SELF-SIMILAR SOLUTIONS OF DECAYING KELLER-SEGEL SYSTEMS FOR SEVERAL POPULATIONS

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Abstract. It is known that solutions of the parabolic elliptic Keller- Segel equations in the two dimensional plane decay, as time goes to infinity, provided the initial data admits sub-critical mass and finite second moments, while such solution concentrate, as $t \to \infty$, in the critical mass. In the sub-critical case this decay can be resolved by a steady, self-similar solution, while no such self similar solution is known to exist for the concentration in the critical case.

This paper is motivated by the Keller-Segel system of several interacting populations, under the existence of an additional drift for each component which decays in time at the rate $O(1/\sqrt{t})$. We show that self-similar solutions always exists in the sub-critical case, while the existence of such self-similar solution in the critical case depends on the gap between the decaying drifts for each of the components. For this, we study the conditions for existence/non existence of solutions for the corresponding Liouville’s systems, which, in turn, is related to the existence/non existence of minimizers to a corresponding Free Energy functional.

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1. Introduction

The Keller-Segel system represents the evolution of living cells under self-attraction and diffusive forces [KS70], [Pat53]. Its general form is given by

$$\frac{\partial \rho}{\partial t} = \Delta \rho - \nabla \cdot \rho \left( a \nabla u - \vec{V} \right) ; \quad (x, t) \in \mathbb{R}^2 \times \mathbb{R}_+$$

(1.1)

where $a > 0$, $\vec{V} = \vec{V}(x, t)$ is a given vector-field, $\rho = \rho(x, t)$ stands for the distribution of living cells and $u = u(x, t)$ the concentration of the chemical substance attracting the cells.

In the parabolic/elliptic limit this concentration is given by the Newtonian potential

$$u(x, t) := -\frac{1}{2\pi} \int_{\mathbb{R}^2} \rho(y, t) \ln |x - y| \, dy , \quad i.e \quad -\Delta u = \rho .$$

(1.2)

Since (1.1) is a parabolic equation of divergence type it follows that the total population number $\int \rho \, dx := \beta > 0$ is conserved in time under suitable boundary conditions at infinity.

If $\vec{V} \equiv 0$ then the steady states of (1.1,1.2) takes the form of Liouville’s Equation

$$\Delta u(x) + \frac{\beta e^{au(x)}}{\int_{\mathbb{R}^2} e^{au(z)} \, dz} = 0 .$$

(1.3)

The spacial dimension 2 which we discuss here was studied by many authors in the case $V \equiv 0$ [BDP06, BKLN06a, BKLN06b, BM08]. The two dimensional case is special in the sense that there is a critical mass $\beta_c = 8\pi/a$. If $\beta < \beta_c$ then, under some natural

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assumptions on the initial data $\rho(x,0) := \rho_0$, the solutions exists globally in time and, moreover, $\lim_{t \to \infty} \rho(x,t) = 0$ locally uniformly on $\mathbb{R}^2$ [BDP06]. In particular, there is no solution of (1.3). If $\beta > \beta_c$ then there is no global in time solution of (1.1, 1.2) [HV96] and, again, no solution of (1.3) exists. In the case $\beta = \beta_c$ there is a family of solutions of (1.3) and the (free-energy) solutions of (1.1, 1.2) exist globally in time. Moreover, if the initial data has finite second moment then any such solution converges asymptotically to the Dirac measure $\beta_c \delta_0$ [BM08], otherwise, any radial solution to (1.1, 1.2) converges asymptotically to one of the solutions of (1.3) [BKLN06b].

In the sub-critical case $\beta \leq \beta_c$ it is natural to ask whether there exists self similar solutions of (1.1,1.2) of the form

$$\rho(x,t) := (2t)^{-1} \bar{\rho} \left( \frac{x}{\sqrt{2t}}, \frac{1}{2} \ln 2t \right), \quad u(x,t) = \bar{u} \left( \frac{x}{\sqrt{2t}}, \frac{1}{2} \ln 2t \right).$$

where $t > 0$.

It follows that

$$\bar{u}(y,t) = -\frac{1}{2\pi} \int_{\mathbb{R}^2} \bar{\rho}(x,t) \ln |x-y| d^2x - \frac{\beta}{2t} \bar{u},$$

in particular $\nabla_x u(x,t) = (2t)^{-1/2} \nabla \bar{u}(x/\sqrt{2t}, 1/2 \ln 2t)$. Substituting in the KS equation we get under the change of variables $x \to \frac{x}{\sqrt{2t}}$, $t \to \frac{1}{2} \ln 2t$,

$$\partial_t \bar{\rho} = \Delta \bar{\rho} - \nabla \cdot \bar{\rho} (a \nabla \bar{u} - x),$$

which corresponds to (1.1) under $V(x,t) = x$. The corresponding steady state of (1.5) is

$$\Delta_x \bar{u} + \frac{\beta e^{a \bar{u}} - |x|^2/2}{\int_{\mathbb{R}^2} e^{a \bar{u}}(z) - |z|^2/2 d^2z} = 0$$

(1.6)

The existence and uniqueness (up to a constant) of the solutions to (1.6) in the sub-critical case $\beta < \beta_c$ was given in [CLMP92, CK94]. In [BF10] the authors considered the existence of such self-similar solution of (1.4) for sub-critical data. Non existence of solutions of (1.6) in the critical case was also proved in [CK94].

In this paper we consider (1.1) with a non-zero, but decaying in time vector field. In particular we assume

$$V(x,t) = -v(2t)^{-1/2}$$

(1.7)

where $v \in \mathbb{R}^2$ is a constant vector. Then we get under the scaling (1.4) the following modification of (1.5):

$$\partial_t \bar{\rho} = \Delta \bar{\rho} - \nabla \cdot \bar{\rho} (\beta \nabla \bar{u} - (x-v)),$$

and the modified Liouville’s equation

$$\Delta_x \bar{u} + \frac{\beta e^{a \bar{u}} - |x-v|^2/2}{\int_{\mathbb{R}^2} e^{a \bar{u}}(z) - |z-v|^2/2 d^2z} = 0$$

(1.9)

Evidently, any solution of (1.6) is transformed into a solution of (1.9) by a shift $x \to x+v$ and v.v. In particular, the self similar solutions of (1.1,1.2) in the case $V = 0$ is translated to the case of $V = -v(2t)^{-1/2}$ by this shift. Thus the non-existence of global, self-similar solutions of the form (1.4) in the case of critical mass $\beta = \beta_c$ [NS04] is obtained under (1.7) as well.

In this paper we are motivated by a generalization of (1.1,1.2) to the case of a system of $n$ populations

$$\frac{\partial \rho_i}{\partial t} = \Delta \rho_i - \nabla \cdot \rho_i \left( \sum_{j=1}^{n} a_{ij} \nabla_x u_j - \bar{V}_i \right); \quad (x,t) \in \mathbb{R}^2 \times \mathbb{R}_+$$

(1.10)
where $A := (a_{ij})_{n \times n}$ is a symmetric and nonnegative (i.e., $a_{ij} \geq 0$ for all $i, j$) matrix and

$$u_i(x, t) := -\frac{1}{2\pi} \int_{\mathbb{R}^2} \rho_i(y, t) \ln|x - y|d^2y. \quad (1.11)$$

In the case $\bar{V}_i = 0$ the stationary solution of such systems, subjected to the initial data satisfying $\int \rho_i(x, 0)d^2x = \beta_i$ solves the Liouville’s systems:

$$\Delta u_i + \frac{\beta_ie^{\sum_j a_{ij}u_j}}{\int_{\mathbb{R}^2} e^{\sum_j a_{ij}u_j(z)}d^2z} = 0. \quad (1.12)$$

Again, such Liouville’s systems have been studied intensively in [CSW97, SW05, Lin11], and the cases where $a_{ij}$ are not necessarily nonnegative (in connection with the chemotactic system known as the conflict case) have also been explored in [Hor11, Wol16].

The solvability of such systems was considered in [CSW97, SW05] and [Wol02]. The criticality condition is determined, in that case, by the functions

$$\Lambda_J(\beta) = \sum_{i \in J} \beta_i \left(8\pi - \sum_{j \in J} a_{ij} \beta_j \right).$$

where $\phi \neq J \subseteq I := \{1, \ldots n\}$. The criticality condition $\beta_c = 8\pi/a$ in the case of single composition is replaced by

$$\Lambda_J(\beta) = 0.$$

In particular it was proved in [CSW97] that an entire solution of (1.12) exists only in the critical case if, in addition, $\Lambda_J(\beta) > 0$ for all $\phi \neq J \subset I$ hold.

In this paper we consider the implementation of

$$\bar{V}_i = -\frac{1}{\sqrt{2}}t^{-1/2}v_i \quad (1.13)$$

in (1.10, 1.11), where $v_i \in \mathbb{R}^2$ are (perhaps different) constant vectors. Under the scaling (1.4) we recover the modified KS system

$$\frac{\partial \bar{\rho}_i}{\partial t} = \Delta \bar{\rho}_i - \nabla \cdot \bar{\rho}_i \left(\sum_{j=1}^n a_{ij} \nabla_x \bar{u}_j - (x - v_i)\right); \quad (x, t) \in \mathbb{R}^2 \times \mathbb{R}_+ \quad (1.14)$$

where

$$\bar{u}_i(x, t) := -\frac{1}{2\pi} \int_{\mathbb{R}^2} \rho_i(y, t) \ln|x - y|d^2y - \frac{\beta_i}{2\pi}t. \quad (1.15)$$

The steady states of (1.14, 1.15) are given by the modified Liouville’s system

$$\Delta_x \bar{u}_i + \frac{\beta_IE^{\sum_j a_{ij} \bar{u}_j - |x - v_i|^2/2}}{\int_{\mathbb{R}^2} E^{\sum_j a_{ij} \bar{u}_j - |x - v_i|^2/2}d^2z} = 0. \quad (1.16)$$

The existence of entire solutions to (1.16) is, thus, directly related to the existence of self-similar solutions of the form (1.4) for (1.10, 1.11) under (1.13).

The modified KS system (1.14, 1.15) and the modified Liouville’s system (1.16) are closely related to the Free energy functional

$$\mathcal{F}_\nu(\bar{\rho}) := \sum_{i=1}^n \int_{\mathbb{R}^2} \bar{\rho}_i(x) \ln \bar{\rho}_i(x)d^2x + \frac{1}{4\pi} \sum_{i=1}^n \sum_{j=1}^n a_{ij} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \bar{\rho}_i(x) \ln|x - y|\bar{\rho}_j(y)d^2xd^2y$$

$$+ \sum_{i=1}^n \frac{1}{2} \int_{\mathbb{R}^2} |x - v_i|^2 \bar{\rho}_i(x)d^2x, \quad (1.17)$$
Theorem 1.2. Suppose \( \beta \) satisfies

\[
\text{Definition 1.1.} \quad \text{Unless otherwise stated, in this article we assume the matrix } \beta \text{ and, in particular, any minimizer is such a solution. Moreover, we expect such minimizers to be a stable stationary solutions of (1.14, 1.15) can be written as a gradient descend system in the Wasserstein sense [AGS05]}
\]

\[
\frac{\partial \bar{\rho}_i}{\partial t} = \nabla \cdot \left( \bar{\rho}_i \nabla \left( \frac{\delta F_v}{\delta \bar{\rho}_i} \right) \right), \quad i = 1, \ldots, n, \quad (1.18)
\]

and, in particular

\[
\frac{d}{dt} F_v(\bar{\rho}) = -\sum_i \int_{\mathbb{R}^2} \bar{\rho}_i \left| \nabla \frac{\delta F_v}{\delta \bar{\rho}_i} \right|^2 . \quad (1.19)
\]

Every critical point of \( F_v \) on \( \Gamma^\beta \) induces a solution of (1.16) [CSW97], [Suz05]. In particular, any minimizer is such a solution. Moreover, we expect such minimizers to be a stable stationary solutions of (1.14, 1.15) and thus to represent stable self similar limit of (1.10, 1.11) under (1.13).

