GLOBAL REGULARITY FOR A MODEL OF NAVIER-STOKES EQUATIONS WITH LOGARITHMIC SUB-DISSIPATION

SHUGUANG SHAO*
College of Applied Sciences, Beijing University of Technology
Beijing 100124, China
and
School of Mathematics and Statistics, Nanyang Normal University
Nanyang 473061, China

SHU WANG
College of Applied Sciences, Beijing University of Technology
Beijing 100124, China

WEN-QING XU
College of Applied Sciences, Beijing University of Technology
Beijing 100124, China
and
Department of Mathematics and Statistics, California State University, Long Beach
Long Beach, CA 90840, USA

(Communicated by Tao Luo)

Abstract. In this paper, we study the global regularity to a three-dimensional logarithmic sub-dissipative Navier-Stokes model. This system takes the form of
\[ \partial_t u + (u \cdot \nabla) u + \nabla p = -A^2 u, \]
where \( D = |\nabla| \) and \( A = |\nabla| \ln^{-1/4}(e + \lambda \ln(e + |\nabla|)) \) with \( \lambda \geq 0 \). The symbols of the \( D \) and \( A \) are \( m(\xi) = |\xi| \) and \( h(\xi) = |\xi|/g(\xi) \) respectively, where \( g(\xi) = \ln^{1/4}(e + \lambda \ln(e + |\xi|)) \), \( \lambda \geq 0 \). It is clear that for the Navier-Stokes equations, global regularity is true under the assumption that \( h(\xi) = |\xi|^\alpha \) for \( \alpha \geq 5/4 \). Here by changing the advection term we greatly weaken the dissipation to \( h(\xi) = |\xi|/g(\xi) \). We prove the global well-posedness for any smooth initial data in \( H^s(\mathbb{R}^3) \), \( s \geq 3 \) by using the energy method.

1. Introduction. The three-dimensional incompressible Navier-Stokes equations reads in the Eulerian coordinates
\[ \begin{align*}
\partial_t u + (u \cdot \nabla) u + \nabla p &= \Delta u, \\
\nabla \cdot u &= 0, \\
u|_{t=0} &= u_0(x)
\end{align*} \tag{1} \]
where \( u : [0, T) \times \mathbb{R}^3 \to \mathbb{R}^3 \) is the velocity field, \( p : [0, T) \times \mathbb{R}^3 \to \mathbb{R} \) is the scalar pressure function and \( u_0 : \mathbb{R}^3 \to \mathbb{R}^3 \) the given initial data. Here we have normalized the kinematic viscosity coefficient to be 1.

2010 Mathematics Subject Classification. Primary: 35A01, 35B65; Secondary: 53C35.
Key words and phrases. Navier-Stokes equations, global regularity, sub-dissipation, energy estimates.

* Corresponding author: Shuguang Shao.
It is well-known that the global regularity problem for the Navier-Stokes equations (1) is a significant open question (see, for instance, [5, 21, 14, 27, 15, 16, 8, 2, 10, 3, 13, 24, 25]). The difficulty in answering this question is that the energy estimate, which is the strongest known coercive \textit{a priori} estimate, is supercritical with respect to the natural scaling of the Navier-Stokes equations.

The fundamental energy identity of the Navier-Stokes equations reads
\begin{equation}
\frac{1}{2}\|u(t, \cdot)\|_{L^2}^2 + \int_0^t \|\nabla u(t', \cdot)\|_{L^2}^2 dt' = \frac{1}{2}\|u_0\|_{L^2}^2, \quad \forall \ t \geq 0. \tag{2}
\end{equation}

The natural scaling of the Navier-Stokes equations is defined by
\begin{equation}
u^\ell(t, x) = \ell u(t/\ell^2, \ell x), \ p^\ell(t, x) = \ell^2 p(t/\ell^2, \ell x), \quad \forall \ \ell > 0. \tag{3}
\end{equation}

The standard scale-transformation methods in [22] have shown that the basic energy equation (2) is not sufficient by itself for an affirmative answer to the global regularity problem of the three-dimensional Navier-Stokes equations. At the same time, the energy identity and the incompressibility condition are very important for preventing negative answers to the problem. In particular, it has been observed that the local structure of the Navier-Stokes equations may lead to a kind of dimension reduction as time approaches the potential singular time [8, 10], due to the incompressibility constraint of the fluids.

We focus on the most recent work [26]. The Navier-Stokes equations on the Euclidean space $\mathbb{R}^3$ can be expressed in the form
\begin{equation}
\partial_t u - \Delta u = -\mathcal{R} \times \mathcal{R} \times [\mathcal{S}(u)u], \quad \mathcal{S}(u) := \nabla u - (\nabla u)^\top \quad \text{and} \quad \mathcal{R} := |\nabla|^{-1}\nabla. \tag{4}
\end{equation}

In [26], the author proposed a three-dimensional Navier-Stokes model
\begin{equation}
\begin{aligned}
\partial_t u + \mathcal{R} \times [\mathcal{S}(u)(\mathcal{R} \times u)] &= -\mathcal{A}^2 u \\
\nabla \cdot u &= 0, \\
|u|_{t=0} &= u_0(x)
\end{aligned} \tag{5}
\end{equation}

where
\begin{equation}
\mathcal{S}(u) := \nabla u - (\nabla u)^\top, \quad \mathcal{R} := |\nabla|^{-1}\nabla \\
\mathcal{A} = \frac{|\nabla|}{\ln^{1/4}(e + \lambda \ln(e + |\nabla|))}, \quad \lambda \geq 0.
\end{equation}

We emphasize that when $\lambda = 0$, model (4) is very close to the original Navier-Stokes equation (1) in the sense that the nonlinear term in (1) can be written as $\mathcal{R} \times \mathcal{R} \times [\mathcal{S}(u)u]$ instead of the one appeared in (4). Moreover, the model obeys the energy identity (2) of the Navier-Stokes equations due to (7) below. Wang [26] proved that model (4) is globally well-posed for any initial data in Sobolev space $H^s$ with $s \geq 3$.

