A GENERAL PETTIS INTEGRAL AND APPLICATIONS TO TRANSITION SEMIGROUPS

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Abstract. Motivated by applications to transition semigroups, we introduce the notion of a norming dual pair and study a Pettis-type integral on such pairs. In particular, we establish a sufficient condition for integrability. We also introduce and study a class of semigroups on such dual pairs which are an abstract version of transition semigroups. Using our results, we prove conditions ensuring that a semigroup consisting of kernel operators is Laplace transformable such that the Laplace transform consists of kernel operators again.

1. Introduction and Summary

The connection between Markov processes and semigroups of linear operators is well known. We recall that associated to a Markov process $X_t$, taking values in a state space $(E, \Sigma)$, there are, in fact, two semigroups which are connected to each other by some sort of duality. The first semigroup $T = (T(t))_{t \geq 0}$ acts on the space $B_b(E)$ of all bounded, measurable functions on $E$ and is used to compute conditional expectations by means of

$$E(f(X_{t+s})|X_s) = (T(t)f)(X_s) \text{ a.e. \quad \forall t, s \geq 0}$$

for all $f \in B_b(E)$. The second semigroup $T = (T'(t))_{t \geq 0}$ acts on the space $\mathcal{M}(E)$ of all bounded complex measures on $E$ and gives the distribution of the random elements $X_t$, i.e.

$$X_s \sim \mu \text{ implies } X_{t+s} \sim T'(t)\mu \quad \forall t, s \geq 0.$$ 

The semigroup on $B_b(E)$, or also its restriction to some invariant subspace, is called the transition semigroup of the Markov process. However, these semigroups are in general not strongly continuous so that the rich theory of strongly continuous semigroups cannot be applied. In fact, the situation is even worse, as the orbits of these semigroups might not be strongly measurable or even weakly measurable, cf. [11].

In order to treat transition semigroups on spaces of bounded continuous functions, several approaches have been proposed in the literature. We mention the theory of weakly continuous semigroups of Cerrai [7], the theory of bi-continuous semigroups of Kühnemund [20], see also [10] [21] for applications...

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in the context of transition semigroups, and the theory of \( \pi \)-semigroups by Priola [24]. However, these approaches make additional assumptions (such as continuity and equicontinuity assumptions) which ensure that a Riemann integral can be used to compute the Laplace transform of the semigroup.

It is not well understood under which conditions in the general situation a Pettis-type integral can be used to compute the Laplace transform of transition semigroups. A first step was taken by Feller [11], who weakened the measurability requirement on a function \( f \), taking values in a Banach space \( X \), from weak measurability to measurability of \( \langle f, y \rangle \) for all \( y \) in a subspace \( Y \) of \( X^* \). However, he uses a Dunford integral rather than a Pettis integral, thus allowing integrals of \( X \)-valued functions to take values in larger spaces than \( X \). Jefferies [16, 17] studied semigroups on locally convex spaces under minimal integrability assumptions but requiring that the Laplace integral yields again values in \( X \). However, there seems to be no convenient criterion to check these assumptions without resorting to stronger notions of integrability. In [18], where this theory is used to study transition semigroups of certain diffusion processes in \( \mathbb{R}^d \), the fact that these assumptions are met was deduced via a monotone class argument from the fact that the transition semigroups leave the space \( C_0(\mathbb{R}^d) \) invariant and that the restriction of the semigroups to that space are strongly continuous and hence have locally Bochner integrable orbits. If we want to consider transition semigroups which do not leave \( C_0 \) invariant or, else, if \( E \) is too big so that \( C_0(E) = \{0\} \) (consider the case where \( E \) is an infinite dimensional Hilbert or Banach space), this strategy does not work. Even though in the monotone class argument we can easily replace \( C_0(E) \) by a Banach space \( X \subset B_b(E) \) from a quite large class of Banach spaces, see Lemma 6.1 we are still left with the problem of whether or not the Laplace transform of \( T(\cdot)f \) for \( f \in X \) can be computed in a reasonable way within \( B_b(E) \) or, even better, within \( X \) itself.

In this paper we study the latter problem in an abstract setting. More precisely, given a Banach space \( X \), we fix a subspace \( Y \) of \( X^* \) which is norm-closed in \( X^* \) and norming for \( X \). In this situation we call \((X, Y)\) a norming dual pair. In applications, we may choose \( X = B_b(E) \) and \( Y = \mathcal{M}(E) \). However, if \( E \) is a completely regular Hausdorff space, it is also possible to replace \( B_b(E) \) with \( C_b(E) \), provided we also replace \( \mathcal{M}(E) \) with \( \mathcal{M}_0(E) \), the space of all bounded Radon measures on \( E \), see Section 2. In Section 4, we introduce a Pettis-type integral on such pairs. Here, in contrast to the usual Pettis integral on the Banach space \( X \), see [22] or Section II.3 in [9], we replace the norm dual \( X^* \) with \( Y \). Naturally, properties of the dual pair \((X, Y)\) become important for the theory of integration in this case. However, in comparison with the Pettis integral on locally convex spaces, see [11], we may use the norm topology on either of the spaces \( X \) and \( Y \). Combining techniques from functional analysis on Banach spaces and the theory of locally convex spaces, we prove a sufficient condition for Pettis integrability, see Theorem 4.3. Our main assumption in that theorem is the existence of a complete, consistent topology on \( X \). On the norming dual pair \((C_0(E), \mathcal{M}_0(E))\), the strict topology, see [5, 6, 25], is a
consistent topology which is complete in many cases. Hence, we may apply our result in this situation.

In Section 5, we study a class of semigroups on norming dual pairs. Though our assumptions are somewhat stronger than those of [16], they are quite natural in the context of transition semigroups. Our main assumption is that the (norm-)adjoint of a semigroup $T$ on $X$ leaves the space $Y$ invariant. Thus, also in this abstract setting, we obtain a second semigroup $T' := T^*|_Y$ on $Y$. We do not only address the question under which conditions such a semigroup is Laplace transformable (we will say that the semigroup is integrable), but also give conditions under which a semigroup is uniquely determined by its Laplace transform. Very weak continuity assumptions on the semigroup will yield this uniqueness, but we will also obtain uniqueness without continuity assumptions.

In the last section we return to our initial question. We will work on the norming dual pair $(\mathcal{C}_b(E), \mathcal{M}_0(E))$, where $E$ is a complete metric space. We prove that every semigroup which is $\sigma(\mathcal{C}_b(E), \mathcal{M}_0(E))$-continuous at 0, is integrable. If $E$ is additionally separable, i.e. $E$ is a Polish space, then this assumption can be weakened.

It is also possible to interpret the continuity assumptions in [7, 20, 24] from our “dual” point of view. This yields interesting new results for semigroups with such continuity properties. We will discuss these questions in a forthcoming paper.

**Notation.** If $(E, \Sigma)$ is a measurable space, then $B_0(E)$ denotes the space of all bounded, measurable functions $f : E \to \mathbb{C}$. This is a Banach space with respect to the supremum norm. By $\mathcal{M}(E)$ we denote the space of all complex measures on $(E, \Sigma)$. The total variation of a measure $\mu$ is defined by

$$|\mu|(A) = \sup_{\mathcal{Z}} \sum_{B \in \mathcal{Z}} |\mu(B)|,$$

where the supremum is taken over all partitions $\mathcal{Z}$ of $A$ into finitely many, disjoint, measurable sets. Endowed with the total variation norm $\|\mu\| := |\mu|(E)$, the space $\mathcal{M}(E)$ is a Banach space. Now suppose that $E$ is a topological space. Then $C_0(E)$ denotes the Banach space of all bounded, continuous functions $f : E \to \mathbb{C}$. The Borel $\sigma$-algebra of $E$ is denoted by $\mathcal{B}(E)$. If we speak about measures or measurable functions on a topological space, this is always to be understood with respect to the Borel $\sigma$-algebra. A positive measure $\mu \in \mathcal{M}(E)$ is a Radon measure if $\mu(A) = \sup\{\mu(K) : K \subset A, K \text{ compact}\}$ for all $A \in \mathcal{B}(E)$. An arbitrary $\mu \in \mathcal{M}(E)$ is called a Radon measure if $|\mu|$ is a Radon measure. We denote the space of all Radon measures on $E$ by $\mathcal{M}_0(E)$. This is a closed subspace of $\mathcal{M}(E)$. By $1_A$ we denote the characteristic function of a set $A$. The Dirac measure in a point $x$ is denoted by $\delta_x$. If $E$ is a metric space, then $B(x, r)$ denotes the open ball of radius $r$ centered at $x$ and $\overline{B}(x, r)$ denotes the closure of that ball. If $X$ is a Banach space, then $X^*$ denotes the norm dual of $X$ and $\langle \cdot, \cdot \rangle_*$ denotes the canonical duality between $X$ and $X^*$, i.e. $\langle x, x^* \rangle_* = x^*(x)$.
2. Norming Dual Pairs

**Definition 2.1.** Let $X$ and $Y$ be nontrivial Banach spaces and $\langle \cdot, \cdot \rangle$ be a duality pairing between $X$ and $Y$. Then $(X, Y, \langle \cdot, \cdot \rangle)$ is called a norming dual pair, if

$$\|x\|_X = \sup\{ |\langle x, y \rangle| : y \in Y, \|y\|_Y \leq 1 \}$$

and

$$\|y\|_Y = \sup\{ |\langle x, y \rangle| : x \in X, \|x\|_X \leq 1 \} .$$

We will write $(X, Y)$ for a norming dual pair if the duality pairing is understood. Note that if $(X, Y)$ is a norming dual pair, then so is $(Y, X)$.

