CUTOFF PHENOMENON FOR CYCLIC DYNAMICS ON HYPERCUBE

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Abstract. The cutoff phenomena for Markovian dynamics have been observed and rigorously verified for a multitude of models, particularly for Glauber-type dynamics on spin systems. However, prior studies have barely considered irreversible chains. In this work, the cutoff phenomenon of certain cyclic dynamics are studied on the hypercube $\Sigma_n = Q^{V_n}$, where $Q = \{1, 2, 3\}$ and $V_n = \{1, \ldots, n\}$. The main feature of these dynamics is the fact that they are represented by an irreversible Markov chain. Based on the couplings modified from the previous study of the cutoff phenomenon for the Curie-Weiss-Potts model, a comprehensive proof is presented.

1. Introduction

This work considers the mixing behavior of irreversible dynamics on the hypercube $\Sigma_n = Q^{V_n}$, where $Q = \{1, 2, 3\}$ and $V_n = \{1, \ldots, n\}$. We consider the hypercube as the structure that assigns the color in $Q$ on each vertex in $V_n$. One of the widely known Markov chains is the discrete time Glauber dynamics for the uniform measure on $\Sigma_n$. At each time step, the vertex $v \in V_n$ is uniformly chosen. Then, we reassign the color of vertex $v$ uniformly on $Q$. The mixing of these dynamics is fully understood, and the sharp convergence exhibited is defined as the cutoff phenomenon.

In this study, the result is extended to the discrete time cyclic dynamics $(\sigma^n_t)_{t=0}^\infty$ iterated by the following rule. At time $t + 1$, the vertex $v \in V_n$ is uniformly chosen. Then, $\sigma^{n+1}_t$ is set as

$$\sigma^{n+1}_t(w) = \begin{cases} \sigma^n_t(w) \text{ w.p. 1 if } w \neq v \\ \sigma^n_t(w) \text{ w.p. } 1 - p \text{ and } \sigma^n_t(w) + 1 \text{ w.p. } p \text{ if } w = v, \end{cases}$$

where $0 < p < 1$. Here, $\sigma^n_t(w)$ is denoted as the color of vertex $w$ on $\sigma^n_t$ and w.p. is an abbreviation of “with probability.” The color of each vertex is evaluated based on modular arithmetic modulo 3. The cutoff phenomenon described below is proved.

The descriptions of the cutoff phenomenon are based on [7]. Let the total variance distance between the two probability distributions $\mu$ and $\nu$ on discrete state space $X$ be defined as

$$\|\mu - \nu\|_{TV} = \max_{A \subseteq X} |\mu(A) - \nu(A)|.$$

Then, consider the Markov chain $(X_t)$ on state space $X$ with the transition matrix $P$ and stationary distribution $\pi$. The maximal distance $d(t)$ of the Markov chain $(X_t)$ is defined as

$$d(t) = \max_{x \in X} \|P^t(x, \cdot) - \pi\|_{TV},$$

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while the $\epsilon$-mixing time is defined as

$$t_{\text{mix}}(\epsilon) = \min\{t : d(t) \leq \epsilon\}.$$ 

The mixing time $t_{\text{mix}}$ is denoted as $t_{\text{mix}}(\frac{1}{4})$ by convention.

For all $\epsilon \in (0, 1)$, suppose that the sequence of Markov chains $\{(X^n_t)\} = (X^1_t), (X^2_t), \ldots$ satisfies

$$\lim_{n \to \infty} \frac{t_{\text{mix}}^{(n)}(\epsilon)}{t_{\text{mix}}^{(n)}(1 - \epsilon)} = 1,$$

where $t_{\text{mix}}^{(n)}(\epsilon)$ is the $\epsilon$-mixing time of the chain $(X^n_t)$. Denote the mixing time of the $n$-th chain as $t_{\text{mix}}^{(n)} = t_{\text{mix}}^{(n)}(\frac{1}{4})$, and the maximal distance as $d(t)$. Then, these Markov chains shows a sharp decrease in the total variance distance from 1 to 0 close to the mixing time. It is said that this sequence exhibits the cutoff phenomenon. Further, it is said to have a window of size $O(w_n)$ if

$$\lim_{n \to \infty} \frac{w_n}{t_{\text{mix}}^{(n)}} = 0,$$

$$\lim_{\alpha \to -\infty} \liminf_{n \to \infty} d(t_{\text{mix}}^{(n)} + \alpha w_n) = 1 \quad \text{and} \quad \lim_{\alpha \to \infty} \limsup_{n \to \infty} d(t_{\text{mix}}^{(n)} + \alpha w_n) = 0.$$

The cutoff phenomenon was first observed in card shuffling, as demonstrated in [1]. Since then, the cutoff phenomena for Markovian dynamics have been observed and rigorously verified with a multitude of models. In recent times, there have been several breakthroughs in the verification of cutoff phenomena for Glauber-type dynamics on spin systems. For example, the cutoff phenomenon for the Glauber dynamics on the Curie-Weiss model, which corresponds to the mean-field Ising model, is proven in [6] for the high temperature regime. This work has been further generalized to [3], where Glauber dynamics for Curie-Weiss-Potts model have been considered. These two outcomes were considered on the mean-field model defined on the complete graph, where geometry is irrelevant.

On the other hand, the cutoff phenomenon for the spin system on the lattices were more complicated. The first development was achieved for the Ising model on the lattice in [9], and in [8], it was extended to a general spin system in the high temperature regime. In [10], a novel method called “information percolation” was developed, and the cutoff for the Ising model with a precise window size was obtained. This information percolation method has also been successfully applied to Swendsen-Wang dynamics for the Potts model and to Glauber dynamics for the random-cluster model in [11] and [4], respectively.

In the present work, the uniform measure, which corresponds to infinite temperature spin systems, is considered. From this perspective, the proposed model is simpler than existing models in which finite temperature has been considered. However, this model has a critical difference in that the dynamics being considered are irreversible. We emphasize here that the cutoff phenomenon for the irreversible chains are known only for few models, e.g., non-backtracking random walks on sparse random graphs [2].

1.1. Main Result. Theorem 1.1 presents the cutoff phenomenon of the cyclic dynamics considered herein and the main result of the current article.
Theorem 1.1. The cyclic dynamics defined on $\Sigma_n$ with probability $0 < p < 1$ exhibit cutoff at mixing time

$$t(n) = \frac{1}{3p} n \log n$$

with a window of size $O(n)$.

As the theorem can be similarly proved for all $0 < p < 1$, the proof is presented for $p = \frac{1}{2}$. In Section 2, the notations are set and the contractions of the proportion chain are provided. The proof of the lower bound of the cutoff is then presented. Section 3 analyzes the coalescence of the proportion and basket chains. Following this, the upper bound of the cutoff is proved.

