and some class of multigraded \(\mathbb{N}\)-ideals of \(P\). This class is defined as follows.

**Definition 3.2.** Let \(J\) be a \(\mathbb{N}\)-ideal of \(P\). We call \(J\) a letterplace ideal or \(L\)-ideal or multilinear \(\mathbb{N}\)-ideal if \(J = \langle J \cap L \rangle_\mathbb{N}\) that is \(J\) is \(\mathbb{N}\)-generated by multilinear elements.

Clearly \(J\) is a multigraded ideal.

The following key result has been proved in [18].

**Proposition 3.3.** Let \(I \subset F\) be a graded ideal and denote \(J = \langle \iota(I) \rangle_\mathbb{N}\). Then \(J \subset P\) is a \(L\)-ideal. Conversely, let \(J \subset P\) be a \(L\)-ideal and denote \(I = \iota^{-1}(J \cap V)\). Then \(I \subset F\) is a graded ideal. Moreover, the mappings \(I \mapsto J\) and \(J \mapsto I\) define a bijective correspondence between graded ideals of \(F\) and letterplace ideals of \(P\).

Hence, we call \(J\) the letterplace analogue of \(I\).

We assume now that the above result is extended to the algebras \(\bar{F}, \bar{P}\). Then, we make use of notations \(\bar{\iota} : \bar{F} \to \bar{P}, \bar{V} = \text{Im} \bar{\iota} \text{ and } \bar{L} = \bigcup_d \bar{V}_d\). Consider the letterplace analogue \(D\) of the ideal \(C = 0^*\). In other words, we have that \(D \subset \bar{P}\)
the \( \mathbb{N} \)-ideal generated by the multilinear elements \( i([x_i, t]) = x_i(1)t(2) - t(1)x_i(2) \), for all \( i \geq 1 \). Note that \( D \) is not a saturated ideal. In fact, the ideal \( D \) contains the element \( t(1)f \), but not \( f = x_1(1)x_2(2) - x_2(1)x_1(2) \). Moreover, its saturation \( \text{Sat}(D) \) is not a \( L \)-ideal that is this ideal is not generated by multilinear elements. For instance, the element \( x_1(1)t(3) - t(1)x_1(3) \notin L \) is contained in \( \text{Sat}(D) \). More generally, the letterplace analogue of a saturated ideal of \( \mathcal{F} \) is not saturated and its saturation is not a letterplace ideal. This suggests that one needs a different notion of saturation for such analogues that are in bijective correspondence with all ideals of \( \mathcal{F} \). To motivate the following definition, note also that if \( f \) and \( t(j)f \) are multilinear elements then necessarily \( j = \deg(f) + 1 \).

**Definition 3.4.** Let \( D \subset J \subset \hat{P} \) be a \( L \)-ideal. We say that \( J \) is \( L \)-saturated or multilinearly saturated if \( t(d+1)f \in J \) with \( f \in L \) and \( d = \deg(f) \) implies that \( f \in J \).

**Proposition 3.5.** Let \( D \subset J \subset \hat{P} \) be a \( L \)-ideal. If we denote \( \text{Sat}_L(J) = \langle f \in L \mid \prod_{d < k \leq d'} t(k)f \in J, \text{ for some } d' \geq d = \deg(f) \rangle_{\mathbb{N}} \) then \( \text{Sat}_L(J) \) is a \( L \)-saturated letterplace ideal containing \( J \). We call \( \text{Sat}_L(J) \) the \( L \)-saturation or multilinear saturation of \( J \) and one has clearly that \( \text{Sat}_L(J) \subset \text{Sat}(J) \).

**Proof.** By definition, one has that \( J' = \text{Sat}_L(J) \) is a \( L \)-ideal that contains \( J \supset D \). Denote \( m_l = \prod_{0 \leq k \leq l} t(k) \) and suppose \( g(d \cdot m_l) \in J' \) with \( g \in \hat{V}_d \) and \( d \geq 0 \). It remains to prove that \( g \in J' \) that is \( J' \) is \( L \)-saturated. By definition of \( J' \) we have that \( g(d \cdot m_l) = \sum_i f_i(d_i \cdot g_i) \) with \( f_i \in \hat{V}_{d_i}, g_i \in \hat{V}_{d-d_i+1} \) and \( f_i(d_i \cdot m_{l_i}) \in J \), for some \( l_i \geq 0 \). If \( l = \max\{l_i\} \) then the element \( g(d \cdot m_{l+1}) = g(d \cdot m_l)((d+1) \cdot m_l) \) is congruent modulo \( D \subset J \) to \( \sum_i f_i(d_i \cdot m_{l+1})((d_i+1) \cdot g_i) \in J \) and therefore \( g \in J' \). \( \square \)

**Proposition 3.6.** Let \( C \subset I \subset \mathcal{F} \) be a saturated ideal and denote \( D \subset J \subset \hat{P} \) the letterplace analogue of \( I \). Then \( J \) is a \( L \)-saturated ideal.

**Proof.** Assume \( gt(d+1) \in J \) with \( g \in \hat{V}_d \), for some \( d \geq 0 \). Then, let \( f \in \hat{V}_d \) such that \( i(f) = g \). We have that \( i(ft) = gt(d+1) \in J \cap \hat{V} \) that is \( ft \in I \) and therefore \( f \in I \) since \( C \subset I \) is a saturated ideal. We conclude that \( g \in J \). \( \square \)

**Proposition 3.7.** Let \( D \subset J \subset \hat{P} \) be a \( L \)-saturated letterplace ideal and put \( I = i^{-1}(J \cap \hat{V}) \). Then \( C \subset I \subset \mathcal{F} \) is a saturated ideal.

**Proof.** It is sufficient to reverse the argument of Proposition 3.6. \( \square \)

We obtain therefore a bijective correspondence between all ideals of \( \mathcal{F} \) and the class of \( L \)-saturated letterplace ideals of \( \hat{P} \). We call this bijection the extended letterplace correspondence.

**Definition 3.8.** Let \( I \) be any ideal of \( \mathcal{F} \) and denote \( D \subset J \subset \hat{P} \) the letterplace analogue of \( C \subset \text{Sat}_L(J) \). We call \( J \) the extended letterplace analogue of \( I \). Clearly, one has \( J = \langle i(f^*) \mid f \in I, f \neq 0 \rangle_{\mathbb{N}} + D \) and \( I = \varphi^{-1}(J \cap \hat{V}) \).

With the notations of the above definition, by Proposition 3.6 we have that \( J = \text{Sat}_L(J) \). Then, it is natural to ask what is the ideal \( \text{Sat}(J) \) extending \( J \).

