Bound state solutions of the Klein-Fock-Gordon equation with the sum of Manning-Rosen potential and Yukawa potential within SUSYQM

A. I. Ahmadov\textsuperscript{1,2}, M. Demirci\textsuperscript{3} and S. M. Aslanova\textsuperscript{1}

\textsuperscript{1} Department of Theoretical Physics, Baku State University, Z. Khalilov st. 23, AZ-1148, Baku, Azerbaijan
\textsuperscript{2} Institute for Physical Problems, Baku State University, Z. Khalilov st. 23, AZ-1148, Baku, Azerbaijan
\textsuperscript{3} Department of Physics, Karadeniz Technical University, TR61080 Trabzon, Turkey
E-mail: ahmadovazar@yahoo.com, mehmetdemirci@ktu.edu.tr, sariyya.aslanova@mail.ru

Abstract. In this study, the bound state solutions of the Klein-Fock-Gordon equation are examined for the sum of Manning-Rosen and Yukawa potential by using a recent improved scheme to deal with the centrifugal term. For any $l \neq 0$, the energy eigenvalues and corresponding radial wave functions are determined under the condition of equal scalar and vector potentials. In order to obtain bound state solutions, we use two different methods called supersymmetric quantum mechanics (SUSYQM) and Nikiforov-Uvarov (NU) methods. The identical expressions for the energy eigenvalues are obtained, and the expression of radial wave functions transformations to each other is revealed via both methods. For arbitrary $l$ states, the energy levels and the corresponding normalized eigenfunctions are given in terms of the Jacobi polynomials. A closed form of the normalized wave function is also obtained. It is seen that the energy eigenvalues and eigenfunctions are sensitive to $n_r$ radial and $l$ orbital quantum numbers.

1. Introduction

Since the early years of quantum mechanics (QM), the exactly solvable problems for physical potentials have received much attention in many branches of physics. In the QM, it is very important to obtain an analytical solution of wave equation since the wave function contains all the necessary information for the exact definition of a quantum system. Furthermore, in the relativistic QM’s applications to particle and nuclear physics, the behavior of particles at high energies can be predicted from the relativistic wave equations. The Klein-Fock-Gordon (KFG) equation \cite{1, 2, 3, 4} is a fundamental relativistic wave equation that is well known to describe the motion of spin-0 particles, for example, pions, Higgs boson and so forth. Thereby, the analytical solutions of the KFG equation with interaction potentials are very important for relativistic QM.

Many techniques have been developed to obtain the solutions of relativistic and non-relativistic wave equations under physical potentials. Some of these are as follows: Nikiforov-Uvarov (NU) method \cite{5}, supersymmetry QM (SUSYQM) \cite{6, 7}, shifted $1/N$ expansion approach \cite{8, 9}, the path integral method \cite{10}, asymptotic iteration method \cite{11}, factorization \cite{12} and Laplace transform approach \cite{13}. With help of these techniques, many works have been carried...
out to obtain exact or approximate solutions with some combined potentials such as: Manning-Rosen plus ring-shaped potential [14], Hulthen plus a Ring-Shaped like potential [15] and Hulthen plus Yukawa potentials [16], and references therein. In the present study, we attempt to study bound state solutions of KFG equation with a potential formed by summing two different potentials which are called as the Manning-Rosen and Yukawa potentials. The Manning-Rosen potential can be used to describe the continuum and bound-states in an interaction system, is given [17]

\[ V_{MR}(r) = \frac{\hbar^2}{2Mb^2} \left[ \frac{\alpha(\alpha - 1)e^{-2r/b}}{(1 - e^{-r/b})^2} - \frac{Ae^{-r/b}}{(1 - e^{-r/b})^2} \right]. \]  

(1)

Our other potential is the Yukawa potential which was proposed by Yukawa [18] as an effective non-relativistic potential describing the strong interactions of nucleons. It is given by:

\[ V(r) = -\frac{V_0 e^{-\delta r}}{r}. \]  

(2)

