Whistleron Gas in Magnetized Plasmas

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(Dated: January 31, 2022)

We have studied the nonlinear dynamics of whistler waves in magnetized plasmas. Since plasmas and beam-plasma systems considered here are assumed to be weakly collisional, the point of reference for the analysis performed in the present paper is the system of hydrodynamic and field equations. We have applied the renormalization group method to obtain dynamical equations for the slowly varying amplitudes of whistler waves. Further, it has been shown that the amplitudes of eigenmodes satisfy an infinite system of coupled nonlinear Schrödinger equations. In this sense, the whistler eigenmodes form a sort of a gas of interacting quasiparticles, while the slowly varying amplitudes can be considered as dynamical variables heralding the relevant information about the system. An important feature of our description is that whistler waves do not perturb the initial uniform density of plasma electrons. The plasma response to the induced whistler waves consists in velocity redistribution which follows exactly the behaviour of the whistlers. In addition, selection rules governing the nonlinear mode coupling have been derived, which represent another interesting peculiarity of our description.

PACS numbers: 52.25.Xz, 53.35.Hr, 52.35.Sb

KEY WORDS: Magnetized Plasma, Renormalization Group, Whistler Waves, Solitary Waves.

I. INTRODUCTION

Over four decades passed since it was first shown that plasmas and beam-plasma systems immersed in an external magnetic field can support travelling electromagnetic waves with specific features. These waves propagate parallel to the applied magnetic field being circularly polarized in a plane transverse to the direction of propagation. It has become conventional in the physics of magnetized plasmas to call such structures waves in the whistler mode.

Although the linear stability properties of the electromagnetic waves in the whistler mode are relatively well studied [1, 2, 3, 4], there is a serious gap in the understanding of their nonlinear behaviour. Chen et al. [5] have shown that electromagnetic whistler waves can be considered as complementary to the nonlinear traveling electrostatic waves, known as the Bernstein-Greene-Kruskal (BGK) modes [6]. While the BGK modes are longitudinal, the whistler modes are transverse, in other words, the components of the electric and magnetic field of the whistler wave parallel to the external magnetic field are both zero. The study of the nonlinear behaviour of whistler waves has been initiated by Taniuti and Washimi [7], who obtained a nonlinear Schrödinger equation for the slowly varying amplitude (see also Reference [8]).

The present paper is aimed at filling the gap in the understanding of the nonlinear evolution of whistler waves. The method adopted here is the renormalization group (RG) method [9, 10]. The basic feature of this approach is that it provides a convenient and straightforward tool to obtain an adequate description of the physically essential properties of self-organization and formation of patterns in complex systems. Coherent structures which result from the nonlinear interaction between plane waves evolve on time and/or spatial scales comparatively large compared to those the fast oscillations occur. The RG method can be considered as a powerful systematic procedure to separate the relatively slow dynamics from the fast one, which is of no considerable physical relevance. In a context similar to that of the present paper, it has been successfully applied by one of the authors [10, 11] to study collective effects in intense charged-particle beams.

The paper is organized as follows. In the next section, we state the basic equations which will be the subject of the renormalization group reduction in section III. Starting from a single equation [see equation (III.6)] for the electromagnetic vector potential, we obtain a formal perturbation expansion of its solution to second order. As expected, it contains secular terms proportional to powers of the time variable which is the only renormalization parameter adopted in our approach. In section IV, the arbitrary constant amplitudes of the perturbation expansion are renormalized such as to eliminate the secular terms. As a result, a set of equations for the renormalized slowly varying amplitudes is obtained, known as the renormalization group equations (RGEs). These equations comprise an infinite system of coupled nonlinear Schrödinger equations. In section V, the latter are analyzed in the simplest case. Finally, section VI is dedicated to discussion and conclusions.
II. FORMULATION OF THE PROBLEM AND BASIC EQUATIONS

Plasmas and beam-plasma systems considered in the present paper are assumed to be weakly collisional. Therefore, the dynamics of plasma species is well described by the hydrodynamic equations coupled with the equations for the electromagnetic self-fields. We start with the equations for plasma in an external constant magnetic field $B_0$, which can be written as follows

$$\frac{\partial n_a}{\partial t} + \nabla \cdot (n_a \mathbf{V}_a) = 0, \quad (II.1)$$

where $n_a$ and $\mathbf{V}_a$ are the density and the current velocity of the species $a$. Furthermore, $m_a$, $q_a$ and $T_a$ are the mass, the relative charge and the temperature, respectively, while $k_B$ is the Boltzmann constant. The substantial derivative on the left-hand-side of equation (II.2) is defined as

$$\frac{D_a V_a}{Dt} = \frac{\partial}{\partial t} + \mathbf{V}_a \cdot \nabla. \quad (II.3)$$

