Sensitivity Analysis of Long-Term Cash Flows

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First version: July 18, 2014
Last version: Nov 11, 2015

Abstract

In this article, a sensitivity analysis of long-term cash flows with respect to perturbations in the underlying process is presented. For this purpose, we employ the martingale extraction through which a pricing operator is transformed into what is easier to address. The method of Fournie will be combined with the martingale extraction. We prove that the sensitivity of long-term cash flows can be represented in a simple form.

Key Words: Long-term cash flows, Martingale extraction, Malliavin calculus

1 Introduction

In finance, we often encounter the quantity of the form:

$$p_T := \mathbb{E}^Q[e^{-\int_0^T r(X_t)dt} f(X_T)] .$$

For example, if $Q$ is a risk-neutral measure and $r(X_t)$ is a short interest rate, then the quantity is the current price of the option with payoff $f(X_T)$ at time $T$. If $Q$ is an objective measure, $f$ is a utility function of an agent and $r(X_t)$ is a discount rate of the agent, then the quantity is the discounted expected utility of the agent. This article examines a sensitivity analysis of the quantity $p_T$ for large $T$ with respect to perturbations in the underlying process $X_t$.

The underlying process $X_t$ in this article is a conservative diffusion process in a Brownian environment. Let $W_t = (W_1(t), W_2(t), \cdots, W_d(t))^\top$ be a standard $d$-dimensional Brownian motion.

Assumption 1. The underlying process $X_t$ is a $d$-dimensional time-homogeneous Markov diffusion process. Assume that $X_t$ satisfies the following stochastic differential equation:

$$dX_t = b(X_t) dt + \sigma(X_t) dW_t , \quad X_0 = \xi .$$

*Most of work in the present article was done when the author was affiliated to Courant Institute of Mathematical Sciences, New York University, NY, USA. The author thanks to Jonathan Goodman and Srinivasa Varadhan for helpful comments.

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Here, $b$ is a $d$-dimensional column vector and $\sigma$ is a $d \times d$ matrix. $b(\cdot)$ and $\sigma(\cdot)$ are continuously differentiable and $\sum_{i,j=1}^{d} \sigma_{ij}(x) v_i v_j > 0$ for all $v \in \mathbb{R}^d - \{0\}$. In addition, we assume that the range of $X_t$ is $\mathbb{R}^d$, that is, the process does not explode in finite time $t$.

**Assumption 2.** $r(\cdot)$ is a continuously differentiable function on $\mathbb{R}^d$.

We explore a sensitivity analysis for the quantity $p_T$ with respect to the perturbation in the underlying process $X_t$. Let $X_t^\epsilon$ be a perturbed process of $X_t$ (with the same initial value $\xi = X_0 = X_0^\epsilon$) of the form:

$$dX_t^\epsilon = b_\epsilon(X_t^\epsilon) dt + \sigma_\epsilon(X_t^\epsilon) dW_t$$

with $b_0(\cdot) = b(\cdot)$ and $\sigma_0(\cdot) = \sigma(\cdot)$. The perturbed quantity is given by

$$p_T^\epsilon := \mathbb{E}^Q[e^{-\int_0^T r(X_s^\epsilon) ds} f(X_T^\epsilon)].$$

(1.1)

For the sensitivity analysis, we compute

$$\frac{\partial}{\partial \epsilon} \bigg|_{\epsilon=0} p_T^\epsilon$$

and investigate the behavior of this quantity for large $T$. The sensitivity with respect to the perturbation of the drift term $b_\epsilon(X_t)$ is called the *rho*, and the sensitivity with respect to the diffusion term $\sigma_\epsilon(X_t)$ is called the *vega*. The sensitivity with respect to the initial value $X_0 = \xi$ is given by

$$\frac{\partial p_T}{\partial \xi} \left( = \frac{\partial}{\partial \xi} \mathbb{E}^Q[e^{-\int_0^T r(X_s) ds} f(X_T)|X_0 = \xi] \right)$$

and is called the *delta*.

The main contribution of this article is the use of the martingale extraction method to the sensitivity analysis. Assume that $(X_t^\epsilon, r)$ admits the martingale extraction that stabilizes $f$ (Definition 2.1 and 2.2), then it can be easily shown that

$$p_T^\epsilon \simeq e^{-\lambda(\epsilon) T} l_\epsilon(\xi)$$

for some number $\lambda(\epsilon)$ and function $l_\epsilon(\xi)$. Here, for two nonzero functions $p_T$ and $q_T$ of $T$, the notation $p_T \simeq q_T$ means that $\lim_{T \to \infty} \frac{p_T}{q_T} = 1$. When $T$ is large, because $e^{-\lambda(\epsilon) T}$ dominates the perturbed quantity $p_T^\epsilon$, we can anticipate that the long-term behavior of $p_T^\epsilon$ is mainly determined by $e^{-\lambda(\epsilon) T}$. We may then expect

$$\frac{\partial}{\partial \epsilon} \bigg|_{\epsilon=0} p_T^\epsilon \simeq -\lambda'(0) T \cdot e^{-\lambda T} l(\xi) + e^{-\lambda T} \frac{\partial}{\partial \epsilon} \bigg|_{\epsilon=0} l_\epsilon(\xi)$$

and we thus obtain the following simple equation:

$$\frac{\partial}{\partial \epsilon} \bigg|_{\epsilon=0} \frac{p_T^\epsilon}{T \cdot p_T} \simeq -\lambda'(0).$$

For the delta, because $\lambda$ is independent of the initial value of $X_t$ as we will see soon - we have

$$\frac{\partial p_T}{\partial \xi} \simeq e^{-\lambda T} l'(\xi).$$
thus we obtain
\[ \frac{\partial p_T}{\partial x} \approx \frac{l'(\xi)}{l(\xi)} . \]

To justify these arguments, we employ the method of Fournie [9], in which there is a remarkable technique for sensitivity analysis. See [2], [28] and [29] as references for the method. Unfortunately, this method cannot be applied to functionals of the following form:
\[ \mathbb{E}^Q[e^{-\int_0^T r(X_s) \, ds} f(X_T)] , \]
and this is the form that interests us. This method (for calculating the delta and vega) is valid only for discretely monitored functionals of the following form:
\[ \mathbb{E}^Q[f(X_{t_1}, X_{t_2}, \ldots, X_{t_m}) ] \]
such that the process \( X_t \) is detected only for finite times up to maturity \( T \). In our case, however, the expectation contains the term
\[ e^{-\int_0^T r(X_s) \, ds} \]
which depends on the entire path of \( X_t \) up to time \( T \). The martingale extraction is useful in overcoming this problem. It is largely because the martingale extraction transforms the functionals depending on the entire path of \( X_t \) up to time \( T \) to the discretely monitored functionals. Thus, while applying the martingale extraction, the Fournie method is able to be successfully applied to our cases.

Another contribution of this article is a generalization of the result of Fournie for the rho. In the paper of Fournie, the perturbation is linear of the form \( b_\epsilon = b + \epsilon \tilde{b} \) and the function \( \tilde{b} \) is bounded. In addition, the diffusion matrix \( \sigma \) satisfies the uniform ellipticity condition and the payoff function satisfies the \( L^2 \)-condition, that is, \( \mathbb{E}^Q[f^2(\cdot)] < \infty \). We slightly generalize these conditions in Proposition C.1 in Appendix. Many financial models including the examples in this paper satisfy the generalized conditions.

Many authors employed the martingale extraction to investigate financial and economic problems. Hansen and Scheinkman explored long-term risk in [14], [15] and [16], in which the martingale extraction was used to show that a pricing operator consists with three components: an exponential term, a martingale and a transient term. They offered financial and economic meanings of the terms.

Borovicka, Hansen, Hendricks and Scheinkman [7] exploit the martingale extraction for a sensitivity analysis. They investigate shock exposure in terms of shock elasticity, which measures the impact of a current shock. Let \( G_t \) be cash flow at time \( t \). It is assumed that \( G_t \) is a multiplicative functional. They consider the following perturbation form, which is somewhat different from the perturbation form in this paper. Set
\[ H_t := e^{\int_0^T \kappa(\epsilon X_s) \, ds + \epsilon \int_0^T \alpha(X_s) \, dW_s} . \]
Here, \( \kappa(\cdot) \) and \( \alpha(\cdot) \) are given functions and define the direction of perturbation. Put the perturbed cash flow by \( q_T := \mathbb{E}^Q[G_T H_T] \). The quantity \( \frac{\partial}{\partial \epsilon}|_{\epsilon=0} q_T \) is called the shock elasticity. The shock elasticity for large \( T \) was analyzed in their work. The shock elasticity is not the same, but is somewhat similar with the notion of delta. Their result coincides with Theorem 4.1 in this article.

We now review the risk elasticity, which is similar to the rho and vega. The perturbed expected return is defined by
\[ R_T := \frac{\mathbb{E}[G_T H_T]}{\mathbb{E}[e^{-\int_0^T r_s \, ds} G_T H_T]} \]
and the quantity $\frac{\partial}{\partial \epsilon} \bigg|_{\epsilon=0} R_T$ is of interest to us and is called the risk elasticity. In their paper, a more general form of discount factor than $e^{-\int_0^T r_s ds}$ is considered. They do not provide a long-term analysis for risk elasticity. Borovicka, Hansen and Scheinkman [6] present more direct way of computing the shock elasticities.

The martingale extraction method is linked to several financial and economic topics. The connection to spectral theory can be found in [8], [12], [20], [21], [22], [23], [24] and [25]. Ross recovery is also closely related to the martingale extraction. Refer to [5], [13], [30], [32] and [33].

The following provides an overview of this article. We present the martingale extraction method in Section 2. In Section 3 and 4, the sensitivity analysis for long-term cash flows is investigated. Sections 5 and 6 present examples, and the last section summarizes the paper. The proofs of main results and the details of examples are in Appendices.

2 Martingale extraction

In this section, we explore the notion of the martingale extraction. Let $\mathcal{L}$ be the infinitesimal generator corresponding to the operator

$$f \mapsto p_T = \mathbb{E}^Q \left[ e^{-\int_0^T r(X_t) dt} f(X_T) \right].$$

Then,

$$\mathcal{L} := \frac{1}{2} \sum_{i,j=1}^d a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^d b_i(x) \frac{\partial}{\partial x_i} - r(x)$$

where $a = \sigma \sigma^\top$. We are interested in an eigenpair $(\lambda, \phi)$ of $\mathcal{L} \phi = -\lambda \phi$ with positive function $\phi$. There are two possibilities.

(i) there is no positive solution $\phi$ for any $\lambda \in \mathbb{R}$, or

(ii) there exists a number $\lambda_0$ such that it has positive solutions for $\lambda \leq \lambda_0$ and has no positive solution for $\lambda > \lambda_0$.

Refer to [31] for proof. In this article, we assume the second case.

Let $(\lambda, \phi)$ be an eigenpair of $\mathcal{L} \phi = -\lambda \phi$ with positive function $\phi$. It is easily checked that

$$M_t := e^{\lambda t - \int_0^t r(X_s) ds} \phi(X_t) \phi^{-1}(\xi)$$

is a local martingale.

**Definition 2.1.** When the local martingale $M_t$ is a martingale, we say that $(X_t, r)$ admits the martingale extraction with respect to $(\lambda, \phi)$.

When $M_t$ is a martingale, we can define a new measure $\mathbb{P}$ by

$$\mathbb{P}(A) := \int_A M_t \ d\mathbb{Q} = \mathbb{E}^Q \left[ \mathbb{I}_A M_t \right] \quad \text{for} \quad A \in \mathcal{F}_t.$$ 

The measure $\mathbb{P}$ is called the transformed measure from $\mathbb{Q}$ with respect to $(\lambda, \phi)$. The definition is well defined: If $A \in \mathcal{F}_t$, then for $0 < t < s$, we have $\mathbb{E}^Q [\mathbb{I}_A M_t] = \mathbb{E}^Q [\mathbb{I}_A M_s]$. Using this transformed measure $\mathbb{P}$, $p_T$ can be expressed by

$$p_T = \mathbb{E}^Q \left[ e^{-\int_0^T r(X_s) ds} f(X_T) \right] = \phi(\xi) e^{-\lambda T} \cdot \mathbb{E}^p \left[ (\phi^{-1} f)(X_T) \right]. \quad (2.1)$$
This relationship implies that the quantity \( p_T \) can be expressed in a relatively more manageable manner. The term \( \mathbb{E}^\mathbb{P}[(\phi^{-1}f)(X_T)] \) depends on the final value of \( X_T \), whereas \( \mathbb{E}^\mathbb{Q}[e^{-\int_0^Tr(X_s)ds}f(X_T)] \) depends on the whole path of \( X_t \) at \( 0 \leq t \leq T \). This advantage makes it easier to analyze the sensitivity of long-term cash flows. As a special case, if the density function of \( X_t \) under \( \mathbb{P} \) is known, one can directly analyze the term \( \mathbb{E}^\mathbb{P}[(\phi^{-1}f)(X_T)] \).

We now observe how the dynamic of \( X_t \) is changed when the underlying measure is changed from the measure \( \mathbb{Q} \) to the transformed measure \( \mathbb{P} \). We know that the Radon-Nikodym derivative of \( \mathbb{Q} \) with respect to \( \mathbb{P} \) on \( \mathcal{F}_t \) is

\[
M_t = e^{\lambda t - \int_0^t r(X_s)ds} \phi(X_t) \phi^{-1}(\xi).
\]

For convenience, let

\[
\varphi := \sigma^\top \cdot \nabla \phi,
\]

where \( \nabla \phi \) is the \( d \times 1 \) gradient vector of \( \phi \). We say \( \varphi \) is the martingale exponent of \( M_t \). According to the Girsanov theorem, we know that a process \( B_t \) defined by

\[
B_t := W_t - \int_0^t \varphi(X_s) ds
\]

is a Brownian motion under \( \mathbb{P} \). Therefore, \( X_t \) follows

\[
dX_t = b(X_t) dt + \sigma(X_t) dW_t \\
= (b(X_t) + \sigma(X_t)\varphi(X_t)) dt + \sigma(X_t) dB_t.
\]

This equation gives us the dynamic of \( X_t \) under \( \mathbb{P} \).

Among all possible martingale extractions, we choose a special one, which will be useful for sensitivity analysis of long-term cash flows. The choice depends on the function \( f \) and is not unique in general.

**Definition 2.2.** Let \( (\lambda, \phi) \) be an eigenpair of \( \mathcal{L} \phi = -\lambda \phi \) with positive \( \phi \). Assume that \( (X_t, r) \) admits the martingale extraction with respect to \( (\lambda, \phi) \). We say the martingale extraction of \( (\lambda, \phi) \) stabilizes \( f \) if

\[
\mathbb{E}^\mathbb{P}[(\phi^{-1}f)(X_T)]
\]

converges to a nonzero constant as \( T \to \infty \), where \( \mathbb{P} \) is the transformed measure with respect to \( (\lambda, \phi) \).

The definition of the term ‘stabilize’ is somewhat different from the meaning used in [15]. It is noteworthy that if \( (\lambda, \phi) \) and \( (\beta, \pi) \) are two eigenpairs that induce the martingale extractions stabilizing the common \( f \), then \( \lambda = \beta \). The stabilizing martingale extraction characterizes the exponential decay (or growth) rate of the quantity \( p_T \) as \( T \to \infty \). If the martingale extraction of \( (\lambda, \phi) \) stabilizes \( f \), then

\[
\lim_{T \to \infty} \ln p_T = -\lambda.
\]

For more about the stabilizing martingale extraction, refer to Appendix A in which there are sufficient conditions for martingale extractions to stabilize the function \( f \).
3 Sensitivity on drift and volatility

We now investigate how the martingale extraction is used for the sensitivity analysis. For the rho and the vega, consider the perturbed process $X_t^\varepsilon$ expressed by
\[ dX_t^\varepsilon = b_t(X_t^\varepsilon) dt + \sigma_t(X_t^\varepsilon) dW_t \]
where $b_0(\cdot) = b(\cdot)$ and $\sigma_0(\cdot) = \sigma(\cdot)$. Assume that the perturbed process $X_t^\varepsilon$ satisfies the conditions in Assumption 1. We slightly generalize the form of perturbed quantity in equation (1.1) to small in some sense, the last three terms converges to zero as $T \to \infty$ for large $X$ the martingale extraction. We assume that $(X_t^\varepsilon, r_t^\varepsilon)$ admits a martingale extraction stabilizing $f_t$. Denote the corresponding eigenpair, the martingale exponent and the transformed measure by $(\lambda(\varepsilon), \phi_\varepsilon)$, $\varphi_\varepsilon$ and $P_\varepsilon$, respectively. Then,
\[ p_T^\varepsilon := \mathbb{E}^Q[e^{-\int_0^T r_s(X_s^\varepsilon)ds} f_t(X_T^\varepsilon)] \]
with $r_0(\cdot) = r(\cdot)$ and $f_0(\cdot) = f(\cdot)$. Then
\[ \frac{\partial}{\partial \varepsilon} \bigg|_{\varepsilon=0} p_T^\varepsilon \]
for large $T$ is of interest to us.

The perturbed quantity $p_T^\varepsilon$ can be expressed in a relatively more manageable manner by using the martingale extraction. We assume that $(X_t^\varepsilon, r_t^\varepsilon)$ admits a martingale extraction stabilizing $f_t$. Then,\[ p_T^\varepsilon = \phi_\varepsilon(\xi) e^{-\lambda(\varepsilon)T} \cdot \mathbb{E}^{P_\varepsilon}[(\phi_\varepsilon^{-1} f_t)(X_T^\varepsilon)] . \] (3.1)

We will explore $\frac{\partial}{\partial \varepsilon} \bigg|_{\varepsilon=0} p_T^\varepsilon$ by analyzing the components $\phi_\varepsilon(\xi)$, $e^{-\lambda(\varepsilon)T}$ and $\mathbb{E}^{P_\varepsilon}[(\phi_\varepsilon^{-1} f_t)(X_T^\varepsilon)]$. Differentiate with respect to $\varepsilon$ and evaluate at $\varepsilon = 0$, then\[ \left. \frac{\partial}{\partial \varepsilon} \right|_{\varepsilon=0} p_T^\varepsilon = -\lambda'(0) + \left. \frac{\partial}{\partial \varepsilon} \right|_{\varepsilon=0} \frac{\phi_\varepsilon(\xi)}{T \cdot \phi(\xi)} \right|_{\varepsilon=0} \mathbb{E}^{P}[\phi_\varepsilon^{-1} f_t(X_T^\varepsilon)] + \frac{\partial}{\partial \varepsilon} \bigg|_{\varepsilon=0} \mathbb{E}^{P_\varepsilon}[(\phi_\varepsilon^{-1} f_t)(X_T^\varepsilon)] . \] (3.2)

Since this is a stabilizing martingale extraction, we know that $\mathbb{E}^{P}[\phi_\varepsilon^{-1} f_t(X_T^\varepsilon)]$ in the denominator in the last two terms converge to a nonzero constant as $T \to \infty$. When the perturbations are small in some sense, the last three terms converges to zero as $T \to \infty$, thus we can anticipate the following simple relationship:
\[ \lim_{T \to \infty} \frac{1}{T} \left. \frac{\partial}{\partial \varepsilon} \right|_{\varepsilon=0} \ln p_T^\varepsilon = \lim_{T \to \infty} \frac{\partial}{\partial \varepsilon} \bigg|_{\varepsilon=0} \frac{p_T^\varepsilon}{T} = -\lambda'(0) . \] (3.3)

We now shift our attention to the four terms in equation (3.2). Only the last term is involved with the perturbation in the underlying process. The main contribution of this article is to control the last term. In the first term, $\lambda(\varepsilon)$ is differentiable at $\varepsilon = 0$ for many financially meaningful cases. In the second term, $\left. \frac{\partial}{\partial \varepsilon} \right|_{\varepsilon=0} \phi_\varepsilon(\xi)$ is independent of $T$. In the third term, $\left. \frac{\partial}{\partial \varepsilon} \right|_{\varepsilon=0} \mathbb{E}^{P}[\phi_\varepsilon^{-1} f_t(X_T^\varepsilon)]$ is just of an ordinary problem of differentiation and integration. Those conditions can be checked case-by-case, thus we do not go further details of the first three terms here. Assume the following conditions.
Condition 1. $\lambda(\epsilon)$ and $\phi(\xi)$ are differentiable at $\epsilon = 0$.

Condition 2. $\mathbb{E}^\mathbb{P}[(\phi^{-1}_\epsilon f:\epsilon)(X_T)]$ is differentiable at $\epsilon = 0$ and $\frac{1}{T} \cdot \frac{\partial}{\partial \epsilon} \bigg|_{\epsilon=0} \mathbb{E}^\mathbb{P}[(\phi^{-1}_\epsilon f:\epsilon)(X_T)] \to 0$ as $T \to \infty$.

