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TWO CONGRUENCES INVOLVING HARMONIC NUMBERS WITH APPLICATIONS

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Abstract. The harmonic numbers $H_n = \sum_{0 < k \leq n} 1/k (n = 0, 1, 2, \ldots)$ play important roles in mathematics. Let $p > 3$ be a prime. With helps of some combinatorial identities, we establish the following two new congruences:

$$\sum_{k=1}^{p-1} \binom{2k}{k} H_k \equiv \frac{1}{3} \left( \frac{p}{3} \right) B_{p-2} \left( \frac{1}{3} \right) \pmod{p}$$

and

$$\sum_{k=1}^{p-1} \binom{2k}{k} H_{2k} \equiv \frac{7}{12} \left( \frac{p}{3} \right) B_{p-2} \left( \frac{1}{3} \right) \pmod{p},$$

where $B_n(x)$ denotes the Bernoulli polynomial of degree $n$. As applications we determine $\sum_{n=1}^{p-1} g_n$ and $\sum_{n=1}^{p-1} h_n$ modulo $p^3$, where

$$g_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k}$$

and

$$h_n = \sum_{k=0}^{n} \binom{n}{k}^2 C_k$$

with $C_k = \binom{2k}{k}/(k + 1)$.

1. Introduction

For $n \in \mathbb{N} = \{0, 1, 2, \ldots\}$, define

$$H_n := \sum_{0 < k \leq n} \frac{1}{k}$$

and

$$H_n^{(2)} := \sum_{0 < k \leq n} \frac{1}{k^2}.$$

Those $H_n$ with $n \in \mathbb{N}$ are classical harmonic numbers, and those $H_n^{(2)}$ with $n \in \mathbb{N}$ are called second-order harmonic numbers.

Let $p > 3$ be a prime. By a classical result of J. Wolstenholme [W], we have

$$H_{p-1} \equiv 0 \pmod{p^2} \text{ and } H_{p-1}^{(2)} \equiv 0 \pmod{p},$$

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which imply that
\[ \frac{1}{2} \binom{2p}{p} = \binom{2p - 1}{p - 1} \equiv 1 \pmod{p^3}. \]

In 2012 Sun [S12a] established some fundamental congruences involving harmonic numbers; for example, he showed that \( \sum_{k=1}^{p-1} H_k/(k2^k) \equiv 0 \pmod{p} \) motivated by the known identity \( \sum_{k=1}^{\infty} H_k/(k2^k) = \pi^2/12 \).

In 2010 Sun and Tauraso [ST10] proved that
\[ p - 1 \sum_{k=1}^{p-1} \left( \binom{2k}{k} \right) H_k \equiv \frac{8}{9} p^2 B_{p-3} \pmod{p^3}, \]
where \( B_0, B_1, B_2, \ldots \) are Bernoulli numbers given by
\[ \frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!} \quad (0 < |x| < 2\pi). \]

In 2011 Sun [S11b] showed that
\[ \sum_{k=1}^{(p-1)/2} \left( \binom{2k}{k} \right) \equiv \left( \frac{-1}{p} \right) \frac{8}{3} p E_{p-3} \pmod{p^2}, \]
where \( \left( \frac{\cdot}{p} \right) \) denotes the Legendre symbol and \( E_{p-3} \) stands for the \((p-3)\)-th Euler number.

In this paper we mainly obtain the following results.

**Theorem 1.1.** Let \( p > 3 \) be a prime. Then
\[ \sum_{k=1}^{p-1} \frac{2^k}{k} H_k \equiv \frac{1}{3} \left( \frac{p}{3} \right) B_{p-2} \left( \frac{1}{3} \right) \pmod{p} \quad (1.1) \]
and
\[ \sum_{k=1}^{p-1} \frac{2^k}{k} H_{2k} \equiv \frac{7}{12} \left( \frac{p}{3} \right) B_{p-2} \left( \frac{1}{3} \right) \pmod{p}. \quad (1.2) \]

Clearly Theorem 1.1 has the following consequence.