The electromagnetic self-fields $\mathbf{E}$ and $\mathbf{B}$ can be obtained in terms of the electromagnetic vector $\mathbf{A}$ and scalar $\varphi$ potentials according to the well-known relations

$$\mathbf{E} = -\nabla \varphi - \frac{1}{c^2} \frac{\partial \mathbf{A}}{\partial t}, \quad \mathbf{B} = \nabla \times \mathbf{A}. \quad (II.4)$$

The latter satisfy the wave equations

$$\Box \mathbf{A} = -\mu_0 \varepsilon \sum_a n_a q_a \mathbf{V}_a, \quad \Box \varphi = -\frac{\varepsilon}{\varepsilon_0} \sum_a n_a q_a, \quad (II.5)$$

in the Lorentz gauge

$$\frac{1}{c^2} \frac{\partial \varphi}{\partial t} + \nabla \cdot \mathbf{A} = 0. \quad (II.6)$$

Here $\Box$ denotes the well-known d’Alembert operator. In what follows, we consider the case of a quasineutral plasma

$$\sum_a n_a q_a = 0, \quad (II.7)$$

in a constant external magnetic field along the $x$-axis $B_0 = (B_0, 0, 0)$. Then, equations (II.1)–(II.6) possess a stationary solution

$$n_a = n_{a0} = \text{const}, \quad \mathbf{V}_a = 0, \quad \mathbf{A} = 0, \quad \varphi = 0. \quad (II.8)$$

The frequency of the wave will be taken as much higher than the ion-cyclotron frequency. Therefore, we can further neglect the ion motion and scale the hydrodynamic and field variables as

$$n_e = n_0 + \varepsilon N, \quad \mathbf{V}_e = \varepsilon \mathbf{V}, \quad \mathbf{A} \rightarrow \varepsilon \mathbf{A}, \quad \varphi \rightarrow \varepsilon \varphi, \quad (II.9)$$

where $\varepsilon$ is a formal small parameter introduced for convenience, which will be set equal to one at the end of the calculations. Thus, the basic equations to be used for the subsequent analysis can be written in the form

$$\frac{\partial N}{\partial t} + n_0 \nabla \cdot \mathbf{V} + \varepsilon \nabla \cdot (N \mathbf{V}) = 0, \quad (II.10)$$

$$\frac{\partial \mathbf{V}}{\partial t} + \varepsilon \mathbf{V} \cdot \nabla \mathbf{V} = -\frac{k_B T}{m(n_0 + \varepsilon N)} \nabla N$$

$$-\frac{\varepsilon}{m} [\mathbf{E} + \mathbf{V} \times (\mathbf{B}_0 + \varepsilon \mathbf{B})], \quad (II.11)$$

$$\Box \mathbf{A} = \mu_0 \varepsilon (n_0 + \varepsilon N) \mathbf{V}, \quad \frac{1}{c^2} \frac{\partial \varphi}{\partial t} + \nabla \cdot \mathbf{A} = 0. \quad (II.12)$$

Before we continue with the renormalization group reduction of the system of equations (II.10)–(II.12), in the next section, let us assume that the actual dependence of the quantities $N$, $\mathbf{V}$, $\mathbf{A}$ and $\varphi$ on the spatial variables is represented by the expression

$$\widehat{\Psi} = \widehat{\Psi}(\mathbf{x}, \mathbf{X}; t), \quad \widehat{\Psi} = (N, \mathbf{V}, \mathbf{A}, \varphi), \quad (III.13)$$

where $\mathbf{X} = \mathbf{e} x$ is a slow spatial variable. Thus, the only renormalization parameter left at our disposal is the time $t$ which will prove extremely convenient and simplify tedious algebra in the sequel.

III. RENORMALIZATION GROUP REDUCTION OF THE MAGNETOHYDRODYNAMIC EQUATIONS

Following the standard procedure of the renormalization group method, we represent $\widehat{\Psi}$ as a perturbation expansion

$$\widehat{\Psi} = \sum_{n=0}^{\infty} \varepsilon^n \widehat{\Psi}_n, \quad (III.1)$$

in the formal small parameter $\varepsilon$. The next step consists in expanding the system of hydrodynamic and field equations (II.10)–(II.12) in the small parameter $\varepsilon$, and obtaining their naive perturbation solution order by order. Note that in all orders the perturbation equations acquire the general form

$$\frac{\partial N_n}{\partial t} + n_0 \nabla \cdot \mathbf{V}_n = \alpha_n, \quad (III.2)$$

$$\frac{\partial \mathbf{V}_n}{\partial t} = \frac{v_f^2}{n_0} \nabla N_n - \frac{e}{m} \mathbf{E}_n - \omega_c \mathbf{V}_n \times \mathbf{e}_x + \mathbf{W}_n, \quad (III.3)$$

where $v_f$ is the fluid velocity.
\[ \Box A_n = \mu_0 e n_0 V_n + U_n, \quad \frac{1}{c^2} \frac{\partial \varphi_n}{\partial t} + \nabla \cdot A_n = \beta_n, \]  

