An explicit high-order single-stage single-step positivity-preserving finite difference WENO method for the compressible Euler equations

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Abstract In this work we construct a high-order, single-stage, single-step positivity-preserving method for the compressible Euler equations. Space is discretized with the finite difference weighted essentially non-oscillatory (WENO) method. Time is discretized through a Lax-Wendroff procedure that is constructed from the Picard integral formulation (PIF) of the partial differential equation. The method can be viewed as a modified flux approach, where a linear combination of a low- and high-order flux defines the numerical flux used for a single-step update. The coefficients of the linear combination are constructed by solving a simple optimization problem at each time step. The high-order flux itself is constructed through the use of Taylor series and the Cauchy Kowalewski procedure that incorporates higher-order terms. Numerical results in one- and two-dimensions are presented.

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1 Introduction

The objective of this work is the development of a new numerical approach to the compressible Euler equations based on modified fluxes resulting from the Picard integral formulation of hyperbolic conservation laws for systems of equations. The Euler equations describe the evolution of density $\rho$, momentum $\rho u$ and energy $\mathcal{E}$ of an ideal gas through

$$
\begin{pmatrix}
\rho \\
\rho u \\
\mathcal{E}
\end{pmatrix}_t + \nabla_x \cdot \begin{pmatrix}
\rho u \\
\rho \parallel u \parallel^2 + p \\
\mathcal{E} + p \parallel u \parallel
\end{pmatrix} = 0.
$$

The energy $\mathcal{E}$ is related to the primitive variables $\rho, u$ and pressure $p$ by:

$$\mathcal{E} = \frac{p}{\gamma - 1} + \frac{1}{2} \rho \parallel u \parallel^2.$$

The ratio of specific heats is the constant $\gamma$.

Numerical difficulties for solving Eqn. (1) include the following:

- Low (1st and 2nd) order methods generally suffer from an inordinate amount of numerical diffusion. However, they are oftentimes more robust, and in some cases they have provable convergence to the correct entropy solution. Historically, 2nd-order schemes [1,2,3,4] have been called “high-resolution” methods when compared to their 1st-order counterparts [5,6,7,8].
- High-order methods [9,10,11] provide greater accuracy and resolution for much less overall computational effort. However, they are oftentimes less robust, and do not necessarily have provable convergence to the correct entropy solution.

In this work, we define a high-order method that obtains the following properties:

- High-order accuracy in space (5th-order) and time (3rd-order). Our method can be extended to arbitrary order in space or time.
- A robust scheme that stems from provable positivity preservation of the pressure and density. Numerical results indicate that high-order accuracy is retained with our positivity-preserving limiter turned on.

The method proposed in this work is a single-stage, single-step method, which means that it obtains the following advantages:

- We only need to solve one optimization problem to retain positivity per time step. Unlike positivity-“preserving” methods that use Runge-Kutta discretizations [12,13], positivity of our solution is guaranteed during the entire simulation because we do not have stages.
- Compared to other positivity-preserving schemes [14,15], we have no additional CFL restrictions.
- Our method is amenable to adaptive mesh refinement (AMR) technology. At present, we aim to lay the necessary foundation that would be required to do so. This will be the subject of a future study.
1.1 An overview of the proposed method

The Euler equations define a system of hyperbolic conservation laws. In 1D, such an equation is given by

\[ q_t + f(q)_x = 0, \]

(2)

where \( q(t,x) : \mathbb{R}^+ \times \mathbb{R} \rightarrow \mathbb{R}^m \) is the unknown vector of \( m \) conserved quantities and \( f : \mathbb{R}^m \rightarrow \mathbb{R}^m \) is a prescribed flux function. The conserved variables for the 1D Euler equations are \( \mathbf{q} = (\rho, \rho u, \mathbf{E})^T \). A typical finite difference solver for (2) discretizes space with a uniform grid of \( m_x \) equidistant points in \( \Omega = [a,b] \),

\[ x_i = a + \left( i - \frac{1}{2} \right) \Delta x, \quad \Delta x = \frac{b - a}{m_x}, \quad i \in \{1, \ldots, m_x\}, \]

(3)

and looks for pointwise approximation \( q^n_i \approx q(t^n, x_i) \) solution to hold at discrete time levels \( t^n \). In a conservative finite difference WENO method, the update of the unknowns is typically defined by

\[ q^{n+1}_i = q^n_i - \Delta t \Delta x \left( \mathbf{F}^n_{i+\frac{1}{2}} - \mathbf{F}^n_{i-\frac{1}{2}} \right), \]

(4)

where the numerical flux \( \mathbf{F}^n_{i+\frac{1}{2}} \) is constructed from a linear combination of the WENO reconstruction procedure applied to stage values from a Runge-Kutta solver.

In this work, we propose the following procedure:

1. Construct a high-order approximation to the time-averaged fluxes \( [9,16] \)

\[ \mathbf{F}^n_i := \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} f(q(t,x_i)) \, dt \]

(5)

at each grid point \( x_i \).

2. Construct a high-order in space and time numerical flux \( \tilde{\mathbf{F}}^n_{i-\frac{1}{2}} \) based upon applying the WENO reconstruction procedure to the time-averaged fluxes \( [16] \).

3. Replace the flux constructed in Step 2 with

\[ \mathbf{F}^n_{i-\frac{1}{2}} := \theta^n_{i-\frac{1}{2}} (\tilde{\mathbf{F}}^n_{i-\frac{1}{2}} - \hat{\mathbf{F}}^n_{i-\frac{1}{2}}) + \hat{\mathbf{F}}^n_{i-\frac{1}{2}}, \]

(6)

where \( \theta^n_{i-\frac{1}{2}} \in [0, 1] \) is found by solving a single optimization problem, and \( \hat{\mathbf{F}}^n_{i-\frac{1}{2}} \) is a low-order flux that guarantees positivity of the solution. This procedure will be described in \( [4] \).

4. Insert the result of Step 3 into Eqn. (4), and update the solution.

Steps 1 and 2 have already been proposed in \( [16] \), where high-order accuracy is obtained through a flux modification that incorporates the high-order temporal discretization. A review of this procedure will be presented in \( [3] \). Step 3 can be thought of as a further flux modification, where an automatic switch adjusts between the high-order non positivity-preserving scheme, and a low-order, positivity-preserving scheme. The original idea is attributed to Harten and Zwas \( [17] \), but has since been extended to high-order WENO schemes \( [12] \). The details of this procedure will be presented in \( [4] \) and \( [5] \).
2 Background

The compressible Euler equations have been an object of study ever since the infancy of numerical methods [5,6,7,8,1]. In recent years, high-order methods have attracted considerable interest because of their ability to obtain higher accuracy on certain problems with an equivalent computational cost of a low-order method. Among many choices of high-order schemes are the classical essentially non-oscillatory (ENO) method [9], the extensions to finite difference (FD) and finite volume (FV) WENO methods [11,18,19], and the discontinuous Galerkin (DG) method [20]. These methods all seek to simultaneously obtain two properties: retain high-order accuracy in smooth regions, and capture shocks without introducing spurious oscillations near discontinuities of the solution. One added difficulty with high-order schemes is the necessity of defining and selecting a high-order time integrator. Runge-Kutta methods applied to the method of lines (MOL) approach is the most widely used discretization for high-order schemes. These methods all treat space and time as separate entities.

