ON A DUAL PROPERTY OF THE MAXIMAL OPERATOR ON WEIGHTED VARIABLE $L^p$ SPACES

ANDREI K. LERNER

Abstract. L. Diening [5] obtained the following dual property of the maximal operator $M$ on variable Lebesgue spaces $L^p(\cdot)$: if $M$ is bounded on $L^p(\cdot)$, then $M$ is bounded on $L^{p'}(\cdot)$. We extend this result to weighted variable Lebesgue spaces.

1. Introduction

Given a measurable function $p : \mathbb{R}^n \to [1, \infty)$, denote by $L^p(\cdot)$ the space of functions $f$ such that for some $\lambda > 0$,
\[
\int_{\mathbb{R}^n} |f(x)/\lambda|^{p(x)} \, dx < \infty,
\]
with norm
\[
\|f\|_{L^p(\cdot)} = \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^n} |f(x)/\lambda|^{p(x)} \, dx \leq 1 \right\}.
\]

Set $p_- \equiv \text{ess inf}_{x \in \mathbb{R}^n} p(x)$ and $p_+ \equiv \text{ess sup}_{x \in \mathbb{R}^n} p(x)$.

Let $M$ be the Hardy-Littlewood maximal operator defined by
\[
Mf(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)| \, dy,
\]
where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ containing the point $x$.

In [5], L. Diening proved the following remarkable result: if $p_- > 1$, $p_+ < \infty$ and $M$ is bounded on $L^p(\cdot)$, then $M$ is bounded on $L^{p'}(\cdot)$, where $p'(x) = \frac{p(x)}{p(x) - 1}$. Despite its apparent simplicity, the proof in [5] is rather long and involved.
In this paper we extend Diening’s theorem to weighted variable Lebesgue spaces $L^p_w$ equipped with norm
$$\|f\|_{L^p_w} = \|fw\|_{L^p}.$$ We assume that a weight $w$ here is a non-negative function such that $w(\cdot)^p$ and $w(\cdot)^{-p}$ are locally integrable. The spaces $L^p_w$ have been studied in numerous works; we refer to the monographs [3, 6] for a detailed bibliography.

Recall that a non-negative locally integrable function $v$ satisfies the Muckenhoupt $A_r$, $1 < r < \infty$, condition if
$$\sup_Q \left( \frac{1}{|Q|} \int_Q v \, dx \right) \left( \frac{1}{|Q|} \int_Q v^{-\frac{1}{r-1}} \, dx \right)^{r-1} < \infty.$$ Set $A_\infty = \cup_{r>1} A_r$.

Our main result is the following.

**Theorem 1.1.** Let $p : \mathbb{R}^n \to [1, \infty)$ be a measurable function such that $p_- > 1$ and $p_+ < \infty$. Let $w$ be a weight such that $w(\cdot)^p \in A_\infty$. If $M$ is bounded on $L^p_w$, then $M$ is bounded on $L^{p'}_{w^{-1}}$.

The relevance of the condition $w(\cdot)^p \in A_\infty$ in this theorem will be discussed in Section 6 below.

Notice that $L^{p'}_{w^{-1}}$ is the associate space of $L^p_w$, namely, $(L^p_w)' = L^{p'}_{w^{-1}}$ (see Sections 2.1 and 2.2). Hence, it is desirable to characterize Banach function spaces $X$ with the property that the boundedness of $M$ on $X$ implies the boundedness of $M$ on $X'$. In Section 3, we obtain such a characterization in terms of an $A_\infty$-type property of $X$.

However, a verification of this property in the case of $X = L^p_w$ is not as simple. In doing so, we use some ingredients developed by L. Diening in [5] (Lemmas 5.1 and 5.2). We slightly simplified their proofs and we give them here in order to keep the paper essentially self-contained.

2. **Preliminaries**

2.1. **Banach function spaces.** Denote by $\mathcal{M}^+$ the set of Lebesgue measurable non-negative functions on $\mathbb{R}^n$.

**Definition 2.1.** By a Banach function space (BFS) $X$ over $\mathbb{R}^n$ equipped with Lebesgue measure we mean a collection of functions $f$ such that
$$\|f\|_X = \rho(|f|) < \infty,$$ where $\rho : \mathcal{M}^+ \to [0, \infty]$ is a mapping satisfying
(i) $\rho(f) = 0 \iff f = 0$ a.e.; $\rho(\alpha f) = \alpha \rho(f), \alpha \geq 0$;
(ii) $\rho(f + g) \leq \rho(f) + \rho(g)$;
(ii) \( g \leq f \) a.e. \( \Rightarrow \rho(g) \leq \rho(f) \);
(iii) \( f_n \uparrow f \) a.e. \( \Rightarrow \rho(f_n) \uparrow \rho(f) \);
(iv) if \( E \subset \mathbb{R}^n \) is bounded, then \( \rho(\chi_E) < \infty \);
(v) if \( E \subset \mathbb{R}^n \) is bounded, then \( \int_E f \, dx \leq c_E \rho(f) \).

Note that it is more common to require that \( E \) is a set of finite measure in (iv) and (v) (see, e.g., [1]). However, our choice of axioms allows us to include weighted variable Lebesgue spaces \( L^{p(\cdot)}_w \) (with the assumption that \( w(\cdot)^p(\cdot), w(\cdot)^{-p'(\cdot)} \in L^{1}_{loc} \)) in a general framework of Banach function spaces. Moreover, it is well known that all main elements of a general theory work with (iv) and (v) stated for bounded sets (see, e.g., [13]). We mention only the next two key properties that are of interest for us. The first property says that if \( X \) is a BFS, then the associate space \( X' \) consisting of \( f \) such that

\[
\|f\|_{X'} = \sup_{g \in X' \mid \|g\|_X \leq 1} \int_{\mathbb{R}^n} |fg| \, dx < \infty
\]

is also a BFS. The second property is the Lorentz-Luxemburg theorem saying that \( X = X'' \) and \( \|f\|_X = \|f\|_{X''} \).

The definition of \( \|f\|_{X'} \) implies that

\[
(2.1) \quad \int_{\mathbb{R}^n} |fg| \, dx \leq \|f\|_X \|g\|_{X'},
\]

and the fact that \( \|f\|_X = \|f\|_{X''} \) yields

\[
(2.2) \quad \|f\|_X = \sup_{g \in X' \mid \|g\|_{X'} \leq 1} \int_{\mathbb{R}^n} |fg| \, dx.
\]

2.2. **Variable \( L^p \) spaces.** It is well known (see [3] or [6]) that if \( p : \mathbb{R}^n \to [1, \infty) \), then \( L^{p(\cdot)} \) is a BFS. Further, if \( p_- > 1 \) and \( p_+ < \infty \), then \( (L^{p(\cdot)})' = L^{p'(\cdot)} \) and

\[
(2.3) \quad \frac{1}{2} \|f\|_{L^{p'(\cdot)}} \leq \|f\|_{(L^{p(\cdot)})'} \leq 2\|f\|_{L^{p'(\cdot)}}
\]

(see [6, p. 78]).

Assume now that \( p : \mathbb{R}^n \to [1, \infty) \) and \( w \) is a weight such that \( w(\cdot)^p(\cdot) \) and \( w(\cdot)^{-p'(\cdot)} \) are locally integrable. The weighted space \( L^{p(\cdot)}_w \) consists of all \( f \) such that

\[
\|f\|_{L^{p(\cdot)}_w} = \|fw\|_{L^{p(\cdot)}} < \infty.
\]

It is easy to see that \( L^{p(\cdot)}_w \) is a BFS. Indeed, axioms (i)-(iii) of Definition 2.1 follow immediately from the fact that the unweighted \( L^{p(\cdot)} \)
is a BFS. Next, (iv) follows from that \( w(\cdot)^{p(-)} \in L^1_{loc} \). Finally, applying (2.1) with \( X = L^{p(\cdot)} \) along with (2.3) yields
\[
\int_E f \, dx \leq 2 \|fw\|_{L^{p(\cdot)}} \|w^{-1}\chi_E\|_{L^{p'(\cdot)}}.
\]
and this proves (v) with \( c_E = 2 \|w^{-1}\chi_E\|_{L^{p'(\cdot)}} < \infty \) (here we have used that \( w(\cdot)^{p(-)} \in L^1_{loc} \)).

