Open Subsets in a Stein Space with Singularities

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Abstract. Serre proved that a domain $Y$ in $\mathbb{C}^n$ is Stein if and only if $H^i(Y, \mathcal{O}_Y) = 0$ for all $i > 0$. Laufer showed that if $Y$ is an open subset of a Stein manifold of dimension $n$ and $H^i(Y, \mathcal{O}_Y)$ is a finite dimensional complex vector space for every $i > 0$, then $Y$ is Stein. Vâjâitu generalized these theorems to singular Stein space of dimension $n$. In this paper, we consider singular Stein spaces $X$ with arbitrary dimension and give necessary and sufficient conditions for an open subset $Y$ in $X$ to be Stein. We show that if $Y$ is an open subset of a reduced Stein space $X$ with arbitrary dimension and singularities, then $Y$ is Stein if and only if $H^i(Y, \mathcal{O}_Y)$ is a finite dimensional complex vector space for every $i > 0$. Without cohomology condition, if $X - Y$ is a closed subspace of $X$, then we show that the geometric condition of the boundary $X - Y$ determines the Steinness of $Y$. More precisely, we show that if $X$ is normal and the boundary $X - Y$ is the support of an effective $\mathbb{Q}$-Cartier divisor, or $X - Y$ is of pure codimension 1 and does not contain any singular points of $X$, then $Y$ is Stein.

1. Introduction

We work over the field $\mathbb{C}$ of complex numbers.

Let $X$ be a Hausdorff topological space. $(X, \mathcal{O}_X)$ is a complex space if every point of $X$ has a neighborhood $U$ such that $(U, \mathcal{O}_U)$ is isomorphic to a closed complex subspace $(A, \mathcal{O}_A)$ of a domain $D \subset \mathbb{C}^m$ for some $m \in \mathbb{N}$, where $A$ is the support of the analytic coherent $\mathcal{O}_D$ sheaf $\mathcal{O}_A = \mathcal{O}_D/I|_A$, and $I \subset \mathcal{O}_D$ is an analytic coherent ideal sheaf.

A complex space $Y$ is Stein if it is both holomorphically convex and holomorphically separable [8, pp. 293—294, Theorem 63.2]. We say that $Y$ is holomorphically convex if for any discrete sequence $\{y_n\} \subset Y$, there is a holomorphic function $f$ on $Y$ such that the supremum of the set $\{|f(y_n)|\}$ is $\infty$. $Y$ is holomorphically separable if for every pair $x, y \in Y$, $x \neq y$, there is a holomorphic function $f$ on $Y$ such that $f(x) \neq f(y)$. By Cartan’s Theorem B, a complex space $Y$ is Stein if and only if $H^i(Y, \mathcal{F}) = 0$ for every analytic coherent sheaf $\mathcal{F}$ on $Y$ and all positive integers $i$ [4, p. 124].

Serre proved that a domain $Y$ in $\mathbb{C}^n$ is Stein if and only if $H^i(Y, \mathcal{O}_Y) = 0$ for all $i > 0$ [17], where $\mathcal{O}_Y$ is the analytic structure sheaf of $Y$. Laufer generalized Serre’s
result to Stein manifolds of dimension \( m \) (see \[4\] pp. 159–160 or \[10\]). In algebraic geometry, Neeman showed that a quasi-compact Zariski open subset \( Y \) of an affine scheme \( X = \text{Spec} \ A \) (with singularities) is affine if and only if \( H^i(Y, \mathcal{O}_Y) = 0 \) for all \( i > 0 \) \[14\], where \( \mathcal{O}_Y \) is the algebraic structure sheaf of \( Y \) and \( A \) may not be noetherian.

In \[10\], Laufer also proved: If \( Y \) is an \( n \)-dimensional Riemann domain over a Stein manifold such that \( \mathcal{O}_Y \) separates points, and for all \( i > 0 \), \( H^i(Y, \mathcal{O}_Y) \) is a finite dimensional complex vector space, then \( Y \) is a Stein manifold. A dimension \( n \) Stein manifold can be biholomorphicall mapped onto a closed complex submanifold of \( \mathbb{C}^{2n+1} \) \[7\] Ch. 5]. Laufer’s proof heavily relies on the local coordinates of a complex manifold which cannot be applied to a singular Stein space.

For an open subset of a Stein space with singularities, we use complex algebraic geometry approach to avoid dealing with singularities directly and show the following result.

**Theorem 1.1.** If \( Y \) is an open subset of a reduced Stein space \( X \) of dimension \( n \), then \( Y \) is Stein if and only if \( H^i(Y, \mathcal{O}_Y) \) is a finite dimensional complex vector space for every \( i > 0 \).

A complex space \( X \) is locally of finite dimension but globally its dimension may not be finite. If a nonempty complex space \( X \) is irreducible, then there is a nonnegative integer \( n \geq 0 \) such that \( \dim_x X = n \) for all \( x \in X \) \[5\] p. 106] and \( n \) is the dimension of \( X \). If \( X \) has infinitely many irreducible components, then the dimension of \( X \): \( \dim X = \sup_{x \in X} \dim_x X \) \[5\] p. 94] may not be finite and there are connected complex spaces with dimension \( \infty \) \[8\] p. 190]. Without the finite dimension condition, we have

**Theorem 1.2.** Let \( Y \) be an open subset of a reduced Stein space \( X \) with arbitrary dimension and singularities. Then \( Y \) is Stein if and only if \( H^i(Y, \mathcal{O}_Y) \) is a finite dimensional vector space over \( \mathbb{C} \) for all \( i > 0 \).

The idea of proof of Theorem 1.2 is the following\(^1\). First, by Sard and Remmert’s theorems, we can construct countably many holomorphic functions \( f_1, f_2, \ldots \) such that each \( f_i \) defines a (smooth) hypersurface \( F_i \) with disconnected components, \( G_i = F_i \cap Y \) is an open subset in \( F_i \) and \( 1, f_1, f_2, \ldots \) are linearly independent (see Lemmas 2.9, 2.11, and 3.4). Here every \( G_i \) is Stein, using the fact that it has disconnected components, Mayer–Vietoris sequence and Theorem 1.1 (Lemma 3.7). Then we can show that for all \( i > 0 \), \( H^i(Y, \mathcal{O}_Y) = 0 \) by Lemma 3.7 and the dimension counting method of vector spaces due to

\(^1\)The author was informed by an anonymous referee that Theorem 1.1 was proved by Vâjâitu for complex spaces (non-reduced) of dimension \( n \) in 2010 and by modifying this proof and not using mathematical induction, the results in \[21\] hold for complex spaces with arbitrary dimension. The key idea in \[21\] to prove holomorphic convexity is to generalize an estimate of Fornæss and Narasimhan and use Wiegmann’s construction to get a proper surjective morphism from a hypersurface to an \( n \)-dimensional Stein space.
Goodman and Hartshorne \[3\] (see Lemmas 2.13 and 3.8). Finally, by Nagel’s theorem on finitely generated property of the ring of holomorphic functions on \(Y\) \[12\], we show that \(Y\) is holomorphically convex (see Theorems 2.3 and 3.9).

In \[22–24\], we investigated a question raised by J.-P. Serre \[17\]: Let \(Y\) be a complex manifold with \(H^i(Y, \Omega^j_Y) = 0\) for all \(j \geq 0\) and \(i > 0\) (where \(\Omega^j_Y\) is the sheaf of holomorphic \(j\)-forms), then what is \(Y\)? Is \(Y\) Stein? If \(Y\) is an algebraic manifold (i.e., an irreducible nonsingular algebraic variety defined over \(\mathbb{C}\)) and \(\Omega^j_Y\) is the sheaf of regular \(j\)-forms, we know that \(Y\) is not an affine variety in general. In fact, if the dimension of \(Y\) is \(d\) and \(X\) is a smooth projective variety containing \(Y\), then \(X\) may have \(d-j\) algebraically independent nonconstant rational functions which are regular on \(Y\), where \(j = 1, 3, \ldots, d-2, d\) if \(d\) is odd or \(j = 0, 2, \ldots, d-2, d\) if \(d\) is even. But the Steinness question is still open except for the trivial case when the dimension is one. By Theorem 1.1, we have

**Corollary 1.3.** If \(Y\) is a nonsingular open subset of a Stein space \(X\) with dimension \(n\) such that \(H^i(Y, \Omega^j_Y) = 0\) for all \(j \geq 0\) and \(i > 0\), then \(Y\) is Stein.

