String Cosmology

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Abstract

“Old’ String Theory is a theory of one-dimensional extended objects, whose vibrations correspond to excitations of various target-space field modes including gravity. It is for this reason that strings present the first, up to now, mathematically consistent framework where quantum gravity is unified with the rest of the fundamental interactions in nature. In these lectures I will give an introduction to low-energy Effective Target-Space Actions derived from conformal invariance conditions of the underlying sigma models in string theory. In this context, I shall discuss cosmology, emphasizing the role of the dilaton field in inducing inflationary scenaria and in general expanding string universes. Specifically, I shall analyse some exact solutions of string theory with a linear dilaton, and discuss their role in inducing expanding Robertson-Walker Universes. I will mention briefly pre-Big-Bang scenaria of String Cosmology, in which the dilaton plays a crucial role. In view of recent claims on experimental evidence (from diverse astrophysical sources) on the existence of cosmic acceleration in the universe today, with a positive non-zero cosmological constant (de Sitter type), I shall also discuss difficulties of incorporating such Universes with eternal acceleration in the context of critical string theory, and present scenaria for a graceful exit from such a phase.

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1 Introduction

Our way of thinking towards an understanding of the fundamental forces in nature, as well as of the structure of matter and that of space-time, has evolved over the last decades of the previous century from that of using point-like structures as the basic constituents of matter, to that employing one-dimensional extended objects (strings [1]), and, recently (from the mid 90’s), higher-dimensional domain-wall like solitonic objects, called (Dirichlet) (mem)branes [2].

The passage from point-like fundamental constituents to strings, in the mid 1980’s, has already revolutionarized our view of space-time and of the unification of fundamental interactions in Nature, including gravity. Although in the framework of point-like field theories, the uncontrollable ultraviolet (short-distance) divergencies of quantum gravity prevented the development of a mathematically consistent unifying theory of all known interactions in Nature, the discovery of one-dimensional fundamental constituents of matter and space-time, called strings, which were in principle free from such divergencies, opened up the way for a mathematically consistent way of incorporating quantum gravity on an equal footing with the rest of the interactions. The existence of a minimum length $\ell_s$ in string theory, in such a way that the quantum uncertainty principle between position $X$ and momenta $P$: $\Delta X \Delta P \geq \hbar$, of point-like quantum mechanics is replaced by: $\Delta X \geq \ell_s$, $\Delta X \Delta P \geq \hbar + O(\ell_s^2) \Delta P^2 + \ldots$, revolutionized the way we looked at the structure of space-time at such small scales. The unification of gravitational interactions with the rest is achieved in this framework if one identifies the string scale $\ell_s$ with the Planck scale, $\ell_P = 10^{-35}$ m, where gravitational interactions are expected to set in. The concept of space-time, as we perceive it, breaks down beyond the string (Planck) scale, and thus there is a fundamental short-distance cutoff built-in in the theory, which results in its finiteness.

The cost, however, for such an achievement, was that mathematical consistency implied a higher-dimensional target space-time, in which the strings propagate. This immediately lead the physicists to try and determine the correct vacuum configurations of string theory which would result into a four-dimensional Universe, i.e. a Universe with four dimensions being “large” compared to the gravitational scale, the Planck length, $10^{-35}$ m, with the extra dimensions compactified on Planckian size manifolds. Unfortunately such consistent ground states are not unique, and there is a huge degeneracy among such string vacua, the lifting of which is still an important unresolved problem in string physics.

In the last half of the 1990’s the discovery of string dualities, i.e. discrete stringy (non-perturbative) gauge symmetries linking various string theories, showed another interesting possibility, which could contribute significantly towards the elimination of the huge degeneracy problem of the string vacua. Namely, many string theories were found to be dual to each other in the sense of exhibiting invariances of their physical spectra of excitations under the action of such discrete symmetries. In fact, by virtue of such dualities one could argue that there exist a sort of unification of string theories, in which all the known string theories (type IIA, type IIB, $SO(32)/Z_2$, Heterotic $E_8 \times E_8$, type I), together with 11 dimensional supergavity (living in one-dimension higher than the critical dimension of superstrings) can be all connected with string dualities, so that one may view them as low
energy limits of a mysterious larger theory, termed \( M \)-theory \(^2\), whose precise dynamics is still not known.

A crucial rôle in such string dualities is played by domain walls, stringy solitons, which can be derived from ordinary strings upon the application of such dualities. Such extended higher-dimensional objects are also excitations of this mysterious M-theory, and they are on a completely equal footing with their one dimensional (stringy) counterparts.

In this framework one could discuss cosmology. The latter is nothing other but a theory of the gravitational field, in which the Universe is treated as a whole. As such, string or M-theory theory, which includes the gravitational field in its spectrum of excitations, seems the appropriate framework for providing analyses on issues of the Early Universe Cosmology, such as the nature of the initial singularity (Big Bang), the inflationary phase and graceful exit from it \textit{etc}, which conventional local field theories cannot give a reliable answer to. It is the purpose of this lectures to provide a very brief, but hopefully comprehensive discussion, on String Cosmology. We use the terminology string cosmology here to discuss Cosmology based on one-dimensional fundamental constituents (strings). Cosmology may also be discussed from the more modern point of view of membrane structures in M-theory, mentioned above, but this will not be covered in these lectures. Other lecturers in the School will discuss this issue.

The structure of the lectures is as follows: in the first lecture we shall introduce the layman into the subject of string effective actions, and discuss how equations of motion of the various low-energy modes of strings are associated with fundamental consistency properties (conformal invariance) of the underlying string theory. In the second lecture we shall discuss various scenaria for String Cosmology, together with their physical consequences. Specifically I will discuss how expanding and inflationary (de Sitter) Universes are incorporated in string theory, with emphasis on describing new futures, not characterizing conventional point-like cosmologies. Finally, in the third lecture we shall speculate on ways of providing possible resolution to various theoretical challenges for string theory, especially in view of recent astrophysical evidence of a current-era acceleration of our Universe. In this respect we shall discuss the application of the so-called \textit{non-critical} (Liouville) string theory to cosmology, as a way of going \textit{off equilibrium} in a string-theory setting, in analogy with the use of non-equilibrium field theories in conventional point-like cosmological field theories of the Early Universe.

2 Lecture 1: Introduction to String Effective Actions

2.1 World-sheet String formalism

In this lectures the terminology “string theory” will be restricted to the “old” concept of \textit{one (spatial) dimensional extended objects}, propagating in target-space times of dimensions higher than four, specifically 26 for Bosonic strings and 10 for Super(symmetric)strings. There are in general two types of such objects, as illustrated in a self-explanatory way in figure 1: \textit{open strings} and \textit{closed strings}. In the first quantized formalism, one is interested
in the propagation of such extended objects in a background space time. By direct extension of the concept of a point-particle, the motion of a string as it glides through spacetime is described by the *world sheet*, a two dimensional Riemann surface which is swept by the extended object during its propagation through spacetime. The world-sheet is a direct extension of the concept of the *world line* in the case of a point particle. The important formal difference of the string case, as compared with the particle one, is the fact that *quantum corrections*, i.e. string loops, are incorporated in a smooth and straightforward manner in the case of string theory by means of summing over Riemann surfaces with non-trivial *topologies* ("genus") (c.f. figure 2). This is allowed because in two (world-sheet) dimensions one is allowed to discuss loop corrections in a way compatible with a (two-dimensional) smooth manifold concept, in contrast to the point-particle one-dimensional case, where a loop correction on the world-line (c.f. figure 2) cannot be described in a smooth way, given that a particle loop does not constitute a manifold. The 'smooth-manifold' property of quantum fluctuating world sheets is essential in analysing target-space quantum corrections within a first-quantization framework, which cannot be done in the one-dimensional particle case. Specifically, as we shall discuss later on, by considering the propagation of a stringy extended object in a curved target space time manifold of higher-dimensionality (26 for Bosonic or 10 for Superstrings), one will be able of arriving at consistency conditions on the background geometry, which are, in turn, interpreted as equations of motion derived from an effective low-energy action constituting the local field theory limit of strings. Summation over genera will describe quantum fluctuations about classical ground states of the strings described by world-sheet with the topology of the sphere (for closed strings) or disc (for open strings).

To begin our discussion we first consider the propagation of a Bosonic string in a flat target space of space-time dimensionality $D$, which will be determined dynamically below by means of certain mathematical self-consistency conditions. From a first quantization viewpoint, such a propagation is described by considering the following world-sheet two-dimensional action:

$$ S_{\sigma} = -\frac{\mathcal{T}}{2} \int_{\Sigma} d^{2}\sigma \sqrt{\gamma} \gamma^{\alpha\beta} \eta_{MN} \partial_{\alpha} X^{M} \partial_{\beta} X^{N}, \quad \alpha, \beta = \sigma, \tau $$

(1)

where $\gamma^{\alpha\beta}$ is the world-sheet metric, and $X^{M}(\sigma, \tau), \; D = 0, \ldots D - 1$ denote a mapping from the world-sheet $\Sigma$ to a target space manifold of dimensionality $D$, of flat Minkowski metric $\eta_{MN}, \; M, N = 0, \ldots D - 1$. The world-sheet zero modes of the $\sigma$-model fields $X^{M}$ are therefore the spacetime coordinates, the 0 index indicating the (Minkowski) time. The action (1) is related to the invariant world-sheet area, in direct extension of the point-particle case, where a particle sweeps out a world line as it glides through space time, and hence its action is proportional to a section of an invariant curve. The quantity $\mathcal{T}$ is the *string tension*, which from a target-space viewpoint is a dimensionful parameter with dimensions of $[\text{length}]^{-2}$. One then denotes

$$ \mathcal{T} = \frac{1}{2\pi\alpha'} $$

(2)
Figure 1: Types of strings and the associated world-sheets swept as the string propagates through a (higher-dimensional) target space time. In the closed-string case, which incorporates gravity, the point-like low-energy field theory limit is obtained by shrinking the size of the external strings (at the tips of the cylinder) to zero, thereby obtaining the topology of a punctured sphere

where $\alpha'$ is the Regge slope. This notation is a result of the original idea for which string theory was invented, namely to explain hadron physics, and in particular the linear dependence of the various hadron resonances of (total) spin $J$ vs Energy, the slope of which was identified with the Regge slope $\sqrt{\alpha'}$.  

The dynamical world-sheet theory based on (1) is a constrained theory. This follows from invariances, which are: (i) the reparametrizations of the world-sheet 

$$(\sigma, \tau) \rightarrow (\sigma', \tau')$$,  

playing the rôle of general coordinate transformations in the two-dimensional world-sheet manifold, and (ii) Weyl invariance, i.e. invariance of the theory under local conformal rescalings of the metric:

$$\gamma_{\alpha\beta} \rightarrow e^{\varphi(\sigma, \tau)} \gamma_{\alpha\beta}$$,  

where $\varphi(\sigma, \tau)$ is a function of $\sigma, \tau$. It should be noted that in two-dimensions the conformal group is infinite dimensional, in contrast to its finite nature in all higher dimensions. It is generated by the Virasoro algebra as we shall discuss later, and plays a crucial rôle for the quantum consistency of the theory (1), with important restrictions on the nature of the target-space time manifold in which the string propagates.
Figure 2: Quantum String Interactions are represented by higher-topologies on the associated world-sheets. The two-dimensional nature of the string world-sheet, which makes it a smooth manifold, should be contrasted with the point-particle world-line case, where loops are not manifolds.

The symmetry under (i) and (ii) above allows one to fix the world-sheet metric into the form:

\[ \gamma_{\alpha\beta} = e^{\rho(\sigma, \tau)} \hat{\gamma}_{\alpha\beta} \]  

(5)

where \( \hat{\gamma}_{\alpha\beta} \) is a fiducial (fixed) metric on the world-sheet. As far as the two-dimensional gravity (world-sheet) theory is concerned, the choice (5) is, in a sense, a “gauge choice”; this is the reason why the ansatz is commonly called a conformal gauge. For most practical purposes the metric \( \hat{\gamma}_{\alpha\beta} \) is taken to be flat \( \eta_{\alpha\beta} \) (plane). However formally this is not quite correct in general, as it depends on the kind of string theory considered. For open strings, whose classical (tree-level) propagation implies world-sheets with the topology of a disc, the fiducial metric is that of a disc, i.e. a manifold with boundary. On the other hand, for closed strings, whose classical (tree-level) propagation implies world-sheets with the topology of a sphere (punctured), for point-like excitations, or cylinder, for stringy excitations, the fiducial metric is taken to be that of a sphere or cylinder. In particular, in the case of low-energy limit of strings, which implies that the external strings have been shrunk to zero size, and hence they are punctures for all practical purposes, the spherical
topology of the fiducial geometry implies an Euler characteristic
\[ \chi = \text{Euler characteristic} = 2 - \text{no. of holes} - 2 \times \text{no. of handles} = \]
\[ 2 = \frac{1}{4\pi} \int_{\Sigma=S(2)} \sqrt{\gamma} \hat{R}^{(2)} \]  
(6)

where \( \hat{R}^{(2)} \) is the two-dimensional curvature. On the other hand, if one used naively a planar fiducial metric, which as mentioned earlier, in many respect is sufficient, such topological properties as (6), would be obscured. The importance of (6) will become obvious later on, when we discuss quantum target-space string corrections (string loops), as opposed to \( \sigma \)-model loops, i.e. world-sheet theory quantum corrections, which will be discussed immediately below.

Under the gauge choice (5) the string equations (i.e. the equations of motion of the fields \( X^M \)) read:
\[ \left( \frac{\partial^2}{\partial \tau^2} - \frac{\partial^2}{\partial \sigma^2} \right) X^M = 0 , \]  
(wave equation) (7)

and are supplemented with the constraint equations arising from vanishing variations with respect to the world-sheet metric field \( \gamma_{\alpha\beta} \) (which should be first varied and then be constrained in the gauge (5)):
\[ \frac{\delta S_{\sigma}}{\delta \gamma_{\alpha\beta}} = 0 \]  
(8)

The constraint (8) is nothing other than the vanishing of the stress-energy tensor \( T_{\alpha\beta} \) of the two-dimensional (world-sheet) field theory, defined as:
\[ T_{\alpha\beta} \equiv -\frac{2}{\sqrt{\gamma}} \frac{\delta S_{\sigma}}{\delta \gamma_{\alpha\beta}} \]  
(9)

The above equations (7),(8) take their simplest form if one uses light-cone coordinates on the world-sheet:
\[ \sigma^\pm = \tau \pm \sigma \]  
(10)

Indeed, in this system of coordinates (8) becomes:
\[ T_{\pm\pm} = \partial_{\pm} X^M \partial_{\pm} X^N \eta_{MN} = 0 , \]
\[ T_{+-} = 0 \]  
(trace of stress tensor) (11)

The vanishing of the trace of the stress tensor of the world-sheet theory implies an important symmetry, that of CONFORMAL INVARIANCE. The maintainence of this classical symmetry at a quantum level is essential for the consistency of the theory, given that above we have used this classical symmetry in order to make the choice (5). In the next subsection we shall turn to a rather detailed discussion on the implications of the requirement of conformal symmetry (which in two-dimensions implies an underlying infinite dimensional (Virasoro) symmetry) at a quantum \( \sigma \)-model level.
Before doing this we simply mention that, in order to understand the existence of an infinite number of conserved quantities, leading to an infinite-dimensional symmetry, in the case of conformal symmetry in two space-time dimensions, it suffices to notice that the conservation of the stress tensor $T_{\alpha\beta}$, which is a consequence of two-dimensional reparametrization invariance, in light-cone coordinates reads: $\partial_- T_{++} + \partial_+ T_{--} = 0$. In view of $T_{+-} = T_{-+} = 0$, then, this implies $\partial_- T_{++} = 0$. If $f(\sigma^+)$ is an arbitrary function of $\sigma^+$, so that $\partial_- f = 0$, then the current $fT_{++}$ is conserved, and hence the spatial integral $Q_f \equiv \int d\sigma f(\sigma^+) T_{++}$ is a conserved charge. The arbitrariness of $f$ implies therefore an infinity of conserved charges. Clearly the argument above holds only in two dimensions. In higher dimensions the conformal symmetry is finite dimensional.

### 2.2 Conformal Invariance and Critical Dimension of Strings

In this subsection we shall discuss the way by which conformal invariance is maintained at a quantum $\sigma$-model level. First of all we should distinguish the quantum $\sigma$-model level, which pertains to quantising the fields $X^M$ of the $\sigma$-model (in, say, a path integral) at a fixed world-sheet topology, but integrating over world-sheet metrics (geometries), from the quantum target-space level, at which one also summs up world-sheet topologies (string loops).

The requirement of vanishing of the trace of the world-sheet stress tensor at a quantum $\sigma$-model level implies important restrictions on the structure of the target space-time of string theory. The first important restriction concerns the dimensionality of target space time. There are various ways in which one can see this. In this lectures we shall follow the covariant path integral quantization, which is most relevant for our purposes. For details on other methods we refer the interested reader in the literature [1].

Consider the free field-theory world-sheet action, describing propagation of a free string in a flat target space time [1].

$$S_\sigma[\gamma, X] = -\frac{1}{4\pi\alpha'} \int_{\Sigma} d^2\sigma \partial\gamma X^M \partial\beta X^N \eta_{MN} \gamma_{\alpha\beta} \sqrt{\gamma}$$  \hspace{1cm} (12)

To quantize in a covariant path-integral way the above world-sheet action one considers the partition function at a fixed world-sheet topology (genus):

$$Z = \int D\gamma DX e^{-iS_\sigma[\gamma, X]}$$  \hspace{1cm} (13)

Formally one should analytically continue to a Euclidean world sheet and go back to the Minkowskian signature world-sheet theory only at the end of the computations. This will be understood in what follows.

We now concentrate on the integration over geometries on the world-sheet, $D\gamma$. This integral is over three independent world-sheet metric components [1]: $\gamma_{++}(\sigma, \tau)$, $\gamma_{--}(\sigma, \tau)$, $\gamma_{-+}(\sigma, \tau)$.

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1. we work in the light-cone coordinate system, whose choice is allowed by postulating invariance under general coordinate transformations of the two-dimensional quantum gravity theory. Notice that in two-dimensions gravity is a renormalizable theory so the quantum path integral over world-sheet metrics is rigorously well defined.
\( \gamma_{+-}(\sigma, \tau) \). An important rôle is played by anomalies, i.e. the potential breakdown of certain symmetries at a quantum world-sheet level, which result in the impossibility of preserving all of the apparent classical symmetries of (13).

As we mentioned earlier there are three ‘gauge invariances’ of the action (1), two reparametrizations of the world-sheet coordinates and a Weyl rescaling. Locally we can use these symmetries to fix the gauge (5). For simplicity, in what follows, and given that we shall work only at a fixed lowest topology on the world sheet, we shall consider the case of flat fiducial metrics; however, the precise discussion on disc (spherical geometries) in case of open (closed strings) should be kept in the back of the reader’s mind as the appropriate procedure when one sums up genera.

In this case the covariant gauge reads:

\[
\gamma_{\alpha\beta} = e^{\rho(\sigma, \tau)} \eta_{\alpha\beta} \tag{14}
\]

In light-cone coordinates then, the condition (14) implies:

\[
0 = \gamma_{++} = \gamma_{--} \tag{15}
\]

Under reparametrizations \( \sigma^\pm \rightarrow \sigma^\pm + \xi^\pm \) the world-sheet metric components in (15) transform as:

\[
\delta \gamma_{++} = 2 \nabla_+ \xi_+ ; \quad \delta \gamma_{--} = 2 \nabla_- \xi_- . \tag{16}
\]

where \( \nabla_\alpha \) denotes covariant world-sheet derivative, with respect to the metric \( \gamma \). To maintain (15) one should constraint the variations (16) to vanish.

