The Generating Formula for The Solutions to The Associativity Equations

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Abstract

An exact formula for the solutions to the WDVV equation in terms of horizontal sections of the corresponding flat connection is found.

1. Introduction. The associativity equations (or WDDV equations) appeared in the classification problem for the topological field theories at the early 90's, see [1], [2]. During the last years these equations has attracted a great interest due to connections with the enumerative geometry (Gromov – Witten invariants [3]), quantum cohomology [4] and the Whitham theory [5].

In [3] it was noticed that the classification problem for the topological field theories is equivalent to the classification of the Egoroff metrics of special type. Egoroff metrics are flat diagonal metrics

\[ ds^2 = \sum_{i=1}^{n} h_i^2(u)(du^i)^2, \quad u = (u^1, \ldots, u^n) \]  

such that \( \partial_i h_j^2(u) = \partial_j h_i^2(u) \), where \( \partial_i = \partial/\partial u^i \). It turns out that for every Egoroff metric with the additional constraints \( \sum_{j=1}^{n} \partial_j h_i = 0 \) the functions

\[ c_{kl}^m(x) = \sum_{i=1}^{n} \frac{\partial x^m}{\partial u^i} \frac{\partial u^i}{\partial x^k} \frac{\partial u^i}{\partial x^l}, \]  

where \( x^k(u) \) are the flat coordinates of the metric [3], satisfy

\[ c_{ij}^k(x)c_{km}^l(x) = c_{jm}^k(x)c_{ik}^l(x), \]  

which are the associativity conditions for the algebra \( \phi_k \phi_l = c_{kl}^m \phi_m \). Moreover, there exists a function \( F(x) \) such that its third derivatives are equal to

\[ \frac{\partial^3 F(x)}{\partial x^k \partial x^l \partial x^m} = c_{klm}(x) = \eta_{mi} c_{kl}^i(x), \quad \text{where} \quad \eta_{pq} = \sum_{i=1}^{n} h_i^2(u) \frac{\partial u^i}{\partial x^p} \frac{\partial u^i}{\partial x^q}. \]  

In the topological field theory with \( n \) primary fields \( \phi_k \) function \( F \) plays role of the partition function. There also exist constants \( r^m \) such that \( \eta_{kl} = r^m c_{klm}(x) \).

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Equations (3) and the existence of the function $F$ such that (4) holds are equivalent to the compatibility conditions of the linear system (see [3])

$$\partial_k \Phi_l - \lambda c_{kl}^m \Phi_m = 0,$$

where $\lambda$ is a spectral parameter.

In few special cases the function $F$ was found explicitly (see [3], [6], [4]). However, in general case an expression for $F$ in terms of the flat sections of the connection $\nabla_k = \partial / \partial x^k - \lambda c_{kl}^m$ has been unknown. The main goal of these notes is to write down such an expression. It was motivated by the results of [8], where explicit formulas for the algebraic-geometrical solutions to the associativity equations were found. Though the general formula of [8] (Theorem 5.1) is correct, in the final formula (Theorem 5.2) one of the terms was missed. Here we take an opportunity to fix it.

2. Let us consider a solution $\beta_{ij}(u) = \beta_{ji}(u)$ to the Darboux–Egoroff system

$$\partial_k \beta_{ij} = \beta_{ik}\beta_{kj}, \quad i \neq j \neq k; \quad \sum_{m=1}^{n} \partial_m \beta_{ij} = 0, \quad i \neq j.$$  

(6)

Following [3] we fix the unique Egoroff metric (1) corresponding to this solution by defining the Lame coefficients $h_i(u)$ from the system

$$\partial_j h_i(u) = \beta_{ij}(u) h_j(u), \quad i \neq j; \quad \partial_i h_i(u) = -\sum_{j \neq i} \beta_{ij}(u) h_j(u),$$  

(7)

and initial conditions $h_i(0) = 1$, $i = 1, \ldots, n$. This system is compatible due to (3). The condition $\beta_{ij} = \beta_{ji}$ implies that the above defined metric is in fact Egoroff metric.

