Projective synchronization of a 4D financial hyper-chaotic system with model uncertainty and external disturbance

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Abstract. This paper investigates the projective synchronization problem of a 4D financial hyper-chaotic system. Firstly, it is proven that projective synchronization problem of such system exists. Moreover, a non-singular transformation is found and used to divide the 4D financial hyper-chaotic system into two subsystems. Secondly, a UDE-based single input controller is proposed and be used to realize this projective synchronization problem. Finally, the correctness and effectiveness of the proposed results is verified by numerical simulations.

1. Introduction

It is well known that Lorenz first proposed the chaotic system in 1963. Chaotic systems and the corresponding control problems has been investigated form all kinds of subjects, especially the social sciences since that time, for details, please see Refs. [1-9]. However, up to date, some important questions still exist and are to be solved in the future. Such as, for a given chaotic system, whether the projective synchronization problem exists or not is a very important foundation question, and it has been not solved completely. How to design a controller to meet the performance that the projective synchronization problem of the studied chaotic systems where the model uncertainty and external disturbance are included. Luckily, the UDE-based control method [8] is an effective method to deal robust control problem of the nonlinear systems, thus we shall apply this method to investigate the robust projective synchronization of those chaotic system with uncertainty and disturbance.

The new 4D financial hyper-chaotic system [3] was firstly proposed in 2001. It has complex dynamics and. there are many results about this system has been published, but the projective synchronization problem of this system has not been investigated carefully, especially when this system has uncertainty and disturbance. Therefore, investigating the projective synchronization problem of the new 4D financial hyper-chaotic system with uncertainty and disturbance is very interesting in both applications and theory, which motivates our present work.

We firstly introduce the projective synchronization problem. For the following system

\[ \dot{w} = h(w) \]  

where \( w \in R^n \) is the state, \( h(w) \in R^n \) is a nonlinear function vector and is assumed to be continuous.

As it is stated in [7], for the system (1), the of projective synchronization problem exists if and only if the system (1) can be divided into the following two subsystems

\[ X = M(Z)X \]

\[ \dot{Z} = N(X, Z) \]  

where \( X, Z \in R^n \) are the state variables of the master and slave systems respectively.
where

\[
\begin{pmatrix}
X \\
Z
\end{pmatrix} = Tw
\]

\( T \in \mathbb{R}^{m \times n} \) and \( |T| \neq 0 \), \( M(Z) \in \mathbb{R}^{m \times n} \) is a matrix about \( Z \), and \( X \in \mathbb{R}^m, 1 \leq m < n, Z \in \mathbb{R}^{n-m} \).

Let the system (1) be the master system, then the slave system is given as

\[
\dot{Y} = M(Z)Y + BU
\]

(4)

where \( Y \in \mathbb{R}^m \) and \( Z \) is given in Eq. (3), \( B \in \mathbb{R}^{m \times r} \) is a constant matrix, \((M(Z), B)\) is assumed to be controllable whatever \( Z \) is, \( U \in \mathbb{R}^r, r \geq 1 \) is the designed controller.

Let \( e = Y - \alpha X \), where \( \alpha \neq 0,1 \), and the error system is given as

\[
\dot{e} = M(Z)e + BU
\]

(5)

where \( e \in \mathbb{R}^m \) is the state.

If the controller \( U \) can make the system (5) be asymptotically stable, then the projective synchronization of the system (1) is realized. When the system (1) has uncertainty and disturbance, how to realize this problem?

Inspired by the aforementioned conclusions, the projective synchronization problem of a 4D financial hyper-chaotic system is investigated. In the first place, it is proven that the projective synchronization problem for such system exists. Moreover, a non-singular transformation is found and used to divide the 4D financial hyper-chaotic system into two subsystems. Secondly, a UDE-based single input controller is proposed. At last, the correctness and effectiveness of the proposed results is verified by numerical simulations.

2. Problem formation

According to [3], the 4D financial hyper-chaotic system is presented as

\[
\dot{x} = f(x) = \begin{pmatrix} (x_2 - 0.9)x_1 + x_3 + x_4 \\ 1 - 0.2x_2 - x_1^2 \\ -x_3 - 1.5x_4 \\ -0.2x_1x_2 - 0.17x_4 \end{pmatrix} + u_d + bu
\]

(6)

where \( x \in \mathbb{R}^4 \) is the state, \( u_d = \Delta A(Z) + d(t) \) is the whole of the model uncertainty and the external disturbance, \( b \in \mathbb{R} \) and \( u \in \mathbb{R} \) is the controller to be designed, which are given as follows

\[
u_d = \begin{pmatrix} 0.1x_1x_3 + 100 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}
\]

(7)

The system (6) is selected as the master system, and the corresponding slave system is

\[
\dot{y} = f(y) = \begin{pmatrix} (y_2 - 0.9)y_1 + y_3 + y_4 \\ 1 - 0.2y_2 - y_1^2 \\ -y_1 - 1.5y_3 \\ -0.2y_1y_2 - 0.17y_4 \end{pmatrix}
\]

(8)

where \( y \in \mathbb{R}^5 \) is the state.

