REVERSES OF THE YOUNG INEQUALITY FOR MATRICES AND OPERATORS

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ABSTRACT. We present some reverse Young-type inequalities for the Hilbert-Schmidt norm as well as any unitarily invariant norm. Furthermore, we give some inequalities dealing with operator means. More precisely, we show that if \( A, B \in \mathfrak{B}(\mathcal{H}) \) are positive operators and \( r \geq 0 \), \( A \triangledown_r B + 2r(A \triangledown B - A^\# B) \leq A^\# - rB \) and prove that equality holds if and only if \( A = B \). We also establish several reverse Young-type inequalities involving trace, determinant and singular values. In particular, we show that if \( A, B \) are positive definite matrices and \( r \geq 0 \), then \( \text{tr}((1 + r)A - rB) \leq \text{tr}(|A|^{1+r}B^{-r}) - r \left( \sqrt{\text{tr}A} - \sqrt{\text{tr}B} \right)^2 \).

1. Introduction and preliminaries

Let \( \mathcal{H} \) be a Hilbert space and let \( \mathfrak{B}(\mathcal{H}) \) be the \( C^* \)-algebra of all bounded linear operators on \( \mathcal{H} \) with the operator norm \( \| \cdot \| \) and the identity \( I_\mathcal{H} \). If \( \dim \mathcal{H} = n \), then we identify \( \mathfrak{B}(\mathcal{H}) \) with the space \( \mathbb{M}_n \) of all \( n \times n \) complex matrices and denote the identity matrix by \( I_n \). For an operator \( A \in \mathfrak{B}(\mathcal{H}) \), we write \( A \geq 0 \) if \( A \) is positive (positive semidefinite for matrices), and \( A > 0 \) if \( A \) is positive invertible (positive definite for matrices). For \( A, B \in \mathfrak{B}(\mathcal{H}) \), we say \( A \geq B \) if \( A - B \geq 0 \). Let \( \mathfrak{B}^+(\mathcal{H}) \) (resp., \( \mathcal{P}_n \)) denote the set of all positive invertible operators (resp., positive definite matrices). A norm \( \| \cdot \| \) on \( \mathbb{M}_n \) is called unitarily invariant norm if \( \| UAV \| = \| A \| \) for all \( A \in \mathbb{M}_n \) and all unitary matrices \( U, V \in \mathbb{M}_n \). The Hilbert-Schmidt norm is defined by \( \| A \|_2 = \left( \sum_{j=1}^n s_j^2(A) \right)^{1/2} \), where \( s(A) = (s_1(A), \ldots, s_n(A)) \) denotes the singular values of \( A \), that is, the eigenvalues of the positive semidefinite matrix \( |A| = (A^*A)^{1/2} \), arranged in the decreasing order with their multiplicities counted. This norm is unitarily invariant. It is known that if \( A = [a_{ij}] \in \mathbb{M}_n \), then \( \| A \|_2 = \left( \sum_{i,j=1}^n |a_{ij}|^2 \right)^{1/2} \).

The weighted operator arithmetic mean \( \triangledown_\nu \), geometric mean \( \sharp_\nu \), and harmonic mean \( !_\nu \), for \( \nu \in [0, 1] \) and \( A, B \in \mathfrak{B}^+(\mathcal{H}) \), are defined as follows:

\[
A \triangledown_\nu B = (1 - \nu)A + \nu B,
\]

\[
A \sharp_\nu B = A^{1/2}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\nu A^{1/2},
\]

\[
A !_\nu B = ((1 - \nu)A^{-1} + \nu B^{-1})^{-1}.
\]

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If \( \nu = 1/2 \), we denote arithmetic, geometric and harmonic mean, respectively, by \( \nabla \), \( \sharp \) and \( \ast \), for brevity.

