UNIVERSITY OF ABERDEEN
Department of Physics

SPACETIME CONFORMAL FLUCTUATIONS AND QUANTUM DEPHASING

A thesis presented for the degree of Doctor of Philosophy
at the University of Aberdeen

Paolo BONIFACIO
Master Degree Università degli studi di Milano, ITALY

2008/09
Abstract

Any quantum system interacting with a complex environment undergoes decoherence and loses the ‘ability’ to show genuine quantum effects. Empty space is filled with vacuum energy due to matter fields in their ground state and represents an underlying environment that any quantum particle has to cope with. In particular quantum gravity vacuum fluctuations should represent a universal source of decoherence. It is important to assess which parameters control such an effect, also in relation to the issue of gaining experimental access to Planck scale physics.

To this end we employ a stochastic approach that models spacetime fluctuations close to the Planck scale by means of a classical, randomly fluctuating metric (random gravity framework). We enrich the classical scheme for metric perturbations over a curved background by also including matter fields and metric conformal fluctuations. We show in general that a conformally modulated metric induces dephasing as a result of an effective nonlinear newtonian potential obtained in the appropriate non-relativistic limit of a minimally coupled Klein-Gordon field. The special case of vacuum fluctuations is considered and a quantitative estimate of the expected effect deduced.

Secondly we address the question of how conformal fluctuations could physically arise. By applying the random gravity framework we first show that standard GR seems to forbid spontaneous conformal metric modulations. Finally we argue that a different result follows within scalar-tensor theories of gravity such as e.g. Brans-Dicke theory. In this case a conformal modulation of the metric arises naturally as a result of the fluctuations in the Brans-Dicke field and quantum dephasing of a test particle is expected to occur. For large negative values of the coupling parameter the conformal fluctuations may also contribute to alleviate the well known problem of the large zero point energy due to quantum matter fields.
Declaration

This thesis has been composed by the candidate. This thesis has not been accepted in any previous application for a degree. The work described in this thesis has been done by the candidate. All quotations have been distinguished by quotation marks and the sources of information are specifically acknowledged.

Chapter 1 is revised and extended from: Paolo M. Bonifacio, Charles H.-T. Wang, J. Tito Mendonca, Robert Bingham 2009 Dephasing of a non-relativistic quantum particle due to a conformally fluctuating spacetime, Class. Quantum Grav. 26 145013

Signed:

June 2009
Acknowledgements

First of all I want to thank my supervisor, Dr Charles Wang, for giving me the chance to do a PhD and for all his support and most valuable scientific advice. I also thank Dr Norval Strachan and the department of Physics for giving me the opportunity to work as a lecturer during the final phase of my PhD research. It has been challenging but highly rewarding. Special thanks also goes to STFC Centre for Fundamental Physics and to Prof Bingham. Various people contributed to make this work possible through their knowledge and stimulating discussions. Without any preferred order I wish to thank: Tito Mendonça, Luigi Galgani, Andrea Carati, Victor Varela, Antonaldo Diaferio, Daniele Bertacca, Rui Reis.

I am deeply grateful to my parents for their support and amore, as well as to all the beautiful friends I met during these years. For their love, company, help and understanding during the good and the tough times. Thanks Dana, Christian, Costantino, Valerio, Cesar & Andrea (or the G5), Mauro, Margaux, Luca, Marko, Jeremy, Fabien, Alice, Ida, Cristian, Lourenço, Annalise, Lavanya, Kenny, Gabriel, Tiziano, Alessio. At last, a sweetly refined thought goes to HannaH, this palindromic special creature that keeps on crossing my path. Thanks to the music, eternal invisible friend and companion.
Sbattuti e stanchi
di guerreggiar tant’anni, e risospinti
ancor da’ fati, i greci condottieri
a l’insidie si dièro; e da Minerva
divinamente instrutti, un gran cavallo
di ben contesti e ben confitti abeti
in sembianza d’un monte edificaro.
Poscia, finto che ciò fosse per voto
del lor ritorno, di tornar sembiante
fecero tal, che se ne sparse il grido.
Dentro al suo cieco ventre e ne le grotte,
che molte erano e grandi, in sí gran mole,
rinchiuser di nascosto arme e guerrieri
a ciò per sorte e per valore eletti.

Virgil, Aeneid Book II, 29-19 BC

To my parents

E C. il ricercatore chiese a F. il musicista:
“Perche’ ti interessi di Fisica e Filosofia della Natura?”
“Ma perche’ e’ fondamentale!”, fu la risposta.
# Contents

Notation and conventions ........................................... 1

Introduction and motivations ..................................... 2
  Probing Quantum Gravity? ........................................ 2
  Brief description and salient points of this work ............... 5

1 Dephasing of a non-relativistic quantum particle due to a conformally fluctuating spacetime ........................................... 9
  1.1 Introduction .................................................. 9
  1.2 Low velocity limit and effective Schrödinger equation .... 11
  1.3 Average quantum evolution .................................... 15
    1.3.1 Dyson expansion for short evolution time ............. 15
  1.4 The conformal field and its correlation properties ......... 17
    1.4.1 Summary of the correlation properties of the conformal fluctuations ............. 18
  1.5 Dephasing calculation outline ................................ 18
    1.5.1 First order terms of the Dyson expansion .............. 18
    1.5.2 Second order terms of the Dyson expansion ............. 19
  1.6 General density matrix evolution for large drift times .... 21
    1.6.1 Correlation time and characteristic function ......... 22
    1.6.2 A remark on the validity of the Dyson expansion ....... 25
  1.7 Explicit dephasing in the case of vacuum fluctuations ...... 26
    1.7.1 Isotropic power spectrum for vacuum conformal fluctuations ............. 26
  1.8 Discussion ................................................... 29
    1.8.1 Probing the particle resolution scale and effective dephasing ............. 29
    1.8.2 Validity of the long drift time regime ................. 30
    1.8.3 Some numerical estimates and outlook ................... 31

2 The nonlinear random gravity framework ....................... 35
  2.1 Seeking a physical basis for the conformal fluctuations .... 35
    2.1.1 The simplest scenario: a conformally modulated universe ............. 36
    2.1.2 The general basis for the model ........................ 37
  2.2 Vacuum and some related problems ........................... 37
  2.3 Nonlinear random gravity .................................... 38
    2.3.1 Stochastic geometry ..................................... 39
    2.3.2 Definition of the relevant scales ....................... 41
    2.3.3 Matter fields stress energy tensor ...................... 42
  2.4 Characterizing vacuum at the random scale .................. 45
2.5 Setting up the expansion scheme ........................................... 47
  2.5.1 Einstein tensor expansion ........................................... 47
  2.5.2 Terms dependent on the conformal factor ............................... 48
  2.5.3 The expansion vacuum equations on a curved background .......... 50
2.6 Considerations on the cosmological constant problem ..................... 50
  2.6.1 Choice of the background geometry ................................... 52
  2.6.2 Simplified form of the expansion equations ............................ 53

3 GR and conformal fluctuations ......................................................... 54
  3.1 Analysis of the linear equation and GWs .................................. 54
  3.2 Analysis of the second order equation .................................... 56
  3.3 Averaged second order equation and zero point energy balance .......... 60
  3.4 Equivalence of the physical and conformal metric formalisms .......... 61
  3.5 Are spontaneous conformal fluctuations compatible with GR? ............ 65
    3.5.1 Attempt for a solution of the second order equation ................. 65
    3.5.2 Second order constraint on the conformal field ..................... 66
    3.5.3 An alternative scenario? conformal metric and GWs ................ 68
  3.6 Summary ................................................................................. 70

4 Scalar-tensor theories and conformal fluctuations ..................................... 72
  4.1 Conformally invariant physics and Bekenstein’s theory .................... 73
    4.1.1 Conformal transformations as local transformation of physical units 73
    4.1.2 CI action for a massive scalar field .................................. 74
    4.1.3 Conformal invariance of Standard Model fields ..................... 75
    4.1.4 Bekenstein’s gauge field and GR as a natural consequence of CI .... 76
    4.1.5 Gravitational equations in an arbitrary conformal frame ............ 78
    4.1.6 Inclusion of matter fields ............................................ 82
  4.2 Scalar-Tensor Theories of gravity ........................................... 84
  4.3 Brans-Dicke theory ..................................................................... 86
  4.4 Random gravity framework and Brans-Dicke theory ......................... 87
    4.4.1 Brans-Dicke field fluctuations at the random scale ................. 89
  4.5 Expansion equations and vacuum solution ................................... 90
    4.5.1 First order solution ................................................. 90
    4.5.2 Second order equation ............................................. 91
  4.6 Second order equation solution ............................................. 93
  4.7 Discussion and outlook ................................................................ 94

A Stochastic scalar waves and generalized Wiener-Khintchine theorem .......... 96
  A.1 General solution to the wave equation ..................................... 96
    A.1.1 Real waves .......................................................... 97
    A.1.2 Decomposition in components traveling along different directions 97
  A.2 Stochastic waves ...................................................................... 97
    A.2.1 Average and correlation properties of the fluctuations .......... 98
    A.2.2 Characterization of the stochastic waves .............................. 99
  A.3 Wiener-Khintchine theorem for a stochastic process ...................... 100
    A.3.1 Standard 3D case ................................................... 100
  A.4 Generalized Wiener-Khintchine Theorem for stochastic waves ............. 101
# Contents

**A.4.1** Evaluation of $\alpha(k)$ and $\beta(k)$ ........................................... 102  
**A.4.2** Real functions ................................................................................. 102  
**A.5** Correlation properties of wave components in different directions ............. 103  
**A.5.1** Real functions ................................................................................. 104  
**A.6** Isotropic power spectrum and field averages ........................................... 104  
**A.7** Treatment of the term $T_4$ in the effective Schrödinger equation ................ 106

**B** Technical derivations related to Chapter 1 ............................................... 107  
**B.1** Separability of the kinetic and potential part in the Dyson expansion ............ 107  
**B.2** Nonlinear part of the potential and density matrix evolution ....................... 110  
**B.3** An integral identity ............................................................................... 115  
**B.4** Fourth order term in the Dyson expansion ............................................. 116

**C** Zero point energy density and pressure of matter fields in the free field approximation ................................................................. 119  
**C.1** Modeling the zero point energy of a massive scalar field ............................ 119

**D** Autoconsistent theory of metric perturbations over a curved background .......... 122  
**D.1** Perturbation theory on a flat background ................................................ 122  
**D.1.1** Linear approximation ........................................................................ 122  
**D.1.2** TT gauge ............................................................................................ 123  
**D.1.3** Second order corrections .................................................................... 124  
**D.2** Non covariant exact definition of gravity ‘stress energy tensor’ .................... 127  
**D.3** Isaacson’s perturbation theory on a curved background geometry ............... 128  
**D.3.1** Slow varying vs. fast varying components: the main assumptions .......... 128  
**D.3.2** GWs propagating on a curved background: autoconsistent case .............. 129  
**D.3.3** GWs stress energy tensor ..................................................................... 131  
**D.3.4** Definition of the spacetime averaging procedure .................................... 135

**E** Quantum physics background ................................................................... 137  
**E.1** State vector and unitary evolution ............................................................ 137  
**E.2** The density operator ............................................................................... 140  
**E.2.1** Unitary evolution of the density operator .............................................. 143  
**E.2.2** Proper and improper mixtures ............................................................... 144  
**E.3** Basic concepts in decoherence theory ...................................................... 146

**Bibliography** ................................................................................................. 156
Notation and conventions

We use metric signature \((-, +, +, +)\) and, unless specified, geometrized unities with \(G = c = 1\). The coordinates in a given system are denoted as \(x = (x^0, x^1, x^2, x^3) \equiv (x^0, \mathbf{x})\), in such a way that the 0-th coordinate represents time. Latin indices from the beginning of the alphabet such as \(a, b, c, d\ldots = 0, 1, 2, 3\) are used for spacetime tensors. Coordinates equations holding for spacial components will be written using latin indices from the middle of the alphabet \(i, j, k, l\ldots = 1, 2, 3\). The physical metric tensor is denoted as \(g_{ab}\) and its inverse is \(g^{ab}\). When we need to highlight that a tensor, say \(T_{ab}\), depends functionally upon other tensors \(A_{a\ldots b\ldots}, B_{a\ldots b\ldots}, \ldots\), we will write \(T_{ab}[A, B, \ldots]\). The notation for a totally symmetric and totally antisymmetric tensor is

\[
T_{(ab)} := \frac{1}{2}(T_{ab} + T_{ba}),
\]

\[
T_{[ba]} := \frac{1}{2}(T_{ab} - T_{ba}).
\]

The Christoffel symbol is defined as

\[
\Gamma^c_{ab} = \frac{1}{2}g^{cd}\left\{\partial_a g_{bd} + \partial_b g_{ad} - \partial_d g_{ab}\right\}.
\]

The sign definition for the Riemann tensor is the same as in Wald [1]

\[
\nabla_a \nabla_b \omega_c - \nabla_b \nabla_a \omega_c = R_{abc}{}^d \omega_d,
\]

where \(\omega_d\) is a covariant vector field. This implies the following expression for the Ricci tensor:

\[
R_{ab} := R_{acb}{}^c = \partial_c \Gamma^c_{ab} - \partial_a \Gamma^c_{cb} + \Gamma^c_{ab} \Gamma^d_{cd} - \Gamma^d_{cb} \Gamma^c_{da}.
\]

The notation \(h_{ab\ldots}^{(n)}\) indicates a \(n\)-th order quantity. Accordingly, when expansions are made, e.g. of the Ricci tensor, the notation \(\hat{R}_{ab}[h^{(1)}]\) indicates the part of the Ricci tensor which is second order in the linear perturbation \(h_{ab}^{(1)}\).
Introduction and motivations

Probing Quantum Gravity?

General relativity (GR) [1] and quantum field theory (QFT) [2] are both very successful theories of nature. Besides providing a deeper understanding of how gravity works by relating it to the curvature of spacetime, GR can be applied to a wide variety of physical domains, ranging from the solar system to the whole universe. So far it has ‘passed’ various crucial experimental tests [3], including anomalous perihelion shift of Mercury, light deflection, as well as energy loss in pulsar binary systems [4] which provided indirect evidence for the existence of gravitational waves (GWs). The large scale application of GR culminated in the standard cosmological model. Through a balanced interplay of theoretical inputs and observations, this is converging in providing a credible description of the visible universe and its evolution. Future direct detection of gravitational radiation, e.g. from projects such as LIGO [5] or LISA [6], could provide further spectacular evidence in favor of GR.

The developments of quantum theory into a relativistic theory of quantum fields has converged into the standard model of particle physics [7]. This shed light onto the nature of elementary particles and, though still plagued by rather technical subtleties, provides a quite a coherent framework which has been so far spectacularly verified experimentally in basically all respects. The only key element that still fails to be detected to date is the Higgs boson, related to the mechanism at the origin of mass. The fact that the newly built LHC [8] could provide future direct evidence of this elusive particle is an exciting possibility which would put the standard model on an even more solid basis.

Both GR and QFT are based on important symmetry principles. The general relativistic framework extends Lorentz invariance between inertial frames to a deeper invariance under arbitrary coordinates transformation (covariance). The standard model is based, beyond the principle of Lorentz invariance, upon a certain number of gauge symmetries, i.e. it enjoys invariance under some special kind of internal (non-spacetime) transformations. An example is the Dirac field for the electron, whose local invariance under arbitrary phase transformations necessarily implies the existence of an extra gauge field with which it interacts. This is clearly the electromagnetic field. The resulting QED theory describes electrons interacting with the quantized radiation field and was the first example of successful gauge theory based upon a \( U(1) \) type symmetry. Other examples are the Weinberg-Salam electroweak theory, where a \( SU(2) \) type symmetry is used to unify the weak and electromagnetic interactions, and the QCD describing the strong interaction between quarks and gluons, based upon the \( SU(3) \) symmetry.

Though successful in their various domains of application, GR and QFT are deeply different at the conceptual level. GR is a classical theory where the evolution of the spacetime metric \( g_{ab} \) stems deterministically from classical field equations. Even though extremely complicated to do in any realistic situation, one can in principle solve Einstein’s equation once all the other classical matter fields and their coupling to gravity are specified. The evolution is determined precisely from the initial data and there are no intrinsic limits to one’s ability to know precisely the matter fields or gravity
configuration at a given spacetime point. An essential feature of the theory is that of being background independent: the spacetime metric is not fixed a priori but it is itself a physical actor, both affected by and affecting the matter fields distribution. In any practically simple situation in which an exact solution of Einstein’s equation can be found analytically, together with the included matter fields, then the past, present and future of the physical system are completely determined and mathematically described by a well defined manifold. These features imply that GR is successful when applied to large scale systems, where the underlying microphysics can be bypassed and described in classical macroscopic terms.

The situation is radically different for QFT and the standard model. These deal with physical entities that are believed to rank among the most elementary constituent of the universe. Even though technically complex, the basic principles at the heart of QFT are those of quantum mechanics. This makes QFT a fundamentally statistical theory, in which possible predictions are probabilities for certain events (such as scattering, particles production from the vacuum, field strengths, number of fields quanta) to happen. The state of a given system is described by a state vector \( \psi \) in a Hilbert space satisfying a Schrödinger -like equation with a suitable Hamiltonian operator. Even though the evolution of the state vector describing a closed system is deterministic, it is impossible, even in principle, to foresee the outcome of a particular measurement. The indeterminism inherent to any quantum theory occurs in a rather subtle way involving the interaction of the physical system under exam and some ‘measuring’ apparatus. It appears that, whenever an interaction and subsequent measurement take place, the state vector collapses, *unpredictably*, to a new state. The related probability is the only information that the quantum theory apparatus enables one to know in advance. Another important feature of QFT that contrasts with GR is that the former is formulated mathematically using a fixed non-dynamical flat spacetime geometry. A quite successful formulation of QFT on a given *fixed* curved background geometry also exists. This allowed to predict important effects such as e.g. particle creation from the vacuum, effects of vacuum fluctuations and polarizations in the early universe and Hawking radiation from black holes. However the passage from a flat Minkowski to a general background involves lots of non trivial issues [9]. Moreover QFT on a curved background neglects the backreaction effect of quantum matter onto the spacetime geometry, which is treated as non-dynamical.

It is widely believed that the ultimate theory of nature should unify GR and QFT into a more comprehensive scheme that could then be applied both to the macroscopic classical world and to the microscopic unpredictable world. The dominant belief is that, within such a unified framework, gravity should be quantized as well. The eventuality that gravity is a classical field coupled to, say, the expectation value of the quantum matter fields leads to theoretical inconsistencies [10, 11] also supported by experiments [12]. The resulting Quantum Gravity (QG) theory [13] is currently the ‘holy grail’ of physics. Beyond esthetic and philosophical reasons, according to which a unified description of nature is very desirable, a valid QG theory will surely prove to be the most valuable tool to unveil new physics. This will probably shed new light upon concepts such as singularities, black holes, the big bang and the origin of the universe, as well as the deeper nature of space and time.

Since the pioneering times of Einstein, Heisenberg and Dirac among the others, people have been trying to build a coherent theory, so far with no success. Various approaches have been attempted, from the early covariant perturbation method (plagued by the issue of non-renormalizability [14, 15]) to effective field theories [16], path integral method, Euclidean path integral approach [11], Penrose twistor approach, supergravity theories, lattice theories, dynamical triangulation [13], canonical geometrodynamics [17]. The two approaches that are being most widely pursued and are believed by many to turn eventually into the right direction are String Theory [18] and Loop Quantum Gravity.
(LQG) [19]. String Theory is closer in spirit to a QFT (though of a rather particular type since the physical particles arise here as excitation states of the string fields). It must recur to perturbative methods over a background in order to deduce the particles spectrum of which the spin 2 massless graviton would be a particular member. On the other hand LQG (based upon a development of the canonical quantization method) sticks closer to the original spirit of GR. It is a less ambitious program in that it does not attempt to unify physics as a whole. However the quantization program is standard in the sense that it is applied to GR once this is recast into Hamiltonian form by suitable canonical transformations. Once the so called Ashtekar variables [20] are introduced to replace the metric by the affine connection and the correct dynamical variables identified, the quantization is relatively straightforward. Within LQG, lengths, areas and volumes become themselves operator whose spectrum has been calculated. This yields a microscopic vision of the spacetime structure in terms of elementary ‘chunks’ of geometry, i.e. areas and volumes which can be described using the spin network theory.

Quite independently of the specific form of the correct QG theory, it is generally believed that spacetime should have non-trivial microscopic properties. These could include complex topologies or an intrinsic discreteness (as emerging in the LQG approach) and, definitely, a non classical behavior [21]. By becoming an operator, the spacetime metric is expected to have unpredictable configurations at the small scale; moreover any viable theory should be able to specify how an appropriate classical limit yielding a smooth metric obeying Einstein’s equation can arise. On simple dimensional grounds QG is expected to alter significantly the structure of spacetime and the related physics at the Planck scale $L_P \approx 10^{-35}$ m. The corresponding Planck energy $E_P \approx 10^{19}$ GeV is far beyond direct reach of even the most powerful particle accelerators such as LHC (even though it is still hoped that this could provide hints for supersymmetry or hidden dimensions). Nonetheless finding experimental access to QG phenomena is a matter of the greatest importance. Various possibilities are being considered, e.g. observing primordial black holes, QG induced anisotropy on the CMB, varying coupling constants, violation of equivalence principle and/or of Lorentz invariance, modified dispersion relations for distant EM radiation due to the discrete microstructure of spacetime [13, 11]. A possibility that also receives attention is that according to which vacuum fluctuations of the spacetime geometry could affect the dynamics of a test quantum particle coupled to gravity: it is expected that microscopic Planck scale physics may be ‘amplified’ at a macroscopic observable scale as a decoherence ‘signal’ [22, 23, 24, 25].

Decoherence is a well known and studied phenomena which occurs whenever a quantum particle is coupled to an environment with many degrees of freedom [26, 27]. Extensive studies have begun in the 70s. Today it is widely believed that decoherence can play an important role in explaining the quantum to classical transition, localization, the measurement problem and addressing the issue of the wave function collapse [28]. It has also been proposed that decoherence could play a role in the emergence of classical spacetime from the full QG regime [26, 29]. In a nutshell, what happens is that the closed system given by the particle plus the environment evolves in the usual unitary way typical of quantum mechanics; this introduces non-trivial correlations between the particle and the environment degrees of freedom; if the latter are ignored, technically by performing a trace that effectively averages over the environment, and the attention is focused on the open system constituted by the particle alone, it is found in general that this loses coherence and the ‘ability’ to show genuine quantum effects. For example pure superposition states decay to mixed states, basically equivalent to simple statistical mixtures and the possibility of interference is suppressed. Decoherence theory recently found brilliant confirmation in interferometry experiments with mesoscopic particles such as fullerene [30]. In these matter waves interferometry experiments heavy molecules are propagated through a Talbot-Lau interferometry apparatus. The environment is mainly represented by the air molecules.
INTRODUCTION AND MOTIVATIONS

Unless the pressure is reduced drastically and all other external influences appropriately controlled, the molecular beam interference pattern's visibility is reduced or canceled. The experimental results are in very good agreement with decoherence theory calculations [31, 32].

Coming back to GQ and the decoherence induced by spacetime vacuum fluctuations, the idea is that these would represent some sort of inevitable environment that any quantum particle has to cope with. It has first been suggested by Percival and collaborators that this could be measured through matter waves interferometry experiments [24, 25, 33]. These ideas have been first explored from the point of view of the Primary State Diffusion theory [24, 34] and further investigated by Power [35] and Power-Percival [23] for the case of wave packets within a standard Quantum Mechanics approach. When trying to study decoherence due to QG effects we face a serious problem: a definite theory describing the quantum behavior of spacetime is still lacking. Usually this is circumvented in the literature by adopting some sort of stochastic treatment. Since we do observe a classical spacetime down to at least the subnuclear scale, it is clear that the true GQ theory must have a classical limit out of which the smooth spacetime manifold emerges. It is usually believed that there should be an intermediate regime, between the Planck scale and the subnuclear scale, where the residual signs of the spacetime quantum nature come in the form of an effective stochastic geometry [23, 36]. In other words the spacetime metric is expected to fluctuate randomly. Given this assumption one can replace the 'hard' fully quantum decoherence problem with the problem of the interaction of a quantum particle with a classical stochastic environment: it is then more appropriate to speak of quantum *dephasing* rather than decoherence since the environment is not treated at the quantum level. The problem of the interaction of a quantum particle with a stochastic gravitational environment emerges even in other domains involving fluctuations of a more general nature than due to vacuum effects. For example Reynaud and collaborators have extensively studied the dephasing induced by stochastic GWs of cosmic origin [37] and argued that these would have a negligible effect on HYPER-like atomic interferometers. More recently the interaction of a quantum particle with a stochastic metric and the induced dephasing have been considered by Breuer *et al.* [38]: in the case of Planck scale fluctuations, this study predicts far a too small effect to be detected with 'ordinary' quantum particles. However the authors do not rule out the possibility of detecting dephasing experimentally by using large quantum composite systems.

Brief description and salient points of this work

The investigations described in this thesis start out of the previous considerations. The original question that we wished to answer is whether decoherence suffered by a quantum particle due to spacetime fluctuations can be detected at present, e.g. by means of matter waves interferometry experiments like those described above for fullerene. In our study we focus on a special class of spacetime fluctuations, namely *conformal fluctuations*. These represent perturbations that change locally the scale of the spacetime geometry. Conformal transformations of the metric play an important role in GR and alternative gravity theories such as scalar-tensor [39]. The main reason why we want to study conformal fluctuations is that these have been advocated in a recent series of papers to be effective in providing a mechanism for quantum dephasing [23, 40, 36]. Power and Percival considered a simple and idealized toy model with conformal fluctuations propagating in 1D over a flat spacetime geometry [23]. Inspired by this work and by a related proposal by Bingham [40], Wang and collaborators further investigated the issue trying to improve over Percival work by incorporating GWs into the analysis [36]. The two main conclusions of this work were: (1) conformal fluctuations can induce dephasing (2) their 'amount' is statistically set by a balancing mechanism according to which for every graviton
quantum there corresponds a ‘conformal quantum’ with a negative compensating energy. Beyond being limited to conformal fluctuations traveling in 1D, the main limit of these analysis stands in the fact that the presence of conformal fluctuations within the framework of GR is somehow ‘postulated’ but not shown to be consistent in a rigorous way.

Our goal here is trying to clarify upon these and some related issues by attempting to answer the following questions:

1. if we model the conformal fluctuations more realistically in 3D, can they still induce dephasing of a quantum particle?
2. if yes, how would this dephasing manifest itself qualitatively?
3. how can we treat consistently the interesting case of vacuum conformal fluctuations?
4. can we make a quantitative theoretical prediction for the amount of expected dephasing on a given quantum particle?
5. are conformal fluctuations physical? In other words, can we identify a theory of gravity which is compatible with the presence of conformal fluctuations?
6. can this theory be GR or shall we recur to alternative theories such as scalar-tensor?
7. can conformal fluctuations really provide a mechanism to balance part or all of the vacuum energy due to GWs and/or matter fields?

Questions 1. and 2. 3. and 4. are addressed in Chapter 1, which is based on the paper [42]. Here we study the dynamics of a quantum particle in the metric $g_{ab} = (1 + A)^2 \eta_{ab}$, where the conformal field $A$ is a stochastic field propagating in 3D space and satisfying the wave equation. We consider this particular metric as given from the outset without asking how it could practically arise: we limit ourselves to studying the quantum mechanics problem of the statistical evolution of the density matrix describing a non relativistic quantum particle coupled to gravity. We derive a dephasing formula generalizing to three space dimensions Percival and Wang’s results. The main result will be that, for an arbitrary power spectrum $S(\omega)$ describing the statistical features of the conformal field, dephasing does indeed occur and it is expected to be roughly two orders of magnitude larger than in the 1D case. Another important feature, in agreement with what found in previous studies, is that dephasing occurs only as a nonlinear effect resulting from a nonlinear effective newtonian potential $\propto A^2$ to which the particle couples. A linear potential $\propto A$ cannot induce dephasing.

In order to address question 4. and the problem of vacuum fluctuations we find inspiration from the work of Boyer in the 70s on random electrodynamics [41]. We simply assume that the classical fluctuations in $A$ mimic the zero point quantum fluctuations of the hypothetical quantum field in its vacuum state. This is implemented in practise by decomposing $A$ into normal modes and by assigning an energy $\hbar \omega/2$ to each mode. This will correspond to choosing $S(\omega) \propto 1/\omega$. In this case the general results can be made explicit and the expected amount of dephasing suffered by a particle of mass $M$ numerically estimated. The conclusion is that the present day technology will not be able to detect any effect, unless very heavy quantum particles are employed, possibly in space based experiments, where the influence of other external environmental factors is minimized.

The second part of the thesis addresses questions 5. 6. and 7. The main goal is to identify a theory of gravity compatible with the presence of conformal fluctuations induced by a conformal field $A$ satisfying the wave equation. In Chapter 2 we establish the main framework by developing a generalization of Isaacson’s theory for metric perturbations on a curved background [43, 44] which can also
incorporate matter fields and conformal fluctuations. The treatment is perturbative and can describe GWs at first order. Since we wish to study phenomena related to vacuum we are ideally seeking for a formalism in which any large scale curvature of the geometry of empty spacetime is induced by the energy content of matter fields, the backreaction due to GWs and, possibly, to the conformal fluctuations. In order to account for this it is necessary to push the analysis to second order. Because of these facts the formalism introduced in Chapter 2 is referred to as the nonlinear random gravity framework. Like in Boyer random electrodynamics the idea is that some of the vacuum properties, namely those related to zero point energy, are modelled by using classical stochastic fields with an appropriate spectrum as described above. This formally allows to implement the passage from the microscopic scale, where fields fluctuate, to the classical scale, where geometry appears smooth, by means of a spacetime averaging procedure. Treating vacuum in this way will raise the issues of whether vacuum energy gravitates or not and how the resulting vacuum energy due to matter fields, GWs and, possibly, conformal fluctuations can be made compatible with the basically vanishing observed value. Thus we will be led to discuss the well known cosmological constant problem [45] and some issues related to the still hypothetical mechanisms that can regularize vacuum energy.

The nonlinear random gravity framework is applied to standard GR in Chapter 3, where we try to find a solution of Einstein’s equation in the form $g_{ab} = [\Omega(A)]^2 \gamma_{ab}$ that may encode GWs and conformal fluctuations. Contrary to what claimed previously in the literature we find that spontaneous conformal fluctuations seem to be incompatible with the structure of GR, at least in the case of the problem considered here dealing with empty spacetime ‘filled’ with fields in their vacuum state. We will show that imposing a metric like $g_{ab} = [\Omega(A)]^2 \gamma_{ab}$ to Einstein’s equation leads to constraint equations for the conformal field $A$ that are unphysical and cannot be satisfied by a short wavelength fluctuating field. Moreover and in relation to question 7. we find that conformal fluctuations within GR cannot offer a mean to provide vacuum energy regularization. This conclusion in particular comes in correction of a technical imprecision in [46] where it had been wrongly found that conformal fluctuations could offer a way to balance the large amount of vacuum energy due to massless fields with a traceless stress energy tensor.

Chapter 4 provides one possible answer to question 6. in relation to alternative theories of gravity. First we correctly re-interpret conformal transformations as a local change in the physical units used in physics. Along these lines we illustrate Bekenstein’s theory of conformal invariant gravity [47], where conformal fluctuations can be introduced in a reasonable way through the fluctuations of Bekenstein’s mass gauge field. This theory is still GR and, though enriched of a new symmetry, conformal fluctuations still cannot induce any measurable effect. Finally we consider the more general class of theories known as scalar-tensor [39], where gravity is described by a rank 2 symmetric tensor and a scalar field which sets the local value of the gravitational constant. We focus in particular on the Brans-Dicke class of theories [48] and apply the nonlinear random gravity framework. The main result is that a vacuum solution in which the spacetime metric also presents a conformal perturbation is indeed possible. The conformal field $A$ in this case represents a first order fluctuation of the Brans-Dicke field and the wave equation is a simple consequence of the general formalism. The main conclusion is that, independently of the value of the Brans-Dicke coupling parameter $\omega$, quantum dephasing due to conformal fluctuations is predicted to occur within the Brans-Dicke framework. A secondary conclusion regards the coupling parameter $\omega$: at second order the structure of spacetime is determined by matter fields, GWs and $A$; this is found to contribute through an usual Klein-Gordon stress energy tensor for a massless field; for a certain range of values for $\omega$, for which the Brans-Dicke field would be a equivalent to a ghost, the conformal fluctuations could indeed provide a vacuum energy balance mechanism. This would serve to balance the traceless part of the matter fields stress energy tensor only. The cosmological constant problem still cannot be addressed.
To conclude we just spend a few words on how the work is organized. We discussed and included all our original results and contributions in the first four chapters, representing the main body of this thesis. We tried, when possible, not to overload the treatment with long derivations or background knowledge. We thus included in the appendices technical derivations and other related material that we deemed would have interrupted the flow of the treatment. In particular, Appendix A introduces the general material for the treatment of stochastic fields and is to be read in conjunction with Chapter 1. There we derive an important theorem which links the power spectral density of a spacetime stochastic process to its autocorrelation function. All technical and lengthy derivations related to Chapter 1 are included in Appendix B, while Appendix C shows how the spectral density $S \propto 1/\omega$ in relation to vacuum can be deduced.

All the background material needed for this work is also shortly discussed in the appendices. In Appendix D we review in great detail Isaacson’s theory of metric perturbations over a curved background. This is the starting point for the material presented in Chapter 2. Finally, the necessary quantum mechanics material, especially in relation to the density matrix formalism and decoherence is briefly illustrated in Appendix E.
Chapter 1

Dephasing of a non-relativistic quantum particle due to a conformally fluctuating spacetime

In this chapter, based on [42], we investigate the dephasing suffered by a nonrelativistic quantum particle within a conformally fluctuating spacetime geometry. Starting from a minimally coupled massive Klein-Gordon field, we derive an effective Schrödinger equation in the non-relativistic limit. The wave function couples to gravity through an effective nonlinear potential induced by the conformal fluctuations. The quantum evolution is studied through a Dyson expansion scheme up to second order. We show that only the nonlinear part of the potential can induce dephasing. This happens through an exponential decay of the off diagonal terms of the particle density matrix. The bath of conformal radiation is modeled in three-dimensions and its statistical properties are described in terms of a general power spectral density. Vacuum fluctuations at a low energy domain are investigated by introducing an appropriate power spectral density and a general formula describing the loss of coherence is derived. This depends quadratically on the particle mass and on the inverse cube of a particle dependent typical cutoff scale. Finally, the possibilities for experimental verification are discussed. It is shown that current interferometry experiments cannot detect such an effect. However this conclusion may improve by using high mass entangled quantum states.

1.1 Introduction

It is generally agreed that the underlying quantum nature of gravity implies that the spacetime structure close to the Planck scale departs from that predicted by GR. Unfortunately the QG domain is still beyond modern particle accelerators such as LHC. Nonetheless, finding experimental ways to test the quantum properties of spacetime would be highly beneficial to the theoretical developments of our fundamental theories of nature. In this respect it is has been suggested that QG could induce decoherence on a quantum particle through its underlying Planck scale spacetime fluctuations [23, 36, 38, 40, 49, 50].

As the sensitivity and performance of matter wave interferometers is increasing [30, 31, 32, 51, 52], it is important to assess the theoretical possibility of a future experimental detection of intrinsic, spacetime induced decoherence. The closely related dephasing effect due to a random bath of classical GWs (e.g. of astrophysical origin) has been extensively studied e.g. in [37]. The problem of
the decoherence induced by spacetime fluctuations is difficult to study: notwithstanding promising progress, mainly in loop quantum gravity and superstring theory [11], a coherent and established QG theoretical framework is still missing. Thus theoretical attempts for a prediction of the decoherence induced by spacetime fluctuations usually exploit some semiclassical framework. Such approaches typically represent the spacetime metric close to the Planck scale by means of fluctuating functions. These are usually supposed to mimic the vacuum quantum properties of spacetime down to some cutoff scale \( \ell = \lambda L_P \), where \( L_P \) is the Planck scale. The dimensionless parameter \( \lambda \) marks the benchmark between the fully quantum regime and the scale where the classical properties of spacetime start to emerge [23, 36]. The fact that classical fluctuating fields can be used to reproduce various genuine quantum effects is well known, e.g. from the work of Boyer [41, 53] in the case of the EM field. Frederick also applied the same technique to model spacetime fluctuations [54]. A classical but fluctuating metric is often exploited in the literature in relation to problems related to the microscopic behavior of the spacetime; e.g. a stochastic metric was employed in [55, 56] to study the problem of gravitational collapse and big bang singularities, while in [57] spacetime metric fluctuations were introduced and their ability to induce a weak equivalence principle violation studied.

A pioneering analysis of the problem of spacetime induced decoherence has been proposed by Power & Percival (PP in the following) [23] in the case of a conformally modulated Minkowski spacetime with conformal fluctuations traveling along one space dimension. This was improved by Wang et al. [36], who extended upon PP work attempting to include the effect of GWs. Conformal fluctuations are interesting as they are relatively easy to treat and offer a convenient way to build ‘toy’ models to assess some of the problem’s features. They have an important role in theoretical physics [58] and are sometimes invoked in the literature also in relation to universal scalar fields [59, 56] that also arise naturally in some modified theories of gravity such as scalar-tensor [60, 48, 39].

Within a semiclassical approach that ‘replaces’ the true quantum environment by classical fluctuating fields we should properly speak of dephasing of the quantum particle rather than decoherence. In a remarkable paper about quantum interference in the presence of an environment [61], Stern and co-authors showed that the fully quantum approach where decoherence is studied by ‘tracing away’ the environment degrees of freedom and that in which the dephasing of the quantum particle is due to a stochastic background field give equivalent results.

Here we consider a conformally modulated four-dimensional spacetime metric of the form \( g_{ab} = (1 + A)^2 \eta_{ab} \). Such a metric has been considered by PP [23], where the dephasing problem was studied in the simple idealized case of a particle propagating in one-dimension. By imposing Einstein’s equation on the metric \( g_{ab} \), PP deduced a wave equation for \( A \). Their procedure to derive an effective newtonian potential interacting with the quantum probe started from the geodesic equation for a test particle. Even though this did not take properly into account the nonlinearity in the conformal factor \((1 + A)^2\), they found correctly that the change in the density matrix is given by \( \delta \rho \propto M^2 T A_0^2 \tau_c \), where \( M \) is the probing particle mass, \( T \) the flight time, \( A_0 \) the amplitude of the conformal fluctuations and \( \tau_c \) their correlation time. This formula was used to set limits upon \( \lambda \). However in doing so they did not treat the statistical properties of the fluctuations properly and this resulted in the wrong estimate \( \lambda \propto (M^2 T/\delta \rho)^{1/7} \), as already noted by Wang et al in [36].

In their work, Wang and co-authors considered a metric of the kind \( g_{ab} = (1 + A)^2 \gamma_{ab} \), where the conformal metric \( \gamma_{ab} \) was supposed to encode GWs. This was done by exploiting the results in [62, 63] where a canonical geometrodynamics approach employing a conformal spacial 3-metric has been studied. By exploiting an energy density balancing mechanism between the conformal and GWs parts of the total gravitational Hamiltonian, the statistical properties of the conformal fluctuations where fixed. This corresponded to assume that each ‘quantum’ of the conformal field possessed a zero point energy \(-\hbar \omega \). Though an improvement over PP work, this approach is still one-dimensional
and too crude to make predictions. Moreover, as it shall be discussed extensively in chapters 3 and 4, the issue of energy balance between conformal fluctuations and GWs is a delicate one and, in fact, it seems to be ruled out within the standard GR framework.

In the following sections we provide a coherent three-dimensional treatment of the problem of a slow massive test particle coupled to a conformally fluctuating spacetime. The conformal field \( A \) is assumed to satisfy a simple wave equation. This will allow for a direct comparison with PP result. We also notice that such a framework is expected to arise naturally within a scalar-tensor theory of gravity, as it will be shown in chapter 4.

The material is organized as follows: in Section 1.2 the non-relativistic limit of a minimally coupled Klein-Gordon field is deduced and an effective newtonian potential depending nonlinearly on \( A \) is identified in the resulting effective Schrödinger equation. In Section 1.3 we set the general formalism to study the average quantum evolution of the particle density matrix \( \rho \) through a Dyson expansion scheme. In Section 1.4, general results derived in Appendix A are used to model the statistical and correlation properties of the fluctuations through a general, unspecified, power spectral density. In Section 1.5 and 1.6 we compute the average quantum evolution and derive a general expression for the evolved density matrix. We show in general that only a nonlinear potential can induce dephasing. The resulting dephasing formula implies an exponential decay of the density matrix off-diagonal elements and is shown to hold in general and independently of the specific spectral properties of the fluctuations. All we assume is that these obey a simple wave equation and that they are a zero mean stochastic process. The overall dephasing predicted within the present three-dimensional model –equation (1.39)– is seen to be about two orders of magnitudes larger than in the one-dimensional case as derived by PP. This result improves over both Percival’s and Wang’s work in that its key ingredients are general enough to be potentially suited for a variety of physical situations. Next we consider in Section 1.7 the problem of vacuum fluctuations. To this end a power spectrum \( S(\omega) \propto 1/\omega \) is introduced and we derive an explicit formula for the rate of change of the density matrix. Finally the discussion in Section 1.8 addresses the question of whether the dephasing due to conformal vacuum spacetime fluctuations could be detected. A possibility would be through matter interferometry employing heavy molecules. We consider this issue in the final part of this chapter by estimating the probing particle resolution scale, setting its ability to be affected by the fluctuations. The resulting formula for the dephasing indicates that the level of the effect is still likely to be beyond experimental reach, even for heavy molecules such as fullerene [51]. A measurable effect could possibly result for larger masses, e.g. if entangles quantum states were employed [64].

## 1.2 Low velocity limit and effective Schrödinger equation

The problem we wish to solve is clearly defined: we consider a scalar field \( A \) inducing conformal fluctuations on an otherwise flat spacetime geometry according to

\[
g_{ab} = \Omega^2 \eta_{ab} = (1 + A)^2 \eta_{ab},
\]

where \( \eta_{ab} = \text{diag}(-1, 1, 1, 1) \) is the Minkowski tensor. We will refer to \( A \) as to the conformal field and this will be assumed to satisfy the wave equation \( \partial^a \partial_a A = 0 \). Solving this equation with random boundary conditions results in a randomly fluctuating field propagating in three-dimensional space. We assume this to be a small first order quantity, i.e. \( |A| = O(\varepsilon \ll 1) \). Equation (1.1) expresses the spacetime metric in the laboratory frame. We also suppose that the typical wavelengths of \( A \) are effectively cut off at a scale set by \( \ell = \lambda L_P \), where \( L_P = (\hbar G/c^3)^{1/2} \approx 10^{-35} \) m is the Planck length. The dimensionless parameter \( \lambda \) represents a structural property of spacetime: below \( \ell \) a full
quantum treatment of gravity would be needed so that, by definition, $\ell$ represents the scale at which a semiclassical approach that treats quantum effects by means of classical randomly fluctuating fields is supposed to be a valid approximation. The value of $\lambda$ is model dependent but it is generally agreed that $\lambda \gtrsim 10^2$ [36], so that $\ell$ is expected to be extremely small from a macroscopic point of view. This motivates the assumption that classical macroscopic bodies, including the objects making up the laboratory frame and also the observers, are unaffected by the fluctuations in $A$. This corresponds to the idea that a physical object is characterized by some typical resolution scale $L_R$ setting its ability to ‘feel’ the fluctuations: if $L_R \gg \ell$ these average out and do not affect the body, which then simply follows the geodesic of the flat background metric. On the other hand a microscopic particle can represent a successful probe of the conformal fluctuations if its resolution scale is small enough.

We are interested in the phase change induced on the wave function of a quantum particle by the fluctuating gravitational field. Various approaches to the problem of how spacetime curvature affects the propagation of a quantum wave exist in the literature; e.g. for a stationary, weak field and a non relativistic particle a Schrödinger-like equation can be recovered [65]. The more interesting case of time varying gravitational fields can be treated e.g. by eikonal methods that are usually restricted to weak fields with $g_{ab} = \eta_{ab} + h_{ab}$ and $|h_{ab}| \ll 1$ [66, 67]. Other approaches, e.g. in [57, 38], are based on the scheme developed by Kiefer [68] for the nonrelativistic reduction of a Klein-Gordon field which is minimally coupled to a linearly perturbed metric.

The approach of PP in [23] and of Wang at el. in [36] was to derive the geodesic equation in the weak field limit. Their treatments were however employing, incorrectly, the usual newtonian limit scheme which is valid only for weak, linear and static perturbations [1]. This cannot be done in the present case as $\Omega^2 = (1 + A)^2$ induces a fast varying, nonlinear perturbation. In alternative to the Newtonian limit approach one could compute the geodesics of a conformally modulated Minkowski metric exactly and without making assumptions on the conformal factor. However this is ideally suitable for a zero size test particle or, more precisely, for a particle whose typical size is much less than the typical scale over which the spacetime geometry varies. This is not suitable for the situation we wish to study, where the spacetime fluctuations are assumed to vary on the very short scale $\ell = \lambda L_P$.

A wave approach that starts from a relativistic KG field does not suffer from the limitations of geodesic approach: this could be suitable for localized particles, while the KG approach does not assume any wave profile. For these reasons we expect the two methods to be inequivalent. Moreover the wave approach is conceptually clearer also because the coupling between gravity and a scalar field is well understood and in the appropriate non-relativistic weak field limit an effective Schrödinger equation emerges naturally. This will be our approach below.

We describe the quantum particle of mass $M$ by means of a minimally coupled Klein-Gordon (KG) field $\phi$:

$$g^{ab}\nabla_a \nabla_b \phi = \frac{M^2 c^2}{\hbar^2} \phi,$$

where $\nabla_a$ is the covariant derivative of the physical metric $g_{ab}$. Using $g_{ab} = \Omega^2 \eta_{ab}$ this equation can easily be made explicit [1] and reads:

$$\left(-\frac{1}{c^2} \frac{\partial^2}{\partial t^2} + \nabla^2\right) \phi = \frac{\Omega^2 M^2 c^2}{\hbar^2} \phi - 2 \partial_a (\ln \Omega) \partial^a \phi,$$

i.e. the wave equation for a massive scalar field plus a perturbation due to $A$ describing the coupling to the conformally fluctuating spacetime. We remark that, had we considered the alternative meaningful scenario of a conformally coupled scalar field, then the equation $g^{ab}\nabla_a \nabla_b \phi - R\phi/6 - M^2 c^2 \phi/\hbar^2 = 0$.
would read explicitly
\[
\left( -\frac{1}{c^2} \frac{\partial^2}{\partial t^2} + \nabla^2 \right) \phi = \frac{\Omega^2 M^2 c^2}{\hbar^2} \phi - 2\partial_a (\ln \Omega) \partial^a \phi - \phi \Omega^{-1} \partial_t \partial_t \Omega.
\]

The extra curvature term reads \(\Omega^{-1} \partial_t \partial_t \Omega = (1 - A) \partial_t \partial_t A + O(\varepsilon^3)\). We see that if \(A\) is assumed to satisfy the wave equation then it has no effect: in this case the minimally and conformally coupled KG equations are equivalent up to second order in \(A\). We also note that, by introducing the auxiliary field \(\Phi := \Omega \phi\), equation (1.2) turns out to be equivalent to:
\[
\left( -\frac{1}{c^2} \frac{\partial^2}{\partial t^2} + \nabla^2 \right) \Phi = \frac{\Omega^2 M^2 c^2}{\hbar^2} \Phi.
\]

In principle, if a solution for \(\Phi\) were known, then the physical scalar field representing the particle would follow formally as \(\phi = \Omega^{-1} \Phi = (1 - A + A^2) \Phi\), up to second order in \(A\). However in studying the dephasing problem we will only find an averaged solution for the average density matrix representing the quantum particle. Therefore, even if a solution in this sense is know in relation to \(\Phi\), it would not be obvious how to obtain the corresponding averaged density matrix related to \(\phi\), which is what we are interested in.

In view of the above considerations we work directly with equation (1.2) and now proceed in deriving its suitable non-relativistic limit. We will make two assumptions:

1. the particle is slow i.e., if \(\tilde{p} = M \tilde{v}\) is its momentum in the laboratory, we have:
   \[\frac{\tilde{v}}{c} \ll 1;\]

2. the effect of the conformal fluctuations is small, i.e. the induced change in momentum \(\delta p = M \delta v\) is small compared to \(M \tilde{v}\):
   \[\frac{\delta v}{\tilde{v}} \ll 1.\]

In view of these assumptions we can write:
\[
\phi = \psi \exp \left( -i M c^2 t / \hbar \right),
\]
where the field \(\psi\) is close to be a plane wave of momentum \(\tilde{p}\). As a consequence we have:
\[
\frac{\partial^2 \psi}{\partial t^2} \approx -\frac{1}{\hbar^2} \left( \frac{\tilde{p}^2}{2 M} \right)^2 \psi.
\]
Using this and multiplying by $\hbar^2/2M$, equation (1.2) yields:

$$
\begin{align*}
\left[ i\hbar \frac{\partial}{\partial t} + \frac{\hbar^2}{2M} \nabla^2 - \frac{Mc^2}{8} \left( \frac{\tilde{v}}{c} \right)^4 \right] \psi &= \left( A + \frac{A^2}{2} \right) Mc^2 \psi - \frac{\hbar^2}{M} \partial^a \phi \partial_a \ln(1 + A) \times \exp \left( iMc^2 t/\hbar \right). \\
&\equiv \left( A + \frac{A^2}{2} \right) Mc^2 \psi - \frac{\hbar^2}{M} \partial^a \phi \partial_a \ln(1 + A) \times \exp \left( iMc^2 t/\hbar \right). \quad (1.3)
\end{align*}
$$

Leaving the term $T_4$ aside for the moment, the orders of the three underlined terms must be carefully assessed. We have:

$$
T_1 \sim M\tilde{v}^2, \quad T_2 \sim Mc^2 \left( \frac{\tilde{v}}{c} \right)^4 \quad \langle T_3 \rangle \sim \varepsilon^2 Mc^2,
$$

where an average has been inserted since $T_3$ is fluctuating. It follows that

$$
\frac{T_2}{T_1} \sim \left( \frac{\tilde{v}}{c} \right)^2 \ll 1.
$$

Thus, in the non-relativistic limit, $T_2$ is negligible in comparison to $T_1$. This is the case in a typical interferometry experiment where it can be $\tilde{v} \approx 10^2$ m s$^{-1}$ [52], so that $(\tilde{v}/c)^2 \sim 10^{-12}$. Next we have:

$$
\frac{\langle T_3 \rangle}{T_1} \sim \left( \frac{\varepsilon c}{\tilde{v}} \right)^2.
$$

The request that the conformal fluctuations have a small effect thus gives the condition

$$
\frac{\langle T_3 \rangle}{T_1} \ll 1 \quad \Leftrightarrow \quad \varepsilon^2 \sim \langle A^2 \rangle \ll \left( \frac{\tilde{v}}{c} \right)^2. \quad (1.4)
$$

That this condition is effectively satisfied can be checked a posteriori after the model is complete. It depends on the statistical properties of the conformal field and the particle ability to probe them. This will be related to a particle resolution scale. At the end of the discussion in Section 1.8.3 we will show that (1.4) is satisfied if, e.g., the particle resolution scale is given by its Compton length.

Under these conditions the non-relativistic limit of equation (1.3) yields:

$$
-\frac{\hbar^2}{2M} \nabla^2 \psi + \left( A + \frac{A^2}{2} \right) Mc^2 \psi + T_4 = i\hbar \frac{\partial \psi}{\partial t}, \quad (1.5)
$$

where

$$
T_4 := -\frac{\hbar^2}{M} \partial^a \phi \partial_a \ln(1 + A) \times \exp \left( iMc^2 t/\hbar \right).
$$

In order to assess the correction due to this term we split it into two contributions by writing separately the time and space derivatives. Using the fact that

$$
\frac{\partial \phi}{\partial t} \approx -\frac{iMc^2}{\hbar} \psi \exp \left( -iMc^2 t/\hbar \right)
$$
it is easy to see that:

$$T_4 = -i\hbar \left( \dot{\hat{A}} - \hat{A}\dot{\hat{A}} \right) \psi - i\hbar \tilde{v} \left( A_x - AA_x \right) \psi,$$

where $\dot{\hat{A}} := \partial A/\partial t$, $A_x := \partial A/\partial x$ and where we assumed that the particle velocity in the laboratory is along the $x$ axis. In Appendix A.7 we show that if $A$ is (i) a stochastic isotropic perturbation and (ii) effectively fast varying over a typical length $\lambda_A = \kappa \hbar/(Mc)$ related to the particle resolution scale, then $T_4$ reduces to:

$$T_4 = \left( -A + A^2 \right) \frac{Me^2}{\kappa} \psi.$$  \hspace{1cm} (1.7)

Here $\kappa \sim 1$ is dimensionless and its precise value is unimportant. The important point is that $T_4$ yields a positive extra nonlinear term in $A$ that adds up to what we already have in (1.5). Finally we get the effective Schrödinger equation

$$-\frac{\hbar^2}{2M} \nabla^2 \psi + V \psi = i\hbar \frac{\partial \psi}{\partial t},$$

where the nonlinear fluctuating potential $V$ is defined by

$$V := (C_1 A + C_2 A^2) Mc^2.$$  \hspace{1cm} (1.8)

The values of the constants $C_1$ and $C_2$ depend on $\kappa$. For $\kappa = 1$ it would be $C_1 = 0$ and $C_2 = 3/2$. For generality we will leave them unspecified and consider $\kappa$ as a constant of order one.

1.3 Average quantum evolution

1.3.1 Dyson expansion for short evolution time

We now have a rather well defined problem: that of the dynamics of a non relativistic quantum particle under the influence of the nonlinear stochastic potential (1.8). The Schrödinger equation describing the dynamics of a free particle is suitable to describe the interference patterns that could result e.g. in an interferometry experiment employing cold molecular beams. When the particle in the beam propagates through an environment, we are dealing with an open quantum system. This in general suffers decoherence, resulting in a loss of visibility in the fringes pattern [32, 52]. This is a well defined macroscopic quantity. In the present semiclassical treatment the environment due to spacetime fluctuations is represented, down to the semiclassical scale $\ell$, by a sea of random radiation encoded in $A$ and resulting in the fluctuating potential $V$. An estimate of the overall dephasing can be obtained by considering the statistical averaged dynamics of a single quantum particle interacting with $V$. In practise we will need (i) to solve for the dynamics of a single particle of mass $M$ and (ii) calculate the averaged wavefunction by averaging over the fluctuations. The outcome of (i) would be some sort of ‘fluctuating’ wavefunction carrying, beyond the information related to the innate quantum behavior of the system, that related to the fluctuations in the potential. The outcome of (ii) is to yield a general statistical result describing what would be obtained in an experiment where many identical particles propagate through the same fluctuating potential.

We thus consider the Hamiltonian operator $\hat{H}(t) = \hat{H}^0 + \hat{H}^1(t)$, where $\hat{H}^0$ is the kinetic part while

$$\hat{H}^1(t) = \int d^3x V(x,t)|x\rangle\langle x|,$$
is the perturbation due to the fluctuating potential energy. Here \(|x\rangle \langle x|\) is the projection operator on the space spanned by the position operator eigenstate \(|x\rangle\). Indicating the state vector at time \(t\) with \(\psi_t\), the related Schrödinger equation reads

\[
\hat{H}(t)\psi_t = i\hbar \frac{\partial \psi_t}{\partial t}.
\]

Using the density matrix formalism and as shown in Appendix E, the general solution can be expressed through a Dyson series as [69]

\[
\rho_T = \rho_0 + \hat{U}_1(T)\rho_0 + \rho_0\hat{U}_1^\dagger(T) + \hat{U}_2(T)\rho_0 + \hat{U}_1(T)\rho_0\hat{U}_1^\dagger(T) + \rho_0\hat{U}_2^\dagger(T) + \ldots, \tag{1.9}
\]

where \(\rho_0\) is the initial density matrix and the propagators \(\hat{U}_1(T)\) and \(\hat{U}_2(T)\) are given by

\[
\hat{U}_1(T) := -\frac{i}{\hbar} \int_0^T \hat{H}(t')dt',
\]

\[
\hat{U}_2(T) := -\frac{1}{\hbar^2} \int_0^T dt' \int_0^{t'} dt'' \hat{H}(t')\hat{H}(t'').
\]

In truncating the series to second order we assume that the system evolves for a time \(T\) such that \(T \ll T^*\), where \(T^*\) is defined as the typical time scale required to have a significant change in the density matrix \(\rho\).

The effect of the environment upon a large collection of identically prepared systems is found by taking the average over the fluctuating potential as explained above. Formally and up to second order we have

\[
\langle \rho_T \rangle = \left\langle \rho_0 + \hat{U}_1(T)\rho_0 + \rho_0\hat{U}_1^\dagger(T) + \hat{U}_2(T)\rho_0 + \hat{U}_1(T)\rho_0\hat{U}_1^\dagger(T) + \rho_0\hat{U}_2^\dagger(T) \right\rangle. \tag{1.10}
\]

The averaged density matrix \(\langle \rho_T \rangle\) will describe the average evolution of the system including the effect of dephasing.

It is straightforward but lengthy to show that, up to second order in the Dyson’s expansion, the kinetic and potential parts of the Hamiltonian give independent, additive contributions to the average evolution of the density matrix, i.e. \(\langle \rho_T \rangle = [\rho_T]_0 + \langle [\rho_T]_1 \rangle\), where

\[
[\rho_T]_0 := \rho_0 + [\hat{U}_1(T)]_0\rho_0 + \rho_0[\hat{U}_1(T)]_0^\dagger + [\hat{U}_2(T)]_0\rho_0 + [\hat{U}_1(T)]_0\rho_0[\hat{U}_1(T)]_0^\dagger + \rho_0[\hat{U}_2(T)]_0^\dagger,
\]

\[
\langle [\rho_T]_1 \rangle := \left\langle \rho_0 + [\hat{U}_1(T)]_1\rho_0 + \rho_0[\hat{U}_1(T)]_1^\dagger + [\hat{U}_2(T)]_1\rho_0 + [\hat{U}_1(T)]_1\rho_0[\hat{U}_1(T)]_1^\dagger + \rho_0[\hat{U}_2(T)]_1^\dagger \right\rangle. \tag{1.11}
\]

Here the kinetic propagators \([\hat{U}_1(T)]_0\) and \([\hat{U}_2(T)]_0\) depend solely on \(\hat{H}_0\), while the potential propagators \([\hat{U}_1(T)]_1\) and \([\hat{U}_2(T)]_1\) depend only on \(\hat{H}_1(t)\). For a proof of this statement we refer to Appendix B.1. In the next section we estimate the dephasing by calculating the term \(\langle [\rho_T]_1 \rangle\) alone.
1.4 The conformal field and its correlation properties

We now set the statistical properties of the conformal field \( A \). This is assumed to represent a real, stochastic process having a zero mean. We further assume it to be isotropic. In Appendix A we review a series of important results concerning stochastic processes, in particular in relation to real stochastic signals satisfying the wave equation. The main quantity characterizing the process is the power spectral density \( S(\omega) \). In the case of an isotropic bath of random radiation, field averages such as \( \langle A^2 \rangle \), \( \langle |\nabla A|^2 \rangle \) and \( \langle (\partial_t A)^2 \rangle \) can be found in terms of \( S(\omega) \), e.g.

\[
\langle A^2 \rangle = \frac{1}{(2\pi)^3} \int d^3k \, S(k),
\]

where \( kc = \omega \). In Appendix A we show how the conformal field can be resolved into components traveling along all possible space directions according to

\[
A(x, t) = \int d\hat{k} \, A_k(\hat{k} \cdot x/c - t),
\]

where \( d\hat{k} \) indicates the elementary solid angle. The capacity of the fluctuations to maintain correlation is encoded in the autocorrelation function \( C(\tau) \). In the same appendix we prove a generalization of the usual Wiener-Khintchine (WK) theorem, valid for the case of a spacetime dependent process satisfying the wave equation, and linking the autocorrelation function to the Fourier transform of the power spectral density according to:

\[
C(\tau) = \frac{1}{(2\pi c)^3} \int d\omega \, \omega^2 S(\omega) \cos(\omega \tau).
\]  

(1.12)

This allows to prove that wave components traveling along independent space directions are uncorrelated, i.e.

\[
\langle A_k(t) A_{k'}(t + \tau) \rangle = \delta(\hat{k}, \hat{k}') \, C(\tau).
\]  

(1.13)

The field mean squared amplitude is related to the correlation function according to \( \langle A^2 \rangle = 4\pi C_0 \), as derived in Appendix A, and where \( C_0 := C(0) \).

