ON THE UNIQUENESS OF CONFORMAL-HARMONIC MAPS

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ABSTRACT. Motivated by the theory of harmonic maps on Riemannian surfaces, conformal-harmonic maps between two Riemannian manifolds $\mathcal{M}$ and $\mathcal{N}$ were introduced in search of a natural notion of "harmonicity" for maps defined on a general even dimensional Riemannian manifold $\mathcal{M}$. They are critical points of a conformally invariant energy functional and reassemble the GJMS operators when the target is the set of real or complex numbers. On a four dimensional manifold, conformal-harmonic maps are the conformally invariant counterparts of the intrinsic bi-harmonic maps and a mapping version of the conformally invariant Paneitz operator for functions.

In this paper, we consider conformal-harmonic maps from certain locally conformally flat 4-manifolds into spheres. We prove a quantitative uniqueness result for such conformal-harmonic maps as an immediate consequence of convexity for the conformally-invariant energy functional. To this end, we are led to prove a version of second order Hardy inequality on manifolds, which may be of independent interest.

1. Introduction

The most prominent and classic problem in calculus of variations for mappings between two Riemannian manifolds $(\mathcal{M}, g)$ and $(\mathcal{N}, h) \hookrightarrow \mathbb{R}^K$ is the study of harmonic maps, which are critical points $u : \mathcal{M} \to \mathcal{N}$ of the Dirichlet energy

$$E_1(u, \mathcal{M}) := \int_{\mathcal{M}} |du|^2 \omega_g,$$

where $\omega_g$ is the volume measure on $\mathcal{M}$ defined by the metric $g$ and $|du|^2$ is the Hilbert-Schmidt norm square of $du$. The conformal invariance of $E_1(u, \mathcal{M})$ on a Riemannian surface $\mathcal{M}$ (with respect to the metric $g$ of $\mathcal{M}$) and its connection to the theory of minimal surfaces make harmonic maps on two dimensional domains the most widely studied topic in the field of geometric analysis ever since the pioneering work of J. Eells and J. Sampson [12], see also [19]. Motivated by the theory of harmonic maps on Riemannian surfaces, the Paneitz operator [38] and GJMS operators [17], in a recent work [2] V. Bérand has shown the existence of an intrinsically defined energy functional for smooth maps between two Riemannian manifolds $(\mathcal{M}^{2m}, g)$ (of even dimension $2m$ where $m \geq 1$) and $\mathcal{N}$, denoted by $E_m(u, \mathcal{M})$, which is conformally invariant with respect to $g$ and coincides with
the above Dirichlet energy $E_1(u, \mathcal{M})$ when $m = 1$. Following the terminology of A. Gastel and A. Nerf in [14] who considered an extrinsic analogue (i.e., a variant dependent of the embedding $\mathcal{N} \hookrightarrow \mathbb{R}^K$) of $E_m(u, \mathcal{M})$, we will call $E_m(u, \mathcal{M})$ an intrinsic Paneitz energy when $m = 2$ and intrinsic Paneitz poly-energy when $m \geq 3$. The critical points of $E_m(u, \mathcal{M})$ are called conformal-harmonic maps or C-harmonic maps, which generalize the harmonic maps on surfaces (i.e., $\text{Ricci curvature of } M$ intrinsic bi-harmonic maps, i.e., the critical points of $E$ and critical points of $\mathcal{N}$ dependent of the embedding extrinsically.

A. Gastel and A. Nerf in [14] who considered an PDEs (c.f. [22, 27, 36]):

where $\tau(u) \in u^*TN$ is the tension field, we denote the scalar curvature and Ricci curvature of $\mathcal{M}$ by $Sc^M$ and $\text{Ric}^M$, respectively. We remark that $\mathcal{E}(u, \mathcal{M})$ is conformally invariant on a four dimensional manifold $\mathcal{M}$ (see Appendix A) and critical points of $\mathcal{E}(u, \mathcal{M})$ are the conformal-invariant counterparts of the intrinsic bi-harmonic maps, i.e., the critical points of the intrinsic bi-energy

$$
\mathcal{F}(u, \mathcal{M}) := \int_{\mathcal{M}} |\tau(u)|^2.
$$

Conformal-harmonic maps are also generalizations of the Paneitz operator in the context of maps, and intrinsically they satisfy the following system of 4-th order PDEs (c.f. [22, 27, 36]):

$$
\Delta^u \tau(u) = R^u (\nabla u, \tau(u)) \nabla u + \frac{2}{3} (Sc^M \tau(u) + \nabla Sc^M \nabla u) - 2 \text{div} (\text{Ric}^M(\nabla u, \cdot)),
$$

where $\Delta^u$ is the induced Laplace operator on the pullback vector bundle $u^*TN$ over $\mathcal{M}$ or equivalently and extrinsically (c.f. [32, equation (1.2)], [31, equation (9)])

$$
\Delta^2 u = -\Delta (\nabla P^\perp \nabla u) - \text{div}(\nabla P^\perp \Delta u) + 2 \nabla P^\perp \nabla (\nabla P^\perp \nabla u) + 2 \nabla P^\perp \nabla \Delta u - (\nabla PP^\perp - P^\perp \nabla P) \nabla \Delta u
$$

\begin{equation}
+ P(\nabla P^\perp \nabla u, D_u(D_u P^\perp) \nabla u) \nabla u - 2 \text{div}(\nabla P^\perp \nabla u, \nabla P^\perp P)
\end{equation}

$$
+ 2 (\nabla P^\perp \nabla u, \nabla P^\perp \nabla P) + \frac{2}{3} Sc^M (\Delta u + \nabla P^\perp \nabla u)
$$

$$
+ \frac{2}{3} \nabla Sc^M \nabla u - 2 \nabla \text{Ric}^M (\nabla u, \cdot) - 2 \text{Ric}^M (P(\nabla^2 u), \cdot),
$$

see Appendix B for a detailed derivation of this equation. Here $R^u$ is the curvature tensor of $\mathcal{N}$, $\nabla$ and $\Delta$ are the Levi-Civita connection and Laplace–Beltrami operator on $\mathcal{M}$ respectively, $\tau(u) = (\Delta u)^T$ is the tension field, $P(u)$ is the orthogonal projection to the tangent plane $T_u \mathcal{N}$, $D$ is the derivative with respect to the standard coordinates of $\mathbb{R}^K$ and $A(\cdot, \cdot)$ denotes the second fundamental
form of $\mathcal{N} \hookrightarrow \mathbb{R}^K$. Note that this conformal-harmonic map equation (1.2) differs from the intrinsic biharmonic map equation by lower order terms which make (1.2) conformally invariant.

Higher order geometric variational problems, including the study of (both extrinsic and intrinsic) biharmonic maps and polyharmonic maps, have attracted much attention in the last two decades, see e.g. [10, 5, 42, 44, 29, 39, 41, 15, 30, 4] for the extrinsic case and [44, 25, 37, 15, 32] for the intrinsic case. The corresponding heat flows have been studied extensively as well, as a tool to prove the existence of biharmonic maps and polyharmonic maps in a given homotopy class, see e.g. [26, 27, 13, 45, 20, 21, 32]. It should be noted that the extrinsic and intrinsic cases come in two different flavors: the intrinsic variants are considered more geometrically natural because they do not depend on the embedding of the target manifold $\mathcal{N} \hookrightarrow \mathbb{R}^K$, although they are less natural from the variational point of view due to the lack of coercivity for the intrinsic energies (and thus they are considerably more difficult analytically and less studied); on the other hand, the extrinsic variants are more natural from the analytical point of view but in turn they do depend on the embedding of $\mathcal{N} \hookrightarrow \mathbb{R}^K$. Among them, the conformally invariant problems (both extrinsic and intrinsic) are considered the most geometric, see e.g., [10, 14]. Most of the results in the current literature concern the regularity and existence of (the weak solutions of) the critical points of the higher order geometric variational problems because they are associated to systems of higher order PDEs with critical growth nonlinearities. However, the uniqueness problem of these critical points have been left largely open. In [32] P. Laurain and the first author proved a version of bienergy convexity (and thus the uniqueness) for weakly intrinsic biharmonic maps in $W^{2,2}(B_1, S^n)$ with small bienergy and prescribed boundary data, where $B_1 \subset \mathbb{R}^4$ is the unit 4-ball and $S^n \subset \mathbb{R}^{n+1}$ is the standard unit sphere. We shall remark that such energy convexity plays an essential role in the discrete replacement min-max construction of geometric objects of interest, see e.g. Colding-Minicozzi [7] and Lamm-Lin [28] (for weakly harmonic maps).

We shall remark that the intrinsic Paneitz energy functional $\mathcal{E}(u, \mathcal{M})$ defined in (1.1) on $S^4$ was already used by T. Lamm in [27] as a tool to prove that every weakly intrinsic biharmonic maps from $\mathbb{R}^4$ into a non-positively curved target manifold with finite bienergy has to be constant. More recently, O. Biquard and F. Madani in [3] used the corresponding heat flow for $\mathcal{E}(u, \mathcal{M})$ to prove an existence result for conformal-harmonic maps from a certain class of 4-manifolds into a non-positively curved manifold $\mathcal{N}$. Here the conformal-harmonic map heat flow (or, the negative $L^2$-gradient flow of $\mathcal{E}(\cdot, \mathcal{M})$) is defined as follow (when
\[ \partial \mathcal{M} = \emptyset: \]
\[
\begin{cases}
\frac{\partial u}{\partial t} + \Delta u \tau(u) = R^N(\nabla u, \tau(u))\nabla u + \frac{2}{3} \left( \text{Sc}^M \tau(u) + \nabla \text{Sc}^M \nabla u \right) \\
- 2 \text{div}(\text{Ric}^M(\nabla u, \cdot)) & \text{on } \mathcal{M} \times [0, T) \\
u = u_0 & \text{on } \mathcal{M} \times \{0\},
\end{cases}
\]
(1.3)
and (when \( \partial \mathcal{M} \neq \emptyset \))
\[
\begin{cases}
\frac{\partial u}{\partial t} + \Delta u \tau(u) = R^N(\nabla u, \tau(u))\nabla u + \frac{2}{3} \left( \text{Sc}^M \tau(u) + \nabla \text{Sc}^M \nabla u \right) \\
- 2 \text{div}(\text{Ric}^M(\nabla u, \cdot)) & \text{on } \mathcal{M} \times [0, T) \\
u = u_0 & \text{on } \mathcal{M} \times \{0\}
\end{cases}
\]
\[
u = u_0, \quad \partial_\nu u = \partial_\nu u_0 & \text{on } \partial \mathcal{M} \times [0, T),
\]
(1.4)
where \( u_0 \in C^\infty \cap W^{2,2}(\overline{\mathcal{M}}, \mathcal{N}) \). More precisely, in \([3]\) O. Biquard and F. Madani proved

**Theorem 1.1.** Let \((\mathcal{M}, g)\) and \((\mathcal{N}, h)\) be compact Riemannian manifolds of four dimensions and \(n\) dimensions respectively. Assume that \(\mathcal{N}\) has non-positive curvature, the Yamabe invariant
\[
\mu(\mathcal{M}, [g]) := \inf_{g' \in [g]} \frac{\int_\mathcal{M} \text{Sc}_{g'} dv_{g'}}{\left( \int_\mathcal{M} dv_{g'} \right)^{1/2}} > 0
\]
and the conformally invariant total \(Q\)-curvature
\[
\kappa(\mathcal{M}, [g]) := \frac{1}{12} \int_\mathcal{M} \left( \text{Sc}^2_g - 3|\text{Ric}_g|^2 \right) dv_g
\]
satisfies \(\kappa + \frac{1}{5}\mu^2 > 0\). Then the conformal-harmonic map heat flow \([3]\) exists and is smooth for all time and converges subsequently to a smooth conformal-harmonic map \(u_\infty(\mathcal{M}, \mathcal{N})\) as \(t \to \infty\). Consequently, there exists a conformal-harmonic map in each homotopy class in \(C^\infty(\mathcal{M}, \mathcal{N})\).