The classical Young inequality states that
\[
a^{\nu}b^{1-\nu} \leq \nu a + (1 - \nu)b,
\]
when \( a, b \geq 0 \) and \( \nu \in [0, 1] \). If \( \nu = \frac{1}{2} \), we obtain the arithmetic-geometric mean inequality
\[
\sqrt{ab} \leq \frac{a + b}{2}.
\]
An operator Young inequality reads as follows:
\[
A^{\nu}B \leq A^{\#\nu}B \leq A^{\nabla\nu}B, \quad \nu \in [0, 1],
\]
where \( A, B \in \mathcal{B}^{+}(\mathcal{H}) \) and \( \nu \in [0, 1] \); cf. [5]. For other generalization of the Young inequality see [15, 16]. A matrix Young inequality due to Ando [1] asserts that
\[
s_j(A^{\nu}B^{1-\nu}) \leq s_j(\nu A + (1 - \nu)B),
\]
in which \( A, B \in \mathcal{M}_n \) are positive semidefinite, \( j = 1, 2, \ldots, n \), and \( \nu \in [0, 1] \). The above singular value inequality entails the unitarily invariant norm inequality
\[
|||A^{\nu}B^{1-\nu}||| \leq |||\nu A + (1 - \nu)B|||,
\]
where \( A, B \in \mathcal{M}_n \) are positive semidefinite and \( 0 \leq \nu \leq 1 \). Kosaki [13] proved that the inequality
\[
\|A^{\nu}XB^{1-\nu}\|_2 \leq \|\nu AX + (1 - \nu)XB\|_2
\]
holds for matrices \( A, B, X \in \mathcal{M}_n \) such that \( A, B \) are positive semidefinite, and for \( 0 \leq \nu \leq 1 \). It should be mentioned here that for \( \nu \neq \frac{1}{2} \) inequality (1.2) may not hold for other unitarily invariant norms. Hirzallah and Kittaneh [7], gave a refinement of (1.2) by showing that
\[
\|A^{\nu}XB^{1-\nu}\|_2^2 + r_0^2\|AX - XB\|_2^2 \leq \|\nu AX + (1 - \nu)XB\|_2^2,
\]
in which \( A, B, X \in \mathcal{M}_n \) are such that \( A, B \) are positive semidefinite, \( 0 \leq \nu \leq 1 \) and \( r_0 = \min\{\nu, 1 - \nu\} \). A determinant version of the Young inequality is also known (see [9, p. 467]):
\[
det(A^{\nu}B^{1-\nu}) \leq \det(\nu A + (1 - \nu)B),
\]
where \( A, B, X \in \mathcal{M}_n \) are such that \( A, B \) are positive semidefinite and \( 0 \leq \nu \leq 1 \). This determinant inequality was recently improved in [10]. Further, Kittaneh [11], proved that
\[
|||A^{1-\nu}XB^{\nu}||| \leq |||AX|||^{1-\nu}|||XB|||^{\nu},
\]
in which \( |||.||| \) is any unitarily invariant norm, \( A, B, X \in \mathcal{M}_n \) are such that \( A, B \) are positive semidefinite and \( 0 \leq \nu \leq 1 \). Conde [2], showed that
\[
2|||A^{1-\nu}XB^{\nu}||| + (|||AX|||^{1-\nu} - |||XB|||^{\nu})^2 \leq |||AX|||^{2(1-\nu)} + |||XB|||^{2\nu},
\]
where \( ||.|| \) is unitarily invariant norm, \( A, B, X \in \mathcal{M}_n \) are such that \( A, B \) are positive semidefinite and \( 0 \leq \nu \leq 1 \). Tominaga [20, 21] employed Specht’s ratio to Young inequality. In addition, some reverses of Young inequality are established in [4].

For \( a, b \in \mathbb{R} \), the number \( x = \nu a + (1 - \nu)b \) belongs to the interval \( [a, b] \) for all \( \nu \in [0, 1] \), and is outside the interval for all \( \nu > 1 \) or \( \nu < 0 \). Exploiting this obvious fact, Fujii [3], showed that if \( f \) is an operator concave function on an interval \( J \), then the inequality

\[
 f(C^*XC - D^*YD) \leq |C|f(V^*XV)| - D^*f(Y)D
\]

holds for all self-adjoint operators \( X, Y \) and operators \( C, D \) in \( \mathfrak{B}(\mathcal{H}) \) with spectra in \( J \), such that \( C^*C - D^*D = I_\mathcal{H}, \sigma(C^*XC - D^*YD) \subseteq J \) and \( C = V|C| \) is the polar decomposition of \( C \).

In this direction, by using some numerical inequalities, we obtain reverses of (1.1), (1.2), (1.3) and (1.4) under some mild conditions. We also aim to give some reverses of the Young inequality dealing with operator means of positive operators. Finally, we present some singular value inequalities of Young-type involving trace and determinant.

2. Reverses of the Young Inequality for the Hilbert-Schmidt Norm

In this section we deal with reverses of the Young inequality for the Hilbert-Schmidt norm. To this end, we need some lemmas.

**Lemma 2.1.** Let \( a, b > 0 \). If \( r \geq 0 \) or \( r \leq -1 \), then

\[
(1 + r)a - rb \leq a^{1+r}b^{-r}.
\]

**Proof.** Let \( f(t) = t^{-r} - (1 + r) + rt \), \( t \in (0, \infty) \). It is easy to see that \( f(t) \) attains its minimum at \( t = 1 \), on the interval \( (0, \infty) \). Hence, \( f(t) \geq f(1) = 0 \) for all \( t > 0 \). Letting \( t = \frac{b}{a} \), we get the desired inequality. \( \square \)

**Remark 2.2.** By virtue of Lemma 2.1, it follows that the inequality

\[
((1 + r)a - rb)^2 \leq (a^{1+r}b^{-r})^2
\]

holds if \( a \geq b > 0 \) and \( r \geq 0 \), or \( b \geq a > 0 \) and \( r \leq -1 \).

**Lemma 2.3.** [22, Theorem 3.4] (Spectral Decomposition) Let \( A \in \mathcal{M}_n \) with eigenvalues \( \lambda_1, \lambda_2, \ldots, \lambda_n \). Then \( A \) is normal if and only if there exists a unitary matrix \( U \) such that

\[
U^*AU = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n).
\]

In particular, \( A \) is positive definite if and only if \( \lambda_j > 0 \) for \( j = 1, 2, \ldots, n \).