As we have done already in the introduction, we will frequently consider $Y$ as a closed subspace of $X^*$, by putting $\langle x, y \rangle_y = \langle x, y \rangle$. With this interpretation, $(X, Y)$ is a norming dual pair if and only if $Y$ is a closed subspace of $X^*$ which is norming in the sense of [3]. For $Y \subset X^*$ to be norming for $X$ it is necessary that $Y$ is weak*-dense in $X^*$. However, not every weak*-dense, closed subspace of $X$ is norming, see [8].

**Example 2.2.** If $X$ is a Banach space, then $(X, X^*)$ (and thus by symmetry also $(X^*, X)$) is a norming dual pair with the canonical duality $\langle x, x^* \rangle_y := x^*(x)$. If $X$ is reflexive, then $Y = X^*$ is the only closed subspace of $X^*$ such that $(X, Y)$ is a norming dual pair. Indeed, if $X$ is reflexive, then the weak and the weak*-topologies on $X^*$ coincide. Thus if $Y \subset X^*$ is norm-closed, it follows from the Hahn-Banach theorem that $Y$ is weakly closed and hence, by reflexivity, weak*-closed. Since $Y$ is weak*-dense, it follows that $Y = X^*$.

**Example 2.3.** Let $(E, \Sigma)$ be a measurable space. Then $(B_b(E), \mathcal{M}(E))$ is a norming dual pair with respect to the duality $\langle \cdot, \cdot \rangle$, given by

$$\langle f, \mu \rangle := \int_E f \, d\mu . \tag{2.1}$$

**Proof.** We clearly have $| \int f \, d\mu | \leq \|f\|_\infty \cdot \|\mu\|$. Considering Dirac measures, we obtain $\|f\|_\infty = \sup\{ |\langle f, \mu \rangle| : \mu \in \mathcal{M}(E), \|\mu\| \leq 1 \}$. Now let $\mu \in \mathcal{M}(E)$. If $Z$ is a partition of $E$ into finitely many, pairwise disjoint, measurable sets, then $f_Z := \sum_{A \in Z} \text{sgn}(\mu(A)) \cdot 1_A$ is a measurable function of norm at most 1. Furthermore, $\langle f_Z, \mu \rangle = \sum_{A \in Z} |\mu(A)|$. Taking the supremum over all such partitions $Z$, it follows that $(B_b(E), \mathcal{M}(E))$ is a norming dual pair. \hfill $\Box$

**Example 2.4.** Let $E$ be a completely regular Hausdorff space. Then, endowed with the duality $\langle 2.1 \rangle$, $(C_b(E), \mathcal{M}_0(E))$ is a norming dual pair.

For a complete, separable metric space $E$, the proof of this statement is implicitly contained in the proof of Theorem 2.2 of [24]. We give the proof in the general case.

**Proof.** It suffices to show that $\|\mu\| \geq \sup\{ |\langle f, \mu \rangle| : f \in C_b(E), \|f\|_\infty \leq 1 \}$. Let $\mu \in \mathcal{M}_0(E)$ be fixed and $Z = \{A_1, \ldots, A_n\}$ be a finite partition of $E$ into measurable sets. As $\mu$ is a Radon measure, given $\varepsilon > 0$, we find compact sets $C_k \subset A_k$ for $k = 1, \ldots, n$ such that $|\mu(A_k) - \mu(C_k)| \leq |\mu|(A_k \setminus C_k) \leq \frac{\varepsilon}{n}$. As $E$
Let Proposition 3.1.

\[ L \text{topology on } X \text{ linear operators on } X \]

Proof. (i) \[ \| \sigma \| \leq 1 \] and \( f|_{C_k} \equiv \text{sgn} \mu(C_k) \). We now have

\[
\left| \int f \, d\mu \right| = \left| \sum_{k=1}^{n} \int_{C_k} f \, d\mu + \sum_{k=1}^{n} \int_{A_k \setminus C_k} f \, d\mu \right|
\geq \sum_{k=1}^{n} |\mu(C_k)| - \sum_{k=1}^{n} |\mu|(A_k \setminus C_k)
\geq \sum_{k=1}^{n} |\mu(A_k)| - 2\varepsilon .
\]

As \( \varepsilon \) was arbitrary, \( \sum_{k=1}^{n} |\mu(A_k)| \leq \sup \{ |\langle f, \mu \rangle | : f \in C_b(E), \| f \| \leq 1 \} \).

Taking the supremum over all such partitions \( Z \) of \( E \), the claim follows. \( \square \)

In the following, we will be interested in locally convex topologies \( \tau \) on \( X \) which are consistent (with the duality). By this we mean that \( (X, \tau)' = Y \), i.e. every \( \tau \)-continuous linear functional \( \varphi \) on \( X \) is of the form \( \varphi(x) = \langle x, y \rangle \) for some \( y \in Y \). By the Mackey-Arens theorem, see [19, 21.4 (2)], a consistent topology is finer than the weak topology \( \sigma(X, Y) \) and coarser than the Mackey topology \( \mu(X, Y) \). To simplify notation, we will write \( \sigma \) for \( \sigma(X, Y) \) and \( \sigma' \) for the \( \sigma(Y, X) \) topology on \( Y \). We will use \( \rightarrow (\rightarrow') \) to indicate convergence with respect to \( \sigma \) (\( \sigma' \)). We will use the name of a topology as a label or prefix to topological notions to indicate that it is to be understood with respect to that topology. Without label or prefix, such notions are always understood with respect to the norm topology.

We now characterize bounded subsets in a norming dual pair.

**Proposition 2.5.** Let \( (X, Y) \) be a norming dual pair and \( \tau \) be a consistent topology on \( X \). For a subset \( M \subset X \), the following are equivalent.

(i) \( M \) is norm-bounded;
(ii) \( M \) is \( \sigma \)-bounded;
(iii) \( M \) is \( \tau \)-bounded.

*Proof.* (i) \( \Rightarrow \) (ii). As \( M \) is \( \sigma \)-bounded iff \( \sup_{x \in M} |\langle x, y \rangle| < \infty \) for all \( y \in Y \), this implication is trivial. (ii) \( \Rightarrow \) (i). If \( M \) is \( \sigma \)-bounded, the uniform boundedness principle in \( Y^* \) implies that \( \sup_{x \in M} \| x \| = \sup_{x \in M} \| x \|_{Y^*} \) is bounded. (ii) \( \Leftrightarrow \) (iii). See §20.11 (7) in [19]. \( \square \)

3. Operators on Norming Dual Pairs

We now turn to continuous, linear operators on norming dual pairs. If \( \tau \) is a locally convex topology on \( X \), we denote the the algebra of \( \tau \)-continuous linear operators on \( X \) by \( L(X, \tau) \). If \( \tau = \| \cdot \| \), we shortly write \( L(X) \) instead of \( L(X, \| \cdot \|) \). For \( T \in L(X, \| \cdot \|) \), we denote its norm-adjoint by \( T^* \). If \( T \in L(X, \sigma) \), then we denote its \( \sigma \)-adjoint by \( T' \).

