On the Thermodynamic Geometry of BTZ Black Holes

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Abstract

We investigate the Ruppeiner geometry of the thermodynamic state space of a general class of BTZ black holes. It is shown that the thermodynamic geometry is flat for both the rotating BTZ and the BTZ Chern Simons black holes in the canonical ensemble. We further investigate the inclusion of thermal fluctuations to the canonical entropy of the BTZ Chern Simons black holes and show that the leading logarithmic correction due to Carlip is reproduced. We establish that the inclusion of thermal fluctuations induces a non zero scalar curvature to the thermodynamic geometry.

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1 Introduction

Over the last few decades, black hole thermodynamics has been one of the most intense topics of research in theoretical physics (for a comprehensive review, see [1]). It is by now well known that black holes are thermodynamical systems, which possess a Bekenstein-Hawking (BH) entropy, and a characteristic Hawking temperature, related to the surface gravity on the event horizon. Indeed, these quantities satisfy the four laws of black hole thermodynamics. However an understanding the microscopic statistical origin of the black hole entropy has been an outstanding theoretical question. Although considerable progress has been made in the recent past, a clear comprehension of the statistical microstates of black holes still remain elusive.

It is well known that equilibrium thermodynamic systems possess interesting geometric features [2]. An interesting inner product on the equilibrium thermodynamic state space in the energy representation was provided by Weinhold [3] as the Hessian matrix of the internal energy with respect to the extensive thermodynamic variables. However there was no physical interpretation associated with this metric structure. The Weinhold inner product was later formulated in the entropy representation by Ruppeiner [4] into a Riemannian metric in the thermodynamic state space. The Ruppeiner geometry was however meaningful in the context of equilibrium thermodynamic fluctuations of the system. The curvature scalar obtained from this geometry signified interactions and was proportional to the correlation volume which diverges at the critical points of phase transitions. The Ruppeiner metric on the thermodynamic state space, is defined as the Hessian of the entropy with respect to the extensive variables and is given by

$$ g_{i,j} = -\partial_i \partial_j S(U, N^a). \quad (1) $$

Here $U$ denotes the internal energy of the system, and $N^a$ are the other extensive thermodynamic variables in the entropy representation. Here, $i, j$ runs over all the extensive variables. It is to be noted that here the volume $V$ is held fixed to provide a physical scale. The Ruppeiner metric is conformal related to the Weinhold metric with the inverse temperature as the conformal factor.

Although, isolated asymptotically flat black holes do not follow the usual precepts of extensive thermodynamic systems it is possible to consider the black hole entropy as an extensive thermodynamic quantity provided the black hole is a part of a larger system with which it is in equilibrium. From this perspective the ge-
ometric notions of thermodynamics may be applied to investigate the nature of the black hole entropy. In particular the investigation of the covariant thermodynamic geometry of Ruppeiner for black holes have elucidated interesting aspects of black hole phase transitions and moduli spaces. This was first explored in the context of charged black hole configurations of $N = 2$ supergravity \cite{5}, and since then, several authors have attempted to understand this connection \cite{6}, \cite{8–10} both for supersymmetric as well as non-supersymmetric black holes. and five dimensional rotating black rings. The simplest black hole system for which it is possible to analyze the thermodynamic geometry is the three dimensional rotating BTZ black hole. Here, one may construct the Ruppeiner metric in terms of the black hole mass and its angular momentum (the non-rotating BTZ black hole is trivial), and it turns out that this metric is flat \cite{10}.

Recently Kraus and Larsen \cite{14} and Solodukhin \cite{15} have shown how various properties of BTZ black holes are affected by the addition of the gravitational Chern-Simons term to the three dimensional Einstein-Hilbert action. In particular, they show that the black hole entropy is modified by the presence of this term and obtain an explicit expression for this modified entropy. In this context, it is imperative to re-examine the Ruppeiner geometry of the BTZ black hole, in the presence of the Chern-Simons term.

