We show that tree level “resonant” $N$ tachyon scattering amplitudes, which define a sensible “bulk” $S$–matrix in critical (super) string theory in any dimension, have a simple structure in two dimensional space time, due to partial decoupling of a certain infinite set of discrete states. We also argue that the general (non resonant) amplitudes are determined by the resonant ones, and calculate them explicitly, finding an interesting analytic structure. Finally, we discuss the space time interpretation of our results.
1. Introduction.

String theory [1] is a prime candidate for a unified quantum description of short distance physics, which naturally gives rise to space-time gravity as well as gauge fields and matter. However, our understanding of this theory is hindered by its complexity, related to the enormous number of space-time degrees of freedom (massive resonances), the proliferation of vacua, and lack of an organizing (non-perturbative) dynamical principle.

In this situation, one is motivated to look for toy models which capture some of the important properties of strings, while allowing for a more complete understanding. In the last year important progress was made in treating such toy models, corresponding to strings propagating in a two dimensional (2D) space-time. The low space-time dimension drastically reduces the number of degrees of freedom, eliminating most of the massive oscillation modes of the string and leaving behind essentially only the center of mass of the string (the ‘tachyon’ field) as a physical field theoretic degree of freedom. Following the seminal work of [2] [3] [4], it was understood that the center of mass in these 2D string theories is described by free fermion quantum mechanics [5] [6] [7] [8]. This remarkable phenomenon has led to rapid progress in the qualitative and quantitative understanding of these theories [3] [10] [11] [12].

This progress was phrased in the language of matrix models of random surfaces [13]; it is important to understand the results and in particular the free fermion structure in the more familiar Polyakov path integral formulation of 2D gravity [14]. If we are to utilize the impressive results of 2D string theory in more physically interesting situations, which are either hard to describe by means of matrix models (e.g. fermionic string theories) or can be described by matrix models which are hard to solve (e.g. $D > 2$ string theories), we must learn how to handle the continuum (Liouville) theory more efficiently. Despite important progress in this direction [15] [16] [17] [18] some aspects of the matrix model results are still mysterious.

The purpose of this paper is to try and probe the continuum string theory in various ways, with the hope of understanding the underlying free fermion structure. We will not be able to get as far as that, but we will see aspects of the simplicity emerging. Most of our analysis will be done on the sphere; the matrix model techniques are (so far) much more powerful in obtaining higher genus results. As a compensation, the spherical structure will be quite well understood; in fact, many of the results described below were not obtained from matrix models (so far).
What can we hope to learn from such an endeavour? The free fermion structure of 2D string theory is highly unlikely to survive in more physically interesting situations. However, there are some features which are expected to survive: the \((2g)!\) growth of the perturbation expansion is expected \([19]\) to be a generic property of all (super–) string theories; issues related to background independence of the string field theory, the form of the (classical) non linear equations of motion in string field theory, and even the right variables in terms of which one should formulate the theory may be studied in this simpler context. The advantage of such a simple solvable framework is to provide a laboratory to quantitatively check ideas in string field theory. The fact that we do not quite understand the matrix model results from the continuum is significant: it suggests that a new point of view on the existing techniques or new techniques are needed for treating strings. Finally, it was argued recently \([20]\) that one can study space-time singularities in string theory using related two dimensional string models. Issues related to gravitational back reaction can be naturally described and studied in the continuum approach.

The paper is organized as follows. In section 2, after an exposition of tachyon propagation in \(D\) dimensional string theory, we discuss in detail 2D bosonic strings, or more precisely \(c \leq 1\) Conformal Field Theory (CFT) coupled to gravity. In the conformal gauge we are led to study (minimal or \(c = 1\)) matter with action \(S_M(g)\) (on a Riemann surface with metric \(g\)), coupled to the Liouville mode. The action is \([14]\):

\[
S = S_L(\hat{g}) + S_M(\hat{g})
\]

(1.1)

where the dynamical metric is \(g_{ab} = e^\phi \hat{g}_{ab}\) and:

\[
S_L(\hat{g}) = \frac{1}{2\pi} \int \sqrt{\hat{g}} [\hat{g}^{ab} \partial_a \phi \partial_b \phi - \frac{Q}{4} \hat{R} \phi + 2\mu e^{\alpha+\phi}]
\]

(1.2)

with \(Q\) and \(\alpha_+\) finitely renormalized parameters \([21]\) (see below). It is very useful to think about the Liouville mode as a target space coordinate, and of \((1.1)\) as a critical string system in a non-trivial background \([22]\). This point of view proves helpful for the analysis of the Liouville dynamics \([17]\), which is given by a non-trivial interacting CFT \((1.2)\). The exponential interaction in \((1.2)\) keeps the Liouville field away from the region where the string coupling \(g_{st} = g_0 e^{-2\phi}\) blows up \((\phi \to -\infty\) in our conventions\). Due to the presence of this ‘Liouville wall’, correlation functions in this theory are non-trivial. To understand them, it is useful to break up the problem into two parts. It is clear that studying the scattering in the bulk of the \(\phi\) volume is a simpler task than that of considering the general
scattering processes. Since such bulk amplitudes are insensitive to the precise form of the wall (as we’ll explicitly see later), they can be calculated using free field techniques. This is the first step which is performed in section 2.

The results for bulk amplitudes are puzzling if one compares them with the well known structures arising in critical string theory. There, bulk scattering is the only effect present, and it is described by a highly non-trivial S – matrix, incorporating duality, an infinite number of massive resonances etc. The main differences between this situation and ours are:

1) The critical string amplitudes are meromorphic in the external momenta. When the integral representation diverges, one calculates the amplitudes by analytic continuation. We will see that in 2D string theory the situation is more involved (this is expected to be a general property of all \( D \neq 26 \) non-critical string theories).

2) The bulk scattering amplitudes in the 2D problem exhibit miraculous symmetries (first noticed in \([23]\)). Most of the tachyon scattering amplitudes vanish. Those which do not, have an extremely simple form which is strongly reminiscent of the corresponding free fermion expressions \([11], [12]\). These phenomena are far from completely understood, and have to do on the one hand with the small number of states and large symmetry in the theory, and on the other with peculiarities of (massless) 2D kinematics. For all \( D > 2 \), the form of the amplitudes is qualitatively similar to that in the critical case \( D = 26 \). Hence, an abrupt change in the behavior of the theory occurs between \( D = 2 \) and \( D > 2 \).

At the second stage, after treating tachyon scattering in the bulk, we proceed and consider the generic scattering amplitudes which probe the structure of the Liouville wall. A direct approach seems unfeasible and we argue instead that one can deduce the general structure of the interactions from their bulk part. The main idea is that the Liouville interaction \([1,2]\) represents (in target space language) a tachyon condensate. If we understand the interaction of tachyons in the bulk, it is reasonable to expect that we can understand the dynamical effect of the wall. This procedure is nevertheless not guaranteed to work apriori, but it does here (in \( 2D \)), and this allows us to obtain the full tree level tachyon scattering matrix. The most remarkable feature of this S – matrix is that one can write down all \( N \) point functions explicitly. Scattering amplitudes are naturally expressed in terms of target space Feynman rules with an infinite number of calculable irreducible \( N \) particle interactions, which can be thought of as arising from integrating out the massive (discrete) modes. One of the main technical results of this section is the evaluation of these irreducible vertices. We also discuss the space-time picture which emerges from this
treatment of the tachyon, and apply the results to calculations of correlation functions in minimal models [24] coupled to gravity, reproducing the results of the KdV formalism [8, 10].

In section 3 we apply the techniques of section 2 to the problem of calculating correlation functions in fermionic string theory (again in 2D). As expected from general arguments, there is little qualitative difference between this case and the simpler bosonic one. The only field theoretic degree of freedom in the Neveu-Schwarz (NS) sector is again the massless “tachyon” (the center of mass of the string); the Ramond (R) sector contains an additional (massless) bosonic space-time field. We find that the massless sector scattering picture in fermionic 2D string theory is similar to the one obtained in the bosonic case. The only difference is in the spectrum of discrete states in the two models; the way it affects the scattering illuminates the role of the latter. We mention the possibility [25] of obtaining stable (tachyon free) superstring theories at $D \geq 2$ by a chiral GSO projection of the fermionic string, and show that the 2D superstring is topological.

Section 4 contains some comments on the physics of discrete massive states in 2D string theory. Those are important from several points of view. First, they represent the only remnants of the infinite tower of massive states – the hallmark of string theory – and it would be interesting to study their dynamics. Second, these discrete operators are instrumental to the question of gravitational back reaction in two dimensional string theory [20], [26], and by understanding their dynamics we may study issues related to gravitational singularities in string theory. Finally they are closely related to the large symmetry of 2D string theory.

Section 5 contains some summarizing remarks. In appendices A,B we compare Liouville results with those of matrix models (given by generalized KdV equations [9] for minimal models) and describe some features of the 1PI tachyon amplitudes.

2. Tachyon Dynamics in Bosonic String Theory.

2.1. The general structure and strategy.

We will concentrate throughout this paper on the situation in string theory in two dimensional space time, where many special features arise. It will be very useful to have in mind the perspective of the higher dimensional situation for comparison. We will describe it in this subsection, in addition to defining some concepts which will be useful later, and describing the procedure which we will use to calculate the $S$ – matrix.
Thus we start with the Polyakov string in flat \( d \) dimensional (Euclidean) space

\[
S_M(X, g) = \frac{1}{2\pi} \int \sqrt{g} g^{ab} \partial_a X^i \partial_b X^i \tag{2.1}
\]

\( i = 1 \ldots d \). The most convenient prescription \([14]\) to quantize this generally covariant two dimensional system is to fix a conformal gauge \( g_{ab} = e^{\phi} \hat{g}_{ab} \), in which the system is described by the Liouville mode \( \phi \) and space coordinates \( X^i \), living in the background metric \( \hat{g} \) (the gauge fixing also introduces reparametrization ghosts \( b, c \) with spins 2, \(-1\) respectively). The action for the system is \((1.1)\) where the Liouville mode is governed by \((1.2)\) and the matter fields \( X^i \) by the free scalar action \((2.1)\) with \( g \rightarrow \hat{g} \), the non dynamical background ("fiducial") metric \([1]\). The parameters in \((1.2)\) are determined by requiring gauge invariance (independence of the arbitrary choice of \( \hat{g} \)). This is equivalent \([21]\) to BRST invariance with \( Q_{BRST} = \oint c T \), \((T = T_L + T_M \) is the total stress tensor of the system\), which fixes

\[
Q = \sqrt{\frac{25 - d}{3}} ; \quad \alpha_+ = -\frac{Q}{2} + \sqrt{\frac{1 - d}{12}} \tag{2.2}
\]

From the critical string point of view, BRST invariance is the requirement that the matter + Liouville system be a consistent background of the \( D = d + 1 \) dimensional critical bosonic string. Thus it is superficially very similar to "compactified" critical string theory, where one also replaces part of the matter system by an arbitrary CFT with the same central charge (here the Liouville CFT). The most important difference is that the density of states of the string theory is \textit{not} reduced by compactification, while it is reduced by Liouville. In other words, although the central charge of the Liouville theory

\[
c_L = 1 + 3Q^2 \tag{2.3}
\]

is in general larger than one, the density of states is that of a \( c = 1 \) system (see \([17], [27]\) for further discussion).

We will concentrate on the dynamics of the center of mass of the string, the tachyon field. Of course, for generic \( D \) there is no reason to focus on the tachyon, both because it

\(^1\) We don’t want to leave the impression that the equivalence of \((2.1)\) and \((1.1), (1.2)\) is well understood. There are subtleties related to the measure of \( \phi \) \([14], [21]\) and the conformal invariance of \((1.2)\). Our point of view is that \((1.2)\) defines a CFT (in a specific regularization to be discussed below), so that we are certainly studying a consistent background of critical string theory. The world sheet physics obtained is also reasonable, thus it is probably the right quantization of \( 2d \) gravity. The relation of \( \phi \) in \((1.2)\) to the conformal factor of \( g_{ab} \) is at best a loose one.
is merely the lowest lying state of the infinite string spectrum, and because it is tachyonic, thus absent in more physical theories. Our justification will come later, when we’ll consider the two dimensional situation, where the tachyon is the only field theoretic degree of freedom, and is massless (we will still call it “tachyon” then). The on shell form of the tachyon vertex operator is

\[ T_k = \exp(ik \cdot X + \beta(k)\phi) \]  \hspace{1cm} (2.4)

where \( k, X \) are \( d \) – vectors, and BRST invariance implies

\[ \frac{1}{2}k^2 - \frac{1}{2}\beta(\beta + Q) = 1 \]  \hspace{1cm} (2.5)

As in critical string theory \((D = d+1 = 26)\), this equation is simply the tachyon mass shell condition; the vertex operator \( T_k \) is related to the wave function \( \Psi \) of the corresponding state through \( T_k = g_{st}\Psi \) so that the wave function has the form (recall \( g_{st} \propto \exp(-Q^2\phi) \))

\[ \Psi(X, \phi) = \exp \left( ik \cdot X + (\beta(k) + \frac{Q}{2})\phi \right) \]  \hspace{1cm} (2.6)

We thus recognize the Liouville momentum (or energy, interpreting Liouville as Euclidean time) \( E = \beta + \frac{Q}{2} \), and space momentum \( p = k \). Eq. (2.5) can be rewritten as

\[ E^2 = p^2 + m^2; \hspace{0.5cm} m^2 = \frac{2-D}{12} \]  \hspace{1cm} (2.7)

reproducing the well known value of the ground state energy of \( D \) dimensional strings.

From the world sheet point of view [17], [18], the region \( \phi \to \infty \) corresponds to small geometries in the dynamical metric \( g \) (2.1). This is also the region where the string coupling constant \( g_{st} \to 0 \) and the Liouville interaction in (1.2) is negligible. From eq. (2.5) we see that on shell states fall into three classes [17], [18]:

1) \( E = \beta + \frac{Q}{2} > 0 \): the wave function \( \Psi \) (2.6) is infinitely peaked at small geometries (in the dynamical metric \( g \) \( \phi \to \infty \)). Insertion of such operators into a correlation function corresponds to local disturbances of the surface.

2) \( E < 0 \): the wave function is infinitely peaked at \( \phi \to -\infty \). Such operators do not correspond to local disturbances of the surface. In [17], [18] it was argued that they do not exist.

3) \( E \) imaginary: \( \Psi(X, \phi) \) is in this case (\( \delta \) function) normalizable. Such states create finite holes in the surface and destroy it if added to the action. Thus they correspond to world sheet instabilities. In space time, such operators correspond to tachyonic string states (real
Euclidean $D$ momentum). It is well known that one can at best make sense of theories with tachyons on the sphere; at higher genus, on shell tachyons in the loops cause IR divergences. The existence of such states in a string theory is in one to one correspondence with existence of a non trivial number of states $[17]$, $[27]$. The cosmological operator in (1.2) corresponds to a macroscopic state for $d > 1$ (2.2).

The main object of interest to us here will be the tachyon S-matrix, the set of amplitudes:

$$A(k_1,..,k_N) = \langle T_{k_1}..T_{k_N} \rangle \quad (2.8)$$

where the average is performed with the action (1.1). Translational invariance in $X$ implies momentum conservation

$$\sum_{i=1}^{N} k_i = 0 \quad (2.9)$$

There is no momentum conservation in the $\phi$ direction due to the interaction, therefore in general all amplitudes (2.8) satisfying (2.9) are non vanishing. The Liouville path integral is complicated, but some preliminary intuition can be gained by integrating out the zero mode of $\phi$, $\phi_0$ $[28]$. Splitting $\phi = \phi_0 + \tilde{\phi}$, where $\int \tilde{\phi} = 0$ and integrating in (2.8) $\int_{-\infty}^{\infty} d\phi_0$, we find:

$$A(k_1,..,k_N) = \left( \frac{\mu}{\pi} \right)^s \Gamma(-s) \langle T_{k_1}..T_{k_N} \left[ \int \exp (\alpha_+ \tilde{\phi}) \right]^s \rangle_{\mu=0} \quad (2.10)$$

In (2.10) the average is understood to exclude $\phi_0$ (and we have absorbed a constant, $\alpha_+$ into the definition of the path integral); note also that it is performed with the free action (1.1), (1.2) : $\mu = 0$. $s$ is the KPZ $[29]$ $[21]$ scaling exponent:

$$\sum_{i=1}^{N} \beta(k_i) + \alpha_+ s = -Q \quad (2.11)$$

The original non linearity manifests itself in (2.10) through the (in general non integer) power of the interaction.

We seem to have gained nothing since for generic momenta $s$ is an arbitrary complex number, and (2.10) is only a formal expression. However now the space time interpretation is slightly clearer. Amplitudes with $s > 0$ (assume $s$ real for simplicity) are dominated by the region $\phi \to \infty$ in the zero mode integral (the region far from the Liouville wall); those

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2 We will be sloppy with integral signs. In $N$ point functions $N - 3$ of the vertices should be integrated over.
with \( s < 0 \) receive their main contribution from the vicinity of the wall. As \( s \to 0 \) we see an apparent divergence in (2.10) (more generally this happens whenever \( s \in Z_+ \)). From the world sheet point of view this is a trivial effect; the Laplace transformed amplitude is finite everywhere:

\[
\mu^s \Gamma(-s) = \int_0^\infty dA A^{-s-1} \exp(-\mu A)
\]

From (2.12) we see that the \( s \to 0 \) divergence at fixed \( \mu \) is a small area divergence in the integral over areas \( A \). From this point of view the right way to interpret (2.10) for \( s \in Z_+ \) is to replace \( \mu^s \Gamma(-s) \to \left(-\frac{\mu}{s!}\right) \log \frac{1}{\mu} \). This so called “scaling violation” is of course in perfect agreement with KPZ scaling of the fixed area amplitudes. In space time the picture is more interesting; at \( s = 0 \) the amplitude balances itself between being exponentially dominated by the boundaries of \( \phi \) space and receives contributions from the bulk of the \( \phi_0 \) integral. Thus such amplitudes represent scattering processes that occur in the bulk of space time, and one would expect them to be insensitive to the precise form of the wall, which from this point of view is a boundary effect. That this is indeed the case is easily seen in (2.10). The coefficient of \( \log \mu \) is given by a free field amplitude – the interaction disappears. Of course, it is natural to interpret \( \log \frac{1}{\mu} \) as the volume of the Liouville coordinate \( \phi \) (remember that the wall effectively enforces \( \phi \geq \log \mu \), and one may introduce a UV cutoff \( \phi \leq \phi_{UV} \)); the bulk amplitudes per unit \( \phi \) volume will be clearly independent of \( \phi_{UV} \), if the latter is large enough, as can be readily verified by repeating the considerations leading to (2.10).