Denote \( Q = K[X(1)] \) the polynomial algebra in the variables \( x_i(1) \) and consider the natural algebra epimorphism \( \eta : F \to Q \) such that \( x_i \mapsto x_i(1) \), for all \( i \geq 1 \). Assume that \( \mathbb{N} \) acts on \( Q \) in the trivial way that is \( j \cdot x_i(1) = x_i(j) \), for any \( j \geq 0 \). Then, one has the \( \mathbb{N} \)-algebra epimorphism \( \theta : P \to Q \) such that \( x_i(j) \mapsto x_i(1) \),
Proposition 3.9. Let $I$ be any ideal of $F$ and put $I' = \theta^{-1}\eta(I)$. Clearly $I' \subset P$ is a $\mathbb{N}$-ideal containing $E$. Denote by $D \subset J \subset \bar{P}$ the extended letterplace analogue of $I$. Then, one has that $\text{Sat}(J) = I''$.

Proof. Since $J$ is a multigraded $\mathbb{N}$-ideal of $\bar{P}$, it is sufficient to show that $\psi(J) = I'$. Consider any element $g' \in I'$. Clearly $g'$ is congruent modulo $E = \ker \theta$ to an element $\eta(f) \in Q \subset P$, for some $f \in I$. If $\eta(f) = 0$ then $g' \in E = \psi(D)$ where $D \subset J$. Otherwise, we have $f \neq 0$ and one can consider $f^* \in I^*$ and hence $g = \bar{i}(f^*) \in J$. It is clear that $\theta \psi(g) = \eta(f)$ that is $\psi(g)$ is congruent modulo $E$ to the element $\eta(f)$. Then, $\psi(g)$ is congruent also to $g'$ that is $g' = \psi(g) + h$ with $h \in E$. Since $E = \psi(D)$ and $D \subset J$, we conclude that $g' \in \psi(J)$. With similar arguments one proves also $\psi(J) \subset I'$.

Assume now one wants to compute the extended letterplace analogue $D \subset J \subset \bar{P}$ of any ideal $I \subset F$. If $I$ is given by a generating set $G$ we may form the graded ideal $I' = C + \langle f^* | f \in G \rangle \subset \bar{F}$ and then its letterplace analogue $D \subset J' \subset P$. One has clearly that $\text{Sat}(I') = I^*$ and $\text{Sat}_K(J') = J$. It is well known that for the commutative case [3, 13] a standard tool to compute saturation consists in performing Gröbner bases with respect to appropriate monomial orderings. Aiming to have a similar method for $L$-saturation, in the next section we review the Gröbner bases theory for letterplace ideals that has been introduced in [18, 19].

4. Gröbner $\mathbb{N}$-bases of letterplace ideals

Since letterplace ideals are a special class of $\mathbb{N}$-ideals, a first step consists in introducing monomial orderings for the polynomial algebra $P$ that are compatible with the action of $\mathbb{N}$. Owing to the Higman’s Lemma, one can provide $P = K[X(\mathbb{N}^*)]$ by monomial orderings even if the set $X(\mathbb{N}^*)$ is infinite. For that purpose, this lemma can be stated in the following way (see for instance [1], Corollary 2.3).

Proposition 4.1. Let $\prec$ be a total ordering on $M = \text{Mon}(P)$ such that

(i) $1 \leq m$ for all $m \in M$;

(ii) $\prec$ is compatible with multiplication on $M$, that is if $m \prec n$ then $tm \prec tn$, for any $m, n, t \in M$.

Then $\prec$ is also a well-ordering that is a monomial ordering of $P$ if and only if the restriction of $\prec$ to the variables set $X(\mathbb{N}^*)$ is a well-ordering.

We can easily assign well-orderings to the countable set $X(\mathbb{N}^*)$ which is in bijective correspondence to $\mathbb{N}^2$. Note that the monoid $\mathbb{N}$ stabilizes the variables set $X(\mathbb{N}^*)$ and hence the monomials set $M$. We have then the following notion.

Definition 4.2. Let $\prec$ be a monomial ordering of $P$. We call $\prec$ a (monomial) $\mathbb{N}$-ordering of $P$ if $m \prec n$ implies that $i \cdot m \prec i \cdot n$, for all $m, n \in M$ and $i \geq 0$.

One defines a main class of $\mathbb{N}$-orderings of $P$ in the following way. Denote $P(j) = K[x_i(j) | i \geq 1]$ and put $M(j) = \text{Mon}(P(j))$. Clearly $P = \bigotimes_{j \geq 1} P(j)$ that is all monomials $m \in M$ can be factorized as $m = m(j_1) \cdots m(j_k)$, where $m(j_s) \in M(j_s)$ and $j_1 > \cdots > j_k$. Let now $\rho : \mathbb{N} \to \text{End}_K(P)$ be the faithful monoid representation corresponding to the action of $\mathbb{N}$ over $P$. For any $j \geq 0$, one
has that the map \( \rho(j) \) defines an isomorphism between the monoids \( M(1), M(j+1) \) and hence between the algebras \( P(1), P(j+1) \).

**Definition 4.3.** Let \( \prec \) be any monomial ordering of the subalgebra \( P(1) \subset P \) and extend it to all subalgebras \( P(j+1) \) \((j \geq 0)\) by the isomorphisms \( \rho(j) \). In other words, we put \( j \cdot m \prec j \cdot n \) if and only if \( m \prec n \), for any \( m, n \in M(1) \). Then, for all \( m, n \in M, m = m(j_1) \cdots m(j_k), n = n(j_1) \cdots n(j_k) \) with \( j_1 > \ldots > j_k \) we define \( m \prec' n \) if and only if \( m(j_s) = n(j_s) \) and \( m(j_t) \prec n(j_t) \), for some \( 1 \leq t \leq k \) and for all \( 1 \leq s < t \). By Proposition 3.7 in [20] one has that \( \prec' \) is a monomial \( \mathbb{N} \)-ordering that we call place \( \mathbb{N} \)-ordering of \( P \) induced by a monomial ordering of \( P(1) \).

Note that if \( X \) is finite then \( P(1) \) is a polynomial algebra in a finite number of variables whose monomial orderings were classified in [26]. If \( X \) is infinite, the algebra \( P(1) \) can be endowed with monomial orderings as in Proposition 4.1 provided that \( x_1(1) \prec x_2(1) \prec \ldots \).

An important feature of the place \( \mathbb{N} \)-orderings is that they are compatible with some special grading of \( P \) which is in turn compatible with the action of \( \mathbb{N} \). Denote \( \hat{\mathbb{N}} = \{ -\infty \} \cup \mathbb{N} \).

**Definition 4.4.** Let \( w : M \to \hat{\mathbb{N}} \) be the unique mapping such that

(i) \( w(1) = -\infty \);
(ii) \( w(mn) = \max(w(m), w(n)) \), for any \( m, n \in M \);
(iii) \( w(x_i(j)) = j \), for all \( i, j \geq 1 \).

We call \( w \) the weight function of \( P \). If \( P(i_1) \subset P \) is the subspace spanned by all monomials of weight \( i \) then \( P = \bigoplus_{i \in \hat{\mathbb{N}}} P(i) \) is grading of \( P \) over the idempotent commutative monoid \( (\hat{\mathbb{N}}, \max) \). Clearly, one has that \( i \cdot P(j) \subset P(i+j) \), for all \( i, j \).