**Corollary 1.1.** For any prime \( p > 3 \) we have
\[ \sum_{k=1}^{p-1} \frac{2^k}{k} (4H_{2k} - 7H_k) \equiv 0 \pmod{p}. \quad (1.3) \]

Motivated by Corollary 1.1, we pose the following further conjecture.
Conjecture 1.1. For any prime \( p > 3 \) we have
\[
\sum_{k=1}^{p-1} \binom{2k}{k} (4H_{2k} - 7H_k) \equiv -14 \frac{H_{p-1}}{p} + \frac{278}{15} p^3 B_{p-5} \pmod{p^4}.
\]

The Franel numbers \( f_n = \sum_{k=0}^{n} \binom{n}{k}^3 \) (\( n = 0, 1, 2, \ldots \)) play important roles in combinatorics and number theory. It is known that \( \sum_{k=0}^{n} \binom{n}{k} f_k = g_n \), where
\[
g_n := \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k}.
\] (1.4)

For any prime \( p > 3 \), Sun [S14] and [S12b, (1.15)] showed that
\[
\sum_{n=1}^{p-1} g_n \equiv \sum_{n=1}^{p-1} h_n \equiv 0 \pmod{p^2},
\]
where
\[
h_n := \sum_{k=0}^{n} \binom{n}{k}^2 C_k
\] (1.5)
and \( C_k \) refers to the Catalan number \( \binom{2k}{k}/(k + 1) = \binom{2k}{k} - \binom{2k}{k+1} \).

Applying Theorem 1.1, we deduce the following result.

Theorem 1.2. Let \( p > 3 \) be a prime. Then
\[
\frac{1}{p^2} \sum_{k=1}^{p-1} g_k \equiv \sum_{k=1}^{p-1} g_k H_k^{(2)} \equiv \frac{5}{8} \left( \frac{p}{3} \right) B_{p-2} \left( \frac{1}{3} \right) \pmod{p},
\] (1.6)
and
\[
\frac{1}{p^2} \sum_{k=1}^{p-1} h_k \equiv \sum_{k=1}^{p-1} h_k H_k^{(2)} \equiv \frac{3}{4} \left( \frac{p}{3} \right) B_{p-2} \left( \frac{1}{3} \right) \pmod{p}.
\] (1.7)

We are going to prove Theorems 1.1 and 1.2 in Sections 2 and 3 respectively. Our proofs make use of some sophisticated combinatorial identities.
2. Proof of Theorem 1.1

Lemma 2.1. For any $n \in \mathbb{N}$, we have

$$\sum_{k=0}^{n} \binom{x}{k} \binom{y}{n-k} = \binom{x+y}{n}, \quad (2.1)$$

$$\sum_{k=0}^{n} \left( \binom{n}{k} \right)^2 H_k = \binom{2n}{n} (2H_n - H_{2n}), \quad (2.2)$$

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{2k}{k} = (-1)^n \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} \binom{2k}{k}. \quad (2.3)$$

The three identities (2.1)-(2.3) are known, see, e.g., [G, (3.1), (3.12 5) and (3.86)].

Proof of Theorem 1.1. By (2.2),

$$\sum_{j=1}^{k} \left( \binom{k}{j} \right)^2 H_j = \binom{2k}{k} (2H_k - H_{2k}) \quad \text{for each} \quad k = 1, \ldots, p - 1.$$

Therefore

$$\sum_{k=1}^{p-1} \frac{2k}{k} H_{2k} = 2 \sum_{k=1}^{p-1} \frac{2k}{k} H_k - \sum_{k=1}^{p-1} \sum_{j=1}^{k} \left( \binom{k}{j} \right)^2 H_j. \quad (2.4)$$

Observe that

$$\sum_{k=1}^{p-1} \frac{1}{k} \sum_{j=1}^{k} \left( \binom{k}{j} \right)^2 H_j = \sum_{j=1}^{p-1} \frac{H_j}{j} \sum_{k=j}^{p-1} \left( \binom{k}{j} \right) \left( k-j \right)$$

and

$$\sum_{k=j}^{p-1} \left( \binom{k}{j} \right) \left( k-j \right)$$

$$= \sum_{i=0}^{p-1-j} \left( \binom{i+j}{i} \binom{i+j-1}{i} \right) = \sum_{i=0}^{p-1-j} \left( \binom{-j-1}{i} \binom{-j}{i} \right)$$

$$= \sum_{i=0}^{p-1-j} \left( \binom{p-j}{i} \binom{p-1-j}{i} \right) = \sum_{i=0}^{p-1-j} \left( \binom{p-j}{i} \binom{p-1-j-i}{i} \right)$$

$$= \binom{2p-2j-1}{p-1-j} \pmod{p}$$
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with the help of the Chu-Vandermonde identity (2.1). Thus

