Static Level Ancestors in Practice

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Abstract. Given a rooted tree $T$, the level ancestor problem aims to answer queries of the form $\text{LA}(v, d)$, which identify the level $d$ ancestor of a node $v$ in the tree. Several algorithms of varied complexity have been proposed for this problem in the literature, including optimal solutions that preprocess the tree $T$ in linear bounded time and proceed to answer queries in constant time. Despite its significance and numerous applications, to date there have been no comparative studies of the performance of these algorithms and few implementations are widely available. In our experimental study we implemented and compared several solutions to the level ancestor problem, including three theoretically optimal algorithms, and examined their space requirements and time performance in practice.

Keywords: level ancestors, algorithm implementations, tree queries

1 Introduction and background

The Level Ancestor (LA) problem is a fundamental problem on trees, defined as follows: Given a rooted tree $T$ with $n$ nodes, and queries of the form $\text{LA}(v, d)$, where $v$ is a tree node and $d$ a non-negative integer, find the depth $d$ ancestor of node $v$, meaning the $d$th vertex on the path from the root to $v$. Naively, such queries can be answered in $O(n)$ without the use of any auxiliary data structure, or alternatively one could precompute all possible queries in $O(n^2)$ time and space, allowing for constant time bounded queries.

The level ancestor problem is related to the extensively studied Least Common Ancestor (LCS) problem. Indeed, Harel and Tarjan\cite{Harel1984-Structure}
use level ancestors as a subroutine of an LCA algorithm. Level ancestors are used as part of space-efficient ordinal trees, which can be used for the representation of XML documents that support XPath queries. Level ancestor queries are part of the primitive operations that are supported by other compressed data structures and are used in range-aggregate queries in trees.

There are several known optimal solutions to the level ancestor problem with $O(n)$ complexity for preprocessing and $O(1)$ query time. Dietz first published an optimal algorithm for the dynamic version of the problem, where Berkman and Vishkin published an optimal parallel (PRAM) algorithm, albeit involving unwieldy constants of the order of $2^{28}$. It is noteworthy that complex algorithms for the dynamic and parallel versions of the level ancestor problem preceded simpler algorithms for the static serial version. A substantially simplified algorithm for the dynamic and static variants was published by Alstrup and Holm and an even simpler optimal algorithm – due to its progressive construction in stages – for the static version was given by Bender and Farach-Colton. Ben-Amram contributed yet another simple optimal algorithm for the static level ancestor problem in a technical report, along with an efficient implementation.

The remainder of this paper is organized as follows. In Section 2 we provide definitions and review the algorithms involved in this study. In section 3 the implementations of each algorithm are briefly explained. In Section 4 we describe the experiments performed and the results obtained. Finally, in Section 5 we draw conclusions about the advantages and limitations of the different algorithms compared.

2 Definitions, experimental design and implementations

Following the notation and definitions in, the depth of a node $v$ in tree $T$, denoted as $\text{depth}(v)$, is the number of edges on the shortest path from $v$ to the root. The root itself has depth 0. The height of a node $v$ in tree $T$, denoted $\text{height}(v)$, is the number of nodes on the path from $v$ to its deepest descendant. Tree leaves have height 1. The level ancestor of node $v$ at depth $d$, denoted as $\text{LA}(v, d)$, is a node
u, such that u is an ancestor of v and \( \text{depth}(u) = d \). If such a node does not exist, then \( \text{LA}(v, d) \) is undefined. The algorithms described in this paper have both preprocessing and query time complexity. An algorithm that has preprocessing time \( f(n) \) and query time \( g(n) \) will be denoted as having complexity \( (f(n), g(n)) \).