The dynamics considered in this article is a Glauber-type (but asymmetric) dynamics on Curie-Weiss-Potts model with three spins at infinite temperature. The cutoff for usual symmetric Glauber dynamics on Curie-Weiss-Potts model has been verified for all the high temperature regime in [3]. For the asymmetric dynamics, the metastability for all the low temperature regime has been thoroughly analyzed in [3] for the three spin case. It is widely believed that the asymmetric dynamics also exhibits cutoff phenomenon at all the entire high temperature regime, but the proof is missing at this moment; the current article investigated the special case of the last problem.

The structure of the proof is similar to the case of the cutoff phenomenon of the Glauber dynamics for the Curie-Weiss-Potts model in high temperature regime presented in [3]. The convergence to the stationary distribution is obtained by successively coalescing the proportion chains and the basket chains with the coupling methods. The major difference with the previous method is the construction of the appropriate couplings to deal with the asymmetric nature of the irreversible dynamics. They are based on the couplings introduced in [3], but more sophisticated constructions are needed in cases where symmetry is starting to break.

The proof cannot be generalized to the cases where the number of colors are larger than three. One of the obstructions is the proof of the Proposition 2.1 which presents the convergence of the proportion chain to stationary distribution in $\ell^2$-norm. The computation is simplified only when the number of colors are three.

We remark that the Glauber dynamics for the uniform measure that corresponds to the current model exhibits the cutoff phenomenon. It is proven in [3] that this reversible dynamics exhibits the cutoff at $\frac{1}{4} n \log n$ with a window of size $O(n)$. For that reversible case, the spectral analysis can be applied to obtain the upper bound (see [3, Chapter 12]). In particular, a direct relationship between the eigenvalues of the transition matrix and the bound of the total variation distance is crucially used. For our irreversible case, we are not able to use spectral analysis and the proof becomes more complex.

Note that when $p > \frac{2}{3}$, the mixing time of the cyclic dynamics considered in this article is smaller than the mixing time of the Glauber dynamics defined above. It shows that the irreversible chain can converge faster into uniform stationary distribution than reversible chain. This study is the part of an attempt to provide the theoretical background in applying the irreversible Markov chains to Markov chain Monte Carlo methods which is believed to be faster than the reversible one.
2. Lower Bound

This section presents the lower bound of the cutoff and the proof is given in Section 2.3. The proof is based on the analyses of the statistical properties of cyclic dynamics described in Section 2.2 and the features evaluated on a stationary distribution in Section 2.3. Prior to the proof, the notations are set, and the proportion chain used throughout this paper is defined.

2.1. Preliminaries. Denote the cyclic dynamics on $\Sigma_n$ as $(\sigma^n_t)_{t=0}^\infty$, and eliminate $n$ for simplicity. When the cyclic dynamics $\sigma_t$ begin at state $\sigma_0$, denote the probability measure as $P_{\sigma_0}$, and the expectation with respect to the probability measure as $E_{\sigma_0}$.

Then, consider the vector $s \in \mathbb{R}^3$ and let its $i$-th element as $s_i$. The $\ell^p$-norm of the vector $s$ is denoted as $\|s\|_p$. Denote the vector $(\bar{s}, \bar{s}, \bar{s}) \in \mathbb{R}^3$ as $\bar{s}$, and let $\tilde{s} = s - \bar{s}$. Consider the $3 \times 3$ matrix $Q$, and let $Q^{i,k}$ be the $(i,k)$ element of matrix $Q$. Let $Q_i$ be its $i$-th row. For $\rho > 0$, the subsets of $\mathbb{R}^3$ are denoted as
\[
S = \{x \in \mathbb{R}^3 : \|x\|_1 = 1\}, \quad S_n = S \cap \frac{1}{n} \mathbb{Z}^3, \\
S^p = \{s \in S : \|s\|_\infty < \rho\}, \quad S^p_n = S^p \cap \frac{1}{n} \mathbb{Z}^3, \\
S^{p+} = \{s \in S : s^k < \frac{1}{3} + \rho, 1 \leq k \leq 3\}, \quad S^{p+}_n = S^{p+} \cap \frac{1}{n} \mathbb{Z}^3.
\]

Now, the proportion chain $(S_t)_{t=0}^\infty$ of the cyclic dynamics $(\sigma_t)_{t=0}^\infty$ is defined as
\[
S_t = (S^1_t, S^2_t, S^3_t),
\]
where
\[
S^k_t = \frac{1}{n} \sum_{v \in V_n} 1_{\{\sigma(v) = k\}} \quad k = 1, 2, 3.
\]

Then, the proportion chain $(S_t)$ is also a Markov chain on state space $S_n$ with jump probability
\[
(S^1_{t+1}, S^2_{t+1}, S^3_{t+1}) = \begin{cases} 
(S^1_t, S^2_t, S^3_t) & \text{w.p. } \frac{1}{2} \\
(S^1_t - \frac{1}{n}, S^2_t + \frac{1}{n}, S^3_t) & \text{w.p. } \frac{1}{2} S^1_t \\
(S^1_t, S^2_t - \frac{1}{n}, S^3_t + \frac{1}{n}) & \text{w.p. } \frac{1}{2} S^2_t \\
(S^1_t + \frac{1}{n}, S^2_t, S^3_t - \frac{1}{n}) & \text{w.p. } \frac{1}{2} S^3_t.
\end{cases}
\]

This formulation is well-defined on $S_n$, because if $S^k_t = 0$ for any $i \in \{1, 2, 3\}$, then the probability of $S^k_t$ decreasing in the next step is zero.

2.2. Statistical Properties of the Chain. This section describes the statistical properties of the proportion chain used in the proof. In particular, the $\ell^2$-norm of $\delta_t$ and the variance of $S_t$ are analyzed.

**Proposition 2.1.** Proportion chain $(S_t)$ of the cyclic dynamics $(\sigma_t)$ has the following $\ell^2$-norm contraction that depends on $n$:
\[
E_{\sigma_0} \|\delta_t\|^2 = \left(1 - \frac{3}{2n}\right)^t \|\delta_0\|^2 + O\left(\frac{1}{n}\right).
\]
This shows the contraction on the expectation of $\ell^2$-norm of $\tilde{S}_t$. In Proposition 2.2, this result is used to evaluate the expectation of $\tilde{S}_t$ at the certain time. Next, the semi-synchronized coupling that contracts the norm between the two proportion chains is defined. It is similar to the synchronized coupling of $\tilde{S}$, but it has more comprehensive cases.

