HILL'S OPERATORS WITH THE POTENTIALS ANALYTICALLY DEPENDENT ON ENERGY

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Abstract. We consider Schrödinger operators on the line with potentials that are periodic with respect to the coordinate variable and real analytic with respect to the energy variable. We prove that if the imaginary part of the potential is bounded in the right half-plane, then the high energy spectrum is real, and the corresponding asymptotics are determined. Moreover, the Dirichlet and Neumann problems are considered. These results are used to analyze the good Boussinesq equation.

1. INTRODUCTION AND MAIN RESULTS

1.1. Introduction. There are a lot of papers about Schrödinger operators with potentials polynomially dependent on energy, see, e.g., the review in [FLM04]. We consider the wider class of potentials analytically dependent on energy. Our motivation is related with the good Boussinesq equation on the circle. McKean [McK81] reduced the third order operator with periodic coefficients, associated with the good Boussinesq equation, to the Hill equation with an energy-dependent potential. This potential is an analytic function of energy in the domain \( \{ \lambda \in \mathbb{C} : |\lambda| > R, |\arg \lambda| < \pi - \delta \} \), where \( R > 0 \) is large enough and \( \delta > 0 \) is small enough. Starting from the famous work of Keldysh [Ke71], operators with a potential polynomially depending on energy were actively studied. At the same time, we know very few works where operators with a potential that is an arbitrary analytic function of the spectral parameter would be considered, see the review below.

We consider Hill’s equation

\[
- y'' + V(x, \lambda)y = \lambda y, \quad \lambda \in \mathcal{D},
\]

on the whole line where the potential \( V(x, \lambda) \) is 1-periodic with respect to \( x \in \mathbb{R} \) and real analytic with respect to \( \lambda \in \mathcal{D} \). Here we assume that \( \mathcal{D} \subset \mathbb{C} \) is a bounded or unbounded domain having a piecewise smooth boundary \( \partial \mathcal{D} \). We study the following spectral problems for this equation:

1) the problem on the whole line,
2) the quasi-periodic problems on the interval \((0, 1)\) including the periodic and antiperiodic problems,
3) the Dirichlet problem \( y(0) = y(1) = 0 \).

Throughout the text, we assume that the potential \( V \) satisfies:

i) For almost every \( x \in \mathbb{R} \) the function \( V(x, \cdot) \) is real analytic in the domain \( \mathcal{D} \),

ii) For each \( \lambda \in \mathcal{D} \) the function \( V(\cdot, \lambda) \) is 1-periodic and \( V(\cdot, \lambda) \in L^1(\mathbb{T}) \), where \( \mathbb{T} = \mathbb{R}/\mathbb{Z} \).
Some of our results are true for the domains $D$ of a quite general form, while others require additional restrictions on the type of the domain. Typically, the appearance of the specific domain $D$ is dictated by the specifics of the problem. For example, in the case of the good Boussinesq equation we are considering, the domain has the form of a complex plane cut along curves lying in the vicinity of the negative half-line, see [McK81] and Fig. 1 a. In any case, as a rule, domains containing a segment of the real axis are of interest, and then we assume that the potential $V$ is a real analytic function.

The problem we are considering arises as a result of the reduction of the spectral problem for a higher order differential operator to a second order one. Such a reduction for the third-order differential operator associated with the good Boussinesq equation on the circle is carried out in the paper of McKean [McK81]. Describe briefly the situation, see the details below in Section 2. The good Boussinesq equation

$$
\begin{align*}
pt &= -\frac{1}{3} p_{xxxx} - \frac{4}{3} (p^2)_{xx}, \\
p_t &= q_x,
\end{align*}
$$

is equivalent to the Lax equation $\dot{L} = LA - AL$, where $A = -\partial^3 - \frac{4}{3}p$ and the operator $L$ has the form $L = \partial^3 + \partial p + p \partial + q$. Recall that the corresponding L-operator for the well-studied Korteweg-de Vries equation is the self-adjoint Schrödinger operator. In contrast to this case, the L-operator for the good Boussinesq is a non-self adjoint third order operator. This non-self-adjointness greatly complicates the application of the inverse problem method, since spectral data become non-real and are more difficult to control. In [McK81] McKean reduces the spectral problem for the operator $L$ to the Schrödinger equation with an energy-dependent potential. The equation obtained by McKean is a special case of equation (1.1) we are considering. The spectrum of the 2-periodic problem is an invariant set with respect to the Boussinesq flow. The Dirichlet spectrum parameterizes the solutions of the Boussinesq equation. The Dirichlet spectrum for the good Boussinesq was the subject of our work [BK19]. In our work [BK11] we made the reduction of the spectral problem for a fourth-order operator to a second order one.

Note that we study here only the case of the good Boussinesq equation on the circle. The associated operator $L$ is non-self-adjoint, however, the high energy spectra for the corresponding Schrödinger equation with an energy-dependent potential localizes near the real axis. The situation for the bad Boussinesq equation is completely different. The associated operator $i\partial^3 + i\partial q + iq \partial + q$ is self-adjoint but the high energy spectra for the corresponding Schrödinger equation with an energy-dependent potential localizes far from the real axis. We considered this operator in our paper [BK15]. The spectral properties of higher order differential operators with periodic coefficients were the subject of Badanin and Korotyaev [BK11], [BK12], Papanicolaou [P95], [P03], see also references therein.

Schrödinger operators with polynomially energy-dependent potentials are also well studied, see, e.g., Alonso [A80], Jaulent and Jean [JJ76], [JJ76x], Kamimura [Ka08], see also the book [Ma12] and references therein, moreover, there is enormous physical and technical literature on this subject. By the well-known technique developed by Keldysh [Ke71], these problems are reduced to vector spectral problems where the potential does not depend on the spectral parameter. We consider a much wider class of problems when the potential is an arbitrary holomorphic function of the spectral parameter. Keldysh’s approach does not work in this case and these problems are much worse studied. In connection with this subject, we mention the papers McKea [McK81] and Badanin–Korotyaev [BK11] for the periodic problems, and
Calogero–Jagannathan [CJ67] for the scattering problems. Note that there are a large number of articles where the certain special classes of holomorphic families of operators with respect to an additional parameter are considered, see Derkach and Malamud [DM89], Gesztesy, Kalton, Makarov and Tsekanovskii [GKMT01] and references therein.

1.2. **The definitions.** We analyze equation \((1.1)\) on the whole line using the direct integral decomposition. In order to describe this decomposition we introduce the operators on \(L^2(0,1)\) given by
\[
H(k, \lambda) = H_o(k) + V(\cdot, \lambda), \quad k \in [0, 2\pi),
\]
where \(\lambda\) belongs to the domain \(\mathcal{D}\) and the unperturbed operators \(H_o(k)\) have the form
\[
H_o(k)y = -y'' \quad \text{under the quasi-periodic boundary conditions}
\]
\[
y(1) = e^{ik}y(0), \quad y'(1) = e^{ik}y'(0), \quad k \in [0, 2\pi).
\]
If \(k = 0\), then the conditions \((1.4)\) are called periodic conditions, if \(k = \pi\), then they are called antiperiodic ones, jointly they are 2-periodic conditions.

Recall the following standard definitions. Let \(k \in [0, 2\pi)\). The point \(\lambda \in \mathcal{D}\) is called the regular point of the operator-valued function \(H(k, \lambda)\), if the resolvent \((H(k, \lambda) - \lambda)^{-1}\) exists and bounded. We denote by \(\rho(H(k, \cdot))\) the set of all regular points of the operator-valued function \(H(k, \lambda)\). The operator-valued function \((H(k, \lambda) - \lambda)^{-1}\) is analytic on the set \(\rho(H(k, \cdot))\). The spectrum \(\sigma(H(k, \cdot))\) of the function \(H(k, \lambda)\) is the set
\[
\sigma(H(k, \cdot)) = \mathcal{D} \setminus \rho(H(k, \cdot)).
\]
The set \(\sigma(H(k, \cdot))\) is closed. The number \(\lambda_o \in \mathcal{D}\) is called the eigenvalue of the operator-valued function \(H(k, \lambda)\), if the equation
\[
H(k, \lambda_o)y_o = \lambda_o y_o
\]
has a non-trivial solution, the corresponding solution \(y_o\) is called the eigenvector. The spectrum \(\sigma(H_o(0)) \cup \sigma(H_o(\pi))\) of the 2-periodic problem for the unperturbed operator \(H_o\) is pure discrete, consists of the simple eigenvalue \(\lambda_o^+ = 0\) and the eigenvalues \(\lambda_n^{o, \pm} = (\pi n)^2, n \in \mathbb{N}\), of multiplicity 2. We show in Theorem [1.1] that the spectrum in the perturbed case is also discrete.

Moreover, introduce the operator-valued function
\[
T(\lambda) = T_o + V(\cdot, \lambda)
\]
in the domain \(\mathcal{D}\), where the unperturbed operator \(T_o\) in \(L^2(0,1)\) has the form \(T_o y = -y''\) with the Dirichlet boundary conditions
\[
y(0) = y(1) = 0.
\]
The point \(\lambda \in \mathcal{D}\) is a regular point of the function \(T(\lambda)\), if the resolvent \((T(\lambda) - \lambda)^{-1}\) exists and bounded. The operator-valued function \((T(\lambda) - \lambda)^{-1}\) is analytic on the set \(\rho(T)\) of all regular points of the operator-valued function \(H(k, \lambda)\). The spectrum \(\sigma(T)\) is the set
\[
\sigma(T) = \mathcal{D} \setminus \rho(T).
\]
The Dirichlet spectrum \(\sigma(T_o)\) for the unperturbed operator \(T_o\) consists of the simple eigenvalues \(m_n^o = (\pi n)^2, n \in \mathbb{N}\).