Such conditions are implemented in the path integral (13) by means of insertion of the identity:

\[
1 = \int Dg(\sigma, \tau) \delta(\gamma^g_{++}) \delta(\gamma^g_{--}) \det \left( \frac{\delta \gamma^g_{++}}{\delta g} \right) \det \left( \frac{\delta \gamma^g_{--}}{\delta g} \right) \tag{17}
\]

where \( Dg \) denotes integration over the group \( G \) of reparametrizations of the string world-sheet, and \( \gamma^g \) denotes the world-sheet metric into which \( \gamma \) is transformed under the action of \( G \). The determinants \( \det (\ldots) \) appearing in (17) are due to the gauge fixing procedure (14). We then have:

\[
Z = \int Dg(\sigma, \tau) \int D\gamma DX e^{-S_\sigma[\gamma, X]} \delta(\gamma^g_{++}) \delta(\gamma^g_{--}) \det \left( \frac{\delta \gamma^g_{++}}{\delta g} \right) \det \left( \frac{\delta \gamma^g_{--}}{\delta g} \right) \tag{18}
\]

Reparametrization invariance implies that \( S_\sigma[\gamma, X] = S_\sigma[\gamma^g, X] \), i.e. that the integrand of the path integral depends on \( \gamma, g \) only through \( \gamma^g \). Making a change of variables from \( \gamma, g \) to \( g \) and \( \gamma' \equiv \gamma^g \), and discarding the \( Dg \) integration, which can be performed trivially yielding an irrelevant constant proportionality (normalization) factor, one arrives at:

\[
\int \int D\gamma^g DX e^{-S_\sigma[\gamma^g, X]} \delta(\gamma^g_{++}) \delta(\gamma^g_{--}) \det \left( \frac{\delta \gamma^g_{++}}{\delta g} \right) \det \left( \frac{\delta \gamma^g_{--}}{\delta g} \right) =
\]

\[
\int D\gamma^g_{++} DX e^{-S_\sigma[\gamma^g, X]} \det \left( \frac{\delta \gamma^g_{++}}{\delta g} \right) \bigg|_{\gamma^g_{++}=0} \det \left( \frac{\delta \gamma^g_{--}}{\delta g} \right) \bigg|_{\gamma^g_{--}=0} \tag{19}
\]
The integration over $\gamma^g_{+-}$ is equivalent to an integration over the function $\rho(\sigma, \tau)$ (c.f. (14)). The determinants in the last expression can be expressed in terms of a set of ‘reparametrization ghost fields of Fadeev-Popov type’, $\{c^\pm, b_{\pm \pm}\}$, of Grassmann statistics:

\[
\begin{align*}
\det \left( \frac{\delta \gamma^g_{++}}{\delta g} \right) |_{\gamma_{++}=0} &= \int Dc^- (\sigma, \tau) Db_-(\sigma, \tau) e^{-\frac{1}{\pi} \int_\Sigma d^2 \sigma c^- \nabla^+ b_-,} \\
\det \left( \frac{\delta \gamma^g_{--}}{\delta g} \right) |_{\gamma_{--}=0} &= \int Dc^+ (\sigma, \tau) Db_+ (\sigma, \tau) e^{-\frac{1}{\pi} \int_\Sigma d^2 \sigma c^+ \nabla^- b_+}. \quad (20)
\end{align*}
\]

Hence one should have as a final result:

\[
Z = \int D\rho (\sigma, \tau) \int DX (\sigma, \tau) DC (\sigma, \tau) DB (\sigma, \tau) e^{-S_{\text{total}}[c, b, X]}, \quad (21)
\]

where $S_{\text{total}} = S_{\sigma} + S_{\text{ghost}}$, with

\[
S_{\text{ghost}} = \frac{1}{2\pi} \int d^2 \sigma \sqrt{\gamma} \gamma^{\alpha \beta} c^\gamma \nabla_\alpha b_{\beta \gamma} \quad (22)
\]

the action for the Fadeev-Popov ghost fields, written in a covariant form for completeness. The $c^\gamma$ ghost field is a contravariant vector, while the ghost field $b_{\beta \gamma}$ is a symmetric traceless tensor. Both fields $b, c$ are of course anticommuting (Grassmann) variables, as mentioned previously.

**Quantization of the Ghost Sector.**

We now proceed to discuss in some detail the quantization of the ghost sector of theory, which has crucial implications for the dimensionality of the target space. From (22), the stress tensor of the ghost sector $T^\text{ghost}_{\alpha \beta} \equiv -\frac{2\pi}{\sqrt{\gamma}} \frac{\delta S_{\text{ghost}}}{\delta \gamma^{\alpha \beta}}$ (imposing the conformal gauge fixing (14) at the end) reads:

\[
T^\text{ghost}_{\alpha \beta} = \frac{1}{2} c^\gamma \nabla_{(\alpha} b_{\beta \gamma)} + \nabla_{(\alpha} c^\gamma b_{\beta \gamma)} - \text{trace} \quad (23)
\]

In the light-cone coordinate system the only non-trivial components of $T^\text{ghost}$ are: $T^\text{ghost}_{++}, T^\text{ghost}_{--}$:

\[
\begin{align*}
T^\text{ghost}_{++} &= \frac{1}{2} c^+ \partial_+ b_{++} + (\partial_+ c^+) b_{++}, \\
T^\text{ghost}_{--} &= \frac{1}{2} c^- \partial_- b_{--} + (\partial_- c^-) b_{--}. \quad (24)
\end{align*}
\]

Canonical quantization of ghost fields imply the following anticommutation relation [1]:

\[
\begin{align*}
\{ b_{++}(\sigma, \tau), c^+(\sigma', \tau) \} &= 2\pi \delta(\sigma - \sigma') , \\
\{ b_{--}(\sigma, \tau), c^-(\sigma', \tau) \} &= 2\pi \delta(\sigma - \sigma') . \quad (25)
\end{align*}
\]

In what follows, for simplicity, we concentrate on the open string case. Comments on the closed strings will be made where appropriate. The interested reader can find details on
this case in the literature \[1\]. In terms of ghost-field oscillation modes:

\[
\begin{align*}
  c^+ &= \sum_{n=-\infty}^{+\infty} c_n e^{-in(\tau+\sigma)} , \\
  c^- &= \sum_{n=-\infty}^{+\infty} c_n e^{-in(\tau-\sigma)} , \\
  b_{++} &= \sum_{n=-\infty}^{+\infty} b_n e^{-in(\tau+\sigma)} , \\
  b_{--} &= \sum_{n=-\infty}^{+\infty} b_n e^{-in(\tau-\sigma)} ,
\end{align*}
\]

one has the following anticommutation relations:

\[
\begin{align*}
  \{c_n, b_m\} &= \delta_{m+n} , \\
  \{c_n, c_m\} &= \{b_n, b_m\} = 0
\end{align*}
\]

(27)

Using the Fourier modes of \( T^{\text{ghost}} \) at \( \tau = 0 \):

\[
L^{\text{ghost}}_m = \frac{1}{\pi} \int_{-\pi}^{\pi} e^{im\sigma} T^{\text{ghost}}_{++}
\]

(28)

we have:

\[
L^{\text{ghost}}_m = \sum_{n=-\infty}^{\infty} [m(J - 1) - n] b_{m+n} c_{-n}
\]

(29)

where \( J \) is the conformal spin of the field \( b \), with \( 1 - J \) that of the field \( c \).

[\text{NB1: For completeness we note that conformal dimensions are defined as follows (open string case for definiteness): consider a local operator on the world sheet } \mathcal{F}(\sigma, \tau). \text{ Set } \sigma = 0 \text{ (or } \sigma = \pi, \text{ the position of the boundaries of the open string) and study } \mathcal{F}(0, \tau) \equiv \mathcal{F}(\tau). \text{ Then, } \mathcal{F}(\tau) \text{ is defined to have conformal dimension (or 'spin') } J \text{ if and only if, under an arbitrary change of variables } \tau \rightarrow \tau'(\tau), \mathcal{F}(\tau) \text{ transforms as:}

\[
\mathcal{F}'(\tau') = \left( \frac{d\tau}{d\tau'} \right)^J \mathcal{F}(\tau)
\]

(30)

The operators \( L^{\text{ghost}}_m \) in (28) are the generators of the infinite-dimensional Virasoro algebra. The action of \( L_m \) on \( \mathcal{F} \) is:

\[
[\mathcal{L}_m, \mathcal{F}(\tau)] = e^{im\tau} \left( -i \frac{d}{d\tau} + mJ \right) \mathcal{F}(\tau)
\]

(31)

or in terms of modes:

\[
[\mathcal{L}_m, \mathcal{F}] = [m(J - 1) - n]\mathcal{F}_{m+n}
\]

(32)

Note for completeness that for closed strings there is a second set of ghost Virasoro generators.\]
The Virasoro algebra of $L_m^{\text{ghost}}$ is defined by the respective commutation relations:

$$[L_m^{\text{ghost}}, L_n^{\text{ghost}}] = (m-n)L_{m+n}^{\text{ghost}} + A(m)^{\text{ghost}}\delta_{m+n}$$

where the second term on the right-hand-side is a "conformal anomaly term", indicating the breakdown of conformal symmetry at a quantum $\sigma$-model level. It can be calculated to be:

$$A(m)^{\text{ghost}} = \frac{1}{12} [1 - 3(2J-1)^2] m^3 + \frac{1}{6} m$$

[NB2: The easiest way to evaluate the anomaly is to look at specific matrix elements, e.g. :

$$A(1)^{\text{ghost}} = \langle 0 | [L_1^{\text{ghost}}, L_{-1}^{\text{ghost}}] | 0 \rangle.$$]

The ghost field $b$ has $J = 2$, so that the anomaly in the ghost sector is:

$$A(m)^{\text{ghost}} = \frac{1}{6} (m - 13m^3)$$

Similar quantization conditions characterize the matter sector of the $\sigma$-model ([3]), pertaining to the fields/coordinates $X^M$. We shall not do the analysis here. The interested reader is referred for details and results in the literature [1]. Adding such ghost and matter contributions, the total conformal anomaly (for a $D$-dimensional target space time) is [1] is found as follows: first we note that the Virasoro generators corresponding to $S_{\text{total}} = S_{\sigma} + S^{\text{ghost}}$, are the Fourier modes $L_m = \frac{1}{\pi} \int_{-\pi}^{\pi} d\sigma e^{im\sigma} T_{++}^{\text{total}}$, where $T_{++}^{\text{total}} = -\frac{2\pi}{\gamma} \delta_{\gamma++} |_{\gamma++=0}$, and $L_m = L_m^{\text{matter}} + L_m^{\text{ghost}} - a\delta_m$, and we have shifted the definition of $L_0$ (related to the Hamiltonian of the string) so that the zeroth-order Virasoro constraint is $L_0 = 0$. Then, following similar mode expansions for the matter sector, as those of the ghost sector outlined above, one arrives at the total conformal anomaly:

$$A(m) = \frac{D}{12} (m^3 - m) + \frac{1}{6} (m - m^3) + 2am$$

where $D$ is the target-space dimensionality (corresponding to the contributions from $D$ $\sigma$-model "mater" fields $X^M$). From (36) one observes that the anomaly VANISHES, and thus conformal world-sheet symmetry is a good symmetry at a quantum $\sigma$-model level, as required for mathematical self-consistency of the theory, if and only if:

$$D_c = 26 \quad \text{(Bosonic String)} \quad a = 1$$

For fermionic (supersymmetric strings (cf below)) the critical space-time dimension is $D_c = 10$.

### 2.3 Some Hints towards Supersymmetric Strings

So far we have examined Bosonic strings. Supersymmetric strings are more relevant for particle phenomenology, because as we shall discuss now, do not suffer from vacuum instabilities like the bosonic counterparts, which are known to contain in their spectrum
tachyons (negative mass squared modes). Moreover such theories are capable of incorporating fermionic target-space backgrounds.

There are two ways to include fermionic backgrounds in a $\sigma$-model string theory, and thus to achieve target-space supersymmetry:

1. The first one is to supersymmetrize the world-sheet theory by introducing fermionic partners $\psi^M(\sigma, \tau)$ to the $X^M(\sigma, \tau)$ fields. There are two kinds of fermions that can be introduced, depending on their boundary conditions (b.c.) on a circle, so that the world-sheet fermion action is invariant under periodic identification on a cylinder $\sigma \rightarrow \sigma + 2\pi$:

$$
\psi^M(\sigma = 0) = -\psi^M(\sigma = 2\pi) \quad \text{antiperiodic b.c.: Neveu - Schwarz (NS)},
$$
$$
\psi^M(\sigma = 0) = \psi^M(\sigma = 2\pi) \quad \text{periodic b.c.: Ramond (R)} \quad (38)
$$

As a result of the presence of these extra degrees of freedom, world-sheet supersymmetry leads to a reduction of the critical target-space dimension, for which the conformal anomaly is absent, from 26 to 10 (i.e. the critical target-space dimensionality of a superstring is 10).

A world-sheet supersymmetric $\sigma$-model does not have manifest supersymmetry in target space; the latter is obtained after appropriate spectrum projection (Goddard, Scherk and Olive [1]).

2. The second way of introducing fermionic backgrounds in string theory is to have bosonic world sheets but with manifest target-space Supersymmetry (Green and Schwarz).

The two methods are equivalent, as far as target-space Supersymmetry is concerned.

Features of Supersymmetric Strings.

- (i) The tachyonic instabilities in the spectrum, which plagued the Bosonic string, are absent in the supersymmetric string case. This stability of the superstring vacuum is one of the most important arguments in favour of (target-space) supersymmetry from the point of view of string theory.

- (ii) From a world-sheet viewpoint, in the Neveu-Schwarz-Ramond formulation of fermionic strings, the world-sheet action becomes a curved two-dimensional locally supersymmetric theory (world-sheet supergravity theory).

- (iii) Target Supersymmetry is broken in general when one considers strings at finite temperatures, obtained upon appropriate compactification of the target-space coordinate. In general, however, the breaking of target supersymmetry at zero temperature, so as to make contact with realistic phenomenologies, is an open issue at present, despite considerable effort and the existence of many scenarios.

### 2.4 Kaluza-Klein Compactification

The fact that the target space-time dimensionality of strings turns out to be higher than four implies the need for compactification of the extra dimensions.
Compactification means that the ground state of string theory has the form:

\[ \mathcal{M}^{(4)} \otimes \mathcal{K} \]  

(39)

where \( \mathcal{M}^{(4)} \) is a four-dimensional non-compact manifold (assumed Minkowski, but in fact it can be any other space-time encountered in four-dimensional general relativity, provided it satisfies certain consistency conditions to be discussed below), and \( \mathcal{K} \) is a compact manifold, six dimensional in the case of superstrings, or 22 dimensional in the case of (unstable) Bosonic strings.

In “old” (conventional) string theory [1], the “size” of the extra dimensions is assumed Planckian, something which in the modern brane version is not necessarily true. For our purposes in these Lectures we shall restrict ourselves to the “old” string theory approach to compactification.

Consider a 26-(or 10-)dimensional metric on \( \mathcal{M}^{(4)} \otimes \mathcal{K} \), \( g_{MN} \), and let \( g_{\mu\nu} \in \mathcal{M}^{(4)} \), and \( g_{ij} \in \mathcal{K} \).

From a four-dimensional point of view \( g_{ij} \) appear as massless spin-one particles, i.e. massless gauge bosons. This is the central point of Kaluza-Klein (KK) approach. Such particles appear if a suitable subgroup of the underlying ten-dimensional general covariance is left unbroken under compactification to \( \mathcal{M}^{(4)} \otimes \mathcal{K} \). Let us see this in some detail.

Consider a general coordinate transformation on the manifold \( \mathcal{K} \):

\[ y^k \rightarrow y^k + \epsilon V^k(y^j) \]  

(40)

where \( \epsilon \) is a small parameter, and \( V^k \) a vector field. In the passive frame, the corresponding change of the metric tensor \( g_{ij} \) is:

\[ \delta g_{ij} = \epsilon (\nabla_i V_j + \nabla_j V_i) \]  

(41)

where \( \nabla_i \) is the gravitational covariant derivative. The metric on \( \mathcal{K} \) is therefore invariant if \( V^k \) obeys a Killing-vector equation:

\[ \nabla_i V_j + \nabla_j V_i = 0 \]  

(42)

Thus, the coordinate transformation (40), generated by the Killing vector \( V^k \), is a symmetry of any generally-covariant equation for the metric of \( \mathcal{K} \). More generally, if one studies an equation involving a coupled system of the metric with some other matter fields (e.g. gauge fields etc.), then one obtains a symmetry if \( V^k \) can be combined with a suitable transformation of the matter fields that leaves their expectation values invariant.

Consider the case in which one has several Killing vector fields \( V^i_a, a = 1, \ldots N \), generating a Lie algebra \( \mathcal{H} \) of some kind:

\[ \left[ V^i_a \partial_i, V^j_b \partial_j \right] = f_{abc} V^k_c \partial_k \]  

(43)

where \( f_{abc} \) are the corresponding structure constants of the Lie algebra that generates a symmetry group \( \mathcal{H} \) on \( \mathcal{K} \).
Consider the transformation

\[(x^\mu, y^k) \rightarrow (x^\mu, y^k + \sum_a \epsilon_a V_a^k)\]  \hspace{1cm} (44)

In the general case one may consider non-constant \(\epsilon_a = \epsilon_a(x^\mu)\) on \(\mathcal{M}^{(4)}\).

At long wavelengths, which are of interest to any low-energy observer, only massless modes are important. Therefore, the transformation (44) will be a symmetry of the theory compactified on \(\mathcal{M}^{(4)} \otimes \mathcal{K}\). From the point of view of the four-dimensional effective low-energy theory the transformations (44) will look like \(\mathcal{M}^{(4)}\)-dependent local gauge transformations with gauge group \(\mathcal{H}\). The effective four-dimensional theory will therefore have massless gauge bosons given by the ansatz:

\[g_{\mu j} = \sum_a A^a_\mu(x^\nu) V_{ja}(y^k)\]  \hspace{1cm} (45)

where \(A^a_\mu(x^\nu)\) are the massless gauge fields that appear in \(\mathcal{M}^{(4)}\). This follows from the fact that under (44), the fields \(A^a_\mu\) in (45) transform as ordinary gauge fields: \(\delta A^a_\mu = \partial_\mu \epsilon^a + f^{abc} \epsilon_b A^c_\mu\).

An interesting question arises at this point as to what symmetry groups can arise via KK compactification. This is equivalent to asking what symmetry groups an \(n\)-dimensional manifold can have.

We consider for completeness the case where \(\mathcal{K}\) has dimension \(n\), which is kept general at this point. The most general answer to the above question is complicated. An interesting question, of phenomenological interest, is for which \(n\) one can get the standard model group \(SU(3) \otimes SU(2) \otimes U(1)\). It can be shown \([1]\) that this happens for \(n = 7\) which is not the case of string theory (superstrings), since in that case \(n = 6\). This is what put off people’s interest in the traditional KK compactification, which was instead replaced by the heterotic string construction, which we shall not analyse here \([1]\). On the other hand, it should be mentioned that in the modern version of string theory, involving branes, KK modes play an important rôle again. For more details we refer the reader to the lectures on brane theory in this School.

### 2.5 Strings in Background Fields

So far we have dealt with flat Minkowski target space times. In general strings may be formulated in curved space times, and, in general, in the presence of non-trivial background fields. In this case conformal invariance conditions of the underlying \(\sigma\)-model theory become equivalent, as we shall discuss below, to equations of motion of the various target-space background fields.

The lowest lying energy multiplet in superstring theory consists (in its bosonic part) of massless states of gravitons \(g_{MN}\) (spin two traceless and symmetric tensor field), dilaton \(\Phi\) (scalar, spin 0) and antisymmetric tensor \(B_{MN}\) field \([\|]\). Target space supersymmetry, of

\[\text{2In the Bosonic states the lowest lying energy state (vacuum) is tachyonic, and the above multiplet occurs at the next level.}\]
course, implies the existence of the supersymmetric (fermionic) partners of the states in this multiplet. In this section we shall discuss the formalism, and its physical consequences, for string propagation in the bosonic part of the massless superstring multiplet, starting from graviton backgrounds, which are discussed next.

2.5.1 Formulation of Strings in Curved Space times-Graviton Backgrounds

The corresponding $\sigma$-model action, describing the propagation of a string in a space time with metric $g_{MN}$ reads:

$$S_{\sigma} = \frac{1}{4\pi\alpha'} \int_{\Sigma} d^{2}\sigma \sqrt{\gamma^{\alpha\beta}} g_{MN}(X^{P}(\sigma, \tau)) \partial_{\alpha}X^{M} \partial_{\beta}X^{N}, \quad \alpha, \beta = \sigma, \tau$$

(46)

One expands around a flat target space time $g_{MN} = \eta_{MN} + h_{MN}(X)$. For $|h_{MN}(X)| \ll 1$ one may expand in Fourier series:

$$h_{MN}(X) = \int \frac{d^{D}k}{(2\pi)^{D}} e^{ikM_{\Sigma}} \tilde{h}_{MN}(k)$$

(47)

in which case the $\sigma$-model action becomes schematically:

$$S_{\sigma} = S^{*} + \frac{1}{4\pi\alpha'} \int_{\Sigma} d^{2}\sigma \sqrt{\gamma^{\alpha\beta}} \partial_{\alpha}X^{M} \partial_{\beta}X^{N} \int \frac{d^{D}k}{(2\pi)^{D}} e^{ikM_{\Sigma}} \tilde{h}_{MN}(k) \equiv$$

$$S^{*} + g^{i} \int_{\Sigma} d^{2}\sigma V_{i}$$

(48)

where $S^{*}$ is the flat space-time action $[1]$, and one has the correspondence $g^{j} \leftrightarrow \tilde{h}_{MN}(k)$, $V_{i} \leftrightarrow \sqrt{\gamma^{\alpha\beta}} \partial_{\alpha}X^{M} \partial_{\beta}X^{N} e^{ikpM} x^{p}$, and $\sum_{i} \leftrightarrow \int \frac{d^{D}k}{(2\pi)^{D}}$.