2.2.1. Semi-Synchronized Coupling. Consider the two cyclic dynamics $(\sigma_t)$ and $(\tilde{\sigma}_t)$ starting from $\sigma_0$, $\tilde{\sigma}_0$. Denote their proportion chains as $(S_t)$ and $(\tilde{S}_t)$. At time $t+1$, the semi-synchronized coupling for the case of $S_t^1 \geq \tilde{S}_t^1$, $S_t^2 \leq \tilde{S}_t^2$, $S_t^3 \leq \tilde{S}_t^3$ is defined as follows:

1. Choose the colors $(I_{t+1}, \tilde{I}_{t+1})$ based on the probability as stated below:

   $$(I_{t+1}, \tilde{I}_{t+1}) = \begin{cases} (1, 1) & \text{w.p. } \tilde{S}_t^1, \\ (2, 2) & \text{w.p. } S_t^2, \\ (3, 3) & \text{w.p. } S_t^3, \\ (1, 2) & \text{w.p. } \tilde{S}_t^2 - S_t^2, \\ (1, 3) & \text{w.p. } \tilde{S}_t^3 - S_t^3. \end{cases}$$

2. Choose the colors $(J_{t+1}, \tilde{J}_{t+1})$ depending on $(I_{t+1}, \tilde{I}_{t+1})$ based on the probability as stated below:

   - $(I_{t+1}, \tilde{I}_{t+1}) = (1, 1) \Rightarrow (J_{t+1}, \tilde{J}_{t+1})$ is $(1, 1)$ w.p. $\frac{1}{4}$, and is $(2, 2)$ w.p. $\frac{1}{2}$.
   - $(I_{t+1}, \tilde{I}_{t+1}) = (2, 2) \Rightarrow (J_{t+1}, \tilde{J}_{t+1})$ is $(2, 2)$ w.p. $\frac{1}{4}$, and is $(3, 3)$ w.p. $\frac{1}{2}$.
   - $(I_{t+1}, \tilde{I}_{t+1}) = (3, 3) \Rightarrow (J_{t+1}, \tilde{J}_{t+1})$ is $(3, 3)$ w.p. $\frac{1}{4}$, and is $(1, 1)$ w.p. $\frac{1}{2}$.
   - $(I_{t+1}, \tilde{I}_{t+1}) = (1, 2) \Rightarrow (J_{t+1}, \tilde{J}_{t+1})$ is $(1, 3)$ w.p. $\frac{1}{4}$, and is $(2, 2)$ w.p. $\frac{1}{2}$.
   - $(I_{t+1}, \tilde{I}_{t+1}) = (1, 3) \Rightarrow (J_{t+1}, \tilde{J}_{t+1})$ is $(1, 1)$ w.p. $\frac{1}{4}$, and is $(2, 3)$ w.p. $\frac{1}{2}$.

3. Choose a vertex that has the color $I_{t+1}$ in $\sigma_t$ uniformly. Then, change its color to $J_{t+1}$ in $\sigma_{t+1}$.
4. Choose a vertex that has the color $\tilde{I}_{t+1}$ in $\tilde{\sigma}_t$ uniformly. Then, change its color to $\tilde{J}_{t+1}$ in $\tilde{\sigma}_{t+1}$.

Semi-synchronized coupling for the other cases can be defined in a similar manner. Let $P^{SC}_{\sigma_0, \tilde{\sigma}_0}$ be the underlying probability measure of this coupling, and $E^{SC}_{\sigma_0, \tilde{\sigma}_0}$ be the expectation with respect to the underlying probability measure. This coupling is constructed to obtain the following $\ell^1$-contraction result.

**Proposition 2.2.** Consider the semi-synchronized coupling of two cyclic dynamics $(\sigma_t)$ and $(\tilde{\sigma}_t)$. Then, the following equation holds:

$$E^{SC}_{\sigma_0, \tilde{\sigma}_0} \| S_t - \tilde{S}_t \|_1 \leq \left( 1 - \frac{1}{2n} \right)^t \| S_0 - \tilde{S}_0 \|_1.$$ 

The following propositions bound the variance of the proportion chain value at time $t$ from the contraction of the norm between two proportion chains. The following theorem presents the relation between the variance and the contraction. Its only difference from Lemma 2.4] is the coefficient $c > 1$.

**Proposition 2.3.** Lemma 2.4] Consider the Markov chain $(Z_t)$ taking values in $\mathbb{R}^d$. When $Z_0 = z$, let $P_z$ and $E_z$ be its probability measure and expectation, respectively. If there exists
0 < \rho < 1 and \epsilon > 0 that satisfies \( \| \mathbb{E}_z[Z_t] - \mathbb{E}_z[Z_t] \|_2 \leq \epsilon \rho \| z - \bar{z} \|_2 \) for every pairs of starting point \((z, \bar{z})\), then

\[ v_t = \sup_{z_0} \text{Var}_{z_0}(Z_t) = \sup_{z_0} \mathbb{E}_{z_0} \| Z_t - \mathbb{E}_{z_0} Z_t \|_2^2 \]

satisfies

\[ v_t \leq \epsilon^2 v_1 \min \{ t, (1 - \rho^2)^{-1} \}. \]

**Proposition 2.4.** For the cyclic dynamics \((\sigma_t)\) starting from \(\sigma_0\) and all \(t \geq 0\),

\[ \text{Var}_{\sigma_0}(S_t) = O(n^{-1}). \]

### 2.3. Statistics of Stationary Distribution

This section presents the proof that \(\mu_n\) is the stationary distribution of the cyclic dynamics. Here, \(\mu_n\) is the uniform probability measure on state space \(\Sigma_n\), i.e.

\[ \mu_n(\sigma) = \frac{1}{\eta^n} \quad \forall \sigma \in \Sigma_n. \]

The underlying probability measure, expectation, and variance are denoted as \(P_{\mu_n}\), \(E_{\mu_n}\), and \(\text{Var}_{\mu_n}\), respectively. First, recall [7, Corollary 1.17], which describes the stationary distribution in the irreducible Markov chain.

**Proposition 2.5.** [7, Corollary 1.17] Let \(P\) be the transition matrix of the irreducible Markov chain. Then, there exists a unique stationary distribution of the chain.

Then, we introduce the product chain suggested in [3, Section 12.4]. For \(j = 1, \ldots, n\), consider the irreducible Markov chain \((Z^n_t)\) on state space \(\mathcal{X}_j\) with transition matrix \(P_j\). Let \(w = (w_1, \ldots, w_n)\) be a probability distribution of \(\{1, \ldots, n\}\), where \(0 < w_j < 1\). Define the product chain on state space \(\mathcal{X} = \mathcal{X}_1 \times \cdots \times \mathcal{X}_n\) with transition matrix \(P\) that has the transition probability as

\[ P(x, y) = \sum_{j=1}^{n} w_j P_j(x, y) \prod_{i: i \neq j} 1_{\{x_i = y_i\}} \]

for any two states \(x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in \mathcal{X}\). For the functions \(f^{(1)}, \ldots, f^{(n)}\), where \(f^{(j)}: \mathcal{X}_j \to \mathbb{R}\), define the product on \(\mathcal{X}\) as

\[ f^{(1)} \otimes f^{(2)} \otimes \cdots \otimes f^{(n)}(x_1, \ldots, x_n) = f^{(1)}(x_1) \cdots f^{(n)}(x_n). \]

**Proposition 2.6.** Consider the product chain of the Markov chains \((Z_1^n), \ldots, (Z_n^n)\) as above. For \(j = 1, \ldots, n\), let \(\pi^{(j)}\) be the stationary distribution of the chain \((Z_j^n)\). Then, \(\pi^{(1)} \otimes \pi^{(2)} \otimes \cdots \otimes \pi^{(n)}\) is the stationary distribution of the product chain.