Similarly, we define the operator \(\mathcal{N}(\lambda)\) of the Neumann problem by
\[
\mathcal{N}(\lambda) = \mathcal{N}_o + V(\cdot, \lambda), \quad \lambda \in \mathcal{D},
\]
where the unperturbed operator $\mathcal{N}_0 y = -y''$ acts on the functions $y$ such that
\[ y'(0) = y'(1) = 0. \tag{1.8} \]

Let us denote by $\sigma(\mathcal{N})$ the spectrum of the operator $\mathcal{N}$. The spectrum $\sigma(\mathcal{N}_0)$ for the unperturbed operator consists of the simple eigenvalues $\lambda_n = (\pi n)^2, n = 0, 1, 2, \ldots$

Introduce the operators $H(\lambda), \lambda \in \mathcal{D}$, acting on $L^2(\mathbb{R})$, by
\[ H(\lambda) = H_0 + V(\cdot, \lambda), \tag{1.9} \]
where the unperturbed operator $H_0$ in $L^2(\mathbb{R})$ has the form
\[ H_0 y = -y''. \]

Now we write the direct integral decomposition for the operator-valued function $H(\lambda)$. Introduce the Hilbert spaces $\mathcal{H}' = L^2([0,1], dt), \mathcal{H} = \int_{(0, 2\pi)} \mathcal{H}' \frac{dk}{2\pi}$ \quad \tag{1.10}
Introduce the unitary operator $U : L^2(\mathbb{R}) \to \mathcal{H}$ by
\[ (Uf)(t) = \sum_{n \in \mathbb{Z}} e^{-ink} f(t + n), \quad (k, t) \in [0, 2\pi) \times [0,1]. \tag{1.11} \]

Now we formulate our preliminary results about the direct integral decomposition of the operator-valued functions $H(\lambda)$ given by (1.9).

**Proposition 1.1.**

i) The operator-valued function $H(\lambda)$ satisfies
\[ UH(\lambda)U^{-1} = \int_{(0, 2\pi)} H(k, \lambda) \frac{dk}{2\pi}, \quad \lambda \in \mathcal{D}, \tag{1.12} \]
where $U$ is defined by (1.11).

ii) The spectra $\sigma(H(k, \cdot))$ for each $k \in [0, 2\pi), \sigma(T)$ and $\sigma(\mathcal{N})$ are pure discrete.

iii) Each eigenvalue $\lambda(k) \in \mathcal{D}$ of the operator $H(k, \cdot)$ is a piecewise analytic and $2\pi$-periodic function of $k \in \mathbb{R}$. Moreover, $\sigma(H(2\pi - k, \cdot)) = \sigma(H(k, \cdot))$ for all $k \in [0, 2\pi)$, counting with multiplicities.

iv) The spectrum $\sigma(H)$ of the operator-valued function $H(\lambda)$ satisfies
\[ \sigma(H) = \bigcup_{k \in [0,\pi]} \sigma(H(k, \cdot)). \tag{1.13} \]

**Remark.**

1) The spectrum $\sigma(H_0)$ of the unperturbed operator $H_0$ on the whole line is pure absolutely continuous, has multiplicity 2, and satisfies $\sigma(H_0) = [0, +\infty)$.

2) We consider the band functions $\lambda_n(k), k \in [0, 2\pi]$ mainly for high energy. Note that if the eigenvalue $\lambda_n(k)$ goes to the boundary of the domain $\mathcal{D}$, then it leaves the spectrum of $H(k, \lambda)$ and, therefore, does not generate the spectrum of $H(\lambda)$.

1.3. **Main results.** Introduce the notations
\[ \lambda = \mu + iv, \quad Q = \text{Im} V, \]
and the norm of the potential
\[ \|V(\cdot, \lambda)\| = \int_0^1 |V(x, \lambda)| dx, \quad \lambda \in \mathcal{D}. \]

Now we formulate our first results about the spectra.
Theorem 1.2. Let $I \subset \mathbb{R}$ be a finite or infinite interval, $I \subset \mathcal{D}$, let for a.e. $x \in \mathbb{R}$ the function $Q = \text{Im} V$ satisfy
\[ Q(x, \cdot) \in C(\mathcal{D}), \quad \frac{\partial Q}{\partial \nu}(x, \cdot) \in C(\mathcal{D}), \]
and
\[ \sup_{(x, \lambda) \in [0,1] \times I} \left| \frac{\partial Q(x, \lambda)}{\partial \nu} \right| < 1. \quad (1.14) \]
Then the spectral set $\mathcal{S}$, defined by
\[ \mathcal{S} = \sigma(H) \cup \sigma(T) \cup \sigma(N), \quad (1.15) \]
for some $\delta > 0$ satisfies
\[ \mathcal{S} \cap (I \times (-\delta, \delta)) \cap (\mathcal{D} \cup I) \subset I. \quad (1.16) \]

Thus, the estimate (1.14) guarantees that the spectrum is real in the vicinity of the real axis. In the following Theorem we obtain the conditions when the spectrum in a half-plane is real. Introduce the domains
\[ \Pi_a = \{ \lambda \in \mathbb{C} : \text{Re} \lambda > a \}, \quad \mathcal{D}_a = \mathcal{D} \cap \Pi_a, \quad a \in \mathbb{R}, \]
and for a domain $\Omega \subset \mathcal{D}$ we introduce the functional
\[ \xi(\Omega) = \sup_{(x, \lambda) \in [0,1] \times \Omega} |Q(x, \lambda)|. \quad (1.17) \]

Theorem 1.3 gives that if $Q$ is bounded on the right half-plane, then the high energy spectra in this half-plane is real.

Theorem 1.3. Let the potential $V$ satisfy the estimate
\[ \xi(\mathcal{D}_a) < \infty \quad (1.18) \]
for some $a \in \mathbb{R}$. Let, in addition,
\[ (a, +\infty) \times (-\rho, \rho) \subset \mathcal{D}, \quad \text{where} \quad \rho = \frac{\xi(\mathcal{D}_a)}{2 - \sqrt{3}}. \quad (1.19) \]
Then the spectra $\sigma(H)$, $\sigma(T)$ and $\sigma(N)$ in the domain $\mathcal{D}_{a+\rho}$ are real:
\[ \mathcal{S} \cap \mathcal{D}_{a+\rho} \subset (a + \rho, +\infty), \quad (1.20) \]
where $\mathcal{S} = \sigma(H) \cup \sigma(T) \cup \sigma(N)$. In particular, if the half-plane $\Pi_a \subset \mathcal{D}$ and $\xi(\Pi_a) < \infty$, then the spectra in the half-plane $\Pi_{a+\rho_1}$ are real:
\[ \mathcal{S} \cap \Pi_{a+\rho_1} \subset (a + \rho_1, +\infty), \quad \rho_1 = \frac{\xi(\Pi_a)}{2 - \sqrt{3}}. \quad (1.21) \]

Remark. 1) The conditions of Theorem 1.3 are more restrictive than the condition (1.14) of Theorem 1.2 in the following sense. Assume that the restrictions of Theorem 1.3 hold true, that is assume that $\xi(\mathcal{D}_a) < \infty$ for some $a \in \mathbb{R}$ and $(a, +\infty) \times (-\rho, \rho) \subset \mathcal{D}$, where $\rho$ is given by (1.19). Then the estimate (4.3) of Lemma 4.2 gives
\[ \sup_{(x, \lambda) \in [0,1] \times (a+\rho_1, +\infty)} \left| \frac{\partial Q(x, \lambda)}{\partial \nu} \right| \leq \frac{2}{\rho} \xi(\mathcal{D}_a) = 2(2 - \sqrt{3}) < 1, \]
that is the restriction (1.14) is fulfilled for the interval $I = (a + \rho, +\infty)$. 
2) This is an open question: is it possible to take the constant in the definitions (1.19) and (1.21) more than \(2 - \sqrt{3}\)?

3) We illustrate Theorems 1.2 and 1.3 with two simple examples in Section 4.4.

### 1.4. High energy asymptotics.

Theorem 1.2 provides that if \((a, +\infty) \times (-r, r) \subset \mathcal{D}\) for some \((a, r) \in \mathbb{R} \times \mathbb{R}_+\) and the potential \(V\) satisfies the condition

\[
\sup_{(x, \lambda) \in [0, 1] \times (a, +\infty)} \left| \frac{\partial Q(x, \lambda)}{\partial \nu} \right| < 1, \tag{1.22}
\]

then the spectra in the half-strip \((a, +\infty) \times (-\delta, \delta)\) for some \(\delta > 0\) are real. In the following theorem we show that the high energy spectra in this case are similar to the spectra for the standard Hill operator with the real potential which does not depend on energy and determine high energy asymptotics of the eigenvalues.

**Theorem 1.4.** Let \((a, +\infty) \times (-r, r) \subset \mathcal{D}\) for some \((a, r) \in \mathbb{R} \times \mathbb{R}_+\). Let the potential \(V\) satisfy the condition (1.22) and let \(b \geq a\) be large enough.

i) Let, in addition, \(\|V(\cdot, \lambda)\| = \lambda^4 o(1)\) as \(\lambda \to +\infty\). Then the eigenvalues \(\lambda^\pm_{2n} \in \mathcal{D}_b\) of the operator \(H(0, \lambda)\) and the eigenvalues \(\lambda^\pm_{2n-1} \in \mathcal{D}_b\) of the operator \(H(\pi, \lambda)\) are real and satisfy

\[
\lambda^+_{N-1} < \lambda^-_N \leq \lambda^+_N < \lambda^-_{N+1} \leq \lambda^+_N < ... \tag{1.23}
\]

for some \(N \in \mathbb{N}\). The spectrum \(\sigma(H)\) in the domain \(\mathcal{D}_b\) is real, absolutely continuous, has multiplicity two, and consists of the intervals \([\lambda^+_{n-1}, \lambda^-_{n}]\), \(n \geq N\), separated by the gaps \((\lambda^-_n, \lambda^+_n)\)

\[
\sigma(H) \cap \mathcal{D}_b = \bigcup_{n \geq N} [\lambda^+_{n-1}, \lambda^-_n] \subset \mathbb{R}. \tag{1.24}
\]

The eigenvalues \(m_n \in \mathcal{D}_b\) of the Dirichlet operator \(T(\lambda)\) are real, simple and satisfy

\[
m_N < m_{N+1} < m_{N+2} < ..., \quad m_n \in [\lambda^-_n, \lambda^+_n], \quad n = N, N + 1, ..., \tag{1.25}
\]

and there are no other Dirichlet eigenvalues in \(\mathcal{D}_b\). The eigenvalues \(n_n \in \mathcal{D}_b\) of the Neumann operator \(\mathcal{N}(\lambda)\) are real, simple and satisfy

\[
n_N < n_{N+1} < n_{N+2} < ..., \quad n_n \in [\lambda^-_n, \lambda^+_n], \quad n = N, N + 1, ..., \tag{1.26}
\]

and there are no other Neumann eigenvalues in \(\mathcal{D}_b\).

ii) Let, in addition, \(\|V(\cdot, \lambda)\| = \lambda^7 o(1)\) as \(\lambda \to +\infty\), and let

\[
\int_0^1 V(s, \lambda) ds = o(1), \tag{1.27}
\]

as \(|\lambda| \to \infty, \lambda \in \mathcal{D}_a\). Then the eigenvalues of the 2-periodic problem satisfy

\[
\lambda^\pm_n = (\pi n)^2 + o(1) \quad \text{as} \quad n \to +\infty. \tag{1.28}
\]

**Remark.** 1) The results, similar to (1.25), for the Neumann eigenvalues hold, see Remark after Lemma 5.2.

2) The condition (1.27) can be written in the slightly more general form \(\hat{V}_n(\lambda) = C + o(1)\) for some \(C \in \mathbb{R}\) independent of \(\lambda\) but the constant \(C\) is removed by shifting the spectral parameter.