Simha proved that an open subset of a normal Stein surface obtained by removing a closed analytic subspace of pure codimension one is a Stein surface \[18\]. This result does not hold for higher dimensional complex spaces with singularities (see an example in Section 3). For a normal Stein space, we have

**Theorem 1.4.** Let \(Y\) be an open subset of a normal Stein space \(X\) such that the complement \(X - Y\) is a closed analytic subspace of \(X\).

1. If \(X - Y\) is the support of an effective \(\mathbb{Q}\)-Cartier divisor, then \(Y\) is Stein.

2. If \(X - Y\) is of pure codimension 1 and does not contain any singular points of \(X\), then \(Y\) is Stein.

In order to prove Theorem 1.2 in Section 3, we first prove Theorem 1.1 in Section 2 by algebraic geometry approach. In Section 2, we also prove Theorem 2.3 for an open subset in a Stein space with arbitrary dimension and several lemmas which will be used in the proof of Theorem 1.2 in Section 3.

**2. Preparations**

A ring \(R\) is local if it has exactly one maximal ideal \(\mathcal{M}\). Every stalk \(\mathcal{O}_x\) of the structure sheaf \(\mathcal{O}_X\) of a complex space \(X\) is a local ring: the maximal ideal \(\mathcal{M}_x \subset \mathcal{O}_x\) consists of all germs at \(x\) which can be represented in a neighborhood of \(x\) by a holomorphic function. In fact, \(\mathcal{O}_x\) is a local \(\mathbb{C}\)-algebra: the composition

\[\phi: \mathbb{C} \cdot 1 \to \mathcal{O}_x \to \mathcal{O}_x / \mathcal{M}_x\]
is an isomorphism of fields (see [5, pp. 5–7] or [8, p. 66, p. 97]).

Let $\mathcal{C}_X$ be the sheaf of germs of complex valued continuous functions on a Hausdorff topological space $X$. Then $\mathcal{C}_X$ is a local $\mathbb{C}$-algebra [5, p. 5]. Since the stalk

$$\mathcal{O}_x = \mathbb{C} \oplus \mathcal{M}_x,$$

every germ $f_x \in \mathcal{O}_x$ can be uniquely written in the form

$$f_x = c_x + m_x,$$

where $c_x$ is the complex value of $f_x$ at $x$ and $m_x \in \mathcal{M}_x$ [5, p. 8]. For a holomorphic function $f$ on an open subset $U$ of $X$, define a function $[f]: U \to \mathbb{C}$ by $[f](x) = c_x$ for $x \in U$. Then $f$ induces a continuous function $[f] \in \mathcal{C}_X(U)$ [5, p. 9].

We need a theorem of Nagel [12].

Let $Y$ be a topological space, and let $\mathcal{A}$ be a sheaf of local $\mathbb{C}$-algebras on $Y$. We assume:

(a) For every $y \in Y$, the maximal ideal of the stalk $\mathcal{A}_y$ is $\mathcal{M}_y$, and the composition

$$\phi: \mathbb{C} \cdot 1 \to \mathcal{A}_y \to \mathcal{A}_y/\mathcal{M}_y$$

is an isomorphism.

(b) For every global section $f \in \Gamma(Y, \mathcal{A})$, the associated complex valued function $[f]$ is continuous, where $[f](y)$ is the residue class of the germ of $f$ at $y$ in $\mathcal{A}_y/\mathcal{M}_y$.

(c) For all $i > 0$, $H^i(Y, \mathcal{O}_Y) = 0$.

**Lemma 2.1** (Nagel). Let $\mathcal{A}$ be a sheaf of local $\mathbb{C}$-algebras on $Y$ such that the above three conditions are satisfied. Suppose that $I$ is an ideal in $\Gamma(Y, \mathcal{A})$, and that there is a finite subset $\{f_1, f_2, \ldots, f_m\} \subset I$, so that for every $y \in Y$, there is an $f_j$ such that $f_j(y) \neq 0$. Then $I = (f_1, f_2, \ldots, f_m) = \Gamma(Y, \mathcal{A})$.

**Lemma 2.2.** Let $Y$ be an open subset of a Stein space $X$ such that $H^i(Y, \mathcal{O}_Y) = 0$ for all $i > 0$. If $h \in H^0(Y, \mathcal{O}_Y)$ is not a zero divisor of the stalk $\mathcal{O}_y$ at every point $y \in Y$, then $H^i(Z, \mathcal{O}_Z) = 0$ for all $i > 0$, where $Z = \{y \in Y, h(y) = 0\}$ is the hypersurface defined by the holomorphic function $h$.

**Proof.** If $h$ is a unit in $H^0(Y, \mathcal{O}_Y)$, then $h$ does not vanish on $Y$ and $Z$ is an empty set. We assume that $h$ is not a unit on $Y$. The multiplication by $h$ defines an injective map from $\mathcal{O}_Y$ to itself. $Z$ is a hypersurface of pure codimension 1 on $Y$ [5, p. 100] and we have a short exact sequence

$$0 \to \mathcal{O}_Y \to \mathcal{O}_Y \to \mathcal{O}_Y/h\mathcal{O}_Y \cong \mathcal{O}_Z \to 0,$$
where the first map is defined by the non-zero divisor (a holomorphic function) \( h \) on \( X \). Since \( H^i(Y, \mathcal{O}_Y) = 0 \) for all \( i > 0 \), the corresponding long exact sequence gives

\[ 0 \to H^0(Y, \mathcal{O}_Y) \to H^0(Y, \mathcal{O}_Y) \to H^0(Y, \mathcal{O}_Z) \to 0, \]

and \( H^i(Z, \mathcal{O}_Z) = 0 \).

A holomorphic map \( f: X \to X' \) is proper if for every compact subset \( K \subset X' \), the inverse image \( f^{-1}(K) \subset X \) is a compact subset in \( X \). Remmert’s Proper Mapping Theorem states that for any proper holomorphic map \( f: X \to X' \) between complex spaces, the image \( f(X) \) is an analytic subset of \( X' \) [5, p. 213].

In Theorem 2.3, the dimension of the Stein space \( X \) is arbitrary.

**Theorem 2.3.** If \( Y \) is an open subset of a Stein space \( X \) such that every accumulation point \( P_0 \in X - Y \) of a discrete sequence in \( Y \) is the only common zero of finitely many holomorphic functions \( f_1, \ldots, f_m \) on \( X \), then \( Y \) is Stein if and only if \( H^i(Y, \mathcal{O}_Y) = 0 \) for all \( i > 0 \).

**Proof.** If \( Y \) is Stein, then for any coherent analytic sheaf \( \mathcal{F} \) on \( Y \) and all \( i > 0 \), by Theorem B, \( H^i(Y, \mathcal{F}) = 0 \). By Oka’s theorem, the structure sheaf \( \mathcal{O}_Y \) is coherent [5, p. 60] so \( H^i(Y, \mathcal{O}_Y) = 0 \). We only need to show that if \( H^i(Y, \mathcal{O}_Y) = 0 \) for all \( i > 0 \), then \( Y \) is Stein.

\( Y \) is holomorphically separable since holomorphic functions on \( X \) separate points on the open subset \( Y \). We will show that \( Y \) is holomorphically convex.

Let \( S = \{P_1, P_2, \ldots, P_k, \ldots\} \) be a discrete sequence in \( Y \). If \( S \) has no accumulation points in \( X \), then there is a holomorphic function \( f \) on \( X \) such that \( f \) is not bounded on \( S \). We are done. We may assume that \( S \) has an accumulation point \( P_0 \in X - Y \).

Since there are finitely many holomorphic functions \( f_1, \ldots, f_m \) on \( X \) such that \( P_0 \notin Y \) is their only common zero, for every point \( y \in Y \), at least one \( f_j \) does not vanish at \( y \). By Lemma 2.1, \( f_1, f_2, \ldots, f_m \) generate the ring \( H^0(Y, \mathcal{O}_Y) = \Gamma(Y, \mathcal{O}_Y) \). Particularly, there are \( g_1, g_2, \ldots, g_m \in H^0(Y, \mathcal{O}_Y) \) such that

\[ f_1g_1 + f_2g_2 + \cdots + f_mg_m = 1 \]
on \( Y \).