Now, our first result reads as follows.
Theorem 2.4. Let $A, B, X \in \mathcal{M}_n$ and let $m, m'$ be positive scalars. If $A \geq mI_n \geq B > 0$ and $r \geq 0$, or $B \geq m' I_n \geq A > 0$ and $r \leq -1$, then the following inequality holds:

$$\| (1+r)AX - rXB \|_2 \leq \| A^{1+r}XB^{-r} \|_2.$$  

Proof. It follows from Lemma 2.3 that there are unitary matrices $U, V \in \mathcal{M}_n$ such that $A = U\Lambda U^*$ and $B = VTV^*$, where $\Lambda = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$, $\Gamma = \text{diag}(\gamma_1, \gamma_2, \ldots, \gamma_n)$, and $\lambda_j, \gamma_j, j = 1, 2, \ldots, n$, are positive. If $Z = U^* XV = [z_{ij}]$, then

$$(1+r)AX - rXB = U \left( (1+r)\Lambda Z - r\Gamma \right)V^* = U \left[ \left( (1+r) \lambda_i - r \gamma_j \right) z_{ij} \right] V^* \quad (2.3)$$

and

$$A^{1+r}XB^{-r} = U\Lambda^{1+r}U^* XV\Gamma^{-r}V^* = U\Lambda^{1+r} \Gamma^{-r}V^* = U \left[ \left( \lambda_i^{1+r} \gamma_j^{-r} \right) z_{ij} \right] V^*. \quad (2.4)$$

Suppose first that $A \geq mI_n \geq B > 0$ and $r \geq 0$. Then, it follows that

$$\lambda_i \geq \gamma_j, \quad 1 \leq i, j \leq n, \quad (2.5)$$

so, utilizing (2.3) and (2.4), we have

$$\| (1+r)AX - rXB \|_2^2 = \sum_{i,j=1}^n \left( (1+r)\lambda_i - r \gamma_j \right)^2 |z_{ij}|^2 \leq \sum_{i,j=1}^n \left( \lambda_i^{1+r} \gamma_j^{-r} \right)^2 |z_{ij}|^2 \quad \text{(by inequality (2.2) and (2.5))}$$

$$= \| A^{1+r}XB^{-r} \|_2^2.$$

The same conclusion can be drawn for the case of $B \geq m' I_n \geq A > 0$ and $r \leq -1$. \hfill $\square$

Recall that a continuous real valued function $f$, defined on an interval $J$, is called operator monotone if $A \leq B$ implies $f(A) \leq g(B)$, for all $A, B \in \mathcal{M}_n$ with spectra in $J$. Now, the following result can be accomplished as an immediate consequence of Theorem 2.4.

Corollary 2.5. Suppose that $A_j, B_j, X \in \mathcal{M}_n$, $1 \leq j \leq n$, with spectra in an interval $J$, and let $m_j, m_j'$, $1 \leq j \leq n$, be positive scalars. If $A_j \geq m_j I_n \geq B_j > 0$, $1 \leq j \leq n$, and $r \geq 0$, or $B_j \geq m_j' I_n \geq A_j > 0$, $1 \leq j \leq n$, and $r \leq -1$, then the inequality

$$\left\| \sum_{j=1}^n \left( (1+r)f(A_j)X - rXf(B_j) \right) \right\|_2 \leq \left\| \left( \sum_{j=1}^n f(A_j) \right)^{1+r} X \left( \sum_{j=1}^n f(B_j) \right)^{-r} \right\|_2$$

holds for any operator monotone function $f$ defined on interval $J$.

Proof. It suffices to set $A = \sum_{j=1}^n f(A_j)$ and $B = \sum_{j=1}^n f(B_j)$ in Theorem 2.4 to get the desired inequality. \hfill $\square$
Generally speaking, Theorem 2.4 does not hold for arbitrary positive definite matrices $A$ and $B$. The reason for this lies in the fact that the inequality (2.2) is not true for arbitrary positive numbers $a, b$. To see this, let $a = 1, b = 4, r = 2$.

Our next intention is to derive a result related to Theorem 2.4 which holds for all positive definite matrices. Observe that the inequality
\[
((1 + r)a - rb)^2 - r^2(a - b)^2 = (1 + 2r)a^2 - 2rab \leq (a^2)^{1+2r}(ab)^{-2r} = (a^{1+r}b^{-r})^2
\]
yields an appropriate relation instead of (2.2), for arbitrary positive numbers $a, b$ and $r \geq 0$ or $r \leq -\frac{1}{2}$, as follows:
\[
((1 + r)a - rb)^2 \leq (a^{1+r}b^{-r})^2 + r^2(a - b)^2 \quad a > 0, r \geq 0 \text{ or } r \leq -\frac{1}{2}.
\]
Note also that if $a = b$, then the equality holds.

Now, utilizing this inequality and the same argument as in the proof of Theorem 2.4, i.e. the spectral theorem for positive definite matrices, we can accomplish the corresponding result.

**Theorem 2.6.** Suppose that $A, B \in \mathcal{P}_n$ and $X \in \mathcal{M}_n$. Then the inequality
\[
\|(1 + r)AX - rXB\|_2^2 \leq \|A^{1+r}XB^{-r}\|_2^2 + r^2 \|AX - XB\|_2^2 \quad (2.6)
\]
holds for $r \geq 0$ or $r \leq -\frac{1}{2}$.

3. **Reverse Young-type inequalities involving unitarily invariant norms**

It has been shown in [8] that the inequality
\[
\|A^{1+r}XB^{1+r}\| \geq \|X\|^{-r}\|AXB\|^{1+r} \quad (3.1)
\]
holds for $A, B \in \mathcal{P}_n$, $0 \neq X \in \mathcal{M}_n$ and $r \geq 0$. Applying inequality (3.1) yields the relation
\[
\|A^{1+r}XB^{-r}\| \geq \|AX\|^{1+r}\|XB\|^{-r}, \quad (3.2)
\]
where $r \geq 0$, $A, B \in \mathcal{P}_n$ and $X \in \mathcal{M}_n$ with $X \neq 0$.

Our next intention is to show that inequality (3.2) holds for every unitarily invariant norm. This can be done by virtue of inequality (1.4). In fact, the following result is, in some way, complementary to inequality (1.4).

