Off-diagonal Bethe ansatz solutions of the anisotropic spin-$\frac{1}{2}$ chains with arbitrary boundary fields

Junpeng Cao$^a$, Wen-Li Yang$^b$, Kangjie Shi$^b$ and Yupeng Wang$^{a,2}$

$^a$Beijing National Laboratory for Condensed Matter Physics, Institute of Physics, Chinese Academy of Sciences, Beijing 100190, China

$^b$Institute of Modern Physics, Northwest University, Xian 710069, China

Abstract

The anisotropic spin-$\frac{1}{2}$ chains with arbitrary boundary fields are diagonalized with the off-diagonal Bethe ansatz method. Based on the properties of the R-matrix and the K-matrices, an operator product identity of the transfer matrix is constructed at some special points of the spectral parameter. Combining with the asymptotic behavior (for XXZ case) or the quasi-periodicity properties (for XYZ case) of the transfer matrix, the extended $T-Q$ ansatzs and the corresponding Bethe ansatz equations are derived.

PACS: 75.10.Pq, 03.65.Vf, 71.10.Pm

Keywords: Spin chain; reflection equation; Bethe Ansatz; $T-Q$ relation

1Corresponding author: wlyang@nwu.edu.cn
2Corresponding author: yupeng@iphy.ac.cn
1 Introduction

The study of exactly solvable models (or quantum integrable systems) \cite{1} has attracted a great deal of interest since Yang and Baxter’s pioneering works \cite{2, 3}. Such exact non-perturbation results have provided valuable insight into the important universality classes of quantum physical systems ranging from modern condensed matter physics \cite{4} to string and super-symmetric Yang-Mills theories \cite{5, 6, 7, 8}. Moreover, quantum integrable models are paramount for the analysis of nano-scale systems where alternative approaches involving mean field approximations or perturbations have failed \cite{9, 10}.

The quantum inverse scattering method \cite{11} (QISM) or the algebraic Bethe ansatz method has been proven to be the most powerful and (probably) unified method to construct exact solutions to the spectrum problem of commuting families of conserved charges (usually called the transfer matrix) in quantum integrable systems. In the framework of QISM, the quantum Yang-Baxter equation (QYBE), which defines the underlying algebraic structure, has become a cornerstone for constructing and solving the integrable models. So far, there have been several well-known methods for deriving the Bethe ansatz (BA) solutions of integrable models: the coordinate BA \cite{12, 13, 14, 15}, the T-Q approach \cite{1, 16}, the algebraic BA \cite{17, 18, 11, 19}, the analytic BA \cite{20}, the functional BA \cite{21} or the separation of variables method among many others \cite{22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37}.

Generally, there are two classes of integrable models: one possesses $U(1)$ symmetry and the other does not. Three well-known examples without $U(1)$ symmetry are the XYZ spin chain \cite{3, 18}, the spin chains with antiperiodic boundary condition \cite{38, 39, 33, 34, 35, 37} and the ones with unparallel boundary fields \cite{24, 25, 26, 27, 32, 33, 34, 35, 36, 37}. It has been proven that most of the conventional Bethe ansatz methods can successfully diagonalize the integrable models with $U(1)$ symmetry. However, for those without $U(1)$ symmetry, only some very special cases such as the XYZ spin chain with even site number \cite{3, 18} and the XXZ spin chain with constrained unparallel boundary fields \cite{19, 25, 26, 40} can be dealt with due to the existence of a proper “local vacuum state” in these special cases. The main obstacle for applying the algebraic Bethe ansatz and Baxter’s method to general integrable models without $U(1)$ symmetry lies in the absence of such a “local vacuum”. A promising method for approaching such kind of problems is Sklyanin’s separation of variables method \cite{21} which has been recently applied to some integrable models \cite{34, 35, 36, 37}. However,
before the very recent work [41], a systematic method was absent to derive the Bethe ansatz

equations for integrable models without $U(1)$ symmetry, which are crucial for studying the

physical properties in the thermodynamic limit.

As for integrable models without $U(1)$ symmetry, the transfer matrix contains not only

the diagonal elements but also some off-diagonal elements of the monodromy matrix. This

breaks down the usual $U(1)$ symmetry. Very recently, a systematic method [41] for dealing

with such kind of models was proposed by the present authors, which has been shown [42] to

successfully construct the exact solutions of the open XXX chain with unparallel boundary

fields and the closed XYZ chain with odd site number. The central idea of the method is

to construct a proper $T-Q$ ansatz with an extra off-diagonal term (comparing with the

ordinary ones) based on the functional relations between eigenvalues $\Lambda(u)$ of the transfer

matrix (the trace of the monodromy matrix) and the quantum determinant $\Delta_q(u)$, (e.g. see

below (4.9) and (5.22)) at some special points of the spectral parameter $u = \theta_j$, i.e.,

$$\Lambda(\theta_j)\Lambda(\theta_j - \eta) \sim \Delta_q(\theta_j).$$

(1.1)

Since the trace and the determinant are two basic quantities of a matrix which are independent

of the representation basis, this method could overcome the obstacle of absence of a reference state which is crucial in the conventional Bethe ansatz methods. Moreover, in this paper we will show that the above relation can be lifted to operator level (see below (3.8)) based on some properties of the R-matrix and K-matrices.

Our primary motivation for this work comes from the longstanding problem of solving the

anisotropic spin-$\frac{1}{2}$ chain with arbitrary boundary fields, defined by the Hamiltonian [43, 44]

$$H = \sum_{j=1}^{N-1} (J_x \sigma_j^x \sigma_{j+1}^x + J_y \sigma_j^y \sigma_{j+1}^y + J_z \sigma_j^z \sigma_{j+1}^z) + \vec{h}(-) \cdot \vec{\sigma}_1 + \vec{h}(+) \cdot \vec{\sigma}_N,$$

(1.2)

where $N$ is the site number of the system; $\sigma_j^\alpha (\alpha = x, y, z)$ is the Pauli matrix on the site $j$

along the $\alpha$ direction; $\vec{h}(-) = (h_x^(-), h_y^(-), h_z^(-))$ and $\vec{h}(+) = (h_x^(+), h_y^(+), h_z^(+))$ are the boundary magnetic fields; $J_\alpha (\alpha = x, y, z)$ are the coupling constants. Solving this problem for generic values of boundary fields is a crucial step for understanding a variety of physical systems without $U(1)$ symmetry. In this paper, we shall use the method developed in [41] to solve the eigenvalue problem of the above Hamiltonian with generic $\vec{h}(\pm)$.

The paper is organized as follows. Section 2 serves as an introduction of our notations and

some basic ingredients. We briefly describe the inhomogeneous spin chains with periodic and
antiperiodic boundary conditions and open spin chains with the most general non-diagonal boundary terms. In Section 3, based on the properties of the $R$-matrix we derive the operator product identities of the transfer matrices for the various closed and open spin chains at some special points of the spectral parameter. In section 4, the $T-Q$ ansatz for the eigenvalues of the transfer matrix and the corresponding Bethe ansatz equations (BAEs) of the open XXZ spin chain are constructed based on the operator product identities of the transfer matrix and its asymptotic behaviors. Section 5 is attributed to the open XYZ spin chain. In section 6, we summarize our results and give some discussions. Some detailed technical proof is given in Appendix A&B.

## 2 Transfer matrix

Throughout, $V$ denotes a two-dimensional linear space. The R-matrix $R(u) \in \text{End}(V \otimes V)$ is a solution of the quantum Yang-Baxter equation (QYBE)

$$R_{12}(u_1 - u_2)R_{13}(u_1 - u_3)R_{23}(u_2 - u_3) = R_{23}(u_2 - u_3)R_{13}(u_1 - u_3)R_{12}(u_1 - u_2). \quad (2.1)$$

It is well-known that there are three standard classes of solutions (or R-matrices), i.e., the elliptic (or the eight-vertex), the trigonometric and rational (or six-vertex) R-matrix. The R-matrix is given by

$$R(u) = \begin{pmatrix} a(u) & b(u) & c(u) & d(u) \\ b(u) & c(u) & d(u) & a(u) \\ c(u) & d(u) & a(u) & b(u) \\ d(u) & a(u) & b(u) & c(u) \end{pmatrix}, \quad (2.2)$$

The non-vanishing matrix elements of the eight-vertex R-matrix are [1]

$$a(u) = \frac{\theta \left[ \frac{0}{2}, (u, 2\tau) \right] \theta \left[ \frac{0}{2}, (u + \eta, 2\tau) \right]}{\theta \left[ \frac{0}{2}, (0, 2\tau) \right] \theta \left[ \frac{1}{2}, (\eta, 2\tau) \right]}, \quad b(u) = \frac{\theta \left[ \frac{1}{2}, (u, 2\tau) \right] \theta \left[ \frac{0}{2}, (u + \eta, 2\tau) \right]}{\theta \left[ \frac{0}{2}, (0, 2\tau) \right] \theta \left[ \frac{1}{2}, (\eta, 2\tau) \right]}, \quad (2.3)$$

$$c(u) = \frac{\theta \left[ \frac{0}{2}, (u, 2\tau) \right] \theta \left[ \frac{0}{2}, (u + \eta, 2\tau) \right]}{\theta \left[ \frac{0}{2}, (0, 2\tau) \right] \theta \left[ \frac{1}{2}, (\eta, 2\tau) \right]}, \quad d(u) = \frac{\theta \left[ \frac{1}{2}, (u, 2\tau) \right] \theta \left[ \frac{1}{2}, (u + \eta, 2\tau) \right]}{\theta \left[ \frac{0}{2}, (0, 2\tau) \right] \theta \left[ \frac{0}{2}, (\eta, 2\tau) \right]}, \quad (2.4)$$

while the non-vanishing matrix elements of the six-vertex R-matrix are [11]

$$a(u) = \frac{\varphi(u + \eta)}{\varphi(\eta)}, \quad b(u) = \frac{\varphi(u)}{\varphi(\eta)}, \quad (2.5)$$

$$c(u) = 1, \quad d(u) = 0. \quad (2.6)$$
Here the generic complex number $\eta$ is the so-called crossing parameter, the definitions of elliptic functions are given in Appendix A and the function $\varphi(u)$ is defined as

$$\varphi(u) = \begin{cases} 
\sinh(u) & \text{for the trigonometric case,} \\
u & \text{for the rational case.}
\end{cases}$$

All the R-matrices possess the following properties,

- **Initial condition**: $R_{12}(0) = P_{12}$,  
  (2.7)
- **Unitarity relation**: $R_{12}(u)R_{21}(-u) = -\xi(u)\text{id}$,  
  (2.8)
- **Crossing relation**: $R_{12}(u) = V_1 R_{12}^t(-u - \eta)V_1$,  
  $V = -i\sigma^y$,  
  (2.9)
- **PT-symmetry**: $R_{12}(u) = R_{21}(u) = R_{12}^t R_{12}^{t_2}(u)$,  
  (2.10)
- **$Z_2$-symmetry**: $\sigma^i_1 \sigma^i_2 R_{1,2}(u) = R_{1,2}(u)\sigma^i_1 \sigma^i_2$, for $i = x, y, z$,  
  (2.11)
- **Antisymmetry**: $R_{12}(-\eta) = -(1 - P) = -2P(-)$.  
  (2.12)

Here $\sigma^i (i = x, y, z)$ is the usual Pauli matrix, $R_{21}(u) = P_{12} R_{12}(u) P_{12}$ with $P_{12}$ being the usual permutation operator and $t_i$ denotes transposition in the $i$-th space. The function $\xi(u)$ is given by

$$\xi(u) = \begin{cases} 
\frac{\sigma(u + \eta)\sigma(u - \eta)}{\sigma(\eta)\sigma(\eta)} & \text{for the eight – vertex case,} \\
\frac{\varphi(u + \eta)\varphi(u - \eta)}{\varphi(\eta)\varphi(\eta)} & \text{for the six – vertex case.}
\end{cases}$$

(2.13)

Here and below we adopt the standard notations: for any matrix $A \in \text{End}(V), A_j$ is an embedding operator in the tensor space $V \otimes V \otimes \cdots$, which acts as $A$ on the $j$-th space and as identity on the other factor spaces; $R_{ij}(u)$ is an embedding operator of R-matrix in the tensor space, which acts as identity on the factor spaces except for the $i$-th and $j$-th ones.

