QUANTIZATION OF SOME POISSON-LIE DYNAMICAL $r$-MATRICES AND POISSON HOMOGENEOUS SPACES

BENJAMIN ENRIQUEZ, PAVEL ETINGOF, AND IAN MARSHALL

Abstract. Poisson-Lie (PL) dynamical $r$-matrices are generalizations of dynamical $r$-matrices, where the base is a Poisson-Lie group. We prove analogues of basic results for these $r$-matrices, namely constructions of (quasi)Poisson groupoids and of Poisson homogeneous spaces. We introduce a class of PL dynamical $r$-matrices, associated to nondegenerate Lie bialgebras with a splitting; this is a generalization of trigonometric $r$-matrices with an abelian base. We prove a composition theorem for PL dynamical $r$-matrices, and construct quantizations of the polarized PL dynamical $r$-matrices. This way, we obtain quantizations of Poisson homogeneous structures on $G/L$ ($G$ a semisimple Lie group, $L$ a Levi subgroup), thereby generalizing earlier constructions.

Introduction

In this paper, we continue the study of Poisson-Lie (PL) dynamical $r$-matrices, which was started in [FM]. We construct new examples of such $r$-matrices, together with their quantizations. We apply this to the quantization of Poisson homogeneous spaces, which were introduced in [DGS].

In Section 1, we define PL dynamical $r$-matrices and give some examples. As in [Lu], PL dynamical $r$-matrices give rise to (quasi)Poisson groupoids.

In Section 2, we introduce the notion of a nondegenerate Lie bialgebra with a splitting $g = l \oplus u$. We associate to this datum a PL dynamical $r$-matrix $\sigma^l$. This construction is a PL analogue of [EE2] (see also [FGP], Proposition 1, and [Xu], Theorem 2.3). The main example is the inclusion of a Levi subalgebra $l$ in a simple Lie algebra $g$. When $l$ coincides with the Cartan subalgebra $h \subset g$, $\sigma^l$ is the standard trigonometric $r$-matrix (see [EV1]). In general, $\sigma^l$ is an ingredient in a composition theorem for PL dynamical $r$-matrices, generalizing [EE2] (see also [EV1], Theorem 3.14 and [FGP], Proposition 1) and [Mu] (who treated the case when $l \subset g$ is the inclusion of a Cartan in a Levi subalgebra).

In Section 3, we introduce the notion of a polarized nondegenerate Lie bialgebra with a splitting. We construct a quantization $\Psi^l$ of $\sigma^l$ in this situation. This construction is a generalization of [EE2], and is based on a nonabelian analogue of the inversion of the Shapovalov pairing; from a representation theoretic viewpoint, this may be formulated in terms of intertwiners, as in [EE2]. This idea is already present in [DM]; however, in order to carry out the analogue of the construction of [EE2], one needs to construct a left coideal $U^l_h(u_+) \subset U^l_h(g)$ (see Sections 3.1, 3.2). We also prove a quantum composition formula for the twists $\Psi^l$ (Section 3.8).

The second part of the paper (Sections 4, 5, 6) is devoted to applying these constructions to the quantization of Poisson homogeneous spaces of the form $G/L$, where $g$ is a simple Lie algebra and $l \subset g$ is a Levi subalgebra.

In Section 4, we recall the classification of the Poisson homogeneous structures on $G/L$ ([DGS]), The set of all these Poisson structures is an algebraic variety $\mathcal{P}$. We introduce
a Zariski open subvariety $P_0 \subset P$, and show that the Poisson structures corresponding to elements of $P_0$ are exactly those which may be obtained either using the dynamical $r$-matrix $r^P_1$, or using its Poisson-Lie analogue $\sigma^P_1$.

In Section 5, we construct quantizations of all the Poisson homogeneous structures corresponding to the elements of $P_0$. For this, we prove an algebraicity result for $\Psi^P$. This result is based on the computation of the Shapovalov pairing for $U_q(\mathfrak{g})$ ([DCK]), and on the quantum composition formula for the twists $\Psi^P$. In Section 6, we compare the quantizations of $G/L$ obtained in Section 5 and in [EE2]. In [EE2], there was constructed the quantization of a family of Poisson structures, indexed by an analytic open subset $U$ of $P_0$; this work is based on the Knizhnik-Zamolodchikov associator and is therefore not purely algebraic, contrary to the construction of Section 5. We show that when the parameter of the Poisson structure belongs to $U$, both quantizations are equivalent. The Poisson homogeneous space $G/L$, equipped with a structure from $P_0$, may be viewed as a dressing orbit of $G^*$. Similarly, in the rational case, the $\mathfrak{g}$-invariant homogeneous structures on $G/L$ are the coadjoint orbits $G/L \rightarrow \mathfrak{g}^*$. In Section 7, we quantize these $G^*$- (or $\mathfrak{g}$-) space embeddings (in the rational case, quantizations were constructed in [AL, EE2]). We also prove that the quantum function algebra of $G/L$ is a subalgebra of $U_\hbar(\mathfrak{g})/\text{Ann}(M^\hbar_{\chi})$ (in the rational case, of $U(\mathfrak{g})/\text{Ann}(M^\hbar_{\chi})[\hbar]$), where $M^\hbar_{\chi}$ (resp., $M^\hbar$) is the generalized Verma module over $U_\hbar(\mathfrak{g})$ (resp., over $U(\mathfrak{g})$). Systems of generators for $\text{Ann}(M^\hbar_{\chi})$, whose classical limits are systems of generators for the defining ideal of the orbit $G^*_\chi \subset \mathfrak{g}^*$, are constructed in [O] when $\mathfrak{g}$ is reductive of classical type.

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1. **Poisson-Lie dynamical $r$-matrices and quasi-Poisson structures: general facts**

In this section, we recall the notion of dressing actions of Poisson-Lie groups. We define Poisson-Lie (PL) dynamical $r$-matrices and give some examples. We show that such $r$-matrices give rise to (quasi)Poisson groupoids, and discuss quantization of these constructions. The material of this section is a PL generalization of basic constructions involving dynamical $r$-matrices.

Our base field is $\mathbb{C}$ (although the results of Sections 1 to Section 3 hold over a field of characteristic 0).

1.1. **Poisson-Lie groups and Lie bialgebras.** A Poisson-Lie group $A$ is a Lie group equipped with a Poisson structure, such that the product map is a Poisson morphism $A \times A \rightarrow A$. This definition makes sense in the formal, algebraic or analytic categories, and we thus obtain the notions of formal, etc., Poisson-Lie groups.

If $A$ is a (finite dimensional) Poisson-Lie group and $\mathfrak{a}$ is its Lie algebra, then $\mathfrak{a}$ identifies with $(\mathfrak{m}/\mathfrak{m}^2)^*$, where $\mathfrak{m}$ is the maximal ideal of the local ring of $A$ at the origin, and the Poisson bracket $\wedge^2(\mathfrak{m}) \rightarrow \mathfrak{m}$ induces a map $\wedge^2(\mathfrak{m}/\mathfrak{m}^2) \rightarrow \mathfrak{m}/\mathfrak{m}^2$, whose dual is a map $\delta : \mathfrak{a} \rightarrow \wedge^2(\mathfrak{a})$. Then $(\mathfrak{a}, \delta)$ is a Lie bialgebra, i.e., $(\delta \otimes \text{id}) \circ \delta(a) + \text{cyclic permutations} = 0$ for any $a \in \mathfrak{a}$, and $\delta([a, b]) = [a \otimes 1 + 1 \otimes a, \delta(b)] + [\delta(a), b \otimes 1 + 1 \otimes b]$ for any $a, b \in \mathfrak{a}$. 
To a Lie bialgebra $\mathfrak{a}$, one associates a formal Poisson-Lie group, i.e., a formal series Hopf algebra $\mathcal{O}(A)$ with a Poisson structure compatible with the coproduct: $\mathcal{O}(A) = U(\mathfrak{a})^*.$

1.2. Dressing actions.

**Definition 1.1.** Let $(\mathfrak{a}, \delta)$ be a Lie bialgebra and let $\mathcal{O}$ be a Poisson algebra. A Poisson action of $\mathfrak{a}$ on $\mathcal{O}$ is a linear map $\mathfrak{a} \to \text{Der}(\mathcal{O})$, $a \to \theta_a$, such that $\theta_a(\{f, g\}) = \{\theta_a(f), g\} + \{a, \theta_a(g)\} + \sum \theta_{a(1)}(f)\theta_{a(2)}(g)$, where $f, g \in \mathcal{O}$, $a \in \mathfrak{a}$, and $\delta(a) = \sum a^{(1)} \otimes a^{(2)}$.

Let $A$ be a Poisson-Lie group with Lie bialgebra $(\mathfrak{a}, \delta)$, let $M$ be a Poisson manifold, and let $A \times M \to M$ be a left Poisson action of $A$ on $M$. Then the map $L : a \to \text{Der}(\mathcal{O}_M)$ defined by $(L_a f)(x) := \frac{d}{d\varepsilon}|_{\varepsilon=0} f(e^{-\varepsilon a}x)$ is a Poisson action of $(\mathfrak{a}, \delta)$ on $\mathcal{O}_M$ (here $\mathcal{O}_M$ is the function algebra of $M$). In the same way, if $M \to A \to N$ is a right Poisson action of $A$ on $M$, then $R : a \to \text{Der}(\mathcal{O}_M)$, defined by $(R_a f)(x) := \frac{d}{d\varepsilon}|_{\varepsilon=0} f(x e^{\varepsilon a})$, is a Poisson action of $(\mathfrak{a}, \delta)$ on $\mathcal{O}_M$. We also use the notation $(L_a(x)f)(x), (R_a(x)f)(x)$ to denote left and right infinitesimal translations by an element of $\mathcal{O}_M \otimes \mathfrak{a}$.

Assume that $\mathfrak{a}$ is finite dimensional. Let $\mathfrak{d}(\mathfrak{a}) = \mathfrak{a} \oplus \mathfrak{a}^*$ be the double Lie bialgebra of $(\mathfrak{a}, \delta)$. The injections $\mathfrak{a} \to \mathfrak{d}(\mathfrak{a}), \mathfrak{a}^* \to \mathfrak{d}(\mathfrak{a})$ are Lie bialgebra morphisms (here cop means the Lie bialgebra with opposite cobracket). Let $A$ be a Poisson-Lie group with Lie bialgebra $\mathfrak{a}$. Assume that the adjoint action of $\mathfrak{a}$ on $\mathfrak{d}(\mathfrak{a})$ extends to $A$ (this is the case e.g. if $A$ is the formal group of $\mathfrak{a}$). The left and right dressing actions are Poisson actions of $\mathfrak{d}(\mathfrak{a})$ on $\mathcal{O}_A$, defined by

$$(\text{dress}_L^a f)(x) = -R_{(\text{Ad}(x^{-1})(\mathfrak{a}))} f(x), \quad (\text{dress}_R^a f)(x) = L_{(\text{Ad}(x)(\mathfrak{a}))} f(x),$$

where $a \in \mathfrak{d}(\mathfrak{a})$, $f \in \mathcal{O}_A$, and $\alpha \to \alpha_a$ is the projection on $\mathfrak{d}$ parallel to $\mathfrak{a}^*$. If $a \in \mathfrak{a}$, we have $\text{dress}_L^a = L_a, \text{dress}_R^a = R_a$.

If $f, g \in \mathcal{O}_A$, we have

$$(f, g) = \sum_i \text{dress}_L^{a_i}(f)L_{e_i}(g) = -\sum_i \text{dress}_R^{a_i}(f)R_{e_i}(g), \tag{1}$$

where $(e_i), (e^i)$ are dual bases of $\mathfrak{a}$ and $\mathfrak{a}^*$.

Assume that $A \to D, A^* \to D$ are Poisson-Lie group morphisms associated with the Lie bialgebra morphisms $\mathfrak{a} \to \mathfrak{d}(\mathfrak{a}), \mathfrak{a}^* \to \mathfrak{d}(\mathfrak{a})$, and that the maps $A \to D/A^*, A \to A^* \setminus D$ are Poisson isomorphisms (these assumptions hold e.g. if $A, A^*, D$ are the formal groups of $\mathfrak{a}, \mathfrak{a}^*, \mathfrak{d}(\mathfrak{a})$). Then dress$_L^a$ (resp., dress$_R^a$) corresponds to the left action of $D$ on $D/A^*$ (resp., right action of $D$ on $A^* \setminus D$), which are Poisson.

If $\mathfrak{b}$ is a Lie algebra and $\mathfrak{a}$ is the Lie bialgebra $\mathfrak{b}^*$ (with zero bracket), then the simply-connected Lie group with Lie algebra $\mathfrak{a}$ is $A = \mathfrak{a}$. Then dress$_L^a = \text{dress}_R^a$ and the common restriction of both actions to $\mathfrak{b} \subset \mathfrak{d}(\mathfrak{a})$ is the adjoint action of $\mathfrak{b}$ on $S'(\mathfrak{b}) = \mathcal{O}_H = \mathcal{O}_A$.

1.3. Poisson-Lie dynamical $r$-matrices. Let $(\mathfrak{g}, \delta)$ be a finite dimensional Lie bialgebra and $\mathfrak{h} \subset \mathfrak{g}$ be a Lie subbialgebra. Let $\mathfrak{h}^*$ be the Lie bialgebra dual to $\mathfrak{h}$ and let $H^*$ be a Poisson-Lie group with Lie bialgebra $\mathfrak{h}^*$. We denote by $\mathcal{O}_{H^*}$ the function ring of $H^*$ (when $H^*$ is a formal group, $\mathcal{O}_{H^*} = U(H^*)^*$ will be denoted $\mathbb{C}[H^*]$; if $H^*$ is an algebraic group, then $\mathcal{O}_{H^*}$ is a Hopf algebra contained in $U(H^*)^*$ and will be denoted $\mathbb{C}[H^*]$).

Define $dL : \mathcal{O}_{H^*} \to \mathfrak{h} \otimes \mathcal{O}_{H^*}$ by $dL(f) = -\sum_i e_i \otimes L_{e_i}(f)$, where $(e_i), (e^i)$ are dual bases of $\mathfrak{h}$. If $\rho = \sum a_\alpha \otimes b_\alpha \otimes \ell_\alpha$, we set $dL(\rho) = \sum a_\alpha \otimes b_\alpha \otimes dL(\ell_\alpha)$.

If $V$ is a vector space, $\wedge^i V$ is the space of totally antisymmetric tensors in $V \otimes i$. If $x \in V \otimes 3$ is antisymmetric w.r.t. a pair of tensor factors, then $\text{Alt}(x)$ is the sum of its cyclic permutations $x + x^{2,3,1} + x^{3,1,2}$. We set $\text{CYB}(\rho) = [\rho^{1,2}, \rho^{1,3}] + [\rho^{1,2}, \rho^{2,3}] + [\rho^{1,3}, \rho^{2,3}]$.

The dressing actions of $\mathfrak{h}$ on $\mathcal{O}_{H^*}$ extend to actions on the localizations $\mathcal{O}_{H^*}[1/P]$, where $P \in \mathcal{O}_{H^*}$ is nonzero.
Definition 1.2. (see [DM]) Let $Z \in \wedge^3(g)^g$ be such that $\text{Alt}(\delta \otimes \text{id}^{S^2})(Z) = 0$. A Poisson-Lie dynamical $r$-matrix for $(H^*, g, Z)$ is an element $\rho \in \wedge^2(g) \otimes \mathcal{O}_H[1/\mathbf{P}]$, which we view as a function $\rho : H^* \to \wedge^2(g)$, such that:

(a) for any $a \in h$, $\text{dress}_L^R(\rho) + [a \otimes 1 + 1 \otimes a, \rho] = 0$
(b) $\text{CYB}(\rho) + \text{Alt}(d^L \rho) + \text{Alt}((\delta \otimes \text{id})(\rho)) = Z$.

This definition is the specialization of the notion from [DM] of a classical dynamical $r$-matrix over a base algebra, when the base is $H^*$.

Remark 1.3. If $\rho$ is as in Definition 1.2, then $h \mapsto \rho(h^{-1})$ satisfies the analogue of Definition 1.2, where $\text{dress}_L$, $d^L$ are replaced by $\text{dress}_R$, $d^R$ ($d^R$ is defined by replacing $L$ with $R$ in the definition of $d^L$).

Remark 1.4. When $h = g$, we may identify $\rho$ with a right-invariant 2-form $\omega$ on $G^*$. Then the term $\text{Alt}(d^L \rho) + \text{Alt}((\delta \otimes \text{id})(\rho))$ identifies with its exterior derivative $d\omega$.

Example 1.5. If $\delta = 0$, then we may take $H^*$ to be the formal group of $h^*$. Then dress is the adjoint action of $h$ on $\mathcal{O}_h^* = \hat{S}(h)$. In that case, $\rho \mapsto -\rho$ yields a bijection between Poisson-Lie (PL) dynamical $r$-matrices for $(H^*, g, Z)$ and (usual) dynamical $r$-matrices for $(h, g, Z)$.

More generally, assume that $\delta_{\text{h}} = 0$, and that $(\mathfrak{g}, \delta)$ is coboundary, i.e., we have $\delta(a) = [a \otimes 1 + 1 \otimes a, r_0]$, where $r_0 \in \wedge^2(g)$ is such that $Z(r_0) := \text{CYB}(r_0) \in \wedge^3(g)^g$. Then the map $\rho \mapsto r_0 - \rho$ is a bijection between $\{\text{PL dynamical } r\text{-matrices for } (H^*, g, Z)\}$ and $\{\text{usual dynamical } r\text{-matrices for } (h, g, Z + Z(r_0))\}$.

In particular, if $g$ is a semisimple Lie algebra and $h \subset g$ is a Cartan subalgebra, then $r_0^g(\lambda) := -\frac{1}{2} \sum_{\alpha \in \Delta_+} (e_\alpha \wedge f_\alpha) \coth(\frac{\Delta_0}{2})$ is a usual dynamical $r$-matrix for $(h, g, Z(r_0))$, where $r_0 = \frac{1}{4} \sum_{\alpha \in \Delta_+} e_\alpha \wedge f_\alpha$. Therefore $r_0 - r(\lambda)$ is a PL dynamical $r$-matrix for $(H^*, g, 0)$, where $g$ is equipped with its standard Lie bialgebra structure $\delta(a) = [a \otimes 1 + 1 \otimes a, r_0]$ (we set $x \wedge y := x \otimes y - y \otimes x$).

Example 1.6. (The Balog-Fehér-Palla $r$-matrix [BFP, FM].) Let $(g, r)$ be a factorizable Lie bialgebra. Recall that this means that $t := r + r^{2,1}$ is a nondegenerate element of $S^2(g)^g$, $\text{CYB}(r) = 0$. The cobracket of $g$ is given by $\delta(a) = [a \otimes 1 + 1 \otimes a, r]$ for any $a \in g$. Set $Z := \{t^{1,2}, t^{2,3}\}$. The relation $\text{Alt}(\delta \otimes \text{id}^{S^2})(Z) = 0$ is satisfied because the twist by $r - r^{2,1}$ of the trivial quasi-Lie bialgebra structure on $g$ is a quasi-Lie bialgebra.

Set $L(\xi) = (\text{id} \otimes \xi)(r)$ and $R(\xi) = (\xi \otimes \text{id})(r)$. Then the maps $R, L : g^* \to g$ are Lie algebra morphisms, and $R \oplus L : g^* \to g \oplus g$ is a Lie algebra injection. We also denote by $R, L$ the corresponding formal group morphisms $G^* \to G$, and by $\lambda : G^* \to G$ the map such that $\lambda(g^*) = L(g^*)R(g^*)^{-1}$. Finally, we define $g^* \mapsto g(g^*)$ to be the map $G^* \to \text{End}(g)$, such that $g(g^*) = \text{Ad}(\lambda(g^*))$.

Define $\rho_{\text{BFP}} : G^* \to \wedge^2(g)$ by

$$\rho_{\text{BFP}}(g^*) = \left( (\nu^2 + g(g^*)^2) \frac{1}{\nu^2 - g(g^*)^2} - \frac{1}{\nu^2 - g(g^*)^2} \right) \otimes \text{id}(t).$$

Then $\rho_{\text{BFP}}$ is a dynamical $r$-matrix for $(G^*, g, Z_\nu)$, where $Z_\nu = (\nu^2 - \frac{1}{4})[t^{1,2}, t^{2,3}]$, and $\nu \in \mathbb{C}$ is an arbitrary constant.

Remark 1.7. A constant PL dynamical $r$-matrix for $(G^*, g, Z)$ is the same as a classical twist between the quasi-Lie bialgebras $(g, -\delta, 0)$ and $(g, -\delta - \delta_{\rho}, -Z)$, where $\delta_{\rho}(x) = [x \otimes 1 + 1 \otimes x, \rho]$ (see [Dr2]).

In Section 2.3, we will construct new families of examples of PL dynamical $r$-matrices.
1.4. Quasi-Poisson structures. If $\mathcal{O}$ is a commutative algebra equipped with an antisymmetric bracket, satisfying the Leibniz identity w.r.t. each variable, a Poisson action of a Lie bialgebra $(\mathfrak{a}, \delta)$ on $\mathcal{O}$ is defined as a Lie algebra morphism $\mathfrak{a} \to \text{Der}(\mathcal{O})$, satisfying the condition of Definition 1.1.

**Proposition 1.8.** Let $(\mathfrak{g}, \delta)$ be a finite dimensional Lie bialgebra, let $Z \in \wedge^3(\mathfrak{g})^3$, and let $\mathfrak{h} \subset \mathfrak{g}$ be a Lie subbialgebra. Let $\rho = \sum \alpha \otimes b_\alpha \otimes \ell_\alpha \in \wedge^2(\mathfrak{g}) \otimes \mathcal{O}_H$. There is a unique bracket on $\mathcal{O}_H \otimes \mathcal{O}_G$, antisymmetric and satisfying the Leibniz identity w.r.t. each variable, such that

$$\forall f_1, f_2 \in \mathcal{O}_H, \quad \{f_1 \otimes 1, f_2 \otimes 1\} = \{f_1, f_2\}_H \otimes 1,$$

$$\forall g \in \mathcal{O}_G, \forall f \in \mathcal{O}_H, \quad \{f \otimes 1, 1 \otimes g\} = -\sum_i \mathbf{L}_{e_i}(f) \otimes \mathbf{L}_{e_i}(g),$$

$$\forall g_1, g_2 \in \mathcal{O}_G, \quad \{1 \otimes g_1, 1 \otimes g_2\} = 1 \otimes \{g_1, g_2\}_G + \sum_\alpha \ell_\alpha \otimes \mathbf{L}_{a_\alpha}(g_1)\mathbf{L}_{b_\alpha}(g_2).$$

Here $(e^i), (e_i)$ are dual bases of $\mathfrak{h}, \mathfrak{h}^*$, and $\{-, -\}_A$ is the Poisson bracket of a Poisson-Lie group $A$.

If $\rho$ is a PL dynamical $r$-matrix for $(H^*, \mathfrak{g}, Z)$, then this bracket satisfies $\{\phi_1, \phi_2, \phi_3\} + \text{c.p} = m((\text{id} \otimes \mathfrak{R}) \otimes \mathfrak{R})(\phi_1 \otimes \phi_2 \otimes \phi_3)$ for any $\phi_1, \phi_2, \phi_3 \in \mathcal{O}_H \otimes \mathcal{O}_G$.

This construction is equipped with the following commuting Poisson actions:

(a) the action of $(\mathfrak{h}, \delta)$ by dress $\otimes \text{id} + \text{id} \otimes \mathfrak{L}$,

(b) the action of $(\mathfrak{g}, \delta)$ by id $\otimes \mathfrak{R}$.

**Proof.** Straightforward. 

The notion of a quasi-Poisson structure ([AKM]) may be generalized as follows: $(\mathfrak{a}, \delta_a, Z_a)$ is a quasi-Lie bialgebra, $M$ is a manifold equipped with a bivector and an action of $\mathfrak{a}$ by vector fields, such that $\theta_a(\{f, g\}) = \{\theta_a(f), g\} + \{f, \theta_a(g)\} + \sum \theta_a(\alpha)(f)\theta_a(\alpha)(g)$ and $\{\{f, g\}, h\} + \text{c.p} = m(\theta^{\otimes 3}(Z_a)(f \otimes g \otimes h))$.

In the situation of Definition 1.2, $(\mathfrak{g}, \delta, Z)$ is a quasi-Lie bialgebra and the structure defined in Proposition 1.8 is quasi-Poisson under the action of $(\mathfrak{g}, \delta, Z)$ by id $\otimes \mathfrak{R}$.

1.5. Poisson groupoids. One checks that $H^* \times G \times H^*$ is equipped with a Poisson groupoid structure, corresponding to the Poisson bracket on $\mathcal{O}_H \otimes \mathcal{O}_G \otimes \mathcal{O}_H$.

$$\{f^{(1)}, f'^{(1)}\} = \{f, f'\}_H, \quad \{f^{(3)}, f'^{(3)}\} = -\{f, f'\}_H, \quad \{f^{(1)}, f'^{(2)}\} = 0,$$

$$\{f^{(1)}, g^{(2)}\} = -\sum_i \mathbf{L}_{e_i}(f)^{(1)}\mathbf{L}_{e_i}(g)^{(2)}, \quad \{f^{(3)}, g^{(2)}\} = -\sum_i \mathbf{R}_{e_i}(f)^{(3)}\mathbf{R}_{e_i}(g)^{(2)},$$

$$\{g^{(2)}, g'^{(2)}\} = \{g, g'\}_G^{(2)} + \sum_\alpha \ell_\alpha^{(1)}(\mathbf{L}_{a_\alpha}(g)\mathbf{L}_{b_\alpha}(g'))^{(2)} - \sum_\alpha \ell_\alpha^{(3)}(\mathbf{R}_{a_\alpha}(g)\mathbf{R}_{b_\alpha}(g'))^{(2)}$$

for any $f, f' \in \mathcal{O}_H$ and $g, g' \in \mathcal{O}_G$. This construction is carried out in [FM] in the setup of Example 1.6.

1.6. Quantization of PL dynamical $r$-matrices. Let $(U_h(\mathfrak{g}), \Delta)$ be a quantized universal enveloping (QUE) algebra. We say that $\Phi \in U_h(\mathfrak{g})^{\otimes 3}$ is an associator for $(U_h(\mathfrak{g}), \Delta)$ if

$$\Phi^{2,3,4}\Phi^{1,2,3,4} = \Phi^{1,2,3,4}\Phi^{12,3,4},$$

and

$$\forall x \in U_h(\mathfrak{g}), \quad [\Delta^{(3)}(x), \Phi] = 0, \quad \Phi = 1 + O(h), \quad \text{alt}(\Phi) = O(h^2).$$

Here alt: $V_{\otimes 3} \to V_{\otimes 3}$ is defined by alt$(x) = \sum_{\sigma \in \mathfrak{S}_3} \text{sgn}(\sigma)\sigma(x)$, and $\otimes$ is the $h$-adically completed tensor product.
Then \((U_h(\mathfrak{g}), \Delta, \Phi)\) is a quasi-Hopf algebra, therefore it gives rise to a quasi-Lie bialgebra \((\mathfrak{g}, \delta, Z)\) (see [Dr2]), where \(Z = (h^{-2}\text{alt}(\Phi) \mod h)\), and

\[
Z \in \wedge^3(\mathfrak{g})^\theta, \quad \text{and } \text{Alt}(\delta \otimes \text{id} \otimes \text{id})(Z) = 0. \tag{4}
\]

Conversely, if \(Z\) satisfies (4), then \(\Phi \in U_h(\mathfrak{g})^{\hat{\otimes} 3}\) satisfying (2), (3) will be called a quantization of \(Z\).