Isotropy implies that all directional components have the same amplitude \( A_0 \). This is found introducing the normalized correlation function \( R(\tau) \) through

\[
R(\tau) := \frac{C(\tau)}{C_0},
\]

so that \( R(0) = 1 \). Equation (1.13) can now be re-written as \( \langle A_k(t) A_{k'}(t + \tau) \rangle = \delta(\hat{k}, \hat{k}') \, C_0 \, R(\tau) \) so that, introducing the normalized directional components, \( f_k(t) := A_k(t) / \sqrt{C_0} \) we have

\[
\langle f_k(t) f_{k'}(t + \tau) \rangle = \delta(\hat{k}, \hat{k}') \, R(\tau).
\]

We now define the constant \( A_0 := \sqrt{C_0} \) which is connected to the squared amplitude per solid angle according to \( A_0^2 = C_0 = \langle A^2 \rangle / 4\pi \). The directional components are given by \( A_k(t) = A_0 \, f_k(t) \) and the general conformal field can finally be expressed as an elementary superposition of the kind

\[
A(x, t) = A_0 \int d\hat{k} \, f_{\hat{k}}(\hat{k} \cdot x/c - t).
\]  

(1.14)
1.4.1 Summary of the correlation properties of the conformal fluctuations

The main statistical properties of the directional stochastic waves $f_k$ are summarized by

$$\langle f_k(t) \rangle = 0,$$

$$\langle f_k(t) f_k'(t') \rangle = \delta(\hat{k}, \hat{k}') R(t - t'),$$

i.e. each component has zero mean and fluctuations traveling along different space directions are perfectly uncorrelated. These two properties imply that odd products of directional components have also a zero mean, i.e.

$$\langle f_{k_1}(t_1) f_{k_2}(t_2) f_{k_3}(t_3) \rangle = 0.$$  

In the following dephasing calculation we will need to evaluate means involving products of four directional components. To this purpose we need to introduce the second order correlation function $R''(t - t')$ according to

$$\langle [f_k(t)]^2 [f_{k'}(t')]^2 \rangle = 1 + \delta(\hat{k}, \hat{k}') [R''(t - t') - 1].$$

This definition is compatible with the fact that the mean is one when components traveling in different direction are involved, i.e. $\langle [f_k(t)]^2 [f_{k'}(t')]^2 \rangle = 1$ if $\hat{k} \neq \hat{k}'$.

1.5 Dephasing calculation outline

To calculate the dephasing suffered by the probing particle we must evaluate the average of all the individual terms in equation (1.11). The relevant propagators are

$$[\hat{U}_1(T)]_1 := -\frac{i}{\hbar} \int_0^T dt' \hat{H}_1(t'),$$

$$[\hat{U}_2(T)]_1 := -\frac{1}{\hbar^2} \int_0^T dt \int_0^t dt' \hat{H}_1(t) \hat{H}_1(t').$$

The interaction Hamiltonian is given by

$$\hat{H}_1(t) = \int d^3x V(x, t) |x\rangle \langle x|,$$

where the potential energy is

$$V(x, t) = C_1 Mc^2 A_0 \int d\hat{k} f_k(t - x \cdot \hat{k}/c) + C_2 Mc^2 A_0^2 \left[\int d\hat{k} f_k(t - x \cdot \hat{k}/c)\right]^2.$$

1.5.1 First order terms of the Dyson expansion

We evaluate the two first order terms in the Dyson expansion. For a more compact notation, we do not show the argument of the directional components $f_k$. The contribution of the linear part of the
potential $C_1Me^2A$ vanishes trivially since $\langle f_k \rangle = 0$. The quadratic part gives:

$$\left\langle \hat{U}_1(T)\rho_0 \right\rangle = -\frac{iC_2Me^2A_0^2}{\hbar} \int_0^T dt \int d^3x |x| \left\langle \left[ \int d\mathbf{k} f_k \right]^2 \right\rangle \rho_0.$$  

Using $A_k(t) = \sqrt{C_0} f_k(t)$ and $\langle A^2 \rangle = 4\pi C_0 \equiv 4\pi A_0^2$ it is seen that the average yields $4\pi$. Since $\int d^3x |x| = \bar{1}$ and integrating over $T$ we find

$$\left\langle \hat{U}_1(T)\rho_0 \right\rangle = -\frac{4\pi C_2iMe^2A_0^2T}{\hbar} \rho_0.$$  

(1.19)

The calculation of the other first order term proceeds in the same way. Since $\hat{U}_1^\dagger(T) = -\hat{U}_1(T)$, it yields the same result as in (1.19) but with the opposite sign (more in general, all the odd terms in the Dyson expansion have an $i$ factor and also yield a vanishing contribution). We thus see that at first order in the Dyson expansion there is no net dephasing and $\langle \hat{U}_1(T)\rho_0 + \rho_0\hat{U}_1^\dagger(T) \rangle = 0$.

### 1.5.2 Second order terms of the Dyson expansion

The second order calculation is more complicated. A fundamental point is that the linear part of the potential does again give a vanishing contribution. Dephasing will be shown to come as a purely nonlinear effect due to the nonlinear potential term $\sim A^2$.

**Non-contribution of the linear part of the potential**

To have an idea of how things work we consider e.g. the average of the term $\hat{U}_2\rho_0$. This has the following structure:

$$\left\langle \hat{U}_2\rho_0 \right\rangle \sim \int dt \int dt' \int d^3y |y\rangle \langle y| \int d^3y' |y'\rangle \langle y'| \langle V(y, t)V(y', t') \rangle \rho_0.$$  

The interesting part is the average $\langle V(y, t)V(y', t') \rangle$. This is:

$$\langle V(y, t)V(y', t') \rangle \sim \langle A(y, t)A(y', t') \rangle + \langle A(y, t)A^2(y', t') \rangle + \langle A^2(y, t)A^2(y', t') \rangle.$$  

The second term vanishes in virtue of property (1.17). This is seen using the directional decomposition (1.14) and writing:

$$\langle A(y, t)A^2(y', t') \rangle = A_0^3 \int d\mathbf{k}_1 \int d\mathbf{k}_2 \int d\mathbf{k}_3 \langle f_{k_1} f_{k_2} f_{k_3} \rangle = 0.$$  

The first term derives from the linear part of the potential. It results in the contribution:

$$\langle A(y, t)A(y', t') \rangle \Rightarrow \int_0^T dt \int_0^t dt' \int d^3y |y\rangle \langle y| \int d^3y' |y'\rangle \langle y'| \int d\mathbf{k} \int d\mathbf{k}' \langle f_{k}(t - y \cdot \hat{k}) f_{k'}(t' - y' \cdot \hat{k}') \rangle \rho_0.$$  

For convenience of notation we set $c = 1$ in the arguments of the directional functions $f_{k'}$. Using equation (1.16) the average yields the 2-point correlation function according to $\delta(\hat{k}, \hat{k}')R(t - t' + y'$. 

$$\langle f_{k}(t - y \cdot \hat{k}) f_{k'}(t' - y' \cdot \hat{k}') \rangle \rho_0.$$  

(1.19)
\(\hat{k}' - y \cdot \hat{k}\). Integrating with respect to \(\hat{k}'\) yields:

\[
\langle A(y, t)A(y', t') \rangle = \int_0^T dt \int_0^t dt' \int d^3 y |y\rangle \int d^3 y' |y'\rangle \langle y' | d\hat{k} R(t - t' + \hat{k} \cdot (y' - y)) \rho_0.
\]

The corresponding matrix element is found by inserting \(\langle x |\) and \(|x'\rangle\) respectively on the left and on the right. Using \(\langle x |y\rangle = \delta(x - y)\) and exploiting the properties of the delta function we find

\[
\langle A(y, t)A(y', t') \rangle \Rightarrow K \times \int_0^T dt \int_0^t dt' \int d\hat{k} R(t - t'),
\]

where \(K\) is a constant given by

\[
K = -\frac{C_1^2 A_0^2 M^2 c^4 \rho_{xx'}(0)}{\hbar^2},
\]

and where \(\rho_{xx'}(0) := \langle x |\rho_0 |x'\rangle\). The similar terms coming from \(\langle \rho_0 \hat{U}_2^\dagger \rangle\) will contribute in the same way as in (1.20), thus yielding an extra factor 2. Finally, through a similar calculation it is found that the terms \(\sim \langle A(y, t)A(y', t') \rangle\) coming from \(\langle \hat{U}_1 \rho_0 \hat{U}_1^\dagger \rangle\) contribute according to:

\[
\langle A(y, t)A(y', t') \rangle \Rightarrow -K \times \int_0^T dt \int_0^t dt' \int d\hat{k} R(t - t' + \hat{k} \cdot (x' - x)).
\]

Bringing all together, the overall contribution deriving from the linear part \(C_1 M c^2 A\) of the effective potential is found to be proportional to the expression:

\[
I := \int_0^T dt \left\{ 2 \int_0^t dt' R(t - t') - \int_0^T dt' R(t - t' + \hat{k} \cdot \Delta x) \right\},
\]

where \(\Delta x := x' - x\). In Appendix B.3 we prove that this vanishes provided \(R(\tau)\) is an even function and the drift time \(T\) is much larger than the time needed by the fluctuations to propagate through the distance \(|\Delta x|\), i.e. if \(T \gg \hat{k} \cdot |\Delta x|\), where \(c = 1\). This condition is certainly satisfied in a typical interferometry experiment where the drift time \(T\) can be of the order of \(\sim 1\) ms and \(cT\) is indeed much larger than the typical space separations \(|\Delta x|\) relevant to quantify the loss of contrast in the measured interference pattern.

Thus we have here the important result that the linear part of the potential doesn’t induce in general any dephasing up to second order in Dyson expansion. In fact we show in the next section that dephasing results purely as an effect of the nonlinear potential term \(C_2 M c^2 A^2\).

**Contribution of the nonlinear part of the potential**

This calculation requires estimating averages of the kind \(\langle A^2(y, t)A^2(y', t') \rangle\), which will bring in the second order correlation function \(R''\) defined in (1.18). This is straightforward but algebraically lengthy. The full calculation is reported in Appendix B.2, where we show that proceeding in a similar way as done above, exploiting the statistical properties (1.15)-(1.18) and the already mentioned result \(I = 0\) in relation to (1.21), then the general result for the density matrix and valid up to second order
1.6. General density matrix evolution for large drift times

In the Dyson expansion follows as:

\[
\rho_{xx}(T) = \rho_{xx}(0) - \frac{32C_2^2\pi^2M^2\epsilon^4A_0\rho_{xx}(0)}{\hbar^2} \times \left[ \int_0^T dt \int_0^T R^2(t - t') \right. \\
\left. - \frac{1}{16\pi^2} \int d\hat{k} \int d\hat{K} \int_0^T dt \int_0^T dt' R(t - t' - \hat{k} \cdot \Delta x/c) R(t - t' - \hat{K} \cdot \Delta x/c) \right].
\]  

\text{(1.22)}

Remarkably, the second order correlation function doesn't play any role: the first order correlation function \( R(\tau) \), and thus the power spectral density \( S(\omega) \), completely determines the system evolution up to second order. Equation (1.22) implies that the diagonal elements of the density matrix are left unchanged by time evolution. This is seen by setting \( \Delta x = 0 \) which yields immediately \( \rho_{xx}(T) = \rho_{xx}(0) \) for every \( T \).

1.6 General density matrix evolution for large drift times

To verify that we have dephasing with an exponential decay of the off diagonal elements we need further simplify the result (1.22) by analyzing its behavior for appropriately large evolution times. To this end we start from the following identity

\[
\int_0^T dt \int_0^T dt' g(t - t') = \frac{1}{2\pi} \int_0^T dt \int_0^T dt' \int_{-\infty}^{\infty} d\omega \tilde{g}(\omega) e^{i\omega(t-t')}
\]

\[
= \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \tilde{g}(\omega) \left[ \frac{\sin(\omega T/2)}{\omega/2} \right]^2 ,
\]  

\text{(1.23)}

where \( \tilde{g}(\omega) \) denotes the Fourier transform of the function \( g(t) \). It is well known that the function in between brackets can be used to define the Dirac delta function through

\[
\frac{1}{2\pi T} \left[ \frac{\sin(\omega T/2)}{\omega/2} \right]^2 \xrightarrow{T \to \infty} \delta(\omega).
\]

It is then clear that, for an appropriately large evolution time, we have approximatively

\[
\int_0^T dt \int_0^T dt' g(t - t') \approx T \int_{-\infty}^{\infty} d\omega \tilde{g}(\omega) \delta(\omega) = \tilde{g}(0)T,
\]  

\text{(1.24)}

where \( \tilde{g}(0) \) is the Fourier transform of \( g \) evaluated at the frequency \( \omega = 0 \). To clarify what we mean by ‘appropriately large evolution time’, we re-write equation (1.23) as

\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \tilde{g}(\omega) \left[ \frac{\sin(\omega T/2)}{\omega/2} \right]^2 = \frac{T}{\pi} \int_{-\infty}^{\infty} dx \tilde{g}(2x/T) \left( \frac{\sin x}{x} \right)^2,
\]

where we defined the dimensionless variable \( x := \omega T/2 \). We now focus on the Fourier transform \( \tilde{g}(\omega) \) and we consider a frequency interval \([0, \Delta \omega]\) in which \( \tilde{g}(\omega) \) varies little and can thus be considered as
basically constant, in such a way that \( \tilde{g}(\omega) \approx \tilde{g}(0) \) for \( \omega \in [0, \Delta \omega] \). Considering now that

\[
\frac{1}{\pi} \int_{-\infty}^{\infty} dx \left( \frac{\sin x}{x} \right)^2 = 1 \tag{1.25}
\]

and

\[
\frac{1}{\pi} \int_{-5}^{5} dx \left( \frac{\sin x}{x} \right)^2 = 0.94 \tag{1.26}
\]

we have the approximate result

\[
\frac{T}{\pi} \int_{-\infty}^{\infty} dx \tilde{g}(2x/T) \left( \frac{\sin x}{x} \right)^2 \approx T \tilde{g}(0) \frac{1}{\pi} \int_{-5}^{5} dx \left( \frac{\sin x}{x} \right)^2 \approx T \tilde{g}(0),
\]

provided that

\[
\frac{1}{T} \lesssim \Delta \omega, \tag{1.27}
\]

guaranteeing that the Fourier transform of the correlation function remains practically constant across the relevant integration interval. We have thus proven the approximate relation

\[
\int_{0}^{T} dt \int_{0}^{T} dt' g(t - t') \approx \tilde{g}(0) T, \quad \text{for} \quad T \gtrsim (\Delta \omega)^{-1}. \tag{1.28}
\]

This can now be used to simplify the equation (1.22) governing the evolution of the density matrix. We remark that for this result to hold the integrand function \( g(t - t') \) need not be an even function and the only requirement is that its Fourier transform is slowly varying over the interval \([0, \Delta \omega]\).

### 1.6.1 Correlation time and characteristic function

Equation (1.28) can now be used to evaluate the time integrals appearing in (1.22). This is done by identifying in one case \( g(t) := R^2(t) \) and in the other \( g_{\tau \tau'}(t) := R(t + \tau)R(t + \tau') \), where \( \tau \) and \( \tau' \) stand respectively for \(-\hat{k} \cdot \Delta x/c\) and \(-\hat{K} \cdot \Delta x/c\), and where the normalized correlation function can be expressed, using the generalized WK theorem (1.12), as

\[
R(\tau) \equiv \frac{C(\tau)}{C_0} = \frac{1}{C_0(2\pi c)^3} \int_{0}^{\omega_c} d\omega \omega^2 S(\omega) \cos(\omega \tau). \tag{1.29}
\]

Notice that the integration frequency has a cutoff at \( \omega_c = \omega_P/\lambda \), where the Planck frequency is \( \omega_P := 2\pi/T_P = 1.166 \times 10^{14} \text{ s}^{-1} \). This is consistent with the fact that below the scale \( \ell = \lambda L_P \) the approximation of randomly fluctuating fields breaks down. In alternative this may simply correspond to the fact that the probing particle is insensitive to the short wavelengths as a result of its own finite resolution scale \( L_R \).

**Correlation time**

Application of (1.28) to \( g(t) := R^2(t) \) yields the result

\[
\int_{0}^{T} dt \int_{0}^{T} dt' R^2(t - t') = \tau_* T, \tag{1.30}
\]
where the correlation time is defined as
\[
\tau_* := \mathfrak{F} \left[ R^2(t) \right] (0) = \pi \int_0^{\infty} d\omega \omega^4 S^2(\omega) \int_0^{\infty} d\omega \omega^2 S(\omega),
\]  

(1.31)

\(\mathfrak{F}\) denoting Fourier transform.

To see this we need to Fourier transform \(R^2(t)\) and evaluate its value for \(\omega = 0\). Explicitly we have:

\[
\mathfrak{F} \left[ R^2(t) \right] (\omega) = \frac{1}{C_0^2 (2\pi)^6} \int_0^{\infty} \int_0^{\infty} dt' dt'' e^{-i\omega t'} e^{i\omega t''} \int_0^{\infty} d\omega' d\omega'' \int_{-\infty}^{\infty} dt \omega'^2 \omega''^2 S(\omega')S(\omega'')e^{-i\omega t} \cos \omega' t \cos \omega'' t
\]

Multiplying the exponential functions and integrating with respect to time we obtain

\[
\mathfrak{F} \left[ R^2(t) \right] (\omega) = \frac{1}{4C_0^2 (2\pi)^6} \int_0^{\infty} \int_0^{\infty} dt' dt'' \int_{-\infty}^{\infty} dt \omega'^2 \omega''^2 S(\omega')S(\omega'')e^{-i\omega t} (e^{i\omega' t} + e^{-i\omega' t}) (e^{i\omega'' t} + e^{-i\omega'' t}).
\]

From the last expression we see that \(\mathfrak{F} \left[ R^2 \right] (\omega) = 0\) for \(|\omega| > 2\omega_c\). The value of the Fourier transform for \(\omega = 0\) follows immediately from the properties of the \(\delta\) function. We have the sum of four integrals of which those containing \(\delta(\omega' + \omega'')\) and \(\delta(-\omega' - \omega'')\) vanish and we are left with the expression

\[
\mathfrak{F} \left[ R^2 \right] (0) = \frac{\pi}{C_0^2 (2\pi)^6} \int_0^{\infty} d\omega' d\omega'' \omega'^2 \omega''^2 S(\omega')S(\omega'') \delta(\omega' - \omega'') \delta(\omega' + \omega'')
\]

where we have taken into account that then integrals containing \(\delta(\omega' - \omega'')\) and \(\delta(-\omega' + \omega'')\) yield the same contribution because of the exchange symmetry between the integration variables \(\omega'\) and \(\omega''\). Carrying on the remaining integration we have

\[
\mathfrak{F} \left[ R^2(t) \right] (0) = \frac{\pi}{C_0^2 (2\pi)^6} \int_0^{\infty} d\omega \omega^4 S^2(\omega).
\]  

(1.32)

Using the explicit expression for the constant \(C_0\) that is obtained from (1.29) for \(\tau = 0\), we obtain the result (1.31), from which it is clear that \(\mathfrak{F} \left[ R^2(t) \right] (0)\) has indeed the dimension of a time.
1.6. General density matrix evolution for large drift times

Characteristic function

We can also apply (1.28) to \( g_{\tau\tau'}(t) := R(t + \tau)R(t + \tau') \), where \( \tau \) and \( \tau' \) stand respectively for \(-\hat{k} \cdot \Delta x/c\) and \(-\hat{K} \cdot \Delta x/c\). Writing

\[
\mathfrak{F}[g_{\tau\tau'}(t)](\omega) = \frac{1}{C_0^2} \int_{-\infty}^{\infty} dt C(t + \tau)C(t + \tau')e^{-i\omega t}
\]

and re-expressing the cosine functions through the exponential function, a calculation similar to the previous one shows that

\[
\mathfrak{F}[g_{\tau\tau'}(t)](\omega) = 0 \quad \text{for} \quad |\omega| > 2\omega_c
\]

and yields the Fourier transform for \( \omega = 0 \) as

\[
\mathfrak{F}[R(t + \tau)R(t + \tau')](0) = \frac{\pi}{C_0^2(2\pi c)^6} \int_{0}^{\omega_c} d\omega \omega S^2(\omega) \cos(\omega(t + \tau)) \cos(\omega(t + \tau')).
\]

This expression reduces of course to (1.32) for \( \tau = \tau' = 0 \) and it is thus convenient to re-write it as

\[
\mathfrak{F}[R(t + \tau)R(t + \tau')](0) = \tau_\ast \Gamma[\omega_c(\tau - \tau')].
\]

This is dimensionless and satisfies the following general properties:

- \( \Gamma(\omega, t) = \Gamma(-\omega, t) \),
- \( \Gamma(0) = 1 \),
- \( \Gamma(\omega, t) < 1 \), for \( t \neq 0 \),
- \( \Gamma(\omega, t) \to 0 \), for \( t \to \infty \).

Notice that both the correlation time \( \tau_\ast \) and the characteristic function \( \Gamma \) solely depend on the fluctuations power spectral density. We can now use equation (1.28) and have at once

\[
\int_0^T dt \int_0^T dt' R(t - t' + \tau)R(t - t' + \tau') = \tau_\ast \Gamma[\omega_c(\tau - \tau')] T,
\]

The results (1.30) and (1.36) can now be used in equation (1.22) to yield the neat result

\[
\rho_{xx'}(T) = \rho_{xx'}(0) \left[ 1 - \frac{32C_0^2\pi^2 M^2c^4A_0^4\tau_\ast T}{\hbar^2} \times F(\Delta x) \right],
\]

where

\[
F(\Delta x) := 1 - \frac{1}{16\pi^2} \int d\hat{k} \int d\hat{K} \Gamma[\omega_c(\hat{K} - \hat{k}) \cdot \Delta x/c].
\]
power spectrum characterizing the fluctuations. Without the need to evaluate the angular integrals, this follows from the properties of the characteristic function \( \Gamma \). The fact that \( \Gamma[\omega_c t] < 1 \) implies \( 0 \leq F(\Delta x) \leq 1 \) with (i) \( F(\Delta x = 0) = 0 \) and (ii) \( F(\Delta x \to \infty) = 1 \) as special limiting cases. As a consequence the diagonal elements are unaffected while the off-diagonal elements decay exponentially according to

\[
\rho_{xx}(0) := \frac{\rho_{xx}(T) - \rho_{xx}(0)}{T} = -\left[ \frac{32C^2 \pi^2 M^2 c^4 A_0^4 \tau_s}{\hbar^2} \times F(\Delta x) \right] \rho_{xx}(0),
\]

providing of course that \( T \) is small enough so that the change in the density matrix is small. Finally, if \( \delta \rho := \rho_{xx}(T) - \rho_{xx}(0) \), we can define the dephasing as \( |\delta \rho/\rho_0| \). Thanks to the property \( F(\Delta x \to \infty) = 1 \), this converges for large spatial separations to the constant maximum value

\[
|\delta \rho/\rho_0| = \frac{32C^2 \pi^2 M^2 c^4 A_0^4 \tau_s T}{\hbar^2}.
\]

This result based on the present three-dimensional analysis of the conformal fluctuations can be compared to the analogue one-dimensional result that PP found in [23]. Using a gaussian correlation function from the outset they found

\[
|\delta \rho/\rho_0|_{1D} = \frac{\sqrt{\pi} M^2 c^4 A_0^4 \tau_g T}{\sqrt{2} \hbar^2},
\]

where \( \tau_g \) stands for some characteristic correlation time of the fluctuations. Identifying approximately \( \tau_s \approx \tau_g \), we have \( (32C^2 \pi^2)/(\sqrt{\pi}/2) \approx 250 \), assuming \( C_2 \sim 1 \). Thus the present three-dimensional analysis is seen to predict a dephasing two orders of magnitude larger than in the idealized one-dimensional case.

1.6.2 A remark on the validity of the Dyson expansion

We have found that the change in the density matrix is given by:

\[
|\delta \rho/\rho_0| \sim \left( \frac{M c^2}{\hbar} \right)^2 \tau_s T \times A_0^4.
\]

In order for the expansion scheme to be effective, the propagation time \( T \) must be short enough to guarantee that \( |\delta \rho/\rho_0| \) is small. How short depends of course on the statistical properties of the fluctuations, encoded in \( \tau_s \), and on the probing particle mass \( M \). A fullerene C\(_{70}\) molecule with \( (MC_{70} \approx 10^{-24} \text{ kg}) \) gives \( Mc^2/\hbar \approx 10^{27} \text{ s}^{-1} \). Therefore the approach is consistent only if the correlation time \( \tau_s \), the flight time \( T \) and field squared amplitude \( A_0^2 \) are appropriately small. We will come back on this issue in Section 1.8.2, where it is shown that, in the case of vacuum fluctuations (introduced in the next section), it is \( \tau_s \sim \lambda T_P \) and \( A_0^2 \sim 1/\lambda^2 \). For a flight time \( T \approx 1 \text{ ms} \), typical of interferometry experiments, this results in \( |\delta \rho/\rho_0| \approx 10^6/\lambda^{-3} \). For any reasonable value of \( \lambda \gtrsim 10^3 \) the density matrix change is indeed small and the Dyson expansion scheme well posed up to second order. In Appendix B.4 we estimate the fourth order term in the expansion, which will also yield a term proportional to \( A_0^4 \). It will be shown that its contribution in fact vanishes under quite general circumstances. This puts the result (1.39) on an even stronger basis.
1.7 Explicit dephasing in the case of vacuum fluctuations

The result (1.37) is quite general. The only ingredients entering the analysis so far have been: (i) a spacetime metric $g_{ab} = (1 + A)^2 \eta_{ab}$ with $|A| = O(\varepsilon \ll 1)$, (ii) a randomly fluctuating conformal field $A$ satisfying the wave equation $\partial^c \partial_c A = 0$, and (iii) isotropic fluctuations characterized by an arbitrary power spectral density $S(\omega)$. The dephasing then occurs as a result of the nonlinearity in the effective potential $V = M c^2 [C_1 A + C_2 A^2]$.

A particularly interesting case, potentially related to the possibility of detecting experimental signs of QG, is that in which the fluctuations in $A$ are the manifestation, at the appropriate semiclassical scale $\ell = \lambda L_P$, of underlying vacuum quantum fluctuations close to the Planck scale. Strictly speaking the presence of the probing particle perturbs the genuine quantum vacuum state. For this reason it would be appropriate to talk of effective vacuum, i.e. up to the presence of the test particle. By its nature, the present semiclassical analysis cannot take into account the backreaction of the system on the environment. Therefore we simply assume that the modifications on the vacuum state can be neglected as long as the probing particle mass is not too large and the evolution time short. We thus model the effective vacuum properties of the conformal field $A$ at the semiclassical scale on the basis of the properties that real vacuum is expected to possess at the same scale. It is a fact that vacuum looks the same to all inertial observers far from gravitational fields. In particular, its energy density content should be Lorentz invariant. This can obtained through an appropriate choice of the power spectrum $S(\omega)$.

1.7.1 Isotropic power spectrum for vacuum conformal fluctuations

According to the above discussion we expect the average properties of $A$ above the scale $\ell$ to be Lorentz invariant. In particular, the interesting quantities derived in Appendix A

\begin{align}
\langle A^2 \rangle & = \frac{1}{(2\pi)^3} \int d^3 k S(k), \\
\langle |\nabla A|^2 \rangle & = \frac{1}{(2\pi)^3} \int d^3 k k^2 S(k), \\
\langle (\partial_t A)^2 \rangle & = \frac{1}{(2\pi)^3} \int d^3 k \omega^2(k) S(k),
\end{align}

should be invariant. As discussed in Appendix A, for a stationary, isotropic signal, the averages $\langle \cdot \rangle$ can in fact be carried out through suitable spacetime integrations over an appropriate averaging scale $L \gg \ell$. In alternative they can be expressed as in the above integrals depending on the power spectrum and adopting a high energy cutoff set by $k_\lambda : = 2\pi/(\lambda L_P)$.

The problem of the Lorentz invariance of the above quantities has been discussed in details by Boyer [53] within his random electrodynamics framework. He showed that the choice $S \propto 1/\omega$ is unique in guaranteeing an energy spectrum $\varrho(\omega) \propto \omega^3$, also shown to be the only possible choice for a Lorentz invariant energy spectrum of a massless field. In the present case we take

\begin{equation}
S(k) := \frac{\hbar G}{2c^2 \omega(k)}.
\end{equation}

The combination of the constants $\hbar$, $G$ and $c$ gives the correct dimensions for a power spectrum (i.e. $L^3$), while the factor $1/2$ guarantees that the resulting energy density is equivalent to that resulting
from the superposition of zero-point contributions $\hbar \omega / 2$—see Appendix (C)—. In particular this element makes the connection between the present stochastic approach and quantum theory.

With this choice for the power spectrum the integrals (1.41) and (1.42) are indeed Lorentz invariant as they are related to the energy density of $A$. The same holds for the integral in (1.40) as $d^3k/\omega(k)$ is a Lorentz invariant measure [2]. A final important point, which should not be overlooked, is that these facts are true provided the cutoff $k_\lambda$ is given by the same number for all inertial observers, as also discussed in details by Boyer. In other words this means that the critical length that sets the border line between the random field approach and the full QG regime is supposed to be the same for any inertial observer. It represents some kind of structural property of spacetime and not an observer dependent property. Accordingly it must not be transformed under Lorentz transformations. It is important to note that this requirement will naturally be satisfied later when we employ an effective cutoff set by the particle Compton length.

Using (1.43) the normalized correlation function can be found explicitly from the generalized WK theorem to be

$$R(\tau) := \frac{C(\tau)}{C_0} = 2 \left[ \frac{\sin \omega_c \tau}{\omega_c \tau} + \frac{\cos \omega_c \tau - 1}{\omega_c^2 \tau^2} \right].$$

(1.44)

The peak of the autocorrelation function is linked to the fluctuations squared amplitude and gives explicitly:

$$C_0 \equiv A_0^2 = \frac{1}{8\pi\lambda^2},$$

(1.45)

implying $\langle A^2 \rangle = 1/(2\lambda)^2$. The correlation time and characteristic function follow from equations (1.31) and (1.35) as:

$$\tau_* = \frac{2}{3} \lambda T_P,$$

(1.46)

**Figure 1.1:** Plot of $R^2(t - t')$. The dimensionless variable $\sigma$ is basically $t - t'$ in units of the correlation time $\tau_*$. It is seen that $\tau \approx \tau_*$ corresponds to the first of the secondary peaks.
and
\[ \Gamma(\sigma) = \frac{3 \sin(\sigma)}{\sigma} + \frac{6 \cos(\sigma)}{\sigma^2} - \frac{6 \sin(\sigma)}{\sigma^3}, \] (1.47)
where \( \sigma = \omega_c t \) is a dimensionless variable. The plot of the squared normalized correlation function \( R^2(t - t') \) is shown in figure 1.1: \( t - t' = \tau_s \) corresponds to the first secondary peak in the curve, where the correlation in the fluctuations is reduced of \( \sim 70\% \). This fully motivates the choice of \( \tau_s \) to represent the correlation time.

The explicit form of the characteristic function can be used in (1.38) to evaluate the remaining angular integrals and find the detailed expression for the density matrix evolution valid for all \((x, x')\). Isotropy implies that the result must depend on \( |x - x'| \) only. For convenience we can choose the reference frame where a given \( \Delta x \) lies along the z axis. Then we can define the dimensionless variable \( \sigma \) through
\[ \frac{\omega_c \Delta x}{c} = \sigma \hat{z}, \] (1.48)
so that
\[ \sigma = \frac{\omega_c |\Delta x|}{c} = \frac{2\pi |\Delta x|}{\ell}. \] (1.49)

With this choice of reference frame the unit vectors \( \hat{k} \) and \( \hat{K} \) have components
\[ \hat{k} = \sin \vartheta \cos \varphi \hat{x} + \sin \vartheta \sin \varphi \hat{y} + \cos \vartheta \hat{z}, \] (1.50)
\[ \hat{K} = \sin \vartheta' \cos \varphi' \hat{x} + \sin \vartheta' \sin \varphi' \hat{y} + \cos \vartheta' \hat{z}. \] (1.51)

Now we have, for example,
\[ \int d\hat{k} = \int d\vartheta \sin \vartheta d\varphi \]
and since the integrand in (1.38) depends only on \( \vartheta \) and \( \vartheta' \) the integrations over \( \varphi \) and \( \varphi' \) are trivial and yield a factor \( 4\pi^2 \). Performing the change of variables
\[ u := -\cos \vartheta, \quad u' := -\cos \vartheta', \] (1.52)
the integration is straightforward and yields the result:
\[ F(\sigma) := 1 - \frac{3}{2\sigma^2} \left( 1 - \frac{\sin \sigma \cos \sigma}{\sigma} \right). \] (1.53)

Substituting the results (1.45), (1.46), (1.47) and (1.54) into (1.37) yields the explicit result for the dephasing, valid for vacuum fluctuations described by \( S \propto 1/\omega \):
\[ \left| \frac{\delta \rho_{xx'}}{\rho_0} \right|^2 = \frac{1}{3\lambda^3} \left( \frac{M}{M_p} \right)^2 \frac{T}{T_p} \times F \left( \frac{2\pi |x - x'|}{\ell} \right), \] (1.55)

where we considered \( C_2 \sim 1 \) and where
\[ M_p := \frac{\hbar}{c^2 T_p} = \sqrt{\frac{\hbar c}{G}} = 2.176 \times 10^{-8} \text{kg} = 1.310 \times 10^{19} \text{amu} \]
is the Planck mass. The function \( F \) is plotted in figure 1.2. It enjoys the properties \( F(0) = 0 \) and \( F(\sigma) \to 1 \) for \( \sigma \gg 1 \), so that for \( |x - x'| \gtrsim 10\ell \) the decoherence rate converges rapidly to its
1.8 Discussion

1.8.1 Probing the particle resolution scale and effective dephasing

Equation (1.55) gives the dephasing in the density matrix of a quantum particle propagating in space under the only action of a randomly fluctuating potential due to spacetime vacuum conformal fluctuations. The fact that it predicts an exponential decay of the off diagonal elements (which is the distinctive feature of quantum decoherence) is interesting as a further confirmation that certain effects involving quantum fluctuations can be mimicked by means of a semi-classical treatment in the spirit of Boyer [53, 41].

A significant feature of our dephasing formula is the quadratic dependence on the probing particle mass $M$, which comes as a consequence of the underlying non linearity. The coefficient $1/(3\lambda^3)$ sets the overall strength of the effect. It is proportional to $A_0^4$ and to the fluctuations correlation time $\tau_\sigma$: the more intense the fluctuations, the larger the dephasing and the longer the various directional components stay correlated, the higher their ability to induce dephasing. We have found $\tau_\sigma \approx \lambda T_P$, in such a way that the correlation time directly depends on the spacetime intrinsic cutoff parameter $\lambda$. According to this picture all the wavelength down to the cutoff $\ell = \lambda L_P$ should be able to affect the probing particle. However an atom or molecule is likely to possess its own resolution scale $L_R$. Thus, whenever $\tau_\sigma c < L_R$, the ability of the fluctuations to affect the particle would be reduced, as they would effectively average out. To characterize this feature of the problem we write, in analogy to $\ell = \lambda L_P$,

$$L_R := \lambda_R L_P,$$

and use $\lambda_R$ as a new, particle dependent, cutoff parameter. In general it is $\lambda_R \geq \lambda$. The new effec-
1.8. Discussion

Figure 1.3: Fourier transform of $R^2(\sigma)$ as a function of the frequency $\omega$ in units of the cutoff frequency $\omega_c$.

The effective correlation time is now given by $\tau_s \approx \lambda_R T_P$. The distance traveled by the fluctuations during a correlation time is $L_s = c \tau_s \equiv 2L_R/3$. Thus the effective correlation distance $L_s$ basically corresponds to the particle resolution scale: short wavelengths that do not keep their correlation up to the scale $L_R$ average out and cannot affect the probing particle. The new, effective dephasing results by substituting $\lambda$ with $\lambda_R$ in (1.55):

$$\frac{\delta \rho_{xx'}}{\rho_0} = \frac{1}{3} \left( \frac{L_P}{L_R} \right)^3 \left( \frac{M}{M_P} \right)^2 \left( \frac{T}{T_P} \right) \times F \left( \frac{2\pi |x - x'|}{L_R} \right).$$

1.8.2 Validity of the long drift time regime

As discussed in Section 1.6, we recall that this result holds for ‘long’ drift times $T$, i.e. when $T \gtrsim (\Delta \omega)^{-1}$, where $\Delta \omega$ is an appropriate frequency range over which the Fourier transforms of $R^2(t)$ and $R(t + \tau)R(t + \tau')$ vary little. We are now in the position to make this precise and define clearly the limits of applicability of the theory. To this end we consider the Fourier transform of $R^2(\omega\tau)$:

$$\mathcal{F}[R^2(\omega\tau)](\omega) = \frac{1}{\omega} \mathcal{F}[R^2(\sigma)](\omega/\omega_c),$$

with $R(\sigma)$ given in (1.44). Its plot is displayed in figure 1.3. The spectrum falls to 0 for $\omega \geq 2\omega_c$. The value of the peak at $\omega = 0$ is precisely $4\pi/3$, verifying that $\tau_s \equiv \mathcal{F}[R^2(\omega\tau)](0) = 4\pi/(3\omega_c)$. The smaller box shows a zoom of the plot in the region $\sigma \in [0, 1/100]$: the curve is slow varying in this range since $\mathcal{F}[R^2(\omega\tau)](0) = 4\pi/3 \approx 4.19$ and $\mathcal{F}[R^2(\omega\tau)](1/100) \approx 4.13$. Similarly it is possible to check that the Fourier transform of $R(\sigma + \eta)R(\sigma + \eta')$, where the dimensionless parameters $\eta$ and
\( \eta' \) depend on space direction and locations as \( \eta := -\omega_c \hat{k} \cdot \Delta x/c \) and \( \eta' := -\omega_c \hat{K} \cdot \Delta x/c \), enjoys a similar property: for every choice of \( \eta \) and \( \eta' \), the resulting Fourier transform is slow varying in the range \( \sigma \in [0, 1/100] \). Following this discussion we choose \( \Delta \omega \approx [0, \omega_c/100] \). We can now quantify the concept of ‘long drift time’ by \( T \gtrsim 100/\omega_c \). From \( \omega_c = 2\pi/(\lambda L_P) = 4\pi/(3 \tau_s) \) this yields the condition

\[
T \gtrsim 25 \tau_s.
\]

### 1.8.3 Some numerical estimates and outlook

In summary we have studied the dephasing on a non-relativistic quantum particle induced by a conformally modulated spacetime \( g_{ab} = (1 + A)^2 \eta_{ab} \), where \( A \) is a random scalar field satisfying the wave equation. The important case of vacuum fluctuations can be characterized by a suitable power spectrum \( S \propto 1/\omega \). If \( L_R = \lambda_R L_P \) is the probing particle resolution scale, the dephasing for \( |x - x'| \gg L_R \) converges rapidly to:

\[
\left| \frac{\delta \rho}{\rho_0} \right| \approx 3 \left( \frac{L_P}{L_R} \right)^3 \left( \frac{M}{M_P} \right)^2 \left( \frac{T}{T_P} \right) .
\]

The effective correlation time of the fluctuations is given by \( \tau_s \approx \lambda_R T_P \). The above result holds for ‘long’ drift times satisfying

\[
T \gtrsim (10 - 10^2) \lambda_R T_P.
\]

To conclude we want to give some numerical estimates of the dephasing that could be expected in a typical matter wave interferometry experiment, e.g. like those described in [32], where fullerene molecules have been employed with drift times of the order of \( T =: T_{ex} \approx 10^{-3} \) s. Consider e.g. a C\(_{70}\) molecule with \( M = M_{C_{70}} \approx 1.24 \times 10^{-24} \) kg. In comparison to the Planck units we have:

\[
T_{ex} \approx 10^{40} T_P, \quad M_{C_{70}} \approx 10^{-17} M_P .
\]

Thus, it is clear that the most critical factor controlling the strength of the effect is set by the probing particle mass, together with the effective resolution cutoff scale. Using these data in (1.56) we can estimate:

\[
\left| \frac{\delta \rho}{\rho_0} \right| \approx 10^6 \frac{\lambda_R^3}{\lambda_R^3} \Leftrightarrow \lambda_R \approx \left( \frac{10^6}{|\delta \rho/\rho_0|} \right)^{\frac{1}{3}} .
\]

This could be used to estimate \( \lambda_R \) if we were able to identify within an experiment a residual amount of dephasing that cannot be explained by other standard mechanisms (e.g. environmental decoherence, internal degrees of freedom). Figure 1.4 plots \( \lambda_R \) against \( |\delta \rho/\rho_0| \): a dephasing due to conformal fluctuations in the range \( 1\% - 0.1\% \) would imply a resolution parameter in the range \( \lambda_R \approx 10^3 - 10^4 \). This would represent a lower bound on \( \lambda_R \), as interferometry experiments will get more and more precise in measuring and modeling environmental decoherence. Estimating the present typical uncertainty of typical interferometry experiments as \( |\delta \rho/\rho_0| \approx 0.01\% \) we get

\[
\lambda_R \gtrsim 10^3, \quad \text{for} \quad \text{C}_{70} .
\]

We remark that such order of magnitudes estimates are consistent with a small change in the density matrix and the second order Dyson expansion approach.

A value for \( \lambda_R \) as small as \( 10^3 \) would probably approach the intrinsic spacetime structural limit set by \( \lambda \), i.e. \( \ell = \lambda L_P \). It is interesting to ask what amount of dephasing our model predicts, inde-
pendently of experimental data. To this end we need to prescribe theoretically the particle resolution scale $L_R$. Though no obvious choice exists, an interesting possibility would be to set it equal to the particle Compton length [57, 70], i.e.

$$L_R = \frac{h}{M c}.$$ 

This choice is obviously Lorentz invariant and also motivated by the fact that the Compton length represents a fundamental uncertainty in the position of a nonrelativistic quantum particle. Indeed, by the Heisenberg uncertainty principle, $\Delta x \approx h/M c$ would imply $\Delta p \gtrsim M c$, implying an uncertainty in the energy of the same order of the rest mass $M c^2$. In such a situation QFT would become relevant. Alternatively it can also be argued that wavelengths shorter than $h/M c$ would have enough energy to create a particle of mass $M$ from the vacuum. With this choice, equation (1.56) becomes

$$\left| \frac{\delta \rho}{\rho_0} \right| = \frac{1}{24\pi^5} \left( \frac{M}{M_P} \right)^5 \left( \frac{T}{T_P} \right).$$ 

(1.57)

This can be used to estimate the amount of dephasing induced by vacuum conformal fluctuations.

In the case of $C_{70}$ the Compton wavelength is $\approx 10^{-18}$ m $\approx 10^{18} L_P$, corresponding to $\lambda_R \approx 10^{18}$. For a propagation time of $\approx 1$ ms this gives a dephasing

$$\left| \frac{\delta \rho}{\rho_0} \right| (M_{C_{70}}, 1\text{ms}) \approx 10^{-44},$$

which would be negligible and far beyond the possibility of experimental detection. Thus, in order to achieve dephasing within the current experimental accuracy, much heavier quantum particles are needed. In atomic mass units $C_{70}$ has a mass $M_{C_{70}} \approx 10^3$ amu. Equation (1.57) applied to a particle

Figure 1.4: Adimensional cutoff parameter as a function of the dephasing for a $C_{70}$ molecule with a drift time of $10^{-3}$ s.
with mass $M \approx 10^{11}$ amu and with a drift time $T \approx 100$ ms gives the estimate:

$$\left| \frac{\Delta \rho}{\rho_0} \right| (10^{11} \text{amu}, 100 \text{ms}) \approx 10^{-2}.$$

A drift time of $\sim 100$ ms could possibly be achieved in a space based experiment. On the other hand, the need of a quantum particle as heavy as $10^{11}$ amu poses an extraordinary challenge. A possibility would be to employ quantum entanglement states. This is already being considered in the literature, e.g. in [64], where entangled atomic states are studied and suggested as a possible improved probe for future detection of spacetime induced dephasing.

In relation to the issue of a possible future experimental detection of dephasing due to vacuum effects, it is important to note that even the vacuum fluctuations of the EM field can influence the fringe visibility of a neutral particle if this has a permanent electric or magnetic dipole moment [71]. The amount of decoherence due to this effect, e.g. in a typical interference experiment, is expected to depend quadratically on the dipole moment of the particle. Fullerene molecules such as C$_{70}$ (or even the spherically shaped C$_{60}$) have symmetric charges distributions and possess no permanent dipoles [72]. In this case no extra effect would be expected. Drugged versions of fullerenes where, e.g., Na or Li atoms are combined with C$_{60}$ to form heavier molecules such as Na$_{18}$C$_{60}$ or Li$_{10}$C$_{60}$ can possess a permanent electric dipole moment between $\sim 10$ and $\sim 20$ Debyes, depending on the number of drugging atoms [73]. While in this case some decoherence due to EM vacuum effects would be theoretically expected, the mass of such molecules is still far too low for any gravitational effect due to conformal fluctuations to be detectable. Only in the case of more complex and heavier quantum systems, e.g. gold clusters or entangled states, with a permanent dipole the two effects would theoretically both contribute to the overall dephasing. However the dephasing due to a permanent dipole moment is in general predicted to be typically of the order of the size of the dipole length in units of the total length of the trajectory [71]. In this sense, as the mass of the quantum probe is increased, we expect the gravitationally induced dephasing to represent the dominant effect.

The last important point that needs verification is that the condition (1.4) given earlier at the beginning of this chapter is indeed verified: that was required in order for the change in momentum due to the fluctuations to be smaller than the laboratory particle momentum $\tilde{p} = M\tilde{v}$. It read: $\varepsilon^2 \sim \langle A^2 \rangle \ll (\tilde{v}/c)^2$. The field effective mean quadratic amplitude interacting with the particle is given by $\langle A^2 \rangle \sim \lambda_R^{-2}$. Thus we have the condition:

$$\frac{1}{\lambda_R} \equiv \frac{L_P}{L_R} \ll \frac{\tilde{v}}{c}.$$

By using the expression for the Planck length and with $L_R$ given by the particle Compton length, this yields a condition on the particle mass $M$:

$$\frac{M}{M_P} \ll 2\pi \frac{\tilde{v}}{c}.$$

For typical laboratory velocities $\tilde{v}/c \approx 10^{-6}$ and, since $M_P \approx 10^{19}$ amu, this condition is met for particle masses up to $M \approx 10^{13}$ amu, including the case of C$_{70}$ molecules or the heavier entangled quantum states discussed above. This limit would be reduced for slower particles.

We conclude by remarking that the theory described until now is quite general, in the sense that as a starting input it only needs a conformally modulated metric and a scalar field satisfying the wave equation. Of course, it is important to identify in concrete which theories of gravity can actually
yield such a scenario. This problem is central to this thesis and is be the object of the next three chapters: in Chapter 2 a general framework for the study of fluctuating fields close to the Planck scale is introduced and the resulting framework applied to standard GR in Chapter 3; in Chapter 4 we consider more general scenarios involving scalar-tensor theories of gravity.
Chapter 2

The nonlinear random gravity framework

In this chapter we introduce the nonlinear random gravity framework. Our goal is modeling some aspects of the low energy physics related to the spacetime metric and matter fields vacuum fluctuations. The approach extends conceptually Boyer’s random electrodynamics to a theory of random geometry but has a somehow richer structure due to the nonlinearity originally inherent to GR. Essentially, Isaacson’s perturbative approach over a curved background is applied to the problem of spacetime fluctuations by introducing randomly fluctuating solutions to the expansion equations. The fluctuations are supposed to mimic some vacuum effects related to zero point energy at an appropriate low energy scale. Isaacson’s original approach is generalized in order to seek for solutions in which the physical metric may exhibit conformal fluctuations. The technique must be extended to second order nonlinearity in order to take into account the backreaction energy due to zero point GWs, as well as matter fields and conformal fluctuations. This reveals interesting connections to the well known cosmological constant problem. Finally the framework is applied to standard GR as a first attempt to identify a theory of gravity which may be compatible with the presence of conformal fluctuations. To this end, the relevant first and second order equations describing GWs and the vacuum backreaction on the spacetime metric within GR are derived.

2.1 Seeking a physical basis for the conformal fluctuations

We denote the spacetime physical metric by $g_{ab}$. This couples as usual to matter fields and determines the geodesics of small classical test particles. Conformal fluctuations can formally be included by writing the physical metric as:

$$g_{ab} = \Omega^2 \gamma_{ab},$$

(2.1)

where $\Omega$ is a conformal factor and $\gamma_{ab}$ will be referred to as the conformal metric. Denoting all matter fields collectively by the symbol $\psi$ and imposing Einstein’s equation

$$G_{ab}[g] = 8\pi T_{ab}[\psi],$$

(2.2)

on the physical metric $g_{ab}$, yields a corresponding equation for the conformal metric. This reads [1]:

$$G_{ab}[\gamma] = 8\pi T_{ab}[\psi] + \Sigma^1_{ab}[\Omega] + \Sigma^2_{ab}[\Omega],$$

(2.3)
where we defined

\[ \Sigma^1_{ab} := 2\nabla_a \nabla_b \ln \Omega - 2\gamma_{ab}\nabla^c \nabla_c \ln \Omega, \] (2.4)

\[ \Sigma^2_{ab} := -(2\nabla_a \ln \Omega \nabla_b \ln \Omega + \gamma_{ab}\nabla^c \ln \Omega \nabla_c \ln \Omega), \] (2.5)

and where \( \nabla_a \) denotes here the covariant derivative of the conformal metric \( \gamma_{ab} \). By definition Einstein’s equation is satisfied if and only if equation (2.3) is satisfied and if the physical and conformal metric are related by (2.1).

The correct point of view to have regarding the problem of whether the physical metric truly has a conformal modulation is the following: we must think of the conformal metric \( \gamma_{ab} \) as having some concrete prescribed structure that does not depend upon \( \Omega \), which implies that (2.3) simply represents an equation constraining the conformal factor. In this sense equation (2.1) could be viewed as an ansatz solution for Einstein’s equation. The problem of having a conformally modulated metric within standard GR thus effectively reduces to that of finding a suitable conformal metric such that the corresponding constraint equation for \( \Omega \) can be satisfied. If the solution for \( \Omega \) is compatible with a conformal factor of the form \( \Omega = 1 + A \) with \( A \) being a randomly fluctuating field then we would have a concrete framework providing a basis for the results derived in Chapter 1.

### 2.1.1 The simplest scenario: a conformally modulated universe

The simplest (and most dramatically restrictive!) choice to constraint the conformal metric is

\[ \gamma_{ab} := \eta_{ab}. \]

This is PP choice in [23]. In this case we have \( \nabla_a = \partial_a \), \( G_{ab}[\eta] = 0 \) and the constraint equation for \( \Omega \) reads

\[ 8\pi T_{ab}[\psi] + \Sigma^1_{ab}[\Omega] + \Sigma^2_{ab}[\Omega] = 0. \] (2.6)

Moreover, in the simplified PP model all matter fields were neglected by simply putting \( T_{ab}[\psi] = 0 \). Such an hypothetical universe containing only conformal fluctuations should then satisfy the equation:

\[ 2 \left( \partial_a \partial_b \ln \Omega - \eta_{ab} \partial^c \partial_c \ln \Omega \right) - 2 \partial_a \ln \Omega \partial_b \ln \Omega + \eta_{ab} \partial^c \ln \Omega \partial_c \ln \Omega = 0. \]

In the interesting case of conformal fluctuations we can write \( \Omega = 1 + \delta \Omega \), where \( \delta \Omega \) is a small fluctuating modulation. Then by expanding \( \ln \Omega \approx 1 + \delta \Omega - \delta \Omega^2 / 2 \) we would have:

\[ 2 \left( \partial_a \partial_b \delta \Omega - \eta_{ab} \partial^c \partial_c \delta \Omega \right) - \left( \partial_a \partial_b \delta \Omega^2 - \eta_{ab} \partial^c \partial_c \delta \Omega^2 \right) - \left( 2 \partial_a \delta \Omega \partial_b \delta \Omega + \eta_{ab} \partial^c \delta \Omega \partial_c \delta \Omega \right) = 0. \]

To linear order, the trace of the equation

\[ \partial_a \partial_b \delta \Omega - \eta_{ab} \partial^c \partial_c \delta \Omega = 0 \] (2.7)

implies that the perturbation \( \delta \Omega \) should satisfy the wave equation, i.e.

\[ \Box \delta \Omega = 0, \] (2.8)

where \( \Box := \partial^c \partial_c \). This is of course what PP found. However their method employed the Hilbert action and completely overlooked upon the fact that, beyond (2.8) being satisfied, Einstein’s equation then implies the very restrictive constraint \( \partial_a \partial_b \delta \Omega = 0 \), as it follows from (2.7) once the wave equation is also taken on board. This constraint is too restrictive and it is hard to see how it could be compatible
with a fluctuating $\delta \Omega$. Beyond this PP model suffers from a further important problem: indeed the second order constraint equation 
$$-(\partial_a \partial_b \delta \Omega^2 - \eta_{ab} \partial_c \partial_c \delta \Omega^2) - (2 \partial_a \delta \Omega \partial_b \delta \Omega + \eta_{ab} \partial_c \delta \Omega \partial_c \delta \Omega) = 0$$

is quadratic and could not be satisfied except in the trivial case of $\delta \Omega = 0$. In conclusion a metric of the form $g_{ab} = (1 + \delta \Omega)^2 \eta_{ab}$ seems to be incompatible with Einstein’s equation. If we insist in wishing to include conformal fluctuations in the formalism we must seek a physical metric with a more general structure.

2.1.2 The general basis for the model

We will try to overcome the above difficulty by improving the model upon three points:

1. allow for the presence of GWs; this will correspond to a larger freedom in choosing the structure of $\gamma_{ab}$,

2. account for the presence of matter fields through $T_{ab}[^{\psi}]$; this is physically relevant as, when studying vacuum, matter fields contribution should not be neglected,

3. in agreement with the nonlinear structure of GR, allow for a nonlinear correction to the conformal factor by writing $\Omega = 1 + A + B$, where $|A| = O(\varepsilon \ll 1)$ and $|B| = O(\varepsilon^2)$, with $\varepsilon \ll 1$.

To implement these points we need to develop a framework allowing to model some of the properties inherent to vacuum in a coherent way. In the next sections we discuss the main features of the random gravity framework. This was first introduced in [46], where the main goal was to show that the inclusion of conformal fluctuations could offer a mechanism for a solution of the cosmological constant problem [45]. This idea could seem at first sight plausible if we look at the quadratic part of the effective stress-energy tensor (2.5) due to the conformal factor: this has a negative definite kinetic term and one might expect that it could serve as a compensating factor to reduce or cancel the large and positive vacuum energy density due to the matter fields and GWs. Even in the case $\gamma_{ab} = \eta_{ab}$, the constraint equation (2.6) seems to motivate this idea. This is also hinted by the canonical analysis of general relativity [17]. In particular the analysis performed by Wang in [62, 63] led to arguing in [36] that hypothetical quanta related to the conformal fluctuations could offer a way to compensate for the vacuum energy associated to GWs. This also was the result in [46]. Unfortunately, as we will show at the end of this chapter, that conclusion was invalid due to the fact that the nonlinear backreaction affect on the spacetime metric due to the conformal fluctuations was not accounted for properly.

Before going into details we need to discuss in the next section some points related to vacuum and the cosmological constant problem.

2.2 Vacuum and some related problems

The nature and properties of vacuum are not fully understood yet [74, 75, 76, 77, 78, 79, 80, 81]. A reasonably clear definition of vacuum exists within QFT for quantum fields on a flat, non-dynamical, Minkowski background. A satisfying theory of quantum fields propagating on an assigned curved background geometry can also be given [9]. In this case the notion of vacuum state is much more subtle, essentially because a generally curved geometry lacks the notion of a global inertial observer, with respect to whom vacuum is usually defined when no gravitational fields are present. As it is well known, even in standard QFT, the ground state vacuum energy receives a contribution from the zero point energies of an infinite amount of harmonic oscillators. This results in a formally infinite amount
of vacuum zero point energy unless some high energy cutoff is applied. Even with a reasonable cutoff, it has been suggested that the resulting large energy density would lead either to a rapid expansion of the universe as discussed by Weinberg [82] or to a drastic collapse of spacetime as noticed by Pauli [78, 79]. Since there is no obvious sign of such a catastrophic breakdown in the spacetime that we can practically observe, this rises the issue of the reality of the zero point energy. In this respect it is important to note that vacuum fluctuations of the electromagnetic field have long received support from experiments. These include the Casimir effect [83, 84, 85], Lamb shift of atomic energy levels [86, 87] and spontaneous emission from atoms [88, 89, 90]. The fundamental fluctuation-dissipation theorem [91] provides a further basis for the vacuum fluctuations, manifested as superconductive current noise in the Josephson junctions [92, 93, 94]. Even so, elements of doubt still surround the physical reality of the quantum vacuum. Some argue that e.g. the Casimir effect could be explained without the vacuum energy [95]. Laboratory measurements of vacuum fluctuations can at best detect differences in the vacuum energy by modifying their boundary conditions [96].

Beyond these considerations, a further important issue related to vacuum energy is that of its backreaction upon the spacetime geometry; i.e. whether vacuum gravitates. It is generally believed that if vacuum energy is real, then it must gravitate in accordance with GR, and that only through the resulting gravity the net vacuum energy could be determined. Recently, it has been shown that the part of vacuum energy responsible for the Casimir effect does indeed gravitate [97]. However, the gravitational consequence of the total vacuum energy remains controversial [98, 99, 100, 101]. Because of the observed Lorentz invariance of vacuum at low energies, after any appropriate regularization, the vacuum energy is expected to couple to Einstein’s equation via an effective cosmological constant only through a pure trace term of the kind $-\Lambda g_{ab}$. There is a common perception that an exceedingly large cosmological constant would built up from all ground states up to the Planck energy density scale [82]. While a sizable cosmological constant could have driven the cosmic inflation in the early universe, observations indicate its present value to be just under the critical density value [102, 45]. This discrepancy between the observed value of the cosmological constant and the contribution that one would expect from QFT constitutes part of what is known in the literature as the cosmological constant problem [103].

Despite the early suggested link between the vacuum energy of elementary particles and the cosmological constant by Zel’dovich [104], a detailed mechanism is still lacking. This motivates the “dark energy” models such as the quintessence fields and their extensions [82, 45]. Leaving aside the original mysterious cancelation of the huge vacuum energy, these models indeed offer arguably the most popular current approach to the “cosmological constant problem”. Nevertheless, efforts to account for the observed cosmic acceleration using cosmological perturbation back-reaction without resorting to dark energy are also being made. The need for a better understanding of nonlinear metric perturbations [105] and QG effects [106] is clearly highlighted by the recent progress in this direction.

### 2.3 Nonlinear random gravity

Our point of view here is that spacetime metric fluctuations are also in fact an integral part of the quantum vacuum. Thus it could be plausible that an attempt to disentangle these difficulties should take them on board. However, a fundamental difficulty to studying these effects is the absence of a consistent quantum theory of gravity with an appropriate classical limit. Recently, progress has been made using the stochastic gravity approach [107, 108, 109, 110]. This generalizes the previous semiclassical approach where gravity is coupled to the expectation value of quantum matter fields stress energy tensor. The stochastic gravity approach attempts to take on board the fluctuations of the
stress energy tensor around its average. By means of suitable statistical considerations, the features of the quantum fluctuations are ‘translated’ into a suitable fluctuating classical, effective tensor that acts back as a source on the spacetime metric, thus inducing a stochastic metric perturbation. The approach is very interesting and it has been applied to the interesting case of the perturbations induced by an arbitrarily coupled scalar field upon Minkowski geometry [108]. This study has shown that Minkowski spacetime is stable upon perturbations. Moreover the induced metric perturbations have been found to be correlated only very close to the Planck scale, in such a way that matter fields tend in fact to suppress the short-scale perturbations for larger scales. Though undoubtedly a very sound theory, the stochastic gravity approach totally ignores the issue of zero point energy. In fact this is done by regularizing from the outset the matter fields stress energy tensor in such a way that, in the vacuum state, it is \( \langle T_{ab}[^\psi] \rangle = 0 \). Various techniques exist to do this on a curved background, as it is discussed in detail by Wald [9]. Without going here into these complexities it is enough to say that, in the case of a flat background, these correspond to the usual practise of normal ordering, by virtue of which all zero point energies are in fact ignored. It follows that if these are believed to be real and a source of gravity, then the stochastic gravity approach is not suitable in its present form.

Another important consideration regards the fact that the stochastic gravity approach accounts for the backreaction on the spacetime geometry due to matter fields only. To our knowledge, possible backreaction effects due to the self energy of GWs have not been investigated yet.

In order to account for zero point energy and GWs, our approach is somehow minimal: we seek to analyze some aspects related to vacuum at an appropriate energy regime, in between the Planck scale and those characterizing the observed classical world; we also wish to incorporate conformal fluctuations explicitly and to account for the backreaction due to GWs zero point energy. The purpose of this approach is thus twofold: (i) investigating whether conformal fluctuations of the physical metric can provide a vacuum energy balance mechanism that may help in relation to the riddle of the cosmological constant problem; (ii) provide a coherent framework that may provide a sound physical basis for the material presented in Chapter 1.