We introduce the “row-to-row” (or one-row) monodromy matrices $T_0(u)$ and $\hat{T}_0(u)$, which are $2 \times 2$ matrices with elements being operators acting on $V^\otimes N$,

$$T_0(u) = R_{0N}(u - \theta_N) R_{0N-1}(u - \theta_{N-1}) \cdots R_{01}(u - \theta_1),$$

(2.14)

$$\hat{T}_0(u) = R_{01}(u + \theta_1) R_{02}(u + \theta_2) \cdots R_{0N}(u + \theta_N).$$

(2.15)

Here $\{\theta_j | j = 1, \cdots, N\}$ are arbitrary free complex parameters which are usually called as inhomogeneous parameters. The transfer matrix $t^{(\nu)}$ of the spin chain with periodic boundary condition (or closed chain) is given by

$$t^{(\nu)}(u) = tr_0(T_0(u)).$$

(2.16)
The QYBE (2.1) leads to the fact that the transfer matrices with different spectral parameters commute with each other [17]: 
\[ [t^{(p)}(u), t^{(p)}(v)] = 0. \]
Then \( t^{(p)}(u) \) serves as the generating functional of the conserved quantities of the corresponding system, which ensures the integrability of the closed spin chain.

Integrable open chain can be constructed as follows [43]. Let us introduce a pair of K-matrices \( K^-(u) \) and \( K^+(u) \). The former satisfies the reflection equation (RE)
\[
R_{12}(u_1 - u_2)K^-_1(u_1)R_{21}(u_1 + u_2)K^-_2(u_2) = K^-_2(u_2)R_{12}(u_1 + u_2)K^-_1(u_1)R_{21}(u_1 - u_2),
\]
and the latter satisfies the dual RE
\[
R_{12}(u_2 - u_1)K^+_1(u_1)R_{21}(-u_1 - u_2 - 2)K^+_2(u_2) = K^+_2(u_2)R_{12}(-u_1 - u_2 - 2)K^+_1(u_1)R_{21}(u_2 - u_1).
\]
For open spin chains, other than the standard “row-to-row” monodromy matrix \( T_0(u) \) (2.14), one needs to consider the double-row monodromy matrix \( \hat{T}_0(u) \)
\[
T_0(u) = T_0(u)K^-_0(u)\hat{T}_0(u).
\]
Then the double-row transfer matrix \( t(u) \) of the spin chain with open boundary (or the open spin chain) is given by
\[
t(u) = tr_0(K^+_0(u)\hat{T}_0(u)).
\]
The QYBE (2.1) and (dual) REs (2.17) and (2.18) lead to the fact that the transfer matrices with different spectral parameters commute with each other [43]: \( [t(u), t(v)] = 0. \) Then \( t(u) \) serves as the generating functional of the conserved quantities of the corresponding system, which ensures the integrability of the open spin chain.

The integrable Hamiltonian (1.2) can be obtained from the transfer matrix as follows. For the XXZ case
\[
H = \sinh \eta \frac{\partial \ln t(u)}{\partial u} |_{u=0, \theta_j=0} - N \cosh \eta - \tanh \eta \sinh \eta
\]
\[
= 2 \sinh \eta \sum_{j=1}^{N-1} P_{j,j+1}R'_{j,j+1}(0) + \sinh \eta \frac{tr_0K^+_0(0)}{tr_0K^+_0(0)} + 2 \sinh \eta \frac{tr_0K^+P_{N,0}R'_{N,0}(0)}{tr_0K^+_0(0)}
\]
\[
+ \sinh \eta \frac{K^-_1(0)}{K^-_1(0)} - N \cosh \eta - \tanh \eta \sinh \eta.
\]
\[ \sum_{j=1}^{N-1} \left[ \sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \cosh \eta \sigma_j^z \sigma_{j+1}^z \right] \]

\[ + \frac{\sinh \eta}{\sinh \alpha_- \cosh \beta_-} (\cosh \alpha_- \sinh \beta_- \sigma_1^z + \cosh \theta_- \sigma_1^x + i \sinh \theta_- \sigma_1^y) \]

\[ + \frac{\sinh \eta}{\sinh \alpha_+ \cosh \beta_+} (-\cosh \alpha_+ \sinh \beta_+ \sigma_N^z + \cosh \theta_+ \sigma_N^x + i \sinh \theta_+ \sigma_N^y). \quad (2.21) \]

The corresponding K-matrices are the most general solutions \( K^\pm(u) \) in \cite{44,45} (see below the Eqs.(4.1)-(4.2)). For the XYZ case,

\[ H = \frac{\sigma(\eta)}{\sigma'(0)} \left\{ \frac{\partial}{\partial u} \ln t(u) \right|_{u=0} - [(N-1)\zeta(\eta) + 2\zeta(2\eta)] \right\} \]

\[ = \sum_{j=1}^{N-1} (J_x \sigma_j^x \sigma_{j+1}^x + J_y \sigma_j^y \sigma_{j+1}^y + J_z \sigma_j^z \sigma_{j+1}^z) + h_{x}^{(-)} \sigma_1^x + h_{y}^{(-)} \sigma_1^y + h_{z}^{(-)} \sigma_1^z \]

\[ + h_x^{(+)} \sigma_N^x + h_y^{(+)} \sigma_N^y + h_z^{(+)} \sigma_N^z. \quad (2.22) \]

with

\[ J_x = \frac{e^{i\pi \eta} \sigma(\eta + \frac{\tau}{2})}{\sigma(\frac{\tau}{2})}, \quad J_y = \frac{e^{i\pi \eta} \sigma(\eta + \frac{1}{2} + \frac{\tau}{2})}{\sigma(\frac{1}{2} + \frac{\tau}{2})}, \quad J_z = \frac{\sigma(\eta + \frac{1}{2})}{\sigma(\frac{1}{2})}, \]

and

\[ h_{x}^{(+)} = \frac{\sigma(\eta)}{\sigma(\frac{\tau}{2})} \prod_{l=1}^{3} \frac{\sigma(\alpha_l^{(-)} - \frac{1}{2})}{\sigma(\alpha_l^{(-)})}, \]

\[ h_{x}^{(-)} = \frac{e^{-i\pi (\sum_{l=1}^{3} \alpha_l^{(-)} - \frac{\tau}{2})}}{\sigma(\frac{\tau}{2})} \prod_{l=1}^{3} \frac{\sigma(\alpha_l^{(+)} - \frac{\tau}{2})}{\sigma(\alpha_l^{(+)})}, \]

\[ h_{y}^{(+)} = \frac{e^{-i\pi (\sum_{l=1}^{3} \alpha_l^{(-)} - \frac{1}{2} - \frac{\tau}{2})}}{\sigma(\frac{1}{2} + \frac{\tau}{2})} \prod_{l=1}^{3} \frac{\sigma(\alpha_l^{(+)} - \frac{1}{2} - \frac{\tau}{2})}{\sigma(\alpha_l^{(+)})}. \quad (2.23) \]

Here \( \sigma(u) \) is the \( \sigma \)-function defined by \[\text{(A.2)}\] and \( \{\alpha_l^{(\pm)}\} \) are parameters contained in the most general K-matrices \cite{46,47} (see below Eqs.(5.1) and (5.2)).

### 3 Operator identity of the transfer matrix

#### 3.1 Closed chains

In order to get some functional relations of the transfer matrix, we evaluate the transfer matrix \( t^{(p)}(u) \) at some particular points such as \( \theta_j \) and \( \theta_j - \eta \). Using the similar procedure
as in [48, 49], we apply the initial condition (2.7) of the R-matrix to express the transfer matrix \( t^{(p)}(\theta_j) \) as

\[
t^{(p)}(\theta_j) = tr_0 \{ R_{0N}(\theta_j - \theta_N) \ldots R_{0j+1}(\theta_j - \theta_{j+1}) P_{0j} R_{0j-1}(\theta_j - \theta_{j-1}) \ldots R_{01}(\theta_j - \theta_1) \}
\]

\[
= R_{j-1}(\theta_j - \theta_{j-1}) \ldots R_{j1}(\theta_j - \theta_1) tr_0 \{ R_{0N}(\theta_j - \theta_N) \ldots R_{0j+1}(\theta_j - \theta_{j+1}) P_{0j} \}
\]

\[
= R_{j-1}(\theta_j - \theta_{j-1}) \ldots R_{j1}(\theta_j - \theta_1) R_{jN}(\theta_j - \theta_N) \ldots R_{j,j+1}(\theta_j - \theta_{j+1}).
\]

In deriving the above equation, we have used the identity: \( tr_0(P_{0j}) = id_j \). The crossing relation (2.9) and \( V^2 = -1 \) enable one to express the transfer matrix \( t^{(p)}(\theta_j - \eta) \) as

\[
t^{(p)}(\theta_j - \eta) = (-1)^N tr_0 \{ R_{0N}^0(-\theta_j + \theta_N) \ldots R_{01}^0(-\theta_j + \theta_1) \}
\]

\[
= (-1)^N tr_0 \{ R_{01}(-\theta_j + \theta_1) \ldots R_{0N}(-\theta_j + \theta_N) \}
\]

\[
= (-1)^N R_{j,j+1}(-\theta_j + \theta_{j+1}) \ldots R_{jN}(-\theta_j + \theta_N)
\]

\[
\times R_{j1}(-\theta_j + \theta_1) \ldots R_{j,j-1}(-\theta_j + \theta_{j-1}).
\]

Using the unitarity relation (2.8), we have

\[
t^{(p)}(\theta_j) t^{(p)}(\theta_j - \eta) = \Delta^{(p)}(\theta_j) = \prod_{l=1}^N \xi(\theta_j - \theta_l) \times id, \tag{3.1}
\]

where the function \( \xi(u) \) is given by (2.13). The commutativity of the transfer matrices with different spectrum implies that they have common eigenstates. Let \( |\Psi\rangle \) be an eigenstate of \( t^{(p)}(u) \), which does not depend upon \( u \), with the eigenvalue \( \Lambda^{(p)}(u) \), i.e.,

\[
t^{(p)}(u)|\Psi\rangle = \Lambda^{(p)}(u)|\Psi\rangle.
\]

The very operator identity (3.1) leads to that the corresponding eigenvalue \( \Lambda^{(p)}(u) \) satisfies the following relation

\[
\Lambda^{(p)}(\theta_j) \Lambda^{(p)}(\theta_j - \eta) = \prod_{l=1}^N \xi(\theta_j - \theta_l), \quad j = 1, \ldots, N. \tag{3.2}
\]

Some remarks are in order. The QYBE (2.1) and the \( Z_2 \)-symmetry (2.11) of the R-matrix imply that the corresponding spin chain with antiperiodic (or twisted ) boundary condition is also integrable, the corresponding transfer matrix \( t^{(ap)}(u) \) can be given by

\[
t^{(ap)}(u) = tr_0(\sigma_0 x T_0(u)). \tag{3.3}
\]
Noticing that $V\sigma^x = -\sigma^x V$, one can easily derive the following relation

$$t^{(ap)}(\theta_j)t^{(ap)}(\theta_j - \eta) = -\Delta^{(p)}_\theta(\theta_j) = -\prod_{l=1}^{N} \xi(\theta_j - \theta_l) \times \text{id}. \quad (3.4)$$

This leads to that the corresponding eigenvalue $\Lambda^{(ap)}(u)$ satisfies the following relation

$$\Lambda^{(ap)}(\theta_j)\Lambda^{(ap)}(\theta_j - \eta) = -\prod_{l=1}^{N} \xi(\theta_j - \theta_l), \quad j = 1, \ldots, N. \quad (3.5)$$

The above relations for the trigonometric case were obtained in $[41]$ by solving some recursion relation. Similar relations were also derived in $[37]$ by the separation of variables method.