These conditions are satisfied for many financially meaningful perturbations as we will see soon. It is noteworthy that we can occasionally interchange the differentiation and the integration:

$$\frac{\partial}{\partial \epsilon} \bigg|_{\epsilon=0} \mathbb{E}^\mathbb{P}[(\phi^{-1}_\epsilon f:\epsilon)(X_T)] = \mathbb{E}^\mathbb{P} \left[ \left( \frac{\partial}{\partial \epsilon} \bigg|_{\epsilon=0} \phi^{-1}_\epsilon f:\epsilon \right)(X_T) \right].$$

This holds, for example, if $h_\epsilon := \phi^{-1}_\epsilon f:\epsilon$ satisfies the hypothesis of Theorem 3.1.

To achieve the relationship in equation (3.3), we have to show that the last part satisfies

$$\frac{1}{T} \frac{\partial}{\partial \epsilon} \bigg|_{\epsilon=0} \mathbb{E}^\mathbb{P}[(\phi^{-1}_\epsilon f:\epsilon)(X_T^\epsilon)] \to 0.$$

The differentiability and the convergence to zero do not look clear. We will find sufficient conditions when this holds. The conditions for the perturbation on the drift $b_\epsilon(\cdot)$ and the volatility $\sigma_\epsilon(\cdot)$ are demonstrated in Section 3.1 and 3.2, respectively.

### 3.1 Rho

In this section, the rho of the quantity $p_T$ is investigated for large $T$. Consider the perturbed process $X^\epsilon_t$ expressed by

$$dX^\epsilon_t = b_\epsilon(X^\epsilon_t)dt + \sigma(X^\epsilon_t)dW_t,$$

where $b_0(\cdot) = b(\cdot)$. Define $k_\epsilon(x) := (\sigma^{-1}b_\epsilon + \phi_\epsilon)(x)$ and $k(x) := k_0(x)$. Assume that $k_\epsilon(x)$ is continuously differentiable at $\epsilon = 0$ for each $x$. Denote the derivative at $\epsilon = 0$ by $\overline{k}(x)$, that is,

$$\overline{k}(x) := \frac{\partial}{\partial \epsilon} \bigg|_{\epsilon=0} k_\epsilon(x).$$

We write the usual $d$-dimensional Euclidean norm by $| \cdot |$. Assume that there exists a function $g : \mathbb{R}^d \to \mathbb{R}$ such that

$$\left| \frac{\partial k_\epsilon(x)}{\partial \epsilon} \right| \leq g(x)$$

on $(\epsilon, x) \in I \times \mathbb{R}^d$ for an open interval $I$ containing 0. Refer to Appendix C, D and E for the proofs of the following theorems and the corollary.

**Theorem 3.1.** Suppose that the following conditions hold.

(i) there exists a positive number $\epsilon_0$ such that

$$\mathbb{E}^\mathbb{P} \left[ \exp \left( \epsilon_0 \int_0^T g^2(X_t) dt \right) \right] = c(T) e^{aT}$$

for some constants $a$ and $c = c(T)$ with $c(T)$ bounded on $0 < T < \infty$.

(ii) for each $T > 0$, there is a positive number $\epsilon_1$ such that $\mathbb{E}^\mathbb{P} \int_0^T g^{2+\epsilon_1}(X_t) dt$ is finite.

(iii) $\frac{1}{T} \cdot \mathbb{E}^\mathbb{P}[(\phi^{-1}f)^2(X_T)] \to 0$ as $T \to \infty$. 
Then, $\mathbb{E}^\varepsilon[(\phi^{-1} f)(X_T^\varepsilon)]$ is differentiable at $\epsilon = 0$ and
\[
\frac{1}{T} \frac{\partial}{\partial \epsilon} \bigg|_{\epsilon=0} \mathbb{E}^\varepsilon\left[(\phi^{-1} f)(X_T^\varepsilon)\right] \to 0 .
\]

In conclusion,
\[
\lim_{T \to \infty} \frac{1}{T} \frac{\partial}{\partial \epsilon} \bigg|_{\epsilon=0} \ln p_T^\varepsilon = -\lambda'(0) .
\]

**Theorem 3.2.** The $L^2$-condition (iii) on $\phi^{-1} f$ in the above theorem can be relaxed if $g$ satisfies a stronger condition. Condition (ii) and (iii) can be replaced by the following way.

(ii)' for each $T > 0$ and $n \in \mathbb{N}$, $\mathbb{E}^T \int_0^T g^n(X_s) \, ds$ is finite,

(iii)' $\sqrt{T} \cdot \mathbb{E}^T[(\phi^{-1} f)^{1+\epsilon_2}(X_T)] \to 0$ as $T \to \infty$ for some positive $\epsilon_2$.

**Corollary 3.3.** We have that $\mathbb{E}^\varepsilon\left[(\phi^{-1} f)(X_T^\varepsilon)\right]$ is differentiable at $\epsilon = 0$ and
\[
\frac{1}{T} \frac{\partial}{\partial \epsilon} \bigg|_{\epsilon=0} \mathbb{E}^\varepsilon\left[(\phi^{-1} f)(X_T^\varepsilon)\right] \to 0
\]
if there exists a positive number $\epsilon_0$ such that $\mathbb{E}[\exp(\epsilon_0 g^2(X_T))]$ is finite on $0 < T < \infty$ and if (iii)' is satisfied.

### 3.2 Vega

#### 3.2.1 The Lamperti transform for univariate processes

In this section, assume that the underlying process $X_t$ is a one-dimensional process. Let $X_t^\varepsilon$ be a perturbed process expressed by
\[
dX_t^\varepsilon = b_\varepsilon(X_t^\varepsilon) \, dt + \sigma_\varepsilon(X_t^\varepsilon) \, dW_t, \quad X_0^\varepsilon = X_0 = \xi ,
\]
with $b_0 = b$ and $\sigma_0 = \sigma$. This form of perturbation covers the vega. The initial value is not perturbed. We are interested in the perturbed quantity $p_t^\varepsilon$ given by equation (3.1). Because it is difficult to analyze the volatility term, we use the Lamperti transform to convert the perturbation of volatility into perturbations of drift. Define a function
\[
u_\varepsilon(x) := \int_\xi^x \sigma_\varepsilon^{-1}(y) \, dy ,
\]
then we have
\[
du_\varepsilon(X_t^\varepsilon) = (\sigma_\varepsilon^{-1}b_\varepsilon - \frac{1}{2} \sigma_\varepsilon^2)(X_t^\varepsilon) \, dt + dW_t , \quad u_\varepsilon(X_0^\varepsilon) = u_\varepsilon(\xi) = 0 .
\]
Here, $\sigma_\varepsilon(x)$ is assumed to be a continuously differentiable function of $x$. We denote the inverse function of $u_\varepsilon(\cdot)$ by $v_\varepsilon(\cdot)$. Set $U_t^\varepsilon := u_\varepsilon(X_t^\varepsilon)$, then
\[
dU_t^\varepsilon = \delta_\varepsilon(U_t^\varepsilon) \, dt + dW_t , \quad U_0^\varepsilon = 0 ,
\]
where $\delta_\varepsilon(\cdot) := (\sigma_\varepsilon^{-1}b_\varepsilon - \frac{1}{2} \sigma_\varepsilon^2) \circ v_\varepsilon(\cdot)$. The perturbation of form (3.4) is transformed into a perturbation in drift.

\[
p_T^\varepsilon := \mathbb{E}^\mathbb{Q}[e^{-\int_0^T r_\varepsilon(U_s^\varepsilon) \, ds} f_t(X_T^\varepsilon)] = \mathbb{E}^\mathbb{Q}[e^{-\int_0^T (r_\varepsilon \circ v_\varepsilon)(U_s^\varepsilon) \, ds} (f_t \circ v_\varepsilon)(U_T^\varepsilon)] 
\]
\[
= \mathbb{E}^\mathbb{Q}[e^{-\int_0^T R_s(U_s^\varepsilon) ds} F_t(U_T^\varepsilon)]
\]
where \( R_\epsilon := r \circ v_\epsilon \) and \( F_\epsilon := f \circ v_\epsilon \). In conclusion, the behavior of the long-term vega is obtained by applying Theorem 3.1, 3.2 or Corollary 3.3 to \( U_\epsilon \), \( R_\epsilon \) and \( F_\epsilon \).

There is an invariant property between \( (X_\epsilon^t, r_\epsilon, f_\epsilon) \) and \( (U_\epsilon^t, R_\epsilon, F_\epsilon) \). Suppose \( (X_\epsilon^t, r_\epsilon) \) admits the martingale extraction stabilizing \( f_\epsilon \) with the eigenpair \( (\lambda(\epsilon), \phi_\epsilon) \) and the martingale exponent \( \varphi_\epsilon \). Then \( (U_\epsilon^t, R_\epsilon) \) admits the martingale extraction stabilizing \( F_\epsilon \) with the eigenpair \( (\lambda(\epsilon), \phi_\epsilon \circ v_\epsilon) \) and the martingale exponent \( \varphi_\epsilon \circ v_\epsilon \).

For the remainder of this section, we introduce a slight variation of the Lamperti transform \([3.5]\). For a real number \( c \), define

\[
u_\epsilon(x) := \int_c^x \sigma_\epsilon^{-1}(y) \, dy.
\]

Denote the inverse function of \( u_\epsilon(\cdot) \) by \( v_\epsilon(\cdot) \). Set \( U_\epsilon^t := u_\epsilon(X_\epsilon^t) \) and \( q(\epsilon) := U_0^\epsilon = \int_c^\epsilon \sigma_\epsilon^{-1}(y) \, dy \), then

\[
dU_\epsilon^t = \delta_\epsilon(U_\epsilon^t) \, dt + dW_t, \quad U_0^\epsilon = q(\epsilon),
\]

where \( \delta_\epsilon(\cdot) := \left((\sigma_\epsilon^{-1} b_\epsilon - \frac{1}{2}(\sigma_\epsilon') \circ v_\epsilon)(\cdot)\right) \). By choosing suitable \( c \), one can find a simple form of \( \delta_\epsilon \), which is useful for the sensitivity analysis. However, different from the previous transform \([3.5]\), the initial value is perturbed. Thus, the sensitivity analysis of the initial value is required. Let \( R_\epsilon := r_\epsilon \circ v_\epsilon, F_\epsilon := f_\epsilon \circ v_\epsilon \) as before and let \( \Phi_\epsilon := \phi_\epsilon \circ v_\epsilon \). Then

\[
p_\epsilon^T = E_\xi^Q\left[e^{-\int_0^T r(X_\epsilon^t) \, ds} f(X_\epsilon^T)\right]
= E_{q(\epsilon)}^Q\left[e^{-\int_0^T R_\epsilon(U_\epsilon^t) \, ds} F_\epsilon(U_\epsilon^T)\right]
= \phi_\epsilon(\xi) \, e^{-\lambda T} \cdot E_{q(\epsilon)}^P \left[ (\Phi_\epsilon^{-1} F_\epsilon)(U_\epsilon^T) \right].
\]

By applying the chain rule, we have

\[
\frac{\partial}{\partial \epsilon} \bigg|_{\epsilon=0} p_\epsilon^T = \frac{\partial}{\partial \epsilon} \bigg|_{\epsilon=0} \phi_\epsilon(\xi) \, e^{-\lambda T} \cdot E_{q(\epsilon)}^P \left[ (\Phi_\epsilon^{-1} F_\epsilon)(U_\epsilon^T) \right]
+ \phi(\xi) \, e^{-\lambda T} \frac{\partial}{\partial \epsilon} \bigg|_{\epsilon=0} E_{q(\epsilon)}^P \left[ (\Phi_\epsilon^{-1} F)(U_T) \right].
\]

The first term can be analyzed by the method in Section 3.1 because the initial value and the volatility are not perturbed. The second term is involved with the perturbation of initial value and can be analyzed, for example, by Theorem 3.1 in Section 4 below.

### 3.2.2 The Fournie method with bounded-derivative coefficients

We present how the Fournie method can be applied to the sensitivity analysis with respect to the perturbation in volatility. In this section, we consider the following perturbed process \( X_\epsilon^t \):

\[
dx_\epsilon^t = b(X_\epsilon^t) \, dt + (\sigma + \epsilon \bar{\sigma})(X_\epsilon^t) \, dW_t
\]

and assume the hypothesis of the paper of Fournie \([9]\). The coefficients \( b, \sigma \) and \( \bar{\sigma} \) are continuously differentiable with bounded derivatives. The diffusion matrix \( \sigma + \epsilon \bar{\sigma} \) satisfies the uniform ellipticity condition for small \( \epsilon \geq 0 \).

Consider the martingale extraction. Under the corresponding transformed measure \( P_\epsilon \), the dynamics of \( X_\epsilon^t \) satisfies

\[
dx_\epsilon^t = (b + (\sigma + \epsilon \bar{\sigma}) \varphi_\epsilon)(X_\epsilon^t) \, dt + (\sigma + \epsilon \bar{\sigma})(X_\epsilon^t) \, dB^\epsilon_t
\]
with a Brownian motion \( B_t^\varepsilon \) on \( \mathbb{P}_\varepsilon \). Thus, the perturbation is induced by two parts: the drift term and the volatility term. We take apart two perturbations by the chain rule. Let \( X_t^\rho \) and \( X_t^\nu \) be the processes corresponding to the perturbations in the drift and in the volatility, respectively:

\[
\begin{align*}
dX_t^\rho &= (b + (\sigma + \rho \varphi))(X_t^\rho) \, dt + \sigma(X_t^\rho) \, dB_t^\rho, \\
dX_t^\nu &= (b + \sigma \varphi)(X_t^\nu) \, dt + (\sigma + \nu \varphi)(X_t^\nu) \, dB_t^\nu.
\end{align*}
\]

Then we have

\[
\frac{\partial}{\partial \varepsilon} \bigg|_{\varepsilon = 0} \mathbb{E}^{\mathbb{P}_\varepsilon}[(\phi^{-1} f)(X_T^\varepsilon)] = \frac{\partial}{\partial \rho} \bigg|_{\rho = 0} \mathbb{E}^{\mathbb{P}_\varepsilon}[(\phi^{-1} f)(X_T^\rho)] + \frac{\partial}{\partial \nu} \bigg|_{\nu = 0} \mathbb{E}^{\mathbb{P}_\nu}[(\phi^{-1} f)(X_T^\nu)].
\]

The perturbation in the drift term can be analyzed by the method in Section 3.1.

We now shift our attention to the perturbation in the volatility term. The main purpose of this section is to use the result of Fournie to investigate when the second term

\[
\frac{1}{T} \frac{\partial}{\partial \nu} \bigg|_{\nu = 0} \mathbb{E}^{\mathbb{P}_\nu}[(\phi^{-1} f)(X_T^\nu)]
\]

goes to zero as \( T \to \infty \). Suppose that \( b + \sigma \varphi \) and \( \phi^{-1} f \) are continuously differentiable with bounded derivatives. Then

\[
\frac{\partial}{\partial \varepsilon} \bigg|_{\nu = 0} \mathbb{E}^{\mathbb{P}_\nu}[(\phi^{-1} f)(X_T^\nu)] = \mathbb{E}^{\mathbb{P}}[\nabla(\phi^{-1} f)(X_T^\nu) Z_T].
\]

Here, \( Z_t \) is the variation process given by

\[
dZ_t = (b + \sigma \varphi)'(X_t) Z_t \, dt + \sigma(X_t) dB_t + \sum_{i=1}^d \sigma_i'(X_t) Z_t \, dB_{i,t}, \quad Z_0 = 0_d
\]

where \( \sigma_i \) is the \( i \)-th column vector of \( \sigma \) and \( 0_d \) is the \( d \)-dimensional zero column vector. From this observation, we have the following theorem.

**Theorem 3.4.** Suppose that \( b + \sigma \varphi \) and \( \phi^{-1} f \) are continuously differentiable with bounded derivatives. If \( \frac{1}{T} \cdot \mathbb{E}^{\mathbb{P}}[|Z_T|] \to 0 \) as \( T \to \infty \), then

\[
\lim_{T \to \infty} \frac{1}{T} \frac{\partial}{\partial \nu} \bigg|_{\nu = 0} \mathbb{E}^{\mathbb{P}_\nu}[(\phi^{-1} f)(X_T^\nu)] = 0.
\]

## 4 Sensitivity on initial value

The sensitivity analysis with respect to the initial perturbation is presented. Set \( p_T := \mathbb{E}^{\mathbb{Q}_\xi}[e^{-\int_0^T r(X_s) \, ds} f(X_T)] \), then the quantity of interest is

\[
\nabla_\xi p_T
\]

for large \( T \). Applying the martingale extraction, by equation (2.1), it follows that

\[
\frac{\nabla_\xi p_T}{p_T} = \frac{\nabla_\xi \phi}{\phi(\xi)} + \frac{\nabla_\xi \mathbb{E}^{\mathbb{P}}[(\phi^{-1} f)(X_T)]}{\mathbb{E}^{\mathbb{P}}[(\phi^{-1} f)(X_T)]}. 
\]
Theorem 4.1. Suppose that $\phi(\xi)$ and $\mathbb{E}_\xi^\mathbb{P}[(\phi^{-1}f)(X_T)]$ are differentiable functions of $\xi$. If $|\nabla_\xi \mathbb{E}_\xi^\mathbb{P}[(\phi^{-1}f)(X_T)]| \leq 0$ as $T \to \infty$, then
\[
\lim_{T \to \infty} \frac{\nabla_\xi p_T}{p_T} = \frac{\nabla_\xi \phi}{\phi(\xi)}.
\]

Corollary 4.2. Assume that the functions $b + \sigma \varphi$ and $\sigma$ are continuously differentiable with bounded derivatives. If $\mathbb{E}_\xi^\mathbb{P}[(\phi^{-1}f)(X_T)]$ and $\mathbb{E}_\xi^\mathbb{P}\|\sigma^{-1}(X_T)Y_T\|^2$ are bounded on $0 < T < \infty$, then $\mathbb{E}_\xi^\mathbb{P}[(\phi^{-1}f)(X_T)]$ is differentiable by $\xi$ and $|\nabla_\xi \mathbb{E}_\xi^\mathbb{P}(\phi^{-1}f)(X_T)| \to 0$ as $T \to \infty$. Here, $\| \cdot \|$ is the matrix $2$-norm and $Y_t$ is the first variation process defined by
\[
dY_t = (b + \sigma \varphi)'(X_t)Y_t \, dt + \sum_{i=1}^d \sigma_i'(X_t)Y_t \, dB_{i,t} , \quad Y_0 = I_d
\]
where $\sigma_i$ is the $i$-th column vector of $\sigma$ and $I_d$ is the $d \times d$ identity matrix.

This theorem is obtained from the result of Fournie [9]. Refer to Appendix E for proof. The $L^2$-condition on $\mathbb{E}_\xi^\mathbb{P}(\phi^{-1}f)^2(X_T)$ can be relaxed when $\mathbb{E}_\xi^\mathbb{P}(\|\sigma^{-1}(X_T)Y_T\|^n)$ is bounded on $0 < T < \infty$ for a larger number $n$.

5 Examples of option prices

5.1 The geometric Brownian motion

Consider the classical Black-Scholes model. The short interest is constant $r$ and the stock price, denoted by $S_t$, follows a geometric Brownian motion:
\[
dS_t = \mu S_t \, dt + \sigma S_t \, dW_t
\]
with $\mu - \frac{1}{2} \sigma^2 > 0$. In this section, we assume that the payoff function $f_\alpha : [0, \infty) \to \mathbb{R}$ for $\alpha > 0$ is a continuous function with growth rate $s^\alpha$ as $s \to \infty$, that is, $\lim_{s \to \infty} f_\alpha(s)/s^\alpha$ exists and is nonzero constant. For example, $f_\alpha(s) = s^\alpha$, $f_\alpha(s) = (s^\alpha - K)_+$ or $f_\alpha(s) = (s - K)^\alpha_+$ for a positive $K$. Set
\[
p_T = \mathbb{E}^\mathbb{Q}[\exp(-rT f_\alpha(S_T))].
\]

We analyze the sensitivity analysis of the long-term option prices with payoff function $f_\alpha$. Consider the corresponding infinitesimal generator
\[
(\mathcal{L}\phi)(s) = \frac{1}{2} \sigma^2 s^2 \phi''(s) + \mu s \phi'(s) - r \phi(s).
\]
It can be shown that the martingale extraction with respect to
\[
(\lambda, \phi(s)) := (r - \mu \alpha - \frac{1}{2} \sigma^2 \alpha (\alpha - 1), s^\alpha)
\]
stabilizes $f_\alpha$. With this $(\lambda, \phi)$, we conclude that
\[
\lim_{T \to \infty} \frac{1}{T} \frac{\partial}{\partial \mu} \ln p_T = -\frac{\partial \lambda}{\partial \mu} = \alpha,
\]
\[
\lim_{T \to \infty} \frac{1}{T} \frac{\partial}{\partial \sigma} \ln p_T = -\frac{\partial \lambda}{\partial \sigma} = \sigma \alpha (\alpha - 1),
\]
\[
\lim_{T \to \infty} \frac{\partial}{\partial S_0} \ln p_T = \frac{\phi'(S_0)}{\phi(S_0)} = \frac{\alpha}{S_0}.
\]
The analysis also can be applied to the expected utility of an investor. Suppose that $Q$ is an objective measure, $f_\alpha(s) = s^\alpha$ with $0 < \alpha < 1$ is the utility function of the investor and $r$ is the discount rate of the investor. Then $p_T$ is the discounted expected utility. Thus we can obtain the sensitivity of the expected utility of the long-term investor.