For the purpose of this study we have implemented eight algorithms for the level ancestor problem, each with distinct time and space complexes. Four of the algorithms are components of the optimal algorithm of Bender and Farach-Colton, described in [4] and are eventually combined. The eight algorithms implemented are listed below, together with their preprocessing and query time bounds:

1. Table algorithm – \( \langle O(n^2), O(1) \rangle \)
2. Jump-Pointers algorithm – \( \langle O(n \log n), O(\log n) \rangle \)
3. Ladder algorithm – \( \langle O(n), O(\log n) \rangle \)
4. Jump-Ladder algorithm – \( \langle O(n \log n), O(1) \rangle \)
5. Macro-Micro-Tree algorithm – \( \langle O(n), O(1) \rangle \)
6. Menghani & Matani algorithm – \( \langle O(n), O(\log n) \rangle \)
7. Ben-Amram algorithm – \( \langle O(n), O(1) \rangle \)
8. Hagerup algorithm – \( \langle O(n), O(1) \rangle \)

Each of these algorithms have their own efficiency/simplicity trends.

3 Level Ancestor Implementations

In all our implementations the tree is stored in an array, based on the Depth First Search (DFS) numbering. This representation, which is equivalent to a pointer-based tree structure for the static Level Ancestor problem, helps avoid a level of indirection when locating a node based on its DFS number. In addition, a DFS procedure only has to visit the table contents in order. This representation can be extended for dynamic approaches, albeit with loss of some of the advantages mentioned. The space utilized by this representation is optimal. Every node stores its parent, left and right child pointers, as well as a field for the depth of the node, which is required by all algorithms we discuss, except for the table algorithm.
3.1 The Table Algorithm

The table algorithm for the level ancestor problem is the naive solution that preprocesses and stores all distinct $O(n^2)$ queries. The simplest of all algorithms presented, it is implemented with dynamic programming, given the Euler tour of the tree, which we will call the tree signature. A 2-dimensional table holds the level ancestors of all nodes. The ancestor list of each node can be progressively generated from the parent node ancestor list, when traversing the tree in the DFS order.

This is an optimal query time algorithm, able to answer level ancestor queries by performing only two memory references per query, an attribute which makes this algorithm superior to any other solution when space is not an issue. The Table algorithm constitutes a compelling solution when dealing with balanced trees or at least trees with logarithmic bounded expected node depth, and the programming language of choice supports ”ragged” arrays (or arrays of arrays), as is the case with C and Java. Preprocessing time is roughly proportional to the space used to store the array.

3.2 The Jump-Pointer Algorithm

The jump-pointer algorithm associates every node in the tree with a list of pointers to its ancestor nodes, that allow to “jump” up the tree by powers of 2. Specifically, for every node $v$ in the tree, there exists a list associated with $v$ that contains pointers to all $2^i$th ancestors of $v$, for $0 \leq i \leq \lceil \log \text{depth}(v) \rceil$.

Using dynamic programming, the pointer lists can be generated in $O(n \log n)$ time by an Euler tour of the tree. Queries can be processed in $O(\log n)$ time by following jump pointers up the tree, covering at least half of the remaining distance to the desired ancestor with each jump.

3.3 The Ladder Algorithm

This algorithm starts with a longest path decomposition of the tree $T$, into non-disjoint paths called ladders. It proceeds by extending the ladders towards the root, up to twice their original size. This latter action creates the property that every node of height $h$ belongs to a
ladder that includes its ancestor of height at least $2h$, or the root of the tree, allowing for queries in $O(\log n)$ time.

The ladder algorithm can be implemented by a bottom-up scan of the tree, in which the heights of the nodes are calculated. Ladders can be constructed recursively, starting from the deepest node and queuing nodes encountered while traversing the current ladder, which are subsequently processed. Careful implementation leads to a linear time and space algorithm, which, as will be shown in the experimental section below, leads to significant space savings, while still being competitive in the time domain.

### 3.4 The Jump-Ladder Algorithm

The jump-ladder algorithm combines the preprocessing benefits of the jump-pointer and ladder algorithms to achieve constant time queries. Since the jump-pointer algorithm makes exponentially decreasing hops up the tree and the ladder algorithm makes exponentially increasing hops up the tree, following a single jump pointer and then climbing a single ladder leads to the desired level ancestor in two steps.