**Lemma 3.1.** Suppose that $A, B \in \mathcal{P}_n$, $X \in \mathcal{M}_n$ are such that $X \neq 0$. If $r \geq 0$ or $r \leq -1$, then the inequality
\[
\|AX\|^{1+r}\|XB\|^{-r} \leq \|A^{1+r}XB^{-r}\|
\]
holds for any unitarily invariant norm $\|\cdot\|$.\[\]
Proof. First, let \( r \geq 0 \). Set \( \alpha = r + 1 \). Utilizing inequality (1.4), it follows that
\[
\|AX\| = \|(A^\alpha)^{\frac{1}{\alpha}}(XB^{1-\alpha})(B^\alpha)\|^\frac{\alpha-1}{\alpha} \leq \|A^\alpha XB^{1-\alpha}\|^\frac{1}{\alpha} \|XB^{\alpha}\|^\frac{\alpha-1}{\alpha},
\]
that is,
\[
\|AX\| \|XB\|^{\frac{1-\alpha}{\alpha}} \leq \|A^\alpha XB^{1-\alpha}\|^\frac{1}{\alpha}.
\]
Hence,
\[
\|AX\|^\alpha \|XB\|^{1-\alpha} \leq \|A^\alpha XB^{1-\alpha}\|,
\]
whence
\[
\|AX\|^{1+r} \|XB\|^{-r} \leq \|A^{1+r}XB^{-r}\|.
\]
On the other hand, if \( r \leq -1 \), set \( \alpha = -r \). By a similar argument, we get the desired result.

Applying Lemmas 2.1 and 3.1 yields the Young-type inequality
\[
(1 + r)\|AX\| - r\|XB\| \leq \|A^{1+r}XB^{-r}\|,
\]
which holds for matrices \( A, B \in \mathcal{P}_n, X \in \mathcal{M}_n \) such that \( X \neq 0 \) and \( r \geq 0 \) or \( r \leq -1 \). It is interesting that the inequality (3.3) can be improved. But first we have to improve the scalar inequality (2.1).

Lemma 3.2. Let \( a, b > 0 \) and \( r \geq 0 \) or \( r \leq -\frac{1}{2} \). Then,
\[
(1 + r)a - rb + r(\sqrt{a} - \sqrt{b})^2 \leq a^{1+r}b^{-r}.
\]

Proof. Due to Lemma 2.1, it follows that
\[
(1 + r)a - rb + r(\sqrt{a} - \sqrt{b})^2 = -2r\sqrt{ab} + (1 + 2r)a \leq (\sqrt{ab})^{-2r}a^{1+2r} = a^{1+r}b^{-r}.
\]

Obviously, if \( r \geq 0 \), inequality (3.4) represents an improvement of inequality (2.1). Finally, we give now an improvement of matrix inequality (3.3).

Theorem 3.3. Let \( A, B \in \mathcal{P}_n, X \in \mathcal{M}_n \) be such that \( X \neq 0 \) and let \( r \geq 0 \). Then the inequality
\[
(1 + r)\|AX\| - r\|XB\| + r(\sqrt{\|AX\|} - \sqrt{\|XB\|})^2 \leq \|A^{1+r}XB^{-r}\|
\]
holds for any unitarily invariant norm \( \| \cdot \| \).
Proof.

\[(1 + r)|||AX||| - r|||XB||| + r(\sqrt{|||AX|||} - \sqrt{|||XB|||})^2 \leq |||AX|||^{1+r}|||XB|||^{-r}\]

(by Lemma 3.2)

\[\leq |||A^{1+r}XB^{-r}|||\]

(by Lemma 3.1).

□

Remark 3.4. It should be noticed here that the Theorem 3.3 is also true in the case of \(r \leq -\frac{1}{2}\). However, in this case, the corresponding inequality is less precise than the relation (3.3) and does not represent its refinement.

4. Reverse Young-type inequalities related to operator means

The matrix Young inequality can be considered in a more general setting. Namely, this inequality holds also for self-adjoint operators on a Hilbert space. The main objective of this section is to derive inequalities which are complementary to mean inequalities in (1.1), presented in the Introduction.

The main tool in obtaining inequalities for self-adjoint operators on Hilbert spaces, is the following monotonicity property for operator functions: If \(X\) is a self-adjoint operator with the spectrum \(\text{sp}(X)\), then

\[f(t) \geq g(t), \ t \in \text{sp}(X) \implies f(X) \geq g(X). \tag{4.1}\]

For more details about this property the reader is referred to [18].

Since \(A, B \in \mathfrak{B}^+(\mathcal{H})\), the expressions \(A\nabla_\nu B\) and \(A^\#_\nu B\) are also well-defined when \(\nu \in \mathbb{R} \setminus [0, 1]\). In this case, we obtain reverse of the second inequality in (1.1).

Theorem 4.1. If \(A, B \in \mathfrak{B}^+(\mathcal{H})\) and \(r \geq 0\) or \(r \leq -1\), then

\[A\nabla_{-r} B \leq A^\#_{-r} B. \tag{4.2}\]

Proof. By virtue of Lemma 2.1, it follows that \(f(x) = x^{-r} + rx - (1 + r) \geq 0, x > 0\). Moreover, since \(B \in \mathfrak{B}^+(\mathcal{H})\), it follows that \(A^{-\frac{1}{2}}BA^{-\frac{1}{2}} \in \mathfrak{B}^+(\mathcal{H})\), that is, \(\text{sp}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}) \in (0, \infty)\).