5.2 The CIR model

We explore the sensitivity analysis of option prices whose underlying process is the Cox–Ingersoll–Ross (CIR) model. Under a risk-neutral measure $Q$, the interest rate $r_t$ follows

$$dr_t = (\theta - ar_t) dt + \sigma \sqrt{r_t} dW_t$$

with $\theta, \sigma > 0$ and $a \in \mathbb{R}$. We assume $2\theta > \sigma^2$ so that the original interest rate process and the perturbation process stay strictly positive for small perturbation.

The quantity

$$p_T := \mathbb{E}_Q[e^{-\int_0^T r_t dt} f(r_T)]$$

for large $T$ is of interest to us. Here, $f$ is a payoff function and we assume that $f(r)$ is a nonnegative continuous function on $r \in [0, \infty)$, which is not identically zero, and that the growth rate at infinity is equal to or less than $e^{mr}$ with $m < \frac{a}{\sigma^2}$. The associated second-order equation is

$$\mathcal{L}\phi(r) = \frac{1}{2} \sigma^2 r \phi''(r) + (\theta - ar) \phi'(r) - r \phi(r) = -\lambda \phi(r).$$

Set $\kappa := \frac{\sqrt{a^2 + 2\sigma^2} - a}{\sigma^2}. \,$ It can be shown that the martingale extraction with respect to

$$(\lambda, \phi(r)) := (\theta \kappa, e^{-\kappa r})$$

stabilizes $f$.

By using this $(\lambda, \phi)$, the sensitivities of the quantity can be analyzed. The sensitivities of the long-term option prices with respect to $\theta$, $a$, $\sigma$ and $r_0$ are given by

$$\lim_{T \to \infty} \frac{1}{T} \frac{\partial}{\partial \theta} \ln p_T = -\frac{\partial \lambda}{\partial \theta} = -\frac{\sqrt{a^2 + 2\sigma^2} - a}{\sigma^2},$$

$$\lim_{T \to \infty} \frac{1}{T} \frac{\partial}{\partial a} \ln p_T = -\frac{\partial \lambda}{\partial a} = \frac{\theta(\sqrt{a^2 + 2\sigma^2} - a)}{\sigma^2 \sqrt{a^2 + 2\sigma^2}},$$

$$\lim_{T \to \infty} \frac{1}{T} \frac{\partial}{\partial \sigma} \ln p_T = -\frac{\partial \lambda}{\partial \sigma} = \frac{\theta(\sqrt{a^2 + 2\sigma^2} - a)^2}{\sigma^3 \sqrt{a^2 + 2\sigma^2}},$$

$$\lim_{T \to \infty} \frac{\partial}{\partial r_0} \ln p_T = \frac{\phi'(r_0)}{\phi(r_0)} = -\frac{\sqrt{a^2 + 2\sigma^2} - a}{\sigma^2}.$$ 

For more details about the sensitivity analysis of the CIR model, refer to Appendix G.

5.3 Quadratic models

We present the sensitivity analysis of short-interest option prices whose underlying process is a quadratic term structure model. This section is indebted to [32]. Suppose $X_t$ is a $d$-dimensional OU process satisfying the SDE under a risk-neutral measure $Q$:

$$dX_t = (b + BX_t) dt + \sigma dW_t$$
where $b$ is a $d$-dimensional column vector, $B$ is a $d \times d$ matrix, and $\sigma$ is a non-singular $d \times d$ matrix, so that $a = \sigma \sigma^T$ is strictly positive definite. The short interest rate is given by
\[ r(x) = \beta + \langle \alpha, x \rangle + \langle \Gamma x, x \rangle \]
where the constant $\beta$, vector $\alpha$ and symmetric positive definite $\Gamma$ are taken to be such that the short interest rate is non-negative for all $x \in \mathbb{R}^d$.

The quantity
\[ p_T = E^Q[e^{-\int_0^T r(X_t) dt} f(X_T)] \]
is the option price with payoff function $f$ and we explore the sensitivity analysis of this quantity with respect to a perturbation of the underlying process $X_t$. Assume that $f$ is bounded and has a bounded support on $\mathbb{R}^d$. Let $V$ be the stabilizing solution of
\[ 2VaV - B^TV - VB - \Gamma = 0 , \]
then it is well-known that $B - 2aV$ is non-singular and the eigenvalues of $B - 2aV$ have negative real parts. Define a vector $u$ by
\[ u = (2Va - B^T)^{-1}(2Vb + \alpha) . \]

It can be shown that the martingale extraction with respect to
\[ (\lambda, \phi(x)) = (\beta - \frac{1}{2} u^T au + tr(aV) + u^T b, e^{-\langle u, x \rangle - \langle Vx, x \rangle}) \]
stabilizes $f$. The sensitivity of the quantity $p_T$ for large $T$ is given by
\[ \lim_{T \to \infty} \frac{1}{T} \frac{\partial}{\partial b_i} \ln p_T = \frac{\partial \lambda}{\partial b_i} , \quad \lim_{T \to \infty} \frac{1}{T} \frac{\partial}{\partial B_{ij}} \ln p_T = \frac{\partial \lambda}{\partial B_{ij}} , \quad \lim_{T \to \infty} \frac{\nabla \xi p_T}{p_T} = -u - 2V \xi \]
for $1 \leq i, j \leq d$. If $f$ is continuously differentiable with compact support, then we have
\[ \lim_{T \to \infty} \frac{1}{T} \frac{\partial}{\partial \sigma_i} \ln p_T = \frac{\partial \lambda}{\partial \sigma_i} . \]

Refer to Appendix H for more details.

6 Examples of expected utilities

6.1 The Heston model

The sensitivity analysis of the expected utility with respect to the parameters of the Heston model is presented. Under the objective measure $Q$, suppose that an asset $X_t$ follows
\[ dX_t = \mu X_t dt + \sqrt{v_t} X_t dZ_t , \]
\[ dv_t = (\gamma - \beta v_t) dt + \delta \sqrt{v_t} dW_t , \]
where $Z_t$ and $W_t$ are two standard Brownian motions with $\langle Z, W \rangle_t = \rho t$ for the correlation $-1 \leq \rho \leq 1$. Assume that $\mu, \gamma, \beta, \delta > 0$ and $2\gamma > \delta^2$. We consider a power utility function of the form
\[ u(c) = c^\alpha , \quad 0 < \alpha < 1 . \]
The sensitivity of the quantity

\[ p_T := \mathbb{E}^Q[u(X_T)] = \mathbb{E}^Q[X_T^\alpha] = \mathbb{E}^Q[e^{\alpha\int_0^T \sqrt{\sigma^2(Z_t)^2 - \frac{\alpha}{2}} f_t^T \mu_t \, dt}] e^{\alpha t} X_0^\alpha \]

for large \( T \) is of interest to us. We have that

\[
\begin{align*}
\lim_{T \to \infty} \frac{1}{T} \frac{\partial}{\partial \mu} \ln p_T &= \alpha \\
\lim_{T \to \infty} \frac{1}{T} \frac{\partial}{\partial \gamma} \ln p_T &= -\frac{1}{2} \alpha(1 - \alpha) \cdot \frac{\sqrt{(\beta - \rho \alpha \delta)^2 + \delta^2 \alpha (1 - \alpha)} - \beta + \rho \alpha \delta}{\delta} \\
\lim_{T \to \infty} \frac{1}{T} \frac{\partial}{\partial \beta} \ln p_T &= \frac{\sqrt{(\beta - \rho \alpha \delta)^2 + \delta^2 \alpha (1 - \alpha)} - \beta + \rho \alpha \delta}{\delta^2 \sqrt{(\beta - \rho \alpha \delta)^2 + \delta^2 \alpha (1 - \alpha)}} \\
&\quad + \frac{(\sqrt{(\beta - \rho \alpha \delta)^2 + \delta^2 \alpha (1 - \alpha)} - \beta + \rho \alpha \delta)^2}{\delta^3 \sqrt{(\beta - \rho \alpha \delta)^2 + \delta^2 \alpha (1 - \alpha)}} \\
\lim_{T \to \infty} \frac{1}{T} \frac{\partial}{\partial \rho} \ln p_T &= -\alpha \cdot \frac{\sqrt{(\beta - \rho \alpha \delta)^2 + \delta^2 \alpha (1 - \alpha)} - \alpha \beta + \rho \alpha^2 \delta}{\delta \sqrt{(\beta - \rho \alpha \delta)^2 + \delta^2 \alpha (1 - \alpha)}} \\
\lim_{T \to \infty} \frac{\partial}{\partial X_0} \ln p_T &= \frac{\alpha}{X_0} \\
\lim_{T \to \infty} \frac{\partial}{\partial v_0} \ln p_T &= -\frac{1}{2} \alpha(1 - \alpha) \cdot \frac{\sqrt{(\beta - \rho \alpha \delta)^2 + \delta^2 \alpha (1 - \alpha)} - \beta + \rho \alpha \delta}{\delta^2}.
\end{align*}
\]

Refer to Appendix I for the details.

6.2 The 3/2 LEFT model

The sensitivity analysis of the expected utility and the return of an exchange-traded fund (ETF) is explored. We investigate the leveraged ETF (LETF), which promises a fixed leverage ratio with respect to a given underlying asset or index process \( X_t \). Assume that \( X_t \) stays positive. A long-leveraged ETF \( L_t \) on \( X_t \) with leverage ratio \( \beta \geq 1 \) is constructed by the following way. At time \( t \), the cash amount of \( \beta L_t \) (\( \beta \) times the fund value) is invested in \( X_t \), and the amount \((\beta - 1)L_t \) is borrowed at the risk-free rate \( r \). For a short-leveraged ETF \( L_t \) with ratio \( \beta \leq -1 \), the cash amount of \(|\beta|L_t \) is shorted on \( X_t \), and \((1 - \beta)L_t \) is kept in the money market account with the risk-free rate \( r \). In practice, most typical leverage ratios are \( \beta = 1, 2, 3 \) (long) and \( \beta = -1, -2, -3 \) (short), thus we assume \(|\beta| \leq 3 \). The LEFT \( L_t \) satisfies

\[
\frac{dL_t}{L_t} = \beta \left( \frac{dX_t}{X_t} \right) - ((\beta - 1)r) \, dt \\
= \left( \beta \left( \frac{\mu(X_t)}{X_t} \right) - (\beta - 1)r \right) \, dt + \beta \left( \frac{\sigma(X_t)}{X_t} \right) \, dB_t
\]

and can be written by

\[
L_t = \left( \frac{X_t}{X_0} \right)^\beta e^{-r(\beta - 1)t - \frac{\beta(\beta - 1)}{2} \int_0^t \sigma^2(X_u)/X_u^2 \, du}.
\]

(6.1)
In this section, we assume $X_0 = L_0 = 1$.

The underlying process $X_t$ and the utility function are as follows. The dynamics of $X_t$ is given by the 3/2 model

$$dX_t = (\theta - aX_t)X_t \, dt + \sigma X_t^{3/2} \, dW_t$$

with $\theta, a, \sigma > 0$ under the objective measure $Q$. This process stays positive and is recurrent. As a practical example, one can consider the leveraged volatility-index EFT. We consider a power function of the form

$$u(c) = c^\alpha, \quad 0 < \alpha \leq 1.$$ 

The sensitivity analysis of the quantity

$$p_T := \mathbb{E}^Q[u(L_T)]$$

is of interest to us. This quantity is the expected utility of $L_T$ if $0 < \alpha < 1$ and is the expected return of $L_T$ if $\alpha = 1$. For the sensitivity on $\theta$,

$$\lim_{T \to \infty} \frac{1}{T} \frac{\partial}{\partial \theta} \ln p_T = -\sqrt{\left(\frac{1}{2} + \frac{a}{\sigma^2}\right)^2 + \alpha \beta (\beta - 1) - \left(\frac{1}{2} + \frac{a}{\sigma^2}\right)}.$$ 

For sensitivities on $a$ and $\sigma$, we have

$$\lim_{T \to \infty} \frac{1}{T} \frac{\partial}{\partial a} \ln p_T = \frac{\theta \left(\sqrt{\left(\frac{1}{2} + \frac{a}{\sigma^2}\right)^2 + \alpha \beta (\beta - 1) - \left(\frac{1}{2} + \frac{a}{\sigma^2}\right)}\right)}{\sigma^2 \sqrt{\left(\frac{1}{2} + \frac{a}{\sigma^2}\right)^2 + \alpha \beta (\beta - 1)}}$$

and

$$\lim_{T \to \infty} \frac{1}{T} \frac{\partial}{\partial \sigma} \ln p_T = \frac{2a \theta \left(\sqrt{\left(\frac{1}{2} + \frac{a}{\sigma^2}\right)^2 + \alpha \beta (\beta - 1) - \left(\frac{1}{2} + \frac{a}{\sigma^2}\right)}\right)}{\sigma^3 \sqrt{\left(\frac{1}{2} + \frac{a}{\sigma^2}\right)^2 + \alpha \beta (\beta - 1)}}$$

when $\frac{a}{\sigma^2} + 1 - \alpha \beta > 0$. Refer to Appendix J for more details.

7 Conclusion

In this article, the sensitivity analysis of long-term cash flows was investigated. We explored the sensitivity of $p_T = \mathbb{E}^Q[e^{-\int_0^T r(X_t) \, dt} f(X_T)]$ with respect to the perturbation on the process $X_t$. Essentially, two types of perturbation were presented. First, we discussed the sensitivities with respect to the perturbation on the drift and the volatility. Under the assumption that the perturbed process $X^\epsilon_t$ and the function $r$ admits the martingale extraction stabilizing $f$, the perturbed quantity $p^\epsilon_T$ was transformed into what is easier to address

$$p^\epsilon_T = \phi^\epsilon(\xi) e^{-\lambda^\epsilon T} \cdot \mathbb{E}^P[\phi^{-1} f(X^\epsilon_T)]$$

with an eigenpair $(\lambda^\epsilon, \phi^\epsilon)$ and the transformed measure $P^\epsilon$. The method of Fournie was useful to analyze the last component in the above expression of $p^\epsilon_T$. We proved that the sensitivity of $p^\epsilon_T$ on $\epsilon$ is expressed in the a simple form for large $T$ under some conditions:

$$\lim_{T \to \infty} \frac{1}{T} \frac{\partial}{\partial \epsilon} \bigg|_{\epsilon=0} \ln p^\epsilon_T = -\lambda'(0).$$
Second, the sensitivity to the initial value $X_0$ was investigated. Assuming that the process $X_t$ and the function $r$ admits the martingale extraction stabilizing $f$, the quantity $p_T$ was expressed by

$$p_T = \phi(\xi) e^{-\lambda T} \cdot \mathbb{E}^P[(\phi^{-1} f)(X_T)]$$

with an eigenpair $(\lambda, \phi)$ and the transformed measure $\mathbb{P}$. It was shown that the sensitivity of $p_T$ is expressed in the following simple form for large $T$ under appropriate conditions:

$$\lim_{T \to \infty} \frac{\nabla_\xi p_T}{p_T} = \frac{\nabla_\xi \phi}{\phi(\xi)}.$$ 

We suggest the following extension for further research. It would be interesting to find the sensitivities of $p_T$ with path-dependent functionals $f(\cdot)$ instead of $f(X_T)$, which depends only on the final time $T$. It is straightforward to extend the results in the paper to discretely monitored functionals $f(X_{t_1}, X_{t_2}, \ldots, X_{t_m})$. However, it will be challenging to find the sensitivities for a general form of functionals including the payoff form of barrier and American options.

### A Stabilizing martingale extractions

We investigate sufficient conditions on $f$ such that the martingale extraction stabilizes $f$ when a martingale extraction is given.

**Theorem A.1.** Suppose that $(X_t, r)$ admits the martingale extraction of $(\lambda, \phi)$. If $X_t$ is positive recurrent under the transformed measure $\mathbb{P}$ with respect to $(\lambda, \phi)$ and if $\phi^{-1} f$ is nonzero and bounded, then the martingale extraction of $(\lambda, \phi)$ stabilizes $f$. In this case,

$$\lim_{T \to \infty} \mathbb{E}^P[(\phi^{-1} f)(X_T)] = \int \phi^{-1} f \, d\nu$$

where $\nu$ is the invariant probability of $X_t$ under $\mathbb{P}$.

The condition that $\phi^{-1} f$ is bounded can be relaxed by using the $L^2$-ergodic property or the Lyapunov criteria. For convenience, put $h := \phi^{-1} f$.

**Theorem A.2.** ($L^2$-ergodicity)

Assume that $X_t$ has an invariant probability $\nu$ under $\mathbb{P}$. For $h \in L^2(\nu)$, we have

$$\lim_{T \to \infty} \mathbb{E}^P[h(X_T)] = \int h \, d\nu$$

$\xi$-pointwise and in $L^2(\nu)$.

**Theorem A.3.** (Lyapunov criteria)

Assume that $X_t$ has an invariant measure $\nu$ (not necessarily a probability) under $\mathbb{P}$. Let $h \geq 0$. If there are constants, $a > 0$ and $b < \infty$, such that

$$\mathcal{L}^\mathbb{P} h(x) \leq -ah(x) + b\mathbb{I}_K(x),$$

where $\mathcal{L}^\mathbb{P}$ is the infinitesimal generator of $X_t$ under $\mathbb{P}$, and $K$ is a compact set, then

$$\lim_{T \to \infty} \mathbb{E}^\mathbb{P}[h(X_T)] = \int h \, d\nu.$$
For more details, refer to [27].

We can apply spectral theory to explore another condition that possesses a martingale extraction that stabilizes $f$ when $X_t$ is a one-dimensional process. Consider the speed measure $\mu$ of $X_t$ under $Q$ defined by $d\mu := w(x)dx$, where

$$w(x) = \frac{1}{\sigma^2(x)}e^{\int x \frac{2b(z)}{\sigma^2(z)}dz}.$$  

It is well known that the infinitesimal generator $L$ is a densely defined symmetric nonpositive operator from $L^2(\mu)$ to itself. Let $A$ be a self-adjoint extension of $-L$. Denote the domain of $A$ by $\text{Dom}(A)$, which is in $L^2(\mu)$.

**Theorem A.4.** Suppose that the operator $A$ has at least one eigenvalue. Assume that the spectral gap is positive when $A$ has a continuum spectrum. Let $\beta$ be the minimum eigenvalue and denote its eigenfunction by $\phi$. Assume that $M_t$, induced by $(\beta, \phi)$, is a martingale. If $f \in \text{Dom}(A)$ and $f \geq 0$, $f \neq 0$, then the martingale extraction with respect to $(\beta, \phi)$ stabilizes $f$. In this case,

$$\lim_{T \to \infty} \mathbb{E}^P[h(X_T)] = \langle f, \phi \rangle / \langle \phi, \phi \rangle$$

where

$$\langle f, g \rangle := \int fg d\mu .$$

Refer to [3], [10], [19] and [35] for more details.

**B  Perturbation of payoff function**

In this section, we are interested in a sufficient condition that the differentiation and expectation are interchangeable:

$$\frac{\partial}{\partial \epsilon} \bigg|_{\epsilon=0} \mathbb{E}^P[h_\epsilon(X)] = \mathbb{E}^P \left[ \frac{\partial}{\partial \epsilon} \bigg|_{\epsilon=0} h_\epsilon(X) \right].$$

The following theorem is a well-known fact and it is noteworthy because we will use this theorem frequently.

**Theorem B.1.** Let $X$ be a random variable and let $h_\epsilon(x)$ be a continuously differentiable function at $\epsilon = 0$ for each $x$. Suppose that there exists a random variable $G$ such that

$$\left| \frac{\partial}{\partial \epsilon} h_\epsilon(X) \right| \leq G \text{ on } (\epsilon, x) \in I \times \mathbb{R}^d \text{ for an open interval } I \text{ containing } 0$$

and

$$\mathbb{E}^P[G] < \infty .$$

Then, $\mathbb{E}^P[h_\epsilon(X)]$ is differentiable at $\epsilon = 0$ and

$$\frac{\partial}{\partial \epsilon} \bigg|_{\epsilon=0} \mathbb{E}^P[h_\epsilon(X)] = \mathbb{E}^P \left[ \frac{\partial}{\partial \epsilon} \bigg|_{\epsilon=0} h_\epsilon(X) \right].$$
C Proof of Theorem 3.1

We first prove the following proposition. This proposition is a generalization of the result of Fournie and gives an implication how to control the last term in equation (3.2) in Section 3.

**Proposition C.1.** Suppose that $\mathbb{E}^P[\exp(\epsilon_1 \int_0^T g^2(X_t) \, dt)]$ and $\mathbb{E}^P \int_0^T g^{2+\epsilon_1}(X_t) \, dt$ are finite for some positive $\epsilon_1$. Then for any given function $f(\cdot)$ with $\mathbb{E}^P[(\phi^{-1} f)^2(X_T)] < \infty$, we have

$$\frac{\partial}{\partial \epsilon} \bigg|_{\epsilon = 0} \mathbb{E}^P[(\phi^{-1} f)(X_T^\epsilon)] = \mathbb{E}^P(\phi^{-1} f)(X_T) \int_0^T k(X_s) \, dB_s.$$

**Proof.** We slightly modify the proof in [9]. A process $B_t^\epsilon$ defined by $dB_t^\epsilon = dB_t - \phi(\epsilon X_t) dt$ is a $d$-dimensional Brownian motion under $\mathbb{P}$.