It should be stressed that implementing a Fourier expansion necessitates an expansion in the neighborhood of the Minkowski space time, so as to be able to define plane waves appropriately. For generic space times one may consider an expansion about an appropriate conformal (fixed point) $\sigma$-model action $S^{*}$, as in the last line of the right-hand-side of (48), but in this case the set of background fields/\sigma-model couplings $\{g^{i}\}$ is found as follows: consider $g_{MN} = g^{*}_{MN} + h_{MN}(X)$, where $g^{*}_{MN}$ a conformal (fixed-point) non-flat metric, and $h_{MN}(X)$ an expansion around it. Then,

$$S = \frac{1}{4\pi\alpha'} \int_{\Sigma} d^{2}\sigma \partial_{\alpha}X^{M} \partial_{\beta}X^{N} g_{MN} \gamma^{\alpha\beta} \sqrt{\gamma} =$$

$$S^{*} + \frac{1}{4\pi\alpha'} \int_{\Sigma} d^{2}\sigma h_{MN}(X) \partial_{\alpha}X^{M} \partial_{\beta}X^{N} \gamma^{\alpha\beta} \sqrt{\gamma} =$$

$$S^{*} + \frac{1}{4\pi\alpha'} \int_{\Sigma} d^{2}\sigma \int d^{D}y \sqrt{g^{*}(y)\delta^{(D)}} \left( g^{M}_{\Sigma} - X^{M}(\sigma, \tau) \right) \left( h_{MN}(y) \partial_{\alpha}X^{M} \partial_{\beta}X^{N} \gamma^{\alpha\beta} \sqrt{\gamma} \equivight.$$}

$$S^{*} + g^{i} \int_{\sigma} d^{2}\sigma V_{i}$$

(49)
where \( g^i \leftarrow \{ h_{MN}(y) \} \), \( V_i \leftarrow \delta^{(D)} \left( y^M - X^M(\sigma,\tau) \right) \partial_\alpha X^M \partial_\beta X^N \gamma^{\alpha\beta} \sqrt{\gamma} \), and \( \Sigma_i \leftarrow \int d^D y \sqrt{g^*}(y) \). As the reader must have noticed, for general backgrounds one pulls out the world-sheet zero mode of \( X^M \) appropriately, which defines the target-space coordinates, and integrates over it, thereby determining the (infinite dimensional) set of \( \sigma \)-model couplings.

### 2.5.2 Other Backgrounds

We continue our discussion on formulating string propagation in non-trivial backgrounds, in the first-quantized formalism, by studying next antisymmetric tensor and dilatons.  

#### Antisymmetric Tensor Background

The antisymmetric tensor backgrounds \( B_{MN} \) are spin one, antisymmetric tensor fields \( B_{MN} = -B_{NM} \). There is an Abelian gauge symmetry which characterizes the corresponding scattering amplitudes (with antisymmetric tensors as external particles),

\[
B_{MN} \rightarrow B_{MN} + \partial_M \Lambda_N
\]

which implies that the corresponding low-energy effective action, which reproduces the scattering amplitudes, will depend only through the field strength of \( B_{MN} \): \( H_{MNP} = \partial_M B_{NP} \).

In a \( \sigma \)-model action the pertinent deformation has the form:

\[
\frac{1}{4\pi \alpha'} \int_{\Sigma} d^2 \sigma B_{MN} V^{(B)MN} = \frac{1}{4\pi \alpha'} \int_{\Sigma} d^2 \sigma B_{MN} \epsilon^{\alpha\beta} \partial_\alpha X^M \partial_\beta X^N
\]

where \( \epsilon^{\alpha\beta} \) is the contravariant antisymmetric symbol.

[NB3: due to its presence there is no explicit \( \sqrt{\gamma} \) factor in (51), as this is incorporated in the contravariant \( \epsilon \)-symbol.]

#### Dilaton Backgrounds and the String Coupling

The dilaton \( \Phi(X) \) is a spin-0 mode of the massless superstring multiplet, which in a \( \sigma \)-model framework couples to the world-sheet scalar curvature \( R^{(2)}(\sigma,\tau) \):

\[
S_\sigma = \frac{1}{4\pi \alpha'} \int_{\Sigma} d^2 \sigma \sqrt{\gamma} \epsilon^{\alpha\beta} \partial_\alpha X^M \partial_\beta X_M + \frac{1}{4\pi} \int_{\Sigma} d^2 \sigma \sqrt{\gamma} \Phi(X) R^{(2)}(\sigma,\tau)
\]

Notice that in the dilaton term there is no \( \alpha' \) factor, which implies that in a (perturbative) series expansion in terms of \( \alpha' \) the dilaton couplings are of higher order as compared with the graviton and antisymmetric tensor backgrounds.

An important rôle of the dilaton is that it determines (via its vacuum expectation value) the strength of the string interactions, the string coupling:

\[
\text{String Coupling } g_s = e^{\langle \Phi \rangle}
\]

where \( \langle \ldots \rangle = \int DXE^{S_\sigma} \) is computed with respect to the string path integral for the \( \sigma \)-model propagating in the background under consideration.
The string coupling is a string-loop counting parameter (c.f. figure 3). This can be seen easily by first recalling the index theorem (6) that connects a geometrical world-sheet quantity like the curvature $R^{(2)}$ to a topological quantity, the Euler characteristic $\chi$, which counts the genus of the surface:

$$\chi = 2 - \text{no. of holes} - 2 \times \text{no. of handles} = \frac{1}{4\pi} \int_{\Sigma} \sqrt{\gamma} R^{(2)}$$

Consider the $\sigma$-model deformation (52) and split the dilaton into a classical (world-sheet coordinate independent) part $<\Phi>$ and a quantum part $\varphi \equiv :\Phi:$, where $:\ldots:\$ denotes appropriate normal ordering of the corresponding operators: $\Phi = <\Phi> + \varphi(\sigma, \tau)$. Using (54), (53), we can then write for the $\sigma$-model partition function summed over surfaces of genus $\chi$:

$$Z = \sum_{\chi} \int D\mathcal{X} e^{-S^{\text{rest}} - \chi <\Phi> - \frac{1}{16\pi} \int_{\Sigma} d^{2}\sigma \sqrt{\gamma} \varphi R^{(2)}} =$$

$$\sum_{\chi} g_s^{-\chi} \int D\mathcal{X} e^{-S^{\text{rest}} - \frac{\chi}{16\pi} \int_{\Sigma} d^{2}\sigma \sqrt{\gamma} \varphi R^{(2)}}$$

(55)
2.6 Conformal Invariance and Background Fields

The presence of $\sigma$-model "deformations" $g^i \int \Sigma V_i$ imply in general deviations from conformal invariance on the world sheet. To ensure conformal invariance we must impose certain conditions on the couplings $g^i$. Such conditions, and their implications will be studied in this section. As we shall see, the conformal invariance conditions are equivalent to equations of motion for the target-space background $g^i$ which are derived from a target-space string effective action. This action constitutes the low-energy (field-theory) limit of strings and will be the main topic of these lectures. String cosmology, which we shall discuss in the second and third lectures, will be based on such string effective actions.

To start with, let us consider a deformed $\sigma$-model action

$$S = S^* + g^i \int \Sigma d^2\sigma V_i$$

which, as we have discussed above, describes propagation (in a first quantized formalism) of a string in backgrounds $\{g^i\} = \{g_{MN}, \Phi, B_{MN}, \ldots\}$.

The partition function of the deformed string may be expanded in an (infinite) series in powers of $g^i$ (assumed weak):

$$Z[g] = \int \mathcal{D}\rho \mathcal{D}X e^{-S^* - g^i \int \Sigma d^2\sigma V_i} = \sum_i \int \cdots \int_{\Sigma} \langle V_{i_1} \ldots V_{i_N} \rangle^* g^{i_1} \ldots g^{i_N} d^2\sigma_1 \ldots d^2\sigma_N$$

where $\langle \ldots \rangle^* = \int \mathcal{D}\rho \mathcal{D}X e^{-S^*}$. We work in the conformal gauge (5), and thus the mode $\rho$ is whatever is left from the integration over world-sheet geometries. In conformal (‘critical’) string theory the quantities $\langle V_{i_1} \ldots V_{i_N} \rangle^*$ are nothing other than the string scattering amplitudes (defining the on-shell S-matrix elements) for the modes corresponding to $\{g^i\}$. It must be stressed that critical string theory is by definition a theory of the S-matrix, and hence this imposes a severe restriction on the appropriate backgrounds. Namely, as we shall discuss in Lecture 3, appropriate string backgrounds are those which can admit asymptotic states, and hence well-defined on-shell S-matrix elements.

As a two-dimensional quantum field theory, the model (56) suffers from world-sheet ultraviolet (short-distance) divergences, which should not be confused with target-space ultraviolet infinities. Such world-sheet infinities arise from short-distance regions

$$\lim_{\sigma_1 \to \sigma_2} \langle V_{i_1}(\sigma_1) \ldots V_{i_2}(\sigma_2) V_{i_3}(\sigma_3) \ldots V_{i_N}(\sigma_N) \rangle^*$$

and they are responsible for the breaking of the conformal invariance at a quantum level, because they require regularization, and regularisation implies the existence of a length (short-distance) cutoff. The presence of such length cutoff regulators break the local (and global) scale invariance in general. Below we shall seek conditions under which the conformal invariance is restored.

\[3\] Eternally accelerating string Universe backgrounds, for instance, which will be the topic of discussion in the last part of our lectures, are incompatible with critical string theory, precisely because of this, namely in such backgrounds one cannot define appropriate asymptotic pure quantum states. We shall discuss how such problems may be overcome in the last part of the lectures.
To this end, we first observe that, according to the general case of renormalizable quantum field theories, one of which is the $\sigma$-model two-dimensional theory (56), such infinities may be absorbed in a renormalization of the string couplings. To this end, one adds appropriate counterterms in the $\sigma$-model action, which have the same form as the original (bare) deformations, but they are renormalization-group scale dependent. Therefore their effect is to ‘renormalise’ the couplings $g_i \rightarrow g_i^R(\ln \mu)$, where $\mu$ is a world-sheet renormalization group scale.

The scale defines the $\beta$-functions of the theory:

$$
\beta^i \equiv \frac{dg^i_R}{d\ln \mu} = \sum_{i_n} C^i_{i_1...i_n} g^{i_1}_R \cdots g^{i_n}_R
$$

One can show in general that the (2d-gravitational) trace $\Theta \equiv T_{\alpha\beta}\gamma^{\alpha\beta}$ of the world-sheet stress tensor in such a renormalized theory can be expressed as:

$$
\langle \Theta \rangle = c R^{(2)} + \beta^i \langle V_i \rangle
$$

where $c$ is the conformal anomaly of the world-sheet theory, and $R^{(2)}$ is the world-sheet curvature. In the case of strings living in their critical dimension, the total conformal anomaly $c$, when Fadeev-Popov contributions are taken into account vanishes, as we have seen in the beginning of this lecture. Thus to ensure conformal invariance in the presence of background fields $g^i$, i.e. $\langle \Theta \rangle = 0$ one must impose

$$
\beta^i = 0
$$

These are the conformal invariance conditions, which in view of (58) imply restrictions on the background fields $g^i$.

A few comments are important at this point before we embark on a discussion on the physical implications for the target-space theory of the conditions (60). The comments concern the geometry of the ‘space of coupling constants $\{g^i\}$’, so called moduli space of strings, or string theory space. As discussed first by Zamolodchikov [3], such a space is a metric space, with the metric being provided by the two-point functions of vertex operators $V_i$ in the deformed theory,

$$
G_{ij} = z^2 \bar{z}^2 \langle V_i(z, \bar{z})V_j(0, 0) \rangle_g
$$

where $z, \bar{z}$ are complex coordinate of a Euclidean world sheet, which is necessary for convergence of our path integral formalism. The notation $\langle \ldots \rangle_g$ denotes path integral with respect to the deformed $\sigma$-model action (59) in the background $\{g^i\}$. The metric (61) acts as a raising and lowering indices operator in $g^i$-space.

An important property of the stringy $\sigma$-model $\beta$-functions is the fact that the ‘covariant’ $\beta$-functions, defined as $\beta_i = G_{ij} \beta^j$, when expanded in powers of $g^i$ have coefficients completely symmetric under permutation of their indices, i.e.

$$
\beta_i = G_{ij} \beta^j = \sum_{i_n} c_{i_1i_2...i_n} g^{i_1} \cdots g^{i_n}
$$
with $c_{i_1i_2...i_n}$ totally symmetric in the indices $i_j$. This can be proven by using specific properties of the world-sheet renormalization group \cite{4}. Such totally symmetric coefficients are associated with dual string scattering amplitudes, as we shall demonstrate explicitly later on.

What (62) implies is a gradient flow property of the stringy $\beta$-functions, namely that

$$\frac{\delta C[g]}{\delta g^i} = G_{ij} \beta^j \quad (63)$$

where $C[g]$ is a target-space space-time integrated functional of the fields $g^i(y)$.

Notice that the conformal invariance conditions (58) are then equivalent to equations of motion obtained from this functional $C[g]$, which thus plays the rôle of a target-space effective action functional for the low-energy dynamics of string theory.

An important note should be made at this point, concerning the rôle of target-space diffeomorphism invariance in stringy $\sigma$-models. As a result of this invariance, which is a crucial target-space symmetry, that makes contact with general relativity in the target manifold, the conformal invariance conditions (58) in the case of strings are slightly modified by terms which express precisely the change of the background couplings $g^i$ under general coordinate diffeomorphisms in target space $\delta g^i$:

$$\beta^i = \beta^i + \delta g^i = 0 \quad (64)$$

in other words conformal invariance in $\sigma$-models implies the vanishing of the modified $\beta$-functions, i.e. it is valid up to general coordinate diffeomorphism terms. This modification plays an important rôle in ensuring the compatibility of the solutions with general coordinate invariance of the target manifold. The modified $\beta$-functions $\beta^i$ are known in the string literature as Weyl anomaly coefficients \cite{1}. In fact, for the stringy $\sigma$-model case, they appear in the expression (59), in place of the ordinary $\beta^i$.

### 2.7 General Methods for Computing $\beta$-functions

In general there are two kinds of perturbative expansions in $\sigma$-model theory.

- **(I) Weak Coupling $g'$-expansion**: in which one assumes weak deformations of conformal $\sigma$-model actions, with $g'$ small enough so as a perturbative series expansion in powers of $g'$ suffices. Usually in this method one deals with Fourier modes (cf below) of background deformations, and hence the results are available in target-momentum space; this is appropriate when one considers scattering amplitudes of strings.

- **(II) $\alpha'$-Regge slope expansion**: in which one considers an expansion of the partition function and correlation functions of $\sigma$-models in powers of $\alpha'$. Given that the Regge slope has dimensions of [length]$^2$, such expansions imply (in Fourier space) appropriate derivative expansions of the string effective actions. It is the second expansion that will be directly relevant for our Cosmological considerations. The Regge slope expansion preserves general covariance explicitly.
It should be stressed that physically the two methods of expansion are completely equivalent. Formally though, as we have mentioned, the various methods may have advantages and disadvantages, compared to each other, depending on the physical problem at hand. For instance when one deals with weak fields, then the first method seems appropriate. In field theory limit of strings, on the other hand, where by definition we are interested in low-energies compared with the string (Planckian $\sim 10^{19}$ GeV) scale, then the second expansion is more relevant. Moreover it is this method that allows configuration-space general covariant expressions for the effective action in arbitrary space-time backgrounds, in which momentum space may not always be a well-defined concept.

Before we turn into an explicit discussion on string effective actions we consider it as instructive to discuss, through a simple but quite generic example, the connection of conformal invariance conditions to string scattering amplitudes through the first method.

2.7.1 String Amplitudes and World-Sheet Renormalization Group

A generic structure of a renormalization-group $\beta$-function in powers of the renormalized couplings $g^i(t)$ is:

$$\beta^i = \frac{dg^i}{dt} = y_i g^i + \alpha^i_{jk} g^j g^k + \gamma^i_{jkl} g^j g^k g^l + \ldots \quad t = \ln \mu$$ \hspace{1cm} (65)

where $y_i$ are the anomalous dimensions, and no summation over the index $i$ is implied in the first term. Summation over repeated indices in the other terms is implied as usual. The bare couplings are the ones for which $t = 0$, $g^i(0) \equiv g^i_0$. The perturbative solution of (65), order by order in a power series in $g^i$, is:

- **First Order**:
  $$g^i(t) = e^{y_i t} g^i(0) .$$ \hspace{1cm} (66)

- **Second Order**
  $$g^i(t) = e^{y_i t} g^i(0) + \delta^i_{jk} g^j(0) g^k(0) ,$$ \hspace{1cm} (67)

  with $\delta^i_{jk} \equiv \frac{d}{dt} \delta^i_{jk} = \alpha^i_{jk} e^{y_j t} e^{y_k t} + y_i \delta^i_{jk}$; \quad $\delta^i_{jk}(0) = 0$, from which:

  $$\delta^i_{jk}(t) = \left( e^{(y_j+y_k)t} - e^{y_i t} \right) \frac{\alpha^i_{jk}}{y_j + y_k - y_i}$$ \hspace{1cm} (68)

and so on. Notice from the expression for the second order terms the resemblance of the anomalous-dimension denominators with “energy denominators” in scattering amplitudes. As we shall discuss below this is not a coincidence; it is a highly non-trivial property of string renormalization group to have a close connection with string scattering amplitudes. We shall explain this through a simplified but quite instructive, and in many respects generic, example, that of an open Bosonic string in a tachyonic background [5].

Open Strings in Tachyonic Backgrounds: Weak Field Expansion
The $\sigma$-model action, for an open string propagating in flat space time in a tachyon background $T(X)$, is:

$$S_{\text{open}} = \frac{1}{4\pi} \int dxdy \eta^{\alpha\beta} \partial_{\alpha} X^M \partial_{\beta} X^N \eta_{MN} + \int_{-\infty}^{\infty} \frac{dx}{a} \int d^{26}k \tilde{T}(k)e^{ikXM}$$

(69)

where we work in units of $\alpha' = 1$, and $a$ is a length scale, which will play the role of a short-distance cut-off scale. Notice that the world-sheet is taken here to be the upper half plane for simplicity. The open string interactions occur at the world-sheet boundary, and this is expressed by the fact that the tachyonic background term is over the real $x$ axis.

We apply the background field method for quantization, according to which we split the fields $X^M = X_0^M + \xi^M$, where $X_0^M$ satisfies the classical equations of motion, and varies slowly with respect to the cut-off scale $a$. The effective action is defined as $S_{\text{eff}}[X_0] = -\ln W[X_0]$, where $W[X_0]$ is the partition function of the $\sigma$-model (69):

$$W[X_0] = \int \mathcal{D}\xi e^{-\frac{1}{4\pi} \int_{y>0} dxdy \eta^{\alpha\beta} \partial_{\alpha} X^M \partial_{\beta} X^N \eta_{MN}} e^{-\int_{-\infty}^{\infty} \frac{dx}{a} \int d^{26}k \tilde{T}(k)e^{ikX_0}e^{ik\xi}}$$

(70)

where $dk \equiv \frac{d^{26}k}{(2\pi)^{26}}$ is the target momentum space integration. Using the free-field contraction, with the scale $a$ as a short-distance regulator,

$$\langle \xi(x_1)\xi(x_2) \rangle_* = -2\ln (|x_1 - x_2| + a)$$

(71)

where * denotes free-field $\sigma$-model action (in flat target space), and expanding the $\sigma$-model partition function in powers of $\tilde{T}(k)$, we obtain the following results, order by order in the weak-field (tachyon) expansion:

**Linear order in $\tilde{T}(k)$:** to this order, the partition function $W[X_0]$ becomes:

$$W[X_0]^{(1)} = -\int_{-\infty}^{\infty} \frac{dx}{a} \int d^{26}k \tilde{T}(k)e^{ikX_0}\langle e^{ik\xi} \rangle_* =$$

$$-\int_{-\infty}^{\infty} dx \int dka^{k^2-1}\tilde{T}(k)e^{ikX_0}$$

(72)

where we used the free-field contraction (71). The scale $a$-dependence may be absorbed in a renormalization of the coupling $\tilde{T}(k)$:

$$\tilde{T}_R(k) \equiv a^{k^2-1}\tilde{T}(k)$$

(73)

Comparing with (69), we observe that one may identify $a = e^{-t}$, $t$ the renormalization-group (RG) scale, from which one obtains the $\beta$-function:

$$\beta^T(k) = -\frac{d\tilde{T}_R(k)}{dlna} = -(k^2 - 1)\tilde{T}_R(k)$$

(74)

Comparison with (63), then, indicates that the anomalous dimension is $k^2 - 1$. 