3) Korotyaev [K99] determined the sharp spectral asymptotics for Schrödinger operators with periodic complex potentials.

The plan of the paper is as follows. We discuss the relations between our second order operator and the third order operator associated with the good Boussinesq equation. In Section
we calculate the resolvent and prove that the spectra $\sigma(H(k, \cdot))$, $k \in [0, 2\pi)$, as well as the spectrum $\sigma(T)$, are sets of zeros of certain functions analytic in the domain $\mathcal{D}$. It follows that the spectra $\sigma(H(k, \cdot))$ and $\sigma(T)$ are discrete and we obtain their description in terms of zeros of the analytic functions. Moreover, in Section 4 we prove Proposition 1.1 on the direct integral decomposition for the operator $H(\lambda)$. In Section 5 we establish the conditions when the spectrum is real and prove Theorems 1.2 and 1.3. In addition, we consider two simple examples there. In Section 6 we study high energy asymptotic behavior of the spectra and prove Theorem 1.4. Moreover, there we prove Corollary 2.1 for the good Boussinesq equation.

2. Relationship with the good Boussinesq equation

2.1. Ramifications and three-point eigenvalues. Recall that the good Boussinesq equation (1.2) is equivalent to the Lax equation $\dot{L} = LA - AL$, where the non-self-adjoint operator $L$, acting on $L^2(\mathbb{R})$, has the form

$$L = \partial^3 + \partial p + p\partial + q.$$

We consider the operator $L$ in the class of real 1-periodic coefficients $p',q \in L^1(\mathbb{T})$. The operator $L$ with smooth coefficients $p,q$ was studied by McKean [McK81]. The following results from [McK81] can be extended from the class of the smooth coefficients onto the class $p',q \in L^1(\mathbb{T})$.

Introduce the fundamental solutions $y_j(x,\zeta), j = 1,2,3$, of the equation

$$y'''' + (pq)' + py' + qy = \zeta y, \quad (x, \zeta) \in \mathbb{R} \times \mathbb{C},$$

satisfying the conditions $y_j^{(k-1)}(0,\zeta) = \delta_{jk}$. Let $M(x,\zeta)$ be the matrix $M = (y_j^{(k-1)})^{3}_{j,k=1}$, $M(0,\zeta) = \Pi_3$ a $3 \times 3$-identical matrix. Each matrix-valued function $M(x, \cdot), x \in \mathbb{R}$, is entire. The matrix $M(1,\zeta)$ is the monodromy matrix. The eigenvalues $\kappa_j, j = 1,2,3,$ of the monodromy matrix are the multipliers, they satisfy the identity $\kappa_1\kappa_2\kappa_3 = 1$. The functions $\kappa_j = \kappa_j(\zeta)$ constitute three branches of the function, analytic on a 3-sheeted multiplier Riemann surface $\mathcal{R}$, see [McK81] (the similar surface for the bad Boussinesq is described in [BK15]). Ramifications of this surface are points where two or all three functions take the same value. They are the zeros of the entire function $(\kappa_1 - \kappa_2)^2(\kappa_1 - \kappa_3)^2(\kappa_2 - \kappa_3)^2$ called the discriminant, see [McK81], [BK14] and [BK15]. There are a finite number of the ramifications in any bounded domain in $\mathbb{C}$. The set $\{r_n^\pm\}_{n \in \mathbb{Z}}$ of ramifications is invariant with respect to the Boussinesq flow.

To each multiplier $\kappa_j(\zeta), j = 1,2,3,$ corresponds the Floquet solution $\psi_j(x,\zeta), (x,\zeta) \in \mathbb{R} \times \mathbb{C}$, satisfying the conditions

$$\psi_j(0,\zeta) = 1, \quad \psi_j(x + 1,\zeta) = \kappa_j \psi_j(x,\zeta).$$

For each $x \in \mathbb{R}$ the functions $\psi_j(x, \cdot)$ constitute three branches of the function, meromorphic on the surface $\mathcal{R}$. The set of poles of the functions $\psi_j(x, \cdot)$ coincides with the spectrum $\{\zeta_n\}_{n \in \mathbb{Z} \setminus \{0\}}$ of the three-point Dirichlet problem

$$y'''' + (pq)' + py' + qy = \zeta y, \quad y(0) = y(1) = y(2).$$

This problem was the subject of our paper [BK19].

In the unperturbed case $p = q = 0$ the ramifications $r_n^0, n \in \mathbb{Z}$, and the three-point eigenvalues $\zeta_n, n \in \mathbb{Z} \setminus \{0\}$, have the form $r_n^0 = \zeta_n = (\frac{2n}{\sqrt{3}})^3, n \in \mathbb{Z} \setminus \{0\}, r_0^0 = 0$, see [McK81]. In the perturbed case the sets of the ramifications and of the three-point Dirichlet
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Figure 1. The domain \( \mathcal{D} \) of analyticity of the function \( \psi_3(x, \cdot) \) (fig. a), the domain \( \mathcal{D}_1 \) of analyticity of the function \( \psi_1(x, \cdot) \) (fig. b), and the slits for the good Boussinesq equation.

eigenvalues are symmetric with respect to the real line. Moreover, the three-point eigenvalues at high energy are real and simple and satisfy \[ \zeta_n = \left( \frac{2\pi n}{\sqrt{3}} \right)^3 - \frac{4\pi n}{\sqrt{3}}p_0 + \frac{2\pi n}{\sqrt{3}}q_0 - \bar{q}_n + O(n^{-\frac{1}{2}}), \]
as \( n \to \pm\infty \), where

\[ \bar{f}_n = \frac{2}{\sqrt{3}} \int_0^1 f(x) \cos \left( 2\pi nx + \frac{\pi}{6} \right) dx, \quad n \in \mathbb{N}. \]

2.2. Transformation to a second order equation. An important problem is to prove that the high energy ramifications are real. In order to solve this problem McKean (referring to J. Moser) reduces the third-order equation (2.1) to a second-order equation with an energy-dependent potential. Now we describe this transformation.

Each function \( \kappa_3 \) and \( \psi_3(x, \cdot), x \in \mathbb{R} \), is analytic in the domain \( \mathcal{D} = \{ \zeta \in \mathbb{C} : |\zeta| > R, |\arg \zeta| < \pi - \delta \} \) (see Fig. 1 a) for any \( \delta > 0 \) small enough and for some \( R > 0 \) large enough. Moreover, if \( \zeta \to \infty \) in \( \mathcal{D} \), then

\[ \kappa_3(\zeta) = e^{\zeta}(1 + O(|\zeta|^{-1})), \quad \psi_3(x, \zeta) = e^{\zeta \frac{2}{3}}(1 + O(|\zeta|^{-1})) \]
uniformly in \( x \in [0, 1] \). Therefore, \( |\kappa_3| \) and \( |\psi_3(x, \cdot)|, x \in \mathbb{R} \), are increasing as \( |\zeta| \to \infty \) in \( \mathcal{D} \).

Using this result we take \( R > 0 \) so large that the function \( \psi_3(x, \zeta) \) does not vanish in \( \mathbb{R} \times \mathcal{D} \).

Let \( \zeta \in \mathcal{D} \). If we take any solution \( y \) of equation (2.1), then the function

\[ f = \psi_3^2 \left( \frac{y}{\psi_3} \right) \]
satisfies the equation

\[ -f'' + \mathcal{V}f = 0, \]
where the energy-dependent potential \( \mathcal{V}(x, \cdot) \) has the form

\[ \mathcal{V} = -2p - \frac{3}{4} \left( 2 \left( \frac{\psi'_3}{\psi_3} \right)' + \left( \frac{\psi'_3}{\psi_3} \right)^2 \right), \]
and satisfies \( \mathcal{V}(\cdot, \zeta) \in L^1(\mathbb{T}) \). Each function \( \mathcal{V}(x, \cdot), x \in \mathbb{R} \), is analytic in the domain \( \mathcal{D} \), real on \( \mathbb{R} \cap \mathcal{D} \), and satisfies

\[ \mathcal{V}(x, \zeta) = -\lambda - p(x) + O(\zeta^{-\frac{1}{2}}), \quad \lambda = \frac{3}{4} \zeta^4, \]
as $|\zeta| \to \infty$, $\zeta \in \mathcal{D}$, uniformly on $x \in \mathbb{T}$. Then equation (2.3) has the form (1.1), where

$$V(x, \lambda) = V(x, \zeta) + \lambda.$$  
(2.6)

For each $x \in \mathbb{R}$ the function $V(x, \cdot)$ is analytic in the domain $\mathcal{D}$ given by

$$\mathcal{D} = \left\{ \lambda \in \mathbb{C} : |\lambda| > \frac{3}{4} R_1^2, \arg \lambda < \frac{2}{3} (\pi - \delta) \right\}.$$
(2.7)

The asymptotics (2.5) shows that $V(x, \lambda) = -p(x) + O(\lambda^{-\frac{3}{2}})$ as $|\lambda| \to \infty$ in $\mathcal{D}$, uniformly in $x \in [0,1]$.

2.3. Results for the Boussinesq equation. Introduce the fundamental solutions $\phi_1(x, \zeta)$, $\phi_2(x, \zeta)$, $(x, \zeta) \in \mathbb{R} \times \mathcal{D}$ of the equation (2.3), satisfying the conditions $\phi_1(0, \zeta) = \phi_2(0, \zeta) = 1$, $\phi_1'(0, \zeta) = \phi_2'(0, \zeta) = 0$. Introduce the fundamental matrix $\Phi = (\phi_j^{(k-1)})_{j,k=1}^2$. The matrix $\Phi(1, \zeta), \zeta \in \mathcal{D}$, is the monodromy matrix. It is analytic in $\mathcal{D}$. It has two eigenvalues $\tau_1(\zeta), \tau_2(\zeta)$, they are the multipliers. The multipliers satisfy the identity $\tau_1 \tau_2 = 1$. The discriminant $(\tau_1 - \tau_2)^2 = 2(\phi_1(1, \cdot) + \phi_2(1, \cdot))$ is an analytic function in $\mathcal{D}$. The zeros of this function are the eigenvalues of the 2-periodic problem for equation (2.3). For each multiplier there are the Floquet solution $f_j(x, \zeta), j = 1, 2$, satisfying the conditions $f_j(x + 1, \zeta) = \tau_j(\zeta)f_j(x, \zeta)$.