Amplitudes with \( s \in Z_+ \) (more precisely the coefficients of \( \mu^s \log \mu \) or, equivalently, the fixed area correlators at integer \( s \)) are also seen to simplify since they too reduce to free field integrals (2.10). The space time interpretation is again clear – these processes correspond to resonances of the scattering particles with the wall – the energy is precisely such that they can scatter against \( s \) zero momentum tachyons (which are the building blocks of the “wall” (1.2)) in the bulk of the \( \phi \) volume. Of course, given all \( s = 0 \) (bulk) amplitudes, the general \( s \in Z_+ \) ones immediately follow by putting some momenta to zero.

After understanding the nature of the difficulties we’re facing, we now turn to the strategy that we’ll use to obtain the amplitudes \( A(k_1,..,k_N) \) (2.10). We will proceed in three stages:

**Step 1:** Calculate (2.10) for \( s \in Z_+ \). For generic \( D \) this step is technically hopeless; the analytic structure of the amplitudes is complicated and it is not known how to perform the free field integrals in (2.10). This is essentially due to the complicated back reaction that occurs when tachyons propagate in space time. For \( D = 2 \) two miracles occur: first,
the kinematics allows a finite region in momentum space where the integral representation \((2.10)\) converges, which is usually not the case for massless/massive particles. It is nice that such a region exists, since unlike critical string theory, the amplitudes here can not be continued analytically: they do not define meromorphic functions of the momenta, because of non conservation of Liouville momentum (energy), associated with the existence of the exponential wall. More importantly, we will be able to actually calculate the integrals \((2.10)\) in the above kinematic region, and find simple results. This will imply that the back reaction is much simpler (and milder) in two dimensions than in general, and will allow us to recover the full dynamical effect of the Liouville wall.

**Step 2:** The result of the first step will be the function \(A(k_1, ..., k_N)\) \((2.8)\) for \(s \in Z_+\) in the kinematic region where the integral representation \((2.10)\) converges. The first remaining question is how to calculate the general \(N\) point functions \((2.8)\) (with \(s \notin Z_+\)) in this kinematic region. It is not known how to make sense of \((2.10)\) in this general case. One expects the qualitative behavior to be different in two dimensions and in \(D > 2\). In the two dimensional case we will argue that one can obtain the result by a physical argument.

We will see that the integer \(s\) tachyon amplitudes are polynomials in momenta (in an appropriate normalization). This will be interpreted as the result of the fact that tachyon dynamics can be described by a local two dimensional field theory (obtained by integrating out the massive discrete string modes), which for large momentum gives algebraic growth of the amplitudes (associated presumably with a UV fixed point). The requirement that all amplitudes must be polynomial in this normalization will fix them uniquely. We would like to stress that the above argument is a phenomenological observation which gives the right result; we do not know why the local tachyon field theory appears.

**Step 3:** After obtaining the amplitudes \((2.8)\) for generic \(s\) in the region where the integral representation converges, we will be faced with the last problem: extending the results to all momenta \(\{k_i\}\). Recall that due to the non trivial background we can not analytically continue. We will see that from general Liouville considerations we expect cuts in amplitudes and will suggest a physical picture based on the above space time field theory, which allows one to calculate all amplitudes. The integrals over moduli space will be split to contributions of intermediate tachyons (coming from regions of degeneration), and an infinite sum over the discrete massive states, which will give irreducible \(N\) point vertices. The tachyon propagator will be seen to be non analytic (containing cuts at zero intermediate momenta), while the vertices will be found to be analytic (in \(\{k_i\}\)). We will give a general procedure for calculating these irreducible 1PI vertices.
The program described above can not be carried out for $D > 2$. We can understand the nature of the difficulties and gain additional intuition by studying the $s = 0$ four point function of tachyons, which can be calculated for all $D$, as in the critical string case \[1\]. Thus we consider $A_{s=0}(k_1, ..., k_4)$, which is given using (2.10), (2.11) by:

$$A_{s=0}(k_1, ..., k_4) = \pi \prod_{i=2}^{4} \frac{\Gamma(k_1 \cdot k_i - \beta_1 \beta_i + 1)}{\Gamma(\beta_1 \beta_i - k_1 \cdot k_i)} \quad (2.13)$$

The amplitude (2.13) exhibits an infinite set of poles at

$$k_1 \cdot k_i - \beta_1 \beta_i + 1 = -n; \quad n = 0, 1, 2, ... \quad (2.14)$$

The meaning of these poles is clear; the $s = 0$ amplitudes have the important property that they conserve Liouville momentum, $\exp(\beta_1 \phi) \exp(\beta_2 \phi) = \exp(\beta_1 + \beta_2) \phi$, as opposed to the general Liouville amplitudes that don’t (due to the existence of the Liouville wall) as explained above\[3\]; this is of course the reason why they are calculable. Thus the intermediate momentum and energy in the $(1, 2)$ channel, say, are $k_{\text{int}} = k_1 + k_2$, $E_{\text{int}} = \beta_1 + \beta_2 + Q/2$ (the shift by $Q/2$ is as in (2.6)). The poles (2.14) occur when

$$E_{\text{int}}^2 - k_{\text{int}}^2 = \frac{2 - D}{12} + 2l \quad (2.15)$$

Thus the poles in (2.13) correspond to on shell intermediate tachyons ($l = 0$), gravitons ($l = 1$), etc \[4\]. They carry the information about the non trivial back reaction of the string to propagation of tachyons in space time. In world sheet terms we learn that trying to turn on a tachyon condensate in the action spoils conformal invariance – switches on a non zero $\beta$ function (infinite correlation functions (2.13) signal logarithmic divergences on the world sheet, as in dimensional regularization). To restore conformal invariance we must correct the tachyon background and turn on the other massless and massive string modes as well.

In space time terms, we conclude that the tachyon background (2.4) while being a solution to the linearized equations of motion of the string is not a solution to the full non linear (classical) equations of motion and must be corrected, both by correcting $T(X, \phi)$

\[3\] Note that Liouville theory seems to exhibit the peculiar property that the OPE depends on the particular correlation function considered (through $s$).

\[4\] The graviton is only massless in $D = 26$ (2.13).
and turning on the other modes \[30\]. This is standard in string theory; we'll see later that while the form \(2.13\) is still correct for \(D = 2\), the physical picture is quite different.

For more than four particles, the \(s = 0\) amplitude \(2.10\) is given by the usual Shapiro – Virasoro integral representation [1]:

\[
A_{s=0}(k_1, \ldots, k_N) = \prod_{i=4}^{N} |z_i|^{2(k_1 \cdot k_i - \beta_1 \beta_i)} |1 - z_i|^{2(k_3 \cdot k_i - \beta_3 \beta_i)} \prod_{4=i<j}^{N} |z_i - z_j|^{2(k_i \cdot k_j - \beta_i \beta_j)} \tag{2.16}
\]

No closed expression for \(2.16\) is known in general. The basic problem in evaluating it is the complicated pole structure of \(A(k_1, \ldots, k_N)\). There are many channels in which poles appear; to analyze them quantitatively one has to consider the region of the moduli integrals in \(2.16\) where some number of \(z_i\) approach each other. For example, to analyze the limit \(z_4, z_5, \ldots, z_{n+2} \rightarrow 0\), it is convenient to redefine

\[
z_4 = \epsilon, \ z_5 = \epsilon y_5, \ldots, z_{n+2} = \epsilon y_{n+2} \tag{2.17}
\]

and consider the contribution of the region \(|\epsilon| << 1\) to \(2.16\). By simple algebra we find an infinite number of poles at \(E = \frac{Q^2}{2} + \sum \beta_i\), \(p = \sum k_i\) (sums over \(i\) run over \(i = 1, 4, 5, 6, \ldots, n + 2\)) satisfying \(E^2 - p^2 = \frac{2 - D}{12} + 2t\) as in \(2.13\). The residues of the poles are related to correlation functions of on shell intermediate string states. Indeed, by plugging \(2.17\) in \(2.16\) it is easy to find the residues explicitly; for the first pole, e.g., we find

\[
A_{s=0}(k_1, \ldots, k_N) \approx \frac{(T_{k_1}T_{k_4} \cdots T_{k_{n+2}})}{(\frac{Q^2}{2} + \sum \beta_i)^2 - (\sum k_i)^2 - \frac{2-D}{12}} \langle T_{\Sigma_i k_i} T_{k_2} T_{k_3} T_{k_{n+3}} \cdots T_{k_N} \rangle \tag{2.18}
\]

where \(\tilde{k} = -\sum k_i\). The generalization of \(2.18\) for the higher poles is straightforward. It is interesting that the amplitudes \(2.16\) have the standard space time interpretation for all \(D\). Poles correspond to on shell intermediate states. One can show decoupling of null states. The only special feature of \(D = 26\) is that in that dimension the vacuum that we are considering is Lorentz invariant. We will use \(2.16\) to study the dynamics of the theory.

Since the residues of the poles in \(2.18\) are in general non zero, we see that \(A(k_1, \ldots, k_N)\) has many poles in all possible channels (corresponding to different ways to cut the space time diagrams). This phenomenon is a reflection of the complicated back reaction in string theory; both the form of the space time equations of motion and their solutions are untractable. Thus in the next subsections we’ll turn to the situation in \(D = 2\) where things are much simpler (but still very interesting).
2.2. Two Dimensional String Theory and Minimal Models.

2.2.1. $d \leq 1$ matter theories.

In the rest of the section we’ll be mainly interested in the theory (2.1) with $D = d + 1 = 2$, which consists of two scalar fields $\phi, X^1 = X$. It will be convenient to generalize slightly by introducing a background charge for $X$:

$$S_M = \frac{1}{2\pi} \int \sqrt{g} \left[ \hat{g}^{ab} \partial_a X \partial_b X + \frac{i\alpha_0}{2} \hat{R} X \right]$$

Introducing $\alpha_0$ shifts the central charge of the matter sector to:

$$c = 1 - 12\alpha_0^2 \quad ; \quad \alpha_0 \in \mathbb{R}$$

and furthermore has the effect of changing the momentum conservation condition in (2.10) to $\sum_{i=1}^{N} k_i = 2\alpha_0(1 - h) (h - \text{genus})$. It is known that in such cases we must insert certain screening charges to make sense of the theory.

There are two main reasons to consider (2.19). First, this allows one to avoid considering zero momentum tachyons in the action: from (2.7) we see that the cosmological term in (1.2) has $E = 0$ at $D = 2$. We will encounter later subtleties at $E = 0$, thus it is convenient to shift $c$ as in (2.20), in which case we have in (2.4) $E = \beta + Q/2, p = k - \alpha_0$ and the on shell condition (2.7) with $m^2 = 0$. Following [17] we choose the solution with positive $E$ (see discussion in section 2.1):

$$\beta + \frac{Q}{2} = |k - \alpha_0|$$

We see that the tachyon is massless for all $\alpha_0$, but $k = 0$ does not correspond to zero momentum ($p = 0$) in general. Thus $\alpha_0$ is a kind of IR regulator. The second reason to study (2.19) is that for rational $\alpha_0^2$ one can restrict the spectrum of $k$’s to a finite set of degenerate Virasoro representations; this is the Feigin Fuchs construction [31], [32] of the BPZ minimal models [24].

The conformal primaries are represented by vertex operators $V_k = e^{ikX}$, with dimensions $\Delta_k = \frac{1}{2} k(k - 2\alpha_0)$. For the minimal models, to evaluate the flat space CFT correlation functions $\langle V_{k_1}...V_{k_N} \rangle$ one has to insert a number of screening operators of dimension 1, $V_{d_-}, V_{d_+}$, integrated over the world sheet; $d_\pm$ are the solutions of:

$$\frac{1}{2} d_\pm (d_\pm - 2\alpha_0) = 1,$$
Momentum conservation implies
\[ \sum_{i=1}^{N} k_i + md_- + nd_+ = 2\alpha_0 \]  
(2.22)

Although the structure for rational \( \alpha_0^2 \) is much richer than the generic one, it is easier to calculate correlators including screening at irrational \( \alpha_0^2 \) and to analytically continue them to rational \( \alpha_0^2 \) (see [32] for details). Furthermore, we will find it convenient to consider generic \( k \)'s (not only those corresponding to degenerate representations). In the application to \( c = 1 \) we are interested in correlators with \( n, m = 0 \) (and generic \( k \)). At the end of the calculation we should take \( \alpha_0 \to 0 \).

What is the space time picture corresponding to string theory with matter given by (2.19)? The action (1.1) takes in this case the form:
\[ S = \frac{1}{2\pi} \int \left[ \partial X \overline{\partial X} + \frac{i\alpha_0}{2} \hat{R}X + \lambda_+ \exp(i\hat{d}X) + \lambda_- \exp(i\hat{d}X) + \partial\phi \overline{\partial\phi} - \frac{Q}{4} \hat{R}\phi + \mu \exp(\alpha_+\phi) \right] \]  
(2.23)

Note the screening charges in the action. The \( X \) zero mode integration enforces (2.22). Naively (2.23) is related in a simple way to the \( d = 1 \) system: by redefining \( \tilde{\phi} = \frac{Q}{2\sqrt{2}} \phi - \frac{i\alpha_0}{\sqrt{2}} X; \tilde{X} = \frac{Q}{2\sqrt{2}} X + \frac{i\alpha_0}{\sqrt{2}} \phi \) we seem to find in terms of \( \tilde{\phi}, \tilde{X} \) a \( d = 1 \) string in a background given by (2.23) (expressed in terms of \( \tilde{\phi}, \tilde{X} \)). We will see later that this is not quite true, but qualitatively (2.23) still describes (before restricting to the minimal models) a solution to 2D critical string theory, and its physics is very similar to that of the \( \alpha_0 = 0 \) theory (see also [33]).

2.2.2. Three point correlators without screening.

We start by considering the simplest case of bulk correlators of three tachyons without screening (\( n = m = 0 \) in (2.22)), with \( s \) zero momentum tachyons (punctures). Here we follow closely [34]; this case will allow us to discuss some important general features of the theory in a relatively simple context, where the results of all the necessary integrals are known. One has to evaluate (2.10):
\[ A(k_1, k_2, k_3) = (-\pi)^3 \left( \frac{d}{\pi} \right)^s \Gamma(-s) \langle T_{k_1}(0) T_{k_2}(\infty) T_{k_3}(1) \left[ \int \exp(\alpha_+\phi) \right] \rangle^s \]  
(2.24)

where we have used \( SL(2,\mathbb{C}) \) invariance to fix the positions of the three tachyons and redefined the path integral by a factor of \( (-\pi)^3 \) for later convenience. The momenta are subject to the conservation laws:
\[ k_1 + k_2 + k_3 = 2\alpha_0 \]
\[ s\alpha_+ + |k_1 - \alpha_0| + |k_2 - \alpha_0| + |k_3 - \alpha_0| = \frac{Q}{2} \] (2.25)

With no loss of generality, we can take \( k_1 \geq \alpha_0, k_2 \geq \alpha_0 \) and \( k_3 \leq \alpha_0 \). The (bulk) amplitude (2.24) for integer \( s \) can be expressed in terms of known integrals [32] (we introduce the notation \( \Delta(x) \equiv \Gamma(x)/\Gamma(1-x) \)):

\[
\langle T_{k_1} T_{k_2} T_{k_3} \left( \int e^{\alpha_+ + \phi} \right) \rangle = \prod_{j=1}^{s} \int d^2 w_j |w_j|^{2\alpha} |1 - w_j|^{2\beta} \prod_{1 \leq i < j \leq s} |w_i - w_j|^{4\rho}
\]

\[
= (s!) (\pi \Delta(-\rho))^s \prod_{i=0}^{s-1} \Delta((i+1)\rho) \Delta(1 + \alpha + i\rho) \Delta(1 + \beta + i\rho) \Delta(-1 - \alpha - \beta - (s + i - 1)\rho)
\]

(2.26)

where we have performed the Wick contractions for the free fields \( X \) and \( \phi \) using the propagators \( \langle X(z)X(0) \rangle = \langle \phi(z)\phi(0) \rangle = -\log |z|^2 \), and:

\[
\alpha = -\alpha_+ \beta(k_1) ; \quad \beta = -\alpha_+ \beta(k_3) ; \quad \rho = -\frac{\alpha_+^2}{2}
\] (2.27)

The on shell kinematics (2.25) implies:

\[
\beta = \begin{cases} 
\rho(1 - s) & \alpha_0 > 0 \\
-1 - \rho s & \alpha_0 < 0,
\end{cases}
\] (2.28)

Plugging (2.28) in (2.26), (2.24) we get (for \( s \geq 1 \)):

\[
\alpha_0 > 0 : A(k_1, k_2, k_3) = 0
\]

\[
\alpha_0 < 0 : A(k_1, k_2, k_3) = -\pi \Delta(-s) [\mu \Delta(-\rho)]^s \prod_{i=1}^{2} (-\pi) \Delta(m_i)
\] (2.29)

where

\[
m_i = \frac{1}{2} \beta_i^2 - \frac{1}{2} k_i^2
\] (2.30)

As discussed above, the apparent infinity due to \( \Gamma(-s) \) is irrelevant at fixed area, and yields a logarithmic correction at fixed \( \mu \). In fact for \( \alpha_0 < 0 \), (2.28) implies that \( m_3 = -s \) so that we can rewrite \( A \) (2.29) as

\[
A(k_1, k_2, k_3) = (\mu \Delta(-\rho))^s \prod_{i=1}^{3} (-\pi) \Delta(m_i)
\] (2.31)

There are two puzzling features in (2.29):
1) We seem to find different results for the two signs of $\alpha_0$; but from (2.19), (2.20) it is clear that physics must be independent of this sign.

