Stiffness Identification Method of Simply Elastic Supported Beam based on First-order Modal Rayleigh Solution

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Abstract. In order to evaluate the support state of bridges with elastic supports, based on first-order mode Rayleigh solution, a stiffness identification method of elastic supports beam is proposed. The superposition displacement of static displacement and rigid body displacement under sinusoidal load is used as the vibration mode function. Based on the mode function, the calculation formula of the fundamental frequency for the simply supported elastic support beam is derived according to the Rayleigh method. The identification method of the support stiffness and beam stiffness of the elastic support beam is derived by using the measured frequency and the mode value. A precast box girder is analyzed by an example. According to the measured results and finite element results, the stiffness of the precast box girder and the stiffness of the laminated rubber bearing are identified by the stiffness identification method established.

1. Introduction

Beam are often used in small and medium-sized highway bridges, but the traditional ideal restraint model[1] is still used in the calculation of internal forces of this kind of beams. However, the actual beam bridge is usually supported by rubber bearing with elastic support performance. At the same time, the pier, pile foundation and soil also have elastic deformation. This kind of elastic deformation will affect the stress of beam bridges[2]. In addition, in the actual use of the beam bridge, due to various external environmental factors, the substructure stiffness of the bridge will change, which will affect the normal use of the beam bridge. Therefore, in order to comprehensively evaluate the working state of the beam bridge, the elastic deformation influence of the rubber bearing and the elastic deformation of the substructure should be considered.

When the beam bridge considers the elastic support boundary condition, it will have different dynamic characteristics from the ideal calculation model of the simply supported beam. Relevant scholars have carried out theoretical research. Yan[3] established its dynamic mechanical model for the simply supported beam according to the general elastic support condition, and derived the theoretical solution of its dynamic response. It is found that under the dynamic load, the natural vibration frequency and amplitude of the elastic support will change with respect to the rigid support. Jiang[4] deduced the coupling vibration equation of bridge and car body with elastic support, analyzed the main vibration of simply elastic supported beam and obtained the main vibration mode analytical formula of the beam. A dynamic analysis method for vehicle bridge coupling of vehicle passing through a simply supported beam under the condition of elastic support is established. The literature [3, 4] to obtain the mode shape function of the elastic support beam by theoretical solution, and then analyze the influence of...
the elastic support on the research characteristics of the beam dynamics. In this method, the self-vibration frequency is often solved by the equation, the calculation process is complicated, and it is difficult to obtain a simple analytical calculation formula, which is not convenient for practical application of the project. Yang[5], Qian[6] assumed the elastic support simply supported beam mode as a superposition of sinusoidal function and rigid body displacement, and studied the dynamic response of the train through the bridge. Song[7], Kang[8], and Wang[9] used similar methods to study different aspects of elastic support beams. Chen[10] conducted a kinetic study by assuming the lateral vibration of the elastically supported rotating beam as a Fourier series. The literature[5-10] analyzes the dynamic response of the elastic support beam by simplifying the mode function of the elastic support beam. The method is simple and easy, but it requires a reasonable elastic support beam shape.

At present, evaluating the working status of an in-service beam bridge has become a hot spot for many scholars, and many methods have been established to study it[11-13]. In terms of stiffness identification, Chen[14] proposed the low-order dynamic test modal results of bridges to directly identify and evaluate the stiffness of bridge structural members with constraints and finite weight least squares solution theory. Moatasem[15] used the measured modal shape to establish modal confidence to correct the established model to identify the stiffness of the structure. Ali[16] used the measured vibration mode and frequency to study the crack problem of prestressed concrete beams under elastic support. However, from the literature found so far, no research has been found on the identification of elastic stiffness directly under the elastic support state of the beam bridge.

It can be seen from the above literature that it is a feasible idea to evaluate the current support state of the bridge by studying the relationship between the elastic effect of the support and the dynamic characteristics of the bridge. Based on this idea, the static deformation curve of simply supported beam and the rigid body displacement of elastic support are used as the vibration mode function of elastic support beam. A stiffness analysis of elastic support beam based on first-order modal Rayleigh solution is proposed method.

2. Simplified Model of Simply Elastic Supported Beam

A calculation model of simple supported beam with elastic support as shown in Figure 1. The length of the beam is \( l \). The cross-sectional area is \( A \). The unit length quality is \( m \). The beam stiffness is \( EI \). The vertical elastic support stiffness on the left and right ends of the beam are \( k_1, k_2 \) respectively.

\[ k_i = \frac{E_i A_i}{h_i} \quad i = 1, 2 \]  

(1)

Where \( k_i \) is vertical stiffness; \( E_i \) is bearing compressive modulus of elasticity; \( h_i \) is the height of the rubber part of the support; \( A_i \) is the cross-sectional area of the rubber bearing.

