Introduction to statistical models and non-extensive statistics

T. S. Biró*

Institute for Theoretical Physics, University of Giessen, D 35392 Heinrich-Buff-Ring 16, Giessen, Germany
and
KFKI Research Institute for Particle and Nuclear Physics, H 1526 Budapest Pf. 49, Hungary

Predictions for occurrence of a quark matter in heavy ion collisions were made up to now in the
framework of extensive thermodynamics. We review here some basic concepts in statistics, kinetic
theory and thermodynamics, in particular models establishing a non-extensive statistics. Possible
connections to particle spectra emerging from high-energy reactions are reviewed.

PACS numbers: 25.75.Nq, 05.20.Dd, 05.90.+m, 02.70.Ns

Keywords: quark matter, Boltzmann equation, non-extensive thermodynamics

INTRODUCTION

Statistical models in general apply to phenomena which appear in a sufficiently large number and can be repeated
independently and (in principle) indefinitely. Whether high energy physics and in particular heavy-ion collisions
provide such phenomena, is a question to be discussed on its own right. We shall try to contribute to an answer to
this question here by reviewing some basic concepts of statistics, kinetic theory and thermodynamics, with the aim
to clarify the limits of the application of statistical models.

First we review the central limit theorem of statistics[1], as the basic law governing the result of very many
independent influences. Then kinetic theory, designed to describe the way towards equilibrium and its maintenance,
is discussed. By doing so we point out quite a few modern applications of statistical methods describing a stationary
state, which obeys non-extensive thermodynamical rules[2]. A review of non-extensive rules, equilibrium one-particle
distributions and entropy density formulas follows.

An underlying particular application we have in mind is a relativistic heavy ion collision[3]. The first touch physics
is probably dominated by nonlinear field dynamics and parton collisions. While some very energetic partons may
escape in form of jets, in a central heavy-ion event most of the beam energy is transformed into a compression of
the nuclear matter and production of relativistic quarks and gluons. How a quark-gluon plasma in a thermal state is
formed in these events and with which properties, is still an objective of the contemporary research.

Assuming i) such a quark matter formation in an intermediate state, ii) ergodization of energy between many
newly produced particles and iii) a relatively fast hadronization, the experimentally measured specific hadron spectra
may reflect statistical, presumably even equilibrium thermal, properties of the precursor matter[4]. Strong final state
interaction on the other hand, a so called hadronic afterburner, may wash out spectral characteristics of earlier
stages of the evolution. Whether this is the case, can in principle be studied by checking quark coalescence rules or
comparing hadron- and lepton-spectra[5]. Finally the late resonance decay[6] during free streaming of hadrons changes
the hydrochemical composition. Fortunately, some properties (e.g. the transverse momentum spectra) are influenced
only partially (at their low end) by this. Since the relativistic energy is given by \( E = m_T \cosh y \)
with transverse mass \( m_T = \sqrt{p_T^2 + m^2} \) and rapidity \( y \) for a particle with mass \( m \), the best way to study statistical equilibrium
distribution of hadrons is the comparison of \( m_T \)-spectra at rapidity \( y = 0 \) for different particles. A universal behavior
\( (m_T\text{-scaling})[7] \) indicates that the one-particle distributions depend on the energy only and not on all momentum
components: a basic feature of generalized and conventional thermal distributions.

In this lecture we review general basic ideas and mathematical formulas related to statistical models. By doing
so emphasis is given to particular cases when the conventional picture of earlier textbook statistical physics does
not apply, instead some generalized concepts have to be considered. Starting with the central limit theorem of
statistics we continue with a discussion of the Brownian motion in the framework of the Langevin and Fokker-Planck
equations. Here the very level of generality is identified that leads to a Tsallis distribution instead of the Gibbs-Boltzmann one. The potential for further generalization is pointed out, too. Then a discussion of the Boltzmann equation follows introducing to modern generalizations capable to describe non-exponential equilibrium one-particle distributions. We close with some remarks about how certain a thermodynamical state may be reconstructed from the observed one-particle energy spectra.

STATISTICS: THE LAW OF BIG NUMBERS

There is a mathematical property behind the applicability of statistical physics: many "normal" distributions of probability fold to a Gaussian with a width scaling down with increasing number of individual constituents. This property is expressed nicely in the central limit theorem. An enlightening simple example of its action is given by the distribution of the sum of uniform random variables in a finite interval. Finally in this section an example not falling under the reign of the central limit theorem, the Lorentzian distribution, is presented.

Let \( x_i \) be random distributed according to \( w_i(x_i) \). We are interested in the distribution of a scaled sum of \( n \) such variables:

\[
P_n(x) = \int \prod_{i=1}^{n} w_i(x_i) \delta\left(x - a_n \sum_{k=1}^{n} x_k\right).
\]

Here we assumed that the joint probability of having \( n \) values is a product of the individual probabilities; this is the requirement of statistical independency. With this assumption the Fourier transform of the soughted probability is a product of properly scaled Fourier transforms of individual probabilities:

\[
\tilde{P}_n(k) = \int dx e^{ikx} P_n(x) = \prod_{i=1}^{n} \tilde{w}_i(a_n k).
\]

From the Taylor expansion of \( \ln \tilde{P}(k) \) around \( k = 0 \) one obtains the central moments (correlations). For the \( \ell \)-th moment the scaling law, \( \sigma_n^{(\ell)} = a_n^{\ell} n^{\sigma^{(\ell)}} \) applies with \( \sigma \) being the finite average of the given central moment of the individual distributions. Whenever \( \sigma_i^{(\ell)} = 0 \) for all \( \ell < \ell_0 \) values and \( \sigma_i^{(\ell_0)} \) is finite, all the higher moments can be scaled down in the folded distribution by choosing \( a_n \propto n^{-1/\ell_0} \). The resulted scaling, \( \sigma_n^{(\ell)} = n^{1-\ell/\ell_0} \sigma^{(\ell)} \), in the \( n \to \infty \) limit leaves us with only one nonzero central moment, the \( \ell_0 \)-th one.

