Perpendicular Diffusion of Energetic Particles: A Complete Analytical Theory

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Abstract

Over the past two decades scientists have significantly improved our understanding of the transport of energetic particles across a mean magnetic field. Due to test-particle simulations, as well as powerful nonlinear analytical tools, our understanding of this type of transport is almost complete. However, previously developed nonlinear analytical theories do not always agree perfectly with simulations. Therefore, a correction factor $a^2$ was incorporated into such theories with the aim to balance out inaccuracies. In this paper a new analytical theory for perpendicular transport is presented. This theory contains the previously developed unified nonlinear transport theory, the most advanced theory to date, in the limit of small Kubo number turbulence. New results have been obtained for two-dimensional turbulence. In this case, the new theory describes perpendicular diffusion as a process that is sub-diffusive while particles follow magnetic field lines. Diffusion is restored as soon as the turbulence transverse complexity becomes important. For long parallel mean-free paths, one finds that the perpendicular diffusion coefficient is a reduced field line random walk limit. For short parallel mean-free paths, on the other hand, one gets a hybrid diffusion coefficient that is a mixture of collisionless Rechester & Rosenbluth and fluid limits. Overall, the new analytical theory developed in the current paper is in agreement with heuristic arguments. Furthermore, the new theory agrees almost perfectly with previously performed test-particle simulations without the need of the aforementioned correction factor $a^2$ or any other free parameter.

Unified Astronomy Thesaurus concepts: Interplanetary turbulence (830); Solar energetic particles (1491); Galactic cosmic rays (567)

1. Introduction

A fundamental problem of modern space and astrophysics is the interaction between electrically charged energetic particles and magnetic turbulence. These interactions are crucial in the theory of cosmic-ray propagation as well as diffusive shock acceleration. This concerns those phenomena in the solar system but also in other scenarios such as the interstellar media of our own and external galaxies.

To understand and describe the interaction between turbulence and particles is difficult due to several reasons. First, turbulence itself is a very complicated topic due to the fact that it is a complex nonlinear dynamical phenomenon (see Matthaeus 2021 for a review). In the theory of energetic particle transport all scales of magnetic turbulence can be significant. For perpendicular transport the large scales of the energy range are particularly important. This is due to the relevance of magnetic field line random walk (FLRW) discussed in the next paragraph.

The simplest picture of perpendicular transport is that of the FLRW limit. Magnetic field lines in turbulence are stochastic curves. The properties of those curves are mostly controlled by the large-scale structure of turbulence (see, e.g., Shalchi & Kourakis 2007b, 2007c). Often it is assumed that magnetic field lines behave diffusively. The term diffusion, however, often causes confusion. In this case, diffusion means a linear increase in the uncertainty in finding a magnetic field line at a certain location of space. In terms of mean square displacements, one has in the diffusive case $\langle (\Delta x)^2 \rangle = 2\kappa_{\text{FL}}|z|$, where $\kappa_{\text{FL}}$ is the field line diffusion coefficient. Here, $z$ is the coordinate along the mean magnetic field and $x$ corresponds to the perpendicular direction, respectively. Non-diffusive FLRW has been discussed in the literature (see, e.g., Zimbardo & Veltri 1995; Zimbardo et al. 1995, 2000; Zimbardo 2005; Shalchi & Kourakis 2007b, 2007c). An understanding of the magnetic field line behavior itself is complicated due to the nonlinearity of the problem (see, e.g., Kadomtsev & Pogutse 1979; Matthaeus et al. 1995; Shalchi & Kourakis 2007b, 2007c). Within the FLRW limit, one assumes that the particles follow magnetic field lines while their motion along the mean magnetic field occurs with constant velocity. If this was true, the perpendicular diffusion coefficient of the particle is given by

$$\kappa_\perp = \frac{v}{2\kappa_{\text{FL}}} \quad \text{(FLRW limit),}$$

where $v$ is the particle speed. This means that perpendicular transport is entirely controlled by the behavior of stochastic magnetic field lines. Sometimes this type of transport is referred to as first diffusion (see, e.g., Matthaeus et al. 2003). However, in reality, the particle motion is not unperturbed in the parallel direction. In laboratory plasmas, for instance, Coulomb collisions lead to parallel diffusion (see, e.g., Rechester & Rosenbluth 1978). In the context of space and astrophysical plasmas, collisions are absent but instead there is pitch-angle scattering due to the interaction between particles and turbulent magnetic fields (see, e.g., Shalchi 2009 for a review). If pitch-angle scattering occurs over an extended period of time, the particle motion becomes diffusive in the parallel direction. As a consequence, perpendicular transport is suppressed to a sub-diffusive level. This type of transport is usually referred to as compound sub-diffusion and can be
described by using analytical theory (see Webb et al. 2006 for the most comprehensive description of this type of transport).

In test-particle simulations one can nicely observe how particle transport is indeed sub-diffusive in some cases, while normal diffusion is restored in other cases (see Qin et al. 2002a, 2002b). Therefore, the question arises as to what triggers the recovery of Markovian diffusion? In the case of Coulomb collisions the particles can get scattered off the original magnetic field line they were tied to. The particle then jumps to a neighboring field line. If magnetic field lines separate, the particle motion becomes more complex leading to normal diffusion instead of compound sub-diffusion. This picture of perpendicular transport forms the basis of the famous Rechester & Rosenbluth (1978) paper. In the context of space and astrophysics, collisions are absent as described above. Furthermore, magnetic field lines do not separate exponentially as assumed by Rechester & Rosenbluth (1978). However, as described in Shalchi (2019a, 2020a, 2020b), Coulomb collisions and exponentially separating field lines are not needed in order to restore diffusion. Pitch-angle scattering itself can scatter particles away from the original field line they were tied to. As long as field lines separate, particle diffusion across the mean field will be restored. All that is needed in order to recover diffusion is a combination of pitch-angle scattering and transverse complexity of the turbulence. The latter effect is directly related to field line separation (see, e.g., Shalchi 2019b). In this scenario, sometimes referred to as second diffusion (see Matthaeus et al. 2003), one has (see Shalchi 2019a) \( \kappa_{\perp}/\kappa_{\parallel} = (\langle \Delta x^2 \rangle^2 / \langle \Delta z^2 \rangle) \approx \ell_{\perp}^2 / L_K^2 \), where we have used the perpendicular correlation scale of the turbulence \( \ell_{\perp} \) as well as the Kolmogorov–Lapunov length \( L_K \). The latter scale indicates the distance one needs to travel along the mean field in order to observe significant transverse structure of the turbulence. Since particles follow field lines until transverse complexity becomes important, one can estimate \( \ell_{\perp}^2 \approx \kappa_{\perp} L_K \).

Using the above to eliminate \( L_K \) leads to

\[
\frac{\kappa_{\perp}}{\kappa_{\parallel}} = \frac{\kappa_{FL}^2}{\ell_{\perp}^2} \quad \text{(CLRR limit),}
\]

which can be understood as collisionless version of Rechester & Rosenbluth (1978), and therefore, we refer to the latter limit as the collisionless Rechester & Rosenbluth (CLRR) limit. According to Equation (2), the ratio of perpendicular and parallel diffusion coefficients is constant, and in most cases, small.