**Proposition 2.7.** The probability measure \(\mu_n\) is a unique stationary distribution of the cyclic dynamics \((\sigma^n_t)\).

Now, define the function \(S: \Sigma_n \to S_n\) as \(S(\sigma) = (S^1(\sigma), S^2(\sigma), S^3(\sigma))\), where

\[ S^k(\sigma) = \frac{1}{n} \sum_{v \in V_n} 1_{\{\sigma(v) = k\}} \quad k = 1, 2, 3. \]

Consider the case where the element \(\sigma \in \Sigma_n\) is distributed according to the probability distribution \(\mu_n\). Because the element \(\sigma\) is uniformly distributed, \(n \cdot S^k(\sigma)\) can be interpreted as the sum of \(n\)
Proof. Note that sufficiently large values of properties in Section 2.3 and application of Chebyshev’s inequality, holds for all \( x > \frac{1}{\sqrt{3}} \) under the probability measure time. The proof of the proposition is based on [3, Section 4.1]. Denote \( t_2.4 \).

2.4. Proof of Lower Bound. This section presents the proof of the lower bound of the mixing time. The proof of the proposition is based on [3, Section 4.1]. Denote \( t(n) = \frac{2}{3} n \log n \) and \( t_\gamma(n) = \frac{2}{3} n \log n + \gamma n \). The main principle is to compare the probability of the event \( \| \hat{S}_{t_\gamma(n)} \|_2 < \frac{r}{\sqrt{n}} \) under the probability measure \( \mathbb{P}_{\sigma_0} \) and under \( \mu_n \).

**Proposition 2.8.** Consider the cyclic dynamics \( (\sigma_t) \) and a fixed constant \( \epsilon > 0 \). Then, for all sufficiently large values of \( n \), there exists a sufficiently large \( -\gamma > 0 \) that satisfies \( t_\gamma(n) \)

\( t_{\text{mix}}(1-\epsilon) \geq t_\gamma(n) \).

Proof. Note that
\[
(1 - \frac{1}{x})^{x-1} > e^{-1} > (1 - \frac{1}{x})^x
\]
holds for all \( x > 1 \). Set the constant \( 0 < \rho < \frac{2}{3} \). Then, choose a configuration \( \sigma_0 \in \Sigma_n \) such that it satisfies \( \rho < \| \hat{S}_0 \|_2 \). Then, for \( t \leq t_\gamma(n) \),
\[
\mathbb{E}_{\sigma_0} \| \hat{S}_t \|_2^2 = \left( 1 - \frac{3}{2n} \right) t \| \hat{S}_0 \|_2^2 + O \left( \frac{1}{n} \right) \geq \frac{1}{n} e^{-\gamma}
\]
holds for all sufficiently large \( n \) and \( -\gamma > 0 \) values depending on \( \rho \).

In addition, because \( \text{Var}_{\sigma_0} (\hat{S}_t) = O(n^{-1}) \) by Proposition 2.4, \( \text{Var}_{\sigma_0} (\hat{S}_t) = O(n^{-1}) \) holds. It leads that
\[
(\mathbb{E}_{\sigma_0} \| \hat{S}_t \|_2^2)^2 \geq \mathbb{E}_{\sigma_0} \| \hat{S}_t \|_2^2 - \text{Var}_{\sigma_0} (\hat{S}_t) \geq \frac{1}{n} e^{-\gamma} - O(n^{-1}),
\]
and it implies that for all sufficiently large \( n \) and \( -\gamma \) values,
\[
\mathbb{E}_{\sigma_0} \| \hat{S}_t \|_2 \geq \frac{1}{\sqrt{n}} e^{-\frac{\gamma}{2}}.
\]

Therefore, for \( 0 < r < e^{-\frac{\gamma}{2}} \) and \( t \leq t_\gamma(n) \), by Chebyshev’s inequality and Proposition 2.4
\[
\mathbb{P}_{\sigma_0} \left( \| \hat{S}_t \|_2 < \frac{r}{\sqrt{n}} \right) \leq \mathbb{P}_{\sigma_0} \left( \mathbb{E}_{\sigma_0} \| \hat{S}_t \|_2 - \| \hat{S}_t \|_2 > \frac{1}{\sqrt{n}} e^{-\frac{\gamma}{2}} - \frac{r}{\sqrt{n}} \right) \leq \frac{\mathbb{V}_{\sigma_0} (\hat{S}_t)}{\left( \frac{1}{\sqrt{n}} e^{-\frac{\gamma}{2}} - \frac{r}{\sqrt{n}} \right)^2} = O \left( \left( e^{-\frac{\gamma}{2}} - r \right)^2 \right).
\]

It follows that
\[
\lim_{\gamma \to -\infty} \lim_{n \to \infty} \mathbb{P}_{\sigma_0} \left( \| \hat{S}_{t_\gamma(n)} \|_2 < \frac{r}{\sqrt{n}} \right) = 0.
\]

Now, consider the cyclic dynamics \( (\sigma_t) \) where \( \sigma_0 \) follows the probability distribution \( \mu_n \). By the properties in Section 2.3 and application of Chebyshev’s inequality,
\[
\mu_n \left( \| \hat{S}_t \|_2 < \frac{r}{\sqrt{n}} \right) \geq 1 - \frac{O(1)}{r^2}
\]
holds for all \( t \geq 0 \). It can be concluded that for all \( r > 0 \),
\[
\lim_{\gamma \to -\infty} \lim_{n \to \infty} \text{d}^{(n)}(t_\gamma(n)) \geq 1 - \frac{O(1)}{r^2}.
\]
Letting $r \to \infty$, the proof is complete. \hfill \blacksquare

3. Upper Bound

This section presents the proof of the upper bound of the mixing time. In [3], it is observed that the cutoff of the upper bound for the Glauber dynamics essentially follows from the precise bound on the coalescing time of the two basket chains. In Section 3.1 semi-coordinatewise coupling is used to analyze the coalescence of the proportion chains. In Section 3.2 basket chain is introduced, and basketwise coupling is used to analyze the coalescence of the basket chains. Based on those analyses, the upper bound of the cutoff is obtained, as presented in Section 3.4.

The following proposition describes the distribution of the proportion chain at time $t(n)$. It bounds the $\ell_\infty$-norm between $S_t$ and $\bar{e}$ over $n^{-\frac{1}{2}}$ scale.