In this paper, we consider the intrinsic Paneitz energy \(E(u, \mathcal{M})\) for \(W^{2,2}\) maps from certain locally conformally flat 4-manifolds with smooth boundaries into spheres and study the (quantitative) uniqueness of the critical points. The first main result of this paper is a version of energy convexity for \(E(u, \mathcal{M})\), more precisely, we prove

**Theorem 1.2.** Let \((M, g)\) be a four dimensional compact locally conformally flat Riemannian manifold with a smooth boundary \(\partial M\) and \(\rho\) be the distance function to the boundary. Suppose one of the following conditions holds:

(i) \(\Delta \rho \leq 0\) on \(\mathcal{M}\) in the support sense; or
(ii) the first eigenfunction \(\psi\) associated to the first eigenvalue of \(-\Delta_g\) on \(\mathcal{M}\) with \(\psi|_{\partial \mathcal{M}} = 0\) satisfies: \(\psi \in C^1(\overline{\mathcal{M}}), \psi > 0\) on \(\mathcal{M}\) and \(|\nabla \psi| \neq 0\) on \(\overline{\mathcal{M}}\),
then there exist $\epsilon_0 > 0, C > 0$ depending only on $\mathcal{M}$ such that for any $u, v \in W^{2,2}(\mathcal{M}, \mathbb{S}^n)$ with $u = v, \partial_\nu u = \partial_\nu v$ on $\partial \mathcal{M}$,

\begin{align}
\int_{\mathcal{M}} |\Delta u|^2 dv_g &\leq \epsilon_0, \\
\int_{\mathcal{M}} |\nabla v|^4 dv_g &\leq \epsilon_0
\end{align}

and $u$ is a weakly conformal-harmonic map, we have

\begin{align}
\frac{1}{C} \int_{\mathcal{M}} |\nabla v - \Delta u|^2 dv_g &\leq \mathcal{E}(v, \mathcal{M}) - \mathcal{E}(u, \mathcal{M}).
\end{align}

In order to prove this theorem, we are led to show a second order Hardy inequality on certain manifolds with boundaries. Inspired by the recent work of D’Ambrosio-Dipierro [8] (which followed the techniques introduced by Mitidieri in [35]), we are able to prove the following version of second order Hardy inequality on certain manifolds which is sufficient for our applications, for details see Section 3.

**Theorem 1.3.** Let $(\mathcal{M}, g)$ be an $n$ dimensional compact Riemannian manifold with a smooth boundary $\partial \mathcal{M}$ and $\rho \in W^{1,2}_{\text{loc}}(\mathcal{M}, \mathbb{R})$ be a nonnegative function such that $\Delta \rho \leq 0$ on $\mathcal{M}$ in the weak sense and $|\nabla \rho| \neq 0$ a.e. on $\mathcal{M}$, then there exits $C := C(\mathcal{M}) > 0$ such that

\begin{align}
\int_{\mathcal{M}} \frac{|w|^2 |\nabla \rho|^2}{\rho^4} dv_g &\leq C \int_{\mathcal{M}} |\Delta w|^2 (1 + |\nabla \rho|^{-2}) dv_g
\end{align}

for any $w \in C^\infty_0(\mathcal{M}, \mathbb{R})$.

It follows immediately from Theorem 1.3 that we have the following

**Theorem 1.4.** Let $(\mathcal{M}, g)$ be an $n$ dimensional compact Riemannian manifold with a smooth boundary $\partial \mathcal{M}$ and $\rho$ be the distance function to the boundary. Suppose one of the following conditions holds:

(i) $\Delta \rho \leq 0$ on $\mathcal{M}$ in the support sense; or

(ii) the first eigenfunction $\psi$ associated to the first eigenvalue of $-\Delta_g$ on $\mathcal{M}$ with $\psi|_{\partial \mathcal{M}} = 0$ satisfies: $\psi \in C^1(\mathcal{M}), \psi > 0$ on $\mathcal{M}$ and $|\nabla \psi| \neq 0$ on $\mathcal{M}$,

then there exits $C := C(\mathcal{M}) > 0$ such that

\begin{align}
\int_{\mathcal{M}} \frac{|w|^2}{\rho^4} dv_g &\leq C \int_{\mathcal{M}} |\Delta w|^2 dv_g
\end{align}

for any $w \in C^\infty_0(\mathcal{M}, \mathbb{R})$.

**Proof.** For case (i), it simply follows from Theorem 1.3 and the fact that $|\nabla \rho| = 1$ a.e. on $\mathcal{M}$ when $\rho$ is the distance function to the boundary $\partial \mathcal{M}$. For case (ii), by assumptions, there exist constants $L, \delta > 0$ such that

$$\psi(x) \leq L \rho(x) \quad \text{and} \quad |\nabla \psi|(x) \geq \delta, \quad x \in \mathcal{M}.$$
Then applying Theorem 1.3 we get
\[ \int_M \frac{|w|^2}{\rho^4} dv_g \leq L^4 \delta^2 \int_M \frac{|w|^2|\nabla \psi|^2}{\psi^4} dv_g \leq \frac{CL^4}{\delta^2} \int_M |\Delta w|^2 dv_g \]
for any \( w \in C_0^\infty (\mathcal{M}, \mathbb{R}) \).

An immediate corollary of Theorem 1.2 is the following

**Corollary 1.5.** Let \((\mathcal{M}, g)\) be a four dimensional compact locally conformally flat Riemannian manifold with a smooth boundary \(\partial \mathcal{M}\). Suppose \(\text{Ric}^\mathcal{M} \geq -3\) and the mean curvature of \(\partial \mathcal{M}\) satisfies \(H \geq 3 \coth h\) where \(h > 1\), then there exist \(\epsilon_0 > 0\) and \(C > 0\) depending only on \(\mathcal{M}\) such that for any \(u, v \in W^{2,2}(\mathcal{M}, S^n)\) with \(u = v\), \(\partial_\nu u = \partial_\nu v\) on \(\partial \mathcal{M}\),

\[ (1.7) \quad \int_M |\Delta u|^2 dv_g \leq \epsilon_0, \quad \int_M |\nabla v|^4 dv_g \leq \epsilon_0 \]

and \(u\) is a weakly conformal-harmonic map, we have

\[ (1.8) \quad \frac{1}{C} \int_M |\Delta v - \Delta u|^2 dv_g \leq \mathcal{E}(v, \mathcal{M}) - \mathcal{E}(u, \mathcal{M}). \]

**Proof.** Let \(\rho\) be the distance function to the boundary. Then by the assumptions on the Ricci and mean curvatures we have in the support sense (see [16] and cf. [46 Proposition 5])

\[ \rho(x) \leq h, \quad \forall x \in \mathcal{M} \]

and

\[ \Delta \rho \leq -3 \coth (h - \rho) < 0. \]

Then we can apply Theorem 1.2.

As another immediate corollary of Theorem 1.2 we get the uniqueness for weakly conformal-harmonic maps from a certain 4-manifold \(\mathcal{M}\) into spheres.

**Corollary 1.6.** Let \((M, g)\) be a four dimensional compact locally conformally flat Riemannian manifold with a smooth boundary \(\partial M\) and \(\rho\) be the distance function to the boundary. Suppose one of the following conditions holds:

(i) \(\Delta \rho \leq 0\) on \(\mathcal{M}\) in the support sense; or

(ii) the first eigenfunction \(\psi\) associated to the first eigenvalue of \(-\Delta_g\) on \(\mathcal{M}\) with \(\psi|_{\partial \mathcal{M}} = 0\) satisfies: \(\psi \in C^1(\overline{\mathcal{M}}), \psi > 0\) on \(\mathcal{M}\) and \(|\nabla \psi| \neq 0\) on \(\overline{M}\),

then there exists \(\epsilon_0 > 0\) depending only on \(\mathcal{M}\) such that for any weakly conformal-harmonic maps \(u, v \in W^{2,2}(\mathcal{M}, S^n)\) with \(u = v\), \(\partial_\nu u = \partial_\nu v\) on \(\partial \mathcal{M}\) and

\[ \int_M |\Delta u|^2 dv_g \leq \epsilon_0, \quad \int_M |\Delta v|^2 dv_g \leq \epsilon_0, \]

we have \(u \equiv v\) on \(\mathcal{M}\).
2. Preliminary

In this section, we will fix some notations and recall a technical theorem (\(\varepsilon\)-regularity) that will be used later. Throughout this section, \((M, g)\) denotes a smooth 4-dimensional compact Riemannian manifold with a smooth boundary \(\partial M\) and \((N, h)\) is an \(n\)-dimensional smooth closed Riemannian manifold which can be embedded into \(\mathbb{R}^k\). As mentioned in the introduction, a weakly conformal-harmonic map \(u\) from \(M\) into \(N\) is a map in \(W^{2,2}(M, \mathbb{R}^k)\) that is a critical point of the conformally invariant energy \(\mathcal{E}(\cdot, M)\) defined in (1.1) and takes values almost everywhere in \(N\).

Note that the dimension 4 is critical for the analysis of weakly conformal-harmonic maps (e.g. a \(W^{2,2}\) map falls in \(L^p\) for any \(p < \infty\) but barely fails to be continuous in dimension 4). Now let \(\Pi : N_\delta \to N\) be the nearest point projection map, which is well defined and smooth for \(\delta > 0\) small enough. Here \(N_\delta = \{y \in \mathbb{R}^k | \text{dist}(y, N) \leq \delta\}\). For \(y \in N\), let

\[P(y) \equiv D\Pi(y) : \mathbb{R}^k \to T_yN\]

be the orthogonal projection onto the tangent plane \(T_yN\), and

\[P^\perp(y) \equiv \text{Id} - D\Pi(y) : \mathbb{R}^k \to (T_yN)^\perp,\]

where \(D\) is the derivative with respect to the standard coordinates of \(\mathbb{R}^k\). In the following, we will write \(P\) (resp. \(P^\perp\)) instead of \(P(y)\) (resp. \(P^\perp(y)\)) and we will identify these linear transformations with their matrix representations in \(M_n\). We also note that these projections are in \(W^{2,2}(M, M_n)\) as soon as \(u\) is in \(W^{2,2}(M, N)\).