In our next article, we will show that the ramifications of the multiplier surface $\mathcal{R}$ coincide with the eigenvalues of the 2-periodic problem for equation (2.3), and the three-point Dirichlet eigenvalues coincide with the Dirichlet eigenvalues for equation (2.3). Here we briefly describe the corresponding arguments.

Let $y = \psi_1$ be the Floquet solution of equation (2.1). Then $f_1 = \psi_3^\frac{3}{2}(\psi_1')'$ is the Floquet solution of equation (2.3) satisfying $f_1(x + 1) = \chi_3^\frac{1}{2}\chi_1 f_1(x)$. Similarly, $f_2 = \psi_3^\frac{3}{2}(\psi_2')'$ is the Floquet solution of equation (2.3) satisfying $f_2(x + 1) = \chi_3^\frac{1}{2}\chi_2 f_2(x)$. Then $\tau_1 = \chi_3^\frac{1}{2}\chi_1$ and $\tau_2 = \chi_3^\frac{1}{2}\chi_2$ are multipliers for equation (2.3).

Let $z \in \mathcal{D}$ be a ramification of the surface $\mathcal{R}$. Recall that in this case at least two function $z_1, z_2, z_3$ take the same value. The identity $z_1 z_2 z_3 = 1$ and the asymptotics $z_n(\zeta) = e^{s(1 + O(|\zeta|^{-1}))}$ show that $z_1(\zeta) = z_2(\zeta)$, which yields $\tau_1(\zeta) = \tau_2(\zeta)$. Therefore, $\zeta$ is an eigenvalue of the 2-periodic problem for equation (2.3).

Furthermore, if $\zeta \in \mathcal{D}$ is an eigenvalue of the three-point Dirichlet problem for equation (2.1), then it is a pole of the Floquet solution $\psi_1$ or $\psi_2$ of equation (2.1), therefore, it is a pole of the Floquet solution $f_1$ or $f_2$ of equation (2.3). Then it is an eigenvalue of the Dirichlet problem for equation (2.3).

In the following corollary of the previous theorems (see the proof in Section 6) we extend McKean’s result that the ramifications are real from the class $p, q \in C^\infty(\mathbb{T})$ onto a wider class of coefficients $p', q \in L^1(\mathbb{T})$.

**Corollary 2.1.** Let $p', q \in L^1(\mathbb{T})$. Then

1) The ramifications and the eigenvalues of the three-point problem in the half-plane $Z_\alpha = \{ \zeta \in \mathbb{C} : \Re \zeta > a \}$ are real for some $a > 0$ large enough. There are exactly two (counting with multiplicity) ramifications $r_n^\pm$ and exactly one simple three-point eigenvalue $\zeta_n$ in each interval $(\alpha_n^-, \alpha_n^+)$ inside this half-plane, where $n \in \mathbb{N}, \alpha_n^\pm = (\frac{n(2n+1)}{4\pi})^\frac{3}{2}$. There are no other ramifications and three-point eigenvalues in the half-plane $Z_\alpha$.

2) The eigenvalues $\zeta_n$ satisfy

$$\zeta_n \in [r_n^-, r_n^+] \subset \mathbb{R},$$
(2.8)
for all } n \in \mathbb{N} \text{ large enough.}

iii) The ramifications } r_n^\pm \text{ satisfy }

\[ r_n^\pm = \left( \frac{2\pi n}{\sqrt{3}} \right)^3 - \frac{4\pi np_0}{\sqrt{3}} + o(n), \]  

\text{as } n \to +\infty, \text{ where } p_0 = \int_0^1 p(x)dx.

\textbf{Remark.} 1) Similarly the negative } \zeta \text{ may be considered. The Floquet solution } \psi_1 \text{ is analytic in the domain } D_1 = \{ \zeta \in \mathbb{C} : |\zeta| > R, |\arg \zeta| > \delta \}, \text{ see Fig. 1b. If we use this function instead of } \psi_3 \text{ in the previous construction, then we obtain the relations similar to (2.9) and (2.8) for } n \to -\infty. 

2) The multiplier Riemann surface and the ramifications for the self-adjoint third order operator associated with the bad Boussinesq equation was the subject of our papers [BK14], [BK15]. The multiplier surface and the ramifications for the good Boussinesq are the subjects of our next paper.

3) The relations (2.8) are proved by McKean [McK81] for the smooth coefficients } p, q. \text{ Our prove is simpler and extends these relations onto the larger class of the coefficients } p', q \in L^1(T).

4) Assuming a higher smoothness of the coefficients, we can improve the asymptotics (2.9) in order to determine a trace formula. This is the subject of our next paper.

5) The previous results may be extended from the class } p', q \in L^1(T) \text{ onto the class } p, q \in L^1(T). \text{ The transformation (2.2) in this case leads to the potential } V \text{ that is the distribution with respect to } x. \text{ Then we have to consider equation (1.1) where the potential } V \text{ is a distribution. We think that our results hold for this case. The corresponding energy-independent potentials were considered by Korotyaev [K03].}

6) The sharp asymptotics of the ramifications for the bad Boussinesq equation is determined in [BK15].

\section{The Lyapunov Function and the Spectra}

\subsection{The fundamental solutions}

Introduce the fundamental solutions } \vartheta(x, \lambda), \varphi(x, \lambda), (x, \lambda) \in \mathbb{R} \times \mathcal{D}, \text{ of equation (1.1) satisfying the initial conditions } \vartheta(0, \lambda) = \varphi'(0, \lambda) = 1, \vartheta'(0, \lambda) = \varphi(0, \lambda) = 0.

The fundamental solutions } \vartheta(x, \lambda), \varphi(x, \lambda) \text{ of the unperturbed equation } -y'' = \lambda y \text{ have the form }

\[ \vartheta_o(x, \lambda) = \cos zx, \quad \varphi_o(x, \lambda) = \frac{\sin zx}{z}, \quad z = \sqrt{\lambda}, \]

\text{here and below } \sqrt{1} = 1. \text{ Each function } \vartheta_o(x, \cdot), \varphi_o(x, \cdot), x \in \mathbb{R}, \text{ is entire.}

Each solution } y(x, \lambda), (x, \lambda) \in \mathbb{R}_+ \times \mathcal{D}, \text{ of equation (1.1) satisfies the following integral equation }

\[ y(x, \lambda) = y(0, \lambda)\vartheta_o(x, \lambda) + y'(0, \lambda)\varphi_o(x, \lambda) + \int_0^x \varphi_o(x - s, \lambda)V(s, \lambda)y(s, \lambda)ds, \quad x \in \mathbb{R}. \]
The standard iterations give
\begin{align}
\vartheta(x, \lambda) &= \sum_{n=0}^{\infty} \vartheta_n(x, \lambda), \quad \vartheta_n(x, \lambda) = \int_{0}^{x} \varphi_o(x-s, \lambda)V(s, \lambda)\vartheta_{n-1}(s, \lambda)ds, \\
\varphi(x, \lambda) &= \sum_{n=0}^{\infty} \varphi_n(x, \lambda), \quad \varphi_n(x, \lambda) = \int_{0}^{x} \varphi_o(x-s, \lambda)V(s, \lambda)\varphi_{n-1}(s, \lambda)ds,
\end{align}
for each \((n, x, \lambda) \in \mathbb{N} \times \mathbb{R}_+ \times \mathcal{D}\).

**Lemma 3.1.** Each function \(\vartheta(x, \cdot), \varphi(x, \cdot), \vartheta'(x, \cdot), \varphi'(x, \cdot), x \in \mathbb{R}\), is analytic in \(\mathcal{D}\). Moreover,
\begin{align}
\sup \left\{ \left| \vartheta(x, \lambda) - \sum_{n=0}^{N} \vartheta_n(x, \lambda) \right|, |z|_1 \left| \varphi(x, \lambda) - \sum_{n=0}^{N} \varphi_n(x, \lambda) \right| \right\}, \quad \left| \varphi'(x, \lambda) - \sum_{n=0}^{N} \varphi'_n(x, \lambda) \right|, \frac{1}{|z|_1} \left| \vartheta'(x, \lambda) - \sum_{n=0}^{N} \vartheta'_n(x, \lambda) \right| \right\} &\leq \frac{\|V(\cdot, \lambda)\|^N e^{\|V(\cdot, \lambda)\| |z|_1}}{|z|_1^{N+1}},
\end{align}
for all \(N \geq 0, (x, \lambda) \in \mathbb{R}_+ \times \mathcal{D}\), where \(|z|_1 = \max\{1, |z|\}\).

**Proof.** The standard arguments, see, e.g., [PT87, Ch 1, Thms 1 and 3], give
\begin{align}
\vartheta_n(x, \lambda) &= \frac{1}{z^n} \int_{0<x_1<\ldots<x_n<x_{n+1}=x} \prod_{k=1}^{n} \sin z(x_{k+1} - x_k)V(x_k, \lambda) \cos zx_1 dx_1 \ldots dx_n,
\end{align}
which yields
\begin{align}
&\quad \left| \vartheta_n(x, \lambda) \right| \leq \frac{1}{|z|_1^n} \int_{0<x_1<\ldots<x_n<x_{n+1}=x} \prod_{k=1}^{n} e^{\|\varphi\| (x_{k+1} - x_k)} |V(x_k, \lambda)| e^{\|\varphi\| |x|} dx_1 \ldots dx_n \\
&\quad \leq \frac{e^{\|\varphi\| |x|}}{|z|_1^n} \int_{0<x_1<\ldots<x_n<x_{n+1}=x} \prod_{k=1}^{n} |V(x_k, \lambda)| dx_1 \ldots dx_n = \frac{e^{\|\varphi\| |x|}}{|z|_1^n} \left( \int_{0}^{x} |V(x, \lambda)|dx \right)^n.
\end{align}
This estimate and the similar estimate for \(\varphi_n(x, \lambda), \varphi'_n(x, \lambda)\) imply
\begin{align}
\max\{\left| \vartheta_n(x, \lambda) \right|, \left| \varphi_n(x, \lambda) \right|, \left| \varphi'_n(x, \lambda) \right| \} \leq \frac{\|V(\cdot, \lambda)\|^{n} e^{\|\varphi\| |z|_1}}{n! |z|_1^n}, \quad \forall (x, \lambda) \in \mathbb{R}_+ \times \mathcal{D}.
\end{align}

Summing the majorants we obtain the estimates (3.2).