### 3.2 Open chains

Now we are in position to compute the transfer matrix $t(u)$ of the open chain at $\theta_j$ and $\theta_j - \eta$. Following the similar procedure as in $[50, 51]$, we have that

$$t(\theta_j) = R_{j,j-1}(\theta_j - \theta_{j-1}) \ldots R_{j1}(\theta_j - \theta_1)K_j^-(\theta_j)R_{ij}(\theta_1 + \theta_j) \ldots R_{j-1,j}(\theta_{j-1} + \theta_j)$$
$$\times \text{tr}_0\{K_0^+(\theta_j)R_{0N}(\theta_j - \theta_N) \ldots R_{0j+2}(\theta_j - \theta_{j+2})R_{0j+1}(\theta_j - \theta_{j+1})$$
$$\times P_{0j}R_{0j}(2\theta_j)R_{j+10}(\theta_{j+1} + \theta_j)R_{j+20}(\theta_{j+2} + \theta_j) \ldots R_{N0}(\theta_N + \theta_j)\}\}

Using QYBE $[2.1]$, we have

$$R_{0j+1}(\theta_j - \theta_{j+1})P_{0j}R_{j0}(2\theta_j)R_{j+10}(\theta_{j+1} + \theta_j)$$
$$= R_{0j+1}(\theta_j - \theta_{j+1})R_{0j}(2\theta_j)R_{j+1j}(\theta_{j+1} + \theta_j)P_{0j}$$
$$= R_{j+1j}(\theta_{j+1} + \theta_j)R_{0j}(2\theta_j)R_{0j+1}(\theta_j - \theta_{j+1})P_{0j}$$
$$= R_{j+1j}(\theta_{j+1} + \theta_j)P_{0j}R_{j0}(2\theta_j)R_{j+1j}(\theta_j - \theta_{j+1}).$$

This gives rise to

$$t(\theta_j) = R_{j,j-1}(\theta_j - \theta_{j-1}) \ldots R_{j1}(\theta_j - \theta_1)K_j^-(\theta_j)R_{ij}(\theta_1 + \theta_j) \ldots R_{j-1,j}(\theta_{j-1} + \theta_j)$$
$$\times R_{j+1j}(\theta_{j+1} + \theta_j) \ldots R_{Nj}(\theta_N + \theta_j)\text{tr}_0\{K_0^+(\theta_j)P_{0j}R_{j0}(2\theta_j)\}$$
$$\times R_{jN}(\theta_j - \theta_N) \ldots R_{j,j+1}(\theta_j - \theta_{j+1}). \quad (3.6)$$

The crossing relation $[2.9]$ of the R-matrix implies

$$t(\theta_j - \eta) = \text{tr}_0\{V_0K_0^+(\theta_j - \eta)V_0R_{0j}(\theta_j + \theta_N) \ldots R_{0j}(\theta_j + \theta_1)$$

$$\times R_{jN}(\theta_j - \theta_N) \ldots R_{j,j+1}(\theta_j - \theta_{j+1}). \quad (3.6)$$

The crossing relation $[2.9]$ of the R-matrix implies
In the derivation of the functional relation (3.8), the following identities have been used

\[ t(\theta_j) t(\theta_j - \eta) = -\frac{\Delta_q^{(o)}(\theta_j)}{\xi(2\theta_j)}. \]  

(3.8)

For generic \{\theta_j\}, the quantum determinant operator is proportional to the identity operator

\[ \Delta_q^{(o)}(u) = \delta(u) \times \text{id}, \]  

(3.9)

The expression of the function \( \delta(u) \) is given by \[52, 53\]

\[ \delta(u) = \text{Det}_q\{T(u)\} \text{ Det}_q\{\tilde{T}(u)\} \text{ Det}_q\{K^-(u)\} \text{ Det}_q\{K^+(u)\}, \]  

(3.10)

and the above various determinants are

\[ \text{Det}_q\{T(u)\} \text{ id} = tr_{12} \left( P_{12}^{(-)} T_1(u - \eta) T_2(u) P_{12}^{(-)} \right) = \prod_{j=1}^{N} \xi(u - \theta_j) \text{id}, \]

\[ \text{Det}_q\{\tilde{T}(u)\} \text{ id} = tr_{12} \left( P_{12}^{(-)} \tilde{T}_1(u - \eta) \tilde{T}_2(u) P_{12}^{(-)} \right) = \prod_{j=1}^{N} \xi(u + \theta_j) \text{id}, \]

\[ \text{Det}_q\{K^-(u)\} = tr_{12} \left( P_{12}^{(-)} K_1^{-} (u - \eta) R_{12}^{(-)} (2u - \eta) K_2^{-} (u) \right), \]  

(3.11)

\[ \text{Det}_q\{K^+(u)\} = tr_{12} \left( P_{12}^{(-)} K_2^{+} (2u - \eta) R_{12}^{(-)} (u - \eta) K_1^{+} (2u - \eta) \right). \]  

(3.12)

In the derivation of the functional relation (3.8), the following identities have been used

\[ K_j^{-} (u) tr_0 \left\{ (V_0 K_0^{-} (u - \eta) V_0)^{t_0} P_{0j} R_{j0}(2u) \right\} = \text{Det}_q \{K^-(u)\} \times \text{id}_j, \]  

(3.13)

\[ tr_0 \left\{ K_0^{+} (u) P_{0j} R_{j0}(2u) \right\} \left\{ V_j K_j^{+} (u - \eta) V_j \right\}^{t_1} = \text{Det}_q \{K^+(u)\} \times \text{id}_j. \]  

(3.14)
The proof of the above equations is relegated to Appendix B.

In the following two sections, we shall demonstrate how to use the above operator identity to construct the off-diagonal Bethe ansatz solutions of the open chains with the most general boundary terms.

4 Results for the open XXZ chain

4.1 Functional relations

The most general solutions \( K^\pm(u) \) to the reflection equation and its dual associated with the trigonometric six-vertex R-matrix (for the rational case see [42]), or the XXZ chain, are given respectively by

\[
K^-(u) = \begin{pmatrix} K_{11}(u) & K_{12}(u) \\ K_{21}(u) & K_{22}(u) \end{pmatrix},
\]

\[
K_{11}(u) = 2 (\sinh(\alpha_-) \cosh(\beta_-) \cosh(u) + \cosh(\alpha_-) \sinh(\beta_-) \sinh(u)),
\]

\[
K_{22}(u) = 2 (\sinh(\alpha_-) \cosh(\beta_-) \cosh(u) - \cosh(\alpha_-) \sinh(\beta_-) \sinh(u)),
\]

\[
K_{12}(u) = e^\theta_- \sinh(2u), \quad K_{21}(u) = e^{-\theta_-} \sinh(2u),
\]

(4.1)

and

\[
K^+(u) = K^-(u - \eta)|_{(\beta_-, \theta_-) \to (-\alpha_+, -\beta_+, \theta_+)}.
\]

(4.2)

Here \( \alpha_+, \beta_+, \theta_+ \) are the boundary parameters which are associated with boundary field terms (see (2.21)). Then the associated function \( \delta(u) \) defined by (3.10) reads [24]

\[
\delta(u) = \frac{-2^4 \sinh(2u - 2\eta) \sinh(2u + 2\eta)}{\sinh^2(\eta)} \sinh(u + \alpha_-) \sinh(u - \alpha_-) \cosh(u + \beta_-) \cosh(u - \beta_-) \times \sinh(u + \alpha_+ \sinh(u - \alpha_+) \cosh(u + \beta_+) \cosh(u - \beta_+)}
\]

\[
\times \prod_{l=1}^N \frac{\sinh(u + \theta_l + \eta) \sinh(u - \theta_l + \eta) \sinh(u + \theta_l - \eta) \sinh(u - \theta_l - \eta)}{\sinh^4(\eta)}.
\]

(4.3)

Following the method in [54, 55] and using the crossing relation of the R-matrix (2.9) and the explicit expressions of the K-matrices (4.1) and (4.2), one can show that the corresponding transfer matrix \( t(u) \) satisfies the following crossing relation

\[
t(-u - \eta) = t(u).
\]

(4.4)
The quasi-periodicity of the R-matrix and K-matrices

\[ R_{12}(u + i\pi) = -\sigma^z_1 R_{12}(u) \sigma^z_1 = -\sigma^z_2 R_{12}(u) \sigma^z_2, \quad K^\pm(u + i\pi) = -\sigma^z K^\pm(u) \sigma^z, \]

and the explicit expression of the K-matrix \( K^-(u) \) given by (4.1) and its special values at \( u = 0, \frac{i\pi}{2} \):

\[ K^-(0) = \frac{1}{2} tr(K^-(0)) \times \text{id}, \quad K^-(\frac{i\pi}{2}) = \frac{1}{2} tr(K^-(\frac{i\pi}{2})\sigma^z) \times \sigma^z, \]

allow one to derive that the associated transfer matrix satisfies the following properties

\[ t(u + i\pi) = t(u), \quad \text{(4.5)} \]

\[ t(0) = -2^3 \sinh \alpha_- \cosh \beta_- \sinh \alpha_+ \cosh \beta_+ \cosh \eta \times \prod_{l=1}^{N} \frac{\sinh(\eta - \theta_l) \sinh(\eta + \theta_l)}{\sinh^2 \eta} \times \text{id}, \quad \text{(4.6)} \]

\[ t(\frac{i\pi}{2}) = -2^3 \cosh \alpha_- \sinh \beta_- \cosh \alpha_+ \sinh \beta_+ \cosh \eta \times \prod_{l=1}^{N} \frac{\sinh(\frac{i\pi}{2} + \theta_l + \eta) \sinh(\frac{i\pi}{2} + \theta_l - \eta)}{\sinh^2 \eta} \times \text{id}, \quad \text{(4.7)} \]

\[ \lim_{u \to \pm\infty} t(u) = -\frac{\cosh(\theta_- - \theta_+) e^{\pm((2N+4)u+(N+2)\eta)}}{2^{2N+1} \sinh^2 \eta} \times \text{id} + \ldots \quad \text{(4.8)} \]

We shall demonstrate how the very identity (3.3), the explicit expression (4.3) of the model, and (4.4)-(4.8) allow us to completely determine the eigenvalues of the corresponding transfer matrix of the open XXZ chain with the most generic K-matrices given by (4.1)-(4.2).