A process $\tilde{X}_t^\epsilon$ defined by

$$d\tilde{X}_t^\epsilon = (\sigma k)(\tilde{X}_t^\epsilon) \, dt + \sigma(\tilde{X}_t^\epsilon) \, dB_t$$

under $\mathbb{P}$ has the same distribution with $X_t^\epsilon$ under $\mathbb{P}$. Thus,

$$\frac{\partial}{\partial \epsilon} \bigg|_{\epsilon = 0} \mathbb{E}^P[(\phi^{-1} f)(X_T^\epsilon)] = \frac{\partial}{\partial \epsilon} \bigg|_{\epsilon = 0} \mathbb{E}^P[(\phi^{-1} f)(\tilde{X}_T^\epsilon)].$$

Because $k_\epsilon$ is continuously differentiable at $\epsilon = 0$, by using the Taylor expansions, we write $k_\epsilon = k + \epsilon \eta_k$ for some $d \times 1$ vector $\eta_k$.

$$d\tilde{X}_t^\epsilon = (\sigma k + \epsilon \sigma \eta_k)(\tilde{X}_t^\epsilon) \, dt + \sigma(\tilde{X}_t^\epsilon) \, dB_t.$$

We now show that $\mathbb{E}^P[(\phi^{-1} f)(\tilde{X}_T^\epsilon)]$ is differentiable at $\epsilon = 0$ and

$$\frac{\partial}{\partial \epsilon} \bigg|_{\epsilon = 0} \mathbb{E}^P[(\phi^{-1} f)(\tilde{X}_T^\epsilon)] = \mathbb{E}^P(\phi^{-1} f)(X_T) \int_0^T k(X_t) \, dB_t.$$  \hspace{1cm} (C.1)

By the assumption that $\mathbb{E}^P[\exp(\epsilon_1 \int_0^T g^2(X_t) \, dt)]$ is finite, we know that

$$Z'(T) := \exp\left(-\epsilon \int_0^T \eta_k(X_t) \, dB_t - \frac{\epsilon^2}{2} \int_0^T |\eta_k|^2(X_t) \, dt\right)$$

is martingale for small $\epsilon$ because the Novikov condition is satisfied. By the Girsanov theorem, we know $\mathbb{E}^P[(\phi^{-1} f)(\tilde{X}_T^\epsilon)] = \mathbb{E}^P[Z'(T)(\phi^{-1} f)(X_T)]$. Thus,

$$\frac{\partial}{\partial \epsilon} \bigg|_{\epsilon = 0} \mathbb{E}^P[(\phi^{-1} f)(X_T^\epsilon)] = \frac{\partial}{\partial \epsilon} \bigg|_{\epsilon = 0} \mathbb{E}^P[Z'(T)(\phi^{-1} f)(X_T)]$$

$$= \lim_{\epsilon \to 0} \mathbb{E}^P[(\epsilon^{-1}(Z'(T) - 1)(\phi^{-1} f)(X_T)]$$

$$= \lim_{\epsilon \to 0} \mathbb{E}^P\left[(\phi^{-1} f)(X_T) \int_0^T Z'(t) \eta_k(X_t) \, dB_t\right].$$
Here, for the last equality, we used $\epsilon^{-1}(Z^\epsilon(T) - 1) = \int_0^T Z^\epsilon(t) \eta_\epsilon(X_t) dB_t$.

To prove equation (2.11), it will be shown that

$$
\lim_{\epsilon \to 0} \mathbb{E}^P \left[ (\phi^{-1}f)(X_T) \int_0^T Z^\epsilon(t) \eta_\epsilon(X_t) - \overline{k}(X_t) \right] = 0 ,
$$

which is obtained from the above equation subtracted by $\mathbb{E}^P[(\phi^{-1}f)(X_T) \int_0^T \overline{k}(X_t) dB_t]$. From the condition $\mathbb{E}^P[(\phi^{-1}f)^2(X_T)] < \infty$, by the Cauchy-Schwarz inequality, it is enough to show that $\int_0^T Z^\epsilon(t) \eta_\epsilon(X_t) - \overline{k}(X_t) dB_t$ converges to zero in $L^2$ as $\epsilon \to 0$. Since we know

$$
\int_0^T Z^\epsilon(t) \eta_\epsilon(X_t) - \overline{k}(X_t) dB_t = \int_0^T (Z^\epsilon(t) - 1) \eta_\epsilon(X_t) dB_t + \int_0^T (\eta_\epsilon - \overline{k})(X_t) dB_t ,
$$

it will be shown that each term on the right hand side converges to zero in $L^2$. For the second term, we use the Lebesgue dominated convergence theorem. Because $|\eta_\epsilon - \overline{k}|^2 \leq 2(|\eta_\epsilon|^2 + |\overline{k}|^2) \leq 4g^2$ and $\mathbb{E}^P \int_0^T g^2(X_t) dt$ is finite, we have that as $\epsilon \to 0$,

$$
\mathbb{E}^P \left( \int_0^T (\eta_\epsilon - \overline{k})(X_t) dB_t \right)^2 = \mathbb{E}^P \int_0^T |\eta_\epsilon - \overline{k}|^2(X_t) dt \to 0 .
$$

We now prove that

$$
\lim_{\epsilon \to 0} \mathbb{E}^P \left( \int_0^T (Z^\epsilon(t) - 1) \eta_\epsilon(X_t) dB_t \right)^2 = 0 .
$$

Choose a positive integer $p$ and a positive number $q$ such that $\frac{1}{p} + \frac{1}{q} = 1$ and $1 < q < 1 + \frac{2}{p}$. Then

$$
\mathbb{E}^P \left( \int_0^T (Z^\epsilon(t) - 1) \eta_\epsilon(X_t) dB_t \right)^2 = \mathbb{E}^P \int_0^T (Z^\epsilon(t) - 1)^2 |\eta_\epsilon|^2(X_t) dt
$$

$$
\leq \left( \int_0^T \mathbb{E}^P (Z^\epsilon(t) - 1)^{2p} dt \right)^{\frac{1}{p}} \cdot \left( \int_0^T \mathbb{E}^P |\eta_\epsilon|^{2q}(X_t) dt \right)^{\frac{1}{q} \cdot \frac{1}{2}}
$$

$$
\leq \left( \int_0^T \mathbb{E}^P (Z^\epsilon(t) - 1)^{2p} dt \right)^{\frac{1}{p}} \cdot \left( \int_0^T \mathbb{E}^P g^{2q}(X_t) dt \right)^{\frac{1}{q} \cdot \frac{1}{2}} .
$$

The second term is finite by the assumption because $2q < 2 + \epsilon_1$. We now prove that the first term converges to zero. Consider $(Z^\epsilon(t) - 1)^{2p} = \sum_{l=0}^{2p} \binom{2p}{l} (-1)^l Z^\epsilon(t)^l$. It is enough to show that $\int_0^T \mathbb{E}^P(Z^\epsilon(t)^l) dt$ converges to $0$ as $\epsilon \to 0$ for $l = 1, 2, \cdots, 2p$, because

$$
\int_0^T \mathbb{E}^P(Z^\epsilon(t) - 1)^{2p} dt = \sum_{l=0}^{2p} \binom{2p}{l} (-1)^l \int_0^T \mathbb{E}^P(Z^\epsilon(t)^l) dt \to T \sum_{l=0}^{2p} \binom{2p}{l} (-1)^l = 0 .
$$

To show this, we use the Lebesgue dominated convergent theorem: prove that $\mathbb{E}^P(Z^\epsilon(t)^l)$ is uniformly bounded for small $\epsilon$ and $0 \leq t \leq T$ and that $\mathbb{E}^P(Z^\epsilon(t)^l)$ converges to $1$ as $\epsilon$ goes to zero for
fixed $t$.

\[ \mathbb{E}^p(Z^\epsilon(t)^l) \]

\[ = \mathbb{E}^p \exp \left( -l\epsilon \int_0^t \eta_h(X_s) dB_s - \frac{l^2\epsilon^2}{2} \int_0^t |\eta_h|^2(X_s) ds \right) \]

\[ = \mathbb{E}^p \exp \left( -l\epsilon \int_0^t \eta_h(X_s) dB_s - l^2\epsilon^2 \int_0^t |\eta_h|^2(X_s) ds \right) \cdot \exp \left( l(l - 1/2)\epsilon^2 \int_0^t |\eta_h|^2(X_s) ds \right) \]

\[ \leq \left( \mathbb{E}^p \exp \left( -2l\epsilon \int_0^t \eta_h(X_s) dB_s - 2l^2\epsilon^2 \int_0^t |\eta_h|^2(X_s) ds \right) \right)^{\frac{1}{2}} \]

\[ \cdot \left( \mathbb{E}^p \exp \left( l(2l - 1)\epsilon^2 \int_0^t |\eta_h|^2(X_s) ds \right) \right)^{\frac{1}{2}} \]

\[ = \left( \mathbb{E}^p \exp \left( l(2l - 1)\epsilon^2 \int_0^t |\eta_h|^2(X_s) ds \right) \right)^{\frac{1}{2}} \cdot \text{ the former term is a martingale for small } \epsilon \]

\[ \leq \left( \mathbb{E}^p \exp \left( l(2l - 1)\epsilon^2 \int_0^t g^2(X_s) ds \right) \right)^{\frac{1}{2}} \]

\[ \leq \left( \mathbb{E}^p \exp \left( \epsilon_1 \int_0^T g^2(X_s) ds \right) \right)^{\frac{1}{2}} \text{ for } t < T \text{ and small } \epsilon \]

\[ < \infty \text{ by assumption.} \]

Thus, $\mathbb{E}^p(Z^\epsilon(t)^l)$ is uniformly bounded for small $\epsilon$ and $0 \leq t \leq T$.

We now show that $\mathbb{E}^p(Z^\epsilon(t)^l)$ converges to 1 as $\epsilon$ goes to zero for fixed $t$. We use the Lebesgue dominated convergent theorem to

\[ \exp \left( l(2l - 1)\epsilon^2 \int_0^t g^2(X_s) ds \right) \]

as $\epsilon$ goes to zero. Because this is dominated pathwise by

\[ \exp \left( \epsilon_1 \int_0^t g^2(X_s) ds \right) \]

whose expectation is finite, we know that

\[ \mathbb{E}^p \exp \left( l(2l - 1)\epsilon^2 \int_0^t g^2(X_s) ds \right) \]

converges to 1 as $\epsilon$ goes to zero.

\[ 1 = \mathbb{E}^p \left( \lim inf_{\epsilon \to 0} Z^\epsilon(t)^l \right) \leq \lim inf_{\epsilon \to 0} \mathbb{E}^p \left( Z^\epsilon(t)^l \right) \leq \lim sup_{\epsilon \to 0} \mathbb{E}^p \left( Z^\epsilon(t)^l \right) \]

\[ \leq \lim_{\epsilon \to 0} \mathbb{E}^p \exp \left( l(2l - 1)\epsilon^2 \int_0^t g^2(X_s) ds \right) = 1. \]

Thus, we obtained the desired result. This completes the proof.

Now we prove Theorem 3.1
Proof. It suffices to show that \( \frac{1}{T} \cdot \mathbb{E}^p[(\phi^{-1} f)(X_T) \int_0^T k(X_s) \, dB_s] \) converges to zero as \( T \) goes to infinity. From the assumption,

\[
\begin{align*}
\mathbb{E}^p \left[ \exp \left( \epsilon_0 \int_0^T g^2(X_t) \, dt \right) \right] & \geq \exp \left( \epsilon_0 \mathbb{E}^p \left[ \int_0^T g^2(X_t) \, dt \right] \right),
\end{align*}
\]

thus we have that

\[
\mathbb{E}^p \left[ \int_0^T |k|^2(X_s) \, ds \right] \leq \mathbb{E}^p \left[ \int_0^T g^2(X_t) \, dt \right] \leq \frac{1}{\epsilon_0} (\log c(T) + aT) \leq a_1 T
\]

for some positive number \( a_1 \) when \( T \) is large.

\[
\begin{align*}
\frac{1}{T} \cdot \mathbb{E}^p \left[ (\phi^{-1} f)(X_T) \int_0^T k(X_s) \, dB_s \right] & \leq \frac{1}{T} \cdot \mathbb{E}^p \left[ (\phi^{-1} f)(X_T) \int_0^T |k|^2(X_s) \, ds \right]^{1/2} \mathbb{E}^p \left[ \int_0^T g^2(X_t) \, dt \right]^{1/2} \leq \sqrt{\frac{a_1}{T}} \cdot \mathbb{E}^p \left[ (\phi^{-1} f)(X_T) \right]^{1/2}.
\end{align*}
\]

This completes the proof. \( \square \)

## D Proof of the Theorem 3.2

We first show the following fact, which is a variation of proposition C.1.

**Proposition D.1.** Let \( g : \mathbb{R}^d \to \mathbb{R} \) be a function such that

\[
\left| \frac{\partial k_s(x)}{\partial \epsilon} \right| \leq g(x)
\]

on \((\epsilon, x) \in I \times \mathbb{R}^d\) for an open interval \( I \) containing 0. Suppose that \( \mathbb{E}^p[\exp(\epsilon_1 \int_0^T g^2(X_t) \, dt)] \) is finite for some positive \( \epsilon_1 \) and \( \mathbb{E}^p \int_0^T g^2(X_t) \, dt \) is finite for all \( n > 0 \). Then for any given function \( f(\cdot) \) with \( \mathbb{E}^p[|\phi^{-1} f|^{1+\epsilon_2}(X_T)] < \infty \) for some positive \( \epsilon_2 \), we have

\[
\frac{\partial}{\partial \epsilon} \bigg|_{\epsilon=0} \mathbb{E}^p \left[ (\phi^{-1} f)(X_T) \right] = \mathbb{E}^p \left[ (\phi^{-1} f)(X_T) \int_0^T k(X_s) \, dB_s \right].
\]

The following theorem will be essentially used for the proof of this proposition.

**Theorem D.1.** Let \( q \) be a positive even integer and let \( Y_t \) be a stochastic process with \( \mathbb{E} \int_0^\infty Y_t^2 \, dt < \infty \). Then

\[
\mathbb{E} \left( \int_0^\infty Y_t \, dB_t \right)^q \leq c_q \cdot \mathbb{E} \left( \int_0^\infty Y_t^2 \, dt \right)^{\frac{q}{2}}
\]

for some positive constant \( c_q \).
For proof, see page 40 in [26].

**Proof.** We now prove proposition D.1. Using the same argument in the proof of Proposition C.1, it will be shown that

\[
\lim_{\epsilon \to 0} \mathbb{E}^p \left[ (\phi^{-1} f)(X_T) \int_0^T Z^\epsilon(t) \eta_\epsilon(X_t) - \overline{k}(X_t) \, dB_t \right] = 0 ,
\]

where \( Z^\epsilon(t) \) and \( \eta_\epsilon \) are as defined in the proof of Proposition C.1. Let \( q' \) be a positive number such that \( 1/q' + 1/(1 + \epsilon_2) = 1 \). From the condition \( \mathbb{E}^p[|\phi^{-1} f|^{1+\epsilon_2}(X_T)] < \infty \), by the Cauchy-Schwarz inequality, it is enough to show that \( \int_0^T Z^\epsilon(t) \eta_\epsilon(X_t) - \overline{k}(X_t) \, dB_t \) converges to zero in \( L^q \) as \( \epsilon \to 0 \) for any positive even integer \( q \) with \( q \ge q' \). Since we know

\[
\int_0^T Z^\epsilon(t) \eta_\epsilon(X_t) - \overline{k}(X_t) \, dB_t = \int_0^T (Z^\epsilon(t) - 1) \eta_\epsilon(X_t) \, dB_t + \int_0^T (\eta_\epsilon - \overline{k})(X_t) \, dB_t ,
\]

it will be shown that each term on the right hand side converges to zero in \( L^q \). For the second term, we use the Lebesgue dominated convergence theorem. Because \(|\eta_\epsilon - \overline{k}|^q \le c \cdot (|\eta_\epsilon|^q + |\overline{k}|^q) \le 2c g^q \) for some positive constant \( c \) and \( \mathbb{E}^p \int_0^T g^q(X_t) \, dt \) is finite, we have that

\[
\mathbb{E}^p \left( \int_0^T (\eta_\epsilon - \overline{k})(X_t) \, dB_t \right)^q \le c_q \cdot \mathbb{E}^p \left( \int_0^T |\eta_\epsilon - \overline{k}|^2(X_t) \, dt \right)^{q/2} \le c_q T^{q/2 - 1} \cdot \mathbb{E}^p \int_0^T |\eta_\epsilon - \overline{k}|^q(X_t) \, dt \to 0
\]
as \( \epsilon \to 0 \) for some constant \( c_q \), which is independent of \( \epsilon \).

We now prove that

\[
\lim_{\epsilon \to 0} \mathbb{E}^p \left( \int_0^T (Z^\epsilon(t) - 1) \eta_\epsilon(X_t) \, dB_t \right)^q = 0 .
\]

It follows that

\[
\mathbb{E}^p \left( \int_0^T (Z^\epsilon(t) - 1) \eta_\epsilon(X_t) \, dB_t \right)^q \le c_q T^{q/2 - 1} \cdot \mathbb{E}^p \int_0^T (Z^\epsilon(t) - 1)^q |\eta_\epsilon|^q(X_t) \, dt \le c_q T^{q/2 - 1} \left( \int_0^T \mathbb{E}^p (Z^\epsilon(t) - 1)^{2q} \, dt \right)^{1/2} \cdot \left( \int_0^T \mathbb{E}^p |\eta_\epsilon|^{2q}(X_t) \, dt \right)^{1/2} \le c_q T^{q/2 - 1} \left( \int_0^T \mathbb{E}^p (Z^\epsilon(t) - 1)^{2q} \, dt \right)^{1/2} \cdot \left( \int_0^T \mathbb{E}^p g^{2q}(X_t) \, dt \right)^{1/2} .
\]

The second term is finite from the assumption. By the same argument in the proof of Proposition C.1, it can be shown that the first term goes to zero as \( \epsilon \) goes to zero. This completes the proof. \( \square \)

We need the following proposition to prove Theorem 3.2.

**Proposition D.2.** Let \( p \) be a positive integer. Then for any positive random variable \( Y \), we have

\[
\mathbb{E}[Y^p]^{1/p} \le \ln \mathbb{E}[e^Y] .
\]
Proof. By direct calculation, we obtain

\[
\mathbb{E}[e^Y] = \sum_{n=0}^{\infty} \frac{\mathbb{E}[Y^n]}{n!} \\
= \sum_{n=0}^{\infty} \frac{\mathbb{E}[Y^{pn}]}{(pn)!} + \sum_{n=0}^{\infty} \frac{\mathbb{E}[Y^{pn+1}]}{(pn + 1)!} + \cdots + \sum_{n=0}^{\infty} \frac{\mathbb{E}[Y^{pn+p-1}]}{(pn + p - 1)!} \\
\geq \sum_{n=0}^{\infty} \frac{\mathbb{E}[Y^{pn}]}{(pn)!} + \sum_{n=0}^{\infty} \frac{\mathbb{E}[Y^{pn+1}]}{(pn + 1)!} + \cdots + \sum_{n=0}^{\infty} \frac{\mathbb{E}[Y^{pn+p-1}]}{(pn + p - 1)!} \\
= \sum_{n=0}^{\infty} \frac{(\mathbb{E}[Y^{pn}])^p}{(pn)!} + \sum_{n=0}^{\infty} \frac{(\mathbb{E}[Y^{pn+1}])^p}{(pn + 1)!} + \cdots + \sum_{n=0}^{\infty} \frac{(\mathbb{E}[Y^{pn+p-1}])^p}{(pn + p - 1)!} \\
= e^{\mathbb{E}[Y^p]^p}
\]

This completes the proof.

