Pure Spinors, Twistors, and Emergent Supersymmetry

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Starting with a classical action whose matter variables are a d=10 spacetime vector $x^m$ and a pure spinor $\lambda^\alpha$, the pure spinor formalism for the superstring is obtained by gauge-fixing the twistor-like constraint $\partial x^m (\gamma_m \lambda)_\alpha = 0$. The fermionic variables $\theta^\alpha$ are Faddeev-Popov ghosts coming from this gauge-fixing and replace the usual $(b, c)$ ghosts coming from gauge-fixing the Virasoro constraint. After twisting the ghost-number such that $\theta^\alpha$ has ghost-number zero and $\lambda^\alpha$ has ghost-number one, the BRST cohomology describes the usual spacetime supersymmetric states of the superstring.
1. Introduction

For computing multiloop scattering amplitudes or constructing quantizable sigma models in Ramond-Ramond backgrounds, the pure spinor formalism is the most convenient description of the superstring \[1\]. This formalism contains the usual \((x^m, \theta^\alpha)\) matter variables of \(d=10\) superspace, as well as bosonic ghost variables \(\lambda^\alpha\) satisfying the pure spinor constraint \(\lambda^\gamma{}^m\lambda = 0\). Physical states are elegantly described by the ghost-number one cohomology of the BRST operator \(Q = \int \lambda^\alpha d_\alpha\) where \(d_\alpha\) is the fermionic Green-Schwarz constraint.

Since \(\{d_\alpha, d_\beta\} = \gamma^m_{\alpha\beta} \Pi_m\) where \(\Pi_m\) is the supersymmetric momentum, \(d_\alpha\) contains both first and second-class constraints. So the nilpotence of \(Q\) does not follow from the Jacobi identity of a first-class constraint algebra but instead requires that the ghost variable \(\lambda^\alpha\) satisfies the pure spinor constraint. This unusual feature has made it difficult to obtain \(Q\) from gauge-fixing a classical action using the standard BRST procedure. Another unusual feature of \(Q\) in the pure spinor formalism is that it does not involve \((b, c)\) worldsheet reparameterization ghosts or the Virasoro constraint.

In this paper, a new interpretation of \(Q\) will be proposed which elegantly explains its origin. Instead of interpreting \((x^m, \theta^\alpha)\) as matter variables and \(\lambda^\alpha\) as ghost variables, \(x^m\) and \(\lambda^\alpha\) will be interpreted as matter variables and \(\theta^\alpha\) as ghost variables. And instead of imposing the usual Virasoro constraint \(\partial x^m \partial x_m + ... = 0\), the matter variables will instead satisfy \(\partial x^m (\gamma^m \lambda)_\alpha = 0\). This constraint implies that \(\partial x^m = \lambda^m \gamma^m h\) for some spinor \(h^\alpha\), so it is a ten-dimensional version of the \(d=4\) twistor constraint \(\partial x^m = \lambda^a \sigma^m_{\dot{a}\dot{b}} \bar{\lambda}^\dot{b}\).\[2\] [3] Quantizing this twistor-like constraint in the standard manner leads to fermionic Faddeev-Popov ghosts \(\theta^\alpha\) and a nilpotent BRST operator. After twisting the ghost-number so that \(\theta^\alpha\) has ghost-number zero and \(\lambda^\alpha\) has ghost-number one, the cohomology of this BRST operator will be related to the cohomology of \(Q = \int \lambda^\alpha d_\alpha\).

Although the classical action has no fermionic variables, the BRST cohomology is spacetime supersymmetric after twisting the ghost charge. The “emergence” of \(d=10\) spacetime supersymmetry after imposing a twistor-like constraint is quite surprising and may explain why twistor methods have been so useful for describing spacetime supersymmetric theories. Although twistors were originally developed by Penrose to describe purely bosonic \(d=4\) theories, their most powerful applications have been for \(d=4\) theories with maximal spacetime supersymmetry.

