Separation Conditions for Iterated Function Systems with Overlaps on Riemannian Manifolds

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Abstract
We formulate the weak separation condition and the finite type condition for conformal iterated function systems on Riemannian manifolds with nonnegative Ricci curvature, and generalize the main theorems by Lau et al. (Monatsch Math 156:325–355, 2009). We also obtain a formula for the Hausdorff dimension of a self-similar set defined by an iterated function system satisfying the finite type condition, generalizing a corresponding result by Jin and Yau (Commun Anal Geom 13:821–843, 2005) and Lau and Ngai (Adv Math 208:647–671, 2007) on Euclidean spaces. Moreover, we obtain a formula for the Hausdorff dimension of a graph self-similar set generated by a graph-directed iterated function system satisfying the graph finite type condition, extending a result by Ngai et al. (Nonlinearity 23:2333–2350, 2010).

Keywords Fractal · Self-conformal measure · Riemannian manifold · Weak separation condition · Finite type condition

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1 Introduction

One form of the weak separation condition (WSC) was introduced by Lau and Ngai [18] to study the multifractal formalism of self-similar measures defined by iterated function systems of contractive similitudes with overlaps. Zerner [31] formulated and proved 10 equivalent forms of (WSC). Although strictly weaker than the well-known open set condition (OSC), the WSC leads to many interesting results, such as the validity of the multifractal formalism (see, e.g., [12–14, 18, 29, 30]), equality of Hausdorff, box, and packing dimensions of self-conformal sets and the computation of these dimensions [8, 11, 22], and conditions on absolute continuity of self-similar and self-conformal measures [20–22]. Equivalent forms of the WSCs have also been studied extensive (see, e.g., [22, 31]).

The finite type condition (FTC) was introduced by Ngai and Wang [25] to calculate the Hausdorff dimension of self-similar sets with overlaps. It was generalized independently by Jin and Yau [16] and Lau and Ngai [19] to include (OSC). Lau and Ngai proved that (FTC) implies (WSC) for IFSs of contractive similitudes. This result was generalized by Lau et al. [22] to conformal iterated function systems (CIFSs).

In 2009, Lau et al. [22] formulated (WSC) for conformal iterated function systems on $\mathbb{R}^n$, and proved the equality of the Hausdorff, box and packing dimensions of the associated self-conformal sets. They also studied the absolute continuity of the associated self-conformal measures. The first goal of this paper is to extend results in [16, 19, 25] to Riemannian manifolds that are locally Euclidean.

Let $M$ be a complete $n$-dimensional smooth Riemannian manifold. Assume that $U \subset M$ is open and connected, and $W \subset U$ is a compact set with $\overline{W} = W$, where $\overline{W}$ is the closure of the interior of $W$. Assume that $\{S_i\}_{i=1}^N$ is a conformal iterated function system (CIFS) on $U$ defined as in Sect. 2. Then there exists a unique nonempty compact set $K \subset W$, called the attractor or self-conformal set, satisfying $K = \bigcup_{i=1}^N S_i(K)$ (see [15]).

Given a probability vector $(p_1, \ldots, p_N)$, i.e., $p_i > 0$ for any $i \in \{1, \ldots, N\}$ and $\sum_{i=1}^N p_i = 1$, there exists a unique Borel probability measure $\mu$, called the self-conformal measure, such that $\mu = \sum_{i=1}^N p_i \mu \circ S_i^{-1}$ and $K = \text{supp} (\mu)$ (see [15]).

Let $\Sigma := \{1, \ldots, N\}$, where $N \in \mathbb{N}$ and $N \geq 2$. Let $\Sigma^* := \bigcup_{k \geq 1} \Sigma^k$, where $k \in \mathbb{N}$. For $u = (u_1, \ldots, u_k) \in \Sigma^k$, let $u^- := (u_1, \ldots, u_{k-1})$ and write $S_u := S_{u_1} \circ \cdots \circ S_{u_k}$, $p_u := p_{u_1} \cdots p_{u_k}$. Let $F$ be an open set, defined as in Sect. 2, such that $\overline{F}$ is compact, $W \subset \overline{F} \subset U$, and $\{S_i\}_{i=1}^N$ satisfies (BDP) on $F$. Define

$$r_u := \inf_{x \in F} |\det S_u'(x)|^{\frac{1}{n}}, \quad r := \min_{1 \leq i \leq N} r_i, \quad R_u := \sup_{x \in F} |\det S_u'(x)|^{\frac{1}{n}}, \quad R := \max_{1 \leq i \leq N} R_i.$$  

If $S = S_u$ for some $u \in \Sigma^*$, we let $R_S = R_u$. For $b \in (0, 1]$, let

$$W_b := \{u = (u_1, \ldots, u_k) \in \Sigma^* : R_u \leq b < R_u^- \} \quad \text{and} \quad A_b := \{S_u : u \in W_b\}.$$
Denote the cardinality of a set $E$ by $\# E$. We say that \( \{S_i\}_{i=1}^N \) satisfies WSC if there exist a constant $\gamma \in \mathbb{N}$ and a subset $D \subset W$, with $D^c \neq \emptyset$, such that for any $b \in (0, 1]$ and $x \in W$,

$$\#\{S \in \mathcal{A}_b : x \in S(D)\} \leq \gamma.$$ 

We remark that if the open set condition (OSC) holds, we can take $D$ to be an OSC set and let $\gamma = 1$ to show that (WSC) also holds. For any $a > 0$ and any bounded subsets $D \subset W$ and $A \subset U$, denote the diameter of $A$ by $|A|$, and let

$$A_{a,A,D} := \{S \in A_{a|A|} : S(D) \cap A \neq \emptyset\}, \quad \gamma_{a,A} := \sup_{A \subset U} \#A_{a,A,D}.$$ 

For $S \in \mathcal{A}_b$, let $p_S := \sum \{p_u : S_u = S, u \in W_b\}$.

Theorems 1.1–1.4 generalize analogous results in [22]. In our proofs, the Lebesgue measure in [22] is changed to the more complicated Riemannian volume measure; properties, such as volume doubling, need not hold. We assume that $M$ is a complete Riemannian manifold with nonnegative Ricci curvature. Under this assumption, the Bishop–Gromov comparison theorem implies that $M$ is a doubling space (see, e.g., [3, 5]), i.e., any $2r$-ball in $M$ can be covered by a finite union of a bounded number of $r$-balls, a property that obviously holds on $\mathbb{R}^n$. Another complication arises on manifolds; unlike Euclidean spaces, it is not easy to calculate the volume of a ball in Riemannian manifolds. For Riemannian manifolds with nonnegative Ricci curvature, we use the Bishop–Gromov inequality (see Lemma 2.4), which says that the Riemannian volume of a ball can be controlled by a ball in $\mathbb{R}^n$ with the same radius. This is crucial in the proof of Theorem 1.1.

For a set $K \subset M$, let $\dim_H(K), \dim_B(K)$ and $\dim_P(K)$ be the Hausdorff, box and packing dimensions, respectively. Let $\mathcal{H}^\alpha(K)$ and $\mathcal{P}^\alpha(K)$ be the Hausdorff and packing measures of $K$, respectively.

**Theorem 1.1** Let $M$ be a complete $n$-dimensional smooth orientable Riemannian manifold with non-negative Ricci curvature, and let $U \subset M$ be open and connect. Assume that $\{S_i\}_{i=1}^N$ is a CIFS on $U$ satisfying (WSC), and $K$ is the associated attractor. Then $\alpha := \dim_H(K) = \dim_B(K) = \dim_P(K)$ and $0 \leq \mathcal{H}^\alpha(K) \leq \mathcal{P}^\alpha(K) < \infty$.

Lau et al. [21] formulated a sufficient condition for a self-similar measure defined by an IFS satisfying (WSC) to be singular. Later, Lau and Wang [20] established the necessity, completing the result on absolute continuity in [21]. We extend these results to Riemannian manifolds.

**Theorem 1.2** Assume the same hypotheses as in Theorem 1.1. Let $K$ be the attractor with $\dim_H(K) = \alpha$. Then a self-conformal measure $\mu$ defined by $\{S_i\}_{i=1}^N$ is singular with respect to $\mathcal{H}^\alpha|_K$ if and only if there exist $b \in (0, 1]$ and $S \in \mathcal{A}_b$ such that $p_S > R_S^\alpha$.

**Theorem 1.3** Assume the same hypotheses as in Theorem 1.1. If a self-conformal measure $\mu$ is absolutely continuous with respect to $\mathcal{H}^\alpha|_K$, then the Radon–Nikodym derivative of $\mu$ is bounded.
In the proof of Theorem 1.3, we use an analogue of the Lebesgue density theorem in metric spaces (see, e.g., [10, Theorem 2.9.8] and [4, Lemma 2.1(i)]), which applies to the collection of Borel sets that forms a Vitali relation (see [10] and a brief summary in Sect. 3). By [10, Definition 2.8.9 and Theorem 2.8.18], we have a collection of open balls in Riemannian manifolds that forms a Vitali relation.

We refer the reader to Sect. 4 for the definition of (FTC).

**Theorem 1.4** Assume the same hypotheses as in Theorem 1.1, and let \( W \subset U \) be a compact set with \( \overline{W} = W \). If \( \{S_i\}_{i=1}^{N} \) is a CIFS on \( U \) satisfying (FTC) on some open sets \( \Omega \subset W \), then \( \{S_i\}_{i=1}^{N} \) satisfies (WSC).

Denote the Riemannian distance in \( M \) by \( d(\cdot, \cdot) \). Let \( W \subset M \) be a compact set. We say that \( \{S_i\}_{i=1}^{N} \) is an IFS of contractions on \( W \) if for any \( i \in \{1, \ldots, N\} \), there exists \( \rho_i \in (0, 1) \) such that for any \( x, y \in W \),

\[
d(S_i(x), S_i(y)) \leq \rho_i d(x, y). \tag{1.1}
\]

If equality in (1.1) holds for all \( i \in \{1, \ldots, N\} \) and all \( x, y \in W \), then say that \( \{S_i\}_{i=1}^{N} \) is an IFS of contractive similitudes on \( W \) and call \( \rho_i \) the contraction ratio of \( S_i \).

We say that a Riemannian manifold \( M \) is locally Euclidean if every point of \( M \) has a neighborhood which is isometric to an open subset of a Euclidean space (see e.g., [17]). By [17, Lemma 2 of Theorem 3.6], contractive similitudes only exist on Riemannian manifolds that are locally Euclidean. In Sect. 5, we obtain the following formula for computing the Hausdorff dimension of a self-similar set defined by a finite type IFS of contractive similitudes on a locally Euclidean Riemannian manifold, extending a result in [16, 19] to locally Euclidean Riemannian manifolds (see Theorem 1.5).

**Theorem 1.5** Let \( M \) be a complete \( n \)-dimensional smooth orientable Riemannian manifold that is locally Euclidean, \( W \subset M \) be compact, and \( \{S_i\}_{i=1}^{N} \) be an IFS of contractive similitudes on \( W \) with attractor \( K \). Let \( \lambda_\alpha \) be the spectral radius of the associated weighted incidence matrix \( A_\alpha \). If \( \{S_i\}_{i=1}^{N} \) satisfies (FTC), then \( \dim_H(K) = \alpha \), where \( \alpha \) is the unique number such that \( \lambda_\alpha = 1 \).

This paper is organized as follows. Section 2 introduces the definition of a CIFS, some properties of (WSC), and gives the proof of Theorem 1.1. In Sect. 3, we study the...
absolute continuity of self-conformal measures on Riemannian manifolds and prove Theorems 1.2 and 1.3. Section 4 is devoted to the proof of Theorem 1.4. In Sect. 5, we study finite type IFSs of contractive similitudes and prove Theorem 1.5. Section 6 is devoted to the proof of Theorem 1.6. Finally, we present some examples of CIFSs and GIFSs satisfying (FTC) and (GFTC) on Riemannian manifolds, respectively.

2 The Weak Separation Condition

Let $M$ be a complete $n$-dimensional smooth orientable Riemannian manifold, $U \subset M$ be open and connected, and let $W \subset U$ be a compact set with $\overline{W} = W$. Recall that a map $S : U \to U$ is called conformal if $S'(x)$ is a similarity matrix for any $x \in U$. We say that $\{S_i\}_{i=1}^N$ is a conformal iterated function system (CIFS) on $U$ if

(a) for any $i \in \Sigma$, $S_i : U \to S_i(U) \subset U$ is a conformal $C^{1+\varepsilon}$ diffeomorphism for some $\varepsilon \in (0, 1)$;
(b) $S_i(W) \subset W$ for any $i \in \Sigma$;
(c) $0 < |\det S_i'(x)| < 1$ for any $i \in \Sigma$ and $x \in U$.

Similar to [28], we can find an open set $F$, instead of $U$ satisfying condition (a), such that $\overline{F}$ is compact and $W \subset F \subset F \subset U$. Moreover, conditions (a)–(c) together imply that the bounded distortion property (BDP) holds on $F$, without assuming any separation condition, i.e., there exists a constant $C_1 \geq 1$ such that for any $u \in \Sigma^*$ and $x, y \in F$,

\[ C_1^{-n} \leq \frac{|\det S_u'(x)|}{|\det S_u'(y)|} \leq C_1^n. \]  

(2.1)

It follows that for any $u \in \Sigma^*$,

\[ C_1^{-1} \leq \frac{r_u}{R_u} \leq \frac{R_u}{r_u} \leq C_1. \]  

(2.2)

Moreover, there exists a constant $C_2 \geq 1$ such that for any $u \in \Sigma^*$ and $x, y \in F$,

\[ C_2^{-1} R_u d(x, y) \leq d(S_u(x), S_u(y)) \leq C_2 R_u d(x, y). \]  

(2.3)

Let $\nu$ be the Riemannian volume measure, and let $A \subset U$ be a measurable set. Denote the Jacobian determinant of a function $f$ by $Jf$. Then

\[ \nu(S_u(A)) = \int_A |J(S_u(x))| d\nu = \int_A |\det S_u'(x)| d\nu. \]  

(2.4)

(see, e.g., [1, Proposition 8.1.10]).
Hence \( R_{uv} \leq R_u R_v \) and \( r_{uv} \geq r_u r_v \). In particular, for \( S = S_u \in A_b, u = (u_1, \ldots, u_k) \in \mathcal{W}_b \),

\[
br \leq C_1 r_{u_1,\ldots,u_{k-1}} r_{u_k} \quad \text{(by (2.2))}
\]

\[
\leq C_1 r_u \quad \text{(by (2.1))}
\]

\[
\leq C_1 R_u,
\]

i.e.,

\[
b < \frac{C_1}{r} R_S \quad \text{for all } S \in A_b.
\]

**Lemma 2.1** Let \( M \) be a complete \( n \)-dimensional smooth orientable Riemannian manifold, \( U \subset M \) be open and connected, \( \{S_i\}_{i=1}^N \) be a CIIF on \( U \), and \( F \subset U \) be an open set described as above such that \( \{S_i\}_{i=1}^N \) satisfies (BDP) on \( F \). Then for any \( b \in (0, 1] \) and any measurable set \( A \subset F \), the following hold.

