AN ALGORITHM FOR ESTIMATING VOLUMES AND OTHER INTEGRALS IN $n$ DIMENSIONS

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Abstract. The computational cost in evaluation of the volume of a body using numerical integration grows exponentially with dimension of the space $n$. The most generally applicable algorithms for estimating $n$-volumes and integrals are based on Markov Chain Monte Carlo (MCMC) methods, and they are suited for convex domains. We analyze a less known alternate method used for estimating $n$-dimensional volumes, that is agnostic to the convexity and roughness of the body. It results due to the possible decomposition of an arbitrary $n$-volume into an integral of statistically weighted volumes of $n$-spheres. We establish its dimensional scaling, and extend it for evaluation of arbitrary integrals over non-convex domains. Our results also show that this method is significantly more efficient than the MCMC approach even when restricted to convex domains, for $n \lesssim 100$. An importance sampling may extend this advantage to larger dimensions.

1. Introduction. Analytic evaluation of volumes is feasible for a relatively small set of symmetric bodies defined in the appropriate coordinate systems. In some cases, the surface of a body may not have a tractable closed form analytical expression and the body may only be defined by a set of inequalities. These challenges in analytical integration were overcome by numerical methods [16]. As the dimension of problems became large, the exponential increase in the cost of numerical methods (NP-hardness) inspired new statistical methods that converge to a reasonable estimate of the volume in polynomial time under certain constraints [1, 20, 17]. In evaluating more general integrals, deterministic sampling methods such as the Quasi-Monte Carlo are very efficient when the integrand can be reduced to a function of a single effective variable [7]. Similarly, the naive Monte Carlo method is largely effective when the limits of the integration are constants, that is, over a domain which is an $n$-orthotope (a rectangle when $n = 2$, cuboid when $n = 3$ etc.). In problems where some function defines the boundary of the domain or its membership, and in problems where the sampled independent variables have an implicit non-uniform probability density, correctly sampling the domain in itself amounts to be NP-hard.

Even for the diminished problem of estimating $n$-volumes, a Markov Chain Monte Carlo (MCMC) sampling is the only tractable approach for large $n$ [19]. This approach is geometrically insightful and involves cancellation of errors in the estimates, resulting in relatively fast convergence for convex volumes. Nevertheless, after improving rapidly from $\mathcal{O}(n^{23})$ scaling in the samples required [9, 8, 15], algorithms using this approach have stagnated at $\mathcal{O}(n^4)$ samples for a given convex shape [18, 14, 12]. Since the cost of evaluating a typical scalar function increases linearly with the number of cardinal directions $n$, the total computing effort in estimating volumes scales as $\mathcal{O}(n^5)$ for these MCMC methods. This general poor scaling of the MCMC approach with the dimension is overcome using specialized algorithms designed for certain forward and inverse problems [4, 5, 10, 21]. Volumes of non-convex bodies can also be evaluated more accurately using semi-definite programming, but they are suited for smaller dimensions [13].

The algorithm presented here is suitable for estimating volumes of both convex and non-convex bodies with fewer exceptions, and for other problems of estimation in continuous spaces. This method also retains the advantages of the naive Monte Carlo sampling such as the full independence of the random samples. The resulting suit-
ability for parallel computation could be of additional significance. The proposed $n$-sphere-Monte-Carlo (NSMC) method decomposes the estimated volume into weighted volumes of $n$-spheres, and these weights are trivially estimated by sampling extents of the domain with respect to an origin. Such a volume preserving transformation was suggested many years ago \[11\]. We also show a straightforward adaptation of this method to estimate arbitrary integrals. Here, the required number of extent samples scale as $O(n)$ for a fixed distribution of extents of the domain, with the corresponding total computing effort scaling as $O(n^2)$ for estimating volumes and as $O(n^3)$ for estimating arbitrary integrals. While estimating volumes using this approach involves only sampling the extents, estimating arbitrary integrals includes sampling the interior of the domain. The proposed approach may have challenges in estimating volumes which are not just highly eccentric but also have a tailed distribution of large extents, such as certain convex shapes. In such cases, the poor scaling in number of samples with $n$ can be reduced by an appropriate importance sampling to capture the tailed extents. The challenges in such sampling of high dimensional sub-spaces along with a potential solution has been described elsewhere \[2, 3\]. In this paper, we limit ourselves to the naive NSMC approach using an unbiased sampling of the extents. The naive algorithm is significantly more efficient than the MCMC approach even when restricted to convex domains, for $n \lesssim 100$.