2) The $\alpha_0 \rightarrow 0 \ (c \rightarrow 1)$ limit is singular since $\Delta(-\rho) \rightarrow 0$.

The resolution of these puzzles is quite instructive. We will see later that (2.31) is the general tachyon three point function. Then it is clear that if nothing special happens, for integer $s = n$ we should have $A(k_1, k_2, k_3) = \mu^n F(k_1, k_2, k_3)$ with some finite $F$. But this is equivalent to a vanishing fixed area amplitude (see (2.12)). In order to have a non zero fixed area amplitude at $s \in Z_+$, $F_{s \rightarrow n}(k_1, k_2, k_3)$ must diverge. The only difference between positive and negative $\alpha_0$ is that for $\alpha_0 > 0$ all factors in (2.31) are finite (for generic $k$’s) while for $\alpha_0 < 0$, $\Delta(m_3) = \Delta(-s)$ supplies the necessary divergence. Eq. (2.29) is an example of a general phenomenon: we will see later that all bulk amplitudes vanish except those for which all $k_i - \alpha_0$ except one have the same sign. Since we chose $k_1, k_2 > \alpha_0$, $k_3 < \alpha_0$ and the $s$ punctures in (2.29) correspond to $k - \alpha_0 > 0$ when $\alpha_0 < 0$ (and vice versa), (2.29) is the natural result. We see that the apparent discrepancy between positive and negative $\alpha_0$ in (2.29) is due to the fact that we impose a “resonance” condition which is discontinuous. Of course, although both signs of $\alpha_0$ are ‘right’, it is more useful to consider $\alpha_0 < 0$. This is what we’ll do below.

The foregoing discussion seems to be at odds with our previous comments. We have argued that if the integral representation (2.10) diverges, we can not continue analytically because of the expected appearance of cuts in amplitudes. But of course for $\alpha_0 > 0$ the integral representation is always divergent; the simplest way to see that is to note that the integrand in (2.29) is positive definite while the integral (2.29) is 0. Shouldn’t we then discard the results in this case?

A useful analogy is critical string tree level scattering. In that case there is no range of momenta where the integral representation converges for massless/massive particles, since the integral representations for the different channels ($s, t, u$ for $N = 4$) converge in different, non overlapping kinematic regions, while the string world sheet integral includes all channels. In that case one splits the world sheet integral into several parts, calculates them at different momenta and analytically continues, using the space time picture as a guide to compute the divergent world sheet integral. Here we do the same. That’s why one can trust the divergent integral (2.29) for $\alpha_0 > 0$. Liouville momentum is conserved for bulk amplitudes; the key assumption is that there is a consistent space time interpretation.

So far we have considered the first puzzle mentioned above. What happens as we take $c \rightarrow 1$? From (2.31) we learn that the operator $\exp(\alpha_+ \phi) = \exp(-\frac{Q}{2} \phi) = \exp(-\sqrt{2} \phi)$
decouples in this case. Notice that its wave function $\Psi$ (2.6) is constant, thus not peaked at $\phi \to \infty$, and does not correspond to a local operator. Its decoupling is consistent with [17], [18]. However, there is another BRST invariant operator $\phi \exp(-\frac{Q}{2}\phi)$ which is a candidate to play the role of the cosmological term (it is known to be interesting [33]). Naively there seems to be a host of difficulties with this operator: it is not clear how to do the $\phi$ zero mode integral in (2.10); and also how to generalize the scaling arguments of [21] to obtain KPZ scaling, and the minisuperspace analysis of [36] to get the Wheeler de Witt equation (both KPZ scaling and the WdW equation are known to be valid at $c = 1$ from matrix models [4], [11], [36]). All these problems are bypassed by turning on a small $\alpha_0$, and considering the cosmological operator $V_c = \frac{1}{\Delta(-\rho)} \exp(\alpha_+ \phi)$. This operator has finite correlators as $c \to 1$. It is indeed equivalent to the previous one since for small $\epsilon = \alpha_+ + \sqrt{2}$, $V_c \simeq \frac{1}{c} \exp(-\sqrt{2}\phi) + \phi \exp(-\sqrt{2}\phi)$. The leading divergent term vanishes inside correlation functions, as remarked above. However, in terms of $V_c$ all the above properties are manifest for all $c$; the singularity at $c = 1$ has been absorbed into an infinite coupling constant renormalization (of $\mu$).

Our final result for the three point functions is (2.31). Remember that it was obtained only for $s \in \mathbb{Z}^+$ for the coefficient of $\mu^s \log \mu$ and is equivalent to (2.29). This completes step 1 in the program of section 2.1.

The amplitudes contain a product of “wave function renormalization” factors $-\pi \Delta(m_i)$ and it seems natural to define ‘renormalized’ operators

$$\tilde{T}_k = \frac{T_k}{(\pi)\Delta\left(\frac{1}{2}\beta^2 - \frac{1}{2}k^2\right)}$$

(2.32)

whose correlators are much simpler. Applying (2.32) in (2.31) and defining $\mu$ as the coefficient of $\tilde{T}_{k=0}$ (which also automatically implements the coupling constant renormalization discussed above, since $\tilde{T}_{k=0} = V_c$) we find:

$$\langle \tilde{T}_{k_1} \tilde{T}_{k_2} \tilde{T}_{k_3} \rangle = \mu^s$$

(2.33)

The second step now is to extend (2.33) to non integer $s$. To do that we must use space time intuition. The main point is that we find here and will see again for higher point functions that correlation functions of $\tilde{T}$ (2.32) are polynomial in momenta

$^5\mu$ is irrelevant: it can be either put to 1 by a shift in $\phi$ or absorbed into the definition (2.32) of $\tilde{T}$ and the path integral.
can calculate them (in the bulk). We would like to argue that this fact is an indication that tachyon dynamics can be described by an effective local two dimensional field theory obtained by integrating out the massive modes. This is not usually the case in string theory; beyond low energy approximations the light string states can not be described (even classically) by a local action. Tachyon amplitudes (e.g.) contain poles corresponding to all the massive modes of the string (2.14). If we integrate out the latter we find a highly non local action. In two dimensions the situation is better. The tachyon is the only field theoretic degree of freedom. It interacts with an infinite set of massive quantum mechanical degrees of freedom, which exist only at particular (discrete) momenta. This interaction is summarized by the normalization factors $\Delta(m_i)$ in (2.31): space time gravity (and in general inclusion of the discrete states) seems to have the mild effect of renormalizing the tachyon field. The renormalized tachyon $\tilde{T}$ is described by a 2D field theory. The fact that its bulk three point function (2.33) is one, and more generally that the bulk correlation functions obtained below are polynomial in momenta is compatible with this suggestion. Thus we are led to postulate that all correlators of $\tilde{T}$ must be polynomial in external momenta. This will allow us to fix them uniquely. E.g. for the three point function we conclude that (2.33) is the general result for all $s$ (since the only polynomial $P(k_i)$ which is 1 whenever $s \in \mathbb{Z}^+$ is $P(k_i) = 1$).

To recapitulate, two dimensional string theory has the striking property that it is described by two consistent S–matrices. The one familiar from critical string theory is that for $T_k$ (2.4), (2.8). It has poles corresponding to all on shell string states and is crucial for the issue of the role of space time gravity in the theory; we will return to it in section 4. However, in two dimensions the role of space time gravity is mild; the renormalized field $\tilde{T}$ is described by a second S–matrix, which follows from a two dimensional field theory action. In fact, the action giving the set of $\tilde{T}$ amplitudes is known from the matrix model approach [6], [7], [8]. In the rest of this section we will describe in detail this S–matrix. It is important to emphasize that despite the simple relation (2.32) the two S–matrices describe genuinely different physics. For example, the $\tilde{T}$ S–matrix does not have bulk scattering unlike that of $T$. In fact, there is nothing special about amplitudes with $s = 0$ at all in this picture. More importantly, gravitational physics is absent in $\tilde{T}$. The new feature in two dimensional string theory is that we seem to be able to turn off space time gravity!

\footnote{In principle there could be tachyon poles in amplitudes, but these would have shown up at integer $s$. We will see later that they turn into cuts, due to non conservation of energy.}
The third step in the general program of section 2.1, involving the extension of (2.33) to regions in $k_i$ where the integral representation diverges is trivial here – we do not expect anything non trivial, since the full effect of Liouville momentum non conservation is not felt in three point functions. It is nevertheless instructive to examine the region of convergence of (2.26) to make contact with the discussion of section 2.1. From integrability as $w_i \to 0, 1$ in (2.27) we find $\alpha + 1 > 0, \beta + 1 > 0$. In terms of the Liouville momenta (and introducing $\alpha_\pm = 2/\alpha_+$, such that $-Q = \alpha_+ + \alpha_-$ and for $\alpha_0 < 0, 2\alpha_0 = \alpha_- - \alpha_+$ ) this implies a restriction on the energies:

$$\beta(k_i) > \frac{\alpha_-}{2} = -\frac{Q}{2} - \frac{\alpha_+}{2} \quad (2.34)$$

Since $\alpha_+$ is negative (2.2) we learn that the integral representation for the correlators (2.26) only converges for states with $E > \frac{\alpha_+}{2}$ in agreement with the physical picture presented in section 2.1 and with the discussion of [17], [18]. Any state with $E > 0$ can be treated by continuing its correlation functions from $c \to -\infty$ ($\alpha_+ \to 0$). The necessity to analytically continue in $c$ (or $\alpha_+$) follows independently in our approach from the precise convergence conditions of (2.26), which are nicely expressed in terms of $m_i$ (2.30): by (2.25) $m_1, m_2$ satisfy $m_1 + m_2 = 1 + \rho s$, and the convergence conditions are $m_1, m_2 > 0$. This is only consistent if $-\rho < \frac{1}{s}$. Convergence of all $s$ amplitudes can only be achieved if $\rho \to 0$ ($c \to -\infty$). On the other hand for $E < 0$ (2.26) is always divergent and one needs additional space time physical input to understand this case. This divergence is presumably related to the fact that for $E < 0$ the corresponding perturbation of the surface is not small.

The three point functions for $c = 1$ ($D = 2$) string theory are thus given in complete generality by (2.31), (2.33) (a $\delta(\sum k_i - 2\alpha_0)$ is understood throughout). We will next consider the three point functions in minimal models, for which we will have to introduce the screening charges $n, m$ in (2.22).

2.2.3. Three point functions with screening (minimal models).

For minimal models, whose free field description was developed in [31], [32], we should include arbitrary numbers of screening charges $V_{d\pm}$ in the ‘matter’ amplitudes. As before, we will choose $k_1, k_2 > \alpha_0, k_3 < \alpha_0$, which is necessary here to ensure that two of the vertices are in one half of the Kac table, while the third is in the other half [32], and choose $\alpha_0 < 0$ for the same reasons as before. In this case we have $d_+ = -\alpha_+, d_- = \alpha_-$. We are interested in the result for $k$’s describing certain degenerate representations and
for rational $\alpha_+^2$, for which $s$ (2.11) is in general non integer. As before we will first tune $k_i, \alpha_+$ such that $s$ is integer, and calculate

$$A_{m,n}(k_1, k_2, k_3) = (-\pi)^3 (\mu \alpha)^s \Gamma(-s) \frac{\lambda^m}{m!} \prod_{i=1}^m \left( \prod_{j=1}^n \left( \int d^2 t_i \right) \prod_{a=1}^s \int d^2 w_a \right)
\langle T_{k_1}(0) T_{k_2}(\infty) T_{k_3}(1) \prod_{i=1}^m T_{\alpha_+}(t_i) \prod_{j=1}^n T_{-\alpha_+}(\tau_j) \prod_{a=1}^s T_0(w_a) \rangle$$

(2.35)

The $t_i, \tau_i$ integrals over the locations of the screenings give the matter correlation function; note the factors of $\frac{\lambda^n}{n!}, \frac{\lambda^m}{m!}$ coming from expanding the action (2.23). The $w_a$ integrals come from Liouville. The various 2D multiple integrals involved here have been computed [32]. Due to the conservation laws:

$$k_1 + k_2 + k_3 + m\alpha_- - n\alpha_+ = 2\alpha_0$$
$$|k_1 - \alpha_0| + |k_2 - \alpha_0| + |k_3 - \alpha_0| + s\alpha_+ = \frac{Q}{2}$$

(2.36)

one obtains (after some algebra):

$$A_{m,n}(k_1, k_2, k_3) = (\mu \Delta(-\rho))^s (-\pi \Delta(-\rho_+))^n (-\pi \Delta(-\rho_-))^m \prod_{i=1}^3 (-\pi \Delta(m_i))$$

(2.37)

where $\rho_\pm = \frac{\alpha_+^2}{2} = -m(\mp\alpha_\pm)$. The result (2.37) is very similar to the case $n = m = 0$ (2.31). By adjusting the coefficients $\lambda_\pm$ of the screening charges in (2.23) to $\lambda_\pm^{-1} = (-\pi) \Delta(-\rho_\pm)$ we can bring (2.37) to the form (2.31). This choice of $\lambda_\pm$ is necessary already in the flat space CFT [32]. It is also very natural from the point of view of (2.32): in terms of $\tilde{T}$ the amplitudes $A_{m,n}$ are given again by (2.33). The continuation to non integer $s$ proceeds now in the same way as for the case without screening, with the same conclusions.

Although (2.33) is our final result for the minimal model correlation functions (rational $\alpha_0^2$), there is a slight subtlety in its interpretation. In that case $k_i = \frac{1}{2}(1 - r_i) \alpha_- - \frac{1}{2}(1 - s_i) \alpha_+ + 1, 2, k_3 = \frac{1}{2}(1 + r_3) \alpha_- - \frac{1}{2}(1 + s_3) \alpha_+, \alpha_+^2 = \frac{2p}{p'}, p < p', r_ip' > s_ip$ and $1 \leq r_i \leq p - 1, 1 \leq s_i \leq p' - 1$. Naively, the only fusion rule for the minimal model three point functions (2.33) is (2.36), which is equivalent to

$$r_1 + r_2 \geq r_3 + 1, s_1 + s_2 \geq s_3 + 1$$

(2.38)
(and a certain $Z_2$ selection rule). Of course this can not be the whole story, since (2.38) is not symmetric under permutations of $(1, 2, 3)$. Even if we symmetrize, it seems that we have lost the truncation of the fusion rules (2.24) $r_1 + r_2 + r_3 \leq 2p - 1$, $s_1 + s_2 + s_3 \leq 2p' - 1$. This is of course not the case; the issue is the correct treatment of the flat space amplitudes. The usual way one gets the three point couplings there is by factorization of four point functions (2.32). It is known that the direct evaluation of the Feigin Fuchs integrals for the three point function (2.35) does not yield the same results; rather one has to symmetrize, by using the symmetry of flipping any two of the three vertices: $V_{r,s} \rightarrow V_{p-r, p'-s}$ (37). This symmetry must be manifest in all $N$ point functions, and, by construction, also after coupling to gravity. Thus we have to apply this symmetry to (2.38). The result is (34):

$$\langle \tilde{T}_{r_1,s_1} \tilde{T}_{r_2,s_2} \tilde{T}_{r_3,s_3} \rangle = \mu^s N_{(r_1,s_1),(r_2,s_2),(r_3,s_3)}$$

(2.39)

where $N_{(r_i,s_i)} \in (0,1)$ are the flat space fusion numbers. Eq. (2.39) is compatible with matrix model results (10) and generalizes them considerably. Similar results for a subset of three point functions were obtained in (38).

2.2.4. $N \geq 4$ point functions.