Another interesting observation is that a projective pure spinor \(\lambda^\alpha\) parametrizes \(SO(10)/U(5)\) which chooses a complex structure of the \(d=10\) (Wick-rotated) spacetime.
So the twistor-like constraint \( \partial x^m (\gamma^m \lambda)_{\alpha} = 0 \) resembles the constraint \( \partial x^I = 0 \) of a \( c = 5 \) topological string where \( I = 1 \) to \( 5 \) labels the holomorphic directions. As pointed out by Nekrasov[4], introducing \( \lambda^\alpha \) to dynamically determine the complex structure enlarges the cohomology of the topological string BRST operator to the full superstring spectrum.

2. Superparticle

2.1. Worldline action and BRST operator

Before discussing this gauge-fixing procedure for the superstring, it will be useful to first discuss the procedure for the superparticle whose spectrum is d=10 super-Yang-Mills. For the superparticle, the classical worldline action will be defined as

\[
S = \int d\tau [P_m \frac{\partial}{\partial \tau} x^m + w_\alpha \frac{\partial}{\partial \tau} \lambda^\alpha + f^\alpha (P_m \gamma^m \lambda)_{\alpha}] 
\]

where \( m = 0 \) to \( 9 \), \( \alpha = 1 \) to \( 16 \), \( \lambda^\alpha \) is a pure spinor satisfying

\[
\lambda \gamma^m \lambda = 0, \tag{2.2}
\]

and \( f^\alpha \) is a Lagrange multiplier for the twistor-like constraint \( P_m (\gamma^m \lambda)_{\alpha} = 0 \).

The first step is to gauge-fix the Lagrange multiplier \( f^\alpha = 0 \) which introduces the fermionic Faddeev-Popov ghosts \( (\theta^\alpha, p_\alpha) \) with worldsheet action \( \int d\tau p_\alpha \frac{\partial}{\partial \tau} \theta^\alpha \). The resulting BRST operator is \( Q = \theta^\alpha P_m (\gamma^m \lambda)_{\alpha} \) where \( (\theta^\alpha, p_\alpha) \) carry ghost-number \((+1, -1)\).

Since only \( 5 \) of the \( 16 \) components of the constraint \( P_m (\gamma^m \lambda)_{\alpha} \) are independent, there are bosonic ghosts-for-ghosts coming from the gauge transformation of the Lagrange multiplier \( \delta f^\alpha = \epsilon^\alpha \) where \( \epsilon^\alpha \) satisfies \( \epsilon^\alpha \gamma^m \lambda^\beta = 0 \). This implies the introduction of bosonic ghost-for-ghosts \( (u^\alpha, v_\alpha) \) with worldsheet action \( \int d\tau v_\alpha \frac{\partial}{\partial \tau} u^\alpha \) where \( u^\alpha \) is constrained to satisfy

\[
u^\alpha \gamma^m \lambda^\beta = 0 \tag{2.3}
\]

and \( (u^\alpha, v_\alpha) \) carry ghost-number \((+2, -2)\). The resulting BRST operator is

\[
Q = \theta^\alpha P_m (\gamma^m \lambda)_{\alpha} + u^\alpha p_\alpha \tag{2.4}
\]

and the gauge-fixed worldline action is

\[
S = \int d\tau [P_m \frac{\partial}{\partial \tau} x^m + w_\alpha \frac{\partial}{\partial \tau} \lambda^\alpha + p_\alpha \frac{\partial}{\partial \tau} \theta^\alpha + v_\alpha \frac{\partial}{\partial \tau} u^\alpha]. \tag{2.5}
\]
2.2. Ghost twisting

At fixed ghost number, the states in the cohomology of $Q$ resemble the states of the topological string. For example, the states at ghost-number one are $V = (\theta \gamma^m \lambda) A_m(x)$ where $\partial_m A_n - \partial_n A_m = 0$ and $A_m \neq \partial_m \Lambda$ for any $\Lambda$. On a surface of trivial topology, this ghost-number one cohomology vanishes. However, note that $Q$ is invariant under the scale transformation generated by

$$\Phi = \lambda^\alpha w_\alpha - \theta^\alpha p_\alpha - u^\alpha v_\alpha$$

which transforms

$$\lambda^\alpha \to \Lambda \lambda^\alpha, \quad \theta^\alpha \to \Lambda^{-1} \theta^\alpha, \quad u^\alpha \to \Lambda^{-1} u^\alpha,$$

$$w^\alpha \to \Lambda^{-1} w^\alpha, \quad p_\alpha \to \Lambda p_\alpha, \quad v_\alpha \to \Lambda v_\alpha.$$

So one can define a twisted ghost number

$$\tilde{G} = G + \Phi = \lambda^\alpha w_\alpha + u^\alpha v_\alpha$$

where $G = \theta^\alpha p_\alpha + 2 u^\alpha v_\alpha$ is the original ghost number. With respect to $\tilde{G}$, $Q$ still carries +1 ghost number but $(\theta^\alpha, p_\alpha)$ now carry zero ghost number.