**Proposition 3.1.** Let \( (X, Y) \) be a norming dual pair. Then \( T \in L(X, \sigma) \) if and only if \( T \in L(X) \) and \( T^* Y \subset Y \). In this case, \( T' = T^* |_Y \). Furthermore, \( \| T \|_{L(X)} = \| T' \|_{L(Y)} \).
Proof. If $T$ is $\sigma$-continuous, then $T$ maps $\sigma$-bounded sets into $\sigma$-bounded sets. By Proposition\textsuperscript{[2,5]} $T$ is a bounded operator on $X$, whence $T \in L(X)$. Furthermore, as $T$ is $\sigma$-continuous, it has a $\sigma$-adjoint $S$. But for $x \in X$ and $y \in Y$ we have $\langle Tx , y \rangle = \langle x , Sy \rangle = \langle x , T^*y \rangle$. It follows that $T^*y = Sy \in Y$, whence $T^*$ leaves $Y$ invariant. Conversely assume that $T \in L(X)$ and $T^*Y \subseteq Y$. Then we have $\langle Tx , y \rangle = \langle x , T^*y \rangle$ for all $x \in X$ and $y \in Y$. Since $T^*y \in Y$ by assumption, it follows that the map $x \mapsto \langle Tx , y \rangle$ is $\sigma$-continuous. As $y$ was arbitrary, $T \in L(X, \sigma)$ follows. Finally we have 
\[
\|T\|_{L(Y)} = \sup_{x} \sup_{y} \|\langle x, T^*y \rangle\| = \sup_{x} \sup_{y} \|\langle Tx , y \rangle\| = \|T\|_{L(X)},
\]
where all suprema are taken over elements of norm at most 1. \hfill $\square$

In the study of transition semigroups, one is mainly interested in positive contraction operators which are kernel operators, as they give the transition probabilities for a Markov process. Let us recall the following definition:

**Definition 3.2.** Let $(E, \Sigma)$ be a measurable space. A **bounded kernel** on $E$ is a mapping $k : E \times \Sigma \rightarrow \mathbb{C}$ such that

(i) $k(x, \cdot)$ is a complex measure on $(E, \Sigma)$ for all $x \in E$;
(ii) $k(\cdot, A)$ is measurable for all $A \in \Sigma$;
(iii) $\sup_{x \in E} |k|(x, E) < \infty$. Here, $|k|(x, \cdot)$ is the total variation of $k(x, \cdot)$.

A linear operator $T$ on a closed subspace $X$ of $B_b(E)$ is called a **kernel operator** (on $X$) if there exists a bounded kernel $k$ on $E$ such that

\[(3.1) \quad (Tf)(x) = \int_{E} f(y) k(x, dy), \quad \forall f \in X .\]

We now prove that for many spaces $X \subseteq B_b(E)$ a kernel operator is the same as a $\sigma$-continuous operator for the norming dual pair $(X, \mathcal{M})$. We need some preparation.

If $S$ is any set of functions, we denote by $\sigma(S)$ the $\sigma$-algebra generated by $S$, i.e. the smallest $\sigma$-algebra such that every $f \in S$ is measurable with respect to this $\sigma$-algebra. If $S$ is a Stonean vector lattice, i.e. a vector lattice of functions such that if $f \in S$ is real then also $\inf\{f, 1\} \in S$, then the system

\[ \mathcal{E}(S) := \{ A : \exists u_n \in S \text{ such that } 0 \leq u_n \uparrow 1_A \text{ pointwise} \} \]

generates $\sigma(S)$ and is closed under finite intersections, see [4, Lemma 39.4].

**Definition 3.3.** Let $(E, \Sigma)$ be a measurable space and $X \subseteq B_b(E)$ be a $\| \cdot \|_{\infty}$-closed subspace of $B_b(E)$ which is a Stonean vector lattice. Further, let $\mathcal{M}_0(E)$ denote either $\mathcal{M}(E)$ or (if $E$ is a completely regular Hausdorff space) $\mathcal{M}_0(E)$. Then $X$ is called a $\mathcal{M}_0(E)$-transition space for $E$ if

(i) $(X, \mathcal{M}_0(E))$ is a norming dual pair (with the canonical duality);
(ii) $\sigma(X) = \Sigma$;
(iii) There exists a sequence $f_n \in X$ such that $0 \leq f_n \uparrow 1$ pointwise.

**Example 3.4.** For every measurable space $(E, \Sigma)$, the space $B_b(E)$ is a $\mathcal{M}(E)$-transition space for $E$. If $E$ is a metric space, then $C_b(E)$ is a $\mathcal{M}_0(E)$-transition space for $E$. Indeed, $\mathcal{E}(C_b(E))$ is the collection of all open $F_\sigma$-sets. However, in a metric space, every open set is an $F_\sigma$-set, whence $\sigma(C_b(E)) = \mathcal{B}(E)$. 

The following is a generalization of Theorem 4.8.1. in [13].

**Proposition 3.5.** Let \((E, \Sigma)\) be a measurable space and \(X\) be a \(\mathcal{M}(0)(E)\)-transition space for \(E\). Denote by \(\sigma\) the \(\sigma(X, \mathcal{M}(0)(E))\)-topology. Consider the following statements:

(i) \(T \in L(X, \sigma)\);

(ii) \(T\) is a kernel operator on \(X\).

Then (i) \(\Rightarrow\) (ii). In this case, \(T\) has a unique extension to a kernel operator on \(B_b(E)\). If \(\mathcal{M}(0)(E) = \mathcal{M}(E)\), then also (ii) \(\Rightarrow\) (i).

**Proof.** (i) \(\Rightarrow\) (ii): If \(T \in L(X, \sigma)\), then \(k(x, \cdot) := T^\prime \delta_x \in \mathcal{M}(0)\). By definition, we have \((Tf)(x) = \langle Tf, \delta_x \rangle = \langle f, T^\prime \delta_x \rangle = \int f(y)k(x, dy)\). Furthermore, \(\sup_x |k(x, E) \leq \|T\| < \infty\). It remains to prove that \(k(\cdot, A)\) is measurable for any \(A \in \Sigma\). Denote the collection of sets \(A\) for which this is true by \(\mathcal{G}\). Then \(\mathcal{E}(X) \subset \mathcal{G}\). Indeed, if \(A \in \mathcal{E}(X)\), then there exists a sequence \(u_n \in X\) with \(0 \leq u_n \uparrow 1_A\). Now the dominated convergence theorem yields

\[
k(x, A) = \langle 1_A, T^\prime \delta_x \rangle = \lim \langle u_n, T^\prime \delta_x \rangle = \lim(Tu_n)(x).
\]

Hence \(k(\cdot, A)\) is measurable as the pointwise limit of measurable functions. As \(\mathcal{E}(X)\) is closed under finite intersections, \(\mathcal{G} = \Sigma\) follows if we prove that \(\mathcal{G}\) is a Dynkin system. By hypothesis, \(E \in \mathcal{E}(X) \subset \mathcal{G}\). If \(A \in \mathcal{G}\), then \(k(x, A^c) = k(x, E) - k(x, A)\) is measurable, whence \(A^c \in \mathcal{G}\). Similarly, if \(A_k\) is a sequence of pairwise disjoint sets in \(\mathcal{G}\), then \(k(x, \bigcup A_k) = \sum k(x, A_k)\) for every \(x \in E\), whence \(k(\cdot, \bigcup A_k)\) is measurable. It follows that \(\bigcup A_k \in \mathcal{G}\). Thus \(\mathcal{G}\) is a Dynkin system and hence \(k\) is a bounded kernel.

(ii) \(\Rightarrow\) (i): By hypothesis, there exists a kernel \(k\) such that (3.1) holds for all \(f \in X\). However, the right hand side of (3.1) also defines a bounded linear operator on \(B_b(E)\) (which we still denote by \(T\)). We may also define an operator \(S\) on \(\mathcal{M}(E)\) by

\[
(S\mu)(A) := \int_E k(x, A) \, d\mu(x).
\]

It is easy to see that \(S \in L(\mathcal{M}(E))\). However, for \(f = 1_A\) and arbitrary \(\mu\), we have

\[
\langle T f, \mu \rangle = \int_E k(x, A) \, d\mu = \langle f, S\mu \rangle.
\]

Using linearity and approximation, we see that the above equation holds for arbitrary \(f \in B_b(E)\). Hence, \(T^\star \mathcal{M} \subset \mathcal{M}\). This implies (i) by Proposition 3.1. \(\square\)

4. A Variant of the Pettis Integral

**Definition 4.1.** Let \((X, Y)\) be a norming dual pair, \((\Omega, \mathcal{F}, m)\) a measure space and \(f : \Omega \to X\) be a function. Then \(f\) is called \(Y\)-measurable (\(Y\)-integrable), if the function \(\omega \mapsto \langle f(\omega), y \rangle\) is measurable (integrable) for every \(y \in Y\).