### 2.3.1 Stochastic geometry

It is generally agreed that, independently of the true underlying QG theory, spacetime structure at very small scales should ‘lose’ its usual classical appearance [13]. This is true in particular within the two main approaches to QG mentioned in the introduction: String Theory and LQG. In the first case the quantum behavior of spacetime is inherent in the presence within the theory of a particular excitation describing the graviton, while one of the main results of the loop quantization program is the discrete, quantized nature of the geometry (lengths, areas, volumes) close to the Planck scale. This is characterized by the fundamental units, introduced by Planck back in 1899:

\[
L_P := \sqrt{\frac{\hbar G}{c^3}} \approx 1.62 \times 10^{-35} \text{ m},
\]

\[
T_P := \frac{L_P}{c} \approx 5.40 \times 10^{-44} \text{ s},
\]

\[
M_P := \frac{\hbar}{L_P c} \approx 2.17 \times 10^{-8} \text{ kg} \approx 1.22 \times 10^{19} \text{ GeV}.
\]

The successful QG theory will have to provide a coherent way to describe how the classical spacetime structure to be described in terms of a smooth Lorentz manifold can emerge from the microphysics at the Planck scale. Even though it is not known exactly at which scale the ‘turbulent’ quantum nature
of spacetime gives in to yield a classical structure, it is fair to state that the assumption of a smooth regular manifold holds down to, at least, the subnuclear scale, i.e. $\approx 10^{-15}$ m. In fact it is likely to hold down to much shorter scales as it is implied e.g. by the fact that when studying gluon-quarks states within QCD, gravity effects can still be ignored. The natural realm of QG effects lies much further away at the bottom and it is still directly inaccessible: even the LHC, with its expected center of mass collision energy of $\sim 10$ TeV, is still not even remotely close to the Planck energy of $10^{19}$ GeV.

It is commonly believed that, between the Planck scale and what we could call the minimal classical scale, there must exist an intermediate regime in which spacetime begins to acquire a classical character, though still inheriting features from the underlying quantum regime [23, 36]. The usual assumption is that these features are manifest as a stochastic behavior of the spacetime geometry. This will also be our point of view. In a nutshell this sums up to the expectation that some of the properties of quantum spacetime can be modeled, at an appropriate low energy scale, by employing a suitable stochastic approach.

That classical stochastic fields can be used to model and reproduce various quantum effects related to the electromagnetic field is well known, e.g. from the work of Boyer [53, 41, 111, 112]. In his theory of Random Electrodynamics (RED) he showed that a large variety of quantum results can be obtained through the classical Maxwell equation supplemented by suitable random boundary conditions. This had already been pointed out by Welton [86]. These can be applied to obtain a classical yet fluctuating solution of Maxwell equations, whose statistical properties can be chosen in order to match those of the true underlying zero point quantum field in such a way to correspond to a Lorentz invariant spectrum. Specifically, it was shown in [111] that QED and RED yield equivalent results for the $N$-points correlation functions of free EM fields and also, e.g., for some (but not all!) expectation values of an oscillator at zero temperature. Beyond Boyer’s work, this kind of classical stochastic behavior has also been advocated by other authors in relation to vacuum related phenomena [113, 114]. York championed a novel gravitational analogue in which black hole entropy and radiance are derived from quasinormal mode metric fluctuations [115]. These fluctuations are prescribed by a classical Vaidya geometry satisfying the Einstein’s equation with amplitudes set to the quantum zero point level. An improved quantum treatment of the problem using path integral is recently provided in [116].

Even though we maintain the view that that quantum theory has a deeper motivation and truth than a classical theory supplemented by boundary conditions, the work of Boyer shows that, in certain regimes, one may use the alternative classical theory to mimic some quantum related behaviors of the system under study. Following these considerations we wish to apply this idea to the interesting problem of spacetime close to the Planck scale. The assumption we make is that this approach should allow us to model some low energy effects due to vacuum metric fluctuations and assess their effect on classical spacetime and other systems at a larger scale.

In this respect a key ingredient we require from the framework is that it allows to implement the passage from the stochastic scale to the classical scale: in other words that it offers a way to recover the smooth classical spacetime. The general idea is to use a perturbation approach and employ suitably defined averaging procedures. Effectively we shall generalize the classic framework due to Isaacson [43, 44] for a theory of metric perturbations over a curved background, including explicitly the conformal fluctuations and applying it to the specific case of vacuum. It was shown by Isaacson that a spacetime averaging procedure allows indeed to treat coherently the backreaction affect due to the local energy content of high frequency GWs, whose effective stress energy tensor then resembles that of a massless spin-2 radiation fluid. As Isaacson work is central to the material developed below, it is reviewed in some details in Appendix D.
2.3. NONLINEAR RANDOM GRAVITY

2.3.2 Definition of the relevant scales

We consider the microscopic structure of spacetime at a scale \( \ell := \lambda L_P \), where \( L_P \approx 10^{-35} \text{ m} \) is the Planck scale and where the dimensionless parameter satisfies \( \lambda \gtrsim 1 \). As discussed in Chapter 1, it sets the benchmark between the full QG domain and a semiclassical domain, in which spacetime properties still inherit traces of the underlying QG physics, though being expected to be treatable by semiclassical means.

Even in empty spacetime the vacuum energy of matter fields is still a source of gravity. Therefore we consider Einstein’s equation for the physical metric \( g_{ab} \)

\[
G_{ab}[g] = 8\pi T_{ab}[\psi, g],
\]

where \( T_{ab} \) will be a model stress-energy tensor describing the overall vacuum energy contributions coming from all matter fields, collectively denoted by \( \psi \). The metric \( g_{ab} \) and the matter fields \( \psi \) are considered to be randomly fluctuating at the scale \( \ell \). Accordingly we shall refer to \( \ell \) as to the random scale and interpret it as the typical scale above which quantum vacuum properties can approximately be described by means of classical stochastic fields. Below \( \ell \) and closer to the Planck scale a full theory of quantum spacetime is required.

We are considering here vacuum metric fluctuations in an otherwise empty universe. These can in practice be assigned as random perturbations about some background metric \( g_{ab}^B \) which we will assume to be set up in an autoconsistent way by the backreaction effect due to vacuum energy. In the following calculations we work locally and with respect to a physical inertial laboratory frame, whose typical scale \( L_{\text{lab}} \) is much larger than the random scale, yet small enough for the background metric to be smooth and slow varying in an appropriate coordinate system. We then expect the vacuum metric to be described by an expansion of the kind

\[
g_{ab} = g_{ab}^B + g_{ab}^{(1)} + g_{ab}^{(2)} + \ldots,
\]

where the \( g_{ab}^{(n)} \) indicate small fluctuating terms. Here and henceforth we follow the standard notation where a superscript \( (n) \) denotes an \( n \)-th order perturbation with reference made to some small parameter \( \varepsilon \ll 1 \). In the subsequent expansion of field equations, the matter fields \( \psi \) will be treated as first order quantities. The classical equation (2.9) will be analyzed explicitly for the first and second order metric perturbations, and suitable Boyer’s type fluctuating boundary conditions will be imposed. Derived physical quantities, e.g. the Einstein tensor, are fluctuating as a result of the fluctuating metric.

Of course vacuum is Lorentz invariant when viewed at some appropriate macroscopic classical scale, in which case we usually speak of empty space. As a result, the only way it can possibly contribute to the Einstein equation is through an effective cosmological constant term. Lorentz invariance also implies that vacuum statistical properties must be the same regardless of space position and direction, i.e. homogeneity and isotropy hold in a statistical sense. In order to include these properties into our formalism we want to recover classical and smooth quantities from the fluctuating fields. This can be obtained as a result of an averaging process. To this end we follow the spacetime averaging procedure described in [44, 117] and reviewed in Appendix D, so that fluctuating tensors average to tensors. This process involves a spacetime averaging over regions whose typical dimensions are large in comparison to the fluctuations typical wavelengths but smaller than the scale over which the background geometry changes significantly. Accordingly we introduce an averaging classical scale \( \mathcal{L} \) such that \( \ell \ll \mathcal{L} \ll L_{\text{lab}} \).

While keeping in mind that we are here only considering ‘apparently’ empty spacetime, a final comment about the involved physical scales is in order. The following hierarchy holds, with
The precise characterization of the classical scale \( \mathcal{L} \) is that of the smallest scale at which classical, locally Lorentz invariant spacetime starts to emerge as a result of the averaging process. Though much larger than \( \ell \), the classical scale \( \mathcal{L} \) is still expected to be very small in comparison to the laboratory scale (Table 2.1). The limits of the presently suggested theory are thus clearly set: (i) below the random scale \( \ell \) the random fields approximation breaks down and a full QG theory would be needed; (ii) spacetime starts to be smooth and classical when viewed at the classical scale \( \mathcal{L} \).

| Scale          | \( L_P \) | \( \ell \) | \( \mathcal{L} \) | \( L_{lab} \) |
|----------------|-------|-----|-------|----------|
| Order of magn. (m) | \( \approx 10^{-35} \) | \( \gtrsim 10^{-35} \) | \( \gg 10^{-35} \) | \( \sim 1 \) |
| Physical domain   | Quantum gravity | Random gravity | Classical gravity | Classical gravity |
| Background       | None | \( g^B_{ab} \) | \( g^B_{ab} + \) small corrections | \( g^B_{ab} + \) small corrections |

Table 2.1: A guide to relevant physical scales.

2.3.3 Matter fields stress energy tensor

In order to have a more precise idea of how the random gravity framework can be applied it is useful to have a closer look at the matter fields stress energy tensor \( T_{ab}[\psi] \). This carries contributions from all sectors of the Standard Model. At this stage we adopt a minimal approach by neglecting the non-gravitational interactions between various matter fields components. Then \( T_{ab} = \sum_j T^{(j)}_{ab} \), where the index \( j \) runs over all matter fields and where \( T^{(j)}_{ab} \) represents the stress energy tensor describing the \( j \)-th matter field as if it was free.

The detailed microscopic expression of the generic component \( T^{(j)}_{ab} \) will depend quadratically upon the corresponding matter field, as well as on the random metric \( g_{ab} = g^B_{ab} + \sum_n g^{(n)}_{ab} \). As a result the stress energy tensor at the random scale \( \ell \) is also a stochastic quantity. The dependence on the \( g^{(n)}_{ab} \) would account for the coupling between gravity fluctuations and matter fields. However, as long as we work up to second order, only the smooth background \( g^B_{ab} \) will appear. We will argue in Section 2.6 that the background can be considered Minkowski to a good approximation. In this sense it is like having a collection of free fluctuating fields on a Minkowski background and the effect of gravity upon the stress energy tensor would only appear as a third order effect.

The corresponding energy density contribution can be defined at the classical scale \( \mathcal{L} \) in a statistical sense through a spacetime averaging procedure. Provided the high frequency components are cut off at the random scale \( \ell = \lambda L_P \), the average will be well defined and finite. Then the quantity \( \langle T_{ab} \rangle \) will represent a macroscopic stress-energy tensor at the classical scale \( \mathcal{L} \). An idea of how this can be done is given in Appendix C, where for simplicity we model each of the matter fields independent degrees of freedom by a Klein-Gordon scalar field. Using the generic notation \( \phi \) for such a degree of freedom, the microscopic structure of the stress energy tensor is then given by the familiar expression

\[
T_{ab}^{\text{KG}}[\phi] = \phi_a \phi_b - \frac{1}{2} \eta_{ab} \left( \phi^c \phi_c + \frac{m^2 c^2}{\hbar^2} \phi^2 \right). \tag{2.11}
\]
We show in Appendix A that, under the assumption that $\phi$ is a zero mean, stationary stochastic process, spacetime averages like e.g. $\langle (\partial_\mu \phi)^2 \rangle$ can be expressed equivalently as an integral of the power spectral density $S(\omega)$ with an high energy cutoff set at the random scale $\ell$. Then we have e.g.

$$\langle (\partial_t \phi)^2 \rangle = \frac{1}{(2\pi)^3} \int d^3 k \omega_k^2 S_\phi(k). \quad (2.12)$$

The average of the stress energy tensor is found to be given by

$$\langle T_{ab}^{\text{KG}} \rangle = \langle \phi_a \phi_b \rangle, \quad (2.13)$$

and can be calculated if a power spectrum is assigned. In particular the energy density is given by

$$\rho = \langle T_{00} \rangle = \frac{1}{c^2} \langle (\partial_t \phi)^2 \rangle = \frac{1}{(2\pi)^3} \int d^3 k \frac{\omega_k^2}{c^2} S_\phi(k). \quad (2.14)$$

The key point to describe zero point vacuum energy is that, in the same way as it was done by Boyer, we can link the random field approach to the quantum properties of vacuum by requiring that the energy density (2.14) matches the zero point energy for a scalar quantum field in its ground state, i.e. we impose

$$\rho := \frac{1}{(2\pi)^3} \int d^3 k \frac{\hbar \omega_k}{2}, \quad (2.15)$$

where in the r.h.s. the contributions from all normal modes of $\phi$ are included. As found by Boyer for the EM field, this implies that the power spectrum must have the following simple form:

$$S_\phi(k) = \frac{\hbar c^2}{2\omega_k}. \quad (2.16)$$

Because of homogeneity and isotropy holding at the classical scale and assuming the stochastic properties of the fluctuations in different spacetime directions to be uncorrelated, the averaged matter fields stress energy tensor will take the perfect fluid form

$$\langle T_{ab} \rangle = \begin{pmatrix} \rho & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{pmatrix} \quad (2.17)$$

where $\rho := \langle T_{00} \rangle$ is the energy density and $p := \frac{1}{3} \langle T_{ii} \rangle$, $i = 1, 2, 3$ is the pressure. The dominant energy condition, i.e. $\rho \geq 0$ and $\rho \geq p$, is normally thought to be valid for all known reasonable forms of matter [118], at least as long as the adiabatic speed of sound $dp/d\rho$ is less than the speed of light. This is true for massless fields since, in this case, $\rho = 3p$. By assuming the vacuum spectral density (2.16), we show in Appendix C that the stronger condition $\rho > 3p$ is satisfied by massive fields in their vacuum state, at least in the ideal case in which interactions can be neglected and the field masses are much smaller than the Planck mass.

To assess $\rho$ and $P$ we use the results (C.17) and (C.18) derived in Appendix C and providing an estimate of the zero point energy $\rho_\phi$ and pressure $p_\phi$ for one scalar field $\phi$ of mass $m$:

$$\rho_\phi = \frac{\pi^2 \rho_P}{\lambda^4} + \frac{\rho_P}{4\lambda^2} \left( \frac{m}{M_P} \right)^2 \quad (2.18)$$
2.3. Nonlinear Random Gravity

\[ p_\phi = \frac{\pi^2 \rho_P}{3 \lambda^4} - \frac{\rho_P}{12 \lambda^2} \left( \frac{m}{M_P} \right)^2. \]  

(2.19)

Describing each independent component of the Standard Model matter fields as a simple scalar field and neglecting fields interactions we can obtain an estimate of the zero point energy and pressure by simply summing up the contributions from all individual components. Indicating with \( N_i \) the number of independent components of the particular field having mass \( m_i \) we thus obtain:

\[ \rho = \frac{N \pi^2 \rho_P}{\lambda^4} + \frac{N \rho_P}{4 \lambda^2} \left( \frac{M}{M_P} \right)^2, \]  

(2.20)

where \( N := \sum_i N_i \) is the total number of independent components for all matter fields and where we defined

\[ M^2 := \frac{\sum_i N_i m_i^2}{N}, \]  

(2.21)

i.e. the weighted average of the squared masses of all matter fields. Notice that for indices \( i \) corresponding to massless fields it is \( m_i = 0 \). The pressure is found to be

\[ p = \frac{N \pi^2 \rho_P}{3 \lambda^4} - \frac{N \rho_P}{12 \lambda^2} \left( \frac{M}{M_P} \right)^2. \]  

(2.22)

These expressions are valid when \( M \ll M_P \), which is certainly the case within the standard model since it is expected \( M \approx 10^2 \text{ GeV} \).

To separate the contributions of the massless and massive fields it is useful to perform a trace decomposition in (2.17). We have

\[ \langle T_{ab}[\psi] \rangle = \begin{pmatrix} \rho_\psi^* & 0 & 0 & 0 \\ 0 & \frac{\rho_\psi^*}{3} & 0 & 0 \\ 0 & 0 & \frac{\rho_\psi^*}{3} & 0 \\ 0 & 0 & 0 & \frac{\rho_\psi^*}{3} \end{pmatrix} - \rho_M \eta_{ab} \]  

(2.23)

where

\[ \rho_\psi^* := \frac{3}{4} (\rho + p) = \frac{N \pi^2 \rho_P}{\lambda^4} + \rho_M, \]  

(2.24)

is the energy density of the traceless part and where the trace part is defined in terms of

\[ \rho_M := \frac{1}{4} (\rho - 3p) = \frac{N \rho_P}{8 \lambda^2} \left( \frac{M}{M_P} \right)^2. \]  

(2.25)

We remark that the trace part corresponds to a cosmological constant term, where the cosmological constant would be defined by

\[ \Lambda_M := \frac{8 \pi G}{c^4} \rho_M. \]  

(2.26)

Defining the Planck cosmological constant as

\[ \Lambda_P := \frac{8 \pi G}{c^4} \rho_P \]  

(2.27)
we find
\[ \Lambda_M = \frac{N}{8\lambda^2} \left( \frac{M}{M_P} \right)^2 \Lambda_P. \quad (2.28) \]

Thus we see that only the massive fields are expected to contribute to the cosmological constant through their zero point energy and the trace part of the matter stress energy tensor. On the other hand this also presents a traceless contribution \( \rho^*_\psi \) which is even more important than \( \rho_M \). Indeed
\[ \frac{\rho_M}{\rho^*_\psi} \propto \left( \frac{M}{M_P/\lambda} \right)^2 \ll 1, \quad (2.29) \]

where the inequality holds within the Standard Model, with \( M \approx 10^2 \) GeV, and for any realistic value of the cutoff parameter \( \lambda \).

Summarizing, the effect of vacuum zero point energy is two fold: (1) the averaged stress energy tensor \( \langle T_{ab}[\psi] \rangle \) would contribute to the overall background geometry curvature while (2) the fluctuations defined by \( T_{ab}[\psi] - \langle T_{ab}[\psi] \rangle \) would further induce extra metric fluctuations. We notice that it is this second element only that is accounted for, though via a different technique, in the stochastic gravity approach. We will discuss some issues related the stress energy tensor average and the cosmological constant in Section 2.6.

The next step is to include GWs backreaction and the conformal fluctuations into the analysis. To this end it is natural to perform an expansion of Einstein’s equation, as we start to illustrate in the next section.

### 2.4 Characterizing vacuum at the random scale

In this section we start applying the nonlinear random gravity framework to standard GR while trying to incorporate conformal fluctuations. To this end we want to seek for a solution of Einstein’s equation \( G_{ab}[g] = 8\pi T_{ab}[\psi] \) at the random scale where the vacuum spacetime physical metric \( g_{ab} \) takes the form
\[ g_{ab} = \Omega^2 \gamma_{ab}, \quad (2.30) \]

where \( \gamma_{ab} \) is a conformal metric and \( \Omega \) a conformal factor. As already discussed above this could make sense if we succeed in imposing some structure on the conformal metric \( \gamma_{ab} \) in such a way that it is not depending on \( \Omega \).

A preliminary formal structure can be imposed on the conformal metric by splitting it into a smooth, slow varying background \( \gamma^B_{ab} \) with superimposed fast varying perturbations according to
\[ \gamma_{ab} = \gamma^B_{ab} + \xi_{ab} + \gamma^{(2)}_{ab} + O(\varepsilon^3), \quad (2.31) \]

where \( \varepsilon \ll 1 \) is a small dimensionless parameter. The perturbations and background typical variations scales are supposed to be set respectively by the random scale \( \ell = \lambda L_P \) and by some macroscopic scale \( L \gg \ell \), where \( L \) is the classical scale (see Table 2.1). As to their typical magnitudes we assume:
\[ |\gamma^B_{ab}| = O(1), \quad |\xi_{ab}| = O(\varepsilon), \quad |\gamma^{(2)}_{ab}| = O(\varepsilon^2). \]

The conformal fluctuations are formally inserted by expressing the conformal factor as
\[ \Omega = 1 + A + B + O(\varepsilon^3), \quad (2.32) \]
where

\[ |A| = O(\varepsilon), \quad |B| = O(\varepsilon^2). \]

In particular, \( A \) will be referred to as the conformal field and will be later on characterized as a zero mean and stationary stochastic process, i.e.

\[ \langle A \rangle = 0, \quad \langle \partial_a A \rangle = 0, \quad \langle A^2 \rangle = \text{const.} \]

Imposing Einstein’s equation \( G_{ab}[g] = 8\pi T_{ab}[\psi] \) yields:

\[ G_{ab}[\gamma] = 8\pi T_{ab}[\psi] + \Sigma_{ab}^{1}[\Omega] + \Sigma_{ab}^{2}[\Omega], \quad (2.33) \]

where

\[ \Sigma_{ab}^{1}[\Omega] := 2\nabla_a \nabla_b \ln \Omega - 2\gamma_{ab} \nabla^c \nabla_c \ln \Omega, \quad (2.34) \]

\[ \Sigma_{ab}^{2}[\Omega] := -(2\nabla_a \ln \Omega \nabla_b \ln \Omega + \gamma_{ab} \nabla^c \ln \Omega \nabla_c \ln \Omega). \quad (2.35) \]

As explained in Section 2.1 this can be viewed as an equation constraining the conformal factor if the conformal metric can somehow be specified. In our attempt to implement this concretely we will tackle equation (2.33) perturbatively, basically extending Isaacson’s framework for small perturbations over a curved background [43, 44] to allow for the presence of conformal fluctuations.

If we succeed, the intuitive picture of vacuum spacetime that would emerge is that of a smooth manifold presenting, at the random scale \( \ell \), metric perturbations due to random stretchings of the scale and GWs. In addition there will be higher order metric perturbations due to all sorts of backreaction effects. A key point is that such a scenario should be autoconsistent: i.e. we are supposing that the background geometry is precisely set up by the contributions due to all vacuum components. We remark thus how, in this sense, Isaacson’s framework provides an ideal basis for our model. Since all these perturbations are small, the physical metric is expected to be also expandable as

\[ g_{ab} = g_{ab}^B + h_{ab} + g_{ab}^{(2)} + O(\varepsilon^3), \quad (2.36) \]

with background, linear and higher order perturbations orders of magnitude characterized by

\[ |g_{ab}^B| = O(1), \quad |h_{ab}| = O(\varepsilon), \quad |g_{ab}^{(2)}| = O(\varepsilon^2). \]

Before embarking into the expansion scheme we now establish some algebraic constraints relating the physical and conformal metric perturbations that will be useful later on. By putting together equations (2.30), (2.31) and (2.32), retaining terms up to second order and comparing with (2.36) we obtain:

\[ g_{ab}^B = \gamma_{ab}^B, \]

\[ h_{ab} = \xi_{ab} + 2A g_{ab}^B, \quad (2.37) \]

\[ g_{ab}^{(2)} = \gamma_{ab}^{(2)} + 2A \xi_{ab} + 2B g_{ab}^B + A^2 g_{ab}^B. \]

In particular the relation \( g_{ab}^B = \gamma_{ab}^B \) shows that we just need introduce a single background geometry. In the following we will use the notation \( g_{ab}^B \)
2.5 Setting up the expansion scheme

We assume that all first order perturbations $h_{ab}, \xi_{ab}$ and $A$ are fast varying with typical variation scale $\ell = \lambda L_P$ set by the random scale. Moreover the smooth slow varying background $g^B$ is characterized by the typical variation scale $L \gg \ell$. By the observed local properties of vacuum spacetime it is in fact fair to assume that $L \gtrsim L_{\text{lab}}$. In this sense, using the laboratory macroscopic scale as a reference, we will consider from now on $L = O(1)$.

The matter fields in their vacuum state are indicated collectively by $\psi$, also assumed to be a first order quantity varying on the typical scale $\ell$, i.e. $|\psi| = O(\varepsilon)$. The stress energy tensor $T_{ab}$ is has a quadratic structure $T \sim \partial \psi \partial \psi$, which yields the order of magnitude estimate:

$$T[\psi] = O(\varepsilon^2/\ell^2). \quad (2.38)$$

As it shown in Appendix D, the effective backreaction energy due to GWs will also be of order $(\varepsilon/\ell)^2$. On the other hand the typical background curvature will be of order $(\partial g^B)^2 \sim 1/L^2$. As it will be shown shortly, even the $A$ dependent terms formally contribute to the backreaction energy with terms of order $(\varepsilon/\ell)^2$. Then the autoconsistent framework, in which by definition all of the background curvature is produced by vacuum components, is characterized by the condition:

$$\varepsilon \sim \frac{\ell}{L}.$$ 

In particular, using $L = O(1)$, this gives the estimate

$$T_{ab}[\psi] = O(1). \quad (2.39)$$

2.5.1 Einstein tensor expansion

In expanding equation (2.33) we start by the Einstein tensor term $G_{ab}[\gamma]$. More details of how this is done are given in Appendix D. Expanding the Ricci tensor $R_{ab}[\gamma]$ gives:

$$R_{ab}[g^B + \xi + \gamma^{(2)}] = R_{ab}^{(0)}[g^B] + R_{ab}^{(1)}[\xi] + R_{ab}^{(1)}[\gamma^{(2)}] + R_{ab}^{(2)}[\xi] + R_{ab}^{(3+)}; \quad (2.40)$$

where $R_{ab}^{(0)}[g^B]$ is the Ricci tensor of the background metric and where the linear operator $R_{ab}^{(1)}[\cdot]$ and nonlinear operator $R_{ab}^{(2)}[\cdot]$ are given explicitly by equations (D.42) and (D.43). The term $R_{ab}^{(3+)}$ represents higher order contributions. As done in Appendix D, by using the information $O(\varepsilon) = O(\ell)$, we have the order of magnitude estimates:

$$R_{ab}^{(0)}[g^B] \sim g^{-1}B \partial^2 g^B \sim 1/L^2 = O(1), \quad (2.41)$$

$$R_{ab}^{(1)}[\xi] \sim g^{-1} \partial^2 \xi \sim \varepsilon/\ell^2 = O(1/\varepsilon), \quad (2.42)$$

$$R_{ab}^{(1)}[\gamma^{(2)}] \sim g^{-1} \partial^2 \gamma^{(2)} \sim \varepsilon^2/\ell^2 = O(1), \quad (2.43)$$

$$R_{ab}^{(2)}[\xi] \sim \xi g^{-2} \partial^2 \xi \sim \varepsilon^2/\ell^2 = O(1), \quad (2.44)$$

together with $R_{ab}^{(3+)} = O(\varepsilon)$. 

The expansion of the Einstein tensor follows as
\[
G_{ab}[\gamma] = G^{(0)}_{ab}[g^B] + G^{(1)}_{ab}[\xi] + G^{(2)}_{ab}[\gamma^{(2)}] + G^{(2)}_{ab}[\xi] + G^{(2)}_{ab}[\xi] + G^{(3)}_{ab}[\xi],
\] (2.45)
where
\[
G_{ab}^{(0)}[g^B] := R_{ab}^{(0)}[g^B] - \frac{1}{2} g^B_{ab} R^{(0)}[g^B],
\] (2.46)
is the background metric Einstein tensor. Moreover the linear tensor \(G_{ab}^{(1)}[\cdot]\) is defined as
\[
G_{ab}^{(1)}[\cdot] := R_{ab}^{(1)}[\cdot] - \frac{1}{2} g^B_{ab} R^{(1)}[\cdot],
\] (2.47)
and the quadratic tensors \(G_{ab}^{(2)}[\cdot]\) and \(G_{ab}^{(\xi\xi)}[\xi]\) by
\[
G_{ab}^{(2)}[\cdot] := R_{ab}^{(2)}[\cdot] - \frac{1}{2} g^B_{ab} R^{(2)}[\cdot],
\] (2.48)
\[
G_{ab}^{(\xi\xi)}[\xi] := \frac{1}{2} g^B_{ab} \xi^{cd} R^{(1)}_{cd}[\xi] - \frac{1}{2} \xi_{ab} R^{(1)}[\xi].
\] (2.49)

### 2.5.2 Terms dependent on the conformal factor

To implement the expansion scheme we must also expand the terms that depend on the conformal factor. These are given in (2.4) and (2.5) as
\[
\Sigma_{ab}^1[\Omega] = 2 \nabla_a \nabla_b \ln \Omega - 2 \gamma_{ab} \nabla^c \nabla_c \ln \Omega,
\] (2.50)
\[
\Sigma_{ab}^2[\Omega] = - (2 \nabla_a \ln \Omega \nabla_b \ln \Omega + \gamma_{ab} \nabla^c \ln \Omega \nabla_c \ln \Omega).
\] (2.51)
The covariant derivative of the conformal metric appears. When the conformal metric is expressed as in (2.31) its Christoffel symbol can be expanded up to linear order in the perturbation as [119]:
\[
\Gamma^a_{bc} = \frac{1}{2} \gamma^{ad} (\partial_c \gamma_{db} + \partial_b \gamma_{dc} - \partial_d \gamma_{bc})
\] (2.52)
\[
= \frac{1}{2} \left( g^B_{ad} - \xi^{ad} \right) \left( \partial_c g^B_{db} + \partial_b g^B_{dc} + \partial_d \xi_{bc} - \partial_d g^B_{bc} - \partial_d \xi_{bc} \right) + O(\varepsilon^2)
\] (2.52)
\[
= \Gamma^B_{bc} + \delta \Gamma^a_{bc} [\xi] + O(\varepsilon^2),
\] (2.52)
where \(\Gamma^B_{bc}\) is the Christoffel symbol of the background metric and the first order correction \(\delta \Gamma^a_{bc} [\xi]\) is given by
\[
\delta \Gamma^a_{bc} := \frac{1}{2} g^B_{ad} \left( \nabla_a \xi_{db} + \nabla_b \xi_{dc} - \nabla^B_{de} \xi_{bc} \right),
\] (2.53)
\(\nabla^B_a\) being the covariant derivative operator associated to the background metric. If \(L = O(1)\) is the typical variation scale of the background geometry we have the orders of magnitudes:
\[
\Gamma^B_{bc} = O(1), \quad \delta \Gamma^a_{bc} [\xi] = O(\varepsilon/\ell).
\] (2.52)
The covariant derivatives in (2.50) and (2.51) act upon \(\ln \Omega\), which must also be expanded up to
second order. Using $\Omega = 1 + A + B$ gives

$$\ln \Omega \equiv \ln(1 + A + B) = A + B - \frac{A^2}{2} + O(\varepsilon^3).$$

(2.54)

Since we are dealing with scalar functions we have e.g.

$$\nabla_a \ln \Omega = \partial_a A + \partial_a B - \frac{1}{2} \partial_a A^2 + O(\varepsilon^3/\ell),$$

(2.55)

where $\ell$ is the typical variation scale of the fluctuations.

The term $\Sigma_{ab}^{(2)}[\Omega]$ being already quadratic in $\ln \Omega$ gives a single contribution quadratic in $A$ which is found immediately to be

$$\Sigma_{ab}^{(2)}[A] := -(2\partial_a \partial_b A + g^{B}_{ab}g^{cd}_{B} \partial_c \partial_d A) = O(\varepsilon^2/\ell^2).$$

(2.56)

Here the suffix $(2)$ stands for ‘the second order terms of’. This notation will be used from now on, e.g. the suffix $(n)$ will indicate ‘the $n$-th order terms of’ for some given quantity whose expansion we are considering.

This is useful for the term $\Sigma_{ab}^{(1)}[\Omega]$ since it will yield both a linear and a quadratic contribution, which we will denote by $\Sigma_{ab}^{(1)}[\Omega]$ and $\Sigma_{ab}^{(2)}[\Omega]$. To find these we have to consider

$$\nabla_a \nabla_b \ln \Omega \equiv \nabla_a [\partial_b A + \partial_a B - \frac{1}{2} \partial_b A^2 + O(\varepsilon^3/\ell)].$$

This gives:

$$\nabla_a \nabla_b \ln \Omega = \nabla_a \partial_b A + \nabla_a \partial_b B - \frac{1}{2} \nabla_a \partial_b A^2 + O(\varepsilon^2/\ell^2)$$

$$= (\partial_a \partial_b A - \Gamma_{ab}^{c} \partial_c A) + (\partial_a \partial_b B - \Gamma_{ab}^{c} \partial_c B) - \frac{1}{2} (\partial_a \partial_b A^2 - \Gamma_{ab}^{c} \partial_c A^2) + O(\varepsilon^3/\ell^2).$$

Using equation (2.52) we have

$$\nabla_a \nabla_b \ln \Omega = \left[ \partial_a \partial_b A - (\Gamma^{Bc}_{ab} + \delta \Gamma^{c}_{ab} [\xi]) \partial_c A \right] + \left[ \partial_a \partial_b B - (\Gamma^{Bc}_{ab} + \delta \Gamma^{c}_{ab} [\xi]) \partial_c B \right]$$

$$- \frac{1}{2} \left[ \partial_a \partial_b A^2 - (\Gamma^{Bc}_{ab} + \delta \Gamma^{c}_{ab} [\xi]) \partial_c A^2 \right] + O(\varepsilon^3/\ell^2).$$

(2.57)

The typical order of these terms are:

$$\partial \partial A = O(\varepsilon/\ell^2), \quad \Gamma ^B \partial A = O(\varepsilon/\ell), \quad \delta \Gamma \partial A = O(\varepsilon^2/\ell^2),$$

(2.58)

$$\partial \partial B = O(\varepsilon^2/\ell^2), \quad \Gamma ^B \partial B = O(\varepsilon^2/\ell), \quad \delta \Gamma \partial B = O(\varepsilon^3/\ell^2),$$

(2.59)

$$\partial \partial A^2 = O(\varepsilon^2/\ell^2), \quad \Gamma ^B \partial A^2 = O(\varepsilon^2/\ell), \quad \delta \Gamma \partial A^2 = O(\varepsilon^3/\ell^2).$$

(2.60)

Using (2.57), $\gamma^{cd} = g^{Bcd} - \xi^{cd} + O(\varepsilon^3)$, and collecting equal order terms according to the above estimates, the term (2.50) is easily found to give the linear contribution

$$\Sigma_{ab}^{(1)}[A] := 2 \partial_a \partial_b A - 2 g^{B}_{ab}g^{cd}_{B} \partial_c \partial_d A = O(1/\varepsilon)$$

(2.61)
and the nonlinear contribution

\[
\Sigma^{(2)}_{ab}[A, B] := \Sigma^{(1)}_{ab}[B] - \left( \partial_a \partial_b A^2 - g_{ab}^B g^{Bcd} \partial_c \partial_d A^2 \right) + S_{ab}[A, \xi] = O(1), \tag{2.62}
\]

where \( S_{ab}[A, \xi] \) is defined by

\[
S_{ab}[A, \xi] := -2\Gamma^c_{ab}[\partial_c A + 2g_{ab}g^{Bcd} \Gamma^e_{cd} \partial_e A \\
-2\delta \Gamma_{ab}[\xi] \partial_c A + 2g_{ab}g^{Bcd} \delta \Gamma^e_{cd}[\xi] \partial_e A \\
+ 2g_{ab} \xi^{cd} \partial_c \partial_d A - 2\xi_{ab}g^{Bcd} \partial_c \partial_d A = O(1). \tag{2.63}
\]

### 2.5.3 The expansion vacuum equations on a curved background

We have found that the random scale Einstein’s equation for the conformal metric up to second order in the perturbations has the structure:

\[
\underbrace{G^{(0)}_{ab}[g^B]}_{O(1)} + \underbrace{G^{(1)}_{ab}[\xi]}_{O(1/\varepsilon)} + \underbrace{G^{(1)}_{ab}[\gamma^{(2)}]}_{O(1)} + \underbrace{G^{(2)}_{ab}[\xi]}_{O(1)} + \underbrace{G^{(1)}_{ab}[\xi]}_{O(1)} = \underbrace{8\pi T^{(2)}_{ab}[\psi]}_{O(1)} + \underbrace{\Sigma^{(1)}_{ab}[A]}_{O(1)} + \underbrace{\Sigma^{(2)}_{ab}[A, B]}_{O(1)} + \underbrace{\Sigma^{(2)}_{ab}[A]}_{O(1)},
\]

where we neglected terms of order \( O(\varepsilon) \).

We can finally write down the expansion equations. To highest order we can collect all terms of order \( O(1/\varepsilon) \) -equations (2.45), (2.61)- to get

\[
G^{(1)}_{ab}[\xi] = \Sigma^{(1)}_{ab}[A],
\]

governing the propagation of the linear first order perturbation \( \xi_{ab} \) upon the curved background \( g_{ab}^B \).

Note that this enters the equation in the r.h.s. \textit{and} through the linear operator \( G^{(1)}_{ab} \) defined in (2.47).

Similarly, we can collect all terms of order \( O(1) \) -equations (2.38), (2.45), (2.56), (2.62)- to get the nonlinear backreaction equation for the background geometry as:

\[
G^{(0)}_{ab}[g^B] + G^{(1)}_{ab}[\gamma^{(2)}] = \underbrace{8\pi T^{(2)}_{ab}[\psi]}_{O(1)} - \underbrace{G^{(2)}_{ab}[\xi]}_{O(1)} - \underbrace{G^{(1)}_{ab}[\xi]}_{O(1)} - \left( 2\partial_a A \partial_b A + g_{ab} g^{Bcd} \partial_c A \partial_d A \right) \\
+ \underbrace{\Sigma^{(1)}_{ab}[B]}_{O(1)} - \left( \partial_a \partial_b A^2 - g_{ab} g^{Bcd} \partial_c \partial_d A^2 \right) + \underbrace{S_{ab}[A, \xi]}_{O(1)}, \tag{2.66}
\]

where \( S_{ab}[A, \xi] \) is given in (2.63).

These equations must be solved simultaneously and generalize Isaacson’s equations to the case where conformal fluctuations described by \( \Omega = 1 + A + B \) are included.

### 2.6 Considerations on the cosmological constant problem

The above equations are very general and are based upon an arbitrary smooth background geometry. By taking the average in (2.66) we obtain the equation for the background as:

\[
G^{(0)}_{ab}[g^B] = \left( 8\pi T^{(2)}_{ab}[\psi] - G^{(2)}_{ab}[\xi] - G^{(1)}_{ab}[\xi] - \left( 2\partial_a A \partial_b A + g_{ab} g^{Bcd} \partial_c A \partial_d A \right) + S_{ab}[A, \xi] \right), \tag{2.67}
\]
where we have used $\langle G^{(0)} \rangle = G^{(0)}$, $\langle \gamma^{(2)} \rangle = 0$, $\langle A^2 \rangle = \text{const}$, $\langle B \rangle = 0$ and the fact that the averaging procedure commutes with linear differential operators such as $G^{(1)}_{ab}$ [119]. This equation establishes the overall, large scale effect of zero point energy on the background spacetime geometry. It is a well known fact that observed vacuum, which we can alternatively call apparently empty space, is locally Lorentz invariant. This implies that it must be:

$$\frac{8\pi T_{ab}^{(2)}[\psi] - C_{ab}^{(2)}[\xi] - C_{ab}^{(\xi\xi)}[\xi]}{c_a A \partial_b A + g_{ab} g^{cd} \partial_c A \partial_d A} + S_{ab}[A, \xi] \equiv -\Lambda_{RG} g_{ab}^B,$$

where $\Lambda_{RG}$ denotes an effective cosmological constant induced at the classical scale by the combined effect of matter fields, GWs and, possibly, conformal fluctuations. The corresponding energy density due to vacuum behavior at the random scale would be

$$\rho_{\text{vacuum}}^{RG} := \frac{c^4 \Lambda_{RG}}{8\pi G},$$

where for clarity we have re-inserted the constants $G$ and $c$.

Now, if we denote by $\rho_{\text{vacuum}}^{QFT}$ the hypothetical true theoretical amount of vacuum energy due to matter fields and QG effects, we reasonably expect $\rho_{\text{vacuum}}^{RG} \lesssim \rho_{\text{vacuum}}^{QFT}$. This is because the random gravity framework provides a mean to model zero point energies down to the random scale $\ell$ but is not expected to model vacuum phenomena that happen closer to the Planck scale or other contributions to vacuum energy such as those due to phase transitions in the electro-weak sector or non linear effects typical of QCD. It is well known that theoretical estimates of $\rho_{\text{vacuum}}^{QFT}$ made within QFT yield a result which is of many order of magnitudes larger than what we expect from observations. This discrepancy between theoretical expectations and observations represents one of the main aspects of the so called cosmological constant problem. In his brilliant review Carroll [45] points out how, taking into account all possible contributions to vacuum energy (scalar fields, ZPE, QCD and EW effects), one finds in fact the planckian value $\rho_{\text{vacuum}}^{QFT} \approx \rho_P$, where $\rho_P \approx 10^{19}$ GeV/$L_P^3$ is the Planck energy density. On the other hand the current estimated amount of observed vacuum energy in the present epoch universe is [45]

$$\rho_{\text{obs, vacuum}} \approx 5 \times 10^{-11} \text{ J m}^{-3} \approx 10^{-125} \rho_P.$$

(2.68)

This value is deduced from the observed large scale properties of the universe, including the approximatively flat geometry, as suggested by CMB anisotropies [120], and the present, small, cosmic acceleration indicated by type Ia supernovae observations [121]. Defining the Planck cosmological constant as

$$\Lambda_P := \frac{8\pi G}{c^4} \rho_P,$$

(2.69)

then (2.68) implies the following observational constraint on the present epoch cosmological constant

$$0 \leq \Lambda_{\text{obs}} \lesssim 10^{-125} \Lambda_P.$$

(2.70)

The upper bound is extremely small, especially in comparison to the QFT theoretical expectations usually reported in the literature. Because of this dramatic discrepancy it is commonly believed that either vacuum energy makes a case on its own in that it does not gravitate or, more likely, that there must exist some sort of balancing mechanism. Such a mechanism is to date unknown. Some possibilities include supersymmetry, extra scalar fields, dark energy models such as quintessence [45] or new symmetry principles such as invariance of the gravitational lagrangian under shift by a constant [122].
2.6.1 Choice of the background geometry

It had been one of the original goals of the present work, as discussed in [46], to verify whether conformal fluctuations could offer a mean to obtain such a balancing mechanism. That this cannot be the case will be shown in the forthcoming sections. For the moment we also recall how the effective averaged stress energy tensor due to matter fields alone in their vacuum state is predicted by the random gravity framework to be

\[
\langle T_{ab}[\psi]\rangle = \begin{pmatrix}
\rho_{\psi}^* & 0 & 0 & 0 \\
0 & \frac{\rho_{\psi}^*}{3} & 0 & 0 \\
0 & 0 & \frac{\rho_{\psi}^*}{3} & 0 \\
0 & 0 & 0 & \frac{\rho_{\psi}^*}{3}
\end{pmatrix} - \rho_M \eta_{ab}. \tag{2.71}
\]

The contribution to the effective cosmological constant coming from the standard model matter fields corresponds to the energy density

\[
\rho_M = \frac{N \rho_p}{8 \pi^2} \left(\frac{M}{M_P}\right)^2,
\]

as given in equation (2.25). In comparison to the usual asserted planckian value, this is reduced by a factor \((M/M_P)^2 \approx 10^{-34}\). The suggested link between vacuum related cosmological constant and mass is interesting but we will not further investigate this issue in the present work. Rather we notice another interesting point, somehow rarely mentioned in the literature, but remarked e.g. by Padmanabhan in [122]. As equation (2.71) shows explicitly, only matter fields whose stress energy tensor has a non vanishing trace are expected to contribute to the cosmological constant. On the other hand massless fields such as the EM field and in general all other forms of matter with a *traceless* component in their stress energy tensor should contribute to vacuum through a term which, beyond attaining a large planckian value, is *not* locally Lorentz invariant. Since such a contribution is obviously not observed the vacuum energy problem appears to be enriched: beyond a mechanism for canceling the contribution to the cosmological constant one appears to need some mechanism to also balance this traceless contribution.

These problems are all very interesting. However providing a deep analysis goes beyond the scope of the present thesis. The approach we will maintain at this point is a pragmatic one. Since what we are really interested in here is whether we can find a solution of Einstein’s equation which presents conformal fluctuations at the random scale, we will simply assume that the theoretical framework can in principle be made general enough to conform to the observational bounds. Thus we assume:

\[
\rho_{\text{vacuum}}^{\text{RG}} \lesssim \rho_{\text{obs}}^{\text{vacuum}} \approx 5 \times 10^{-11} \text{ J m}^{-3}.
\]

This corresponds to the assumption that the hypothetical missing ingredient that can (and hopefully will) provide the energy balance mechanism is actually thought to be included within the collection of matter fields \(\psi\). An interesting alternative which we will investigate later on is whether the conformal fluctuations \(A\) can actually represent the missing ingredient.

To conclude this discussion, we have quite a strong bound on the magnitude of the averaged stress energy tensor that we expect to obtain from the random scale physics. The limiting value is so close to zero that many authors believe it is natural that the actual bulk contribution of vacuum to the large scale energy density is actually *exactly* zero. In this respect, the other and non trivial aspect of the cosmological constant problem is to find a mechanism that yields an almost, but not quite vanishing, value of the cosmological constant. The problem is all the more relevant since in terms of the critical cosmological energy density, the cosmological constant contribution is roughly 70%. Leaving this aspect of the problem aside our point of view here will be the simplest: i.e. the missing theoretical ingredient to the vacuum energy can provide an *exact* balance at every macroscopic scale. In any case
we can reasonably expect that, at the laboratory scale $L_{\text{lab}}$, we should have

$$\left\langle 8\pi T_{ab}^{(2)}[\psi] - G_{ab}^{(2)}[\xi] - G_{ab}^{(\xi\xi)}[\xi] \right. - \left. \left( 2\partial_a A \partial_b A + g_{ab}^B g^{Bcd} \partial_c A \partial_d A \right) + S_{ab}[A, \xi] \right) = 0. \tag{2.72}$$

As a result, a natural choice for the background geometry that neglects secular evolution on long time scales and appears to be a good approximation in line with observations is a flat Minkowski background geometry, i.e. we put

$$g_{ab}^B := \eta_{ab}. \tag{2.73}$$

2.6.2 Simplified form of the expansion equations

With this choice of the background geometry, the perturbation equations (2.65) and (2.66) simplify a lot. Indeed we now have

$$\Gamma^{B \alpha}_{\beta \gamma} = 0, \quad g^{B \alpha \beta} = \eta^{\alpha \beta}, \quad \nabla^B_a = \partial_a, \quad G_{ab}^{(0)}[\eta] = 0, \tag{2.74}$$

and we can use $\eta_{ab}$ to rise and lower indices without ambiguity. Equations (2.53) and (2.63) now read respectively:

$$\delta \Gamma^{a}_{\beta \gamma}[\xi] = \frac{1}{2} \eta^{ad} (\partial_c \xi_{db} + \partial_b \xi_{dc} - \partial_d \xi_{bc}), \tag{2.75}$$

and

$$S_{ab}[A, \xi] = -2\delta \Gamma^{c}_{ab}[\xi] \partial_c A + 2\eta_{ab} \eta^{cd} \delta \Gamma^{e}_{cd}[\xi] \partial_e A + 2\eta_{ab} \xi^{cd} \partial_c \partial_d A - 2\xi_{ab} \partial^f \partial_f A. \tag{2.76}$$

We can now re-write the first and second order equations as

$$G_{ab}^{(1)}[\xi] = 2\partial_a \partial_b A - 2\eta_{ab} \partial^f \partial_f A, \tag{2.77}$$

and

$$G_{ab}^{(1)}[\gamma^{(2)}] = 8\pi T_{ab}^{(2)}[\psi] - G_{ab}^{(2)}[\xi] - G_{ab}^{(\xi\xi)}[\xi] \right. - \left. \left( 2\partial_a A \partial_b A + \eta_{ab} \partial^f \partial_f A \right) + (2\partial_a \partial_b \eta - 2\eta_{ab} \partial^f \partial_f \eta) + S_{ab}[A, \xi]. \tag{2.78}$$

These are to be considered together with (2.75) and (2.76) and are compatible with the condition (2.72). Notice that $T_{ab}^{(2)}[\psi]$ is quadratic in $\psi$ and contains otherwise just the Minkowski tensor $\eta_{ab}$. The same is true for the expressions (2.47), (2.48), (2.49) defining $G_{ab}^{(1)}$, $G_{ab}^{(2)}$ and $G_{ab}^{(\xi\xi)}$, which now all contain $\eta_{ab}$. The algebraic constraints (2.37) now read

$$h_{ab} = \xi_{ab} + 2A \eta_{ab}, \tag{2.79}$$

$$g_{ab}^{(2)} = \gamma_{ab}^{(2)} + 2A \xi_{ab} + 2B \eta_{ab} + A^2 \eta_{ab}. \tag{2.80}$$

Equations (2.77) and (2.78) are the starting point of the next chapter, where we can finally provide the answer to the question of whether conformal fluctuations can play any role within standard GR.
Chapter 3

GR and conformal fluctuations

In this chapter we study the perturbation equations derived in the previous chapter following the application of the random gravity framework to standard GR. An attempt is made to build explicit solutions in which the physical metric presents true conformal fluctuations. A secondary but equally important goal is to assess whether conformal fluctuations can offer a balancing mechanism that may cancel part of the large vacuum energy related to matter fields and GWs, as it was recently claimed in [46], and previously in [36]. We show that a technical imprecision affected the conclusion in [46]: in particular, the nonlinear backreaction effect due to the conformal fluctuations was not treated properly. When this is fixed for, the correct second order equations show that no vacuum energy balance in fact occurs. What is more, we show quite in general that conformal fluctuations cannot have any physical effect within the standard GR framework: the attempt to build a vacuum solution where the physical metric has a true conformal modulation leads to unphysical conditions for the conformal field. If GR is the correct theory of gravitation down to at least the random scale, the main conclusion is that dephasing of a quantum particle due to gravity related conformal fluctuations is not expected to occur.

3.1 Analysis of the linear equation and GWs

We now proceed to study the perturbation equations derived in the previous chapter. We start the analysis from the linear equation (2.77)

\[ G_{ab}^{(1)}[\xi] = 2 \partial_a \partial_b A - 2 \eta_{ab} \partial_c \partial_c A, \]

(3.1)

which will allow us to introduce GWs. The linear operator \( G_{ab}^{(1)} \) is given by

\[ G_{ab}^{(1)}[\xi] = R_{ab}^{(1)}[\xi] - \frac{1}{2} \eta_{ab} R^{(1)}[\xi], \]

(3.2)

where the linear part of the Ricci tensor is

\[ R_{ab}^{(1)}[\xi] = \frac{1}{2} \partial_c \partial_b \xi_{ac} + \frac{1}{2} \partial_c \partial_a \xi_{bc} - \frac{1}{2} \partial_c \partial_c \xi_{ab} - \frac{1}{2} \partial_a \partial_b \xi. \]

(3.3)

Equation (3.1) can be re-written in a very simple form. By using equations (3.2) and (3.3) it is
3.1. Analysis of the Linear Equation and GWs

It is straightforward to verify that

\[ G_{ab}^{(1)}[-2A\eta] = 2\partial_a\partial_b A - 2\eta_{ab}\partial^c\partial_c A. \]  

(3.4)

From (2.79) we know that \( \xi_{ab} + 2A\eta_{ab} = h_{ab} \), i.e. the first order perturbation of the physical metric \( g_{ab} \). Since \( G_{ab}^{(1)} \) is a linear operator, equation (3.1) can be written in an equivalent way as

\[ G_{ab}^{(1)}[\xi + 2A\eta] \equiv G_{ab}^{(1)}[h] = 0. \]  

(3.5)

This is precisely the linear equation which would result if the expansion scheme had been applied directly to Einstein’s equation expressed in terms of \( g_{ab} \). At first order we thus have two equivalent equations:

\[ G_{ab}^{(1)}[h] = 0 \iff G_{ab}^{(1)}[\xi] = 2\partial_a\partial_b A - 2\eta_{ab}\partial^c\partial_c A, \]  

(3.6)

and one algebraic constraint

\[ h_{ab} = \xi_{ab} + 2A\eta_{ab}, \]  

(3.7)

linking the physical metric and conformal metric linear perturbations \( h_{ab} \) and \( \xi_{ab} \).

The key point now is to decide how to represent the GWs. We introduce the notation \( h_{ab}^{GW} \) to indicate GWs, i.e. a metric perturbation satisfying the linear equation \( G_{ab}^{(1)}[h^{GW}] = 0 \). The above equations seem to suggest the obvious choice \( h_{ab}^{GW} := h_{ab} \). This choice was also made in [46] and we will explore its consequences below. We point out however that one could also, at least formally, set \( h_{ab}^{GW} := \xi_{ab} \) by imposing \( G_{ab}^{(1)}[\xi] = 0 \). This would provide an example of what we mean by forcing a structure on the conformal metric. It would imply the constraint \( \partial_a\partial_b A - \eta_{ab}\partial^c\partial_c A = 0 \) on the conformal field. This choice would lead to an alternative formalism in which the conformal 4-metric encodes GWs and it will be explored further in 3.5.3.

We now proceed in studying the scenario in which the physical metric linear perturbation \( h_{ab} \) describes GWs. In practise this is achieved by choosing a solution for the homogeneous linear equation \( G_{ab}^{(1)}[h] = 0 \), to which we apply fluctuating boundary conditions as explained by Boyer in the case of the zero point classical fluctuations of the EM field [41]. It is also convenient to put the perturbation in the TT gauge, as this will slightly simplify some formulas later. Then vacuum fluctuations due to GWs are represented by:

\[ h_{ab}^{GW} := h_{ab}. \]  

(3.8)

From now on the notation \( h_{ab}^{GW} \) indicates a concrete fluctuating, TT gauge, solution of the homogeneous linearized Einstein’s equation. Concretely and coherently with the properties of vacuum as described in the previous chapter, \( h_{ab}^{GW} \) would have to be given by a superposition of random phase waves traveling in all space directions and having suitable statistical properties that should be yielding a homogeneous, isotropic and Lorentz invariant character to all averaged quantities we may derive from \( h_{ab}^{GW} \). In the following this is understood but we won’t need to make these properties formally explicit.

The conformal metric linear perturbation now follows from the general constraint (3.7) as

\[ \xi_{ab} = h_{ab}^{GW} - 2A\eta_{ab}, \]  

(3.9)

in such a way that the equation

\[ G_{ab}^{(1)}[\xi] = -G_{ab}^{(1)}[2A\eta_{ab}] \equiv 2\partial_a\partial_b A - 2\eta_{ab}\partial^c\partial_c A \]  

(3.10)
3.2. Analysis of the second order equation

The second order equation is central to the random gravity framework as it describes how matter fields \( \psi \), GWs given by \( h_{ab}^{GW} \) and conformal fluctuations \( \tilde{A} \) build up the background spacetime curvature through their energy content. As discussed in the previous chapter, it is our assumption that if all the ingredients entering the matter fields symbol \( \psi \) were known and explicitly included, then it would be possible to ‘see’ concretely how the overall backreaction energy at the classical scale due to zero point energy is basically vanishing, in agreement with observations. Thanks to this assumption it was possible to select a flat Minkowski background. Our two main objectives at this point are:

1. verifying whether it is possible to find a vacuum solution where the physical metric \( g_{ab} \) depends nonlinearly on the conformal fluctuations, e.g. through a term like \( \tilde{A}^2 \eta_{ab} \) which, according to the analysis in chapter 1 would induce dephasing;

2. in relation to the issue of zero point vacuum energy balance, verifying whether the conformal fluctuations \( \tilde{A} \) themselves can actually provide the (or one) ingredient for a total (or partial) balancing mechanism.
3.2. Analysis of the second order equation

The second order equation for the nonlinear conformal 4-metric perturbation $\gamma_{ab}^{(2)}$ with a source term depending on $\psi$, $\tilde{A}$ and $h_{ab}^{GW}$ is given in (2.78) as:

$$G_{ab}^{(1)}[\gamma^{(2)}] = 8\pi T_{ab}^{(2)}[\psi] - G_{ab}^{(2)}[\xi] - G_{ab}^{(\xi\xi)}[\xi] - \left(2\partial_a \tilde{A} \partial_b \tilde{A} + \eta_{ab} \partial^c \tilde{A} \partial_c \tilde{A}\right)$$
$$+ (2\partial_a \partial_b B - 2\eta_{ab} \partial^c \partial_c B) - \left(\partial_a \partial_b \tilde{A}^2 - \eta_{ab} \partial^c \partial_c \tilde{A} \right) + S_{ab}[\tilde{A}, \xi], \tag{3.17}$$

where $S_{ab}[A, \xi]$ is defined in (2.76). The quadratic part of the Einstein tensor is defined by

$$G_{ab}^{(2)}[\cdot] := R_{ab}^{(2)}[\cdot] - \frac{1}{2} \eta_{ab} R^{(2)}[\cdot], \tag{3.18}$$

with

$$R_{ab}^{(2)}[\xi] := \frac{1}{2} \xi^{cd} \partial_a \partial_b \xi_{cd} - \xi^{cd} \partial_c \partial_d (\partial_a \xi_{b}) + \frac{1}{4} \left(\partial_a \xi_{cd}\right) \partial_b \xi_{cd} + (\partial^d \xi^c) \partial_d \xi_{ca} + \frac{1}{2} \partial_d (\xi^d \partial_c \xi_{ab})$$
$$- \frac{1}{4} (\partial^c \xi) \partial_c \xi_{ab} - (\partial_d \xi^{cd} - \frac{1}{2} \partial^c \xi) \partial_d (\partial_a \xi_{b}). \tag{3.19}$$

Substituting the first order solution (3.12) into (3.17) we have

$$G_{ab}^{(1)}[\gamma^{(2)}] = 8\pi T_{ab}^{(2)}[\psi] - G_{ab}^{(2)}[h^{GW} - 2\tilde{A}\eta] - G_{ab}^{(\xi\xi)}[h^{GW} - 2\tilde{A}\eta] - \left(2\partial_a \tilde{A} \partial_b \tilde{A} + \eta_{ab} \partial^c \tilde{A} \partial_c \tilde{A}\right)$$
$$+ (2\partial_a \partial_b B - 2\eta_{ab} \partial^c \partial_c B) - \left(\partial_a \partial_b \tilde{A}^2 - \eta_{ab} \partial^c \partial_c \tilde{A} \right) + S_{ab}[\tilde{A}, h_{ab}^{GW} - 2\tilde{A}\eta], \tag{3.20}$$

where we have used (3.4) to re-express the terms depending on $B$ and $A^2$ more compactly. Symbolically, the r.h.s. will have the following structure:

$$\text{r.h.s.} \sim (\partial \tilde{A})(\partial \tilde{A}) + (\partial h^{GW})(\partial h^{GW}) + (\partial \psi)^2 + (\partial \tilde{A} \partial h^{GW}). \tag{3.21}$$

The average of the cross terms $(\partial \tilde{A} \partial h^{GW})$ will vanish thanks to property (3.16). On the other hand the three quadratic contributions will describe, in order, the backreaction energy due to conformal fluctuations, GWs and matter fields (possibly including the relevant unknown exotic components that provide the energy balance). To proceed we must thus evaluate each term in equation (3.20) explicitly.