The flat coordinates $x^1, \ldots, x^n$ of this metric can be found from the linear system

$$\partial_i \partial_j x^k = \Gamma_{ij}^l \partial_i x^l + \Gamma_{ji}^l \partial_j x^l, \quad i \neq j; \quad \partial_i \partial_i x^k = \sum_{j=1}^{n} \Gamma_{ii}^j \partial_j x^k,$$

(8)

where $\Gamma_{ij}^k$ are Christoffel symbols: $\Gamma_{ij}^k = \partial_j h_i / h_i$, $\Gamma_{ii}^j = (2\delta_{ij} - 1)(h_i \partial_j h_i)/(h_j^2)$. We choose the following initial conditions: $x^k(0) = 0$, $\sum_{k,l} \eta_{kl} \partial_i x^k(0) \partial_j x^l(0) = \delta_{ij}$. Here $\eta_{kl}$ is the fixed symmetric nondegenerate matrix.

The Darboux–Egoroff system can be regarded as the compatibility conditions of the following linear system:

$$\partial_j \Psi_i(u, \lambda) = \beta_{ij}(u) \Psi_j(u, \lambda), \quad i \neq j$$

$$\partial_i \Psi_i(u, \lambda) = \lambda \Psi_i(u, \lambda) - \sum_{k \neq i} \beta_{ik}(u) \Psi_k(u, \lambda),$$

(9) (10)

Here $\Psi_i = (\Psi_i^1, \ldots, \Psi_i^n)$, $i = 1, \ldots, n$, are vector-functions which are formal power series in $\lambda$. There exists a unique solution of this linear system with the initial conditions $\Psi_i^k(0, \lambda) = \lambda \partial_i x^k(0)$.

It follows from (9), (10) that the expansion of $\Psi_i$ has the form

$$\Psi_i(u, \lambda) = \sum_{s=0}^{\infty} \frac{\partial_i x^k(u)}{h_i(u)} \lambda^s,$$

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where $\xi_0^k = r^k$ are constants (we will define them later), $\xi^k_1(u) = x^k(u)$ are flat coordinates and $\xi^k_s$ for $s \geq 2$ obey the following equations

\begin{align}
\partial_i\partial_j\xi^k_s &= \Gamma^i_{ij}\partial_i\xi^k_s + \Gamma^j_{ij}\partial_j\xi^k_s, \\ \partial_i\partial\xi^k_s &= \sum_{j=1}^n \Gamma^j_{ii}\partial_j\xi^k_s + \partial_i\xi^k_{s-1},
\end{align}

with the initial conditions $\xi^k_s(0) = 0$, $\partial_i\xi^k_s(0) = 0$. The equations

\begin{equation}
\frac{\partial^2\xi^m_s}{\partial x^k\partial x^l} = \sum_{p=1}^n c^{p}_{kl} \frac{\partial\xi^m_{s-1}}{\partial x^p},
\end{equation}

where $c^{p}_{kl}$ are defined by (2), follow immediately from (11), (12). We will denote $\xi^k_2(u)$ and $\xi^k_3(u)$ by $y^k(u)$ and $z^k(u)$, respectively.

3. Let $\psi$ be given by the formula

\begin{equation}
\lambda\psi(u, \lambda) = \sum_{i=1}^n h^i(u)\Psi_i(u, \lambda).
\end{equation}

It’s straightforward to check that $\partial_i\psi(u, \lambda) = h^i(u)\Psi_i(u, \lambda)$. As a formal power series on $\lambda$ the $k$-th component of the vector-function $\psi(u, \lambda)$ has the form

\begin{equation}
\psi^k(u, \lambda) = r^k + x^k(u)\lambda + y^k(u)\lambda^2 + z^k(u)\lambda^3 + \sum_{s=4}^\infty \xi^k_s(u)\lambda^s.
\end{equation}

Note that (13) implies that the functions $\Phi_k(x) = \partial\psi(x)/\partial x^k$ satisfy (3). Moreover,

\begin{equation}
\lambda\psi(x) = \sum_{k=1}^n r^k\Phi_k(x)
\end{equation}

and thus the function $\psi$ can be regarded as a generating function for the flat sections of the above-defined connection $\nabla_k$.