$$\sum_{k=1}^{p-1} \frac{1}{k} \sum_{j=1}^{k} \binom{k}{j}^2 H_j \equiv \sum_{j=1}^{p-1} \frac{H_j}{j} \cdot \binom{2p-2j-1}{p-1-j} \equiv \sum_{j=1}^{p-1} \frac{H_j}{j} \cdot \binom{-2j-1}{p-1-j} = \sum_{j=1}^{p-1} \frac{H_j}{j} \cdot \left( \frac{p+j-1}{2j} \right)^j \equiv \sum_{j=1}^{p-1} \frac{H_j}{j} \cdot \frac{(p-1)j!}{(2j)!} \prod_{i=1}^{j} (p^2 - i^2)$$

$$\equiv p \sum_{j=1}^{p-1} \frac{H_j}{j^2} \equiv p \sum_{j=(p+1)/2}^{p-1} \frac{H_j}{j^2} \pmod{p}.$$ 

By [S11a, Lemma 2.1],

$$j \left( \binom{2j}{j} \binom{2(p-j)}{p-j} \right) \equiv 2p \pmod{p^2} \quad \text{for all } j = \frac{p+1}{2}, \ldots, p-1.$$

Therefore

$$\sum_{k=1}^{p-1} \frac{1}{k} \sum_{j=1}^{k} \binom{k}{j}^2 H_j \equiv \frac{1}{2} \sum_{j=(p+1)/2}^{p-1} H_j \cdot \binom{2(p-j)}{p-j} = \frac{1}{2} \sum_{k=1}^{(p-1)/2} \frac{(2k)H_{p-k}}{p-k} \pmod{p}.$$ 

Since

$$H_{p-k} = H_{p-1} - \sum_{0<j<k} \frac{1}{p-j} \equiv H_{k-1} = H_k - \frac{1}{k} \pmod{p}$$

for all $k = 1, \ldots, p-1$, from the above we obtain

$$\sum_{k=1}^{p-1} \frac{1}{k} \sum_{j=1}^{k} \binom{k}{j}^2 H_j \equiv - \frac{1}{2} \sum_{k=1}^{p-1} \frac{(2k)H_k}{k^2} + \frac{1}{2} \sum_{k=1}^{p-1} \frac{(2k)}{k^2} \pmod{p}.$$ 

Combining this with (2.4) we get

$$\sum_{k=1}^{p-1} \frac{(2k)}{k} H_{2k} \equiv \frac{5}{2} \sum_{k=1}^{p-1} \frac{(2k)}{k} H_k - \frac{1}{2} \sum_{k=1}^{p-1} \frac{(2k)}{k^2} \pmod{p}. \quad (2.5)$$

For each $k = 1, \ldots, p-1$, clearly

$$\binom{p}{k} = \prod_{0<j<k} \frac{p-j}{j} \equiv (-1)^{k-1} \frac{p}{k} (1-pH_{k-1}) \pmod{p^3}.$$
Thus
\[ p^{-1} \sum_{k=1}^{p-1} (-1)^k \binom{p}{k} \binom{2k}{k} \equiv -p \sum_{k=1}^{p-1} \frac{2k}{k} (1-pH_{k-1}) \quad (\text{mod } p^3). \]

On the other hand, by (2.3) we have
\[ \sum_{k=0}^{p} (-1)^k \binom{p}{k} \binom{2k}{k} = (-1)^p \sum_{k=0}^{(p-1)/2} \binom{p}{2k} \binom{2k}{k} = -1 + p \sum_{k=1}^{(p-1)/2} \frac{1-pH_{2k-1}}{2k} \binom{2k}{k} \quad (\text{mod } p^3). \]

Therefore
\[ -p \sum_{k=1}^{p-1} \frac{(2k)}{k} \left( 1-pH_k + \frac{p}{k} \right) - \binom{2p}{p} + 1 \equiv -1 + p \sum_{k=1}^{(p-1)/2} \frac{1-pH_{2k-1}}{2k} \binom{2k}{k} \quad (\text{mod } p^3). \]

(2.6)