(i) Analysis of the term $-G_{ab}^{(\xi\xi)}[h_{ab}^{GW} - 2\tilde{A}\eta]$

From the general expression (2.49) with $g_{ab}^B = \eta_{ab}$ we have

$$-G_{ab}^{(\xi\xi)}[h_{ab}^{GW} - 2\tilde{A}\eta] = \frac{1}{2} \left(h_{ab}^{GW} - 2\tilde{A}\eta_{ab}\right) R_{ab}^{(1)}[\xi] - \frac{1}{2} \eta_{ab} \left(h_{ab}^{GW \cd} - 2\tilde{A}\eta \cd\right) R_{cd}^{(1)}[\xi].$$

Using the fact that $R_{ab}^{(1)}[\cdot]$ is a linear operator we have

$$R_{ab}^{(1)}[\xi] = R_{ab}^{(1)}[h_{ab}^{GW}] - R_{ab}^{(1)}[2\tilde{A}\eta] = -R_{ab}^{(1)}[2\tilde{A}\eta]$$
since $h_{ab}^{GW}$ satisfies the linear GWs equation, equivalent to $R^{(1)}_{ab} [h^{GW}] = 0$. Exploiting this we find

$$-G^{(ξξ)}_{ab}[ξ] = \frac{1}{2} \left( 2A_{ab} - h_{ab}^{GW} \right) R^{(1)}_{ab} [2A] + \frac{1}{2} \eta_{ab} \left( h^{GWcd} - 2\tilde{A}^{cd} \right) R^{(1)}_{cd} [2A],$$

$$= 2 \left( A_{ab} R^{(1)} [2A] - \tilde{A}_{ab} \tilde{R}^{(1)} [2A] \right) + \left( \eta_{ab} h^{GWcd} R^{(1)}_{cd} [2A] - h_{ab}^{GW} R^{(1)} [2A] \right).$$

The first bracketed term is zero since $η^{cd} R^{(1)}_{cd} [2A] = R^{(1)} [2A]$ and we see that only a cross term of the kind $\partial \tilde{A} \partial h^{GW}$ survives. Because of the statistical property $\langle h^{GW}_{ab}, \ldots, \tilde{A}, \ldots \rangle = 0$, the average of such a term will vanish, in such way that it cannot affect the large scale vacuum energy balance. However its fluctuations will induce extra fluctuations on the spacetime geometry and it is thus useful for later purposes to make it explicit. We can now use the expression (3.3) to find, in any gauge:

$$R^{(1)}_{ab} [2A] = -\partial_a \partial_b \tilde{A} - \frac{1}{2} \eta_{ab} \partial^c \partial_c \tilde{A}, \quad (3.22)$$

implying

$$R^{(1)} [2A] = -3 \partial^c \partial_c \tilde{A}. \quad (3.23)$$

Using these we get the general result

$$-G^{(ξξ)}_{ab}[ξ] = 3h_{ab}^{GW} \partial^c \partial_c \tilde{A} - \eta_{ab} h^{GWcd} \partial_c \partial_d \tilde{A} - \frac{1}{2} h^{GW}_{cd} \eta_{ab} \partial^c \partial_d \tilde{A}. \quad (3.24)$$

We exploit now the fact that the GWs perturbation $h_{ab}^{GW}$ is in the TT gauge, i.e. its components in the laboratory reference frame satisfy:

$$h^{GW}_{\mu\nu} := η^{\mu\nu} h_{\mu\nu}^{GW} = 0; \quad h_{\delta\mu}^{GW} = 0; \quad \partial^\mu h_{\mu\nu}^{GW} = 0. \quad (3.25)$$

The expression (3.24) now simplifies to

$$-G^{(ξξ)}_{ab}[ξ] = 3h_{ab}^{GW} \partial^c \partial_c \tilde{A} - \eta_{ab} h^{GWcd} \partial_c \partial_d \tilde{A}. \quad (3.26)$$

Using the wave equation for $\tilde{A}$ this reduces to

$$-G^{(ξξ)}_{ab}[ξ] = -\eta_{ab} h^{GWcd} \partial_c \partial_d \tilde{A}, \quad (3.27)$$

which is valid in an arbitrary gauge.

(ii) Analysis of the term $S_{ab}[\tilde{A}, h^{GW} - 2\tilde{A}]$:

From equation (2.76) we have explicitly

$$S_{ab} = -η^{cd} (\partial_d ξ_{ab} + \partial_b ξ_{ad} - \partial_a ξ_{bd}) \partial_c \tilde{A} + \eta_{ab} η^{de} η^{ef} (\partial_d ξ_{ef} + \partial_e ξ_{df} - \partial_f ξ_{de}) \partial_c \tilde{A}$$

$$- 2ξ_{ab} \partial^c \partial_c \tilde{A} + 2η_{ab} ξ^{cd} \partial_d \partial_c \tilde{A}. \quad (3.28)$$

Using $ξ_{ab} = h_{ab}^{GW} - 2\tilde{A}_{ab}$ to eliminate $ξ_{ab}$, we get a series of terms involving cross products $h_{ab}^{GW} \tilde{A}$, whose corresponding average will vanish, plus terms quadratic in $\tilde{A}$. Explicitly and in an arbitrary
gauge we find:

\[ S_{ab}[\tilde{A}, h^{GW}] = 4\partial_a\tilde{A}\partial_b\tilde{A} + 2\eta_{ab}\partial^cA\partial_cA \]

\[ - \left[ \partial^dA \left( \partial_a h^G_{bd} + \partial_b h^G_{ad} - \partial_d h^G_{ab} \right) - \eta_{ab}\partial^fA \left( \partial^c h^G_{cf} + \partial^d h^G_{df} - \partial_f h^G \right) \\
+ 2h^G_{ab}\partial^cA - 2\eta_{ab}h^{GWd}e\partial_d\partial_c\tilde{A} \right]. \]

(3.29)

Exploiting the TT gauge properties we have

\[ S_{ab}[\tilde{A}, h^{GW}] = 4\partial_a\tilde{A}\partial_b\tilde{A} + 2\eta_{ab}\partial^c\tilde{A}\partial_c\tilde{A} \]

\[ - \left[ \partial^d\tilde{A} \left( \partial_a h^G_{bd} + \partial_b h^G_{ad} - \partial_d h^G_{ab} \right) + 2h^G_{ab}\partial^c\tilde{A} - 2\eta_{ab}h^{GWd}e\partial_d\partial_c\tilde{A} \right]. \]

(3.30)

Using the wave equation for \( \tilde{A} \) this simplifies to

\[ S_{ab}[\tilde{A}, h^{GW}] = 4\partial_a\tilde{A}\partial_b\tilde{A} + 2\eta_{ab}\partial^c\tilde{A}\partial_c\tilde{A} \]

\[ - \left[ \partial^d\tilde{A} \left( \partial_a h^G_{bd} + \partial_b h^G_{ad} - \partial_d h^G_{ab} \right) - 2\eta_{ab}h^{GWd}e\partial_d\partial_c\tilde{A} \right]. \]

(3.31)

(iii) Analysis of the term \(-G^{(2)}_{ab}[h^{GW} - 2\tilde{A}\eta]\

The analysis of this term is relatively straightforward but particularly lengthy due to the nonlinear structure of the operator \( G^{(2)}_{ab}[\cdot] \). This is where the error in [46] was made.

More explicitly, the structure of this nonlinear term is:

\[-G^{(2)}_{ab}[h^{GW} - 2\tilde{A}\eta] = -G^{(2)}_{ab}[2\tilde{A}\eta] - G^{(2)}_{ab}[h^{GW}] + [\partial\tilde{A}\partial h^{GW}]. \]

(3.32)

where the symbolic notation \([\partial\tilde{A}\partial h^{GW}] \) stands for the collection of all the cross term involving \( \tilde{A} \) and \( h^{GW} \) that will result and will be analyzed in details in Section 3.4. This term has a zero average and, again, it cannot affect the net amount of large scale vacuum energy. We recall now that \( h^{GW}_{ab} \) satisfies the linearized Einstein’s equation \( G^{(1)}_{ab}[h^{GW}] = 0 \). Then, Isaacson’s general results summarized in Appendix D guarantee that the averaged quantity \( \langle -G^{(2)}_{ab}[h^{GW}] \rangle \) is given by equations (D.64) and (D.65) as

\[
\langle -G^{(2)}_{ab}[h^{GW}] \rangle = \left\langle \frac{1}{4} \nabla_a\tilde{h}_{cd}\nabla_b\tilde{h}^{cd} - \frac{1}{8} \nabla_a\tilde{h}\nabla_b\tilde{h} - \frac{1}{2} \nabla_a\tilde{h}^{cd}\nabla_{(a}\tilde{h}_{b)c} \right\rangle. \]

(3.33)

Isaacson showed that this is (1) gauge invariant, (2) positive definite and, for \( h^{GW}_{ab} \) given by an arbitrary superposition of plane waves, it is also seen to be (3) traceless. Because of these reasons it can represent the GWs stress energy tensor. When considering the averaged second order equation this contributes, together with the other standard matter fields in \( \psi \), to build up a positive, large, zero point vacuum energy.

The mistake in [46] came about as follows: there, the expression (3.33) was used to evaluate the averaged quantity \( \langle -G^{(2)}_{ab}[2\tilde{A}\eta] \rangle \). The result was \(-6 \left\langle \partial_a\tilde{A}\partial_b\tilde{A} \right\rangle \). When this is added to the other contributions from all other terms quadratic in \( \tilde{A} \), the net result was that a negative definite, traceless, effective tensor quadratic in \( \tilde{A} \) given by \(-4\partial_a\partial_b\tilde{A} - \eta_{ab}\partial^c\partial_c\tilde{A} \) appeared on the r.h.s. of averaged
second order Einstein’s equation. By suitably adjusting the statistical amplitude of the fluctuations in \( \tilde{A} \) this was then used to cancel out the positive vacuum energy contribution coming from GWs plus the massless matter fields in \( \psi \). Unfortunately the flaw in the above reasoning is that, in fact, the expression (3.33) cannot be used to evaluate \( \langle -G^{(2)}_{ab}[2\tilde{A}\eta]\rangle \). As explained in detail in Appendix D at page 133, the reason for this is that (3.33) holds only when the general operator \( G^{(2)}_{ab}[\cdot] \) is acting on a perturbation satisfying itself the linearized Einstein’s equation. However, in the problem we are studying, the perturbation upon which \( G^{(2)}_{ab}[\cdot] \) is acting on is \( 2\tilde{A}\eta \). This does not satisfy the linearized equation and it is instead

\[
G^{(1)}_{ab}[-2\tilde{A}\eta] = 2\partial_a\partial_b\tilde{A} - 2\eta_{ab}\partial^c\partial_c\tilde{A} = 2\partial_a\partial_b\tilde{A} \neq 0. 
\] (3.34)

From these considerations it follow that we must calculate \( -G^{(2)}_{ab}[2\tilde{A}\eta] \) exactly and before any average is done, using the full expression of the nonlinear operator \( G^{(2)}_{ab}[\cdot] \). To do this we estimate first \( R^{(2)}_{ab}[2\tilde{A}\eta] \) starting from equation (3.19) and find:

\[
R^{(2)}_{ab}[2\tilde{A}\eta] = 4\tilde{A}\partial_a\partial_b\tilde{A} + 6\partial_a\tilde{A}\partial_b\tilde{A} + 2\tilde{A}\eta_{ab}\partial^c\partial_c\tilde{A}. 
\]

After finding the trace \( R^{(2)}[2\tilde{A}\eta] = 12\tilde{A}\partial^c\partial_c\tilde{A} + 6\partial^c\partial_c\tilde{A} \) we get

\[
G^{(2)}_{ab}[2\tilde{A}\eta] = R^{(2)}_{ab}[2\tilde{A}\eta] - \frac{1}{2}\eta_{ab}R^{(2)}[2\tilde{A}\eta] = 4\tilde{A}\partial_a\partial_b\tilde{A} + 6\partial_a\tilde{A}\partial_b\tilde{A} - 4\tilde{A}\eta_{ab}\partial^c\partial_c\tilde{A} - 3\eta_{ab}\partial^c\partial_c\tilde{A}. 
\] (3.35)

This can be conveniently written as the sum of two contributions, one of which is just a total derivative and will have a zero average. Using the identity \( \partial_a(\tilde{A}\partial_b\tilde{A}) = \partial_a\tilde{A}\partial_b\tilde{A} + \tilde{A}\partial_a\partial_b\tilde{A} \) and its contracted form we finally find

\[
-G^{(2)}_{ab}[2\tilde{A}\eta] = -2\partial_a\tilde{A}\partial_b\tilde{A} - \eta_{ab}\partial^c\partial_c\tilde{A} - 4\left[ \partial_a(\tilde{A}\partial_b\tilde{A}) - \eta_{ab}\partial^c(\tilde{A}\partial_c\tilde{A}) \right], 
\] (3.36)

where the terms in brackets have a vanishing average. Using the wave equation for \( \tilde{A} \) then \( \partial^c(\tilde{A}\partial_c\tilde{A}) = \partial^c\tilde{A}\partial_c\tilde{A} \) and it is

\[
-G^{(2)}_{ab}[2\tilde{A}\eta] = -2\partial_a\tilde{A}\partial_b\tilde{A} + 3\eta_{ab}\partial^c\partial_c\tilde{A} - 4\partial_a(\tilde{A}\partial_b\tilde{A}). 
\] (3.37)

### 3.3 Averaged second order equation and zero point energy balance

We are finally in the position to present the second order equation (3.20) in a more explicit form. This will allow to address the question of whether conformal fluctuations can function as a balancing agent for the large amount of zero point energy. By collecting together the results (3.24), (3.29), (3.32) and (3.36) we find

\[
G^{(1)}_{ab}[\gamma^{(2)}] = 8\pi \left( T^{(2)}_{ab}[\psi] - \frac{1}{8\pi} G^{(2)}_{ab}[h^{GW}] \right) + 4\left[ \eta_{ab}\partial^c(\tilde{A}\partial_c\tilde{A}) - \partial_a(\tilde{A}\partial_b\tilde{A}) \right] + G^{(1)}_{ab}[-2B\eta + \tilde{A}^2\eta] + \left[ \text{terms } (\partial\tilde{A}\partial h^{GW}) \right], 
\] (3.38)
which, as long as we do not consider the explicit structure of the cross terms, is valid in any gauge. Taking the spacetime average these vanish thanks to the property \( \langle h^{GW}_{ab} \dots \rangle = 0 \). All the other terms containing \( A \) or \( B \) are total derivatives. Their average also vanishes since all fluctuations are assumed to be stationary. Thus the averaged, classical scale equation is:

\[
G_{ab}^{(1)} \langle \gamma^{(2)} \rangle = 8\pi \left( \langle T^{(2)}_{ab} [\psi] \rangle + T_{ab}^{GW} [h^{GW}] \right),
\]

(3.39)

where, as defined in Appendix D, the GWs stress energy tensor is

\[
T_{ab}^{GW} := -\frac{1}{8\pi} \langle G_{ab}^{(2)} [h^{GW}] \rangle.
\]

(3.40)

This is traceless and contributes together with other matter fields to build up vacuum energy at the classical scale. This results shows that conformal fluctuations cannot provide, within GR, a mechanism to obtain vacuum energy cancelation. This would seem to imply that either zero point energy is not real and it does not gravitate or else that there should be a insofar undiscovered hypothetical ingredient for cancelation. In this case, this would be thought to be included within the collection of matter fields \( \psi \).

### 3.4 Equivalence of the physical and conformal metric formalisms

We now come to the important matter of building an explicit microscopic solution for the spacetime physical metric. The formalism developed so far was based on the idea of choosing an ansatz metric \( g_{ab} = [\Omega(A)]^2 \gamma_{ab} \) first, on which Einstein’s equation is imposed; secondly the perturbation scheme is applied using \( \gamma_{ab} = \eta_{ab} + \xi_{ab} + \gamma_{ab}^{(2)} \). We will refer to this procedure as to the Conformal Metric Scheme, i.e. impose ansatz solution first-expand second. An alternative procedure would be to expand formally Einstein’s equation in terms of the metric \( g_{ab} = \eta_{ab} + h_{ab} + g_{ab}^{(2)} \) first and after impose the ansatz solution directly on the linear and second order perturbations \( h_{ab} \) and \( g_{ab}^{(2)} \) by using the expressions (2.79). We will refer to this procedure as to the Physical Metric Scheme, i.e. expand first-impose ansatz solution second. The analysis in Section 3.1 shows that these two procedures are totally equivalent at first order. This is implied by the result

\[
G_{ab}^{(1)} [h] = 0 \iff G_{ab}^{(1)} [\xi] = 2\partial_a \partial_b A - 2\eta_{ab} \partial^c \partial_c A,
\]

(3.41)

that holds together with the linear algebraic constraint

\[
h_{ab} = \xi_{ab} + 2A\eta_{ab}.
\]

(3.42)

An important question now is whether the two procedures are in fact equivalent even to second order. We stress that this is not a priori obvious. Indeed, when carrying out the expansion scheme, it may well be that the strong nonlinearity inherent to Einstein’s equation could lead in the two cases to physically inequivalent results. In fact this belief had been central to the work in [46]. Ironically, this had been fueled by the fact that the already cited mistake actually led to vacuum energy cancelation; a result which would have indeed been physically inequivalent to what one finds within the Physical Metric Scheme!\(^1\) Contrary to our early expectations, we now show that the two procedures are in fact

\(^1\)In this footnote we leave the pluralis maestatis aside to highlight some details regarding the historic development of
3.4. EQUIVALENCE OF THE PHYSICAL AND CONFORMAL METRIC FORMALISMS

Equivalent, even to second order. The fact that conformal fluctuations cannot provide a mechanism to balance vacuum energy at the large scale also depends heavily on this equivalence.

This equivalence will simplify the structure of the second order equation in a very severe way: at the end of this section we will show that it is still possible, at least formally, to impose a suitable microscopic condition on the second order conformal fluctuations, in such a way that the physical metric seems to depend on \( \tilde{A}^2 \); however we will show how the constraint that must be imposed on \( \tilde{A} \) to achieve this is so strict that the solution must be ruled out as unphysical. The conclusion will be that standard GR does seem not to allow for the possibility of a conformally modulated metric that may in turn induce quantum dephasing.

To see how this conclusion can be achieved let us see how the physical metric scheme would work. We start from the physical metric \( g_{ab} = \eta_{ab} + h_{ab} + g_{ab}^{(2)} \) and expand Einstein’s equation \( G_{ab}[g] = 8\pi T_{ab}[\psi] \) up to second order. This is a standard procedure and results in the two equations:

\[
G^{(1)}_{ab}[h] = 0, \tag{3.43}
\]

\[
G^{(1)}_{ab}[g^{(2)}] = 8\pi \left( T^{(2)}_{ab}[\psi] - \frac{1}{8\pi} G^{(2)}_{ab}[h] \right). \tag{3.44}
\]

Next we bring in the information related to the conformal fluctuations, i.e. we use \( g_{ab} = (1 + A + B)^2 \gamma_{ab} \). This translates in the algebraic constraints:

\[
h_{ab} = \xi_{ab} + 2A\eta_{ab}, \tag{3.45}
\]

\[
g^{(2)}_{ab} = \gamma^{(2)}_{ab} + 2A\xi_{ab} + 2B\eta_{ab} + A^2\eta_{ab}, \tag{3.46}
\]

which must now be substituted into equations (3.43) and (3.44) starting from first order. We will thus get two equations for the conformal 4-metric perturbations in the Physical Metric Scheme.

In order to parallel the formalism developed above we need to solve, with Boyer’s boundary conditions, equation (3.43) first. This simply gives

\[
h_{ab} = h_{ab}^{\text{GW}}. \tag{3.47}
\]

Then the constraint (3.45) yields again

\[
\xi_{ab} = h_{ab}^{\text{GW}} - 2A\eta_{ab}. \tag{3.48}
\]

This completes the first order analysis which yields of course the same results as we found above with the Conformal Metric Scheme.

To find the second order equation for \( \gamma^{(2)} \) we substitute the first order solution (3.47)-(3.48) and the constraint (3.46) into (3.44) to find:

\[
G^{(1)}_{ab}[\gamma^{(2)} + 2Ah^{\text{GW}} - 3A^2\eta + 2B\eta] = 8\pi \left( T^{(2)}_{ab}[\psi] - \frac{1}{8\pi} G^{(2)}_{ab}[h^{\text{GW}}] \right). \]

this PhD work. Notwithstanding the plausible hints, it was towards the end of my first attempt to write up my PhD results that I came to look at the possibility that the two expansion schemes are physically inequivalent as suspicious. That was mainly because the identity of the two formalisms at linear order was so obvious and clear that I found surprising it would not hold at second order. I came to a point in which I believed there must be some error in the treatment of the second order terms in [46]. This effectively allowed me to spot the mistake described in the last part of section 3.2. This then quickly led to the results presented in the reminder of this chapter.
This is better rearranged as
\[
G_{ab}^{(1)}[\gamma^{(2)}] = 8\pi \left( T_{ab}^{(2)}[\psi] - \frac{1}{8\pi} G_{ab}^{(2)}[h^{GW}] \right) + G_{ab}^{(1)}[2A^2\eta] - G_{ab}^{(1)}[2Ah^{GW}] - G_{ab}^{(1)}[2B\eta - A^2\eta].
\] (3.49)

By comparing with the previous result (3.38) we notice that the two equations have at least the same structure. The two equations (i.e. the two alternative schemes) are truly equivalent if it is true that
\[
G_{ab}^{(1)}[2A^2\eta] \equiv 4 [\eta_{ab}\partial^c(A\partial_c A) - \partial_a(A\partial_b A)]
\] (3.50)
and
\[
-G_{ab}^{(1)}[2Ah^{GW}] \equiv [\text{terms } (\partial A\partial h^{GW})].
\] (3.51)

We now show that these are in fact two identities.

**Verification for the term** \(G_{ab}^{(1)}[2A^2\eta]:\)

This is straightforward. We start by noting the identity
\[
4 [\eta_{ab}\partial^c(A\partial_c A) - \partial_a(A\partial_b A)] = 2\eta_{ab}\partial^c\partial_c A^2 - 2\partial_a\partial_b A^2.
\]
Then, by simply looking at (3.34), we see immediately that
\[
G_{ab}^{(1)}[2A^2\eta] = 2\eta_{ab}\partial^c\partial_c A^2 - 2\partial_a\partial_b A^2,
\] (3.52)
so that (3.50) is verified in an arbitrary gauge.

**Verification for the term** \(-G_{ab}^{(1)}[2Ah^{GW}]:\)

This is more tricky as there are a lot of cross terms to check. For this reason it will simplify things a bit if we work with \(h_{ab}^{GW}\) in the TT gauge. We start with the r.h.s. of (3.51), i.e. by putting together all the cross terms we found in the previous section. To have shorter formulas we will use here the notation \(h\) instead of \(h^{GW}\). Equations (3.26), (3.30), (3.32) yield
\[
[\text{terms } (\partial A\partial h^{GW})] = \left[ 3h_{ab}\partial^c\partial_c A - \eta_{ab}h^{cd}\partial_c \partial_d A \right]
- \left[ \partial^d A \left( \partial_a h_{bd} + \partial_b h_{ad} - \partial_d h_{ab} \right) + 2h_{ab}\partial^c\partial_c A - 2\eta_{ab}h^{de}\partial_d \partial_e A \right]
+ \left[ \partial A\partial h \right]_{G(2)},
\] (3.53)
where we recall that \([\partial A\partial h]_{G(2)}\) indicates all the cross terms resulting from \(-G_{ab}^{(2)}[h^{GW} - 2A\eta]\).

To find these we start from the expression (3.19) for \(R_{ab}^{(2)}[\cdot]\). We act on \(h_{ab}^{GW} - 2A\eta_{ab}\) and collect all the cross terms. By using the TT gauge conditions, \(\partial^a h_{ab} = 0\), \(\eta^{ab} h_{ab} = 0\) and the wave equation \(\partial^c \partial_c h_{ab} = 0\) we find:
\[
R_{ab}^{(2)}[h_{ab}^{GW} - 2A\eta_{ab}] \Rightarrow \partial_a(h^{c}e_{b}\partial_c A) - \partial_c A\partial_e h_{ab} + \partial^d A\partial_d h_{eb} - \partial^c A\partial_d h_{eb} - \partial^c A\partial_e h_{ab} - \eta_{ab}h^{de}\partial_d \partial_e A + h^{c}e_{a}\partial_c A.
\]
3.4. EQUIVALENCE OF THE PHYSICAL AND CONFORMAL METRIC FORMALISMS

In evaluating the trace, many terms vanish because of the TT conditions. We have:

\[ R^{(2)}[h_{ab}^{GW} - 2A\eta_{ab}] \Rightarrow \partial^b(h^c_v\partial_c A) - 4h^{dc}\partial_d\partial_c A + h^{ch}\partial_c\partial_b A = -2h^{cd}\partial_d\partial_c A. \]

Finally collecting all the cross terms in \(-G^{(2)}_{ab}[h_{ab}^{GW} - 2A\eta]\) \(\equiv -R^{(2)}_{ab} + \frac{1}{2}\eta_{ab}R^{(2)}\) yields

\[ [\partial A\partial h]_{G^{(2)}} = -\partial_a(h^c_v\partial_c A) + \partial^d A\partial_d h_{ab} - \partial^d A\partial_b h_{da} + \partial^e A\partial_e h_{ac} \]

Going now back to equation (3.53) and putting all terms together gives the neat result:

\[ [\text{terms } (\partial A\partial h^{GW})] = h_{ab}\partial^d \partial_d A + \eta_{ab} h^{dc}\partial_d \partial_c A + 2\partial_d h_{ab}\partial^d A - \partial_a(h_{cb}\partial^c A) - \partial_b(h_{ca}\partial^c A), \tag{3.54} \]

which should be compared to \(G_{ab}^{(1)}[-2Ah]\). This is very straightforward. Using (3.2) and (3.3) and the TT gauge conditions for \(h_{ab}\) we have

\[ G_{ab}^{(1)}[-2Ah] = h_{ab}\partial^d \partial_d A + \eta_{ab} h^{dc}\partial_d \partial_c A + 2\partial_d h_{ab}\partial^d A - \partial_a(h_{cb}\partial^c A) - \partial_b(h_{ca}\partial^c A). \tag{3.55} \]

We see that the r.h.s. is indeed the same as in (3.54). We have thus verified equation (3.51), at least in the TT gauge, i.e. it is indeed true that

\[ -G_{ab}^{(1)}[2Ah^{GW}] = [\text{terms } (\partial A\partial h^{GW})]. \tag{3.56} \]

To summarize, this proves that the Conformal and Physical Metric Schemes are in fact equivalent procedures as they result in the same set of equations. Namely, in the case in which \(G_{ab}^{(1)}[h] = 0\) is solved first and \(h_{ab}^{GW} = h_{ab}\), we have by either procedure:

\[ G_{ab}^{(1)}[\xi] = -G_{ab}^{(1)}[2A\eta_{ab}] \quad \Leftrightarrow \quad G_{ab}^{(1)}[h^{GW}] = 0, \tag{3.57} \]

and

\[ G_{ab}^{(1)}[\gamma^{(2)}] = 8\pi T_{ab}^{\text{eff}}[\psi, h_{ab}^{GW}] + G_{ab}^{(1)}[3A^2\eta - 2B\eta - 2Ah^{GW}] \quad \Leftrightarrow \quad G_{ab}^{(1)}[\gamma^{(2)}] = 8\pi T_{ab}^{\text{eff}}[\psi, h_{ab}^{GW}], \tag{3.58} \]

where

\[ T_{ab}^{\text{eff}}[\psi, h_{ab}^{GW}] := \left( T_{ab}^{(2)}[\psi] - \frac{1}{8\pi}G_{ab}^{(2)}[h_{ab}^{GW}] \right) \tag{3.59} \]

and

\[ h_{ab}^{GW} = \xi_{ab} + 2A\eta_{ab}, \tag{3.60} \]

\[ g_{ab}^{(2)} = \gamma_{ab}^{(2)} + 2A\xi_{ab} + 2B\eta_{ab} + A^2\eta_{ab}. \tag{3.61} \]

In particular this implies that, when one tries to assess the overall amount of vacuum energy at the classical scale by taking the average of the second order equation(s), it is

\[ G_{ab}^{(1)}[\langle \gamma^{(2)} \rangle] = 8\pi \langle T_{ab}^{\text{eff}}[\psi, h_{ab}^{GW}] \rangle \quad \Leftrightarrow \quad G_{ab}^{(1)}[\langle g^{(2)} \rangle] = 8\pi \langle T_{ab}^{\text{eff}}[\psi, h_{ab}^{GW}] \rangle, \tag{3.62} \]

showing again that, if implemented as done so far, conformal fluctuations do not contribute to the net vacuum energy amount. Either the averaged physical or the conformal metric can serve to describe the large scale structure of vacuum spacetime. The two averaged metrics can be related as \(\langle \gamma_{ab}^{(2)} \rangle = \).
3.5. Are spontaneous conformal fluctuations compatible with GR?

Although the two expansion schemes are formally equivalent and lead to no vacuum energy balance mechanism, the question remains to be addresses about which metric between $g_{ab}$ and $\gamma_{ab}$ has a true dependence on $A^2$ at the microscopic level. Indeed all we know, in general, is that the constraint (3.61) must hold. Answering whether $g^{(2)}_{ab}$ or rather $\gamma^{(2)}_{ab}$ really depends on $A^2$ is an important matter as this affects the theoretical occurrence of quantum dephasing. This is so because, within the standard GR framework we are analyzing now, a test particle couples to the physical metric $g_{ab}$. This must present a genuine dependence on $A^2$ if the particle wave function is to suffer dephasing. As we discussed for the first order equation, the question now boils down to which equation is solved first in (3.58). If one solves first the equation on the right then the physical metric will not depend on $A$ up to second order. In this case the conformal field $A$ would just represent a formal device that has no physical effect at all. As anticipated earlier the key lies now in trying to fix the second order structure of the conformal metric in such a way that it does not contain any $A^2$ dependence. This would in turn induce a constraint on the conformal field $A$ and its nonlinear correction $B$. If this constraint can be satisfied, then the algebraic constraint (3.61) would automatically imply that the physical metric would present a true nonlinear modulation $A^2\eta_{ab}$. We turn to this central issue in the next section.

3.5.1 Attempt for a solution of the second order equation

In order to implement these ideas a more detailed analysis of the second order equation for the conformal metric nonlinear perturbation is required. This is in general composed of a smooth component $\langle \gamma^{(2)}_{ab} \rangle$ and a randomly fluctuating component $\Delta \gamma^{(2)}_{ab}$:

$$\gamma^{(2)}_{ab} = \langle \gamma^{(2)}_{ab} \rangle + \Delta \gamma^{(2)}_{ab}.$$  \hfill (3.63)

Thus the second order equation gives rise to two separate equations. One determines the background geometry and can be found by averaging (3.58). It reads:

$$G_{ab}^{(1)}(\langle \gamma^{(2)} \rangle) = 8\pi \langle T^{\text{eff}}_{ab}[\psi, h^{\text{GW}}] \rangle \quad \Leftrightarrow \quad G_{ab}^{(1)}(\langle g^{(2)} \rangle) = 8\pi \langle T^{\text{eff}}_{ab}[\psi, h^{\text{GW}}] \rangle,$$  \hfill (3.64)

from which one can calculate the smooth perturbation induced over the flat Minkowski. The second equation determines how the fluctuations in the effective stress energy tensor induce extra metric perturbations. It can be found by subtracting equation (3.64) from (3.58). It reads:

$$G_{ab}^{(1)}[\Delta \gamma^{(2)}] = 8\pi \left( T^{\text{eff}}_{ab}[\psi, h^{\text{GW}}] - \langle T^{\text{eff}}_{ab}[\psi, h^{\text{GW}}] \rangle \right) + G_{ab}^{(1)}[3A^2\eta - 2B\eta - 2Ah^{\text{GW}}],$$  \hfill (3.65)

which is equivalent to

$$G_{ab}^{(1)}[\Delta g^{(2)}] = 8\pi \left( T^{\text{eff}}_{ab}[\psi, h^{\text{GW}}] - \langle T^{\text{eff}}_{ab}[\psi, h^{\text{GW}}] \rangle \right).$$  \hfill (3.66)
3.5. ARE SPONTANEOUS CONFORMAL FLUCTUATIONS COMPATIBLE WITH GR?

In both these equations it is

\[ T^{\text{eff}}_{ab}[\psi, h^{\text{GW}}] = \left( T^{(2)}_{ab}[\psi] - \frac{1}{8\pi} G^{(2)}_{ab}[h^{\text{GW}}] \right) \]  (3.67)

and

\[ \langle T^{\text{eff}}_{ab}[\psi, h^{\text{GW}}] \rangle = \langle T^{(2)}_{ab}[\psi] \rangle + T_{ab}^{\text{GW}}. \]  (3.68)

Moreover

\[ h^{\text{GW}}_{ab} = \xi_{ab} + 2A\eta_{ab}, \]  (3.69)

\[ \langle g^{(2)}_{ab} \rangle + \Delta g^{(2)}_{ab} = \langle \gamma^{(2)}_{ab} \rangle + \Delta \gamma^{(2)}_{ab} + 2Ah^{\text{GW}}_{ab} + 2B\eta_{ab} - 3A^2\eta_{ab}. \]  (3.70)

As discussed in Section 2.6 we assume

\[ \langle T^{\text{eff}}_{ab}[\psi, h^{\text{GW}}] \rangle = 0, \]  (3.71)

as implied by observations. The second order equations thus take the form:

\[ G^{(1)}_{ab}[\langle \gamma^{(2)} \rangle] = 0 \iff G^{(1)}_{ab}[\langle g^{(2)} \rangle] = 0, \]  (3.72)

for the background correction, and

\[ G^{(1)}_{ab}[\Delta \gamma^{(2)}] = 8\pi T^{\text{eff}}_{ab}[\psi, h^{\text{GW}}] + G^{(1)}_{ab}[3A^2\eta - 2B\eta - 2Ah^{\text{GW}}], \]  (3.73)

equivalent to

\[ G^{(1)}_{ab}[\Delta g^{(2)}] = 8\pi T^{\text{eff}}_{ab}[\psi, h^{\text{GW}}], \]  (3.74)

for the second order fluctuations. Equations (3.72) for the background correction imply that \( \langle \gamma^{(2)} \rangle \) (and \( \langle g^{(2)} \rangle \)) could in general be given by any smooth GWs perturbation. Since we are focusing here on the random scale structure of vacuum only, we can make the simple choice \( \langle \gamma^{(2)} \rangle = \langle g^{(2)} \rangle = 0. \)

3.5.2 Second order constraint on the conformal field

We now proceed to build explicitly a solution in which the physical metric seems to depend formally on the conformal fluctuations. To this end we start from equation (3.73) and first re-write it as

\[ G^{(1)}_{ab}[\Delta \gamma^{(2)} + 2Ah^{\text{GW}}] = 8\pi T^{\text{eff}}_{ab}[\psi, h^{\text{GW}}] + G^{(1)}_{ab}[3A^2\eta - 2B\eta]. \]  (3.75)

The main idea now is that \( \gamma^{(2)} \) would not depend on \( A^2 \) if we could somehow manage to impose the condition

\[ G^{(1)}_{ab}[3A^2\eta - 2B\eta] = 0. \]  (3.76)

From equation (3.4) this reads explicitly

\[ G^{(1)}_{ab}[3A^2\eta - 2B\eta] = \eta_{ab}\partial^c \partial_c (3A^2 - 2B) - \partial_a \partial_b (3A^2 - 2B) = 0. \]  (3.77)

Thus it is satisfied provided that

\[ \partial_a \partial_b B = \frac{3}{2} \partial_a \partial_b A^2. \]  (3.78)
3.5. ARE SPONTANEOUS CONFORMAL FLUCTUATIONS COMPATIBLE WITH GR?

As it had to be expected, this shows that the hope to be able to build a solution where the physical metric contains a true physical dependence on the conformal fluctuations is doomed to failure. Indeed we have two options to satisfy equation (3.78). The most obvious is to set

\[ B = \frac{3}{2} A^2. \]

In this case however we see from equation (3.70) that the physical metric wouldn’t have in fact any dependence on either \( A^2 \) or \( B! \) Its fluctuating part would simply be given by

\[ \Delta g_{ab}^{(2)} = \Delta \gamma_{ab}^{(2)} + 2Ah_{ab}^{GW}. \]  

(3.79)

The term \( \Delta \gamma_{ab}^{(2)} \) represents the fluctuations induced upon the Minkowski background by the fluctuations in the effective stress energy tensor. We notice that this is precisely the same kind of problem that Verdaguer and collaborators studied in [108] using the stochastic gravity approach. As we already mentioned at the beginning of Section 2.3 their result is that the matter fields tend to suppress the induced metric perturbations above the Planck scale. Too see that even our classical random approach leads to a qualitatively similar conclusion let us consider the fluctuations backreaction second order equation:

\[ G^{(1)}_{ab}[\Delta \gamma^{(2)} + 2Ah^{GW}] = 8\pi T^{\text{eff}}_{ab}[\psi, h^{GW}]. \]  

(3.80)

This could in principle be solved for \( E_{ab} := \Delta \gamma^{(2)} + 2Ah^{GW} \). Working in the Lorentz gauge this is a simple inhomogeneous wave equation for the trace reversed perturbation \( \bar{E}_{ab} \)

\[ \partial^\nu \partial_\nu \bar{E}_{ab} = -16\pi T^{\text{eff}}_{ab}[\psi, h^{GW}]. \]

The general solution can be written in terms of the retarded Green’s function [1]. In coordinates components it reads:

\[ \bar{E}_{\mu\nu}(x) = 4 \int_\Sigma \frac{T^{\text{eff}}_{\mu\nu}(x')}{|x - x'|} dS(x'), \]

where \( \Sigma \) denotes the past light cone of the point \( x \) and the volume element on the light cone is \( dS = r^2 dr d\Omega \). Without going into the details that would not affect anyway the main conclusion, it is clear that, being the source a zero average term, the solution for \( \bar{E}_{ab} \) will be slow varying and smoother than the fields \( \psi \) and \( h^{GW} \) inducing the fluctuations in the source. Thus is because performing the spacetime integral of the zero average fast varying source will tend to smooth out and suppress the fluctuations. For our present purposes we do not need to calculate the induced fluctuations \( E_{ab} \) in any detail, and we simply need to know that these would be characterized by larger variations scales and would have a more regular behavior than the fluctuations in \( \psi \) or \( h^{GW}_{ab} \). We thus write the solution of equation (3.80) symbolically as

\[ \Delta \gamma^{(2)} = \bar{E}_{ab} - 2Ah^{GW}, \]

where the notation \( \bar{E}_{ab} \) indicates a slow varying, long wavelength second order perturbation. The physical metric thus would follow as

\[ g_{ab} = \eta_{ab} + h^{GW}_{ab} + \bar{E}_{ab}. \]

This is the result that one expects from a standard perturbation scheme applied to Einstein’s equation without even trying to introduce any conformal fluctuations: the Minkowski background here presents a linear perturbation \( h^{GW}_{ab} \) describing GWs and a slow varying perturbation \( \bar{E}_{ab} \) induced by
the fluctuations in the backreaction effective stress energy tensor. The conformal fluctuations have disappear altogether and do not play any role.

An alternative possibility to satisfy the constraint (3.77) would appear to require

\[ B = 0, \quad \text{and} \quad \partial_a \partial_b A^2 = 0. \]

Proceeding exactly as before to deduce \( \Delta \gamma^{(2)} = E_{ab} - 2A h^{GW}_{ab} \) we would now find that the physical metric is formally given by

\[ g_{ab} = \eta_{ab} + h^{GW}_{ab} - 3A^2 \eta_{ab} + E_{ab}, \tag{3.81} \]

which appears to possess the desired formal dependence on \( A^2 \). However, even though the solution we have built seems to be mathematically feasible, it is based on the very severe constraint \( \partial_a \partial_b A^2 = 0 \). This looks very restrictive and artificial and would imply a very unphysical spacetime dependence for the conformal field which, we recall, has also been assumed to satisfy \( \partial_c \partial_c A = 0 \).

The conclusion is quite strong: all the elements suggest that within the random gravity framework, a vacuum solution of Einstein’s equation for standard GR is not compatible with the presence of spontaneous metric conformal fluctuations. Surely they do not lead to any vacuum energy cancelation and the attempt to build a solution with a true conformal modulation leads to formal inconsistences.

### 3.5.3 An alternative scenario? conformal metric and GWs

The treatment given above was based on the fact that the physical metric was assumed to represent GWs through its linear perturbation \( h_{ab} \). As anticipated earlier we now wish to explore whether the alternative formalism in which GWs are described through the conformal 4-metric linear perturbation \( \xi_{ab} \) can be given any sense. The idea that the conformal 3-metric may represent GWs in the sense of carrying GR ‘true’ degrees of freedom was proposed by York \[123, 124\] in his study of the canonical formulation of the theory and related initial value problem. More recently this also been advocated in \[62\], where the usual geometrodynamic approach has been extended in such a way that the conformal metric and related conformal factor play a central role. Moreover this is the approach in another paper by Wang and co-workers \[36\], where a metric of the form \( g_{ab} = (1 + A)^2 \gamma_{ab} \) was considered and where \( \gamma_{ab} \) was claimed to encode the GWs perturbation. In the same work the authors claimed to find a vacuum energy balancing mechanism between a negative contribution due to \( A \) and the usual positive contribution due to the GWs.

To reconsider this possibility in detail we start from the first order equation (3.1)

\[ G^{(1)}_{ab}[\xi] = 2\partial_a \partial_b A - 2\eta_{ab} \partial_c \partial_c A. \]

The requirement \( h^{GW}_{ab} := \xi_{ab} \), i.e. \( G^{(1)}_{ab}[\xi] = 0 \), implies the constraint equation on \( A \):

\[ \partial_a \partial_b A = 0. \tag{3.82} \]

By taking the trace we have

\[ \partial_c \partial_c A = 0. \tag{3.83} \]

Technically the next step would be to find the general solution of the wave equation with Boyer’s fluctuating boundary conditions and then restrict its form appropriately so that (3.82) is satisfied. Again we denote the result of this procedure by the symbol \( \tilde{A} \).

Next, after having solved the equation \( G^{(1)}_{ab}[\xi] = 0 \) in the TT gauge, leading to \( \xi_{ab} = h^{GW}_{ab} \), the
3.5. ARE SPONTANEOUS CONFORMAL FLUCTUATIONS COMPATIBLE WITH GR?

physical metric linear perturbation follows from the algebraic constraint (3.7) as

\[ h_{ab} = h_{ab}^{GW} + 2 \tilde{\eta} \eta_{ab}. \quad (3.84) \]

We remark that \( h_{ab} \) also satisfies the linearized Einstein’s equation equation \( G^{(1)}_{ab} [h] = 0 \). However this comes in this case only as a consequence of \( h_{ab}^{GW} \) and \( \tilde{A} \) being already fixed through their respective equations. In particular we can e.g. assume that \( h_{ab}^{GW} \) has been found in such a way as to satisfy the TT gauge conditions. We then see that the physical metric perturbation would present a randomly fluctuating trace \( \tilde{h} \) and given by

\[ \tilde{h} := \eta^{ab} h_{ab} = 8 \tilde{A}. \quad (3.85) \]

Notice that one cannot think of performing a further gauge transformation to put \( h_{ab} \) in a TT gauge, thus reabsorbing the effect of \( \tilde{A} \) in the definition of a new coordinate frame. We observe indeed that to do so one should consider a change of coordinates \( x^\mu \mapsto x^\mu + \delta x^\mu \) whose generator \( \delta x^\mu \) would have to be itself fluctuating. This would correspond to pass from the macroscopic observer frame to an effective microscopic fluctuating frame. We simply assume that this cannot be done as long as we just wish to describe physics from the point of view of a macroscopic observer. As a consequence of the above construction we now have that the conformal 4-metric linear perturbation and the conformal fluctuations are independent and, as a consequence, statistically uncorrelated, i.e. we have

\[ \left\langle \xi_{ab},...,\tilde{A},..., \right\rangle \equiv \left\langle h_{ab},...,\tilde{A},... \right\rangle = 0. \quad (3.86) \]

We now consider the second order equation (3.17)

\[ G_{ab}^{(1)} [\gamma^{(2)}] = 8 \pi T_{ab}^{(2)} [\psi] - G_{ab}^{(2)} [\xi] - G_{ab}^{(2)} [\xi] - \left( 2 \partial_a \tilde{A} \partial_b \tilde{A} + \eta_{ab} \partial^c \tilde{A} \partial_c \tilde{A} \right) + (2 \partial_a \partial_b B - 2 \eta_{ab} \partial^c \partial_c B) \left( \partial_a \partial_b \tilde{A}^2 - \eta_{ab} \partial^c \partial_c \tilde{A}^2 \right) + S_{ab}[\tilde{A}, \xi], \quad (3.87) \]

and we simply use \( \xi_{ab} = h_{ab}^{GW} \). This leads to a much simpler formalism than before. Indeed \( G_{ab}^{(1)} [h_{GW}] = 0 \) implies that it is simply \( G_{ab}^{(2)} [h_{GW}] = 0 \) -see equation (2.49)- and that \( -G_{ab}^{(2)} [h_{GW}] \) satisfies all of Isaacson’s property so that it will represent, after average, the GWs stress energy tensor. We have

\[ G_{ab}^{(1)} [\gamma^{(2)}] = 8 \pi T_{ab}^{\text{eff}} - \left( 2 \partial_a \tilde{A} \partial_b \tilde{A} + \eta_{ab} \partial^c \tilde{A} \partial_c \tilde{A} \right) + (2 \partial_a \partial_b B - 2 \eta_{ab} \partial^c \partial_c B) \left( \partial_a \partial_b \tilde{A}^2 - \eta_{ab} \partial^c \partial_c \tilde{A}^2 \right) + S_{ab}[\tilde{A}, h_{GW}], \quad (3.88) \]

where we defined again

\[ T_{ab}^{\text{eff}} := \left( T_{ab}^{(2)} [\psi] - \frac{1}{8 \pi} G_{ab}^{(2)} [h_{GW}] \right). \]

If we take the average we obtain now

\[ G_{ab}^{(1)} [\langle \gamma^{(2)} \rangle] = 8 \pi \left\langle T_{ab}^{\text{eff}} - \frac{1}{4 \pi} \partial_a \tilde{A} \partial_b \tilde{A} \right\rangle, \quad (3.89) \]

where we have used \( \langle \partial_c \tilde{A} \partial_c \tilde{A} \rangle = 0 \), as implied by \( \partial^c \partial_c \tilde{A} = 0 \), and \( \left\langle h_{ab},...,\tilde{A},... \right\rangle = 0 \). At first
3.6. Summary

In this and the previous chapter we have provided a general framework which allows to model some of the effects related to vacuum down to the random scale $\ell = \lambda L_P$ by using classical fluctuating fields. The main motivation was to provide a concrete framework in which conformal fluctuations of the metric could enter as a new physical actor and thus be seen as a cause of dephasing through the nonlinear effect they would have on the spacetime metric. This was attempted within standard GR by looking for a solution of Einstein’s equation in the form $g_{ab} = (1 + A + B)^2 \gamma_{ab}$, where $A$ and $B$ encode the conformal fluctuations. The main idea was to fix somehow the structure of the conformal metric $\gamma_{ab}$ in such a way to induce a true dependence of the physical metric on the conformal field $A$. This can be formally done by inserting the ansatz solution form for $g_{ab}$ into Einstein’s equation, thus obtaining a linear and second order constraint equations for $A$ and $B$. The hope that such an
approach could lead to physically viable results was based upon the possibility that the two expansion schemes of Einstein’s equation based on the metric $g_{ab}$ or $\gamma_{ab}$ were in fact inequivalent to second order due to the nontrivial nonlinear features of the problem. However we have verified that the expansion schemes are in fact equivalent both to first and second order. This implies that the set of constraints that one would need to impose to build a solution in which the physical metric contains an $A^2$ dependence are mathematically too restrictive and unphysical. Formally, there are two main scenarios: one in which the physical metric encodes the GWs and an alternative scenario in which one seeks to describe GWs through the conformal metric linear perturbation. We have shown that in both cases acceptable solutions for the physical metric do not contain a true dependence on $A^2$. Moreover we have shown that, contrary to previous claims, conformal metric fluctuations within standard GR do not offer a viable way for a vacuum energy balance mechanism. It is our main conclusion at this point that, though an interesting possibility, the idea that conformal fluctuations of the metric may represent a true physical actor of the microscopic physics close to the Planck scale is not tenable within GR. The next chapter is devoted to investigating whether and how such conclusions may change within some modified theories of gravity such e.g. scalar-tensor.
Chapter 4

Scalar-tensor theories and conformal fluctuations

In this chapter we study conformal transformations from the deeper point of view of local units transformations. In this case all lengths, times and masses also transform along with the metric tensor. As noted by Bekenstein and other authors before him this implies that all the fields of the standard model have in fact a conformally invariant dynamics. For this to be true the transformation properties of rest masses are essential. Requiring that gravity be also conformally invariant leads to Bekenstein action for the metric tensor and a novel gauge field which couples directly to rest masses. As pointed out by Bekenstein, this theory of conformally invariant gravity is in fact physically equivalent to GR, the equivalence being manifest in the particular conformal frame where the mass field is constant. Reinterpreting Bekenstein theory within the random gravity framework, the new gauge field is also expected to possess spontaneous fluctuations at the random scale. These are mathematically equivalent to conformal fluctuations and this fact allows to re-interpret conformal fluctuations of the metric as fluctuations of the mass field. In this sense the framework of conformally invariant gravity offers a natural way to introduce conformal fluctuations. Of course, this theory is still essentially GR and these cannot have any physically observable effect, in agreement with the results of the previous chapter. However the renewed point of view makes clear how the action for gravity should be modified in order for conformal fluctuations to play a role. This leads us to consider scalar-tensor theories. After reviewing some general facts, the analysis focuses on the special class of theories due to Brans and Dicke. Application of the random gravity scheme provides a valid vacuum solution where the physical metric does possess a conformal modulation. This is induced by the Brans-Dicke field spontaneous linear fluctuations, which naturally satisfy a standard wave equation. This facts happen independently of the specific value of the Brans-Dicke coupling parameter and provide a physical basis for the dephasing of a quantum particle studied in Chapter 1. Together with matter fields and GWs, the Brans-Dicke field linear fluctuations consistently contribute to a correction of the background geometry through a standard Klein-Gordon stress-energy tensor whose coupling to gravity depends linearly on the Brans-Dicke parameter. In particular, for large negative values the theory would provide a ghost field which could alleviate the usual problem of the large zero point energy.
4.1 Conformally invariant physics and Bekenstein’s theory

We start the analysis by showing that, even if they do not seem to lead to physically observable effects, conformal fluctuations within GR can at least be introduced in a more satisfactory way through the random fluctuations of a novel gauge scalar field. This is important in that it sheds light on how one should properly interpret conformal transformations (CT) of the metric. As already noted by Dicke in his second paper on the Brans-Dicke theory [48], these have to be seen as local change of physical units in such a way that all dimensional quantities, including lengths, times and masses must be transformed simultaneously. The failure to do this leads to the usually asserted fact that masses break the conformal invariance of physics. In fact, as it was already noted by Bekenstein in his 1980 remarkable paper [47] Conformal invariance, microscopic physics, and the nature of gravitation, physics is indeed conformally invariant, in the sense of being invariant under local changes of units. This fact has been more recently acknowledged by a number of authors [125, 128, 129]. Bekenstein’s very lucid treatment shows that the actions of all matter fields in the standard model of particles are conformally invariant in the sense specified above. Moreover the requirement that gravity be itself conformally invariant leads to the introduction of a new gauge field that couples directly to, and in fact defines, particle masses. Because of its importance for our results in relation to scalar-tensor theories we will now review Bekenstein’s theory in some detail. This will allow us to show that the conformal fluctuations introduced somehow ‘by hand’ in the previous chapters can be interpreted more elegantly as natural fluctuations in the Bekenstein’s gauge field.

4.1.1 Conformal transformations as local transformation of physical units

We start by considering a CT for the metric tensor

\[ \bar{g}_{ab} = \Omega^2 g_{ab}, \]  

(4.1)

where \( \Omega^2 \) is an arbitrary, positive and smooth dimensionless spacetime function. In the following all the conformally transformed quantities will be denoted with a bar.

To understand the role played by the mass in guaranteeing a conformally invariant physics, it is instructive to consider the example of a scalar field \( \phi \). It is well known that the standard minimally coupled action is not conformally invariant. On the other hand conformal invariance (CI) arises naturally if \( \phi \) couples to the metric scalar curvature. A straightforward calculation shows that [1]

\[ \left( g^{ab} \nabla_a \nabla_b - \frac{1}{6} \bar{R} \right) \left( \Omega^{-1} \phi \right) = \Omega^{-3} \left( \bar{g}^{ab} \nabla_a \nabla_b - \frac{1}{6} \bar{R} \right) \phi, \]  

(4.2)

where all the barred quantities refer to the conformally transformed metric \( \bar{g}_{ab} \). Defining the conformally transformed field

\[ \tilde{\phi} := \Omega^{-1} \phi, \]  

(4.3)

(4.2) says that the equation

\[ \left( g^{ab} \nabla_a \nabla_b - \frac{1}{6} \bar{R} \right) \phi = 0 \]  

(4.4)

is conformally invariant.

We consider now a massive field, satisfying the equation \( \left( g^{ab} \nabla_a \nabla_b - \frac{1}{6} \bar{R} \right) \phi = m^2 \phi \). Now equa-
tion (4.2) gives
\[
\left( \bar{g}^{ab} \nabla_a \nabla_b - \frac{1}{6} \bar{R} - m^2 \Omega^{-2} \right) \bar{\phi} = \Omega^{-3} \left( g^{ab} \nabla_a \nabla_b - \frac{1}{6} R - m^2 \right) \phi. \quad (4.5)
\]

Because of the \( \Omega \) dependent term in the left hand side it is usually concluded that mass spoils CI. However, as discussed very clearly by Bekenstein [47], this conclusion is a misconception deriving from a common confusion between CT and scale transformations. The latter require to transform the metric according to (4.1) and to rescale the fields by powers of \( \Omega \) and are equivalent to an enlargement of the physical system under study. On the other side, as originally discussed by Weyl [126], Dicke [48], Hoyle and Narlikar [127], CT should properly be interpreted as *local units transformations*. In this case all lengths, time intervals and masses are stretched by a factor that depends on the spacetime location. All lengths and durations are multiplied by a factor \( \Omega \) in such a way that velocities are unchanged, in particular the speed of light. In order to deduce the transformation law for the masses the key point here is to note that Compton wavelengths make no exception and must also be transformed as any other length: Bekenstein refers to them properly as “fundamental metersticks in physics”. As any other length, they must transform under a CT like (4.1) as
\[
\bar{\lambda}_c = \lambda_c \Omega. \quad (4.6)
\]
Considering that
\[
\lambda_c := \frac{\hbar}{mc}, \quad (4.7)
\]
where \( c \) is the (conformally invariant) speed of light and \( m \) the rest mass of the particle under exam, and assuming that \( \hbar \to \hbar \) under a CT, then equation (4.6) implies the following transformation law for the rest mass
\[
\bar{m} = m \Omega^{-1}. \quad (4.8)
\]
Including the transformation of rest mass, equation (4.5) now reads
\[
\left( \bar{g}^{ab} \nabla_a \nabla_b - \frac{1}{6} \bar{R} - \bar{m}^2 \right) \bar{\phi} = \Omega^{-3} \left( g^{ab} \nabla_a \nabla_b - \frac{1}{6} R - \bar{m}^2 \right) \phi, \quad (4.9)
\]
showing that the massive scalar field equation
\[
\left( g^{ab} \nabla_a \nabla_b - \frac{1}{6} R \right) \phi = m^2 \phi \quad (4.10)
\]
is indeed conformally invariant once the conformal transformation is interpreted as a local change of all units.

### 4.1.2 CI action for a massive scalar field

The relativistic action that leads to equation (4.10) is
\[
S_{\phi} = -\frac{1}{2} \int \left( \phi,_{a} \phi^{a} + \frac{1}{6} R \phi^{2} + m^2 \phi^2 \right) \sqrt{-\bar{g}} \, d^4 x, \quad (4.11)
\]
where \( g \) is the metric determinant and \( \partial_a \) indicates coordinate partial differentiation. We can verify that this is CI by considering the transformation laws

\[
g_{ab} \rightarrow \bar{g}_{ab} = \Omega^2 g_{ab}
\]

\[
\sqrt{-g} \rightarrow \sqrt{-\bar{g}} = \sqrt{-g} \Omega^4,
\]

\[
\phi \rightarrow \bar{\phi} = \phi \Omega^{-1},
\]

\[
m \rightarrow \bar{m} = m \Omega^{-1}.
\]

The action in the conformally transformed frame is

\[
\bar{S}_{\bar{\phi}} = -\frac{1}{2} \int \left( \bar{\phi}_a \overline{\phi}^a + \frac{1}{6} \bar{R} \bar{\phi}^2 + \bar{m}^2 \bar{\phi}^2 \right) \sqrt{-\bar{g}} \, d^4x. \tag{4.12}
\]

Using the above transformation laws one gets

\[
\sqrt{-\bar{g}} \bar{\phi}_a \overline{\phi}^a = \sqrt{-g} \phi_a \phi^a + \sqrt{-g} \left( \Omega^{-2} \phi^2 \Omega_{,a} g^{ab} \Omega_{,b} - 2 \Omega^{-1} \phi \phi_{,a} g^{ab} \Omega_{,b} \right), \tag{4.13}
\]

\[
\bar{m}^2 \bar{\phi}^2 \sqrt{-\bar{g}} = m^2 \phi^2 \sqrt{-g}. \tag{4.14}
\]

The scalar curvature term transforms as

\[
\sqrt{-\bar{g}} \frac{1}{6} \bar{R} \bar{\phi}^2 = \sqrt{-g} \frac{1}{6} R \phi^2 - \sqrt{-g} \phi^2 \Omega^{-1} \square_g \Omega, \tag{4.15}
\]

where \( \square_g := g^{ab} \partial_a \partial_b \). Performing an integration by parts on the second term and neglecting the vanishing boundary term gives

\[
- \int d^4x \sqrt{-g} \phi^2 \Omega^{-1} \square_g \Omega = \int d^4x \frac{1}{2} \Omega^{-1} \phi_{,a} g^{ab} \Omega_{,b}
\]

\[
= \int d^4x \sqrt{-g} \left( 2 \Omega^{-1} \phi \phi_{,a} g^{ab} \Omega_{,b} - \Omega^{-2} \phi^2 \Omega_{,a} g^{ab} \Omega_{,b} \right).
\]

This cancels the \( \Omega \) dependent terms in (4.13) so that \( \bar{S}_{\bar{\phi}} = S_{\phi} \) as required.

### 4.1.3 Conformal invariance of Standard Model fields

In fact all the physical fields of the Standard Model are conformally invariant once the transformation laws (4.6), (4.8) are adopted together with (4.1) [47, 129]. Bekenstein shows this explicitly for the actions of abelian and non-abelian gauge fields, massive vector field, Dirac field, scalar field, Higgs field, including all the relevant coupling terms. In particular, a physical field with dimensions \( L^a T^b M^c \) must transform under a CT with \( \Omega^{a+b-c} \). The fact that all the microscopic physics of the standard model is CI is remarkable. To quote Bekenstein’s own words

*The extension of units transformations to the local level (CT’s), and the requirement of CI of physical laws then parallel the promotion of gauge and internal invariances to the local level by the introduction of gauge fields.*
In this sense the conformal transformation \( g_{ab} \rightarrow \Omega^2 g_{ab} \) is like a gauge transformation. As explained in the next section, the requirement of local invariance under this kind of gauge transformation implies the existence of a new gauge field, deeply related to mass.

### 4.1.4 Bekenstein’s gauge field and GR as a natural consequence of CI

As by the usual practise we use the term *conformal frame* (CF) to indicate an arbitrary system of units, *locally defined at each point in the spacetime manifold*. If physics is to be CI, an important consequence is that the rest mass \( m_i \) of any given particle field must be itself a spacetime field. This is because, in an arbitrary CF, the rest mass of a given physical field will vary over spacetime. If one insists on the fact that there is no preferred system of units and that the laws of physics must be the same within any arbitrary CF, then there is no other choice but to admit that the rest mass corresponds to a physical field. A priori there are as many mass fields as there are different types of fundamental rest masses. However, if the ratios of all possible kind of rest masses are strictly constant, i.e. \( m_i/m_j = \text{cont} \), it follows that there must exist *only one* mass field. This must be a scalar and defines fields masses through

\[
m_i := \frac{g^c_i h}{c} \mathcal{M},
\]

where \( g^c_i \) is a dimensionless mass coupling constant for the physical field labeled by \( i \). It follows that \( \mathcal{M} \) has the dimensions of \( L^{-1} \) and must transform under a CT according to

\[
\mathcal{M} \rightarrow \bar{\mathcal{M}} = \mathcal{M} \Omega^{-1}.
\]

We will call \( \mathcal{M} \) the *Bekenstein mass field*. It plays the role of gauge field associated with CTs. As all the other gauge fields in physics, even the mass field \( \mathcal{M} \) should have a dynamics. Bekenstein identifies the proper action \( S_{\mathcal{M}} \) by requiring the following properties:

1. \( S_{\mathcal{M}} \) is covariant;
2. \( S_{\mathcal{M}} \) depends only on \( \mathcal{M} \), \( g_{ab} \) and their derivatives;
3. the dynamical equation for \( \mathcal{M} \) can contain up to second derivatives;
4. \( S_{\mathcal{M}} \) is CI.

In particular the last requirement is important and it is put forward by Bekenstein mainly for esthetic reasons, in analogy with the fact that all the other standard model fields are known to have a CI dynamics. It corresponds to the idea that *the whole of physics should be CI*. The most general action satisfying these requirements comes in the form [47]

\[
S_{\mathcal{M}} = \pm \frac{1}{2} \hbar c \int \left( \mathcal{M}_\alpha \mathcal{M}^\alpha + \frac{1}{6} R \mathcal{M}^2 + \lambda \mathcal{M}^4 \right) \sqrt{-g} d^4x,
\]

where \( \lambda \) is a dimensionless parameter, \( d^4x = c dt d^3x \) and \( \hbar c \) guarantees that \( S_{\mathcal{M}} \) has the dimensions of a relativistic action. Apart for the \( \pm \) sign, this action has the same form as that in (4.11) with the quartic self-interaction term playing the role of the mass term.

