FORMALIZATION OF BOHR’S CONTEXTUALITY
WITHIN THE THEORY OF OPEN QUANTUM SYSTEMS

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Abstract

In quantum physics, the notion of contextuality has a variety of interpretations, which are typically associated with the names of their inventors, say, Bohr, Bell, Kochen, Specker, and recently Dzhafarov. In fact, Bohr was the first who pointed to contextuality of quantum measurements as a part of formulation of his principle of complementarity. (Instead of “contextuality,” he considered dependence on “experimental conditions.”) Unfortunately, the contextuality counterpart of the complementarity principle was overshadowed by the issue of incompatibility of observables. The interest for contextuality of quantum measurements rose again only in connection with the Bell inequality. The original Bohr’s contextuality, as contextuality of each quantum measurement, was practically forgotten. It was highlighted in our works with applications both to physics and cognition. In this note, the theory of open quantum systems is applied to formalization of Bohr’s contextuality within the framework of indirect measurements. This scheme is widely used in quantum information theory and leads to the Davis–Lewis–Ozawa theory of quantum instruments. In this scheme, Bohr’s viewpoint on contextuality of quantum measurements is represented within the formal mathematical framework.

Keywords: Bohr’s contextuality, indirect measurement scheme, open quantum systems, measurement apparatus, state of system, state of apparatus, interaction between system and apparatus, quantum instruments.

1. Introduction

Nowadays, contextuality defined via violation of the Bell-type inequalities [1, 2] or generally noncontextuality inequalities [3, 4] is a hot topic in quantum physics. Although this notion is well formalized within the mathematical framework, its physical meaning (beyond nonlocality) is unclear. The same can be said about Kochen–Specker contextuality [5]. The main problem is how to separate contextuality from incompatibility. For quantum observables, incompatibility is the necessary condition of violation of noncontextuality inequalities [6]. Moreover, for the Clauser–Horne–Shimony–Holt (CHSH) inequality (and its noncontextual inequality counterpart), incompatibility is also the sufficient condition of its violation [6, 7], at least for observables having the tensor product structure.* So, the problem whether the

*In [6], I tried to find intrinsic contextual component within the CHSH framework—beyond the tensor-structured observables. It was shown that generally contextuality without incompatibility may have some physical content. A mathematical constraint extracting the Bell-contextuality component from incompatibility was found. However, its physical meaning is the subject of further studies.
Bell contextuality is reduced to incompatibility or not is still open. Unfortunately, the Bell contextuality community simply ignores this complex problem and proceed further with mathematical studies.\(^1\)

However, at the very beginning of quantum mechanics, Bohr considered the notion of contextuality having the clear physical meaning\(^2\):

“Strictly speaking, the mathematical formalism of quantum mechanics and electrodynamics merely offers rules of calculation for the deduction of expectations pertaining to observations obtained under well-defined experimental conditions specified by classical physical concepts.”

Instead of “contextuality,” Bohr considered the dependence on “experimental conditions.”\(^3\) But original Bohr’s contextuality was completely forgotten,\(^4\) and Bell, Kochen, Specker, and their followers considered a variety of its “derivatives” related to the very special case of joint measurement of pairs of compatible observables. Heuristically, this sort of contextuality is represented as the following one:

**Joint-measurement contextuality:** If \(A, B,\) and \(C\) are three quantum observables, such that \(A\) is compatible with \(B\) and \(C\), a measurement of \(A\) might give different result depending upon whether \(A\) is measured with \(B\) or with \(C\).

This formulation is based on counterfactual argument and, from our viewpoint, it cannot be tested experimentally, so it has no relation to physics.\(^5\)

Bohr’s contextuality is experimentally tested through incompatibility and theoretically it is formulated in terms of commutators. The basic test is based on the Heisenberg uncertainty relation in its general form of the Schrödinger–Robertson inequality structure. Bell’s contextuality can be tested experimentally in experiments by demonstration of violation of various Bell-type inequalities.

