APPLICATIONS OF EXTREMAL FUNCTIONS IN STUDIES OF NORMALIZED SOLUTIONS TO LOWER CRITICAL CHOQUARD EQUATION

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Abstract. This paper studies the applications of the extremal functions associated with the Gagliardo-Nirenberg inequality and the Hardy-Littlewood-Sobolev inequality in the existence and non-existence of normalized solutions to the lower critical Choquard equation with a local perturbation

\begin{equation}
\begin{aligned}
-\Delta u + \lambda u &= \gamma(I_{\alpha} * |u|^{\frac{N+\alpha}{N}})|u|^{\frac{N+\alpha}{N}} - 2u + \mu|u|^{q-2}u, \quad \text{in } \mathbb{R}^N, \\
\int_{\mathbb{R}^N}|u|^2dx &= c^2,
\end{aligned}
\end{equation}

where \(\gamma, \mu, c\) are given positive numbers, \(2 < q \leq 2 + \frac{4}{N}\), and \(\lambda \in \mathbb{R}\) is an unknown parameter that appears as a Lagrange multiplier. The results of this paper settle some open questions proposed by Yao, Chen, Rădulescu and Sun [Siam J. Math. Anal., 54(3) (2022), 3696-3723].

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1. Introduction and main results

In this paper, we study the applications of the extremal functions to the Gagliardo-Nirenberg inequality (see [15])

\begin{equation}
\begin{aligned}
\left(\int_{\mathbb{R}^N}|u|^r dx\right)^{\frac{1}{r}} &\leq \tilde{S}^{\frac{1}{r}} \left(\int_{\mathbb{R}^N}|u|^2 dx\right)^{\frac{1-r}{2}} \left(\int_{\mathbb{R}^N}|\nabla u|^2 dx\right)^{\frac{r}{2}}, \quad u \in H^1(\mathbb{R}^N), \\
2 < r < \left(\frac{2N}{N-2}\right)^{\frac{1}{2}}, \quad \eta_r := \frac{N}{2} - \frac{N}{r},
\end{aligned}
\end{equation}

and the equality holds if and only if \(u\) is the unique positive radial solution to the equation \(-\Delta u + u = |u|^{r-2}u\)

and the extremal functions to the Hardy-Littlewood-Sobolev inequality (see [11])

\begin{equation}
\begin{aligned}
\left(\int_{\mathbb{R}^N}(I_{\alpha} * |u|^{\frac{N+\alpha}{N}})|u|^{\frac{N+\alpha}{N}} dx\right)^{\frac{N}{N-\alpha}} &\leq \int_{\mathbb{R}^N}|u|^2dx, \\
\text{whose extremal functions are } u(x) = C \left(\frac{\epsilon}{|x|^2}\right)^{N/2}, \quad C \in \mathbb{R}, \epsilon > 0
\end{aligned}
\end{equation}

in the existence and non-existence of normalized solutions to the lower critical Choquard equation with a local term

\begin{equation}
\begin{aligned}
-\Delta u + \lambda u &= \gamma(I_{\alpha} * |u|^{\frac{N+\alpha}{N}})|u|^{\frac{N+\alpha}{N}} - 2u + \mu|u|^{q-2}u, \quad \text{in } \mathbb{R}^N, \\
\int_{\mathbb{R}^N}|u|^2dx &= c^2,
\end{aligned}
\end{equation}

Key words and phrases. Normalized solutions; Existence and non-existence; Extremal functions; Lower critical Choquard equation; Variational methods.

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where $N \geq 1$, $\gamma, \mu, c > 0$, $2 < q \leq 2 + \frac{4}{N}$, $\lambda \in \mathbb{R}$ is an unknown parameter that appears as a Lagrange multiplier, $\alpha \in (0, N)$, and $I_\alpha$ is the Riesz potential defined for every $x \in \mathbb{R}^N \setminus \{0\}$ by

$$I_\alpha(x) := \frac{A_\alpha(N)}{|x|^{N-\alpha}}, \quad A_\alpha(N) := \frac{\Gamma\left(\frac{N-\alpha}{2}\right)}{\Gamma\left(\frac{N}{2}\right)\pi^{N/2}}$$

with $\Gamma$ denoting the Gamma function (see [14]).