(III.4)

where \( \alpha_n, \beta_n, U_n \) and \( W_n \) are quantities, that have been already determined from previous orders. Here

\[ v_T^2 = \frac{k_B T}{m}, \quad \omega_e = \frac{e B_0}{m} \]  

(III.5)

are the thermal velocity of electrons and the electron-cyclotron frequency, respectively and \( e_x = (1, 0, 0) \) is the unit vector in the \( x \)-direction. Manipulating in an obvious manner equations (III.2)–(III.4), it is possible to obtain a single equation for \( A_n \). The latter reads as

\[ \Box \frac{\partial^2 A_n}{\partial t^2} - v_T^2 \Box \nabla (\nabla \cdot A_n) + \omega_e \Box \frac{\partial A_n}{\partial t} = \mu_0 e n_0 \frac{\partial W_n}{\partial t} + \frac{\partial^2 U_n}{\partial t^2} \]

\[ -\frac{\omega_p^2}{c^2} \frac{\partial^2 A_n}{\partial t^2} + \omega_p^2 \nabla (\nabla \cdot A_n) = \mu_0 e n_0 \frac{\partial W_n}{\partial t} + \omega_2 \frac{\partial U_n}{\partial t} + \omega_p \nabla \beta_n, \]  

(III.6)

where

\[ \omega_p^2 = \frac{e^2 n_0}{\epsilon_0 m}, \]  

(III.7)

is the electron plasma frequency. Note that the thermal velocity \( v_T \) as defined by equation (III.6) can be alternatively expressed according to the expression

\[ v_T = \omega_p r_D, \quad r_D = \frac{e_0 k_B T}{e^2 n_0}, \]  

(III.8)

where \( r_D \) is the electron Debye radius. Equation (III.6) represents the starting point for the renormalization group reduction, the final goal of which is to obtain a description of the relatively slow dynamics leading to formation of patterns and coherent structures.

Let us proceed order by order. We assume that the dependence on the fast spatial variables \( x = (x, y, z) \) is through the longitudinal (parallel to the external magnetic field \( B_0 \)) \( x \)-coordinate only. The solution to the zero-order perturbation equations (III.6) can be written as

\[ A_0 = \sum_k A_k^{(0)} A_k e^{i \psi_k}, \]  

(III.9)

where

\[ \psi_k(x; t) = k x - \omega_k t, \]  

(III.10)

and \( A_k \) is an infinite set of constant complex amplitudes, which will be the subject of the renormalization procedure in the sequel. Here “constant” means that the amplitudes \( A_k \) do not depend on the fast spatial variable \( x \) and on the time \( t \), however, it can depend on the slow spatial variables \( X \). The summation sign in equation (III.9) and throughout the paper implies summation over the wave number \( k \) in the case where it takes discrete values, or integration in the continuous case. From the dispersion equation

\[ D(k; \omega_k) = \omega_k^2 \left[ \Box - \frac{\omega_p^2}{c^2} - \omega_e \Box \right] = 0, \]  

(III.11)

it follows that the wave frequency \( \omega_k \) can be expressed in terms of the wave number \( k \), where the Fourier-image \( \Box_k \) of the d’Alembert operator can be written according to

\[ \Box_k = \frac{\omega_k^2}{c^2} - k^2. \]  

(III.12)

Moreover, it can be verified in a straightforward manner that the constant vector \( A_k^{(0)} \) can be expressed as

\[ A_k^{(0)} = (0, 1, -i \text{sgn}(k)), \]  

(III.13)

where \( \text{sgn}(k) \) is the well-known sign-function. Details concerning the derivation of the dispersion law (III.11) and equation (III.13) can be found in the Appendix. Note that equation (III.13) is an alternative representation of the solvability condition (III.3). It is important to emphasize that

\[ \omega_{-k} = -\omega_k, \quad A_{-k} = A_k^*, \]  

(III.14)

where the asterisk denotes complex conjugation. The latter assures that the vector potential as defined by equation (III.1) is a real quantity. The zero-order current velocity \( V_0 \) obtained directly from the first equation (III.4) can be written as

\[ V_0 = \sum_k V_k^{(0)} A_k e^{i \psi_k}, \quad V_k^{(0)} = \frac{\Box_k}{\mu_0 e n_0} A_k^{(0)} . \]  

(III.15)

In addition, the zero-order density, scalar potential and magnetic field are represented by the expressions

\[ N_0 \equiv 0, \quad \varphi_0 \equiv 0, \quad B_0 = \sum_k B_k^{(0)} A_k e^{i \psi_k}, \]  

