Phase operator of the quantum supersymmetric harmonic oscillator

Gavriel Segre

After a brief introduction recalling how, in the limit in which the mass and the electric charge of the electron and the positron tend to zero, Quantum Electrodynamics reduces to a collection of uncoupled quantum supersymmetric harmonic oscillators, the phase operator of the quantum fermionic harmonic oscillator and of the quantum supersymmetric harmonic oscillator are introduced and their properties analyzed.

It is then shown that the phase operator of a supersymmetric harmonic oscillator is a Goldstone operator at any strictly positive temperature (finite or infinite).
| Contents                                                                 |   |
|------------------------------------------------------------------------|---|
| I. Acknowledgements                                                   | 3 |
| II. Introduction                                                      | 4 |
| I  Theory at zero temperature.                                        |   |
| III. Phase operator of the quantum bosonic oscillator                  | 6 |
| IV. Phase operator of the quantum fermionic oscillator                 | 11|
| V. Phase properties of the quantum supersymmetric oscillator           | 15|
| II  Theory at strictly positive temperature.                          |   |
| VI. A brief review of Umezawa’s thermofield dynamics                  | 19|
| VII. Expectation value of the bosonic phase operator                   | 22|
| VIII. Expectation value of the fermionic phase operator                | 23|
| IX. The supersymmetric phase operator as a Goldstone operator at strictly positive temperature | 24|

References

27
I. ACKNOWLEDGEMENTS

I would like to thank strongly Vittorio de Alfaro for his friendship and his moral support, without which I would have already given up.

Then I would like to thank strongly Jack Morava for many precious teachings.

Finally I would like to thank strongly Andrei Khrennikov and the whole team at the International Center of Mathematical Modelling in Physics and Cognitive Sciences of Växjö for their very generous informatics’ support.

Of course nobody among the mentioned people has responsibilities as to any (eventual) error contained in these pages.
II. INTRODUCTION

A free field theory is equivalent to a collection of harmonic oscillators. The quantization of such a field theory simply reduces to the quantization of these oscillators. The bosonic or fermionic nature of the involved field theory determines whether the resulting quantum harmonic oscillators are bosonic or fermionic.

A longstanding issue in the framework of Quantum Optics concerns the definition of the phase operator for a bosonic harmonic oscillator and the resulting phase properties of the quantum electromagnetic field (see for instance [1], the section 2.8 ”Phase Properties of the Field” of [2], the 4th chapter ”Phase Operator” of [3] as well as [4], [5], [6], [7], [8], [9]).

Such an issue is deeply linked with the issue concerning the impossibility of defining a time operator in Quantum Mechanics [10] and is, hence, deeply linked with the previous research about time we performed in [11] and in [12].

Though considering the recent wonderful book edited by Stephen M. Barnett and John A. Vaccaro with no doubt the best existing resource concerning the issue of defining the phase operator for a quantum bosonic oscillator, we must confess that we don’t agree with the viewpoint of the authors since we think that the Pegg-Barnett operator, introduced by D.T. Pegg and S.M. Barnett in their 1988-1989’s papers (now available as [13], [14], [15], [16]) recovers the self-adjointness lacking to the Susskind-Glogover operator (introduced by Leonard Susskind and Jonathan Glogover in their 1964’s paper now available as [17] and supported by R. Loudon in the 7th chapter of the first 1973’s edition of his manual of Quantum Optics now available as [18]) only at the prize of a considerable decrease in the formal elegance and beauty.

Furthermore nowadays it has become generally accepted to consider as the set of the physical observables of a (closed) quantum system something bigger than the set of all the self-adjoint operators on the system’s Hilbert space $H_{\text{system}}$ commuting with all the superselection charges.

For instance unsharp observables (i.e. positive-operator valued measures that are not projection valued measures and hence are not equivalent, via the Spectral Theorem [19], to a self-adjoint operator over $H_{\text{system}}$) are nowadays generally accepted.

For this reasons we believe that the correct phase operator for a bosonic harmonic oscillator is the Susskind-Glogover operator to which will refer from here and beyond as the bosonic phase operator.

Let us now remark that the following analog problems, i.e.:

1. to define the phase operator for the quantum fermionic harmonic oscillator

2. to define the phase operator for the quantum supersymmetric harmonic oscillator

haven’t been, at least up to our knowledge, investigated yet.

This is curious since, in the framework of Relativistic Quantum Mechanics, we are used nowadays to think that we know everything concerning Quantum Electrodynamics [21], [22], [23], [24].

Anyway it is sufficient to consider the limit in which the electric charge and the mass of the electron and of the positron tend to zero of the QED’s quantum field theory having lagrangian density $^2$:

$$\mathcal{L}_{QED} := -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\gamma^{\mu}D_{\mu} - m)\psi$$

$$D_{\mu} := \partial_{\mu} - ieA_{\mu}$$

$^1$ Let us recall that, by Naimark’s Theorem, a positive-operator valued measure over $H_{\text{system}}$ may be seen as a projection valued measure on a suitably enlarged Hilbert space, though this fact, together with the acceptance of unsharp observables (and of non-projective measurements) doesn’t solve the Measurement Problem of Quantum Mechanics contrary to what it is sometimes believed [20].

$^2$ adopting the usual notation where:

$$F_{\mu\nu} := \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}$$

$$\{\gamma^{0}, \gamma^{1}, \gamma^{2}, \gamma^{3}\} \text{ are Dirac matrices, i.e. } 4 \times 4 \text{ matrices satisfying the condition:}$$

$$\{\gamma^{\mu}, \gamma^{\nu}\} = 2\eta^{\mu\nu}$$

where $(\mathbb{R}^4, \eta := \eta_{\mu\nu}dx^{\mu} \otimes dx^{\nu})$ is the Minkowski spacetime with $\eta_{\mu\nu} := \text{diag}(1, -1, -1, -1)$, where $\psi$ is a 4-component spinor, where $\bar{\psi} := \psi^{\dagger}\gamma^{0}$, where Einstein’s convention of sum over repeated indices is assumed and where indices are raised and lowered by contraction with the Minkowskian metric tensor.
having quantum hamiltonian \(^3\): 

\[ H = H_B + H_F \]  

(2.5)

where:

\[ H_B := \sum_{\vec{k} \in \mathbb{Z}^3} \sum_{r=0}^{3} \omega_{B;\vec{k},r} N_{B;\vec{k},r} \]  

(2.6)

\[ \omega_{B;\vec{k}} := |\vec{k}| \]  

(2.7)

\[ N_{B;\vec{k},r} := \zeta_r a_{B;\vec{k},r}^\dagger a_{B;\vec{k},r} \]  

(2.8)

\[ \zeta_r := \begin{cases} -1, & \text{if } r = 0 \\ 1, & \text{if } r \in \{1,2,3\} \end{cases} \]  

(2.9)

\[ [a_{B;\vec{k},r}, a_{B;\vec{k}',s}^\dagger] = \delta_{r,s} \delta_{\vec{k},\vec{k}'} \]  

(2.10)

\[ H_F := \sum_{\vec{k} \in \mathbb{Z}^3} \sum_{r \in \{1,2\}} \omega_{F;\vec{k}} (N_{F,+;\vec{k},r} + N_{F,-;\vec{k},r}) \]  

(2.12)

\[ \omega_{F;\vec{k}} := \sqrt{\vec{k}^2 + m^2} \]  

(2.13)

\[ N_{F,+;\vec{k},r} := a_{F,+;\vec{k},r}^\dagger a_{F,+;\vec{k},r} \]  

(2.14)

\[ N_{F,-;\vec{k},r} := a_{F,-;\vec{k},r}^\dagger a_{F,-;\vec{k},r} \]  

(2.15)

\[ \{a_{F,+;\vec{k},r}, a_{F,+;\vec{k}',s}^\dagger\} = \delta_{\vec{k},\vec{k}'} \delta_{r,s} \]  

(2.16)

\[ \{a_{F,-;\vec{k},r}, a_{F,-;\vec{k}',s}^\dagger\} = \delta_{\vec{k},\vec{k}'} \delta_{r,s} \]  

(2.17)

\[ \{a_{F,+;\vec{k},r}, a_{F,+;\vec{k}',s}\} = \{a_{F,+;\vec{k},r}^\dagger a_{F,+;\vec{k}',s}^\dagger\} = 0 \]  

(2.18)

\[ \{a_{F,-;\vec{k},r}, a_{F,-;\vec{k}',s}\} = \{a_{F,-;\vec{k},r}^\dagger a_{F,-;\vec{k}',s}^\dagger\} = 0 \]  