2. Frequently used terms and symbols.

$S^{n-1}$ is the set of all points on the surface of the unit sphere.

**membership function** It is a function $\mathbb{R}^n \mapsto \{0, 1\}$ that maps a point in space, to 0 if that point lies outside the body, or 1 if that point lies inside the body.

**extent** The extent of a body is the distance between the origin of the coordinate system and a point on the surface of the body. If $r$ is the extent of a body along the direction vector $\hat{s}$, then $r\hat{s}$ lies on the surface of the body.

**extent function** The extent function of a body is a function $S^{n-1} \mapsto \mathbb{R}$ that maps a direction vector to a corresponding extent of the body.

**extent density** The extent density of a body is the probability density function of extents obtained when direction vectors are randomly sampled from a uniform distribution on $S^{n-1}$.

$s_n$ is the surface area of the $n$ dimensional unit sphere given by

\[
s_n = \frac{n\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2} + 1)}
\]

$v_n$ is the volume of the $n$ dimensional unit sphere given by

\[
v_n = \frac{s_n}{n} = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2} + 1)}
\]

3. Estimation of volume.

3.1. Problem statement. Given a closed body containing the origin, specified by an extent function $S$ with an extent density function $f_R$, estimate the volume enclosed by the body.

We assume $S$ to be single valued for clarity of the paper but it need not be continuous. The constraint of $S$ being single valued leaves out some non-convex geometries such as in Figure 3.1b. This constraint can be relaxed by a simple generalization of the extent of such a body as shown in Appendix A. Also, in many cases, the extent
function may not be available explicitly and only a membership function may be available. In such cases, we can construct an extent function that estimates the extent in the given direction by repeatedly invoking the membership function for points along that direction, say using a bisection search.

3.2. Solution. We repose the problem of \( n \)-dimensional integration for volume in spherical coordinates, as an estimation of the relative weights for the volumes of spheres of varying radii that add up to the volume of the given body. This approach allows a simple statistical estimation of the volume of even arbitrary non-convex bodies and requires no lower bounds on the smoothness of the body. The two dimensional illustration in Figure 3.2 serves as a simple example.

The volume of a body in spherical coordinates, with \( \rho \) being the radial coordinate and \( d\hat{s} \) being the surface element of the unit sphere, is

\[
V = \oint_{S^{n-1}} \int_0^{\rho(S\hat{s})} \rho^{n-1} d\rho \, d\hat{s} = \frac{1}{n} \oint_{S^{n-1}} r^n d\hat{s}
\]

While the above form is convenient for analytic integration when \( S \) is tractable and known, it is best avoided otherwise. But this form is well suited for a statistical estimation by uniform sampling on the surface of the unit sphere as given below.

If \( R \) is a random variable representing the extent obtained when sampling direction vectors uniformly distributed on \( S^{n-1} \), (3.1) can be rewritten using the expectation of \( R^n \) in terms of the extent density \( f_R \) of the body.

\[
V = \left\{ \frac{1}{n} \oint_{S^{n-1}} d\hat{s} \right\} \left\{ \int_0^\infty r^n f_R(r) \, dr \right\}
\]

Expressing in terms of the surface area \( s_n \) and volume \( v_n \) of the unit sphere,

\[
V = \frac{s_n}{n} \int_0^\infty r^n f_R(r) \, dr = v_n \mathbb{E}[R^n]
\]
Fig. 3.2: Consider the 2-dimensional body in Figure 3.2a consisting of two semicircles of radius $r_1$ and $r_2$ attached to each other. The 2-volume (or area) of this composite body is $\pi \left( \frac{1}{2} r_1^2 + \frac{1}{2} r_2^2 \right)$. Likewise, the 2-volume of the body in Figure 3.2b is $\pi \left( \frac{3}{4} r_1^2 + \frac{1}{4} r_2^2 \right)$. We may observe that given extents $R$ in all directions, the 2-volume of an arbitrary body is simply the mean of $R^2$ with a multiplying front constant $\pi$; this front constant depends on the dimension of space. Note that the angular sectors with identical radii need not be contiguous, and even the body of Figure 3.2c has the same 2-volume as that of Figure 3.2a.

For the purpose of volume estimation, classifying bodies based on their extent densities is more convenient especially for non-convex and non-symmetric bodies.