Let

\[
\text{Remark 1.} \quad \text{Unless otherwise stated, in this article we assume the matrix } A = (a_{ij})_{n \times n} \text{ satisfies}
\]

\[
(H) \quad A \text{ is symmetric and nonnegative},
\]

and \( \beta \) satisfies

\[
\begin{cases}
\Lambda_J(\beta) \geq 0, \text{ for all } \emptyset \neq J \subseteq I, \\
\text{if, for some } J \neq \emptyset , \Lambda_J(\beta) = 0, \text{ then } a_{ii} + \Lambda_{J \setminus \{i\}} > 0, \forall i \in J.
\end{cases}
\]

The main result of this article is:

**Theorem 1.1.** Suppose \( A \) satisfies (H) and \( \beta \) satisfies (1.21). Then

(a) \( F_v \) is bounded from below on \( \Gamma^\beta \).

(b) If \( \Lambda_J(\beta) > 0 \) for all \( \emptyset \neq J \subseteq I \), then there exists a minimizer of \( F_v \) on \( \Gamma^\beta \), for all \((v_1, \ldots, v_n) \in (\mathbb{R}^2)^n\).

(c) If \( \Lambda_I(\beta) = 0 \) and \( \text{Var}(v_1, \ldots, v_n) = 0 \) then there is no minimizer of \( F_v \) in \( \Gamma^\beta \).

(d) If \( n = 2 \) and \( \Lambda_{\{1,2\}}(\beta) = 0, \Lambda_{\{1\}}(\beta), \Lambda_{\{2\}}(\beta) > 0 \) and \( |v_1 - v_2| \) is large enough then there exists a minimizer of \( F_v \) on \( \Gamma^\beta \).

For a given such matrix \( A \), we define

**Definition 1.1.**

- \( \beta \) is sub-critical if \( \Lambda_J(\beta) > 0 \) for any \( \emptyset \neq J \subseteq I \).
- \( \beta \) is critical if \( \Lambda_I(\beta) = 0 \) and \( \Lambda_J(\beta) > 0 \) for any \( \emptyset \neq J \subseteq I \).

**Theorem 1.2.**

(a) There exists a solution of (1.16) for any sub-critical \( \beta \) and any \( v_1, \ldots, v_n \in \mathbb{R}^2 \).

(b) If \( \beta \) is critical, \( \text{Var}(v_1, \ldots, v_n) = 0 \), and \( A \) is invertible and irreducible, then there is no solution to (1.16).

(c) There exists a solution of (1.16) for \( n = 2 \) in the critical case provided \( |v_1 - v_2| \) is large enough.

**Remark 1.**

- Theorem 1.2-a,c follows immediately from Theorem 1.1-a,b,d.
Theorem 1.1-c implies the non-existence of minimizers in the critical case. The non-existence of solutions in the critical case (Theorem 1.2-c) follows from a different argument.

The results of Theorem 1.1-d and Theorem 1.2-c can be easily extended to the case $n > 2$, provided $\text{Var}(v_1, \ldots, v_n)$ is large enough. It is not known if $\text{Var}(v_1, \ldots, v_n) \neq 0$ is sufficient for existence of solutions of (1.16) in the critical case for any $n \geq 2$.

Our organization of the article is as follows: in Section 2 we discussed the boundedness from below of the functional $F_\alpha$ over $\Gamma^\beta$. Section 3 is devoted to the basic lemmas required for the proof of our main theorem. In Section 4 we proved the existence of minimizers for subcritical $\beta$. The critical case has been analyzed in Sections 5 and 6 and we established an if and only if criterion (Proposition 6.1) for the existence of minimizers. More precisely, we proved that either a minimizer exists or equality holds in (5.3). At the end of this article we exhibited certain examples (when $\text{Var}(v_1, v_2)$ large) for which the minimum is actually attained and proved the nonexistence result (Theorem 2(b)) when $\text{Var}(v_1, \ldots, v_n) = 0$.

2. Boundedness from below

Since we can shift $(v_1, \ldots, v_n)$ by any constant vector we can set $v_1 = v_2 = \ldots = v_n = 0$ if $\text{Var}(v_1, \ldots, v_n) = 0$. The functional $F_\alpha$ will be denoted by $F_0$ in that case. Also, we omit the bars from $\bar{\rho}_i$ from now on.

We will actually prove the boundedness from below of a little more general functional. For $\alpha := (\alpha_1, \cdots, \alpha_n) \in (\mathbb{R}_+)^n$, (where $\mathbb{R}_+$ is the set of all positive real numbers) define

$$F_{v,\alpha}(\rho) := \sum_{i=1}^n \int_{\mathbb{R}^2} \rho_i(x) \ln \rho_i(x) d^2 x + \frac{1}{4\pi} \sum_{i=1}^n \sum_{j=1}^n a_{ij} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \rho_i(x) \ln |x-y|/\beta_j(y) d^2 x d^2 y$$

$$+ \sum_{i=1}^n \alpha_i \int_{\mathbb{R}^2} |x-v_i|^2 \rho_i(x) d^2 x. \quad (2.1)$$

When $v_i = 0$ for all $i$, it will be denoted by $F_{0,\alpha}$.

**Theorem 2.1.** Condition (1.21) is necessary and sufficient condition for the boundedness from below of $F_{v,\alpha}$ on $\Gamma^\beta$.

**Proof.** First we recall [CSW97, SW05] that if $\rho$ is supported in a given bounded set then $F_{v,0}$ is bounded from below iff (1.21) is satisfied. This implies the necessary part. For the sufficient part we know from the same references that (1.21) together with the condition $\Lambda_1(\beta) = 0$ imply that $F_{v,0}$ is bounded from below. We only need to show that for any positive $\alpha$ we still obtain the bound from below in the case $\Lambda_1(\beta) > 0$. Note also that since $|x-v|^2 > |x|^2/2 - C$ for any $x \in \mathbb{R}^2$ and $C$ depending on $|v|$ it is enough to prove the sufficient condition for $v = 0$.

The proof is a straight forward adaptation of the corresponding proof in [SW05] without the potential $|x|^2$. For $\rho = (\rho_1, \cdots, \rho_n) \in \Gamma^\beta$ let $\rho^*_i$ be the symmetric decreasing rearrangement of $\rho_i$. Then clearly we have

$$\int_{\mathbb{R}^2} \rho_i \ln \rho_i = \int_{\mathbb{R}^2} \rho^*_i \ln \rho^*_i, \quad \int_{\mathbb{R}^2} \rho_i |\ln \rho_i| = \int_{\mathbb{R}^2} \rho^*_i |\ln \rho^*_i|, \quad \int_{\mathbb{R}^2} |x|^2 \rho^*_i \leq \int_{\mathbb{R}^2} |x|^2 \rho_i.$$

Thus if we define $\rho^* = (\rho^*_1, \cdots, \rho^*_n)$ then $\rho^* \in \Gamma^\beta$. Furthermore, we have (see [CL92, SW05])

$$\int_{\mathbb{R}^2} \rho^*_i(x) \ln |x-y| \rho^*_j(y) \leq \int_{\mathbb{R}^2} \rho_i(x) \ln |x-y| \rho_j(y), \; \forall i, j.$$
and hence $\mathcal{F}_{0,\alpha}(\rho^*) \leq \mathcal{F}_{0,\alpha}(\rho)$. Therefore it is enough to prove the theorem for radially symmetric decreasing function of $|x|$. Let $\rho \in \mathbb{T}^\beta$ be a radially symmetric decreasing function of $r = |x|$. As in [CSW97, SW05] we define

$$m_i(r) = 2\pi \int_0^r \tau \rho_i(\tau) \, d\tau, \quad r \in (0, \infty),$$

$$u_i(x) = -\frac{1}{2\pi} \int_{\mathbb{R}^2} \ln |x - y| \rho_i(y) \, d^2y.$$

Then we get using $\int_{\mathbb{R}^2} |x|^2 \rho_i < \infty$ and [SW05, equation (5.6)]

$$\begin{align*}
\lim_{R \to \infty} \left[ u_i(R) + \frac{\beta_i}{2\pi} \ln R \right] &= 0, \\
\lim_{R \to \infty} (\beta_i - m_i(R)) R^2 &= 0.
\end{align*}$$

Furthermore, if we define

$$\mathcal{F}_{0,\alpha,R}(\rho) := \sum_{i=1}^n \int_{B(0,R)} \rho_i \ln \rho_i - \frac{1}{2} \sum_i \sum_j a_{ij} \int_{B(0,R)} \rho_i u_j + \sum_{i=1}^n \alpha_i \int_{B(0,R)} |x|^2 \rho_i(x),$$

then by dominated convergence theorem we have

$$\mathcal{F}_{0,\alpha}(\rho) = \lim_{R \to \infty} \mathcal{F}_{0,\alpha,R}(\rho).$$

Again following [SW05], we define $w_i(s) = m_i(e^s)$. Then

$$\sum_{i=1}^n \alpha_i \int_{B(0,R)} |x|^2 \rho_i(x) \, d^2x = \sum_{i=1}^n 2\pi \alpha_i \int_0^R r^3 \rho_i(r) \, dr$$

$$= \sum_{i=1}^n \alpha_i \int_0^R r^2 m_i(r) \, dr$$

$$= -2 \sum_{i=1}^n \alpha_i \int_0^R r m_i(r) \, dr + \sum_{i=1}^n \alpha_i m_i(R) R^2$$

$$= -2 \sum_{i=1}^n \alpha_i \int_{-\infty}^{\ln R} e^{2s} w_i(s) \, ds + \sum_{i=1}^n \alpha_i m_i(R) R^2$$

and therefore we can write $\mathcal{F}_{0,\alpha,R}(\rho) = G_R(w) - (\ln 2\pi) \sum_{i=1}^n m_i(R)$, where

$$G_R(w) = \int_{-\infty}^{\ln R} \sum_{i=1}^n w_i' \ln w_i' \, ds + \int_{-\infty}^{\ln R} \left[ 2 \sum_{i=1}^n w_i - \frac{1}{4\pi} \sum_{i,j=1}^n a_{ij} w_i w_j \right] \, ds$$

$$- 2 \sum_{i=1}^n \alpha_i \int_{-\infty}^{\ln R} e^{2s} w_i \, ds - \sum_{i=1}^n m_i(R) \left( 2 \ln R + \frac{1}{2} \sum_{j=1}^n a_{ij} u_j(R) - \alpha_i R^2 \right).$$

Now define $\nu_i = 2 - \frac{1}{4\pi} \sum_{j=1}^n a_{ij} \beta_j$. Using the identity $\frac{\Lambda_1(\beta)}{4\pi} = \sum_{i=1}^n \nu_i \beta_i$ and (2.2) we get

$$- \sum_{i=1}^n m_i(R) \left[ 2 \ln R + \frac{1}{2} \sum_{j=1}^n a_{ij} u_j(R) \right] + \sum_{i=1}^n 2\nu_i \beta_i \ln R = \frac{\Lambda_1(\beta)}{4\pi} \ln R + o_R(1),$$

where $o_R(1)$ stands for a quantity going to zero as $R \to \infty$. Utilizing (2.3), we can decompose $G_R(w)$ as follows

$$G_R(w) = J_{-\infty}(w) + J_{\infty}(w) + E_R(w) + o_R(1),$$
where

\[
J_{-\infty}(w) = \int_{-\infty}^{0} \sum_{i=1}^{n} w'_i \ln w'_i \, ds + \int_{-\infty}^{0} \left[ 2 \sum_{i=1}^{n} w_i - \frac{1}{4\pi} \sum_{i,j=1}^{n} a_{ij} w_i w_j \right] \, ds \\
- 2 \sum_{i=1}^{n} \alpha_i \int_{-\infty}^{0} e^{-2s} w_i \, ds,
\]

\[
J_{\infty}(w) = \int_{0}^{\ln R} \sum_{i=1}^{n} w'_i \ln w'_i \, ds + \int_{0}^{\ln R} \left[ \sum_{i=1}^{n} 2(1 - \nu_i) w_i - \frac{1}{4\pi} \sum_{i,j=1}^{n} a_{ij} w_i w_j + \frac{A_{ij}(\beta)}{4\pi} \right] \, ds
\]

\[
E_R(w) = -2 \sum_{i=1}^{n} \alpha_i \int_{0}^{\ln R} e^{-2s} w_i \, ds + \sum_{i=1}^{n} 2 \nu_i \int_{0}^{\ln R} w_i \, ds - 2 \left( \sum_{i=1}^{n} \nu_i \beta_i \right) \ln R
\]

\[+ \sum_{i=1}^{n} \alpha_i m_i(R) R^2.\]