The commutativity of the transfer matrix \( t(u) \) implies that one can find the common eigenstates of \( t(u) \), which indeed do not depend upon \( u \). Suppose \( |\Psi\rangle \) is an eigenstate of \( t(u) \) with an eigenvalue \( \Lambda(u) \), namely,

\[ t(u)|\Psi\rangle = \Lambda(u)|\Psi\rangle. \]

The very operator identity (3.3) of the six-vertex model implies the corresponding eigenvalue \( \Lambda(u) \) satisfies the similar relation

\[ \Lambda(\theta_j)\Lambda(\theta_j - \eta) = \frac{\delta(\theta_j) \sinh \eta \sinh \eta}{\sinh(\eta - 2\theta_j) \sinh(\eta + 2\theta_j)}, \quad j = 1, \ldots, N, \quad \text{(4.9)} \]

where the function \( \delta(u) \) is given by (4.3). Similar relation for the open XXX chain was also previously obtained in [56] by the separation of variables method. The special case of
the identity (4.9), when one of K-matrices $K^\pm(u)$ is diagonal or lower triangle matrix, was derived in [57] by the separation of variables method.

The properties of the transfer matrix $t(u)$ given by (4.4)-(4.8) imply that the corresponding eigenvalue $\Lambda(u)$ satisfies the following relations:

$$
\Lambda(-u - \eta) = \Lambda(u), \quad \Lambda(u + i\pi) = \Lambda(u),
$$
(4.10)

$$
\Lambda(0) = -2^3 \sinh \alpha_- \cosh \beta_- \sinh \alpha_+ \cosh \beta_+ \cosh \eta \times \prod_{l=1}^{N} \frac{\sinh(\eta - \theta_l) \sinh(\eta + \theta_l)}{\sinh^2 \eta},
$$
(4.11)

$$
\Lambda\left(\frac{i\pi}{2}\right) = -2^3 \cosh \alpha_- \sinh \beta_- \cosh \alpha_+ \sinh \beta_+ \cosh \eta \times \prod_{l=1}^{N} \frac{\sinh\left(\frac{i\pi}{2} + \theta_l + \eta\right) \sinh\left(\frac{i\pi}{2} + \theta_l - \eta\right)}{\sinh^2 \eta},
$$
(4.12)

$$
\lim_{u \to \pm \infty} \Lambda(u) = -\frac{\cosh(\theta_- - \theta_+) e^{\pm i(2N+4)\eta u + (N+2)\eta}}{2^{2N+1} \sinh^{2N} \eta} + \ldots.
$$
(4.13)

The asymptotic behavior (4.13), the second relation of (4.10), and the analyticity of the R-matrix and K-matrices and independence on $u$ of the eigenstate lead to the fact that the eigenvalue $\Lambda(u)$ further possesses the property

$$
\Lambda(u), \text{ as an entire function of } u, \text{ is a trigonometric polynomial of degree } 2N + 4. \quad (4.14)
$$

Therefore the very functional relations (4.9)-(4.14) completely determine the function $\Lambda(u)$. Here we construct the solutions of these equations in terms of a generalized $T-Q$ ansatz formulism developed in [41]. For this purpose, we introduce the following functions

$$
\bar{A}(u) = \prod_{l=1}^{N} \frac{\sinh(u - \theta_l + \eta) \sinh(u + \theta_l + \eta)}{\sinh^2 \eta},
$$

$$
\bar{a}(u) = -2^2 \frac{\sinh(2u + 2\eta)}{\sinh(2u + \eta)} \sinh(u - \alpha_-) \cosh(u - \beta_-) \times \sinh(u - \alpha_+) \cosh(u - \beta_+) \bar{A}(u),
$$
(4.15)

$$
\bar{d}(u) = \bar{a}(-u - \eta).
$$
(4.16)

### 4.2 $T-Q$ ansatz for even $N$

Motivated by the results of [41][42], we introduce

$$
\Lambda(u) = \bar{a}(u) \frac{Q_1(u - \eta)}{Q_2(u)} + \bar{d}(u) \frac{Q_2(u + \eta)}{Q_1(u)} \quad \text{for even } N.
$$

\[\text{3}\text{Some deformed } T-Q \text{ ansatz for the eigenvalues of the graded XXZ open chain was introduced in [58].}\]
where the functions $Q_1(u)$ and $Q_2(u)$ are some trigonometric polynomials parameterized by $N$ Bethe roots $\{\mu_j|j = 1, \ldots, N\}$ as follows,

\[ Q_1(u) = \prod_{j=1}^{N} \frac{\sinh(u - \mu_j)}{\sinh(\eta)}, \]

\[ Q_2(u) = \prod_{j=1}^{N} \frac{\sinh(u + \mu_j + \eta)}{\sinh(\eta)} = Q_1(-u - \eta). \]

the parameters $\bar{c}$ is determined by the boundary parameters and $\mu_j$

\[ \bar{c} = \cosh((N + 1)\eta + \alpha_- + \beta_- + \alpha_+ + \beta_+ + 2 \sum_{j=1}^{N} \mu_j) - \cosh(\theta_- - \theta_+). \]

The above relation leads to the fact that the asymptotic behavior (4.13) of the eigenvalue $\Lambda(u)$ is automatically satisfied. Let us evaluate the function (4.17) at points $\theta_j$ and $\theta_j - \eta$

\[ \Lambda(\theta_j) = \bar{a}(\theta_j) \frac{Q_1(\theta_j - \eta)}{Q_2(\theta_j)}, \quad \Lambda(\theta_j - \eta) = \bar{b}(\theta_j - \eta) \frac{Q_2(\theta_j)}{Q_1(\theta_j - \eta)}, \quad j = 1, \ldots, N, \]

which yields that

\[ \Lambda(\theta_j)\Lambda(\theta_j - \eta) = \bar{a}(\theta_j)\bar{b}(\theta_j - \eta), \quad j = 1, \ldots, N. \]

This implies that the function $\Lambda(u)$ indeed satisfies the required identities (4.9). If the $N$ parameters $\{\mu_j|j = 1, \ldots, N\}$ satisfy the following Bethe ansatz equations (BAEs)

\[ \frac{2\bar{c}\sinh(2\mu_j)\sinh(2\mu_j + 2\eta) \bar{A}(\mu_j)\bar{A}(-\mu_j - \eta)}{d(\mu_j)Q_2(\mu_j)Q_2(\mu_j + \eta)} = -1, \quad j = 1, \ldots, N, \]

with the following selection rule for the roots of the above equations

\[ \mu_j \neq \mu_l \quad \text{and} \quad \mu_j \neq -\mu_l - \eta, \]

the function $\Lambda(u)$ becomes the solution of (4.9)-(4.14).

The Bethe ansatz equation (4.21) in the homogeneous limit $\theta_j = 0$ reads

\[ \frac{\bar{c}\sinh(2\mu_j + \eta)\sinh(2\mu_j + 2\eta)}{2\sinh(\mu_j + \alpha_- + \eta)\cosh(\mu_j + \beta_- + \eta)\sinh(\mu_j + \alpha_+ + \eta)\cosh(\mu_j + \beta_+ + \eta)} = \prod_{l=1}^{N} \frac{\sinh(\mu_j + \mu_l + \eta)\sinh(\mu_j + \mu_l + 2\eta)}{\sinh(\mu_j + \eta)\sinh(\mu_j + \eta)}, \quad j = 1, \ldots, N. \]
The eigenvalue of the Hamiltonian is given by

\[
E = -\sinh \eta \left[ \coth(\alpha_-) + \tanh(\beta_-) + \coth(\alpha_+) + \tanh(\beta_+) \right] \\
-2 \sinh \eta \sum_{j=1}^{N} \coth(\mu_j + \eta) + (N - 1) \cosh \eta.
\]

(4.24)

Some remarks are in order. In [42], a more general form of \(T - Q\) ansatz was proposed, which involves parameters \(\lambda_j, \mu_j, \text{ and } \nu_j\). However, the numerical analysis of the solutions of the associated Bethe ansatz equations (the rational version of the Bethe ansatz equation (4.21)) for some small sites \(N\) strongly suggests that a fixed \(M\) may give a complete set of solutions of the transfer matrix. In such a sense, different \(M\) in [42] just give different parametrization of the eigenvalues but not different states. This suggests us in this paper to adopt the above parametrization of the \(T - Q\) relations (4.17) and the following ones such as (4.29), (5.33) and (5.39). Moreover, numerical solutions of the BAEs (4.21) for small size with random choices of \(\eta\) and the boundary parameters \(\alpha_\pm, \beta_\pm\) and \(\theta_\pm\) strongly suggest that the BAEs would give the complete solutions of the model (namely, the eigenvalues calculated from the BAEs coincide exactly to those obtained from exact diagonalization).

The numerical results for the \(N = 4\) case with the parameters: \(\eta = 0.5\), \(\alpha_+ = 1\), \(\alpha_- = 0.8\), \(\beta_+ = 0.4\), \(\beta_- = 0.3\), \(\theta_+ = 0.7i\) and \(\theta_- = 0.9i\) are shown in TABLE 1.

It follows from (4.20) that the parameter \(c\) does depend up not only the boundary parameters but also the parameters \(\{\mu_j\}\) (such a dependence on the parameters \(\{\mu_j\}\) also appears in the open XYZ chain (see section 5 below)). The vanishing condition of \(c\), i.e. \(c = 0\), will leads to the constraint among the boundary parameters found in [25, 40], where one could find a proper “local vacuum” to apply the conventional Bethe ansatz. The Bethe ansatz equations (4.23) imply that for this case the parameters \(\{\mu_j\}\) have to form two types of pairs:

\[(\mu_i, -\mu_i - \eta), \quad (\mu_i, -\mu_i - 2\eta)\]

Suppose the number of the first type pairs is \(M\), the resulting \(T - Q\) relation (4.17) becomes the conventional one

\[
\Lambda(u) = \bar{a}(u) \frac{\bar{Q}(u - \eta)}{\bar{Q}(u)} + \bar{d}(u) \frac{\bar{Q}(u + \eta)}{\bar{Q}(u)},
\]

(4.25)

with

\[
\bar{Q}(u) = \prod_{j=1}^{M} \frac{\sinh(u - \mu_i) \sinh(u + \mu_i + \eta)}{\sinh^2 \eta}.
\]
The constraint (4.20) allows us to fix the integer $M$ by the following condition which the boundary parameters must obey

$$(N - 1 - 2M)\eta = \alpha_- + \beta_+ + \alpha_+ + \beta_+ \pm (\theta_- - \theta_+) = k\eta, \mod(i\pi).$$

(4.26)

It is exactly the constrained boundary parameters for which a proper “local vacuum” exists [25]. If the boundary parameter obey the constraint (4.26), there exists another solution to the Bethe ansatz equations (4.23) which also corresponds to $c = 0$

$$(\mu_l - \mu_l - \eta), \quad (-\alpha_+ - \eta, -\alpha_+ - \eta, -\beta_- - \eta + \frac{i\pi}{2}, -\beta_- - \eta + \frac{i\pi}{2}), \quad (\mu_l, -\mu_l - 2\eta).$$