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The conformal invariance conditions (30) amount to the vanishing of the $\beta$-function, which thus turns out to be equivalent to the \textit{on-shell} condition for tachyons:

$$-(k^2 - 1)\tilde{T}_R(k) = 0 \rightarrow k^2 = 1$$

(75)

This is the first important indication that the conformal invariance conditions of the stringy $\sigma$-model imply important restrictions for the dynamics of the background over which it propagates.

Less trivial consequences for the background become apparent if one examines the next order in the expansion in powers of $\tilde{T}$.

**Quadratic Order in $\tilde{T}$:** to this order the partition function $W[X_0]$ reads:

$$W[X_0]^{(2)} = \int_{-\infty}^{+\infty} \frac{dx_1}{a} \int_{-\infty}^{+\infty} \frac{dx_2}{a} \int dk_1 \tilde{T}(k_1) \int dk_2 \tilde{T}(k_2) \cdot e^{ik_1 \cdot X_0(x_1) + ik_2 \cdot X_0(x_2)} \langle e^{ik_1 \cdot \xi(x_1) + ik_2 \cdot \xi(x_2)} \rangle.$$ 

(76)

Since $X_0$ varies slowly, one may expand à la Taylor: $X_0(x_2) = X_0(x_1) + (x_2 - x_1)X_0'(x_1) + \ldots \simeq X_0(x_1)$ to a good approximation.

Implementing the free-field contraction (71), and performing straightforward algebraic manipulations, we arrive at integrals of the form:

$$a^{k_1^2 + k_2^2 - 2} \int_{-\infty}^{x_1} dx_2 (x_1 - x_2 - a)^{2k_1 \cdot k_2} = -a^{(k_1 + k_2)^2 - 1} \frac{1}{2k_1 \cdot k_2}$$

(77)

The integral converges for

$$2k_1 \cdot k_2 - 1 < 0$$

(78)

Absorbing the scale dependence in renormalized tachyons, as before, one obtains to second order:

$$\tilde{T}_R(k) = a^{k^2 - 1} \left( \tilde{T}_R(k) + \int dk_1 \int dk_2 \frac{\tilde{T}_R(k_1)\tilde{T}_R(k_2)}{2k_1 \cdot k_2 + 1} \delta^{(26)}(k_1 + k_2 - k) \right)$$

(79)

Comparing (77), (78) with (79), we observe that we are missing the term $a^{-y_1 - y_2} = a^{k_1^2 + k_2^2 - 2}$. To the order we are working, this discrepancy can be justified as follows: removing the cut-off, i.e. going to a non-trivial fixed point $t \rightarrow \infty$, and taking into account the convergence region (78), which implies $y_1 + y_2 < y$, with $y = k^2 - 1$ the anomalous dimension, we observe that in the regime $t \rightarrow \infty$ the missing term is negligible compared with the one which is present, and thus the above computation is consistent with the generic renormalization group analysis, near a non-trivial fixed point. One then defines the $\beta$-functions of the theory, away from a fixed point (in the entire (target) momentum space) by analytic continuation.

Comparing the above results with (77),(78) we then find that to second order:

$$2k_1 \cdot k_2 + 1 = y_1 + y_2 - y, \quad (y_i = 1 - k_i^2),$$

$$\alpha_{k_1, k_2}^k = -\delta^{(26)}(k_1 + k_2 - k)$$

(80)
Conformal Invariance Condition:

\[ g^i_1 = - \frac{1}{y_i} \alpha^i_{jk} g^j_0 g^k_0 \]

Conformal Invariance Condition:

\[ g^i_2 = - \left( \frac{\alpha^i_{jn} \alpha^m_{kn}}{y_n} \right) \gamma^{i j k m} g^j_0 g^k_0 g^m_0 \]

Figure 4: Schematic representation of the equivalence of conformal invariance conditions (vanishing of world-sheet renormalization group \( \beta \)-functions) and on-shell string scattering amplitudes in the case of an open string in a tachyonic background.

The corresponding conformal invariance condition can be found by iterating the one at previous order as follows: the first order result yields \( y_i g^i_0 = 0 \); to second order we write for the coupling \( g^i = g^i_0 + g^i_1 \), which then, on account of the vanishing of the \( \beta \)-function \( \beta^i = y_i g^i + \alpha^i_{jk} g^j g^k + \ldots = 0 \), yields:

\[ g^i_1 = - \frac{1}{y_i} \alpha^i_{jk} g^j_0 g^k_0 \]  \[ (81) \]

The situation is depicted in figure 4. It represents a three-tachyon scattering amplitude, with two external legs set on-shell, and with one propagator pole at \( y_i = 0 \). If one sets this third leg on shell two, then the residue of the pole is the three-on-shell tachyon scattering amplitude.

**Higher orders in** \( \tilde{T}_R(k) \): at the next level one obtains a highly non-trivial demonstration of the above-mentioned equivalence between conformal invariance conditions and on-shell \( S \)-matrix elements. We shall not give details here, as these can be found by the interested reader in the literature [5]. Below we shall only outline the results. Schematically the situation is depicted in fig. 4.
Following a similar treatment as before, but encountering significantly more complex mathematical manipulations, one obtains as a solution of the conformal invariance conditions to this order:

\[ g^i_j = \frac{1}{y_i} \left( \frac{\alpha_j m \alpha_k^m}{y_m} - \gamma^i_{jk\ell} \right) g^k_0 g^\ell_0 g^j_0 \]

where, in the tachyonic background open string case, the contact terms of the graph are:

\[ \gamma^i_{jk\ell} g^j_0 g^k_0 g^\ell_0 = \left( -D^i_{jk\ell} \frac{2\alpha_j m \alpha_k^m}{y_j + y_m - y_i} \right) g^j_0 g^k_0 g^\ell_0 \]

where, in the tachyonic background open string case, the contact terms of the graph are:

\[ D^i_{k_1 k_2 k_3} = \delta^{[26]}(k_1 + k_2 + k_3 - k) \frac{3F_2(1, -1 - B - C, -C, -1 - A - B - C, -B - C; 1)}{y_j + y_m - y_i} \]

with \( 3F_2 \) denoting a hypergeometric function, and \( A = 2k_1 \cdot k_2, B = 2k_1 \cdot k_3, C = 2k_1 \cdot k_3 \), and \( 2 + A + B + C = y_j + y_k + y_\ell - y_i \), with \( y_i \) the anomalous dimensions defined above.

This completes the demonstration on the equivalence of the conformal invariance conditions of a stringy \( \sigma \)-model with string scattering amplitudes. As we have discussed above such amplitudes can be reproduced by a target space diffeomorphism invariant effective action. The form of this action can be most easily obtained if one follows the second method of perturbative expansion for computing the \( \beta \)-functions, the so-called Regge-slope \( \alpha' \) expansion, which from now on we shall restrict ourselves upon. For simplicity, in these lectures we shall restrict ourselves to \( \mathcal{O}(\alpha') \) in this expansion. This will be sufficient for our cosmological considerations. Some comments on higher orders will be made where appropriate.

2.7.2 Regge-slope (\( \alpha' \)) expansion: \( \mathcal{O}(\alpha') \)-Weyl anomaly Coefficients

The second method of perturbative \( \sigma \)-model expansion, which we shall make use of in the context of the present lectures, consists of expanding the partition function, correlation functions and \( \beta \)-functions in powers of \( \alpha' \), or rather in the dimensionless quantity \( \alpha' k^2 \), where \( k^M \) is a target momentum contravariant vector (for open strings the expansion is actually made in powers of \( \sqrt{\alpha' k} \)). The Regge slope \( \alpha' \)-expansion is independent of the \( g^i_j \)-expansion, studied above, but formally it is equivalent to that, in the sense that the exact expressions (resummed to all orders) of the pertinent \( \sigma \)-model partition function in both expansion methods contain the same physical information. In practice, the \( \alpha' \) expansion is appropriate if one is interested, as we are in the cosmological context of these lectures, in long-wavelength (compared to Planck scales) effective actions. In such a case the first few orders in the \( \alpha' \) expansion (actually up to and including \( \mathcal{O}(\alpha') \) ) will suffice to provide an adequate description of the observed Universe, as we shall discuss in Lecture 2.

In these lectures we shall not discuss in detail the very interesting techniques underlying the \( \alpha' \)-expansion of \( \sigma \)-model renormalization-group analysis. The interested reader may find details on this in the vast literature [1]. For our purposes here, we shall merely quote the results for the \( \mathcal{O}(\alpha') \) Weyl anomaly coefficients for Bosonic (or better the bosonic part of) \( \sigma \)-model backgrounds of graviton, antisymmetric tensor and dilaton fields.

For such backgrounds in the Bosonic string case (for definiteness) we have:
• **Graviton:** For the Weyl anomaly coefficient of the graviton background one has:

\[
\hat{\hat{\beta}}^g_{MN} = \alpha' \left( R_{MN} - \frac{1}{4} H^P_{M} H^Q_{NP} + 2 \nabla_{(M} \partial_{N)} \Phi \right)
\]  

where the last part (depending on \( \Phi \)) may be attributed to the diffeomorphism \( \delta g^i \) part of the Weyl anomaly coefficient.

• **Antisymmetric Tensor:** For the antisymmetric tensor backgrounds one finds:

\[
\hat{\beta}_B^B_{MN} = \frac{\alpha'}{2} \left( -\nabla_P H^P_{MN} + 2(\partial_P \Phi) H^P_{MN} \right)
\]  

where again the dilaton (\( \Phi \)) dependent part is attributed to target-space diffeomorphism parts.

• **Dilaton Fields:** For dilaton fields it is convenient, for reasons that will become clear below, to define a Weyl anomaly coefficient with the (target-space) gravitational trace of graviton Weyl anomaly coefficient subtracted:

\[
\tilde{\hat{\beta}}^\Phi = \hat{\beta}^\Phi - \frac{1}{4} g^{MN} \beta^g_{MN} = \frac{\alpha'}{4} \left( -4(\partial_M \Phi)^2 + 4 \nabla^2 \Phi + R - \frac{1}{12} H^2_{MNP} - \frac{2(D - 26)}{3\alpha'} \right)
\]  

Notice in the last expression (85) that the appearance of the scalar curvature is an exclusive consequence of the presence of the trace of the graviton Weyl anomaly coefficient in \( \tilde{\hat{\beta}}^\Phi \). The dilaton Weyl anomaly coefficient, to \( \mathcal{O}(\alpha') \) does not depend on the target-space curvature, only on derivatives of the dilaton field. Moreover, we also notice that to zeroth order in \( \alpha' \), the dilaton Weyl anomaly coefficient does depend on the conformal anomaly \( D - 26 \), which is absent for critical dimension strings. This term, if present, would act as an exponential dilaton potential (or equivalently vacuum energy). In the critical dimension \( D_c = 26 \) (for bosonic strings) is absent. We shall come back to this important issue in our third lecture, when we discuss the issue of cosmological constant in the context of string theory. For superstrings the \( D - 26 \) term is replaced by \( D - 10 \), and the vacuum energy term is absent for the case of critical superstring space-time dimension \( D_c = 10 \).

We now notice that, as can be shown straightforwardly, the vanishing of the above expressions (i.e. the conformal invariance conditions for this set of background fields) corresponds to **equations of motion** of a low-energy \( \mathcal{O}(\alpha') \) target-space effective action:

\[
I_{\text{eff}} = -\frac{1}{2\kappa^2} \int d^D X \sqrt{g} e^{-2\Phi} \left( R + 4(\partial_M \Phi)^2 - \frac{1}{12} H^2_{MNP} - \frac{2(D - 26)}{3\alpha'} + \ldots \right)
\]  

(86)
where $\kappa^2$ is the Gravitational constant in $D$ target space time dimensions (related appropriately to the Planck (or string) mass scale $M_s$).

In fact, as mentioned earlier in the context of $g^i$ weak field expansion, it can also be shown explicitly within the $\alpha'$ expansion \[3\], that the above-mentioned Weyl anomaly coefficients $\hat{\beta}^i$ are gradient flows in $g^i$ space of $I_{\text{eff}}$:

$$ \frac{\delta I_{\text{eff}}}{\delta g^i} = G_{ij} \hat{\beta}^j $$

where, up to appropriate field redefinitions, which are irrelevant from the point of view of scattering amplitudes, as they leave them invariant, the function $G_{ij}$ coincides with the Zamolodchikov metric \[61\].

### 2.7.3 World-Sheet Renormalizability Constraints on the $\hat{\beta}$-functions

The world-sheet renormalizability of the $\sigma$-model action, deformed by background fields $g^i$, i.e. the fact that this two-dimensional theory has ultraviolet divergencies which can be absorbed in appropriate redefinition of its coupling/fields $g^i$, without the necessity for introducing new types of interactions that do not exist in the bare theory, is expressed simply in terms of the renormalization-group scale invariance of the components of the world-sheet stress tensor of the theory:

$$ \frac{d}{d \ln \mu} T_{\alpha\beta} = 0, \quad \alpha, \beta = \sigma, \tau $$

where $\ln \mu$ is the renormalization-group scale.

Equations of the type \[88\] implies severe constraints among the $\hat{\beta}$-functions which, after some elegant $\sigma$-model renormalization-group analysis, are expressed by means of the Curci-Paffuti equation \[7\]. To order $\alpha'$ this equation reads:

$$ \nabla_N \hat{\beta}^\Phi = 2g e^{-2\Phi} \nabla_N \left( e^{-2\Phi} \hat{\beta}^g_{MP} \right) + O(\hat{\beta}^B) $$

An immediate consequence of this equation is that not all of the $\hat{\beta}^i = 0$ equations are independent. In particular, at a fixed point of the renormalization-group on the world-sheet, for which $\hat{\beta}^g_{MN} = \hat{\beta}^B_{MN} = 0$, one obtains from \[89\] that the dilaton Weyl anomaly coefficient is constant, not necessarily zero. In the particular case of strings in Bosonic massless backgrounds, for instance, this constant is simply the conformal anomaly $D-26$ (Bosonic Strings) or $D-10$ (Superstrings).

When discussing equations of motion the Curci-Paffuti constraint \[89\] should always be taken into account. Although the constraint may seem trivial in case one is interested in solutions of the conformal invariance conditions \[64\], $\hat{\beta}^i = 0$, this is not the case when one encounters non-trivial $\hat{\beta}^i \neq 0$ away from the fixed points of the renormalization group on the world sheet. Such a situation (non-critical Strings) may be of interest in non-equilibrium cosmological situations, and we shall discuss it briefly in Lecture 3.
2.7.4 A note about “Frames”

The action (86) is derived in the so-called \( \sigma \)-model frame, because it is derived directly from expressions obtained in \( \sigma \)-model renormalization-group analysis. Such a terminology *should not be confused* with the general coordinate frames in general relativity. The terminology “frame” here is used to mean a given background metric configuration. In string theory, the perturbative string S-matrix elements (scattering amplitudes) are invariant under *local redefinition of the background fields* \( g^i \) ("equivalence theorem"), which simply corresponds to a particular renormalization-group scheme choice.

The \( \sigma \)-model frame metric corresponds to one such configuration. One may *redefine* the metric field so as to pass to an effective action, where the curvature scalar term in the action will have the standard (from the point of view of General relativity) coefficient \( 1/\kappa^2 \), *without* the dilaton conformal factor \( e^{-2\Phi} \) in front. In other words, it will have the canonically normalized Einstein action form. Such a “frame”, termed *Einstein* (or “physical”) frame, is obtained upon redefining the \( \sigma \)-model background metric as follows (the superscript \( E \) denotes quantities in the Einstein frame):

\[
g_{MN} \rightarrow g^E_{MN} = e^{-\frac{4}{D-2}\Phi} g_{MN} \quad (90)
\]

In this frame, then, the effective action (86) acquires, as mentioned already, its canonical Einstein form, as far as the gravitational parts are concerned:

\[
I^E_{\text{eff}} = -\frac{1}{2\kappa^2} \int d^D X \sqrt{g^E} \left( R^E - \frac{4}{D-2} (\partial_M \Phi)^2 - \frac{1}{12} e^{-\frac{4}{D-2}\Phi} H^2_{MNP} - e^{-\frac{4}{D-2}\Phi} \frac{2(D-26)}{3\alpha'} + \ldots \right) \quad (91)
\]

where the \( \ldots \) denote higher-order terms, as well as other fields, such as gauge-boson terms (in the case of heterotic string) etc. Notice the change of relative sign between the curvature and dilaton kinetic terms in the Einstein frame.

From the point of view of discussing physical low-energy applications of string theory, such as cosmological models based on strings, the Einstein frame is the “physical” one, where the astrophysical observations are made. This will always be understood when we discuss string cosmology in Lectures 2 and 3.

2.7.5 Higher orders in \( \alpha' \)

Corrections to General Relativity occur at the next order in \( \alpha' \), at which one can show, for instance, that the graviton \( \beta \)-function has the form (ignoring the contributions from other backgrounds for simplicity):

\[
\beta^g_{MN}(X^P) = -\alpha' \left( R_{MN} + \frac{\alpha'}{2} R_{MKLP} R_N^{KLP} \right) \quad (92)
\]
The higher-curvature terms will result in corrections to the Einstein term in the target-space effective action. Such action terms have some ambiguities concerning their coefficients, since the scattering $S$-matrix elements one derives from an effective action correspond to more than one set of these coefficients (the equivalence theorem, mentioned earlier in the Lecture). The amplitudes are invariant under local redefinitions of the graviton field (in this case): $g_{MN} \rightarrow g_{MN} + \alpha' R_{MN}$, where $c$ a constant coefficient. Such redefinitions affect the higher order in $\alpha'$ terms of the target-space effective action, in such a way that one can always cast it in the Gauss-Bonnet (gravitational ghost-free) combination:

$$S = \int d^Dx \sqrt{g} \left( R + \alpha \alpha' \left( R_{MNP}^2 - 4R_{MN}^2 + R^2 \right) + \ldots \right) \quad (93)$$

where the coefficient $\alpha$ is determined by comparison with string tree amplitudes. It is found to be: $\alpha = 1/4$ (Bosonic String), $\alpha = 0$ (Superstring type II), and $\alpha = 1/8$ (Heterotic string).

The fact that stringy higher-order corrections to the low-energy effective actions of string theories are free from gravitational ghosts, in the sense that the effective action can always be cast, under local field redefinitions, in the ghost-free Gauss-Bonnet combination, is consistent with the unitarity of the underlying string theory.

The higher-order corrections to Einstein’s general relativity are in principle an infinite series of terms, which become stronger at high energies (short distances). From a cosmological viewpoint, the higher-curvature terms may thus have effects at very early stages of our Universe, but such effects are negligible at redshifts $z \sim 1$ and lower, where we shall concentrate most of our discussion in these lectures. One should notice that the presence of higher-curvature correction terms of Einstein’s general relativity leads some times to highly non-trivial effects. For instance, one may have black hole solutions with (secondary) dilaton hair in such models, which do not exist in standard Einstein’s relativity. Such objects may play a rôle in the Early Universe.

3 Lecture 2: String Cosmology

3.1 An Expanding Universe in String Theory and the rôle of the Dilaton Background

As has already been discussed in the cosmology lectures in this School, the Observed Universe is, to a good approximation, homogeneous and isotropic. From the point of view of string theory, therefore, one is interested in describing the propagation of strings in such homogeneous backgrounds, i.e. space-time geometries whose metric tensors depend only on time, and thus have no spatial dependence.