(2.19)

\[ \{a_{F,+;\vec{k},r}, a_{F,-;\vec{k}',s}\} = \{a_{F,+;\vec{k},r}^\dagger a_{F,-;\vec{k}',s}\} = \{a_{F,+;\vec{k},r}^\dagger a_{F,-;\vec{k}',s}^\dagger\} = 0 \]  

(2.20)

\(^3\) imposing for simplicity periodic boundary conditions on the walls of a cube of side \(L\).
to obtain a system of uncoupled quantum supersymmetric oscillators whose possible physical states are the rays of the \( \mathbb{Z}_2 \)-graded Hilbert space \( \mathcal{H}_B^{\text{physical}} \otimes \mathcal{H}_F \), where \( \mathcal{H}_B^{\text{physical}} \) is the subspace of \( \mathcal{H}_B \) obtained imposing the Lorentz gauge condition \( \partial_{\mu} A^{\mu} = 0 \) in the Gupta-Bleuer form:

\[
\mathcal{H}_B^{\text{physical}} := \{ |\psi > \in \mathcal{H}_B : (a_{B;\vec{k},3} - a_{B;\vec{k},0})|\psi > = 0 \ \forall \vec{k} \in \frac{2\pi}{L} \mathbb{Z}^3 \} \quad (2.21)
\]

as it appears evident as soon as one expresses the restriction of the quantum hamiltonian \( H \) to \( \mathcal{H}_B^{\text{physical}} \otimes \mathcal{H}_F \) as:

\[
H|_{\mathcal{H}_B^{\text{physical}} \otimes \mathcal{H}_F} = \sum_{\vec{k} \in \frac{2\pi}{L} \mathbb{Z}^3} \sum_{r \in \{1,2\}} |\vec{k}| (N_{B;\vec{k},r} + N_{F;\vec{k},r})
\]

\[
N_{F;\vec{k},r} := N_{F;+;\vec{k},r} + N_{F;--;\vec{k},r} \quad \vec{k} \in \frac{2\pi}{L} \mathbb{Z}^3, r \{1,2\} \quad (2.22)
\]

Curiously the phase properties of such a collection of uncoupled quantum supersymmetric harmonic oscillators have not been investigated yet.

In this paper we introduce the \textit{fermionic phase operator}, i.e. the \textit{phase operator} for a fermionic harmonic oscillator, and the \textit{supersymmetric phase operator}, i.e. the \textit{phase operator} for a supersymmetric harmonic oscillator.

Furthermore we show that the \textit{supersymmetric phase operator} is a Goldstone operator at any strictly positive temperature.
Part I
Theory at zero temperature.

III. PHASE OPERATOR OF THE QUANTUM BOSONIC OSCILLATOR

Let us consider a quantum bosonic oscillator having, hence, hamiltonian:

\[ H_B := \omega N_B \] (3.1)

where \( a_B \) and \( a_B^\dagger \) are respectively the annihilation and the creation operators:

\[ [a_B, a_B] = [a_B^\dagger, a_B^\dagger] = 0 \] (3.2)

\[ [a_B, a_B^\dagger] = 1 \] (3.3)

and where \( N_B \) is the number operator:

\[ N_B := a_B^\dagger a_B \] (3.4)

The equation 3.2 and the equation 3.3 imply that:

\[ [N_B, a_B] = -a_B \] (3.5)

\[ [N_B, a_B^\dagger] = a_B^\dagger \] (3.6)

that imply that:

\[ a_B|n> = \begin{cases} 0, & \text{if } n = 0; \\ \sqrt{n}|n-1>, & \text{if } n \in \mathbb{N}_+. \end{cases} \] (3.7)

\[ a_B^\dagger|n> = \sqrt{n+1}|n+1> \ \forall n \in \mathbb{N} \] (3.8)

\[ |n> = \frac{(a_B^\dagger)^n}{\sqrt{n!}}|0> \ \forall n \in \mathbb{N} \] (3.9)

\[ N_B|n> = n|n> \ \forall n \in \mathbb{N} \] (3.10)

\[ H_B|n> = E_B(n)|n> \ \forall n \in \mathbb{N} \] (3.11)

where:

\[ E_B(n) := \omega n \ n \in \mathbb{N} \] (3.12)

Let us now introduce the bosonic angle states:

\[ |\theta> := \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \exp(in\theta)|n> \ \theta \in [0,2\pi) \] (3.13)

Remark III.1
Let us remark that:

\[ \langle \theta_1 | \theta_2 \rangle = \frac{1}{2\pi} \sum_{n=0}^{\infty} \exp[in(\theta_2 - \theta_1)] \neq \delta(\theta_1 - \theta_2) \ \forall \theta_1, \theta_2 \in [0, 2\pi) \quad (3.14) \]

though clearly:

\[ \langle \theta | \theta \rangle = +\infty \ \forall \theta \in [0, 2\pi) \quad (3.15) \]

Though not orthonormal, the bosonic angle states are complete:

\[ \int_0^{2\pi} d\theta \langle \theta | \theta \rangle = \frac{1}{2\pi} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \int_0^{2\pi} d\theta \exp[i(n - m)\theta] |n \rangle \langle m| = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \delta_{n,m} |n \rangle \langle m| = \sum_{n=0}^{\infty} |n \rangle \langle n| = 1 \quad (3.16) \]

where we have used the fact that:

\[ \int_0^{2\pi} d\theta \exp[i(n - m)\theta] = 2\pi\delta_{n,m} \ \forall n, m \in \mathbb{N} \quad (3.17) \]

Let us introduce the bosonic exponential phase operator:

\[ \exp(i\hat{\theta}) := \sum_{n=0}^{\infty} |n \rangle \langle n+1| \quad (3.18) \]

whose name is justified by the fact that:

\[ \exp(i\hat{\theta}) |\theta\rangle = \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} |n \rangle \langle n+1| |m \rangle \langle m+1| \exp(i\theta)|n\rangle = \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \delta_{m,n+1} \exp(i\theta)|n\rangle = \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \exp[i(n+1)\theta]|n\rangle = \exp(i\theta)|\theta\rangle \ \forall \theta \in [0, 2\pi) \quad (3.19) \]

**Remark III.2**

Let us remark that the exponential bosonic phase operator \( \exp(i\hat{\theta}) \) is not unitary and, hence, the bosonic phase operator \( \hat{\theta} \) is not self-adjoint.

In fact:

\[ \exp(i\hat{\theta})(\exp(i\hat{\theta})^\dagger = (\sum_{n=0}^{\infty} |n \rangle \langle n+1|)(\sum_{m=0}^{\infty} |m+1 \rangle \langle m|) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} |n \rangle \langle n+1|m+1 \rangle \langle m| = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \delta_{m+1,n+1} |n \rangle \langle m| = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \delta_{m,n} |n \rangle \langle m| = \sum_{n=0}^{\infty} |n \rangle \langle n| = 1 \quad (3.20) \]

but:

\[ (\exp(i\hat{\theta})^\dagger \exp(i\hat{\theta}) = (\sum_{n=0}^{\infty} |n+1 \rangle \langle n|)(\sum_{m=0}^{\infty} |m \rangle \langle m+1|) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} |n+1 \rangle \langle n|m \rangle \langle m+1| = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \delta_{m,n} |n+1 \rangle \langle m+1| = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} |n+1 \rangle \langle n+1| = \sum_{n=0}^{\infty} |n \rangle \langle n| - |0 \rangle \langle 0| = 1 - |0 \rangle \langle 0| \]

\[ (3.21) \]

Given a generic normalized state:

\[ |\psi\rangle := \sum_{n=0}^{\infty} c_n |n\rangle \quad (3.22) \]
the probability that a measurement of the bosonic phase operator $\hat{\theta}$ when the oscillator is in the state $|\psi>\rangle$ gives as result $\theta \in [0, 2\pi)$ is:

$$Pr_{|\psi>\rangle}(\theta) := |<\theta|\psi>\rangle|^2 = \frac{1}{2\pi} \sum_{n=0}^{\infty} c_n \exp(-in\theta)|^2$$ (3.24)

Obviously:

$$\int_0^{2\pi} d\theta Pr_{|\psi>\rangle}(\theta) = \frac{1}{2\pi} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} c_n \overline{c_m} \int_0^{2\pi} d\theta \exp[i(n-m)\theta] = \frac{1}{2\pi} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} c_n \overline{c_m} 2\pi \delta_{n,m} = \sum_{n=0}^{\infty} |c_n|^2 = 1$$ (3.25)

where we have used the equation 3.17.