Usually the \( \ell = 0 \) central moment is zero due to the normalization of the probability: \( \ln \tilde{P}(0) = \ln 1 = 0 \). The first moment (\( \ell = 1 \)) can be made zero due to symmetry or by a simple shift in the variables \( x_i \). The second central moment is then the first nonzero value. As a consequence the resulted distribution of the scaled sum has a second moment with all higher moments vanishing, therefore \( \ln \tilde{P}(k) \) is quadratic in \( k \), so \( \tilde{P}(k) \) and with that \( P(x) \) is Gaussian.

A nice, simple example is given for \( x_i \)-s uniformly distributed in the interval \((-1, +1)\). The distribution of

\[
x = \sqrt{3/n} \sum_{i=1}^{n} x_i
\]

approaches the Gaussian:

\[
\lim_{n \to \infty} \tilde{P}_n(k) = \lim_{n \to \infty} \left(\frac{\sin(k\sqrt{3/n})}{k\sqrt{3/n}}\right)^n = \exp(-k^2/2).
\]

A counterexample is given by the Lorentzian distribution, having a Fourier transform \( \tilde{w}_i(k) = \exp(-|k|) \). Now there is a non-Gaussian limiting distribution with an altered scaling for

\[
x = \frac{1}{n} \sum_{i=1}^{n} x_i.
\]
It is itself a Lorentzian:
\[
\lim_{n \to \infty} \tilde{P}_n(k) = \lim_{n \to \infty} \left( e^{-|k|/n} \right)^n = \exp(-|k|).
\] (6)

This is a special case of the Lévy distribution[12].

**KINETIC THEORY**

In a brief review of concepts distilled from the kinetic theory approach to thermodynamics we shall rely on the notion of "noise" heavily. The sum of random influences (forces) is itself a random variable. It shows relationship to the sum of random numbers; its distribution under quite general circumstances can be considered as a Gaussian distribution. The independency of individual influences is assumed first of all in time: such a view deals with uncorrelated stochastic changes in the parameters describing a physical system.

The simplest physical theory of such a system is that of the Brownian motion[13], considering a free, massive particle under the influence of forces acting as an uncorrelated, Gaussian (white) noise. A description of the motion of such a particle is given by the classical Langevin equation, or equivalently a statistics over possible such motions is described by the Fokker-Planck equation. A balance between damping and accelerating forces leads to a stationary state, with vivid microscopical dynamics, but macroscopically (on the average over many particles) presenting a thermodynamical equilibrium state. General statements about this balance are comprised in the fluctuation-dissipation theorem. In the next section we review generalizations of the Boltzmann equation, a somewhat more complicated kinetic theory. The effects of such generalizations on the the concept and definition of entropy and on equilibrium distributions will be an issue of a further section.

**General Langevin problem**

Let us consider a simple, one degree of freedom motion. The change of momentum \( p \) in time is given by a force depending on this momentum and on a noise variable \( z \):
\[
\dot{p} = F(p, z).
\] (7)

Following the method pioneered by Ornstein and Uhlenbeck[14] a distribution of many possible values of \( p \) at a time \( t \) is considered. This \( f(p, t) \) distribution governs average values of a smooth but otherwise arbitrary test function \( R(p) \). The same integral over \( p \) can be expressed at the time \( t + dt \) assuming a statistical average over the noise \( z \) in the time interval passed since \( t \):
\[
\int dp R(p)f(p, t + dt) = \int dp \langle R(p + dt F(p, z)) \rangle f(p, t).
\] (8)

One assumes that the averaging of the force \( F \) over the noise \( z \) gives the result:
\[
\langle F \rangle = -G(p), \quad \langle FF \rangle - \langle F \rangle \langle F \rangle = 2D(p)/dt.
\] (9)

The above scaling of the correlation with \( dt \) follows from the Gaussian nature of the noise \( z \). Expanding now the equation up to terms linear in \( dt \) one arrives at:
\[
\int dp R(p) \frac{\partial f}{\partial t}(p, t) = \int dp \left[ -G(p)R'(p) + D(p)R''(p) \right] f(p, t).
\] (10)

After partial integration and considering arbitrary \( R(p) \) one gets the Fokker-Planck equation:
\[
\frac{\partial f}{\partial t} = \frac{\partial}{\partial p} (G f) + \frac{\partial^2}{\partial p^2} (D f).
\] (11)
Particular Langevin problem

The above Langevin and Fokker-Planck problem is still quite general. Damping and diffusion coefficients depend on the momentum $p$ in a general way. Ergodization in phase space is achieved on the other hand when constant energy surfaces are covered. In such a situation the distribution $f$, as well as the coefficients $G$ and $D$ (the latter related to the noise), depends on the energy $E(p)$ only. Note, however, that they are not constant.