Now let the number of the first type pairs be $\bar{M}$, the constraint (4.20) allows us to fix the integer $\bar{M}$ by

$$(-N + 1 + 2\bar{M})\eta = \alpha_- + \beta_+ + \alpha_+ + \beta_+ \pm (\theta_- - \theta_+) = k\eta, \mod(i\pi).$$

(4.27)

The resulting $T - Q$ relation (4.17) becomes another conventional one (cf. (4.28))

$$\Lambda(u) = \tilde{a}(u) \frac{Q(u - \eta)}{Q(u)} + \tilde{a}(u) \frac{Q(u + \eta)}{Q(u)},$$

(4.28)
with
\[\tilde{a}(u) = -2^2 \frac{\sinh(2u + 2\eta)}{\sinh(2u + \eta)} \sinh(u + \alpha_-) \cosh(u + \beta_-) \times \sinh(u + \alpha_+) \cosh(u + \beta_+) \tilde{A}(u), \quad \tilde{d}(u) = \tilde{a}(-u - \eta),\]
\[\tilde{Q}(u) = \prod_{j=1}^{M} \frac{\sinh(u - \mu_i) \sinh(u + \mu_i + \eta)}{\sinh^2 \eta}.\]

Then the two resulting conventional T-Q relations (4.25) and (4.28) with the numbers constraints (4.26) and (4.27) recover the Bethe ansatz solutions [59, 16] of the open XXZ chain when boundary parameters satisfy the constraint (4.26). Moreover, one can check that when the K-matrices are diagonal ones which correspond to the cases of \( \alpha_{\pm} \) or \( \beta_{\pm} \to \infty \), there is no constrain for the choices of the integer \( M \). Hence the resulting \( T - Q \) ansatz is reduced to the usual form parameterized by a discrete \( M = 0, \ldots, N \).

4.3 \( T - Q \) ansatz for odd \( N \)

For the case of \( N \) being odd, we introduce
\[\Lambda(u) = \tilde{a}(u) \frac{Q_1(u - \eta)}{Q_2(u)} + \tilde{d}(u) \frac{Q_2(u + \eta)}{Q_1(u)} + \frac{2\tilde{c} \sinh(2u) \sinh(2u + 2\eta) \sinh u \sinh(u + \eta)}{Q_1(u)Q_2(u)} \frac{\tilde{A}(u)\tilde{A}(-u - \eta)}{\sinh^2 \eta}, \quad (4.29)\]
where the functions \( Q_1(u) \) and \( Q_2(u) \) are some trigonometric polynomials parameterized by \( N + 1 \) Bethe roots \( \{\mu_j|j = 1, \ldots, N + 1\} \) as follows,
\[Q_1(u) = \prod_{j=1}^{N+1} \frac{\sinh(u - \mu_j)}{\sinh(\eta)}, \quad (4.30)\]
\[Q_2(u) = \prod_{j=1}^{N+1} \frac{\sinh(u + \mu_j + \eta)}{\sinh(\eta)} = Q_1(-u - \eta). \quad (4.31)\]
the parameters \( \tilde{c} \) is determined by the boundary parameters and \( \mu_j \)
\[\tilde{c} = \cosh((N + 3)\eta + \alpha_- + \beta_- + \alpha_+ + \beta_+ + 2 \sum_{j=1}^{N+1} \mu_j) - \cosh(\theta_- - \theta_+). \quad (4.32)\]
The above relation leads to the fact that the asymptotic behavior (4.13) of the eigenvalue \( \Lambda(u) \) is automatically satisfied. Then the function \( \Lambda(u) \) given by (4.29) becomes the solution
of (4.9)-(4.14) provided that the $N+1$ parameters $\{\mu_j|j=1,\ldots,N+1\}$ satisfy the following Bethe ansatz equations

\[
\frac{2\bar{c}\sinh(2\mu_j)\sinh(2\mu_j+2\eta)\bar{A}(\mu_j)\bar{A}(-\mu_j-\eta)}{d(\mu_j)Q_2(\mu_j)Q_2(\mu_j+\eta)} = -\frac{\sinh^2\eta}{\sinh \mu_j \sinh(\mu_j+\eta)},
\]

\[j = 1, \ldots, N+1,
\]

(4.33)

with the very selection rule (4.22) for the roots of the above equations. In the homogeneous limit $\theta_j = 0$, the Bethe ansatz equations (4.33) can be written as

\[
\frac{\bar{c}\sinh(2\mu_j+\eta)\sinh(2\mu_j+2\eta)\sinh(\mu_j+\eta)\sinh^{2N}(\mu_j+\eta)}{2\sinh(\mu_j+\alpha_++\eta)\cosh(\mu_j+\beta_++\eta)\sinh(\mu_j+\alpha_-+\eta)\cosh(\mu_j+\beta_-+\eta)}
\]

\[= \prod_{l=1}^{N+1} \sinh(u_j+u_l+\eta) \sinh(u_j+u_l+2\eta), \ j = 1, \ldots, N+1.
\]

(4.34)

The eigenvalue of the Hamiltonian reads

\[
E = -\sinh \eta [\coth(\alpha_-) + \tanh(\beta_-) + \coth(\alpha_+ + \tanh(\beta_+))]
\]

\[-2\sinh \eta \sum_{j=1}^{N+1} \coth(\mu_j+\eta) + (N-1) \cosh \eta.
\]

(4.35)

Numerical solutions of the BAEs (4.34) for small size with random choices of $\eta$ and the boundary parameters $\alpha_\pm, \beta_- \pm$ and $\theta_\pm$ strongly suggest that the BAEs might give the complete solutions of the model. The numerical results for the $N = 3$ case with the parameters: $\eta = 0.5$, $\alpha_+ = 1$, $\alpha_- = 0.8$, $\beta_+ = 0.4$, $\beta_- = 0.3$, $\theta_+ = 0.7i$ and $\theta_- = 0.9i$ are shown in TABEL 2.

5 Results for the open XYZ chain

5.1 Operator identity

Now we consider the most general solutions $K^\pm(u)$ [46] of the reflection equation and its dual equation associated with the eight-vertex R-matrix given by (2.3)-(2.4),

\[
K^-(u) = \frac{\sigma(2u)}{2\sigma(u)} \left\{ \text{id} + \frac{\epsilon_z(-)\sigma(u)e^{-i\pi u}}{\sigma(u+\frac{1}{2})}\sigma^x + \frac{\epsilon_y(-)\sigma(u)e^{-i\pi u}}{\sigma(u+\frac{1}{2})}\sigma^y + \frac{\epsilon_z(-)\sigma(u)}{\sigma(u+\frac{1}{2})}\sigma^z \right\},
\]

(5.1)

\[
K^+(u) = K^-(u-\eta)|_{\epsilon_z(-)\rightarrow\epsilon_z(+)},
\]

(5.2)
Table 2: Numerical solutions of the BAEs for the $N = 3$ case with the parameters: $\eta = 0.5$, $\alpha_+ = 1$, $\alpha_- = 0.8$, $\beta_+ = 0.4$, $\beta_- = 0.3$, $\theta_+ = 0.7i$ and $\theta_- = 0.9i$. $E$ is the eigenvalues of the Hamiltonian. The eigenvalues are exactly the same as those from the exact diagonalization.

\[
\begin{array}{cccccc}
\mu_1 & \mu_2 & \mu_3 & \mu_4 & E & n \\
-0.5276 - 0.3652i & -0.5276 + 0.3652i & -0.2481 - 0.1756i & -0.2481 + 0.1756i & -4.8590 & 1 \\
-2.9056 + 0.0000i & -1.1969 - 0.0000i & -0.2500 - 0.1261i & -0.2500 + 0.1261i & -3.5939 & 2 \\
-0.6974 - 0.5166i & -0.6974 + 0.5166i & -0.2826 - 0.4381i & -0.2826 + 0.4381i & -0.1251 & 3 \\
-0.9296 - 0.0000i & -0.2637 - 0.4333i & -0.2637 + 0.4333i & 0.6919 + 1.5708i & 0.0479 & 4 \\
-0.9741 + 4.7124i & -0.7424 - 4.7124i & -0.5001 - 0.5664i & -0.5001 + 0.5664i & 1.1449 & 5 \\
-1.1498 + 0.0000i & -0.5212 - 2.5079i & -0.5212 + 0.6337i & 0.0230 + 1.5708i & 1.8855 & 6 \\
-1.1060 - 0.1659i & -1.1060 + 0.1659i & -0.5216 - 1.5708i & 0.3535 + 0.0000i & 2.5676 & 7 \\
-1.5205 + 0.0000i & -1.2965 - 0.0000i & -1.1030 + 1.5708i & 0.8003 - 1.5708i & 3.0278 & 8
\end{array}
\]

where the constants $\{c_i^{(\mp)}\}$ are expressed in terms of boundary parameters $\{\alpha_i^{(\mp)}\}$ as follows:

\[
c_{x}^{(\mp)} = e^{-i\pi(\sum_{i} \alpha_i^{(\mp)} - \frac{\pi}{2})} \prod_{i=1}^{3} \frac{\sigma(\alpha_i^{(\mp)} - \frac{\pi}{2})}{\sigma(\alpha_i^{(\mp)})}, \quad c_{y}^{(\mp)} = \prod_{i=1}^{3} \frac{\sigma(\alpha_i^{(\mp)} - \frac{1}{2} - \frac{\pi}{2})}{\sigma(\alpha_i^{(\mp)})},
\]

(5.3)

Direct calculation [54] shows that

\[
\text{Det}_q(K^-(u)) = \frac{\sigma(2u - 2\eta)}{\sigma(\eta)} \prod_{i=1}^{3} \frac{\sigma(\alpha_i^{(-)} + u)\sigma(\alpha_i^{(-)} - u)}{\sigma(\alpha_i^{(-)})\sigma(\alpha_i^{(-)})},
\]

(5.4)

\[
\text{Det}_q(K^+(u)) = -\frac{\sigma(2u + 2\eta)}{\sigma(\eta)} \prod_{i=1}^{3} \frac{\sigma(\alpha_i^{(+)}) + u)\sigma(\alpha_i^{(+)} - u)}{\sigma(\alpha_i^{(+)})\sigma(\alpha_i^{(+)})}.
\]

(5.5)

This leads to that for the eight-vertex model the function $\delta(u)$ defined by (3.10) reads

\[
\delta(u) = -\frac{\sigma(2u + 2\eta)}{\sigma(\eta)} \prod_{i=1}^{3} \frac{\sigma(u + \alpha_i^{(\gamma)})\sigma(u - \alpha_i^{(\gamma)})}{\sigma(\alpha_i^{(\gamma)})\sigma(\alpha_i^{(\gamma)})} \times \prod_{l=1}^{N} \frac{\sigma(u + \theta_l + \eta)\sigma(u + \theta_l - \eta)\sigma(u - \theta_l + \eta)\sigma(u - \theta_l - \eta)}{\sigma(\eta)\sigma(\eta)\sigma(\eta)\sigma(\eta)}.
\]

(5.6)

Following the method in [54, 55] and using the crossing relation of the R-matrix (2.9) and the explicit expressions of the K-matrices (5.1) and (5.2), one can show that the corresponding transfer matrix $t(u)$ satisfies the following crossing relation

\[
t(-u - \eta) = t(u).
\]