As we have discussed in the previous lecture, conformal invariance conditions of the associated $\sigma$-model will imply target-space equations of motion for the background fields, which will determine the dynamics. This is, in general terms, what String Cosmology is about. The pertinent dynamics will be described by means of string effective actions for
the various (time dependent only) modes. Of course, this is a first order approximation. Spatial Inhomogeneities can be incorporated by allowing spatial dependence of the various \( \sigma \)-model couplings/background fields.

It is the purpose of this part of the lectures to discuss how one can incorporate expanding Universe scenarios in the above string context. We shall start with the simplest scenario, that of a linearly expanding non-accelerating Universe. Subsequently we shall discuss more complicated models, including inflationary scenarios in string theory, and mechanisms for graceful exit from it. Due to lack of time, the discussion will be relatively brief. For more details, the interested reader will be referred to the literature, which is vast, and still growing. In the last two lectures I will try to give whatever details, and technical aspects, I believe are essential for introducing the layman into the subject of string cosmology and make him/her understand the various subtleties involved. It should be stressed that string cosmology is not a physically well established subject, and part of the third lecture will be devoted to discussing open issues, motivated by recent astrophysical observations on the possibility of a currently accelerating phase of the Universe, and the way such issues can be tackled in the framework of string theory.

In all stringy cosmological scenarios of expanding Universes that we shall examine here the dilaton plays a crucial rôle, as being directly responsible for providing consistent time-dependent backgrounds in string theory. This is an important feature which differentiates string cosmology from conventional one (this feature is, of course, in addition to the fact that the target-space dimensionality of string theory is higher than four).

We commence our discussion by considering the \( \sigma \)-model action of a string propagating in time-dependent backgrounds of graviton \( g_{MN} \), antisymmetric tensor \( B_{MN} \) and dilatons \( \Phi \). Although given in Lecture 1, for completeness we give again the action explicitly (in this subsection we work in units of \( \alpha' = 2 \) (closed strings) for convenience, and we follow the normalization of [9] for the dilaton field, which implies that the dilaton field here equals twice the dilaton field in the previous section):

\[
S_\sigma = \int_{\Sigma} \frac{1}{4\pi} d^2\sigma \left( \sqrt{\gamma} \gamma^{\alpha\beta} g_{MN}(X^0) \partial_\alpha X^M \partial_\beta X^N + B_{MN}(X^0) \epsilon^{\alpha\beta} \partial_\alpha X^M \partial_\beta X^N + \sqrt{\gamma} \Phi(X^0) R^{(2)} \right)
\]

(94)

where \( M, N = 0, \ldots, D - 1 \), and \( X^0 \) denotes the target time. The reader is required to remember that the dilaton coupling is of one order in \( \alpha' \) higher than the rest of the terms in (94). As already mentioned, the time dependence of the backgrounds is appropriate for a discussion of isotropic and homogeneoeus cosmological solutions of the conformal invariance conditions (54), which we now turn to.

### 3.1.1 Linear Dilaton Background Conformal Field Theory

Consider the \( \sigma \)-model background [3]:

\[
g_{MN} = \eta_{mn}, \quad B_{MN} = 0, \quad \Phi = -2QX^0, \quad Q = \text{const}
\]

(95)
in which the dilaton is growing linearly with the target time. We observe that this is an exact solution of the $\sigma$-model conformal invariance conditions (64), (89), for the Weyl anomaly coefficients, which, for the problem at hand, and to $\mathcal{O}(\alpha')$ are given by (83), (84), (85). Hence it is an acceptable background in string theory.

Let us describe the basic features of this conformal theory. We wish to determine first the central charge (conformal anomaly). To this end we need to compute the world-sheet stress tensor [1]. As we have discussed in Lecture 1, the latter is defined by the response (9) of the world-sheet action (94) to a variation of the world-sheet metric. The presence of the dilaton term results in the following form:

$$T_{zz} = -\frac{1}{2} \partial_z X^M \partial_z X^N g_{MN}(X^0) + Q \partial_z^2 X^0$$

(96)

where $z$ is the complexified world-sheet coordinate (we work in a Euclidean world-sheet, appropriate for the convergence of the $\sigma$-model path integral formalism we adopt here).

[ NB4: For completeness we sketch below the derivation of the $Q \partial_z^2 X^0$ term in (96). This comes from varying the world-sheet curvature/dilaton term with respect to the world-sheet metric, and setting at the end $\gamma^{\alpha \beta} = \delta^{\alpha \beta}$ (for Euclidean world sheets): $\delta_{\gamma} \int_\Sigma R^{(2)} Q X^0(\sigma, \tau)$. Noticing that only contributions from the second derivatives of the world-sheet metric in $R^{(2)}$ survive this procedure, we obtain: $\int_\Sigma d^2 \sigma' \sqrt{\gamma^2} \delta^{(2)}(\sigma - \sigma') Q t(\sigma') = -\int_\Sigma \delta^{(2)}(\sigma - \sigma') Q \partial_z t = -Q \partial_z^2 t$, where partial integration has been made in order to arrive at the last equality ].

From (96) it is straightforward to compute the conformal anomaly $c$. From basic conformal field theory we recall that the latter is given by [1]):

$$\lim_{z \to 0} 2z^4 \langle T_{zz}(\sigma) T_{zz}(0) \rangle = c$$

(97)

Regulating the short-distance behaviour of the theory by replacing $z \to 0$ by $z \to a$, where $a$ is a short-distance cutoff scale, and using the free-field contractions for two-point correlators on the world-sheet (71) (with $\xi(x_1) \leftrightarrow X^M(\sigma)$) it is straightforward to derive [3]:

$$c = D - 12Q^2$$

(98)

where $D$ is the dimensionality of the target-space time.

This is an important result. In the conformal field theory of a non-trivial linear dilaton background, and flat $\sigma$-model target spacetime, the conformal anomaly is no longer given by the target-space dimensionality $D$ alone, which was the case of Minkowski space times, as we have seen in Lecture 1.

The cancellation of the Weyl anomaly implies $c = 26$ (for bosonic strings, which we restrict ourselves from now on for definiteness, unless otherwise stated). This, therefore, means that the critical dimensionality of the string is $D > 26$. This string theory is termed supercritical [9].

The non-trivial issue is to demonstrate the mathematical consistency of such string theories, by demonstrating unitarity of the physical spectrum, and modular invariance, associated with string loops [1]. We note that both of these properties have been shown to

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be valid for the linear dilaton background (95). We shall not demonstrate them here due to lack of time. We refer the interested reader to the literature [9] for a detailed verification.

We next proceed to discuss the target-space time interpretation of the linear dilaton background (95). As we have mentioned previously, the ‘physical’ metric, appropriate for cosmological considerations in theories with non-trivial dilaton fields, is provided by the Einstein frame target-space metric tensor (90), corresponding to a low-energy effective action (91) with canonically-normalized Einstein curvature term, without dilaton conformal factors. For the background (95), therefore, the Einstein-metric invariant line element reads:

\[ ds_E^2 = e^{4QX^0} \eta_{MN} dX^M dX^N \] (99)

Upon redefining the time \( X^0 \rightarrow t \):

\[ t = \frac{D - 2}{2Q} e^{\frac{2Q}{D - 2} X^0} \] (100)

we observe that the Einstein (“physical”) metric may be cast into a Robertson-Walker (RW) form [9]:

\[ ds_E^2 = -(dt)^2 + t^2 dX^i dX^j \delta_{ij} \] (101)

with a linearly expanding in time, non accelerating scale factor \( a(t) = t, \quad \ddot{a}(t) \equiv d^2 a(t)/dt^2 = 0 \). The RW Universe (101) has zero spatial curvature, i.e. is flat.

In these coordinates the dilaton field has a logarithmic dependence on time:

\[ \Phi(t) = (2 - D) \ln \left( \frac{2Qt}{D - 2} \right) \] (102)

One may accommodate more general RW backgrounds with non-trivial spatial curvature in the above framework, by including non-trivial antisymmetric tensor backgrounds [9]. This is what we shall discuss below.

3.1.2 The antisymmetric tensor field and More General Cosmological Backgrounds

First of all we concentrate our attention to (98). We assume that in our model there are \( d = 4 \) “large” (non-compact) target-space time dimensions, one of which is the Minkowski time. The rest of the target dimensions (6 in the case of superstring, or 22 in the case of Bosonic strings) are replaced by an appropriate “internal” conformal field theory with a central charge \( c_I \):

\[ c = d + c_I - 12Q^2 = 4 + c_I - 12Q^2 \] (103)

Notice that the total central charge \( c \) is required to equal its critical value (26 for Bosonic strings, 10 for superstrings) so as to ensure target-space diffeomorphism invariance (i.e. to cancel the Fadeedv-Popov reparametrization ghost contributions to the conformal anomaly),
and also conformal invariance of the complete theory. These two requirements are essential for giving string theory a space-time interpretation. Then (103) leads to (for Bosonic strings for brevity):

$$c_{I} = 22 + 12Q^2 \equiv 22 + \delta c$$

(104)

where $\delta c$ is known as the central charge deficit. For a critical dimension string theory, $\delta c = 0$.

From now on we shall ignore the details of the compact internal theory, and simply assume it is there to ensure the above properties, and hence consistency, of the string theory at hand. One can show that non-trivial internal conformal field theories can indeed be constructed with the desired properties \[9\]. We therefore consider $d = 4$-dimensional backgrounds $g_{\mu\nu}(x)$, $B_{\mu\nu}(x)$, $\Phi(x)$, where $\mu, \nu = 0, \ldots, 3$, and $x^\mu$ are four-dimensional spacetime coordinates.

In four space time dimensions the antisymmetric tensor field strength may be written in terms of a pseudoscalar field $b(x)$ to be identified with the axion field \[l, \beta\]:

$$H^\lambda_{\mu\nu} = e^{2\Phi} \epsilon^\lambda_{\mu\rho} \partial^\rho b$$

(105)

The conformal invariance conditions (64), then, corresponding to the four-dimensional Weyl anomaly coefficients (83), (84) read:

graviton: \quad $R_{\mu\nu} = \frac{1}{2} \partial_{\mu} \Phi \partial_{\nu} \Phi + \frac{1}{2} g_{\mu\nu} \nabla^2 \Phi + \frac{1}{2} e^{2\Phi} [\partial_{\mu} b \partial_{\nu} b - g_{\mu\nu} (\partial b)^2]$,

antisym. tensor: \quad $\nabla^2 b + 2 \partial_\lambda b \partial^\lambda \Phi = 0$ (106)

The fact that the total central charge is 26 (for Bosonic strings, or 10 for superstrings) implies the dilaton equation \[l\]:

$$\delta c = 12Q^2 = -3e^{-\Phi} \left[ -R + \nabla^2 \Phi + \frac{1}{2} (\partial \Phi)^2 - \frac{1}{2} e^{2\Phi} (\partial b)^2 \right]$$

(107)

(in units $\alpha' = 1$), where $\delta c$ is defined in (104). We stress again that in the case of critical strings $\delta c$ would vanish. Here, as a result of the Bianchi identity $\nabla^\mu R_{\mu\nu} = \frac{1}{2} \nabla_\nu R$, the equations (106) imply the consistency of (107), i.e. that the right-hand-side is a constant, consistent with $\delta c = \text{const}$. This consistency is nothing other than the Curci-Paffuti equation (83), stemming from renormalizability of the world-sheet $\sigma$-model theory.

The four-dimensional effective low-energy action obtained from (106),(107), in the Einstein frame, is:

$$I_{\text{eff}} = \int d^4x \sqrt{-g^E} \left[ R - \frac{1}{2} (\partial \Phi)^2 - \frac{1}{2} e^{2\Phi} (\partial b)^2 - \frac{1}{3} e^\Phi \delta c \right]$$

(108)

Note that, as a result of the non-trivial central-charge deficit $\delta c \neq 0$, there is a non-vanishing potential for the “internal” fields, which implies a non-trivial vacuum energy term for a four-dimensional observer.

We note now that the linear dilaton background (93) is indeed a special case of the equations (106),(107), leading in the Einstein frame, to spatially flat RW linearly expanding
Universes, as we have seen above, . For non trivial axion fields $b$ one has more general RW backgrounds, with spatial curvature. Indeed, it can be shown that the equations (106), (107) admit as solution [9]

$$ds_E^2 = -dt^2 + a(t)^2 \tilde{g}_{ij} dx^i dx^j, \ i, j = 1, 2, 3$$ (109)

where $\tilde{g}_{ij}$ is a three-dimensional maximally symmetric metric:

$$\tilde{g}_{ij} dx^i dx^j = \frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2)$$ (110)

where $t$ is the physical time (100), and the RW parameter $k$, related to spatial curvature, is to be determined below.

The Hubble parameter is given by: $H(t) \equiv \frac{\dot{a}(t)}{a(t)}$, with the dot denoting derivative w.r.t. $t$. With the ansatz (109), (110) the antisymmetric tensor/axion equation in (106) is solved by [9]:

$$\dot{b} = b_0 \frac{e^{-2\Phi}}{a(t)^3}, \quad b_0 = \text{const.}$$ (111)

and the dilaton equation (107) implies for the central-charge deficit:

$$\delta c = 6e^{-\Phi} \left( \dot{H} + 3H^2 + \frac{2k}{a(t)^2} \right)$$ (112)

The graviton equations have in principle two independent components:

$$\mu = \nu = t : \quad -6(\dot{H} + H^2) = \ddot{\Phi} + 3H \dot{\Phi} + (\dot{\Phi})^2,$$

$$\mu, \nu = i, j : -2(\dot{H} + 3H^2 + \frac{2k}{a(t)^2}) = \ddot{\Phi} + 3H \dot{\Phi} - \left( \frac{b_0^2}{a(t)^6} \right) e^{-2\Phi}$$ (113)

However, since the dilaton equation (107) is an identity (up to an irrelevant constant), one observes actually that there is only one independent equation for the graviton. Indeed, solving (112) for the dilaton and substituting into (113), and subtracting these two equations we obtain:

$$\left( \frac{\ddot{H} + 6\dot{H}H - (4k/a(t)^2)H}{\dot{H} + 3H^2 + 2k/a(t)^2} \right)^2 = -4\dot{H} + \frac{4k}{a(t)^2} - \left( \frac{\delta c}{36a(t)^6} \right) \frac{b_0^2}{(\dot{H} + 3H^2 + (2k/a(t)^2))^2}$$ (114)

This equation can in principle be solved, yielding the Hubble parameter for the string Universe under consideration.

**Asymptotic Solutions of (114):** There are two kinds of asymptotic solutions, of (114), which can be obtained analytically:
• (I) $H \to 0$, as $t \to +\infty$, $\Phi = \phi_0 =$constant, $\dot{b} = b_0 e^{-2\phi_0}$ and the space curvature 
obeys
\[ k = \frac{1}{4} b_0^2 e^{-2\phi_0} \geq 0 \tag{115} \]
and thus is non negative. This Universe is therefore closed. The central charge deficit, in this case is determined via the dilaton equation (107) to be: $\delta c = 12 e^{-\phi_0} k$. This asymptotic Universe is therefore a static Einstein Universe with non-negative spatial curvature.

• (II) A linearly expanding Universe $a(t) = t$ with metric:
\[ ds_E^2 = -(dt)^2 + t^2 \left[ \frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right] \tag{116} \]
with Hubble parameter relaxing to zero as $t \to \infty$ as: $H(t) \sim 1/t$, and hence one has:
\[ \Phi = -2 \ln t + \phi_0, \quad b = 2 e^{-\phi_0} \sqrt{k} t \tag{117} \]
with $k$ again non-negative. The four-dimensional curvature is $R = 6(1 + k)/t^2$, and the central charge deficit is $\delta c = 12 e^{-\phi_0} (1 + k)$.

Conformal Field Theories corresponding to the asymptotic solutions (I) and (II)

The asymptotic solutions, found above to leading order in $\alpha'$, can become exact solutions if one manages to construct explicitly the corresponding conformal field theories (CFT) on the world sheet.

This has been done in some detail in [9]. Below we only describe the main results. For the static Einstein Universe the corresponding CFT is a two-dimensional Wess-Zumino model on a $O(3)$ group manifold, with a time coordinate which is a free world-sheet field. The corresponding central charge is:
\[ c = 1 + \frac{3 \tilde{\kappa}}{\tilde{\kappa} + 1} = 4 - \frac{3}{\tilde{\kappa} + 1} \tag{118} \]
where $\tilde{\kappa}$ is the Wess-Zumino level parameter of the $O(3)$ Kac-Moody algebra. The central charge deficit (104) is in this case:
\[ \delta c = \frac{3}{\tilde{\kappa} + 1} = 12 e^{-\phi_0} k + \ldots \tag{119} \]
where $\ldots$ denote higher orders. The important point to notice is that the level parameter $\tilde{\kappa}$ is an integer for topological reasons (equivalently, this result follows from unitarity of the spectrum and modular invariance of the underlying string theory [3]). Thus, (119) implies that the central charge deficit is quantized.

The conformal field theory corresponding to the second asymptotic solution (II) of (114), that of a linearly expanding Universe, can be found most conveniently if we go
back to the \( \sigma \)-model frame: 
\[ g_{\mu\nu} = e^{\Phi} g_{\mu\nu}^E \]
and the \( \sigma \)-model coordinate time \( X^0 \) \((100)\), in which the dilaton is linear:
\[ \Phi = -2e^{-\phi_0/2} X^0 + \phi_0 \equiv -2QX^0 + \phi_0 \] 
(120)
Thus we observe that \( Q \equiv e^{-\phi_0/2} \) plays the rˆole of a “charge at infinity” in similar spirit to the Coulomb-gas conformal models \( \text{[1]} \), an analogy prompted by the form of the corresponding stress tensor \( \text{[9]} \).

The corresponding world-sheet conformal field theory is again a Wess-Zumino model on a group manifold, in which \( g_{ij} \) is the metric, and \( H_{ij\ell} = \nabla[i B_{j\ell}] \) is the volume element.

The model has again a time coordinate but with a charge \( Q \) at infinity, as we have just mentioned. The (two-dimensional) Lagrangian of the model is:
\[ \mathcal{L}^{(2)} = -(\partial X^0)^2 - QX^0 R^{(2)} + \mathcal{L}_{WZW}(O(3)) \] 
(121)
where \( R^{(2)} \) is the world-sheet curvature. The central charge is: \( c = 1 - 12Q^2 + c_{WZW} \) with the level parameter \( \tilde{\kappa} \) being related to the spatial curvature \( k \) as follows:
\[ k = \frac{1}{2Q^2 \tilde{\kappa}} \] 
(122)
Since \( \tilde{\kappa} \in \mathbb{Z}^+ \cup \{0\} \) for topological (or, equivalently unitarity and modular invariance) reasons, then \( k > 0 \) and the four-dimensional Universe is again closed. The 4-d curvature is found again to be \( R = 6(1 + k)/\ell^2 \).

### 3.1.3 The spectrum of the Linear-Dilaton Strings: Mass Shifts

Consider the conformal invariant solution \( \text{[53]} \). The corresponding Virasoro operators, i.e. the moments of the world-sheet stress tensor, as we have discussed in the first Lecture, are \( \text{[9]} \):
\[ L_n = \frac{1}{2} \sum_j \eta_{\mu\nu} \partial_{\alpha} x^\mu_{n-j} \partial^\nu x^\alpha_j + iQ(n + 1)x^0_n \] 
(123)
where \( x^\mu_n \) are moments of the world-sheet operators \( i\partial_{\alpha} x^\mu \), satisfying:
\[ [x^\mu_m, x^\nu_n] = mn^{\mu\nu} \delta_{m+n,0} , \]
\[ x^\mu_n = x^\mu_n + 2iQn^{\mu\alpha} \delta_{n,0} \] (since \( L_n^\dagger = L_{-n} \)) 
(124)
This implies that the 0-th (time) component of Minkowski space-time momentum has a fixed imaginary part \( \text{[4]} \)
\[ p^0 = E + iQ \] 
(125)
where the real part \( E \) corresponds to “energy”.

Consider for definiteness a bosonic scalar mode, e.g. a tachyon, which is the lowest lying energy state of a Bosonic string (ground state):
\[ |p\rangle = e^{-p_n x^\mu(0)} |0\rangle \] 
(126)
annihilated by all $x_n^\mu$ ($n > 0$). The corresponding mass-shell condition is:

$$\frac{1}{2} p_{\mu} p^\mu + i Q p^0 = -\frac{1}{2} (E^2 + Q^2 - \vec{p}^2) = 1$$ (127)

where $A$ denotes three-dimensional vectors. Thus, from (127) one observes that there is a shifted mass for the tachyonic mode:

$$\delta m_T^2 = m^2 - Q^2$$ (128)

in such linear dilaton backgrounds.

