On the Divergence and Vorticity of Vector Ambit Fields

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Abstract

This paper studies the asymptotic behavior of the flux and circulation of a subclass of random fields within the family of 2-dimensional vector ambit fields. We show that, under proper normalization, the flux and the circulation converge stably in distribution to certain stationary random fields that are defined as line integrals of a Lévy basis. A full description of the rates of convergence and the limiting fields is given in terms of the roughness of the background driving Lévy basis and the geometry of the ambit set involved. We further discuss the connection of our results with the classical Divergence and Vorticity Theorems. Finally, we introduce a class of models that are capable to reflect stationarity, isotropy and null divergence as key properties.

Keywords: Ambit fields, divergence, vorticity, Lévy bases, infinite divisibility, stationary and isotropic fields, 2-dimensional turbulence, Stoke’s Theorem.

1 Introduction

A classical result of vector calculus, namely Stokes’ Theorem, allows to express the vorticity (also known as curl) and divergence operators in terms of the circulation and the flux of a vector field. In 2 dimensions, it states that

\[ \int_D \nabla \cdot u(x,y) \, dx \, dy = \oint_{\partial D} u(s) \cdot n_D(s) \, ds; \]

\[ \int_D \nabla \times u(x,y) \, dx \, dy = \oint_{\partial D} u(s) \cdot n_D(s) \, ds, \]

with \( \nabla := (\partial_x,\partial_y)' \), \( \nabla \times := (-\partial_y,\partial_x)' \), \( u : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) a continuously differentiable field, \( D \) a compact set on \( \mathbb{R}^2 \) with area \( |D| > 0 \) and smooth boundary \( \partial D \), \( n_D \) the outwards unitary vector on \( \partial D \) and \( n_D^\perp \) is the unitary vector which is perpendicular (counterclockwise) to \( n_D \).

The quantities obtained by normalizing the right-hand side of (1) and (2) by \(|D|\) are termed the 2-dimensional mean circulation and mean flux, respectively. In fluid mechanics, the mean circulation measures the degree of rotation and the mean flux measures the degree of incompressibility. If we let \( u \) be the 2-dimensional velocity field of a streaming fluid and \( D \) a disk, then the mean circulation and the mean flux will measure the movement of the fluid along and through the region \( D \), respectively. The more the fluid is aligned to \( \partial D \) (the larger the mean circulation), the more the motion is of rotational type. A large positive (negative) mean flux describes the situation where more (less) fluid is entering \( D \), which implies that the density of the fluid is increasing (decreasing). Hence, when the radius of \( D \) is small, the mean circulation and mean flux quantify the pointwise rotation/vorticity and the pointwise change of density of the fluid, respectively. The concept of incompressibility expresses the fact that the density of the fluid is constant and this property is a common assumption in many turbulence studies. Likewise, vorticity and the related concept of vortex merging is believed to be a main dynamic process for 2-dimensional turbulent flows. See for instance [15] and [39]. Having in mind application to turbulence modeling, it is therefore crucial to understand the behavior of the limits

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The main goal of this paper is to study the limits appearing in (3) for a certain class of non-smooth random fields belonging to the family of ambit fields. The class of ambit fields was introduced originally in [8] as a potential way to study the velocity field in a turbulent flow. A distinctive characteristic of the ambit stochastics approach, which distinguishes this from others, is that it specifically incorporates additional inputs referred to as volatility or intermittency. Another special feature is the presence of ambit sets that delineate which part of space-time may influence the value of the field at any given point in space-time. More specifically, a random field \( Y \) is said to be an ambit field if it admits the following dynamics

\[
Y_t(p) = \mu + \int_{A_t(p)} F(t, s, p, q)\sigma_s(q)L(dsdq) + \int_{B_t(p)} G(t, s, p, q)\chi_s(q)dtdq,
\]

where \( t \) denotes time while \( p \) gives the position in \( d \)-dimensional Euclidean space. Further, \( A_t(p) \) and \( B_t(p) \) are subsets of \( \mathbb{R} \times \mathbb{R}^d \), termed ambit sets, \( F \) and \( G \) are deterministic weight functions, and \( \sigma \) and \( \chi \) are stochastic fields. Finally, \( L \) denotes a Lévy basis (i.e., an independently scattered and infinitely divisible random measure). For surveys on ambit fields and their relation to turbulence modeling, we refer to [25], [4], [4] and reference therein. In this paper, we will focus on purely spatial stationary ambit fields of the form

\[
Y(p) = \int_{\mathbb{R}^d} F(p - q)V(q)L(dq),
\]

with \( F \) a vector-valued function, \( \mathcal{R} \) a compact set in \( \mathbb{R}^2 \), \( V \) a real-valued measurable random field, and \( L \) a real-valued homogeneous Lévy basis.

The null-space version of ambit fields are called Lévy semistationary processes (LSS for short) which are stochastic processes on a filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{R}}, \mathbb{P})\) that are described by the formula

\[
Y_t = \theta + \int_{-\infty}^t q(t-s)\sigma_sds + \int_{-\infty}^t q(t-s)\sigma_sds, \quad t \in \mathbb{R},
\]

where \( \theta \in \mathbb{R} \), \( L \) is a Lévy process, \( g \) and \( q \) are deterministic functions such that \( g(x) = q(x) = 0 \) for \( x \leq 0 \), and \( \sigma \) and \( a \) are adapted processes. When \( L \) is a two-sided Brownian motion, \( Y \) is called a Brownian semistationary process (BSS). As further references to theory and applications of LSS, see for instance [33], [6], [35] and therein references. For recent results on limit theorems see [19] and [10].

The relations (1) and (2) are particular cases of Stokes’ Theorem which, in its standard form, is stated for smooth manifolds. We refer to [31] for an interesting historical review on this topic. Several extensions of Stokes’ Theorem can be found in the literature, mainly those involving surface/line integrals of smooth forms over non-smooth regions, e.g. fractals or paths of stochastic processes. See for instance [26], [28] and therein references. A non-stochastic approach is described in [23], where the authors proved a version of Stokes’ Theorem for non-smooth manifolds. This is done by introducing a certain type of surface/line integral of smooth forms over what is called chainlets, which turned out to be a general class of regions that contains, among others, smooth sub-manifolds, fractals and vector fields. In contrast, very little has been done in the other direction, i.e. to consider line/surface integrals of non-smooth forms over smooth manifolds. This paper intends to develop some results in that direction. To our knowledge, the only existing work in relation to non-smooth forms is [44], in which the author, by employing Young’s approach (see [43]), introduced an integral for non-smooth forms over Lipschitz manifolds. However, the stated version of Stokes’ Theorem requires the form to be constant.

The organization of the present work is as follows: In Section 2 we introduce the basic notations as well as the basic assumptions. We also recall several results and concepts related to stable convergence of r.v.s, Lévy bases and infinite divisibility. We further give some geometrical preliminaries. Our main results, concerning the asymptotic behavior of the flux and circulation, is stated in Section 3. Specifically, we show that under proper normalization, the flux and the circulation of a purely spatial stationary ambit field converge stably to certain random fields that are defined in terms of a separable Lévy basis whose control measure is the 1-dimensional Hausdorff measure. We postpone their proof to Section 5. As an application of our results, we introduce in Section 4 a class of purely spatial and \( \mathbb{R}^2 \)-valued ambit fields which have stationary and isotropic increments. Such a family of fields was originally introduced jointly with Ole E. Barndorff-Nielsen and Jürgen Schmiegel as a potential modeling framework for 2-dimensional turbulent flows. Moreover, these fields are rotational and have the property of incompressibility. We
also include two appendixes. Appendix A provides a Steiner-type formula for closed sets, and briefly describes the convergence of stochastic integrals with respect to Lévy bases. Appendix B focus on technical results that are used in the proof of our main results.

2 Preliminaries and basic notation

This part is devoted to introduce the basic notations as well as to recall several basic results and concepts that will be used through this paper.

2.1 Stable convergence

For the rest of this paper we will consider $(\Omega, \mathcal{F}, \mathbb{P})$ to be a complete probability space. As usual, the notation $\xrightarrow{p}$ means convergence in probability and the notation $X_n = o_p(Y_n)$ means that $X_n/Y_n \xrightarrow{p} 0$ when $n \to \infty$. Given a sub-$\sigma$-field $\mathcal{G} \subseteq \mathcal{F}$ and a sequence of random variables (r.v.’s for short) $(\xi_n)_{n \geq 1}$ on $(\Omega, \mathcal{F}, \mathbb{P})$, the notation $\xi_n \xrightarrow{\mathcal{G}} \xi$ will mean that as $n \to \infty$, $\xi_n$ converges $\mathcal{G}$-stably in distribution towards a random variable (r.v. for short) $\xi$ (defined possibly on an extension of $(\Omega, \mathcal{F}, \mathbb{P})$), that is, for any $F \subseteq \mathcal{G}$, with $\mathbb{P}(F) > 0$, conditioned on the event $F$, $\xi_n \xrightarrow{\mathcal{G}} \xi$, weakly. In the same framework, if $(X_n(\mathbf{p}))_{\mathbf{p} \in \mathbb{R}^d, \mathbf{n} \in \mathbb{N}}$ is a family of random fields, we will write $X_n \xrightarrow{\mathcal{G}} X$ if the finite-dimensional distributions (f.d.d. for short) of $X$ converge $\mathcal{G}$-stably toward the f.d.d. of $X$. We refer the reader to [24] for a concise exposition of stable convergence.

2.2 Lévy bases and infinite divisibility

The symbols $D_r(p)$ and $rS^{d-1}(p)$ will denote the closed disk and the sphere with center $p$ and radius $r$. When $p = 0$, we will just write $D_r$ and $rS^{d-1}$ instead of $D_r(0)$ and $rS^{d-1}(p)$, respectively. For any $A \subseteq \mathbb{R}^d$, we let $-A = \{-x : x \in A\}$. Furthermore, we denote by $A, \overline{A}, \partial A$ and $A^c$ the interior, the closure, the boundary and the complement of $A$, respectively and we put $A^* = \overline{A}^c$. The inner product and the norm of vectors $x, y \in \mathbb{R}^d$ will be represented by $x \cdot y$ and $\|x\|$, respectively. Let $\mu$ be a measure on $B(\mathbb{R}^d)$, the Borel sets on $\mathbb{R}^d$, and let $\mathcal{B}_0^c(\mathbb{R}^d) := \{A \in \mathcal{B}(\mathbb{R}^d) : \mu(A) < \infty\}$. The family $L = \{L(A) : A \in \mathcal{B}_0^c(\mathbb{R}^d)\}$ of real-valued r.v.’s will be called a Lévy basis if it is an infinitely divisible (ID for short) independently scattered random measure, that is, $L$ is $\sigma$-additive almost surely and such that for any $A, B \in \mathcal{B}_0^c(\mathbb{R}^d)$, $L(A)$ and $L(B)$ are ID r.v.’s that are independent whenever $A \cap B = \emptyset$. The cumulant of a r.v. $\xi$, in case it exists, will be denoted by $C(z \upharpoonright \xi) := \log \mathbb{E}(e^{iz\xi})$. We will say that $L$ is separable with control measure $\mu$, if

$$C(z \upharpoonright L(A)) = \mu(A)\psi(z), \ A \in \mathcal{B}_0^c(\mathbb{R}^d), z \in \mathbb{R},$$

where

$$\psi(z) := i\gamma z - \frac{1}{2}b^2 z^2 + \int_{\mathbb{R}\setminus\{0\}} (e^{ixz} - 1 - i\pi x1_{|x|\leq 1})\nu(dx), \ z \in \mathbb{R},$$

with $\gamma \in \mathbb{R}$, $b \geq 0$ and $\nu$ is a Lévy measure, i.e. $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}\setminus\{0\}} (1 \wedge |x|^2)\nu(dx) < \infty$. When $\mu = \text{Leb}$, in which $\text{Leb}$ represents the Lebesgue measure on $\mathbb{R}^d$, $L$ is called homogeneous. The ID r.v. associated to the characteristic triplet $(\gamma, b, \nu)$ is called the Lévy seed of $L$ and will be denoted by $L'$. As usual, $(\gamma, b, \nu)$ will be called the characteristic triple of $L$ and $\psi$ its characteristic exponent. In this paper, the sigma field generated by $L$ is denoted by $\mathcal{F}_L$.

For any Lévy measure $\nu$, we associate the functions $\nu^+ : (0,1) \to \mathbb{R}^+$, defined as $\nu^+(x) := \nu(x, \infty)$ and $\nu^-(x) := \nu(-\infty,-x)$. Let $K_+, K_- \geq 0$ and $0 < \beta \leq 2$. A separable Lévy basis is called strictly $\beta$-stable with parameters $(K_+, K_-, \beta, \gamma)$ if its Lévy seed is distributed according to a strictly $\beta$-stable distribution, that is, $L'$ is Gaussian if $\beta = 2$, while for $\beta < 2$ the characteristic triplet of $L'$ has no Gaussian component ($b = 0$), its Lévy measure satisfies

$$\frac{\nu(dx)}{dx} = K_+ |x|^{-1-\beta}1_{\{x>0\}} + K_- |x|^{-1-\beta}1_{\{x<0\}},$$

and $\gamma = (K_+-K_-)/(1-\beta)$ if $\beta \neq 1$, and $\gamma$ arbitrary with $K_+ = K_-$ when $\beta = 1$. 
2.3 Geometrical preliminaries

Fix $A \subseteq \mathbb{R}^d$ a closed set and denote by $\mathcal{H}^n$ the $n$th-dimensional Hausdorff measure. The normal cone of $A$ at $p \in A$ is defined as

$$\text{nor}(A,p) := \{ u \in \mathbb{R}^d : u \cdot v \leq 0, \forall v \in \text{Tan}(A,p) \},$$

where $\text{Tan}(A,p)$ denotes the set of all tangent vectors to $A$ at $p$, that is, $v \in \text{Tan}(A,p)$ if and only if there is a sequence $(p_n) \subseteq A \setminus \{p\}$ such that $p_n \rightarrow p$ and $\frac{p_n-p}{\|p_n-p\|} \rightarrow \frac{v}{\|v\|}$ as $n \rightarrow \infty$. We recall that a Jordan curve $\emptyset \neq C \subseteq \mathbb{R}^d$ is a curve in $\mathbb{R}^d$ parametrized by $\varphi : [0,1] \rightarrow C$, such that $\varphi$ is continuous and injective on $(0,1)$ and $\varphi(0) = \varphi(1)$. We will say that a compact set $A \subseteq \mathbb{R}^2$ is a Jordan domain if $A$ is totally connected and $\partial A \neq \emptyset$ is a Jordan curve.

In this framework, we say that a Jordan domain $A$ has Lipschitz-regular boundary if the parametrization of $\partial A$ is Lipschitz and for $\mathcal{H}^2$-a.a. $q \in \partial A$ there is $u_A(q) \in \mathbb{S}^{d-1}$ orthogonal to $\text{Tan}(\partial A,q)$, such that

$$\text{nor}(A,q) = \{ \lambda u_A(q) : \lambda \geq 0 \}; \quad \text{nor}(A^c,p) := \{ -\lambda u_A(q) : \lambda \geq 0 \}.$$

In other words, a Jordan domain with Lipschitz boundary is regular if it has unique outwards and inwards unit vectors almost everywhere.

The metric projection on $A$, $\Pi_A : \mathbb{R}^d \rightarrow A$, is the set function

$$\Pi_A(q) := \{ p \in A : d_A(q) = \|p-q\| \},$$

where $d_A(q) := \inf_{p \in A} \|p-q\|$. We set

$$\text{Unp} A := \{ q \in \mathbb{R}^d : \exists p \in A \text{ s.t. } d_A(q) = \|p-q\| \}.$$

Under the previous notation, the reduced normal bundle and the reach function of $A$ are given, respectively, by

$$\mathcal{N}(A) = \{ (\Pi_A(q), q - \Pi_A(q)\|q - \Pi_A(q)\|) : q \in \text{Unp}(A) \setminus A \},$$

and, $\delta_A(q,u) := 0$ for $(q,u) \in N(A)^c$ and for $(q,u) \in N(A)$

$$\delta_A(q,u) := \inf \{ t \geq 0 : q + tu \in \text{Unp}(A)^c \}.$$

For $r \geq 0$, the $r$-parallel set of $A$ is defined as

$$A_{\perp r} := \{ q \in \mathbb{R}^d : d_A(q) \leq r \}.$$

3 Divergence and Vorticity Theorems for Ambit fields

For the rest of this section we will be interested in the asymptotic behavior of the following functionals

$$\mathcal{E}_r(p; X) := \int_{r \mathbb{S}^1(p)} X(q) \cdot dq = r \int_0^{2\pi} X(p + ru(\theta)) \cdot u^\perp(\theta) d\theta, \quad p \in \mathbb{R}^2, \ r > 0, \quad (5)$$

$$\mathcal{D}_r(p; X) := \int_{r \mathbb{S}^1(p)} X(s) \cdot n(s) ds = r \int_0^{2\pi} X(p + ru(\theta)) \cdot u(\theta) d\theta, \quad p \in \mathbb{R}^2, \ r > 0, \quad (6)$$

as $r \downarrow 0$. Above $X$ is a vector-valued random field, $n$ is the outward unit vector in $r \mathbb{S}^1(p)$, that is, $n(p + ru(\theta)) = u(\theta)$, with $u(\theta) := (\cos(\theta), \sin(\theta))'$ for $0 \leq \theta \leq 2\pi$, and $(x, y)^\perp = (-y, x)$.