(a) For any \( u \in \mathcal{W}_b \),

\[
\left( \frac{br}{C_1} \right)^n v(A) \leq v(S_u(A)) \leq b^n v(A).
\]

(b) For any \( u, v \in \mathcal{W}_b \),

\[
\left( \frac{r}{C_1} \right)^n v(S_v(A)) \leq v(S_u(A)) \leq \left( \frac{C_1}{r} \right)^n v(S_v(A)).
\]

**Proof** (a) Let \( u = (u_1, \ldots, u_k) \in \mathcal{W}_b \). Then by the definition of \( \mathcal{W}_b \),

\[
\sup_{x \in F} |\det S'_u(x)| \leq b^n \leq \sup_{x \in F} |\det S'_{u-}(x)|. \quad (2.6)
\]

Hence for any \( x \in F \),

\[
b^n \geq |\det S'_u(x)| = |\det S'_{u-}(S_{u_k}(x))| \cdot |\det S'_{u_k}(x)| \quad \text{(by (2.6))}
\]

\[
\geq r^n \sup_{x \in F} |\det S'_{u-}(S_{u_k}(x))| \quad \text{(by (2.1))}
\]

\[
> \left( \frac{br}{C_1} \right)^n \quad \text{(by (2.6)).}
\]

It follows that

\[
\int_A b^n d\nu \geq \int_A |\det S'_u(x)| d\nu \geq \int_A \left( \frac{br}{C_1} \right)^n d\nu,
\]
which proves (a) by (2.4).

(b) Making use of part (a), we have

\[ \nu(S_u(A)) \leq b^n \nu(A) \leq \left( \frac{C_1}{r} \right)^n \nu(S_v(A)). \]

Similarly,

\[ \nu(S_v(A)) \leq b^n \nu(A) \leq \left( \frac{C_1}{r} \right)^n \nu(S_u(A)), \]

which proves (b). \( \square \)

Let \( \mathcal{F} := \{ S_v S_u^{-1} : u, v \in \Sigma^* \} \). It is possible that \( \tau = S_v S_u^{-1} \) can be simplified to \( S_v' S_u^{-1} \). Denote the domain of \( \tau \) by \( \text{Dom}(\tau) := S_u(F) \) (containing \( S_u(F) \)). The proof of the following lemma is similar to that of [22, Lemma 2.2] and is omitted.

**Lemma 2.2** Assume the same hypotheses of Lemma 2.1. Then for any \( u, v \in \Sigma^* \) and any \( x, y \in \text{Dom}(S_v' S_u^{-1}) \), we have

\[ \frac{|\det(S_v S_u^{-1})'(x)|}{|\det(S_v S_u^{-1})'(y)|} \leq C_1^{2n}. \]

**Lemma 2.3** Assume the same hypotheses of Lemma 2.1. Let \( \tau = S_v S_u^{-1} = S_v' S_u^{-1} \in \mathcal{F} \) with \( \text{Dom}(\tau) = S_u(F) \). Then the following hold.

(a) For any measurable subset \( A \subseteq \text{Dom}(\tau) \),

\[ \left( \frac{r_v}{R_u} \right)^n \nu(A) \leq \nu(\tau(A)) \leq \left( \frac{R_v}{r_u} \right)^n \nu(A). \]

(b) For any \( A, B \) belonging to some collection \( \mathcal{C} \) of measurable subsets of \( F \), suppose \( C \geq 1 \) is a constant such that

\[ C^{-1} \nu(B) \leq \nu(A) \leq Cv(B). \]

Then for any \( A, B \in \mathcal{C} \) such that \( A, B \subseteq \text{Dom}(\tau) \),

\[ C^{-1} C_1^{-2n} \nu(\tau(B)) \leq \nu(\tau(A)) \leq CC_1^{2n} \nu(\tau(B)). \]

**Proof** (a) For \( x \in \text{Dom}(\tau) \), let \( y = S_u^{-1}(x) \in F \). Then

\[ |\det \tau'(x)| = |\det(S_v S_u^{-1})'(x)| = |\det S_v'(S_u^{-1}(x))| \cdot |\det(S_u^{-1})'(x)| = \frac{|\det S_v'(y)|}{|\det S_u'(y)|}. \]
Hence
\[
\left( \frac{r'}{R'} \right)^n \leq |\det \tau'(x)| \leq \left( \frac{R'}{r'} \right)^n.
\]

Thus,
\[
\int_A \left( \frac{r'}{R'} \right)^n d\nu \leq \int_A |\det \tau'(x)| d\nu \leq \int_A \left( \frac{R'}{r'} \right)^n d\nu,
\]
which proves (a) by (2.4).

(b) Note that
\[
\nu(\tau(A)) \leq \left( \frac{R'}{r'} \right)^n \nu(A) \quad \text{(by part (a))}
\]
\[
\leq C \left( \frac{R'}{r'} \right)^n \nu(B) \quad \text{(by assumption)}
\]
\[
\leq C \left( \frac{R'}{r'} \right)^n \left( \frac{R'}{r'} \right)^n \nu(\tau(B)) \quad \text{(by part (a))}
\]
\[
\leq CC_1^{2n} \nu(\tau(B)) \quad \text{(by (2.2)).}
\]

On the other hand,
\[
\nu(\tau(A)) \geq \left( \frac{r'}{R'} \right)^n \nu(A) \quad \text{(by part (a))}
\]
\[
\geq C^{-1} \left( \frac{r'}{R'} \right)^n \nu(B) \quad \text{(by assumption)}
\]
\[
\geq C^{-1} \left( \frac{r'}{R'} \right)^n \left( \frac{r'}{R'} \right)^n \nu(\tau(B)) \quad \text{(by part (a))}
\]
\[
\geq C^{-1} C_1^{-2n} \nu(\tau(B)) \quad \text{(by (2.2)).}
\]

Combining the above proves (b). \qed

Throughout this paper, we denote the open ball centered on \( x \) with radius \( r > 0 \) by \( B_r(x) \). We remark that the following lemma also is true for closed balls.

**Lemma 2.4** (Bishop–Gromov inequality (see, e.g., [2, 5, 6])) Let \( M \) be a complete \( n \)-dimensional Riemannian manifold with non-negative Ricci curvature, and \( B_r(x) \) be an \( r \)-ball in \( M \). Then
\[
\nu(B_r(x)) \leq c_n r^n,
\]
where \( c_n = \pi^{\frac{n}{2}} / \Gamma\left(\frac{n}{2} + 1\right) \) is the volume of the unit ball in \( \mathbb{R}^n \).
We use some arguments in [22, Proposition 3.1], Lemma 2.4 and the doubling property of Riemannian manifolds with non-negative Ricci curvature to prove the following proposition.

**Proposition 2.5**  Let $M$ be a complete $n$-dimensional orientable Riemannian manifold with non-negative Ricci curvature. Assume the same hypotheses of Lemma 2.1. Then the following are equivalent:

(a) $\{S_i\}_{i=1}^N$ satisfies (WSC);
(b) there exist $a > 0$ and a nonempty subset $D \subset W$ such that $\gamma_{a,D} < \infty$;
(c) for any $a > 0$ and any nonempty subset $D \subset W$, $\gamma_{a,D} < \infty$;
(d) for any $D \subset W$ there exists $\gamma = \gamma(D)$ (depending only on $D$) such that for any $b \in (0, 1]$ and $x \in W$,

$$\# \{ S \in A_b : x \in S(D) \} \leq \gamma.$$  

**Proof** (a) $\Rightarrow$ (b) To obtain (b), we will prove that there exist $\gamma' \in \mathbb{N}$ and $D \subset W$ with $D^o \neq \emptyset$ such that for any $x \in W$ and $b \in (0, 1]$ satisfying $B_b(x) \subset U$,  

$$\# \{ S \in A_b : S(D) \cap B_b(x) \neq \emptyset \} \leq \gamma'.$$

Let $D \subset W$ be defined as in the definition of WSC (see Sect. 1). For $x \in W$, $b \in (0, 1]$ satisfying $B_b(x) \subset U$, we assume that $S \in A_b$ such that $S(D) \cap B_b(x) \neq \emptyset$, and $x' \in D$ such that $S(x') \in B_b(x)$. Then for any $y \in D$,

$$d(S(y), x) \leq d(S(y), S(x')) + d(S(x'), x) \quad \text{(triangle inequality)}$$

$$\leq C_2 R_S d(y, x') + b \quad \text{(by (2.3))}$$

$$\leq (1 + C_2 |D|)b$$

$$=: \eta,$$

and thus $S(D) \subset \overline{B_\eta(x)}$. It follows from (a) that each point in $W$ is covered by at most $\gamma$ elements of $\{S(D) : S \in A_b\}$. Hence

$$\sum \{ v(S(D)) : S \in A_b, S(D) \cap B_b(x) \neq \emptyset \} \leq \gamma v(B_\eta(x)). \quad (2.7)$$

By Lemma 2.1(a), we have

$$\left( \frac{b r}{C_1} \right)^n v(D) \cdot \# \{ S \in A_b : S(D) \cap B_b(x) \neq \emptyset \}$$

$$\leq \sum \{ v(S(D)) : S \in A_b, S(D) \cap B_b(x) \neq \emptyset \}$$

$$\leq \gamma v(B_\eta(x)) \quad \text{(by (2.7))}$$

$$\leq \gamma c n \eta^n \quad \text{(by Lemma 2.4)}$$

$$=: \gamma c b^n.$$
where \( c := c_n(1 + C_2|D|)^n \). Hence there exists \( \gamma' \in \mathbb{N} \) such that

\[
\#\{S \in \mathcal{A}_b : S(D) \cap B_b(x) \neq \emptyset\} \leq \frac{\gamma cC^n}{r^n \nu(D)} \leq \gamma'.
\]

(b) \( \Rightarrow \) (c) Use (2.3), the space doubling property, and a technique in the proof of [22, Proposition 3.1 (b) \( \Rightarrow \) (c)]. We omit the details.

(c) \( \Rightarrow \) (d) Let \( D \subset W \). Then for any \( x \in W \) and \( b \in (0, 1] \) satisfying \( B_{b/2}(x) \subset U \),

\[
\#\{S \in \mathcal{A}_b : x \in S(D)\} \leq \#\{S \in \mathcal{A}_b, S(D) \cap B_{b/2}(x) \neq \emptyset\}
= \#A_{1, b/2}(x), D
\leq \gamma_1, D < \infty.
\]

(d) \( \Rightarrow \) (a) is trivial.

Proof of Theorem 1.1 Use the above lemmas and propositions, as in [22, Theorem 3.2]; we omit the details.

As an important consequence of Theorem 1.1, we obtain the following estimate for \( \#A_b \) and a formula for \( \dim_H(F) \). Making use of (2.3), (2.5) and Proposition 2.5, we obtain the following analogue of [22, Corollary 3.3]. The proof is similar and is omitted.

**Corollary 2.6** Let \( \{S_i\}_{i=1}^N \) and \( \alpha \) be as in Theorem 1.1. Then there exists a constant \( C_3 > 0 \) such that for any \( b \in (0, 1] \),

\[
C_3^{-1} b^{-\alpha} \leq \#A_b \leq C_3 b^\alpha.
\]

Consequently,

\[
\alpha = \dim_H(K) = \dim_B(K) = \dim_P(K) = -\lim_{b \to 0^+} \frac{\log \#A_b}{\log b}.
\]

### 3 Absolute Continuity of Self-conformal Measures

Throughout this section, we let \( M \) be a complete \( n \)-dimensional smooth orientable Riemannian manifold with non-negative Ricci curvature, \( U \subset M \) be open and connected, \( W \subset U \) be a compact set with \( \overline{W} = W \), and \( \{S_i\}_{i=1}^N \) be a CIFS with attractor \( K \subset W \) on \( U \) defined as in Sect. 2. The proof of the following proposition is similar to that of [22, Proposition 4.1] and is omitted.

**Proposition 3.1** Assume that \( \{S_i\}_{i=1}^N \) satisfies (WSC). Then for any finite subset \( \Lambda \subset \Sigma^*, \) the family \( \{S_v : v \in \Lambda\} \) also satisfies (WSC).

For \( u \in \Sigma^*, \) let \( [u] := \{u' \in \Sigma^* : S_u = S_{u'}\} \). The following lemma can be proved by using Corollary 2.6 and the argument in [22, Lemma 4.2].
Lemma 3.2 Assume that \( \{ S_i \}_{i=1}^N \) satisfies (WSC). Let \( \{ p_i \}_{i=1}^N \) be the associated probability weights, and let \( K \) be the attractor with \( \dim_H(K) = \alpha \). For \( b \in (0, 1] \) and \( \Lambda \subset W_b \), let

\[
\widetilde{\Lambda} := \left\{ u \in \Lambda : \sum_{u' \in [u] \cap \Lambda} p_{u'} > \frac{b^\alpha}{4C_3} \right\},
\]

where \( C_3 \) is as in Corollary 2.6. Then \( P(\Lambda) > 1/2 \) implies \( P(\widetilde{\Lambda}) > 1/4 \).

**Proof of Theorem 1.2** The proof of Theorem 1.2 follows by using Proposition 2.5, Lemma 3.2, and a technique in the proofs of [21, Theorem 1.1] and [20, Theorem 1.1]; we omit the details. \( \square \)

We state the definition of Vitali relation (see [10, Definition 2.8.16]). Let \( X \) be a metric space. Any subset of

\[
\{(x, A) : x \in A \subset X\}
\]

is called a covering relation. Let \( C \) be a covering relation and \( Z \subset X \), we let

\[
C(Z) := \{ A \subset X : (x, A) \in C \text{ for some } x \in Z \}.
\]

We say \( C \) is fine at \( x \) if

\[
\inf\{|A| : (x, A) \in C\} = 0.
\]

We say \( C(Z) \) is fine on \( Z \) if for any \( x \in Z \), \( C(x) \) is fine at \( x \).

Following [10], we say that a Borel measure \( \phi \) is regular on \( X \) if each subset \( A \) of \( X \) is contained in a Borel subset \( B \) for which \( \phi(A) = \phi(B) \). It is known that \( s \)-dimensional Hausdorff measure is regular (see, e.g., [23]). Let \( \phi \) be a regular Borel measure on \( X \). A covering relation \( V \) is called a \( \phi \) Vitali relation if \( V(X) \) is a family of Borel sets, \( V \) is fine on \( X \), and moreover, whenever \( C \subset V \), \( Z \subset X \), and \( C \) is fine on \( Z \), then \( C(Z) \) has a countable disjoint subfamily covering \( \phi \) almost all of \( Z \).

By making use of [10, Definition 2.8.9 and Theorem 2.8.18], we see that \( \{ B_r(x) : x \in K, 0 < r < \infty \} \) is a \( \phi \) Vitali relation on \( K \). Assume that \( \mu \) is absolutely continuous with respect to \( \phi \) and let \( f \) be the Radon–Nikodym derivative. By [10, Theorem 2.9.8] or [4, Lemma 2.1(i)], we have

\[
\lim_{r \to 0} \frac{\mu(B_r(x))}{\phi(B_r(x))} = \lim_{r \to 0} \frac{1}{\phi(B_r(x))} \int_{B_r(x)} f \, d\phi = f(x)
\]

for \( \phi \) almost all \( x \in K \). Assume that \( A \subset K \) and \( f = \chi_A \). It follows that

\[
\lim_{r \to 0} \frac{\phi(A \cap B_r(x))}{\phi(B_r(x))} = \lim_{r \to 0} \frac{1}{\phi(B_r(x))} \int_{B_r(x)} \chi_A \, d\phi = 1
\]

(3.1)
for $\phi$ almost all $x \in A$.