If the extent density $f_R$ is known, one can integrate (3.3) using a numerical quadrature, and this scales only as $O(n)$ in the total computing effort. But in practice, for an unknown body, the estimation and the integration of the extent density are implemented as a single algorithm represented by (3.3) and shown in Algorithm 3.1.

**Algorithm 3.1 Estimate volume**

```
procedure ESTIMATE_VOLUME(S)
    \[ V \leftarrow 0 \]
    \[ \text{for } i = 1 : N \text{ do} \]
    \[ \hat{s}_i \leftarrow \text{unit vector in random direction} \]
    \[ R_i \leftarrow S(\hat{s}_i) \]
    \[ V \leftarrow V + R_i^n \]
    \[ \text{end for} \]
    \[ V \leftarrow \frac{2}{N} V \]
    \[ \text{return } V \]
end procedure
```

There are two significant advantages to this statistical estimation.

1. For a body with a given extent density, the number of random samples required for the convergence of the $n^{th}$ moment of the extent density, i.e., the $n$-dimensional volume of the body, has an upper-bound that varies as $O(n)$. This is proved in section 4.
2. The independence of the random samples is maintained, and hence it is suitable for parallel computing approaches.
The simplest extent density is $f_R(r) = \delta(r - r_0)$ for a sphere of radius $r_0$. Some convex bodies, such as the cube, are well defined by their symmetries for all dimensions, while their extent densities change with dimension. Conversely, different bodies, including their different orientations, can result in the same extent density. Different reference points or origins can result in different extent densities for the same body, and thus affect the convergence weakly but not the order of convergence with $n$. Also, note that iterating the point of reference to the nominal centre of the body requires only $O(n)$ extent samples, in any case. Further analysis of this algorithm and numerical results for demonstration follow in the later sections.

4. Analysis. Approximating the expectation in (3.3) using a Monte Carlo estimate $V_N$ of $N$ samples,

$$V_N = \frac{v_n}{N} \sum_{i=1}^{N} R_i^n$$

The expected root-mean-square (RMS) error of this estimate can then be written as

$$\varepsilon = \sqrt{\text{Var}[V_N]} = v_n \sqrt{\text{Var}[R^n]} \frac{1}{\sqrt{N}}$$

We normalize this RMS error with the true volume from (3.3) to obtain the relative error $\bar{\varepsilon}$.

$$\bar{\varepsilon} = \frac{\varepsilon}{V} = \frac{\sqrt{\text{Var}[R^n]}}{\mathbb{E}[R^n]} \frac{1}{\sqrt{N}}$$

We then pose the analysis of the relative error as derivation of a bounds for the variance-to-square-mean ratio of the $n^{th}$ moment of a random variable in a Hausdorff moment problem. Using the above, we establish the scaling of the number of samples $N$ for any given relative RMS error $\bar{\varepsilon}$ in terms of the number of dimensions $n$ in the volume estimation.

4.1. Scale invariance of relative error in the volume estimate. Suppose extents of an $n$-dimensional body were scaled by a factor $a$,

$$\bar{\varepsilon} = \frac{\sqrt{\text{Var}[(aR)^n]}}{\mathbb{E}[(aR)^n]} \frac{1}{\sqrt{N}} = a^n \frac{\sqrt{\text{Var}[R^n]}}{\mathbb{E}[R^n]} \frac{1}{\sqrt{N}} = \frac{\sqrt{\text{Var}[R^n]}}{\mathbb{E}[R^n]} \frac{1}{\sqrt{N}}$$

Thus, relative error in the volume estimate is invariant under a scaling of the body. Without any loss of generality of our analysis, it is sufficient to only consider bodies with extents ranging from $\frac{1}{\lambda}$ to 1, where $\lambda > 1$ represents the ratio of the largest to the smallest extent of the body. Hence, in our analysis, we only consider extent densities with compact support $[\frac{1}{\lambda}, 1]$. Likewise, the convergence of the algorithm itself is not affected by the scale of the body; only the distribution of relative extents matters.

4.2. Scaling of relative error with dimension. Given that the extent density of interest has been reduced to a compact support $[\frac{1}{\lambda}, 1]$, we have the following theorems on moments of $R$ and their variance-to-square-mean ratio. We consider boundaries given by a continuous extent function $S$, where the extent density is also continuous, bounded and greater than zero in the interval $[\frac{1}{\lambda}, 1]$. If the boundary
is defined by a function $S$ that is not continuous, all the possible relative extents in $[\frac{1}{\lambda}, 1]$ need not exist and the extent density can indeed be discontinuous or zero at points within the interval. The following theorems nevertheless apply to such extent densities in a piece-wise manner with rescaling, thus we incur no loss of generality in the bodies considered.