The preprocessing stage combines the jump-pointer and ladder algorithm preprocessing procedures, resulting in $O(n \log n)$ complexity in time and space, where the actual space utilized is the sum of the space required by both component algorithms.

### 3.5 The Macro-Micro-Tree Algorithm

The Macro-Micro-Tree algorithm combines and extends the jump-pointer and ladder algorithms. Since ladders can be preprocessed in linear time, they are used without modifications. Jump pointers though are not assigned to all nodes – since the preprocessing would then require $O(n \log n)$ space and time – but only to a specific set of nodes, which are called jump nodes. Ancestors of jump nodes can use their jump pointers, and are called macro nodes, forming a connected subtree of $T$ called the macrotree. Descendants of jump nodes form disconnected microtrees. By applying the standard data structural technique in $\mathbb{R}$ and limiting the size of the microtrees to $\lceil \log n/4 \rceil$, the number of jump nodes is bounded by $n/\log n$ and the total number of
jump pointers becomes $O(n)$. Due to their limited size, it has been shown that microtrees adopt a limited number of shapes, $O(\sqrt{n})$, which can be precomputed and stored using the table algorithm, also in linear bounded time.

The Macro-Micro algorithm has optimal time complexity, but also significant implementation complexity, with quite a few details and subtleties on top of the other four implementations. In addition to the node depth and height pre-calculations, the weight of each node, as in the size of the subtree rooted at a node, has to be calculated in order to characterize Micro and Macro nodes. Overall there are four distinct types of nodes, the Micro, the Macro, the Micro-Root (roots of the Micro trees) and the Jump nodes, the latter been associated with jump pointers, which can be utilized by their Macro ancestors. Extra space is required to store auxiliary pointers for each node, in addition to data structures that hold the tables for the Micro trees.

To locate appropriate entries in the tables of the microtrees, the microtree signatures are enumerated and stored, being indexed when probed by level ancestor queries. The queries themselves have multi-case functionality, using a different approach to locate the level ancestor of a node based on its type.

### 3.6 Menghani & Matani algorithm

One of the simplest algorithms for the static level ancestor problem was developed by Gaurav Menghani and Dhruv Matani.\(^7\) The algorithm performs a pre-order traversal of a rooted tree to assign to each node a unique label in the order that it is visited, with the root starting as zero. Node labels are unique (no duplicates) and values range from 0 to $(n - 1)$, for a tree with $n$ nodes. The algorithm takes advantage of a property of this tree that for any given node $v$, all descendants of $v$ have label values greater than that of $v$ and smaller than the label of $v$'s closest right-sibling node. This property allows us to find a depth $d$ ancestor of any given node $v$ by locating the largest label value node at depth $d$ less than $v$’s label value. In practice, this is accomplished using a two-dimensional array for every depth $d$ containing every node at that depth in ascending label value order. To answer level-ancestor query $LA(v, d)$, a modified bi-
nary search is performed on the array for depth $d$ that uses $v - 1$ as the goal value and keeps track of the maximum value less than the goal value found. If the goal value is found, then the associated node with that label is the answer to the level ancestor query, otherwise it is the node associated with the maximum value found. This results in a sub-optimal query time of $O(\log n)$, though it can be improved to $O(\log \sqrt{n})$ for sufficiently large arrays using a faster binary search technique.

3.7 Ben-Amram’s Find-Smaller algorithm

Another relatively simple optimal algorithm for the static level ancestor problem was presented by Ben-Amram, which is also utilizes the microset technique. His algorithm uses the Euler tour representation of a rooted tree and is a simplification of Berkman and Vishkin’s PRAM algorithm. It reduces the level ancestor problem to the Find-Smaller problem, the goal of which is, given an array $A$, an index $v$ and an integer $d$, to find the minimal $u > v$ such that $A_u \leq d$. Auxiliary array data structures are created to support efficient queries and the microset technique is used to sparsify these data structures.