We note that if \(f\) is \(Y\)-integrable, then, for any \(A \in \mathcal{F}\), the linear functional

\[
\varphi_A : y \mapsto \int_A \langle f(\omega), y \rangle \, dm
\]
Lemma 4.4. If \( f \) is \( Y \)-integrable, then the element \( \varphi_A \) of \( Y^* \) is called the \( Y \)-integral of \( f \) over \( A \). We write \( \int_A f \, dm := \varphi_A \). If \( \varphi_A \in X \subset Y^* \), for every \( A \in \mathcal{F} \), we say that \( f \) is Pettis integrable (on \((X,Y)\)).

In the rest of this paper, the term Pettis integrability is always understood as Pettis integrability on some norming dual pair \((X,Y)\). Our main result about Pettis integrability is the following:

**Theorem 4.3.** Let \((X,Y)\) be a norming dual pair, \((\Omega, \mathcal{F}, m)\) be a measure space and assume that there exists a consistent topology \( \tau \) on \( X \) such that \((X,\tau)\) is complete. Then every almost \( \tau \)-separably valued, \( Y \)-integrable function \( f : \Omega \to X \), such that \( \|f\| \) is majorized by an integrable function, is Pettis integrable. Here, \( f \) is called almost \( \tau \)-separably valued if there exists a null set \( N \) and a \( \tau \)-separable subspace \( X_0 \) of \( X \) such that \( f(\Omega \setminus N) \subset X_0 \).

We state part of the proof as a lemma, since we will also use it independently of the theorem.

**Lemma 4.4.** Let \((X,Y)\) be a norming dual pair and \( f : \Omega \to X \) be \( Y \)-measurable. Suppose that \( \|f\| \) is majorized by an integrable function \( g \). Then \( f \) is \( Y \)-integrable and the \( Y \)-integral of \( f \) over any \( A \in \mathcal{F} \) is sequentially \( \sigma' \)-continuous and satisfies the estimate

\[
(4.2) \quad \left\| \int_A f \, dm \right\|_{Y^*} \leq \int_A g(\omega) \, dm(\omega).
\]

**Proof.** As \( f \) is \( Y \)-measurable and satisfies the estimate \( |\langle f(\cdot), y \rangle| \leq g(\cdot)\|y\| \), it follows that \( f \) is \( Y \)-integrable. Integrating this inequality and taking the supremum over \( y \) with \( \|y\| \leq 1 \), the estimate \((4.2)\) follows. Now, let \((y_n)\) be a sequence in \( Y \) and assume \( y_n \rightharpoonup y \in Y \). Then \( \langle f, y_n \rangle \to \langle f, y \rangle \) pointwise on \( \Omega \).

By Proposition 2.5, \( |\|y_n\|| \) is bounded, say by \( M \). Hence \( |\langle f, y_n \rangle| \leq M \cdot g \). Thus \( \varphi_A \) is sequentially \( \sigma' \)-continuous by the dominated convergence theorem. \( \square \)

**Proof of Theorem 4.3.** We assume without loss that the set \( N \) is empty, otherwise changing \( f \) on a set of measure 0. We may furthermore assume that \((X,\tau)\) is separable. If this is not the case, we replace \( X \) by \( X_1 := \overline{X_0}^\tau \) and \( Y \) by \( Y/X_1^\perp \). Since \( \|\cdot\| \) is finer than \( \tau \), the space \( X_1 \) is norm closed and hence a Banach space. Furthermore, \((X_1,\tau|\tau_1)\) is a complete, locally convex space and, as a consequence of the Hahn-Banach theorem, we have \((X_1,\tau|\tau_1)' = Y/X_1^\perp \).

By Lemma 1.1, the \( Y \)-integral \( \varphi_A \in Y^* \) of \( f \) over \( A \) is sequentially \( \sigma' \)-continuous. In order to finish the proof, we have to show that \( \varphi_A \) is \( \sigma' \)-continuous, since then \( \varphi_A \in (Y,\sigma') = X \), i.e. \( f \) is Pettis integrable. Since \((X,\tau)\) is complete, it suffices to prove that \( \varphi_A \) is \( \sigma' \)-continuous on every \( \tau \)-equicontinuous subset of \( Y \). This follows from Grothendieck’s completeness theorem, see §21.9 (4) in \[19\]. However, since \((X,\tau)\) is separable, the \( \sigma' \)-topology is metrizable on \( \tau \)-equicontinuous subsets of \( Y \), cf §21.9 (5) in \[19\]. Hence a functional on \( Y \) is \( \sigma' \)-continuous if and only if it is sequentially \( \sigma' \)-continuous. This finishes the proof. \( \square \)
Remark 4.5. If $Y = X^*$, then the norm topology on $X$ is a complete consistent topology. In this case, the assumptions that $f$ is $X^*$-measurable and almost $\| \cdot \|$-separably valued imply that $f$ is strongly measurable. This is the Pettis measurability theorem, see Theorem 2 in II.1 of [9]. However, in Pettis measurability theorem the assumption of $X^*$-measurability can be weakened to $Y$-measurability, see Corollary 4 in II.1 of [9]. Since we only ask that the range of $f$ is almost $\tau$-separable in Theorem 4.3, we do not implicitly require that $f$ is strongly measurable.

Example 4.6. Let $E$ be a completely regular Hausdorff space and consider the norming dual pair $(C_b(E), M_0(E))$. Then, by Section 7.6 of [15], the strict topology is a consistent topology on $X$. It is complete if and only if $C(E)$, the space of all continuous functions on $E$, is complete with respect to the compact-open topology, see Section 3.6 of [15]. If $E$ is metrizable or locally compact, this is certainly the case.

The following theorem is useful in establishing Pettis integrability.

Theorem 4.7. Let $(\Omega, \mathcal{F}, m)$ be a measurable space such that $m$ is $\sigma$-finite and let $\mathcal{E}$ be a generator of $\mathcal{F}$ which is closed under finite intersections. Furthermore, let $(X, Y)$ be a norming dual pair and $f : \Omega \to X$ be a $Y$-measurable function with the following properties:

(i) There exists a measurable function $g$ such that $\|f\| \leq g$;
(ii) There exists a sequence $(\Omega_n) \subset \mathcal{F}$ with $m(\Omega_n) < \infty$ and $\bigcup \Omega_n = \Omega$ such that the function $g$ from (i) satisfies $g\mathbf{1}_{\Omega_n} \in L^1(m)$ for all $n \in \mathbb{N}$;
(iii) For every $A \in \mathcal{E} \cup \{\Omega_n : n \in \mathbb{N}\}$ there exists $x_A \in X$ such that

$$\langle x_A, y \rangle = \int_A \langle f, y \rangle \, dm.$$

Then for every measurable function $\alpha : \Omega \to \mathbb{C}$ with $|\alpha|g \in L^1(m)$ the function $\alpha f$ is Pettis integrable.

Proof. By Lemma 4.3 $\alpha f$ is $Y$-integrable on $\Omega$. It suffices to prove that its $Y$-integral over $\Omega$ belongs to $X$, as we can clearly replace $\alpha$ by $\alpha \cdot \mathbf{1}_A$ for any $A \in \mathcal{F}$. We proceed in several steps.

Step 1: Let $n \in \mathbb{N}$ be arbitrary and define

$$\mathcal{D}_n := \left\{ A \in \mathcal{F} : \exists x_A \in X \text{ s.t. } \langle x_A, y \rangle = \int_{\Omega_n \cap A} \langle f, y \rangle \, dm \quad \forall y \in Y \right\}.$$ 

We claim that $\mathcal{D}_n = \mathcal{F}$. By assumption (iii), $\mathcal{E} \subset \mathcal{D}_n$. Hence the claim follows if we prove that $\mathcal{D}_n$ is a Dynkin system. Clearly, $\Omega \in \mathcal{D}_n$ and $\mathcal{D}_n$ contains $A^c$ if it contains $A$. Now let a sequence $(A_k)$ of pairwise disjoint sets in $\mathcal{D}_n$ be given and put $A = \bigcup A_k$. It is easy to see that for every $N \in \mathbb{N}$ we have $\bigcup_{k=1}^N A_k \in \mathcal{D}_n$. Define $\beta_N := \mathbf{1}_{\bigcup_{k=1}^N A_k}$. Then $\beta_N$ converges pointwise to $\mathbf{1}_A$ and, using (4.2), we obtain

$$\left\| \int_{\Omega_n} \beta_N f \, dm - \int_{\Omega_n} \mathbf{1}_A f \, dm \right\| \leq \int_{\Omega_n} |\beta_N - \mathbf{1}_A| g \, dm \to 0.$$
as $N \to \infty$ by the dominated convergence theorem. It follows that

$$\int_{\Omega_n} 1_A f \, dm = \|1_A\| \lim_{N \to \infty} \int_{\Omega_n} \beta_N f \, dm \in X$$

as $X$ is norm-closed in $Y^*$. This shows that $A \in D_n$, whence $D_n$ is a Dynkin system. This proves the claim.

