A Note on Y-energies of Knots

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Abstract

We study a 1-form which can be given by a vector in a conformally invariant way. We then study conformally invariant functionals associated to a “Y-diagram” on the space of knots which are made from the 1-form.

Key words and phrases. Geometric knot theory, conformal geometry, Möbius transformation

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1 Introduction

The study was motivated to merge the following two:

In [O1], we defined the energy of a knot. It can be considered as the normalization of modified electrostatic energy of a charged knot. It measures geometric complexity of knots. Later on, it was proved to be conformally invariant ([FHW]).

On the other hand, Lin and Wang gave functionals associated to chord diagrams on the space of knots using a 1-form which comes from the Gauss integral formula for the linking number ([LW]).

The energy of a knot can be considered as the integration of the interaction between a pair of points on a knot. Let us proceed to study functionals so that more than two points on a knot are involved in the integrands. For this purpose, we study functionals associated to a chord diagram of the shape of “Y”.

We first show that a vector in $\mathbb{R}^3$ can give, in a conformally invariant way, a vector field, or equivalently, a 1-form on the complement of the point from which the vector starts. This can be done by a conformal transportation of the vector, which is generalization of the notion introduced in [LO]. We can play a similar game to that in [LW] by substituting this conformally invariant 1-form for the 1-form coming from the Gauss formula for the linking number. But unfortunately, it does not work well: It turns out that our functionals are either identically 0 or identically $+\infty$ for any knot.

This article serves as an errata to the author’s talk at the AMS Spring Western Sectional Meeting at San Francisco in May 2003. He presented four kinds of
Y-energies; two are conformally invariant and the other two not. But the first one turns out to be 0, the second one diverges, and the third one was already given in Lin and Wang’s paper, and the fourth one can be obtained from the third one by replacing the integrand by its absolute value.

2 The results by Lin and Wang

Let us start with introducing the study by Lin and Wang [LW].

2.1 The linking number

Let \( L = K_1 \cup K_2 \) be a 2-component link in \( \mathbb{R}^3 \) with \( K_1 = f(S^1) \) and \( K_2 = g(S^1) \). Put

\[
\varphi : \mathbb{R}^3 \times \mathbb{R}^3 \setminus \Delta \ni (x,y) \mapsto \frac{x-y}{|x-y|} \in S^2.
\]

(1)

Define a map \( \varphi_L \) from a torus \( K_1 \times K_2 \) to \( S^2 \) by \( \varphi_L = \varphi_{|K_1 \times K_2} \). The linking number \( Lk(K_1, K_2) \) of \( K_1 \) and \( K_2 \) is equal to the degree of \( \varphi_L \).

Let \( \omega_{S^2} \) be the unit volume form of \( S^2 \):

\[
\omega_{S^2} = \frac{1}{4\pi} x_1 dx_2 \wedge dx_3 + x_2 dx_3 \wedge dx_1 + x_3 dx_1 \wedge dx_2.
\]

(2)

Define a 2-form \( \omega \) on \( \mathbb{R}^3 \times \mathbb{R}^3 \setminus \Delta \) by \( \omega = \varphi^* \omega_{S^2} \). Then the linking number can be expressed by the Gauss integral:

\[
\text{GI}(f,g) = \int_{K_1 \times K_2} \varphi_L^* \omega_{S^2} = \int_{K_1 \times K_2} \omega(x,y)
= \frac{1}{4\pi} \int_{S^1 \times S^1} \frac{\det(f'(s),g'(t),f(s) - g(t))}{|f(s) - g(t)|^3} ds dt,
\]

(3)

where \( f(s), g(t) \) are considered as column vectors and \( \times \) denotes the vector product in \( \mathbb{R}^3 \).

2.2 Integration associated with chord diagrams

The formulae (2) and (3) can be considered as the integrals of the interactions between a pair of points on a knot or a link. We can generalize them by taking into account more complex combinations of points.

For this purpose, it is natural to use so-called chord diagrams which are used in the study of the Kontsevich integral of the Vassiliev invariant. Then (2) and (3) can be considered to be associated with the “\( \theta \)-graph” as illustrated in Figure 1 and with the “handcuffs graph”.

