MATHEMATICAL ANALYSIS OF A CLOUD RESOLVING MODEL INCLUDING THE ICE MICROPHYSICS

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Abstract. We extend our study for the warm cloud model in [13] to the analysis of a more general cloud model including the ice microphysics in [28]. The moisture variables comprise water vapor, cloud condensates (cloud water, cloud ice), and cloud precipitations (rain, snow), with respective mass ratios $q_v, q_c$ and $q_p$. A typical assumption in [13] for the calculation of condensation rate is that the warm clouds are exactly at water saturation with no supersaturation in general. When the ice microphysics are included, the situation becomes more complicated. We have to consider both the saturation mixing ratio with respect to water ($q_{vw}$) and the saturation with respect to ice ($q_{vi}$) when the temperature $T$ is below the freezing point $T_w$ but above the threshold $T_i$ for homogeneous ice nucleation. A remedy, acceptable from the physical and mathematical viewpoints, is to define the overall saturation mixing ratio $q_{vs}$ as a convex combination of $q_{vw}$ and $q_{vi}$. Under this setting, supersaturation can still be avoided and we have the constraint $q_v \leq q_{vs}$ with $q_{vs}$ depending itself on the state. Mathematically, we are led to a system of equations and inequalities involving some quasi-variational inequalities for which we prove the global existence and regularity of solutions.

1. Introduction. Clouds, which consist of various forms of water droplets and/or ice crystals, have been the greatest source of uncertainty regarding the current numerical weather and climate prediction models. The primitive equations (PEs) provide the classical model for the study of climate and weather prediction, describing the motion of the atmosphere when the hydrostatic assumption is enforced (see e.g., [33], [34], [51]). To the best of our knowledge, the mathematical analysis of the humid atmosphere equations appearing in [27], [51] has been initiated in [46]. Over the last three decades, this research field has developed rapidly and attracted numerous researchers.

In [46] and the more recent articles [31, 32], only the air vapor concentration $q_v$ is considered and the equation for $q_v$ is a simple transport equation when the

2010 Mathematics Subject Classification. Primary: 35K86, 49J40, 76D03; Secondary: 35K55, 86A10.

Key words and phrases. Humid atmosphere, penalization, regularization, variational and quasi variational inequalities.

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saturation of water vapor in the air is not being accounted for. The references [18], [16], and [10] are among the first articles to have accounted for the water saturation, with the two additional simplifying assumptions that the fluid velocity $u$ is known and that the saturation concentration $q_{vs}$ is constant, as $q_{vs}$ does not vary too much from the physical viewpoint. In the references [18, 16, 10], a major mathematical difficulty is from the source term of the equation for the water vapor $q_v$, which is modeled via a Heaviside function as a switching term between the saturated and unsaturated regions. More precisely, in cloud microphysics parameterizations it is often assumed that the vapor-to-cloud water conversion is instantaneous, i.e., that either the air is saturated, that is, the water vapor content matches its saturation value, $q_v = q_{vs}$, and the cloud water droplets can exist after condensation, or the air is unsaturated, i.e., $q_v < q_{vs}$, in which case the cloud water evaporates immediately; see e.g., [28]. The involvement of thresholds for the cloud water condensation and evaporation then leads to the introduction of a Heaviside function, which in turn brings nonlinearity, discontinuity and non-monotonicity to the equations for $q_v$ and the temperature $T$. Nevertheless, the above mentioned references managed to establish results of existence, uniqueness and regularity of solutions. Then in [17], without assuming that the velocity $u$ is known, the authors carried out a study of the humid atmosphere model with phase change coupled to the primitive equations by combining the methods in [16, 18] with the methods for the 3-dimensional PEs in [14, 40]. Other equations of geophysics involving a discontinuous Heaviside function can be found, for instance, in [20, 21], and [24, 25, 27].

In trying to extend the study to the case where $q_{vs}$ is not constant [58], it was found that the equations of the humid atmosphere in the classical references, e.g., [33, 34, 54], are inconsistent for the extreme cases $q_v = 0$ and $q_v = 1$, whether $q_{vs}$ is constant or not. Here $q_v = 0$ corresponds to a totally dry atmosphere, $q_v = 1$ corresponds to a totally humid atmosphere. A physically satisfactory resolution of this difficulty is proposed in [58] by formulating the equation for $q_v$ as a variational inequality. Still assuming $q_{vs}$ to be constant, results of existence, maximum principle and regularity of solutions for the variational inequality are derived in [60]. A numerical study of the systems in [60] follows in [59].

A more recent article [13] further generalized the model studied in [60] and [58] by considering the more realistic situation where the saturation mixing ratio $q_{vs}$ itself depends on the temperature $T$ and the humid atmosphere comprises three components instead of only one, namely water vapor $q_v$, rain water $q_r$ and cloud water $q_c$. In line with the hypothesis that the vapor-to-cloud water conversion is instantaneous, the water vapor $q_v$ in general satisfies the nonlinear constraint $q_v \leq q_{vs}$. When the constraint $q_v \leq q_{vs}$ itself depends on the solution $T$, the authors are led to introduce and handle a system of equations and inclusions involving a so-called quasi-variational inequality for which they prove the existence of solutions using penalization techniques. Penalization has been introduced by R. Courant [19] and it is very common in Optimization Theory (see e.g., [15] and [53]). For general results on (quasi-)variational inequalities and their utilization in economics, mechanics and physics, see among a vast literature [12, 2, 6, 7, 5, 8, 9, 4, 22, 23, 26, 41, 36, 38, 44, 47, 50, 49, 48].

While the above mentioned references [18, 16, 10, 58, 60, 59, 13] all adopt the classical setting that the vapor-to-cloud water conversion is instantaneous for the parameterization of the condensation of water vapor to cloud water and the inverse evaporation process, another bulk microphysics description was proposed in [39],
in which the authors did not assume this limiting vapor-to-cloud water conversion behavior from the outset and demonstrated how it may be derived in a consistent asymptotic framework given large but finite condensation rates. More recently in [35], the authors proved the global existence and uniqueness of uniformly bounded solutions of the model introduced in [39], where the parameterization of the microphysics terms are essentially of power law type.

Despite the differences in the parametrization of the condensation, the humidity atmosphere models studied in [13] and [35] both follow the spirit of the pioneering warm rain bulk microphysics model by Kessler [37]. The approach of Kessler has been extended to include the ice phase ever since the 1980s, see e.g., [42, 55] and [28]. For a detailed introduction of the evolution of cloud microphysics modeling, we may refer the readers to the review paper [29] and the references therein.

The purpose of this article is to extend our previous work in [13] to a cloud model which includes the ice microphysics. We still assume the velocity is given and study the model for water vapor $q_v$, cloud condensates $q_c$ (including cloud water and cloud ice) and cloud precipitation $q_p$ (including rain and snow) coupled to the thermodynamic equation through the latent heat in the setting of [28]. In [28] a simple extension of the bulk warm rain model is proposed for the cloud resolving simulations of large-scale circulations. The proposed model attempts to capture the essential aspects of the cloud physics at subfreezing temperatures without including additional moisture variables like graupel and heavily rimed particles. The general strategy of the model is to focus only on two classes of the condensed water: the cloud condensate $q_c$ and the precipitation $q_p$. Cloud condensate follows the motion of air and there is no sedimentation, while cloud precipitation will fall from the cloud in the form of rain or snow. It is worth mentioning that, although we do not account for some moisture species like graupel and heavily rimed particles in the present model, our analysis can be easily extended to the more sophisticated state-of-the-art ice bulk microphysics models, e.g., [42, 55], since the mathematical form of the governing equations for other moisture species is essentially the same as that of the equations we consider here. Under the setting of [28], the cloud condensate $q_c$ represents either cloud water $q_w$ or cloud ice $q_i$, depending on the temperature $T$. More precisely, we assume $q_c$ to be in the form of water $q_w$ for temperatures warmer than $T_w$ and in the form of ice $q_i$ for temperatures colder than $T_i$. For temperatures between $T_w$ and $T_i$, $q_c$ is a mixture of $q_w$ and $q_i$, with the relative contribution of the cloud water $q_w$ linearly decreasing with the temperature $T$ approaching $T_i$. In other words, we define the function $\alpha(T)$ as follows:

$$\alpha(T) = \begin{cases} 
0 & \text{for } T \leq T_i, \\
\frac{T-T_i}{T_w-T_i} & \text{for } T \in (T_i, T_w], \\
1 & \text{for } T > T_w.
\end{cases}$$

And we can retrieve $q_w$ and $q_i$ from $q_c$ by the equation $q_w = \alpha(T)q_c$ and $q_i = (1-\alpha(T))q_c$. Similarly for the precipitation $q_p$, we have $q_r = \alpha(T)q_p$ for rain and $q_s = (1-\alpha(T))q_p$ for snow. Then for the source and sink terms for $q_v$, $q_c$ and $q_p$, this model contains closures for the condensation from water vapor, the autoconversion of cloud condensate into precipitation, and the accretion of cloud condensate by precipitation, as well as the source (resp. sink) of precipitation due to deposition (resp. evaporation) of water vapor on (from) precipitation particles.

When we extend the analysis for the warm clouds model in [13] to the ice-bearing cloud model that we study here, one critical question is how we define the saturation
condition and calculate the condensation rate when both water and ice phases are to be considered. In [13] we have the assumption that cloud condensate evaporates instantaneously in the unsaturated conditions. In the warm cloud case, the above assumption that clouds are exactly at water saturation is fairly accurate because of the abundance of cloud condensation nuclei and very low supersaturations. In our current study considering the ice microphysics, this situation becomes more complicated. When the temperature is between -20°C and 0°C, ice nuclei are scarce and the conditions inside ice-bearing clouds can vary between saturation with respect to water and saturation with respect to ice, depending on the balance between the water vapor, ice clouds and precipitation particles that are available for growth. Fortunately, for temperatures lower than -20°C, the number of ice nuclei strongly increases with decreasing temperatures and the typical conditions inside cold clouds are generally close to the saturation with respect to ice. Therefore, as far as formation of the cloud condensate is concerned, it is still acceptable from the physical viewpoint to extend the classical warm rain approach to cold clouds. Accordingly, the saturated conditions are determined by the saturation mixing ratio \( q_{vw} \) with respect to the plane water for temperatures above a threshold temperature \( T_w \) (e.g., 0°C) and by the saturation mixing ratio \( q_{vi} \) with respect to the plane ice for temperatures below the threshold temperature \( T_i \) (e.g., -20°C). For temperatures between \( T_w \) and \( T_i \), the saturation mixing ratio is defined using a linear combination of the water and ice saturation mixing ratios so that the transition between warm (i.e., water saturated) and very cold (ice saturated) regimes is smooth. Then the overall saturation mixing ratio \( q_{vs} \) is defined by \( q_{vs} = \alpha(T)q_{vw} + (1 - \alpha(T))q_{vi} \). The condensation rate can then be calculated using the typical saturation adjustment method with \( q_{vs} \) being the threshold for condensation, which will introduce the discontinuous Heaviside function in the form of \( H(q_v - q_{vs}) \) in the source term.

Based on the discussions about the saturation conditions above, we will still have the nonlinear constraint \( q_v \leq q_{vs} \) which will lead to the study of a quasi-variational inequality as in the previous work [13], since \( q_{vs} \) depends on \( T \) and hence on the solution. The main theorem of our article is to prove the global existence of the solution to this quasi-variational inequality. The mathematical difficulty which comes from the discontinuity of the source terms and from the nonlinear constraint \( q_v \leq q_{vs} \) can be overcome after we carefully extended the regularization and penalization techniques introduced in [13] to our current ice-bearing cloud model. The new challenges mainly come from two aspects. On the one hand, the saturation mixing ratio \( q_{vs} \) requires more delicate treatment when we derive its a priori estimates because of its new formulations. On the other hands, we need to modify our a priori estimates to insert the anelastic framework used by the model in [28].

The rest of the article is organized as follows. In Section 2, we introduce the formulation of the equations and the specific forms of the source terms given in [28]. Then in Sections 3 and 4 we develop the mathematical setting for these equations. Section 3 is devoted to presenting the general mathematical setting, the initial and boundary conditions, and the regularized extensions of the source terms outside its physical relevant range of values. In Section 4, we account for the constraint \( q_v \leq q_{vs} \) and introduce the quasi-variational inequality that we intend to study, that is, prove the existence of its solution. To this aim, we introduce, in Section 4, a penalization procedure, by which we approximate the quasi variational inequality by a relatively standard nonlinear problem which can be treated by classical tools found in many
references, including e.g., [45, 56, 57]. Note that the use of the penalization method is just a convenient mathematical tool and we do not try to give a physical meaning to the penalized problem. Then we prove some a priori estimates for the penalized and \( \varepsilon \)-regularized solution, and finally pass to the limit as \( \varepsilon \to 0 \) to end up with the existence of the solution for the initial non regularized problem. The passage to the limit relies on using some classical compactness results and convex analysis tools. Lastly, in Section 5, we illustrate the theory studied in the previous sections with some numerical simulations done in a slightly different setting in a 2D space (with coordinates \( x \) and \( z \)), where we simulate the formation of snow on one or two mountains. The viscosity terms are omitted as they are not significant for short term forecasts, although they are important for the mathematical analysis.

2. Formulation and setting of the exact problem. We let \( M \subset \mathbb{R}^3 \) be the spatial domain for our study and a typical point in \( M \) is denoted by \( x = (x, y, z) \). The thermodynamics variables we consider are the potential temperature \( \theta \), the water vapor mixing ratio \( q_v \), the cloud condensate mixing ratio \( q_c \), and the precipitation water mixing ratio \( q_p \). The conservation equations for the thermodynamic variables in the anelastic frame work are written as follows

\[
\frac{\partial (\rho_o \theta)}{\partial t} + \nabla_3 \cdot (\rho_o \mathbf{u} \theta) = \frac{L_v \theta_e}{c_p T_e} (\text{CON} + \text{DEP}) + D_\theta, \tag{2}
\]

\[
\frac{\partial (\rho_o q_v)}{\partial t} + \nabla_3 \cdot (\rho_o \mathbf{u} q_v) = -\text{CON} - \text{DEP} + D_{q_v}, \tag{3}
\]

\[
\frac{\partial (\rho_o q_c)}{\partial t} + \nabla_3 \cdot (\rho_o \mathbf{u} q_c) = \text{CON} - \text{ACC} - \text{AUT} + D_{q_c}, \tag{4}
\]

\[
\frac{\partial (\rho_o q_p)}{\partial t} + \nabla_3 \cdot [\rho_o (\mathbf{u} - \mathbf{V}_T k) q_p] = \text{ACC} + \text{AUT} + \text{DEP} + D_{q_p}. \tag{5}
\]

Here, \( \mathbf{u} = (u, v, w) \) is the prescribed air velocity; the \( D \) terms on the right hand side of (2)-(5) are the usual dissipation terms (like the 3D Laplacian \( \Delta_3 \)), symbolizing subgrid-scale turbulence parameterization terms as well as gravity wave absorbers employed in the vicinity of the model boundaries; \( L_v \), \( c_p \), and \( V_T \) denote the latent heat of condensation, specific heat at constant pressure, and mass-weighted terminal velocity of precipitation particles, respectively; and \( \mathbf{k} = (0, 0, 1)^t \) is the unit vector in the vertical direction. The source terms on the right hand side of (2)-(5) describe the microphysics processes of cloud condensate from water vapor (\( \text{CON} \)), autoconversion of cloud condensate into precipitation (\( \text{AUT} \)), accretion of cloud condensate by precipitation (\( \text{ACC} \)), and source (sink) of precipitation due to deposition (evaporation) of water vapor on (from) precipitation particles (\( \text{DEP} \)). The expressions for the above source terms will be given in the following subsection.