Ben Amram’s algorithm was implemented in C by Victor Buchnik and can be obtained by contacting the authors. We used an obtained implementation to compare to our own algorithm implementations described in the other subsections. We will refer to Ben-Amram’s algorithm as the Find-Smaller algorithm.

3.8 Torben Hagerup’s Find-Larger algorithm

The Find-Larger algorithm created by Torben Hagerup is a variation of the previous algorithm which reduces the static level ancestor problem to the Find-Smaller problem. The algorithm inverts the values of the Find-Smaller array $A$ such that the level ancestor problem reduces to finding the minimal $u > v$ such that $A_u \geq d$. The algorithm then constructs a series of data structures from the Find-Larger array which can be represented as a two-dimensional line graph of peaks and valleys, with the root being the tallest peak, and the greatest depth nodes being the lowest valleys. Ladder data
structures are created for each index of the Find-Larger array and contain answers for level ancestor queries, with the deepest valleys containing the largest ladders, in some cases answering queries all the way up to the root node. The algorithm then answers a level ancestor query for node \( v \) by consulting \( v \)'s ladder, and if the answer is not contained within, jumping to \( v \)'s deepest reachable valley and using it's ladder structure to find the answer.

### 3.9 Tree representation and random generation

All level ancestor algorithms work on rooted trees, which consist part of the input of the algorithm. To represent rooted trees, we decided to use the Euler representation, a unique mapping of a tree to an array which is compact and is derived from an Euler traversal of the tree, commonly referred to as depth first search (DFS). For our implementations we selected to use a binary representation of the Euler traversal, where ‘1’ represents a forward downward traversal of an edge towards a descendant node, and ‘0’ represents an upward traversal of an edge toward an ancestor node. A rooted tree with \( n \) nodes can therefore be represented by \( 2^n - 2 \) binary digits, which we call the signature of the tree. It should be noted that, for ordinal binary trees of large size, this is the most compact representation. For the purpose of our study, we have used an ASCII representation of the binary digits of the Euler traversal, in order to facilitate visual inspection and debugging. Given a rooted tree signature, it is trivial to construct a pointer linked or table based tree structure in linear time on the size of the tree.

To generate random trees for our experiments, we use the split subtree method, where, in order to create a tree with \( n \) nodes, we generate a real-valued random variable \( x \) in the unit interval \((0, 1)\), assign \( \lfloor xn \rfloor \) nodes to the left subtree, one node to the root, and the rest to the right subtree. The process continues recursively for each subtree. This method generates trees equivalent to ones generated by the random binary search tree process by using random permutations, where any node has an equal probability to be chosen as root.

This method has several advantages when constructing random trees. It can be used in a straightforward manner to construct the
Euler traversal sequence (signature) of a random tree with $n$ nodes. First we allocate a table with $2n - 2$ positions and select a random number $k = \lfloor x n \rfloor$ between 1 and $n$. We can then insert ‘1’ in positions 0 and $2k$, ‘0’ in positions $2k - 1$ and $2n - 3$, and recursively process the two subtrees and corresponding subarrays (or single subtree/subarray when $k = n$). The expected average depth of a node of a random tree created with the split subtree method is $O(\log n)$, while the expected depth of the tree is also bounded logarithmically. These observations have been validated in the experiments described in the next section. The split subtree method also is easily modifiable to accommodate skewed unbalanced trees, by varying the range of values that the random variable takes. Using that property we have generated skewed trees with higher average node depth, as described in the next section.

4 Algorithm Evaluation

All eight algorithms described in the previous section were implemented in the programming language C, using standard libraries. All programs were compiled with gcc version 7.2.0, on the CentOS Linux 7.9.2009 operating system, running on an AMD EPYC 7702P Zen2 CPU (incorporates logic fabricated TSMC 7nm process and I/O fabricated on GlobalFoundries 14nm process) at 2 GHz, on a single core and thread. Each core has 32KB of L1 cache, 512KB of L2 cache, and shares 8MB of L3 cache with the other cores. The machine used has 512GB of main memory and test cases have been limited in size in order for program instructions and data to reside in main memory. All programs were compiled with the default ‘-O0’ optimization level set.