It is important to note that the requirement of CI (with the metric transformation \( g_{ab} \rightarrow \Omega^2 g_{ab} \)) automatically implies that one must consider a general curved geometry which then enters the action through the scalar curvature \( R. \) This fact has a fundamental consequence. Indeed, by the principle
of CI one can always perform a CT to a conformal frame where the mass field is represented by a constant spacetime function, i.e. $\mathcal{M} = \mathcal{M}_0$. In this CF particle masses are also constant and the action reads

$$\bar{S}_{\mathcal{M}_0} = \pm \kappa \int (\bar{R} - 2\Lambda_0) \sqrt{-\bar{g}} \, d^4x,$$

(4.19)

where we defined

$$\kappa := \frac{\hbar c M_0^2}{12},$$

(4.20)

and

$$\Lambda_0 := -3\lambda M_0^2.$$

(4.21)

The masses of the physical fields are constant and given by

$$m_i = \frac{g^i_c h}{c} \mathcal{M}_0.$$

(4.22)

Selecting the plus sign, one recognizes in (4.19) the usual Hilbert Einstein action in empty space with a bare cosmological constant term. Performing the functional derivative with respect to the inverse metric yields indeed

$$\delta \bar{S}_{\mathcal{M}_0} \delta \bar{g}^{ab} = \bar{R}_{ab} - \frac{1}{2} \bar{g}_{ab} \bar{R} + \Lambda_0 \bar{g}_{ab} = 0.$$

(4.23)

Since conventionally $\kappa = c^4/(16\pi G)$ the gravitational constant $G$ follows from (4.20) as

$$G = \frac{3c^3}{4\pi \hbar M_0^2}.$$

(4.24)

The variability of $G$ due to its dependence upon the arbitrary constant $\mathcal{M}_0$ simply reflects the variation in its numerical value in different global systems of units. To check whether there is any intrinsic variation that could lead to physically observable effects one has to look at the dimensionless gravitational strength, defined as the ratio between the gravitational and electric forces that two protons exert on each other. If

$$F_g = \frac{Gm_p^2}{d^2}, \quad F_e = \frac{1}{4\pi \varepsilon_0} \frac{e^2}{d^2},$$

(4.25)

this is given by

$$\frac{F_g}{F_e} = \frac{Gm_p^2}{e^2/4\pi \varepsilon_0},$$

(4.26)

where $m_p$ is the proton mass. On the other hand

$$\frac{e^2}{4\pi \varepsilon_0} = \alpha \hbar c,$$

(4.27)

where $\alpha \approx 1/137$ is the fine structure constant. The strength of gravity is accordingly defined as

$$\gamma := \frac{Gm_p^2}{\hbar c}.$$

(4.28)

A real, intrinsic variation of the strength of gravity would manifest itself as a variation of $\gamma$. Within
the present CI theoretical framework and in the CF introduced above with \( \mathcal{M} = \mathcal{M}_0 \), equations (4.22) and (4.24) yield

\[
\gamma = \frac{3c^3}{4\pi \hbar m_0^2} \times g_p^2 m_0^2 \left( \frac{\hbar}{c} \right)^2 \times \frac{1}{\hbar c} = \frac{3g_p^2}{4\pi}.
\]

(4.29)

Since the mass coupling constants are assumed to be constant it follows that \( \gamma \) is constant in every CF. Thus, in the CF where particle masses are constant, the action

\[
S_M = \frac{1}{2} \hbar c \int \left( \mathcal{M}_a \mathcal{M}^a + \frac{1}{6} R \mathcal{M}^2 + \lambda \mathcal{M}^4 \right) \sqrt{-g} \, d^4x,
\]

(4.30)

is manifestly equivalent to the standard Hilbert action of GR with no matter sources. However, since this action is CI, the physical predictions of the theory will be the same in every CF, in such a way that \( S_M \) should be regarded, in general, as the action for gravity. It is remarkable that the principle of CI together with that of General Covariance, according to which the laws of physics must be invariant under general coordinate transformations and local change of units, basically naturally imply that gravity be described by GR.

**Bare cosmological constant**

The term \( \lambda \mathcal{M}^4 \) plays in (4.30) the role of mass term. The mass of the mass field would then follow as

\[
\mu := \sqrt{\lambda} \hbar \frac{c}{m}.
\]

(4.31)

This formula is consistent with the general equation (4.16) holding for all the other massive fields so that \( \sqrt{\lambda} \) plays the role of a self-coupling constant for the mass field \( \mathcal{M} \). It follows that a bare cosmological constant emerges in the theory if and only if the mass field suffers a self-interaction. The simplest possibility is that with no self-interaction and \( \lambda = \Lambda_0 = \mu = 0 \). In this case the action reads

\[
S_M = \frac{1}{2} \hbar c \int \left( \mathcal{M}_a \mathcal{M}^a + \frac{1}{6} R \mathcal{M}^2 \right) \sqrt{-g} \, d^4x,
\]

(4.32)

and, as far as \( \mathcal{M} \) is concerned, it is equivalent to the action for a massless scalar field. The corresponding dynamical equation for \( \mathcal{M} \) is of course

\[
\frac{\delta S_M}{\delta \mathcal{M}} = \left( \Box - \frac{1}{6} R \right) \mathcal{M} = 0.
\]

(4.33)

**4.1.5 Gravitational equations in an arbitrary conformal frame**

In an arbitrary CF, and with \( \lambda = 0 \) the field equations for the metric follow from

\[
\frac{\delta S_M}{\delta g^{ab}} = 0.
\]

(4.34)

As we will see more in detail in the next section, as long as the coupling to matter is neglected, the action \( S_M \) is formally equivalent to that of gravity non minimally coupled to a scalar field \( \phi \).
Capozziello et al. [130] consider a general action of the form

$$S = \int \left( \frac{1}{2} \phi_{,a} \phi^{,a} + F(\phi) R - V(\phi) \right) \sqrt{-g} \, d^4 x, \quad (4.35)$$

where, for simplicity of notation we set $\hbar = c = 1$. This arises naturally within scalar-tensor theories of gravity [39] and is also closely related to the action for Brans-Dicke theories [131]. Upon functional variation with respect to $g^{ab}$, this yields the equation

$$F(\phi) G_{ab} = -\frac{1}{2} \Xi_{ab} - g_{ab} \Box F(\phi) + F(\phi) R_{ab}, \quad (4.36)$$

where $\phi_{,a}$ indicates the covariant derivative and where the tensor $\Xi_{ab}$ is defined as

$$\Xi_{ab} := \phi_{,a} \phi_{,b} - \frac{1}{2} g_{ab} \phi_{,a} \phi_{,b} + g_{ab} V(\phi). \quad (4.37)$$

The mass field action (4.32) formally corresponds to (4.35) with the choice

$$F(\phi) = \frac{\phi^2}{12}, \quad V(\phi) = 0. \quad (4.38)$$

It follows that, in an arbitrary CF, the dynamical equation for the spacetime metric corresponding to $S_M$ takes the form

$$G_{ab} = \frac{12}{M^2} \left[ F(M)_{,ab} - g_{ab} \Box F(M) - \frac{1}{2} \left( M_{a,c} M_{b,c} - \frac{1}{2} g_{ab} M_{c,d} M^{cd} \right) \right]. \quad (4.39)$$

Considering that

$$F(M)_{,ab} = \left[ \partial_a \left( \frac{M^2}{12} \right) \right]_{,b} = \frac{1}{6} \left( M M_{,ab} + M_{,a} M_{,b} \right) \quad (4.40)$$

and

$$\Box F(M) = g^{ab} F(M)_{,ab} = \frac{1}{6} \left( M \Box M + M_{,c} M^{cd} \right) \quad (4.41)$$

we obtain

$$G_{ab} = 2 \left( \frac{M_{,ab}}{M} - g_{ab} \Box M \right) - \left( \frac{4}{M^2} ( M_{a,c} M_{b,d} - g_{ab} M_{c,d} ) \right). \quad (4.42)$$

Defining now a smooth scalar field $\alpha$ through

$$\alpha = \ln \frac{M}{M_0}, \quad (4.43)$$

where $M_0$ is an arbitrary constant with dimensions of $L^{-1}$ yields

$$M = e^\alpha M_0. \quad (4.44)$$
The following relations hold

\[
\begin{align*}
\frac{\mathcal{M}_a}{\mathcal{M}} &= \alpha_{,a}, \\
\frac{\mathcal{M}_{ab}}{\mathcal{M}} &= \alpha_{,ab} + \alpha_{,a}\alpha_{,b}, \\
\frac{\Box\mathcal{M}}{\mathcal{M}} &= \Box\alpha + \alpha_{,c}\alpha^{,c}.
\end{align*}
\] (4.45) (4.46) (4.47)

The field equation expressed in terms of \(\alpha\) reads

\[
G_{ab} = 2(\alpha_{,ab} - g_{ab}\Box\alpha) - 2\left(\alpha_{,a}\alpha_{,b} + \frac{1}{2}g_{ab}\alpha_{,c}\alpha^{,c}\right).
\] (4.48)

We can re-write this as

\[
G_{ab}[g] = 2\left[\left(\ln \frac{\mathcal{M}}{\mathcal{M}_0}\right)_{,ab} - g_{ab}\Box\left(\ln \frac{\mathcal{M}}{\mathcal{M}_0}\right)\right] - 2\left[\left(\ln \frac{\mathcal{M}}{\mathcal{M}_0}\right)_{,a}\left(\ln \frac{\mathcal{M}}{\mathcal{M}_0}\right)_{,b} + \frac{1}{2}g_{ab}\left(\ln \frac{\mathcal{M}}{\mathcal{M}_0}\right)_{,c}\left(\ln \frac{\mathcal{M}}{\mathcal{M}_0}\right)^{,c}\right],
\] (4.49)

which, apart for the matter fields not being included yet, is formally equivalent to equation (2.3) with \(\mathcal{M}/\mathcal{M}_0\) playing the role of \(\Omega\).

We close this section by pointing out that the dynamical equation (4.33) for \(\mathcal{M}\) that one derives by varying the action \(S_{\mathcal{M}}\) with respect to \(\mathcal{M}\) is automatically satisfied if the Einstein’s equation is satisfied, for an arbitrary spacetime function \(\mathcal{M}\). We can easily verify this by taking the trace of equation (4.42). Since \(g^{ab}G_{ab} = -R\) we have

\[
-R\mathcal{M} = 2\Box\mathcal{M} - 8\Box\mathcal{M} - 4\frac{\mathcal{M}_c\mathcal{M}^{,c}}{\mathcal{M}} + 4\frac{\mathcal{M}_c\mathcal{M}_c}{\mathcal{M}},
\] (4.50)

which yields \((\Box - R/6)\mathcal{M} = 0\).

**From the Jordan frame to the Einstein frame**

To reveal clearly the connection to our treatment of conformal fluctuations presented in the previous chapters we need to analyze more closely the properties of the explicit CT that allows to connect an arbitrary CF to the CF with \(\mathcal{M} = \mathcal{M}_0 = \text{const}\). In this section we shall refer to the arbitrary CF where \(\mathcal{M}\) is a non-constant function of the spacetime coordinates non minimally coupled to gravity as to the Jordan frame. The CF in which \(\mathcal{M} = \mathcal{M}_0 = \text{const}\) will be referred to as the Einstein frame. We remark that, within the present framework based on the CI of physics, it is clear a priori that these two frames are equally good in providing a description of any given physical system. In particular they must lead to the same physical predictions.

Imagine to select any Jordan frame where \(\mathcal{M}\) is not constant, in such a way that the action (4.32) yields the dynamical equation (4.49) for the Jordan frame metric \(g_{ab}\). We then perform a CT to a new...
CF, i.e. we transform the metric and the mass field as

\[ g_{ab} \rightarrow \bar{g}_{ab} = \Omega^2 g_{ab}, \]
\[ \bar{M} \rightarrow \bar{\bar{M}} = M \Omega^{-1}. \]

(4.51)

(4.52)

We now require that this CT allows to pass to an Einstein frame, i.e. a frame where the mass field and all particle masses are constant. Since it was introduced arbitrarily, there is no loss of generality in requiring that, in the transformed frame, one has

\[ \bar{\bar{M}} \equiv \bar{M}_0. \]

(4.53)

Comparing to equation (4.44) we see that it must be

\[ \Omega = e^\alpha \equiv \frac{\bar{M}}{\bar{M}_0}. \]

(4.54)

The action in the transformed frame (Einstein frame) takes of course the usual Hilbert action form and yields the Einstein’s equation in empty space for the metric \( \bar{g}_{ab} \) as discussed in Section 4.1.4. In the Einstein frame particle masses are constant and the mass field doesn’t appear explicitly in the equations of motion. Thus it appears that the transformation \( \bar{g}_{ab} = e^{2\alpha} g_{ab} \) is simply that CT that connects the particular Jordan frame with metric \( g_{ab} \) and spacetime dependent mass field \( M(x) = e^{\alpha(x)} M_0 \) to the Einstein frame with metric \( \bar{g}_{ab} \) and constant mass field \( \bar{\bar{M}} = \bar{M}_0 \).

We thus see how, within this new perspective of a conformally invariant theory of gravity, conformal fluctuations would formally be equivalent to the spontaneous fluctuations in the mass field \( \bar{M} \).

From the Einstein frame to the Jordan frame

Conversely one can start from the action expressed in an arbitrary Einstein frame with metric \( \bar{g}_{ab} \) and constant mass function \( \bar{\bar{M}} = \bar{M}_0 \), namely

\[ \mathcal{S}_{\bar{g}_{0}} = \int \frac{\bar{M}_0^2}{12} \bar{R} \sqrt{-\bar{g}} \, d^4 x. \]

(4.55)

Perform now the CT

\[ \bar{g}_{ab} \rightarrow g_{ab} = \omega^2 \bar{g}_{ab}, \]
\[ \bar{M}_0 \rightarrow \bar{M} = \bar{M}_0 \omega^{-1}, \]

(4.56)

(4.57)

where \( \omega \) is an arbitrary and non-constant function of the spacetime coordinates. To find the action in the new CF (Jordan frame with variable mass function) we simply have to express the ‘old’ quantities \( (\bar{M}_0, \bar{R}, \sqrt{-\bar{g}}) \) in terms of the ‘new’ related quantities \( (\bar{M}, \bar{R}, \sqrt{-\bar{g}}) \) and substitute directly in (4.55). We have

\[ \bar{M}_0 = \omega \bar{M}, \]
\[ \sqrt{-\bar{g}} = \omega^{-4} \sqrt{-\bar{g}}. \]

(4.58)

(4.59)
The corresponding equation for $\bar{R}$ is [129]

$$\bar{R} = \omega^2 R + 6\omega \Box \omega - 12\omega_c \omega^c. \quad (4.60)$$

The action (4.55) now yields

$$\bar{S}_{\mathcal{M}_0} = \int \frac{\mathcal{M}^2}{12} \bar{R} \sqrt{-g} \, d^4x + \int \mathcal{M}^2 \left( \frac{1}{2} \frac{\Box \omega}{\omega} - \frac{\omega_c \omega^c}{\omega} \right) \sqrt{-g} \, d^4x =: S_1 + S_2, \quad (4.61)$$

where, in the second term $S_2$, indices are raised by the new inverse metric $g^{ab}$. Using equation (4.58) this term can be re-written in terms of $\mathcal{M}$ only as

$$S_2 = \int \sqrt{-g} \, d^4x \left[ \frac{1}{2} \mathcal{M}^3 \Box (\mathcal{M}^{-1}) - \mathcal{M}^4 (\mathcal{M}^{-1})_c (\mathcal{M}^{-1})^c \right]. \quad (4.62)$$

The terms containing the derivatives of the mass field yield

$$-\mathcal{M}^4 (\mathcal{M}^{-1})_c (\mathcal{M}^{-1})^c = -\mathcal{M}_c \mathcal{M}^c \quad (4.63)$$

$$\frac{1}{2} \mathcal{M}^3 \Box (\mathcal{M}^{-1}) = \mathcal{M}_c \mathcal{M}^c - \frac{1}{2} \partial_c \partial^c \mathcal{M}, \quad (4.64)$$

so that

$$S_2 = -\frac{1}{2} \int \sqrt{-g} \, d^4x \mathcal{M} \Box \mathcal{M} = \frac{1}{2} \int \sqrt{-g} \, d^4x \mathcal{M}_c \mathcal{M}^c \quad (4.65)$$

after an integration by parts. Finally, the action in the new Jordan frame takes the desired form

$$\bar{S}_{\mathcal{M}_0} = \int \left( \frac{\mathcal{M}^2}{12} \bar{R} + \frac{1}{2} \mathcal{M}_c \mathcal{M}^c \right) \sqrt{-g} \, d^4x \equiv \bar{S}_{\mathcal{M}}. \quad (4.66)$$

### 4.1.6 Inclusion of matter fields

We have shown that the action $\bar{S}_{\mathcal{M}}$ provides a CI framework for GR at the expense of introducing a mass field $\mathcal{M}$ that plays the role of a new gauge field in the theory. This come as a consequence of the fact that CT should properly be interpreted as local units transformations that one must carry out simultaneously on the metric $g_{ab}$, the mass field $\mathcal{M}$ and all the other matter fields that one wishes to include in the formalism. We stress again the fact that, since the mass field couples to particle masses, its transformation law implies that particle masses themselves transform under CT. In particular, in an arbitrary Jordan frame, particle masses will be varying from point to point in spacetime in proportion to the variations in $\mathcal{M}$. The usual Einstein frame corresponds to the particular situation in which the mass field, and thus particle masses, are constant.

Moreover we have shown how fluctuations in the mass field could equivalently be interpreted as conformal fluctuations of the metric. In this sense, if one accepts that gravity is conformally invariant and by applying the random gravity framework to study vacuum properties at the random scale, it would follow that conformal fluctuations are indeed expected to be there after all! However the question of whether these can actually affect test particles is doomed to yield a negative answer. This depends entirely on how one prescribes now the coupling of matter to gravity.

To see this let us notice that, as mentioned above, in the original work by Bekenstein it is shown that all the fields of the particle Standard Model have a CI action once one transforms their masses accordingly. In fact particle masses are directly coupled to the mass field $\mathcal{M}$ through $m_i = (g^i_c / h_c) \mathcal{M}$. 
It follows that, in the presence of matter fields, the total conformally invariant action one should consider in an arbitrary Jordan frame is

$$S_{\text{tot}}^{\text{JF}} := S_{\mathcal{M}}[g_{ab}, \mathcal{M}] + S^{(m)}[\psi, g_{ab}, \mathcal{M}], \quad (4.67)$$

where $S^{(m)}$ is the action of matter fields, described collectively by $\psi$. We stress here the important point that, in an arbitrary Jordan frame, only the massive matter fields included in $S^{(m)}$ get a dependence on the mass field $\mathcal{M}$ through their rest masses. Apart from this fact, $\psi$ couples otherwise to the metric $g_{ab}$ in the usual way.

The equation resulting from the above action follows upon functional differentiation with respect to $g_{ab}$. In particular, matter fields stress energy tensor will be defined as usual as

$$T_{ab}^{(m)}[\psi, g, \mathcal{M}] := -\frac{2}{\sqrt{-g}} \frac{\delta S^{(m)}[\psi, g_{ab}, \mathcal{M}]}{\delta g_{ab}}. \quad (4.68)$$

We notice from equations (4.36) and (4.38) that the variation of $S_{\mathcal{M}}[g_{ab}, \mathcal{M}]$ yields on the l.h.s. the Einstein tensor multiplied by $\mathcal{M}^2/12$. It follows that the conformal invariant gravitational equation in an arbitrary conformal frame take the form [129]:

$$G_{ab}[g] = 2(\alpha_{,ab} - g_{ab}\Box \alpha) - 2\left(\alpha_{,a}\alpha_{,b} + \frac{1}{2}g_{ab}\alpha_{,c}\alpha_{,c}\right) + \frac{6}{\mathcal{M}^2}T_{ab}^{(m)}[\psi, g, \mathcal{M}], \quad (4.69)$$

where

$$\alpha = \ln \frac{\mathcal{M}}{\mathcal{M}_0}. \quad (4.70)$$

We thus see that the whole formalism of the previous chapters is recovered if one inserts fluctuations in the mass field by prescribing $\alpha = 1 + A + B$. The extra term $6/\mathcal{M}^2$ that couples to $T_{ab}^{(m)}$ defines the gravitational constant and contains fluctuations that would appear as higher than second order corrections. Thus, up to second order, equation (4.69) is formally equivalent to equation (2.3). Now however $g_{ab}$ indicates the physical metric which necessarily couples to the intrinsic fluctuations in the mass field. In this sense our vacuum solution would imply that $g_{ab}$ does indeed depend upon the fluctuations in $\mathcal{M}$. Unfortunately a massive test particle would still be not affected by such fluctuations. The reason is that if it is true that the metric depends on $\alpha$, so does the particle mass, which also fluctuates. This happens in such a way that there cannot be any observable effect. This must be so exactly because the theory is conformally invariant. As shown above one can always transform away from the Jordan frame to an Einstein frame in which the new $\bar{\mathcal{M}}$ is constant. The corresponding Einstein frame action would be

$$S_{\text{tot}}^{\text{EF}} := S_{\bar{\mathcal{M}}_0}[\bar{g}_{ab}] + S^{(m)}[\bar{\psi}, \bar{g}_{ab}], \quad (4.71)$$

where rest masses are exactly constant and the dependence on the constant $\bar{\mathcal{M}}_0$ is simply absorbed in the particular *global* units one employs for the gravitational constant $G$. In this frame the field equation is manifestly Einstein’s equation with a source represented by the stress energy tensor of matter coupled to the metric $\bar{g}_{ab}$. Indeed, when $\mathcal{M} = \mathcal{M}_0 = \text{const}$, equation (4.69) above reduces to the standard Einstein’s equation with gravitational constant

$$G = \frac{3}{4\pi \mathcal{M}_0^2}, \quad (4.72)$$
in agreement with what we had defined in equation (4.24) with \( c = \hbar = 1 \). The physical prediction in this gauge is that a massive quantum particle would not suffer dephasing. By conformal invariance this must then also be the result in the general Jordan frame where the mass field is fluctuating.

Our previous result that conformal fluctuations are not effective within GR is thus confirmed. However the present deeper point view also provides hints of how the situation could change. Indeed it is clear that an action of the kind \( S = S_M[g_{ab}, \mathcal{M}] + S^{(m)}[\psi, g_{ab}] \), where matter fields do not couple directly to \( \mathcal{M} \) would indeed lead to a different physical result. Of course such a theory would exit the realm of standard GR: the above action would have to be re-interpreted as a special case of a theory of gravity coupled to an external field. In fact we will see in the next section that such an action actually corresponds to a Brans-Dicke theory with coupling parameter \( \omega = -3/2 \). Even though such a value seems to be ruled out by observations within the solar system [131], this suggests considering scalar-tensor theories of gravity more in detail and verifying what our random gravity framework can then predict in relation to the problem of dephasing of a quantum particle.

### 4.2 Scalar-Tensor Theories of gravity

Scalar tensor theories refer to alternative theories of gravity where the gravitational field is described by a rank 2 symmetric tensor \( g_{ab} \) and one spin 0 scalar field \( \phi \). The first examples of such theories were considered by Jordan [132] and by Brans and Dicke [60] in the attempt to incorporate Mach principle within GR and find a general framework which could accommodate for a varying gravitational ‘constant’ \( G \) whose value was related to the local value of the scalar field \( \phi \), in turn related to the energy-matter distribution of the universe. The original class of Brans-Dicke theories was extended in the 70s by Wagoner [39], who considered a more general action for gravity and a scalar field. Such theories go under the name of scalar-tensor. Since their original introduction, they have been constrained by many observational tests, mainly within the solar system. From the theoretical point of view it is known that gravity description must include a rank 2 symmetric tensor. However there are not valid reasons to exclude the theoretical possibility for an extra scalar field. More recently, more general theories with nonlinear terms in \( R \) included in the Einstein Hilbert action have also been considered [130], in particular in relation to their cosmological consequences. Beyond the original motivation that had inspired Brans and Dicke’s work, an extra universal scalar field could have an important role in relation to dark energy, the cosmological constant and inflation. Moreover String Theory in its low energy limit also leads to the prediction of a scalar field (dilaton) whose dynamics is governed by an action in all respect similar to Brans-Dicke’s [133, 131].

Wagoner action for a scalar-tensor theory is [128]:

\[
S_W = \int d^4x \sqrt{-g} \left[ \frac{f(\phi)}{2} R - \frac{\omega(\phi)}{2} g^{ab} \nabla_a \phi \nabla_b \phi - V(\phi) \right] + \int d^4x \sqrt{-g} \alpha_m \mathcal{L}^{(m)}[g, \psi],
\]

(4.73)

where \( \mathcal{L}^{(m)}[g, \psi] \) is the matter fields lagrangian and \( \phi \) is the Brans-Dicke scalar field. The functions \( f, \omega \) and \( V \) are arbitrary. Matter fields are only coupled to the metric \( g_{ab} \) and their stress-energy tensor is defined as usual as

\[
T_{(m)}^{ab} := \frac{-2}{\sqrt{-g}} \delta S^{(m)} \delta g^{ab}.
\]

(4.74)

It is possible to show that this is conserved in the usual sense:

\[
\nabla^a T_{(m)}^{ab} = 0,
\]
4.2. Scalar-Tensor Theories of Gravity

which also implies that point-like test particles follow the geodesic of the physical metric $g_{ab}$, so that the WEP is satisfied. Experimental constraints on $\omega$, deduced from measurements of the frequency shift of radio signals to and from the Cassini-Huygens spacecraft, suggest [134, 135]:

$$|\omega(\phi_0)| > 40000,$$

where $\phi_0$ represents the present value of the scalar field. Such a form of the action defines the theory in the so called Jordan Frame (JF) in terms of the gravitational variables $(g_{ab}, \phi)$. In the Jordan frame the scalar field is non-minimally coupled to the metric and matter fields suffer gravity through direct interaction with the metric $g_{ab}$ only. On the other hand the scalar field contributes, together with other form of matter, to set up the physical metric. Such a theory is characterized by a varying gravitational constant defined by

$$G_{\text{eff}} := \frac{1}{8\pi f(\phi)}.$$

It is well known that the theory can be equivalently formulated in such a way that the action goes over to the usual Hilbert-Einstein with a standard, minimally coupled scalar field. This can be achieved by means of the following CT [39, 125, 128]:

$$\bar{g}_{ab} = \Omega^2 g_{ab}, \quad \text{with} \quad \Omega = \sqrt{f(\phi)};$$

$$\bar{\phi} = \int \frac{d\phi}{f(\phi)} \sqrt{f(\phi) + \frac{3}{2} \left( \frac{df}{d\phi} \right)^2}$$

with allows to pass to the Einstein Frame (EF), in terms of the variables $(\bar{g}_{ab}, \bar{\phi})$. The action then takes the form:

$$S_W = \int d^4x \sqrt{-\bar{g}} \left[ \frac{\bar{R}}{12\pi} - \frac{1}{2} \bar{g}^{ab} \nabla_a \bar{\phi} \nabla_b \bar{\phi} - \bar{V}(\bar{\phi}) + (G\phi)^{-2} \mathcal{L}^{(m)}[\bar{g}, \psi] \right],$$

where

$$\bar{V} := \frac{V}{\Omega^2}.$$

In this frame the action resembles the standard HE action with a canonically coupled scalar field $\bar{\phi}$. However this theory is not GR since the matter field lagrangian is now coupled to $\bar{\phi}$ through the coupling constant $[G\phi(\phi)]^{-2}$. In particular test particles do not follow the geodesic of the metric $\bar{g}_{ab}$.

As already shown by Dicke [48] and as discussed in the previous section in relation to Bekenstein’s theory, the two frames are physically equivalent [128, 125] provided that the units of mass, length, time, and quantities derived therefrom scale with appropriate powers of the conformal factor $\Omega$ in the Einstein frame. In this sense it is still true that \textit{physics is invariant under choice of the units}. In this point of view the Einstein Frame contains a different system of units at each spacetime location and the symmetry group of classical physics is enlarged to include conformal transformations with the associated rescaling of units. These must transform according to:

$$d\bar{t} = \Omega dt, \quad d\bar{x}^i = \Omega dx^i, \quad \bar{m} = \Omega^{-1} m.$$

The lucid discussion of Flanagan [125] and Faraoni [128] make this point clear while providing many examples of calculations in the literature that employ different conformal frames and actually do yield the same physical prediction because the scaling of units under CTs was properly taken into account.
Summarizing one has: in the *Jordan frame* \( h, c, \) particle masses \( m \) and all coupling constant of physics are constant, while the gravitational ‘constant’ \( G \) varies and test particles follows metric geodesic; in the *Einstein frame* \( G, h, c \) are constant while particle masses, fields couplings and units of length, time and mass vary with spacetime location.

### 4.3 Brans-Dicke theory

The original Brans-Dicke theory [60] is a special case of scalar-tensor theory and is recovered with the choice:

\[
\begin{align*}
  f(\phi) &= 2\phi; \\
  \omega(\phi) &= \frac{2\omega}{\phi}, \quad \omega = \text{const}; \\
  V(\phi) &= 0.
\end{align*}
\]

Equation (4.73) then yields the Brans-Dicke action:

\[
S_{BD} = \int d^4x \sqrt{-g} \left[ \phi R - \frac{\omega}{\phi} g^{ab} \nabla_a \phi \nabla_b \phi \right] + \int d^4x \sqrt{-g} \alpha_m \mathcal{L}^{(m)}[g, \psi].
\] (4.77)

By applying the usual variational principle with respect to the metric this yields the dynamical equation [133]:

\[
G_{ab}[g] = \frac{1}{2\phi} T^{(m)}_{ab} + \frac{1}{\phi} \left( \nabla_a \nabla_b \phi - g_{ab} \nabla^c \nabla_c \phi \right) + \frac{\omega}{\phi^2} \left( \nabla_a \phi \nabla_b \phi - \frac{1}{2} g_{ab} \nabla^c \phi \nabla_c \phi \right)
\] (4.78)

for the spacetime metric \( g_{ab} \). By varying with respect to the scalar field yields instead:

\[
\nabla^c \nabla_c \phi = \frac{1}{4\omega + 6} T^{(m)},
\] (4.79)

where \( T^{(m)} \) denotes the trace of the matter fields stress energy tensor. In this theory the gravitational constant \( G \) is related to \( 1/\phi \) and in general varies from point to point. Its local values is determined through \( \phi \) in response to the matter-energy distribution throughout the universe, in agreement with Mach’s principle.

The Brans Dicke action can be written in an equivalent way that will be useful for us by defining [131]:

\[
\epsilon := \text{Sign}(\omega), \quad \xi := \frac{\epsilon}{4\omega} > 0, \quad \phi := \frac{1}{2} \xi \phi^2.
\]

Then we have:

\[
S_{BD} = \int d^4x \sqrt{-g} \left[ \frac{1}{2} \xi \phi^2 R - \frac{\epsilon}{2} g^{ab} \nabla_a \phi \nabla_b \phi \right] + \int d^4x \sqrt{-g} \alpha_m \mathcal{L}^{(m)}[g, \psi].
\] (4.80)

This reveals that, as long as matter fields are neglected, Bekenstein’s action (4.32) corresponds to a Brans Dicke model with \( \epsilon = -1 \) and \( \xi = 1/6 \), i.e. \( \omega = -3/2 \).

This form of the action is also useful to prove the equation for the scalar field (4.79). Starting from equation (4.78) for the metric and taking the trace we have:

\[
\Box \phi - \frac{1}{3} R \phi + \frac{\omega}{3} \frac{\nabla^c \phi \nabla_c \phi}{\phi} = \frac{T^{(m)}}{6}.
\] (4.81)
On the other hand, varying the Brans-Dicke action in its form (4.80) with respect to the field \( \varphi \) yields

\[ \epsilon \Box \varphi + \xi \varphi R = 0. \]

Multiplying by \( \varphi \) and using \( \phi = \xi \varphi^2/2 \) gives

\[ \epsilon \varphi \Box \varphi + 2 \phi R = 0. \] (4.82)

It is also

\[ \frac{\xi}{2} \Box \varphi^2 \equiv \Box \phi = \xi (\varphi \Box \varphi + \nabla^c \varphi \nabla_c \varphi) \] (4.83)

and

\[ \frac{\nabla^c \phi \nabla_c \phi}{\phi} = 2 \xi \nabla^c \varphi \nabla_c \varphi. \] (4.84)

Using (4.82) and (4.84) into (4.83) gives:

\[ \frac{\nabla^c \phi \nabla_c \phi}{\phi} = 2 \Box \phi + 4 \phi R \xi \epsilon^{-1} = 2 \Box \phi + \frac{\phi R}{\omega}. \] (4.85)

Substituting into equation (4.81) gives

\[ \Box \phi - \frac{1}{3} R \phi + \frac{\omega}{3} \left( 2 \Box \phi + \frac{\phi R}{\omega} \right) = \frac{T^{(m)}}{6}. \] (4.86)

Thus we see that the curvature terms simplify and we obtain

\[ \Box \phi = \frac{T^{(m)}}{6 + 4 \omega}, \] (4.87)

which is indeed equation (4.79).

### 4.4 Random gravity framework and Brans-Dicke theory

We now apply our random gravity framework to the problem of vacuum fluctuations within the special class of scalar-tensor theories given by the Brans-Dicke action. We work in the Jordan frame, in such a way that the scalar field is non-minimally coupled and test particles follow geodesics of the physical metric \( g_{ab} \). The relevant dynamical equation for the spacetime geometry is obtained upon variation of the Brans-Dicke action as given by Fujii [136]

\[ \delta S_{BD} = \delta \int d^4x \sqrt{-g} \left[ \frac{1}{2} \xi \varphi^2 R - \frac{\epsilon}{2} g^{ab} \nabla_a \varphi \nabla_b \varphi + \alpha_m \mathcal{L}^{(m)}[g, \psi] \right] = 0. \] (4.88)

\footnote{This is obviously so since it is already known that that action must yield the conformally invariant Klein-Gordon equation when \( \xi = 1/6 \) and \( \epsilon = -1 \).}
Apart for the constant $\epsilon = \pm 1$ this corresponds to the action we already considered in (4.35). It is then straightforward to verify that the modified Einstein’s equation results in:

$$
G_{ab}[g] = \frac{1}{\xi \phi^2} T_{ab}^{(m)}[g, \psi] + \frac{1}{\phi^2} \left( \nabla_a \nabla_b \phi^2 - g_{ab} \nabla^c \nabla_c \phi^2 \right) + \frac{\epsilon}{\xi \phi^2} \left( \nabla_a \phi \nabla_b \phi - \frac{1}{2} g_{ab} \nabla^c \phi \nabla_c \phi \right),
$$

where we recall that the field $\phi$ is related to the Brans-Dicke field $\psi$ by

$$
\phi = \frac{1}{2} \xi \phi^2
$$

and where the matter fields stress energy tensor is defined by

$$
T_{ab}^{(m)} := -\frac{2}{\sqrt{-g}} \frac{\delta S^{(m)}}{\delta g^{ab}}.
$$

It is immediate to verify that (4.89) is indeed equivalent to (4.78). We also remark that in the action (4.88) we didn’t include a potential term $V(\phi)$, which could appear in more general scalar-tensor theories having implications for the cosmological constant.

We now wish to apply the random gravity framework to equation (4.89) and study its implications at the level of the random scale $\ell = \lambda L_P$. The situation is now different from what we had considered in the previous chapters. Indeed the scalar field $\phi$ (or equivalently $\psi$) now affects the spacetime metric together with the other matter fields. It is thus expected that, in the problem of vacuum at the random scale, the physical metric will depend on the scalar field. A test particle or any physical field couple indeed to the metric but are nonetheless influenced indirectly by the scalar field since this does influence the metric. That this must be the case can also be inferred by the fact that the physical predictions in the Jordan frame we are using must be equivalent to those that one would find in the Einstein frame after a local units transformation (CT): in the Einstein frame the scalar field minimally couples to gravity and also couples directly to all matter fields; as a result of this a test particle would deviate from geodesic motion through the effect of an effective force induced by the scalar field [128].

Ad discussed in Chapter 2 we expect some effects related to the quantum fluctuations of the various fields in their vacuum state to be mimicked at the random scale by classical but randomly fluctuating fields. To see how these fluctuations manifest themselves we start by re-writing equation (4.89) in a more convenient way. Using $\nabla_a \nabla_b \phi^2 = 2(\phi \nabla_a \nabla_b \phi + \nabla_a \phi \nabla_b \phi)$ the term containing $\phi^2$ becomes:

$$
\frac{1}{\phi^2} \left( \nabla_a \nabla_b \phi^2 - g_{ab} \nabla^c \nabla_c \phi^2 \right) = 2 \left( \frac{\phi_{,ab}}{\phi} - g_{ab} \Box \phi \right) + 2 \left( \frac{\varphi_a \phi_{,b}}{\phi^2} - g_{ab} \frac{\varphi_c \phi_{,c}}{\phi^2} \right),
$$

where $\Box = \nabla^c \nabla_c$. We define now

$$
\alpha := \ln \frac{\varphi}{\varphi_0},
$$

where $\varphi_0$ is an arbitrary constant. The Brans-Dicke field $\phi$ is then represented as

$$
\phi = \phi_0 e^{2\alpha}, \quad \text{with} \quad \phi_0 := \frac{1}{2} \xi \varphi_0^2,
$$

in such a way that the field $\alpha$ can later be employed to describe the vacuum random fluctuations at
the random scale. We have
\[ \frac{\phi_a}{\phi} = \alpha_a \] and \[ \frac{\phi_{ab}}{\phi} = \alpha_{ab} + \alpha_a \alpha_{b}. \]

Then equation (4.92) becomes
\[ \frac{1}{\varphi^2} \left( \nabla_a \nabla_b \varphi^2 - g_{ab} \nabla^c \nabla_c \varphi^2 \right) = 2 \left( \alpha_{ab} - g_{ab} \Box \alpha \right) + 4 \left( \alpha_a \alpha_b - g_{ab} \alpha^c \alpha_c \right). \]

In terms of \( \alpha \) the modified Einstein’s equation now reads:
\[ G_{ab}[g] = 2 \left( \alpha_{ab} - g_{ab} \Box \alpha \right) + \frac{e^{-2\alpha}}{2\phi_0} T_{ab}^{(m)}[g, \psi] + \left( \frac{4\xi + \epsilon}{\xi} \right) \left[ \alpha_a \alpha_b - \frac{1}{2} \left( \frac{8\xi + \epsilon}{4\xi + \epsilon} \right) \alpha^c \alpha_c \right] \]
\[ \tag{4.94} \]
which, as anticipated, correctly reduces to (4.69) for \( \epsilon = -1 \) and \( \xi = 1/6 \). The constant \( \phi_0 \) is related to the value of the gravitational constant in the particular units that one wishes to employ.

### 4.4.1 Brans-Dicke field fluctuations at the random scale

Equation (4.94) provides the basis to study vacuum at the random scale. We will employ the same technique that we used in our study of conformal fluctuations within GR and we will perform an expansion scheme up to second order. The fluctuations in the Brans-Dicke scalar field will be encoded in \( \alpha \), which we assume to be a small quantity.

As a first step we consider the wave equation that holds for the Brans-Dicke field:
\[ \Box \phi = \frac{1}{4\omega + 6} T^{(m)}. \]  
\[ \tag{4.95} \]
Expanding the exponential to second order gives
\[ \phi = \phi_0 e^{2\alpha} = \phi_0 \left[ 1 + 2\alpha + 2\alpha^2 + O(\epsilon^3) \right]. \]
Equation (4.95) now reads in terms of \( \alpha \):
\[ \Box \phi_0 \left[ 1 + 2\alpha + 2\alpha^2 + O(\epsilon^3) \right] = \frac{1}{4\omega + 6} T^{(m)}, \]
which can conveniently be rearranged as
\[ \Box \alpha = \frac{1}{2\phi_0(4\omega + 6)} T^{(m)} - \Box \alpha^2, \]
and where we neglected third order corrections. Using now \( \Box \alpha^2 = 2(\alpha \Box \alpha + \alpha^c \alpha_{c,e}) \) we have
\[ (1 + 2\alpha) \Box \alpha = \frac{1}{2\phi_0(4\omega + 6)} T^{(m)} + 2\alpha^c \alpha_{c,e}. \]
Multiplying by \( (1 + 2\alpha)^{-1} \) and retaining only terms up to second order we find
\[ \Box \alpha = \frac{1}{2\phi_0(4\omega + 6)} T^{(m)} + 2\alpha^c \alpha_{c,e}, \]
\[ \tag{4.96} \]
showing that the fluctuations field $\alpha$ satisfies a wave equation with a source that depends on the trace $T^{(m)}$ of the matter fields stress energy tensor and on an auto-interaction term $2\alpha^c \alpha_c$.

It is in the spirit of the random gravity framework that the Brans-Dicke field should have spontaneous vacuum fluctuations at the random scale due to its supposed quantum nature. These do not depend on other matter fields or energy sources. Accordingly we express the fluctuations field $\alpha$ as

$$\alpha = A + B. \quad (4.97)$$

Here $|A| = O(\varepsilon)$ is a first order field satisfying the homogeneous wave equation, while $|B| = O(\varepsilon^2)$ is a nonlinear correction that depends on the sources:

$$\Box A = 0 \quad \text{and} \quad \Box B = \frac{1}{2\phi_0(4\omega + 6)} T^{(m)} + 2A^c A_c. \quad (4.98)$$

The first of these equation can be solved with Boyer random boundary conditions to yield a fluctuating field which we will denote by $\tilde{A}$. Thus the Brans-Dicke field at the random scale and up to second order is given by

$$\phi = \phi_0 e^{2\alpha} = \phi_0 \left[1 + 2\tilde{A} + 2B + 2\tilde{A}^2 + O(\varepsilon^3)\right]. \quad (4.99)$$

### 4.5 Expansion equations and vacuum solution

We now proceed in expanding the equation

$$G_{ab}[g] = 2(\alpha_{ab} - g_{ab} \Box \alpha) + \frac{e^{-2\alpha}}{2\phi_0} T^{(m)}_{ab}[g, \psi] + \left(\frac{4\xi + \epsilon}{\xi}\right) \left[\alpha_{a,b} - \frac{1}{2} \left(\frac{8\xi + \epsilon}{4\xi + \epsilon}\right) \alpha^c \alpha_c\right], \quad (4.100)$$

taking into account that $\alpha = \tilde{A} + B$. This equation establishes how the spacetime metric is affected at the random scale by the fluctuations in the Brans-Dicke field $\phi$ and in all other matter fields $\psi$. In general we expect that $g_{ab}$ will indeed depend upon $\tilde{A}$ nonlinearly. We thus write down an ansatz solution of the kind

$$g_{ab} = \eta_{ab} + h_{ab} + g_{ab}^{(2)} + O(\varepsilon^3) \quad (4.101)$$

Here the constants $k_1, k_2, k_3, k_4$ and the metric perturbations $\xi_{ab}$ and $\gamma_{ab}^{(2)}$ are to be determined by imposing (4.100). The background has been chosen equal to $\eta_{ab}$ in virtue of the discussion in Section 2.6.

#### 4.5.1 First order solution

Using the same notation of Chapter 2 we define $\Sigma_{ab}^{(1)}[\alpha] := 2(\alpha_{ab} - g_{ab} \Box \alpha)$. This will be giving rise to both first order and second order terms which we denote by $\Sigma_{ab}^{(1)}$ and $\Sigma_{ab}^{(2)}$. Then the first order expansion equation that follow from (4.100) reads:

$$G_{ab}^{(1)}[h] = \Sigma_{ab}^{(1)}[\tilde{A}], \quad (4.102)$$
where
\[ \Sigma_{ab}^{1(1)}[\tilde{A}] := 2 \left( \partial_a \partial_b \tilde{A} - g_{ab} \partial^c \partial_c \tilde{A} \right). \]

We know from equation (3.4) that
\[ 2 \partial_a \partial_b \tilde{A} - 2 \eta_{ab} \partial^c \partial_c \tilde{A} = G_{ab}^{(1)}[-2\tilde{A}]. \]
Then, using \( h_{ab} = \xi_{ab} + k_1 \tilde{A} \eta_{ab} \) we have
\[ G_{ab}^{(1)}[\xi + (k_1 + 2)\tilde{A}] = 0. \]

This can be easily satisfied by:
\[ k_1 = -2, \quad \text{and} \quad G_{ab}^{(1)}[\xi] = 0. \]

In the TT gauge we thus have a wave equation for \( \xi \). As done previously this can be solved with fluctuating boundary conditions and we have the solution
\[ \xi_{ab} = h_{ab}^{GW} \]
representing vacuum fluctuations of GWs at the random scale. To first order the spacetime physical metric is thus:
\[ g_{ab} = \eta_{ab} + \left( h_{ab}^{GW} - 2 \tilde{A} \eta_{ab} \right) + O(\epsilon^2), \]
with
\[ h_{ab} = h_{ab}^{GW} - 2 \tilde{A} \eta_{ab}. \]

Moreover the GWs and Brans-Dicke field linear fluctuations are totally uncorrelated, implying
\[ \left\langle h_{ab}^{GW}, \ldots \tilde{A}, \ldots \right\rangle = 0. \]

### 4.5.2 Second order equation

The second order equation is very similar to what we already studied in Chapter 3. Expanding the Einstein tensor and selecting the second order terms coming from \( \Sigma_{ab}^{1}[\alpha = A + B] \) gives:
\[ G_{ab}^{(1)}[g^{(2)}] = -G_{ab}^{(2)}[h] - G_{ab}^{(hh)}[h] + \Sigma_{ab}^{1(1)}[B] + \Sigma_{ab}^{1(2)}[\tilde{A}, h] + \Sigma_{ab}^{2(2)}[\tilde{A}] + \frac{1}{2 \phi_0} T_{ab}^{(m)(2)}[\psi], \]
where
\[ \Sigma_{ab}^{2(2)}[\tilde{A}] := \left( \frac{4 \xi + \epsilon}{\xi} \right) \left[ \tilde{A}_{,a} \tilde{A}_{,b} - \frac{1}{2} \left( \frac{8 \xi + \epsilon}{4 \xi + \epsilon} \right) \tilde{A}_{,c} \tilde{A}_{,c} \right] \]
and where \( T_{ab}^{(m)(2)}[\psi] \) is quadratic in the matter fields \( \psi \) and contains otherwise the Minkowski tensor; note that the factor \( \exp(-2\tilde{A}) \) does not add extra contributions to second order.

To proceed we simply have to use the first order solution \( h_{ab} = h_{ab}^{GW} - 2 \tilde{A} \eta_{ab} \) and make the various terms in the r.h.s. explicit. These are the same we already found in Chapter 3. Equations (3.26), (3.32) and (3.36) tell us immediately:
\[ -G_{ab}^{(hh)}[h^{GW} - 2 \tilde{A} \eta] = 3 h_{ab}^{GW} \partial^c \partial_c \tilde{A} - \eta_{ab} h_{cd}^{GW} \partial_c \partial_d \tilde{A}, \]
and

\[-G^{(2)}_{ab}[h^G - 2\tilde{A}\eta] = - G^{(2)}_{ab}[h^G] - \left(2\partial_a\tilde{A}\partial_b\tilde{A} + \eta_{ab}\varphi^f\tilde{A}\partial_c\tilde{A}\right)
- 4\left[\partial_a(\tilde{A}\partial_b\tilde{A}) - \eta_{ab}\varphi^f(\tilde{A}\partial_c\tilde{A})\right] + \left[\partial\tilde{A}\partial h^G\right]_{G^{(2)}},\]

(4.112)

where the first equation holds for \(h^G\) in the TT gauge and where \([\partial\tilde{A}\partial h^G]\) indicates all the cross terms deriving from \(G^{(2)}\).

The terms deriving from \(\Sigma^{(2)}_{ab}[\tilde{A}, h]\) can be found by keeping just the second order terms in the expression

\[\Sigma^{(2)}_{ab} = 2\nabla_a\partial_b\tilde{A} - 2g_{ab}\nabla^c\partial_c\tilde{A} \]

Using \(\nabla_a\partial_b\tilde{A} = \partial_a\partial_b\tilde{A} - \Gamma^c_{ab}[g]\partial_c\tilde{A}\) and the linearized connection

\[\Gamma^c_{ab} = \frac{1}{2}\eta^{cd}(\partial_d h_{bd} + \partial_b h_{ad} - \partial_d h_{ab}),\]

(4.113)

we have

\[\Sigma^{(2)}_{ab} = -\eta^{cd}(\partial_d h_{bd} + \partial_b h_{ad} - \partial_d h_{ab})\partial_c\tilde{A} + \eta_{ab}\eta^{de}\eta^{ef}(\partial_d h_{cf} + \partial_c h_{df} - \partial_f h_{dc})\partial_e\tilde{A}
- 2h_{ab}\varphi^f\partial_c\tilde{A} + 2\eta_{ab}h^{dc}\partial_d\partial_e\tilde{A},\]

(4.114)

where the last two terms arise because \(g_{ab}\nabla^c = g_{ab}\eta^{de}\nabla_d \approx (\eta_{ab} + h_{ab})(\eta^{de} - h^{de})\nabla_d\). Using \(h_{ab} = h^{GW}_{ab} - 2\tilde{A}\eta_{ab}\) to eliminate \(h_{ab}\), we get a series of terms involving cross products \(h^{GW}_{ab}\tilde{A}\) plus terms quadratic in \(\tilde{A}\). Explicitly and in an arbitrary gauge:

\[\Sigma^{(2)}_{ab}[\tilde{A}, h^{GW}] = 4\partial_a\tilde{A}\partial_b\tilde{A} + 2\eta_{ab}\varphi^f\tilde{A}\partial_c\tilde{A}
- \left[\varphi^d\tilde{A}\left(\partial_d h^{GW}_{bd} + \partial_b h^{GW}_{ad} - \partial_d h^{GW}_{ab}\right)
- \eta_{ab}\varphi^f \tilde{A}\left(\varphi^c h^{GW}_{cf} + \varphi^d h^{GW}_{df} - \partial_f h^{GW}\right)\right]
+ 2h^{GW}_{ab}\varphi^c\partial_c\tilde{A} - 2\eta_{ab}h^{GWde}\partial_d\partial_e\tilde{A}.\]

(4.115)

Specializing to the TT gauge we have

\[\Sigma^{(2)}_{ab}[\tilde{A}, h^{GW}] = 4\partial_a\tilde{A}\partial_b\tilde{A} + 2\eta_{ab}\varphi^f\tilde{A}\partial_c\tilde{A}
- \left[\varphi^d\tilde{A}\left(\partial_d h^{GW}_{bd} + \partial_b h^{GW}_{ad} - \partial_d h^{GW}_{ab}\right)
+ 2h^{GW}_{ab}\varphi^c\partial_c\tilde{A} - 2\eta_{ab}h^{GWde}\partial_d\partial_e\tilde{A}\right],\]

(4.116)

and we see that this coincides with the result (3.30) for \(S_{ab}[\tilde{A}, h^{GW}]\) in Chapter 3.

By putting together the results (4.111), (4.112) and (4.116) we get

\[\Sigma^{(2)}_{ab}[\tilde{A}, h^{GW}] - G^{(hh)}_{ab}[h^{GW}, \tilde{A}] - G^{(2)}_{ab}[h^{GW}, \tilde{A}] =
= -G^{(2)}_{ab}[h^{GW}] + \left(2\partial_a\tilde{A}\partial_b\tilde{A} + \eta_{ab}\varphi^f\tilde{A}\partial_c\tilde{A}\right) + 4\left[\eta_{ab}\varphi^f(\tilde{A}\partial_c\tilde{A}) - \partial_a(\tilde{A}\partial_b\tilde{A})\right] + \left[\text{terms } (\partial\tilde{A}\partial h^{GW})\right],\]

(4.117)
4.6. SECOND ORDER EQUATION SOLUTION

and

\[ \left[ \text{terms } (\partial \tilde{A} \partial h^{GW}) \right] = -G_{ab}^{(1)} [2\tilde{A}h^{GW}]. \] (4.119)

Collecting together all these results, the second order equation (4.109) reads:

\[ G_{ab}^{(1)} [g^{(2)}] = G_{ab}^{(1)} [2\tilde{A} \eta] - G_{ab}^{(1)} [2\tilde{A}h^{GW}] - G_{ab}^{(1)} [2B \eta] + \]
\[ + \frac{1}{2\phi_0} T_{ab}^{(m)(2)} [\psi] - G_{ab}^{(2)} [h^{GW}] + \left(6 + \frac{\epsilon}{\xi}\right) \left(\partial_a \tilde{A} \partial_b \tilde{A} - \frac{1}{2} \eta_{ab} \partial^c \tilde{A} \partial_c \tilde{A}\right) \] (4.120)

where we used \( \Sigma_{ab}^{(1)} [B] = 2 (\partial_a \partial_b B - g_{ab} \partial^c \partial_c B) \equiv G_{ab}^{(1)} [-2B \eta]. \)

4.6 Second order equation solution

To solve equation (4.120) we consider the ansatz for the second order metric perturbation:

\[ g_{ab}^{(2)} = k_2 B \eta_{ab} + k_3 \tilde{A}^2 \eta_{ab} + k_4 \tilde{A} h_{ab}^{GW} + \gamma_{ab}^{(2)}. \]

Then the second order equation can be rearranged as

\[ G_{ab}^{(1)} [(k_2 + 2)B \eta + (k_3 - 2)\tilde{A}^2 \eta + (k_4 + 2)\tilde{A} h^{GW} + \gamma^{(2)}] = \]
\[ = \frac{1}{2\phi_0} T_{ab}^{(m)(2)} [\psi] - G_{ab}^{(2)} [h^{GW}] + \left(6 + \frac{\epsilon}{\xi}\right) \left(\partial_a \tilde{A} \partial_b \tilde{A} - \frac{1}{2} \eta_{ab} \partial^c \tilde{A} \partial_c \tilde{A}\right). \] (4.121)

We see immediately that this can be satisfied with the choice

\[ k_2 = -2, \quad \text{and} \quad k_3 = 2, \quad k_4 = -2. \]

In this case the physical metric up to second order is given by:

\[ g_{ab} = \eta_{ab} + h_{ab}^{GW} - 2\tilde{A} \eta_{ab} + 2\tilde{A}^2 \eta_{ab} - 2\tilde{A} h_{ab}^{GW} + \gamma_{ab}^{(2)} - 2B \eta_{ab}, \] (4.122)

where the second order metric correction satisfies the equation:

\[ G_{ab}^{(1)} [\gamma^{(2)}] = \frac{1}{2\phi_0} T_{ab}^{\text{eff}} [\psi, h^{GW}, \tilde{A}]. \] (4.123)

The effective stress energy tensor representing the effect of vacuum at the random scale is defined by

\[ T_{ab}^{\text{eff}} [\psi, h^{GW}, \tilde{A}] := T_{ab}^{(m)(2)} [\psi] - 2\phi_0 G_{ab}^{(2)} [h^{GW}] + 2\phi_0 \left(6 + \frac{\epsilon}{\xi}\right) \left(\partial_a \tilde{A} \partial_b \tilde{A} - \frac{1}{2} \eta_{ab} \partial^c \tilde{A} \partial_c \tilde{A}\right). \] (4.124)

We see that, remarkably, the vacuum spontaneous fluctuations at the random scale of Brans-Dicke field, described by \( \tilde{A} \), contribute through the usual Klein-Gordon stress energy tensor for a massless field and with coupling constant

\[ C_{BD} := 2\phi_0 \left(6 + \frac{\epsilon}{\xi}\right) = 8\phi_0 \left(\frac{3}{2} + \omega\right). \]

This is consistent with the fact that the equation \( \partial^c \partial_c \tilde{A} = 0 \) holds.
Following the discussion of Section 2.6 we will assume that \[ \langle T^{\text{eff}}[\psi, h^{\text{GW}}, \tilde{A}] \rangle = 0 \] (4.125)

at the classical scale. This implies that the yet unknown ingredient that can achieve the vacuum energy balance and bring it down to the observed almost vanishing value is included within the collection of matter fields \( \psi \). As a result, considering that \( \gamma^{(2)} \) is not expected to have its own zero point spontaneous fluctuation, it will simply represent an extra second order induced perturbation in response of the microscopic behavior of all the sources. As noticed in Section 3.5.2 \( \gamma_{ab} \) can be found from the usual retarded Green’s function and is expected to be slow varying in comparison to \( \tilde{A}, h^{\text{GW}} \) and \( \psi \). The same kind of argument holds for the Brans-Dicke field fluctuations second order correction \( B \). From (4.98) this satisfies

\[
\Box B = \frac{1}{2\phi_0(4\omega + 6)} T^{(m)} + 2\tilde{A}^c \tilde{A}_{,c} = \frac{T^{(m)}}{C_{BD}} + 2\tilde{A}^c \tilde{A}_{,c}. \tag{4.126}
\]

The source term has again a zero average. Indeed \( \langle \tilde{A}^c \tilde{A}_{,c} \rangle = 0 \) because \( \partial^c \partial_{,c} \tilde{A} = 0 \). On the other hand we know from the discussion in Section 2.3.3 that the average of the trace of matter fields stress energy tensor is related to the effective cosmological constant. This again will be basically vanishing if the unknown ingredient for vacuum energy balance is included into \( \psi \). As a result \( B \) will also be a slow varying perturbation in comparison to \( \tilde{A}, h^{\text{GW}} \) and \( \psi \).

To indicate specific, slow varying solutions for \( \gamma_{ab} \) and \( B \) we employ the symbol \( \sim \). Thus the explicit solution for the spacetime physical metric within a Brans-Dicke model, and quite independently of the details of the model as set by \( \xi \) (or \( \omega \)), is given by:

\[
g_{ab} = \eta_{ab} - 2\tilde{A}\eta_{ab} + 2\tilde{A}^2 \eta_{ab} + h^{\text{GW}}_{ab} - 2\tilde{A}h^{\text{GW}}_{ab} + \tilde{\gamma}^{(2)}_{ab} - 2\bar{B}\eta_{ab}, \tag{4.127}
\]

where \( \partial^c \partial_{,c} \tilde{A} = 0 \) and \( \partial^c \partial_{,c} h^{\text{GW}}_{ab} = 0 \) for the GWs perturbation described by \( h^{\text{GW}}_{ab} \) in the TT gauge. Moreover, noticing that \( \exp(-2\tilde{A}) = 1 - 2\tilde{A} + 2\tilde{A}^2 + O(\varepsilon^3) \) we can re-write the result more compactly as:

\[
g_{ab} = \exp(-2\tilde{A})\eta_{ab} + h^{\text{GW}}_{ab} - 2\tilde{A}h^{\text{GW}}_{ab} + \tilde{\gamma}^{(2)}_{ab} - 2\bar{B}\eta_{ab} + O(\varepsilon^3), \tag{4.128}
\]

where it is understood the the exponential is to be considered only up to second order. The emerging physical picture is that of a spacetime which presents a conformal modulation around a Minkowski background induced by the Brans-Dicke field linear perturbation; superimposed to this there is a totally independent and uncorrelated perturbation due to GWs. Extra perturbations include a cross term due to some form of interaction between \( \tilde{A} \) and \( h^{\text{GW}} \) plus second order slow varying corrections that depend on the average behavior of the vacuum source defined in term of the matter fields \( \psi, h^{\text{GW}} \) and \( \tilde{A} \).

### 4.7 Discussion and outlook

The main important conclusion is that the random gravity framework applied to Brans-Dicke theory predicts that the microscopic structure of the spacetime metric should present a conformal modulation in response to the linear part of the fluctuations in the Brans-Dicke field \( \phi = \phi_0 \exp[2(A + B)] \), where \( |A| = O(\varepsilon) \) and \( \partial^c \partial_{,c} A = 0 \). The vacuum solution representing spacetime geometry has the structure \( g_{ab} = \exp(-2\tilde{A})\eta_{ab} + [h^{\text{GW terms}}] \). The extra terms, also including second order slow varying
corrections, depend on $h_{ab}^{GW}$, describing gravitational waves fluctuations, are uncorrelated to the terms in $A$. As far as the dephasing problem studied in Chapter 1 is concerned this is important because, in a first approximation, we can assume that the conformal term $\exp(-2A)\eta_{ab}$ and the extra terms affect a test particle independently. The prediction is thus that, beyond a possible dephasing effect due to interaction with GWs, a quantum particle should suffer extra dephasing due to the conformal modulation of the metric. We stress that, as proven in Chapter 3, this is not expected to occur within standard GR. Thus it is a precise prediction of the Brans-Dicke theory (and possibly of more general scalar-tensor theories) that dephasing of a quantum particle due to spacetime conformal fluctuations should occur.

The treatment has now gone all the way back to where we started: in Chapter 1 we proved quite in general that a quantum particle interacting with a conformally modulated metric $g_{ab} = \Omega^2 \eta_{ab}$ where $\Omega^2 = 1 + 2A + A^2$ and with $A$ satisfying the wave equation should suffer dephasing. In the case of vacuum fluctuations at the random scale this is quantified by equation (1.56). The key feature behind this result is the fact that the conformal fluctuation induce an effective nonlinear newtonian potential given in equation (1.8) as $V := (C_1A + C_2A^2) Mc^2$.