A solution $u$ to the problem (1.3) corresponds to a critical point of the functional

$$E_q(u) := \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx - \frac{\gamma N}{2(N+\alpha)} \int_{\mathbb{R}^N} (I_\alpha * |u|^\frac{N+\alpha}{N-\alpha})|u|^\frac{N+\alpha}{N-\alpha} dx - \frac{\mu}{q} \int_{\mathbb{R}^N} |u|^q dx$$

restricted to the sphere

$$S(c) := \{ u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} |u|^2 dx = c^2 \}.$$

It is well known that $E_q \in C^1(H^1(\mathbb{R}^N), \mathbb{R})$ and

$$E_q(u)\varphi = \int_{\mathbb{R}^N} \nabla u \nabla \varphi dx - \gamma \int_{\mathbb{R}^N} (I_\alpha * |u|^\frac{N+\alpha}{N-\alpha})|u|^\frac{N+\alpha}{N-\alpha} \varphi dx - \mu \int_{\mathbb{R}^N} |u|^{q-2} u \varphi dx$$

for any $\varphi \in H^1(\mathbb{R}^N)$.

One motivation driving the search for normalized solutions to (1.3) is the non-linear equation

(1.4) \quad $i\partial_t \psi - \Delta \psi - \gamma (I_\alpha * |\psi|^p)|\psi|^{p-2}\psi - \mu |\psi|^{q-2}\psi = 0,$ \quad $(t, x) \in \mathbb{R} \times \mathbb{R}^N,$

where $\frac{N+\alpha}{N} \leq p \leq \frac{N+\alpha}{(N-2)\gamma}$ and $2 < q \leq \frac{2N}{(N-2)\gamma}$. Searching for standing wave solution $\psi(t, x) = e^{-i\lambda t}u(x)$ of (1.4) leads to

(1.5) \quad $-\Delta u + \lambda u = \gamma (I_\alpha * |u|^p)|u|^{p-2}u + \mu |u|^{q-2}u,$ \quad in $\mathbb{R}^N$.

The research of finding normalized solutions to (1.5) (i.e., solutions to (1.5) restricted to $S(c)$) is meaningful from the physical point of view, as the $L^2$-norm is a preserved quantity of the evolution and the variational characterization of such solutions is often a strong help to analyze their orbital stability, see [4] and the references therein. Recently, Li [8, 9] studied the existence, multiplicity, orbital stability and instability of the normalized solutions to the upper critical Choquard equation (1.5), i.e., $p = \frac{N+\alpha}{N-2}$. As to the lower critical Choquard equation, the lower critical exponent $p = \frac{N+\alpha}{N}$ seems to be a new feature for the Choquard equation, which is related to a new phenomenon of “bubbling at infinity” (see [13]). So compared with the study developed for the upper critical Choquard equation, the lower critical problem seems to be more challenging. In a recent paper [16], the authors explored new methods and ideas to study the normalized solutions to the lower critical Choquard equation, meanwhile they proposed some open questions. In this paper, by using the extremal functions corresponding to the equalities (1.1) and (1.2), we settle parts of the open questions. See [3, 5, 6, 7, 10, 12, 17, 18] for more results about the studies of normalized solutions to the Choquard equation.

The main results of this paper are as follows.

**Theorem 1.1.** Let $\gamma, \mu > 0$, $2 < q < 2 + \frac{4}{N}$. Then the infimum

$$\sigma(c) := \inf_{S(c)} E_q(u) < -\frac{\gamma N}{2(N+\alpha)} S_\alpha^{-\frac{N+\alpha}{N}} e^{\frac{2(N+\alpha)}{N}}$$

is achieved by $u_0 \in S(c)$ with the following properties:
(i) $u_0$ is a real-valued positive function in $\mathbb{R}^N$, which is radially symmetric and non-increasing;

(ii) $u_0$ is a ground state of (1.3) with some $\lambda_e > \frac{\gamma N}{N + \alpha} S_{\alpha}^{-\frac{N + \alpha}{N}} e^{\frac{2(N + \alpha)}{N}}$. A ground state $v$ of (1.3) is defined as follows:

$$E_q |_{S(c)}(v) = 0 \text{ and } E_q(v) = \inf\{E_q(w) : w \in S(c), E_q |_{S(c)}(w) = 0\}.$$

Theorem 2.1. Let $\gamma, \mu > 0$, $q = 2 + \frac{4}{N}$. If $0 < \mu c^{4/N} < \frac{N + 2}{N S}$, then (1.3) has no solutions.