(III.16)

where

\[ B_k^{(0)} = -k A_k^{(0)} \text{sgn}(k) = (0, -k \text{sgn}(k), ik). \]  

(III.17)

It has been mentioned that the first-order “source terms” on the right-hand-side of equation (III.6) can be expressed via quantities already known from zero order. Thus, we have

\[ \alpha_1 = -n_0 \hat{\nabla} \cdot V_0, \quad \beta_1 = -\hat{\nabla} \cdot A_0, \]  

(III.18)

\[ U_1 = -2 \nabla \cdot \hat{\nabla} A_0, \quad W_1 = \frac{e}{m} V_0 \times B_0, \]  

(III.19)
where the shorthand notation
\[ \hat{\nabla} = \frac{\partial}{\partial \mathbf{X}} \]  
(III.20)

has been introduced. Note that the vector \( \mathbf{W}_1 \) representing the zero-order Lorentz force has the only nonzero component along the external magnetic field, that is
\[ \mathbf{W}_1 = e_x \sum_{k,l} \alpha_{kl} A_k e^{i(\psi_k + \psi_l)}, \]  
(III.21)

where
\[ \alpha_{kl} = -\frac{i}{2\mu_0 n_0 m}(k\square + l\square)[1 - \text{sgn}(k)\text{sgn}(l)]. \]  
(III.22)

Equation (III.6) has now two types of solutions. The first is a secular solution linearly dependent on the time variable in the first-order approximation. As a rule, the highest power in the renormalization parameter of the secular terms contained in the standard perturbation expansion is equal to the corresponding order in the small perturbation parameter. The second solution of equation (III.6) arising from the nonlinear interaction between waves in the first order, is regular. Omitting tediums but standard algebra, we present here only the result
\[ A_1 = \sum_k \tilde{A}_k^{(1)} A_k e^{i\psi_k} + e_x \sum_{k,l} A_k^{(1)} A_l e^{i(\psi_k + \psi_l)}, \]  
(III.23)

where
\[ \tilde{A}_k^{(1)} = \left( \tilde{A}_k^{(1)}, t\tilde{A}_k^{(1)}, -i t\tilde{A}_k^{(1)} \text{sgn}(k) \right), \]  
(III.24)

Some of the details of the calculations are presented in the Appendix. In explicit form, the components of the vector operator \( \tilde{A}_k^{(1)} \) and those of the infinite matrix \( A_k^{(1)} \) are given by the expressions
\[ \tilde{A}_k^{(1)} = -\frac{i k \beta_k}{\gamma_k \chi_k} \hat{\nabla}_k, \quad \nabla_k = A_k^{(0)} \hat{\nabla}, \]  
(III.25)

\[ \tilde{A}_k^{(1)} = -\frac{F_k}{2\omega_k \alpha_k \text{sgn}(k) + \omega_k \chi_k}, \]  
(III.26)

\[ A_k^{(1)} = \frac{e}{2mv_T D_{kl}} \frac{\omega_k + \omega_l}{k\square + l\square} [1 - \text{sgn}(k)\text{sgn}(l)], \]  
(III.27)

where
\[ F_k = 2k\omega_k [\omega_k \text{sgn}(k) + \omega_k] \hat{\nabla}_k, \]  
(III.28)

\[ \square_k = \frac{(\omega_k + \omega_l)^2}{c^2} - (k + l)^2, \]  
(III.29)

\[ D_{kl} = \frac{(\omega_k + \omega_l)^2}{c^2} - (k + l)^2 - \frac{1}{r_D^2}. \]  
(III.30)

In addition, the constants \( \alpha_k, \beta_k, \gamma_k \) and \( \chi_k \) entering the expressions above are given by
\[ \alpha_k = \square_k + \frac{\omega_k^2 - \omega_l^2}{c^2}, \quad \beta_k = \square_k - \frac{1}{r_D}, \]  
(III.31)

\[ \gamma_k = \frac{\omega_k^2}{v_T^2} - k^2 - \frac{1}{r_D}, \quad \chi_k = \square_k + \frac{2\omega_k^2}{c^2}. \]  
(III.32)

Furthermore, the first-order current velocity can be expressed as
\[ \mathbf{V}_1 = \sum_k \hat{\nabla}_k^{(1)} A_k e^{i\psi_k} + e_x \sum_{k,l} V_{kl}^{(1)} A_k A_l e^{i(\psi_k + \psi_l)}, \]  
(III.33)

where
\[ \hat{\nabla}_k^{(1)} = \left( \hat{\nabla}_k^{(1)}, \hat{\nabla}_k^{(1)}, -i \hat{\nabla}_k^{(1)} \text{sgn}(k) \right). \]  
(III.34)