When the mapping $p \mapsto X(p)$ is smooth almost surely, by the usual Stokes’ Theorem, it holds that

$$\lim_{r \downarrow 0} \frac{1}{2\pi r^2} \mathcal{E}_r(p; X) \xrightarrow{a.s.} \nabla^\perp \cdot X(p), \quad p \in \mathbb{R}^2;$$

$$\lim_{r \downarrow 0} \frac{1}{2\pi r^2} \mathcal{D}_r(p; X) \xrightarrow{a.s.} \nabla \cdot X(p), \quad p \in \mathbb{R}^2.$$

where $\nabla := (\partial_x, \partial_y)'$ and $\nabla^\perp := (-\partial_y, \partial_x)'$. However, not surprisingly, such a result does not hold anymore when one consider fields of the form of $\mathcal{E}_r$. Furthermore, as expected, when the kernel $F$ is smooth enough, the rates of convergence for $\mathcal{E}_r$ and $\mathcal{D}_r$ depend entirely on the ambit set and background driving Lévy basis. Before presenting our main results we first explain the intuition behind them.
3.1 Some intuitive description

Previously, we mentioned that when the kernel involved in the definition of \( [\text{4}] \) is smooth, then the asymptotic behavior of \( \zeta_t \) and \( \bar{\mathcal{R}}_t \) would be determined by the background driving Lévy basis and the ambit set. In this subsection we will give an intuitive description of why this would be the case. As a motivation and starting point in our analysis, let us first describe what is known in the one-dimensional case. For \( s_0 > 0 \) and \( t \in \mathbb{R} \), let \( \mathcal{R}(t) := [-s_0 + t, t] \) and put

\[
X_t := \int_{\mathcal{R}(t)} f(t-s) dL_s,
\]

where \( L \) denotes a Lévy process on \( \mathbb{R} \) with characteristic triplet \((\gamma, b, \nu)\), and \( f \) a real-valued function. In this case for \( r > 0 \), we get that

\[
\int_{r \mathbb{S}^0(t)} X_s \cdot ds = X_{t+r} - X_{t-r} = \int_{r \mathbb{S}^0(t)} \partial X_s \cdot ds + \int_{r \mathbb{S}^0(t)} \hat{X}_s \cdot ds,
\]

where

\[
\partial X_t := f(0)L_t - f(s_0)L_{t-s_0}; \hat{X}_t := X_t - \partial X_t.
\]

The notation \( \partial X, \hat{X} \) is not by chance, many properties of \( \partial X \) and \( \hat{X} \) are completely determined by the interaction of \( f \) and \( L \) on \( \partial \mathcal{R}(t) \) and \( \hat{\mathcal{R}}(t) \), respectively. We first observe that \( \hat{X} \) admits the representation

\[
\hat{X}_t = \int_{\mathcal{R}(t)} g(t-s) dL_s,
\]

where \( g \) is absolutely continuous and with \( g(\cdot) \big|_{\partial \mathcal{R}} \equiv 0 \), and that

\[
\int_{r \mathbb{S}^0(t)} \partial X_s \cdot ds = \int_{(\partial \mathcal{R}(t))_{(0)}} h(t-s) dL_s,
\]

for some measurable function \( h \). Additionally, we have that if \( f \) is continuously differentiable, then the path \( t \mapsto \hat{X}_t \) is almost surely absolutely continuous (see \[13], cf. \[16] and \[12] for more details) in such a way that

\[
\frac{1}{2r} \int_{r \mathbb{S}^0(t)} \hat{X}_s \cdot ds \quad \overset{p}{\rightarrow} \quad \int_{\mathcal{R}(t)} f'(t-s) dL_s, \quad t \geq 0.
\]

On the other hand, \( \hat{f}_{r \mathbb{S}^0(t)} \partial X_s \cdot ds \) consists of the increments of size \( r > 0 \) of \( L \) around \( \partial \mathcal{R}(t) \). Therefore, under proper normalization, \( \hat{f}_{r \mathbb{S}^0(t)} \partial X_s \cdot ds \) has a non-trivial limit if and only if the same property holds for the increments of \( L \). In connection to the former, it is well known that when \( b > 0 \), the increments of \( L \) are totally dominated by its Gaussian component. Moreover, if \( L \) is of bounded variation, then the increments of \( L \) are totally dominated by the drift component. For these facts we refer to \[14\] p. 16). Ultimately, when \( b = 0 \) and \( L \) is of unbounded variation, typically the increments of \( L \) are in the domain of attraction of a strictly \( \beta \)-stable distribution with \( L_{t+r} - L_{t-r} = O_p(r^{1/\beta}) \), for some \( 1 \leq \beta < 2 \). All in all then give us the following asymptotics for \( \hat{f}_{r \mathbb{S}^0(t)} X_s \cdot ds \)

1. **Gaussian regime:** If \( f \big|_{\partial \mathcal{R}} \neq 0 \) and \( b > 0 \), then \( \hat{f}_{r \mathbb{S}^0(t)} X_s \cdot ds = O_p(r^{1/2}) \) with Gaussian limit.

2. **Stable regime:** If \( f \big|_{\partial \mathcal{R}} \neq 0 \), \( b = 0 \), \( L \) is of unbounded variation, then \( \hat{f}_{r \mathbb{S}^0(t)} X_s \cdot ds = O_p(r^{1/\beta}) \) with strictly \( \beta \)-stable limit, for some \( 1 \leq \beta < 2 \).

3. **“Classical” regime:** If \( L \) is of bounded variation or if \( f \big|_{\partial \mathcal{R}} = 0 \) with arbitrary \( L \), then \( \hat{f}_{r \mathbb{S}^0(t)} X_s \cdot ds = O_p(r) \).

As a final remark for the one-dimensional framework, we note that when \( f \) is not smooth enough, the regimes previously stated are not valid anymore. For this situation, we refer to \[19\], \[11\] and references therein.

Now, when we consider the ID field given by

\[
X(p) := \int_{\mathcal{R} + p} F(p - q)L(dq),
\]
with $F$ a vector valued function, $\mathcal{R}$ a compact set in $\mathbb{R}^2$ and $L$ a real-valued homogeneous Lévy basis, we may in principle try to follow the same reasoning as in the temporal case and expect to recover similar results. Thus, we may try to decompose $X$ as

$$X(p) = \partial X(p) + \hat{X}(p),$$  \hspace{1cm} (10)

for some fields $\partial X$ and $\hat{X}$ whose trajectories are totally determined by the behavior of $L$ and $F$ on $\partial \mathcal{R}(p)$ and $\mathcal{R}(p)$, respectively. Unfortunately, to our knowledge, there is no a general identification of the fields $\partial X$ and $\hat{X}$ appearing in (10), except in very special cases (see for instance [17]). However, our proof is based on an analogous decomposition to that in (7). Specifically, we decompose

$$\int_{\mathcal{R}(p)} X \cdot nds = \int_{(\partial \mathcal{R}(p))_{\partial r}} H(p - q)L(dq) + E_r(p),$$  \hspace{1cm} (11)

for a certain smooth function $H$ and $E_r(p)$ a random field satisfying that as $r \downarrow 0$

$$\frac{1}{|D_r(p)|} E(p) \rightarrow \int_{\mathcal{R}(p)} \nabla \cdot F(p - q)L(dq).$$

Therefore, as in the one-dimensional case, understanding the asymptotic behavior of $\int_{\mathcal{R}(p)} X \cdot nds$ requires a full knowledge of the asymptotic behavior of $L$ on the $r$-parallel sets of $\mathcal{R}(p)$.

### 3.2 The divergence and vorticity theorems

In this part we present our main results about $\mathcal{C}_r$ and $\mathcal{D}_r$ for the stationary ID field

$$X(p) := \int_{\mathcal{R} + p} F(p - q)L(dq), \hspace{0.5cm} p \in \mathbb{R}^2,$$  \hspace{1cm} (12)

with $F$ continuously differentiable on $-\mathcal{R}$ and $L$ a real-valued homogeneous Lévy basis with characteristic triplet $(\gamma, b, \nu)$. In the next subsection we will study the case in which an additional stochastic field $V$ is included in (12).

For the rest of this paper we will be working under the following assumption on the ambit set $\mathcal{R}$.

**Assumption 1.** The ambit set $\mathcal{R} \subseteq \mathbb{R}^2$ can be written as

$$\mathcal{R} = \mathcal{R}_1 \setminus \bigcup_{i=2}^n \mathcal{R}_i,$$

where $\mathcal{R}_1, \ldots, \mathcal{R}_n$ are Jordan domains with Lipschitz-regular boundary satisfying that $\mathcal{R}_i \cap \mathcal{R}_j = \emptyset$ and $\mathcal{R}_i \subset \mathcal{R}_1$ for $i, j = 2, \ldots, n$ and $i \neq j$. Furthermore, for $i = 1, \ldots, n$

$$\int_{\mathbb{R}^2} \left\{ \sum_{q:(q,u) \in N(\mathcal{R}_i)} (\delta_{\mathcal{R}_i}(q, u) \wedge 1) + \sum_{q:(q,u) \in N(\mathcal{R}_i^c)} (\delta_{\mathcal{R}_i^c}(q, u) \wedge 1) \right\} \mathcal{H}^1(du) < \infty.$$

Some examples of sets satisfying Assumption 1 will be presented in the next section. However, the type of sets we have in mind can be visualized in Figure 1.

**Remark 1.** Assumption 1 allows to extended the definition of $u_\mathcal{R}$ (the outward unit vector) to the whole $\partial \mathcal{R}$ by letting $u_\mathcal{R}(q) \equiv 0$ in the irregular points of $\partial \mathcal{R}(p)$. Furthermore, since $\mathcal{R}(p)$ is just a translation of $\mathcal{R}$, we get that $\mathcal{R}(p)$ satisfies Assumption 1 if and only if $\mathcal{R}$ does and in this case $u_{\mathcal{R}(p)}(q + p) = u_\mathcal{R}(q)$ for $q \in \partial \mathcal{R}$.

As in Subsection 3.1 our analysis will be divided into three different scenarios. For the sake of exposition, all the proofs of this section will be postponed to Section 6. Let us start with the case in which the Gaussian part of $L$ dominates the asymptotics. In order to improve the presentation of this result let us first introduce the limiting fields. Recall the notation $(x,y)^\perp = (-y,x)$. Given $b > 0$, $F$ as above and $\mathcal{R}$ as in Assumption 1 the fields $\{\mathcal{C}_W(p; F, \mathcal{R}), \mathcal{D}_W(p; F, \mathcal{R})\}_{p \in \mathbb{R}^2}$ will denote a collection of stationary Gaussian fields defined for any $p \in \mathbb{R}^2$ as

$$\mathcal{D}_W(p; F, \mathcal{R}) := \int_{\partial \mathcal{R}(p)} F(p - q) \cdot u_\mathcal{R}(p)(q)W_{\mathcal{H}^1}(dq),$$

$$\mathcal{C}_W(p; F, \mathcal{R}) := \int_{\partial \mathcal{R}(p)} F(p - q) \cdot u_\mathcal{R}^\perp(p)(q)W_{\mathcal{H}^1}(dq).$$
where \( u_R(p) \) is as in Remark 1. Furthermore, \( W_{\mathcal{H}^1} \) is a separable Gaussian Lévy basis (see Subsection 2.2) defined on an extension of \((\Omega, \mathcal{F}, \mathbb{P})\) having the following properties: 1) Its Lévy seed has a centered Gaussian distribution with variance \( b^2 \); 2) Its control measure is \( \mathcal{H}^1 \), the 1-dimensional Hausdorff measure; 3) \( W_{\mathcal{H}^1} \) is independent of \( L \).

In this setting, using the notation \( v_\beta := 2 \left\{ \int_{-\infty}^{\gamma_1} (1 - s^2)^{\beta/2} ds \right\}^{1/\beta} \), we have that:

**Theorem 1 (Gaussian attractor).** Let \( R \subset \mathbb{R}^2 \) be as in Assumption 1. Consider \( X \) as in (12) with \( F|_{\partial R} \neq 0 \). If \( b > 0 \), then as \( r \downarrow 0 \)

\[
\frac{1}{v_\beta r^{1+1/2}} \mathcal{C}_r(p; X) \overset{p}{\to} \mathcal{C}_r(p; b, F, R);
\]

\[
\frac{1}{v_\beta r^{1+1/2}} \mathcal{D}_r(p; X) \overset{p}{\to} \mathcal{D}_r(p; b, F, R).
\]

On the other hand, as expected, when \( L \) is of bounded variation the rate of convergence for \( \mathcal{C}_r \) and \( \mathcal{D}_r \) are the classical ones, i.e. of order \( r^2 \). More precisely:

**Theorem 2 ("Classical" regime).** Let \( R \subset \mathbb{R}^2 \) be as in Assumption 1 and \( X \) as in (12). Then the following convergence holds as \( r \downarrow 0 \)

\[
\frac{1}{\pi r^2} \mathcal{C}_r(p; X) \overset{p}{\to} \omega(p), \quad p \in \mathbb{R}^2;
\]

\[
\frac{1}{\pi r^2} \mathcal{D}_r(p; X) \overset{p}{\to} \sigma(p), \quad p \in \mathbb{R}^2,
\]

if one of the following (not-necessarily mutually exclusive) cases holds:

i) \( b = 0 \) and \( \int_{\mathbb{R}} (1 \wedge |x|) \nu(dx) < \infty \);

ii) \( F|_{\partial R} = 0 \);

where the limiting processes are defined as

\[
\omega(p) := \int_{\mathbb{R}^d} \nabla_\perp \cdot F(p - q) \tilde{L}(dq),
\]

\[
\sigma(p) := \int_{\mathbb{R}^d} \nabla \cdot F(p - q) \tilde{L}(dq).
\]

with \( \nabla := (\partial_x, \partial_y)' \), \( \nabla_\perp := (-\partial_y, \partial_x)' \), \( \tilde{L} = L - \gamma_d \text{Leb} \), where \( \gamma_d = \gamma - \int_{|x| \leq 1} x \nu(dx) \) for case i) while \( \gamma_d = \gamma \) for case ii) and iii).

There is another situation in which the classical rate appears. Before presenting this last case we introduce the limiting fields: For \( 0 < \beta < 2 \), let \( (K_+, K_-, \beta, \gamma) \) be the parameters of a strictly \( \beta \)-stable r.v. (see Subsection
We proceed now to make some remarks about the previous theorems.

**Remark 2.** In Theorems 1 and 3, the limiting fields possess very irregular path properties. For example, if \( \partial \mathcal{R} \) is strictly convex, then necessarily for any \( p \in \mathbb{R}^2 \), as \( r \downarrow 0 \)

\[
\frac{1}{v_{\beta}r^{1+1/\beta}} \mathcal{C}_r(p; X) \xrightarrow{F.d.} \mathcal{C}_\beta(p; F, \mathcal{R}), \quad p \in \mathbb{R}^2;
\]

\[
\frac{1}{v_{\beta}r^{1+1/\beta}} \mathcal{D}_r(p; X) \xrightarrow{F.d.} \mathcal{D}_\beta(p; F, \mathcal{R}), \quad p \in \mathbb{R}^2,
\]

where \( \hat{\gamma} = \gamma - \text{PV} \int_{-1}^{1} x \nu(dx) \), and \( \omega \) and \( \sigma \) as in Theorem 2.

We proceed now to make some remarks about the previous theorems.

**Remark 3.** By the previous remark, we have that the convergence in Theorems 1 and 3 cannot in general be strengthened to functional convergence. Moreover, since the convergence is stable and the limit is independent of the background driving Lévy basis, we deduce that the convergence cannot take place in probability either.

**Remark 4.** The rates of convergence for \( \mathcal{C}_r \) and \( \mathcal{D}_r \) can be seen as an \( L^\beta \) norm of a certain parametrization of a disk. Indeed, let \( D_r(p) \) be a disk of radius \( r > 0 \) and center \( p \), and put \( g(s, \rho; \beta) := 2\sqrt{1 - s^2(1+\beta)/2}\rho \) for \( 1 \leq \beta \leq 2 \).

Then

\[
|D_r(p)| = \int_{-r}^{r} \int_{-1}^{1} |g(s, \rho; 1)| \, ds \, d\rho.
\]

Moreover

\[
r^{1+1/\beta} v_{\beta} = \left( \int_{-r}^{r} \int_{-1}^{1} |g(s, \rho; \beta)|^{1/\beta} \, ds \, d\rho \right)^{1/\beta},
\]

so \( r^{1+1/\beta} v_{\beta} \) can be thought as an \( L\beta \) norm of \( g \).

---

1 Recall that the Cauchy principal value of an integral around 0 is defined as the limit (in case it exists)

\[
\text{PV} \int_{-1}^{1} f(x) \nu(dx) := \lim_{a \to 0} \left[ \int_{-1}^{-a} f(x) \nu(dx) + \int_{a}^{1} f(x) \nu(dx) \right].
\]
3.3 Examples

To clarify the results and the assumptions of Theorems 13 in this part we present several examples.