From a target-spacetime view point, this can be easily understood considering the Lagrangian for a scalar mode $\varphi$ in the background (95) in the Einstein frame (90):

$$L_\varphi = e^{2Qx^0} \left( -\eta^{\mu\nu} \partial_{\mu} \varphi \partial_{\nu} \varphi - m^2 \varphi^2 \right)$$ (129)

Indeed, rescaling the field $\varphi \rightarrow \tilde{\varphi} = e^{Qx^0} \varphi$, so as to have a canonical kinetic term, one obtains a mode that obeys a free scalar-field wave equation, in flat space-time, with shifted mass (128).

This result can be extended to include all the other bosonic modes [9], including graviton and dilaton. All such bosonic modes therefore will have a mass shift of tachyonic type in supercritical strings:

$$\delta m_B^2 = -Q^2 < 0$$ (130)

For target-space Supersymetric strings, including the phenomenologically relevant Heterotic string [1], in linear-dilaton backgrounds, one observes that there are no mass shifts for the fermionic target space time modes [9].

This can be readily seen, for instance, by noting first that the anomaly condition for superstrings becomes:

$$c_I + d - 8Q^2 = 10$$ (131)

This is due to the additional stress tensor contributions on the world-sheet pertaining to fermionic backgrounds $T_F = -\psi_{\mu} \partial_\mu x^\mu + 2Q \partial_5 \psi^0$.

The lowest-lying fermionic excitations are massless, since superstrings do not have tachyonic instabilities. Consider for simplicity the case $c_I = 0$ and concentrate on the lowest-energy Ramond state. Consider the moments of $T_F$:

$$G_n = i \sum_n \psi^\mu_{n-m} x_{\mu,n} - 2Q(n+1) \psi^0_n$$ (132)

When acting on the highest-weight state, one has:

$$G_0 = -i (\gamma_0 E - \vec{\gamma} \cdot \vec{p})$$ (133)

This is precisely the massless Dirac operator in flat space. Thus one observes that there is no mass shift for the fermionic modes.
From a field-theoretic viewpoint this can be seen from the quadratic part of the target-space Lagrangian for fermionic modes Ψ, in the background (95):

\[ L_{\text{fermion}} = e^{2Qx^0} \left( \overline{\Psi} \gamma^\mu \Psi + m \overline{\Psi} \Psi + \ldots \right) \] (134)

It is easily seen that the rescaled fermion field \( \tilde{\Psi} = e^{Qx^0} \Psi \) obeys the free Dirac equation without a mass shift.

Thus, in a linear dilaton background, which leads to a linearly expanding Universe in Einstein frame, there is no target-space fermionic-mode mass shift:

\[ \delta m_F^2 = 0 \] (135)

So far, our considerations pertain to tree-level world-sheet \( \sigma \)-models, i.e. world-sheets with the topology of a disc (open strings) or sphere (closed strings). String loop corrections do affect the \( \beta \)-functions of the theory, and actually they do result in the appearance of non-trivial dilaton potentials \( \delta V(\Phi) \), whose effects we now come to discuss, from the point of view of Cosmological backgrounds, which are of interest to us in the context of these lectures.

3.2 String Loop Corrections and De Sitter (Inflationary) Space Times

The string loop corrections, i.e. effects coming from higher world-sheet topologies, are non-trivial and they do modify the tree-level \( \beta \)-functions of the theory through the so-called Fischler-Susskind mechanism [10]. To understand qualitatively the rôle of such effects let us consider the indicative example of a \( \sigma \)-model partition function on a world-sheet torus. As one sums up over tori geometries, with handles of variable size, one encounters extra divergencies, as compared to the case of world-sheet spheres, arising from pinched tori, as indicated in fig. 6.

Such infinities (modular) are equivalent to considering tori with handles of sizes below the ultraviolet cutoff on the world sheet. Such degenerate higher-genus surfaces cannot be distinguished from those of spherical topology. Thus, in a regularization procedure the effect of the presence of these surfaces is to induce new types of counterterms for the spherical topology regularized \( \sigma \)-model action, which result in the string-loop modifications of the \( \sigma \)-model \( \beta \)-functions. For technical details, the interested reader is referred to the literature [10, 4].

For our purposes here, we note that these string-loop corrections induce a dilaton potential \( \delta V(\Phi) \) in the four-dimensional string effective action, whose contributions to the conformal invariance conditions (64), for the \( \sigma \)-model (94), can be summarized as follows:

\[
R_{\mu\nu} = R_{\mu\nu}^{\text{old}} + \frac{1}{2} g_{\mu\nu} [\delta V(\Phi) - \delta V'(\Phi)] ,
\]

\[
\delta c = \delta c^{\text{old}} - 3e^{-\Phi} [2\delta V(\Phi) - \delta V'(\Phi)]
\] (136)
Figure 5: Extra world-sheet partition function divergencies arising from pinched tori. Regularizing such pinched surfaces modifies the $\beta$-function of the theory at lower genera, since it introduces new types of $\sigma$-model counterterms. This is the essence of the Fischler-Susskind mechanism.

where the suffix “old” denotes the right-hand-sides of the tree-level “graviton” equation in (106), and that of the dilaton equation (107), and the prime denotes differentiation with respect to the dilaton field $\Phi$, $\delta V'(\Phi) \equiv \frac{2}{\Phi} \delta V(\Phi)$.

From (136) we observe again that $\delta c$ is a $c$-number (constant), as required by consistency, for arbitrary dilaton potential $\delta V(\Phi)$. In string-loop perturbation theory the dilaton potential can be computed order by order, and has the generic form:

$$\delta V(\Phi) = \sum_{n \geq 1} a_n e^{(n+1)\Phi}$$

where we remind the reader that $g_s = e^{\phi_0/2}$ is the string coupling constant, which is a string-loop counting parameter, as explained in Lecture 1.

An important physical consequence of the presence of a dilaton potential due to string loop corrections is the possibility of having $De Sitter$ (inflationary) solutions in string theory, i.e. solutions in which the Hubble parameter is constant in time $H(t) = \text{const}$. This implies an $exponentially expanding Universe$, with scale factor

$$a(t) \sim e^{Ht}$$

The constancy of $H$ can be achieved by selecting constant values for the dilaton and axion fields $\phi_0 = \text{const}, b = b_0 = \text{const}$, and non-trivial values for the dilaton potential, induced by string loops:

$$R = 12H^2 = 2 [\delta V(\Phi) - \delta V'(\Phi)] ,$$

$$\delta c = -3\delta V'(\Phi)e^{-\Phi}$$

One should emphasize the crucial rôle of the constant value of the dilaton field in determining both the value of the Hubble constant during the inflationary period of the string Universe, and the string coupling $g_s = e^{\phi_0/2}$. 

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The physically interesting issue is how one can exit from the inflationary phase in string Universes. In the simplified background considered above this cannot be possible in a smooth continuous way. The rest of the lectures will be therefore devoted to a rather brief, but to the point, discussion of more complicated string backgrounds and scenarios that might achieve such a graceful exit from the inflationary period. We shall also point out some essential problems that an eternal de Sitter Universe poses for critical string theory in general, namely for a proper definition of scattering amplitudes which is an essential feature of any critical string theory.

3.3 De Sitter Universes and pre-Big Bang scenarios: the crucial rôle of the Dilaton Field

3.3.1 Life before the Big Bang in string theory?

As we have seen above, a non-trivial dilaton field $\Phi$ is an important ingredient for providing inflationary, and in general expanding, Universes in string theory. As argued by Veneziano and collaborators [11], the presence of a non-trivial dilaton potential may result in scenarios for expanding Universes in which there is no initial singularity (Big Bang), since in such cases the “singularity” is replaced by a (yet not fully known) non-perturbative strongly-coupled region of string theory, in which $g_s = e^{\Phi/2} \gg 1$. This is the so-called Pre Big-Bang scenario (PBB) of the string Universe, which we now proceed to discuss in general terms. For details we refer the interested reader in the relevant literature [11].

In PBB scenarios one is typically encountering the situation for a dilaton potential depicted in figure 6. In generic PBB models the string Universe has a (weak string coupling) “life”, before one reaches the “big bang”, which is not a singularity, but a potential barrier separating the weak phase from that at which the string theory becomes strongly coupled. The weakly coupled string-theory (pre Big Bang) region can be treated analytically by means of Einstein-type low-energy effective actions, of the form (91). In this region one considers homogeneous Bianchi type I solutions of the equations of motion obtained from the string-effective action [11].

Let $t = 0$ be the “Big-Bang” time moment, i.e. the time moment for which the dilaton potential has its maximum height (see figure 6). The pre Big Bang (weakly coupled) solution occurs for $t < 0$, and has the form (111):

$$
\begin{align*}
    ds_E^2 &= -(dt)^2 + \sum_i (-t)^{2a_i} dx_i dx^j \eta_{ij}, \\
    \Phi &= -(1 - \sum_i a_i) \ln(-t), \quad \sum_i a_i^2 = 1, \ t < 0
\end{align*}
$$

It is customary [11] to use a redefined dilaton field, shifted by the logarithm of the determinant of the spatial part of the metric,

$$
\overline{\Phi} \equiv \Phi - \frac{1}{2} \ln \det(g_{ij}) = -\ln(-t)
$$
Figure 6: A typical dilaton potential encountered in pre-Big-Bang scenario for string Universe. In such scenario the initial (Big-Bang) singularity of standard cosmology is absent, and is replaced by a non-perturbative region of string theory, in which the string coupling, being given by the exponential of the dilaton field, is very strong. The arrows indicate flow of cosmic time. The dilaton today is at (or close to) the minimum of its potential. At present the rigorous derivation of such potentials from exact string theory models is lacking.

Notice that in PBB scenario it is the early times regime that is characterized by a weakly coupled string theory, and dilaton potential which asymptotes to zero. This has to be contrasted with the situation in ref. [9], where it is the late times region which has these features, as we have seen in the previous subsection.

Inhomogeneities are introduced in a straightforward manner [11]:

\[
 ds_E^2 = -(dt)^2 + \sum_b e^b_i(x)e^b_j(x)(-t)^{2\alpha_b(x)}dx^i dx^j,
\]

\[
 \Phi = -(1 - \sum_i a_i)\ln(-t), \quad \sum_i a_i^2 = 1, \quad t < 0
\]  

3.3.2 Stringy Dilaton Driven Inflation in PBB scenario

In a PBB scenario, like the one depicted in figure 6, the dilaton continues to grow (as time evolves) in such a way that the string coupling \( g_s = e^\Phi \) becomes strong and, hence, perturbative solutions like \((140),(142)\) are no longer possible. In strong string coupling situations the resummation of string world-sheet genera has to be performed, something which at present is not feasible. Moreover, many physicists believe that in such strong string-coupling situations even the concept of a \( \sigma \)-model breaks down, and one encounters a fully non-perturbative stringy situation which is far from being understood at present. It
is in this regime that non-perturbative concepts like branes, M-theory etc., are applicable, and one would hope to find appropriate dualities which would map the strongly-coupled string theory to a dual theory which could be treated perturbatively in an analytic way. At present, despite effort, this issue is still open in our opinion, and this prevents one from providing analytic arguments in support of the crossing of the potential “Big-Bang” barrier.

However, the lack of analytic treatment does not prevent one from making a qualitative description as to how the situation is expected to be [11]. After crossing the barrier one expects to have an inflationary phase, driven by the dilaton field, and eventually a graceful exit from it, so as to reach the present era of our power-law expanding Universe. Schematically, the PBB scenario and its post big-bang inflationary phase is represented by means of “wine-glass” space-time diagrams [11]. In figure 7, which is a crude reproduction of the original figure suggested by Veneziano [11], the PBB scenario for a string Universe, together with its post BB evolution, is compared, in terms of the corresponding space time diagrams, with that of a standard Big-Bang Cosmology. A physical picture of what it is envisaged in a PBB situation, including the dilaton-driven inflation is given in figure 8. Our Universe starts as a small (Planckian) fluctuation of the string vacuum, and then turns
Before closing this subsection it is worthy of pointing out that in the Einstein-frame PBB scenaria the issue of dilaton-driven inflation becomes equivalent to that of studying gravitational collapse \[11\], in the sense of the Einstein-metric spatial volume element being shrunk to zero size at a certain moment, as time goes backwards. The reason is simple: in this frame, one observes from (91) that the dynamics of the problem are those of a minimally coupled scalar field $\Phi$ to Einstein gravity. Such a situation is characterized by positive pressure, as can be trivially verified, and thus it cannot lead to inflation. However, at these singularities the dilaton also blows up, and one can verify that in PBB scenaria
the stringy metric, related to the Einstein one via (90), also blows up there, leading to stringy inflation. Such a situation is depicted in figs. 7, 8. For more details on such issues in the context of PBB scenaria we refer the reader to the literature [11].

3.4 Some Phenomenological Implications of String Cosmology

The string cosmologies we have discussed so far have a far richer spectrum of physical excitations, as compared with standard cosmologies. The quantum fluctuations of these stringy excitations are expected to undergo amplification under inflation, which is expected to lead to a rich unconventional phenomenology, not characterizing the case of conventional cosmologies.

In PBB scenaria one can actually show [11] that some “pump” fields, a terminology to be defined immediately below, tend to grow during the PBB inflation in contrast to the situation encountered in standard (conventional, field-theoretic) inflationary scenaria, where they tend to decay.

Consider a generic perturbation \(\Psi\) in the low-energy limit of string theory with action (86) in the \(\sigma\)-model frame (e.g. metric, dilaton, axion fluctuation etc). We assume the theory has been appropriately compactified to four space-time dimensions. As mentioned previously, in the context of our generic discussion in this lectures, we shall not bother with explicit details of the internal dimensions. The effective action of this perturbation has the generic form:

\[
I_{\text{eff, pert}} = \int d\eta d^3x s(\eta) \left[ \Psi'^2 - (\nabla \Psi)^2 \right]
\]  

(143)

where \(\eta\) is the conformal time, defined by \(d\eta = dt/a(t)\), with \(a(t)\) the scale factor of the Universe (in the \(\sigma\)-model frame), and the prime denotes differentiation with respect to \(\eta, \partial/\partial \eta\). The function \(s(\eta)\) is a function of the scale factor \(a(\eta)\) and other scalar fields (dilaton, moduli-i.e. fields related to the internal dimensions etc), which characterize the string background under consideration. The function \(s(\eta)\) is called a “pump” field, since a \(s(\eta) \neq \text{const} \) couples non-trivially to the perturbation \(\Psi\) and leads to the production of pairs of quanta of this perturbation.

The pump fields are crucial in determining the evolution of the perturbation. Let \(\hat{\Psi}_k\) be a Fourier component of such a perturbation. Then one may define: \(\hat{\Psi}_k \equiv s^{1/2}(\eta) \Psi_k\), which can be shown to satisfy a Schrödinger type equation [11]:

\[
\hat{\Psi}_k'' + \left[ k^2 - \left( s^{1/2}(\eta) s^{-1/2} \right) \right] \hat{\Psi}_k = 0
\]  

(144)

where the prime denotes differentiation w.r.t. the conformal time \(\eta\).

In string cosmology, and in particular PBB scenaria, the most interesting perturbations correspond to the following pump fields [11]:

- Gravity waves, dilaton: \(s(\eta) = a^2 e^{-\Phi}\),
- Heterotic gauge bosons: \(s(\eta) = e^{-\Phi}\),
- \(B_{\mu\nu}\) Kalb – Ramond field, (axion): \(s(\eta) = a^{-2} e^{-\Phi}\).  

(145)
where $a$ is the RW scale factor in the $\sigma$-model frame, related to the scale factor in the Einstein frame $a_E$ by $a_E = ae^{-\Phi/2}$. These are found easily by looking at the corresponding terms of the low-energy string effective action (in these lectures we only exhibited explicitly the gravitational part of the effective action (86), (11) (or (108)), and not the gauge and other parts. The interested reader is referred to the literature for explicit forms of such background fields [1]. For example, looking at the axion term in the Einstein frame effective action (108) it is immediate to see that the axion $b$ perturbations will have a pump field $a^2 e^{\Phi}$. On the other hand, when expressed in terms of the field strength of the Kalb-Ramond field $B_{MN}$, $H_{\mu\nu\rho} = e^{2\Phi} \epsilon_{\mu\nu\rho\lambda} \partial^{\lambda} b$, such axion terms lead to effective action $H$-terms of the form (11), and therefore to the Kalb-Ramond pump field indicated in (145).

After amplification during PBB inflation, such perturbations may lead to observable effects. Below we shall briefly catalogue the claimed effects. The interested reader may find more detailed discussion in the literature [1].

- **Tensor Perturbations**: such perturbations are associated with gravitational field perturbations, and may have effects in the observable cosmic gravitational radiation background (gravity waves). Such effects are though extremely tiny, due to the weakness of the interaction. Conventional models of inflation also have such perturbations, and it will be very difficult to disentangle the stringy situations from the conventional ones, as far as tensor perturbations are concerned, even if the gravitational radiation is observed.

- **Dilaton Perturbations**: since the dilaton plays the rôle of the *inflaton* in string cosmology, as it drives string inflation, as discussed above, it is the natural source for adiabatic scalar perturbations. One would expect it to lead quite naturally to a quasi scale invariant Harrison-Zeldovich spectrum of adiabatic perturbations. This would be desirable in explaining the observed cosmic microwave background (CMB) anisotropies. Unfortunately, however, detailed studies in the PBB scenario [11] have revealed that both scalar and tensor perturbations remain exceedingly small at large scales, so CMB data cannot be explained by the dilaton inflation-amplified perturbations.

- **Gauge-Field Perturbations**: in standard cosmology there is no amplification of vacuum fluctuations of gauge fields. This is due to the fact that the inflaton in such cases makes the metric conformally flat, and in such metrics, the gauge fields decouple from geometry in $D = 3 + 1$ dimensions. In contrast, in PBB stringy scenarios, the effective gauge coupling, being related proportionally to the string coupling $g_s = e^{\Phi/2}$, grows together with the inflated space. This is an exclusive feature of stringy models. In this sense, one would expect [11] that PBB, or in general stringy inflationary scenario, could provide an explanation for the origin of primordial seeds of the observed galactic magnetic fields. This, however, still remains a theoretically unsolved problem. Gauge perturbations interact considerably with the hot plasma of the early post big Bang Universe, and hence converting the primordial seeds into those that may have existed in the era of galaxy formation is a non-trivial and still unresolved task.
Axion Perturbations: As we have discussed above, in four space-time dimensions, the field strength of the antisymmetric tensor field of the $\sigma$-model is related to the axion field $b$: $H_{\mu\nu\rho} = e^{2\Phi} \epsilon_{\mu\nu\rho\lambda} \partial^\lambda b$. It must be stressed that the spectrum of the axion field perturbations is very sensitive to the cosmological behaviour of the internal (compactified) dimensions during the string inflationary era, thereby making axions a window to extra dimensions. On the other hand, the axion spectrum is flat even red (tilted towards large scales).

With these brief comments we finish our discussion on the string cosmological scenario. We only glanced the surface of a huge subject here, and the interested reader is strongly advised to seek further details in the literature. As we have seen, there are many issues that need further exploration, both theoretical and experimental ones. There are important differences from standard cosmology. However, de Sitter Universes in string theory pose serious theoretical challenges as well, which we did not discuss so far. This, and ways of incorporating such backgrounds in a mathematically consistent string-theory framework, will be the (speculative) topic of the third Lecture, which we now turn to.

4 Lecture 3: Challenges in String Cosmology and Speculations on their Treatment

4.1 Exit from Inflationary Phase: a theoretical challenge for String Theory

An important, and still unresolved issue, in stringy inflationary cosmology is the graceful exit from the De Sitter (inflation) phase. As we have seen previously, an important ingredient for inflation is the existence of a dilaton potential, which in critical (conformal) string theories is absent at tree world-sheet level, and can only be generated by resumming string loops (higher genera). In PBB scenario, during the inflationary period one is dealing with a strongly coupled phase of string theory, and hence analytic arguments on such a resummation cannot be provided. Exit mechanisms have been proposed though at a qualitative level by many groups [11, 12, 13, 14]; some of them involve non-local dilaton potentials [13], which however lack a good motivation within the framework of string theory; others implement exit via quantum tunnelling [14] through the dilaton potential barrier (see figure 6), which exploits the associated Wheeler-de-Witt equation, without modification of the low-energy string effective actions. Unfortunately, although in such tunnelling scenarios the quantum probability of a classically forbidden exit turns out to be suppressed by $e^{-\Phi}$ factor, and a priori looks to be a promising scenario, however this suppression exist throughout the three-space, and thus in such scenarios only tiny regions have a reasonable chance of tunnelling.