At fixed twisted ghost number, the cohomology of $Q$ contains non-topological states describing the super-Maxwell spectrum. These states are described by the vertex operator

$$V = \lambda^\alpha A_\alpha(x, \theta) \delta(u - \lambda)$$

where

$$\delta(u - \lambda) \equiv (\lambda^3 h^{11}) \Pi_{I=1}^{11} \delta(h_{I\beta}(u^\beta - \lambda^\beta)),$$

$$(\lambda^3 h^{11}) \equiv \epsilon^{\alpha_1 \ldots \alpha_{16}} h_{1\alpha_1} \ldots h_{11\alpha_{11}} (\lambda \gamma^m)_{\alpha_12} (\lambda \gamma^n)_{\alpha_13} (\lambda \gamma^p)_{\alpha_14} (\gamma_{mnp})_{\alpha_{15}\alpha_{16}}$$

and $h_{I\alpha}$ are any 11 spinors such that $(\lambda^3 h^{11})$ is nonzero. Note that $\delta(u - \lambda)$ of (2.10) is independent of the choice of $h_{I\alpha}$ since it is invariant under

$$\delta h_{I\alpha} = (\gamma_m \lambda)_{\alpha} \Lambda^m_I$$

for any $\Lambda^m_I$.

To verify that $V$ describes the super-Maxwell spectrum, define $U^\alpha \equiv u^\alpha - \lambda^\alpha$ where $U^\alpha$ satisfies $U^\gamma m^\lambda = 0$. Then using $P_m = -i \partial_m$ and $p_\alpha = \frac{\partial}{\partial \theta^\alpha}$, $Q = \lambda^\alpha D_\alpha + U^\alpha \frac{\partial}{\partial \theta^\alpha}$
and \( V = \lambda^\alpha A_\alpha(x, \theta) \delta(U) \) where \( D_\alpha = \frac{\partial}{\partial \theta_\alpha} - i (\gamma^m \theta)_\alpha \partial_m \). \( QV = 0 \) implies the super-Maxwell equations \( \gamma_{m_1 \ldots m_5}^\alpha D_\alpha A_\beta = 0 \) and \( \delta V = Q[\Omega(x, \theta) \delta(U)] \) implies the super-Maxwell gauge transformation \( \delta A_\alpha = D_\alpha \Omega \) where \( A_\alpha(x, \theta) \) is the super-Maxwell spinor gauge superfield \( \textbf{[5]} \). So even though the classical action contains no fermionic variables, the BRST cohomology after twisting the ghost-number is spacetime supersymmetric and describes super-Maxwell.

Note that there are other states in the BRST cohomology in addition to \((2.9)\). For example, the states

\[
(\lambda \frac{\partial}{\partial \lambda} - u \frac{\partial}{\partial u} - \theta \frac{\partial}{\partial \theta}) V \quad \text{and} \quad [2\lambda \gamma^{mn} \frac{\partial}{\partial u} - \lambda \gamma^{mn} \frac{\partial}{\partial \lambda} - u \gamma^{mn} \frac{\partial}{\partial u} + \theta \gamma^{mn} \frac{\partial}{\partial \theta} + i (\theta \gamma^{mnp} \theta) \partial_p] V
\]

are also in the cohomology where \( V \) is defined in \((2.9)\). However, as will be discussed in the last section, these states are eliminated after truncating to a “small” Hilbert space.

