One loop renormalization for the axial Ward-Takahashi identity in Domain-wall QCD

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Abstract

We calculate one-loop correction to the axial Ward-Takahashi identity given by Furman and Shamir in domain-wall QCD. It is shown perturbatively that the renormalized axial Ward-Takahashi identity is satisfied without fine tuning and the “conserved” axial current receives no renormalization, giving $Z_A = 1$. This fact will simplify the calculation of the pion decay constant in numerical simulations since the decay constant defined by this current needs no lattice renormalization factor.
I. INTRODUCTION

The lack of chirally invariant fermion formulations is one of the most uncomfortable points theoretically and practically in lattice QCD. For example, in the Wilson fermion formulation, which is popularly used in numerical simulations, the chiral limit can be realized only by the fine tuning of bare mass parameter, which compensates the additive quantum correction to the quark mass.

Recently the domain-wall fermion formulation \[1,2\], which was originally proposed for lattice chiral gauge theories \[3\], has been employed in lattice QCD simulations \[4\] and has shown its superiority over other formulations: there seems no need of the fine tuning to realize the chiral limit while there is no restriction to the number of flavors. In particular massless mode which presents at the tree level seems stable against the quantum correction. This property of the domain-wall fermion seems to suggest the existence of a kind of axial symmetry. In Ref. \[2\] Furman and Shamir have defined the axial transformation and the current in the domain-wall fermion. They have shown that although their transformation is not a symmetry of the action, the axial Ward-Takahashi identity is satisfied in Green’s functions between the current operators and physical quark fields made of boundary fermions.

Although it has already argued in the same reference that the axial Ward-Takahashi identity seems to hold nonperturbatively, it is still interesting to see a definite form of one loop correction to Green’s functions in the identity since renormalized form of the Green’s functions is nontrivial. In this paper we calculate the one loop correction to the Green’s functions with the axial current operators and the quark fields. It is shown that all the divergence are renormalized into the same Green’s functions without any mixing between these physical operators and unphysical heavy fermion operators. The renormalized axial Ward-Takahashi identity is satisfied without any fine tuning and the renormalization factor of the axial current becomes unity, \(Z_A = 1\).

This paper is organized as follows. In Sec. II we introduce the action of domain-wall QCD. The axial transformation and its Ward-Takahashi identity is given according to Ref. \[2\]. In Sec. III we calculate one loop correction to the Ward-Takahashi identity. The Feynman rules relevant to our calculations are defined in our previous paper \[5\] and are briefly given in the appendix. Our conclusion is given in Sec. IV.

In this paper we set the lattice spacing \(a = 1\) and take \(SU(N_c)\) gauge group with the gauge coupling \(g\).

II. AXIAL CURRENT AND WARD-Takahashi IDENTITY

The domain-wall fermion is a 4+1 dimensional Wilson fermion with a “mass term” which depends on the coordinate in the extra dimension. For the explicit form of the “mass term” we adopt the Shamir’s one \[1\] and the corresponding action becomes

\[
S_{DW} = \sum_{n} \sum_{s=1}^{N} \left[ \frac{1}{2} \sum_{\mu} \left( \overline{\psi}(n)_s (-r + \gamma_\mu) U_\mu(n) \psi(n + \mu)_s + \overline{\psi}(n)_s (-r - \gamma_\mu) U^\dagger_\mu(n - \mu) \psi(n - \mu)_s \right) \\
+ \frac{1}{2} \left( \overline{\psi}(n)_s (1 + \gamma_5) \psi(n)_{s+1} + \overline{\psi}(n)_s (1 - \gamma_5) \psi(n)_{s-1} \right) + (M - 1 + 4r) \overline{\psi}(n)_s \psi(n)_s \right]
\]
\[ + m \sum_n \left( \bar{\psi}(n)_N P_+ \psi(n)_1 + \bar{\psi}(n)_1 P_- \psi(n)_N \right), \]  

(II.1)

where \( n \) is a 4 dimensional space-time coordinate and \( s \) is an extra fifth dimensional or flavor index, the Dirac “mass” \( M \) is a parameter of the theory which we set \( 0 \leq M \leq 2 \) to realize the massless fermion at tree level, \( m \) is a physical quark mass, and the Wilson parameter is set to \( r = -1 \). It is important to notice that we have boundaries for the flavor space; \( 1 \leq s \leq N \). \( P_{\pm} \) is a projection operator \( P_{\pm} = (1 \pm \gamma_5)/2 \).

The remarkable property of the domain-wall fermion is that there exists a massless fermion mode in the \( N \to \infty \) limit at small momentum at \( m = 0 \). This massless fermion stays near the boundaries of the flavor space with the left and the right mode on the opposite side. At tree level the massless mode \( \chi_0 \) is given explicitly in zero momentum limit as

\[ \chi_0 = \sqrt{1 - w_0^2} \left( P_+ w_0^{s-1} \psi_s + P_- w_0^{N-s} \bar{\psi}_s \right), \]

(II.2)

where \( w_0 = 1 - M \). Although the zero mode is stable against quantum correction \[3\], the damping factor \( w_0 \) is renormalized due to the additive correction to the Dirac mass \( M \). Thus in numerical simulations it is more convenient to use the interpolating “physical” quark field defined by the boundary fermions:

\[ q(n) = P_+ \psi(n)_1 + P_- \psi(n)_N, \]
\[ \bar{q}(n) = \bar{\psi}(n)_N P_+ + \bar{\psi}(n)_1 P_. \]  

(II.3)