In the previous subsections we have obtained the three point function of the tachyon field. Remarkably, in two dimensional string theory one can calculate all $N$ point functions. The miraculous cancellations encountered above will be seen here to be due to an interesting structure of the bulk $N$ point functions. For reasons to be explained below we will restrict ourselves to $N$ point functions without screening charges $^7$, where the conservation laws take the form:

$$\sum_{i=1}^{N} k_i = 2\alpha_0$$

$$s\alpha_+ + \sum_{i=1}^{N} |k_i - \alpha_0| = (\frac{N}{2} - 1)Q$$

(2.40)

The correlator reads then:

$$A(k_1,..,k_N) = (-\pi)^3 \left( \frac{\alpha}{\pi} \right)^s \Gamma(-s) \prod_{a=1}^{s} \int d^2 w_a \prod_{i=4}^{N} \int d^2 z_i$$

$$\langle T_{k_1}(0)T_{k_2}(\infty)T_{k_3}(1) \prod_{a=1}^{s} T_{0}(w_a) \prod_{i=4}^{N} T_{k_i}(z_i) \rangle$$

(2.41)

$^7$ Although we believe the general case is not much harder. It is also interesting, e.g. for the study of factorization in 2D gravity coupled to minimal matter.
where we have fixed again the positions of three tachyons by $SL(2, \mathbb{C})$ invariance, and tuned $k_i, \alpha_0$ such that $s$ is integer. The free field correlator is:

$$
\langle T_{k_1} T_{k_2} T_{k_3} \prod_{i=4}^{N} T_{k_i} \right| \int T_0 \rangle = \prod_{a=1}^{s} \int d^2 w_a \hspace{1em} \prod_{i=4}^{N} d^2 z_i |w_a|^{2\delta_i} |1 - w_a|^{2\theta_i,j} \prod_{i < j} |z_i - z_j|^{2\theta_i,j} \prod_{i,a} |z_i - w_a|^{2\delta_i} \int d^2 w_a \frac{N}{s} \prod_{i=4}^{N} |w_a - w_b|^4 \rho
$$

\hspace{5em} (2.42)

where:

$$
\delta_i = -2\alpha_+ \beta(k_i) \hspace{1em} \theta_{i,j} = k_i k_j - \beta(k_i) \beta(k_j) \hspace{1em} \rho = -\frac{\alpha_+^2}{2} \hspace{1em} (2.43)
$$

Our experience from the previous cases suggests to study the $(N - 1, 1)$ kinematics:

$$
k_1, k_2, ..., k_{N-1} > \alpha_0 \hspace{1em} k_N < \alpha_0 < 0 \hspace{1em} (2.44)
$$

The conservation laws (2.40) lead to:

$$
k_N = \frac{N + s - 3}{2} \alpha_+ + \frac{\alpha_-}{2} \hspace{1em} (2.45)
$$

Anticipating the form of the result, we choose to parametrize the momenta by the variables $m_i = \frac{1}{2} (\beta(k_i)^2 - k_i^2)$, in terms of which:

$$
\delta_i = \rho - m_i \hspace{1em} i < N
$$
$$
\delta_N = -1 - (N + s - 3) \rho
$$
$$
\theta_{i,j} = -m_i - m_j \hspace{1em} i < j < N
$$
$$
\theta_{i,N} = -1 + (N + s - 3)m_i
$$

Now (2.42) does not look particularly simple. In fact, it is a special case of the $N$ point amplitudes (2.16), which are certainly complicated. As we saw in section 2.1, the main reason for the complications is the presence of poles (2.18) in all possible channels. We seem to have the same problem here: upon observation, (2.42) seems to have similar poles. The main difference between (2.42) and its higher dimensional analogues is that in two dimensions the residues of most of these poles vanish! Consider for example the $(1, 4)$ channel. The first pole occurs when $\theta_{1,4} = -1$ (compare to (2.14)). The residue of the pole

\hspace{5em} \footnote{All other $(n,m)$ kinematic regions with $n, m > 1$ give zero; this can be shown by similar techniques to those used below.}
(using (2.18)) involves the correlation function of an intermediate tachyon at \( k = k_1 + k_4 \) and \( \beta = \beta_1 + \beta_4 \). Plugging the on shell condition \( \beta_i = k_i + \alpha_+ \) into (2.43) we find

\[
k = \frac{\alpha}{2} - \alpha_+; \quad \beta = \frac{\alpha}{2} + \alpha_+ \tag{2.47}
\]

Near the pole (2.42) has the form (see (2.18)):

\[
\langle T_{k_1}...T_{k_N}\rangle \simeq \frac{1}{\theta_{1,4} + 1} \langle T_{k_1}\rangle \tag{2.48}
\]

Now we proceed inductively. Suppose we have shown that all \( M \) point functions with \( M \leq N - 1 \) satisfy

\[
\langle T_{k_1}...T_{k_M}\rangle = \prod_{i=1}^{M} \Delta(m_i) P(k_1, ..., k_M) \tag{2.49}
\]

with some polynomial in the momenta \( P(k_i) \). We will soon show the same for \( M = N \), but in the meantime we can use (2.49) to show that the residue (2.48) vanishes: \( k \) (2.47) satisfies \( m(k) = 2 \) and since \( \Delta(2) = 0 \), plugging (2.49) in (2.48) we indeed get zero for the residue of the pole at \( \theta_{1,4} = -1 \). In other words, by two dimensional kinematics the on shell tachyon is automatically at one of the (discrete) values of the momentum for which the “renormalization factor” \( \Delta(k) \) (2.32) vanishes. Therefore, the residue (2.48) is zero. This is of course markedly different from the situation in higher dimensions. One can argue similarly for the higher poles at \( \theta_{1,4} = -n, n \geq 2 \); for those we need a similar property of the discrete oscillator states which we will derive in section 4.

The general poles were discussed in section 2.1. It is easy to see that the residue (2.18) (with \( \bar{k} = 2\alpha_0 - \sum_i k_i \)) is almost always zero. For example, focussing on the (first) poles that occur when some of the \( z_i \to 0 \) we have two classes of poles:

1) A subset of \( z_i, \ i = 4, ..., N - 1 \) approach zero. In this case, \( T_{\Sigma, k_i} \) in (2.18) has the property that \( m(\Sigma, k_i) \) is a positive integer so that \( \Delta(m(\Sigma, k_i)) = 0 \) and the second term in the residue (2.18) vanishes (using the induction hypothesis (2.49)). The first term is finite, thus the residue is zero.

2) A subset of \( z_i, \ i = 4, ..., N - 1 \) and \( z_N \) approach zero. Here the second term in the residue (2.18) is finite but the first term vanishes, (again by (2.49)), except when the subset of \( z_i, \ i = 4, ..., N - 1 \) is empty. A similar structure occurs for the massive poles in all channels (see section 4).

We see that the phenomenon underlying the vanishing of the residues of the above poles is the special role of the states at the discrete momenta (\( \sqrt{2}k \in \mathbb{Z} \) for \( c = 1 \)). All

22
the on shell intermediate states in (2.42) occur at these momenta in the wrong branch ($\beta < -\frac{Q}{2}$). Their vanishing, advocated in [17] is therefore crucial for the simplicity of the amplitudes in 2D string theory. There seems to be a large symmetry relating these states to each other which underlies this.

Thus the only poles with non vanishing residues in (2.42) are those coming from $z_N$ approaching one of the other vertices; since $k_N$ is fixed (2.43), this implies that although the interpretation of the poles is the standard Veneziano one, they occur only as a function of individual external momenta $k_i$ (or equivalently $m_i$) and not more complicated kinematic invariants. I.e. the positions of poles in $m_1$, say, depends at most on $s$ and not on the other $m_i$. The poles from $z_N \to \infty$ depend on $m_2$, which is a function of $m_1, m_3, ..m_{N-1}$ through the kinematic relation

$$\sum_{i=1}^{N-1} m_i = 1 + \rho s$$

(2.50)

One could ask, why aren’t the residues of the poles in $m_i$ zero as well, since as for all other poles, they can be seen to involve discrete states in the wrong branch. The answer is that as we mentioned above, the decoupling of these states is only partial. In the presence of enough discrete states from the ‘right branch’ it no longer occurs. Indeed, the residue of the poles in $m_i$ involves three point functions of the form $\langle V^-(T_1 T_2) \rangle$ where $V^-$ is a discrete state in the wrong branch (see section 4). The point is that the tachyons $T_1, T_2$ are forced by kinematics to be at one of the discrete momenta in the ‘right branch’, hence the residue is in general non vanishing. We will return to the “competition” between $V^+$ and $V^-$ in section 4.

Where are these poles located? Naively, from (2.42) first order poles in $m_1$ e.g. seem to appear when $\theta_{1,N} = -l$ ($l = 1, 2, ..$). However, one can convince oneself that the residues always vanish except when $m_1 = -n$ (and $-l = \theta_{1,N} = -1 - (N + s - 3)n$ (2.46)). This can be shown either by noticing that the location of the poles is independent of $k_3, .., k_{N-1}$, so we can take them to zero and use (2.31), or by showing that only when $m_1 = -n$, does the intermediate state describe an on shell physical string state. The residue of the corresponding pole (2.18) is the correlation function of $T_{k_2}, .., T_{k_{N-1}}$ with one of the string states at level $l - 1 = n(N + s - 3)$. A simple consistency check is that on shell discrete states appear precisely when $l = -1 - (N + s - 3)n, n = 1, 2, 3, 4, ..$ (see section 4).

---

9 It is known [39] that in the minimal models ($c < 1$) there are no “discrete oscillator states” in addition to the “tachyon” (although there are other new states [31]). We find such oscillator
Similarly, from the region $z_N \to \infty$ we find poles at $m_2 = -n$ or expressing $m_2$ in terms of the other $m_i$ through (2.50), the location of the poles is $m_1 = 1 + \rho s - \sum_{i=3}^{N-1} m_i + n$.

Summarizing, if we consider $A(k_1, \ldots, k_N)$ as a function of $k_1$, the pole structure consists of first order poles at

$$m_1 = -n \quad n = 0, 1, 2, 3, \ldots$$

$$m_1 = \rho s + 1 - \sum_{i=3}^{N-1} m_i + n \quad n = 0, 1, 2, 3, \ldots$$  \hspace{1cm} (2.51)

In view of the result (2.49) we are trying to prove, it is natural to consider

$$f(m_1, m_3, \ldots m_{N-1}) = \frac{\langle T_{k_1} \ldots T_{k_N} \rangle}{\prod_{i=1}^{N} \Delta(m_i)}$$  \hspace{1cm} (2.52)

We now know (2.51) that all the poles of the numerator on the r.h.s. of (2.52) are matched by similar poles of the denominator. Thus if $f$ (2.52) is to have any poles, they must come from zeroes of the denominator, which are not matched by similar zeroes of the numerator. Of course, the denominator has simple zeroes at $m_i = l$ ($l = 1, 2, 3, \ldots$). We will next show that the correlator (2.41) also vanishes for these values of the momenta. The simplest way to see that is to use (2.49) recursively. We know that $A(k_1, \ldots, k_N)$ has the form $A = \prod \Gamma(m_i) g(k_1, \ldots, k_N)$. We have to prove that $g = 0$ whenever $m_i = l$. To do that we can use the OPE of, say, $T_{k_{N-1}}$ and $T_{k_N}$ and focus on the residue of the pole at $m_{N-1} = -n$ which is given by an $N - 1$ point function, which vanishes for $m_1 = l$ by (2.49).

From another point of view, by the standard DDK argument [21], the vanishing of $T_{\alpha_o}$ (which is the first case $l = 1$ where we want $A$ to vanish) is equivalent to KPZ scaling of correlation functions involving the operator $\phi \exp(-Q^2 \phi + i\alpha_0 X)$. Thus it is good news that $T_{\alpha_o}$ vanishes. To show vanishing of $T^{(l)} \equiv T(m_k = l)$ for $l > 1$ given vanishing of states in intermediate channels here at generic $\alpha_0$ because we couple to gravity the Feigin Fuchs model [31], [32]. The momentum $k$ is continuous and there are discrete states as in the $c = 1$ model. These states can be obtained by rotating the $c = 1$ spectrum, as discussed below eq. (2.28). A peculiar feature of these states is that they do not have the form $V_m e^{\beta \phi}$, but rather depend non trivially on $\phi, X$. This is one of the indications that these models should not be taken seriously, except as rotated $c = 1$ [33].

\[10\] In $D > 2$ this is no longer the case: the operator $\exp(ik \cdot X - \frac{Q^2}{2} \phi)$ (with appropriate $k^2$) does not vanish in (2.13); KPZ scaling breaks down, and the structure of the theory is more complicated.
\( T^{(1)} = T_{\alpha_0} \), one can use the fact that \( T^{(l)} = \frac{T_{\alpha}}{T} \) recursively. We leave the details of this argument to the reader.

This concludes the proof of vanishing of \( A(k_1, \ldots, k_N) \) whenever \( m_i = l(= 1, 2, 3, \ldots) \). Returning to (2.52), we have shown that \( f \) is an entire function of the \( m_i \). To completely fix it we show that it is bounded as \( |m_i| \to \infty \). Consider, e.g. the \( m_1 \) dependence. For large \( |m_1| \), (2.42) is dominated by \( z_i, w_a \simeq 1 \). To blow up this region we redefine \( z_i = \exp(\frac{y}{m_1}) \) and \( w_a = \exp(\frac{w_a}{m_1}) \) and estimate (2.42). We find \( f(m_1, \ldots) \to \text{const.} \) Thus considered as a function of \( m_1 \), \( f \) is analytic everywhere and bounded at \( \infty \). Therefore, it is independent of \( m_1 \). Repeating the argument for the other \( m_i \) (or by symmetry) \( f \) is independent of all \( m_i \).

It may only depend on \( N, s \). But since it is independent of \( k_i \), we can set \( k_3, \ldots, k_{N-1} = 0 \) keeping \( s, N \) fixed, and calculate \( f(s, N) \) from (2.26). Plugging the result back in (2.52) we finally find the \( N \) point function of the renormalized fields \( \tilde{T} \) (2.32):

\[
\langle \tilde{T}_{k_1} \ldots \tilde{T}_{k_N} \rangle = (\partial_{\mu})^{N-3} \mu^{s+N-3} \quad (2.53)
\]

Notice that (2.53) has the form (2.49) as promised. As explained above, for \( s \not\in \mathbb{Z}_+ \) we still get (2.53), assuming as before that the exact result is a polynomial in momenta \( k_i \).

Eq. (2.53) completes the first two steps of the procedure described in section 2.1. Its region of validity is tied to the region of convergence of (2.42). One can show that the latter converges whenever \( m_i > 0, i = 1, \ldots, N - 1 \) (with the relation (2.50)). In the next subsection we will describe the correct continuation of (2.53) to all momenta and find an interesting kinematic structure.

Finally, (2.53) can be compared to matrix model results. We do that in Appendix A and find agreement between the different approaches.

### 2.2.5. The analytic structure of the \( N \) point functions.

So far we have treated the amplitudes in non critical string theory using critical string techniques. We have found that the Shapiro – Virasoro amplitudes (2.16), which are defined for arbitrary \( D \), are actually calculable for \( D = 2 \) due to simplifications in the dynamics of the theory. This involved two elements: we have used the fact that Liouville momentum is conserved in the bulk, and continued the amplitudes analytically from the region where they converge. In particular, in the process we have ignored the requirement \( \beta > \frac{Q}{2} (E > 0) \) discussed above. We have shown that the amplitudes thus obtained have a standard space time interpretation, although there are interesting symmetries special to two dimensions, which make them simple. The set of bulk \((s = 0)\) amplitudes defines
therefore a consistent S–matrix in the sense of critical string theory. We have also begun in previous subsections to extend the result to non integer \( s \), obtaining (2.53). Since no direct methods to evaluate such amplitudes are available, we had to invoke a space time principle; the assumption that the \( \tilde{T} \) amplitudes are described by a local two dimensional field theory, and are polynomial in momenta. As it turns out, this assumption rules out a naive analytic continuation of (2.53) to all momenta. Our next task is to understand the general structure of the correlators by deriving constraints which Liouville theory places on the form of this space time field theory and propose the general correlators.

The basic property we will use is that when \( s \not\in \mathbb{Z}_+ \) the Liouville interaction is crucial, as explained in section 2.1, and momentum is not conserved; we have:

\[
\exp(\beta_1 \phi) \exp(\beta_2 \phi) = \int d\beta \exp(\beta \phi) f(\beta, \beta_1, \beta_2)
\]  

(2.54)

\( f \) is an OPE coefficient. We did not specify the contour of integration over \( \beta \) in (2.54) since it is not known. In [17] it has been argued (based on space time considerations) that the amplitudes should be defined by summing over macroscopic states \( \beta = -\frac{Q}{2} + ip, p \in \mathbb{R} \).

We will adopt this procedure here.

Consider\(^{11}\) the region of the moduli integrals in a generic tachyon amplitude (2.8) where, say, \( T_{k_1}(z) \rightarrow T_{k_2}(0) \). The contribution of the region \( z \rightarrow 0 \) to the amplitude is given by [17]:

\[
\int_{|z|<\epsilon} d^2 z \int_{-\infty}^{\infty} dp (z \bar{z})^\frac{1}{2} p^2 + \frac{1}{2} (k_1 + k_2 - \alpha_0)^2 - 1 f(p, \beta_1, \beta_2) \quad (2.55)
\]

Assuming that we may interchange the order of integration over \( z, p \) we obtain\(^{12}\):

\[
\int d^2 z \langle T_{k_1}(z) T_{k_2}(0) \ldots \rangle \simeq \int_{-\infty}^{\infty} dp \frac{f(p, \beta_1, \beta_2)}{p^2 + (k_1 + k_2 - \alpha_0)^2} \langle T_{k_1+k_2} \ldots \rangle \quad (2.56)
\]

Now for fixed \( p \), (2.56) has the familiar form from critical string theory; we find a pole corresponding to the intermediate state \( T_{k_1+k_2} \). The fact that Liouville momentum is not conserved and we have to sum over all \( p \)'s may turn this pole into a cut: (2.56) depends on \(|k_1 + k_2 - \alpha_0|\). Thus we expect cuts whenever some of the momenta \( \{k_i\} \) in (2.8) satisfy\(^{13}\)

\(^{11}\) We thank N. Seiberg for discussions on this issue.

\(^{12}\) More generally, if the matter sector OPE produces an intermediate state of dimension \( \Delta \), we have: \( \int \frac{df(p, \beta_1, \beta_2)}{p^2 + 2(\Delta - \frac{c}{24})} \langle V_\Delta \ldots \rangle \). The only singularities occur at \( E = \sqrt{2(\Delta - \frac{c}{24})} \rightarrow 0 \).

\(^{13}\) Note incidentally that the integral representation always diverges before any intermediate momentum gets to \( \alpha_0 \): if e.g. \( \sum_{i=1}^{n} k_i = \alpha_0, \sum_{i=1}^{n} m_i = 1 - \rho(n-1) \). Using (2.50) we find that \( \sum_{i=n+1}^{N} m_i = \rho(s + n - 1) < 0 \). But the integral representation converges iff all \( m_i > 0 \). Thus the integral representation is not useful to study the behaviour near the cuts.
\[ \sum_i k_i \equiv p \rightarrow \alpha_0. \] How can we make this more precise? A major clue comes from comparing an amplitude with an insertion of a puncture \( P = T_{k=0} \) to the amplitude without it. By KPZ scaling (2.10), (2.11) we have:

\[
\langle P T_{k_1} \ldots T_{k_N} \rangle = \left[ -\frac{\alpha}{2} \sum_{i=1}^{N} |k_i - \alpha_0| - \left( \frac{N}{2} - 1 \right)(1 + \frac{\alpha^2}{2}) \right] \langle T_{k_1} \ldots T_{k_N} \rangle \tag{2.57}
\]

Thinking of (2.57) as a relation between tree amplitudes in the purported space time field theory reveals its essential features: we can insert the puncture \( T_{k=0} \) into the tree amplitude \( \langle T_{k_1} \ldots T_{k_N} \rangle \) either by attaching it to one of the \( N \) external legs, thus adding an internal propagator of momentum \( k_i + 0 = k_i \) or inside the diagram. The first term (the sum) on the r.h.s. of (2.57) corresponds to the first possibility; we can read off the propagator \( -\frac{\alpha}{2} |k - \alpha_0| \). The second term corresponds to the second possibility, and our remaining goal is to make it too more explicit.