In addition, the subscripts \( e \) for \( T_e \) and \( \theta_e \) in (2) refer to profiles of environment reference state. For temperature \( T \) and potential temperature \( \theta \), we classically have

\[
T = \theta \left( \frac{p_e}{p_{oo}} \right)^{R_d/c_p} = \theta \Pi, \tag{6}
\]

where \( p_e \) is the environment pressure depending on the height \( z \), \( R_d \) is the gas constant for the dry air, \( p_{oo} = 10^5 \text{Pa} \), and \( \Pi = (p_e/p_{oo})^{R_d/c_p} \) is the non-dimensional pressure. The anelastic reference density profile \( \rho_o = \rho_o(z) \) is a linear function
depending on $z$. More specifically, $\frac{d\rho_o}{dz} = \kappa \frac{N^2}{g}$, where $N$ is the (constant) Brunt-Väisälä frequency, $g$ is the acceleration of gravity and $\kappa$ is some constant. (In [30], the chosen value for $\frac{N^2}{g}$ is $1 \times 10^{-5}$ m$^{-1}$).

2.1. The source terms. In this section, we will give the specific expressions for the source terms $\text{CON}$, $\text{AUT}$, $\text{ACC}$ and $\text{DEP}$. Most important for our study is the term $\text{CON}$, which represents the cloud condensation rate from water vapor. The calculation of $\text{CON}$ usually involves the saturation adjustment and the contribution of this term only applies when the water vapor $q_v$ reaches the saturation threshold $q_{vs}$. From the mathematical viewpoint, the formulation of $\text{CON}$ introduces the difficulty of dealing with a nonlinear and discontinuous right hand side. As the $\text{CON}$ term and the saturation mixing ratio $q_{vs}$ play a crucial role in our analysis, we will give their detailed expressions in this section. For the other terms, since their expressions are more regular, we only give their simplified expressions here and refer the interested reader to the original formulations in [28] and in the appendix of this article. We will see that each source term is made of two parts, one term accounts for the contribution of humidity quantities from the above freezing regime (e.g., water and rain), the other accounts for the contribution of those quantities from the below freezing regime (e.g., ice and snow). We will use the subscripts $w$ and $i$ to represent water and ice, respectively. Similarly, the subscripts $r$ and $s$ correspond to the situations of rain and snow, respectively.

$\text{CON}$: cloud bulk condensation rate from water vapor. The expression for $\text{CON}$ is adapted from [61], which is a function depending on the saturation mixing ratio $q_{vs}$ and the temperature $T$. The term $\text{CON}$ has two parts: the condensation of the cloud water and the condensation of the cloud ice, namely

\[
\text{CON} = \text{CON}_w + \text{CON}_i,
\]

with

\[
\text{CON}_w = C_{\text{conw}} \rho_o \alpha(T) \mathcal{H}(q_v - q_{vs}) \left(1 + \frac{L_v^2 q_{vw}}{c_p R_v T^2}\right)^{-1}
\]

\[
\text{CON}_i = C_{\text{coni}} \rho_o (1 - \alpha(T)) \mathcal{H}(q_v - q_{vs}) \left(1 + \frac{L_s^2 q_{vi}}{c_p R_v T^2}\right)^{-1}
\]

(7)

Here $C_{\text{conw}}$ and $C_{\text{coni}}$ are some nondimensional positive constants, $\mathcal{H}(r)$ is the multi-valued Heaviside function with $\mathcal{H}(r) = 0$ for $r < 0$, $\mathcal{H}(r) = [0, 1]$ for $r = 0$ and $\mathcal{H}(r) = 1$ for $r > 0$. $\mathcal{H}(r)$ is introduced in view of the assumption that cloud condensate is formed instantly to avoid supersaturation.

The other functions and parameters involved in the expression of $\text{CON}$ are defined as follows.

- $L_v = 2.53 \times 10^6$ J kg$^{-1}$ and $L_s = 2.84 \times 10^6$ J kg$^{-1}$ are the latent heat for condensation and sublimation, respectively.
- The overall saturation ratio is defined as

\[
q_{vs} = \alpha(T) q_{vw} + (1 - \alpha(T)) q_{vi}.
\]

(8)

Here $q_{vw}$ and $q_{vi}$ are the saturated water vapor mixing ratios with respect to water and ice, respectively. They can be approximated by (see [54])

\[
q_{vw} = \frac{0.622 e_{vw}}{p_e - e_{vw}}, \quad q_{vi} = \frac{0.622 e_{vi}}{p_e - e_{vi}}.
\]

(9)
The value 0.622 in (9) is calculated from the ratio $R_d/R_v$ of the gas constant for the dry air over the gas constant for the water vapor. The environment pressure $p_e$ is an decreasing exponential function depending on $z$.

- $e_{vw}$ and $e_{vi}$ are the saturated water vapor pressures over water and over ice, respectively. Their values are given by

$$e_{vw}(T) = 611 \exp \left[ \frac{L_v}{R_v} \left( \frac{1}{T_{oo}} - \frac{1}{T} \right) \right], \quad e_{vi}(T) = 611 \exp \left[ \frac{L_v}{R_v} \left( \frac{1}{T_{oo}} - \frac{1}{T} \right) \right]$$

with $T_{oo} = 273.16$ K.

To simplify the notations, we write below $\alpha$ instead of $\alpha(T)$, except where we need to emphasis the dependence of $\alpha$ on $T$. We also define the function $F_{con} = F_{con}(T, z)$ as

$$F_{con} = \alpha C_{conw} \rho_o \left( 1 + \frac{L_v^2 q_{vw}}{c_p R_v T^2} \right)^{-1} + (1 - \alpha) C_{coni} \rho_o \left( 1 + \frac{L_v^2 q_{vi}}{c_p R_v T^2} \right)^{-1}$$

then $CON$ can be simply written as $CON = F_{con} H(q_v - q_{vs})$.

**Remark 1.** The Heaviside function $H(q_v - q_{vs})$ is introduced to the condensation rate (7) since the condensation only occurs when the saturation concentration $q_{vs}$ is reached. The Heaviside function $H(q_v - q_{vs})$ describes the change of phase when $q_v$ reaches the threshold value $q_{vs}$. It is also worth noting that, within the physical range of values for $p_e$ and $e_{vw}$, $e_{vi}$, $p_e \gg e_{vw} \geq e_{vi}$, so $q_{vw}$ and $q_{vi}$ are strictly positive.

**AUT:** autoconversion of cloud condensate into precipitation. The expression for AUT reads

$$AUT = AUT_r + AUT_s,$$

$$AUT_r = \frac{\rho_o^3 (\alpha q_c)^3}{C_{autr} + C'_{autr} \rho_o (\alpha q_c)},$$

$$AUT_s = \frac{\rho_o \cdot [(1 - \alpha) q_c]}{\tau_a}.$$  \hspace{1cm} (12)

Here, $\alpha q_c$ and $(1 - \alpha)q_c$ correspond to the cloud water and cloud ice part of the cloud condensates, respectively; $C_{autr}$ and $C'_{autr}$ are dimensionless positive constants; and $\tau_a$ is the conversion time scale assumed equal to a time required to grow an ice crystal by diffusion of water vapor in water saturated conditions up to a size of small precipitation particle (mass of $10^{-9}$ kg), which can be approximated by

$$\tau_a(T) = -800 e^{-(T+15)^2} + 1000$$

**ACC:** accretion of cloud condensation by precipitation. The expression for the ACC term is

$$ACC = ACC_r + ACC_s,$$

$$ACC_r = C_{accr} \rho_o^{13/8} \cdot (\alpha q_c) \cdot (\alpha q_p)^{5/8},$$

$$ACC_s = C_{accs} \rho_o^{7/4} \cdot (1 - \alpha) q_c \cdot [(1 - \alpha) q_p]^{3/4}.$$  \hspace{1cm} (13)

As before, $C_{accr}$ and $C_{accs}$ are dimensionless positive constants; and $\alpha q_p$ and $(1 - \alpha) q_p$ correspond to the rain and snow parts of the precipitation respectively.
DEP: source (sink) of precipitation due to deposition (evaporation) of water vapor on (from) precipitation particles. Similarly as the above terms, DEP can be separated into two parts

\[
\text{DEP} = \text{DEP}_r + \text{DEP}_s,
\]

with

\[
\text{DEP}_r = C_{dpr} (\rho_o \alpha q_p)^{1/4} \left( 1 + C'_{dpr} (\rho_o \alpha q_p)^{7/16} \right) \left( \frac{q_v}{q_{ew}} - 1 \right) G,
\]

\[
\text{DEP}_s = C_{dps} [\rho_o (1 - \alpha) q_p]^{1/3} \left( 1 + C'_{dps} [\rho_o (1 - \alpha) q_p]^{5/24} \right) \left( \frac{q_v}{q_{vi}} - 1 \right) G.
\]

In the above expression, \( G = G(T_e) \) is a thermodynamic function depending on the environment temperature \( T_e \), which we will treat as a known parameter (for more details, see Appendix A).

Lastly, for the mass-weighted terminal velocity \( V_T \), we can calculate it as

\[
V_T = \alpha V_{Ts} + (1 - \alpha) V_{Ts},
\]

\[
V_{Tr} = C_{VT} (\rho_o \alpha q_p)^{1/2}, \quad \text{and} \quad V_{Ts} = C_{VT} [\rho_o (1 - \alpha) q_p]^{1/2}.
\]

3. Change of phase and boundary value problem. In this section and the next one, we will consider the above equations which will be supplemented with initial and boundary conditions. We let \( \mathcal{M} \subset \mathbb{R}^3 \) be the spatial domain for our study in the \( x, y, z \) variables and a typical point in \( \mathcal{M} \) is denoted by \( \mathbf{x} = (x, y, z) \).

We assume that \( \mathcal{M} = \mathcal{M}' \times (z_0, z_1) \), where \( \mathcal{M}' \subset \mathbb{R}^2 \) is smooth and bounded, and \( 0 < z_0 \leq z \leq z_1 \) is the range of values of \( z \) that we consider; here \( z_0, z_1 \) are two fixed real numbers. We use "\( n \)" to denote the outward unit normal vector field to the boundary \( \partial \mathcal{M} \) of \( \mathcal{M} \), where the boundary is decomposed as \( \partial \mathcal{M} = \Gamma_i \cup \Gamma_u \cup \Gamma_l \) corresponding respectively to the bottom, top and lateral boundaries of \( \mathcal{M} \). We assume the boundaries are sufficiently regular (e.g., of class \( C^3 \) each individually).

We set \( \nabla = (\partial_x, \partial_y) \) and \( \Delta = \partial_x^2 + \partial_y^2 \) to be the horizontal gradient and horizontal Laplace operators, respectively and \( \nabla_3 = (\nabla, \partial_z) \), \( \Delta_3 = \Delta + \partial_z^2 \) to be the 3D gradient and Laplace operators, respectively. In this way, the heat and vapor diffusion operators \( \mathcal{A}_0 \) and \( \mathcal{A}_q \) corresponding to the \( \mathcal{D} \)-terms in the original system (2)-(5) are described as

\[
\mathcal{A}_0 = -\mu_0 \Delta - \nu_0 \frac{\partial^2}{\partial z^2}, \quad \mathcal{A}_q = -\mu_q \Delta - \nu_q \frac{\partial^2}{\partial z^2},
\]

where \( \mu_q, \nu_q \) \((q \in \{v_0, q_v, q_p\})\), \( \mu_0, \nu_0 \) are positive constants.

We set \( U = (q_v, q_c, q_p, \theta) \). Before we move on to introduce in details the boundary value problem for each of the quantities under consideration, we make some observations here.

Firstly, under the anelastic frame work, we have \( \nabla_3 \cdot (\rho_o \mathbf{u}) = 0 \). Then we have

\[
\nabla_3 \cdot (\rho_o \mathbf{u} U) = \nabla_3 \cdot (\rho_o \mathbf{u}) U + (\rho_o \mathbf{u}) \cdot \nabla_3 U = (\rho_o \mathbf{u}) \cdot \nabla_3 U.
\]

In addition, we notice that there is discontinuity in the Heaviside function \( \mathcal{H}(q_v - q_{vs}) \) from the expression of \( \text{CON} \), as the condensation only occurs when the water vapor mixing ratio \( q_v \) reaches the saturation concentration \( q_{vs} \). Viewing \( q_{vs} \) as a threshold, this precisely describes the change of phase which \( q_v \) obeys. As the
Heaviside function in source term requires special treatment, we will write the continuous parts and the discontinuous parts of the source term separately.

3.1. The equation for \( q_v \). The equation for \( q_v \) is written as

\[
\frac{\partial (\rho_o q_v)}{\partial t} + A_{q_v} q_v + (\rho_o u) \cdot \nabla q_v \in -F_{\text{con}} \mathcal{H}(q_v - q_{vs}) + f_{q_v}(q_v, q_c, q_r, \theta) \\
= -F_{\text{con}} \mathcal{H}(q_v - q_{0v}) + f_{q_v}(U),
\]

where \( F_{\text{con}} = F_{\text{con}}(T, z) \) as defined in (11) and

\[
f_{q_v}(U) = f_{q_v}(q_v, q_c, q_r, \theta) = -\text{DEP}.
\]

We consider the following boundary conditions to be associated with the above equation:

\[
\partial_n q_v = \beta_v (q_{vs} - q_v) \text{ on } \Gamma_i, \quad \partial_n q_v = 0 \text{ on } \Gamma_u, \quad \partial_n q_v = 0 \text{ on } \Gamma_i,
\]

where \( \partial_n = \partial_{n_{A_{q_v}}} \) is the co-normal derivative associated with \( A_{q_v} \) which reduces on \( \Gamma_i \) to

\[
-\mu_{q_v} n_H \cdot \nabla q_v,
\]

where \( n_H \) is the horizontal component of the unit outward normal \( n \) on \( M \) (that is the unit outward normal on \( \Gamma_i \)).

We also associate with (18) the following initial condition

\[
q_v(x, y, z, 0) = q_{v0}(x, y, z).
\]

In (20), \( q_{vs} = q_{vs}(x, y, t) \) is a specific humidity distribution at the bottom of the atmosphere and \( \beta_v \) is a given positive constant.

3.2. The equation for \( q_c \). The equation for \( q_c \) is written as

\[
\frac{\partial (\rho_o q_c)}{\partial t} + A_{q_c} q_c + (\rho_o u) \cdot \nabla q_c \in F_{\text{con}} \mathcal{H}(q_c - q_{cs}) + f_{q_c}(q_v, q_c, q_r, \theta) \\
= F_{\text{con}} \mathcal{H}(q_c - q_{0s}) + f_{q_c}(U),
\]

where \( F_{\text{con}} \) is defined below (18) and

\[
f_{q_c}(U) = f_{q_c}(q_v, q_c, q_r, \theta) = -\text{ACC} - \text{AUT}
\]

We supplement the above equation with the following natural boundary conditions

\[
\partial_z q_c = \beta_c (q_{cs} - q_c) \text{ on } \Gamma_i, \quad \partial_z q_c = 0 \text{ on } \Gamma_u, \quad \partial_n q_c = 0 \text{ on } \Gamma_i,
\]

and the initial condition

\[
q_c(x, y, z, 0) = q_{c0}(x, y, z).
\]

In (25), \( q_{cs} = q_{cs}(x, y, t) \) is a critical specific humidity distribution at the bottom of the atmosphere and \( \beta_c \) is a given positive constant, and \( \partial_n q_c \) is defined as \( \partial_n q_v \) in (21).
3.3. The equation for $q_r$. The equation for $q_r$ is written as

$$
\frac{\partial (\rho_o q_r)}{\partial t} + A q_r q_p + (\rho_o u) \cdot \nabla q_r = f_{q_r}(U, \nabla_3 U),
$$

where $f_{q_r}(U, \nabla_3 U) = -\frac{\partial}{\partial z}(\rho_o q_r V_T) + \text{ACC} + \text{AUT} + \text{DEP}$, as we have moved the term $\nabla_3 \cdot [\rho_o (-V_T k q_p)]$ in (5) to the right hand side.