Such a particular Langevin equation is given by

$$
\dot{p} = z - G(E) \frac{\partial E}{\partial p}
$$

(12)

containing an energy dependent damping proportional to the general velocity $v = \partial E / \partial p$, and a zero-average noise, $\langle z(t) \rangle = 0$, with a correlation

$$
\langle z(t)z(t') \rangle = 2D(E)\delta(t-t').
$$

(13)

The Fokker-Planck equation contains in this case the factors $D(p) = D(E)$ and $G(p) = -G(E)\partial E / \partial p$. Its stationary solution is given by

$$
f(p) = \frac{A}{D(E)} \exp \left( - \int \frac{G(E)}{D(E)} dE \right) = A \exp \left( - \int \frac{dE}{T(E)} \right).
$$

(14)

This result is not readily the Boltzmann-Gibbs distribution, $\exp(-E/T)$, only in the case of energy-independent damping and noise coefficients. A general equilibrium is able to feature almost any other distribution of the energy of a single degree of freedom picked out of its environment. Instead of a constant temperature, $T$, in the general case a sliding inverse logarithmic slope is characteristic to such states. From $1/T(E) = -d \ln f(E)/dE$ its relation to the damping and diffusion coefficients follows:

$$
T(E) = \frac{D(E)}{G(E) + D'(E)}.
$$

(15)

The low-energy limit of this expression, pretending as $G(E)$ and $D(E)$ were constant, leads to an experimentally feasible definition of the Gibbs temperature, $T_{Gibbs} = D(0)/G(0)$. From the viewpoint of the Brownian motion another temperature may be used, the Einstein temperature $T_{Einstein} = \lim_{E \to \infty} D(E)/G(E)$. None of these two approximations are, however, coincident with the sliding slope of particle spectra given by eq. (15).

An important and historically mostly considered particular case is given by the constant slope distribution, the Boltzmann-Gibbs distribution: $T(E) = T$. A modern, non-classical distribution, the Tsallis distribution seems to be the next simplest, having a linear inverse slope – energy relation:

$$
T(E) = T/q + E(1 - 1/q).
$$

(16)

The corresponding equilibrium distribution in this case turns to be an exponential of a logarithm, which is a power-law:

$$
f(p) = \frac{1}{Z} \left( 1 + (q - 1) \frac{E}{T} \right)^{-\frac{q}{1}}
$$

(17)

It has the interesting property, that the parameter $T$ is the fixed point of the sliding (linear) slope: $T(T) = T$. This parameter shall be referred to as the Tsallis temperature.

Fluctuation-dissipation theorem

In a realistic system there are many microscopic degrees of freedom to be considered. Denoting a point in the $6N$-dimensional phase space by $p_i$, the Langevin equation can be implemented in the form

$$
\dot{p}_i = (S_{ij} - G_{ij}) \nabla_j E + z_i.
$$

(18)
The symplectic coefficient, $S_{ij}$, does not change the total energy of the system, $E(p)$, it causes conservative motion inside a given energy shell only. For the sake of energy distribution it is therefore not interesting and shall be omitted in the followings. The damping and dissipation terms with the respective symmetric coefficients $G_{ij}$ and $D_{ij}$ keep balance on the long term. An ergodized equilibrium distribution can be a single function of the energy $E$ only, therefore these coefficient matrices have to be connected by a single function of energy, too. This gives rise to a general fluctuation dissipation theorem:

$$D_{ij}(E) = T(E) \left( G_{ij}(E) + D_{ij}^0(E) \right).$$

(19)

It is highly nontrivial that two high-dimensional matrix functions of the phase space coordinates $p_i$ would be related by a single scalar function of energy only! Recalling that $T(E)$ is the inverse logarithmic slope of the equilibrium distribution, the fluctuation dissipation relation can be expressed using $f(E)$, too:

$$D_{ij}(E) = \frac{1}{f(E)} \int_E^\infty G_{ij}(x) f(x) \, dx.$$

(20)

Still, quite general diffusion and damping coefficient matrices are allowed, but $D_{ij}$ is connected to $G_{ij}$ via an energy dependent scalar, the equilibrium energy distribution, $f(E)$. Particular cases of this relation are i) the use of the Gibbs distribution, $f(E) \propto \exp(-E/T)$, leading to $D_{ij} = TG_{ij}$ with constant matrices (the usual textbook case), or ii) the use of a Tsallis distribution giving rise to $D_{ij}(E) = (T + (q-1)E) G_{ij}$ with constant $G_{ij}$ but linearly energy dependent diffusion coefficient, $D_{ij}$. A further interesting case is presented by assuming that both the damping and the diffusion coefficient matrix is a linear function of the energy, but their ratio (the Einstein temperature) is constant. This assumption is typical to field theory calculations. Due to eq. (19) the sliding slope is, however, not constant; it rather interpolates between a linear rise at low energy and a constant at high energy. From $D = TG = \gamma T(E/E_c)$ it follows $1/T(E) = 1/T + 1/(E + E_c)$.

**Additive and multiplicative noise**

Another approach to reach a non-exponential stationary energy distribution from a kinetics described by the Langevin equation considers a stochastic damping coefficient. The Langevin equation is kept linear,

$$\dot{p} = \zeta - \gamma p,$$

(21)

but, more general than in the classical approach, both $\zeta$ and $\gamma$ are stochastic variables. With constant mean values, $\langle \zeta \rangle = F$ and $\langle \gamma \rangle = G$, and white-noise correlations,

$$\langle \gamma(t)\gamma(t') \rangle = 2C \delta(t - t'), \quad \langle \zeta(t)\gamma(t') \rangle = 2B \delta(t - t'), \quad \langle \zeta(t)\zeta(t') \rangle = 2D \delta(t - t'),$$

(22)

the equivalent Fokker-Planck equation contains a damping factor $G(p) = Gp - F$ and a diffusion factor $D(p) = Cp^2 - 2Bp + D$. For $B \neq 0$ the two noisy coefficients are cross-correlated. For a single degree of freedom the stationary distribution can be obtained analytically:

$$f(p) = A \left( \frac{D}{D(p)} \right)^v \exp \left( -\frac{\alpha}{\theta} \text{atan} \left( \frac{p\theta}{D - Bp} \right) \right)$$