Although the statistical origin of black hole entropy is still elusive it stands to reason that a black hole in equilibrium with the thermal Hawking radiation at a fixed Hawking temperature is described by a canonical ensemble. The thermodynamic geometry of the black hole entropy function has been determined with reference to the canonical ensemble. However its well known that thermal fluctuations in the canonical ensemble generates logarithmic corrections to the entropy. These corrections vanish in the thermodynamic limit where the canonical and the microcanonical entropy are identical. For black holes such logarithmic corrections to the canonical entropy have been obtained in \cite{11}. It is a natural question as to whether the thermodynamic geometry of black holes are sensitive to these fluctuations. As we will show in the next few sections, thermal fluctuations indeed modify the Ruppeiner geometry of the BTZ black-holes with and without the Chern-Simons term.

In another interesting development, Sahoo and Sen \cite{16} have computed the

\footnote{Ricci flatness in two dimensions implies a flat space.}
\footnote{See Ref. \cite{9} and \cite{10} for a classification of the nature of the Ruppeiner metrics for black holes in various dimensions.}
BTZ black hole entropy in the presence of the Chern-Simons and higher derivative terms [17]. A variant of the attractor mechanism involving the use of the Sen entropy function was applied to an effective two-dimensional theory that results upon making the angular coordinate of the BTZ solution as a compact direction for this analysis. It is indeed natural to examine the thermodynamic geometry of BTZ Chern Simons black holes with higher derivative terms and investigate the effect of thermal fluctuations to this geometry. It is to be emphasized here that the thermal fluctuations in the canonical ensemble may be analysed through purely thermodynamic considerations. In contrast the corrections to the black hole entropy from the $\alpha'$ corrections of higher derivative terms in the effective action needs to be analysed through gravitational considerations. Although the structures of the corrections are similar, they may enter with opposing signs leading to a cancellation. In addition, there should be quantum corrections following from purely quantum gravitational effects. We note that it is not meaningful to analyse corrections due to thermal fluctuations over and above corrections due to quantum effects.

It is the above considerations, that we set out to explore in this paper. Our main result is that the thermodynamic geometry is flat for the rotating BTZ black hole in the presence of the Chern-Simons and higher derivative terms. We show that inclusion of thermal fluctuations non-trivially modify the thermodynamic geometry of the BTZ black hole both with and without the Chern-Simons and the higher derivative corrections. As a by-product of our results, we show that the leading order correction to the canonical entropy of the BTZ black hole due to thermal fluctuations are reproduced in the presence of Chern-Simons terms also illustrating further the universality of these corrections.

The article is organized as follows. In section 2, we first review some known facts about the thermodynamic geometry for BTZ black holes, mainly to set the notations and conventions used in this paper, and then examine the thermodynamic geometry of BTZ black holes including small thermodynamic fluctuations. In section 3, we examine the Ruppeiner geometry of the BTZ black hole in the presence of the Chern-Simons term [15] and show that including small fluctuations in the analysis, the leading order correction to the entropy turns out to be the same as that of [12]. We then calculate the Ruppeiner curvature scalar and verify the bound on the Chern-Simons coupling, as predicted by Solodukhin. Section 4 contains some comments on higher derivative corrections to the BTZ black hole entropy, and discussions and directions for future investigations. Some
of the calculations are unfortunately too long to reproduce here, and wherever necessary, we have used numerical techniques to highlight and illustrate our results.

2 Thermodynamic Geometry of BTZ black holes

In this section, we study certain aspects of the thermodynamic (Ruppeiner) geometry of BTZ black holes. We will use the units $8G_N = \hbar = c = 1$ and start by reviewing the results for the rotating BTZ black hole and then examine the role of small thermal fluctuations.

2.1 Rotating BTZ black holes

The purpose of this subsection is mainly to set the notations and conventions that will be followed in the rest of the paper. We start with the BTZ metric

$$ds^2 = -N(r)dt^2 + \frac{1}{N(r)}dr^2 + r^2 (N^\phi dt + d\phi)^2$$

(2)

where $N$ and $N^\phi$ are the (squared) lapse and shift functions defined by

$$N(r) = \frac{J^2}{4r^2} + \frac{r^2}{l^2} - M; \quad N^\phi = -\frac{J}{2r^2}$$