Over the past decade, there has been a rejuvenation of interest in high-order single-stage, single-step methods for hyperbolic conservation laws, including the compressible Euler equations. All of these methods are typically based upon a Taylor temporal discretization that use the Cauchy Kovalewski procedure to exchange temporal for spatial derivatives. Lax and Wendroff performed this very procedure in 1960 [1], and this technique has since been called the Lax-Wendroff procedure within the numerical analysis community. Methods for defining a second and higher-order single-step version of Godunov’s method were investigated in the 1980’s [21,22,23]. The original high-order ENO method of Harten et. al [9] used Taylor series for their temporal discretization, although most of the attention they have received is for their emphasis on the spatial discretization. In 2001, the preliminary definitions for the so-called Arbitrary DERivative (ADER) methods [24,25,26] were put in place. Additionally, various FD WENO methods with Lax-Wendroff time discretizations have been constructed and tested on the Euler equations [27,28,16]. Recent ADER methods have been defined by Balsara and collaborators for hydrodynamics and magnetohydrodynamics [29,30], and have later been extended to an adaptive mesh refinement (AMR) setting [31]. Other recent work in single-stage, single-step methods for Euler equations includes Lax-Wendroff time stepping coupled with DG [32,33,34], and high-order Lagrangian schemes [35]. The present work is an extension of the Taylor discretization of the Picard integral formulation that uses finite differences for its spatial discretization [16], which falls into this single-stage, single-step class of methods.

Defining high-order numerical schemes that retain positivity of the solution for hydrodynamics (or magnetohydrodynamics) simulations is genuinely a non-trivial task. This has been an ongoing subject of study even for low and the so-called “high-resolution” schemes [36,37,38,39,40,41,42,43]. All methods that are second or higher order share the same disadvantage that without care, they may violate a natural weak stability condition that the density and pressure need to keep positive, which is necessary to satisfy an entropy inequality. Of some of the early positivity works, Perthame and Shu proposed a general reconstruction approach to obtain a high-order positivity-preserving finite volume schemes from a low-order scheme [39]. In addition, they proved that the explicit Lax-Friedrichs scheme is positivity-preserving with a CFL number up to 0.5. Later on, a more general result extended the positivity-preserving property to CFL numbers up to 1 for both explicit and implicit Lax-Friedrichs methods [42]. Using those results as building blocks, a positivity-preserving limiter has been proposed for DG schemes [44,45,46,47] and FD and FV WENO schemes [14,15] for the Euler equations. In [48], a flux cut-off limiter is also applied to FD WENO
schemes to retain positivity. In addition to gas dynamics, plasma physics is another area where retaining positivity of numerical solutions is critical, and therefore has seen recent attention in the literature [49][50][51]. For example, collision operators for Vlasov equations require a positive distribution function in order to avoid creating artificial singularities.

Our method is based upon a paramaterized flux limiter that can be dated back to at least 1972 with the work of Harten and Zwas [17]. There, the authors proposed a second-order shock-capturing self adjusting hybrid scheme through a simple linear combination of low- and high-order fluxes that is identical to Eqn. (6). The original idea was to combine a “high-order” flux with a first-order flux such that it has better accuracy in smooth region and produces a smooth profile around shock regions. A similar approach called flux-corrected transport is proposed by Boris and Book [52][53][54][55], where the purpose of limiting the high-order flux is to control overshoots and undershoots around shock regions. Sod has performed an extensive review of these and other classical finite difference methods [2]. Xu and Lian have recently extendeded this work to WENO methods that maintain the maximum-principle-preserving property for scalar hyperbolic conservation laws with the so-called “parametrized flux limiter” [56][57]. Later on, these limiters have been applied to FD WENO schemes on rectangular meshes [58] and FV WENO schemes on triangular meshes [13] for the Euler equations. These limiters have also been applied to magnetohydrodynamics with a constrained transport framework [59]. The basic idea for all of these methods has been the same: modify the high-order non positivity-preserving numerical flux by a linear combination of a low- and high-order flux in order to retain positivity of the solution. The modification is carefully designed so that the high-order accuracy remains.

The purpose of this work is to define a single-stage, single-step finite difference WENO method that is provably positivity-preserving for the compressible Euler equations. Our method begins with the Taylor discretization of the Picard integral formulation of finite difference schemes [16], and then applies recently developed flux limiters [58][13]. One advantage of the chosen limiter is that positivity is preserved without introducing additional CFL restrictions. However, our primary contribution is that the present methods is the first scheme to simultaneously obtain high-order, single-stage, single-step and provable positivity preservation.

The outline of this paper is as follows. In §3 we will briefly review the high-order single-stage single-step method based on the Picard integral formulation and WENO reconstruction. In §4 and §5 we will present the positivity-preserving limiter for PIF-WENO schemes applied to the compressible Euler system in single and multiple dimensions. Numerical examples of the positivity-preserving PIF-WENO scheme applied to the problems with low density and low pressure will be shown in §6. Conclusions and future work will be given in §7.

3 A single-stage single-step finite difference WENO method

The numerical method that is the subject of this work is based upon the Taylor discretization of the Picard integral formulation [16]. Our focus is on the finite difference WENO method based on a Taylor discretization of the time-averaged fluxes because it easily lends itself to the positivity-preserving limiters that will be presented in §4. In this section, we review the minimal details presented in [16] that are necessary to reproduce the present work. In addition, this section serves to set the notation that will be used in upcoming sections.

Without loss of generality, we review the Picard integral formulation for a generic hyperbolic conservation law in multiple dimensions. In 2D, a hyperbolic conservation law is
defined by a flux function with two components,

$$q_t + f(q)_x + g(q)_y = 0,$$  \hspace{1cm} (7)

where \(q(t,x,y) : \mathbb{R}^+ \times \mathbb{R}^2 \rightarrow \mathbb{R}^n\) is the vector of conserved variables, and \(f,g : \mathbb{R}^n \rightarrow \mathbb{R}^n\) are the two components of the flux function. The Euler equations are an example of a set of equations from this class of problems.