Every holomorphic function \( f_i \) is continuous on \( X \) and \( f_i(P_0) = 0 \), \( i = 1, 2, \ldots, m \), so its limit at \( P_0 \in X - Y \) is 0. By the equation, at least one \( g_j \) has limit infinity at \( P_0 \). This implies that \( g_j \) is not bounded on the discrete sequence \( S \) since \( P_0 \in X - Y \) is an accumulation point of \( S \subset Y \).

We show that \( Y \) is holomorphically convex so it is Stein. \( \square \)

**Lemma 2.4.** Let \( Y \) be an open subset of a Stein space \( X \) with dimension \( n \) such that \( H^i(Y, \mathcal{O}_Y) = 0 \) for all \( i > 0 \). Then \( Y \) is holomorphically convex.
Proof. Since $X$ is a Stein space of dimension $n$, there is a one-to-one, proper holomorphic map from $X$ to $\mathbb{C}^{2n+1}$ [13].

Let $S = \{P_1, P_2, \ldots, P_k, \ldots\}$ be a discrete sequence in $Y$. As in the proof of Theorem 2.3, we may assume that $S$ has an accumulation point $P_0 \in X - Y$. By Narasimhan’s theorem, let $\psi: X \to \mathbb{C}^{2n+1}$ be a one-to-one, proper holomorphic map which is regular at every uniformizable point [13]. By Remmert’s Proper Mapping Theorem, $\psi(X)$ is a closed subspace of $\mathbb{C}^{2n+1}$. By affine algebraic geometry, there are $m$ polynomials $f_1, f_2, \ldots, f_m$ in $\mathbb{C}^{2n+1}$ such that the point $\psi(P_0)$ in $\psi(X)$ (not in $\psi(Y)$) is the only point in the zero set
\[
\{x \in \psi(X), f_1(x) = f_2(x) = \cdots = f_m(x) = 0\}.
\]

Pull these polynomials back to $X$ by the proper injective holomorphic map $\psi$, we receive $m$ holomorphic functions (still denoted by $f_i$ for simplicity) $f_1, f_2, \ldots, f_m$ in $X$ such that their only common zero is the point $P_0 \in X - Y$. So for every point $y \in Y$, at least one $f_j$ does not vanish at $y$. By Lemma 2.1, $f_1, f_2, \ldots, f_m$ generate $\Gamma(Y, \mathcal{O}_Y)$ and the rest of the proof is the same as proof of Theorem 2.3.

We show that $Y$ is holomorphically convex so it is Stein. □

By Lemma 2.4, we have

**Theorem 2.5.** If $Y$ is an open subset of a Stein space $X$ of dimension $n$, then $Y$ is Stein if and only if $H^i(Y, \mathcal{O}_Y) = 0$ for all $i > 0$.

**Definition 2.6.**  
(1) If $I$ is an ideal of a ring $R$, then the set
\[
\sqrt{I} = \{r \in R, r^j \in I, j \in \mathbb{N}\}
\]
is an ideal of $R$ called the radical of $I$ in $R$.

(2) An element $r \in R$ is an nilpotent element if there is a positive integer $n$ such that $r^n = 0$.

(3) The radical $N = \sqrt{0}$ is called the nilradical of $R$.

(4) The ideal $I$ in a commutative ring $R$ is reduced if for $r \in R$, there is an integer $m \in \mathbb{N}$, $r^m \in I$, then $r \in I$.

**Definition 2.7.** The radical sheaf $\mathcal{N} = \sqrt{0}$ of the zero ideal in the structure sheaf $\mathcal{O}_X$ of a complex space $X$ is called the nilradical of $\mathcal{O}_X$.

By Definition 2.7, the stalk $\mathcal{N}_x$ is the ideal of all nilpotent germs in the stalk $\mathcal{O}_x$. For a complex space $X$, the nilradical $\mathcal{N}$ is a coherent ideal sheaf of $\mathcal{O}_X$ [5, p. 86]. A complex space $X$ is reduced at a point $x_0 \in X$ if the stalk $\mathcal{O}_{x_0}$ is reduced: $\mathcal{O}_{x_0}$ has no nilpotent
elements. \( X \) is reduced if for all points \( x \in X \), all stalks \( \mathcal{O}_x \) are reduced rings, i.e., if \( f_x \in \mathcal{O}_x \) and \( f_x^m = 0 \) for some \( m \in \mathbb{N} \) (\( m \) relies on the point \( x \) and the function \( f \)), then \( f_x = 0 \) [8, p. 151].

**Definition 2.8.** A germ \( f_x \in \mathcal{O}_x \) at a point \( x \) in a complex space \( X \) is active if for every \( g_x \in \mathcal{O}_x \), with \( f_x g_x \in \mathcal{N}_x \), we have \( g_x \in \mathcal{N}_x \).

The set of points of a complex space \( X \) where \( X \) is not reduced is an analytic subset of \( X \) [5, p. 88]. A holomorphic function \( f \) on \( X \) is active at a point \( x \in X \) if there is an open neighborhood \( U \) of \( x \) such that \( f \) does not vanish at every irreducible component of \( X \) in \( U \) [5, p. 98].

**Lemma 2.9.** Let \( \{1, f_1, f_2, \ldots, f_m, \ldots\} \subset V \) be linearly independent in a vector space \( V \) over \( \mathbb{C} \). Then for any constant \( a_i \in \mathbb{C} \), \( \{1, f_1 - a_1, f_2 - a_2, \ldots, f_m - a_m, \ldots\} \) is also linearly independent in \( V \).

**Proof.** For any \( m \in \mathbb{N} \), let \( c_i \in \mathbb{C} \), \( i = 0, 1, 2, \ldots, m \) and

\[
0 + c_1(f_1 - a_1) + c_2(f_2 - a_2) + \cdots + c_m(f_m - a_m) = 0.
\]

Then

\[
(c_0 - c_1a_1 - c_2a_2 - \cdots - c_ma_m) + c_1 f_1 + c_2 f_2 + \cdots + c_m f_m = 0.
\]

Since \( 1, f_1, f_2, \ldots, f_m \) are linearly independent, we have

\[
c_0 - c_1a_1 - c_2a_2 - \cdots - c_ma_m = c_1 = c_2 = \cdots = c_m = 0.
\]

So \( c_0 = c_1 = c_2 = \cdots = c_m = 0 \) and \( 1, f_1 - a_1, f_2 - a_2, \ldots, f_m - a_m \) are linearly independent. Similarly, we can show that any finite subset of \( \{1, f_1 - a_1, f_2 - a_2, \ldots, f_m - a_m, \ldots\} \) is linearly independent. Therefore, \( \{1, f_1 - a_1, f_2 - a_2, \ldots, f_m - a_m, \ldots\} \) is linearly independent. \( \square \)

**Lemma 2.10.** If \( X \) is a Stein space of dimension at least 1, then the dimension \( h^0(X, \mathcal{O}_X) \) of the vector space \( H^0(X, \mathcal{O}_X) \) over \( \mathbb{C} \) is not finite.