**Definition 4.5.** Let \( \prec \) be a monomial \( \mathbb{N} \)-ordering of \( P \). We say that \( \prec \) is a weighted ordering if \( w(m) < w(n) \) implies that \( m \prec n \), for all \( m, n \in M \).

By Proposition 5.11 in [20] one has that all place \( \mathbb{N} \)-orderings are weighted ones. Note also that for multilinear monomials \( m \in M \cap L \) one has that \( w(m) = \deg(m) \).

**Definition 4.6.** Let \( \prec \) be a well-ordering of \( W = \text{Mon}(F) \). We call \( \prec \) a monomial ordering of \( F \) if \( m \prec n \) implies that \( unv \prec unv \), for all \( m, n, u, v \in W \). In particular, we say that \( \prec \) is a graded ordering if \( \deg(m) < \deg(n) \) implies that \( m \prec n \), for any \( m, n \in W \).

**Proposition 4.7.** Let \( \prec \) be a weighted \( \mathbb{N} \)-ordering of \( P \) and define a total ordering \( \prec' \) of \( W \) by putting \( m \prec' n \) if and only if \( \iota(m) \prec \iota(n) \), for all \( m, n \in W \). Then, the ordering \( \prec' \) is a graded monomial ordering of \( F \) that we call induced by \( \prec \).

**Proof.** It is clear that \( \prec' \) is a well-ordering since the same holds for the restriction of \( \prec \) to \( M \cap L \). Let \( m', n', u', v' \in W \) and denote by \( m, n, u, v \in M \cap L \) their images under the map \( \iota \). If \( \deg(m') < \deg(n') \) then \( w(m) < w(n) \) and hence \( m \prec n \) that is \( m' \prec' n' \). Assume now \( m' \prec' n' \). If \( \deg(m') < \deg(n') \) we have that \( \deg(u'm'n'v') < \deg(u'n'v') \) and hence \( u'm'n'v' \prec' u'n'v' \). If \( d' = \deg(m') = \deg(n') \) and \( d = \deg(u') \) one obtains that \( d \cdot m \prec d \cdot n \) since \( \prec \) is a \( \mathbb{N} \)-ordering. We conclude that \( \iota(u'm'n'v') = u(d \cdot m)((d + d') \cdot v) \prec u(d \cdot n)((d + d') \cdot v) = \iota(u'n'v') \) that is \( u'm'n'v' \prec' u'n'v' \). \( \square \)

The above result implies that a class of graded monomial orderings of \( F = K\langle X \rangle \) can be obtained from the class of weighted \( \mathbb{N} \)-orderings of \( P \) by restriction to \( L \). In particular, one has the following result.
Proposition 4.8. Let \( \prec \) be any monomial ordering of \( P(1) \) and extend it to a place \( \mathbb{N} \)-ordering of \( P \). Moreover, denote by \( \prec' \) the graded monomial ordering of \( F \) induced by \( \prec \). Then \( \prec' \) is the graded right lexicographic order that is for any \( m = x_{i_1} \cdots x_{i_k}, n = x_{j_1} \cdots x_{j_k} \in W \) one has \( m \prec' n \) if and only if \( k < l \) or \( k = l, i_s = j_s \) and \( i_t < j_t \), for some \( 1 \leq t \leq k \) and for all \( t < s \leq k \).

Proof. Note that if \( X \) is an infinite set then necessarily \( x_1(i) \prec x_2(i) \prec \ldots \) and \( x_1 \prec' x_2 \prec' \ldots \) because \( \prec, \prec' \) are well-orderings. Then, one has that \( \iota(m) = x_{i_k}(k) \cdots x_{i_1}(1), \iota(n) = x_{j_k}(k) \cdots x_{j_1}(1) \) and \( \iota(m) \prec \iota(n) \) if and only if \( x_{j_s}(s) = x_{j_t}(s) \) and \( x_{i_t}(t) \prec x_{j_t}(t) \) that is \( i_s = j_s \) and \( i_t < j_t \), for some \( 1 \leq t \leq k \) and for all \( t < s \leq k \). \( \square \)

We start now introducing Gröbner bases in the context of \( \mathbb{N} \)-ideals. Fix \( \prec \) any \( \mathbb{N} \)-ordering of \( P \). Let \( f = \sum_i c_i m_i \in P \) with \( m_i \in M, c_i \in K, c_i \neq 0 \). We denote \( \text{lm}(f) = m_k = \max_\prec \{m_i\}, \text{lc}(f) = c_k \) and \( \text{lt}(f) = \text{lc}(f) \text{lm}(f) \). Let \( f, g \in P, f, g \neq 0 \) and put \( \text{lt}(f) = cm, \text{lt}(g) = dn \) with \( m, n \in M \) and \( c, d \in K \). If \( l = \text{lc}(m, n) \) we define as usual the \( S \)-polynomial \( \text{spoly}(f, g) = (l/cm)f - (l/dn)g \). Finally, if \( G \subset P \) we put \( \text{lm}(G) = \{\text{lm}(f) \mid f \in G, f \neq 0\} \) and we define \( \text{LM}(G) \) the ideal of \( P \) generated by \( \text{lm}(G) \). The following results were proved in [18, 19].

Proposition 4.9. Let \( G \subset P \). Then \( \text{lm}(N \cdot G) = N \cdot \text{lm}(G) \). In particular, if \( I \) is a \( \mathbb{N} \)-ideal of \( P \) then \( \text{LM}(I) \) is also \( \mathbb{N} \)-ideal.

Definition 4.10. Let \( I \subset P \) be a \( \mathbb{N} \)-ideal and \( G \subset I \). We call \( G \) a Gröbner \( \mathbb{N} \)-basis of \( I \) if \( \text{lm}(G) \) is a \( \mathbb{N} \)-basis of \( \text{LM}(I) \). In other words, \( N \cdot G \) is a Gröbner basis of \( I \) as an ideal of \( P \).

Definition 4.11. Let \( f \in P, f \neq 0 \) and \( G \subset P \). If \( f = \sum_i f_i g_i \) with \( f_i \in P, g_i \in G \) and \( \text{lm}(f) \succeq \text{lm}(f_i)\text{lm}(g_i) \) for all \( i, \) we say that \( f \) has a Gröbner representation with respect to \( G \).

Proposition 4.12. Let \( G \) be a \( \mathbb{N} \)-basis of a \( \mathbb{N} \)-ideal \( I \subset P \). Then, \( G \) is a Gröbner \( \mathbb{N} \)-basis of \( I \) if and only if for all \( f, g \in G, f, g \neq 0 \) and for any \( i \geq 0 \) the \( S \)-polynomial \( \text{spoly}(f, i \cdot g) \) have a Gröbner representation with respect to \( N \cdot G \).