Since \( \|fw^{-1}\|_{(L^{p(\cdot)})'} = \|f\|_{(L^{p(\cdot)})'} \), we obtain from (2.3) that if \( p_- > 1 \) and \( p_+ < \infty \), then \( (L^{p(\cdot)}_w)' = L^{p'(\cdot)}_w \) and
\[
\frac{1}{2} \|f\|_{L^{p'(\cdot)}_w} \leq \|f\|_{(L^{p(\cdot)}_w)'} \leq 2 \|f\|_{L^{p'(\cdot)}_w}.
\]

Denote \( \varrho(f) = \int_{\mathbb{R}^n} |f(x)|^{p(x)} \, dx \). We will frequently use the following lemma (see [3, p. 25]).

**Lemma 2.2.** Let \( p : \mathbb{R}^n \to [1, \infty) \) and \( p_+ < \infty \). If \( \|f\|_{L^{p(\cdot)}} > 1 \), then
\[
\varrho(f)^{1/p_+} \leq \|f\|_{L^{p(\cdot)}} \leq \varrho(f)^{1/p_-}.
\]
If \( \|f\|_{L^{p(\cdot)}} \leq 1 \), then
\[
\varrho(f)^{1/p_-} \leq \|f\|_{L^{p(\cdot)}} \leq \varrho(f)^{1/p_+}.
\]

### 2.3. Dyadic grids and sparse families.

The standard dyadic grid in \( \mathbb{R}^n \) consists of the cubes
\[
2^{-k}([0,1)^n + j), \quad k \in \mathbb{Z}, j \in \mathbb{Z}^n.
\]
Following its basic properties, we say that a family of cubes \( \mathcal{D} \) is a general dyadic grid if (i) for any \( Q \in \mathcal{D} \) its sidelen\( \ell_Q \) is of the form \( 2^k, k \in \mathbb{Z} \); (ii) \( Q \cap R \in \{Q, R, \emptyset\} \) for any \( Q, R \in \mathcal{D} \); (iii) for every \( k \in \mathbb{Z} \), the cubes of a fixed sidelen\( \ell^k \) form a partition of \( \mathbb{R}^n \).

Given a dyadic grid \( \mathcal{D} \), consider the associated dyadic maximal operator \( M^\mathcal{D} \) defined by
\[
M^\mathcal{D} f(x) = \sup_{Q : x, Q \in \mathcal{D}} \frac{1}{|Q|} \int_Q |f(y)| \, dy.
\]
On one hand, it is clear that \( M^\mathcal{D} f \leq Mf \). However, this inequality can be reversed, in a sense, as the following lemma shows (its proof can be found in [10, Lemma 2.5]).

**Lemma 2.3.** There are \( 3^n \) dyadic grids \( \mathcal{D}_\alpha \) such that for every cube \( Q \subset \mathbb{R}^n \), there exists a cube \( Q_\alpha \in \mathcal{D}_\alpha \) such that \( Q \subset Q_\alpha \) and \( |Q_\alpha| \leq 6^n |Q| \).
We obtain from this lemma that for all \( x \in \mathbb{R}^n \),

\[
Mf(x) \leq 6^n \sum_{\alpha=1}^{3^n} M^{D\alpha} f(x).
\]

Given a cube \( Q_0 \), denote by \( D(Q_0) \) the set of all dyadic cubes with respect to \( Q_0 \), that is, the cubes from \( D(Q_0) \) are formed by repeated subdivision of \( Q_0 \) and each of its descendants into \( 2^n \) congruent subcubes. Consider the local dyadic maximal operator \( M_{Q_0}^d \) defined by

\[
M_{Q_0}^d f(x) = \sup_{Q \ni x, Q \in D(Q_0)} \frac{1}{|Q|} \int_Q |f(y)|dy.
\]

Denote \( f_Q = \frac{1}{|Q|} \int_Q f \). The following lemma is a standard variation of the Calderón-Zygmund decomposition (see, e.g., [8, Theorem 4.3.1]). We include its proof for the reader convenience.

**Lemma 2.4.** Suppose \( \mathcal{D} \) is a dyadic grid. Let \( f \in L^p(\mathbb{R}^n), 1 \leq p < \infty \), and let \( \gamma > 1 \). Assume that

\[
\Omega_k = \{ x \in \mathbb{R}^n : M^\mathcal{D} f(x) > \gamma^k \} \neq \emptyset \quad (k \in \mathbb{Z}).
\]

Then \( \Omega_k \) can be written as a union of pairwise disjoint cubes \( Q_j^k \in \mathcal{D} \) satisfying

\[
|Q_j^k \cap \Omega_{k+l}| \leq 2^n (1/\gamma)^l |Q_j^k| \quad (l \in \mathbb{Z}_+).
\]

The same property holds in the local case for the sets

\[
\Omega_k = \{ x \in Q_0 : M_{Q_0}^d f(x) > \gamma^k |f|_{Q_0} \} \quad (f \in L^1(Q_0), k \in \mathbb{Z}_+).
\]

**Proof.** Consider the case of \( \mathbb{R}^n \), the same proof works in the local case. Let \( Q_j^k \) be the maximal cubes such that \( |f|_{Q_j^k} > \gamma^k \). Then, by maximality, they are pairwise disjoint and \( |f|_{Q_j^k} \leq 2^n \gamma^k \). Also, \( \Omega_k = \bigcup_j Q_j^k \).

Therefore,

\[
|Q_j^k \cap \Omega_{k+l}| = \sum_{Q_i^{k+l} \subset Q_j^k} |Q_i^{k+l}| < (1/\gamma)^{k+l} \int_{Q_j^k} |f| \leq 2^n (1/\gamma)^l |Q_j^k|.
\]

\( \square \)

**Definition 2.5.** Let \( \mathcal{D} \) be a dyadic grid, and let \( 0 < \eta < 1 \). We say that a family of cubes \( \mathcal{S} \subset \mathcal{D} \) is \( \eta \)-sparse if for every cube \( Q \in \mathcal{S} \), there is a measurable subset \( E(Q) \subset Q \) such that \( \eta |Q| \leq |E(Q)| \) and the sets \( \{ E(Q) \}_{Q \in \mathcal{S}} \) are pairwise disjoint.
Lemma 2.6. Let $\mathcal{D}$ be a dyadic grid, and let $0 < \eta < 1$. For every non-negative $f \in L^p(\mathbb{R}^n), 1 \leq p < \infty$, there exists an $\eta$-sparse family $\mathcal{S} \subset \mathcal{D}$ such that for all $x \in \mathbb{R}^n$,

$$M^\mathcal{D} f(x) \leq \frac{2^n}{1 - \eta} \sum_{Q \in \mathcal{S}} f_Q \chi_{E(Q)}(x).$$

Proof. For $k \in \mathbb{Z}$, set $\Omega_k = \left\{ x \in \mathbb{R}^n : M^\mathcal{D} f(x) > \left( \frac{2^n}{1 - \eta} \right)^k \right\}$. Then, by Lemma 2.4, $\Omega_k = \bigcup_j Q^k_j$, and $|Q^k_j \cap \Omega_{k+1}| \leq (1 - \eta)|Q^k_j|$. Therefore, setting $E(Q^k_j) = Q^k_j \setminus \Omega_{k+1}$, we obtain that $\eta|Q^k_j| \leq |E(Q^k_j)|$, and the sets $\{E(Q^k_j)\}$ are pairwise disjoint. Further,

$$M^\mathcal{D} f \leq \sum_{k \in \mathbb{Z}} (M^\mathcal{D} f) \chi_{\Omega_k \setminus \Omega_{k+1}} \leq \frac{2^n}{1 - \eta} \sum_{k \in \mathbb{Z}} \left( \frac{2^n}{1 - \eta} \right)^k \chi_{\Omega_k \setminus \Omega_{k+1}} \leq \frac{2^n}{1 - \eta} \sum_{j,k} f_Q \chi_{E(Q^k_j)},$$

which completes the proof with $\mathcal{S} = \{Q^k_j\}$. $\square$

2.4. $A_p$ weights. Given a weight $w$ and a measurable set $E \subset \mathbb{R}^n$, denote $w(E) = \int_E w dx$. Given an $A_p, 1 < p < \infty$, weight, its $A_p$ constant is defined by

$$[w]_{A_p} = \sup_Q \left( \frac{1}{|Q|} \int_Q w dx \right) \left( \frac{1}{|Q|} \int_Q w^{-1/(p-1)} dx \right)^{p-1}. $$