**Proof.** We will construct infinitely many holomorphic functions on \( X \) which are linearly independent.

Let \( C \) be an irreducible analytic curve in \( X \) and \( \mathcal{I}_C \) be its ideal sheaf in \( X \) such that \( C \cap Y \) is an open subset of \( C \) and contains smooth points in \( X \). Then \( \mathcal{I}_C \) is coherent analytic sheaf on \( X \) [5, p. 84]. We have a short exact sequence

\[
0 \to \mathcal{I}_C \to \mathcal{O}_X \to \mathcal{O}_X/\mathcal{I}_C = \mathcal{O}_C \to 0.
\]

Since \( X \) is Stein, \( H^1(X, \mathcal{I}_C) = 0 \) and we have a surjective map \( H^0(X, \mathcal{O}_X) \to H^0(C, \mathcal{O}_C) \).
Let $P_0$ be a smooth point of the curve $C$ and $X$ and $z$ be the local coordinate at $P_0$ on $C$. Now $C$ is a Stein curve so there is a nonconstant holomorphic function $f$ on $C$ such that $f(z) = z + z^2g(z)$ near $P_0$ [4, p. 151], where $g(z)$ is holomorphic near $P_0$. Then $1, f, f^2, \ldots, f^m, \ldots$ are holomorphic functions on $C$ and linearly independent over $\mathbb{C}$. Since $h^0(C, \mathcal{O}_C) = \infty$, we have $h^0(X, \mathcal{O}_X) = \infty$. $\square$

**Lemma 2.11.** If $X$ is a reduced Stein space of arbitrary dimension, then there are infinitely many holomorphic functions $1, f_1 - a_1, f_2 - a_2, \ldots, f_m - a_m, \ldots$ on $X$ such that they are linearly independent in $H^0(Y, \mathcal{O}_Y)$ and each of $f_1 - a_1, f_2 - a_2, \ldots, f_m - a_m, \ldots$ defines a reduced hypersurface on $Y$.

**Proof.** By Lemma 2.10 there are holomorphic functions $1, f_1, f_2, \ldots, f_m, \ldots$ on $X$ such that they are linearly independent in the vector space $H^0(X, \mathcal{O}_X)$ over $\mathbb{C}$. Each function $f_i$ gives a nonconstant holomorphic map from $X$ to $\mathbb{C}$. By open mapping theorem, if $f_i$ is not a constant near a point $p \in X$, then the map $f_i: X \to \mathbb{C}$ is open near $p$ [5, p. 109]. So the image $f_i(X)$ contains an open subset $V$ of $\mathbb{C}$. By the Sard type theorem, there is a countable subset $B \subset \mathbb{C}$ such that for every point $a_i \in \mathbb{C} - B$, the fiber (hypersurface) $X_{a_i} = f_i^{-1}(a_i)$ is reduced [11]. We may choose suitable $a_i$ such that each $f_i - a_i$ defines a reduced hypersurface in $Y$.

By the construction in the proof of Lemma 2.10, we may choose these holomorphic functions so that they are linearly independent on an irreducible curve $C$ (i.e., linearly independent in $H^0(C, \mathcal{O}_C)$) in $X$ such that $C \cap Y$ is an open subset of $C$, then they are linearly independent in $H^0(Y, \mathcal{O}_Y)$. This is because if we have

$$c_0 + c_1f_1 + c_2f_2 + \cdots + c_mf_m = 0,$$

on $Y$, then $c_0 + c_1f_1 + c_2f_2 + \cdots + c_mf_m = 0$ on the curve $C \cap Y$. By the Identity Theorem [5, p. 170], the equation holds on the irreducible curve $C$. But these functions are linearly independent on $C$, we have $c_0 = c_1 = \cdots = c_m = 0$ and they are linearly independent on $Y$. By Lemma 2.9, $\{1, f_1 - a_1, f_2 - a_2, \ldots, f_m - a_m, \ldots\}$ is linearly independent in $H^0(Y, \mathcal{O}_Y)$. $\square$

**Lemma 2.12.** Let $Y$ be an open subset of a Stein space $X$ of arbitrary dimension such that $H^i(Y, \mathcal{O}_Y)$ is a finite dimensional vector space over $\mathbb{C}$ for all $i > 0$, then the dimension $h^i(Z, \mathcal{O}_Z) < \infty$ for every hypersurface $Z$ defined by a holomorphic function $h$ on $Y$ which is not a zero divisor of $\mathcal{O}_y$ at every point $y \in Y$.

**Proof.** Since $h$ is not a zero divisor on $Y$, we have a short exact sequence

$$0 \to \mathcal{O}_Y \to \mathcal{O}_Y \to \mathcal{O}_Y/h\mathcal{O}_Y \cong \mathcal{O}_Z \to 0,$$
where the first map is defined by $h$. The corresponding long exact sequence gives
\[
0 \longrightarrow H^0(Y, \mathcal{O}_Y) \longrightarrow H^0(Y, \mathcal{O}_Y) \longrightarrow H^0(Z, \mathcal{O}_Z) \longrightarrow H^1(Y, \mathcal{O}_Y) \longrightarrow H^1(Y, \mathcal{O}_Y) \\
\alpha \longrightarrow H^1(Z, \mathcal{O}_Z) \longrightarrow H^2(Y, \mathcal{O}_Y) \longrightarrow H^2(Y, \mathcal{O}_Y) \longrightarrow H^2(Z, \mathcal{O}_Z) \longrightarrow \cdots.
\]

The sequence is exact at $H^1(Z, \mathcal{O}_Z)$ so the relationship between the image of $\alpha$ and kernel of $\beta$ is
\[
\text{im}(\alpha) = \alpha(H^1(Y, \mathcal{O}_Y)) = \ker(\beta) \subset H^1(Z, \mathcal{O}_Z)
\]
and
\[
\dim \mathbb{C} \ker(\beta) = \dim \mathbb{C} \text{im}(\alpha) \leq h^1(Y, \mathcal{O}_Y) < \infty.
\]

$\beta$ is a homomorphism from the vector space $H^1(Z, \mathcal{O}_Z)$ to the vector space $H^2(Y, \mathcal{O}_Y)$ [16, pp. 627–629], so the image vector space $\text{im}(\beta)$ is a subspace of $H^2(Y, \mathcal{O}_Y)$. This implies
\[
\dim \mathbb{C} \text{im}(\beta) \leq h^2(Y, \mathcal{O}_Y) < \infty.
\]

By the rank theorem in linear algebra, these two inequalities give $h^1(Z, \mathcal{O}_Z) < \infty$. Using the fact that the sequence is exact at $H^i(Z, \mathcal{O}_Z)$ and $h^i(Y, \mathcal{O}_Y) < \infty$ for all $i > 0$, $h^i(Z, \mathcal{O}_Z) < \infty$ can be similarly proved.

**Lemma 2.13.** Let $Y$ be an open subset of a reduced Stein space $X$ of dimension $n$ such that $H^i(Y, \mathcal{O}_Y)$ is a finite dimensional vector space over $\mathbb{C}$ for all $i > 0$, then $H^i(Y, \mathcal{O}_Y) = 0$.

**Proof.** For every holomorphic function $f \in H^0(Y, \mathcal{O}_Y)$, the multiplication by $f$ induces a homomorphism:
\[
f^{*i}: H^i(Y, \mathcal{O}_Y) \longrightarrow H^i(Y, \mathcal{O}_Y)
\]
and the map $f \to f^{*i}$ is a $\mathbb{C}$-homomorphism (see [3] or [16, pp. 627–629]):
\[
H^0(Y, \mathcal{O}_Y) \longrightarrow \text{End}_\mathbb{C}(H^i(Y, \mathcal{O}_Y)),
\]
where $\text{End}_\mathbb{C}(V)$ is the set of all vector homomorphisms (linear transformations) from a vector space $V$ over $\mathbb{C}$ to itself. Since $H^i(Y, \mathcal{O}_Y)$ is a finite dimensional vector space over $\mathbb{C}$ for all $i > 0$, $\text{End}_\mathbb{C}(H^i(Y, \mathcal{O}_Y))$ is also a finite dimensional vector space over $\mathbb{C}$ for all $i > 0$.

By Lemma 2.11, there are infinitely many holomorphic functions $1, f_1, f_2, \ldots, f_m, \ldots$ on $X$ such that they are linearly independent and each of them defines a reduced hypersurface on $X$. Each $f_j$ defines a homomorphism $f_j^{*i}$ from the vector space $H^i(Y, \mathcal{O}_Y)$ to itself. But $\text{End}_\mathbb{C}(H^i(Y, \mathcal{O}_Y))$ is a finite dimensional vector space over $\mathbb{C}$ for all $i > 0$, so for each $i$, there is an $f_i \subset \{f_1, f_2, \ldots, f_m, \ldots\}$ such that it induces a zero map from $H^i(Y, \mathcal{O}_Y)$ to itself. By the choice of the functions, $f_i$ defines a reduced hypersurface $Z_i$. 
We will use mathematical induction on the dimension of $X$ to show that for all $i > 0$, $H^i(Y, \mathcal{O}_Y) = 0$.