(3)

with $M$ and $J$ being the mass and the angular momentum of the black hole, and $l^2$ represents the Cosmological constant term. The BTZ black hole has two horizons, located at

$$r_\pm = \sqrt{\frac{1}{2}Ml^2 (1 \pm \Delta)}$$

(4)

where

$$\Delta = \sqrt{1 - \frac{J^2}{M^2l^2}}$$

(5)

The mass and angular momentum of the black hole may be expressed in terms of $r_\pm$ of eq. (4) as;

$$M = \frac{r_+^2 + r_-^2}{l^2}; \quad J = \frac{2r_+r_-}{l}$$

(6)

The BH entropy of the ordinary BTZ black hole is given by

$$S = 4\pi r_+$$

(7)
The Ruppeiner metric is two dimensional, and is a function of the black hole mass $M$ and angular momentum $J$. Explicitly, the metric is given by

$$g_{ij} = -\left( \frac{\partial^2 S}{\partial J^2} \frac{\partial^2 S}{\partial J \partial M^2} \right)$$

with $i, j \equiv J, M$.

We will use this general form of the Ruppeiner metric throughout this paper.

A simple calculation shows that the Christoffel symbols are given by

$$\Gamma_{JJJ} = -\frac{1}{2} \frac{\partial^3 S}{\partial J^3} \Gamma_{MMM} = -\frac{1}{2} \frac{\partial^3 S}{\partial M^3} \Gamma_{JMM} = -\frac{1}{2} \frac{\partial^3 S}{\partial M \partial J^2} \Gamma_{MJM} = -\frac{1}{2} \frac{\partial^3 S}{\partial M \partial J^2} \Gamma_{JMM} = -\frac{1}{2} \frac{\partial^3 S}{\partial M^2 \partial J} \Gamma_{MMJ} = -\frac{1}{2} \frac{\partial^3 S}{\partial M^2 \partial J}$$

with the symmetries relating the other components. The only non-vanishing component of the Riemann-Christoffel curvature tensor is

$$R_{JMJM} = \frac{N}{D},$$

where

$$N = \frac{\partial^2 S}{\partial J^2} \left[ \left( \frac{\partial^3 S}{\partial M \partial J^2} \left( \frac{\partial^3 S}{\partial M^3} \right) - \left( \frac{\partial^3 S}{\partial J \partial M^2} \right)^2 \right) \right]$$

$$+ \frac{\partial^2 S}{\partial M^2} \left[ \left( \frac{\partial^3 S}{\partial J \partial M^2} \left( \frac{\partial^3 S}{\partial J^3} \right) - \left( \frac{\partial^3 S}{\partial M \partial J^2} \right)^2 \right) \right]$$

$$+ \frac{\partial^2 S}{\partial J \partial M} \left[ \left( \frac{\partial^3 S}{\partial M \partial J^2} \left( \frac{\partial^3 S}{\partial J \partial M^2} \right) - \left( \frac{\partial^3 S}{\partial J \partial M} \right)^2 \right) \right]$$

and

$$D = 4 \left[ \left( \frac{\partial^2 S}{\partial J^2} \left( \frac{\partial^2 S}{\partial M^2} \right) - \left( \frac{\partial^2 S}{\partial J \partial M} \right)^2 \right) \right]$$

The Ricci scalar is

$$R = \frac{2}{\det g} R_{JMJM}$$

It is easy to compute the Ricci scalar by using

$$r_+ = \frac{1}{2} \left[ \sqrt{l(Ml + J)} + \sqrt{l(Ml - J)} \right]$$

Using eq. (13) in eqs. (7), (12), (10) and (11), it can be easily shown that the Ricci scalar vanishes identically [10].

We might point out here that in [6], from considerations of the laws of black hole thermodynamics, the authors have argued that the internal energy of a

$^3$Our notation is $\Gamma_{ijk} = g_{ij,k} + g_{ik,j} - g_{jk,i}$
charged or rotating black hole might not always be equal to its mass. Although we are not in full agreement with the arguments of [6], we have checked nevertheless that a modification of the internal energy of the rotating BTZ black hole in lines with [6] does not change the observation above.