The corresponding operators and matrix coefficients can be written explicitly according to the expressions
\[ \hat{\nabla}_k^{(1)} = \frac{2\mu_0 e_n}{\mu_0 e_0} A_k^{(1)} A_k^{(1)}, \]  
(III.35)

\[ \hat{\nabla}_k^{(1)} = \frac{2i}{\mu_0 e_n} \left[ i\square_k A_k^{(1)} + 2i(e_k^2 A_k^{(1)} + k\hat{\nabla}_k) \right]. \]  
(III.36)

Calculating the first-order density \( N_1 \) from equation (III.2), we obtain
\[ N_1 = \sum_k \hat{N}_k^{(1)} A_k e^{i\psi_k} + \sum_{k,l} N_{kl}^{(1)} A_k A_l e^{i(\psi_k + \psi_l)}, \]  
(III.37)

\[ \hat{N}_k^{(1)} = \frac{k + l}{2\mu_0 e_n v_T D_{kl}} [k\square_k + l\square_k] [1 - \text{sgn}(k)\text{sgn}(l)]. \]  
(III.38)

\[ N_{kl}^{(1)} = \frac{e c^2}{\mu_0 e_n} \hat{N}_k^{(1)} \hat{N}_l^{(1)} = \frac{e^2}{\omega_k} \left( k\tilde{A}_k^{(1)} + n\tilde{A}_k^{(1)} \right). \]  
(III.39)

Analogously, for the first-order scalar potential \( \varphi_1 \), we find
\[ \varphi_1 = \sum_k \varphi_k^{(1)} A_k e^{i\psi_k} + \sum_{k,l} \varphi_{kl}^{(1)} A_k A_l e^{i(\psi_k + \psi_l)}, \]  
(III.40)

\[ \varphi_k^{(1)} = \frac{e}{\mu_0 e_n} \hat{N}_k^{(1)} \hat{N}_l^{(1)} = \frac{e}{\omega_k} \left( k\tilde{A}_k^{(1)} + n\tilde{A}_k^{(1)} \right). \]  
(III.41)

\[ \varphi_{kl}^{(1)} = \frac{ec^2}{\mu_0 e_n v_T D_{kl}} [k\square_k + l\square_k] [1 - \text{sgn}(k)\text{sgn}(l)]. \]  
(III.42)
Finally, the first-order magnetic field is calculated to be

$$ B_1 = \sum_k \hat{B}_k^{(1)} A_k e^{i\psi_k}, \quad (\text{III.43}) $$

where

$$ \hat{B}_k^{(1)} = \left(-\text{sgn}(k)\vec{\nabla}_k, -i\hat{B}_k^{(1)} \text{sgn}(k)\right), \quad (\text{III.44}) $$

$$ \hat{B}_k^{(1)} = -\text{sgn}(k) \left(tk\hat{A}_k^{(1)} - i\vec{\nabla}_X\right). \quad (\text{III.45}) $$

A couple of interesting features of the zero and first-order perturbation solution are noteworthy to be commented at this point. First of all, the zero-order density $N_0$ vanishes which means that no density waves are induced by the whistler eigenmodes. The second terms in the expressions for the first-order density $N_1$ and current velocity $V_1$ [see equations (III.33) and (III.37)] imply contribution from nonlinear interaction between waves according to the nonlinear Lorentz force. It will be shown in the remainder that these terms give rise to nonlinear terms in the renormalization group equation and describe solitonic wave behaviour of the whistler mode.

**IV. THE RENORMALIZATION GROUP EQUATION**

Passing over to the final stage of our renormalization group procedure, we note that in second order the quantities $U_2$ and $W_2$ entering the right-hand-side of equation (IV.1) can be written as

$$ U_2 = -2\nabla \cdot \vec{\nabla} A_1 - \vec{\nabla}^2 A_0 + \mu_0 e N_1 V_0, \quad (\text{IV.1}) $$

$$ W_2 = \frac{e}{m} \vec{\nabla} \varphi_1 - \frac{v_0^2}{m_0} \vec{\nabla} N_1 - V_1 \cdot \vec{\nabla} V_0 - \frac{e}{m} V_1 \times B_0, \quad (\text{IV.2}) $$

Since we are interested only in the secular terms in second order, appearing in the expressions for the $y$ and $z$ components of the electromagnetic vector potential $\mathbf{A}_2$, contributions in the source vectors $U_2$ and $W_2$ leading to such terms are sufficient for completing the renormalization group procedure. Thus, we can write

$$ A_2 = \sum_k \left( \hat{A}_k^{(2)} + t^2 \hat{C}_k \right) A_k e^{i\psi_k} $$

$$ + t \sum_k \hat{D}_k^{(2)} A_k e^{i\psi_k} + t \sum_{k,l} \Gamma_{kl} |A_l|^2 A_k e^{i\psi_k}. \quad (\text{IV.3}) $$