The idea of this paper is in spirit similar to that in [26]. In this work, we take another angle of attack to the global regularity problem. We propose the following model
\begin{equation}
\begin{aligned}
\partial_t u + (D^{-1/2} u) \cdot \nabla u + \nabla p &= -\mathcal{A}^2 u, \\
\nabla \cdot u &= 0, \\
|u|_{t=0} &= u_0(x)
\end{aligned} \tag{5}
\end{equation}

where
\begin{equation}
D = |\nabla|
\end{equation}
and
\[ A = \frac{|\nabla|}{\ln^{1/4}(e + \lambda \ln(e + |\nabla|))}, \quad \lambda \geq 0. \]

The symbols of the \( D \) and \( A \) are \( m(\xi) = |\xi| \) and \( h(\xi) = |\xi|/g(\xi) \) respectively, where \( g(\xi) = \ln^{1/4}(e + \lambda \ln(e + |\xi|)), \lambda \geq 0 \). If \( \lambda = 0 \), then \( -A^2 \) is the usual Laplacian operator. If \( \lambda > 0 \), then the model (5) is sub-dissipative. Using the standard energy method [23] and [26], it can be proved that the model (5) is globally well-posed for any initial data in Sobolev space \( H^s \) with \( s \geq 3 \).

The proof of Theorem 2.1 is motivated by Tao’s work [23] and Wang’s work [26]. Let \( d \geq 3 \). In [23] Tao considered the global Cauchy problem for the generalized Navier-Stokes system
\[
\begin{align*}
\partial_t u + (u \cdot \nabla) u &= -A^2 u - \nabla p, \\
\nabla \cdot u &= 0, \\
\quad u(0, x) &= u_0(x)
\end{align*}
\]
for \( u : \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R}^d \) and \( p : \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R} \), where \( u_0 : \mathbb{R}^d \to \mathbb{R}^d \) is smooth and divergence free, and
\[ A = \frac{|\nabla|^{(d+2)/4}}{\ln^{1/4}(2 + |\nabla|^2)} \]
is a Fourier multiplier whose symbol \( h : \mathbb{R}^d \to \mathbb{R}^+ \) is nonnegative (the case \( h(\xi) = |\xi| \) is essentially Navier-Stokes). For the hyper-dissipative Navier-Stokes equations, it is folklore (e.g. [11]) that one has global regularity in the critical and subcritical hyper-dissipation regimes \( h(\xi) = |\xi|^{\alpha} \) for \( \alpha \geq (d+2)/4 \). Tao improved this by establishing global regularity under the slightly weaker condition that \( h(\xi) \geq |\xi|^{(d+2)/4}/g(|\xi|) \) for all sufficiently large \( \xi \) and some non-decreasing function \( g : \mathbb{R}^+ \to \mathbb{R}^+ \) such that \( \int_1^\infty \frac{ds}{sg(s)^{\alpha}} = +\infty \). In particular, the results apply for the logarithmically supercritical dissipation
\[ h(\xi) := \frac{|\xi|^{(N+2)/4}}{\ln^{1/4}(2 + |\xi|^2)}. \]

These results demonstrate that finer structures of the nonlinearity term in the Navier-Stokes equations are more crucial for the study of this system, in addition to the validity of the energy identity and incompressibility which are the most fundamental properties of Navier-Stokes equations.

The purpose of this paper is to generalize the result for the critical and sub-critical hyper-dissipative Navier-Stokes equations to the slightly supercritical and sub-dissipative Navier-Stokes system (the case \( \lambda > 0 \)). We emphasize that model (5) is very close to the original Navier-Stokes equations (1). Moreover, the model obeys energy identity (2) of the Navier-Stokes equations due to the energy identity (7) below. Before going any further, we give our motivation to consider this model. Just as the classical 3D Navier-Stokes system, (5) has a scale transform. Indeed, if \( \lambda = 0 \), under the following scaling transformation
\[ u^\ell(t, x) = \ell^{3/2} u(\ell^2 t, \ell x), \quad p^\ell(t, x) = \ell^{5/2} p(\ell^2 t, \ell x), \quad \forall \ell > 0, \]
system (5) is invariant. In this sense, as the 2D incompressible Navier-Stoke equations, the \( L^2 \) energy is the critical case with respect to the scaling of (5) when \( \lambda = 0 \). This motivates us to expect that model (5) admits a large global solution.
if the initial data belongs to some Lebesgue or Sobolev spaces without smallness condition.

Before ending this introduction, let us mention that, in [7] Hou and Lei suggested a three-dimensional model of the Navier-Stokes equations which obeys a similar energy law to (2). That model is formulated in terms of a set of new variables related to the angular velocity, the angular vorticity, and the angular stream function. The only difference between the 3D model and the reformulated Navier-Stokes equations in terms of these new variables is that certain convection term in the model is neglected. That 3D model preserves almost all the properties of the full 3D Euler or Navier-Stokes equations. If one could construct singular solutions starting from well-prepared initial data for the model (for instance, see [7] and [9]), then it might yield a more convincing conclusion that finer structures, in particular, the effect of convection terms of the Navier-Stokes equations have to be taken into account for a positive answer of its global regularity problem.