Sets with positive reach

Let $A$ be a Jordan domain such that $\inf_{N(\partial A)} \delta_{\partial A} > 0$. Then the integrability condition in Assumption 1 is satisfied. Indeed, we have in particular that $\delta_A$ and $\delta_A^*$ are bounded from below by, let’s say $\varepsilon > 0$. Thus, according to Theorems 7 and 8

$$\int_{\mathbb{S}^1} \sum_{q(u,v)\in N(A)} (\delta_A(q,u) \wedge 1)\mathcal{H}^i(du) \leq \int_{\mathbb{S}^1} \sum_{q(u,v)\in N(A)} 1_{\delta_A(q,u) \geq \varepsilon}\mathcal{H}^i(du) < \infty.$$ 

Our claim follows by replacing $A$ by $A^*$ in the previous equation. Sets with the property $\inf_{N(\partial A)} \delta_{\partial A} > 0$ are known as sets with positive reach, see [21] for more details. It was shown in [38] that simple curves have positive reach if and only if are of class $C^{1,1}$, i.e. differentiable with Lipschitz derivative. Therefore, Jordan domains with boundary of positive reach satisfy Assumption 1.

Piecewise $C^{1,1}$ curves

Let $A$ be a Jordan domain whose boundary is piecewise $C^{1,1}$. This class of sets have indeed Lipschitz-regular boundary. However, as we saw above, $\partial A$ cannot have positive reach. Actually if $q_0 \in \partial A$ is a corner then necessarily $\delta_{\partial A}(q_0, v) = 0$ for any $v \in \mathbb{S}^1$. Nevertheless, Assumption 1 remains valid in this case. To see this, for simplicity assume that there is only one corner, say $q_0 \in \partial A$. We can find $\rho > 0$, such that the points in $D_{\rho}(q_0) \cap \partial A$ has null-curvature, i.e. two straight lines intersecting in $q_0$. Then outside of $D_{\rho}(q_0) \cap \partial A$, $\delta_{\partial A}$ is bounded (Corollary 8.9 in [38]) from below and there are $u_1, u_2 \in \mathbb{S}^1$ such that

$$\mathcal{H}^0\{q \in D_{\rho}(q_0) \cap \partial A \setminus \{q_0\} : (q, u) \in N(A)\} = 0, \quad u \in \mathbb{S}^1 \setminus \{u_1, u_2\}.$$ 

The integrability condition in Assumption 1 follows from these observations. More generally, Jordan domains with piecewise $C^{1,1}$ boundary are within Assumption 1.

Stable distributions

Let $L$ be a homogeneous Lévy basis with characteristic triplet $(\gamma, b, \nu)$. Assume that the Lévy seed $L'$ has a $\beta$-stable distribution for $0 < \beta \leq 2$. Thus if $\beta = 2$, we have that $L'$ is a Gaussian r.v. with mean $\gamma$ and variance $b^2$ meaning that $L$ satisfies the assumptions of Theorem 1. On the other hand, if $\beta < 2$

$$\nu^{\pm}(x) = \beta^{-1}K_{\pm}x^{-\beta}, \quad x > 0,$$

meaning that, as $x \downarrow 0$, $\nu^{\pm}(x) \sim \beta^{-1}K_{\pm}x^{-\beta}$. Consequently, for $1 < \beta < 2$, $L$ is within the framework of Theorem 3i., while for $\beta = 1$, $L$ satisfies the assumptions of Theorem 3ii. if and only if $K_+ = K_-$. Furthermore, for $\beta < 1$, $L$ fulfills the requirements of Theorem 2.

Generalized Hyperbolic distributions

The family of Generalized hyperbolic distributions, originally introduced in [2], constitutes a rich class of infinitely divisible normal mean-variance mixture distributions. A r.v. $\xi$ is said to have generalized hyperbolic distribution with parameters $\lambda, \mu \in \mathbb{R}$, $\delta > 0$ and $0 \leq |\theta| < \alpha$, and we write $\xi \sim GH(\lambda, \alpha, \theta, \delta, \mu)$, if for $u \in \mathbb{R}$

$$C(u \uparrow \xi) = \mu \theta\delta \left\{ \frac{K_{\lambda}(\delta \sqrt{\alpha^2 - \theta^2})}{\sqrt{\alpha^2 - \theta^2}K_{\lambda}(\delta \sqrt{\alpha^2 - \theta^2})} \right\} iu$$

$$+ \int_{\mathbb{R}\setminus\{0\}} (e^{iux} - 1 - iux)v_{GH(\lambda, \alpha, \theta, \delta, \mu)}(x)dx,$$

where $K_\zeta$ denotes the modified Bessel function of second kind with index $\zeta$ and

$$v_{GH(\lambda, \alpha, \theta, \delta, \mu)}(x) = \frac{e^{\theta x}}{|x|}k_{\lambda, \alpha, \delta}(|x|)1_{|x| > 0},$$

9
with \( k_{\lambda,\alpha,\delta} \) satisfying
\[
k_{\lambda,\alpha,\delta}(x) = \frac{\delta}{\pi} x^{-1} + o(x^{-1}), \quad \text{as } x \downarrow 0.
\] (14)

For a closed form of \( k_{\lambda,\alpha,\delta} \) the reader may consult [36]. This means that the Lévy measure of \( \xi \) satisfies that
\[
\int_{\mathbb{R}} (1 \wedge |x|) \nu_{\text{GH}}(\lambda,\alpha,\delta,\mu) (x) dx = +\infty \quad \text{and} \quad \nu^\pm (x) \sim \frac{\delta}{\pi} x^{-1} \quad \text{as } x \downarrow 0.
\]
Moreover, for any \( y > 0 \)
\[
\int_y^1 x \nu_{\text{GH}}(\lambda,\alpha,\delta,\mu) (x) dx + \int_{-1}^{-y} x \nu_{\text{GH}}(\lambda,\alpha,\delta,\mu) (x) dx = \int_y^1 [e^{\theta x} - e^{-\theta x}] k_{\lambda,\alpha,\delta}(x) dx,
\]
so by the Monotone Convergence Theorem and (14) we have that, as \( y \downarrow 0, \)
\[
\int_y^1 [e^{\theta x} - e^{-\theta x}] k_{\lambda,\alpha,\delta}(x) dx \to \int_0^1 [e^{\theta x} - e^{-\theta x}] k_{\lambda,\alpha,\delta}(x) dx < \infty.
\]

Thus, \( L \) satisfies the assumptions of Theorem 3.

**Non-negative Lévy bases**

Any non-negative homogeneous Lévy basis satisfies that \( b = 0, \int_\mathbb{R} (1 \wedge |x|) \nu(dx) < \infty, \) \( \nu([-\infty,0]) = 0, \) and \( \gamma_0 = \gamma = \int_0^1 x \nu(dx) \geq 0. \) For a proof of this fact we refer to [7]. In this case we have that \( L \) can be written as
\[
L(A) = \gamma_0 \text{Leb}(A) + \int_A \int_0^\infty xN(dx,dq), \quad A \in \mathcal{B}_b(\mathbb{R}^2),
\]
where \( N \) is a Poisson random measure with intensity \( \nu(dx)dq. \) Hence, any non-negative Lévy basis fulfills the condition of Theorem 2.

**Isotropic kernels**

Let \( R_\phi \) be the rotation matrix on \( \mathbb{R}^2 \) and \( f \) a continuous function. Put
\[
F_{\phi,f}(q) = f(\|q\|)R_\phi q, \quad q \in \mathbb{R}^2.
\]
In the next section we will show that when the ambit set is isotropic, meaning that it can be written as
\[
\mathcal{R} = \{ q \in \mathbb{R}^2 : h(\|q\|) \in A \},
\]
for some measurable function \( h \) and \( A \subset \mathbb{R}, \) then the ambit field of the form of \( \mathcal{R} \) induced by \( F_{\phi,f} \) and \( \mathcal{R}, \) has isotropic increments (see next section for a precise definition). Now, if we let \( h(x) = x \) and \( A = [a,b], \) with \( 0 \leq a < b, \) then \( \mathcal{R} \) is an annulus, meaning that Assumption 1 is satisfied. Moreover, if \( f(a) = f(b) = 0, \) then \( F|_{-\theta \mathcal{R}} \equiv 0, \) or in other words, the conclusion of Theorem 2 holds for any Lévy basis, whenever \( f \) is continuously differentiable on \([a,b].\) On the other hand, if we put
\[
f(x) = x^{-2}, \quad x > 0,
\]
then \( F_{\phi,f} \) is continuously differentiable on \( \mathcal{R} \) if and only if \( a > 0. \)

### 4 A class of incompressible and rotational ambit fields

The main goal of this section is to build a class of ambit fields that have homogeneous and isotropic increments as well as being rotational and having the property of incompressibility. Let us introduce the formal definition of these concepts.

**Definition 1.** An \( \mathbb{R}^2 \)-valued random field \( (Y(p))_{p \in \mathbb{R}^2} \) is said to have *homogeneous* and *isotropic* increments if respectively, the following two conditions hold
- For any \( p_0 \in \mathbb{R}^2 \) the field \( (Y(p + p_0) - Y(p))_{p \in \mathbb{R}^2} \) is stationary;
- For any \( p_0 \in \mathbb{R}^2 \) and \( \theta \in [0, 2\pi) \) we have that
  \[
  \{ R_\theta^{-1} [Y(R_\theta(p + p_0)) - Y(R_\theta p)] \}_{p \in \mathbb{R}^2} \overset{d}{=} \{ [Y(p + p_0) - Y(p)] \}_{p \in \mathbb{R}^2}.
  \]
Furthermore, we will say that it is incompressible if for any \( p \in \mathbb{R}^2 \)
\[
\lim_{r \downarrow 0} \frac{1}{\pi r^2} \int_{r^2(p)} Y \cdot n ds \rightarrow 0,
\]
and it will be called rotational if the following limit exists and it is not constantly zero
\[
\mathbb{P} - \lim_{r \downarrow 0} \frac{1}{\pi r^2} \int_{r^2(p)} Y \cdot n ds.
\]

Now, given a real-valued continuous function \( f \) and \( \phi \in [0, 2\pi) \), let
\[
F_{\phi,f}(q) := R_{\phi} q f(||q||)
\]
and consider the class of ambit fields than can be written as
\[
V
\]
where \( L \) is a homogeneous Lévy basis with characteristic triplet \((\gamma, b, \nu)\) and \( V \) a predictable real-valued field. Moreover, we let
\[
\mathcal{R} = \{ q \in \mathbb{R}^2 : h(||q||) \in A \},
\]
for some measurable function \( h \) and \( A \subset \mathbb{R} \), in such a way that \( \mathcal{R} \) is compact. As the following result shows, this family of ambit fields is well defined and has isotropic and homogeneous increments.

**Proposition 1.** Suppose that \( V \) is predictable (see Appendix A) and locally bounded. Then \( Y_{\phi,f} \) as in (15) is well defined. If in addition we have that \( V \) is bounded in \( \mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P}) \), then \( Y_{\phi,f} \) is continuous in probability. Finally, if \( V \) is independent of \( L \) and almost surely
\[
\int_{\mathcal{R}} \int_{|x V(q)| > 1} |x V(q)| \nu(dx) dq < \infty,
\]
then \( Y_{\phi,f} \) is stationary with finite first moment and possesses isotropic and homogeneous increments.

**Proof.** From Proposition 5 and its subsequent remark in Appendix A, we have that \( Y_{\phi,f} \) is well defined if and only if a.s.
\[
\int_{\mathcal{R}} \Phi_{L}^0(\|F_{\phi,f}(-q)\|) |V(p + q)| dq < \infty.
\]
Since \( V \) is locally bounded, we have that almost surely \( \|F_{\phi,f}(q)\| |V(p - q)| \leq M_p \) for any \( q \in \mathcal{R} \), for some r.v. \( M_p \) only depending on \( p \). Hence, by Lemma 2.1.5 in [10], we have
\[
\int_{\mathcal{R}} \Phi_{L}^0(\|F_{\phi,f}(q)\|) |V(p - q)| dq \leq 2 \Phi_{L}^0(M_p) L eb(\mathcal{R}) < \infty,
\]
showing this the well definiteness of \( Y_{\phi,f} \). Now, if \( V \) is bounded on \( \mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P}) \), the stochastic continuity of \( Y_{\phi,f} \) can be shown in a similar way as in the proof of Lemma 2 below.

In general, \( Y_{\phi,f} \) is stationary whenever \( V \) is stationary and independent of \( L \), so it has homogeneous increments in this situation. On the other hand if \( V \) is independent of \( L \), and for \( p_0, p_1, \ldots, p_n \in \mathbb{R}^n \), we let \( Y_{\phi,f}(\overrightarrow{p}, p_0) := \{ Y(p_i + p_0) - Y(p_i) \}_{i=1}^{n} \), then
\[
\mathbb{E}(\exp[i \langle z, Y_{\phi,f}(\overrightarrow{p}, p_0) \rangle]) = \mathbb{E}\exp \left( \int_{\mathbb{R}^2} C(\langle z, F_{\phi,f}(\overrightarrow{p}, p_0, q) \rangle V(q) dq \right),
\]
where
\[
F_{\phi,f}(\overrightarrow{p}, p_0, q) = \{ F_{\phi,f}(p_i + p_0 - q)1_{\mathcal{R} + p_i + p_0}(q) - F_{\phi,f}(p_i - q)1_{\mathcal{R} + p_i}(q) \}_{i=1}^{n}.
\]
Hence, if \( \theta \in [0, 2\pi) \) then
\[
\int_{\mathbb{R}^2} C(\langle z, R_{\theta}^{-1} F_{\phi,f}(R_{\theta} \overrightarrow{p}, p_0, q) \rangle V(q) dq = \int_{\mathbb{R}^2} C(\langle z, R_{\theta}^{-1} F_{\phi,f}(R_{\theta} \overrightarrow{p}, p_0, R_{\theta} q) \rangle V(R_{\theta} q) dq.
\]
where we have done the change of variable \( q = R_\theta q' \). But in view of
\[
R_\theta^{-1} F_{\phi,f}(R_\theta \vec{p}', p_0, R_\theta q) = F_{\phi,f}(\vec{p}', p_0, q),
\]
we get that
\[
E(\exp\{i \left(z, R_\theta^{-1} Y_{\phi,f}(R_\theta \vec{p}', p_0)\right)\}) = E(\exp\{i \left(z, F_{\phi,f}(\vec{p}', p_0, q)\right) V(R_\theta q)\|dq\})
\]
\[
= E(\exp\{i \left(z, Y_{\phi,f}(\vec{p}', p_0)\right)\}),
\]
which shows that \( Y_{\phi,f} \) has isotropic increments. Finally, assume that (16) holds. Since \( V \) is independent of \( L \), then, conditioned on \( V, Y_{\phi,f}(0) \) is an ID random variable. Suppose for a moment that almost surely \( \int_{|y|>1} \|y\| \nu_{\phi,f}(dy) < \infty \). Then \( E(\|Y_{\phi,f}(0)\| |V) < \infty \) a.s. and (see Example 25.12 in [42])
\[
E(Y_{\phi,f}(0) | V) = \gamma \int_\mathcal{R} F_{\phi,f}(-q) V(q) dq + \int_\mathcal{R} \int_{|x|>1} x F_{\phi,f}(-q) V(q) \nu(dx) dq ds,
\]
which would imply immediately the existence of \( E(Y_{\phi,f}(0)) \). Therefore, it only remains to show that almost surely \( \int_{|y|>1} \|y\| \nu_{\phi,f}(dy) < \infty \). Observe that (16) together with the stationarity and the local boundedness of \( V \), implies that there is \( \Omega \in \mathcal{F} \) with \( P(\Omega) = 1 \), such that for any \( \omega \in \Omega \), \( \int_{|x|V(q)(\omega)>1} |x| \nu(dx) < \infty \) for almost all \( q \in \mathcal{R} \), and \( 0 < \sup_{\mathcal{R}} |V(q)| (\omega) \leq M(\omega) < \infty \) for some \( M(\omega) > 0 \). Thus, for \( \omega \in \Omega \)
\[
\int_{|y|>1} \|y\| \nu_{\phi,f}(dy)(\omega) \leq M(\omega) M_{F} Leb(\mathcal{R}) \int_{|x|M_{F}|M_{F}|>1} |x| \nu(dx) < \infty,
\]
where \( 0 < M_{F} = \sup_{\mathcal{R}} \|F_{\phi,f}(-q)\| < \infty \). This concludes the proof. \( \square \)

**Remark 5.** Observe that, besides the isotropy of the increments, all the stated conclusions of the previous proposition are valid for the class of ambit fields of the form of
\[
Y(p) = \int_{\mathcal{R}+p} F(p-q) V(q) L(dq), \tag{17}
\]
where \( \mathcal{R} \) is compact and \( F \) continuously differentiable.

Next, we proceed to characterize incompressible fields of the form of (15) under the framework of Assumption \( 1 \). This is done by extending the results of Theorem 2 to the context of ambit fields. The proof of the following result will be presented in the Section 5.

**Theorem 4.** Let \( V \) satisfy the whole assumptions of the previous propositions and independent of \( L \). If \( Y \) is as in (17), then Theorem 2 remains valid when we replace \( L(dq) \) by \( V(q)L(dq) \) and \( \hat{L} \) by \( L \).