Let \((U_h(\mathfrak{g}), \Delta)\) be a QUE algebra, \(\Phi \in U_h(\mathfrak{g})^{\hat{\otimes} 3}\) be an associator, let \(U_h(\mathfrak{h}) \subset U_h(\mathfrak{g})\) be a QUE subalgebra and let \(\Psi \in U_h(\mathfrak{g})^{\hat{\otimes} 2 \hat{\otimes} U_h(\mathfrak{h})}\) be such that

\[
\Psi^{2,3,4} \psi^{1,23,4} \Phi^{1,2,3} = \psi^{1,2,34} \Psi^{12,3,4},
\]

\[
h^{-1}(\Psi - 1) \in U_h(\mathfrak{g})^{\hat{\otimes} 2 \hat{\otimes} \mathbb{C}[[H^*]]_h}, \quad \forall y \in U_h(\mathfrak{h}), [\Psi, \Delta^{(3)}(y)] = 0. \tag{5}
\]

Here \(\mathbb{C}[[H^*]]_h \subset U_h(\mathfrak{h})\) is defined as \(\{a \in U_h(\mathfrak{h})| \forall n \geq 0, (id - \eta \circ \varepsilon)^\otimes_n \circ \Delta^{(n)}(f) = O(h^n)\}\); it is a flat deformation of \(\mathbb{C}[[H^*]] := U(\mathfrak{h})^*\) (see [Dr1, Gav]).

Set

\[
\rho := (h^{-1}(\Psi - \Psi^{2,1,3}) \mod h). \tag{7}
\]

Then \(\rho \in \wedge^2(\mathfrak{g}) \otimes \mathbb{C}[[H^*]]\).

**Proposition 1.9.** Let \((\mathfrak{g}, \delta)\) be the classical limit of \((U_h(\mathfrak{g}), \Delta)\) (so for \(x \in \mathfrak{g}, \delta(x) = (h^{-1}(\Delta(\bar{x}) - \Delta(\bar{x})^{(1)} \mod h) \) for any \(\bar{x}\) such that \((\bar{x} \mod h) = x\). Then \(\rho\) is a PL dynamical \(r\)-matrix for \((H^*, \mathfrak{g}, Z)\).

**Proof.** The proof relies on two facts:
(a) if \(f \in \mathbb{C}[[H^*]]_h\) and if we view \(\Delta(f)\) as an element of \(U_h(\mathfrak{g})^{\hat{\otimes} \mathbb{C}[[H^*]]_h}\), then \((h^{-1}(\Delta(f) - 1 \otimes f) \mod h) = d^x(f_0),\) where \(f_0 = (f \mod h) \in \mathbb{C}[[H^*]];\)
(b) if \(a \in \mathbb{C}[[H^*]]_h, x, y \in U_h(\mathfrak{g}), f \in \mathbb{C}[[H^*]]_h,\) then

\[
\left(h^{-1}[\Delta^{(3)}(a), x \otimes y \otimes f] \mod h\right) = \sum \left[\left[\epsilon, x_0 \otimes y_0 \otimes f_0 + x_0 \otimes \epsilon, y_0 \otimes f_0 + x_0 \otimes y_0 \otimes \text{dress}_{\epsilon}(f_0)\right](1 \otimes 1 \otimes \textbf{L}_{\epsilon^2}(a_0)),
\]

where \(x_0, \ldots, a_0\) are the reductions modulo \(h\) of \(x, \ldots, a.\)

(a) follows from [EGH], appendix. (b) follows from (a) and (1).

An element \(\Psi \in U_h(\mathfrak{g})^{\hat{\otimes} 2 \hat{\otimes} U_h(\mathfrak{h})}\) satisfying (5), (6) and (7) will be called a quantization of \(\rho\) compatible with \(\Phi\).

### 1.7. Relation with twists of QUE algebras.

Assume that \((\mathfrak{g}, \delta)\) is coboundary, and \(\mathfrak{h} \subset \mathfrak{g}\) is such that \(\delta_{\mathfrak{h}} = 0\). Let \(r_0 \in \wedge^2(\mathfrak{g})\) be such that \(\delta(a) = [a \otimes 1 + 1 \otimes a, r_0]\) and \(Z(r_0) \in \wedge^3(\mathfrak{g})^\theta\).

Let \(r'\) be a dynamical \(r\)-matrix for \((\mathfrak{h}, \mathfrak{g}, 0)\); then \(r_0 - r'\) is a PL dynamical \(r\)-matrix for \((H^* = \mathfrak{h}^*, \mathfrak{g}, -Z(r_0))\) (see Example 1.5). This correspondence may be quantized as follows.

Assume that \(\Phi \in U(\mathfrak{g})^{\otimes 3}([h])\) is an associator quantizing \(Z(r_0)\) and let \(J_0 \in U(\mathfrak{g})^{\otimes 2}([h])\) be a \(\mathfrak{h}\)-invariant quantization of \(r_0\) such that

\[
J_0^{12,3} J_0^{23} = \Phi J_0^{12,3} J_0^{1,2}.
\]

Let \(J' \in U(\mathfrak{g})^{\otimes 2} \otimes U(1)\) \([h])\) be a pseudotwist quantizing \(r'\), i.e.,

\[
J'^{12,3,4} J'^{11,2,3,4} = (\Phi^{-1})^{1.2,3} J^{11,2,3,4} J'^{2.3,4}.
\]

Define \(U_h(\mathfrak{g})\) as the QUE algebra \((U(\mathfrak{g})\) \([h]), \text{Ad}(J_0^{12}) \circ \Delta_0).\) Then

\[
\Psi := (J')^{-1} J_0^{1,2} \tag{8}
\]

is a quantization of \(r_0 - r'\), associated with the quantum group \(U_h(\mathfrak{g})\).
In the particular case where \( g \) is semisimple, \( h \subset g \) is the Cartan subalgebra, and \( r' \) is as in Example 1.3, a quantization \( J' \) of \( r' \) was obtained in [EE2]. On the other hand, a quantization \( \Psi \) of the PL dynamical \( r \)-matrix \( r_0 - r' \) was obtained in [EV2] (we are going to generalize this construction in Section 3). In [EE2], Remark 3.7, it was also conjectured that the relation between \( \Psi \) and \( J' \) is given by (8).

1.8. Quantization of quasi-Poisson structures. If \( M \) is a (formal or algebraic) manifold with a quasi-Poisson structure under the action of a quasi-Lie bialgebra \( (a, \delta, Z_a) \), then a quantization of \( M \) is defined as the data of (a) a quantization \((U_h(a), \Delta_a, \Phi_a)\) of \((a, \delta_a, Z_a)\), (b) a \( \mathbb{C}[[h]]\)-module \( O_{M,h} \), isomorphic to \( O_M[[h]] \), equipped with a bilinear map \( \mu : O_{M,h} \otimes O_{M,h} \to O_{M,h} \), deforming \( \mu \) by \( \Delta, S^{\otimes 3}(\Phi)^{-1} \) is also a quantization of \((g, \delta, Z)\) (here \( S \) is the antipode of \( U_h(g) \)).

Set \( \mathbb{C}[[G]]_{h} := U_h(g)^* \); this is an algebra, equipped with commuting actions \( L, R \) of \( U_h(g) \), where \( (L(a)f)(x) = f(xa), (R(a)f)(x) = f(S^{-1}(a)x) \).

**Proposition 1.10.** Let \( A := U_h(g) \otimes \mathbb{C}[[G]]_{h} \), and define \( \mu : A^{\otimes 2} \to A \) by

\[
\mu((a \otimes f) \otimes (b \otimes g)) = \sum_{\alpha} L_{a}(a^{(2)}b \otimes L_{B_{\alpha^{(1)}}}(f)),
\]

for \( R : U_h(g) \otimes A \to A \) by \( R(x)(a \otimes f) = a \otimes R(x)(f) \). Then \( B := \mathbb{C}[[H^*]]_{h} \otimes \mathbb{C}[[G]]_{h} \subset A \) is stable under \( \mu \) and \( \Theta \), and is a quantization of the quasi-Poisson structure of Proposition 1.8, compatible with the quantization \((U_h(g), \Delta, S^{\otimes 3}(\Phi)^{-1})\) of \((g, \delta, Z)\).

Here \( \mathbb{C}[[a_1, \ldots, a_d, h]] \otimes \mathbb{C}[[b_1, \ldots, b_d, h]] \) means \( \mathbb{C}[[a_1, \ldots, b_d]] \).

**2. A composition theorem for Poisson-Lie dynamical \( r \)-matrices**

In this section, we introduce a PL dynamical \( r \)-matrix \( \sigma^P_1 \), associated to a nondegenerate Lie bialgebra with a splitting. This construction is a generalization of [EE2] (see also [FGP], Proposition 1, [Xu], Theorem 2.3). When \( l \) is a Cartan subalgebra of a semisimple Lie algebra \( g \), then \( \sigma^P_1 \) is a trigonometric \( r \)-matrix. In general, \( \sigma^P_1 \) is the basic ingredient of a composition theorem for PL dynamical \( r \)-matrices (Section 2.4), generalizing [EE2], Proposition 0.1 (also [FGP], Proposition 1).

2.1. Lie bialgebras with a splitting. Let \( (g, \delta) \) be a finite dimensional Lie bialgebra and let \( l \subset g \) be a Lie subalgebra. Assume that we have a decomposition \( g = l \oplus u \), such that \( l, u \subset u \) and \( \delta(u) \subset ((l \otimes u) \oplus (u \otimes l)) \oplus (u \otimes u) \). Equivalently, \( g^* = l^* \oplus u^* \), where \( l^* \) is a Lie subalgebra and \( u^* \) is an ideal of \( g^* \). We will call such a quadruple \( (g, \delta, l, u) \) a Lie bialgebra with a splitting.

Assume further that \( g = l \oplus u \) is nondegenerate as a Lie algebra with a splitting (see [EE2]); in other words, we assume that for \( \lambda : l^* \) generic, the pairing \( \omega_{\lambda}(l \times u \to \mathbb{C}, (u, v) \mapsto \lambda(\langle u, v \rangle)) \) is nondegenerate (here \( z \mapsto z_1 \) is the projection on \( l \parallel u \)).

As we explained in [EE2], the nondegeneracy assumption may be expressed as follows. We associate an element \( p_1^l \in S^d(l) \) \((d := \dim(u))\) to \( g = l \oplus u \), well defined up to multiplication by a nonzero scalar: fix a linear isomorphism \( i : u \to u^* \), define \( a(\lambda) \in \text{End}(u) \) by \( a(\lambda)(u) = \)
i^{-1}(\omega_0(\lambda)(u, -))$ for any $\lambda \in \mathfrak{g}^*$, then $p^\mathfrak{g}_L(\lambda) = \det(a(\lambda))$. The nondegeneracy assumption is that $p^\mathfrak{g}_L \neq 0$.

Example 2.1. $(\mathfrak{g}, \delta)$ is a semisimple Lie algebra with its standard Lie bialgebra structure, $I \subset \mathfrak{g}$ is a Levi subalgebra, $u = u_+ \oplus u_-$, $u_+ = \oplus_{\alpha \in \Delta_+(\mathfrak{g}) \setminus \Delta_+(I)} \mathfrak{g}_\alpha$. More generally, $\mathfrak{g} = \mathfrak{l} \oplus u_+ \oplus u_-$ is a polarized nondegenerate Lie algebra (see [EE2]), and $\delta(x) = [\mathfrak{o} \otimes 1 + 1 \otimes x, r]$, where $r \in (1 \otimes I) \oplus (u_+ \otimes u_-)$ is such that $r + r^2$ is invariant and CYB$(r) = 0$.

2.2. Localizations. Assume that $L^* \subset G^*$ is a subgroup ring associated with these Lie algebras; we denote by $\mathbb{C}[[L^*]] = U(\mathfrak{g})^*$ the function ring of the formal group $L^*$. If $x \in L^*$, define $\omega_x : u \times u \to \mathbb{C}$ by

$$\omega_x(u, v) = ((\text{Ad}(x^{-1})(u))_\mathfrak{g}, (\text{Ad}(x^{-1})(v))_\mathfrak{g}),$$

where $(-, -)$ is the canonical bilinear form of $\mathfrak{d}(\mathfrak{g}) = \mathfrak{g} \oplus \mathfrak{g}^*$, and $x \mapsto x_\mathfrak{g}, x_{\mathfrak{g}^*}$ are the projections on $\mathfrak{g}, \mathfrak{g}^*$. Since the bilinear form on $\mathfrak{d}(\mathfrak{g})$ is $\text{Ad}$-invariant and $u$ is an isotropic subspace of $\mathfrak{d}(\mathfrak{g})$, $\omega_x$ is antisymmetric.

If $x \in L^*, 1$ define $a(x) \in \text{End}(u)$ by $a(x)(u) = i^{-1}(\omega_x(u, -))$ and set

$$P_1^\mathfrak{g}(x) := \det(a(x)).$$

Then $P_1^\mathfrak{g}(x)$ is an element of $\mathbb{C}[[L^*]]$, well-defined up to multiplication by a nonzero scalar. Let $\mathfrak{m}_{L^*}$ be the maximal ideal of $\mathbb{C}[[L^*]]$ (it coincides with the kernel of the counit map). Then $P_1^\mathfrak{g} \in \mathfrak{m}_{L^*}$. Recall that the associated graded ring of $\mathbb{C}[[L^*]]$ w.r.t. its filtration by the powers of $\mathfrak{m}_{L^*}$ is $S(1)$. Then the class of $P_1^\mathfrak{g}$ in $\mathfrak{m}_{L^*}/\mathfrak{m}^2_{L^*}$ coincides with $P_1^\mathfrak{g}$. Therefore $P_1^\mathfrak{g}$ is nonzero. The localized ring $\mathbb{C}[[L^*]][1/P_1^\mathfrak{g}]$ is also filtered by $\text{val}(f/P_1^\mathfrak{g}) = \text{val}(f) - kd$, and the associated graded ring is $S(0)[1/P_1^\mathfrak{g}]$.

Any linear lift of the projection map $\mathfrak{m}_{L^*} \to \mathfrak{m}_{L^*}/\mathfrak{m}^2_{L^*} = I$ induces a (noncanonical) identification of $\mathbb{C}[[L^*]]$ with the formal series completion $\hat{S}(1)$ of the symmetric algebra of $I$, and an injection of $\mathbb{C}[[L^*]][1/P_1^\mathfrak{g}]$ in the degree completion of $S(1)[1/P_1^\mathfrak{g}]$, which extends to an identification of the valuation completion of $\mathbb{C}[[L^*]][1/P_1^\mathfrak{g}]$ with this degree completion.

2.3. The $r$-matrix $\sigma_1^\mathfrak{g}$. If $x \in L^*$ is such that $P_1^\mathfrak{g}(x)$ is invertible, then $\omega_x \in \wedge^2(u)^*$ is invertible, and we set $\sigma_1^\mathfrak{g}(x) := (\omega_x)^{-1}$. Then

$$\sigma_1^\mathfrak{g} \in \wedge^2(u) \otimes \mathbb{C}[[L^*]][1/P_1^\mathfrak{g}].$$

$\sigma_1^\mathfrak{g}$ is uniquely determined by the following equivalent conditions (we set $\sigma_1^\mathfrak{g} = \sum_\alpha a_\alpha \otimes b_\alpha \otimes \ell_\alpha$):

1) if $u, v \in u$, then $\omega_x(u, v) = \sum_\alpha \omega_x(u, a_\alpha)\omega_x(v, b_\alpha)\ell_\alpha(x)$ (equality in $\mathbb{C}[[L^*]]$)

2) for any $v \in \mathfrak{g}$, $\sum_\alpha a_\alpha \otimes \omega_x(v, a_\alpha)\ell_\alpha(x) = v \otimes 1$ (equality in $u \otimes \mathbb{C}[[L^*]]$).

Theorem 2.2. $\sigma_1^\mathfrak{g}$ is a PL dynamical $r$-matrix for $(L^*, \mathfrak{g}, 0)$.

Proof. Let us first prove the $1$-invariance of $\sigma_1^\mathfrak{g}$. We first prove the identity:

$$\forall u, v \in u, \forall a \in I, \quad \text{dress}^\mathfrak{g}_0 \omega_x(u, v) + \omega_x([u, a], v) + \omega_x(u, [v, a]) = 0. \quad (11)$$

Set $a' = \text{Ad}(x^{-1})(a)$ (so $a' \in I \oplus \mathfrak{g}^*$, $u' = \text{Ad}(x^{-1})(u)$, $v' = \text{Ad}(x^{-1})(v)$, then

$$\text{dress}^\mathfrak{g}_0 \omega_x(u, v) = \langle ([a', a'], u')_{\mathfrak{g}} \rangle_{\mathfrak{g}^*} + \langle u'_{\mathfrak{g}^*}, ([a', a']_{\mathfrak{g}})_{\mathfrak{g}^*} \rangle,$$

and

$$\omega_x([u, a], v) + \omega_x(u, [v, a]) = \langle ([u', a']_{\mathfrak{g}})_{\mathfrak{g}^*} \rangle_{\mathfrak{g}^*} + \langle u'_{\mathfrak{g}^*}, ([v', a']_{\mathfrak{g}})_{\mathfrak{g}^*} \rangle.$$
The sum of these terms is
\[ \langle [u', a'_1], v'_1 \rangle + \langle u'_g, [v', a'_g] \rangle. \] (12)

Here \( a'_g \) is the projection of \( a' \) on \( l \) parallel to \( l^* \). Let us prove that (12) is zero. We have
\[ [l, u \oplus u^*] \subset u \oplus u^*, \]
so \( u', v' \in u \oplus u^* \). If \( v' \in u \), (12) is clearly 0; if \( u' \in u^* \), (12) is
\[ \langle [u', a'_1], v'_1 \rangle = \langle [u', a'_1], v'_1 \rangle = \langle a'_1, [v'_1, u'] \rangle, \] which is zero since \( u^* \) is an ideal of \( g^* \). If \( u' \in u \)
and \( v' \in u^* \), then the vanishing of (12) follows from the invariance of \((-,-)\). This proves (11).

Let us now prove that
\[ \forall a \in l, \quad \text{dress}_a \sigma^a_1 + [a^1 + a^2, \sigma^a_2] = 0. \] (13)

Write \( \sigma^a_1 = \sum_{\alpha} a_\alpha \otimes b_\alpha \otimes \ell_\alpha \). Then (11) implies that for any \( u, v \in u \), we have
\[ \sum_{\alpha} \omega_x([u, a_\alpha]\omega_x(v, b_\alpha)\ell_\alpha(x) + \omega_x(u, a_\alpha)\omega_x([v, a_\alpha], b_\alpha)\ell_\alpha(x) = \text{dress}_a \omega_x(u, v), \]
and using again (11), we get
\[ \sum_{\alpha} \left( - \text{dress}_a \omega_x(u, a_\alpha)\omega_x(v, b_\alpha)\ell_\alpha(x) - \omega_x(u, a_\alpha)\text{dress}_a \omega_x(v, b_\alpha)\ell_\alpha(x) \right) \\
+ \omega_x(u, [a, a_\alpha])\omega_x(v, b_\alpha)\ell_\alpha(x) + \omega_x(u, a_\alpha)\omega_x([v, a_\alpha], b_\alpha)\ell_\alpha(x) = - \text{dress}_a \omega_x(u, v). \] (14)

Applying \( \text{dress}_a \) to the identity \( \sum_{\alpha} \omega_x(u, a_\alpha)\omega_x(v, b_\alpha)\ell_\alpha(x) = \omega_x(u, v) \), and adding the resulting
identity to (14), we get
\[ \sum_{\alpha} \omega_x(u, a_\alpha)\omega_x(v, b_\alpha)\text{dress}_a \ell_\alpha(x) + \omega_x(u, [a, a_\alpha])\omega_x(v, b_\alpha)\ell_\alpha(x) + \omega_x(u, a_\alpha)\omega_x([v, a_\alpha], b_\alpha)\ell_\alpha(x) = 0. \] (15)

This is the result of pairing (13) with \( u \otimes v \) using \( \omega_x \otimes \omega_x \). This proves (13).

Let us now show that \( \sigma^a_1 \) satisfies the Poisson-Lie CDYBE. The nontrivial components of this
identity lie in the direct sum of \( \wedge^3(u) \otimes \mathbb{C}[L^*] \) and of \( ((1 \otimes \wedge^2(u)) \otimes \mathbb{C}[L^*]) \otimes \mathbb{C}[L^*] \).
Since the l.h.s. of the PL CDYBE is obviously invariant under the cyclic permutations of the
first three tensor factors, it suffices to show the identities
\[ (p^3_u \otimes \text{id})(l. h. s. \, \text{of the PL CDYBE}) = 0 \] (16)
and
\[ (p_l \otimes p^2_u \otimes \text{id})(l. h. s. \, \text{of the PL CDYBE}) = 0. \] (17)

Here \( p_l, p_u \) are the projections \( g \to l, g \to u \) corresponding to \( g = l \oplus u \).

Let us prove (16). Let \( u, v, w \) be arbitrary elements of \( u \). For \( z \in \{u, v, w\} \), we set \( \overline{z} := \text{Ad}(z^{-1})(z) \), which we decompose as \( \overline{z} = z' + z'' \), \( z' \in g, z'' \in g^* \). Then we have \( \langle [\overline{u}, \overline{v}], w \rangle = \langle [u, v], w \rangle = 0 \) (since \((-,-)\) is Ad-invariant and \( g \) is isotropic). On the other hand, \( \langle [\overline{u}, \overline{v}], \overline{w} \rangle \) can be expanded (again using isotropies and invariances) as \( \langle [u', v'], w'' \rangle + \langle [u''', v'''], w' \rangle + \text{cyl. perm.} \). Therefore
\[ \langle [u', v'], w'' \rangle + \langle [u''', v'''], w' \rangle + \text{cyl. perm.} = 0. \] (18)

Let us now extend \( \omega_x \) to a bilinear map \( \overline{\omega}_x : \partial(g) \times \partial(g) \to \mathbb{C} \) using formula (9).
Since \( \text{Ad}(z^{-1})(l) \subset l \oplus l^* \) and \( \text{Ad}(z^{-1})(u) \subset u \oplus u^* \), we get
\[ \overline{\omega}_x(\ell, a) = 0 \quad \text{whenever} \quad \ell \in l \quad \text{and} \quad a \in u. \] (19)
Now
\[ \sum_{\alpha} \sum_{\beta} \omega_x(p_a(a^{(1)}_{\alpha}), u) \omega_x(p_a(a^{(2)}_{\alpha}), v) \omega_x(b_{\alpha}, w) \ell_{\alpha}(x) \]
\[ = \sum_{\alpha} \sum_{\beta} \omega_x(p_a(u^{(1)}_{\alpha}), u) \omega_x(p_a(u^{(2)}_{\alpha}), v) = \sum_{\alpha} \omega_x(w^{(1)}_{\alpha}, u) \omega_x(w^{(2)}_{\alpha}, v) \] (using (19))
\[ = \sum \langle (Ad(x^{-1})(w^{(1)}_{\alpha})), (Ad(x^{-1})(u)) \rangle \langle (Ad(x^{-1})(w^{(2)}_{\alpha})), (Ad(x^{-1})(v)) \rangle \]
\[ = \langle (Ad(x^{-1})(w^{(1)}_{\alpha})), (Ad(x^{-1})(u)) \rangle \langle (Ad(x^{-1})(w^{(2)}_{\alpha})), (Ad(x^{-1})(v)) \rangle \] (since \( g^* \) is isotropic)
\[ = \langle (Ad(x^{-1})(w, Ad(x^{-1})(u))), p_{g^*} \rangle (Ad(x^{-1})(v)) \rangle \] (since \( Ad(x) \) is an automorphism)
\[ = \langle [w, u], p_{g^*} \rangle \] (since \( g^* \) is isotropic).

Here \( p_{g^*} \) is the projection \( g \oplus g^* \rightarrow g^* \), \( p_u \) is the projection \( g = I \oplus u \rightarrow u \), and we write \( \delta(w) = w^{(1)} \otimes w^{(2)} \).

On the other hand,
\[ \sum_{\alpha, \beta} \omega_x(p_a([a_{\alpha}, a_{\beta}]), u) \omega_x(b_{\alpha}, v) \omega_x(b_{\beta}, w) \ell_{\alpha \beta}(x) \]
\[ = \omega_x(p_a([v, w]), u) = \omega_x([v, w], u) \] (using (19))
\[ = \langle [Ad(x^{-1})(v), Ad(x^{-1})(w)], p_{g^*} \rangle (Ad(x^{-1})(u)) \] (since \( g^* \) is isotropic)
\[ = \langle [w, u], p_{g^*} \rangle \] (since \( g^* \) is isotropic).

Using the invariance of \( (-,-) \), taking the difference of these results, we get
\[ \sum_{\alpha, \beta} \omega_x(p_a([a_{\alpha}, a_{\beta}]), u) \omega_x(b_{\alpha}, v) \omega_x(b_{\beta}, w) \ell_{\alpha \beta}(x) = \sum_{\alpha} \omega_x(p_a(a^{(1)}_{\alpha}), u) \omega_x(p_a(a^{(2)}_{\alpha}), v) \omega_x(b_{\alpha}, w) \ell_{\alpha}(x) \]
\[ = \langle [v', u'], w'' \rangle + \langle [v', w''], u' \rangle + \langle v, [w'', u''] \rangle + \langle w', [v'', u''] \rangle \]
\[ + \text{cycl. perm.} = \langle [v', u'], w'' \rangle + \langle v', [w'', u''] \rangle + \langle v, w'' \rangle + \langle w, u'' \rangle \]
\[ + \text{cycl. perm.} = 0. \] (20)

Where the last equality follows from (18). Now the l.h.s. of (20) is the pairing of \( (p_u \otimes id)(CYB(\rho) + \text{Alt}(\delta \otimes id)(\rho)) \) with \( u \otimes v \otimes w \) (using \( \omega_x^{\otimes 3} \)). Therefore this pairing is zero, for any \( u, v, w \in u \). It follows that
\[ (p_u \otimes id)(CYB(\rho) + \text{Alt}(\delta \otimes id)(\rho)) = 0. \]
This proves (16).