An important remark is in order at this point. From the solvability condition (A.3), it follows that the complex amplitude $A_k$ must satisfy the complex Poisson equation

$$ \vec{\nabla}_k^2 A_k = 0. \quad (\text{IV.4}) $$

The latter imposes additional restrictions on the dependence of the wave amplitudes $A_k$ on the slow transverse independent variables $Y$ and $Z$. Straightforward calculations yield (see the Appendix for details)

$$ \hat{A}_k^{(2)} = -\frac{\text{sgn}(k)}{2\omega_k \alpha_k \text{sgn}(k) + \omega_k \chi_k} \left( \hat{B}_k^{(2)} \hat{A}_k^{(1)2} - \hat{C}_k \right), \quad (\text{IV.5}) $$

$$ \hat{D}_k^{(2)} = \frac{i \nu_0^2 \beta_k \text{sgn}(k)}{2\omega_k \alpha_k \text{sgn}(k) + \omega_k \chi_k} \left( 1 + \frac{k^2 \beta_k}{\gamma_k \Box_k} \right) \vec{\nabla}_Y \vec{\nabla}_k, \quad (\text{IV.6}) $$

$$ \hat{C}_k = \frac{1}{2} \hat{A}_k^{(1)2}, \quad (\text{IV.7}) $$

where

$$ \beta_k = \alpha_k + \frac{4\omega_k^2}{c^2} + \frac{3\omega_k \omega_k}{c^2} \text{sgn}(k), \quad (\text{IV.8}) $$

$$ \hat{G}_k = \omega_k \text{sgn}(k) [\omega_k \text{sgn}(k) + \omega_k] \vec{\nabla}_k^2. \quad (\text{IV.9}) $$

The matrix coefficient $\Gamma_{kly}$ determining the nonlinear contribution represented by the second term in equation (IV.3) reads explicitly as

$$ \Gamma_{kly} = -\frac{1 - \text{sgn}(k) \text{sgn}(l) i \omega_k \Box_l (k \Box_l + l \Box_k) \text{sgn}(k)}{\mu_0 n_0 n_l \omega_k \Box_k} \frac{1 - \text{sgn}(k) \text{sgn}(l) i \omega_k \Box_l (k \Box_l + l \Box_k) \text{sgn}(k)}{2\omega_k \alpha_k \text{sgn}(k) + \omega_k \chi_k} \times [\omega_k (l \omega_k - k \omega_k) \text{sgn}(l) + (k + l) \omega_k \omega_l]. \quad (\text{IV.10}) $$

Following the standard procedure [10] of the RG method, we finally obtain the desired RG equation

$$ \frac{\partial \hat{A}_k}{\partial t} - \epsilon \hat{A}_k^{(1)} \hat{A}_k $$

$$ = \epsilon^2 \left( \hat{A}_k^{(2)} + \hat{D}_k^{(2)} \right) \hat{A}_k + \epsilon^2 \sum_l \Gamma_{kly} |\hat{A}_l|^2 \hat{A}_k, \quad (\text{IV.11}) $$

where now $\hat{A}_k$ is the renormalized complex amplitude [10]. Thus, the renormalized solution for the electromagnetic vector potential acquires the form

$$ \mathbf{A} = \sum_k A_k^{(0)} \hat{A}_k e^{i\psi_k}. \quad (\text{IV.12}) $$

Analogously, for the electric and magnetic field of the whistler wave, one can obtain in a straightforward manner the following expressions

$$ \mathbf{B} = \sum_k B_k^{(0)} \hat{A}_k e^{i\psi_k}, \quad \mathbf{E} = i \sum_k \omega_k A_k^{(0)} \hat{A}_k e^{i\psi_k}. \quad (\text{IV.13}) $$

It is important to mention that the plasma density remains unchanged ($N = 0$) contrary to the case of electrostatic waves, where the evolution of the induced electrostatic waves follows the evolution of the density waves.
V. SYSTEM OF COUPLED NONLINEAR SCHRÖDINGER EQUATIONS

The simplest case of the validity of the solvability condition (V.3) consists in the assumption that the slow wave amplitudes \( A_k \) do not depend on the transverse coordinates. Setting \( \epsilon = 1 \) in equation (V.4), we obtain the following system of coupled nonlinear Schrödinger equations

\[
\text{sgn} (k) \frac{\partial A_k}{\partial t} + i \nu_k \text{sgn} (k) \frac{\partial A_k}{\partial x} = \lambda_k \frac{\partial^2 A_k}{\partial x^2} + i \sum l \mu_{kl} |A_l|^2 A_k, \tag{V.1}
\]

where for simplicity the tilde-sign over the renormalized amplitudes has been dropped. Moreover, the coefficients \( \nu_k, \lambda_k \) and \( \mu_{kl} \) are given by the expressions