We use some arguments in [22, Theorem 1.3] and (3.1) to prove Theorem 1.3.

**Proof of Theorem 1.3** Let $\phi := \mathcal{H}^\alpha|_K$ and $f$ be the Radon–Nikodym derivative of $\mu$ with respect to $\phi$. Suppose that $f$ is unbounded. For any $m > 0$, let $A = A(m) := \{t \in K : f(t) > m\}$. Then for any $\delta > 0$, by (3.1), there exist $x \in K$ and $b > 0$ such that $B_{b\delta}(x) \subset U$ and

$$\phi(A \cap B_{b\delta}(x)) > \frac{1}{2} \phi(B_{b\delta}(x)). \quad (3.2)$$

Hence

$$\mu(B_{b\delta}(x)) = \int_{B_{b\delta}(x)} f(t) d\phi(t) \geq \int_{A \cap B_{b\delta}(x)} f(t) d\phi(t) \geq m \phi(A \cap B_{b\delta}(x)) > \frac{1}{2} m \phi(B_{b\delta}(x)) \quad (by (3.2)). \quad (3.3)$$

Let $\delta := (C_2 + 1)|K|$ in (3.3). Note that $x \in K = \bigcup_{S \in A_b} S(K)$. Hence there exists $S \in A_b$ such that $x \in S(K)$. By (2.3), we have

$$|S(K)| \leq C_2 R_S |K| < b\delta,$$

and thus

$$S(K) \subset B_{b\delta}(x). \quad (3.4)$$

For $S = S_{u_1 \ldots u_k} \in A_b$, we have $R_S = R_u \leq b < R_u^\delta$. Hence by (2.2),

$$br < R_{u_1 \ldots u_{k-1} r_{u_k}} \leq C_1 r_{u_1 \ldots u_{k-1} r_{u_k}} \leq C_1 r_u \leq C_1 R_u. \quad (3.5)$$

Combining (2.3), (3.4) and (3.5), we have

$$\phi(B_{b\delta}(x)) \geq \phi(S(K)) = \mathcal{H}^\alpha(S(K)) \geq C_2^{-\alpha} R_S^\alpha \mathcal{H}^\alpha(K) > \left(\frac{r}{C_1 C_2}\right)^{\alpha} b^\alpha \phi(K). \quad (3.6)$$

Combining (3.3) and (3.6), we obtain

$$\mu(B_{b\delta}(x)) > \frac{1}{2} m \left(\frac{r}{C_1 C_2}\right)^{\alpha} b^\alpha \phi(K). \quad (3.7)$$
On the other hand,
\[
\mu(B_{b\delta}(x)) = \sum_{S \in A_b, S(K) \cap B_{b\delta}(x) \neq \emptyset} p_S \mu \circ S^{-1}(B_{b\delta}(x)) \\
\leq \sum_{S \in A_b, S(K) \cap B_{b\delta}(x) \neq \emptyset} p_S.
\]

Let $S_0 \in A_b$ such that $p_{S_0}$ is the maximum among all the summands in the above summation. Then
\[
\mu(B_{b\delta}(x)) \leq p_{S_0} \cdot \#\{S \in A_b : S(K) \cap B_{b\delta}(x) \neq \emptyset\} \leq p_{S_0} \cdot \gamma_{1/(2\delta), K}. \quad (3.8)
\]

We choose $m$ such that
\[
\frac{1}{2} m \left( \frac{r}{C_1 C_2} \right)^{\alpha} \phi(K) > \gamma_{1/(2\delta), K}. \quad (3.9)
\]

It follows from (3.7)–(3.9) that there exists $S \in A_b$ such that
\[
b^\alpha \cdot \gamma_{1/(2\delta), K} < \mu(B_{b\delta}(x)) \leq p_{S_0} \cdot \gamma_{1/(2\delta), K}.
\]

Hence
\[
R_{S_0}^\alpha \leq b^\alpha < p_{S_0}.
\]

It now follows from Theorem 1.2 that $\mu$ is singular with respect to $\phi$, a contradiction. This proves the theorem. \(\square\)

## 4 The Finite Type Condition

In this section we extend (FTC) to Riemannian manifolds and prove Theorem 1.4. We follow the set-up in [19, 22]. We begin by defining a partial order $\preceq$ on $\Sigma^*$. For $u, v \in \Sigma^*$, we denote by $u \preceq v$ if $u$ is an initial segment of $v$ or $u = v$. We denote by $u \not\preceq v$ if $u \preceq v$ does not hold. Let $\{M_k\}_{k=0}^\infty$ be a sequence of index sets such that for any $k \geq 0$, $M_k$ is a finite subset of $\Sigma^*$. We say that $M_k$ is an antichain if for any $u, v \in M_k$, $u \not\preceq v$ and $v \not\preceq u$. Let
\[
m_k = m_k(M_k) := \min\{|u| : u \in M_k\}, \quad \overline{m}_k = \overline{m}_k(M_k) := \max\{|u| : u \in M_k\},
\]

where $|u|$ is the length of $u$.

**Definition 4.1** We say that $\{M_k\}_{k=0}^\infty$ is a sequence of nested index sets if it satisfies the following conditions:

1. both $\{m_k\}$ and $\{\overline{m}_k\}$ are nondecreasing, and $\lim_{k \to \infty} m_k = \lim_{k \to \infty} \overline{m}_k = \infty$;
(b) for any \( k \geq 0 \), \( \mathcal{M}_k \) is an antichain in \( \Sigma^* \);
(c) for any \( v \in \Sigma^* \) with \( |v| > m_k \), there exists \( u \in \mathcal{M}_k \) such that \( u \leq v \);
(d) for any \( v \in \Sigma^* \) with \( |v| < m_k \), there exists \( u \in \mathcal{M}_k \) such that \( v \leq u \);
(e) there exists a positive integer \( L \), independent of \( k \), such that for any \( u \in \mathcal{M}_k \) and \( v \in \mathcal{M}_{k+1} \) with \( u \leq v \), we have \( |v| - |u| \leq L \).

Note that \( \mathcal{M}_k \) can intersect \( \mathcal{M}_{k+1} \), and \( \{\Sigma^k\}_{k=0}^\infty \) is an example of a sequence of nested index sets. For each integer \( k \geq 0 \), let \( \mathcal{V}_k \) be the set of \( k \)-th level vertices defined as

\[
\mathcal{V}_0 := \{(u, 0)\} \quad \text{and} \quad \mathcal{V}_k := \{(S_u, k) : u \in \mathcal{M}_k\} \text{ for any } k \geq 1.
\]

We write \( \omega_{\text{root}} := (u, 0) \) and call it the root vertex. Let \( \mathcal{V} := \bigcup_{k \geq 0} \mathcal{V}_k \) be the set of all vertices. For \( \omega = (S_u, k) \), we define \( S_\omega := S_u \). Let \( W \subset M \) be a compact set. For an IFS \( \{S_i\}_{i=1}^N \) on \( W \), let \( \Omega \subset W \) be a nonempty open set that is invariant under \( \{S_i\}_{i=1}^N \). Such a set exists if \( \{S_i\}_{i=1}^N \) are contractions on \( W \). We say that two \( k \)-th level vertices \( \omega, \omega' \in \mathcal{V}_k \) are neighbors if \( S_\omega(\Omega) \cap S_{\omega'}(\Omega) \neq \emptyset \). Let \( \mathcal{N}(\omega) := \{\omega' : \omega' \in \mathcal{V}_k \text{ is a neighbor of } \omega\} \), which is called the neighborhood of \( \omega \).

**Definition 4.2** For any two vertices \( \omega \in \mathcal{V}_k \) and \( \omega' \in \mathcal{V}_{k'} \), let

\[
\tau := S_{\omega'}S_\omega^{-1} : \bigcup_{\sigma \in \mathcal{N}(\omega)} S_\sigma(W) \to W.
\]

We say \( \omega \) and \( \omega' \) are equivalent, i.e., \( \omega \sim \omega' \), if the following conditions hold

(a) \( \{S_{\sigma'} : \sigma' \in \mathcal{N}(\omega')\} = \{\tau \circ S_\sigma : \sigma \in \mathcal{N}(\omega)\} \);
(b) for \( \sigma \in \mathcal{N}(\omega) \) and \( \sigma' \in \mathcal{N}(\omega') \) such that \( S_{\sigma'} = \tau \circ S_\sigma \), and for any positive integer \( k_0 \geq 1 \), \( u \in \Sigma^* \), \( u \) satisfies \( (S_\sigma \circ S_u, k + k_0) \in \mathcal{V}_{k+k_0} \) if and only if it satisfies \( (S_{\sigma'} \circ S_u, k' + k_0) \in \mathcal{V}_{k'+k_0} \).

It is easy to see that \( \sim \) is an equivalence relation. Denote the equivalence class of \( \omega \) by \( [\omega] \), and call it the neighborhood type of \( \omega \).

Let \( \omega = (S_u, k) \in \mathcal{V}_k \) and \( \sigma = (S_\nu, k + 1) \in \mathcal{V}_{k+1} \). Suppose that there exists \( w \in \Sigma^* \) such that

\[
v = (u, w).
\]

Then we connect a directed edge from \( \omega \) to \( \sigma \), and denote this as \( \omega \rightarrow \sigma \). We call \( \omega \) a parent of \( \sigma \) and \( \sigma \) an offspring of \( \omega \). Define a graph \( \mathcal{G} := (\mathcal{V}, \mathcal{E}) \), where \( \mathcal{E} \) is the set of all directed edges. We first remove from \( \mathcal{G} \) all but the smallest (in the lexicographic order) directed edges going to a vertex. After that, we remove all vertices that do not have any offspring, together with all vertices and edges leading only to them. The
resulting graph is called the reduced graph. Denote it by \( \mathcal{G}_R := (\mathcal{V}_R, \mathcal{E}_R) \), where \( \mathcal{V}_R \) and \( \mathcal{E}_R \) are the sets of all vertices and all edges, respectively.

The proof of the following proposition is similar to that of [19, Proposition 2.4]; we omit the details.

**Proposition 4.3** Let \( \omega \) and \( \omega' \) be two vertices in \( \mathcal{V} \) with offspring \( u_1, \ldots, u_m \) and \( u'_1, \ldots, u'_s \) in \( \mathcal{G}_R \), respectively. Suppose \( [\omega] = [\omega'] \). Then
\[
\{ [u_i] : 1 \leq i \leq m \} = \{ [u'_i] : 1 \leq i \leq s \}
\]
counting multiplicity. In particular, \( m = s \).

**Definition 4.4** Let \( \{ S_i \}_{i=1}^N \) be an IFS on \( W \) consisting of injective contractions, and let \( \mathcal{V}/\sim := \{ [\omega] : \omega \in \mathcal{V} \} \). We say that \( \{ S_i \}_{i=1}^N \) satisfies the finite type condition (FTC) if there exists a nonempty invariant open set \( \Omega_1 \subset W \) with respect to some sequence of nested index sets \( \{ M_k \}_{k=0}^\infty \) and such that
\[
\#(\mathcal{V}/\sim) < \infty.
\]

Such a set \( \Omega_1 \) is called a finite type condition set (FTC set).

Obviously, if \( \{ S_i \}_{i=1}^N \) satisfies (OSC), then \( \#(\mathcal{V}/\sim) = 1 \), and thus \( \{ S_i \}_{i=1}^N \) satisfies (FTC). We assume that \( \{ S_i \}_{i=1}^N \) is a CIFS in the rest of this section.

**Lemma 4.5** Let \( M \) be a complete \( n \)-dimensional smooth orientable Riemannian manifold, \( U \subset M \) be open and connected, \( W \subset U \) be a compact set with \( \overline{W} = W \), \( \{ S_i \}_{i=1}^N \) be a CIFS on \( U \), and \( F \subset U \) be open described as in Sect. 2 such that \( \{ S_i \}_{i=1}^N \) satisfies (BDP) on \( F \). Assume that \( \{ S_i \}_{i=1}^N \) satisfies (FTC) with \( \Omega_1 \subset W \) being an FTC set. Then there exists a constant \( C_4 \geq 1 \) such that for any two neighboring vertices \( \omega_1 \) and \( \omega_2 \), we have
\[
C_4^{-1} \leq \frac{\nu(S_{\omega_1}(\Omega))}{\nu(S_{\omega_2}(\Omega))} \leq C_4.
\]

**Proof** Let \( T \) be a neighborhood type, and \( \omega \) be a vertex such that \( [\omega] = T \). Let
\[
\mathcal{N}(\omega) = \{ \omega_0, \omega_1, \ldots, \omega_m \},
\]
where \( \omega_0 = \omega \). Substituting \( S_{\omega_0}(\Omega) = A \) and \( S_{\omega_1}S_{\omega_0}^{-1} = \tau \) into Lemma 2.3(a) and using (2.2), we see that there exists a constant \( c_1 \geq 1 \) such that for any \( i \in \{0, 1, \ldots, m\} \),
\[
c_1^{-1} \nu(S_{\omega_0}(\Omega)) \leq \nu(S_{\omega_i}(\Omega)) \leq c_1 \nu(S_{\omega_0}(\Omega)). \tag{4.1}
\]
Let \( \omega \sim \omega' \), \( \tau = S_{\omega'}S_{\omega}^{-1} \in \mathcal{F} \), and
\[
\mathcal{N}(\omega') = \{ \omega'_0, \omega'_1, \ldots, \omega'_m \},
\]
where $\omega'_0 = \omega'$. Without loss of generality, for any $i \in \{0, 1, \ldots, m\}$ we can assume $S_{\omega'_0} = \tau \circ S_{\omega_i}$. It follows from the definition of $\tau$ that $S_{\omega_i}(\Omega) \subset \operatorname{Dom}(\tau)$. Making use of (4.1) and substituting $S_{\omega_i}(\Omega) = A$, $S_{\omega_0}(\Omega) = B$ and $S_{\omega'_i}S_{\omega_i}^{-1} = \tau$ into Lemma 2.3(b), we see that for any $i \in \{0, 1, \ldots, m\}$,

$$c_1^{-1}C_1^{-2n} \nu(S_{\omega'_0}(\Omega)) \leq \nu(S_{\omega'_i}(\Omega)) = \nu(\tau \circ S_{\omega_i}(\Omega)) \leq c_1C_1^{-2n} \nu(S_{\omega'_0}(\Omega)).$$

Hence the lemma holds for any two neighboring vertices $\omega_1, \omega_2$ with one of them being of type $T$. Since there are only finitely many distinct neighborhood types, the result follows.