Finally, let’s look at some of the previous results for various models for which the energy identity is not available (for example, see [6, 18]). In a slightly different direction, finite time blowup was established in [17] for a complexified version of the Navier-Stokes equations for which the energy equality was again unavailable. Further models of the Navier-Stokes type, which obey an energy identity, can be found in [7, 12, 19, 20]. We also mention that for the general 3D incompressible Navier-Stokes equations which possess hyper-dissipation in the horizontal direction, Fang and Han in [4] obtain the global existence result when the initial data belongs to some anisotropic Besov spaces.

The rest of this paper is organized as follows. In section 2, we state the main result of this paper. In section 3, we present the proof of global regularity for model (5).

Notation. For $a \lesssim b$, we mean that there is a “harmless” positive constant $C$ which may be different from line to line such that $a \leq Cb$.

2. The main result. The main result of this paper is stated as follows:

**Theorem 2.1.** Suppose $u_0 \in H^s(\mathbb{R}^3), s \geq 3$ with $\text{div } u_0 = 0$, then the model system (5) for the incompressible three-dimensional Navier-Stokes equations is globally well-posed.

The theorem demonstrates that, showing finite time singularities for a Navier-Stokes model may not be sufficiently convincing to conjecture that the Navier-Stokes equations itself can develop finite time singularities starting from smooth initial data, even though the model is well-designed so that it satisfies energy identity and incompressibility. Any attempt to negatively resolve the Navier-Stokes global regularity problem in three dimensions also has to consider finer structures on the nonlinear portion of the equations than is provided by the energy identity and incompressibility. At last, it seems an interesting question whether there is a possibility that as time approaches the potential singular time, our model could serve as an approximation of the Navier-Stokes equations for computations and simulations.

Furthermore, those results demonstrate that finer structures of the nonlinearity in the Navier-Stokes equations are crucial for the study of this model system, beyond the validity of the energy identity and incompressibility which are the most fundamental properties of the Navier-Stokes equation. Without further understanding of
the structure of nonlinearity, any attempt to positively resolve the Navier-Stokes global regularity problem in three dimensions is impossible due to [23] and [26], and any attempt to negatively resolve the same problem for a Navier-Stokes model is also not convincing to yield a negative answer to the global regularity problem of the original Navier-Stokes equations due to the example given in this paper.

3. Proof of Theorem 2.1. Without loss of generality, we only consider the Sobolev space $H^s$ for $s = 3$. Once the theorem is proved for the case of $s = 3$, it is not difficult to get higher regularity of solutions for any $t > 0$. The local well-posedness of the Navier-Stokes model (5) with initial data $u_0 \in H^3$ is standard (for example, we may apply the well-known Friedrichs approximation scheme in [1]). As a result of local regularity, we have, for any initial data $u_0 \in H^3$, there exists a unique solution $u$ to (5) which satisfies

$$\|u(t)\|_{H^3} \leq c\|u_0\|_{H^3}$$

at least on a small time interval $[0, T]$.

We start the proof of Theorem 2.1 by establishing the following a priori estimate. Using the classical energy method, taking the $L^2$ inner product of the model equation (5) with $u$ yields

$$\int_{\mathbb{R}^3} \left( \partial_t u \cdot u + ((D^{\frac{1}{2}} u) \cdot \nabla) u \cdot u + \nabla p \cdot u + A^2 u \cdot u \right) dx = 0.$$

We obtain

$$\frac{1}{2} \frac{d}{dt} \|u\|^2_{L^2} + \|Au\|^2_{L^2} = 0,$$

which gives that

$$\|u(t, \cdot)\|^2_{L^2} + 2 \int_0^t \|Au(\tau, \cdot)\|^2_{L^2} d\tau = \|u_0\|^2_{L^2} \quad (7)$$

Taking the $L^2$ inner product of the model equation (5) with $\Delta u$ and integrating over $\mathbb{R}^3$, we get

$$\int_{\mathbb{R}^3} \left( \partial_t \partial u \cdot \partial u + \partial((D^{\frac{1}{2}} u) \cdot \nabla u) \cdot \partial u + \partial \nabla p \cdot \partial u + A^2 \partial u \cdot \partial u \right) dx = 0.$$

That is

$$\frac{1}{2} \frac{d}{dt} \|\partial u\|^2_{L^2} + \|A\partial u\|^2_{L^2} = - \int \partial u \cdot \partial((D^{\frac{1}{2}} u) \cdot \nabla u) dx \quad (8)$$

Integrating by parts and using the divergence free property, one has

$$A_2 = 0.$$

Now, we estimate $A_1$. We can write

$$A_1 = - \int A\partial u \cdot A^{-1}(\partial D^{\frac{1}{2}} u \cdot \nabla u) dx.$$

Then using Cauchy-Schwarz inequality, we obtain the bound

$$|A_1| \lesssim \|A\partial u\|_{L^2} \|A^{-1}(\partial D^{\frac{1}{2}} u \cdot \nabla u)\|_{L^2} \quad (9)$$
By the arithmetic mean-geometric mean inequality we then have
\[ |A_1| \lesssim \gamma \|A \partial u\|_{L^2}^2 + \gamma^{-1} \|A^{-1}(\partial D^{-\frac{1}{2}} u \cdot \nabla u)\|_{L^2}^2. \tag{10} \]

