Exact $\beta$-function in Abelian and non-Abelian $\mathcal{N} = 1$ supersymmetric gauge models and its analogy with QCD $\beta$-function in C-scheme.

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Abstract

For $\mathcal{N} = 1$ supersymmetric Yang-Mills theory without matter it is demonstrated that there is a class of renormalization schemes, in which the exact Novikov, Shifman, Vainshtein, and Zakharov (NSVZ) formula for the renormalization group $\beta$-function, defined in terms of the renormalized coupling constant, is valid. This class is described by finite renormalizations forming a one-parameter commutative subgroup of general renormalization group transformations. The analogy

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between the exact $\beta$-function in $\mathcal{N} = 1$ supersymmetric Yang-Mills theory without matter and the $\beta$-function of the C-scheme in quantum chromodynamics is discussed.

1. According to the renormalization group method [1], the evolution of the renormalized coupling constant with the change of scale $\mu$ is described by the $\beta$-function:

$$\beta (a_s) = \left. \frac{d a_s (a_{s0}, \mu^2/\Lambda^2)}{d \ln \mu^2} \right|_{a_s} = - (\beta_0 a_s^2 + \beta_1 a_s^3 + \beta_2 a_s^4 + O (a_s^5)), \quad (1)$$

where $a_s \equiv \alpha_s/\pi$, $a_{s0}$ is the bare coupling constant and $\Lambda$ is the dimensionful parameter, introduced in the theory by regularization. The definition (1) is written in terms of the renormalized coupling constant. The $\beta$-function coefficients $\beta_i$ (for $i \geq 2$) depend on the renormalization procedure. They may be changed by the definite finite renormalization prescriptions for the coupling constant.

The characteristic feature of $\mathcal{N} = 1$ supersymmetric (SUSY) gauge theories is the existence of the exact $\beta$-functions [2]. For example, in $\mathcal{N} = 1$ supersymmetric quantum electrodynamics (SQED) with $N_f$ number of matter superfields the corresponding exact $\beta$-function, expressed in terms of the renormalized coupling constant, reads [3]:

$$\beta (a) = a^2 N_f \left( \frac{1}{2} - \gamma (a) \right), \quad (2)$$

This formula relates the $\beta$-function and the anomalous dimension of matter

$$\gamma (a) = \left. \frac{d \ln Z (a (a_{0}, \mu^2/\Lambda^2), \mu^2/\Lambda^2)}{d \ln \mu^2} \right|_{a_0}, \quad (3)$$

where $Z (a, \mu^2/\Lambda^2)$ is the renormalization constant of matter superfields. It was shown in [4], that the relation (2) stays valid in the class of the renormalization prescriptions, which are called the NSVZ-type schemes, following the terminology of [5].

When the theory is regularized with the help of higher derivatives (HD) [6, 7] (see [8] as well) in their supersymmetric form [9, 10], the equation (2) is valid in all orders of the perturbation theory (PT) in terms of the bare coupling constant $a_0$ [11]. This was explicitly verified in the 3-loop approximation in [11], and also in [12] within a bit different approach. For
this equation to be satisfied in the renormalized language, special boundary conditions $a = a_0$ and $Z = 1$, imposed at a fixed value of $\mu$, were formulated in [13]. In the case of $\mu = \Lambda$ this scheme is called the procedure of minimal subtraction of logarithms (MSL). In [14] it was demonstrated that the HD + MSL prescription also provides the NSVZ-type relation in renormalized softly broken SQED. Further on it was shown in [15] that the relation (2) is also satisfied exactly in all orders of the PT in on mass shell (OS) subtraction scheme (used previously in [16]).

More frequently for the regularization of SUSY models, instead of the dimensional regularization [17], the method of dimensional reduction (DRED) is used [18]. The renormalization procedure in this case is the DR prescription, which is analogous to the MS scheme. Its application leads to the violation of the expression (2) for the exact $\beta$-function starting from the three-loop approximation [3]. As was mentioned above, the the equation (2) can be restored using the finite renormalization of the coupling constant, which should be fine tuned at every subsequent order of the PT. In general, this redefinition may be fixed by the boundary conditions, analogous to the MSL-type procedure, applied at the 3-loop order in [19]. The possibility of restoring the NSVZ relation in SQED, regularized by DRED, was discussed earlier in [20].