A very critical parameter in the theory of three-dimensional turbulence is the so-called Kubo number, which was introduced by Kubo (1963) and is defined via

\[
K = \frac{\ell_{\perp} B_0}{\kappa_{\perp} B_0} \quad \text{(Kubo number),}
\]

where we have used the turbulence correlation scales in parallel and perpendicular directions, the \( x \)-component of the turbulent magnetic field \( \delta B_x \), and the mean field \( B_0 \). One expects to find the FLRW limit as given by Equation (1) for very long parallel mean-free paths and CLRR diffusion as given by Equation (2) for short parallel mean-free paths. Analytical theories, on the other hand, provide in the limit of short parallel mean-free paths and large Kubo numbers the result

\[
\frac{\kappa_{\perp}}{\kappa_{\parallel}} = \frac{\delta B^2}{B_0^2} \quad \text{(fluid limit),}
\]

as, for instance, derived in Shalchi et al. (2004), Zank et al. (2004), and Shalchi et al. (2010, 2015). In this case, particles follow ballistic magnetic field lines while their parallel motion is diffusive (see Shalchi 2019a, 2020a, 2020b). A remaining puzzle in the theory of perpendicular transport is how analytical theories can be altered so that they provide Equation (2) and explain how CLRR and fluid limits are related to each other in the large Kubo number regime. Furthermore, heuristic arguments cannot be very accurate, and in some cases, it is unclear how they need to be used in order to construct a perpendicular diffusion coefficient. This is, in particular, the case for two-component turbulence.

In reality, perpendicular transport is a combination of first and second diffusion. If pitch-angle scattering is weak, corresponding to a very long parallel mean-free path, perpendicular transport is mostly controlled by the random walk of magnetic field lines and one finds \( \kappa_{\perp} \approx \kappa_{FL} \nu / 2 \). If pitch-angle scattering is very strong, equivalent to very short parallel mean-free paths, the perpendicular diffusion coefficient is close to \( \kappa_{\perp} / \kappa_{\parallel} = \text{const} \). Qualitatively, this behavior of perpendicular transport is universal, meaning that it does not depend on the detailed properties of magnetic turbulence (see Hussein et al. 2015). However, in turbulence without transverse structure, usually referred to as slab turbulence, second diffusion cannot be established due to the lack of field-line separation. In this case, compound sub-diffusion is the final state of the transport. It is important to note that in turbulence with transverse complexity, diffusion will eventually be restored. However, a long time can pass before the transverse structure of the turbulence starts to influence the particle motion in the perpendicular direction. Therefore, after the initial ballistic regime, one often observes sub-diffusive behavior for a long time before diffusion is finally restored. The sequence ballistic transport \( \to \) sub-diffusion \( \to \) normal diffusion, is in particular prevalent in turbulence with small Kubo numbers (see Shalchi 2019a, 2020a, 2020b).

Although our qualitative understanding of perpendicular transport is complete, one still needs an analytical theory describing the physics outlined above. Such a theory, in order to be reliable, needs to be in good agreement with test-particle simulations for a variety of turbulence configurations. It is clear that due to the complexity of the problem, simple concepts such as quasi-linear theory or hard-sphere scattering models will fail completely in the general case. Major progress has been achieved by Matthaeus et al. (2003), where after employing a set of approximations, a nonlinear integral equation has been derived. The latter theory provided some promising results in particular in the context of solar wind observations (see, e.g., Bieber et al. 2004). However, the aforementioned theory cannot be the final solution to the problem. First of all, the theory does not provide a vanishing diffusion coefficient for slab turbulence as observed in simulations (see Qin et al. 2002a). Furthermore, the theory does not provide the correct FLRW limit if pitch-angle scattering is suppressed. In other words, the Matthaeus et al. (2003) theory disagrees with the Matthaeus et al. (1995) theory of FLRW. In order to address these two problems, the unified nonlinear transport (UNLT)
theory has been developed in Shalchi (2010). The theory provides $\kappa_\perp = 0$ for slab turbulence and it contains the Matthaeus et al. (1995) theory as special limit. Therefore, it is a unified transport theory for magnetic field lines and energetic particles. A time-dependent version of UNLT theory was developed in Shalchi (2017) and Lasuik & Shalchi (2017). The latter theory was able to provide a full time-dependent description of the transport, including the initial ballistic regime, compound sub-diffusion, as well as the recovery of diffusion due to transverse complexity. In fact, it was the first theory that was able to explain why and how diffusion is restored. However, all the aforementioned systematic theories have one problem in common. In the strong scattering regime, the parallel mean-free path is short, and for turbulence with large Kubo numbers (this includes the case of two-component turbulence), the theories do not agree very well with simulations. In fact, they provide a perpendicular diffusion coefficient that is roughly a factor of 3 too large (see, for instance, Figures 4–9). In order to balance out this discrepancy, a correction factor $\alpha^2 \approx 1/3$ was included in previous transport theories. It is the purpose of this paper to develop a theory that does not require this type of correction factor but has all the advantages of UNLT theories.

The remainder of this paper is organized as follows. In Section 2, we provide an overview of previously developed analytical descriptions of perpendicular transport. This includes a discussion of the FLRW limit, diffusive UNLT theory, and its time-dependent version. In Section 3, a new theory is systematically developed and in the following Sections 4–6 special cases and limits are considered. Section 7 focuses on a comparison with previously performed test-particle simulations for the well-known and accepted two-component turbulence model. In Section 8, we summarize and conclude.

2. Previously Developed Descriptions of Perpendicular Transport

In order to achieve a further improvement of the analytical theory of perpendicular transport, it is essential to briefly review existing theories.

2.1. The FLRW Limit

The simplest picture of perpendicular transport is that of the FLRW limit. In this case, particles are tied to magnetic field lines while their parallel motion occurs unperturbed (with constant velocity). Since the random walk of magnetic field lines is often assumed to behave diffusively, described by the field line diffusion coefficient $\kappa_{FLR}$, the particles move diffusively in the perpendicular direction. The corresponding particle diffusion coefficient is then given by Equation (1). To understand the FLRW itself is complicated due to the nonlinearity of the problem. Only in slab turbulence, corresponding to turbulence without any transverse complexity, the theory of FLRW is exact (see Shalchi 2020a for a detailed explanation). However, based on Corrsin’s independence hypothesis (see Corrsin 1959), sometimes referred to as random phase approximation, Kadomtsev & Pogutse (1979) as well as Matthaeus et al. (1995) developed a theory for the field line diffusion coefficient. The latter theory provides the following nonlinear integral equation:

$$\kappa_{FLR} = \frac{1}{B_0} \int d^3k \frac{P_{x\perp}(k)}{k_{\parallel}^2 + (\kappa_{FLR} k_{\perp}^2)^2}. \quad (5)$$