Remark III.3

The issue concerning the definition of the phase operator is deceptively similar but essentially different from two other issues:

1. the quantization of the dynamical system consisting of a spinless boson of unary mass having as configuration space the circle $(S^1, \delta := d\theta \otimes d\theta)$ and hence having lagrangian $L : TS^1 \rightarrow \mathbb{R}$:

$$L(\theta, \dot{\theta}) := \frac{|\dot{\theta}|^2}{2} = \frac{\dot{\theta}^2}{2}$$ (3.26)

that furnishes the prototypical example of the topological superselection rule with superselection charge $\in\text{Hom}(H_1(\text{configuration space}, \mathbb{Z}), U(1))$ (discovered independently by Cecile Morette De Witt and by Larry Schulman at the end of the sixthes and the beginning of the seventhes and nowadays commonly founded in the literature: see for instance the 23rd chapter of [25], the 7th chapter of [26] and the 8th chapter of [27] as to its implementation, at different levels of mathematical rigor, in the path-integration’s formulation, as well as the 8th chapter of [28], the 3rd chapter of [29] and the section 6.8 of [30] for its formulation in the operatorial formulation).

Actually, in such a prototypical example, using the Hurewicz isomorphism:

$$H_1(M, \mathbb{Z}) = \frac{\pi_1(M)}{[\pi_1(M), \pi_1(M)\rangle}$$ (3.27)

holding for an arbitrary differentiable manifold M, and where:

$$[G, G] := \{a \cdot b \cdot a^{-1} \cdot b^{-1} \mid a, b \in G\}$$ (3.28)

is the commutator subgroup of an arbitrary group G, if follows that:

$$\pi_1(S^1) = \mathbb{Z}$$ (3.29)

$$[\pi_1(S^1), \pi_1(S^1)] = 0$$ (3.30)

and hence the involved superselection charge is simply a phase $\in U(1)$, the distinct superselection sectors simply corresponding to different self-adjoint extension of $-\frac{1}{2} \frac{d^2}{dr^2} : C_0^\infty(S^1) \rightarrow C_0^\infty(S^1)$, where $C_0^\infty(M)$ denotes the set of all the smooth functions with compact support over an arbitrary differentiable manifold M.

2. the Bloch theory concerning the lattice $a\mathbb{Z}$ [31] (considered for simplicity in the tight binding approximation), i.e. the Quantum Mechanics of a spinless boson of unary mass living on the euclidean real line $(\mathbb{R}, \delta := dx \otimes dx)$ under the influence of a field’s force with energy potential $V(x)$ periodic of period $a \in (0, +\infty)$ and hence such that:

$$\tau(a)^{\dagger}V(x)\tau(a) = V(x + a) = V(x)$$ (3.31)
(where \( \tau(l) \) is operator of translation by \( l \in \mathbb{R} \)).

Obviously the group \( \{ \tau(x), x \in a\mathbb{Z} \} \) of the translations by vectors belonging to the lattice \( a\mathbb{Z} \) is a symmetry of the system:

\[
[H, \tau(x)] = 0 \; \forall x \in a\mathbb{Z} \tag{3.32}
\]

so that \( H \) and \( \tau(a) \) may be diagonalized simultaneously.

Denoting with \( |n> \) a state localized in the \( n^{th} \) cell \( |na, (n+1)a\rangle \) and hence such that:

\[
\tau(a)|n> = |n+1>
\tag{3.33}
\]

the tight binding approximation imposes that there exists a \( \Delta \in (0, +\infty) \) such that:

\[
<n|H|m> = -(\delta_{m,n-1} + \delta_{m,n+1})\Delta + E_0\delta_{n,m} \; \forall n, m \in \mathbb{Z} \tag{3.34}
\]

and hence:

\[
H|n> = E_0|n> - \Delta|n-1> - \Delta|n+1>
\tag{3.35}
\]

Introduced the angle states:

\[
|\theta> := \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{+\infty} \exp(in\theta)|n> \; \theta \in [0, 2\pi) \tag{3.36}
\]

it follows that:

\[
\tau(a)|\theta> = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{+\infty} \exp(in\theta)|n+1> = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{+\infty} \exp[i(n-1)\theta]|n> = \exp(-i\theta)|\theta> \; \forall \theta \in [0, 2\pi) \tag{3.37}
\]

\[
H|\theta> = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{+\infty} \exp(in\theta)H|n> =
\]

\[
\frac{E_0}{\sqrt{2\pi}} \sum_{n=-\infty}^{+\infty} \exp(in\theta)|n> - \frac{\Delta}{\sqrt{2\pi}} \sum_{n=-\infty}^{+\infty} \exp(in\theta)|n+1> - \frac{\Delta}{\sqrt{2\pi}} \sum_{n=-\infty}^{+\infty} \exp(in\theta)|n-1> =
\]

\[
\frac{E_0}{\sqrt{2\pi}} \sum_{n=-\infty}^{+\infty} \exp(in\theta)|n> - \frac{\Delta}{\sqrt{2\pi}} \sum_{n=-\infty}^{+\infty} \exp[i(n-1)\theta]|n> - \frac{\Delta}{\sqrt{2\pi}} \sum_{n=-\infty}^{+\infty} \exp[i(n+1)\theta]|n> = (E_0 - 2\Delta \cos(\theta))|\theta> \; \forall \theta \in [0, 2\pi) \tag{3.38}
\]

The formal similarities between the issue of defining the phase operator for a bosonic harmonic oscillator and each of these two other issues are, anyway, deceptive since:

1. the configuration space of the harmonic oscillator is the real line having trivial fundamental group. So no topological superselection rule exists in this case.

2. the sum in the angle states of the periodic one-dimensional potential runs from \( -\infty \) to \( +\infty \) while the sum in the bosonic angle states of the bosonic phase operator runs only from 0 to \( +\infty \).

Therefore in the case of the one dimensional particle in a periodic energy potential it follows that:

\[
\sum_{n=-\infty}^{+\infty} |n+1><n+1| = \sum_{n=-\infty}^{+\infty} |n><n| = 1 \tag{3.39}
\]

while in the case of the bosonic phase operator we have seen in the equation 3.21 how the fact that:

\[
\sum_{n=0}^{+\infty} |n+1><n+1| \neq \sum_{n=0}^{+\infty} |n+1><n+1| \tag{3.40}
\]

is responsible of the fact that the bosonic exponential phase operator is not unitary.
IV. PHASE OPERATOR OF THE QUANTUM FERMIONIC OSCILLATOR

Let us consider a quantum fermionic oscillator having, hence, hamiltonian:

\[ H_F := \omega N_F \]  \hspace{1cm} (4.1)

where \( a_F \) and \( a_F^\dagger \) are respectively the annihilation and the creation operators:

\[ \{a_F, a_F\} = \{a_F^\dagger, a_F^\dagger\} = 0 \]  \hspace{1cm} (4.2)

\[ \{a_F, a_F^\dagger\} = 1 \]  \hspace{1cm} (4.3)

and where \( N_F \) is the fermionic number operator:

\[ N_F := a_F^\dagger a_F \]  \hspace{1cm} (4.4)

The equation (4.2) and the equation (4.3) imply that:

\[ [N_F, a_F] = -a_F \]  \hspace{1cm} (4.5)

\[ [N_F, a_F^\dagger] = a_F^\dagger \]  \hspace{1cm} (4.6)

that imply that:

\[ a_F|n> = \begin{cases} 0, & \text{if } n = 0; \\ |0>, & \text{if } n = 1. \end{cases} \]  \hspace{1cm} (4.7)

\[ a_F^\dagger|n> = \begin{cases} |1>, & \text{if } n = 0; \\ 0, & \text{if } n = 1. \end{cases} \]  \hspace{1cm} (4.8)

\[ |n> = (a_F^\dagger)^n|0> \quad \forall n \in \{0, 1\} \]  \hspace{1cm} (4.9)

\[ N_F|n> = n|n> \quad \forall n \in \{0, 1\} \]  \hspace{1cm} (4.10)

\[ H_F|n> = E_F(n)|n> \quad \forall n \in \{0, 1\} \]  \hspace{1cm} (4.11)

where:

\[ E_F(n) := \omega n \quad n \in \{0, 1\} \]  \hspace{1cm} (4.12)

It appears natural, mimicking the approach of the section III, to define the fermionic angle states as:

\[ |\theta> := \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{1} \exp(in\theta)|n> = \frac{1}{\sqrt{2\pi}}(|0> + \exp(i\theta)|1>) \quad \theta \in [0, 2\pi) \]  \hspace{1cm} (4.13)

Remark IV.1

Let us remark that:

\[ <\theta_1|\theta_2> = \frac{1}{2\pi}\{1 + \exp[i(\theta_2 - \theta_1)]\} \quad \forall \theta_1, \theta_2 \in [0, 2\pi) \]  \hspace{1cm} (4.14)
and hence in particular:

\[ < \theta | \theta > = \frac{1}{\pi} \forall \theta \in [0, 2\pi) \]  \hspace{1cm} (4.15)