Every $A_p$ weight satisfies the reverse Hölder inequality (see, e.g., [9, Theorem 9.2.2]), namely, there exist $c > 0$ and $r > 1$ such that for any cube $Q$,

$$\left( \frac{1}{|Q|} \int_Q w^r dx \right)^{1/r} \leq c \frac{1}{|Q|} \int_Q w dx. \tag{2.6}$$

It follows from this and from Hölder’s inequality that for every $Q$ and any measurable subset $E \subset Q$,

$$\frac{w(E)}{w(Q)} \leq c \left( \frac{|E|}{|Q|} \right)^{1/r'} \tag{2.7}$$

Notice also that the following converse estimate

$$\frac{w(Q)}{w(E)} \leq \left( \frac{|Q|}{|E|} \right)^p [w]_{A_p} \quad (E \subset Q, |E| > 0) \tag{2.8}$$
holds for all $p > 1$. Indeed, by Hölder’s inequality,

$$|E|^p \leq \left( \int_E w \, dx \right) \left( \int_E w^{-1/(p-1)} \, dx \right)^{p-1},$$

which along with the definition of $[w]_{A_p}$ implies (2.8).

### 3. Maximal operator on associate spaces

Since $(L^p_w)^{\prime} = L^{p^\prime}_{w^{-1}}$, the statement of Theorem 1.1 leads naturally to a question about conditions on a BFS $X$ such that $M : X \to X \Rightarrow M : X^{\prime} \to X^{\prime}$. The result below provides a criterion in terms of sparse families and an $A_\infty$-type condition. Its proof is based essentially on the theory of $A_p$ weights.

**Theorem 3.1.** Let $X$ be a BFS such that the Hardy-Littlewood maximal operator $M$ is bounded on $X$. Let $0 < \eta < 1$. The following conditions are equivalent:

(i) $M$ is bounded on $X^{\prime}$;

(ii) there exist $c, \delta > 0$ such that for every dyadic grid $\mathcal{D}$ and any finite $\eta$-sparse family $S \subset \mathcal{D}$,

$$\left\| \sum_{Q \in S} \alpha_Q \chi_{G_Q} \right\|_X \leq c \left( \max_{Q \in S} \frac{|G_Q|}{|Q|} \right)^{\delta} \left\| \sum_{Q \in S} \alpha_Q \chi_Q \right\|_X,$$

where $\{\alpha_Q\}_{Q \in S}$ is an arbitrary sequence of non-negative numbers, and $\{G_Q\}_{Q \in S}$ is any sequence of pairwise disjoint measurable subsets $G_Q \subset Q$.

**Proof.** Let us first prove (i) $\Rightarrow$ (ii). Let $g \geq 0$ and $\|g\|_{X^{\prime}} = 1$. We use the standard Rubio de Francia algorithm [14], namely, set

$$Rg = \sum_{k=0}^{\infty} \frac{M^k g}{(2\|M\|_{X^{\prime}})^k},$$

where $M^k$ denotes the $k$-th iteration of $M$ and $M^0 g = g$. Then $g \leq Rg$ and $\|Rg\|_{X^{\prime}} \leq 2$. Also,

$$M(Rg)(x) \leq 2\|M\|_{X^{\prime}} Rg(x).$$

Therefore, $Rg \in A_1$. 


Using the properties of $Rg$ along with (2.7) and Hölder inequality (2.1), we obtain that there exist $c, \delta > 0$ such that

$$\int_{\mathbb{R}^n} \left( \sum_{Q \in S} \alpha_Q \chi_{G_Q} \right) g \, dx \leq \sum_{Q \in S} \alpha_Q \int_{G_Q} Rg \, dx \leq c \sum_{Q \in S} \alpha_Q \left( \frac{|G_Q|}{|Q|} \right)^{\delta} \int_{G_Q} Rg \, dx \leq c \left( \max_{Q \in S} \frac{|G_Q|}{|Q|} \right)^{\delta} \left\| \sum_{Q \in S} \alpha_Q \chi_Q \right\|_{X'} \cdot$$

It remains to take here the supremum over all $g \geq 0$ with $\|g\|_{X'} = 1$ and to use (2.2).

Turn to the proof of (ii) $\Rightarrow$ (i). By (2.4), it suffices to prove that the dyadic maximal operator $M^\delta$ is bounded on $X'$. Let us show that there is $c > 0$ such that for every $f \in L^1 \cap X'$,

$$\|M^\delta f\|_{X'} \leq c \|f\|_{X'} \quad (3.1)$$

Notice that (3.1) implies the boundedness of $M^\delta$ on $X'$. Indeed, having (3.1) established, for an arbitrary $f \in X'$ we apply (3.1) to $f_N = f \chi_{\{|x| \leq N\}}$ (clearly, $f_N \in L^1 \cap X'$). Letting then $N \to \infty$ and using the Fatou property ((iii) of Definition 2.1), we obtain that (3.1) holds for any $f \in X'$.

In order to prove (3.1), by Lemma 2.6 it suffices to show that the operator

$$\mathcal{M}_S f = \sum_{Q \in S} f_Q \chi_{E(Q)}$$

satisfies

$$\|\mathcal{M}_S f\|_{X'} \leq c \|f\|_{X'}$$

for every non-negative $f \in L^1 \cap X'$ with $c > 0$ independent of $f$ and $S$. Notice that here $S = \{Q^k_j\}$, and $Q^k_j$ are maximal dyadic cubes forming the set

$$\Omega_k = \left\{ x \in \mathbb{R}^n : M^\delta f(x) > \left( \frac{2^n}{1 - \eta} \right)^k \right\}.$$

By duality, it is enough to obtain the uniform boundedness of the adjoint operator

$$\mathcal{M}_S^* f = \sum_{j,k} \left( \frac{1}{|Q^k_j|} \int_{E(Q^k_j)} f \right) \chi_{Q^k_j}$$
on $X$. Using the Fatou property again, one can assume that $\mathcal{S}$ is finite. Take $\nu \in \mathbb{N}$ such that

$$2^{n\delta}c \sum_{l=\nu}^{\infty} \left(\frac{1-\eta}{2^n}\right)^{l\delta} \leq 1/2,$$

where $c$ and $\delta$ are the constants from condition (ii). Denote $\alpha_{j,k} = \frac{1}{|Q^j_k|} \int_{E(Q^j_k)} f$. Then, using that $\bigcup_j Q^j_k \setminus \Omega_{k+\nu} = \bigcup_{l=0}^{\nu-1} \Omega_{k+l} \setminus \Omega_{k+l+1}$, we obtain

$$\mathcal{M}_S^* f \leq \sum_{j,k} \alpha_{j,k} \chi_{Q^j_k \setminus \Omega_{k+\nu}} + \sum_{j,k} \alpha_{j,k} \chi_{Q^j_k \cap \Omega_{k+l}},$$

Therefore, applying (2.5) along with condition (ii), we obtain

$$\|\mathcal{M}_S^* f\|_X \leq \nu \|Mf\|_X + \sum_{l=\nu}^{\infty} \sum_{j,k} \alpha_{j,k} \chi_{Q^j_k \cap (\Omega_{k+l} \setminus \Omega_{k+l+1})} \|\chi_{Q^j_k \cap (\Omega_{k+l} \setminus \Omega_{k+l+1})}\|,$$

Since $\mathcal{S}$ is finite, by (iv) of Definition 2.1 we obtain that $\|\mathcal{M}_S^* f\|_X < \infty$. Hence,

$$\|\mathcal{M}_S^* f\|_X \leq 2\nu \|M\|_X \|f\|_X,$$

and this completes the proof of (ii) $\Rightarrow$ (i).

