Σ-SEMI-COMPACT RINGS AND MODULES

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Abstract. In this paper several characterizations of semi-compact modules are given. Among other results, we study rings whose semi-compact modules are injective. We introduce the property Σ-semi-compact for modules and we characterize the modules satisfying this property. In particular, we show that a ring $R$ is left Σ-semi-compact if and only if $R$ satisfies the ascending (resp. descending) chain condition on the left (resp. right) annihilators. Moreover, we prove that every flat left $R$-module is semi-compact if and only if $R$ is left Σ-semi-compact. We also show that a ring $R$ is left Noetherian if and only if every pure projective left $R$-module is semi-compact. Finally, we consider rings whose flat modules are finitely (singly) projective. For any commutative arithmetical ring $R$ with quotient ring $Q$, we prove that every flat $R$-module is semi-compact if and only if every flat $R$-module is finitely (singly) projective if and only if $Q$ is pure semisimple. A similar result is obtained for reduced commutative rings $R$ with the space $\text{Min } R$ compact. We also prove that every $(\aleph_0, 1)$-flat left $R$-module is singly projective if $R$ is left Σ-semi-compact, and the converse holds if $R^N$ is an $(\aleph_0, 1)$-flat left $R$-module.

Introduction

We shall assume that all rings are associative with identity and all modules are unitary. Let $R$ be a ring and $M$ be a left $R$-module. For any left ideal $I$ of $R$, set $M[I] = \{m \in M \mid Im = 0\}$. It is a subgroup of $M$. As in [15] and [16], $M$ is said to be semi-compact if every finitely solvable set of congruences $x \equiv x_\alpha \pmod{M[I_\alpha]}$ ($\alpha \in \Lambda$, $x_\alpha \in M$ and $I_\alpha$ is a left ideal of $R$ for each $\alpha \in \Lambda$) has a simultaneous solution in $M$. Also, we say that $M$ is Σ-semi-compact if all direct sums of copies of $M$ are semi-compact. Semi-compactness was introduced by Matlis in [15] (and also in [16]) for modules over commutative rings.

In the present article, we shall study semi-compact and Σ-semi-compact modules over arbitrary rings (not necessarily commutative). In Section 1 we consider some basic properties of semi-compact modules, their relationship to other concepts such as injectivity and pure-injectivity, and some rings characterized by semi-compactness. Several characterizations of semi-compact modules are given in Proposition 1.2 and Theorem 1.1. For instance, it is shown that a left $R$-module $M$ is semi-compact if and only if every finite solvable system of equations of the form $r_jx = a_j \in M$, $r_j \in R$ has a global solution in $M$, if and only if, every pure extension of $M$ is cyclically pure. So, it follows that the semi-compact modules are exactly the singly pure-injective modules introduced by Azumaya in [2]. It is easy to see that every semi-compact left $R$-module is injective if and only if $R$ is a von-Neumann regular ring (see Theorem 1.3). In Section 2, we introduce and study...
Σ-semi-compact modules. It is shown that a left $R$-module $M$ is Σ-semi-compact if and only if $M^{(\aleph_0)}$ is semi-compact if and only if $M$ satisfies the descending chain condition (d.c.c.) on the subgroups of $M$ which are annihilators of (finitely generated) left ideals of $R$ (see Proposition 2.1 and Theorem 2.1). It is also shown that every pure projective left $R$-module is semi-compact if and only if $R$ is left Noetherian (see Theorem 2.2). In Theorem 2.3 we show that a ring $R$ is left Σ-semi-compact if and only if every flat left $R$-module is semi-compact. If $R$ is a commutative ring, we prove that each flat $R$-module is semi-compact if the quotient ring $Q$ is Noetherian (see Proposition 2.2).

In Section 3, for a ring $R$ we compare the following conditions:

- each flat left $R$-module is semi-compact;
- each flat left $R$-module is finitely projective;
- each flat left $R$-module is singly projective.

There are many examples of rings for which these conditions are equivalent. For instance, if $R$ is a commutative ring and $Q$ its quotient ring, and if $R$ is either arithmetical or reduced with Min $R$ compact, it is proven that these conditions are satisfied if and only if $Q$ is pure-semisimple. But, if $R$ is a self left FP-injective ring, the two first conditions are not equivalent: the first holds if and only if $R$ is quasi-Frobenius, and the second if and only if $R$ is left perfect. We give an example of a self FP-injective commutative perfect ring which is not quasi-Frobenius.

It is also shown that each $(\aleph_0, 1)$-flat left $R$-module is singly projective if $R$ is Σ-semi-compact as left module (see Proposition 3.1) and the converse holds if $R^{\aleph_0}$ is a $(\aleph_0, 1)$-flat left $R$-module.

In Section 4 we investigate the rings $R$ for which each semi-compact left $R$-module is pure-injective. We get only some partial results. However, if $R$ is a reduced commutative ring, then $R$ satisfies this condition if and only if $R$ is von Neumann regular.

1. SEMI-COMPACT MODULES

**Definition 1.1.** Let $M$ be a left $R$-module. For any subset (subgroup) $X$ of $M$, $\perp X = \{ r \in R \mid rX = 0 \}$ is a left ideal of $R$. The set of such left ideals will be denoted by $A_l(R, M)$. For any subset (left ideal) $X$ of $R$, $M[X] = \{ m \in M \mid XM = 0 \}$ is a subgroup of $M$. The set of such subgroups will be denoted by $A_r(R, M)$.

Since $X \mapsto \perp X$ is an order antiisomorphism between $A_r(R, M)$ and $A_l(R, M)$, then one satisfies the a.c.c. if and only if the other satisfies the d.c.c. In the special case $M = R$, the elements of $A_r(R, R)$ (respectively $A_l(R, R)$) are called right (respectively left) annihilators of $R$: in this case we denote $X^\perp$ the right annihilator of $X$. As in [15] and [16] $M$ is said to be semi-compact if every finitely solvable set of congruences $x = x_\alpha \ (\text{mod } M[I_\alpha])$ (where $\alpha \in \Lambda$, $x_\alpha \in M$ and $I_\alpha$ is a left ideal of $R$ for each $\alpha \in \Lambda$) has a simultaneous solution in $M$.

**Lemma 1.1.** Let $M$ be a left $R$-module and $I$ and $J$ left ideals of $R$. Then

$$(M[I])[J] = (M[J])[I] = M[I + J] = M[I] \cap M[J].$$

**Proposition 1.1.** Let $R$ be a ring and $M$ a semi-compact left $R$-module. Then $M[I]$ and $M/M[I]$ are semi-compact for each two-sided ideal $I$ of $R$.