This paper is motivated by [16], where Theorem 1.1 is obtained if $\mu$ is larger than some positive constant, and Theorem 1.2 is obtained if $0 < \mu c^{4/N} < \frac{N + 2}{N S}$. In this paper, by using the extremal functions of (1.1) and (1.2), we can improve their results. Moreover, we give a different proof for Theorem 1.2 and further give some insights for other cases (see Remark 2.3).

Notation: For $t \geq 1$, the $L^t$-norm of $u \in L^t(\mathbb{R}^N)$ is denoted by $\|u\|_t$. $o_n(1)$ denotes a real sequence with $o_n(1) \to 0$ as $n \to +\infty$. `$\rightarrow$' denotes strong convergence and `−' denotes weak convergence. $B_r(x_0) := \{x \in \mathbb{R}^N : |x - x_0| < r\}$.

2. Proof of the main results

2.1. The case $2 < q < 2 + \frac{4}{N}$.

Lemma 2.1. Let $\gamma, \mu > 0$ and $2 < q < 2 + \frac{4}{N}$. Then

(1) The functional $E_q$ is bounded below and coercive on $S(c)$;

(2) $\sigma(c) < -\frac{\gamma N}{2(N + \alpha)} S_{\alpha}^{-\frac{N + \alpha}{N}} e^{\frac{2(N + \alpha)}{N}}$, where $\sigma(c)$ is defined in Theorem 1.1;

(3) For $0 < c_1 < c_2$, there holds $\frac{c_1}{c_2} \sigma(c_2) < \sigma(c_1)$.

Proof. (1) By (1.1) and (1.2), for any $u \in S(c)$, one has

$$E_q(u) \geq \frac{1}{2} \|\nabla u\|_2^2 - \frac{\gamma N}{2(N + \alpha)} S_{\alpha}^{-\frac{N + \alpha}{N}} e^{\frac{2(N + \alpha)}{N}} - \frac{\mu}{q} S_{\alpha}^{(1 - \eta_0)} \|\nabla u\|_2^q,$$

which implies that $E_q$ is bounded below and coercive on $S(c)$ for $\eta \delta q < 2$.

(2) By (1.2), we choose $v$ such that

$$S_{\alpha} \left( \int_{\mathbb{R}^N} (J_{\alpha} * |v|^{\frac{N + \alpha}{\alpha}})|v|^{\frac{N + \alpha}{\alpha}} dx \right)^{\frac{\alpha}{N + \alpha}} = \int_{\mathbb{R}^N} |v|^2 dx.$$

Define $u := \frac{v}{\|v\|_2}$ and $u_{\tau}(x) := \tau^{N/2} u(\tau x)$ for $\tau > 0$, then $u_{\tau} \in S(c)$ for all $\tau > 0$. By direct calculations, we deduce that

$$E_q(u_{\tau}) = \frac{1}{2} \tau^2 \|\nabla u\|_2^2 - \frac{\gamma N}{2(N + \alpha)} \int_{\mathbb{R}^N} (J_{\alpha} * |u|^{\frac{N + \alpha}{\alpha}})|u|^{\frac{N + \alpha}{\alpha}} dx - \frac{\mu}{q} \tau^{\eta_n} \|u\|_q^q$$

$$= \frac{1}{2} \tau^2 \|\nabla u\|_2^2 - \frac{\mu}{q} \tau^{\eta_n} \|u\|_q^q - \frac{\gamma N}{2(N + \alpha)} S_{\alpha}^{-\frac{N + \alpha}{N}} e^{\frac{2(N + \alpha)}{N}}$$