**Lemma 4.1.** If $X$ is a random variable whose probability density function $f$ is supported on $[\frac{1}{\lambda}, 1]$ where $\lambda \in (1, \infty)$, and $f$ is bounded as $f_{\text{max}} \geq f(x) \geq f_{\text{min}} > 0$ for all $x \in [\frac{1}{\lambda}, 1]$, then for all $k \in \mathbb{N}$

$$
\frac{f_{\text{max}}}{k + 1} \left( 1 - \frac{1}{\lambda^{k+1}} \right) \geq \mathbb{E} [X^k] \geq \frac{f_{\text{min}}}{k + 1} \left( 1 - \frac{1}{\lambda^{k+1}} \right)
$$

**Proof.** When $f$ is bounded as $f_{\text{max}} \geq f(x) \geq f_{\text{min}} > 0$ for all $x \in [\frac{1}{\lambda}, 1]$, its moments can be trivially bounded by zeroth order approximations as given below.

(4.5) \[
\int_{\frac{1}{\lambda}}^{1} x^k f_{\text{max}} \, dx \geq \int_{\frac{1}{\lambda}}^{1} x^k f(x) \, dx \geq \int_{\frac{1}{\lambda}}^{1} x^k f_{\text{min}} \, dx
\]
resulting in

(4.6) \[
\frac{f_{\text{max}}}{k + 1} \left( 1 - \frac{1}{\lambda^{k+1}} \right) \geq \mathbb{E} [X^k] \geq \frac{f_{\text{min}}}{k + 1} \left( 1 - \frac{1}{\lambda^{k+1}} \right)
\]

Using bounds of Lemma 4.1, we can now establish that the variance-to-square-mean ratio relating the number of samples and corresponding error, varies as $O(n)$ for a volume in $n$ dimensions.

**Theorem 4.2.** If $X$ is a random variable whose probability density function $f$ is supported on $[\frac{1}{\lambda}, 1]$ where $\lambda \in (1, \infty)$, and $f$ is bounded as $f_{\text{max}} \geq f(x) \geq f_{\text{min}} > 0$ for all $x \in [\frac{1}{\lambda}, 1]$, then, for $k \gg \frac{1}{1 - \lambda^{-1}}$ and $k \in \mathbb{N}$, there exists some $c \in \mathbb{R}$ such that

$$
\sqrt{\frac{\text{Var} [X^k]}{\mathbb{E} [X^k]}} \leq c \sqrt{k}
$$

**Proof.**

(4.7) \[
\frac{\text{Var} [X^k]}{\left\{ \mathbb{E} [X^k] \right\}^2} = \frac{\mathbb{E} [X^{2k}] - \left\{ \mathbb{E} [X^k] \right\}^2}{\left\{ \mathbb{E} [X^k] \right\}^2}
\]

(4.8) \[
\frac{\text{Var} [X^k]}{\left\{ \mathbb{E} [X^k] \right\}^2} = \frac{\mathbb{E} [X^{2k}]}{\left( \mathbb{E} [X^k] \right)^2} - 1
\]

The two bounds on moments in Lemma 4.1 applied to maximize the above ratio, gives us

(4.9) \[
\frac{\text{Var} [X^k]}{\left\{ \mathbb{E} [X^k] \right\}^2} \leq \frac{f_{\text{max}} (k + 1)^2}{f_{\text{min}}^2} \frac{1}{2k + 1} \left( \frac{1}{\lambda^{k+1}} \right)^2 - 1
\]
and further for all $k \gg \frac{1}{\lambda - 1}$.

\begin{equation}
\sqrt{\frac{\text{Var} \left[ X^k \right]}{\mathbb{E} \left[ X^k \right]}} \leq c \sqrt{k}
\end{equation}

where

\begin{equation}
c \approx \frac{\sqrt{f_{\text{max}}}}{f_{\text{min}}}
\end{equation}

In the proposition below, we use the variance-to-square-mean ratio of the $k^{\text{th}}$ moment varying as $O(k)$, to derive the expected number of samples for a given error.

**Proposition 4.3.** For a given extent density and a relative RMS error $\tilde{\varepsilon}$ in the volume estimate $V_N$, the required number of samples $N$ increases linearly with dimension $n$.