Since \( \binom{2p}{p} \equiv 2 \quad (\text{mod } p^3) \) by Wolstenholme’s theorem, and
\[ \sum_{k=1}^{p-1} \frac{(2k)}{k} \equiv 0 \quad (\text{mod } p^2) \]

by [ST10], from (2.6) we get
\[ \sum_{k=1}^{p-1} \frac{(2k)}{k} H_k \equiv \frac{1}{2p} \sum_{k=1}^{(p-1)/2} \frac{(2k)}{k} - \frac{1}{2} \sum_{k=1}^{(p-1)/2} \frac{(2k)}{k} H_{2k} + \frac{5}{4} \sum_{k=1}^{(p-1)/2} \frac{(2k)}{k^2} \quad (\text{mod } p). \]

(2.8)

(Note that \( \sum_{k=1}^{p-1} \frac{(2k)}{k^2} \equiv \sum_{k=1}^{(p-1)/2} \frac{(2k)}{k^2} \quad (\text{mod } p). \)).

Clearly,
\[ pH_{2p-2k} = p \sum_{j=1}^{2p-2k} \frac{1}{j} + 1 \equiv 1 \quad (\text{mod } p) \]

for all \( k = 1, \ldots, (p-1)/2 \). So we have
\[ \sum_{k=1}^{(p-1)/2} \frac{1}{k^2} \frac{(2k)}{k} \equiv \sum_{k=1}^{(p-1)/2} \frac{pH_{2p-2k}}{k^2} \binom{2k}{k} = \sum_{j=(p+1)/2}^{p-1} \frac{pH_{2j}}{(p-j)^2} \binom{2(p-j)}{p-j} \]
\[ = \frac{1}{2} \sum_{j=(p+1)/2}^{p-1} \frac{(2j)}{j} H_{2j} \quad (\text{mod } p) \]
with the help of [S11b, Lemma 2.1]. By [S11b, (1.2) and (1.3)],
\( \frac{1}{p} \sum_{k=1}^{p-1} \frac{(2k)}{k} \frac{H_{2k}}{k} + \sum_{k=1}^{(p-1)/2} \frac{2}{k^2(2k)} \equiv 0 \pmod{p}. \)

Therefore
\[
\sum_{k=1}^{p-1} \frac{(2k)}{k} \frac{H_{2k}}{k} \equiv \sum_{k=1}^{(p-1)/2} \frac{(2k)}{k} H_{2k} - \frac{1}{p} \sum_{k=1}^{(p-1)/2} \frac{(2k)}{k} H_{2k} \pmod{p}.
\]

Combining this with (2.8) we get
\[
\sum_{k=1}^{p-1} \frac{(2k)}{k} \frac{H_{2k}}{k} \equiv \frac{5}{4} \sum_{k=1}^{p-1} \frac{(2k)}{k^2} - \frac{1}{2} \sum_{k=1}^{p-1} \frac{(2k)}{k} H_{2k} \pmod{p}. \quad (2.9)
\]

(2.5) and (2.9) together imply that
\[
\sum_{k=1}^{p-1} \frac{(2k)}{k} \frac{H_{2k}}{k} \equiv 2 \sum_{k=1}^{p-1} \frac{(2k)}{k^2} \pmod{p}
\text{ and }
\sum_{k=1}^{p-1} \frac{(2k)}{k} \frac{H_{2k}}{k} \equiv \frac{7}{6} \sum_{k=1}^{p-1} \frac{(2k)}{k^2} \pmod{p}.
\]

It is known that
\[
\sum_{k=1}^{p-1} \frac{(2k)}{k} \frac{H_{2k}}{k^2} \equiv \frac{1}{2} \left( \frac{p}{3} \right) B_{p-2} \left( \frac{1}{3} \right) \pmod{p} \quad (2.10)
\]
(cf. [MT]). So we get the desired (1.1) and (1.2).

**Remark 2.1.** In [S11b] the second author proved that
\[
- \frac{1}{p} \sum_{k=1}^{(p-1)/2} \frac{(2k)}{k^2} \frac{H_{2k}}{k} \equiv \sum_{k=1}^{(p-1)/2} \frac{2}{k^2(2k)} \equiv \left( \frac{-1}{p} \right) \frac{8}{3} E_{p-3} \pmod{p}.
\]

### 3. Proof of Theorem 1.2

**Lemma 3.1.** For any nonnegative integers \( m \) and \( n \), we have
\[
\sum_{k=0}^{n} \binom{x+k}{m} = \binom{n+x+1}{m+1} - \binom{x}{m+1}. \quad (3.1)
\]

and
\[
\sum_{k=0}^{n} \binom{n}{k}^2 \binom{x+k}{2n} = \binom{x}{n}^2. \quad (3.2)
\]