This equation contains two asymptotic limits depending on what the Kubo number is. The first asymptotic limit is obtained for small Kubo numbers for which Equation (5) turns into the quasi-linear limit

$$\kappa_{FLR} = L_\parallel \frac{\delta B_{\parallel}}{B_0} \quad (for \ K \ll 1), \quad (6)$$

where we have used the parallel integral scale $L_\parallel$ of the turbulence. For large Kubo numbers, on the other hand, one obtains

$$\kappa_{FLR} = L_U \frac{\delta B_{\parallel}}{B_0} \quad (for \ K \gg 1), \quad (7)$$

where we have used the ultra-scale $L_U$. By combining Equations (1) and (5), one can easily derive an integral equation for the perpendicular diffusion coefficient of the charged particles, namely,

$$\kappa_{\perp} = \frac{v^2}{3B_0^2} \int d^3k \frac{P_{x\perp}(k)}{v^2 k_{\parallel}^2 / (3\kappa_{\parallel} k_{\perp}^2) + (4/3) \kappa_{\perp}^2 k_{\perp}^2}. \quad (8)$$

In the following we refer to Equation (8) either as the FLRW limit or the weak (pitch-angle) scattering limit. This corresponds to the formal limit where the parallel mean-free path is $\lambda_{\parallel} \to \infty$.

2.2. Diffusive UNLT Theory

The basic equation describing the particle motion in a magnetic system is the Newton–Lorentz equation. However, a significant simplification can be achieved by using guiding center coordinates (see, e.g., Schlickeiser 2002) so that the equation of motion for perpendicular transport becomes

$$V_{x}(t) = v_x(t) \frac{\delta B_{\perp}(x(t))}{B_0}, \quad (9)$$

where $V_{x}(t)$ is the perpendicular velocity of the guiding center. A very similar equation can be obtained for $V_{y}(t)$ but this is not necessary due to the usual assumption of axisymmetry. In the following we perform some steps in order to rewrite Equation (9). These steps are standard in transport theory (see, e.g., Shalchi 2020a for a review). First, we replace the turbulent magnetic field in Equation (9) by a usual Fourier representation,

$$\delta B_{\perp}(x) = \int d^3k \delta B_{\perp}(k)e^{ikx}. \quad (10)$$

Using this in the equation of motion (9) yields

$$V_{x}(t) = \frac{1}{B_0} \int d^3k \frac{\delta B_{\perp}(k)}{v_x(t)}e^{ikx}. \quad (11)$$
From this one can derive the velocity autocorrelation function as
\[
\langle V_i(t)V_i(0) \rangle = \frac{1}{B_0} \int d^3k \int d^3k' \left( \delta B_i(k) \delta B_i^*(k') \right) \times \langle v_i(t) v_i(0) e^{i(k \cdot x(t) - k' \cdot x(0))} \rangle. \tag{12}
\]
Therein one can easily see the emergence of higher order correlation functions. In order to simplify this, we employ Corrsin’s independence hypothesis (see Corrsin 1959) allowing us to write
\[
\langle V_i(t)V_i(0) \rangle = \frac{1}{B_0} \int d^3k \int d^3k' \left( \delta B_i(k) \delta B_i^*(k') \right) \times \langle v_i(t) v_i(0) e^{i(k \cdot x(t) - k' \cdot x(0))} \rangle. \tag{13}
\]
To continue we assume homogeneous turbulence to find
\[
\langle V_i(t)V_i(0) \rangle = \frac{1}{B_0} \int d^3k \ P_{\perp}(k) \times \langle v_i(t) v_i(0) e^{i(k \cdot x(t) - k' \cdot x(0))} \rangle, \tag{14}
\]
where we have used the spectral tensor \( P_{\perp}(k) \). As the next step one can employ the Taylor–Green–Kubo (TGK) formulation (see Taylor 1922; Green 1951, and Kubo 1957)
\[
d_{\perp}(t) = d_{\perp}(0) + \Re \int_0^t dt' \langle V_i(t)V_i(0) \rangle, \tag{15}
\]
where we have used the running perpendicular diffusion coefficient defined via
\[
d_{\perp}(t) = \frac{1}{2} \frac{d}{dt} \langle (\Delta x)^2 \rangle. \tag{16}
\]
In Equation (15) we have allowed for a nonvanishing initial diffusion coefficient \( d_{\perp}(0) \). However, perpendicular transport is initially ballistic, and therefore, \( d_{\perp}(0) = 0 \) in our case. In the limit \( t \to \infty \) we find the usual time-independent diffusion coefficient
\[
\kappa_{\perp} = \lim_{t \to \infty} d_{\perp}(t) = \Re \int_0^\infty dt \langle V_i(t)V_i(0) \rangle. \tag{17}
\]
After combining Equations (14) and (17), we obtain
\[
\kappa_{\perp} = \frac{v^2}{B_0} \int d^3k \ P_{\perp}(k) T(k), \tag{18}
\]
where we have used
\[
T(k) = \frac{1}{v^2} \int_0^\infty dt \langle v_i(t) v_i(0) e^{i(k \cdot x(t) - x(0))} \rangle. \tag{19}
\]
The key problem is to determine the function \( T(k) \). It is important to note here that velocities and positions are highly correlated. Therefore, a pitch-angle dependent transport equation is needed. In Shalchi (2010), the following Fokker–Planck equation was used in order to determine the analytical form of \( T(k) \):
\[
\frac{\partial f}{\partial t} + v_j \frac{\partial f}{\partial x_j} = \frac{\partial}{\partial \mu} \left[ D_{\mu \nu} \frac{\partial f}{\partial \mu} \right] + D_1 \left[ \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \right]. \tag{20}
\]
The latter equation and more general forms are discussed in Skilling (1975), Schlickeiser (2002), and Zank (2014). In Equation (20) we have used the particle distribution function \( f \), time \( t \), the particle speed \( v \), the pitch-angle cosine \( \mu \), and the position in three-dimensional space \((x, y, z)\). Furthermore, the equation contains the pitch-angle Fokker–Planck coefficient \( D_{\mu \nu} \) as well as the Fokker–Planck coefficient of perpendicular diffusion \( D_1 \). The quantity \( T(k) \) defined via Equation (19) can be derived from Equation (20) as shown in Shalchi (2010). After some lengthy algebra one obtains the following form:
\[
T(k) = \frac{1}{3} \frac{v}{v/\lambda_0 + (v k_0^2)/(3\kappa_{\perp} k_1^2) + (4/3)\kappa_{\perp} k_1^2}. \tag{21}
\]
Combining this with Equation (18) leads to the following integral equation:
\[
\kappa_{\perp} = \frac{2v^2}{3B_0^2} \int d^3k \ P_{\perp}(k) \frac{P_{\perp}(k)}{v/\lambda_0 + (v k_0^2)/(3\kappa_{\perp} k_1^2) + (4/3)\kappa_{\perp} k_1^2}, \tag{22}
\]
where we have introduced the correction factor \( a^2 \) to balance out inaccuracies of the used approximations. It will be the aim of this paper to develop an analytical theory that does not require this type of factor.