The axial transformation is defined to rotate the right and left mode of the Dirac fermion oppositely. In the domain-wall fermion the massless mode is localized at the boundary of the flavor space with the right and left mode in the different side. The axial transformation of the domain-wall fermion is given by the flavor dependent vector like rotation of the fermion field which transform the different boundary fermions with opposite charge \[2\],

\[ \delta^a \psi(n)_s = \epsilon \left( \frac{N}{2} + \frac{1}{2} - s \right) i \lambda^a \psi(n)_s, \]
\[ \delta^a \bar{\psi}(n)_s = -\epsilon \left( \frac{N}{2} + \frac{1}{2} - s \right) i \bar{\psi}(n)_s \lambda^a, \]  

(II.4)

(II.5)

where \( \lambda^a \) is a SU\((N_f)\) generator of the rotation where \( N_f \) is a number of “real” quark flavors, and \( \epsilon(s) \) is a step function,

\[ \epsilon(s) = \begin{cases} 
1, & s \geq 0, \\
-1, & s < 0. 
\end{cases} \]

(II.6)

This transformation acts on the quark field as a usual axial rotation,

\[ \delta^a q(n) = i \lambda^a \gamma_5 q(n), \quad \delta^a \bar{q}(n) = i \bar{q}(n) \gamma_5 \lambda^a. \]  

(II.7)

The corresponding axial current is given by the sum of the conserved vector current in the Wilson fermion over the flavor index,

\[ A^a_\mu(n) = \sum_s \epsilon \left( \frac{N}{2} + \frac{1}{2} - s \right) J^a_\mu(n, s), \]  

(II.8)

\[ J^a_\mu(n, s) = \frac{1}{2} \left( \bar{\psi}(n)_s (1 + \gamma_\mu) U_\mu(n) \lambda^a \psi(n_\mu)_s - \bar{\psi}(n_\mu) \lambda^a \psi(n)_s (1 - \gamma_\mu) U^\dagger_\mu(n) \right). \]  

(II.9)
Unfortunately this transformation is not a symmetry of the action and therefore the axial current is not conserved. The divergence of the axial current becomes

$$\nabla_\mu A_\mu^a(n) = 2m\overline{\psi}(n)\lambda^a\gamma_5 q(n) - 2X^a(n), \tag{II.10}$$

where $\nabla_\mu$ is a backward derivative and $X^a$ is the explicit breaking term characteristic for the domain-wall fermion,

$$X^a(n) = \overline{\psi}(n)\frac{\lambda^a\psi(n)}{2} - \overline{\psi}(n)\frac{\lambda^a\psi(n)}{2+1}.$$

In Ref. [2] it is argued that the proper axial Ward-Takahashi identity is realized in $N \rightarrow \infty$ limit if we consider the Green’s functions with physical quark operators,

$$\nabla_\mu \langle A_\mu^a(n)\mathcal{O}\rangle - 2m \langle \overline{\psi}(n)\lambda^a\gamma_5 q(n)\mathcal{O}\rangle + i \langle \delta^a\mathcal{O}\rangle = 0, \tag{II.12}$$

where operator $\mathcal{O}$ is made of the quark field (II.3). The reason for this is that the explicit breaking term $X^a$ and the physical operator $\mathcal{O}$ is separated by $N/2$ in the flavor space and the Green’s function $\langle X^a(n)\mathcal{O}\rangle$ is suppressed by a factor $e^{-\alpha N/2}$, where $\alpha$ is some positive number.

In this paper we set $\mathcal{O} = q(y)\overline{\psi}(z)$ and calculate the one loop correction to the Ward-Takahashi identity.

## III. ONE LOOP CORRECTION TO WARD-TAKAHASHI IDENTITY

In this section we consider the following Ward-Takahashi identity with $\mathcal{O} = q(y)\overline{\psi}(z)$,

$$0 = \nabla_\mu \langle A_\mu^a(n)q(y)\overline{\psi}(z)\rangle - 2m \langle \overline{\psi}(n)\lambda^a\gamma_5 q(n)\rangle q(y)\overline{\psi}(z) + 2 \langle X^a(n)q(y)\overline{\psi}(z)\rangle$$

$$+ \delta_{n,y}\lambda^a\gamma_5 \langle q(n)\overline{\psi}(z)\rangle + \delta_{n,z} \langle q(y)\overline{\psi}(n)\rangle \gamma_5 \lambda^a. \tag{III.1}$$

Here the axial current for domain-wall quarks is given by a sum of the conserved vector current of the Wilson fermion system. One may wonder how this “vector” current turns out to be the continuum axial vector current $\overline{\psi}\gamma_\mu\gamma_5\psi$ in the $a \rightarrow 0$ limit. To see this we first study the continuum form of the Green’s function $\langle A_\mu^a(q(y)\overline{\psi}(z)) \rangle$ at tree level and in $N \rightarrow \infty$ limit before the one loop calculation. In momentum space the tree level Green’s function is given by Fig. 4 and has the following form in the continuum limit,

$$\langle A_\mu^a(q(k)\overline{\psi}(p)) \rangle_0 = \delta_{\mu,0} \langle q(-k)\overline{\psi}(k, s)\rangle \gamma_\mu \epsilon(s, t) \langle \psi(-p, t)\overline{\psi}(p)\rangle \gamma_\mu \epsilon(s, t) \tag{III.2}$$

where $\epsilon(s, t)$ is a diagonal matrix defined by

$$\epsilon(s, t) = \epsilon\left(\frac{N}{2} + \frac{1}{2} - s\right) \delta_{s, t}. \tag{III.3}$$