Proof. We now prove the Theorem 3.2. It suffices to show that

\[
\frac{1}{T} \cdot \mathbb{E}^p[(\phi^{-1} f)(X_T) \int_0^T \kappa(X_s) \, dB_s]
\]

converges to zero as \( T \) goes to infinity. Let \( q' \) be the positive number such that \( 1/q' + 1/(1+\epsilon_2) = 1 \) and let \( q \) be any positive even integer \( q \) with \( q \geq q' \). From the assumption and the proposition above,

\[
\ln c(T) + aT = \ln \mathbb{E}^p \left[ \exp \left( \epsilon_0 \int_0^T g^2(X_t) \, dt \right) \right] \\
\geq \left( \mathbb{E}^p \left( \epsilon_0 \int_0^T g^2(X_t) \, dt \right)^{\frac{q}{q'}} \right)^{\frac{q'}{q}}
\]

it is obtained that

\[
\mathbb{E}^p \left( \int_0^T |\kappa|^2(X_s) \, ds \right)^{\frac{q}{2}} \leq \mathbb{E}^p \left( \int_0^T g^2(X_s) \, ds \right)^{\frac{q}{2}} \leq \left( \frac{\ln c(T) + aT}{\epsilon_0} \right)^{\frac{q}{2}} \leq a_1 T^{\frac{q}{2}}
\]

for some positive number \( a_1 \) when \( T \) is large.

\[
\frac{1}{T} \cdot \mathbb{E}^p \left[ (\phi^{-1} f)(X_T) \int_0^T \kappa(X_s) \, dB_s \right] \\
\leq \frac{1}{T} \cdot \mathbb{E}^p \left[ |\phi^{-1} f|^{1+\epsilon_2}(X_T) \right]^{\frac{1}{1+\epsilon_2}} \cdot \mathbb{E}^p \left[ \left( \int_0^T \kappa(X_s) \, dB_s \right)^q \right]^{\frac{1}{q}} \\
\leq \frac{c_2}{T} \cdot \mathbb{E}^p \left[ |\phi^{-1} f|^{1+\epsilon_2}(X_T) \right]^{\frac{1}{1+\epsilon_2}} \cdot \mathbb{E}^p \left[ \left( \int_0^T |\kappa|^2(X_s) \, ds \right)^{\frac{q}{2}} \right]^{\frac{1}{q}} \\
\leq \frac{c_2 a_1^{\frac{q}{2}}}{\sqrt{T}} \cdot \mathbb{E}^p \left[ |\phi^{-1} f|^{1+\epsilon_2}(X_T) \right]^{\frac{1}{1+\epsilon_2}} \\
\leq \frac{c_2 a_1^{\frac{q}{2}}}{\sqrt{T}} \cdot \mathbb{E}^p \left[ |\phi^{-1} f|^{1+\epsilon_2}(X_T) \right].
\]
This completes the proof.

**E  Proof of Corollary 3.3**

*Proof.* It suffices to show the condition (i) and (ii)’. First show that

$$
\mathbb{E}^P \left[ \exp \left( \frac{1}{T} \int_0^T \epsilon_0 g^2(X_t) \, dt \right) \right]
$$

is bounded on $0 < T < \infty$. Let $c_1$ be such that $\mathbb{E}^P[\exp(\epsilon_0 g^2(X_t))] \leq c_1$ for all $t > 0$. Then,

$$
\mathbb{E}^P \left[ \exp \left( \frac{1}{T} \int_0^T \epsilon_0 g^2(X_t) \, dt \right) \right] \leq \mathbb{E}^P \left[ \frac{1}{T} \int_0^T \exp(\epsilon_0 g^2(X_t)) \, dt \right] = \frac{1}{T} \int_0^T \mathbb{E}^P[\exp(\epsilon_0 g^2(X_t))] \, dt \leq c_1,
$$

which is the desired result. Now we prove that

$$
\mathbb{E}^P \left[ \exp \left( \epsilon_0 \int_0^T g^2(X_t) \, dt \right) \right] = c(T) e^{\alpha T}
$$

for some constants $a$ and $c = c(T)$ with $c(T)$ bounded on $T > 0$. For any positive integer $n$, by using Proposition [D.2] we have that

$$
\left( \mathbb{E} \left( \frac{1}{T} \int_0^T \epsilon_0 g^2(X_t) \, dt \right)^n \right)^{\frac{1}{n}} \leq \ln \mathbb{E}^P \left[ \exp \left( \frac{1}{T} \int_0^T \epsilon_0 g^2(X_t) \, dt \right) \right] \leq \ln c_1.
$$

Thus,

$$
\mathbb{E}^P \left( \int_0^T \epsilon_0 g^2(X_t) \, dt \right)^n \leq (T \ln c_1)^n.
$$

We obtain

$$
\mathbb{E}^P \left[ \exp \left( \epsilon_0 \int_0^T g^2(X_t) \, dt \right) \right] \leq \sum_{n=0}^{\infty} \frac{1}{n!} \mathbb{E}^P \left( \int_0^T \epsilon_0 g^2(X_t) \, dt \right)^n \leq \sum_{n=0}^{\infty} \frac{(T \ln c_1)^n}{n!} = e^{T \ln c_1}.
$$

Thus, condition (i) is proved and condition (ii)’ is trivial. This completes the proof.

**F  Proof of Corollary 4.2**

*Proof.* From Proposition 3.2 in Fournie [9], we have that

$$
\nabla_{\xi} \mathbb{E}^P_F(\phi^{-1}f)(X_T) = \frac{1}{T} \mathbb{E}^P \left[ (\phi^{-1}f)(X_T) \int_0^T (\sigma^{-1}(X_t)Y_t)^\top dB_t \right].
$$

By the Cauchy-Schwarz inequality, it follows that

$$
|\nabla_{\xi} \mathbb{E}^P_F(\phi^{-1}f)(X_T)| \leq \frac{1}{T} \left( \mathbb{E}^P(\phi^{-1}f)^2(X_T) \right)^{\frac{1}{2}} \cdot \left( \mathbb{E}^P \int_0^T \|\sigma^{-1}(X_t)Y_t\|^2 dt \right)^{\frac{1}{2}}
$$

which gives the desired result.

□
G The CIR model

G.1 The martingale extraction

We explore the sensitivity analysis of option prices whose underlying process is the Cox–Ingersoll–Ross (CIR) model. Under a risk-neutral measure \( \mathbb{Q} \), the interest rate \( r_t \) follows

\[
dr_t = (\theta - ar_t) \, dt + \sigma \sqrt{r_t} \, dW_t
\]

with \( \theta, a, \sigma > 0 \). We assume \( 2\theta > \sigma^2 \) so that the original interest rate process and the perturbation process stay strictly positive for small perturbation. The associated second-order equation is

\[
\mathcal{L}\phi(r) = \frac{1}{2} \sigma^2 r \phi''(r) + (\theta - ar) \phi'(r) - r \phi(r) = -\lambda \phi(r) .
\]

Set \( \kappa := \sqrt{a^2 + 2\sigma^2 - a} \).

We explore the martingale extraction with respect to \( (\lambda, \phi(r)) := (\theta \kappa, e^{-\kappa r}) \).

First, by direct calculation, it can be shown that this is an eigenpair. The dynamics of the diffusion process induced by this pair satisfies equation \( (G.1) \) below and is recurrent, thus this pair admits a martingale extraction. Let \( \mathbb{P} \) be the transformed measure with respect to \( (\theta \kappa, e^{-\kappa r}) \). The corresponding martingale exponent is \( \varphi(r) := -\sigma \kappa \sqrt{r} \). We know that a process \( B_t \) defined by

\[
\text{dB}_t = dW_t + \sigma \kappa \sqrt{r_t} \, dt
\]

is a Brownian motion under \( \mathbb{P} \). The interest rate \( r_t \) follows

\[
dr_t = \left( \theta - \sqrt{a^2 + 2\sigma^2} \, r_t \right) \, dt + \sigma \sqrt{r_t} \, dB_t . \tag{G.1}
\]

We see that this martingale extraction stabilizes \( f \). Here, \( f(r) \) is a nonnegative continuous function on \( r \in [0, \infty) \), which is not identically zero, and whose growth rate at infinity is equal to or less than \( e^{mr} \) with \( m < \frac{a}{\sigma^2} \). Even more, it can be shown that \( \mathbb{E}^\mathbb{P}[(\phi^{-1} f)^2(r_t)] \) is convergent as \( t \) approaches to infinity. To achieve this, it is enough to prove that \( \mathbb{E}^\mathbb{P}[e^{c} \varphi] \) is convergent as \( t \) goes to infinity for \( c < \frac{2\sqrt{a^2 + 2\sigma^2}}{\sigma^2} \). Set \( b := \sqrt{a^2 + 2\sigma^2} \). We consider the density function of \( r_t \) under \( \mathbb{P} \). The CIR process has an explicit formula of the density function:

\[
g(r; t) = h_t \, e^{-u-v} \left( \frac{v}{u} \right)^{q/2} \, I_q(2\sqrt{uv})
\]

where \( I_q \) is the modified Bessel function of the first kind of order \( q \) and

\[
h_t = \frac{2b}{\sigma^2(1-e^{-bt})} , \quad q = \frac{2\theta}{\sigma^2} - 1 , \quad u = h_t r_0 e^{-bt} , \quad v = h_t r .
\]

After rewriting slightly, we find

\[
g(r; t) = k_t \, h_t \, e^{-h_t r_t q/2} I_q(2h_t e^{-bt/2} \sqrt{r_0 r}) .
\]
Here, \( k_t = e^{-ht}r_0e^{-ht}(r_0e^{-ht})^{-q/2} \) and

\[
I_q(z) = \frac{(z/2)^q}{\pi^{1/2} \Gamma(q + 1/2)} \int_0^\pi (e^z \cos u \sin^2 u) du \leq \frac{\pi^{1/2}(z/2)^q e^z}{\Gamma(q + 1/2)} .
\]

For large \( t \), we have

\[
g(r; t) \leq Be^{-ht} e^{2ht} \sqrt{r_0}
\]

for some constant \( B \). Because \( c < \frac{2b}{\sigma^2} < h_t \), we know that \( e^{ct} g(r; t) \) is dominated by

\[
Be^{\left(c - \frac{2b}{\sigma^2}\right) r_0 e^{2ht} \sqrt{r_0}},
\]

whose integration over \((0, \infty)\) is finite. By the Lebesgue dominated convergent theorem, we have that \( \mathbb{E}^P [e^{ct}] \) is convergent and the limit is

\[
\int_0^\infty e^{ct} g(r; \infty) \, dr
\]

where \( g(r; \infty) = \lim_{t \to \infty} g(r; t) \), which is equal to the invariant density function of \( r_t \) under \( P \). For more details of the density of the CIR model, refer to [4].

### G.2 Sensitivity on \( \theta \)

Now, we see the sensitivity analysis with respect to \( \theta \) of long-term option prices. Consider the perturbed process \( r^\epsilon_t \) with respect to \( \theta \) :

\[
dr^\epsilon_t = ((\theta + \epsilon) - ar^\epsilon_t) \, dt + \sigma \sqrt{r^\epsilon_t} \, dB^\epsilon_t .
\]

We already know

\[
(\lambda(\epsilon), \phi(\epsilon)) := ((\theta + \epsilon)\kappa, e^{e^{\kappa r}})
\]

stabilizes \( f \) described above. The dynamics of \( r^\epsilon_t \) follows

\[
dr^\epsilon_t = \left((\theta + \epsilon) - \sqrt{a^2 + 2\sigma^2 r^\epsilon_t}\right) \, dt + \sigma \sqrt{r^\epsilon_t} \, dB^\epsilon_t
\]

where \( B^\epsilon_t \) is a Brownian motion under the corresponding measure \( P^\epsilon \).

We apply Theorem 3.1 to conclude that

\[
\lim_{T \to \infty} \frac{1}{T} \frac{\partial}{\partial \theta} \ln p_T = -\frac{\sqrt{a^2 + 2\sigma^2} - a}{\sigma^2} .
\]

First, Condition 1 and 2 in Section 3 are clearly satisfied. Now it will be shown in the following proposition that one of the conditions of the theorem is satisfied. Using

\[
k_\epsilon(r) = \frac{\theta + \epsilon}{\sigma \sqrt{r}} - \frac{\sqrt{a^2 + 2\sigma^2}}{\sigma} \sqrt{r},
\]

we know

\[
\frac{\partial}{\partial \epsilon} k_\epsilon(r) = \frac{1}{\sigma \sqrt{r}} ,
\]

and thus

\[
\bar{k}(r) = \frac{1}{\sigma \sqrt{r}} .
\]

Since \( \frac{\partial}{\partial \epsilon} k_\epsilon(r) \) is independent of \( \epsilon \), we set \( g(r) := \bar{f}(r) \) in the theorem. The following proposition is enough to confirm one of the conditions of the theorem.
Proposition G.1. For \( \epsilon_0 \) with \( \epsilon_0 \leq \frac{1}{2} \left( \frac{a}{2} - \frac{b}{a} \right)^2 \), we have

\[
\mathbb{E}^P \left[ \exp \left( \epsilon_0 \int_0^T r_t^{-1} \, dt \right) \right] \leq c(T) e^{aT}
\]

for some constants \( a \) and \( c(T) \) with \( c(T) \) bounded on \( T \).

Proof. The main idea of the proof is from \([I]\). We know that \( r_t \) satisfies \( dr_t = (\theta - b r_t) \, dt + \sigma \sqrt{r_t} \, dB_t \) where \( b = \sqrt{a^2 + 2\sigma^2} \). Define \( X_t := r_t^{-1} \). By direct calculation, we have

\[
dX_t = ((\sigma^2 - \theta)X_t + b)X_t \, dt - \sigma X^{3/2} \, dB_t .
\]

We find a positive function \( V(x, t) \) on \((x, t) \in \mathbb{R}^+ \times [0, T] \) such that \( V(X_t, t) \exp \left( \epsilon_0 \int_0^t X_t \, dt \right) \) is a local martingale and \( V(x, T) \) is a constant function of \( x \). It follows that

\[
V_t + \frac{1}{2} \sigma^2 x^3 V_{xx} + ((\sigma^2 - \theta)x + b)x V_x + \epsilon_0 x V = 0 .
\]

Try this form: \( V(x, t) = f(y)y^\gamma \) where \( y = a(t)/x \).

\[
V_x = -\frac{1}{a(t)} f'(y)y^{\gamma+1} - \frac{\gamma}{a(t)} f(y)y^{\gamma+1}
\]

\[
V_{xx} = \frac{1}{a^2(t)} f''(y)y^{\gamma+4} + \frac{2(\gamma + 1)}{a^2(t)} f'(y)y^{\gamma+3} + \frac{\gamma(\gamma + 1)}{a^2(t)} f(y)y^{\gamma+2}
\]

\[
V_t = \frac{a'(t)}{a(t)} f'(y)y^{\gamma+1} + \frac{a'(t)}{a(t)} \gamma f(y)y^{\gamma} .
\]

Then we have

\[
\frac{1}{2} \sigma^2 a(t)y^{\gamma+1} f''(y) + \left( \frac{a'(t)}{a(t)} y^{\gamma+1} - by^{\gamma+1} - (\sigma^2 - \theta)a(t)y^{\gamma} + \sigma^2(\gamma + 1)a(t)y^{\gamma} \right) f'(y)
\]

\[
+ \left( \frac{a'(t)}{a(t)} \gamma y^{\gamma} - b\gamma y^{\gamma} + \frac{1}{2} \sigma^2 \gamma(\gamma + 1)a(t)y^{\gamma-1} - (\sigma^2 - \theta)\gamma a(t)y^{\gamma-1} + \epsilon_0 a(t)y^{\gamma-1} \right) f(y) = 0
\]

Let

\[
\left\{ \begin{array}{l}
\frac{a'(t)}{a(t)} - b = a(t) \\
\frac{1}{2} \sigma^2 \gamma(\gamma + 1) - (\sigma^2 - \theta)\gamma + \epsilon_0 = 0 .
\end{array} \right.
\]

It follows that

\[
\frac{1}{2} \sigma^2 y f''(y) + (y + \sigma^2 \gamma + \theta) f'(y) + \gamma f(y) = 0 .
\]

Define a new variable \( z \) by \( y = -\frac{1}{2} \sigma^2 z \) and set \( g(z) := f(y) \). We have

\[
zg''(z) + (\kappa - z)g'(z) - \gamma g(z) = 0
\]

where \( \kappa = 2 \left( \gamma + \frac{2}{3} \right) \). A solution of this equation is the standard confluent hypergeometric function:

\[
f(y) = g(z) = M(\gamma, \kappa; z) .
\]

27
We now find an explicit expression for $V(x, t)$. From $\frac{a'(t)}{a(t)} - b = a(t)$, we obtain

$$a(t) = \frac{b}{e^{b(T-t)} - 1}$$

for $t < T$. Solving $\frac{1}{2}\sigma^2 \gamma (\gamma + 1) - (\sigma^2 - \theta) \gamma + \epsilon_0 = 0$ yields

$$\gamma = \frac{1}{2} - \frac{\theta}{\sigma^2} + \sqrt{\left(\frac{1}{2} - \frac{\theta}{\sigma^2}\right)^2 - \frac{2\epsilon_0}{\sigma^2}},$$

which is a real number by the assumption on $\epsilon_0$. We also have that $\gamma < 0$ and $\kappa = 2 \left( \gamma + \frac{\theta}{\sigma^2} \right) > 0$. The solution $V(x, t)$ is given by

$$V(x, t) = f(y) y^\gamma = g(z) \left( -\frac{1}{2} \sigma^2 z \right)^\gamma$$

$$= \left( \frac{1}{2} \sigma^2 \right)^\gamma M(\gamma, \kappa; z) (-z)^\gamma$$

$$= \left( \frac{1}{2} \sigma^2 \right)^\gamma M(\kappa - \gamma, \kappa; -z)(-z)^\gamma e^z$$

with

$$z = -\frac{2y}{\sigma^2} = -\frac{2a(t)}{\sigma^2 x} = -\frac{2b}{\sigma^2 (e^{b(T-t)} - 1)x}.$$  

Here, we used $M(\gamma, \kappa; z) = M(\kappa - \gamma, \kappa; -z)e^z$.

We show that

$$V(x, T) := \lim_{t \to T} V(x, t) = \left( \frac{1}{2} \sigma^2 \right)^\gamma \Gamma(\kappa) \Gamma(\kappa - \gamma).$$

It is obtained by

$$\lim_{t \to T} V(x, t) = \left( \frac{1}{2} \sigma^2 \right)^\gamma \lim_{z \to -\infty} M(\kappa - \gamma, \kappa; -z)(-z)^\gamma e^z$$

$$= \left( \frac{1}{2} \sigma^2 \right)^\gamma \lim_{u \to \infty} M(\kappa - \gamma, \kappa; u) u^\gamma e^{-u}$$

$$= \left( \frac{1}{2} \sigma^2 \right)^\gamma \frac{\Gamma(\kappa)}{\Gamma(\kappa - \gamma) \Gamma(\gamma)} \lim_{u \to \infty} u^\gamma e^{-u} \int_0^1 e^{us} s^{\kappa - \gamma - 1} (1 - s)^{\gamma - 1} ds$$

$$= \left( \frac{1}{2} \sigma^2 \right)^\gamma \frac{\Gamma(\kappa)}{\Gamma(\kappa - \gamma) \Gamma(\gamma)} \lim_{u \to \infty} u^\gamma \int_0^1 e^{-us} (1 - s)^{\kappa - \gamma - 1} s^{\gamma - 1} ds$$

$$= \left( \frac{1}{2} \sigma^2 \right)^\gamma \frac{\Gamma(\kappa)}{\Gamma(\kappa - \gamma) \Gamma(\gamma)} \int_0^\infty e^{-t(1 - t/u)^{\kappa - \gamma - 1} t^{\gamma - 1} dt}$$

$$= \left( \frac{1}{2} \sigma^2 \right)^\gamma \frac{\Gamma(\kappa)}{\Gamma(\kappa - \gamma) \Gamma(\gamma)} \int_0^\infty e^{-t} t^{\gamma - 1} dt$$

$$= \left( \frac{1}{2} \sigma^2 \right)^\gamma \frac{\Gamma(\kappa)}{\Gamma(\kappa - \gamma) \Gamma(\gamma)} \cdot$$

On the other hand, we have

$$V(x, 0) = c_1(T; x) \cdot e^{-\gamma b T}$$

28
with $c_1(T; x)$ bounded for large $T$. It is because

$$V(x, 0) = c_2(T; x) \cdot \left( \frac{\sigma^2(1 - e^{-bT})x}{2b} \right)^{-\gamma} \cdot e^{-\gamma bT}$$

where

$$c_2(T; x) = \left( \frac{1}{2} \sigma^2 \right)^{\gamma} \cdot M \left( \kappa - \gamma, \kappa; \frac{2b}{\sigma^2(e^{bT} - 1)x} \right) \exp \left( -\frac{2b}{\sigma^2(e^{bT} - 1)x} \right)$$

and $c_2(T; x)$ is bounded for large $T$. It is known that $\lim_{u \to 0} M(\kappa - \gamma, \kappa, u) = 1$.