The propagator we find is related to the two point function as usual; it should be proportional to the inverse two point function (recall that the correlation functions (2.8) have the external propagators stripped). Indeed, by integrating (2.33) once (first putting \( k_2 = 0 \)), we find:

\[
\langle \tilde{T}_k \tilde{T}_{2\alpha_0-k} \rangle = -\frac{1}{\alpha_- |k - \alpha_0|} \tag{2.58}
\]

so that the propagator in (2.57) is \( \frac{1}{2} (\langle \tilde{T}_k \tilde{T}_{2\alpha_0-k} \rangle)^{-1} \). Now that we understand the propagator, the only remaining problem is the specification of the vertices in the space time field theory. The three point vertex is 1 by (2.33). To find the higher irreducible vertices we have to use the world sheet – space time correspondence. Consider, for example, the four point function

\[
\tilde{A}(k_1, \ldots, k_4) = \int d^2 z \langle \tilde{T}_{k_1}(0) \tilde{T}_{k_2}(\infty) \tilde{T}_{k_3}(1) \tilde{T}_{k_4}(z) \rangle \tag{2.59}
\]

To integrate out the massive string states we separate the \( z \) integral in (2.59) into two pieces. One is a sum of three contributions of intermediate tachyons from the regions \( z \rightarrow 0, 1, \infty \). By (2.55), (2.56) we expect to get \( -\frac{\alpha}{2} |k_1 + k_i - \alpha_0| \) from those. The rest of the \( z \) integral is the contribution of massive states; it gives a new irreducible four particle interaction (which we will denote by \( A_1^{(4)} \)) for the tachyons. The crucial observation that allows one to calculate this term is that the contribution of the massive modes is analytic in external momenta. This can be understood from several different points of view; from Liouville, (2.55), (2.56) imply that only intermediate states with \( E \rightarrow 0 \) cause
non analyticities of the amplitudes. But the massive states occur only at discrete momenta (and energies) which are never close to zero.

This observation allows us to calculate $A_{1PI}^{(4)}$; we write

$$A(k_1, ..., k_4) = -\frac{1}{2} \alpha_-(|k_1 + k_2 - \alpha_0| + |k_1 + k_3 - \alpha_0| + |k_1 + k_4 - \alpha_0|) + A_{1PI}^{(4)}$$

(2.60)

and now use the fact that we actually know $\tilde{A}(k_1, ..., k_4)$ whenever, say, $k_1, k_2, k_3 > 0, k_4 < \alpha_0$. In that kinematic region we can compare the result (2.53) with (2.60) and find

$$A_{1PI}^{(4)} = -\frac{1}{2} (1 + \frac{\alpha^2}{2})$$

(2.61)

But now, for $A_{1PI}^{(4)}$ we know that we can use analytic continuation through the zero energy cuts, since by general arguments it must be analytic in $k_i$. Of course this immediately implies that (2.61) is the correct irreducible four tachyon interaction everywhere. This concludes the derivation of the tachyon four point function (2.59). A few comments about (2.60), (2.61) are in order:

1) The irreducible vertices for three and four tachyons were found to be constant. This is not general. We will soon see that for $N \geq 5$ $A_{1PI}^{(N)}$ is a highly non trivial (analytic) function of the momenta.

2) For $c = 1$ (2.60) agrees with matrix model results [11], [12].

3) It is interesting to consider the cuts (2.60) in the case of the bulk amplitudes $s = 0$ (since then the Liouville momentum is conserved). For $d = c = 1$ ($\alpha_0 = 0$) the only non zero amplitudes are those with (e.g.) $k_1, k_2, k_3 > 0, k_4 < 0$. We can never pass through $k_i + k_j = 0$ because of kinematics. Therefore the cuts (2.60) are invisible in the bulk. This is no longer the case for $c \neq 1$. There we have $k_1, k_2, k_3 > \alpha_0, k_4 < \alpha_0 < 0$ and (e.g.) $k_1 + k_2 = \alpha_0$ is not on the boundary of this region. What is the interpretation of the cuts then? We no longer have (2.54)– the Liouville momentum is conserved in the bulk. However, as explained above, the integral representation starts diverging before we get to $k_1 + k_2 = \alpha_0$ (from $k_1 + k_2 > \alpha_0$). This is crucial for consistency; we learn that when the integral representation diverges we shouldn’t use the naive continuation but rather use the space time field theory as a guide, a point of view emphasized above.

4) We can now come back to the relation between the $\alpha_0 \neq 0$ model and the two dimensional string mentioned below (2.23). We see (2.60) that even for $s = 0$ where naively the amplitudes in the two cases are related by a rotation, this is not the case; the region $k_i > \alpha_0$ is transformed to $\tilde{k}_i > 0$ but the amplitudes (2.61) do not transform accordingly.

28
When the integral representations diverge they are defined in a different way in the two cases. However we see that both situations are described by essentially the same two dimensional field theory in space time.

5) Another curious feature of the $c < 1 \ (\alpha_0 \neq 0)$ models is that the screening charges $V_{d_\pm}$ in (2.23) are not treated on the same footing as the tachyon, despite the fact that they are naively tachyon vertices of momenta $d_\pm$. To see that one can compare the three point functions with screenings to the $N$ point functions without screenings. For example, comparing (2.37) with $n = 1, m = 0$ to the four point function (2.60) with one of the momenta equal to $d_+$ we find that in general the two differ. Again, this is consistent, since the screening charges lie outside of the region of convergence of the integral representation (2.42), however the full implications of this observation are unclear to us. These complications are also the reason why $N \geq 4$ point functions with screening are harder to obtain.

It is now clear how to proceed in the case of $N$ point functions. We assume that we know already $A^{(4)}_{1P1},.., A^{(N-1)}_{1P1}$. Then we write all possible tree graphs with $N$ external legs, propagator $-\frac{\alpha}{2} |k - \alpha_0|$ and vertices $A^{(n)}_{1P1} \ (n \leq N - 1)$ and add an unknown new irreducible vertex $A^{(N)}_{1P1}(k_1,..,k_N)$. The interpretation in terms of integrating out massive states is as before. $A^{(N)}_{1P1}$ is again analytic in $\{k_i\}$ and we can fix it by comparing the sum of exchange amplitudes (reducible graphs) and $A^{(N)}_{1P1}$ to the full answer (2.53) in the appropriate kinematic region (2.44). This fixes $A^{(N)}_{1P1}$ in the above kinematic region. Then we use analyticity of $A^{(N)}_{1P1}$ to fix it everywhere. The outcome of this process is the determination of the amplitudes in all kinematic regions given their values in one kinematic region.

In principle, the procedure we have given above can be implemented to find $A^{(N)}_{1P1}$, in very much the same fashion as we have derived $A^{(4)}_{1P1}$ above. However, it is more convenient to use a different technique, which we will describe next.

2.2.6. Irreducible $N$ point functions.

We are faced with a kind of “inverse problem”: given the set of amplitudes $\langle \tilde{T}_{k_1}..\tilde{T}_{k_N} \rangle$ (2.53) in the kinematic region $k_1,..,k_{N-1} > 0, k_N < 0$, find the set of irreducible vertices which together with the propagator $\frac{|k|}{\sqrt{2}}$ give these amplitudes in the appropriate kinematic region.

---

We will restrict ourselves to the case $c = 1$ in this subsection.
region. It is important that the vertices are analytic in \( \{ k_i \} \). It is very useful to Legendre transform: the generating functional \( G(j) \) for connected Green's functions has the form

\[
e^{-G(j)} = \int \mathcal{D} \psi e^{-S(\psi) + \int j \psi}
\]

where the action \( S \) is given by

\[
S(\psi) = -\sum_{n=2}^{\infty} \frac{1}{n!} \int dk_1..dk_n \psi(k_1)\psi(k_n)\delta(k_1 + .. + k_n)A^{(n)}_{1PI}(k_1, .., k_n)
\]

(2.63)

and \( A^{(2)}_{1PI} = -\frac{\sqrt{2}}{|k|} \). At tree level the function \( G(j) \) reads

\[
G(j) = -\sum_{n=2}^{\infty} \frac{1}{n!} \int dk_1..dk_n j(k_1)\ldots j(k_n)\delta(k_1 + .. + k_n)\langle \psi(k_1)\ldots\psi(k_n) \rangle_c
\]

(2.64)

The connected correlators \( \langle \psi(k_1)\ldots\psi(k_n) \rangle_c \) are equal to \( \tilde{A}(k_1, .., k_n) = \langle \tilde{T}_{k_1}\ldots\tilde{T}_{k_n} \rangle \) up to insertion of external propagators, which appear in the former and are stripped off in the latter. It is natural to redefine \( j(k) \rightarrow \frac{\sqrt{2}j(k)}{|k|} \) on the r.h.s. of (2.64) after which

\[
G(j) = -\sum_{n=2}^{\infty} \frac{1}{n!} \int dk_1..dk_n j(k_1)\ldots j(k_n)\delta(k_1 + .. + k_n)\tilde{A}(k_1, .., k_n)
\]

(2.65)

It is well known that the saddle approximation in (2.62) is exact at tree level; therefore

\[
-G(j) = -S(\psi) + \int \frac{\sqrt{2}}{|k|} j \psi \frac{\sqrt{2}}{|k|} j = S'(\psi)
\]

(2.66)

By duality of the Legendre transform we also have

\[
-S(\psi) = -G(j) - \int \frac{\sqrt{2}}{|k|} j \psi \frac{\sqrt{2}}{|k|} \psi = -G'(j),
\]

(2.67)

which implies that \( S(\psi) \) is the generating functional for connected tree level Green's functions arising from the action (2.63). In other words, the irreducible amplitudes \( A^{(n)}_{1PI} \) in (2.63) play now the role of amplitudes generated by Feynman rules with propagator of opposite sign \(- \frac{1}{|k|} \sqrt{2} \) and the full amplitudes \( \tilde{A}(k_1, .., k_N) \) playing the role of vertices. To use our knowledge of \( \tilde{A} \) in the kinematic region \( k_1, .., k_{N-1} > 0 \) we can now calculate these “dual” amplitudes in that region of momentum space. Of course, we must first verify that if the external momenta lie in this kinematic region, then for all internal vertices in all
possible Feynman diagrams there are precisely \( n - 1 \) positive and 1 negative incoming momenta (since otherwise the “dual vertices” are unknown). One can easily convince oneself that this is the case. After calculating \( A^{(N)}_{1PI} \) from these Feynman rules, we can continue them analytically to all \( k \) using their analyticity.

The problem of calculating \( A^{(N)}_{1PI} \) has been reduced to evaluation of tree level Feynman diagrams. The general expressions are complicated; we discuss them in Appendix B. Here we will illustrate the kind of results one gets by giving two typical examples:

\[
A^{(N)}_{1PI}(k_1, k_2, k_3, k_4 = 0, \ldots, k_N = 0) = (\partial_\mu)^{N-3} \left\{ \frac{1}{\mu} \prod_{i=1}^{3} \frac{1}{\cosh(\frac{k_i}{\sqrt{2}} \log \mu)} \right\} \bigg|_{\mu=1} \tag{2.68}
\]

\[
A^{(N)}_{1PI}(k_1, k_2, k_3, k_4, 0, \ldots, 0) = (\partial_\mu)^{N-4} \mu^{-2} \left( \prod_{i=1}^{4} \frac{1}{\cosh(\frac{k_i}{\sqrt{2}} \log \mu)} \right) \left( -1 - \mu \partial_\mu \log \prod_{1 \leq i < j \leq 3} \cosh(\frac{k_i + k_j}{\sqrt{2}} \log \mu) \right) \bigg|_{\mu=1} \tag{2.69}
\]

Notice that as expected, the irreducible amplitudes (2.68), (2.69) are analytic in \( \{k_i\} \). A general proof of this statement is given in Appendix B. The discussion in this subsection is closely related to the structure at \( k = 0 \) discussed in [10].

### 3. Two Dimensional Fermionic String Theory.

#### 3.1. The model.

We will not repeat the general considerations of section 2.1 for the fermionic case as they are quite similar. Instead, we will turn directly to the situation in two dimensions which is the case of interest to us here.

The matter system is in this case one superfield

\[
X = x + \theta \psi_x + \bar{\theta} \bar{\psi}_x + i \theta \bar{\theta} F_x \tag{3.1}
\]

which we want to couple to supergravity. As in the bosonic case it is convenient to generalize by turning on a background charge for \( x \), which is therefore governed by the action (2.19). The fermions \( \bar{\psi}_x (\psi_x) \) are free, left (right) moving. Similarly, we have a Liouville superfield

\[
\Phi = \phi + \theta \psi + \bar{\theta} \bar{\psi} + i \theta \bar{\theta} F_l \tag{3.2}
\]
related to the conformal factor of the metric and the gravitino field in superconformal
gauge. Φ is governed by the action \[14\]:

\[
S_{SL} = \frac{1}{2\pi} \int d^2z \int d^2\theta \left[ D\Phi \bar{D}\Phi + 2\mu \exp(\alpha_+\Phi) \right]
\] (3.3)

where \( D = \partial_\theta + \theta \partial_z \), and we have dropped curvature couplings \[12\]. The central charge
of \( X \) \[3.1\] is \( \hat{c} = \frac{2}{3}c = 1 - 8\alpha_0^2 \) and that of Φ, \( \hat{c}_{SL} = 1 + 2Q^2 \), where \[11\]:

\[
Q = \sqrt{\frac{9}{2} - \frac{\hat{c}}{2}} ; \quad \alpha_+ = -\frac{Q}{2} + |\alpha_0| \quad (3.4)
\]

Since we are making a non chiral GSO projection, we have two sectors in the theory: (NS, NS) and (R, R) \[1\]. The (NS, NS) sector contains one field theoretic degree of freedom, the “tachyon” center of mass of the string, whose vertex operator is given by

\[
T_k = \int d^2\theta \exp(ikX + \beta\Phi) ; \quad \beta + \frac{Q}{2} = |k - \alpha_0| \quad (3.5)
\]

Bulk correlation functions of \[3.5\] are calculated by integrating over the locations of \( N - 3 \) of the \( T_k \), and inserting two of the remaining vertices as lower components:

\[
A(k_1..., k_N) = (-\pi)^3 \int d^2\theta_1 \prod_{j=4}^N \int d^2z_j \int d^2\theta_1 \langle \exp(ik_1X(0) + \beta_1\Phi(0)) \exp(ik_2x(\infty) + \beta_2\phi(\infty)) \exp(ik_3x(1) + \beta_3\phi(1)) \exp(ik_jX(z_j) + \beta\Phi(z_j)) \rangle
\] (3.6)

As before, the cosmological term in the action \[3.3\] is the zero momentum tachyon. This
presents the following subtlety. We can write \( T_k \) in components as:

\[
T_k = \exp(ikx + \beta\phi) \left[ (ik\psi_x + \beta\psi)(ik\bar{\psi}_x + \beta\bar{\psi}) + i\beta F - kF_x \right] \quad (3.7)
\]

The auxiliary fields \( F_x, F \) have delta function propagators (in the free theory \[3.3\]); this
can cause divergences of the form \( \delta^2(z)|z|^a \) in the OPE of the fields \( T_k \) \[3.7\]. This is a
familiar issue in fermionic string theory \[12\]; we have two possible ways to proceed:
1) Calculate everything at generic momenta. In this case we can set the auxiliary fields
\( F = 0 \), since we can continue analytically from a region in momentum space where the
contact terms do not contribute.
2) If we must calculate at some given momentum, we have to carefully regulate the
divergences in a way compatible with world sheet supersymmetry (SUSY). In particular we
must keep \( F \) \[12\].
The second procedure is in general difficult to implement, especially in the presence of Ramond fields. Therefore, we will use the first one. Note that in this case we will not be able to perform the generalization of (2.24) here.\footnote{Indeed, we are not aware of the existence of (analogous) calculations for the Feigin Fuchs representations of the supersymmetric minimal models \cite{43}.}

The Ramond (R, R) sector gives rise to another (massless) field theoretic degree of freedom, whose vertex operator can be constructed using \cite{42}. First, we bosonize the fermions $\psi_x, \psi$ as:

\[
\psi = \frac{1}{\sqrt{2}}(e^{i h} + e^{-i h}); \quad \psi_x = \frac{1}{\sqrt{2}}(ie^{i h} - ie^{-i h})
\]

where $\langle h(z)h(w) \rangle = - \log|z - w|$, and similar expressions hold for the left movers (which we will suppress below). The R vertex is given by

\[
V_{-\frac{1}{2}} = \exp\left(-\frac{1}{2}\sigma + \frac{i}{2}\epsilon h + \epsilon k x + \beta \phi\right); \quad \beta = -\frac{Q}{2} + |k - \alpha_0| \quad (3.9)
\]

$V_{-\frac{1}{2}}$ is the fermion vertex in the “$-\frac{1}{2}$ picture”. There is an infinite number of versions of $V$ in different pictures (see \cite{42}). $\sigma$ in (3.9) is the bosonized ghost current and $\epsilon = \pm 1$. The mass shell condition for $\beta$ in (3.9) does not ensure BRST invariance in this case. Imposing invariance w.r.t. the susy BRST charge, $Q_{\text{susy}} = \oint \gamma T_F$ with $T_F = \psi_x \partial x + \psi \partial \phi + Q \partial \psi - 2i \alpha_0 \partial \psi_x$, we find

\[
\beta + \frac{Q}{2} = -\epsilon(k - \alpha_0) \quad (3.10)
\]

This is the two dimensional Dirac equation in space time. Correlation functions involving Ramond fields are constructed using standard rules \cite{42}. Defining $T_k^{(-1)} = \exp(-\sigma + i k x + \beta \phi)$, correlation functions with two (R, R) fields have the general form

\[
A_{2V}(k_1, \ldots, k_N) = \langle V_{-\frac{1}{2}} V_{-\frac{1}{2}} T^{(-1)}_{k_1} T_{k_2} \ldots T_{k_N} \rangle \quad (3.11)
\]

those with four (R, R) fields\footnote{Only correlators with an even number of Ramond fields can be non zero due to a $Z_2$ symmetry.}

\[
A_{4V}(k_1, \ldots, k_N) = \langle V_{-\frac{1}{2}} V_{-\frac{1}{2}} V_{-\frac{1}{2}} V_{-\frac{1}{2}} T_{k_1} \ldots T_{k_N} \rangle \quad (3.12)
\]

where $N - 3$ of the vertices are always integrated. For correlators with more than four (R, R) fields we need $V_{+\frac{1}{2}}$; we will not consider those here, but give its form for completeness:

\[
V_{\frac{1}{2}} = (2\epsilon k + Q) \exp\left(\frac{\sigma}{2} + \frac{3\epsilon}{2} i h + ik x + \beta \phi\right) + (\partial \phi - i \epsilon \partial x + 2 \alpha_0 - \epsilon Q) \exp\left(\frac{\sigma}{2} - \frac{\epsilon}{2} i h + ik x + \beta \phi\right) \quad (3.13)
\]
3.2. The massless S – Matrix.