and $\langle q(-k)\overline{\psi}(k, s)\rangle$, $\langle \psi(-p, t)\overline{\psi}(p)\rangle$ are the external quark line propagator [3]. A summation over the same flavor index is taken implicitly in this paper. Hereafter we will omit the SU($N_f$) generator $\lambda^a$ for simplicity.
We first notice the fact that the quark external line propagator is written as a product of the physical quark propagator and some damping factor in the continuum,

$$\langle q(-p)\bar{\psi}(p, s) \rangle \rightarrow \frac{1 - w_0^2}{i\gamma + (1 - w_0^2)m} \left[ (w_0^{(N-s)} P_+ + w_0^{(s-1)} P_-) - \frac{w_0}{1 - w_0^2} i\gamma \left( w_0^{(s-1)} P_+ + w_0^{(N-s)} P_- \right) \frac{w_0}{1 - w_0^2} i\gamma \right] \frac{1 - w_0^2}{i\gamma + (1 - w_0^2)m}. \tag{III.4}$$

$$\langle \psi(-p, s)\bar{\tau}(p) \rangle \rightarrow \left[ (w_0^{(N-s)} P_+ + w_0^{(s-1)} P_-) - \left( w_0^{(s-1)} P_+ + w_0^{(N-s)} P_- \right) \frac{w_0}{1 - w_0^2} i\gamma \right] \frac{1 - w_0^2}{i\gamma + (1 - w_0^2)m}. \tag{III.5}$$

We have expanded the propagator to the next to leading order in the quark external momentum and mass, both of which give the leading order contribution in the one loop corrections. Now we can use the formula

$$\langle X^a(q)q(k)\bar{\tau}(p) \rangle_0 = \sum_{s, t} \langle q(-k)\bar{\psi}(k, s) \rangle \left( P_+\delta_{s, s} \gamma_5 \delta_{t, t} + P_+\delta_{s, s+1} \gamma_5 \delta_{t, t} \right) \times \langle \psi(-p, t)\bar{\tau}(p) \rangle. \tag{III.9}$$

It is easily shown that this is suppressed by $w_0^{N/2}$ in $N \to \infty$ limit because of the damping factor in (III.4) and (III.5).

Now we calculate the one loop corrections to each terms in the Ward-Takahashi identity (III.3). We set the external quark momentum and the quark mass to the physical scale, which is much smaller than the cut-off. We make an expansion in these variables and pick up the leading order terms which are relevant for renormalization. The next to leading terms are higher order errors in the lattice spacing.

The one loop correction to the quark propagator and the pseudo scalar density vertex is already evaluated [3][4]. The one loop level full quark propagator and the Green’s function with pseudo scalar density become

$$\langle q(-p)\bar{\tau}(p) \rangle_1^{\text{full}} = \frac{(1 - w_0^2) Z_w Z_2}{i\gamma + (1 - w_0^2) Z_w Z^{-1} m}. \tag{III.10}$$

$$\langle (\bar{\tau}\gamma_5 q)q(k)\bar{\tau}(p) \rangle_1^{\text{full}} = \frac{1}{i\gamma + (1 - w_0^2) Z_w Z^{-1} m} \left( 1 - w_0^2 \right) Z_w Z_{\text{lat}}^\gamma_5 \frac{(1 - w_0^2) Z_w Z_2}{i\gamma + (1 - w_0^2) Z_w Z^{-1} m}. \tag{III.11}$$
where $Z_2$, $Z_m$, $Z_P^{lat}$ and $Z_w$ are lattice renormalization factors for the quark wave function, the quark mass, the pseudoscalar density operator, and the overall factor $w_0$, respectively. ($\lambda^a$ is still omitted.) Their definitions are given in Ref. [5].

A one loop contribution to the Green’s function $\langle A_\mu^a(q)q(k)\overline{q}(p)\rangle$ is given by the diagrams in Figs. 2 and 3 with vertices $A_i \cdot \epsilon$ which come from the axial vector current. The one loop diagrams in Fig. 2 contribute to the current vertex correction,

$$\langle A_\mu^a(q)q(k)\overline{q}(p)\rangle^{\text{vertex}}_1 = \frac{1 - w_0^2}{i\not{k} + (1 - w_0^2)m} - \frac{T_A^{(1)}}{1 - w_0^2} \gamma_\mu \gamma_5 \frac{1 - w_0^2}{i\not{k} + (1 - w_0^2)m},$$

where $T_A^{(1)}$ is obtained by multiplying the vertex correction with the external propagator damping factor,

$$T_A^{(1)} = \gamma_\mu \gamma_5 = \left( w_0^{(N-s)} P_+ + w_0^{(s-1)} P_- \right) \left( \bar{\Gamma}_\mu^{(0)} + \bar{\Gamma}_\mu^{(2)} + \bar{\Gamma}_\mu^{(3)} \right) (s, t) \times \left( w_0^{(N-t)} P_+ + w_0^{(t-1)} P_- \right).$$