3. Rayleigh Solution of First-order Frequency

3.1. The Formula of First-order Frequency

The Rayleigh method derives the natural frequency of the structure based on the principle of conservation of energy. For the beam model shown in Figure 1, the vibration displacement equation of the elastic system is
\[ y(x, t) = \varphi(x)q(t) = \varphi(x)\sin(\omega t + \theta) \]  
\[ (2) \]

In equation (2): \( x \) is the coordinate value established along the longitudinal direction of the beam; \( y(x, t) \) is the vertical vibration displacement of the elastic support beam; \( \varphi(x) \) is a vibration mode function, \( q(t) \) is time coordinates. The kinetic energy of the system at a certain moment is

\[ T(t) = \frac{1}{2} m \int_0^l [y'(x, t)]^2 dx = \frac{1}{2} m \omega^2 \cos^2 (\omega t + \theta) \int_0^l \varphi^2(x) dx \]
\[ (3) \]

when \( \cos(\omega t + \theta) = 1 \), the system has the largest kinetic energy. that is

\[ T_{\text{max}} = \frac{1}{2} m \omega^2 \int_0^l \varphi^2(x) dx \]
\[ (4) \]

The potential energy of the beam at a certain moment is \( V(t) \).

\[ V(t) = V_i(t) + V_k(t) \]
\[ (5) \]

\( V(t) \) consists of two parts: \( V_i(t) \) and \( V_k(t) \). \( V_i(t) \) is bending strain energy of the beam. \( V_k(t) \) is the elastic deformation energy of the springs at both ends.

\[ V_i(t) = \frac{1}{2} \int_0^l EI[y(x, t)]^2 dx \]
\[ (6) \]

\[ V_k(t) = \frac{1}{2} k_1 y^2(0, t) + \frac{1}{2} k_2 y^2(l, t) \]
\[ (7) \]

Where \( EI \) is bending stiffness of the beam. According to the law of conservation of energy, there is

\[ T_{\text{max}} = V_{\text{max}} \]
\[ (8) \]

Using this equation (8), the formula for calculating the first-order frequency can be obtained

\[ \omega^2 = \frac{EI \int_0^l [\varphi'(x)]^2 dx + k_1 \varphi^2(0) + k_2 \varphi^2(l)}{m \int_0^l \varphi^2(x) dx} \]
\[ (9) \]

3.2. The First-order Mode Function for the Elastic Support Beam

According to the derivation of the previous section, the displacement of the first-order mode function is expressed as \( y(x, t) \). The displacement at any point on the beam can be expressed as

\[ y(x, t) = \varphi(x)q(t) = (u(x) + v(x))q(t) \]
\[ (10) \]

In equation (10), it is assumed that the displacement is composed of \( u(x)q(t) \) and \( v(x)q(t) \). \( u(x)q(t) \) is the deformation displacement of the rigid simply supported beam. \( v(x)q(t) \) is the rigid body displacement when the beam stiffness is infinite. Here, \( v(x) \) is assumed to be a linear function. \( u(x) \) is assumed a deformation displacement under sinusoidal load for simply supported beams (as shown in Figure 2).
After calculation, the $\phi(x)$ can be expression as the following formula

$$\phi(x) = u(x) + v(x) = \frac{l^4}{EI\pi^4} \sin \left( \frac{\pi x}{l} \right) + \left( \frac{1}{\pi k_2} - \frac{1}{\pi k_1} \right)x + \frac{l}{\pi k_1} \tag{11}$$

3.3. The first-order frequency for the elastic support beam

Substituting the equation (11) into equation (9), the first-order frequency for the elastic support beam can be obtained:

$$\omega^2 = \frac{3l + 6a^2(K_1 + K_2)}{3l + 12a^2(K_1 + K_2) + 2la^2[K_2^2 + K_1K_2 + K_1^2]} \frac{El a^4}{m} \tag{12}$$

where $K_1 = \frac{EI}{k_1}, K_2 = \frac{EI}{k_2}, a = \frac{\pi}{l}$. The equation (12) can be expressed as

$$\omega = \alpha \sqrt{\frac{a^4 EI}{m}} \tag{13}$$

where $\alpha = \frac{3l + 6a^2(K_1 + K_2)}{\sqrt{3l + 12a^2(K_1 + K_2) + 2la^2[K_2^2 + K_1K_2 + K_1^2]}}$. $\alpha$ represents the influence of the stiffness at two ends of the elastic simply supported beam. When the bearing stiffness is infinite, the equation (13) becomes the first-order natural frequency expression of the general simply supported beam. If the bearing stiffness at both ends $K_1 = K_2 = \frac{EI}{k}$, the first-order natural frequency can be simplified to the following

$$\omega^2 = \frac{l + 4a^2K}{l + 8a^2K + 2la^2K^2} \frac{El a^4}{m} \tag{14}$$

4. Stiffness Identification of Elastic Support Beam Based on Rayleigh Method

For the general medium-small bridge elastic with simply elastic supported beam, it is easy to get its first-order natural frequency and mode shape. Using the measured frequency and mode shape, we can evaluate the stiffness support. Suppose the measured first-order natural frequency of the beam is $\omega_1$. And the measured mode shape displacements at two end and mid-span are $y_0, y_f, y_{as}$, respectively. From the mode shape function $\phi(x)$, the following formula can be obtained

$$\lambda_1 = \frac{y_0}{y_{as}} = \frac{2\pi^2 El k_2}{2k_l k_f l^3 + \pi^2 EI(k_2 + k_1)} \tag{15}$$

$$\lambda_2 = \frac{y_f}{y_{as}} = \frac{2\pi^2 El k_2}{2k_l k_f l^3 + \pi^2 EI(k_2 + k_1)} \tag{16}$$