4. PROOF OF THEOREM 1.1

Take $X = L^{p(-)}_w$ in Theorem 3.1. All we have to do is to check condition (ii) in this theorem. In order to do that, we need a kind of the reverse Hölder property for the weights $(tw(x))^p(x)$. The following key lemma provides a replacement of such a property which is enough for our purposes.

**Lemma 4.1.** Let $1 < p_- \leq p_+ < \infty$. Assume that $w(\cdot)^p(\cdot) \in A_\infty$ and that $M$ is bounded on $L^{p(-)}_w$. Then there exist $\gamma > 1$ and $c, \eta > 0$, and there is a measure $b$ on $\mathbb{R}^n$ such that for every cube $Q$ and all $t > 0$
such that $t\|\chi_Q\|_{L^p_w} \leq 1$ one has

$$
(4.1) \quad |Q| \left( \frac{1}{|Q|} \int_Q (tw(x))^{\gamma p(x)} dx \right)^{1/\gamma} \leq c \int_Q (tw(x))^{p(x)} dx + 2 t^{\eta b(Q)} \chi(0,1)(t),
$$

and for every finite family of pairwise disjoint cubes $\pi$, $\sum_{Q \in \pi} b(Q) \leq c$.

The proof of this lemma is rather technical, and we postpone it until the next Section. Let us see now how the proof of Theorem 1.1 follows.

**Proof of Theorem 1.1.** Let $D$ be a dyadic grid, and let $S \subset D$ be a finite $\frac{1}{2}$-sparse family. Let $\{G_Q\}_{Q \in S}$ be a family of pairwise disjoint sets such that $G_Q \subset Q$. Take any sequence of non-negative numbers $\{\alpha_Q\}_{Q \in S}$ such that

$$
(4.2) \quad \left\| \sum_{Q \in S} \alpha_Q \chi_Q \right\|_{L^p_w} = 1.
$$

By Lemma 2.2 and Theorem 3.1 it suffices to show that there exist absolute constants $c, \delta > 0$ such that

$$
(4.3) \quad \sum_{Q \in S} \int_{G_Q} (\alpha_Q w(x))^{p(x)} dx \leq c \left( \max_{Q \in S} \frac{|G_Q|}{|Q|} \right)^{\delta}.
$$

It follows from (4.2) that $\alpha_Q \|\chi_Q\|_{L^p_w} \leq 1$ for every $Q \in S$. Therefore, if $\alpha_Q \geq 1$, by Lemma 4.1 and Hölder’s inequality along with (4.2) we obtain

$$
(4.4) \quad \sum_{Q \in S: \alpha_Q \geq 1} \int_{G_Q} (\alpha_Q w(x))^{p(x)} dx
$$

$$
\leq \sum_{Q \in S: \alpha_Q \geq 1} |Q| \left( \frac{|G_Q|}{|Q|} \right)^{1/\gamma'} \left( \frac{1}{|Q|} \int_Q (\alpha_Q w(x))^{\gamma p(x)} dx \right)^{1/\gamma}
$$

$$
\leq c \sum_{Q \in S: \alpha_Q \geq 1} \left( \frac{|G_Q|}{|Q|} \right)^{1/\gamma'} \int_Q (\alpha_Q w(x))^{p(x)} dx \leq c \left( \max_{Q \in S} \frac{|G_Q|}{|Q|} \right)^{1/\gamma'}.
$$

The case when $\alpha_Q < 1$ is more complicated because of the additional term on the right-hand side of (4.1). We proceed as follows. Denote

$$
S_k = \{Q \in S : 2^{-k} \leq \alpha_Q < 2^{-k+1} \} \quad (k \in \mathbb{N}).
$$

Let $Q^k_i$ be the maximal cubes from $S_k$ such that every other cube $Q \in S_k$ is contained in one of them. Then the cubes $Q^k_i$ are pairwise
disjoint (for \( k \) fixed). Set
\[
\psi_{Q_i^k}(x) = \sum_{Q \in S_k: Q \subseteq Q_i^k} \chi_{G_Q}(x).
\]

Then
\[
\sum_{Q \in S: \alpha_Q < 1} \int_{G_Q} (\alpha_Q w(x))^{p(x)} \, dx = \sum_{k=1}^{\infty} \sum_{Q \in S_k} \int_{G_Q} (\alpha_Q w(x))^{p(x)} \, dx
= \sum_{k=1}^{\infty} \sum_{i} \sum_{Q \in S_k: Q \subseteq Q_i^k} \int_{G_Q} (\alpha_Q w(x))^{p(x)} \, dx
\leq 2^{p^+} \sum_{i,k} \int_{Q_i^k} (\alpha_{Q_i^k} w(x))^{p(x)} \psi_{Q_i^k}(x) \, dx.
\]

By Hölder’s inequality,
\[
\sum_{i,k} \int_{Q_i^k} (\alpha_{Q_i^k} w(x))^{p(x)} \psi_{Q_i^k}(x) \, dx
\leq \sum_{i,k} |Q_i^k| \left( \frac{1}{|Q_i^k|} \int_{Q_i^k} (\alpha_{Q_i^k} w(x))^{\gamma p(x)} \, dx \right)^{\frac{1}{\gamma'}} \left( \frac{1}{|Q_i^k|} \int_{Q_i^k} \psi_{Q_i^k}(x)^{\gamma'} \, dx \right)^{\frac{1}{\gamma'}}.
\]

Since \( S \) is \( \frac{1}{2} \)-sparse,
\[
\int_{Q_i^k} \psi_{Q_i^k}(x)^{\gamma'} \, dx = \sum_{Q \in S_k: Q \subseteq Q_i^k} |G_Q| \leq \left( \max_{Q \in S} \frac{|G_Q|}{|Q|} \right) \sum_{Q \in S_k: Q \subseteq Q_i^k} |Q| \leq 2 \left( \max_{Q \in S} \frac{|G_Q|}{|Q|} \right) |Q_i^k|.
\]

Combining this with the two previous estimates yields
\[
\sum_{Q \in S: \alpha_Q < 1} \int_{G_Q} (\alpha_Q w(x))^{p(x)} \, dx
\leq c \left( \max_{Q \in S} \frac{|G_Q|}{|Q|} \right)^{\frac{1}{\gamma'}} \sum_{i,k} |Q_i^k| \left( \frac{1}{|Q_i^k|} \int_{Q_i^k} (\alpha_{Q_i^k} w(x))^{\gamma p(x)} \, dx \right)^{\frac{1}{\gamma'}}.
\]

By Lemma 4.1 along with (4.2), and Lemma 2.2
\[
\sum_{i,k} |Q_i^k| \left( \frac{1}{|Q_i^k|} \int_{Q_i^k} (\alpha_{Q_i^k} w(x))^{p(x)} \, dx \right)^{\frac{1}{\gamma}} \leq c + 2 \sum_{i,k} \alpha_{Q_i^k}^2 b(Q_i^k).
\]
Since for every fixed $k$, the cubes $\{Q_i^k\}$ are pairwise disjoint,
\[ \sum_{i,k} \alpha^k_i b(Q_i^k) \leq 2^n \sum_{k=1}^\infty 2^{-k\eta} \sum_i b(Q_i^k) \leq c \sum_{k=1}^\infty 2^{-k\eta} \leq c. \]
This, combined with the two previous estimates implies
\[ \sum_{Q \in S; \alpha_Q < 1} \int_{G_Q} (\alpha_Q w(x))^{p(x)} dx \leq \left( \max_{Q \in S} \frac{|G_Q|}{|Q|} \right)^{1/\gamma'}, \]
which along with (4.4) proves (4.3).

\[ \Box \]

5. Proof of Lemma 4.1

We split the proof of Lemma 4.1 into several pieces. Lemmas 5.1 and 5.2 below are due to L. Diening [5]. We give slightly shortened versions of their proofs for the sake of completeness. Notice that these lemmas hold for arbitrary weights $w$ such that $w(\cdot) p(\cdot)$ is locally integrable. Lemma 5.3 is new. The assumption that $w(\cdot) p(\cdot) \in A_\infty$ is essential there. Throughout this section, we assume that $p_\oplus > 1$ and $p_\oplus < \infty$.