**Lemma 4.6** Assume the same hypotheses of Lemma 4.5. Then for any $u \in \Sigma^k$, $k \in \mathbb{N}$, we have

$$r^{kn} \leq \frac{\nu(S_u(\Omega))}{\nu(\Omega)} \leq R^{kn}. \quad (4.2)$$

**Proof** Let $x \in F$. Then by the definition of $R$, we have

$$|\det S'_u(x)| = |\det S'_{u^i}(S_{u^i}(x))| \cdot |\det S'_{u_k}(x)| \leq R^R_{u^i} - R^R_{u_k} \leq R^R_{u^i} \cdots R^R_{u_k} \leq R^{kn}.$$  

Note that $\Omega \subset F$. Then by (2.4), we have

$$\nu(S_u(\Omega)) = \int_\Omega |\det S'_u(x)|d\nu(x) \leq R^{kn} \nu(\Omega).$$

This proves the right side of (4.2). On the other hand, if $x \in F$, then by the definition of $r$, we have

$$|\det S'_u(x)| = |\det S'_{u^i}(S_{u^i}(x))| \cdot |\det S'_{u_k}(x)| \geq r^R_{u^i} - r^R_{u_k} \geq r^R_{u^i} \cdots r^R_{u_k} \geq r^{kn}.$$  

Consequently,

$$\nu(S_u(\Omega)) = \int_\Omega |\det S'_u(x)|d\nu(x) \geq r^{kn} \nu(\Omega).$$

This proves the left side of (4.2).  

We now prove Theorem 1.4.  

\(\square\) Springer
Proof of Theorem 1.4 For \( b \in (0, 1] \) and \( x \in W \), let \( S(\Omega) := \{S \in A_b : x \in S(\Omega)\} \).

List all elements of \( S(\Omega) \) as \( S_{u_1}, \ldots, S_{u_m} \). For any \( j \in \{1, \ldots, m\} \), note that \( u_j \in M_k \) for some \( k \). Let \( \tilde{u}_j = (\tilde{u}_j, \tilde{v}_j) \), where \( \tilde{u}_j \) is the longest initial segment of \( u_j \) such that \( \tilde{u}_j \in M_k \).

Without loss of generality, we assume

\[
k_1 = \min\{k_j : 1 \leq j \leq m\} = k.
\]

Then \( \tilde{u}_1 \in M_k \). For any \( j \in \{1, \ldots, m\} \), let \( u'_j \) be the initial segment of \( u_j \) such that \( u'_j \in M_k \). Note that \( u'_1 = \tilde{u}_1 \). Since \( x \in S(\Omega) \) for any \( S \in S(\Omega) \), we see that

\[
\omega_2 = (S_{u_2'}, k), \ldots, \omega_m = (S_{u_m'}, k)
\]

are neighbors of \( \omega_1 = (S_{u_1'}, k) \). It follows from the definition of (FTC) that there exists a positive integer \( c_2 \) independent of \( x \) and \( b \) such that

\[
\#\{\omega_1, \ldots, \omega_m\} \leq c_2.
\] (4.3)

By Lemma 4.5, for any \( j \in \{2, \ldots, m\} \), we have

\[
C_4^{-1} \leq \frac{v(S_{u_j'}(\Omega))}{v(S_{u_j}(\Omega))} \leq C_4. \tag{4.4}
\]

For any \( j \in \{2, \ldots, m\} \), making use of Lemma 2.1(b), we have

\[
\left( \frac{r}{C_1} \right)^n \leq \frac{v(S_{u_j}(\Omega))}{v(S_{u_1}(\Omega))} \leq \left( \frac{C_1}{r} \right)^n. \tag{4.5}
\]

It follows from (4.4) and (4.5) that

\[
C_4^{-1} \left( \frac{r}{C_1} \right)^n \leq \frac{v(S_{u_j}(\Omega))}{v(S_{u_j'}(\Omega))} \cdot \frac{v(S_{u_j'}(\Omega))}{v(S_{u_1}(\Omega))} \leq C_4 \left( \frac{C_1}{r} \right)^n. \tag{4.6}
\]

For each \( j \in \{1, \ldots, m\} \), write \( u_j = u'_j \cdot v_j \). Let \( S_1 \) be the identity map, and \( \tau = S_{u_j'} S_1^{-1} \).

Then

\[
\frac{v(S_{u_j}(\Omega))}{v(S_{u_j'}(\Omega))} = \frac{v(\tau \circ S_{v_j}(\Omega))}{v(\tau(\Omega))} = \frac{R_{u_j}^n v(S_{v_j}(\Omega))}{r_{u_j}^n v(\Omega)} \leq \frac{C_2^n v(S_{v_j}(\Omega))}{v(\Omega)} \tag{by Lemma 2.3 (a))}
\]

(by (2.2)).
It follows that for any \( j \in \{1, \ldots, m\}, \)
\[
\frac{\nu(S_{u_1}(\Omega))}{\nu(S_{u_j}(\Omega))} \leq C_4 \left( \frac{C_1}{r} \right)^n \frac{\nu(S_{v_j}(\Omega))}{\nu(\Omega)} \quad \text{(by (4.6))}
\]
\[
\leq C_4 \left( \frac{C_2}{r} \right)^n \frac{\nu(S_{v_j}(\Omega))}{\nu(\Omega)} \quad \text{(by (4.7))}
\]
\[
\leq C_4 \left( \frac{C_2}{r} \right)^n R^{\nu_j|n|} \quad \text{(by (4.2))}.
\]

On the other hand, similar to (4.8) and by (4.2), we have
\[
\frac{\nu(S_{u_1}(\Omega))}{\nu(S_{u_j}(\Omega))} \geq C_1^{-n} \frac{\nu(S_{v_1}(\Omega))}{\nu(\Omega)} \geq C_1^{-n} r^{|v_j|n}.
\]

Recall that \( u_1 = u'_1 v_1 \), where \( u'_1 \) is the longest initial segment of \( u_1 \) such that \( u'_1 \in M_k \). In view of Definition 4.1(c), there exists \( v'_1 \in \Sigma^* \) such that \( u_1 v'_1 = u'_1 v_1 v'_1 \in M_{k+1} \). By Definition 4.1(e), we have
\[
|v_1| \leq |v_1| + |v'_1| \leq L.
\]

Combining (4.8), (4.9) and (4.10) shows that there exists a constant \( c_3 > 0 \) such that for any \( j \in \{1, \ldots, m\}, \)
\[
c_3 \leq R^{|v_j|}.
\]

In particular, we can take \( c_3 = r^{L+1}/(C_4^1 C_4^{1/n}) \). Let \( c_4 := \lceil \log c_3 / \log R \rceil + 1 \). Then \( |v_j| \leq c_4 \). Combining these and (4.3) yields
\[
\#\{S \in A_b : x \in S(\Omega)\} \leq c_2 N^{c_4},
\]
which implies that \( \{S_i\}_{i=1}^N \) satisfies (WSC).

\[\square\]

5 Hausdorff Dimension of Self-similar Sets

In this section, we assume that \( M \) is a complete \( n \)-dimensional smooth orientable Riemannian manifold that is locally Euclidean, i.e., every point of \( M \) has a neighborhood which is isometric to an open subset of a Euclidean space. Let \( U \subset M \) be open and connected, \( W \subset U \) be a compact subset with \( W^c = W \), and let \( \{S_i\}_{i=1}^N \) be an IFS of contractive similitudes on \( W \) such that \( S_i(W) \subset W \) for any \( i \in \{1, \ldots, N\} \). Denote the attractor of \( \{S_i\}_{i=1}^N \) by \( K \subset W \). Assume that \( \{S_i\}_{i=1}^N \) satisfies (FTC) on some open sets \( \Omega \subset W \). Recall that \( \rho_i \) denotes the contraction ratio of \( S_i \). We define
\[
\rho := \min\{\rho_i : 1 \leq i \leq N\}, \quad \rho_{\max} := \max\{\rho_i : 1 \leq i \leq N\}.
\]
Denote the neighborhood types of \( \{ S_i \}_{i=1}^N \) by \( \{ T_1, \ldots, T_q \} \). Fix a vertex \( \omega \in \mathcal{V}_R \) such that \( [\omega] \in T_i \), where \( i \in \{1, \ldots, q\} \). Let \( \sigma_1, \ldots, \sigma_m \) be the offspring of \( \omega \) in \( \mathcal{G}_R \), and let \( w_k \) be the unique edge in \( \mathcal{G}_R \) connecting \( \omega \) to \( \sigma_k \) for \( 1 \leq k \leq m \). Define the weighted incidence matrix \( A_\alpha := (A_\alpha(i, j))_{i,j=1}^q \) as

\[
A_\alpha(i, j) := \sum_{k=1}^m \left( \rho_{w_k}^\alpha : \omega \xrightarrow{w_k} \sigma_k, [\sigma_j] = T_j \right).
\]

We remark that the definition of \( A_\alpha \) is independent of the choice of \( \omega \) above. We denote by \( \omega \rightarrow_R \sigma \) if \( \omega, \sigma \in \mathcal{V}_R \) and \( \sigma \) is an offspring of \( \omega \) in \( \mathcal{G}_R \). We define an (infinite) path in \( \mathcal{G}_R \) to be an infinite sequence \( (\omega_0, \omega_1, \ldots) \) such that for any \( k \geq 0 \),

\[
\omega_k \in \mathcal{V}_k \quad \text{and} \quad \omega_k \rightarrow_R \omega_{k+1},
\]

where \( \omega_0 = \omega_{\text{root}} \). Let \( \mathbb{P} \) be the set of all paths in \( \mathcal{G}_R \). If the vertices \( \omega_0 = \omega_{\text{root}}, \omega_1, \ldots, \omega_k \) satisfy

\[
\omega_j \rightarrow_R \omega_{j+1} \quad \text{for} \quad 1 \leq j \leq k - 1,
\]

then the set

\[
I_{\omega_0, \omega_1, \ldots, \omega_k} = \{(\sigma_0, \sigma_1, \ldots) \in \mathbb{P} : \sigma_j = \omega_j \text{ for any } 0 \leq j \leq k\}
\]

is called a cylinder. Note that the path from \( \omega_0 \) to \( \omega_k \) is unique in \( \mathcal{G}_R \). We let

\[
I_{\omega_k} := I_{\omega_0, \omega_1, \ldots, \omega_k}.
\]

For any cylinder \( I_{\omega_k} \), where \( \omega_k \in \mathcal{V}_k \) and \( [\omega_k] = T_i \), let

\[
\hat{\mu}(\omega_{\text{root}}) = a_1 = 1 \quad \text{and} \quad \hat{\mu}(\omega_k) = \rho_{\omega_k}^\alpha a_i,
\]

where \( [a_1, \ldots, a_q]^T \) is a 1-eigenvector of \( A_\alpha \), normalized so that \( a_1 = 1 \). We will show that \( \hat{\mu} \) is a measure on \( \mathbb{P} \). Note that for two cylinders \( I_\omega \) and \( I'_\omega \) with \( \omega \in \mathcal{V}_k, \omega' \in \mathcal{V}_\ell \) and \( k \leq \ell \), \( I_\omega \cap I'_\omega \neq \emptyset \) iff either \( \omega' = \omega \) in the case \( k = \ell \) or \( \omega' \) is a descendant of \( \omega \) in the case \( k < \ell \). Hence \( I'_\omega \subset I_\omega \). Let \( \omega \in \mathcal{V}_R \) and \( \mathcal{D} := \{ \sigma_k \}_{k=1}^m \) be the set of all offspring of \( \omega \) in \( \mathcal{G}_R \). For \( k \in \{1, \ldots, m\} \), let \( \omega \xrightarrow{w_k} \sigma_k \). Then

\[
\sum_{\sigma \in \mathcal{D}} \hat{\mu}(I_\sigma) = \sum_{j=1}^q \sum_{\sigma \in \mathcal{D}, [\sigma] = T_j} \hat{\mu}(I_\sigma) = \sum_{j=1}^q \sum_{\sigma \in \mathcal{D}, [\sigma] = T_j} \rho_{\sigma}^\alpha a_j
\]
\[= \rho_{\omega}^{\alpha} \sum_{j=1}^{q} \sum_{\sigma \in D, |\sigma| = T_j} \rho_{\omega_k}^\alpha a_j \]

\[= \rho_{\omega}^{\alpha} \sum_{j=1}^{q} A_\alpha(i_j, j) a_j \]

\[= \rho_{\omega}^{\alpha} a_i = \hat{\mu}(I_\omega).\]

Combining these with \(\hat{\mu}(\mathbb{P}) = \hat{\mu}(\omega_{\text{root}}) = 1\) shows that \(\hat{\mu}\) is indeed a measure on \(\mathbb{P}\). Define \(f : \mathbb{P} \to W\) by letting \(f(\omega_0, \omega_1, \ldots)\) be the unique point in \(\bigcap_{k=0}^{\infty} S_{\omega_k}(K)\). It is clear that \(f(\mathbb{P}) = K\). Let \(\tilde{\mu} := \hat{\mu} \circ f^{-1}\). Then \(\tilde{\mu}\) is a measure on \(K\).

Recall that \(\Omega_1 \subset W\) is an open FTC set and invariant under \(\{S_i\}_{i=1}^{N}\). For any bounded Borel set \(Q \subset U\), let

\[B_k(Q) := \{I_{\omega_k} = I_{\omega_0}, \ldots, I_{\omega_k} : |S_{\omega_k}(\Omega)| \leq |Q| < |S_{\omega_k-1}(\Omega)|\} \text{ and } Q \cap S_{\omega_k}(\Omega) \neq \emptyset\].

Making use of some arguments of [19, Lemma 4.1] and Lemma 2.4, we have the following lemma.

**Lemma 5.1** Let \(B_k(Q)\) be defined above. Then there exists a constant \(C_5 > 0\) such that for any bounded Borel set \(Q \subset U\) and any \(k \in \mathbb{N}\), we have \(\#B_k(Q) \leq C_5\).

**Proof** Define

\[\tilde{B}_k(Q) := \{\omega_k \in V_k : |S_{\omega_k}(\Omega)| \leq |Q| < |S_{\omega_k-1}(\Omega)|\} \text{ and } Q \cap S_{\omega_k}(\Omega) \neq \emptyset\} \]

\[= \{\omega_k \in V_k : \rho_{\omega_k} \leq |Q|/|\Omega| < \rho_{\omega_k-1} \text{ and } Q \cap S_{\omega_k}(\Omega) \neq \emptyset\}.\]

Since the relation between \(I_{\omega_k}\) and \(\omega_k\) is one-to-one, we have \(\#B_k(Q) = \#\tilde{B}_k(Q)\).

Let \(b := |Q|/|\Omega|\) and \(\omega_k \in \tilde{B}_k(Q)\). Then there exists a unique \(u \in M_k\) such that \(\omega_k = (S_u, k)\). Let \(u' \preccurlyeq u\) such that \(S_{u'} \in A_b\). Then

\[\rho_{u'} \leq b = |Q|/|\Omega| < \rho_{\omega_k-1}.\]

Thus \(u' \in M_{k-1}\) or \(M_k\). Combining these and Definition 4.1(e), we have \(|u| - |u'| \leq L\). Note that \(Q \cap S_{\omega_k}(\Omega) \neq \emptyset\) implies that \(Q \cap S_{u'}(\Omega) \neq \emptyset\). Since \(S_{u'} \in A_b\), we have

\[|S_{u'}(\Omega)| = \rho_{u'}|\Omega| \leq b|\Omega|\].