2
2.3 Gauss integral associated with chord diagrams

It is convenient to give the domain of the integral associated with a chord diagram as a subset of a configuration space. Let \( p_1, \ldots, p_n \) be points on an oriented knot. We write \( p_1 \prec \cdots \prec p_n \) if the cyclic order of \( p_1, \ldots, p_n \) coincides with the orientation of the knot. Put

\[
\text{Conf}_{3,1}(K, \mathbb{R}^3) = \{(y_1, y_2, y_3, x) \mid y_1, y_2, y_3 \in K, x \in \mathbb{R}^3 \setminus K, y_j \neq y_k (j \neq k)\},
\]

\[
U_Y = \{(y_1, y_2, y_3, x) \in \text{Conf}_{3,1}(K, \mathbb{R}^3) \mid y_1 \prec y_2 \prec y_3\}.
\]

(4)

**Definition 2.1** ([LW]) Let \( \omega = \varphi^* \omega_{S^2} \) as before. Define the \( X \)-Gauss integral and \( Y \)-Gauss integral of a knot \( K \) by

\[
GI_X(K) = \int_{y_1 \prec y_2 \prec y_3 \prec y_4} \omega(y_1, y_3) \wedge \omega(y_2, y_4),
\]

(5)

\[
GI_Y(K) = \int_{U_Y} \omega(x, y_1) \wedge \omega(x, y_2) \wedge \omega(x, y_3).
\]

(6)

Theorem 2.2 ([BT], [LW]) There holds

\[
\frac{1}{4} GI_X(K) - \frac{1}{3} GI_Y(K) + \frac{1}{24} = v_2(K),
\]

where \( v_2(K) \) is the second coefficient of the Conway polynomial.
2.4 Expression in terms of 1-form

In order to generalize GI$_Y$($K$) in a conformal geometric way in Subsection 5 we give another expression of GI$_Y$($K$).

For a vector $v \in T_y \mathbb{R}^3$ we define a vector field $X_G(y; v)$ on $\mathbb{R}^3 \setminus \{y\}$ by

$$X_G(y; v)(x) = \frac{1}{4\pi} \cdot \frac{x-y}{|x-y|^3} \times v = \frac{1}{4\pi} v \times \nabla \left( \frac{1}{|x-y|} \right) \quad (x \in \mathbb{R}^3 \setminus \{y\}),$$

and a 1-form $\lambda_G(y; v)$ on $\mathbb{R}^3 \setminus \{y\}$ by

$$\lambda_G(y; v)(u) = u \cdot X_G(y; v)(x) = \frac{\det(x-y, v, u)}{4\pi|x-y|^3} \quad (u \in T_x \mathbb{R}^3).$$

Let $v_y$ denote a unit tangent vector to $K$ at $y$. Then GI$_Y$($K$) can be expressed in terms of this 1-form, or equivalently, by the vector field as follows (LW):

$$GI_Y(K) = \int_{U_y} dy_1 dy_2 dy_3 \lambda_G(y_1; v_1) \wedge \lambda_G(y_2; v_2) \wedge \lambda_G(y_3; v_3)$$

$$= -\int_{U_y} \det(X_G(y_1; v_1)(x), X_G(y_2; v_2)(x), X_G(y_3; v_3)(x)) \ dy_1 dy_2 dy_3 dvolx. \quad (7)$$

**Definition 2.3** Let $K = f(S^1)$ be a knot. Define a 1-form $\lambda_K$ on the complement of the knot $K$ by

$$\lambda_K(u) = \int_K \lambda_G(y; v_y)(u) dy = \frac{1}{4\pi} \int_K \frac{\det(x-y, v_y, u)}{|x-y|^3} dy$$

for $u \in T_x \mathbb{R}^3 (x \notin K)$, where $v_y$ denotes the unit tangent vector to $K$ at $y$.

Although $\lambda_G(y; v_y)$ is not closed,

**Lemma 2.4** The 1-form $\lambda_K$ is closed.

**Proof:** Suppose the knot $K$ is given by $K = f(S^1)$. Then

$$4\pi \lambda_K(u) = u \cdot \int_{S^1} \frac{x - f(s)}{|x - f(s)|^3} \times f'(s) \ ds.$$ 

Therefore, the 2-form $d\lambda_K$ vanishes if and only if

$$\nabla \times \left( \int_{S^1} \frac{x - f(s)}{|x - f(s)|^3} \times f'(s) \ ds \right) = 0,$$
which is the consequence of

\[
\nabla \times \left( \frac{x - f(s)}{|x - f(s)|^3} \times f'(s) \right) = -(f'(s) \cdot \nabla) \left( \frac{x - f(s)}{|x - f(s)|^3} \right) + \nabla^2 \left( \frac{1}{|x - f(s)|} \right) f'(s) = \frac{f'(s)}{|x - f(s)|^3} - 3 \frac{f'(s) \cdot (x - f(s))}{|x - f(s)|^5} (x - f(s)) = -\frac{d}{ds} \left( \frac{x - f(s)}{|x - f(s)|} \right).
\]