Because $V(X_t, t) \exp \left( \epsilon_0 \int_0^t X_t dt \right)$ is a positive local martingale, it is a supermartingale. Thus, we have

$$\left( \frac{1}{2} \sigma^2 \right)^{\gamma} \frac{\Gamma(\kappa)}{\Gamma(\kappa - \gamma)} \cdot \mathbb{E} \left[ \exp \left( \epsilon_0 \int_0^T X_t dt \right) \right] = \mathbb{E} \left[ V(X_T, T) \exp \left( \epsilon_0 \int_0^T X_t dt \right) \right] \leq V(X_0, 0) = c_1(T; X_0) \cdot e^{-\gamma b T}.$$ 

By setting $c(T) := \left( \frac{1}{2} \sigma^2 \right)^{-\gamma} \frac{\Gamma(\kappa - \gamma)}{\Gamma(\kappa)} c_1(T; r_0^{-1})$ and $a = -\gamma b$, we obtain the desired result. \hfill \Box

We now prove that the other conditions in Theorem 3.1 are satisfied. Let $\epsilon_1$ be a positive number with $\frac{\epsilon_1}{2} < \frac{2b}{\sigma^2} - 1$. We show that $\mathbb{E}^p[(1/\sqrt{r_t})^{2+\epsilon_1}]$ is convergent to a constant as $t$ approaches to infinity. From equation (G.2), we know that

$$\mathbb{E}^p[(1/\sqrt{r_t})^{2+\epsilon_1}] = \int_0^\infty r_t^{-1-\frac{\epsilon_1}{2}} g(r; t) dr \leq B \int_0^\infty e^{-h_1 r} \frac{2b}{\sigma^2} - 1 \frac{1}{\sqrt{r_t}} \frac{e^{2h_1 \sqrt{r_t}}}{\sqrt{r_t}} dr .$$

For large $t$, the integrand is dominated by $e^{-2b \frac{2b}{\sigma^2} - 1} \frac{1}{\sqrt{r_t}} e^{2h_1 \sqrt{r_t}}$, whose integration over $(0, \infty)$ is finite because $\frac{2b}{\sigma^2} - 2 - \frac{\epsilon_1}{2} > -1$. By the Lebesgue dominated convergent theorem, we obtain the desired result. We already showed that the condition $\frac{1}{T} \cdot \mathbb{E}^p[(\phi^{-1} f)^2(X_T)] \to 0$ as $T \to \infty$ is satisfied with $f$ described above.

### G.3 Sensitivity on $a$

Now, we explore the sensitivity of variable $a$ in the drift coefficient. The perturbed process $r_t^\epsilon$ with respect to $a$ is

$$dr_t^\epsilon = (\theta - (a + \epsilon) r_t^\epsilon) dt + \sigma \sqrt{r_t^\epsilon} dW_t .$$

We know that

$$(\lambda(\epsilon), \phi(\epsilon)) := (\theta \kappa(\epsilon), e^{-\kappa(\epsilon)r})$$

stabilizes $f$ described above, where

$$\kappa(\epsilon) = \sqrt{a + \epsilon^2} + 2\sigma^2 - (a + \epsilon) .$$

The dynamics of $r_t^\epsilon$ follows

$$dr_t^\epsilon = \left( \theta - \sqrt{(a + \epsilon)^2 + 2\sigma^2 r_t^\epsilon} \right) dt + \sigma \sqrt{r_t^\epsilon} dB_t .$$

29
under the corresponding measure $\mathbb{P}_\epsilon$.
First, we check Condition 1 and 2 in Section 3. The first condition is clear, thus the second condition will be proved. We will use Theorem [B.1], so the function $G$ in the theorem is constructed. By direct calculation,
\[
\frac{\partial}{\partial \epsilon} (\phi_{\epsilon}^{-1} f)(r) = \kappa'(\epsilon) re^{\kappa(\epsilon) r} f(r).
\]
Since
\[
\kappa(0) = \kappa = \frac{\sqrt{a^2 + 2\sigma^2} - a}{\sigma^2} < \frac{2\sqrt{a^2 + \sigma^2} - a}{\sigma^2},
\]
there exists $\eta > 0$ such that for $|\epsilon| < \eta$,
\[
\kappa(\epsilon) < \frac{2\sqrt{a^2 + \sigma^2} - a}{\sigma^2}.
\]
Set
\[
G = G(r_T) = C r_T e^{\frac{2\sqrt{a^2 + \sigma^2} - a}{\sigma^2} r_T} f(r_T)
\]
where $C = \sup_{|\epsilon| < \eta} |\kappa'(\epsilon)|$. Clearly,
\[
\sup_{|\epsilon| < \eta} \left| \frac{\partial}{\partial \epsilon} (\phi_{\epsilon}^{-1} f)(r_T) \right| \leq G.
\]
We now see that
\[
\mathbb{E}^\mathbb{P}[G] < \infty.
\]
It was assumed that the growth rate of $f(r)$ is less than $e^{mr}$ with $m < \frac{a}{\sigma^2}$, so the growth rate of $G(r)$ is less than $re^{\frac{2\sqrt{a^2 + \sigma^2}}{\sigma^2} r}$, which is less than $e^{\frac{2\sqrt{a^2 + \sigma^2}}{\sigma^2} r}$. We showed in Appendix G.1 that $\mathbb{E}^\mathbb{P}[e^{c r_T}]$ is convergent as $T$ goes to infinity for $c < \frac{2\sqrt{a^2 + \sigma^2}}{\sigma^2}$. In particular, we have $\mathbb{E}^\mathbb{P}[G] < \infty$. Therefore, we obtain that $\mathbb{E}^\mathbb{P}[(\phi_{\epsilon}^{-1} f)(r_T)]$ is differentiable at $\epsilon = 0$ and
\[
\frac{1}{T} \cdot \frac{\partial}{\partial \epsilon} \left|_{\epsilon=0} \mathbb{E}^\mathbb{P}[(\phi_{\epsilon}^{-1} f)(r_T)] \right| = \mathbb{E}^\mathbb{P} \left[ \left( \frac{\partial}{\partial \epsilon} \left|_{\epsilon=0} \phi_{\epsilon}^{-1} f \right) (r_T) \right] \right] = \kappa'(0) \cdot \mathbb{E}^\mathbb{P}[r_T e^{\kappa r_T} f(r_T)].
\]
By the same argument above, $\mathbb{E}^\mathbb{P}[r_T e^{\kappa r_T} f(r_T)]$ converges to a constant as $T$ approaches infinity because the growth rate of $r e^{\kappa r} f(r)$ is less than $e^{c r}$ with $c < \frac{2\sqrt{a^2 + \sigma^2}}{\sigma^2}$. In conclusion, we have that
\[
\frac{1}{T} \cdot \frac{\partial}{\partial \epsilon} \left|_{\epsilon=0} \mathbb{E}^\mathbb{P}[(\phi_{\epsilon}^{-1} f)(r_T)] \right| = \frac{k'(0)}{T} \cdot \mathbb{E}^\mathbb{P}[r_T e^{\kappa r_T} f(r_T)] \to 0
\]
as $T$ approaches infinity.
We apply Theorem [3.1] to conclude that
\[
\lim_{T \to \infty} \frac{1}{T} \frac{\partial}{\partial a} \ln p_T = -\theta \kappa'(0) = \frac{\theta(\sqrt{a^2 + 2\sigma^2} - a)}{\sigma^2 \sqrt{a^2 + 2\sigma^2}}.
\]
Using that
\[
k_{\kappa}(r) = \frac{\theta}{\sigma \sqrt{r}} \frac{\sqrt{(a + \epsilon)^2 + 2\sigma^2}}{\sigma \sqrt{r}},
\]

we know
\[ \frac{\partial}{\partial \epsilon} k_\epsilon(r) = -\frac{(a + \epsilon)\sigma}{\sqrt{(a + \epsilon)^2 + 2\sigma^2}} \sqrt{r}, \]
and thus
\[ k(r) = -\frac{a\sigma}{\sqrt{a^2 + 2\sigma^2}} \sqrt{r}. \]

Define \( g(r) = \sigma \sqrt{r} \). We show that \( g \) satisfies the hypothesis of the theorem. First, it is trivial that
\[ \left| \frac{\partial}{\partial \epsilon} k_\epsilon(r) \right| \leq \sigma \sqrt{r} = g(r) \]
because \( \frac{|(a+\epsilon)\sigma|}{\sqrt{(a+\epsilon)^2 + 2\sigma^2}} \leq \sigma \). Now it suffices to prove that
\[ E\mathbb{P} \left[ \exp \left( \int_0^T r_t dt \right) \right] \leq c(T) e^{dT} \]
for some constants \( d \) and \( c(T) \) with \( c(T) \) bounded on \( T \). For proof, refer to Lemma 3.1 on page 6 in [34]. For another condition, let \( \epsilon_1 = 2 \). It can be easily shown that \( E\mathbb{P}[r_T^2] \) is convergent to a constant as \( t \) approaches to infinity. We already showed that the condition \( \frac{1}{T} E\mathbb{P}[\Phi^{-1} f(X_T)] \to 0 \) as \( T \to \infty \) is satisfied with \( f \) described above.

### G.4 Sensitivity on \( \sigma \)

The sensitivity analysis of variable \( \sigma \) in the diffusion coefficient is explored. The perturbed process \( r_\epsilon^T \) follows
\[ dr_\epsilon^t = (\theta - ar_\epsilon^t) dt + (\sigma + \epsilon)\sqrt{r_\epsilon^t} dW_t. \]
The initial value is not perturbed, that is, \( r_\epsilon^0 = r_0 \). We know that
\[ (\lambda(\epsilon), \phi_\epsilon(r)) := (\theta \ell(\epsilon), e^{-\ell(\epsilon)r}) \]
stabilizes \( f \) described above, where
\[ \ell(\epsilon) := \frac{\sqrt{a^2 + 2(\sigma + \epsilon)^2} - a}{(\sigma + \epsilon)^2}. \]
Motivated by the discussion in Section 3.2.1 define \( u_\epsilon(r) = \frac{2}{\sigma + \epsilon} \sqrt{r} \) and \( U_\epsilon = u_\epsilon(r_\epsilon^T) \), then we have
\[ dU_\epsilon^t = \left( \left( \frac{2\theta}{(\sigma + \epsilon)^2} - \frac{1}{2} \right) U_\epsilon^t + \frac{a}{2} U_\epsilon^t \right) dt + dB_\epsilon^t, \]
\[ U_\epsilon^0 = \frac{2}{\sigma + \epsilon} \sqrt{r_0} \]
where \( b(\epsilon) = \sqrt{a^2 + 2(\sigma + \epsilon)^2} \). Here, \( B_\epsilon^t \) is a Brownian motion under the corresponding transformed measure \( \mathbb{P}_\epsilon \). The quantity \( p_T^\epsilon \) can be expressed by
\[ p_T^\epsilon := E\mathbb{P}^\epsilon[e^{-\int_0^T r_s^t ds} f(r_T^\epsilon)] = E\mathbb{Q}\left[e^{-\int_0^T R_s(U_\epsilon^s) ds} F_\epsilon(U_\epsilon^T)\right] \]
\[ = e^{-\ell(\epsilon) r_0} e^{-\theta(\epsilon) T} \cdot E\mathbb{P}_\epsilon^\epsilon[(\Phi^{-1}_\epsilon F_\epsilon)(U_\epsilon^T)] \]

31
where \( R_\varepsilon(u) := (\sigma + \varepsilon)^2 u^2 / 4 \), \( F_\varepsilon(u) := f((\sigma + \varepsilon)^2 u^2 / 4) \), \( \Phi_\varepsilon(u) = \phi_{\varepsilon}((\sigma + \varepsilon)^2 u^2 / 4) \) and \( q(\varepsilon) = \frac{2}{\sigma + \varepsilon} \sqrt{r_0} \).

Differentiate with respect to \( \varepsilon \) and evaluate at \( \varepsilon = 0 \), then

\[
\frac{\partial}{\partial \varepsilon} \left|_{\varepsilon=0} \frac{P_T}{T \cdot p_T} \right. = -\theta \ell'(0) - \frac{\ell'(0) r_0}{T} \frac{\partial}{\partial \varepsilon} \left|_{\varepsilon=0} \right. \mathbb{E}^p \left[ (\Phi_\varepsilon^{-1} F_\varepsilon)(U_T) \right] + \frac{\partial}{\partial \varepsilon} \left|_{\varepsilon=0} \right. \mathbb{E}^p \left[ (\Phi^{-1} F)(U_T) \right]. \tag{G.3}
\]

We now prove that

\[
\lim_{T \to \infty} \frac{1}{T} \frac{\partial}{\partial \sigma} \ln p_T = -\theta \ell'(0) = \frac{\theta (\sqrt{a^2 + 2\sigma^2} - a)^2}{\sigma^3 \sqrt{a^2 + 2\sigma^2}}
\]

by showing that the third and the last terms go to zero as \( T \) goes to infinity. For the last term, it is enough to show that the conditions in Theorem 3.1 are satisfied. To check the condition of the theorem, define

\[
k_\varepsilon(u) = \left( \frac{2\theta}{(\sigma + \varepsilon)^2} - \frac{1}{2} \right) \frac{1}{u} - \frac{b(\varepsilon)}{2} u.
\]

By direct calculation of \( \frac{\partial}{\partial \varepsilon} k_\varepsilon(u) \), it can be shown that there exists a number \( C > 0 \) such that

\[
\left| \frac{\partial}{\partial \varepsilon} k_\varepsilon(u) \right| \leq C \left( \frac{1}{u} + u \right)
\]

for \( \varepsilon \) near 0 and for all \( u > 0 \). Set \( g(u) := C \left( \frac{1}{u} + u \right) \). Because

\[
g^2(U_t) = C^2 \left( \frac{1}{U_t} + U_t \right)^2 \leq 2C^2 \left( \frac{1}{U_t^2} + U_t^2 \right) = 2C^2 \left( \frac{\sigma^2}{4r_t} + \frac{4r_t}{\sigma^2} \right) \leq C_1 \left( \frac{1}{r_t} + r_t \right)
\]

for sufficiently large \( C_1 > 0 \), to confirm the condition of the theorem, it suffices to show that for a small positive number \( \epsilon_0 \),

\[
\mathbb{E}^p \left[ \exp \left( \epsilon_0 \int_0^T (r_t + r_t^{-1}) \, dt \right) \right] \leq c(T) e^{aT}
\]

for some constants \( a \) and \( c(T) \) with \( c(T) \) bounded on \( T \). This was proven in section G.2 and G.3. We already showed that the other conditions in the theorem are satisfied.

Now it will be proven that the third term of equation (G.3) goes to zero as \( T \) goes to infinity. We will use that \( \frac{\partial}{\partial \varepsilon} \left|_{\varepsilon=0} \right. \frac{\partial}{\partial \sigma} \). The parameter \( \sigma \) is involved with both \( \Phi^{-1} F \) and the dynamics of \( U_T \). However, in the third term, the differentiation is involved only with the parameter \( \sigma \) in \( \Phi^{-1} F \). To distinguish the parameter \( \sigma \) in \( \Phi^{-1} F \) with that in the dynamics of \( U_T \), we will use parameter \( s \). Define

\[
\eta(s) = \frac{\sqrt{a^2 + 2s^2} - a}{s^2},
\]

\[
\pi_s(r) = e^{-\eta(s)r},
\]

\[
\Pi_s(u) = \pi_s(s^2 u^2 / 4),
\]

\[
G_s(u) = f(s^2 u^2 / 4),
\]

\[
\zeta(s) = \frac{2 \sqrt{r_0}}{s}.
\]

32
Then
\[ \frac{\partial}{\partial \epsilon} \bigg|_{\epsilon=0} \mathbb{E}^P_{\epsilon}(\Phi^{-1}F_\epsilon)(U_T) = \frac{\partial}{\partial s} \bigg|_{s=\sigma} \mathbb{E}^P_{\sigma}(\Pi^{-1}_s G_s)(U_T) . \]

It suffices to show that \( \frac{1}{T} \cdot \frac{\partial}{\partial s} \bigg|_{s=\sigma} \mathbb{E}^P_{\sigma}(\Pi^{-1}_s G_s)(U_T) \to 0 \) as \( T \to \infty \).

**Proposition G.2.**

\[ \frac{\partial}{\partial s} \bigg|_{s=\sigma} \mathbb{E}^P_{\sigma}(\Pi^{-1}_s G_s)(U_T) \]

is bounded on \( 0 < T < \infty \).

**Proof.** It is clear that \( (\Pi^{-1}_s G_s)(U_T) = (\pi^{-1}_s f)(s^2U_T^2/4) = (\pi^{-1}_s f)(Z_t) \) where \( Z_t = Z_t(s) = s^2U_T^2/4 \).

By direct calculation, we have
\[ dZ_t = \left( \frac{\theta s^2}{\sigma^2} - bZ_t \right) dt + s\sqrt{Z_t} dB_t , \quad Z_0 = r_0 . \]

It is noteworthy that two parameters \( \sigma \) and \( s \) are involved in the dynamics. One of nice properties of this process is that the initial value is not perturbed. We know that \( Z_t \) is a CIR process and the density function is given by
\[ g(z; t) = g_s(z; t) = h_t e^{-u-v \left( \frac{v}{u} \right)^{q/2}} I_q(2\sqrt{uv}) \]

where \( I_q \) is the modified Bessel function of the first kind of order \( q \) and
\[ h_t = \frac{2b}{s^2(1-e^{-bt})} , \quad q = \frac{2\theta}{\sigma^2} - 1 , \quad u = h_t z_0 e^{-bt} , \quad v = h_t z . \]

After rewriting slightly, we find
\[ g(z; t) = e^{-h_t z_0 e^{-bt}} (z_0 e^{-bt})^{-q/2} h_t e^{-h_t z q/2} I_q(2h_t e^{-bt/2} \sqrt{z_0 z}) . \]

We prove that
\[ \frac{\partial}{\partial s} \int_0^\infty (\pi^{-1}_s f)(z) g(z; t) dz = \int_0^\infty f(z) \frac{\partial}{\partial s} \pi^{-1}_s(z) g(z; t) dz \quad (G.4) \]

by using theorem B.1. It is enough to show that there exists a function \( G(z) \) such that
\[ \int_0^\infty G(z) dz < \infty , \]

and for \( s \) near \( \sigma \) and for all \( z > 0 \),
\[ \left| f(z) \frac{\partial}{\partial s} \pi^{-1}_s(z) g(z; t) \right| \leq G(z) . \]

Using \( \frac{\partial}{\partial s} h_t = -2h_t/s \), we have
\[ \frac{\partial}{\partial s} g(z; t) = \frac{2}{s} h_t z_0 e^{-bt} g(z; t) - \frac{2}{s} g(z; t) + \frac{2}{s} h_t g(z; t) \]
\[ - \frac{2}{s} e^{-h_t z_0 e^{-bt}} z_0^{(q+1)/2} e^{(q-1)bt/2} h_t^2 e^{-h_t z (q+1)/2} (I_q-1 + I_{q+1}) . \]
Here, we used $I_q' (\cdot) = \frac{1}{2} (I_{q-1} (\cdot) + I_{q+1} (\cdot))$.

Now we can find the decay rate of $|f(z) \frac{\partial}{\partial s} \pi_s^{-1}(z) g(z; t)|$. For large $t$ and for $s$ near $\sigma$, each term of $\frac{\partial}{\partial s} g(z; t)$ above is dominated by, up to constant multiples, one of

$$
\frac{\partial}{\partial s} g(z; t), \ z g(z; t), \ e^{-h(t)} z^{(q+1)/2} (I_{q-1} + I_{q+1}).
$$

The decay rate of each term is $e^{-h(t)}$ up to polynomial decay or growth rate. Thus, the decay rate of $|f(z) \frac{\partial}{\partial s} \pi_s^{-1}(z) g(z; t)|$ is less than or equal to $e^{(\eta(s)-h(t)) z}$ up to polynomial decay or growth rate. It was assumed that the growth rate of $f(z)$ is less than $e^{m z}$ with $m < \frac{a}{\sigma}$. We obtain that the decay rate of $|f(z) \frac{\partial}{\partial s} \pi_s^{-1}(z) g(z; t)|$ is less than $e^{(a/\sigma^2 + \eta(s)-h(t)) z}$, whose exponent satisfies

$$
\frac{a}{\sigma^2} + \eta(s) - h(t) = \frac{a}{\sigma^2} + \frac{\sqrt{a^2 + 2s^2} - a}{s^2} - \frac{2b}{s^2 (1 - e^{-bt})} < \frac{a}{\sigma^2} + \frac{\sqrt{a^2 + 2s^2} - a}{s^2} - \frac{2b}{s^2} = \frac{a}{\sigma^2} + \frac{\sqrt{a^2 + 2s^2} - a}{s^2} - \frac{2\sqrt{a^2 + 2\sigma^2}}{s^2}.
$$

When $s = \sigma$, the last term is $-\frac{\sqrt{a^2 + 2\sigma^2}}{s^2}$, thus, less than half of which the last term is for $s$ near $\sigma$ by the continuity argument. We have that for $s$ near $\sigma$,

$$
\frac{a}{\sigma^2} + \eta(s) - h(t) < -\frac{\sqrt{a^2 + 2\sigma^2}}{2\sigma^2}.
$$

Thus by setting $G(z) := C e^{-\frac{\sqrt{a^2 + 2\sigma^2}}{2\sigma^2} z}$ for sufficiently large $C$, we obtain the equation (G.4). Moreover, from this observation, we know that

$$
\int_0^\infty f(z) \frac{\partial}{\partial s} \bigg|_{s=\sigma} \pi_s^{-1}(z) g(z; T) \ dz
$$

is finite and converges to a constant as $T$ goes to infinity. Since

$$
\frac{\partial}{\partial s} \bigg|_{s=\sigma} \mathbb{E}_\pi^P [ \pi_s^{-1}(z) (\Pi_s^{-1} G_s) (U_T) ] = \frac{\partial}{\partial s} \bigg|_{s=\sigma} \mathbb{E}_\pi^P [ (\pi_s^{-1} f)(Z_T) ]
$$

$$
= \frac{\partial}{\partial s} \bigg|_{s=\sigma} \int_0^\infty (\pi_s^{-1} f)(z) g(z; T) \ dz = \int_0^\infty f(z) \frac{\partial}{\partial s} \bigg|_{s=\sigma} \pi_s^{-1}(z) g(z; T) \ dz,
$$

we conclude that $\frac{\partial}{\partial s} |_{s=\sigma} \mathbb{E}_\pi^P [ (\pi_s^{-1} f)(s^2 U_T^2/4) ]$ is bounded on $T$, which is the desired result. \qed

### G.5 Sensitivity on $r_0$

The sensitivity on the initial value $r_0$ is presented in this section. From the discussion in Section 4 we can write the quantity $\frac{\partial}{\partial r_0} p_T$ by

$$
\frac{\partial}{\partial r_0} p_T = -\kappa + \frac{\partial}{\partial r_0} \mathbb{E}_{r_0}^P [ (\phi^{-1} f)(r_T) ] \frac{\partial}{\partial r_0} \mathbb{E}_{r_0}^P [ (\phi^{-1} f)(r_T) ].
$$
It is well-known that the density function of $r_t$ from Appendix G.1

$$g(r; t) = e^{-h_t r_0 e^{-bt}} (r_0 e^{-bt})^{-q/2} h_t e^{-h_t r_0 q/2} I_q(2h_t e^{-bt/2} \sqrt[2]{r_0 r}).$$

It is enough to show that $\partial \frac{\partial}{\partial r_0} \mathbb{E}_{r_0}^p [(\phi^{-1} f)(r_T)] \rightarrow 0$ as $T \rightarrow \infty$. Corollary 4.2 cannot be used because the drift and the volatility do not satisfy the conditions of the corollary. However, the expectation depends only on the final value $r_T$, which is the beauty of the martingale extraction, we can easily calculate the expectation. Recall the density function of $r_0$, one can expect that

$$\lim_{t \rightarrow \infty} \frac{\partial g(r; t)}{\partial r_0} = 0$$

and the proof is as follows.