(23)

with the power $v = 1 + G/2C$, the exponent factor $\alpha = GB/C - F$ and the variable $\theta = \sqrt{DC - B^2}$. For $F = 0$ a characteristic momentum scale in this distribution is given by $p_c^2 = D/C$, the ratio of the additive and multiplicative noise widths. For $F = 0$ and $B = 0$, i.e. for two independent noises, the Tsallis distribution arises as a stationary solution:

$$f(p) = A \left( 1 + \frac{C}{D} p^2 \right)^{-v}.$$

(24)

Utilizing the energy formula for a free, massive, non-relativistic particle it reads as

$$f(p) = A \left( 1 + (q-1) \frac{E}{T} \right)^{-v}.$$  

(25)
with the Tsallis index \( q = 1 + 2C/G \) and the Tsallis temperature coinciding with that of the classical Brownian motion, \( T = D/mG \). In the small momentum limit, \( p \ll p_c \), the Tsallis distribution is nearly Gaussian, independent from the properties of the multiplicative noise, \( f = A \exp(-Gp^2/2D) \) (or expressed with the energy a Gibbs distribution, \( f = A \exp(-E/T) \)). In the opposite limit at high energy it is a power-law distribution, \( f = A(p/p_c)^{-2v} = A(E/E_c)^{-v} \). It is interesting to note, that a definite relation arises between the energy scale, \( E_c \), the temperature parameter, \( T \), and the tail power, \( v \):

\[
v = 1 + E_c/T.
\]

This relation can be experimentally tested.

**NON-EXTENSIVE BOLTZMANN EQUATION**

The heart of kinetic theory is the classical Boltzmann equation. It does not only describe dynamical evolution of large systems microscopically, resulting in stationary distributions on which thermodynamics can be based on, but it also offers a microscopical foundation to the key quantity entropy. It has, of course, also quite a few assumptions built in the theory; dropping one or other of them may lead to a generalization of the classical approach. The most recognized assumption, the micro-reversibility of the transition probability, establishes H-theorem and the definition of entropy; it should not be dropped. Less explicit assumptions, like taking the two-particle probability as a product of one-particle probabilities, or taking the total energy of a colliding pair as the sum of the respective one-particle energies for freely moving (asymptotic) particles, can be more readily generalized.

We review these two generalizations of the Boltzmann equation in this section: the generalization of the product rule for probabilities (dropping statistical independency) leads to a non-linear Boltzmann equation (NLBE), while considering two-particle energies composed by an extended addition rule mounds in the non-extensive Boltzmann equation (NEBE). Resulting stationary distributions, the H-theorem and the main characteristics of generalized thermodynamicses following from this will be presented.

**NLBE: generalized product**

The general structure of the Boltzmann equation describes the evolution of the one-particle phase space density (interpreted as finding probability of a particle or a microstate) via an integral over all possible transitions to and from other states:

\[
\dot{f}_1 = \int_{234} w_{1234} (f_{34,12} - f_{12,34}).
\]

Here the dot denotes a total time derivative (Vlasov operator) comprising the essential evolution of the one-particle phase space density, \( f_1 \). The indices 1234 refer to two particles before and after a microcollision. The transition probability, \( w_{1234} \) reflects microreversibility and partner symmetry in the permutation of these indices. It is positive and it contains some conditions on conserving physical quantities; at least momentum and energy.

\[
w_{1234} = M^2_{1234} \delta((\vec{p}_1 + \vec{p}_2) - (\vec{p}_3 + \vec{p}_4)) \delta(E_{12} - E_{34}),
\]

with \( E_{12} \) total two-particle energy before and \( E_{34} \) after the collision. The particle density factors, \( f_{12,34} \) and \( f_{34,12} \) weight the transition yields for a \( 3 + 4 \to 1 + 2 \) and for a \( 1 + 2 \to 3 + 4 \) process, respectively.

In the NLBE approach the traditional simple additivity of energy is kept,

\[
E_{12} = E_1 + E_2,
\]
but the classical product formula of Boltzmann, \( f_{12,34} = f_1 f_2 \), or the supplement with blocking factors due to Uehling and Uhlenbeck \( f_{12,34} = f_1 f_2 (1 \pm f_3) (1 \pm f_4) \) is generalized. The generalization still reflects particle separation property (particle 1 goes into particle 3 and particle 2 goes into particle 4 classically), but abandons the linearity:

\[
f_{12,34} = \gamma(f_1, f_3) \cdot \gamma(f_2, f_4).
\]

This intends to simulate statistical correlations between initial and final states and a nonlinearity of transition yields. Further requirement is that the phase space density factor, \( \gamma \), factorizes to a production factor \( a \), to a blocking factor \( b \) (depending on the respective initial and final phase space densities only) and to a factor symmetric in both:

\[
\gamma(x, y) = a(x) b(y) c(x, y)
\]

with \( c(x, y) = c(y, x) \).