**Lemma 1.** The vector-functions $\Psi_i(u, \lambda)$ satisfy the equations

\begin{equation}
\left\langle \Psi_i(u, \lambda), \Psi_j(u, -\lambda) \right\rangle = -\delta_{ij}\lambda^2,
\end{equation}

the scalar product given by the matrix $\eta_{kl}$: $\langle A, B \rangle = \eta_{kl}A^kB^l$.

**Lemma 2.** The scalar product

\begin{equation}
\left\langle \frac{1}{\lambda}\partial_i\psi(u, \lambda), \psi(u, -\lambda) \right\rangle
\end{equation}

does not depend on $\lambda$.  

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Both lemmas can be proved in the similar way. First we establish the required properties of the scalar products (16), (17) at the point $u = 0$. Then we show that they satisfy certain differential equations. The Uniqueness Theorem implies both statements.

**Corollary.** The functions $x^k(u)$, $y^k(u)$ and $z^k(u)$ satisfy the following relation:

$$\sum_{q=1}^{n} \eta_{kq} y^q = \sum_{p,q=1}^{n} \eta_{pq} \left( x^q \frac{\partial y^p}{\partial x^k} - r^q \frac{\partial z^p}{\partial x^k} \right).$$

(18)

**Proof.** Consider the scalar product $\langle \frac{1}{\lambda} \frac{\partial \psi}{\partial x^k}(u, \lambda), \psi(u, -\lambda) \rangle$. Since $\partial \psi / \partial x^k$ is the linear combination of $\partial \psi / \partial u^i$ Lemma 2 implies that this scalar product is independent of $\lambda$. On the other hand, it can be presented as a power series in $\lambda$

$$\langle \frac{1}{\lambda} \frac{\partial \psi}{\partial x^k}, \psi^\sigma \rangle = \frac{1}{\lambda} \sum_{p,q=1}^{n} \eta_{pq} \left( \delta^p_k \lambda + \frac{\partial y^p}{\partial x^k} \lambda^2 + \frac{\partial z^p}{\partial x^k} \lambda^3 + \ldots \right) \left( r^q - x^q \lambda + y^q \lambda^2 - z^q \lambda^3 + \ldots \right)$$

(19)

(here $\psi^\sigma(u, \lambda) = \psi(u, -\lambda)$). Therefore, all but the first coefficients of the series (19) should equal zero. Applying this argument to the coefficient of $\lambda^2$ we obtain (18).

**Theorem.** The function $F(x) = F(u(x))$ defined by the formula

$$F(u) = \frac{1}{2} \sum_{p,q=1}^{n} \eta_{pq} \left( x^q(u) y^p(u) - r^q z^p(u) \right)$$

(20)

satisfies the equation (4).

**Proof.** Let us notice that Corollary implies $\partial F / \partial x^k = \sum_{q=1}^{n} \eta_{kq} y^q$. Now the statement of the Theorem is direct implication of (4) and (13) for $s = 2$.

From the previous formulas it follows that $F$ satisfies the renormalization group type equation:

$$F(x) - \sum_{k=1}^{n} x^k \frac{\partial F}{\partial x^k} = - \sum_{p,q=1}^{n} \eta_{pq} r^q z^p.$$

In our next paper we hope to obtain more general formula, where the function $F$ depends on infinitely many variables corresponding to gravitational descendants of the primary fields $\phi_k$.

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