$$< - \frac{\gamma N}{2(N + \alpha)} S_{\alpha}^{-\frac{N + \alpha}{N}} e^{\frac{2(N + \alpha)}{N}}$$

for $\tau > 0$ small enough. This proves (2).
We first verify that there exists $\phi \in X$ such that
\[ \int_{\mathbb{R}^N} (I_{\alpha} * |u_n|^{\frac{N+\alpha}{N}})|u_n|^{\frac{N-\alpha}{N}} dx \leq 4X\text{INFU LI, JIANGUANG BAO, WENGUANG TANG}
\]
which implies that $E_{q}(\nu u_n) < \nu^2 E_{q}(u_n)$. Consequently, $\sigma(\nu c_{1}) \leq \nu^2 \sigma(c_{1})$, where the equality holds if and only if
\[ \int_{\mathbb{R}^N} (I_{\alpha} * |u_n|^{\frac{N+\alpha}{N}})|u_n|^{\frac{N-\alpha}{N}} dx = 0 \text{ as } n \to +\infty. \]
But this is not possible, since otherwise we find that
\[ 0 > \sigma(c_{1}) \leftarrow E_{q}(u_n) \geq \frac{1}{2} \|\nabla u_n\|_{2}^2 \geq 0. \]
Hence $\sigma(\nu c_{1}) < \nu^2 \sigma(c_{1})$, that is, $\frac{\nu^2}{c^2} \sigma(c_{2}) < \sigma(c_{1})$. The proof is complete. \( \square \)

**Lemma 2.2.** Assume $\gamma, \mu > 0$ and $2 < q < 2 + \frac{4}{N}$. Let $\{u_n\} \subset H^1(\mathbb{R}^N)$ be a sequence such that $\|u_n\|_{2} \to c$ and $E_{q}(u_n) \to \sigma(c)$. Then the sequence $\{u_n\}$ is relatively compact in $H^1(\mathbb{R}^N)$ up to translations.

**Proof.** It follows from Lemma 2.1(1) that the sequence $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$. We first verify that there exists $\beta > 0$ such that
\[ \lim_{n \to +\infty} \sup_{y \in \mathbb{R}^N} \int_{B_R(y)} |u_n|^2 dx \geq \beta, \text{ for some } R > 0. \]
Suppose the contrary. Then $u_n \to 0$ in $L^s(\mathbb{R}^N)$ for $2 < s < \frac{2N}{(N-2)+2}$. Using this together with (1.1) and (1.2) gives
\[ \sigma(c) + o_n(1) = E_{q}(u_n) \]
\[ = \frac{1}{2} \|\nabla u_n\|_{2}^2 - \frac{\gamma N}{2(N+\alpha)} \int_{\mathbb{R}^N} (I_{\alpha} * |u_n|^{\frac{N+\alpha}{N}})|u_n|^{\frac{N-\alpha}{N}} dx + o_n(1) \]
\[ \geq \frac{1}{2} \|\nabla u_n\|_{2}^2 - \frac{\gamma N}{2(N+\alpha)} S_{\alpha}^{\frac{N-\alpha}{N}} c^{\frac{(N+\alpha)}{N}} + o_n(1), \]
which contradicts Lemma 2.1(2). So (2.1) holds. By (2.1), there exists $\{y_n\} \subset \mathbb{N}$ such that $u_n(x + y_n) \to u \neq 0$ in $H^1(\mathbb{R}^N)$. Set $v_n = u_n(x + y_n) - u$. If $\|u\|_{2} = b \neq c$, then $b \in (0, c)$. Setting $d_n = \|v_n\|_{2}$, and by using
\[ \|u_n(x + y_n)\|_{2}^2 = \|v_n\|_{2}^2 + \|u\|_{2}^2 + o_n(1), \]
we obtain that $\|v_n\|_{2} \to d$ and $d_n \in (0, c)$ for $n$ large enough, where $c^2 = d^2 + b^2$. Using the Brézis-Lieb lemma (see [1, 2]), and Lemma 2.1(3), we obtain that
\[ \sigma(c) + o_n(1) = E_{q}(u_n(x + y_n)) \geq E_{q}(v_n) + E_{q}(u) + o_n(1) \]
\[ \geq \sigma(d_n) + \sigma(b) + o_n(1) \]
\[ \geq \frac{d^2}{c^2} \sigma(c) + \sigma(b) + o_n(1). \]
Letting $n \to +\infty$ and using again Lemma 2.1(3), we find that
\[ \sigma(c) \geq \frac{d^2}{c^2} \sigma(c) + \sigma(b) > \frac{d^2}{c^2} \sigma(c) + \frac{b^2}{c^2} \sigma(c) = \sigma(c), \]
which is a contradiction. So \( \|u\|_2 = c \) and \( u_n(x + y_n) \to u \) in \( L^2(\mathbb{R}^N) \) and then \( u_n(x + y_n) \to u \) in \( L^s(\mathbb{R}^N) \) for \( 2 \leq s < \frac{2N}{N-2}^+ \). Consequently, we obtain that
\[
\sigma(c) = \lim_{n \to +\infty} E_q(u_n) = \lim_{n \to +\infty} E_q(u_n(x + y_n)) \geq E_q(u) \geq \sigma(c).
\]
This shows that \( E_q(u) = \sigma(c) \) and \( \|\nabla u_n(x + y_n)\|_2 \to \|\nabla u\|_2 \), that is, \( u \in S(c) \) is a minimizing sequence such that \( u \to u \) in \( H^1(\mathbb{R}^N) \). The proof is complete. \[ \square \]