For the sake of completeness, we recall also the notion of Gröbner bases for ideals of the free associative algebra. For any subset \( G \subset F \), define \( \text{lm}(G) \) and \( \text{LM}(G) \) as we have done for \( P \).

Definition 4.13. Let \( I \subset F \) be an ideal and \( G \subset I \). We call \( G \) a Gröbner basis of \( I \) if \( \text{lm}(G) \) is a basis of \( \text{LM}(I) \). In other words, for any \( f \in I, f \neq 0 \) one has that \( \text{lm}(f) = u\text{lm}(g) \) for some \( g \in G, g \neq 0 \) and \( u, v \in W \).

From now on, assume that \( P \) is endowed with a weighted \( \mathbb{N} \)-ordering and \( F \) with the induced graded monomial ordering. By abuse of notation, we will denote both these orderings as \( \prec \). We mention finally the following key result proved in [18] for Gröbner \( \mathbb{N} \)-bases of letterplace ideals.

Proposition 4.14. Let \( I \subset F \) be a graded ideal and denote \( J \subset P \) its letterplace analogue. If \( G \) is a multihomogeneous Gröbner \( \mathbb{N} \)-basis of \( J \) then \( \iota^{-1}(G \cap L) \) is a homogeneous Gröbner basis of \( I \).

This result together with the Proposition 4.12 implies the following algorithm for the computation of homogeneous noncommutative Gröbner bases that is alternative to the classical method developed in [15, 24, 27, 28].
Algorithm 4.1 HFreeGBasis

Input: $H$, a homogeneous basis of a graded ideal $I \subset F$.
Output: $\iota^{-1}(G)$, a homogeneous Gröbner basis of $I$.

$G := \iota(H)$;
$B := \{(f, g) \mid f, g \in G\}$;

while $B \neq \emptyset$ do
  choose $(f, g) \in B$;
  $B := B \setminus \{(f, g)\}$;
  for all $i \geq 0$ s.t. $\gcd(\text{lm}(f), \text{lm}(i \cdot g)) \neq 1$, $\text{lcm}(\text{lm}(f), \text{lm}(i \cdot g)) \in L$ do
    $h := \text{Reduce}(\text{spoly}(f, i \cdot g), N \cdot G)$;
    if $h \neq 0$ then
      $B := B \cup \{(h, h), (h, k), (k, h) \mid k \in G\}$;
      $G := G \cup \{h\}$;
    end if;
  end for;
end while:
return $\iota^{-1}(G)$.

Note that the iteration “for all $i \geq 0$ s.t. ...” runs over a finite number of integers since condition $\gcd(\text{lm}(f), \text{lm}(i \cdot g)) \neq 1$ implies that $i < w(f) = \deg(f)$. Moreover, by multihomogeneity of the elements of $P$ involved in the computation, one has that the condition $\text{lcm}(\text{lm}(f), \text{lm}(i \cdot g)) \in L$ is equivalent to require that the element $h = \text{Reduce}(\text{spoly}(f, i \cdot g), N \cdot G)$ is multilinear. Note finally that there are a finite number of elements of the infinite set $N \cdot G$ that may be involved in such reduction. Owing to Non-Noetherianity of the free associative algebra $F$ or of the polynomial algebra $P$ that has an infinite number of variables even if the set $X$ is finite, it is clear that one has termination only for truncated computations up to some fixed degree $d$, provided that the ideal $I \subset F$ is finitely generated up to $d$. For more details about the above algorithm we refer to [18, 19].

5. Gröbner L-bases and L-saturation

The fact that letterplace ideals are $N$-generated by multilinear elements and Proposition 4.14 suggest that for such ideals one needs a notion of Gröbner basis that involves only multilinear elements.

Definition 5.1. Let $J$ be a $L$-ideal of $P$ and let $H \subset J \cap L$ be a subset of multilinear elements. If $H$ is a $N$-basis of $J$ then we call $H$ a $L$-basis or multilinear $N$-basis of $J$.

Definition 5.2. Let $J \subset P$ be a $L$-ideal and denote $\text{LM}_L(J) = \langle \text{lm}(f) \mid f \in J \cap L \rangle_N$. Let $G \subset J \cap L$ be a subset of multilinear elements. We call $G$ a Gröbner $L$-basis or Gröbner multilinear $N$-basis of $J$ if $\text{lm}(G)$ is an $N$-basis of $\text{LM}_L(J)$ that is for all multilinear elements $f \in J \cap L$ one has that $i \cdot \text{lm}(g)$ divides $\text{lm}(f)$, for some $g \in G$ and $i \geq 0$. Clearly, all Gröbner $L$-bases are also $L$-bases of letterplace ideals.

If $I$ is a graded ideal of $F$ and $J \subset P$ is its letterplace analogue, by Proposition 4.14 one has that $G \subset J \cap L$ is a Gröbner $L$-basis of $J$ if and only if $\iota^{-1}(G)$ is a homogeneous Gröbner basis of $I$. In this sense, we may say that Gröbner $L$-bases are “letterplace analogues” of homogeneous Gröbner bases of the free associative
algebra. Another interesting feature of Gröbner $L$-bases is that they can be obtained as complete multihomogeneous Gröbner $\mathbb{N}$-bases of suitable ideals.

**Definition 5.3.** Denote $N = \langle x_i(1)x_j(1) \mid i, j \geq 1 \rangle_\mathbb{N} \subset P$. A monomial $m = x_{i_1}(j_1) \cdots x_{i_d}(j_d) \in M$ is said normal modulo $N$ if $j_1 \neq \ldots \neq j_d$. A polynomial $f \in P$ is in normal form modulo $N$ if all its monomials are normal modulo $N$.

**Definition 5.4.** Let $N \subset J \subset P$ be a $\mathbb{N}$-ideal and let $G \subset J$ be a subset of polynomials in normal form modulo $N$. We say that $G$ is a Gröbner $\mathbb{N}$-basis of $J$ modulo $N$ if $G \cup \{x_i(1)x_j(1) \mid i, j \geq 1\}$ is a Gröbner $\mathbb{N}$-basis of $J$.

**Proposition 5.5.** Let $J$ be a $L$-ideal of $P$ and let $G \subset J \cap L$. Then $G$ is a Gröbner $L$-basis of $J$ if and only if $G$ is a multihomogeneous Gröbner $\mathbb{N}$-basis of $J + N$ modulo $N$.

**Proof.** It is sufficient to prove that there is a Gröbner $\mathbb{N}$-basis of $J + N$ modulo $N$ whose elements are all multilinear. Then, consider to apply the Buchberger algorithm to a $L$-basis of $J$. By the product criterion and multihomogeneity of the computation, it is clear that for the monomials $m = \text{lcm}(\text{lm}(f), i \cdot \text{lm}(g))$ where $f, g$ are elements of the current $\mathbb{N}$-basis, one has that either $m$ is multilinear or $m \in N$.