The linear combination of these two potentials can be used to examine the interactions of deformed-pair of the nucleus and spin-orbit coupling for a particle in the potential field. Another fascinating perspective of this potential can be utilized in the description of vibrations in the side of the hadronic system. Inspired by all motivations and works, we consider a sum of Manning-Rosen and Yukawa potentials:

\[ V(r) = \frac{\hbar^2}{2Mb^2} \left[ \frac{\alpha(\alpha - 1)e^{-2r/b}}{(1 - e^{-r/b})^2} - \frac{Ae^{-r/b}}{(1 - e^{-r/b})^2} \right] - \frac{V_0 e^{-\delta r}}{r}. \]  

(3)

In this study, our aim is to examine such a large quantum system. Therefore, we can investigate the considered quantum system at a long distance. To achieve this goal, we use both NU method and SUSYQM [19, 20], taking account of a recent improved scheme to deal with the centrifugal term. Thereby, the radial wave functions and energy eigenvalues are found for any orbital angular momentum \( l \). However, the same problem have been investigated within ordinary quantum mechanics in Refs. [21, 22] as well, but our results disagree with their results.

The remainder of the paper is organized as follows: In Sec. 2, the KFG equation for the Manning-Rosen potential plus the Yukawa potential is presented. In Sec. 3, the exact analytic bound state energies and the corresponding radial solution of KFG equation are derived in the context of NU method (Sec. 3.1) and SUSYQM method (Sec. 3.2), respectively. Finally, the results are summarized and discussed in Sec. 4.

2. The KFG Equation with the Manning-Rosen potential plus the Yukawa potential

Two different type potentials could be presented into KFG equation since it includes two objects; the scalar rest mass and the four-vector linear momentum operator. These are a scalar potential (S) (which introduce via scalar coupling) and a vector potential (V) (which introduce via minimal coupling). In the spherical coordinates system, the KFG equation with scalar potential and vector potential is formulated by

\[ \left[-\nabla^2 + (M + S(r))^2\right] \psi(r, \theta, \phi) = \left|E - V(r)\right|^2 \psi(r, \theta, \phi), \]  

(4)

where \( M \) is the rest mass of a scalar particle and \( E \) denotes the relativistic energy of the system.

Substituting this into Eq.(4), the radial part of KFG equation is obtained as

\[ \chi''(r) + \left[ (E^2 - M^2) - 2(M \cdot S(r) + E \cdot V(r)) + V^2(r) - S^2(r) \right] - \frac{l(l + 1)}{r^2} \chi(r) = 0. \]  

(5)
In present work, the vector potential is assumed to be equal to scalar potential, namely, \( V(r) = S(r) \). Hence, equation (5) becomes

\[
\chi''(r) + \left[ (E^2 - M^2) - 2(M + E) \cdot V(r) - \frac{l(l + 1)}{r^2} \right] \chi(r) = 0. \tag{6}
\]

The solution of KFG equation (6) with the potential (3) can be exactly obtained by using suitable approximation scheme to overcome the centrifugal term. In order to effectively use the Yukawa potential in this system, the following improved approximation scheme must be used for its centrifugal term. For \( \delta r << 1 \), the approximation scheme [23, 24, 25] is given by,

\[
\frac{1}{r} \approx 2\delta \left[ \frac{e^{-\delta r}}{1 - e^{-2\delta r}} \right] \quad \text{and} \quad \frac{1}{r^2} \approx 4\delta^2 \frac{e^{-2\delta r}}{(1 - e^{-2\delta r})^2}. \tag{7}
\]

In the Manning-Rosen potential we assume \( 1/b = 2\delta \) and hence one gets:

\[
V_{\text{MR}}(r) = \frac{\hbar^2}{2MB^2} \left[ \frac{\alpha(a - 1)e^{-2r/b}}{(1 - e^{-r/b})^2} - \frac{Ae^{-r/b}}{(1 - e^{-r/b})} \right] = \frac{V_{01}e^{-4\delta r}}{(1 - e^{-2\delta r})^2} - \frac{V_{02}e^{-2\delta r}}{1 - e^{-2\delta r}}, \tag{8}
\]

where \( V_{01} = \frac{2\delta^2 A(a - 1)}{M} \) and \( V_{02} = \frac{2\delta^2 A}{M} \).