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1. Each commutative reduced pure-semisimple ring is semisimple.
Proof. Let \( x \equiv x_\alpha \pmod{(M[I])[J_\alpha]} = M[I + J_\alpha] \) be a finitely solvable set of congruences (where \( \alpha \in \Lambda, x_\alpha \in M[I] \) and \( J_\alpha \) is a left ideal of \( R \) for each \( \alpha \in \Lambda \)). Since \( M \) is semi-compact, there exists \( m \) in \( M \) such \( m - x_\alpha \in M[I + J_\alpha] \subseteq M[I] \) for each \( \alpha \in \Lambda \). Since \( x_\alpha \in M[I] \), \( m \) is. Therefore, \( M[I] \) is semi-compact. Now, let \( x \equiv x_\alpha + M[I] \pmod{(M/M[I])[J_\alpha]} \) be a finitely solvable set of congruences where \( \alpha \in \Lambda, x_\alpha \in M \) and \( J_\alpha \) is an ideal of \( R \) for each \( \alpha \in \Lambda \). Obviously, \( x \equiv x_\alpha \pmod{(M[I]J_\alpha)} \) is a finitely solvable set of congruences and so has a global solution \( m \in M \). Thus \( J_\alpha(m - x_\alpha) \in M[I] \). Therefore, \( m + M[I] - x_\alpha + M[I] \in (M/M[I])[J_\alpha] \) for each \( \alpha \in \Lambda \). □

Let \( M \) be a left \( R \)-module. Then the system of equations \( \sum_{i \in I} r_{ij}x_i = m_j \in M, j \in J \) is called compatible if, for any choice of \( s_j \in R, j \in J \), where only a finite number of \( s_j \) are nonzero, the relations \( \sum_{j \in J} s_jr_{ij} = 0 \) for each \( i \in I \) imply that \( \sum_{j \in J} s_jm_j = 0 \) (see [8] Chapter 18) for more details about systems of equations). Throughout this paper, all systems of equations are assumed to be compatible. The following proposition is crucial in our investigation.

**Proposition 1.2.** Let \( M \) be a left \( R \)-module. Then the following statements are equivalent:

1. \( M \) is semi-compact;
2. Every finitely solvable set of congruences \( x \equiv x_\alpha \pmod{M[I_\alpha]} \) (where \( \alpha \in \Lambda, x_\alpha \in M \) and \( I_\alpha \) is a finitely generated left ideal of \( R \) for each \( \alpha \in \Lambda \)) has a simultaneous solution in \( M \);
3. Every finitely solvable system of equations of the form \( r_jx = m_j \in M, j \in J, r_j \in R \), is solvable in \( M \).
4. For each left ideal \( I \) of \( R \), every homomorphism \( h : I \to M \), for which the restriction to any finitely generated left subideal \( I_0 \) of \( I \) can be extended to \( R \), extends itself to a homomorphism \( R \to M \).

Proof. (1)⇒(2) is clear.

(2)⇒(3). Let \( r_jx = m_j \in M, j \in J, r_j \in R \), be a finitely solvable system of equations. For each finite subset \( J_\alpha \) of \( J \), let \( I_\alpha \) be the left ideal generated by \( \{r_j \mid j \in J_\alpha \} \). Let \( x_\alpha \in M \) be a solution of the finite system of equations \( r_jx = m_j \in M, j \in J_\alpha \). Obviously, the set of congruences \( x \equiv x_\alpha \pmod{M[I_\alpha]} \) is finitely solvable and by hypothesis has a global solution \( z \in M \). Therefore, \( z \) is a solution of the above system of equations.

(3)⇒(1). Let \( x \equiv x_\alpha \pmod{M[I_\alpha]} \) be a finitely solvable system of congruences in \( M \), where \( x_\alpha \in M \) and \( I_\alpha \) is a left ideal of \( R \) for each \( \alpha \in \Lambda \). Consider the following system of equations.

\[ r_{\alpha j}x = m_{\alpha j}, r_{\alpha j} \in I_\alpha, m_{\alpha j} = r_{\alpha j}x_\alpha \in M. \] 

Since \( x \equiv x_\alpha \pmod{M[I_\alpha]} \) is finitely solvable, so is the system of equations (*)\), and by hypothesis it has a global solution \( z \in M \). So, \( z \) is a solution of \( x \equiv x_\alpha \pmod{M[I_\alpha]} \).

(3)⇒(4) is clear. □

Let \( S \) be a class of finitely presented left \( R \)-modules. We say that an exact sequence of left \( R \)-modules is \( S \)-pure if each module of \( S \) is projective relatively to it. Each left \( R \)-module which is injective relatively to each \( S \)-pure exact sequence is said to be \( S \)-pure injective. When \( S \) contains all finitely presented left \( R \)-modules we say respectively "pure exact sequence" and "pure-injective module". And, when
S contains all finitely presented cyclic left $R$-modules we say respectively ”$(\aleph_0, 1)$-pure exact sequence” and ”$(\aleph_0, 1)$-pure-injective module”. For any class $S$ of finitely presented left $R$-modules, each pure-exact sequence of left modules is $S$-pure exact, whence each $S$-pure injective left $R$-module is pure injective. So, we prove the following results by using Proposition 1.2(3) and [9, Theorem 1.35(d)] for the first result.

**Example 1.1.** Let $R$ be a ring. Then:

(i) For each class $S$ of finitely presented left $R$-modules, every $S$-pure injective module is semi-compact.

(ii) If $R$ is a domain, then every torsion-free (and so every flat) left $R$-module is semi-compact.

**Remark 1.1.** Let $R$ be a ring. Recall that a left $R$-module $M$ is semi-injective (or $f$-injective) if for each finitely generated left ideal $I$, every $R$-homomorphism $f : I \rightarrow M$ can be extended to an $R$-homomorphism from $R$ into $M$ (see [16]). By Proposition 1.2, a left $R$-module $M$ is injective if and only if $M$ is semi-injective and semi-compact. Since every direct sum of semi-injective left $R$-modules is semi-injective, Bass’s theorem implies that every direct sum of semi-compact left $R$-modules is semi-compact if and only if $R$ is left Noetherian.

Recall that a submodule $A$ of $B$ is pure (resp. $(\aleph_0, 1)$-pure) if and only if the exact sequence $0 \rightarrow A \hookrightarrow B \xrightarrow{\cdot a_j} B/A \rightarrow 0$ is pure (resp. $(\aleph_0, 1)$-pure). In this case we say that $B$ is a pure extension (resp. $(\aleph_0, 1)$-pure extension) of $A$. It is known that $A$ is a $(\aleph_0, 1)$-pure submodule of $B$ if and only if for each $n \in \mathbb{N}$, any system of equations $r_ixa_j = a_j \in A$ ($r_j \in R$, $1 \leq j \leq n$) is solvable in $A$ whenever it is solvable in $B$ (see [21], [9], [11]). As in [2], a left $R$-module $B$ is called a single extension of $M$ if the factor module $B/M$ is cyclic, i.e., there is a cyclic submodule $A$ of $B$ such that $B = A + M$. We say that $B$ is a single pure extension (resp. single $(\aleph_0, 1)$-pure extension) of $M$, if $B$ is a pure (resp. $(\aleph_0, 1)$-pure) extension and a single extension of $M$.

**Lemma 1.2.** Let $M$ be a left $R$-module and

$$r_jx = m_j, j \in J, r_j \in R, m_j \in M$$

be a finitely solvable system of equations in $M$. Then there exists a singly pure extension $B$ of $M$ such that the system of equations (*) has a solution $b \in B$.