The main goal of this paper is to design a controller \( u \) to meet the following performance
\[
\lim_{t \to \infty} ||e(t)|| = \lim_{t \to \infty} ||y(t) - \alpha x(t)|| = 0
\]
where \(\alpha\) is a scalar and \(\alpha \neq 0,1\).

3. Main results
Firstly, we prove that the projective synchronization problem of the system (6) exists and present a conclusion.

**Theorem 1** The system (6) with \(u_d = 0\) is divided into the following two subsystems
\[\dot{X} = M(Z)X + U_d + BU\]
\[\dot{Z} = N(X,Z)\]
where \(X \in \mathbb{R}^3, Z \in \mathbb{R}^2\), and
\[
X = \begin{pmatrix} x_1 \\ x_3 \\ x_4 \end{pmatrix}, Z = x_2, M(Z) = \begin{pmatrix} Z - 0.9 & 1 & 1 \\ -1 & -1.5 & 0 \\ -0.2Z & 0 & -0.17 \end{pmatrix}
\]
\[
B = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, U_d = \begin{pmatrix} 0.1X_1X_3 + 100 \\ 0 \\ 0 \end{pmatrix}
\]
\[
N(X,Z) = 1 - 0.2Z + X_1^2
\]
which implies that the projective synchronization problem of the system (6) exists. It is noted that \((A(Z), B)\) is controllable whatever \(Z\) is.

**Proof** It is easy to determine that
\[
\gamma = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{pmatrix} = \begin{pmatrix} \alpha \\ \alpha \\ \alpha \\ 1 \end{pmatrix}
\]
is a solution of the following equations about \(\beta\)
\[
\begin{align*}
\beta_1\beta_3 &= \beta_1 \\
\beta_1\beta_2 &= \beta_3 \\
\beta_3 &= \beta_4
\end{align*}
\]
Thus the conclusion holds, which completes the proof.

Secondly, the system (11) is set as the master system, and the controlled slave system is presented as follows
\[\dot{Y} = M(Z)Y\]
where \(Y \in \mathbb{R}^3\) is the state and \(Z \in \mathbb{R}\) is presented in Eq. (11).

Let \(e = Y - \alpha X\), where \(|\alpha| \neq 0,1\), and the error system is described as follows
\[\dot{e} = M(Z)e + U_d + BU\]
where \(E \in \mathbb{R}^3\) is the state, \(B, U_d\) are presented in Eq. (12).

It is noted that the error system (18) has a simple form, we design a single input controller and present the following conclusion.
Theorem 2 For the error system (18). If a designed filter \( g_f(t) \) meets the following performance:

\[
\hat{U}_d = \hat{U}_d - U_d, \quad t \to \infty
\]  

(18)

where \( \hat{U}_d = (\hat{e} - F(e) - BU_{ude}) \ast g_f(t) \), then the single input UDE-based controller \( U \) is proposed as follows

\[
U = U_z + U_{ude}
\]  

(19)

\[
U_z = -e_i Z - e_2 - e_3
\]  

(20)

\[
U_{ude} = B^\ast \left\{ \ell^{-1} \left[ \frac{G_f(s)}{1 - G_f(s)} \right] \ast (-0.9 e_i) - \ell^{-1} \left[ \frac{s G_f(s)}{1 - G_f(s)} \right] \ast e_i \right\}
\]  

(21)

\[
F(X,e) = M(Z)X + BU_z
\]  

(22)

Proof. Substituting the controller \( U \) in Eq. (19) into the system (17), we get

\[
\dot{e} = M(Z)e + BU_z + U_d + BU_{ude} = F(X,e) + U_d + BU_{ude}
\]

Noticing that \( \dot{e} = F(e) \) is globally stable and \( BU_{ude} = \hat{U}_d \), therefore the system (17) is globally stable, which implies that the projective synchronization between the system (9) and the system (16) is realized by the above controller.

4. Numerical simulations

Choosing the initial conditions: \( X_0 = [5 \ -4 \ 6], Y_0 = [-2 \ 5.2 \ -4.6], Z(0) = 9, \alpha = 2 \). From Figure 1, we observed that the states of the system (17) converge to origin. Figure 2 shows that the projective synchronization between the system (9) and the system (16) is achieved. It can be seen that the two figures in Figure 2 are same, but the axis of the lower is twice the same as the axis of the upper one. Figure 3 shows that \( \hat{u}_d \) converges to \( u_d \) as \( t \to \infty \).

![Fig 1: the states the system (17) converge to origin](image-url)
Fig 2: the phase portrait of the system (9) and the system (16)

Fig 3 $\hat{u}_d$ converges to $u_d$ as $t \rightarrow \infty$

5. Conclusions
The projective synchronization problem of a 4D financial hyper-chaotic system has been investigated. In the first place, it has proven that the projective synchronization problem of such system existed, and the 4D financial hyper-chaotic system has been into two subsystems by a non-singular transformation. Secondly, a UDE-based single input controller has been proposed to realize the projective synchronization problem. At last, the correctness and effectiveness of the proposed results has been verified by numerical simulations.

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