$\bar{\Gamma}^{(i)}$’s are contributions from each diagrams represented by the massless fermion propagator $S_F$, fermion-gluon vertex $V_{1\mu}$, gluon propagator $G_{\mu\nu}^{ab}$ and axial current vertex $A_\mu^{(i)}(l)$ given in the appendix,

$$\bar{\Gamma}_\mu^{(0)}(s, t) = \int_{-\pi}^{\pi} \frac{d^4l}{(2\pi)^4} \sum_{\nu p} V_{1\nu}^a(0, l) S_F(l)_{s, \nu} A_\mu^{(0)}(l) \epsilon(u, u') S_F(l)_{u', \nu} V_{1\rho}(-l, 0) G_{\nu\rho}^{ab}(l),$$

$$\bar{\Gamma}_\mu^{(1)}(s, t) = \int_{-\pi}^{\pi} \frac{d^4l}{(2\pi)^4} \sum_{\nu} V_{1\nu}^a(0, l) S_F(l)_{s, \nu} A_\mu^{(1)ab}(l) G_{\nu\mu}^{ab}(l) \epsilon(u, t),$$

$$\bar{\Gamma}_\mu^{(2)}(s, t) = \int_{-\pi}^{\pi} \frac{d^4l}{(2\pi)^4} \sum_{\nu} \epsilon(s, u) A_\mu^{(2)ab}(l) S_F(l)_{u, \nu} V_{1\nu}(-l, 0) G_{\mu\nu}^{ab}(l),$$

$$\bar{\Gamma}_\mu^{(3)}(s, t) = \int_{-\pi}^{\pi} \frac{d^4l}{(2\pi)^4} A_\mu^{(3)ab} G_{\mu\nu}^{ab}(l) \epsilon(s, t).$$

Here we have set the external quark momentum $k_{\mu}$, $p_{\mu}$ and the quark mass $m$ to be zero in the loop because the terms dependent on them are $O(a)$ and do not affect the renormalization.

We can easily see that $\epsilon(s, t)$ is multiplied directly by the damping factors in (III.13) for $\bar{\Gamma}_\mu^{(1)}$, $\bar{\Gamma}_\mu^{(2)}$ and $\bar{\Gamma}_\mu^{(3)}$ and we can use the formula (III.7). For $\bar{\Gamma}_\mu^{(0)}$ we have a relation for massless fermion propagator,

$$S_F(l)_{t, s} \left( w_0^{(N-s)} P_+ + w_0^{(s-1)} P_- \right) = \left[ P_+ \left(a_1 w_0^{N-t} + a_2 e^{-\alpha(N-t)} \right) + P_- \left(b_1 w_0^{N-t} + b_2 e^{-\alpha(N-t)} \right) \right] (-i\gamma_\mu \sin \theta) \left[ P_+ \left(c_1 w_0^{N-t} + c_2 e^{-\alpha(N-t)} \right) + P_- \left(d_1 w_0^{N-t} + d_2 e^{-\alpha(N-t)} \right) \right],$$

where $\alpha$ and the coefficients of each term are explicitly given in the appendix. By neglecting $e^{-\alpha N/2}$ order terms together with $w_0^{N/2}$ we can make use of the formula similar to (III.7). Finally the effective vertex becomes
\[
T_3^{(1)}(s,t) = \left( w_0^{(N-s)} P_+ + w_0^{(s-1)} P_- \right) \left( \Gamma_\mu^{(0)} + \Gamma_\mu^{(1)} + \Gamma_\mu^{(2)} + \Gamma_\mu^{(3)} \right) (s,t) \gamma_5 \\
= \left( w_0^{(N-t)} P_- + w_0^{(t-1)} P_+ \right)
\]

with

\[
\Gamma_\mu^{(0)}(s,t) = \int_{-\pi}^\pi d^4l \int_{-\pi}^\pi \frac{d^4l}{(2\pi)^4} \sum_{\nu} V_{1\nu}(0,l) S_F(l)_{s,u} A_\mu^{(0)}(l) S_F(l)_{u,t} V_{1\nu}(-l,0) G_{\nu\nu}(l),
\]

\[
\Gamma_\mu^{(1)}(s,t) = \int_{-\pi}^\pi d^4l \int_{-\pi}^\pi \frac{d^4l}{(2\pi)^4} \sum_{\nu} V_{1\nu}(0,l) S_F(l)_{s,t} A_\mu^{(1)}(l) G_{\nu\nu}(l),
\]

\[
\Gamma_\mu^{(2)}(s,t) = \int_{-\pi}^\pi d^4l \int_{-\pi}^\pi \frac{d^4l}{(2\pi)^4} \sum_{\nu} A_\mu^{(2)}(l) S_F(l)_{s,t} V_{1\nu}(-l,0) G_{\nu\nu}(l),
\]

\[
\Gamma_\mu^{(3)}(s,t) = \int_{-\pi}^\pi d^4l \int_{-\pi}^\pi \frac{d^4l}{(2\pi)^4} A_\mu^{(3)} G_{\nu\nu}(l) \delta(s,t).
\]

Here we have used the fact that \( \Gamma^{(i)}_\mu \) are proportional to \( \gamma_\mu \) and therefore anticommute with \( \gamma_5 \).

The half-circle and tadpole diagrams in Fig. 3 contribute to the wave function renormalization,

\[
\langle A_\mu^a(q)q(k)\bar{q}(p) \rangle_1^{\text{wave}} = \frac{1 - w_0^2}{i \not{q} + (1 - w_0^2)m} \left( w_0^{(N-s)} P_+ + w_0^{(s-1)} P_- \right) \gamma_\mu \epsilon(s,t) S_F(p)_{t,u} \Sigma(p, m)_{t',u} \\
\times \left[ \left( w_0^{(N-u)} P_- + w_0^{(u-1)} P_+ \right) - \left( w_0^{(u-1)} P_- + w_0^{(N-u)} P_+ \right) \frac{w_0}{1 - w_0^2 i \not{q}} \right] \frac{1 - w_0^2}{i \not{q} + (1 - w_0^2)m} \\
+ \frac{1 - w_0^2}{i \not{q} + (1 - w_0^2)m} \left[ \left( w_0^{(N-s)} P_+ + w_0^{(s-1)} P_- \right) - \frac{w_0}{1 - w_0^2} i \not{q} \left( w_0^{(s-1)} P_+ + w_0^{(N-s)} P_- \right) \right] \\
\times \Sigma(k, m)_{s,t} S_F(k)_{t',u} \gamma_\mu \epsilon(t', u) \left( w_0^{(N-u)} P_- + w_0^{(u-1)} P_+ \right) \frac{1 - w_0^2}{i \not{q} + (1 - w_0^2)m}.
\]