Thus, applying the monotonicity property (4.1) to the above function \(f\), we have that

\[\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{-r} + rA^{-\frac{1}{2}}BA^{-\frac{1}{2}} - (1 + r)I_{\mathcal{H}} \geq 0.\]

Finally, multiplying both sides of this relation by \(A^\frac{1}{2}\), we have

\[A^\frac{1}{2} \left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{-r} A^\frac{1}{2} + rB - (1 + r)A \geq 0,\]

and the proof is completed. □
If \( A, B \in \mathfrak{B}^+(\mathcal{H}) \) are such that \( A \leq B \), the expression \( A!_{-r} B \) is well defined for \( r \geq 0 \). Namely, due to operator monotonicity of the function \( h(x) = -\frac{1}{x} \) on \((0, \infty)\) (for more details, see [18]), \( A \leq B \) implies that \( B^{-1} \leq A^{-1} \), so that \((r + 1)A^{-1} - rB^{-1} \in \mathfrak{B}^+(\mathcal{H})\). Therefore, the operator \( A!_{-r} B = ((r + 1)A^{-1} - rB^{-1})^{-1} \) is well-defined for \( r \geq 0 \).

Now, we give the reverse of the first inequality in (1.1).

**Corollary 4.2.** Let \( A, B \in \mathfrak{B}^+(\mathcal{H}) \) be such that \( A \leq B \). If \( r \geq 0 \), then \( A!_{-r} B \leq A!_{-r} B \).

**Proof.** Theorem 4.1 with operators \( A \) and \( B \) replaced by \( A^{-1} \) and \( B^{-1} \), respectively, follows that
\[
A^{-1}\nabla_{-r} B^{-1} \leq A^{-1}_{-r} B^{-1}.
\]
(4.3)

Now, applying operator monotonicity of the function \( h(x) = -\frac{1}{x} \), \( x \in (0, \infty) \), to relation (4.3), we have that \((A^{-1}_{-r} B^{-1})^{-1} \leq (A^{-1}\nabla_{-r} B^{-1})^{-1} \). Finally, the result follows since \((A^{-1}_{-r} B^{-1})^{-1} = A!_{-r} B \).

Kittaneh et al. obtained in [12] the following relation (see also [14]):
\[
2 \max\{\nu, 1 - \nu\}(A\nabla B - A^{\sharp}_{\nu} B) \geq A\nabla B - A^{\sharp}_{\nu} B \geq 2 \min\{\nu, 1 - \nu\}(A\nabla B - A^{\sharp}_{\nu} B).
\]
(4.4)
Clearly, the left inequality in (4.4) represents the converse, while the right inequality represents the refinement of arithmetic-geometric mean operator inequality in (1.1).

Our next goal is to derive refinement of inequality (4.2) which is, in some way, complementary to above relations in (4.4). Clearly, this will be carried out by virtue of Lemma 3.2.

**Theorem 4.3.** If \( A, B \in \mathfrak{B}^+(\mathcal{H}) \) and \( r \geq 0 \), then the following inequality holds
\[
A\nabla_{-r} B + 2r(A\nabla B - A^{\sharp}_{\nu} B) \leq A^{\sharp}_{-r} B.
\]
(4.5)

**Proof.** By virtue of Lemma 3.2, it follows that
\[
(1 + r) - rx + r(x - 2\sqrt{x} + 1) \leq x^{-r}
\]
(4.6)
holds for all \( x > 0 \). Now, applying the functional calculus, i.e. the property (4.1) to this scalar inequality, we have
\[
(1 + r)I_{\mathcal{H}} - rA^{-\frac{1}{2}}BA^{-\frac{1}{2}} + r(A^{-\frac{1}{2}}BA^{-\frac{1}{2}} - 2\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{-\frac{1}{2}})^{\frac{1}{2}} + I_{\mathcal{H}} \leq \left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{-r}.
\]
Finally, multiplying both sides of this operator inequality by \( A^{\frac{1}{2}} \), we obtain (4.5).

**Corollary 4.4.** Let \( A, B \in \mathfrak{B}^+(\mathcal{H}) \) and \( r > 0 \). Then, \( A\nabla_{-r} B = A^{\sharp}_{-r} B \) if and only if \( A = B \).

**Proof.** It follows from Theorem 4.3 and the fact that \( A\nabla B = A^{\sharp} B \) if and only if \( A = B \).
Remark 4.5. Having in mind that scalar inequality (4.6) holds also for \( r \leq -\frac{1}{2} \) (see Lemma 3.2), it follows that inequality (4.5) holds also for \( r \leq -\frac{1}{2} \). However, if \( r < -1 \), relation (4.5) is less precise than the original inequality (4.2) and does not represent its refinement. On the other hand, it is interesting to consider the case when \( -1 \leq r \leq -\frac{1}{2} \). Namely, denoting \( \nu = -r \), where \( \frac{1}{2} \leq \nu \leq 1 \), (4.5) reduces to

\[
A \nabla \nu B - 2\nu(A \nabla B - A^\sharp \nu B) \leq A^\sharp \nu B,
\]

and this relation coincides with the converse of the arithmetic-geometric mean inequality, that is, with the left inequality in (4.4).