Note that the above proposition provides another insight into the relationships between noncommutative structures and their commutative analogues subjected to the shift action of the monoid $\mathbb{N}$. Let us extend now the results of Section 4 and the previous ones to the algebras $F, P$. In what follows, **assume** the polynomial algebra $P$ be endowed with a place $\mathbb{N}$-ordering induced by a monomial ordering of $P(1)$ such that $t(1) \prec x_1(1) \prec x_2(1) \prec \ldots$. Therefore, the free associative algebra $F$ is provided with the graded right lexicographic ordering such that $t \prec x_1 \prec x_2 \prec \ldots$. One obtains immediately the following result.

**Proposition 5.6.** The elements $\bar{\iota}(\{t, x_i\}) = t(1)x_i(2) - x_i(1)t(2)$ $(i \geq 1)$ are a Gröbner $L$-basis of the $L$-ideal $D$ that is the commutators $\{t, x_i\}$ are a homogeneous Gröbner basis of the graded ideal $C$. Then, a multilinear element $f \in \bar{L}$ is said in normal form modulo $D$ if it is such with respect to the above Gröbner $L$-basis.

**Definition 5.7.** Let $D \subset J \subset \bar{P}$ be a $L$-ideal and let $G \subset J \cap \bar{L}$ be a subset of multilinear elements in normal form modulo $D$. We say that $G$ is a Gröbner $L$-basis of $J$ modulo $D$ if $G \cup \{\bar{\iota}(\{t, x_i\}) \mid i \geq 1\}$ is a Gröbner $L$-basis of $J$.

A natural characterization of the $L$-saturation of a letterplace ideal containing $D$ is the following one.

**Proposition 5.8.** Let $J \subset \bar{P}$ be a $L$-ideal containing $D$. Then a Gröbner $L$-basis of $\text{Sat}_L(J)$ modulo $D$ is given by the elements $\psi(f)^*$ for all $f \in J \cap \bar{L}$ in normal form modulo $D$.

**Proof.** It is sufficient to note that if $f \in \bar{L}$ is in normal form modulo $D$ then $\psi(f) \in V$ and $g = \psi(f)^* \in \bar{L}$. Moreover, it is clear that $f = \prod_{d < k \leq d'} t(k)g$ where $\text{deg}(f) = d' \geq d = \text{deg}(g)$.

**Proposition 5.9.** Let $D \subset J \subset \bar{P}$ be a $L$-ideal and denote $J' = \text{Sat}_L(J)$ its $L$-saturation. Moreover, let $G$ be a Gröbner $L$-basis of $J$ modulo $D$. Then $G' = \psi(G)^* = \{\psi(g)^* \mid g \in G\}$ is a Gröbner $L$-basis of $J'$ modulo $D$. 

Proof. Note that if \( f' \in \bar{L} \) is in normal form modulo \( D \) and \( f = \psi(f') \in V \) then \( \text{lm}(f^*) = \text{lm}(f) \in M \) by definition of the monomial ordering of \( \bar{P} \). Now, let \( f' \in J \cap \bar{L} \) be an element in normal form modulo \( D \). Hence, there is \( g' \in G \) and \( h \geq 0 \) such that \( h \cdot \text{lm}(g') \) divides \( \text{lm}(f') \). Put \( f = \psi(f'), g = \psi(g') \) and \( m_i = \prod_{0 \leq j \leq t(j)} t(j) \).

Then, one has that \( f' = f' \cdot (i - m_j), g' = g' \cdot (k - m_l) \) where \( i = \deg(f), k = \deg(g) \) and \( j, l \geq 0 \). From \( h \cdot \text{lm}(g') \) divides \( \text{lm}(f') \) if follows that \( h + k \leq i \) and hence \( k \cdot \text{lm}(g^*) \) divides \( \text{lm}(f^*) \). We conclude that \( \psi(G^*) \) is a Gröbner \( \bar{L} \)-basis of \( \text{Sat}_L(J) \) modulo \( D \).

From the above result one obtains immediately an algorithm for computing Gröbner \( \bar{L} \)-bases of \( \bar{L} \)-saturated letterplace ideals of \( \bar{P} \) containing \( D \). This is especially relevant since such bases are in correspondence with homogeneous Gröbner bases of saturated ideals of \( \bar{F} \) containing \( C \). In fact, the Gröbner bases of any ideal \( I \subset F \) are in correspondence with the homogeneous ones of its homogenization \( I^* \).

Definition 5.10. A homogeneous element \( f \in \bar{F} \) is said in normal form modulo \( C \) if it is such with respect to the Gröbner basis \( \{[t, x_i] \mid i \geq 1 \} \). In other words, \( \bar{i}(f) \in \bar{L} \) is in normal form modulo \( D \).

Note that \( \bar{i}(f^*) = \bar{i}(f)^* \) for all \( f \in F, f \neq 0 \). Moreover, if \( f \in \bar{F} \) is a homogeneous element in normal form modulo \( C \) then we have also that \( \bar{i}(\varphi(f)) = \bar{i}(\psi(f)) \).

Definition 5.11. Let \( C \subset I \subset \bar{F} \) be a graded ideal and let \( G \subset I \) be a subset of homogeneous elements in normal form modulo \( C \). We say that \( G \) is a Gröbner basis of \( I \) modulo \( C \) if \( G \cup \{[t, x_i] \mid i \geq 1 \} \) is a Gröbner basis of \( I \). In other words, \( \bar{i}(G) \subset \bar{L} \) is a Gröbner \( \bar{L} \)-basis modulo \( D \) of the letterplace analogue of \( I \).

The following result can be found also in \[23, 29\].

Proposition 5.12. Let \( I \subset F \) be any ideal and let \( G \) be any Gröbner basis of \( I \). Then \( G^* = \{g^* \mid g \in G\} \) is a homogeneous Gröbner basis of \( I^* \) modulo \( C \). Moreover, one has that \( \text{lm}(G^*) = \text{lm}(G) \).

Proof. Let \( f' \in I^* \) be a homogeneous element in normal form modulo \( C \) and put \( f = \varphi(f') \). Then \( f' = f^* t_i \) for some \( i \geq 0 \) and \( \text{lm}(f) = \text{ulm}(g)v \) for some \( g \in G \) and \( u, v \in W \). Since \( \text{lm}(f^*) = \text{lm}(f), \text{lm}(g^*) = \text{lm}(g) \) we conclude that \( \text{lm}(f^*) = \text{ulm}(g^*)vt_i \). \( \square \)

The above result, together with Proposition \[5.9\], implies an alternative algorithm to compute Gröbner bases of nongraded noncommutative ideals of the free associative algebra via homogeneous commutative computations in their extended letterplace analogues.
Algorithm 5.1 FreeGBasis