**Step 2:** Now we prove the assertion for a simple function $\alpha$. By Step 1 and linearity, the $Y$-integral of $\alpha f$ over $\Omega_n$ is an element $I_n$ of $X$. By a dominated convergence argument as in Step 1, we see that $\int_{\Omega} \alpha f \, dm$ is the norm limit of the $I_n$ and hence also an element of $X$.

**Step 3:** Now let $\alpha$ be arbitrary. Then there exists a sequence of step functions $\alpha_k$ such that $|\alpha_k| \leq |\alpha|$ and $\alpha_k \to \alpha$ pointwise. By Step 2, $\int_{\Omega} \alpha_k f \, dm \in X$ for every $k$. Appealing to the dominated convergence theorem once more, we see that $\int_{\Omega} \alpha f \, dm$ converges in norm to $\int_{\Omega} \alpha f$. This implies that $\int \alpha f \, dm \in X$. \hfill \Box

We end this section with the following

**Lemma 4.8.** Let $(\Omega, \mathcal{F}, m)$ be a measure space, $(X, Y)$ be a norming dual pair and $f : \Omega \to X$ be $Y$-integrable such that $\int_{\Omega} f \, dm \in X$. Then, for $T \in L(X, \sigma)$, the function $T f$ is $Y$-integrable and we have

$$T \int_{\Omega} f \, dm = \int_{\Omega} T f \, dm .$$

In particular, $\int_{\Omega} T f \, dm \in X$.

**Proof.** Let $y \in Y$. Then $T'y \in Y$ since $T \in L(X, \sigma)$. We obtain

$$\left\langle T \int_{\Omega} f \, dm , y \right\rangle = \left\langle \int_{\Omega} f \, dm , T'y \right\rangle = \int_{\Omega} \left\langle f , T'y \right\rangle \, dm = \int_{\Omega} \left\langle T f , y \right\rangle \, dm .$$

\hfill \Box

5. **Semigroups and Their Laplace Transforms**

**Definition 5.1.** Let $(X, Y)$ be a norming dual pair. A semigroup on $(X, Y)$ is a family of operators $T = (T(t))_{t \geq 0} \subset L(X, \sigma)$ such that $T(t+s) = T(t)T(s)$ for all $t, s \geq 0$ and $T(0) = id_X$. A semigroup is called exponentially bounded, if there exist $M \geq 1$ and $\omega \in \mathbb{R}$ such that $\|T(t)\| \leq Me^{\omega t}$. In this case, we say that $T$ is of type $(M, \omega)$. An integrable semigroup is a semigroup of some type $(M, \omega)$ such that for every $\text{Re} \lambda > \omega$, there exists $R(\lambda) \in L(X, \sigma)$ such that

$$(5.1) \quad \langle R(\lambda)x , y \rangle = \int_0^\infty e^{-\lambda t} \langle T(t)x , y \rangle \, dt ,$$

for all $x \in X$ and $y \in Y$. In particular, we assume that the integrals on the right hand side of (5.1) all exist. The family $(R(\lambda))_{\text{Re} \lambda > \omega}$ is called the Laplace transform of $T$.

We note that the definition of an integrable semigroup is symmetric, i.e. if $T$ is an integrable semigroup on $(X, Y)$, then the $\sigma$-adjoint semigroup $T'$ is an integrable semigroup on $(Y, X)$. 


Lemma 5.2. Let \( T \) be a semigroup on the norming dual pair \((X, Y)\). Then \( T \) is exponentially bounded if and only if \( \sup_{0 \leq t \leq 1} \|T(t)\| < \infty \). In particular, if \( T \) is \( \sigma \)-continuous at 0, i.e. \( T(t)x \to x \) as \( t \downarrow 0 \) for all \( x \in X \), then \( T \) is exponentially bounded.

Proof. Let us first prove that \( \sigma \)-continuity at 0 implies \( \sup_{0 \leq t \leq 1} \|T(t)\| < \infty \). To that end, observe that for any \( x \in X \) there exists \( \delta_x \) such that \( A_x := \{ \|T(t)x\| : 0 \leq t \leq \delta_x \} \) is bounded. Indeed, if this is wrong, there exists a sequence \( t_n \downarrow 0 \) such that \( \|T(t_n)x\| \) is unbounded. However, as \( T(t_n)x \to x \), the set \( \{T(t_n)x\} \) has to be \( \sigma \)-bounded and hence, by Proposition 2.5, norm-bounded. A contradiction. Now the semigroup law implies that \( \{T(t)x : 0 \leq t \leq 1\} \subseteq A_x \cup T(\delta_x)A_x \cup \cdots \cup T(\delta_x)^k A_x \) for some \( k \in \mathbb{N} \). As all operators \( T(t) \) are bounded, it follows that \( \{T(t)x : 0 \leq t \leq 1\} \) is bounded. By the uniform boundedness principle, \( \sup_{0 \leq t \leq 1} \|T(t)\| =: M < \infty \). Now let \( \omega = \log M \). For \( t \geq 0 \) split \( t = n + r \) for some \( n \in \mathbb{N}_0 \) and \( r \in [0, 1) \). Then 
\[
\|T(t)\| = \|T(r)T(1)^n\| \leq M e^{\omega n} \leq M e^{\omega t}.
\]

By definition, an integrable semigroup has a Laplace transform. However, as we did not require any continuity of the maps \( t \mapsto (T(t)x, y) \), we cannot expect the Laplace transform to be injective. In particular, it is in general not the resolvent of an operator. Instead, we obtain a pseudoresolvent, cf. [11]. Recall that if \( \Omega \subseteq \mathbb{C} \) is non empty and \( X \) is a Banach space, then a pseudoresolvent is a map \( R : \Omega \to L(X) \) such that for \( \lambda, \mu \in \Omega \) we have
\[
R(\lambda) - R(\mu) = (\mu - \lambda)R(\lambda)R(\mu).
\]
In this case, there exists a unique multivalued (m.v. for short) operator \( A \) such that \( R(\lambda) = (\lambda - A)^{-1} \), see Appendix A in [12]. In this situation, one could call \( A \) the generator of \( T \). However, we want to reserve this name for the situation where \( \Re \lambda > \omega \) is injective, i.e. \( A \) is single-valued.

Proposition 5.3. Let \( T \) be an integrable semigroup of type \((M, \omega)\) with Laplace transform \( (R(\lambda))_{\Re \lambda > \omega} \). Then
\[
\begin{align*}
(\text{i}) & \quad (R(\lambda))_{\Re \lambda > \omega} \text{ is a pseudoresolvent;} \\
(\text{ii}) & \quad \text{For } \Re \lambda > \omega \text{ and } k \in \mathbb{N} \text{ we have } \|(\Re \lambda - \omega)^k R(\lambda)^k\| \leq M; \\
(\text{iii}) & \quad R(\lambda) \text{ commutes with } T(t) \text{ for every } \Re \lambda > \omega \text{ and every } t \geq 0.
\end{align*}
\]

We extract from the proof the following lemma.

