LOG-CONCAVITY OF THE OVERPARTITION FUNCTION

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Abstract. We prove that the overpartition function \( p(n) \) is log-concave for all \( n \geq 2 \). The proof is based on Sills Rademacher type series for \( p(n) \) and inspired by DeSalvo and Pak’s proof for the partition function.

Keywords. Integer Partition, Overpartition Function, Log-Concave Sequence, Circle Method

MSC Keywords. 05A17, 11P82, 11F20, 11F37

1. Introduction and Statement of Results

A partition of a positive integer \( n \) is a non-increasing sequence of positive integers \( a_1, \ldots, a_l \) whose sum is equal to \( n \). The number of partitions of a positive integer is commonly denoted by \( p(n) \). For example, the partitions of 4 are

\[
\{1, 1, 1, 1\}, \{1, 1, 2\}, \{2, 2\}, \{1, 3\}, \{4\}.
\]

Determining this number might seem like a very simple objective, a priori. However, in order to prove strong results about \( p(n) \), one needs significant amounts of theory such as complex analysis, the theory of modular forms, and knowledge about Kloosterman sums and Bessel functions. For example, with the circle method, one can obtain series representations or investigate the asymptotic behavior of the partition function, as well as prove other results.

Much of the modern study of properties of \( p(n) \) begins with Ramanujan and Hardy, who found among many other results the asymptotic behavior of \( p(n) \):

\[
p(n) \sim \exp \left( \frac{\pi \sqrt{n}}{4n^{3/2}} \right).
\]

In 1917, Littlewood, Ramanujan and Hardy invented in [3] the circle method to obtain (1).

Based on their results, Rademacher extended [1] to an exact formula in 1937 [4]. In the same year, D. H. Lehmer published a formula with an error term [2]:

\[
p(n) = \frac{\sqrt{12}}{24n - 1} \sum_{k=1}^{N} A_k(n) \left[ \left(1 - \frac{k}{\mu} \right) e^{\mu/k} + \left(1 + \frac{k}{\mu} \right) e^{-\mu/k} \right] + R_2(n, N),
\]

Here \( \mu := \mu(n) = \frac{\pi}{6} \sqrt{24n - 1} \) and

\[
A_k(n) := \sqrt{k} A_k^*(n) = \sqrt{k} \sum_{\substack{h, k \in \mathbb{Z}/k \mathbb{Z}^* \times \omega(h,k) e \left( \frac{hn}{k} \right),}}
\]

with \( e(x) := \exp(2\pi ix) \).

The author is grateful for useful advice and guidance from Professor Bringmann, Dr. Krauel, Dr. Li, Dr. Mertens and Dr. Rolen.
The term $\hat{R}_2/(n,N)$ is Lehmer’s reminder Term. Finally,

$$\omega(h,k) := \exp \left( \pi^2 \sum_{r=1}^{k-1} \frac{r}{k} \frac{hr}{k} - \left\lfloor \frac{hr}{k} \right\rfloor - \frac{1}{2} \right)$$

for positive integers $h$ and $k$.

We come now to the main result of this paper. Therefore, we recall what it means for a function to be log-concave. For a real valued function $f$ and positive integer $n$, define $C(f(n))$ by

$$C(f(n)) := \log(f(n+1)) - 2\log(f(n)) + \log(f(n-1)).$$

**Definition.** A function $f$ is log-concave for a positive integer $n$ if $C(f(n)) \leq 0$.

In 2013, Desalvo and Pak utilize (2) in [5] to develop the following result about the log-concavity of $C_n$ and positive integers $h$ and $k$.

**Theorem (Desalvo and Pak).** The sequence $p(n)$ is log-concave for all $n > 25$.

They show this by reorganizing $p(n)$ into a main part $T(n)$ and an error term $R(n)$. Then they use (1.1) to get an upper and lower bound for $C(T(n))$ which leads to upper and lower bound for $C(p(n))$. In this paper we will derive a similar Statement for overpartitions by a careful analysis of the Rademacher-type series given by Sills [1].