Input: $H$, a basis of an ideal $I \subset F$.
Output: $\varphi(\bar{i}^{-1}(G))$, a Gröbner basis of $I$.
$\begin{align*}
G & := \bar{i}(H^* \cup \{[t, x_i] \mid i \geq 1\}); \\
B & := \{(f, g) \mid f, g \in G\}; \\
\text{while } B \neq \emptyset & \text{ do} \\
& \quad \text{choose } (f, g) \in B; \\
& \quad B := B \setminus \{(f, g)\}; \\
& \quad \text{for all } i \geq 0 \text{ s.t. } \gcd(\text{lm}(f), \text{lm}(i \cdot g)) \neq 1, \text{lcm}(\text{lm}(f), \text{lm}(i \cdot g)) \in \bar{L} \text{ do} \\
& \quad & \quad h := \text{Reduce}(\text{spoly}(f, i \cdot g), \mathbb{N} \cdot G); \\
& \quad & \quad \text{if } h \neq 0 \text{ then} \\
& \quad & \quad \quad h := \psi(h)^* \\
& \quad & \quad \quad B := B \cup \{(h, h), (h, k), (k, h) \mid k \in G\}; \\
& \quad & \quad \quad G \ := G \cup \{h\}; \\
& \quad \text{end if;} \\
& \quad \text{end for;} \\
& \quad \text{end while;} \\
& \quad \text{return } \varphi(\bar{i}^{-1}(G)).
\end{align*}$

Proposition 5.13. The algorithm FreeGBasis is correct.

Proof. Let $D \subset J \subset \bar{P}$ be the extended letterplace analogue of $I$. At each step of the procedure FreeGBasis, the set $G$ is clearly a $L$-basis of an ideal $D \subset J \subset \bar{P}$ such that $\text{Sat}_L(J') = J$. Moreover, since the elements $\bar{i}([t, x_i])$ initially belong to $G$ we have automatic normalization modulo $D$. Recall now that if $h \in \bar{L}$ is a multilinear element in normal form modulo $D$ then $h' = \psi(h)^*$ divides $h$. This implies that if an $S$-polynomial can be reduced to zero by adding $h$ to the basis $G$, the same holds if we substitute $h$ with $h'$. In case of termination, one has therefore that the set $G$ is a Gröbner $L$-basis of $J'$ whose elements satisfy $h = \psi(h)^*$. By Proposition 5.3, we conclude that $J'$ is $L$-saturated that is $J' = J$. Then $G' = \bar{i}^{-1}(G)$ is homogeneous Gröbner basis of $I'$ that is $\varphi(G')$ is a Gröbner basis of $I$ by Proposition 5.12. □

Note that the above algorithm has neither general termination nor just termination up to some fixed degree $d$. The reason is that even if all computations are homogeneous, because of the saturation $h = \psi(h)^*$ that may decrease the degree we cannot be sure at some suitable step that we will not get additional elements of degree $\leq d$ in the steps that will follow. This agrees with the well known fact that the word-problem is generally undecidable for nongraded associative algebras even if these are finitely generated. Nevertheless, if an ideal of the free associative algebra has a finite Gröbner basis then the algorithm FreeGBasis is able to compute it in a finite number of steps.

Definition 5.14. Let $G \subset F$ be any subset. We call $G$ a minimal Gröbner basis if $\text{lm}(G)$ is a minimal basis of $\text{LM}(G)$ that is $\text{lm}(f) \neq u\text{lm}(g)v$, for all $f, g \in G, f \neq g$ and for any $u, v \in W$.

By the choice of the monomial ordering of $\bar{F}$ and the property that the elements are kept in normal form modulo $C$ we have clearly that if $G'$ is a minimal Gröbner basis of $I^*$ modulo $C$ then $\varphi(G')$ is also a minimal Gröbner basis of $I$ since $\text{lm}(G') = \text{lm}(\varphi(G'))$. This is the main advantage to compute on the fly the homogenization.
Moreover, it is easy to see that all multilinear S-polynomials between basis $\phi_i$ of $I^*$ instead of working with any graded ideal $C \subset I^* \subset F$ such that $\varphi(I^*) = I$. In fact, the ideal $I^*$ may have an infinite minimal Gröbner basis even if $I$ has a finite one and more generally this basis has elements in higher degrees than the basis of $I^*$. In other words, to work without saturation is usually very inefficient. Such strategy is described in [29] in the context of classical algorithm and called “rabbit strategy” or “cancellation rule”.

Note that actual computations with the algorithm FreeGBasis are performed by bounding the weight of the variables of $P$ that is in a (Noetherian) polynomial algebra with a finite number of variables. This may result in an incomplete computation because some of the S-polyomials may be not defined owing to this bound. Since the S-polynomials $s = spoly(f, i \cdot g)$ such that $gcd(lm(f), lm(i \cdot g)) \neq 1$ that are considered in the procedure are multilinear elements, it is clear that $w(s) = \deg(s) \leq 2d - 1$ where $d = \max\{\deg(f) \mid f \in G\}$ and $G$ is the current basis. We conclude that an actual computation is certified complete if the weight bound fixed for the variables of $P$ is $\geq 2d - 1$, where $d$ is the maximal degree occurring in the output generators.

6. An illustrative example

With the aim of showing a concrete computation with the algorithm FreeGBasis, we present here a simple application to finitely presented groups. Consider the Klein group $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ that can be presented (as a Coxeter group) in the following way

$$G = \langle x, y \mid x^2 = y^2 = (xy)^2 = 1 \rangle.$$  

Define the free associative algebra $F = K\langle x, y \rangle$ and consider the elements

$$f_1 = x^2 - 1, f_2 = y^2 - 1, f_3 = (xy)^2 - 1 \in F.$$  

Then, the group algebra $KG$ is clearly isomorphic to the quotient algebra $F/I$ where $I = \langle f_1, f_2, f_3 \rangle$. A next step is to consider the free commutative $\mathbb{N}$-algebra $P = K[x(1), y(1), t(1), x(2), y(2), t(2), \ldots]$ and to encode the noncommutative algebra $F/I$ in the letterplace way that is by defining the extended letterplace analogue $J \subset P$ of the two-sided ideal $I \subset F$. As explained in the comments at the end of Section 3, we consider therefore the polynomials

$$d_1 = \bar{i}(t, x) = t(1)x(2) - x(1)t(2), d_2 = \bar{i}(t, y) = t(1)y(2) - y(1)t(2),$$

$$g_1 = \bar{i}(f'_1) = x(1)x(2) - t(1)t(2), g_2 = \bar{i}(f'_2) = y(1)y(2) - t(1)t(2),$$

and we define the L-ideal $J' = \langle d_1, d_2, g_1, g_2, g_3 \rangle_N$. In fact, one has that $J = \text{Sat}_L(J')$ and to perform this ideal operation one needs a Gröbner basis computation. Then, we fix the lexicographic monomial ordering on $P$ with $t(1) < y(1) < x(1) < t(2) < y(2) < x(2) < \ldots$

which is clearly a place $\mathbb{N}$-ordering inducing the graded right lexicographic ordering on $F$ with $y < x$. Then, to compute $\text{Sat}_L(J')$ one has to reduce multilinear S-polynomials between generators and performing the saturation of new generators arising by such reductions. At the end of computation, whenever $I$ admits a finite Gröbner basis, one obtains a (saturated) Gröbner L-basis $G \subset J$ that is a Gröbner basis $\varphi(r^{-1}(G))$ of $I$, as prescribed by the algorithm FreeGBasis.