**Proof.** Set $B = (M \oplus F)/S$, where $F$ is the free module with the basis $\{x\}$ and $S$ is the submodule of $M \oplus F$ generated by $\{(m_j, -r_jx) \mid j \in J\}$. Obviously,

$$S = \{(\sum_{k=1}^n z_km_k, -\sum_{k=1}^n z_kr_kx) \mid n \in \mathbb{N}, z_k \in R\}.$$  

Clearly, the map $\alpha : M \rightarrow B$ defined by $\alpha(m) = (m, 0) + S$ ($m \in M$) is an $R$-homomorphism. We claim that $\alpha$ is a monomorphism. To see this, let $\alpha(m) = (m, 0) + S = 0$ for some $m \in M$. Then $m = \sum_{k=1}^n z_km_k + \sum_{k=1}^n z_kr_kx = 0$ for some $z_1, \ldots, z_n \in R$. Since the system of equations (*) is compatible, we conclude that $m = 0$ and so $\alpha$ is a monomorphism. One can easily see that $b = (0, x) + S \in B$
is a solution of the system of equations $r_j X = (m_j, 0) + S \in \alpha(M)$. We claim that \(\alpha(M)\) is a pure submodule of \(M\). Let

\[
\sum_{k=1}^{n} c_{lk} y_k = (m'_l, 0) + S \in \alpha(M), \quad 1 \leq l \leq w, c_{lk} \in R, m'_l \in M \quad (**)
\]

be a system of equations with the solution \(\{(a_k, t_k x) + S\}_{k=1}^{n} \subseteq B\), where \(a_k \in M\)

and \(t_k \in R\). Then \(\sum_{k=1}^{n} c_{lk} a_k - m'_l - \sum_{k=1}^{n} c_{lk} t_k x \in S\) for each \(1 \leq l \leq w\). Therefore, for each \(l\) \((1 \leq l \leq w)\), there exist \(n_l \in \mathbb{N}, z_{l1}, \ldots, z_{ln_l} \in R\) such that

\[
\sum_{k=1}^{n} c_{lk} a_k - m'_l = \sum_{s=1}^{n_l} z_{ls} m_{ls} \quad (1)
\]

and

\[
\sum_{k=1}^{n} c_{lk} t_k = - \sum_{s=1}^{n_l} z_{ls} r_{ls} \quad (2).
\]

Since the system of equations (*) is finitely solvable, there exists \(m' \in M\) such that \(r_{ls} m'_l = m_{ls}\) for some finite subset \(\{l_s\} \subseteq J\). In view of (1) and (2) we conclude that

\[
\sum_{k=1}^{n} c_{lk} t_k m'_l = - \sum_{s=1}^{n_l} z_{ls} r_{ls} m'_l = - \sum_{s=1}^{n_l} z_{ls} m_{ls} = -(\sum_{k=1}^{n} c_{lk} a_k - m'_l).
\]

Therefore, \(\sum_{k=1}^{n} c_{lk} (a_k - t_k m'_l) = m'_l\). Thus \(\{(a_k - t_k m'_l, 0) + S\}_{k=1}^{n} \subseteq \alpha(M)\) is a solution of the system (**). It means that \(\alpha(M)\) is a pure submodule of \(B\). \[\square\]

As in [2], a left \(R\)-module \(M\) is singly split in \(B\) if, for every submodule \(A\) of \(B\) which is a single extension of \(M\), \(M\) is a direct summand of \(A\), and \(M\) is said to be singly pure-injective if \(M\) is singly split in any pure extension of itself. Recall that an exact sequence \(\varepsilon : 0 \to A \to B \to C \to 0\) of left \(R\)-modules is cyclically pure if every cyclic left \(R\)-module has the projective property relative to \(\varepsilon\) (see [19]). Proposition [L2] leads us to obtain the following characterizations of semi-compact left \(R\)-modules.

**Theorem 1.1.** Let \(M\) be a left \(R\)-module. Then the following statements are equivalent:

1. \(M\) is semi-compact.
2. \(M\) has the injective property relative to every \((R_0, 1)\)-pure exact sequence \(0 \to A \to B \to C \to 0\) where \(C\) is a cyclic left \(R\)-module.
3. \(M\) is a direct summand of every single \((R_0, 1)\)-pure extension.
4. Every pure extension of \(M\) is cyclically pure.
5. \(M\) is a direct summand of every module \(B\) if \(B\) contains \(M\) as an \((R_0, 1)\)-pure submodule and if \(B/M\) is a direct summand of a direct sum of cyclic left \(R\)-modules.
6. \(M\) has the injective property relative to every \((R_0, 1)\)-pure exact sequence \(0 \to A \to B \to C \to 0\) where \(C\) is a direct summand of a direct sum of cyclic left \(R\)-modules.
7. \(M\) is singly pure-injective.
Proof. (1)⇒(2). Since C is cyclic then B = A + Rb for some b ∈ B. Let f : A → M be a homomorphism of left R-modules. Let \{r_j\}_{j \in J} ⊆ R be the set of all elements of R such that r_j b ∈ A. Since A is an \((\mathbb{N}_0, 1)\)-pure submodule of B, the system of equations r_j x = f(r_j b) ∈ M is finitely solvable and by hypothesis it has a solution m ∈ M. Obviously, φ : B → M defined by φ(a + sb) = f(a) + sm for each a ∈ A and s ∈ R is an extension of f.

(2)⇒(3) is clear.

(3)⇒(4). Let B be a pure extension of M and A ⊆ B be a singly extension of M. Then M is an \((\mathbb{N}_0, 1)\)-pure submodule of A. By hypothesis, M is a summand of A and so M is singly split in B. Therefore, \[3\] Theorem 3 implies that the sequence 0 → M → B → B/M → 0 is cyclically pure.

(4)⇒(5). It is well-known that every direct summand of a direct sum of cyclic modules has the projective property relative to each cyclically pure exact sequence.

(5)⇒(6). Let f : A → M be a homomorphism and u : A → B the inclusion map. We consider the following pushout diagram:

\[
\begin{array}{ccc}
A & \xrightarrow{u} & B \\
\downarrow{f} & & \downarrow{g} \\
M & \xrightarrow{v} & T
\end{array}
\]

By \[22\] 33.4(2)] v is an \((\mathbb{N}_0, 1)\)-pure monomorphism. Since coker v ≅ C then v is a split monomorphism. So, f extends to a homomorphism from B into M.

(6)⇒(3) is clear.

(3)⇒(1). Let r_j x = m_j, j ∈ J, r_j ∈ R, m_j ∈ M, be a finitely solvable system of equations in M. By Lemma \[12\] there exists a single pure extension B of M such that the system of equations \((*)\) contains a solution b ∈ B. By hypothesis, M is a direct summand of B. Thus there exists a submodule A of B such that B = M ⊕ A. Therefore, there exist m ∈ M and a ∈ A such that b = m + a. Since r_j b = m_j for each j ∈ J, we conclude that r_j m - r_j a = m_j. Thus r_j a = r_j m - m_j ∈ A \cap M = 0.

