Wilson Loop on a Light-Cone Cylinder

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Abstract

QCD without matter and quantized on a light-cone spatial cylinder is considered. For the gauge group SU(N) the theory has $N - 1$ quantum mechanical degrees of freedom, which describe the color flux that circulates around the spatial cylinder. In 1+1 dimensions this problem can be solved analytically. I use the solution for SU(2) to compute the Wilson loop phase on the surface of the cylinder and find that it is equal to $g^2 \text{area}/4$. This result is different from the well known result for flat space. I argue that for SU(N) the Wilson loop phase for a contour on a light-cone spatial cylinder is $g^2 (\text{area})(N - 1)/4$. The underlying reason for this result is that only the $N - 1$ dimensional Cartan subgroup of SU(N) is dynamical in this problem.
I. INTRODUCTION

The expectation value of the Wilson loop is an important quantity in gauge theories. This phase is a gauge invariant quantity that provides information about the long range behavior of a theory, however it is beyond the reach of weak coupling perturbation theory calculations. On the other hand for non-perturbative approaches the Wilson loop is an important object to consider. While for QCD in 3+1 dimensions the calculation of the Wilson loop is quite difficult, in 1+1 dimension it is much more tractable and in some problems can be calculated exactly.

The problem of pure glue QCD in 1+1 dimensions in the gauge SU(N) with periodic boundary conditions can be solved exactly [1,2] since it has only \( N - 1 \) degrees of freedom which are independent of space and the problem is therefore a quantum mechanical rather than a true field theory problem. Nevertheless the problem is very interesting from a number of points of view. First the degrees of freedom are simply color flux loops that circulate around the entire spatial cylinder and as such they rely on the fact that the problem is formulated on a cylindrical topology. This problem is particularly interesting to people studying light-cone field theory since it is the only known gauge theory where the Hamiltonian takes exactly the same functional form in both the light-cone and equal-time formulations.

We will briefly review the formulation and solution of this problem, here using the light-cone gauge and light-cone quantization in the gauge \( SU(2) \). We will solve for the wave-functions and the energy eigenvalue of the problem. We will solve the equations of motion for the vector potential and use the solution to calculate the path integral of the vector potential around a closed loop on the surface of the cylinder which makes up space time in this problem. We then calculate the Wilson loop by taking the vacuum expectation value of this loop calculation.

There is an exact general result for the Wilson loop expectation value in 1+1 dimensional QCD in the absence of matter. The result we find here by direct calculation for \( SU(2) \) on a cylinder does not agree with this result. This is perhaps not surprising because of the special topology of the space we consider. Based on our result for \( SU(2) \) we suggest a general result for the value of the Wilson loop phase for \( SU(N) \) on a cylinder. Our conjecture agrees with the general result in the large \( N \) limit as would be suggested by the work of Gross [4].

II. GAUGE FIXING

The Lagrangian density for SU(2) non-Abelian gauge theory in 1+1 dimensions is,

\[
\mathcal{L} = \frac{1}{2} \text{Tr} (F_{\mu\nu} F^{\mu\nu})
\]

where \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - g [A_\mu, A_\nu] \). We consider the theory on a finite interval, \( x^- \) from \(-L\) to \( L\), and we impose periodic boundary conditions on all gauge potentials \( A_\mu \).

We now show that the light-cone gauge \( A^+ = 0 \) which is the one that one normal would prefer to use for light-cone quantization cannot be reached. A gauge transformation \( U \) bringing a gauge potential \( B^\mu \), itself in some arbitrary gauge configuration, to some other gauge configuration \( A^\mu \) is
\[ gA^\mu = \partial_\muUU^{-1} + gUB^\muUU^{-1}. \] (2.2)

Here \( g \) is the coupling constant and \( U \) is an element of the Lie algebra of SU(2). Clearly \( U \) given by

\[ U = P \exp \left[ -g \int_{-L}^{x^-} dy^- B^+(y^-) \right] \] (2.3)

will bring us to the gauge \( A^+ = 0 \).

We appear to have been successful in getting the light-cone gauge. However, the element \( U \) through which we wish to achieve the gauge condition must satisfy \( Z_2 \) periodic boundary condition, \( U(x^-) = (\pm)U(x^- + 2L) \). This is so, because gauge fixing is usually done with trivial elements of the gauge group — namely transformations generated by the Gauss law operator via the classical brackets or corresponding quantum commutators. However for this to be carried through, one needs to be able to discard surface terms. With nonvanishing boundary conditions this can only be realized by \( Z_2 \) periodic elements \( U \). Clearly Eq.(2.3) does not satisfy these boundary conditions. So in fact the attempt has failed.