An application of this result gives us that:

**Proposition 2.** Let \( V \) and \( L \) be as in the previous proposition and \( Y_{\phi,f} \) be as in (15), \( \mathcal{R} \) fulfilling Assumption \( 1 \) and \( F_{\phi,f} \) being continuously differentiable in \( \mathcal{R} \). Then we have the following:

1. \( Y_{\phi,f} \) is incompressible if either \( \phi = \frac{\pi}{2}, \frac{3\pi}{2} \) and \( f \) arbitrary, or \( \phi \neq \frac{\pi}{2}, \frac{3\pi}{2} \) and for some constant \( K \in \mathbb{R} \)
   \[
f(x) = K x^{-2}, \quad x > 0.
   \]

2. \( Y_{\phi,f} \) is irrotational if either \( \phi = 0, \pi \) and \( f \) arbitrary, or \( \phi \neq 0, \pi \) and for some constant \( K \in \mathbb{R} \)
   \[
f(x) = K x^2, \quad x > 0.
   \]

**Proof.** From Theorem 4 we have that
\[
\frac{1}{\pi r^2} \int_{r \mathbb{S}^1(p)} Y_{\phi,f} \cdot n ds \rightarrow_{p} \int_{\mathcal{R}+p} \nabla \cdot F_{\phi,f}(p-q) V(q) L(dq);
\]
\[
\frac{1}{\pi r^2} \int_{r \mathbb{S}^1(p)} Y \cdot n ds \rightarrow_{p} \int_{\mathcal{R}+p} \nabla^\perp \cdot F_{\phi,f}(p-q) V(q) L(dq).
\]
Therefore, $Y_{\phi,f}$ is incompressible if and only if the divergence of $F_{\phi,f}$ vanishes. In an similar way we see that $Y_{\phi,f}$ is irrotational if and only if the curl of $F_{\phi,f}$ vanishes. The conclusions of this proposition follow then from the previous observations and the fact that
\[
\nabla \cdot F_{\phi,f}(q) = 2\cos(\phi) f(||q||) + \cos(\phi) f'(||q||) ||q||;
\]
\[
\nabla \times F_{\phi,f}(q) = 2\sin(\phi) f(||q||) - \sin(\phi) f'(||q||) ||q||.
\]

Remark 6. The previous proposition together with Proposition [1] show that it is always possible to construct an ambit field that has homogeneous and isotropic increments as well as being incompressible and rotational. Then a natural question arises: Is there another class of kernels different from $F_{\phi,f}$ having these properties?

5 Proofs

In this part we present the proofs of our main results. We will proceed as follow: First, we establish some preliminary results on the existence and the representation of the functionals $\mathcal{D}_r(\cdot; X)$ and $\mathcal{C}_r(\cdot; X)$ when $X$ is of the form (12). We use such a representation to formalize the decomposition (11) discussed in Subsection 3.1. We apply this to identify the part of $\mathcal{D}_r(\cdot; X)$ and $\mathcal{C}_r(\cdot; X)$ that dominates the asymptotics. We then invoke Assumption 1 to fully describe the asymptotic rates of such a dominating part. The conclusions of Theorems 1-3 will follow from this. The proof of Theorem 4 will basically be a consequence of Theorem 2 and Proposition 5 in Appendix A. Furthermore, since our reasoning is independent to whether we use $n$ or $n^{-1}$, we will only present the proof of those results concerned to $\mathcal{D}_r(\cdot; X)$.

5.1 Preliminary results and remarks

In what follows $L$ will denote a real-valued homogeneous Lévy basis on $\mathbb{R}^d$ with characteristic triplet $(\gamma, b, \nu)$ and $X$ as in (12). Let us start by a very simple observation: $\mathcal{D}_r(\cdot; X)$ vanishes when $L$ is deterministic. Indeed, suppose that $L(A) = \gamma \text{Leb}(A)$ for some $\gamma \in \mathbb{R}$. Then for any $p \in \mathbb{R}^2$, $X(p) \equiv X(0) \equiv \gamma \int_{\mathbb{R}} F(-q)dq$, meaning this that
\[
\mathcal{D}_r(p; X) = r \int_{0}^{2\pi} X(0) \cdot u(\theta)d\theta = 0,
\]
as claimed. In fact, a deterministic homogeneous field is necessarily constant. Hence, we can conclude for this and the Lévy-Itô decomposition for Lévy bases (see [24]) that for any $p \in \mathbb{R}^2$
\[
\mathcal{D}_r(p; X) = \mathcal{D}_r(p; \hat{X}),
\]
where
\[
\hat{X}(p) = \int_{\mathbb{R}^+} F(p - q)\hat{L}(dq),
\]
and $\hat{L}$ as in Theorem [2].

Now, by using Lemma 3 in Appendix B, we formalize (11).

Lemma 1. Let $\mathcal{R} \subset \mathbb{R}^2$ be compact and $F : \mathbb{R}^2 \to \mathbb{R}^2$ a measurable function that is continuously differentiable on $-\mathcal{R}$. Then the field
\[
X(p) := \int_{\mathbb{R}^+} F(p - q)L(dq), \quad p \in \mathbb{R}^2,
\]
is well defined and continuous in probability. Moreover, the functionals $\mathcal{C}_r(\cdot; X)$ and $\mathcal{D}_r(\cdot; X)$ given in [3] and [6], are well defined and they can be decomposed as
\[
\mathcal{C}_r(p; X) = \hat{\mathcal{C}}_r(p; X) + \partial \mathcal{C}_r(p; X); \quad (20)
\]
\[
\mathcal{D}_r(p; X) = \hat{\mathcal{D}}_r(p; X) + \partial \mathcal{D}_r(p; X), \quad (21)
\]
where
\[
\dot{\mathcal{D}}_r(p;X) := \int_{\mathcal{R}(p) \setminus (\partial\mathcal{R}(p))_{\beta r}} \int_{D_r(p-q)} \nabla \cdot F(y) dy L(\text{d}q);
\]
\[
\partial \mathcal{D}_r(p;X) := \int_{(\partial\mathcal{R}(p))_{\beta r}} \int_{r^2 \mathbb{S}^1(q)} F(p-q) \cdot n \text{d}s L(\text{d}q),
\]
and
\[
\dot{\mathcal{C}}_r(p;X) := \int_{\mathcal{R}(p) \setminus (\partial\mathcal{R}(p))_{\beta r}} \int_{D_r(p-q)} \nabla \cdot \mathcal{F}(y) dy L(\text{d}q);
\]
\[
\partial \mathcal{C}_r(p;X) := \int_{(\partial\mathcal{R}(p))_{\beta r}} \int_{r^2 \mathbb{S}^1(q)} F(p-q) \cdot \mathbb{1}_{\mathcal{R}(p)}(q) \cdot \text{d}u L(\text{d}q).
\]

Proof. The proof will be divided in three steps. First we prove the existence and the stochastic continity of $X$. Second, we show that $\mathcal{D}_r(p;X)$ is well defined by verifying that the conditions of Lemma 3 in Appendix B, are satisfied. The decomposition (21) will follow from this.

Existence and continuity in probability: From Lemma 2.1.5 in [40], we have that
\[
\int_{\mathcal{R}} \Phi^0_L(\|F(-q)\|) \text{d}q \leq 2\text{Leb}(\mathcal{R}) \Phi^0_L(c_F) < \infty,
\]
where $c_F = \sup_{\mathcal{R}} \|F(-q)\| < \infty$. This shows that $X$ is well defined. Now, pick $p_n \to 0$. Using the same lemma, we have that
\[
\Phi^0_L \left( \|F(p_n-q) \mathbb{1}_{\mathcal{R}(p_n)} - F(-q) \mathbb{1}_{\mathcal{R}} \| \right) \leq 2c_{F,\Phi} \mathbb{1}_{\mathcal{R} \cup \mathcal{R}(p_n)},
\]
where $c_{F,\Phi} = \max\{\Phi^0_L(c_F), \Phi^0_L(2c_F)\}$. Thus, by the Dominated Convergence Theorem, we deduce that, as $n \to \infty$,
\[
\int_{\mathbb{R}^2} \Phi^0_L \left( \|F(p_n-q) \mathbb{1}_{\mathcal{R}(p_n)} - F(-q) \mathbb{1}_{\mathcal{R}} \| \right) \text{d}q \to 0,
\]
which according to Theorem 3 in Appendix B, shows that $X$ is continuous in probability.

Application of Stochastic Fubini Theorem: We now verify that 1.-3. in Lemma 3 in Appendix B are satisfied for the function $f_{\varphi}(p,q) := \mathbb{1}_{\mathcal{R}+p} F(p-q) \cdot n(p)$ where we consider $K = [0, 2\pi], \varphi(\theta) = p + ru(\theta)$ with $u(\theta) = (\cos \theta, \sin \theta)'$ and $|D\varphi(\theta)| = r$. By the continuity of $F$, one can deduce that $|\gamma| r \int_{0}^{2\pi} \int_{\mathcal{R}} |F(-q) \cdot u(\theta)| \text{d}qd\theta < \infty$ and $b^2 r \int_{0}^{2\pi} \int_{\mathcal{R}} |F(-q) \cdot u(\theta)| \text{d}qd\theta < \infty$. Similarly, we get that
\[
\int_{r^2 \mathbb{S}^1} \int_{\mathcal{R}} \int_{|z| \leq 1} |xF(-q) \cdot u(\theta)|^2 \nu(\text{d}x) \text{d}qd\theta < \infty.
\]
Thus, 1. and 2. hold. Hence, it only remains to verify 3. Before doing this, let us first note that for a fixed $r > 0$, $\mathbb{1}_{\mathcal{R}+p+ru(\theta)}(q) = 0$ if $d_{\mathcal{R}(p)}(q) > r$. In view of the previous observation, we get that $\chi(q) := \int_{K} f_{\varphi}(\varphi(\theta),q) |D\varphi(\theta)| \text{d}\theta = 0$, whenever $q \in \mathcal{I}_r(p)^c$, where
\[
\mathcal{I}_r(p) := \mathcal{R}(p) \cup (\partial\mathcal{R}(p))_{\beta r}.
\]
Consequently
\[
\int_{\mathcal{R}} \int_{|z| > 1} (|z\chi(q)| \wedge 1) \nu(\text{d}x) \text{d}q \leq \int_{\mathcal{R}} \int_{|z| > 1} \nu(\text{d}x) \text{d}q < \infty,
\]
i.e. 3. is satisfied.

Decomposition: We now proceed to show that (21) is valid. By the previous part and Lemma 3 in Appendix B, we can conclude that the functional $\mathcal{D}_r(p;X)$ is well defined and almost surely
\[
\mathcal{D}_r(p;X) = r \int_{\mathcal{I}_r(p)} \int_{0}^{2\pi} f_{\varphi}(p+ru(\theta),q) \text{d}\theta L(\text{d}q), \quad r > 0, p \in \mathbb{R}^2.
\]
with \( I_r(p) \) as in (22). On the other hand, by the definition of \((\partial R(p))_\ominus r\), one get that

\[
\oint_{rS^1(p-q)} F 1_{-R} \cdot nds = \pi r^2 \int_0^{2\pi} f_\varphi(p + ru(\theta), q) d\theta
\]

and the usual Stokes Theorem, we have

\[
\oint_{rS^1(p-q)} F \cdot nds = \int_{D_r(p-q)} \nabla \cdot F(y) dy, \quad q \in R(p) \setminus (\partial R(p))_\ominus r.
\]

The decomposition appearing in (21) follows from this.

5.2 Identification of the rates

In this part, by using the decomposition obtained in Lemma 1, we identify the dominating part of \( \hat{\mathcal{C}}_r(p; X) \) and \( \hat{\mathcal{D}}_r(p; X) \).

Asymptotics for \( \hat{\mathcal{C}}_r(p; X) \) and \( \hat{\mathcal{D}}_r(p; X) \)

The following lemma reveals that, as in the temporal case, the asymptotic rate for \( \hat{\mathcal{C}}_r(p; X) \) and \( \hat{\mathcal{D}}_r(p; X) \) is the “classical” one, i.e. \( \pi r^2 \).

**Proposition 3.** Let \( X, \hat{\mathcal{C}}_r(p; X) \) and \( \hat{\mathcal{D}}_r(p; X) \) be as in Lemma 1. Then for every \( p \in \mathbb{R}^2 \) we have that as \( r \downarrow 0 \)

\[
\frac{1}{\pi r^2} \hat{\mathcal{C}}_r(p; X) \xrightarrow{\mathcal{P}} \hat{\sigma}(p); \quad \frac{1}{\pi r^2} \hat{\mathcal{D}}_r(p; X) \xrightarrow{\mathcal{P}} \hat{\omega}(p).
\]

where

\[
\hat{\sigma}(p) := \int_{R^+_p} \nabla \cdot F(p - q)L(dq); \quad \hat{\omega}(p) := \int_{R^+_p} \nabla \cdot F(p - q)L(dq)
\]

**Proof.** Let us first note that since \( F \) is continuously differentiable on \( -R \), we can argue as in the first step of the proof of Lemma 1 in order to deduce that

\[
\int_R \Phi^0_L(|\nabla \cdot F(-q)|) + \Phi^0_L(|\nabla \cdot F(-q)|) dq < \infty,
\]

which shows that the fields \( \omega \) and \( \sigma \) are well defined. In order to finish the proof, thanks to Theorem 9 in Appendix A, the previous lemma and stationarity, we only need to verify that as \( r \downarrow 0 \)

\[
\int_{R \setminus (\partial R(p))_\ominus r} \Phi^0_L(h_{r,\varphi}(q)) dq \to 0, \quad (23)
\]

where for \( q \in R \setminus (\partial R(p))_\ominus r \)

\[
h_{r,\varphi}(q) := \frac{1}{\pi r^2} \int_{D_r(-q)} \{ \nabla \cdot F(y) - \nabla \cdot F(-q) \} dy.
\]

In the present framework, the mapping \( y \mapsto \nabla \cdot F(y) \) is continuous on \( -R \). Thus, since \( R \) is compact, the Tietze Extension Theorem allow us to extend continuously \( h_{r,\varphi} \) on \( \mathbb{R}^2 \), in such a way

\[
|h_{r,\varphi}(q)| \leq 2 \sup_{\mathbb{R}^2} |\nabla \cdot F(q)| < \infty, \quad q \in \mathbb{R}^2.
\]

Therefore, by Lemma 2.1.5 in [40], we obtain that

\[
\Phi^0_L(h_{r,\varphi}(q)) \mathbf{1}_{R \setminus (\partial R(p))_\ominus r}(q) \leq 2\Phi^0_L(c_1),
\]

where \( c_1 = 2 \sup_{\mathbb{R}^2} |\nabla \cdot F(q)| \). Equation (23) then follows as an application of the Lebesgue Differentiation Theorem and the Dominated Convergence Theorem.

Proposition 3 formalizes the intuition discussed in Subsection 3.1, i.e. we have shown that if \( X \) is of the form of [19], then the limit behavior of \( \hat{\mathcal{D}}_r(p; X) \) is completely determined by \( \partial \mathcal{D}_r(p; X) \), which is in turn depending on \( L, F \) and the geometry of \( \partial R \).
The functionals $\partial \mathcal{E}_r(p;X)$ and $\partial \mathcal{D}_r(p;X)$ are negligible in the compound Poisson case

Not surprisingly, as the following lemma shows, when the Lévy seed of $L$ has a compound Poisson distribution we get that $\partial \mathcal{E}_r(p;X)$ and $\partial \mathcal{D}_r(p;X)$ are negligible.

**Proposition 4.** Let $X$, $\partial \mathcal{E}_r(p;X)$ and $\partial \mathcal{D}_r(p;X)$ be as in Lemma 1 and assume that $L'$, the Lévy seed of $L$, satisfies

$$\mathcal{C}(z \uparrow L') = \int_{\mathbb{R}} (e^{izx} - 1) \nu(dx), \quad z \in \mathbb{R},$$

where $\nu(\mathbb{R}) < \infty$. Let $(a_r)_{r>0}$ be any collection of real number such that $a_r \to 0$ as $r \downarrow 0$. Then for all $p \in \mathbb{R}^2$ we have that

$$a_r^{-1} \partial \mathcal{E}_r(p;X) \overset{p}{\to} 0; \quad a_r^{-1} \partial \mathcal{D}_r(p;X) \overset{p}{\to} 0.$$

**Proof.** By Lemma 1 and our assumption, as $r \downarrow 0$

$$|\mathcal{C}(z \uparrow a_r^{-1} \partial \mathcal{E}_r(p;X))| \leq 2c_L' \text{Leb}(\partial \mathcal{R}(p)_{\mathbb{R}^2}) \to 0,$$

where $c_{L'} = \sup_{\mathbb{R}} |\mathcal{C}(z \uparrow L')| < \infty$. \hfill ■

**Remark 7.** Observe that at this point none of the results of this section have used Assumption 1. We can therefore infer that Theorem 2 holds for any field of the form of (19) for which $F$ is continuously differentiable, $\mathcal{R}$ is compact and the Lévy seed of $L$ is distributed according to a compound Poisson distribution. Assumption 1 is crucial when this is not the case.