If $X$ is a Stein curve, then $Y$ is an open subset of $X$ and for any coherent sheaf $\mathcal{F}$ on $Y$ and all $i > 0$, $H^i(Y, \mathcal{F}) = 0$ \[19\] so $H^i(Y, \mathcal{O}_Y) = 0$ for all $i > 0$.

If $X$ is a Stein surface and $h^i(Y, \mathcal{O}_Y) < \infty$ for all $i > 0$, then the hypersurface $Z$ defined by a holomorphic function (non-zero divisor) $f$ on $Y$ is Stein \[19\] so $H^i(Z, \mathcal{O}_Z) = 0$ and $H^2(Z, \mathcal{O}_Z) = 0$. Since $Y$ is an open surface, $H^i(Y, \mathcal{O}_Y) = 0$ for all $i > 1$ \[19\].

By the above long exact sequence, \[ f^*: H^1(Y, \mathcal{O}_Y) \to H^1(Y, \mathcal{O}_Y) \] is surjective. Now for all $j \in \mathbb{N}$, we have infinitely many surjective group homomorphisms $f_{ij}^*$ of a finite dimensional vector space $H^1(Y, \mathcal{O}_Y)$

\[ f_{ij}^*: H^1(Y, \mathcal{O}_Y) \to H^1(Y, \mathcal{O}_Y) \]

induced by each $f_j \in \{f_1, f_2, \ldots, f_m, \ldots\}$. Because $\text{End}_\mathbb{C}(H^1(Y, \mathcal{O}_Y))$ is a finite dimensional vector space over $\mathbb{C}$ for all $i > 0$, by counting the dimensions of vector spaces, we see that there is a $k \in \mathbb{N}$ such that $f_{ij}^* = 0$ \[3\]. But $f_{ij}^*$ is a surjective map from $H^1(Y, \mathcal{O}_Y)$ to itself. We see $H^1(Y, \mathcal{O}_Y) = 0$.

We receive $H^i(Y, \mathcal{O}_Y) = 0$ for all $i > 0$.

By mathematical induction, we may assume that if dimension of $X$ is $n - 1$, and $h^i(Y, \mathcal{O}_Y) < \infty$ for all $i > 0$, then $H^i(Y, \mathcal{O}_Y) = 0$ for all $i > 0$.

Let $X$ be a Stein space of dimension $n$ in the lemma and $h^i(Y, \mathcal{O}_Y) < \infty$ for all $i > 0$. By Lemma 2.12 any reduced hypersurface $Z$ defined by a holomorphic function satisfies $h^i(Z, \mathcal{O}_Z) < \infty$ for all $i > 0$. By inductive assumption, $H^i(Z, \mathcal{O}_Z) = 0$ for all $i > 0$. Using the long exact sequence, for every $f_j \in \{f_1, f_2, \ldots, f_m, \ldots\}$ we have infinitely many surjective maps

\[ f_{ij}^*: H^1(Y, \mathcal{O}_Y) \to H^1(Y, \mathcal{O}_Y) \]

and isomorphisms

\[ f_{ij}^*: H^i(Y, \mathcal{O}_Y) \to H^i(Y, \mathcal{O}_Y) \]

for $i > 1$. By counting the dimensions of the vector spaces, we see that for all $i > 0$, $H^i(Y, \mathcal{O}_Y) = 0$ because $\text{End}_\mathbb{C}(H^1(Y, \mathcal{O}_Y))$ is a finite dimensional vector space over $\mathbb{C}$ for all $i > 0$.

**Lemma 2.14.** Let $Y$ be an open subset of a reduced Stein space $X$ of dimension $n$ such that $H^i(Y, \mathcal{O}_Y)$ is a finite dimensional vector space over $\mathbb{C}$ for all $i > 0$, then the ring of holomorphic functions on $Y$ is finitely generated: there are holomorphic functions $f_1, f_2, \ldots, f_m$ on $Y$ such that they generate $H^0(Y, \mathcal{O}_Y)$.\[\square\]
Proof. By Lemma 2.13, $H^i(Y, \mathcal{O}_Y) = 0$ for all $i > 0$. By Lemma 2.1 and proof of Lemma 2.4, $H^0(Y, \mathcal{O}_Y)$ is generated by finitely many holomorphic functions $f_1, f_2, \ldots, f_m$ on $Y$.

**Theorem 2.15.** Let $Y$ be an open subset of a reduced Stein space $X$ with dimension $n$. Then $Y$ is Stein if and only if $H^i(Y, \mathcal{O}_Y)$ is a finite dimensional complex vector space for every $i > 0$.

Proof. By Lemma 2.13, if $H^i(Y, \mathcal{O}_Y)$ is a finite dimensional complex vector space for every $i > 0$, then $H^i(Y, \mathcal{O}_Y) = 0$. By Theorem 2.5, $Y$ is Stein.

In Section 3, we will prove that Theorem 2.15 holds if the dimension of $X$ is not finite.

3. Spaces with arbitrary dimension

We will first prove Theorem 1.2 in this section.

By Remmert’s Proper Mapping Theorem (see [5, p. 213] or [15, Satz 23]), if $f: X \to Y$ is a proper holomorphic map, then the image of any analytic set in $X$ is again analytic in $Y$. If $f$ is not proper, this is not true.

**Definition 3.1.** A subset $A$ of a complex space $X$ is said to be analytically meagre if $A \subset \bigcup_{i \in \mathbb{N}} A_i$, where each $A_i$ is a locally analytic subset of $X$ with codimension at least 1.

An analytically meagre subset of a curve is a countable set [11]. If $f$ is not proper, Remmert proved (see [11] or [15, Satz 20]).

**Lemma 3.2** (Remmert). If $f: X \to Y$ is a holomorphic map between complex spaces and $Z$ is an analytic subset of $X$, then $f(Z)$ is a countable union of locally analytic subsets of $Y$. In particular, if the interior of $f(Z)$ is empty, then $f(Z)$ is analytically meagre.

An analytic subset $Z$ in a complex space $X$ is always nowhere dense in $X$ if $Z$ is at least 1 codimensional in $X$ and $Z$ contains interior points of $X$ if $Z$ contains an irreducible component of $X$ [5, pp. 102–103].

**Lemma 3.3** (Sard). If $X$ is a complex manifold and $f: X \to \mathbb{C}$ is a holomorphic function, then there exists a countable subset $A \subset \mathbb{C}$ such that for each $c \in \mathbb{C} - A$, the fiber $X_c = f^{-1}(c)$ is a manifold.

The following construction is inspired by Remmert and Sard’s theorems.

**Lemma 3.4.** Let $Y$ be an open subset of a reduced Stein space $X$, then there is a holomorphic function $h$ on $X$ such that for any $a \in \mathbb{C} - A$, the hypersurface defined by $h - a$ on $X$ is a complex manifold $H = H_1 \cup H_2 \cup \cdots$ of codimension 1 in $X$ and for all $i \neq j$, $H_i \cap H_j = \emptyset$, where $A$ is a countable subset in $\mathbb{C}$.
Proof. Since $X$ is reduced, the singular locus $X_{\text{sing}}$ of $X$ is a nowhere dense analytic subset of $X$ (i.e., for every open subset $U$ in $X$, $U \cap X_{\text{sing}}$ is not dense in $U$) such that the local dimension at $x$: $\dim_x X_{\text{sing}} < \dim_x X$ [5, p. 117]. Let $X = X_1 \cup X_2 \cup \cdots$ be the decomposition of $X$ into irreducible components [4, p. 19]. The singular locus $X_{\text{sing}}$ consists of all intersection points of $X_i \cap X_j$, $i \neq j$ and the singular points of each component $X_i$ [5, p. 117].