Let \(\delta := 2b|\Omega|\) and fix any \(x_0 \in Q\). Then

\[S_{u'}(\Omega) \subset \overline{B_\delta(x_0)}.\]

Since (FTC) implies (WSC), there exists a constant \(\gamma > 0\) (independent of \(b\)) such that for any \(x \in W\),

\[\#\{S \in A_b : x \in S(\Omega)\} \leq \gamma.\]
Note that the contraction ratio of \( S_u \) is \( \rho_u = \sqrt[\frac{1}{n}]{\det S_u'(x)} \) for any \( x \in W \). Let \( A \subset W \) be a measurable set. Then by Lemma 2.1(a), we have
\[
\nu(S_u(A)) \geq (b \rho)^n \nu(A).
\]
Combining these we have
\[
(b \rho)^n \nu(\Omega) \cdot \#\{S_u' : Q \cap S_u'(\Omega) \neq \emptyset\} \leq \sum \{\nu(S_u'(\Omega)) : Q \cap S_u'(\Omega) \neq \emptyset\}
\leq \gamma \nu(B_\delta(x_0))
\leq \gamma c_n \delta^n \quad \text{(by Lemma 2.4)}
=: \gamma c_1 b^n,
\]
where \( c_n \) is the volume of the unit ball in \( \mathbb{R}^n \) and \( c_1 := c_n 2^n |\Omega|^n \). Thus,
\[
\#\{S_u' : Q \cap S_u'(\Omega) \neq \emptyset\} \leq \frac{\gamma c_1}{\rho^n \nu(\Omega)} =: c_2.
\]
Hence
\[
\#B_k(Q) = \#\tilde{B}_k(Q) \leq N^L \cdot \#\{S_u' : Q \cap S_u'(\Omega) \neq \emptyset\} \leq c_2 N^L.
\]
The lemma follows by letting \( C_5 := c_2 N^L \).

**Proof of Theorem 1.5** Use of Lemma 5.1 and the properties of the measure \( \tilde{\mu} \) on \( K \), as in [19, Theorem 1.2]; we omit the details.

### 6 Hausdorff Dimension of Graph Self-similar Sets

In this section, we define graph self-similar sets on Riemannian manifolds, and derive a formula for computing the Hausdorff dimension of such sets. We assume that \( M \) is a complete \( n \)-dimensional Riemannian manifold that is locally Euclidean.

Let \( G = (V, E) \) be a graph, where \( V = \{1, \ldots, t\} \) is the set of vertices and \( E \) is the set of all directed edges. We assume that there is at least one edge between two vertices. It is possible that the initial and terminal vertices are the same. A directed path in \( G \) is a finite string \( e = e_1 \cdots e_p \) of edges in \( E \) such that the terminal vertex of each \( e_i \) is the initial vertex of the edge \( e_{i+1} \). For such a path, denote the length of \( e \) by \( |e| = p \). For any two vertices \( i, j \in V \) and any positive integer \( p \), let \( E^{i,j}_p \) be the set of all directed edges from \( i \) to \( j \), \( E^{i,j}_p \) be the set of all directed paths of length \( p \) from \( i \) to \( j \), \( E_p \) be the set of all directed paths of length \( p \), and \( E^* \) be the set of all directed paths, i.e.,
\[
E_p := \bigcup_{i,j=1}^{p} E^{i,j}_p \quad \text{and} \quad E^* := \bigcup_{p=1}^{\infty} E_p.
\]
For any edge \( e \in E \), we assume that there corresponds a contractive similitude \( S_e \) with ratio \( \rho_e \) on an open and connected subset of \( M \). For \( e = e_1 \cdots e_p \in E^* \), let
\[
S_e = S_{e_1} \circ \cdots \circ S_{e_p} \quad \text{and} \quad \rho_e = \rho_{e_1} \cdots \rho_{e_p}.
\]

Let \( \{ \Omega_i \}_{i=1}^t \) be a family of nonempty bounded open subsets of \( M \). Assume that there exists an isometry \( \phi : \Omega_i \rightarrow \tilde{\Omega}_i \) for any \( i \in \{1, \ldots, t\} \),
\[
\text{(6.1)}
\]
where \( \tilde{\Omega}_i \subset \mathbb{R}^n \) is a nonempty bounded open set. Then \( \{ \tilde{\Omega}_i \}_{i=1}^t \) is a family of compact sets with nonempty interior. Let \( \tilde{\Omega} := \bigcup_{i=1}^t \tilde{\Omega}_i \). For any edge \( e \in E \), we define
\[
\tilde{S}_e := \phi \circ S_e \circ \phi^{-1} : \tilde{\Omega} \rightarrow \tilde{\Omega}.
\]
\[
\text{(6.2)}
\]
Then \( \{ \tilde{S}_e \}_{e \in E} \) is a family of contractive similitudes on \( \tilde{\Omega} \) with the same contraction ratios as \( \{ S_e \}_{e \in E} \). Assume that \( \tilde{S}_e \) can be extended to contractive similitudes \( \tilde{\Omega}_i \) for any \( e \in E \) and \( i, j \in \{1, \ldots, t\} \).

Then there exists a unique family of nonempty compact sets \( \tilde{K}_1, \ldots, \tilde{K}_t \) satisfying
\[
\tilde{K}_i = \bigcup_{j=1}^t \bigcup_{e \in E^{i,j}} \tilde{S}_e(\tilde{K}_j), \quad i \in \{1, \ldots, t\},
\]
(see e.g., [7, 9, 24, 27]). Define
\[
\tilde{K} := \bigcup_{i=1}^t \tilde{K}_i \quad \text{and} \quad K := \phi^{-1}(\tilde{K} \cap \tilde{\Omega}).
\]
\[
\text{(6.3)}
\]
We call \( \tilde{K} \) and \( K \) the graph self-similar sets defined by \( G = (V, E) \) associated to \( \{ \tilde{S}_e \}_{e \in E} \) and \( \{ S_e \}_{e \in E} \), respectively. We call \( G = (V, E) \) the graph-directed iterated function system (GIFS) associated to \( \{ S_e \}_{e \in E} \).

Substituting \( E^* \) for \( \Sigma^* \) in Definition 4.1, we define a sequence of nested index sets \( \{ \mathcal{F}_k \}_{k=1}^\infty \) of directed paths. Note that \( \{ E_k \}_{k=1}^\infty \) is an example of a sequence of nested index sets of directed paths. Fix a sequence \( \{ \mathcal{F}_k \}_{k=1}^\infty \) of nested index sets. For \( i, j \in \{1, \ldots, t\} \), we let \( \mathcal{F}_k^{i,j} \) be a partition of \( \mathcal{F}_k \) defined as
\[
\mathcal{F}_k^{i,j} := \mathcal{F}_k \cap \left( \bigcup_{p \geq 1} E_p^{i,j} \right) = \{ e = e_1 \cdots e_p \in \mathcal{F}_k : e \in E_p^{i,j} \text{ for some } p \geq 1 \}.
\]
Note that \( \mathcal{F}_k = \bigcup_{i, j=1}^t \mathcal{F}^{i, j}_k \). For \( i, j \in \{1, \ldots, t\} \), \( k \geq 1 \), let \( \mathcal{V}_k \) be the set of \( k \)-th level vertices defined as

\[
\mathcal{V}_k := \{(S_e, i, j, k) : e \in \mathcal{F}^{i, j}_k, 1 \leq i, j \leq t\}.
\]

For \( e \in \mathcal{F}^{i, j}_k \), we call \( (S_e, i, j, k) \) (or simply \( (S_e, k) \)) a vertex. For a vertex \( \omega = (S_e, i, j, k) \in \mathcal{V}_k \) with \( k \geq 1 \), let

\[
S_\omega = S_e \quad \text{and} \quad \rho_\omega = \rho_e.
\]

Let \( \mathcal{F}_0 = \{1, \ldots, t\} \) and \( \mathcal{V}_0 = \{\omega^{\text{root}}_1, \ldots, \omega^{\text{root}}_t\} \), where \( \omega^{\text{root}}_i = (I, i, i, 0) \) and \( I \) is the identity map on \( M \). We say \( \mathcal{V}_0 \) is the set of root vertices, and \( \{\mathcal{F}_k\}_{k=0}^\infty \) is a sequence of nested index sets if \( \mathcal{F}_k \cap \mathcal{V}_0 = \emptyset \). Let \( \mathcal{V} := \bigcup_{k \geq 0} \mathcal{V}_k \) be the set of all vertices, and \( \pi : \bigcup_{k \geq 0} \mathcal{F}_k \to \mathcal{V} \) be defined as

\[
\pi(e) := \begin{cases} 
(S_e, i, j, k), & \text{if } e \in \mathcal{F}^{i, j}_k \text{ and } k \geq 1, \\
\omega_i^{\text{root}}, & \text{if } e = i \in \mathcal{F}_0.
\end{cases}
\]

Let \( \omega \in \mathcal{V}_k \) and \( \omega' \in \mathcal{V}_{k+1} \). Suppose that there exist directed paths \( e \in \mathcal{F}_k, e' \in \mathcal{F}_{k+1} \) and \( k \in E^* \) such that \( \pi(e) = \omega, \pi(e') = \omega' \) and \( e' = ek \). Then we connect a directed edge \( k \) from \( \omega \) to \( \omega' \), and denote this as \( \omega \xrightarrow{k} \omega' \). We call \( \omega \) and \( e \) a parent of \( \omega' \) and \( e' \), \( \omega' \) and \( e' \) an offspring of \( \omega \) and \( e \), respectively. Define a graph \( \mathcal{G} := (\mathcal{V}, \mathcal{E}) \), where \( \mathcal{E} \) is the set of all directed edges of \( \mathcal{G} \). Let \( \mathcal{G}_R := (\mathcal{V}_R, \mathcal{E}_R) \) be the reduced graph of \( \mathcal{G} \), defined in a similar fashion as the reduced graph in Sect. 4, with \( \mathcal{V}_R \) and \( \mathcal{E}_R \) being the sets of all vertices and all directed edges, respectively.

Let \( \Omega := \{\Omega_i\}_{i=1}^t \), where \( \Omega_i \subset M \) is a nonempty bounded open set for any \( i \in \{1, \ldots, t\} \). We say that \( \Omega \) is an invariant family under the GIFS \( G = (V, E) \) if

\[
\bigcup_{i \in E^{i, j}} S_e(\Omega_j) \subset \Omega_i, \quad i, j \in \{1, \ldots, t\}.
\]

Since \( S_e \) is a contractive similitude for any \( e \in E^{i, j} \), such a family always exists. Fix an invariant family \( \Omega = \{\Omega_i\}_{i=1}^t \) of \( G = (V, E) \). Let \( \omega = (S_e, i, j, k) \in \mathcal{V}_k \) with \( e \in \mathcal{F}^{i, j}_k \) and \( \omega' = (S_{e'}, i', j', k) \in \mathcal{V}_k \) with \( e' \in \mathcal{F}^{i', j'}_k \). We say that two vertices \( \omega, \omega' \) are neighbors (with respect to \( \Omega \)) if

\[
i = i' \quad \text{and} \quad S_e(\Omega_j) \cap S_{e'}(\Omega_{j'}) \neq \emptyset.
\]

Let

\[
\mathcal{N}(\omega) := \{\omega' : \omega' \in \mathcal{V}_k \text{ is a neighbor of } \omega\}
\]

and call it the neighborhood of \( \omega \) (with respect to \( \Omega \)).
Definition 6.1 For any two vertices $\omega = (S_{e_\omega}, i_\omega, j_\omega, k) \in \mathcal{V}_k$ and $\omega' = (S_{e_\omega'}, i_{\omega'}, j_{\omega'}, k') \in \mathcal{V}_k'$, let $\sigma = (S_{e_\sigma}, i_\sigma, j_\sigma, k) \in \mathcal{N}(\omega)$ and $\sigma' = (S_{e_{\sigma'}}, i_{\omega'}, j_{\omega'}, k') \in \mathcal{N}(\omega')$. Assume that

$$\tau = S_{\omega'} S_{\omega}^{-1} : \bigcup_{\sigma \in \mathcal{N}(\omega)} S_{\sigma}(\Omega_{j_{\sigma}}) \to \bigcup_{i=1}^{t} \Omega_{i},$$

induces a bijection $g_{\tau} : \mathcal{N}(\omega) \to \mathcal{N}(\omega')$ defined by

$$g_{\tau}(\sigma) = g_{\tau}(S_{e_\sigma}, i_\omega, j_\sigma, k) = (\tau \circ S_{e_{\sigma}}, i_{\omega'}, j_{\omega'}, k').$$  \hspace{1cm} (6.4)

We say $\omega$ and $\omega'$ are equivalent, i.e., $\omega \sim \omega'$, if the following conditions hold:

(a) $\#\mathcal{N}(\omega) = \#\mathcal{N}(\omega')$ and $j_\sigma = j_{\omega'}$ in (6.4);
(b) for $\sigma \in \mathcal{N}(\omega)$ and $\sigma' \in \mathcal{N}(\omega')$ such that $g_{\tau}(\sigma) = \sigma'$, and for any positive integer $k_0 \geq 1$, a directed path $e \in E^*$ satisfies $(S_{\sigma} \circ S_{e}, k + k_0) \in \mathcal{V}_{k+k_0}$ if and only if it satisfies $(S_{\omega'} \circ S_{e}, k' + k_0) \in \mathcal{V}_{k'+k_0}$.

It is easy to check that $\sim$ is an equivalence relation. Denote the equivalence class of $\omega$ by $[\omega]$, and call it the neighborhood types of $\omega$ (with respect to $\Omega$).

For a graph $G = (\mathcal{V}, \mathcal{E})$, we can prove that $G$ satisfies Proposition 4.3 as in [27, Proposition 2.5]; we omit the details. We now define the graph finite type condition.

Definition 6.2 Let $M$ be a complete $n$-dimensional Riemannian manifold that is locally Euclidean, $G = (V, E)$ be a GIFS with $V = \{1, \ldots, t\}$, and let $\{S_{e}\}_{e \in E}$ be a family of contractive similitudes defined on an open and connected subset of $M$. If there exists an invariant family of nonempty bounded open sets $\Omega = \{\Omega_i\}_{i=1}^{t}$ with respect to some sequence of nested index sets $\{\mathcal{F}_k\}_{k=0}^{\infty}$ such that

$$\#\mathcal{V}/\sim < \infty,$$

then we say that $G = (V, E)$ satisfies the graph finite type condition (GFTC). We call such an invariant family $\Omega$ a graph finite type condition family of $G$.

By assuming a GIFS satisfies (GFTC), we get a formula for $\dim_{H}(K)$.

Proof of Theorem 1.6 The proof is similar to that of [27, Theorem 1.1] and the definition of a weighted incidence matrix is the same as that in Sect. 5; we omit the details. \hfill \Box

7 Examples

In this section, we construct some examples of CIFSs and GIFs on Riemannian manifolds satisfying (FTC) and (GFTC), respectively.