(4)⇒(7) by \[2\] Theorem 10].

Remark 1.2. Recall that a submodule A of a left R-module B is cyclically pure if and only if the exact sequence 0 → A ⊆ B \xrightarrow{g} B/A → 0 is cyclically pure. By \[11\] Proposition 1.2, A is a cyclically pure submodule of B if and only if for each index set J, any system of equations r_j x = a_j ∈ A \(r_j \in R, j \in J\) is solvable in A whenever it is solvable in B.

Corollary 1.1. Let R be a ring and \(\{M_i\}_{i \in I}\) be a set of semi-compact left R-modules. Then \(\oplus_{i \in I} M_i\) is semi-compact if and only if \(\oplus_{i \in I} M_i\) is a cyclically pure submodule of \(\prod_{i \in I} M_i\).

Proof. (⇒) is clear by Theorem 1.2

(⇐). It is sufficient to show that every finitely solvable system of equations of the form r_j x = a_j ∈ \(\oplus_{i \in I} M_i\) \(r_j \in R, j \in J\) is solvable in A. Since \(\prod_{i \in I} M_i\) is semi-compact, there exists an element \(b \in \prod_{i \in I} M_i\) such that r_j b = a_j for each j ∈ J. Since \(\oplus_{i \in I} M_i\) is a cyclically pure submodule of \(\prod_{i \in I} M_i\), by Remark 1.2 there is an element \(a \in \oplus_{i \in I} M_i\) such that r_j a = a_j for each j ∈ J.

From Theorem 1.1 and Remark 1.2 we deduce the following corollary.

Corollary 1.2. Let R be a ring. Then the following statements are equivalent:
(1) $R$ is left Noetherian.
(2) Every left $R$-module is semi-compact.
(3) Every direct sum of semi-compact left $R$-modules is semi-compact.

Proof. (1)⇒(2). Since every cyclic left $R$-module is finitely presented and so pure projective, so, by Theorem 1.1 (3), every $R$-module is semi-compact.
(2)⇒(3) is obvious and (3)⇒(1) holds by Remark 1.1.

A ring $R$ is called left pure-semisimple if each left $R$-module is pure-injective. Recall that a ring $R$ is left perfect if each flat left $R$-module is projective.

Theorem 1.1 implies that there exists a semi-compact left $R$-module which is not pure injective. It is easy to see that a left Noetherian ring $R$ is pure-semisimple if and only if every semi-compact left $R$-module is pure injective. In the next theorem we show that a domain $R$ is a division ring if and only if the class of semi-compact $R$-modules and the class of pure injective $R$-modules coincide.

Theorem 1.2. Let $R$ be a domain (not necessarily commutative). Then $R$ is a division ring if and only if every semi-compact $R$-module is pure injective.

Proof. If $R$ is a division ring, it is obvious that each semi-compact module is pure injective. Conversely, let $M$ be a flat left $R$-module. Then there exists a free left $R$-module $F$ and submodule $K$ of $F$ such that $\varepsilon : 0 \to K \to F \to M \to 0$ is pure exact. Since $R$ is domain, $F$ and $K$ are torsion-free and so semi-compact. By hypothesis, $K$ is pure injective and $\varepsilon$ splits. Therefore, $M$ is projective and so $R$ is left perfect. This implies that $R$ is a division ring.

Recall that a ring $R$ is von-Neumann regular if for each element $a$ of $R$ there exists $b \in R$ such that $a = aba$. It is equivalent to the fact that every finitely generated left ideal is a summand of $R$. Also, it is known that a ring $R$ is von-Neumann regular if and only if every pure injective left $R$-module is injective. In the next theorem we have another characterization of von-Neumann regular rings.

Theorem 1.3. Let $R$ be a ring. Then the following statements are equivalent:

(1) $R$ is von-Neumann regular.
(2) Every pure injective left $R$-module is injective.
(3) Every semi-compact left $R$-module is injective.

Proof. (3)⇒(2) is obvious.
(2)⇒(1). Each left module is semi-injective because it is a pure submodule of a pure injective module which is injective. So, $R$ is von Neumann regular since each left module is semi-injective.
(1)⇒(3). In this case each left $R$-module is semi-injective. So, by [10, Lemma 5.5] a left $R$-module is injective if and only if it is semi-compact.

Rings whose flat modules are semi-compact

Definition 2.1. Let $R$ be a ring and $M$ a left $R$-module. We say that $M$ is $\Sigma$-semi-compact if all direct sums of copies of $M$ are semi-compact. By Example 1.1(2) every torsion-free module over an integral domain is $\Sigma$-semi-compact.
Faith in [10], proved that an injective left $R$-module $M$ is $\Sigma$-injective if and only if $R$ satisfies the a.c.c. on the left ideals in $\mathcal{A}_l(R, M)$ (equivalently, $M$ satisfies the d.c.c. on the subgroups in $\mathcal{A}_r(R, M)$). We need the following proposition of [10] to characterize $\Sigma$-semi-compact left $R$-modules.

**Proposition 2.1.** [10] Proposition 1 Let $M$ be a left $R$-module. Then $\mathcal{A}_l(R, M)$ satisfies the a.c.c., equivalently, $M$ satisfies the d.c.c. on the subgroups in $\mathcal{A}_r(R, M)$, if and only if for each left ideal $I$ of $R$, there exists a finitely generated subideal $I_1$ such that $M[I] = M[I_1]$.

**Theorem 2.1.** Let $R$ be a ring and $M$ a left $R$-module. Then the following statements are equivalent:

1. $M^{(n)}$ is semi-compact;
2. $R$ satisfies the a.c.c. on the left ideals in $\mathcal{A}_l(R, M)$ ($M$ satisfies the d.c.c. on the subgroups in $\mathcal{A}_r(R, M)$);
3. $M$ satisfies the d.c.c. on the subgroups of $M$ which are annihilators of finitely generated left ideals of $R$;
4. $M$ is $\Sigma$-semi-compact.

**Proof.** (1)$\Rightarrow$(2). Let $M[I_1] \supset M[I_2] \supset M[I_3] \ldots$ be a strictly descending chain, where $I_n$ is a left ideal for each integer $n \geq 1$, and let $y_n \in M[I_n] \setminus M[I_{n+1}]$. Obviously, $x \equiv x_i \pmod {M^{(n)}[I_i]} = M^{(n)}[I_i]$ is finitely solvable where $x_i = (y_1, \ldots, y_i, 0, 0, \ldots)$. So it has a simultaneous solution in $M^{(n)}$ since $M^{(n)}$ is semi-compact. But each $m = (s_1, s_2, \ldots, s_i, 0, 0, \ldots) \in M^{(n)}$ cannot be a solution of the above system, since $a - x_{i+2} = (s_1 - y_1, \ldots, s_i - y_i, y_{i+1}, y_{i+2}) \notin M^{(n+2)}$, a contradiction.

(2)$\Rightarrow$(3) is clear.