First, we claim that there is a holomorphic function $h$ on $X$ such that it is not a constant on $Y$ and $h(X_{\text{sing}})$ is nowhere dense in $C$. In fact, let $C$ be a holomorphic curve in $X$ such that $C \cap Y$ is an open subset of $C$ and $C \cap X_{\text{sing}}$ is empty (that is, $C$ contains no singular points of $X$). Let $B = C \cup X_{\text{sing}}$, then $B$ is a closed subspace of $X$ locally defined by holomorphic functions [5, p. 15]. Let $\mathcal{I}_B$ be the ideal sheaf of $B$, then we have a short exact sequence

$$0 \rightarrow \mathcal{I}_B \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_{C \cup X_{\text{sing}}} \rightarrow 0.$$ 

Since $X$ is Stein and $\mathcal{I}_B$ is a coherent ideal sheaf, $H^1(X, \mathcal{I}_B) = 0$. We have a surjective map $H^0(X, \mathcal{O}_X) \rightarrow H^0(B, \mathcal{O}_{C \cup X_{\text{sing}}})$. By the fact that $C$ and $X_{\text{sing}}$ are disconnected, we may construct a holomorphic function $h$ on $B$ such that $h$ is not a constant on $C$ and $h(X_{\text{sing}})$ is nowhere dense in $C$ (for example, we may choose $h$ such that it is a constant on every connected component of $X_{\text{sing}}$).

By Lemma 3.2 $h(X_{\text{sing}}) = A_1$ is a countable union of locally analytic subsets so is a countable subset of $C$. For any $a \in C - A_1$, the fiber $X_a = h^{-1}(a)$ has no intersection points with the singular locus $X_{\text{sing}}$ of $X$. But the hypersurface $X_a \subset X - X_{\text{sing}}$ may have singular points as a closed subspace of $X$. Now the restriction $h: X - X_{\text{sing}} \rightarrow C$ is a holomorphic function on the complex manifold $X - X_{\text{sing}}$. By Sard’s Theorem, there exists a countable subset $A_2 \subset C$ such that for each $c \in C - A_2$, the fiber

$$X_c \cap (X - X_{\text{sing}}) = h^{-1}(c) \cap (X - X_{\text{sing}}) \subset X - X_{\text{sing}}$$ 

is a manifold. $h$ may be a constant at some irreducible component of $X$. Since $X$ has at most a countably many irreducible components [4, p. 19], there is a countable subset $A_3 \subset C$ such that for every $a \in C - A_3$, the fiber $X_a = h^{-1}(a)$ is of pure codimension 1 in $X$. Let $A = A_1 \cup A_2 \cup A_3$, then $A$ is the union of three countable subsets so is a countable subset of $C$. For all $a \in C - A$, the fiber $X_a = h^{-1}(a)$ is of pure codimension 1 in $X$, smooth and can be decomposed into the union of disjoint complex manifolds. Therefore $H = X_a = H_1 \cup H_2 \cup \cdots$ is a smooth hypersurface in $X$, each $H_i$ is irreducible and for all $i \neq j$, $H_i \cap H_j = \emptyset$. \qed

Lemma 3.5. Let $Y$ be an open subset of a reduced Stein space $X$ such that $H^i(Y, \mathcal{O}_Y)$ is a finite dimensional vector space over $\mathbb{C}$ for all $i > 0$. In above lemma, for every irreducible
component $H_j$ of hypersurface $H$, let $Z = Y \cap H$ and $Z_j = Y \cap H_j$, then $Z_j$ is an open subset in $H_j$ and

$$h^i(Z_j, \mathcal{O}_Z) < \infty.$$  

Proof. We may assume that $X$ contains no isolated points since $X$ is Stein then every connected component of $X$ is Stein [4, p. 125]. Let $Z = H \cap Y = Z_1 \cup Z_2 \cup \cdots$, where $Z_i = Y \cap H_i$ is either empty or an open subset in $H_i$ (the subspace topology on $H_i$ is induced from the topology on $X$ since $Y$ is open in $X$ and $H_i$ is a closed subspace of $X$).

If $Z_j$ is an empty set or a set of points, then the inequality is true. We may assume that the dimension of $Z_j$ is at least one. Let $Z = Z_j \cup Z'_j$, where $Z'_j = Z - Z_j$ is the complement of $Z_j$ in $Z$. By the construction in Lemma 3.4, $Z_j \cap Z'_j$ is empty and by Mayer–Vietoris sequence [1, p. 30], we have

$$
0 \longrightarrow H^0(Z, \mathcal{O}_Z) \longrightarrow H^0(Z_j, \mathcal{O}_Z) \oplus H^0(Z'_j, \mathcal{O}_Z) \longrightarrow H^0(Z_j \cap Z'_j, \mathcal{O}_Z)
$$

$$
\longrightarrow H^1(Z, \mathcal{O}_Z) \longrightarrow H^1(Z_j, \mathcal{O}_Z) \oplus H^1(Z'_j, \mathcal{O}_Z) \longrightarrow H^1(Z_j \cap Z'_j, \mathcal{O}_Z)
$$

$$
\longrightarrow H^2(Z, \mathcal{O}_Z) \longrightarrow H^2(Z_j, \mathcal{O}_Z) \oplus H^2(Z'_j, \mathcal{O}_Z) \longrightarrow H^2(Z_j \cap Z'_j, \mathcal{O}_Z) \longrightarrow \cdots.
$$

Since $Z_j \cap Z'_j = \emptyset$, $H^i(Z_j \cap Z'_j, \mathcal{O}_Z) = 0$ for all $i \geq 0$, we have

$$H^i(Z, \mathcal{O}_Z) \cong H^i(Z_j, \mathcal{O}_Z) \oplus H^i(Z'_j, \mathcal{O}_Z).$$

By Lemma 2.12 for all $i > 0$,

$$h^i(Z, \mathcal{O}_Z) < \infty,$$

so $h^i(Z_j, \mathcal{O}_Z) < \infty$, and $h^i(Z'_j, \mathcal{O}_Z) < \infty$. \hfill \Box

Lemma 3.6. In above lemma, for every irreducible component $H_j$ of hypersurface $H$ such that $Z_j = Y \cap H_j \neq \emptyset$, $Z_j$ is a Stein subset in $H_j$.

Proof. The hypersurface $H$ in $X$ is Stein [4, p. 125]. Since $H = H_1 \cup H_2 \cup \cdots$ and for all $i \neq j$, $H_i \cap H_j = \emptyset$, every irreducible (thus connected) component $H_i$ is Stein [4, p. 125]. For each irreducible component $H_i$, its dimension is a constant [5, p. 169] even though the dimension of $H$ may not be finite. By Lemma 3.5 and Theorem 2.15, the nonempty open subset $Z_j$ in $H_j$ is a Stein open subset in $H_j$. \hfill \Box

Lemma 3.7. In above lemmas, the hypersurface $Z = H \cap Y$ in the open subset $Y$ is holomorphically convex therefore is Stein.

Proof. Let $S = \{P_1, P_2, \ldots, P_k, \ldots\}$ be a discrete sequence in $Z = Z_1 \cup Z_2 \cup \cdots$, where $Z_i = H_i \cap Y$. As in the proof of Theorem 2.3, we may assume that it has an accumulation point $P_0$ in $X$. 

If there is an irreducible hypersurface $Z_j \subset H_j \subset H$ such that $Z_j \cap S \subset H_j \cap S$ contains infinitely many points of $S$, then there is a holomorphic function $f$ on $Z_j$ such that $f$ is not bounded on $Z_j \cap S$. By Mayer–Vietoris sequence,

$$H^i(Z, \mathcal{O}_Z) \cong H^i(Z_j, \mathcal{O}_Z) \oplus H^i(Z'_j, \mathcal{O}_Z),$$

we can extend $f$ to the complement $Z'_j$ of $Z_j$ in $Z$ by zero since $Z_j$ and $Z'_j$ are disconnected. In this way, we receive a holomorphic function $f$ on $Z$ such that it is not bounded on $S$.