We set the external quark momentum and mass to physical scale and extract the leading order terms in these variables. The fermion self-energy is the same as that is given in our previous paper [1, 2]. The difference is the existence of the fermion propagator in the internal line, which is expanded as follows together with the external line damping factor,

\[
\left( w_0^{(N-s)} P_- + w_0^{(s-1)} P_+ \right) S_F(p)_{s,t} \\
\to \frac{1}{i \not{q} + (1 - w_0^2)m} \left[ \left( w_0^{N-t} P_+ + w_0^{t-1} P_- \right) + mw_0 \left( w_0^{t-1} P_+ + w_0^{N-t} P_- \right) \right],
\]

\[
S_F(p)_{t,s} \left( w_0^{(N-s)} P_+ + w_0^{(s-1)} P_- \right) \\
\to \left[ \left( w_0^{t-1} P_+ + w_0^{N-t} P_- \right) + mw_0 \left( w_0^{N-t} P_+ + w_0^{t-1} P_- \right) \right] \frac{1}{i \not{q} + (1 - w_0^2)m}.
\]
With this expansion we have

\[
\langle A_\mu^0(q)q(k)\overline{\psi}(p)\rangle^\text{wave}_1 = \frac{1}{ik^\mu + (1 - w_0^2)m} \gamma_\mu \gamma_5 (1 - w_0^2) \Sigma_v(p, m) \frac{1}{i\not\phi + (1 - w_0^2)m} \gamma_\mu \gamma_5 (1 - w_0^2) \not\phi + (1 - w_0^2)m.
\]

(III.28)

where \( \Sigma_v \) is defined by

\[
\Sigma_v(p, m) = \left[ \left( w_0^{N-s} P_+ + w_0^{s-1} P_- \right) + m w_0 \left( w_0^{s-1} P_+ + w_0^{N-s} P_- \right) \right] \Sigma(p, m)_{s,t} \times \left[ \left( w_0^{N-t} P_- + w_0^{t-1} P_+ \right) - \left( w_0^{t-1} P_+ + w_0^{N-t} P_+ \right) \frac{w_0}{1 - w_0^2} \not\phi \right].
\]

(III.29)

The quark self-energy \( \Sigma_{st} \) is expanded in terms of \( p_\mu \) and \( m \),

\[
\Sigma(p, m)_{st} = \Sigma(0)_{st} + \frac{\partial \Sigma(0)_{st}}{\partial p_\mu} p_\mu + \frac{\partial \Sigma(0)_{st}}{\partial m} m + O(p^2, m^2, pm),
\]

(III.30)

and \( \Sigma_v \) becomes

\[
\Sigma_v(p, m) = \frac{1}{1 - w_0^2} i\not\phi \left( Z_2^{(1)} - \frac{1}{2} Z_w^{(1)} \right) + m \left( Z_m^{(1)} + \frac{1}{2} Z_w^{(1)} \right),
\]

(III.31)

where

\[
\frac{1}{1 - w_0^2} Z_2^{(1)} = \left( w_0^{N-s} P_+ + w_0^{s-1} P_- \right) \frac{1}{4} \text{tr} \left( \gamma_\mu \frac{\partial \Sigma(0)_{st}}{\partial p_\mu} \right) \left( w_0^{N-t} P_- + w_0^{t-1} P_+ \right),
\]

(III.32)

\[
Z_m^{(1)} = \left( w_0^{N-s} P_+ + w_0^{s-1} P_- \right) \frac{\partial \Sigma(0)_{st}}{\partial m} \left( w_0^{N-t} P_- + w_0^{t-1} P_+ \right),
\]

(III.33)

\[
Z_w^{(1)} = 2 w_0 \left( w_0^{N-s} P_+ + w_0^{s-1} P_- \right) \Sigma(0)_{st} \left( w_0^{t-1} P_+ + w_0^{N-t} P_- \right).
\]

(III.34)

Summing up all the contribution we have one loop level full Green’s function,

\[
\langle A_\mu(n)q(y)\overline{\psi}(z)\rangle^\text{full}_1 = \langle A_\mu(n)q(y)\overline{\psi}(z)\rangle^0_0 + \langle A_\mu(n)q(y)\overline{\psi}(z)\rangle^\text{vertex}_1 + \langle A_\mu(n)q(y)\overline{\psi}(z)\rangle^\text{wave}_1
\]

\[
= \frac{(1 - w_0^2)(Z_mA Z_2^{(1)})^{\frac{1}{2}}}{Z_A + Z_2^{(1)} + T_A^{(1)}} \frac{Z_A}{Z_2^{(1)} + Z_2^{(1)}} \frac{(1 - w_0^2)(Z_w Z_2^{(1)})^{\frac{1}{2}}}{Z_w + Z_m^{(1)} + Z_2^{(1)}} \frac{Z_m^{(1)} - Z_m^{(1)} + Z_2^{(1)}}{1 - w_0^2 Z_m^{(1)} m},
\]

(III.35)

where

\[
Z_A = 1 + Z_2^{(1)} + T_A^{(1)},
\]