2.2 Inclusion of thermal fluctuations

We will now discuss the Ruppeiner geometry of BTZ black holes including thermal fluctuations about the equilibrium. As is well known, any thermodynamical system, considered as a canonical ensemble has logarithmic and polynomial corrections to the entropy [7]. These considerations apply to black holes as well (considered as a canonical ensemble), and the specific forms of the logarithmic and polynomial corrections has been calculated for a wide class of black holes in [11]. It is to be noted that the applicability of this analysis presupposes that the canonical ensemble is thermodynamicaly stable. This requires a positive specific heat or correspondingly the Hessian of the entropy function must be negative definite.

The microcanonical entropy for any thermodynamical system, incorporating such corrections, is [7]

\[ S = S_0 - \frac{1}{2} \ln (CT^2) \]  \hspace{1cm} (14)

where \( S_0 \) is the entropy calculated in the canonical ensemble, and \( S \) is the corrected microcanonical entropy. \( C \) is the specific heat, and it is understood that appropriate factors of the Boltzmann’s constant are included to make the logarithm dimensionless. The approximation is valid only in the regime where thermal fluctuations are much larger than quantum fluctuations. In [11], the BTZ black hole was analysed in this framework and eq. (14) reproduces the leading order correction to the entropy as obtained in [12]. It is then a natural question as to how the Ruppeiner geometry for the BTZ black hole is modified due to the thermal fluctuations in the canonical ensemble and this is what we will analyse in the rest of this section.

The Ruppeiner metric for the corrected entropy for the BTZ black hole of eq. (14) can be calculated using the equations (10), (11) and (12). Since the expressions involved are lengthy, we will set the cosmological constant \( \Lambda = 1 \).

The Hawking temperature of the BTZ black hole is given by

\[ T_H = \frac{1}{2\pi} \left[ \frac{r_+^2 - r_-^2}{r_+} \right] \]  \hspace{1cm} (15)
which can be readily expressed in terms of the entropy of eq. (7) as

\[ T_H = \frac{S}{8\pi^2} - \frac{8\pi^2 J^2}{S^3} \]  

(16)

The specific heat is

\[ C = \left( \frac{\partial M}{\partial T} \right)_J = \frac{S (S^4 - 64\pi^4 J^2)}{S^4 + 192\pi^4 J^2} \]  

(17)

The specific heat is positive and this ensures that the stability of the corresponding canonical ensemble. Alternatively the Hessian of the internal energy (ADM mass) with respect to the extensive variables in the energy representation is given as

\[ \| \frac{\partial^2 M}{\partial X_i \partial X_j} \| = \frac{1}{S^2 T^2} - \frac{64\pi^4 J^2}{S^6} \]

This is positive provided \( \frac{J}{S} < 1 \) ensuring the thermodynamic stability of the corresponding BTZ black hole. It is to be noted that this condition also governs the situation away from extremality. Substituting the expressions of (16) and (17) in (14), we obtain the corrected entropy of the BTZ black hole, and the Ruppeiner metric for this entropy. The expression of the curvature scalar of this metric is far too complicated to present here, so we present the results numerically.

First, we consider the Ruppeiner metric with just the leading logarithmic correction of [12]. In this case, the analysis is simplified and (14) reduces to

\[ S = S_0 - \frac{3}{2} \ln S_0 \]  

(18)

Figure (1) shows the curvature scalar of the Ruppeiner metric, \( R \), plotted against the angular momentum \( J \) for \( M = 100 \), where we have taken only the logarithmic correction of eq. (18) to the entropy into account. We have restricted to small values of \( J \), so that we are far from extremality, i.e in the regime where these results are valid. Indeed, for near extremal BTZ black holes (i.e for very low temperatures), our analysis is not valid [11]. From fig. (1), we see that in this case, the curvature scalar is not positive definite, and indeed, by extending the values of \( J \), it is seen that the curvature scalar goes to zero at \( J \) increases towards its extremal value. However, we must point out that our calculations that lead to this result can only be trusted when the black hole is far from extremality. Also note that even at zero angular momentum, there is a small but finite value
Figure 1: The Ricci scalar $R$ of the Ruppeiner metric for the BTZ black hole, as a function of the angular momentum $J$, with only the logarithmic correction to the entropy (eq. 18) being taken into account. The mass $M$ has been set to 100.

of the curvature scalar. This indicates that even at zero angular momentum, the statistical system is interacting, once small fluctuations are included. This should be contrasted with the non-rotating BTZ black hole which is a non-interacting system even when small fluctuations are included. We have checked that increasing the value of $M$, the value of the curvature scalar becomes smaller, while preserving the shape of the graph.