Let us prove (17). For this, we will prove that for any \( u, v \in u \), the pairing (using \( \omega_x^{\otimes 2} \)) of the r.h.s. of (17) with \( id \otimes u \otimes v \) is zero. This pairing is
\[ \sum_{\alpha, \beta} p_1([a_{\alpha}, a_{\beta}]) \otimes \omega_x(b_{\alpha}, u) \omega_x(b_{\beta}, v) \ell_{\alpha \beta}(x) - \sum_{i, \alpha} e_i \otimes \omega_x(a_{\alpha}, u) \omega_x(b_{\alpha}, v) \ell_{\alpha}(x) \]
\[ + \sum_{\alpha} p_1(a^{(1)}_{\alpha}) \otimes \omega_x(p_a(a^{(2)}_{\alpha}), u) \omega_x(b_{\alpha}, v) \ell_{\alpha}(x) - \sum_{\alpha} p_1(a^{(1)}_{\alpha}) \otimes \omega_x(p_a(a^{(2)}_{\alpha}), v) \omega_x(b_{\alpha}, u) \ell_{\alpha}(x). \] (21)

The first sum of (21) is equal to \( p_1([u, v]) \otimes 1. \)
The second sum of (21) is equal to
\[
- \sum_{i,\alpha} e_i \otimes L_{x^i} (\omega_x(a_\alpha, u)w_x(b_\alpha, v)\ell_\alpha(x)) + \sum_{i,\alpha} e_i \otimes L_{x^i} (\omega_x(b_\alpha, v)w_x(a_\alpha, u)\ell_\alpha(x))
\]
\[
+ \sum_{i,\alpha} e_i \otimes \omega_x(a_\alpha, u)L_{x^i} (\omega_x(b_\alpha, v)\ell_\alpha(x)) = - \sum_{i,\alpha} e_i \otimes L_{x^i} (\omega_x(u, v))
\]
\[
+ \sum_{i,\alpha} e_i \otimes \omega_x(a_\alpha, u)\omega_x(b_\alpha, v)\ell_\alpha(x) + \sum_{i,\alpha} e_i \otimes \omega_x(a_\alpha, [\varepsilon^i, u])\omega_x(b_\alpha, v)\ell_\alpha(x)
\]
\[
+ \sum_{i,\alpha} e_i \otimes \omega_x(a_\alpha, u)\omega_x([\varepsilon^i, b_\alpha], v)\ell_\alpha(x) + \sum_{i,\alpha} e_i \otimes \omega_x(a_\alpha, u)\omega_x([\varepsilon^i, v])\ell_\alpha(x)
\]
\[
= - \sum_{i} e_i \otimes \omega_x([\varepsilon^i, u], v) - \sum_{i} e_i \otimes \omega_x(u, [\varepsilon^i, v]) - \sum_{i} e_i \otimes \omega_x([\varepsilon^i, v], u) - \sum_{i} e_i \otimes \omega_x(v, [\varepsilon^i, u])
\]
\[
+ \sum_{i} e_i \otimes \omega_x([\varepsilon^i, u], v) + \sum_{i} e_i \otimes \omega_x(u, [\varepsilon^i, v]) = - \sum_{i} e_i \otimes \omega_x([\varepsilon^i, v], u) - \sum_{i} e_i \otimes \omega_x(v, [\varepsilon^i, u])
\]
(22)
where both equalities follows from the identity
\[
\forall e \in \mathfrak{b}^*, \forall \alpha, \beta \in \mathfrak{d}(\mathfrak{g}), L_x(\omega_x(\alpha, \beta)) = \omega_x([\varepsilon, \alpha, \beta]) + \omega_x(\alpha, [\varepsilon, \beta]).
\]
The two last sums of (21) give the contribution
\[
- \sum_{i} p_t(u^{(1)}) \otimes \omega_x(p_u(u^{(2)}), u) + \sum_{i} p_t(u^{(1)}) \otimes \omega_x(p_u(u^{(2)}), v).
\]
Now if \(x \in u\), then
\[
\sum_{i} x^{(1)} \otimes x^{(2)} = \delta(x) = \sum_{i} [x, e_i] \otimes \varepsilon^i + \sum_{i} e_i \otimes [x, \varepsilon^i] + \sum_{i} [x, u_j] \otimes \lambda^j + \sum_{j} u_j \otimes [x, \lambda^j],
\]
where \((u_j), (\lambda^j)\) are dual bases of \(u\) and \(u^*\). Therefore
\[
\sum_{i} p_t(x^{(1)}) \otimes p_u(x^{(2)}) = \sum_{i} e_i \otimes p_u([x, \varepsilon^i]),
\]
so the contribution of the two last sums of (21) is
\[
- \sum_{i} e_i \otimes \omega_x(p_u([v, \varepsilon^i]), u) + \sum_{i} e_i \otimes \omega_x(p_u([u, \varepsilon^i]), v).
\]
(23)
The sum of (22) and (23) is equal to
\[
\sum_{i} e_i \otimes \omega_x(p_u([u, \varepsilon^i]), v) + \sum_{i} e_i \otimes \omega_x(p_u([v, \varepsilon^i]), u) - \sum_{i} e_i \otimes \omega_x(v, [\varepsilon^i, u]).
\]
(24)
Now \(\omega_x(\alpha, \beta) = 0\) whenever \(\alpha \in \mathfrak{g}^*\), so the first term of (24) vanishes; we have
\[
\omega_x(p_u([u, \varepsilon^i]), v) = \langle p_{\mathfrak{g}} \circ \text{Ad}(x^{-1}) \circ p_u([u, \varepsilon^i]), p_{\mathfrak{g}} \circ \text{Ad}(x^{-1})(v) \rangle
\]
\[
= \langle p_{\mathfrak{g}} \circ \text{Ad}(x^{-1})([u, \varepsilon^i]), p_{\mathfrak{g}} \circ \text{Ad}(x^{-1})(v) \rangle
\]
(since \(\text{Ad}(x^{-1})(\mathfrak{g}^*) \subset \mathfrak{g}^*\) and \(p_{\mathfrak{g}}(\mathfrak{g}^*) = 0\), and
\[
-\omega_x(v, [\varepsilon^i, u]) = \langle p_{\mathfrak{g}} \circ \text{Ad}(x^{-1})(v), p_{\mathfrak{g}} \circ \text{Ad}(x^{-1})([u, \varepsilon^i]) \rangle;
\]
adding up these equalities, we get
\[
(24) = \sum_{i} e_i \otimes (\text{Ad}(x^{-1})(v), \text{Ad}(x^{-1})([u, \varepsilon^i])) = \sum_{i} e_i \otimes ([v, u], \varepsilon^i) = p_t([v, u]) \otimes 1.
\]
Therefore (21) = 0, which proves (17). \(\square\)
Remark 2.3. The fact that $\sigma^g_1$ is equivariant implies that $\text{dress}^L_P(\rho^g_1) = \chi(a)P^g_1$, where $\chi : I \to \mathbb{C}$ is the character defined by $\chi(a) = \text{tr}(\text{ad}(a)_u)$.

Remark 2.4. Let $C[[L^*]][1/P^g_1]_{\geq 1}$ be the valuation $\geq i$ part of $C[[L^*]][1/P^g_1]$. Then $\sigma^g_1$ belongs to $\wedge^2(u) \otimes C[[L^*]][1/P^g_1]_{\geq 1}$. Its image in $\wedge^2(u) \otimes \text{gr}C[[L^*]][1/P^g_1]_{\geq -1} = \wedge^2(u) \otimes S(0)[1/P^g_1]_{-1}$ is $-r^g_1$, where $r^g_1$ is the $r$-matrix introduced in [EE2]. Taking the images in $\text{gr}(C[[L^*]][1/P^g_1]_{-1})$ (resp., $\text{gr}(C[[L^*]][1/P^g_1]_{-2})$) of the invariance (resp., CDYBE) identities for $\sigma^g_1$, we find the invariance (resp., CDYBE) identities for $\rho^g_1$, i.e., Proposition 1.1 in [EE2].

Example 2.5. If $(g, \delta)$ is as in Example 2.1 and $I = \mathfrak{h}$ is the Cartan subalgebra of $\mathfrak{g}$, then $\sigma^g_1$ is given by
\[
\forall \lambda \in \mathfrak{h}^*, \quad \sigma^g_1(e^\lambda) = \sum_{\alpha \in \Delta_+(g)} \frac{e^{\alpha} \wedge f_\alpha}{1 - e^{-(\lambda, \alpha)}},
\]
which is the solution of Example 1.5.

Example 2.6. Assume that $(\mathfrak{g}, \delta)$ is semisimple and $I \subset \mathfrak{g}$ is a Levi subalgebra as in Example 2.1. The Cartan decomposition of $I$ is $I = \mathfrak{h} \oplus \mathfrak{m}_+ \oplus \mathfrak{m}_-$, where $\mathfrak{m}_+ = \oplus_{\alpha \in \Delta_+(g)} \mathfrak{g}_\alpha, \mathfrak{m}_- = \oplus_{\alpha \in \Delta_-(g)} \mathfrak{g}_\alpha$. Let $G$ be a (formal or algebraic) group with Lie algebra $\mathfrak{g}$, $H \subset L_+ \subset L \subset G$ be subgroups corresponding to $\mathfrak{h} \subset (\mathfrak{h} \oplus \mathfrak{m}_+) \subset I \subset \mathfrak{g}$. Then $L^* = \{ (x_+, x_-) \in L_+ \times L_, h_+(x_+)h_-(x_-) = 1 \}$, where $h_\pm : L_\pm \to H$ are the group morphisms corresponding to the projections $(\mathfrak{h} \oplus \mathfrak{m}_+) \to \mathfrak{h}$. If $x \in L$, then $\text{Ad}(x)$ restricts to an automorphism of $u$. For a generic $x \in L$, $\text{Ad}(x) - \text{id}$ is invertible, when viewed as an endomorphism of $u$. Then $\sigma^g_1 : L^* \to \wedge^2(u)$ is defined by
\[
\sigma^g_1(x_+, x_-) = \sum_{\alpha \in \Delta_+(g) \setminus \Delta_+(I)} (\text{id} - \text{Ad}(x_+ x_-^{-1}))^{-1}(e^\alpha) \wedge f_\alpha.
\]

2.4. The composition theorem.

Theorem 2.7. Let $(g, l, u, \delta)$ be a nondegenerate Lie bialgebra with a splitting (see Section 2.1). Let $(a, \delta)$ be a Lie bialgebra equipped with a Lie bialgebra inclusion $\mathfrak{g} \subset \mathfrak{a}$. Let $Z \in \wedge^3(\mathfrak{a})$. Then the map $\rho \mapsto \rho|_{L^*} + \sigma^g_1$ takes $\{ \text{PL dynamical r-matrices for } (G^*, a, Z) \}$ to $\{ \text{PL dynamical r-matrices for } (L^*, a, Z) \}$.

Proof. This is a consequence of the following statement: if $\tau : G^* \to \wedge^2(a)$ is such that $\text{dress}_a^3(\tau) + [a^1 + a^2, \tau] = 0$ for any $a \in \mathfrak{g}$, then
\[
(dL^*)|_{L^*} = dL^*(\tau|_{L^*}) + [(\tau|_{L^*})^{1,2}, (\sigma^g_1)^{1,3} + (\sigma^g_1)^{2,3}].
\]

(25)

Here the map $\mathfrak{g}^\otimes 3 \otimes C[[G^*]] \to \mathfrak{g}^\otimes 3 \otimes C[[L^*]], \omega \mapsto \omega|_{L^*}$ is the tensor product of $\text{id}^\otimes 3$ with the restriction map $C[[G^*]] \to C[[L^*]]$.

Let us prove (25). Let $(u_j), (\lambda^j)$ be dual bases of $u$ and $u^*$. Set $\tau = \sum_{\alpha} x_{\alpha} \otimes y_{\alpha} \otimes m_{\alpha}$. Then
\[
(dL^*)|_{L^*} - dL^*(\tau|_{L^*}) = - \sum_{\alpha, \beta} x_{\alpha} \otimes y_{\alpha} \otimes u_j \otimes L_{\lambda_j}(m_{\alpha})|_{L^*} = \sum_{\alpha} x_{\alpha} \otimes y_{\alpha} \otimes u_j \otimes R_{\text{Ad}(x^{-1}(\lambda_j))(m_{\alpha})}|_{L^*}.
\]

On the other hand,
\[
[(\tau|_{L^*})^{1,2}, (\sigma^g_1)^{1,3} + (\sigma^g_1)^{2,3}] = \sum_{\alpha, \beta} \left[ x_{\alpha}, a_{\beta} \right] \otimes y_{\alpha} \otimes b_{\beta} \otimes (m_{\alpha} \ell_{\beta})|_{L^*} = x_{\alpha} \otimes \left[ y_{\alpha}, a_{\beta} \right] \otimes b_{\beta} \otimes (m_{\alpha} \ell_{\beta})|_{L^*}.
\]

\[
= \sum_{\alpha, \beta} x_{\alpha} \otimes y_{\alpha} \otimes b_{\beta} \otimes (\ell_{\beta} \text{dress}^L_{\alpha}(m_{\alpha}))|_{L^*},
\]

(25)
Therefore (25) follows from
\[
\sum_{\beta} \left( (\text{Ad}(x^{-1})(a_{\beta}))_g \otimes b_{\beta} \right) \ell_{\beta}(x) = - \sum_{j} \text{Ad}(x^{-1})(\lambda_j) \otimes u_j. \tag{26}
\]

Let us prove (26). Both sides of this equation belong to \( g^* \otimes u \otimes \mathbb{C}[[L^*]] \). We will show that the pairings of both sides of (26) with \( b \otimes \text{id} \otimes \text{id} \) are the same, for any \( b \in g \). As \( x \) is formal near the origin, the map \( p_0 \circ \text{Ad}(x^{-1}) : g \to g \) is bijective, so we have \( b = p_0 \circ \text{Ad}(x^{-1})(a) \), where \( a \in g \). Pairing the l.h.s. of (26) with \( b \otimes \text{id} \otimes \text{id} \), we get
\[
\sum_{\beta} b_{\beta} \otimes (p_0^* \circ \text{Ad}(x^{-1})(a_{\beta}))_g \otimes \ell_{\beta}(x) = \sum_{\beta} b_{\beta} \otimes \tilde{\omega}_x(a_{\beta})_g \otimes \ell_{\beta}(x) = -pu(a) \otimes 1,
\]
and pairing the r.h.s. of (26) with the same element, we get
\[
- \sum_{j} u_j \otimes (\text{Ad}(x^{-1})(\lambda_j))_g \otimes \ell_{\beta}(x) = - \sum_{j} u_j \otimes (\text{Ad}(x^{-1})(\lambda_j), \text{Ad}(x^{-1})(a_{\beta}))_g \otimes \ell_{\beta}(x) \quad \text{(as \( g^* \) is isotropic)}
\]
\[
= - \sum_{j} (\lambda_j, a) u_j \otimes 1 = -pu(a) \otimes 1.
\]

This proves (26). \( \square \)

Remark 2.8. Assume that \( a = \mathfrak{g} \oplus \mathfrak{v} \) is a nondegenerate Lie bialgebra with a splitting. Then \( \sigma^g = (\sigma^g)_{\mathfrak{L}^*} + \sigma^g_{\mathfrak{P}} \), in other words, the map of Theorem 2.7 takes \( \sigma^g \) to \( \sigma^g_{\mathfrak{L}^*} \) (here \( Z = 0 \)).

Example 2.9. Applying Theorem 2.7 when \((\mathfrak{g}, \delta)\) is semisimple, \( \mathfrak{l} \subset \mathfrak{g} \) is a Levi subalgebra as in Example 2.1, and \( \rho \) is the Balog-Fehér-Palla \( r \)-matrix, we get the following result. Define \( \rho_{\text{BFP}}^\mathfrak{l,\mathfrak{g}} : L^* \to \wedge^2(\mathfrak{g}) \) by
\[
\rho_{\text{BFP}}^\mathfrak{l,\mathfrak{g}}(x_+, x_-) = \nu(\frac{id + g^{2\nu}}{id - g^{2\nu}} \otimes \text{id})(t) - \frac{1}{2} (\frac{id + g}{id - g} \otimes \text{id})(t_1) + \frac{1}{2} s - \frac{1}{2} s^{2,1}, \tag{27}
\]
where \( g = \text{Ad}(x_+ x_-^{-1}) \in \text{End}(\mathfrak{g}) \), \( t \in S^2(\mathfrak{g})^g \), \( t_1 \in S^2(\mathfrak{g})^l \) are the Casimir elements of \( \mathfrak{g}, \mathfrak{l} \), and \( s = \sum_{\alpha \in \Delta_+} c_{\alpha} f_\alpha + \sum_{\alpha \in \Delta_-} c_{\alpha} f_\alpha \), so that \( t_1 = t_1 + s + s^{2,1} \). Then \( \rho_{\text{BFP}}^\mathfrak{l,\mathfrak{g}} \) is a PL dynamical \( r \)-matrix for \((L^*, g, Z_\rho)\).

Example 2.10. Rewriting (27) using \( t_1 = t_1 + s + s^{2,1} \), we get in the case \( \mathfrak{l} = \mathfrak{h} \) (the Cartan subalgebra of \( \mathfrak{g} \)) that
\[
\rho_{\text{BFP}}^\mathfrak{h,\mathfrak{g}}(e^\lambda) = \frac{1}{2} \sum_{\alpha \in \Delta_+} \left( 1 + 2\nu \coth(\nu(\lambda, \alpha)) \right) (e_\alpha \wedge f_\alpha)
\]
is a PL dynamical \( r \)-matrix for \((H^*, \mathfrak{g}, (4\nu^2 - 1)Z(r_0))\). Under the correspondence of Example 1.5, this solution corresponds to the usual dynamical \( r \)-matrix \( \lambda \mapsto 2\nu r_0^g(2\nu \lambda) \) for \((\mathfrak{h}, \mathfrak{g}, 4\nu^2 Z(r_0))\).

2.5. A PL dynamical \( r \)-matrix for \((G^*, \mathfrak{g}, (4\nu^2 - 1)Z(r_0))\). The proof of Theorem 2.7 implies the following result: let \( \rho : G^* \to \wedge^2(\mathfrak{g}) \) be a \( g \)-equivariant function, such that \( \rho|_{L^*} + \sigma^g_{\mathfrak{P}} \) is a PL dynamical \( r \)-matrix for \((L^*, \mathfrak{g}, Z)\), then \( \rho \) is a PL dynamical \( r \)-matrix for \((G^*, \mathfrak{g}, Z)\). This result leads to the following \( r \)-matrix (an analogue of the \( r \)-matrix in [EE2], Remark 1.8).

Let \( \mathfrak{g} \) be a semisimple Lie algebra and \( t \in S^2(\mathfrak{g})^g \) be nondegenerate. Let \( \mathfrak{h} \) be a Cartan subalgebra of \( \mathfrak{g} \) and \( \mathfrak{b}_\pm \) be opposite Borel subalgebras containing \( \mathfrak{h} \), \( G \) a (formal or algebraic) Lie group with Lie algebra \( \mathfrak{g}, H \subset B_\pm \subset G \) Lie subgroups corresponding to \( \mathfrak{h} \subset \mathfrak{b}_\pm \subset \mathfrak{g} \). Then \( G^* = \{ (x_-, x_+) \in B_- \times B_+ | x_- h_- x_+(x_+) = 1 \} \). The map \( G^* \to G, g^* = (x_-, x_+) \mapsto x_+ x_-^{-1} = \lambda(g^*) \) is equivariant for the left dressing action of \( \mathfrak{g} \) on \( G^* \) and its adjoint action on \( G \).
Let $G_{reg} \subset G(R)$ be the subset of regular real semisimple elements, and define $G^*_{reg} := \lambda^{-1}(G_{reg})$. Let $\vartheta : [0, 1] \times \mathbb{R} \to [1, -1]$ be defined by $\vartheta_{|0, 1]} = -1$, $\vartheta_{|1, \infty]} = +1$. If $g^\prime \in G_{reg}$, let $h(g') = \{x \in g | \text{Ad}(g')(x) = 0\}$ be the Cartan subalgebra of $g$ associated to $g'$, and if $h' \subset g$ is a Cartan subalgebra, let $t_{h'}$ be the part of $t$ corresponding to $h'$. For $\nu \in \mathbb{C}$, define $\rho_{\nu} : G^*_{reg} \to \wedge^2(g)$ by

$$\rho_{\nu}(g^\prime) = -\left(\frac{1}{2} \text{Ad}(g) + \text{id} + \nu \theta(\text{Ad}(g)) \otimes \text{id}\right)(t - t_{h'(g')})$$

where $g = \lambda(g^\prime)$. Here $\theta(\text{Ad}(g))$ is the operator acting by $\theta(\lambda)$ on the $\lambda$-eigenspace of $\text{Ad}(g)$.

**Proposition 2.11.** $\rho_{\nu}$ is a PL dynamical $\nu$-matrix for $(G^*, g, (4\nu^2 - 1)Z(r_0))$.

**Proof.** $\rho_{\nu}$ is clearly equivariant, $(\rho_{\nu})(H) + \sigma^g_{h^\prime} = (1 + 2\nu)r_0$, and $\text{CYB}((1 + 2\nu)r_0) + \text{Alt}(\delta \otimes \text{id})(1 + 2\nu) = (4\nu^2 - 1)Z(r_0)$.

\[\square\]

### 3. Quantization of $\sigma^g_{h^\prime}$

We introduce the notion of a polarized Lie bialgebra with a splitting. We will quantize $\sigma^g_{h^\prime}$ when $(g, \delta, l, u)$ is polarized. This yields a quantization in the case of Example 2.1.

#### 3.1. A construction for Hopf algebras

Let $A, B$ be Hopf algebras, $i : B \hookrightarrow A$ and $\pi : A \to B$ be Hopf algebra morphisms, such that $\pi \circ i = \text{id}_B$. Define maps

$$\alpha : A \to A, \beta : A \to A \otimes B$$

by $\alpha(a) = \sum a^{(1)}(i \circ \pi \circ S)(a^{(2)})$, and $\beta(a) = (\text{id} \otimes \pi) \circ \Delta(a) - a \otimes 1$.

**Proposition 3.1.** (see [Rad], Theorem 3)

1) $\text{Im}(\alpha) = \text{Ker}(\beta)$. Set $C := \text{Im}(\alpha) = \text{Ker}(\beta)$.

2) $C$ is a subalgebra and a left coideal of $A$.

3) $C$ is stable under the adjoint action of $A$ on itself.

**Example 3.2.** Let $p, l$ be Lie algebras, and assume that we have Lie algebra morphisms $l \hookrightarrow p$ and $p \twoheadrightarrow l$, such that $\pi \circ i = \text{id}_l$. This means that $p = l \oplus u$, $l$ is a Lie subalgebra of $p$, $u$ is an ideal of $p$. If we set $A = U(p)$, $B = U(l)$, we get $C = U(u)$. This is proved using the isomorphism $U(u) \otimes U(l) \to U(p)$, $u \otimes l \to ul$.

**Example 3.3.** In the situation of Example 3.2, let $A = \mathbb{C}[[P]]$, $B = \mathbb{C}[[L]]$ (here $\mathbb{C}[[X]] = U(p)^*$ if $r$ is a Lie algebra, and $X$ is the corresponding formal group). Then $C = \mathbb{C}[[P/L]]$ is the ring of right $l$-invariant formal functions on $P$. The natural map $U \to P/L$ is an isomorphism of formal manifolds, so we will denote $\mathbb{C}[[P/L]]$ as $\mathbb{C}[[U]]$.

#### 3.2. Lie bialgebras with a splitting

Let $(l, \delta)$ and $(p, \delta)$ be Lie bialgebras, equipped with Lie bialgebra morphisms $i : l \hookrightarrow p$ and $\pi : p \twoheadrightarrow l$, such that $\pi \circ i = \text{id}_l$. We set $u := \text{Ker}(\pi)$. Then $u$ is a Lie algebra ideal of $p$, such that $\delta(u) \subset \left((l \otimes u) \oplus (u \otimes l)\right) \oplus \wedge^2(u)$.

**Example 3.4.** $p = p_+$, a parabolic subalgebra of a semisimple Lie bialgebra $g$, and $l \subset p$ is the corresponding Levi subalgebra.

Assume that we have constructed a pair of QUE algebras $U_h(l)$, $U_h(p)$ quantizing $l, p$, and a sequence of morphisms $U_h(l) \hookrightarrow U_h(p) \twoheadrightarrow U_h(l)$, whose composition is the identity, quantizing $l \hookrightarrow p \twoheadrightarrow l$. This can be achieved: (a) in the general case, by applying a quantization functor to $l \hookrightarrow p \twoheadrightarrow l$ (see [EK]): (b) in the situation of Example 2.1, let $I \subset [1, r]$ be the set of indices of $p$. Then $U_h(p)$ is the subalgebra of $U_h(g)$ generated by the $h_i, e_i, i \in [1, r]$, and $f_i, i \in I$. 

$U_h(l) \subset U_h(p)$ is generated by the $e_i, f_i, i \in I$, and $h_i, i \in [1, r]$. The morphism $\pi$ is defined by $\pi(x) = x$ for $x = e_i, f_i, i \in I$, and $x = h_i, i \in [1, r]$, and $\pi(e_i) = 0$ if $i \notin [1, r]$.

Then we are in the situation of Section 3.1, with $A = U_h(p)$, $B = U_h(l)$. Define $U_h(u)$ to be the $C$ defined in this section.

**Proposition 3.5.** $U_h(u) \subset U_h(p)$ is a flat deformation of $U(u) \subset U(p)$.

**Proof.** $U_h(u)$ is defined as $\text{Ker}(\beta)$, therefore it is a complete, saturated submodule of $U_h(p)$ (here saturated means that if $x \in U_h(p)$ is such that $hx \in U_h(u)$, then $x \in U_h(u)$). So it remains to show that the morphism of reduction modulo $h$, $(\mod h) : U_h(p) \to U(p)$, satisfies $(\mod h)(U_h(u)) = U(u)$.

Let $\alpha_0, \beta_0$ be the reductions modulo $h$ of $\alpha, \beta$. According to Example 2.1, $\text{Im}(\alpha_0) = \text{Ker}(\beta_0) = U(u)$.

Now $(\mod h)(U_h(u)) = (\mod h)(\text{Im}(\alpha)) \supset \text{Im}(\alpha_0)$, and $(\mod h)(U_h(u)) = (\mod h)(\text{Ker}(\beta)) \subset \text{Ker}(\alpha_0)$. Therefore $(\mod h)(U_h(u)) = U(u)$.

We now describe a quantized formal series Hopf (QFSH) algebra version of the above constructions. Recall that if $a$ is a finite dimensional Lie bialgebra, and if $U_h(a)$ is a QUE algebra quantizing $a$, then the corresponding quantized formal series Hopf (QFSH) algebra is

$$\mathbb{C}[[A^*]]_h := \{ x \in U_h(a) | \forall n \geq 0, \delta_n(x) \in h^n U_h(a)^{\otimes n} \}.$$ 

Here $\delta_n = (\text{id} - \eta \circ \epsilon)^{\otimes n} \circ \Delta^{(n)}$. Then $\mathbb{C}[[A^*]]_h$ is a flat deformation of $\mathbb{C}[[A^*]] = (a^*)^*$, the function ring of the formal group with Lie algebra $a^*$.