To estimate the second term \( \|A^{-1}(\partial D^{-\frac{1}{2}} u \cdot \nabla u)\|_{L^2}^2 \), we introduce a parameter \( N = N(t) \geq 500 \), a big number to be determined later. Let us divide \( \|A^{-1}(\partial D^{-\frac{1}{2}} u \cdot \nabla u)\|_{L^2}^2 \) into the high frequency part and the low frequency part.
\[ \|A^{-1}(\partial D^{-\frac{1}{2}} u \cdot \nabla u)\|_{L^2}^2 \lesssim \|P_{>N}A^{-1}(\partial D^{-\frac{1}{2}} u \cdot \nabla u)\|_{L^2}^2 \]
\[ + \|P_{\leq N}A^{-1}(\partial D^{-\frac{1}{2}} u \cdot \nabla u)\|_{L^2}^2 \tag{11} \]

where \( P_{>N} \) and \( P_{\leq N} \) are the Fourier projections to the regions \( \{\xi : |\xi| > N\} \) and \( \{\xi : |\xi| \leq N\} \) defined by
\[ P_{>N}f = \mathcal{F}^{-1}(1_{|\xi|>N}\mathcal{F}(f)), \quad P_{\leq N}f = \mathcal{F}^{-1}(1_{|\xi|\leq N}\mathcal{F}(f)). \]

We first deal with the high-frequency part. By Sobolev embedding and Plancherel’s Theorem, we obtain that
\[ \|P_{>N}A^{-1}(\partial D^{-\frac{1}{2}} u \cdot \nabla u)\|_{L^2}^2 \lesssim N^{-2} \ln^2 (\ln N) \|\nabla A^{-1}(\partial D^{-\frac{1}{2}} u \cdot \nabla u)\|_{L^2}^2 \]
\[ \lesssim N^{-2} \ln^2 (\ln N) \|\partial D^{-\frac{1}{2}} u\|_{L^\infty}^2 \|\nabla u\|_{L^2}^2 \tag{12} \]
\[ \lesssim N^{-2} \ln^2 (\ln N) \|u\|_{H^3}^2. \]

Next, we turn to the low-frequency part. Using the Plancherel’s Theorem, Sobolev imbedding and Hölder inequality, we have
\[ \|P_{\leq N}A^{-1}(\partial D^{-\frac{1}{2}} u \cdot \nabla u)\|_{L^2}^2 \lesssim \ln^\frac{1}{2}(\ln N) \|\nabla A^{-1}(\partial D^{-\frac{1}{2}} u \cdot \nabla u)\|_{L^2}^2 \]
\[ \lesssim \ln^\frac{1}{2}(\ln N) \|\partial D^{-\frac{1}{2}} u \cdot \nabla u\|_{L^2}^2 \]
\[ \lesssim \ln^\frac{1}{2}(\ln N) \|\nabla u\|_{L^2}^2 \|\nabla u\|_{H^3}^2 \tag{13} \]
\[ \lesssim \ln^\frac{1}{2}(\ln N) \|\nabla u\|_{L^2}^2 \|u\|_{H^3}^2. \]

We further estimate \( \|\nabla u\|_{L^2}^2 \). Similarly, we divide \( \|\nabla u\|_{L^2}^2 \) into the high-frequency part and the low-frequency part. For the high frequency part, one has
\[ \|P_{>N}\nabla u\|_{L^2}^2 \lesssim N^{-4} \|\nabla^3 u\|_{L^2}^2 \lesssim N^{-4} \|u\|_{H^3}^2. \]

For the low frequency part, we have
\[ \|P_{\leq N}\nabla u\|_{L^2}^2 \lesssim \ln^\frac{1}{2}(\ln N) \|Au\|_{L^2}^2. \]

So, we obtain that
\[ \|\nabla u\|_{L^2}^2 \lesssim \ln^\frac{1}{2}(\ln N) \|Au\|_{L^2}^2 + N^{-4} \|u\|_{H^3}^2. \tag{14} \]

Substituting (14) into (13), we infer that
\[ \|P_{\leq N}A^{-1}(\partial D^{-\frac{1}{2}} u \cdot \nabla u)\|_{L^2}^2 \]
\[ \lesssim \ln^\frac{1}{2}(\ln N) \left( \ln^\frac{1}{2}(\ln N) \|Au\|_{L^2}^2 + N^{-4} \|u\|_{H^3}^2 \right) \|u\|_{H^3}^2 \tag{15} \]
\[ \lesssim \ln(\ln N) \|Au\|_{L^2}^2 \|u\|_{H^3}^2 + N^{-4} \ln^\frac{1}{2}(\ln N) \|u\|_{H^3}^4. \]
Then, we estimate $A$ and the divergence free property. Therefore, we obtain that
\begin{equation}
\|A^{-1}(\partial D^{-\frac{1}{2}} u \cdot \nabla u)\|_{L^2}^2 \lesssim N^{-2} \ln^\frac{3}{4}(\ln N)\|u\|_{H^3}^4
+ \ln(\ln N)\|Au\|_{L^2}^2\|u\|_{H^3}^3 + N^{-4} \ln^\frac{1}{2}(\ln N)\|u\|_{H^3}^4
\end{equation}
Putting (11),(12),(15) all together, we conclude that
\begin{equation}
\nonumber
\text{and Plancherel’s Theorem to obtain that}
\end{equation}
\begin{equation}
\nonumber
\text{Similarly as in (12), we can estimate the high frequency part by Sobolev embedding}
\end{equation}
Inserting (16) into (10), we have
\begin{equation}
\nonumber
\text{Substituting (17) into (8), we finally arrive at}
\end{equation}
\begin{equation}
\nonumber
\text{Next, taking the } L^2 \text{ inner product of the model equation (5)_1 with } \Delta^2 u \text{ and then}
\end{equation}
\begin{equation}
\nonumber
\text{integrating over } \mathbb{R}^3, \text{ we have}
\end{equation}
\begin{equation}
\nonumber
\text{Applying the Leibniz rule to } \partial^2(D^{-\frac{1}{2}} u \cdot \nabla u), \text{ there is one term involving third-order}
\end{equation}
\begin{equation}
\nonumber
\text{derivatives of } u, \text{ but the contribution of that term vanishes by integration by parts}
\end{equation}
\begin{equation}
\nonumber
\text{and the divergence free property. Therefore, we obtain that}
\end{equation}
\begin{equation}
\nonumber
\text{Then, we estimate } A_3 \text{ and } A_4. \text{ Similar to } A_1, \text{ we have}
\end{equation}
\begin{equation}
\nonumber
\text{and}
\end{equation}
\begin{equation}
\nonumber
\text{Similarly as in (12), we can estimate the high frequency part by Sobolev embedding}
\end{equation}
\begin{equation}
\nonumber
\text{and Plancherel’s Theorem to obtain that}
\end{equation}
\[ \| P_{\leq N} \mathcal{A}^{-1} (\partial^2 D^{-\frac{1}{2}} u \cdot \nabla u) \|_{L^2}^2 \lesssim N^{-2} \ln \frac{1}{N} \| (\partial^2 D^{-\frac{1}{2}} u \cdot \nabla u) \|_{L^2}^2 \]