When \( K' \) is a knot in \( \mathbb{R}^3 \setminus K \) then

\[
\int_{K'} \lambda_K = \frac{1}{4\pi} \iint_{K \times K'} \frac{\det(x - y, v_y, v_x)}{|x - y|^3} \, dx \, dy
\]
is the linking number of \( K \) and \( K' \).

### 3 Preliminaries from conformal geometry

**Definition 3.1** (Doyle and Schramm) Let \( C(x, y, y) \) \( (x \neq y) \) be the circle which is tangent to \( K \) at \( y \) that passes through \( x \), and let \( v_x \) be the unit tangent vector to \( K \) at \( x \). Define the conformal angle \( \theta_K(x, y) \) \( (0 \leq \theta_K(x, y) \leq \pi) \) by the angle between \( C(x, y, y) \) and \( v_x \) at \( x \).

**Lemma 3.2** ([LO]) The conformal angle \( \theta_K(x, y) \) is of the order of \(|x - y|^2\) near the diagonal. To be precise,

\[
\theta_K(x, y) = \frac{\sqrt{\kappa'^2 + \kappa^2 \tau^2}}{6} |x - y|^2 + O(|x - y|^3)
\]
near the diagonal, where \( s, \kappa, \tau \) denote the arc-length, curvature, and torsion of \( K \) respectively, and \( \kappa' \) means \( \frac{d\kappa}{ds} \).

**Definition 3.3** Let \( T \) be a Möbius transformation of \( \mathbb{R}^3 \cup \{\infty\} \). Define \(|T'(r)|\) for \( p \in \mathbb{R}^3 \) by

\[
|T'(r)| = |\det dT(p)|.
\]

**Lemma 3.4** (1) If \( I_0(p) \) is an inversion in a sphere of radius \( r \) with center the origin then

\[
|I_0(p)'(p)| = \frac{r}{|p|}.
\]
We have
\[ |T(p) - T(q)| = |T'(p)||T'(q)||p - q| \] (9)
for a pair of points \( p, q \) in \( \mathbb{R}^3 \) and
\[ |T_*v| = |T'(p)|^2|v| \] (10)
for a vector \( v \) in \( T_p\mathbb{R}^3 \).

**Corollary 3.5** The 2-form \( \frac{dxdy}{|x - y|^2} \) on \( K \times K \setminus \Delta \) is conformally invariant. In other words, if \( T \) is a Möbius transformation and \( \tilde{x} \) and \( \tilde{y} \) denote \( T(x) \) and \( T(y) \) then
\[ T^* \left( \frac{d\tilde{x}d\tilde{y}}{|\tilde{x} - \tilde{y}|^2} \right) = \frac{dxdy}{|x - y|^2}. \] (11)

4 Conformally invariant 1-form via vector field

We shall construct a vector field on \( \mathbb{R}^3 \setminus \{y\} \) for a vector in \( T_y\mathbb{R}^3 \) in a conformally invariant manner. It gives a conformally invariant 1-form on \( \mathbb{R}^3 \setminus \{y\} \). We can define conformally invariant functionals associated with the \( Y \)-graph in terms of this 1-form.

We make use of tangent circles at a given vector in \( T\mathbb{R}^3 \) since a Möbius transformation maps a circle into a circle and tangent curves into tangent curves.

**Definition 4.1** Let \( x \) and \( y \) be a pair of points \( (x \neq y) \) and \( v \) a vector in \( T_y\mathbb{R}^3 \). Let \( C(y; v, x) \) denote the circle through \( x \) and \( y \) which is tangent to \( v \) at \( y \) whose orientation is given by \( v \).