The stationary state of the NLBE is governed by the ratio \( \kappa(x) = a(x)/b(x) \). It is easy to see from the following derivation of the generalized H-theorem. We seek for a quantity called entropy in form of a one-particle phase space integral, \( S = \int \sigma(f_1) \). The question is what \( \sigma(f) \) functional form guarantees macro-irreversibility, i.e. \( \dot{S} \geq 0 \). Using the general Boltzmann equation \( \Sigma \) one writes

\[
\dot{S} = \int \sigma'(f_1) = \int w_{1234} c_{13} c_{24} \sigma'_1 \cdot \sigma'_2 \cdot \gamma_3 \gamma_4 - \sigma'_1 \cdot \sigma'_3 \gamma_2 - \sigma'_1 \cdot \sigma'_4 \gamma_2.
\]

with the corresponding indices referring to arguments of the functions \( a \), \( b \) and \( c \). Now we explore particle permutation symmetries of this expression. After the exchange of particle 1 with 2 and simultaneously particle 3 with 4 we describe the same process. An exchange of the initial with the final state is done by 1 ↔ 3 and 2 ↔ 4 (micro-reversibility). It amounts to a relative minus sign in the total rate by exchanging gain and loss terms. Finally the combined operation of both also contributes with a minus sign in total. Dressing now the transition rate with factors carrying the same symmetry we use \( \dot{w}_{1234} = w_{1234} c_{13} c_{24} b_1 b_2 b_3 b_4 / 4 \). With the notation \( \kappa_i = a_i / b_i \) for the production to blocking ratios we arrive at

\[
\dot{S} = \int \dot{w}_{1234} (\sigma'_1 - \sigma'_3 - \sigma'_4 - \kappa_1 \kappa_2) .
\]

This already reminds to the structure of the H-theorem result of the classical Boltzmann equation. The correct entropy density can be read off as satisfying

\[
\sigma'(f) = -\ln \kappa(f).
\]

The stable equilibrium, where – due to ergodization – \( f(p) \) can be expressed as a solely function of the corresponding energy, satisfies a product rule for the \( \kappa(f_i) \)-s while the energy is additive. The only solution is that the canonical equilibrium of the NLBE is given by

\[
\kappa(f) = \frac{1}{Z} \exp(-E/T).
\]

As particular cases the well-known Boltzmann-Gibbs, Fermi-Dirac or Bose-Einstein distributions are recovered, but this result is far more general. The corresponding general entropy formula can be constructed founding a generalized thermodynamics with (in general) non-extensive entropy composition rules when merging large subsystems.

**NEBE: generalized sum**

Another, recently pursued way is to keep the statistical independency, \( f_{12,34} = f_1 f_2 \), but to generalize the energy addition formula to a nontrivial composition rule

\[
E_{12} = h(E_1, E_2).
\]
Rules not being a simple sum, \( h(x, y) \neq x + y \), present a non-extensive energy composition. Latest in the thermodynamical limit, only associative rules are physical: \( h(h(x, y), z) = h(x, h(y, z)) \). Due to a mathematical theorem for functional equations the general solution of the associativity requirement is a strict monotonous mapping to the simple addition:

\[
X(h) = X(x) + X(y). \tag{37}
\]

This is unique up to a constant factor.

As a consequence for any microcollision \( X(E_1) + X(E_2) = X(E_3) + X(E_4) \) holds. The stationary solution of NEBE is hence given by

\[
f(p) = \frac{1}{Z} \exp(-X(E)/T) \tag{38}
\]

allowing again for a general, non-exponential energy dependence. It is noteworthy that to each associative non-extensive composition rule \( h(x, y) \) there exist a mapping function \( X(E) \). An in-medium dispersion relation, or quasi-energy is obtained this way: the total sum,

\[
X(E_{tot}) = \sum_i X(E_i) \tag{39}
\]

is conserved by the NEBE.

In order to connect the approaches NLBE and NEBE (and both with the non-extensive thermodynamics) the non-extensive composition rule of the energy has to be extended to composition rules of other, traditionally "extensive" quantities. Most important is the entropy composition rule. And, as putting a corner stone into the right place, the scaling law

\[
X(E)/T = X_s(E/T) \tag{40}
\]

allows us to relate the equilibrium distributions by a one-variable functional relation,

\[
Z f_{eq}(E) = \exp(-X_s(E/T)) = \kappa^{-1}(\exp(-E/T)). \tag{41}
\]

At the same time the generalized entropy density satisfies

\[
\sigma'(f) = -\ln \kappa(Zf) = X_s^{-1}(-\ln Zf) \tag{42}
\]

in equilibrium. This gives rise to an interpretation of the H-theorem for NEBE where the never-decreasing total entropy is given by the same strict monotonic back-mapping from the original Boltzmann-entropy:

\[
S_B = X_s(S_{tot}) = \int -f \ln f. \tag{43}
\]

The use of a non-extensive entropy composition rule, \( h_s(x, y) \), mapped to additivity by \( X_s(t) \) leads to the following mapping of the individual entropy-density: \( s(f) = f X_s^{-1}(-\ln f) \). The usual non-extensive entropy is then defined by \( S_T = \int s(f) \).

**THERMODYNAMICSES**

It is enlightening to list some particular cases of non-extensive composition laws and corresponding thermodynamics. The trivial law, \( h(x, y) = x + y \), is mapped by the identity, \( X(E) = E \), and in (canonical) equilibrium the Boltzmann-Gibbs distribution, \( \exp(-E/T) \), emerges. The entropy density is given by \( s(f) = -f \ln f \).

Tsallis’ non-extensive thermodynamics relies on the composition rule, \( h(x, y) = x + y + a xy \), connected to the mapping

\[
X_s(S) = \frac{1}{a T} \ln (1 + a T S), \tag{44}
\]
first presented by Abe\cite{Abe}. The canonical equilibrium distribution,
\[ f_{eq}(E) = \frac{1}{Z} \frac{1}{(1 + aE)^{-1/aT}}, \quad (45) \]
can easily be connected to the Tsallis distribution\cite{Tsallis} by \( q = 1 - aT \). Finally an entropy density following Tsallis’ original suggestion (encountered also earlier by several authors\cite{Renyi}): \( s(f) = \frac{1}{q-1} f^q - f^{q-1} \), can be obtained.