Though not orthonormal, the fermionic angle states are complete:

\[ \int_0^{2\pi} d\theta |\theta > < \theta | = \frac{1}{2\pi} \int_0^{2\pi} d\theta (|0 > < 0| + \exp(-i\theta)|0 > < 1| + \exp(i\theta)|1 > < 0| + |1 > < 1|) = 1 \]  \hspace{1cm} (4.16)

Always mimicking the approach of the section III, it would then appear natural to define the fermionic exponential phase operator as the operator $|0 > < 1|$. Anyway:

\[ |0 > < 1| \theta > = 1/\sqrt{2\pi} (|0 > < 1|0 > + \exp(i\theta)|0 > < 1|1 >) = \frac{\exp(i\theta)}{\sqrt{2\pi}}|0 \neq \exp(i\theta)| > \]  \hspace{1cm} (4.17)

and hence $|\theta >$ is not an eigenstate of $|0 > < 1|$. Let us then proceed in a different way expressing the exponential phase operator in the more general way:

\[ \exp(i\hat{\theta}) := c_{00}|0 > < 0| + c_{01}|0 > < 1| + c_{10}|1 > < 0| + c_{11}|1 > < 1| \]  \hspace{1cm} (4.18)

and imposing the condition:

\[ \exp(i\hat{\theta})|\theta > = \exp(i\theta)|\theta > \forall \theta \in [0, 2\pi) \]  \hspace{1cm} (4.19)

and hence that:

\[ \exp(i\hat{\theta})|\theta > = 1/\sqrt{2\pi} (\exp(i\theta)|0 > + \exp(i2\theta)|1 >) \forall \theta \in [0, 2\pi) \]  \hspace{1cm} (4.20)

Since:

\[ \exp(i\hat{\theta})|\theta > = 1/\sqrt{2\pi} [(c_{00} + \exp(i\theta))|0 > + (c_{10} + c_{11} \exp(i\theta))|1 >] \]  \hspace{1cm} (4.21)

it follows that:

\[ c_{00} + c_{01} \exp(i\theta) = \exp(i\theta) \]  \hspace{1cm} (4.22)

\[ c_{10} + c_{11} \exp(i\theta) = \exp(i2\theta) \]  \hspace{1cm} (4.23)

The imposition of the unitarity of the fermionic exponential phase operator leads to the constraints:

\[ |c_{0,0}|^2 + |c_{1,0}|^2 = 1 \]  \hspace{1cm} (4.24)

\[ |c_{0,1}|^2 + |c_{1,1}|^2 = 1 \]  \hspace{1cm} (4.25)

\[ \overline{c_{0,0}}c_{0,1} + \overline{c_{1,0}}c_{1,1} = 0 \]  \hspace{1cm} (4.26)

\[ \overline{c_{0,0}}c_{0,1} + \overline{c_{1,0}}c_{1,1} = 0 \]  \hspace{1cm} (4.27)

\[ |c_{0,0}|^2 + |c_{0,1}|^2 = 1 \]  \hspace{1cm} (4.28)

\[ |c_{1,0}|^2 + |c_{1,1}|^2 = 1 \]  \hspace{1cm} (4.29)

\[ \overline{c_{0,0}}c_{1,0} + \overline{c_{0,1}}c_{1,1} = 0 \]  \hspace{1cm} (4.30)
\[ c_{0,0}c_{1,0} + c_{1,1}c_{0,1} = 0 \] (4.31)

We will now show that there don’t exist four complex numbers \( c_{00}, c_{01}, c_{10}, c_{11} \) satisfying simultaneously the equation 4.22, the equation 4.23, the equation 4.24, the equation 4.25, the equation 4.26, the equation 4.27, the equation 4.28, the equation 4.29, the equation 4.30 and the equation 4.31.

Given a normalized state:
\[ |\psi\rangle := d_0|0\rangle + d_1|1\rangle \] (4.32)
\[ <\psi|\psi\rangle = |d_0|^2 + |d_1|^2 = 1 \] (4.33)

it follows that:
\[ <\psi|(\exp(i\hat{\theta}))^\dagger \exp(i\hat{\theta})|\psi\rangle = \int_0^{2\pi} d\theta \int_0^{2\pi} d'\theta' <\psi|\theta><\theta|(\exp(i\hat{\theta}))^\dagger \exp(i\hat{\theta})|\theta'><\theta'|\psi\rangle = \int_0^{2\pi} d\theta \int_0^{2\pi} d'\theta' \exp[i(\theta' - \theta)] <\psi|\theta><\theta'|\psi\rangle \] (4.34)

where we have used the completeness of the fermionic angle states stated by the equation 4.16.

Since:
\[ <\psi|\theta> = (<0|d_0 + <1|d_1> \frac{1}{\sqrt{2\pi}}(|0\rangle + \exp(i\theta)|1\rangle) = \frac{1}{\sqrt{2\pi}}(d_0 + \exp(i\theta)d_1) \] (4.35)
\[ <\theta'|\psi\rangle = \frac{1}{\sqrt{2\pi}}(d_0 + \exp(-i\theta')d_1) \] (4.36)

it follows that:
\[ <\psi|(\exp(i\hat{\theta}))^\dagger \exp(i\hat{\theta})|\psi\rangle = \frac{1}{(2\pi)^2} \int_0^{2\pi} d\theta \int_0^{2\pi} d'\theta' \{\exp[i(\theta' - \theta)] + \exp[2i(\theta' - \theta)]\} |d_0|^2 + d_0d_1 \exp(-i\theta') + d_0d_1 \exp(i\theta) + |d_1|^2 \exp[i(\theta - \theta')] \] = \[ \frac{|d_1|^2}{(2\pi)^2} \neq 1 \] (4.37)

(where we have used the equation 4.14) and hence the fermionic exponential phase operator \( \exp(i\hat{\theta}) \) is not unitary and the fermionic phase operator \( \hat{\theta} \) is not self-adjoint.

As we have seen this is equivalent to the fact that there don’t exist four complex numbers \( c_{00}, c_{01}, c_{10}, c_{11} \) satisfying simultaneously the equation 4.22, the equation 4.23, the equation 4.24, the equation 4.25, the equation 4.26, the equation 4.27, the equation 4.28, the equation 4.29, the equation 4.30 and the equation 4.31.

Imposing only the equation 4.22 and the equation 4.23 one obtains many possible solutions among which there is the choice:
\[ c_{00} := 0 \] (4.38)
\[ c_{01} := 1 \] (4.39)
\[ c_{10} := \exp(i\theta) \] (4.40)
\[ c_{11} := 1 \] (4.41)

determining the fermionic exponential phase operator:
\[ \exp(i\hat{\theta}) = |0><1| + \exp(i\theta)|1><0| + |1><1| \] (4.42)
Given the generic normalized state given by the equation \[4.3\] and the equation \[4.33\] the probability that a measurement of the fermionic phase operator \(\hat{\theta}\) when the oscillator is in the state \(|\psi\rangle\) gives as result \(\theta \in [0, 2\pi)\) is:

\[
Pr_{|\psi\rangle}(\theta) := |\langle \theta |\psi \rangle|^2 = \frac{1}{2\pi}(1 + d_0d_1 \exp(i\theta) + \overline{d_0}d_1 \exp(-i\theta))
\]  

(4.43)

Obviously:

\[
\int_0^{2\pi} d\theta \, Pr_{|\psi\rangle}(\theta) = \frac{1}{2\pi} + (2\pi + d_0d_1 \int_0^{2\pi} d\theta \exp(i\theta) + \overline{d_0}d_1 \int_0^{2\pi} d\theta \exp(i\theta)) = 1
\]  

(4.44)
V. PHASE PROPERTIES OF THE QUANTUM SUPERSYMMETRIC OSCILLATOR

Let us now consider the quantum supersymmetric oscillator (see for instance the 6th chapter "Supersymmetry" of [32]) having hamiltonian:

$$ H := H_B + H_F $$

(5.1)

where $H_B$ and $H_F$ are the hamiltonians of, respectively, the quantum bosonic oscillator and the quantum fermionic oscillator given, respectively, by the equation 3.1 and the equation 4.1 and where:

$$ [a_{B}, a_{F}] = [a_{B}, a_{F}] = [a_{B}, a_{F}^\dagger] = 0 $$

(5.2)

Clearly:

$$ H|n_B, n_F > = E(n_B, n_F)|n_B, n_F > \quad \forall n_B \in \mathbb{N}, \forall n_F \in \{0, 1\} $$

(5.3)

where:

$$ E(n_B, n_F) := E_B(n_B) + E_F(n_F) \quad n_B \in \mathbb{N}, n_F \in \{0, 1\} $$

(5.4)

$$ |n_B, n_F > = \frac{(a_{B})^{n_B}}{\sqrt{n_B!}} (a_{F}^\dagger)^{n_F} |0 > \quad \forall n_B \in \mathbb{N}, \forall n_F \in \{0, 1\} $$

(5.5)

with $E_B(n_B)$ and $E_F(n_F)$ defined, respectively, by the equation 3.12 and the equation 4.12.