Let $l, p$ be as in the beginning of this section. Then we have Lie algebra morphisms $l^* \hookrightarrow p^*$, $p^* \to l^*$ such that $l^* \to p^* \to l^*$ is the identity, so $u^*$ is an ideal of $l^*$. We are again in the situation of Section 3.1, with $A = \mathbb{C}[[P^*]]_h$, $B = \mathbb{C}[[L^*]]_h$. Define $\mathbb{C}[[U^*]]_h$ to be the algebra $C$ defined in this section.

**Proposition 3.6.** $\mathbb{C}[[U^*]]_h \subset \mathbb{C}[[P^*]]_h$ is a flat deformation of $\mathbb{C}[[U^*]] \subset \mathbb{C}[[P^*]]$.

**Proof.** Similar to the proof of Proposition 3.5, using Example 3.3. □

### 3.3. Polarized Lie bialgebras with a splitting.

**Definition 3.7.** $(g, \delta, l, u)$ is a polarized Lie bialgebra with a splitting if $(g, \delta)$ is a Lie bialgebra, such that $l \subset g$ is a Lie subbialgebra, $u = u_+ \oplus u_-$, where $u_\pm$ are Lie subalgebras of $g$, and $\delta(u_\pm) \subset ((1 \otimes u_\pm) \oplus (u_\pm \otimes 1)) \oplus \wedge^2(u_\pm)$.

**Example 3.8.** See Example 2.1.

Set $p_\pm = l \oplus u_\pm$, then $p_\pm$ are Lie subalgebras of $(g, \delta)$. Then we have Lie bialgebra morphisms $l \subset p_\pm \subset g$ and $p_\pm \to l$, such that the composed map $l \hookrightarrow p_\pm \to l$ is the identity. Applying a quantization functor to this situation, we get QUE algebras $U_h(l)$, $U_h(p_\pm)$ and $U_h(g)$ quantizing $l$, $p_\pm$ and $g$, together with QUE algebra inclusions $U_h(l) \subset U_h(p_\pm) \subset U_h(g)$, and QUE algebra morphisms $U_h(p_\pm) \pi_+ \to U_h(l)$, such that the composed map $U_h(l) \hookrightarrow U_h(p_\pm) \pi_+ \to U_h(l)$ is the identity. In the situation of Example 2.1, $U_h(l) \subset U_h(p_\pm) \subset U_h(g)$ is as above, $U_h(p_-)$ is the subalgebra of $U_h(g)$ generated by the $h_i, f_i, i \in [1, r]$, and $e_i, i \in I$ and $\pi_\pm : U_h(p_\pm) \to U_h(l)$ are as above.

Let us now set $U_h(u_\pm) := \{ x \in U_h(p_\pm) | (\text{id} \otimes \pi_\pm) \circ \Delta(x) = x \otimes 1 \}$, then we have seen that $U_h(u_\pm)$ is a subalgebra and a left coideal of $U_h(p_\pm)$, and $U_h(u_\pm) \subset U_h(p_\pm)$ is a flat deformation of $U(u_\pm) \subset U(p_\pm)$. 
Then the tensor product of inclusions followed by the product $U_h(u_+) \otimes U_h(p_-) \to U_h(g)$ is an isomorphism of $C[[\hbar]]$-modules. Therefore
\[ \text{PBW} : U_h(u_+) \otimes U_h(l) \otimes U_h(u_-) \to U_h(g) \]
is an isomorphism of $C[[\hbar]]$-modules. We define $H : U_h(g) \to U_h(l)$ as the composed map $(\varepsilon \otimes \text{id} \otimes \varepsilon) \circ \text{PBW}^{-1}$. Then we have
\[ \forall x_+ \in U_h(u_+), \forall x_0 \in U_h(l), \quad H(x_+ x_0 x_-) = \varepsilon(x_+) \varepsilon(x_-) x_0 \]
and since $U_h(u_{\pm})$ is stable under the adjoint action of $U_h(p_{\pm})$ on itself,
\[ \forall x_+ \in U_h(p_{\pm}), \forall x \in U_h(g), \quad H(x_+ x) = \varepsilon(x_+) H(x), \quad H(x x_-) = \varepsilon(x_-) H(x). \]

In the same way, we set $C[[U^\pm_{\hbar}]] := \{ x \in C[[P^\pm_{\hbar}]] \mid (\text{id} \otimes \pi_{\pm}) \circ \Delta(x) = x \otimes 1 \} = U_h(u_{\pm}) \cap C[[P^\pm_{\hbar}]],$
then $C[[U^\pm_{\hbar}]]_{\hbar}$ is a subalgebra of a left coalgebra $C[[P^\pm_{\hbar}]]_{\hbar}$ and $C[[U^\pm_{\hbar}]]_{\hbar} \subset C[[P^\pm_{\hbar}]]_{\hbar}$ is a flat deformation of $C[[U^\pm_{\hbar}]] \subset C[[P^\pm_{\hbar}]].$ Therefore PBW restricts to an isomorphism of $C[[\hbar]]$-modules
\[ \text{PBW} : C[[U^+_{\hbar}]]_{\hbar} \otimes C[[L^+_{\hbar}]]_{\hbar} \otimes C[[U^-_{\hbar}]]_{\hbar} \to C[[G^*]]_{\hbar}. \]
$H$ restricts to a map $C[[G^*]]_{\hbar} \to C[[L^+_{\hbar}]]_{\hbar}$ with the same properties as above (replacing $U_h(\mathfrak{g})$ by $C[[L^+_{\hbar}]]_{\hbar}$ everywhere).

3.4. Localization of QFSH algebras. Let $a$ be a finite dimensional Lie bialgebra, let $U_h(a)$ be a QUE algebra quantizing $a$ and let $C[[A^*]]_{\hbar}$ be the corresponding QFSH algebra. Recall that $C[[A^*]]_{\hbar}$ is a flat deformation of $C[[A^*]] = U(a^*)^*.$

Let $D \in S^d(a)$ be a nonzero element. We denote the localization of $C[[A^*]]$ w.r.t. $D$ by $C[[A^*]]_{D}.$ This is a Poisson algebra, equipped with left and right coproduct morphisms
\[ C[[A^*]]_{D} \to C[[A^*]] \otimes C[[A^*]]_{D}, \quad C[[A^*]]_{D} \to C[[A^*]]_{D} \otimes C[[A^*]], \quad (28) \]
satisfying natural compatibility rules; in particular, $C[[A^*]]_{D}$ is a Poisson base algebra over $C[[A^*]]$ (see [DM], Section 4.1). Here $\otimes$ has the following meaning: if $V$ is a vector space, then $V \otimes C[a_1, \ldots, a_d] = C[a_1, \ldots, a_d] \otimes V = V[[a_1, \ldots, a_d]].$

**Proposition 3.9.** 1) There is a unique quantization $C[[A^*]]_{D,\hbar}$ of $C[[A^*]]_{D},$ containing $C[[A^*]]_{\hbar}.$

2) The coproduct morphism $C[[A^*]]_{\hbar} \to C[[A^*]]_{D,\hbar} \otimes C[[A^*]]_{D} \otimes C[[A^*]]_{D}$ extends uniquely to morphisms $\Delta_{L} : C[[A^*]]_{D,\hbar} \to C[[A^*]]_{D,\hbar} \otimes C[[A^*]]_{D}$ and $\Delta_{R} : C[[A^*]]_{D,\hbar} \to C[[A^*]]_{D,\hbar} \otimes C[[A^*]]_{D,\hbar},$ quantizing (28), satisfying natural axioms (see [DM]).

**Proof.** In the same way as in [ER], Proposition 3.1, one proves the following statement. Let $R$ be a Poisson ring and $S$ be a multiplicative part of $R.$ Then the localization $R_S$ has a unique Poisson structure, extending the Poisson structure of $R.$ Let $R_{\hbar}$ be a quantization of $R$ (i.e., $R_{\hbar}$ is isomorphic to $R[[\hbar]]$ as a $C[[\hbar]]$-module, it is a topological algebra, with classical limit $R$). Then there is a unique quantization $R_{S,\hbar}$ of $R_S,$ containing $R_{\hbar}.$ If $s \in S$ and $s \in R_{\hbar}$ is a lift of $s$ in $R_{\hbar},$ then $s$ is invertible in $R_{S,\hbar}.$ The algebra $R_{S,\hbar}$ may be characterized by the following universal property: if $A_{\hbar}$ is a quasi-commutative algebra (i.e., $A_{\hbar}$ is a topological $C[[\hbar]]$-algebra, whose reduction modulo $\hbar$ is commutative), any morphism $R_{\hbar} \to A_{\hbar}$ of topological algebras, whose reduction modulo $\hbar$ extends to a morphism $R_S \to A_S/\hbar A_S,$ extends in a unique way to a morphism $R_{S,\hbar} \to A_{\hbar}$ of topological algebras. We call $R_{S,\hbar}$ the localization of $R_{\hbar}$ w.r.t. $S.$ Then $C[[A^*]]_{D,\hbar}$ corresponds to $R = C[[A^*]]_{\hbar},$ $S = \{\text{powers of } D\}.$

Let us now construct $\Delta_{L}, \Delta_{R}.$ We have $\Delta(D) = D \otimes 1 + \sum_i a_i \otimes b_i,$ where $b_i \in m(A^*)_{\hbar} = \text{Ker}(\varepsilon : C[[A^*]]_{\hbar} \to C[[\hbar]])$. Therefore $\Delta(D)$ is invertible in $C[[A^*]]_{D,\hbar} \otimes C[[A^*]]_{D,\hbar},$ with image
\[ D^{-1} \otimes 1 - \sum_i D^{-1} a_i D^{-1} \otimes b_i + \sum_{i,j} D^{-1} a_i D^{-1} a_j D^{-1} \otimes b_i b_j - \cdots ; \]
this sum is convergent since \( b_1, \ldots, b_n \in \mathfrak{m}(A^*)^p \). Applying the universal property to \( \Delta : C[[A^*]]_h \to C[[A^*]]^2_h \to C[[A^*]]^2_h \otimes C[[A^*]]_h \), we obtain the existence of \( \Delta_t \). The morphism \( \Delta_t \) is constructed in the same way. \(\square\)

In the case when \( U_h(a) = U(a)[[h]] \) (so the Lie cobracket of \( a \) is zero) and \( D \in C[[A^*]] \) is homogeneous, we now relate the algebra \( C[[A^*]]^2_h \otimes C[[A^*]]_h \) to the algebra \( \overline{U(a)} \) introduced in [EE2]. \( \overline{U(a)} \) is the microlocalization of \( U(a) \) corresponding to \( D \). It is a \( \mathbb{Z} \)-filtered algebra, whose associated graded is isomorphic to \( S'(A)_D \).

**Lemma 3.10.** Let \( \mathcal{A} \) be the subspace of \( \overline{U(a)}((h)) \) consisting in all series \( \sum_{i \in \mathbb{Z}} h^i x_i \), where \( x_i \) vanishes for \( i \ll 0 \), and \( \deg(x_i) \leq i \) for any \( i \in \mathbb{Z} \). Then \( \mathcal{A} \) is a subalgebra of \( \overline{U(a)}((h)) \), isomorphic to \( C[[A^*]]^2_h \).

Here \( X((h)) = X[[h]][h^{-1}] \).

**Proof.** \( \mathcal{A} \) is a topologically free \( C[[h]] \)-module and a subalgebra of \( \overline{U(a)}((h)) \); one checks that \( \mathcal{A}/h\mathcal{A} \) is isomorphic to the degree completion of \( S'(A)_D \). \( C[[A^*]]^2_h \subset U(a)[[h]] \) identifies with the space of all series \( \sum_{i \geq 0} h^i x_i \), with \( \deg(x_i) \leq i \). So we have an injection \( C[[A^*]]^2_h \subset \mathcal{A} \).

One checks that a lift of \( D \) in \( C[[A^*]]^2_h \) is taken to an invertible element of \( \mathcal{A} \). This shows that \( C[[A^*]]_h \subset \mathcal{A} \) extends to an injection \( \mathcal{O}(A^*)_h \subset \mathcal{A} \), whose reduction modulo \( h \) is the identity. This implies that \( C[[A^*]]^2_h \otimes C[[A^*]]^2_h \) is an isomorphism. \(\square\)

**Remark 3.11.** We do not know a quantization of \( \overline{U(a)} \) without assuming \( U_h(a) = U(a)[[h]] \) and \( D \) homogeneous.

3.5. **Construction of \( \Upsilon \).** Let \( \mathfrak{g} \) be a nondegenerate polarized Lie bialgebra with a splitting. Let \( p \in S^2(\mathfrak{g}) \) be as in Section 2.1. Then there exists a unique \( \overline{\sigma}^p \in u_+ \otimes u_- \otimes C[[L^*]]^p \), such that

\[
\sigma^p = \overline{\sigma}^p - (\overline{\sigma}^p)^{2,1}.
\]

Let \( U_h(t), U_h(u_\pm)), U_h(p_\pm), U_h(g) \colon C[[L^*]]^p \rightarrow C[[U^\pm_+]]^p \), \( C[[P^\pm]]^p \), \( C[[G^*]]^p \), and the morphisms between these algebras, be as in Section 3.2.

**Theorem 3.12.** There exists a unique element \( \Upsilon \in U_h(u_\pm) \otimes U_h(u_-) \otimes C[[L^*]]^p \), which we denote \( \Upsilon = \sum_{i} A_i \otimes B_i \otimes L_i \), such that for any \( x_\pm \in U_h(u_\pm) \), we have

\[
\sum_i H(x_- A_i) L_i H(B_i x_+) = H(x_- x_+).
\]

Here we use the injection \( U_h(t) \subset C[[L^*]]^p \). Then this identity also holds when \( x_\pm \in U_h(p_\pm) \).

\( \Upsilon \) is a series \( \sum_{n \geq 0} \frac{h^n}{n!} \Upsilon_n \), where \( \Upsilon_n \in U_h(u_\pm) \otimes U_h(u_-) \otimes C[[L^*]]^p, \) and \( \deg(x_i) \leq i \). It is such that its reduction modulo \( h \) is an element of \( U_h(u_\pm) \otimes U_h(u_-) \otimes C[[L^*]]^p, \) and the image of this reduction in \( S^2(u_\pm) \otimes S^2(u_-) \otimes C[[L^*]]^p \) is equal to \( (\overline{\sigma}^p)^n \).

Moreover, \( \Upsilon \) (resp., \( \Upsilon^{2,1,3} \)) belongs to \( U_h(u_\pm) \otimes C[[U^\pm_+]]^p \otimes C[[L^*]]^p \).

If \( V = E[[h]] \) and \( W = C[[a_1, \ldots, a_d]][1/p][[h]] \), then \( V \otimes W = E[[a_1, \ldots, a_d]][1/p][[h]] \).

**Proof.** We first prove some properties of \( H : C[[G^*]]^p \rightarrow C[[L^*]]^p \). Let \( \mathfrak{m}(U^*_\pm)_h \subset C[[U^\pm_+]]^p \) be the kernel of the composed map \( C[[U^\pm_+]]^p \rightarrow C[[h]] \rightarrow C \). Let \( \mathfrak{m}(U^\pm_\pm) \) be the kernel of \( C[[U^\pm_\pm]]^p \rightarrow C \). Then \( \mathfrak{m}(U^*_\pm) \rightarrow \mathfrak{m}(U^\pm_\pm)/\mathfrak{m}(U^\pm_\pm)_h \).
Lemma 3.13. Define a bilinear map \((-,-): \mathbb{C}[[U^*_+]]_h \times \mathbb{C}[[U^*_-]]_h \to \mathbb{C}[[L^*]]_h \) by \((f_-, f_+) := H(f_- f_+)\). Then if \(f_\pm \in \mathfrak{m}(U^*_\pm)_h^{\mathfrak{m}+} \), we have \((f_-, f_+) = O(h^{\max(n, n_-)})\).

Thus, \(h^{-n}(-,-)\) restricts to a bilinear map \(\mathfrak{m}(U^*_+)^n \times \mathfrak{m}(U^*_-)^n \to \mathbb{C}[[L^*]]_h\). This map induces a bilinear map \(\mathfrak{m}(U^*_+) \times \mathfrak{m}(U^*_-) \to \mathbb{C}[[L^*]]_h\), which factors through a bilinear map \((-,-)_n : S^n(u_-) \times S^n(u_+) \to \mathbb{C}[[L^*]]_h\) (as \(S^n(u_\pm) = \mathfrak{m}(U^*_\pm)^n / \mathfrak{m}(U^*_\pm)^{n+1}\)). \((-,-)_n\) is the \(n\)th symmetric power of the bilinear map \((-,-)_1\), taking \((u_-, u_+)_n \in u_- \times u_+\) to the formal function \(\ell^* \mapsto \omega_{\ell^*}(u_+, u_-)\) \((\ell^* \in \mathfrak{g}^*)\).

Proof of Lemma. We have \(\mathfrak{g}^* = \mathfrak{t}^* \oplus u_+^* \oplus u_-^*\), where \(\mathfrak{t}^*\) is a Lie subalgebra, \([\mathfrak{t}^*, u^*_\pm] \subseteq u^*_\pm\), \([u^*_+, u^*_-] = 0\). We have \(p^*_\pm = \mathfrak{t}^* \oplus u^*_\pm\), and the natural projections \(\mathfrak{g}^* \to p^*_\pm\) are Lie algebra morphisms.

Let compute the classical limit of the maps PBW and \(H\) (relative to QFS algebras). The classical limit of PBW is an algebra morphism \(\text{PBW}_0: \mathbb{C}[[U^*_+]] \otimes \mathbb{C}[[L^*]] \otimes \mathbb{C}[[U^*_-]] \to \mathbb{C}[[G^*]]\), induced by the inclusions \(\mathbb{C}[[U^*_+]] = \mathbb{C}[[P^*_+] / L^*] \subseteq \mathbb{C}[[P^*_+]] \subseteq \mathbb{C}[[G^*]]\) followed by multiplication. It follows that PBW is the dual of the coalgebra isomorphism \(U(\mathfrak{g}^*)^0 \to \mathfrak{u}(\mathfrak{t}^*) \otimes U(\mathfrak{t}^*) \otimes U(\mathfrak{u}^*)\), whose inverse takes \(x^*_+ \otimes x^*_- \to x^*_+ x^*_- x_-^*\). Therefore the classical limit of \(H\) is the algebra morphism \(H_0: \mathbb{C}[[G^*]] \to \mathbb{C}[[L^*]]\) induced by the restriction \((\ell^* \in \mathfrak{t}^*\) is a Lie subalgebra of \(\mathfrak{g}^*\), thus \(L^*\) is a formal subgroup of \(G^*\).

Let us now prove the Lemma. Let \(f_\pm \in \mathfrak{m}(U^*_\pm)_h\) and \(\tilde{f}_\pm \in \mathfrak{m}(U^*_\pm)_h\) be the image of \(f_\pm\) in \(\mathbb{C}[[U^*_\pm]]_h\). Then \(H(f_- f_+) = H(f_- f_-) + H(f_- f_+)\). We have \(H(f_- f_+) = O(h^2)\). On the other hand, \((\text{mod } h) \tilde{h}^{-1}([f_- f_+]) = \{\tilde{f}_-, \tilde{f}_+\}\). Therefore, \(H(f_- f_+) = hH_0(\{\tilde{f}_-, \tilde{f}_+\}) + O(h^2)\). This proves the first statement when \(n = 1\).

In the general case, the first statement of the Lemma follows from a computation based on the fact that \(\mathbb{C}[[G^*]]_h\) is quasimultiplicative. The same computation shows that if \(f_\pm \in \mathfrak{m}(U^*_\pm)_h\), \(i \in [1, r]\), then
\[
(f_-1 \cdots f_-n, f_{+1} \cdots f_{+n}) = \sum_{\sigma \in \mathfrak{S}_n} \prod_{i=1}^n (f_-i, f_{+\sigma(i)}) + O(h^{1-n}).
\]

It remains to compute the map \(\mathfrak{m}(U^*_+) \otimes \mathfrak{m}(U^*_-) \to \mathbb{C}[[L^*]]\), \(\tilde{f}_- \otimes \tilde{f}_+ \mapsto H_0(\{\tilde{f}_-, \tilde{f}_+\})\). This map clearly factors through \((\mathfrak{m}(U^*_+) / \mathfrak{m}(U^*_+)^2) \otimes (\mathfrak{m}(U^*_-) / \mathfrak{m}(U^*_-)^2) = u_- \otimes u_+\). Let \(u_\pm \in u_\pm\) be the images of \(\tilde{f}_\pm\). Let \(x \in \mathfrak{t}^*\), then
\[
\langle \{\tilde{f}_-, \tilde{f}_+\}, x^n \rangle = \langle \tilde{f}_- \otimes \tilde{f}_+, \delta(x^n) \rangle.
\]

Now \(\langle \tilde{f}_+, U(\mathfrak{g}^*)^0 \rangle = 0\), so the only contribution is that of \(\langle \tilde{f}_- \otimes \tilde{f}_+, \Delta_0(x)^{n-1}\delta(x) \rangle\), i.e., \((\tilde{f}_- \otimes \tilde{f}_+, \text{ad}(x \otimes 1 + 1 \otimes x))^{n-1}\langle \delta(x) \rangle\). Let \(x\) be formal near 0 in \(\mathfrak{t}^*\), then
\[
\{\tilde{f}_-, \tilde{f}_+\}(e^x) = \sum_{n \geq 0} \frac{1}{n!} (u_- \otimes u_+, \text{ad}(x \otimes 1 + 1 \otimes x))^{n-1}\langle \delta(x) \rangle
\]
\[
= \sum_{k, l \geq 0} \frac{1}{(k + l + 1)k!l!} \{[\text{ad}(x)^k(u_+)], \text{ad}(x)^l(u_-)\},
\]
\[
= \sum_{k, l \geq 0} \frac{(-1)^{l+1}}{(k + l + 1)k!l!} (u_+, \text{ad}(x)^k \circ p_{\mathfrak{g}} \circ \text{ad}(x) \circ p_{\mathfrak{g}} \circ \text{ad}(x)^l(u_-)),
\]
where \(p_{\mathfrak{g}}, p_{\mathfrak{g}^*}\) are the projections of \(\delta(\mathfrak{g}) = \mathfrak{g} \oplus \mathfrak{g}^*\) on \(\mathfrak{g}\) and \(\mathfrak{g}^*\).
On the other hand,
\[
\omega_{e^x}(u_+, u_-) = \sum_{k,l,\alpha \geq 0} \frac{(-1)^k + l}{k!l!} (p_\theta \circ \text{ad}(x)^k (u_+), p_{\theta^*} \circ \text{ad}(x)^l (u_-))
\]
\[
= \sum_{k,l,\alpha \geq 0} \frac{(-1)^l}{k!l!} \langle u_+, \text{ad}(x)^k \circ p_{\theta^*} \circ \text{ad}(x)^l (u_-) \rangle.
\]
We have \(\text{ad}(x)(g) \subset g \oplus g^*\), \(\text{ad}(x)(g^*) \subset g^*\), therefore
\[
p_{\theta^*} \circ \text{ad}(x)^l (u_-) = \sum_{\alpha = 0}^{l-1} \text{ad}(x)\alpha \circ p_{\theta^*} \circ \text{ad}(x) \circ p_\theta \circ \text{ad}(x)^{l-1-\alpha} (u_-).
\]
So
\[
\omega_{e^x}(u_+, u_-) = \sum_{k,l,\alpha \geq 0} \frac{(-1)^l}{k!l!} \langle u_+, \text{ad}(x)^{k+\alpha} \circ p_{\theta^*} \circ \text{ad}(x) \circ p_\theta \circ \text{ad}(x)^{l-1-\alpha} (u_-) \rangle
\]
\[
= \sum_{k,l,\alpha \geq 0} \left( \sum_{\alpha = 0}^{l-1} \frac{(-1)^l + \alpha + 1}{(l' + \alpha + 1)!} \right) \langle u_+, \text{ad}(x)^{k'} \circ \text{ad}(x) \circ p_\theta \circ \text{ad}(x)^{l'} (u_-) \rangle.
\]
Now
\[
\sum_{\alpha \geq 0} \frac{(-1)^l + \alpha + 1}{(k' - \alpha)!(l' + \alpha + 1)!} = \frac{(-1)^l + 1}{(k' + l' + 1)!} \left( \binom{k' + l' + 1}{l' + 1} - \binom{k' + l' + 1}{l' + 2} + \cdots + (-1)^k \binom{k' + l' + 1}{k' + l' + 1} \right)
\]
\[
= \frac{(-1)^{k+l+1}}{(k' + l' + 1)!} \left( \binom{k' + l' + 1}{1} - \binom{k' + l' + 1}{2} + \cdots + (-1)^k \binom{k' + l' + 1}{k'} \right)
\]
\[
= \frac{(-1)^{k+l+1}}{(k' + l' + 1)!} (-1)^k \binom{k' + l'}{k'} = \frac{(-1)^{l+1}}{(k' + l' + 1)!} k^l !.
\]
It follows that \(\{ f_-, f_+ \} (e^x) = \omega_{e^x}(u_+, u_-)\).

This ends the proof of the Lemma.

\[\square\]

**Lemma 3.14.** The bilinear map \((-,-)\) of Lemma 3.13 extends to a bilinear map \((-,-) : \mathbb{C}[[U^\ast]]_h \times U_h(u_+) \to \mathbb{C}[[L^\ast]]_h\), whose reduction modulo \(h\) is a bilinear map \((-,-)_0 : \mathbb{C}[[U^\ast]] \times U(u_+) \to \mathbb{C}[[L^\ast]]\), uniquely defined by the conditions that \((f g, x)_0 = \sum (f, x)^{(1)}(0)(g, x)^{(2)}(0)\) for any \(x \in U(u_+)\) and \(f, g \in \mathbb{C}[[U^\ast]]\), and \((f, u_+)_0\) is the formal function \(\ell^* \to \omega_{e^x}(u_+, f)\) if \(f \in \mathfrak{m}(U^\ast_+)\) and \(f\) is its image in \(u_+\).

In particular, we have \((f, x)_0 = 0\) if \(\text{val}(x) > \text{deg}(f)\). When \(\text{val}(x) = \text{deg}(f) = n\), the pairing \((-,-)_0\) factors through a bilinear map \(S^n(u_-) \times S^n(u_+) \to \mathbb{C}[[L^\ast]]\), which coincides with the map described in Lemma 3.13.