### 3.2. Asymptotics of the Lyapunov function.
Introduce the Lyapunov function by
\begin{align}
\Delta(\lambda) &= \frac{1}{2} \left( \vartheta(1, \lambda) + \varphi'(1, \lambda) \right), \quad \lambda \in \mathcal{D}.
\end{align}
The function \(\Delta\) is analytic in \(\mathcal{D}\).
In the following Lemma we prove estimates for the solution $\varphi(1, \lambda)$ and the Lyapunov function $\Delta(\lambda)$. Introduce the functions

$$\Delta_j(\lambda) = \frac{1}{2} (\vartheta_j(1, \lambda) + \varphi_j'(1, \lambda)), \quad j \in \mathbb{N}, \quad \lambda \in \mathcal{D}. \quad (3.4)$$

**Lemma 3.2.** i) The functions $\Delta_j, j = 1, 2$, satisfy

$$\Delta_1(\lambda) = \frac{\sin z}{2z} \hat{V}_0(\lambda), \quad (3.5)$$

$$\Delta_2(\lambda) = \frac{1}{4z^2} \left( \cos z \left( \int_0^1 ds \int_0^s \cos 2z(s-t)W(s,t,\lambda)dt - \frac{\hat{V}_0^2(\lambda)}{2} \right) + \sin z \int_0^1 ds \int_0^s \sin 2z(s-t)W(s,t,\lambda)dt \right), \quad (3.6)$$

for all $\lambda \in \mathcal{D}$, where $\hat{V}_0(\lambda) = \int_0^1 V(s, \lambda)ds$ and $W(s,t,\lambda) = V(s,\lambda)V(t,\lambda)$.

ii) The following estimates hold true:

$$\left| \varphi(1, \lambda) - \frac{\sin z}{z} \right| \leq e_1(\lambda)/|z|^1, \quad (3.7)$$

$$\left| \Delta(\lambda) - \cos z \right| \leq e_1(\lambda), \quad (3.8)$$

$$\left| \Delta(\lambda) - \cos z - \Delta_1(\lambda) \right| \leq e_2(\lambda), \quad (3.9)$$

$$\left| \Delta(\lambda) - \cos z - \Delta_1(\lambda) - \Delta_2(\lambda) \right| \leq e_3(\lambda), \quad (3.10)$$

for all $\lambda \in \mathcal{D}$, where

$$e_j(\lambda) = \frac{\|V(\cdot,\lambda)\|_{\mathcal{L}}}{|z|^j} e^{\int_{1}^{\infty} \frac{|\int V(\cdot,\lambda)|}{|z|}} + |\int V(\cdot,\lambda)|, \quad j \geq 0.$$

**Proof.** i) The definitions (3.1) imply

$$\vartheta_0(1) = \frac{1}{z} \int_0^1 V(s) \sin z(1-s) \cos z s ds, \quad \varphi_0'(1) = \frac{1}{z} \int_0^1 V(s) \cos z(1-s) \sin z s ds,$$

here and below in this proof $\vartheta(x) = \vartheta(x,\lambda), V(x) = V(x,\lambda), W(s,t) = W(s,t,\lambda), ...$ Substituting these identities into the definition (3.1) we obtain (3.5). Moreover,

$$\vartheta_1(1) = \frac{1}{z^2} \int_0^1 ds \int_0^s W(s,t) \sin z(1-s) \sin z(s-t) \cos z t dt,$$

$$\varphi_1'(1) = \frac{1}{z^2} \int_0^1 ds \int_0^s W(s,t) \cos z(1-s) \sin z(s-t) \sin z t dt,$$

therefore,

$$2\Delta_1 = \vartheta_2(1) + \varphi_2'(1) = \frac{1}{z^2} \int_0^1 ds \int_0^s W(s,t) \sin z(1-s+t) \sin z(s-t)dt$$

$$= \frac{1}{2z^2} \left( \sin z \int_0^1 ds \int_0^s \sin 2z(s-t) W(s,t) dt - \cos z \int_0^1 ds \int_0^s (1 - \cos 2z(s-t)) W(s,t) dt \right),$$

which yields (3.6).

ii) Let $\lambda \in \mathcal{D}$. The estimates (3.2) give (3.7) and

$$\max \left\{ \left| \vartheta(1, \lambda) - \cos z \right|, \left| \varphi'(1, \lambda) - \cos z \right| \right\} \leq e_1(\lambda),$$
These estimates together with the definitions (3.4) yield (3.8)–(3.10).

The spectrum of the 2-periodic problem has the form

\[ \lambda \in \mathbb{R} \quad \text{satisfies} \quad \psi(x, \lambda) = 0, \quad \forall \ x \in \mathbb{R}, \]

where

\[ R_T(x, s; \lambda) = \frac{1}{\varphi(1, \lambda)} \begin{cases} \varphi(s, \lambda)(\vartheta(1, \lambda)\varphi(x, \lambda) - \vartheta(x, \lambda)\varphi(1, \lambda)), & s < x \\ \varphi(x, \lambda)(\vartheta(1, \lambda)\varphi(s, \lambda) - \vartheta(s, \lambda)\varphi(1, \lambda)), & s > x. \end{cases} \]

The spectrum \( \sigma(T) \) of the operator \( T(\lambda) \) is discrete and coincides with the set

\[ \sigma(T) = \{ \lambda \in \mathcal{D} : \vartheta(1, \lambda) = 0 \}. \]
Proof. i) Direct calculations show that $R_H(x, s; k, \lambda)$ satisfies the standard properties of Green’s functions for equation (1.1) and the conditions (1.4). This yields the identity (3.12). The identity (3.12) shows that the resolvent is a bounded operator for all $\lambda \in \mathcal{D}$ such that $\Delta(\lambda) \neq \cos k$, therefore the set of the regular points has the form (3.11) and then the spectrum satisfies (3.13). Let $\lambda \in \mathcal{D}$ be a zero of the function $\varphi(1, \lambda)\eta(x, \lambda) - (\varphi(1, \lambda) - e^{ik})\varphi(x, \lambda)$ satisfies equation (1.1) and the conditions (1.4). Therefore, $\lambda$ is an eigenvalue. Thus, the spectrum is pure discrete. The identity (3.13) yields (3.14).

ii) The function $R_T(x, s, \lambda)$ is the Green function for the problem (1.1), (1.6). This yields the identity (3.16). This identity gives that the resolvent is a bounded operator for all $\lambda \in \mathcal{D}$: $\phi(1, \lambda) \neq 0$, therefore the set of the regular points is given by (3.15) and the spectrum satisfies (3.17). Let $\lambda \in \mathcal{D}$ be a zero of the function $\varphi(1, \lambda)$. Then the function $\phi(x, \lambda)$ is an eigenfunction of the problem (1.1), (1.6) with the eigenvalue $\lambda$. This yields that the spectrum is pure discrete.

iii) The proof is similar.

The maximum number of linearly independent eigenvectors associated with an eigenvalue, is referred to as the eigenvalue’s geometric multiplicity. The non-trivial solutions $y_1, y_2, ..., y_{m-1}$ of the equations

$$\sum_{j=0}^{n} \frac{1}{j!}(H(k, \lambda) - \lambda)^{(j)}|_{\lambda=\lambda_o}y_{n-j} = 0, \quad n = 1, 2, ..., m - 1,$$

are called the adjoined vectors to the eigenvector $y_o$ and the number $m$ is called the algebraic multiplicity of the eigenvalue $\lambda_o$.

The identity (3.13) shows that the spectrum $\sigma(H(k, \cdot))$ of the quasiperiodic problem consists of eigenvalues that are zeros of the function $\Delta(\lambda) - \cos k$ analytic in $\mathcal{D}$. Similarly, the identity (3.17) yields that the spectrum $\sigma(H_d)$ of the Dirichlet problem consists of eigenvalues that are zeros of the function $\phi(1, \cdot)$ analytic in $\mathcal{D}$. The multiplicity of the zero is the algebraic multiplicity of the corresponding eigenvalue. The algebraic and geometric multiplicity of the eigenvalue can be different from each other.

We are ready to prove our results about the direct integral decomposition and the spectrum of the operator $H(\lambda)$.

Proof of Proposition $\text{1.1}$ i) The proof of the identity (1.12) is standard, see [RS78, Ch XIII.16].

ii) The statement is proved in Lemma 3.3.

iii) The identity (3.13) and the analyticity of the function $\Delta$ on the domain $\mathcal{D}$ yield the statement.

iv) The decomposition (1.12) and the statement iii) yield (1.13). ■

4. Conditions when the spectra are real.

4.1. Local conditions. Introduce the function

$$\eta(x, \lambda) = Q(x, \lambda) - \nu,$$

recall that $\lambda = \mu + i\nu$ and $Q = \text{Im} V$. For each $x \in [0, 1]$ the function $\eta(x, \cdot)$ is harmonic in $\mathcal{D}$. Below we need the following auxiliary result.
Lemma 4.1. Let \( \lambda_0 = \mu_0 + iv_0 \in \mathcal{S} \). Then there exists (maybe not unique) point \( x_0 = x_0(\lambda_0) \in (0, 1) \) such that \( \eta(x_0, \lambda_0) = 0 \). Moreover, in this case
\[
|\nu_0| \leq \sup_{x \in [0, 1]} |Q(x, \lambda_0)|.
\]

Proof. We consider the spectrum \( \sigma(H(k, \cdot)) \) of the quasi-periodic problem. The proofs for the Dirichlet spectrum \( \sigma(T) \) and for the Neumann spectrum \( \sigma(\mathcal{N}) \) are similar. Let \( \lambda_0 \in \sigma(H(k, \cdot)) \). The corresponding eigenfunction \( y \) yields
\[
0 = \int_0^1 \overline{y}(x, \lambda_0)(-y''(x, \lambda_0) + (V(x, \lambda_0) - \lambda_0)y(x, \lambda_0))dx
= \int_0^1 (|y'(x, \lambda_0)|^2 + (V(x, \lambda_0) - \lambda_0)|y(x, \lambda_0)|^2)dx,
\]
which yields \( \int_0^1 \eta(x, \lambda_0)|y(x, \lambda_0)|^2dx = 0 \). This identity shows that \( \eta(x, \lambda_0) \) vanishes at least at one point in the interval \( x \in (0, 1) \).