By using Eq.(7), then we can write Yukawa potential in the following form:

\[
V_Y(r) = -\frac{2\delta Be^{-2\delta r}}{1 - e^{-2\delta r}} = -\frac{V_{03}e^{-2\delta r}}{1 - e^{-2\delta r}} \tag{9}
\]

where \( V_{03} = 2\delta B \).

As a result, sum of Manning-Rosen potential and Yukawa potential is written as

\[
V(r) = V_{\text{MR}}(r) + V_Y(r) = \frac{V_{01}e^{-4\delta r}}{(1 - e^{-2\delta r})^2} - \frac{V_{02} + V_{03})e^{-2\delta r}}{(1 - e^{-2\delta r})^2} = \frac{V_{01}e^{-4\delta r}}{(1 - e^{-2\delta r})^2} - \frac{V_{023}e^{-2\delta r}}{1 - e^{-2\delta r}}, \tag{10}
\]

where \( V_{023} = (V_{02} + V_{03}) \). Substituting the potential (10) into Eq.(6), we have

\[
\chi''(r) + \left[ (E^2 - M^2) - 2(M + E) \left( \frac{V_{01}e^{-4\delta r}}{(1 - e^{-2\delta r})^2} - \frac{V_{023}e^{-2\delta r}}{1 - e^{-2\delta r}} \right) - 4l(l + 1)e^{-2\delta r} \right] \chi(r) = 0. \tag{11}
\]

Thereby, the effective potential is defined as:

\[
V_{\text{eff}}(r) = 2 \left( (M + E)\frac{V_{01}e^{-4\delta r}}{(1 - e^{-2\delta r})^2} - \frac{V_{023}e^{-2\delta r}}{1 - e^{-2\delta r}} \right) + 4l(l + 1)e^{-2\delta r}. \tag{12}
\]

3. Bound State Solutions of the Radial Klein-Fock-Gordon Equation

3.1. Implementation of Nikiforov-Uvarov Method

To apply the NU method, we should transform Eq.(11) to the following form:

\[
\chi''(s) + \frac{\tau}{\sigma} \chi'(s) + \frac{\tilde{\sigma}}{\sigma} \chi(s) = 0. \tag{13}
\]

Equation (11) can be further simplified by using a new variable \( s = e^{-2\delta r} \), and \( r \in [0, \infty) \) and \( s \in [0, 1] \). For bound states, it can be used the following notations:

\[
\varepsilon = \sqrt{M^2 - E^2} > 0, \quad \alpha = \sqrt{2(E + M)V_{01}} > 0, \quad \beta = \sqrt{2(E + M)V_{023}} > 0, \tag{14}
\]
where $E$ and $\varepsilon$ should be, respectively, smaller and bigger than $M$ and zero. The boundary conditions are $\chi(0) = 0$ and $\chi(\infty) \to 0$. Using Eq. (7) and Eq. (14), the Eq. (11) can be rewritten with the transformation $s = e^{-2\tau}$ as follows:

$$\chi''(s) + \frac{1-s}{s(1-s)} \chi'(s) + \left[ \frac{1}{s(1-s)} \right]^2 \left[ -\varepsilon^2 (1-s)^2 - \alpha^2 s^2 + \beta^2 s(1-s) - l(l+1)s \right] \chi(s) = 0. \quad (15)$$

As we expected, this equation has a suitable form to apply the NU method. By comparing Eq.(15) with Eq.(13) we get:

$$\tilde{\tau}(s) = 1-s, \quad \sigma(s) = s(1-s), \quad \tilde{\sigma}(s) = -\varepsilon^2 (1-s)^2 - \alpha^2 s^2 + \beta^2 s(1-s) - l(l+1)s. \quad (16)$$