**Lemma 1.1 (DeSalvo and Pak).** Suppose $f(x)$ is a positive, increasing function with two continuous derivatives for all $x > 0$, and that $f'(x) > 0$ and decreasing and $f''(x) < 0$ is increasing for all $x \geq 1$. Then for all $x > 1$

$$f''(x-1) < f(x+1) - 2f(x) + f(x-1) < f''(x+1).$$

In this paper we extend this result to overpartitions. An overpartition of a positive integer $n$ is a partition of $n$ in which the last occurrence of a number can be distinguished, which we do by overlining it. For example, the overpartitions of 4 are

$$\{1, 1, 1, 1\}, \{1, 1, 1, \overline{1}\}, \{1, 1, 2\}, \{1, \overline{1}, 2\}, \{1, \overline{1}, \overline{2}\}, \{2, 2\}, \{2, \overline{2}\}, \{1, 3\}, \{1, \overline{3}\}, \{\overline{3}, \overline{3}\}, \{4\} \text{ and } \{\overline{4}\}.$$

Commonly, the number of overpartitions of a positive integer $n$ is denoted by $\overline{p}(n)$. Sills [1] rediscovered Zuckermann’s [6] formula for the overpartition and pointed out that it is indeed a Rademacher-type series

$$\overline{p}(n) = \frac{1}{2\pi} \sum_{k \geq 1} \sqrt{k} \sum_{\omega \leq h \leq k, \omega \neq h, \omega \neq k} \frac{\omega(h,k)^2}{\omega(h,k)} \frac{(-nh/k)}{d} \left( \frac{\sinh(\pi\sqrt{n}/k)}{\sqrt{n}} \right)^{1/2},$$

by using the generating function for the overpartition

$$\sum_{n \geq 1} \overline{p}(n)q^n = \prod_{n \geq 1} \frac{1 + q^n}{1 - q^n},$$

and the circle method.

In this paper, our main result is the following.

**Theorem 1.2.** The function $\overline{p}(n)$ is log-concave for $n \geq 2$.

Table [1] illustrates the Theorem 1.2 and the behavior of $\overline{p}(n)$ and $p(n)$ for increasing $n$.

This paper is a condensed version of my Diploma thesis at the University of Cologne in 2014 which was supervised by Professor K. Bringmann and Dr. L.
Table 1. Comparison of \( p(n) \) and \( \overline{p}(n) \) and their image under \( C \).

| \( n \) | \( p(n) \) | \( \overline{p}(n) \) | \( C(p(n)) \) | \( C(\overline{p}(n)) \) |
|---|---|---|---|---|
| 2 | 2 | 4 | -0.287682 | 0 |
| 3 | 3 | 8 | 0.105361 | -0.133531 |
| 4 | 5 | 14 | -0.174353 | -0.0206193 |
| 5 | 7 | 24 | 0.115513 | -0.0281709 |
| 6 | 11 | 40 | -0.14183 | -0.040822 |
| 7 | 15 | 64 | 0.0728373 | -0.0237165 |
| 8 | 22 | 100 | -0.0728373 | -0.0145047 |
| 9 | 30 | 154 | 0.0263173 | -0.0219976 |
| 10 | 42 | 232 | -0.0487902 | -0.0158805 |
| 15 | 176 | 1472 | 0.0067245 | -0.0103469 |
| 20 | 627 | 7336 | -0.0129263 | -0.0062498 |
| 25 | 1958 | 31064 | 0.000765534 | -0.0047666 |
| 30 | 5604 | 116624 | -0.00546266 | -0.0037539 |
| 35 | 14883 | 398640 | -0.000934141 | -0.00212943 |
| 40 | 37338 | 1263272 | -0.00273332 | -0.00250508 |
| 45 | 89134 | 3759240 | -0.00120242 | -0.00212943 |
| 50 | 204226 | 10605564 | -0.00173137 | -0.0018351 |

Rolen and is structured as follows. First, we elaborate on an error bound for \( p(n) \), and then we split \( p(n) \) into the main term \( \hat{T}(n) \) and a reminder term \( \hat{R}(n) \) and show that \( \hat{T}(n) \) is itself log-concave. Finally, we deduce the log-concavity of \( p(n) \) from the log-concavity of \( \hat{T}(n) \) and bounding \( \hat{R}(n) \).

2. The error term of \( \overline{p}(n) \)

In this section, we provide an analogous error term for the overpartition function as Lehmer did in [2] for the partition function, which is labeled \( R_2(n, N) \) in this paper (see (2)).