Formal integration of (7) in time over \(t \in [t^n,t^{n+1}]\) defines the 2D Picard integral formulation [16] as

$$q^{n+1} = q^n - \Delta t (F^n(x,y))_x - \Delta t (G^n(x,y))_y,$$  \hspace{1cm} (8a)

where the time-averaged flux is defined as

$$F^n(x,y) := \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} f(q(t,x,y)) \, dt, \quad G^n(x,y) := \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} g(q(t,x,y)) \, dt.$$  \hspace{1cm} (8b)

The basic idea of the Picard integral formulation of WENO (PIF-WENO) [16] is to first approximate the time-averaged fluxes (5) at each grid point using some temporal discretization, and then approximate spatial derivatives in (8a) by applying the conservative WENO reconstruction procedure to the resulting time-averaged fluxes. In this work, we approximate Eqn. (8a) with the finite difference WENO method, and we use a third-order Taylor discretization for (8b). We remark that the positivity-preserving limiter proposed in [4] can be generally applied to any form of the Picard integral formulation, including Runge-Kutta time discretizations.

Given a domain \(\Omega = [a_x,b_x] \times [a_y,b_y]\), a finite difference approximation seeks pointwise approximations \(q^n_{i,j} \approx q(t^n,x_i,y_j)\) to hold at each

$$x_i = a_x + \left(i - \frac{1}{2}\right) \Delta x, \quad \Delta x = \frac{b_x - a_x}{m_x}, \quad i \in \{1, \ldots, m_x\},$$

$$y_j = a_y + \left(j - \frac{1}{2}\right) \Delta y, \quad \Delta y = \frac{b_y - a_y}{m_y}, \quad j \in \{1, \ldots, m_y\},$$  \hspace{1cm} (9a)

for discrete values of \(t = t^n\). The 2D PIF-WENO scheme [16] solves Eqn. (7) with a conservative form

$$q_{i,j}^{n+1} = q^n_{i,j} - \lambda_x \left( \hat{F}_{i+\frac{1}{2},j}^n - \hat{F}_{i-\frac{1}{2},j}^n \right) - \lambda_y \left( \hat{G}_{i,j+\frac{1}{2}}^n - \hat{G}_{i,j-\frac{1}{2}}^n \right),$$  \hspace{1cm} (10)

where \(\lambda_x = \Delta t / \Delta x\), \(\lambda_y = \Delta t / \Delta y\), and \(\hat{F}_{i+\frac{1}{2},j}^n\) and \(\hat{G}_{i,j+\frac{1}{2}}^n\) are high-order fluxes obtained by applying the classical WENO reconstruction to the time-averaged fluxes in place of a typical “frozen-in-time” approximation to the fluxes. This requires a total of two steps: construct a time-averaged flux, followed by applying the WENO reconstruction procedure to the resulting modified fluxes.

We first define numerical time-averaged fluxes at each grid point \((x_i,y_j)\) through Taylor expansions. After taking temporal derivatives of \(f\) and \(g\), we can integrate the resulting Taylor polynomials over \(t^n, t^{n+1}\) to yield

$$F^n_{f}(x,y) := f(q(t^n,x,y)) + \Delta t \frac{df}{dt}(q(t^n,x,y)) + \frac{\Delta t^2}{2!} \frac{d^2 f}{dt^2}(q(t^n,x,y))$$  \hspace{1cm} (11a)

$$G^n_{f}(x,y) := g(q(t^n,x,y)) + \Delta t \frac{dg}{dt}(q(t^n,x,y)) + \frac{\Delta t^2}{2!} \frac{d^2 g}{dt^2}(q(t^n,x,y)).$$  \hspace{1cm} (11b)
The temporal derivatives that appear can be found via the Cauchy-Kowalewski procedure. For example, the first two time derivatives of the first component of the flux function are given by
\[ \frac{df}{dt} = -\frac{\partial f}{\partial q} \cdot (f_x + g_y), \tag{12a} \]
and
\[ \frac{d^2 f}{dt^2} = \frac{\partial^2 f}{\partial q^2} \cdot (f_x + g_y \cdot f_x + g_y) - \frac{\partial f}{\partial q} \cdot (f_x + g_y)_t. \tag{12b} \]

The last time derivative can be further simplified to
\[ (f_x + g_y)_t = \frac{\partial^2 f}{\partial q^2} \cdot (q_y, f_x + g_y) + \frac{\partial f}{\partial q} \cdot (f_{xx} + g_{yy}) + \frac{\partial^2 g}{\partial q^2} \cdot (q_y, f_x + g_y) + \frac{\partial g}{\partial q} \cdot (f_{xy} + g_{yx}). \tag{13} \]

Temporal derivatives of \( g \) have a similar structure, and can be found in [16].

We approximate each \( \partial_x, \partial_{xx} \) and \( \partial_y, \partial_{yy} \) in (12a) and (12b) by applying the 5-point finite difference formulae
\[
\begin{align*}
\frac{\partial u}{\partial x} &= \frac{1}{2\Delta x} (u_{i+1} - u_{i-1}) + O(\Delta x^2), \\
\frac{\partial^2 u}{\partial x^2} &= \frac{1}{12\Delta x} (-u_{i+2} + 8u_{i+1} - 8u_{i-1} + u_{i-2}) + O(\Delta x^4),
\end{align*}
\tag{14a}
\]

\[
\begin{align*}
\frac{\partial u}{\partial y} &= \frac{1}{2\Delta y} (u_{i+1} - u_{i-1}) + O(\Delta y^2), \\
\frac{\partial^2 u}{\partial y^2} &= \frac{1}{12\Delta y} (-u_{i+2} + 8u_{i+1} - 8u_{i-1} + u_{i-2}) + O(\Delta y^4),
\end{align*}
\tag{14b}
\]
in each direction. In order to retain a compact stencil, we compute the cross derivatives, \( \partial_{xy} \) with a second-order approximation
\[ u_{i,j} := \frac{1}{4\Delta x \Delta y} (u_{i+1,j+1} - u_{i-1,j+1} - u_{i+1,j-1} + u_{i-1,j-1}). \tag{15} \]

After defining these higher derivatives, we define numerical fluxes by \( F_{i,j} := F^o_T(x_i, y_j) \) and \( G_{i,j} := G^o_T(x_i, y_j) \). We then apply the conservative WENO reconstruction procedure in a dimension by dimension fashion to each component of the flux to construct interface values \( F^n_{i-\frac{1}{2},j} \) and \( G^n_{i,j+\frac{1}{2}} \). The complete description of this process can be found in [16].