### 3. Superstring

#### 3.1. Worldsheet action and BRST operator

In this section, the gauge-fixing procedure will be repeated for the superstring. The classical worldsheet action is constructed from the variables \( x^m \) and the left and right-moving pure spinors \( \lambda^\alpha \) and \( \hat{\lambda}^{\hat{\alpha}} \) satisfying \( \lambda^\gamma \lambda = \hat{\lambda}^{\hat{\gamma}} \hat{\lambda} = 0 \) where \( \lambda^\alpha \) and \( \hat{\lambda}^{\hat{\alpha}} \) have the same spacetime chirality for the Type IIB superstring and opposite spacetime chirality for the Type IIA superstring. In addition to the left and right-moving twistor-like constraints \( \partial x_m (\gamma^m \lambda)_\alpha = 0 \) and \( \bar{\partial} x_m (\gamma^m \hat{\lambda})_{\hat{\alpha}} = 0 \) which replace the left and right-moving Virasoro constraints, one will also need to include the left and right-moving constraints \( \partial \lambda^\alpha = 0 \) and \( \bar{\partial} \hat{\lambda}^{\hat{\alpha}} = 0 \). These new constraints are necessary to close the first-class algebra since

\[
[\partial x_m (\gamma^m \lambda)_\alpha, \partial x_n (\gamma^n \lambda)_\beta] = (\gamma_m \lambda)_{[\alpha} (\gamma^m \partial \lambda)_{\beta]} , \quad (3.1)
\]

\[
[\bar{\partial} x_m (\gamma^m \hat{\lambda})_{\hat{\alpha}}, \bar{\partial} x_n (\gamma^n \hat{\lambda})_{\hat{\beta}}] = (\gamma_m \hat{\lambda})_{[\hat{\alpha}} (\gamma^m \bar{\partial} \hat{\lambda})_{\hat{\beta}]} . \quad (3.2)
\]

The classical worldsheet action is defined as

\[
S_{\text{classical}} = \int d^2 z [\partial x^m \bar{\partial} x_m + w_\alpha \bar{\partial} \lambda^\alpha + \hat{w}_{\hat{\alpha}} \partial \hat{\lambda}^{\hat{\alpha}} + f^\alpha \partial x_m (\gamma^m \lambda)_\alpha + g_\alpha \partial \lambda^\alpha + \hat{f}^{\hat{\alpha}} \bar{\partial} x_m (\gamma^m \hat{\lambda})_{\hat{\alpha}} + \hat{g}_{\hat{\alpha}} \bar{\partial} \hat{\lambda}^{\hat{\alpha}}]
\]
where \( \partial = \frac{\partial}{\partial z} \) and \( \bar{\partial} = \frac{\partial}{\partial \bar{z}} \), \( f^\alpha \) and \( \hat{f}^{\hat{\alpha}} \) are Lagrange multipliers for the twistor-like constraints \( \partial x_m (\gamma^m \lambda)_{\alpha} = 0 \) and \( \bar{\partial} x_m (\gamma^m \hat{\lambda})_{\hat{\alpha}} = 0 \), and \( g_\alpha \) and \( \hat{g}_{\hat{\alpha}} \) are Lagrange multipliers for the constraints \( \partial \lambda^\alpha = 0 \) and \( \bar{\partial} \hat{\lambda}^{\hat{\alpha}} = 0 \).

The first step in quantizing this action is to gauge-fix the Lagrange multipliers \( f^\alpha = \hat{f}^{\hat{\alpha}} = g_\alpha = \hat{g}_{\hat{\alpha}} = 0 \). This introduces the fermionic Faddeev-Popov ghosts \((\theta^\alpha, p_\alpha), (\hat{\theta}^{\hat{\alpha}}, \hat{p}_{\hat{\alpha}}), (c_\alpha, b^\alpha)\) and \((\hat{c}_{\hat{\alpha}}, \hat{b}^{\hat{\alpha}})\) with the worldsheet action

\[
S_{\text{ghost}} = \int d^2 z (p_\alpha \bar{\partial} \theta^\alpha + b^\alpha \bar{\partial} c_\alpha + \hat{p}_{\hat{\alpha}} \bar{\partial} \hat{\theta}^{\hat{\alpha}} + \hat{b}^{\hat{\alpha}} \bar{\partial} \hat{c}_{\hat{\alpha}}).
\]

The resulting left-moving BRST operator is

\[
Q = \int dz [\theta^\alpha \partial x_m (\gamma^m \lambda)_\alpha + c_\alpha \partial \lambda^\alpha + \frac{1}{2} (b \gamma^m \theta) (\lambda \gamma_m \theta)]
\]

where the last term comes from the constraint algebra of (3.1). The right-moving BRST operator is similarly constructed from the right-moving variables which will sometimes be suppressed.

