Alternative solution to a quantum-mechanical four-particle system in one dimension

Francisco M. Fernández∗
INIFTA, DQT, Sucursal 4, C.C 16, 1900 La Plata, Argentina

Abstract

We propose an alternative solution to a quantum-mechanical four-particle system in one dimension with two- and three-particle interactions. The solution of the eigenvalue equation in center-of-mass and Jacobi coordinates is considerably simpler than a procedure proposed recently based on spherical coordinates.

1 Introduction

In this paper we propose an alternative solution to an exactly-solvable quantum-mechanical four-particle model in one dimension with two- and three-particle interactions. This model was recently proposed by Bakhshi and Khoshdooni (BK) [1]. In order to solve the eigenvalue equation they resorted to center-of-mass and “Jacobi” coordinates [2] in order to remove the free motion of the center of mass and obtain a new operator with discrete spectrum. BK showed that the resulting Hamiltonian is separable in spherical coordinates and provided solutions for the radial and angular parts of the eigenvalue equation.

∗fernande@quimica.unlp.edu.ar
In addition to providing the alternative solution just mentioned we carry out an analysis of BK’s results. In section 2 we introduce the model, in section 3 we outline and analyze BK’s results, in section 4 we rederive those results in a somewhat clearer way, in section 5 we solve the problem directly in Jacobi coordinates and, finally, in section 6 we outline the main results and draw conclusions.

2 The model

BK proposed the following four-particle Hamiltonian in one dimension [1]

\[ H = -\frac{1}{2} \sum_{i=1}^{4} \frac{\partial^2}{\partial x_i^2} + \frac{\omega^2}{8} \sum_{i<j}^{4} (x_i - x_j)^2 + \frac{g_1^2}{(x_1 + x_2 - 2x_3)^2}, \]

and argued that they chose \(h = 2m = 1\) which is obviously inconsistent with the factor \(1/2\) in the kinetic-energy term (see reference [3] for a pedagogical introduction to dimensionless quantum-mechanical equations). The strength \(g_1^2\) of the Wolfes’ interaction [4] is expected to be a real and positive model parameter. BK stated that the spectrum of this operator is real which is not the case unless one removes the motion of the center of mass.

By means of the change of variables \(\tilde{x}_i = \sqrt{\omega} x_i\) one can easily prove that \(H(\omega, g_1) = \omega H(1, g_1)\) so that one can choose \(\omega = 1\) without loss of generality (see reference [3] for other cases in which the number of model parameters can be reduced).

3 BK’s solution in spherical coordinates

In order to solve the Schrödinger equation BK resorted to the center-of-mass and translation-invariant Jacobi coordinates [2]

\[ X_1 = \frac{1}{\sqrt{2}} (x_1 - x_2), \]
\[ X_2 = \frac{1}{\sqrt{6}} (x_1 + x_2 - 2x_3), \]
\[
X_3 = \frac{1}{\sqrt{12}} (x_1 + x_2 + x_3 - 3x_4),
\]
\[
X = \frac{1}{2} (x_1 + x_2 + x_3 + x_4),
\]
(2)

and the spherical coordinates
\[
\begin{align*}
X_1 &= r \sin \theta \cos \phi, \\
X_2 &= r \sin \theta \sin \phi, \\
X_3 &= r \cos \theta,
\end{align*}
\]
(3)

Curiously, BK chose the obviously incorrect domains \(0 \leq \theta \leq 2\pi, \ 0 \leq \phi \leq \pi\).