Where $\lambda_1$ and $\lambda_2$ are deflection ratio. After derivation from equations (15), (16), the following formulas can be obtained
From equation (13), who can get the bending stiffness

\[ k_i = \frac{(2 - \lambda_i^2 - \lambda_j^2) \pi^3 EI}{2l^3 \lambda_i^3} \]  
(17)

\[ k_j = \frac{(2 - \lambda_i^2 - \lambda_j^2) \pi^3 EI}{2l^3 \lambda_j^3} \]  
(18)

Substituting (19) into (17), (18) yields

\[ EI = \frac{m\omega_0^2}{\alpha^2} \]  
(19)

Under the condition of measuring the first-order natural frequency and mode shape (the measured value \( \omega_0, y_0, y_i, y_{0i} \)), who can use (19), (20), (21) to identify the stiffness of the elastic support beam.

5. Numerical Example

A precast box girder is tested for numerical example. The basic parameters of the section for the beam are shown in Figure 3. The test beam test layout is shown in Figure 4. \( l = 30m; c50 \) concrete; main beam elastic modulus \( E = 3.45 \times 10^3 Pa \) Reinforced concrete’s density \( \rho = 2500 kg/m^3 \). The girder support is a double support, using gyz450×99(nr) type natural plate rubber bearing.

![Figure 3. Basic dimensions of precast beam section (unit: cm).](image3)

![Figure 4. Layout of the test beam](image4)

The support is placed on the foundation pad beam. The center distance between the two supports is 27 meters. According to the requirements of the test task, the acceleration sensor is placed in the support and the mid-span position. The influence of the substructure such as the foundation pad beam on the vibration mode needs to be removed. Therefore, it is necessary to arrange the sensor on the pad beam. The test uses environmental excitation; the acceleration sensor is used to pick up the vibration; Figure5 shows the field test. Figure 6 shows an acceleration test curve.

![Figure 5. On-site test photos.](image5)

![Figure 6. Acceleration test curve.](image6)
Table 1. Displacement and frequency of first order mode.

| Modal value | $y_0$ | $y_{pad}$ | $y_{max}$ | $\omega$ | $\alpha$ ($s^{-1}$) |
|-------------|-------|-----------|-----------|---------|------------------|
| Measured value | 0.080 | 0.070 | 1 | 0.077 | 0.068 | 4.60 |
| Finite element value | 0.010 | 1 | 0.010 | 4.65 | |

Figure 7. Finite Element Model of precast box beam.  
Figure 8. The cloud solution for first order mode.

The finite element model of the simply supported girder is established by the software of ANSYS. The girder is simulated by a solid three-dimensional solid unit; the support is simulated by a COMBIN14 spring unit. Figure 7 shows the finite element model of the girder. Figure 8 shows the first-order finite element mode cloud image of this girder. Table 1 shows the first-order mode displacement values and frequency values of the measured and correspondingly established finite element model.

Table 2. Identification stiffness of elastic support beam.

| Stiffness | Theoretical value | Measured value | Error 1 | Finite element value | Error 2 |
|-----------|-------------------|----------------|---------|----------------------|---------|
| $k_1$     | $2.41*10^9$       | $2.55*10^9$    | 5.8%    | $2.45*10^9$         | 1.6%    |
| $k_2$     | $2.41*10^9$       | $2.83*10^9$    | 17.4%   | $2.45*10^9$         | 1.6%    |
| $EI$      | $1.39*10^{10}$    | $1.70*10^{10}$ | 22.3%   | $1.57*10^{10}$      | 12.9%   |

The identification stiffness of the simply elastic supported girder is show in Table 2. The “Measured value” and the “Finite element value” are identification stiffness obtained by the proposed method. The “Theoretical value” of $k_1$ and $k_2$ are calculated by equation (1); $EI$ is calculated by elementary beam theory. Error 1 is the percentage between the “Measured value” and the “Theoretical value”; Error 2 is the percentage between the “Finite element value” and the “Theoretical value”. It can be seen from the results of Table 2 that the finite element identification accuracy is higher than the measured value, and the measured value is larger than the theoretical value.

6. Conclusion
Based on the measured data, this paper establishes a method based on the first-order modal Rayleigh solution to identify the stiffness of elastic support beam. Through the analysis of the example, some useful conclusions are obtained:

- Under the condition that the first-order natural frequency and vibration mode are measured, the stiffness and support stiffness of the elastic support beam can be identified by the proposed method. The method is simple and easy.
- It can be seen from the example that there is a certain error in the actual bridge stiffness identification using this method. This kind of error should come from the accumulation of measurement error, modal identification error, structural parameter error, etc. In the future, the recognition accuracy of this method should be improved from these three aspects.
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