Lemma 5.4. Let \( T \) be an integrable semigroup of type \((M, \omega)\) and \( R(\lambda) \) be defined by (5.1). Then for all \( \Re \lambda > \omega \) and \( x \in X \) we have
\[
\int_{0}^{h} e^{-\lambda t} T(t)x \, dt = R(\lambda)x - e^{-\lambda h} T(h)R(\lambda)x,
\]
where the integral is understood as a \( Y \)-integral. In particular, the integral on the left hand side belongs to \( X \).
Proof. Let \( x \in X \) and \( y \in Y \) be given. Using Lemma 4.8 and the semigroup law we obtain

\[
\langle T(h)R(\lambda)x , y \rangle = \int_0^\infty e^{-\lambda t} \langle T(t+h)x , y \rangle \, dt
\]

\[
= e^{\lambda h} \int_h^\infty e^{-\lambda r} \langle T(r)x , y \rangle \, dr
\]

\[
= e^{\lambda h} \left( \langle R(\lambda)x , y \rangle - \int_0^h e^{-\lambda r} \langle T(r)x , y \rangle \, dr \right).
\]

As \( y \) was arbitrary, (5.3) is proved. \( \square \)

Proof of Proposition 5.3. (i). Assume that \( \omega < \Re \mu < \Re \lambda \). Using Lemma 5.4 and Fubini’s theorem, we obtain

\[
(\mu - \lambda)R(\lambda x) = (\mu - \lambda) \int_0^\infty e^{-\lambda t} T(t)R(\mu)x \, dt
\]

\[
= (\mu - \lambda) \int_0^\infty e^{(\mu - \lambda)t} \left( R(\mu)x - \int_0^t e^{-\mu r} T(r)x \, dr \right) dt
\]

\[
= -R(\mu)x - (\mu - \lambda) \int_0^\infty T(r)e^{-\mu r} \int_r^\infty e^{(\mu - \lambda)t} dt dr
\]

\[
= R(\lambda)x - R(\mu)x.
\]

This proves that \( R \) is a pseudoresolvent.

(ii). It easily follows from the (5.2) that \( R(\lambda) \) is analytic in \( \lambda \) and \( \frac{d^k}{d\lambda^k} R(\lambda) = (-1)^k k! R(\lambda)^{k+1} \). Interchanging differentiation and integration and using exponential boundedness, we obtain

\[
|\langle R(\lambda)^k x , y \rangle| = \frac{1}{(k-1)!} \left| \frac{d^{k-1}}{d\lambda^{k-1}} \langle R(\lambda)x , y \rangle \right|
\]

\[
= \frac{1}{(k-1)!} \left| \int_0^\infty t^{k-1} e^{-\lambda t} \langle T(t)x , x \rangle \, dt \right|
\]

\[
\leq \frac{1}{(k-1)!} \int_0^\infty t^{k-1} e^{-\Re \lambda t} M e^{\omega t} \|x\| \cdot \|y\| \, dt
\]

\[
= \frac{M}{(\Re \lambda - \omega)^k} \|x\| \cdot \|y\|.
\]

As \( x \) and \( y \) are arbitrary, (ii) follows. (iii) follows from Lemma 4.8. \( \square \)

We now address the question of whether a semigroup on a norming dual pair is uniquely determined by its Laplace transform. Without any further assumptions this is not the case, see [23]. We need the following

**Definition 5.5.** Let \( X \) be a Banach space and \( M \) be a subspace of \( X \). A subset \( W \subset X^* \) is said to separate points in \( M \) if for every \( x \in M \setminus \{0\} \) there exists \( w \in W \) with \( \langle x, w \rangle \neq 0 \). A norming dual pair \((X,Y)\) is said to be countably separated if there exists a countable subset of \( X \) separating points in \( Y \) and there exists a countable subset of \( Y \) separating points in \( X \).
**Theorem 5.6.** Let $T, S$ be integrable semigroups on $(X, Y)$ with Laplace transforms $R_T(\lambda) = (\lambda - A)^{-1} = R_S(\lambda)$ for sufficiently large $\lambda$. Then $T(t) = S(t)$ for all $t \geq 0$, provided one of the following conditions is satisfied:

(i) $D(A)$ is $\sigma$-dense in $X$;
(ii) $(X, Y)$ is countably separated.

The proof uses the following lemma which is taken from [2, Lemma 3.16.5].

**Lemma 5.7.** Let $M \subset (0, \infty)$ be a set of Lebesgue measure 0 and assume that $t, s \not\in M$ implies $t + s \not\in M$. Then $M = \emptyset$.

**Proof of Theorem 5.6.** By assumption, we have

$$
\int_0^\infty e^{-\lambda t} \langle T(t)x, y \rangle dt = \int_0^\infty e^{-\lambda t} \langle S(t)x, y \rangle dt
$$

for every $\lambda$ with $\text{Re}\lambda > \max\{\omega_T, \omega_S\}$ and arbitrary $x \in X$ and $y \in Y$. By the uniqueness theorem for Laplace transforms (see [2, Theorem 1.7.3]), there exists a null set $N(x, y)$ such that

$$
\langle T(t)x, y \rangle = \langle S(t)x, y \rangle \quad \forall t \not\in N(x, y).
$$

First assume (i). For $x \in D(A) = \text{rg}R_T(\lambda)$, say $x = R_T(\lambda)z$, and arbitrary $y \in Y$ we have

$$
\langle T(t)x, y \rangle = e^{\lambda t} \left( \langle x, y \rangle - \int_0^t e^{-\lambda r} \langle T(r)z, y \rangle dr \right)
$$

by Lemma 5.7 when $t \mapsto \langle T(t)x, y \rangle$ is continuous. The same applies to $S(t)$ and we find $N(x, y) = \emptyset$. Thus $T(t)x = S(t)x$ for every $t \geq 0$ and $x \in D(A)$.

However, if the $\sigma$-continuous linear operators $T(t)$ and $S(t)$ coincide on the $\sigma$-dense subspace $D(A)$, then they are equal.

Now assume that (ii) is satisfied. Let $\{x_n\}_{n \in \mathbb{N}} \subset X$ be a countable subset separating points in $X$ and $\{y_n\}_{n \in \mathbb{N}} \subset Y$ be a countable subset separating points in $Y$. Fix $x \in X$ and put $N(x) = \bigcup_{n \in \mathbb{N}} N(x, y_n)$, where $N(x, y_n)$ is the null set from above. Then $N(x)$ is a set of measure 0 and we have

$$
\langle T(t)x, y_n \rangle = \langle S(t)x, y_n \rangle \quad \forall t \not\in N(x), \quad n \in \mathbb{N}.
$$

Since $\{y_n\}$ separates points in $X$, it follows that $T(t)x = S(t)x$ for all $t \not\in N(x)$. In particular, $(T(t)x, y) = (S(t)x, y)$ for all $t \not\in N(x)$ and all $y \in Y$.

Now fix $y \in Y$ and put $N = \bigcup_{n \in \mathbb{N}} N(x_n)$. Then $N$ has measure 0 and for $t \not\in N$ and $n \in \mathbb{N}$ we have

$$
\langle x_n, T(t)'y \rangle = \langle T(t)x_n, y \rangle = \langle S(t)x_n, y \rangle = \langle x_n, S(t)'y \rangle.
$$

As $\{x_n\}$ separates points, it follows that $T(t)'y = S(t)'y$ for all $t \not\in N$. Since $y$ was arbitrary, we obtain $T(t) = S(t)$ for all $t \not\in N$. Now let $M = \{t : T(t) \neq S(t)\}$. Then $M \subset N$ and it follows that $M$ has measure 0. However, if $t, s \not\in M$ then, by the semigroup law, $t + s \not\in M$. Thus Lemma 5.7 implies $M = \emptyset$. This finishes the proof. \qed

**Remark 5.8.** The proof of Theorem 5.6 shows that if $T$ and $S$ are $\sigma$-continuous, or even only $\sigma$-continuous at 0, then they are equal if they have the same Laplace transform. However, if $T$ is $\sigma$-continuous at 0, it can be proved that
$D(A)$ is $\sigma$-dense in $X$. Thus Theorem 5.6 shows that uniqueness holds already if one of the semigroups $T$ or $S$ is assumed to be $\sigma$-continuous at 0.

We now prove a generalization of a well known result from the theory of strongly continuous semigroup, cf. [2, Proposition 3.1.9]. All integrals in the next Proposition are understood as $Y$-integrals.

**Proposition 5.9.** Let $T$ be an integrable semigroup on $(X, Y)$ and let $A$ be the unique m.v. operator such that $R(\lambda) = (\lambda - A)^{-1}$.

(i) The following are equivalent.