The definition (4.1) gives
\[
|\eta(x, \lambda)| \geq |\nu| - |Q(x, \lambda)| \quad \forall (x, \lambda) \in \mathbb{R} \times \mathcal{D}.
\]
If \( |\nu| > \sup_{x \in [0, 1]} |Q(x, \lambda)| \), then \( |\eta(x, \lambda)| > 0 \) for all \( x \in [0, 1] \). Therefore, \( \eta(x, \lambda) \) may vanish only if \( |\nu| \leq \sup_{x \in [0, 1]} |Q(x, \lambda)| \). This yields the estimate (4.2).

We prove our first main results about the spectra.

Proof of Theorem 1.2. Let \( x \in \mathbb{R} \). The function \( V(x, \cdot) \) is real analytic in \( \mathcal{D} \) and \( Q(x, \cdot) \in C(\mathcal{D}) \), then \( \eta(x, \cdot) \) is harmonic in \( \mathcal{D} \), each \( \eta(x, \cdot), \partial_\nu \eta(x, \cdot) \in C(\mathcal{D}) \), and \( \eta(x, \mu) = 0 \) for all \( \mu \in I \). Moreover,
\[
\partial_\nu \eta(x, \lambda) = \partial_\nu Q(x, \lambda) - 1, \quad \partial_\nu = \frac{\partial}{\partial \nu}
\]
which yields
\[
|\partial_\nu \eta(x, \lambda)| \geq |1 - |\partial_\nu Q(x, \lambda)||,
\]
for all \( (x, \lambda) \in \mathbb{R} \times \mathcal{D} \). The estimate (1.14) implies \( \sup_{x \in [0, 1]} |\partial_\nu Q(x, \mu + iv)|_{\nu=0} < 1 \), which yields \( \inf_{x \in [0, 1]} |\partial_\nu \eta(x, \mu + iv)|_{\nu=0} > 0 \) for all \( \mu \in I \).

Thus, we have \( \eta(x, \mu) = 0 \) for all \( (x, \mu) \in \mathbb{R} \times I \), and \( \inf_{x \in [0, 1]} |\partial_\nu \eta(x, \mu + iv)|_{\nu=0} > 0 \) for all \( \mu \in I \). Then the asymptotics
\[
\eta(x, \mu + iv) = \partial_\nu \eta(x, \mu + iv)|_{\nu=0} + O(\nu^2), \quad \nu \to 0,
\]
yields \( |\eta(x, \mu + iv)| > 0 \) for all \( x \in \mathbb{R}, \mu \in I \) and \( |\nu| < \delta \) for some \( \delta \) small enough. Then Lemma 4.1 shows that there are no the spectra \( \sigma(H(k, \cdot)) \) for each \( k \in [0, 2\pi) \) and the spectrum \( \sigma(T) \) in the domain \( (\mu, \nu) \in I \times (-\delta, \delta) \), which yields (1.16).

4.2. Auxiliary estimate. Below we search for the conditions for the potential, when the high energy spectrum is real. In our proofs we use the arguments from [McKS1].

Below we need the following auxiliary result.

Lemma 4.2. Let the function \( f \) be harmonic in the disc \( \mathcal{D}_\mu(r) = \{ \lambda \in \mathbb{C} : |\lambda - \mu| < r \} \) for some \( r > 0, \mu \in \mathbb{R} \). Then
\[
|\partial_\nu (\mu + iv)| \leq \frac{2r}{(r - |\nu|)^2} \max_{\lambda \in \mathcal{D}_\mu(r)} |f(\lambda)|, \quad \forall \ \nu \in (-r, r).
\]
If, in addition, \(-\phi r \leq \nu \leq \phi r\) for some \(\phi \in (0, 1)\) and
\[
\max_{\lambda \in \mathbb{D}_\mu(r)} |f(\lambda)| < \frac{r(1-\phi)^2}{2},
\] (4.4)
then
\[
\left| \frac{\partial f(\mu + iv)}{\partial \nu} \right| < 1.
\] (4.5)

**Proof.** Consider the case \(\nu > 0\). Poisson’s formula for the disc \(\mathbb{D}_\mu(r)\) gives
\[
f(\mu + iv) = \frac{1}{2\pi} \int_0^{2\pi} \frac{r(2-\nu^2)f(\mu + re^{i\theta})}{r^2 + \nu^2 - 2r\nu \sin \theta} d\theta,
\]
which yields
\[
\frac{\partial f(\mu + iv)}{\partial \nu} = \frac{r}{\pi} \int_0^{2\pi} \frac{\nu^2 \sin \theta + \nu^2 \sin \theta - 2\nu r}{(r^2 + \nu^2 - 2r\nu \sin \theta)^2} f(\mu + re^{i\theta}) d\theta.
\] (4.6)
The estimates
\[
r^2 + \nu^2 - 2r\nu \sin \theta = (r - \nu)^2 + 2r\nu(1 - \sin \theta) \geq (r - \nu)^2,
\]
\[
\nu^2 \sin \theta + \nu^2 \sin \theta - 2\nu r \leq (\nu - r)^2
\]
give
\[
\left| \frac{\partial f(\mu + iv)}{\partial \nu} \right| \leq \frac{2r}{(r - \nu)^2} \max_{\theta \in [0, 2\pi]} |f(\mu + re^{i\theta})|,
\]
which yields (4.3) for the case \(\nu > 0\). The arguments for the case \(\nu < 0\) are similar. Let \(\nu \to 0\) in the identity (4.6), then we obtain
\[
\left. \frac{\partial f(\mu + iv)}{\partial \nu} \right|_{\nu = 0} = \frac{1}{\pi r} \int_0^{2\pi} \frac{f(\mu + re^{i\theta}) \sin \theta d\theta}{(r - \nu)^2},
\]
which yields (4.3) for the case \(\nu = 0\). The estimate (4.3) gives (4.5). \(\blacksquare\)

### 4.3. Global conditions.

Introduce the domains in \(\mathbb{C}\):
\[
\Pi_{a,b}(r) = (a, b) \times (-r, r), \quad \Pi_a(r) = (a, +\infty) \times (-r, r), \quad a, b \in \mathbb{R}, \quad a < b, \quad r > 0.
\]
Now we prove that the spectra are real under some specific restriction on the potential.

**Lemma 4.3.** Let \(\Pi_a(r) \subset \mathcal{D}\) for some \((a, r) \in \mathbb{R} \times \mathbb{R}_+\).

i) Let, in addition, \(b > a + 2r\) and let \(|Q(x, \lambda)|\) be bounded in \([0, 1] \times \Pi_{a,b}(r)\) and satisfy
\[
x(\Pi_{a,b}(r)) \leq \frac{r(1-\phi)^2}{2}
\] (4.7)
for some \(\phi \in (0, 1)\), where the functional \(x\) is given by (1.17). Then for all \((x, \lambda) \in \mathbb{R} \times \Pi_{a+r,b-r}(\phi r)\) the function \(\eta(x, \lambda) = Q(x, \lambda) - \nu\) can vanish only for real \(\lambda\). Moreover, the spectra \(\sigma(H), \sigma(T)\) and \(\sigma(N)\) in the rectangle \(\Pi_{a+r,b-r}(\phi r)\) are real:
\[
\mathcal{G} \cap \Pi_{a+r,b-r}(\phi r) \subset \mathbb{R}.
\] (4.8)

ii) Let, in addition, \(|Q(x, \lambda)|\) be bounded in \([0, 1] \times \Pi_a(r)\) and satisfy
\[
x(\Pi_a(r)) \leq \frac{r(1-\phi)^2}{2}
\] (4.9)
for some \( \phi \in (0, 1) \). Then the spectra in the half strip domain \( \Pi_{a+r}(\phi r) \) are real:
\[
\mathcal{S} \cap \Pi_{a+r}(\phi r) \subset (a + r, +\infty).
\] (4.10)

Moreover, if
\[
\xi(\Pi_a(r)) \leq (2 - \sqrt{3})r,
\] (4.11)
then
\[
\mathcal{S} \cap \Pi_{a+r}(2 - \sqrt{3})r \subset (a + r, +\infty).
\] (4.12)

**Proof.** i) Let \( x \in \mathbb{R} \). Due to \( V(x, \lambda) \) is real for \( \lambda \in (a, b) \), we have \( \eta(x, \lambda) = 0 \) as \( \lambda \in (a, b) \).

Let, in addition, \( \mu \in (a + r, b - r) \). The function \( Q(x, \cdot) \) is harmonic in \( \Pi_{a,b}(r) \) and satisfies \( (1.17) \), then the estimate (4.5) shows that
\[
\left| \frac{\partial Q(x, \mu + i\nu)}{\partial \nu} \right| < 1, \quad \forall \ |\nu| \leq \phi r,
\]
which yields
\[
\frac{\partial \eta(x, \mu + i\nu)}{\partial \nu} > 0 \quad \text{or} \quad \frac{\partial \eta(x, \mu + i\nu)}{\partial \nu} < 0, \quad \forall \ |\nu| \leq \phi r.
\]
Consider the case \( \partial \eta/\partial \nu > 0 \). Then \( \eta(x, \mu + i\nu) > 0 \), if \( \nu \in (0, \phi r) \), and \( \eta(x, \mu + i\nu) < 0 \), if \( \nu \in (-\phi r, 0) \). The similar arguments for the case \( \partial \eta/\partial \nu < 0 \) hold. Thus for all \( x \in \mathbb{R} \) and for all \( \lambda \in \Pi_{a+r,b-r}(\phi r) \) the function \( \eta(x, \lambda) \) can vanish only for real \( \lambda \).

Let \( \lambda_o \in \mathcal{S} \). Lemma 4.1 i) yields that \( \eta(x_o, \lambda_o) = 0 \) for some \( x_o \in (0, 1) \). If, in addition, \( \lambda_o \in \Pi_{a+b-r}(\phi r) \), then the statement i) implies \( \lambda_o \in \mathbb{R} \). The relation (4.8) follows.

ii) Taking \( b \to +\infty \) in (4.7) and (4.8) we obtain (4.9) and (4.10). If \( \phi = 2 - \sqrt{3} \), then \((1 - \phi)^2 = 2\phi \). The relations (4.9) and (4.10) imply (4.11) and (4.12).