By [SW05] we have \(J_{-\infty}\) and \(J_{\infty}\) are bounded from below on \(\Gamma^\beta\), once we observe that

\[
\int_{-\infty}^{0} e^{-2s} w_i \leq \beta_i \int_{-\infty}^{0} e^{-2s} \leq \frac{\beta_i}{2}.
\]

Therefore, we only need to show that \(E_R(w)\) is bounded from below. We can rewrite \(E_R(w)\) in the following way

\[
E_R(w) = \int_{0}^{\ln R} \left[ \sum_{i=1}^{n} 2 \left( \nu_i - \alpha_i e^{2s} \right) w_i - 2 \sum_{i=1}^{n} \nu_i \beta_i + 2 \sum_{i=1}^{n} \alpha_i \beta_i e^{2s} \right] \, ds
\]

\[
= \int_{0}^{\ln R} \left[ 2 \sum_{i=1}^{n} (\beta_i - w_i(s))(\alpha_i e^{2s} - \nu_i) \right] \, ds.
\]

Now \(w_i(s) \leq \beta_i\) for all \(s\) and \(\alpha_i > 0, \nu_i\) are being fixed numbers, we can find a \(R_0 > 0\), independent of \(w_i\) such that \((\beta_i - w_i(s))(\alpha_i e^{2s} - \nu_i) \geq 0\) for all \(s \geq \ln R_0\). Again since

\[
\left| \int_{0}^{\ln R_0} \left[ 2 \sum_{i=1}^{n} (\beta_i - w_i(s))(\alpha_i e^{2s} - \nu_i) \right] \, ds \right| \leq \sum_{i=1}^{n} 4\beta_i \left( \frac{\alpha_i}{2} R_0^2 - \nu_i \ln R_0 - \frac{\alpha_i}{2} \right),
\]

we have \(E_R(w) \geq -|E_{R_0}(w)| \geq -C\). This proves the sufficiency of the condition (1.21). \(\square\)

3. Basic Lemmas

In this section we will recall a few definitions and lemmas and also prove some basic ingredients required for the proof of our main results. We define the space \(\mathbb{L} \ln L(\mathbb{R}^2)\) as the Orlicz space determined by the N-function \(N(t) = (1 + t) \ln (1 + t) - t, t \geq 0:\)

\[
\mathbb{L} \ln L(\mathbb{R}^2) := \left\{ \rho : \mathbb{R}^2 \to \mathbb{R} \text{ measurable} : \int_{\mathbb{R}^2} \left[ (1 + |\rho|) \ln (1 + |\rho|) - |\rho| d^2x \right] < \infty \right\}.
\]

Then \(\mathbb{L} \ln L(\mathbb{R}^2)\) is a Banach space with respect to the Luxemburg norm (because \(N(t)\) satisfies the \(\Delta_2\) condition: \(N(2t) \leq 2 N(t)\) for all \(t \geq 0\)).

The dual space of \(\mathbb{L} \ln L(\mathbb{R}^2)\) is the Orlicz space determined by the N-function \(M(t) = (e^t - t - 1), t \geq 0.\) It is important to remark that \(\mathbb{L} \ln L(\mathbb{R}^2)\) is not reflexive (because
M(t) does not satisfy the $\Delta_2$ condition). However, there is a notion of weak convergence which is slightly weaker than the usual weak convergence in Banach spaces. A sequence $\rho_m \in \mathbb{L}\ln\mathbb{L}(\mathbb{R}^2)$ is said to converge $L_M$-weakly to $\rho$ if

$$\int_{\mathbb{R}^2} \rho_m \phi \to \int_{\mathbb{R}^2} \rho \phi, \text{ for all bounded measurable functions } \phi \text{ with bounded support}.$$  

It is well known from the general Orlicz space theory [KR61] that $\mathbb{L}\ln\mathbb{L}(\mathbb{R}^2)$ is $L_M$-weakly compact. To simplify our notations we will denote the weak convergence (in the above sense) by $\rho_m \rightharpoonup \rho$.

We begin with the following elementary lemma whose proof can be found in [BF10]:

**Lemma 3.1.** For $1 \leq i \leq n$ let $\rho_i \in L^1(\mathbb{R}^2)$ be such that $\rho_i \geq 0$ and satisfies

$$\int_{\mathbb{R}^2} \rho_i \ln \rho_i \leq C_0, \int_{\mathbb{R}^2} |x|^2 \rho_i \leq C_0.$$  

Then

$$\sum_{i=1}^{n} \int_{\mathbb{R}^2} \rho_i \ln \rho_i \leq \sum_{i=1}^{n} \int_{\mathbb{R}^2} \rho_i \ln \rho_i + 2 \ln 2\pi \left( \sum_{i=1}^{n} \int_{\mathbb{R}^2} \rho_i \right) + 2 \sum_{i=1}^{n} \int_{\mathbb{R}^2} |x|^2 \rho_i + 2ne^{-1}.$$  

**Lemma 3.2.** Let $\{\rho_m\}$ be a sequence in $\mathbb{L}\ln\mathbb{L}(\mathbb{R}^2)$ such that

$$\int_{\mathbb{R}^2} \rho_m \ln \rho_m \leq C_0, \int_{\mathbb{R}^2} \rho_m = \beta, \int_{\mathbb{R}^2} |x|^2 \rho_m \leq C_0.$$  

Then there exists $\rho \in \mathbb{L}\ln\mathbb{L}(\mathbb{R}^2)$ such that up to a subsequence $\rho_m \rightharpoonup \rho$ in the topology of $\mathbb{L}\ln\mathbb{L}(\mathbb{R}^2)$ and satisfies

$$\int_{\mathbb{R}^2} \rho \ln \rho \leq \liminf_{n \to \infty} \int_{\mathbb{R}^2} \rho_m \ln \rho_m. \quad (3.1)$$  

**Remark 2.** The conclusion of the lemma is false without the assumption on the uniform boundedness of $\int_{\mathbb{R}^2} |x|^2 \rho_m$. As a counter example, let $\phi \in C_c^\infty(\mathbb{R}^2)$ be a smooth cutoff function such that $0 \leq \phi \leq 1 - \delta$, for some $\delta \in (0, 1)$. Let $x_m$ be a sequence in $\mathbb{R}^2$ such that $|x_m| \nearrow \infty$ and define the sequence

$$\rho_m(x) = \phi(x + x_m).$$  

Then it is easy to check that $\int_{\mathbb{R}^2} |x|^2 \rho_m \to \infty$, and

$$\int_{\mathbb{R}^2} \rho_m \ln \rho_m = \int_{\mathbb{R}^2} \phi \ln \phi < 0, \text{ for all } m.$$  

But $\rho_m \rightharpoonup \rho \equiv 0$ in $\mathbb{L}\ln\mathbb{L}(\mathbb{R}^2)$ and hence $\int_{\mathbb{R}^2} \rho \ln \rho = 0$. Therefore the assumption $\int_{\mathbb{R}^2} |x|^2 \rho_m$ bounded is a necessary condition for the Fatou’s type Lemma (3.1) to hold true.

We need some supplementary lemmas to prove Lemma 3.2.

**Lemma 3.3.** Let $\{\rho_m\}$ be a sequence in $\mathbb{L}\ln\mathbb{L}(\mathbb{R}^2)$ such that

$$\int_{\mathbb{R}^2} \rho_m \ln \rho_m \leq C_0, \int_{\mathbb{R}^2} \rho_m = \beta, \int_{\mathbb{R}^2} |x|^2 \rho_m \leq C_0. \quad (3.2)$$  

Then there exists a $\rho \in L^1(\mathbb{R}^2, (1 + |x|^2)d^2x)$ such that (up to a subsequence) $\rho_m \rightharpoonup \rho$ weakly in $L^1(\mathbb{R}^2)$, i.e.,

$$\int_{\mathbb{R}^2} \rho_m g \to \int_{\mathbb{R}^2} \rho g, \text{ for all } g \in L^\infty(\mathbb{R}^2).$$
Proof. By Lemma 3.1, the assumption (3.2) implies that
\[ \int_{\mathbb{R}^2} |\rho_m| \ln |\rho_m| \leq C. \]
for some constant \( C \), and hence \( \int_{\mathbb{R}^2} [(1 + \rho_m) \ln(1 + \rho_m) - \rho_m] \) is uniformly bounded. Since \( \int_{\mathbb{R}^2} \rho_m = \beta \) by weak* compactness in \( L^1 \) there exists a finite measure \( \mu \) on \( \mathbb{R}^2 \) such that
\[ \int_{\mathbb{R}^2} \rho_m \phi \to \int_{\mathbb{R}^2} \phi \, d\mu, \text{ for all } \phi \in C_0(\mathbb{R}^2). \]
Furthermore, the uniform boundedness of \( \int_{\mathbb{R}^2} [(1 + \rho_m) \ln(1 + \rho_m) - \rho_m] \) implies \( \mu \) has a density \( \rho \in L^1_{\text{loc}}(\mathbb{R}^2) \). Now we claim that \( \int_{\mathbb{R}^2} |x|^2 \rho < +\infty \). To prove it we let \( \phi \in C_0(\mathbb{R}^2) \) such that \( \phi(x) = |x|^2 \) in \( B(0,R), 0 \leq \phi \leq |x|^2 \) in \( \mathbb{R}^2 \). Then by (3.2) and \( L^1 \) weak* convergence we get
\[ \int_{\{|x| < R\}} |x|^2 \rho \leq \int_{\mathbb{R}^2} \rho \phi = \lim_{m \to \infty} \int_{\mathbb{R}^2} \rho_m \phi \leq C_0. \]
Letting \( R \to \infty \) we reach at the desired claim. Moreover, the assumption \( \int_{\mathbb{R}^2} |x|^2 \rho_m \leq C_0 \) gives \( \int_{\mathbb{R}^2} \rho = \beta \). Therefore, by Portmanteau’s theorem
\[ \int_{\mathbb{R}^2} \rho_m \phi \to \int_{\mathbb{R}^2} \rho \phi, \quad (3.3) \]
for all bounded continuous functions \( \phi \) on \( \mathbb{R}^2 \). Using Lusin’s theorem and Tietz’s extension theorem we can extend this result to \( \phi \in L^\infty(\mathbb{R}^2) \).

Lemma 3.4. The set
\[ \mathcal{S} := \left\{ \rho \in L^1(\mathbb{R}^2) : \rho \geq 0, \int_{\mathbb{R}^2} \rho \ln \rho \leq \alpha, \int_{\mathbb{R}^2} \rho = \beta, \int_{\mathbb{R}^2} |x|^2 \rho \leq C_0 \right\} \]
is a weakly closed subset in \( L^1(\mathbb{R}^2) \).