Our first step is to calculate the derivative in (3):

\[
\frac{d}{dn} \left( \frac{\sinh(\hat{\mu}/k)}{\sqrt{n}} \right) = \frac{\pi}{2kn} \left( \cosh \left( \frac{\hat{\mu}}{k} \right) - k \frac{\hat{\mu}}{k} \sinh \left( \frac{\hat{\mu}}{k} \right) \right),
\]

where \( \hat{\mu} := \hat{\mu}(n) = \pi \sqrt{n} \). We then let

\[
\hat{A}_k(n) := \sum_{0 \leq h < k \cap (h,k)=1} \left( \frac{\omega(h,k)^2}{\omega(2h,k)} \right) e(-nh/k),
\]

and split up (3) so that for any integer \( N \geq 1 \),

\[
\overline{p}(n) = \frac{1}{2\pi} \sum_{k \geq 1 \cap 2 \mid k} \sqrt{k} \hat{A}_k(n) \frac{d}{dn} \left( \frac{\sinh(\hat{\mu}/k)}{\sqrt{n}} \right)
\]

(5)

\[
= \frac{1}{2\pi} \sum_{k \geq 1 \cap 2 \mid k} \sqrt{k} \hat{A}_k(n) \frac{d}{dn} \left( \frac{\sinh(\hat{\mu}/k)}{\sqrt{n}} \right) + \frac{1}{4n} \sum_{k \geq 1 \cap 2 \mid k} \hat{A}_k(n) \sqrt{k} \left( \cosh \left( \frac{\hat{\mu}}{k} \right) - k \frac{\hat{\mu}}{k} \sinh \left( \frac{\hat{\mu}}{k} \right) \right).
\]

To get a useful error bound, we want to estimate \( |R_2(n, N)| \) by comparing with an elementary function. This is done in two steps. First, we state two lemmas,
Lemma 2.1. We have \( f_n(k) \geq 0 \) for all \( k \in \mathbb{R} \setminus \{0\} \).

The previous Lemma follows immediately by rewriting \( \cosh(x) \) and \( \sinh(x) \) in there series representation. The next step is to take a closer look at \( \frac{\omega(h,k)^2}{\omega(2h,k)} e(-nh/k) \). To this end, we use that \( \omega(h,k) \) is a certain 24th root of unity, and the fact that \( \frac{\omega(h,k)^2}{\omega(2h,k)} e(-nh/k) = 1 \). Thus, we can trivially bound \( \hat{A}_k(n) \) from above by \( k \).

Using this and Lemma 2.1, we have

\[
\sum_{k \geq N+1} \frac{\hat{A}_k(n)}{\sqrt{k}} f_n(k) \leq \sum_{k \geq N+1} |\sqrt{k} f_n(k)| = \sum_{k \geq N+1} \sqrt{k} f_n(k) =: S.
\]

To simplify further, we need the absolute convergence of \( S \), which is true if \( S \) is convergent because of Lemma 2.1.

Lemma 2.2. The sum \( S \) is convergent.

It follows from Lemmas 2.1 and 2.2 that we can rearrange terms and obtain

\[
\sum_{k \geq N+1} \sum_{m \geq 1} \sqrt{k} \left( \frac{\mu}{n} \right)^{2m} (2m)! = \sum_{m \geq 1} \sum_{k \geq N+1} \sqrt{k} \left( \frac{\mu}{n} \right)^{2m} (2m+1)!.
\]

Finally, we put this all together and prove the following theorem.

Theorem 2.3. Let \( \mathcal{P}(n) \) be defined as in (3). Then

\[
\mathcal{P}(n) = \frac{1}{2\pi} \sum_{k \geq 1} N \int \frac{\sinh(\mu/k)}{\sqrt{n}} + R_2(n,N),
\]

where \( |R_2(n,N)| \leq \frac{\pi^{3/2}}{n!} \sinh \left( \frac{\mu}{n} \right) \).