In this work, we further modify the fluxes to obtain a provably positivity preserving method for Euler equations, which we now describe.

### 4 The positivity preserving method: the 1D case

In this section, we apply the positivity-preserving limiter to the Taylor discretization of the 1D PIF-WENO scheme. Recall that the update for the vector of conserved variables is given by Eqn. (4) for the 1D conservation law defined in (2). We denote \( \tilde{F}^{n}_{i-\frac{1}{2}} \) as the numerical flux that is high order accurate in time and space constructed from the 1D Taylor discretization of the PIF-WENO [16]. We also denote \( F^\theta_{i-\frac{1}{2}} \) as the low-order flux constructed from the Lax-Friedrichs scheme that is provably positivity preserving [39]. In this work, we modify the high-order flux by
\[ \tilde{F}^{n}_{i-\frac{1}{2}} := \theta^n_{i-\frac{1}{2}} (\tilde{F}^{n}_{i-\frac{1}{2}} - F^\theta_{i-\frac{1}{2}}) + F^\theta_{i-\frac{1}{2}}, \tag{16} \]
where the limiting parameter satisfies \( \theta^n_{i-\frac{1}{2}} \in [0, 1] \). In this section, we prove
The proof of this theorem is by construction, and it follows a two step procedure: i) guarantee positivity of the density, then ii) guarantee positivity of the pressure. The details of this procedure will be spelled out in the following subsections.

We observe that if $\theta_{i-\frac{1}{2}} = 0$, the scheme reduces to the first-order Lax-Friedrich's scheme, which is positivity preserving. If $\theta_{i-\frac{1}{2}} = 1.0$, the scheme reduces to the high-order scheme, but does not guarantee positivity of the numerical solution. In order to retain high-order accuracy, we would like to choose $\theta_{i-\frac{1}{2}}$ as close to 1.0 as possible without violating positivity of the density and pressure.

4.1 Step 1: maintain positivity of the density

This discussion will focus on the first component of the flux. We define $\hat{f}^{n,\rho}$ to denote the first component of the low-order flux $f^{n,\rho}$, for the first component of the high-order flux $\hat{F}^{n}$, and

$$f_i^{m,\rho} := \theta_i^{m,\rho} \left( f_i^{m,\rho} - f_i^{m,\rho}_{i-\frac{1}{2}} \right) + \hat{f}_i^{m,\rho}_{i+\frac{1}{2}}$$

as the first component of the modified flux $\hat{F}^{n}$ from Eqn. \(16\).

We observe that after the low- and high-order fluxes have been computed, the update for the density at a single grid point $x_i$ only depends on two values of $\theta_{i-\frac{1}{2}}$ through

$$\rho_i^{n+1} \left( \theta_{i-\frac{1}{2}}^{n}, \theta_{i+\frac{1}{2}}^{n} \right) = \rho_i^{n} - \lambda \left( f_i^{n,\rho}_{i+\frac{1}{2}} - f_i^{n,\rho}_{i-\frac{1}{2}} \right), \quad \lambda = \frac{\Delta t}{\Delta x}, \quad i \in \{1, 2, \ldots, m\}.$$  \(18\)

This, and each of the conserved variables are linear functions with respect to the variable $\left( \theta_{i-\frac{1}{2}}^{n}, \theta_{i+\frac{1}{2}}^{n} \right) \in [0, 1]^2$. In order to guarantee $\rho_i^{n+1} \geq 0$, as a function of the limiting parameter, we seek bounds $\Lambda_{\frac{1}{2}, \frac{1}{2}}$ such that whenever

$$\left( \theta_{i-\frac{1}{2}}^{n}, \theta_{i+\frac{1}{2}}^{n} \right) \in \left[ 0, \Lambda_{\frac{1}{2}, \frac{1}{2}} \right] \times \left[ 0, \Lambda_{\frac{1}{2}, \frac{1}{2}} \right] \subseteq [0, 1]^2,$$

we have

$$\rho_i^{n+1} \left( \theta_{i-\frac{1}{2}}^{n}, \theta_{i+\frac{1}{2}}^{n} \right) = \rho_i^{n} - \lambda \left( f_i^{n,\rho}_{i+\frac{1}{2}} - f_i^{n,\rho}_{i-\frac{1}{2}} \right) \geq 0.$$  \(19\)

The purpose of defining such a set is that in Step 2 of \(12\), we will further limit the fluxes to maintain positivity of the pressure.

To continue, we consider the low-order update for the density. We assume that $\rho_i^{n}$ is positive, and we define the low-order update for the density as $I_i := \rho_i^{n} - \lambda (f_i^{n,\rho}_{i+\frac{1}{2}} - f_i^{n,\rho}_{i-\frac{1}{2}})$. We observe $I_i \geq 0$ because of the positivity of low-order flux \(39\). We insert this definition of $I_i$ into Eqn. \(19\) to see

$$I_i - \lambda \left\{ \theta_i^{n,\rho} \left( f_i^{n,\rho}_{i+\frac{1}{2}} - f_i^{n,\rho}_{i-\frac{1}{2}} \right) - \theta_i^{n,\rho} \left( f_i^{n,\rho}_{i-\frac{1}{2}} - f_i^{n,\rho}_{i+\frac{1}{2}} \right) \right\} \geq 0,$$  \(20\)

Theorem 1 The numerical flux in Eqn. \(16\) preserves positivity of the solution with a suitable choice of the limiting parameters $\theta_{i-\frac{1}{2}}$. 

### Theorem 1

The numerical flux in Eqn. \(16\) preserves positivity of the solution with a suitable choice of the limiting parameters $\theta_{i-\frac{1}{2}}$. 

which is equivalent to

\[ \theta_n^i \Delta f_{i-\frac{1}{2}} - \theta_n^{i+1} \Delta f_{i+\frac{1}{2}} \geq -\Gamma, \tag{21} \]

where \( \Delta f_{i-\frac{1}{2}} := \lambda (f_{i-\frac{1}{2}}^\rho - f_{i-\frac{1}{2}}^{\tilde{\rho}}) \) is a measure of the deviation of the high- and low-order fluxes.