As in the superparticle, the constraints \( \partial x_m (\gamma^m \lambda)_{\alpha} = 0 \) are not all independent and require the introduction of left-moving ghost-for-ghosts \((u^\alpha, v_\alpha)\) satisfying the constraints

\[
u \gamma^m \lambda = 0.
\]

Furthermore, only 11 of the 16 components of the \( \partial \lambda^\alpha = 0 \) constraint are independent since \( \lambda \gamma^m \partial \lambda = 0 \). To remove the unnecessary ghosts, \((b^\alpha, c_\alpha)\) will be required to satisfy a similar constraint

\[
\hat{b} \gamma^m \hat{\lambda} = 0,
\]

so that only 11 of the 16 \( b^\alpha \)’s are independent. The worldsheet action and left-moving BRST operator are modified by these ghost-for-ghosts to

\[
S = \int d^2 z [\partial x_m \bar{\partial} x_m + w_\alpha \bar{\partial} \lambda^\alpha + \hat{w}_{\hat{\alpha}} \bar{\partial} \hat{\lambda}^{\hat{\alpha}}
\]

\[
+p_\alpha \bar{\partial} \theta^\alpha + b^\alpha \bar{\partial} c_\alpha + v_\alpha \bar{\partial} u^\alpha + \hat{p}_{\hat{\alpha}} \bar{\partial} \hat{\theta}^{\hat{\alpha}} + \hat{b}^{\hat{\alpha}} \bar{\partial} \hat{c}_{\hat{\alpha}} + \hat{v}_{\hat{\alpha}} \bar{\partial} \hat{u}^{\hat{\alpha}}],
\]

\[
Q = \int dz [\theta^\alpha \partial x_m (\gamma^m \lambda)_\alpha + c_\alpha \partial \lambda^\alpha + \frac{1}{2} (b \gamma^m \theta) (\lambda \gamma_m \theta) + u^\alpha p_\alpha].
\]
3.2. Ghost twisting

As in the superparticle, the BRST operator is invariant under global scale transformations generated by

$$\Phi = \int dz (\lambda^\alpha w_\alpha - \theta^\alpha p_\alpha - u^\alpha v_\alpha - c_\alpha b^\alpha)$$

(3.9)

which transform

$$\lambda^\alpha \rightarrow \Lambda \lambda^\alpha, \quad \theta^\alpha \rightarrow \Lambda^{-1} \theta^\alpha, \quad u^\alpha \rightarrow \Lambda^{-1} u^\alpha, \quad c_\alpha \rightarrow \Lambda^{-1} c_\alpha$$

(3.10)

$$w^\alpha \rightarrow \Lambda^{-1} w^\alpha, \quad p_\alpha \rightarrow \Lambda p_\alpha, \quad v_\alpha \rightarrow \Lambda v_\alpha, \quad b^\alpha \rightarrow \Lambda b^\alpha.$$  

To obtain a nontrivial cohomology, physical states will be defined to have fixed twisted ghost-number with respect to $\tilde{G}$ where

$$\tilde{G} = G + \Phi = \int dz (\lambda^\alpha w_\alpha + u^\alpha v_\alpha),$$

(3.11)

and $G = \int dz (\theta^\alpha p_\alpha + c_\alpha b^\alpha + 2u^\alpha v_\alpha)$ is the original ghost number.