One can easily see that in the asymptotic limit \( \lambda_0 \to \infty \), Equation (22) reduces indeed to the FLRW limit given by Equation (8). Furthermore, for slab turbulence, we have \( k_1 \to 0 \), and Equation (22) provides \( \kappa_{\perp} = 0 \) as the solution as required. Compared to other theories such as the nonlinear guiding center (NLGC) theory of Matthaeus et al. (2003), the main advantage of Equation (22) is that it provides the correct result for slab turbulence but it also works for three-dimensional turbulence with small Kubo numbers (see Shalchi 2020a). For two-dimensional turbulence and three-dimensional turbulence with large Kubo numbers, on the other hand, Equation (22) and NLGC theory are almost identical (see the next subsection).

2.3. UNLT and NLGC Theories for Two-dimensional Turbulence

Solar wind fluctuations contain a subpopulation with wavevector orientations nearly perpendicular to the mean magnetic field (Matthaeus et al. 1990). As also emphasized in Matthaeus et al. (1990), the corresponding density fluctuations are small corresponding to the nearly incompressible case. This type of turbulence is usually referred to as two-dimensional (2D) turbulence. Two-dimensional turbulence is also familiar in laboratory plasma studies such as tokamaks (see, e.g., Robinson & Rusbridge 1971 as well as Zweibel et al. 1979). The theoretical framework for this type of turbulence is provided by reduced magnetohydrodynamics (see Zank & Matthaeus 1992). For two-dimensional magnetic fluctuations, the corresponding spectral tensor is given by
\[
P_{nm}(k) = g^2D(k_1) \frac{\delta(k_0)}{k_1} \left[ \delta_{nm} - \frac{k_n k_m}{k_1^2} \right], \tag{23}
\]
where we have used \( n, m = x, y \). All other tensor components are assumed to be zero. This is a consequence of considering nearly incompressible turbulence where \( B_0 \approx 0 \). The function \( g^2D(k_1) \) corresponds to the two-dimensional spectral function. The analytical form of the spectrum will be discussed later in this paper. With Equation (23) diffusive UNLT theory,
represented by Equation (22), becomes

$$\kappa_{\perp} = \frac{\pi a^2 v^2}{3 B_0^2} \int d^3 k \ g^{D} (k_\perp) \frac{1}{\sqrt[3]{\lambda d} + (4/3) \kappa_{\perp} k_\perp^2}. \quad (24)$$

It needs to be emphasized that for this very special case, the latter equation agrees with the NLGC theory of Matthaeus et al. (2003) except for the factor 4/3 in the denominator. Although this factor needs to be there in order to find agreement with the FLRW limit given by Equation (8), this is a minor difference.

### 2.4. Time-dependent UNLT Theory

A further improvement of the theory of perpendicular transport was achieved due to the development of time-dependent UNLT theory (see Shalchi 2017; Lasuik & Shalchi 2017). This approach is discussed in the following. As before we use Equation (9) as a starting point and as in diffusive UNLT theory we employ the same steps until we arrive at Equation (12). The fundamental approximation leading to time-dependent UNLT theory is

$$\langle V_\perp(t) V_\perp(0) \rangle = a^2 \langle V_\perp(t) V_\perp(0) \rangle \langle \delta B^2_{\perp}(k') e^{i(k_\perp \cdot \hat{x}_\perp)} e^{i(k_\perp \cdot \hat{x}_\perp)} \rangle \langle \delta B^2_{\perp}(k') e^{i(k_\perp \cdot \hat{x}_\perp)} e^{i(k_\perp \cdot \hat{x}_\perp)} \rangle. \quad (25)$$

This means that we group together magnetic fields, all particle properties associated with their parallel motion, and all particle properties associated with their perpendicular motion. In Equation (25) we have used again the parameter $a^2$ to balance out inaccuracies arising from the approximations used. Using Equation (25) in Equation (12) allows us to write the velocity autocorrelation function as

$$\langle V_\perp(t) V_\perp(0) \rangle = \frac{a^2}{B_0^2} \int d^3 k \ P_{\perp \perp}(k, t) \langle \delta B^2_{\perp}(k) \rangle \langle \delta B^2_{\perp}(k') \rangle \langle e^{i(k_\perp \cdot \hat{x}_\perp)} e^{i(k_\perp \cdot \hat{x}_\perp)} \rangle, \quad (26)$$

where we have used the parallel correlation function

$$\langle \delta B_{\perp}(k) \rangle = \langle V_\perp(t) V_\perp(0) \rangle \langle e^{i(k_\perp \cdot \hat{x}_\perp)} \rangle. \quad (27)$$

For the perpendicular characteristic function in Equation (26), we employ

$$\langle e^{i(k_\perp \cdot \hat{x}_\perp)} \rangle = e^{-\frac{1}{2} \langle (\Delta x^2) \rangle k_\perp^2}. \quad (28)$$

The latter form corresponds to a Gaussian function with zero mean. In Lasuik & Shalchi (2018), alternative distributions, such as kappa distributions, were considered but it was demonstrated that the assumed statistics has only a very small influence on the resulting perpendicular diffusion coefficient. If we combine Equations (26) and (28), we find for the velocity correlation function

$$\langle V_\perp(t) V_\perp(0) \rangle = \frac{a^2}{B_0^2} \int d^3 k \ P_{\perp \perp}(k, t) \langle \delta B^2_{\perp}(k) \rangle e^{-\frac{1}{2} \langle (\Delta x^2) \rangle k_\perp^2}. \quad (29)$$

From Equations (15) and (16), it follows that

$$\frac{d^2}{dt^2} \langle (\Delta x^2) \rangle = 2 \langle V_\perp(t) V_\perp(0) \rangle, \quad (30)$$

and therefore, we can write Equation (29) as

$$\frac{d^2}{dt^2} \langle (\Delta x^2) \rangle = \frac{2a^2}{B_0^2} \int d^3 k \ P_{\perp \perp}(k, t) \langle \delta B^2_{\perp}(k) \rangle e^{-\frac{1}{2} \langle (\Delta x^2) \rangle k_\perp^2}. \quad (31)$$

The solution of this integro-differential equation is the mean square displacement $\langle (\Delta x^2) \rangle$ of the energetic particle. This solution looks like depends on the spectral tensor $P_{\perp \perp}(k, t)$, describing magnetic turbulence, and the parallel correlation function $\langle \delta B_{\perp}(k) \rangle$. It has been shown before (see Shalchi 2020a for a systematic explanation) that

$$\xi(k_\perp, t) = \frac{v^2}{3} \frac{1}{\omega_\perp + \omega_\parallel} \left[ \omega_\perp e^{\omega_\perp t} - \omega_\parallel e^{\omega_\parallel t} \right] \quad (32)$$

with the parameters $\omega_\perp$ given by

$$\omega_\perp = -\frac{v}{2 \lambda_{||}} \pm \sqrt{\left( \frac{v}{2 \lambda_{||}} \right)^2 - \frac{1}{3} (vk_\perp)^2}, \quad (33)$$

which requires knowledge of the parallel mean-free path $\lambda_{||}$. Note, this result was obtained by solving Equation (20) without a perpendicular diffusion term. The parallel correlation function given by Equations (32) and (33) describes parallel transport as a process that is initially ballistic and at later times diffusive. Diffusive UNLT theory, represented by Equation (22), can be derived by combining Equation (31) with a diffusion approximation of the form $\langle (\Delta x^2) \rangle = 2 \kappa_{\perp} t$ and time integrating. From this, one can derive

$$\kappa_{\perp} = \frac{a^2 v^2}{3 B_0^2} \int d^3 k \ P_{\perp \perp}(k, t) \langle \delta B^2_{\perp}(k) \rangle e^{-\frac{1}{2} \langle (\Delta x^2) \rangle k_\perp^2}. \quad (34)$$

This result is almost identical to Equation (22). The only difference is the factor $4/3$ in the denominator of Equation (22). The latter equation is more accurate because the factor $4/3$ is needed in order to find the correct FLRW limit as given by Equation (8). However, this factor does not change the perpendicular diffusion coefficient significantly. The reason why this factor is not part of time-dependent UNLT theory is the subspace approximation used to derive Equation (32).