**Proof.** By direct calculation, we have

$$\frac{\partial g}{\partial r_0} = \left(-h_t e^{-bt} - \frac{q}{2r_0} + \frac{1}{2} \sqrt{\frac{r}{r_0}} h_t e^{-bt/2} \frac{I_{q-1}(z) + I_{q+1}(z)}{I_q(z)}\right) g$$

where $z = 2h_t e^{-bt/2} \sqrt{r_0}$. Here, we used $\frac{I_q'}{I_q} = \frac{1}{2}(I_{q-1} + I_{q+1})$. Observe that $z \rightarrow 0$ when $t \rightarrow \infty$. It is well-known that the modified Bessel function $I_q$ of order $q$ satisfies

$$\lim_{z \rightarrow 0} \frac{I_q(z)}{z^{(q+1)/2}} = 1.$$

We have that

$$\lim_{t \rightarrow \infty} h_t e^{-bt/2} \frac{I_{q-1}(z) + I_{q+1}(z)}{I_q(z)} = \lim_{t \rightarrow \infty} h_t e^{-bt/2} \frac{(h_t e^{-bt/2} \sqrt{r_0})^{q-1}}{\Gamma(q)} + \frac{(h_t e^{-bt/2} \sqrt{r_0})^{q+1}}{\Gamma(q+2)} = \frac{q}{\sqrt{r_0 r}}.$$

Thus, $\lim_{t \rightarrow \infty} \frac{\partial g}{\partial r_0} = 0$.

Now we prove that $\partial \frac{\partial}{\partial r_0} \mathbb{E}_{r_0}^p [(\phi^{-1} f)(r_T)] \rightarrow 0$ as $T \rightarrow \infty$. It follows that

$$\lim_{T \rightarrow \infty} \frac{\partial}{\partial r_0} \mathbb{E}_{r_0}^p [(\phi^{-1} f)(r_T)] = \lim_{T \rightarrow \infty} \frac{\partial}{\partial r_0} \int_0^\infty (\phi^{-1} f)(r) g(r; T) \, dr$$

$$= \lim_{T \rightarrow \infty} \int_0^\infty (\phi^{-1} f)(r) \frac{\partial g(r; T)}{\partial r_0} \, dr$$

$$= \int_0^\infty (\phi^{-1} f)(r) \lim_{T \rightarrow \infty} \frac{\partial g(r; T)}{\partial r_0} \, dr$$

$$= 0$$

which is the desired result. The interchangeability of the differentiation with the integration and the limit with the integration can be easily justified.
The quadratic term structure model

The martingale extraction with respect to

\[(\lambda, \phi(x)) = (\beta - \frac{1}{2} u^\top a u + tr(aV) + u^\top b , e^{-\langle u, x \rangle - \langle V, x \rangle})\]

stabilizes \(f\). The dynamics of \(X_t\) follows

\[dX_t = (b - au + (B - 2aV)X_t) dt + \sigma dB_t\]  \hspace{1cm} (H.1)

where \(B_t\) is a Brownian motion under the corresponding transformed measure.

H.1 Sensitivity on \(b\)

We present the sensitivity analysis of the quantity \(p_T\) with respect to \(b = (b_1, b_2, \cdots, b_d)^\top\). Consider the following perturbed process \(X_t^\epsilon\ :

\[dX_t^\epsilon = (b + \epsilon \bar{b} + BX_t^\epsilon) dt + \sigma dW_t\]

for some \(d\)-dimensional column vector \(\bar{b}\). By the chain rule, we may assume that \(\bar{b} = (1, 0, 0, \cdots, 0)^\top\). We already know the martingale extraction with respect to

\[(\lambda(\epsilon), \phi_\epsilon(x)) = (\beta - \frac{1}{2} u_\epsilon^\top a u_\epsilon + tr(aV) + u_\epsilon^\top b_\epsilon , e^{-\langle u_\epsilon, x \rangle - \langle V, x \rangle})\]

stabilizes \(f\), where \(b_\epsilon := b + \epsilon \bar{b}\) and \(u_\epsilon = (2Va - B^\top)^{-1}(2Vb_\epsilon + \alpha) = u + \epsilon(2Va - B^\top)^{-1}(2V\bar{b} + \alpha)\). The dynamics of \(X_t^\epsilon\) follows

\[dX_t^\epsilon = (b_\epsilon - au_\epsilon + (B - 2aV)X_t^\epsilon) dt + \sigma dB_t^\epsilon\]

where \(B_t^\epsilon\) is a Brownian motion under the corresponding transformed measure \(\mathbb{P}_\epsilon\).

We apply Theorem 3.1 to conclude that

\[\lim_{T \to \infty} \frac{1}{T} \frac{\partial}{\partial b_1} \ln p_T = -\lambda'(0) .\]

First, we check Condition 1 and 2 in Section 3. The first condition is clear by the above observation. For the second condition, we use Theorem 4.1. By direct calculation,

\[\frac{\partial}{\partial \epsilon} (\phi^{-1}_\epsilon f) = \left(\frac{\partial}{\partial \epsilon} u_\epsilon, x\right) \cdot e^{\langle u_\epsilon, x \rangle + \langle V, x \rangle} f(x) = \langle (2Va - B^\top)^{-1}(2V\bar{b} + \alpha), x \rangle \cdot e^{\langle u_\epsilon, x \rangle + \langle V, x \rangle} f(x) .\]

Since \(f\) is bounded and has a bounded support, we know that \(\left|\frac{\partial}{\partial \epsilon} (\phi^{-1}_\epsilon f)\right|\) is bounded by a constant for \(\epsilon\) near 0 and all \(x\). It follows that \(\mathbb{E}^\mathbb{P}[(\phi^{-1}_\epsilon f)(X_T)]\) is differentiable at \(\epsilon = 0\) and \(\frac{\partial}{\partial \epsilon} \mathbb{E}^\mathbb{P}[(\phi^{-1}_\epsilon f)(X_T)]\) is bounded on \(T\). This gives the desired result. We now check the conditions of the theorem. Let

\[k_\epsilon(x) = \sigma^{-1}b_\epsilon - \sigma^\top u_\epsilon + (\sigma^{-1}B - 2\sigma^\top V)x .\]

Then, \(\frac{\partial}{\partial \epsilon} k_\epsilon(x)\) is a constant vector independent of \(\epsilon\). Define \(g(x) = c_1\) for sufficiently large constant \(c_1\) so that \(\left|\frac{\partial}{\partial \epsilon} k_\epsilon(x)\right| \leq c_1\). Since \(g\) is a constant function, clearly \(g\) satisfies the conditions of the theorem. It is clear that \(\mathbb{E}^\mathbb{P}[(\phi^{-1} f)^2(X_T)]\) is bounded on \(0 < T < \infty\) by considering the Gaussian density of \(X_T\) because \(f\) is bounded and has bounded support.
H.2 Sensitivity on $B$

We investigate the sensitivity analysis of the quantity $p_T$ with respect to the matrix $B$. Consider the following perturbed process $X^\epsilon_t$:

$$
dX^\epsilon_t = (b + (B + \epsilon B)X^\epsilon_t) dt + \sigma dW_t
$$

for some $d \times d$ matrix $\overline{B}$. For convenience, set $B_\epsilon = B + \epsilon \overline{B}$. Let $V_\epsilon$ be the stabilizing solution of

$$
2V_\epsilon aV_\epsilon - B_\epsilon^T V_\epsilon - V_\epsilon B_\epsilon - \Gamma = 0 ,
$$

and let $u_\epsilon := (2V_\epsilon a - B_\epsilon^T)^{-1}(2V_\epsilon b + \alpha)$. We know the martingale extraction with respect to

$$(\lambda(\epsilon), \phi_\epsilon(x)) = (\beta - \frac{1}{2} u_\epsilon^T a u_\epsilon + tr(aV_\epsilon) + u_\epsilon^T b, e^{-(u_\epsilon, x)-(V_\epsilon, x)})$$

stabilizes $f$. The dynamics of $X^\epsilon_t$ follows

$$
dX^\epsilon_t = (b - au_\epsilon + (B_\epsilon - 2aV_\epsilon)X^\epsilon_t) dt + \sigma dB^\epsilon_t
$$

(H.2)

where $B^\epsilon_t$ is a Brownian motion under the corresponding transformed measure $P_\epsilon$.

We apply Corollary 3.3 to conclude that

$$
\lim_{T \to \infty} \frac{1}{T} \frac{\partial}{\partial \epsilon} \ln p^\epsilon_T = -\lambda'(0).
$$

First, we check Condition 1 and 2 in Section 3. For the first condition, it is enough to show that $V_\epsilon$ and $u_\epsilon$ are differentiable at $\epsilon = 0$. Here, the differentiability of a matrix means that all components are differentiable.

Proof. This proof is indebted to Appendix D in [11]. Consider the stabilizing solution $V_\epsilon$ of

$$
2V_\epsilon aV_\epsilon - B_\epsilon^T V_\epsilon - V_\epsilon B_\epsilon - \Gamma = 0 .
$$

The solution $V_\epsilon$ can be expressed by the following way. Define

$$
H_\epsilon = \begin{pmatrix}
B_\epsilon & -2a \\
-\Gamma & -B_\epsilon^T
\end{pmatrix}.
$$

Since a similarity transformation preserves the eigenvalues, the eigenvalues of $H_\epsilon$ are the same as those of $-H_\epsilon^T$. On the other hand, the eigenvalues of $H_\epsilon$ and $H_\epsilon^T$ must be same. Hence the spectral set of $H_\epsilon$ is the union of two sets $\Lambda^a_\epsilon$ and $\Lambda^b_\epsilon$ such that if $\beta \in \Lambda^a_\epsilon$, then $-\beta \in \Lambda^b_\epsilon$. According to the continuous-time algebraic Riccati equation theory, $H$ does not contain any eigenvalue on the imaginary axis when $\Gamma$ is positive definite. We can form $\Lambda^a_\epsilon$ such that it contains only the eigenvalues of $H_\epsilon$ that lie in the open left-half plane. Then there always exists a nonsingular matrix $P_\epsilon$ such that

$$
P^{-1}_\epsilon H_\epsilon P_\epsilon = \begin{pmatrix}
H^a_\epsilon & 0 \\
0 & H^b_\epsilon
\end{pmatrix}
$$

where $H^a_\epsilon$ and $H^b_\epsilon$ are diagonal matrices with eigenvalues sets $\Lambda^a_\epsilon$ and $\Lambda^b_\epsilon$, respectively. Write

$$
P_\epsilon = \begin{pmatrix}
P_{\epsilon,11} & P_{\epsilon,12} \\
P_{\epsilon,21} & P_{\epsilon,22}
\end{pmatrix},
$$

37
then \( V_\epsilon = P_{\epsilon,21}P_{\epsilon,11}^{-1} \) is the stabilizing solution.

Form this observation, we can prove that \( V_\epsilon \) is differentiable at \( \epsilon = 0 \). Since the eigenvalues of a matrix are continuously differentiable by the linear-perturbation in the components (see [IS]), we know that \( H^\epsilon \) and \( H^2 \) are differentiable, so \( P_\epsilon \) is also differentiable. Hence \( V_\epsilon \) is differentiable, which induces that \( u_\epsilon \) is also differentiable. This gives the desired result. \( \square \)

The second condition can be proven by the same way in the previous section. We now check the conditions of the theorem. Let

\[
k_\epsilon(x) = \sigma^{-1}b - \sigma^\top u_\epsilon + (\sigma^{-1}B_\epsilon - 2\sigma^\top V_\epsilon)x .
\]

Since \( V_\epsilon \) and \( u_\epsilon \) are continuously differentiable at \( \epsilon = 0 \), there exist sufficiently large constants \( c_1 \) and \( c_2 \) such that \( |\frac{\partial}{\partial \epsilon}k_\epsilon(x)| \leq c_1 + c_2|x| \) for \( \epsilon \) near 0 and for all \( x \in \mathbb{R}^d \). Define \( g(x) = c_1 + c_2|x| \).

To check the hypothesis of the corollary with \( \sigma \), it suffices to show that there exists a positive \( \epsilon_0 \) such that \( \mathbb{E}[\exp(\epsilon_0 |X_T|^2)] \) is finite on \( 0 < T < \infty \). Consider the density function of \( X_T \), which is a multivariate normal random variable.

\[
\mathbb{E}[\exp(\epsilon_0 |X_T|^2)] = \frac{1}{\sqrt{(2\pi)^d |\Sigma_T|}} \int_{\mathbb{R}^d} e^{\epsilon_0|z|^2 - \frac{1}{2}(z-\mu_T)^\top \Sigma_T^{-1}(z-\mu_T)} d\mu
\]

where \( \mu_T \) and \( \Sigma_T \) are the mean vector and the covariance matrix of \( X_T \), respectively. Under \( \mathbb{P} \), the coefficient of \( X_t \) in the drift term of equation (H.1) is \( B - 2\sigma V \), all of whose eigenvalues have negative real parts. Thus, the distribution of \( X_T \) is convergent to an invariant distribution, which is a non-degenerate multivariate normal random variable. Let \( \Sigma_\infty \) be the covariance matrix of the invariant distribution. Choose \( \epsilon_0 \) less than the smallest eigenvalue of \( \Sigma_\infty^{-1} \), then the above integral converges to a constant as \( T \to \infty \). Lastly, it is clear that \( \mathbb{E}[\exp(\epsilon_0 |X_T|^2)] \) is bounded on \( 0 < T < \infty \) by considering the Gaussian density of \( X_T \) because \( f \) is bounded and has bounded support.

### H.3 Sensitivity on \( \sigma \)

We investigate the sensitivity analysis of the quantity \( p_T \) with respect to the volatility matrix \( \sigma \). Consider the following perturbed process \( X^\epsilon_t \):

\[
dX^\epsilon_t = (b + BX^\epsilon_t) \, dt + (\sigma + \epsilon\overline{\sigma}) \, dB_t
\]

Set \( a_\epsilon = (\sigma + \epsilon\overline{\sigma})(\sigma + \epsilon\overline{\sigma})^\top \). Let \( V_\epsilon \) be the stabilizing solution of

\[
2V_\epsilon a_\epsilon V_\epsilon - B^\top V_\epsilon - V_\epsilon B - \Gamma = 0 ,
\]

and let \( u_\epsilon := (2V_\epsilon a_\epsilon - B^\top)^{-1}(2V_\epsilon b + \alpha ) \). We know the martingale extraction with respect to

\[
(\lambda(\epsilon), \phi(\epsilon)(x)) = (\beta - \frac{1}{2}u_\epsilon^\top a_\epsilon u_\epsilon + tr(a_\epsilon V_\epsilon) + u_\epsilon^\top b, e^{-\langle u_\epsilon, x \rangle - \langle V_\epsilon, x \rangle})
\]

stabilizes \( f \). The dynamics of \( X^\epsilon_t \) follows

\[
dX^\epsilon_t = (b - a_\epsilon u_\epsilon + (B - 2a_\epsilon V_\epsilon)X^\epsilon_t) \, dt + (\sigma + \epsilon\overline{\sigma}) \, dB^\epsilon_t
\]

where \( B^\epsilon_t \) is a Brownian motion under the corresponding transformed measure \( \mathbb{P}_\epsilon \).
We apply Theorem 3.4 to conclude that
\[
\lim_{T \to \infty} \frac{1}{T} \left. \frac{\partial}{\partial \epsilon} \right|_{\epsilon=0} \ln p_T^\epsilon = -\lambda'(0) .
\]

Condition 1 and 2 in Section 3 can be proven by the same way in the previous section. Define \( X_t^\rho \) and \( X_t^\nu \) as in Section 3.2.2 and recall the equation (3.8):
\[
\left. \frac{\partial}{\partial \epsilon} \right|_{\epsilon=0} \mathbb{E}^{P^\epsilon}[(\phi - 1 f)(X_T^\epsilon)] = \mathbb{E}^{P^\rho}[(\phi - 1 f)(X_T^\rho)] + \mathbb{E}^{P^\nu}[(\phi - 1 f)(X_T^\nu)] .
\]

We want to show \( \lim_{T \to \infty} \left. \frac{\partial}{\partial \epsilon} \right|_{\epsilon=0} \mathbb{E}^{P^\epsilon}[(\phi - 1 f)(X_T^\epsilon)] = 0 \).

For the first term, it can be shown that \( \lim_{T \to \infty} \left. \frac{\partial}{\partial \rho} \right|_{\rho=0} \mathbb{E}^{P^\rho}[(\phi - 1 f)(X_T^\rho)] = 0 \) by the same argument in Section H.1 and H.2 because \( X_t^\rho \) is involved with the perturbation in the drift. For the second term, it will be proven that
\[
\lim_{T \to \infty} \left. \frac{\partial}{\partial \nu} \right|_{\nu=0} \mathbb{E}^{P^\nu}[(\phi - 1 f)(X_T^\nu)] = 0 .
\]

Since \( X_t^\nu \) satisfies
\[
dX_t^\nu = \left( b - au + (B - 2aV)X_t^\nu \right) dt + \sigma dB_t^\nu ,
\]
the corresponding process \( Z_t \) is given by
\[
dZ_t = (B - 2aV)Z_t dt + \sigma dB_t .
\]

The process \( Z_t \) is an OU process and we know that all eigenvalues of \( B - 2aV \) have negative real parts, thus \( \mathbb{E}^\nu[|Z_T|^2] \) is convergent as \( T \) goes to infinity. Thus the conditions of Theorem 3.4 are satisfied for \( f \) continuously differentiable with compact support.

H.4 Sensitivity on \( \xi \)

We apply Corollary 4.2 to show that
\[
\lim_{T \to \infty} \frac{\nabla \xi p_T}{p_T} = \frac{\nabla \xi \phi(\xi)}{\phi(\xi)} = -u - 2V \xi .
\]

It is enough to show that \( \mathbb{E}^\xi[|Y_T|^2] \) is bounded on \( 0 < T < \infty \). The first variation process \( Y_t \) is given by \( dY_t = (B - 2aV)Y_t dt \) with \( Y_0 = I_d \), where \( I_d \) is the \( d \times d \) identity matrix. Thus,
\[
\mathbb{E}_\xi^\xi[|Y_T|^2] = \|Y_T\|^2 = \|e^{(B-2aV)T}\|^2
\]
and since all eigenvalues of \( B - 2aV \) have negative real parts, we obtain the desired result.