We now see how the Brans-Dicke theory can provide such a physical scenario, independently of the value of the Brans-Dicke parameter $\omega$ specifying the model. In particular the wave equation for $A$ is a direct consequence of the wave equation governing the dynamics of the Brans-Dicke field $\phi$. The fact that the conformal factor is now mathematically given by $\Omega = \exp(-A)$ rather than $1 + A$ as we considered in Chapter 1 doesn’t affect in any major way the conclusion that leads to dephasing. Indeed the explicit form of the Klein-Gordon equation for a minimally coupled massive Klein-Gordon $\phi$ still reads, like in (1.2),

$$\left(-\frac{1}{c^2} \frac{\partial^2}{\partial t^2} + \nabla^2\right) \phi = \frac{\Omega^2 M^2 c^2}{h^2} \phi - 2\partial_a (\ln \Omega) \partial^a \phi. \quad (4.129)$$

The squared conformal factor $\Omega^2 = \exp(-2A)$ still leads to a quadratic potential. We remark that the situation is now actually simper than before. Indeed the extra term containing $\ln \Omega$ now simply contributes to the effective potential through an extra linear term in $A$. The analysis in Chapter 1 showed in general that the linear part of the potential does not contribute to the dephasing at all. Following the derivation of the potential in Chapter 1 and using $\Omega = \exp(-A)$ we can now conclude that, within the Brans-Dicke framework, the effective potential that can induce dephasing is

$$V = A^2 Mc^2. \quad (4.130)$$

The maximum dephasing is thus given, as in equation (1.55), by

$$\left|\frac{\delta \rho_{xx'}}{\rho_0}\right| = \frac{1}{3\lambda^3} \left(\frac{M}{M_P}\right)^2 \left(\frac{T}{T_P}\right). \quad (4.131)$$

To conclude we observe that the structure of the effective stress energy tensor in equation (4.124), including the backreaction due to the conformal fluctuations, seems to suggest that a possible zero point energy balance mechanism could be obtained for a suitably negative and large value of the Brans-Dicke coupling parameter $\omega$, in which case the Brans-Dicke field would be equivalent to a ghost. This of course would require fine tuning, but indicates nonetheless that further research in this direction, possibly exploring scenarios involving more general scalar tensor theories than Brans-Dicke and modeling the matter fields contribution to vacuum energy more realistically, will be worth pursuing.
Appendix A

Stochastic scalar waves and generalized Wiener-Khintchine theorem

In this appendix we work out some general results used in Chapter 1 and related to scalar stochastic waves. We focus on the case of interest of a scalar, real stochastic field satisfying the wave equation in three-dimensional space. The main result is the generalization to a spacetime defined random process of the Wiener-Khintchine theorem, relating the autocorrelation function to its power spectral density $S(\omega)$. We also derive general relations allowing to estimate a variety of interest statistical quantities, such as the field mean squared amplitude, through suitable integrals involving $S(\omega)$.

A.1 General solution to the wave equation

Let $\phi$ be a scalar field defined on spacetime. Working in units with $c = 1$, the solution to the wave equation $(\nabla^2 - \partial_t^2)\phi(x, t) = 0$ can be written as

$$\phi(x, t) = \frac{1}{(2\pi)^3} \int d^3k \tilde{\phi}(k, t)e^{ik\cdot x}, \quad (A.1)$$

where

$$\tilde{\phi}(k, t) = \int d^3x \phi(x, t)e^{-ik\cdot x} \quad (A.2)$$

and where $\phi \in L^2(\mathbb{R}^3)$. The Fourier coefficients take the general form

$$\tilde{\phi}(k, t) = a(k)e^{-ikt} + b(k)e^{ikt} \quad (A.3)$$

for some complex functions $a(k)$ and $b(k)$ and where $k := |k|$. These can be obtained by Fourier transforming $\phi(x, 0)$ and $\phi_t(x, 0) := \partial_t \phi(x, 0)$. It is indeed straightforward to show that

$$a(k) = \frac{1}{2} \int d^3x \left[ \phi(x, 0) + \frac{i}{k} \phi_t(x, 0) \right] e^{-ik\cdot x}, \quad (A.4)$$

$$b(k) = \frac{1}{2} \int d^3x \left[ \phi(x, 0) - \frac{i}{k} \phi_t(x, 0) \right] e^{-ik\cdot x}. \quad (A.5)$$
The general solution can thus be written as

\[ \phi(x, t) = \frac{1}{(2\pi)^3} \int d^3k \left[ a(k)e^{i(k \cdot x - kt)} + b(k)e^{i(k \cdot x + kt)} \right]. \]  
(A.6)

### A.1.1 Real waves

It is readily shown that \( \phi \) is real, that is \( \phi = \phi^* \), if the following condition is satisfied

\[ a(k) = b^*(-k). \]  
(A.7)

It follows from equations (A.4) and (A.5) that, if \( \phi(x, 0) \) and \( \phi_t(x, 0) \) are real, then \( \phi(x, t) \) is real for any \( t \). In this case the solution takes the form

\[ \phi(x, t) = \frac{1}{(2\pi)^3} \int d^3k \left[ a(k)e^{i(k \cdot x - kt)} + a^*(-k)e^{i(k \cdot x + kt)} \right] \in \mathbb{R}. \]  
(A.8)

Taking the real part it is easy to show that

\[ \phi(x, t) \equiv \text{Re} \phi(x, t) = \frac{1}{(2\pi)^3} \int d^3k \left\{ [2 \text{ Re} a(k)] \cos(k \cdot x - kt) + [2 \text{ Im} a(-k)] \sin(k \cdot x + kt) \right\}. \]  
(A.9)

### A.1.2 Decomposition in components traveling along different directions

Considering the general solution (A.6), swapping \( k \) to \( -k \) in the second term and splitting the integration over \( d^3k \) into the solid angle part \( d\hat{k} \) in the 2 dimensional space and the magnitude part \( k \) we have

\[ \phi(x, t) = \frac{1}{(2\pi)^3} \int d\hat{k} \int_0^\infty dk k^2 \left[ a(k)e^{i(k \cdot x - kt)} + b(-k)e^{-i(k \cdot x - 0)} \right]. \]  
(A.10)

Using spherical coordinates \( \hat{k}(\vartheta, \varphi, k) \) in momentum space and employing the notation \( a_{\hat{k}}(k) := a(\vartheta, \varphi; k) \) and \( b_{-\hat{k}}(k) := b(\pi - \vartheta, \varphi + \pi; k) \), we define the directional wave along \( \hat{k}(\vartheta, \varphi) \) as

\[ \phi_{\hat{k}}(\hat{k} \cdot x/c - t) := \frac{1}{(2\pi)^3} \int_0^\infty dk k^2 \left[ a_{\hat{k}}(k)e^{i(k \cdot x/c - t)} + b_{-\hat{k}}(k)e^{-i(k \cdot x/c - t)} \right], \]  
(A.11)

where for clarity we have restored the speed of light. The general solution can thus be written according to the directional decomposition

\[ \phi(x, t) = \int d\hat{k} \phi_{\hat{k}}(\hat{k} \cdot x/c - t). \]  
(A.12)

### A.2 Stochastic waves

The scalar field satisfies the wave equation but has a stochastic nature. We are now going to extend the general results of the previous section to this interesting case. The advantage of the directional decomposition (A.12) is that the wave traveling in 3D space can be seen as the superposition of signals propagating in all possible directions. Once a given direction is chosen we are practically dealing with
a function of just one time variable $\hat{k} \cdot x/c - t$. In this section we thus consider some elementary properties of stochastic signals of one single variable $t$.

### A.2.1 Average and correlation properties of the fluctuations

Consider a generic stochastic process $\phi(t)$. The precise outcome of the variable $\phi$ at time $t$ is unpredictable. Nevertheless its properties can be well defined in a statistical sense. At time $t$ we can ideally think of an ensemble of values for $\phi(t)$, distributed according to a probability density $p_t(\phi)$ in general depending on time and such that $p_t(\phi)d\phi$ gives the probability that the actual value $\phi(t)$ at time $t$ is found to be between $\phi$ and $\phi + d\phi$. The average value of $\phi$ at time $t$ is defined as

$$\langle \phi(t) \rangle := \int p_t(\phi)\phi d\phi.$$  \hfill (A.13)

The variance is defined as

$$\langle [\phi(t)]^2 \rangle := \int p_t(\phi)\phi^2 d\phi.$$  \hfill (A.14)

Equations (A.13) and (A.14) are not enough to completely describe a stochastic field in a statistical sense. It is also important to know whether and how the value of $\phi$ at time $t' > t$ is related to the actual value $\phi$ had at time $t$. In principle the probability of having the particular outcomes $\phi(t)$ and $\phi(t')$ is governed by a joint distribution function $p_{tt'}(\phi, \phi')$. Then the quantity $p_{tt'}(\phi, \phi')d\phi d\phi'$ gives the probability for $\phi$ to have values lying between $\phi$ and $\phi + d\phi$ at time $t$ and between $\phi'$ and $\phi' + d\phi'$ at time $t'$. The first order autocorrelation function of $\phi(t)$ can then be defined as

$$R(t - t') := \langle \phi(t)\phi(t') \rangle \equiv \int p_{tt'}(\phi, \phi')\phi\phi' d\phi d\phi'.$$  \hfill (A.15)

If the probability of having the value $\phi(t')$ is completely independent from the previous outcome $\phi(t)$ then

$$p_{tt'}(\phi, \phi') = p_t(\phi)p_{t'}(\phi')$$  \hfill (A.16)

and the stochastic process described by $\phi(t)$ is said to be perfectly uncorrelated. In this case

$$\langle \phi(t)\phi(t') \rangle = \langle \phi(t) \rangle \langle \phi(t') \rangle.$$  \hfill (A.17)

Higher order autocorrelation functions can also be defined. The second order correlation function is given by

$$R''(t - t') := \langle [\phi(t)]^2[\phi(t')]^2 \rangle \equiv \int p_{tt'}(\phi, \phi')\phi^2\phi'^2 d\phi d\phi'.$$  \hfill (A.18)

Finally, if two different stochastic processes $\phi_1(t)$ and $\phi_2(t)$ are given, their correlation function is defined as

$$g(t - t') = \langle \phi_1(t)\phi_2(t') \rangle \equiv \int q_{tt'}(\phi_1, \phi_2)\phi_1\phi_2 d\phi_1 d\phi_2,$$  \hfill (A.19)

where $q_{tt'}(\phi_1, \phi_2)$ is the joint probability distribution for the two stochastic processes. The processes are perfectly uncorrelated if

$$\langle \phi_1(t)\phi_2(t') \rangle = \langle \phi_1(t) \rangle \langle \phi_2(t') \rangle.$$  \hfill (A.20)

All the mean quantities defined in this section through the underlying distribution functions are called ensemble or statistical averages.
A.2.2 Characterization of the stochastic waves

We now go back to the scalar wave equation but we consider the situation in which \( \phi \) is stochastic wave. For our purposes, the interesting statistical properties of the process are specified once the average value at different spacetime points and the autocorrelation properties are known. As explained above it is impossible to predict the value that the field will have at the location \( x \) at time \( t \). Nonetheless it is conceptually possible to consider many measurements of the values that the stochastic wave takes at nearby spacetime points. This would ideally give an a posteriori knowledge of a particular realization of the process. We indicate this with \( \phi(x, t) \), representing a concrete sample function of the underlying stochastic process. Since we are interested in the case in which this models the conformal fluctuations of vacuum spacetime, we will assume the stochastic process to be stationary, i.e. all mean properties do not depend on time. Moreover, because of Lorentz invariance, all the mean properties cannot depend on the space location either. In the non-relativistic formalism employed in Chapter 1 the space and time continua are split: in the following we assume the stochastic process underlying the conformal fluctuations to be ergodic with respect to time and 3 dimensional space. By this we mean that ensemble averages are assumed to be equal to time or space averages taken on any given sample function representing the process. The time and space averages are thought to be obtained through integrals of the kind \((\int d^3x \ldots)/L^3\) and \((\int dt \ldots)/T\), where \( T := L/c \). The scale \( L \) is small from a macroscopic point of view but still supposed to contain many wavelengths. It can be identified with the classical scale defined in chapter 1 (see table 2.1).

Our main goal is to derive a generalization of the Wiener-Khintchine theorem linking the power spectrum to the first order autocorrelation function. This will enable us to evaluate expectation values of the field at different spacetime points like \( \langle \phi(x_1, t_1)\phi(x_2, t_2) \rangle \). We proceed by steps, reviewing the standard theorem holding for a stochastic functions of three variables first, and generalizing it to the case of a stochastic wave propagating in 4D spacetime. We assume that a particular sample function representing the stochastic process can be written in analogy to (A.12) according to

\[
\phi(x, t) = \int d\mathbf{k} \phi_k(\mathbf{k} \cdot \mathbf{x}/c - t), \tag{A.21}
\]

in such a way that the wave equation is satisfied. Some care must be taken in using the Fourier expansions relations of the previous section. Indeed, for a given \( t \), \( \phi \) is not in general a square integrable function belonging to \( L^2(\mathbb{R}^3) \). To circumvent this problem, given a sample function of the process and for any given \( t \), we define

\[
\tilde{\phi}_L^t(k, t) := \int_{D_L} d^3x \phi(x, t)e^{-i\mathbf{k} \cdot \mathbf{x}}, \tag{A.22}
\]

where \( D_L \) is a 3D cubic domain of side \( L \). Then, the function

\[
\tilde{\phi}^L(x, t) := \frac{1}{(2\pi)^3} \int d^3k \tilde{\phi}_L^t(k, t)e^{i\mathbf{k} \cdot \mathbf{x}}, \tag{A.23}
\]

satisfies the wave equation provided that

\[
\tilde{\phi}_L^t(k, t) = a^L(k)e^{-ikt} + b^L(k)e^{ikt}. \tag{A.24}
\]

These expressions will be used later while taking the limit with \( L \to \infty \).
A.3 Wiener-Khintchine theorem for a stochastic process

A.3.1 Standard 3D case

Consider a complex stationary stochastic function $\phi(x)$ in 3-dimensions. Its Fourier transform over the compact domain $D_L$ of volume $L^3$ is given by

$$\tilde{\phi}^L(k) := \int_{D_L} d^3x \phi(x) e^{-ik \cdot x}. \quad (A.25)$$

The mean power spectral density over this interval is defined by

$$S^L(k) := \frac{1}{L^3} \langle \tilde{\phi}^L(k)^* \tilde{\phi}^L(k) \rangle. \quad (A.26)$$

In the limit $L \to \infty$ we have

$$S(k) := \lim_{L \to \infty} S^L(k). \quad (A.27)$$

The autocorrelation function of $\phi(x)$ for any two values $x_1$ and $x_2 = x_1 + \xi$ is given by

$$C(\xi) = \langle \phi(x_1)^* \phi(x_2) \rangle \quad (A.28)$$

satisfying

$$C(-\xi) = C(\xi)^*. \quad (A.29)$$

Consider now

$$\langle \tilde{\phi}^L(k)^* \tilde{\phi}^L(k) \rangle = \int_{D_L} d^3x_1 \int_{D_L} d^3x_2 \langle \phi(x_1)^* \phi(x_2) \rangle e^{-ik \cdot (x_2 - x_1)} \quad (A.30)$$

$$= \int_{D_L} d^3x_1 \int_{D_L} d^3\xi C(\xi)e^{-ik \cdot \xi},$$

where (A.25) and (A.28) have been used. Since $\int_{D_L} d^3x_1 = L^3$, we have

$$\langle \tilde{\phi}^L(k)^* \tilde{\phi}^L(k) \rangle = L^3 \int d^3\xi C(\xi)e^{-ik \cdot \xi}. \quad (A.30)$$

In the limit $L \to \infty$, we see that

$$S(k) = \int d^3\xi C(\xi)e^{-ik \cdot \xi}. \quad (A.31)$$

That is, the power spectral density $S(k)$ is the Fourier transform of the autocorrelation function $C(\xi)$. Conversely,

$$C(\xi) = \frac{1}{(2\pi)^3} \int d^3k S(k)e^{ik \cdot \xi}. \quad (A.32)$$

These two equations express the well known WK theorem.
### A.4 Generalized Wiener-Khintchine Theorem for stochastic waves

**Real functions**

If \( \phi(x) \) is real it follows from (A.28) that the correlation function \( C(\xi) \) is also real, and hence (A.29) becomes

\[
C(-\xi) = C(\xi). \tag{A.33}
\]

Inserting this into (A.31), we get

\[
S(k) = \int d^3\xi \, C(\xi) \cos(k \cdot \xi). \tag{A.34}
\]

This implies that the spectral density \( S(k) \) is even

\[
S(-k) = S(k). \tag{A.35}
\]

Therefore, (A.32) becomes

\[
C(\xi) = \frac{1}{(2\pi)^3} \int d^3k S(k) \cos(k \cdot \xi). \tag{A.36}
\]

**A.4 Generalized Wiener-Khintchine Theorem for stochastic waves**

Consider now a complex, ergodic time-dependent stochastic function \( \phi(x, t) \) in an 3-dimensional space with time \( t \), satisfying the wave equation \( (\partial_t^2 - \nabla^2)\phi(x, t) = 0 \). We now assume that the autocorrelation function of \( \phi(x, t) \) for any two events \( (x_1, t_1) \) and \( (x_2, t_2) = (x_1 + \xi, t_1 + \tau) \) is a function of \( \xi \) and \( \tau \) given by

\[
C(\xi, \tau) = \langle \phi(x_1, t_1)^* \phi(x_2, t_2) \rangle, \tag{A.37}
\]

and having the property

\[
C(-\xi, -\tau) = C(\xi, \tau)^*. \tag{A.38}
\]

For any fixed choice of \( x_1 \) and \( t_1 \), it follows from the definition (A.37) that \( C(\xi, \tau) \) also satisfies the wave equation, i.e.

\[
(\partial_\tau^2 - \nabla^2_\xi) C(\xi, \tau) = 0. \tag{A.39}
\]

Assuming that, for any \( \tau \), \( C(\xi, \tau) \) belongs to \( L^2(\mathbb{R}^3) \) and using equation (A.6) we have,

\[
C(\xi, \tau) = \frac{1}{(2\pi)^3} \int d^3k \left[ \alpha(k)e^{-ik\tau} + \beta(k)e^{ik\tau} \right] e^{ik\cdot\xi}. \tag{A.40}
\]

The correlation function satisfies

\[
C(-\xi, -\tau) = \frac{1}{(2\pi)^3} \int d^3k \left[ \alpha(k)e^{ik\tau} + \beta(k)e^{-ik\tau} \right] e^{-ik\cdot\xi}, \tag{A.41}
\]

\[
C(\xi, \tau)^* = \frac{1}{(2\pi)^3} \int d^3k \left[ \alpha(k)^*e^{ik\tau} + \beta(k)^*e^{-ik\tau} \right] e^{-ik\cdot\xi}. \tag{A.42}
\]

From (A.38), (A.41), (A.42) we see that

\[
\alpha(k)^* = \alpha(k), \quad \beta(k)^* = \beta(k) \tag{A.43}
\]
i.e. both $\alpha(k)$ and $\beta(k)$ are real. Considering that the process is stationary the power spectrum cannot depend on time. Setting $\tau = 0$, we have that (A.40) correctly reduces to (A.32) if

$$S(k) = \alpha(k) + \beta(k). \quad (A.44)$$

### A.4.1 Evaluation of $\alpha(k)$ and $\beta(k)$

To determine $\alpha(k)$ and $\beta(k)$ let us consider the stochastic process for some fixed time $t_0$. From equations (A.23) and (A.24) we have

$$\phi^L(x, t_0) = \frac{1}{(2\pi)^3} \int d^3k \tilde{\phi}^L(k, t_0) e^{ik \cdot x}, \quad (A.45)$$

with

$$\tilde{\phi}^L(k, t_0) = a^L(k)e^{-ikt_0} + b^L(k)e^{ikt_0}. \quad (A.46)$$

We are thus dealing with a 3 dimensional stochastic process and we can use the results of Section A.3.1. Exploiting the fact that the stochastic process is stationary we define mean power spectral density as

$$S(k) := \lim_{L,T \to \infty} \frac{1}{T} \int_0^T dt_0 \frac{1}{L^3} \left\langle \tilde{\phi}^L(k, t_0)^* \tilde{\phi}^L(k, t_0) \right\rangle. \quad (A.47)$$

Substituting equations (A.45) and (A.46) we get

$$S(k) = \lim_{L,T \to \infty} \frac{1}{T} \int_0^T dt_0 \frac{1}{L^3} \left\langle [a^L(k)^* e^{ikt_0} + b^L(k)^* e^{-ikt_0}] [a^L(k)e^{-ikt_0} + b^L(k)e^{ikt_0}] \right\rangle$$

$$= \lim_{L,T \to \infty} \frac{1}{T} \int_0^T dt_0 \frac{1}{L^3} \left\langle |a^L(k)|^2 + |b^L(k)|^2 + 2\text{Re}[a^L(k)b^L(k)^* e^{-2ikt_0}] \right\rangle$$

$$= \lim_{L \to \infty} \frac{1}{L^3} \left\langle |a^L(k)|^2 + |b^L(k)|^2 \right\rangle. \quad (A.48)$$

Comparing (A.48) with (A.44) we have

$$\alpha(k) = \lim_{L \to \infty} \frac{1}{L^3} \left\langle |a^L(k)|^2 \right\rangle \quad (A.49)$$

$$\beta(k) = \lim_{L \to \infty} \frac{1}{L^3} \left\langle |b^L(k)|^2 \right\rangle \quad (A.50)$$

and

$$S(k) = \lim_{L \to \infty} \frac{1}{L^3} \left\langle |a^L(k)|^2 + |b^L(k)|^2 \right\rangle. \quad (A.51)$$

### A.4.2 Real functions

If $\phi(x, t)$ is real we know from (A.7) that $a^L(k)^* = b^L(-k)$ and equation (A.40) becomes

$$C(\xi, \tau) = \frac{1}{(2\pi)^3} \lim_{L \to \infty} \frac{1}{L^3} \left\langle \int d^3k [a^L(k)^2 e^{-ik\tau} + |a^L(-k)|^2 e^{ik\tau}] e^{ik \cdot \xi} \right\rangle \quad (A.52)$$
and

\[ S(k) = \lim_{L \to \infty} \frac{1}{L^3} \left< |a^L(k)|^2 + |a^L(-k)|^2 \right>, \quad (A.53) \]

implying that \( S(k) \) is even, i.e. \( S(k) = S(-k) \). Moreover, from (A.52) we see that \( \alpha^*(k) = \beta(-k) \), implying that \( C(\xi, \tau) \) is real. Swapping \( k \) to \( -k \) in the second integral in (A.52) we have

\[
C(\xi, \tau) = \frac{1}{(2\pi)^3} \lim_{L \to \infty} \frac{1}{L^3} \left< \int d^3k \left| a^L(k) \right|^2 e^{i(k \cdot \xi - k\tau)} + \left| a^L(-k) \right|^2 e^{-i(k \cdot \xi - k\tau)} \right>.
\]

This reduces to (A.36) for \( \tau = 0 \) if

\[ S(k) = \lim_{L \to \infty} \frac{2}{L^3} \left< |a^L(k)|^2 \right>. \quad (A.54) \]

To summarize the main result of this section, we have found that the generalized Wiener-Khintchine theorem for a real, stationary stochastic scalar wave takes the form

\[ C(\xi, \tau) = \frac{1}{(2\pi)^3} \int d^3k S(k) \cos(k \cdot \xi - kc\tau), \quad (A.55) \]

where we have restored the speed of light and where the mean power spectrum is defined in (A.54).

### A.5 Correlation properties of wave components in different directions

The results that we have come to establish allow to show that wave components traveling in different directions are uncorrelated. We evaluate \( \langle \phi^*(x_1, t_1) \phi(x_2, t_2) \rangle \) using equation (A.21) and we have

\[
C(\xi, \tau) = \langle \phi^*(x, t) \phi(x + \xi, t + \tau) \rangle = \int d\hat{k} \int d\hat{k}' \left< \phi_{\hat{k}}^*(\hat{k} \cdot x/c - t) \phi_{\hat{k}'}([\hat{k}' \cdot x/c - t] + [\hat{k}' \cdot \xi/c - \tau] \right>. \quad (A.56)
\]

Using the Wiener-Khintchine theorem in its form as given by (A.40), restoring the speed of light, swapping \( k \) to \( -k \) in the second integral and splitting the \( d^3k \) integral in its angular and magnitude parts we can write as well

\[
C(\xi, \tau) = \frac{1}{(2\pi)^3} \int d^3k [\alpha(k)e^{-ikc\tau} + \beta(k)e^{ikc\tau}] e^{ik \cdot \xi} = \frac{1}{(2\pi)^3} \int d^3k [\alpha(k)e^{i(k \cdot \xi - k\tau)} + \beta(-k)e^{-i(k \cdot \xi - k\tau)}] = \frac{1}{(2\pi)^3} \int d\hat{k} \int_0^\infty dk k^2 [\alpha(k)e^{ikc(\hat{k} \cdot \xi/c - \tau)} + \beta(-k)e^{-ikc(\hat{k} \cdot \xi/c - \tau)}].
\]
Therefore

\[ C_\xi(\tau) = \int d\hat{k} C_\hat{k}(\hat{k} \cdot \xi/ c - \tau), \tag{A.57} \]

where we defined the correlation function in the direction \( \hat{k} \) as

\[ C_\hat{k}(\hat{k} \cdot \xi/ c - \tau) := \frac{1}{(2\pi)^3} \int_0^\infty dk k^2 \left[ \alpha_{\hat{k}}(k) e^{i k c (\hat{k} \cdot \xi / c - \tau)} + \beta_{-\hat{k}}(k) e^{-i k c (\hat{k} \cdot \xi / c - \tau)} \right], \tag{A.58} \]

with \( \alpha_{\hat{k}}(k) := \alpha(k) \) and \( \beta_{-\hat{k}}(k) := \beta(-k) \). The two equations (A.56) and (A.57) must be equivalent.

This implies at once the following equation

\[ \langle \phi_\hat{k}^*(\hat{k} \cdot x / c - t) \phi_\hat{k}'([\hat{k}' \cdot x / c - t] + [\hat{k}' \cdot \xi / c - \tau]) \rangle = \delta(\hat{k}, \hat{k}') C_\hat{k}(\hat{k} \cdot \xi / c - \tau) \tag{A.59} \]

or, equivalently, since \( \hat{k} \cdot x / c \) has the dimensions of a time,

\[ \langle \phi_\hat{k}^*(t) \phi_\hat{k}'(t + \tau) \rangle = \delta(\hat{k}, \hat{k}') C_\hat{k}(\tau) \tag{A.60} \]

We thus see that the fluctuating field can be resolved into components along different directions represented by completely uncorrelated functions of *just one time variable*.

### A.5.1 Real functions

If \( \phi \) is real equation (A.60) translates to

\[ \langle \phi_{\hat{k}}(t) \phi_{\hat{k}'}(t + \tau) \rangle = \delta(\hat{k}, \hat{k}') C_{\hat{k}}(\tau), \tag{A.61} \]

with

\[ C_{\hat{k}}(\tau) = \frac{1}{(2\pi)^3} \int_0^\infty dk k^2 S(k) \cos(k c \tau), \tag{A.62} \]

as deduced from equation (A.55).

### A.6 Isotropic power spectrum and field averages

In general wave components traveling in different directions can have different autocorrelation properties. However, in the case relevant to us and connected to vacuum conformal fluctuations, the isotropy of space implies that the spectral density is also isotropic, i.e. \( S(k) \equiv S(\hat{k}) \). In this case we can introduce a single *isotropic correlation function* defined as

\[ C(\tau) := \frac{1}{(2\pi)^3} \int_0^\infty dk k^2 S(k) \cos(k c \tau) \tag{A.63} \]

and (A.61) simply becomes

\[ \langle \phi_{\hat{k}}(t) \phi_{\hat{k}'}(t + \tau) \rangle = \delta(\hat{k}, \hat{k}') C(\tau). \tag{A.64} \]
We conclude this section by deducing a useful formula for the mean squared amplitude. Using equations (A.64) together with (A.12), this follows as

\[
\langle \phi^2 \rangle = \langle \int d\mathbf{k} \phi_k(x, t) \int d\mathbf{k}' \phi_{k'}(x, t) \rangle = C_0 \int d\mathbf{k} \int d\mathbf{k}' \delta(\mathbf{k}, \mathbf{k}').
\]  

(A.65)

Thus we have

\[
\langle \phi^2 \rangle = 4\pi C_0,
\]

(A.66)

where \( C_0 \), the peak of the autocorrelation function, follows from (A.63) as

\[
C_0 := \frac{1}{(2\pi)^3} \int_0^\infty dk k^2 S(k).
\]

(A.67)

The power spectrum appears to be the most fundamental quantity related to the stochastic waves. Indeed, once \( S(k) \) is known, the correlation properties, as well as all other sort of averages of the fluctuating field can be calculated. For instance, equations (A.66) and (A.67) can be combined to yield

\[
\langle \phi^2 \rangle = \frac{1}{(2\pi)^3} \int d^3k S(k).
\]

(A.68)

An analogue result holds more in general for a complex and nonisotropic stochastic process.

It is also useful to deduce two expressions for the averaged time and space derivatives of the field. Exploiting the fact that the stochastic process is ergodic and stationary we can write (for arbitrary complex and non isotropic signal)

\[
\langle \partial_t \phi^* \partial_t \phi \rangle := \lim_{L,T \to \infty} \frac{1}{L^3} \int d^3x \frac{1}{T} \int_0^T dt \langle \partial_t \tilde{\phi}^L(k_1, t)^* \partial_t \tilde{\phi}^L(k_2, t) \rangle e^{i\mathbf{k} \cdot (\mathbf{k}_2 - \mathbf{k}_1)},
\]

(A.69)

Using equation (A.23) the right hand side gives

\[
\text{r.h.s.} = \frac{1}{(2\pi)^6} \lim_{L,T \to \infty} \frac{1}{L^3} \frac{1}{T} \int d^3k_1 \int d^3k_2 \int \int d^3k \langle \partial_t \tilde{\phi}^L(k_1, t)^* \partial_t \tilde{\phi}^L(k_2, t) \rangle \delta(k_2 - k_1)
\]

(A.70)

where the statistical average \( \langle \rangle \) has been inserted to deal with the stochasticity in the Fourier coefficients \( \tilde{\phi}^L(k_1, t) \). Integrating over the space variable and using the properties of the \( \delta \) function this gives

\[
\text{r.h.s.} = \frac{1}{(2\pi)^3} \lim_{L,T \to \infty} \frac{1}{L^3} \frac{1}{T} \int d^3k \langle \partial_t \tilde{\phi}^L(k, t)^* \partial_t \tilde{\phi}^L(k, t) \rangle.
\]

(A.71)

Finally using equation (A.24) to express the Fourier coefficients we have

\[
\text{r.h.s.} = \frac{1}{(2\pi)^3} \lim_{L,T \to \infty} \frac{1}{L^3} \frac{1}{T} \int_0^T dt \int d^3k \langle [a^L(k)e^{ikt} + b^L(k)e^{-ikt}]^* \partial_t [a^L(k)e^{-ikt} + b^L(k)e^{ikt}] \rangle.
\]

(A.72)
Performing the derivatives and the time integration we finally get the result
\[
\langle \partial_t \phi^* \partial_t \phi \rangle = \left( \frac{2\pi}{3} \right)^3 \int d^3k \, k^2 S(k).
\]
(A.74)

A perfectly analogue calculation allows to find
\[
\langle \nabla \phi^* \cdot \nabla \phi \rangle = \left( \frac{2\pi}{3} \right)^3 \int d^3k \, k^2 S(k).
\]
(A.75)

A.7 Treatment of the term $T_4$ in the effective Schrödinger equation

In this last section we sketch a semi-qualitative argument by which we can infer the behavior of the extra term $T_4$ appearing in the effective Schrödinger equation (1.5).

This term reads
\[
T_4 = -i\hbar \left( A - \dot{A} \right) \psi - i\hbar \bar{v} (A_x - \dot{A} A_x) \psi,
\]
as derived in Section 1.2. Let us write the conformal field $A$, in the isotropic and real case, as:
\[
A \approx \int d\hat{k} \int_0^{2\pi/L_R} dk \, k^2 a(k) e^{ik(\hat{k} \cdot x - ct)},
\]
where the upper cutoff is set by the particle resolution scale $L_R$. The power spectrum is basically proportional to the square of the Fourier component $a(k)$. For $S(k) \sim 1/k$ we have $a(k) \sim k^{-1/2}$, so that the effective coefficient appearing in the expansion is $k^2 a(k) \sim k^{3/2}$. Thus the short wavelengths close to the cutoff give the most important contribution. For this reason we can approximate the field as
\[
A \approx \int d\hat{k} \int_0^{2\pi/L_R} dk \, k^{3/2} \Delta(k - k_A) e^{ik(\hat{k} \cdot x - ct)},
\]
(A.77)

where the function $\Delta(k - k_A)$ is peaked around a typical wave number $k_A$. This can in principle be selected in such a way that the average properties of (A.76) are equivalent to those of (A.77). Effectively we get:
\[
A \approx \int d\hat{k} k_A^{3/2} e^{ik_A(\hat{k} \cdot x - ct)},
\]
(A.78)

so that the conformal field is approximated as a fast varying and isotropic random signal characterized by a single typical wavelength $\lambda_A \equiv 2\pi/k_A$. In relation to the fluctuations ability to affect the particle, this will be close to the particle resolution scale, i.e. we put $\lambda_A \equiv \kappa L_R$, with $\kappa \gtrsim 1$. From (A.78) we now have:
\[
\dot{A} \approx -ik_A c A = \frac{2\pi i}{\kappa L_R} c A = -\frac{i M c^2}{\kappa \hbar} A,
\]
where we used $L_R = \hbar/Mc$. The space derivatives yield:
\[
A_{,x} \approx ik_A^{5/2} \int d\hat{k} k_x e^{ik_A(\hat{k} \cdot x - ct)} = 0.
\]

Using these two relations in (1.6) yields the result (1.7).
Appendix B

Technical derivations related to Chapter 1

This Appendix reports some technical derivations related to Chapter 1. In the first part B.1 we show that the kinetic and potential parts of the Hamiltonian give separate, additive contributions to the Dyson expansion and can be considered separately. In B.2 we report the details of the second order calculation leading to the most general expression (1.22) for the density matrix evolution. In B.3 we prove an approximate integral identity which allows to achieve an important simplification of the final result. Finally, in B.4 we briefly outline the analysis of the fourth order term in the Dyson expansion and show how this is expected to yield a vanishing contribution, at least in the case of vacuum fluctuations.

B.1 Separability of the kinetic and potential part in the Dyson expansion

The potential energy of a particle of mass $M$ due to its coupling to the conformal fluctuations is $V(x, t) = Mc^2(C_1 A(x, t) + C_2 A^2)$. Using the spectral theorem for hermitian operators the corresponding abstract operator representing the interaction part of the hamiltonian can be written as

$$
\hat{H}^1(t) = \int d^3x V(x, t) \hat{\Pi}(x),
$$

where $\hat{\Pi}(x) = |x\rangle\langle x|$ is the projection operator on the space spanned by the position operator eigenstate $|x\rangle$.

We now proceed to show that, up to second order in the Dyson’s expansion, the kinetic and conformal part of the hamiltonian give additive contributions to the average evolution of the density matrix. This fact provides an important simplification and the kinetic part is simply responsible for the free evolution while the dephasing effect solely depends upon the fluctuating part of the Hamiltonian. Since $\hat{H}(t) = H^0 + \hat{H}^1(t)$, the linear terms of equation (1.10) containing $\hat{U}_1(T)$ and $\hat{U}_1^\dagger(T)$ do not

\footnote{Indeed, using the $\delta$ function properties, it is easily verified that $\hat{H}^1(t)|x\rangle = V(x, t)|x\rangle$. This implies $\hat{H}^{1}_{xx'}(t) \equiv \langle x|\hat{H}^1(t)|x'\rangle = V(x, t)\delta(x - x')$.}

pose any problem. Turning to the second order terms we have

\[
\left\langle \hat{U}_2(T) \rho_0 \right\rangle = -\frac{1}{\hbar^2} \left\langle \int_0^T dt \int_0^t dt' \left[ \int d^3y V(y, t) \Pi(y) + \hat{H}_0 \right] \times \left[ \int d^3y' V(y', t') \Pi(y') + \hat{H}_0 \right] \rho_0 \right\rangle. \quad (B.2)
\]

This expression can be simplified using the standard properties of the projectors operators. Going to the position representation and exploiting the fact that

\[
\langle x| \left[ \int d^3y V(y, t) |y\rangle \langle y \right] = \int d^3y V(y, t) \delta(x - y) \langle y \right| = \langle x|V(x, t), \quad (B.3)
\]

the matrix elements \([\langle \hat{U}_2(T) \rho_0 \rangle]_{xx'} = \langle x| \left\langle \hat{U}_2(T) \rho_0 \right\rangle |x'\rangle\) read

\[
\left\langle \hat{U}_2(T) \rho_0 \right\rangle]_{xx'} = -\frac{1}{\hbar^2} \left\langle \int_0^T dt \int_0^t dt' \left[ \langle x|V(x, t) + \langle x|\hat{H}_0 \right] \times \left[ \int d^3y' V(y', t') \Pi(y') + \hat{H}_0 \right] \rho_0|x'\rangle \right]\]

\[
= -\frac{1}{\hbar^2} \int_0^T dt \int_0^t dt' \left\{ \langle V(x, t)V(x, t')\rangle \rho_{xx'}(0) + \langle x|\hat{H}_0^2\rho_0|x'\rangle
\]

\[
+ \langle V(x, t)|x\rangle \hat{H}_0 \rho_0|x'\rangle + \langle x| \hat{H}_0 \int d^3y' \langle y', t'| \Pi(y') \rho_0|x'\rangle \right\}, \quad (B.4)
\]

where \(\rho_{xx'}(0) = \langle x|\rho_0|x'\rangle\). Since the statistical properties of the conformal fluctuations \(A\) are stationary and spatially homogeneous and the potential \(V\) is quadratic, its average will take on a finite value, say \(\langle V(x, t)\rangle =: V_0\). As we will discuss more in details later on, the precise value depends on the spectral properties of the conformal fluctuations, as well as on the cutoff parameter that sets the boundaries of the random scale. We have

\[
\left\langle \hat{U}_2(T) \rho_0 \right\rangle]_{xx'} = -\frac{1}{\hbar^2} \int_0^T dt \int_0^t dt' \left\{ \langle V(x, t)V(x, t')\rangle \rho_{xx'}(0) + \langle x|\hat{H}_0^2\rho_0|x'\rangle
\]

\[
+ V_0\langle x|\hat{H}_0 \rho_0|x'\rangle + V_0\langle x|\hat{H}_0 \left[ \int d^3y' \Pi(y') \right] \rho_0|x'\rangle \right\}
\]

\[
= -\frac{1}{\hbar^2} \int_0^T dt \int_0^t dt' \left\{ \langle V(x, t)V(x, t')\rangle \rho_{xx'}(0) + \langle x|\hat{H}_0^2\rho_0|x'\rangle + 2V_0\langle x|\hat{H}_0 \rho_0|x'\rangle \right\}, \quad (B.5)
\]

where in the last line we used the fact that the position eigenstates \(|x\rangle\) form a complete set, as expressed by the projectors identity \(\int d^3y \Pi(y') = 1\). Performing the time integrals in the last term and going back to the abstract operators notation we finally have

\[
\left\langle \hat{U}_2(T) \rho_0 \right\rangle = [\hat{U}_2(T)]_0 \rho_0 + \left[ \hat{U}_2(T) \right]_1 \rho_0 - \frac{V_0 T^2}{\hbar^2} \hat{H}_0 \rho_0, \quad (B.6)
\]
where we defined the second order *kinetic and potential propagator* as

\[
[\hat{U}_2(T)]_0 := -\frac{1}{\hbar^2} \int_0^T dt \int_0^t dt' \hat{H}^0 \hat{H}^0 = -\frac{T^2}{2\hbar^2} (\hat{H}^0)^2, \tag{B.7}
\]

\[
[\hat{U}_2(T)]_1 := -\frac{1}{\hbar^2} \int_0^T dt \int_0^t dt' H^1(t) H^1(t'). \tag{B.8}
\]

Of course, in equation (B.6), the statistical average appears only in the term depending upon the potential propagator \([\hat{U}_2(T)]_1\).

The calculation of the term \(\langle \rho_0 \hat{U}_2^\dagger(T) \rangle\) can be carried on in the same way since \(\hat{U}_2(T) = \hat{U}_2^\dagger(T)\) and yields the final result

\[
\langle \rho_0 \hat{U}_2^\dagger(T) \rangle = \rho_0[\hat{U}_2(T)]_0 + \rho_0[\hat{U}_2(T)]_1^\dagger - \frac{V_0 T^2}{\hbar^2} \rho_0 \hat{H}^0. \tag{B.9}
\]

The last term \(\langle \hat{U}_1(T) \rho_0 \hat{U}_1^\dagger(T) \rangle\) must be considered separately

\[
\langle \hat{U}_1(T) \rho_0 \hat{U}_1^\dagger(T) \rangle = \frac{1}{\hbar^2} \left\langle \int_0^T dt \left[ \int d^3 y V(y,t) \Pi(y) + \hat{H}^0 \right] \times \rho_0 \times \int_0^T dt \left[ \int d^3 y' V(y',t') \Pi(y') + \hat{H}^0 \right] \right\rangle. \tag{B.10}
\]

Taking the matrix elements we have

\[
\left\langle \hat{U}_1(T) \rho_0 \hat{U}_1^\dagger(T) \right\rangle_{xx'} = \frac{1}{\hbar^2} \left\langle \int_0^T dt \int_0^T dt' \langle x|V(x,t) + \hat{H}^0\rangle \rho_0 \langle x'|V(x',t') + \hat{H}^0\rangle \right\rangle
\]

\[
= \frac{1}{\hbar^2} \int_0^T dt \int_0^T dt' \left\{ \langle x|V(x,t)V(x',t')\rangle \rho_{xx'}(0) + \langle x|\hat{H}^0\rho_0\hat{H}^0\rangle \right\}
\]

\[
+ \langle x|\rho_0\hat{H}^0|x'\rangle + \langle x|\hat{H}^0\rho_0|x'\rangle \langle V(x',t')\rangle. \tag{B.11}
\]

Performing the time integrals in the last term and going back to the abstract operator notation yields

\[
\langle \hat{U}_1(T) \rho_0 \hat{U}_1^\dagger(T) \rangle = [\hat{U}_1(T)]_0 \rho_0 [\hat{U}_1(T)]_0^\dagger + \langle [\hat{U}_1(T)]_0 \rho_0 [\hat{U}_1(T)]_1^\dagger \rangle + \frac{V_0 T^2}{\hbar^2} \rho_0 \hat{H}^0 + \frac{V_0 T^2}{\hbar^2} \hat{H}^0 \rho_0, \tag{B.12}
\]

where we defined the first order kinetic and potential propagator

\[
[\hat{U}_1(T)]_0 := -\frac{i}{\hbar} \int_0^T \hat{H}^0 dt' = -\frac{i}{\hbar} \hat{H}^0 T, \tag{B.13}
\]

and

\[
[\hat{U}_1(T)]_1 := -\frac{i}{\hbar} \int_0^T H^1(t') dt'. \tag{B.14}
\]

Substituting now equations (B.6), (B.9) and (B.12) into equation (1.10) we see that the cross terms
containing the operators $\rho_0 \hat{H}^0$ and $\hat{H}^0 \rho_0$ cancel out and the average evolution for the density matrix follows as

$$\rho_T = [\rho_T]_0 + [\rho_T]_1,$$

where

$$[\rho_T]_0 := \rho_0 + [\hat{U}_1(T)]_0 \rho_0 + \rho_0 [\hat{U}_1(T)]_1 + [\hat{U}_2(T)]_0 \rho_0 + [\hat{U}_1(T)]_0 \rho_0 [\hat{U}_1(T)]_1 + \rho_0 [\hat{U}_2(T)]_0,$$

$$[\rho_T]_1 := \left\langle \rho_0 + [\hat{U}_1(T)]_1 \rho_0 + \rho_0 [\hat{U}_1(T)]_1 + [\hat{U}_2(T)]_1 \rho_0 + [\hat{U}_1(T)]_1 \rho_0 [\hat{U}_1(T)]_1 + \rho_0 [\hat{U}_2(T)]_1 \right\rangle.$$

We thus see that, at least up to second order in the Dyson expansion, the kinetic and the potential parts contributions to the density matrix evolution can be calculated separately. We also remark that this result is independent of the precise form of the operator $\hat{H}^0$. In our present case though we have $\hat{H}^0 = \hat{p}^2/2M$ and equation (B.16) simply describes free evolution. Equation (B.17) on the other hand will describe dephasing.

## B.2 Nonlinear part of the potential and density matrix evolution

In this section we prove the result (1.22) for the general evolution of the density matrix. In Chapter 1 we showed that only the nonlinear part of the potential, i.e. $C_2 M c^2 A^2$, gives a non vanishing contribution, coming from the second order terms in the Dyson expansion (B.17). This means we need to evaluate explicitly the terms $\hat{U}_2(T) \rho_0$, $\hat{U}_1(T) \rho_0 \hat{U}_1^\dagger(T)$ and $\rho_0 \hat{U}_2^\dagger(T)$, where the propagators are given in (B.8) and (B.14) (for simplicity of notation here we do not indicate $[ ]_1$ for the potential propagators).

Let us start to evaluate $\langle \hat{U}_2(T) \rho_0 \rangle$ for $C_2 M c^2 A^2$. We have explicitly

$$\langle \hat{U}_2(T) \rho_0 \rangle = -\frac{C_2 M c^2 A^2}{\hbar^2} \int_0^T dt \int d^3 y'|y| \int_0^t dt' \int d^3 y'|y'| \times

\times \left\langle \left[ \int d\hat{k} f_k(t - y \cdot \hat{k}) \right]^2 \times \left[ \int d\hat{k}' f_{k'}(t' - y' \cdot \hat{k}') \right]^2 \right\rangle \rho_0.$$

(B.18)

To carry on the mean we write

$$\left[ \int d\hat{k} f_k(t - y \cdot \hat{k}) \right]^2 = \int d\hat{k} d\hat{K} f_k(t - y \cdot \hat{k}) f_{\hat{K}}(t - y \cdot \hat{K}),$$

(B.19)

in such a way that

$$\left\langle \left[ \int d\hat{k} f_k(t - y \cdot \hat{k}) \right]^2 \times \left[ \int d\hat{k}' f_{k'}(t' - y' \cdot \hat{k}') \right]^2 \right\rangle =$$

$$= \int \int \int d\hat{k} d\hat{K} d\hat{k}' d\hat{K}' \left\langle f_k(t - y \cdot \hat{k}) f_{\hat{K}}(t - y \cdot \hat{K}) f_{k'}(t' - y' \cdot \hat{k}') f_{\hat{K}'}(t' - y' \cdot \hat{K}') \right\rangle.$$

The four angular integration variables $\hat{k}, \hat{K}, \hat{k}', \hat{K}'$ are independent and care must be used in evaluating the multiple mean while applying the statistical properties listed in Section 1.4.1. While perform-
ing the integral, the four angular variables will take on all possible values and the explicit form of
the averaged integrand will depend upon these. In most of cases, for example if \( \mathbf{k} \neq (\mathbf{K}, \mathbf{k}' \text{and } \mathbf{K}') \),
the integrand simply vanishes since it involves at least one uncorrelated directional component. For
a correct evaluation, the quadruple integral must be split into a sum of various integrals where the
integrant is properly expressed in order to take into account all possible reciprocal values of angular
variables. Below we list all the possible combinations that can occur. For a more compact notation
we use \( \tau_k := t - y \cdot \mathbf{k}, \tau'_{k'} := t' - y' \cdot \mathbf{k}' \) and \( \tau_K := t - y \cdot \mathbf{K} \) and \( \tau'_{K'} := t' - y' \cdot \mathbf{K}' \). Keeping in mind
that the directional components characterized by \( \mathbf{k} \) and \( \mathbf{K} \) depend on the unprimed coordinates \( t, y \)
while those characterized by \( \mathbf{k}' \) and \( \mathbf{K}' \) depend on the primed coordinates \( t', y' \), there are three main
sets of possibilities:

**S1.** If \( \mathbf{k} = \mathbf{K} \text{ and } \mathbf{k}' = \mathbf{K}' \) then we need the second order correlation function defined in (1.18) as
we are dealing with the squares of two directional components at the same point. At the same
time, if \( \mathbf{k} \neq \mathbf{K} \text{ or } \mathbf{k}' \neq \mathbf{K}' \) then the mean yields zero thanks to the statistical properties (1.15)
and (1.17). The occurrence of all these possibilities is summarized by expressing the integrand as
\[
\delta(\mathbf{k}, \mathbf{K})\delta(\mathbf{k}', \mathbf{K}') \langle [f_k(\tau)]^2[f_{k'}(\tau')]^2 \rangle = \delta(\mathbf{k}, \mathbf{K})\delta(\mathbf{k}', \mathbf{K}') \left\{ 1 + \delta(\mathbf{k}, \mathbf{K}) \left[ R''(\tau_k - \tau_{k'}) - 1 \right] \right\}.
\]
(B.21)

**S2.** If \( \mathbf{k} = \mathbf{K}' \text{ and } \mathbf{k} = \mathbf{K} \) then we are dealing with means of products involving a given direc-
tional component at different points. In this case we need the autocorrelation function and the
integrand can be expressed with the help of (1.16) as
\[
\langle f_k(\tau)f_{k'}(\tau') \rangle \langle f_{k}(\tau)f_{k'}(\tau') \rangle = \delta(\mathbf{k}, \mathbf{K}')\delta(\mathbf{K}, \mathbf{K}') R(\tau_k - \tau_{k'}) R(\tau_K - \tau'_{K'}) \left[ 1 - \delta(\mathbf{k}, \mathbf{K}) \right].
\]
(B.22)

**S3.** If \( \mathbf{k} = \mathbf{K}' \text{ and } \mathbf{k} = \mathbf{K} \) then we have in a similar way
\[
\langle f_k(\tau)f_{k'}(\tau') \rangle \langle f_{k}(\tau)f_{k'}(\tau') \rangle = \delta(\mathbf{k}, \mathbf{K}')\delta(\mathbf{K}, \mathbf{k}') R(\tau_k - \tau_{k'}) R(\tau_K - \tau'_{K'}) \left[ 1 - \delta(\mathbf{k}, \mathbf{K}) \right].
\]
(B.23)

We remark that the cases S2 and S3 are indeed complementary to S1, since they occur when \( \mathbf{k} \neq \mathbf{K} \)
and \( \mathbf{k}' \neq \mathbf{K}' \) (in which case S1 yields zero) while giving a non vanishing contribution. We also notice
that the extra delta function factor \( [1 - \delta(\mathbf{k}, \mathbf{K})] \) in (B.22) and (B.23) has been introduced in order
to have a vanishing integrand when \( \mathbf{k} = \mathbf{K} \text{ and } \mathbf{k}' = \mathbf{K}' \). Indeed this situation involving only one
directional component must be properly expressed with the second order correlation function and, as
such, it is already included in S1.

Given these rules the multiple integral in (B.20) must be correctly split into the sum of three
where we also re-expressed explicitly the arguments of all the correlation functions and with $I_1, I_2$ and $I_3$ defining respectively the three integrals corresponding to the possibilities S1, S2 and S3. These integrals must be evaluated one by one. Since the integrand in $I_1$ doesn’t depend upon $\hat{K}$ or $\hat{K}'$ we use the fact that $\int d\hat{K} \delta(\hat{K}, \hat{K}) = 1$ and $\int d\hat{K}' \delta(\hat{K}', \hat{K}') = 1$ to get

$$I_1 = \iint d\hat{k} d\hat{k}' \left\{ 1 + \delta(\hat{k}, \hat{k}') \left[ R''(t - t' - \hat{k} \cdot \hat{y}, \hat{y}') - 1 \right] \right\}.$$  

We can now integrate with respect to $\hat{k}'$ and we get three terms. Since $\int d\hat{k}' = 4\pi$ we find

$$I_1 = \int d\hat{k} \left\{ 4\pi + R''[t - t' - \hat{k} \cdot (\hat{y} - \hat{y}')] - 1 \right\}. \quad (B.24)$$

To evaluate $I_2$ we start by integrating over $\hat{k}'$ and $\hat{K}'$ to obtain

$$I_2 = \iint d\hat{k} d\hat{K} R[t - t' - \hat{k} \cdot (\hat{y} - \hat{y}')] \times R[t - t' - \hat{K} \cdot (\hat{y} - \hat{y}')] \left[ 1 - \delta(\hat{k}, \hat{K}) \right].$$

Carrying on the integration with respect to $\hat{K}$ this yields two contributions

$$I_2 = \int d\hat{k} \left\{ \iint d\hat{K} R[t - t' - \hat{k} \cdot (\hat{y} - \hat{y}')] \times R[t - t' - \hat{K} \cdot (\hat{y} - \hat{y}')] - R^2[t - t' - \hat{k} \cdot (\hat{y} - \hat{y}')] \right\}.$$

The integral $I_3$ gives exactly the same result and we can write

$$I_2 + I_3 =
2\int d\hat{k} \left\{ \iint d\hat{K} R[t - t' - \hat{k} \cdot (\hat{y} - \hat{y}')] \times R[t - t' - \hat{K} \cdot (\hat{y} - \hat{y}')] - R^2[t - t' - \hat{k} \cdot (\hat{y} - \hat{y}')] \right\}. \quad (B.25)$$

To further simplify equation (B.18) and perform the space integration with respect to $y$ and $y'$, it is useful to go to the position representation and evaluate $\langle \hat{U}_2(T) \rho_0 \rangle_{xx'} := \langle x | \hat{U}_2(T) \rho_0 | x' \rangle$. The
density matrix $\rho_0$ operates on the ket $|x'\rangle$ while the projection operators act from the right on the bra $\langle x|$ producing delta functions. By investigating the structure of the right hand side in (B.18) it is easily seen that the effect of integrating with respect to $y$ first and $y'$ next is to transform all the $y$ and $y'$ comparing in the arguments of the correlation functions into $x$, in such a way that all the factors containing the difference $y - y'$ disappear. Therefore the following results hold

$$\langle x| \left[ \int d^3y |y\rangle \int d^3y' |y'\rangle \langle y'| I_1 \right] = \langle x| \int d\hat{k} \left\{ 4\pi - 1 + R''(t - t') \right\},$$

and

$$\langle x| \left[ \int d^3y |y\rangle \int d^3y' |y'\rangle \langle y'| (I_2 + I_3) \right] = \langle x| \int d\hat{k} 2(4\pi - 1)R^2(t - t').$$

We can finally plug these back into equation (B.18) which gives, in the position representation

$$\langle \hat{U}_2(T) \rho_0 \rangle_{xx'} = -\frac{C^2_2 M^2 c^4 A^4_0 \rho_{xx'}(0)}{\hbar^2} \int_0^T dt \int_0^t dt' \int d\hat{k} \left\{ 4\pi - 1 + R''(t - t') + 2(4\pi - 1)R^2(t - t') \right\},$$

(B.26)

where $\rho_{xx'}(0) := \langle x|\rho_0|x'\rangle$. Since $\hat{U}_2(T) = \hat{U}_2(T)\dagger$, the evaluation of the second order term $\langle \rho_0 \hat{U}_2(T)\dagger \rangle_{xx'}$ would proceed in exactly the same way to yield

$$\langle \rho_0 \hat{U}_2(T)\dagger \rangle_{xx'} = \langle \hat{U}_2(T) \rho_0 \rangle_{xx'} \quad \text{(B.27)}$$

Finally we must consider the remaining second order term $\langle \hat{U}_1(T) \rho_0 \hat{U}_1(T)\dagger \rangle_{xx'}$. This reads explicitly

$$\langle \hat{U}_1(T) \rho_0 \hat{U}_1(T)\dagger \rangle = \frac{C^2_2 M^2 c^4 A^4_0}{\hbar^2} \left\langle \int_0^T dt \int_0^T dt' \int d^3y |y\rangle \int d\hat{k} f_k(t - y \cdot \hat{k}) \right\rceil^2 \times \rho_0 \times \left\lfloor \int d^3y' |y'\rangle \int d\hat{k}' f_{k'}(t' - y' \cdot \hat{k}') \right\rfloor^2 \right)$$

and it should be noticed that the both time integrals run from 0 to $T$. The main difference when compared to (B.18) is that the density matrix is now ‘squeezed’ in between space and directional integrals defining the potential. For this reason it is convenient to go to the position representation and carry out the integrations with respect to $y$ and $y'$ and evaluate the statistical average afterwards. This gives

$$\langle \hat{U}_1(T) \rho_0 \hat{U}_1(T)\dagger \rangle_{xx'} = \frac{C^2_2 M^2 c^4 A^4_0 \rho_{xx'}(0)}{\hbar^2} \left\langle \int_0^T dt \int_0^T dt' \left\langle \left[ \int d\hat{k} f_k(t - x \cdot \hat{k}) \right]^2 \times \left[ \int d\hat{k}' f_{k'}(t' - x' \cdot \hat{k}') \right]^2 \right. \right.$$}

The statistical average has the same structure as in (B.20) and the result can be read off equations (B.24) and (B.25) by simply replacing $y$ and $y'$ with $x$ and $x'$. By (B.28) defining

$$\Delta x := x - x'$$
we have
\[
\left\langle \hat{U}_1(T) \rho_0 \hat{U}_1(T)^\dagger \right\rangle_{xx'} = \frac{C_2^2 M^2 c^4 A_0^4 \rho_{xx'}(0)}{\hbar^2} \int_0^T \int_0^T d\hat{k} \int_0^T d\hat{k}' \left\{ 4\pi - 1 + R''(t - t' - \hat{k} \cdot \Delta x) \right\}
\]
\[
+ 2 \int d\hat{K} R(t - t' - \hat{k} \cdot \Delta x) \times R(t - t' - \hat{K} \cdot \Delta x) - 2 R^2(t - t' - \hat{k} \cdot \Delta x) \right\}.
\]

(B.29)

Bringing together the result (B.26), (B.27) and (B.29) into the genera Dyson expansion expression, we get a formula for the evolved density matrix at time \(t\):
\[
\rho_{xx'}(T) = \rho_{xx'}(0) - \frac{C_2^2 M^2 c^4 A_0^4 \rho_{xx'}(0)}{\hbar^2} \int d\hat{k}
\]
\[
\left\{ 2 \int_0^T dt \int_0^t dt' [(4\pi - 1) + R''(t - t') - 2 R^2(t - t') + 8\pi R^2(t - t')] \right. \]
\[
- \left. \int_0^T dt \int_0^T dt' [(4\pi - 1) + R''(t - t' - \hat{k} \cdot \Delta x) - 2 R^2(t - t' - \hat{k} \cdot \Delta x)] \right. \]
\[
+ 2 \int d\hat{K} R(t - t' - \hat{k} \cdot \Delta x) \times R(t - t' - \hat{K} \cdot \Delta x) \right\}.
\]

(B.30)

This seemingly complicated expression, involving multiple time integrals and directional integrals of the first and second order correlation functions can still be simplified significantly using the result proven in Section B.3:
\[
2 \int_0^T dt \int_0^t dt' f(t - t') - \int_0^T dt \int_0^T dt' f(t - t' - \hat{k} \cdot \Delta x) = 0,
\]

(B.31)

holding for an even function \(f\) and for \(T \gg \hat{k} \cdot \Delta x\).

Since the first and second order correlation functions are even functions of their argument we see that the first three time integrals in (B.30) involving the quantities \((4\pi - 1)\), \(R''(t - t')\) and \(-2 R^2(t - t')\) are precisely in the form (B.31). Therefore their contribution vanishes and the general formula for the density matrix evolution finally reduces to
\[
\rho_{xx'}(T) = \rho_{xx'}(0) - \frac{2 C_2^2 M^2 c^4 A_0^4 \rho_{xx'}(0)}{\hbar^2} \int d\hat{k} \int_0^T dt \int_0^T dt' \left\{ 4\pi R^2(t - t') - \int d\hat{K} R(t - t' - \hat{k} \cdot \Delta x/c) \times R(t - t' - \hat{K} \cdot \Delta x/c) \right\}.
\]

(B.32)
This can be written in a slightly more compact form as
\[
\rho_{xx}(T) = \rho_{xx}(0) - \frac{32C_s^2 \pi^2 M^2 c^4 A^4}{h^2} \times \left[ \frac{T}{0} dt \int_0^T dt' R^2(t-t') - \frac{1}{16\pi^2} \int d\hat{k} \int d\hat{K} \int_0^T dt \int_0^T dt' R(t-t' - \hat{k} \cdot \Delta x/c) \times R(t-t' - \hat{K} \cdot \Delta x/c) \right], \tag{B.33}
\]
which is precisely the expression (1.22).