In order to obtain the exact solution to Eq.(13), the wave function $\chi(s)$ is set as follows:

$$\chi(s) = \phi(s)y(s), \quad (17)$$

for the appropriate function $\phi(s)$. Then, substituting Eq.(17) into Eq.(13) reduces Eq.(13) into the well known hypergeometric-type equation,

$$\sigma(s)y''(s) + \tau(s)y'(s) + \bar{\lambda}y(s) = 0. \quad (18)$$

The appropriate function $\phi(s)$ should satisfy the condition as follows

$$\frac{\phi'(s)}{\phi(s)} = \frac{\pi(s)}{\sigma(s)}, \quad (19)$$

where $\pi(s)$, which can be at most first-order polynomial, is given by

$$\pi(s) = \frac{\sigma' - \tilde{\tau}}{2} \pm \sqrt{\left( \frac{\sigma' - \tilde{\tau}}{2} \right)^2 - \tilde{\sigma} + k\sigma}. \quad (20)$$

Consequently, the equation, where $y(s)$ is one of its solutions, takes the form of hypergeometric-type, if the polynomial $\tilde{\sigma}(s) = \bar{\sigma}(s) + \pi^2(s) + \pi(s)[\tilde{\tau}(s) - \sigma'(s)] + \pi'(s)\sigma(s)$ can be divisible by $\sigma(s)$, i.e., $\bar{\sigma} = \lambda \sigma(s)$. The constant $\bar{\lambda}$ and polynomial $\tau(s)$ in Eq.(13) are given by

$$\bar{\lambda} = k + \pi'(s), \quad \tau(s) = \tilde{\tau}(s) + 2\pi(s), \quad (21)$$

respectively. For our problem, $\pi(s)$ is obtained as

$$\pi(s) = \frac{-s}{2} \pm \sqrt{s^2[a - k] - s[b - k] + c} \quad (22)$$

where $a = \frac{1}{2} + \varepsilon^2 + \alpha^2 + \beta^2$, $b = 2\varepsilon^2 + \beta^2 - l(l+1)$ and $c = \varepsilon^2$. The parameter $k$ could be determined by using the condition that the discriminant of the expression Eq.(22) under the square root is equal to zero. Therefore, we get:

$$k_{1,2} = (b - 2c) \pm 2\sqrt{c^2 + c(a-b)} \quad (23)$$

We can obtain four different expressions of $\pi(s)$ from Eq.(22) for the above four different values of $k$.

$$\pi(s) = \frac{-s}{2} \pm \left\{ \begin{array}{l} \sqrt{c} - \sqrt{c + a - b} s \pm \sqrt{c} \quad \text{for} \quad k = (b - 2c) + 2\sqrt{c^2 + c(a-b)}, \\ \sqrt{c} + \sqrt{c + a - b} s - \sqrt{c} \quad \text{for} \quad k = (b - 2c) - 2\sqrt{c^2 + c(a-b)} \end{array} \right. \quad (24)$$
However, we choose only one of the above expressions so that the function $\tau(s)$ has the negative derivative. Another expressions have no physical meaning.

Given a nonnegative integer $n_r$, the hypergeometric-type equation has an unique polynomial solution of degree $n_r$, if

$$\bar{\lambda} = \bar{\lambda}_{n_r} = -n_r\tau' - \frac{n_r(n_r - 1)}{2}\sigma'', (n_r = 0, 1, 2...),$$

and $\bar{\lambda}_m \neq \bar{\lambda}_n$ for $m = 0, 1, 2, ..., n_r - 1$, it follows that,

$$\bar{\lambda}_{n_r} = b - 2c - 2\sqrt{c^2 + c(a - b)} - \left[\frac{1}{2} + \sqrt{c + \sqrt{c + a - b}}\right]$$

$$= 2n_r\left[1 + \left(\sqrt{c + \sqrt{c + a - b}}\right)\right] + n_r(n_r - 1).$$