Let us now introduce the operators:

$$ Q := a_B^\dagger a_F $$

(5.6)

$$ \bar{Q} := Q^\dagger = a_F^\dagger a_B $$

(5.7)

Since:

$$ [Q, H] = [\bar{Q}, H] = 0 $$

(5.8)

$Q$ and $\bar{Q}$ are symmetries of the quantum supersymmetric oscillator and:

$$ H(Q|n_B, n_F >) = QH|n_B, n_F > = E(n_B, n_F)(Q|n_B, n_F >) \quad \forall n_B \in \mathbb{N}, \forall n_F \in \{0, 1\} $$

(5.9)

$$ H(\bar{Q}|n_B, n_F >) = \bar{Q}H|n_B, n_F > = E(n_B, n_F)(\bar{Q}|n_B, n_F >) \quad \forall n_B \in \mathbb{N}, \forall n_F \in \{0, 1\} $$

(5.10)

It may be, furthermore, easily verified that:

$$ [N_B, \bar{Q}] = -\bar{Q} $$

(5.11)

$$ [N_B, Q] = Q $$

(5.12)

$$ [N_F, Q] = -Q $$

(5.13)

$$ [N_F, \bar{Q}] = \bar{Q} $$

(5.14)

from which it follows that:

$$ Q|n_B, n_F > = \begin{cases} \sqrt{n_B + 1}|n_B + 1, n_F - 1 >, & \text{if } n_F = 1; \\ 0, & \text{if } n_F = 0. \end{cases} $$

(5.15)

$$ \bar{Q}|n_B, n_F > = \begin{cases} \sqrt{n_B}|n_B - 1, n_F + 1 >, & \text{if } n_B \in \mathbb{N}_+ \text{ and } n_F = 0; \\ 0, & \text{if } n_B = 0 \text{ or } n_F = 1. \end{cases} $$

(5.16)
Since, informally speaking, one can say that $Q$ transforms a "fermionic quantum" into a "bosonic quantum" while $\bar{Q}$ transforms a "bosonic quantum" into a "fermionic quantum", $Q$ and $\bar{Q}$ are called supersymmetric charges.

The approach followed in the section III and in the section IV leads naturally to define the supersymmetric angle states as:

$$|\theta> := \frac{1}{\sqrt{2}}(|\theta_B> \otimes |\theta_F>) = \frac{1}{(2)^{2\pi}} \sum_{n_B=0}^{\infty} \sum_{n_F=0}^{1} \exp[i(n_B + n_F)\theta]|n_B, n_F> \quad \theta \in [0, 2\pi)$$

(5.17)

where $|\theta_B>$ and $|\theta_F>$ are, respectively, the bosonic angle state and the fermionic angle state defined, respectively, by the equation 3.13 and by the equation 4.13.

**Remark V.1**

Let us remark that:

$$<\theta_1|\theta_2> = \frac{1}{\delta\pi} \sum_{n_B=0}^{\infty} \sum_{n_F=0}^{1} \exp\{i[(n_B + n_F)(\theta_2 - \theta_1)]\} \neq \delta(\theta_1 - \theta_2)$$

(5.18)

though obviously:

$$<\theta|\theta> = +\infty$$

(5.19)

Though not orthonormal, the supersymmetric angle states are complete:

$$\int_0^{2\pi} d\theta|\theta><\theta| = \int_0^{2\pi} d\theta(\theta_B><\theta_B \otimes 1_F) + \int_0^{2\pi} d\theta(1_B \otimes \theta><\theta_F) = 1$$

(5.20)

where we have used the completeness condition of, respectively, the bosonic angle states and the fermionic angle states given, respectively, by the equation 3.16 and by the equation 4.16.

It would appear natural to define the supersymmetric exponential phase operator as $\sum_{n_B=0}^{\infty} |n_B, 0><n_B + 1, 1|$. Anyway the same considerations concerning the fermionic exponential phase operator and condensed in the equation 4.42 lead us to observe that:

$$\sum_{n_B=0}^{\infty} |n_B, 0><n_B + 1, 1| \theta> = \frac{1}{(2)^{2\pi}} \sum_{n_B=0}^{\infty} \exp[i(n_B + 2)\theta]|n_B, 0> \neq \exp(i\theta)|\theta>$$

(5.21)

Since in the last section we have, indeed, seen that the correct fermionic exponential phase operator is given by the equation 4.42 it follows that the supersymmetric exponential phase operator is:

$$\exp(i\hat{\theta}) := \exp(i\hat{\theta}_B) \otimes \exp(i\hat{\theta}_F) = \left(\sum_{n=0}^{\infty} |n><n + 1| \otimes |0><1| + \exp(i\theta)|1><0| + |1><1|\right) =$$

$$\sum_{n=0}^{\infty} (|n, 0><n + 1, 1| + \exp(i\theta)|n, 1><n + 1, 0| + |n, 1><n + 1, 1|)$$

(5.22)

since it obeys the equation:

$$\exp(i\theta)|\theta> = \frac{1}{\sqrt{2}}[\exp(i\hat{\theta}_B) \otimes \exp(i\hat{\theta}_F)]|\theta_B><\theta_B \otimes |\theta_F> = \exp(i\theta)|\theta> \quad \forall \theta \in [0, 2\pi)$$

(5.23)

**Remark V.2**
Let us remark that, by construction, the supersymmetric exponential phase operator $\exp(i\hat{\theta})$ is not unitary and hence the supersymmetric phase operator $\hat{\theta}$ is not self-adjoint.

Given a normalized product state:

$$|\psi\rangle := |\psi\rangle_B \otimes |\psi\rangle_F$$  \hspace{1cm} (5.24)

$$|\psi\rangle_B := \sum_{n_B=0}^{\infty} c_{n_B} |n_B\rangle$$  \hspace{1cm} (5.25)

$$\sum_{n_B=0}^{\infty} |c_{n_B}|^2 = 1$$  \hspace{1cm} (5.26)

$$|\psi\rangle_F := \sum_{n_F=0}^{1} c_{n_F} |n_F\rangle$$  \hspace{1cm} (5.27)

$$\sum_{n_F=0}^{1} |c_{n_F}|^2 = 1$$  \hspace{1cm} (5.28)

$$\langle \psi |\psi\rangle = \langle \psi |B|\psi\rangle_B < \psi |F|\psi\rangle_F = 1$$  \hspace{1cm} (5.29)

$$\langle \theta |\psi\rangle = \langle \theta |B|\psi\rangle_B < \theta |F|\psi\rangle_F$$  \hspace{1cm} (5.30)

let us introduce the following two events:

- $EV_B(|\psi\rangle, \theta) := \text{"a measurement of the bosonic phase operator $\hat{\theta}_B$, when the supersymmetric oscillator is in the state $|\psi\rangle$, gives as result $\theta \in [0, 2\pi]$"
- $EV_F(|\psi\rangle, \theta) := \text{"a measurement of the fermionic phase operator $\hat{\theta}_F$, when the supersymmetric oscillator is in the state $|\psi\rangle$, gives as result $\theta \in [0, 2\pi]$"

The fact that $|\psi\rangle$ is a product state implies that $EV_B(|\psi\rangle, \theta)$ and $EV_F(|\psi\rangle, \theta)$ are independent events and hence:

$$Pr[EV_B(|\psi\rangle, \theta) \land EV_F(|\psi\rangle, \theta)] = Pr[EV_B(|\psi\rangle, \theta)] \cdot Pr[EV_F(|\psi\rangle, \theta)] = Pr_{|\psi\rangle_B}(\theta) \cdot Pr_{|\psi\rangle_F}(\theta) = |\langle \theta |B|\psi\rangle_B|^2 \cdot |\langle \theta |F|\psi\rangle_F|^2 = |\langle \theta |\psi\rangle|^2 \forall \theta \in [0, 2\pi]$$  \hspace{1cm} (5.31)

Obviously:

$$\int_0^{2\pi} d\theta_1 \int_0^{2\pi} d\theta_2 Pr[EV_B(|\psi\rangle, \theta_1) \land EV_F(|\psi\rangle, \theta_2)] = \int_0^{2\pi} d\theta_1 \int_0^{2\pi} d\theta_2 Pr[EV_B(|\psi\rangle, \theta_1)] \cdot Pr[EV_F(|\psi\rangle, \theta_2)] =$$

$$= (\int_0^{2\pi} d\theta_1 Pr[EV_B(|\psi\rangle, \theta_1)]) \cdot (\int_0^{2\pi} d\theta_2 Pr[EV_F(|\psi\rangle, \theta_2)]) = [\int_0^{2\pi} d\theta_1 Pr_{|\psi\rangle_B}(\theta_1)] \cdot [\int_0^{2\pi} d\theta_2 Pr_{|\psi\rangle_F}(\theta_2)] = 1$$  \hspace{1cm} (5.32)

where we have used the equation (5.25) and the equation (4.44)

When the state $|\psi\rangle$ is entangled, $EV_B(|\psi\rangle, \theta)$ and $EV_F(|\psi\rangle, \theta)$ are not independent events so that:

$$Pr[EV_B(|\psi\rangle, \theta) \land EV_F(|\psi\rangle, \theta)] \neq Pr[EV_B(|\psi\rangle, \theta)] \cdot Pr[EV_F(|\psi\rangle, \theta)]$$  \hspace{1cm} (5.33)

and the situation is more complex.
Let us now finish to consider separately measurements of the *bosonic phase operator* and of the *fermionic phase operator* and let us take into account directly measurement of the *supersymmetric phase operator*.