**Proof.** Applying the bound on the variance-to-square-mean ratio in Theorem 4.2 into (4.3) for relative RMS error, gives us the following.

\begin{equation}
\tilde{\varepsilon} \leq \sqrt{\frac{f_{\text{max}}}{f_{\text{min}}}} \frac{\sqrt{N}}{\sqrt{N}}
\end{equation}

\begin{equation}
N \leq \frac{f_{\text{max}}}{f_{\text{min}}} \frac{n}{\varepsilon^2}
\end{equation}

It can be shown that the number of samples required for a relative RMS error $\tilde{\varepsilon}$ is significantly smaller than this upper bound for probability density functions supported on $[0, 1]$ that do not have a tail along large extents. Exact relations for moments of a few distributions are shown in the following corollary, and other demonstrations are shown in section 6. For extent densities $f_R$ that are tailed along large extents, an importance sampling can limit the number of samples to a reasonable value.

**Corollary 4.4.** We present a few distributions where exact analytical relations for the variance-to-square-mean ratio of the $k^{\text{th}}$ moment can be derived.

**Proof.** For the uniform distribution on the interval $[0, 1],

\begin{equation}
f(x) = \begin{cases} 
1 & x \in [0, 1] \\
0 & \text{elsewhere}
\end{cases}
\end{equation}

\begin{equation}
\sqrt{\frac{\text{Var} \left[ X^k \right]}{\mathbb{E} \left[ X^k \right]}} = \frac{k}{\sqrt{2k + 1}}
\end{equation}

For the distribution with the polynomial probability density function given by the following with $m \neq -1$.

\begin{equation}
f(x) = \begin{cases} 
(m + 1)x^m & x \in [0, 1] \\
0 & \text{otherwise}
\end{cases}
\end{equation}

\begin{equation}
\sqrt{\frac{\text{Var} \left[ X^k \right]}{\mathbb{E} \left[ X^k \right]}} = \frac{k}{\sqrt{(m + 1)(m + 2k + 1)}}
\end{equation}
For a U-quadratic distribution with a probability density function given by

\begin{equation}
 f(x) = \begin{cases} 
 12 \left( x - \frac{1}{2} \right)^2 & x \in [0, 1] \\
 0 & \text{otherwise}
\end{cases}
\end{equation}

(4.17)

\begin{equation}
 \frac{\sqrt{\text{Var}[X^k]}}{\mathbb{E}[X^k]} = \sqrt{\frac{(2k^2 + k + 1)(k + 1)(k + 2)(k + 3)^2}{3(2k + 1)(2k + 3)(k^2 + k + 2)^2}} - 1
\end{equation}

(4.18)

For large $k$,

\begin{equation}
 \frac{\sqrt{\text{Var}[X^k]}}{\mathbb{E}[X^k]} \approx \sqrt{\frac{k}{6}}
\end{equation}

(4.19)

5. Estimation of arbitrary integrals. In this section we extend the proposed algorithm to estimate arbitrary integrals.

5.1. Problem statement. Given a function $h$ defined over an arbitrary domain specified by an extent function $S$, estimate the integral of $h$ over the domain.

5.2. Solution. The required integral in spherical coordinates, with $\rho$ being the radial coordinate and $d\hat{s}$ being the surface element of the unit sphere, is

\begin{equation}
 I = \iint_{S(n-1)} S(\hat{s}) \rho^{n-1} h(\rho \hat{s}) \, d\rho \, d\hat{s}
\end{equation}

(5.1)

Let $i(\hat{s})$ be the integral along $\rho$ for a given $\hat{s}$.

\begin{equation}
 i(\hat{s}) = \int_{0}^{S(\hat{s})} \rho^{n-1} h(\rho \hat{s}) \, d\rho
\end{equation}

(5.2)

Then, the integral $I$ over the arbitrary domain is

\begin{equation}
 I = \int_{S(n-1)} i(\hat{s}) \, d\hat{s} = s_n \mathbb{E}[i(\hat{s})]
\end{equation}

(5.3)

An algorithm implementing this expectation is shown in Algorithm 5.1. Note that any importance sampling applied to $R^n$ in estimating volumes, can also be extended to $i(\hat{s})$ in the problem of integration over a domain. In this work, we present results of a hybrid approach to the problem of $n$ dimensional integration, where one dimensional integration of $i(\hat{s})$ along the radial direction is performed using deterministic quadrature schemes such as Gaussian quadrature, while the high dimensional partial integral over the angular coordinates is estimated statistically using the naive NSMC approach. Alternative approaches for integration using the naive NSMC are possible.