We determine bounds on \( \Lambda_{\rho}^{i-\frac{1}{2}}, \Lambda_{\rho}^{i+\frac{1}{2}} \) through a case-by-case discussion based on the signs of \( \Delta f_{i-\frac{1}{2}} \) and \( \Delta f_{i+\frac{1}{2}} \). This analysis has already been performed for single [56] and multidimensional [57] scalar problems. There are a total of four cases of Eqn. (21) to consider:

- **Case 1.** If \( \Delta f_{i-\frac{1}{2}} \geq 0 \) and \( \Delta f_{i+\frac{1}{2}} \leq 0 \), then we set

  \[ (\Lambda_{\rho}^{i-\frac{1}{2}}, \Lambda_{\rho}^{i+\frac{1}{2}}) = (1, 1). \]

- **Case 2.** If \( \Delta f_{i-\frac{1}{2}} \geq 0 \) and \( \Delta f_{i+\frac{1}{2}} > 0 \), then we define

  \[ (\Lambda_{\rho}^{i-\frac{1}{2}}, \Lambda_{\rho}^{i+\frac{1}{2}}) := \left(1, \min \left(1, \frac{-\Gamma_i}{\Delta f_{i+\frac{1}{2}}} \right)\right). \]

- **Case 3.** If \( \Delta f_{i-\frac{1}{2}} < 0 \) and \( \Delta f_{i+\frac{1}{2}} \leq 0 \), then we set

  \[ (\Lambda_{\rho}^{i-\frac{1}{2}}, \Lambda_{\rho}^{i+\frac{1}{2}}) := \left(\min \left(1, \frac{-\Gamma_i}{\Delta f_{i-\frac{1}{2}}} \right), 1\right). \]

- **Case 4.** If \( \Delta f_{i-\frac{1}{2}} < 0 \) and \( \Delta f_{i+\frac{1}{2}} > 0 \), there are two sub-cases to consider.

  - **Case 4a.** If the inequality (21) is satisfied with \( (\theta_n^{i-1}, \theta_n^{i+1}) = (1, 1) \) then we set

    \[ (\Lambda_{\rho}^{i-\frac{1}{2}}, \Lambda_{\rho}^{i+\frac{1}{2}}) := (1, 1). \]

  - **Case 4b.** Otherwise, we choose

    \[ (\Lambda_{\rho}^{i-\frac{1}{2}}, \Lambda_{\rho}^{i+\frac{1}{2}}) := \left(\frac{-\Gamma_i}{\Delta f_{i-\frac{1}{2}} - \Delta f_{i+\frac{1}{2}}}, \frac{-\Gamma_i}{\Delta f_{i-\frac{1}{2}} - \Delta f_{i+\frac{1}{2}}} \right). \]

After considering each of the above cases at each grid element \( x_i \), we define the following set

\[ S_i^\rho := \left[0, \Lambda_{\rho}^{i-\frac{1}{2}}\right] \times \left[0, \Lambda_{\rho}^{i+\frac{1}{2}}\right]. \tag{22} \]

The obtained set has the property that \( \rho_i^{n+1} (\theta_n^{i-\frac{1}{2}}, \theta_n^{i+\frac{1}{2}}) \geq 0 \) for any \( (\theta_n^{i-\frac{1}{2}}, \theta_n^{i+\frac{1}{2}}) \in S_i^\rho \).
4.2 Step 2: maintain positivity of the pressure

The second step will focus on the pressure $p^{n+1}$. We begin with the following Lemma, which has already been used in the past [4,12].

**Lemma 1** The pressure function satisfies

$$p \left( q^n \left( \alpha \bar{\theta}^1 + (1 - \alpha) \bar{\theta}^2 \right) \right) \geq \alpha p \left( q^n \left( \bar{\theta}^1 \right) \right) + (1 - \alpha) p \left( q^n \left( \bar{\theta}^2 \right) \right)$$

for any $\alpha \in [0, 1]$ and $\bar{\theta}^1, \bar{\theta}^2 \in S_p^n$.

**Proof** Provided $\rho > 0$, the pressure function

$$p(q) := (\gamma - 1) \left( \mathcal{E} - \frac{\|\rho u\|^2}{2\rho} \right)$$

is concave with respect to the conserved variables $q = (\rho, \rho u, \mathcal{E})$ [40,44,12]. By definition of the limiting parameter, each of the conserved variables are linear functions of $\bar{\theta}$, which means

$$q^n \left( \alpha \bar{\theta}^1 + (1 - \alpha) \bar{\theta}^2 \right) = \alpha q^n \left( \bar{\theta}^1 \right) + (1 - \alpha) q^n \left( \bar{\theta}^2 \right).$$

Together, and as a result of the construction in Step 1, these imply

$$p \left( q^n \left( \alpha \bar{\theta}^1 + (1 - \alpha) \bar{\theta}^2 \right) \right) = p \left( \alpha q^n \left( \bar{\theta}^1 \right) + (1 - \alpha) q^n \left( \bar{\theta}^2 \right) \right),$$

$$\geq \alpha p \left( q^n \left( \bar{\theta}^1 \right) \right) + (1 - \alpha) p \left( q^n \left( \bar{\theta}^2 \right) \right).$$

We define $p_i(\bar{\theta}) := p \left( q^n \left( \bar{\theta} \right) \right)$ for any $\bar{\theta} \in [0, 1]^2$ in order to simplify the notation for the ensuing discussion.

We consider the following subset

$$S^p_i := \left\{ (\theta^n_{i-\frac{1}{2}}, \theta^n_{i+\frac{1}{2}}) \in S^p_i : p_i \left( \theta^n_{i-\frac{1}{2}}, \theta^n_{i+\frac{1}{2}} \right) \geq 0 \right\} \subseteq S^p_i,$$  \tag{23}

and we observe that $S^p_i$ is convex, thanks to the result of Lemma 1. We do not attempt to find the entire boundary of $S^p_i$ because that would be computationally intractable. Instead, we define a single rectangle $R^{i,j}_p$ inside of $S^p_i$ that will define bounds on the limiting parameters.

To do this, we consider finitely many points on the boundary of $S^p_i$. To begin, denote the four vertices of $S^p_i$ as $A^{k_1,k_2} := (k_1A^p_{-\frac{1}{2}}, k_2A^p_{\frac{1}{2}})$, with $k_1, k_2$ being 0 or 1. For each $(k_1, k_2)$, we define $B^{k_1,k_2}$ based on two cases:

- **Case 1.** If $p_i(A^{k_1,k_2}) \geq 0$, we put $B^{k_1,k_2} := A^{k_1,k_2}$. The origin always falls into this case.
- **Case 2.** Otherwise, we solve the quadratic equation $p_i(rA^{k_1,k_2}) = 10^{-13}$ for the unknown variable $r \in [0, 1]$, and define $B^{k_1,k_2} := rA^{k_1,k_2}$.