First of all, note that no multilinear S-polynomial is defined for the elements $d_i$. Moreover, it is easy to see that all multilinear S-polynomials between $d_i$ and any
saturated element can be reduced to zero. For instance, one has the S-polynomial 
\[ \text{spoly}(d_1, 1 \cdot g_1) = -x(1) t(2) x(3) + t(1) t(2) t(3) \] 
that is reduced modulo \( 1 \cdot d_1 \) to the element 
\[ x(1) x(2) t(3) - t(1) t(2) t(3) = g_1 t(3). \]
Consider now the S-polynomial 
\[ \text{spoly}(g_1, 1 \cdot g_1) = -t(1) t(2) x(3) + x(1) t(2) t(3) \]
that can be clearly reduced to zero modulo the set \( \mathbb{N} \cdot d_1 \). In the same way, one obtains that \( \text{spoly}(g_2, 1 \cdot g_2) \) reduces to zero. Then, we define the S-polynomial 
\[ \text{spoly}(g_3, 3 \cdot g_2) = -x(1) y(2) x(3) t(4) t(5) + t(1) t(2) t(3) t(4) t(5) \]
that is reduced modulo the set \( \mathbb{N} \cdot g_2 \) to the element 
\[ g_4 = x(1) y(2) x(3) t(4) - y(1) t(2) t(3) t(4) t(5). \]
This polynomial cannot be further reduced by the current \( \mathbb{N} \)-basis and hence one adds to this set the saturated element
\[ g_4 = \psi(g_4)^* = x(1) y(2) x(3) - y(1) t(2) t(3). \]
Then, we consider 
\[ \text{spoly}(g_3, g_4) = y(1) t(2) t(3) y(4) - t(1) t(2) t(3) t(4) \]
that can be reduced to zero modulo \( \mathbb{N} \cdot \{d_2\} \). Consider now the next S-polynomial
\[ \text{spoly}(g_1, 1 \cdot g_4) = -t(1) t(2) y(3) x(4) + x(1) y(2) t(3) t(4) \]
By applying the set \( \mathbb{N} \cdot \{g_3\} \) one obtains the element
\[ g_5 = y(1) x(2) t(3) t(4) - x(1) y(2) t(3) t(4) \]
and hence its saturation
\[ g_5 = \psi(g_5)^* = y(1) x(2) - y(1) y(2). \]
enters the \( \mathbb{N} \)-basis of the current \( L \)-ideal. All remaining S-polynomials reduce to zero which means that such ideal is \( L \)-saturated and therefore coincides with \( J = \text{Sat}_L(J') \). Since the sequence of leading monomials of the polynomials \( g_i \) is 
\[ \text{lm}(g_1) = x(1) x(2), \text{lm}(g_2) = y(1) y(2), \text{lm}(g_3) = x(1) y(2) x(3) y(4), \]
\[ \text{lm}(g_4) = x(1) y(2) x(3), \text{lm}(g_5) = y(1) x(2) \]
and one has that \( 1 \cdot \text{lm}(g_5) \) divides \( \text{lm}(g_4) \) that divides \( \text{lm}(g_3) \), we conclude that a minimal Gröbner \( L \)-basis of the ideal \( J \) is given by the set 
\[ G = \{d_1, d_2, g_1, g_2, g_5\}. \]
Because \( J \subset B \) is exactly the extended letterplace analogue of the two-sided ideal \( I \subset F \), we obtain that the set \( \{x^2 - 1, y^2 - 1, yx - xy\} \) is a minimal Gröbner basis of \( I \) with respect to graded right lexicographic ordering. In other words, we have found the canonical presentation
\[ G = \langle x, y \mid x^2 = y^2 = 1, yx = xy \rangle \]
of the group \( G \) as a direct product of cyclic groups.

7. Implementations and testing

In this section we present an experimental implementation of the algorithm FreeGBasis that has been developed in the language of Maple. We have obtained such implementation by modifying the algorithm SigmaGBasis introduced and experimented in [20] for the computation of Gröbner bases for finite difference ideals. Precisely, the letterplace computations are a special case of the ordinary difference ones. The main modifications to obtain FreeGBasis consist in adding the commutators \([t, x_i]\) to the elements introduced by homogenizing the initial non-commutative generators and in encoding all such elements in the letterplace way. Moreover, it is necessary to add to the procedure the “multilinearity criterion” that is the condition \( \text{lcm}(\text{lm}(f), \text{lm}(i \cdot g)) \in L \) when considering the S-polynomial \( \text{spoly}(f, i \cdot g) \). Finally, one has to implement the saturation of the elements that are obtained by reducing these S-polynomials. Note that according to Proposition 5.6
the multilinearity criterion, that is essential to have tractable computations, can be obtained simply by adding the set of monomials $\mathcal{N} = \{x_i(1)x_j(1), t(1)^2, t(1)x_i(1)\}$ to the initial letterplace basis. This option is a useful trick if one wants to obtain the algorithm $\text{FreeGBasis}$ by means of a standard implementation of the Buchberger procedure for commutative Gröbner bases.

To the purpose of studying the impact of different strategies used in $\text{FreeGBasis}$, we have tested also two variants of this algorithm that are indicated in the examples with the suffix $\text{noc}$ (no-criterion) and $\text{bas}$ (basic). Both these variants make use of the saturation step $h := \psi(h)^*$ since it is well known that mere homogenization of the initial generators is generally inefficient and may lead to an infinite Gröbner basis for the corresponding graded but not saturated ideal even if the input ideal have a finite one \[29\]. The variant $\text{noc}$ is obtained simply by suppressing the “shift criterion” that is all S-polynomials $\text{spoly}(i \cdot f, j \cdot g)$ ($i, j \in \mathbb{N}$) have to be considered for reduction. In the variant $\text{bas}$ we suppress also the shifting of the new generators obtained from the reduction of the S-polynomials. In other words, one applies shift operators just to the input letterplace generators. This is correct since the different shifted versions of the generators that are necessary to the reduction process will be created in any case from the S-polynomials provided that the shift criterion is off. Up to the saturation step, the basic version can be obtained therefore by applying the Buchberger algorithm to the set of shifted elements of the initial letterplace basis joined to the set of monomials $\mathcal{N}$. We apply this trick on some examples where no saturation arises, in order to have computing times with standard routines of $\text{Singular}$ that estimate approximately the speed-up that one may obtain moving from the Maple interpreter to the kernel of a computer algebra system. Note that an implementation of noncommutative Gröbner bases in the library $\text{LETTERPLACE}$ of $\text{Singular}$ is currently under development.