An interesting endeavor is to consider the composition rule, \( h(x, y) = (x^b + y^b)^{1/b} \). Mapping the energy to its power, \( X(E) = (aE)^b/a \), it gives rise in equilibrium to a Lévy distribution\cite{Levy}, a.o. known from anomalous diffusion problems:
\[ f_{eq}(E) = \frac{1}{Z} \exp \left( -\frac{1}{aT} \frac{aE}{aT} \right). \quad (46) \]
The corresponding expression for the entropy is an incomplete gamma function.

The pure multiplicative law, \( h(x, y) = axy \), which may be considered as the high-energy limit of Tsallis’ construction with fixed energy scale \( E_c = 1/a \), is mapped by \( X(E) = \frac{1}{a} \ln(aE) \) and leads to a pure power-law behavior, \( Zf_{eq} = (aE)^{-1/aT} \). The entropy formula due to \( X_s(t) = \frac{1}{aT} \ln(aT t) \), \( sf = \frac{1}{1-q} f^q \) reminds to the original suggestion by Rényi\cite{Renyi}. In fact defining \( S_R = X_s(S_T) \) the (Tsallis) entropy is mapped to the extensive quantity:
\[ S_R = \frac{1}{1-q} \ln \left( \int f^q \right). \quad (47) \]

### Canonical and Extended Equilibrium

Summarizing this part it seems to be useful to spell out the general conditions leading to the traditional and to the generalized treatment of canonical equilibrium. Three conditions are important:

1. In the equilibrium state the energy shell is covered uniformly (ergodic assumption),
2. The average energy exchange between the system under study and its environment is zero (not a driven system),
3. The fluctuations in the system’s energy are negligible compared to the average value of its energy (canonical limit).

The third requirement has its physical basis in the fact that the characteristic range of forces acting in the system is much less than the system size. So in the thermodynamical limit of large systems, \( V \gg A\ell \), with a finite interaction range, \( \ell \), all surface effects (proportional to \( A \)) in a large volume, \( V \), become negligible. In this case the entropy is quite generally additive, \( S_{12} = h(S_1, S_2) = S_1 + S_2 \), and the equilibrium canonical distribution is of Boltzmann-Gibbs type, \( Zf = \exp(-E/T) \). Note that already the quantum statistical distributions violate this requirement: the exchange interaction (Pauli blocking or Bose enhancement) is long-ranged.

In an extended equilibrium state, non-extensive thermodynamics and generalized kinetic descriptions deal with, the third requirement is not fulfilled. Either long range forces, or large fluctuations (related to each other) keep the non-extensivity parameter \( a \sim A\ell/V \) finite. Applying this idea to the time direction, also long-term memory effects spoil the traditional picture of canonical equilibrium and textbook thermodynamics. The unusual large fluctuations are treated in the framework of superstatistics\cite{Beck}, where the traditionally intensive thermodynamical parameters, like temperature, have a probability distribution instead of a fixed value. This view, of course, also transforms the Boltzmann-Gibbs distribution into an arbitrary one:
\[ f_{eq}(E) = \int d\beta \mathcal{P}(\beta) \frac{1}{Z(\beta)} e^{-\beta E}. \quad (48) \]
For example, the Tsallis distribution can be obtained by considering a Gamma-distributed inverse temperature, \( \beta = 1/T \). Some preview about a numerical simulation of NEBE for the Tsallis distribution can be obtained from a poster presented by G. Purcsel at this conference.
Statistical models have been often applied to hadron physics. Starting with Rolf Hagedorn’s statistical model of meson resonances\[21\], several attempts occurred to describe hadron multiplicities in elementary collisions by means of statistical distributions. The very idea of a phase transition between confined and deconfined quark matter relies on traditional equilibrium thermodynamics. With the dawn of relativistic heavy ion physics a search for the nuclear and quark matter equation of state began assuming a local thermal equilibrium in an otherwise exploding fireball. Such models and their predictions for experimentally observable properties will be reviewed in other lectures\[22, 23\].

Particle spectra and equation of state

Here, following the above introduction to non-extensive thermodynamics, just some basic ideas about the interpretation of the power-law tails of nearly exponential spectra shall be discussed. Both exponential and power-law regions are observed in hadron spectra stemming from pp or heavy-ion collisions. The $m_T$-scaling property, which reveals in particular after taking into account Doppler blue shift like corrections due to a transverse flow, indicates a thermodynamical origin. Although the traditional approach explains the power-law tail at very high $p_T$ values by pQCD calculations\[24\], the non-extensive statistics provides a unified view for the whole spectrum. Certainly, jets also contribute to the hard part of light hadron spectra, but their angular distribution is far from uniform. In principle, these contributions may be filtered out. The question is, whether the rest still shows a (now statistical) power-law behavior.

Another point can be made by inspecting the minimum bias pion $p_T$-spectrum form RHIC AuAu collisions at 200 GeV (Fig.1 in Ref. [15]). To this a non-exponential fit can already be made at the $p_T$-region between 1 and 4 GeV. The extrapolation of this fit almost coincides with the fit to the whole observed range between 1 and 12 GeV. So one concludes that the power-law behavior is not only a very hard scale physics. Furthermore in the non-extensive statistical approach there is a connection between the soft properties (temperature $T$) and the hard properties (power $v$, transition scale $E_c$) cf. eq.(26).