After checking each vertex of $S^p_i$, we define

$$R^{i,j}_p := \left[ 0, A_{-\frac{1}{2},\delta} \right] \times \left[ 0, A_{\frac{1}{2},\delta} \right] \subseteq S^p_i \subseteq S^p_i,$$  \tag{24}
where

\[
\Lambda_{-\frac{1}{2}h} := \min_{k=0,1} \left( B^{1,k} \right), \quad \Lambda_{\frac{1}{2}h} := \min_{k=0,1} \left( B^{k,1} \right).
\]  

(25)

After performing this two step process at each grid cell \( x_i \), we finally define the limiting parameter as

\[
\theta^n_{i-\frac{1}{2}} := \min \left( \Lambda_{\frac{1}{2}h}, \Lambda_{-\frac{1}{2}h} \right).
\]  

(26)

This finishes the discussion for the 1D scheme.

**Remark 1** The positivity of the solution is guaranteed for the entire simulation.

One consequence of being a single-stage, single-step method is that we do not have stages where the density or pressure can go negative. In fact, when our method is compared to previous finite difference or finite volume WENO schemes with Runge-Kutta (RK) time discretizations \([12,13]\), the authors indicate they need to take the speed of the sound as \( c = \sqrt{|\gamma| \rho} \) for the intermediate stages of the RK method, because the limiter is only applied to the final stage of the RK method. Although it does not affect the refinement study in \([12,13]\), this treatment may lead to a potential numerical instability for some extreme cases. A similar issue has been observed during the numerical experiment of the ideal MHD equations \([59]\).

### 5 The positivity preserving method: the 2D case

In this section, we apply the positivity-preserving limiter to the two-dimensional case. Extensions to a general multi-D case follow from what will be provided here.

Recall that our single-stage, single-step update is given by Eqn. \((10)\). Similar to the 1D case, we use \( \tilde{f}^n_{i-\frac{1}{2}j} \) and \( \tilde{g}^n_{i,j-\frac{1}{2}} \) to denote the low-order positivity-preserving fluxes, and our numerical method uses modified fluxes through

\[
\tilde{F}^n_{i-\frac{1}{2}j} := \theta^n_{i-\frac{1}{2}j} \left( \tilde{f}^n_{i-\frac{1}{2}j} - \tilde{f}^n_{i-\frac{1}{2}j} \right) + \tilde{f}^n_{i-\frac{1}{2}j},
\]  

(27a)

\[
\tilde{G}^n_{i,j-\frac{1}{2}} := \theta^n_{i,j-\frac{1}{2}} \left( \tilde{g}^n_{i,j-\frac{1}{2}} - \tilde{g}^n_{i,j-\frac{1}{2}} \right) + \tilde{g}^n_{i,j-\frac{1}{2}}.
\]  

(27b)

Identical to the single-dimensional case, the positivity-preserving limiting procedure consists of two steps.

#### 5.1 Step 1: maintain positivity of the density

Our first step is to find local bounds \( \Lambda^\rho_{L,i,j} \), \( \Lambda^\rho_{R,i,j} \), \( \Lambda^\rho_{U,i,j} \) and \( \Lambda^\rho_{D,i,j} \), such that for any \((\theta^n_{i-\frac{1}{2}j}, \theta^n_{i+\frac{1}{2}j}, \theta^n_{i,j-\frac{1}{2}}, \theta^n_{i,j+\frac{1}{2}}) \in S^\rho_{i,j}\), we have

\[
\rho_{i,j}^{n+1} = \rho_{i,j}^n - \lambda^\rho \left( \tilde{f}_{i+\frac{1}{2}j}^n - \tilde{f}_{i-\frac{1}{2}j}^n \right) - \lambda^\rho \left( \tilde{g}_{i,j+\frac{1}{2}}^n - \tilde{g}_{i,j-\frac{1}{2}}^n \right) \geq 0,
\]  

(28)

where

\[
S^\rho_{i,j} := \left[ 0, \Lambda^\rho_{L,i,j} \right] \times \left[ 0, \Lambda^\rho_{R,i,j} \right] \times \left[ 0, \Lambda^\rho_{U,i,j} \right] \times \left[ 0, \Lambda^\rho_{D,i,j} \right].
\]  

(29)
Again, we have used the notation \( g^\rho \) to refer to the first component of the flux function, \( g \).
We define the low-order update as
\[
\Gamma_{i,j} := \rho^\alpha_{i,j} - \lambda_s \left( \frac{\hat{f}^{\alpha\rho}_{i-\frac{1}{2},j} - \hat{f}^{\alpha\rho}_{i+\frac{1}{2},j}}{\hat{f}^\rho_{i+\frac{1}{2},j} - \hat{f}^\rho_{i-\frac{1}{2},j}} \right) - \lambda_h \left( \frac{\hat{g}^{\alpha\rho}_{i+\frac{1}{2},j+\frac{1}{2}} - \hat{g}^{\alpha\rho}_{i+\frac{1}{2},j-\frac{1}{2}}}{\hat{g}^\rho_{i+\frac{1}{2},j+\frac{1}{2}} - \hat{g}^\rho_{i+\frac{1}{2},j-\frac{1}{2}}} \right),
\]
and observe that it satisfies \( \Gamma_{i,j} \geq 0 \) for all \( i,j \), provided the density is positive at time \( t^n \).

Similar to Eqn. (21), we rewrite Eqn. (28) as
\[
\theta^n_{i+\frac{1}{2},j} \Delta f_{i-\frac{1}{2},j} - \theta^n_{i+\frac{1}{2},j} \Delta f_{i+\frac{1}{2},j} + \theta^n_{i,j-\frac{1}{2}} \Delta g_{i,j-\frac{1}{2}} - \theta^n_{i+\frac{1}{2},j} \Delta g_{i,j+\frac{1}{2}} \geq -\Gamma_{i,j},
\]
where we have defined the deviation between the high- and low-order fluxes as
\[
\begin{align*}
\Delta f_{i-\frac{1}{2},j} &:= \lambda_s (f^{\alpha\rho}_{i-\frac{1}{2},j} - f^{\alpha\rho}_{i+\frac{1}{2},j}), \\
\Delta f_{i+\frac{1}{2},j} &:= \lambda_s (f^{\alpha\rho}_{i+\frac{1}{2},j} - f^{\alpha\rho}_{i-\frac{1}{2},j}), \\
\Delta g_{i,j-\frac{1}{2}} &:= \lambda_h (g^{\alpha\rho}_{i,j-\frac{1}{2}} - g^{\alpha\rho}_{i,j+\frac{1}{2}}), \\
\Delta g_{i,j+\frac{1}{2}} &:= \lambda_h (g^{\alpha\rho}_{i,j+\frac{1}{2}} - g^{\alpha\rho}_{i,j-\frac{1}{2}}).
\end{align*}
\]