To relate $Q$ of (3.8) with the pure spinor BRST operator, perform the similarity transformation $Q \rightarrow e^{-\int dz(c\partial\theta)} Q e^{\int dz(c\partial\theta)}$ so that

$$Q = \int dz [(\lambda^m \gamma^\alpha) \partial x^m + \frac{1}{2} (\partial \gamma^m \theta)(\lambda^\alpha \gamma_m \theta) + u^\alpha p_\alpha + c_\alpha (\partial \lambda^\alpha - \partial u^\alpha) + \frac{1}{2} (b^m \gamma^\alpha)(\lambda^\alpha \gamma_m \theta)]$$

(3.12)

$$= \int dz [\lambda^\alpha d_\alpha + U^\alpha (p_\alpha + \partial c_\alpha) + \frac{1}{2} (b^m \gamma^\alpha)(\lambda^\alpha \gamma_m \theta)]$$

where

$$U^\alpha = u^\alpha - \lambda^\alpha, \quad d_\alpha = p_\alpha + \partial x^m (\gamma^m \theta)_{\alpha} - \frac{1}{2} (\gamma^m \theta)_{\alpha} (\theta \gamma^m \partial \theta).$$

(3.13)

After performing the similarity transformation, the constraint of (3.6) needs to be modified since it no longer anticommutes with $Q$. A suitable modified constraint is

$$b^m \gamma^\alpha \lambda = U^m (b + \partial \theta),$$

(3.14)

which is BRST-invariant with respect to (3.12) and which coincides with the original constraint when $U^\alpha = 0$. 

6
3.3. Physical states

In analogy with the super-Maxwell vertex operator \( V = \lambda^\alpha A_\alpha(x, \theta)\delta(U) \) of the previous section, the naive guess for the general vertex operator is

\[
V = V_0(\lambda, x, \theta, p, w)\delta(U)
\]  

where \( V_0 \) is in the cohomology of \( \int dz \lambda^\alpha d_\alpha \). However, this is not quite right since the terms \( \int dz U^\alpha p_\alpha \) and \( \int dz \frac{1}{2} (b^m \gamma^\theta)(\lambda \gamma_m \theta) \) in \( Q \) might not annihilate \( V \) if \( V_0 \) involves \( \partial \theta^\alpha \) or \( p_\alpha \).

To ensure that the vertex operator is annihilated by \( Q \), consider

\[
V = V_0(\lambda, x, \theta, p, w)Y\delta(U)
\]  

where \( Y\delta(U) \) is defined by

\[
Y\delta(U) = \prod_{I=1}^{11} (h_{1\alpha} b^\alpha) \delta(h_{I\rho} U^\rho) \delta(h_{I\beta} \partial U^\beta).
\]

As before, \( Y\delta(U) \) is independent of the choice of \( h_{1\alpha} \) since it is invariant under \( (2.11) \). Furthermore, \( QV = 0 \) for all states where \( p_\alpha \) has at most a double pole and \( \frac{1}{2} (b^m \gamma^\theta)(\lambda \gamma_m \theta) \) has at most a single pole with \( V_0 \).

To describe a general massive state where \( p_\alpha \) and \( \frac{1}{2} (b^m \gamma^\theta)(\lambda \gamma_m \theta) \) can have more singular poles, define

\[
V = V_0(\lambda, x, \theta, p, w) \lim_{n \to \infty} Y^n\delta(U)
\]  

where

\[
Y^n\delta(U) = \prod_{I=1}^{11} (h_{1\alpha} b^\alpha)\cdots(h_{I\beta} \partial^{n-1} b^\beta) \delta(h_{I\rho} U^\rho) \delta(h_{I\gamma} \partial U^\gamma)\cdots\delta(h_{I\delta} \partial^n U^\delta).
\]

In other words,

\[
V = V_0(\lambda, x, \theta, p, w)|0\rangle
\]  

where \( |0\rangle \) is annihilated by all positive and negative modes of \( U^\alpha \) and \( b^\alpha \) and is not annihilated by any modes of \( v_\alpha \) or \( c_\alpha \). Note that \( |0\rangle \) carries zero conformal weight because of cancellation between the bosonic variables \( (U^\alpha, v_\alpha) \) and fermionic variables \( (b^\alpha, c_\alpha) \).
4. Scattering Amplitudes

4.1. Non-minimal variables

As in the pure spinor formalism, scattering amplitude computations are simplified by introducing “non-minimal” left-moving bosonic worldsheet variables $\bar{\lambda}_\alpha$ satisfying $\bar{\lambda}\gamma^m\lambda = 0$ and its conjugate momentum $\bar{w}^\alpha$. Formally, $\bar{\lambda}_\alpha$ can be interpreted as the complex conjugate of $\lambda^\alpha$ in Euclidean signature, and it is natural to identify both $\lambda^\alpha$ and $\bar{\lambda}_\alpha$ as classical variables. One should also introduce the classical constraint $\bar{w}^\alpha = 0$ so that $\bar{\lambda}_\alpha$ has no effect on the cohomology.