**Proposition 3.1.** Consider the cyclic dynamics $(\sigma_t)$ starting at $\sigma_0$ and its proportion chain $(S_t)$. For all $r > 0$ and $\sigma_0 \in \Sigma_n$, it holds that
$$\mathbb{P}_{\sigma_0}(S_{t(n)} \notin S^{\bar{e}}_r) = O\left( r^{-1} \right).$$

3.1. Coalescing Proportion Chains. For the two cyclic dynamics $(\sigma_t)$ and $(\tilde{\sigma}_t)$, the proportion chains $S_t$ and $\tilde{S}_t$ are made to coalesce with high probability. First, $S_t - \bar{e}$ is bound with the $n^{-\frac{1}{2}}$ scale. Then, $S_t$, $\tilde{S}_t$ is matched via coupling under certain condition.

3.1.1. Preliminaries. Here, the two well-known theorems used in the current section are introduced.

**Proposition 3.2.** [3, Lemma 2.1 (2)] Consider the discrete time process $(X_t)_{t \geq 0}$ adapted to filtration $(F_t)_{t \geq 0}$ that starts at $x_0 \in \mathbb{R}$. Let the underlying probability measure as $\mathbb{P}_{x_0}$, and let $\tau_{x} = \inf\{t : X_t \geq x\}$. Then, if the process $(X_t)$ satisfies the below two conditions, the following statement holds:

(a) $\exists \delta \geq 0 : \mathbb{E}_{x_0}[X_{t+1} - X_t | F_t] \leq -\delta$ on $\{X_t \geq 0\}$ for all $t \geq 0$.

(b) $\exists R > 0 : |X_{t+1} - X_t| \leq R, \forall t \geq 0$.

If $x_0 \leq 0$, then for $x_1 > 0$ and $t_2 \geq 0$,
$$\mathbb{P}_{x_0}(\tau_{x_1} \leq t_2) \leq 2 \exp\left\{ -\frac{(x_1 - R)^2}{8t_2R^2} \right\}.$$

**Proposition 3.3.** [3, Lemma 2.3] Suppose that the non-negative discrete time process $(Z_t)_{t \geq 0}$ adapted to $(G_t)_{t \geq 0}$ is a supermartingale. Let $N$ be a stopping time. If $(Z_t)$ satisfies the below three conditions:

(a) $Z_0 = z_0$

(b) $|Z_{t+1} - Z_t| \leq B$

(c) $\exists \sigma > 0$ such that $\mathbb{V}ar( Z_{t+1} | G_t ) > \sigma^2$ on the event $\{N > t\}$,

and $u > 4B^2 / (3\sigma^2)$, then $\mathbb{P}_{z_0}(N > u) \leq 4z_0 / (\sigma \sqrt{u})$.

3.1.2. Restriction of Proportion Chain.
Proposition 3.4. Consider the cyclic dynamics \((\sigma_t)\) and its proportion chain \((S_t)\). For the fixed constant \(r_0 > 0\) and \(\gamma > 0\), there exist \(C, c > 0\) that satisfies the following statement:

For all sufficiently large \(n\) and \(r > \max\{3r_0, 2\}\), let \(t = \gamma n\), \(\rho_0 = \frac{r_0}{n}\) and \(\rho = \frac{r}{n}\). Then,

\[
P_{\sigma_0} \left( \exists 0 < u \leq t : S_u \notin S_{\rho_0}^+ \right) \leq C e^{-\gamma r^2}
\]

holds for all \(\sigma_0\) such that \(S_0 \in S_{\rho_0}^0\).

3.1.3. Semi-Coordinatewise Coupling. This section introduces the semi-coordinatewise coupling of two cyclic dynamics \((\sigma_t)\) and \((\tilde{\sigma}_t)\). This type of coupling is used in Proposition 3.5 to prove that \(\|S_t - \tilde{S}_t\|_1\) is a supermartingale.

First, semi-independent and coordinatewise coupling are defined as in [3]. For the two probability distributions \(\nu\) and \(\tilde{\nu}\) on \(\Omega = \{1, 2, 3\}\) and \(i \in \{1, 2, 3\}\), \(\{i\}\)-semi-independent coupling of \(\nu\) and \(\tilde{\nu}\) is a pair of random variables \((X, \tilde{X})\) on \(\Omega \times \Omega\) as follows:

1. Pick \(U\) uniformly on \([0, 1]\).
2. If \(U \leq \min\{\nu(i), \tilde{\nu}(i)\}\), choose \((X, \tilde{X})\) as \((i, i)\).
3. If \(U > \min\{\nu(i), \tilde{\nu}(i)\}\), choose \(X\) and \(\tilde{X}\) independently according to the following rules: In the case of \(X\), if \(U < \nu(i)\), choose \(i\). Otherwise, choose \(i + 1\) with probability \(\frac{\nu(i+1)}{\nu(i+1)+\nu(i+2)}\) and choose \(i + 2\) with probability \(\frac{\nu(i+2)}{\nu(i+1)+\nu(i+2)}\). In the case of \(\tilde{X}\), if \(U < \tilde{\nu}(i)\), choose \(i\). Otherwise, choose \(i + 1\) with probability \(\frac{\tilde{\nu}(i+1)}{\tilde{\nu}(i+1)+\tilde{\nu}(i+2)}\), and choose \(i + 2\) with probability \(\frac{\tilde{\nu}(i+2)}{\tilde{\nu}(i+1)+\tilde{\nu}(i+2)}\).

It is evident from the construction that random variables \(X\) and \(\tilde{X}\) follow the distributions \(\nu\) and \(\tilde{\nu}\), respectively. Now, the \(\{i\}\)-coordinatewise coupling of two cyclic dynamics is defined as follows:

1. Choose two colors \(I_{t+1}\) and \(\tilde{I}_{t+1}\) based on \(\{i\}\)-semi-independent coupling of \(S_t\) and \(\tilde{S}_t\).
2. Choose two colors \(J_{t+1}\) and \(\tilde{J}_{t+1}\) based on \(\{i\}\)-semi-independent coupling of \(\nu\) and \(\tilde{\nu}\). Here, \(\nu(I_{t+1}) = \frac{1}{2}, \nu(I_{t+1} + 1) = \frac{1}{2}, \nu(I_{t+1} + 2) = 0, \tilde{\nu}(\tilde{I}_{t+1}) = \frac{1}{2}, \tilde{\nu}(\tilde{I}_{t+1} + 1) = \frac{1}{2}\) and \(\tilde{\nu}(\tilde{I}_{t+1} + 2) = 0\).
3. Uniformly choose the vertex of color \(I_{t+1}\) in \(\sigma_t\) and change its color to \(J_{t+1}\), and uniformly choose the vertex of color \(\tilde{I}_{t+1}\) in \(\tilde{\sigma}_t\) and change its color to \(\tilde{J}_{t+1}\).