Given a normalized state:

\[
|\psi> := \sum_{n_B=0}^{\infty} \sum_{n_F=0}^{1} c_{n_B,n_F} |n_B,n_F>
\]

(5.34)

\[
\sum_{n_B=0}^{\infty} \sum_{n_F=0}^{1} |c_{n_B,n_F}|^2 = 1
\]

(5.35)

satisfying the following mysterious constraint:

\[
\sum_{n_B=0}^{\infty} \Re[c_{n_B,1}c_{n_B+1,0}] = 0
\]

(5.36)

the interpretation of \(|<\theta|\psi>|^2\) as the probability that a measurement of the *supersymmetric phase operator*, when the supersymmetric oscillator is the state \(|\psi>\), gives as result \(\theta \in [0,2\pi]\):

\[
Pr|\psi>(\theta) := |<\theta|\psi>|^2 = \sum_{n_B=0}^{\infty} \sum_{n_F=0}^{1} \sum_{n_B'=0}^{1} \sum_{n_F'=0}^{1} \exp[i\theta(n_B+n_F-n_B'-n_F')]c_{n_B,n_F}c_{n_B',n_F'}
\]

(5.37)

is consistent since:

\[
\int_{0}^{2\pi} d\theta Pr|\psi>(\theta) = \sum_{n_B=0}^{\infty} \sum_{n_B'=0}^{1} \int_{0}^{\infty} d\theta \exp[i(n_B-n_B')\theta](c_{n_B,0}c_{n_B',0}+c_{n_B,1}c_{n_B',1}) + \sum_{n_B=0}^{\infty} \sum_{n_B'=0}^{1} \int_{0}^{\infty} d\theta \exp[i(n_B-n_B'-1)\theta]c_{n_B,1}c_{n_B',0} + \sum_{n_B=0}^{\infty} \sum_{n_B'=0}^{1} \int_{0}^{\infty} d\theta \exp[i(n_B-n_B'-1)\theta]c_{n_B,0}c_{n_B',1} = \sum_{n_B=0}^{\infty} (|c_{n_B,0}|^2 + |c_{n_B,1}|^2) + \sum_{n_B=0}^{\infty} (c_{n_B,1}c_{n_B+1,0} + c_{n_B+1,0}c_{n_B,1}) = 1
\]

(5.38)

where we have used the equation \(3.17\) the equation \(5.35\) and the mysterious constraint of the equation \(5.36\).

Contrary, if the mysterious constraint of the equation \(5.36\) is not satisfied, such a probabilistic interpretation is not consistent.

Let us introduce the set of the states of \(\mathcal{H} := \mathcal{H}_B \otimes \mathcal{H}_F\) satisfying such a constraint:

\[
\mathcal{H}_{\text{constraint}} := \{ |\psi> = \sum_{n_B=0}^{\infty} \sum_{n_F=0}^{1} c_{n_B,n_F} |n_B,n_F> \in \mathcal{H} : \sum_{n_B=0}^{\infty} \Re[c_{n_B,1}c_{n_B+1,0}] = 0 \}
\]

(5.39)

It may be easily verified that:

1. \(\mathcal{H}_{\text{constraint}}\) is not a linear subspace of \(\mathcal{H}\).

2. its complement \(\mathcal{H} - \mathcal{H}_{\text{constraint}}\) contains both *product states* and *entangled states*, i.e.:

\[
\mathcal{H}_{\text{constraint}} \cap \mathcal{H}_{\text{product}} \neq \emptyset
\]

(5.40)

\[
\mathcal{H}_{\text{constraint}} \cap \mathcal{H}_{\text{entangled}} \neq \emptyset
\]

(5.41)

where obviously:

\[
\mathcal{H}_{\text{product}} := \{ |\psi>_B \otimes |\psi>_F \ |\psi>_B \in \mathcal{H}_B, |\psi>_F \in \mathcal{H}_F \}
\]

(5.42)

\[
\mathcal{H}_{\text{entangled}} := \mathcal{H} - \mathcal{H}_{\text{product}}
\]

(5.43)
Part II

Theory at strictly positive temperature.

VI. A BRIEF REVIEW OF UMEZAWA’S THERMOFIELD DYNAMICS

Among the different existing approaches available to study quantum field theories at strictly positive temperature [33], [34], H. Umezawa’s approach, usually called thermofield dynamics, is particularly adapted to the discussion of symmetry breaking issues, as we will briefly recall following closely the 3th chapter ”Thermofield Dynamics” of [35] and [36].

Given a quantum system having an hamiltonian $H$ (being of course a self-adjoint operator over a suitable Hilbert space $\mathcal{H}$) with discrete spectrum:

$$H|n> = E_n|n>$$

being in thermodynamical equilibrium with a thermal bath at temperature $T > 0$, let us define a thermal vacuum at inverse temperature $\beta := \frac{1}{T}$ as a state $|0;\beta>$ such that the expectation value $<0;\beta|A|0;\beta>$ of an arbitrary observable $A$ is equal to the statistical average of $A$ over the canonical ensemble, i.e.:

$$<0;\beta|A|0;\beta> = \frac{\text{Tr} \exp(-\beta H) A}{Z(\beta)} = \sum_n \exp(-\beta E_n) <n|A|n> \frac{1}{Z(\beta)}$$

where:

$$Z(\beta) := \text{Tr} \exp(-\beta H)$$

is the canonical partition function.

Using the completeness condition for the eigenvectors of the hamiltonian:

$$\sum_n |n><n| = 1$$

we obtain that:

$$|0;\beta> = \sum_n |n><n|0;\beta>$$

$$<0;\beta| = \sum_m <0;\beta|m><m|$$

and hence we can write the expectation value of the observable $A$ over the thermal vacuum as:

$$<0;\beta|A|0;\beta> = \sum_n \sum_m <n|A|n><n|0;\beta><0;\beta|m><m|A|n>$$

that since:

$$<0;\beta|m> = <m|0;\beta>$$

becomes:

$$<0;\beta|A|0;\beta> = \sum_n \sum_m <n|A|n><n|0;\beta><m|0;\beta><m|A|n>$$

Comparing the equation (6.3) with the equation (6.10) we see that a thermal vacuum $|0;\beta>\in \mathcal{H}$ should satisfy the impossible condition:

$$<n|0;\beta><m|0;\beta> = \frac{\exp(-\beta E_n) \delta_{n,m}}{Z(\beta)}$$
Hence a thermal vacuum $|0; \beta \rangle \in \mathcal{H}$ doesn’t exist.

It follows that, if we insist on looking for a thermal vacuum, we have to search it in a suitably enlarged Hilbert space.

The simplest choice is the doubled Hilbert space $\mathcal{H} \otimes \tilde{\mathcal{H}}$ where $\tilde{\mathcal{H}} := \mathcal{H}$ is a copy of $\mathcal{H}$.

Let us denote with $|\tilde{n} \rangle$ the identical copy of the vector $|n \rangle$ but belonging to the copy Hilbert space $\tilde{\mathcal{H}}$.