(III.36)

\[
Z_2 = 1 + Z_2^{(1)},
\]

(III.37)

\[
Z_w = 1 - Z_2^{(1)},
\]

(III.38)

\[
Z_m^{(1)} = 1 - Z_m^{(1)} + Z_2^{(1)}.
\]

(III.39)

In the following we will show the relation
\[
\frac{\partial \Sigma(0)_{st}}{\partial ip_{\mu}} = -\left( \Gamma_{\mu}^{(0)} + \Gamma_{\mu}^{(1)} + \Gamma_{\mu}^{(2)} + \Gamma_{\mu}^{(3)} \right) (s, t), \tag{III.40}
\]
and consequently \( Z_2^{(1)} = -T_A^{(1)} \). We start by writing \( \Sigma(p, m) \) explicitly

\[
\Sigma(p, m)_{st} = \int_{-\pi}^{\pi} \frac{d^4 l}{(2\pi)^4} \sum_{\nu \rho} V_{1\nu}^{a}(p, l + p) S_F(p + l)_{s,t} V_{1\rho}^{b}(-l - p, p) G_{\nu\rho}^{ab}(l) + \int_{-\pi}^{\pi} \frac{d^4 l}{(2\pi)^4} \gamma^{ab}_{2\nu\nu}(-p, p) G_{\nu\nu}^{ab}(l) \delta_{s,t}. \tag{III.41}
\]

A derivative with \( p_{\mu} \) is given by

\[
\frac{\partial \Sigma(0)_{st}}{\partial ip_{\mu}} = \left( I_{\mu}^{(0)} + I_{\mu}^{(1)} + I_{\mu}^{(2)} + I_{\mu}^{(3)} \right) (s, t), \tag{III.42}
\]

where

\[
I_{\mu}^{(0)} = \int_{-\pi}^{\pi} \frac{d^4 l}{(2\pi)^4} \sum_{\nu \rho} V_{1\nu}^{a}(0, l) \frac{\partial S_F(p + l)_{s,t}}{\partial ip_{\mu}} \bigg|_{p_{\mu} = 0} V_{1\rho}^{b}(-l, 0) G_{\nu\rho}^{ab}(l) = -\Gamma_{\mu}^{(0)} (s, t), \tag{III.43}
\]

\[
I_{\mu}^{(1)} = \int_{-\pi}^{\pi} \frac{d^4 l}{(2\pi)^4} \sum_{\nu \rho} \frac{\partial V_{1\nu}^{a}(-p, l + p)}{\partial ip_{\mu}} \bigg|_{p_{\mu} = 0} S_F(l)_{s,t} V_{1\rho}^{b}(-l, 0) G_{\nu\rho}^{ab}(l) = -\Gamma_{\mu}^{(1)} (s, t), \tag{III.44}
\]

\[
I_{\mu}^{(2)} = \int_{-\pi}^{\pi} \frac{d^4 l}{(2\pi)^4} \sum_{\nu \rho} V_{1\nu}^{a}(0, l) S_F(l)_{s,t} \frac{\partial V_{1\rho}^{b}(-l - p, p)}{\partial ip_{\mu}} \bigg|_{p_{\mu} = 0} G_{\nu\rho}^{ab}(l) = -\Gamma_{\mu}^{(2)} (s, t), \tag{III.45}
\]

\[
I_{\mu}^{(3)} = \int_{-\pi}^{\pi} \frac{d^4 l}{(2\pi)^4} \frac{\partial V_{2\nu\nu}^{ab}(-p, p)}{\partial ip_{\mu}} \bigg|_{p_{\mu} = 0} G_{\nu\nu}^{ab}(l) \delta_{s,t} = -\Gamma_{\mu}^{(3)} (s, t). \tag{III.46}
\]

For the second equality in each equation we use the relation

\[
\left. \frac{\partial S_F(p + l)_{s,t}}{\partial ip_{\mu}} \right|_{p_{\mu} = 0} = -S_F(l)_{su} A_{\mu}(0) S_F(l)_{ut}, \tag{III.47}
\]

\[
\left. \frac{\partial V_{1\nu}^{a}(-p, l + p)_{s,t}}{\partial ip_{\mu}} \right|_{p_{\mu} = 0} = -A_{\mu}^{(1)a}(l) \delta_{\nu\mu} \delta_{s,t}, \tag{III.48}
\]

\[
\left. \frac{\partial V_{1\rho}^{b}(-l - p, p)_{s,t}}{\partial ip_{\mu}} \right|_{p_{\mu} = 0} = -A_{\mu}^{(2)b}(l) \delta_{\rho\mu} \delta_{s,t}, \tag{III.49}
\]

\[
\left. \frac{\partial V_{2\nu\nu}^{ab}(-p, p)_{s,t}}{\partial ip_{\mu}} \right|_{p_{\mu} = 0} = -\frac{1}{2} \left( A_{\mu}^{(2)ab} + A_{\mu}^{(2)ba} \right) \delta_{\mu\nu} \delta_{s,t}. \tag{III.50}
\]

The Eq. \( \text{(III.47)} \) is given by differentiating the definition of the propagator

\[
\left[ i\gamma_{\mu} \sin p_{\mu} + W^+(p) P_+ + W^-(p) P_- \right] S_F(p) = 1 \tag{III.51}
\]
and multiplying \( S_F(p) \) from the left.