Finally, note that the second fundamental form \(A(\cdot)(\cdot, \cdot)\) of \(N \subset \mathbb{R}^k\) is defined by

\[A(y)(Y, Z) := DY P^\perp(y)(Z), \quad \forall y \in N \quad \text{and} \quad Y, Z \in T_yN.\]

We know that \(u = (u^1, ..., u^k) \in W^{2,2}(M, N)\) is an intrinsic bi-harmonic map if it satisfies the fourth order PDE (see [43] and [31] for details)

\[
\begin{align*}
\Delta^2 u &= -\Delta(\nabla P^\perp \nabla u) - \text{div}(\nabla P^\perp \Delta u) + 2\nabla P^\perp \nabla(\nabla P^\perp \nabla u) \\
&\quad + 2\nabla P^\perp \nabla P^\perp \Delta u - (\nabla PP^\perp - P^\perp \nabla P) \nabla \Delta u \\
&\quad + P(\nabla P^\perp \nabla u, D_u(D_u P^\perp) \nabla u \nabla u) - 2\text{div}(\nabla P^\perp \nabla u, \nabla P^\perp P) \\
&\quad + 2(\nabla P^\perp \nabla u, \nabla P^\perp P) \tag{2.1}
\end{align*}
\]

Here \(D_u P^\perp = D_y P^\perp(y)|_{y=u}\). Note that

\[
\nabla P^\perp = D_u P^\perp \nabla u, \tag{2.2}
\]

and the following two terms in (2.1) are equivalent to:

\[
\nabla P^\perp P = D_u P^\perp \nabla u P = A(u)(\nabla u, P) \tag{2.3}
\]
and

\begin{equation}
\nabla P \cdot \nabla P = D_u P \cdot \nabla u \nabla P = A(u)(\nabla u, \nabla P).
\end{equation}

When \( \mathcal{N} = S^n \), we have (note that \( \Delta u = \tau(u) - A(u)(\nabla u, \nabla u) \)).

\begin{equation}
P^\perp(\Delta u) = -\nabla P \cdot \nabla u = -u|\nabla u|^2,
\end{equation}

and therefore

\begin{equation}
A(u)(\nabla u, \nabla u) = \nabla P \cdot \nabla u = u|\nabla u|^2.
\end{equation}

In particular, when \( \mathcal{N} = S^n \), the intrinsic bi-harmonic map equation can be rewritten as (see e.g. Lamm-Rivièere [29])

\begin{equation}
\Delta^2 u = \Delta(V \cdot \nabla u) + \text{div} (w \nabla u) + W \cdot \nabla u,
\end{equation}

where

\begin{align}
V^{ij} &= u^i \nabla u^j - u^j \nabla u^i \\
w^{ij} &= \text{div} (V^{ij}) \\
W^{ij} &= \nabla w^{ij} + 2 [\Delta u^i \nabla u^j - \Delta u^j \nabla u^i + |\nabla u|^2 (u^i \nabla u^j - u^j \nabla u^i)].
\end{align}

**Remark 2.1.** In local coordinates, the terms in (2.1) read as

\[
P(\langle \nabla P^\perp \nabla u, D_u D_u P^\perp \rangle \nabla u \nabla u) = \sum_{\alpha,\beta,\gamma,i,j,k,m} P_{ik} \nabla_\alpha (P^\perp)_{ij} \nabla_\alpha u^j D_{u^k} D_{u^\beta}(P^\perp)_{im} \nabla_\gamma u^\beta \nabla_\gamma u^m;
\]

\[
\langle \nabla P^\perp \nabla u, \nabla P^\perp \nabla P \rangle = \sum_{\alpha,\beta,i,j,k} \nabla_\alpha (P^\perp)_{ij} \nabla_\alpha u^j \nabla_\beta (P^\perp)_{ik} \nabla_\beta P_{kl};
\]

\[
\text{div} \langle \nabla P^\perp \nabla u, \nabla P^\perp \nabla P \rangle = \sum_{\alpha,\beta,i,j,k} \nabla_\beta [\nabla_\alpha (P^\perp)_{ij} \nabla_\alpha u^j \nabla_\beta (P^\perp)_{ik} P_{kl}],
\]

where \( \nabla_\alpha := \nabla e_\alpha \) and \( \{e_\alpha\}_{\alpha=1,\ldots,4} \) is a local orthonormal frame on \( \mathcal{M} \).

To end this section, let us recall the following version of \( \varepsilon \)-regularity for approximate intrinsic and extrinsic bi-harmonic maps into spheres, which will be useful later. Throughout the rest of this paper, \( \nabla \) and \( \Delta \) will denote the connection and Laplacian with respect to the Euclidean metric.

**Theorem 2.2.** ([33, Theorem A.4], c.f. [29, 31]) Let \( B_1 \subset \mathbb{R}^4 \) be the unit 4-ball. There exist \( \varepsilon > 0 \), \( 0 < \delta < 1 \), \( \alpha > 0 \) and \( C > 0 \) independent of \( u \) such that if \( u \in W^{2,2}(B_1, S^n) \) is a solution of

\begin{equation}
\tilde{\Delta}^2 u = \tilde{\Delta}(V \nabla u) + \text{div} (w \nabla u) + \nabla \omega \nabla u + F \nabla u,
\end{equation}

and therefore

\begin{equation}
A(u)(\nabla u, \nabla u) = \nabla P \cdot \nabla u = u|\nabla u|^2.
\end{equation}
where $V \in W^{1,2}(B_1, M_{n+1} \otimes \Lambda^1 \mathbb{R}^4)$, $w \in L^2(B_1, M_{n+1})$, $\omega \in L^2(B_1, so_{n+1})$ and $F \in L^2 \cdot W^{1,2}(B_1, M_{n+1} \otimes \Lambda^1 \mathbb{R}^4)$, which satisfy

$$|V| \leq C|\nabla u|,$$

(2.10) $$|F| \leq C|\nabla u| \left(|\nabla^2 u| + |\nabla u|^2\right),$$

$$|w| + |\omega| \leq C \left(|\nabla^2 u| + |\nabla u|^2\right)$$

almost everywhere (where $C > 0$ is a constant depending only on $N$) and

(2.11) $$\|\bar{\Delta} u\|_{L^2(B_1)} \leq \varepsilon,$$

then we have $u \in W^{3,4/3}_{loc}(B_1, \mathbb{R}^{n+1})$ and

$$\|\nabla^3 u\|_{L^2(B_1, \rho)} + \|\nabla^2 u\|_{L^2(B_1, \rho)} + \|\nabla u\|_{L^1(B_1, \rho)} \leq C \rho^a \|\bar{\Delta} u\|_{L^2(B_1)}$$

for all $p \in B_1 \rho$ and $0 \leq \rho \leq \delta$. Moreover, $u \in W^{3,\infty}(B_{1/4}, \mathbb{R}^{n+1})$ and for $l = 1, 2, 3$ we have

(2.12) $$|\nabla^l u(0)| \leq C_l \|\bar{\Delta} u\|_{L^2(B_1)}$$

for some constant $C_l > 0$. In particular, by rescaling we have for $x \in B_1$ and $l = 1, 2, 3$

(2.13) $$|\nabla^l u(x)| \leq C_l (1 - |x|)^{-l} \|\bar{\Delta} u\|_{L^2(B_1)}.$$

3. Second-order Hardy inequality on manifolds

The first order Hardy inequality on Euclidean domains is well-known. Let’s recall the second order Hardy inequality on Euclidean domains (see e.g. [11, 23]).

**Theorem 3.1.** ([11, Theorem 2]) Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. There exists a constant $C > 0$ depending only on $\Omega$ such that if $w \in W^{2,2}_{0}(\Omega, \mathbb{R}^K)$, then we have

(3.1) $$\int_\Omega |w|^2 (dist(x, \partial \Omega))^{-4} dx \leq C \int_\Omega |\bar{\Delta} w|^2 dx.$$

Recently, in [8] D’Ambrosio and Dipierro extended the first order Hardy inequality to the Riemannian manifold setting (c.f. [4, 24]), more precisely, among other things they proved:

**Theorem 3.2.** ([8, Theorem 2.1]) Let $(\mathcal{M}, g)$ be an $n$ dimensional compact Riemannian manifold with a smooth boundary $\partial \mathcal{M}$. Let $\rho \in W^{1,2}_{loc}(\mathcal{M}, \mathbb{R})$ be a non-negative function such that $\Delta \rho \leq 0$ on $\mathcal{M}$ in the weak sense, then we have

$$\frac{|\nabla \rho|^2}{\rho^2} \in L^1_{loc}(\mathcal{M}, \mathbb{R})$$

and

$$\int_{\mathcal{M}} \frac{|w|^2}{\rho^2} |\nabla \rho|^2 dv_g \leq 4 \int_{\mathcal{M}} |\nabla w|^2 dv_g$$

for any $w \in C^\infty_0(\mathcal{M}, \mathbb{R})$. 

Inspired by the work of D’Ambrosio-Dipierro [8], which follows the techniques introduced by Mitidieri in [35], we can prove a version of second order Hardy inequality on certain manifolds that is sufficient for the applications in our setting in this paper, namely, Theorem 1.4 and Theorem 1.3. Since Theorem 1.4 is a direct consequence of Theorem 1.3, we only give a proof of Theorem 1.3.

Proof. (of Theorem 1.3) First note that by the assumption $\Delta \rho \leq 0$ we have in the weak sense that
\[
\Delta \left( \frac{1}{2\rho^2} \right) = \text{div} \left( -\frac{\nabla \rho}{\rho^3} \right) = -\frac{\Delta \rho}{\rho^3} + \frac{3|\nabla \rho|^2}{\rho^4} \geq \frac{3|\nabla \rho|^2}{\rho^4}.
\]
Therefore, for any $w \in C^\infty_0(M, \mathbb{R})$ we have
\[
3 \int_M |w|^2 \frac{|\nabla \rho|^2}{\rho^4} dv_g \leq \int_M |w|^2 \Delta \left( \frac{1}{2\rho^2} \right) dv_g = -2 \int_M w \nabla w \cdot \nabla \left( \frac{1}{2\rho^2} \right) dv_g = 2 \int_M (|\nabla w|^2 + w \Delta w) \left( \frac{1}{2\rho^2} \right) dv_g \leq \left( \int_M \frac{3|w|^2 |\nabla \rho|^2}{\rho^4} dv_g \right)^{1/2} \left( \int_M \frac{|\Delta w|^2}{3|\nabla \rho|^2} dv_g \right)^{1/2} + \int_M \frac{|\nabla w|^2}{\rho^2} dv_g.
\]
Now apply the first order Hardy inequality (Theorem 3.2) to the last term in (3.2) and use the Kato’s inequality, Reilly’s formula and Poincaré inequality, we get
\[
\int_M |\nabla w|^2 dv_g \leq 4 \int_M |\nabla^2 w|^2 dv_g \leq C \int_M |\Delta w|^2 dv_g.
\]
Inserting this back to (3.2) we get
\[
\int_M |w|^2 \frac{|\nabla \rho|^2}{\rho^4} dv_g \leq C \int_M |\Delta w|^2 (1 + |\nabla \rho|^{-2}) dv_g
\]
for some $C := C(M) > 0$. 