We can solve Eq.(26) explicitly for $c$ by using the relation $c = \varepsilon^2$ which brings

$$\varepsilon^2 = \frac{\beta^2 - \delta(l + 1) - 1/2 - n_r(n_r + 1) - (2n_r + 1)\sqrt{1/4 + \alpha^2 + l(l + 1)}}{2n_r + 1 + 2\sqrt{1/4 + \alpha^2 + l(l + 1)}}.$$

After inserting $\varepsilon^2$ into Eq.(14) for energy levels we get:

$$M^2 - E^2_{n_r,l} = \left[\frac{\beta^2 - \delta(l + 1) - 1/2 - n_r(n_r + 1) - (2n_r + 1)\sqrt{1/4 + \alpha^2 + l(l + 1)}}{n_r + 1/2 + \sqrt{1/4 + \alpha^2 + l(l + 1)}}\right] \times \delta.$$

The energy levels $E_{n_r,l}$ can be obtained from the energy equation Eq.(28), which is rather complicated than the square equation.

Let us now begin to obtain the radial eigenfunctions by using the NU-method. After substituting $\pi(s)$ and $\sigma(s)$ into Eq.(19), the function $\phi(s)$ is easily obtained as a solution of the first order differential equation:

$$\phi(s) = s^k(1 - s)^K,$$

where $K = 1/2 + \sqrt{1/4 + \alpha^2 + l(l + 1)}$. Beyond that, the other part of the wave function, $y_n(s)$, is the hypergeometric-type function and its polynomial solutions are given by Rodrigues relation:

$$y_n(s) = \frac{C_n}{\rho(s)} \frac{d^n}{ds^n} [\sigma^n(s)\rho(s)],$$

where $C_n$ is a normalization constant and $\rho(s)$ denotes the weight function. Note that $\rho(s)$ is a solution of the Pearson differential equation. The Pearson differential equation and $\rho(s)$ for this problem are given by

$$(\sigma\rho)' = \tau\rho,$$

$$\rho(s) = (1 - s)^{2K - 1}s^{2\varepsilon}.$$  

By substituting the Eq.(32) into Eq.(30), one get

$$y_n(s) = C_{n_r}(1 - s)^{1 - 2K}s^{-2\varepsilon} \frac{d^{n_r}}{d{s^{n_r}}} \left[s^{2\varepsilon + n_r}(1 - s)^{2K - 1 + n_r}\right].$$
Then by using the definition of the Jacobi polynomials [26] and substituting Eq. (33) into the Eq. (17) then for \( \chi_{nr}(s) \) obtain:

\[
\chi_{nr}(s) = C_{nr} s^\varepsilon (1 - s)^K P_{nr}^{(2\varepsilon, 2K - 1)}(s).
\]  

(34)

Also \( \chi_{nr}(s) \) written in terms of the hypergeometric function as follows [26]

\[
\chi_{nr}(s) = C_{nr} s^\varepsilon (1 - s)^K \frac{\Gamma(n_r + 2\sqrt{\varepsilon} + 1)}{n_r \Gamma(2\varepsilon + 1)} F(-n_r, 2\varepsilon + 2K + n_r, 1 + 2\varepsilon; s).
\]  

(35)

The constant \( C_{nr} \) can be calculated from the following normalization condition

\[
\int_0^\infty |R(r)|^2 r^2 dr = \int_0^\infty |\chi(r)|^2 dr = \frac{1}{2\delta} \int_0^1 \frac{1}{s} |\chi(s)|^2 ds = 1,
\]  

(36)

After simple calculations, the normalization constant is obtained by

\[
C_{nr} = \sqrt{ \frac{2\delta n_r! (n_r + K + \varepsilon) \Gamma(2\varepsilon + 1) \Gamma(n + 2\varepsilon + K - 1)}{(n_r + K) \Gamma(2\varepsilon) \Gamma(n_r + 2K)}}.
\]  

(37)

3.2. Implementation of SUSY Quantum Mechanics

According to SUSYQM, the eigenfunction of ground state \( \chi_0(r) \) is given by

\[
\chi_0(r) = N \exp \left( - \int W(r) dr \right),
\]  