Figure 2 shows the Ruppeiner curvature scalar plotted against the angular momentum, calculated using eq. (14). Interestingly, in this case, the Ruppeiner scalar is positive definite. Again, we have restricted ourselves to values of $J$ small compared to $M$ (i.e. far from extremality) where our results can be trusted.

3 BTZ black holes with the Chern-Simons term

Recently, Kraus and Larsen [14] and Solodukhin [15] have studied gravitational anomalies for three-dimensional gravity in the presence of the Chern-Simons term. Indeed, the BTZ black hole is a bonafide solution to the gravitational action that included both the Einstein-Hilbert and the Chern-Simons term. We will henceforth refer to the BTZ black hole with the Chern-Simons term as the BTZ-
CS black hole. In [14], [15], the entropy of BTZ-CS black holes have been analysed, and these authors have derived an expression for the entropy, which differs from the entropy of the “usual” BTZ black hole, eq. (7). The modified entropy for the BTZ-CS black hole is

\[ S = 4\pi \left( r_+ - \frac{K}{l} r_- \right) \]  

(19)

where \( K \) is the Chern-Simons coupling. The extra term in eq. (19) as compared to eq. (7) is the contribution from the Chern-Simons term and has very interesting properties. In particular, [15] predicts a stability bound

\[ |K| \leq l \]  

(20)

on the Chern-Simons coupling, from physical considerations. In view of the above, it is natural to ask what type of Ruppeiner geometry is seen by the BTZ black hole in the presence of the Chern-Simons term and it is this issue that we address in this section.

It is important to remember here that the usual mass and angular momentum of the BTZ black hole is modified in the presence of the Chern-Simons term. This may be calculated by integrating the modified stress tensor of the theory using

Figure 2: The Ricci scalar \( R \) of the Ruppeiner metric for the BTZ black hole, as a function of the angular momentum \( J \), including small fluctuations (eq. [14]). The mass \( M \) has been set to 100.
the Fefferman-Graham expansion of the BTZ metric and reads [15]

\[ M = M_0 - \frac{K}{l^2} J_0; \quad J = J_0 - K M_0 \]  \hspace{1cm} (21)

where \( M_0 \) and \( J_0 \) are the mass and angular momentum of the usual BTZ black hole of eqn. (6). We have calculated the Ruppeiner metric for the BTZ black hole (with the thermodynamic coordinates now being \( M \) and \( J \), rather than \( M_0 \) and \( J_0 \)) in the presence of the Chern-Simons term, taking into account the modifications of the mass and angular momentum as in eq. (21). \(^4\) Writing the entropy as

\[ S = 2\pi \left[ \sqrt{(1 - K)(M + J)} + \sqrt{(1 + K)(M - J)} \right] \]  \hspace{1cm} (22)

it is easy to calculate the geometric quantities. The expressions leading to the calculation of the Ricci scalar are not important, and we simply point out that the curvature scalar for this geometry turns out to be zero, i.e, the Ruppeiner geometry of the BTZ-CS black hole is flat showing that it is a non interacting statistical system. This is the main result of this subsection.

### 3.1 BTZ-CS black holes with small fluctuations

We will now discuss some thermodynamic properties of the BTZ-CS black holes, treating the system as a canonical ensemble. We allow for small thermal fluctuations of the system considered as a canonical ensemble, and study the thermodynamic geometry of the BTZ-CS black hole in lines with our treatment of the usual rotating BTZ black hole described earlier.