The classical worldsheet action involving the left and right-moving non-minimal variables is

$$S = \int d^2z [\partial x^m \partial x_m + w_\alpha \partial \lambda^\alpha + \bar{w}_\dot{\alpha} \partial \dot{\lambda}^{\dot{\alpha}} + \bar{w}^\alpha \partial \bar{\lambda}_\alpha + \bar{\dot{w}}^{\dot{\alpha}} \partial \dot{\lambda}_\dot{\alpha}]$$

(4.1)

where the Lagrange multipliers $e_\alpha$ and $\hat{e}_{\dot{\alpha}}$ are constrained to satisfy $e\gamma^m\bar{\lambda} = 0$ and $\hat{e}\gamma^m\bar{\dot{\lambda}} = 0$. Gauge-fixing $e_\alpha = 0$ and $\hat{e}_{\dot{\alpha}} = 0$ introduces the fermionic Faddeev-Popov ghosts $(r_\alpha, s^\alpha)$ and $(\hat{r}_{\dot{\alpha}}, \hat{s}^{\dot{\alpha}})$ satisfying the constraints $r\gamma^m\bar{\lambda} = 0$ and $\hat{r}\gamma^m\bar{\dot{\lambda}} = 0$ with the worldsheet action

$$\int d^2z (s^\alpha \bar{\partial} r_\alpha + \hat{s}^{\dot{\alpha}} \bar{\partial} \hat{r}_{\dot{\alpha}})$$

(4.2)

and modifies the left-moving BRST operator to

$$Q = \int dz [\theta^\alpha \partial x_m (\gamma^m \lambda)_\alpha + c_\alpha \partial \lambda^\alpha + \frac{1}{2} (b\gamma^m \theta) (\lambda\gamma^p \theta) + u^\alpha p_\alpha + \bar{w}^\alpha r_\alpha].$$

(4.3)

4.2. Small Hilbert space

Since $c_\alpha$ only appears with derivatives in the BRST operator, it is consistent to remove its zero mode from the Hilbert space as one does with the $\xi$ zero mode of the RNS bosonized ghosts [6]. So physical states $V$ will be required to be in the “small” Hilbert space, i.e. they need to satisfy $b_\alpha^\alpha V = 0$.

This requirement is necessary since, as in the pure spinor formalism, the measure factor for tree level amplitudes will be defined as $\langle (\lambda^3\theta^5) \rangle = 1$ where

$$(\lambda^3\theta^5) \equiv (\lambda\gamma^m \theta)(\lambda\gamma^n \theta)(\lambda\gamma^p \theta)(\theta\gamma_{mnp} \theta).$$

(4.4)

For amplitudes to be BRST invariant, $(\lambda^3\theta^5)$ must be in the cohomology of $Q$. However, $(\lambda^3\theta^5) = Q\Omega$ where $\Omega = (c\gamma^m \lambda)(\lambda\gamma^p \theta)(\theta\gamma_{mnp} \theta)$. Since $b_\alpha^\alpha \Omega$ is nonzero, this does not cause problems if one restricts physical states to the small Hilbert space.
Interestingly, the “picture-raising” operator associated with the \( c_\alpha \) zero mode is
\[
\{ Q, c_\alpha \} = \frac{1}{2} (\gamma^m \theta)_\alpha (\lambda \gamma^m \theta)
\] (4.5)
which is the unintegrated vertex operator for the zero-momentum gluino. The integrated vertex operator of this state is the spacetime supersymmetry generator, so “picture-raising” is related to spacetime supersymmetry.

In the next subsection, it will be argued that the path integrals over the bosonic \((U^\alpha, v_\alpha)\) and fermionic \((b^\alpha, c_\alpha)\) variables should cancel each other. In order for these path integrals to cancel, one needs to impose a second restriction on states in the “small” Hilbert space that they satisfy \( U^\alpha_0 V = 0 \) in addition to \( b^\alpha_0 V = 0 \). This second restriction truncates out the states of (2.12) and reduces the cohomology to the superstring spectrum.