\[ \lesssim N^{-2} \ln \frac{1}{N} \| \partial^2 D^{-\frac{1}{2}} u \|_{L^2}^2 \| \nabla u \|_{L^\infty}^2 \]

\[ \lesssim N^{-2} \ln \frac{1}{N} \| u \|_{H^{\frac{1}{2}}}^2 \| u \|_{H^3}^2 \]

\[ \lesssim N^{-2} \ln \frac{1}{N} \| u \|_{H^3}^4, \quad (23) \]

For the low frequency parts, similarly to (13), we have

\[ \| P_{\leq N} \mathcal{A}^{-1} (\partial D^{-\frac{1}{2}} u \cdot \nabla u) \|_{L^2}^2 \lesssim \ln \frac{1}{N} \| \nabla^{-1} (\partial D^{-\frac{1}{2}} u \cdot \nabla u) \|_{L^2}^2 \]

\[ \lesssim \ln \frac{1}{N} \| (\partial D^{-\frac{1}{2}} u \cdot \nabla u) \|_{H^{-1}}^2 \]

\[ \lesssim \ln \frac{1}{N} \| (\partial D^{-\frac{1}{2}} u \cdot \nabla u) \|_{L^{6/5}}^2 \quad (24) \]

\[ \lesssim \ln \frac{1}{N} \| \partial D^{-\frac{1}{2}} u \|_{L^3}^2 \| \nabla u \|_{L^2}^2 \]

\[ \lesssim \ln \frac{1}{N} \| \nabla u \|_{L^2}^2 \| \nabla^2 u \|_{L^2}^2. \]

Hence, substituting (14) into (24) and (25), we can bound the two estimates by

\[ \ln \ln \frac{1}{N} \| Au \|_{L^2}^2 \| u \|_{H^3}^2 + N^{-4} \ln \frac{1}{N} \| u \|_{H^3}^4. \quad (26) \]

Consequently, by (20), (22) and (26) all together, we finally arrive at

\[ |A_3| \lesssim \gamma \| A \partial^2 u \|_{L^2}^2 + \gamma^{-1} N^{-2} \ln \frac{1}{N} \| u \|_{H^3}^4 \]

\[ + \gamma^{-1} \ln \ln \frac{1}{N} \| Au \|_{L^2}^2 \| u \|_{H^3}^2 + \gamma^{-1} N^{-4} \ln \frac{1}{N} \| u \|_{H^3}^4. \quad (27) \]

Similarly, by (21), (23) and (26) all together, we have that

\[ |A_4| \lesssim \gamma \| A \partial^2 u \|_{L^2}^2 + \gamma^{-1} N^{-2} \ln \frac{1}{N} \| u \|_{H^3}^4 \]

\[ + \gamma^{-1} \ln \ln \frac{1}{N} \| Au \|_{L^2}^2 \| u \|_{H^3}^2 + \gamma^{-1} N^{-4} \ln \frac{1}{N} \| u \|_{H^3}^4. \quad (28) \]

It follows from (19), (27), (28), we finally obtain

\[ \frac{1}{2} \frac{d}{dt} \| \partial^2 u \|_{L^2}^2 + \| A \partial^2 u \|_{L^2}^2 \lesssim \gamma \| A \partial^2 u \|_{L^2}^2 + \gamma^{-1} N^{-2} \ln \frac{1}{N} \| u \|_{H^3}^4 \]