Proof. We will show that the set \( \mathcal{S} \) is a convex and strongly closed subset of \( L^1(\mathbb{R}^2) \). Then by Mazur’s lemma it will imply the weakly closeness of \( \mathcal{S} \). Again by the convexity of \( t \ln t \) we only need to show that \( \mathcal{S} \) is strongly closed in \( L^1(\mathbb{R}^2) \). Let \( \{\rho_m\}_m \) be a sequence in \( L^1(\mathbb{R}^2) \) such that \( \rho_m \to \rho \) in \( L^1(\mathbb{R}^2) \). Let \( \rho_m^*, \rho^* \) be the symmetric decreasing rearrangement of \( \rho_m \) and \( \rho \) respectively. Then \( \rho_m^* \to \rho^* \) in \( L^1(\mathbb{R}^2) \) and up to a subsequence \( \rho_m \) (respectively \( \rho_m^* \)) converges point wise a.e. in \( \mathbb{R}^2 \). By strong convergence and Fatou’s lemma we have
\[ \int_{\mathbb{R}^2} \rho = \beta, \int_{\mathbb{R}^2} |x|^2 \rho \leq C_0. \]
Furthermore, by Lemma 3.1 and the point wise convergence we obtain
\[ \int_{\mathbb{R}^2} \rho |\ln \rho| < +\infty. \]
To conclude the proof of the lemma we will show \( \int_{\mathbb{R}^2} \rho^* \ln \rho^* \leq \alpha \). Using Fatou’s lemma we get
\[ \int_{B(0,R)} \rho^* \ln \rho^* \leq \liminf_{m \to \infty} \int_{B(0,R)} \rho_m^* \ln \rho_m^*, \]
for any \( R > 0 \). Now to estimate for \( |x| > R \) we will use the bound \( 0 \leq \rho^*(|x|) \leq \frac{\beta}{\pi |x|^2} \). The bound follows from
\[ \beta = \int_{\mathbb{R}^2} \rho = \int_{\mathbb{R}^2} \rho^* = 2\pi \int_0^\infty s \rho^*(s)ds \geq 2\pi \int_0^R s \rho^*(s)ds \geq \pi r^2 \rho^*(r). \]
Choosing $\epsilon \in (0, \frac{1}{2})$ and using $\ln(1/t) \leq 1/t$ for $t < 1$ we get, after multiplying by $\epsilon$ and using $\rho^*(x) < 1$ for sufficiently large $R$

$$
\int_{\{|x| > R\}} \rho^*_m \ln \rho^*_m \leq \frac{1}{\epsilon} \int_{\{|x| > R\}} \rho^*_m \frac{1}{(\rho^*_m)^\epsilon} = \frac{1}{\epsilon} \int_{\{|x| > R\}} (\rho^*_m)^{1-\epsilon},
$$

$$
= \frac{1}{\epsilon} \int_{\{|x| > R\}} \left(\frac{|x|^2 \rho^*_m}{|x|^{2(1-\epsilon)}}\right)^{1-\epsilon},
$$

$$
\leq \frac{1}{\epsilon} \left(\int_{\{|x| > R\}} |x|^2 \rho^*_m\right)^{1-\epsilon} \left(\int_{\{|x| > R\}} |x|^{2(1-\frac{1}{\epsilon})}\right)^\epsilon,
$$

$$
= O\left(\frac{1}{R^{2(\frac{1}{\epsilon} - 2)}}\right).
$$

Thus we obtain

$$
\int_{B(0, R)} \rho^*_m \ln \rho^*_m \leq \int_{\mathbb{R}^2} \rho^*_m \ln \rho^*_m + O\left(\frac{1}{R^{2(\frac{1}{\epsilon} - 2)}}\right),
$$

and hence

$$
\int_{B(0, R)} \rho^* \ln \rho^* \leq \liminf \int_{\mathbb{R}^2} \rho^*_m \ln \rho^*_m + O\left(\frac{1}{R^{2(\frac{1}{\epsilon} - 2)}}\right).
$$

Letting $R \to \infty$ we get the desired result. \hfill \square

**Proof of Lemma 3.2:**

*Proof.* Define $\alpha = \liminf \int_{\mathbb{R}^2} \rho_m \ln \rho_m + \epsilon$, where $\epsilon > 0$ is a small fixed number. Let $\rho_{m_k}$ be a subsequence such that $\lim \int_{\mathbb{R}^2} \rho_{m_k} \ln \rho_{m_k} = \liminf \int_{\mathbb{R}^2} \rho_m \ln \rho_m$. By Lemma 3.3, up to a subsequence $\rho_{m_k}$ converges to some $\rho$ weakly in $L^1(\mathbb{R}^2)$. Since for sufficiently large $k, \rho_{m_k} \in S$, which is weak $L^1$-closed by Lemma 3.4, we conclude that $\rho \in S$ and hence

$$
\int_{\mathbb{R}^2} \rho \ln \rho \leq \liminf \int_{\mathbb{R}^2} \rho_{m_k} \ln \rho_{m_k} + \epsilon.
$$

Since $\epsilon > 0$ is arbitrary the proof of the lemma is completed. \hfill \square

**Lemma 3.5.** Let $\rho \in L^1(\mathbb{R}^2)$ satisfies

$$
\int_{\mathbb{R}^2} \rho \ln \rho \leq C_0, \int_{\mathbb{R}^2} \rho = \beta, \int_{\mathbb{R}^2} |x|^2 \rho \leq C_0.
$$

Define

$$
u(x) = -\frac{1}{2\pi} \int_{\mathbb{R}^2} \ln |x - y| \rho(y) \, d^2y, \quad \text{for } x \in \mathbb{R}^2.
$$

Then there exists a constants $C, R$ depending only on $C_0$ and $\beta$ such that

$$
\left| \nu(x) + \frac{\beta}{2\pi} \ln |x| \right| \leq C, \quad \text{for all } |x| > R.
$$

*Proof.* The proof goes in the same line as in Chen and Li [CL93] with slight modifications. As in [CL93] we write

$$
u(x) = \frac{\beta}{2\pi} \int_{\mathbb{R}^2} \frac{\ln |x - y| - \ln |x|}{\ln |x|} \rho(y) \, d^2y = I_1 + I_2 + I_3,
$$
where the integral $I_1$ is over the domain $\{ |x - y| < 1 \}$, $I_2$ is over the domain $\{ |x - y| > 1, |y| \leq \frac{|x|}{2} \}$ and $I_3$ is over the domain $\{ |x - y| > 1, |y| > \frac{|x|}{2} \}$. We want to show that each $I_j$ is bounded by $C(\beta, C_0)/\ln |x|$. Now

$$|I_1| \leq \int_{\{ |x - y| < 1 \}} \rho(y) \, d^2y + \frac{1}{\ln |x|} \int_{\{ |x - y| < 1 \}} |\ln |x - y|| \rho(y) \, d^2y$$

(3.5)

Since $\{ |x - y| < 1 \} \subset \{ |y| > |x| - 1 \}$, and $\int_{\mathbb{R}^2} |x|^2 \rho \leq C_0$ the first integral in (3.5) is bounded by $C(\beta, C_0)/(|x| - 1)^2$. To estimate the second integral in (3.5) we divide it into two parts $\{ |x - y| > 1, \rho \leq 1 \}$ and $\{ |x - y| > 1, \rho > 1 \}$. Clearly,

$$\int_{\{ |x - y| < 1, \rho \leq 1 \}} |\ln |x - y|| \rho(y) \, d^2y \leq \frac{C(\beta, C_0)}{\ln |x|}.$$

Choose $\epsilon \in (0, 1)$ then

$$\int_{\{ |x - y| < 1, \rho > 1 \}} |\ln |x - y|| \rho(y) \, d^2y \leq \int_{\{ |x - y| < 1, \ln \rho < \epsilon \ln \frac{1}{|x - y|} \}} |\ln |x - y|| \rho(y) \, d^2y$$

$$+ \int_{\{ |x - y| < 1, \ln \rho > \epsilon \ln \frac{1}{|x - y|} \}} |\ln |x - y|| \rho(y) \, d^2y$$

$$\leq \int_{\{ |x - y| < 1 \}} \ln \frac{1}{|x - y|} e^{\epsilon \ln \frac{1}{|x - y|}}$$

$$+ \frac{1}{\epsilon} \int_{\{ |x - y| < 1 \}} \rho \ln \rho$$

$$\leq \int_{\{ |x - y| < 1 \}} \ln \frac{1}{|x - y|} \frac{1}{|x - y|^\epsilon}$$

$$+ \frac{1}{\epsilon} \int_{\{ |x - y| < 1 \}} \rho \ln \rho$$

$$\leq C(\beta, C_0).$$

Combining all we get the estimate

$$|I_1| \leq C(\beta, C_0) \left[ \frac{1}{\ln |x|} + \frac{1}{(|x| - 1)^2} \right].$$

To estimate $I_2$ we see that on the domain $\{ |x - y| > 1, |y| \leq \frac{|x|}{2} \}, |\ln |x - y| - \ln |x|| \leq 1$. Thus

$$|I_2| \leq \frac{1}{\ln |x|} \int_{|y| \leq \frac{|x|}{2}} \rho(y) \, d^2y \leq \frac{C(\beta)}{\ln |x|}.$$

Now on $I_3, \ln |x - y| \geq 0, |x - y| \leq 3|y|$ and hence $|\ln |x - y| - \ln |x|| \leq \ln 3|y| + \ln |x|$. Therefore

$$|I_3| \leq \frac{1}{\ln |x|} \int_{|y| \geq \frac{|x|}{3}} \ln 3|y| \rho(y) \, d^2y$$

$$+ \int_{|y| \geq \frac{|x|}{3}} \rho(y) \, d^2y$$

$$\leq C(\beta, C_0) \left[ \frac{1}{|x| \ln |x|} + \frac{1}{|x|^2} \right].$$

We end this section with the following compactness lemma whose proof can be found in [ST13].
1.1.1.3. Theorem A. Suppose we have a sequence \( \{u_m\} \subset H^1(B(0,2R)) \) of weak solutions to
\[-\Delta u_m = f_m, \quad \text{in } B(0,2R),\]
and \( \{f_m\} \subset L \ln L(B(0,2R)) \). Suppose there exists a constant \( C < +\infty \) such that
\[ \|u_m\|_{H^1(B(0,2R))} + \|f_m\|_{L \ln L(B(0,2R))} \leq C. \]
Then there exists \( u \in H^1_{\text{loc}}(B(0,2R)) \) such that
\[ \|u_m - u\|_{H^1(B(0,R))} \to 0, \quad \text{as } m \to \infty. \]

In [ST13], the authors actually proved the above compactness theorem for \( R = \frac{1}{2} \) but for more general inhomogeneity \( \Omega_m, \nabla u_m + f_m \) under some smallness condition on \( \Omega_m \). For our purpose we can take \( \Omega_m \equiv 0 \), and the general \( R \) can be dealt with through a simple scaling argument. To be meticulous, define \( \tilde{u}_m(x) = u_m(2Rx) \) and \( \tilde{f}_m(x) = (2R)^2 f_m(2Rx) \). Then one can easily verify that (3.6),(3.7) holds with \( u_m, f_m \) replaced by \( \tilde{u}_m, \tilde{f}_m \) in the domain \( B(0,1) \). Hence by compactness theorem there exists \( \tilde{u} \in H^1_{\text{loc}}(B(0,1)) \) such that \( \tilde{u}_m \to \tilde{u} \) in \( H^1(B(0,\frac{1}{2})) \). Scaling back to the original variable we see that \( u_m(\cdot) \to u(\cdot) := \tilde{u}(\frac{2x}{R}) \) in \( H^1(B(0,R)) \). We refer the reader to [ST13] for more details.

4. Existence of Minimizers: Sub-critical Case

In this section we assume \( \beta \) is sub-critical (Definition 1.1).

Theorem 4.1. If \( \beta \) is sub-critical then for all \( \nu = (v_1,\cdots,v_n) \in (\mathbb{R}^2)^n \) there exists a minimizer of \( F_\nu \) on \( \Gamma^\beta \).

Proof. Let \( \rho^m = (\rho_1^m,\cdots,\rho_n^m) \) be a minimizing sequence for \( F_\nu \) on \( \Gamma^\beta \). Since \( \beta \) is sub-critical we can choose \( \epsilon \in (0,\frac{1}{2}) \) small such that
\[ \sum_{i \in J} \beta_i \left( 8\pi - \sum_{j \in J} (a_{ij} + \epsilon) \beta_j \right) > 0, \quad \text{for all } \emptyset \neq J \subset I. \]

Step 1: \( \int_{\mathbb{R}^2} |x - v_i|^2 \rho_i^m \) is uniformly bounded by some constant \( C_0 \).