As before, we would like to analyse the Ruppeiner metric for the BTZ-CS black hole, with the entropy now being given by eq. (19). Again, for ease of notation, we set the cosmological constant \( l = 1 \). We begin by expressing the outer and inner horizons of the BTZ-CS black hole as

\[ r_+ = \frac{1}{2} \left[ \sqrt{M_0 + J_0} + \sqrt{M_0 - J_0} \right] \]
\[ r_- = \frac{1}{2} \left[ \sqrt{M_0 + J_0} - \sqrt{M_0 - J_0} \right] \]  \hspace{1cm} (23)

When expressed in terms of the corrected mass and angular momentum of eq. (21), these expressions become

\[ r_+ = \frac{1}{2} \left[ \sqrt{\frac{M + J}{1 - K}} + \sqrt{\frac{M - J}{1 + K}} \right] \]

\(^4\)We have set the cosmological constant \( l = 1 \) for simplicity.
\[ r_- = \frac{1}{2} \left[ \sqrt{\frac{M + J}{1 - K}} - \sqrt{\frac{M - J}{1 + K}} \right] \]  

(24)

The equation for the entropy, given by (19), can now be solved to obtain the mass \( M \) in terms of \( S \) and \( J \), and gives

\[ M = \frac{1}{2K^2} \left[ \left( 2KJ + \frac{S^2}{4\pi^2} \right) + \left[ \left( 2KJ + \frac{S^2}{4\pi^2} \right)^2 - 4K^2 \left( \frac{S^4}{64\pi^4} + \frac{S^2KJ}{4\pi^2} + J^2 \right) \right]^{\frac{1}{2}} \right] \]

(25)

The temperature of the BTZ-CS black hole, given by \( \left( \frac{\partial M}{\partial S} \right)_J \), may be obtained from the expression for \( M \), and is given by

\[ T = \frac{SK^2 \left[ S^2 (1 - K^2) + 8JK\pi^2 (1 - K^2) + [S^2 (1 - K^2) (S^2 + 16\pi^2JK)]^{\frac{1}{2}} \right]}{4\pi^2 \left[ S^2 (1 - K^2) (16\pi^2JK + S^2) \right]} \]

(26)

The specific heat may be calculated from the expression

\[ C = \left( \frac{\partial M}{\partial T} \right)_J = \frac{T}{\left( \frac{\partial T}{\partial S} \right)_J} \]

(27)

and is evaluated as

\[ C = \frac{S\alpha \left[ \beta + (8KJ\pi^2 + S^2) (1 - K^2) \right]}{\alpha\beta + S^2 (1 - K^2) (S^2 + 24KJ\pi^2)} \]

(28)

where \( \alpha = 16\pi^2KJ + S^2 \) and \( \beta = (S^2 (1 - K^2) \alpha)^{\frac{1}{2}} \). It may be checked that the specific heat is positive ensuring local thermodynamic stability. Using eq. (28), we calculate the correction to the canonical entropy including small thermal fluctuations of the statistical system and this leads to,

\[ S = S_0 - \frac{1}{2} \ln CT^2 \]

(29)

where \( S_0 \) is the entropy (19) of the BTZ-CS in the canonical ensemble. We approximate (29) in the limit of large entropy, following [11]. It may be easily examined that in the limit of \( S \gg J^2 \) which is the stability bound, the above formula reduces to

\[ S = S_0 - \frac{3}{2} \ln S_0 \]

(30)

It is interesting to note that the factor of \( \frac{3}{2} \), first calculated in [12] is reproduced for the BTZ-CS black hole as well. Illustrating the seeming universality of this factor. This is one of the main result of this subsection.
Figure 3: The Ricci scalar $R$ of the Ruppeiner metric for the BTZ-CS black hole, as a function of the Chern-Simons coupling $K$, with only the logarithmic correction to the entropy being taken into account. The mass $M$ has been set to 100 and we have set $J = 1$ to ensure that we are far from extremality.

We now calculate the Ruppeiner geometry corresponding to the modified entropy of the BTZ-CS black hole with thermal fluctuations. As in the last section, we first present the numerical result for the Ricci scalar using the leading order correction of eq. (30). This is depicted in fig. (3). In this analysis, we have set $M = 100$ and $J = 1$, to ensure that we are far from extremality. The Ricci scalar is positive definite in this case. The Ricci scalar diverges for $|K| = 1$. More appropriately, since we had set the cosmological constant to unity, it is not difficult to see that the bound on $K$ from the Ruppeiner geometry is $|K| \leq l$ where $l$ is the cosmological constant. This is of course as expected, since the entropy (22) becomes unphysical beyond this limit.