\[ + \gamma^{-1} \ln \ln \frac{1}{N} \| Au \|_{L^2}^2 \| u \|_{H^3}^2 + \gamma^{-1} N^{-4} \ln \frac{1}{N} \| u \|_{H^3}^4. \quad (29) \]
Finally, taking the $L^2$ inner product of the model equation (5) with $\Delta^3 u$, similar to (19), we obtain
\[
\frac{1}{2} \frac{d}{dt} \| \partial^3 u \|_{L^2}^2 + \| A \partial^3 u \|_{L^2}^2 = - \int_{\mathbb{R}^3} \partial^3 u \cdot \partial^3 \Delta^{-\frac{1}{2}} u \cdot \nabla u \, dx
\]
\[
- \int_{\mathbb{R}^3} \partial^3 u \cdot \partial \Delta^{-\frac{1}{2}} u \cdot \partial^2 \nabla u \, dx
\]
\[
- \int_{\mathbb{R}^3} \partial^3 u \cdot \partial^2 \Delta^{-\frac{1}{2}} u \cdot \partial \nabla u \, dx
\]
\[
= A_5 + A_6 + A_7 \tag{30}
\]
Now, we estimate $A_5$, $A_6$ and $A_7$. Similarly as in (10), we have
\[
|A_5| \lesssim \gamma \| A \partial^3 u \|_{L^2}^2 + \gamma^{-1} \| A^{-1}(\partial^3 \Delta^{-\frac{1}{2}} u \cdot \nabla u) \|_{L^2}^2 \tag{31}
\]
\[
|A_6| \lesssim \gamma \| A \partial^3 u \|_{L^2}^2 + \gamma^{-1} \| A^{-1}(\partial \Delta^{-\frac{1}{2}} u \cdot \partial^2 \nabla u) \|_{L^2}^2 \tag{32}
\]
and
\[
|A_7| \lesssim \gamma \| A \partial^3 u \|_{L^2}^2 + \gamma^{-1} \| A^{-1}(\partial^2 \Delta^{-\frac{1}{2}} u \cdot \partial \nabla u) \|_{L^2}^2 \tag{33}
\]
Similarly, for the high frequency part, we have
\[
\| P_{> N} A^{-1}(\partial^3 \Delta^{-\frac{1}{2}} u \cdot \nabla u) \|_{L^2}^2 \lesssim N^{-2} \ln^\frac{1}{2} (\ln N) \| (\partial^3 \Delta^{-\frac{1}{2}} u \cdot \nabla u) \|_{L^2}^2
\]
\[
\lesssim N^{-2} \ln^\frac{1}{2} (\ln N) \| \partial^3 \Delta^{-\frac{1}{2}} u \|_{L^\infty}^2 \| \nabla u \|_{L^2}^2
\]
\[
\lesssim N^{-2} \ln^\frac{1}{2} (\ln N) \| u \|_{H^3}^4 \tag{34}
\]
and
\[
\| P_{> N} A^{-1}(\partial \Delta^{-\frac{1}{2}} u \cdot \partial^2 \nabla u) \|_{L^2}^2 \lesssim N^{-2} \ln^\frac{1}{2} (\ln N) \| (\partial \Delta^{-\frac{1}{2}} u \cdot \partial^2 \nabla u) \|_{L^2}^2
\]
\[
\lesssim N^{-2} \ln^\frac{1}{2} (\ln N) \| \partial \Delta^{-\frac{1}{2}} u \|_{L^\infty}^2 \| \partial^2 \nabla u \|_{L^2}^2
\]
\[
\lesssim N^{-2} \ln^\frac{1}{2} (\ln N) \| u \|_{H^3}^2 \| u \|_{H^3}^2
\]
\[
\lesssim N^{-2} \ln^\frac{1}{2} (\ln N) \| u \|_{H^3}^4 \tag{35}
\]
and
\[
\| P_{> N} A^{-1}(\partial^2 \Delta^{-\frac{1}{2}} u \cdot \partial \nabla u) \|_{L^2}^2 \lesssim N^{-2} \ln^\frac{1}{2} (\ln N) \| (\partial^2 \Delta^{-\frac{1}{2}} u \cdot \partial \nabla u) \|_{L^2}^2
\]
\[
\lesssim N^{-2} \ln^\frac{1}{2} (\ln N) \| \partial^2 \Delta^{-\frac{1}{2}} u \|_{H^1}^2 \| \nabla^2 u \|_{H^\frac{1}{2}}^2
\]
\[
\lesssim N^{-2} \ln^\frac{1}{2} (\ln N) \| u \|_{H^3}^4 \tag{36}
\]
For the low frequency parts of $\| A^{-1}(\partial^3 \Delta^{-\frac{1}{2}} u \cdot \nabla u) \|_{L^2}^2$ and $\| A^{-1}(\partial \Delta^{-\frac{1}{2}} u \cdot \partial^2 \nabla u) \|_{L^2}^2$, using the Plancherel’s Theorem, Sobolev imbedding and Hölder inequality, we have
\[
\| P_{\leq N} A^{-1}(\partial^3 \Delta^{-\frac{1}{2}} u \cdot \nabla u) \|_{L^2}^2 \lesssim \ln^\frac{1}{2} (\ln N) \| \nabla^{-1}(\partial^3 \Delta^{-\frac{1}{2}} u \cdot \nabla u) \|_{H^{-1}}^2
\]
\[
\lesssim \ln^\frac{1}{2} (\ln N) \| (\partial^3 \Delta^{-\frac{1}{2}} u \cdot \nabla u) \|_{L^2}^2
\]
\[
\lesssim \ln^\frac{1}{2} (\ln N) \| (\partial^3 \Delta^{-\frac{1}{2}} u \cdot \nabla u) \|_{L^{6/5}}^2
\]
\[
\lesssim \ln^\frac{1}{2} (\ln N) \| \partial^3 \Delta^{-\frac{1}{2}} u \|_{L^2}^2 \| \nabla u \|_{L^2}^2
\]
\[
\lesssim \ln^\frac{1}{2} (\ln N) \| \nabla u \|_{L^2}^2 \| u \|_{H^3}^2 \tag{37}
\]
and

\[ \| P_{\leq N} A^{-1}(\partial D^{-\frac{1}{2}} u \cdot \partial^2 \nabla u) \|_{L^2}^2 \lesssim \ln^{\frac{1}{2}}(\ln N) \| \nabla^{-1}(\partial D^{-\frac{1}{2}} u \cdot \partial^2 \nabla u) \|_{L^2}^2 \]

\[ \lesssim \ln^{\frac{1}{2}}(\ln N) \| (\partial D^{-\frac{1}{2}} u \cdot \partial^2 \nabla u) \|_{L^2}^2 \lesssim \ln^{\frac{1}{2}}(\ln N) \| (\partial D^{-\frac{1}{2}} u \cdot \partial^2 \nabla u) \|_{L^2}^2 \]