![Figure 4: \( v \) and \( \hat{v} \)](image)

(1) Let \( \hat{v}(x) \) be a tangent vector to \( C(y; v, x) \) at \( x \) with the same norm as \( v \). As \( \hat{v}(x) \) is symmetric to \( v \) in the line joining \( x \) and \( y \), it is given by
\[ \hat{v}(x) = \left\{ 2 \left( \frac{v \cdot x - y}{|x - y|} \right) \frac{x - y}{|x - y|} - v \right\}. \] (12)
We call the correspondence \( v \mapsto \hat{v}(x) \) the *unit conformal transportation* from \( y \) to \( x \) ([LO]).
Let $v$ be a vector in $T_y\mathbb{R}^3$. Define a 1-form $\omega = \omega(y; v)$ on $\mathbb{R}^3 \setminus \{y\}$ by

$$\omega(u) = u \cdot \hat{v}(x) \quad (x \in \mathbb{R}^3 \setminus \{y\}, \ u \in T_x\mathbb{R}^3).$$

Remark: The 1-form $\hat{\omega} = \hat{\omega}(y; v)$ on $\mathbb{R}^3 \setminus \{y\}$, where $v$ is a unit vector in $T_y\mathbb{R}^3$, defined by

$$\hat{\omega}(u) = u \cdot \hat{v}(x) \quad (x \in \mathbb{R}^3 \setminus \{y\}, \ u \in T_x\mathbb{R}^3)$$

was used by Hélein to show the isoperimetric inequality (H). Then the vector field and the 1-form can be expressed in terms of an inversion in a 2-sphere.

**Lemma 4.3** Let $I_p$ ($p \in \mathbb{R}^3$) denote an inversion in the 2-sphere with center $p$ and radius 1. Let $v$ be a vector in $T_y\mathbb{R}^3$ and $x$ a point in $\mathbb{R} \setminus \{y\}$.

1. $\hat{v}(x) = -I_{x*}(v)$ as a vector.
2. $\hat{\omega}(y; v)(u) = -v \cdot (I_{y*}u)$ for $u \in T_x\mathbb{R}^3$.

**Proof:**

1. Since $v$ is tangent to the circle $C(y; v, x)$, the vector $I_{x*}v$ is tangent to $I_x(C(y; v, x))$. Since the tangent vector to $C(y; v, x)$ at $x$ is equal to a positive multiple of $\hat{v}(x)$, $I_x(C(y; v, x))$ is a line whose direction vector is equal to a positive multiple of $-\hat{v}(x)$. Therefore, $I_{x*}v$ is a positive multiple of $-\hat{v}(x)$. Then the conclusion comes from $|I_{x*}v| = \frac{|v|}{|x - y|^2} = |\hat{v}(x)|$, which is implied by Lemma 3.4.

2. The image $I_y(C(y; v, x))$ is the line through $I_y(x)$ whose direction vector is equal to a positive multiple of $-v$. Since $\hat{v}(x)$ is tangent to $C(y; v, x)$, it follows that $I_{y*}\hat{v}(x)$ is a positive multiple of $-v$. Since $\angle I_{y*}$ is a conformal mapping, we have the following equality between angles:

$$\angle I_{y*}u \cdot (-v) = \angle I_{y*}u \cdot I_{y*}\hat{v}(x) = \angle u \cdot \hat{v}(x).$$

Since $|I_{y*}u| = \frac{|u|}{|x - y|^2}$ by Lemma 3.4 we have

$$\omega(y; v)(u) = u \cdot \hat{v}(x) = (I_{y*}u) \cdot (-v).$$

As a corollary we have
Proposition 4.4 Define a map \( \psi : \mathbb{R}^3 \setminus \{y\} \rightarrow \mathbb{R} \) by

\[
\psi(x) = -I_y(x) \cdot v.
\]

Then \( \tilde{\omega}(y; v) = d\psi \), i.e. the 1-form \( \tilde{\omega}(y; v) \) is exact.

**Proof:** Suppose \( x \in \mathbb{R} \setminus \{y\} \) and \( u \in T_x\mathbb{R}^3 \). Then Lemma 4.3 (2) implies

\[
d\psi(u) = -(I_y \ast u) \cdot v = \tilde{\omega}(y; v)(u).
\]

We remark that the 1-form \( \lambda_G(y; v_y) \) is not necessarily closed.

Proposition 4.5 The \( \tilde{\omega}(y; v) \) is conformally invariant, i.e.

\[
\tilde{\omega}(T(y); T \ast v)(T \ast u) = \tilde{\omega}(y; v)(u)
\]

for any \( u \in T_x\mathbb{R}^3 \) (\( x \in \mathbb{R}^3 \setminus \{y\} \)) and Möbius transformation \( T \) of \( \mathbb{R}^3 \cup \{\infty\} \).