We are ready to prove Theorem 1.3.

**Proof of Theorem 1.3.** The definition (1.19) gives
\[
\xi(D_{a+\rho}) \leq \xi(D_a) = \phi \rho, \quad \phi = 2 - \sqrt{3}.
\]
Then the estimate (4.2) shows that if \( \lambda_o = \mu_o + i\nu_o \in \mathcal{S} \cap D_{a+\rho} \), then
\[
|\nu_o| \leq \xi(D_{a+\rho}) \leq \phi \rho,
\]
therefore, \( \lambda_o \in \Pi_{a+\rho}(\phi \rho) \). This yields
\[
\mathcal{S} \cap D_{a+\rho} \subset \Pi_{a+\rho}(\phi \rho).
\] (4.13)

The relations (4.12) and (4.13) give (1.20), which yields (1.21).

4.4. **Examples.** The following examples illustrate Theorems 1.2 and 1.3. We consider the potentials \( V \) of the forms
\[
V_1(x, \lambda) = \sum_{n=1}^{N} q_n(x)e^{-\kappa_n \lambda},
\] (4.14)
\[
V_2(x, \lambda) = \sum_{n=1}^{N} q_n(x)\cos(\kappa_n \lambda),
\] (4.15)
for all \( (x, \lambda) \in \mathbb{R} \times \mathbb{C} \), where
\[
0 < \kappa_1 < \kappa_2 < \ldots < \kappa_N, \quad q_n \in L^\infty_{\text{real}}(\mathbb{T}), \quad n = 1, \ldots, N, \quad N \geq 1.
\] (4.16)
Introduce the norm \( \|f\|_\infty = \sup_{x \in [0,1]} |f(x)| \).


**Proposition 4.4.** Let $\kappa_n, n = 1, \ldots, N, N \in \mathbb{N},$ be positive numbers and let $q_n$ be real functions satisfying (4.16).

i) If $V = V_1$, then

a) The spectra $\sigma(H)$, $\sigma(T)$ and $\sigma(N)$ in the strip $\mathbb{R} \times (-\delta, \delta)$ for some $\delta > 0$ are real:

$$\mathcal{S} \cap (\mathbb{R} \times (-\delta, \delta)) \subseteq \mathbb{R}. \quad (4.17)$$

b) The spectra $\sigma(H)$, $\sigma(T)$ and $\sigma(N)$ in the half-plane $\text{Re} \lambda > \mu_1$, are real:

$$\mathcal{S} \cap \Pi_{\mu_1} \subseteq \mathbb{R}, \quad (4.18)$$

where

$$\mu_1 = \frac{1}{2 - \sqrt{3}} \sum_{n=1}^{N} \|q_n\|_{\infty}. \quad (4.19)$$

ii) If $V = V_2$, then

a) The spectra $\sigma(H)$, $\sigma(T)$ and $\sigma(N)$ in the strip $\mathbb{R} \times (-\delta, \delta)$ for some $\delta > 0$ are real:

$$\mathcal{S} \cap (\mathbb{R} \times (-\delta, \delta)) \subseteq \mathbb{R}. \quad (4.19)$$

b) For any $\nu_0 > 0$ the spectra $\sigma(H)$, $\sigma(T)$ and $\sigma(N)$ in the half-strip $(\mu_0, +\infty) \times (-\nu_0, \nu_0)$ are real:

$$\mathcal{S} \cap \Pi_{\mu_0}(\nu_0) \subseteq \mathbb{R}, \quad (4.20)$$

where

$$\mu_0 = \frac{1}{2 - \sqrt{3}} \sum_{n=1}^{N} \|q_n\|_{\infty} \sinh(\kappa_n \nu_0). \quad (4.21)$$

**Proof.** i) Let the potential have the form (4.14). Then

$$Q(x, \lambda) = - \sum_{n=1}^{N} q_n(x) e^{-\kappa_n \mu} \sin(\kappa_n \nu), \quad \lambda = \mu + i\nu$$

which yields

$$\sup_{(x,\lambda) \in [0,1] \times \mathbb{R}} \left| \frac{\partial Q(x, \lambda)}{\partial \nu} \right| \leq \sum_{n=1}^{N} \kappa_n \|q_n\|_{\infty},$$

$$\xi(\Pi_0) \leq \sum_{n=1}^{N} \|q_n\|_{\infty}. \quad (4.17)$$

Theorem 1.2 gives (4.17). Theorem 1.3 yields (4.18).

ii) Let the potential have the form (4.15). Then

$$Q(x, \lambda) = - \sum_{n=1}^{N} q_n(x) \sin(\kappa_n \mu) \sinh(\kappa_n \nu).$$

which yields

$$\sup_{(x,\lambda) \in [0,1] \times \mathbb{R}} \left| \frac{\partial Q(x, \lambda)}{\partial \nu} \right| \leq \sum_{n=1}^{N} \kappa_n \|q_n\|_{\infty},$$

Theorem 1.2 implies (4.19).
Consider the half-strip domain \( \Pi_0(\nu_0), \nu_0 > 0 \). Then
\[
\xi(\Pi_0(\nu_0)) \leq \sum_{n=1}^{N} \|q_n\|_\infty \sinh(\kappa_n\nu_0).
\]

Theorem 1.3 yields (4.20).

5. High energy spectrum

5.1. Spectral properties. Theorem 1.2 shows that if \( \Pi_a(r) = (a, +\infty) \times (-r, r) \subset \mathcal{D} \) for some \( (a, r) \in \mathbb{R} \times \mathbb{R}_+ \) and the potential \( V \) satisfies the condition (1.22), then the spectrum \( \sigma(H) \) on the half-strip \( \Pi_a(\delta) \) for some \( \delta > 0 \) is real. Using this result and the standard arguments based on the analyticity of the Lyapunov function \( \Delta(\lambda) \), see [Kr83], we obtain the following results about this function.

**Lemma 5.1.** Let \( \Pi_a(r) \subset \mathcal{D} \) for some \( (a, r) \in \mathbb{R} \times \mathbb{R}_+ \), let \( Q = \text{Im} \, V \) satisfy the estimate (1.22) and let \( \lambda \in (a, +\infty) \). Then if \( \Delta(\lambda) \in (-1, 1) \), then \( \Delta'(\lambda) \neq 0 \). Moreover, if \( \Delta(\lambda) = \pm 1 \) and \( \Delta'(\lambda) = 0 \), then \( \Delta(\lambda)\Delta''(\lambda) < 0 \).

**Proof.** We give the proof by the method of “on the contrary”. Assume that \( \lambda_0 \in (a, +\infty) \) satisfies \( \Delta(\lambda_0) \in (-1, 1) \) and \( \Delta'(\lambda_0) = 0 \). Then \( \Delta(\lambda) = \Delta(\lambda_0) + 1/2 \Delta''(\lambda_0)(\lambda - \lambda_0)^2 + O((\lambda - \lambda_0)^3) \)
as \( \lambda \to \lambda_0 \). Consider the mapping \( \lambda \to \Delta(\lambda) \) in some neighborhood of the point \( \lambda_0 \). Any angle made by lines started from the point \( \lambda_0 \), is transformed onto the angle two or more times grater. Then the segment \( [\Delta(\lambda_0) - \delta, \Delta(\lambda_0) + \delta] \subset [-1, 1] \) for some \( \delta > 0 \) small enough has the pre-image, that cannot entirely lie on the real axis. The identity (3.13) gives that \( \sigma(H(k, \cdot)) \) is non-real for some \( k \in [0, 2\pi) \). The identity (1.13) implies that \( \sigma(H) \) is non-real that contradicts to Theorem 1.2. Thus, \( \Delta'(\lambda_0) 
eq 0 \), which proves the first statement. The proof of the second one is similar.

In the unperturbed case \( V = 0 \) the spectrum \( \sigma(H_0(k)), k \in [0, \pi] \), consists of the eigenvalues
\[
\lambda_n^0 = \frac{2\pi}{k}n, \quad n = 0, 1, 2, \ldots, \quad \lambda_n^0 = \frac{2\pi}{k}n - k^2, \quad n \in \mathbb{N},
\]
and
\[
\lambda_n^0(k) \leq \lambda_n^0(k) \leq \lambda_n^0(k) \leq \lambda_n^0(k) < \ldots
\]
If \( k \in (0, \pi) \), then all eigenvalues are simple. If \( k = 0 \), then \( \lambda_n^0(0) \) is simple and all other eigenvalues have multiplicity 2. If \( k = \pi \), then all eigenvalues have multiplicity 2. Moreover, using \( \sigma(H(2\pi - k, \cdot)) = \sigma(H(k, \cdot)) \) for all \( k \in [0, 2\pi) \), we put \( \lambda_n^0(2\pi - k) = \lambda_n^0(k) \), \( n \in \mathbb{N} \).