A fundamental decomposition of $\partial \mathcal{E}_r(p;X)$ and $\partial \mathcal{D}_r(p;X)$

Under the framework of Assumption 1 we have that $\partial \mathcal{R} = \bigcup_{i=1}^n \partial \mathcal{R}_i$ where all the $\partial \mathcal{R}_i$’s are disjoint and closed. This implies, in particular, that for $r$ small enough the parallel sets of $\partial \mathcal{R}$ satisfies that

$$(\partial \mathcal{R})_{\mathbb{R}^2} = \bigcup_{i=1}^n (\partial \mathcal{R}_i)_{\mathbb{R}^2},$$

with $(\partial \mathcal{R}_i)_{\mathbb{R}^2}$ disjoint. Consequently, by Lemma 1 almost surely

$$\partial \mathcal{D}_r(p;X) = \int_{(\partial \mathcal{R}_i + p)_{\mathbb{R}^2}} \int_{rS^1(q)} F(p - \cdot)1_{\mathcal{R}_i(p)} \cdot ndsL(dq)$$

$$\sum_{j=1}^m \int_{(\partial \mathcal{R}_j + p)_{\mathbb{R}^2}} \int_{rS^1(q)} F(p - \cdot)1_{\mathcal{R}_j(p)} \cdot ndsL(dq),$$

where the summands are independent. An analogous decomposition holds for $\partial \mathcal{D}_r(p;X)$, $\partial \mathcal{E}_r(p;X)$ and $\partial \mathcal{D}_r(p;X)$ consist of independent sums of functionals of the type of

$$\mathcal{T}_{r,v}^i(p;F,A,L) := \int_{(\partial A+p)_{\mathbb{R}^2}} \int_{rS^1(q)} F(p - u)1_{B^i(p)}(u) \cdot duL(dq);$$

$$\mathcal{T}_{r,d}^i(p;F,A,L) := \int_{(\partial A+p)_{\mathbb{R}^2}} \int_{rS^1(q)} F(p - \cdot)1_{B^i(p)} \cdot ndsL(dq),$$

with $B^1 = A$, $B^2 = A^\ast$. Hence, we proceed to study these functionals in more detail. To improve the presentation of the following results, we introduce a simplified version of Assumption 1.

**Assumption 2.** Let $A \subseteq \mathbb{R}^2$ be a Jordan domain with Lipschitz-regular boundary satisfying that

$$\int_{S^1} \left\{ \sum_{q:(q,u) \in N(A)} (\delta_A(q,u) \wedge 1) + \sum_{q:(q,u) \in N(A^\ast)} (\delta_{A^\ast}(q,u) \wedge 1) \right\} \mathcal{H}^1(du) < \infty,$$

and $F : \mathbb{R}^2 \to \mathbb{R}^2$ a measurable function that is continuous on either $A \cap (\partial A)_{\mathbb{R}^2}$ or $A^\ast \cap (\partial A)_{\mathbb{R}^2}$ for some $r_0 > 0$.

As for the case of $\mathcal{E}_r(p;X)$ and $\mathcal{D}_r(p;X)$, we will only proof our statements for $\mathcal{T}_{r,d}^i(p;F,A,L)$. 

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Negligibility of $\partial \mathcal{C}_r(p; X)$ and $\partial \mathcal{R}_r(p; X)$ in the case when $F|_{-\partial R} = 0$ and $L$ arbitrary

**Theorem 5.** Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be such that $F(-\cdot)$ and $A \subseteq \mathbb{R}^2$ are as in Assumption 2. Then for $r > 0$ small, the functionals $\mathcal{J}_{r,v}(\cdot; F, A, L)$ and $\mathcal{J}_{r,d}(\cdot; F, A, L)$ introduced in (25) and (26) are well defined. Furthermore, if $F(-\cdot)$ is continuously differentiable on $B^1 \cap (\partial A)_{\sup r}$ and $F|_{-\partial A} = 0$, then for any $p \in \mathbb{R}^2$, it holds that for $i = 1, 2$, as $r \downarrow 0$,

$$r^{-2} \mathcal{J}_{r,v}(p; F, C, L) \xrightarrow{p} 0; \quad r^{-2} \mathcal{J}_{r,d}(p; F, C, L) \xrightarrow{p} 0. \quad (27)$$

**Proof.** We first note that by Lemma 4 and its subsequent remark in Appendix B, $c_1 = \sup_{r \leq r_0, q \in (\partial A)_{\sup r}} \left| r^{-2} \int_{\mathbb{R}^3(q)} F(\cdot)1_{B^1} \cdot n \, dx \right| < \infty$.

If $c_1 = 0$, then the result is trivial, so assume that $c_1 > 0$. An application of Lemma 2.1.5 in [40] gives us that for $i = 1, 2$,

$$\int_{(\partial A)_{\sup r}} \Phi^0_L \left( r^{-2} \int_{\mathbb{R}^3(q)} F(\cdot)1_{B^1} \cdot n \, dx \right) \, dq \leq 2 Leb((\partial A)_{\sup r})\Phi^0_L(c_1) \to 0,$$

which by stationarity and Theorem 3 in Appendix A, give all the conclusions in this theorem. \hfill \blacksquare

**Corollary 1.** Let Assumption 1 holds. If $F|_{-\partial R} = 0$, then for any $p \in \mathbb{R}^2$, when $r \downarrow 0$, we have that $r^{-2} \partial \mathcal{C}_r(p; X) \xrightarrow{p} 0; \quad r^{-2} \partial \mathcal{R}_r(p; X) \xrightarrow{p} 0$.

Asymptotics of $\mathcal{J}_{r,v}(p; F, C, L)$ and $\mathcal{J}_{r,d}(p; F, C, L)$ in the case when $F|_{-\partial R} \neq 0$

Let us now concentrate on the situation when $F|_{-\partial R} \neq 0$. Recall that $v_\beta = 2\{2 \int_0^1 (1 - s^2)^{\beta/2} ds\}^{1/\beta}$ and that $(x, y)^{\perp} = (-y, x)$.

**Theorem 6.** Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be such that $F(-\cdot)$ and $A \subseteq \mathbb{R}^2$ are as in Assumption 2. Take $\mathcal{J}_{r,v}(\cdot; F, A, L)$ and $\mathcal{J}_{r,d}(\cdot; F, A, L)$ as in (25) and (26), respectively, with $L$ a homogeneous Lévy basis with characteristic triplet $(\gamma, b, \nu)$. Assume that $F|_{-\partial A} \neq 0$. Then,

1. If $b > 0$,

$$\frac{1}{v_2 r^{1+1/2}} \mathcal{J}_{r,v}(p; F, A, L) \xrightarrow{f,d} \int_{\partial A(p)} F(p - q) \cdot u_A(p)(q)W_\gamma^1(q) \, dq;$$

$$\frac{1}{v_2 r^{1+1/2}} \mathcal{J}_{r,d}(p; F, A, L) \xrightarrow{f,d} \int_{\partial A(p)} F(p - q) \cdot u_A(p)(q)W_\gamma^1(q) \, dq;$$

where $u_A(p)$ as in Assumption 2 and $W_\gamma^1$ as in Theorem 4.

2. If $b = 0$ and $\int_0^1 (1 + |x|) \nu(dx) < \infty$, then as $r \downarrow 0$

$$\frac{1}{\pi r^2} \mathcal{J}_{r,v}(p; A, L) \xrightarrow{p} (-1)^{\gamma_0} \int_{\partial A(p)} F(p - q) \cdot u_A(p)(q)H_0^1(q) \, dq;$$

$$\frac{1}{\pi r^2} \mathcal{J}_{r,d}(p; A, L) \xrightarrow{p} (-1)^{\gamma_0} \int_{\partial A(p)} F(p - q) \cdot u_A(p)(q)H_0^1(q) \, dq;$$

where $\gamma_0 = -\int_{|x| \leq 1} x \nu(dx)$.

3. If $L$ satisfies the assumptions of Theorem 3 for some $2 > \beta \geq 1$, then, as $r \downarrow 0$,

$$\frac{1}{v_\beta r^{1+1/\beta}} \mathcal{J}_{r,v}(p; F, A, L) \xrightarrow{f,d} (-1)^{\gamma_0} \int_{\partial A(p)} F(p - q) \cdot u_A(p)(q)M^{\beta}_H^1(q) \, dq;$$

$$\frac{1}{v_\beta r^{1+1/\beta}} \mathcal{J}_{r,d}(p; F, A, L) \xrightarrow{f,d} (-1)^{\gamma_0} \int_{\partial A(p)} F(p - q) \cdot u_A(p)(q)M^{\beta}_H^1(q) \, dq;$$

where $M^{\beta}_H$ as in Theorem 3.
Proof. The proof is divided in two steps. First we show the convergence of the finite-dimensional distributions. After that, we conclude the proof by showing the stable convergence.

**Finite-dimensional convergence:** We start by observing that if \( L \) has characteristic exponent \( \psi \) then by Theorem 2 in [29] as \( r \downarrow 0 \)

\[
  r\psi(r^{-1/\beta}z) \to \psi_\beta(z) = \begin{cases} 
  -\frac{1}{3}b^2z^2 & \text{if } b > 0 \text{ and } \beta = 2; \\
  i\gamma_0z & \text{if } b = 0, \int_{\mathbb{R}} (1 \wedge |x|) \nu(dx) < \infty \text{ and } \beta = 1; \\
  \psi_{K_\pm,\beta,\hat{\gamma}}(z) & \text{under 3. and } 1 \leq \beta < 2;
\end{cases}
\]

where \( \psi_{K_\pm,\beta,\hat{\gamma}} \) denotes the characteristic exponent of a strictly \( \beta \)-stable distribution whose parameters \( K_\pm \) and \( \hat{\gamma} \) are defined as in Theorem [5]. Therefore, by Lemma [5] in Appendix B

\[
  \frac{1}{\psi^{r+1/\beta}} \mathcal{F}_r^i(p; F, A, L) \equiv \frac{1}{\psi^{r+1/\beta}} \mathcal{F}_r^i(p; F, A, L_\beta) + o_r(1),
\]

where \( \equiv \) stand by equality of the finite-dimensional distributions, and \( L_\beta \) is a homogeneous Lévy basis with characteristic exponent \( \psi_\beta \). Hence, it is enough to show that the asymptotics in 1.-3. holds for \( \mathcal{F}_r^i(\cdot; F, A, L_\beta) \).

For \( i = 1, 2, z \in \mathbb{R}, r \leq r_0, \) put

\[
  z_r^i(q) := (v_\beta r)^{-1}z \int_{r^{3i}(q)} \mathcal{F}(-\cdot)1_{B_{1i}} \cdot n dx 1_{(\partial A)_{\beta, r}}(q).
\]

By relation [36] in Appendix A, and the strict stability of \( \psi_\beta \), we have that

\[
  C \left[ z_r^i(q) \right] = \frac{1}{r} \left\{ \int_{A_{\beta, r} \setminus A} \psi_\beta(z_r^i(q)) dq + \int_{A_{\beta, r} \setminus A} \psi_\beta(z_r^i(q)) dq \right\}.
\]

For the notation involved below, we refer to the reader to Appendix B. We want to apply Theorems [7] and [8] to the previous relation. From Assumption [3], we have that \( \mathcal{H}^1(\partial^{++}A \cap \partial^{++}A^* \setminus \partial A) = 0 \). Consequently, in this case Theorems [7] and [8] read as

\[
  \frac{1}{r} \int_{A_{\beta, r} \setminus A} \psi_\beta(z_r^i(q)) dq = \int_{\partial A} \int_0^1 \psi_\beta \left[ z_r^i(q + rsu_A(q)) \right] 1_{(\delta_A(q, u) > rs)} \mathcal{H}^1(dq)\]

\[
  \quad + \frac{w_1}{r} \int_{N(A)} \left[ \int_0^r s \psi_\beta \left[ z_r^i(q + su_A(q)) \right] 1_{(\delta_A(q, u) > s)} ds \right] \mu_0(A; d(q, u)).
\]

Lemma [3] in Appendix B, guarantees that for almost all \( q \in \partial A \), as \( r \downarrow 0 \)

\[
  z_r^i(q + rsu_A(q)) \to v_\beta^{-1}z(-1)^i 2\sqrt{1 - s^2} F(q) \cdot u_A(q).
\]

and

\[
  \int_0^1 \int_{\partial A} \| \psi_\beta(v_\beta^{-2} s z_r^i, q) 1_{(\delta_A(q, u) > rs)} \| \mathcal{H}^1(dq) ds \leq K 2 \mathcal{H}^1(\partial A) < \infty,
\]

for some constant \( K > 0 \). Hence, the Dominated Convergence Theorem can be applied to get that as \( r \downarrow 0 \)

\[
  \int_{\partial A} \int_0^1 \psi_\beta \left[ z_r^i(q + rsu_A(q)) \right] 1_{(\delta_A(q, u) > rs)} \mathcal{H}^1(dq) ds \to \frac{1}{2} \int_{\partial A} \psi_\beta \left[ (-1)^i F(q) \cdot u_A(q) \right] \mathcal{H}^1(dq).
\]

We claim that the last integral in (28) vanishes when \( r \downarrow 0 \). Indeed, as before we can choose \( K > 0 \) such that

\[
  \left| \frac{1}{r} \int_{N(A)} \int_0^r s \psi_\beta \left[ z_r^i(q + su_A(q)) \right] 1_{(\delta_A(q, u) > s)} ds \mu_0(A; d(q, u)) \right| \leq K \int_{N(A)} \int_0^r s 1_{(\delta_A(q, u) > s)} ds \delta_0(A; d(q, u)).
\]

On the other hand, for any \( r \leq 1 \)

\[
  \frac{1}{r} \int_0^r s 1_{(\delta_A(q, u) > s)} ds \leq [1 \wedge \delta_A(q, u)]^2 + 1 \wedge \delta_A(q, u).
\]
Our claim then follows by the integrability condition in Assumption 2, Theorem 7 in the appendix, and the Dominated Convergence Theorem. In a similar way it is possible to verify that as \( r \downarrow 0 \)

\[
\frac{1}{r} \int_{A^c \setminus A^*} \psi_{\beta}(z^*_L(q))dq \rightarrow \frac{1}{2} \int_{\partial A} \psi_{\beta}[-(1)^{i}F(q) \cdot u_{A}(q)]H^1(dq).
\]

All above give us the pointwise convergence in 1.-3. The finite-dimensional convergence can be shown using similar arguments as above as well as an application of the Cramér–Wold methodology and the Inclusion-Exclusion principle.

**Stable convergence:** To avoid extra notation we only show that the stable convergence holds when \( b > 0 \). For the other cases a similar argument can be applied.

Let \( A \) be a bounded Borel set. Since for \( i = 1, 2 \), \( J_{i}^{r}(\cdot; F, A, L) \) is \( F_{L} \)-measurable and thanks to Theorem 3.2 in [24], it is sufficient for the \( F \)-stable convergence of the finite-dimensional distributions of \( \left\{ \frac{1}{\sqrt{v_{1} + \ldots + v_{r}}} J_{i}^{r}(p; F, C, L) \right\}_{p \in \mathbb{R}^{2}} \) to show that for any \( p_{1}, \ldots, p_{m}, \) \( \left\{ \frac{1}{\sqrt{v_{1} + \ldots + v_{r}}} J_{i}^{r}(p; F, A, L) \right\}_{j = 1}^{m}, L(A) \) converges in distribution towards

\[
\left\{ \int_{\partial A + p_{j}} F(p - q) \cdot u_{A(p_{j})}(q)W_{H_{1}}(dq) \right\}_{j = 1}^{m}, L(A) \right) .
\]

For \( z = (z_{1}, \ldots, z_{m+1}) \in \mathbb{R}^{m+1} \), let \( C \left[ z \downarrow \left\{ \frac{1}{\sqrt{v_{1} + \ldots + v_{r}}} J_{i}^{r}(p_{j}; F, A, L) \right\}_{j = 1}^{m}, L(A) \right] \) be the characteristic exponent of the random vector \( \left\{ \frac{1}{\sqrt{v_{1} + \ldots + v_{r}}} J_{i}^{r}(p_{j}; F, A, L) \right\}_{j = 1}^{m}, L(A) \). For the former, we are going to show that for \( z \neq 0 \), it converges as \( r \downarrow 0 \) to \( C \left( z \downarrow \left\{ \int_{\partial A + p_{j}} F(p - q) \cdot u_{A(p_{j})}(q)W_{H_{1}}(dq) \right\}_{j = 1}^{m} + C(z_{m+1} \downarrow L(A)) \right) \).

Let \( A_{m,r} = \bigcup_{i=1}^{m+1} (\partial A + p_{j}) \}_{\partial A} \). If \( A \cap (\partial A + p_{j}) = \emptyset \), for any \( i = 1, \ldots, m \), then for \( r \) small enough \( A_{m,r} \cap A = \emptyset \) and \( 29 \) follows by independently scattered property of \( L \). Suppose then that \( A_{m,r} \cap A \neq \emptyset \). Then, almost surely

\[ L(A) = L(A \cap A_{m,r}) + L(A \setminus A_{m,r}). \]

Once again, in view that \( L \) is independently scattered, we get that \( C \left[ z \downarrow \left\{ \frac{1}{\sqrt{v_{1} + \ldots + v_{r}}} J_{i}^{r}(p_{j}; F, A, L) \right\}_{j = 1}^{m}, L(A) \right] \) equals

\[
\log \mathbb{E} \left\{ i \sum_{j=1}^{m} \frac{z_{j}}{\sqrt{v_{1} + \ldots + v_{r}}} J_{i}^{r}(p_{j}; F, C, L) + z_{m+1}L(A \cap A_{m,r}) \right\} + C(z_{m+1} \downarrow L(A \setminus A_{m,r})). \]

Since \( L(A \cap A_{m,r}) \rightarrow 0 \) and \( L(A \setminus A_{m,r}) \rightarrow L(A) \), equation \( 20 \) follows by the previous relation and Slutsky’s Theorem.