Randomly generated trees were varied in size between 100K and 1B nodes, with sizes increasing by a factor of 10 for each step. For each size step, 5 random trees were generated and the preprocessing and query time results were averaged between 5 runs of each algorithm on each random tree. The number of random level ancestor queries generated and run for each experiment were kept constant at 100M, which we deemed sufficient for experiments to run long enough not to be influenced by process swapping, interrupts, and other operating system events. We decided not to vary the number
of queries for each experiment, since query execution commences after static structures are in place following their preprocessing, and we expect the execution time to be a linear function of the number of queries. Tree depth and average node depth for the random trees generated are depicted in Figure 1(a). Each algorithm’s preprocessing and query time is calculated using the clock_gettime() function included in the standard C library time.h header file. We timed algorithms using both elapsed real-time and CPU clock time and, since we had exclusive use of the CPU, we determined there was a negligible difference between the two and ultimately tracked the elapsed real-time for the experiments.

![Graph of tree properties](chart.png)

**Fig. 1.** Tree properties. (a) Randomly generated split trees. (b) Randomly generated split trees with bounded subtree size ratio.

Applications of the level ancestor problems quite often involve trees that are not balanced and may have properties, such as depth, that are not well represented by the expected evenly distributed nodes of random trees described above. To better examine the behavior of the algorithms when run on unbalanced trees, we created a set of skewed rooted trees with 1M nodes each, varying the size ratio of the subtrees at each node. Using the same split subtree method, we limited the range of the random variable to a fraction of the unit interval. The properties of these skewed trees generated for varying ratios are shown in Figure 1(b).
4.1 Preprocessing

Preprocessing involves the time required to build necessary auxiliary data structures that support each algorithm, and includes the time it takes to load the tree in main memory. Experimental preprocessing times for all eight algorithms and varying tree sizes, for the randomly generated trees with expected logarithmic height, are shown in Table 1.

Table 1. Preprocessing times (seconds) for evenly distributed randomly generated trees

| Nodes (1000x) | Table & Pointer | Jump-Ladder | Ladder | Jump-Micro-Tree | Macro-Tree | Menghani & Matani | Smaller | Larger |
|--------------|-----------------|-------------|--------|-----------------|------------|-------------------|--------|--------|
| 100          | 0.017           | 0.011       | 0.011  | 0.016           | 0.015      | 0.006             | 0.022  | 0.014  |
| 1000         | 0.140           | 0.082       | 0.097  | 0.143           | 0.140      | 0.044             | 0.169  | 0.106  |
| 10000        | 1.500           | 0.778       | 0.946  | 1.439           | 1.343      | 0.408             | 1.641  | 1.008  |
| 100000       | 16.560          | 7.911       | 9.339  | 14.422          | 13.201     | 4.000             | 16.374 | 10.091 |
| 1000000      | 183.366         | 80.510      | 93.745 | 145.759         | 131.124    | 40.259            | 163.571| 101.523|

All algorithms can preprocess trees of size up to 100M nodes in less than 17 seconds, with the Menghani & Matani algorithm outperforming the rest (e.g. taking about 4 seconds to preprocess 100M nodes vs about 8 seconds for the next best). When tree sizes reach 1B nodes the differing preprocessing bounds become even more apparent as several algorithms preprocessing times take more than 100 seconds, with the Table algorithm being the worst at around 180 seconds for preprocessing. The Menghani & Matani algorithm continues to perform the best in terms of preprocessing speed as it takes about 40 seconds to preprocess a tree with 1B nodes. While the differences between algorithms seem significant, preprocessing is a one-time cost for static trees; no algorithm seems to have a clear disadvantage based on preprocessing time alone when tree size remains at or below 1B nodes. In the case of skewed trees, all algorithms except the Table algorithm perform equally well (data not shown). The Table algorithm experiences preprocessing time increases proportional to its space utilization, as analyzed in Table 1.
4.2 Space Requirements