We supplement equation (27) with the following boundary conditions and initial conditions:

$$
\partial_z q_p = \beta_r (q_{ps} - q_p) \text{ on } \Gamma_i, \quad \partial_z q_p = 0 \text{ on } \Gamma_u, \quad \partial_n q_p = 0 \text{ on } \Gamma_l, \quad q_p(x, y, z, 0) = q_{p0}(x, y, z).
$$

Here $q_{ps} = q_{ps}(x, y, t)$ is a specific humidity distribution at the bottom of the atmosphere; $\beta_r$ is a given positive constant. Also $\partial_n q_p$ is defined as $\partial_n q_v$ in (21).

3.4. The equation for $\theta$. The potential temperature $\theta$ satisfies the following equation

$$
\frac{\partial (\rho_o \theta)}{\partial t} + A \theta + (\rho_o u) \cdot \nabla \theta = \frac{L_v}{c_p \Pi} \text{F}_{\text{con}} \mathcal{H}(q_v - q_{vs}) + f_\theta(q_v, q_c, q_p, \theta) = \frac{L_v}{c_p \Pi} \text{DEP},
$$

where $f_\theta(U) = f_\theta(q_v, q_c, q_r, \theta) = \frac{L_v}{c_p \Pi} \text{DEP}$. (31)

We consider the boundary conditions

$$
\partial_z \theta = \beta_\theta (\theta_s - \theta) \text{ on } \Gamma_i, \quad \partial_z \theta = 0 \text{ on } \Gamma_u, \quad \partial_n \theta = 0 \text{ on } \Gamma_l,
$$

and initial condition

$$
\theta(x, y, z, 0) = \theta_0(x, y, z).
$$

Here the function $\theta_s = \theta_s(x, y, t)$ is a typical potential temperature; $\beta_\theta$ is a given positive constant, and $\partial_n \theta$ is defined as $\partial_n q_v$ in (21).

Let $A = \text{diag}(A_{q_r}, A_{q_c}, A_{q_p}, A_{\theta})$, $F = (-F_{\text{con}}, F_{\text{con}}, 0, \frac{L_v}{c_p \Pi} F_{\text{con}})^t$ and

$$
f(U, \nabla_3 U) = (f_{q_v}(U), f_{q_c}(U), f_{q_p}(U, \nabla_3 U), f_{\theta}(U))^t.
$$

Then we can combine the equations for $q_v, q_c, q_p$ and $\theta$ together and rewrite an equation in the compact form in terms of $U$:

$$
\frac{\partial (\rho_o U)}{\partial t} + A U + (\rho_o u) \cdot \nabla_3 U \in f(U, \nabla_3 U) + F \mathcal{H}(q_v - q_{vs})
$$

If we adopt the following notations for $U_0 = U_0(x, y, z)$, $U_s = U_s(x, y, z)$ and $U = U(x, y, z)$

$$
U_0 = (q_{v0}, q_{c0}, q_{p0}, \theta_0)^t, \quad U_s = (q_{vs}, q_{cs}, q_{ps}, \theta_s)^t,
$$

and define the coefficient matrix $C = \text{diag}\{\beta_c, \beta_e, \beta_r, \beta_\theta\}$, then the initial and boundary conditions associated with the system (34) can be written as follows

$$
\begin{align*}
U(x, y, z, 0) &= U_0(x, y, z), \\
\partial_z U &= C(U_s - U) \text{ on } \Gamma_i, \quad \partial_n U = 0 \text{ on } \Gamma_u \cup \Gamma_l,
\end{align*}
$$

In the following subsection, we will discuss the regularity of the boundary datum $U_s$ and the source term $f(U, \nabla_3 U) + F \mathcal{H}(q_v - q_{vs})$. 

3.5. **Regularity of the source terms and the boundary data.** We can observe from the Section 2.1 that the microphysics terms $\text{CON, ACC, AUT, DEP}$ and $V_T$ are smooth functions of $U = (q_v, q_c, q_p, \theta)$ when $U$ takes its physical relevant values. We can slightly modify some terms in a way which simplifies the mathematical study but does not modify the physical relevance of the equations. For example, after a suitable extension of $q_v, q_c, q_p, \theta$ outside the range of their physically relevant values, all we need to assume is that $f_g(U), f_{q_v}(U), f_{q_c}(U)$ and $f_{q_p}(U, \nabla U)$ are continuous bounded functions of $U$. More precisely, considering the fact that $q_v, q_c, q_p$ are relative mass fractions ratios that take their values in the interval $[0, 1]$, we can replace $q (q \in \{q_v, q_c, q_p\})$ in (13)-(15) and the term $\frac{\partial}{\partial T}(\rho_o q_p V_T)$ by $\tau(q)$ where $\tau(q) = 0$ if $q \leq 0$; $= q$ if $0 \leq q \leq 1$; and $= 1$ if $q \geq 1$.

Next, to avoid the possible singularity in (10) and (11), we replace $T$ by $\varphi(T)$, where $\varphi$ is a smooth (e.g. $C^2$) positive real function with $\varphi(T)$:

$$\begin{cases}
= T & \text{for } T_* \leq T \leq T_{**}, \\
\geq T_*/2 & \text{for } T \leq T_*, \\
= 0 & \text{for } T \geq 2T_{**}.
\end{cases}$$

Here $T_*$ is smaller than any temperature on earth (e.g. 100K) and $T_{**}$ is larger than any temperature on earth (e.g. 355K).

We also smooth the function $\alpha(T)$ at $T_w$ and $T_1$. Let $\alpha_1(T)$ and $\alpha_2(T)$ be two $C^2$ functions such that,

$$\alpha_1(T_1) = 0, \quad \alpha_1(T_1 + 0.1) = \frac{1}{10(T_w - T_1)};$$

$$\alpha_2(T_w) = 1, \quad \alpha_2(T_w - 0.1) = 1 - \frac{1}{10(T_w - T_1)};$$

$$\frac{\partial}{\partial T}\alpha_1(T)|_{T = T_1 + 0.1} = \frac{\partial}{\partial T}\alpha_2(T)|_{T = T_w - 0.1} = \frac{1}{T_w - T_1}.$$

Then we redefine $\alpha(T)$ as

$$\alpha(T) = \begin{cases}
0 & \text{for } T \leq T_i, \\
\alpha_1(T) & \text{for } T_i < T \leq T_i + 0.1, \\
\frac{T - T_i}{T_w - T_i} & \text{for } T_i + 0.1 < T \leq T_w - 0.1, \\
\alpha_2(T) & \text{for } T_w - 0.1 < T \leq T_w, \\
1 & \text{for } T > T_w.
\end{cases}$$

It is easy to check that $\alpha(T)$ is a bounded continuous function of $T$ which also has continuous first order derivative with respect to $T$.

Moreover, with $T$ being replaced by $\varphi(T)$ in (10) and $\alpha(T)$ redefined in (38), it is easy to check that the saturation mixing ratio $q_{vs}$ in (8) is a bounded positive $C^1$ function of $T$ and $z$. Indeed, for $T_* \leq T \leq T_{**}$,

$$\frac{\partial q_{vs}}{\partial T}(T, z) = \frac{\partial \alpha(T)}{\partial T}(q_{vw} - q_{vi}) + \alpha(T) \frac{\partial q_{vw}}{\partial T} + (1 - \alpha(T)) \frac{\partial q_{vi}}{\partial T},$$

$$\frac{\partial q_{vw}}{\partial T} = C \frac{p_e}{(p_e - e_{vw})^2} T^2 e_{vw}, \quad \frac{\partial q_{vi}}{\partial T} = C \frac{p_e}{(p_e - e_{vi})^2} T^2 e_{vi},$$

where $e_{vw}$ and $e_{vi}$ are the relative humidity of the water vapor and ice, respectively.
\[
\frac{\partial q_{vs}}{\partial z}(T, z) = \frac{\partial \alpha(T) \partial T}{\partial z}(q_{vw} - q_{vi}) + \alpha(T) \frac{\partial q_{vw}}{\partial T} + (1 - \alpha(T)) \frac{\partial q_{vi}}{\partial T},
\]
\[
\frac{\partial q_{vw}}{\partial z} = C \frac{p_e}{(p_e - e_{vw})^2} \frac{1}{T^2} \frac{\partial T}{\partial z} + C_1 e_{vw} \frac{\partial p_e}{\partial z},
\]
\[
\frac{\partial q_{vi}}{\partial z} = C \frac{p_e}{(p_e - e_{vi})^2} \frac{1}{T^2} e_{vi} \frac{\partial T}{\partial z} + C_1 e_{vi} \frac{\partial p_e}{\partial z},
\]
where we have represented the constant coefficients by the notation \(C\) and \(C_1\). By the expression (38), \(\frac{\partial \alpha(T)}{\partial T}\) is continuous and bounded for all values of \(T\). With the extension \(\phi(T)\) of \(T\) outside its physical relevant values, and using the fact that \(\partial p_e/\partial z\) is bounded, we have that \(\partial q_{vw}/\partial T\) (resp. \(\partial q_{vi}/\partial T\)) and \(\partial q_{vw}/\partial z\) (resp. \(\partial q_{vi}/\partial z\)) are uniformly bounded and \(\partial q_{vw}/\partial z\) is continuous and uniformly bounded for all values of \(T\) and \(z \in [z_0, z_1]\). Then it follows that \(\partial q_{vs}/\partial T\) and \(\partial q_{vs}/\partial z\) are continuous and uniformly bounded for all values of \(T\) and \(z \in [z_0, z_1]\). It is not difficult to see that \(q_{vs}\) has bounded second order derivative with respect to \(T\) as well.

We can also infer from (11) that the function \(F_{con}(T, z)\) is a positive function that is Lipschitz continuous and uniformly bounded in \(T\) and \(z\). In addition, we also have the following estimates for \(F = (-F_{con}, F_{con}, 0, \frac{L}{\epsilon_{con}} F_{con})\) and \(f(U, \nabla U)\),
\[
|f(U, \nabla U)|^2_{L^2} + |F|^2_{L^2} \leq \kappa_1 |U|^2_{L^2} + \kappa_2 |\nabla U|^2_{L^2} + \kappa_3
\]
for some positive constants \(\kappa_1, \kappa_2\) and \(\kappa_3\).

For the boundary datum, throughout the paper we assume \(U_* = (q_{vs}, q_{cw}, q_{iw}, \theta_*)\) satisfy
\[
U_* \in L^2(0, t_1; L^2(\Gamma_1)^4). \tag{42}
\]
Moreover, set \(\bar{U}_* = (q_{cw}, q_{iw}, \theta_*)\), we need impose higher regularity assumptions on \(\bar{U}_*\) for the homogenization of the Robin boundary(36) conditions in the a priori estimates in the following section, where we assume
\[
\bar{U}_* \in L^2(0, t_1; H^1_0(\Gamma_1)^3), \quad \partial \bar{U}_* / \partial t \in L^2(0, t_1; H^1_0(\Gamma_1)^3). \tag{43}
\]

4. Variational and weak formulation of the problem. As mentioned in Section 2, the model we use in this article is an extension of the warm rain model which was studied earlier in [13], and a similar difficulty we have to deal with is the constraint \(q_v \leq q_{vs}\). We recall that the expression of \(q_{os}\ (\text{8} - \text{10}))\ is a function depending itself on \(T\) and \(z\). Hence the constraint \(q_v \leq q_{os}\ appears as a quasi variational inequality where the solution \(U\) is subject to belonging to a convex set which depends itself on the solution:
\[
U \in K = K(U).
\]
Notice that in our model with ice physics, the overall saturation mixing ratio \(q_{os}\ is a linear combination of the saturation mixing ratio with respect to water and the saturation mixing ratio with respect to ice, which will make our analysis more complicated comparing with the warm rain case in [13].

Quasi variational inequalities have been introduced in [9], [4] by Bensoussan and Lions, motivated by the study of economical problems; see also [2], [3], [6], [7], [5] and [8]. Subsequently quasi variational inequalities have been used for problems in mechanics, physics and imagery, see e.g., [41], [36], [47] and [44].

We start in Section 4.1 by giving the weak form of the problem and then in Section 4.3 we account for the constraint \(U \in K(U)\ and introduce the quasi-variational inequality.
Remark 2. Before going any further, we make some simple observations here. Beside the nonlinear constraint \( q_v \leq q_{vs} \) that depends itself on the solution, the quantities \( q_v, q_c, q_p \) are relative mass fractions ratios, thus also satisfying the constraint \( q_v, q_c, q_p \geq 0 \). However, since the proofs for the nonnegativity of \( q_v, q_c, q_p \) can be accomplished via more standard methods (e.g., Stampacchia method), we will only discuss the nonlinear constraint \( q_v \leq q_{vs}(T, z) \) which requires special treatment in this article, and refer the readers to [60, 17] and [35] for the proofs of more regular constraints on the humidity quantities and temperature.

4.1. Notations. We denote as usual \( H = L^2(M), V = H^1(M) \) and we set \( \mathbb{H} = H \times H \times H \) and \( \mathbb{V} = V \times V \times V \). We use \((\cdot,\cdot)_{L^2}\) (regarded the same as \((\cdot,\cdot)_H\) and \(|\cdot|_{L^2}\) to denote the usual scalar product and induced norm in \( H \). In the space \( V \), we will use \((\cdot,\cdot)\) to denote the scalar product adapted to the problem under investigation

\[
((\varphi, \phi)) := (\nabla \varphi, \nabla \phi) + (\partial_z \varphi, \partial_z \phi) + \int_{\Gamma_i} \varphi \phi \, d\Gamma_i,
\]

and the induced norm is denoted \( ||\cdot|| \). The symbol \((\cdot,\cdot)\) will denote the duality pair between a Banach space \( E \) and its dual space \( E^* \). Although the velocity field is no longer divergence free under the anelastic framework, we still use the following standard notations associated with the Navier-Stokes equations:

\[
H = \{ u \in H \times H \times H \mid div (\rho_u) = 0 \text{ and } u \cdot n = 0 \text{ on } \partial M \},
\]

\[
V = \{ u \in V \times V \times V \mid div (\rho_u) = 0 \text{ and } u \cdot n = 0 \text{ on } \partial M \},
\]

which will serve as the natural function spaces for the vector field \( u \). In fact we will assume that

\[
u \in L^\infty(0, t_1; H^1(M))^3 \cap L^\infty((0, t_1) \times M).
\]

In view of deriving the weak (variational) formulation of the boundary value problem, we multiply e.g. the expression \( A_\theta \theta \) by a test function \( \theta^b \). Assuming smoothness and taking into account the boundary conditions (32) for \( \theta \) we find

\[
\langle A_\theta \theta, \theta^b \rangle = \left( -\mu_\theta \Delta \theta - \nu_\theta \partial^2 \partial_z \theta^b \right)
\]

\[
:= \mu_\theta (\nabla \theta, \nabla \theta^b)_H + \nu_\theta \int_M \partial_z \theta^b \, dM
\]

\[
+ \nu_\theta \int_{\Gamma_i} \beta_\theta (\theta - \theta_i) \theta^b \, d\Gamma_i.
\]

We do the same for \( q \in \{q_v, q_c, q_p\} \) and thus

\[
\langle A_q q, q^b \rangle = \mu_q (\nabla q, \nabla q^b)_H + \nu_q \int_M \partial_z q^b \, dM
\]

\[
+ \nu_q \int_{\Gamma_i} \beta_q (q - q_i) q^b \, d\Gamma_i.
\]

Consequently, we define the following bilinear forms

\[
a_\theta(\theta, \theta^b) = \mu_\theta (\nabla \theta, \nabla \theta^b)_H + \nu_\theta \int_M \partial_z \theta^b \, dM + \nu_\theta \beta_\theta \int_{\Gamma_i} \theta \theta^b \, d\Gamma_i,
\]

\[
a_q(q, q^b) = \mu_q (\nabla q, \nabla q^b)_H + \nu_q \int_M \partial_z q^b \, dM + \nu_q \beta_q \int_{\Gamma_i} q q^b \, d\Gamma_i.
\]
which correspond to the constant terms in $A$ are both bounded linear invertible operators.