The dilemma between the statistical and pQCD explanation can be well expressed in the following formula for the particle yield:

$$\frac{(2\pi\hbar)^3}{V} \frac{d^3N}{dk^3} = \int_0^\infty d\omega \omega \rho(\omega, k) f(\omega/T).$$

The observed spectra, even not considering a collective flow or an extended source, are already convolutions of a spectral function, $\rho(\omega, k)$, and statistical-environmental effects, $f(\omega/T)$. Both these factors may embrace non-trivial effects, distorting this way a naively expected Boltzmann-Gibbs exponential. Both may contain a further energy scale; the spectral function $\Lambda_{QCD}$, the thermodynamical weight $E_c$. Even in the case, when $\rho$ would contain a very sharp peak at a given dispersion relation, $\omega_k = \omega(k)$, only, the result is a composite function, $\exp(-\omega_k/T)$, undistinguishable from a deformed thermodynamical equilibrium, $\exp(-X(|k|)/T)$. Quite generally in this quasiparticle picture, whenever the effective dispersion relation, $\omega(k)$, contains parameters depending e.g. on the temperature, $T$, an assumption of an exponential equilibrium distribution like $\exp(-\omega(k, T)/T)$ questions the physical meaning of the temperature. If it is an environmental parameter, the inverse slope is not necessarily equal to this, if it is an inverse slope, it is not necessarily the enviromental parameter governing an effective dispersion relation.

Finally we note, that non-exponential distributions can also be interpreted in the framework of so called superstatistics\[20\]. Here a distribution of the intensive thermodynamical parameters is assumed instead of a fixed value. The physical reason behind maybe the finiteness of the observed subsystem, as well as extraordinarily large fluctuations not scaling down with the system size alike the ones governed by the Gaussian distribution would do. The Tsallis distribution, for example, can be given by a continuos distribution of the inverse temperature according
to a Gamma distribution:
\[
(1 + x/c)^{-(c+1)} = \frac{1}{\Gamma(c+1)} \int_0^\infty dt t^c e^{-t} e^{-xt/c}.
\] (50)

There are several possibilities to interpret such a distribution. Among others a heat conduction equation with multiplicative noise \([25]\) or taking into account an energy imbalance in two-body collisions due to a presence of further participant agents \([26]\) leads to the required result.

**Limiting temperature with Tsallis distribution**

Another probe for the Tsallis distribution than the one-particle \(p_T\)-spectra may be given by considering the spectrum of heavy hadronic resonances. An exponentially growing mass spectrum, originally proposed by Hagedorn and recently checked again latest experimental data in Ref.\([27]\), with its famous consequence of having a limiting (or Hagedorn-) temperature for such a system, can be reconstructed on the basis of Tsallis distributed quark constituents. This approach\([28]\) assumes that the Tsallis distribution of the quarks and antiquarks is folded into mesonic and baryonic distributions of the conserved total energy satisfying \(X(E) = \sum_i X(E_i)\). In general for an \(N\)-fold convolution of massless constituents with \(d\) dimensional momenta one easily obtains that the average energy satisfies
\[
\langle X(E) \rangle = \frac{N \sum_{j=1}^d \frac{TE_c}{E_c - jT}}{E_c - jT}.
\] (51)

This expression diverges first as the temperature reaches the limiting value, \(T_H = E_c/d\) starting from zero (for positive \(E_c = 1/a\), i.e. for repulsive modifications of the extensive energy composition rule). This way Hagedorn hadrons emerge from Tsallis partons. This model also explains naturally, why the baryonic and mesonic mass spectrum seems to have a different rise: they contain different polynomial coefficients in front of three exponential factors.

**CONCLUSION**

In conclusion we have reviewed basic ideas of statistical physics and kinetic theory which may lead to a non-extensive thermodynamics. In particular the role of a multiplicative noise in the linear Fokker-Planck and Langevin approach has been emphasized. Generalizations of the Boltzmann equation towards non-factorizing yield factors or non-additive energy composition rules were then shown to lead again to non-extensive entropy definitions and thermodynamics. Finally these ideas have been related to hadron spectra observed in relativistic heavy ion collisions. We pointed out that not only the omnipresence and \(p_T\)-scaling of power-law tails in particle spectra, but also a possible interpretation of the Hagedorn spectrum of heavy resonances may support the presence of a non-exponential equilibrium distribution in hot quark matter.

**Acknowledgment**

Enlightening discussions and common work with Drs. Géza Győrgyi at Eötvös University and Antal Jakovác at the Technical University Budapest are hereby gratefully acknowledged. Further collaborations with Berndt Müller, André Peshier and Gábor Purcsel contributed essentially to the development of ideas presented above. This work has been supported by the Hungarian National Science Fund OTKA (T037689) and the Deutsche Forschungsgemeinschaft.

[1] R. Botet, M. Ploszajczak: Universal Fluctuations, World Scientific Lect. Notes vol.65, 2002; http://de.wikipedia.org/wiki/Central_limit_theorem T.A.Trainer, hep-ph/0001148
[2] C. Tsallis, J.Stat.Phys.52, 50, 1988; Physica A 221, 277, 1995; V. Latora, A. Rapisarda, C. Tsallis, Phys.Rev.E 64, 056134, 2001; C. Antenedo, C. Tsallis, Physica A 324, 89, 2003; G. Kaniadakis, cond-mat/0507311

[3] L. P. Csernai, Introduction to Relativistic Heavy Ion Collisions, Wiley & Sons, Chichester, 1994.

[4] F. Becattini et.al. Z.Phys.C 72, 491, 1996; J.Phys.G 23, 1933, 1997; Eur.Phys.J. C 5, 143, 1998; J.Phys.G 25, 287, 1999; P. Braun-Munzinger, J. Stachel, J. P. Wessels, N. Xu, Phys.Lett.B 344, 43, 1995; Phys.Lett.B 365, 1, 1996; J. Letessier, J. Rafelski, A. Tounsi, Phys.Lett.B 328, 499, 1999; G. Torrieri, J. Rafelski, New J.Phys. 3, 12, 2003; J. Cleymans, H. Oeschler, K. Redlich, Phys.Lett.B 485, 27, 2000; K. Redlich, S. Hamieh, A. Tounsi, J.Phys.G 27, 143, 2001.