Space is probably the most critical factor in selecting an appropriate level ancestor algorithm. Queries can be performed in constant or near-constant time and preprocessing is expedient even for very large trees. Since most solutions will be satisfactory when processing small trees in modern workstations, it is large trees and the extra storage space requirements which will influence most significantly the selection of one solution over another. The space utilization of the eight algorithm implementations after preprocessing is concluded is presented in Tables 2 and 3. This data was collected using the maximum resident set size statistic value displayed by the Linux /usr/bin/time command using the −v GNU option.

| Nodes (1000x) | Table Pointer | Jump-Ladder | Jump-Ladder | Macro-Micro-Tree | Menghani & Matani Smaller | Find-Smaller | Find-Larger |
|---------------|---------------|-------------|-------------|-----------------|---------------------------|--------------|-------------|
| 100           | 13            | 6           | 5           | 9               | 3                         | 6            | 15          |
| 1000          | 136           | 55          | 48          | 87              | 61                        | 32           | 55          | 146         |
| 10000         | 1569          | 548         | 461         | 852             | 592                       | 210          | 531         | 1457        |
| 100000        | 17495         | 5470        | 4586        | 8494            | 5840                      | 1966         | 5132        | 14594       |
| 1000000       | 199845        | 53697       | 45860       | 84934           | 57815                     | 19547        | 50198       | 146175      |

The Menghani & Matani algorithm performs the best when it comes to space utilization. It consistently pulls ahead of the more complex constant query time algorithms, an effect we primarily attribute due to its particularly simple data structures.

Notably, all algorithms have a linear space utilization dependence on the size of the tree, as demonstrated in the results shown in Table 2 with the exception for the Table algorithm, which shows \( O(n \log h) \) space utilization, where \( h \) is the average node depth of the tree. The last test case set with trees of 1B nodes forced the Table algorithm to use approximately 200GB of memory, which significantly affected the preprocessing time of the algorithm as well.

In the case of unbalanced trees, as observed in Figure 3 the Menghani & Matani and Ladder algorithms exhibit the lowest space
Table 3. Space usage (MB) of level ancestor algorithms using skewed random trees

| Ratio | Table | Jump-Pointer | Ladder | Jump-Ladder | Macro-Micro-Tree | Menghani & Matani | Find-Smaller | Find-Larger |
|-------|-------|--------------|--------|-------------|-----------------|-------------------|--------------|-------------|
| 1/2   | 137   | 55           | 45     | 84          | 59              | 32                | 55           | 147         |
| 1/5   | 198   | 56           | 42     | 82          | 58              | 32                | 56           | 147         |
| 1/10  | 295   | 63           | 41     | 87          | 57              | 32                | 56           | 147         |
| 1/20  | 448   | 68           | 40     | 92          | 57              | 32                | 57           | 147         |
| 1/50  | 868   | 70           | 38     | 92          | 51              | 32                | 58           | 148         |
| 1/100 | 1479  | 71           | 39     | 94          | 54              | 32                | 58           | 148         |

utilization, followed by the Macro-Micro-Tree and Hagerup algorithms. An interesting trend is noticeable in the space utilization of the Ladder algorithm, which seems to slightly improve with increased imbalance. This is not surprising, since unbalanced trees are expected to have fewer ladders as a result of the path decomposition, and fewer additional nodes on average when they are extended. Additionally, the Mengani & Matani algorithm is not affected by increased imbalance. This can be attributed to the simple data structure used by the algorithm which generally stores a single integer for each node and an integer pointer for each depth value, quantities that do not significantly change with increased imbalance.