(38)

where \( W(r) \) is called as superpotential function and \( N \) is a normalization constant. The relationships between the superpotential \( W(r) \) and the supersymmetric partner potentials \( V_-(r), V_+(r) \) are defined as [6, 7]:

\[
V_-(r) = W^2(r) - W'(r), \quad V_+(r) = W^2(r) + W'(r).
\]  

(39)

The particular solution of the Riccati equation Eq. (39) is given by

\[
W(r) = \left( F - \frac{Ge^{-2\delta r}}{1 - e^{-2\delta r}} \right),
\]  

(40)

where \( G \) and \( F \) are unknown constants. By using \( V_-(r) = V_{\text{eff}}(r) - (E^2 - M^2) \), the expressions (12) and (40) are inserted into the Eq. (39). Then, for \( G \) and \( F \), we get the following relations from comparison of compatible quantities in the left and right sides of the equation:

\[
F^2 = 4\delta^2 \varepsilon^2,
\]  

(41)

\[
2FG + 2\delta G = 4\delta^2 \beta^2 - 4\delta^2 l(l + 1), \quad G^2 - 2\delta G = 4\delta^2 \alpha^2 + 4\delta^2 l(l + 1).
\]  

(42)

If taking into account of extremity conditions for wave functions, it reaches \( G > 0 \) and \( F < 0 \). Solving Eq. (42) gives

\[
G = \frac{2\delta \pm \sqrt{4\delta^2 + 16\delta^2 (\alpha^2 + l(l + 1))}}{2} = \delta \pm 2\delta \sqrt{\frac{1}{4} + l(l + 1) + \alpha^2}.
\]  

(43)
When consider \( G > 0 \), from Eq. (42) we find

\[
2FG + G^2 = 4\delta^2(\alpha^2 + \beta^2) \Rightarrow F = \frac{G}{2} + \frac{2\delta^2(\alpha^2 + \beta^2)}{G}. \tag{44}
\]

Next, from Eq. (41) and Eq. (44), we find

\[
\varepsilon^2 = \frac{1}{4\delta^2} \left[ -\frac{\delta + 2\delta \sqrt{\frac{1}{4} + \alpha^2 + l(l+1)}}{2} - \frac{2\delta^2(\alpha^2 + \beta^2)}{\delta + 2\delta \sqrt{\frac{1}{4} + \alpha^2 + l(l+1)}} \right]^2. \tag{45}
\]

After inserting Eq. (45) into the Eq. (14) for energy eigenvalue, we obtain

\[
M^2 - E^2 = \left[ -\frac{\delta + 2\delta \sqrt{\frac{1}{4} + \alpha^2 + l(l+1)}}{2} - \frac{2\delta(\alpha^2 + \beta^2)}{1 + 2\sqrt{\frac{1}{4} + \alpha^2 + l(l+1)}} \right]^2. \tag{46}
\]

When \( r \to \infty \), the superpotential \( W(r) \to F \). Inserting the Eq. (40) into Eq. (39), the supersymmetric partner potentials \( V_-(r) \) and \( V_+(r) \) can be found in the following forms.

\[
V_+(r) = W^2(r) + W'(r) = \left[ F^2 - \frac{(2FG + 2\delta G)e^{-2\delta r}}{1 - e^{-2\delta r}} + \frac{(G^2 + 2\delta G)e^{-4\delta r}}{1 - e^{-2\delta r}} \right]^2. \tag{47}
\]