**Remark 3.1.** Both (3.1) and (3.2) can be found in [G, (1.48) and (6.30)].
Lemma 3.2. For any nonnegative integer \( n \), we have
\[
\sum_{k=0}^{n} \binom{n}{k}^2 \binom{x+k}{2n+1} = \frac{1}{(4n+2)^{\binom{2n}{n}}} \sum_{k=0}^{n} (2x-3k)^2 \binom{x}{k}^2 \binom{2k}{k}.
\] (3.3)

Proof. Let \( F(x) \) and \( G(x) \) denote the left-hand side and the right-hand side of (3.3). With the help of (3.2), we see that
\[
F(x+1) - F(x) = \binom{x}{n}^2.
\]
Applying the Zeilberger algorithm (cf. [PWZ, pp. 101-119]) via Mathematica, we find that
\[
G(x+1) - G(x) = \binom{x}{n}^2 \quad \text{for all} \ x = 0, 1, 2, \ldots.
\]
So, by induction \( F(x) = G(x) \) for all \( x \in \mathbb{N} \). As \( F(x) \) and \( G(x) \) are polynomials in \( x \) of degree \( 2n+1 \), we have the desired (3.3). \( \square \)

Proof of Theorem 1.2. (i) With the help of Lemma 3.1, we have
\[
\sum_{n=0}^{p-1} g_n = \sum_{n=0}^{p-1} \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k} = \sum_{k=0}^{p-1} \binom{2k}{k} \sum_{n=k}^{p-1} \binom{n}{k}^2
\]
\[
= \sum_{k=0}^{p-1} \binom{2k}{k} \sum_{n=k}^{p-1} \sum_{j=0}^{n} \binom{k}{j}^2 \binom{n+j}{2k}
\]
\[
= \sum_{k=0}^{p-1} \binom{2k}{k} \sum_{j=0}^{k} \binom{k}{j}^2 \sum_{n=k}^{p-1} \binom{n+j}{2k}
\]
\[
= \sum_{k=0}^{p-1} \binom{2k}{k} \sum_{j=0}^{k} \binom{k}{j}^2 \binom{p+j}{2k+1}.
\]

Thus, by applying Lemma 3.2 we get
\[
\sum_{k=0}^{p-1} g_k = \sum_{k=0}^{p-1} \frac{1}{4k+2} \sum_{j=0}^{k} (2p-3j) \binom{p}{j}^2 \binom{2j}{j}
\]
\[
= \frac{1}{2} \sum_{j=0}^{p-1} (2p-3j) \binom{p}{j}^2 \binom{2j}{j} \left( \sum_{k=0}^{p-1} \frac{1}{2k+1} - \sum_{0 \leq i<j} \frac{1}{2i+1} \right)
\]
\[
= \frac{1}{2} \sum_{j=0}^{p-1} (2p-3j) \binom{p}{j}^2 \binom{2j}{j} \left( H_{2p-1} - \frac{H_{p-1}}{2} - H_{2j} + \frac{H_j}{2} \right).
\]
Note that \( p H_{p-1} \equiv 0 \pmod{p^3} \) and

\[
pH_{2p-1} = 1 + p \sum_{j=1}^{p-1} \left( \frac{1}{p - j} + \frac{1}{p + j} \right) = 1 + p \sum_{j=1}^{p-1} \frac{2p}{p^2 - j^2}
\]

\[
\equiv 1 - 2p^2 H_{p-1}^{(2)} \equiv 1 \pmod{p^3}.
\]

Therefore

\[
\sum_{k=0}^{p-1} g_k \equiv \sum_{j=0}^{p-1} \frac{2p - 3j}{2p} \frac{\left( \frac{p}{j} \right)^2 \left( \frac{2j}{j} \right)}{\left( \frac{2j}{j} \right)}
\]

\[
+ \sum_{j=0}^{p-1} \frac{2p - 3j}{2} \frac{p^2}{j^2} \frac{\left( \frac{p}{j} \right)^2 \left( \frac{2j}{j} \right) \left( \frac{H_j}{2} - H_{2j} \right)}{\left( \frac{2j}{j} \right)} \pmod{p^3}
\]

and hence

\[
\sum_{k=1}^{p-1} g_k \equiv \sum_{j=1}^{p-1} \frac{2p - 3j}{2p} \frac{p^2}{j^2} \frac{\left( \frac{p}{j} \right)^2 \left( \frac{2j}{j} \right)}{\left( \frac{2j}{j} \right)}
\]