Remark 4.6. In [12], the authors considered operator version of the classical Heinz mean, i.e., the operator

\[
H_\nu(A, B) = \frac{A^\sharp \nu B + A^{\sharp 1-\nu} B}{2},
\]

(4.7)

where \( A, B \in \mathfrak{B}^+(\mathcal{H}) \), and \( \nu \in [0, 1] \). Like in the real case, this mean interpolates in between arithmetic and geometric mean, that is,

\[
A^\sharp B \leq H_\nu(A, B) \leq A \nabla B.
\]

(4.8)

On the other hand, since \( A, B \in \mathfrak{B}^+(\mathcal{H}) \), the expression (4.7) is also well-defined for \( \nu \in \mathbb{R} \setminus [0, 1] \). Moreover, due to Theorem 4.1, we obtain the inequality

\[
H_{-r}(A, B) = \frac{A^\sharp -r B + A^{\sharp 1+r} B}{2} \geq \frac{A \nabla -r B + A \nabla 1+r B}{2} = A \nabla B, \quad r \geq 0 \text{ or } r \leq -1,
\]

complementary to (4.8).

In order to conclude this section, we mention yet another inequality closely connected to the Young inequality. Namely, in [6], it has been shown the equivalence between the Young inequality and the Hölder-McCarthy inequality which asserts that

\[
\langle Ax, x \rangle^{-r} \leq \langle A^{-r} x, x \rangle, \quad x \in \mathcal{H}, \quad \|x\| = 1,
\]

(4.9)

holds for all \( A \in \mathfrak{B}^+(\mathcal{H}) \) and \( r > 0 \) or \( r < -1 \). If \( -1 < r < 0 \), then the sign of inequality in (4.9) is reversed.

Now, we give a refinement of the Hölder-McCarthy, once again by exploiting Lemma 3.2.

**Theorem 4.7.** Let \( A \in \mathfrak{B}^+(\mathcal{H}) \) and \( r > 0 \). Then the inequality

\[
0 \leq 2r \left( 1 - \langle A^{\frac{1}{r}} x, x \rangle \langle A x, x \rangle^{-\frac{1}{r}} \right) \leq \langle A^{-r} x, x \rangle \langle A x, x \rangle^r - 1
\]

(4.10)

holds for any unit vector \( x \in \mathcal{H} \).

**Proof.** By virtue of (4.6), it follows that the inequality \( 2r (1 - \sqrt{x}) \leq x^{-r} - 1 \) holds for all \( x > 0 \). Now, applying the functional calculus to this inequality and the positive operator \( \lambda^{\frac{1}{r}} A, \lambda > 0 \), we have

\[
2r \left( I_{\mathcal{H}} - \lambda^{\frac{1}{r}} A^{\frac{1}{r}} \right) \leq \lambda^{-1} A^{-r} - I_{\mathcal{H}}.
\]
Further, fix a unit vector \( x \in H \). Then we have
\[
2r \left(1 - \lambda^{\frac{1}{2r}} \langle A^\frac{1}{2} x, x \rangle\right) \leq \lambda^{-1} \langle A^{-r} x, x \rangle - 1.
\]
Finally, putting \( \lambda = \langle Ax, x \rangle - r \) in the last inequality, we obtain second inequality in (4.10). Clearly, the first inequality sign in (4.10) holds due to (4.9) since \( \langle A^\frac{1}{2} x, x \rangle \leq \langle Ax, x \rangle^\frac{1}{2} \).

\[ \square \]

Remark 4.8. Since relation (4.6) holds for \( r \leq -\frac{1}{2} \), it follows that the second inequality in (4.10) holds also for \( r \leq -\frac{1}{2} \). Clearly, the case of \( r < -1 \) is not interesting since in this case we obtain less precise relation than the original Hölder-McCarthy inequality (4.9). On the other hand, the case of \(-1 < r < \frac{1}{2} \) yields a converse of (4.9).

5. Reverse Young-type inequalities for the trace and the determinant

In this section we derive some Young-type inequalities for the trace and the determinant of a matrix. The starting point for this direction is already used Lemma 3.2.

In [10], Kittaneh and Manasrah obtained the inequality
\[
\text{tr} \left| A^\nu B^{1-\nu} \right| + r_0 \left(\sqrt{\text{tr} A} - \sqrt{\text{tr} B}\right)^2 \leq \text{tr} \left( \nu A + (1 - \nu) B \right),
\]
which holds for positive semidefinite matrices \( A, B \in M_n, 0 \leq \nu \leq 1 \), and \( r_0 = \min\{\nu, 1-\nu\} \).

By virtue of Lemma 3.2, we can accomplish the inequality complementary to (5.1). To do this, we also need the following inequality regarding singular values of complex matrices:
\[
\sum_{j=1}^{n} s_j(A)s_{n-j+1}(B) \leq \sum_{j=1}^{n} s_j(AB) \leq \sum_{j=1}^{n} s_j(A)s_j(B).
\]

Now, we have the following result:

Theorem 5.1. If \( A, B \in P_n \) and \( r \geq 0 \), then the following inequality holds:
\[
\text{tr}((1 + r)A - rB) \leq \text{tr} \left| A^{1+r} B^{-r} \right| - r \left(\sqrt{\text{tr} A} - \sqrt{\text{tr} B}\right)^2.
\]