**5.3 Proof of Theorems 1-3**

In this part we present the proof of Theorems 1-3 by combining the results obtained in the previous subsections.

**Proof.** First observe that in general, if \( L \) has characteristic triplet \( (\gamma, b, \nu) \), then from (18), Lemma 1 and the Lévy-Itô decomposition, we get that a.s.

\[
\mathcal{P}_{r}(p; X) = \partial \mathcal{P}_{r}(p; \tilde{X}) + \mathcal{P}_{r}(p; \tilde{X}),
\]

where

\[
\tilde{X}(p) = \int_{\mathbb{R}^{p}} F(p - q)L(dq),
\]

and \( \tilde{L} \) as in Theorem 2. Assume now that \( F|_{-\partial \mathcal{R}} = 0 \). In this case, by Corollary 1, we have that

\[
\mathcal{P}_{r}(p; X) = \hat{\mathcal{P}}_{r}(p; \tilde{X}) + o(r^{2}).
\]

The convergence in Theorem 2 follows immediately from this and Proposition 3. Suppose now that \( F|_{-\partial \mathcal{R}} \neq 0 \). If \( b = 0 \) and \( \int_{\mathbb{R}} (1 \wedge |x|)\nu(dx) < \infty \), by equation (24) and Theorem 6 we still get that relation (31) holds in this case.
Lemma 3 in Appendix B are satisfied almost surely for $\beta > 0$. Similar arguments used in the proof of Theorem 6
\[ \mathcal{D}_r(p; X) \xrightarrow{fd} \partial \mathcal{D}_r(p; X^W) + o_p(r^{1+1/2}), \]
where $X^W$ is defined as $\tilde{X}$ but we replace $\tilde{L}$ by a Gaussian Lévy basis with variance $b^2$. Another application of (24) and Theorem 6 conclude the proof of Theorem 1. Finally, let the assumptions of Theorem 3 hold. Analogously as the preceding argument, we deduce that if $1 < \beta < 2$
\[ \mathcal{D}_r(p; X) \xrightarrow{fd} \partial \mathcal{D}_r(p; X^\beta) + o_p(r^{1+1/\beta}), \]
while for $1 = \beta$
\[ \mathcal{D}_r(p; X) \xrightarrow{fd} \partial \mathcal{D}_r(p; X^\beta) + \hat{D}_r(p; X) + o_p(r^2), \]
where $X^W$ is defined as $\tilde{X}$ but $\tilde{L}$ is substituted by a strictly $\beta$-stable Lévy basis with the parameters given in Theorem 3. The conclusions of Theorem 3 then follows from this approximation and Theorem 6.

5.4 Proof of Theorem 4
As an application of Theorem 2 and Proposition 5 in Appendix A, we present a proof for Theorem 4.

Proof. Note first that from Proposition 1 and Remark 5 we can find a measurable modification of $Y$, which will be also denoted by $Y$, satisfying that
\[ \mathbb{E} \left[ \int_0^{2\pi} |Y(p + ru(\theta)) \cdot u(\theta)| \, d\theta \right] \leq 2\pi \mathbb{E}[\|Y(0)\|] < \infty, \]
meaning that the field $\mathcal{D}_r(p; Y)$ is well defined for any $p \in \mathbb{R}^2$ and $r > 0$. The key step for the rest of the proof consists in showing that
\[ \{ \mathcal{D}_r(p; Y) \}_{p \in \mathbb{R}^2} \overset{d}{=} \{ \hat{\mathcal{D}}_r(p; Y) + \partial \mathcal{D}_r(p; Y) \}_{p \in \mathbb{R}^2}, \]
Relation (32) then follows immediately from this.

5.5 Final remarks and generalizations
To conclude this section we further discuss other possible asymptotic rates for $\mathcal{D}_r(p; X)$. We also briefly discuss some generalizations on the ambit set. First observe that the only possible limit fields for $\mathcal{D}_r(p; X)$ are the one appearing in Theorems 1-3. Indeed, we have established in Proposition 3 that in general
\[ (\pi r^2)^{-1} \hat{\mathcal{D}}_r(p; X) \xrightarrow{p} \int_{\mathbb{R}^2} \nabla \cdot F(p - q) L(dq). \]
Thus, in the framework of Assumption 1 ($n = 1$) let $a_r \to 0$ as $r \downarrow 0$. Then
\[ C \left[ z \frac{1}{r} \mathcal{D}_r(p; X) \right] = \frac{1}{r} \int_{(\partial A)_{a_r}} r\psi(a_r^{-1} z_{1/2}(q)) dsdq, \]
where $z^i_r(q)$ is as in the proof of Theorem 6. We have shown that $r^{-1}\text{Leb}((\partial A)_{\oplus r})$ and $z^i_r(q)$ are uniformly bounded. Thus, we deduce that the sequence $\frac{\partial F_r(X)}{ra(r)}$ is bounded in probability whenever $r \psi(a_r^{-1})$ is. This is achieved for instance when $r \psi(a_r^{-1})$ is convergent, which according to [22], occurs if and only if $a_r = r^{1/\beta} l_r$, for some $0 < \beta \leq 2$ and a slowly varying function $l_r$ at 0. In this framework, Lemma 5 in Appendix B remains valid, leading this to the conclusion that Theorems 1-3 still hold when we replace $r^{1/\beta}$ by $a_r$.

Let us now discuss some feasible generalizations that can be considered for further research. Our proofs are based on two key results, namely Theorem 7 and Lemma 4, in Appendix A and B, respectively. The former deals with the asymptotic behaviour of the mapping $q \mapsto \frac{\partial F_r(q)}{\text{Leb}(\partial A)_{\oplus r}}$ on $(\partial A)_{\oplus r}$ when $r \downarrow 0$ which is in general extremely dependent on the geometry/smoothness of $\partial A$. Thus, a natural generalization of our framework is to consider non-smooth curves, e.g. fractals. Note that assumption 1 implies in particular that the limit $\lim_{r \downarrow 0} r^{-1}\text{Leb}((\partial A)_{\oplus r})$ exists and is finite. The later property is called 1-dimensional Minkowski measurability. More generally, a set $A \subseteq \mathbb{R}^d$ is said to be s-dimensional Minkowski measurable if there is $s \leq d$ such that the limit $\lim_{r \downarrow 0} r^{-(d-s)}\text{Leb}(A_{\oplus r})$ exists, is finite and different from zero. Therefore, if $\partial A$ is $s$-dimensional Minkowski measurable, then we can deduce

$$\partial F_r(p; X) = O_r(r^{1+\frac{d-s}{s}}).$$

Minkovski measurability holds for a big class of self similar curves (see [22]). The main challenge in this framework relies on the identification of the limit (in case it exists) of $\frac{\partial F_r(q)}{\text{Leb}(\partial A)_{\oplus r}}$.

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Appendix A

For a self-contained presentation, we present in this appendix some results related to the stochastic integration with respect to Lévy bases and formulas for the parallel sets of a closed set known as Steiner formula.

A Steiner-type formula for closed sets

This appendix gives a Steiner formula for closed sets in terms of the so-called reach measures of $A$. We refer to [27] for more details. Such Steiner formula reads as follow:

**Theorem 7.** For any non-empty closed set $A \subseteq \mathbb{R}^d$, there exist uniquely determined reach measures $\mu_0(A; \cdot), \ldots, \mu_0(A; \cdot)$ defined on $N(A)$ satisfying that for all $j = 0, \ldots, d - 1$, $r > 0$ and any compact set $B \subseteq \mathbb{R}^d$

$$\int_{N(A)} 1_B(x)(r \wedge \delta_A(q, u))^{d-j} |\mu_j| (A; d(q, u)) < \infty.$$ 

Moreover, if $w_j$ denotes the surface area of $\in S^j$ and $f : \mathbb{R}^d \rightarrow \mathbb{R}$ is measurable with compact support, it holds

$$\int_{A_{\oplus r} \setminus A} f(x) dx = \sum_{j=0}^{d-1} u_{d-j} \int_{N(A)} \int_0^r s^{d-1-j} 1_{\{\delta_A(q, u) > t\}} \times f(q + su) ds \mu_j(A; d(q, u)).$$

The reach measures $\mu_{d-1}(A; \cdot)$ and $\mu_0(A; \cdot)$ can be written in an explicit way. To do this, we introduce some extra notation. The positive boundary of $A$ is defined as

$$\partial^+ A := \{q \in \partial A : (q, u) \in N(A)\},$$

and setting $N(A, q) := \{u \in S^{d-1} : (q, u) \in N(A)\}$, we write

$$\partial^{++} A := \{q \in \partial^+ A : N(A, q) = \{u(q)\} \text{ or } N(A, q) = \{u(q), -u(q)\}; u(q) \in S^{d-1}\},$$
Theorem 8. For any non-empty closed set \( A \subseteq \mathbb{R}^d \), it holds that for any measurable and bounded function \( g : N(A) \rightarrow \mathbb{R} \) with compact support,
\[
\int_{N(A)} g(q, u) \mu_{d-1}(A; d(q, u)) = \frac{1}{2} \int_{\partial^+A} \sum_{u(q, y) \in N(A)} g(q, u) \mathcal{H}^{d-1}(dq);
\]
\[
w_d \int_{N(A)} g(q, u) \mu_0(A; d(q, u)) = \int_{\partial^+A} \sum_{x(q, y) \in N(A)} g(q, u)(-1)^d A(q, u) \mathcal{H}^{d-1}(du),
\]
for a measurable function taking values in \( \{0, 1, \ldots d-1\} \).

Limits for sequences of stochastic integrals w.r.t. Lévy bases.

Below we will present some results concerning the existence and the convergence of sequences of stochastic integrals with respect to Lévy bases. We refer the reader to [37], [9], and [18]. Fix a filtered probability space \( (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{R}}, \mathbb{P}) \) satisfying the usual conditions. Recall that a function \( \tau : \mathbb{R}^m \rightarrow \mathbb{R}^m \) is said to be a truncation function if it is bounded and \( \tau(x) = x \) in a neighborhood of 0. Denote by \( B_b (\mathbb{R} \times \mathbb{R}^d) \) the bounded Borel sets on \( \mathbb{R} \times \mathbb{R}^d \). Let \( \left( L(A) : A \in B_b (\mathbb{R} \times \mathbb{R}^d) \right) \) be a real-valued homogeneous Lévy basis with characteristic triplet \((\gamma, b, \nu)\) relative to a continuous truncation function \( \tau \), that is, the Lévy seed of \( L \) satisfies that
\[
C(z \downarrow L') = i\gamma z - \frac{1}{2} b^2 z^2 + \int_{\mathbb{R}\setminus\{0\}} |e^{izx} - 1 - i\tau(x)| \nu(dx).
\]
A real-valued random field of the form
\[
\xi(\omega, s, q) = \sum_{i=1}^{n} a_i 1_{F_i}(\omega) 1_{(u_i, t_i]}(s) 1_{A_i}(q), \tag{33}
\]
where \( A_i \in B_b (\mathbb{R}^d) \), \( u_i < t_i \), \( F_i \in \mathcal{F}_{u_i} \), and \( a_i \in \mathbb{R} \), is called a simple predictable random field. More generally, if \( \mathcal{P} \) denotes the predictable \( \sigma \)-algebra associated to \((\mathcal{F}_t)_{t \in \mathbb{R}} \), a random field is said to be predictable if it is \( \mathcal{P} \otimes B(\mathbb{R}^d) \)-measurable. When \( \xi \) is a simple random field as in (33), the stochastic integral of \( \xi \) w.r.t. \( L \) is defined as
\[
I_L(\xi) := \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \xi(s, q) L(dqds) := \sum_{i=1}^{n} a_i 1_{F_i} L((u_i, t_i] \times A_i). \tag{34}
\]
In stochastic integration theory one is usually looking for a linear extension of \( I_L \) into a rich enough linear subspace of predictable random fields, let’s say \( D(I_L) \), such that \( I_L(\xi) \) can be approximated by simple stochastic integrals of the form (33) as well as satisfying a Dominated Convergence Theorem, that is, if \( (\xi_n)_{n \in \mathbb{N}} \) is a sequence of simple functions such that \( \xi_n \rightarrow \xi \) point-wise, then \( I_L(\xi_n) \rightharpoonup I_L(\xi) \). Moreover, if \( (\xi_n)_{n \in \mathbb{N}} \subset D(I_L) \) such that \( \xi_n \rightarrow \xi \) point-wise and \(|\xi_n| \leq \xi^* \) for some \( \xi^* \in D(I_L) \), then \( I_L(\xi_n) \rightharpoonup I_L(\xi) \). When \( c(\{s\} \times A) = 0 \) for every \( t \in \mathbb{R} \), we will choose \( D(I_L) \) to be the space of predictable random fields such that almost surely
\[
\begin{align*}
1. & \int_{\mathbb{R}} \int_{\mathbb{R}^d} |\gamma(\xi(s, q)) + \int_{\mathbb{R}} \tau(\xi(s, q) x) - \xi(s, q) \tau(x) | \nu(dx) dq ds < \infty; \\
2. & \int_{\mathbb{R}} \int_{\mathbb{R}^d} \xi^2(s, q) b^2 dq ds < \infty; \\
3. & \int_{\mathbb{R}} \int_{\mathbb{R}^d} \int_{\mathbb{R}} \left( 1 + |\xi(s, q) x|^2 \right) \nu(dx) dq ds < \infty.
\end{align*} \tag{35}
\]
where \( \tau \) is a continuous truncation function. [18] have shown that \( D(I_L) \) is actually the biggest closed linear subspace of predictable random fields in which \( I_L \) can be defined in the previously explained sense. In the case when \( \xi \) is deterministic, we have that \( I_L(\xi) \) is ID, and
\[
C \left[ z \downarrow \int_{\mathbb{R}} \int_{\mathbb{R}^d} \xi(s, q) L(dqds) \right] := \int_{\mathbb{R}} \int_{\mathbb{R}^d} C(z \xi(s, q) \downarrow L') dq ds, \ z \in \mathbb{R}. \tag{36}
\]
\footnote{In the case when there is no temporal component, \( \mathcal{P} \) is replaced by \( \mathcal{F} \) and all the results presented in this appendix remain valid.}
Proof. By Lemma 2, we get that
\[ 9 = 9. \] ■
All the conclusions of this proposition then follow easily by this, the L0 provided that
\[ \gamma \leq 2. \]
\[ \xi \geq 0, \] (\[9\])

Theorem 9 [9]. Fix a continuous truncation function \( \tau \) and let \( L \) be a homogeneous Lévy basis with characteristic triplet \( (\gamma, b, \nu) \). Suppose that \( (\xi_n)_{n \in \mathbb{N}} \subset \mathcal{D}(IL) \). Then as \( n \to \infty \)
\[ \int_{\mathbb{R}} \int_{\mathbb{R}^d} \xi_n(s, q) L(dqds) \xrightarrow{\mathbb{P}} 0 \quad \text{if and only if} \quad \Psi_0(\xi_n) \to 0. \] (40)
In particular, if the \( \xi_n \)'s are deterministic, this is equivalent to having that
\[ \int_{\mathbb{R}} \int_{\mathbb{R}^d} \Phi_0^\omega [\xi_n(s, q)] dqds \to 0. \]

Based on the previous theorem, it is possible to find a sufficient condition for the convergence of sequences of the ambit-type. To do that the following lemma is crucial and it was originally stated in Lemma 2.1.5 in [40].

Lemma 2. For \( \tau(x) = \frac{x}{1 + |x|} \) let \( \Phi_0^\omega \) be as in (37). Then \( \Phi_0^\omega \) is continuous, even, and satisfies
\[ \Phi_0^\omega (x + y) \leq 3[\Phi_0^\omega (x) + \Phi_0^\omega (y)], \]
and
\[ \Phi_0^\omega (Kx) \leq (K^2 \lor 2) \Phi_0^\omega (x), \]
for any \( x, y, K \in \mathbb{R} \).

Recall that a random field \( V \) is said to be bounded in \( \mathcal{L}^p(\Omega, \mathcal{F}, \mathbb{P}) \) for \( p > 0 \), if \( \sup_{s,q} \mathbb{E} (|V_s(q)|^p) < \infty \). Under this terminology we have the following result.