**Proof.** The first statement follows from the fact that \(\mathbb{C}[[U^\ast]]_h\) is the \(h\)-adic completion of \(\sum_{n \geq 0} h^{-n} \mathfrak{m}(U^\ast_+)\). The rest follows from the formula
\[
(f, u_1 \cdots u_n)_0 = \{ f, f_{u_1}, \ldots, f_{u_n} \}_{L^\ast},
\]
where \(f_u \in \mathfrak{m}(U^\ast_+)\) is a lift of \(u \in u_+ = \mathfrak{m}(U^\ast_+)/\mathfrak{m}(U^\ast_+)^2\) and the results of Lemma 3.13. \[\square\]

**End of proof of Theorem 3.12.** The properties of \((-,-)_0\), together with the nondegeneracy of \(\omega_{e^x}(u_+, u_-)\), imply that there exists a unique element \(\Upsilon_0 \in U(u_+) \otimes \mathbb{C}[[L^\ast]]_p^p \otimes \mathbb{C}[[U^\ast]]\), which we write \(\Upsilon_0 = \sum A_{i,0} \otimes L_{i,0} \otimes B_{i,0}\), such that for any \(f_0 \in \mathbb{C}[[U^\ast]]\), \(x_0 \in U(u_+)\), we have
\[
\sum_i (f_0, A_{i,0}) L_{i,0} (B_{i,0}, x_0)_0 = (f_0, x_0)_0.
\]
The dynamical twist \( \Upsilon_0 \) satisfies the relation \( \Upsilon_0^{1,2,3,4} = \Upsilon_0^{1,3,4} \Upsilon_0^{2,3,4} \), which implies that it has the form \( \Upsilon_0 = \exp(v_0) \), with \( v_0 \in u_+ \otimes \mathbb{C}[L^*]_{P^\#} \otimes \mathbb{C}[L^*]_{\ell} \); moreover, the properties of \((-,-)_0\) imply that the projection of \( v_0 \) in \( u_+ \otimes \mathbb{C}[L^*]_{P^\#} \otimes u_- \) is \( (\overline{\sigma_\phi})^{1,3,2,4} \).

Let \( \Upsilon_1 \) be a lift of \( \Upsilon_0 \) to \( U_h(u_+) \otimes \mathbb{C}[L^*]_{P^\#} \otimes \mathbb{C}[L^*]_{\ell} \), which we write \( \Upsilon_1 = \sum_i A_{i,1} \otimes L_{i,1} \otimes B_{i,1} \). Define \( \varphi : \mathbb{C}[L^*] \to \mathbb{C}[L^*]_{P^\#} \otimes \mathbb{C}[L^*]_{\ell} \) by

\[
\varphi(f) := \sum_i (f, A_{i,1})L_{i,1} \otimes B_{i,1}.
\]

Then the reduction modulo \( \ell \) of \( \varphi \) is \( \mathbb{C}[L^*] \to \mathbb{C}[L^*]_{P^\#} \otimes \mathbb{C}[L^*]_{\ell} \), \( f_0 \mapsto f_0 \otimes 1 \).

\( \varphi \) gives rise to an endomorphism \( \overline{\varphi} \) of \( \mathbb{C}[L^*]_{P^\#} \otimes \mathbb{C}[L^*]_{\ell} \), defined uniquely by \( \overline{\varphi}(\ell \otimes f) = (\ell \otimes 1)\varphi(f) \) for \( \ell \in \mathbb{C}[L^*]_{P^\#} \), \( f \in \mathbb{C}[L^*]_{\ell} \). The reduction modulo \( \ell \) of \( \overline{\varphi} \) is the identity, so \( \varphi \) is invertible. Let \( \psi \) be its inverse. Define \( \Upsilon \) by

\[
\Upsilon^{1,3,2} := \sum_i (A_{i,1} \otimes L_{i,1} \otimes 1)(1 \otimes (1 \otimes B_{i,1})).
\]

We write \( \Upsilon = \sum_i A_i \otimes L_i \otimes B_i \). Then if \( f \in \mathbb{C}[L^*]_{\ell} \), we have

\[
\sum_i (f, A_{i,1})L_{i,1} \otimes B_i = \sum_i ((f, A_{i,1})L_{i,1} \otimes 1)\psi(1 \otimes B_{i,1}) = \sum_i (\varphi(f) \otimes 1)\psi(1 \otimes \varphi(f)) = \psi \circ \varphi(f) = \psi \circ \overline{\varphi}(1 \otimes f) = 1 \otimes f.
\]

Here we write \( \varphi(f) = \sum \varphi(f^{(1)}) \otimes \varphi(f^{(2)}) \); the third equality follows from the fact that \( \psi \) satisfies \( \psi(\ell \otimes f) = (\ell \otimes 1)\psi(1 \otimes f) \). The other properties of \( \Upsilon \) follow from the fact that \( (\text{mod } \ell)(\psi) \) is the identity.

Finally, the fact that \( \Upsilon^{2,1,3,4} \) is in \( U_h(u_+) \otimes \mathbb{C}[L^*]_{P^\#} \otimes \mathbb{C}[L^*]_{\ell} \) is proven by exchanging the indices +, −.

\[\square\]

### 3.6. The dynamical twist \( \Psi_1^\# \)

Set now

\[
\Psi_1^\# = \left( \sum_i A_i \otimes B_i S(L_i^{(2)}) \otimes S(L_i^{(1)}) \right)^{-1}.
\]

Then \( \Psi_1^\# \) lies in \( U_h(u_+) \otimes \mathbb{C}[L^*]_{P^\#} \otimes \mathbb{C}[L^*]_{\ell} \), therefore in \( (U_h(u_+) \otimes U_h(p_-)) \otimes \mathbb{C}[L^*]_{P^\#} \).

**Theorem 3.15.** \( \Psi_1^\# \) satisfies the dynamical twist equation

\[
(\Psi_1^\#)^{1,2,3,4} (\Psi_1^\#)^{1,3,4} = (\Psi_1^\#)^{2,3,4} (\Psi_1^\#)^{1,2,3,4}.
\]

Moreover \( \Psi_1^\# = 1 + O(\ell) \), and \( (\text{mod } \ell)(\Psi_1^\# - 1) = \overline{\sigma_\phi} \). So \( \Psi_1^\# \) is a quantization of \( \sigma_1^\# \).

**Proof.** Similar to the proof of Proposition 2.5 in [EE2]. \[\square\]

**Remark 3.16.** When the cobracket of \( \ell \) is zero, \( \ell^* = L^* \) and \( P_1^\# \) defined by (10) is a homogeneous element of \( \mathbb{C}[L^*] = \tilde{S}(1) \). So Lemma 3.10 implies that the construction of Theorems 3.12, 3.15 are generalizations of [EE2], Propositions 0.1 and 1.1. Theorem 3.15 may also be viewed as a generalization of the construction of [EV2] to the case of a nonabelian base.
3.7. Localized Harish-Chandra maps. Let $\mathcal{O}(X)_{\hbar}$ be QFS algebra, i.e., a flat deformation of a formal series algebra $\mathcal{O}(X)$. To each $\tilde{D} \in \mathcal{O}(X)$, we associate the completed localization $\mathcal{O}(X)_{D,\hbar}$.

Proposition 3.17. Let $\tilde{D} \in \mathcal{O}(X)_{\hbar}$ be a lift of $D$. For each $\alpha > 0$, the subspace $\mathcal{O}_\alpha(\tilde{D}) := \{ \sum_{i \geq 0} \hbar^i x_i \in \tilde{D}^{-i} \mathcal{O}(X)_{\hbar} \} \subset \mathcal{O}(X)_{D,\hbar}$ is independent of a choice of $\tilde{D}$. We denote it $\mathcal{O}_\alpha$.

Set $\mathcal{O}_0 := \mathcal{O}(X)_{\hbar}$. Then we have $\mathcal{O}_0 \subset \mathcal{O}_1 \subset \mathcal{O}_2 \subset \ldots$, and $\mathcal{O}_\alpha \mathcal{O}_\beta \subset \mathcal{O}_{\alpha+\beta}$, for any $\alpha, \beta \geq 0$.

Proof. For $a, b \in \mathcal{O}(X)_{\hbar}$, set $\text{ad}_\hbar(a)(b) = \hbar^{-1}[a, b]$. Then $\text{ad}_\hbar(a)(b) \in \mathcal{O}(X)_{\hbar}$. Then we have:

$$\forall a \in \mathcal{O}(X)_{\hbar}, \ a \tilde{D}^{-1} = \tilde{D}^{-1} a - \hbar \tilde{D}^{-2} \text{ad}_\hbar(\tilde{D})(a) + \hbar^2 \tilde{D}^{-3} \text{ad}_\hbar(\tilde{D})^2(a) - \cdot \cdot \cdot.$$ (30)

Using this identity, one shows that $\mathcal{O}_\alpha(\tilde{D}) \mathcal{O}_\beta(\tilde{D}) \subset \mathcal{O}_{\alpha+\beta}(\tilde{D})$ for any $\alpha, \beta$.

Let us now prove the independence statement. Let $\tilde{D}$ be another lift of $D$. Then $\tilde{D} = \tilde{D} + ha$, where $a \in \mathcal{O}(X)_{\hbar}$. Then:

$$\tilde{D}^{-1} = \tilde{D}^{-1} - \hbar \tilde{D}^{-2} a \tilde{D}^{-1} + \hbar^2 \tilde{D}^{-3} a \tilde{D}^{-1} a \tilde{D}^{-1} - \cdot \cdot \cdot.$$ Using (30), we see that $\tilde{D}^{-1} \in \mathcal{O}_1(\tilde{D})$. In the same way, one shows that if $x \in \mathcal{O}(X)_{\hbar}$, then $\hbar^i \tilde{D}^{-i} x \in \mathcal{O}_i(\tilde{D})$, therefore $\mathcal{O}_1(\tilde{D}) \subset \mathcal{O}_1(D)$. So $\mathcal{O}_1(\tilde{D}) = \mathcal{O}_1(D)$. In the same way, one shows that $\mathcal{O}_\alpha(\tilde{D}) = \mathcal{O}_\alpha(D)$.

□

Let $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}$ be a Lie bialgebra with a splitting. Then we have a Hopf algebra inclusion $\mathbb{C}[[L^*]] \subset \mathbb{C}[[P^*]]$ and an algebra inclusion $\mathbb{C}[[U^*]] \subset \mathbb{C}[[P^*]]$. Let $\Pi \in \mathbb{C}[[L^*]]$ be a nonzero element.

Let $U_h(\mathfrak{l}) \subset U_h(\mathfrak{p}) \subset U_h(\mathfrak{l})$ be a quantization of $\mathfrak{l} \rightarrow \mathfrak{p} \rightarrow \mathfrak{l}$. It gives rise to a QFHS algebra inclusion $\mathbb{C}[[L^*]]_h \subset \mathbb{C}[[P^*]]_h$ and a left coideal inclusion $\mathbb{C}[[U^*]]_h \subset \mathbb{C}[[P^*]]_h$. The tensor product of inclusions followed by the product yields isomorphisms of topological vector spaces

$$\mathbb{C}[[L^*]]_h \hat{\otimes} \mathbb{C}[[U^*]]_h \rightarrow \mathbb{C}[[P^*]]_h, \ \mathbb{C}[[U^*]]_h \hat{\otimes} \mathbb{C}[[L^*]]_h \rightarrow \mathbb{C}[[P^*]]_h.$$

Proposition 3.18. The tensor product of inclusions followed by the product also gives rise to isomorphisms

$$\alpha : \mathbb{C}[[L^*]]_{\Pi,\hbar} \hat{\otimes} \mathbb{C}[[U^*]]_h \rightarrow \mathbb{C}[[P^*]]_{\Pi,\hbar}, \ \beta : \mathbb{C}[[U^*]]_h \hat{\otimes} \mathbb{C}[[L^*]]_{\Pi,\hbar} \rightarrow \mathbb{C}[[P^*]]_{\Pi,\hbar}.$$ (31)

Proof. The inclusion $\mathbb{C}[[L^*]]_h \rightarrow \mathbb{C}[[P^*]]_{\Pi,\hbar}$ obviously extends to an inclusion $\mathbb{C}[[L^*]]_{\Pi,\hbar} \rightarrow \mathbb{C}[[P^*]]_{\Pi,\hbar}$. Then the reductions modulo $\hbar$ of the maps (31) coincide with the product map $\mathbb{C}[[L^*]]_{\Pi,\hbar} \otimes \mathbb{C}[[U^*]]_h \rightarrow \mathbb{C}[[P^*]]_{\Pi,\hbar}$, and with its composition with the permutation, which are isomorphisms. Therefore the maps (31) are isomorphisms.

□

Let now $\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{u}_+ \oplus \mathfrak{u}_-$ be a polarized Lie bialgebra with a splitting. Let $\Pi \in \mathbb{C}[[L^*]]$ be a nonzero element and let $U_h(\mathfrak{l}) \subset U_h(\mathfrak{p}_+) \subset U_h(\mathfrak{g})$. $U_h(\mathfrak{p}_+) \rightarrow U_h(\mathfrak{l})$ be quantizations of the natural Lie bialgebra morphisms. Define a product on $\mathbb{C}[[L^*]]_{\Pi,\hbar} \hat{\otimes} (\mathbb{C}[[U^*_+]]_h \hat{\otimes} \mathbb{C}[[U^*_-]]_h)$ by

$$\mu = (132) \circ (m_{\mathbb{C}[[U^*_+]]_h} \otimes m_{\mathbb{C}[[L^*]]_{\Pi,\hbar}} \otimes m_{\mathbb{C}[[U^*_-]]_h}) \circ (\text{id} \otimes e_+ \otimes \text{id} \otimes e_- \otimes \text{id}) \circ ((132) \otimes (132)),$$

where $m_A$ is the product map of an algebra $A$, $m_{(3)} = (m_A \otimes \text{id}) \circ m_A$, $e_{\pm} : \mathbb{C}[[L^*]]_{\Pi,\hbar} \hat{\otimes} \mathbb{C}[[U^*_\pm]]_h \rightarrow \mathbb{C}[[U^*_\pm]]_h \hat{\otimes} \mathbb{C}[[L^*]]_{\Pi,\hbar}$ is the composition $\beta_{\pm}^{-1} \circ \alpha_{\pm}$ (see (31)),

$$\pi : \mathbb{C}[[U^*_-]]_h \hat{\otimes} \mathbb{C}[[U^*_+]]_h \rightarrow \mathbb{C}[[U^*_+]]_h \hat{\otimes} \mathbb{C}[[L^*]]_h \hat{\otimes} \mathbb{C}[[U^*_-]]_h$$

is the composition of the product $\mathbb{C}[[U^*_-]]_h \hat{\otimes} \mathbb{C}[[U^*_+]]_h \rightarrow \mathbb{C}[[G^*]]_h$ with the inverse of PBW (see Section 3.3).
Proposition 3.19. \( \mu \) is an associative product, extending the transport of the product on \( \mathbb{C}[[G^*]]h \) by PBW. Let \( \Pi' \in \mathbb{C}[[G^*]] \) be a nonzero element, such that \( \Pi'|_{L^*} = \Pi \). Then there is a unique algebra morphism \( \text{PBW}^{-1}: \mathbb{C}[[G^*]][\Pi', h] \to (\mathbb{C}[[L^*]]_{\Pi, h} \otimes \mathbb{C}[[U^*_+]] \otimes \mathbb{C}[[U^*_-]]h, \mu) \), extending \( \text{PBW}^{-1} \).

Proof. The maps \( \alpha_\pm, \beta_\pm \) satisfy identities implying the associativity of \( \mu \). The reduction of \( \mu \) modulo \( h \) is the standard product on \( \mathbb{C}[[L^*]]_{\Pi, h} \otimes \mathbb{C}[[U^*_+]] \otimes \mathbb{C}[[U^*_-]]h \). So it remains to prove that the image of \( \Pi' \in \mathbb{C}[[G^*]] \) in this algebra is invertible. This image is \( \Pi \otimes 1 \otimes 1 + \sum_i a_i \otimes b_i \otimes c_i \), where \( b_i \otimes c_i \) belongs to the maximal ideal \( \mathfrak{m}(U^*_+ \times U^*_-) \) of \( \mathbb{C}[[U^*_+]] \otimes \mathbb{C}[[U^*_-]]h \). Then the inverse of this inverse is \( \sum_{\alpha > 0} (-1)^\alpha (\Pi \otimes 1 \otimes 1)\alpha + (\sum_i a_i \otimes b_i \otimes c_i)\alpha \); the term corresponding to the index \( \alpha \) belongs to \( \mathfrak{m}(U^*_+ \otimes U^*_-)\alpha \), so the series converges. \( \square \)

Remark 3.20. Set \( H := (\varepsilon \otimes \text{id} \otimes \varepsilon) \circ \text{PBW}^{-1} \), then \( H \) is a flat deformation of the restriction map \( \mathbb{C}[[G^*]] \to \mathbb{C}[[L^*]] \). \( H \) is also an analogue of the Harish-Chandra map. Let us define \( \tilde{H} \) as the composed map \( \mathbb{C}[[G^*]]_{\Pi', h} \to \mathbb{C}[[L^*]]_{\Pi, h} \otimes \mathbb{C}[[U^*_+]]h \otimes \mathbb{C}[[U^*_-]]h \overset{\text{id} \otimes \varepsilon}{\to} \mathbb{C}[[L^*]]_{\Pi, h} \), then \( \tilde{H} \) is an extension of \( H \). We will denote \( H, \tilde{H} \) by \( H^\theta, \tilde{H}^\theta \) to avoid confusions.

3.8. The quantum composition formula. We now prove a Poisson-Lie analogue of Proposition 2.15 in [EE2]. Namely, we prove a composition formula for the twists \( \Psi^\theta \). It may be viewed as a quantization of Remark 2.8. We will use it to determine \( \Psi^\theta \) using \( \Psi^\theta_h \) and \( \Psi^\theta_{l'} \) in the situation of Example 2.1.

Assume that \( \mathfrak{g} = l \oplus \mathfrak{u}_+ \oplus \mathfrak{u}_- \) and \( l = t \oplus \mathfrak{m}_+ \oplus \mathfrak{m}_- \) are nondegenerate polarized Lie bialgebras. Set \( v_+ = u_+ \oplus \mathfrak{m}_+ \). Then \( \mathfrak{g} = t \oplus v_+ \oplus \mathfrak{v}_- \) is a nondegenerate polarized Lie bialgebra (the nondegeneracy follows from [EE2], Lemma 2.13). We fix compatible quantizations of these polarized Lie bialgebras.

We denote by \( P_1^\theta \in \mathbb{C}[[L^*]] \) and \( P_1^t, P_0^\theta \in \mathbb{C}[[K^*]] \) the elements associated to the decompositions \( \mathfrak{g} = l \oplus \mathfrak{u}_+ \oplus \mathfrak{u}_- \), \( l = t \oplus \mathfrak{m}_+ \oplus \mathfrak{m}_- \), \( g = t \oplus v_+ \oplus \mathfrak{v}_- \) according to (10). Then \( P^\theta = (P^\theta)_1^t P^\theta_1 \).

It follows that we have natural injections
\[
\mathbb{C}[[K^*]]_{(P^\theta_1)_{K, h}} \hookrightarrow \mathbb{C}[[K^*]]_{P^\theta_1, h} \quad \text{and} \quad \mathbb{C}[[K^*]]_{P^\theta_1, h} \hookrightarrow \mathbb{C}[[K^*]]_{P^\theta_1, h}
\]
Define the linear map
\[
\eta: (U_h(u_+ \otimes U_h(p_-))) \otimes \mathbb{C}[[L^*]]_{P^\theta_1, h} \to (U_h(v_+) \otimes U_h(p_-))) \otimes \mathbb{C}[[K^*]]_{P^\theta_1, h}
\]
by \( \eta(\alpha \otimes \beta \otimes \lambda) = \sum_i \alpha S(\lambda_{+i}^{(2)}) \otimes \beta S(\lambda_{-i}^{(1)}) \otimes \varepsilon(\lambda_{-i}) \lambda_{0i}, \) where \( \text{PBW}^{-1}(\lambda) = \sum_i \lambda_{+i} \otimes \lambda_{0i} \otimes \lambda_{-i}, \) with \( \lambda_{+i}, \lambda_{0i}, \lambda_{-i} \in \mathbb{C}[[M^*_+]]h \) and \( \lambda_{0i} \in \mathbb{C}[[K^*]]_{(P^\theta_1)_{K, h}} \hookrightarrow \mathbb{C}[[K^*]]_{P^\theta_1, h} \), so that \( \lambda = \sum_i \lambda_{+i} \lambda_{0i} \lambda_{-i}. \)

Proposition 3.21. We have
\[
(\Psi^\theta_{l'})^{-1} = \eta((\Psi^\theta_{l'})^{-1})(\Psi^\theta_t)^{-1}
\]
(equality in \( (U_h(v_+) \otimes U_h(p_-)) \otimes \mathbb{C}[[K^*]]_{P^\theta_1, h}. \))

Proof. Similar to the proof of [EE2], Proposition 2.15. \( \square \)

Remark 3.22. As in [EE2], Remark 2.16, this formula enables one to recover \( \Psi^\theta \) from \( \Psi^\theta_t, \Psi^\theta_{l'} \). Indeed, let
\[
\eta': (U_h(v_+) \otimes U_h(p_-)) \otimes \mathbb{C}[[K^*]]_{P^\theta_1, h} \to (U_h(u_+) \otimes U_h(p_-)) \otimes \mathbb{C}[[K^*]]_{P^\theta_1, h}[h^{-1}]
\]
be the linear map taking \( u_+ \lambda_+ \otimes p_- k \) to \( u_+ \otimes p_- S(\lambda_+) \otimes k \), where \( u_+ \in U_h(u_+) \), \( \lambda_+ \in U_h(m_+) \), \( p_- \in U_h(p_-) \), \( k \in \mathbb{C}[[K^*]]_{P^\theta_1, h} \). Then \( \eta' \circ \eta = \text{id} \otimes \text{id} \otimes H_1^\theta \). Now \( \Psi^\theta \) may be recovered uniquely from its image \( \text{id} \otimes \text{id} \otimes H_1^\theta \) using its \( L \)-invariance, as in [EE2].
4. Poisson homogeneous structures on \( G/L \)

We first complement Section 1 by studying reductions of the (quasi)Poisson structures associated with dynamical \( r \)-matrices.

We then restrict ourselves to the case when \( G \) is simple and \( L \subset G \) is a Levi subgroup. The Poisson homogeneous structures on \( G/L \) were classified in [DGS]. The set of all these Poisson structures is an algebraic variety \( P \), which we study in Section 4.2. We introduce a Zariski open subset \( P_0 \subset P \) and show that the Poisson structures corresponding to elements of \( P_0 \) can be expressed using PL dynamical \( r \)-matrices, and can therefore be viewed as examples of the construction of Section 4.1 (Section 4.3).

4.1. Reductions of Poisson structures. Let \( I \subset g \) be an inclusion of finite dimensional Lie bialgebras. Let \( L^*_\text{alg} \) be a Poisson-Lie algebraic group with Lie bialgebra \( \ast \), let \( \mathbb{C}[L^*] \) be its Hopf algebra of regular functions. Let \( \sigma = \sum_\alpha a_\alpha \otimes b_\alpha \otimes \ell_\alpha \in \wedge^2 (g) \otimes \mathbb{C}[L^*][1/P] \) be a PL dynamical \( r \)-matrix for \( (L^*, g, 0) \). Then the quasi-Poisson structure of Section 1.4 is actually Poisson.

Define \( G/L \) as the quotient of the formal group \( G \) by \( L \); its function algebra is \( \mathbb{C}[[G/L]] = \{ f \in \mathbb{C}[[G]][\forall a \in I, R_0(f) = 0] \} \).

The Poisson structure of Section 1.4 may be reduced to define a Poisson homogeneous structure on \( G/L \), as follows. Assume that \( \chi_{L^*} \in L^*_\text{alg} \) is a point where the Poisson bivector vanishes and where \( \sigma \) is defined (\( \chi_{L^*} \) may be viewed as a character of \( \mathbb{C}[L^*] \), such that \( \chi_{L^*}(\{f, g\}) = 0 \) for any \( f, g \in \mathbb{C}[L^*] \)). Set \( \sigma(\chi_{L^*}) := \sum_\alpha \ell_\alpha (\chi_{L^*}) a_\alpha \otimes b_\alpha \). Define a bilinear map on \( \mathbb{C}[[G]] \) by

\[
\{g_1, g_2\}_{\chi_{L^*}} := \{g_1, g_2\}_G - m(R^\otimes 2(\sigma(\chi_{L^*}))(g_1 \otimes g_2)).
\]

This map restricts to a bilinear map \( \mathbb{C}[[G/L]]^2 \to \mathbb{C}[[G/L]] \), because for any \( a \in I \), the vector field \( a \) vanishes at \( \chi \). One also checks that this is a Poisson structure on \( G/L \), which is Poisson homogeneous under the right action of the Poisson-Lie group \( G \).

The reduction outlined here is a generalization of [Lu].

4.2. The Poisson homogeneous structures on \( G/L \).

**Proposition 4.1.** Let \( (g, r_0) \) be a coboundary Lie bialgebra, i.e., \( r_0 \in \wedge^2 (g) \) is such that \( Z(r_0) = \text{CYB}(r_0) \in \wedge^3 (g) \). Let \( I \subset g \) be a Lie subbialgebra and assume that \( g = I \oplus u \), where \( [I, u] \subset u \). Let \( P \) be the set of all elements \( \rho \in \wedge^2 (u) \), such that the equality \( \text{CYB}(\rho) = Z(r_0) \) holds in \( \wedge^3 (g/u) \). The Poisson homogeneous structures on \( G/L \) correspond bijectively to the elements of \( P \).

**Proof.** The proof is straightforward. The Poisson structure corresponding to \( \rho \) is \( \{f, g\}_\rho = -m((L^\otimes 2(r_0) + R^\otimes 2(\rho))(f \otimes g)) \). \( \square \)

**Remark 4.2.**

1) If \( r(\lambda) \) is a dynamical \( r \)-matrix for \( (I, g, Z(r_0)) \), and \( \chi \in I^* \) is a character such that \( r(\chi) \) is defined and belongs to \( \wedge^2 (u) \), then \( r(\chi) \) belongs to \( P \).

2) The cobracket of \( g \) is \( \delta(a) = [a^1 + a^2, r_0] \). Assume that \( r_0 = r_{0,1} + r_{0,u} \), where \( r_{0,1} \in \wedge^2 (l) \), \( r_{0,u} \in \wedge^2 (u) \). If \( L^*_{\text{alg}} \) is an algebraic PL group with Lie bialgebra \( \ast \); if \( \sigma : L^*_{\text{alg}} \to \wedge^2 (g) \) is a PL dynamical \( r \)-matrix for \( (L^*_{\text{alg}}, g, 0) \); if \( \chi_{L^*} \in L^*_{\text{alg}} \) is a point where the Poisson structure vanishes, such that \( \sigma(\chi_{L^*}) \) is defined and belongs to \( \wedge^2 (u) \), then \( \sigma(\chi_{L^*}) - r_{0,u} \) also belongs to \( P \). \( \square \)

When \( g \) is a simple Lie algebra, \( I \subset g \) is a Levi subalgebra and \( r_0 \) is the standard \( r \)-matrix of \( g \), the authors of [DGS] described the set of all Poisson homogeneous structures on \( G/L \) explicitly as follows.
Let $\mathfrak{h} \subset \mathfrak{l} \subset \mathfrak{g}$ be a Cartan subalgebra. Let $I$ be the index set of $\mathfrak{l}$ and let $\mathfrak{z}$ be the center of $\mathfrak{l}$. Then $\mathfrak{z} \subset \mathfrak{h}$.