4.3. Tree amplitude prescription

The states of (3.18) are constructed out of a ground state \(|0\rangle\) annihilated by both positive and negative modes of \( U^\alpha \) and \( b^\alpha \). So the path integral over the \((U^\alpha, v_\alpha)\) and \((b^\alpha, c_\alpha)\) variables is not the usual one. However, since the operators \( V_0(\lambda, x, \theta, p, w) \) appearing in vertex operators are independent of the \((U^\alpha, v_\alpha)\) and \((b^\alpha, c_\alpha)\) variables, one never needs to evaluate correlation functions for these variables and only needs to know their partition functions. It will be assumed that the partition function of the bosonic \((U^\alpha, v_\alpha)\) variables cancels the partition function of the fermionic \((b^\alpha, c_\alpha)\) variables, so the scattering amplitude computation reduces to the path integral over the pure spinor non-minimal variables \((x, \theta, \lambda, \bar{\lambda}, r, p, w, \bar{w}, s)\).

As shown in [7], the path integral over non-minimal variables reproduces the tree-level measure factor of \( \langle (\lambda^3 \theta^3) \rangle = 1 \). So the N-point tree amplitude prescription is simply
\[
\langle V_0^{(1)}(z_1)V_0^{(2)}(z_2)V_0^{(3)}(z_3) \prod_{r=4}^N dz_r U_0^{(r)}(z_r) \rangle
\] (4.6)
where the integrated vertex operator is \( U^{(r)} = U_0^{(r)}|0\rangle \) and \( U_0^{(r)} \) satisfies
\[
[ \int \lambda^\alpha d_\alpha, U_0^{(r)}(z_r) ] = \partial V_0^{(r)}(z_r).
\] (4.7)
Assuming the partition functions over the \((U^\alpha, v_\alpha)\) and \((b^\alpha, c_\alpha)\) variables cancel each other, the prescription of (4.6) therefore reproduces the usual pure spinor tree amplitude prescription.
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References

[1] N. Berkovits, *Super-Poincare covariant quantization of the superstring*, JHEP 0004 (2000) 018, [hep-th/0001035](http://arxiv.org/abs/hep-th/0001035).

[2] L.P. Hughston, *The Wave Equation in Even Dimensions*, Further Advances in Twistor Theory, vol. 1, Research Notes in Mathematics 231, Longman, pp. 26-27, 1990; L.P. Hughston, *A Remarkable Connection between the Wave Equation and Pure Spinors in Higher Dimensions*, Further Advances in Twistor Theory, vol. 1, Research Notes in Mathematics 231, Longman, pp. 37-39, 1990; L.P. Hughston and L.J. Mason, *A Generalized Kerr-Robinson Theorem*, Classical and Quantum Gravity 5 (1988) 275.

[3] N. Berkovits and S. Cherkis, *Higher-dimensional twistor transforms using pure spinors*, JHEP 0412 (2004) 049, [hep-th/0409243](http://arxiv.org/abs/hep-th/0409243); N. Berkovits, *Ten-dimensional super-twistors and super-Yang-Mills*, JHEP 1004 (2010) 067, [arXiv:0910.1684](http://arxiv.org/abs/0910.1684).

[4] N. Nekrasov, KITP lecture “Pure spinors, beta-gammas, super-Yang-Mills and Chern-Simons”, [http://online.kitp.ucsb.edu/online/strings09/nekrasov2/](http://online.kitp.ucsb.edu/online/strings09/nekrasov2/), January 2009.

[5] P.S. Howe, *Pure spinors lines in superspace and ten-dimensional supersymmetric theories*, Phys. Lett. B258 (1991) 141.

[6] D. Friedan, E. Martince and S. Shenker, *Conformal invariance, supersymmetry and string theory*, Nucl. Phys. B271 (1986) 93.

[7] N. Berkovits, *Pure spinor formalism as an N=2 topological string*, JHEP 0510 (2005) 089, [hep-th/0509120](http://arxiv.org/abs/hep-th/0509120).