Similar to the 1D case, we solve (31) based on the signs of \( \Delta f_{i\pm\frac{1}{2},j} \) and \( \Delta g_{i,j\pm\frac{1}{2}} \) at each node \((x_i, y_j)\). The basic idea requires a total of two steps:

1. Identify the negative values of each of the four numbers
\[
\left\{ \Delta f_{i-\frac{1}{2},j}, -\Delta f_{i+\frac{1}{2},j}, \Delta g_{i,j-\frac{1}{2}}, -\Delta g_{i,j+\frac{1}{2}} \right\}.
\]

2. Corresponding to the collective negative values, we can define the upper bounds of limiting parameters by solving Eqn. (31) for each value of \( \theta \) after neglecting any positive values found. For example, if \( \Delta f_{i-\frac{1}{2},j} - \Delta f_{i+\frac{1}{2},j} < 0 \) and \( \Delta g_{i,j-\frac{1}{2}} - \Delta g_{i,j+\frac{1}{2}} \geq 0 \), then we define
\[
\begin{align*}
\Lambda^\rho_{R,L,j} &:= \Lambda^\rho_{R,L,j} := \min \left( \frac{-\Gamma_{i,j}}{\Delta f_{i-\frac{1}{2},j} - \Delta f_{i+\frac{1}{2},j}}, 1 \right), \\
\Lambda^\rho_{U,D,L,j} &:= \Lambda^\rho_{U,D,L,j} := 1.
\end{align*}
\]
Likewise, if \( -\Delta f_{i+\frac{1}{2},j}, \Delta g_{i,j-\frac{1}{2}} < 0 \) and \( \Delta f_{i-\frac{1}{2},j} - \Delta g_{i,j+\frac{1}{2}} \geq 0 \), then we define
\[
\begin{align*}
\Lambda^\rho_{R,U,L,j} &:= \Lambda^\rho_{R,U,L,j} := \min \left( \frac{-\Gamma_{i,j}}{-\Delta f_{i+\frac{1}{2},j} + \Delta g_{i,j+\frac{1}{2}}}, 1 \right), \\
\Lambda^\rho_{U,L,D,j} &:= \Lambda^\rho_{U,L,D,j} := 1.
\end{align*}
\]
There are a total of 16 cases. Each follows similarly, and are omitted for brevity.
5.2 Step 2: maintain positivity of the pressure

Using the same construction from §4.2 we identify a rectangle \( R_{i,j}^{p,r} \subseteq S_{i,j}^p \) where the pressure satisfies \( p_{i,j}(\theta_{i,j}^{n}, \theta_{i,j}^{n+1}, \ldots) \geq 0 \). Again, we consider the vertices of the region that were computed in the first step. In 2D, we define them as

\[
A_{k_1,k_2,k_3,k_4} = (k_1 A_{L,j}^p, k_2 A_{R,l}^p, k_3 A_{D,l}^p, k_4 A_{U,j}^p), \quad k_1, k_2, k_3, k_4 \in \{0, 1\}.
\]

We rescale each vertex in an identical manner to the 1D case presented in subsection 4.2 which follows two cases:

- **Case 1.** If \( p_{i,j}(A_{k_1,k_2,k_3,k_4}) \geq 0 \), we define the vertex \( B_{k_1,k_2,k_3,k_4} := A_{k_1,k_2,k_3,k_4} \).
- **Case 2.** Otherwise we solve the quadratic equation \( p_{i,j}(r A_{k_1,k_2,k_3,k_4}) = 10^{-13} \) for the unknown \( r \in [0, 1] \) and put \( B_{k_1,k_2,k_3,k_4} := r A_{k_1,k_2,k_3,k_4} \).

In the final step, we identify a rectangular box inside \( S_{i,j}^p \) through

\[
R_{i,j}^{p,r} := [0, A_{L,i,j}] \times [0, A_{R,j,i}] \times [0, A_{D,l,i}] \times [0, A_{U,j,i}],
\]

where

\[
A_{L,i,j} := \min_{k_2, k_3, k_4 \in \{0, 1\}} \left( B_{k_1,k_2,k_3,k_4} \right), \quad A_{R,i,j} := \min_{k_1, k_3, k_4 \in \{0, 1\}} \left( B_{k_1,k_2,k_3,k_4} \right),
\]

\[
A_{D,l,i} := \min_{k_1, k_2, k_4 \in \{0, 1\}} \left( B_{k_1,k_2,k_3,k_4} \right), \quad A_{U,j,i} := \min_{k_1, k_2, k_3 \in \{0, 1\}} \left( B_{k_1,k_2,k_3,k_4} \right). \]

After repeating this procedure for each node \((x, y)\), we finish by defining the scaling parameter as

\[
\theta_{i,j}^{n} := \min(A_{R,j,i}, A_{L,i,j}), \quad \theta_{i,j}^{n+1} := \min(A_{U,j,i}, A_{D,l,i}),
\]

and insert the result into Eqn (27) to define our modified fluxes. This finishes the discussion for the 2D scheme.

6 Numerical results

In this section, we perform numerical simulations with our proposed positivity-preserving method on 1D and 2D compressible Euler equations. Unless otherwise stated, the gas constant is \( \gamma = 1.4 \) and the CFL number is 0.35.

6.1 Accuracy test

To test the accuracy of this limiter in PIF-WENO schemes, we use the smooth vortex problem with low density and low pressure [12,13]. Initially, we have a mean flow

\[
(\rho, u_1, u_2, u_3, p) = (1, 1, 1, 0, 1),
\]

with perturbations on \( u_1, u_2 \) and the temperature \( T = p/\rho \):

\[
(\delta u_1, \delta u_2) = \frac{\varepsilon}{2\pi} e^{\frac{5(1-r^2)}{2}}(-y, x),
\]

\[
\delta T = -\frac{(\gamma - 1)\varepsilon^2}{8\gamma \pi^2} e^{1-r^2}.
\]
Table 1  Accuracy test of the 2D smooth vortex. Shown are the $L_1$-errors and $L_{\infty}$-errors at time $t = 0.01$ of the density. The solutions converge at fifth-order accuracy.

| Mesh   | $L_1$-Error | Order | $L_{\infty}$-Error | Order |
|--------|-------------|-------|--------------------|-------|
| $80 \times 80$ | 2.970e-06 | -     | 2.494e-04          | -     |
| $160 \times 160$ | 1.627e-07 | 4.190 | 2.442e-05          | 3.353 |
| $320 \times 320$ | 7.384e-09 | 4.462 | 1.390e-06          | 4.135 |
| $640 \times 640$ | 2.428e-10 | 4.927 | 4.718e-08          | 4.881 |

Here $r^2 = x^2 + y^2$. The initial pressure and density are determined by keeping the entropy $S = p/\rho^\gamma$ constant. The domain is $(x, y) \in [-5, 5] \times [-5, 5]$ with periodic boundary condition on all sides. The vortex strength $\varepsilon$ is set as $10^{0.0828}$ such that the lowest density and lowest pressure in the center of the vortex are $7.8 \times 10^{-15}$ and $1.7 \times 10^{-20}$ respectively.