4. Proof of the main result

In this section, we prove Theorem 1.2. Let us first fix some notations. Since $M$ is compact and locally conformally flat with a smooth boundary $\partial M$, we can choose a smooth atlas
\[
\{ \Phi_i : B_{2r_i}(p_i) \to \mathbb{R}^4 \}_{i=1}^N
\]
for $M$ such that $\{p_i\}_{i=1}^k \subset M$, $\{p_i\}_{i=k+1}^N \subset \partial M$ and $B_{2r_i}(p_i)$ is conformally flat for each $i$. We assume that
\[
g = e^{2\phi_i} \tilde{g} \quad \text{on} \quad B_{2r_i}(p_i),
\]
where $\tilde{g}$ is the Euclidean metric. Here $B_{2r_i}(p_i) \subset M$ denotes the set
\[
B_{2r_i}(p_i) := \{ y \in M \cup \partial M : \text{dist}_g(y, p_i) < 2r_i \}, \quad i = 1, \ldots, N.
\]
Moreover,
\[ \mathcal{M} \cup \partial \mathcal{M} \text{ is covered by } \{ B_r(p_i) \}_{i=1}^N \]
and every point in \( \mathcal{M} \cup \partial \mathcal{M} \) is covered by \( \{ B_{2r}(p_i) \}_{i=1}^N \) at most \( L \) times, see e.g. [40] Lemma 3.3. For \( i = 1, \ldots, N \), let
\[ U_i := \Phi_i(B_r(p_i)) \quad \text{and} \quad 2U_i := \Phi_i(B_{2r}(p_i)). \]
When choosing the smooth atlas, we additionally require that there exists \( \delta > 0 \) depending only on \( \mathcal{M} \) and \( \partial \mathcal{M} \) such that for any \( i = k + 1, \ldots, N \) and any point \( x \in \{ y \in \mathcal{M} : \text{dist}(y, \partial \mathcal{M}) \leq \delta \} \cap B_r(p_i) \) we have
\[ \Phi_i(B_{\rho(x)}(x)) \subset U_j \quad \text{for some } j = k + 1, \ldots, N, \]
where \( \rho(x) = \text{dist}_g(x, \partial \mathcal{M}) \). Moreover,
\[ \{ y \in \mathcal{M} : \text{dist}(y, \partial \mathcal{M}) \geq \delta \} \subset \bigcup_{i=1,\ldots,k} B_r(p_i). \]
Let \( \Omega_1 := B_{r_1}(p_1) \) and
\[ \Omega_{i+1} := \left( \bigcup_{j=1,\ldots,i+1} B_r(p_j) \right) \setminus \left( \bigcup_{j=1,\ldots,i} \Omega_j \right), \quad i = 1, \ldots, N - 1 \]
be a disjoint partition of \( \mathcal{M} \). Now for any \( u \in W^{2,2}(\mathcal{M}, \mathbb{S}^n) \) we define
\[ \bar{u}_i(x) := u(\Phi_i^{-1}(x)), \quad x \in 2U_i, \quad i = 1, \ldots, N. \]

**Remark 4.1.** Note that by Proposition [A.1] we know that if \( u \) is a conformal-harmonic map on \( \mathcal{M} \), then \( \bar{u}_i \) defined in (4.5) is an intrinsic biharmonic map on \( 2U_i, i = 1, \ldots, N \).

In what follows, we will denote \( | \cdot | \) the norm with respect to the Euclidean metric in \( \mathbb{R}^k \). As before, \( \nabla \) and \( \Delta \) will denote the connection and Laplacian with respect to the Euclidean metric \( \bar{g} \), but note that on each \( \Omega_i, i = 1, \ldots, N \) the information of \( \Phi_i \) is embedded in these two operators and we do not differentiate the notations.

1To see this, let \( \Sigma_i = \partial B_r(p_i) \cap \partial \mathcal{M}, i = k + 1, \ldots, N \) be the projection of \( B_r(p_i) \) onto \( \partial \mathcal{M} \) by normal geodesics. \( \Gamma_i = \partial \Sigma_i \) denotes its boundary in \( \partial \mathcal{M} \). We choose the smooth atlas of \( \mathcal{M} \) in such a way that every \( \Sigma_i \) is covered by finitely many \( \Sigma_{j\alpha} \)'s and all the intersecting points \( q_{st} = \Gamma_s \cap \Gamma_t \cap \Sigma_i \) stay in the interior of \( \Sigma_i \), where \( \alpha = 1, \ldots, \alpha_i, j_{\alpha} \in [k + 1, N], l \neq s, t \) and \( s, t, l = i, j_1, \ldots, j_{\alpha_i} \). Then there exists a constant \( C_i > 0 \) such that for any \( y_0 \in \Sigma_i \), there holds \( \text{dist}(y_0, \Gamma_i) \geq C_i \) for some \( l = i, j_1, \ldots, j_{\alpha_i} \). Now, for any \( x \in \{ y \in \mathcal{M} : \text{dist}_g(y, \partial \mathcal{M}) \leq \delta \} \), there exists \( x_0 \in \partial \mathcal{M} \) such that \( x_0 \in \Sigma_i \) for some \( i \in [k+1, N] \) and \( \rho(x) = \text{dist}_g(x, \partial \mathcal{M}) = \text{dist}_g(x, x_0) \). Since \( B_{\rho(x)}(x) \subset B_{\delta}(x) \subset B_{2\delta}(x_0) \), we have \( B_{\rho(x)}(x) \subset B_r(p_i) \) for some \( i = k + 1, \ldots, N \) if we choose \( 2\delta < \min_{i=k+1,\ldots,N} \{ C_i \} \).
Lemma 4.2. There exists $\epsilon_0 > 0$ depending only on $\mathcal{M}$ such that if $u, v$ are as in Theorem 1.2, then we have

\[
\int_{\mathcal{M}} |\Delta (v - u)|^2 dv_g \leq 4 \int_{\mathcal{M}} |\vec{\tau}(v) - \vec{\tau}(u)|^2 dv_g,
\]

where $\vec{\tau}(u) = (\Delta u)^T$ is the tension field of $u$ with respect to the flat connection.

Proof. Let $\bar{u}_i, \bar{v}_i$ be defined as in (4.5), then use the conformal change of Laplacian (c.f. (A.6)), for $i = 1, \ldots, N$ we get

\[
\int_{2U_i} |\Delta \bar{u}_i|^2 dx = \int_{B_{2r_i}(p_i)} |\Delta u|^2 dv_g
\]

\[
\leq C_i \int_{B_{2r_i}(p_i)} (|\Delta u|^2 + |\nabla u|^2) dv_g \leq C_i \sqrt{\epsilon_0}
\]

and

\[
\int_{2U_i} |\nabla \bar{v}_i|^4 dx = \int_{B_{2r_i}(p_i)} |\nabla v|^4 dv_g = \int_{B_{2r_i}(p_i)} |\nabla v|^4 dv_g \leq \epsilon_0,
\]

where $C_i = C_i(\Phi_i, \phi_i) > 0$ are positive constants. Moreover, using (4.7), (4.8), (4.5) and

\[
\int_{\mathcal{M}} |\Delta u|^2 dv_g = \int_{\mathcal{M}} |\tau(u)|^2 dv_g + \int_{\mathcal{M}} |\nabla u|^4 dv_g,
\]

we have

\[
\int_{\mathcal{M}} |\Delta u|^2 dv_g \leq C \int_{\mathcal{M}} (|\Delta u|^2 + |\nabla u|^2) dv_g \leq C \sqrt{\epsilon_0}
\]

and

\[
\int_{\mathcal{M}} |\nabla v|^4 dv_g = \int_{\mathcal{M}} |\nabla v|^4 dv_g \leq \epsilon_0,
\]

where $C > 0$ may depend on all $\phi_i$ and $\Phi_i, i = 1, \ldots, N$. Now using the decomposition $\Delta u = \vec{\tau}(u) - u|\nabla u|^2$, we have

\[
\int_{\mathcal{M}} |\Delta (v - u)|^2 dv_g = \int_{\mathcal{M}} |\vec{\tau}(v) - \vec{\tau}(u) - v|\nabla v|^2 + u|\nabla u|^2|^2 dv_g
\]

\[
\leq 2 \int_{\mathcal{M}} |\vec{\tau}(v) - \vec{\tau}(u)|^2 dv_g + 2 \int_{\mathcal{M}} |v|\nabla v|^2 - u|\nabla u|^2|^2 dv_g
\]

\[
= 2 \int_{\mathcal{M}} |\vec{\tau}(v) - \vec{\tau}(u)|^2 dv_g + 2 \int_{\mathcal{M}} |v|(|\nabla v|^2 - |\nabla u|^2) + (v - u)|\nabla u|^2|^2 dv_g
\]

\[
\leq 2 \int_{\mathcal{M}} |\vec{\tau}(v) - \vec{\tau}(u)|^2 dv_g + 4 \int_{\mathcal{M}} |\nabla v|^2 - |\nabla u|^2|^2 dv_g + 4 \int_{\mathcal{M}} |v - u|^2|\nabla u|^4 dv_g
\]

\[
\leq 2 \int_{\mathcal{M}} |\vec{\tau}(v) - \vec{\tau}(u)|^2 dv_g + 4 \left( \int_{\mathcal{M}} |\nabla (v + u)|^4 dv_g \right)^{\frac{1}{2}} \left( \int_{\mathcal{M}} |\nabla (v - u)|^4 dv_g \right)^{\frac{1}{2}} dv_g
\[ \leq 2 \int_{\mathcal{M}} |\bar{\tau}(v) - \bar{\tau}(u)|^2 d\bar{v}_g + C \sqrt{\rho_0} \int_{\mathcal{M}} |\bar{\Delta}(v-u)|^2 d\bar{v}_g + 4 \int_{\mathcal{M}} |v-u|^2 |\nabla u|^4 d\bar{v}_g, \]

where we have used (1.5), (4.9), (4.10), (4.13) holds.

(4.11) \[ \int_{\mathcal{M}} |\nabla u|^4 d\bar{v}_g \leq \int_{\mathcal{M}} |\bar{\Delta} u|^2 d\bar{v}_g \leq C \sqrt{\epsilon_0} \]

and

(4.12) \[ \left( \int_{\mathcal{M}} |\nabla (v-u)|^4 d\bar{v}_g \right)^{\frac{1}{4}} \leq C \left( \int_{\mathcal{M}} |\bar{\Delta}(v-u)|^2 d\bar{v}_g \right)^{\frac{1}{2}}, \]

where \( C > 0 \) is a universal constant. For the last term above, we claim that

(4.13) \[ \int_{\mathcal{M}} |v-u|^2 |\nabla u|^4 d\bar{v}_g \leq C_0 \epsilon_0 \int_{\mathcal{M}} |v-u|^2 \rho^4 d\bar{v}_g, \]

where \( \rho(x) = \text{dist}_{\bar{g}}(x, \partial \mathcal{M}) \) and \( C_0 = C_0(\mathcal{M}) > 0 \). If the claim (4.13) is true, then the second Hardy inequality (Theorem 1.4) imply that (since \( v-u \in W^{2,2}_{0}(\mathcal{M}, \mathcal{S}^n) \) and \( \frac{\partial (v-u)}{\partial \nu} = 0 \) on \( \partial \mathcal{M} \))

\[ \int_{\mathcal{M}} |v-u|^2 |\nabla u|^4 d\bar{v}_g \leq C \epsilon_0 \int_{\mathcal{M}} |\bar{\Delta}(v-u)|^2 d\bar{v}_g \]

\[ = C \epsilon_0 \sum_{i=1, \ldots, N} \int_{\Omega_i} |\Delta(v-u)|^2 d\nu_g \]

\[ \leq C \epsilon_0 \sum_{i=1, \ldots, N} \int_{\Omega_i} \left( |\bar{\Delta}(v-u)|^2 + |\nabla(v-u)|^2 \right) d\bar{v}_g \]

(4.14) \[ \leq C \epsilon_0 \int_{\mathcal{M}} \left( |\bar{\Delta}(v-u)|^2 + |\nabla(v-u)|^2 \right) d\bar{v}_g \leq C \epsilon_0 \int_{\mathcal{M}} |\bar{\Delta}(v-u)|^2 d\bar{v}_g. \]

where we used \( \int_{\Omega_i} |\Delta(v-u)|^2 d\nu_g \leq C_i \int_{\Omega_i} \left( |\bar{\Delta}(v-u)|^2 + |\nabla(v-u)|^2 \right) d\bar{v}_g \) for each \( i \), which is similar to (4.7). Therefore, we obtain (4.6) by choosing \( \epsilon_0 \) sufficiently small. To complete the proof of this lemma, it remains to show that the claim (4.13) holds.