With a modification of Eq.(2.3),

\[ U(x) = e^{gxx^+} Pe^{-g \int_{-L}^{x^-} dy^- B^+(y^-)} . \] (2.4)

where \( V^+ \) is the integral \( B^+ \) over space normalized by the length of the spatial cylinder, sometimes called the ”zero mode”, this is an allowed gauge transformation. However it does not completely bring us to the light-cone gauge. We find instead

\[ A^+ = V^+ . \] (2.5)

In other words, we cannot eliminate the zero mode of the gauge potential. The reason is evident: it is invariant under periodic gauge transformations. But of course we can always perform a rotation in color space. In line with other authors \([10]\), we choose this so that \( V^+ = v(x^+)\tau_3 \) is the only non-zero element, since in our representation only \( \tau_3 \) is diagonal.

In addition, we can impose the subsidiary gauge condition that the zero mode of \( A^- \) is zero. This would appear to have enabled complete fixing of the gauge. This is still not so. Gauge transformations

\[ G = \exp \{ ix^- \left( \frac{n\pi}{L} \right) \tau_3 \} \] (2.6)

generate shifts, according to Eq.(2.2), in the zero mode component

\[ v(x^+) \rightarrow v(x^+) + \frac{n\pi}{gL}. \] (2.7)

All of these possibilities, labelled by the integer \( n \), of course still satisfy \( \partial_- A^+ = 0 \), but as one sees \( n = 0 \) should not really be included. One notes that the transformation is \( x^- \)-dependent and \( Z_2 \) periodic. It is thus a simple example of a Gribov copy \([9]\) in 1+1 dimensions. We follow the conventional procedure by demanding
v(x^+) \neq \frac{n\pi}{gL}, \ n = \pm 1, \pm 2, \ldots \ . \ \ (2.8) \\

This eliminates singularity points at the Gribov ‘horizons’ which in turn correspond to a vanishing Faddeev-Popov determinant [8].

The equations of motion for the theory are

\[ [D^\mu, F_{\mu\nu}] = \partial^\mu F_{\mu\nu} - g[A^\mu, F_{\mu\nu}] = 0 . \ \ (2.9) \]

For our purposes it is convenient to break this equation up into color components \( A^\mu_a \). Color will always be the lower index. Rather than the three color fields \( A^\mu_1, A^\mu_2 \) and \( A^\mu_3 \) we will use chiral notation with \( A^\mu_+ = A^\mu_1 + iA^\mu_2 \) and \( A^\mu_- = A^\mu_1 - iA^\mu_2 \). With the above gauge conditions the \( \nu = + \) equations are

\[
\begin{align*}
(i\partial^+)^2 A^-_3 &= 0, \ \ (2.10) \\
(i\partial^+ + g v(x^+))^2 A^-_- &= 0 \ \ (2.11)
\end{align*}
\]

These equation are of course easily solvable. The solution for \( A^-_- \) is zero up to a constant which is the zero mode. Earlier we used our gauge freedom to set this zero mode to zero. The operator in the equation for \( A^-_- \) is in fact not singular in a particular Gribov region and is therefore invertible giving \( A^-_- = 0 \).

The only remaining equation of motion that is not totally trivial is the equation for \( v \)

\[ \partial^2_+ v(x^+) = 0 \ \ (2.12) \]

The solution is of course

\[ v(x^+) = \frac{g^2}{2\pi} \Pi_z x^+ + v(0) \ \ (2.13) \]

and where \( \Pi_z \) is the canonical momentum defined below.