(5.7)
The quasi-periodicity of $\sigma$-function (A.5) allows one to derive the following properties of the R-matrix and K-matrices:

$$R_{12}(u + 1) = -\sigma_y^x R_{12}(u) \sigma_y^x = -\sigma_y^x R_{12}(u) \sigma_y^x, \quad K^\tau(u + 1) = -\sigma^x K^\tau(u) \sigma^x, \quad (5.8)$$

$$R_{12}(u + \tau) = -e^{-2\pi i (u + \frac{\eta}{2} + \frac{\tau}{2})} \sigma_1^x R_{12}(u) \sigma_1^x = -e^{-2\pi i (u + \frac{\eta}{2} + \frac{\tau}{2})} \sigma_2^x R_{12}(u) \sigma_2^x, \quad (5.9)$$

$$R_{12}(u + 1 + \tau) = e^{-2\pi i (u + \frac{\eta}{2} + \frac{\tau}{2})} \sigma_1^y R_{12}(u) \sigma_1^y = e^{-2\pi i (u + \frac{\eta}{2} + \frac{\tau}{2})} \sigma_2^y R_{12}(u) \sigma_2^y, \quad (5.10)$$

$$K^-(u + \tau) = -e^{-2\pi i (3u + \frac{\tau}{2})} \sigma^x K^-(u) \sigma^x, \quad (5.11)$$

$$K^-(u + 1 + \tau) = e^{-2\pi i (3u + \frac{\tau}{2})} \sigma^y K^-(u) \sigma^y, \quad (5.12)$$

$$K^+(u + \tau) = -e^{-2\pi i (3u + 3\eta + \frac{\tau}{2})} \sigma^x K^+(u) \sigma^x, \quad (5.13)$$

$$K^+(u + \tau) = e^{-2\pi i (3u + 3\eta + \frac{\tau}{2})} \sigma^y K^+(u) \sigma^y. \quad (5.14)$$

From these relations one obtains the quasi-periodic properties of the transfer matrix $t(u)$,

$$t(u + 1) = t(u), \quad t(u + \tau) = e^{-2\pi i (N + 3)(2u + \eta + \tau)} t(u). \quad (5.15)$$

With the help of the explicit expression (5.1) of the K-matrix $K^-(u)$, one may derive that

$$K^-(0) = \frac{1}{2} tr(K^-(0)) \times \text{id}, \quad K^-(\frac{1}{2}) = \frac{1}{2} tr(K^-(\frac{1}{2}) \sigma^z) \times \sigma^z, \quad (5.16)$$

$$K^-(\frac{\tau}{2}) = \frac{1}{2} tr(K^-(\frac{\tau}{2}) \sigma^x) \times \sigma^x, \quad K^-(\frac{1 + \tau}{2}) = \frac{1}{2} tr(K^-(\frac{1 + \tau}{2}) \sigma^y) \times \sigma^y. \quad (5.17)$$

Then we can evaluate the transfer matrix $t(u)$ at these particular points:

$$t(0) = \frac{1}{2} tr(K^+(0)) tr(K^-(0)) \prod_{l=1}^{N} \frac{\sigma(\eta + \theta_l) \sigma(\eta - \theta_l)}{\sigma(\eta) \sigma(\eta)} \times \text{id}, \quad (5.18)$$

$$t(\frac{1}{2}) = \frac{1}{2} tr(K^+(\frac{1}{2}) \sigma^z) tr(K^-(\frac{1}{2}) \sigma^z) (-1)^N \times \prod_{l=1}^{N} \frac{\sigma(\eta + \frac{1}{2} + \theta_l) \sigma(\eta - \frac{1}{2} - \theta_l)}{\sigma(\eta) \sigma(\eta)} \times \text{id}, \quad (5.19)$$

$$t(\frac{\tau}{2}) = \frac{1}{2} tr(K^+(\frac{\tau}{2}) \sigma^x) tr(K^-(\frac{\tau}{2}) \sigma^x) (-1)^N e^{-2\pi i (\frac{\tau}{2} - \sum_{j=1}^{N} \theta_j)} \times \prod_{l=1}^{N} \frac{\sigma(\eta + \frac{\tau}{2} + \theta_l) \sigma(\eta - \frac{\tau}{2} - \theta_l)}{\sigma(\eta) \sigma(\eta)} \times \text{id}, \quad (5.20)$$

$$t(\frac{1 + \tau}{2}) = \frac{1}{2} tr(K^+(\frac{1 + \tau}{2}) \sigma^y) tr(K^-(\frac{1 + \tau}{2}) \sigma^y) e^{-2\pi i (\frac{1}{2} - \sum_{j=1}^{N} \theta_j)} (-1)^N \times \prod_{l=1}^{N} \frac{\sigma(\eta + \frac{1 + \tau}{2} + \theta_l) \sigma(\eta - \frac{1 + \tau}{2} - \theta_l)}{\sigma(\eta) \sigma(\eta)} \times \text{id}. \quad (5.21)$$
5.2 Functional relations

The very operator identity (3.3) implies the corresponding eigenvalue \(\Lambda(u)\) also satisfies the similar relations

\[
\Lambda(\theta_j)\Lambda(\theta_j - \eta) = \frac{\delta(\theta_j)\sigma(\eta)\sigma(\eta)}{\sigma(\eta - 2\theta_j)\sigma(\eta + 2\theta_j)}, \quad j = 1, \ldots, N,
\]

where for the XYZ open spin chain the function \(\delta(u)\) is given by (5.6). The properties of the transfer matrix \(t(u)\) given by (5.7) and (5.18)-(5.21) imply that the corresponding eigenvalue \(\Lambda(u)\) satisfies the following relations:

\[
\Lambda(-u-1) = \Lambda(u),
\]

\[
\Lambda(0) = \frac{1}{2} tr(K^+(0))tr(K^-(0)) \prod_{i=1}^N \frac{\sigma(\eta + \theta_i)\sigma(\eta - \theta_i)}{\sigma(\eta)\sigma(\eta)},
\]

\[
\Lambda\left(\frac{1}{2}\right) = \frac{1}{2} tr(K^+(\frac{1}{2})\sigma^x)tr(K^-(\frac{1}{2})\sigma^x)(-1)^N
\times \prod_{i=1}^N \frac{\sigma(\eta + \frac{1}{2} + \theta_i)\sigma(\eta - \theta_i)}{\sigma(\eta)\sigma(\eta)},
\]

\[
\Lambda\left(\frac{\tau}{2}\right) = \frac{1}{2} tr(K^+(\frac{\tau}{2})\sigma^x)tr(K^-(\frac{\tau}{2})\sigma^x)(-1)^N e^{-2\pi i(\frac{N}{2}\eta - \sum_{j=1}^N \theta_j)}
\times \prod_{i=1}^N \frac{\sigma(\eta + \frac{\tau}{2} + \theta_i)\sigma(\eta - \frac{\tau}{2} - \theta_i)}{\sigma(\eta)\sigma(\eta)},
\]

\[
\Lambda\left(\frac{1+\tau}{2}\right) = \frac{1}{2} tr(K^+(\frac{1+\tau}{2})\sigma^y)tr(K^-(\frac{1+\tau}{2})\sigma^y)e^{-2\pi i(\frac{N}{2}\eta - \sum_{j=1}^N \theta_j)}(-1)^N
\times \prod_{i=1}^N \frac{\sigma(\eta + \frac{1+\tau}{2} + \theta_i)\sigma(\eta - \frac{1+\tau}{2} - \theta_i)}{\sigma(\eta)\sigma(\eta)}.
\]

The quasi-periodic properties (5.15) of the transfer matrix \(t(u)\) allow us to derive the following quasi-periodic properties of the corresponding eigenvalue

\[
\Lambda(u+1) = \Lambda(u), \quad \Lambda(u+\tau) = e^{-2\pi i(N+3)(2u+\eta+\tau)} \Lambda(u).
\]

The analyticity of the R-matrix and K-matrices and independence on \(u\) of the eigenstate lead to that the eigenvalue \(\Lambda(u)\) further possesses the property

\[
\Lambda(u), \text{ as an entire function of } u, \text{ is an elliptic polynomial of degree } 2N + 6.
\]

Therefore the values of \(\Lambda(u)\) at generic \(2N+6\) points in the fundamental domain of the elliptic functions suffice to determine the function uniquely. Actually, we have already obtained
the corresponding values of \( \Lambda(u) \) at points \( u = 0, \frac{1}{2}, \frac{\tau}{2}, \) as well as their crossing points \(-\eta, -\frac{1}{2} - \eta, -\frac{\tau}{2} - \eta\) via the relation (5.23). With the help of the relation (5.22), one can further obtain the values of \( \Lambda(u) \) at other \( 2N \) points \( \{\theta_j | j = 1, \ldots, N\} \) and their crossing points \( \{-\theta_j - \eta | j = 1, \ldots, N\} \). Then we can completely determine the eigenvalue function \( \Lambda(u) \).

Let us introduce some functions \( A(u), a(u) \) and \( d(u) \)

\[
A(u) = \prod_{j=1}^{N} \frac{\sigma(u + \theta_j + \eta) \sigma(u - \theta_j + \eta)}{\sigma(\eta) \sigma(\eta)}, \tag{5.30}
\]

\[
a(u) = -\frac{\sigma(2u + 2\eta)}{\sigma(2u + \eta)} \prod_{\gamma = \pm} \prod_{l=1}^{3} \frac{\sigma(u - \alpha_{\gamma}^{(l)})}{\sigma(\alpha_{\gamma}^{(l)})} A(u), \tag{5.31}
\]

\[
d(u) = a(-u - \eta). \tag{5.32}
\]

5.2.1 \( T - Q \) ansatz for even \( N \)

For the case of \( N \) being even, we can construct the solutions of (5.22)-(5.29) by the following ansatz

\[
\Lambda(u) = a(u) \frac{Q_{1}(u - \eta)}{Q_{2}(u)} + d(u) \frac{Q_{2}(u + \eta)}{Q_{1}(u)}
+ \frac{c \sigma(2u) \sigma(2u + 2\eta)}{Q_{1}(u) Q_{2}(u)} A(u) A(-u - \eta). \tag{5.33}
\]

The functions \( Q_{1}(u) \) and \( Q_{2}(u) \) are parameterized by \( N + 1 \) Bethe roots \( \{\mu_j | j = 1, \ldots, N+1\} \) as follows,

\[
Q_{1}(u) = \prod_{j=1}^{N+1} \frac{\sigma(u - \mu_j)}{\sigma(\eta)}, \tag{5.34}
\]

\[
Q_{2}(u) = \prod_{j=1}^{N+1} \frac{\sigma(u + \mu_j + \eta)}{\sigma(\eta)}. \tag{5.35}
\]