Proof. We have already seen that \( |R_2(n,N)| \leq S \). Using (3) and the fact \( \frac{1}{n!} \) is monotonically decreasing and positive in \([N,\infty)\) for an \( \alpha > 0 \), we obtain

\[
\left| \sum_{m \geq 1} \sum_{k \geq N+1} \sqrt{k} \left( \frac{\mu}{n} \right)^{2m} (2m) \right| \leq \sum_{m \geq 1} \int_N^\infty \sqrt{x} \left( \frac{\mu}{x} \right)^{2m} (2m) d\mu.
\]

Since \( \sqrt{x} \left( \frac{\mu}{x} \right)^{2m} (2m) \) is decreasing and continuous for \( x \in [N,\infty) \), we find

\[
\int_N^\infty \sqrt{x} \left( \frac{\mu}{x} \right)^{2m} (2m) d\mu = \left[ \frac{-4m\mu^{2m+3/2-2m}}{(4m-3)(2m+1)!} \right]_N^\infty.
\]

For \( m > \frac{4}{3} \), we find

\[
\lim_{x \to \infty} \frac{-4m\mu^{2m+3/2-2m}}{(4m-3)(2m+1)!} = 0.
\]
Finally, we have

\[ |R_2(n, N)| \leq \frac{1}{4n} \sum_{m \geq 1} \frac{4m}{4m - 3} \left( \frac{\pi}{4} \right)^{2m} \frac{N^{3/2}}{n} \leq \frac{N^{3/2}}{n} \sum_{m \geq 0} \left( \frac{\pi}{4} \right)^{2m} \frac{N^{3/2}}{(2m + 1)!} \]

\[ = \frac{N^{5/2}}{n \hat{\mu}} \sinh \left( \frac{\hat{\mu}}{N} \right) , \]

which completes the proof.

3. Bounding the main term of \( \bar{P}(n) \)

This section provides the final expression of \( \bar{P}(n) \) as the sum of two functions \( \hat{T}(n) \) and \( \hat{R}(n) \) and concludes with a proof of the log-concavity of \( \hat{T}(n) \). In order to proceed, we now evaluate \( \frac{d}{dn} \left( \frac{\sinh(\hat{\mu})}{\sqrt{n}} \right) \) in another way by recalling the identity

\[ \sinh(z) = \frac{e^z - e^{-z}}{2} . \]

This shows that

\[ \frac{d}{dn} \left( \sinh \left( \frac{\hat{\mu}}{\sqrt{n}} \right) \right) = \frac{d}{dn} \left( \frac{\hat{\mu}}{2k \sqrt{n}} \right) \left( e^{\hat{\mu} t} + e^{-\hat{\mu} t} \right) - \frac{\left( e^{\hat{\mu} t} - e^{-\hat{\mu} t} \right)}{4n \hat{\mu}} . \]

If we now use \( \frac{d}{dn} \hat{\mu} = \frac{\pi}{2\sqrt{n}} \), then (7) simplifies to

\[ \frac{\pi}{4nk} \left( e^{\hat{\mu} t} + e^{-\hat{\mu} t} \right) - \frac{1}{4n^2} \left( e^{\hat{\mu} t} - e^{-\hat{\mu} t} \right) = \frac{\pi}{4n} \left( \frac{1}{k} \left( e^{\hat{\mu} t} + e^{-\hat{\mu} t} \right) - \frac{1}{\hat{\mu}} \left( e^{\hat{\mu} t} - e^{-\hat{\mu} t} \right) \right) \]

\[ = \frac{\pi}{4nk} \left( \frac{1}{k} \left( e^{\hat{\mu} t} + e^{-\hat{\mu} t} \right) - \frac{k}{\hat{\mu}} \left( e^{\hat{\mu} t} - e^{-\hat{\mu} t} \right) \right) \]

\[ = \frac{\pi}{4nk} \left( \frac{1}{1 + \frac{k}{\hat{\mu}}} \left( e^{\hat{\mu} t} + e^{-\hat{\mu} t} \right) + \left( 1 - \frac{k}{\hat{\mu}} \right) e^{\hat{\mu} t} \right) . \]

Substituting (5) into (5) gives us

\[ P(n) = \frac{1}{8n} \sum_{\substack{k=1 \atop (k, 2) = 1}}^{N} \frac{\hat{A}_k(n)}{\sqrt{k}} \left( \left( 1 + \frac{k}{\hat{\mu}} \right) e^{\hat{\mu} t} + \left( 1 - \frac{k}{\hat{\mu}} \right) e^{-\hat{\mu} t} \right) + R_2(n, N) \]

\[ = \hat{T}(n) + \hat{R}(n) , \]

where

\[ \hat{A}_k(n) := \sum_{0 \leq h < k, h = 1} \frac{\omega(h, k)^2}{\omega(2h, k)} \left( \frac{n h}{k} \right) . \]