By the equality and the trace theorem, whose detail shall be omitted here.

It is worth noting that, thanks to the condition $\lambda$ exist universal positive constants above functionals.

We introduce the multilinear forms for $A$ and $B$ such that $(\theta, \theta^b)$ by

We also define the linear functionals:

More precisely, we have the following lemma concerning the boundedness of the above functionals.

We introduce the multilinear forms for $U$ and $U^b = (q_v, q_v^b, q_p, \theta^b)$

We also define the linear functionals:

It is easy to see that

It is worth noting that, thanks to the condition $\nabla_3^-(\rho_o) = 0$, we have $b(\rho_o, \psi, \psi) = 0, \forall \psi \in V$.

Before we continue, we first recall the following well-known estimates.

More precisely, we have the following lemma concerning the boundedness of the above functionals.

**Lemma 4.1.** Assume $U = (q_v, q_v^b, q_p^b, \theta, \theta^b) \in V$ and $u \in V$. There exist universal positive constants $\lambda$ and $\kappa$ such that (If denotes $q$ here)

We have the following lemma concerning the boundedness of the above functionals.

**Lemma 4.1.** Assume $U = (q_v, q_v^b, q_p, \theta) \in V$ and $u \in V$. There exist universal positive constants $\lambda$ and $\kappa$ such that (If denotes $q$ here)

The proof of Lemma 4.1 is based on a routine use of the Cauchy-Schwarz inequality and the trace theorem, whose detail shall be omitted here.

It is well-known that the linear operators $A_\theta, A_q : V \to V^*$ defined through the relations

are both bounded linear invertible operators.

Similarly, the operator $B(\rho_o u, U) = (b(\rho_o u, q), b(\rho_o u, \theta)) : V \times V \to V^*$ is defined by

where $V^*$ is the dual space of $V$. 

4.2. **Weak formulation of the problem.** We recall our equation (34) and replace the multivalued Heaviside function $\mathcal{H}(q_v - q_{vs})$ by a representative $h_{q_v} \in \mathcal{H}(q_v - q_{vs})$ taking values in $[0, 1]$, the initial and boundary value problem now becomes

$$
\frac{\partial (\rho_o U)}{\partial t} + A U + (\rho_o \mathbf{u}) \cdot \nabla_3 U = f(U, \nabla_3 U) + \mathcal{F} h_{q_v},
$$

(60)

$$
U(x, y, z, 0) = U_0(x, y, z),
$$

(61)

$$
\partial_t U = \mathcal{C}(U_* - U) \text{ on } \Gamma_t, \quad \partial_{n\mathbf{u}} U = 0 \text{ on } \Gamma_u \cup \Gamma_t,
$$

(62)

\[
\text{at} \quad t \leq t_1, \quad \mathbf{v} \in U, \quad \partial_t \mathbf{v} = \mathcal{L}(\mathbf{v}), \quad \mathbf{v} \in \mathbb{R}^3.
\]

For the weak formulation we will treat differently the equations for $\bar{U} = (q_v, q_b, \theta)$ and the equation for $q_v$ which is subjected to the constraint $q_v \leq q_{vs}$.

For $\bar{U}$, we consider the equations (23), (27), (30) for $q_v, q_b, \theta$, respectively, and multiply them by test functions $q_v^b, q_b^b, \theta^b$. Assuming smoothness as before, we obtain in view of (45)--(46),

$$
\int_0^{t_1} \left[ (\rho_o \partial_t \bar{U}; \bar{U}^b) + \bar{a}(\bar{U}, \bar{U}^b) + \bar{b}(\rho_o \mathbf{u}, \bar{U}, \bar{U}^b) - \bar{l}(\bar{U}^b) \right] dt
$$

(63)

$$
= \int_0^{t_1} (\bar{f}(U, \nabla_3 U) + \bar{\mathcal{F}} h_{q_v}, \bar{U}^b) dt,
$$

for all $\bar{U}^b \in L^2(0, t_1; (H^1)^3)$ and

$$
\bar{U}(t = 0) = \bar{U}_0.
$$

(64)

Recall again that here $\bar{l}$ represents the constant part of the operator $\bar{A}$ (with respect to $\bar{U}$) and $\bar{\mathcal{F}}$ represents the vector ($\mathcal{F}_{con}, 0, \frac{l_3}{\ell_3} \mathcal{F}_{con}$).

With the constraint $q_v \leq q_{vs}$, and analogy with what was done in [58] when $q_{vs}$ is constant and $0 \leq q_v \leq q_{vs}$, we can weaken (18) in the form:

$$
\mathcal{L}(q_v) \leq f_{q_v}(U, \nabla_3 U) - \mathcal{F}_{con} h_{q_v},
$$

where $\mathcal{L}(q_v)$ is the left hand side (LHS) of (18). Take now a test function $q_v^b \leq q_{vs}$. We see that pointwise

$$
(\mathcal{L}(q_v) - f_{q_v}(U, \nabla_3 U) + \mathcal{F}_{con} h_{q_v})(q_v^b - q_{vs}) \geq 0,
$$

in all cases, that is if $q_v^b = q_{vs}$ or $q_v^b < q_{vs}$.

This leads us to the formulation of (18)-(19) as a quasi-variational inequality : $q_v \in L^\infty(0, t_1; L^2(\mathcal{M})) \cap L^2(0, t_1; H^1(\mathcal{M}))$, $q_v \leq q_{vs}(T, z)$ and

$$
\int_0^{t_1} \left[ (\rho_o \partial_t q_v, q_v^b - q_v) + a_q(q_v, q_v^b - q_v) + b(q_o \mathbf{u}, q_v, q_v^b - q_v) - l_q(q_v^b - q_v) \right] dt
$$

$$
\geq \int_0^{t_1} (f_q(U, \nabla_3 U) - \mathcal{F}_{con} h_{q_v}, q_v^b - q_v) dt,
$$

(65)

for all $q_v^b \in L^\infty(0, t_1; H^1)$ with $q_v^b \leq q_{vs} = q_{vs}(T, z)$.

In addition,

$$
q_v(t = 0) = q_{v0}.
$$

(66)

At this point, let us introduce what we will call here a solution of (60) in the weak sense. Let $U_0 \in \mathbb{V}$ be such that $0 \leq q_{v0} \leq q_{vs}(t = 0)$ and let $t_1 > 0$ be an arbitrary but fixed constant. A vector $U = (U(t) = (q_v, \bar{U}) \in L^2(0, t_1; K) \cap C([0, t_1]; \mathbb{V})$ with $\partial_t \bar{U} \in L^2(0, t_1; (V^3)^*)$, $\partial_t q_v \in L^{5/3}(0, t_1; V^*)$ is a solution to the initial-boundary value problem (60)-(61)-(62), if, for almost every $t \in [0, t_1]$ and for every $U^b \in \mathcal{K}$, we have (63) and (65) satisfied.
4.3. **The penalized and regularized problem.** To deal with the inequality constraint \( q_v \leq q_{vs} \) and the discontinuity of the Heaviside function \( H \), we introduce a penalized and regularized version of the problem associated with the parameters \( \varepsilon_1, \varepsilon_2 > 0 \). The penalization is introduced below by introduction of the term \( \varepsilon_1^{-1}(h_v - q_{vs})^{3/2} \). We address the discontinuity of the Heaviside function as in \([16]\) and \([18]\). Recall the multi-valued Heaviside function

\[
H(r) = \begin{cases} 
0 & \text{for } r < 0, \\
[0,1] & \text{for } r = 0, \\
1 & \text{for } r > 0,
\end{cases} \tag{67}
\]

and the single-valued function \( h_{q_v} \) where \( h_{q_v} \in H(q_v - q_{vs}) \). Following \([60]\), we can characterize \( h_{q_v} \in H(q_v - q_{vs}) \) by

\[
(h_v - q_{vs})^+ - (h_v - q_{vs})^- \geq h_{q_v} \quad \text{for a.e. } t \in [0, t_1], \quad \forall q_v \in V. \tag{68}
\]

Now we approximate \( h_{q_v} \) by \( H_{\varepsilon_2}(q_v - q_{vs}) \) for \( \varepsilon_2 > 0 \), where \( H_{\varepsilon_2}(r) \) is defined as

\[
H_{\varepsilon_2}(r) = \begin{cases} 
0 & \text{for } r \leq 0, \\
\frac{r}{\varepsilon_2} & \text{for } r \in (0, \varepsilon_2], \\
1 & \text{for } r > \varepsilon_2.
\end{cases} \tag{69}
\]

In this setting, \( F_{\text{con}}h_{q_v} \) (\( \sim F_{\text{con}}H \)) in the right hand side of (18) and (65) are replaced by \( F_{\varepsilon_2}(q_v - q_{vs}) \). Similarly, the \( h_{q_v} \) in the equations for \( \theta, q_c \) and \( q_b \) are replaced by \( H_{\varepsilon_2}(q_v - q_{vs}) \) as well. Here the regularized \( f(U, \nabla U) + F_{\varepsilon_2}(q_v - q_{vs}) \) has the same boundedness as the original one. Now the related penalized and regularized system of equations reads

\[
\begin{align*}
\partial_t \rho \partial_t q_v^c + A_c q_v^c + \rho_0 u \cdot \nabla q_v^c &= \frac{(q_v - q_{vs})^+}{\varepsilon_1^{3/2}} = f_{\text{con}}(U^\varepsilon) - F_{\text{con}}H_{\varepsilon_2}(q_v - q_{vs}), \\
\partial_t \rho_0 q_v^c + A_c q_v^c + \rho_0 u \cdot \nabla q_v^c &= f_{\text{con}}(U^\varepsilon) + F_{\text{con}}H_{\varepsilon_2}(q_v - q_{vs}), \\
\partial_t \rho_0 q_v^p + A_p q_v^p + \rho_0 u \cdot \nabla q_v^p &= f_{\text{con}}(U^\varepsilon, \nabla U^\varepsilon), \\
\partial_t \rho_0 q_v^c + A_\theta q_v^c + \rho_0 u \cdot \nabla q_v^c &= f_{\text{con}}(U^\varepsilon) + f_{\text{con}}H_{\varepsilon_2}(q_v - q_{vs}).
\end{align*} \tag{70}
\]

The unknown functions above depend on the small parameter \( \varepsilon = (\varepsilon_1, \varepsilon_2) \) and they are written as \( q_v^{c, \varepsilon}, q_v^c, q_v^p, \theta^\varepsilon \), and we write \( U^\varepsilon = (q_v^{c, \varepsilon}, q_v^c, q_v^p, \theta^\varepsilon) \). The initial and boundary conditions associated with (70) are the same as those for \( U \):

\[
U^\varepsilon(x, y, z, 0) = U_0(x, y, z), \tag{71}
\]

\[
\partial_t U^\varepsilon = C(U^\varepsilon - U) \quad \text{on } \Gamma_i, \quad \partial_{n_A} U^\varepsilon = 0 \quad \text{on } \Gamma_u \cup \Gamma_i. \tag{72}
\]

By analogy with (63), the weak formulation of this problem is to find a function \( U^\varepsilon = U^\varepsilon(t) \in L^2(0, t_1; V) \) with \( \partial_t U^\varepsilon \in L^2(0, t_1; (V^3)^*) \) and \( \partial q_v^c \in L^{5/3}(0, t_1; V^*) \), such that

\[
\int_0^{t_1} \left[ (\rho_0 \partial_t U^\varepsilon, U^b) + a(U^\varepsilon, U^b) + b(\rho_0 u, U^\varepsilon, U^b) - l(U^b) \right] dt \\
+ \frac{1}{\varepsilon_1 \varepsilon_1^{-1}} \langle (q_v^c - q_{vs})^+, q_v^b \rangle dt \\
= \int_0^{t_1} \left( f(U^\varepsilon, \nabla U^\varepsilon) + F_{\varepsilon_2}(q_v^c - q_{vs}), U^b \right) dt, \tag{73}
\]

\( f(U^\varepsilon, \nabla U^\varepsilon) + F_{\varepsilon_2}(q_v^c - q_{vs}), U^b \)
for all $\bar{U}^b \in L^2(0,t_1;V^3)$ and $q_v^b \in L^\infty(0,t_1;V)$.

$$U^\varepsilon(0) = U_0.$$ (74)

4.4. The formal a priori estimates and existence of solution for (73). From now on, aiming to simplify the presentation, we will omit the dependence on $\varepsilon$ of $U^\varepsilon$ that we will denote instead by $U$; the superscript $\varepsilon$ will be reintroduced when it is necessary.

We set $U^b = U$ in (73) and obtain

$$\langle \rho_o \partial_t U, U \rangle + a(U, U) + b(\rho_o \mathbf{u}, U^\varepsilon, U) - l(U) + \frac{1}{\varepsilon_1} \langle ((q_v - q_{vs})^+)^{3/2}, q_v \rangle$$

$$= (f(U, \nabla_3 U) + \mathcal{F}_{\varepsilon_2}(q_v - q_{vs}), U),$$ (75)

We then deduce an energy equality which is in fact obtained by adding the corresponding energy equalities for each component of $U^\varepsilon$, namely $q_v^\varepsilon, q_{vs}^\varepsilon, q_p^\varepsilon$ and $\theta^\varepsilon$.

The first term in the LHS of (75) satisfies

$$\langle \rho_o \partial_t U^\varepsilon, U \rangle = \frac{d}{dt} \| \rho_o^{1/2} U \|_{L^2}^2$$ (76)

For the term $b(\rho_o \mathbf{u}, U, U)$, we calculate it using integration by part formula

$$b(\rho_o \mathbf{u}, U, U) = \int_M (\rho_o \mathbf{v} \cdot \nabla U + (\rho_o w) \partial_z U) U dM,$$

$$= (\text{since } \mathbf{u} \cdot n = 0 \text{ on } \partial M)$$

$$= -\frac{1}{2} \int_M \text{div} (\rho_o \mathbf{v}) U^2 + \frac{\partial (\rho_o w)}{\partial z} U^2 dM$$ (77)

$$= (\text{since } \nabla \cdot (\rho_o \mathbf{u}) = 0)$$

$$= 0$$

The penalization term $\frac{1}{\varepsilon_1} \langle ((q_v - q_{vs})^+)^{3/2}, q_v \rangle$ satisfies

$$\frac{1}{\varepsilon_1} \int_0^{t_1} \int_M ((q_v - q_{vs})^+)^{3/2} q_v dM ds = \frac{1}{\varepsilon_1} \int_0^{t_1} \int_M ((q_v - q_{vs})^+)^{5/2} dM ds +$$

$$\frac{1}{\varepsilon_1} \int_0^{t_1} \int_M ((q_v - q_{vs})^+)^{3/2} q_{vs} dM ds \geq 0,$$ (78)

Then we apply Lemma 4.1 to left terms on the LHS of (75) and deduce that

$$a(U, U) \geq \lambda \| U \|^2,$$ (79)

and

$$l(U) \leq \kappa \| U \| \leq \frac{\lambda}{4} \| U \|^2 + \frac{\kappa^2}{\lambda}.$$ (80)

By (41) and Cauchy-Schwarz inequality, the RHS of (75) can be bounded by

$$(f(U, \nabla_3 U) + \mathcal{F}_{\varepsilon_2}(q_v - q_{vs}), U) \leq \frac{1}{2c_1} \| f(U, \nabla_3 U) \|^2_{L^2} + \frac{1}{2c_1} \| \mathcal{F} \|^2_{L^2} + c_1 \| U \|^2_{L^2}$$

$$\leq \frac{\kappa_2}{2c_1} \| \nabla_3 U \|^2_{L^2} + (\frac{\kappa_1}{2c_1} + c_1) \| U \|^2_{L^2} + \frac{\kappa_3}{2c_1},$$ (81)

where $c_1$ is some positive constant such that

$$\frac{\kappa_2}{2c_1} \| \nabla_3 U \|^2_{L^2} \leq \frac{\lambda}{4} \| U \|^2.$$
In (79)-(81), the constants $\kappa_1, \kappa_2, \kappa_3$ are from the estimate of the source term (41), and the constants $\lambda, \kappa$ are from the Lemma 4.1.