**Lemma 5.1.** Assume that $M$ is bounded on $L^{p(\cdot)}$. Then there exist $r, c > 1$ such that for every family of pairwise disjoint cubes $\pi$ and for every sequence of non-negative numbers $\{t_Q\}_{Q \in \pi}$,
\[ \sum_{Q \in \pi} \int_Q (t_Q w(x))^{p(x)} dx \leq 1 \Rightarrow \sum_{Q \in \pi} |Q| \left( \frac{1}{|Q|} \int_Q (t_Q w(x))^{r p(x)} dx \right)^{1/r} \leq c. \]

**Proof.** Given a family $\pi$ and a sequence $\{t_Q\}_{Q \in \pi}$, denote $v_Q(x) = (t_Q w(x))^{p(x)}$ and $\alpha_Q = \frac{1}{|Q|} \int_Q v_Q dx$.

By Lemma 2.4, write the set
\[ \Omega_k(Q) = \{ x \in Q : M_{Q}^{\delta} v_Q(x) > (2^{n+1})^k \alpha_Q \} \quad (k \in \mathbb{N}) \]
as a union of pairwise disjoint cubes $P_j^k(Q)$ satisfying $|E_j^k(Q)| \geq \frac{1}{2} |P_j^k(Q)|$, where $E_j^k(Q) = P_j^k(Q) \setminus \Omega_{k+1}(Q)$. From this,
\[ \sum_{Q \in \pi} t_Q \chi_{\Omega_k(Q)} \leq 2M \left( \sum_{Q \in \pi} t_Q \chi_{\Omega_k(Q) \setminus \Omega_{k+1}(Q)} \right), \]
and hence,
\[ \left\| \sum_{Q \in \pi} t_Q \chi_{\Omega_k(Q)} \right\|_{L^{p(\cdot)}_w} \leq 2 \left\| M \right\|_{L^{p(\cdot)}_w} \left\| \sum_{Q \in \pi} t_Q \chi_{\Omega_k(Q) \setminus \Omega_{k+1}(Q)} \right\|_{L^{p(\cdot)}_w}. \]
ON A DUAL PROPERTY OF THE MAXIMAL OPERATOR 13

Setting $t'_Q = \frac{t_Q}{\| \sum_{Q \in \pi} t_Q \chi_{\Omega_k(Q)} \|_{L^p_w}}$, this inequality yields

$$1 \leq 2 \| M \|_{L^p_w} \| \sum_{Q \in \pi} t'_Q \chi_{\Omega_k(Q) \setminus \Omega_{k+1}(Q)} \|_{L^p_w}.$$  

Since

$$\left\| \sum_{Q \in \pi} t'_Q \chi_{\Omega_k(Q) \setminus \Omega_{k+1}(Q)} \right\|_{L^p_w} \leq \left\| \sum_{Q \in \pi} t'_Q \chi_{\Omega_k(Q)} \right\|_{L^p_w} = 1,$$

Lemma 2.2 along with the previous estimate implies,

$$\frac{1}{(2 \| M \|_{L^p_w})^{p_+}} \leq \sum_{Q \in \pi} \int_{\Omega_k(Q) \setminus \Omega_{k+1}(Q)} (t'_Q w(x))^{p(x)} dx \leq 1 - \sum_{Q \in \pi} \int_{\Omega_{k+1}(Q)} (t'_Q w(x))^{p(x)} dx,$$

which in turn implies (again, by Lemma 2.2)

$$\left\| \sum_{Q \in \pi} t'_Q \chi_{\Omega_{k+1}(Q)} \right\|_{L^p_w} \leq \beta,$$

where $\beta = \left(1 - \frac{1}{(2 \| M \|_{L^p_w})^{p_+}}\right)^{1/p_+}$. Hence,

$$\left\| \sum_{Q \in \pi} t_Q \chi_{\Omega_{k+1}(Q)} \right\|_{L^p_w} \leq \beta \left\| \sum_{Q \in \pi} t_Q \chi_{\Omega_k(Q)} \right\|_{L^p_w},$$

and thus, $\| \sum_{Q \in \pi} t_Q \chi_{\Omega_k(Q)} \|_{L^p_w} \leq \beta^{k-1}$, which by Lemma 2.2 implies

$$\sum_{Q \in \pi} \int_{\Omega_k(Q)} (t_Q w(x))^{p(x)} dx \leq \beta^{p-(k-1)}.$$  

(5.1)

Denote $\Omega_0(Q) = Q$. Then, for $\varepsilon > 0$ we have

$$\int_Q (t_Q w(x))^{(1+\varepsilon)p(x)} dx = \sum_{k=0}^{\infty} \int_{\Omega_k(Q) \setminus \Omega_{k+1}(Q)} (t_Q w(x))^{(1+\varepsilon)p(x)} dx \leq \alpha_Q \sum_{k=0}^{\infty} 2^{(n+1)(k+1)\varepsilon} \int_{\Omega_k(Q)} (t_Q w(x))^{p(x)} dx.$$
Take $\varepsilon > 0$ such that $\sum_{k=0}^{\infty} (2^{(n+1)\varepsilon p_-})^k < \infty$. Then, combining the previous estimate with (5.1) and Hölder’s inequality, we obtain

$$\sum_{Q \in \pi} |Q| \left( \frac{1}{|Q|} \int_Q (t_Q w(x))^{(1+\varepsilon)p(x)} dx \right)^{\frac{1}{1+\varepsilon}}$$

$$\leq \sum_{Q \in \pi} v_Q(Q)^{\frac{1}{1+\varepsilon}} \left( \sum_{k=0}^{\infty} 2^{(n+1)(k+1)\varepsilon} v_Q(Q) \right)^{\frac{1}{1+\varepsilon}}$$

$$\leq \left( 2^{n+1}\varepsilon + \sum_{k=1}^{\infty} 2^{(n+1)(k+1)\varepsilon} \beta p_-(k-1) \right)^{\frac{1}{1+\varepsilon}} \leq c,$$

and therefore, the proof is complete.

\[ \square \]

**Lemma 5.2.** Assume that $M$ is bounded on $L_{w}^{p(\cdot)}$. Then there exist $r, k > 1$, and a measure $b$ on $\mathbb{R}^n$ such that the following properties hold: if $\int_{Q}(tw(x))^{p(x)}dx \leq 1$, then

$$|Q| \left( \frac{1}{|Q|} \int_Q (tw(x))^{rp(x)} dx \right)^{1/r} \leq k \int_Q (tw(x))^{p(x)} dx + b(Q),$$

and for every finite family of pairwise disjoint cubes $\pi$, \( \sum_{Q \in \pi} b(Q) \leq 2k.$

**Proof.** Let $r$ and $c$ be the constants from Lemma 5.1. Set $k = 2^{\frac{p_+}{r} + 1} c$.

Given a cube $Q$, denote by $A(Q)$ the set of $t > 0$ such that

$$\int_Q (tw(x))^{p(x)} dx \leq 1$$

and

$$|Q| \left( \frac{1}{|Q|} \int_Q (tw(x))^{rp(x)} dx \right)^{1/r} > k \int_Q (tw(x))^{p(x)} dx.$$

Let $t_Q = \sup A(Q)$ (if $A(Q) = \emptyset$, set $t_Q = 0$). Then

$$\int_Q (t_Q w(x))^{p(x)} dx < 1.$$

Indeed, if $\int_Q (t_Q w(x))^{p(x)} dx = 1$, we obtain

$$|Q| \left( \frac{1}{|Q|} \int_Q (t_Q w(x))^{rp(x)} dx \right)^{1/r} \geq k,$$
and this would contradict Lemma 5.1. Further, we have
\begin{equation}
|Q| \left( \frac{1}{|Q|} \int_Q (t_Q w(x))^{rp(x)} dx \right)^{1/r} = k \int_Q (t_Q w(x))^{p(x)} dx,
\end{equation}
since otherwise (5.3) holds with $t = t_Q$, and by continuity, using also (5.4), we would obtain that $t_Q + \varepsilon \in A(Q)$ for some $\varepsilon > 0$, which contradicts the definition of $t_Q$.