Finally, the semi-coordinatewise coupling of two cyclic dynamics is defined as follows:

1. min \(\{|S_t - \tilde{S}_t|, |S_t^2 - \tilde{S}_t^2|, |S_t^3 - \tilde{S}_t^3|\} \geq \frac{2}{n}\): Update the chains independently.
2. min \(\{|S_t^1 - \tilde{S}_t^1|, |S_t^2 - \tilde{S}_t^2|, |S_t^3 - \tilde{S}_t^3|\} = \frac{1}{n}\): There exists \(i \in \{1, 2, 3\}\) such that \(|S_t^i - \tilde{S}_t^i| = \frac{1}{n}\).
   Choose a minimum \(i\) value that satisfies the condition, and update the chains based on \(\{i\}\)-coordinatewise coupling.
3. min \(\{|S_t^1 - \tilde{S}_t^1|, |S_t^2 - \tilde{S}_t^2|, |S_t^3 - \tilde{S}_t^3|\} = 0\): Find \(i \in \{1, 2, 3\}\) such that \(|S_t^i - \tilde{S}_t^i| = 0\). We may assume that \(S_t^{i+1} \geq \tilde{S}_t^{i+1}\). Then,
(a) Choose the colors \((I_{t+1}, \tilde{I}_{t+1})\) based on the probability, as stated below:

\[
(I_{t+1}, \tilde{I}_{t+1}) = \begin{cases} 
(i, i) & \text{w.p. } S_t^i = \tilde{S}_t^i \\
(i+1, i+1) & \text{w.p. } \min(S_t^{i+1}, \tilde{S}_t^{i+1}) \\
(i+2, i+2) & \text{w.p. } \min(S_t^{i+2}, \tilde{S}_t^{i+2}) \\
(i+1, i+2) & \text{w.p. } S_t^{i+1} - \tilde{S}_t^{i+1}.
\end{cases}
\]

(b) Choose the colors \((J_{t+1}, \tilde{J}_{t+1})\) according to the discrete random variable \(X\) dependent on \((I_{t+1}, \tilde{I}_{t+1})\), as stated below:

- \((I_{t+1}, \tilde{I}_{t+1}) = (i, i)\): Select \(X\) uniformly from \(\{(i, i), (i+1, i+1)\}\).
- \((I_{t+1}, \tilde{I}_{t+1}) = (i+1, i+1)\): Select \(X\) uniformly from \(\{(i+1, i+1), (i+1, i+2), (i+2, i+1), (i+2, i+2)\}\).
- \((I_{t+1}, \tilde{I}_{t+1}) = (i+2, i+2)\): Select \(X\) uniformly from \(\{(i, i), (i+2, i+2)\}\).
- \((I_{t+1}, \tilde{I}_{t+1}) = (i+1, i+2)\): Select \(X\) uniformly from \(\{(i+1, i), (i+2, i+2)\}\).

(c) Choose a vertex uniformly that has the color \(I_{t+1}\) in \(\sigma_t\). Then, change its color to \(J_{t+1}\) in \(\sigma_{t+1}\).

(d) Choose a vertex uniformly that has the color \(\tilde{I}_{t+1}\) in \(\tilde{\sigma}_t\). Then, change its color to \(\tilde{J}_{t+1}\) in \(\tilde{\sigma}_{t+1}\).

The coupling for the case of \(S_t^{i+1} \leq \tilde{S}_t^{i+1}\) can be similarly defined.

Let \(\mathbb{P}_{\sigma_0, \tilde{\sigma}_0}^{CC}\) be the underlying measure, and \(\mathbb{P}_{\sigma_0, \tilde{\sigma}_0}^{CC}\) be the expectation of semi-coordinatewise coupling. This coupling is used to prove that the \(\ell^1\)-norm between two proportion chains is a supermartingale. It is also used to guarantee the lower bound of its variance.

On Proposition 3.6, we limit the \(\ell^1\)-norm between \(S_t\) and \(\tilde{S}_t\) with \(n^{-1}\) scale. The semi-coordinatewise coupling is used on Proposition 3.3 to make this norm as a supermartingale. Proposition 3.3 provides the bound of the time spent to limit the norm.

**Proposition 3.5.** Consider the two cyclic dynamics \((\sigma_t)\) and \((\tilde{\sigma}_t)\), and the corresponding proportion chains \((S_t)\) and \((\tilde{S}_t)\). Let \(d_t = \|S_{t+1} - \tilde{S}_{t+1}\|_1 - \|S_t - \tilde{S}_t\|_1\). Suppose that \(\|S_t - \tilde{S}_t\|_1 \geq \frac{10}{n}\) for some \(t \geq 0\). If semi-coordinatewise coupling is applied in the following step, then

\[
\mathbb{E}_{\sigma_0, \tilde{\sigma}_0}^{CC}[d_t | F_t] \leq 0.
\]

Consider the two cyclic dynamics \((\sigma_t)\) and \((\tilde{\sigma}_t)\) and its proportion chains. Define the time \(T_1 = \min\{ t : \|S_t - \tilde{S}_t\|_1 < \frac{10}{n} \}\). In the next proposition, we prove that \(T_1\) is bounded with high probability. Moreover, if the value of the proportion chains \(S_t\) and \(\tilde{S}_t\) are bounded with \(n^{-\frac{1}{2}}\) scale at time zero, then the two chains are bounded until time \(T_1\) with high probability.

**Proposition 3.6.** Consider the two cyclic dynamics \((\sigma_t)\) and \((\tilde{\sigma}_t)\) that satisfy \(\sigma_0, \tilde{\sigma}_0 \in \mathcal{S}^{10n}\) for some \(n_0 > 0\). For a fixed value of \(\epsilon > 0\), the following statement holds:

There exist constant \(\gamma, r > 0\) such that

\[
\mathbb{P}_{\sigma_0, \tilde{\sigma}_0}^{CC}\left( T_1 < \gamma n, \max\left( \|S_t - \bar{c}\|_2, \|\tilde{S}_t - \bar{c}\|_2 \right) < \frac{r}{\sqrt{n}} \forall t \leq T_1 \right) \geq 1 - \epsilon
\]

holds for all sufficiently large \(n\) that is bigger than \(100 r^2\).
Because the ℓ₁-norm between two proportion chains is bounded, the coalescence of these chains can be proved with semi-synchronized coupling.