(3)$\Rightarrow$(4). First we show that $M$ is semi-compact. Let $x \equiv x_i \pmod {M[I_\alpha]}$, where $\alpha \in \Lambda$, be a finitely solvable system. By Proposition 2.1 we may assume that $I_\alpha$ is finitely generated for each $\alpha \in \Lambda$. There exists $V = M[I_{\alpha_1}] \cap \ldots \cap M[I_{\alpha_n}]$ which is minimal among the set of all finite intersections of the $M[I_\alpha]$ for $\alpha \in \Lambda$. Obviously, $V \subseteq M[I_\alpha]$ for each $\alpha \in \Lambda$. Let $y_i - x_{i+1} \in M[I_{\alpha_1}]$ for each $1 \leq i \leq n$. Let $y_i$ be a solution of $x \equiv x_i \pmod {M[I_{\alpha_i}]}$ and $x \equiv x_i \pmod {M[I_{\alpha_i}]}$ where $1 \leq i \leq n$. It is easy to check that $y_1 - y_i \subseteq V \subseteq M[I_{\beta_i}]$. Thus $y_i - x_{i+2} - y_{i+2} - x_{i+2} \in M[I_{\beta_i}]$. Therefore, $y_i$ is a simultaneous solution in $M$. So, $M$ is semi-compact. Let $J$ be an index set, $I$ a left ideal of $R$ and $h : I \to M^{(j)}$ a homomorphism such that, for any finitely generated left ideal $I_0$ of $I$, there exists $m_0 \in M^{(j)}$ such that $h(r_0) = r_0m_0$ for each $r_0 \in I_0$. Let $I_1 = Rs_1 + \ldots + Rs_n$ be the finitely generated subideal of $I$ given by Proposition 2.1 such that $M[I] = M[I_1]$. Therefore, there exists $m_1 = (m_{1j})_{j \in J} \in M^{(j)}$ such that $h(r_1) = r_1m_1$ for each $r_1 \in I_1$. Since $M^{(j)}$ is semi-compact, there exists an element $m' = (m'_{ij})_{j \in J} \in M'$ such that $h(r) = rm'$ for each $r \in I$. Thus for each $j \in J$, $m'_{ij} - m_{1j} \in M[I_1] = M[I]$. We conclude that $h(r) = rm_1$ for each $r \in I$.

(4)$\Rightarrow$(1) is clear. □

**Corollary 2.1.** Every submodule of a $\Sigma$-semi-compact module is $\Sigma$-semi-compact.

**Corollary 2.2.** Let $R$ be a ring and $I$ a two-sided ideal of $R$. If $M$ is a $\Sigma$-semi-compact left $R$-module, then so is $M/M[I]$.

**Proof.** It is clear by Theorem 2.1 and Corollary 2.1. □
Corollary 2.3. Let \( R \) be a ring. Then the following statements are equivalent:

1. \( R \) satisfies the a.c.c. on the left annulets.
2. \( R \) satisfies the d.c.c. on the right annulets.
3. \( R \) is \( \Sigma \)-semi-compact as left \( R \)-module.

Corollary 2.4. Let \( M \) be a semi-injective left \( R \)-module. Then \( M \) is \( \Sigma \)-injective if and only if it is \( \Sigma \)-semi-compact.

The following theorem is a generalization of Corollary 1.2.

Theorem 2.2. Let \( R \) be a ring. Then every pure projective left \( R \)-module is semi-compact if and only if \( R \) is left Noetherian.

Proof. Let \( C \) be a cyclic left \( R \)-module. By [22, Theorem 33.5], there exists a pure exact sequence \( \varepsilon : 0 \to K \to P \to C \to 0 \) where \( P \) is a pure projective left module. By Corollary 2.1, \( K \) is \( \Sigma \)-semi-compact. Therefore, \( \varepsilon \) splits and so \( C \) is a direct summand of \( P \). Since \( C \) is cyclic, it is a direct summand of a finite direct sum of finitely presented \( R \)-modules. It follows that \( C \) is finitely presented. Hence \( R \) is left Noetherian. The converse is clear by Corollary 1.2. \( \square \)

Theorem 2.3. Let \( R \) be a ring. Then the following statements are equivalent:

1. Every flat left \( R \)-module is semi-compact.
2. \( R \) is \( \Sigma \)-semi-compact as left \( R \)-module.
3. \( R \) satisfies the a.c.c. on the left annulets (\( R \) satisfies the d.c.c. on the right annulets).

Proof. (1)\( \Rightarrow \) (2) is clear.
(2)\( \Leftrightarrow \) (3) by Corollary 2.3.
(2)\( \Rightarrow \) (1). Let \( M \) be a flat left \( R \)-module. Then \( M = F/K \) where \( F \) is a free \( R \)-module and \( K \) is a pure submodule of \( F \). Suppose that \( M \) is not \( \Sigma \)-semi-compact. Then by Theorem 2.1 there exists a strict descending chain \( M[I_1] \supset M[I_2] \supset \ldots \). By Proposition 2.1, we may assume that for each \( i \in \mathbb{N} \), \( I_i \) is finitely generated. For each \( i \in \mathbb{N} \), let \( a_i + K \in M[I_i] \setminus M[I_{i+1}] \) and let \( r_{i,1}, \ldots, r_{i,m_i} \) be generators of \( I_i \) where \( m_i \in \mathbb{N} \). Therefore, \( r_{i,j}a_i \in K \) for \( j = 1, \ldots, m_i \). Since \( K \) is pure submodule of \( F \), there exist \( k_i \in K \) such that \( I_i(a_i - k_i) = 0 \) for each \( i \in \mathbb{N} \). One can easily see that \( I_{i+1}(a_i - k_i) \neq 0 \) for each \( i \in \mathbb{N} \). Thus we get the strict descending chain \( F[I_1] \supset F[I_2] \supset \ldots \supset F[I_n] \supset \ldots \). This contradicts that \( F \) is \( \Sigma \)-semi-compact. \( \square \)

In [4], Björk proved that a left semi-injective ring \( R \) is quasi-Frobenius if and only if it satisfies the a.c.c. on the left annulets. Therefore we have the following evident corollary by Theorem 2.3.

Corollary 2.5. Let \( R \) be a self left semi-injective ring. Then each flat left \( R \)-module is semi-compact if and only if \( R \) is a quasi-Frobenius ring.

Proposition 2.2. Let \( R \) be a ring. Assume that \( R \) is a subring of a left Noetherian ring \( S \). Then each flat left \( R \)-module is semi-compact.

Proof. As left \( S \)-module, \( S \) is \( \Sigma \)-semi-compact. If \( A \) is a left ideal of \( R \) and \( A' = SA \) then it is easy to check that \( S[A] = S[A'] \). So \( S \) is a left \( \Sigma \)-semi-compact \( R \)-module. It follows that so is \( R \) by Corollary 2.1. \( \square \)

From these last two propositions we deduce the following corollary.
Corollary 2.6. Let $R$ be a commutative ring and $Q$ its quotient ring. Assume that $Q$ is semi-injective. Then each flat $R$-module is semi-compact if and only if $Q$ is quasi-Frobenius.