**Proof of Theorem 1.1.** In view of Lemma 2.1(1), let \( \{u_n\} \subset S(c) \) be a sequence such that \( E_q(u_n) \to \sigma(c) \). Then by Lemma 2.2, there exist a subsequence, still denoted by \( \{u_n\} \), a sequence of points \( \{y_n\} \subset \mathbb{R}^N \), and a function \( u \in S(c) \) such that \( u_n(\cdot + y_n) \to u \) in \( H^1(\mathbb{R}^N) \). Thus \( E_q(u) = \sigma(c) \).

Let \( u_0 \) denote the Schwartz rearrangement of \( |u| \). By using the Riesz rearrangement inequality (see [11])
\[
\|\nabla u_0\|_2 \leq \|\nabla u\|_2, \; \|u_0\|_r = \|u\|_r \text{ for any } r \in [2, \frac{2N}{N-2}^+),
\]
\[
\int_{\mathbb{R}^N} (I_\alpha * |u_0|^{N+\alpha})|u_0|^{\frac{N+\alpha}{N}}dx \geq \int_{\mathbb{R}^N} (I_\alpha * |u|^{N+\alpha})|u|^{\frac{N+\alpha}{N}}dx,
\]
we deduce that \( E_q(u_0) = \sigma(c) \), that is, \( \sigma(c) \) is achieved by the real-valued positive and radially symmetric non-increasing function \( u_0 \in S(c) \).

It is obvious that \( u_0 \) is a ground state to (1.3) and there exists \( \lambda_c \in \mathbb{R} \) such that \( E_q'(u_0) + \lambda_c u_0 = 0 \). Then
\[
\lambda_c c^2 = -\|\nabla u_0\|_2^2 + \gamma \int_{\mathbb{R}^N} (I_\alpha * |u_0|^{N+\alpha})|u_0|^{\frac{N+\alpha}{N}}dx + \mu \|u_0\|_q^q
\]
\[
= -2\sigma(c) + \frac{\gamma \alpha}{N+\alpha} \int_{\mathbb{R}^N} (I_\alpha * |u_0|^{N+\alpha})|u_0|^{\frac{N+\alpha}{N}}dx + \frac{\mu(q-2)}{q} \|u_0\|_q^q
\]
\[
\geq -2\sigma(c),
\]
which implies that \( \lambda_c \geq \frac{2N}{N+\alpha} c^{\frac{\alpha}{N}} \), where we have used Lemma 2.1(2). The proof is complete.