These \( N + 1 \) parameters \( \mu_j \) (which are different from each other) and \( c \) should satisfy the following \( N + 2 \) equations

\[
\sum_{\gamma = \pm} \sum_{l=1}^{3} \alpha_{\gamma}^{(l)} + (N + 3)\eta + 2 \sum_{j=1}^{N+1} \mu_j = 0 \mod(1), \tag{5.36}
\]

\[
\frac{c \sigma(2\mu_j) \sigma(2\mu_j + 2\eta) A(\mu_j) A(-\mu_j - \eta)}{d(\mu_j) Q_{2}(\mu_j) Q_{2}(\mu_j + \eta)} = -1, \ j = 1, \ldots, N + 1, \tag{5.37}
\]
with the very selection rule (4.22) for the roots of the above equations. In the homogeneous limit, the Bethe ansatz equations (5.37) become

\[
\begin{align*}
&c \sigma(2\eta)\sigma(2\mu_j + \eta)\sigma(2\mu_j + 2\eta)\sigma^{2N}(\mu_j + \eta) \\
&= \prod_{\gamma=\pm} \prod_{l=1}^{3} \frac{\sigma(\mu_j + \alpha^{(\gamma)}_l + \eta)}{\sigma(\alpha^{(\gamma)}_l)} \prod_{l=1}^{N+1} \sigma(\mu_j + \mu_l + \eta) \sigma(\mu_j + \mu_l + 2\eta), \\
&\quad j = 1, \ldots, N+1.
\end{align*}
\]

The eigenvalues \( E \) of the Hamiltonian (2.22) read

\[
E = \frac{\sigma(\eta)}{\sigma'(0)} \left\{ 2 \sum_{j=1}^{N+1} (\zeta(u_j) - \zeta(u_j + \eta)) + (N - 1)\zeta(\eta) + \sum_{\gamma=\pm} \sum_{l=1}^{3} \zeta(\alpha^{(\gamma)}_l) \right\}. \tag{5.38}
\]

### 5.2.2 \( T-Q \) ansatz for odd \( N \)

For the case of \( N \) being odd, we need to construct the solutions of (5.22)-(5.29) by the following ansatz

\[
\Lambda(u) = a(u) \frac{Q_1(u - \eta)}{Q_2(u)} + d(u) \frac{Q_2(u + \eta)}{Q_1(u)} + \frac{c \sigma(2u)\sigma(2u + 2\eta)\sigma(u)\sigma(u + \eta)}{Q_1(u)Q_2(u)\sigma(\eta)\sigma(\eta)} A(u) A(-u - \eta). \tag{5.39}
\]

The functions \( Q_1(u) \) and \( Q_2(u) \) are parameterized by \( N+2 \) Bethe roots \( \{\mu_j|j = 1, \ldots, N+2\} \) as follows,

\[
\begin{align*}
Q_1(u) &= \prod_{j=1}^{N+2} \frac{\sigma(u - \mu_j)}{\sigma(\eta)}, \tag{5.40} \\
Q_2(u) &= \prod_{j=1}^{N+2} \frac{\sigma(u + \mu_j + \eta)}{\sigma(\eta)}. \tag{5.41}
\end{align*}
\]

These \( N+2 \) parameters \( \mu_j \) (which are different from each other) and \( c \) should satisfy the following \( N+3 \) equations

\[
\begin{align*}
&\sum_{\gamma=\pm} \sum_{l=1}^{3} \alpha^{(\gamma)}_l + (N + 5)\eta + 2 \sum_{j=1}^{N+1} \mu_j = 0 \text{ mod}(1), \tag{5.42} \\
&\frac{c \sigma(2\mu_j)\sigma(2\mu_j + 2\eta)\sigma(\mu_j + \eta)\sigma(\mu_j + \eta) A(\mu_j) A(-\mu_j - \eta)}{d(\mu_j)Q_2(\mu_j)Q_2(\mu_j + \eta)\sigma(\eta)\sigma(\eta)} = -1, \quad j = 1, \ldots, N+2. \tag{5.43}
\end{align*}
\]
with the very selection rule (4.22) for the roots of the above equations. In the homogeneous limit, the Bethe ansatz equations (5.43) read

\[ c \sigma^2(\eta) \sigma(2\mu_j + \eta) \sigma(2\mu_j + 2\eta) \sigma^{2N+1}(\mu_j + \eta) = \prod_{\gamma=\pm} \prod_{l=1}^{3} \frac{\sigma(\mu_j + \alpha_{l}^{(\gamma)} + \eta)}{\sigma(\alpha_{l}^{(\gamma)})} \prod_{l=1}^{N+2} \sigma(\mu_j + \mu_l + \eta) \sigma(\mu_j + \mu_l + 2\eta), \]

\[ j = 1, \ldots, N + 2. \]

and the eigenvalue of the Hamiltonian is

\[ E = \frac{\sigma(\eta)}{\sigma'(0)} \left\{ 2 \sum_{j=1}^{N+2} (\zeta(u_j) - \zeta(u_j + \eta)) + (N - 1)\zeta(\eta) + \sum_{\gamma=\pm} \sum_{l=1}^{3} \zeta(\alpha_{l}^{(\gamma)}) \right\}. \] 

(5.44)

6 Conclusions

The anisotropic spin-$\frac{1}{2}$ chains with arbitrary boundary fields (i.e., there is not any constrain to the boundary parameters, cf. [19, 25]) defined by (1.2), which includes the most general open XXZ chain and open XYZ chain, are studied by the off-diagonal Bethe ansatz proposed in [41]. The eigenvalues of the transfer matrix are given in terms of generalized $T-Q$ ansatzs (4.17), (4.29), (5.33) and (5.39). The corresponding Bethe ansatz equations are given by (4.23), (4.34), (5.36)-(5.37), (5.42)-(5.43) respectively.

The different forms of BAEs indicate different topological natures for even $N$ and odd $N$ cases, which was firstly observed in the closed XYZ chain. This phenomenon is also quite similar to that appeared in the XXZ model, where the periodic and antiperiodic boundary conditions also induce quite different Bethe ansatz equations and indeed show different topological behaviors [41].

As for integrable models without $U(1)$ symmetry, most of conventional Bethe ansatz methods failed because of the lack of a proper “local vacuum”. The off-diagonal Bethe ansatz method overcomes this obstacle by using functional relations to construct the $T-Q$ ansatz, which is completely independent of the representation basis and thus does not need any information of the states. Although the functional relations between eigenvalues $\Lambda(u)$ and the quantum determinant $\delta(u)$ at some particular points can be obtained in different ways for some special cases [41, 42, 37, 34], our $T-Q$ ansatz would play an important role to construct manageable Bethe ansatz equations. In the present case, based on the relation (4.9) (or (5.22)) and some properties (4.10)-(4.14) (or (5.23)-(5.29)) of $\Lambda(u)$ we can give
a generalized $T - Q$ ansatz, which modifies the usual $T - Q$ relation by adding an extra off-diagonal term. Such an extra term encodes the contribution of the off-diagonal element of the associated $K$-matrix.

In fact, the functional relation between the eigenvalue $\Lambda(u)$ and the quantum determinant $\delta(u)$ is a direct consequence of the operator identity (3.8) which is obtained only via some properties of the $R$-matrix and $K$-matrices. As we demonstrated, similar operator relation holds for arbitrary integrable boundaries (no matter periodic, anti-periodic or open boundaries). In such a sense, our method would greatly simplify the process of Bethe ansatz and would provide an unified procedure for approaching the integrable models both with and without $U(1)$ symmetry.

Acknowledgments

The financial support from the National Natural Science Foundation of China (Grant Nos. 11174335, 11075126, 11031005, 11375141, 11374334), the National Program for Basic Research of MOST (973 project under grant No. 2011CB921700) and the State Education Ministry of China (Grant No. 20116101110017 and SRF for ROCS) are gratefully acknowledged. Two of the authors (W.-L. Yang and K. Shi) would like to thank IoP/CAS for the hospitality and they enjoyed during their visit there. We also would like to acknowledge Y.Z. Jiang and S. Cui for their numerical helps.

Appendix A: Elliptic functions

In this appendix, we give the definitions of elliptic functions which appear in our study related to the XYZ models and some identity relations between the functions.

Let us fix $\tau$ as such that $\text{Im}(\tau) > 0$. We introduce the following elliptic functions

\[
\theta \left[ \begin{array}{c} a \\ b \end{array} \right] (u, \tau) = \sum_{m=-\infty}^{\infty} \exp \left\{ i\pi \left[ (m+a)^2 \tau + 2(m+a)(u+b) \right] \right\}, \tag{A.1}
\]

\[
\sigma(u) = \tau \left[ \begin{array}{c} u \\ \tau \end{array} \right], \quad \zeta(u) = \frac{\partial}{\partial u} \{ \ln \sigma(u) \}. \tag{A.2}
\]
Among them the $\sigma$-function satisfies the Riemann-identity:

$$\sigma(u + x)\sigma(u - x)\sigma(v + y)\sigma(v - y) - \sigma(u + y)\sigma(u - y)\sigma(v + x)\sigma(v - x)$$
$$= \sigma(u + v)\sigma(u - v)\sigma(x + y)\sigma(x - y),$$

(A.3)

and other identities which have been used in this paper

$$\sigma(2u) = \frac{2\sigma(u)\sigma(u + \frac{1}{2})\sigma(u + \frac{3}{2})\sigma(u - \frac{1}{2} - \frac{3}{2})}{\sigma(\frac{1}{2})\sigma(\frac{3}{2})\sigma(-\frac{1}{2} - \frac{3}{2})},$$

(A.4)

$$\sigma(u + 1) = -\sigma(u), \quad \sigma(u + \tau) = e^{-2i\pi(u + \frac{3}{2})}\sigma(u),$$

(A.5)

$$\frac{\sigma(u)}{\sigma(\frac{3}{2})} = \theta \left[ \begin{array}{c} 0 \\ \frac{1}{2} \end{array} \right] (u, 2\tau) \theta \left[ \begin{array}{c} \frac{5}{4} \\ \frac{1}{2} \end{array} \right] (u, 2\tau),$$

(A.6)

$$\theta \left[ \begin{array}{c} \frac{1}{2} \\ \frac{7}{4} \end{array} \right] (2u, 2\tau) = \theta \left[ \begin{array}{c} \frac{1}{2} \\ \frac{7}{4} \end{array} \right] (\tau, 2\tau) \times \frac{\sigma(u)\sigma(u + \frac{1}{2})}{\sigma(\frac{3}{2})\sigma(\frac{3}{2} + \frac{5}{2})},$$

(A.7)

$$\theta \left[ \begin{array}{c} 0 \\ \frac{1}{2} \end{array} \right] (2u, 2\tau) = \theta \left[ \begin{array}{c} 0 \\ \frac{1}{2} \end{array} \right] (0, 2\tau) \times \frac{\sigma(u - \frac{3}{2})\sigma(u + \frac{1}{2} + \frac{3}{2})}{\sigma(\frac{3}{2})\sigma(\frac{3}{2} + \frac{5}{2})}.$$  

(A.8)

**Appendix B: The proof of (3.13)-(3.14)**

In this appendix, we give the proof of (3.13)-(3.14) for the eight-vertex model. For the case of the six-vertex model, the equations can be proven in a similar way (or by taking the corresponding limits of the eight-vertex case).