Set now
\[ b(Q) = |Q| \left( \frac{1}{|Q|} \int_Q (t_Q w(x))^{rp(x)} dx \right)^{1/r}. \]
Then (5.2) holds trivially.

Let $\pi$ be any finite family of pairwise disjoint cubes. Let $\pi' \subseteq \pi$ be a maximal subset such that $\sum_{Q \in \pi'} \int_Q (t_Q w(x))^{p(x)} dx \leq 2$ (maximal in the sense of the number of elements; this set is not necessarily unique, in general). We claim that $\pi' = \pi$. Indeed, assume that $\pi' \neq \pi$. Then we have $\sum_{Q \in \pi'} \int_Q (t_Q w(x)/2^{1/p-})^{p(x)} dx \leq 1$, and by Lemma 5.1
\begin{equation}
\sum_{Q \in \pi'} |Q| \left( \frac{1}{|Q|} \int_Q (t_Q w(x))^{rp(x)} dx \right)^{1/r} \leq 2^{\frac{p+}{p-}} c.
\end{equation}
From this and from (5.3),
\[ \sum_{Q \in \pi'} \int_Q (t_Q w(x))^{p(x)} dx = \frac{1}{k} \sum_{Q \in \pi'} |Q| \left( \frac{1}{|Q|} \int_Q (t_Q w(x))^{rp(x)} dx \right)^{1/r} \leq \frac{1}{2}. \]
Therefore, if $P \in \pi \setminus \pi'$, we obtain
\[ \sum_{Q \in \pi' \cup \{P\}} \int_Q (t_Q w(x))^{p(x)} dx \leq \frac{3}{2}, \]
which contradicts the maximality of $\pi'$. This proves that $\pi' = \pi$. Hence,
\[ \sum_{Q \in \pi} b(Q) = k \sum_{Q \in \pi} \int_Q (t_Q w(x))^{p(x)} dx \leq 2k, \]
which completes the proof.

\begin{lemma}
Assume that $w(.)^{p(.)} \in A_\infty$ and that $M$ is bounded on $L^{p(.)}_w$. There exist $\gamma, c > 1$ and $\varepsilon > 0$ such that if
\begin{equation}
(5.6) \quad t \in \left[ \min \left( 1, 1/\|\chi_Q\|_{L^{p(.)}_w}^{1+\varepsilon} \right), \max \left( 1, 1/\|\chi_Q\|_{L^{p(.)}_w}^{1+\varepsilon} \right) \right],
\end{equation}
then
\begin{equation}
(5.7) \quad \left( \frac{1}{|Q|} \int_Q (tw(x))^{rp(x)} dx \right)^{1/\gamma} \leq c \frac{1}{|Q|} \int_Q (tw(x))^{p(x)} dx.
\end{equation}
\end{lemma}
Proof. By the definition of $A_{\infty}$, there is an $s > 1$ such that $w(\cdot)^{p_{\gamma}(\cdot)} \in A_s$. By (2.6), $w(\cdot)^{p_{\gamma}(\cdot)}$ satisfies the reverse Hölder inequality with an exponent $\nu > 1$. Let $r > 1$ be the exponent from Lemma 5.1. Take any $\gamma$ satisfying $1 < \gamma < \min(\nu, r)$. Set $\varepsilon = \frac{r-1}{\gamma(1+(s-1)r)}$.

For every $\alpha > 0$,
$$
\left(\frac{1}{|Q|} \int_Q (tw(x))^{\gamma p(x)} dx \right)^{1/\gamma} = \left(\frac{1}{|Q|} \int_Q t^{\gamma(p(x)-\alpha)} w(x)^{\gamma p(x)} dx \right)^{1/\gamma} t^\alpha.
$$

Next, by (5.6), for all $x \in Q$,
$$
t^{\gamma(p(x)-\alpha)} \leq 1 + \|xQ\|_{L_p^w(\cdot)}^{\gamma(1+\varepsilon)} (1/\|xQ\|_{L_p^w(\cdot)}^{1+\varepsilon}) \gamma p(x),
$$
and hence,
$$
\int_Q t^{\gamma(p(x)-\alpha)} w(x)^{\gamma p(x)} dx \leq \int_Q w(x)^{\gamma p(x)} dx
$$
$$
\quad + \|xQ\|_{L_p^w(\cdot)}^{\alpha(1+\varepsilon)} \left( \frac{1}{|Q|} \int_Q \left( \frac{w(x)}{\|xQ\|_{L_p^w(\cdot)}^{1+\varepsilon}} \right)^{\gamma p(x)} dx \right)^{1/\gamma} t^\alpha.
$$

Combining this with the previous estimates yields
$$
(5.8) \quad \left(\frac{1}{|Q|} \int_Q (tw(x))^{\gamma p(x)} dx \right)^{1/\gamma} \leq \left(\frac{1}{|Q|} \int_Q w(x)^{\gamma p(x)} dx \right)^{1/\gamma} t^\alpha
$$
$$
\quad + \|xQ\|_{L_p^w(\cdot)}^{\alpha(1+\varepsilon)} \left( \frac{1}{|Q|} \int_Q \left( \frac{w(x)}{\|xQ\|_{L_p^w(\cdot)}^{1+\varepsilon}} \right)^{\gamma p(x)} dx \right)^{1/\gamma} t^\alpha.
$$

Let $\alpha = m_p(Q)$ be a median value of $p$ over $Q$, that is, a number satisfying
$$
\max \left( \frac{|\{x \in Q : p(x) > m_p(Q)\}|}{|Q|}, \frac{|\{x \in Q : p(x) < m_p(Q)\}|}{|Q|} \right) \leq \frac{1}{2}.
$$

Set $E_1 = \{x \in Q : p(x) \leq m_p(Q)\}$ and $E_2 = \{x \in Q : p(x) \geq m_p(Q)\}$.
Then $|E_1| \geq \frac{1}{2}|Q|$ and $|E_2| \geq \frac{1}{2}|Q|$.