\[
+ \sum_{j=1}^{p-1} \frac{2p - 3j}{2} \frac{p^2}{j^2} \frac{\left( \frac{p}{j} \right)^2 \left( \frac{2j}{j} \right) \left( \frac{H_j}{2} - H_{2j} \right)}{\left( \frac{2j}{j} \right)}
\]

\[
\equiv p^2 \sum_{j=1}^{p-1} \frac{\left( \frac{2j}{j} \right)}{j^2} - \frac{3p}{2} \sum_{j=1}^{p-1} \frac{\left( \frac{2j}{j} \right)}{j^2} \left( \frac{p - 1}{j - 1} \right)^2
\]

\[
- \frac{3}{2} p^2 \sum_{j=1}^{p-1} \frac{\left( \frac{2j}{j} \right)}{j} \left( \frac{H_j}{2} - H_{2j} \right) \pmod{p^3}.
\]

(Note that for \( \frac{\left( \frac{2j}{j} \right)}{j} H_{2j} \) is \( p \)-adic integral for all \( j = 1, \ldots, p-1 \).) Clearly,

\[
\left( \frac{p - 1}{j - 1} \right)^2 \equiv (1 - pH_{j-1})^2 \equiv 1 - 2pH_{j-1} \pmod{p^2}.
\]

Thus

\[
\frac{1}{p^2} \sum_{k=1}^{p-1} g_k \equiv \sum_{k=1}^{p-1} \frac{\left( \frac{2k}{k} \right)}{k^2} - \frac{3}{2} \left( \frac{1}{p} \sum_{k=1}^{p-1} \left( \frac{2k}{k} \right) - 2 \sum_{k=1}^{p-1} \frac{\left( \frac{2k}{k} \right)}{k} \left( H_k - \frac{1}{k} \right) \right)
\]

\[
- \frac{3}{2} \left( \frac{1}{2} \sum_{k=1}^{p-1} \frac{\left( \frac{2k}{k} \right)}{H_k} - \sum_{k=1}^{p-1} \frac{\left( \frac{2k}{k} \right)}{k} H_{2k} \right)
\]

\[
\equiv - 2 \sum_{k=1}^{p-1} \frac{\left( \frac{2k}{k} \right)}{k^2} + \frac{9}{4} \sum_{k=1}^{p-1} \frac{\left( \frac{2k}{k} \right)}{k} H_k + \frac{3}{2} \sum_{k=1}^{p-1} \frac{\left( \frac{2k}{k} \right)}{k} H_{2k} \pmod{p}
\]
with the help of (2.7). Now, applying Theorem 1.1 and (2.10) we immediately get that
\[ \frac{1}{p^2} \sum_{k=1}^{p-1} g_k \equiv \frac{5}{8} \left( \frac{p}{3} \right) B_{p-2} \left( \frac{1}{3} \right) \pmod{p}. \quad (3.5) \]

(ii) Observe that
\[ p - 1 \sum_{n=0}^{p-1} (2g_n - h_n) = \sum_{n=0}^{p-1} \sum_{k=0}^{n} \binom{n}{k}^2 \left( 2 - \frac{1}{k+1} \right) \binom{2k}{k} \]
\[ = \sum_{k=0}^{p-1} \frac{2k+1}{k+1} \binom{2k}{k} \sum_{n=k}^{p-1} \binom{n}{k}. \]

Similar to the proof in part (i), we have
\[ \sum_{n=0}^{p-1} (2g_n - h_n) = \frac{1}{2} \sum_{j=0}^{p-1} (2p - 3j) \binom{p}{j} \binom{2j}{j} \left( H_{p-1} + \frac{1}{p} - H_j \right) \]
and thus
\[ \sum_{k=1}^{p-1} (2g_k - h_k) \equiv \frac{1}{2} \sum_{j=1}^{p-1} (2p - 3j) \frac{p^2}{j^2} \binom{p-1}{j-1}^2 \binom{2j}{j} \left( \frac{1}{p} - H_j \right) \]
\[ \equiv p^2 \sum_{j=1}^{p-1} \frac{(2j)}{j^2} - 3p \sum_{j=1}^{p-1} \frac{(2j)}{j} \left( 1 - 2pH_{j-1} \right) \left( 1 - pH_j \right) \pmod{p^3} \]
\[ \equiv p^2 \sum_{j=1}^{p-1} \frac{(2j)}{j^2} - 3p \sum_{j=1}^{p-1} \frac{(2j)}{j} \left( 1 + p \left( \frac{2}{j} - 3H_j \right) \right) \]
\[ = -2p^2 \sum_{j=1}^{p-1} \frac{(2j)}{j^2} - 3p \sum_{j=1}^{p-1} \frac{(2j)}{j} + \frac{9}{2} p^2 \sum_{j=1}^{p-1} \frac{(2j)}{j} H_j \pmod{p^3}. \]