**Proof:** Lemma 4.4 implies

\[
|T \ast u| = |T'(x)|^2 |u|,
\]

\[
|T \ast v| = |T'(y)|^2 |v|.
\]

\[
|\tilde{\omega}_T(v(T(x)))| = \frac{|T \ast v|}{|T(y) - T(x)|^2} = \frac{|v|}{|T'(x)|^2 |x - y|^2},
\]

where \( \tilde{\omega}_T(v(T(x))) \) denotes the vector \( T \ast v \) conformally transported to \( T(x) \) (Definition 4.1).

We have

\[
\angle T \ast u \cdot \tilde{\omega}_T(v(T(x))) = \angle u \cdot \tilde{v}(x).
\]

This is because the right hand side is equal to the angle between \( u \) and \( C(y; v, x) \) at \( x \) as \( \tilde{v}(x) \) is tangent to the circle \( C(y; v, x) \) at \( x \), whereas the left hand side is equal to the angle between \( T \ast u \) and \( T(C(y; v, x)) \) at \( T(x) \) as \( \tilde{\omega}_T(v(T(x))) \) is tangent to the circle \( T(C(y; v, x)) \) at \( T(x) \).

Putting (13), (14), and (15) together we have

\[
\tilde{\omega}(T(y); T \ast v)(T \ast u) = (T \ast u) \cdot \tilde{\omega}_T(v(T(x))) = u \cdot \tilde{v}(x) = \tilde{\omega}(y; v)(u).
\]

Let us play the same game as in the Subsection 2 by substituting the conformally invariant 1-form \( \tilde{\omega} \) for the 1-form \( \lambda_G \) coming from the Gauss formula for the linking number. But unfortunately, it turns out that this attempt is not successful.

A similar construction of \( \lambda_K \) from \( \lambda_G(y; v_y) \) gives a trivial 2-form for \( \tilde{\omega}(y; v_y) \).
Proposition 4.6 Let \( \bar{\omega}_K \) be a 1-form on \( \mathbb{R}^3 \setminus K \) defined by
\[
\bar{\omega}_K(u) = \int_K \bar{\omega}(y; v_y)(u) \, dy = \int_K u \cdot \hat{v}_y(x) \, dy = u \cdot \int_{S^1} \hat{f}'(s)(x) \, ds
\]
for \( u \in T_x \mathbb{R}^3 \), where \( v_y \) is the unit tangent vector to \( K \) at \( y \). Then \( \bar{\omega}_K \) vanishes for any knot \( K \).

Proof: Lemma 4.3 indicates that
\[
\int_{S^1} \hat{f}'(s)(x) \, ds = -\int_{S^1} I_{x*}(f'(s)) \, ds = -\int_{S^1} \frac{d}{ds}(I_x \circ f(s)) \, ds = 0. \tag{16}
\]
\( \square \)

5 Conformally invariant \( Y \)-energy

Using the conformally invariant 1-form given in the previous section we can consider the same construction of \( GI_Y \) from \( \lambda_K \) [7]. But unfortunately, it gives a trivial functional.

Let \( v_{y_i} \) (\( i = 1, 2, 3 \)) be a unit tangent vector at \( y_i \) and \( \hat{v}_{y_i} = \hat{v}_{y_i}(x) \) be the image of \( v_{y_i} \) under the unit conformal transportation from \( y_i \) to \( x \) (Definition 4.1). Let \( \bar{\omega}(y; v) \) be the 1-form given in Definition 4.2. Let \( U_Y \) be the domain in \( K^3 \times \mathbb{R}^3 \) given by \{1\}.

Proposition 5.1 Put
\[
E^Y_K(K) = \int_{U_Y} dy_1 dy_2 dy_3 \bar{\omega}(y_1; v_{y_1}) \wedge \bar{\omega}(y_2; v_{y_2}) \wedge \bar{\omega}(y_3; v_{y_3})
\]
\[
= \int_{U_Y} \frac{\det(\hat{v}_{y_2}(x), \hat{v}_{y_2}(x), \hat{v}_{y_3}(x))}{|y_1 - x|^2|y_2 - x|^2|y_3 - x|^2} dy_1 dy_2 dy_3 d_{vol} x. \tag{17}
\]
Then \( E^Y_K(K) = 0 \) for any knot \( K \).

It is enough to show the following.