(a) $x \in D(A)$ and $z \in Ax$;
(b) For any $t > 0$ we have

$$\int_0^t T(s)x \, ds = T(t)x - x .$$

(ii) For $x \in X$ and $t > 0$ we have $\int_0^t T(s)x \, ds \in D(A)$ and

$$T(t)x - x \in A \int_0^t T(s)x \, ds .$$

**Proof.** (i). (a) $\Rightarrow$ (b): (a) is equivalent to $x = R(\lambda)(\lambda x - z)$. Now fix $t > 0$ and $y \in Y$. Define the analytic functions $f, g : \mathbb{C} \to \mathbb{C}$ by

$$f(\lambda) := \lambda \int_0^t e^{-\lambda s} \langle T(s)x, y \rangle \, ds - \int_0^t e^{-\lambda s} \langle T(s)z, y \rangle \, ds$$

$$g(\lambda) := \langle x, y \rangle - e^{-\lambda t} \langle T(t)x, y \rangle .$$

By Lemma 5.4, we have $f(\lambda) = g(\lambda)$ for all $\text{Re}\, \lambda > \omega$. The uniqueness theorem for analytic functions yields $f(0) = g(0)$. As $t, x$ and $y$ were arbitrary, (b) is proved.

(b) $\Rightarrow$ (a): If $\int_0^t T(s)z \, ds = T(t)x - x$, then

$$x = \lambda R(\lambda)x - \lambda R(\lambda)x + x = \lambda R(\lambda)x - \int_0^\infty \lambda e^{-\lambda t} (T(t)x - x) \, dt$$

$$= \lambda R(\lambda)x - \int_0^\infty \lambda e^{-\lambda t} \int_0^t T(s)x \, ds \, dt$$

$$= \lambda R(\lambda)x - \int_0^\infty \int_0^\infty \lambda e^{-\lambda s} T(s)z \, dt \, ds$$

$$= \lambda R(\lambda)x - \int_0^\infty e^{-\lambda s} T(s)z \, ds = R(\lambda)(\lambda x - z) .$$

This is equivalent with $x \in D(A)$ and $z \in Ax$.

(ii). Let $y \in Y$. We have:

$$\int_0^t \langle T(s)x, y \rangle \, ds = \int_0^t \langle T(s)(\lambda - A)R(\lambda)x, y \rangle \, ds$$

$$= \lambda \int_0^t \langle T(s)R(\lambda)x, y \rangle \, ds - \int_0^t \langle T(s)AR(\lambda)x, y \rangle \, ds$$

$$= \lambda \int_0^t \langle T(s)R(\lambda)x, y \rangle \, ds + \langle (R(\lambda)x - T(t)R(\lambda)x, y \rangle .$$
where we have used $R(\lambda)x \in D(A)$ and part (i) in the last step. Furthermore, in slight abuse of notation, we wrote $AR(\lambda)x$ in place of any element in this set. Appealing again to part (i), we see that there exists an element $x_0 \in X$ (independent of $y$) such that $\int_0^t \langle T(s)R(\lambda)x, y \rangle \, ds = \langle x_0, y \rangle$. It follows from Lemma 4.8 that $x_0 = R(\lambda) \int_0^t T(s)x \, ds$. Plugging this in the above equation, we obtain

$$\int_0^t T(s)x \, ds = R(\lambda) \left( \lambda \int_0^t T(s)x \, ds + x - T(t)x \right),$$

which is equivalent with (ii). \qed

**Theorem 5.10.** Let $T$ be a semigroup of type $(M, \omega)$ on the norming dual pair $(X, Y)$. The following are equivalent:

(i) $T$ is an integrable semigroup;

(ii) For every $x \in X$ and $y \in Y$ the orbits $T(\cdot)x$ and $T(\cdot)'y$ are locally Pettis integrable, i.e. these functions are Pettis integrable on every bounded interval $(a, b) \subset (0, \infty)$

**Proof.** (i) $\Rightarrow$ (ii): As a consequence of Proposition 5.9 (ii), $\int_a^b T(t)x \, dt \in X$ for every $x \in X$ and $0 \leq a < b < \infty$. As such intervals generate the Borel $\sigma$-algebra on $(0, \infty)$ and are closed under finite intersections, it follows from Theorem 4.7 that $T(\cdot)x$ is locally Pettis integrable. Applying the same arguments to $T(\cdot)'y$ for every $y \in Y$, (ii) follows.

(ii) $\Rightarrow$ (i): Fix $\lambda$ with $\text{Re} \lambda > \omega$. It follows from (ii) and Theorem 4.7 that there exists an element $R(\lambda)x \in X$ such that $R(\lambda)x = \int_0^\infty e^{-\lambda t}T(t)x \, dt$. It remains to prove that $R(\lambda) \in L(X, \sigma)$. It is easy to see that $R(\lambda)$ is linear. Furthermore, using the exponential boundedness of $T$ and the dominated convergence theorem, it follows that $R(\lambda) \in L(X)$. However, arguing similar, it follows that there exists $V(\lambda) \in L(Y)$ such that $V(\lambda)y = \int_0^\infty e^{-\lambda t}T(t)'y \, dt$. It is easily seen that $(R(\lambda)x, y) = \langle x, V(\lambda)y \rangle$, whence $V(\lambda) = R(\lambda)^*|_Y$. Proposition 3.1 implies $R(\lambda) \in L(X, \sigma)$. This proves (ii). \qed

### 6. Integrable semigroups on $(C_b(E), M_0(E))$

We now return to the question of integrability of transition semigroups. As we will not use positivity or contractivity, we will consider general semigroups of kernel operators. Taking Theorem 5.10 into account, this is exactly the same as a semigroup on the norming dual pair $(B_b(E), M(E))$. Our first result states that measurability and integrability extends from $(X, M_0(E))$ to $(B_b(E), M_0(E))$ if $X$ is a $M_0(E)$-transition space for $E$.

**Lemma 6.1.** Let $(\Omega, F, m)$ be a $\sigma$-finite measure space, $(E, \Sigma)$ be a measurable space and let $M_0(E)$ denote either $M(E)$ or (if $E$ is a completely regular Hausdorff space) $M_0(E)$. We write $\sigma$ for $\sigma(B_b(E), M_0(E))$. Let $T : \Omega \to L(B_b(E), \sigma)$ and $X$ be a $M_0(E)$-transition space for $E$.

(i) $T(\cdot)f$ is $M_0(E)$-measurable for every $f \in B_b(E)$ if and only if $T(\cdot)f$ is $M_0(E)$-measurable for every $f \in X$. 

...
(ii) Assume additionally, that \( \|T\| \) is majorized by an integrable function. Then \( T(\cdot)f \) is Pettis integrable for every \( f \in B_b(E) \) if and only if \( T(\cdot)f \) is Pettis integrable for every \( f \in X \). In both cases, Pettis integrability means Pettis integrability on \( (B_b(E), \mathcal{M}_0(E)) \).

**Proof.** (i). We only need to prove that \( \mathcal{M}_0 \)-measurability of \( T(\cdot)f \) for all \( f \in X \) implies \( \mathcal{M}_0 \)-measurability of \( T(\cdot)f \) for all \( f \in B_b(E) \), the converse being trivial. To that end, define

\[
\mathcal{G} := \{ A \in \Sigma : T(\cdot)1_A \text{ is } \mathcal{M}_0 \text{-measurable} \}.
\]

If \( 1_A = \sup f_n \) for some sequence \( f_n \in X \), then \( T(\omega)f_n \to T(\omega)f \) for all \( \omega \in \Omega \) by the \( \sigma \)-continuity of \( T(\omega) \). Hence, for any \( \mu \in \mathcal{M}(E) \), we have \( \langle T(\cdot)1_A, \mu \rangle = \lim \langle T(\cdot)f_n, \mu \rangle \). This proves that \( \langle T(\cdot)1_A, \mu \rangle \) is measurable. It follows that \( \mathcal{E}(X) \subset \mathcal{G} \). It is easy to see that \( \mathcal{G} \) is a Dynkin system, whence \( \mathcal{G} = \Sigma \). It now follows from linearity that \( T(\cdot)f \) is \( \mathcal{M}_0 \)-measurable for any simple function \( f \). Approximating an arbitrary function by a sequence of simple functions and using the \( \sigma \)-continuity of the operators \( T(\cdot) \) again, the assertion follows.

(ii). The \( \mathcal{M}_0 \)-measurability of \( T(\cdot)f \) for all \( f \in B_b(E) \) follows from (i). To prove Pettis integrability, we proceed as in (i). Define

\[
\mathcal{G} := \{ A \in \Sigma : T(\cdot)1_A \text{ is Pettis integrable} \}.
\]

If \( 1_A = \sup f_n \) for a sequence \( f_n \in X \), then we see, similar as in the proof of Theorem 4.7, that for any \( S \in \mathcal{F} \) the limit \( g_S := \lim \int_X T(\omega)f_n dm(\omega) \) exists in the norm sense. In particular, \( g_S \in B_b(E) \). It follows from the dominated convergence theorem, that \( \int_X T(\omega)1_A dm(\omega) = g_S \). This proves that \( \mathcal{E}(X) \subset \mathcal{G} \). The rest of the proof is similar as in (1).