\[
\nu_k = \frac{2k \omega_k [\omega_k \text{sgn} (k) + \omega_c]}{2 \omega_k \alpha_k \text{sgn} (k) + \omega_c \chi_k}, \tag{V.2}
\]

\[
\lambda_k = \frac{\omega_k [\omega_k \text{sgn} (k) + \omega_c]}{2 \omega_k \alpha_k \text{sgn} (k) + \omega_c \chi_k}
\]

\[
\left\{ \frac{4k^2 \omega_k \beta_k^2 (\omega_k \text{sgn} (k) + \omega_c)}{2 \omega_k \alpha_k \text{sgn} (k) + \omega_c \chi_k} - \text{sgn} (k) \right\}, \tag{V.3}
\]

\[
\mu_{kl} = \frac{1 - \text{sgn} (k) \text{sgn} (l) \omega_k \square_l (k \square_l + l \square_k)}{2 \omega_k \alpha_k \text{sgn} (k) + \omega_c \chi_k}
\]

\[
\times [\omega_c (k \omega_l - l \omega_k) + (k + l) \omega_k \omega_l]. \tag{V.4}
\]

Interestingly enough, the infinite matrix of coupling coefficients \( \mu_{kl} \) represents a sort of selection rules. Clearly,

\[
\mu_{kk} = 0, \quad \mu_{k,-k} = 0, \tag{V.5}
\]

and

\[
\mu_{kl} = 0, \quad \text{for } \text{sgn} (k) \text{sgn} (l) = 1. \tag{V.6}
\]

This means that a generic mode with a wave number \( k \) cannot couple with itself, neither can it couple with another mode with a wave number of the same sign. Note that this feature is a consequence of the vector character of the nonlinear coupling between modes and is due to the nonlinear Lorentz force. Therefore, for a given mode \( k \) the simplest nontrivial reduction of the infinite system of coupled nonlinear Schrödinger equations consists of minimum two coupled equations.

Without loss of generality, we can assume in what follows that the sign of an arbitrary mode \( k \) under consideration is positive \((k > 0)\). Suppose that for a particular whistler mode with a positive wave number \( k \) there exist a mode with wave number \(-l\) for which the coupling coefficient \( \mu_{k,-l} \) is maximum. Neglecting all other modes but the modes \( k \) and \(-l\), we can write

\[
i \frac{\partial A_k}{\partial t} + i \nu_k \frac{\partial A_k}{\partial x} = \lambda_k \frac{\partial^2 A_k}{\partial x^2} + \mu_1 |A_l|^2 A_k, \tag{V.7}
\]

\[
i \frac{\partial A_l}{\partial t} + i \nu_l \frac{\partial A_l}{\partial x} = \lambda_l \frac{\partial^2 A_l}{\partial x^2} + \mu_2 |A_k|^2 A_l, \tag{V.8}
\]

where

\[
\mu_1 = \frac{2}{\mu_0 n_0 m v_f^2 \omega_l D_{k,-l}} \chi_k \omega_k \omega_l \chi_k
\]

\[
\times [\omega_c (k \omega_l - l \omega_k) + (k + l) \omega_k \omega_l], \tag{V.9}
\]

\[
\mu_2 = \frac{2}{\mu_0 n_0 m v_f^2 \omega_k D_{k,-l}} \chi_k \omega_k \omega_l \chi_k
\]

\[
\times [\omega_c (k \omega_l - l \omega_k) + (k + l) \omega_k \omega_l]. \tag{V.10}
\]

The system of coupled nonlinear Schrödinger equations (V.7) and (V.8) is non integrable in general [14]. It represents an important starting point for further investigations on the nonlinear dynamics and evolution of whistler waves in magnetized plasmas.

VI. DISCUSSION AND CONCLUSIONS

We studied the nonlinear dynamics of whistler waves in magnetized plasmas. Since plasmas and beam-plasma systems considered here are assumed to be weakly collisional, the point of reference for the analysis performed in the present paper is the system of hydrodynamic and field equations. We apply the renormalization group method to obtain dynamical equations for the slowly varying amplitudes of whistler waves. As a result of the investigation performed, it has been shown that the amplitudes of eigenmodes satisfy an infinite system of coupled nonlinear Schrödinger equations. In this sense, the whistler eigenmodes form a sort of a gas of interacting quasiparticles, while the slowly varying amplitudes can be considered as dynamical variables heralding the relevant information about the system.

An important feature of our description is that whistler waves do not perturb the initial uniform density of plasma electrons. The plasma response to the induced whistler waves consists in velocity redistribution which follows exactly the behaviour of the whistlers. Another interesting peculiarity are the selection rules governing the nonlinear mode coupling. According to these rules modes with the same sign do not couple, which is a direct consequence of the vector character of the interaction.
Careful inspection shows that the initial source of the nonlinear interaction between waves in the whistler mode is the zero-order Lorentz force [see equation (III.2)]. Since the quantity $W_1$ is proportional to $\mathbf{A}_k^{(0)} \times \mathbf{A}_l^{(0)}$, the above mentioned selection rules follow directly, provided the only case in which the cross product does not vanish is the case, where modes $k$ and $l$ have different sign.