Combining the estimates we have in (76)-(81), (75) now becomes

$$\frac{1}{2} \frac{d}{dt} \|\rho_0^{1/2} U\|_{L^2}^2 + \frac{\lambda}{2} \|U\|^2 \leq (\frac{\kappa_1}{2\varepsilon} + c_1)\|U\|_{L^2}^2 + (\frac{\kappa_3}{2\varepsilon} - \frac{\kappa_2}{\varepsilon})$$  \hspace{1cm} (82)

As the density $\rho_0 = \rho_0(z)$ is strictly positive and uniformly bounded on $\mathcal{M}$, we can write (82) as

$$\frac{1}{2} \frac{d}{dt} \|\rho_0^{1/2} U\|_{L^2}^2 + \frac{\lambda}{2} \|U\|^2 \leq C(U, U_0, t_1)\|\rho_0^{1/2} U\|_{L^2}^2 + C(U, U_0, t_1),$$  \hspace{1cm} (83)

for some positive generic constant $C(U, U_0, t_1)$ independent of $\varepsilon$. Applying the Gronwall inequality to (83) we find

$$|\rho_0^{1/2} U^\varepsilon|_{L^\infty(0,t_1; L^2(\mathcal{M}^t))} \leq C(U, U_0, t_1),$$

$$|U^\varepsilon|_{L^2(0,t_1;\mathcal{V})} \leq C(U, U_0, t_1).$$

Using again the boundedness of $\rho_0$, we actually have

$$|U^\varepsilon|_{L^\infty(0,t_1; L^2(\mathcal{M}^t))} \leq C(U, U_0, t_1),$$

$$|U^\varepsilon|_{L^2(0,t_1;\mathcal{V})} \leq C(U, U_0, t_1).$$

4.5. A priori estimates on the time derivative of $U$. We now aim to derive a priori estimates for the time derivatives of $U$ in view of obtaining a strong convergence result for these functions and especially $\theta$ ($\sim T$), by application of a compactness theorem.

More precisely, we prove in this subsection some a priori estimates for the solution $U$ of the system (70) associated with the initial and boundary conditions (71) and (72), respectively. The intent is to show that the time derivative of $\bar{U} = \bar{U}^\varepsilon = (q_\varepsilon^\varepsilon, \varphi_\varepsilon^\varepsilon, \theta^\varepsilon)$ and $q_\varepsilon^\varepsilon$, recalling here the dependence of the solution $U$ on $\varepsilon$, are bounded independently of $\varepsilon$. The estimate for the time derivative of $q_\varepsilon^\varepsilon$ is more subtle due to the presence of the penalization term which contains the large factor $\frac{1}{\varepsilon}$. So we will estimate time derivative of $\bar{U}$ first and treat $dq_\varepsilon^\varepsilon/dt$ differently later on.

To begin with, we need we need to homogenize the boundary conditions on $\bar{U}$. We introduce $\bar{U}_s$, the solution of the stationary problem associated with (60). Namely,

$$\bar{A}\bar{U}_s = 0,$$

$$\bar{U}_s^0 = \bar{U}_0(x, y, z),$$

$$\partial_2 \bar{U}_s = \bar{C}(\bar{U}_s - \bar{U}_s) \text{ on } \Gamma_i, \quad \partial_4 \bar{U}_s = 0 \text{ on } \Gamma_u \cup \Gamma_l,$$

By our assumptions on $\bar{U}_s$ in (43), the above stationary problem (86)-(88) has a solution $\bar{U}_s \in H^2(\mathcal{M})^3$ satisfying

$$\|\bar{U}_s\|_{H^2(\mathcal{M})} \leq C\|\bar{U}_s\|_{H^1(\Gamma_s)},$$

thanks to Theorem 4.5 of [52]. In addition, as we have also assumed $\partial \bar{U}_s/\partial t \in L^2(0, t_1; H^0_0(\Gamma_s)^3)$ in (43), we can differentiate (86)-(88) with respect to time $t$ and apply Theorem 4.5 of [52] again to deduce that $\partial \bar{U}_s/\partial t \in H^2(\mathcal{M})^3$ and

$$\|\partial \bar{U}_s/\partial t\|_{H^2(\mathcal{M})} \leq C\|\bar{U}_s/\partial t\|_{H^1(\Gamma_s)} \leq C(U, U_0, t_1).$$

We then consider the function $\bar{U}_h = \bar{U} - \bar{U}_s$ which satisfies the following equation

$$\frac{\partial (\rho_0 \bar{U}_h)}{\partial t} + \mathcal{A}\bar{U}_h = - (\rho_0 \varphi) \cdot \nabla_3 \bar{U} + \tilde{f}(U, \nabla_3 U) + \mathcal{P}^\varepsilon(q_\varepsilon - q_\varepsilon^\varepsilon) - \frac{\partial (\rho_0 \bar{U}_s)}{\partial t} \hspace{1cm} (91)$$
with homogeneous boundary conditions

\[ \partial_t \bar{U}_h + \mathcal{C} \bar{U}_h = 0 \text{ on } \Gamma, \quad \partial_{n, A} \bar{U}_h = 0 \text{ on } \Gamma_u \cup \Gamma_l. \]  

(92)

Multiplying (91) with \( \frac{\partial U_h}{\partial t} \) and integrating on \( \mathcal{M} \), we have

\[
\min_{z} |\rho_o(z)| \left| \frac{\partial U_h}{\partial t} \right|_{L^2}^2 + \langle \tilde{A} \bar{U}_h, \frac{\partial U_h}{\partial t} \rangle \\
\leq \left( -\langle \rho_o \mathbf{u} \cdot \nabla_3 \bar{U} + \tilde{f}(U, \nabla_3 U) + \mathcal{F}_{\mathcal{H}_2}(q_v - q_{vs}) - \frac{\partial (\rho_o \bar{U}_s)}{\partial t}, \frac{\partial U_h}{\partial t} \rangle \right) \\
\leq J_1 + \frac{1}{2} \min_{z} |\rho_o(z)| \left| \frac{\partial U_h}{\partial t} \right|_{L^2}^2,
\]

with

\[
J_1 = C \left| -\langle \rho_o \mathbf{u} \cdot \nabla_3 \bar{U} + \tilde{f}(U, \nabla_3 U) + \mathcal{F}_{\mathcal{H}_2}(q_v - q_{vs}) - \frac{\partial (\rho_o \bar{U}_s)}{\partial t} \rangle \right|_{L^2}^2. 
\]

(93)

Since \( U_h \) satisfies the homogeneous boundary condition (92), we can use the symmetry of the operator \( \mathcal{A} \) and obtain

\[ \langle \tilde{A} \bar{U}_h, \frac{\partial U_h}{\partial t} \rangle = \frac{1}{2} \frac{d}{dt} \langle \tilde{A} \bar{U}_h, \bar{U}_h \rangle. \]

We then estimate the components of \( J \). With the assumption \( \mathbf{u} \in L^\infty(\mathcal{M} \times (0, T)) \) and using (85), we obtain

\[ |\langle \rho_o \mathbf{u} \cdot \nabla_3 \bar{U} \rangle|_{L^2}^2 \leq |\rho_o \mathbf{u}|^2_{L^\infty(\mathcal{M} \times (0, T))} |\nabla_3 \bar{U}|_{L^2}^2 \leq \mathcal{G}(t), \]

where \( \mathcal{G}(t) \) is a generic function of time \( t \), bounded in \( L^1(0, t_1) \).

By (41), we also have

\[ |\tilde{f}(U, \nabla_3 U) + \mathcal{F}_{\mathcal{H}_2}(q_v - q_{vs})|^2 \leq \mathcal{G}(t). \]

For the last term in \( J_1 \), we recall (90) and bound it as

\[ \left| \frac{\partial (\rho_o \bar{U}_s)}{\partial t} \right|_{L^2}^2 \leq \max_{z} |\rho_o|^2 \left| \frac{\partial \bar{U}_s}{\partial t} \right|_{L^2}^2 \leq C \left\| \partial \bar{U}_s / \partial t \right\|_{H^1(\Gamma_l)}^2 \leq \mathcal{G}(t). \]

Hence (93) implies

\[ \min_{z} |\rho_o(z)| \left| \frac{\partial U_h}{\partial t} \right|_{L^2}^2 + \frac{d}{dt} \langle \tilde{A} \bar{U}_h, \frac{\partial U_h}{\partial t} \rangle \leq \mathcal{G}(t), \]

(95)

with \( \mathcal{G}(t) \) being a generic function belonging to \( L^1(0, t_1) \).

Now we integrate (95) from 0 to \( t_1 \), drop the positive term \( \langle \tilde{A} \bar{U}_h(t_1), \bar{U}_h(t_1) \rangle \) in the LHS of the equation and use the fact that \( \bar{U}_{h0} = \bar{U}_0 - \bar{U}_s(t = 0) = 0 \) to deduce that

\[ \int_0^{t_1} \min_{z} |\rho_o(z)| \left| \frac{\partial U_h}{\partial t} \right|_{L^2}^2 \leq C(\mathbf{u}, U_0, t_1). \]

With the uniform boundedness of \( \rho_o \) and the relationship \( \bar{U} = \bar{U}_s + \bar{U}_h \), it then follows that

\[ \left| \frac{\partial \bar{U}}{\partial t} \right|_{L^2(0,t_1;L^2(\mathcal{M}))}^2 \leq \left| \frac{\partial \bar{U}_h}{\partial t} \right|_{L^2(\mathcal{M})^3}^2 + \left| \frac{\partial \bar{U}_s}{\partial t} \right|_{L^2(\mathcal{M})^3}^2 \leq C(\mathbf{u}, U_0, t_1), \]

(96)

where \( C(\mathbf{u}, U_0, t_1) \) is some generic constant independent of \( \varepsilon \).
By similar argument as what we have done above, we can actually prove higher-order uniform estimates for $\bar{U}_h$ ($\sim U$). Firstly, with $\bar{U}_0 \in V^3$, integrating (95) from 0 to $t$ for any $t \in [0, t_1]$, we can also infer that

$$\bar{U}_h \text{ is bounded independently of } \varepsilon \text{ in } L^\infty(0, t_1; V^3). \quad (97)$$

Moreover, if we multiply (91) by $\bar{A} \bar{U}_h$, following the similar steps as what we did in (93)-(95) for $\frac{d}{dt} \bar{U}_h$, we obtain

$$\int_0^{t_1} |\bar{A} \bar{U}_h|^2_{L^2} \leq C(u, U_0, t_1) < +\infty, \quad (98)$$

In other words,

$$\bar{A} \bar{U}_h \text{ is bounded independently of } \varepsilon \text{ in } L^2(0, t_1; L^2(M)^3). \quad (99)$$

Because $\bar{U} = \bar{U}_h + \bar{U}_s$, thanks to (89) and (90), the above estimates (100) and (101) also hold with $\bar{U}_h$ being replaced by $\bar{U}$. Namely, we have

$$\bar{U} \text{ is bounded independently of } \varepsilon \text{ in } L^\infty(0, t_1; V^3), \quad (100)$$

and

$$\bar{A} \bar{U} \text{ is bounded independently of } \varepsilon \text{ in } L^2(0, t_1; L^2(M)^3). \quad (101)$$

**Estimate of $dq_v/\partial t$.** With the help of the higher order a priori estimates on $\bar{U}$, we are now aiming to show that the time derivative for $q_v$ is bounded independently of $\varepsilon$ in $L^{5/3}(0, t_1; V^*)$ so that we can apply a suitable version of the Aubin-Lions compactness theorem [43] to extract a convergent subsequence afterwards. We first establish the following technical lemma in order to control the penalization term which contains the “large” factor $\frac{1}{\varepsilon^3}$.

**Remark 3.** Before we continue, we shall add some remarks about the choice of the power $5/3$, which appears in the space $L^{5/3}(0, t_1; V^*)$ and in (102) below. Due to the “large” factor $\frac{1}{\varepsilon^3}$ in the penalization term, we are unable to bound the time derivative of $q_v$ in the more standard space $L^2(0, t_1; V^*)$. Fortunately we only need to show that $dq_v/\partial t$ remains in a bounded set of the space $L^2(0, t_1, V^*)$ for some $\beta > 1$ in view of using the Aubin-Lions compactness theorem. As we will see from the proof of the following technical lemma which controls the penalization term, the power $5/3$ is actually the largest value we can choose for $\beta$, based on the conditions of the Gagliardo-Nirenberg interpolation inequality and the regularity of $\bar{U}$ in $L^\infty(0, t_1; V^3) \cap L^2(0, t_1; H^2(M)^3)$, see (108) and (109). Moreover, the power $3/2$ in the penalization term $\frac{1}{\varepsilon^3}((q_v - q_v^*)^3)^{3/2}$ is chosen accordingly to facilitate the a priori estimates of $dq_v/\partial t$ in $L^{5/3}(0, t_1; V^*)$.