3. Finite projectivity and $\Sigma$-semi-compacitiveness

As in [2], a left $R$-module $M$ is called finitely projective (respectively singly projective) if any homomorphism from a finitely generated (respectively cyclic) left $R$-module into $M$ factors through a free left $R$-module. If $m, n$ are positive integers, a right $R$-module is said to be $(m, n)$-flat if, for each $n$-generated left submodule $K$ of $R^m$, the homomorphism $M \otimes_R K \rightarrow M \otimes_R R^m$ deduced of the inclusion map is injective. We say that $M$ is $\langle N_0, 1 \rangle$-flat if it is $(m, 1)$-flat for each integer $m > 0$. In [13] Theorem 5] Shenglin proved that every flat left module is singly projective if, for each descending chain of finitely generated right ideals $I_1 \supseteq I_2 \supseteq I_3 \supseteq \ldots$, the ascending chain $\overset{1}{I}_1 \subseteq \overset{2}{I}_2 \subseteq \overset{3}{I}_3 \subseteq \ldots$ terminates. Therefore by Corollary 2.3(1), we deduce that every flat left module is singly projective if $R$ is $\Sigma$-semi-compact as left module. The following proposition generalizes this result.

Proposition 3.1. Let $R$ be a ring which is $\Sigma$-semi-compact as left $R$-module. Then each $\langle N_0, 1 \rangle$-flat left $R$-module is singly projective.

Proof. Let $M$ be a $\langle N_0, 1 \rangle$-flat left $R$-module and let $\pi : F \rightarrow M$ be an epimorphism with $F$ a free left $R$-module. Let $C$ be a left cyclic module generated by $c$, let $f : C \rightarrow M$ be a homomorphism and let $A = 1 + \langle c \rangle$. Since $F$ is $\Sigma$-semi-compact, so by Proposition 2.1 there exists a finitely generated left subideal $B$ of $A$ such that $F[A] = F[B]$. Let $g : R/B \rightarrow M$ be the homomorphism defined by $g(1 + B) = f(c)$. Since $M$ is $\langle N_0, 1 \rangle$-flat, so, by [14] Proposition 4.1 and Theorem 1.1, there exists a homomorphism $h : R/B \rightarrow F$ such that $g = \pi \circ h$. Let $x = h(1 + B)$. Then $x \in F[B] = F[A]$. So, the homomorphism $\phi : C \rightarrow F$ defined by $\phi(c) = x$ satisfies $f = \phi \circ \pi$. \hfill $\square$

Corollary 3.1. Let $R$ be a ring such that $R^{(n)}$ is $\langle N_0, 1 \rangle$-flat as left module. Then the following conditions are equivalent:

1. $R$ is $\Sigma$-semi-compact as left module.
2. Each $\langle N_0, 1 \rangle$-flat left $R$-module is singly projective.

Proof. (1)⇒(2) follows from Proposition 3.1.

(2)⇒(1). From $R^{(n)}$ $\langle N_0, 1 \rangle$-flat and $R^{(n)}$ pure submodule of $R^{(n)}$ we deduce that $R^{(n)}/R^{(n)}$ is $\langle N_0, 1 \rangle$-flat. We conclude by [14] Proposition 1] and Corollary 2.3 \hfill $\square$

An $R$-module is called uniserial if if the set of its submodules is totally ordered by inclusion. A commutative ring $R$ is a chain ring (or a valuation ring) if it is a uniserial $R$-module.

The following example satisfies the equivalent conditions of the previous corollary but the hypothesis does not hold.

Example 3.1. Let $R$ be a chain ring whose quotient ring $Q$ is Artinian and not reduced. We assume that Spec $R$ is finite and $R \neq Q$. By [2] Corollary 36 each ideal is countably generated. Each flat $R$-module is semi-compact by Proposition 2.2 and singly projective by [6] Theorem 7]. Since $R \neq Q$, $R$ is not FP-injective as $R$-module, so, by [5] Theorem 37(6)] $R^{(n)}$ is not $\langle N_0, 1 \rangle$-flat.\footnote{Over a chain ring each $\langle N_0, 1 \rangle$-flat module is flat.}
If \( R \) is a commutative ring, then we consider on \( \text{Spec} \, R \) the equivalence relation \( \mathcal{R} \) defined by \( LR \mathcal{L}' \) if there exists a finite sequence of prime ideals \( (L_k)_{1 \leq k \leq n} \) such that \( L = L_1, \ L' = L_n \) and \( \forall k, \ 1 \leq k \leq (n-1), \) either \( L_k \subseteq L_{k+1} \) or \( L_k \supseteq L_{k+1}. \) We denote by \( \text{pSpec} \, R \) the quotient space of \( \text{Spec} \, R \) modulo \( \mathcal{R} \) and by \( \lambda_R : \text{Spec} \, R \to \text{pSpec} \, R \) the natural map. The quasi-compactness of \( \text{Spec} \, R \) implies the one of \( \text{pSpec} \, R, \) but generally \( \text{pSpec} \, R \) is not \( T_1: \) see [13, Propositions 6.2 and 6.3].

**Lemma 3.1** ([12, Lemma 2.5]). Let \( R \) be a commutative ring and let \( C \) a closed subset of \( \text{Spec} \, R \). Then \( C \) is the inverse image of a closed subset of \( \text{pSpec} \, R \) by \( \lambda_R \) if and only if \( C = V(A) \) where \( A \) is a pure ideal. Moreover, in this case, \( A = \cap_{P \in C} 0_P \) (where \( 0_P \) is the kernel of the canonical homomorphism \( R \to R_P \)).

A commutative ring \( R \) is called arithmetical if \( R_L \) is a chain ring for each maximal ideal \( L. \)

**Theorem 3.1.** Let \( R \) be a commutative arithmetical ring and \( Q \) its quotient ring. Then the following conditions are equivalent:

1. \( Q \) is pure-semisimple;
2. each flat \( R \)-module is semi-compact;
3. each flat \( R \)-module is finitely projective;
4. each flat \( R \)-module is singly projective.

**Proof.** (1) \( \Rightarrow \) (2) is a consequence of Proposition 2.2. (1) \( \Rightarrow \) (3) follows from [6, Theorem 7], (2) \( \Rightarrow \) (4) holds by Proposition 3.1 and (3) \( \Rightarrow \) (4) is obvious.

(4) \( \Rightarrow \) (1). First we show that \( \text{Min} \, R \) is finite. Since each prime ideal contains a unique minimal prime ideal, then each point of \( \text{pSpec} \, R \) is of the form \( V(L) \) where \( L \) is a minimal prime ideal. By Lemma 3.1 there exists a pure ideal \( A \) such that \( V(L) = V(A). \) Since \( R/A \) is flat, it is projective, whence \( A = Re, \) where \( e \) is an idempotent of \( R. \) Hence each single subset of \( \text{pSpec} \, R \) is open. From the quasi-compactness of \( \text{pSpec} \, R, \) we deduce that \( \text{Min} \, R \) is finite. Let \( P \) be a maximal ideal of \( R. \) By using [6, Proposition 6] we get that \( R_P \) satisfies (3). By [6, Theorem 33] the quotient ring \( Q(R_P) \) of \( R_P \) is artinian. It follows that \( Q(R_P) = R_L \) where \( L \) is the minimal prime ideal contained in \( P. \) Let \( s \) be an element of \( R \) which does not belong to any minimal prime ideal. If \( a \in R \) satisfies \( sa = 0 \) then it is easy to check that \( \frac{a}{1} = 0 \) in \( R_P \) for each maximal ideal \( P. \) So, \( a = 0 \) and \( s \) is regular. We deduce that \( Q \cong \prod_{L \in \text{Min} \, R} R_L. \) Hence \( Q \) is pure-semisimple.