Here, \( \hat{T}(n) \) and \( \hat{R}(n) \) are defined as

\[ \hat{T}(n) := \frac{1}{8n} \left\{ e^{-\hat{\mu} t} + \left( 1 - \frac{1}{\hat{\mu}} \right) e^{\hat{\mu} t} \right\} , \]

\[ \hat{R}(n) := \frac{e^{-\hat{\mu} t}}{8n\hat{\mu}} + R_2(n, 3) . \]

In order to show the log-concavity of \( \hat{T}(n) \), we use Lemma 1.1 and set

\[ f(x) := \log(\hat{T}(x)) . \]

First, we show that \( f(x) \) fulfills the necessary preliminaries of Lemma 1.1.

**Lemma 3.1.** The function \( f(x) \) satisfies the following properties:

(i) If \( x \geq 1 \), then \( f(x) \) is increasing and positive.

(ii) For all \( x > 1 \), the function \( f'(x) \) is positive.
(iii) If $x \geq 1$, then $f'(x)$ is decreasing.
(iv) For all $x \geq 1$, the function $f''(x)$ is negative.
(v) If $x \geq 3$, then $f''(x)$ is increasing.

Proof. (i) + (ii): We notice that
\begin{equation}
\hat{T}(x) > \frac{1}{2}e^{\hat{\mu}(x)} > 1 \text{ for all } x > 0,
\end{equation}
and hence
\begin{equation}
f(x) > \hat{\mu}(x) - \log(2) > 0 \text{ for all } x > 1 > \left(\frac{\log(2)}{\pi}\right)^2,
\end{equation}
with $\hat{\mu}(x) = \pi \sqrt{x}$. This establishes positivity.

As the composition of a couple twice differentiable function, $f(x)$ is also twice differentiable for all $x > 0$.

We show now that $f(x)$ is increasing by proving that $f'(x) > 0$ for all $x \geq 1$. By simple calculus, we obtain
\begin{equation}
f'(x) = \frac{e^{2\hat{\mu}(x)} - 2\hat{\mu}(x)^2}{2x(\hat{\mu}(x) + e^{2\hat{\mu}(x)}(\hat{\mu}(x) - 1))}.
\end{equation}
Next we notice that for $x \geq 1 > (1/\pi)^2$,
\begin{equation}
\hat{\mu}(x) > 1
\end{equation}
and so
\begin{equation}
2x\left(\hat{\mu}(x) + e^{2\hat{\mu}(x)}(\hat{\mu}(x) - 1)\right) > 0.
\end{equation}
To show the denominator of (10) is positive, we notice
\begin{equation}
\exp(2\hat{\mu}(x)) = \sum_{n \geq 0} \frac{(2\hat{\mu}(x))^n}{n!} > 1 + 2\hat{\mu}(x) + 2\hat{\mu}(x)^2
\end{equation}
and obtain
\begin{equation}
\exp(3\hat{\mu}(x)) - 2\hat{\mu}(x)^2 \geq 1 + 2\hat{\mu}(x) + 2\hat{\mu}(x)^2 - 2\hat{\mu}(x)^2 > 0
\end{equation}
for all $x \in \mathbb{R}$, and so we have that $f(x)$ is increasing for $x \geq 1$.