We believe that the results obtained in the present paper might have a wide class of possible applications ranging from laboratory experiments to observations of a variety of effects relevant to space plasmas.

Acknowledgments

It is a pleasure to thank B. Baizakov for many interesting and useful discussions concerning the subject of the present paper.

APPENDIX A: DETAILS CONCERNING THE DERIVATION OF THE PERTURBATION EXPANSION

Under the assumption that whistler waves propagate parallel to the external magnetic field $\mathbf{B}_0$ (dependence on the longitudinal x-coordinate and the time $t$ only), the equation (III.6) for the longitudinal component $A_{nx}$ decouples from the equations for the other two components. In zero order it has an obvious solution $A_{nx} = 0$. The perturbation equations for the transverse components of the electromagnetic vector potential can be written as

$$\nabla^2 A_{0y} + \omega_c \nabla A_{0z} - \frac{\omega_p^2}{c^2} \nabla^2 A_{0y} = 0, \quad (A.1)$$

$$\nabla^2 A_{0z} - \omega_c \nabla A_{0y} - \frac{\omega_p^2}{c^2} \nabla^2 A_{0z} = 0. \quad (A.2)$$

To solve the above system of equations, we use the ansatz (III.9). Then, the dispersion equation (III.11) can be obtained in a straightforward manner as a condition for vanishing of the determinant of the linear system consisting of the components $A_{ky}^{(0)}$ and $A_{kz}^{(0)}$. Moreover, from the dispersion equation it follows that $A_{kz}^{(0)}$ is proportional to $A_{ky}^{(0)}$, that is

$$A_{kz}^{(0)} = -\text{sgn}(k)A_{ky}^{(0)}. \quad (A.3)$$

Equation (A.3) should be regarded as a solvability condition and must be satisfied order by order.

The first order is characterized by the presence of resonant terms (proportional to $e^{i\psi k}$) on the right-hand-side of equation (III.6). These yield a secular solution to the perturbation equations linear (in first order) with respect to the time $t$. Taking into account the resonant terms alone, it can be verified that the operators $A_{ky}^{(1)}$ and $A_{kz}^{(1)}$ satisfy the following system of linear equations

$$2i\omega_k \alpha_k A_{ky}^{(1)} - \omega_c \chi_k A_{kz}^{(1)} = -\text{sgn}(k)\widehat{F}_k, \quad (A.4)$$

$$2i\omega_k \alpha_k A_{kz}^{(1)} + \omega_c \chi_k A_{ky}^{(1)} = -\widehat{F}_k. \quad (A.5)$$

The solvability condition (A.3) which also holds in first order yields immediately equations (III.23) and (III.26). Non resonant terms can be handled in a straightforward manner, yielding the second term on the right-hand-side of equation (III.23).

In second order the right-hand-side of equation (III.6) contains resonant terms proportional to $e^{i\psi k}$, as well as terms proportional to $t e^{i\psi k}$. Contributions to the second-order solution of the first type can be handled in a way similar to that already discussed in first order. To deal with the second type of resonant contributions, we write the second-order perturbation equations as

$$\nabla^2 A_{2y} + \omega_c \nabla A_{2z} - \frac{\omega_p^2}{c^2} \nabla^2 A_{2y} = \text{i} t \text{sgn}(k) H_k A_k e^{i\psi k}, \quad (A.6)$$

$$\nabla^2 A_{2z} - \omega_c \nabla A_{2y} - \frac{\omega_p^2}{c^2} \nabla^2 A_{2z} = t H_k A_k e^{i\psi k}. \quad (A.7)$$

It is straightforward to verify that the solution to equations (A.6) and (A.7) is of the form

$$A_{2y,z} = (t^2 C_{ky,z} + t L_{ky,z}) A_k e^{i\psi k}, \quad (A.8)$$

where the coefficients $C_{ky,z}$ and $L_{ky,z}$ can be written as

$$C_{ky} = -\frac{H_k}{2(\omega_k \alpha_k \text{sgn}(k) + \omega_c \chi_k)}, \quad (A.9)$$

$$L_{ky} = -i\frac{2\beta(2) C_{ky} \text{sgn}(k)}{2 \omega_k \alpha_k \text{sgn}(k) + \omega_c \chi_k}. \quad (A.10)$$

Equation (A.11) should be regarded as a solvability condition and must be satisfied order by order.

Substituting the appropriate form of $H_k$ and collecting similar terms proportional to $t$ and $t^2$, we readily obtain equations (IV.9) – (IV.11).
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