Now we assume that every nonempty component $Z_i$ only contains finitely many points of $S$ and $S$ has an accumulation point $P_0$ in $X$. Choose a subsequence $\{P_{n_i}\}_{i=1}^{\infty}$ in $S$ such that $P_0 \in X - Z$ is its limit point. Let the holomorphic function $h$ define the hypersurface $H$ in $X$. Since $h(P_i) = 0$ for all $i$, we have $h(P_0) = 0$. This implies that $P_0$ is a point on some irreducible component $H_k$ of $H$. By Lasker–Noether Decomposition Theorem, there is an open subset $U \ni P_0$ in $X$ such that in $U$, $H$ has only finitely many components: $H \cap U = H_{i_1} \cup H_{i_2} \cup \cdots \cup H_{i_m}$ \cite[pp. 78–79]{5}. But each irreducible component in $H \cap U$ contains only finitely many points of $S$. $P_0$ cannot be an accumulation point of $S$. The contradiction implies that if $S = \{P_1, P_2, \ldots, P_k, \ldots\} \subset Z$ has an accumulation point in $X$, then there is a component $H_j$ such that $H_j \cap S$ is not a finite set. By the above proof, we show that there is a holomorphic function $f$ on $Z$ such that it is not bounded on $S$. So the hypersurface $Z$ in the open subset $Y$ is holomorphically convex. \hfill \Box

**Lemma 3.8.** Let $Y$ be an open subset of a reduced Stein space $X$ such that $H^i(Y, \mathcal{O}_Y)$ is a finite dimensional vector space over $\mathbb{C}$ for all $i > 0$, Then

$$H^i(Y, \mathcal{O}_Y) = 0.$$

*Proof.* By the construction in Lemmas \ref{2.11} and \ref{3.4} let $f_1, f_2, \ldots, f_m, \ldots$ be holomorphic functions on $X$ such that $1, f_1, f_2, \ldots, f_m, \ldots$ are linearly independent in $H^0(Y, \mathcal{O}_Y)$ and for every $i$, each image $f_i(X_{\text{sing}})$ in $\mathbb{C}$ is nowhere dense. By Lemma \ref{3.4}, choose $a_j \in \mathbb{C}$ such that each fiber $X_{a_j} = f_i^{-1}(a_j)$ defines a pure codimension 1 complex manifold $X_{a_j}$ in $X$. By Lemma \ref{2.9}, $1, f_1 - a_1, f_2 - a_2, \ldots, f_m - a_m, \ldots$ are linearly independent in $H^0(Y, \mathcal{O}_Y)$. By Lemmas \ref{3.4} \ref{3.7} each $Y_{a_j} = Y \cap X_{a_j}$ is a smooth Stein hypersurface on $Y$, so $h^i(Y_{a_j}, \mathcal{O}_{Y_{a_j}}) = 0$ for all $i > 0$. Using the idea of the proof of Lemma \ref{2.13}, multiplicitating by each $f_j - a_j$ from $\mathcal{O}_Y$ to itself for all $j \in \mathbb{N}$, we have infinitely many surjective $\mathbb{C}$-homomorphisms $(f_j - a_j)^{\ast 1}$ of a finite dimensional vector space $H^1(Y, \mathcal{O}_Y)$

$$(f_j - a_j)^{\ast 1}: H^1(Y, \mathcal{O}_Y) \to H^1(Y, \mathcal{O}_Y)$$

and infinitely many $\mathbb{C}$-isomorphisms

$$(f_j - a_j)^{\ast i}: H^i(Y, \mathcal{O}_Y) \to H^i(Y, \mathcal{O}_Y),$$
which are induced by each \( f_j - a_j \in \{ f_1 - a_1, f_2 - a_2, \ldots, f_m - a_m, \ldots \} \) for \( i > 1 \). Comparing the dimensions of vector spaces, for all \( i > 0 \), we have \( H^i(Y, \mathcal{O}_Y) = 0 \).

**Theorem 3.9.** Let \( Y \) be an open subset of a reduced Stein space \( X \) such that \( H^i(Y, \mathcal{O}_Y) \) is a finite dimensional vector space over \( \mathbb{C} \) for all \( i > 0 \). Then \( Y \) is holomorphically convex therefore is Stein.

**Proof.** Let \( S = \{ P_1, P_2, \ldots, P_k, \ldots \} \) be a discrete sequence in \( Y \) and \( P_0 \) be its accumulation in \( X \). Since \( X \) is Stein, \( X \) is holomorphically spreadable, that is, there exist finitely many holomorphic functions \( f_1, f_2, \ldots, f_m \) on \( X \) such that \( P_0 \) is an isolated point in the zero set \( A = \{ x \in X, f_1(x) = f_2(x) = \cdots = f_m(x) = 0 \} \) [8, pp. 293–294]. We can write \( A = B \cup \{ P_0 \} \) then \( B \cap \{ P_0 \} \) is an empty set. Let \( \mathcal{I}_A \) be the ideal generated by \( f_1, f_2, \ldots, f_m \) in \( X \), then we have a short exact sequence

\[
0 \rightarrow \mathcal{I}_A \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_X/\mathcal{I}_A = \mathcal{O}_A \rightarrow 0.
\]

The ideal sheaf \( \mathcal{I}_A \) is coherent on the Stein space \( X \) [5, p. 84]. The long exact sequence and \( H^1(X, \mathcal{I}_A) = 0 \) give

\[
0 \rightarrow H^0(X, \mathcal{I}_A) \rightarrow H^0(X, \mathcal{O}_X) \rightarrow H^0(A, \mathcal{O}_A) \rightarrow 0.
\]

Let \( f_{m+1} \in H^0(A, \mathcal{O}_A) \) such that \( f_{m+1}(P_0) = 0 \) and \( B \cap \{ x \in X, f_{m+1}(x) = 0 \} = \emptyset \). Then there is a holomorphic function (still denoted by \( f_{m+1} \)) on \( X \) such that it vanishes at \( P_0 \) and does not vanish at every point of \( B \).

Now \( f_1, f_2, \ldots, f_m, f_{m+1} \) are holomorphic on \( X \) and have a unique common zero \( P_0 \) on \( X \). They have no common zeros on \( Y \). By Lemma 3.8 for all \( i > 0 \), \( H^i(Y, \mathcal{O}_Y) = 0 \). By Theorem 2.3, \( Y \) is Stein.

We have proved

**Theorem 3.10.** Let \( Y \) be an open subset of a reduced Stein space \( X \) with arbitrary dimension and singularities. Then \( Y \) is Stein if and only if \( H^i(Y, \mathcal{O}_Y) \) is a finite dimensional vector space over \( \mathbb{C} \) for all \( i > 0 \).

Next we will prove Theorem 1.4

**Definition 3.11.** A Weil divisor on a reduced complex space \( X \) is a locally finite linear combination with integral coefficients of irreducible reduced analytic subspaces of codimension 1 in \( X \) such that every subspace is not contained in the singular locus of \( X \).

The set of all Weil divisors form an abelian group. If \( D \) is a Weil divisor, then we can write \( D = \sum_{i=1}^{\infty} n_i D_i \), where \( n_i \in \mathbb{Z} \) and each \( D_i \) is an irreducible reduced analytic
subspace of codimension 1 in $X$ which is not contained in the singular locus of $X$ (see [2], [4, pp. 139–140], [6, pp. 130–143], or [20, pp. 35–36]).

The support of a Weil divisor $D$ is the union of all closed subspaces $D_i$ such that $n_i \neq 0$. $D$ is an effective divisor, written $D > 0$, if every coefficient $n_i \geq 0$ and $D$ is not a zero divisor. Two Weil divisors $D \geq D'$ if $D - D' \geq 0$, i.e., $D - D'$ is an effective divisor or a zero divisor in the space. When every coefficient $n_i = 1$, $D = \sum D_i$ is called a reduced divisor.

A reduced point $x \in X$ is a normal point of $X$ if the stalk $O_x$ is integrally closed in its quotient ring. A reduced complex space is normal if every point in the space is a normal point [5, p. 8]. If $X$ is a compact normal reduced complex space, then a Weil divisor $D$ is a finite sum on $X$: $D = \sum_{i=1}^{N} n_i D_i$ [20, p. 35].

If $X$ is normal, then the singular locus of $X$ is a closed subspace of codimension at least 2 in $X$ [5, p. 128]. A Weil divisor is well-defined as a linear combination of irreducible codimension one closed subspaces on a normal complex space $X$.