**Lemma 4.2.** The following bound holds:

$$\frac{1}{\varepsilon_1^{5/3}} \int_0^{t_1} |(q_v - q_v^*)^+|^{5/2}_{L^{5/3}(M)} dt \leq C(u, U_0, t_1). \quad (102)$$

**Proof.** This lemma can be proved similarly as Lemma 4.2 in [13]. However, as we are using a different coordinate system and we need to deal with the anelastic reference density $\rho_o(\cdot)$, we still give the details of the proof tailored to our current situation. We multiply (70) by $(q_v - q_{\varepsilon v})^+$ and integrate on $M$, we find

$$\left(\rho_o \frac{\partial q_v}{\partial t}, (q_v - q_{\varepsilon v})^+\right) + (A_{q_v}(q_v - q_{\varepsilon v})^+ \cdot (q_v - q_{\varepsilon v})^+) + \frac{1}{\varepsilon_1} \int_M ((q_v - q_{\varepsilon v})^+)^{5/2} dM$$

$$= (f_{q_v}(U) - F_{q_v}^h \varepsilon^2 (q_v - q_{\varepsilon v}) - \rho_o u \cdot \nabla \varepsilon g_{q_v} (q_v - q_{\varepsilon v})^+). \quad (103)$$
The first two terms in the LHS can be rewritten as
\[(\rho_o \partial_t q_v, (q_v - q_{vs})^+) = (\rho_o \partial_t (q_v - q_{us}), (q_v - q_{vs})^+) + (\rho_o \partial_t q_{us}, (q_v - q_{us})^+)\]
\[= \frac{1}{2} \frac{d}{dt} |\rho_o^{1/2} (q_v - q_{us})^+|^2_{L^2} + \frac{1}{\varepsilon_1} |((q_v - q_{us})^+)|^{5/2}_{L^{5/2}}\]
\[= (A_v q_v, (q_v - q_{vs})^+) = (A_v(q_v - q_{us}), (q_v - q_{us})^+) + (A_v q_{us}, (q_v - q_{us})^+)\]
\[= (A_v(q_v - q_{us})^+, (q_v - q_{us})^+) + (A_v q_{us}, (q_v - q_{us})^+).\]

Dropping the positive term: \((A_v(q_v - q_{us})^+, (q_v - q_{us})^+)\) on the LHS, we can deduce from (103) that
\[\frac{1}{2} \frac{d}{dt} |\rho_o^{1/2} (q_v - q_{us})^+|^2_{L^2} + \frac{1}{\varepsilon_1} |((q_v - q_{us})^+)|^{5/2}_{L^{5/2}}\]
\[\leq |(A_v q_{us}, (q_v - q_{us})^+) + (\rho_o \partial_t q_{us}, (q_v - q_{us})^+) + (\rho_o \mathbf{u} \cdot \nabla q_v, (q_v - q_{us})^+)\]
\[- (f_q(U) - F_{con} \mathcal{H}_2 (q_v - q_{us})) (q_v - q_{us})^+]|. \quad (104)\]

Using the Hölder and Young inequalities, the RHS of (104) can be estimated in the following way:
\[|(\rho_o \partial_t q_{us}, (q_v - q_{us})^+)| = \int_{\mathcal{M}} \varepsilon_1^{2/5} \rho_o \partial_t q_{us} \cdot \frac{(q_v - q_{us})^+}{\varepsilon_1^{2/5}} d\mathcal{M}\]
\[\leq \varepsilon_1^{2/5} |\rho_o \partial_t q_{us}|_{L^{5/3}} \left| \frac{(q_v - q_{us})^+}{\varepsilon_1^{2/5}} \right|_{L^{5/2}}\]
\[\leq C \varepsilon_1^{2/3} (\max \varepsilon_1^{5/3} |\partial_t q_{us}|^{5/3}_{L^{5/3}} + \frac{1}{10 \varepsilon_1} |(q_v - q_{us})^+|^{5/2}_{L^{5/2}}\]
\[\leq C \varepsilon_1^{2/3} |\partial_t q_{us}|^{5/3}_{L^{5/3}} + \frac{1}{10 \varepsilon_1} |(q_v - q_{us})^+|^{5/2}_{L^{5/2}}.\]

The other terms can be addressed similarly. Then (104) becomes
\[\frac{1}{2} \frac{d}{dt} |\rho_o^{1/2} (q_v - q_{us})^+|^2_{L^2} + \frac{1}{\varepsilon_1} |((q_v - q_{us})^+)|^{5/2}_{L^{5/2}}\]
\[\leq C \varepsilon_1^{2/3} (|A_v q_{us}|^{5/3}_{L^{5/3}} + |\partial_t q_{us}|^{5/3}_{L^{5/3}} + |\nabla q_v|^{5/3}_{L^{5/3}} + |q_v|^{5/3}_{L^{5/3}}\]
\[+ C_1) + \frac{1}{2 \varepsilon_1} |(q_v - q_{us})^+|^{5/2}_{L^{5/2}}. \quad (105)\]

Compared with the RHS of (104), as \(\rho_0\) and \(\mathbf{u}\) are bounded in \(L^\infty(\mathcal{M} \times (0, t_1))\), these terms are absorbed in the constant \(C\) in (105). In addition, the source term \(|f_q(U) - F_{con} \mathcal{H}_2 (q_v - q_{us})|^ {5/3}_{L^{5/3}}\) are bounded by \(C(|q_v|^{5/3}_{L^{5/3}} + C_1)\) in (105).

Integrating now (105) in time on \((0, t_1)\), we have
\[\frac{1}{2} \rho_o |(q_v(t_1) - q_{us}(t_1))|^2_{L^2} - \frac{1}{2} \rho_o |(q_v - q_{us}(0))|^2_{L^2} + \frac{1}{2 \varepsilon_1} \int_0^{t_1} |(q_v - q_{us})^+|^{5/2}_{L^{5/2}} dt\]
\[\leq C \varepsilon_1^{2/3} \int_0^{t_1} (|A_v q_{us}|^{5/3}_{L^{5/3}} + |\partial_t q_{us}|^{5/3}_{L^{5/3}} + |\nabla q_v|^{5/3}_{L^{5/3}} + |q_v|^{5/3}_{L^{5/3}} + C_1) dt. \quad (106)\]

The first term in the LHS of (106) is positive and the second term is 0 because of the constraint on the initial value \(q_{00} \leq q_{us}(t = 0)\).

To reach the desired bound (102) on the penalization term, we will bound the integral in the RHS of (106) independently of \(\varepsilon\), drop the positive term in the LHS.
and divide both sides of (106) by $\varepsilon_1^{2/3}$. We now estimate each term in the RHS of (106).

Both $|q_v|$ and $|\nabla_3 q_v|$ are bounded in $L^{5/3}((0, t_1) \times \mathcal{M})$, thanks to (84), (85) and the fact that $L^2((0, t_1) \times \mathcal{M}) \subset L^{5/3}((0, t_1) \times \mathcal{M})$.

Then for $\partial_t q_{vs}$, we see that $\partial_t q_{vs} = \frac{\partial q_{vs}}{\partial T} (T, z) \cdot \partial_t T$. Because of the relationship between $T$ and $\theta$ and recalling that $\partial_t \theta$ has already been bounded in $L^2((0, t_1) \times \mathcal{M})$, $\partial_T T$ is bounded in $L^2((0, t_1) \times \mathcal{M})$. Also, $\partial q_{vs}/\partial T$ is uniformly bounded by (39). Thus we have $|\partial_t q_{vs}|$ bounded in $L^{5/3}((0, t_1) \times \mathcal{M})$.

The most problematic term is $|A_{v} q_{vs}|_{L^{5/3}}$. We begin by exploring the relationship between $\Delta_3 q_{vs}$ and $T$.

$$
\frac{\partial^2 q_{vs}}{\partial x^2} = \frac{\partial}{\partial x} \left( \frac{\partial q_{vs}(T, z)}{\partial T} \frac{\partial T}{\partial x} \right) = \frac{\partial}{\partial x} \left( \frac{\partial q_{vs}(T, z)}{\partial T} \right) \frac{\partial T}{\partial x} + \frac{\partial q_{vs}(T, z)}{\partial T} \frac{\partial^2 T}{\partial x^2}. 
$$

(107)

Using the fact that $\partial q_{vs}(T, z)/\partial T$ is uniformly bounded, we can easily deduce that

$$
\left| \frac{\partial}{\partial x} \left( \frac{\partial q_{vs}(T, z)}{\partial T} \right) \right| \leq C \left| \frac{\partial T}{\partial x} \right|,
$$

for some generic constant $C$ that does not depend on $\varepsilon$.

Recalling (107), we can further deduce that, pointwisely,

$$
\left| \frac{\partial^2 q_{vs}}{\partial x^2} \right| \leq C \left( \left| \frac{\partial T}{\partial x} \right|^2 + \left| \frac{\partial^2 T}{\partial x^2} \right| \right), \quad \left| \frac{\partial^2 q_{vs}}{\partial y^2} \right| \leq C \left( \left| \frac{\partial T}{\partial y} \right|^2 + \left| \frac{\partial^2 T}{\partial y^2} \right| \right).
$$

The second derivative of $q_{vs}$ with respect to $z$ is slightly different with $\partial^2 q_{vs}/\partial x^2$ and $\partial^2 q_{vs}/\partial y^2$, as $q_{vs}$ depends on $z$ explicitly. Following the calculation in (40), and differentiating the three equations in (40) one more time with respect to $z$, we can deduce that, after some algebra:

$$
\left| \frac{\partial^2 q_{vs}}{\partial z^2} \right| \leq C \left( \left| \frac{\partial T}{\partial z} \right| + \left| \frac{\partial^2 T}{\partial z^2} \right| + C_1 \right).
$$

Noticing that the notations $C$ and $C_1$ are used to represent generic constants whose values may vary in different cases.

We can then obtain

$$
|A_{v} q_{vs}|_{L^{5/3}(\mathcal{M})} \leq C \left( |\Delta_3 T|_{L^{5/3}(\mathcal{M})}^{5/3} + |\nabla_3 T|_{L^{10/3}(\mathcal{M})}^{10/3} + C_1 \right). \quad (108)
$$

By Gagliardo-Nirenberg’s interpolation inequality, we have

$$
|\nabla_3 T|_{L^{10/3}(\mathcal{M})} \leq C \left( |\nabla_3 T|_{L^{3}(\mathcal{M})}^{10/3} + |\nabla_3 T|_{L^{2}(\mathcal{M})}^{4/3} |\Delta_3 T|_{L^{2}(\mathcal{M})}^{2} \right). \quad (109)
$$

Note that we also have $\nabla_3 T \in L^{\infty}(0, t_1; V)$ and $\Delta_3 T \in L^2((0, t_1) \times \mathcal{M})$ by (100) and (101), and thus

$$
\int_0^{t_1} |A_{v} q_{vs}|_{L^{5/3}(\mathcal{M})} \, dt \leq C \int_0^{t_1} \left( |\Delta_3 T|_{L^{5/3}}^{5/3} + |\nabla_3 T|_{L^{10/3}}^{10/3} + C_1 \right) \, dt.
$$
By now all the terms in the integral in the RHS of (106) have been bounded independently of \( \varepsilon \), this finishes the proof of Lemma 4.2.

With the help of Lemma 4.2, we are in a position to derive the needed estimate of \( \partial t \rho_q \) in \( L^{5/3}(0, t_1; V^*) \). We multiply (70) by \( q_v^b \in L^{5/2}(0, t_1; V) \) and integrate on \( M \):

\[
\langle \partial_t \rho_{q_v}, q_v^b \rangle + \langle A_v q_v, q_v^b \rangle + \langle \rho_v \cdot \nabla_3 q_v, q_v^b \rangle + \langle \frac{1}{\varepsilon} ((q_v - q_{v^\varepsilon})^+) \rangle^{3/2} q_v^b 
= (f_{q_v}(U^-) - F_{\text{con}} \mathcal{H}_{z_2} (q_v^\varepsilon - q_v^\varepsilon_0), q_v^b).
\]

Rearranging the above equation, we have

\[
|\langle \partial_t \rho_{q_v}, q_v^b \rangle| = |a_{q_v}(q_v, q_v^b) - b(\rho_v, q_v, q_v^b) - \langle \frac{1}{\varepsilon} ((q_v - q_{v^\varepsilon})^+) \rangle^{3/2} q_v^b
+ f_{q_v}(U^-) - F_{\text{con}} \mathcal{H}_{z_2} (q_v^\varepsilon - q_v^\varepsilon_0), q_v^b) |
\leq C(\|q_v\|_V + \|u\|_V \|q_v\|_V + \frac{1}{\varepsilon} |(q_v - q_{v^\varepsilon})^+|_{L^{5/3}} + \|q_v\|_{L^2} + C_1) \|q_v^b\|_V.
\]

Here we have used the Lemma 4.1, the uniform boundedness of \( \rho_v, \partial \rho_{q_v}/\partial z \) and the fact that

\[
\frac{1}{\varepsilon} \int_M ((q_v - q_{v^\varepsilon})^+) \frac{3/2}{q_v} dM \leq \frac{1}{\varepsilon} |(q_v - q_{v^\varepsilon})^+|_{L^{5/3}} \|q_v^b\|_{L^{5/3}} 
\leq (V \subset L^{5/2}(M) \text{ in } \mathbb{R}^3) 
\leq \frac{1}{\varepsilon} |(q_v - q_{v^\varepsilon})^+|_{L^{5/2}} \|q_v^b\|_V.
\]

Hence,

\[
\min_z \rho_{q_v}(z) \|\partial_t q_v\|_{V^*} \leq C(\|q_v\|_V + \|u\|_V \|q_v\|_V + \frac{1}{\varepsilon} |(q_v - q_{v^\varepsilon})^+|_{L^{5/2}} + \|q_v\|_{L^2} + C_1), 
\]

\[
\|\partial_t q_v\|_{V^*} \leq C(\|q_v\|_{V^*}^{5/3} + \|u\|^{5/3}_V \|q_v\|_V^{5/3} + \frac{1}{\varepsilon} |(q_v - q_{v^\varepsilon})^+|_{L^{5/2}} + \|q_v\|_{L^2} + C_1).
\]

Then thanks to (84),(85) and Lemma 4.2,

\[
\int_0^{t_1} \|\partial_t q_v\|_{V^*}^{5/3} dt \leq C(u, U_0, t_1),
\]

where \( C(u, U_0, t_1) \), as before, is a constant independent of \( \varepsilon \). We have thus derived an a priori bound for \( \partial_t q_v \) in \( L^{5/3}(0, t_1; V^*) \) as promised before.

Finally, we summarize all the estimates that we obtained above for \( U^\varepsilon, \bar{U}^\varepsilon \) and \( \partial_t q_v^\varepsilon \). For some positive constant \( C = C(u, U_0, t_1) \) which is independent of \( \varepsilon \), we now have
\[ |U^\varepsilon|_{L^\infty(0, t_1; \mathbb{H})} \leq C, \quad \|U^\varepsilon\|_{L^2(0, t_1; V)} \leq C, \quad \|\bar{U}^\varepsilon\|_{L^2(0, t_1; H^2(\mathcal{M})^3)} \leq C, \]
\[ \|\partial_t U^\varepsilon\|_{L^2(0, t_1; L^2(\mathcal{M})^3)} \leq C, \quad \|\bar{U}^\varepsilon\|_{L^\infty(0, t_1; H^1(\mathcal{M})^3)} \leq C, \]
\[ \text{and} \quad \|\partial_t q^\varepsilon\|_{L^{5/3}(0, t_1; V')} \leq C. \]  

**Remark 4.** As usual by implementing a Galerkin approximation for the problem (70)-(74) we can obtain an a priori estimates similar to the above estimates for the Galerkin approximation. Then passing to the lower limit we obtain these very estimates (independent of \( \varepsilon \)) for the actual solution \( U^\varepsilon \) of (70)-(74). We state this existence result in the following theorem, but we will skip the proof since it is straightforward after the analysis above on the a priori estimates.

**Theorem 4.3.** Let \( \varepsilon > 0 \) be fixed and assume that \( u \in L^\infty((0, t_1) \times \mathcal{M}) \) and \( U_0 \in \mathcal{V} \) are given. Then, the system (70) associated with the initial and boundary conditions (71) and (72), respectively, has a solution \( U^\varepsilon \) such that

\[ U^\varepsilon \in L^\infty(0, t_1; \mathbb{H}) \cap L^2(0, t_1; \mathcal{V}), \]  

and

\[ \bar{U}^\varepsilon \in L^2(0, t_1; H^2(\mathcal{M})^3), \quad \partial_t \bar{U}^\varepsilon \in L^2(0, t_1; L^2(\mathcal{M})^3), \quad \partial_t q^\varepsilon \in L^{5/3}(0, t_1; V^*). \]  

Furthermore the norms of \( U^\varepsilon, \bar{U}^\varepsilon \) and \( \partial_t \bar{U}^\varepsilon \) in the corresponding spaces are bounded independently of \( \varepsilon \) by quantities which depend on the norm of \( U_0 \) in \( \mathbb{H} \) and on the other data.

**4.6. Passage to the limit.** In the following we will pass to the limit, as \( \varepsilon \to 0 \), in the penalized system (70), and to avoid a possible confusion we reintroduce here the dependence on \( \varepsilon \). First, using (111) and Aubin-Lions compactness theorem, we deduce the existence of a subsequence, still denoted \( U^\varepsilon = (q^\varepsilon, q^\varepsilon, q^\varepsilon, \theta^\varepsilon) \), and a function \( U = (q_c, q_c, q_c, \theta) \) both verifying (112), (113), such that, as \( \varepsilon \to 0 \),

(i) \( U^\varepsilon \rightharpoonup U \) weakly in \( L^2(0, t_1; \mathcal{V}) \) and weak-* in \( L^\infty(0, t_1; \mathbb{H}) \),
(ii) \( \partial_t U^\varepsilon \rightharpoonup \partial_t U \) weakly in \( L^2(0, t_1; L^2(\mathcal{M})^3) \),
(iii) \( \partial_t q^\varepsilon \rightharpoonup \partial_t q \) weakly in \( L^{5/3}(0, t_1; V^*) \),
(iv) \( U^\varepsilon \rightharpoonup U \) strongly in \( L^2(0, t_1; H^1(\mathcal{M})^3) \) and weakly in \( L^2(0, t_1; H^2(\mathcal{M})^3) \),
(v) \( q^\varepsilon \rightharpoonup q \) strongly in \( L^2(0, t_1; H) \) and weakly in \( L^2(0, t_1; V) \),
(vi) \( (q^\varepsilon - q_{\varepsilon})^2 \rightharpoonup 0 \) strongly in \( L^{5/2}((0, t_1) \times \mathcal{M}) \), as a result of Lemma 4.2,
(vii) \( \mathcal{H}(q^\varepsilon - q_{\varepsilon}) \rightharpoonup h_q \), weak-* in \( L^\infty((0, t_1) \times \mathcal{M}) \) for \( h_q \in \mathcal{H}(q_e - q_{es}) \).