Equalities in \( \text{(III.43)} \) to \( \text{(III.46)} \) prove eq. \( \text{(III.40)} \), which leads to \( Z_A = 1 \). This means that the axial vector current does not receive renormalization.
The one loop correction to the explicit breaking term is given by the first diagram in Fig. 2 and those in Fig. 3 whose current vertex is now replaced by

\[ X^a_{st} = \left( P_+ \delta_{s,N/2} \delta_{t,N/2+1} - P_- \delta_{s,N/2+1} \delta_{t,N/2} \right) \lambda^a. \] (III.52)

In the diagram of Fig. 3 the vertex \( X^a \) is directly multiplied by the damping factor in external quark line and gives the suppression factor \( w_0^{N/2} \). A contributions form the remaining part of the diagram is \( O(1) \) at most and the total contribution, which is a product of the two, vanish in \( N \to \infty \) limit. For the first diagram in Fig. 2 the damping factor appears in the contracted fermion propagator, whose explicit form is given in the appendix. Since the vertex \( X^a \) is localized around \( s, t \simeq N/2 \), the damping factor produces suppressions such as \( w_0^{N/2} \) or \( e^{-\alpha N/2} \), and therefore the contribution from Fig. 2 again vanishes in the large \( N \) limit.

Consequently the renormalized axial Ward-Takahashi identity is satisfied without fine tuning,

\[
0 = i(-k_\mu + p_\mu) \frac{1}{ik^\mu + (1 - w_0^2)Z_w Z_m^{-1}m} \lambda^a Z_A \gamma_\mu \gamma_5 \frac{(1 - w_0^2)Z_w Z_2}{i\gamma_\mu + (1 - w_0^2)Z_w Z_m^{-1}m} \\
- 2 \frac{1}{ik^\mu + (1 - w_0^2)Z_w Z_m^{-1}m} \lambda^a \left( 1 - w_0^2 \right) Z_w Z_m^{-1}m \gamma_5 \frac{(1 - w_0^2)Z_w Z_2}{i\gamma_\mu + (1 - w_0^2)Z_w Z_m^{-1}m} \\
+ \lambda^a \gamma_5 \frac{(1 - w_0^2)Z_w Z_2}{i\gamma_\mu + (1 - w_0^2)Z_w Z_m^{-1}m} + \frac{(1 - w_0^2)Z_w Z_2}{i\gamma_\mu + (1 - w_0^2)Z_w Z_m^{-1}m} \gamma_5 \lambda^a, \tag{III.53}
\]

together with conditions such that \( Z_A = 1 \) and \( Z_P = Z_m^{-1} \).

**IV. CONCLUSION**

In this paper we have calculated the one loop correction to the axial Ward-Takahashi identity and a renormalization factor for the “conserved” axial vector current in domain-wall QCD. Starting from the Green’s function with the operators constructed from the physical quark field, we find that the axial Ward-Takahashi identity holds exactly without fine tuning at this order of the perturbation theory, and consequently the axial vector current defined by Furman and Shamir receives no renormalization, \( Z_A = 1 \). As discussed in Ref. [2], one may expect that this property holds in all orders of the perturbation theory as long as zero modes exist. So one should use this almost conserved axial current to extract the pion decay constant in numerical simulations.

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APPENDIX A. FEYNMAN RULES

The fermion propagator is defined by

\[ S_F(p)_{st} = \left(-i\gamma_\mu \sin p_\mu + W^-\right)_{su} G_R(u, t) P_+ + \left(-i\gamma_\mu \sin p_\mu + W^+\right)_{su} G_L(u, t) P_- \]  

(IV.54)

with

\[
W^+_{s, t} = \begin{pmatrix}
-W & 1 & \cdots & 1 \\
-1 & W & \cdots & \cdots \\
\vdots & \vdots & \ddots & \vdots \\
m & \cdots & \cdots & W
\end{pmatrix},
\]

\[
W^-_{s, t} = \begin{pmatrix}
-W & -W & \cdots & m \\
1 & -W & \cdots & \cdots \\
\vdots & \vdots & \ddots & \vdots \\
1 & \cdots & \cdots & -W
\end{pmatrix},
\]

(IV.55)

\[
W = 1 - M - r \sum_\mu (1 - \cos p_\mu),
\]

(IV.56)

\[
G_R(s, t) = \frac{A}{F} \left[ -(1 - m^2) \left( 1 - We^{-\alpha} \right) e^{\alpha(-2N+s+t)} - (1 - m^2) \left( 1 - We^{-\alpha} \right) e^{-\alpha(s+t)} - 2W \sinh(\alpha) m \left( e^{\alpha(-N+s-t)} + e^{\alpha(-N+s+t)} \right) \right] + A e^{-\alpha|s-t|},
\]

(IV.57)

\[
G_L(s, t) = \frac{A}{F} \left[ -(1 - m^2) \left( 1 - We^{\alpha} \right) e^{\alpha(-2N+s+t-2)} - (1 - m^2) \left( 1 - We^{\alpha} \right) e^{\alpha(-s-t+2)} - 2W \sinh(\alpha) m \left( e^{\alpha(-N+s-t)} + e^{\alpha(-N-s+t)} \right) \right] + A e^{-\alpha|s-t|},
\]

(IV.58)

\[
cosh(\alpha) = \frac{1 + W^2 + \sum_\mu \sin^2 p_\mu}{2W},
\]

(IV.59)

\[
A = \frac{1}{2W \sinh(\alpha)},
\]

(IV.60)

\[
F = 1 - e^\alpha W - m^2 \left( 1 - We^{-\alpha} \right).
\]

(IV.61)

The physical quark propagator is given by

\[
\langle q(-p)\bar{q}(p) \rangle = \frac{-i\gamma_\mu \sin p_\mu + \left( 1 - W e^{-\alpha} \right) m}{-\left( 1 - e^\alpha W \right) + m^2 \left( 1 - W e^{-\alpha} \right)}.
\]