Denote by $\Delta(\mathfrak{l}) \subset \Delta(\mathfrak{g}) \subset \mathfrak{h}^*$ the sets of roots of $\mathfrak{l}$ and $\mathfrak{g}$. Define $\Delta(\mathfrak{g},\mathfrak{l}) \subset \mathfrak{z}^*$ as the set of all elements of $\mathfrak{z}^*$ which are restrictions to $\mathfrak{z}$ of elements of $\Delta(\mathfrak{g})$, and are nonzero. The elements of $\Delta(\mathfrak{g},\mathfrak{l})$ are called quasi-roots.

**Lemma 4.3.** For any $\alpha \in \Delta(\mathfrak{g})$, the restriction of $\alpha$ to $\mathfrak{z}$ is nonzero iff $\alpha \notin \Delta(\mathfrak{l})$.

**Proof.** Let us fix a system $\alpha_s, s \in [1, r]$ of simple roots. Recall that $I \subset [1, r]$ is the set of indices of $\mathfrak{l}$. Then $\mathfrak{h}^* = \bigoplus_{i=1}^r \mathbb{C} \alpha_i$ decomposes as $\bigoplus_{i \in I} \mathbb{C} \alpha_i \bigoplus \bigoplus_{i \notin I} \mathbb{C} \alpha_i$. The annihilator of the first space is $\mathfrak{z}$, so the second space identifies with $\mathfrak{z}^*$. Let $\alpha \in \Delta(\mathfrak{g})$ be such that $\alpha_{|\mathfrak{l}} = 0$. Let us show that $\alpha \in \Delta(\mathfrak{l})$. Set $\alpha = \sum_{s \in [1, r]} n_s \alpha_s$. If $s \in S$, then $(\alpha_s)_{|\mathfrak{z}} = 0$. Therefore $\alpha_{|\mathfrak{z}} = \sum_{s \notin I} n_s (\alpha_s)_{|\mathfrak{z}}$. Since $(\alpha_s)_{|\mathfrak{z}}$, $s \notin I$, is a basis of $\mathfrak{z}^*$, we get $n_s = 0$ for any $s \notin I$, so $\alpha \in \Delta(\mathfrak{l})$.

It follows that $\Delta(\mathfrak{g},\mathfrak{l})$ is the image of $\Delta(\mathfrak{g}) \setminus \Delta(\mathfrak{l})$ by the restriction map, which is therefore a surjection $\Delta(\mathfrak{g}) \setminus \Delta(\mathfrak{l}) \twoheadrightarrow \Delta(\mathfrak{g},\mathfrak{l})$.

We fix a system $\alpha_s, s \in [1, r]$ of simple roots and denote by $\Delta_+(\mathfrak{g})$, $\Delta_+(\mathfrak{l})$ the corresponding systems of positive roots of $\mathfrak{g}, \mathfrak{l}$. We define $\Delta_+(\mathfrak{g},\mathfrak{l})$ as the image of $\Delta_+(\mathfrak{g}) \setminus \Delta_+(\mathfrak{l})$ by the restriction map.

**Lemma 4.4.** $\Delta(\mathfrak{g},\mathfrak{l})$ is the disjoint union $\Delta_+(\mathfrak{g},\mathfrak{l}) \coprod (-\Delta_+(\mathfrak{g},\mathfrak{l}))$.

**Proof.** It suffices to show that the union is disjoint. Assume that $\alpha, \beta \in \Delta(\mathfrak{g},\mathfrak{l})$ are such that $\alpha + \beta = 0$. Let $\alpha, \beta \in \Delta_+(\mathfrak{g}) \setminus \Delta_+(\mathfrak{l})$ be preimages of $\alpha, \beta$. Set $\alpha = \sum_{s=1}^r a_s \alpha_s$, $\beta = \sum_{s=1}^r b_s \alpha_s$, then $(\alpha + \beta)_{|\mathfrak{z}} = 0$, which means that $a_s + b_s = 0$ for any $s \notin I$. Since the $a_s, b_s, s \in I$ are all $\geq 0$, we get $a_s = b_s = 0$ for any $s \notin I$. So $\alpha, \beta \in \Delta_+(\mathfrak{l})$, a contradiction.

If $\alpha \in \Delta(\mathfrak{g}) \setminus \Delta(\mathfrak{l})$, we denote by $\overline{\alpha} \in \Delta(\mathfrak{g},\mathfrak{l})$ its image by the restriction map.

For $\alpha \in \Delta_+(\mathfrak{g})$, we fix elements $e_\alpha \in \mathfrak{g}_\alpha$, $f_\alpha \in \mathfrak{g}_{-\alpha}$ such that $\{e_\alpha, f_\alpha\} = 1$ ($\langle - , - \rangle$ is the Killing form).

**Proposition 4.5.** (see [DGS]) $\mathcal{P}$ consists of all the elements of the form

$$\rho = \sum_{\theta \in \Delta_+(\mathfrak{g},\mathfrak{l})} c(\theta) \sum_{\alpha \in \Delta_+(\mathfrak{g}) \setminus \Delta_+(\mathfrak{l})} e_\alpha \vee f_\alpha,$$

where $c : \Delta_+(\mathfrak{g},\mathfrak{l}) \rightarrow \mathbb{C}$ such that for any $\theta, \theta' \in \Delta_+(\mathfrak{g},\mathfrak{l})$ such that $\theta + \theta' \in \Delta_+(\mathfrak{g},\mathfrak{l})$, $c(\theta + \theta') = c(\theta) c(\theta') + \frac{1}{4}$.

$\mathcal{P}$ therefore identifies with an algebraic subvariety of $\mathbb{C}^{\Delta_+(\mathfrak{g},\mathfrak{l})}$.

Define $\mathcal{P}_0 \subset \mathcal{P}$ as the Zariski open subset defined by the condition that for any $\theta \in \Delta_+(\mathfrak{g},\mathfrak{l})$, we have $c(\theta) \neq \pm 1/2$.

**Lemma 4.6.** Any element of $\Delta_+(\mathfrak{g},\mathfrak{l}) \subset \mathfrak{z}^*$ is a linear combination of the $\alpha_s, s \notin I$; the corresponding coefficients are nonnegative integers, which do not vanish simultaneously.

**Proof.** If $\theta \in \Delta_+(\mathfrak{g},\mathfrak{l})$ and $\alpha \in \Delta_+(\mathfrak{g}) \setminus \Delta_+(\mathfrak{l})$ is such that $\alpha = \theta$, then $\alpha = \sum_{s \in [1, r]} n_s \alpha_s$, where $n_s \geq 0$ and the $n_s, s \notin I$ are not all zero. Therefore $\theta = \sum_{s \notin I} n_s \alpha_s$, where the $n_s$ are $\geq 0$, integers, and not all zero.

Let $\tilde{I} := [1, r] \setminus I$. If $\theta \in \Delta_+(\mathfrak{g},\mathfrak{l})$, we denote by $(n_s(\theta))_{s \in \tilde{I}}$ the integers such that $\theta = \sum_{s \notin I} n_s(\theta) \alpha_s$. We denote by $\chi_\theta : (\mathbb{C}^\times)^I \rightarrow \mathbb{C}^\times$ the group morphism taking $(t_s)_{s \in \tilde{I}}$ to $\prod_{s \in \tilde{I}} t_s^{n_s(\theta)}$. If $t \in \mathbb{C}^\times \setminus \{1\}$, we set $\kappa(t) := \frac{1 + t}{1 + t^4}$. 


Proposition 4.7. There is a unique isomorphism of algebraic varieties
\[ \iota : (\mathbb{C}^x)^f \setminus \bigcup_{\theta \in \Delta_+(g; l)} \ker(\chi_\theta) \to \mathcal{P}_0, \]
 Taking \((t_s)_{s \in I}\) to \((c(\theta))_{\theta \in \Delta_+(g; l)})\), where \(c(\theta) = \kappa \circ \chi_\theta((t_s)_{s \in I})\).

Proof. Let us first show that \(\iota\) is well defined. If \(t = (t_s)_{s \in I}\) belongs to \((\mathbb{C}^x)^f \setminus \bigcup_{\theta \in \Delta_+(g; l)} \ker(\chi_\theta)\), then \(\chi_\theta(t) \neq 0\) so \(\kappa \circ \chi_\theta(t) \in \mathbb{C}\) is well defined. Moreover, the family \((\kappa \circ \chi_\theta(t))_{\theta \in \Delta_+(g; l)}\) satisfies the defining relations of \(\mathcal{P}\), because of the identity \(\kappa(x + x')(\kappa(x) + \kappa(x')) = \frac{1}{2} + \kappa(xx')\) for any \(x, x' \in \mathbb{C}^x\) such that \(x, x', xx' \neq 1\). Finally, \(\kappa \circ \chi_\theta(t)\) belongs to the image of \(\kappa\), hence is not equal to 1/2 or -1/2. It follows that \((\kappa \circ \chi_\theta(t))_{\theta \in \Delta_+(g; l)}\) belongs to \(\mathcal{P}_0\).

\(\iota\) is clearly a morphism of algebraic varieties. To show that it is an isomorphism, we will construct the inverse morphism. We claim that there is a unique morphism \(\iota' : \mathcal{P}_0 \to (\mathbb{C}^x)^f \setminus \bigcup_{\theta \in \Delta_+(g; l)} \ker(\chi_\theta)\), taking \((c(\theta))_{\theta \in \Delta_+(g; l)}\) to \(t = (t_s)_{s \in I}\), defined by \(t_s = \kappa^{-1}(c(\alpha_s))\) (we have \(\kappa^{-1}(y) = \frac{2y-1}{2y+1}\)). Let us show that \(\iota'\) is well defined. Since each \(c(\alpha_s)\) is \(\neq \pm 1/2\), each \(t_s\) is defined and belongs to \(\mathbb{C}^x\). Moreover, using the fact that \((c(\theta))_{\theta \in \Delta_+(g; l)}\) obeys the defining relations of \(\mathcal{P}\), one proves by induction on \(\text{ht}(\theta) = \sum_{s \in I} n_s(\theta)\) that \(1 + \chi_\theta(t) = 2(1 - \chi_\theta(t))c(\theta)\). So \(\chi_\theta(t) \neq 1\) for any \(\theta \in \Delta_+(g; l)\). So \(\iota'\) is well defined.

It is straightforward to check that \(\iota\) and \(\iota'\) are inverse to each other. \(\square\)

If \(t \in (\mathbb{C}^x)^f \setminus \bigcup_{\theta \in \Delta_+(g; l)} \ker(\chi_\theta)\), we denote by \(\rho_t\) the element of \(\mathcal{P}_0\) corresponding to \(t\) under \(\iota\). Then
\[ \rho_t = \frac{1}{2} \sum_{\theta \in \Delta_+(g; l)} \sum_{\alpha \in \Delta_+(g; l)} \frac{1 + \chi_\theta(t)}{1 - \chi_\theta(t)} e_\alpha \cdot f_\alpha. \]

4.3. The Poisson structures from \(\mathcal{P}_0\) are reductions. In this section, we show that the elements of \(\mathcal{P}_0\) can be obtained
(a) using a dynamical \(r\)-matrix \(r_1^\theta\) and a character \(\chi \in \mathfrak{l}^*\), as in Remark 4.2, 1) (this result was also obtained in [KMST], Theorem 15);
(b) using the PL dynamical \(r\)-matrix \(r_1^{\sigma}\) and an element \(\chi_{L,*} \in L_{\text{alg}}^*\), as in Remark 4.2, 2).

This shows that any Poisson structure from \(\mathcal{P}_0\) is
(a) a reduction of a Poisson structure on \(\mathfrak{l}^* \times G\), as in [Lu]
(b) a reduction of a Poisson structure on \(L_{\text{alg}}^* \times G\), of the type described in Section 4.1.

4.3.1. \(\mathcal{P}_0\) and \(r_1^\theta\). It follows from [FGP, EE2] that there is a unique dynamical \(r\)-matrix \(r_1^\theta : \mathfrak{l}^* \to \wedge^2(\mathfrak{g})\) for \((\mathfrak{l}, \mathfrak{g}, Z(\mathfrak{r}_0))\) (see Example 1.5), such that
\[ \forall \lambda \in \mathfrak{h}^*, \ r_1^\theta(\lambda) = -\frac{1}{2} \sum_{\alpha \in \Delta_+(g; l)} \coth \left( \frac{\lambda, \alpha}{2} \right) e_\alpha \wedge f_\alpha + \sum_{\alpha \in \Delta_+(g; l)} \varphi(\lambda, \alpha) e_\alpha \wedge f_\alpha, \]
where \(\varphi(x) = -\frac{1}{2} \coth(x/2) + 1/x\).

Recall that \(\{\text{characters of } \mathfrak{l}\} = \mathfrak{g}^*\). If \(\lambda\) is a character of \(\mathfrak{l}\), then \((\beta, \lambda) = 0\) for any \(\beta \in \mathfrak{g}^*\), therefore \((\alpha, \lambda) = 0\) for any \(\alpha \in \Delta_+(l; l)\), and if \(\alpha \in \Delta_+(g; l) \setminus \Delta_+(l; l)\), then \((\alpha, \lambda)\) depends only on \(\bar{\alpha} \in \Delta_+(g; l)\). We denote it by \((\bar{\alpha}, \lambda)\).

Definition 4.8. A character \(\lambda\) of \(\mathfrak{l}\) is called regular iff for any \(\theta \in \Delta_+(g; l)\), \((\theta, \lambda) \notin 2\pi i \mathbb{Z}\).

There is a unique map \(\{\text{regular characters of } \mathfrak{l}\} \to (\mathbb{C}^x)^f \setminus \bigcup_{\theta \in \Delta_+(g; l)} \ker(\chi_\theta)\), taking \(\lambda\) to \(t(\lambda) = (t_s)_{s \in I}\), where \(t_s = c(\bar{\alpha}, \lambda)\). The map \(\lambda \mapsto t(\lambda)\) is a covering with group \(\mathbb{Z}^f\).
Lemma 4.9. If $\chi$ is a regular character of $I$, we have $\rho_{t(\chi)} = r^I_t(\chi)$. 

Proof. We have $(\alpha, \chi) = 0$ if $\alpha \in \Delta_+(I)$, which simplifies the expression of $r^I_t(\chi)$. 

It follows from [Lu] that the bilinear map on $\mathbb{C}[G]$ defined by 

$$g_1, g_2 \in L^\ast \text{ such that the denominators are nonzero.}$$ 

Corollary 4.10. The Poisson structures $\{-, -\}_{\text{Ln}}$ and $\{-, -\}_{\rho_t}$ coincide.

4.3.2. $\mathcal{P}_0$ and $\sigma^0_\chi$. Let us restrict $\sigma^0_\chi : L^\ast_{\text{alg}} \to \wedge^2(\mathfrak{g})$ to $H_{\text{alg}}^\ast \subset L^\ast_{\text{alg}}$. $H_{\text{alg}}^\ast$ is a torus with Lie algebra $\mathfrak{h}^\ast$. We have 

$$\sigma^0_\chi(\epsilon) = \sum_{\alpha \in \Delta_+(I) \setminus \Delta_+^{\ast}(g)} \frac{e_\alpha \wedge f_\alpha}{1 - e^{-(\chi, \alpha)}}$$

for any $\chi \in \mathfrak{h}^\ast$ such that the denominators are nonzero.

Lemma 4.11. There is a unique subtorus $Z_{\text{alg}}^\ast \subset H_{\text{alg}}^\ast$ with Lie algebra $\mathfrak{z}^\ast$. If $t \in (\mathbb{C}^\times)^I \setminus \cup_{\theta \in \Delta_-(g)} \text{Ker}(\theta)$, there exists a regular $\chi \in \mathfrak{z}^\ast$ such that $e^{-(\chi, \alpha)} = t_\alpha$ for any $\alpha \in I$. Set $\chi_{\mathfrak{z}^\ast} = e^{\chi}$. Then the Poisson structure of $L_{\text{alg}}^\ast$ vanishes at $\chi_{\mathfrak{z}^\ast}$, and $\rho_t = \sigma^0_\chi(\chi_{\mathfrak{z}^\ast}) - r_{0, u}$. 

Corollary 4.12. The Poisson brackets $\{-, -\}_{\chi_{\mathfrak{z}^\ast}}$ and $\{-, -\}_{\rho_t}$ (see Section 4.1) coincide.

Proof. If $g_1, g_2 \in \mathbb{C}[G]$, then 

$$\{g_1, g_2\}_{\chi_{\mathfrak{z}^\ast}} = m((\mathbb{R}^\mathfrak{g} \otimes \mathbb{R}^\mathfrak{g})(r_0)(g_1 \otimes g_2)),$$

so

$$\{g_1, g_2\}_{\chi_{\mathfrak{z}^\ast}} = -m(\mathbb{R}^\mathfrak{g}(r_0)(g_1 \otimes g_2)) - m(\mathbb{R}^\mathfrak{g}(\sigma^0_\chi(\epsilon) - r_0)(g_1 \otimes g_2))$$

$$= \{g_1, g_2\}_{\rho_t} + m(\mathbb{R}^\mathfrak{g}(r_0)(g_1 \otimes g_2))$$

since $g_1$ and $g_2$ are $I$-invariant. 

5. Quantization of Poisson homogeneous structures on $G/L$

In this section, we construct quantizations of all the Poisson homogeneous structures on $G/L$, corresponding to elements of $\mathcal{P}_0$. For this, we prove algebraicity results for $\Psi^0_\mathfrak{h}$ and $\Psi^I_\mathfrak{h}$, relying on the computation of [DCK] of the determinant of the Shapovalov pairing for $U_q(\mathfrak{g})$. We can then construct the element $(\Psi^I_\mathfrak{h})^{-1}(\chi)$, which serves to quantize $G/L$ in the usual way.

5.1. Algebraicity of $\Psi^0_\mathfrak{h}$. Let $U_h(\mathfrak{g}) \subset U_h(\mathfrak{g})$ be the subalgebra generated by the $h_i$. Then $U_h(\mathfrak{h}) = \mathbb{C}[h_i, i = 1, \ldots, r][[\mathfrak{h}]]$. The intersection of this algebra with $\mathbb{C}[G]$ is $\mathbb{C}[[H^\ast]]_h = \mathbb{C}[[hh_i, h]]$.

Set $q = e^h$, $k_i = e^{hh_i}$. If $\beta = \sum_i n_i \alpha_i$ is a positive root of $\mathfrak{g}$, we set $k_\beta = \prod_{i=1}^r k_i^{n_i}$. Set $\mathbb{C}[H^\ast]_g = \mathbb{C}[k_i^{\pm 1}, i = 1, \ldots, r][[[\mathfrak{h}]]]$, where the index means that we invert all the $k_\beta^2 - 1$, where $\beta \in \Delta_+(\mathfrak{g})$.

Proposition 5.1. 1) $\Upsilon^0_\mathfrak{h}$ belongs to $U_h(n^\ast) \otimes U_h(-n^\ast) \otimes \mathbb{C}[H^\ast]_g$.

2) $\Psi^0_\mathfrak{h}$ belongs to $U_h(n^\ast) \otimes U_h(-n^\ast) \otimes \mathbb{C}[H^\ast]_g$.

Proof. Let us prove 1). $\Upsilon^0_\mathfrak{h}$ is the sum $\sum_{\alpha \in \Delta_+(\mathfrak{g})} (\Upsilon^0_\mathfrak{h})_{\alpha}$, where $(\Upsilon^0_\mathfrak{h})_{\alpha}$ is inverse to the quantum Shapovalov pairing $U_h(n^\ast)_{\alpha} \otimes U_h(-n^\ast)_{-\alpha} \to U_h(\mathfrak{h})$, $x \otimes y \mapsto h(yx)$. Here $U_h(n^\pm) \subset U_h(\mathfrak{g})$ are the subalgebras generated by the $e_i$ (resp., the $f_i$), and the index $\pm \alpha$ means the degree $\pm \alpha$ part.

Let $U_q(\mathfrak{g}) \subset U_h(\mathfrak{g})$ be the $\mathbb{C}(q)$-subalgebra generated by the $e_i, f_i$ and $k_i$, and $U_q(n^\pm)$, $U_q(\mathfrak{h})$ the $\mathbb{C}(q)$-subalgebras of $U_q(\mathfrak{g})$ generated by the $e_i$ (resp., the $f_i$, the $k_i^{\pm 1}$). Then the
Shapovalov pairing restricts to a pairing \( U_q(n_+) \otimes U_q(n_-) \to U_q(\mathfrak{h}) \), the determinant of which was computed in [DCK], Proposition 1.9, (a). Let \( \det_\alpha \) be this determinant; then
\[
(\Psi^\alpha)_\alpha \in U_q(n_+) \otimes U_q(n_-) \otimes U_q(\mathfrak{h}) [1/ \det_\alpha],
\]
where the tensor products are over \( \mathbb{C}(q) \). It follows from [DCK] that if we identify \( \det_\alpha \) with its image in \( \mathbb{C}[k_i^{\pm 1}, i = 1, \ldots, r][[h]] \), then we have
\[
\det_\alpha = u_\alpha h^{- \sum_{n \geq 0} P(\alpha - n \beta)} \prod_{\beta \in \Delta_+} (k^2 - 1) \sum_{n \geq 0} P(\alpha - n \beta)(1 + O(h)),
\]
where \( P(\alpha) \) is the number of decompositions \( \alpha = \sum_{\beta \in \Delta_+(\mathfrak{g})} k_\beta \beta \), with \( k_\beta \in \mathbb{Z}_+ \) and \( u_\alpha \) is invertible. Therefore \( \det_\alpha \) is invertible in \( \mathbb{C}[H^*]_\mathfrak{g} \). So \( (\Psi^\alpha)_\alpha \in U_h(n_+) \otimes U_h(n_-) \otimes \mathbb{C}[H^*]_{\text{loc}}. \)

We also know that the \( h \)-adic valuation of \( (\Psi^\alpha)_\alpha \) tends to infinity with \( \alpha \), which implies 1).

2) then follows from the fact that the "left coproduct" \( \Delta_l \) takes \( \mathbb{C}[H^*]_\mathfrak{g} \) to \( \mathbb{C}[H^*]_\mathfrak{g} \otimes \mathbb{C}[h, h^{-1}]. \)

5.2. \( (\Psi^\alpha)^{-1}(\chi) \) and its properties. Set \( \Lambda = \bigoplus_{\alpha \in \Delta_+(\mathfrak{g})} \mathbb{C} c_\alpha, L = \bigoplus_{\alpha \in \Delta_+(\mathfrak{g})} \mathbb{C} f_\alpha. \) Denote by \( \hat{H}^\alpha : \mathbb{C}[L^*]_\mathfrak{p}^{\mathfrak{p}, h} \to \mathbb{C}[H^*]_\mathfrak{p}^{\mathfrak{p}, h, \alpha} \) the Harish-Chandra map as in Section 3.3.

Set \( \mathbb{C}[H^*]_{\mathfrak{g}1} := \mathbb{C}[k_i^{\pm 1}, i = 1, \ldots, r][[h]], \) where the index means that we invert all the \( k_i^{\pm 1}, i \in \Delta_+ \setminus \Delta_+(\mathfrak{g}). \)

**Lemma 5.2.** Set \( \Xi := (\text{id} \otimes \text{id} \otimes \hat{H}^\alpha)(\Psi^\alpha)^{-1}. \) Then \( \Xi \in U_h(n_+) \otimes U_h(p_-) \otimes \mathbb{C}[H^*]_{\mathfrak{g}1} \).

**Proof.** We already know that \( (\Psi^\alpha)^{-1} \in U_h(n_+) \otimes U_h(p_-) \otimes \mathbb{C}[L^*]_\mathfrak{p}^{\mathfrak{p}, h}. \) Therefore
\[
(\text{id} \otimes \text{id} \otimes \hat{H}^\alpha)(\Psi^\alpha)^{-1}) \in U_h(n_+) \otimes U_h(p_-) \otimes \mathbb{C}[H^*]_\mathfrak{p}^{\mathfrak{p}, h, \alpha}
\]
where we recall that \( (\mathfrak{p}_h^{\mathfrak{p}})_{H^*} = \prod_{\beta \in \Delta_+(\mathfrak{g}) \setminus \Delta_+(\mathfrak{g}) (k^2 - 1). \)

On the other hand, Proposition 3.21 implies that \( \eta(\Phi^\alpha)^{-1} \) belongs to \( U_h(\mathfrak{g}) \otimes \mathbb{C}[H^*]_\mathfrak{g}. \) Therefore \( \eta' \circ \eta(\Phi^\alpha)^{-1} = (\text{id} \otimes \text{id} \otimes \hat{H}^\alpha)(\Phi^\alpha)^{-1} \) belongs to the same space.