By (i) and (iii), we also have

\[ \text{(114)} \]

\[ q^\varepsilon(t_1) \rightharpoonup q(t_1) \text{ weakly in } L^2(\mathcal{M}). \]

For the inequality constraint on \( q_c \), once we show that \( q^\varepsilon_{cs} = q_{es}(T^\varepsilon, z) \to q_{es}(T, z) \) in \( L^2(0, t_1; V) \) (see Lemma 4.4 below), (vi) implies in particular that \( q_c \leq q_{es} \).

We also note that the strong convergence in \( L^2(0, t_1; \mathbb{H}) \) is actually available in \( L^p(0, t_1; \mathbb{H}) \), for all \( p \geq 1 \), thanks to the continuity of \( U^\varepsilon \in C([0, t_1]; \mathbb{H}) \).

Recalling the continuous parts of the source terms \( f(U^\varepsilon, \nabla U^\varepsilon) = (f_{qc}(U^\varepsilon), f_{qc}(U^\varepsilon), f_{qc}(U^\varepsilon, \nabla U^\varepsilon), f_0(U^\varepsilon)) \), using the strong convergence in (iv) and (v), we have

\[ f(U^\varepsilon, \nabla U^\varepsilon) \rightharpoonup f(U, \nabla U) \text{ strongly in } L^2(0, t_1; L^2(\mathcal{M})^4). \]  

Moreover, the coefficient function \( F_{\text{con}}(T^\varepsilon, z) \) in the discontinuous part of the source terms also satisfies

\[ F_{\text{con}}(T^\varepsilon, z) \rightharpoonup F_{\text{con}}(T, z) \text{ strongly in } L^2(0, t_1; L^2(\mathcal{M})). \]
Then due to the boundedness of \( \mathcal{F} \mathcal{H}_{c_2}(q_v - q_{vs}) \) and using (vii) and [43, Lemma 1.3], we have
\[
\mathcal{F}(T^\varepsilon, z)\mathcal{H}_{c_2}(q_v - q_{vs}) \rightharpoonup \mathcal{F}(T, z) h_{q_v} \quad \text{weakly in } L^2(0, t_1; L^2(M)^4),
\]
where \( \mathcal{F} = (-F_{con}, F_{con}, 0, \frac{L}{c_p h} F_{con})^t \).

Therefore one can pass to the limit, as \( \varepsilon \to 0 \), in (70)\(_{2,3,4} \) in a straightforward manner (see [60] and [59]).

For the passage to the limit \( \varepsilon \to 0 \) in the \( q_v \) equation, we still need the following results which will be used in the proof of the convergence of the penalized term, namely (70)\(_1 \).

**Lemma 4.4.** If \( T^\varepsilon \) converges to \( T \) strongly in \( L^2(0, t_1; V) \), then \( q_{vs}(T^\varepsilon, z) \) as expressed in (8), converges to \( q_{vs}(T, z) \) strongly in \( L^2(0, t_1; V) \).

**Proof.** The proof goes along the lines of the proof of [13, Lemma 4.5], but with the vertical variable \( p \) being replaced by \( z \). Therefore, here we only provide a sketch of the proof. By the expressions (8)-(10), and smooth extension \( \varphi(T) \) outside the physical relevant values of \( T \), it is easy to see that \( q_{vs}(T^\varepsilon, z) \), as a smooth function of \( T \), converges to \( q_{vs}(T, z) \) strongly in \( L^2(0, t_1; L^2) \) using the fact that \( \bar{U}^\varepsilon \) converges to \( \bar{U} \) strongly in \( (L^2(0, t_1; V))^3 \). To show that \( \nabla_3 q_{vs}^\varepsilon \) converges to \( \nabla_3 q_{vs} \) in \( L^2(0, t_1; L^2) \), we write
\[
\nabla_3 q_{vs}^\varepsilon = \frac{\partial q_{vs}}{\partial T}(T^\varepsilon, z) \cdot \nabla_3 T^\varepsilon, \quad \nabla_3 q_{vs} = \frac{\partial q_{vs}}{\partial T}(T, z) \cdot \nabla_3 T.
\]
Taking the differences of these two terms, we have
\[
\left| \frac{\partial q_{vs}}{\partial T}(T^\varepsilon, z) \cdot \nabla_3 T^\varepsilon - \frac{\partial q_{vs}}{\partial T}(T, z) \cdot \nabla_3 T \right|_{L^2} \leq (117)
\]
\[
= \left| \frac{\partial q_{vs}}{\partial T}(T^\varepsilon, z)(\nabla_3 T^\varepsilon - \nabla_3 T) \right|_{L^2} + \left| (\frac{\partial q_{vs}}{\partial T}(T^\varepsilon, z) - \frac{\partial q_{vs}}{\partial T}(T, z)) \nabla_3 T \right|_{L^2}.
\]
We can show that the RHS of (117) converges to zero by the strong convergence (iv) and the uniform boundedness and continuity of the function \( \frac{\partial q_{vs}}{\partial T}(T^\varepsilon, z) \). Then we have \( q_{vs}^\varepsilon \) converges to \( q_{vs} \) strongly in \( L^2(0, t_1; V) \).

**Lemma 4.5.** For all \( q_v^b \in K = K(U) \), we consider \( q_v^{bc} = q_v^b - (q_v^b - q_{vs}^b)^+ \). Then \( q_v^{bc} \) converges to \( q_v^b \) strongly in \( L^2(0, t_1; V) \).

**Proof.** We first observe, using the definitions of \( q_v^{bc} \) and of the set \( K(U) \), that \( q_v^{bc} \) converges to \( q_v^b \) in \( L^2(0, t_1; L^2) \). Then we see that the derivative of \( q_v^{bc} \) with respect to the space variable \( \nabla_3 q_v^{bc} = \nabla_3 q_v^b - 1_{\{q_v^b > q_{vs}^b\}} \nabla_3 (q_v^b - q_{vs}^b) \). Using Lemma 4.4 we conclude that \( q_v^{bc} \) converges to \( q_v^b \) strongly in \( L^2(0, t_1; V) \).

**Remark 5.** From the proof of Lemma 4.4, we see that \( |\nabla_3 q_v^{bc}| \leq C|\nabla_3 T^\varepsilon| + C_1 \). Then noting that \( \nabla_3 T^\varepsilon \in L^\infty(0, t_1; L^2(M)) \) by (100), here \( q_v^{bc} \) actually lies in a bounded set of \( L^\infty(0, t_1; V) \). And by our assumption, \( q_v^{bc} \in L^\infty(0, t_1; V) \). Therefore, \( q_v^{bc} = \min(q_v^b, q_{vs}^b) \) lies in a bounded set in \( L^\infty(0, t_1; V) \) as well. Also \( q_v^{bc} \) converges to \( q_v^b \) almost everywhere in \( V \) for \( t \in [0, t_1] \). Applying Lemma 4.5 as well as Lebesgue’s dominated convergence theorem, we infer that
\[
q_v^{bc} \rightarrow q_v^b \quad \text{strongly in } L^p(0, t_1; V) \quad \text{for any } p > 1.
\]
In particular, we will use the result with \( p = \frac{5}{2} \) for passing to limit in the \( q_v \)-equation.
Now, we are ready to pass to the limit in the \(q_v\)-equation \((70)_1\), which contains the penalization term. Let us first rewrite as follows the weak formulation of the penalized equation \((70)_1\) in view of \((65)\). For all \(q_v^\varepsilon \in \mathcal{K}(U)\), we consider \(q_v^{bc} = q_v^b - (q_v^b - q_v^{es})^+ = \min(q_v^b, q_v^{es}) \leq q_v^{es}\). We then write the first equation \((q_v\text{-equation})\) of \((73)\) with \(q_v^b\) replaced by \(q_v^{bc}\), and we find

\[
\langle \rho_0 \partial_t q_v^b, q_v^{bc} \rangle + a(q_v^b, q_v^{bc} - q_v^b) + b(\rho_0, q_v^b, q_v^{bc} - q_v^b) - l(q_v^{bc} - q_v^b)
\]

\[
+ \frac{1}{\varepsilon_1} \langle ((q_v^\varepsilon - q_v^{es})^+)\rangle^{3/2}, q_v^{bc} - q_v^b \rangle = \langle f_{q_v}(U^\varepsilon) - F_{\text{con}} \mathcal{H}_{\varepsilon_2}(q_v^\varepsilon - q_v^{es}), q_v^{bc} - q_v^b \rangle,
\]

Regarding \((119)\), we first observe that

\[
\langle ((q_v^\varepsilon - q_v^{es})^+)\rangle^{3/2}, q_v^{bc} - q_v^b \rangle \leq 0
\]

After integrating in time on \((0, t_1)\) and using \((120)\), we rewrite \((119)\) as follows:

\[
\int_0^{t_1} \langle \rho_0 \partial_t q_v^b, q_v^{bc} \rangle + a(q_v^b, q_v^{bc} - q_v^b) + b(\rho_0, q_v^b, q_v^{bc} - q_v^b) dt
\]

\[
- \int_0^{t_1} \int q_v^b(q_v^{bc} - q_v^b) dt \geq \int_0^{t_1} \langle f_{q_v}(U^\varepsilon) - F_{\text{con}} \mathcal{H}_{\varepsilon_2}(q_v^\varepsilon - q_v^{es}), q_v^{bc} - q_v^b \rangle dt.
\]

The passage to the limit \(\varepsilon \to 0\) in \((121)\) can be carried out in a very similar way as what was done in \([13]\), thus we only show the details of the proof for some typical terms in \((121)\).

Integrating by parts the first term in the LHS of \((121)\), we have

\[
\int_0^{t_1} \langle \rho_0 \partial_t q_v^\varepsilon, q_v^{es} \rangle dt = -\frac{1}{2} \int_0^{t_1} \frac{d}{dt} |\rho_0^{1/2} q_v^\varepsilon|^2 dt L^2 = -\frac{1}{2} |\rho_0^{1/2} q_v^\varepsilon(t_1)|^2 L^2 + \frac{1}{2} |\rho_0^{1/2} q_v^0|^2 L^2,
\]

By the semi lower continuity of the \(L^2\) norm, it follows that

\[
\limsup_{\varepsilon \to 0} \int_0^{t_1} \langle \rho_0 \partial_t q_v^\varepsilon, -q_v^\varepsilon \rangle dt = -\liminf_{\varepsilon \to 0} \int_0^{t_1} |\rho_0^{1/2} q_v^\varepsilon(t_1)|^2 L^2 + \frac{1}{2} |\rho_0^{1/2} q_v^0|^2 L^2
\]

\[
\leq -\frac{1}{2} |\rho_0^{1/2} q_v(t_1)|^2 L^2 + \frac{1}{2} |\rho_0^{1/2} q_v^0|^2 L^2
\]

\[
= -\int_0^{t_1} \langle \rho_0 \partial_t q_v, q_v \rangle dt.
\]

In addition, by (iii), Lemma 4.5 and Remark 5, we have

\[
\langle \rho_0 \partial_t q_v^\varepsilon, q_v^{bc} \rangle \to \langle \rho_0 \partial_t q_v^0, q_v^b \rangle, \text{ as } \varepsilon \to 0.
\]

Therefore,

\[
\limsup_{\varepsilon \to 0} \int_0^{t_1} \langle \rho_0 \partial_t q_v^\varepsilon, q_v^{bc} - q_v^\varepsilon \rangle dt \leq \int_0^{t_1} \langle \rho_0 \partial_t q_v^b, q_v^b - q_v \rangle dt.
\]

We see that \((122)\) also keeps the inequality \((121)\) in the desired direction.
Then we investigate the trilinear form \( b(\rho_o u, q_v^\varepsilon, q_v^{b\varepsilon} - q_v^\varepsilon) \).

\[
\left| \int_0^{t_1} \left[ b(\rho_o u, q_v^\varepsilon - q_v, q_v^{b\varepsilon} - q_v^\varepsilon) dt - b(\rho_o u, q_v - q_v, q_v^{b\varepsilon} - q_v^\varepsilon) dt \right] \right| \leq \left| \int_0^{t_1} \left[ b(\rho_o u, q_v^\varepsilon - q_v, q_v^{b\varepsilon} - q_v^\varepsilon) dt + b(\rho_o u, q_v - q_v, q_v^{b\varepsilon} - q_v^\varepsilon) dt \right] - b(\rho_o u, q_v^\varepsilon - q_v) dt \right| \leq \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3,
\]

where

\[
\mathcal{I}_1 = \int_0^{t_1} b(\rho_o u, q_v^\varepsilon - q_v, q_v^{b\varepsilon} - q_v^\varepsilon) dt, \quad \mathcal{I}_2 = \int_0^{t_1} b(\rho_o u, q_v, q_v^{b\varepsilon} - q_v^\varepsilon) dt
\]

and

\[
\mathcal{I}_3 = \int_0^{t_1} b(\rho_o u, q_v, q_v^\varepsilon - q_v) dt.
\]

In view of Lemma 4.1,

\[
\mathcal{I}_1 \leq \|\rho_o u\|_{L^\infty(0,t_1;V)} \|q_v^\varepsilon - q_v\|^{1/2}_{L^2(0,t_1;V)} \|q_v^{b\varepsilon} - q_v^\varepsilon\|^{1/2}_{L^2(0,t_1;V)},
\]

\[
\mathcal{I}_2 \leq \|\rho_o u\|_{L^\infty(0,t_1;V)} \|q_v\|_{L^2(0,t_1;V)} \|q_v^{b\varepsilon} - q_v^\varepsilon\|_{L^2(0,t_1;V)},
\]

both of which tend to zero, thanks to (v) and Lemma 4.5. \( \mathcal{I}_3 \) also converges to zero due to the weak convergence of \( q_v^\varepsilon \) to \( q_v \) in \( L^2(0,t_1;V) \). Hence, we obtain

\[
\int_0^{t_1} b(\rho_o u, q_v^\varepsilon, q_v^{b\varepsilon} - q_v^\varepsilon) dt \to \int_0^{t_1} b(\rho_o u, q_v, q_v^{b\varepsilon} - q_v^\varepsilon) dt, \quad \varepsilon \to 0.
\]

The passage to the limit \( \varepsilon \to 0 \) in the other terms are exactly the same as what was done in [13], thus we omit the details here. After we pass to limit \( \varepsilon \to 0 \) in (119) and retrieve (65), we still need to check that \( h_{q_v} \) belongs to \( \mathcal{H}(q_v - q_{v_0}) \), i.e., to prove (68). The proof for this part also follows in lines as what we did in [13], so we omit the details and move on to conclude the main theorem of this paper.