The monomial $\mathbb{N}$-ordering that is considered for the polynomial algebra $\bar{P}$ is the lexicographic ordering with

$$t(1) \prec x_n(1) \prec \ldots \prec x_1(1) \prec t(2) \prec x_n(2) \prec \ldots \prec x_1(2) \prec \ldots$$

that is clearly a place $\mathbb{N}$-ordering. Then, one has that the free associative algebra $\bar{F} = F\langle x_1, \ldots, x_n, t \rangle$ is endowed with the graded left lexicographic ordering with $t \prec x_n \prec \ldots \prec x_1$ by means of a reversing letterplace embedding $\bar{i} : y_1 \cdots y_d \mapsto y_d(1) \cdots y_1(d)$, where $y_k = x_{i_k}$ or $y_k = t$.

The parameters that are considered in the experiments are the number of respectively input generators, output Gröbner generators, elements of a minimal Gröbner basis, pairs (S-polynomials) that are actually reduced and saturations steps. The last parameter is the computing time which is given in the format “minutes:seconds”. Attached to the number of elements of a basis, after the letter “d” we indicate the maximum degree of such elements. Note that for the variant $\text{bas}$ the input and output numbers count all the shifted versions of the basis elements. Moreover, for all variants we have that the pairs number includes the initial generators since we actually treat them as S-polynomials in order to interreduce. All examples have been computed with Maple 12 running on a server with a four core Intel Xeon at 3.16GHz and 64 GB RAM.
The performance of the different variants of the algorithm FreeGBasis have been studied on a test set based on presentations of relevant classes of noncommutative algebras. The examples $g_{3332}$ and $g_{444}$ refer to the presentation of group algebras of presented groups. Precisely, such groups belong to the classes $G_{(l,m,n,q)} = \langle r,s \mid r^l,s^m,(rs)^n,[r,s]^q \rangle$ and $G_{(m,n,p)} = \langle a,b,c \mid a^m,b^n,c^p,(ab)^2,(bc)^2, (ca)^2, (abc)^2 \rangle$. The examples hecke are the presentation of the Hecke algebras defined by the following Coxeter matrices

$$A = \begin{pmatrix} 1 & 3 & 2 & 3 \\ 3 & 1 & 3 & 2 \\ 2 & 3 & 1 & 3 \\ 3 & 2 & 3 & 1 \end{pmatrix};$$
$$D = \begin{pmatrix} 1 & 3 & 2 & 2 \\ 3 & 1 & 3 & 2 \\ 2 & 3 & 1 & 2 \\ 3 & 2 & 3 & 1 \end{pmatrix};$$
$$E = \begin{pmatrix} 1 & 2 & 3 & 2 & 2 \\ 2 & 1 & 3 & 2 & 2 \\ 3 & 2 & 1 & 3 & 2 \\ 2 & 3 & 1 & 3 & 2 \\ 2 & 2 & 3 & 1 & 3 \end{pmatrix}. $$

For the noncommutative polynomials defining the relations of the considered Hecke algebras, the quantity "$q$" is assumed a parameter. The examples indicated as lie refer to the universal enveloping algebra of two indecomposable nilpotent Lie
algebras, namely
\[
\text{lie5} : [x_1, x_2] - x_3, [x_1, x_3] - x_4, [x_2, x_5] - x_4;
\]
\[
\text{lie7} : [x_1, x_2] - x_3, [x_1, x_3] - x_4, [x_1, x_5] - x_6,
[x_2, x_3] - \frac{1}{2} x_4 - \frac{1}{4} x_5 + \frac{1}{8} x_6 + \frac{1}{2} x_7, [x_2, x_4] - \frac{1}{2} x_5 - \frac{1}{4} x_6,
[x_2, x_5] - x_6, [x_2, x_7] - \frac{1}{2} x_5 + \frac{1}{4} x_6, [x_3, x_4] + \frac{1}{2} x_6, [x_3, x_7] - \frac{1}{4} x_6.
\]
Finally, the examples \text{templieb8}, \text{templieb9} are the defining relations of the Temperley-Lieb algebras \cite{17} respectively in 7 and 8 variables. The quantity “δ” used in the definition of such algebras is considered a parameter. In all the names of the tests, we indicate after the letter “d” the bounded degree within the computation is performed that is the maximal weight allowed for the variables of \(P\).

The experiments show in a sufficiently clear way that the standard version of the algorithm \textsc{FreeGBasis} is generally the most efficient one. In fact, this procedure is able to decrease relevantly the number of S-polynomial reductions that are usually time-consuming. For instance, this emerges in a dramatic way for the example \text{lie7}. Note that for the examples \text{g3332}, \text{g444} the basic variant results very competitive. This can be explained as the result of a low cost for the S-polynomial reductions (binomial generators) compared to the cost of applying shifting to letterplace polynomials. As previously remarked, a noncommutative Gröbner basis computed within a fixed bounded weight is certified complete if such bound is \(\geq 2d-1\), where \(d\) is the maximal degree of the output generators. This happens for instance for the examples \text{g3332}, \text{g444}, \text{heckeD} and \text{lie7}. In particular, one obtains a computational proof that the ideal of the relations defining the Hecke algebra of the example \text{heckeD} is a saturated one.

The computing times obtained with the implementation of \textsc{FreeGBasis} in the language of Maple are useful to evaluate the possible different variants of this algorithm but they are not especially relevant when compared to other implementations of noncommutative Gröbner bases developed in the kernel of highly efficient computer algebra systems. Among these fast implementations, one has to mention the one of \textsc{Magma} \cite{4} that makes use of a noncommutative version of the Faugère’s F4 method. To the purpose of estimating the speed-up that may be achieved with a kernel implementation, we compute then the timings of some examples with the basic variant of \textsc{FreeGBasis} obtained by using the function “std” of \textsc{Singular} that implements the Buchberger algorithm. For the examples \text{heckeAd15}, \text{heckeEd10} and \text{teli9d9} such computing times are respectively 0.26, 0.34 and 1.01 sec. Keeping into account that the variant \text{bas} shows to be the less efficient, we believe that these data, together with all experiments performed in \cite{18,19}, indicate that letterplace approach is feasible for both the homogeneous and inhomogeneous case.

8. Conclusion and future directions

The theory and the methods proposed in this paper and in the previous ones \cite{18,19} proves that commutative and noncommutative Gröbner bases and the related algorithms can be unified in a general theory for Gröbner bases of commutative ideals that are invariant under the action of suitable algebra endomorphisms \cite{5,18,19,20}. We believe that this idea will have not only consequences in the development of new algorithmic methods but also in the reformulation in the letterplace language of structures and problems of noncommutative nature. It is sufficient in
fact to mention that the notion of Gröbner basis is a key ingredient for the description and computation of many fundamental invariants. The experiments show that the letterplace methods are computationally practicable and hence new noncommutative tasks can be achieved now by commutative computer algebra systems. Future research directions may consist in investigating relationships between commutative and noncommutative invariants based on Gröbner bases and in developing optimized libraries for their computation.

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