In fig. (4), we present the result for the Ricci scalar of the Ruppeiner geometry taking into account the full correction of eq. (29). Again, as a function of $K$, the curvature scalar is positive definite and the graph has the same qualitative features as in fig. (3).

For the sake of completeness, we have also numerically evaluated the Ricci scalar of the Ruppeiner metric for the BTZ-CS black hole as a function of the
Figure 4: The Ricci scalar $R$ of the Ruppeiner metric for the BTZ-CS black hole, as a function of the Chern-Simons coupling $K$. The mass $M$ has been set to 100 and we have set $J = 1$ to ensure that we are far from extremality.

angular momentum, and studied its behaviour. The plots in this case are qualitatively the same as in fig. 1 and fig. 2 and we do not discuss them further.

4 Discussions and Conclusions

In this article, we have mainly investigated the thermodynamic geometry of a class of BTZ black holes, both with and without the Chern-Simons term. We have shown that the Ruppeiner geometry remains flat even with the introduction of the Chern-Simons term, as it was without this term. However, introducing small thermal fluctuations in the analysis produces a non-zero Ricci scalar for the thermodynamic geometry, and we have calculated this quantity for some special cases. As a byproduct of our calculations, we have shown that the leading logarithmic correction to the canonical entropy of the BTZ-CS black hole retains the same form as for the ordinary rotating BTZ black hole thus illustrating the universality of this correction. We should mention here that the validity of this analysis depends on the local thermodynamic stability which is ensured by a positive specific heat for the BTZ and the BTZ-CS black holes. This is also generally true for charged and rotating charged black holes. It would be interesting to extend
our analysis to other black holes and investigate the subtle interplay between the corrections due to thermal fluctuations and $\alpha^\prime$ corrections resulting from higher derivative terms. It is expected that the corresponding thermodynamic geometries would be sensitive to these corrections. Furthermore thermodynamic geometries provide a direct way to analyze critical points of black hole phase transitions which is an area of current interest. This may have important implications for black holes in string theory and the geometry of moduli spaces. Some of these issues will be investigated in future.

A few comments are in order here. It is clear that our analysis will be similar for BTZ black holes with higher derivative corrections. As shown in [14] and [16], the form of the entropy for the BTZ-CS black hole remains the same in the presence of the higher derivative corrections, and it is the central charge of the underlying conformal field theory that is modified. Hence, we expect qualitatively similar results for the thermodynamic geometry of BTZ-CS black holes with higher derivative corrections. We have explicitly verified this.

As we have pointed out earlier, leading logarithmic correction to the black hole entropy arises from various sources. The black hole considered as a canonical ensemble admits such corrections to the entropy due to standard thermal fluctuations. The effect of such fluctuations may be analyzed from purely thermodynamic considerations. It is to be understood that these fluctuations vanish in the thermodynamic limit of large systems where the canonical and the microcanonical entropy becomes identical. Apart from these the black hole entropy also admits logarithmic corrections due to presence of higher derivative terms to the gravitational action from the perspective of low energy effective field theories resulting from some underlying theory of quantum gravity. These higher derivative corrections are accessible to analysis through purely gravitational considerations like Wald's formula or through gravitational anomalies. It is a meaningful exercise to analyze these two corrections simultaneously and in certain cases leads to a cancellation. However corrections due to purely quantum effects must be considered separately. Lacking a viable fundamental theory of quantum gravity these quantum corrections still need to be elucidated.

We should also remark here that as pointed out in [15] that the modification of the entropy due to the gravitational Chern-Simons term being dependent on the radius of the inner horizon seems to probe the black hole interior. This is in contrast to the higher derivative $\alpha^\prime$ corrections which are only dependent on the radii of the outer horizon. This seems to indicate that contrary to the
existing point of view certain degrees of freedom may be associated with the black hole interior. This may have implications for space time holography and is an important issue for future investigations.
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