Since Bohr’s formulation of the complementarity principle, including the contextuality counterpart, quantum measurement theory was essentially developed. The most powerful quantum measurement formalism is based on the theory of *open quantum systems*\(^6\) describing a system \(S\) interacting with the surrounding environment \(E\). In the particular application of this theory to the description of measurements, the role of environment is played by a measurement apparatus \(M\) used to measure some observable \(A\) on \(S\). One of the basic mathematical frameworks for modeling this situation is the *scheme of indirect measurements*\(^7\). The outcomes of \(A\) are represented as outcomes of apparatus’ pointer \(M_A\).

The process of measurement is described as the interaction between \(S\) and \(M\). This interaction generates the dynamics of the state of the compound system \(S + M\). The state evolution is unitary; for pure states, it is described by the Schrödinger equation and for mixed states given by density operators, by the von Neumann equation. Finally, the probability distribution for pointer’s outcomes is extracted from the compound state with the trace operation.

The scheme of indirect measurements is closely coupled to the theory of quantum instruments\(^8\), but in this paper we will not discuss this issue.

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\(^1\)It has to be stressed that this paper is devoted to physics. Outside of physics, noncontextual inequalities can be violated even for compatible observables. The best example is the Suppes–Zanotti inequality\(^9\) for three compatible observables; see\(^10\) for violation of Bell-type inequalities for observables in cognitive experiments. However, for such applications these inequalities have to be modified to take into account the presence of signaling. The latter is typical for experimental data in cognitive and psychological studies. Such inequalities are derived in the novel approach known as contextuality per default\(^11\).

\(^2\)We remark that neither Bell nor Kochen and Specker operated with the notion of contextuality. It was invented later; see\(^12\) for details.

\(^3\)It was highlighted only in our works with applications both to physics\(^13\) and cognition\(^14\).

\(^4\)We remark: Svozil\(^15\) and Griffiths\(^16\) claim that they elaborated experimental tests for joint-measurement contextuality.
We start with structuring Bohr’s complementarity principle [14,36] and highlighting its contextuality component in Sec. 2. Then we present the indirect measurement scheme in Sec. 3 and finally in Sec. 4 formalize Bohr’s contextuality within this framework.

2. Contextuality Component of Bohr’s Principle of Complementarity

Here, we follow our previous works devoted to Bohr’s principle of complementarity and its contextual component [7,22,24,25]. We start with the well-known citation of Bohr [14]:

“This crucial point ... implies the impossibility of any sharp separation between the behavior of atomic objects and the interaction with the measuring instruments, which serve to define the conditions under which the phenomena appear. In fact, the individuality of the typical quantum effects finds its proper expression in the circumstance that any attempt of subdividing the phenomena will demand a change in the experimental arrangement introducing new possibilities of interaction between objects and measuring instruments which, in principle, cannot be controlled. Consequently, evidence obtained under different experimental conditions cannot be comprehended within a single picture, but must be regarded as complementary in the sense that only the totality of the phenomena exhausts the possible information about the objects.”

The contextual component of this statement can be formulated as

**Principle 1 (Contextuality)** The output of any quantum observable is indivisibly composed of the contributions of the system and the measurement apparatus.

There is no reason to expect that all experimental contexts can be combined and all observables can be measured jointly. Hence, incompatible observables (complementary experimental contexts) may exist. Moreover, they should exist, otherwise the contextuality principle would have the empty content. In fact, if all experimental contexts can be combined into single context $C$, and all observables can be jointly measured in this context, then the outputs of such joint measurements can be assigned directly to a system. To be more careful, we have to say: “assigned to a system and context $-C$.” But the latter can be omitted, since this is the same context for all observables. This reasoning implies:

**Principle 2 (Incompatibility)** There exist incompatible observables (complementary experimental contexts).

Since both principles, contextuality and incompatibility, are so closely interrelated, it is natural to unify them into a single principle, the **contextuality–incompatibility principle**. This is our understanding of the Bohr’s complementarity principle.