Let $\{S_i\}_{i=1}^{N}$ be a CIFS on an open and connected set $\tilde{U} \subset \mathbb{R}^n$, i.e., for any $i \in \{1, \ldots, N\}$, $S_i : \tilde{U} \to \tilde{S}_i(\tilde{U}) \subset \tilde{U}$ is a $C^{1+\varepsilon}$ contractive injective conformal map with $0 < \varepsilon < 1$, there exists a compact set $\tilde{W} \subset \tilde{U}$ such that $\tilde{S}_i(\tilde{W}) \subset \tilde{W}$, and...
0 < |det $\tilde{S}_i'(x)| < 1$ for any $x \in \tilde{U}$. Let $M$ be a complete $n$-dimensional smooth Riemannian manifold. Assume that there exists a $C^{1+\varepsilon}$ diffeomorphism

$$\varphi : U \rightarrow \tilde{U},$$

where $U \subset M$ is open and connected. Define

$$S_i := \varphi^{-1} \circ \tilde{S}_i \circ \varphi : U \rightarrow S_i(U) \quad \text{for any } i \in \{1, \ldots, N\}. \quad (7.1)$$

By [26, Proposition 7.1], $\{S_i\}_{i=1}^N$ is a contractive CIFS on $U$.

Fix a sequence of nested index sets $\{M_k\}_{k=0}^\infty$. Let $\hat{\mathcal{V}}$ and $\mathcal{V}$ be the sets of all vertices, with respect to $\{M_k\}_{k=0}^\infty$, of $\{\tilde{S}_i\}_{i=1}^N$ and $\{S_i\}_{i=1}^N$, respectively. For $\tilde{\omega} = (\tilde{S}_u, k) \in \hat{\mathcal{V}}_k$ and $\omega = (S_u, k) \in \mathcal{V}_k$, where $u \in M_k$ and $k \geq 0$, we define $\tilde{S}_\omega := \tilde{S}_u$ and $S_\omega := S_u$. It follows from (7.1) that

$$S_\omega = \varphi^{-1} \circ \tilde{S}_\omega \circ \varphi. \quad (7.2)$$

For $\tilde{\omega}, \tilde{\omega}' \in \hat{\mathcal{V}}$, let $\tilde{N}(\tilde{\omega})$ be the neighborhood of $\tilde{\omega}$ and $\tilde{\omega} \sim \tilde{\omega}'$ be defined as in Definition 4.2. We have the following proposition.

**Proposition 7.1** Let $S_i$ be defined as in (7.1), where $i \in \{1, \ldots, N\}$. If $\{\tilde{S}_i\}_{i=1}^N$ satisfies (FTC), then so does $\{S_i\}_{i=1}^N$.

**Proof** For any $\tilde{\omega}, \tilde{\omega}' \in \hat{\mathcal{V}}$ satisfying $\tilde{\omega} \sim \tilde{\omega}'$, similar to Definition 4.2, there exist $\tilde{\sigma} \in \tilde{N}(\tilde{\omega})$ and $\tilde{\sigma}' \in \tilde{N}(\tilde{\omega}')$ such that

$$\tilde{S}_{\tilde{\omega}}^{-1} \circ \tilde{S}_{\tilde{\sigma}} = \tilde{S}_{\tilde{\omega}'}^{-1} \circ \tilde{S}_{\tilde{\sigma}'} \quad (7.3)$$

For any $\omega, \omega' \in \mathcal{V}$, $\sigma \in N(\omega)$ and $\sigma' \in N(\omega')$, we have

$$S_{\omega}^{-1} \circ S_{\sigma'} = \varphi^{-1} \circ \tilde{S}_{\omega'}^{-1} \circ \varphi \circ \varphi^{-1} \circ \tilde{S}_{\sigma'} \circ \varphi \quad \text{by (7.2)}$$

$$= \varphi^{-1} \circ \tilde{S}_{\omega}'^{-1} \circ \tilde{S}_{\sigma} \circ \varphi \quad \text{by (7.3)}$$

$$= \varphi^{-1} \circ \tilde{S}_{\omega}'^{-1} \circ \varphi \circ \varphi^{-1} \circ \tilde{S}_{\sigma} \circ \varphi$$

$$= S_{\omega}^{-1} \circ S_{\sigma}. \quad (\text{by } \# \tilde{\mathcal{V}}/\sim < \infty)$$

It follows from Definition 4.2 that $\omega \sim \omega'$. Since $\{\tilde{S}_i\}_{i=1}^N$ satisfies (FTC), we have $\# \mathcal{V}/\sim < \infty$. Thus, $\# \mathcal{V}/\sim < \infty$. This proves the proposition. \hfill \square

Let

$$\mathbb{S}^n := \left\{ (x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : \sum_{i=1}^{n+1} x_i^2 = 1 \right\},$$

$$\mathbb{D}^n := \left\{ (x_1, \ldots, x_n) \in \mathbb{R}^n : \sum_{i=1}^{n} x_i^2 < 1 \right\}. \quad \text{Springer}$$
Let $S^n_+$ be the upper hemisphere of $S^n$, and define the stereographic projection $\varphi: S^n_+ \to \mathbb{D}^n$ as

$$\varphi(x_1, \ldots, x_{n+1}) = \frac{1}{1 + x_{n+1}} (x_1, \ldots, x_n) := (y_1, \ldots, y_n).$$

Then

$$\varphi^{-1}(y_1, \ldots, y_n) = \frac{1}{|y|^2 + 1} (2y_1, \ldots, 2y_n, 1 - |y|^2),$$

where $|y|^2 = y_1^2 + \cdots + y_n^2$.

For convenience, we consider the case of $n = 2$. We will give some actual examples illustrating Proposition 7.1. The first example below satisfies (OSC), while the other two satisfy (FTC) but not (OSC).

**Example 7.2** Let $\{\widetilde{S}_i\}_{i=1}^3$ be a Sierpinski gasket on $\mathbb{R}^2$, i.e., for $x \in \mathbb{R}^2$,

$$\widetilde{S}_1(x) = \frac{1}{2} x + \left(0, \frac{1}{2}\right), \quad \widetilde{S}_2(x) = \frac{1}{2} x + \left(-\frac{1}{4}, 0\right), \quad \widetilde{S}_3(x) = \frac{1}{2} x + \left(0, \frac{1}{4}\right).$$

Let $S_i$ be defined as in (7.1), where $i \in \{1, 2, 3\}$. Then $\{S_i\}_{i=1}^3$ is a CIFS satisfying (OSC) on $S^2_+$ (see Fig. 1(a)).

**Example 7.3** Let $\{\widetilde{S}_i\}_{i=1}^4$ be a CIFS defined as in [19] satisfying (FTC), i.e., for $x \in \mathbb{R}^2$,

$$\widetilde{S}_1(x) = \rho x + \left(\frac{1}{2} \rho, 0\right), \quad \widetilde{S}_2(x) = r x + \left(\rho - \rho r + \frac{1}{2} r, 0\right),$$

$$\widetilde{S}_3(x) = r x + \left(1 - \frac{1}{2} r, 0\right), \quad \widetilde{S}_4(x) = r x + \left(\frac{1}{2} r, 1 - r\right),$$

where $0 < \rho, r < 1$ and $\rho + 2r - \rho r \leq 1$. Let $S_i$ be defined as in (7.1), where $i \in \{1, 2, 3, 4\}$. By Proposition 7.1, $\{S_i\}_{i=1}^4$ is a CIFS satisfying (FTC) on $S^2_+$ (see Fig. 1(b)).

**Example 7.4** Let $\{\widetilde{S}_i\}_{i=1}^3$ be a golden Sierpinski gasket defined as in [25] satisfying (FTC), i.e., for $x \in \mathbb{R}^2$,

$$\widetilde{S}_1(x) = \rho x, \quad \widetilde{S}_2(x) = \rho x + (\rho^2, 0), \quad \widetilde{S}_3(x) = \rho^2 x + (\rho, \rho),$$

where $\rho = (\sqrt{5} - 1)/2$. Let $S_i$ be defined as in (7.1), where $i \in \{1, 2, 3\}$. By Proposition 7.1, $\{S_i\}_{i=1}^3$ is a CIFS satisfying (FTC) on $S^2_+$ (see Fig. 1(c)).

Let $M$ be a complete $n$-dimensional smooth Riemannian manifold that is locally Euclidean. Now, we construct an example of GIFS on $M$ satisfying (GFTC) to illustrate Theorem 1.6. Let $G = (V, E)$ with $V = \{1, \ldots, t\}$ be a GIFS of contractive

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similitudes \( \{S_e\}_{e \in E} \) defined on \( \mathbb{R}^n \), and \( \tilde{\Omega} := \{\tilde{\Omega}_i\}_{i=1}^t \), where \( \tilde{\Omega}_i \subset \mathbb{R}^n \), be an invariant family of nonempty bounded open sets under \( G = (V, E) \). Recall that \( E^{i,j} \) is the set of all directed edges from \( i \) to \( j \), and \( E^* \) is the set of all directed paths. Then

\[
\bigcup_{e \in E^{i,j}} \tilde{S}_e(\tilde{\Omega}_j) \subset \tilde{\Omega}_i, \quad i, j \in \{1, \ldots, t\}. \tag{7.4}
\]

For \( e = e_1 \cdots e_p \in E^* \), let \( \tilde{S}_e = \tilde{S}_{e_1} \circ \cdots \circ \tilde{S}_{e_p} \) and denote the contractive ratio of \( \tilde{S}_e \) by \( \rho_e = \rho_{e_1} \cdots \rho_{e_p} \).

Let \( \{F_k\}_{k=1}^\infty \) be a sequence of nested index sets of directed paths, \( F_k^{i,j} \) be a partition of \( F_k \) defined as in Sect. 5. Similar to Sect. 5, we define the set of \( k \)-th level vertices associated to \( \{S_e\}_{e \in E} \) as

\[
\tilde{V}_k := \{ (\tilde{S}_e, i, j, k) : e \in F_k^{i,j}, 1 \leq i, j \leq t \},
\]

for \( i, j \in \{1, \ldots, t\}, k \geq 1 \). For \( e \in F_k^{i,j} \), we call \( (\tilde{S}_e, i, j, k) \) a vertex. For a vertex \( \tilde{\omega} = (\tilde{S}_e, i, j, k) \in \tilde{V}_k \) with \( k \geq 1 \), let

\[
\tilde{S}_{\tilde{\omega}} = \tilde{S}_e \quad \text{and} \quad \rho_{\tilde{\omega}} = \rho_e.
\]

Recall that \( F_0 = \{1, \ldots, t\} \). Let \( \tilde{V}_0 := \{\tilde{\omega}_\text{root}^1, \ldots, \tilde{\omega}_\text{root}^t\} \), where \( \tilde{\omega}_\text{root}^i = (\tilde{I}, i, i, 0) \) and \( \tilde{I} \) is the identity map on \( \mathbb{R}^n \). We say \( \tilde{V}_0 \) is the set of root vertices associated to \( \{S_e\}_{e \in E} \). Let \( \tilde{\nabla} := \bigcup_{k \geq 0} \tilde{V}_k \) be the set of all vertices, and \( \tilde{\pi} : \bigcup_{k \geq 0} F_k \to \tilde{\nabla} \) be defined as

\[
\tilde{\pi}(e) := \begin{cases} 
(\tilde{S}_e, i, j, k), & \text{if } e \in F_k^{i,j} \text{ and } k \geq 1, \\
\tilde{\omega}_\text{root}^i, & \text{if } e = i \in F_0.
\end{cases}
\]

For \( \tilde{\omega} \in \tilde{\nabla}_k \) and \( \tilde{\omega}' \in \tilde{\nabla}_{k+1} \), if there exist directed paths \( e \in F_k, e' \in F_{k+1} \) and \( k \in E^* \) such that \( \tilde{\pi}(e) = \tilde{\omega}, \tilde{\pi}(e') = \tilde{\omega}' \) and \( e' = ek \), then we say \( \tilde{\omega} \) is a parent of \( \tilde{\omega}' \) and \( \tilde{\omega}' \) is offspring of \( \tilde{\omega} \). Denote it by \( \tilde{\omega} \xrightarrow{k} \tilde{\omega}' \). Moreover, we say \( e \) is a parent of \( e' \) and \( e' \) is offspring of \( e \).
Let \( \tilde{\omega} = (\tilde{S}_e, i, j, k) \in \tilde{\mathbb{V}}_k \) with \( e \in \mathcal{F}^{i,j}_k \) and \( \tilde{\omega}' = (\tilde{S}_{e'}, i', j', k) \in \tilde{\mathbb{V}}_k \) with \( e' \in \mathcal{F}^{i',j'}_k \). We say that two vertices \( \tilde{\omega}, \tilde{\omega}' \) are neighbors if
\[
i = i' \quad \text{and} \quad \tilde{S}_e(\tilde{\Omega}_j) \cap \tilde{S}_{e'}(\tilde{\Omega}_{j'}) \neq \emptyset.
\]

Define the neighborhood of \( \tilde{\omega} \) as
\[
\tilde{N}(\tilde{\omega}) := \{ \tilde{\omega}' : \tilde{\omega}' \in \tilde{\mathbb{V}}_k \text{ is a neighbor of } \tilde{\omega} \}.
\]

Let \( \tilde{\omega} \sim \tilde{\omega}' \) be defined similar to Definition 6.1. Note that \( \sim \) is an equivalence relation. Denote the equivalence class of \( \tilde{\omega} \) by \([\tilde{\omega}]\), and call it the neighborhood types of \( \tilde{\omega} \) (with respect to \( \tilde{\mathbb{V}} \)). We say that \( G = (V, E) \) associated to \( \{\tilde{S}_e\}_{e \in E} \) satisfies (GFTC) if \(#\tilde{\mathbb{V}}/#\sim < \infty\).