Obviously:

$$\sum_n |n > < n| = 1_{\mathcal{H}} \quad (6.12)$$

$$\sum_n |\tilde{n} > < \tilde{n}| = 1_{\tilde{\mathcal{H}}} \quad (6.13)$$

$$\sum_n \sum_m |n, \tilde{m} > < n, \tilde{m}| = 1_{\mathcal{H} \otimes \tilde{\mathcal{H}}} \quad (6.14)$$

$$< n, \tilde{m}|n', \tilde{m}' > = \delta_{n,n'}\delta_{\tilde{m},\tilde{m}'} \quad (6.15)$$

Hence we can express the putative thermal vacuum as:

$$|0; \beta \rangle = \sum_n \sum_{\tilde{m}} |n, \tilde{m} > < n, \tilde{m}|0; \beta \rangle \quad (6.16)$$

Let us now observe that since the copy orthonormal basis $\{|\tilde{n} \rangle\}$ of the copy Hilbert space $\tilde{\mathcal{H}}$ is identical to the basis $\{|n \rangle\}$ of $\mathcal{H}$, it follows that:

$$< n, \tilde{m}|0; \beta > = \delta_{n,\tilde{n}} < n, \tilde{m}|0; \beta > \quad (6.17)$$

and hence:

$$|0; \beta \rangle = \sum_n |\tilde{n} > < n, \tilde{n}|0; \beta \rangle \quad (6.18)$$

**Remark VI.1**

Let us remark that given an observable of our system, i.e. a self-adjoint operator $A$ over $\mathcal{H}$:

$$< n, \tilde{m}|A|n', \tilde{m}' > = < n|A|n' > < \tilde{m}|\tilde{m}' > = < n|A|n' > \delta_{\tilde{m},\tilde{m}'} \quad (6.19)$$

Considering instead the corresponding operator $\tilde{A}$ over the copy Hilbert space $\tilde{\mathcal{H}}$:

$$< n, \tilde{m}|\tilde{A}|n', \tilde{m}' > = < n|\tilde{A}|n' > < \tilde{m}|\tilde{A}|\tilde{m}' > = \delta_{n,n'} < \tilde{m}|\tilde{A}|\tilde{m}' > \quad (6.20)$$

Given an observable $A$ of our system we have then that:

$$< 0; \beta|A|0; \beta > = \sum_n \sum_m < 0; \beta|n, \tilde{n} > < m, \tilde{m}|0; \beta > < n, \tilde{n}|A|m, \tilde{m} > =$$

$$\sum_n \sum_m < 0; \beta|n, \tilde{n} > < m, \tilde{m}|0; \beta > < n|A|m > \delta_{n,m} \quad (6.21)$$

where in the last passage we have used the equation [6.19]

Hence:

$$< 0; \beta|A|0; \beta > = \sum_n |< n, \tilde{n}|0; \beta > |^2 < n|A|n > \quad (6.22)$$
So the equation defining a thermal vacuum is satisfied by the vector $|0; \beta > \in \mathcal{H} \otimes \tilde{\mathcal{H}}$ if and only if:

$$| < n, \tilde{n}|0; \beta > |^2 = \frac{\exp(-\beta E_n)}{Z(\beta)}$$  \hspace{1cm} (6.23)

that admits many solutions among which the simpler one may be obtained imposing that $< n, \tilde{n}|0; \beta > \in \mathbb{R}$:

$$< n, \tilde{n}|0; \beta > := \frac{\exp(-\frac{\beta E_n}{2})}{\sqrt{Z(\beta)}}$$  \hspace{1cm} (6.24)

**Remark VI.2**

Up to this point the introduction of the notion of a thermal vacuum may appear an unjustified complication. Its power appears as soon as one analyzes the phenomenon of symmetry breaking and symmetry restoration at strictly positive temperature.

Let us, first of all, review the definition of symmetry breaking at zero temperature.

Let us suppose to have a strongly continuous one-parameter unitary group $U_\alpha := \exp(-i\alpha Q)$ that is a symmetry of the system, i.e.:

$$U_\alpha H U_\alpha^\dagger = H \quad \forall \alpha \in \mathbb{R}$$  \hspace{1cm} (6.25)

and hence:

$$[Q, H] = 0$$  \hspace{1cm} (6.26)

We will say that such a symmetry is broken at zero temperature (i.e. at $\beta = +\infty$) whether:

$$Q|0 > \neq 0$$  \hspace{1cm} (6.27)

where $|0 >$ is the vacuum state.

Let us define a Goldstone operator at zero temperature (i.e. at $\beta = +\infty$) as an operator $A$ such that:

$$< 0|[Q, A]|0 > \neq 0$$  \hspace{1cm} (6.28)

Clearly the symmetry is broken at zero temperature if and only if there exists a Goldstone operator at zero temperature.

Let us now consider the same system in thermodynamical equilibrium with a thermal bath at strictly positive temperature.

We will say that the symmetry is broken at inverse temperature $\beta \in [0, +\infty)$ whether:

$$Q|0; \beta > \neq 0$$  \hspace{1cm} (6.29)

where $|0; \beta >$ is a thermal vacuum.

Let us define a Goldstone operator at inverse temperature $\beta \in [0, +\infty)$ as an operator $A$ such that:

$$< 0; \beta|[Q, A]|0; \beta > \neq 0$$  \hspace{1cm} (6.30)

Clearly the symmetry is broken at inverse temperature $\beta \in [0, +\infty)$ if and only if there exists a Goldstone operator at inverse temperature $\beta$. 
VII. EXPECTATION VALUE OF THE BOSONIC PHASE OPERATOR

Given the bosonic oscillator with hamiltonian given by the equation 3.1, the condition of equation 6.24 determines the following thermal vacuum:

\[
|0; \beta > := \sqrt{1 - \exp(-\beta\omega)} \sum_{n=0}^{\infty} \exp(-\frac{n\beta\omega}{2})|n, \tilde{n} >
\]  

(7.1)

where we have used the fact that:

\[
\sum_{n=0}^{\infty} x^n = \frac{1}{1 - x} \quad \forall x \in [0, 1)
\]  

(7.2)

Introduced the self-adjoint operator:

\[
Q(\phi_B) := -i\phi_B(\beta)(\tilde{a}a - a^\dagger \tilde{a}^\dagger)
\]  

(7.3)

and the unitary operator:

\[
U(\phi_B) := \exp(-iQ(\phi_B))
\]  

(7.4)

it follows that the thermal vacuum may be obtained by the Bogoliubov transformation:

\[
|0; \beta > = U(\phi_B)|0, \tilde{0} >
\]  

(7.5)

provided:

\[
\cosh \phi_B(\beta) = \frac{1}{\sqrt{1 - \exp(-\beta\omega)}}
\]  

(7.6)

\[
\sinh \phi_B(\beta) = \frac{\exp(-\frac{\beta\omega}{2})}{\sqrt{1 - \exp(-\beta\omega)}}
\]  

(7.7)

Clearly:

\[
< N_B |_\beta = < 0; \beta | N_B | 0; \beta > = [1 - \exp(-\beta\omega)] \sum_{n=0}^{\infty} n \exp(-\beta\omega n) = \frac{\exp(-\beta\omega)}{1 - \exp(-\beta\omega)} = \sinh^2 \phi_B(\beta)
\]  

(7.8)

where we have used the fact that:

\[
\sum_{n=0}^{\infty} nx^n = \frac{x}{(1 - x)^2} \quad \forall x \in [0, 1)
\]  

(7.9)

Furthermore:

\[
< \exp(i\hat{\theta}) |_\beta = < 0; \beta | \exp(i\hat{\theta}) | 0; \beta > = (1 - \exp(-\beta\omega)) \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \exp(-\frac{(n + m + \beta\omega)}{2}) \delta_{n,k} \delta_{m,k+1} \delta_{n,m} = 0
\]  

(7.10)

as it can be checked computing the expectation value of the bosonic exponential phase operator directly, i.e. avoiding the thermofield dynamics’ approach:

\[
< \exp(i\hat{\theta}) |_\beta = \frac{\sum_{n=0}^{\infty} \exp(-\beta E_n) < n | \exp(i\hat{\theta}) | n >}{Z(\beta)} = (1 - \exp(-\beta\omega)) \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \exp(-\beta\omega n) \delta_{n,k} \delta_{n,k+1} = 0
\]  

(7.11)
VIII. EXPECTATION VALUE OF THE FERMIONIC PHASE OPERATOR

Given the fermionic oscillator with hamiltonian given by the equation 4.1, the condition of equation 6.24 determines the following thermal vacuum:

\[ |0; \beta > := \frac{1}{\sqrt{1 + \exp(-\beta \omega)}} (|0, \hat{0} > + \exp(-\frac{\beta \omega}{2})|1, \hat{I} >) \] (8.1)