3. Foundations of Quantum Formalism

In quantum theory, it is postulated that every quantum system $S$ corresponds to a complex Hilbert space $\mathcal{H}$; denote the scalar product of two vectors by the symbol $\langle \psi_1 | \psi_2 \rangle$. Throughout this paper, we assume $\mathcal{H}$ is finite dimensional. States of the quantum system $S$ are represented by density operators acting in $\mathcal{H}$. Denote this state space by the symbol $S(\mathcal{H})$. Observables are represented by Hermitian operators in $\mathcal{H}$. These are just symbolic expressions of physical observables, say, the position, momentum,
or energy. Each Hermitian operator $A$ can be represented as
\[ A = \sum_x x E^A(x), \] (1)
where $x$ labels the eigenvalues, and $E^A(x)$ is the spectral projection of the observable $A$ corresponding to the eigenvalue $x$.

The operator $A$ can be considered as the compact mathematical representation for probabilities of outcomes of the physical observable. These probabilities are given by the *Born rule* that states – If an observable $A$ is measured in a state $\rho$, then the probability distribution $\text{Pr}\{A = x \| \rho\}$ of the outcome of the measurement is given by
\[ \text{Pr}\{A = x \| \rho\} = \text{Tr}[E^A(x)\rho] = \text{Tr}[E^A(x)\rho E^A(x)]. \] (2)

4. **Indirect Measurement Scheme: Apparatus with Meter Interacting with a System**

The scheme of indirect measurements represents the framework which was emphasized by Bohr, namely, the outcomes of quantum measurements are created in the complex process of the interaction of a system $S$ with a measurement apparatus $M$. The latter is combined of a complex physical device interacting with $S$ and a pointer showing the outcomes of measurements; for example, it can be the “spin up or spin down” arrow. The system $S$ by itself is not approachable by the observer, who can see only the pointer of $M$. Then the observer associates pointer’s outputs with the values of measured observable $A$ for the system $S$.

Can the outputs of the pointer be associated with the “intrinsic properties” of $S$ or not? This is one of the main questions of disturbing the quantum foundations during the last 100 years.

The indirect measurement scheme can be represented as a block of following interrelated components:

- The states of the systems $S$ and the apparatus $M$; they are represented in complex Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, respectively.
- The unitary operator $U$ representing the interaction dynamics for the compound system $S + M$.
- The meter observable $M_A$ giving outputs of the pointer of the apparatus $M$.

In the indirect measurement scheme, it is assumed that the compound system $S + M$ is isolated. The dynamics of pure states of the compound system is described by the Schrödinger equation
\[ i \frac{d}{dt} |\Psi\rangle(t) = H |\Psi\rangle(t), \quad |\Psi\rangle(0) = |\Psi\rangle_0, \] (3)
where $H$ is its Hamiltonian (generator of the evolution) of $S + M$. The state $|\Psi\rangle(t)$ evolves as
\[ |\Psi\rangle(t) = U(t)|\Psi\rangle_0, \]
where $U(t)$ is the unitary operator represented by $U(t) = e^{-itH}$. The Hamiltonian (evolution generator) describing information interactions reads
\[ H = H_S \otimes I + I \otimes H_M + H_{S,M}, \]
where $H_S : \mathcal{H} \rightarrow \mathcal{H}$ and $H_M : \mathcal{K} \rightarrow \mathcal{K}$ are Hamiltonians of $S$ and $M$, respectively, and $H_{S,M} \in \mathcal{H} \otimes \mathcal{K} \rightarrow \mathcal{H} \otimes \mathcal{K}$ is the Hamiltonian of interaction between systems $S$ and $M$.

The Schrödinger equation implies that the evolution of the density operator $R(t)$ of the system $S + M$ is described by the von Neumann equation

$$\frac{dR}{dt}(t) = -i[H, R(t)], \quad R(0) = R_0.$$  \hspace{1cm} (4)

However, the state $R(t)$ is too complex to be handled consistently – the apparatus includes many degrees of freedom.

Assume that we want to measure an observable on the system $S$, which is represented by Hermitian operator $A$ acting in system’s state space $\mathcal{H}$. The indirect measurement model for measurement of the observable $A$ was introduced by Ozawa [33] as a “(general) measuring process”; this is a quadruple $(K, \sigma, U, M_A)$ consisting of a Hilbert space $K$, a density operator $\sigma \in S(K)$, a unitary operator $U$ on the tensor product of the state spaces of $S$ and $M$, $U : \mathcal{H} \otimes \mathcal{K} \rightarrow \mathcal{H} \otimes \mathcal{K}$, and a Hermitian operator $M_A$ on $K$. Here, $K$ represents the states of the apparatus $M$, $U$ describes the time evolution of system $S + M$, $\sigma$ describes the initial state of the apparatus $M$ before the measurement starts, and the Hermitian operator $M_A$ is the meter observable of the apparatus $M$ (say, the pointer of $M$). This operator represents indirectly outcomes of an observable $A$ for the system $S$.