Using (4.7), Remark 4.1 and Theorem 2.2 and choosing \( \epsilon_0 \) small enough we have

(4.15) \[ |\nabla^l \bar{u}_i(x)| \leq C (\text{dist}_{\bar{g}}(x, \partial(2U_i)))^{-l} \epsilon_0^{1/4}, \]

for any \( x \in 2U_i, i = 1, \ldots, k \) and \( l = 1, 2, 3 \), where \( C > 0 \) depends on the \( C_i \) above. In particular, define

\[ r_0 := \min_{i=1, \ldots, k} \text{dist}_{\bar{g}}(U_i, \partial(2U_i)), \]
then we have
\begin{equation}
|\nabla^l u_i(x)| \leq C r_0^{-l} \epsilon_0^{1/4} \quad \text{for any } x \in U_i, i = 1, \ldots, k \text{ and } l = 1, 2, 3, \tag{4.16}
\end{equation}
\begin{equation}
|\nabla^l u|(x) \leq C r_0^{-l} \epsilon_0^{1/4} \quad \text{for any } x \in B_{\epsilon_0}(p_i), i = 1, \ldots, k \text{ and } l = 1, 2, 3, \tag{4.17}
\end{equation}
and therefore
\begin{equation}
|\nabla^l u|(x) \leq C r_0^{-l} \epsilon_0^{1/4} (\text{diam}_g(M))^l \rho^{-l}(x) \leq C_1 \epsilon_0^{1/4} \rho^{-l}(x) \tag{4.18}
\end{equation}
for any \( x \in B_{\epsilon_0}(p_i), i = 1, \ldots, k \) and \( l = 1, 2, 3 \), where the \( C > 0 \) in (4.17) and (4.18) may depend on \( \Phi_i, i = 1, \ldots, k \), and
\[
C_1 := \max_{l=1,2,3} \left\{ C r_0^{-l} (\text{diam}_g(M))^l \right\}.
\]

Now by the choice of the smooth atlas, in particular, (4.2), with a similar argument as above, we have
\begin{equation}
|\nabla^l u|(x) \leq C_2 \epsilon_0^{1/4} \rho^{-l}(x) \tag{4.19}
\end{equation}
for any \( x \in B_{\epsilon_0}(p_i) \cap \{ y \in M : \text{dist}(y, \partial M) \leq \delta \}, i = k+1, \ldots, N \) and \( l = 1, 2, 3 \).

Here \( C_2 > 0 \) may depend on all \( \Phi_i \) and \( \phi_i, i = k, \ldots, N \). Then with a argument similar to (4.14) we get (4.13). This completes the proof of the lemma. \( \square \)

**Lemma 4.3.** There exists \( \epsilon_0 > 0 \) depending only on \( M \) such that if \( u, v \) are as in Theorem 1.2, then we have
\begin{equation}
\int_M |\Delta v|^2 dv_g - \int_M |\Delta u|^2 dv_g - \int_M |\Delta (v - u)|^2 dv_g \geq - C \epsilon_0^{1/4} \int_M |\Delta (v - u)|^2 dv_g + 4 \int_M |\nabla u|^2 \nabla u \cdot \nabla (v - u) dv_g. \tag{4.20}
\end{equation}

**Proof.** By Remark 4.11 we know that \( u \) is an intrinsic biharmonic map on each \( \Omega_i, i = 1, \ldots, N \), with respect to the flat connection \( \nabla \), namely, we have
\[
\Delta^2 u = P(\nabla^2 u) + P\langle \nabla P^\perp \nabla u, D_u(D_u P^\perp) \nabla u \nabla u \rangle - 2 \text{div}_g \langle \nabla P^\perp \nabla u, \nabla P^\perp \nabla \nabla u \rangle + 2 \langle \nabla P^\perp \nabla u, \nabla P^\perp \nabla P \rangle.
\]

Since \( u = v, \partial_u u = \partial_v v \) on \( \partial M \), we have
\[
\int_M |\Delta v|^2 dv_g - \int_M |\Delta u|^2 dv_g - \int_M |\Delta (v - u)|^2 dv_g = 2 \int_M \langle \Delta^2 u, v - u \rangle dv_g
\]
\[
= 2 \int_M \langle P^\perp (\Delta^2 u), v - u \rangle dv_g + 2 \int_M \langle P(\nabla P^\perp \nabla u, D_u(D_u P^\perp) \nabla u \nabla u \rangle, v - u \rangle dv_g
\]
\[
- 4 \int_M \langle \text{div}_g \langle \nabla P^\perp \nabla u, \nabla P^\perp \nabla \nabla u \rangle, v - u \rangle dv_g + 4 \int_M \langle \nabla P^\perp \nabla u, \nabla P^\perp \nabla P \rangle, v - u \rangle dv_g.
\]
\[= 2 \int_M \langle P^\perp(\Delta^2 u), v - u \rangle \, dv_g + 2 \int_M \langle \nabla P^\perp \nabla u, Du(D_u P^\perp) \nabla u \rangle, P(v - u) \rangle \, dv_g \]
\[+ 4 \int_M \langle \nabla P^\perp \nabla u, \nabla P^\perp P \rangle, \nabla (v - u) \rangle \, dv_g + 4 \int_M \langle \nabla P^\perp \nabla u, \nabla P^\perp \nabla P \rangle, v - u \rangle \, dv_g \]
\[= \mathbf{I} + \mathbf{II} + \mathbf{III} + \mathbf{IV}. \]

For term \( \mathbf{I} \), we can use (1.4) a similar argument as in the proof of (4.13) and the second order Hardy inequality (Theorem 1.4) to get
\[
2 \int_M \langle P^\perp(\Delta^2 u), v - u \rangle \, dv_g \geq -C \int_M |(v - u)^\perp| \cdot |P^\perp(\Delta^2 u)| \, dv_g \\
\geq -C\epsilon_0^{1/4} \int_M |v - u|^2 \rho(x)^{-4} \, dv_g \\
\geq -C\epsilon_0^{1/4} \int_M |\Delta(v - u)|^2 \, dv_g \\
(4.21)
\]

For term \( \mathbf{II} \), we note that the integrand reads as
\[
\sum_{\alpha, \beta, \gamma, i, j, k, m, l} P_{ik} \nabla_\alpha(P^\perp)_{ij} \nabla_\alpha u^j D_{u^k}D_{u^\beta}(P^\perp)_{im} \nabla_\gamma u^\beta \nabla_\gamma u^m (v - u)^l.
\]
Now since the target manifold is \( S^n \), we know that \( u \) is the unit normal vector at the point \( u \in S^n \) and \( P^\perp(v) = \langle v, u \rangle u \) for any vector \( v \in T_u(R^{n+1}) \) so that
\[
P^\perp_{ij} = u^i u^j \quad \text{and} \quad P^\perp(\Delta u) = -\nabla P^\perp \nabla u = -|\nabla u|^2.
\]
Therefore, in this case the integrand in term \( \mathbf{II} \) becomes
\[
\sum_{\beta, \gamma, i, k, m, l} P_{ik} u^i |\nabla u|^2 (\delta_{i\beta}\delta_{mk} + \delta_{i\beta}\delta_{mk}) \nabla_\gamma u^\beta \nabla_\gamma u^m (v - u)^l = 0 \quad \text{.}
\]

For term \( \mathbf{III} \), we have (using again \( P^\perp_{ij} = u^i u^j \))
\[
\langle \langle \nabla P^\perp \nabla u, \nabla P^\perp P \rangle, \nabla (v - u) \rangle = |\nabla u|^2 \nabla u \cdot \nabla (v - u). \]

For term \( \mathbf{IV} \), we use again \( P^\perp_{ij} = u^i u^j \), \( P + P^\perp = \text{Id} \) and also \( u \cdot \nabla u = 0 \) on \( S^n \) to get
\[
\int_M \langle \langle \nabla P^\perp \nabla u, \nabla P^\perp \nabla P \rangle, v - u \rangle \, dv_g = \int_M \nabla P^\perp_{ij} \nabla u^i \nabla P^\perp_{jk} \nabla P_{ks}(v - u)^s \, dv_g \\
= -\int_M |\nabla u|^4 u \cdot (v - u) \, dv_g \geq -2 \int_M |\nabla u|^4 |(v - u)^\perp| \, dv_g \\
\geq -C \int_M |v - u|^2 |\nabla u|^4 \, dv_g.
\]
Then by (4.13) and (4.14) we have

\[ IV \geq -C\epsilon_0^{1/4} \int_M |\Delta(v - u)|^2 dv_g. \]

Now (4.20) follows directly by combining the estimates above. \(\square\)

**Lemma 4.4.** There exists \(\epsilon_0 > 0\) depending only on \(M\) such that if \(u, v\) are as in Theorem 1.2 then we have

\[ \int_M |\bar{\tau}(v)|^2 dv_g - \int_M |\bar{\tau}(u)|^2 dv_g - \int_M |\bar{\tau}(v) - \bar{\tau}(u)|^2 dv_g \geq -C\epsilon_0^{1/4} \int_M |\bar{\Delta}(v - u)|^2 dv_g. \]

**Proof.** It follows from Lemma 4.3 and (4.20) that

\[ \psi = \int_M |\bar{\tau}(v)|^2 dv_g - \int_M |\bar{\tau}(u)|^2 dv_g - \int_M |\bar{\tau}(v) - \bar{\tau}(u)|^2 dv_g \]
\[ = \int_M |\bar{\Delta}v|^2 dv_g - \int_M |\bar{\Delta}u|^2 dv_g - \int_M |\bar{\Delta}(v - u)|^2 dv_g \]
\[ - \int_M |\bar{\nabla}v|^4 - |\bar{\nabla}u|^4 dv_g + \int_M |v|\bar{\nabla}v|^2 + |u|\bar{\nabla}u|^2 |dv_g \]
\[ + 2 \int_M |\bar{\nabla}u|^2 u\bar{\Delta}v + |\bar{\nabla}v|^2 v\bar{\Delta}u dv_g \]
\[ \geq -C\epsilon_0^{1/4} \int_M |\bar{\Delta}(v - u)|^2 dv_g + 4 \int_M |\bar{\nabla}u|^2 \bar{\Delta}u\bar{\nabla}(v - u) dv_g \]
\[ - \int_M |\bar{\nabla}v|^4 - |\bar{\nabla}u|^4 dv_g + \int_M |v|\bar{\nabla}v|^2 + |u|\bar{\nabla}u|^2 |dv_g \]
\[ + 2 \int_M |\bar{\nabla}u|^2 u\bar{\Delta}v + |\bar{\nabla}v|^2 v\bar{\Delta}u dv_g. \]