Locality of reference of an algorithm is another significant factor when selecting a specific implementation, as algorithms which repeatedly access the same data in answering level ancestor queries are expected to suffer less from cache misses. The locality of reference of each examined algorithm is partially illustrated in Figure 2, which shows how often each algorithm suffers a last-level cache (LLC) miss and reads data from main memory. There seems to be a correlation between algorithm query time and LLC misses when compared to query times in Figure 3. Even though the number of queries is fixed for each experiment, query times continue to grow as LLC miss percentage approaches 100%. Additionally, the Menghani & Matani algorithm’s query time isn’t nearly as impacted as the query time of all of the other algorithms when tree size steps from 100k to 1 million nodes. This can possibly be explained by the algo-
rithm’s superior locality of reference and having 96.4% of all cache references result in a cache hit at a tree size of 1 million nodes.

![Graph showing Last-Level Cache Miss percentage vs. Number of tree nodes](image)

**Fig. 2.** Percentage of last-level cache (LLC) references that result in a cache miss

### 4.3 Query time performance

Level ancestor queries are often called as subroutines in other algorithms. As such, it is critical that they are expediently executed. In Figure 3 we can observe that the Table algorithm outperforms all others in query execution time as long as its data structures can be stored in main memory in their entirety, which in our experiments was possible due to the available 512GB of main memory. In scenarios where the table algorithm cannot fit its data structures in main memory, a significant number of page faults will drastically affect the query time of the algorithm. All algorithms, including the

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1. Note: Figure 2 shows data for trees having up to 100 million nodes; the AMD-based machine used for the rest of the experiments did not provide access to hardware event data such as cache references and misses, so this experiment was conducted on an Intel-based machine. As this latter machine had smaller main memory, experimental tree size was limited in order to maintain trees in memory. Otherwise, the behavior of the examined algorithms on the machines with different processor architectures was comparable.
query-optimal ones, experience increased query execution times as a function of tree size, which may be explained by increased cache misses, once their data structures exceed the size of the different levels of cache of the processor. Surprisingly, the Menghani & Matani algorithm with its sub-optimal query time outperforms all but the Table algorithm once the tree size reaches 1M nodes. One possible explanation would involve the small, frequently accessed data structures, leading to higher cache memory utilization.

![Fig. 3. Time to run 100M queries on randomly generated trees](image)

On unbalanced trees, the Ladder and Menghani & Matani algorithms perform particularly well, with query times that improve as imbalance increases. Once again, the Ladder algorithm is expected to form fewer ladders as a result of tree imbalance, which can lead to a reduced expected number of hops between ladders when executing queries. For the Menghani & Matani algorithm, the smaller arrays at each depth in an imbalanced tree can lead to faster binary search resolutions, which in turn speed up queries for the algorithm. The performance of all other algorithms seem to remain relatively stable.
Fig. 4. Time to run 100M queries on unbalanced randomly generated trees with 1M nodes. The x axis represents the maximum ratio between subtree sizes of node children. with increased imbalance, except for the Jump-Pointer algorithm, which seems to require an increasing number of jumps for nodes to reach their ancestors, as the average node depth of the tree increases.

5 Conclusions

Each algorithm presented, implemented, and tested in this paper has distinct merits and shortcomings. No one algorithm has a clear advantage over the others in every aspect, although the Menghani & Matani algorithm outperforms most in several areas. The Table algorithm outperforms – as expected – all other solutions in query execution time, if one can afford its worst case quadratic space requirement, which can be significant even when the input tree has logarithmically bounded depth. Space utilization, which is possibly the most critical component influencing algorithm choice, seems to correlate with the simplicity of algorithms with data structures that use linear bounded space, with the Menghani & Matani, Jump-Pointer-Ladder, and Find-Smaller algorithms outperforming all others, and
the first two demonstrating increased benefits when dealing with unbalanced trees.