**Theorem 4.6.** Let \( U_0 \in V \), \( t_1 > 0 \) be given and assume that \( u \in L^\infty((0,t_1) \times \mathcal{M}) \) is given. Then, the system (63)-(65) associated with the initial and boundary conditions (71) and (72), respectively, has a solution \( U \) such that

\[
U \in L^\infty(0,t_1;\mathbb{H}) \cap L^2(0,t_1;V),
\]

Furthermore, we have additional regularity results

\[
\bar{U} \in L^2(0,t_1;H^2), \quad \partial_t \bar{U} \in L^2(0,t_1;L^2), \quad \partial_t q_v \in L^{5/3}(0,t_1;V^*),
\]

where \( U = (q_v, q_{\cdot v}, q_{\cdot p}, \theta) \), \( \bar{U} = (q_v, q_{\cdot p}, \theta) \).

5. **Numerical simulations.** In this section, we illustrate the theory above with some numerical simulations done in a slightly different setting, easier in some sense, and more challenging in some sense.

We consider a two-dimensional problem with directions \( x \) (west-east) and \( z \) (the altitude), and the domain is not rectangular, corresponding to the geometry above one or two mountains. In the simulations, we ignore the dissipation terms (the \( D \)
terms) in (2)-(5), which are important in the analysis but have negligible effect on short term simulations. Hence we use the following system of equations.

\[
\frac{\partial (\rho \theta)}{\partial t} + \nabla \cdot (\rho \theta \mathbf{u}) = \frac{L_e \theta}{c_p T_e} (\text{CON} + \text{DEP}), \tag{127}
\]

\[
\frac{\partial (\rho q_v)}{\partial t} + \nabla \cdot (\rho q_v \mathbf{u}) = -\text{CON} - \text{DEP}, \tag{128}
\]

\[
\frac{\partial (\rho q_c)}{\partial t} + \nabla \cdot (\rho q_c \mathbf{u}) = \text{CON} - \text{ACC} - \text{AUT}, \tag{129}
\]

\[
\frac{\partial (\rho q_p)}{\partial t} + \nabla \cdot [\rho (\mathbf{u} - \mathbf{V_T}) q_p] = \text{ACC} + \text{AUT} + \text{DEP}. \tag{130}
\]

The source terms \( \text{CON} \), \( \text{AUT} \), \( \text{ACC} \), and \( \text{DEP} \) are defined in (7), (12), (13), and (14), respectively. All quantities are expressed in the metric system. A west-east prevailing wind is used in the simulations.

The domain represents the atmosphere above one or two mountains. We set the domain as \( z_0(x) \leq z \leq z_f \), \( x \in [x_0, x_f] \), where \( x_0 = 0 \), \( x_f = 9 \times 10^4 \text{ m} \), and \( z_f = 1.6 \times 10^4 \text{ m} \), \( z = z_f \) being the height of the domain under consideration. The function \( z_0(x) \) defines the topography along the mountains.

- In the first simulation, the topography \( z_0 \) is set to be

\[ z_0(x) = 2500 \exp \left[ - \left( \frac{x - 45000}{6000} \right)^2 \right] \]

representing a mountain of height 2500 m.

- In the second simulation, \( z_0 \) is set to be

\[
z_0(x) = \begin{cases} 
2500 \exp \left[ - \left( \frac{x - 22500}{6000} \right)^2 \right], & \text{if } x \leq 45000, \\
1500 \exp \left[ - \left( \frac{x - 67500}{6000} \right)^2 \right], & \text{if } x > 45000. 
\end{cases}
\]

It represents a mountain of height 2500 meters on the left and a lower mountain of height 1500 meters on the right.

- In the third simulation, \( z_0 \) is set to be

\[
z_0(x) = \begin{cases} 
1500 \exp \left[ - \left( \frac{x - 22500}{6000} \right)^2 \right], & \text{if } x \leq 45000, \\
2500 \exp \left[ - \left( \frac{x - 67500}{6000} \right)^2 \right], & \text{if } x > 45000. 
\end{cases}
\]

In this simulation, we have a lower mountain (1500 m) on the left and a higher mountain (2500 m) on the right.

The air velocity \( \mathbf{u} \) is assumed to be given in the form \( \mathbf{u} = (a \cos \xi, a \sin \xi) \). Here, \( a \) is the magnitude of the velocity, which assumes a constant value of 1 m \( \cdot \) s\(^{-1} \) everywhere in the domain. The direction of the air velocity is controlled by \( \xi = c(x)(z - z_f) \), where \( z_f \) is as above and \( \xi |_{x=z_0(x)} = c(x)(z_0(x) - z_f) = \arctan \left( \frac{z_0'(x)}{c(x)} \right) \). That is, the air velocity is horizontal at the top of the atmosphere and tangent to the mountain surface at the bottom of the atmosphere. This ensures the non-penetration conditions on the top and bottom sides of the domain.
For the initial conditions, we use the data

\[ T(x, z, t = 0) = \bar{T}(p) = T_0 - \left(1 - \frac{p}{p_{oo}}\right)\Delta T, \quad (131) \]

and

\[ p = p_{oo} \cdot \left(1 - \frac{g}{c_p T_0} \cdot \frac{z}{\frac{M}{R_0}}\right) \quad (132) \]

where \( c_p = 1.0047 \times 10^3 \text{Jkg}^{-1}\text{K}^{-1}, \ R_0 = 8.3158 \text{Jmol}^{-1}\text{K}^{-1}, \ p_{oo} = 10^5 \text{Pa}, \ T_0 = 300 \text{K}, \) and \( \Delta T = 50 \text{K}. \) The initial condition for \( \theta \) is computed using

\[ \theta(x, z, t = 0) = T \left( \frac{p}{p_{oo}} \right)^{-R_d/c_p}. \]

The initial condition for \( q_v \) is

\[ q_v(x, z, t = 0) = \gamma q_{vs}, \]

where \( q_{vs} \) is computed by (8) and the saturation \( \gamma \in [0, 1] \) is taken to be 0.9 in our simulations.

We use zero initial conditions for \( q_c \) and \( q_p, \) i.e.,

\[ q_c(x, z, t = 0) = 0, \]
\[ q_p(x, z, t = 0) = 0. \]

The boundary conditions are

\[
\begin{align*}
\frac{\partial z_0}{\partial x} &= 0, \quad \text{at } x = 0, x_f, \\
\Theta &= G(p), \quad \text{at } x = 0, \\
\frac{\partial \Theta}{\partial n} &= 0, \quad \text{at } x = x_f, \\
\frac{\partial \omega}{\partial n} &= 0, \quad \text{at } x = 0, x_f,
\end{align*}
\]

where \( \Theta = (\theta, q_v, q_c, q_p) \) is the solution, and \( G = (\theta, q_v, q_c, q_p) \) defines the boundary values of the solution on the left boundary (at \( x = 0 \)) for \( \theta, q_v, q_c, \) and \( q_p, \) respectively. The following definition of \( G \) is used in our simulations.

\[ g_\theta = \bar{T}, \quad g_{q_v} = q_{vs}(\bar{T}, p_{e}), \quad g_q = 0, \quad g_{q_p} = 0. \]

In the numerical simulations, we used the upwind Godunov scheme with an \( n \times n \) mesh in the spatial domain and the 4th order Runge-Kutta method for the time discretization, similar to the numerical scheme used in [11] and in [1], where a simpler atmosphere model was considered.

Now, we give the results of the numerical simulations. All the 3 simulations were computed with a spatial mesh of size \( 200 \times 200 \) and a time step of \( \Delta t = 0.5s. \) For our ice-bearing model, we focus on the formation of snow in the simulations. For each simulation, we plot the amount of snow \( q_s \) at 4 different times. The snow \( q_s \) is computed by

\[ q_s = (1 - \alpha(T))q_p. \]

It is the part of the precipitation (i.e., the precipitation water mixing ratio \( q_p \)) that is in the form of ice, measured in \( \text{g} \cdot \text{kg}^{-1}. \) In the following contour plots on \( q_s, \) areas in the atmosphere with brighter color (i.e., towards the yellow end on the color bar) receives more snow, whereas areas with darker color (i.e., towards the
blue end on the color bar) receive less snow. As we will see below, most of the atmosphere shows no presence of snow (dark blue areas in the contour plots), due to higher than freezing point temperature or low level of humidity.

Simulation 1

Snow \( q_s \) at \( t = 100 \) s

Snow \( q_s \) at \( t = 1000 \) s

Snow \( q_s \) at \( t = 2000 \) s

Snow \( q_s \) at \( t = 4000 \) s

Figure 1. In the first simulation, we have a single mountain of height 2500 meters. The amount of snow \( q_s \) is measured in g·kg\(^{-1}\). Brighter color indicates higher quantity (i.e., more snow) in the contour plots. Solid deep blue represents total absence of snow. The dashed red line shows the separation between rain and snow. We can see that the area that receives the most snow is the part of the atmosphere slightly to the left of the peak above the mountain. Note that the air flows from left to right. These results are coherent with the physical context.
Simulation 2

Figure 2. In the second simulation, there are two mountains of heights 2500 m and 1500 m with the taller mountain on the left. The amount of snow $q_s$ is measured in $\text{g} \cdot \text{kg}^{-1}$. Brighter color indicates higher quantity of snow. Solid deep blue represents total absence of snow. The dashed red line shows the separation between rain and snow. The air flows from left to right as before. In this simulation, the taller mountain on the left blocks the passing of moist, resulting in less snow around the lower mountain on the right. These results are coherent with the physical context.
Simulation 3

Figure 3. In the third simulation, we have two mountains of heights 1500 m and 2500 m with the taller mountain on the right. The amount of snow is measured in g·kg\(^{-1}\) and brighter color indicates higher quantity of snow. Solid deep blue represents total absence of snow. The dashed red line shows the separation between rain and snow. The air flows from left to right as before. In this simulation, there is snow in the atmosphere above both mountains, as the mountain on the left does not block as much moist as in the previous simulation. These results are also coherent with the physical context.

Appendices.

Appendix A. The detailed formulation of the microphysics terms. Here we give the detailed expressions of AUT (the autoconversion of cloud condensate into precipitation), ACC (the accretion of cloud condensate by precipitation), and DEP (source (sink) of precipitation due to deposition (evaporation) of water vapor on (from) precipitation particles), based on the reference [28].
**AUT:** autoconversion of cloud condensate into precipitation

\[ \text{AUT} = \text{AUT}_r + \text{AUT}_s, \]

\[ \begin{align*}
\text{AUT}_r &= 1.67 \times 10^{-5} \psi^2 \left( 5 + \frac{0.036 N_d}{D_d \psi} \right)^{-1}, \\
\text{AUT}_s &= \frac{\rho_o \cdot \left[ (1 - \alpha) q_p \right]}{\tau_a},
\end{align*} \] \hspace{1cm} (133)

- \( \psi = 10^3 \rho_o \cdot (\alpha q_p) \) is the density of precipitation water expressed in g \( \cdot \) m\(^{-3}\).
- \( N_d = 200 \text{cm}^{-3} \) is the concentration of cloud droplets.
- \( D_d = 0.146 - 5.964 \times 10^{-2} \ln \frac{N_d}{2000} \) is the relative dispersion of cloud droplet population.
- \( \tau_a \) is the conversion time scale assumed equal to a time required to grow an ice crystal by diffusion of water vapor in water saturated conditions up to a size of small precipitation particle (mass of \( 10^{-9} \text{kg} \)), which can be approximated as

\[ \tau_a(T) = -800e^{-(T+15)^2} + 1000 \]

**ACC:** accretion of cloud condensation by precipitation

\[ \text{ACC} = \text{ACC}_r + \text{ACC}_s, \]

\[ \begin{align*}
\text{ACC}_r &= \frac{0.8\pi}{4} \cdot \overline{D}_r \cdot v_{tr}(\overline{D}_r) \rho_o \cdot \alpha q_p, \\
\text{ACC}_s &= \frac{0.06\pi}{4} \cdot \overline{D}_s \cdot v_{ts}(\overline{D}_s) \rho_o \cdot (1 - \alpha) q_p.
\end{align*} \] \hspace{1cm} (134)

The \( \text{ACC} \) terms are estimated by growth rates of the mean particle due to accretion of the cloud condensate. The involving parameters are explained below.

- \( \overline{D} \) denotes the diameter of a particle with average mass. As usual, the subscript \( r \) represents the rain particle and the subscript \( s \) represents the snow particle.

\[ \overline{D}_r = \left( \frac{\lambda_r \rho_o (\alpha q_p)}{\frac{\pi}{6} \rho_w N_0} \right)^{1/3}, \quad \overline{D}_s = \left( \frac{40 \rho_o (1 - \alpha) q_p |\lambda_s|}{N_0} \right)^{1/2}. \]

- \( N_0 = 10^7 \text{m}^{-4} \) is a fixed parameter of the Marshall–Palmer size distribution.
- The parameter \( \lambda \) is the slope of the distribution. It depends on the mixing ratio of precipitation particle. Now in our case, we use \( \lambda_r \) as the slope for rain and \( \lambda_s \) for snow particles.

\[ \begin{align*}
\lambda_r &= \lambda_r(T, q_p) = \left( \frac{\pi}{6} \rho_w N_0 \Gamma(4) \right)^{1/4}, \\
\lambda_s &= \lambda_s(T, q_s) = \left( \frac{2.5 \times 10^{-2} \cdot N_0 \Gamma(3)}{\rho_o (1 - \alpha) q_p} \right)^{1/3}.
\end{align*} \] \hspace{1cm} (135)

- \( v_{tr}(\overline{D}_r) \) and \( v_{ts}(\overline{D}_s) \) are the sedimentation velocities (terminal velocities) with respect to the rain and snow particles of average mass.

\[ v_{tr}(\overline{D}_r) = 130 \overline{D}_r^{1/2}, \quad v_{ts}(\overline{D}_s) = 4 \overline{D}_s^{1/4}. \]

- \( \rho_w = 10^3 \text{kg m}^{-3} \) is the water density.
DEP: source (sink) of precipitation due to deposition (evaporation) of water vapor on (from) precipitation particles

\[ \text{DEP} = \text{DEP}_r + \text{DEP}_s, \]

\[ \text{DEP}_r = \frac{4\pi}{2} D_r \left( \frac{q_v}{q_{vw}} - 1 \right) F_r G, \]

\[ \text{DEP}_s = \frac{4\pi}{3} D_s \left( \frac{q_v}{q_{vi}} - 1 \right) F_s G. \]

\(136\)

- \( F_r \) and \( F_s \) are ventilation factors, for raindrops and ice particles, respectively.

\[ F_r = 0.78 + 0.27 R_{er}^{1/2}, \quad F_s = 0.65 + 0.39 R_{es}^{1/2}, \]

where \( R_{er} \) and \( R_{es} \) are Reynolds numbers defined by

\[ R_{er} = \frac{D_r v_{er}}{\nu}, \quad R_{es} = \frac{D_s v_{es}}{\nu}, \]

and \( \nu \approx 2 \times 10^{-5} \text{m}^2\text{s}^{-1} \) is the kinematic viscosity of air.

- \( G = G(T_e) \) is the thermodynamic function, defined as

\[ G = G(T_e) = A \left( \frac{2.2 T_e}{e_{vs}(T_e)} + \frac{2.2 \times 10^2}{T_e} \right)^{-1} \]

\(137\)

where \( A = 10^{-7} \text{kg m}^{-1}\text{s}^{-1} \) and SI units are assumed in (137). The term \( e_{vs}(T_e) \) is calculated as

\[ e_{vs}(T_e) = \begin{cases} e_{vw}(T_e), & \text{if } T_e \geq 0^\circ \text{C}, \\ e_{vi}(T_e), & \text{if } T_e \leq 0^\circ \text{C}. \end{cases} \]

Compared with the detailed expressions above, in Section 2.1 we omit the specific forms of constants and other given parameters, and only keep the terms involving the four variables \( q_v, q_c, q_p \) and \( \theta(\sim T) \) when developing the simplified form of the source terms.

Acknowledgments. This work was partially supported by the National Science Foundation under the grants NSF-DMS-1510249 and by the Research Fund of Indiana University.

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Received for publication January 2020.

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