### 3. The Field Line Particle Decorrelation (FLPD) Theory

Diffusive and time-dependent UNLT theories have been successful in describing perpendicular transport for a wide range of parameters and turbulence models (see Shalchi 2020a for a review). The aforementioned theories contain previously developed approaches as special limits. For instance, the FLRW limit (see Equation (8)) is obtained by considering $\lambda_{||} \to \infty$. Compound sub-diffusion for slab turbulence (see, e.g., Webb et al. 2006) is automatically obtained for this specific turbulence model. For two-dimensional turbulence diffusive UNLT theory agrees with the Matthaeus et al. (2003) theory (except for the factor $4/3$ discussed above). Even a collisionless version of Rechester & Rosenbluth (1978) can be obtained if the assumption of strong pitch-angle scattering is combined with small Kubo number turbulence (see Shalchi 2015). However, UNLT theory deviates from test-particle simulations in some cases. This is, in particular, the case where pitch-angle scattering is strong and if turbulence with larger Kubo numbers is considered. This numerical disagreement (roughly a factor of 3) was previously balanced out by incorporating a correction factor $a^2 \approx 1/3$ as originally suggested by Matthaeus et al. (2003). It was finally explained in Shalchi (2019a) why this factor is needed and why previously developed theories fail...
in the aforementioned case. In the context of diffusive UNLT theory the reason is that Equation (20) is not valid in the general case. This problem is often overlooked in the literature. Equation (20) is based on the assumption that perpendicular transport becomes diffusive instantaneously. After some time has passed, corresponding to the pitch-angle isotropization timescale, parallel transport becomes diffusive as well. Therefore, Equation (20) is valid as long as perpendicular transport becomes diffusive before parallel transport. However, this is not always true. In particular, for short parallel mean-free paths, parallel transport becomes diffusive before perpendicular transport. If particles follow magnetic field lines while parallel transport is diffusive, perpendicular transport is suppressed to a sub-diffusive level. Perpendicular transport is then restored later as soon as the transverse complexity of the turbulence becomes significant. This scenario is not described by Equation (20). Therefore, it is important that non-diffusive transport equations are explored (see, e.g., Webb et al. 2006; Zimbardo et al. 2017; Strauss & Effenberger 2017).

In the context of time-dependent UNLT theory, the problem is that approximation (25) does not work in the general case. In reality, particle trajectories and particle velocities are highly correlated. If this correlation is neglected, resulting theories deviate from test-particle simulations as demonstrated in detail in Qin & Shalchi (2016). The assumption that trajectories and velocities are uncorrelated is also the reason why the NLGC theory of Matthaeus et al. (2003) needs the correction factor $a^2$ and why it does not work for slab and small Kubo number turbulence.

### 3.1. Generalized Compound Sub-diffusion

A detailed analysis of compound sub-diffusion was presented in Webb et al. (2006), who developed a description for this type of transport based on the so-called Chapman–Kolmogorov equation (see, e.g., Gardiner 1985),

$$f_{\parallel}(x, y; t) = \int_{-\infty}^{+\infty} dz \ f_{\perp}(x, y; z) f_{\parallel}(z; t), \tag{35}$$

where the particle distribution in the perpendicular direction $f_{\perp}(x, y; t)$ is given as convolution integral of the parallel distribution function $f_{\parallel}(z; t)$ and the field line distribution function $f_{\perp}(x, y; z)$. The parallel distribution function describes the probability of finding the particle at the (parallel) position $z$ at a certain time $t$. In the past (see Webb et al. 2006), a Gaussian distribution with vanishing mean has been used to model the parallel motion and the function $f_{\parallel}(z; t)$. However, parallel transport is a process that is initially ballistic and becomes diffusive (Gaussian) at later times. Therefore, to model the function $f_{\parallel}(z; t)$ is not trivial. More details about the analytical form of the parallel distribution and its Fourier transform can be found below. Equation (35) corresponds to the assumption that particles follow magnetic field lines and that the particle distribution in the perpendicular direction depends only on parallel transport and the FLRW.

From Equation (35) we can consider the second moment to find (see, e.g., Shalchi & Kourakis 2007a)

$$\langle (\Delta x)^2 \rangle_p(t) = \int_{-\infty}^{+\infty} dz \ \langle (\Delta x)^2 \rangle_{\perp}(z) f_{\parallel}(z; t). \tag{36}$$

As in Equation (35) we have used the parallel distribution function of the particles therein. The quantity $\langle (\Delta x)^2 \rangle_p$ corresponds to the perpendicular mean square displacement of the energetic particles and $\langle (\Delta x)^2 \rangle_{\perp}$ to the mean square displacement of the magnetic field lines. The parallel distribution function in Equation (36) can be replaced by the Fourier representation

$$f_{\parallel}(z; t) = \int_{-\infty}^{+\infty} dk_{\parallel} \ F(k_{\parallel}; t) e^{ik_{\parallel}z}, \tag{37}$$

where we have used the notation $k_{\parallel}$ to distinguish the latter parameter from the wave number used later in this paper. Using Equation (37) in Equation (36) yields

$$\langle (\Delta x)^2 \rangle_p(t) = \int_{-\infty}^{+\infty} dz \ \langle (\Delta x)^2 \rangle_{\perp}(z) \int_{-\infty}^{+\infty} dk_{\parallel} \ F(k_{\parallel}; t) e^{ik_{\parallel}z} \tag{38}$$

$$= \int_{-\infty}^{+\infty} dk_{\parallel} \ F(k_{\parallel}; t) \int_{-\infty}^{+\infty} dz \ \langle (\Delta x)^2 \rangle_{\perp}(z) e^{ik_{\parallel}z},$$

where in the last line, we just changed the order of the integrals. We can further rewrite this via

$$\langle (\Delta x)^2 \rangle_p(t) = -\int_{-\infty}^{+\infty} dz \ \langle (\Delta x)^2 \rangle_{\perp}(z) \int_{-\infty}^{+\infty} dk_{\parallel} \ F(k_{\parallel}; t) e^{ik_{\parallel}z} \tag{39}$$

$$\times \int_{-\infty}^{+\infty} dz \ \left[ \frac{d^2}{dz^2} \langle (\Delta x)^2 \rangle_{\perp}(z) \right] e^{ik_{\parallel}z},$$

where we have omitted the time dependence of $\langle (\Delta x)^2 \rangle_p$. We continue by using integration by parts twice to find

$$\langle (\Delta x)^2 \rangle_{\perp}(z) = \frac{d^2}{dz^2} \langle (\Delta x)^2 \rangle_{\perp}(z) \tag{40}$$