The $L_1$-errors and $L_{\infty}$-errors of the density at $t = 0.01$ are shown in Table 1. We observe the fifth-order accuracy of the proposed scheme. The result is also comparable with those given in [12,15]. In [15], the authors took $\Delta t = \Delta x^5$ so as to make the spatial error dominating the numerical error. We find this treatment does not affect the result so much when the time $t$ is as small as 0.01. So in the table, we only present the case when a constant CFL number of 0.35 is taken. If the proposed limiter is not used for PIF-WENO scheme, we observed negative density and negative pressure appear in the center of vortex with the same setup.

6.2 1D Sedov blast wave problem

The first 1D problem we considered is the 1D Sedov blast problem originally from the book by Sedov [60]. The problem describes an intense explosion in a gas where the disturbed air is separated from the undisturbed air by a shock wave. Initially, we deposit a quantity of energy $\delta = 3200000$ into the center cell of the domain with the length of $\Delta x$, while the energy in the other cell is simply set as $10^{-12}$. The other quantities is initialize as a constant with $\rho = 1$ and $u_1 = 0$. The numerical results are shown in Fig. 1 where we can see the shock wave is captured well with the proposed limiter used. In Fig. 1, we use the exact solution given in Sedov’s book [60] as the reference solution. Our results are in good agreement with recent work [44,15,12].

6.3 1D double rarefaction problem

The second 1D problem we considered is the double rarefaction problem. It is a Riemann problem with an initial condition of $\rho_L = \rho_R = 7$, $u_1^L = -1$, $u_1^R = 1$ and $p_L = p_R = 0.2$. The exact solution consists of two rarefaction wave traveling in opposite directions. This results in a vacuum in the center of the domain. With the proposed limiter, we are able to solve this low density and low pressure problem. The numerical results are shown in Fig. 2. We used the same resolution of $\Delta x = 1/200$ as those in reference [44,15,12]. We can see the result agrees well with the exact solution and those in the reference. Here, the exact solution is a highly resolved solution with $\Delta x = 1/1000$. Other Riemann problems have been investigated, and our method gives similar results [38,40].
6.4 2D Sedov blast wave problem

We also considered a 2D version of Sedov blast wave problem. In 2D case, the problem has an exact self-similar solution and we expect the numerical result has a similar structure. In the simulation, we only computed one quadrant of the whole domain, i.e., the computation domain is \((x, y) \in [0, 1.1] \times [0, 1.1]\). Similar as the 1D case, we deposit a quantity of energy \(\mathcal{E} = 0.244816\) into the lower left corner cell, while let the energy in the other cells be \(10^{-12}\). The other initial values are the same as the 1D case. Solid wall boundary conditions are used at the left and bottom boundary such that the setup is equivalent with computing the whole domain \([-1.1, 1.1] \times [-1.1, 1.1]\) with \(\mathcal{E} = 0.979264\). The computed density are presented in Fig. 3, from which we can see the result has a good self-similar structure. Additionally, it is observed that the density cut at \(x = 0\) matches with the exact solution very well. Again, our results are in agreement with other recent works [44,15,12].

6.5 2D Shock diffraction problem

The second 2D problem we considered is the shock diffraction problem. The computational domain is \([0, 1] \times [6, 11] \cup [1, 13] \times [0, 11]\). There is a shock wave of Mach number 5.09 initially located at \(\{x = 0.5, 6 \leq y \leq 11\}\). As time evolves, the wave moves into undisturbed air with a density of 1.4 and pressure of 1. We use an inflow boundary condition at \(\{x = 0, 6 \leq y \leq 11\}\), and an outflow boundary condition at \(\{x = 13, 0 \leq y \leq 11\}\), \(\{1 \leq x \leq 13, y = 0\}\) and \(\{0 \leq x \leq 13, y = 11\}\). For the other parts of the boundary where \(\{0 \leq x \leq 1, y = 6\}\) and \(\{x = 1, 0 \leq y \leq 6\}\), solid wall boundary conditions are used. As the shock hit the corner, negative density and negative pressure can be observed if the proposed limiter is not used for the PIF-WENO scheme. The problem can be successfully solved with the limiter used. The density and pressure at time \(t = 2.3\) are presented in Figures 4. The results are consistent with those solutions solved by different schemes in [44,15,12,13].

7 Conclusions and future work

In this paper we proposed a high-order, single-stage, single-step, positivity-preserving method for the compressible Euler equations. The base scheme is using a Lax-Wendroff procedure constructed by the Picard integral formulation coupled with a single finite difference WENO reconstruction per time step. A positivity-preserving limiter is used in such a way that the positivity of the solution is preserved for the entire simulation, which makes our scheme robust. In addition, we have no excessive CFL restriction in order to retain positivity. This makes our new method more efficient compared to recent positivity-preserving methods. We demonstrated the effectiveness and efficiency of the positivity-preserving schemes on one- and two-dimensional problems with low density and pressure. Numerical results indicate that high-order accuracy is retained with our positivity preserving limiter. The future work will include applying the proposed methods to other systems such as magnetohydrodynamics and incorporating our method into an AMR framework.

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Fig. 1 1D Sedov blast wave problem. Shown in these panels are plots at time $t = 0.001$ of (a) the density, (b) the pressure and (c) $u^1$. The solid lines are the exact solutions. The solution was obtained on a mesh with $\Delta x = 1/200$.

Fig. 2 1D double rarefaction problem. Shown in these panels are plots at time $t = 0.6$ of (a) the density, (b) the pressure and (c) $u^1$. The solid lines are the exact solutions. The solution was obtained on a mesh with $\Delta x = 1/200$. 
Fig. 3 2D Sedov blast wave problem. Shown in these panels are plots at time $t = 1$ of (a) the density, (b) the density cut at $y = 0$. The solid line in (b) is the exact solution. The solution was obtained on a $160 \times 160$ mesh.

Fig. 4 2D Shock diffraction problem. Shown in these panels are plots at time $t = 2.3$ of (a) the density, (b) the pressure. 20 equally spaced contour lines from $\rho = 0.0662$ to 7.07 are used for (a). 40 equally spaced contour lines from $p = 0.091$ to 37 are used for (b). The solution was obtained on a mesh with $\Delta x = \Delta y = 1/30$.

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