For the low frequency parts of \( \| A^{-1}(\partial^2 D^{-\frac{1}{2}} u \cdot \partial \nabla u) \|_{L^2}^2 \), we use Sobolev imbedding

\[ \| uv \|_{H^s} \leq \| u \|_{H^{s_1}} \| v \|_{H^{s_2}}, \quad s = s_1 + s_2 - \frac{3}{2}, \quad s_1, s_2 < \frac{3}{2} \] (39)

Furthermore, by using (14), we can bound the above three estimates by

\[ \ln(\ln N) \| A u \|_{L^2}^2 \| u \|_{H^3}^2 + N^{-4} \ln^{\frac{1}{2}}(\ln N) \| u \|_{H^3}^4 \] (41)

Now, by (31), (34), (37) and (41) all together, we have that

\[ |A_5| \lesssim \gamma \| A \partial^3 u \|_{L^2}^2 + \gamma^{-1} (\ln N) \| u \|_{H^3}^4 + \gamma^{-1} \ln(\ln N) \| A u \|_{L^2}^2 \| u \|_{H^3}^2 + \gamma^{-1} N^{-4} \ln^{\frac{1}{2}}(\ln N) \| u \|_{H^3}^4 \] (42)

By (32), (35), (38) and (41) all together, we have

\[ |A_6| \lesssim \gamma \| A \partial^3 u \|_{L^2}^2 + \gamma^{-1} (\ln N) \| u \|_{H^3}^4 + \gamma^{-1} \ln(\ln N) \| A u \|_{L^2}^2 \| u \|_{H^3}^2 + \gamma^{-1} N^{-4} \ln^{\frac{1}{2}}(\ln N) \| u \|_{H^3}^4 \] (43)

And by (33), (36), (40) and (41) all together, we obtain

\[ |A_7| \lesssim \gamma \| A \partial^3 u \|_{L^2}^2 + \gamma^{-1} (\ln N) \| u \|_{H^3}^4 + \gamma^{-1} \ln(\ln N) \| A u \|_{L^2}^2 \| u \|_{H^3}^2 + \gamma^{-1} N^{-4} \ln^{\frac{1}{2}}(\ln N) \| u \|_{H^3}^4 \] (44)

It follows from (30), (42), (43) and (44) that

\[ \frac{1}{2} \frac{d}{dt} \| \partial^3 u \|_{L^2}^2 + \| A \partial^3 u \|_{L^2}^2 \lesssim \gamma \| A \partial^3 u \|_{L^2}^2 + \gamma^{-1} N^{-2} \ln^{\frac{1}{2}}(\ln N) \| u \|_{H^3}^4 + \gamma^{-1} \ln(\ln N) \| A u \|_{L^2}^2 \| u \|_{H^3}^2 + \gamma^{-1} N^{-4} \ln^{\frac{1}{2}}(\ln N) \| u \|_{H^3}^4 \] (45)
Now, putting (18), (29), (45) all together, we conclude that
\[
\frac{1}{2} \frac{d}{dt} \| \partial u \|_{L^2}^2 + \frac{1}{2} \frac{d}{dt} \| \partial^2 u \|_{L^2}^2 + \frac{1}{2} \frac{d}{dt} \| \partial^3 u \|_{L^2}^2 \\
+ \| A \partial u \|_{L^2}^2 + \| A \partial^2 u \|_{L^2}^2 + \| A \partial^3 u \|_{L^2}^2 \lesssim \gamma \| A \partial u \|_{L^2}^2 + \gamma \| A \partial^2 u \|_{L^2}^2 \\
+ \gamma^{-1} \| A \partial^3 u \|_{L^2}^2 \\
+ \gamma^{-1} N^{-2} \ln^2 (\ln N) \| u \|_{H^3}^4 \\
+ \gamma^{-1} \ln (\ln N) \| A u \|_{L^2}^2 \| u \|_{H^3}^2 \\
+ \gamma^{-1} N^{-1} \ln^2 (\ln N) \| u \|_{H^3}^4.
\] (46)

Taking \( \gamma \) to be sufficiently small and
\[
N(t) = \gamma^{-1} (500 + \| u \|_{H^3}^2),
\]
using Cauchy-Schwarz inequality, by (46) and (7) we obtain
\[
\frac{d}{dt} \| u(t) \|_{H^3}^2 \lesssim (1 + \| A u \|_{L^2}^2) \| u \|_{H^3} \ln (500 + \| u \|_{H^3}^2) \ln \ln (500 + \| u \|_{H^3}^2)
\]
Finally by Gronwall’s inequality, we have
\[
\| u(t) \|_{H^3}^2 \lesssim C,
\]
where the positive function \( C \) depends only on \( t \) and \( \| u_0 \|_{H^3} \) which is finite for any \( 0 \leq t < \infty \).

We complete the proof of Theorem 2.1.

Acknowledgments. Part of this work was done when Shu-guang Shao was visiting the Department of Mathematics of Fudan University. He would like to thank Prof. Zhen Lei for his encouragement and guidance, and also thank Prof. Fanghua Lin and Prof. Yuli Ge for helpful discussions. The authors are supported by Key Fund of the Beijing Education Committee of China, NSFC (No.11371042), BNSF (No.1132006, 1164010), Collaborative Innovation Center on Beijing Society-building and Social Governance, China Postdoctoral Science Foundation funded project, Government of Chaoyang District Postdoctoral Research Foundation and Beijing University of Technology Foundation funded project, Natural Science Fund of Henan Province (No.162300410084), the Key Research Fund (No.16A110019) and the Key Youth Teacher Foundation (2011GGJS-210) of Department Education of Henan Province (No.162300410084).