Shalchi & Kourakis (2007b) derived the following second-order differential equation for the field line mean square displacement:

$$\frac{d^2}{dz^2} \langle (\Delta x)^2 \rangle_{\perp} = \frac{2}{B_0^2} \int d^3k \ P_{\alpha\beta}(k) \cos(k_{\alpha}z) e^{-\frac{1}{2}(\Delta x)^2_{\perp}k_{\alpha}k_{\alpha}}, \tag{41}$$

The latter equation describes the random walk of magnetic field lines as a process that is initially ballistic and becomes diffusive at larger distances of $z$. In order to simplify Equation (41), we can use a diffusion approximation for the field lines, meaning that we replace $\langle (\Delta x)^2 \rangle_{\perp} = 2k_{\perp}^2z^2$ on the right-hand side. If we would additionally integrate Equation (41) over all $z$, we would obtain Equation (5). Using the field line diffusion approximation and the nonlinear field line Equation (41) in Equation (40) yields

$$\langle (\Delta x)^2 \rangle_p = -\frac{2}{B_0^2} \int_{-\infty}^{+\infty} dk_{\parallel} \ F(k_{\parallel}; t) k_{\parallel}^{-2} \tag{42}$$

$$\times \int d^3k \ P_{\alpha\beta}(k) \cos(k_{\alpha}z)e^{-\frac{1}{2}(\Delta x)^2_{\perp}k_{\alpha}k_{\alpha}z}.$$
The integral therein can be solved via
\[ \int_{-\infty}^{+\infty} dz \cos (k_i z) e^{-\kappa_{FL} z} = \frac{\kappa_{FL} k_i^2}{(k_i + k'_{i})^2 + (\kappa_{FL} k_i^2)^2} + \frac{\kappa_{FL} k_i^2}{(k_i - k'_{i})^2 + (\kappa_{FL} k_i^2)^2}. \]  (43)

For convenience we define the resonance function as
\[ R_{\pm}(k', k_i) := \frac{\kappa_{FL} k_i^2}{(k_i \pm k'_{i})^2 + (\kappa_{FL} k_i^2)^2}. \]  (44)

In the literature, this type of resonance function is usually referred to as a Cauchy distribution. Note, the resonance function found here describes the coupling between particles and magnetic field lines. Using Equations (43) and (44) in Equation (42) allows us to write
\[
\langle (\Delta x)^2 \rangle = -\frac{2}{B_0^2} \int_0^{+\infty} dk_i' F(k_i', t) k_i'^{-2} \times \int d^3k' P_{xx}(k)[R_+(k, k_i') + R_-(k, k_i')].
\]  (45)

Note, Equation (45) is just a different way of writing down Equation (36), which, in turn, corresponds to the second moment of Equation (35). Furthermore, we have omitted the subscript \( P \), meaning that the mean square displacement in Equation (45) is, of course, the perpendicular mean square displacement of the particles.

3.2. Velocity Correlations and Parallel Distribution Functions

Because of reasons that will become clearer later on, we want to derive a relation for the perpendicular velocity correlation function, which can be related to mean square displacements via Equation (30). However, we keep the mean square displacement in our equation but consider the second time derivative so that we derive from Equation (45)
\[ \frac{d^2}{dt^2} \langle (\Delta x)^2 \rangle = -\frac{2}{B_0^2} \int_0^{+\infty} dk_i' \frac{d^2}{dt^2} F(k_i', t) k_i'^{-2} \times \int d^3k' P_{xx}(k)[R_+(k, k_i') + R_-(k, k_i')]. \]  (46)

We also need to think about the parallel distribution function in Fourier space \( F(k_{||}, t) \). Parallel transport of energetic particles is described via the Fokker–Planck equation
\[ \frac{\partial f}{\partial t} + v \frac{\partial f}{\partial x} = \frac{\partial}{\partial \mu} \left[ D_{\mu \nu} \frac{\partial f}{\partial \mu} \right], \]  (47)

which is a special case of Equation (20). As demonstrated, for instance, in Shalchi (2020a), Equation (47) describes the transport as a process that is initially ballistic. If pitch-angle scattering occurs over an extended period of time, one finds a pitch-angle isotropization process leading to parallel diffusion. To solve Equation (47) analytically is difficult, but a two-dimensional subspace approximation method has been developed in the past (see Shalchi 2020a for more details). The latter method allows one to derive a simple but accurate solution for the Fourier-transformed parallel distribution function, namely,
\[ F(k_{||}, t) = \frac{1}{2\pi} \frac{1}{\omega_+ - \omega_-} \left[ \omega_+ e^{\omega_+ t} - \omega_- e^{\omega_- t} \right]. \]  (48)

The latter solution depends on the two parameters \( \omega_{\pm} \), which are given by Equation (33). Note this is related to but not identical compared to Equation (32). The second time derivative of Equation (48) is
\[ \frac{d^2}{dt^2} F(k_{||}, t) = \frac{1}{2\pi} \frac{\omega_+ \omega_-}{\omega_+ - \omega_-} \left[ \omega_+ e^{\omega_+ t} - \omega_- e^{\omega_- t} \right]. \]  (49)

Therein, we can use
\[ \omega_+ - \omega_- = \frac{1}{3} v^2 k_{||}^2, \]  (50)

which follows from Equation (33). Using these two relations as well as Equation (32) in Equation (46) gives us
\[
\frac{d^2}{dt^2} \langle (\Delta x)^2 \rangle = \frac{1}{\pi B_0^4} \int_0^{+\infty} dk_i' \xi(k_i', t) \times \int d^3k' P_{xx}(k)[R_+(k, k_i') + R_-(k, k_i')].
\]  (51)

One can still understand Equation (51) as a generalized compound transport model where parallel transport is initially ballistic and becomes diffusive at later times. The magnetic field lines have similar properties. A slightly different derivation of Equation (51), based on velocity correlation functions, is presented in Appendix A. However, so far a very important ingredient is missing. In the approach presented above, we do not allow the particles to get scattered away from the original magnetic field line they have been tied to. In order to include this essential effect, we need to incorporate the transverse complexity of the turbulence.

3.3. Transverse Complexity of the Turbulence

In order to achieve a complete analytical description of perpendicular transport, we need to include the effect of transverse structure of the turbulence in Equation (51). The simplest way to do this is by including the factor
\[ e^{-\frac{1}{2} \langle (\Delta x)^2 \rangle k_{\perp}^2}, \]  (52)

where \( \langle (\Delta x)^2 \rangle \) is the particle mean square displacement in the perpendicular direction. The factor used here corresponds to a Gaussian distribution of the particles with vanishing mean. If this distribution is combined with a diffusion approximation (see below), this factor is equal to the Fourier-transformed solution of a usual diffusion equation. In fact, the factor used here can also be derived from transport equations such as Equation (20). The mathematical details can be found in the literature (see, e.g., Shalchi 2020a).