By Theorem 2.1
\[ \sum_{i=1}^n \int_{\mathbb{R}^2} \rho_i^m \ln \rho_i^m + \frac{1}{4\pi} \sum_{i=1}^n \sum_{j=1}^n a_{ij} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \rho_i^m(x) \ln |x-y| \rho_j^m(y) \]
\[ + \sum_{i=1}^n (\frac{1}{2} - \epsilon) \int_{\mathbb{R}^2} |x - v_i|^2 \rho_i^m \geq -C \]
which implies \( F_\nu(\rho^m) - \epsilon \sum_{i=1}^n \int_{\mathbb{R}^2} |x - v_i|^2 \rho_i^m \geq -C \). Since along a minimizing sequence \( F_\nu(\rho^m) \) is bounded above, the conclusion of Step 1 is proved.

Step 2: \( \rho_i^m \) are uniformly bounded in \( L \ln L \).

Define
\[ I_{ij}^m := \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \rho_i^m(x) \ln |x-y| \rho_j^m(y). \]

Using Step 1 and the following inequality
\[ \ln |x-y| \leq \frac{1}{2} \ln(1 + |x|^2) + \frac{1}{2} \ln(1 + |y|^2), \quad |x|^2 > \ln(1 + |x|^2) \]

...
we see that \( I_{ij}^m \leq \frac{C}{|x|} (\beta_i + \beta_j) \). Since \( \beta \) satisfies (4.1) we obtain by Theorem 2.1
\[
\sum_{i=1}^{n} \int_{\mathbb{R}^2} \rho_i^m \ln \rho_i^m + \frac{1}{4\pi} \sum_{i=1}^{n} \sum_{j=1}^{n} (a_{ij} + \epsilon) \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \rho_i^m(x) \ln |x - y| \rho_j^m(y) \\
+ \sum_{i=1}^{n} \frac{1}{2} \int_{\mathbb{R}^2} |x - v_i|^2 \rho_i^m \geq -C.
\]
Therefore we have
\[
F_v(\rho^m) + \frac{\epsilon}{4\pi} \sum_{I_{ij}^m > 0} I_{ij}^m - \frac{\epsilon}{4\pi} \sum_{I_{ij}^m < 0} |I_{ij}^m| \geq -C.
\]
Since along a minimizing sequence \( F_v(\rho^m) \) is bounded we obtain \( \sum_{I_{ij}^m < 0} |I_{ij}^m| \) is uniformly bounded. Hence \( \sum_{i=1}^{n} \int_{\mathbb{R}^2} \rho_i^m \ln \rho_i^m \) is upper bounded and hence by Lemma 3.1 we get the uniform bound of \( \rho^m \) in \( L^\infty \ln L \).

Step 3: Existence of a limit.

By Lemma 3.2 there exists \( \rho_i \in L^\infty \ln L(\mathbb{R}^2) \) such that up to a subsequence \( \rho_i^m \to \rho_i \) in the topology of \( L^\infty \ln L \) and satisfies the inequality
\[
\int_{\mathbb{R}^2} \rho_i \ln \rho_i \leq \liminf \int_{\mathbb{R}^2} \rho_i^m \ln \rho_i^m, \text{ for all } i.
\]
Furthermore, it also follows from the proof of Lemma 3.2 that
\[
\sum_{i=1}^{n} \frac{1}{2} \int_{\mathbb{R}^2} |x - v_i|^2 \rho_i \leq \liminf \sum_{i=1}^{n} \int_{\mathbb{R}^2} \rho_i^m \ln |x - v_i|^2, \quad \int_{\mathbb{R}^2} \rho_i = \beta_i,
\]
and hence \( \rho := (\rho_1, \cdots, \rho_n) \in \Gamma^\beta \). To complete the proof of the theorem we need to show that
\[
\int_{\mathbb{R}^2} \rho_i^m u_j^m \to \int_{\mathbb{R}^2} \rho_i u_j \text{ for all } 1 \leq i, j \leq n,
\]
where \( u_j, u_j^m \) are defined by (3.4) via \( \rho_j, \rho_j^m \) respectively.

By Lemma 3.5 we have for \( R \) large
\[
\left| \int_{\{|x| > R\}} \rho_i^m u_j^m \right| \leq C \left[ \int_{\{|x| > R\}} \rho_i^m \ln |x| + \int_{\{|x| > R\}} \rho_i^m \right]
\leq \frac{C}{R}
\]
(4.5)
For \( \{|x| \leq R\} \) we will use Theorem A to prove the convergence. For that we need to show that \( u_i^m \in H^1_{loc}(\mathbb{R}^2) \) and \( ||u_i^m||_{L^2(B(0,2R))} \) is uniformly bounded for all \( i = 1, \cdots, n \):
\[
\int_{\{|x| < 2R\}} |u_i^m| \leq \frac{1}{2\pi} \int_{\{|x| < 2R\}} \int_{\mathbb{R}^2} |\ln |x - y|| \rho_i^m(y) \ d^2y \ d^2x
\leq \int_{\{|x| < 2R\}} \int_{\{|y| < 4R\}} |\ln |x - y|| \rho_i^m(y) \ d^2y \ d^2x
+ \int_{\{|x| < 2R\}} \int_{\{|y| > 4R\}} |\ln |x - y|| \rho_i^m(y) \ d^2y \ d^2x
\leq \int_{\{|y| < 4R\}} \rho_i^m(y) \int_{\{|x| < 2R\}} |\ln |x - y|| \ d^2x \ d^2y + C(R) \int_{\mathbb{R}^2} |y|^2 \rho_i^m(y) \ d^2y
\leq C(R).
\]
By compactness Theorem A, there exists \( \tilde{u}_i \in H^1(B(0, R)) \) such that \( u_i^m \) converges to \( u_i \) in \( H^1(B(0, R)) \). Therefore \( u_i^m \) converges to \( \tilde{u}_i \) in the strong topology of Orlicz space determined by the \( N \)-function \( (e^t - t - 1) \). By duality

\[
\int_{B(0, R)} \rho_i^m u_j^m \to \int_{B(0, R)} \rho_i u_j.
\]  
(4.6)

Hence by (4.4) and (4.6) we see that

\[
\int_{\mathbb{R}^2} \rho_i^m u_j^m \to \int_{\mathbb{R}^2} \rho_i u_j, \text{ for all } i, j.
\]  
(4.7)

Therefore by (4.2), (4.3) and (4.7) we have \( \rho \in \Gamma^\beta \) and

\[
\mathcal{F}_v(\rho) \leq \lim \inf \mathcal{F}_v(\rho^m) = \inf_{\Gamma^\beta} \mathcal{F}_v.
\]

This completes the proof of the theorem.

\[\square\]

**Remark 3.** It follows from the proof of Theorem 4.1 that if a minimizing sequence is bounded in the \( \mathbb{L} \ln \mathbb{L} \) topology and has bounded second moment then the minimizing sequence converges and the limit is a minimizer. More precisely, if \( \rho^m \) is minimizing sequence that satisfies

\[
\sum_{i=1}^{n} \int_{\mathbb{R}^2} \rho_i^m \ln \rho_i^m \leq C_0, \text{ and } \sum_{i=1}^{n} \int_{\mathbb{R}^2} |x|^2 \rho_i^m \leq C_0
\]

for some constant \( C_0 \) independent of \( m \) then there exists \( \rho_0 \in \Gamma^\beta \) such that \( \rho^m \to \rho_0 \) in the topology of \( \mathbb{L} \ln \mathbb{L} \) and \( \mathcal{F}_v(\rho_0) = \inf_{\rho \in \Gamma^\beta} \mathcal{F}_v(\rho) \).

5. The Critical case

Recall the definition of the functional \( \mathcal{F}_v(\rho) \) (1.17). In this section we assume the critical case

\[
\Lambda_I(\beta) = 0 , \Lambda_J(\beta) > 0 \ \forall J \subset I, J \neq I, \emptyset.
\]  
(5.1)

**Lemma 5.1.** If \( \beta \) satisfies (5.1) then \( \mathcal{F}_0 \) does not attain its infimum on \( \Gamma^\beta \).

**Proof.** Let \( \rho_m \) be a minimizing sequence. Define

\[
\tilde{\rho}_i^m(x) = R^2 \rho_i^m(Rx), \ x \in \mathbb{R}^2, R > 0.
\]

Direct computation gives

\[
\mathcal{F}_0(\tilde{\rho}_m) = \mathcal{F}_0(\rho^m) + \left( \frac{1}{R^2} - 1 \right) \sum_{i=1}^{n} \frac{1}{2} \int_{\mathbb{R}^2} |x|^2 \rho_i^m.
\]

Thus we have (using \( \lim \inf(a_m + b_m) = \lim a_m + \lim \inf b_m \), if \( a_m \) converges)

\[
\inf_{\rho \in \Gamma^\beta} \mathcal{F}_0(\rho) \leq \lim \inf \mathcal{F}_0(\rho^m) + \lim \inf \left( \frac{1}{R^2} - 1 \right) \sum_{i=1}^{n} \frac{1}{2} \int_{\mathbb{R}^2} |x|^2 \rho_i^m.
\]

which gives

\[
\lim \inf \left( \frac{1}{R^2} - 1 \right) \sum_{i=1}^{n} \frac{1}{2} \int_{\mathbb{R}^2} |x|^2 \rho_i^m \geq 0.
\]  
(5.2)

Choosing \( R < 1 \) in (5.2) we get \( \lim \inf \sum_{i=1}^{n} \frac{1}{2} \int_{\mathbb{R}^2} |x|^2 \rho_i^m \) \( \geq 0 \), while \( R > 1 \) gives \( \lim \sup \left( \sum_{i=1}^{n} \frac{1}{2} \int_{\mathbb{R}^2} |x|^2 \rho_i^m \right) \leq 0 \) and hence

\[
\lim \left( \sum_{i=1}^{n} \frac{1}{2} \int_{\mathbb{R}^2} |x|^2 \rho_i^m \right) = 0.
\]
Therefore all the components of \( \rho^m \) concentrates at the origin and hence there does not exists a minimizer of \( F_0 \) on \( \Gamma^B \).

5.1. A Functional inequality:

**Lemma 5.2.** The following inequality holds true

\[
\inf_{\rho \in \Gamma^B} F_v(\rho) \leq \inf_{\rho \in \Gamma^B} F_0(\rho) + \min_{x_0 \in \mathbb{R}^2} \sum_{i=1}^n \frac{1}{2} \beta_i |x_0 - v_i|^2. \tag{5.3}
\]

**Proof.** Let \( \rho_m \) be a minimizing sequence for \( \inf_{\rho \in \Gamma^B} F_v(\rho) \). Define for \( x_0 \in \mathbb{R}^2 \),

\[
\tilde{\rho}_m(x) = \rho_m(x - x_0), \quad x \in \mathbb{R}^2.
\]

Then a direct computation gives

\[
\inf_{\rho \in \Gamma^B} F_v(\rho) \leq F_v(\tilde{\rho}_m) = F_0(\rho^m) + \sum_{i=1}^n \frac{1}{2} \int_{\mathbb{R}^2} (|x + x_0 - v_i| - |x|^2) \rho^m_i \\
= F_0(\rho^m) + \sum_{i=1}^n \frac{1}{2} \int_{\mathbb{R}^2} (x_0 - v_i) \rho^m_i + \sum_{i=1}^n \frac{1}{2} \beta_i |x_0 - v_i|^2. \tag{5.4}
\]

Since by Lemma 5.1 \( \lim \left( \sum_{i=1}^n \frac{1}{2} \int_{\mathbb{R}^2} |x|^2 \rho^m_i \right) = 0 \) we get

\[
\sum_{i=1}^n \frac{1}{2} \int_{\mathbb{R}^2} (x_0 - v_i) \rho^m_i \leq \sum_{i=1}^n \frac{1}{2} \beta_i |x_0 - v_i| \left( \int_{\mathbb{R}^2} |x|^2 \rho^m_i \right)^{\frac{1}{2}} \to 0,
\]

as \( m \to \infty \). Therefore letting \( m \to \infty \) in (5.4) we get

\[
\inf_{\rho \in \Gamma^B} F_v(\rho) \leq \inf_{\rho \in \Gamma^B} F_0(\rho) + \sum_{i=1}^n \frac{1}{2} \beta_i |x_0 - v_i|^2.
\]

Since \( x_0 \in \mathbb{R}^2 \) is arbitrary the proof of the lemma is completed. \( \square \)

**Remark 4.** If the equality occurs in (5.3) then every minimizing sequence for \( F_v \) on \( \Gamma^B \) is also a minimizing sequence for \( F_0 \) on \( \Gamma^B \). Hence, for any such minimizing sequence we get \( \sum_{i=1}^n \int_{\mathbb{R}^2} \rho^m_i \ln \rho^m_i \to \infty \). Otherwise, as in Theorem 4.1 (Remark 3) we can prove the existence of a minimizer of \( F_0 \) on \( \Gamma^B \), which contradicts Lemma 5.1.