Introduced the self-adjoint operator:

\[ Q(\phi_F) := -i\phi_F(\beta)(\hat{a}\hat{a} - \hat{a}^\dagger \hat{a}^\dagger) \] (8.2)

and the unitary operator:

\[ U(\phi_F) := \exp(-iQ(\phi_F)) \] (8.3)

it follows that the thermal vacuum may be obtained by the Bogoliubov transformation:

\[ |0; \beta > = U(\phi_F)|0, \hat{0} > \] (8.4)

provided:

\[ \cos \phi_F(\beta) = \frac{1}{\sqrt{1 + \exp(-\beta \omega)}} \] (8.5)

\[ \sin \phi_F(\beta) = \frac{\exp(-\frac{\beta \omega}{2})}{\sqrt{1 + \exp(-\beta \omega)}} \] (8.6)

Clearly:

\[ < N_F >_\beta = < 0; \beta|N_F|0; \beta > = \frac{1}{1 + \exp(-\beta \omega)} [< 0, \hat{0} > + \exp(-\frac{\beta \omega}{2}) < 1, \hat{I} >] = \frac{1}{1 + \exp(-\beta \omega)} [< 0, \hat{0} > + \exp(-\frac{\beta \omega}{2}) < 1, \hat{I} >] \exp(-\frac{\beta \omega}{2})|1, \hat{I} > = \frac{\exp(-\beta \omega)}{1 + \exp(-\beta \omega)} = \sin^2 \phi_F(\beta) \] (8.7)

\[ < \exp(i\hat{\theta}) >_\beta = < 0; \beta|\exp(i\hat{\theta})|0; \beta > = \frac{\exp(-\beta \omega)}{1 + \exp(-\beta \omega)} = \frac{\exp(-\beta \omega)}{1 + \exp(-\beta \omega)} = \sin^2 \phi_F(\beta) \] (8.8)
IX. THE SUPERSYMMETRIC PHASE OPERATOR AS A GOLDSTONE OPERATOR AT STRICTLY
POSITIVE TEMPERATURE

Given the supersymmetric oscillator with hamiltonian given by the equation \[5.1\] let us introduce the self-adjoint operator:

\[ G(\phi_B, \phi_F) := -i\phi_B(\beta)(\bar{a}_B a_B - a_B^\dagger \bar{a}_B^\dagger) - i\phi_F(\beta)(\bar{a}_F a_F - a_F^\dagger \bar{a}_F^\dagger) \] (9.1)

and the unitary operator:

\[ U(\phi_B, \phi_F) := \exp(-iG(\phi_B, \phi_F)) \] (9.2)

The thermal vacuum determined by the equation \[6.24\] may be obtained by the Bogoliubov transformation:

\[ |0; \beta > = U(\phi_B, \phi_F)|0, \tilde{0} > \] (9.3)

provided:

\[ \tanh \phi_B(\beta) = \tan \phi_F(\beta) = \exp(-\frac{\beta \omega}{2}) \] (9.4)

Clearly:

\[ < N_B >_\beta = < 0; \beta |N_B|0; \beta > = \sinh^2 \phi_B(\beta) \] (9.5)

\[ < N_F >_\beta = < 0; \beta |N_F|0; \beta > = \sin^2 \phi_F(\beta) \] (9.6)

so that the internal energy is:

\[ U(\beta) = < 0; \beta |H|0; \beta > = \omega[\sinh^2 \phi_B(\beta) + \sin^2 \phi_F(\beta)] \] (9.7)

Furthermore:

\[ Q|0; \beta > = a_B^\dagger a_F|0; \beta > = \cosh \phi_B(\beta) \sin \phi_F(\beta)|n_B(\beta) = 1, n_F(\beta) = 0; \tilde{n}_B(\beta) = 0, \tilde{n}_F(\beta) = 1 > \] (9.8)

\[ \bar{Q}|0; \beta > = a_F^\dagger a_B|0; \beta > = \sinh \phi_B(\beta) \cos \phi_F(\beta)|n_B(\beta) = 0, n_F(\beta) = 1; \tilde{n}_B(\beta) = 1, \tilde{n}_F(\beta) = 0 > \] (9.9)

and hence:

\[ Q|0; \beta > \begin{cases} = 0, & \text{if } \beta = +\infty; \\ \neq 0, & \text{if } \beta \in (0, +\infty). \end{cases} \] (9.10)

\[ \bar{Q}|0; \beta > \begin{cases} = 0, & \text{if } \beta = +\infty; \\ \neq 0, & \text{if } \beta \in (0, +\infty). \end{cases} \] (9.11)

from which we can infer that:

• the supersymmetry is unbroken at zero temperature

• the supersymmetry is broken at every temperature \( T > 0 \) (finite or infinite).

Remark IX.1
Supersymmetry breaking is usually analyzed in terms of the Witten index (defined as the difference between the number of bosonic and fermionic zero-energy states).

Indeed, in his 1982’s fundamental paper, Edward Witten showed that the vanishing of the Witten index is a necessary (though not sufficient) condition for having Susy broken.

Unfortunately, in the case of the supersymmetric oscillator, the computation of Witten index involves subtle regularization’s issues that we have preferred to avoid (see for instance the 4th chapter “SUSY Breaking, Witten Index and Index Condition” of [37] and the references therein indicated).

We will now show that the supersymmetric phase operator is a Goldstone operator at every temperature $T > 0$.

Let us observe, first of all, that:

$$Z(\beta) = \sum_{n_B=0}^{\infty} \sum_{n_F=0}^{1} \exp[-\beta \omega (n_B + n_F)] = \sum_{n_B=0}^{+\infty} \exp(-\beta \omega n_B) + \exp(-\beta \omega (n_B + 1)) = \frac{1 + \exp(-\beta \omega)}{1 - \exp(-\beta \omega)} \quad (9.12)$$

Furthermore some trivial computation leads to:

$$< n_B, n_F | [Q, \exp(i\hat{\theta})] | n_B, n_F > = \exp(i\theta) \sum_{n=0}^{\infty} \sqrt{n + 1} \delta_{n_B,n+1} \delta_{n_F,0} \quad (9.13)$$

Therefore:

$$< 0; \beta | [Q, \exp(i\hat{\theta})] | 0; \beta > = \frac{\text{Tr} \exp(-\beta H) [Q, \exp(i\hat{\theta})]}{Z(\beta)} \sum_{n_B=0}^{\infty} \sum_{n_F=0}^{1} \exp[-\beta \omega (n_B + n_F)] < n_B, n_F | [Q, \exp(i\hat{\theta})] | n_B, n_F > = \frac{1 - \exp(-\beta \omega)}{1 + \exp(-\beta \omega)} \exp(i\theta) Li_{-\frac{1}{2}}[\exp(-\beta \omega)] \neq 0 \ \forall \beta \in [0, +\infty) \quad (9.14)$$

where:

$$Li_n(x) := \sum_{k=1}^{\infty} \frac{x^k}{k^n} \quad (9.15)$$

is the polylogarithmic function.

In a similar way one gets that:

$$< n_B, n_F | [\bar{Q}, \exp(i\hat{\theta})] | n_B, n_F > = -\exp(i\theta) \sum_{n=0}^{\infty} \sqrt{n + 1} \delta_{n_B,n} \delta_{n_F,1} \quad (9.16)$$

and hence:

$$< 0; \beta | [\bar{Q}, \exp(i\hat{\theta})] | 0; \beta > = \frac{\text{Tr} \exp(-\beta H) [\bar{Q}, \exp(i\hat{\theta})]}{Z(\beta)} \sum_{n_B=0}^{\infty} \sum_{n_F=0}^{1} \exp[-\beta \omega (n_B + n_F)] < n_B, n_F | [\bar{Q}, \exp(i\hat{\theta})] | n_B, n_F > = -\frac{1 - \exp(-\beta \omega)}{1 + \exp(-\beta \omega)} \exp(i\theta) Li_{-\frac{1}{2}}[\exp(-\beta \omega)] \neq 0 \ \forall \beta \in [0, +\infty) \quad (9.17)$$

**Remark IX.2**

Let us remark that the equation 9.14 and the equation 9.17 contemplate also the case $\beta = 0$ corresponding to infinite temperature.

In fact, it can be easily checked that, in the limit $\beta \rightarrow 0$, the divergence of $Li_{-\frac{1}{2}}[\exp(-\beta \omega)]$ wins against the convergence to zero of $\frac{1 - \exp(-\beta \omega)}{1 + \exp(-\beta \omega)}$.

Therefore the supersymmetric phase operator is a Goldstone operator at infinite temperature.
Remark IX.3

Let us remark that since

\[ \lim_{\beta \to +\infty} < 0; \beta | (Q, \exp(i\hat{\theta})) | 0; \beta > = \lim_{\beta \to +\infty} < 0; \beta | (\overline{Q}, \exp(i\hat{\theta})) | 0; \beta > = 0 \] (9.18)

it follows that the \textit{supersymmetric phase operator} is not a Goldstone operator at zero temperature, as we already knew by the fact that the supersymmetry is unbroken at zero temperature.