Proof. By Lemma 3.2, we have
\[
(1 + r)s_j(A) - rs_{n-j+1}(B) \leq s_j^{1+r}(A)s_{n-j+1}^{-r}(B) - r \left(\sqrt{s_j(A)} - \sqrt{s_{n-j+1}(B)}\right)^2,
\]
for \( j = 1, 2, \ldots, n \).
Now, utilizing the above inequality and (5.2), as well as the properties of the trace functional, it follows that
\[
\text{tr}((1 + r)A - rB) = (1 + r)\text{tr}A - r\text{tr}B
\]
\[
= \sum_{j=1}^{n} ((1 + r)s_j(A) - rs_{n-j+1}(B))
\]
\[
\leq \sum_{j=1}^{n} s_j^{1+r}(A)s_{n-j+1}^{-r}(B)
\]
\[
- r \sum_{j=1}^{n} \left( s_j(A) + s_{n-j+1}(B) - 2\sqrt{s_j(A)s_{n-j+1}(B)} \right)
\]
\[
= \sum_{j=1}^{n} s_j(A^{1+r})s_{n-j+1}(B^{-r})
\]
\[
- r \left( \text{tr}A + \text{tr}B - 2\sum_{j=1}^{n} \sqrt{s_j(A)s_{n-j+1}(B)} \right)
\]
\[
\leq \sum_{j=1}^{n} s_j(A^{1+r}B^{-r}) - r \left( \text{tr}A + \text{tr}B - 2\sum_{j=1}^{n} \sqrt{s_j(A)s_{n-j+1}(B)} \right).
\]
Moreover, by virtue of the well-known Cauchy-Schwarz inequality, we have
\[
\sum_{j=1}^{n} \sqrt{s_j(A)s_{n-j+1}(B)} \leq \left( \sum_{j=1}^{n} s_j(A) \right)^{\frac{1}{2}} \left( \sum_{j=1}^{n} s_{n-j+1}(B) \right)^{\frac{1}{2}} = \sqrt{\text{tr}A\text{tr}B},
\]
so that
\[
\text{tr}((1 + r)A - rB) \leq \text{tr} \left| A^{1+r}B^{-r} \right| - r \left( \text{tr}A + \text{tr}B - 2\sqrt{\text{tr}A\text{tr}B} \right).
\]
This completes the proof. \(\Box\)

**Remark 5.2.** Although the proof of Theorem 5.1 seems to be very interesting, it can be accomplished in a much simpler way, if we take into account Theorem 3.3. Namely, considering Theorem 3.3 with \(X = I_n\) and with the trace norm \(\| \cdot \|_1\), that is, \(\|A\|_1 = \sum_{i=1}^{n}s_j(A) = \text{tr}|A|\), it follows that
\[
(1 + r)\|A\|_1 - r\|B\|_1 + r(\sqrt{\|A\|_1} - \sqrt{\|B\|_1})^2 \leq \|A^{1+r}B^{-r}\|_1.
\]
Now, since \(A, B \in \mathcal{P}_n\), it follows that \(\|A\|_1 = \text{tr}A\) and \(\|B\|_1 = \text{tr}B\), that is, \((1 + r)\|A\|_1 - r\|B\|_1 = \text{tr}(1 + r)A - rB\), so we retain the inequality (5.3).

Our next intention is to obtain an analogous reverse relation for the determinant of a matrix. In [10], the authors obtained inequality
\[
\det( A^\nu B^{1-\nu}) + r_n^\nu \det(2A\nabla B - 2A^\# B) \leq \det(\nu A + (1 - \nu)B),
\]
where $0 \leq \nu \leq 1$, $r_0 = \min\{\nu, 1 - \nu\}$, and $A, B$ are positive definite matrices. The corresponding complementary result can also be established by virtue of Lemma 3.2.

**Theorem 5.3.** Let $r \geq 0$ and let $A, B \in \mathcal{P}_n$ be such that $A \geq \frac{r}{r+1} B$. Then the following inequality holds:

$$
\det \left( (1 + r)A - rB \right) \leq \det \left( A^{r+1}B^{-r} \right) - r^n \det \left( 2ANB - 2A^\#B \right). 
$$

(5.4)

**Proof.** The starting point is Lemma 3.2 with $a = s_j \left( B^{-\frac{1}{2}}AB^{-\frac{1}{2}} \right)$ and $b = 1$, i.e. the inequality

$$
s_j^{r+1} \left( B^{-\frac{1}{2}}AB^{-\frac{1}{2}} \right) \geq (1 + r)s_j \left( B^{-\frac{1}{2}}AB^{-\frac{1}{2}} \right) - r \left( s_j^{\frac{1}{2}} \left( B^{-\frac{1}{2}}AB^{-\frac{1}{2}} \right) - 1 \right)^2.
$$

Furthermore, since $A \geq \frac{r}{r+1} B$, it follows that $B^{-\frac{1}{2}}AB^{-\frac{1}{2}} \geq \frac{r}{r+1} I_n$, which means that $s_j \left( B^{-\frac{1}{2}}AB^{-\frac{1}{2}} \right) \geq \frac{r}{r+1}$. Consequently, we have that

$$(1 + r)s_j \left( B^{-\frac{1}{2}}AB^{-\frac{1}{2}} \right) - r \geq 0.$$ 

Hence, by virtue of the above two relations and the well-known determinant properties, we have

$$
\det \left( B^{-\frac{1}{2}}AB^{-\frac{1}{2}} \right)^{r+1} = \prod_{j=1}^{n} s_j^{r+1} \left( B^{-\frac{1}{2}}AB^{-\frac{1}{2}} \right) 
\geq \prod_{j=1}^{n} \left[ (1 + r)s_j \left( B^{-\frac{1}{2}}AB^{-\frac{1}{2}} \right) - r + r \left( s_j^{\frac{1}{2}} \left( B^{-\frac{1}{2}}AB^{-\frac{1}{2}} \right) - 1 \right)^2 \right] 
\geq \prod_{j=1}^{n} \left[ (1 + r)s_j \left( B^{-\frac{1}{2}}AB^{-\frac{1}{2}} \right) - r \right] 
+ r^n \prod_{j=1}^{n} \left[ \left( s_j^{\frac{1}{2}} \left( B^{-\frac{1}{2}}AB^{-\frac{1}{2}} \right) - 1 \right)^2 \right] 
= \det \left( (1 + r)B^{-\frac{1}{2}}AB^{-\frac{1}{2}} - rI_n \right) 
+ r^n \det \left( \left( B^{-\frac{1}{2}}AB^{-\frac{1}{2}} \right)^{\frac{1}{2}} - I_n \right)^2.
$$