We now consider the situation of semigroups of kernel operators on \( C_b(E) \).

We have

**Theorem 6.2.** Let \( E \) be a complete metric space. Then every semigroup on \( (C_b(E), \mathcal{M}_0(E)) \) which is \( \sigma \)-continuous at 0 is integrable.

**Proof.** By Lemma 5.2, the semigroup \( T \) is exponentially bounded, say of type \( (M, \omega) \).

**Claim 1:** We prove that for every \( f \in C_b(E) \) the orbit \( T(\cdot)f \) is locally Pettis integrable. Fix \( f \in C_b(E) \). As every operator \( T(t) \) is \( \sigma \)-continuous, the semigroup law and the \( \sigma \)-continuity at 0 imply that \( t \mapsto \langle T(t)f, \mu \rangle \) is right continuous for every \( \mu \in \mathcal{M}_0 \). In particular, the orbit \( T(\cdot)f \) is \( \mathcal{M}_0 \)-measurable. Furthermore, by right continuity the range of this function is \( \sigma \)-separable and hence, as a consequence of the Hahn-Banach theorem, separable with respect to any consistent topology. Since there exists a complete, consistent topology on \( C_b(E) \), see Example 4.6, local Pettis integrability of \( T(\cdot)f \) follows from Theorem 4.3.

**Claim 2:** We prove that for every \( \mu \in \mathcal{M}_0(E) \) the orbit \( T(\cdot)\mu \) is locally Pettis integrable. Fix \( \mu \in \mathcal{M}_0(E) \). As \( C_b(E) \) is a \( \mathcal{M}_0 \)-transition space for \( E \), every \( T(t) \) is a kernel operator by Proposition 3.5. In particular, it has a unique extension to an operator \( \hat{T}(t) \in L(B_b(E), \sigma) \). We thus infer from Lemma 6.1 (i), that \( t \mapsto \langle f, T(t)\mu \rangle = \langle \hat{T}(t)f, \mu \rangle \) is measurable for every \( f \in B_b(E) \). Now
let $S \subset [0, \infty)$ be a bounded interval. By Lemma 4.3, $\varphi(f) := \int_S \langle f, T(t)\mu \rangle \, dt$ exists as a $B_b(E)$-integral and is sequentially $\sigma(M, B_b)$-continuous. If we put $\rho(A) = \varphi(1_A)$, then it follows from sequential continuity that $\rho$ is a measure. Clearly $\varphi(f) = \int_S f \, dp$. It remains to prove that $\rho \in M_0(E)$. Since $E$ is a complete metric space, a measure on $E$ is a Radon measure if and only if it has separable support. By assumption, the measure $T(t)\mu$ is a Radon measure for every $t \in S$, whence we find a separable set $E_t$ such that $T(t)\mu(A) = 0$ for all $A \subset E \setminus E_t$. Define

$$E_0 := \bigcup_{r \in S \cap \mathbb{Q}} E_r.$$  

Then $E_0$ is a separable set. We claim that $\rho$ is supported in $E_0$. Let $A \subset E \setminus E_0$ be an open set. Then $A$ is an $F_\sigma$-set, say $A = \bigcup F_n$ for an increasing sequence of closed sets. By Tieze’s extension theorem, there exist functions $f_n$ such that $f|_{F_n} \equiv 1$ whereas $f|_{A^c} \equiv 0$. By right continuity of the paths obtain

$$\int_E f_n \, dT(t)\mu = \langle f_n, T(t)\mu \rangle = \lim_{r \uparrow t, r \in \mathbb{Q}} \langle f_n, T(r)\mu \rangle = 0,$$

for all $n \in \mathbb{N}$. Integrating over $S$ yields $\int_S \langle f_n, T(t)\mu \rangle \, dt = 0$. Now the dominated convergence theorem implies that $\rho(A) = \lim_{n \to \infty} \int_S \langle f_n, T(t)\mu \rangle \, dt = 0$. This proves that $\rho$ is supported in $E_0$ and hence a Radon measure. Thus Claim 2 is proved.

Now the statement follows from Theorem 5.10. □

In applications to transition semigroups, the assumption that a semigroup on the norming dual pair $(C_b(E), M(E))$ is $\sigma$-continuous at $0$ is easy to verify. Indeed, this merely means that for every $f \in C_b$ we have $T(t) \to f$ pointwise as $t \downarrow 0$. In Theorem 6.2, this assumption was (amongst others) used to prove that the orbits of $T$ have $\sigma$-separable range. If $E$ is a Polish space, then this is automatic.

**Theorem 6.3.** Let $(E, B(E))$ be a separable metric space equipped with its Borel $\sigma$-algebra. Then the norming dual pair $(C_b(E), M_0(E))$ is countably separated.

**Proof.** Let $D := \{ x_m : m \in \mathbb{N} \}$ be a countable, dense subset of $E$. Then $\{ \delta_{x_m} : m \in \mathbb{N} \} \subset M_0(E)$ separates points in $C_b(E)$ as continuous functions which agree on a dense subset are equal. To find a sequence in $C_b(E)$ which separates points in $M_0(E)$, we proceed as follows. For $n, m \in \mathbb{N}$, choose $f_{n,m} \in C_b(E)$ such that

$$\mathbbm{1}_{B(x_m, n^{-1})} \leq f_{n,m} \leq \mathbbm{1}_{B(x_m, n^{-1})^c}.$$  

If $J \subset \mathbb{N}$ is a finite subset, we put $f_{n,J} := \max\{ f_{n,m} : m \in J \}$ and define

$$M := \{ f_{n,J} : n \in \mathbb{N}, J \subset \mathbb{N} \text{ finite} \}.$$  

Then $M$ is a countable set. We claim that $M$ separates points in $M_0(E)$. To that end, let $\mu \in M_0(E)$ satisfy $\int f \, d\mu = 0$ for all $f \in M$. We have to prove that $\mu = 0$. Since $\mu$ is a Radon measure, it suffices to prove that $\mu(K) = 0$ for all compact sets $K$. So let a compact set $K \neq \emptyset$ be given. As $D$ is dense in $E$, the set $K$ is covered by $\{ B(x_m, (n+1)^{-1}) : m \in \mathbb{N} \}$ for every $n \in \mathbb{N}$. Since $K$ is compact, there exist $m_1, \ldots, m_k$ such that $K$ is already covered
by $\mathcal{B}_n := \{ B(x_m, (n + 1)^{-1}) : i = 1, \ldots, k_n \}$. We may assume without loss that every ball in $\mathcal{B}_n$ intersects $K$. Define $f_n := f_{n, \{m_1, \ldots, m_k\}} \in M$. Then $f_n$ is a bounded sequence which converges pointwise to $1_K$. As $\int f_n \, d\mu = 0$ by assumption, the dominated convergence theorem yields $\mu(K) = \lim f_n \, d\mu = 0$. As $K$ was arbitrary, $\mu = 0$. 

**Corollary 6.4.** Let $(E, B(E))$ be a separable metric space and $(T(t))_{t \geq 0}$ and $(S(t))_{t \geq 0}$ be integrable semigroups on $(C_b(E), M_0(E))$. If $T$ and $S$ have the same Laplace transform, then $T(t) = S(t)$ for all $t \geq 0$.

**Proof.** This follows immediately from Theorems 5.6 and 6.3. 

**Corollary 6.5.** Let $(\Omega, \Sigma, m)$ be a $\sigma$-finite measure space and $E$ be a Polish space. If $F : \Omega \to C_b(E)$ is $M(E)$-measurable and $\|F\|$ is majorized by an integrable function, then $F$ is Pettis integrable on $(C_b(E), M(E))$. If $T$ is an exponentially bounded semigroup on $(C_b(E), M(E))$ such that $t \mapsto \langle T(t) f, \mu \rangle$ is measurable for all $f \in C_b(E)$ and $\mu \in M(E)$, then $T$ is integrable.

**Proof.** The first statement follows from Theorem 6.3 and Theorem 4.3. The second statement follows by reworking the proof of Theorem 6.2 using the first part in the proof of Claim 1. Note that on a Polish space we have $M(E) = M_0(E)$.  

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