Suppose, for instance, that $\|xQ\|_{L_p^w(\cdot)} \leq 1$. Then $t \geq 1$. Let us estimate the first term on the right-hand side of (5.8). Since $\gamma < \nu$, the reverse Hölder inequality implies
$$
\left(\frac{1}{|Q|} \int_Q w(x)^{\gamma p(x)} dx \right)^{1/\gamma} \leq c \frac{1}{|Q|} \int_Q w(x)^p dx.
$$
By (2.8) and since \(|E_2| \geq \frac{1}{2}|Q|\),
\[
\int_Q w(x)^{p(x)} dx \leq c \int_{E_2} w(x)^{p(x)} dx.
\]

Using also that \(t \geq 1\), we obtain
\[
(5.9) \left( \frac{1}{|Q|} \int_Q w(x)^{\gamma p(x)} dx \right)^{1/\gamma} t^{m_p(Q)} \leq \frac{c}{|Q|} \int_{E_2} w(x)^{p(x)} dx
\]
\[
\leq \frac{c}{|Q|} \int_Q (tw(x))^{p(x)} dx.
\]

Turn to the second term on the right-hand side of (5.8). The boundedness of \(M\) on \(L^p_w(\cdot)\) implies \(\|\chi_Q\|_{L^p_w(\cdot)} \leq c \|\chi_{E_1}\|_{L^p_w(\cdot)}\). By Lemma 2.2 (to be more precise, we use here a local version of Lemma 2.2; see [3, p. 25] for details),
\[
\|\chi_{E_1}\|_{L^p_w(\cdot)} \leq \left( \int_{E_1} w(x)^{p(x)} dx \right)^{1/p_+(E_1)} \leq \left( \int_{E_1} w(x)^{p(x)} dx \right)^{1/m_p(Q)},
\]
where \(p_+(E_1) = \text{ess sup}_{x \in E_1} p(x)\). As previously, by (2.8), \(\int_{E_1} w(x)^{p(x)} dx \leq c \int_{E_2} w(x)^{p(x)} dx\). Therefore, combining the previous estimates yields
\[
(5.10) \quad \|\chi_Q\|_{L^p_w(\cdot)} \leq c \left( \int_{E_2} w(x)^{p(x)} dx \right)^{1/m_p(Q)}.
\]

Let \(q = \frac{1+r(s-1)}{1+\gamma(s-1)}\) and \(q' = \frac{q}{q-1}\). Then \(q(1+\varepsilon)\gamma = r\) and \(q'\varepsilon\gamma = \frac{1}{s-1}\). Hence, Hölder’s inequality with the exponents \(q\) and \(q'\) along with Lemma 5.1 implies
\[
\frac{1}{|Q|} \int_Q \left( \frac{w(x)}{\|\chi_Q\|_{L^p_w(\cdot)}^{1+p(x)}} \right)^{\gamma p(x)} dx
\]
\[
\leq \left( \frac{1}{|Q|} \int_Q \left( \frac{w(x)}{\|\chi_Q\|_{L^p_w(\cdot)}} \right)^{r p(x)} dx \right)^{1/q} \left( \frac{1}{|Q|} \int_Q w(x)^{-\frac{1}{q} p(x)} dx \right)^{1/q'}
\]
\[
\leq c \frac{1}{|Q|^{\frac{r-1}{q}+1}} \left( \int_Q w(x)^{-\frac{1}{q} p(x)} dx \right)^{1/q'}.
\]

Notice that
\[
\frac{r-1}{q} + 1 = \frac{r}{q} + \frac{1}{q'} = (1+\varepsilon)\gamma + \varepsilon\gamma(s-1) = \gamma(s\varepsilon + 1).
\]
Therefore, from the previous estimate and from (5.10),

\[
\|\chi_Q\|_{L^p_w(Q)}^{m_p(Q)} \left( \frac{1}{|Q|} \int_Q \left( \frac{w(x)}{\|\chi_Q\|_{L^p_w(Q)}^{1+\varepsilon}} \right)^{\gamma p(x)} \, dx \right)^{1/\gamma} 
\leq c \frac{|Q|^s}{|Q|} \left( \frac{1}{|Q|} \int_Q w(x)^{p(x)} \, dx \right)^\varepsilon \left( \frac{1}{|Q|} \int_Q w(x)^{-\frac{1}{s-1} p(x)} \, dx \right)^{\varepsilon(s-1)}
\leq c [w(\cdot)^{p(\cdot)}]_{A_s} \frac{1}{|Q|}
\]

From this, using (5.10) again, we obtain

\[
\|\chi_Q\|_{L^p_w(Q)}^{m_p(Q)(1+\varepsilon)} \left( \frac{1}{|Q|} \int_Q \left( \frac{w(x)}{\|\chi_Q\|_{L^p_w(Q)}^{1+\varepsilon}} \right)^{\gamma p(x)} \, dx \right)^{1/\gamma} t^{m_p(Q)}
\leq \frac{c}{|Q|} t^{m_p(Q)} \int_{E_2} w(x)^{p(x)} \, dx \leq \frac{c}{|Q|} \int_Q (tw(x))^{p(x)} \, dx.
\]

This along with (5.8) and (5.9) proves (5.7).

Finally, we note that the proof in the case when \( \|\chi_Q\|_{L^p_w(Q)} \geq 1 \) is the same, with reversed roles of the sets \( E_1 \) and \( E_2 \).

**Proof of Lemma 4.1.** Assume that \( t \|\chi_Q\|_{L^p_w(Q)} \leq 1 \). If \( t \geq 1 \), then the conclusion of Lemma 4.1 follows immediately from Lemma 5.3. Therefore, it remains to consider the case when \( t < 1 \).

We may keep all main settings of Lemma 5.3, namely, assume that \( w(\cdot)^{p(\cdot)} \in A_s \), and take the same numbers \( \gamma \) and \( \varepsilon = \frac{r-\gamma}{\gamma(1+(s-1)r)} \).

If

\[
\left( \frac{1}{|Q|} \int_Q (tw(x))^{\gamma p(x)} \, dx \right)^{1/\gamma} \leq A \frac{1}{|Q|} \int_Q (tw(x))^{p(x)} \, dx,
\]

where \( A > 0 \) will be determined later, then (4.1) is trivial. Suppose that

\[
(5.11) \quad \frac{1}{|Q|} \int_Q (tw(x))^{p(x)} \, dx < \frac{1}{A} \left( \frac{1}{|Q|} \int_Q (tw(x))^{\gamma p(x)} \, dx \right)^{1/\gamma}.
\]
As in the proof of Lemma 5.3, take \( q = \frac{1 + r(s-1)}{1 + \gamma(s-1)} \) and apply Hölder’s inequality with the exponents \( q \) and \( q' \). We obtain

\[
\left( \frac{1}{|Q|} \int_Q (tw(x))^{\gamma p(x)} dx \right)^{1/\gamma} \leq \left( \frac{1}{|Q|} \int_Q (t^{1+\varepsilon} w(x))^{p(x)} dx \right)^{\frac{1+\varepsilon}{1+\varepsilon}} \left( \frac{1}{|Q|} \int_Q w^{-\varepsilon} p(x) dx \right)^{(s-1)\varepsilon}.
\]

From this, applying Hölder’s inequality again along with (5.11) yields

\[
\left( \frac{1}{|Q|} \int_Q (t^{1+\varepsilon} w(x))^{p(x)} dx \right)^{1/r} \leq \left( \frac{1}{|Q|} \int_Q (tw(x))^{p(x)} dx \right)^{\frac{1}{1+\varepsilon}} \left( \frac{1}{|Q|} \int_Q w^{\gamma p(x)} dx \right)^{\frac{\varepsilon}{1+\varepsilon}} \left( \frac{1}{|Q|} \int_Q (t^{1+\varepsilon} w(x))^{p(x)} dx \right)^{1/r}.
\]

Further, from (5.11) and from Lemma 5.3, \( t^{1+\varepsilon} \leq \frac{1}{\|\chi_Q\|_{L^p(A)}} \) (here we assume that \( A \geq c \), where \( c \) is the constant from Lemma 5.3). Hence, by Lemma 5.2,

\[
|Q| \left( \frac{1}{|Q|} \int_Q (t^{1+\varepsilon} w(x))^{p(x)} dx \right)^{1/r} \leq k \int_Q (t^{1+\varepsilon} w(x))^{p(x)} dx + b(Q),
\]

where \( b(Q) \) is defined in the proof of Lemma 5.2. Thus, taking \( A = \max((2k)^{1+\varepsilon}[w(\cdot)^{p(\cdot)}]_{A_x}, c) \), where \( c \) is the constant from Lemma 5.3 and applying (5.12), we obtain

\[
|Q| \left( \frac{1}{|Q|} \int_Q (t^{1+\varepsilon} w(x))^{p(x)} dx \right)^{1/r} \leq 2b(Q),
\]

which implies

\[
|Q| \left( \frac{1}{|Q|} \int_Q (tw(x))^{p(x)} dx \right)^{1/r} \leq 2t^{\frac{\varepsilon}{1+\varepsilon}} - b(Q).
\]