Therefore, using \(u|_{\partial M} = v|_{\partial M}\) and \(\partial_v u|_{\partial M} = \partial_v v|_{\partial M}\), we have

\[ \psi \geq -C\epsilon_0^{1/4} \int_M |\bar{\Delta}(v - u)|^2 dv_g - 2 \int_M \bar{\nabla}|\bar{\nabla}u|^2 u \bar{\nabla}v dv_g - 2 \int_M |\bar{\nabla}u|^2 u\bar{\Delta}v dv_g \]
\[ - 2 \int_M \bar{\nabla}|\bar{\nabla}u|^2 v \bar{\nabla}u dv_g - 2 \int_M |\bar{\nabla}u|^2 v\bar{\Delta}u dv_g - 4 \int_M |\bar{\nabla}u|^4 dv_g \]
\[ - \int_M (|\bar{\nabla}v|^4 - |\bar{\nabla}u|^4) dv_g + \int_M |v|\bar{\nabla}v|^2 + |u|\bar{\nabla}u|^2 |dv_g \]
\[ + 2 \int_M |\bar{\nabla}u|^2 u\bar{\Delta}v + |\bar{\nabla}v|^2 v\bar{\Delta}u dv_g \]
\[ = -C\epsilon_0^{1/4} \int_M |\bar{\Delta}(v - u)|^2 dv_g + \int_M \bar{\nabla}|\bar{\nabla}u|^2 |v - u|^2 dv_g \]
Proof of Theorem 1.2.
From (4.14), we have
\[ \Delta u(|\nabla v|^2 - |\nabla u|^2)dv \geq 2\int_M |\nabla u|^2 uv(|\nabla v|^2 - |\nabla u|^2)dv \]
\[ -\int_M |v - u|^2 |\nabla u|^4 dv \,, \]
where we have used \(|u|^2 = |v|^2 = 1\) so that \(1 - uv = \frac{1}{2}|v - u|^2\) and \(u\nabla v + v\nabla u = -\frac{1}{2}\nabla |v - u|^2\). Thus, we have
\[ \psi \geq -C \varepsilon_0^{1/4} \int_M |\Delta (v - u)|^2 dv + 2 \int_M (\Delta u + u|\nabla u|^2)v(|\nabla v|^2 - |\nabla u|^2)dv \]
\[ = -C \varepsilon_0^{1/4} \int_M |\Delta (v - u)|^2 dv + 2 \int_M \nabla (u)(v - u)\cdot (\nabla (v + u), \nabla (v - u)) dv \geq -C \varepsilon_0^{1/4} \int_M |\Delta (v - u)|^2 dv \]
\[ - 2 \left( \int_M |\nabla (v + u)|^2 dv \right)^{1/2} \left( \int_M |\nabla (v - u)|^4 dv \right)^{1/4} \]
\[ \geq -C \varepsilon_0^{1/4} \int_M |\Delta (v - u)|^2 dv \,, \]
where we used (4.12), (4.18), (4.19), Hardy inequality (Theorem 1.4), (4.10) and (4.11).

Proof of Theorem 1.2. From (4.14), we have
\[ \int_M |\Delta v - \Delta u|^2 dv \leq C \int_M |\Delta v - \Delta u|^2 dv \, . \]
Combining (4.6) and (4.25) we get
\[ \int_M |\bar{\tau}(u)|^2 dv \geq \int_M |\bar{\tau}(v)|^2 dv + \int_M |\bar{\tau}(v) - \bar{\tau}(u)|^2 dv \]
\[ \leq C \varepsilon_0^{1/4} \int_M |\bar{\tau}(v) - \bar{\tau}(u)|^2 dv \, . \]
Now choosing \(\varepsilon_0\) sufficiently small we get (using (4.26) and (4.27))
\[ \int_M |\Delta v - \Delta u|^2 dv \leq C \int_M |\Delta v - \Delta u|^2 dv \]
\[ \leq 4C \int_M |\bar{\tau}(v) - \bar{\tau}(u)|^2 dv \leq 8C \left( \int_M |\bar{\tau}(v)|^2 dv - \int_M |\bar{\tau}(v)|^2 dv \right) \]
\[ = 8C \sum_{i=1,\ldots, N} \left( \int_{\Omega_i} |\bar{\tau}(v)|^2 dv - \int_{\Omega_i} |\bar{\tau}(u)|^2 dv \right) = 8C \sum_{i=1,\ldots, N} (E(v, \Omega_i) - E(u, \Omega_i)) \]
\[ = 8C (E(v, \mathcal{M}) - E(u, \mathcal{M})) \, . \]
This completes the proof of Theorem 1.2. \(\square\)
Appendix A. Conformal invariance of $E(u, \mathcal{M})$

In this appendix we give a quick check of the conformal invariance of $E(u, \mathcal{M})$ in four dimensions which does not seem to be in the literature. Let $\mathcal{M}$ be a smooth Riemannian 4-manifold with or without boundary and $\mathcal{N}$ be another smooth Riemannian manifold. Given $u \in C^2(\mathcal{M}, \mathcal{N})$, we denote by $u^*TN$ the pull-back on $(\mathcal{M}, g)$ of the tangent bundle of $(\mathcal{N}, h) \hookrightarrow \mathbb{R}^K$. Then the tension field $\tau(u) \in u^*TN$ is defined by

\[
\tau(u) = \sum_{\alpha} \tilde{\nabla}_e du(e_{\alpha}),
\]

where $\{e_{\alpha}\}$ is an orthonormal frame of $T\mathcal{M}$ and $\tilde{\nabla}$ is the metric connection on $T^*\mathcal{M} \otimes u^*TN$. Fix local coordinates $\{x^\alpha\}$ and $\{y^i\}$ of $\mathcal{M}$ and $\mathcal{N}$ respectively, we have

\[
\tau^i(u) = g^{\alpha\beta} \frac{\partial^2 u^i}{\partial x^\alpha \partial x^\beta} - g^{\alpha\beta} \Gamma^\gamma_{\alpha\beta} \frac{\partial u^i}{\partial x^\gamma} + g^{\alpha\beta} \bar{\Gamma}^i_{jk} \frac{\partial u^j}{\partial x^\alpha} \frac{\partial u^k}{\partial x^\beta},
\]

where $\Gamma$ and $\bar{\Gamma}$ are the Christoffel symbols on $\mathcal{M}$ and $\mathcal{N}$ respectively. Define

\[
A^i := g^{\alpha\beta} \frac{\partial^2 u^i}{\partial x^\alpha \partial x^\beta} + g^{\alpha\beta} \bar{\Gamma}^i_{jk} \frac{\partial u^j}{\partial x^\alpha} \frac{\partial u^k}{\partial x^\beta}.
\]

Then

\[
\tau^i(u) = A^i - g^{\alpha\beta} \Gamma^\gamma_{\alpha\beta} \frac{\partial u^i}{\partial x^\gamma} = A^i - g^{\alpha\beta} \langle \nabla X_\alpha X_\beta, du^i \rangle,
\]

where $\nabla = \nabla_g$ is the connection on $\mathcal{M}$ and $X_\alpha := \frac{\partial}{\partial x^\alpha}$. Now under the conformal change $\bar{g} = e^{2\phi} g$ (with $\phi \in C^\infty(\mathcal{M}, \mathbb{R})$ if $\partial \mathcal{M} \neq \emptyset$), we have

\[
\bar{A}^i = e^{-2\phi} A^i
\]

and

\[
\bar{g}^{\alpha\beta} \langle (\nabla_{\bar{g}}) X_\alpha X_\beta, du^i \rangle = e^{-2\phi} g^{\alpha\beta} \langle \nabla X_\alpha X_\beta, du^i \rangle - 2e^{-2\phi} \langle \nabla \phi, du^i \rangle,
\]

where we have used

\[
(\nabla_{\bar{g}}) X = \nabla_X Y + X(\phi) Y + Y(\phi) X - g(X, Y) \nabla \phi.
\]

Therefore, combing (A.4) and (A.5) together, we get

\[
\bar{\tau}^i(u) = e^{-2\phi} \tau^i(u) + 2e^{-2\phi} \langle \nabla \phi, du^i \rangle
\]

and

\[
|\bar{\tau}(u)|^2 - e^{-4\phi} |\tau(u)|^2 = 4e^{-4\phi} h_{ij}(u) \tau^i(u) \langle \nabla \phi, du^i \rangle + 4e^{-4\phi} \langle \nabla \phi, du \rangle^2.
\]

Recall the following formulas for the conformal change of Ricci tensor and scalar curvature:

\[
\bar{\text{Ric}} = \text{Ric} - 2(\text{Hess}(\phi) - \nabla \phi \otimes \nabla \phi) - (\Delta \phi + 2|\nabla \phi|^2) g
\]
and

\[ \mathcal{S}c = e^{-2\phi}(Sc - 6\Delta \phi - 6|\nabla \phi|^2), \]  

where \( \text{Hess}(\phi) \) denotes the Hessian of \( \phi \). Consequently, we obtain

\[ \frac{2}{3} \mathcal{S}c|du|_g^2 - \frac{2}{3} e^{-4\phi} \mathcal{S}c|du|_g^2 = \frac{2}{3} e^{-4\phi}(-6\Delta \phi - 6|\nabla \phi|^2)|du|_g^2 = -4e^{-4\phi}(\Delta \phi + |\nabla \phi|^2)|du|_g^2 \]  

and

\[ 2\text{Ric}(du, du) - 2e^{-4\phi}\text{Ric}(du, du) \]

\[ = 2\check{R}_{\alpha\beta}g^{\alpha\gamma}g^{\beta\sigma} \left( \frac{\partial u}{\partial x_\gamma} \frac{\partial u}{\partial x_\sigma} \right) \]  

\[ = 2e^{-4\phi}g^{\alpha\gamma}g^{\beta\sigma} \left( (-\Delta \phi - 2|\nabla \phi|^2)g_{\alpha\beta} - 2(\nabla_\beta \phi_\alpha - \phi_\alpha \phi_\beta) \right) \langle u_\gamma, u_\sigma \rangle \]

\[ = -4e^{-4\phi}|\nabla \phi|^2|du|_g^2 + 4e^{-4\phi}|(\nabla \phi, du)|_g^2 - 2e^{-4\phi}\Delta \phi|du|_g^2 \]

\[ - 4e^{-4\phi} \sum_{\alpha,\beta} \nabla^2 \phi \langle \nabla_\alpha u, \nabla_\beta u \rangle , \]

where Einstein summation convention was used and we used different subscripts to distinguish different metrics on the bundle \( T^*M \otimes u^*TN \). Combining these we have

\[ \int_M \left[ (A.8) + (A.11) - (A.12) \right] dv_g \]

\[ = \sum_{\alpha,\beta} \int_M 2e^{-4\phi} \left( 2h_{ij}(u)\tau^j(u)(\nabla \phi, du^i) - \Delta \phi|du|^2 + 2\nabla^2 \phi(\nabla_\alpha u, \nabla_\beta u) \right) dv_g \]

\[ = \sum_{\alpha,\beta} \int_M 4\nabla_\alpha \phi(\Delta u, \nabla_\alpha u) - 2\Delta \phi|du|^2 + 4\nabla^2 \phi(\nabla_\alpha u, \nabla_\beta u) dv_g \]

\[ = \sum_{\alpha,\beta} \int_M -2\Delta \phi|du|^2 - 4\nabla_\beta \phi(\nabla_\alpha u, \nabla^2 \phi) dv_g \]

\[ = \sum_{\alpha,\beta} \int_M 2\nabla_\beta \phi \nabla_\beta (\nabla_\alpha u, \nabla_\alpha u) - 4\nabla_\beta \phi(\nabla_\alpha u, \nabla^2 \phi) dv_g = 0. \]