A Cartier divisor $D$ on a complex space $X$ is a global section of the sheaf $\mathcal{M}_X^*/\mathcal{O}_X^*$, where $\mathcal{M}_X^*$ is the sheaf of germs of not identically vanishing meromorphic functions on $X$ and $\mathcal{O}_X^*$ is the sheaf of germs of nowhere vanishing holomorphic functions on $X$. A Cartier divisor $D$ on a complex space $X$ can be described by an appropriate open cover $\{U_i\}_{i \in I}$ of $X$ and a collection of meromorphic functions $f_i$ on $U_i$, $i \in I$ such that on $U_i \cap U_j \neq \emptyset$, $\frac{f_i}{f_j}$ and $\frac{f_j}{f_i}$ are holomorphic (see [4, p. 138] or [20, p. 30]). $D$ is an effective Cartier divisor, written $D > 0$, if every $f_i$ is a holomorphic function and at least one of them has zeros [20, p. 31].

Every Cartier divisor on a normal reduced complex space $X$ defines a Weil divisor and if $X$ is nonsingular, then every Weil divisor is Cartier, i.e., locally it is defined by one equation. But if $X$ is not a complex manifold, then the Weil divisor $D$ is not a Cartier divisor in general, i.e., it is not locally defined by one equation [20, p. 36].

A Weil divisor $D$ is $\mathbb{Q}$-Cartier if there is an $n \in \mathbb{N}$ such that $nD$ is a Cartier divisor, i.e., $nD$ is locally defined by one equation.

**Example 3.12.** Let $X \in \mathbb{C}^4$ be a quadric threefold defined by

$$X = \{z = (z_1, z_2, z_3, z_4) \in A^4_k, p(z) = z_1^2 + z_2^2 + z_3^2 + z_4^2 = 0\}.$$ 

The structure sheaf

$$\mathcal{O}_X = \mathcal{O}_{\mathbb{C}^4}/p(z)\mathcal{O}_{\mathbb{C}^4}.$$ 

$X$ is a normal Stein variety with a unique isolated singularity at 0. Let $H$ be a hypersurface through 0 defined by

$$H = \{z = (z_1, z_2, z_3, z_4) \in X, z_1 = iz_2, z_3 = iz_4\}.$$
Example 3.12 shows that the open subset \( Y = X - A \) in a normal Stein space \( X \) obtained by removing a pure codimension 1 subspace \( A \) of \( X \) is not Stein in general if the dimension of \( X \) is at least 3. We give a sufficient condition:

**Theorem 3.13.** If \( Y \) is an open subset of a normal Stein space \( X \) such that the complement \( X - Y \) is a closed analytic subspace of \( X \) and the support of an effective \( \mathbb{Q} \)-Cartier divisor, then \( Y \) is Stein.

**Proof.** Let \( D' \) be the effective \( \mathbb{Q} \)-Cartier divisor with support \( X - Y \) on \( X \). Then there is an \( n \in \mathbb{N} \) such that \( D = nD' \) is an effective Cartier divisor with support \( X - Y \) on \( X \) [20, pp. 36–38].

Let \( \{ U_i \}_{i \in I} \) be a Stein open cover of \( X \) and let \( f_i \) be the holomorphic function on \( U_i \) defining \( D|_{U_i} \). Then for every point \( x \in U_i \), the stalk of the invertible sheaf (coherent) \( \mathcal{O}_X(D) \) is defined by [20, p. 30]

\[
\mathcal{O}_X(D)_x = \frac{1}{f_i} \mathcal{O}_x \cong \mathcal{O}_x.
\]

Let \( S = \{ P_1, P_2, \ldots \} \subset Y \) be a discrete sequence on \( Y \) with an accumulation point \( P_0 \in (X - Y) \cap U_i \) for some \( i \in I \). Since \( \mathcal{O}_X(D) \) is a coherent sheaf on \( X \), by Cartan’s Theorem A [4, p. 124], the module of global sections \( H^0(X, \mathcal{O}_X(D)) \) generates every stalk \( \mathcal{O}_X(D)_x \). There is a meromorphic function \( f \in H^0(X, \mathcal{O}_X(D)) \) (holomorphic on \( Y \) with poles in \( X - Y \)) and a local holomorphic function \( g \in \mathcal{O}_{P_0} \) such that near \( P_0 \) [4, p. 129],

\[
\frac{1}{f_i} = fg.
\]

Now \( f_i \) is a holomorphic function on \( U_i \cap Y \) and vanishes at \( P_0 \). So \( f(f_i g) = 1 \) near \( P_0 \) in \( Y \). From \( f_i(P_0) = 0 \), we see that \( f \) is not bounded near \( P_0 \) on the sequence \( S \) in \( Y \). We show that \( Y \) is holomorphically convex therefore is Stein.

**Remark 3.14.** A Stein open subset of an algebraic affine variety is not an algebraic affine variety in general. For example, let \( X = \mathbb{C}^n \), let \( Z \) be the closed analytic subvariety of \( X \) defined by \( f(z) = \sin z_1 \), where \( (z_1, z_2, \ldots, z_n) \) are coordinates in \( \mathbb{C}^n \). Then \( Y = X - Z \) is Stein but not an algebraic variety.

Surprisingly, Neeman constructed an example: there is a scheme \( U \) of finite type over \( \mathbb{C} \) such that \( U \) is a Zariski open subset of an affine scheme and the associated analytic complex space \( U' \) of \( U \) is a Stein space, but \( U \) is not an affine scheme [14].

**Theorem 3.15.** If \( Y \) is an open subset of a Stein space \( X \) such that the complement \( X - Y \) is a closed analytic subspace of \( X \) with pure codimension 1 and \( X - Y \) does not contain any singular points of \( X \), then \( Y \) is Stein.
Proof. By Reduction Theorem [4, p. 154], X is Stein if and only if its reduction is Stein. The normalization of a reduced complex space is a finite surjective holomorphic map [4, p. 22]. So a complex space is Stein if and only if its normalization is Stein [8, p. 313, Prop. 73.1]. The normalization \( \tilde{X} \) of X is a disjoint union of irreducible components and it is Stein if and only if every irreducible component is Stein [8, p. 308, Cor. 71.14]. Therefore we may assume that X is an irreducible normal (reduced) Stein space.

Let \( X_{\text{sing}} \) be the set of singular points of X. Then \( X_{\text{sing}} \) is of codimension at least 2 in X [5, p. 128], and \( X - Y \subset X - X_{\text{sing}} \) is a closed subspace of pure codimension 1 in the complex manifold \( X - X_{\text{sing}} \). Since every point in \( X - Y \) is smooth in X, \( (X - Y) \cap X_{\text{sing}} \) is an empty set. So \( X - Y \) is support of an effective Cartier divisor D in the complex manifold \( X - X_{\text{sing}} \) [20, p. 36].

Let \( \{(U_i, f_i)\}_{i \in I} \) be a representative of D in the complex manifold \( X - X_{\text{sing}} \), where \( \{U_i\}_{i \in I} \) is a Stein open cover of the complex manifold \( X - X_{\text{sing}} \), each \( f_i \) is a holomorphic function on \( U_i \), at least one \( f_i \) has zeros, and \( f_i/f_j \) is a holomorphic function on \( U_i \cap U_j \) for all \( i, j \in I \).

Let \( \{V_j\}_{j \in J} \) be a Stein open cover of \( X_{\text{sing}} \) in \( Y : X_{\text{sing}} \subset \bigcup_j V_j \subset Y \). On each \( V_j \cap U_i \neq \emptyset \), \( f_i|_{V_j \cap U_i} \) is nowhere zero. In particular, on every \( V_j - V_j \cap X_{\text{sing}} \), we have [20, p. 36, Thm. 4.13]

\[
\mathcal{O}_{V_j - V_j \cap X_{\text{sing}}}(D) \cong \mathcal{O}_{V_j - V_j \cap X_{\text{sing}}}.
\]

Now the codimension of \( V_j \cap X_{\text{sing}} \) is at least 2 in \( V_j \), therefore the invertible sheaf \( \mathcal{O}_{V_j - V_j \cap X_{\text{sing}}}(D) \) can be extended to \( V_j \) uniquely [0]. This implies that we have an invertible sheaf \( \mathcal{O}_X(D) \) on X, i.e., D is an effective Cartier divisor on X [6, p. 144, Prop. 6.13]. By Theorem 3.13 Y is Stein.

Corollary 3.16. If \( Y \) is an open subset of a Stein manifold \( X \) such that the complement \( X - Y \) is a closed subspace of \( X \) with pure codimension 1, then \( Y \) is Stein.

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