[5] T. S. Biro, P. Levai, J. Zimanyi, Phys.Lett.B 347, 6, 1995; J. Zimanyi, T. S. Biro, T. Csorgo, P. Levai, APH Heavy Ion Physics 4, 15, 1996.

[6] K. S. Lee, M. Rhoades-Brown, U. W. Heinz, Phys.Lett.B 174, 123, 1986; J. Sollfrank, P. Koch, U. Heinz, Phys.Lett.B 252, 256, 1990; J. Sollfrank, U. Heinz, Phys.Lett.B 289, 132, 1992; S. Ochs, U. Heinz, Phys.Rev.C 54, 3199, 1996; J. Rafelski, J. Letessier, G. Torrieri, Phys.Rev.C 64, 054907, 2001; G. Torrieri, J. Rafelski, Phys.Lett.B 509, 239, 2001; J.Phys.G 28, 1911, 2002; Phys.Rev.C 68, 034912, 2003; SHARE (G. Torrieri, S. Steinke, W. Broniowski, J. Letessier, J. Rafelski) nucl-th/0404083; W. Broniowski, W. Florkowski, B. Hiller, Phys.Rev.C 68, 034911, 2003; Acta Phys.Hung. A22, 259, 2005; F. Karsch, K. Redlich, A. Tawfik, Phys.Lett.B 571, 67, 2003; Eur.Phys.J. C 29, 549, 2003.

[7] J. Schaffner-Bielich, D. Kharzeev, L. McLerran, R. Venugopalan, Nucl.Phys.A 705, 494, 2002.

[8] T. S. Biro, G. Purcsel, J.Phys.G 31, s759, 2005.

[9] G. Kaniadakis, Physica A 296, 405, 2001; Phys.Rev.E 66, 056125, 2002.

[10] T. S. Biro, G. Purcsel, hep-ph/0503204.

[11] T. S. Biro, B. Muller, Phys.Lett.B 578, 78, 2004.

[12] http://en.wikipedia.org/wiki/Levy_skew_alpha-stable_distribution; T. Csorgo, S. Hegyi, W. A. Zajc, Eur.Phys.J. C 36, 67, 2004.

[13] T. S. Biro, C. Greiner, Phys.Rev.Lett. 79, 3138, 1997.

[14] G. E. Uhlenbeck, L. S. Ornstein, Phys.Rev. 36, 823, 1930.

[15] T. S. Biro, G. Purcsel, G. Gyorgyi, A. Jakovac, G. Purcsel, hep-ph/0409157.

[16] T. S. Biro, A. Jakovac, Phys.Rev.Lett. 94, 132302, 2005.

[17] E. Castillo, A. Iglesias, R. Ruiz-Cobo, Functional Equations in Applied Sciences, Elsevier, Amsterdam, 2005.

[18] T. S. Biro, G. Kaniadakis, Contribution to the NEXT Sigma Phi Conference, Kolymbia, Crete, Aug.12-18, 2005.

[19] A.Rényi, MTA Mat.Kut.Int.Közl. 1, 9, 1956; Acta Math.Acad.Sci.Hung. 10, 193, 1959; Calcul de probabilité, Dunod, Paris, 1966.

[20] C. Beck, E. G. D. Cohen, Physica A 322, 267, 2003; C. Beck, cond-mat/0303288.

[21] R. Hagedorn: Nucl.Phys.B 24, 93, 1970; C. J. Hamer, S. C. Frautschi: Phys.Rev.D 24, 2125, 1971; R. Hagedorn, J. Ranft: Nucl.Phys.B 48, 157, 1972; R. Hagedorn: Riv. Nuovo Cimento 6N 10, 1, 1984; R. Hagedorn, K. Redlich: Z.Phys.C 27, 541, 1985.

[22] G. Torrieri, Lecture for students, QM2005, Budapest, Aug.3. 2005.

[23] L. P. Csernai, Lecture for students, QM2005, Budapest, Aug.3. 2005.

[24] R. J. Fries, B. Muller, C. Nonaka, S. A. Bass, Phys.Rev.Lett. 90, 202303, 2003; C. Nonaka, R. J. Fries, S. A. Bass, Phys.Lett.B 583, 73, 2004; R. J. Fries, S. A. Bass, B. Muller, Phys.Rev.Lett. 94, 122301, 2005; D. Molnar, M. Gyulassy, Phys.Rev.C 62, 054907, 2000; D. Molnar, S. A. Voloshin, Phys.Rev.Lett. 91, 092301, 2003; D. Molnar, P.Huovinen, Phys.Rev.Lett. 94, 012302, 2005; D. Molnar, nucl-th/0503051.

[25] G. Wilk, Z. Wlodarczyk, Physica A 305, 227, 2002; Chaos, Solitons and Fractals 13, 581, 2002; Phys.Rev.Lett. 84, 2770, 2000.

[26] T. J. Sherman, J. Rafelski, physics/0204011.

[27] W. Broniowski, W. Florkowski, L. Ya. Glozman: Phys.Rev.D 117503, 2004.

[28] T. S. Biro, A. Peshier, hep-ph/0506132.

[29] S. Abe, Physica A 300, 417, 2001; Phys.Rev.E 63, 061105, 2001.

[30] J. S. Havrda, F. Charvat, Kybernetica 3, 30, 1967; Z. Daroczy, Inf.Control 16, 36, 1970;