In the case of the renormalized $\mathcal{N} = 1$ SUSY Yang-Mills theory without matter the exact $\beta$-function has the geometric series related form [21]

$$\beta(a_s) = \frac{-3C_2a_s^2}{4 - 2C_2a_s},$$

(4)

where $C_2$ is the Casimir operator in the adjoint representation.

Let’s clarify, how Eq. (4) was obtained. In SUSY theories the related to axial and conformal anomalies operators should enter the same supermultiplet and ought to be renormalized in the same way. It is known, that the renormalization of the trace of the energy-momentum tensor is proportional to the conformal anomaly factor $(\beta(a_s)/a_s)$, while in SUSY gauge models the axial anomaly operator should stay non-renormalized due to the validity of the Adler-Bardeen theorem [22] in these theories as well [23]. On the first glance these facts are difficult to reconcile (see, e.g., [24] and references there). However, in [21] it was shown that they are consistent with each other when the the $\beta$-function is given by (4).

This expression doesn’t agree with the 3-loop result [26, 27], obtained in the DR scheme. They are starting to agree with each other after restoring
the equation finite renormalization of the coupling constant, which was fixed in in the 3-loop order. Note, that another solution of the considered in Ref. puzzle was given in by applying finite renormalizations of the special form. It may be interesting to understand whether these finite renormalizations are related to HD + MSL prescription used in \( \mathcal{N} = 1 \) SUSY Yang-Mills models with matter in [28].

In this paper it is demonstrated that there is a class of renormalization schemes in \( \mathcal{N} = 1 \) SUSY Yang-Mills theory without matter, in which the expression is exactly valid. The group structure of the transformations acting in this class is investigated. Their analogy to the finite renormalizations, conserving the \( \beta \)-function of the C-scheme in non-supersymmetric quantum chromodynamics (QCD), is considered.

2. Let us remind, that in \( \mathcal{N} = 1 \) SQED the change of the renormalization scheme is performed in the following way:

\[
a' (a_0, \mu/\Lambda) = a' (a (a_0, \mu/\Lambda)), \quad Z' (a' (a), \mu/\Lambda) = z (a) Z (a, \mu/\Lambda),
\]

(5)

where \( Z \) and \( Z' \) are the renormalization constants of matter in the schemes under consideration, while the finite functions \( a' (a) \) and \( z (a) \) can be chosen arbitrarily. Choosing the renormalization procedures of the equations within the class of NSVZ schemes, one arrives to the following condition:

\[
\frac{1}{a' (a)} - \frac{1}{a} - N_f \ln z (a) = \pi B = - \frac{N_f}{2} \ln \frac{\mu'^2}{\mu^2},
\]

(6)

which is valid in all orders of the PT and the parameter \( B \) doesn’t depend on \( a \). This condition relates the particular representatives of the class of NSVZ schemes: HD + MSL [13], HD + OS [15], DRED + DR + (special finite renormalization) [19] prescriptions among others.

In general, the transformations within the class of all these schemes are parameterized by the set \( \{ a' (a), z (a), B \} \), where \( a' (a) \) and \( z (a) \) conserve the form of the exact \( \beta \)-function (2). The function \( z (a) \), performing the finite renormalization of matter, and the variable \( B \) are convenient to choose as the independent quantities.

Consider now the two transformations, parameterized by the set \( \{ a_i (a), z_i (a), B^{(i)} \} \) (with \( i = 1, 2 \)), which satisfy the restriction (6). Note, that the second order expansion of the function \( z_i (a) \) has the form

\[
z_i (a) = 1 + D_{1}^{(i)} a + D_{2}^{(i)} a^2 + O (a^3),
\]

(7)
while the related PT expansion for \( a_i(a) \) can be found from the following equation:

\[
\frac{1}{a_i(a)} = \frac{1}{a} + \pi B(i) + N_f \ln z_i(a).
\]  

(8)

Thus, to describe the transformations within the considered class in the three-loop approximation it is necessary to fix three coefficients, namely \( B(i), D_1(i) \) and \( D_2(i) \).