Lemma 5.2 Suppose \( y_1 \neq y_2 \neq y_3 \neq y_1 \). Then
\[
\int_{\mathbb{R}^3 \setminus \{y_1, y_2, y_3\}} \bar{\omega}(y_1; v_{y_1}) \wedge \bar{\omega}(y_2; v_{y_2}) \wedge \bar{\omega}(y_3; v_{y_3}) = 0.
\]

Proof: Define a map \( \psi_i : \mathbb{R}^3 \setminus \{y_i\} \to \mathbb{R} \) (\( i = 1, 2, 3 \)) by
\[
\psi_i(x) = -I_{y_i}(x) \cdot v_{y_i},
\]
where \( I_{y_i} \) is an inversion in a sphere with radius 1 and center \( y_i \). Proposition 4.4 implies that
\[
\omega(y_i; v_{y_i}) = d\psi_i.
\]
Let \( \Sigma_i (i = 1, 2, 3) \) be a sphere with radius \( \varepsilon (\varepsilon \ll 1) \) and center \( y_i \), and \( \Sigma_0 \) a sphere with radius \( R (R \gg 1) \) and center the origin. Let \( \Omega_{\varepsilon,R} \) be the domain in \( \mathbb{R}^3 \) bounded by \( \Sigma_0, \Sigma_1, \Sigma_2, \) and \( \Sigma_3 \). Then

\[
\int_{\mathbb{R}^3 \setminus \{y_1, y_2, y_3\}} \tilde{\omega}(y_1; v_{y_1}) \land \tilde{\omega}(y_2; v_{y_2}) \land \tilde{\omega}(y_3; v_{y_3}) = \lim_{\varepsilon \to +0, R \to +\infty} \int_{\Omega_{\varepsilon,R}} d(\psi_1 d\psi_2 \land d\psi_3).
\]

(i) On \( \Sigma_0 \) we have \( |\psi_1| = O\left(\frac{1}{R}\right) \) and \( |d\psi_2 \land d\psi_3| = O\left(\frac{1}{R^2}\right) \) whereas \( \text{Area}(\Sigma_0) = O(R^2) \). Therefore

\[
\lim_{R \to +\infty} \int_{\Sigma_0} \psi_1 d\psi_2 \land d\psi_3 = 0.
\]

(ii) On \( \Sigma_2 \) we have \( \psi_1 = C + O(\varepsilon) \) for some constant \( C \). Therefore

\[
\lim_{\varepsilon \to +0} \int_{\Sigma_2} \psi_1 d\psi_2 \land d\psi_3 = C \lim_{\varepsilon \to +0} \int_{\Sigma_2} d\psi_2 \land d\psi_3 = C \lim_{\varepsilon \to +0} \int_{\Sigma_2} d(\psi_2 \land d\psi_3) = 0.
\]

The same argument works for \( \Sigma_3 \).

(iii) On \( \Sigma_1 \) we have \( |\psi_1| = O\left(\frac{1}{\varepsilon}\right) \) and

\[
d\psi_2 \land d\psi_3 = \omega' + O(\varepsilon)
\]

for some constant 2-form \( \omega' \), whereas \( \text{Area}(\Sigma_1) = O(\varepsilon^2) \). Therefore

\[
\lim_{\varepsilon \to +0} \int_{\Sigma_1} \psi_1 d\psi_2 \land d\psi_3 = 0.
\]

\[\square\]

**Proposition 5.3** Put

\[
AE_\gamma^0(K) = \int_{U' \setminus \Omega} \frac{|\det(\hat{\psi}_{y_1}(x), \hat{\psi}_{y_2}(x), \hat{\psi}_{y_3}(x))|}{|y_1 - x|^2 |y_2 - x|^2 |y_3 - x|^2} \, dy_1 dy_2 dy_3 d\text{vol}_x.
\]

Then \( AE_\gamma^0 \) diverges for any knot \( K \).

**Proof:** The integrand blows up near the diagonal of \( K^3 \times \mathbb{R}^3 \), which is stratified into strata of different (co)dimensions. The contribution of the neighborhood of

\[
\mathcal{N}_2 = \{(y_1, y_2, y_3, x) \in K^3 \times \mathbb{R}^3 \mid x = y_1 = y_2 \neq y_3\}
\]
makes the integral diverge.