### 2.2. The case \( q = 2 + \frac{4}{N} \)

**Proof of Theorem 1.2.** If \( u \) is a solution to (1.3), by the Pohozaev identity, \( u \) satisfies
\[
Q_q(u) := \int_{\mathbb{R}^N} |\nabla u|^2 dx - \mu q \int_{\mathbb{R}^N} |u|^q dx = 0,
\]
see [16]. So we must find solutions to (1.3) in the set
\[
M_q(c) := \{ u \in S(c) : Q_q(u) = 0 \}.
\]

For any \( u \in M_q(c) \), using the Gagliardo-Nirenberg inequality (1.1), we obtain that
\[
\int_{\mathbb{R}^N} |\nabla u|^2 dx = \mu q \int_{\mathbb{R}^N} |u|^q dx \leq \mu q \tilde{S} \|u\|^{q \eta_q}_2 \|u\|^{2(1-\eta_q)}_2
\]
\[
= \frac{\mu N}{N+2} \tilde{S} c^{A/N} \|\nabla u\|_2^2
\]
for \( q = 2 + \frac{4}{N} \), which implies that \( M_q(c) = \emptyset \) if \( \frac{\mu N}{N+2} \tilde{S} c^{A/N} < 1 \). Hence, (1.3) does not have a solution if \( \frac{\mu N}{N+2} \tilde{S} c^{A/N} < 1 \).
If \( \frac{\mu N}{N+2} \bar{S} c^{4/N} = 1 \), then in (2.2), the equality holds in the Gagliardo-Nirenberg inequality. So by (1.1), \( u \in S(c) \) must be the unique positive radial solution to the equation

\[
-\Delta u + u = |u|^{4/N} u.
\]

Obviously, such \( u \) is not a solution to (1.3). Thus, (1.3) does not have a solution in the case \( \frac{\mu N}{N+2} \bar{S} c^{4/N} = 1 \).

Remark 2.3. For any \( u \in \mathcal{M}_q(c) \), by using the Hardy-Littlewood-Sobolev inequality (1.2), we obtain that for \( q = 2 + \frac{4}{N} \)

\[
E_q(u) = -\frac{\gamma N}{2(N + \alpha)} \int_{\mathbb{R}^N} (I_{\alpha} * |u|^{\frac{2(N + \alpha)}{N}}) |u|^{\frac{2(N + \alpha)}{N}} dx
\]

\[
\geq -\frac{\gamma N}{2(N + \alpha)} S_{\alpha}^{-\frac{N + \alpha}{N}} \|u\|_2^{\frac{2(N + \alpha)}{N}}
\]

\[
= -\frac{\gamma N}{2(N + \alpha)} S_{\alpha}^{-\frac{N + \alpha}{N}} c^{\frac{2(N + \alpha)}{N}}.
\]

The equality holds if and only if there exist \( \epsilon, \bar{C} > 0 \) such that

\[
u = \bar{C} \left( \frac{\epsilon}{\epsilon^2 + |x|^2} \right)^{N/2} \quad \text{and} \quad u \in \mathcal{M}_q(c),
\]

which is equivalent to

\[
u = c \left( \int_{\mathbb{R}^N} \frac{1}{(1 + |x|^2)^N} dx \right)^{-1/2} \left( \frac{\epsilon}{\epsilon^2 + |x|^2} \right)^{N/2}
\]

and

\[
\left( \int_{\mathbb{R}^N} \frac{1}{(1 + |x|^2)^N} dx \right)^{(q-2)/2} \int_{\mathbb{R}^N} \frac{|x|^2}{(1 + |x|^2)^{N+2}} dx
\]

\[
= \frac{\mu c^{q-2}}{N(N + 2)} \int_{\mathbb{R}^N} \frac{1}{(1 + |x|^2)^{Nq/2}} dx.
\]

That is, under the assumption (2.4), \( u \) defined in (2.3) belongs to \( \mathcal{M}_q(c) \) and is the unique minimizer of \( E_q|_{\mathcal{M}_q(c)} \). However, \( u \) defined in (2.3) is not a solution to (1.3), so we can not obtain a solution to (1.3) by minimizing \( E_q|_{\mathcal{M}_q(c)} \) if \( c, \mu \) satisfy (2.4).

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