(iii) + (iv): We show that $f'(x)$ is decreasing by observing that $f''(x) < 0$ for all $x \geq 1$. We calculate $f''(x)$ directly as
\begin{equation}
f''(x) = \frac{2\hat{\mu}(x)^3 + e^{4\hat{\mu}(x)}(2 - 3\hat{\mu}(x)) + e^{2\hat{\mu}(x)}\hat{\mu}(x)(-3 + 2\hat{\mu}(x) - 2\hat{\mu}(x)^2 + 4\hat{\mu}(x)^3)}{4\left(x^{3/2}\pi + e^{2\hat{\mu}(x)}x(\hat{\mu}(x) - 1)^2\right)^2}.
\end{equation}
Note that the denominator is positive for all $x \in \mathbb{R}^+$. Next, we take into account that
\begin{equation}
2 - 3\hat{\mu}(x) < -2\hat{\mu}(x) < 0 \text{ for } x \geq 1 > \left(\frac{2}{\pi}\right)^2.
\end{equation}
Using (12), we obtain
\begin{equation}
-3 + 2\hat{\mu}(x) - 2\hat{\mu}(x)^2 + 4\hat{\mu}(x)^3 < 4\hat{\mu}(x)^3 - 3 \text{ for } x \geq 1 \geq (1/\pi)^2,
\end{equation}
which brings us to
\begin{equation}
0 \leq -3 + 4\hat{\mu}(x)^3 \leq 5\hat{\mu}(x)
\end{equation}
for $x \geq 1 > (3/4)^{2/3} \frac{1}{\pi^2}$. With (12) and (13), we have
\begin{equation}
2\hat{\mu}(x)^3 + e^{4\hat{\mu}(x)}(2 - 3\hat{\mu}(x)) + e^{2\hat{\mu}(x)}\hat{\mu}(x)(-3 + 2\hat{\mu}(x) - 2\hat{\mu}(x)^2 + 4\hat{\mu}(x)^3)
\leq 2\hat{\mu}(x)^3 - 2\hat{\mu}(x)e^{4\hat{\mu}(x)} + 5\hat{\mu}(x)^6 e^{2\hat{\mu}(x)} \leq 0,
\end{equation}
for $x \geq 1$. Therefore, we have (iii) and (iv).
We note that the denominator of (15) is positive and nonzero for all \( x \geq 1 \). We compute

\[
 f''(x) = \frac{e^{\hat{6}(x)p_1(x)} + e^{2\hat{\mu}(x)\hat{\mu}(x)} p_2(x) + e^{4\hat{\mu}(x)} p_3(x) - 3\hat{\mu}(x)^4}{8(x^{3/2} + e^{2\hat{\mu}(x)(\hat{\mu}(x) - 1))}}
\]

with

\[
 p_1(x) := 8 - 24\hat{\mu}(x) + 24\hat{\mu}(x)^2 - 9\hat{\mu}(x)^3 + 3\hat{\mu}(x)^4,
 p_2(x) := 15 - 21\hat{\mu}(x) + 21\hat{\mu}(x)^2 - 20\hat{\mu}(x)^3 + 8\hat{\mu}(x)^4,
 p_3(x) := -21\hat{\mu}(x) + 45\hat{\mu}(x)^2 - 30\hat{\mu}(x)^3 + 7\hat{\mu}(x)^4 + 4\hat{\mu}(x)^5 - 8\hat{\mu}(x)^6.
\]

We note that the denominator of (15) is positive and nonzero for all \( x \geq 1 \). Next, we will show that the numerator of (15) is positive. For this, we need

\[
 p_1(x) > \hat{\mu}(x)^4, \quad p_2(x) > 4\hat{\mu}(x)^4, \quad p_3(x) > -8\hat{\mu}(x)^6.
\]

We begin with the proof for (16). For \( x \geq 2 > \frac{1}{\sqrt{\pi}} \), we have \( \hat{\mu}(x) - 1 > 0 \), and we find

\[
 8 + 24\hat{\mu}(x)(\hat{\mu}(x) - 1) + \hat{\mu}(x)^3(2\hat{\mu}(x) - 9) > 0,
\]

which is equivalent to

\[
 8 - 24\hat{\mu}(x) + 24\hat{\mu}(x)^2 - 9\hat{\mu}(x)^3 + 2\hat{\mu}(x)^4 > 0,
\]

and can be rewritten as \( p_1(x) > \hat{\mu}(x)^4 \).

Next, we prove (17). If \( x > 1 > \frac{2}{\sqrt{\pi}} \), then we find

\[
 2\hat{\mu}(x) > 5 \quad \text{and} \quad \hat{\mu}(x) - 1 > 0
\]

which is equivalent to

\[
 15 + 21\hat{\mu}(x)(\hat{\mu}(x) - 1) + 4\hat{\mu}(x)^3(2\hat{\mu}(x) - 5) > 0
\]

and leads to \( p_2(x) > 4\hat{\mu}(x)^3 \).

Finally, we prove (18). For \( x \geq 1 \) we use \( \hat{\mu}(1) = \pi > 3 \), so that \( \hat{\mu}(x)^2 > \hat{\mu}(x) \).