6. Examples and demonstrations. The NSMC algorithm was used to estimate the volumes of bodies with various extent densities, and to estimate various other integrals. The relative error between the estimate and the true value obtained from a known analytical expression was used as a stopping criterion. The number of samples on $S^{n-1}$ required for 1000 consecutive estimates to achieve a relative tolerance of 0.05, 0.1 and 0.2, is plotted against the dimension of the problem in Figures 6.1a to 6.1f. A direct comparison with an implementation [6] of a simulated annealing MCMC method [18] to estimate the volume of certain convex bodies is shown in Figures 6.2a and 6.2b.
Fig. 6.1: The number of samples on $S^{n-1}$ required for 1000 consecutive estimates to achieve a relative tolerance (tol) of 0.05, 0.1 and 0.2, is plotted against the dimension of the problem. Figures 6.1a to 6.1c show respectively the number of samples required to estimate the volume of a body with uniform(0, 1), beta($\alpha = 2, \beta = 2$) and arcsine extent densities. Figures 6.1d to 6.1f show respectively the number of samples required to estimate the integral of the radially symmetric Gaussian integrand of (6.5), the radially symmetric oscillatory polynomial integrand of (6.8) and the radially asymmetric x-coordinate integrand of (6.12), over domains with uniform(0,1) extent density. Note that this domain is highly eccentric and can have an arbitrary geometry. The number of samples in Figures 6.1a to 6.1c was averaged over 100 trials, and the number of samples in Figures 6.1d to 6.1f was averaged over 1000 trials.
Algorithm 5.1 Estimate arbitrary integral

\textbf{procedure} \textsc{estimate arbitrary integral}(i)
\begin{align*}
& I \leftarrow 0 \\
& \text{for } k = 1 : N \text{ do} \\
& \quad \hat{s}_k \leftarrow \text{unit vector in random direction} \\
& \quad I \leftarrow I + i(\hat{s}_k) \\
& \text{end for} \\
& I \leftarrow \frac{i}{N} I \\
& \text{return } I \\
\text{end procedure}
\end{align*}

6.1. Estimation of the volume represented by extent densities. For estimation of the volumes of bodies represented by various extent densities, see Figures 6.1a to 6.1c.

6.1.1. Uniform extent density. The estimation of the volume of a body with extents uniformly distributed between 0 and 1 is shown in Figure 6.1a. The extent density and true volume of a body with extents uniformly distributed between \(a\) and \(b\) are

\begin{align*}
   f_R(r) &= \begin{cases} 
   \frac{1}{b-a} & r \in [a,b] \\
   0 & \text{otherwise}
   \end{cases} \\
   V &= \frac{v_n}{n+1} \sum_{k=0}^{n} a^{k} b^{n-k}
\end{align*}

6.1.2. Beta extent density. The estimation of the volume of a body with a beta(\(\alpha = 2, \beta = 2\)) distribution of extents is shown in Figure 6.1b. The probability density of the general beta(\(\alpha = 2, \beta = 2\)) distribution and the true volume of a body with extents distributed as the general beta(\(\alpha = 2, \beta = 2\)) distribution are

\begin{align*}
   f_R(r) &= \begin{cases} 
   \frac{x^{\alpha-1} (1-x)^{\beta-1}}{B(\alpha,\beta)} & x \in [0,1] \\
   0 & \text{otherwise}
   \end{cases} \\
   V &= v_n \prod_{i=0}^{k-1} \frac{\alpha + i}{\alpha + \beta + i}
\end{align*}

6.1.3. Arcsine extent density. The estimation of the volume of a body with an arcsine distribution of extents is shown in Figure 6.1c. The arcsine distribution is a special case of the beta distribution with \(\alpha = \beta = \frac{1}{2}\).