Most of the features of the discussion of the wave function (2.6), the $\phi$ zero mode integration (2.10) and its space time interpretation, can be borrowed for the supersymmetric case. The only modification of (2.10) needed is replacing bosonic correlators by fermionic ones (replacing fields by superfields (3.1), (3.2), moduli by supermoduli, etc) as well as adding the new (R, R) field $V$. Since, as explained above, we are forced to analytically continue in momenta in order to ignore contact terms, we concentrate below on the case $s = 0$ in (2.10) (which is in any case the most general bulk amplitude). In the next two subsections we first consider the S – matrix of the tachyon $T$ and, then that of the Ramond field $V$.

3.2.1. Tachyon scattering in fermionic 2D string theory.

It is useful to start with (3.10) for the case $N = 4$ (and $s = 0$); putting $F = F_x = 0$ in (3.7) we find:

$$A_{s=0}(k_1,..,k_4) = \pi^4 \prod_{i=1}^{3} \frac{\Gamma(k_4 \cdot k_i - \beta_4 \beta_i + 1)}{\Gamma(\beta_4 \beta_i - k_4 \cdot k_i)}$$

(3.14)

This formula, which is superficially identical to (2.13), is of course true (as there) for all values of the dimension of space time. The poles reflect again the presence of massive string states, which in two dimensions are restricted to special momenta ($k \in \mathbb{Z}$). To study the simplifications in $D = 2$, we use (3.5) and find that:

1) In the “(2,2)” kinematics $k_1, k_2 > \alpha_0$, $k_3, k_4 < \alpha_0$, the amplitude (3.14) vanishes. This seems peculiar, since we expect poles with finite residues in the $s, t, u$ channels (as in (2.18)). However, the poles in the (say) $u$ channel are absent because the intermediate momentum is fixed by kinematics, while those in the $s, t$ channel cancel among themselves (precisely as in the bosonic case).

2) For (3,1) kinematics, $k_1, k_2, k_3 > \alpha_0$, $k_4 < \alpha_0$ (or vice versa): $A_{s=0}(k_1,..,k_4) = \prod_{i=1}^{3} \Delta(\tilde{m}_i)$, where

$$\tilde{m}_i = \frac{1}{2} \beta^2_i - \frac{1}{2} k^2_i + \frac{1}{2}$$

(3.15)

In fact, since in this case kinematics forces $\tilde{m}_4 = 0$, we can, as in (2.31), absorb the log $\mu$ into an infinite factor in the amplitude and write:

$$A_{s=0}(k_1,..,k_4) = \prod_{i=1}^{4} (-\pi) \Delta(\tilde{m}_i)$$

(3.16)
Now, eq. (3.16) is equivalent to (3.14) in all kinematic regions (recall that a finite \( A \) (3.16) is interpreted as zero in the bulk – we need a pole to produce the log \( \mu \) implicit in (3.14)). The form of (3.16) is suggestive (compare e.g. to (2.31)). We recognize many of the familiar features from the bosonic case; e.g. the first zero at \( \tilde{m}_i = 1 \) occurs at \( \beta = -\frac{Q}{2} \) (zero energy) and has a similar interpretation. The poles at \( \tilde{m}_i = 0, -1, -2, \ldots \) occur (for \( \hat{c} = 1 \)) at \( |k| = 1, 2, 3, \ldots \), which is again the set of momenta where oscillator states exist (see section 4). Our next goal is to show that the simple structure of (3.16) persists for higher point functions.

Thus we return to the \( N \) point function (3.6) with \( s = 0 \). It is clear from the discussion of the four point function above that the interesting kinematics to consider is \((N - 1, 1)\) (the rest will vanish identically). We choose it to be the same as in the bosonic case (2.44); other regions can be treated similarly. Energy/momentum conservation leads to

\[
k_N = \frac{N-3}{2} \alpha_+ + \frac{1}{2} \alpha_- \quad \text{(here we defined } \alpha_- \equiv \frac{1}{\alpha_+})
\]

or by (3.15), \( \tilde{m}_N = -\frac{1}{2} (N - 4) \). We expect to get the bulk divergence from an infinity of \( \Gamma(\tilde{m}_N) \), which happens only for even \( N \). This is consistent with (3.6): due to the (global) \( Z_2 \) \( R \) – symmetry \( \psi \to -\psi, \bar{\psi} \to \bar{\psi} \), (3.6) is indeed zero identically\(^{17}\) for odd \( N \). Therefore, we replace \( N \to 2N \) in (3.6) and proceed. We have constructed the arguments in section 2 in such a way that the generalizations are trivial. First one has to show that the residues of most of the apparent poles in (3.6) as groups of \( \{z_i\} \) get close, vanish. These residues have to do as before (2.18) with correlators involving physical states at the discrete momenta \( k \in \mathbb{Z} \) and in the wrong branch. Therefore we have to show decoupling of such states; this works precisely as in the bosonic case (see section 4). Assuming that, we have again only poles coming from \( z_{2N} \) approaching other \( z_i \). Their locations are easily verified to be \( \tilde{m}_i = -l \) \( (l = 0, -1, \ldots) \) corresponding to intermediate states of mass \( m^2 = (2l + 1)(2N - 3) \); only odd masses appear due to the \( Z_2 \) \( R \) – symmetry mentioned above \((\psi \to -\psi)\) under which the tachyon and all other states with even \( m^2 \) are odd.

We define, in analogy with (2.52),

\[
\tilde{f}(\tilde{m}_1, \tilde{m}_3, \ldots, \tilde{m}_{2N-1}) = \frac{A(k_1, \ldots, k_{2N})}{\prod_{i=1}^{2N} \Delta(\tilde{m}_i)} \quad (3.17)
\]

All the poles of the numerator \( A \) are matched by poles of the denominator; it is again necessary to show that \( A \) vanishes whenever (say) \( \tilde{m}_1 = 1, 2, 3, \ldots \). This is the case for

\(^{17}\) To avoid misunderstanding, we emphasize that this does not necessarily mean that correlators of an odd number of tachyons vanish, but only that they vanish in the bulk.
we can proceed recursively as in the bosonic case, or use a symmetry argument relating vanishing of $T(\tilde{m}_1 = l)$ to that of $T(\tilde{m}_1 = l + 1)$ (see discussion after (2.52)). Therefore, $\tilde{f}$ (3.17) is an entire function of $\tilde{m}_i$. One can also show in complete parallel with the bosonic case that $\tilde{f}$ is bounded as $|\tilde{m}_1| \to \infty$ (say). To do this we redefine $z_i = e^{\xi_i}$ in (3.6) and (after some algebra) find that $\tilde{f} \to \text{const}$ as $\tilde{m}_1 \to \infty$. Since an entire function which is bounded at infinity is constant, we conclude that $\tilde{f}$ depends at most on $N$ and the central charge.

This concludes the evaluation of the bulk tachyon amplitudes; the final result is (3.17); $A(k_1, \ldots, k_{2N})$ is proportional to a product of “leg factors” up to a function $\tilde{f}$ of $N$, $\hat{c}$. In the bosonic case we could fix the function $f$ (2.52), the analog of $\tilde{f}$, by using (2.31). This is not available to us here, but we can still determine $\tilde{f}$ by a space time argument analogous to the one made in the bosonic case.

The point is that regardless of whether we know $\tilde{f}(N)$ or not, we have to perform now steps 2,3 of the general program of section 2.1. We again make the assumption (which is plausible, but was not derived neither in the bosonic case nor here) that the massless amplitudes are governed by a 2D field theory (which now has two fields), and furthermore that correlators in this theory are algebraic in momenta. Eq. (3.17) (with $\tilde{f} = \tilde{f}(N, \hat{c})$) is a highly non trivial check of this idea. Using the above assumption, we can find $\tilde{f}$ by calculating the two point function $\langle T_k T_{2\alpha_0-k} \rangle$ for all $k$. The two point function is (up to an unimportant constant) the inverse propagator, which we can obtain by using KPZ scaling as in (2.57). Repeating the same argument here we find the propagator $-\alpha_\perp |k-\alpha_0|$. Thus the two point function (in a convenient normalization) is $\langle T_k T_{2\alpha_0-k} \rangle = -\frac{1}{2\alpha_\perp |k-\alpha_0|}$. This translates in (3.17) to

$$\tilde{f} = (-\pi)^{2N}(2N-3)!$$

The constant can be determined by comparing to (3.16). It would be nice to verify this result directly by computing $\tilde{f}(N)$ from the integrals (3.6) (for $N = 2$ we have checked this form above (3.16)).

As in the bosonic case, we can now obtain the general $N$ point functions (any $s$). In fact, redefining

$$\tilde{T}_k = \frac{T_k}{(-\pi)\Delta(\frac{1}{2}\beta(k))^2 - \frac{1}{2}k^2 + \frac{1}{2}}$$

(3.19)
we find that $\tilde{T}_k$ scattering is described by the same $S$-matrix as that of the bosonic tachyon \((2.32)\). Some examples:

\[
\langle \tilde{T}_{k_1} \tilde{T}_{k_2} \tilde{T}_{k_3} \rangle = 1
\]
\[
\langle \tilde{T}_{k_1} \tilde{T}_{k_2} \tilde{T}_{k_3} \tilde{T}_{k_4} \rangle = -\alpha_-(|k_1 + k_2 - \alpha_0| + k_1 + k_3 - \alpha_0) + |k_1 + k_4 - \alpha_0| - \frac{1}{2}(1 + \alpha_-^2)
\]  
(3.20)

etc. The cuts at $k_i + k_j = \alpha_0$ correspond to intermediate tachyons, as in the bosonic theory. Eq. \((3.20)\) coincides with \((2.60), (2.61)\) after making the identification $k_{\text{fermionic}} = \frac{1}{\sqrt{2}} k_{\text{bosonic}}, (\alpha_-, \alpha_+, \alpha_0)_{\text{fermionic}} = \frac{1}{\sqrt{2}} (\alpha_-, \alpha_+, \alpha_0)_{\text{bosonic}}$. The only difference is in the external leg factors \((2.32), (3.19)\) reflecting a different spectrum of oscillator states. This is reminiscent of earlier ideas \([45]\) relating bosonic and fermionic strings in two dimensional space time (although clearly one needs much more information for a complete comparison of the two theories). In the next subsection, we will study one aspect of the fermionic theory which certainly has no counterpart in the bosonic one: the dynamics of the Ramond field $V$.

3.2.2. Scattering of the Ramond field

We follow again the same steps as for the tachyon field $T_k$. First we consider four point functions. In order to have a non zero bulk four point function of two $R$ fields and two tachyons, we must choose both $R$ particles to move in the same direction, say to the right $k > \alpha_0$. Then the amplitude \((3.11)\) can be evaluated to give:

\[
A = \frac{\pi^4(\beta_4^2 - k_4^2)}{\Gamma(k_1 k_4 - \beta_1 \beta_4 + \frac{1}{2}) \Gamma(k_2 k_4 - \beta_2 \beta_4 + \frac{1}{2}) \Gamma(k_3 k_4 - \beta_3 \beta_4 + 1)}
\]
\[
\frac{\Gamma(k_1 k_4 - \beta_1 \beta_4 - k_1 k_4 + \frac{1}{2}) \Gamma(k_2 k_4 - \beta_2 \beta_4 - k_2 k_4 + \frac{1}{2}) \Gamma(k_3 k_4 - k_3 k_4 + 1)}
\]  
(3.21)

If both tachyons move left $k_3, k_4 < \alpha_0$ ((2,2) kinematics), \((3.21)\) vanishes, while if the signature is (3,1) we find again \((3.16)\) with one modification; $m_i$ has the form \((3.15)\) for the NS particles ($i = 3, 4$) while for the Ramond field $V$:

\[
m_i = \frac{1}{2}(\beta_i^2 - k_i^2)
\]  
(3.22)

In complete parallel with the previous cases, \((3.16)\) can now be verified to describe all bulk four point functions involving an arbitrary combination of $R$ and NS fields (provided the

\[\text{Note that in the interacting theory, the symmetry } \psi \rightarrow -\psi \text{ is broken by the interaction in } (3.3); \text{ therefore, although the tachyon is odd under this symmetry, we do have a non zero tachyon three point function, tachyon intermediate states in the four point functions, etc.}\]
correct $\tilde{m}_i$ (3.13), (3.22) are used). One has to remember that in fermionic string theory in addition to the trivial momentum conservation $\delta(\sum_i k_i - 2\alpha_0)$, which is implied in all amplitudes, we also have a $Z_2$ selection rule: a Kronecker $\delta$ of the number of R fields modulo two: only correlation functions with an even number of $V$’s can be non zero. A non trivial check of (3.16) is the four R field scattering: according to (3.16) we should get zero identically in the bulk. This can be verified directly by computing the integrals.

The form (3.16), (3.22) of Ramond scattering has the following interesting feature: the zero energy ($k = \alpha_0$) states (3.9) do not decouple, unlike the case of the tachyon (3.13), despite the fact that their wave function (2.6) is not peaked at $\phi \rightarrow \infty$. We saw in section 2 (see discussion following eq. (2.52)) that one way to understand the decoupling of the zero energy tachyon is KPZ scaling. At $k = \alpha_0$ there is an additional BRST invariant tachyon state $\phi \exp(-\frac{Q}{2}\phi + i\alpha_0 X)$; KPZ scaling of its correlation functions is equivalent to vanishing of the operator $\exp(-\frac{Q}{2}\phi + i\alpha_0 X)$. That argument goes through in the supersymmetric case: the operator $\phi \exp(-\sigma - \frac{Q}{2}\phi + i\alpha_0 x)$ is BRST invariant, therefore $T_{\alpha_0}$ (3.3) must decouple. In the Ramond sector on the other hand, the operator with an insertion of $\phi$ at $\beta = -\frac{Q}{2}$ is not BRST invariant, as is easy to verify. Therefore, $V_{k=\alpha_0}$ need not (and does not) vanish. One can also understand the difference between the situation between the NS and R sectors from a different point of view\(^\text{19}\). The exact wave functions of the various states satisfy the WdW equation \([36]\). In the NS sector, the form of this equation is such that if as $\phi \rightarrow \infty$, $\Psi(\phi) \rightarrow \text{const}$, then in the IR, ($\phi \rightarrow -\infty$), $\Psi(\phi)$ blows up. This means that the operator $\exp(-\sigma - \frac{Q}{2}\phi + i\alpha_0 x)$ behaves like the operators with $E < 0$ ($\beta < -\frac{Q}{2}$, see section 2) and should decouple. In the Ramond sector, the form of the WdW equation allows a zero energy solution which is constant at large $\phi$, decays at $\phi \rightarrow -\infty$, and is normalizable. Thus in this case the zero energy state behaves like the macroscopic states \([17]\) and need not decouple.

We now turn to $N$ point functions (3.11), (3.12). All the steps are as in the previous two cases, so we will be brief. The main issue is the analysis of poles and zeroes. This is performed precisely as before: the residues of most of the poles vanish by using (3.16) recursively (as well as properties of the discrete states). The only poles occur at $\tilde{m}_i \in Z_-$ (with the notation (3.15) (NS), (3.22) (R)) and correspond to on shell intermediate states. The zeroes are also treated as before; we leave the details to the reader. We find again that $\tilde{f}$ (3.17) is an analytic function of momenta ($\tilde{m}_i$); in a by now standard fashion we also

\(^{19}\) We thank N. Seiberg for this argument.
show that it is bounded as $|\tilde{m}_i| \to \infty$, hence it is independent of the $\{\tilde{m}_i\}$. To determine $\tilde{f}$ we use space time arguments, as for the tachyon. KPZ scaling (2.11) allows us to read off the propagator for the Ramond field, $-\alpha_-|k - \alpha_0|$, and consequently the two point function $\langle V_k V_{2\alpha_0 - k} \rangle = -\frac{1}{2\alpha_- |k - \alpha_0|}$. This fixes $\tilde{f}$ to be the same as before (3.18).

We now have all the correlation functions involving Ramond fields (we actually checked those involving up to four R fields, but showed how to obtain all of them, and conjecture that the results are going to agree as well). For example, after absorbing the external leg factors as in (3.19) (and for the R field as well), we have:

$$\langle V_{k_1} V_{k_2} T_{k_3} \rangle = 1$$
$$\langle V_{k_1} V_{k_2} T_{k_3} T_{k_4} \rangle = -\alpha_- (|k_1 + k_2 - \alpha_0| + |k_1 + k_3 - \alpha_0| + |k_1 + k_4 - \alpha_0|) - \frac{1}{2}(1 + \alpha_-^2)$$
$$\langle V_{k_1} V_{k_2} V_{k_3} V_{k_4} \rangle = \langle V_{k_1} V_{k_2} T_{k_3} T_{k_4} \rangle$$

(3.23)

The space time interpretation is as before. The cuts correspond to massless intermediate states (with $VV \to T$, $VT \to V$, $TT \to T$), and the contact terms to a new irreducible interaction.