Exit from inflationary phase is a generic challenge for critical string theory, not only of PBB scenario. This problem becomes even more serious today, where there seems to be experimental evidence [13] (from high-redshift supernovae Ia data, supported by
complementary observations of CMB data \([13]\) that our Universe today is in an *accelerating phase*, \(\ddot{a}(t) > 0\), which, within the Friedman cosmological model, implies also a non-trivial positive cosmological constant \(\Lambda > 0\). In fact there is evidence that 70% of the total available energy density is *dark energy component*, not matter, which could be an honest cosmological constant, or, even, a relaxing to zero time-dependent energy component of a quintessence field \([17]\). This may be said differently: our Universe is still in a de Sitter phase, which if true may imply *eternal acceleration*, given that in such a phase, with a non-zero positive cosmological constant, eventually the vacuum energy due to \(\Lambda\) will become dominant over the matter, whose density decays with the scale factor as \(a(t)^{-3}\). In such a vacuum-dominated Universe, the scale factor of a Friedman model varies exponentially with the Robertson-Walker time \(t\):

\[
a(t) \sim e^{\sqrt{\frac{8\pi G N}{3}\Lambda t}}, \quad \Lambda = \text{const} > 0
\]

(146)

Such eternally accelerating Universes are plagued by the presence of finite *cosmic horizons* \(\delta_H\):

\[
\delta_H \propto \int_{t_0}^{\infty} \frac{dt}{a(t)} < \infty
\]

(147)

If the Universe does not exit from the inflationary (de Sitter) phase, then the inevitable existence of horizons will imply the *impossibility* of defining properly asymptotic states, and hence a scattering matrix \(S = e^{-iHt}\), where \(H\) the Hamiltonian operator of the Universe.

The situation is somewhat analogous to that of having space-time boundaries, e.g. due to the existence of microscopic black hole fluctuations in certain scenarios of quantum gravity. There is “information” loss in such a situation for asymptotic observers, due to modes crossing these boundaries, which implies that an asymptotic observer *cannot* define pure quantum states \(|\psi\rangle\), but only *mixed states*, defined by a density matrix \(\rho = \text{Tr}_M|\psi\rangle\langle\psi|\), obtained by tracing over unobserved degrees of freedom \(\{M\}\).

We recall that in a unitary quantum theory, the S-matrix connects asymptotic *in* states to asymptotic *out* states:

\[
|\text{out}\rangle = S|\text{in}\rangle
\]

(148)

On the other hand, if one encounters mixed states, as is the case of *open quantum mechanical systems*, non-equilibrium systems, or systems with space time boundaries, such as gravitational theories with local or global (cosmological) horizons, then the concepts of “in” and “out states” should be replaced by those of “in” and “out density matrices”, given that pure states evolve to mixed ones, as depicted in figure 9.

In such a case, as suggested first by Hawking \([18]\), one can still link the in and out density matrices using not the \(S\)-matrix, but another object, called \(\mathcal{S}\) matrix:

\[
\rho_{\text{out}} = \mathcal{S}\rho_{\text{in}}
\]

(149)

The operator \(\mathcal{S}\) factorizes into a product \(SS^\dagger\) *only* in pure state quantum mechanics without unobserved degrees of freedom, in which \(\rho = |\psi\rangle\langle\psi|\). In general, however, once there are
unobserved degrees of freedom, thereby opening up the system, as is the case of horizons (local or global) (see figure 9), the factorization property of $\mathcal{S}$ is lost:

$$\mathcal{S} \neq \mathcal{S} \mathcal{S}^\dagger$$

(150)

In such systems one cannot define on-shell scattering amplitudes.

This is a serious theoretical challenge for string theory. As we have discussed in Lecture 1, string theory is by construction a theory of on-shell $S$-matrix, based on scattering amplitudes which are reproduced by the appropriate conformal invariance conditions. Thus, de-Sitter Universes (with eternal acceleration) pose a challenge which needs to be resolved [19, 20].

In the remaining of these lectures I shall present some speculations as to how this problem can be tackled. A straightforward possibility would be to demonstrate the existence of string backgrounds which allow graceful exit both from the de-Sitter as well as the accelerating phases. Within critical (conformal) string theory such a scenario has not yet been achieved. However, I will discuss an alternative possibility to such critical string theory models, known as Liouville Strings [21], where graceful exit may be a realistic possibility. Such string theories are supposed to be mathematically consistent attempts to formulate $\sigma$-models away from their world-sheet renormalization-group fixed (conformal) points, i.e. $\sigma$-models for which the conditions (54) are not valid. The topic, however, is by no means as well established as critical strings, and therefore the treatment requires extreme caution. Nevertheless, as I will try to argue in this part of the Lectures, Liouville strings have some nice and quite interesting features which certainly support further studies and are worthy of discussion.
4.2 Cosmological Backgrounds in String Theory and World-Sheet Renormalization-Group Flow

We shall introduce the reader into the topic of Liouville strings by first elaborating further on time-dependent (cosmological) backgrounds of the $\sigma$-model theory. Consider for definiteness a $(d + 1)$-dimensional target space $\sigma$-model, describing propagation of a Bosonic closed string on a background consisting of the massless string multiplet of graviton $g_{MN}(\vec{x}, t)$, antisymmetric tensor $B_{MN}(\vec{x}, t)$ and dilaton $\Phi(\vec{x}, t)$ fields. Here $\vec{x}$ span a $d$-dimensional Euclidean space $(x^i, i = 1, \ldots d)$ and $t$ is the time. Assume the $(d + 1)$-dimensional $\sigma$-model at its conformal point, at which the conformal invariance conditions (154) are satisfied.

Target-space diffeomorphism invariance and the Abelian gauge symmetry associated with $B_{MN}$, discussed in Lecture 1, can be used to ensure:

$$ B_{0i} = 0, \quad G_{00} = -1, \quad G_{0i} = 0, \quad i = 1, \ldots d \quad (151) $$

A $(d + 1)$-dimensional string solution can be represented as a trajectory $f^A(t)$ in the space of $d$-dimensional $\sigma$-model fields $x^i$, with the time $t$ being a parameter along the trajectory:

$$ f^A = \{ g_{ij}(\vec{x}(t)), B_{ij}(\vec{x}(t)), \Phi(\vec{x}(t)) \} , \quad i, j = 1, \ldots d \quad (152) $$

[NB5: in this section we shall use the notation $M, N, \ldots$ for $(d+1)$-dimensional spacetimes, and $i, j \ldots$ for $d$-dimensional space]

The set of fields $f^A$ can be viewed as couplings of a $\sigma$-model in $d$-dimensional target space: \cite{22}

$$ S_\sigma = \frac{1}{4\pi \alpha'} \int d^2\sigma \left[ \sqrt{\gamma} g_{ij}(\vec{x}) \partial_\alpha x^i \partial^\alpha x^j + ie^{\alpha\beta} B_{ij}(\vec{x}) \partial_\alpha x^i \partial_\beta x^j + \alpha' \sqrt{\gamma} R^{(2)} \Phi(\vec{x}) \right] \quad (153) $$

As we shall discuss now, the orbits $f^A$ resemble standard world sheet renormalization group (RG) trajectories in the space of couplings $g^i$ of the two-dimensional theory \cite{153}. This is a very important feature which goes beyond a simple analogy \cite{23}, as we shall discuss later on in the lectures.

For the moment we note that the string theory \cite{153} lives necessarily in a non-critical dimension, since the $(d + 1)$-dimensional theory has been assumed critical. Thus, the couplings of \cite{153} lie away from their fixed point, and hence must have non-trivial RG flows (and therefore non-trivial Weyl anomaly coefficients). Their flows are to be identified with the flow in the real time $t$, as we shall discuss now \cite{23}.

Consider the $O(\alpha')$ $\tilde{\beta}$-functions of the $d$-dimensional $\sigma$-model theory:

$$ \tilde{\beta}_{g}^{(d)} = \alpha' \left( R_{ij} + 2 \nabla_i \nabla_j \Phi - \frac{1}{4} H_{imn} H_{j}^{mn} \right) , $$

$$ \tilde{\beta}_{B}^{(d)} = \alpha' \left( -\frac{1}{2} \nabla_m H_{ij}^{m} + H_{ij}^{m} \partial_m \Phi \right) , $$

$$ \tilde{\beta}_{\Phi}^{(d)} = \beta^{(d)} - \frac{1}{4} g_{ij} \tilde{\beta}_{g}^{(d)} = \frac{1}{6} \left[ c^{(d)}(\vec{x}) - 26 \right] , $$
\[
c^{(d)}(\vec{x}) = d - \frac{3\alpha'}{2} \left[ R - \frac{1}{12} H^2 - 4(\nabla \Phi)^2 + 4 \nabla^2 \Phi \right]
\]

(154)

Above the superscript \((d)\) denotes \(d\)-dimensional quantities, and \(c^{(d)}(\vec{x})\) is the Zamolodchikov running central charge [3] of the non-critical theory (153). This determines the \(d\)-dimensional target-space effective action [4, 6]:

\[
I_{\text{eff}, \text{off-shell}}^{(d)} = \int d^d x \sqrt{|g|} e^{-2\Phi(\vec{x})} \left[ c^{(d)}(\vec{x}) - 26 \right]
\]

(155)

The off-shell variations of (155) yield the \(\hat{\beta}\)-functions (154). It must be stressed that the non-criticality of the \(d\)-dimensional \(\sigma\)-model (153) implies that the \(\hat{\beta}\)-functions in (154) are non-vanishing (so the conformal invariance conditions (64) are not satisfied for the \(d\)-dimensional theory).

Let us now consider the corresponding \((d+1)\)-dimensional \(\hat{\beta}^i\), which should be set to zero, on account of the criticality assumption of the \((d+1)\)-dimensional theory. Let us then split the equations into temporal and spatial \((d\)-dimensional) parts. The result is facilitated if one uses a shifted dilaton:

\[
\varphi \equiv 2\Phi - \ln \sqrt{g}
\]

(156)

Then we have [22]:

\[
\begin{align*}
0 &= \hat{\beta}_{00}^{(d+1)} = 2\bar{\varphi} - \frac{1}{2} g^{ik} G^{j\ell} \left( \dddot{g}_{ij} \dddot{g}_{k\ell} + \dddot{B}_{ij} \dddot{B}_{k\ell} \right), \\
0 &= \hat{\beta}_{ij}^{(d+1)} = \hat{\beta}_{ij}^{(d)} - g^{00} \left[ \dddot{g}_{ij} - \dddot{\varphi} \dddot{g}_{ij} - g^{mn} \left( \dddot{g}_{im} \dddot{g}_{jn} - \dddot{B}_{im} \dddot{B}_{jn} \right) \right], \\
0 &= \hat{\beta}_{ij}^{B(d+1)} = \hat{\beta}_{ij}^{B(d)} - g^{00} \left( \dddot{B}_{ij} - \dddot{\varphi} \dddot{B}_{ij} - 2 g^{k\ell} \dddot{g}_{k\ell} \dddot{B}_{ij} \right), \\
0 &= c^{(d+1)} - 26 = c^{(d)} - 25 - 3 g^{00} \left( \dddot{\varphi} - (\dddot{\varphi})^2 \right)
\end{align*}
\]

(157)

where the superscript \((d+1)\) denotes critical dimension \(d+1 = 26\) Bosonic string quantities, and the indices 0 denote temporal components.

It is straightforward to show that these equations are derived from the action [22]:

\[
I_{\text{eff}}^{(d+1)} = \int dt d^d x \sqrt{|g^{00}|} e^{-\varphi} \left( c^{(d)}(\vec{x}) - 26 + 3 g^{00} \left( \varphi^2 - \frac{1}{4} g^{ik} g^{j\ell} \left( \dddot{g}_{ij} \dddot{g}_{k\ell} + \dddot{B}_{ij} \dddot{B}_{k\ell} \right) \right) \right)
\]

(158)

In addition, one also should satisfy the conditions (151), which imply the constraints:

\[
\begin{align*}
0 &= \beta_{0i}^{(d+1)} = \nabla_k \left( g^{k\ell} \dddot{g}_{ki} \right) - \frac{1}{2} \dddot{B}_{i\ell} H^{k\ell}_i + 2 \partial_i \dddot{\varphi} - g^{k\ell} \dddot{g}_{ki} \partial_k \varphi , \\
0 &= \beta_{0i}^{B(d+1)} = - g_{ik} \partial_j \left( g^{k\ell} g^{in} \dddot{B}_{j\ell} \right) + 2 \dddot{B}_{ij} \partial^j \varphi
\end{align*}
\]

(159)

which, to \(\mathcal{O}(\alpha')\), can be shown not to have any important consequences other than restricting the initial values of the fields and their derivatives [22].
To proceed with our cosmological solutions one should define quantities integrated over spatial coordinates $\vec{x}$, which thus have only a time dependence. In this spirit we define \[22\]:

$$\varphi_0(t) \equiv -\ln \left( \int d^d x e^{-\varphi(\vec{x},t)} \right)$$

This allows a splitting of the dilaton field $\varphi(\vec{x}, t)$ into $\vec{x}$-dependent and $\vec{x}$-independent parts:

$$\varphi(\vec{x}, t) = \varphi_0(t) + \tilde{\varphi}(\vec{x}, t)$$

From (156), then, we have:

$$\varphi_0(t) = -\ln \left[ \int d^d x \sqrt{g} e^{-2\Phi(\vec{x},t)} \right] \equiv -\ln V(d)$$

where $V(d)$ is the proper volume of the $d$-dimensional space.

One can also define the space-average of a function $f(\vec{x}, \ldots)$ as \[22\]:

$$\langle \langle f(\vec{x}, \ldots) \rangle \rangle = \frac{\int d^d x f(\vec{x}, \ldots) e^{-\varphi(\vec{x}, \ldots)}}{\int d^d x e^{-\varphi(\vec{x}, \ldots)}}$$

From this we observe that

$$\dot{\varphi}_0(t) \equiv -Q(t) = \langle \langle \dot{\varphi}(\vec{x}, t) \rangle \rangle$$

The time-dependent function $Q(t)$ is related to the central charge deficit, and hence to the $Q$ of the linear dilaton background (153), of Lecture 2, as follows: At the “fixed points” of the $t$-flow: $\dot{g}_{ij} = \dot{B}_{ij} = \ddot{\varphi} = \ddot{\Phi} = 0$, it follows that $Q = Q_0$-constant, and in fact:

$$\varphi_0(t) = -\frac{1}{2} Q_0 \ t + \text{const}$$

which is a linear dilaton background, analogous to that examined in \[22\], corresponding to conformal field theory models with central-charge deficits $Q_0$.

In general, however, away from the “fixed-pointas” of the $t$-flow, $Q(t)$ is a function of $t$. Integrating the dilaton equation in (last of) (157), and taking the space average (163), we obtain \[22\]:

$$\dot{Q}(t) + Q^2(t) = -\frac{1}{3} g^{00}(\bar{\tau} - 25) ,$$

$$\bar{\tau} = \tau(g, B, \varphi) \equiv \langle \langle \epsilon(d)(\vec{x}) \rangle \rangle = \frac{T_{\text{eff-off-shell}}^{(d)} V(d)}{V(d)} + 26$$

The function $\tau$ plays the rôle of the ‘running central charge’ of a non-conformal world-sheet field theory away from the fixed points, in the presence of non-constant dilatons. Notice that in case $\dot{Q} = 0$ then (166) becomes just the definition of the central charge deficit appearing in the linear dilaton background (95), or in standard non-critical strings [9, 21].
Using the first of equations (157) one may compute $\dot{Q}$:

$$
\dot{Q}(t) = \langle\langle (-\ddot{\varphi} + \dot{\varphi}^2 - Q^2) \rangle\rangle = \langle\langle (\dot{\varphi} - \langle\langle \dot{\varphi} \rangle\rangle)^2 \rangle\rangle = 
\langle\langle (\dot{\varphi} - \langle\langle \dot{\varphi} \rangle\rangle)^2 - \frac{1}{4} g^{ik} g^{jl}(g_{ij} g_{kl} + \dot{B}_{ij} \dot{B}_{kl}) \rangle\rangle
$$

(167)

We also notice that the first three of (157) can be written in a compact form [22]:

$$
\ddot{g} + Q(t) \dot{g} = g^{00} \ddot{\beta} + O(\dot{g}^2),
\tilde{g} = \{g_{ij}(\vec{x}), B_{ij}(\vec{x})\},
$$

(168)

with

$$
Q^2(t) = -\frac{1}{3} g^{00} [\tau(\tilde{g}) - 25] + \frac{1}{4} \dot{g}^2,
\dot{g}^{00} = -1 \text{ if } \tau(\tilde{g}) > 25,
\dot{g}^{00} = +1 \text{ if } \tau(\tilde{g}) < 25.
$$

(169)

The equations (168) are sufficient to describe the theory in the vicinity of fixed-points (with respect to the $t$-flow) in the space of couplings $\{\tilde{g}\}$ of the $\sigma$-model (153). Notice the “friction” form of these equations, due to the presence of a non-trivial dilaton (164).

### 4.3 Liouville Strings and Time as a world-sheet RG flow parameter

The similarity of the $t$-flow with the two-dimensional renormalization-group flow is more than a mere analogy, and if made [23], it results in some important consequences for the underlying physics [4]. In that case, i.e. after identifying the target time $t$ with a renormalization-group flow parameter on the world sheet of the $\sigma$-model (153), the $t$-dependence of $Q(t)$ is identified with the RG scale dependence of the running Zamolodchikov central charge [3] of this two-dimensional non-conformal theory.

Notice that the equations (168) refer to couplings of a non-conformal $\sigma$-model, in a $d$-dimensional target space, which however can become conformal in one target-space dimension higher, i.e. by making the trajectory parameter $t$ a fully-fledged quantum field in the $\sigma$-model. In this sense, the equations (168) may be thought of as a generalization of the conformal invariance conditions $\ddot{\beta}^i = 0$ (14) of a critical (fixed point) theory. This is precisely the principle of Liouville Strings [21].

From this point of view the equations (168) stem from the following fact: As just said, Liouville theory [21] restores conformal invariance of $\sigma$-models which are away from their fixed points, by coupling them with an extra fully fledged world-sheet quantum field $\rho(\sigma, \tau)$, the Liouville mode. If a vertex deformation $V_i$ is not a conformal (marginal) operator of the $\sigma$-model, then the “Liouville-dressed” operator:

$$
V_i^L \equiv e^{\alpha \rho(\sigma, \tau)} V_i
$$

(170)

It should be stressed, though, that this is not the interpretation adopted by the authors of ref. [22].
is a marginal operator, in the two-dimensional renormalization group sense. The quantity \( \alpha_i \) is known as the ‘gravitational anomalous dimension’ [21], and it satisfies the equation (for \( \tau \geq 25 \) we are interested in here):

\[
\alpha_i (\alpha_i + Q) = \Delta_i \quad \text{no sum over } i
\]  

(171)

where \( Q \) is a ‘charge at infinity’, with \( Q^2 \) denoting the central charge deficit, and \( \Delta_i = h_i - 2 \) is the anomalous dimension of the operator \( V_i \), with \( h_i \) its conformal dimension. We repeat that eq. (171) is nothing other than the condition that the Liouville dressed operator \( V^L_i \) have vanishing anomalous dimension [21].

Consider, now, a Liouville-dressed deformation of the \( \sigma \)-model (170). The gravitationally-renormalized couplings can be read off directly from this expression as: \( g_i^L \equiv g_i e^{\alpha_i \rho} \). Considering the second derivative of \( g_i^L \) with respect to the world-sheet zero mode of the Liouville field, \( \rho_0 \), and using (171), one can arrive [23] at equations of the form (168), with the overdot denoting differentiation with respect to \( \rho_0 \). In such equations the \( O(\dot{g}^2) \) terms stem from possible \( \rho_0 \) dependence of \( Q \), as in our case.