The following proposition is a slight generalization of [6, Proposition 6] and the proof is similar.

**Proposition 3.2.** Let \( \phi : R \to S \) be a right flat epimorphism of rings. Then:

1. For each singly (respectively finitely) projective left \( R \)-module \( M, \ S \otimes_R M \)
   is singly (respectively finitely) projective over \( S; \)
2. Let \( M \) be a singly (respectively finitely) projective left \( S \)-module. If \( \phi \) is injective then \( M \) is singly (respectively finitely) projective over \( R. \)

**Theorem 3.2.** Let \( R \) be a ring. Assume that \( R \) has a right flat epimorphic extension \( S \) which is von Neumann regular. Then the following conditions are equivalent:

1. \( S \) is semisimple;
2. each flat left \( R \)-module is semi-compact;
(3) each flat left $R$-module is finitely projective;
(4) each flat left $R$-module is singly projective.

Proof. (1) $\Rightarrow$ (3) is an immediate consequence of [18, Corollary 7] and (3) $\Rightarrow$ (4) is obvious.
(1) $\Rightarrow$ (2) is an immediate consequence of Proposition 2.2, and (2) $\Rightarrow$ (4) holds by Proposition 3.1.
(4) $\Rightarrow$ (1). First we show that each left $S$-module $M$ is singly projective. Every left $S$-module $M$ is flat over $S$ and $R$. So, $M$ is singly projective over $R$. It follows that $M \cong S \otimes_R M$ is singly projective over $S$ by Proposition 3.2(1). Now let $A$ be a left ideal of $S$. Since $S/A$ is singly projective, it is projective. So, $S/A$ is finitely presented over $S$. Hence $S$ is semisimple. □

Corollary 3.2. Let $R$ be a commutative reduced ring and $Q$ its quotient ring. Assume that the space $\text{Min} R$ of minimal prime ideals of $R$ is compact in its Zariski topology. Then the following conditions are equivalent:

1. $Q$ is semisimple;
2. each flat $R$-module is semi-compact;
3. each flat $R$-module is finitely projective;
4. each flat $R$-module is singly projective.

Proof. We use the assumption that $\text{Min} R$ is compact. By [20, Theorem 3.14.1] and [17, Proposition 1] $Q$ is a subring of a von Neumann regular ring $S$ such that the inclusion map $R \rightarrow S$ is a flat epimorphism ($S$ is the maximal flat epimorphic extension of $R$). If either $Q$ or $S$ is semisimple, then $Q = S$. □

Given a ring $R$, a left $R$-module $M$ and $x \in M$, the content ideal $c(x)$ of $x$ in $M$, is the intersection of all right ideals $A$ for which $x \in AM$. We say that $M$ is a content module if $x \in c(x)M$, $\forall x \in M$. We say that $M$ is FP-injective if $\text{Ext}_R^1(F, M) = 0$ for each finitely presented left $R$-module $F$. It is easy to see that each FP-injective module is semi-injective, but we do not know if the converse holds, except for some classes of rings.

Proposition 3.3. Let $R$ be a self left FP-injective ring. Then each flat left $R$-module is finitely projective if and only if $R$ is left perfect.

Proof. Let $M$ be a flat left $R$-module. Since it is finitely projective, so it is FP-injective and a content module by [6, Proposition 3(2)]. We conclude that $R$ is left perfect by [6, Theorem 2]. □

Corollary 3.3. Let $R$ be a ring. Assume that $R$ has a right flat epimorphic extension $S$ which is self left FP-injective. Then each flat left $R$-module is finitely projective if and only if $S$ is left perfect.

Proof. If $S$ is left perfect we conclude by [18, Corollary 7]. Conversely, first we show that each flat left $S$-module is finitely projective. The proof is similar to that of (4) $\Rightarrow$ (1) of Theorem [17,2] and then we use the previous proposition. □

By [24, Corollary 16] (a result due to Jensen) each semiprimary ring with square of the Jacobson radical zero is $\Sigma$-pure-injective (hence $\Sigma$-semi-compact) on either side. Since these rings are left and right perfect then each flat module is projective.

Remark 3.1. Consider the following two conditions:
(1) each flat left $R$-module is semi-compact;
(2) each flat left $R$-module is finitely projective.

Let us observe that there are many examples of rings satisfying the two conditions. We shall see that they are not equivalent. Does the first condition imply the second?

Let $R$ be a ring which is a FP-injective left module. Then $R$ is left perfect if and only if $R$ satisfies the second condition by Proposition 3.3. By Corollary 2.5 $R$ satisfies the first condition if and only if $R$ is quasi-Frobenius. It remains to give an example of a left perfect ring which is self left FP-injective and which is not quasi-Frobenius.

**Proposition 3.4.** Let $R$ be a local commutative ring of maximal ideal $P$ such that $P^2$ is the only minimal non-zero ideal of $R$. Then:

1. $R$ is perfect and self FP-injective;
2. $R$ is quasi-Frobenius if and only if $P$ is finitely generated if and only if $R^N$ is $(1,1)$-flat.

**Proof.** (1). Since $R$ is local and $P$ nilpotent ($P^3 = 0$), $R$ is perfect. For each $R$-module $M$ we put $M^* = \text{Hom}_R(M, R)$. By [12, Theorem 2.3] to show that $R$ is self FP-injective it is enough to prove that the evaluation map $\phi_M : M \to M^{**}$ is injective for each finitely presented $R$-module $M$. We consider a finitely presented module $M$. We have the following exact sequence $0 \to K \xrightarrow{u} F \xrightarrow{\pi} M \to 0$ where $F$ is a free $R$-module of finite rank, $K$ a finitely generated submodule of $F$ and $u$ the inclusion map. We may assume that $K \subseteq PF$. We have the following commutative diagram with exact horizontal sequences:

$$
\begin{array}{c}
K \xrightarrow{u} F \xrightarrow{\pi} M \\
\phi_K \downarrow \quad \phi_F \downarrow \quad \phi_M \downarrow \\
K^{**} \xrightarrow{u^{**}} F^{**} \xrightarrow{\pi^{**}} M^{**}
\end{array}
$$

Since $\phi_F$ is an isomorphism and $u$ a monomorphism then $\phi_K$ is injective. On the other hand, if $E = E(R/P)$ then $E \cong E(R)$. If $N$ is a module of finite length, denoted by $\ell(N)$, then $\ell(N) = \ell(\text{Hom}_R(N, E))$ (this can be proved by induction on $\ell(N)$). We have $PK \subseteq P^2F$. So, $PK$ is a semisimple module of finite length. Since so is $K/PK$, it follows that $K$ is of finite length too. From $K^* \subseteq \text{Hom}_R(K, E)$ we deduce that $\ell(K^*) \leq \ell(K)$. In the same way we get that $\ell(K^{**}) \leq \ell(K)$. Whence $\ell(K^{**}) = \ell(K)$, $\phi_K$ is an isomorphism and $u^{**}$ is injective. From snake Lemma we deduce that $\phi_M$ is a monomorphism.