I The Heston model

The sensitivity of the quantity
\[
p_T := \mathbb{E}^Q[u(X_T)] = \mathbb{E}^Q[X_T^\alpha] = \mathbb{E}^Q[e^{\alpha \int_0^T \sqrt{v_t} dZ_t - \frac{1}{2} \alpha^2 \int_0^T v_t dt}] e^{\alpha \mu T} X_0^\alpha
\]
for large $T$ is of interest to us. Let $\mathbb{L}$ be a measure defined by

$$
\frac{d\mathbb{L}}{d\mathbb{Q}} |_{\mathcal{F}_T} = e^{\frac{1}{2} \int_0^T \sqrt{v_t} \sqrt{\alpha \mu_t} - \frac{1}{2} \int_0^T v_t \, dt},
$$

then, using the Girsanov theorem, a process $U_t$ given by $dU_t = -\alpha \sqrt{v_t} \, dt + dZ_t$ with $U_0 = 1$ is a Brownian motion under $\mathbb{L}$. It follows that

$$
p_T = \mathbb{E}^\mathbb{L}[e^{-\int_0^T \frac{1}{2} (1-\alpha) \int_0^T v_t \, dt}] e^{\alpha T} X_0^\alpha.
$$

For convenience, put $q_T := \mathbb{E}^\mathbb{L}[e^{-\int_0^T \frac{1}{2} (1-\alpha) \int_0^T v_t \, dt}]$. For some Brownian motion $Z_t$ independent of $Z_t$, we have

$$
dv_t = (\gamma - \beta v_t) \, dt + \delta \sqrt{v_t} \, dW_t = (\gamma - \beta v_t) \, dt + \rho \delta \sqrt{v_t} \, dZ_t + \sqrt{1 - \rho^2 \delta^2} \sqrt{v_t} \, dZ_t.
$$

It follows that

$$
dv_t = (\gamma - (\beta - \alpha \rho \delta) v_t) \, dt + \rho \delta \sqrt{v_t} \, dU_t + \sqrt{1 - \rho^2 \delta^2} \sqrt{v_t} \, dZ_t
$$

for a Brownian motion $B_t$ under $\mathbb{L}$. Define $r_t = \frac{1}{2} \alpha (1-\alpha) v_t$, then $r_t$ is a CIR model expressed by

$$
dr_t = \left( \frac{1}{2} \alpha (1-\alpha) \gamma - (\beta - \alpha \rho \delta) r_t \right) \, dt + \sqrt{\frac{2}{\alpha} (1-\alpha)} \sqrt{r_t} \, dB_t
$$

and $q_T$ is written by $q_T = \mathbb{E}^\mathbb{L}[e^{-\int_0^T r_t \, dt}]$. We already analyzed the sensitivity of $q_T$ for large $T$ in Section 5.2 by using the martingale extraction. To apply the chain rule, put $\theta = \frac{1}{2} \alpha (1-\alpha) \gamma$, $a = \beta - \rho \alpha \delta$, $\sigma = \sqrt{\frac{2}{\alpha} (1-\alpha)}$ and $r_0 = \frac{1}{2} \alpha (1-\alpha) v_0$, then $r_t$ is expressed by

$$
dr_t = (\theta - a r_t) \, dt + \sigma \sqrt{r_t} \, dB_t.
$$

In conclusion,

$$
\lim_{T \to \infty} \frac{1}{T} \frac{\partial}{\partial \mu} \ln q_T = \alpha
$$

$$
\lim_{T \to \infty} \frac{1}{T} \frac{\partial}{\partial \gamma} \ln q_T = \lim_{T \to \infty} \frac{1}{T} \frac{\partial}{\partial \gamma} \ln q_T = \lim_{T \to \infty} \frac{1}{T} \frac{\partial}{\partial \theta} \ln q_T \cdot \frac{\partial \theta}{\partial \gamma}
$$

$$
= \frac{1}{2} \alpha (1-\alpha) \lim_{T \to \infty} \frac{1}{T} \frac{\partial}{\partial \theta} \ln q_T
$$

$$
= \frac{1}{2} \alpha (1-\alpha) \cdot \frac{\sqrt{a^2 + 2\sigma^2} - a}{\sigma^2}
$$

$$
= \frac{1}{2} \alpha (1-\alpha) \cdot \frac{\sqrt{(\beta - \rho \alpha \delta)^2 + \delta^2 \alpha (1-\alpha)} - \beta + \rho \alpha \delta}{\delta^2}
$$

$$
\lim_{T \to \infty} \frac{1}{T} \frac{\partial}{\partial \beta} \ln q_T = \lim_{T \to \infty} \frac{1}{T} \frac{\partial}{\partial \beta} \ln q_T = \lim_{T \to \infty} \frac{1}{T} \frac{\partial}{\partial a} \ln q_T \cdot \frac{\partial a}{\partial \beta}
$$

$$
= \lim_{T \to \infty} \frac{1}{T} \frac{\partial}{\partial a} \ln q_T
$$

$$
= \frac{\theta (\sqrt{a^2 + 2\sigma^2} - a)}{\sigma^2 \sqrt{a^2 + 2\sigma^2}}
$$

$$
= \frac{\sqrt{(\beta - \rho \alpha \delta)^2 + \delta^2 \alpha (1-\alpha)} - \beta + \rho \alpha \delta}{\delta^2 \sqrt{(\beta - \rho \alpha \delta)^2 + \delta^2 \alpha (1-\alpha)}}
$$

40
\[
\lim_{T \to \infty} \frac{1}{T} \left( \frac{\partial}{\partial \alpha} \ln q_T \cdot \frac{\partial a}{\partial \delta} + \frac{\partial}{\partial \sigma} \ln q_T \cdot \frac{\partial \sigma}{\partial \delta} \right) = -\rho \alpha \lim_{T \to \infty} \frac{1}{T} \frac{\partial}{\partial a} \ln q_T + \sqrt{2} \alpha(1 - \alpha) \lim_{T \to \infty} \frac{1}{T} \frac{\partial}{\partial \sigma} \ln q_T
\]

\[
\lim_{T \to \infty} \frac{1}{T} \frac{\partial}{\partial \rho} \ln p_T = \lim_{T \to \infty} \frac{1}{T} \frac{\partial}{\partial \rho} \ln q_T = -\alpha \lambda \lim_{T \to \infty} \frac{1}{T} \frac{\partial}{\partial a} \ln q_T
\]

\[
\lim_{T \to \infty} \frac{\partial}{\partial \nu_0} \ln p_T = \lim_{T \to \infty} \frac{\partial}{\partial \nu_0} \ln q_T = \lim_{T \to \infty} \frac{\partial}{\partial r_0} \ln q_T \cdot \frac{\partial r_0}{\partial \nu_0} = \frac{1}{2} \alpha(1 - \alpha) \lim_{T \to \infty} \frac{\partial}{\partial r_0} \ln q_T
\]

\[
= -\frac{1}{2} \alpha(1 - \alpha) \cdot \sqrt{\alpha^2 + 2 \sigma^2} - a
\]

\[
= -\frac{1}{2} \alpha(1 - \alpha) \cdot \frac{\sqrt{(\beta - \rho \alpha \delta)^2 + \delta^2 \alpha(1 - \alpha) - \beta + \rho \alpha \delta}}{\delta^2}
\]

\[
\textbf{J The 3/2 model}
\]

Denote a perturbed process of \(X_t\) and the induced perturbed process of \(L_t\) by \(X'_t\) and \(L'_t\), respectively. The quantity

\[
\frac{\partial}{\partial \epsilon} \bigg|_{\epsilon = 0} \mathbb{E}^Q[u(L'_T)]
\]

will be investigated for large \(T\). We now find the corresponding martingale extraction. Using equation (6.1), the quantity \(p_T\) can be expressed by

\[
p_T := \mathbb{E}^Q[u(L_T)] = \mathbb{E}^Q[e^{-\frac{\alpha \beta (\beta - 1) \sigma^2}{2} \int_0^T X_u \, du} X_T^{\alpha \beta} \cdot e^{-r_0(\beta - 1)T}].
\]

Consider the operator

\[
f \mapsto \mathbb{E}^Q[e^{-\frac{\alpha \beta (\beta - 1) \sigma^2}{2} \int_0^T X_u \, du} f(X_t)]
\]
The corresponding infinitesimal generator is
\[ \frac{1}{2} \sigma^2 x^3 \phi''(x) + (\theta - ax)x \phi'(x) - \frac{1}{2} \alpha \beta (\beta - 1) \sigma^2 x \phi(x) = - \lambda \phi(x). \]

Set
\[ \ell := \sqrt{\left( \frac{1}{2} + \frac{a}{\sigma^2} \right)^2 + \alpha \beta (\beta - 1) - \left( \frac{1}{2} + \frac{a}{\sigma^2} \right)}. \]

It can be shown that the martingale extraction with respect to
\[ (\lambda, \phi(x)) := (\theta \ell, x^{-\ell}) \]
stabilizes \( f(x) := x^{\alpha \beta} \). Then, \( X_t \) follows
\[ dX_t = (\theta - (a + \sigma^2 \ell)X_t)X_t \, dt + \sigma X_t^{3/2} \, dB_t \]
where \( B_t \) is a Brownian motion under the corresponding transformed measure \( \mathbb{P} \).

### J.1 Sensitivity on \( \theta \)

Now, we see the sensitivity analysis of the expected utility and the return of \( L_t \) with respect to \( \theta \).

Consider the following perturbed process \( X^\epsilon_t \):
\[ dX^\epsilon_t = ((\theta + \epsilon) - a X^\epsilon_t)X^\epsilon_t \, dt + \sigma X^\epsilon_t^{3/2} \, dB^\epsilon_t \]
where \( B^\epsilon_t \) is a Brownian motion under the corresponding transformed measure \( \mathbb{P}^\epsilon \).

Theorem 3.2 will be applied to conclude that
\[ \lim_{T \to \infty} \frac{1}{T} \frac{\partial}{\partial \theta} \ln p_T = - \lambda'(0) = - \ell. \]

Condition 1 and 2 in Section 3 are clearly satisfied. For the second condition, \( \phi_\epsilon(x) = x^{-\ell} \) is independent of \( \epsilon \). We now show that the conditions of the theorem are satisfied. Using that
\[ k_\epsilon(x) = \left( \frac{\theta + \epsilon}{\sigma} \right) \frac{1}{\sqrt{x}} - \left( \frac{a}{\sigma} + \sigma \ell \right) \frac{1}{\sqrt{x}}, \]
we know \( \frac{\partial}{\partial x} k_\epsilon(x) = \frac{1}{\sigma \sqrt{x}} \), and thus \( k(x) = \frac{1}{\sigma \sqrt{x}} \). Define \( g(x) = \frac{1}{\sigma \sqrt{x}} \). It suffices to prove that
\[ \mathbb{E}^{\mathbb{P}} \left[ \exp \left( \int_0^T \frac{1}{X_t} \, dt \right) \right] \leq c(T) \ e^{aT} \]
for some constants \( a \) and \( c(T) \) with \( c(T) \) bounded on \( T \). Define \( r_t = 1/X_t \), then \( r_t \) is the CIR model and we already proved this condition is satisfied. For the CIR process \( r_t \), it is well-known that \( \mathbb{E}^{\mathbb{P}}[r^n_T] \) is convergent to a constant as \( T \to \infty \) for any \( n \in \mathbb{N} \). By considering the density function of \( r_t = 1/X_t \), it can be easily checked that for small \( \epsilon_2 > 0 \),
\[ \mathbb{E}^{\mathbb{P}}[(\phi^{-1} f)^{1+\epsilon_2}(X_T)] = \mathbb{E}^{\mathbb{P}}[X_T^{(1+\epsilon_2)(\alpha \beta + \ell)}] \]
is convergent as \( T \to \infty \).
J.2 Sensitivity on \(a\)

In this section, the sensitivity analysis of the expected utility and the return of \(L_t\) with respect to \(a\) is explored. Consider the following perturbed process \(X_t^{\epsilon}\):

\[
dX_t^{\epsilon} = (\theta - (a + \epsilon)X_t^{\epsilon})X_t^{\epsilon} \, dt + \sigma X_t^{\epsilon 3/2} \, dW_t.
\]

We already know

\[
(\lambda(\epsilon), \phi(\epsilon)) := (\theta \ell(\epsilon), x^{-\ell(\epsilon)})
\]
stabilizes \(f(x) = x^{\alpha \beta}\), where

\[
\ell(\epsilon) := \sqrt{\left(\frac{1}{2} + \frac{a + \epsilon}{\sigma^2}\right)^2 + \alpha \beta (\beta - 1) - \left(\frac{1}{2} + \frac{a + \epsilon}{\sigma^2}\right)}.
\]

The dynamics of \(X_t^{\epsilon}\) follows

\[
dX_t^{\epsilon} = (\theta - (a + \epsilon + \sigma^2 \ell(\epsilon))X_t^{\epsilon})X_t^{\epsilon} \, dt + \sigma X_t^{\epsilon 3/2} \, dB_t^{\epsilon}
\]

where \(B_t^{\epsilon}\) is a Brownian motion under the corresponding transformed measure \(\mathbb{P}_\epsilon\).

Theorem 3.1 will be applied to conclude that

\[
\lim_{T \to \infty} \frac{\partial \theta T}{\partial a} = -\theta \ell'(0) = \frac{\left(\sqrt{\left(\frac{1}{2} + \frac{a}{\sigma^2}\right)^2 + \alpha \beta (\beta - 1) - \left(\frac{1}{2} + \frac{a}{\sigma^2}\right)}\right) \theta}{\sigma^2 \sqrt{\left(\frac{1}{2} + \frac{a}{\sigma^2}\right)^2 + \alpha \beta (\beta - 1)}}.
\]

We first check Condition 1 and 2 in Section 3. The first condition is trivial. For the second condition, Theorem 3.1 is used. By direct calculation,

\[
\left| \frac{\partial}{\partial \epsilon} \left( \phi^{-1} f(X_t) \right) \right| = \left| \ell'(\epsilon) X_t^{\alpha \beta + \ell(\epsilon)} \ln X_t \right| \leq c_2 X_t^{\alpha \beta + \ell + 1}
\]

near \(\epsilon = 0\) for some positive constant \(c_2\). Since \(\mathbb{E}^{\mathbb{P}}[X_t^{\alpha \beta + \ell + 1}]\) is finite by considering the density of \(r_t := 1/X_t\), we obtain the desired result. We now show that the conditions of the theorem are satisfied. Using that

\[
k_\epsilon(x) = \frac{\theta}{\sigma \sqrt{x}} - \left(\frac{a + \epsilon}{\sigma} + \sigma \ell(\epsilon)\right) \sqrt{x},
\]

we know \(\frac{\partial}{\partial \epsilon} k_\epsilon(x) = -\left(\frac{1}{\sigma} + \sigma \ell'(\epsilon)\right) \sqrt{x}\), and thus \(k(x) = -\left(\frac{1}{\sigma} + \sigma \ell'(0)\right) \sqrt{x}\). For sufficiently large \(c_1 > 0\), we have that

\[
\left| \frac{\partial}{\partial \epsilon} k_\epsilon(x) \right| \leq c_1 \sqrt{x}
\]

near \(\epsilon = 0\) and for all \(x > 0\). Define \(g(x) = c_1 \sqrt{x}\). It suffices to prove that there exists a positive \(\epsilon_0\) such that

\[
\mathbb{E}^{\mathbb{P}} \left[ \exp \left( \epsilon_0 \int_0^T X_t \, dt \right) \right] \leq c(T) e^{aT}
\]

for some constants \(a\) and \(c(T)\) with \(c(T)\) bounded on \(T\). Define \(r_t = 1/X_t\), then \(r_t\) is the CIR model and we already proved this condition in Appendix G.2. For another condition, it is enough to show that \(\mathbb{E}^{\mathbb{P}}[(1/\sqrt{r_T})^{2+\epsilon_1}]\) is convergent as \(T \to \infty\) and it was shown in Appendix G.2. For the last condition, by considering the density function of \(r_t\), we know that for \(m \in \mathbb{N}\) with \(0 < m < \frac{2a}{\sigma^2} + 2\ell + 2\), \(\mathbb{E}^{\mathbb{P}}[X_t^{m}]\) converges to a constant. Because \((\phi^{-1} f)^2(X_t) = X_t^{2\alpha + 2\ell}\), it follows that the condition is satisfied when \(\frac{a}{\sigma^2} + 1 - \alpha \beta > 0\).
J.3 Sensitivity on $\sigma$

The sensitivity analysis of variable $\sigma$ in the diffusion coefficient is explored. The perturbed process $X_t^\epsilon$ follows

$$dX_t^\epsilon = (\theta - a X_t^\epsilon) X_t^\epsilon \, dt + (\sigma + \epsilon) X_t^{\epsilon^2/2} \, dW_t^\epsilon, \quad X_0^\epsilon = \xi.$$  

We know that

$$(\lambda(\epsilon), \phi(\epsilon)) := (\theta \ell(\epsilon), x^{-\ell(\epsilon)})$$  

stabilizes $f(x) = x^{\alpha \beta}$, where

$$\ell(\epsilon) := \sqrt{\left(\frac{1}{2} + \frac{a}{(\sigma + \epsilon)^2}\right)^2 + \alpha \beta (\beta - 1) - \left(\frac{1}{2} + \frac{a}{(\sigma + \epsilon)^2}\right)}.$$  

The dynamics of $X_t^\epsilon$ follows

$$dX_t^\epsilon = (\theta - (a + (\sigma + \epsilon)^2 \ell(\epsilon)) X_t^\epsilon) X_t^\epsilon \, dt + (\sigma + \epsilon) X_t^{\epsilon^2/2} \, dB_t^\epsilon$$  

where $B_t^\epsilon$ is a Brownian motion under the corresponding transformed measure $P_{\epsilon}$.

Motivated by the discussion in section 3.2.1, define $u_\epsilon(x) = \frac{2}{(\sigma + \epsilon)^{\sqrt{2}}} + u_\epsilon(X_t^\epsilon)$, then we have

$$dU_t^\epsilon = \left(\left(\frac{2a}{(\sigma + \epsilon)^2} + 2 \ell(\epsilon) + \frac{3}{2}\right) \frac{1}{U_t^\epsilon} - \frac{\theta}{2} U_t^\epsilon\right) dt - dB_t^\epsilon,$$

$$U_0^\epsilon = \frac{2}{(\sigma + \epsilon)^{\sqrt{2}}}.$$  

Here, $B_t^\epsilon$ is a Brownian motion under the corresponding transformed measure $P_{\epsilon}$. The quantity $p_T^\epsilon$ can be expressed by

$$p_T^\epsilon := e^{-\sigma(\beta - 1)T} \cdot E^\epsilon [e^{-\alpha \beta(\beta - 1)(\epsilon + \epsilon^2)/2} \int_0^T X_t^\epsilon \, dt + X_T^\epsilon].$$  

$$= \xi^{-\ell(\epsilon)} e^{-\sigma(\beta - 1)T} \cdot E^\epsilon [((\sigma + \epsilon) U_t^\epsilon/2)^{-2\beta - 2\ell(\epsilon)}].$$  

$$= \xi^{-\ell(\epsilon)} e^{-\sigma(\beta - 1)T} \cdot E^\epsilon [((\Phi^{-1} F_{\epsilon}) (U_T^\epsilon)].$$  

where $F_{\epsilon}(u) := ((\sigma + \epsilon) u^2/2)^{-2\alpha \beta}$, $\Phi_{\epsilon}(u) = \phi_{\epsilon}(((\sigma + \epsilon) u^2/2)^{-2}) = ((\sigma + \epsilon) u^2/2)^{-2\ell(\epsilon)}$ and $q(\epsilon) = 2/(\sigma + \epsilon)^{\sqrt{2}}$. Differentiate the above equation with respect to $\epsilon$ and evaluate at $\epsilon = 0$, then

$$\frac{\partial}{\partial \epsilon} \bigg|_{\epsilon=0} p_T^\epsilon = -\theta \ell(0) - \frac{-\ell'(0) \ln \xi}{T}$$  

$$+ \frac{\partial}{\partial \epsilon} \bigg|_{\epsilon=0} E^\epsilon [((\Phi^{-1} F_{\epsilon}) (U_T^\epsilon)] + \frac{\partial}{\partial \epsilon} \bigg|_{\epsilon=0} E^\epsilon [((\Phi^{-1} F) (U_T^\epsilon)] \cdot T.$$  

We now prove that

$$\lim_{T \to \infty} \frac{\partial p_T}{\partial \sigma} = -\theta \ell'(0) = \frac{2a \theta}{\sigma^3} \left(\sqrt{\left(\frac{1}{2} + \frac{a}{\sigma^2}\right)^2 + \alpha \beta (\beta - 1) - \left(\frac{1}{2} + \frac{a}{\sigma^2}\right)}\right)$$  

$$\sigma^3 \left(\sqrt{\left(\frac{1}{2} + \frac{a}{\sigma^2}\right)^2 + \alpha \beta (\beta - 1)}\right)$$
by showing that the third and the last terms go to zero as \( T \to \infty \). Using the same method in Proposition [G.2] it can be proven that the third term goes to zero as \( T \) goes to infinity. For the last term, Theorem 3.1 is applied. Define

\[
    k_\epsilon(u) = \left( \frac{2a}{(\sigma + \epsilon)^2} + 2\ell(\epsilon) + \frac{3}{2} \right) \frac{1}{u} - \frac{\theta}{2u}.
\]

By direct calculation of \( \frac{\partial}{\partial \epsilon} k_\epsilon(u) \), we have that there exists a number \( c_1 > 0 \) such that

\[
    \left| \frac{\partial}{\partial \epsilon} k_\epsilon(u) \right| \leq \frac{c_1}{u}
\]

for \( \epsilon \) near 0 and for all \( u > 0 \). Set \( g(u) := \frac{a}{u} \). Using \( g^2(U_t) = \frac{a^2}{U_t} = c_2X_t \) for sufficiently large \( c_2 > 0 \), it can be shown that the conditions of the theorem are satisfied by the same method in Appendix J.2 when \( \frac{a}{\sigma^2} + 1 - \alpha\beta > 0 \).

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