\section*{B.3 An integral identity}

In this appendix we prove that the result
\[
I := \int_0^T dt \left[ 2 \int_0^t dt' f(t-t') - \int_0^T dt' f(t-t' - \hat{k} \cdot \Delta x) \right] = 0,
\]
used in Section 1.5.2, holds for an arbitrary even function $f$ and in the limit $\hat{k} \cdot \Delta x / T \to 0$.

For simplicity let $\Delta := \hat{k} \cdot \Delta x$. Defining the variable $\tau := t - t'$ we have
\[
\int_0^t dt' f(t-t') = \int_0^t d\tau f(\tau),
\]
while, with $\tau := t - t' - \Delta$,
\[
\int_0^T dt' f(t-t' - \Delta) = \int_{t-\Delta-T}^{t-\Delta} d\tau f(\tau) = \int_0^{t-\Delta} d\tau f(\tau) - \int_{t-\Delta}^{t-\Delta-T} d\tau f(\tau).
\]
Introducing the primitive of $f(t)$
\[
F(t) := \int_0^t d\tau f(\tau)
\]
we can re-write $I$ as
\[
I = 2 \int_0^T dt F(t) - \int_0^T dt F(t - \Delta) + \int_0^T dt F(t - \Delta - T).
\]
Performing a further change of variable $z := t - \Delta$ the second integral reads
\[
\int_0^T dt F(t - \Delta) = \int_{-\Delta}^{T-\Delta} dz F(z) := \int_0^T dz F(z) - \int_{-\Delta}^0 dz F(z).
\]
Performing a similar operation on the third integral we obtain
\[
I = 2 \int_0^T dt F(t) - \int_0^T dz F(z) + \int_0^{T-\Delta} dz F(z) + \int_0^{T-\Delta} dz F(z) - \int_{-\Delta}^{T-\Delta} dz F(z).
\]
B.4 Fourth order term in the Dyson expansion

Now we use the information $T \gg \Delta$ and $f(t) = f(-t)$. As an elementary consequence we have that $F(t) = -F(-t)$, implying that

$$\int_0^\chi d\tau F(\tau) = \int_0^{-\chi} d\tau F(\tau).$$

Approximating $T \pm \Delta \approx T$ and swapping the sign of the upper integration bound appropriately we have

$$I = 2 \int_0^T dt F(t) - \left[ \int_0^T dz F(z) - \int_0^{\Delta} dz F(z) \right] - \left[ \int_0^T dz F(z) - \int_0^{\Delta} dz F(z) \right].$$

The integrals from 0 to $\Delta$ can all be neglected exploiting again the fact that $T \gg \Delta$ and we obtain

$$I \approx 2 \int_0^T dt F(t) - 2 \int_0^T dt F(t) = 0.$$

The result is exact in the limit $\Delta/T \to 0$.

### B.4 Fourth order term in the Dyson expansion

The fourth order propagator in the Dyson expansion (1.10) would be given by

$$\hat{U}_4(T) := \left( \frac{-i}{\hbar} \right)^4 \int_0^T dt^{(1)} \int_0^{t^{(1)}} dt^{(2)} \int_0^{t^{(2)}} dt^{(3)} \int_0^{t^{(3)}} dt^{(4)} \hat{H}(t^{(1)}) \hat{H}(t^{(2)}) \hat{H}(t^{(3)}) \hat{H}(t^{(4)}),$$

implying a fourth order term in the expression for the density matrix evolution given by

$$\langle \hat{U}_4 \rho_0 \rangle \sim \int_0^T dt^{(1)} \int_0^{t^{(1)}} dt^{(2)} \int_0^{t^{(2)}} dt^{(3)} \int_0^{t^{(3)}} dt^{(4)} \int d^3y^{(1)}|y^{(1)}\rangle \langle y^{(1)}| \cdots \int d^3y^{(4)}|y^{(4)}\rangle \langle y^{(4)}| \times \langle V(y^{(1)}, t^{(1)})V(y^{(2)}, t^{(2)})V(y^{(3)}, t^{(3)})V(y^{(4)}, t^{(4)}) \rangle \rho_0.$$

Since the potential is $V = Mc^2(C_1 A + C_2 A^2)$ the average yields one term proportional to $A^4_0$:

$$\langle V(y^{(1)}, t^{(1)})V(y^{(2)}, t^{(2)})V(y^{(3)}, t^{(3)})V(y^{(4)}, t^{(4)}) \rangle \Rightarrow \left( \frac{Mc^2 A_0}{\hbar} \right)^4 \int d\vec{k}^{(1)} \cdots \int d\vec{k}^{(4)} \langle f_{\vec{k}^{(1)}}(\tau^{(1)}) f_{\vec{k}^{(2)}}(\tau^{(2)}) f_{\vec{k}^{(3)}}(\tau^{(3)}) f_{\vec{k}^{(4)}}(\tau^{(4)}) \rangle,$$

where $\tau^{(i)} := t^{(i)} - \mathbf{y}^{(i)} \cdot \hat{\mathbf{v}}^{(i)}$. This requires knowledge of the four-point correlation function, involving the average of the product of four directional components evaluated at different points. For a real random process having a zero mean and gaussian distribution the four-points function reduces to [137, 138]:

$$\langle f_{\vec{k}^{(1)}}(\tau^{(1)}) f_{\vec{k}^{(2)}}(\tau^{(2)}) f_{\vec{k}^{(3)}}(\tau^{(3)}) f_{\vec{k}^{(4)}}(\tau^{(4)}) \rangle = \langle f_{\vec{k}^{(1)}}(\tau^{(1)}) f_{\vec{k}^{(2)}}(\tau^{(2)}) \rangle \langle f_{\vec{k}^{(3)}}(\tau^{(3)}) f_{\vec{k}^{(4)}}(\tau^{(4)}) \rangle + \langle f_{\vec{k}^{(1)}}(\tau^{(1)}) f_{\vec{k}^{(3)}}(\tau^{(3)}) \rangle \langle f_{\vec{k}^{(2)}}(\tau^{(2)}) f_{\vec{k}^{(4)}}(\tau^{(4)}) \rangle + \langle f_{\vec{k}^{(1)}}(\tau^{(1)}) f_{\vec{k}^{(4)}}(\tau^{(4)}) \rangle \langle f_{\vec{k}^{(2)}}(\tau^{(2)}) f_{\vec{k}^{(3)}}(\tau^{(3)}) \rangle.$$
We can now use equation (1.16) to express the 2-point correlations:

\[
\langle f_{\hat{k}(1)}(\tau^{(1)})f_{\hat{k}(2)}(\tau^{(2)})f_{\hat{k}(3)}(\tau^{(3)})f_{\hat{k}(4)}(\tau^{(4)}) \rangle = \\
\delta(\hat{k}^{(1)},\hat{k}^{(2)})R(\tau^{(1)} - \tau^{(2)})\delta(\hat{k}^{(3)},\hat{k}^{(4)})R(\tau^{(3)} - \tau^{(4)}) + \\
\delta(\hat{k}^{(1)},\hat{k}^{(3)})R(\tau^{(1)} - \tau^{(3)})\delta(\hat{k}^{(2)},\hat{k}^{(4)})R(\tau^{(2)} - \tau^{(4)}) + \\
\delta(\hat{k}^{(1)},\hat{k}^{(4)})R(\tau^{(1)} - \tau^{(4)})\delta(\hat{k}^{(2)},\hat{k}^{(3)})R(\tau^{(2)} - \tau^{(3)}).
\]

This implies that the term \( T_4[A^4_0] \) deriving from \( \langle \hat{U}_4 \rho_0 \rangle \) has the structure:

\[
T_4[A^4_0] \sim 3 \left[ \int d\hat{k}^{(1)} \int d\hat{k}^{(2)} \int_0^T dt^{(1)} \int_0^T dt^{(2)} \delta(\hat{k}^{(1)},\hat{k}^{(2)})R(\tau^{(1)} - \tau^{(2)}) \right]^2,
\]

where all upper bounds in the time integration can be set equal to \( T \) by appropriate normal ordering [69]. By carrying out one of the two angular integrations we have:

\[
T_4[A^4_0] \sim 3 \left[ \int d\hat{k}^{(1)} \int_0^T dt^{(1)} \int_0^T dt^{(2)} R(\hat{k}^{(1)} - \hat{k}^{(2)})(\hat{y}^{(1)} - \hat{y}^{(2)}) \right]^2.
\]

The double time integral can be simplified using the general result (1.28) and we have

\[
T_4[A^4_0] \sim 3 \left[ \int d\hat{k}^{(1)} \tilde{\mathcal{F}} \left[ R(t + \tau) \right](0)T \right]^2,
\]

where \( \tilde{\mathcal{F}} \) denotes Fourier transform and \( \tau := -\hat{k}^{(1)} \cdot (\hat{y}^{(1)} - \hat{y}^{(2)}) \). The Fourier transform can be evaluated using that \( R(t + \tau) = C(t + \tau)/C_0 \). Then

\[
\tilde{\mathcal{F}} \left[ R(t + \tau) \right](\omega) = \frac{1}{C_0} \int_{-\infty}^{\infty} dtC(t + \tau)e^{-i\omega t}
\]

\[
= \frac{1}{C_0(2\pi)^{3}} \int_{0}^{\omega_c} d\omega' \int_{-\infty}^{\infty} dt \omega'^2 S(\omega')e^{-i\omega t} \cos \omega'(t + \tau)
\]

\[
= \frac{1}{2C_0(2\pi)^{3}} \int_{0}^{\omega_c} d\omega' \int_{-\infty}^{\infty} dt \omega'^2 S(\omega')e^{-i\omega t} \left[ e^{i\omega'(t+\tau)} + e^{-i\omega'(t+\tau)} \right]
\]

\[
= \frac{1}{2C_0(2\pi)^{3}} \int_{0}^{\omega_c} d\omega' \omega'^2 S(\omega') \int_{-\infty}^{\infty} dt \left[ e^{i(\omega' - \omega)t} e^{i\omega' \tau} + e^{-i(\omega' - \omega)t} e^{-i\omega' \tau} \right].
\]

Integrating with respect to \( t \) gives

\[
\tilde{\mathcal{F}} \left[ R(t + \tau) \right](\omega) = \frac{\pi}{C_0(2\pi)^{3}} \int_{0}^{\omega_c} d\omega' \omega'^2 S(\omega') \left[ \delta(\omega' - \omega)e^{i\omega' \tau} + \delta(-\omega' - \omega)e^{-i\omega' \tau} \right].
\]

This vanishes for \( \omega > \omega_c \). For \( \omega = 0 \) we have:

\[
\tilde{\mathcal{F}} \left[ R(t + \tau) \right](0) = \frac{\pi}{C_0(2\pi)^{3}} \int_{0}^{\omega_c} d\omega' \omega'^2 S(\omega') \left[ \delta(\omega')e^{i\omega' \tau} + \delta(-\omega')e^{-i\omega' \tau} \right].
\]

Carrying out the frequency integral and using the properties of the \( \delta \) function we have:

\[
\tilde{\mathcal{F}} \left[ R(t + \tau) \right](0) = \frac{2\pi}{C_0(2\pi)^{3}} \lim_{\omega \to 0} \omega^2 S(\omega).
\]
In the interesting case $S \propto 1/\omega$ this tends to 0, in such a way that $T_4[A_0^4] \to 0$ and the fourth order Dyson expansion term doesn’t give any contribution.
Appendix C

Zero point energy density and pressure of matter fields in the free field approximation

In this Appendix we estimate the zero point energy density and pressure of a typical matter field in its vacuum state as described within the random gravity framework. The special case of a scalar massive field with a Klein-Gordon stress energy tensor is considered as a simple model and used to derive the expression for the vacuum power spectral density introduced in Section 1.7.

C.1 Modeling the zero point energy of a massive scalar field

Estimating the total amount of vacuum energy in a rigorous way is a nontrivial QFT problem: within the Standard Model, vacuum energy density is estimated to be given by, at least, three main contributions [103]: (i) vacuum zero-point energy plus virtual particles fluctuations, (ii) QCD gluon and quark condensates (iii) Higgs field. With the following calculation we want to get a first approximation estimate, by neglecting fields interactions and by describing the free-field configuration as a collection of decoupled harmonic oscillators of frequency

$$\omega_k = \sqrt{c^2 k^2 + m^2 c^4},$$  \hspace{1cm} (C.1)

where $k$ is the norm of the spatial wave vector $k$ and $m$ is the mass of the field quanta under examination. This is quite accurate for the EM field but it is likely to be quite a crude approximation for e.g. the QCD sector. By doing so we are neglecting higher order contributions to the vacuum energy density as well as the nonlinear and strong coupling effects of QCD and a detailed treatment of the Higgs fields. Nonetheless we expect to obtain a meaningful lower bound estimate to the vacuum energy.

We consider an arbitrary matter field of mass $m$ within the Standard Model. We calculate a lower bound to the associated vacuum energy density by estimating the overall energy density resulting from the summation of the zero point energies of all frequency modes, up to the random scale cutoff set by $\ell = \lambda L_p$.

We model each independent field component as a scalar field $\phi$ with mass $m$ and associated Klein-Gordon stress-energy tensor

$$T_{ab}^{KG}[\phi] = \phi_a \phi_b - \frac{1}{2} \eta_{ab} \left( \phi^c \phi_c + \frac{m^2 c^2}{\hbar^2} \phi^2 \right).$$  \hspace{1cm} (C.2)
C.1. Modeling the Zero Point Energy of a Massive Scalar Field

From this we see that $\phi^2$ has the dimension of force. Regarding $\phi$ as in its zero-point fluctuating state, we shall assume it to be stationary over spacetime and having an isotropic spectral density $S_\phi(k)$. In Appendix A we have shown that the mean squared field is given by

$$\langle \phi^2 \rangle = \frac{1}{(2\pi)^3} \int d^3k S_\phi(k). \quad (C.3)$$

Furthermore, the mean squared time derivative $\phi, t = c\phi, 0$ satisfies

$$\langle \phi^2, t \rangle = \frac{1}{(2\pi)^3} \int d^3k \omega_k^2 S_\phi(k), \quad (C.4)$$

and the mean squared gradient $(\nabla \phi)_i := \phi, i$ for $i = 1, 2, 3$ satisfies

$$\langle |\nabla \phi|^2 \rangle = \frac{1}{(2\pi)^3} \int d^3k k^2 S_\phi(k). \quad (C.5)$$

It follows from (C.1), (C.3), (C.4), (C.5) that

$$\langle \phi^a \phi_a \rangle = -\frac{1}{c^2} \langle \phi^2 \rangle + \langle |\nabla \phi|^2 \rangle = -\frac{m^2 c^2}{\hbar^2} \langle |\phi|^2 \rangle. \quad (C.6)$$

Using (C.4), (C.5) and (C.6) we see that the mean stress-energy tensor given by (C.2) becomes

$$\langle T_{ab}^{KG} \rangle = \langle \phi_a \phi_b \rangle, \quad (C.7)$$

which yields the effective energy density

$$\rho = \langle T_{00} \rangle = \frac{1}{c^2} \langle \phi^2, t \rangle = \frac{1}{(2\pi)^3} \int d^3k \frac{\omega_k^2}{c^2} S_\phi(k), \quad (C.8)$$

and the effective pressure

$$p = \frac{1}{3} \langle T_{i}^{i} \rangle = \frac{1}{3} \langle |\nabla \phi|^2 \rangle = \frac{1}{3(2\pi)^3} \int d^3k k^2 S_\phi(k). \quad (C.9)$$

A useful combination follows from (C.1), (C.8) and (C.9) as

$$\rho - 3p = \frac{1}{(2\pi)^3} \frac{m^2 c^2}{\hbar^2} \int d^3k S_\phi(k). \quad (C.10)$$

The spectral density $S_\phi(k)$ itself can be determined through the well-known zero point energy density expression

$$\rho = \frac{1}{(2\pi)^3} \int d^3k \frac{\hbar \omega_k}{2}, \quad (C.11)$$

that adds up contributions from all wave modes of $\phi$. This amounts to neglecting, in a first approximation, the virtual particles non linear terms due to the matter field interactions. Comparing (C.8) and (C.11) we see that the spectral density $S_\phi(k)$ takes the form

$$S_\phi(k) = \frac{\hbar c^2}{2\omega_k}. \quad (C.12)$$
Substituting (C.1) and (C.12) into (C.8) and (C.9) and integrating using \[ \int d^3k = 4\pi \int dk k^2 \] up to the cutoff value \( k_\lambda = \frac{2\pi}{M_P\lambda} \), we obtain
\[
\rho = \frac{\hbar}{4\pi^2} \int_0^{k_\lambda} dk k^2 \sqrt{c^2k^2 + \frac{m^2c^4}{\hbar^2}}, \tag{C.13}
\]
and
\[
p = \frac{\hbar c^2}{12\pi^2} \int_0^{k_\lambda} dk k^4 \sqrt{c^2k^2 + \frac{m^2c^4}{\hbar^2}}. \tag{C.14}
\]
Introducing the dimensionless variable \( y := \frac{k\hbar}{mc} \), we can rewrite (C.13) and (C.14) as
\[
\rho = \frac{\rho_P}{4\pi^2} \left( \frac{m\lambda}{\lambda} \right)^4 \int_0^{2\pi/m\lambda} dy y^2 \sqrt{1 + y^2}, \tag{C.15}
\]
and
\[
p = \frac{\rho_P}{12\pi^2} \left( \frac{m\lambda}{\lambda} \right)^4 \int_0^{2\pi/m\lambda} dy \frac{y^4}{\sqrt{1 + y^2}}. \tag{C.16}
\]
in terms of the Planck energy density \( \rho_P \) and the effective mass of the field in units of \( M_P/\lambda \), i.e. \( m_\lambda := \frac{m}{(M_P/\lambda)} \).

For \( m_\lambda \ll 1 \) we can approximate (C.15), (C.16) by
\[
\rho = \frac{\pi^2\rho_P}{\lambda^3} + \frac{\rho_P}{4\lambda^2} \left( \frac{m}{M_P} \right)^2, \tag{C.17}
\]
and
\[
p = \frac{\pi^2\rho_P}{3\lambda^4} - \frac{\rho_P}{12\lambda^2} \left( \frac{m}{M_P} \right)^2, \tag{C.18}
\]
up to \( O([m\lambda/M_P]^4) \) terms. This approximation is physically well justified. The heaviest particles in the Standard Model are the quark top, with \( m_t \approx 173 \text{ GeV} \), the weak bosons \( W^\pm \), with \( m_{W^\pm} \approx 80 \text{ GeV} \) and the \( Z \) boson with \( m_Z \approx 91 \text{ GeV} \). All the other particles have \( m \lesssim 1 \text{ GeV} \). Being the Planck mass of the order of \( 10^{19} \text{ GeV} \), it will be \( m_\lambda \ll 1 \) as long as the cutoff parameter satisfies \( \lambda \lesssim 10^{15} \). This is a safe upper bound, as the stochastic classical conformal fluctuations are expected to have a cutoff which should not exceed \( \lambda \approx 10^2 - 10^5 \) [23, 36]. Within this approximation, by subtracting (C.17) and (C.18) we get
\[
\rho - 3p = \frac{\rho_P}{2\lambda^2} \left( \frac{m}{M_P} \right)^2. \tag{C.19}
\]
This relation can also be obtained directly from the right hand side of (C.10) by following through the steps leading from (C.13) to (C.18).
Appendix D

Autoconsistent theory of metric perturbations over a curved background

In this Appendix we review the theory of small perturbation over a general curved background geometry. We start by the standard results holding for linear perturbations upon a Minkowski background and then we look in detail at the more general case, originally studied by Isaacson [43, 44], in which fast varying small perturbations propagate upon a smooth slow varying background that they themselves contribute to set up through their energy density backreaction. This material is at the base of the material presented in Chapter 2 and should be read prior or in conjunction to it.

D.1 Perturbation theory on a flat background

D.1.1 Linear approximation

We start by reviewing the standard linearized gravity theory over a flat background [1, 139, 119]. In a situation where gravity is week one can write the metric as

\[ g_{ab} = \eta_{ab} + h_{ab} \]  

(D.1)

where the metric perturbation \( h_{ab} \) is small, i.e. it is possible to find some global inertial coordinate system of \( \eta_{ab} \) where its components satisfy \( |h_{\mu\nu}| \ll 1 \). To first order in the perturbation the inverse metric is given by \( g^{ab} = \eta^{ab} - h^{ab} + O(2) \), where indices are raised by \( \eta_{ab} \), e.g. \( h^{ab} = \eta^{ac}\eta^{bd}h_{cd} \).

The Christoffel symbol and the Ricci tensor are in general given by

\[ \Gamma^c_{\ ab} = \frac{1}{2} \eta^{cd} \left\{ \partial_a g_{bd} + \partial_b g_{ad} - \partial_d g_{ab} \right\}, \]  

(D.2)

\[ R_{ab} = \partial_c \Gamma^c_{\ ab} - \partial_a \Gamma^c_{\ cb} + \Gamma^c_{\ ab} \Gamma^d_{\ cd} - \Gamma^c_{\ ad} \Gamma^d_{\ cb}. \]  

(D.3)

Then to linear order in \( h_{ab} \) it is

\[ \Gamma^c_{\ ab}^{(1)} := \frac{1}{2} \eta^{cd} \left\{ \partial_a h_{bd} + \partial_b h_{ad} - \partial_d h_{ab} \right\}, \]  

(D.4)

\[ R_{ab}^{(1)} := \partial_c \Gamma^c_{\ ab}^{(1)} - \partial_a \Gamma^c_{\ cb}^{(1)} = \frac{1}{2} \partial^c \partial_b h_{ac} + \frac{1}{2} \partial^c \partial_a h_{bc} - \frac{1}{2} \partial^c \partial_c h_{ab} - \frac{1}{2} \partial_a \partial_b h, \]  

(D.5)
where \( h := \eta^{ab} h_{ab} \) is the trace of the perturbation.

The linearized Ricci scalar is given by \( R^{(1)} := \eta^{ab} R^{(1)}_{ab} \) and the linear part of the Einstein tensor follows as

\[
G^{(1)}_{ab} := R^{(1)}_{ab} - \frac{1}{2} \eta_{ab} R^{(1)} = \frac{1}{2} \partial^c \partial_b h_{ac} + \frac{1}{2} \partial^c \partial_a h_{bc} - \frac{1}{2} \partial_a \partial_b h - \frac{1}{2} \eta_{ab} \left( \partial^c \partial^d h_{cd} - \partial^c \partial_c h \right). \tag{D.6}
\]

This defines a linear operator whose action on an arbitrary symmetric tensor \( \chi_{ab} \) of type \((0,2)\) (i.e. with two covariant indices) will be indicated through the notation \( G^{(1)}_{ab}[\chi] \).

This framework is appropriate to describe the propagation of GWs whose self-energy is so small that it doesn’t produce appreciable curvature. In empty space Einstein’s equation reads \( G_{ab} = 0 \). In the linear approximation one has the usual equation for the small metric perturbation \( h_{ab}, G^{(1)}_{ab}[h] = 0 \). By defining the trace reversed perturbation \( \bar{h}_{ab} := h_{ab} - \frac{1}{2} \eta_{ab} h \), with \( h = -i \), this simplifies to

\[
G^{(1)}_{ab}[\bar{h}] = -\frac{1}{2} \partial^c \partial_a \bar{h}_{ab} + \frac{1}{2} \partial^c \partial_b \bar{h}_{ac} + \frac{1}{2} \partial^c \partial_a \bar{h}_{bc} - \frac{1}{2} \eta_{ab} \partial^c \partial^d \bar{h}_{cd} = 0. \tag{D.7}
\]

As it is well known this simplifies to the free wave equation when the Lorentz gauge condition \( \partial^a \bar{h}_{ab} = 0 \) is imposed. This can always be done by exploiting the gauge freedom of GR related to the group of diffeomorphisms. In the linear approximation the gauge freedom is given by

\[
h_{ab} \to h'_{ab} := h_{ab} + \partial_a v_b + \partial_b v_a, \tag{D.8}
\]

where \( v_a \) is an arbitrary vector field that generates the coordinate transformation. The tensors \( h_{ab} \) and \( h'_{ab} \) represent the same physical perturbation since their components are related, to first order, by a coordinate transformation. The linearized classical vacuum Einstein’s equation in the Lorentz gauge reads

\[
\partial^c \partial_c \bar{h}_{ab} = 0. \tag{D.9}
\]

As it is well known, this equation admits a class of homogeneous solutions which are superpositions of elementary plane waves, i.e.

\[
\bar{h}_{ab}(x; k) := A_{ab}(k) e^{i(kx - \omega t)} \equiv A_{ab}(k) e^{ika}, \tag{D.10}
\]

where \( k_a = (\omega, k) \) is the wave vector. The wave equation implies that this is a null vector, i.e. \(|k| = \omega\). The complex amplitude \( A_{ab}(k) \) satisfies the constraint \( k^a A_{ab} = 0 \), due to the Lorentz gauge.

### D.1.2 TT gauge

With the metric perturbation put in the Lorentz gauge it looks as if all the degrees of freedom of the metric perturbation are radiative, as they all obey a wave equation. However this is an artifact due to the choice of gauge. It can be shown that only the traceless and transverse part of the metric perturbation involves radiative, physical degrees of freedom. The remaining components obey Poisson like equations and are thus not radiative. For a proof see e.g. [119]. In particular, for a globally classically vacuum spacetime, it is always possible to exploit the residual gauge freedom to impose the TT gauge, in which the metric perturbation satisfies the conditions:

\[
h_{0\mu} = 0, \quad (\mu = 0, 1, 2, 3); \quad h = h^i_i = 0; \quad \partial^i h_{ij} = 0. \tag{D.11}
\]
Because of the traceless condition, in the TT gauge, it is $h_{ab} \equiv \mathring{h}_{ab}$. This gauge shows clearly that GWs only have two degrees of freedom, linked to their two possible polarization components. This can be seen clearly, e.g. by considering a plane wave propagating in the $z$ direction, for which the TT metric perturbation has the form
\[ h_{ab} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & \mathring{h}_x & 0 \\ 0 & \mathring{h}_x & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \] (D.12)
where the two quantities $h_+ := h_{xx}(t - z)$ and $\mathring{h}_x := h_{xy}(t - z)$ represent the two polarization components of the wave.

### D.1.3 Second order corrections

The inclusion of second order terms in the Ricci tensor becomes necessary when one wants to assess the backreaction energy on spacetime due to the presence of GWs. This is not a trivial problem, the main reason being that in GR the spacetime metric plays both roles of ‘stage’, or background for physical processes, and ‘actor’. Any formalism in which one wishes to identify the actor with GWs only, then will also require to identify somehow a background. A satisfactory answer to this problem was given by Isaacson [43, 44], whose formalism is presented in Section D.3.

We notice that the approach of this section, based on the linear approximation $g_{ab} = \eta_{ab} + h_{ab}$, has a flat Minkowski background. As we now show, one can still carry the analysis to second order, thus identifying a background correction due to the presence of linear GWs. Since these however satisfy the flat spacetime wave equation, this approach gives a good approximation only when this correction is so small that the related curvature effects upon the GWs propagation can be neglected.

Substituting $g_{ab} = \eta_{ab} + h_{ab}$ into (D.3) and retaining terms up to second order yields [1, 139]
\[ R_{ab}^{(2)} := \frac{1}{2} h^{cd} \partial_a \partial_b h_{cd} - h^{cd} \partial_c \partial_d (h_{ab}) + \frac{1}{4} (\partial_a h_{cd}) \partial_b h^{cd} + (\partial^d h^c_b) \partial_d h_{cd} \partial_a h_{ab} + \frac{1}{2} \partial_d (h^{dc} \partial_c h_{ab}) 
- \frac{1}{4} (\partial^d h) \partial_d h_{ab} - (\partial_d h^{cd} - \frac{1}{2} \partial^d h) \partial_d h_{bc}, \] (D.13)
where $()$ and $[]$ denote respectively symmetrization and antisymmetrization.

The classical vacuum Einstein’s equation $R_{ab}[g] = 0$ now would read
\[ R_{ab}^{(1)}[h] + R_{ab}^{(2)}[h] + O(3) = 0. \] (D.14)
Since the first order equation gives $R_{ab}^{(1)}[h] = 0$, (D.14) cannot be satisfied but for the trivial case $h_{ab} = 0$. This simply means that one is forced to introduce a second order metric perturbation $h_{ab}^{(2)}$ and consider a metric of the form
\[ g_{ab} = \eta_{ab} + h_{ab} + h_{ab}^{(2)} . \] (D.15)
Then the second order expansion of the Ricci tensor yields
\[ R_{ab} = R_{ab}^{(1)}[h] + R_{ab}^{(1)}[h^{(2)}] + R_{ab}^{(2)}[h] + O(3) = 0, \] (D.16)
which implies that the second order metric perturbation satisfies
\[ R_{ab}^{(1)}[h^{(2)}] = -R_{ab}^{(2)}[h]. \] (D.17)

**General second order expansion of the Einstein tensor**

To put equation (D.17) into a form that resembles Einstein’s equation we can find the trace of equation (D.16) for the Ricci tensor:
\[
R := g^{ab} R_{ab} \approx (\eta^{ab} - h^{ab})(R_{ab}^{(1)}[h] + R_{ab}^{(1)}[h^{(2)}] + R_{ab}^{(2)}[h])
\]
\[
= \eta^{ab} R_{ab}^{(1)}[h] + \eta^{ab} R_{ab}^{(1)}[h^{(2)}] + \eta^{ab} R_{ab}^{(2)}[h] - h^{ab} R_{ab}^{(1)}[h] + O(3)
\]
\[
= R^{(1)}[h] + R^{(1)}[h^{(2)}] + R^{(2)}[h] - h^{ab} R_{ab}^{(1)}[h] + O(3),
\] (D.18)

where we defined the \( n \)-th order Ricci scalars by \( R^{(n)} := \eta^{ab} R_{ab}^{(n)}. \) By using \( G_{ab} = R_{ab} - \frac{1}{2} g_{ab} g^{cd} R_{cd} \) and keeping terms up to second order the Einstein tensor can be expanded as:
\[
G_{ab} = G_{ab}^{(1)}[h] + G_{ab}^{(1)}[h^{(2)}] + G_{ab}^{(2)}[h] + G_{ab}^{(hh)}[h] + O(3),
\] (D.19)

where the *linear tensor* \( G_{ab}^{(1)} \) is defined as
\[
G_{ab}^{(1)}[\cdot] := R_{ab}^{(1)}[\cdot] - \frac{1}{2} \eta_{ab} R^{(1)}[\cdot],
\] (D.20)

and the *quadratic tensors* \( G_{ab}^{(2)} \) and \( G_{ab}^{(hh)} \) by
\[
G_{ab}^{(2)}[\cdot] := R_{ab}^{(2)}[\cdot] - \frac{1}{2} \eta_{ab} R^{(2)}[\cdot],
\] (D.21)
\[
G_{ab}^{(hh)}[h] := \frac{1}{2} \eta_{ab} h^{cd} R_{cd}^{(1)}[h] - \frac{1}{2} h_{ab} R^{(1)}[h].
\] (D.22)

The linear Einstein tensor general expression is given in equation (D.6) in terms of the trace reversed linear metric perturbation, while the second order part can be built starting from equation (D.13) for the second order terms of the Ricci tensor. For convenience we re-write those expressions below:
\[
G_{ab}^{(1)}[h] = -\frac{1}{2} \partial^a \partial_c h_{ab} + \frac{1}{2} \partial^a \partial_b h_{ac} + \frac{1}{2} \partial^a \partial_c h_{bc} - \frac{1}{2} \eta_{ab} \partial^a \partial^b h_{cd}
\] (D.23)
\[
R_{ab}^{(2)}[h] := \frac{1}{2} h^{cd} \partial_a \partial_b h_{cd} - h^{cd} \partial_c \partial_d h_{ab} + \frac{1}{4} (\partial_a h_{cd}) \partial_b h^{cd} + (\partial^d h^c_a) \partial_d h_{cb} + \frac{1}{2} \partial_d (h^{de} \partial_e h_{ab})
\]
\[
- \frac{1}{4} (\partial^d h) \partial_d h_{ab} - (\partial_d h^{cd} - \frac{1}{2} \partial^d h) \partial_d (h_{ab} h_{bc}).
\] (D.24)

It is important to note that the above expansion (D.19) for the Einstein tensor is general. It also holds in the general case where matter enters Einstein’s equation on the r.h.s. through its stress energy tensor \( T_{ab}. \) However, in the case of classical vacuum it is \( T_{ab} = 0, \) so that \( G_{ab}[g] = 0 \) implies the
following set of equations:

\[ G^{(1)}_{ab}[h] = 0 \iff R^{(1)}_{ab}[h] = 0, \]

\[ G^{(1)}_{ab}[h^{(2)}] = -G^{(2)}_{ab}[h]. \]  

Note that, in this case, it is \( G^{(hh)}_{ab}[h] = 0 \) because \( h_{ab} \) satisfies the classical vacuum linearized Einstein’s equation.

Note that, up to second order in perturbation, the second order equation can be cast into the more suggestive form

\[ G_{ab}[\eta_{ab} + h^{(2)}_{ab}] = 8\pi t_{ab}[h], \]  

where we have defined the ‘source’ tensor

\[ t_{ab}[h] := \frac{1}{8\pi} G^{(2)}_{ab}[h]. \]

This suggests that one may interpret \( t_{ab}[h] \) as the GWs backreaction stress energy tensor causing a correction to the background geometry. Once again, it is important to note that GWs propagate on the flat background according to the linear equation, in such a way that this scheme is satisfactory only when the metric correction \( h^{(2)}_{ab} \) is very small and its influence on the propagation of the GWs can be ignored.

**Properties of the backreaction effective stress energy tensor**

The main properties of \( t_{ab}[h] \) that would suggest considering it as the effective stress energy tensor of GWs are: (1) it is quadratic in \( h_{ab} \), (2) symmetry and (3) conservation with respect to the flat background, i.e. \( \partial^a t_{ab}[h] = 0 \). In particular this second property is true if \( h_{ab} \) satisfies the classical vacuum linearized Einstein’s equation.

However the main problem towards such an interpretation is that \( t_{ab}[h] \) is not gauge invariant, i.e. under a gauge transformation \( h_{ab} \rightarrow h_{ab} + 2\partial_{(a}v_{b)} \), the expression for \( t_{ab}[h] \) does not remain unchanged. This means that, for two observers whose coordinates and metric perturbation components are respectively \( x^\mu, h_{\mu\nu} \) and \( x'^\mu := x^\mu + \delta v^\mu, h'_{\mu\nu} := h_{\mu\nu} + 2\partial_{(\mu}\delta v_{\nu)} \), where \( \delta v_a \) is an infinitesimal vector field, the quantities \( t_{\mu\nu}[h] \) and \( t_{\mu\nu}[h'] \) are not the components of the same tensor.

While some **global** quantities such as the total energy \( E = \int_\Sigma t_{00}[h]d^3x, \) where \( \Sigma \) is a spacelike hypersurface, can be shown to be gauge invariant for asymptotically flat metrics \( \eta_{ab} + h_{ab} \) [1], one would in general wish to be able to define some meaningful **local** notion of stress energy tensor associated with GWs.

This can be achieved, e.g., by defining a suitably averaged tensor as it is done, among other authors, by Isaacson in his important work of 1968 on *Gravitational radiation in the limit of high frequency* [43, 44]. This is the object of Section D.3. Before that, we review in the next section a more general formalism for an arbitrarily large perturbation over Minkowski spacetime.
D.2 Non covariant exact definition of gravity ‘stress energy tensor’

Further insight into the nonlinear structure of GR and the inherent backreaction effect can be obtained by writing the exact Einstein’s equation in a non manifestly covariant form [139]. This is done by choosing a quasi-Minkowskian coordinates system, in which the metric components are written as

\[ g_{\mu\nu} = \eta_{\mu\nu} + H_{\mu\nu}. \]  \hspace{1cm} (D.28)

The perturbation component \( H_{\mu\nu} \) are not assumed to be small but they fall off to zero at infinity. The part of the Ricci tensor linear in \( H_{\mu\nu} \) is given of course by \( R^{(1)}_{\mu\nu}[H] \), where \( R^{(1)}_{\mu\nu}[\cdot] \) is the operator defined in (D.5). In this coordinate system, the exact Einstein’s equation with a matter source, \( G_{\mu\nu}[g] = 8\pi T_{\mu\nu} \), can be written as

\[ G^{(1)}_{\mu\nu}[H] = 8\pi \{ T_{\mu\nu} + t_{\mu\nu}[H] \} := 8\pi \tau_{\mu\nu}[H], \]  \hspace{1cm} (D.29)

where we defined

\[ t_{\mu\nu}[H] := -\frac{1}{8\pi} \{ G_{\mu\nu}[\eta + H] - G^{(1)}_{\mu\nu}[H] \} \]  \hspace{1cm} (D.30)

and with the linear operator \( G^{(1)}_{\mu\nu}[\cdot] \) defined in (D.6).

Equation (D.29) has the expected form for a spin 2 field \( H_{\mu\nu} \) suffering a self-interaction as dictated by the ‘source’ term \( \tau_{\mu\nu}[H] := T_{\mu\nu} + t_{\mu\nu}[H] \), which depends nonlinearly upon \( H_{\mu\nu} \). As Weinberg suggests, if one decided to view \( H_{\mu\nu} \) as the gravitational field, then the tensor \( t_{\mu\nu}[H] \) would be the candidate to represent gravitation’s stress energy tensor. We remark that the above dynamical equation is exact, even though not manifestly covariant. Indeed it is based upon an a-priori choice of one coordinate system in which the metric components \( g_{\mu\nu} \) are simply re-written as \( \eta_{\mu\nu} + H_{\mu\nu} \), thereby defining the observer-related gravitational field \( H_{\mu\nu} \).

The main properties that the matter plus gravitation ‘stress energy tensor’ \( \tau_{\mu\nu}[H] \) enjoys are: (1) symmetry, (2) conservation, i.e. \( \partial_\sigma \tau_{\mu\nu}[H] = 0 \), as implied by the linearized Bianchi identity obeyed by \( G^{(1)}_{\mu\nu}[H] \), (3) if one computes \( t_{\mu\nu}[H] \) as a formal power series, the first term is quadratic in \( H_{\mu\nu} \) and given by

\[ t_{\mu\nu}[H] = -\frac{1}{8\pi} \{ G^{(2)}_{\mu\nu}[H] + G^{HH}_{\mu\nu}[H] \} + O(H^3), \]  \hspace{1cm} (D.31)

where are the operators \( G^{(2)}_{\mu\nu}[H] \) and \( G^{HH}_{\mu\nu}[H] \) are defined by equations (D.21) and (D.22). Of course the third and higher order terms in \( H_{\mu\nu} \) would account for gravity self interaction. A fourth important property enjoyed in general by \( \tau_{\mu\nu}[H] \) is that of being (4) Lorentz invariant [139].

Though not manifestly covariant the above formalism is exact. The link to the perturbation approach over a flat non-dynamical background of Section D.1 can be found when gravity is weak and \( |H_{\mu\nu}| \ll 1 \), by expanding

\[ H_{\mu\nu} = h_{\mu\nu} + h^{(2)}_{\mu\nu} + h^{(3)}_{\mu\nu} + \ldots \]  \hspace{1cm} (D.32)

Since \( G^{(1)}_{\mu\nu}[\cdot] \) is linear, equation (D.29) reads

\[ G^{(1)}_{\mu\nu}[h] + G^{(1)}_{\mu\nu}[h^{(2)}] + O(h^3) = 8\pi \left\{ T_{\mu\nu} - \frac{1}{8\pi} \{ G^{(2)}_{\mu\nu}[h] + G^{hh}_{\mu\nu}[h] \} + O(h^3) \right\}, \]  \hspace{1cm} (D.33)

and where the metric tensor expansion also appears implicitly in the expression for \( T_{\mu\nu} \). In the classi-
cal vacuum case $T_{\mu\nu} = 0$ and $G^{hh}_{\mu\nu}[h] = 0$. By equating equal order terms, the above equation yields of course

$$G^{(1)}_{\mu\nu}[h] = 0,$$

$$G^{(1)}_{\mu\nu}[h^{(2)}] = -G^{(2)}_{\mu\nu}[h].$$

(D.34) (D.35)

D.3 Isaacson’s perturbation theory on a curved background geometry

The material in this section is based upon Isaacson papers [43, 44] as well as the review paper by Flanagan [119] and Gravitation [140] by Wheeler and co-authors.

D.3.1 Slow varying vs. fast varying components: the main assumptions

The perturbation approach over Minkowski spacetime given in the previous sections suffers of a serious limitation problem: in the weak field limit and for classical vacuum, the GWs as described by $h_{ab}$ and satisfying the linear wave equation $G^{(1)}_{ab}[h] = 0$ propagate on a flat background; the formalism is therefore suitable only to describe situations in which the spacetime geometry is essentially flat to 0-th order.

However, there are situations in which classical vacuum spacetime geometry may be highly curved by itself, even at the lowest order of approximation. This can be the case for, e.g., the neighborhood of black holes, a neutron stars or a collapsing supernovae. If GWs are also present, one would require a formalism allowing to study their propagation on an arbitrary curved background geometry. This also seems to be desirable to study the vacuum random Einstein’s equation (Section 2.3) in situations where the background curvature due to the net vacuum energy amount plus standard non-vacuum matter in a given spacetime region cannot be ignored.

Isaacson devised a suitable method that works well in the case of high frequency GWs and which is based upon a suitable split of the spacetime geometry $g_{ab}$ into a background $g^B_{ab}$ and a perturbation $h_{ab}$. The split defines the background geometry as the smooth, slow varying part of $g_{ab}$, while the GWs are defined as a high frequency, fast varying superimposed perturbation.

A meaningful definition for the local energy content of the high frequency perturbation can be given by means of a suitable spacetime averaging procedure, originally introduced by Brill and Hartle [141]. Isaacson shows that the resulting stress energy tensor is quadratic in $h_{ab}$, gauge invariant and can be identified with that of massless spin-2 field if the perturbation $h_{ab}$ satisfies the GWs equation on the curved background. This result is important for the work presented in this thesis and the details are reported in D.3.3.

The key assumption behind Isaacson paper is that the GWs perturbation must have high frequency, in such a way that their typical small wavelength $\ell$ can be used as a formal expansion parameter. A perturbation is defined to have a high frequency whenever its wavelength is much shorter than the typical radius of curvature $L$ of the background geometry. It is important to note that the case $g^B_{ab} = \eta_{ab}$ implies an infinite radius of curvature, so that the standard flat background expansion scheme presented in Section D.1 is a very special case of Isaacson more general framework. In that case ‘high frequency’ simply means ‘all frequencies’!

To help visualize the physical situation involved, one may think of an orange’s skin: the overall shape represents the curved background geometry, while the small scale ripples represent the GWs.
The amplitude of the GWs is assumed to be small but this doesn’t imply that their energy content also must be small: in fact the formalism is also suited to describe situations in which the GWs energy content is the only cause of the background geometry curvature. We refer to this as to the autoconsistent case.

D.3.2 GWs propagating on a curved background: autoconsistent case

Isaacson framework can be implemented by splitting the spacetime metric as

$$g_{ab} = g^B_{ab} + \varepsilon h_{ab}, \quad (D.36)$$

where the slow varying background geometry $g^B_{ab}$ varies on a typical scale $L$, while the high frequency component $h_{ab}$, representing the GWs, varies on a typical scale $\ell \ll L$. The fact that the GWs have a small amplitude is embodied in the choice $\varepsilon \ll 1$. This ensures that the laboratory geometry has only microscopic fluctuations.

Orders of magnitude estimates

The background metric components are assumed to be of order $g^B_{ab} = O(1)$. The metric derivatives then have the typical magnitudes $\partial g^B \sim g^B / L$ and $\partial h \sim h / \ell$. To lowest order the effective energy connected to the GWs and acting as a source in Einstein’s equation is of order $(\varepsilon / \ell)^2$, while the background curvature is of order $(1 / L)^2$. Because of Einstein’s equation and the fact that other sources beyond GWs in general may curve the background one has

$$ (1 / L)^2 \geq (\varepsilon / \ell)^2, \quad (D.37) $$

implying that

$$ \varepsilon \leq \ell / L \ll 1. \quad (D.38) $$

The case in which no other causes of curvature are present beside GWs implies the equality $\varepsilon = \ell / L$. This relation can be used to estimate the order of magnitude of the various quantities involving derivatives that define the Ricci tensor. Since it is $L = O(1)$, $g^B_{ab} = O(1)$ and $h_{ab} = O(1)$, it follows that $\ell$ and $\varepsilon$ are small parameters of the same order and one can deduce the following orders of magnitudes estimates:

$$ \partial_a g^B_{bc} = O(1), \quad \partial_a \partial_b g^B_{cd} = O(1), \quad \partial_a h_{bc} = O(1 / \varepsilon) \quad \partial_a \partial_b h_{cd} = O(1 / \varepsilon^2). \quad (D.39) $$

The autoconsistent framework implies that we want to study a situation in which all of the background curvature is produced precisely by the backreaction energy of the GWs. We thus set $\varepsilon = \ell / L$.

Ricci tensor expansion

By expanding the Ricci tensor $R_{ab}[g] = R_{ab}[g^B + \varepsilon h]$ in powers of $\varepsilon$ one obtains, in a similar way to what happens in the flat background case,

$$ R_{ab}[g^B + \varepsilon h] = R_{ab}^{(0)} + \varepsilon R_{ab}^{(1)} + \varepsilon^2 R_{ab}^{(2)} + \varepsilon^3 R_{ab}^{(3+)} , \quad (D.40) $$

---

1By definition, $f = O(\varepsilon^n)$ means that one can find a constant $0 < C < \infty$ such that $f < C \varepsilon^n$ as $\varepsilon$ goes to zero. Then $f = O(1)$ simply means that $f$ has a finite value that does not depend on the smallness parameter $\varepsilon$. 

---
where
\[ R^{(0)}_{ab} := R_{ab}[g^B] \] (D.41)
is the full Ricci tensor of the smooth, slow varying background geometry while
\[ R^{(1)}_{ab} := \frac{1}{2} (\nabla^e \nabla_b h_{ae} + \nabla^e \nabla_a h_{be} - \nabla^e \nabla_c h_{ab} - \nabla_b \nabla_a h) \] (D.42)
and
\[ R^{(2)}_{ab} := \frac{1}{2} h^{cd} \nabla_b \nabla_a h_{cd} - h^{cd} \nabla_e \nabla_{(a} h_{b)d} + \frac{1}{4} (\nabla_a h_{cd}) \nabla_b h^{cd} + (\nabla^d h_c^e) \nabla_{(d} h_{e)a} + \frac{1}{2} \nabla_d (h^{cd} \nabla_e h_{ab}) - \frac{1}{4} (\nabla^e h) \nabla_c h_{ab} - (\nabla_d h^{cd} - \frac{1}{2} \nabla^e h) \nabla_{(a} h_{b)c}. \] (D.43)

In the above expressions \( \nabla_a \) is the covariant derivative of the background geometry \( g^B_{ab} \), which is also used to raise or lower all indices, in such a way that, e.g., \( \nabla^c = g^Ba^c \nabla_a \) and \( h = g^B_{ab} h_{ab} \). The linear operator \( R^{(1)}_{ab} \) reduces to the flat background expression (D.5) when \( g^B_{ab} \to \eta_{ab} \) and \( \nabla \to \partial \) and the same happens for the quadratic operator \( R^{(2)}_{ab} \) which reduces to the expression (D.13). Note however that in the general background case the order in which second covariant derivatives appear is important. Finally, the higher order term \( R^{(3+)}_{ab} \) is simply defined by equation (D.40).

\( R^{(0)}_{ab} \) is the only term not to scale with \( \epsilon^2 \), and so it is important to assess its relative order. Indeed, the various \( R^{(n)}_{ab} \) terms are not of the same magnitude, and this must be carefully assessed before one can write down the approximate Einstein’s equation order by order. By using \( \epsilon = \ell/L \), the estimates (D.39), and by inspecting the structure of the various Ricci tensor components it is found:

\[ R^{(0)}_{ab} \sim g^{B^{-1}} \partial^2 g^B \sim 1/L^2 = O(1), \] (D.44)

\[ \epsilon R^{(1)}_{ab} \sim g^{B^{-1}} \partial^2 (\epsilon h) \sim \epsilon / \ell^2 = O(1/\epsilon), \] (D.45)

\[ \epsilon^2 R^{(2)}_{ab} \sim \epsilon h g^{B^{-2}} \partial^2 (\epsilon h) \sim \epsilon^2 / \ell^2 = O(1). \] (D.46)

Moreover it is possible to verify that \( \epsilon^3 R^{(3+)}_{ab} = O(\epsilon) \) so that it truly represents a small correction. The important point about the above estimates is that they show how the background curvature and \( \epsilon^2 \) will be connected to the GWs local energy density have the same order of magnitude. In particular, even though \( \epsilon h_{ab} \) is small, the associated energy needs not be small. Note also that, correctly \( \epsilon^2 R^{(2)}_{ab} \) is smaller than \( \epsilon R^{(1)}_{ab} \) by a factor \( \epsilon \).

**Einstein’s equation expansion**

Einstein’s equation in classical vacuum reads \( R_{ab}[g] = 0 \). Collecting together terms of the same order in the Ricci tensor expansion (D.40) yields:

\[ R^{(1)}_{ab}[h] = 0, \] (D.47)
governing the GWs propagation upon the background \( g^B_{ab} \) which they themselves produce according to

\[ R^{(0)}_{ab}[g^B] = -R^{(2)}_{ab}[h], \] (D.48)

where for convenience we have set \( \epsilon = 1 \).
Isaacson showed that, in the high frequency limit, the theory enjoys gauge invariance similar to that of the linearized weak field limit and given by $h_{ab} \rightarrow h_{ab} - \nabla (a) v_b)$. In order to exploit the gauge freedom the linear equation (D.47) is re-expressed in terms of the trace reversed perturbation $\tilde{h}_{ab} := h_{ab} - \frac{1}{2} g^{ab} h$. By exploiting the following properties of the Riemann tensor and of the covariant derivative, holding for any scalar function $f$ and any tensor $T_{ab}$:

$$\nabla_a \nabla_b f = \nabla_b \nabla_a f,$$

$$(\nabla_a \nabla_b - \nabla_b \nabla_a) T_{cd} = R_{abc}^e T_{cd} + R_{abed} T_{ce},$$

$$R_{abcd} = R_{cdab}, \quad R_{abcd} = - R_{bacd}, \quad R_{abdc} = - R_{abad},$$

equation (D.42) yields

$$\nabla^c \nabla_c \tilde{h}_{ab} - \frac{1}{2} g^{cd} \nabla_c \tilde{h}_{cd} - 2 \nabla_b \nabla^c \tilde{h}_{ac} - 2 R_{c(0)}^{(0)} h_{dc}^{(0)} - 2 R_{a(0)}^{(0)} h_{bc}^{(0)} = 0.$$  \hfill (D.50)

The gauge freedom can now be exploited to obtain a traceless and transverse perturbation satisfying $\tilde{h} = 0$ and $\nabla^c \tilde{h}_{ac} = 0$. In this gauge $h_{ab} \equiv \tilde{h}_{ab}$ and the wave equation simplifies to:

$$\nabla^c \nabla_c h_{ab} - 2 R_{c(0)}^{(0)} h_{dc}^{(0)} - 2 R_{a(0)}^{(0)} h_{bc}^{(0)} = 0.$$  \hfill (D.51)

This obviously reduces to equation (D.9) in the flat background limit.

The curved background GWs propagation equation couples the perturbation to the background curvature. This causes a gradual evolution in the properties of the wave. In the high frequency limit this evolution can be described using the formalism of geometric optics, showing that GWs travel along null geodesics with slowly evolving amplitudes and polarizations [43].

### D.3.3 GWs stress energy tensor

#### Second order equations

The analysis of the second order equation (D.48) is central to this thesis since it provides information on the local energy content of GWs. It can be re-written as

$$G^{(0)}_{ab} [g^B] = 8 \pi \left( \frac{1}{8 \pi} G^{(2)}_{ab} [h] \right),$$

where $G^{(2)}_{ab} := R^{(2)}_{ab} [h] - \frac{1}{2} g^{ab} g^{cd} R^{(2)}_{cd} [h]$, with $R^{(2)}_{ab} [h]$ given in (D.43), and where we have used the linear equation $R^{(1)}_{ab} [h] = 0$. The tensor in between brackets on the r.h.s. is the curved background generalization of $t_{ab} [h]$ defined in (D.27). Even though it enjoys various properties as observed above, the main obstacle in interpreting it as the stress energy tensor of GWs is that it is not invariant under a general change of coordinates. This implies that one can in principle always find a suitable system of coordinate where, locally at some spacetime event $P$, the total physical metric $g_{ab}$ is Minkowski. For such an observer $t_{ab} [h] \big|_P$ would vanish. Such considerations lead to expect that in order to define a meaningful concept of local energy density for GWs one should recur to some kind of spacetime averaging over a region containing many wavelengths.

Another reason for this emerges by looking at (D.52). The background Einstein tensor is smooth
and varies on a typical scale $L \gg \ell$. This means that it contains no fluctuations. Yet, on the r.h.s. we have a tensor built upon the high frequency perturbation $h_{ab}$. This also leads to think that it is some suitable smooth average of $G_{ab}^{(2)}[h]$ that should act as a source for the smooth background curvature.

To see this better we can recall that $R_{ab}[g] = R_{ab}^{(0)} + \varepsilon R_{ab}^{(1)} + \varepsilon^2 R_{ab}^{(2)} + \varepsilon^3 R_{ab}^{(3+)} = 0$. Once the linear equation is imposed by $R_{ab}^{(1)} = 0$, the classical vacuum Einstein’s equation implies $R_{ab}^{(0)} + R_{ab}^{(2)} = 0$. Setting again $\varepsilon = 1$ for convenience, we can re-write this as

$$R_{ab}^{(0)}[g^B] + \left\langle R_{ab}^{(2)}[h] \right\rangle + \left\{ R_{ab}^{(2)}[h] - \left\langle R_{ab}^{(2)}[h] \right\rangle \right\} = 0,$$

where $\langle \cdot \rangle$ denotes some average to be defined later. The first two terms are now both smooth and slow varying so that we can set

$$R_{ab}^{(0)}[g^B] = - \left\langle R_{ab}^{(2)}[h] \right\rangle.$$  \hfill (D.54)

The term in curly bracket however defines a high frequency perturbation with zero average. In order to maintain a solution of Einstein’s equation at second order one must introduce a higher order metric perturbation $g_{ab}^{(2)}$ by writing from the start $g_{ab} = g^B_{ab} + h_{ab} + g_{ab}^{(2)}$. This way an extra term $R_{ab}^{(1)}[g^{(2)}]$ appears in (D.53) on the l.h.s. and we get the extra equation

$$R_{ab}^{(1)}[g^{(2)}] = \left\langle R_{ab}^{(2)}[h] \right\rangle - R_{ab}^{(2)}[h].$$  \hfill (D.55)

Summarizing we have the following interpretation: the GWs are represented by a high frequency perturbation $h_{ab}$ of a smooth, slow varying background $g^B_{ab}$. Their propagation is described by the linear equation $R_{ab}^{(1)}[h] = 0$ which, in an arbitrary gauge, reads

$$\nabla^c \nabla_c h_{ab} + \nabla_b \nabla_a h - \nabla^c \nabla_b h_{ac} - \nabla^c \nabla_a h_{bc} = 0.$$  \hfill (D.56)

The smooth background is produced by a suitable average of a backreaction tensor quadratic in the perturbation, according to the effective Einstein’s equation

$$G_{ab}^{(0)}[g^B] = 8\pi T_{ab}^{GW},$$  \hfill (D.57)

where the GWs stress energy tensor will be defined by an expression like

$$T_{ab}^{GW} := - \frac{1}{8\pi} \left\{ R_{ab}^{(2)}[h] - \frac{1}{2} g_{ab} \left\langle R_{ab}^{(2)}[h] \right\rangle \right\} \equiv - \frac{1}{8\pi} \left\langle G_{ab}^{(2)}[h] \right\rangle.$$  \hfill (D.58)

Finally, due to the the fact that gravity auto-interacts with itself, the linear GWs are themselves source of extra, higher order perturbations that obey a wave equation with a source given by

$$R_{ab}^{(1)}[g^{(2)}] = \left\langle R_{ab}^{(2)}[h] \right\rangle - R_{ab}^{(2)}[h].$$  \hfill (D.59)

This can account for phenomena such as e.g. wave-wave scattering or higher harmonics [140].

**General definition and properties of the GWs stress energy tensor**

The second order part of the Einstein tensor, giving the microscopic structure of what will become the GWs stress energy tensor, can be found starting from (D.43). It is straightforward to show that
D.3. Isaacson’s Perturbation Theory on a Curved Background Geometry

this can be written as [43]:

\[ G^{(2)}_{ab} = \frac{1}{2} (\nabla_c S_{ab}^c - Q_{ab}), \]  

(D.60)

where

\[ S_{ab}^c := \delta_b^e h^{ed} \nabla_a h_{ed} + h^{ed} (\nabla_d h_{ab} - \nabla_b h_{da} - \nabla_a h_{db}) + g^B_{ab} \left[ h^{cd} (\nabla_c h_{de} - \frac{1}{2} \nabla_d h) - \frac{1}{2} h_{ed} \nabla_c h^{ed} \right], \]  

(D.61)

and

\[ Q_{ab} := \frac{1}{2} \nabla_a h^{cd} \nabla_b h_{cd} - \nabla^c h_b^d (\nabla_c h_{da} - \nabla_d h_{ca}) - \frac{1}{2} \nabla^d h (\nabla_b h_{da} + \nabla_a h_{db} - \nabla_d h_{ab}) + \frac{1}{2} g^B_{ab} \left[ \frac{1}{2} \nabla^c h^{ed} \nabla_e h_{cd} - \nabla^c h^{ed} \nabla_d h_{ec} + \nabla^d h (\nabla_c h_{de} - \frac{1}{2} \nabla_d h) \right]. \]  

(D.62)

Symbolically the simple structure emerges \( G^{(2)} \sim (\partial h)^2 + \partial (h \partial h) \), which implies that only the part quadratic in \( \partial h \) will survive an averaging procedure over extended spacetime regions.

Similarly to what is done to study macroscopic electric fields inside a dielectric, one can define the energy content of GWs by neglecting the fine details due to the fast fluctuations and recovering an average expression which is suitable to represent the local energy content of GWs. The procedure used by Isaacson is the same as that introduced by Arnowitt, Deser and Misner in [117]. The notation \( \langle \cdot \rangle \) then denotes a spacetime integral average over a region whose characteristic size \( L_{(\cdot)} \) contains many fluctuations wavelengths: i.e. it must be small compared to the scale \( L \) over which the background varies, yet still of order \( O(1) \) and much larger of \( \ell \), the typical fluctuations scale, and we have

\[ [\ell = O(\epsilon)] \ll [L_{(\cdot)} = O(1)] \ll [L = O(1)]. \]  

(D.63)

The procedure is defined in such a way that the average of a tensor still yields a tensor. Further details are given at the end of this section.

The GWs stress energy tensor is then given by

\[ T^{GW}_{ab} = -\frac{1}{8\pi} \left\langle G^{(2)}_{ab}[h] \right\rangle \equiv \frac{1}{16\pi} \left\langle Q_{ab} - \nabla_c S_{ab}^c \right\rangle. \]  

(D.64)

When performing the average many terms can be simplified or neglected thanks to the following rules [44, 140]:

1. covariant derivatives of \( h_{ab} \) commute as \( \varepsilon \to 0 \) and one can use \( \langle h \nabla_a \nabla_c h_{ab} \rangle = \langle h \nabla_c \nabla_d h_{ab} \rangle \);

2. averages of a divergence, e.g. \( \langle \nabla_c S_{ab}^c \rangle \), are reduced by a factor \( \varepsilon \) and can also be neglected;

3. for the same reason it is possible to integrate by parts under integrals so that, ignoring extra terms smaller by a factor \( \varepsilon \), one can use e.g. \( \langle h \nabla_c \nabla_d h_{ab} \rangle = -\langle \nabla_d h \nabla_c h_{ab} \rangle \).

Using these prescriptions as well as [140]:

4. the fact that \( h_{ab} \) satisfies the linearized GWs equation \( G^{(1)}_{ab}[h] = 0 \) given in (D.56)

it is possible to show that [140, 119]

\[ T^{GW}_{ab} = \frac{1}{32\pi} \left\langle \nabla_a \bar{h}_{cd} \nabla_b \bar{h}^{cd} - \frac{1}{2} \nabla_a \bar{h} \nabla_b \bar{h} - 2 \nabla_a \bar{h}^{cd} \nabla_c \bar{h}_{b} d \right\rangle + O(\varepsilon), \]  

(D.65)
which is an expression valid in any gauge.