6.2. Estimation of the volume of convex bodies. For estimation of the volumes of various convex bodies, see Figures 6.2a and 6.2b. We compare against an implementation [6] of the simulated annealing MCMC method [18]. While we compare the number of samples in each method, note that the cost of each sample in simulated annealing MCMC is much higher than the cost of each sample in NSMC. Iterating the point of reference to the nominal centre of the body, using \(\sim n\) pairs of extents of...
Fig. 6.2: The number of samples required in NSMC and an implementation [6] of simulated annealing MCMC [18] to estimate volume to a relative tolerance of 0.1, is plotted against the dimension of the problem. For NSMC, the algorithm was stopped when 1000 consecutive estimates fell within the required relative tolerance of the analytically known true volume. Figures 6.2a and 6.2b show respectively the number of samples required to estimate the volume of a cube with edge 1.0, and ellipsoid with lengths of axes spaced uniformly between 0.5 and 1.0. NSMC was averaged over 10 trials, each trial estimating the volume from a random center of reference. In Figure 6.2a, the random center of reference was uniformly distributed in a co-centered sphere with a diameter 6.25% of the edge of the cube. In Figure 6.2b, the random center of reference was uniformly distributed in an co-centered sphere with a diameter 10% of the longest axis of the ellipsoid. While we compare the number of samples in each method, note that the cost of each sample in simulated annealing MCMC is much higher than the cost of each sample in NSMC.

6.3. Estimation of arbitrary integrals. For estimation of various integrals, see Figures 6.1d to 6.1f. The irregular domain chosen has a very large eccentricity with its extents distributed uniformly between 0 and 1. The Gaussian integrand is in itself radially symmetric, but note that it is sensitive to any small errors in sampling such an eccentric domain when \( n \) is large. The next example given by a polynomial includes an additional oscillatory behavior, but the proposed hybrid approach is robust for such integrands as well. On the other hand, the example called the x-coordinate integrand is highly asymmetric radially. The integrals chosen have exact analytical expressions to confirm convergence for all dimensions, as they can indeed be reduced to functions of a single effective variable. Note that in these figures, the number of samples required in NSMC and an implementation [6] of simulated annealing MCMC [18] to estimate volume to a relative tolerance of 0.1, is plotted against the dimension of the problem. For NSMC, the algorithm was stopped when 1000 consecutive estimates fell within the required relative tolerance of the analytically known true volume. Figures 6.2a and 6.2b show respectively the number of samples required to estimate the volume of a cube with edge 1.0, and ellipsoid with lengths of axes spaced uniformly between 0.5 and 1.0. NSMC was averaged over 10 trials, each trial estimating the volume from a random center of reference. In Figure 6.2a, the random center of reference was uniformly distributed in a co-centered sphere with a diameter 6.25% of the edge of the cube. In Figure 6.2b, the random center of reference was uniformly distributed in an co-centered sphere with a diameter 10% of the longest axis of the ellipsoid. While we compare the number of samples in each method, note that the cost of each sample in simulated annealing MCMC is much higher than the cost of each sample in NSMC.

a body in directions \( \hat{s} \) and \( -\hat{s} \) is relatively trivial in NSMC, and for bodies of higher reflection symmetries this convergence is faster. The presented results average over varying origins uniformly distributed in a sphere co-centred with the convex body.

Since the extent density of a convex shape is a function of dimension \( n \), the expected samples required by NSMC need not a monotonic function of \( n \). For example, in the case of cube, there is a reduction of the required number of the extent samples by a factor \( \sim 2^n \) due to its symmetry, but the \( n^{th} \) moment of the extent density increases approximately as \( (\sqrt{\frac{\pi}{2}})^n \). This results in very favorable comparison of the naive NSMC with MCMC up to moderate values of \( n \), and this advantage over MCMC is lost as \( n \) becomes larger than 100 where the extent density becomes tailed.
samples indicates the number of direction vectors sampled. This does not include the cost of the deterministic quadrature in evaluating $i(\hat{s})$ along a direction. The precise cost of this quadrature depends on the integrand, but note that the cost of evaluating a given scalar function $h$ increases as $\mathcal{O}(n)$ with the number of cardinal directions $n$, and the number of evaluations of the integrand $\rho^{n-1}h$ required for the quadrature also increase approximately as $n$, making the computing effort in evaluating $i(\hat{s})$ scale at most as $\mathcal{O}(n^2)$. The examples demonstrate the $\mathcal{O}(n)$ scaling of the number of random samples required on $S^{n-1}$ with the dimension $n$ of the non-convex domain of an arbitrary integral, with the overall computing effort thus scaling as $\mathcal{O}(n^3)$ at most. Some problems of integration where domains represent a tailed distribution of large extents with appropriately aligned highly asymmetric integrands, can render the above approach ineffective. Such special cases require an important sampling of the partial integral $i(\hat{s})$, and they will be addressed elsewhere.