After carrying out such transformations and removing the free motion of the center of mass the Hamiltonian operator becomes
\[
H = -\frac{1}{2r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{\omega^2}{2} r^2 + \frac{K}{2r^2},
\]
\[
K = -\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} + \frac{F}{\sin^2 \theta},
\]
\[
F = -\frac{\partial^2}{\partial \phi^2} + \frac{g_1^2}{3 \sin^2 \phi}.
\]
(4)

In order to solve the eigenvalue equation for \(H\) BK resorted to the method of separation of variables and, consequently, chose a solution of the form
\[
\psi(r, \theta, \phi) = R(r) \Theta(\theta) \Phi(\phi).
\]
(5)

However, they assumed that \(\Theta(\theta)\) is a solution to \(K \Theta_l(\theta) = k_l^2 \Theta_l(\theta)\), \(l = 0, 1, \ldots\), which is incorrect because \(K\) is a two-variable operator. They compared the resulting radial equation with that for a harmonic oscillator and concluded that \(k_l^2 = l(l + 1)\) and \(E = \omega \left(2n + l + \frac{3}{2}\right)\). BK did not specify the values of \(n\) but we may safely assume that \(n = 0, 1, \ldots\). At this point it is worth noticing that BK’s energies are independent of \(g_1\) which is most suspicious. In fact, the Hellmann-Feynman theorem \([5, 6]\) states that
\[
\frac{dE}{dg_1} = \left\langle \frac{1}{(x_1 + x_2 - 2x_3)^2} \right\rangle > 0.
\]
(6)
In their section 4 BK appeared to solve the eigenvalue equation in a more reasonable way starting from \( F\Phi_m(\phi) = f_m^2 \Phi_m(\phi), \ m = 0, 1, \ldots \). Upon comparing their equations (4.4) and (4.5) they concluded that \( f_m = \lambda \) and \( k_l^2 = n(n + 1) \). BK did not specify the values of \( n \) but it is clear that \( n = -l - 1 \) or \( n = l \) in order to be consistent with the previous result for \( k_l^2 \). After some judicious manipulation of their equations BK concluded that \( g_1^2 = 3(\lambda^2 - \frac{1}{4}) \) from which we derive \( f_m^2 = \frac{1}{3}g_1^2 + \frac{1}{4} \). In other words: given a value of the model parameter \( g_1 \) all the eigenvalues of \( F \) are equal, which does not seem to be reasonable.

Somewhat later BK obtained the expression \( f_m = n^2 + \frac{1}{2} \) which is intriguing because it relates the quantum numbers \( m \) and \( n \). From all those equations we conclude that \( g_1^2 = 3n(n + 1) = 3l(l + 1) \) which makes no sense because the model parameter \( g_1 \) (the strength of the Wolfes’ interaction) cannot depend on the quantum numbers. In the next section we outline the solution of the eigenvalue equation for \( H \) in spherical coordinates in a more reasonable way.

4 Correct solution in spherical coordinates

In order to solve the eigenvalue equation for \( H \) in spherical coordinates we first solve
\[
F\Phi_m(\phi) = \left(-\frac{\partial^2}{\partial \phi^2} + \frac{g_1^2}{3\sin^2 \phi}\right)\Phi_m(\phi) = f_m^2 \Phi_m(\phi), \ m = 0, 1, \ldots, \tag{7}
\]
and remove \( \Phi_m(\phi) \) from the eigenvalue equation for \( H \). Second, we solve
\[
\left(-\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} + \frac{f_m^2}{\sin^2 \theta}\right)\Theta_{lm}(\theta) = k_{lm}^2 \Theta_{lm}(\theta), \tag{8}
\]
and remove \( \Theta_{lm}(\theta) \) from the eigenvalue equation for \( H \). Finally, we are left with the radial equation
\[
\left(-\frac{1}{2r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{\omega^2}{2} r^2 + \frac{k_{lm}^2}{2r^2}\right)R_{nlm}(r) = E_{nlm} R_{nlm}(r). \tag{9}
\]

It is not difficult to verify that the eigenvalues of this radial equation are given by
\[
E_{nlm} = \omega \left(2n + s + \frac{3}{2}\right), \ s = \frac{1}{2} \left(\sqrt{k_{lm}^2 + 1} - 1\right), \tag{10}
\]
that resembles BK's equation (3.6), except for the fact that \(s\) depends on \(g_1\) as well as on the quantum numbers \(l\) and \(m\). In order to obtain the exact analytical expression we have to solve equations (7) and (8). However it is not necessary, in our opinion, because there is a much simpler approach outlined in the following section.