6. Blow up analysis: Brezis Merle type argument

We pose the following alternatives

**Proposition 6.1.** Suppose \( \beta \) satisfies (5.1), then one of the following alternative holds: either

(a) there exists a minimizer of \( F_v \) over \( \Gamma^B \), or
(b) equality holds in the functional inequality (5.3).

For the proof of this Proposition will need the two Lemmas below:

Let \( \beta_m \) be a sequence such that \( \beta_m \not\rightarrow \beta \) and satisfies

\[
\Lambda_J(\beta_m) > 0, \quad \text{for all } \phi \not\in J \subset I.
\]

One can indeed choose such sequence \( \beta_m \), see for example [CSW97, Lemma (5.1) and equation (5.4)]. By Theorem 4.1 the infimum \( \inf_{\rho \in \Gamma^B} F_v(\rho) \) is attained. Let us denote the minimizer by \( \rho^m \in \Gamma^B \).
Lemma 6.2. The following holds
\[ \sup_m \int_{\mathbb{R}^2} |x|^2 \rho_i^m < +\infty, \text{ for all } i. \]

Proof. For each fixed \( m \) define
\[ \tilde{\rho}_i^m(x) = R_m^2 \rho_i^m(R_m x), \]
where \( R_m > 0 \) to be decided later. A direct computations gives
\[ F_v(\tilde{\rho}_m) = F_v(\rho^m) + f_m(R_m), \]
where \( f_m : (0, \infty) \to \mathbb{R} \) is defined by
\[ f_m(t) = a_m \ln t + \frac{b_m}{2t^2} + \frac{2c_m}{t} + d_m, \]
and \( a_m, b_m, c_m, d_m \) are defined as follows:
\[ a_m = \frac{1}{4\pi} \Lambda_f(\beta_m) \to 0, \quad 2c_m = -\sum_{i=1}^n \int_{\mathbb{R}^2} \langle x, v_i \rangle \rho_i^m, \]
\[ b_m = \sum_{i=1}^n \int_{\mathbb{R}^2} |x|^2 \rho_i^m, \quad d_m = -\frac{1}{2} \sum_{i=1}^n \int_{\mathbb{R}^2} |x - v_i|^2 \rho_i^m + \frac{1}{2} |v_i|^2 \beta_i^m. \]

One can easily verify that the following inequalities hold:
\[ |c_m| \leq \left( \sup_i \frac{\sqrt{n}}{2} |v_i| \beta_i^\frac{1}{2} \right) b_m^\frac{1}{2}, \]  
\[ -\frac{1}{4} \sum |v_i|^2 \beta_i^m - d_m \leq b_m \leq -4d_m + 4 \sum |v_i|^2 \beta_i^m. \]

Since \( \rho^m \) minimizes \( F_v \) over \( \Gamma_{\beta_m} \) we have
\[ \inf_{\rho \in \Gamma_{\beta_m}} F_v(\rho) \leq F_v(\tilde{\rho}_m) = F_v(\rho^m) + f_m(R_m) = \inf_{\rho \in \Gamma_{\beta_m}} F_v(\rho) + f_m(R_m), \]
and therefore \( f_m(R_m) \geq 0 \). We now choose \( R_m \) such that \( f_m(R_m) \) is the minimum of \( f_m(t) \) over \( (0, \infty) \). Since \( f_m(t) \to \infty \) as \( t \to 0 \) and \( t \to \infty \) we have \( R_m > 0 \) and satisfies \( f_m'(R_m) = 0 \), i.e.,
\[ \frac{a_m}{R_m} - \frac{b_m}{R_m^2} - \frac{2c_m}{R_m^3} = 0. \]

Therefore we have from (6.3)
\[ f_m(R_m) = a_m \ln R_m + a_m - \frac{b_m}{2R_m^2} + d_m \]
\[ = a_m \ln a_m R_m - a_m \ln a_m + a_m - \frac{b_m}{2R_m^2} + d_m. \]

Now assume, in contradiction, that \( b_m \to \infty \). Then we deduce from (6.3) and the bound on \( c_m \) (6.1) that \( a_m R_m \leq C b_m^\frac{1}{2} \), for some positive constant \( C \) independent of \( m \) and hence we have
\[ a_m \ln a_m R_m \leq a_m \ln C + \frac{a_m}{2} \ln b_m, \]

Since \( b_m \geq 0 \), (6.4) and (6.5) and the estimates of \( d_m \) in terms of \( b_m \) gives
\[ f_m(R_m) \leq (a_m \ln C - a_m \ln a_m + a_m) + \frac{a_m}{2} \ln b_m + d_m. \]
Proof. We first prove inequality (6.6). Let $\rho \in \Gamma^\beta$ be a fixed element. Choose $R_i^m > 0$ such that $\int_{B(0,R_i^m)} \rho_i = \tilde{\rho}_i^m$ and define $\tilde{\rho}_i^m = \rho_i \chi_{B(0,R_i^m)}$. Then $\rho^m \in \Gamma^{\beta_m}$ and by dominated convergence theorem
\[
\lim_{m \to \infty} \mathcal{F}_v(\rho^m) = \mathcal{F}_v(\rho).
\]
Thus we have
\[
\lim_{m \to \infty} \inf_{\rho \in \Gamma^{\beta_m}} \mathcal{F}_v(\rho) \leq \lim_{m \to \infty} \mathcal{F}_v(\rho^m) = \mathcal{F}_v(\rho).
\]
Since $\rho \in \Gamma^\beta$ is arbitrary, we have proved the inequality (6.6). Next we prove (6.7). Thanks to (6.6), we only need to show $\lim_{m \to \infty} \inf_{\rho \in \Gamma^{\beta_m}} \mathcal{F}_0(\rho) \geq \inf_{\rho \in \Gamma^\beta} \mathcal{F}_0(\rho)$. This step is a little bit technical and therefore we divide the proof into several parts.

(1) By Theorem 4.1, there exists $\rho^m \in \Gamma^{\beta_m}$ such that
\[
\mathcal{F}_0(\rho^m) = \inf_{\rho \in \Gamma^{\beta_m}} \mathcal{F}_0(\rho).
\]
Furthermore, $\tilde{\rho}_i^m$ are radially symmetric and decreasing function of $r = |x|$. By abuse of notations, we will also denote the radial function by $\tilde{\rho}_i^m(r)$.

(2) A simple adoption of the proof of Lemma 5.1 gives $\int_{\mathbb{R}^2} |x|^2 \tilde{\rho}_i^m(x) \to 0$ as $m \to \infty$. Therefore for any $r \in (0, \infty)$
\[
o_m(1) = \int_{\mathbb{R}^2} |x|^2 \tilde{\rho}_i^m(x) = 2\pi \int_0^\infty s^3 \tilde{\rho}_i^m(s) \, ds \geq 2\pi \int_0^r s^3 \tilde{\rho}_i^m(s) \, ds \geq \frac{\pi}{2} r^4 \tilde{\rho}_i^m(r),\]
where $o_m(1)$ denotes a quantity going to 0 as $m \to \infty$. Thus we have $\sup_{r \in (0, \infty)} r^4 \tilde{\rho}_i^m(r) = o_m(1)$ as $m \to \infty$.

A similar argument using $\int_{\mathbb{R}^2} \rho_i^m = \beta_i^m$ gives $\sup_{r \in (0, \infty)} r^2 \rho_i^m(r) \leq \frac{\beta_i^m}{2}$.

(3) Let $\phi$ be a smooth, nonnegative, radial, compactly supported function such that $\int_{\mathbb{R}^2} \phi = 1$. Define $\epsilon_i^m = \beta_i - \beta_i^m > 0$ and
\[
\tilde{\rho}_i^m(x) = \rho_i^m(x) + \epsilon_i^m \phi(x), \quad x \in \mathbb{R}^2.
\]
Then $\tilde{\rho}_i^m \in \Gamma^\beta$ for all $m$ and hence $\inf_{\rho \in \Gamma^\beta} \mathcal{F}_0(\rho) \leq \mathcal{F}_0(\tilde{\rho}_i^m)$. Now we will estimate each term of $\mathcal{F}_0(\tilde{\rho}_i^m)$ and show that
\[
\mathcal{F}_0(\tilde{\rho}_i^m) = \mathcal{F}_0(\rho^m) + o_m(1).
\]
On the set \( \{ \rho \leq \epsilon \} \) we see that \( \rho \) is bounded on any compact subset of \([0, \infty)\). Clearly we have

\[
\int B(0, \delta_m) \left( \rho_i^m \ln \rho_i^m - \tilde{\rho}_i^m \ln \rho_i^m \right) \chi(\rho_i^m \leq 2) = o_m(1),
\]

because \( t \ln t \) is bounded on any compact subset of \([0, \infty)\). Now using mean value theorem we get

\[
\int B(0, \delta_m) \left( \rho_i^m \ln \rho_i^m - \tilde{\rho}_i^m \ln \rho_i^m \right) \chi(\rho_i^m > 2) \]

\[
= 2\pi \epsilon_i^{(m)} \int_0^1 \int_0^{\delta_m} \left[ 1 + \ln(\rho_i^m(r) + t\epsilon_i^{(m)} \phi(r)) \right] \phi(r) \chi(\rho_i^m > 2) \, dt \, dr
\]

\[
= o_m(1) + 2\pi \epsilon_i^{(m)} \int_0^1 \int_0^{\delta_m} r \ln(\rho_i^m(r) + t\epsilon_i^{(m)} \phi(r)) \phi(r) \chi(\rho_i^m > 2) \, dr \, dt
\]

On the set \( \{ \rho_i^m > 2 \} \), we have \( \rho_i^m(r) + t\epsilon_i^{(m)} \phi(r) > 1 \) for \( m \) large enough. Moreover, using the estimate of (2) we see that \( \rho_i^m(r) + t\epsilon_i^{(m)} \phi(r) \leq C \) where \( C \) is some positive constant. Therefore \( 0 \leq r \ln(\rho_i^m(r) + t\epsilon_i^{(m)} \phi(r)) \leq r \ln C \) and hence

\[
2\pi \epsilon_i^{(m)} \int_0^1 \int_0^{\delta_m} r \ln(\rho_i^m(r) + t\epsilon_i^{(m)} \phi(r)) \phi(r) \chi(\rho_i^m > 2) \, dr \, dt = o_m(1),
\]

which gives

\[
\int B(0, \delta_m) \left( \tilde{\rho}_i^m \ln \tilde{\rho}_i^m - \rho_i^m \ln \rho_i^m \right) \chi(\rho_i^m > 2) = o_m(1).
\]

Now let us estimate \( \int B(0, \delta_m) \rho_i^m \ln \tilde{\rho}_i^m \).

\[
\left| \int B(0, \delta_m) \rho_i^m \ln \tilde{\rho}_i^m \right| = \left| 2\pi \int_{\delta_m}^{\infty} r \rho_i^m(r) \ln \rho_i^m(r) \, dr \right|
\]

\[
\leq 2\pi \int_{\delta_m}^{\infty} \frac{\ln(\rho_i^m(r))}{r^3} \, dr + 8\pi \int_{\delta_m}^{\infty} \frac{\rho_i^m(r) \ln(\rho_i^m(r))}{r^3} \, dr
\]

\[
\leq 2\pi |k_m \ln k_m| \int_{\delta_m}^{\infty} \frac{dr}{r^3} + 8\pi |k_m| \int_{\delta_m}^{\infty} \frac{\ln r \, dr}{r^3}
\]

\[
\leq \frac{2\pi |k_m \ln k_m|}{\delta_m^2} + C \frac{k_m}{\delta_m^{2-\epsilon}}, \text{ for some } \epsilon > 0
\]

\[
= o_m(1).
\]

In an entirely similar way we can verify that \( \left| \int B(0, \delta_m) \tilde{\rho}_i^m \ln \tilde{\rho}_i^m \right| = o_m(1) \), and hence we have proved (6.8).