With the help of Equation (30), we can easily see that Equation (51) corresponds to the velocity correlation function. Including the transverse complexity factor means that the velocity correlation function decays rapidly as soon as the particles experience the transverse structure of the turbulence.
We finally find the following differential equation:

\[
\frac{d^2}{dt^2} \langle (\Delta x)^2 \rangle = \frac{1}{\pi B_0^2} \int_{-\infty}^{\infty} dk_\perp \xi(k_\perp', t) \times \int d^3k \; P_{xx}(k) \left[ R_+ (k, k_\perp') + R_- (k, k_\perp') \right] \times e^{-\frac{1}{2} (\Delta x \|^2) k_\perp^2},
\]

(53)

where the parallel correlation function \( \xi(k_\perp', t) \) is given by Equation (32), the resonance functions \( R_\pm(k, k_\perp') \) are given by Equation (44), and \( P_{xx}(k) \) is the xx-component of the spectral tensor as usual. Equation (53) is the central result of this paper. It contains the complete physics necessary to describe perpendicular transport. The function \( \xi(k_\perp', t) \) describes parallel transport and contains the initial ballistic regime as well as the diffusive regime. The spectral tensor \( P_{xx}(k) \) contains the information about the properties of the magnetic fluctuations, and the resonance functions \( R_\pm(k, k_\perp') \) describe the FLRW and the coupling between field lines and particles. The exponential in Equation (53) describes the effect of transverse complexity leading to the particles leaving the original magnetic field lines they were tied to. In principle, we can solve Equation (53) to obtain the mean square displacement \( \langle (\Delta x)^2 \rangle \) as a function of time. Compared to time-dependent UNLT theory, however, we need to evaluate an extra integral. We call the new approach developed here Field Line Particle Decorrelation (FLPD) theory. Equation (53) corresponds to the time-dependent version of this theory. A diffusive version, where Equation (53) is combined with a diffusion approximation, is discussed in Section 5.

4. Special Limits and Cases

In the next few paragraphs, we consider some limits in order to simplify Equation (53). This will also help the reader to better understand Equation (53) and the physics it describes.

4.1. The Limit of Small Kubo Numbers

Equation (53) depends on the resonance function given by Equation (44). If the spectral tensor corresponds to three-dimensional turbulence with small Kubo numbers, we have by definition \( k_\perp \to 0 \). Therefore, we can use the relation

\[
\lim_{\alpha \to 0} \frac{\alpha}{\alpha^2 + x^2} = \pi \delta(x)
\]

(54)

and our resonance functions (44) become in the considered limit

\[
R_\pm(k_\perp', k_\perp) = \pi \delta(k_\parallel \pm k_\perp').
\]

(55)

Using this in Equation (53) yields

\[
\frac{d^2}{dt^2} \langle (\Delta x)^2 \rangle = \frac{1}{B_0^2} \int_{-\infty}^{\infty} dk_\perp \xi(k_\perp', t) \times \int d^3k \; P_{xx}(k) \left[ \delta(k_\parallel + k_\perp') + \delta(k_\parallel - k_\perp') \right] \times e^{-\frac{1}{2} (\Delta x \|^2) k_\perp^2}
\]

(56)

Due to the two Dirac deltas, the \( k_\parallel \) integral can be evaluated to find

\[
\frac{d^2}{dt^2} \langle (\Delta x)^2 \rangle = \frac{2}{B_0^2} \int d^3k \; P_{xx}(k) \xi(k_\perp', t) e^{-\frac{1}{2} (\Delta x \|^2) k_\perp^2},
\]

(57)

where we assumed that the spectral tensor is symmetric in \( k_\parallel \). After comparing this result with Equation (31), we conclude that for three-dimensional turbulence with small Kubo numbers, our new approach agrees perfectly with time-dependent UNLT theory. This is a very important result because it has been demonstrated in the past that UNLT theory agrees very well with simulations performed for small Kubo number turbulence without the need for the correction factor \( a^2 \) (see, e.g., Shalchi 2020a). For pure slab turbulence, Equation (57) describes compound sub-diffusion, and normal diffusion is never restored. In the next section we shall see that for the opposite case, where the Kubo number is large, a result is found that is different compared to UNLT theory. The relation between different theories for perpendicular transport is depicted in Figure 1.

4.2. Two-dimensional Turbulence

In the previous section we have shown that our new equation for perpendicular transport, namely, Equation (53) is identical compared to time-dependent UNLT theory for small Kubo number turbulence. In this section, we consider the opposite case, namely, two-dimensional (2D) turbulence corresponding to turbulence with large Kubo numbers. In the case of 2D turbulence, the corresponding spectral tensor is given by Equation (23). The analytical form of the spectrum \( \tilde{R}^\parallel_\parallel(k_\perp) \) thereon will be discussed later in this paper. With Equation (23) our equation for perpendicular
transport (53) becomes
\[
\frac{d^2}{dt^2} \langle (\Delta x)^2 \rangle = \frac{1}{B_0^2} \int_{-\infty}^{+\infty} dk_\parallel \xi(k_\parallel, t)
\times \int_0^{\infty} dk_p g^{2D}(k_p) \left[ R_+(k_p) + R_-(k_p) \right]
\times e^{-\frac{i}{2}(\Delta x)^2 k_p^2},
\]
where now
\[
R_\pm(k) = \frac{\kappa_{\text{FL}} k_\perp^2}{k_\parallel^2 + (\kappa_{\text{FL}} k_\perp^2)^2}.
\] (59)

It needs to be emphasized that in two-dimensional turbulence there is no parallel wave number. The parameter \(k_\parallel\) in the resonance function, given by Equation (59), comes from the Fourier transform of the parallel particle distribution function and the coupling of the particles with the magnetic field lines. Using Equation (59) explicitly in Equation (58) allows us to write
\[
\frac{d^2}{dt^2} \langle (\Delta x)^2 \rangle = \frac{2}{B_0^2} \int_{-\infty}^{+\infty} dk_\parallel \xi(k_\parallel, t)
\times \int_0^{\infty} dk_p g^{2D}(k_p) \frac{\kappa_{\text{FL}} k_\perp^2}{k_\parallel^2 + (\kappa_{\text{FL}} k_\perp^2)^2}
\times e^{-\frac{i}{2}(\Delta x)^2 k_p^2}. \] (60)

Note that compared to time-dependent UNLT theory, as given by Equation (31), the new theory provides a very different equation for perpendicular transport in two-dimensional as well as large Kubo number turbulence. A further simplification of Equation (60) can be achieved if a diffusion approximation is employed. This is done in the next section.

5. Reestablishing the Diffusion Approximation

In Matthaeus et al. (2003) and Shalchi (2010) a diffusion approximation was used in order to derive equations much simpler compared to Equation (53). This type of approximation can also be combined with the new theory. This is done in the following. Thereafter, we consider again the limits of small and large Kubo numbers, respectively.