Combining this with (1.1), (2.7) and (2.10), we obtain that
\[ \sum_{k=1}^{p-1} (2g_k - h_k) \equiv \frac{p^2}{2} \left( \frac{p}{3} \right) B_{p-2} \left( \frac{1}{3} \right) \pmod{p^3}. \]

Thus,
\[ \sum_{k=1}^{p-1} h_k \equiv \frac{3}{4} p^2 \left( \frac{p}{3} \right) B_{p-2} \left( \frac{1}{3} \right) \pmod{p^3} \quad (3.6) \]
with the help of (3.5).

(iii) By [S14, Theorem1.1],
\[
\sum_{k=1}^{p-1} g_k \equiv p^2 \sum_{k=1}^{p-1} g_k H_k^{(2)} + \frac{7}{6} p^3 B_{p-3} \pmod{p^4}.
\]
(3.7)

Therefore
\[
\sum_{k=1}^{p-1} g_k H_k^{(2)} \equiv \frac{1}{p^2} \sum_{k=1}^{p-1} g_k \pmod{p}
\]
and hence (1.6) holds in view of (3.5).

From [S14, Theorem1.1] we know that
\[
\sum_{k=0}^{p-1} g_k(x) \left(1 - p^2 H_k^{(2)}\right) \equiv \sum_{k=0}^{p-1} \frac{p}{2k+1} \left(1 - 2p^2 H_k^{(2)}\right) x^k \pmod{p^4}.
\]
(3.8)

Therefore, the left-hand side of (3.8) minus the right-hand side of (3.8) can be written as \( p^4 P(x) \), where \( P(x) \) is a polynomial of degree at most \( p-1 \) with \( p \)-adic integer coefficients. Since

\[
h_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k} C_k = \int_0^1 g_n(x) dx \quad \text{for } n = 0, 1, 2, \ldots,
\]

we deduce that
\[
\sum_{k=0}^{p-1} h_k \left(1 - p^2 H_k^{(2)}\right)
\]
\[
= \int_0^1 \sum_{k=0}^{p-1} g_k(x) \left(1 - p^2 H_k^{(2)}\right) dx
\]
\[
= \sum_{k=0}^{p-1} \frac{p}{2k+1} \left(1 - 2p^2 H_k^{(2)}\right) \int_0^1 x^k dx + p^4 \int_0^1 P(x) dx
\]
\[
\equiv \sum_{k=0}^{p-1} \frac{2p}{2k+1} \left(1 - 2p^2 H_k^{(2)}\right) - \sum_{k=0}^{p-1} \frac{p}{k+1} \left(1 - 2p^2 H_k^{(2)}\right) \pmod{p^3}.
\]

Combining (3.7) and (3.8) we see that
\[
1 + \frac{7}{6} p^3 B_{p-3} \equiv \sum_{k=0}^{p-1} g_k \left(1 - p^2 H_k^{(2)}\right) \equiv \sum_{k=0}^{p-1} \frac{p}{2k+1} \left(1 - 2p^2 H_k^{(2)}\right) \pmod{p^4}.
\]
Therefore
\[\sum_{k=0}^{p-1} h_k (1 - p^2 H_k^{(2)}) \equiv 2 + \frac{7}{3} p^3 B_{p-3} - \sum_{k=0}^{p-1} \frac{p}{k+1} + 2p^3 \sum_{k=0}^{p-1} \frac{H_k^{(2)}}{k+1} \]
\[\equiv 2 - 1 - p H_{p-1} + 2p^2 H_{p-1}^{(2)} \equiv 1 \pmod{p^3}\]
which implies that
\[\sum_{k=1}^{p-1} h_k (1 - p^2 H_k^{(2)}) \equiv 0 \pmod{p^3}.
\]
Combining this (3.6) we obtain the desired (1.7).
So far we have completed the proof of Theorem 1.2. □

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