Let us fix $y_1$ and $y_3$ ($y_1 \neq y_3$). We may assume without loss of generality that the knot $K$ is parametrized by the arc-length as $K = \gamma([0, 1])$ and that $y_1 = \gamma(0) = 0$. Let $\Omega(\varepsilon)$ be a domain given by

$$\Omega(\varepsilon) = \left\{(y_2, x) \in K \times \mathbb{R}^3 : y_2 = \gamma(s), x \neq y_2, \frac{\varepsilon}{2} < s, |x| \leq \varepsilon \right\}.$$ 

Since $v_{y_3}(x) = v' + O(\varepsilon)$ on $\Omega(\varepsilon)$ for some constant vector $v'$, we have the similarity between the integrands on $\Omega(\varepsilon)$ and $\Omega(\frac{\varepsilon}{2})$:

$$\frac{|\det(\hat{\nu}_0(\frac{\varepsilon}{2}), \hat{\nu}_{y_1}(\frac{\varepsilon}{2}), \hat{\nu}_{y_3}(\frac{\varepsilon}{2}))|}{|\varepsilon|^2|\gamma(\frac{\varepsilon}{2}) - \frac{\varepsilon}{2}| |y_3 - \frac{\varepsilon}{2}|^2} = 2^4 \frac{|\det(\hat{\nu}_0(x), \hat{\nu}_{y_1}(x), \hat{\nu}_{y_3}(x))|}{|x|^2|\gamma(s) - x|^2 |y_3 - x|^2} + O(\varepsilon^{-3}).$$

Since the volume of $\Omega(\varepsilon)$ is of order $\varepsilon^4$ we have

$$\int_{\Omega(\frac{\varepsilon}{2})} \frac{|\det(\hat{\nu}_0(x), \hat{\nu}_{y_2}(x), \hat{\nu}_{y_3}(x))|}{|x|^2|y_2 - x|^2 |y_3 - x|^2} \, dy_2 d_{\text{vol}}x$$

$$= \int_{\Omega(\varepsilon)} \frac{|\det(\hat{\nu}_0(x), \hat{\nu}_{y_2}(x), \hat{\nu}_{y_3}(x))|}{|x|^2|y_2 - x|^2 |y_3 - x|^2} \, dy_2 d_{\text{vol}}x + O(\varepsilon).$$

It follows that there is a positive constant $C'$ such that

$$\int_{\Omega(2^{-n})} \frac{|\det(\hat{\nu}_0(x), \hat{\nu}_{y_2}(x), \hat{\nu}_{y_3}(x))|}{|x|^2|y_2 - x|^2 |y_3 - x|^2} \, dy_2 d_{\text{vol}}x \geq C'.$$

for all $n \in \mathbb{N}$. Therefore

$$AE_{\varepsilon}^\gamma(K) \geq \int \left( \sum_{n=1}^{\infty} \int_{\Omega(2^{-n})} \frac{|\det(\hat{\nu}_y(x), \hat{\nu}_{y_2}(x), \hat{\nu}_{y_3}(x))|}{|y_1 - x|^2|y_2 - x|^2 |y_3 - x|^2} \, dy_2 d_{\text{vol}}x \right) \, dy_1 dy_3$$

$$= \infty.$$

**Remark:** There is a well-defined non-trivial conformally invariant “trilocal” functional on the space of knots ([LO2]). Although it is not associated to a Y-diagram, the integrand involves a triplet $(x, v_x), (y, v_y)$, and $(z, v_z)$.

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**References**

[BT] R. Bott and C. Taubes, *On the self-linking of knots*, J. of Math. Physics 35 (1994), 5247–5287.

[FHW] M.H. Freedman, Z-X. He and Z. Wang, *Möbius energy of knots and unknots*, Ann. of Math. 139 (1994), 1–50.
[H] F. Hélein, *Isoperimetric inequalities and calibrations*, “Progress in Partial Differential Equations: the Metz surveys”, M. Chipot and I. Shafrir ed., Pitman Research Notes in Mathematics, Series 345, Longman (1996).

[LO] R. Langevin and J. O’Hara, *Conformally invariant energies of knots*, J. Institut Math. Jussieu. 4 (2005), 219–280.

[LO2] R. Langevin and J. O’Hara. *Extrinsic Conformal Geometry of Curves and Surfaces*. in preparation.

[LW] X. -S. Lin. and Z. Wang, *Integral geometry of plane curves and knot invariants*, J. Diff. Geom. 44 (1996), 74–95.

[O1] J. O’Hara, *Energy of a knot*, Topology 30 (1991), 241–247.

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