For a graph \( \tilde{\mathbb{G}} := (\tilde{\mathbb{V}}, \tilde{E}) \), where \( \tilde{E} \) is the set of all directed edges of \( \tilde{\mathbb{G}} \). Let \( \tilde{\mathbb{G}}_R := (\tilde{\mathbb{V}}_R, \tilde{E}_R) \) be the reduced graph of \( \tilde{\mathbb{G}} \) defined analogous to \( \mathbb{G}_R \) (see Sect. 7). Denote the neighborhood types of \( \tilde{G} = (V, E) \) associated to \( \{\tilde{S}_e\}_{e \in E} \) by \( \{\tilde{T}_1, \ldots, \tilde{T}_q\} \). Fix a vertex \( \tilde{\omega} \in \tilde{\mathbb{V}}_R \) such that \([\tilde{\omega}]\) \( \in \tilde{T}_i \), where \( i \in \{1, \ldots, q\} \). Let \( \tilde{\sigma}_1, \ldots, \tilde{\sigma}_m \) be the offspring of \( \tilde{\omega} \) in \( \tilde{\mathbb{G}}_R \), and let \( w_k \) be the unique edge in \( \tilde{\mathbb{G}}_R \) connecting \( \tilde{\omega} \) to \( \tilde{\sigma}_k \) for \( 1 \leq k \leq m \). Recall that the weighted incidence matrix \( \tilde{A}_\alpha = (\tilde{A}_\alpha(i, j))_{i,j=1}^q \) is defined as
\[
\tilde{\alpha}(i, j) := \sum_{k=1}^m (\rho_{w_k}^q : \tilde{\omega} \xrightarrow{w_k} \tilde{\sigma}_k, [\tilde{\sigma}_k] = \tilde{T}_j). \tag{7.5}
\]

Assume that there exists an isometry
\[
\psi : \tilde{\Omega}_i \to \Omega_i \quad \text{for any } i \in \{1, \ldots, t\}, \tag{7.6}
\]
where \( \Omega_i \subset M \) is a nonempty bounded open set. Let \( \Omega := \bigcup_{i=1}^t \Omega_i \). For any edge \( e \in E \), define
\[
S_e := \psi \circ \tilde{S}_e \circ \psi^{-1} : \Omega \to \Omega. \tag{7.7}
\]

Then \( \{S_e\}_{e \in E} \) is a family of contractive similitudes on \( \Omega \), and thus for any \( e \in E \), the contraction ratios of \( S_e \) and \( \tilde{S}_e \) are the same. It follows from (7.4), (7.6) and (7.7) that for \( i, j \in \{1, \ldots, t\} \),
\[
\bigcup_{e \in E^{i,j}} S_e(\Omega_j) = \bigcup_{e \in E^{i,j}} \psi \circ \tilde{S}_e \circ \psi^{-1}(\Omega_j) = \bigcup_{e \in E^{i,j}} \psi \circ \tilde{S}_e(\tilde{\Omega}_j) \subset \psi(\tilde{\Omega}_i) = \Omega_i.
\]

Hence \( \{\Omega_i\}_{i=1}^t \) is an invariant family of \( \{S_e\}_{e \in E} \) under \( G = (V, E) \), and thus \( G = (V, E), \{\Omega_i\}_{i=1}^t \), and \( \{S_e\}_{e \in E} \) together form a GIFS on \( \Omega \). For \( \omega = (\tilde{S}_e, i, j, k) \in \tilde{\mathbb{V}}_k \) and \( \omega = (S_e, i, j, k) \in \mathbb{V}_k \) with \( e \in \mathcal{F}^{i,j}_k \), recall that \( \tilde{S}_\omega := \tilde{S}_e \) and \( S_\omega := S_e \). It
follows from (7.7) that

$$S_\omega = \psi \circ \tilde{S}_{\omega} \circ \psi^{-1}. \quad (7.8)$$

The proof of the following proposition is similar to that of Proposition 7.1.

**Proposition 7.5** Use the above notation and setup. Let $M$ be a complete $n$-dimensional smooth Riemannian manifold that is locally Euclidean. Let $G = (V, E)$ be a GIFS defined on $\mathbb{R}^n$ satisfying (GFTC) with $\{\Omega_i\}_{i=1}^I$ being a GFTC-family and $\{\tilde{S}_e\}_{e \in E}$ being an associated family of contractive similitudes. For any $e \in E$, let $S_e$ be a similitude defined as in (7.7). Then the GIFS $G = (V, E)$ defined on $\Omega$ satisfies (GFTC) with $\{\Omega_i\}_{i=1}^I$ being a GFTC-family and $\{S_e\}_{e \in E}$ being an associated family of contractive similitudes. Moreover, for such two GIFSs that are connected by a diffeomorphism, there is an one-to-one correspondence between the neighborhood types; consequently the weighted incidence matrices and the Hausdorff dimension of the corresponding graph self-similar sets are the same.

**Proof** For $k, k' \in \mathbb{N}$, let $\tilde{\omega} := (\tilde{S}_{e_1}, i, j_1, k) \in \tilde{\mathbb{V}}_k$ with $e_1 \in \mathcal{F}_k^{i,j_1}$ and $\tilde{\omega} := (\tilde{S}_{e_1}', i', j_1', k') \in \tilde{\mathbb{V}}_{k'}$ with $e_1' \in \mathcal{F}_{k'}^{i',j_1'}$ satisfying $\tilde{\omega} \sim \tilde{\omega}'$. Then there exist $\tilde{\sigma} \in \tilde{N}(\tilde{\omega})$ and $\tilde{\sigma}' \in \tilde{N}(\tilde{\omega}')$ such that

$$\tilde{S}_{\tilde{\omega}}^{-1} \circ \tilde{S}_{\tilde{\omega}'} = \tilde{S}_{\tilde{\omega}}^{-1} \circ \tilde{S}_{\tilde{\sigma}}. \quad (7.9)$$

We write $\tilde{\sigma} := (\tilde{S}_{e_2}, i, j_2, k)$ and $\tilde{\sigma}' := (\tilde{S}_{e_2}', i', j_2', k')$, where $e_2 \in \mathcal{F}_k^{i,j_2}$, $e_2' \in \mathcal{F}_{k'}^{i',j_2'}$. Assume that $\omega := (S_{e_1}, i, j_1, k), \sigma := (S_{e_2}, i, j_2, k) \in \mathbb{V}_k$ and $\omega' := (S_{e_1}', i', j_1', k'), \sigma' := (S_{e_2}', i', j_2', k') \in \mathbb{V}_{k'}$ are defined by $\tilde{\omega}, \tilde{\sigma}, \tilde{\omega}'$ and $\tilde{\sigma}'$ as in (7.8), respectively. Then we claim that $\sigma \in \mathcal{N}(\omega)$ and $\sigma' \in \mathcal{N}(\omega')$. To prove $\sigma \in \mathcal{N}(\omega)$, we make use of $\tilde{\omega} \in \tilde{N}(\tilde{\omega})$ that

$$\tilde{S}_{e_1}(\tilde{\Omega}_{j_1}) \cap \tilde{S}_{e_2}(\tilde{\Omega}_{j_2}) \neq \emptyset. \quad (7.9)$$

Hence

$$S_{e_1}(\Omega_{j_1}) \cap S_{e_2}(\Omega_{j_2}) = \psi \circ \tilde{S}_{e_1} \circ \psi^{-1}(\Omega_{j_1}) \cap \psi \circ \tilde{S}_{e_2} \circ \psi^{-1}(\Omega_{j_2}) \quad (by \ (7.7))$$

$$= \psi \circ \tilde{S}_{e_1}(\tilde{\Omega}_{j_1}) \cap \psi \circ \tilde{S}_{e_2}(\tilde{\Omega}_{j_2}) \quad (by \ (7.6))$$

$$= \psi(\tilde{S}_{e_1}(\tilde{\Omega}_{j_1}) \cap \tilde{S}_{e_2}(\tilde{\Omega}_{j_2}))$$

$$\neq \emptyset \quad (by \ (7.9)).$$

This proves $\sigma \in \mathcal{N}(\omega)$. That $\sigma' \in \mathcal{N}(\omega')$ can be proved similarly. Similar to Proposition 7.1, by (7.8) and (7.9) we can prove

$$S_{\tilde{\sigma}}^{-1} \circ S_{\tilde{\sigma}'} = S_{\tilde{\sigma}}^{-1} \circ S_{\tilde{\sigma}}, \quad \text{and thus } \omega \sim \omega'. \quad \text{Since } G = (V, E) \text{ is a GIFS associated to } \{\tilde{S}_e\}_{e \in E} \text{ satisfying (GFTC)} \text{ with } \{\tilde{\Omega}_i\}_{i=1}^I \text{ being a GFTC-family, we have } \#\tilde{\mathbb{V}}/\sim < \infty. \text{ Thus, } \#\mathbb{V}/\sim < \infty,$$
i.e., the GIFS $G = (V, E)$ associated to $\{S_e\}_{e \in E}$ satisfies (GFTC) with $\{\Omega_t\}_{t=1}^l$ being a GFTC-family.

Let $\{\tilde{T}_1, \ldots, \tilde{T}_q\}$ be the neighborhood types of $G = (V, E)$ associated to $\{\tilde{S}_e\}_{e \in E}$. Since $\tilde{\omega} \sim \tilde{\omega}'$ implies $\omega \sim \omega'$, for $[\tilde{\omega}] = [\tilde{\omega}]_l$, we can let $[\omega] = \tilde{T}_l$, where $l \in \{1, \ldots, q\}$. Assume that $\tilde{\omega} = (\tilde{S}_{e_1}, i, j, k) \in \tilde{V}_R$ with $[\tilde{\omega}] = \tilde{T}_l$, and $\tilde{\sigma}_1, \ldots, \tilde{\sigma}_m$ are the offspring of $\tilde{\omega}$ in $\tilde{G}_R$ with $\tilde{\sigma}_t = (\tilde{S}_{e_{1t}}, i_t, j_{1t}, k_t) \in \tilde{V}_R$, where $t \in \{1, \ldots, m\}$ and $e_{1t} \in \mathcal{F}^{i_t, j_{1t}}$. Let $w_t$ be the unique edge in $\tilde{G}_R$ connecting $\tilde{\omega}$ to $\tilde{\sigma}_t$ for $t \in \{1, \ldots, m\}$, i.e., $e_{1t} = e_1w_t$. Hence the offspring of $e_1$ are $w_1, \ldots, w_m$. Note that for $\omega = (S_{e_1}, i, j, k) \in V_R$, we have $[\omega] = \tilde{T}_l$. It follows from $(7.5)$ that $A_\alpha(i, j) = \tilde{A}_\alpha(i, j)$ for any $i, j \in \{1, \ldots, q\}$, and thus $A_\alpha = \tilde{A}_\alpha$. The proposition now follows by applying Theorem 1.6 and [27, Theorem 1.1(a)].

We now describe how to generate the weighted incidence matrix by using the same method as in [19, 27]. Assume that $G = (V, E)$ is a GIFS of contractive similitudes defined on $\Omega$ satisfying (GFTC). Let $T_1, \ldots, T_q$ be all the distinct neighborhood types with $T_i = [\omega^i_{root}]$ for any $i \in \{1, \ldots, t\}$. Fix a vertex $\omega \in V_R$ such that $[\omega] \in T_i$, where $i \in \{1, \ldots, q\}$. Let $\sigma_1, \ldots, \sigma_m$ be the offspring of $\omega$ in $G_R$, $k_\ell$ be the unique edge in $G_R$ connecting $\omega$ to $\sigma_\ell$ for $1 \leq \ell \leq m$, and

$$C_{ij} := \{\sigma_\ell : 1 \leq \ell \leq m, \ [\sigma_\ell] = T_j\}.$$

Note that for two edges $k_\ell$ and $k_{\ell'}$ connecting $\omega$ to two distinct $\sigma_\ell$ and $\sigma_{\ell'}$ satisfying $[\sigma_\ell] = [\sigma_{\ell'}] = T_j$, the contraction ratios $\rho_{\sigma_\ell}$ and $\rho_{\sigma_{\ell'}}$ may be different. We can partition $C_{ij} := C_{ij}(1) \cup \cdots \cup C_{ij}(n_{ij})$ by using $\rho_{\sigma_t}$, where for $s \in \{1, \ldots, n_{ij}\}$,

$$C_{ij}(s) := \{\sigma_\ell \in C_{ij} : \rho_{\sigma_\ell} = \rho_{ij,s}\},$$

and the $\rho_{ij,s}$ are distinct. Thus, for any entry $A_\alpha(i, j)$ of the weighted incidence matrix,

$$A_\alpha(i, j) = \sum_{s=1}^{n_{ij}} C_{ij}(s) \rho_{ij,s}.$$

Moreover, we can write symbolically

$$T_i \longrightarrow \sum_{j=1}^{q} \sum_{s=1}^{n_{ij}} C_{ij}(s) T_j(\rho_{ij,s}),$$

where the $T_j(\rho_{ij,s})$ are defined in an obvious way. We say that $T_i$ generates $\#C_{ij}(s)$ neighborhoods of type $T_j$ with contraction ratio $\rho_{ij,s}$.

Now consider the following example. Let $T^2 = S^1 \times S^1$ be a 2-torus, viewed as $[0, 1] \times [0, 1]$ with opposite sides identified, and $T^2$ be endowed with the Riemannian

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metric induced from $\mathbb{R}^2$. We consider the following IFS with overlaps on $\mathbb{R}^2$:

\[
\begin{align*}
\tilde{h}_1(x) &= \frac{1}{2}x + \left(0, \frac{1}{4}\right), \\
\tilde{h}_2(x) &= \frac{1}{2}x + \left(\frac{1}{2}, \frac{1}{4}\right), \\
\tilde{h}_3(x) &= \frac{1}{2}x + \left(\frac{1}{2}, \frac{1}{4}\right), \\
\tilde{h}_4(x) &= \frac{1}{2}x + \left(\frac{1}{4}, \frac{1}{4}\right).
\end{align*}
\]

Iterations of $\{\tilde{h}_i\}_{i=1}^4$ induce iterations $\{h_t\}_{t=1}^4$ on $\mathbb{T}^2 = \mathbb{R}^2 / \mathbb{Z}^2$, defined as $h_t(x) := \tilde{h}_t(x) \mod \mathbb{Z}^2$, $x \in \Omega_0 := [0, 1) \times [0, 1)$. Iterations $\{h_t\}_{t=1}^4$ on $\Omega_0$ generates a compact set $K \subset \mathbb{T}^2$ defined as

\[
K := \bigcap_{k=0}^{\infty} \bigcup_{t \in \{1, \ldots, 4\}^k} h_t(\Omega_0).
\]

We call $K$ the **attractor** of $\{h_t\}_{i=1}^4$ on $\mathbb{T}^2$ (see Fig. 3). We are interested in computing the Hausdorff dimension of $K$. However, the image of $\Omega_0$ under the relation induced by $h_4$ is a connected rectangle in $\mathbb{T}^2$, but it is divided into two disjoint rectangles in $\mathbb{R}^2$. It is easy to see that the relation induced by $h_4$ on $\mathbb{T}^2$ is not a well-defined function and not contractive under the metric of $\mathbb{T}^2$. Note also that $\Omega_0$ is not compact. All these make it awkward to apply the theory of IFSs developed in Sect. 5. To overcome this difficulty, we need to use the framework of a GIFS.