It was shown in [4], that transformations, discussed above, constitute a subgroup of general renormalization group transformations. Indeed, the composition of finite renormalizations, parameterized by the sets \( \{ a_1(a), z_1(a), B^{(1)} \} \) and \( \{ a_2(a), z_2(a), B^{(2)} \} \), which satisfies the following condition

\[
a'(a) = a_2(a_1(a)), \quad z(a) = z_2(a_1(a)) z_1(a), \quad B = B^{(1)} + B^{(2)},
\]  

(9)

provides the validity of (6) as well. For every transformation, characterized by the set of functions \( \{ a'(a), z(a), B \} \), there is an inverse one, namely:

\[
\{ a(a'), 1/z(a(a')), -B \}.
\]  

(10)

The identical element in the group of finite renormalizations, by definition, is

\[
a'(a) = a, \quad z(a) = 1, \quad B = 0.
\]  

(11)

One can verify that the finite renormalizations (10-11) satisfy the restriction (6). Similarly to the equations (7) and (8) one can write down the PT expansion for the functions \( z(a), a'(a) \) and find the inverse one for the latter, i.e. \( a(a') \).

Note, that in the case of \( \mathcal{N} = 1 \) SQED the given subgroup is non-commutative. Indeed, let us compare the following composition of transformations

\[
a'(a) = a_1(a_2(a)), \quad z(a) = z_1(a_2(a)) z_2(a), \quad B = B^{(1)} + B^{(2)}
\]  

(12)

with the result of the finite renormalization (9) and then substitute the expansion (7) in both the compositions (9) and (12), employing the formula (8). The coincidence of these compositions is only achieved when the coefficients in (7) and (8) are related as

\[
B^{(1)} D^{(2)}_1 = B^{(2)} D^{(1)}_1.
\]  

(13)
In general the coefficients $B^{(i)}$ and $D^{(i)}$ are arbitrary and don’t have to satisfy this condition. For this reason the compositions (9) and (12) provide different results and thus the subgroup, conserving the equation (2) in $N = 1$ SQED, is non-commutative (non-Abelian).

3. Consider now $N = 1$ SUSY Yang-Mills theory without matter. In this theory there are also transformations, which conserve the form of its exact $\beta$-function, given in (4). They are defined by finite renormalizations of the coupling constant $a'_{s}(a_{s})$, satisfying the condition, analogous to SQED equation (6), namely

$$\frac{1}{a'_{s}(a_{s})} - \frac{1}{a_{s}} + \frac{\beta_{1}}{\beta_{0}} \ln z_{\alpha}(a_{s}) = \pi \hat{B} \equiv \beta_{0} \ln \frac{\mu^{2}}{\mu'^{2}},$$

(14)

where $z_{\alpha}(a_{s}) \equiv a'_{s}(a_{s})/a_{s}$ and the first two coefficients of the $\beta$-function (4), which were calculated in [29], read:

$$\beta_{0} = \frac{3}{4} C_{2}, \quad \beta_{1} = \frac{3}{8} C_{2}^{2}.$$  

(15)

Comparing the restrictions (6) and (14) we conclude that the latter is more strict. Indeed, it contains only one unfixed parameter $B$, which is similar to $\hat{B}$ in the equation (6).

Finite renormalizations, defined by the condition (14), correspond to the change of the scale $\mu$ and form a one-parameter commutative (Abelian) subgroup of general renormalization group transformations. Let us verify this explicitly. The composition of sequential renormalizations $\{a_{s1}(a_{s}), \hat{B}^{(1)}\}$ and $\{a_{s2}(a_{s}), \hat{B}^{(2)}\}$, which satisfies the condition (14), does not depend on the order of action:

$$a'_{s}(a_{s}) = a_{s2}(a_{s1}(a_{s})) = a_{s1}(a_{s2}(a_{s})), \quad (16)$$

and provides the validity of (14) with $\hat{B} = \hat{B}^{(1)} + \hat{B}^{(2)}$. The neutral element of the identical transformation $\{a_{s}(a_{s}), \hat{B} = 0\}$ belongs to the studied subgroup. For an arbitrary finite renormalization, there is an inverse transformation $\{a_{s}(a_{s}'), -\hat{B}\}$, which is analogous to (10). Thus, the set of finite renormalizations, satisfying the restriction (14), is a commutative subgroup. In contrast, the subgroup, discussed in section 2, is non-commutative due to the effects of matter renormalization.
Note also that the corresponding to the change of $\mu$ transformation properties of reflexivity, transitivity and symmetry were considered earlier in [30]. In the case of $N = 1$ SUSY Yang-Mills theory the subgroup of these transformations is theoretically distinguished, since they conserve the form of the exact $\beta$-function [4].