(2). The first equivalence is obvious and the second is a consequence of Corollaries 2.5 and 3.1 and [11, Theorem 4.11].

**Example 3.2.** Let $K$ be a field, $\Lambda$ an index set and $\alpha \in \Lambda$. Let $R$ be the factor ring of the polynomial ring $K[X_\lambda \mid \lambda \in \Lambda]$ modulo the ideal generated by

$$
\{ X_\lambda^2 - X_\mu^2 \mid \lambda \in \Lambda \} \cup \{ X_\lambda X_\mu \mid \lambda, \mu \in \Lambda, \lambda \neq \mu \}.
$$

Then $R$ satisfies the assumptions of Proposition 3.4. Consequently, if $\Lambda$ is not a finite set then $R$ verifies the second condition of Remark 3.1 but not the first.
4. Semi-compactness and pure-injectivity

By Example 1.1(1) each pure-injective module is semi-compact. By Theorem 1.3 the converse holds over every von Neumann regular ring. From Corollary 1.2 we deduce the following:

**Corollary 4.1.** Let $R$ be a left Noetherian ring. Then each semi-compact left $R$-module is pure-injective if and only if $R$ is left pure-semisimple.

Now, we investigate rings for which each semi-compact left module is pure-injective. We shall give a partial answer.

Recall that a left $R$-module $M$ is cotorsion if $\text{Ext}^1_R(F, M) = 0$ for each flat left $R$-module $F$. It is easy to check that every pure-injective module is cotorsion, and, by [23, Proposition 3.3.1] a ring $R$ is left perfect if and only if each left $R$-module is cotorsion. So, if $R$ is left Artinian, then each left $R$-module is semi-compact and cotorsion, but each left module is pure-injective if and only $R$ is left pure-semisimple.

The following theorem completes Theorem 1.3.

**Theorem 4.1.** For any ring $R$ the following conditions are equivalent:

1. $R$ is von Neumann regular;
2. each cotorsion left (right) $R$-module is injective.

*Proof. (1) $\Rightarrow$ (2).* Since each left $R$-module is flat, so each cotorsion left module is injective.

(2) $\Rightarrow$ (1). Let $M$ be a left $R$-module. We shall prove that $M$ is flat. By [3, Theorem 3] and [23, Lemma 2.1.1], there exists an exact sequence

$$0 \to K \to F \to M \to 0,$$

where $F$ is flat and $K$ cotorsion. Since $K$ is injective, the sequence splits and we deduce that $M$ is flat. $\square$

**Proposition 4.1.** Let $R$ be a commutative ring. Then:

1. if each semi-compact $R$-module is cotorsion (respectively pure-injective) then, for each multiplicative subset $S$ of $R$, each semi-compact $S^{-1}R$-module is cotorsion (respectively pure-injective).
2. if each semi-compact $R$-module is pure-injective then each prime ideal of $R$ is maximal.

*Proof. (1).* Let $M$ be a semi-compact $S^{-1}R$-module. Then $M$ is semi-compact over $R$ too. It follows that $M$ is cotorsion (respectively pure-injective) as $R$-module. We easily check that it satisfies this property as $S^{-1}R$-module.

(2). We apply Theorem 1.2 to $R/L$ where $L$ is a prime ideal. $\square$

**Theorem 4.2.** Let $R$ be a commutative reduced ring. Then the following conditions are equivalent:

1. $R$ is von Neumann regular;
2. each semi-compact $R$-module is pure-injective.

*Proof. It easy to prove that (1) $\Rightarrow$ (2).

(2) $\Rightarrow$ (1). By Proposition 4.1(2), each prime ideal is maximal. It follows that $R_P$ is a field for each maximal ideal $P$ of $R$. Hence $R$ is von Neumann regular. $\square$
Proposition 4.2. Let $R$ be a commutative local ring of maximal ideal $P$. Assume that $P \neq P^2$. Then $R$ is pure-semisimple if each semi-compact $R$-module is pure-injective.

Proof. By Proposition 4.1 $P$ is the only prime ideal of $R$. So, it suffices to show that $\dim_{R/P} P/P^2 = 1$. By way of contradiction suppose that $\dim_{R/P} P/P^2 > 1$. After replacing $R$ by a suitable factor, we may assume that $1 < \dim_{R/P} P/P^2 < \infty$. So, $R$ is Artinian but not pure-semisimple. Then each $R$-module is semi-compact (and cotorsion) but there exists a module which is not pure-injective, whence a contradiction. □

Proposition 4.3. Let $R$ be ring and $J$ its Jacobson radical. Assume that $R/J$ is von Neumann regular and $J$ nilpotent. Then each semi-compact left (right) $R$-module is cotorsion.

Proof. If $J = 0$, then we apply Theorem 1.3. Let $n$ be the smallest integer satisfying $J^n = 0$. We proceed by induction on $n$. Let $M$ be a semi-compact module. We consider the following exact sequence:

$$0 \to M[J^p] \to M[J^{p+1}] \to M[J^{p+1}]/M[J^p] \to 0.$$ 

We assume that the theorem holds if $n = p$. For any proper two-sided ideal $A$, a left $R/A$-module is cotorsion as $R$-module if so is as $R/A$-module (see [23, Proposition 3.3.3]). On the other hand $M[J^p] = M[J^{p+1}][J^p]$. By using Proposition 1.1 and the fact that the class of cortorsion modules is closed by extension, we get that $M[J^{p+1}]$ is cortorsion, because $M[J^p]$ and $M[J^{p+1}]/M[J^p]$ are left modules over $R/J^p$ and they are semi-compact by Proposition 1.1. □

Corollary 4.2. Let $R$ be a commutative ring and $N$ its nilradical. Then:

1. $R_P$ is perfect for each maximal ideal $P$ if $N$ is $T$-nilpotent and if every semi-compact $R$-module is cotorsion.
2. $R_P$ is pure-semisimple for each maximal ideal $P$ if $N$ is $T$-nilpotent and if every semi-compact $R$-module is pure-injective.
3. Every semi-compact $R$-module is cotorsion if $N$ is nilpotent and if each prime ideal is maximal.

Proof. (1). By Proposition 1.1(2) each prime ideal is maximal. So, for each maximal ideal $P$ the Jacobson of $R_P$ is $T$-nilpotent, whence $R_P$ is perfect. (2). As in (1) we prove that $R_P$ is perfect for each maximal ideal $P$. So, $PR_P \neq (PR_P)^2$. We use Proposition 4.2 to conclude. (3). It is easy to see that $N$ is the Jacobson radical of $R$ and that $R/N$ is von Neumann regular. We conclude by Proposition 1.3. □

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