**Lemma 5.2.** Let \( \Pi_a(r) \subset \mathcal{D} \) for some \( (a, r) \in \mathbb{R} \times \mathbb{R}_+ \). Let the potential \( V \) satisfy the condition (1.22) and let \( \|V(\cdot, \lambda)\| = o(1) \) as \( \lambda \to +\infty, \quad z = \lambda^{1/2} \to +\infty \). Let \( b > a \) be large enough. Then
\begin{enumerate}
  \item There exist exactly two (counting with multiplicity) eigenvalues \( \lambda_{2n}^\pm \) of the operator \( H(0, \lambda) \) in each interval \( ((2n - 1)^2, (2n + 1)^2) \subset \mathcal{D}_b \), \( n \in \mathbb{N} \), and exactly two (counting with multiplicity) eigenvalues \( \lambda_{2n}^\pm \) of the operator \( H(\lambda, \pi) \) in each interval \( ((2n - 2)^2, (2n)^2) \subset \mathcal{D}_b \), \( n \in \mathbb{N} \), and there are no other eigenvalues in \( \mathcal{D}_b \).
  \item The eigenvalues \( \lambda_n(\pi) \in \mathcal{D}_b, k \in (0, \pi) \), of the operator \( H(\lambda, \pi) \) are simple and satisfy
\[
\lambda_n(k) < \lambda_{n+1}(k) < \lambda_{n+2}(k) < \lambda_{2n+1}(k) < \ldots,
\]
for some \( N \in \mathbb{N} \). Moreover,
\[
\lambda_{2n}(k) \in (\lambda_{4n-3}^+, \lambda_{4n-2}^-), \quad \frac{d\lambda_{2n}(k)}{dk} < 0, \quad \lambda_{2n}(0) = \lambda_{4n-2}^-, \quad \lambda_{2n}(\pi) = \lambda_{4n-3}^+.
\]
\end{enumerate}
\[ \lambda_{2n-1}(k) \in \left[ \lambda_{2n-4}^+, \lambda_{2n-3}^- \right], \quad \frac{d\lambda_{2n-1}(k)}{dk} > 0, \quad \lambda_{2n-1}(0) = \lambda_{2n-4}^+, \quad \lambda_{2n-1}(\pi) = \lambda_{2n-3}^- \tag{5.4} \]
where \( n = \frac{k}{2}, \frac{k}{2} + 1, \frac{k}{2} + 2, \ldots \), and recall \( \lambda_n(k) = \lambda_n(2\pi - k) \). There are no other eigenvalues of the operator \( H(k, \lambda) \) in \( \mathcal{D}_b \).

iii) There exists exactly one simple eigenvalue \( \mathbf{m}_n \) of the Dirichlet operator \( T(\lambda) \) and exactly one simple eigenvalue \( \mathbf{n}_n \) of the Neumann operator \( \mathcal{N}(\lambda) \) in each interval \( ((\pi(n - \frac{1}{2})^2, (\pi(n + \frac{1}{2})^2)) \subset \mathcal{D}_b, n \in \mathbb{N}, \) and there are no other eigenvalues in \( \mathcal{D}_b \).

iv) The Lyapunov function satisfies
\[ -1 < \Delta(\lambda) < 1, \quad \text{as} \quad \lambda \in (\lambda_{2n-1}^+, \lambda_{2n}^-), \]
\[ \Delta(\lambda) > 1, \quad \text{as} \quad \lambda \in (\lambda_{2n}^-, \lambda_{2n}^+), \]
\[ \Delta(\lambda) < -1, \quad \text{as} \quad \lambda \in (\lambda_{2n-1}^-, \lambda_{2n-1}^+), \tag{5.5} \]
for all \( n > N \), where \( N \in \mathbb{N} \) is large enough.

v) The relations (1.25) hold true.

**Proof.** i) We consider the eigenvalues of the periodic problem, the proof for the anti-periodic ones is similar. We proved in Theorem 1.3 that the eigenvalues of the periodic problem in \( \mathcal{D}_b \) are real and the identity (3.14) shows that they are zeros of the function \( \Delta(\lambda) - 1 \). The estimate (3.8) implies
\[ \Delta(\lambda) = \cos z + o(1), \quad \Delta(\lambda) - 1 = -\sin^2 \frac{z}{2} + o(1), \quad \lambda \to +\infty. \tag{5.6} \]
This asymptotics shows that for each \( n \in \mathbb{N} \) large enough there exists exactly two (counting with multiplicity) eigenvalue of the periodic problem in the interval \( ((2n-1)^2, (2n+1)^2) \) and there are no other eigenvalues in \( \mathcal{D}_b \).

ii) Lemma 5.1 and the asymptotics (5.10) show that the Lyapunov function at high energies behaves in the similar way as in the case of the Schrodinger operator with a potential, which does not depend on energy. Exactly, it oscillates as follows. It increases from -1 to 1, then either immediately starts to decrease, or first it becomes more than 1, and then it goes back to the value 1. After that, if decreases from 1 to -1, then it either immediately starts to increase, or first it becomes less than -1, and then returns back to -1. Further, the process is repeated again and again to infinity. Thus, the zeros \( \lambda_n(k) \) of the function \( \Delta - \cos k \) (the eigenvalues of the problem (1.1), (1.2)) at high energy satisfy (5.2)–(5.4).

iii) We have proved in Theorem 1.3 that the eigenvalues of the Dirichlet problem in \( \mathcal{D}_b \) are real and the identity (3.17) shows that they are zeros of the function \( \varphi(1, \lambda) = 0 \). The estimate (5.7) gives \( \varphi(1, \lambda) = \frac{1}{2}(\sin z + o(1)) \) as \( \lambda \to +\infty \). This asymptotics shows that for each \( n \in \mathbb{N} \) large enough there exists exactly one simple eigenvalue of the Dirichlet problem in the interval \( ((\pi(n - \frac{1}{2})^2, (\pi(n + \frac{1}{2})^2) \) and there are no other eigenvalues in \( \mathcal{D}_b \). The proof for the Neumann operator is similar.

iv) The identities (3.13) and (3.14), Lemma 5.1 and the asymptotics (5.6) imply (5.5).

v) We have the identities
\[ \Delta(\lambda)^2 = \left( \frac{\varphi(1, \lambda) + \varphi'(1, \lambda)}{2} \right)^2 = \left( \frac{\varphi(1, \lambda) - \varphi'(1, \lambda)}{2} \right)^2 + \varphi(1, \lambda)\varphi'(1, \lambda) \tag{5.7} \]
for all \( \lambda \in \mathcal{D} \). Let \( \lambda \in \sigma(T) \). Then \( \varphi(1, \lambda) = 0 \) and we obtain \( \Delta^2(\lambda) \geq 1 \). The estimates (5.5) give (1.25). The proof for the Neumann eigenvalues is similar. ■
Remark. Similarly we can consider the mix problems $y(0) = y'(1) = 0$ and $y'(0) = y(1) = 0$ for equation (1.11). The spectra are discrete and the large eigenvalues are simple and belong to the intervals $[\lambda_{n+1}, \lambda_n]$, $n = N, N + 1, \ldots$

5.2. Spectral asymptotics. Now we determine high energy eigenvalue asymptotics for the operator $H(k, \lambda)$.

Proposition 5.3. Let $\Pi_a(r) \subset \mathcal{D}$ for some $(a, r) \in \mathbb{R} \times \mathbb{R}_+$. Let the potential $V$ satisfy the conditions (1.22) and (1.27) and let $\|V(\cdot, \lambda)\| = z^2 o(1)$ as $\lambda \to +\infty$. Then the eigenvalues of the operator $H(k)$ satisfy

$$\lambda_n(k) = \lambda_n^0(k) + o(1), \quad \forall \ k \in (0, \pi),$$

as $n \to +\infty$, where $\lambda_n^0(k)$ are given by (5.7).

Proof. Let $0 < k < \pi$. We prove the asymptotics (5.8) for $\lambda_{2n+1}(k)$, the proof for $\lambda_{2n}(k)$ is similar. Let $\lambda = \lambda_{2n+1}(k)$ for some $n \in \mathbb{N}$ large enough. Then $z = \lambda_{2n+1}^\frac{1}{2} = 2\pi n + k + \delta, \delta = \delta_n = O(1)$, as $n \to +\infty$, and the estimate (3.39) gives

$$\Delta(\lambda) - \cos k = -2 \sin \frac{\delta}{2} \sin \left(k + \frac{\delta}{2}\right) + \frac{\sin(k + \delta)}{4\pi n} \hat{V}_o(\lambda) + \frac{o(1)}{n}. \quad (5.9)$$

The identity $\Delta(\lambda) - \cos k = 0$ gives $\delta = O(n^{-1})$ and using (1.27) and (5.9) again we obtain

$$\Delta(\lambda) - \cos k = -2 \sin \frac{\delta}{2} \sin k + \frac{o(1)}{n}.$$

Now the identity $\Delta(\lambda) - \cos k = 0$ gives $\delta = o(n^{-1})$. Then $z = 2\pi n + k + o(n^{-1})$, which yields the asymptotics (5.8). ■

Now we prove our results about the high energy asymptotics of the spectra of the operator $H(\lambda)$.

Proof of Theorem 1.4. i) Due to Theorem 1.2 the spectra are real. Lemma 5.2 ii) and the identity (1.13) give (1.24). The relation (1.25) is proved in Lemma 5.2 v).

ii) Let $\lambda = \lambda_{2n}^\pm$. Then $z = \lambda_{2n}^\frac{1}{2} = 2\pi n + \delta, \delta = \delta_n = O(1)$, as $n \to +\infty$, and the estimate (3.10) gives

$$\Delta(\lambda) - 1 = -2 \sin \frac{\delta}{2} + \frac{\sin \delta}{4\pi n} \hat{V}_o(\lambda) + \frac{o(1)}{n^2} = -2 \left(\sin \frac{\delta}{2} - \cos \frac{\delta}{4\pi n} \hat{V}_o(\lambda)\right)^2 + \frac{O(1)}{n^2}. \quad (5.10)$$

The identity $\Delta(\lambda) - 1 = 0$ implies $\delta = O(n^{-1})$. Using the asymptotics (3.10) again we obtain

$$\Delta(\lambda) - 1 = -\frac{\delta^2}{2} + \frac{\delta}{4\pi n} \hat{V}_o(\lambda) + o(n^{-2}) = -\frac{1}{2} \left(\delta - \frac{\hat{V}_o(\lambda)}{4\pi n}\right)^2 + o(n^{-2}).$$

The identity $\Delta(\lambda) - 1 = 0$ and the condition (1.27) give $\delta = o(n^{-1})$, which yields (1.28) for $n$ even. The proof for $n$ odd is similar. ■

Now we prove the results about the good Boussinesq equation.

Proof of Corollary 2.1. Let $p', q \in L^1(\mathbb{T})$. Then the solution $\psi_q$ of equation (2.1) satisfies $\psi_q''(\cdot, \zeta) \in L^1(\mathbb{T})$ for all $\zeta \in \mathcal{D}$. The definition (2.4) and the asymptotics (2.5) show that the function $V$, given by (2.6), satisfies: $\|V(\cdot, \lambda)\|$ is uniformly bounded in $\mathcal{D}$ and $|\text{Im} V(x, \zeta)|$ is uniformly bounded in $[0, 1] \times \mathcal{D}$, where the domain $\mathcal{D}$ has the form (2.7). The relation
(1.21) yields that the ramifications \( r_n^+ \) and the three-point eigenvalues \( \zeta_n \) in the half-plane \( Z_n \) are real. Lemma 5.2 gives that there are exactly two ramifications \( r_n^\pm \) and exactly one simple eigenvalue \( \zeta_n \) in each interval \((\alpha_n^-, \alpha_n^+))\) inside this half-plane. Moreover, the estimate (1.14) holds true for all \( \lambda > 0 \) large enough (see Remarks to Theorem 1.3). Then the asymptotics (1.28) implies (2.9). The relations (1.25) give (2.8).

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