4. In QCD the special C-scheme has recently been proposed [31]. It was used in [32, 33, 34]) to study the analytical contributions to QCD perturbative series of the terms, proportional to Riemann functions $\zeta(n)$ with even integer arguments ($n = 4, 6, \ldots$). By definition, the $\beta$-function in the C-scheme [31] is

$$\beta(a_s) = \frac{-\beta_0 a_s^2}{1 - (\beta_1/\beta_0) a_s}.$$

The coefficients $\beta_0$ and $\beta_1$ were evaluated in [35, 36] and [37, 38] respectively. They read

$$\beta_0 = \frac{11}{12} C_2 - \frac{1}{3} T_F n_f, \quad \beta_1 = \frac{17}{24} C_2^2 - \frac{5}{12} C_2 T_F n_f - \frac{1}{4} C_F T_F n_f,$$

where $n_f$ is the number of quark flavours, $C_F$ and $C_2$ are the Casimir operators in the fundamental and adjoint representations of the gauge group, $T_F$ is the Dynkin index.

The expression (17) is consistent with the equation (4) if instead of the coefficients $\beta_0$ and $\beta_1$ given in (18) one takes their values (15) for $N = 1$ SUSY Yang-Mills theory. In the latter case $\beta_1$ is divided by $\beta_0$ in the denominator of (17) without the remainder. Moreover, the same feature also takes place in QCD without quarks (gluodynamics), when $\beta_0$ and $\beta_1$ are defined by the analog of (18) with $T_F n_f = 0$.

To relate the renormalization group quantities in the $\overline{\text{MS}}$ and the C-scheme, it is required to perform the following finite renormalization:

$$a_s' = a_s + \left( \frac{\beta_1}{\beta_0} - \frac{\beta_2}{\beta_0^2} \right) a_s^3 + O(a_s^4).$$

In the case of $\overline{\text{MS}}$ scheme $\beta_2$ was analytically computed in [39, 40]. In gluodynamics the corresponding expression is $\beta_2 = (2857/3456) C_2^3$ and is divided by $\beta_0$ in (19) without the remainder, similarly to the case of $N = 1$ SUSY Yang-Mills theory without matter. Therefore, the transformation (19) is analogous to the renormalization $a_s'(a_s)$, which leads to the exact $\beta$-function.
from the DR scheme result, and the C-scheme $\beta$-function in gluodynamics is analogous to the exact $\beta$-function itself.

Now return to the case of QCD with quarks. Any finite renormalizations, which conserve the form of (17), satisfy the exact restriction:

$$
\frac{1}{a'_s} - \frac{1}{a_s} + \frac{\beta_1}{\beta_0} \ln\frac{a'_s}{a_s} = \beta_0 C \equiv \beta_0 \ln \frac{\mu'}{\mu^2}.
$$

(20)

The similar expression was derived in [31], where the appearance of the parameter $C$ explains the term “C-scheme”. In $\mathcal{N} = 1$ SUSY Yang-Mills theory the formula (14) has the same form and meaning as the condition (20), and the parameters $\tilde{B}$ and $C$ are related:

$$
\tilde{B} = \frac{3C^2}{4\pi}.
$$

(21)

Note that the variable $\Delta$, analogous to $C$, was introduced earlier in [41] while solving the renormalization group equations for the two-loop $\beta$-function of a general asymptotically-free theory. In the case of QCD these variables are identical:

$$
\Delta = \ln \frac{\mu^2}{\mu^2} = C.
$$

(22)

The difference is that in the C-scheme this parameter appears when the corresponding renormalization group equations are solved in higher orders of the PT with the $\beta$-function defined by the formula (17).

Satisfying the condition (20) finite renormalizations, in analogy to those discussed in the section 3, form a commutative subgroup of general renormalization group transformations. This subgroup is entirely characterized by the variable $C$.

**Conclusion**

In this paper it has been demonstrated, that in $\mathcal{N} = 1$ SUSY Yang-Mills theory without matter superfields there is a class of renormalization schemes, in which the exact $\beta$-function [4] is valid in terms of the renormalized coupling constant. Acting within this class finite renormalizations form a one-parameter commutative subgroup of transformations, corresponding to the change of the scale $\mu$. It has been shown, that the analogous transformations in QCD conserve the form of the C-scheme $\beta$-function. In contrast, the
subgroup of finite renormalizations, conserving the equation (2) in $\mathcal{N} = 1$ SQED, in general, is non-commutative.

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