By using the superpotential \( W(r) \) in Eq. (40), the radial eigenfunction \( \chi_0(r) \) is obtained as

\[
\chi_0(r) = N \exp \left[ -\int W(r)dr \right] = Ne^{Fr} \exp \left[ \frac{G}{\delta} \int \frac{d(1 - e^{-2\delta r})}{1 - e^{-2\delta r}} \right] = Ne^{Fr}(1 - e^{-2\delta r})^{\frac{G}{2\delta}}, \tag{48}
\]

where \( r \to 0 \); \( \chi_0(r) \to 0 \), \( G > 0 \) and \( r \to \infty \); \( \chi_0(r) \to 0 \), \( F < 0 \). The potentials \( V_-(r) \) and \( V_+(r) \) differ from each other with additive constants. They have the same functional form called the invariant potentials \([19, 20]\). The invariant forms of potentials \( V_-(r) \) and \( V_+(r) \) in Eq. (39) are given by

\[
R(G_1) = V_+(G, r) - V_-(G_1, r) = \left[ \frac{G}{2} + \frac{2\delta^2(\alpha^2 + \beta^2)}{G} \right]^2 + \left[ \frac{G + 2\delta}{2} + \frac{2\delta^2(\alpha^2 + \beta^2)}{G + 2\delta} \right]^2, \tag{49}
\]

\[
R(G_i) = V_+[G + (i - 1) \times 2\delta, r] - V_-[G + i \times 2\delta, r]
= \left( -\frac{G + (i - 1) \times 2\delta}{2} + \frac{2(\alpha^2 + \beta^2)\delta^2}{G + (i - 1) \times 2\delta} \right)^2 - \left( \frac{G + i \times 2\delta}{2} - \frac{2(\alpha^2 + \beta^2)\delta^2}{G + i \times 2\delta} \right)^2, \tag{50}
\]

where the reminder \( R(G_i) \) is independent of \( r \). If this procedure is continued by using \( G_0 = G_{n=1} + 2\delta = G + 2n_r\delta \), then the whole discrete spectrum of Hamiltonian \( H_-(G) \) can be written as:

\[
E_{n_r}^2 = E_0^2 + \sum_{i=1}^{n_r} R(G_i) = M^2 - \left( -\frac{G + 2n_r\delta}{2} + \frac{2\delta^2(\alpha^2 + \beta^2)}{G + 2n_r\delta} \right)^2. \tag{51}
\]

Substitute Eq. (43) into Eq. (51) then finally for energy eigenvalues as in ordinary quantum mechanics, we found the following form:

\[
M^2 - E_{n_r,l}^2 = \left[ \frac{\beta^2 - l(l+1) - 1/2 - n_r(n_r + 1) - (2n_r + 1)\sqrt{\frac{1}{4} + \alpha^2 + l(l+1)}}{n_r + \frac{1}{2} + \sqrt{\frac{1}{4} + \alpha^2 + l(l+1)}} \right]^2. \tag{52}
\]

As seen from Eq. (52), it is exactly identical with Eq. (28).
4. Results and Discussion
In this study, we have used alternative two method to solve the KFG equation for the Manning-Rosen plus Yukawa potential by using the developed scheme to overcome the centrifugal part. The energy eigenvalues and corresponding eigenfunctions of a mentioned quantum system have been analytically obtained for any \( l \) and \( n_r \) quantum numbers by using both NU and SUSYQM methods. The same expressions for the energy eigenvalues are obtained, and the expression of radial wave functions transformations to each other is revealed via both methods. It is seen that the energy eigenvalues and eigenfunctions are sensitive to the quantum numbers \( n_r \) and \( l \).

A few important points should be emphasized as follows: When \( l = 0 \), the approximation centrifugal term is \( l(l + 1)\delta^2 \frac{e^{−2\delta r}}{(1−e^{−2\delta r})^2} = 0 \). If we let \( l = 0 \) in Eq.\((28)\) and Eq.\((35)\), they describe the unnormalized radial wave functions and the exact energy spectrum formula for the bound states of \( s \)-wave KFG equation. It is obvious from Eq.\((28)\) that the bound states are more stable for Manning-Rosen plus Yukawa potential compared to each of these potentials (Manning-Rosen and Yukawa potentials). Two important cases are seen at the combined potential: In the first case which \( V_{03} = 0 \) the potentials turn to central Manning-Rosen potential. In the second case, if \( V_{01} = V_{02} = 0 \) then the potentials turn to central Yukawa potential.

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