(5) Next we estimate

\[
\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \tilde{\rho}_i^m(x) \ln |x - y| \tilde{\rho}_j^m(y)
\]
Combining (6.8), (6.9) and (6.10) we get
\[ \inf_{\rho \in \Gamma^m} \mathcal{F}_0(\rho) \leq \mathcal{F}_0(\tilde{\rho}_m) = \mathcal{F}_0(\rho^m) + o_m(1) = \inf_{\rho \in \Gamma^\beta_m} \mathcal{F}_0(\rho) + o_m(1). \]

Letting \( m \to \infty \), we reach at the desired conclusion. This completes the proof of the lemma. \( \square \)

6.1. Proof of Proposition 6.1. Recall that \( \rho^m \) is a minimizer of \( \mathcal{F}_\nu \) over \( \Gamma^\beta_m \), where \( \beta_m \not\succ \beta \). Define the Newtonian potentials
\[ u_i^m(x) = -\frac{1}{2\pi} \int_{\mathbb{R}^2} \ln |x-y| \rho_i^m(y) \, dy, \quad x \in \mathbb{R}^2. \]

By variational principle and Lemma 6.2, \( u_i^m \) satisfies the following equation:
\[ \begin{cases} -\Delta u_i^m(x) = \mu_i^m e^{\sum_{j=1}^n a_{ij} u_j^m(x)} - \frac{1}{2} |x-v_i|^2, & \text{in } \mathbb{R}^2, \\ \mu_i^m \int_{\mathbb{R}^2} e^{\sum_{j=1}^n a_{ij} u_j^m(x)} = \beta_i^m, \\ \mu_i^m \int_{\mathbb{R}^2} |x|^2 e^{\sum_{j=1}^n a_{ij} u_j^m(x)} - \frac{1}{2} |x-v_i|^2 \leq C_0, \end{cases} \]

where \( C_0 \) is a constant independent of \( m \). Define
\[ v_i^m(x) = \ln \mu_i^m + \sum_{j=1}^n a_{ij} u_j^m(x), \quad x \in \mathbb{R}^2. \]

Let us consider the two cases:

Case (A): Suppose there exists \( R > 0 \) such that
\[ \max_{1 \leq i \leq n} \sup_{x \in B(0,R)} u_i^m(x) \to \infty, \quad \text{as } m \to \infty. \]  

Case (B): For any \( R > 0 \) there exists a constant \( C(R) \) such that
\[ \max_{1 \leq i \leq n} \sup_{x \in B(0,R)} v_i^m(x) \leq C(R). \]

We first prove:

Lemma 6.4. Under the assumption of Case (A), the following equality holds:
\[ \inf_{\rho \in \Gamma^\beta} \mathcal{F}_\nu(\rho) = \inf_{\rho \in \Gamma^\beta_m} \mathcal{F}_0(\rho) + \min_{x_0 \in \mathbb{R}^2} \sum_{i=1}^n \frac{1}{2} \beta_i |x_0 - v_i|^2. \]  

Proof. By definition \( v_i^m, 1 \leq i \leq n \) satisfies the equation
\[ \begin{cases} -\Delta v_i^m(x) = \sum_{j=1}^n a_{ij} e^{v_j^m(x)} - \frac{1}{2} |x-v_i|^2, & \text{in } \mathbb{R}^2, \\ \int_{\mathbb{R}^2} e^{v_i^m} - \frac{1}{2} |x-v_i|^2 = \beta_i^m, \\ \int_{\mathbb{R}^2} |x|^2 e^{v_i^m} - \frac{1}{2} |x-v_i|^2 \leq C_0. \end{cases} \]
Furthermore, the following relation holds:
\[
\rho^m_i(x) = \mu^m_i e^{\sum_{j=1}^n a_{ij} v^m_j(x) - \frac{1}{2} |x - v_i|^2} = e^{\nu^m_i(x) - \frac{1}{2} |x - v_i|^2}, \quad x \in \mathbb{R}^2. \tag{6.13}
\]
After passing to a subsequence if necessary we may assume the supremum in (6.11) is attained by \(v^m_1\) for all \(m\). That is, there exists \(x_m \in B(0, \bar{R})\) such that
\[
v^m_1(x_m) = \max_i \sup_{x \in B(0, \bar{R})} v^m_i(x) \to \infty, \quad \text{as } m \to \infty.
\]
Let \(x_m \to x_0\) for some \(x_0 \in \overline{B(0, \bar{R})}\), and choose a \(\bar{R} > 0\) large enough so that \(\overline{B(0, \bar{R})} \subset B(x_0, \bar{R})\). Since \(v^m_1(x_m) \to \infty\) we have
\[
\sup\{v^m_i(x) + 2 \ln(\bar{R} - |x - x_0|) : x \in B(x_0, \bar{R}), 1 \leq i \leq n\} \to \infty, \quad \text{as } m \to \infty. \tag{6.14}
\]
Again after passing to a subsequence we may assume \(y_m \in B(x_0, \bar{R})\) be the point and \(i_0\) be the index such that the supremum in \((6.14)\) is attained for all \(m\). Since \(2 \ln(\bar{R} - |x - x_0|)\) is bounded above on \(B(x_0, \bar{R})\) we have \(v^m_{i_0}(y_m) \to \infty\).

Define \(\delta_m = e^{-\frac{v^m_{i_0}(y_m)}{2}}\), then \(\delta_m \to 0\) and it follows from \((6.14)\) that
\[
\left(\frac{\bar{R} - |y_m - x_0|}{\delta_m}\right) \to \infty, \quad \text{as } m \to \infty. \tag{6.15}
\]
Now define
\[
\tilde{v}^m_i(x) = v^m_i(y_m + \delta_m(x - x_0)) + 2 \ln \delta_m.
\]
We note that \(\tilde{v}^m_{i_0}(x_0) = 0\) for all \(m\). Furthermore, it follows from \((6.15)\) that for any \(M > 0\) fixed and \(x \in B(x_0, M), y_m + \delta_m(x - x_0) \in B(x_0, \bar{R})\) for large \(m\). Now \(\tilde{v}^m_i(x)\) satisfies the equation
\[
\begin{cases}
-\Delta \tilde{v}^m_i(x) = \sum_{j=1}^n a_{ij} e^{\tilde{v}^m_j(x) - \frac{1}{2} |y_m + \delta_m(x - x_0) - v_j|^2} \quad \text{in } \mathbb{R}^2, \\
\int_{\mathbb{R}^2} e^{\tilde{v}^m_i(x) - \frac{1}{2} |y_m + \delta_m(x - x_0) - v_i|^2} = \beta^m_i.
\end{cases} \tag{6.16}
\]
Let \(y_m \to y_0 \in B(x_0, \bar{R})\). Since \(\tilde{v}^m_{i_0}(x_0) = 0\) either \(\tilde{v}^m_i\) converges to some \(\tilde{v}_i\) in \(C^0_{\text{loc}}(\mathbb{R}^2)\) for all \(i\) or \(\tilde{v}^m_i\) converges to \(-\infty\) uniformly on compact subsets of \(\mathbb{R}^2\) for some \(i \neq i_0\).

Let \(I' \subset I\) be the set of indices such that \(\tilde{v}_i \neq -\infty\) iff \(i \in I'\). Then \(\tilde{v}^m_i\) converges to \(\tilde{v}_i\) in \(C^0_{\text{loc}}(\mathbb{R}^2)\) for \(i \in I'\) and, by \((6.16)\)
\[
\begin{cases}
-\Delta \tilde{v}_i = \sum_{j \in I'} a_{ij} e^{\tilde{v}_j - \frac{1}{2} |y_0 - v_j|^2} \quad \text{in } \mathbb{R}^2, \\
\int_{\mathbb{R}^2} e^{\tilde{v}_i - \frac{1}{2} |y_0 - v_i|^2} = \beta_i,
\end{cases} \tag{6.17}
\]
Letting \(z_i(x) = \tilde{v}_i(x) - \frac{1}{2} |y_0 - v_i|^2\) we obtain
\[
\begin{cases}
-\Delta z_i = \sum_{j \in I'} a_{ij} e^{z_j} \quad \text{in } \mathbb{R}^2, \\
\int_{\mathbb{R}^2} e^{z_i} = \tilde{\beta}_i.
\end{cases} \tag{6.18}
\]
holds for \(i \in I'\) for some \(\tilde{\beta}_i \leq \beta_i\).

A necessary condition for the existence of solution to \((6.18)\) is \(\Lambda_f(\tilde{\beta}) = 0\) ([CSW97], see also [LZ11, PT14]). Since we assumed \(\Lambda_f(\beta) = 0\) this implies \(I' = I\) and \(\beta = \beta_i\). (see [CSW97]).

It follows that, in Case (A), \(\rho^m_i\) concentrates at some point \(y_0 \in \mathbb{R}^2\). In particular
\[
\lim_{m \to \infty} \int_{\mathbb{R}^2} |x - v_i|^2 \rho^m_i(x) \, dx \geq \beta_i |y_0 - v_i|^2 \quad \text{for all } 1 \leq i \leq n. \tag{6.19}
\]
We want to show that \( y_0 \) is the global minima of \( \sum_{i=1}^{n} \frac{1}{2} \beta_i |x - v_i|^2 \) on \( \mathbb{R}^2 \). Let us define \( \tilde{\rho}_m \) as

\[
\tilde{\rho}_m(x) = \frac{1}{\delta^2} \rho^m \left( \frac{x}{\delta} - y_0 \right).
\]

Then

\[
\mathcal{F}_0(\tilde{\rho}_m) = \mathcal{F}_v(\rho^m) - \frac{\Lambda_f(\beta_m)}{4\pi} \ln \delta + \delta^2 \sum_{i=1}^{n} \frac{1}{2} \int_{\mathbb{R}^2} |x + y_0|^2 \rho_i^m
\]

\[- \sum_{i=1}^{n} \frac{1}{2} \int_{\mathbb{R}^2} |x - v_i|^2 \rho_i^m .
\]

Therefore we obtain

\[
\inf_{\rho \in \Gamma^m} \mathcal{F}_0(\rho) \leq \inf_{\rho \in \Gamma^m} \mathcal{F}_v(\rho) + \delta^2 O(1) - \sum_{i=1}^{n} \frac{1}{2} \int_{\mathbb{R}^2} |x - v_i|^2 \rho_i^m .
\]

Letting \( m \to \infty \) and using (6.19) and Lemma 6.3(b) we get

\[
\inf_{\rho \in \Gamma^m} \mathcal{F}_0(\rho) \leq \inf_{\rho \in \Gamma^m} \mathcal{F}_v(\rho) + \delta^2 O(1) - \sum_{i=1}^{n} \frac{1}{2} \beta_i |y_0 - v_i|^2 .
\]

Since \( \delta > 0 \) is arbitrary, by (5.3) we get \( y_0 \) is the global minima of \( \sum_{i=1}^{n} \frac{1}{2} \beta_i |x - v_i|^2 \) on \( \mathbb{R}^2 \) and (6.12) holds true.