Finally, multiplying both sides of the obtained inequality by $\det(B^{\frac{1}{2}})$ and utilizing the well-known Binet-Cauchy theorem, we obtain (5.4), as claimed. \(\square\)
6. Reverses of the Young Inequality dealing with singular values

Let \( x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in \mathbb{R}^n \) be such that \( 0 \leq x_1 \leq \cdots \leq x_n \) and \( 0 \leq y_1 \leq \cdots \leq y_n \). Then \( x \) is said to be log majorized by \( y \), and denoted by \( x \prec_{\text{log}} y \), if

\[
\prod_{j=1}^k x_j \leq \prod_{j=1}^k y_j \quad (1 \leq k < n) \quad \text{and} \quad \prod_{j=1}^n x_j = \prod_{j=1}^n y_j.
\]

For \( X \in \mathcal{M}_n \) and \( k = 1, \ldots, n \), the \( k \)-th compound of \( X \) is defined as the \( \binom{n}{k} \times \binom{n}{k} \) complex matrix \( C_k(X) \), whose entries are defined by \( C_k(X)_{r,s} = \det X[(r_1, r_2, \cdots, r_k)](s_1, s_2, \cdots, s_k) \), where \( (r_1, r_2, \cdots, r_k), (s_1, s_2, \cdots, s_k) \in P_{k,n} = \{(x_1, \cdots, x_k) \mid 1 \leq x_1 < \cdots < x_k \leq n\} \) are arranged in a lexicographical order and \( (r_1, r_2, \cdots, r_k) \) and \( (s_1, s_2, \cdots, s_k) \) are the \( r \)-th and \( s \)-th element in \( P_{k,n} \), respectively. \( X[r,s] \) is the \( k \times k \) matrix that contains the elements in the intersection of rows \( (r_1, r_2, \cdots, r_k) \in P_{k,n} \) and columns \( (s_1, s_2, \cdots, s_k) \in P_{k,n} \) (for more details, see [17]). For example, if \( n = 3 \) and \( k = 2 \), then \( (1, 2), (1, 3) \) and \( (2, 3) \) are the first, the second and the third element of \( P_{k,n} \), respectively. So,

\[
C_2(X) = \begin{pmatrix}
\det X[1,2][1,2] & \det X[1,2][1,3] & \det X[1,2][2,3] \\
\det X[1,3][1,2] & \det X[1,3][1,3] & \det X[1,3][2,3] \\
\det X[2,3][1,2] & \det X[2,3][1,3] & \det X[2,3][2,3]
\end{pmatrix}.
\]

In a general case, for \( A, B \in \mathcal{M}_n \), we have

\[
C_k(AB) = C_k(A)C_k(B) \quad \text{and} \quad s_1(C_k(A)) = \prod_{j=1}^k s_j(A) \ (1 \leq k \leq n). \tag{6.1}
\]

Finally we use the corresponding ideas from [19] to present our last result.

**Theorem 6.1.** Suppose that \( A, B \in \mathcal{P}_n \) and \( X \in \mathcal{M}_n \). If \( r \geq 0 \), then

\[
\text{(i)} \quad s(A^{1+r}XB^{1+r}) \succ_{\text{log}} s^{1+r}(AXB)s^{-r}(X),
\]

\[
\text{(ii)} \quad s(A^{1+r}XB^{-r}) \succ_{\text{log}} s^{1+r}(AX)s^{-r}(XB).
\]

**Proof.** (i) Let \( C_k(X) \in \mathbb{C}^{\binom{n}{k} \times \binom{n}{k}} \) denote the \( k \)-th component of \( X, 1 \leq k \leq n \). Then, we have

\[
\prod_{i=1}^k s_i(A^{1+r}XB^{1+r}) = s_1(C_k(A^{1+r}XB^{1+r})) \quad \text{by (6.1)}
\]

\[
= s_1(C_k(A)^{1+r}C_k(X)C_k(B)^{1+r}) \quad \text{by (6.1)}
\]

\[
\geq s_1^{-r}(C_k(X))s_1^{1+r}(C_k(AXB)) \quad \text{(by inequality (3.1))}
\]

\[
= \prod_{i=1}^k s_i^{-r}(X)\prod_{i=1}^k s_i^{1+r}(AXB).
\]

Moreover, if \( k = n \), we have

\[
\prod_{i=1}^n s_i(A^{1+r}XB^{1+r}) = |\det(A^{1+r}XB^{1+r})| = (\det A)^{1+r}|\det X|(\det B)^{1+r}
\]
and
\[
\prod_{i=1}^{n} s_i(X)^{-r} \prod_{i=1}^{n} s_i(AXB)^{1+r} = |\det X^{-r}| |\det(AXB)^{1+r}| = (\det A)^{1+r} |\det X| (\det B)^{1+r}.
\]

(ii) The second conclusion can be accomplished by a similar argument as in (i) and by utilizing the inequality (3.2). □

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