The probability distribution $\Pr\{A = x \parallel \rho\}$ in the system state $\rho \in S(\mathcal{H})$ is given by

$$\Pr\{A = x \parallel \rho\} = \text{Tr}\[(I \otimes E^{M_A}(x))U(\rho \otimes \sigma)U^*\],$$  \hspace{1cm} (5)

where $E^{M_A}(x)$ is the spectral projection of $M_A$ for the eigenvalue $x$.

We recall that operator $M_A$ is Hermitian. In the finite-dimensional case, it can be represented in the form

$$M_A = \sum_k x_k E^{M_A}(x_k),$$  \hspace{1cm} (6)

where $(x_k)$ is the set of its eigenvalues, and $E^{M_A}(x_k)$ is the projector on the subspace of eigenvectors corresponding to eigenvalue $x_k$.

The change of the state $\rho$ of the system $S$ caused by the measurement for the outcome $A = x$ is represented with the aid of the map $I_A(x)$ in the space of density operators defined as

$$I_A(x)\rho = \text{Tr}_K[(I \otimes E^{M_A}(x))U(\rho \otimes \sigma)U^*],$$  \hspace{1cm} (7)

where $\text{Tr}_K$ is the partial trace over $\mathcal{K}$. The map $x \mapsto I_A(x)$ is a quantum instrument. We remark that conversely any quantum instrument can be represented via the indirect measurement model; see Ozawa [33].

5. Bohr’s Contextuality from the Indirect Measurement Scheme

We take the basic part of the aforementioned citate of Bohr: “... the impossibility of any sharp separation between the behavior of atomic objects and the interaction with the measuring instruments” and establish correspondence with the indirect measurement scheme:

- “atomic object” $\rightarrow$ state $\rho$;
“measuring instrument” \rightarrow \text{state } \sigma; \quad \text{“interaction” } \rightarrow \text{unitary operator } U.

Within this framework, the triple \( C = (\rho, \sigma, U) \) represents the complex of the “experimental conditions,” the context of measurement. Within this framework, the relation between contextuality (in Bohr’s sense) and incompatibility is completely clear — incompatibility is so to say the “derivative” of contextuality. There is no reason to expect that any pair of contexts, \( C_1 = (\rho_1, \sigma_1, U_1) \) and \( C_2 = (\rho_2, \sigma_2, U_2) \), can be unified in the joint measurement scheme, even if \( \rho_1 = \rho_2 = \rho \).

6. Concluding Remarks

Recently, the Bell and Kochen–Specker contextualities attracted a lot of attention; interesting studies were performed, and they definitely provide the mathematical structures of these contextualities more complete. However, the physical meaning of these contextualities is still not clear. The common way to connect the Bell contextuality with physics is to refer to nonlocality and spooky action at the distance. However, many authors think that the nonlocality issue is not crucial [37–49]. The other way to connect the Bell contextuality with physics is to couple it to incompatibility [7,24,37,44] and generally to Bohr’s complementarity principle. As was mentioned in the introduction, for the CHSH scheme (and “natural observables”), the Bell contextuality is identical to incompatibility. This is really the alarming signal for those who use violation of Bell-type inequalities to quantify contextuality. (Of course, interrelation incompatibility–contextuality has to be investigated in more details, especially for noncontextual inequalities with \( n \geq 5 \) observables.)

At the same time, the original Bohr’s viewpoint on contextuality of quantum measurements and its connection with incompatibility is practically forgotten. In this paper, Bohr’s viewpoint was refreshed (Sec. 2.) in the form of Principle 1 (contextuality) and Principle 2 (incompatibility). Bohr’s contextuality was formalized within the scheme of indirect measurements.

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