The $Z_2$-symmetry (2.11) of the R-matrix implies that

$$\text{tr}_0 \left\{ (V_0K_0^{-}(u - \eta)V_0)^{t_0} P_{0j} R_{j0}(-2u) \right\}$$
$$= \text{tr}_0 \left\{ V_0 (K_0^{-}(u - \eta))^{t_0} V_0 P_{0j} V_j R_{j0}(-2u) V_0 V_j \right\}$$
$$= -V_j \text{tr}_0 \left\{ V_0 (K_0^{-}(u - \eta))^{t_0} P_{0j} R_{j0}(-2u) V_0 \right\} V_j$$
$$= V_j \text{tr}_0 \left\{ P_{0j} R_{j0}(-2(u - \eta) - 2\eta)(K_0^{-}(u - \eta))^{t_0} \right\} V_j.$$  

(B.1)

The following useful relations of the K-matrices given by (5.1) and (5.2) were proven in [54] (for details we refer the reader to see (4.11), (4.12) and (A.1) in the Ref.[54])

$$\text{tr}_0 \left\{ (V_0K_0^{-}(u - \eta)V_0)^{t_0} P_{0j} R_{j0}(-2u) \right\} = \frac{\sigma(2u - 2\eta)}{\sigma(\eta)} K_j^{-}(-u),$$

(B.2)

\footnote{Our $\sigma$-function is the $\vartheta$-function $\vartheta_1(u)$ in [60]. It has the following relation with the Weierstrassian $\sigma$-function if one denotes it by $\sigma_w(u)$: $\sigma_w(u) \propto e^{\eta_1 u^2} \sigma(u)$, $\eta_1 = \pi^2(\frac{1}{b^2} - 4 \sum_{n=1}^\infty \frac{n q^{2n}}{1 - q^{2n}})$ and $q = e^{i\tau}$.}
\[
\begin{align*}
tr_0 \{ P_{0j} R_{j0}(2u) K_0^+(u)^t \} &= -\frac{\sigma(2u + 2\eta)}{\sigma(\eta)} V_j K_j^+(-u - \eta)V_j, \quad (B.3) \\
K_j^-(u) K_j^+(-u) &= \prod_{i=1}^{3} \frac{\sigma(\alpha_i^{(-)} + u)\sigma(\alpha_i^{(-)} - u)}{\sigma(\alpha_i^{(-)})\sigma(\alpha_i^{(-)})} \times \text{id}_j. \quad (B.4)
\end{align*}
\]

The above relations give rise to
\[
K_j^-(u) tr_0 \left\{ (V_0 K_0^{-}(u - \eta) V_0)^t P_{0j} R_{j0}(-2u) \right\} = \frac{\sigma(2u - 2\eta)}{\sigma(\eta)} K_j^-(u) K_j^+(-u) \\
= \frac{\sigma(2u - 2\eta)}{\sigma(\eta)} \prod_{i=1}^{3} \frac{\sigma(\alpha_i^{(-)} + u)\sigma(\alpha_i^{(-)} - u)}{\sigma(\alpha_i^{(-)})\sigma(\alpha_i^{(-)})} \times \text{id}_j \\
= \text{Det}_q(K^-(u)) \times \text{id}_j, \quad (B.5)
\]

and
\[
\begin{align*}
tr_0 \left\{ K_0^+(u) P_{0j} R_{j0}(2u) \right\} \left\{ V_j K_j^+(u - \eta)V_j \right\}^t \\
= -\frac{\sigma(2u + 2\eta)}{\sigma(\eta)} \prod_{i=1}^{3} \frac{\sigma(\alpha_i^{(+)} + u)\sigma(\alpha_i^{(+)} - u)}{\sigma(\alpha_i^{(+)})\sigma(\alpha_i^{(+)})} \times \text{id}_j \\
= \text{Det}_q \left\{ K^+(u) \right\} \times \text{id}_j. \quad (B.6)
\end{align*}
\]

This completes the proof of (3.13)-(3.14).

References

[1] R.J. Baxter, *Exactly Solved Models in Statistical Mechanics* (Academic Press, 1982).

[2] C. N. Yang, *Phys. Rev. Lett.* 19 (1967), 1312; *Phys. Rev.* 168 (1968), 1920.

[3] R. J. Baxter, *Phys. Rev. Lett.* 26 (1971), 832; 26 (1971), 834; *Ann. Phys. (N.Y.)* 70 (1972), 323.

[4] M. A. Kastner, R. J. Birgeneau, G. Shirane and Y. Endoh, *Rev. Mod. Phys.* 70 (1998), 897.

[5] J. M. Maldacena, *Adv. Theor. Math. Phys.* 2 (1998), 231.

[6] L. Dolan, C. R. Nappi and E. Witten, *JHEP* 0310 (2003), 017.

[7] B. Chen, X. J. Wang and Y. S. Wu, *Phys. Lett.* B 591 (2004), 170.
[8] N. Beisert et al, *Lett. Math. Phys.* 99 (2012), 1.

[9] C. T. Black, D. C. Ralph and M. Tinkham, *Phys. Rev. Lett.* 76 (1996), 688; *ibid* 78 (1997), 4087.

[10] J. Dukelsky, S. Pittel and G. Sierra, *Rev. Mod. Phys.* 76 (2004), 643.

[11] V. E. Korepin, N. M. Bogoliubov and A. G. Izergin, *Quantum Inverse Scattering Method and correlation Function*, Cambridge Univ. Press, Cambridge, 1993.

[12] H. Bethe, *Z. Phys.* 71 (1931), 205.

[13] F. C. Alcaraz, M. N. Barber, M. T. Batchelor, R. J. Baxter and G. R. W. Quispel, *J. Phys.* A 20 (1987), 6397.

[14] S. Belliard, N. Crampe and E. Ragoucy, *Lett. Math. Phys.* 103 (2013), 493.

[15] N. Crampe and E. Ragoucy, *Nucl. Phys.* B 858 (2012), 502.

[16] W. -L. Yang, R. I. Nepomechie and Y. -Z. Zhang, *Phys. Lett.* B 633 (2006), 664.

[17] E. K. Sklyanin and L. D. Faddeev, Sov. Phys. Dokl. 23 (1978), 902.

[18] L. A. Takhtadzhan and L. D. Faddeev, Rush. Math. Surveys 34 (1979), 11.

[19] H. Fan, B. -Y. Hou, K. -J. Shi and Z. -X. Yang, *Nucl. Phys.* B 478 (1996), 723.

[20] N. Yu. Reshetikhin, *Sov. Phys. JETP* 57 (1983), 691.

[21] E. K. Sklyanin, *Lect. Notes Phys.* 226 (1985), 196; *J. Sov. Math.* 31 (1985), 3417; *Prog. Theor. Phys. Suppl.* 118 (1995), 35.

[22] G. E. Andrews, R. J. Baxter and P. J. Forrester, *J. Stat. Phys.* 35 (1984), 193.

[23] V.V. Bazhanov and N. Yu. Reshetikhin, *Int. J. Mod. Phys.* A 4 (1989), 115.

[24] R. I. Nepomechie, *J. Phys.* 34 (2001), 9993 [hep-th/0110081]; *Nucl. Phys.* B 622 (2002), 615 [hep-th/0110116]; *J. Stat. Phys.* 111 (2003), 1363 [hep-th/0211001]; *J. Phys. A* 37 (2004), 433 [hep-th/0304092].

[25] J. Cao, H. -Q. Lin, K. -J. Shi and Y. Wang, *Nucl. Phys.* B 663 (2003), 487.

28
[26] W.-L. Yang, Y.-Z. Zhang and M. Gould, Nucl. Phys. B 698 (2004), 503 [hep-th/0411048].

[27] J. de Gier and P. Pyatov, J. Stat. Mech. (2004), P03002; A. Nichols, V. Rittenberg and J. de Gier, J. Stat. Mech. (2005), P05003; J. de Gier, A. Nichols, P. Pyatov and V. Rittenberg, Nucl. Phys. B 729 (2005), 387.

[28] J. de Gier and F. H. L. Essler, Phys. Rev. Lett. 95 (2005), 240601; J. Stat. Mech. (2006), P 12011.

[29] C. S. Melo, G. A. P. Ribeiro and M. J. Martins, Nucl. Phys. B 711 (2005), 565.

[30] A. Doikou and P. P. Martins, J. Stat. Mech. (2006), P 06004; A. Doikou, J. Stat. Mech. (2006), P 09010.

[31] Z. Bajnok, J. Stat. Mech. (2006), P06010.

[32] P. Baseilhac and K. Koizumi, J. Stat. Mech. (2007), P09006.

[33] W. Galleas, Nucl. Phys. B 790 (2008), 524.

[34] H. Frahm, J. H. Grelik, A. Seel, and T. Wirth, J. Phys. A 44 (2011), 015001.

[35] S. Niekamp, T. Wirth, and H. Frahm, J. Phys. A 42 (2009), 195008.

[36] A. M. Grabinski and H. Frahm, J. Phys. A 43 (2010), 045207.

[37] G. Niccoli, Nucl. Phys. B 870 (2013), 397; J. Phys. A 46 (2013), 075003.

[38] C.M. Yung and M.T. Batchelor, Nucl. Phys. B 446 (1995), 461.

[39] M.T. Batchelor, R. J. Baxter, M. J. O’Rourke, and C. M. Yung, J. Phys. A 28 (1995), 2759.

[40] W.-L. Yang and Y.-Z. Zhang, JHEP 04 (2007), 044; Nucl. Phys. B 789 (2008), 591.

[41] J. Cao, W.-L. Yang, K.-J. Shi and Y. Wang, Phys. Rev. Lett. 111 (2013), 137201 [arXiv:1305.7328].

[42] J. Cao, W.-L. Yang, K.-J. Shi and Y. Wang, Nucl. Phys. B 875 (2013), 152 [arXiv:1306.1742; arXiv:1307.0280].
[43] E. K. Sklyanin, *J. Phys.* A **21** (1988), 2375.

[44] H. J. de Vega and A. González-Ruiz, *J. Phys.* A **26** (1993), L519.

[45] S. Ghoshal and A.B. Zamolodchikov, *Int. J. Mod. Phys.* A **9** (1994), 3841 [hep-th/9306002].

[46] T. Inami and H. Konno, *J. Phys.* A **27** (1994), L913.

[47] B. Y. Hou, K. J. Shi, H. Fan and Z.-X. Yang, *Commun. Theor. Phys.* **23** (1995), 163.

[48] N. Kitanine, J. M. Maillet and V. Terras, *Nucl. Phys.* B **554** (1999), 647.

[49] J. M. Maillet and V. Terras, *Nucl. Phys.* B **575** (2000), 627.

[50] K. Hikami, *J. Phys.* A **28** (1996), 4997.

[51] Y. S. Wang, *J. Phys.* A **33** (2000), 4009.

[52] L. Mezincescu, R. I. Nepomechie and V. Rittenberg, *Phys. Lett.* A **147** (1990), 70.

[53] Y. K. Zhou, *Nucl. Phys.* B **458** (1996), 504 [hep-th/9510095].

[54] W.-L. Yang and Y.-L. Zhang, *Nucl. Phys.* B **744** (2006), 312-329.

[55] L. Mezincescu and R. I. Nepomechie, *Nucl. Phys.* B **372** (1992), 597; R. E. Behreud, P. Pearce and D. L. O’Brien, *J. Stat. Phys.* **84** (1996), 1.

[56] H. Frahm, A. Seel and T. Wirth, *Nucl. Phys.* B **802** (2008), 351.

[57] G. Niccoli, *J. Stat. Mech.* (2012), P10025;

[58] N. Karaiskos, A. Grabinski and H. Frahm, [arXiv:1304.2659](https://arxiv.org/abs/1304.2659).

[59] R. I. Nepomechie and F. Ravanini, *J. Phys.* A **36** (2003), 11391; Addendum, *J. Phys.* A **37** (2004), 1945.

[60] E. T. Whittaker and G. N. Watson, *A course of modern analysis: 4th edn.*, Cambridge University Press, Cambridge, 2002.