Now \( \mathbb{C}[H^*]_\mathfrak{p}^{\mathfrak{p}, h, \alpha} \cap \mathbb{C}[H^*]_\mathfrak{g} = \mathbb{C}[H^*]_\mathfrak{g}, \) which implies the lemma. \( \square \)

Let us denote by \( \hat{H} : \mathbb{C}[L^*]_\mathfrak{p}^{\mathfrak{p}, h} \otimes \mathbb{C}[L^*]_{\mathfrak{g}1} \to \mathbb{C}[H^*]_\mathfrak{g} \) the map \( \pi \to \pi \rightarrow \hat{H}^\alpha(\pi y) \) (here \( \mathfrak{p}_h \) denotes the augmentation ideal of an augmented algebra \( \mathfrak{A}). \)

**Lemma 5.3.** The identity
\[
\Xi^{1,3,4} \Xi^{1,2,3,4} = (\text{id} \otimes \hat{H}^\alpha)(\Phi^\alpha)^{-1})^{1,2,3,4} \Xi^{1,2,3,4}
\]
holds in \( U_h(\mathfrak{g}) \otimes \mathbb{C}[H^*]_{\mathfrak{g}1} \). Here \( I \subset \mathbb{C}[H^*]_\mathfrak{p}^{\mathfrak{p}, h, \alpha} \) is the ideal generated by \( \text{Im}(\hat{H}) \) and \( J := I \cap \mathbb{C}[H^*]_\mathfrak{g}1. \)

**Proof.** Apply \( \text{id} \otimes \hat{H}^\alpha \) to the equation derived from the dynamical twist equation (29) by taking the inverses of both sides. We have \( (\Phi^\alpha)^{-1} = \Xi + \Theta, \) where \( \Theta = \sum_i a_i \otimes b_i \otimes c_i \ell_i d_i, \) and \( a_i, b_i \in U_h(\mathfrak{g}), \ell_i \in \mathbb{C}[H^*]_\mathfrak{p}^{\mathfrak{p}, h, \alpha}, c_i \in \mathbb{C}[L^*]_\mathfrak{p}^{\mathfrak{p}, h}, d_i \in \mathbb{C}[L^*]_\mathfrak{p}^{\mathfrak{p}, h}, \) and \( (\varepsilon(c_i), \varepsilon(d_i)) = 0. \) Then if \( \xi \in \mathbb{C}[H^*]_\mathfrak{g}, \) then \( H^\alpha_\pi(\varepsilon(c_i) \ell_i d_i) = H^\alpha_\pi(c_i \ell_i d_i) = \varepsilon(c_i) \varepsilon(d_i) = 0, \) and \( H^\alpha_\pi(\varepsilon(c_i) \ell_i d_i) = H^\alpha_\pi(c_i \ell_i d_i) = \varepsilon(c_i) \varepsilon(d_i) = 0. \) So the \( \Theta - \Xi \) contributions vanish, and the \( \Theta - \Xi \) contribution involves only pairs \( (i, j) \) such that \( \varepsilon(c_i) \varepsilon(d_i) \neq 0; \) if \( (i, j) \) is such a pair, then \( \varepsilon(d_i) = \varepsilon(c_j) = 0; \) so this contribution belongs to \( I = \text{Im}(\hat{H}). \) So (32) holds in \( U_h(\mathfrak{g}) \otimes \mathbb{C}[H^*]_\mathfrak{p}^{\mathfrak{p}, h, \alpha}/I. \) Now both sides of (32) belong to \( U_h(\mathfrak{g}) \otimes \mathbb{C}[H^*]_{\mathfrak{g}1}, \) which implies the lemma. \( \square \)

Let \( \chi \) be a character of \( I. \) As \( \chi \) is uniquely determined by its restriction of \( \mathfrak{h}, \) we identify it with an element of \( \mathfrak{h}^*. \) Recall that \( \chi \) is regular if for any \( \beta \in \Delta_+ \setminus \Delta_+(\mathfrak{g}), \) we have

\[\Xi \in \mathbb{C}[H^*]_\mathfrak{g}1, \]

holds in \( U_h(\mathfrak{g}) \otimes \mathbb{C}[H^*]_{\mathfrak{g}1} \).
\( \chi(h_3) \notin 2\pi i \mathbb{Z} \). Here for \( \beta = \sum_{s=1}^{r} n_s \alpha_s \in \Delta_+(g) \), we set \( h_3 = \sum_{s=1}^{r} n_s h_s \). It follows that \( \chi \) is regular iff \( \sigma_1^\beta \) is defined at \( e^{\lambda} \).

Then if \( \chi \) is regular, there is a unique character \( \hat{\chi} : C[H^*]_{g,h} \to C[[h]] \), taking each \( k_s \) to \( \exp(\chi(h_s)/2) \). We generalize these definitions to the case of a character \( \chi : L \to C[[h]] \) as follows: \( \chi \) is regular iff \( (\chi(h_3) \mod h) \notin 2\pi i \mathbb{Z} \) for any \( \beta \in \Delta_+(g) \setminus \Delta_+(1) \); then \( \hat{\chi} \) is defined in the same way.

Recall from Lemma 5.3 that \( J \) is an ideal of \( C[H^*]_{g,1} \).

**Lemma 5.4.** If \( \chi \) is a regular character of \( \mathfrak{l} \), then \( J \subset \text{Ker}(\hat{\chi}) \).

**Proof.** Let \( \mathfrak{g} \) be the center of \( \mathfrak{l} \), and set \( \mathfrak{h} := \mathfrak{g} \cap \mathfrak{h} \), where \( \mathfrak{g} = [\mathfrak{l}, \mathfrak{l}] \). Then \( \mathfrak{h} = \mathfrak{h}_I \oplus \mathfrak{g} \). \( \mathfrak{h}_I \) is spanned by the \( h_s \), where \( s \in [1, r] \) is an index of \( \mathfrak{l} \).

\[ C[[H^*]]_{(P_{g,h}^\beta), h} \] is the \( h \)-adic completion of \( C[[h_3, h, s = 1, \ldots, r]][(k_2^\beta - 1)^{-1}, \beta \in \Delta_+(g) \setminus \Delta_+(1)] \), and \( I \) is the complete ideal of \( C[[H^*]]_{(P_{g,h}^\beta), h} \) generated by \( \mathfrak{h}_I \), i.e., by the \( h_s \), an index of \( \mathfrak{l} \).

Recall that \( C[[H^*]]_{g,1} = C[k^{\pm 1}, s = 1, \ldots, r][(k_2^\beta - 1)^{-1}, \beta \in \Delta_+(g) \setminus \Delta_+(1) [[h]]] \); then \( J \) is a complete ideal of \( C[[H^*]]_{g,1} \), generated by the \( k_s - 1 \), an index of \( \mathfrak{l} \). Then if \( \chi \) is a regular character of \( \mathfrak{l} \), and \( s \) is an index of \( \mathfrak{l} \), then \( h_s \in [\mathfrak{l}, \mathfrak{l}] \), therefore \( \chi(h_s) = 0 \) and so \( \hat{\chi}(h_s) = \exp(\chi(h_s)/2) = 1 \), so \( \hat{\chi}(J) = 0 \).

**Proposition 5.5.** If \( \chi \) is a regular character of \( \mathfrak{l} \), set \( \Xi(\chi) := (\text{id} \otimes \text{id} \otimes \hat{\chi})(\Xi) \). Then

\[ \Xi(\chi)^{2,3} \Xi(\chi)^{1,23} = (\text{id}^{2,3} \otimes \hat{\chi} \circ H_0^1)((\Psi_{\mathfrak{l}}^{1,2,34})^{12,3}) \Xi(\chi)^{12,3}. \]

**Proof.** This follows immediately from Lemma 5.3 and Lemma 5.4.

### 5.3. Quantized Poisson homogeneous structures.

In this section, we construct quantizations of all the Poisson structures of Section 4.3.

As before, \( g \) is a semisimple Lie algebra, \( \mathfrak{l} \subset g \) is a Levi subalgebra, and \( \chi \in \mathfrak{l}^* \) is a regular character of \( \mathfrak{l} \). A quantization of the formal Poisson manifold \( \mathfrak{l} \setminus G \), equivariant under \( U_h(g) \), is constructed as follows.

Set \( C[[G]]_h := \text{Hom}(U_h(g), C[[h]]) \) and define the left and right actions of \( U_h(g) \) on \( C[[G]]_h \) by \( (L_a f)(x) = f(S(a) x) \), \( (R_a f)(x) = f(x a) \) for any \( a, x \in U_h(g) \). Then we set

\[ C[[G/L]]_h = \{ f \in C[[G]]_h \mid \forall a \in U_h(l), R_a f = \varepsilon(a) f \}. \]

\( C[[G/L]]_h \) is a subalgebra of \( C[[G]]_h \); we denote its product by \( m_h \).

The right action \( R \) restricts to an action of \( U_h(g) \) on \( C[[G/L]]_h \). If \( f_1, f_2 \in C[[G/L]]_h \), we set

\[ f_1 * f_2 = m_h((R \circ S)^{\otimes 2}(\Xi(\chi))(f_1 \otimes f_2)). \]

**Theorem 5.6.** \( (C[[G/L]]_h, \ast) \) is a quantization of \( G/L \) with its Poisson structure described in Section 4.3, equivariant under \( U_h(g) \).

**Proof.** Let us prove that \( \ast \) is associative. If \( x \in C[[L^*]]_{P_{g,h}^\beta} \) is such that \( (\text{id} \otimes \hat{H}_0^\beta) \circ \Delta(x) \in U_h(l) \hat{\otimes} C[H^*]_{g,1} \), then applying \( \varepsilon \) to the first factor we get \( \hat{H}_0^\beta(x) \in C[H^*]_{g,1} \). Then \( (\text{id} \otimes (\hat{\chi} \circ \hat{H}_0^\beta)) \circ \Delta(x) \in U_h(l) \). Therefore, if \( h \in C[[G/L]]_h \), we get

\[ R \left( (\text{id} \otimes (\hat{\chi} \circ \hat{H}_0^\beta)) \circ \Delta(x) \right)(h) = (\varepsilon \otimes (\hat{\chi} \circ \hat{H}_0^\beta)) \circ \Delta(x) h = (\hat{\chi} \circ \hat{H}_0^\beta)(x) h. \]
It follows that if \( f, g, h \in \mathbb{C}[[G/L]]_h \), then
\[
\begin{align*}
((R \circ S)^{\otimes 2} \otimes R \otimes \text{id}) & \left( \left( \text{id}^{\otimes 3} \otimes (\chi \circ \tilde{R}^\psi) \right) \left( (\Psi^\psi)^{-1} \right)^{1,2,3,4} \right) (f \otimes g \otimes h) \\
= (R \circ S)^{\otimes 2} \otimes \text{id} & \left( \left( \text{id} \otimes \text{id} \otimes (\chi \circ \tilde{R}^\psi) \right) \left( (\Psi^\psi)^{-1} \right) \right) (f \otimes g \otimes h) \\
= (R \circ S)^{\otimes 2} \otimes \text{id} & (\Xi(\chi)^{1,2})(f \otimes g \otimes h).
\end{align*}
\]

Using this identity together with Proposition 5.5, we obtain the associativity of \( \ast \). The classical limit of \( \ast \) is then the Poisson structure of Section 4.3. One shows easily that \( R \) restricts to an action of \( U_h(\mathfrak{g}) \) on \( \mathbb{C}[[G/L]]_h \), compatible with \( \ast \).

\[ \square \]

6. Relation with [EE2]

In this section, we compare the quantizations of the Poisson homogeneous structures on \( G/L \) obtained in Section 4 and in [EE2]. We first recall the construction of [EE2] (Section 6.1). We then prove a rigidity result satisfied by the \( \sigma^\chi(\chi) \) (Section 6.3). Finally, we compare both constructions (Section 6.4). As before, \( \mathfrak{g} \) is a semisimple Lie algebra and \( \mathfrak{l} \subset \mathfrak{g} \) is a Levi subalgebra.

6.1. The construction of [EE2]. Let \( \Phi \) be a Drinfeld associator, \( t_0 \in S^2(\mathfrak{g})^\# \) be the Casimir element, and set \( \Phi_\mathfrak{g} : = \Phi(h_0^{1,2}, h_0^{2,3}) \). According to [Dr2], there exists \( J_0 \in U(\mathfrak{g})^{\otimes 2}[[h]] \), such that \( J_0 = 1 + h_{\mathfrak{r}} + O(h^2) \), and
\[
J_0^{12,3,4} J_0^{1,2} = \Phi_\mathfrak{g}^{-1} J_0^{1,23} J_0^{2,3}. \tag{33}
\]

Then \( U(\mathfrak{g})[[h]], \text{Ad}(J_0^{-1}) \circ \Delta_0 \) is isomorphic to \( (U_h(\mathfrak{g}), \Delta_h) \).

In [EE2], we construct a solution \( J \in U(\mathfrak{g})^{\otimes 2} \hat{\otimes} U(\mathfrak{l})[[h]] \) of the dynamical pseudotwist equation
\[
J^{12,3,4} J^{1,2,3,4} = (\Phi_\mathfrak{g}^{-1})^{1,2,3} J^{1,23,4} J^{2,3,4}.
\]

Here \( \hat{U}(\mathfrak{l}) \) is the microlocalization of \( U(\mathfrak{l}) \) w.r.t. \( D_\mathfrak{l}^\# \), which may be defined as the unique element of \( S^{\dim(\mathfrak{u}_\mathfrak{l})}(\mathfrak{l}) \) whose restriction to \( \mathfrak{h} \) coincides with \( \prod_{\beta \in \Delta_+(\mathfrak{g}) \setminus \Delta_+(\mathfrak{l})} \beta \).

Recall that \( \{ \text{characters of } \mathfrak{l} \} = \mathfrak{z}^* \). Then if \( \chi \) belongs to an analytic open subset \( U_{\mathfrak{z}^*} \subset \mathfrak{z}^* \), then \( J(\chi) = (\text{id}^{\otimes 2} \otimes h^{-1})c(\chi)(J) \) is well defined. For \( f_1, f_2 \in \mathbb{C}[[G/L]][[h]] \), we set
\[
f_1 \star f_2 = m_0(L^{\otimes 2}(J_0)R^{\otimes 2}(J(\chi)^{-1}))(f_1 \otimes f_2).
\]

We define \( R' \) by \( R \circ S \). This defines a star-product on \( G/L \) equivariant under the left action of \( (U(\mathfrak{g})[[h]], \text{Ad}(J_0^{-1}) \circ \Delta_0) \simeq U_h(\mathfrak{g}) \), quantizing the Poisson structure of Section 4.1.

Let us study the effect of a gauge transformation of \( J_0 \) on this construction. Let \( u \in U(\mathfrak{g})[[h]] \) be an invertible element, and set \( u \ast J_0 = \Delta_0(u) J_0(u \otimes u)^{-1} \). Then \( u \ast J_0 \) is a solution of (33); we denote by \( \ast_u \) the corresponding star-product. According to the Belavin-Drinfeld classification of solutions of the CYBE, \( \rho_0 \) is conjugate to \( r_0 + h_0 \), where \( h_0 \in \mathfrak{h} \wedge \mathfrak{l}(\mathfrak{h})[[h]] \). Therefore \( J_0 = v_1 \ast J^\Phi_{\text{EK}}(r_0 + h_0) \), where \( v_1 \in U(\mathfrak{g})[[h]] \).

Now the QUE algebra \( (U(\mathfrak{g})[[h]], \text{Ad}(J^\Phi_{\text{EK}}(r_0 + h_0)^{-1}) \circ \Delta_0) \) is isomorphic
to \((U_\hbar(\g), \Delta_\hbar)\), and therefore contains \(r = \text{rank}(\g)\) QUE subalgebras isomorphic to \(U_\hbar(\sl_2)\), corresponding to each simple root.

Applying the quantization functor corresponding to the associator \(\Phi\), we obtain the Lie bialgebra \((\g[[h]], \mu, \text{ad}(r_0 + h_0))\); this Lie bialgebra should contain \(r\) Lie subbialgebras isomorphic to \((\sl_2[[h]], \text{standard structure})\), whose reductions modulo \(\hbar\) are the subbialgebras corresponding to the simple roots of \(\g\). One checks that such Lie subbialgebras can exist only if \(h_0 = 0\). Therefore \(J_0\) is gauge-equivalent to \(J^\hbar_{\text{EK}}(r_0)\).

It follows that the star-products \(*\) and \(*_u\) are gauge-equivalent. We will now assume that \(J_0 = J^\hbar_{\text{EK}}(r_0)\).

**Remark 6.2.** Assume that \(\g\) is simple. According to [Dr2], we have a bijection \{\(\g\)-invariant solutions of the pentagon equation in \(U(\g)^{\otimes 3}[[h]]\)\}/(invariant twists)\(\approx \wedge^3(\g)^\hbar[[h]] \approx \mathbb{C}[[h]]\), since \(\g\) is simple. It follows that all the \(\Phi_\hbar\) coincide, when \(\Phi\) runs over all associators, up to a rescaling of \(\hbar\). Moreover, the \(\Phi_\hbar\) all satisfy the same hexagon relation, so they all coincide up to twist. Then Proposition 6.1 implies that if \(\Phi, \Phi'\) are two associators, there exists an invertible \(T \in (U(\g)^{\otimes 2})\hbar[[h]]\), such that \(J^\hbar_{\text{EK}}(r_0)\) and \(TJ^\hbar_{\text{EK}}(r_0)\) are gauge-equivalent.

**6.2. Comparison of Levi subalgebras.**

**Theorem 6.3.** There exists an invertible element \(u \in U(\g)^\hbar[[h]]\), such that the subalgebra \(\text{Ad}(u)(U(\ell)([h])) \subset U(\g)([h])\) identifies with \(U_\hbar(\ell) \subset U_\hbar(\g)\) under \((U(\g)([h]), \text{Ad}(J^\hbar_{\text{EK}}(r_0)^{-1}) \circ \Delta_\hbar) \approx (U_\hbar(\g), \Delta_\hbar)\).

**Proof.** Let \(Q\) be the quantization functor associated to \(\Phi\) by [EK]. Apply \(Q\) to the sequence of inclusions of Lie bialgebras \(\ell \subset \g \subset \g\). We get the sequence of inclusions \(U_\hbar(\ell) \subset U_\hbar(\ell) \subset U_\hbar(\g)\).

We have an isomorphism \(U_\hbar(\ell) \approx U(\ell)([[h]])\), such that the composed map \(\ell \subset U_\hbar(\ell) \approx U(\ell)([[h]])\) is the standard inclusion (see [EK2]). We have also an isomorphism \(U_\hbar(\g) \approx U(\g)([[h]])\) such that \(\ell \subset U_\hbar(\g) \approx U(\g)([[h]])\) is the standard inclusion.

Using these isomorphisms, \(U_\hbar(\ell) \subset U_\hbar(\g)\) identifies with an injection \(U(\ell)([[h]]) \hookrightarrow U(\g)([[h]])\), such that the composed map \(\ell \subset U(\ell)([[h]]) \hookrightarrow U(\g)([[h]])\) is the standard inclusion. The theorem now follows from the fact that such an injection is necessarily of the form \(\text{Ad}(u) \circ i\), where \(u \in U(\g)^\hbar[[h]]\) is invertible and \(i\) is the inclusion \(U(\ell)([[h]]) \subset U(\g)([[h]])\).

This last statement is a consequence of:

**Lemma 6.4.** Any derivation \(\delta : \ell \to U(\g)\), such that \(\delta |_{\ell} = 0\), is inner, i.e., of the form \(x \mapsto [v, x]\), where \(v \in U(\g)^b\).

**Proof of Lemma.** Since \(\ell = \g \oplus \ell'\), we have an injection \(H^1(\ell, U(\g)) \subset H^1(\g, U(\g)) \oplus H^1(\ell', U(\g))\). Since \(\ell'\) is semisimple and \(U(\g)\) is a semisimple \(\ell'\)-module, the last cohomology group vanishes. It follows that there exists \(w \in U(\g)\) and \(\delta_0 \in H^1(\g, U(\g)) = \g^* \otimes U(\g)^3\), such that \(\delta(z + \ell') = \delta_0(z) + [w, \ell']\) for any \(z \in \g\) and \(\ell' \subset \ell'\). Now \(\delta_0 = 0\), so \(\delta_0 = 0\), so \(\delta_0 = 0\).

On the other hand, \(\delta\) is a derivation of \(\ell\), hence \(0 = \delta([z, \ell']) = [z, \delta(\ell')] = [z, w, \ell'] = [[[z, w], \ell']\) for any \(z \in \g\), \(\ell' \subset \ell'\), i.e., \([z, w] \in U(\ell'[[h]])\) for any \(z \in \g\).

The decomposition of \(U(\g)\) as a \(\ell'\)-module has the form \(U(\g) = U(\g)^{\ell'\oplus \bigoplus_{\rho \in \text{Irr}(\ell')} M_{\rho} \otimes V_{\rho}\), where \(\text{Irr}(\ell')\) is the set of irreducible finite dimensional \(\ell'\)-modules, and \(1\) is the trivial \(\ell'\)-module. Let \(w = w_0 + \sum_{\rho \in \text{Irr}(\ell'), \rho \neq 1} M_{\rho} \otimes V_{\rho}\), be the corresponding decomposition of \(w\). Then if \(z \in \g\), we have \([z, w_0] \in U(\g)^{\ell'}\), and \([z, w_{\rho}] \in M_{\rho} \otimes V_{\rho}\), so \([z, w_{\rho}] = 0\) for any \(\rho \neq 1\). We then set \(v := \sum_{\rho \in \text{Irr}(\ell'), \rho \neq 1} M_{\rho} \otimes V_{\rho}\). We have for any \(\ell' \subset \ell'\), \(\delta(\ell') = [w, \ell'] = [w - w_0, \ell'] = [v, \ell']\) since \(w_0 \in U(\g)^{\ell'}\), and \(\delta(z) = 0 = \sum_{\rho \in \text{Irr}(\ell'), \rho \neq 1} [w_{\rho}, z] = [v, z]\) for any \(z \in \g\). Hence \(\delta(l) = [v, l]\) for any \(l \in \ell\). Since \(\delta_{\ell} = 0\), we also have \(v \in U(\g)^b\).
We will now assume that \( J_0 = u^{-1} * J_{\mathrm{EEFC}}(r_0) \), therefore \( U(\theta)([h]) \) is a QUE subalgebra of \((U(\theta)[[h]], \Ad(J_0^{-1}) \circ \Delta_0)\), which is the image of \( U(\theta)(l) \subset U(\theta)(g) \) under \( U(\theta)(l) \simeq (U(\theta)[[h]], \Ad(J_0^{-1}) \circ \Delta_0) \).

6.3. \textbf{Deformations of} \( \sigma_1^0(\chi) \). Proposition 4.7 can be generalized to a scheme-theoretic setup as follows. If \( R \) is a ring containing \( C \), define \( \mathcal{P}(R) \) as the set of all \( (c(\theta))_{\theta \in \Delta_+(g; l)} \subset \Delta_+(g; l) \), such that for any \( \theta, \theta' \in \Delta_+(g; l) \) such that \( \theta + \theta' \in \Delta_+(g; l) \), we have \( c(\theta + \theta')(c(\theta) + c(\theta')) = c(\theta)c(\theta') + \frac{1}{2} \). Define \( \mathcal{P}_0(R) \) as the subset of \( \mathcal{P}(R) \) of all \( (c(\theta))_{\theta \in \Delta_+(g; l)} \), such that for any \( \theta \in \Delta_+(g; l) \), \( c(\theta) \pm \frac{1}{2} \) is invertible in \( R \).

Define \( \chi_\theta : (R^x)^I \to R^x \) by \( \chi_\theta(t) = \prod_{s \in I} t_s^\theta(s) \) if \( t = (t_s)_{s \in I} \), and define \( X(R) \subset (R^x)^I \) as the set of all \( t \), such that \( \chi_\theta(t) - 1 \) is invertible in \( R \) for any \( \theta \in \Delta_+(g; l) \).

Then the isomorphism \( \iota \) generalizes to an isomorphism between \( \mathcal{P}_0(R) \) and \( X(R) \).

When \( R \) is the ring \( \mathbb{C}[[\varepsilon]/(\varepsilon^2)] \) of dual numbers, \( \mathcal{P}_0(\mathbb{C}) \) is the preimage of \( \mathcal{P}(\mathbb{C}) \) under the natural map \( \mathcal{P}(R) \to \mathcal{P}(\mathbb{C}) \). In the same way, \( X(R) \) is the preimage of \( (\mathbb{C}^x)^I \setminus \cup_{\theta \in \Delta_+(g; l)} \ker(\chi_\theta) \) under \( (\mathbb{C}^x)^I \to (\mathbb{C}^x)^I \).

We denote by \((a, b) \mapsto \text{CYB}(a, b)\) the bilinear map derived from the quadratic map \( a \mapsto \text{CYB}(a) \).

**Proposition 6.5.** If \( t \in (\mathbb{C}^x)^I \setminus \cup_{\theta \in \Delta_+(g; l)} \ker(\chi_\theta) \), and \( \rho' \in \Lambda^2(u)^I \) is such that \( \text{CYB}(\rho_t, \rho') = 0 \) in \( \Lambda^3(g; l) \), then there exists a unique family \((\tau_s)_{s \in I} \) in \( \mathbb{C}^I \), such that

\[
\rho' = \sum_{s \in I} \frac{\partial \rho_t}{\partial t_s} \tau_s.
\]

**Proof.** \( \rho_t + \varepsilon \rho' \) is an element of \( \mathcal{P}_0(R) \), where \( R = \mathbb{C}[[\varepsilon]]/(\varepsilon^2) \). Its image by the generalization of \( \iota \) is a family of \( (R^x)^I \), which has the form \((t_s + \varepsilon \tau_s)_{s \in I} \), with \( \tau_s \in \mathbb{C} \). \( \square \)

**Corollary 6.6.** If \( \chi \in \mathfrak{s}^* \) is a regular character of \( l \), and \( \sigma' \in \Lambda^2(u)^I \) is such that \( \text{CYB}(\sigma_1^0(e^{\chi}), \sigma') = 0 \) in \( \Lambda^3(g; l) \), then there exists a unique element \( \chi' \in \mathfrak{s}^* \), such that

\[
\sigma' = \frac{d}{d\varepsilon} \big|_{\varepsilon = 0} \sigma_1^0(e^{\chi + \varepsilon \chi'}).
\]

**Proof.** We have a sequence of coverings \( \mathfrak{s}^* \to Z_{\mathfrak{sl}_g} \to (\mathbb{C}^x)^I \), with composition \( \chi \mapsto t(\chi) \), such that \( t_s = e^{-(\chi, a_s)} \). Then \( \sigma_1^0(e^{\chi}) = \rho_t(\chi) \). The statement is then a consequence of Proposition 6.5. \( \square \)

6.4. \textbf{Comparison with} \[\text{EE2}]. Let us reexpress the algebra \((\mathbb{C}[[G/L]]_h, *)\) defined in Section 5.3. The isomorphism of \( U(\theta)(g) \) with \((U(\theta)[[h]], \Ad(J_0^{-1}) \circ \Delta_0)\) allows to express the product of \( \mathbb{C}[[G]]_h \) as follows: \( \mathbb{C}[[G]][[h]] \) is isomorphic to \( \mathbb{C}[[G]][[[h]]] \), and \( f * g = m(L^\otimes 2(J_0)R^\otimes 2(J_0^{-1})(f \otimes g)) \).

Moreover, by virtue of the remark following Theorem 6.3, the image of \( \Delta(\mathbb{C}[[G/L]]_h) \) under \( \mathbb{C}[[G]][[[h]]] \to \mathbb{C}[[G/L]][[[h]]] \). Under this isomorphism, the product of \((\mathbb{C}[[G/L]]_h, *)\) is transported to the product

\[
f_1 * \chi f_2 = m_0(L^\otimes 2(J_0)R^\otimes 2(S^\otimes 2(\Xi(\chi))(J_0^{-1})(f_1 \otimes f_2)),
\]

for any \( f_1, f_2 \in \mathbb{C}[[G/L]][[[h]]] \) (here we view \( S^\otimes 2(\Xi(\chi)) \) as an element of \((U(\theta)[[h]]))).