Proposition 5. Let \( \tau(x) = \frac{x}{1 + |x|} \) and consider \( L \) to be a homogeneous Lévy basis with characteristic triplet \( (\gamma, b, \nu) \). Put
\[ \xi_n(s, q) := f_n(s, q) V_s(q), \quad (s, q) \in \mathbb{R} \times \mathbb{R}^d, \]
where \( (f_n)_{n \in \mathbb{N}} \) is a sequence of deterministic functions, and \( V \) a predictable random field which is bounded in \( \mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P}) \). Then \( (\xi_n)_{n \in \mathbb{N}} \subset \mathcal{D}(IL) \) provided that \( (f_n)_{n \in \mathbb{N}} \subset \mathcal{D}(IL) \). Moreover, as \( n \to \infty \), \( \int_{\mathbb{R}} \int_{\mathbb{R}^d} \xi_n(s, q) L(dqds) \xrightarrow{\mathbb{P}} 0 \) provided that \( \int_{\mathbb{R}} \int_{\mathbb{R}^d} f_n(s, q) L(dqds) \xrightarrow{\mathbb{P}} 0 \).

Proof. By Lemma 2 we get that
\[ \Psi_{\Phi_0^\omega}(\xi_n) \leq \int_{\mathbb{R}} \int_{\mathbb{R}^d} \mathbb{E}(V_s(q)^2 \lor 2) \Phi_0^\omega [f_n(s, q)] dqds. \]
All the conclusions of this proposition then follow easily by this, the \( \mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P}) \) boundedness of \( V \) and Theorem 9. ■

Remark 9. Integration of \( \mathbb{R}^m \)-valued predictable fields can be done entry by entry, that is, if \( \xi = (\xi_i)_{i=1}^n \), in which \( \xi_i \in \mathcal{D}(IL) \), then we define
\[ \int_{\mathbb{R}} \int_{\mathbb{R}^d} \xi(s, q) L(dqds) := \left\{ \int_{\mathbb{R}} \int_{\mathbb{R}^d} \xi_i(s, q) L(dqds) \right\}_{i=1}^n. \]
Finally, observe that \( \xi \in D (I_L) \) is equivalent to
\[
\int_{\mathbb{R}} \int_{\mathbb{R}^d} \Phi^0_L (||\xi (s, q)||) dq ds < \infty.
\]
In particular, when \( \xi \) is deterministic, we have that the random vector \( X = \int_{\mathbb{R}} \int_{\mathbb{R}^d} \xi (s, q) L (dq ds) \) is ID and has characteristic triplet \((\Gamma, B, \nu)\) relative to some \( \mathbb{R}^m \)-valued truncation function \( \tau \), given by
\[
\Gamma_X = \gamma \int_{\mathbb{R}} \int_{\mathbb{R}^d} \xi (s, q) dq ds + \int_{\mathbb{R}} \int_{\mathbb{R}^d} \int_{\mathbb{R}} (\tau (x) - \xi (s, q) \tau (x)) \nu (dx) dq ds;
\]
\[
B_X = b^2 \int_{\mathbb{R}} \int_{\mathbb{R}^d} \xi (s, q) \xi (s, q) ' dq ds;
\]
\[
\nu_X (A) = \int_{\mathbb{R}} \int_{\mathbb{R}^d} 1_A (x \xi (s, q)) \nu (dx) dq ds.
\]

Appendix B
This supplementary appendix contains several technical results that are used through the proofs of Theorems 1-3.

A Stochastic Fubini Theorem
By using the Lévy-Itô decomposition of ID fields introduced in [1], cf. [3], and a small refinement in the arguments of Theorem 3.1 in [5], cf. Lemma 4.9 in [12], we obtain a stochastic Fubini’s Theorem for surface integrals and Lévy bases. We recall that a random field \((Z(t))_{t \in T}\) is said to be separable in probability if there exist \( T_0 \subseteq T \) countable such that for any \( t \in T \) it is possible to extract \( \{t_n\} \subseteq T_0 \) satisfying that \( Z(t_n) \overset{p}{\to} Z_t \).

Lemma 3. Fix \( n \leq m \). Let \( K \subset \mathbb{R}^n \) be compact and \( \varphi : K \to \mathbb{R}^m \) a continuously differentiable function with Jacobian \( D \varphi \). Given a measurable function \( f : \mathbb{R}^n \times \mathbb{R}^d \to \mathbb{R} \) and a homogeneous Lévy basis \( L \) with characteristic triplet \((\gamma, b, \nu)\), assume that the ID field
\[
Z (p) = \int_{\mathbb{R}^d} f (p, q) L (dq), \quad p \in \mathbb{R}^m,
\]
is well defined and separable in probability on \( \varphi (K) \). Suppose in addition that
1. \( \int_K \left[ \int_{\mathbb{R}^d} |\gamma f (\varphi (u), q)| + |b f (\varphi (u), q)|^2 \right] |D \varphi (u)| du < \infty; \)
2. \( \int_K \left[ \int_{\mathbb{R}^d} \int_{|x| \leq 1} |x f (\varphi (u), q)| |x f (\varphi (u), q)|^2 \nu (dx) dq \right] |D \varphi (u)| du < \infty; \)
3. \( \int_{\mathbb{R}^d} \int_{|x| > 1} (|x \chi (q)| \wedge 1) \nu (dx) dq < \infty, \)
where \( |D \varphi| := |Det(D \varphi ' D \varphi)^{1/2} \) and \( \chi (q) := \int_K |f (\varphi (u), q)| |D \varphi (u)| du. \) Then the random field \( \{Z(\varphi (u))\}_{u \in K} \) can be chosen measurable and
\[
\int_K Z (\varphi (u)) |D \varphi (u)| du = \int_{\mathbb{R}^d} \int_K f (\varphi (u), q) |D \varphi (u)| du L (dq),
\]
in the sense that both integrals exist and are equal almost surely.

Proof. Arguing as in [5], we can always choose a measurable modification of \( Z \), meaning that \( \{Z (\varphi (u))\}_{u \in K} \) can be assumed to be measurable. Now, by the Lévy-Itô decomposition for Lévy bases (see [3]) \( L \) can be written as
\[
L (A) = \gamma \text{Leb} (A) + W (A) + M (A) + J (A), \quad A \in \mathcal{B}_0 (\mathbb{R}^d),
\]
where \( W, M \) and \( J \) are three independent homogeneous Lévy bases with characteristic triplets \((0, b, 0), (0, 0, \nu |_{[-1,1]^d})\) and \((0, 0, \nu |_{[-1,1]^d}) \wedge 1\), respectively, where the notation \( \nu |_B \) represents the restriction of the measure \( \nu \) to the set \( B \). Thus, \( Z \) can be written as
\[
Z = Z^\gamma + Z^W + Z^M + Z^J,
\]
where $Z^*, Z^W, Z^M$ and $Z^J$ are independent fields defined as in [41] when we replace $L$ by $\gamma \text{Leb}, W, M$ and $J$, respectively. Thus, it is enough to verify that (42) holds when we replace $Z$ by any of these fields. Now, it is clear that due to 1., the usual Fubini’s Theorem can be applied to $Z$. Furthermore, since for every $A \in \mathcal{B}_b(\mathbb{R}^d)$, $\mathbb{E}[|W(A)|] < \infty$, $\mathbb{E}[|M(A)|] < \infty$, we have that $W$ and $M$ are within the framework of Theorem 3.1 in [3]. Consequently, in view that $\int K |D\varphi(x)| \, dx < \infty$, and 1. and 2. are satisfied, we have that the stochastic Fubini’s Theorem in [3] can be applied to get that (42) is fulfilled for $Z^W$ and $Z^M$. Therefore, it only remains to show that (42) holds for $Z^J$. Since $Z$ is separable we may assume that the same holds for $Z^J$, which by Theorem 3.2 in [41] implies that for every $p \in \mathbb{R}^m$, almost surely
\[
Z^J(p) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(p, q) xN(dx, dq),
\]
where $N$ is a Poisson random measure with intensity $\mu(dx, dq) = \nu([-1,1]^d) \otimes dq$. Since $N(dx, dq)(\omega)$ is a $\sigma$-finite measure for every $\omega \in \Omega$, then $\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(p, q) xN(dx, dq)$ can be understood as a Lebesgue integral $\omega$ by $\omega$. Thus, to finish the proof, it is sufficient to show that on a set of probability one, it holds that
\[
\int K \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |f(\varphi(u), q)x| N(dx, dq) |D\varphi(u)| \, du < \infty,
\]
(44) because in this case the usual Fubini’s Theorem can be applied $\omega$ by $\omega$ to obtain (42). By Tonelli’s Theorem, almost surely
\[
\int K \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |f(\varphi(u), q)x| N(dx, dq) |D\varphi(u)| \, du = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \chi(q) |x| N(dx, dq).
\]
Equation (44) follows from the previous equation, 3. and Lemma 12.13 in [30].

Remark 10. Observe that under the assumptions of the previous lemma, if $\varphi$ is one-to-one (up to a null set) the area formula (see [20]) shows that almost surely
\[
\int K \, Z(\varphi(u)) |D\varphi(u)| \, du = \int C \, Z(y) \mathcal{H}^n(dy) = \int_{\mathbb{R}^d} \int C \, f(y, q) \mathcal{H}^n(dy)L(dq),
\]
where $\mathcal{H}^n$ is the $n$-dimensional Hausdorff measure on $\mathbb{R}^m$.

Line integrals and Jordan domains

In this part we deal with the asymptotic behaviour of certain type of line integrals. Recall the notation $(x, y)^\perp = (-y, x)$.

Lemma 4. Let $A \subseteq \mathbb{R}^2$ be a Jordan domain with Lipschitz-regular boundary and $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ a measurable function that is continuous on $B^i \cap (\partial A)_{\subseteq r_0^1}$ for some $r_0^1 > 0$, where for $i = 1, 2$, $B^1 = A$ and $B^2 = A^*$. Up to a null set, define for almost all $q \in \partial A$
\[
G^r_{r, \varphi}(s, q; F, A) := \oint_{r^{1}(q + r u_A(q))} F1_{B^i} \cdot ndx;
\]
(45)
\[
G^r_{r, \varphi}(s, q; F, A) := \oint_{r^{1}(q + r u_A(q))} F(u)1_{B^i} \cdot du,
\]
(46)
where $u_A(q)$ it the outward vector at $q$. Then

i) We have that
\[
\sup_{r \leq r_0, (s, q) \in [-1,1] \times \partial A} |r^{-1} G^r_{r, \varphi}(s, q; F, A)| < \infty;
\]
and for $\mathcal{H}^1$-almost all $q \in \partial A$, as $r \downarrow 0$
\[
r^{-1} G^i_{r, \varphi}(s, q; F, A) \rightarrow (-1)^i 2 \sqrt{1 - |s|^2} F(q) \cdot u_A(q);
\]
(47)
\[
r^{-1} G^j_{r, \varphi}(s, q; F, A) \rightarrow (-1)^j 2 \sqrt{1 - |s|^2} F(q) \cdot v_A(q).
\]
(48)
ii) If in addition $F$ is continuously differentiable on $B^i \cap (\partial A)_{\varnothing r_0}$ and $F|_{\partial A} = 0$, then
\[
\sup_{r \leq r_0, (s, q) \in [-1, 1] \times \partial A} \left| r^{-2} G_{r, \varnothing}^i (s, q; F, A) \right| < \infty; \quad \sup_{r \leq r_0, (s, q) \in [-1, 1] \times \partial A} \left| r^{-2} G_{r, \varnothing}^i (s, q; F, A) \right| < \infty,
\]
Furthermore, for $H^1$-almost all $q \in \partial A$ and $|s| < 1$, the following limits hold as $r \downarrow 0$
\[
r^{-2} G_{r, \varnothing}^i (s, q; F, A) \to \begin{cases} 
[DF(q)_{11} + DF(q)]_{22} (\pi + |s| \sqrt{1 - s^2} - \arccos(|s|)) & \text{if } i = 1; \\
[DF(q)_{11} + DF(q)]_{22} (\arccos(|s|) - |s| \sqrt{1 - s^2}) & \text{if } i = 2;
\end{cases}
\]
\[
r^{-2} G_{r, \varnothing}^i (s, q; F, A) \to \begin{cases} 
[DF(q)_{12} - DF(q)]_{12} (\pi + |s| \sqrt{1 - s^2} - \arccos(|s|)) & \text{if } i = 1; \\
[DF(q)_{12} - DF(q)]_{12} (\arccos(|s|) - |s| \sqrt{1 - s^2}) & \text{if } i = 2;
\end{cases}
\]
where $DF$ denotes the Jacobian of $F$.

**Proof.** We will only concentrate on $G_{r, \varnothing}^i$. Observe that $G_{r, \varnothing}^i$ and is well defined and measurable for any $r \leq r_0$. Now, by the continuity of $F$, we have that
\[
\left| G_{r, \varnothing}^i (s, q; F, A) \right| \leq 2\pi r \sup_{B'(\partial A)_{\varnothing r_0}} \|F(q)\| < \infty,
\]
which is the first conclusion in i). On the other hand, $F$ is continuously differentiable on $B^i \cap (\partial A)_{\varnothing r_0}$ and $F|_{\partial A} = 0$, then by the Mean Value Theorem, we have that for $H^1$-a.a. $q \in \partial A$ and $|s| \leq 1$
\[
\left| G_{r, \varnothing}^i (s, q; F, A) \right| \leq r \int_0^{2\pi} \|F(q + rsu_A(q) + ru(\theta)) - F(q) \cdot u(\theta)\| d\theta
\]
\[
= r^2 \int_0^{2\pi} |A_r(q, s, \theta)(su_A(q) + u(\theta)) \cdot u(\theta)| d\theta,
\]
where $A_r(q, s, \theta) = \int_0^1 DF(q + x[su_A(q) + u(\theta)])dx$ with $DF$ the Jacobian matrix of $F$. Since $F$ is continuously differentiable on $B^i \cap (\partial A)_{\varnothing r_0}$ and $u_A(q)$ and $u(\theta)$ are unitary, we get
\[
\left| G_{r, \varnothing}^i (s, q; F, A) \right| \leq r^2 4\pi \sup_{B'(\partial A)_{\varnothing r_0}} \|DF(q)\| < \infty.
\]
The first part of ii) is obtained from this. In what follows, we fix $q \in \partial A$ from which the $u_A(q)$ is well defined and all the limits appearing below are taken when $r \downarrow 0$. If $s = \pm 1$, we have that for $r$ small enough, $rS^1(q + rsu_A(q)) \cap B^i = \{q\}$ for any $i = 1, 2$. In this case [47] and [48] follow trivially, so for the rest of the proof we consider only the

![Figure 2: The figure illustrates a typical shape of $rS^1(q + rsu_A(q)) \cap \partial A$](image)
case when $|s| < 1$. Assume first that $s > 0$, $F|_{∂A} \neq 0$ and $i = 2$. Let $φ_{q}$ be the angle of $u_{A}(q)$ and parametrize $rS^{1}(q + rsu_{A}(q))$ as

$$\varphi_{q,r,s}(θ) := q + rsu_{A}(q) + ru(θ + φ_{q}), \quad 0 \leq θ < 2π,$$

in such a way that

$$G_{r,Θ}(s, q; F, A) = r \int_{Θ} F(\varphi_{q,r,s}(θ)) \cdot u(θ + φ_{q})dθ,$$

where $Θ = \{0 \leq θ < 2π : \varphi_{q,r,s}(θ) ∈ A^{1}\}$. We note that $Θ$ is based on arcs between the elements of $rS^{1}(q + rsu_{A}(q)) \cap ∂A$, so we now proceed to describe it. Let $θ(x)$ be such that $x = x_{q,r,s}(θ(x))$. The mapping $x \mapsto θ(x)$ is well defined and continuous on the closed set $rS^{1}(q + rsu_{A}(q)) \cap ∂A$. Indeed, if we denote by $a(x)$ the angle between $rsu_{A}(q)$ and $x - (q + rsu_{A}(q))$, see Figure 2, we get by the cosine law and Taylor’s Theorem that

$$θ(x) = π ± a(x) = π ± \arccos\left(\frac{x - q}{r} \cdot u_{A}(q)\right), \quad x ∈ rS^{1}(q + rsu_{A}(q)) \cap ∂A,$$

according to whether $θ(x) ∈ [π, 2π)$ or $θ(x) ∈ [0, π)$. The continuity then follows from this. Hence, we have that there are $x_{max,r}^{-}, x_{min,r}^{-}, x_{max,r}^{+}, x_{min,r}^{+} ∈ rS^{1}(q + rsu_{A}(q)) \cap ∂A$

$$\begin{align*}
θ(x_{min,r}^{-}) &= \inf\{θ ∈ [0, π] : \varphi_{q,r,s}(θ) ∈ ∂A\}; \quad θ(x_{max,r}^{-}) = \sup\{θ ∈ [0, π] : \varphi_{q,r,s}(θ) ∈ ∂A\}, \\
θ(x_{min,r}^{+}) &= \inf\{θ ∈ [π, 2π] : \varphi_{q,r,s}(θ) ∈ ∂A\}; \quad θ(x_{max,r}^{+}) = \sup\{θ ∈ [π, 2π] : \varphi_{q,r,s}(θ) ∈ ∂A\},
\end{align*}$$

and $0 ≤ θ(x_{min,r}^{-}) ≤ θ(x_{max,r}^{-}) ≤ π ≤ θ(x_{max,r}^{+}) ≤ θ(x_{min,r}^{+}) ≤ 2π$. Now suppose for a moment that $r^{-1}\|x_{min,r}^{-} - q\|$ and $r^{-1}\|x_{max,r}^{+} - q\| are bounded over $r$. If this were true, then by the Lipschitz-regularity condition on $∂A$, we would have $θ(x_{min,r}^{-}) → π ± \arccos(s)$ and $θ(x_{max,r}^{+}) → π ± \arccos(s)$ which, together with the fact that $u_{A}(q) = u(φ_{q})$, would give us that

$$r^{-1}G_{r,Θ}(s, q; F, A) = \int_{θ(x_{min,r}^{-})}^{θ(x_{max,r}^{-})} F(\varphi_{q,r,s}(θ)) \cdot u(θ + φ_{q})dθ + \int_{Θ ∩ [θ(x_{min,r}^{+}), θ(x_{max,r}^{+})]} F(\varphi_{q,r,s}(θ)) \cdot u(θ + φ_{q})dθ + \int_{Θ ∩ [θ(x_{min,r}^{-}), θ(x_{max,r}^{-})]} F(\varphi_{q,r,s}(θ)) \cdot u(θ + φ_{q})dθ + \int_{θ(x_{min,r}^{-})}^{θ(x_{max,r}^{-})} F(θ) \cdot u(θ + φ_{q})dθ = 2\sqrt{1 - s^{2}}F(q) \cdot u_{A}(q),$$

which is (47). We will only check that $r^{-1}\|x_{min,r}^{-} - q\|$ is bounded since the boundness of the other quantities can be shown in a similar way. To see the former holds, let $γ(x_{min,r}^{-})$ be as in Figure 3 and note that by the cosine law

$$2r\|x_{min,r}^{-} - q\|cos(γ(x_{min,r}^{-})) = \|x_{min,r}^{-} - q\|^{2} + r^{2}(1 - s^{2}) ≥ \|x_{min,r}^{-} - q\|^{2} > 0,$$

as claimed. The case $i = 1$ follows by noting that

$$r^{-1}G_{r,Θ}(s, q; F, A) = r^{-1} \left\{ \int_{rS^{1}(q + rsu_{A}(q))} F \cdot ndx - G_{r,Θ}(s, q; F, A) \right\} \rightarrow -2\sqrt{1 - s^{2}}F(q) \cdot u_{A}(q). \quad (52)$$