Some of the simpler non-optimal algorithms, such as the Menghani & Matani, lead to simple efficient implementations, that are easier to comprehend, maintain, optimize and extend. On the other hand, with ever increasing problem sizes and machine capabilities, the optimal algorithms, especially the Find-Smaller, may be the solutions of choice once tree size sufficiently large, but certainly much larger than 1B nodes. Overall, the Menghani & Matani and Ben-Amram’s algorithms seem to be the most versatile solutions, combining excellent query execution times with a smaller space footprint and reasonable preprocessing requirements.

The Table, Jump-Pointer, Ladder, Jump-Ladder, Macro-Micro- Tree, Menghani & Matani, and Hagerup algorithm implementations can be downloaded at [www.tcnj.edu/~papamicd/level_ancestor](http://www.tcnj.edu/~papamicd/level_ancestor). The Find-Smaller algorithm implementation can be obtained from its authors.

### Acknowledgments.

The authors would like to acknowledge the assistance of Shawn Sivy who graciously provided critical hardware to perform our experiments. The authors also acknowledge use of the ELSA high performance computing cluster at The College of New Jersey for all computational experiments performed. This cluster is funded by the NSF grant OAC-1828163.

### References

1. David Aldous. Probability distributions on cladograms. In In Random Discrete Structures, pages 1–18. Springer, 1996.
2. Stephen Alstrup and Jacob Holm. Improved algorithms for finding level ancestors in dynamic trees. In Automata, Languages and Programming, 27th International Colloquium, ICALP 2000, number 1853 in LNCS, pages 73–84. Springer Verlag, 2000.
3. Amir M. Ben-Amram. The euler path to static level-ancestors. CoRR, abs/0909.1030, 2009.
4. Michael A. Bender and Martín Farach-Colton. The level ancestor problem simplified. Theor. Comput. Sci., 321(1):5–12, June 2004.
5. Omer Berkman and Uzi Vishkin. Finding level-ancestors in trees. Journal of Computer and System Sciences, 48(2):214 – 230, 1994.
6. Luc Devroye. A note on the height of binary search trees. *J. ACM*, 33(3):489–498, May 1986.
7. Luc Devroye and Paul Kruszewski. The botanical beauty of random binary trees. In Franz Brandenburg, editor, *Graph Drawing*, volume 1027 of *Lecture Notes in Computer Science*, pages 166–177. Springer Berlin / Heidelberg, 1996. 10.1007/BFb0021801.
8. Paul Dietz. Finding level-ancestors in dynamic trees. In Frank Dehne, Jörg-Rüdiger Sack, and Nicola Santoro, editors, *Algorithms and Data Structures*, volume 519 of *Lecture Notes in Computer Science*, pages 32–40. Springer Berlin / Heidelberg, 1991. 10.1007/BFb0028247.
9. Harold N. Gabow and Robert Endre Tarjan. A linear-time algorithm for a special case of disjoint set union. In *Proceedings of the fifteenth annual ACM symposium on Theory of computing*, STOC ’83, pages 246–251, New York, NY, USA, 1983. ACM.
10. Richard F. Geary, Rajeev Raman, and Venkatesh Raman. Succinct ordinal trees with level-ancestor queries. *ACM Trans. Algorithms*, 2(4):510–534, October 2006.
11. Dov Harel and Robert Endre Tarjan. Fast algorithms for finding nearest common ancestors. *SIAM J. Comput.*, 13(2):338–355, May 1984.
12. C.C. McGeoch. *A Guide to Experimental Algorithmics*. Cambridge University Press, 2012.
13. Kunihiko Sadakane and Roberto Grossi. Squeezing succinct data structures into entropy bounds. In *Proceedings of the seventeenth annual ACM-SIAM symposium on Discrete algorithm*, SODA ’06, pages 1230–1239, New York, NY, USA, 2006. ACM.
14. Hao Yuan and Mikhail J. Atallah. Efficient data structures for range-aggregate queries on trees. In *Proceedings of the 12th International Conference on Database Theory*, ICDT ’09, pages 111–120, New York, NY, USA, 2009. ACM.
Preprocessing time (sec) vs. Number of tree nodes (1000x)