REFERENCES

[1] H. Bahouri, J. Y. Chemin and R. Danchin, Fourier Analysis and Nonlinear Partial Differential Equations, Grundlehren Math. Wiss. (Fundamental Principles of Mathematical Sciences), vol. 343, Springer, Heidelberg, 2011, xvi+523pp.
[2] J.-Y. Chemin and I. Gallagher, Wellposedness and stability results for the Navier-Stokes equations in \( \mathbb{R}^3 \), Ann. Inst. H. Poincaré Anal. Non Linéaire, 26 (2009), 599–624.
[3] C. R. Doering and J. D. Gibbon, Bounds on moments of the energy spectrum for weak solutions of the three-dimensional Navier-Stokes equations, Phys. D, 165 (2002), 163–175.
[4] D. Fang and B. Han, Global solution for the generalized anisotropic Navier-Stokes equations with large data, Mathematical Modeling and Analysis, 20 (2015), 205–231.
[5] C. L. Fefferman, Existence and smoothness of the Navier-Stokes equation, in: J. Carlson, et al. (Eds.), The Millennium Prize Problems, Clay Math. Inst., (2006), 57–67.
[6] I. Gallagher and M. Paicu, Remarks on the blow-up of solutions to a toy model for the Navier-Stokes equations, Proc. Amer. Math. Soc., 137 (2009), 2075–2083.
[7] T. Y. Hou and Z. Lei, On the stabilizing effect of convection in three-dimensional incompressible flows, Comm. Pure Appl. Math., 62 (2009), 501–564.
[8] T. Y. Hou, Z. Lei and C. M. Li, Global regularity of the 3D axi-symmetric Navier-Stokes equations with anisotropic data, Comm. Partial Differential Equations, 33 (2008), 1622–1637.

[9] T. Y. Hou, Z. Lei, G. Luo, S. Wang and C. Zou, On finite time singularity and global regularity of an axisymmetric model for the 3D Euler equations, Arch. Ration. Mech. Anal., 212 (2014), 683–706.

[10] T. Y. Hou and R. Li, Dynamic depletion of vortex stretching and non-blowup of the 3-D incompressible Euler equations, J. Nonlinear Sci., 16 (2006), 639–664.

[11] N. Katz and N. Pavlović, A cheap Caffarelli-Kohn-Nirenberg inequality for the Navier-Stokes equation with hyper-dissipation, Geom. Funct. Anal., 12 (2002), 355–379.

[12] N. H. Katz and N. Pavlovic, Finite time blow-up for a dyadic model of the Euler equations, Trans. Amer. Math. Soc., 357 (2005), 695–708.

[13] H. Koch and D. Tataru, Well-posedness for the Navier-Stokes equations, Adv. Math., 157 (2001), 22–35.

[14] Z. Lei and F. H. Lin, Global mild solutions of Navier-Stokes equations, Comm. Pure Appl. Math., 64 (2011), 1297–1304.

[15] Z. Lei, F. H. Lin and Y. Zhou, Structure of helicity and global solutions of incompressible Navier-Stokes equation, Arch. Ration. Mech. Anal., 218 (2015), 1417–1430.

[16] Z. Lei, E. A. Navas and Q. S. Zhang, A priori bound on the velocity in axially symmetric Navier-Stokes equations, Comm. Math. Phys., 341 (2016), 289–307.

[17] D. Li and Ya. Sinai, Blow ups of complex solutions of the 3d-Navier-Stokes system and renormalization group method, J. Eur. Math. Soc. (JEMS) 10 (2008), 267–313.

[18] S. Montgomery-Smith, Finite time blow up for a Navier-Stokes like equation, Proc. Amer. Math. Soc., 129 (2001), 3025–3029.

[19] P. Plechac and V. Severak, singular and regular solutions of a nonlinear parabolic system, Nonlinearity, 16 (2003), 2083–2097.

[20] P. Plechac and V. Severak, On self-similar singular solutions of the complex Ginzburg-Landau equation, Comm. Pure Appl. Math., 54 (2001), 1215–1242.

[21] T. Tao, Localisation and compactness properties of the Navier-Stokes global regularity problem, Anal. PDE, 6 (2013), 25–107.

[22] T. Tao, Structure and Randomness: Pages from Year One of a Mathematical Blog, American Mathematical Society, 2008.

[23] T. Tao, Global regularity for a logarithmically supercritical hyperdissipative Navier-Stokes equation, Anal. PDE, 2 (2009), 361–366.

[24] T. Tao, Nonlinear Dispersive Equations: Local and Global Analysis, CBMS Regional Conference Series in Mathematics 106, Conference Board of the Mathematical Sciences, Washington, DC, 2006.

[25] T. Tao, A quantitative formulation of the global regularity problem for the periodic Navier-Stokes equation, Dyn. Partial Differ. Equ., 4 (2007), 293–302.

[26] K. Y. Wang, Global regularity for a model of three-dimensional Navier-Stokes equation, J. Differential Equations, 258 (2015), 2969–2982.

[27] Y. Zhou and Z. Lei, Logarithmically improved criteria for Euler and Navier-Stokes equations, Commun. Pure Appl. Anal., 12 (2013), 2715–2719.

Received October 2016; revised December 2016.

E-mail address: ssg@emails.bjut.edu.cn
E-mail address: wangshu@bjut.edu.cn
E-mail address: xwq@bjut.edu.cn