For the situation when $-1 < s < 0$, we observe that

$$G_{r,Θ}(s, q; F, A) = \int_{rS^{1}(q + rsu_{A}(q)) \cap B^{i}} F \cdot ndx,$$

where $u_{A}(q)$ is the outer vector to $A^{1}$ at $q$. Thus (48) follows by replacing $u_{A}(q)$ by $u_{A}(q)$ in the preceding arguments. This concludes the proof of (47). The proof of (48) is done by changing $u$ by $u^{±}$ in the previous
Figure 3: The figure shows the construction of the angle $\gamma(x_{\min,r}^-)$. 

reasoning. Now suppose that $F|_{\partial A} = 0$, $i = 2$ and $1 > s > 0$. Under the notation used above we get that in this case

$$r^{-2}G_{r,q}^A(s,q;F,A) = \int_{\theta(\pm x_{\min,r})}^{\theta(x_{\max,r})} A_r(q,s,\theta)[su_A(q) + u(\theta + \phi_q)] \cdot u(\theta + \phi_q) d\theta$$

$$+ \int_{\theta_r \cap \theta(\pm x_{\min,r})} A_r(q,s,\theta)[su_A(q) + u(\theta + \phi_q)] \cdot u(\theta + \phi_q) d\theta$$

$$+ \int_{\theta_r \cap \theta(\pm x_{\min,r})} A_r(q,s,\theta)[su_A(q) + u(\theta + \phi_q)] \cdot u(\theta + \phi_q) d\theta$$

$$\rightarrow \int_{\arccos(s)}^{\arccos(s)-\pi} DF(q)[su_A(q) + u(\theta + \phi_q)] \cdot u(\theta + \phi_q) d\theta$$

$$= [DF(q)_{11} + DF(q)_{22}](\pi + s\sqrt{1-s^2} - \arccos(s)).$$

Since

$$r^{-2} \int_{r^2(q+x_{\min,r}(q))} B \cdot n dx = \int_{0}^{2\pi} A_r(q,s,\theta)[su_A(q) + u(\theta + \phi_q)] \cdot u(\theta + \phi_q) d\theta$$

$$\rightarrow \pi[DF(q)_{11} + DF(q)_{22}],$$

we can then use a similar argument as in (52) to show that (49) holds when $i = 1$ and $s \geq 0$. Moreover, due to equation (53), the case $s < 0$ in (49) can be obtained using our former reasoning. Finally, to complete the argument of the proof, we note that the interchange of all the previous limits with the integral sign is possible due to the uniform bounds shown at the beginning of the proof and the Dominated Convergence Theorem.

Remark 11. Arguing as in the proof of the previous lemma, it is possible to show that if $F$ is a vector-valued function that is continuous on $B^i \cap (\partial A)_{\oplus r_0}$ for some $r_0 > 0$, then

$$\sup_{r \leq r_0, q \in (\partial A)_{\oplus r}} \left| r^{-1} \int_{r^2(q)} F1_{B^i} \cdot n dx \right| < \infty, \quad \text{and} \quad \sup_{r \leq r_0, q \in (\partial A)_{\oplus r}} \left| r^{-1} \int_{r^2(q)} F(u)1_{B^i} \cdot du \right| < \infty.$$ 

If in addition $F$ is continuously differentiable on $B^i \cap (\partial A)_{\oplus r_0}$ and $F|_{\partial A} = 0$, then

$$\sup_{r \leq r_0, q \in (\partial A)_{\oplus r}} \left| r^{-2} \int_{r^2(q)} F1_{B^i} \cdot n dx \right| < \infty, \quad \text{and} \quad \sup_{r \leq r_0, q \in (\partial A)_{\oplus r}} \left| r^{-2} \int_{r^2(q)} F(u)1_{B^i} \cdot du \right| < \infty.$$ 

28
An approximation

The following lemma allows to replace $L$ in the definitions of $C_r$ and $D_r$ by a strictly $\beta$-stable homogeneous Lévy basis.

Lemma 5. Let $A \subseteq \mathbb{R}^2$ be a Jordan domain with Lipschitz-regular boundary and $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be such that $F(\cdot)$ is as in Lemma 4. Suppose that $\psi$ is the characteristic exponent of an ID distribution on $\mathbb{R}$ and that there exists $1 \leq \beta \leq 2$ for which $r\psi(r^{-1/\beta}) \rightarrow \psi_{\mu}(\cdot)$, as $r \downarrow 0$, where $\psi_{\mu}$ is the characteristic exponent of an ID distribution $\mu$. Then $\mu$ is (possibly deformed) strictly $\beta$-stable. Furthermore, if $n \in \mathbb{N}$, $(p_j)^n_{j=1} \subseteq \mathbb{R}^2$, $(z_j)^n_{j=1} \subseteq \mathbb{R}$, then for $i = 1, 2$, as $r \downarrow 0$

\[
\left| \int_{\cup_{j=1}^n (\partial A + p_j) \oplus r} [\psi(r^{-1/\beta}z_{r,d,n}(q)) - \psi_{\mu}(r^{-1/\beta}z_{r,d,n}(q))] \, dq \right| \rightarrow 0, \\
\left| \int_{\cup_{j=1}^n (\partial A + p_j) \oplus r} [\psi(r^{-1/\beta}z_{r,c,n}(q)) - \psi_{\mu}(r^{-1/\beta}z_{r,c,n}(q))] \, dq \right| \rightarrow 0,
\]

where

\[z_{r,d,n}(q) := r^{-1} \sum_{j=1}^n \mathbf{1}_{(\partial A + p_j) \oplus r}(q)z_j \sum_{j=1}^n \int_{r^2}(q) F(p_j - \cdot) \mathbf{1}_{B}, \, ndx;\]

\[z_{r,c,n}(q) := r^{-1} \sum_{j=1}^n \mathbf{1}_{(\partial A + p_j) \oplus r}(q)z_j \sum_{j=1}^n \int_{r^2}(q) F(p_j - u) \mathbf{1}_{B}, \, du.\]

Proof. The fact that $\mu$ is strictly stable follows by Theorem 1 in [1], cf. [2]. Observe that the convergence $r\psi(r^{-1/\beta}) \rightarrow \psi_{\mu}(\cdot)$ can always be strengthened to uniform convergence on compacts. Furthermore, thanks to Remark 11 for any $i = 1, 2$, we can choose $a < b$ in such a way that $b \leq z_{r,d}(q) \leq a$, for every $r \leq r_0$ and $q \in \cup_{j=1}^n (\partial A + p_j) \oplus r$. Thus

\[\int_{\cup_{j=1}^n (\partial A + p_j) \oplus r} [\psi(r^{-1/\beta}z_{r,d,n}(q)) - \psi_{\mu}(r^{-1/\beta}z_{r,d,n}(q))] \, dq \leq n \frac{\text{Leb}(\partial A) \oplus r}{r} \times \sup_{a \leq u \leq b} r \left| \psi(r^{-1/\beta}u) - \psi_{\mu}(r^{-1/\beta}u) \right|\]

Since $\partial A$ is Lipschitz, by Theorem 5 and Corollary 1 in [1], we get that as $r \downarrow 0$, $\frac{\text{Leb}(\partial A) \oplus r}{r} \rightarrow 2\mathcal{H}^1(\partial A) < \infty$. A combination of this and the previous estimate, give us (54). The approximation in (55) is shown in a similar way. 

References

[1] Luigi Ambrosio, Andrea Colesanti, and Elena Villa. Outer minkowski content for some classes of closed sets. *Mathematische Annalen*, 342(4):727–748, 2008.

[2] O. E. Barndorff-Nielsen. Hyperbolic distributions and distributions on hyperbolae. *Scandinavian Journal of Statistics*, 5(3):151–157, 1978.

[3] O. E. Barndorff-Nielsen and A. Basse-O’Connor. Quasi Ornstein-Uhlenbeck processes. *Bernoulli*, 17:916–941, 2011.

[4] O. E. Barndorff-Nielsen, F. Benth, and A. Veraart. Ambit processes and stochastic partial differential equations. In *Advanced Mathematical Methods for Finance*, pages 35–74. Springer Berlin Heidelberg, 2011.

[5] O. E. Barndorff-Nielsen, F. Benth, and A. Veraart. Recent advances in ambit stochastics with a view towards tempo-spatial stochastic volatility/intermittency. *Banach Center Publications*, 104:25–60, 2015.

[6] O. E. Barndorff-Nielsen, F. E. Benth, and A. Veraart. Modelling energy spot prices by volatility modulated Lévy-driven Volterra processes. *Bernoulli*, 19(3):803–845, 2013.
[7] O. E. Barndorff-Nielsen and J. Pedersen. Meta-times and extended subordination. *Theory of Probability & Its Applications*, 56(2):319–327, 2012.

[8] O. E. Barndorff-Nielsen and J. Schmiegel. Ambit processes; with applications to turbulence and cancer growth. In Fred Espen Benth, Giulia Di Nunno, Tom Lindstrom, Bernt Oksendal, and Tusheng Zhang, editors, *Stochastic Analysis and Applications: The Abel Symposium 2005*. Springer Berlin Heidelberg, 2007.

[9] A. Basse-O’Connor, S.E. Graversen, and J. Pedersen. Stochastic integration on the real line. *Theory of Probability and Its Applications*, 58:355–380, 2013.

[10] A. Basse-O’Connor, Claudio Heinrich, and Mark Podolskij. On limit theory for Lévy semi-stationary processes. To appear in *Bernoulli*, 2017.

[11] A. Basse-O’Connor, R. Lachièze-Rey, and Mark Podolskij. Power variation for a class of stationary increments Lévy driven moving averages. To appear in *Annals of Probability*, 2017.

[12] A. Basse-O’Connor and Jan Pedersen. Lévy driven moving averages and semimartingales. *Stochastic Processes and their Applications*, 119(9):2970 – 2991, 2009.

[13] A. Basse-O’Connor and J. Rosiński. Characterization of the finite variation property for a class of stationary increment infinitely divisible processes. *Stochastic Processes and their Applications*, 123(6):1871–1890, 2013.

[14] J. Bertoin. *Lévy Processes*. Cambridge Tracts in Mathematics. Cambridge University Press, UK., 1998.

[15] Guido Boffetta and Robert E. Ecke. Two-dimensional turbulence. *Annual Review of Fluid Mechanics*, 44(1):427–451, 2012.

[16] Michael Braverman and Gennady Samorodnitsky. Symmetric infinitely divisible processes with sample paths in Orlicz spaces and absolute continuity of infinitely divisible processes. *Stochastic Processes and their Applications*, 78(1):1–26, 1998.

[17] R. Cairoli and John B. Walsh. Stochastic integrals in the plane. *Acta Math.*, 134:111–183, 1975.

[18] Carsten Chong and Claudia Klüppelberg. Integrability conditions for space–time stochastic integrals: Theory and applications. *Bernoulli*, 210(4):2190–2216, 2015.

[19] José Manuel Corcuera, Emil Hedevang, Mikko S. Pakkanen, and Mark Podolskij. Asymptotic theory for Brownian semi-stationary processes with application to turbulence. *Stochastic Processes and their Applications*, 123(7):2552–2574, 2013.

[20] L. C. Evans and R. F. Gariepy. *Measure Theory and Fine Properties of Functions*. Studies in Advanced Mathematics. CRC Press, Florida, 1992.

[21] Herbert Federer. Curvature measures. *Trans. Amer. Math. Soc.*, 93:418–491, 1959.

[22] Dimitris Gatzouras. Lacunarity of self-similar and stochastically self-similar sets. *Transactions of the American Mathematical Society*, 352(5):1953–1983, 2000.

[23] J Harrison. Flux across nonsmooth boundaries and fractal Gauss/Green/Stokes’ theorems. *Journal of Physics A: Mathematical and General*, 32(28):5317, 1999.

[24] Erich Häusler and Harald Luschgy. *Stable Convergence and Stable Limit Theorems*. Probability Theory and Stochastic Modelling. Springer International Publishing Switzerland, 2015.

[25] E. Hedevang and J. Schniegel. A Lévy based approach to random vector fields: With a view towards turbulence. *International Journal of Nonlinear Sciences and Numerical Simulation*, 15(7-8):411–435, 2014.

[26] Elton P. Hsu. *Stochastic Analysis on Manifolds*. Graduate Studies in Mathematics 38. American Mathematical Society, 2002.

[27] Daniel Hug, Günter Last, and Wolfgang Weil. A local steiner-type formula for general closed sets and applications. *Mathematische Zeitschrift*, 246(1):237–272, 2004.
[28] Nobuyuki Ikeda and Shojiro Manabe. Integral of differential forms along the path of diffusion processes. *Publications of the Research Institute for Mathematical Sciences*, 15(3):827–852, 1979.

[29] J. Ivanovs. Zooming in on a Lévy process at its supremum. *ArXiv e-prints*, 2016.

[30] O. Kallenberg. *Foundations of Modern Probability*. Probability and Its Applications. Springer New York, 2002.

[31] Victor J. Katz. The history of Stokes’ theorem. *Mathematics Magazine*, 52(3):146–156, 1979.

[32] John Lamperti. Semi-stable stochastic processes. *Trans. Amer. Math. Soc.*, 104:62–78, 1962.

[33] MS Pakkanen. Brownian semistationary processes and conditional full support. *International Journal of Theoretical and Applied Finance*, 14:579–586, 2011.

[34] J. Pedersen. The Lévy-Itô decomposition of an independently scattered random measure. *MaPhySto preprint MPS-RR*, 2003.

[35] Jan Pedersen and Orimar Sauri. On Lévy semistationary processes with a gamma kernel. In *XI Symposium on Probability and Stochastic Processes*, volume 69 of *Progress in Probability*, pages 217–239. Springer International Publishing Switzerland, 2015.

[36] S. Raible. *Lévy processes in finance: Theory, numerics, and empirical facts*. PhD thesis, University of Freiburg, 2000.

[37] Balram S. Rajput and Jan Rosiński. Spectral representations of infinitely divisible processes. *Probability Theory and Related Fields*, 82(3):451–487, 1989.

[38] Jan Rataj and Luděk Zajíček. On the structure of set with positive reach. *Math. Nachr.*, 290(11-12):1806–1829, 2017.

[39] Michael Rivera, Xiao-Lun Wu, and Chuck Yeung. Universal distribution of centers and saddles in two-dimensional turbulence. *Phys. Rev. Lett.*, 87:044501, 2001.

[40] Jan Rosiński. Lévy and related jump-type infinitely divisible processes. Lecture Notes, Cornell University, 2007.

[41] Jan Rosiński. Representations and isomorphism identities for infinitely divisible processes. *ArXiv e-prints*, 2016.

[42] K. Sato. *Lévy Processes and Infinitely Divisible Distributions*. Cambridge University Press, UK, 1st edition, 1999.

[43] L.C. Young. An inequality of the Hölder type, connected with Stieltjes integration. *Acta Mathematica*, 67(1):251–282, 1936.

[44] Roger Züst. Integration of Hölder forms and currents in snowflake spaces. *Calculus of Variations and Partial Differential Equations*, 40(1):99–124, 2011.