**Proposition 3.7.** Consider the two cyclic dynamics \((σ_t)\) and \((\tilde{σ}_t)\) where \(\|S_0 - \tilde{S}_0\|_1 < \frac{10}{n}\) holds. For the fixed constant \(ε > 0\), there exists a sufficiently large \(γ > 0\) such that if \(t ≥ γn\),

\[
P_{SC}^{σ_0,\tilde{σ}_0} (S_t = \tilde{S}_t) ≥ 1 - ε.
\]

3.2. **Coalescing Basket Chains.** For the two cyclic dynamics \((σ_t)\) and \((\tilde{σ}_t)\), the basket chains are made to coalesce with high probability. The theorems and proofs presented throughout this section are similar to \([3]\); however, because the conditions are slightly different, detailed analyses are provided here for completeness.

First, define \(B\) as a 3-partition of \(V_n = \{1, \ldots, n\}\), and let \(B = (B_m)_m=1\). We call these \(B_m\) partitions as baskets, and denote \(B\) as the \(λ\)-partition if \(|B_m| > λn\) holds for all \(m\). For the configuration \(σ ∈ Σ_n\), define \(3 \times 3\) matrix \(S(σ)\) representing the proportions of the number of the vertices in each basket, i.e.

\[
S^{m,k}(σ) = \frac{1}{|B_m|} \sum_{v ∈ B_m} 1_{\{σ(v) = k\}} \quad m, k ∈ \{1, 2, 3\}.
\]

The basket chain \((S_t)\) of the cyclic dynamics \((σ_t)\) is defined as \(S_t = S(σ_t)\). Note that the basket chain is a Markov chain.

Recalling the sets \(S, S^0, S^{0+}\) in Section 2.1, \(S\) is defined as the set of \(3 \times 3\) matrices where each row of each matrix exist in \(S\). This set is denoted as \(\prod_{m=1}^3 S\), and the sets \(S^0 = \prod_{m=1}^3 S^0\) and \(S^{0+} = \prod_{m=1}^3 S^{0+}\) are similarly defined.

Proposition 3.8 provides the contraction of the basket chains. Proposition 3.9 limits the distribution of the basket chains.

**Proposition 3.8.** Suppose that \(B\) is a \(λ\)-partition for some \(λ > 0\). Consider the basket chain \((S_t)\) and the proportion chain \((S_t)\) of the cyclic dynamics \((σ_t)\), and define the \(3 \times 3\) matrix \(Q_t\) as \(Q^{m,k}_t = S^{m,k}_t - S^k_t\). Then, the following holds:

\[
E_{σ_0} \left[ \left( Q^{m,k}_{t+1} \right)^2 \right] = \left( 1 - \frac{1}{n} \right) E_{σ_0} \left[ \left( Q^{m,k}_t \right)^2 \right] + \frac{1}{n} E_{σ_0} \left[ Q^{m,k}_{t-1} Q^{m,k}_t \right] + O \left( \frac{1}{n^2} \right).
\]

**Proposition 3.9.** For the cyclic dynamics \((σ_t)\), consider the \(λ\)-partition basket \(B\) for some \(λ > 0\). Assume that either of these conditions are satisfied:

1. \(t ≥ t(n)\).
2. \(S_0 ∈ S^{0+}\) and \(t ≤ γ_0 n\) for some constant \(r_0, γ_0 > 0\).

Then, for sufficiently large \(r > 0\), the following holds for all sufficiently large \(n\):

\[
P_{σ_0} (S_t ∉ S^{0+}) = O \left( r^{-2} \right).
\]

3.2.1. **Basketwise Coupling.** The basketwise coupling introduced in \([3]\) is utilized herein. The objective of basketwise coupling is to enable the two basket chains to coalesce. This coupling is used in Proposition 3.10 to prove that \(∥S^n_t - \tilde{S}^n_t∥_1\) is a supermartingale.

Consider the two cyclic dynamics \((σ_t)\) and \((\tilde{σ}_t)\), where \(S_0 = \tilde{S}_0\). The coupling begins at \(t = 0\), \(m = 1\). While \(S^n_t ≠ \tilde{S}^n_t\),
(1) Choose the color \( I_{t+1} = \tilde{I}_{t+1} \) according to the distribution \( S_t = \tilde{S}_t \).
(2) Choose the color \( J_{t+1} = \tilde{J}_{t+1} \) as \( I_{t+1} = \tilde{I}_{t+1} \) with probability \( \frac{1}{2} \), and \( I_{t+1} + 1 = \tilde{I}_{t+1} + 1 \) with probability \( \frac{1}{2} \).
(3) Uniformly choose the vertex \( V_{t+1} \) that has the color \( I_{t+1} \) in \( \sigma_t \).
(4) Choose the vertex \( \tilde{V}_{t+1} \) based on the following rules:
   (a) If \( V_{t+1} \in B_{m_0} \) for some \( m_0 < m \), then uniformly choose \( \tilde{V}_{t+1} \) in \( B_{m_0} \) that has the color \( I_{t+1} \).
   (b) If \( V_{t+1} \in B_{m_0} \) for some \( m_0 \geq m \), \( S_t^{m_0, I_{t+1}} \neq \tilde{S}_t^{m, \tilde{I}_{t+1}} \) and \( S_t^{m_0, J_{t+1}} \neq \tilde{S}_t^{m, \tilde{J}_{t+1}} \), then uniformly choose \( \tilde{V}_{t+1} \) in \( B_{[m,3]} \) that has the color \( \tilde{I}_{t+1} \) in \( \tilde{\sigma}_t \). \( B_{[m,3]} \) is defined as \( \bigcup_{i=m}^3 B_i \).
   (c) In other cases, let \( \{ v_i \} = v_1, v_2, \ldots \) be an enumeration of the vertices in \( B_{[m,3]} \) with the color \( I_{t+1} \) in \( \sigma_t \). It is first ordered based on the index of the basket it belongs to, and then based on its index in \( V \). Let \( \{ \tilde{v}_1, \tilde{v}_2, \ldots \} \) be the enumeration of the vertices in \( B_{[m,3]} \) with the color \( \tilde{I}_{t+1} \) in \( \tilde{\sigma}_t \) having the same rule. Then, as \( V_{t+1} \in \{ v_i \} \), there exists \( j \) that satisfies \( V_{t+1} = v_j \). Let \( \tilde{V}_{t+1} \) as \( \tilde{v}_j \in \{ \tilde{v}_i \} \).
(5) Change the color of the vertex \( V_{t+1} \) to \( J_{t+1} \) in \( \sigma_t \), and change the color of the vertex \( \tilde{V}_{t+1} \) to \( \tilde{J}_{t+1} \) in \( \tilde{\sigma}_t \).

When \( S_t^1 = \tilde{S}_t^1 \) is reached, repeat the process with \( m = 2 \). Note that if \( S_t^1 = \tilde{S}_t^1 \) and \( S_t^2 = \tilde{S}_t^2 \), then \( S_t^3 = \tilde{S}_t^3 \). Denote \( \mathbb{P}^{BC}_{\sigma_0, \tilde{\sigma}_0} \) as the underlying probability measure of the coupling, and let \( \mathbb{E}^{BC}_{\sigma_0, \tilde{\sigma}_0} \) be the expectation and \( \mathbb{V}^{BC}_{\sigma_0, \tilde{\sigma}_0} \) be the variance with respect to the probability measure. The following proposition proves the coalescence of the basket chains with high probability.