5 Exact textbook solution

The Hamiltonian operator in the center-of-mass and Jacobi coordinates becomes

\[
H = H_d - \frac{1}{2} \frac{\partial^2}{\partial X^2}, \quad H_d = -\frac{1}{2} \sum_{i=1}^{3} \frac{\partial^2}{\partial X_i^2} + \frac{\omega^2}{2} \sum_{i=1}^{3} X_i^2 + \frac{g_1^2}{6X_2^2}.
\]

(11)

We appreciate that the spectrum of \(H\) is continuous while that of \(H_d\) is discrete. In fact, in the two preceding sections we discussed the solutions to the eigenvalue equation for \(H_d\) in spherical coordinates. It is clear that \(H_d\) is separable in Jacobi coordinates

\[
H_d = H_1 + H_2 + H_3,
\]

\[
H_i = -\frac{1}{2} \frac{\partial^2}{\partial X_i^2} + \frac{\omega^2}{2} X_i^2, \quad i = 1, 3,
\]

\[
H_2 = -\frac{1}{2} \frac{\partial^2}{\partial X_2^2} + \frac{\omega^2}{2} X_2^2 + \frac{g_1^2}{6X_2^2}.
\]

(12)

Note that \(H_1\) and \(H_3\) are one-dimensional harmonic oscillators (HO), whereas \(H_2\) is the so-called singular harmonic oscillator (SHO). Therefore, if we try a solution of the form

\[
\psi_{n_1n_2n_3}(X_1, X_2, X_3) = \varphi_{n_1}^{HO}(X_1) \varphi_{n_2}^{SHO}(X_2) \varphi_{n_3}^{HO}(X_3),
\]

(13)

we conclude that

\[
H_i \varphi_{n_i}^{HO} = E_{n_i}^{HO} \varphi_{n_i}^{HO}, \quad i = 1, 3, \quad H_2 \varphi_{n_2}^{SHO} = E_{n_2}^{SHO} \varphi_{n_2}^{SHO},
\]

\[
E_{n_i}^{HO} = \omega \left(n_i + \frac{1}{2}\right), \quad n_i = 0, 1, \ldots,
\]

\[
E_{n_1n_2n_3}^{HO} = E_{n_1}^{HO} + E_{n_2}^{SHO} + E_{n_3}^{HO}.
\]

(14)
The eigenvalues of the singular harmonic oscillator can also be calculated analytically [7]:

\[ E_{n_2}^{S_{\text{HO}}} = \omega \left( 2n_2 + \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{g^2}{3}} \right). \] (15)

6 Conclusions

The main conclusion of this paper is that the problem can be solved more easily and straightforwardly in Jacobi coordinates which makes it unnecessary to resort to spherical ones. However, we have outlined an earlier solution of the problem in spherical coordinates in order to compare those and present results. In section 4 we outlined a more rigorous and clearer solution in terms of spherical coordinates and in section 5 the simpler procedure in terms of Jacobi coordinates proposed here.

References

[1] Z. Bakhsia and S. Khoshdooni, Eur. Phys. J. Plus 136, 374 (2021).

[2] F. Calogero, J. Math. Phys. 12, 419 (1971).

[3] F. M. Fernández, Dimensionless equations in non-relativistic quantum mechanics, [arXiv:2005.05377] [quant-ph].

[4] J. Wolfes, J. Math. Phys. 15, 1420 (1974).

[5] P. Güttinger, Z. Phys. 73, 169 (1932).

[6] R. P. Feynman, Phys. Rev. 56, 340 (1939).

[7] G. Palma and U. Raff, Am. J. Phys. 71, 247 (2003). Addendum: Am. J. Phys. 71, 956 (2003)