(IV.62)

The gluon propagator can be written as

\[
G^{ab}_{\mu\nu}(p) = \frac{1}{4 \sin^2 p/2} \left[ \delta_{\mu\nu} - (1 - \alpha) \frac{4 \sin p_\mu/2 \sin p_\nu/2}{4 \sin^2 p/2} \right] \delta_{ab},
\]

(IV.63)

where \( \sin^2 p/2 = \sum_\mu \sin^2 p_\mu/2 \). The fermion-gluon interaction vertices which are relevant for the one loop calculation are given by

\[
V_{1\mu}^a(k, p)_{st} = -i g T^a \left\{ \gamma_\mu \cos \frac{1}{2}(-k_\mu + p_\mu) - i r \sin \frac{1}{2}(-k_\mu + p_\mu) \right\} \delta_{st},
\]

(IV.64)

\[
V_{2\mu
u}^{ab}(k, p)_{st} = \frac{1}{2} g^2 \frac{1}{2} \left\{ T^a, T^b \right\} \left\{ i \gamma_\mu \sin \frac{1}{2}(-k_\mu + p_\mu) - r \cos \frac{1}{2}(-k_\mu + p_\mu) \right\} \delta_{\mu\nu} \delta_{st}.
\]

(IV.65)

The current vertices used in Eqs. (III.14) to (III.17) are given by expanding the axial vector current in terms of the gauge field. Only three kinds of them are relevant for one loop calculation,
The explicit form of the contracted propagator with the damping factor is given by

\[ (w_0^{(N-s)} P_- + w_0^{(s-1)} P_+) S_F(p, t) \]

\[ = -i\gamma \sin \eta \left( w_0^{(N-s)} G_R(s, t) P_+ + w_0^{(s-1)} G_L(s, t) P_- \right) + \left( (w_0 - W(p)) w_0^{(s-1)} G_R(s, t) + mG_r(N, t) \right) P_+ \]

\[ + \left( (w_0 - W(p)) w_0^{N-s} G_L(s, t) + mG_L(1, t) \right) P_- , \]

where

\[ w_0^{s-1} G_R(s, t) = \frac{A}{F} \left[ (1 - m^2) (1 - We^\alpha) \frac{1}{1 - w_0 e^{-\alpha}} e^{-\alpha(t-1)} \right. \]

\[ - 2W \sinh(\alpha) me^{-\alpha} \frac{1}{1 - w_0 e^{-\alpha}} e^{-\alpha(N-t)} \]

\[ + \frac{A}{1 + w_0^2 - 2w_0 \cosh \alpha} \]

\[ w_0^{N-s} G_R(s, t) = \frac{A}{F} \left[ (1 - m^2) (1 - We^{-\alpha}) \frac{1}{1 - w_0 e^{-\alpha}} e^{-\alpha(N-t)} \right. \]

\[ - 2W \sinh(\alpha) me^{-\alpha} \frac{1}{1 - w_0 e^{-\alpha}} e^{-\alpha(t-1)} \]

\[ + \frac{e^{-\alpha(N-t)} (1 - w_0 e^{-\alpha}) - 2w_0^{(t-1)} w_0 \sinh \alpha}{1 + w_0^2 - 2w_0 \cosh \alpha} , \]

\[ w_0^{s-1} G_L(s, t) = \frac{A}{F} \left[ (1 - m^2) (1 - We^{-\alpha}) \frac{1}{1 - w_0 e^{-\alpha}} e^{-\alpha(t-1)} \right. \]

\[ - 2W \sinh(\alpha) me^{-\alpha} \frac{1}{1 - w_0 e^{-\alpha}} e^{-\alpha(N-t)} \]

\[ + \frac{A}{1 + w_0^2 - 2w_0 \cosh \alpha} \]

\[ w_0^{N-s} G_L(s, t) = \frac{A}{F} \left[ (1 - m^2) (1 - We^\alpha) \frac{1}{1 - w_0 e^{-\alpha}} e^{-\alpha(N-t)} \right. \]

\[ - 2W \sinh(\alpha) me^\alpha \frac{1}{1 - w_0 e^{-\alpha}} e^{-\alpha(t-1)} \]

\[ w_0^{N-s} G_L(s, t) = \frac{A}{F} \left[ (1 - m^2) (1 - We^\alpha) \frac{1}{1 - w_0 e^{-\alpha}} e^{-\alpha(N-t)} \right. \]

\[ - 2W \sinh(\alpha) me^\alpha \frac{1}{1 - w_0 e^{-\alpha}} e^{-\alpha(t-1)} \]
\[ G_R(N,t) = \frac{A}{F} \left[ -(1 - m^2) \left( 1 - W e^{-\alpha} \right) e^{-\alpha(N-t)} ight. \\
+ \left. 2W \sinh(\alpha)e^{-\alpha(t-1)} \right] + Ae^{-\alpha(N-t)}, \] (IV.75)

\[ G_L(1,t) = \frac{A}{F} \left[ -(1 - m^2) \left( 1 - W e^{-\alpha} \right) e^{-\alpha(t-1)} ight. \\
+ \left. 2W \sinh(\alpha)e^{-\alpha(N-t)} \right] + Ae^{-\alpha(t-1)}. \] (IV.76)
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FIG. 1. Tree level Green’s function with axial current vertex.

FIG. 2. One loop diagrams which contribute to the operator vertex.
FIG. 3. One loop diagrams which contribute to the quark wave function.