3.3. Chiral GSO projection.

In fermionic 2D string theory we have the option to make a chiral GSO projection [24]. For $D > 2$ this is useful to construct stable (tachyon free) string theories with space time fermions. In $D = 2$ there are no tachyons, but one may still make the projection. This is useful as a toy model for higher dimensional (non) critical superstrings. We will briefly review the construction of [25] in $D = 2$ and discuss some of the emerging properties.

We start with the observation [25] that the 2D fermionic string system, which consists of two superfields (3.1), (3.2) has a natural global $N = 2$ superconformal symmetry. The $U(1)$ generator, which connects the two supercurrents is $J(z) = i\partial h + 2i\partial x$ (the cosmological term in (3.3) breaks this symmetry). There is a well known procedure in the critical string implementing the GSO projection in the presence of such a symmetry [48], which we imitate here. We define

$$I(z) = \exp\left(-\frac{1}{2} \sigma(z) - \frac{i}{2} h(z) + ix(z)\right)$$

(3.24)

---

20 This subsection is based on [46] (see also [47]). We will put $\alpha_0 = 0$ for simplicity.
$I(z)$ is a holomorphic operator ($\bar{\partial}I = 0$). Note that it is BRST invariant (3.10). We now project out all operators (3.3), (3.9) etc, which do not have a local OPE with $I(z)$ (3.24). This removes some states from the existing (NS, NS), (R, R) Hilbert spaces. By acting on the remaining states with $I(z)$ we generate two new sectors, (R, NS) and (NS, R), which contain space time fermions. Geometrically, the chiral GSO projection corresponds to enlarging the gauge group on the world sheet by a certain $Z_2$ R – symmetry [46]. The operator
\[ Q = \oint I(z); \quad Q^2 = 0 \] (3.25)
generates target SUSY. Due to the low dimension (and lack of time translation invariance) the SUSY generator $Q$ is a kind of BRST operator (in higher dimensions one would find a “space SUSY” algebra in the transverse directions [25]). How does the spectrum look after the projection in $D = 2$? It is convenient to analyze it chirally:

**NS sector**: Requiring locality of the ‘tachyon’ (3.5) with $I(z)$ (3.24), we find that only $T_k$ with $k \in \mathbb{Z} + \frac{1}{2}$ survive. In addition we have all the discrete states with odd $m^2$, starting with $\partial x$.

**R sector**: Imposing locality of (3.9) with (3.24) we find two solutions: a) $\epsilon = -1$, $k \in \mathbb{Z}_+$, b) $\epsilon = +1$, $0 > k \in \mathbb{Z} + \frac{1}{2}$.

The cosmological constant operator $T_{k=0}$ (3.3) has been projected out of the spectrum; it is very natural [25] to set the scale with $T_{\frac{1}{2}}$, which preserves the $N = 2$ symmetry. If we add it to the action with coefficient $\mu$, all the operators left in the theory have the interesting property that their correlation functions scale as integer powers of $\mu$. This is very reminiscent of the topological theory of $c = -2$ matter coupled to ordinary (bosonic) gravity [49]. Superficially there are problems with a topological interpretation of our theory: by using (3.17), (3.22) for the Ramond correlators we see that for half of the R states (those with $\epsilon = +1$), most correlation functions blow up. Also, the fact that only integer powers of $\mu$ appear in correlation functions is spoiled by addition of $\partial x \bar{\partial} x$ to the action (the scaling dimensions change continuously with the radius). Despite these problems, there probably is a topological theory here. The point is that we have not used the BRST like properties of the operator $Q$ (3.25). In [49], the topological theory had in addition to the usual string BRST another gauged topological symmetry. Perhaps we should add $Q$ to our $N = 1$ superconformal BRST charge. Doing that, requiring that
\[ Q|_{\text{phys}} = 0 \] (3.26)
we find that both problems mentioned above disappear. $\partial x \bar{x}$ is removed from the spectrum, as are all R operators with $\epsilon = +1$ and the NS operators with $k < 0$. We are left with the operators

$$T_n = \exp(-\sigma + i(n + \frac{1}{2})x + (n - \frac{1}{2})\phi); \quad V_n = \exp(-\frac{\sigma}{2} - \frac{i}{2}h + inx + (n - 1)\phi); \quad n = 0, 1, 2, ..$$

(3.27)

The correlation functions are now all bulk, and we have to divide them by $\log \mu$. Assuming that the correct prescription to calculate $N$ point functions is by inserting $N - 1$ operators (3.27) and one conjugate operator (with $k < 0$), the amplitudes are very simple to obtain from the discussion above. At $\mu = 0$ we have, e.g. (after redefining the operators as usual):

$$\langle T_{n_1}..T_{n_N} \rangle = (N - 3)!$$

(3.28)

It is amusing that after restricting to (3.26), all space time fermions are projected out of the spectrum. The reason is that the Liouville momentum must satisfy $p_{\text{left}} = p_{\text{right}}$, which is only possible (3.27) in the (NS, NS) and (R,R) sectors. We don’t know whether this observation is more general. This theory deserves a more detailed examination. Finally we would like to mention that the conjecture that the model we are discussing is topological is due to E. Martinec [50].

4. Oscillator states and gravitational degrees of freedom.

In the previous sections we have shown that the simple scattering pattern in two dimensional string theory is related to decoupling of the string states at certain discrete momenta. We start this section by reviewing their form and then discuss some of their properties. For simplicity, we restrict to $c = 1$ ($\alpha_0 = 0$).

At values of the momenta $\sqrt{2}k \in Z$, the Virasoro representations degenerate. Hence the spectrum is richer [21], [23], [24]. Parametrizing $k = \frac{r_1 - r_2}{\sqrt{2}}$ ($r_1, r_2 \in Z_+$), we have physical states of the form:

$$V_{r_1, r_2}^{(\pm)} = [\partial^{r_1 r_2} X + ...] \exp \left( i \frac{r_1 - r_2}{\sqrt{2}} X + \beta^{(\pm)}_{r_1, r_2} \phi \right)$$

(4.1)

at level $\frac{1}{2}m^2 = r_1 r_2$. $\beta_{r_1, r_2}$ can take as usual [21] two values: $\beta^{(\pm)}_{r_1, r_2} = -\sqrt{2} \pm \frac{r_1 + r_2}{\sqrt{2}}$. In section 2 we used the fact that $V_{r_1, r_2}^{(-)}$ decouple in correlation functions of tachyons. More precisely, all bulk correlation functions of the form $\langle V_{r_1, r_2}^{(-)} T_{k_1}..T_{k_N} \rangle$ where $k_1, .., k_{N-1} > 0$
are generic and \( k_N < 0 \), vanish. Here we will sketch the proof of this statement. It is in fact more convenient to prove vanishing of correlators containing any number of \( V(−) \) and tachyons of generic momenta(\( \sqrt{2} k \notin Z \)):

\[
\langle V_{r_1, r_2}^{(−)} ... V_{r_{2n−1}, r_{2n}}^{(−)} T_{k_1} ... T_{k_{N−n}} \rangle = 0 \tag{4.2}
\]

inductively in \( N(\geq 4) \). First one has to check this for \( N = 4 \): consider \( (k_1, k_2 > 0, k_3 < 0) \):

\[
\int d^2 z \langle V_{r_1, r_2}^{(−)}(0) T_{k_3}(z) \rangle
\]

By plugging in the kinematics, one may easily check that the result is a sum of integrals of the form \( \int d^2 z z^n \bar{z}^m (1 − z)^\alpha (1 − \bar{z})^\beta \) where \( n, m \in \mathbb{Z}_+ \) and \( \alpha, \beta \notin \mathbb{Z} \) (for generic \( k_1, k_2 \)). These integrals vanish by the standard analytic continuation. Hence, \( \langle V^{(−)}TTT \rangle = 0 \).

Similarly one checks that \( \langle V^{(−)}V^{(−)}T \rangle = 0 \) as well. Now suppose we have shown (4.2) for all \( N < N_0 \); we want to prove it for \( N = N_0 \). The strategy involves as before examining the poles of the integral representation of (4.2). The residues of the poles can be checked by a short calculation to be given by lower point functions of the form (4.2) again, which vanish by hypothesis. Therefore, the \( N = N_0 \) point function (4.2) has the property that it has no poles as a function of the tachyon momenta \( k_i \). As before, one can also estimate the large \( k \) behaviour, and find that this (entire) function of \( k_i \) vanishes at infinity (for a range of values of the other \( \{k_i \} \)). Hence, it is zero everywhere (4.2). This concludes the proof of decoupling of \( V_{r_1, r_2}^{(−)} \).

We would like next to make several comments about this result:

1) Decoupling of states with \( \beta < −\frac{Q}{2} \) was advocated in [17], from the point of view of 2D gravity. Our results, while probably related, are not identical: we proved a statement about bulk amplitudes, where the Liouville wall, which plays a major role in the considerations of [17], is irrelevant; we used an analytic continuation of the amplitudes (as a very useful technical tool), which as we saw above is not valid for generic Liouville amplitudes. Also, the decoupling we find is not complete: if enough of the tachyons in (4.2) are at the discrete momenta \( \sqrt{2} k \in Z \) (in the right branch), the amplitudes need not vanish; and tachyons of generic \( \beta < −\frac{Q}{2} \) do not decouple.

2) The dynamics of \( V_{r_1, r_2}^{(−)} \) becomes crucial in the 2D black hole solution of [20]. It was shown in [26] that this theory is identical to the \( c = 1 \) model described here with the cosmological term replaced by \( \mu V_{1,1}^{(−)} \). \( \mu \) is related to the mass of the black hole.
3) In a recent paper \[53\] it was shown that the \(c = 1\) matrix model possesses a large symmetry algebra, closely related to the discrete states \(V_{r_1,r_2}\). Here, on the other hand, we have seen that the simplicity of the amplitudes is directly due to the decoupling of \(V_{r_1,r_2}^{-}\).

The two observations should be related. A symmetry\[21\] would explain e.g. why decoupling of all \(V_{r_1,r_2}^{-}\) is implied by that of \(V_{r_1,0}^{-}, V_{0,r_2}^{-}\) (tachyons at special momenta).

As explained in section 2, the poles that do appear in the final answer for the \(N\) point functions, correspond to intermediate states in the \((i,N)\) channel. We can now check which of the states \(V^{(+)}_{r_1,r_2}\) appear in this channel. Straightforward algebra leads to the conclusion that the pole of \(\Gamma(m_i)\) \[2.32\] at \(m_i = -r_1\) corresponds to the intermediate state \(V^{(+)}_{r_1,r_2} = N - 3\). Thus, for given \(N\) we see in intermediate channels all states with \(m^2 = 2r_1(N - 3)\), as noted in section 2. As we vary \(N\), we find contributions of all physical states. The reason why only intermediate states with fixed \(r_2\) appear for fixed \(N\) is actually purely kinematical: \(\langle T_{k_1}..T_{k_M} V^{(+)}_{r_1,r_2}\rangle\) with all \(k_i > 0\) (which arise as residues of poles in \[2.41\], see \[2.18\]) can only be non zero if \(r_2 = M - 1\) (by momentum conservation and the resonance condition).

We see that the bulk \(S\) – matrix for the tachyon field describes reasonable space time physics. The (massless scalar) tachyon field couples to an infinite set of massive higher spin fields, which are essentially pure gauge (except at particular momenta). The gauge symmetry of string theory corresponding to decoupling of BRST commutators is responsible for the restriction of the (on shell) massive fields to discrete momenta. However, the simplicity of the results \[2.32\], \[2.53\] is due in addition to decoupling of half of the remaining states \(V_{r_1,r_2}^{-}\), which is not explained by these symmetries. This implies a further simplification in the dynamics, and in particular is responsible for the fact that the poles in the \(S\) – matrix occur as a function of external momenta alone\[22\]. The solvability of the matrix model is probably closely related to this phenomenon. One of the most interesting remaining problems is the realization and implications of this “new symmetry” on the space time equations of motion in two dimensional string theory. It appears that the discrete momenta must play a special role in the space time action. There are several properties of the results, which point to this, all essentially related to the decoupling of \(V_{r_1,r_2}^{-}\) (which

\[21\] The standard matter \(SU(2)\) can not be used since it is not a symmetry for generic radius (e.g. \(R = \infty\)).

\[22\] We have mainly discussed the \(S\) – matrix for \((N - 1,1)\) kinematics, but one can easily see that the same decoupling of the ‘wrong branch’ discrete states leads to vanishing of the bulk amplitudes in all other kinematic regions.

43
we emphasize is not automatically related to the fact that by gauge invariance massive physical states occur only at the above discrete momenta). In particular, applying the logic of [30] to our situation, it seems that a tachyon background $T_k$ \textsuperscript{(2.4)} which satisfies the linearized equations of motion of the string (is marginal), also solves the exact non-linear equations of motion (is truly marginal), as long as $\sqrt{2}k \not\in \mathbb{Z}$. This would imply that gravitational back reaction is only possible for discrete momenta $k$ (this is not a field redefinition invariant statement, nevertheless, if true, it would be important).

Thus, it is important to understand the dynamics of the operators $V^{(+)}_{r_1,r_2}$. The scattering formulae of section 2 diverge as the tachyon momentum $k \rightarrow n/\sqrt{2}$, due to the divergence of the ‘leg factors’ \textsuperscript{(2.32)}. One way to interpret this divergence is to note that by KPZ scaling and \textsuperscript{(2.32)}, an insertion of $T_k$ into a correlator multiplies it by

$$\Omega(k) = \frac{\Gamma(1 - \sqrt{2}|k|)}{\Gamma(\sqrt{2}|k|)} \frac{\mu^{\frac{k}{\sqrt{2}}}}{\mu} - 1$$ \quad (4.3)

As $k \rightarrow \frac{n+1}{\sqrt{2}}$, we can interpret the divergence of \textsuperscript{(4.3)} as a scaling violation:

$$\Omega_n \simeq (-)^n \frac{\mu^{\frac{n}{2}}}{n!} \log \mu$$ \quad (4.4)

Indeed, the bulk correlation functions considered above have precisely one insertion of $\log \mu$ corresponding to the unique discrete momentum tachyon. In general, if more than one momentum goes to $n/\sqrt{2}$, there are higher powers of $\log \mu$; of course such powers of $\log \mu$ can occur for any $s$.

In fact, one can convince oneself that the appearance of powers of $\log \mu$ is a generic property of all $V^{(+)}_{r_1,r_2}$. In particular bulk amplitudes this can be easily verified by factorization of tachyon bulk amplitudes in appropriate channels. Hence we have in general:

$$\langle V^{(+)}_{r_1,r_2} \cdots V^{(+)}_{r_{2n-1},r_{2n}} T_{k_1} \cdots T_{k_{N-n}} \rangle \propto (\log \mu)^n$$

(for generic $k_1,..k_{N-n}$). One can derive the equivalent of \textsuperscript{(4.4)} for all $V^{(+)}_{r_1,r_2}$; we will not do that here. Similarly, a natural way to interpret the vanishing of amplitudes involving $V^{(-)}_{r_1,r_2}$ is \textsuperscript{(4.3)} as factors of $\frac{1}{\log \mu}$ accompanying each $V^{(-)}$. Therefore, in general we have:

$$\langle V^{(+)}_{r_1,r_2} \cdots V^{(+)}_{r_{2n-1},r_{2n}} V^{(-)}_{s_1,s_2} \cdots V^{(-)}_{s_{2l-1},s_{2l}} T_{k_1} \cdots T_{k_{N}} \rangle \propto (\log \mu)^{n-l}$$ \quad (4.5)

Correlators which behave as negative powers of $\log \mu$ are interpreted as vanishing. From eq. \textsuperscript{(4.5)} one can see precisely the interplay of $V^{(-)}$ and $V^{(+)}$. For bulk amplitudes, for example, we find zero if $n \leq l$; this is consistent with all the results described above.
Another (inequivalent) way to define correlation functions of $V^{(+)}_{r_1,r_2}$ is to follow the critical string logic. We illustrate this procedure with the example of $V^{(+)}_{1,1} = \partial X \bar{\partial} X$. The log $\mu$ divergence discussed above is due in this case to the fact that turning on $\partial X \bar{\partial} X$ shifts the dimensions of the exponentials $T_k$, and we have to compensate by adjusting the momenta $k_i$. Then inserting $\partial X \bar{\partial} X$ into a correlation function (2.8) corresponds to $\sum_i k_i \frac{\partial}{\partial k_i}$. A similar procedure can probably be followed for all the discrete states.

The world sheet supersymmetric case is again very similar. At momenta of the form $k = \frac{r_1 - r_2}{2}$, where $r_1, r_2 \in \mathbb{Z}_+$ and $r_1 - r_2 \in 2\mathbb{Z}$ corresponding to NS states, while $r_1 - r_2 \in 2\mathbb{Z} + 1$ are in the R sector, we have discrete states at level $\frac{1}{2}m^2 = \frac{1}{2}r_1 r_2$. Thus in the NS sector the discrete momenta are $k \in \mathbb{Z}$ while for R states it’s $k \in \mathbb{Z} + \frac{1}{2}$ (in agreement with (3.13), (3.22)). The Liouville dressing takes the form $\beta^{(\pm)}_{r_1, r_2} = -1 \pm \frac{r_1 + r_2}{2}$. As in the bosonic case, $V^{(-)}_{1,1}$ vanish inside correlation functions of tachyons (4.2), and Ramond fields $V$ (4.2). The derivation is completely parallel to the one in the bosonic case and we leave it to the reader.

5. Comments.

There is a large number of open problems related to our work. We will mention here a few.