Thus,

\[
 p_3(x) + 8\hat{\mu}(x)^6 > -21\hat{\mu}(x) + 45\hat{\mu}(x) - 30\hat{\mu}(x)^3 + 21\hat{\mu}(x)^3 + 4\hat{\mu}(x)^5
 > 24\hat{\mu}(x) - 9\hat{\mu}(x)^3 + 36\hat{\mu}(x)^3 = 24\hat{\mu}(x) + 27\hat{\mu}(x)^3 > 0.
\]

It remains to show that

\[
 e^{6\hat{\mu}(x)}\hat{\mu}(x)^4 + e^{2\hat{\mu}(x)\hat{\mu}(x)} p_2(x) - 3\hat{\mu}(x)^4 > 3\hat{\mu}(x)^4 - 3\hat{\mu}(x)^4 = 0
\]

and

\[
 e^{6\hat{\mu}(x)}\hat{\mu}(x)^4 - 8\hat{\mu}(x)^6 e^{4\hat{\mu}(x)} = \hat{\mu}(x)^4 e^{4\hat{\mu}(x)} \left( e^{2\hat{\mu}(x)} - \hat{\mu}(x)^6 \right) > 0.
\]

We show (20) by first noticing that \( \exp(2z)/z^6 \) and \( \hat{\mu}(x) \) are increasing for \( x \geq 1 \). Moreover, \( \frac{\exp(2z)}{z^6} \to \infty \), and so \( \hat{\mu}(x) \to \infty \) for \( x, z \to \infty \). Hence, there exists a \( z_0 \in \mathbb{R} \) with

\[
 \exp \left( \frac{2z}{z^6} \right) \geq 1
\]
for \( z \geq z_0 \). There also exists an \( x_0 \) so that \( \exp(2\hat{\mu}(x))/\hat{\mu}(x)^6 > 1 \) for \( x \geq x_0 \), and when we have \( \exp(2\hat{\mu}(3))/\hat{\mu}(3)^6 > 1 \), we find
\[
\frac{\exp(2\hat{\mu}(x))}{\hat{\mu}(x)^6} > 1
\]
which is equivalent to
\[
\exp(2\hat{\mu}(x)) - \hat{\mu}(x)^6 > 1
\]
for all \( x \geq 3 \geq x_0 \). This shows \([19]\), which indicates \( f''(x) \) is increasing for \( x \geq 3 \). This concludes the proof. \( \square \)

Using Lemma 3.1, we can now prove the following theorem.

**Theorem 3.2.** The function \( \hat{T}(n) \) is log-concave for all \( n \geq 3 \). In particular,
\[
C(\hat{T}(n)) \leq \frac{2\hat{\mu}^3 - 2\hat{\mu}^4\hat{\nu} + 5\hat{\nu}^2\hat{\mu}}{4n^2e^{4\hat{\mu}(\hat{\mu} - 1)^2}}
\]
for all \( n \geq 3 \).

**Proof.** To show the log-concavity of \( \hat{T}(n) \), we have to show that
\[
\left(\log\left(\hat{T}(n)\right)\right)' < 0.
\]

By Lemma 3.1 we have the preliminaries for Lemma 1.1 are fulfilled by \( \hat{T}(n) \), and may now use (14) to complete the proof. \( \square \)

In order to establish the log-concavity of \( p(n) \), we use the estimates
\[
\hat{T}(n) \left(1 - \frac{\hat{R}(n)}{\hat{T}(n)}\right) \leq p(n) \leq \hat{T}(n) \left(1 + \frac{\hat{R}(n)}{\hat{T}(n)}\right)
\]
and the following lemma.

**Lemma 3.3.** If \( y_n := \frac{\hat{R}(n)}{\hat{T}(n)} \), then \( \log\left(\frac{(1-y_n)(1-y_n)}{(1-y_{n-1})(1-y_{n+1})}\right) \geq \frac{9}{4n^2} e^{-2\hat{\mu}/3} \) for \( n \geq 2 \).