**Lemma 6.7.** If \( A(\chi) \) is either the elements \( J(\chi)^{-1} \) and \( S^\otimes 2(\Xi(\chi))J_0^{-1} \) of \((U(\theta)[[h]])) \), then \( A(\chi) \) satisfies the identities:

(i) the image of \( A(\chi) \) in \((U(\theta)[[h]]) \) is \( l \)-invariant;

(ii) the image of \( A(\chi) \) in \((U(\theta)[[h]]) \) is zero under the projection \((U(\theta)[[h]])) \to (U(\theta)/U(\theta)[[h]]) \);

(iii) the image of \( A(\chi) - (1 + h\Phi_\theta(\chi)) \) is \( O(h^2) \) under the projection \((U(\theta)[[h]])) \to (U(\theta)/U(\theta)[[h]]) \).
Proof. The arguments are the same as those used to prove that $*$ and $*_\chi$ are associative products on $\mathbb{C}[[G/L]]$. \hfill \square

**Proposition 6.8.** Let $A(\chi), B(\chi)$ be analytic functions $U_\ast \to U(\mathfrak{g})\otimes^2[[\hbar]]$, satisfying the conditions (i), (ii), (iii) of Lemma 6.7. Then there exists an analytic function $u(\chi) : U_\ast \to 1 + \hbar U(\mathfrak{g})[[\hbar]]$, and analytic function $\chi_n : U_\ast \to \mathfrak{g}^* (n \geq 1)$, such that the image of

$$A(\chi) - (u(\chi) \otimes u(\chi))B(\chi(\chi))\Delta_0(u(\chi))^{-1}$$

in $(U(\mathfrak{g})/U(\mathfrak{g}))\otimes^2[[\hbar]]$ is zero. Here we set $\chi'(\chi) = \chi + \sum_{n \geq 1} \hbar^n \chi_n(\chi)$.

Proof. There is a unique map

$$d : ((U(\mathfrak{g})/U(\mathfrak{g}))\otimes^2)[[\hbar]] \to ((U(\mathfrak{g})/U(\mathfrak{g}))\otimes^3)[[\hbar]]$$

taking the class of $A$ to the class of $A^{1,2}A^{1,23} - A^{2,3}A^{1,23} \Phi$, where $A \in U(\mathfrak{g})\otimes^2[[\hbar]]$ is a representative of $A$. The group $1 + \hbar U(\mathfrak{g})/U(\mathfrak{g})$ acts on $((U(\mathfrak{g})/U(\mathfrak{g}))\otimes^2)[[\hbar]]$ as follows: $\bar{u} \ast \bar{A}$ is the class of $(u \otimes u)A\Delta_0(u^{-1})$, where $u \in U(\mathfrak{g})[[\hbar]]$, $A$ are representatives of $\bar{u}, \bar{A}$. Then

$$d(\bar{u} \ast \bar{A})$$

is the class of $u^{\otimes 3}d(\bar{A})\Delta_0(u^{-1})$, where $d(\bar{A})$ is a representative of $d(\bar{A})$; in particular, $d(\bar{A}) = 0$ iff $d(\bar{u} \ast \bar{A}) = 0$.

The classes $\bar{A}, \bar{B}$ are analytic functions $U_\ast \to ((U(\mathfrak{g})/U(\mathfrak{g}))\otimes^2)[[\hbar]]$, such that $d(\bar{A}(\chi)) = d(\bar{B}(\chi)) = 0$. Assume that we have constructed $\bar{u}_p(\chi) : U_\ast \to (U(\mathfrak{g})/U(\mathfrak{g}))^l$ and $\chi_p : U_\ast \to \mathfrak{g}^*$, $(p = 1, \ldots, n - 1)$, such that if $\chi_{n-1}(\chi) = \chi + \sum_{p=1}^{n-1} \hbar^p \chi_p(\chi)$ and $u_{n-1}(\chi) = 1 + \sum_{p=1}^{n-1} \hbar^p u_p(\chi)$, then

$$\bar{A}(\chi) - u_{n-1}(\chi) \ast \bar{B}(\chi_{n-1}(\chi)) = \mathcal{O}(\hbar^{n-1}). \tag{34}$$

The co-Hochschild differential $d_{\text{co-Hoch}}(A) = A^{1,2}A^{1,23} - A^{2,3}A^{1,23} + A^{1,2}$ induces a differential $((U(\mathfrak{g})/U(\mathfrak{g}))\otimes^2)[[\hbar]] \to ((U(\mathfrak{g})/U(\mathfrak{g}))\otimes^3)[[\hbar]]$, and identifies with the standard co-Hochschild differential $\langle S(\mathfrak{u})\otimes^2[[\hbar]] \langle S(\mathfrak{u})\otimes^3[[\hbar]]$.

Let $\bar{C}(\chi)$ be the class modulo $\hbar$ of $\hbar^{1-n}(\bar{A}(\chi) - u_{n-1}(\chi) \ast \bar{B}(\chi_{n-1}(\chi)))$. Then $d_{\text{co-Hoch}}(\bar{C}(\chi)) = 0$. So $\bar{C}(\chi) = d_{\text{co-Hoch}}(u_{n-1}(\chi)) + \rho(\chi)$, where $u_{n-1}(\chi), \rho(\chi)$ are analytic functions $U_\ast \to (U(\mathfrak{g})/U(\mathfrak{g}))^l$ and $U_\ast \to \Lambda^2(\mathfrak{g}/\mathfrak{l})^l$.

Since $\rho_\ast(\mathfrak{g}/\mathfrak{l})$ is $\mathfrak{l}$-invariant, the map $\rho \mapsto \text{CYB}(\rho_\ast(\mathfrak{g}/\mathfrak{l}), \rho)$ induces a linear map $\Lambda^2(\mathfrak{g}/\mathfrak{l})^l \to \Lambda^3(\mathfrak{g}/\mathfrak{l})^l$. Proposition 6.6 says that the kernel of this map consists of the $\frac{\partial}{\partial \varepsilon}|_{\varepsilon = 0}(\rho + \varepsilon \lambda')$, $\lambda' \in \mathfrak{g}^*$.

Now the vanishing of the coefficient of $\hbar^n$ in $d(\bar{A}(\chi)) = d(u_{n-1}(\chi) \ast \bar{B}(\chi_{n-1}(\chi)))$ yields CYB($\rho_\ast(\mathfrak{g}/\mathfrak{l}), \rho(\chi)$) + $d_{\text{co-Hoch}}(\bar{D}(\chi)) = 0$, where $\bar{D}(\chi)$ is the coefficient of $\hbar^n$ in $\bar{A}(\chi) - \bar{u}_{n-1}(\chi) \ast \bar{B}(\chi_{n-1}(\chi))$. It follows that $\text{CYB}(\rho_\ast(\mathfrak{g}/\mathfrak{l}), \rho(\chi)) = 0$, so $\rho(\chi) = \frac{\partial}{\partial \varepsilon}|_{\varepsilon = 0} \rho_\ast(\mathfrak{g} + \varepsilon \lambda)(\chi), \lambda(\chi) \in \mathfrak{g}^*$. So we have determined $u_{n-1}(\chi)$ and $\lambda(\chi)$, such that (34) holds at order $n$. \hfill \square

**Corollary 6.9.** $f \mapsto R_{u(\chi)}(f)$ is an isomorphism between the products $\ast$ and $\ast_\chi$ on $\mathbb{C}[[G/L]][[\hbar]]$.

7. Quantization of $G/L \to \mathfrak{g}^*$, $G/L \leftrightarrow G^*$ and Verma modules

Let $\mathfrak{g}$ be a semisimple Lie algebra, $\mathfrak{l} \subset \mathfrak{g}$ be a Levi subalgebra, $g = \mathfrak{l} \oplus \mathfrak{u}_+ \oplus \mathfrak{u}_-$ be a decomposition of $\mathfrak{g}$. Let $\chi \in \mathfrak{l}^*$ be a character of $\mathfrak{l}$; we view $\chi$ as an element of $\mathfrak{g}^*$ by defining it to be zero on $\mathfrak{u}_\pm$. We assume that $\chi$ is regular, which means that the centralizer of $\chi$ is $\mathfrak{l}$. Let $G_{\text{alg}}$ be the adjoint group of $\mathfrak{g}$, $L_{\text{alg}} \subset G_{\text{alg}}$ the subgroup corresponding to $\mathfrak{l}$. Then the orbit $G_{\text{alg}} \chi \subset \mathfrak{g}^*$ identifies with $G_{\text{alg}}/L_{\text{alg}}$. We have a sequence of maps

$$G_{\text{alg}} \to G_{\text{alg}}/L_{\text{alg}} = G_{\text{alg}}\chi \subset p^{-1}(p(\chi)) \subset \mathfrak{g}^*; \tag{35}$$
here \( p : \mathfrak{g}^* \to \mathfrak{g}^*/\mathfrak{g}_{\text{alg}} = \mathfrak{h}^*/W \) is the natural projection. \( p^{-1}(p(\chi)) \) is a union of coadjoint orbits on \( \mathfrak{g}^* \), and \( \mathfrak{g}_{\text{alg}} \chi \) is its only closed orbit ([Ko]). In (35), all varieties and maps except the first one are Poisson.

We now construct the quantum version of (35).

7.1. Poisson algebras. The function algebra of \( \mathfrak{g}_{\text{alg}} \) is \( \mathbb{C}[G] = \bigoplus_{V \in \text{Irr}} V^* \otimes V \). Here \( \text{Irr} \) is the set of simple objects in the subcategory of the tensor category of finite dimensional \( \mathfrak{g} \)-modules, generated by the adjoint representation \(^3\). Its product is defined by \( (\xi \otimes v) \ast (\eta \otimes w) = \varpi((\xi \otimes \eta) \otimes (v \otimes w)) \). Here \( \xi \otimes v \in V^* \otimes V, \eta \otimes w \in W^* \otimes W \), and \( \varpi : (V \otimes W)^* \otimes (V \otimes W) \to \oplus_{Z \in \text{Irr}} Z^* \otimes Z \) is the composition of the map induced from \( V \otimes W = \oplus_{Z \in \text{Irr}} Z \otimes \mathfrak{g}_{\text{alg}}^* \) with the maps \( \text{tr} : M_{V,W}^Z \otimes (M_{W,V}^Z)^* \to \mathbb{C} \) and the zero map on \( M_{V,W}^Z \otimes (M_{W,V}^Z)^* \) if \( Z^* \neq Z \).

The map \( \mathbb{C}[G] \subset \mathbb{C}[G] = \mathbb{C}[G]^{1} = \bigoplus_{V \in \text{Irr}} V^* \otimes V^* \). Its Poisson structure is defined by

\[
\{ \xi \otimes v, \eta \otimes w \} = \varpi((\xi \otimes \eta) \otimes \rho_{V \otimes W}(r^1_0(\chi))(v \otimes w)).
\]

This is a Poisson subalgebra of \( \mathbb{C}[[G/L]] = (U(\mathfrak{g})/U(\mathfrak{g}))^{*} \).

The dual of the map \( \mathfrak{g}_{\text{alg}}/\mathfrak{g}_{\text{alg}} \to \mathfrak{g}^* \) is the map \( \mathbb{C}[] = S(\mathfrak{g}) \to \mathbb{C}[G/L] \), defined as the unique algebra morphism taking \( x \in \mathfrak{g} \) to \( x \otimes \chi \in \mathfrak{g} \otimes (\mathfrak{g}^*)^1 \). This is a Poisson algebra morphism, which factors through a morphism

\[
S(\mathfrak{g}) \to S(\mathfrak{g}) \otimes Z(S(\mathfrak{g})) \mathbb{C}[\chi] \to \mathbb{C}[G/L].
\]

Here \( Z(S(\mathfrak{g})) = S(\mathfrak{g})^0 \) is the Poisson center of \( S(\mathfrak{g}) \), and \( p(\chi) \) is the character of \( Z(S(\mathfrak{g})) \) obtained by restricting the character \( S(\mathfrak{g}) \to \mathbb{C} \) induced by \( \chi \).

Indeed, an element \( P \in S(\mathfrak{g})^0 \) is mapped by \( P \otimes \chi : \mathfrak{g} \) then maps to \( P(\chi)1 \otimes 1 \).

The composed map \( S(\mathfrak{g}) \to (U(\mathfrak{g})/U(\mathfrak{g}))^{*} \) is given by the algebra-coalgebra pairing \( S(\mathfrak{g}) \otimes (U(\mathfrak{g})/U(\mathfrak{g})^l) \to \mathbb{C} \), taking \( x \otimes T \to \chi(\text{ad}(S(T)))(x) \) (here \( S \) is the antipode, \( \text{ad} \) is the adjoint action of \( U(\mathfrak{g}) \) on \( S(\mathfrak{g}) \), and \( \chi : S(\mathfrak{g}) \to \mathbb{C} \) is the character corresponding to \( \chi \).

\( \mathfrak{g}_{\text{alg}}/\mathfrak{g}_{\text{alg}} \) is a subvariety of \( \mathfrak{g}^* \), and its function ring is \( \mathbb{C}[G/L] \), hence the map \( \mathfrak{g} \to \mathbb{C}[G/L] \) is surjective.

7.2. Quantization of \( G/L \to \mathfrak{g}_{\text{alg}}/\mathfrak{g}_{\text{alg}} \to \mathfrak{g}^* \). In [EE2], we constructed an \( \mathfrak{l} \)-invariant element \( J := J^0(\hbar^{-1}\chi) \in U(\mathfrak{u}_+) \otimes U(\mathfrak{p}_-)\mathbb{C}[[\hbar]] \), such that \( J = 1 + O(\hbar), \hbar^{-1}(J - J^{2,1}) = r^0(\chi) \) mod \( \hbar \), and \( (U(\mathfrak{g})/U(\mathfrak{g})^l)[[\hbar]] \), equipped with \( \Delta(f) := \Delta_0(f)J^{2,1} \) is a coassociative coalgebra (we set \( \mathfrak{p}_+ = \mathfrak{l} \oplus \mathfrak{u}_+ \)). Its dual algebra \( (U(\mathfrak{g})/U(\mathfrak{g})^l)^*[\hbar] \) is a quantization of \( \mathbb{C}[[G/L]] \), which we denote by \( \mathbb{C}[[G/L]] \).

Define \( \mathbb{C}[G/L]_h \) as \( \bigoplus_{V \in \text{Irr}} V^* \otimes V^*[\hbar] \), equipped with the product

\[
(\xi \otimes v) \ast (\eta \otimes w) = \varpi((\xi \otimes \eta) \otimes \rho_{V \otimes W}(J)(v \otimes w)).
\]

Then \( \xi \otimes v \mapsto (T \mapsto \xi \rho_T(v)) \) is an algebra embedding \( \mathbb{C}[G/L]_h \subset \mathbb{C}[[G/L]]_h \).

Define \( \mathbb{C}[\mathfrak{g}^*]_h \) as the enveloping algebra of \( \mathfrak{g}[[\hbar]] \), equipped with the bracket \( [x, y]_h := [h, x, y] \). Then \( \mathbb{C}[\mathfrak{g}^*]_h \) is a flat deformation of \( \mathbb{C}[[\mathfrak{g}^*]] = S(\mathfrak{g}) \). \( \mathbb{C}[\mathfrak{g}^*]_h \) is a subalgebra of \( U(\mathfrak{g})[[\hbar]] \) under \( x \mapsto x_h \).

If \( \chi_0 \in \mathfrak{l}^* \) is a character, define \( \chi_0 : U(\mathfrak{g}) \to \mathbb{C} \) as the unique linear map such that \( \chi_0 : (x_+ x_0 x_-) = \varepsilon(x_+)\varepsilon(x_-)\chi_0(x_0) \), where \( x_\pm \in U(\mathfrak{u}_+) \) and \( x_0 \in U(0) \). This definition extends to the case when \( \chi_0 \in \mathfrak{l}^*[[\hbar]] \). In particular, \( \hbar^{-1} \chi \) is a character \( U(\mathfrak{g})[[\hbar]] \to \mathbb{C}[[\hbar]] \); one checks that it restricts to a character \( \chi : \mathbb{C}[\mathfrak{g}^*]_h \to \mathbb{C}[[\hbar]] \).

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\(^3\text{Irr} \) may also be described as the set of simple \( \mathfrak{g} \)-modules \( V \) with highest weight in the root lattice, or such that \( V[0] \neq 0 \).
Proposition 7.1. There is a unique pairing
\[
(U(\mathfrak{g})/U(\mathfrak{g})) \otimes \mathbb{C}[\mathfrak{g}^*]_h \rightarrow \mathbb{C}[[h]],
\]
taking \([T] \otimes x\) to \(\tilde{\chi}(\text{ad}(S^{-1}(T))(x))\). It induces an algebra morphism \(\mathbb{C}[\mathfrak{g}^*]^{\text{op}}_h \rightarrow \mathbb{C}[G/L]_h\). This morphism factors through a morphism \(\mathbb{C}[\mathfrak{g}^*]^{\text{op}} \rightarrow \mathbb{C}[G/L]_h \subset \mathbb{C}[G/L]_h\), equivariant under the adjoint action of \(U(\mathfrak{g})\) on \(\mathbb{C}[\mathfrak{g}^*]_h\) and its left action on the two other algebras.

Proof. The fact that the pairing is well-defined follows from the \(\ell\)-invariance of \(\tilde{\chi}\). Let us denote it by \([T] \otimes x \mapsto (T,x)\) and check that it is a coalgebra-algebra pairing. We first prove that if \(x, y \in \mathbb{C}[\mathfrak{g}^*]_h\), then
\[
\tilde{\chi}(xy) = \sum_i \tilde{\chi}(\text{ad}(S^{-1}(\alpha_i))(x))\tilde{\chi}(\text{ad}(S^{-1}(\beta_i))(y)),
\]
where \(J = \sum \alpha_i \otimes \beta_i\). Recall that \(J = \sum a_i \otimes (b_i \otimes \ell_i)\), where \(K = \sum a_i \otimes b_i \otimes \ell_i\) is the element of \(U(\mathfrak{g}) \otimes U(\mathfrak{u}) \otimes \hat{U}(\mathfrak{l})\) such that the identity \(\sum_i H(x_a_i)\ell_i H(b_i y) = H(xy)\) for any \(x, y \in U(\mathfrak{g})\) holds; here \(H : U(\mathfrak{g}) \rightarrow \hat{U}(\mathfrak{l})\) is the linear map taking \(x \otimes x_0 y\) to \(\varepsilon(x_+ \varepsilon(x-) x_0)\), where \(x_+ \in U(\mathfrak{u})_+, x_0 \in U(\mathfrak{l})\); we again denote by \(\tilde{\chi}\) the unique extension of \(\tilde{\chi}\) to a character of \(\hat{U}(\mathfrak{l})\) (it exists because \(\chi\) is nondegenerate).

Using the identities \(\tilde{\chi}(x+ x) = \varepsilon(x_+) \tilde{\chi}(x), \tilde{\chi}(x x_-) = \tilde{\chi}(x) \varepsilon(x_-)\) for \(x \in U(\mathfrak{g}), x \in U(\mathfrak{u})_\pm\), and the \(\ell\)-invariance of \(\tilde{\chi}\), we identify the r.h.s. of (36) with \(\sum_i \tilde{\chi}(\alpha_i(x))\tilde{\chi}(\text{ad}(S^{-1}(\beta_i))(y))\). This is \(\sum_i \tilde{\chi}(\alpha_i)\tilde{\chi}(\ell_1(x))\tilde{\chi}(\ell_2(y)), \) i.e., \(\sum_i \tilde{\chi}(\alpha_i)\tilde{\chi}(\ell_1)\tilde{\chi}(b_2(y))\) (as \(\chi\) is a regular character of \(\ell\), we have \((\chi \otimes \chi) \circ \Delta = \chi\), where \(\Delta : \hat{U}(\mathfrak{l}) \rightarrow \hat{U}(\mathfrak{l}) \otimes \hat{U}(\mathfrak{l})\) is the “right coproduct” of \(\tilde{\chi}\)).

If \(T \in U(\mathfrak{g})\) and \(x, y \in \mathbb{C}[\mathfrak{g}^*]_h\), we have
\[
(T, xy) = \tilde{\chi}(\text{ad}(S^{-1}(\ell_2))(x))\tilde{\chi}(\text{ad}(S^{-1}(\ell_1))(y)) = \sum_i \tilde{\chi}(\text{ad}(S^{-1}(\alpha_i)))\tilde{\chi}(\text{ad}(S^{-1}(\ell_2))(x))\tilde{\chi}(\text{ad}(S^{-1}(\ell_1)))\tilde{\chi}(\text{ad}(S^{-1}(\beta_i))(y)) \quad \text{(by (36))}
\]
\[
= \sum_i \tilde{\chi}(T^{(2)} \alpha_i, x)\tilde{\chi}(T^{(1)} \beta_i, y).
\]

If \(x \in U(\mathfrak{g})\) is an element of degree \(d\), then the map \(T \mapsto \tilde{\chi}(\text{ad}(S^{-1}(x)))\) is the matrix coefficient of \(x \otimes \tilde{\chi}(\varepsilon)_\leq d \in U(\mathfrak{g})_{\leq d} \otimes (U(\mathfrak{g})_{\leq d})^{\text{op}}\); so the image of \(\mathbb{C}[\mathfrak{g}^*]^{\text{op}} \rightarrow \mathbb{C}[G/L]_h\) is contained in \(\mathbb{C}[G/L]_h\).

7.3. Relation with generalized Verma modules. Let \((M_\chi, \pi_\chi)\) be the \(U(\mathfrak{g})\)-module \(\text{Ind}_{\mathfrak{p}^-}(\mathbb{C}((h))\chi)\), where \(\chi\) is the character \(\mathfrak{p}^- \rightarrow I^{h^{-1}x} \mathbb{C}((h))\).

Proposition 7.2. The morphism \(\mathbb{C}[\mathfrak{g}^*]_h \rightarrow \text{End}(M_\chi)[[h]]\) induced by \(\pi_\chi\) factors through \(\mathbb{C}[\mathfrak{g}^*]_h \xrightarrow{\alpha} \mathbb{C}[G/L]_h \xrightarrow{\beta} \text{End}(M_\chi)[[h]]\), where \(\alpha\) is surjective and \(\beta\) is injective. It follows that \(\mathbb{C}[G/L]_h \simeq \mathbb{C}[\mathfrak{g}^*]_h/(\mathbb{C}[\mathfrak{g}^*]_h \cap \text{Ker}(\pi_\chi))\).

Proof. Let us construct an algebra morphism \(\beta : \mathbb{C}[G/L]_h \rightarrow \text{End}(M_\chi)[[h]]\). If \(V \in \text{Irr}\), we define a linear map \(V^* \otimes V^* \rightarrow \text{End}(M_{\chi_v})[[h]]\) by dualizing the linear map \(M_\chi \rightarrow M_\chi \otimes V \otimes (V^i)^*[[h]]\) equal to \(\sum_{b \in B} \Phi_b \otimes b^*\). Here \(B\) is a basis of \(V^i\), \((b^*)^*\) is the dual basis of \((V^i)^* \simeq (V^i)^\dagger\), and \(\Phi_b : M_\chi \rightarrow M_\chi \otimes V[[h]]\) is the intertwiner with expectation value \(b\). Using the fact that \(\Phi_b(1) = \sum \alpha_i \ell_1 \otimes \beta_i b\), one can prove that \(\mathbb{C}[G/L]_h \rightarrow \text{End}(M_{\chi_v})[[h]]\) is an algebra morphism.

Let us prove that \(\beta\) is injective. Recall that \(\beta\) takes \(v \otimes v^* \otimes \Phi_v(m)\) of \(\text{End}(M_{\chi_v})[[h]]\). Therefore, \(\beta\) extends to a linear map \(\beta' : \text{End}(M_{\chi_v})[[h]]\).
(U(g)/Lalg)∗ → End(Mχ)[[ℏ]], taking ξ to the endomorphism x+1χ → ∑i(ξi,x+1(βi))x+2(αi)1χ (here x+ ∈ U(u+)).

We have a linear isomorphism (U(g)/U(g))∗ ∼= U(u+)∗ ⊗ U(u−)∗, whose inverse takes ξ+ ⊗ ξ− to the form x+ x− → (ξ+, x+)ξ−(x−, x−), where x± ∈ U(u±).

Let π : U(p+) → U(u+) be the linear mapping taking x+ x0 to x+x(x0), where x ∈ U(u+), x0 ∈ U(l).

The map β′′ : U(u+)∗ ⊗ U(u−)∗ → End(Mχ)[[ℏ]] induced by β′ takes ξ+ ⊗ ξ− to the endomorphism

x+1χ → ∑i(ξi,x+1(βi)x+2(ξ−, π(βi)))αi1χ.

Assume that ∑iξ+ α ⊗ ξ− α is in Ker(β′′). Then

∀x ∈ U(u+), ∑i,α(ξ+ α,x+1(βi)x+2(ξ−, π(βi)))αi = 0. (37)

Since ∑iαi ⊗ π(βi) coincides with exp(ℏ(χ)) up to lower degree terms, the linear map U(u−)∗ → U(u+), ξ → ∑i(ξi(βi))αi is injective.

Then using (37) with x = 1, we find ∑iξ(ξ+, α)x(ξ−, α) = 0. Let us prove by induction on d that for any x ∈ U(u+)x, ∑iξ(ξ+, α)x(ξ−, α) = 0. Assume that this holds at order d and let x ∈ U(u+)x. Then the induction hypothesis implies ∑iξ(ξ+, α)x(ξ−, α) = 0. The injectivity of ξ → ∑iξ(ξ+, α)x(ξ−, α) then implies that ∑iξ(ξ+, α)x(ξ−, α) = 0, as wanted. Therefore ∑iξ+ α ⊗ ξ− α = 0. Hence β is injective.

Let us now show that β ∘ α = πχ[c]∗[[ℏ]]. Let x ∈ U(g), then α(x) is the form U(g)/U(g) → C, T → χ(ad(S−1(T))(x)). Then β′′(α(x)) is the endomorphism of Mχ[[ℏ]] taking x+1χ to

∑i(α(x), x+1βi)x+2αi1χ = ∑iχ(S−1(βi2)S−1(x+2βi1)x+3αi1χ = ∑iχ(S−1(βi)S−1(x+2x+1βi)x+3αi1χ. (38)

Now we have ∑iαiχ(S−1(βi)X+X0X−) = X+X(0)ξ(X−) for X± ∈ U(u±) and X0 ∈ U(l). Hence if S−1(x+2x+1) = ∑iX+X0X−, then

(38) = x+3(∑αX+X0ξ(X−)ξ(X−))1χ.

On the other hand,

xx+1χ = x+3S−1(x+2x+1)1χ = x+3∑αX+X0X−ξ(X−)1χ

= x+3∑αX+X0ξ(X−)1χ,

which is (38). Hence (38) = xx+1χ, so β ∘ α = πχ[c]∗[[ℏ]].

7.4. PL versions. Let t, χ, ηL be as in Lemma 4.11. The dressing orbit of ηL∗ ∈ G∗alg is G∗algL∗ ∼= G∗alg/Lalg. Moreover, the Poisson structure of G∗algL∗ identifies with {−, −}ηL∗.

We have therefore a Poisson G-space embedding G∗alg/Lalg ∼= G∗algL∗ → G∗alg. As before, we can construct a morphism C[G∗]h → C[G/L]h quantizing this embedding; here C[G∗]h is a subalgebra of Uh(g). Let Mhχ be the generalized Verma module over Uh(g) corresponding to χ.

As before, the morphism πχh : Uh(g) → End(Mhχ) restricts to a morphism C[G∗]h → End(Mhχ), which factors through an injective morphism C[G/L]h → End(Mhχ) which can be defined using intertwiners. So C[G/L]h ∼= C[G∗]h/[C[G∗]h ∩ Ker(πχh)]
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