6.3.1. Gaussian integrand. The estimation of the integral of the following radially symmetric Gaussian integrand, where $r$ is the radial coordinate, over a domain with extents distributed uniformly between 0 and 1 is shown in Figure 6.1d.

(6.5) 
$$h(r) = \exp\left(-\frac{r^2}{2}\right)$$

If $\gamma$ is the lower incomplete gamma function, then the true partial integral along a direction $\hat{s}$ and the true integral over a domain with extents uniformly distributed between 0 and $r_0$ are

(6.6) 
$$i(\hat{s}) = 2^{\frac{n+1}{2}}\gamma\left(\frac{n}{2}, \frac{(S(\hat{s}))^2}{2}\right)$$

(6.7) 
$$I = \frac{2^{\frac{n+1}{2}}s_n}{r_0} \left[ \frac{r_0}{\sqrt{2}} \gamma\left(\frac{n}{2}, \frac{r_0^2}{2}\right) - \gamma\left(\frac{n+1}{2}, \frac{r_0^2}{2}\right) \right]$$

6.3.2. Polynomial integrand. The estimation of the integral of the following radially symmetric oscillatory polynomial integrand, where $r$ is the radial coordinate, over a domain with extents distributed uniformly between 0 and 1 is shown in Figure 6.1e.

(6.8) 
$$h(r) = (r - 0.25)(r - 0.50)(r - 0.75) = r^3 - 1.5r^2 + 0.6875r - 0.09375$$

For a general polynomial of the form below with $a_k$ as its coefficients,

(6.9) 
$$h(r) = \sum_{k=0}^{m-1} a_k r^k$$

the true partial integral along a direction $\hat{s}$ and the true integral over a domain with extents uniformly distributed between 0 and $r_0$ are

(6.10) 
$$i(\hat{s}) = \sum_{k=0}^{m-1} \frac{a_k}{n+k} (S(\hat{s}))^{n+k}$$

(6.11) 
$$I = s_n r_0^n \sum_{k=0}^{m-1} \frac{a_k}{(n+k)(n+k+1)} r_0^k$$
6.3.3. x-coordinate integrand. The estimation of the integral of the following radially asymmetric integrand that maps a vector \( \vec{x} \) to the absolute value of its coordinate along the first cardinal direction, over a domain with extents uniformly distributed between 0 and 1 is shown in Figure 6.1f. Here, \( \hat{x}_1 \) is the unit vector along the first cardinal direction.

\[
\begin{align*}
(6.12) & \quad h(\vec{x}) = |\vec{x} \cdot \hat{x}_1| \\
(6.13) & \quad i(\hat{s}) = \frac{\{S(\hat{s})\}^{n+1}}{2\pi^2} \frac{s_{n+3}}{s_n} \\
(6.14) & \quad I = \frac{s_{n+3}}{2\pi^2(n + 2)} r_0^{n+1}
\end{align*}
\]

Appendix A. Multi-valued extent function.

In case the extent function \( S \) is multi-valued (see Figure 3.1b), the volume of a body, whether it is simply connected or not, is

\[
(A.1) \quad V = \oint_{S^{n-1}} \left\{ \sum_{\text{odd } j} \int_{S_{j-1}(\hat{s})}^{S_j(\hat{s})} \rho^{n-1} \, d\rho \right\} \, d\hat{s}
\]

with \( j = 1, 2, 3, \ldots \), and the above can again be reduced to the statistical estimate of the volume as

\[
(A.2) \quad V = \frac{1}{n} \oint_{S^{n-1}} r^n \, d\hat{s} = \left\{ \frac{1}{n} \oint_{S^{n-1}} d\hat{s} \right\} \left\{ \int_0^\infty r^n f_R(r) \, dr \right\}
\]

with the random extent \( R \) now generalized as

\[
(A.3) \quad R^n = \sum_j (-1)^{j+1} R_j^n
\]

where \( R_1 < R_2 < R_3 \ldots \) are the random extents representing multiple values \( S^j \) for a given direction \( \hat{s} \), and \( S^0 = 0 \) always. Note that the largest natural number \( j \) representing number of extents in a given direction, is always odd for a closed body defined by a bounding surface \( S \) around the origin of reference. In case the origin is outside the closed body, the number of extents is even valued and this can be treated by a simple negation of signs in the above equation defining the generalized extents.

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