**Proof.** To begin, we observe that
\[
\left|\frac{e^{-\hat{\mu}}}{8n\hat{\mu}} + R_2(n, 3)\right| \leq \frac{e^{-\hat{\mu}}}{8n\hat{\mu}} + \left|R_2(n, 3)\right| \leq \frac{e^{-\hat{\mu}}}{8n\hat{\mu}} + \frac{3^5/2}{n\hat{\nu}} \sinh\left(\frac{\hat{\nu}}{3}\right)
\]
\[
\leq \left(\frac{e^{-2\hat{\mu}/3}}{2\hat{\mu}} - 1\right) e^{-\hat{\mu}/3} + \frac{3^5/2 \hat{\nu}^2/3}{2n\hat{\nu}} \leq \frac{3^5/2}{2n\hat{\nu}}\hat{\nu}^{2/3}.
\]
(21)

Using (9) and (21), we find
\[
y_n \leq \frac{3^{5/2}}{n\hat{\mu}} e^{-2\hat{\mu}/3} \quad \text{for} \quad n \geq 1.
\]

For \( n \geq 2 \), we have \( 0 \leq x_n < 1 \) and
\[
\log\left(\frac{(1-y_n)(1-y_n)}{(1-y_{n-1})(1-y_{n+1})}\right) = 2\log(1-y_n) - \log(1-y_{n-1}) - \log(1-y_{n+1})
\]
\[
\geq 2\log(1-y_n) \geq -\frac{2y_n}{1-y_n} \geq -\frac{2y_n}{1-y_2} \geq -3y_n.
\]

Therefore, \( \log\left(\frac{(1-y_n)(1-y_n)}{(1-y_{n-1})(1-y_{n+1})}\right) \geq -\frac{9}{4n^2} e^{-2\hat{\mu}/3} \), as desired. \( \square \)

The last lemma needed to complete the proof of Theorem 3.3 is the following.

**Lemma 3.4.** If the two functions \( a(n), b(n) \) are log-concave for \( n > n_0 \), the following is true:
(i) They satisfy \( C(a(n)b(n)) = C(a(n)) + C(b(n)) \).
(ii) The product \( a(n)b(n) \) is also log-concave for \( n > n_0 \).
If we now combine Theorem 3.2 with Lemmas 3.3 and 3.4, we can complete the proof of Theorem 3.5, which implies 1.2 after a short calculation.

**Theorem 3.5.** The overpartition function $p(n)$ is log-concave for $n \geq 4$. More precisely, for $n > 8$

$$C(p(n)) \leq \frac{2\mu^3 - 2\mu e^{4\mu} + 5\mu^6 e^{2\mu}}{4n^2 e^{4\mu}(\mu - 1)^2} + \frac{9}{n\mu} e^{-2\mu/3}.$$  

**Proof.** To begin, we combine the fact $p(n) = \hat{T}(n) \left(1 + \frac{\hat{R}(n)}{\hat{T}(n)}\right)$ with Lemma 3.4 (ii) to obtain

$$C(p(n)) = C\left(\hat{T}(n)\right) + C\left(1 + \frac{\hat{R}(n)}{\hat{T}(n)}\right).$$

Next, we use Theorem 3.2 and Lemma 3.3 to obtain

$$C(p(n)) \leq \frac{2\mu^3 - 2\mu e^{4\mu} + 5\mu^6 e^{2\mu}}{4n^2 e^{4\mu}(\mu - 1)^2} + \frac{9}{n\mu} e^{-2\mu/3}$$

(22)

$$= \frac{1}{4n\mu} \left(36e^{-2/3\mu} - \frac{\pi^2 (2 - 2e^{-4\mu} \mu^2 - 5e^{-2\mu} \mu^5)}{(\mu - 1)^2}\right).$$

In order to show that (22) is negative, we only have to show that the inner of the brackets is negative for a certain $n_0 \in \mathbb{N}$. Therefore we can neglect the factor $\frac{1}{4n\mu}$ because it is strictly positive. And now we proof the equivalent condition

$$\frac{36(\mu - 1)^2}{\pi^2} e^{-2/3\mu} + 2e^{-4\mu} \mu^4 + 5e^{-2\mu} \mu^5 < 2.$$  

(23)

$$=: f(n) > 0 \text{ for } n \in \mathbb{N}$$

The function $f(n)$ is monotone decreasing and so it suffice to find a $n_0 \in \mathbb{N}$ so that $f(n_0) < 2$. This is given for $n_0 \geq 4$ and so it follows that

$$C(p(n)) \leq 0$$

for $n \geq 4$.

\[\square\]

Theorem 1.2 now follows for large $n$ from Theorem 3.5 and for small $n$ from Table 1.

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