Hence, the energy functional \( E(u, \mathcal{M}) \) is conformally invariant in four dimensions. Now using the notations in (4.1), on each \( \Omega_i, i = 1, \ldots, N \) we have

\[ g = e^{2\phi} \bar{g}, \]

where \( \bar{g} \) is the standard Euclidean metric. By the above conformal change of \( E(\cdot, \mathcal{M}) \), for any \( w \in C^2(\mathcal{M}, \mathcal{N}) \) we get

\[ E(w, (\mathcal{M}, g)) \]
\[
= \sum_i \mathcal{E}(w, (\Omega_i, \bar{g})) + \sum \int_{\Omega_i} 4 \nabla_\alpha \phi_i \langle \bar{\Delta} w, \nabla_\alpha w \rangle - 2 \bar{\Delta} \phi_i |\nabla w|^2 + 4 \nabla_\alpha \phi_i \langle \nabla_\alpha w, \nabla_\beta w \rangle dv_g.
\]

Thus, for any variation \(w(t)\) of \(w\) such that \(\frac{d}{dt} \big|_{t=0} w(t) = P\phi \in T_u\mathcal{N}\) where \(\phi \in \mathcal{C}_0^\infty(M, R^K)\), the first variation formula for \(\mathcal{E}(w, (M, g))\) is (cf. Appendix B)

\[
\frac{d}{dt} \bigg|_{t=0} \mathcal{E}(w(t), (M, g)) = 2 \int_{\Omega_j} e^{-4\phi_i} \langle P(\bar{\Delta}^2 u) - P(A(\bar{\nabla} u, \bar{\nabla} u)D_u A(\bar{\nabla} u, \bar{\nabla} u)) + 2\text{div}(A(\bar{\nabla} u, \bar{\nabla} u)A(\bar{\nabla} u, P)) - 2A(\bar{\nabla} u, \bar{\nabla} u)A(\bar{\nabla} u, \bar{\nabla} P), \phi \rangle_g dv_g.
\]

(A.14)

\[
\frac{d}{dt} \bigg|_{t=0} \mathcal{E}(w(t), (M, g)) = 2 \int_{\Omega_j} e^{-4\phi_i} \langle P(\bar{\Delta}^2 u) - P(A(\bar{\nabla} u, \bar{\nabla} u)D_u A(\bar{\nabla} u, \bar{\nabla} u)) + 2\text{div}(A(\bar{\nabla} u, \bar{\nabla} u)A(\bar{\nabla} u, P)) - 2A(\bar{\nabla} u, \bar{\nabla} u)A(\bar{\nabla} u, \bar{\nabla} P), \phi \rangle_g dv_g.
\]

(A.15)

In particular, for any fixed \(j = 1, \ldots, N\) and any \(\phi \in \mathcal{C}_0^\infty(\Omega_j, R^K)\) we have

\[
\frac{d}{dt} \bigg|_{t=0} \mathcal{E}(w(t), (M, g)) = 2 \int_{\Omega_j} e^{-4\phi_i} \langle P(\bar{\Delta}^2 u) - P(A(\bar{\nabla} u, \bar{\nabla} u)D_u A(\bar{\nabla} u, \bar{\nabla} u)) + 2\text{div}(A(\bar{\nabla} u, \bar{\nabla} u)A(\bar{\nabla} u, P)) - 2A(\bar{\nabla} u, \bar{\nabla} u)A(\bar{\nabla} u, \bar{\nabla} P), \phi \rangle_g dv_g,
\]

noting that since \(\phi \in \mathcal{C}_0^\infty(\Omega_j, R^K)\), the terms in \textcolor{red}{[A.14]} and \textcolor{red}{[A.15]} cancel out to zero using integration by parts over \(\Omega_j\). Therefore we have

**Proposition A.1.** If \(u\) is a conformal-harmonic map on a locally conformally flat 4-manifold \((M, g)\), then \(u\) is locally an intrinsic biharmonic map on \(M\) (with respect to the flat connection) up to a conformal change of \(g\).

**Appendix B. Conformal-harmonic map equation**

For the sake of completeness, in this appendix we include a derivation of the conformal-harmonic map equation \textcolor{red}{[1.2]}, i.e., the Euler-Lagrangian equation for \(\mathcal{E}(u, M)\). Consider a variation \(u(t)\) of \(u\) such that \(\frac{d}{dt} \big|_{t=0} u(t) = P\phi \in T_u\mathcal{N}\) where \(\phi \in \mathcal{C}_0^\infty(M, R^K)\). Then

\[
\frac{d}{dt} \bigg|_{t=0} \mathcal{E}(u(t), M) = \int_M \frac{d}{dt} \bigg|_{t=0} |\tau(u(t))|^2 + 2 \frac{3}{3} \text{Sc}^M \frac{d}{dt} \bigg|_{t=0} |du(t)|^2 - 2 \frac{d}{dt} \bigg|_{t=0} \text{Ric}^M(du(t), du(t)).
\]
(B.1) \( := I + II + III. \)

We compute the three terms in (B.1) as follow. Note that the first term \( I \) is the first variation of the intrinsic bienergy \( \int_M |\tau(u)|^2 \) and therefore yields exactly the intrinsic biharmonic map equation.

\[
I = \frac{d}{dt} \bigg|_{t=0} \int_M |\Delta u(t)|^2 - |A(u(t))(\nabla u(t), \nabla u(t))|^2 \\
= 2 \int_M \langle \Delta u, \Delta (P\phi) \rangle - A(\nabla u, \nabla u)(D_u A(\nabla u, \nabla u) P\phi + 2A(\nabla u, \nabla (P\phi))) \\
= 2 \int_M \langle \Delta^2 u, P\phi \rangle - A(\nabla u, \nabla u)D_u A(\nabla u, \nabla u), P\phi \\
+ 2\langle \nabla (A(\nabla u, \nabla u)A(\nabla u, \cdot)), P\phi \rangle
\]

Now let’s look at terms \( II \) and \( III \) which give the additional lower order terms to the intrinsic biharmonic map equation that make the conformal-harmonic map equation (1.2) conformally invariant.

\[
II = 2 \int_M \MakeMath{\text{Sc}^M \frac{d}{dt}}_{t=0} |du(t)|^2 = \frac{4}{3} \int_M \langle \text{Sc}^M \nabla u, \nabla (P\phi) \rangle \\
= \frac{4}{3} \int_M \langle \text{Sc}^M \nabla_{\epsilon_\alpha} u, \nabla_{\epsilon_\alpha} (P\phi) \rangle = -\frac{4}{3} \int_M \langle \nabla_{\epsilon_\alpha} (\text{Sc}^M \nabla_{\epsilon_\alpha} u), P\phi \rangle \\
= -\frac{4}{3} \int_M \langle \text{Sc}^M \Delta u + \nabla \text{Sc}^M \nabla u, P\phi \rangle
\]

\[
III = -2 \int_M \frac{d}{dt} \bigg|_{t=0} \text{Ric}^M (du(t), du(t)) \\
= -4 \int_M \text{Ric}^M (du, d(\phi)) \\
= -4 \int_M \langle \text{Ric}^M (du, \cdot), d(\phi) \rangle \\
= 4 \int_M \langle \nabla_{\epsilon_\beta} (\text{Ric}^M)^{\alpha\beta} \nabla_{\epsilon_\alpha} u + (\text{Ric}^M)^{\alpha\beta} \nabla^2_{\alpha\beta} u, P\phi \rangle \\
= 4 \int_M \langle \nabla_{\beta} (\text{Ric}^M)^{\alpha\beta} \nabla_{\alpha} u + (\text{Ric}^M)^{\alpha\beta} P(\nabla^2_{\alpha\beta} u, \phi) \rangle
\]
where we used the subscripts $\alpha, \beta$ to denote $e_\alpha, e_\beta$ for short.

Combining these together, we get

\[
\left. \frac{d}{dt} \right|_{t=0} \mathcal{E}(u(t), \mathcal{M}) = 2 \int_{\mathcal{M}} \left( P(\nabla^2 u) - P(A(\nabla u, \nabla u) D_u A(\nabla u, \nabla u)) \right. \\
+ 2 \text{div}(A(\nabla u, \nabla u) A(\nabla u, P)) - 2 A(\nabla u, \nabla u) A(\nabla u, \nabla P) \\
\left. - \frac{2}{3} \text{Sc}^\mathcal{M}(\Delta u + A(\nabla u, \nabla u)) - \frac{2}{3} \nabla \text{Sc}^\mathcal{M} \nabla u \\
+ 2 \nabla_\beta (\text{Ric}^\mathcal{M})^{\alpha\beta} \nabla_\alpha u + 2 (\text{Ric}^\mathcal{M})^{\alpha\beta} P(\nabla^2_{\alpha\beta} u, \phi) \right).
\]

Thus, the critical point $u$ of $\mathcal{E}$ satisfies the fourth order PDE:

\[
P(\nabla^2 u) = P(A(\nabla u, \nabla u) D_u A(\nabla u, \nabla u)) - 2 \text{div}(A(\nabla u, \nabla u) A(\nabla u, P)) \\
+ 2 A(\nabla u, \nabla u) A(\nabla u, \nabla P) + \frac{2}{3} \text{Sc}^\mathcal{M}(\Delta u + A(\nabla u, \nabla u)) \\
+ \frac{2}{3} \nabla \text{Sc}^\mathcal{M} \nabla u - 2 \nabla_\beta (\text{Ric}^\mathcal{M})^{\alpha\beta} \nabla_\alpha u - 2 (\text{Ric}^\mathcal{M})^{\alpha\beta} P(\nabla^2_{\alpha\beta} u, \cdot).
\]

Note that

\[
P^\perp(\nabla^2 u) = \text{div}(P^\perp \nabla \Delta u) - \nabla P^\perp \nabla \Delta u \\
\quad = - \Delta A(\nabla u, \nabla u) - \Delta P^\perp \Delta u - 2 \nabla P^\perp \nabla \Delta u \\
\quad = - \Delta A(\nabla u, \nabla u) + \Delta P \Delta u + 2 \nabla P \nabla \Delta u \\
\quad = - \Delta (A(\nabla u, \nabla u)) - \Delta P \Delta u + 2 \text{div}(\nabla P \Delta u) ,
\]

and

\[
\Delta^2 u = P(\nabla^2 u) + P^\perp(\nabla^2 u) \\
\quad = P(\nabla^2 u) - \Delta (A(\nabla u, \nabla u)) - \Delta P \Delta u + 2 \text{div}(\nabla P \Delta u).
\]