Isaacson studied the gauge invariance properties of this quantity. He showed that, under a general gauge transformation \( h_{ab} \rightarrow h_{ab} + \nabla (a v_b) \), it is

\[
G^{(2)}_{ab} \rightarrow G^{(2)}_{ab}' = G^{(2)}_{ab} + \nabla U_{a b}^c [v] + O(\varepsilon). \tag{D.66}
\]

The extra term \( \nabla U_{a b}^c [v] \) is of order \( O(\varepsilon) \) if the coordinate transformation vector \( v_a \) contains only low frequency components and it could be neglected as such. When \( v_a \) includes high frequency modes it is instead \( \nabla U_{a b}^c [v] = O(1) \), showing that the un-averaged \( G^{(2)}_{ab} \) is not indeed gauge invariant. However, since divergences are reduced under averages, it follows that the averaged tensor is gauge invariant up to smaller terms of order \( \varepsilon \), i.e.

\[
T_{ab}^{\text{GW}} \rightarrow T_{ab}^{\text{GW}'} = T_{ab}^{\text{GW}} + O(\varepsilon). \tag{D.67}
\]

The gauge invariance of \( T_{ab}^{\text{GW}} \) being shown, one can always go to the traceless-transverse gauge for \( \tilde{h}_{ab} \), where it takes the very simple form

\[
T_{ab}^{\text{GW}} = \frac{1}{32\pi} \left\langle \nabla_a h_{cd} \nabla_b h^{cd} \right\rangle + O(\varepsilon). \tag{D.68}
\]

Note from the background Einstein’s equation \( G^{(0)}_{ab} [g^B] = 8\pi T_{ab}^{\text{GW}} \) that this is conserved with respect to \( g^B_{ab} \), i.e.

\[
\nabla^a T_{ab}^{\text{GW}} = 0 + O(\varepsilon). \tag{D.69}
\]

GWs stress energy tensor in the geometric optic approximation

In the geometric optic approximation, suitable to describe GWs propagation on the curved background, one has an explicit representation of \( h_{ab} \) as

\[
h_{ab} = \mathcal{A} e_{ab} e^{i\phi}, \tag{D.70}
\]

where the polarization \( e_{ab} \) is defined to satisfy \( e_{ab} e^{ab} = 1 \), while \( \phi \) is a rapidly fluctuating phase with large first derivatives but negligible higher derivatives and \( k_a := \partial_a \phi \) represents ray vectors normal to the surfaces of constant phase. The effect of the curvature is to induce a slow variation in the amplitude \( \mathcal{A} \), polarization \( e_{ab} \) and propagation vector \( k_a \) of the wave. Then imposition of the transverse-traceless gauge yields

\[
g^{Bab} e_{ab} = 0, \tag{D.71}
\]

\[
k_a e^{ab} = 0, \tag{D.72}
\]

while imposing the wave equation yields:

\[
k_a k^a = 0 \tag{D.73}
\]

and

\[
\nabla_a \left( \mathcal{A}^2 k^a \right) = 0, \quad \nabla_c e_{ab} k^c = 0. \tag{D.74}
\]

The first condition gives the variation of the amplitude once the integral null curves along which waves propagate are known, while the second shows that the polarization tensor is parallel-transported along such curves. By using this information the GWs stress energy tensor (D.65) reduces to the very simple
form
\[ T_{ab}^{\text{GW-GO}} := \frac{A^2}{64\pi} k_a k_b. \] (D.75)

Note that, as it happens for a radiation fluid, it is traceless
\[ T_{ab}^{\text{GW-GO}} = g_b^{b'} T_{ab}^{\text{GW-GO}} = \frac{A^2}{64\pi} k_a k_a = 0. \] (D.76)

We summarize now some important points which will be crucial to the development of this thesis:

1. the non-averaged expressions (D.60)-(D.62) are general and simply define the non linear operator:
\[ G^{(2)}_{ab}[\cdot] = \frac{1}{2} \left( \nabla_c S_{ab}^c[\cdot] - Q_{ab}[\cdot] \right); \] (D.77)

2. if acting on a symmetric tensor \( h_{ab} \) describing GWs, i.e. satisfying the wave equation (D.56), its average yields the expression
\[ T_{ab}^{\text{GW}} = \frac{1}{32\pi} \left\langle \nabla_a \bar{h}_{cd} \nabla_b \bar{h}_{cd} - \frac{1}{2} \nabla_a \bar{h} \nabla_b \bar{h} - 2 \nabla_a \bar{h}^c d \nabla_{(a} \bar{h}_{b)c} \right\rangle; \] (D.78)

3. finally, if \( h_{ab} \) also satisfies the geometric optics approximation, this further reduces to
\[ T_{ab}^{\text{GW-GO}} = \frac{A^2}{64\pi} k_a k_b. \] (D.79)

### D.3.4 Definition of the spacetime averaging procedure

In general, averages of tensors over a curved geometry do not yield a tensor as the integration involves tensors defined at different spacetime points which, therefore, have different transformation properties. To define a meaningful average whose result is a true tensor one must devise a method to transport the tensors entering the average at the same spacetime point, where they can be summed in the ordinary manner [44, 140].

This can be done in a unique manner by defining the \textit{bivector of geodesic parallel displacement} \( g_{a'}^a \), discussed by De Witt [142] and Synge [143]. Its definition follows by observing that, in a small macroscopic region containing many perturbation’s wavelengths, there exists a unique geodesic \( \gamma_{PP'} \) connecting any two events \( P \) and \( P' \). Then, if \( E \) denotes an arbitrary tensor field, one can define
\[ E(P') \rightarrow P := \text{parallel transport of } E(P') \text{ to } P \text{ along } \gamma_{PP'}. \] (D.80)

Next one defines a scalar weighting function \( f(P, P') \) such that
\[ f(P, P') \rightarrow 0 \text{ when } d(P, P') \geq L(\ell), \] (D.81)

\[ \int f(P, P') \sqrt{-gB^a d^4x'} = 1, \] (D.82)

where \( L(\ell) \), such that \( \ell \ll L(\ell) \ll L \), defines the averaging scale containing many wavelengths and \( d(P, P') \) is the distance between \( P \) and \( P' \). The average of the tensor field \( E(P') \) about event \( P \) is
then defined as
\[
\langle E \rangle_{P} := \int E(P') \cdot \mathcal{P'} f(P, P') \sqrt{-gB} d^4 x' .
\] (D.83)

Given these definition it is possible to show that \[140\] an object $g^B_a a'$ can be defined such that it transform as a tensor at $P'$ with respect to the index $a'$ and as a tensor at $P$ with respect to the index $a$. This is the bivector of geodesic parallel displacement mentioned above. In the case of $E$ having two covariant indices, this allows to express its parallel transport as
\[
E_{ab}(P') \rightarrow P = g^B_a a' g^B_b b' E_{a'b'}(P').
\] (D.84)

Then, when expressed explicitly in the coordinate system, the average (D.83) yields
\[
\langle E_{\mu\nu} \rangle_x = \int g^B_{\mu'} (x, x') g^B_{\nu'} (x, x') E_{\mu'\nu'} (x') f(x, x') \sqrt{-gB(x')} d^4 x'.
\] (D.85)

Finally Isaacson paper \[44\] shows how to derive the rules 1., 2. and 3. listed at page 133 starting from this expression.
Appendix E

Quantum physics background

In this Appendix we give a very synthetic summary of some quantum mechanics concepts that we employ in Chapter 1. In particular we illustrate how the density matrix formalism can be introduced as an alternative to the well known state vector formalism. In the last section we illustrate some ideas related to decoherence theory.

E.1 State vector and unitary evolution

In quantum mechanics the state of a system is described by a normalized vector $\psi$ in an abstract complex Hilbert space. In the $\textit{Schrödinger picture}$ dynamical variables are represented by fixed hermitian operators while the state vector evolves in time according to Schrödinger equation

$$\dot{H}\psi_t = i\hbar \frac{d\psi_t}{dt}, \hspace{1cm} (E.1)$$

where $\hat{H}$ is the hamiltonian operator. The fact that the operators are hermitian guarantees that they admit a complete set of orthogonal eigenvectors with real eigenvalues. One important postulate of quantum mechanics states the possible results of an experiment devoted to measure, say, an observable $A$ can be only the eigenvalues of the corresponding quantum operator $\hat{A}$. If $\hat{A}v_n = a_nv_n$ is the eigenvalue equation for the operator $\hat{A}$ then the set $\{v_n\}_n$ of orthogonal eigenvectors allows to expand a generic state $\psi$ as

$$\psi = \sum_n c_nv_n,$$

where $c_n = (v_n, \psi)$ and $(, )$ denotes the Hilbert space scalar product. The components $c_n$ are related to the probabilities that a single measurement of the observable $A$ yields the value $a_n$, i.e.$^1$

$$P(A = a_n) = |c_n|^2 = |(v_n, \psi)|^2. \hspace{1cm} (E.2)$$

By the usual properties of probability this implies that it must be

$$\sum_n |c_n|^2 = 1.$$

$^1$For simplicity we assume the eigenvalues to be not degenerate.
The expectation or average value of the observable $A$ after many identical experiments on identically prepared systems is given by
\[ \langle A \rangle = (\psi, \hat{A}\psi) = \sum_n a_n |c_n|^2. \]

Since the state vector changes in time, the expectation values of an observable also change in general. From\[ \langle A \rangle_t = (\psi_t, \hat{A}\psi_t), \]using the equation of motion (E.1), and if $\hat{A}$ doesn’t depend on time, we have
\[ \frac{d}{dt} \langle A \rangle_t = \frac{i}{\hbar} (\hat{H}\psi, \hat{A}\psi) - \frac{i}{\hbar} (\psi, \hat{A}\hat{H}\psi) = (\psi, [\hat{H}, \hat{A}]\psi), \]
where $[\ , \ ]$ denotes the commutator and where we have exploited the fact that the $\hat{H}$ is hermitian. Thus
\[ \frac{d}{dt} \langle A \rangle_t = \frac{i}{\hbar} \langle [\hat{H}, \hat{A}] \rangle_t. \]

The spectrum of an operator can be discrete or continuous as, e.g., it is the case for the position operator $\hat{x}$. Considering for simplicity a single particle in one-dimension the eigenvalue equation reads $\hat{x}u_x = x u_x$, where $x \in \mathbb{R}$. An arbitrary state $\psi$ could then be expanded as
\[ \psi = \int_{\mathbb{R}} \psi(x) u_x dx, \]
where
\[ \psi(x) := (u_x, \psi). \]

The complex function $\psi(x)$ describes the quantum state in the well known position representation. In this case the scalar product has the explicit representation
\[ (\psi, \phi) = \int_{\mathbb{R}} dx \psi^*(x) \phi(x), \]
where $^*$ denotes complex conjugation, and the Hilbert space can be identified with $L^2(\mathbb{R})$. In the position representation $|\psi(x)|^2$ can be interpreted as the position probability density to measure the particle position at the location $x$.

Going back to the abstract level, the state vector evolution in the Hilbert space can be explicitly written in terms of the temporal evolution operator $\hat{U}(t, t_0)$ as
\[ \psi_t = \hat{U}(t, t_0)\psi_0, \]
where $\psi_0$ represents the initial state of the system at $t = t_0$. The operators $\hat{U}(t, t_0)$ are unitary and form a group
\[ \hat{U}(t, t_0) = \hat{U}(t, t_1)\hat{U}(t_1, t_0), \]
\[ \hat{U}^{-1}(t, t_0) = \hat{U}(t_0, t), \]
\[ \hat{U} \hat{U}^\dagger = \hat{U}^\dagger \hat{U} = \hat{I}, \]
where $^\dagger$ denotes the adjoint operator and $\hat{I}$ is the identity operator. Substituting equation (E.7) into
equation (E.1) we find

\[ \frac{\hbar}{i} \frac{\partial \hat{U}(t, t_0)}{\partial t} + \hat{H}\hat{U}(t, t_0) = 0, \quad \hat{U}(t_0, t_0) = \hat{I}. \]  

(E.11)

Equation (E.11) can be cast in the equivalent integral form

\[ \hat{U}(t, t_0) = \hat{I} - \frac{i}{\hbar} \int_{t_0}^{t} \hat{H}\hat{U}(t', t_0) dt'. \]  

(E.12)

A formal solution of (E.11), valid when \( \hat{H} \) does not depend itself on time, is

\[ \hat{U}(t) = e^{-\frac{\hbar}{i} \hat{H}t}, \]  

where we set \( t_0 = 0 \).

In many situations, and as it also happens for the problem we study in Chapter 1, the Hamiltonian operator can have an explicit time dependence. This is typically the case when \( \hat{H} \) represents some time dependent small perturbation acting on the system. Then a solution to equation (E.12) can be attempted through a Dyson's perturbation series. As the 0-th approximation we have

\[ \hat{U}^{(0)}(t, t_0) = \hat{I}. \]  

(E.13)

Substituting back into equation (E.12) we find the first order approximation

\[ \hat{U}^{(1)}(t, t_0) = \hat{I} - \frac{i}{\hbar} \int_{t_0}^{t} \hat{H}(t') dt'. \]  

(E.14)

In the same manner, the second order approximation results in

\[ \hat{U}^{(2)}(t, t_0) = \hat{I} - \frac{i}{\hbar} \int_{t_0}^{t} \hat{H}(t') dt' + \left( \frac{-i}{\hbar} \right)^2 \int_{t_0}^{t} \int_{t_0}^{t'} \hat{H}(t') \hat{H}(t'') dt'' dt'. \]  

(E.15)

If the procedure converges\(^2\) the approximation process can be continued in the same fashion and we get the series expansion

\[ \hat{U}(t, t_0) = \sum_{n=0}^{\infty} \hat{U}_n(t, t_0), \]  

where

\[ \hat{U}_n(t, t_0) := \left( \frac{-i}{\hbar} \right)^n \int_{t_0}^{t} \int_{t_0}^{t'} \cdots \int_{t_0}^{t^{(n-1)}} \hat{H}(t') \hat{H}(t'') \cdots \hat{H}(t^{(n)}) dt'' \cdots dt'. \]  

(E.17)

Then \( N \)-th order truncated series for the evolution operator is

\[ \hat{U}^{(N)}(t, t_0) = \sum_{n=0}^{N} \hat{U}_n(t, t_0). \]  

(E.18)

If the hamiltonian depends upon some small parameter \( \varepsilon \) then \( \hat{U}^{(N)}(t, t_0) \) will be of order \( \varepsilon^N \)

Going back to the case of time independent Hamiltonian for simplicity, an important feature of

---

\(^2\)It is known that whenever \( \hat{H} \) is a bounded operator the series does indeed converge.
quantum mechanics is that the so called unitary evolution described by

$$\psi_t = e^{-i\hat{H}t}\psi_0$$

is deterministic and continuous. The indeterminism comes in only when a measurement is performed. In that case only one of eigenvalues $a_n$ of the observable $A$ under exam results unpredictably. The usual view is that if the state before the measurement was $\psi = \sum_n c_n v_n$, then the state after the measurement is $\psi' = v_n$. The transition $\psi \rightarrow v_n$ goes under the name of state vector reduction or collapse of the state vector. Within the standard framework of quantum mechanics it is supposed to happen instantly and it thus represents a non-unitary, discontinuous element in the system evolution.

It is important to realize that the transition $\psi \rightarrow v_n$ cannot happen by virtue of unitary evolution alone. This is because Schrödinger equation is linear and the operators of quantum mechanics are linear. In fact if the system initial state is $\psi_0 = \sum_n c_n^{(0)} v_n$, where the $v_n$ are the eigenvectors of some hermitian operator $\hat{A}$, then the evolved state at time $t$ is

$$\psi_t = \hat{U}(t) \left( \sum_n c_n^{(0)} v_n \right) = \sum_n c_n^{(0)} \hat{U}(t) v_n,$$

which is still given by a superposition of states.

The unitary evolution maintains the correlations that may be present in the state vector describing some system. This is important for phenomena such as, e.g., quantum interference to happen. We can consider for simplicity a superposition of two states $\psi = c_1 \psi_1 + c_2 \psi_2$, which may describe the wave function of a single particle emerging behind two slits like in the standard Young interference experiment. Until the particle hits the photographic plate and a position measurement is performed, unitary evolution maintains the superposition state intact. This implies that the position probability density at the screen location is given by $|\psi_1 + \psi_2|^2 = |c_1|^2 |\psi_1|^2 + |c_2|^2 |\psi_2|^2 + c_1^* c_2 \psi_1^* \psi_2 + c_1 c_2^* \psi_1 \psi_2^*$ and it is well known that interference is due to the cross terms. It is only after the measurement that the particle wave functions collapses to some position eigenstate according to $\psi \rightarrow u_x$. The important feature of a so called pure state like the one considered above is that, prior to measurement, the particle is neither in the state $\psi_1$ nor $\psi_2$. It is really in a superposition of the two. The two states are there simultaneously and interact giving rise to the typical interference pattern. This particular pure state could be described in words as $\psi_1$ AND $\psi_2$.

In some situations one may however encounter a situation in which it is $\psi_1$ OR $\psi_2$. This would corresponds to the particle state being really either given by $\psi_1$ or $\psi_2$. The ‘doubt’ standing solely in our lack of knowledge on the ‘real’ state of the system. In this case one usually speaks of a mixed state or statistical mixture. Notice that no interference is possible because the two states $\psi_1$ and $\psi_2$ cannot interact; they simply are not there simultaneously! In the next section we introduce the density matrix formalism which allows to treat the case of mixed states in a very natural way.

### E.2 The density operator

A quantum system which is not in pure state can be described by a statistical mixture of pure states $\psi^{(i)}$, each one having a probability $w^{(i)}$ to occur. The $\psi^{(i)}$ are normalized but do not have to be necessarily orthogonal. Choosing any maximal set of commuting operators, any vector of the Hilbert
space can be expanded in the orthonormal basis of their common eigenvectors \( u_n \). In particular

\[
\psi^{(i)} = \sum_n c_n^{(i)} u_n, \tag{E.19}
\]

where

\[
(u_n, u_m) = \delta_{nm}, \quad \sum_n |c_n^{(i)}|^2 = 1. \tag{E.20}
\]

The expectation value of an observable \( \hat{A} \) in the pure state \( \psi^{(i)} \) is

\[
\langle A \rangle_i = (\psi^{(i)}, \hat{A} \psi^{(i)}) = \sum_{n,m} c_n^{(i)*} c_m^{(i)} (u_n, \hat{A} u_m) = \sum_{n,m} c_n^{(i)*} c_m^{(i)} A_{nm}, \tag{E.21}
\]

where \( A_{nm} := (u_n, \hat{A} u_m) \) are the matrix elements of the operator \( \hat{A} \). Since the system has a statistical probability \( w^{(i)} \) to be described by the pure state \( \psi^{(i)} \), the statistical grand average of the observable in the mixed state can be defined as

\[
\langle A \rangle := \sum_i w^{(i)} \langle A \rangle_i = \sum_{nm} A_{nm} \sum_i w^{(i)} c_n^{(i)*} c_m^{(i)}. \tag{E.22}
\]

Defining the density matrix elements by

\[
\rho_{mn} := \sum_i w^{(i)} c_n^{(i)*} c_m^{(i)} \tag{E.23}
\]

then

\[
\langle A \rangle = \sum_{nm} \rho_{nm} A_{nm} = \text{Tr}(\rho \hat{A}), \tag{E.24}
\]

where \( \rho \) indicates the abstract density matrix operator and \( \text{Tr} \) denotes the trace.

Dirac’s bra-ket notation allows to get an useful representation for the density matrix. Denoting the base states \( u_n \) by the kets \( |n\rangle \) and the pure states \( \psi^{(i)} \) by \( |i\rangle \) then by definition

\[
c_n^{(i)} = \langle n|i \rangle, \tag{E.25}
\]

so that

\[
\rho_{mn} = \sum_i \langle m|i \rangle w^{(i)} \langle i|n \rangle \tag{E.26}
\]

and the density operator itself can be written as

\[
\rho = \sum_i w^{(i)} |i\rangle \langle i|. \tag{E.27}
\]

Thus \( \rho \) appears as a statistical sum of the projector operators \( |i\rangle \langle i| \) upon the pure states \( |i\rangle \).

In the special case in which the system happens to be in the pure state \( |k\rangle \) then \( w^{(i)} = \delta_{ik} \) and \( \rho \) reduces to be a simple projection operator

\[
\rho^P = |k\rangle \langle k|, \tag{E.28}
\]
where $^P$ stands for pure. Equation (E.24) for the expectation value would read

$$\langle A \rangle = \sum_m (\rho^P \hat{A})_{mm} = \sum_m \langle m | k \rangle \langle k | \hat{A} | m \rangle = \sum_m c_m^{(k)} | k \rangle \langle m | = \langle k | \hat{A} \sum_m c_m^{(k)} | m \rangle = \langle k | \hat{A} | k \rangle,$$

(E.29)
as it would result from the usual quantum mechanics pure states formalism.

This example shows that the density matrix formalism includes as a special case the more usual pure states formalism. At a formal level, equation (E.24) can be used as a general definition of the density operator, in the sense that $\rho$ is some operator characterizing the system (whether mixed or pure), allowing to get the expectation value of any observable $A$ through $\langle A \rangle = \text{Tr}(\rho \hat{A})$.

In the density matrix formalism the rule (E.2) for the probability that the measurement of $A$ yields the value $a_n$ reads:

$$P(A = a_n) = \text{Tr} [\rho \hat{P}(a_n)],$$

(E.30)

Here the projector operator $\hat{P}(a_n)$ into the subspace spanned by the eigenvectors of $\hat{A}$ corresponding to the eigenvalue $a_n$ is given by $\hat{P}(a_n) = |a_n\rangle\langle a_n|$. Equation (E.30) is general and it can be easily checked that it reduces to (E.2) in the special case in which $\rho^P = |\psi\rangle \langle \psi|$ describes a pure state:

$$\text{Tr} [\rho^P \hat{P}(a_n)] = \sum_m \langle m | \psi \rangle \langle \psi | a_n \rangle \langle a_n | m \rangle = |\langle a_n | \psi \rangle|^2,$$

where we have used the completeness relation $\sum_m |m\rangle\langle m| = \hat{1}$.

The following formal properties of a density operator are important. For a proof we refer to [144]:

1. $\rho$ is Hermitian;
2. $\rho$ is positive definite, i.e. $\langle u | \rho | u \rangle \geq 0$ for any $|u\rangle$;
3. $\text{Tr} \rho = 1$, where $\text{Tr} \rho = \sum_n \langle n | \rho | n \rangle$ denotes the Trace;
4. The diagonal elements of $\rho$ in any representation are nonnegative;
5. The eigenvalues $p_n$ satisfy $0 \leq p_n \leq 1$;
6. If $\rho$ is a projection operator it projects into a one dimensional subspace;
7. $\text{Tr} (\rho^2) \leq 1$, and the equality holds only if $\rho$ is a projector operator;
8. With $\rho$ written as $\rho = \sum_i w^{(i)} |i\rangle \langle i|$, a necessary and sufficient condition for it to be a projection operator is that all the $|i\rangle$ should be identical up to a phase factor. In this case the above sum reduces to only one term.

Thus we have the important property that $\text{Tr} (\rho^2) = 1$ only when $\rho$ describes a pure state. Statistical mixtures are thus characterized by the important property $\text{Tr} (\rho^2) < 1$.

An elementary but instructive example that shows the utility of the density matrix formalism in relation to the problem of decoherence is the following. Let $|\psi\rangle := c_1 |1\rangle + c_2 |2\rangle$ be a pure state and let us consider the special case in which the two states $|1\rangle$ and $|2\rangle$ are orthogonal and make part of complete set $\{|n\rangle\}_n$. As it has been discussed above the ket $|\psi\rangle$ is a superposition of states that can interfere with each other. In the density matrix formalism the same system would be described by the projector operator

$$\rho^P = |\psi\rangle \langle \psi| = |c_1|^2 |1\rangle \langle 1| + |c_2|^2 |2\rangle \langle 2| + c_1^* c_2 |2\rangle \langle 1| + c_1 c_2^* |1\rangle \langle 2|.$$
The matrix elements would follow form \( \rho_{nm} = \langle n | \rho | m \rangle \). The only non-vanishing matrix element would be
\[
\rho_{11}^P = |c_1|^2, \quad \rho_{22}^P = |c_2|^2, \quad \rho_{12}^P = c_1 c_2^* = \rho_{21}^P = c_2 c_1^*.
\] (E.31)

In particular the possibility of interference between the pure state two components is expressed in the matrix element formalism by the fact that the off-diagonal terms are not zero.

On the other hand a mixed state describing a system which is either \(|1\rangle\) or \(|2\rangle\) and where interference cannot happen would be described by the density matrix
\[
\rho = w^{(1)} |1\rangle \langle 1 | + w^{(2)} |2\rangle \langle 2 |.
\]

Using as above the set of orthonormal vectors \( \{|n\rangle\}_n \) as a base, in this case only the diagonal elements are present, while the off-diagonal components vanish. Quite in general we have that in the density matrix formalism the absence of off-diagonal components implies the inability of the system to show interference.

### E.2.1 Unitary evolution of the density operator

We now derive the equation for the time evolution of the density matrix. To this end we can consider an arbitrary observable \( A \) and write the time derivative of its expectation value as
\[
\frac{d}{dt} \langle A \rangle = \frac{d}{dt} \text{Tr}(\rho \hat{A}) = \text{Tr}(\dot{\rho} \hat{A}).
\] (E.32)

Moreover we have
\[
\frac{i}{\hbar} \langle [\hat{H}, \hat{A}] \rangle = \frac{i}{\hbar} \text{Tr}(\rho [\hat{H}, \hat{A}]) = \frac{i}{\hbar} \text{Tr}(\rho \hat{H} \hat{A} - \hat{A} \hat{H} \rho) = \frac{i}{\hbar} \text{Tr}(\rho \hat{H} \hat{A} - \hat{A} \hat{H} \rho) = \text{Tr}(\frac{i}{\hbar} [\rho, \hat{H}] \hat{A}),
\] (E.33)

where we have used the fact that the trace is additive and unchanged under cyclic permutation of the factors. Comparing equations (E.5), (E.32), (E.33) we get
\[
\dot{\rho} = \frac{i}{\hbar} [\rho, \hat{H}],
\] (E.34)

which is the analogue of the Schrödinger equation and describes unitary evolution.

This equation is general but we are also interested in having an explicit formal expression for the evolved density matrix \( \rho_t \) of an initial density operator \( \rho_0 \) through some kind of temporal evolution operator as it has been done for the state vector in equation (E.7). If the system is in a mixed state then it can conveniently be described by a density operator given by
\[
\rho_t = \sum_i w^{(i)}_{\text{t}} |i\rangle_t \langle i|_t
\] (E.35)
in terms of the time dependent pure states \( |i\rangle_t \). The time evolution of the density operator stems from the time evolution of the individual pure states. For these we have
\[
|i\rangle_t = \hat{U}(t, t_0) |i\rangle_0.
\] (E.36)
The bra $\langle i | t \rangle$ corresponding to the ket $| i \rangle_t$ must evolve according to

$$\langle i | t \rangle = \langle i | 0 \rangle U^\dagger(t, t_0). \tag{E.37}$$

This guarantees indeed the normalization of the pure states at every time

$$\langle i | i \rangle_t = \langle i | 0 \rangle U^\dagger(t, t_0) \hat{U}(t, t_0) | i \rangle_0 = \langle i | i \rangle_0 = 1. \tag{E.38}$$

The time evolution of the density matrix follows as

$$\rho_t = \sum_i \hat{U}(t, t_0) | i \rangle_0 \langle i |_0 \hat{U}^\dagger(t, t_0) = \hat{U}(t, t_0) \rho_0 \hat{U}^\dagger(t, t_0). \tag{E.39}$$

Then the expansion (E.16) can be used to express $\hat{U}(t, t_0)$ and obtain an explicit expression for the evolved density matrix.

Let us think now of $\hat{H}$ as a small perturbation (which is indeed the case in our case of interest where it represents the tiny fluctuations in the gravitational field). In the expansion series for $\hat{U}$ we retain then only the terms up to second order and the evolved density matrix is

$$\rho_t = [\hat{I} + \hat{U}_1(t, 0) + \hat{U}_2(t, 0)] \rho_0 [\hat{I} + \hat{U}_1^\dagger(t, 0) + \hat{U}_2^\dagger(t, 0)]. \tag{E.40}$$

Keeping terms up to second order gives

$$\rho_t = \rho_0 + \hat{U}_1(t, 0) \rho_0 + \rho_0 \hat{U}_1^\dagger(t, 0) + \hat{U}_2(t, 0) \rho_0 + \hat{U}_1(t, 0) \rho_0 \hat{U}_1^\dagger(t, 0) + \rho_0 \hat{U}_2^\dagger(t, 0), \tag{E.41}$$

with

$$\hat{U}_1(t, 0) = -\frac{i}{\hbar} \int_0^t \hat{H}(t') dt', \tag{E.42}$$

$$\hat{U}_2(t, 0) = -\frac{1}{\hbar^2} \int_0^t \int_0^{t'} \hat{H}(t') \hat{H}(t'') dt'' dt'. \tag{E.43}$$

These are the expressions (1.9) used in chapter 1.

### E.2.2 Proper and improper mixtures

In many important situations a quantum system can be decomposed into various subsystems. The state vector (or density matrix) describing the whole system encodes information about all the subparts and their correlations. This applies e.g. to the relatively simple case of a system composed of $N$ particles or, which is relevant to the issue of decoherence, to the case of one particle and a complex quantum environment with which it interacts. One of the key and striking features of QM is that, because there is only one overall state vector for the whole system, its subparts are often entangled; i.e. systems that interacts once and get correlated maintain their correlation even at later times. A typical simple example includes EPR-like systems [145], e.g. when a particle with, say, zero spin decays into two particles that then propagate far from each other: the time evolution of the whole system is unitary until a spin measurement is performed on one of the particles, in which case the part of the wave function describing the second particle, possibly causally separated from the first, also collapses, instantaneously to a new appropriate state. This example shows the non-local nature of QM, as it is also implicit in the violation of Bell inequalities [146] and as it has also been confirmed by experiments [147].
For sake of illustration we can consider e.g. a composite quantum system \( \Sigma = S + E \), where the subsystems \( S \) and \( E \) could be described individually through their relevant Hilbert spaces \( \mathcal{H}^{(S)} \) and \( \mathcal{H}^{(E)} \). The total system Hilbert is given by the tensor product \( \mathcal{H} = \mathcal{H}^{(S)} \otimes \mathcal{H}^{(E)} \) and a pure state could be generally written as

\[
|\psi\rangle = \sum_{ij} c_{ij} |v_i\rangle |u_j\rangle,
\]

(E.44)

where \( \{|v_i\rangle\} \) and \( \{|u_i\rangle\} \) are two complete sets of orthonormal kets in \( \mathcal{H}^{(S)} \) and \( \mathcal{H}^{(E)} \). Such a situation could typically occur as a result of some past interaction between \( S \) and \( E \). The specific example of the two particles having opposite spins could e.g. be described as

\[
|\psi\rangle = \frac{1}{\sqrt{2}} |\uparrow\rangle |\downarrow\rangle + \frac{1}{\sqrt{2}} |\downarrow\rangle |\uparrow\rangle.
\]

If a measurement, say of particle 1, yields a spin up alignment as described by \( |\uparrow\rangle \), then the entanglement would force the wave function related to particle 2 to collapse to a spin down alignment, as described by \( |\downarrow\rangle \).

Let us go back now to the general case (E.44) and suppose we are interested in describing the subsystem \( S \) only, which could be a quantum particle propagating through an environment described by \( E \). Let \( A \) be an observable pertaining to \( S \). The expectation value of \( A \) is

\[
\langle A \rangle = \langle \psi | \hat{A} | \psi \rangle = \sum_{ijrs} c_{ij}^* c_{rs} \langle v_i | \langle u_j | \hat{A} | v_r \rangle | u_s \rangle.
\]

Since \( A \) belongs to the system \( S \), the operator \( \hat{A} \) does not affect the vectors in \( \mathcal{H}^{(E)} \) and the previous equation can be rewritten as

\[
\langle A \rangle = \langle \psi | \hat{A} | \psi \rangle = \sum_{ijr} c_{ij}^* c_{rj} \langle v_i | A | v_r \rangle = \sum_{ir} \rho_{ri} A_{ir} = \text{Tr} (\rho \hat{A}),
\]

where we used \( \langle u_j | u_s \rangle = \delta_{js} \) and where \( A_{ir} = \langle v_i | A | v_r \rangle \) are the matrix elements of the operator \( \hat{A} \) in the base \( \{|v_i\rangle\} \) while

\[
\rho_{ri} = \sum_j c_{rj} c_{ij}^* \quad \text{(E.45)}
\]

are the matrix elements of the operator

\[
\rho = \sum_{rs} |v_r\rangle \rho_{ri} \langle v_i|.
\]

(E.46)

The density operator

\[
\rho^{(\Sigma)} = \sum_{rsij} |v_r\rangle |u_s\rangle \rho^{(\Sigma)}_{rs;ij} \langle v_i| \langle u_j|
\]

describing the whole system \( \Sigma = S + E \) has matrix elements given by \( \rho^{(\Sigma)}_{rs;ij} = c_{rs} c_{ij}^* \). Equations (E.45) and (E.46) can thus be rewritten as

\[
\rho_{ri} = \sum_j \rho^{(\Sigma)}_{rj;ij},
\]
which corresponds to
\[ \rho = \text{Tr}^{(E)} \rho^{(\Sigma)} := \sum_n \langle u_n | \rho | u_n \rangle \]
where the partial trace \( \text{Tr}^{(E)} \) means that the operation of taking the trace of the operator is carried over only with respect to \( \mathcal{H}^{(E)} \).

It is possible to show that the operator \( \rho \) defined above satisfies all the property enjoyed by any density operator as listed above. Moreover it can be verified that
\[ \text{Tr} \rho^2 < 1, \]
implying that, as long as it is considered on its own, the subsystem \( S \) resembles a statistical mixture. This fact is remarkable and motivates the convention which extends the name of mixtures also to the systems like \( S \) considered here. Such an emerging mixture should be more properly called an improper mixture; the reason being that the full system \( \Sigma = S + E \) is still described by a pure state. The system \( S \) gains mixtures properties only due to the fact that in tracing out the properties of the environment \( E \) one loses information about the correlation that are still present in the overall system.

It is important to note that the fact that the density matrix related to system \( S \) turns into the appropriate form to describe mixtures cannot stem out of unitary evolution. Indeed, from \( \rho_t = \hat{U}(t,t_0) \rho_0 \hat{U}^\dagger(t,t_0) \), we can observe that:
\[ \rho_t^2 = \hat{U}(t,t_0) \rho_0 \hat{U}^\dagger(t,t_0) \hat{U}(t,t_0) \rho_0 \hat{U}^\dagger(t,t_0) = \hat{U}(t,t_0) \rho_0^2 \hat{U}^\dagger(t,t_0). \]
We see that, if \( \rho_0^2 = \rho_0 \), then \( \rho_t^2 = \rho_t \) for every \( t \). Thus a pure state can never evolve into a mixture if the time evolution is governed by Schrödinger equation.

### E.3 Basic concepts in decoherence theory

Many excellent reviews articles analyzing decoherence, its roots, its meaning, and its consequences can be found in the literature [27, 148, 149, 150, 26]. We refer to these for a comprehensive review. With the following we aim to describe briefly some essential features that are important in relation to the present work.

As succinctly expressed by Joss in [150]

Decoherence is the irreversible formation of quantum correlations of a system with its environment. These correlations lead to entirely new properties and behavior compared to that shown by isolated objects.

The motivation behind this statement is expressed in a nutshell by the example of the previous section. Quantum mechanics unitary evolution applies to the closed system \( \Sigma = S + E \). However because of the entanglement from interaction, as expressed by \( |\psi\rangle = \sum_{ij} c_{ij} |v_i\rangle |u_j\rangle \), the system of interest \( S \) appears to gain new properties: namely it becomes analogue to a statistical mixture, thus losing the coherence properties that are needed to show genuine quantum effects such as, e.g., interference. Whenever the environment \( E \) has many degrees of freedom the entanglement is practically irreversible and the effects of decoherence on \( S \) can be quite dramatic.

Decoherence is an ubiquitous quantum phenomenon and it is important to understand that it represents the rule more than the exception. Indeed a perfectly isolated and closed system does not exist, exception made, perhaps, for the whole universe. Of course the degree of decoherence suffered by a
quantum system coupled to its environment will depend upon the complexity of the environment and the strength of the coupling. Decoherence is strongly believed to play an important role in determining the quantum to classical transition and helping alleviating the measurement problem of quantum mechanics. Starting from the 90s it received important experimental verifications in [151], where decoherence of mesoscopic superposition of quantum states involving radiation fields was observed, and in [152], when interference with fullerene molecules was first obtained.

According to Von Neumann’s measurement process description, in the interaction with a macroscopic quantum apparatus, the state $|n\rangle$ of a quantum system gets correlated with the so called macroscopic pointer state $|\Phi(n)\rangle$. If $|\Phi(0)\rangle$ represents the initial pointer position prior to interaction, an ideal measurement should look like

$$|n\rangle|\Phi(0)\rangle \rightarrow |n\rangle|\Phi(n)\rangle.$$  

The roots of the measurement problem are in the fact that if the particle is initially in a pure superposition state then, by linearity, we have:

$$\left( \sum_n c_n |n\rangle \right) |\Phi(0)\rangle \rightarrow \sum_n c_n |n\rangle |\Phi(n)\rangle,$$

i.e. because of the interaction the system and the apparatus become correlated in such a way that the superposition state also affects the macroscopic apparatus. Clearly macroscopic superpositions of classical objects have never been observed and the problem stands in asking why this is so and why the measurement always yields one definite answer, corresponding to one definite pointer state. The problem is somehow alleviated if one takes the environment into account and considers the resulting decoherence. The coupling of the apparatus to the environment dislocalizes the phase relations to the enlarged total system made up of quantum particle + apparatus + environment according to

$$\left( \sum_n c_n |n\rangle |\Phi(n)\rangle \right) |E_0\rangle \rightarrow \sum_n c_n |n\rangle |\Phi(n)\rangle |E_n\rangle := \psi,$$

where the kets $|E_n\rangle$ denote orthogonal environment states. Taking the partial trace of the total density matrix $\rho := |\psi\rangle \langle \psi|$ with respect to the environment states yields the reduced density matrix describing the particle + apparatus as

$$\rho^{(S+A)} = \sum_n |c_n|^2 |n\rangle \langle n| \otimes |\Phi_n\rangle \langle \Phi_n|.$$

This density matrix describes an improper mixture with no correlations; off-diagonal terms are absent and, in this sense, the system $S + A$ acquires classical behavior.

Another example is the localization process of macroscopic objects. In this case one can consider the scattering of microscopic environmental particles off a body that represents the system of interest. It can be shown that, as a result of many scattering processes, the reduced density matrix describing the object suffers an exponential damping of spacial coherence, i.e. in the position representation one finds [153]

$$\rho_t(x, x') = \rho_0(x, x') \exp \left[ -\Lambda t (x - x')^2 \right],$$

where the localization rate $\Lambda$ depends on the flux and momenta of the environment particles and on scattering cross section. Numerical estimates indicate that macroscopic particles such as dust grains or even relatively large molecules ($d \approx 10^{-6}$ cm) have a very large localization rate (and thus suffer
fast decoherence) even for a typical laboratory vacuum environment, estimated as containing roughly $10^3$ particles per cm$^3$.

We conclude this short summary by remarking that the reduced density matrix describing a system that suffers decoherence usually obeys some master equation that doesn’t describe unitary evolution such as in (E.34). In this localization example the appropriate master equation takes the form:

$$\dot{\rho} = \frac{i}{\hbar} [\rho, \hat{H}] + \dot{\rho}_{\text{scattering}}, \quad (E.47)$$

where $\hat{H}$ describes in this case the free Schrödinger evolution while the extra term results as an effect of the scattering process. In the position representation and in one dimension this equation reads ($\hbar = 1$):

$$i\dot{\rho}_t(x, x') = \frac{1}{2m} \left( \frac{\partial^2}{\partial x'^2} - \frac{\partial^2}{\partial x^2} \right) \rho - i\Lambda(x - x')^2 \rho.$$
Bibliography

[1] Wald R M 1984 *General Relativity* The University of Chicago Press

[2] Ryder L H 1985 *Quantum Field Theory* Cambridge University Press

[3] Clifford M W 1981 *Theory and Experiment in Gravitational Physics* Cambridge University Press

[4] Taylor J H, Fowler L A and McCulloch P M 1979 Measurement of general reativistic effects in the binary pulsar PSR 1913 + 16 *Nature* 277 437–40

[5] Abbott B et al 2008 Joint search for gravitational-wave bursts in LIGO and GEO 600 data *Class. Quantum Grav.* 25 245008

[6] Araujo H et al 2007 LISA and LISA PathFinder, the endeaveour to detect low frequency GWs *J. Phys. Conf. Ser.* 66 012003

[7] Weinberg S 2005 *The Quantum Theory of Fields* Cambridge University Press

[8] Han T, Logan H E, McElrath B and Wang L-T 2003 Phenomenology of the little Higgs model *Phys. Rev. D* 67 095004

[9] Wald R M 1994 *Quantum Field Theory in Curved Spacetime and Black Hole Thermodynamics* The University of Chicago Press

[10] Kiefer C 2004 *Quantum Gravity* Clarendon Press Oxford

[11] Kiefer C 2005 Quantum Gravity: General Introduction and Recent Developments *Annalen Phys.* 15 129–48

[12] Page D N and Geilker C D 1981 Indirect evidence for quantum gravity *Phys. Rev. Lett.* 47(14) 979–82

[13] Giulin D, Kiefer C and Lämmertzahl C (Eds.) 2003 *Quantum Gravity: from Theory to Experimental Research* Lecture Notes in Physics, Springer

[14] Rosenfeld L 1930 Z. Phys. 65 589

[15] Goroff M H and Sagnotti A 1985 *Phys. Lett. B* 160 81

[16] Donoghue J F 1994 General relativity as an effective field theory: the leading quantum corrections *Phys. Rev. D* 50(6) 3874–88

[17] DeWitt B S 1967 Quantum Theory of Gravity. I. The Canonical Theory *Phys. Rev.* 160 1113–48
[18] Polchinski P 1998 *String Theory*, two volumes, Cambridge University Press

[19] Rovelli C 2004 *Quantum Gravity* Cambridge University Press

[20] Ashtekar A 1986 New variables for classical and quantum gravity *Phys. Rev. Lett.* 57(18) 2244–48

[21] Kimberly D and Magueijo J 2005 Phenomenological Quantum Gravity (*Preprint* gr-qc/0502110v1)

[22] Kok P and Yurtsever U 2003 Gravitational decoherence *Phys. Rev. D* 68 085006

[23] Power W L and Percival I C 2000 Decoherence of quantum wave packets due to interaction with conformal space-time fluctuations *Proc. R. Soc. Lond. A* 456 955

[24] Percival I C 1994 Primary state diffusion *Proc. R. Soc. Lond. A* 447 189–209

[25] Percival I C 1995 Quantum spacetime fluctuations and primary state diffusion *Proc. R. Soc. Lond. A* 451 503–13

[26] Kiefer C and Joos E 1998 Decoherence: Concepts and Examples (*Preprint* quant-ph/9803052v1)

[27] Zeh H D 2006 *Roots and Fruits of Decoherence*, in: Quantum Decoherence, Duplantier B, Raimond J-M and Rivasseau V (Eds) Birkhäuser 151–75

[28] Giulini D, Joos E, Kiefer C, Kupsch J, Stamatescu I-O, and Zeh H D 1996 *Decoherence and the Appearance of a Classical World in Quantum Theory* Springer Berlin

[29] Whelan J T 1998 Modeling the decoherence of spacetime *Phys. Rev. D* 57 768–797

[30] Hackermüller L, Utenthaler S, Hornberger K, Reiger E, Brezger B, Zeilinger A and Arndt M 2003 Wave nature of biomolecules and fluorofullerenes *Phys. Rev. Lett.* 91 9

[31] Hornberger K and Sipe J E 2003 Collisional decoherence reexamined *Phys. Rev. A* 68 012105

[32] Hornberger K, Sipe J E and Arndt M 2004 Theory of decoherence in a matter wave Talbot-Lau interferometer *Phys. Rev. A* 70 053608

[33] Percival I C and Strunz W T 1997 Detection of spacetime fluctuations by a model matter interferometer *Proc. R. Soc. Lond. A* 453 431–46

[34] Percival I C 1998 *Quantum State Diffusion* Cambridge University Press

[35] Power W L 1999 The interaction of propagating space-time fluctuations with wave packets *Proc. R. Soc. Lond. A* 455 991–1002

[36] Wang C H–T, Bingham R and Mendonça J T 2006 Quantum gravitational decoherence of matter waves *Class. Quantum Grav.* 23 L59–L65

[37] Reynaud S, Lamine B, Lambrecht A, Neto P M and Jaekel M-T 2004 HYPER and gravitational decoherence *General Relativity and Gravitation* 36 10
[38] Breuer H–P, Göklü E and Lämmerzahl C 2008 Metric fluctuations and decoherence (Preprint gr–qc/0812.0420)

[39] Wagoner R V 1970 Scalar tensor theory and gravitational wave Phys. Rev. D 1 12 3209

[40] Bingham R 2005 The search for quantum gravity using matter interferometers Phys. Scr. T116 132-34

[41] Boyer T H 1975 Random electrodynamics: the theory of classical electrodynamics with classical electromagnetic zero point radiation Phys. Rev. D 11 12 3209

[42] Bonifacio P M, Wang C H–T, Mendonça J T and Bingham R 2009 Dephasing of a non-relativistic quantum particle due to a conformally fluctuating spacetime, To appear in Class. Quantum Grav. (Preprint gr–qc/arXiv:0903.1668v1)

[43] R. A. Isaacson 1968 Gravitational Radiation in the Limit of High Frequency. I. The Linear Approximation and Geometric Optics Phys. Rev. 166 1263

[44] R. A. Isaacson 1968 Gravitational Radiation in the Limit of High Frequency. II. Nonlinear Terms and the Effective Stress Tensor Phys. Rev. 166 1272

[45] S. M. Carrol 2001 The cosmological constant Living Rev. Rel. 4 1

[46] Wang C H–T, Bonifacio P M, Mendonça J T and Bingham R 2008 Nonlinear random gravity. I. Stochastic gravitational waves and spontaneous conformal fluctuations due to the quantum vacuum (Preprint gr–qc/arXiv:0806.3042)

[47] Bekenstein J D and Meisels A 1980 Conformal invariance, microscopic physics, and the nature of gravitation Phys. Rev. D 22(6) 1313–24

[48] Dicke R H 1962 Mach’s principle and invariance under transformation of units Phys. Rev. 125 6 2163

[49] Amelino-Camelia G, Lämmerzahl C, Macias M and Müller H 2005 The search for quantum gravity signals Preprint gr–qc/0501053v1

[50] Anastopoulos C and Hu B L 2007 Decoherence in quantum gravity: issues and critiques J. Phys. Conf. Ser. 67 012012 (Preprint gr–qc/0703137v1)

[51] Hackermüller L, Hornberger K, Brezger B, Zeilinger A and Arndt M 2004 Decoherence of matter waves by thermal emission of radiation Nature 427 711

[52] Stibor A, Hornberger K, Hackermüller L, Zeilinger A and Arndt M 2005 Laser Physics 15(1) 10-17

[53] Boyer T H 1969 Derivation of the blackbody radiation spectrum without quantum assumptions Phys. Rev. 182(5) 1374–83

[54] Frederick C 1976 Stochastic spacetime and quantum theory Phys. Rev. D 13(12) 3183

[55] Moffat J W 1997 Stochastic gravity Phys. Rev. D 56(10) 6264
[56] Miller S D 1999 Einstein-Langevin and Einstein-Fokker-Planck equations for Oppenheimer-Snyder gravitational collapse in a spacetime with conformal vacuum fluctuations Class. Quantum Grav. 16 33813403

[57] Göklü E and Lämmerzahl C 2008 Metric fluctuations and the weak equivalence principle Class. Quantum Grav. 25 105012

[58] Kastrup H A 2008 On the advancements of conformal transformations and their associated symmetries in geometry and theoretical physics Ann. Phys. 17 631 (Preprint physics.hist-ph/0808.2730v1)

[59] Nelson E 1985 Quantum Fluctuations Princeton University Press

[60] Brans C and Dicke R H 1961 Mach’s principle and relativistic theory of gravitation Phys. Rev. 124 3 925

[61] Stern A, Aharonov Y and Imry Y 1990 Phase uncertainty and loss of interference Phys. Rev. A 41 7 3436

[62] Wang C H–T 2005 Conformal geometrodynamics: true degrees of freedom in a truly canonical structure Phys. Rev. D 71 124026

[63] Wang C H–T 2005 Unambiguous spin-gauge formulation of canonical general relativity with conformorphism invariance Phys. Rev. D 72 087501

[64] Everitt M S, Jones M L and Varcoe B T H 2008 Dephasing of entangled atoms as an improved test of quantum gravity Preprint gr–qc/0812.3052v2

[65] DeWitt B S 1966 Phys. Rev. Lett. 16 1092

[66] Linet B and Tourrenc P 1976 Changement de phase dans un champ de gravitation: possibilité de détection interférentielle Can. J. Phys. 54 1129

[67] Cai Y Q and Papini G 1989 Particle interferometry in weak gravitational fields Class. Quantum Grav. 6 407–18

[68] Kiefer C and Singh T P 1991 Quantum gravitational corrections to the functional Schrödinger equation Phys. Rev. D 44(4) 1067

[69] Roman P 1965 Advanced Quantum Theory; an Outline of the Fundamental Ideas Addison-Wesley Pub. Co.

[70] Aloisio R, Galante A, Grillo A, Liberati S, Luzio E and Mendez F 2006 Phys. Rev. D 74 085017

[71] Mazzitelli F D, Paz J P and Villanueva A 2003 Decoherence and recoherence from vacuum fluctuations near a conducting plate Phys. Rev. A 68 062106

[72] Khudiakov D V, Rubtsov I V, Nadtochenko V A and Moravskii A P 1996 Orientational dynamics of C\textsubscript{70} molecules in chlorobenzene Russian Chemical Bulletin 45(3) 601–04

[73] Rabilloud F, Antoine R, Broyer M, Compagnon I, Dugourd P, Rayane D and Calvo F 2007 Electric dipoles and susceptibilities of alkali clusters/fullerene complexes: experiments and simulations J. Phys. Chem. C 111 17795–803
[74] Unruh W G 1976 Notes on black-hole evaporation *Phys. Rev. D* **14** 870–92

[75] Ford L H 1975 Quantum vacuum energy in general relativity *Phys. Rev. D* **11** 3370–77

[76] Ford L H 1993 Electromagnetic vacuum fluctuations and electron coherence *Phys. Rev. D* **47** 5571

[77] Haisch B, Rueda A and Puthoff H E 1994 Inertia as a zero-point-field Lorentz force *Phys. Rev. A* **49** 678

[78] Straumann N 1999 The mystery of the cosmic vacuum energy density and the accelerated expansion of the universe *Eur. J. Phys.* **20** 419–27

[79] Antoniadis I, Mazur P O and Mottola E 2007 Cosmological dark energy: prospects for a dynamical theory *New J. Phys* **9** 11

[80] Milonni P W 1994 *The quantum vacuum: An Introduction to Quantum Electrodynamics* Academic Press, New York

[81] Saunders S 2002 *Is the Zero Point Energy Real?* in: Kuhlmann M, Lyre H, Wayne A (Eds.) *Ontological Aspects of Quantum Field Theory*, World Scientific Singapore

[82] Weinberg S 1989 The cosmological constant problem *Rev. Mod. Phys.* **61** 1-23

[83] Casimir H B G 1948 *Proc. K. Ned. Akad. Wet.* **51** 792

[84] Casimir H B G and Polder D 1948 The Influence of Retardation on the London-van der Waals Forces *Phys. Rev.* **73** 360–72

[85] Mohideen U and Roy A 1998 Precision Measurement of the Casimir Force from 0.1 to 0.9 µm *Phys. Rev. Lett.* **81** 4549–52

[86] Welton T A 1948 Some Observable Effects of the Quantum-Mechanical Fluctuations of the Electromagnetic Field *Phys. Rev.* **74** 1157–67

[87] Beiersdorfer P, Chen H, Thorn D B and Träbert E 2005 Measurement of the Two-Loop Lamb Shift in Lithiumlike U\(^{89+}\) *Phys. Rev. Lett.* **95** 233003

[88] Louisell W H 1973 *Quantum Statistical Properties of Radiation* Wiley New York

[89] Gea-Banacloche J, Scully M O and Zubairy M S 1988 Vacuum Fluctuations and Spontaneous Emission in Quantum Optics *Phys. Scr.* **T21** 81–85

[90] Wódkiewicz K 1988 Stochastic description of vacuum fluctuations *Phys. Rev. A* **38** 2932–36

[91] Callen H B and Welton T A 1951 Irreversibility and Generalized Noise *Phys. Rev.* **83** 34–40

[92] Josephson B D 1962 Possible new effects in superconductive tunnelling *Phys. Lett.* **1** 251

[93] Koch R H, van Harlingen D and Clarke J 1980 Quantum-Noise Theory for the Resistively Shunted Josephson Junction *Phys. Rev. Lett.* **45** 2132

[94] Koch R H, van Harlingen D, Clarke J 1982 Measurements of quantum noise in resistively shunted Josephson junctions *Phys. Rev. B* **26** 74–87
[95] Jaffe R L 2005 The Casimir Effect and the Quantum Vacuum Phys. Rev. D 72 021301(R)

[96] Doran M and Jaeckel J 2006 What measurable zero point fluctuations can(not) tell us about dark energy J. Cosmol. Astropart. Phys. JCAP08 010

[97] Fulling S A, Milton K A, Parashar P, Romeo A, Shajesh K V, and Wagner J 2007 How Does Casimir Energy Fall? Phys. Rev. D 76, 025004

[98] Beck C, Mackey M C 2005 Could dark energy be measured in the lab? Phys. Lett. B 605 295

[99] Jetzer P and Straumann N 2005 Has dark energy really been discovered in the Lab? Phys. Lett. B 606 77

[100] Jetzer P and Straumann N 2006 Josephson junctions and dark energy Phys. Lett. B 639 57

[101] Mahajan G, Sarkar S and Padmanabhan T 2006 Casimir effect confronts cosmological constant Phys. Lett. B 641 6

[102] Mottola E 1986 Quantum fluctuation-dissipation theorem for general relativity Phys. Rev. D 33 2136

[103] Rugh S E and Zinkernagel H 2002 The Quantum Vacuum and the Cosmological Constant Problem Studies in History and Philosophy of Modern Physics 33 663–705

[104] Zel’dovich Y B 1967 Cosmological Constant and Elementary Particles JETP letters 6 316–17

[105] Kolb E W, Matarrese S and Riotto A 2006 On cosmic acceleration without dark energy New J. Phys. 8 322

[106] Li N, Seikel M and Schwarz D J 2008 Is dark energy an effect of averaging? Fortschr. Phys. 56 465–74

[107] Campos A and Verdaguer E 1996 Stochastic semiclassical equations for weakly inhomogeneous cosmologies Phys. Rev. D 53 1927–37

[108] Martín R and Verdaguer E 2000 Stochastic semiclassical fluctuations in Minkowski spacetime Phys. Rev. D 61 124024

[109] Hu B L, Roura A and Verdaguer E 2004 Induced quantum metric fluctuations and the validity of semiclassical gravity Phys. Rev. D 70

[110] Hu B L and Verdaguer E 2004 Stochastic Gravity: Theory and Applications Living Rev. Relativity 7 3

[111] Boyer T H 1975 General connection between random electrodynamics and quantum electrodynamics for free electromagnetic fields and for dipole oscillator systems Phys. Rev. D 11 809

[112] Boyer T H 1980 Thermal effects of acceleration through random classical radiation Phys. Rev. D 21 2137–48

[113] Cavalleri G 1981 The propagator of stochastic electrodynamics Phys. Rev. D 23 363–72

[114] M. Ibison and B Haisch 1996 Quantum and classical statistics of the electromagnetic zero-point field Phys. Rev. A 54 2737
[115] York J W 1983 Dynamical origin of black-hole radiance Phys. Rev. D 28 2929–45

[116] York J W and Schmekel B S 2005 Path integral over black hole fluctuations Phys. Rev. D 72 024022

[117] Arnowitt R, Deser S and Misner C W 1961 Phys. Rev. 121 1556

[118] Hawking S W and Ellis G F R 1973 The large scale structure of space-time Cambridge University Press

[119] Flanagan E F and Hughes S A 2005 The basics of gravitational wave theory New J. Phys. 7 204

[120] Melchiorri A et al. 2000 A measurement of Omega from the North American test flight of BOOMERANG Astrophys. J. 536 L63-L66

[121] Riess A G et al. 1998 Observational Evidence from Supernovae for an Accelerating Universe and a Cosmological Constant Astron. J. 116 1009–38

[122] Padmanabhan T 2008 Dark energy and its implications for gravity (Preprint arXiv:0807.2356v1)

[123] York J W 1971 Gravitational degrees of freedom and the initial-value problem Phys. Rev. Lett. 26 1656

[124] York J W 1972 Role of the conformal three-geometry in the dynamics of gravitation Phys. Rev. Lett. 28 1082

[125] Flanagan E E 2004 Class. Quant. Grav. 21 3817

[126] Weyl H 1952 Space-Time-Matter Dover New York

[127] Hoyle F and Narlikar J V 1974 Action at a Distance in Physics and Cosmology Freeman San Francisco

[128] Faraoni V and Nadeau S 2007 The (pseudo)issue of the conformal frame revisited Phys. Rev. D 75 023501

[129] Dąbrowski M P, Garecki J and Blaschke D B 2008 Conformal transformations and conformal invariance in gravitation (Preprint arXiv:0806.2683v2)

[130] Capozziello S, de Ritis R and Marino A A 1997 Some aspects of the cosmological conformal equivalence between “Jordan Frame” and “Einstein Frame” Class. Quant. Grav. 14 3243–58

[131] Fujii Y 2004 Some aspects of the scalar-tensor theory (Preprint gr–qc/0410097v1)

[132] Jordan F 1959 Z. Physik 157 112

[133] Brans C H 2005 The roots of scalar-tensor theory: an approximate history (Preprint: gr–qc/0506063v1)

[134] Bertotti B, Iess L and Tortora P 2003 A test of general relativity using radio links with the Cassini spacecraft Nature 425 374
[135] Shapiro S S, Davis J L, Lebach D E and Gregory J S 2004 Measurement of the Solar Gravitational Deflection of Radio Waves using Geodetic Very-Long-Baseline Interferometry Data, 1979-1999 Phys. Rev. Lett. 92 121101

[136] Fujii Y and Maeda K 2003 The Scalar Tensor Theory of Gravitation Cambridge University Press

[137] Adler R 1981 The Geometry of Random Fields John Wiley & Sons.

[138] Eriksen H K, Banday A J and Górski K M 2002 The N-point Correlation Functions of the COBE-DMR Maps Revisited Astron. Astrophys. 395 409–416

[139] Weinberg S 1972 Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity John Wiley & Sons New York

[140] Misner C W, Thorne K S and Wheeler J A 1973 Gravitation W. H. Freeman San Francisco

[141] Brill D R and Hartle J B 1964 Method of self-consistent field in general relativity and its application to the gravitational geon Phys. Rev. 135 B271

[142] De Witt B S and Brehme R W 1960 Ann. Phys. 90 220

[143] Synge J L 1960 Relativity: the General Theory North Holland Publishing Co. Amsterdam

[144] d’Espagnat B 1976 Conceptual Foundations of Quantum Mechanics Perseus Books Publishing

[145] Einstein A, Podolsky B and Rosen N 1935 Can Quantum-Mechanical description of physical reality be considered complete? Phys. Rev. 47 777–80

[146] Bell J S 1987 Speakable and Unspeakable in Quantum Mechanics Cambridge University Press

[147] Aspect A, Granigier P and Roger G 1935 Experimental test of Einstein-Podolsky-Rosen Gedankenexperiment: a new violation of Bell’s inequalities Phys. Rev. Lett. 48 91-94

[148] Zeh H D 1996 What is achieved by decoherence? (arXiv:quant-ph/9610014v1)

[149] Zureck W H 2003 Decoherence and the transition from quantum to classical (arXiv:quant-ph/0306072v1)

[150] Joss E 1999 Elements of environmental decoherence, in proceedings of the Bielefeld conference on Decoherence: Theoretical, Experimental, and Conceptual Problems edited by P. Blanchard, D. Giulini, E. Joos, C. Kiefer, and I.-O. Stamatescu, Springer (arXiv:quant-ph/9908008v1)

[151] Brune M, Hagley E, Dreyer J, Maître X, Moali Y, Wunderlich C, Raimond J M and Haroche S 1996 Observing the progressive decoherence of the ‘meter’ in a quantum measurement Phys. Rev. Lett. 77 4887

[152] Arndt M, Nairz O, Vos-Andreae J, Keler C, van der Zouw G and Zeilinger A 1999 Waveparticle duality of C60 molecules Nature 401 680

[153] Joos E, Zeh H D 1985 The emergence of classical properties through interaction with the environment Z. Phys. B59 223–43.