**Proposition 3.10.** For the two cyclic dynamics \( (\sigma_t) \) and \( (\tilde{\sigma}_t) \), suppose that \( S_0 = \tilde{S}_0 \) and \( S_0, \tilde{S}_0 \in S^{\infty} \) for some constant \( r > 0 \). Let \( B \) be a \( \lambda \)-partition, where \( \lambda > 0 \). Then, for a given \( \epsilon > 0 \), there exists sufficiently large \( \gamma \) that satisfies

\[
\mathbb{P}^{BC}_{\sigma_0, \tilde{\sigma}_0} \left( S_{\gamma n} = \tilde{S}_{\gamma n} \right) \geq 1 - \epsilon.
\]

Now, the overall coupling, which is a combination of the coupling methods proposed in the previous sections, is introduced. With this coupling, the coalescence of the two cyclic dynamics is obtained with high probability, and the proof of the upper bound is completed.

### 3.3. Overall Coupling.

**The overall coupling** of the two cyclic dynamics \( (\sigma_t) \) and \( (\tilde{\sigma}_t) \) is denoted with the parameters \( \gamma_1, \gamma_2, \gamma_3, \gamma_4 > 0 \). These parameters are taken from Proposition 2.1, Proposition 3.6, Proposition 3.7, and Proposition 3.10 respectively. The first cyclic dynamics \( (\sigma_t) \) starts at \( \sigma_0 \) and the second cyclic dynamics \( (\tilde{\sigma}_t) \) starts at \( \tilde{\sigma}_0 \), where \( \tilde{\sigma}_0 \) is determined according to the distribution \( \mu_n \). The coupling is evolved through the following procedure:

1. Iterate two chains independently until time \( t(1)(n) = \gamma_1 n \).
2. Configure the baskets \( B = \bigcup_{k=1}^3 B_k \) with the colors in \( \sigma_t(n), \) \ie \( B_k = \{ v : \sigma_t(n)(v) = k \} \), \( k = 1, 2, 3 \).
3. Iterate two chains independently until time \( t(2)(n) = t(1)(n) + t(n) \).
4. Iterate two chains with semi-coordinatewise coupling until time \( t(3)(n) = t(2)(n) + \gamma_2 n \).
(5) Iterate two chains with semi-synchronized coupling until time \( t(4) = t(3) + \gamma_3n \).

(6) Iterate two chains with basketwise coupling until time \( t(5) = t(4) + \gamma_4n \).

Denote \( P^{\text{OC}}_{\sigma_0} \) as the underlying probability measure.

3.4. **Proof of Upper Bound.** Here, the proof of the upper bound of the mixing time is demonstrated using the overall coupling. The proof is similar to that in [3, Section 4.7]; however, because the conditions are slightly different, the detailed proof is provided here to ensure completeness.

**Proposition 3.11.** For the cyclic dynamics \((\sigma_t)\) and the constant \(\epsilon > 0\), and for all sufficiently large \(n\), there exists \(\gamma > 0\) such that

\[
\| P_{\sigma_0} (\sigma_t(n) \in \cdot) - \mu_n \|_{TV} \leq \epsilon.
\]

**Proof.** First, the overall coupling is applied via seven steps:

1. Choose \(\rho > 0\). By Proposition 2.1 a sufficiently large \(\gamma_1\) satisfying \(S_{t(1)}(n) \in S^\rho\) with probability \(1 - \epsilon/6\) for all large \(n\) can be chosen.

2. Then, \(B\) (defined in step 2 of the overall coupling procedure) can be considered as a \((\frac{1}{3} - \rho)\) partition.

3. By Proposition 3.1 for some \(r > 0\), \(S_{t(2)}(n) \in S^{\sigma_0}\) with a minimum probability of \(1 - \epsilon/6\).

4. By the proof of Proposition 3.9 if \(r\) is sufficiently large, \(\tilde{S}_{t(2)}(n) \in S^{\sigma_0}\) with a minimum probability of \(1 - \epsilon/6\).

5. By Propositions 3.6 and 3.7 \(S_{t(4)}(n) = \tilde{S}_{t(4)}(n)\) with a minimum probability of \(1 - 3\epsilon/6\).

6. By Proposition 3.10 for sufficiently large \(r_1 > 0\), \(S_{t(4)}(n), \tilde{S}_{t(4)}(n) \in S^{\sigma_0}\) with a minimum probability of \(1 - \epsilon/6\).

7. By Proposition 3.11 \(S_{t(5)}(n) = \tilde{S}_{t(5)}(n)\) with a minimum probability of \(1 - 5\epsilon/6\).

When \(t \geq t(1)\) and \(F_{t(1)}\) are given, by the manner in which the baskets \(B\) were defined, the distribution of \(\sigma_t\) is the same under the permutations of the vertices on each basket \(B_m\). As the probability measure \(\mu_n\) is uniformly distributed in \(\Sigma_n\), the same notion can be applied to \(\tilde{\sigma}_t\). Thus,

\[
\| P^{\sigma_0}_{\sigma_0} \left( \sigma_{t(5)}(n) \in \cdot \mid F_{t(1)}(n), S_{t(1)}(n) \in S^\rho \right) - \mu_n \|_{TV} \\
= \| P^{\sigma_0}_{\sigma_0} \left( S_{t(5)}(n) \in \cdot \mid F_{t(1)}(n), S_{t(1)}(n) \in S^\rho \right) - \mu_n \circ S^{-1} \|_{TV} \\
\leq \| P^{\sigma_0}_{\sigma_0} \left( S_{t(5)}(n) \neq \tilde{S}_{t(5)}(n) \mid F_{t(1)}(n), S_{t(1)}(n) \in S^\rho \right) \leq 5\epsilon/6.
\]

Therefore,

\[
\| P_{\sigma_0} (\sigma_{t(5)}(n) \in \cdot) - \mu_n \|_{TV} \\
\leq \| P^{\sigma_0}_{\sigma_0} (\sigma_{t(5)}(n) \in \cdot \mid F_{t(1)}(n)) - \mu_n \|_{TV} | S_{t(1)}(n) \in S^\rho | + \| P^{\sigma_0}_{\sigma_0} (S_{t(1)}(n) \notin S^\rho) \leq \epsilon.
\]

Finally, let \(\gamma\) be \(\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4\). This completes the proof. \(\square\)

3.5. **Proof of Theorem 1.1.** In Theorem 1.1 the lower bound of the mixing time is guaranteed by the Proposition 2.3, and the upper bound of the mixing time is guaranteed by the Proposition 3.11. Therefore, the cutoff phenomenon of cyclic dynamics is proved.
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