Therefore, (B.6) can be rewritten as

\[
\Delta^2 u = - \Delta (A(\nabla u, \nabla u)) - \Delta P \Delta u + 2 \text{div}(\nabla P \Delta u) \\
+ P(A(\nabla u, \nabla u) D_u A(\nabla u, \nabla u)) - 2 \text{div}(A(\nabla u, \nabla u) A(\nabla u, P)) \\
+ 2 A(\nabla u, \nabla u) A(\nabla u, \nabla P) + \frac{2}{3} \text{Sc}^\mathcal{M}(\Delta u + A(\nabla u, \nabla u)) \\
+ \frac{2}{3} \nabla \text{Sc}^\mathcal{M} \nabla u - 2 \nabla \text{Ric}^\mathcal{M}(\nabla u, \cdot) - 2 \text{Ric}^\mathcal{M}(P(\nabla^2 u), \cdot)
\]

or equivalently (c.f. [32], equation (1.2), [31], equation (9))

\[
\Delta^2 u = - \Delta (\nabla P^\perp \nabla u) - \text{div}(\nabla P^\perp \Delta u) + 2 \nabla P^\perp \nabla (\nabla P^\perp \nabla u) \\
+ 2 \nabla P^\perp \nabla P^\perp \Delta u - (\nabla P P^\perp - P^\perp \nabla P) \nabla \Delta u \\
\quad + P(\nabla P^\perp \nabla u, D_u (D_u P^\perp) \nabla u) - 2 \text{div}(\nabla P^\perp \nabla u, \nabla P^\perp P) \\
\quad + 2 (\nabla P^\perp \nabla u, \nabla P^\perp P) + \frac{2}{3} \text{Sc}^\mathcal{M}(\Delta u + \nabla P^\perp \nabla u)
\]
\[ + \frac{2}{3} \nabla \text{Sc}^\mathcal{M} \nabla u - 2 \nabla \text{Ric}^\mathcal{M}(\nabla u, \cdot) - 2 \text{Ric}^\mathcal{M}(P(\nabla^2 u), \cdot), \]

where

\[ \nabla \text{Ric}^\mathcal{M}(\nabla u, \cdot) := \nabla_\beta (\text{Ric}^\mathcal{M})^{\alpha \beta} \nabla_\alpha u \]

and

\[ \text{Ric}^\mathcal{M}(P(\nabla^2 u), \cdot) := (\text{Ric}^\mathcal{M})^{\alpha \beta} P(\nabla^2 u)^{\alpha \beta}. \]

References

1. Wanjun Ai and Hao Yin, \textit{Neck analysis of extrinsic polyharmonic maps}, Ann. Global Anal. Geom. \textbf{52} (2017), no. 2, 129–156. MR 3690012
2. Vincent Bérard, \textit{Un analogue conforme des applications harmoniques}, C. R. Math. Acad. Sci. Paris \textbf{346} (2008), no. 17-18, 985–988. MR 2449641
3. Olivier Biquard and Farid Madani, \textit{A construction of conformal-harmonic maps}, Comptes Rendus Mathematique \textbf{350} (2012), no. 21, 967–970.
4. G. Carron, \textit{Inégalités de Hardy sur les variétés riemanniennes non-compactes}, J. Math. Pures Appl. (9) \textbf{76} (1997), no. 10, 883–891. MR 1489943
5. S. A. Chang, L. Wang, and P. C. Yang, \textit{A regularity theory of biharmonic maps}, Comm. Pure Appl. Math. \textbf{52} (1999), no. 9, 1113–1137. MR 1692148
6. Sun-Yung A. Chang and Paul C. Yang, \textit{On a fourth order curvature invariant}, Spectral problems in geometry and arithmetic (Iowa City, IA, 1997), Contemp. Math., vol. 237, Amer. Math. Soc., Providence, RI, 1999, pp. 9–28. MR 1710786
7. T. H. Colding and W. P. Minicozzi, \textit{Width and finite extinction time of Ricci flow}, Geom. Topol. \textbf{12} (2008), no. 5, 2537–2586. MR 2460871
8. Lorenzo D’Ambrosio and Serena Dipierro, \textit{Hardy inequalities on Riemannian manifolds and applications}, Ann. Inst. H. Poincaré C Anal. Non Linéaire \textbf{31} (2014), no. 3, 449–475. MR 3208450
9. Frédéric Louis de Longueville and Andreas Gastel, \textit{Conservation laws for even order systems of polyharmonic map type}, Calc. Var. Partial Differential Equations \textbf{60} (2021), no. 4, Paper No. 138, 18. MR 4279397
10. Frank Duzaar and Ernst Kuwert, \textit{Minimization of conformally invariant energies in homotopy classes}, Calc. Var. Partial Differential Equations \textbf{6} (1998), no. 4, 285–313. MR 1624288
11. D. E. Edmunds and J. Rákosník, \textit{On a higher-order Hardy inequality}, Math. Bohem. \textbf{124} (1999), no. 2-3, 113–121. MR 1780685
12. James Eells, Jr. and J. H. Sampson, \textit{Harmonic mappings of Riemannian manifolds}, Amer. J. Math. \textbf{86} (1964), 109–160. MR 164306
13. A. Gastel, \textit{The extrinsic polyharmonic map heat flow in the critical dimension}, Adv. Geom. \textbf{6} (2006), no. 4, 501–521. MR 2267035
14. Andreas Gastel and Andreas J. Nerf, \textit{Minimizing sequences for conformally invariant integrals of higher order}, Calc. Var. Partial Differential Equations \textbf{47} (2013), no. 3-4, 499–521. MR 3070553
15. Andreas Gastel and Christoph Scheven, \textit{Regularity of polyharmonic maps in the critical dimension}, Comm. Anal. Geom. \textbf{17} (2009), no. 2, 185–226. MR 2520907
16. Jian Ge, \textit{Comparison theorems for manifolds with mean convex boundary}, Commun. Contemp. Math. \textbf{17} (2015), no. 5, 1550010, 12. MR 3404748
17. C. Robin Graham, Ralph Jenne, Lionel J. Mason, and George A. J. Sparling, \textit{Conformally invariant powers of the Laplacian. I. Existence}, J. London Math. Soc. (2) \textbf{46} (1992), no. 3, 557–565. MR 1190438
18. Weiyoung He, Ruiqi Jiang, and Longzhi Lin, *Existence of polyharmonic maps in critical dimensions*, Preprint (2019).

19. F. Hélein, *Harmonic maps, conservation laws and moving frames*, second ed., Cambridge Tracts in Mathematics, vol. 150, Cambridge University Press, Cambridge, 2002, Translated from the 1996 French original, With a foreword by James Eells. MR 1913803

20. J. Hineman, T. Huang, and C. Wang, *Regularity and uniqueness of a class of biharmonic map heat flows*, Calc. Var. Partial Differential Equations 50 (2014), no. 3-4, 491–524. MR 3216822

21. F. Hélein, *Harmonic maps, conservation laws and moving frames*, second ed., Cambridge Tracts in Mathematics, vol. 150, Cambridge University Press, Cambridge, 2002, Translated from the 1996 French original, With a foreword by James Eells. MR 1913803

22. J. Hineman, T. Huang, and C. Wang, *Regularity and uniqueness of a class of biharmonic map heat flows*, Calc. Var. Partial Differential Equations 50 (2014), no. 3-4, 491–524. MR 3216822

23. G. Jiang, 2-harmonic isometric immersions between Riemannian manifolds, Chinese Ann. Math. Ser. A 7 (1986), no. 2, 130–144, An English summary appears in Chinese Ann. Math. Ser. B 7 (1986), no. 2, 255.

24. Ismail Kombe and Murad Özaydin, Improved Hardy and Rellich inequalities on Riemannian manifolds, Trans. Amer. Math. Soc. 361 (2009), no. 12, 6191–6203. MR 2538592

25. Y. Ku, Interior and boundary regularity of intrinsic biharmonic maps to spheres, Pacific J. Math. 234 (2008), no. 1, 43–67. MR 2375314

26. T. Lamm, *Heat flow for extrinsic biharmonic maps with small initial energy*, Adv. Calc. Var. 2 (2009), no. 1, 1–16. MR 2494504

27. Tobias Lamm and Changyou Wang, Boundary regularity for polyharmonic maps in the critical dimension, Adv. Calc. Var. 2 (2009), no. 1, 1–16. MR 2494504

28. T. Lamm and T. Rivi`ere, Conservation laws for fourth order systems in four dimensions, Comm. Partial Differential Equations 33 (2008), no. 1-3, 245–262. MR 2398228

29. Tobias Lamm and Changyou Wang, Boundary regularity for polyharmonic maps in the critical dimension, Adv. Calc. Var. 2 (2009), no. 1, 1–16. MR 2494504

30. P. Laurain and T. Rivi`ere, Energy quantization for biharmonic maps, Adv. Calc. Var. 6 (2013), no. 2, 191–216. MR 3043576

31. Lei Liu and Hao Yin, Neck analysis for biharmonic maps, Math. Z. 283 (2016), no. 3-4, 807–834. MR 3519983

32. Paul Laurain and Longzhi Lin, Energy convexity of intrinsic bi-harmonic maps and applications I: Spherical target, J. Reine Angew. Math. 772 (2021), 53–81. MR 4227593

33. ______, Energy convexity of intrinsic bi-harmonic maps and applications I: Spherical target, J. Reine Angew. Math. 772 (2021), 53–81. MR 4227593

34. E. Mitidieri, A simple approach to Hardy inequalities, Mat. Zametki 67 (2000), no. 4, 563–572. MR 1769903

35. R. Moser, The blowup behavior of the biharmonic map heat flow in four dimensions, IMRP Int. Math. Res. Pap. (2005), no. 7, 351–402.

36. Roger Moser, A variational problem pertaining to biharmonic maps, Comm. Partial Differential Equations 33 (2008), no. 7-9, 1654–1689. MR 2450176

37. Stephen M. Paneitz, A quartic conformally covariant differential operator for arbitrary pseudo-Riemannian manifolds (summary), SIGMA Symmetry Integrability Geom. Methods Appl. 4 (2008), Paper 036, 3. MR 2393291

38. Christoph Scheven, Dimension reduction for the singular set of biharmonic maps, Adv. Calc. Var. 1 (2008), no. 1, 53–91. MR 2402212
40. Michael Struwe, *On the evolution of harmonic mappings of Riemannian surfaces*, Comment. Math. Helv. 60 (1985), no. 4, 558–581. MR 826871

41. ______, *Partial regularity for biharmonic maps, revisited*, Calc. Var. Partial Differential Equations 33 (2008), no. 2, 249–262. MR 2413109

42. Paweł Strzelecki, *On biharmonic maps and their generalizations*, Calc. Var. Partial Differential Equations 18 (2003), no. 4, 401–432. MR 2020368

43. C. Wang, *Stationary biharmonic maps from \( \mathbb{R}^m \) into a Riemannian manifold*, Comm. Pure Appl. Math. 57 (2004), no. 4, 419–444. MR 2026177

44. Changyou Wang, *Biharmonic maps from \( \mathbb{R}^4 \) into a Riemannian manifold*, Math. Z. 247 (2004), no. 1, 65–87. MR 2054520

45. ______, *Heat flow of biharmonic maps in dimensions four and its application*, Pure Appl. Math. Q. 3 (2007), no. 2, Special Issue: In honor of Leon Simon. Part 1, 595–613. MR 2340056

46. Xiaodong Wang, *On compact Riemannian manifolds with convex boundary and Ricci curvature bounded from below*, J. Geom. Anal. 31 (2021), no. 4, 3988–4003. MR 4236549

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