The Hamiltonian for this quantum mechanics problem is easily obtained from the above Lagrangian and we find,

\[ P^- = L\partial^2_+ v(x^+) \ \ (2.14) \]

This leads to a set of properly normalized conjugate variables,

\[ z = \frac{gLv}{\pi} \quad \Pi_z = \frac{2\pi}{g} \partial_+ v \ \ (2.15) \]

which satisfy the canonical commutation relation \([z, \Pi] = i\) in the fundamental modular domain \(-1 < z < 0\). The Schrödinger equation is straightforward to solve and wavefunction and energy eigenvalues are

\[ \psi_n(z) = C_n \sin(n\pi z) \quad E_n = \frac{g^2L(n^2 - 1)}{4} \ \ (2.16) \]

where we have renormalized the ground state \((n = 1)\) energy to zero and \( C_n \) are normalization constants. The wavefunction must vanish at \( z = 0 \) and \(-1\) the Gribov horizons [2].
III. WILSON LOOP

The vacuum expectation value of the Wilson loop provides information about the large distance behavior of a theory which is not accessible to perturbation theory calculation. The well established lore associated with the Wilson loop is that if the phase goes like the area of the enclosed contour the theory is confining, whereas if the phase goes like the perimeter of enclosed contour the theory is not confining.

In 1+1 dimensions the general result for $SU(N)$ QCD without matter is \[ W \propto e^{i g^2 \frac{(N^2-1)}{8} A}, \] where $A$ is the area enclosed by the Wilson loop. The Wilson loop for the problem we are considering here can be written as the vacuum expectation value of the Wilson loop phase factor, \[ W = \langle \psi_1 | Tr P e^{i g \oint A \cdot dx} | \psi_1 \rangle. \] The vacuum expectation value here takes the form of the expectation value with the ground state wavefunction. The contour that we will choose for the path integral consists of straight lines connecting the following points in $(x^-, x^+)$ space on the surface of the light-cone space-time cylinder.

\[ (0, 0) \rightarrow (l, 0) \rightarrow (l, t) \rightarrow (0, t) \rightarrow (0, 0) \]

The only component of $A^\mu$ that is non-zero is $A^+ = v(x^+) \tau_3$; therefore the contour integral yields $(v(0) - v(t)) l \tau_3$. Now using the solution of the equation of motion for $v(x^+)$ we find for the contour integral, \[ ig \oint A \cdot dx = -ig^2 2\pi A \Pi_2 \tau_3 \] where $A$ is the area of the enclosed contour. This leads to the following expression for the Wilson loop $W$,

\[ W = Tr \int_{-1}^{0} dz \sin(\pi z)(\cos(\theta) + i \tau_3 \sin(\theta)) \sin(\pi z) \]

where \[ \theta = \frac{1}{4\pi} \frac{g^2 A}{dz} \]

The momentum operator acts on the ground state wave function to the right. The expansion of the $\sin(\theta)$ gives an odd number of derivatives leaving an integral of $\sin(\pi z) \cos(\pi z)$ which vanishes when integrated over the Gribov region $-1 < z < 0$. This leaves only the $\cos(\theta)$ function and after some algebra I find

\[ W \propto \cos\left(\frac{g^2 A}{4}\right) \] (3.5)

The general result Eqn(3.1) for the Wilson loop when evaluated for $N = 2$ gives $\frac{g^2 A}{8}$. \[ }
IV. DISCUSSION

Let us first summarize the essential points. I analyzed pure glue non-Abelian gauge theory in a compact spatial volume with periodic boundary conditions on the gauge potentials. Working in the light-cone Hamiltonian approach, I demonstrated how one carefully fixes the gauge. The quantum field theory problem then reduces to a quantum mechanical problem which can be solved exactly. Given this exact non-perturbative result for the vacuum state and vector potential it becomes a straightforward calculation to evaluate the Wilson loop and the result for the gauge group $SU(2)$, $\frac{g^2 A^4}{4}$ does not agree with the general result.

How can we understand these different results? The natural explanation seems to be that on the cylinder the gauge field only has support on the abelian Cartan subalgebra whereas the general result gets contributions from all color components. We can speculate about the extension of this calculation to $SU(N)$ where the vector potential only has support only $N - 1$ dimensional abelian Cartan sub-algebra. Since the contributions are abelian we expect the phases to simply add for each additional field component and therefore the Wilson loop phase should be

$$W \propto e^{ig^2(N-1)A_4}.$$  \hspace{1cm} (4.1)

In a rather different context Gross [5] has identified the topological expansion of the space on which the Wilson loop is calculated with the $1/N$ expansion of the result. This then allows us to connect our calculation on a cylindrical topology with the large $N$ expansion of the general result Eqn(3.1). We see that in the large $N$ limit Eqn(3.1) and Eqn(4.1) agree. Thus a possible interpretation of this calculation might be the explicit verification of the Gross [5] result.

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