1) We do not feel that the issue of states with negative energy ($E = \beta + \frac{Q}{2} < 0$) is well understood. We have shown here that the bulk S–matrix, which is the only part of Liouville correlators which is well understood, has a sensible interpretation which includes such states. It is true that the discrete states with $E < 0$ partially decouple, but this is not true for tachyons of generic momentum, and also breaks down if we turn on discrete states with $E > 0$. States with $E < 0$ do not correspond to small deformation of the world sheet surface from the point of view of 2D gravity, but they should still play an important role in the dynamics (e.g. the black hole [20], [26]).

2) One would like to have a useful description of the space time physics described by the amplitudes we have found – perhaps a simple action principle for the tachyon and massive degrees of freedom. As discussed above, this should be different from the existing string field theories [4], [7], [8]. In particular, it would be interesting to incorporate the partial decoupling of $V^{(-)}$ and understand whether there are new symmetries (perhaps related to those of [53], [54]) which are responsible for this. Of course, such a formulation would be useful to study gravitational back reaction and other issues in this theory.
3) There are extensions and applications of our results which may be interesting.

a) It is important to derive our results for the extension from the “bulk” to the “boundary”
correlation functions, which we got by using space time arguments, directly from the world
sheet Liouville theory. This should shed some light on the origin of the local action for the
tachyon field.

b) We have restricted our attention to genus zero (tree level) amplitudes. From matrix
models [1], [2] we know that the results for higher genus are almost as simple, and it
would be nice to understand them too from the continuum. We would like to point out
in this context, that one may have problems of convergence of the appropriate integral
representations (which are again trivial generalizations of the 26 dimensional ones [1]): the
sum rule \( \sum_i m_i = 1 \) (2.50) (for \( s = 0 \)), is replaced for genus \( h \) by
\[
\sum_{i=1}^{N-1} m_i = 1 - 2h
\]
and since one still expects divergences when \( m_i < 0 \), there is probably no region where the
integral representation converges. The space time picture should be useful here, as in the
spherical case, and we expect a similar analysis to give the results of [1].

c) It would be interesting to see what properties survive in more “realistic” string theories.
The natural candidates to consider are the non critical superstrings [25], where one can
increase the number of degrees of freedom in a controlled way, without losing stability of
the vacuum. We have seen that in two dimensions the theory of [25] is topological. Its
properties should be elucidated further. One may study the related heterotic theories,
which are probably topological as well; they comprise a large class of theories which are
probably completely solvable.

d) We saw that the 2D fermionic string is described in space time by a field theory with two
(bosonic) fields, whose tree level S – matrix is exactly known. One approach to calculate
higher genus corrections would be to try to write a space time theory similar to the Das-
Jevicki one [3], now with two fields; hopefully the tree level structure, which we have found
explicitly, will determine it uniquely. Then one can use this action in the standard way
[12] to get all order results. This should (among other things) shed light on [13].

e) We have treated here two dimensional strings with \( N = 0,1 \) SUSY. For \( N = 2 \) two
dimensional string theory is critical, and has been recently shown to possess some interest-
ing features [33]. We saw that the cases \( N = 0,1 \) give similar space time physics and
are closely related to the \( c = 1 \) matrix model. The situation is reminiscent of the relation
between the $N = 0, 1, 2$ minimal models of [24], [43] in flat space. Using our techniques, it is easy to show that all $N \geq 4$ point functions in critical $N = 2$ string theory vanish, in agreement with [55]. The reason is that as emphasized in [55] the theory is really four dimensional, but there is again only one field theoretic degree of freedom. Unlike the $N = 0, 1$ cases, here the four dimensional kinematics implies vanishing of the amplitudes. It would be interesting to understand the connection between the work of [55] and the theories described here.

4) One interesting application is to the two dimensional black hole of [20]. To understand that, we have [26] to replace the cosmological term $T_{k=0}$ by $V_{1,1}^{(-)}$. First, it is clear that all bulk amplitudes ($s = 0$) of tachyons are the same as in the black hole solution of [20] and the usual $c = 1$ case considered in this paper. Also, we saw that bulk amplitudes containing tachyons of generic $k$ and $V_{1,1}^{(-)}$ vanish. We see again that to solve the black hole theory we must understand the dynamics of the discrete states $V_{r_1, r_2}^{(+)}$, since only they couple to $V^{(-)}$. The resulting picture of gravitational back reaction in 2D string theory should be fascinating.

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\footnote{Since all discrete states of the $c = 1$ model appear as intermediate states in such amplitudes (see sections 2, 4) we immediately conclude that all the $c = 1$ discrete states must be physical in the black hole background as well. Furthermore, no additional discrete states appear in intermediate channels. This suggests that all other states (e.g. those of [26]) decouple in the bulk, and perhaps also in general. This phenomenon was demonstrated in [26].}
Appendix A. Comparison to KdV.

The solution of random multimatrix chain-interacting models can be expressed in terms of certain differential operators

\[ Q = D^m - \frac{u}{2} D^{m-2} + \ldots \quad \text{and} \quad P = D^p - \frac{u}{2} D^{p-2} + \ldots, \]

satisfying the ‘string equation’\[ [P, Q] = 1 \] (A.1)

The solution of this system of coupled differential equations for the coefficients of \( P \) and \( Q \) yields in particular the string susceptibility \( u = \partial_x^2 \log Z = \langle PP \rangle \). In the following we restrict ourselves to the ‘unitary case’ \( \deg(P) = n + 1, \deg(Q) = n \), where the explicit solution of (A.1) on the sphere is known\[ L = Q_1^+ \quad P = L_{n+1}^+ \]. Operators are defined by generalized KdV flows:

\[ \partial_t \phi_j L = [L_j^+, L] \quad j = 1, 2, 3, \ldots \] (A.2)

or in terms of the partition sum: \( \partial_t u = \langle \phi_j \mathcal{P} \mathcal{P} \rangle = -2 (\text{Res}L_j^j)' \). The only feature of the solution of \[ L \] that we will need is, that \( L \) satisfies (see \[ L \] for notation and derivations):

\[ L_j^j = -\frac{u}{2} j D^{-j} + O(D^{-j-1}) \] (A.3)

Using (A.3) it was shown in \[ L \] that:

\[ \langle \phi_j \phi_m \rangle = -2 \int \text{Res}[L_j^+, L_m^-] = j x^{2\Delta_j - \gamma_{\text{str}}} \delta_{j,m} \quad j, m < n. \]

\[ \langle \phi_j \phi_l \phi_m \rangle = 2 \int \text{Res}[[L_j^+, L_j^-], L_m] - [L_j^+, [L_j^+, L_m^m]] \]

\[ = j l m x^{\Delta_j + \Delta_l + \Delta_m - \gamma_{\text{str}} - 1} N_{jlm} \quad j, l, m < n. \] (A.4)

where the scaling dimensions \( \Delta_j = \frac{j-1}{2n} \) and string susceptibility exponent \( \gamma_{\text{str}} = -\frac{1}{n} \) are the KPZ exponents\[ L \] and \[ L \] for the unitary CFT \((n + 1, n)\) coupled to gravity. Note the appearance of the CFT fusion coefficients \( N_{jlm} \in \{0, 1\} \) for the three point functions. The operators \( \phi_j \) with \( j < n \) were singled out in the calculation: they correspond to the order parameters of the theory, whose definition is unambiguous\[ L \]. The results (A.4) agree with (2.39).

In section 2 we also considered \( N \) point functions without screening. For the order parameters, which are the only operators that are simple to treat using KdV (A.2) we have, \( k_p = \alpha_0 (1 - j_p), \quad p = 1, \ldots, N - 1 \) and \( k_N = \alpha_0 (1 + j_N) \). Note that \( \alpha_0 < 0 \) therefore
$k_p > 0$; hence we don’t have to worry about the zero energy cuts in $N$ point functions (e.g. (2.60)), and the Liouville result we have to compare to is (2.53). The sum rule (2.40) takes the form: $j_1 + j_2 + .. + j_{N-1} = j_N + N - 2$. Thus $N$ point functions without screenings correspond to ‘the boundary of the fusion rules’. In this case, using (A.3) it is easy to calculate all $N$ point functions. Indeed:

\[
\langle \phi_{j_1}...\phi_{j_N} \rangle = -2 \int \text{Res}[L^{j_1}_+, [L^{j_2}_+, [..L^{j_{N-1}}_+, L^{j_N}_-[..]..]

= -2j_1j_2...j_N \int \text{Res}(L^{j_1+j_2+..+j_{N-1}-N+1}[L, [L, [..[L, L^{j_N}_{j_N}..]])

= j_1j_2...j_N F_N(j_N) \tag{A.5}
\]

The second line results from the fact that we work on the sphere, where each commutator acts with one derivative only. In (A.5) we have strongly used eq. (A.3). Note the close correspondence between (A.5) and the Liouville calculation: after we factor a product of normalization factors (which are of course different in the two cases, compare to (2.52)) we are left with a function of $s$ or $j_N$, only. As in the Liouville case, the function of $j_N$, $F$, is now determined by putting $N - 3$ of the $j_i$ to 1. Then we can use the result for the three point function (A.4), to find $F_N = (\partial_x)^{N-3}x^{s+N-3}$, where $s = \sum_{p=1}^{N} \Delta j_p - \gamma_{str} - N + 2$, the correct KPZ scaling for the $N$ point function, and finally:

\[
\langle \phi_{j_1}...\phi_{j_N} \rangle = j_1j_2...j_N(\partial_x)^{N-3}x^{s+N-3} \tag{A.6}
\]

In agreement (up to a different normalization of the operators) with the Liouville result (2.53).

**Appendix B. 1PI calculus.**

This appendix is devoted to various calculations of 1PI vertices at $c = 1$ ($\alpha_0 = 0$). In sect. 2.2.6, we have shown how to compute the general 1PI vertices $A^{(N)}_1(k_1,..,k_N)$ directly: it is the sum over all tree graphs with $N$ external legs carrying the momenta $k_1,..,k_{N-1} > 0$, $k_N < 0$, and the following Feynman rules:

1) propagators: $-|k|/\sqrt{2}$ for each internal leg carrying the total momentum $k$ (momentum is conserved at the vertices).
2) vertices: $\tilde{A}(l_1,..,l_n) = (\partial_{\mu})^{n-3} \mu^{\sqrt{2}|l|-1}|_{\mu=1}$ for each $n$-legged vertex with incoming momenta $l_1,..,l_n$, $l$ denoting the only negative momentum among these.

49
To illustrate the procedure, let us calculate $A_{1PI}^{(4)}$ again, using the new method: there are four trees with external momenta $k_1,..,k_4$, the $s$, $t$, $u$ channels and the maximal star of one 4-legged vertex. Adding up the four contributions we find:

$$A_{1PI}^{(4)}(k_1,..,k_4) = -\frac{1}{\sqrt{2}} \left[ |k_1 + k_2| + |k_1 + k_3| + |k_2 + k_3| \right] + (\sqrt{2}|k_4| - 1)$$

where obvious use of the conservation law $-k_4 = k_1 + k_2 + k_3$ has been made.

Repeating the same procedure for $N = 5, 6$ yields:

$$A_{1PI}^{(5)} = 2 - \frac{1}{2} \sum_{i=1}^{5} k_i^2$$

$$A_{1PI}^{(6)} = -6 + 3 \sum_{i=1}^{6} k_i^2$$

Note that the irreducible vertices are no longer constants. The main problem with these computations is that they involve writing all tree graphs with $N$ external legs whose number grows very quickly (26 in the case $N = 5$, 236 in the case $N = 6$). We will present below a simple recursive way of generating arbitrary 1PI vertices.

The first simple object one can look at is the vertex with, say $p$ non-zero momenta and $N-p$ zero momenta $A_{1PI}^{(N)}(k_1,..,k_p,0,..,0)$. Using the method described in the beginning of this appendix, it is easy to see the effect of adding one zero-momentum external leg to such a vertex: due to the form of the propagator $\pi(k) = -\frac{\sqrt{2}}{2} |k|$, the only non-zero contributions to the sum over trees come from either an addition on a leg carrying a non-zero momentum $k$ (multiplication by $-\frac{\sqrt{2}}{2} |k|$), or an addition on a vertex $V_n(k) = (\partial_\mu)^{n-3} \mu^{\sqrt{2}|k|^{-1}} |_{\mu=1}$, which simply changes it into $V_{n+1}$. By recursion, it is straightforward to show that:

$$A_{1PI}^{(N)}(k_1,..,k_p,0,..,0) = (\partial_\mu)^{N-p} \prod_{i=1}^{p} \frac{2}{1 + \mu^{\sqrt{2}|k_i|}}$$

$$\sum_{\text{trees}(k_1,..,k_p)} (\pi(k) = -\frac{\sqrt{2}|k|}{1 + \mu^{\sqrt{2}|k|}}; V_n(k) = (\partial_\mu)^{n-3} \mu^{\sqrt{2}|k|^{-1}} |_{\mu=1})$$

where the sum extends to all trees with external momenta $k_1 > 0,..,k_{p-1} > 0$ and $k_p < 0$; the notation $(\pi(k) = ..; V_n(k) = ..)$ means that a weight $\pi(k)$ has to be attached to each internal leg carrying the momentum $k$, and a weight $V_n(k)$ has to be attached to each
an $n$-legged vertex whose only negative external momentum is $k$. Note that the external legs receive a weight $2/1 + \mu \sqrt{2}|k|$. It is an easy exercise to see that a differentiation w.r.t. $\mu$ exactly reproduces the above additions.

As an example, in the case $p = 2$, eqn. (B.3) yields:

$$A^{(N)}_{1PI}(k, -k, 0, \ldots, 0) = -\frac{\sqrt{2}}{|k|} (\partial_{\mu})^{N-2} \left. \frac{2}{1 + \mu \sqrt{2}|k|} \right|_{\mu=1}$$

$$= -\frac{\sqrt{2}}{|k|} (\partial_{\mu})^{N-2} (1 - \tanh(\frac{|k|}{\sqrt{2} \log \mu})) \bigg|_{\mu=1} \quad (B.4)$$

from which we get immediately:

$$A^{(3)}_{1PI} = 1$$
$$A^{(4)}_{1PI} = -1$$
$$A^{(5)}_{1PI} = 2 - k^2$$
$$A^{(6)}_{1PI} = -6 + 6k^2$$
$$A^{(7)}_{1PI} = 24 - 35k^2 + 4k^4$$
$$A^{(8)}_{1PI} = -120 + 225k^2 - 60k^4$$
$$A^{(9)}_{1PI} = 720 - 1624k^2 + 700k^4 - 34k^6$$

In the case $p = 3$, (B.3) is still very simple because the sum reduces to only one term, with weight $\mu \sqrt{2}|k_3|$, so that:

$$A^{(N)}_{1PI}(k_1, k_2, k_3, 0, \ldots, 0) = (\partial_{\mu})^{N-3} \left. \left( \mu \sqrt{2}|k_3| \prod_{i=1}^{3} \frac{2}{1 + \mu \sqrt{2}|k_i|} \right) \right|_{\mu=1}$$

or, by redistributing the power $\sqrt{2}|k_3| = \frac{1}{\sqrt{2}} (|k_1| + |k_2| + |k_3|)$ onto the individual leg factors, this can be put in the form (2.68).

In fact, the general expression (B.3) can be improved as follows:

$$A^{(N)}_{1PI}(k_1, \ldots, k_p, 0, \ldots, 0) = (\partial_{\mu})^{N-p} \prod_{i=1}^{p} \frac{1}{\cosh(\frac{k}{\sqrt{2} \log \mu})}$$

$$\sum_{\text{trees}(k_1, \ldots, k_p)} (\pi(k)) = -\mu \partial_{\mu} \log \cosh(\frac{k}{\sqrt{2} \log \mu}); V_n = \mu^{2-n} A^{(n)}_{1PI} \bigg|_{\mu=1}$$

(B.7)
which yields (2.68), (2.69) in the particular cases \( p = 3, 4 \). To get (B.7) from (B.3), we reabsorbed a factor \( \mu \sqrt{|k|} \) into each leg around a vertex, yielding the product of external leg weights prefactor, and a propagator

\[
\pi(k) = -\mu \partial_\mu \log(1 + \mu \sqrt{|k|}) = -\frac{|k|}{\sqrt{2}} - \mu \partial_\mu \log(\frac{k}{\sqrt{2}} \log \mu),
\]

and performed the partial sums corresponding to the \( -\frac{|k|}{\sqrt{2}} \) piece of the propagator, yielding the vertices \( V_n = \mu^{2-n} A^{(n)}_{1P I} \).

In the case \( p = N - 1 \), the expression (B.7) gives rise to a very simple recursion relation:

\[
A^{(N)}_{1P I}(0, k_1, .., k_{N-1}) = (3 - N) A^{(N-1)}_{1P I}(k_1, .., k_{N-1}) - \\
- \sum_{2 \leq p \leq \frac{N}{2}} \frac{l^2}{2} A^{(p+1)}_{1P I}(k_{\sigma(1)}, .., k_{\sigma(p)}, l) A^{(N-p)}_{1P I}(l, k_{\sigma(p+1)}, .., k_{\sigma(N-1)})
\]

(B.8)

where for each \( p \) the sum extends over the permutations \( \sigma \) of \( \{1, .., N-1\} \) yielding distinct sets \( \{\sigma(1), .., \sigma(p)\} \) (the symmetric term \( N - p = p + 1 \) is counted only once), and \( l \) denotes the intermediary momentum fixed by the conservation law. This expression shows explicitly that \( A^{(N)}_{1P I} \) with one zero external momentum is a polynomial in the variables \( k^2_i = (\sum_{i \in I_j} k_i)^2, I_j \subset \{1, .., N-1\} \), with total degree \( N - 4 + (N \mod 2) \). The general vertex is then obtained by symmetrization of (B.8) w.r.t. \( k_N \). As an example we quote the case \( N = 7 \):

\[
A^{(7)}_{1P I} = 24 - \frac{35}{2} \sum_{i=1}^{7} k_i^2 + \left( \sum_{i=1}^{7} k_i^2 \right)^2 + \frac{1}{4} \sum_{1 \leq i < j \leq 7} (k_i + k_j)^2 [(k_i + k_j)^2 - k_i^2 - k_j^2],
\]

(B.9)

valid for all momenta.
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