Consider the GIFS $G = (V, E)$ with $V = \{1, 2\}$, $E = \{e_1, \ldots, e_8\}$, and invariant family $\{\Omega_i\}_{i=1}^2$, where $\Omega_1 = (0, 1) \times (0, 1/2)$ and $\omega_2 = (0, 1) \times (1/2, 1)$, are the lower and upper pieces of the interior of $\mathbb{T}^2$, respectively, and

\[
e_1, e_2, e_3 \in E^{1.1}, \quad e_4 \in E^{1.2}, \quad e_5, e_6, e_7 \in E^{2.2}, \quad e_8 \in E^{2.1}
\]

(see Fig. 2). Let $\Omega := \Omega_1 \cup \Omega_2$. The associated similitudes $S_e$, $e \in E$, are defined as $S_e(x) = x/2 + d_i$, where $x \in \Omega$ and

\[
\begin{align*}
d_1 &= \left(0, \frac{1}{4}\right), \\
d_2 &= \left(\frac{1}{4}, \frac{1}{4}\right), \\
d_3 &= \left(\frac{1}{2}, \frac{1}{4}\right), \\
d_4 &= \left(\frac{1}{4}, \frac{1}{2}\right), \\
d_5 &= (0, 0), \\
d_6 &= \left(\frac{1}{4}, 0\right), \\
d_7 &= \left(\frac{1}{2}, 0\right), \\
d_8 &= \left(\frac{1}{4}, \frac{3}{4}\right).
\end{align*}
\]

Let $\tilde{\Omega}_1 := (0, 1) \times (0, 1/2) \subset \mathbb{R}^2$, $\tilde{\Omega}_2 := (0, 1) \times (1/2, 1) \subset \mathbb{R}^2$, and $\tilde{\Omega} := \tilde{\Omega}_1 \cup \tilde{\Omega}_2$. Note that there exists an isometry

\[
\phi : \Omega_i \rightarrow \tilde{\Omega}_i \quad \text{for} \ i \in \{1, 2\},
\]

defined as $\phi(x, y) = (x, y)$. For any $e \in E$, let $\tilde{S}_e$ be defined as in (6.2). Note that $\{\tilde{S}_e\}_{e \in E}$ can be extended to $\tilde{\Omega}$, and $\{\tilde{\Omega}_i\}_{i=1}^2$ is an invariant family of the GIFS $G = $
Let
\[ \tilde{K} := \bigcap_{k=0}^{\infty} \bigcup_{i,j=1}^{2} \tilde{S}_{e}(\Omega_{j}) \]  
(7.11)

be the graph self-similar sets generated by the GIFS \( G = (V, E) \) associated to \( \{\tilde{S}_{e}\}_{e \in E} \).

Note that
\[
\begin{align*}
    h_{1}|_{\Omega_{1}} &= S_{e_{3}}|_{\Omega_{1}}, & h_{2}|_{\Omega_{1}} &= S_{e_{2}}|_{\Omega_{1}}, & h_{3}|_{\Omega_{1}} &= S_{e_{3}}|_{\Omega_{1}}, & h_{4}|_{\Omega_{1}} &= S_{e_{4}}|_{\Omega_{2}}, \\
    h_{1}|_{\Omega_{2}} &= S_{e_{3}}|_{\Omega_{2}}, & h_{2}|_{\Omega_{2}} &= S_{e_{6}}|_{\Omega_{2}}, & h_{3}|_{\Omega_{2}} &= S_{e_{7}}|_{\Omega_{2}}, & h_{4}|_{\Omega_{2}} &= S_{e_{8}}|_{\Omega_{1}}.
\end{align*}
\]  
(7.12)

Then
\[
\bigcup_{t=1}^{4} \overline{h_{t}(\Omega_{0})} = \bigcup_{t=1}^{4} \overline{h_{t}(\Omega_{1}) \cup h_{t}(\Omega_{2})} = \bigcup_{i,j=1}^{2} \overline{S_{e}(\Omega_{j})}.
\]

Using this and (7.12), one can prove by induction that for all \( k \geq 1 \),
\[
\bigcup_{t \in \{1, \ldots, 4\}^{k}} \overline{h_{t}(\Omega_{0})} = \bigcup_{i,j=1}^{2} \overline{S_{e}(\Omega_{j})}.
\]  
(7.13)

For \( k \in \mathbb{N} \), define the following open subset of \( \tilde{\Omega} \):
\[
\Lambda_{k} := \bigcup_{i,j=1}^{2} \tilde{S}_{e}(\tilde{\Omega}_{j}).
\]  
(7.14)

We claim that
\[
\bigcap_{k=0}^{\infty} \overline{\Lambda_{k} \cap \Omega} = \left( \bigcap_{k=0}^{\infty} \overline{\Lambda_{k}} \right) \cap \tilde{\Omega}.
\]  
(7.15)

To see this, we first note that for two open sets \( A, B \subset \mathbb{R}^{n} \) with \( A \subset B \), we have \( \overline{A} = \overline{A \cap B}. \) Thus,
\[
\overline{\Lambda_{k}} = \overline{\Lambda_{k} \cap \Omega}
\]  
(7.16)

and hence
\[
\bigcap_{k=0}^{\infty} \overline{\Lambda_{k}} = \bigcap_{k=0}^{\infty} \overline{\Lambda_{k} \cap \Omega}.
\]  
(7.17)
Note that for any $i, j \in \{1, 2\}, k \geq 2, k \in \mathbb{N}$ and any $e \in E_{k}^{i, j}$, it follows from definition that there exist $i', j' \in \{1, 2\}$ and $e' \in E_{k}^{i', j'}$ such that

$$\tilde{S}_{e}(\Omega_{j}) \cap \tilde{S}_{e'}(\Omega_{j'}) \neq \emptyset \quad \text{and} \quad \tilde{S}_{e}(\Omega_{j'}) \subset \tilde{\Omega}.$$ 

Since $|\tilde{S}_{e}(\Omega_{j})| \to 0$ as $k \to \infty$, we see that $\tilde{K} \cap \tilde{\Omega}$ is dense in $\tilde{K}$. Hence $\tilde{K} \cap \tilde{\Omega} = \tilde{K}$, i.e.,

$$\bigcap_{k=0}^{\infty} \lambda_{k} = \left( \bigcap_{k=0}^{\infty} \lambda_{k} \right) \cap \tilde{\Omega}. \quad (7.18)$$

By combining (7.17) and (7.18), we obtain (7.15). It follows that

$$K = \bigcap_{k=0}^{\infty} \bigcup_{i, j=1}^{2} \bigcup_{e \in E_{k}^{i, j}} \tilde{S}_{e}(\Omega_{j}) \quad \text{(by (7.10) and (7.13))}$$

$$= \bigcap_{k=0}^{\infty} \bigcup_{i, j=1}^{2} \bigcup_{e \in E_{k}^{i, j}} \phi^{-1}(\tilde{S}_{e}(\Omega_{j})) \quad \text{(by (6.1) and (6.2))}$$

$$= \bigcap_{k=0}^{\infty} \phi^{-1}(\tilde{\lambda}_{k} \cap \tilde{\Omega}) \quad \text{(by (7.14) and (7.16))}$$

$$= \phi^{-1}(\tilde{K} \cap \tilde{\Omega}) \quad \text{(by (7.11) and (7.15))}.$$ 

Recall from (6.3) that $K$ is the graph self-similar set generated by the GIFS $G = (V, E)$ associated to $\{S_{e}\}_{e \in E}$.

**Example 7.6** Let $\mathbb{T}^{2}, \{h_{t}\}_{t=1}^{4}, \{S_{e}\}_{e \in E}$ and $\{\tilde{S}_{e}\}_{e \in E}$ be defined as above, and let $K$ and $\tilde{K}$ be the graph self-similar set generated by the GIFS $G = (V, E)$ associated to $\{S_{e}\}_{e \in E}$ and $\{\tilde{S}_{e}\}_{e \in E}$, respectively. Then $\dim_{H}(K) = \dim_{H}(\tilde{K}) = \log(2 + \sqrt{2})/\log 2 = 1.77155 \ldots$.

**Proof** For convenience, we write $\tilde{S}_{e_{i}} = \tilde{S}_{i}$ for any $i \in \{1, \ldots, 8\}$. Let $F_{k} = E_{k}$ for $k \geq 1$, and let $\tilde{F}_{1}$ and $\tilde{F}_{2}$ be the neighborhood types of the root neighborhoods.
[Ω̃₁] and [Ω̃₂], respectively. Iterations of the root vertices are shown in Fig. 3(b,c). All neighborhood types are generated after three iterations. To construct the weighted incidence matrix in the reduced graph G_R, we note that

\[ \tilde{V}_1 = \{ (S_1, 1), \ldots, (S_8, 1) \}. \]

Denote by \( \tilde{\omega}_1, \ldots, \tilde{\omega}_8 \) the vertices in \( \tilde{V}_1 \) according to the above order. Then \( [\tilde{\omega}_4] = \tilde{T}_2 \) and \([\tilde{\omega}_8] = \tilde{T}_1 \). Let \( \tilde{T}_3 := [\tilde{\omega}_1], \tilde{T}_4 := [\tilde{\omega}_2], \tilde{T}_5 := [\tilde{\omega}_3], \tilde{T}_6 := [\tilde{\omega}_5], \tilde{T}_7 := [\tilde{\omega}_6], \) \( \tilde{T}_8 := [\tilde{\omega}_7] \). Then

\[ \tilde{T}_1 \longrightarrow \tilde{T}_2 + \tilde{T}_3 + \tilde{T}_4 + \tilde{T}_5 \]

and

\[ \tilde{T}_2 \longrightarrow \tilde{T}_1 + \tilde{T}_6 + \tilde{T}_7 + \tilde{T}_8, \]

where we write \( \tilde{T}_i = \tilde{T}_i (\frac{1}{2}) \) for convenience. Since \( \tilde{S}_{13} = \tilde{S}_{21} \), the edge \( e_1 e_3 \) is removed in \( \tilde{G}_R \). Hence \( \tilde{\omega}_1 \) has three offspring, namely

\[ (\tilde{S}_{11}, 2), (\tilde{S}_{12}, 2), (\tilde{S}_{14}, 2) \in \tilde{V}_2, \]

which are of neighborhood types \( \tilde{T}_3, \tilde{T}_4, \tilde{T}_2 \), respectively. Iterating \( (\tilde{S}_1, 1) \) gives

\[ \tilde{T}_3 \longrightarrow \tilde{T}_2 + \tilde{T}_3 + \tilde{T}_4. \]

Since \( \tilde{S}_{23} = \tilde{S}_{31} \), the edge \( e_2 e_3 \) is removed in \( \tilde{G}_R \). Hence \( \tilde{\omega}_2 \) has three offspring, namely,

\[ (\tilde{S}_{21}, 2), (\tilde{S}_{22}, 2), (\tilde{S}_{24}, 2) \in \tilde{V}_2. \]

Note that \( [(\tilde{S}_{22}, 2)] = \tilde{T}_4 \) and \( [(\tilde{S}_{24}, 2)] = \tilde{T}_2 \). Let \( \tilde{\omega}_9 := (\tilde{S}_{22}, 2) \) and \( \tilde{T}_9 := [\tilde{\omega}_9] \). Then

\[ \tilde{T}_4 \longrightarrow \tilde{T}_2 + \tilde{T}_4 + \tilde{T}_9. \]

Note that \( \tilde{\omega}_3 \) has four offspring, namely,

\[ (\tilde{S}_{31}, 2), (\tilde{S}_{32}, 2), (\tilde{S}_{33}, 2), (\tilde{S}_{34}, 2) \in \tilde{V}_2, \]

with \( [(\tilde{S}_{32}, 2)] = \tilde{T}_4, [(\tilde{S}_{33}, 2)] = \tilde{T}_5 \) and \( [(\tilde{S}_{34}, 2)] = \tilde{T}_2 \). Let \( \tilde{T}_{10} := (\tilde{S}_{31}, 2) \) and \( \tilde{T}_{10} := [\tilde{\omega}_{10}] \). Then

\[ \tilde{T}_5 \longrightarrow \tilde{T}_2 + \tilde{T}_4 + \tilde{T}_5 + \tilde{T}_{10}. \]

Since \( \tilde{S}_{213} = \tilde{S}_{221} \), the edge \( e_2 e_1 e_3 \) is removed in \( \tilde{G}_R \). Hence \( \tilde{\omega}_9 \) has three offspring, namely,

\[ (\tilde{S}_{211}, 2), (\tilde{S}_{212}, 2), (\tilde{S}_{214}, 2) \in \tilde{V}_3, \]
Fig. 3  a $\{h_t\}_{t=1}^4$ on $\mathbb{R}^2/\mathbb{Z}^2$. b The first iteration of $G = (V, E)$ on $\mathbb{R}^2$ associated to the relations induced by $\{h_t\}_{t=1}^4$. c The second iteration. d The corresponding graph self-similar set

which are of neighborhood types $\tilde{T}_{10}$, $\tilde{T}_{4}$, $\tilde{T}_{2}$, respectively. Iterating $(\tilde{S}_{22}, 2)$ gives

$$\tilde{T}_9 \longrightarrow \tilde{T}_2 + \tilde{T}_4 + \tilde{T}_{10}.$$ 

Since $\tilde{S}_{313} = \tilde{S}_{321}$, the edge $e_3e_1e_3$ is removed in $\tilde{G}_R$. Hence $\tilde{\omega}_{10}$ has three offspring, namely,

$$(\tilde{S}_{311}, 2), (\tilde{S}_{312}, 2), (\tilde{S}_{314}, 2) \in \tilde{\mathcal{V}}_3,$$
which are of neighborhood types $\tilde{T}_10, \tilde{T}_4, \tilde{T}_2$, respectively. Iterating $(\tilde{S}_{31}, 2)$ gives

$$\tilde{T}_10 \rightarrow \tilde{T}_2 + \tilde{T}_4 + \tilde{T}_10.$$ 

Let $\tilde{T}_{11} := [(\tilde{S}_{65}, 2)]$ and $\tilde{T}_{12} := [(\tilde{S}_{75}, 2)]$. Using the same argument, it can be checked directly that

$$\tilde{T}_6 \rightarrow \tilde{T}_1 + \tilde{T}_6 + \tilde{T}_7, \quad \tilde{T}_7 \rightarrow \tilde{T}_1 + \tilde{T}_7 + \tilde{T}_{11},$$
$$\tilde{T}_8 \rightarrow \tilde{T}_1 + \tilde{T}_7 + \tilde{T}_8 + \tilde{T}_{12}, \quad \tilde{T}_{11} \rightarrow \tilde{T}_1 + \tilde{T}_7 + \tilde{T}_{12}, \quad \tilde{T}_{12} \rightarrow \tilde{T}_1 + \tilde{T}_7 + \tilde{T}_{12}.$$

Hence the weighted incidence matrix is

$$\tilde{A}_\alpha = \left(\frac{1}{2}\right)^\alpha \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} =: \left(\frac{1}{2}\right)^\alpha \tilde{A}_\alpha,$$

and the maximal eigenvalue of $\tilde{A}_\alpha'$ is $2 + \sqrt{2}$. The GIFS $G = (V, E)$ defined on $\mathbb{R}^2$ satisfies (GFTC) with $\{\tilde{\Omega}_i\}_{i=1}^2$ being a GFTC-family and $\{\tilde{S}_e\}_{e \in E}$ being an associated family of contractive similitudes. By Proposition 7.5, the GIFS $G = (V, E)$ defined on $\mathbb{T}^2$ satisfies (GFTC) with $\{\tilde{\Omega}_i\}_{i=1}^2$ being a GFTC-family and $\{\tilde{S}_e\}_{e \in E}$ being an associated family of contractive similitudes. By Proposition 7.5 and Theorem 1.6,

$$\dim_H(K) = \dim_H(\tilde{K}) = \log(2 + \sqrt{2})/\log 2 = 1.77155 \ldots.$$  

**Remark 7.7** Unlike $\mathbb{T}^2$, the family of contractive similitudes associated to a GIFS defined on $\mathbb{R}^2$ can be characterized explicitly. In fact, Proposition 7.5 is not needed in the proof of Example 7.6. We may use Theorem 1.6 and a similar argument as in the proof of Example 7.6 to obtain the same result.

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**Declarations**

**Conflict of interest** The authors hereby declare that there is no conflict of interest regarding this work.
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