**Lemma 6.5.** Under the assumption of Case (B) there exists a minimizer of \( \mathcal{F}_v \) in \( \Gamma^\beta \). In particular

\[
\inf_{\rho \in \Gamma^\beta} \mathcal{F}_v(\rho) < \inf_{\rho \in \Gamma^\beta} \mathcal{F}_0(\rho) + \min_{x_0 \in \mathbb{R}^2} \sum_{i=1}^{n} \frac{1}{2} \beta_i |x_0 - v_i|^2 .
\]

**Proof.** Under this assumption, we have from (6.13) that \( ||\rho_i^m||_{L^\infty(B(0,R))} \leq C_0 \), for some constant \( C_0 \) independent of \( m \). In the proof \( C_0 \) will stand for some universal constant independent of \( m \) but may depend on \( R \). Then

\[
\left| \sum_{i=1}^{n} \int_{B(0,R)} \rho_i^m(x) \ln \rho_i^m(x) \ d^2 x \right| \leq C_0 .
\]

Now let

\[
\tilde{u}_i^m(x) := - \frac{1}{2\pi} \int_{\mathbb{R}^2} \ln |x - y| \rho_i^m(y) \chi_{B(0,R)}(y) \ d^2 y ,
\]

then it follows from Lemma 3.5 (using the fact \( ||\rho_i^m||_{L^\infty(B(0,R))} \leq C_0 \) that

\[
|\tilde{u}_i^m(x)| \leq \begin{cases} 
C_0 , & \text{if } |x| \leq 1 , \\
C_0(1 + \ln |x|) , & \text{if } |x| > 1 ,
\end{cases}
\]

Thus we have

\[
\left| \int_{B(0,R)} \int_{B(0,R)} \rho_i^m(x) \ln |x - y| \rho_i^m(y) \ d^2 y d^2 x \right| \leq \int_{\mathbb{R}^2} \rho_i^m(x) |\tilde{u}_i^m(x)| \ d^2 x
\]

\[
\leq C_0 \left[ \int_{\mathbb{R}^2} \rho_i^m \ d^2 x + \int_{\{|x| > 1\}} \ln |x| \rho_i^m \ d^2 x \right]
\]

\[
\leq C_0 \left[ \rho_i^m + \int_{\mathbb{R}^2} |x|^2 \rho_i^m \ d^2 x \right] \leq C_0 .
\]
Let us define \( \hat{\rho}_m^R(x) = \rho^n(x)\chi_{B(0,R)}(x) \). Let

\[
F_{v,R}(\rho) := \sum_{i=1}^{n} \int_{B(0,R)} \rho_i \ln \rho_i + \frac{1}{4\pi} \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} \int_{B(0,R)} \int_{B(0,R)} \rho_i^m(x) \ln |x-y|\rho_j^m(y) \\
+ \frac{1}{2} \sum_{i=1}^{n} \int_{B(0,R)} |x-v_i|^2 \rho_i . \tag{6.23}
\]

We can write \( F_v(\rho^m) \) as

\[
F_v(\rho^m) = F_{v,R}(\rho^m) + F_v(\hat{\rho}_m^R) \\
+ \frac{1}{2\pi} \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} \int_{B(0,R)} \int_{B(0,R)} \rho_i^m(x) \ln |x-y|\rho_j^m(y)d^2xd^2y . \tag{6.24}
\]

Since \( \|\rho^m\|_{L^\infty(B(0,R))} \leq C_0 \) we obtain that \( F_{v,R}(\rho^m) = O(1) \). Also, (6.22) implies that the second line in (6.24) is \( O(1) \) as well. Since \( F_v(\rho^m) \) is a bounded sequence (as \( \rho^m \) is a minimizer of \( \inf_{\beta_{n,m}} F_v \), see Lemma 6.3(a)) this implies that

\[
F_v(\hat{\rho}_m^R) = O(1) \tag{6.25}
\]

uniformly in \( m \).

Next, observe that we can choose \( R \) large enough for which \( \int_{\mathbb{R}^2} \hat{\rho}_m^R < \beta/2 \). Indeed, since \( \int_{\mathbb{R}^2} |x|^2 \rho_i^m \leq C \) then \( \int_{\{ |x| > R \}} \hat{\rho}_m^R \leq R^{-2} \int_{\{ |x| > R \}} |x|^2 \rho_i^m \leq C/R^2 \). For such \( R \), \( \hat{\rho}_m \) is sub-critical, uniformly in \( m \), thus

\[
F_v(\hat{\rho}_m^R) \geq C \int_{\mathbb{R}^2} \hat{\rho}_m^R \ln \hat{\rho}_m^R .
\]

From (6.25) we obtain that \( \hat{\rho}_m^R \) has a uniform bound in \( L \ln L \). Since \( \|\rho^m\|_{L^\infty(B(0,R))} = O(1) \) by assumption we obtain that \( \rho^m \) is bounded in \( L \ln L \) as well.

Proceeding as in the sub critical case (Theorem 4.1, see Remark 3) we can prove the existence of a minimizer of \( F_v \) over \( \Gamma^\beta \).

\section{Case of \( \text{Var}(v_1, \ldots, v_n) \) large: Proof of Theorem 1.1-d}

According to Proposition 6.1 we only have to exclude case A. We show it in the case \( n = 2 \). The general case follows similarly.

\begin{lemma}
Suppose \( \beta \) satisfies (5.1). Then there exists a constant \( \kappa(\beta) \) such that whenever \( |v_1 - v_2| > \kappa \), then (6.20) holds.
\end{lemma}

\begin{proof}
Let \( \bar{\rho} \) be any non-negative, bounded function of compact support (say \( \bar{\rho}(x) = 0 \) if \( |x| > 1 \)) such that \( \int_{\mathbb{R}^2} \bar{\rho} = 1 \). Define \( \rho_i(x) := \beta_i\bar{\rho}(x-v_i) \) so that \( \rho \in \Gamma^\beta \).

Then we immediately see that

\[
\int_{\mathbb{R}^2} \rho_i \ln \rho_i = O(1), \quad \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \rho_i(x) \ln |x-y|\rho_i(y) = O(1),
\]

\[
\int_{\mathbb{R}^2} |x-v_i|^2 \rho_i = O(1),
\]

for all \( i = 1, 2 \), where \( O(1) \) denotes a quantity independent of \( v_i \). Now

\[
\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \rho_1(x) \ln |x-y|\rho_2(y) = \beta_1\beta_2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \bar{\rho}(x) \ln |x-y+(v_1-v_2)|\bar{\rho}(y) . \tag{6.26}
\]

One can easily estimate that $|\ln |x-y+(v_1-v_2)|-\ln |v_1-v_2|| \leq \frac{2}{|v_1-v_2|}$, for all $x,y \in (0,1)$ provided $|v_1-v_2| > 2$ (this condition on $|v_1-v_2|$ is unnecessary, because we can choose the support of $\tilde{\rho}$ accordingly). Since $\tilde{\rho}$ has support in $B(0,1)$ we get
\[
\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \tilde{\rho}(x) \ln |x-y+(v_1-v_2)| \tilde{\rho}(y) - \ln |v_1-v_2| = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \tilde{\rho}(x) \ln |x-y+(v_1-v_2)| - \ln |v_1-v_2| \tilde{\rho}(y) = O(1). \tag{6.27}
\]
Thus we obtain from (6.26) and (6.27),
\[
\inf_{\rho \in \Gamma^\beta} \mathcal{F}_\nu(\rho) \leq \mathcal{F}_\nu(\tilde{\rho}) = O(1) + \frac{a_1^2}{2\pi} \beta_1 \beta_2 \ln |v_1-v_2|. \tag{6.28}
\]
While the right hand side of (5.3) becomes
\[
\inf_{\rho \in \Gamma^\beta} \mathcal{F}_0(\rho) + \min_{x_0 \in \mathbb{R}^2} \sum_{i=1}^n \frac{\beta_i}{2} |x_0-v_i|^2 = O(1) + \frac{\beta_1 \beta_2}{2(\beta_1 + \beta_2)} |v_1-v_2|^2. \tag{6.29}
\]
We see from (6.28) and (6.29) that the equality can not occur in (5.3) provided $|v_1-v_2|$ is very large. Hence by Proposition 6.1, there exists a minimizer of $\mathcal{F}_\nu$ on $\Gamma^\beta$. This completes the proof of the lemma.

**Proof of Theorem 1.2:**

*Proof.* The proof of (a) and (c) follows from Theorem 1.1 (b) and (d) respectively. We only need to prove (b). Since $A$ is invertible and all the $v_i$ are equal by translating and adding constants to the solution we can assume $u_i, 1 \leq i \leq n$ satisfies
\[
\begin{cases}
-\Delta u_i = e^{\sum_{j=1}^n a_{ij} u_j - \frac{1}{2} |x|^2}, & \text{in } \mathbb{R}^2, \\
\int_{\mathbb{R}^2} e^{\sum_{j=1}^n a_{ij} u_j - \frac{1}{2} |x|^2} = \beta_i.
\end{cases} \tag{6.30}
\]
Again using the invertibility and irreducibility of $A$ we get by [CSW97, Proposition 4.1] with $V_i(x) = e^{-\frac{|x|^2}{2}}$ that $u_i$ in (6.30) are radially symmetric with respect to the origin. By abuse of notation we still denote the radial function by $u_i(r), r = |x|$. Then $u_i$ satisfies
\[
-\frac{1}{r} \frac{d}{dr} \left( r \frac{du_i}{dr} \right) = e^{\sum_{j=1}^n a_{ij} u_j(r) - \frac{r^2}{2}}, r \in (0, \infty). \tag{6.31}
\]
Define
\[
m_i(r) = 2\pi \int_0^r se^{\sum_{j=1}^n a_{ij} u_j(s) - \frac{s^2}{2}} ds = -2\pi r \frac{du_i}{dr}, r \in (0, \infty), i = 1, \cdots, n.
\]
Then $m_i$ satisfies
\[
\lim_{r \to 0+} m_i(r) = 0, \quad \lim_{r \to \infty} m_i(r) = \beta_i, \quad \text{and } m_i \text{ are non decreasing.} \tag{6.32}
\]
Furthermore, since $u_i$ has log decay at infinity i.e., $|u_i(r) + \frac{\beta_i}{2\pi} \ln r| = O(1)$ as $r \to \infty$ (see [CSW97, Proposition 3.1]) we see that
\[
\lim_{r \to \infty} r^2 m'_i(r) = 0. \tag{6.33}
\]
Now define $w_i(s) = m_i(e^s), s \in (-\infty, \infty)$ then it follows from (6.32), (6.33) that $w_i$ is non decreasing and satisfies
\[
\lim_{s \to -\infty} w_i(s) = 0, \quad \lim_{s \to \infty} w_i(s) = \beta_i, \quad \lim_{s \to -\infty} e^{-s} w'_i(s) = 0, \quad \int_{-\infty}^\infty e^s w'_i(s) ds < \infty.
\]
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Therefore using the equation (6.31) we see that \( w_i \) satisfies
\[
 w''_i(s) = w'_i(s) \left[ 2 - \frac{1}{2\pi} \sum_{j=1}^{n} a_{ij} w_j(s) - e^s \right].
\] (6.34)

Summing over all \( i \) we can rewrite (6.34) as
\[
 \left( \sum_{i=1}^{n} w'_i(s) \right)' = \left[ 2 \sum_{i=1}^{n} w_i(s) - \frac{1}{4\pi} \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} w_i(s) w_j(s) \right]' - \sum_{i=1}^{n} e^s w'_i(s).
\] (6.35)

Since \( \lim_{s \to -\infty} \sum_{i=1}^{n} w_i(s) = \sum_{i=1}^{n} \beta_i \), \( w_i \) are non decreasing we can find a sequence \( s_m \) converging to \( \infty \) such that \( \sum_{i=1}^{n} w'_i(s_m) \to 0 \) as \( m \to \infty \). Therefore integrating (6.35) from \( -\infty \) to \( s_m \) and letting \( m \to \infty \) we obtain
\[
 2 \sum_{i=1}^{n} \beta_i - \frac{1}{4\pi} \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} \beta_i \beta_j = \sum_{i=1}^{n} \int_{-\infty}^{\infty} e^s w'_i(s) ds
\]
which implies \( \Lambda_f(\beta) > 0 \), contradicting our assumption. This completes the proof of the corollary.

\[\square\]

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