This along with Hölder’s inequality (since \( \gamma < r \)) proves (4.1). \( \square \)
6. Concluding remarks and open questions

6.1. About the assumption $w(\cdot)^{p(\cdot)} \in A_\infty$. We start with the following question.

**Question 6.1.** Is it possible to remove completely the assumption $w(\cdot)^{p(\cdot)} \in A_\infty$ in Theorem 1.1?

Several remarks related to this question are in order. Denote by $\mathcal{L}_H(\mathbb{R}^n)$ the class of exponents $p(\cdot)$ with $p_- > 1, p_+ < \infty$ and such that

$$|p(x) - p(y)| \leq \frac{c}{\log(e + 1/|x - y|)} \quad \text{and} \quad |p(x) - p_\infty| \leq \frac{c}{\log(e + |x|)}$$

for all $x, y \in \mathbb{R}^n$, where $c > 0$ and $p_\infty \geq 1$. Also denote by $A_{p(\cdot)}(\cdot)$ the class of weights such that

$$\sup_Q |Q|^{-1} \|\chi_Q\|_{L^p(w)} \|\chi_Q\|_{L^{p(\cdot)}_{w^{-1}}} < \infty.$$

It was shown in [2, 4] that if $p(\cdot) \in \mathcal{L}_H(\mathbb{R}^n)$, then $M$ is bounded on $L^p(w)$ if and only if $w \in A_{p(\cdot)}$. An important ingredient in the proof in [4] is the fact that if $p(\cdot) \in \mathcal{L}_H(\mathbb{R}^n)$ and $w \in A_{p(\cdot)}$, then $w(\cdot)^{p(\cdot)} \in A_\infty$. Since the boundedness of $M$ on $L^p_w$ implies the $A_{p(\cdot)}$ condition trivially, we see that if $p(\cdot) \in \mathcal{L}_H(\mathbb{R}^n)$, then the assumption that $w(\cdot)^{p(\cdot)} \in A_\infty$ in Theorem 1.1 is superfluous. However, we do not know whether this assumption can be removed (or at least weakened) in general.

It is well known (see, e.g., [3, Th. 3.16]) that if $p(\cdot) \in \mathcal{L}_H(\mathbb{R}^n)$, then $M$ is bounded on $L^p$. This fact raises the following questions.

**Question 6.2.** Suppose that $M$ is bounded on $L^p$ and $w \in A_{p(\cdot)}$. Does this imply $w(\cdot)^{p(\cdot)} \in A_\infty$?

**Question 6.3.** Is it possible to replace in Theorem 1.1 the assumption $w(\cdot)^{p(\cdot)} \in A_\infty$ by the boundedness of $M$ on $L^p$?

Question 6.3 is closely related to another open question stated in [7] and [3] p. 275: is it possible to deduce the equivalence $M : L^p_w(\cdot) \to L^p_w(\cdot) \Leftrightarrow w \in A_{p(\cdot)}$ assuming only that $M$ is bounded on $L^p(\cdot)$?

6.2. An application. It is a well known principle that if $M$ is bounded on a BFS $X$ and on $X'$, then some other basic operators in harmonic analysis are also bounded on $X$. Consider, for instance, a Calderón-Zygmund operator $T$. By this we mean that $T$ is an $L^2$ bounded integral operator represented as

$$Tf(x) = \int_{\mathbb{R}^n} K(x, y) f(y) dy, \quad x \notin \text{supp } f,$$
with kernel $K$ satisfying $|K(x, y)| \leq \frac{c}{|x-y|^n}$ for all $x \neq y$, and for some $0 < \delta \leq 1$,

$$|K(x, y) - K(x', y)| + |K(y, x) - K(y, x')| \leq c \frac{|x - x'|^{\delta}}{|x - y|^{n+\delta}},$$

whenever $|x - x'| < |x - y|/2$. It was shown in [12] that

$$\int_{\mathbb{R}^n} |Tf(x)g(x)| \, dx \leq \int_{\mathbb{R}^n} Mf(x)Mg(x) \, dx.$$  

This estimate along with (2.1) and (2.2) implies that if $M$ is bounded on a BFS $X$ and on $X'$, then $T$ is bounded on $X$. Hence, Theorem 1.1 yields the following corollary.

**Corollary 6.4.** Let $p : \mathbb{R}^n \to [1, \infty)$ be a measurable function such that $p_- > 1$ and $p_+ < \infty$. Let $w$ be a weight such that $w(\cdot)^{p(\cdot)} \in A_\infty$. If $M$ is bounded on $L_w^{p(\cdot)}(\mathbb{R}^n)$, then $T$ is bounded on $L_w^{p(\cdot)}$.

As we have mentioned above, it was shown in [4] that if $p(\cdot) \in LH(\mathbb{R}^n)$ and $w \in A_{p(\cdot)}$, then $w(\cdot)^{p(\cdot)} \in A_\infty$ and $M$ is bounded on $L_w^{p(\cdot)}$. Therefore, Corollary 6.4 implies the following less general result.

**Corollary 6.5.** If $p(\cdot) \in LH(\mathbb{R}^n)$ and $w \in A_{p(\cdot)}$, then $T$ is bounded on $L_w^{p(\cdot)}$.

Notice that a closely related result was very recently proved in [11].

**Acknowledgement.** I am grateful to Alexei Karlovich for valuable remarks on an earlier version of this paper. Also I would like to thank the anonymous referee for detailed comments that improved the presentation.

**References**

[1] C. Bennett and R. Sharpley, Interpolation of operators, Academic Press, New York, 1988.

[2] D. Cruz-Uribe, L. Diening and P. Hästö, The maximal operator on weighted variable Lebesgue spaces, Fract. Calc. Appl. Anal. 14 (2011), no. 3, 361-374.

[3] D. Cruz-Uribe and A. Fiorenza, Variable Lebesgue spaces. Foundations and harmonic analysis. Birkhäuser/Springer, Heidelberg, 2013.

[4] D. Cruz-Uribe, A. Fiorenza and C.J. Neugebauer, Weighted norm inequalities for the maximal operator on variable Lebesgue spaces, J. Math. Anal. Appl. 394 (2012), no. 2, 744-760.

[5] L. Diening, Maximal function on Musielak-Orlicz spaces and generalized Lebesgue spaces, Bull. Sci. Math. 129 (2005), no. 8, 657-700.

[6] L. Diening, P. Harjulehto, P. Hästö and M. Růžička, Lebesgue and Sobolev spaces with variable exponents. Lecture Notes in Mathematics, 2017. Springer, Heidelberg, 2011.
[7] L. Diening and P. Hästö, *Muckenhoupt weights in variable exponent spaces*, preprint. Available at [http://www.helsinki.fi/~hasto/pp/p75_submit.pdf](http://www.helsinki.fi/~hasto/pp/p75_submit.pdf)

[8] L. Grafakos, Classical Fourier Analysis. Second edition. Graduate Texts in Mathematics, 249. Springer, New York, 2009.

[9] L. Grafakos, Modern Fourier Analysis. Second edition. Graduate Texts in Mathematics, 250. Springer, New York, 2009.

[10] T. Hytönen, M. Lacey and C. Pérez, *Sharp weighted bounds for the $q$-variation of singular integrals*, Bull. Lond. Math. Soc. **45** (2013), no. 3, 529-540.

[11] M. Izuki, E. Nakai and Y. Sawano, *Wavelet characterization and modular inequalities for weighted Lebesgue spaces with variable exponent*, Ann. Acad. Sci. Fenn. Math. **40** (2015), 551–571.

[12] A.K. Lerner, *Weighted norm inequalities for the local sharp maximal function*, J. Fourier Anal. Appl., **10** (2004) no. 5, 465–474.

[13] W.A.J. Luxemburg, Banach function spaces. Thesis, Delft Institute of Technology, Assen (Netherlands), 1955.

[14] J.L. Rubio de Francia, *Factorization theory and $A_p$ weights*, Amer. J. Math. **106** (1984), 533–547.