The Liouville mode \( \rho(\sigma, \tau) \) is nothing other than a dynamical \( \sigma \)-model field mode, which appears in the sum over geometries of a non-conformal \( \sigma \)-model through, e.g. the conformal gauge fixing (14). In a conformal field theory the Liouville mode decouples from the world-sheet path integral. This is not the case, however, in a non-conformal \( \sigma \)-model, and this is what we demonstrated above with our simplified example of stringy cosmology. In such non-conformal cases, the Liouville mode becomes a fully fledged \( \sigma \)-model field in order to restore the lost conformal invariance of the \( \sigma \)-model. From a physical point of view the reader’s attention is drawn to the property (169) of the central charge deficit in a Liouville theory. As we have seen above, the \((d+1)\)-dimensional target-space time (after taking into account the Liouville field as a time coordinate) has a Minkowskian signature for supercritical strings, i.e. \( \tau > 25 \) [1, 23], and Euclidean signature for subcritical strings, i.e. \( \tau < 25 \).

In other words, the above-described “Liouville dressing” procedure implies a temporal signature for the Liouville field, which can thus be identified with the time \( t \), only in the case where the central-charge deficit of the non-conformal \( \sigma \)-model theory is supercritical [1, 23]. By construction (14), the Liouville mode may be viewed [23] as a local world-sheet renormalization-group scale, since it enters the expression of a covariant cutoff distance in space, necessary for regulating ultraviolet divergencies in curved space in a way compatible with two-dimensional general covariance [6]. The target-time then is nothing other than the world-sheet zero mode of the Liouville field [23].

In this interpretation of target time as a world-sheet renormalization group scale there is hidden an important property, which makes the Liouville coordinate different from the rest of \( \sigma \)-model coordinates. That of its irreversibility [23]. This stems from the fact that a world-sheet RG flow encodes information loss due to the presence of an ultraviolet cutoff in the theory, and as such is irreversible. This irreversibility can be expressed in terms of the irreversibility of the flow of the running central charge of the non-conformal cut-off theory [3] (Zamolodchikov’s c-theorem), \( \dot{c} \leq 0 \) towards a non-trivial infrared fixed point. We shall come back to this important point later on.
Notice that the central charge has been argued to count physical target-space degrees of
cost of a stringy $\sigma$-model [24], and hence its decrease along a RG trajectory
is in perfect agreement with the loss of degrees of freedom in a cosmological situation with
horizons as the time (RG scale) evolves. It is for this reason that Liouville strings with the
time identified with a world-sheet RG scale are viewed as sort of non-equilibrium string
theories, with the conformal strings corresponding to equilibrium points [23]. What we shall
do in the remainder of the lecture, then, is to discuss some important physical features of
Liouville strings, such as time-dependent vacuum energy for the Liouville Universe, as well
as the impossibility of defining a proper on-shell scattering matrix for a Liouville string.
We shall also revisit the de Sitter string Universes from this point of view, and present
various possibilities for a graceful exit from the de Sitter, or in general, the accelerating
phase in the context of string theory.

4.4 Liouville String Universe and time-dependent Vacuum Energy

The presence of a time dependent central charge deficit $Q(t)$ in Liouville strings on cosmo-
logical backgrounds, with the time identified with the world-sheet RG scale [23], implies
- from the point of view of the corresponding effective target-space action (158)- a time-
dependent dilaton potential, already at tree level world-sheet topologies [23]:

$$I^{(d+1)} \equiv \int \frac{dtd^dx}{\sqrt{g(\vec{x},t)}} e^{-2\Phi(\vec{x},t)} \left[ -g^{00}Q^2(t) \right]$$

One should compare this term with the corresponding term (108) of the model of [9] (after
appropriate metric redefinitions to go to the Einstein frame). In that case, $\delta c$ came from
the internal conformal field theory (Wess-Zumino model), and this is why it turned out to
be constant. In contrast, in (172), which represents a more general situation, the deficit
depends on the RG scale $t$, since the underlying $\sigma$-model theory is considered away from
its fixed point (unlike the situation in [11]).

One may construct consistent examples of string theories, compactified appropriately
to four-dimensional cosmological backgrounds [25], in which the theory flows to a linear
dilaton conformal field theory background of [3] asymptotically, as $t \to \infty$ (which here plays
the role of the infrared fixed point). The non-conformality of the original theory is then
attributed to some sort of fluctuations of the geometry, which result in departure from
equilibrium of the corresponding string theory.

Such non-critical string theories allow for relaxing to zero vacuum energies, asymptoti-
cally in time. Indeed, in the Einstein frame, the respective vacuum energy densities have the form [25]:

$$\sqrt{g(\vec{x},t)} \Lambda^E = \sqrt{g(\vec{x},t)} e^{2\Phi(\vec{x},t)} Q^2(t) \to \sqrt{g(\vec{x},t)} \frac{Q^2_0}{t^2}, \quad t \to \infty$$

which is a consequence of the fact that, as $t \to \infty$, the theory flows to a conformal field
theory of ref. [3], i.e. $Q^2(t) \to Q^2_0=$constant, and $\Phi \to -\ln t$ in the Einstein frame, with
t the Robertson-Walker time, discussed previously in the Lectures. Such vacuum energies are compatible with recent observations [15, 16], and in fact there is a similarity here with *quintessence models* [17], where the rôle of the quintessence field is played by the dilaton [23, 25].

### 4.5 No Scattering Matrix for Liouville Strings

When consider a Liouville string, which as discussed above represents a mathematically consistent description of a string theory away from its conformal point, the concept of a string scattering amplitude breaks down. Below, I shall not give a detailed discussion of this important issue, but I would rather sketch the main reason behind it in a simple way. For details the interested reader is referred to the literature [23, 20].

Consider a generic correlation function among $n$ vertex operators $V_i$ of a Liouville string. In a critical string theory, this can be associated with appropriate on-shell scattering amplitudes. In Liouville strings, though, with the target-time identified as the Liouville (RG) mode, this association cannot be made. Let us see briefly why. In such a case the correlator reads:

$$\langle V_{i_1} \cdots V_{i_n} \rangle_g = \int D\rho D\Sigma e^{-S^{\ast} - g^i \int_\Sigma d^2\sigma V_i + Q^2 \partial \rho \bar{\partial} \rho - Q^2 \rho \int_\Sigma d^2\sigma \rho R^{(2)}} V_{i_1} \cdots V_{i_n}$$  \hspace{1cm} (174)$$

where $\rho$ is the Liouville mode, and $Q^2$ denotes the central charge deficit, quantifying the departure of the non-conformal theory from criticality [21].

![Diagram of contour integration](attachment:Contour.png)

**Figure 10:** Contour of integration for a proper definition of Liouville field path integration. The quantity $A$ denotes the (complex) world-sheet area, which is identified with the logarithm of the Liouville (world-sheet) zero mode. This is known in the literature as the Saalschutz contour, and has been used in conventional quantum field theory to relate dimensional regularization to the Bogoliubov-Parasiuk-Hepp-Zimmermann renormalization method. Upon the interpretation of the Liouville field as target time, this curve resembles closed-time-paths in non-equilibrium field theories.

A detailed analysis [23] shows that, upon performing the world-sheet zero-mode $\rho_0$ integration of the Liouville mode $\rho$ in (174), one obtains that the dominant contributions
to the path integral can be represented by a steepest-descent contour of $\rho_0$ as indicated in fig. 10. The interpretation of the Liouville zero mode as the target time implies a direct analogy of this contour with closed time like paths in non-equilibrium field theories [23].

When consider infinitesimal Weyl shifts of the world-sheet metric of the correlators $\delta_w \langle V_{i_1} \ldots V_{i_n} \rangle$, then a straightforward but rather tedious world-sheet analysis shows that [23]:

$$\delta_w \langle V_{i_1} \ldots V_{i_n} \rangle \propto O\left(\frac{s}{A}\right) \langle V_{i_1} \ldots V_{i_n} \rangle + A \text{ - independent terms} \quad (175)$$

where $s = \sum_i \alpha_i/Q$ is the sum of the corresponding Liouville anomalous dimensions of the vertex operators $V_i$ [21], and $Q^2$ is the corresponding central charge deficit. The $\alpha_i$ are defined such that, if $V_i$ is not a conformal (marginal) operator, then the “Liouville-dressed” operator $V_i^L \equiv e^{\alpha_i \rho(\sigma,\tau)} V_i$ is a marginal operator.

In the scenario of [23], the identification of the world-sheet area (covariant scale) $A$ with $e^{-t}$, where $t$ is the target time, implies therefore, on account of (175), that these correlators do exhibit time-dependence, and as such cannot be associated with on-shell S-matrix elements. Such an association can only be made at the infrared fixed point of the world-sheet flow, $A \to \infty$, where the string reaches its equilibrium position. It should be mentioned though that the definition of the correlators (174) on the closed-time-like contour of fig 10 implies that they represent $\mathcal{S}$ elements, associated with density matrices.

To understand better this last point, it suffices to mention that the world sheet partition function $Z$ of a conformal $\sigma$-model, resummed (in general) over world-sheet topologies, is related to the wavefunctional $\Psi[g]$ of the underlying string theory:

$$Z[g] \equiv e^{-I_{\text{eff}}[g]} \leftarrow \Psi[g] \quad (176)$$

where $I_{\text{eff}}[g] = \int dt d\vec{X} \mathcal{L}[g]$, with $t$ the time, and $\vec{X}$ spatial coordinates, is the target-space effective action of the backgrounds $g$, which is the appropriate Legendre transform of the generating functional of connected correlators in target space.

In the non-critical string approach of [23], discussed here, the time $t$ is nothing other but the world-sheet zero mode of the Liouville field $\rho(\sigma,\tau)$. As we have discussed above, the proper definition of Liouville correlators necessitates an integration of this time variable over the closed-time-like path of fig. 10. Due to the different sense of the two branches of this contour, it is then straightforward to see that, upon analytic continuation to the target-space Minkowski formalism, the middle side of (176) becomes “almost” the product of $\Psi \Psi^\dagger$ (with $\Psi, \Psi^\dagger$ associated with, say, the lower (upper) branch of the curve of fig. 10). We say “almost”, because, as discussed in some detail in [23], there are world-sheet infinities around the turning (ultraviolet) point of the curve ($A \sim 0$), whose regularization (dashed curve in fig. 10) prevents such a complete factorisation. In this sense, the world-sheet Liouville correlation functions are associated with $\mathcal{S}$-matrix elements, linking density-matrices instead of pure quantum states.

In this respect, one might conjecture [21] that an eternally accelerating Universe can be represented by a (non-equilibrium) Liouville rather than critical string, with the target time variable being identified with the world-sheet zero mode of the Liouville field. This is
consistent with the previous discussion in the beginning of this section, on the impossibility of constructing a proper $S$-matrix in such situation, but rather a $\mathcal{S}$ matrix, non factorizable in $SS^\dagger$.

4.6 Graceful Exit from Inflation in Liouville Strings

In the previous section we have argued on the equivalence of a Liouville string theory with a non-equilibrium dynamical system, for which asymptotic states cannot be defined properly. From a physical point of view, one of the most interesting applications are the eternally accelerating Universes, characterized by cosmic (global) horizons beyond which an observer cannot “see”, and hence the system is open.

Another interesting possibility, however, can arise in the context of non-critical strings, namely that of a graceful exit from the de Sitter or in general the accelerating phase. Such a possibility has been discussed in detail in [24], in the context of a specific cosmological model based on the so-called type 0-string theory [27]. Such models involve three-dimensional branes worlds (appropriate stringy domain walls), playing the role of our observable Universe. We shall not discuss details here, but outline the main results of that work. Due to the specific choice of a background flux field characterizing the type 0 strings [27], the internal dimensions freeze out after inflation in different sizes in such a way that one dimension (along the chosen flux background) freezes out to a much larger size than the others, thereby implying an effectively five-dimensional model. In such a model the departure from criticality is provided by quantum fluctuations of the three-dimensional brane worlds.

The model has an inflationary (de Sitter type) phase, characterized by a positive dilaton potential, and then a smooth exit from it. It is crucial, for consistency of the theory that the central charge deficit, quantifying the departure from criticality, depends on time. Immediately after the inflationary period the Universe enters a decelerating phase, which is succeeded by an accelerating one [25]. The important feature of this model is that, asymptotically, for large times, it tends to a non-accelerating conformal field theory with a linear dilaton in the $\sigma$-model frame [9] (or, equivalently logarithmic dilaton in the Einstein frame, depicted in fig. 14). Asymptotically, the dilaton potential, which plays the role of an (equilibrium) vacuum energy, relaxes to zero as a quintessence like field (173), the role of the quintessence field being provided by the dilaton. However, we stress again, here one encounters a non-eternally accelerating quintessence model. During such phases the behaviour of the central charge $Q^2(t)$ is as indicated in figure 11 in the Einstein frame. Notably, due to the Minkowski signature of the target time (“non-unitary” $\sigma$-model field) there is some oscillation of the central charge before relaxing into its asymptotic infrared fixed-point value. There is a conformal metastable point at which momentarily the theory becomes critical ($Q^2 = 0$), and after this there is some oscillatory behaviour until the theory settles in its final infrared fixed point. The existence of the conformal metastable point is a result of the fact that the theory asymptotes to that of a linear dilaton. In such a case the dilaton equation forces $Q$ to change sign at a certain stage of the evolution [25]. Despite the oscillatory behavior, however, there is an overall decrease of the central charge.
Figure 11: The behaviour of the central charge deficit (upper) and the dilaton (continuous line) and scale factor (dashed line) (lower), in the Einstein frame, during the various evolutionary phases of the cosmological non-critical type-0 string theory of ref. [25]. The central charge relaxes asymptotically to a constant value, when the model asymptotes, for large times, to a conformal field theory of the type of ref. [9], describing a non-accelerating Universe with a negative (logarithmically divergent) dilaton. The diagram inside the box on the right shows the cosmic acceleration for late Einstein times, indicating the passage from a decelerating phase after inflation, to an accelerating one, with asymptotic exit from it.

as it flows from the Gaussian (UV) fixed point value (Big-Bang? Early Universe) to the infrared one (far future). Unfortunately, the perturbative $\mathcal{O}(\alpha')$ calculations of [25] (solid line in fig. 11) cannot give sufficient information on the value of the UV fixed point (dashed line) at present, but we conjectured in [25] that the initial fixed point (constant) value of $Q^2$ is also finite, corresponding to a given conformal field theory.

It is interesting to remark that in this model, at late stages of the evolution, the string coupling $g_s = e^\Phi \ll 1$, and thus perturbation theory applies. This is due to the fact that the dilaton asymptotes to $-\infty$ for large times. This situation has to be contrasted with the pre-Big-Bang scenario [14] where the weak field regime occurs for early (pre-Big-Bang) Universes.

The $\mathcal{O}(\alpha')$ analysis of [25] implies initial singularities (Big-Bang type), but, as mentioned already, this may be an artifact of the lowest-order truncation. Summing up higher orders of $\alpha'$ corrections, as well as world-sheet topologies, in other words going to a fully non perturbative string level, may indeed lead to the removal of such singularities. For
instance, this is known to be the case in some stringy cosmological models with curvature-squared corrections of $\mathcal{O}(\alpha'^2)$, in the string effective action $[28]$. The latter effects are known to be induced by string loops.

The asymptotic exit from the accelerating phase, and the absence of cosmic horizons in the model of $[23]$ is a very welcome feature from the point of view of the possibility of defining asymptotic states $[19, 20]$, and hence a proper $S$-matrix (for this, however, a resolution of the initial singularities will be desirable, if not essential). In this respect, our work is somewhat similar in spirit to the arguments of $[29]$, where eternal quintessence was argued not to occur in perturbative string theory, which thus was conjectured to exhibit exit from de Sitter phase, and have a proper $S$-matrix, calculated though by purely non-perturbative methods.

The basic argument of $[29]$, which however, we stress, should not be considered as a rigorous proof, can be summarized as follows: in perturbative string cosmology, like the case examined in $[9, 23]$, but not in PBB scenaria $[11]$ (see fig. 6), the dilaton potential $V_{\text{dil}}$ vanishes asymptotically in time, together with the energy $E$ of the dilaton field $\Phi$, which, in this context, plays the rôle of a quintessence field. In the framework of (low-energy) perturbative string-inspired Friedmann-Robertson-Walker Cosmologies, involving the (minimal) coupling of the dilaton field to gravity, it can be shown that the existence of cosmic horizons $[47]$ depends on how fast $V_{\text{dil}}$ approaches zero as compared with $E$. In critical strings, as we have discussed in Lecture 2, a non trivial dilaton potential is generated through string loops via the Fischler-Susskind mechanism $[10]$ (dilaton tadpoles), and as such it is given by infinite sums of the form $[137]$, being proportional to various powers of the string coupling $g_s \sim e^\Phi$. In the case of a non-perturbative string potential, then, one expects such resummations to exponentiate, and in this case $V_{\text{dil}}$ would be the exponential function of an exponential of the dilaton field $\Phi$. On the other hand, general arguments $[29]$ can be given in support of the fact that in perturbative string theory, i.e. in regimes where the string coupling is weak, so that $\sigma$-model perturbation theory is valid, $E$ has at most a power-law dependence on $g_s$. Thus, as $\Phi \to -\infty$, one has that $E \gg V_{\text{dil}}$ and, therefore, there will be no cosmic horizon, in the sense that the integral $[147]$ would diverge in the limit $t \to \infty$.

Notice, however, one important difference of the non-critical string approach of $[23]$ from that of $[24]$. As just mentioned, in standard critical string theory, a positive cosmological constant in the effective action, as required by the de Sitter phase, is obtained through string loops. In contrast, as we have discussed in this Lecture, the non-criticality of the stringy model of $[25]$ introduces a vacuum energy (dilaton potential) already at a tree $\sigma$-model level.

There are many open issues that are left undiscussed in the non-critical string approach, regarding the phase after inflation, such as reheating etc. These are open issues for future work. I must stress though that, although the non-critical string approach to cosmology appears promising, and already gave physically interesting results, such as the possibility of graceful exit from de Sitter (and in general accelerating) Universe phase, nevertheless it is still very far from being considered as well established. So far we have treated the departure from criticality at a “phenomenological” level, by treating the time dependence of
the central charge deficit as being determined by consistency with the rest of the Liouville conditions (168), which replace the conformal invariance (64) conditions of the critical strings. To be complete one should discuss explicitly the internal conformal field theory (pertaining to the extra dimensions), whose ‘flow’ between fixed points results in the $Q(t)$ under consideration. Moreover, from the physical viewpoint one should also examine the rôle of supersymmetric target-spaces in cosmological scenaria. Note that even in the case of type-0 strings, with explicitly broken supersymmetry, fermionic target-space backgrounds do exist, given that the original underlying theory is a superstring [27]. These issues present important theoretical challenges, awaiting further studies, which, in my personal opinion, is something that should be done.

5 Conclusions

In these lectures I have tried to give a brief account of interesting cosmological scenaria from the point of view of string theory. As we have seen, there are amusing possibilities, such as a pre-Big-Bang life of the Universe, graceful exit from accelerating Universe phases etc., which do not seem to be characterizing conventional cosmological models.

Recent experimental developments in the field of astrophysics, concerning for instance the possibility for the current era of the Universe to be an accelerating phase, present important theoretical challenges for string theory, which probably necessitate a fresher look at string cosmology. One such possibility might be the representation of a cosmological (time-dependent) background of string theory as a non-critical (non-conformal), non-equilibrium situation. Although speculative, such a possibility seems, at least to the me, a mathematically viable one, if the non-conformal nature of the background is seen from the point of view of a renormalization-group flow between fixed (equilibrium) points in string theory space.

In this context it should be mentioned that there are many explicit models one can construct, which exhibit graceful exit from de Sitter, or, in general, accelerating phases. One of them was presented in [25], and analysed briefly in these lectures. Additional non-critical string models with such exit properties can be found in toy two-dimensional non-critical stringy cosmologies [30], where the non-criticality is induced by initial fluctuations of matter backgrounds. Moreover, in higher-dimensional theories, one encounters such graceful exit properties in cases of intersecting brane cosmologies. For instance, it can be shown that if one represents our universe as a three-brane domain wall, punctured with D-particles (point-like solitonic defects) [31], then recoil of these D-particle during scattering with macroscopic numbers of closed string states propagating on the brane can also lead to space-time back reaction, which is sufficient to induce exit from an accelerating phase, so that the final equilibrium theory will again asymptote to a conformal field theory of the type of ref. [9].

In general, there are many issues in the context of string cosmology that remain open, apart from the exit problem. Issues like reheating after the inflationary phase, the rôle of supersymmetry in inflationary scenaria etc, are some of them. We have not touched
such issues here, but we believe that we have presented enough material in this admittedly brief and by far not complete exposition, which would motivate the interested reader to do further research in the exciting directions opened up by string cosmology.

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