5.1. The Diffusion Approximation for the General Case

In order to find a simpler approach compared to Equation (53), we can employ a diffusion approximation. Using \(\langle (\Delta x)^2 \rangle = 2\kappa_\perp t\) and integrating Equation (53) over all times yields
\[
\kappa_\perp = \frac{1}{2\pi B_0^2} \int_{-\infty}^{+\infty} dk_\parallel \xi(k_\parallel, t)
\times \int d^3k P_{\alpha\beta}(k)[R_+(k, k_\parallel') + R_-(k, k_\parallel')]
\times \int_0^{\infty} dt \xi(k_\parallel', t)e^{-\kappa_\perp k_\parallel'^2}.
\] (61)

Employing Equation (32) to replace \(\xi(k_\parallel', t)\) allows us to solve the time integral in Equation (61). We find
\[
\int_0^{\infty} dt \xi(k_\parallel', t)e^{-\kappa_\perp k_\parallel'^2} = v^2 \frac{1}{3} \frac{\omega_- - \omega_+}{\omega_- - \kappa_\perp k_\parallel'^2 - \omega_+}
\int_0^{+\infty} \frac{\kappa_\perp k_\parallel'^2}{\omega_- - \kappa_\perp k_\parallel'^2}
\times \left( \omega_+ - \kappa_\perp k_\parallel'^2 \right) \left( \omega_- - \kappa_\perp k_\parallel'^2 \right)
\times \left( \omega_+ - \omega_- - \kappa_\perp k_\parallel'^2 \right)^2.
\] (62)

Using this in Equation (61) yields
\[
\kappa_\perp = \frac{1}{2\pi B_0^2} \frac{1}{3} \int_{-\infty}^{+\infty} dk_\parallel \xi(k_\parallel, t)
\times \int d^3k P_{\alpha\beta}(k)[R_+(k, k_\parallel') + R_-(k, k_\parallel')]
\times \frac{\kappa_\perp k_\parallel'^2}{\omega_+ - \omega_- - \kappa_\perp k_\parallel'^2}.
\] (63)

From Equation (33) we can easily derive
\[
\omega_+ + \omega_- = -v/\lambda_\parallel, \quad \omega_- = v/\lambda_\parallel, \quad \omega_+ = v/\lambda_\parallel,
\] (64)
as well as
\[
\omega_- = v^2 k_\parallel'^2. \quad \omega_+ = v^2 k_\parallel'^2. \quad \omega_- = v^2 k_\parallel'^2. \quad \omega_+ = v^2 k_\parallel'^2.
\] (65)

Using this in Equation (62) yields, after some straightforward algebra,
\[
\kappa_\perp = \frac{1}{2\pi B_0^2} \frac{1}{3} \int_{-\infty}^{+\infty} dk_\parallel \xi(k_\parallel, t)
\times \int d^3k P_{\alpha\beta}(k)[R_+(k, k_\parallel') + R_-(k, k_\parallel')]
\times \frac{1}{v/\lambda_\parallel + v^2 k_\parallel'^2 / (3\kappa_\perp k_\parallel'^2) + \kappa_\perp k_\parallel'^2}.
\] (66)

The latter equation corresponds to the diffusive version of Equation (53). Therefore, we call Equation (66) the diffusive FLPD theory. It needs to be emphasized that Equation (53) is more general and accurate since it is not based on this additional approximation. However, Equation (66) is easier to apply and more convenient for finding pure analytical results.

5.2. Small Kubo Number Turbulence

We can simplify Equation (66) by considering small Kubo number turbulence. In this limit the resonance functions \(R_\pm\) can be replaced by using Equation (55). After evaluating the \(k_\parallel\) integral, we then find
\[
\kappa_\perp = \frac{v^2}{3B_0^2} \int d^3k \frac{P_{\alpha\beta}(k)}{v/\lambda_\parallel + v^2 k_\parallel'^2 / (3\kappa_\perp k_\parallel'^2) + \kappa_\perp k_\parallel'^2}.
\] (67)

The latter formula agrees with diffusive UNLT theory (see Equation (22)). However, this agreement is only obtained for turbulence with small Kubo numbers. Note, we could also derive Equation (67) by directly combining the diffusion
approximation with Equation (57). Diagram 1 shows these two paths from Equation (53) to (67).

5.3. Two-dimensional Turbulence

We now consider the case of two-dimensional turbulence but with diffusion approximation. In Equation (66) we can use the spectral tensor corresponding to two-dimensional turbulence (23) to find

\[ \kappa_\parallel = \frac{v^2}{3B_0^2} \int_0^\infty dk_\perp g^{2D}(k_\perp) \int_{-\infty}^{+\infty} dk_\| \frac{\kappa_{\perp}(k_\perp)}{\kappa_\parallel(\kappa_\perp)^2} \]  
\[ \times \frac{1}{\sqrt{\lambda_\parallel + v/\lambda_\perp + \kappa_\perp^2 + \kappa_\parallel^2}}. \tag{68} \]

The \( \kappa_\parallel \) integral therein can be solved by (see, e.g., Gradsteyn & Ryzhik 2000)

\[ \int_{-\infty}^{+\infty} dk_\parallel \frac{1}{k_\parallel^2 + b^2c k_\parallel^4 + d} = \frac{\pi}{b} \frac{1}{\sqrt{cd + d}}. \tag{69} \]

Using this in combination with \( b = \kappa_{\perp}L_{\perp}^2, \ c = v^2/3, \) and \( d = v/\lambda_\parallel \kappa_\perp^2 + (\kappa_\parallel \kappa_\perp)^2 \) allows us to write Equation (68) as

\[ \kappa_\parallel = \frac{v^2}{3B_0^2} \int_0^\infty dk_\perp g^{2D}(k_\perp) \times \frac{1}{\sqrt{\lambda_\parallel + v/\lambda_\parallel + \kappa_\perp^2 + \kappa_\parallel^2}}. \tag{70} \]

After comparing this result with equations derived from previous theories (see, e.g., Equation (24)), we can see that the two terms \( v/\lambda_\parallel \) and \( \kappa_\perp \kappa_\parallel^2 \) are as before. If the term \( v/\lambda_\parallel \) is dominant, this is, in particular, the case in the formal limit \( \lambda_\parallel \rightarrow 0 \), the solution would be independent of the turbulence spectrum, namely, \( \kappa_\parallel/\kappa_\| = \delta B^2_\perp/B_0^2 \) corresponding to the fluid limit as given by Equation (4). In this case, particles interact predominantly with ballistic magnetic field lines while their parallel motion is diffusive. If the term \( \kappa_\parallel \kappa_\perp^2 \) in Equation (70) is dominant, on the other hand, one finds the usual FLRW limit, where the perpendicular diffusion coefficient is given by \( \kappa_\perp \approx \nu_{\perp}L_{\perp}^2/2 \). As in some other results shown in this paper, the factor \( 4/3 \) is also missing in front of the \( \kappa_\perp \kappa_\parallel^2 \) term in Equation (70). As before, this is due to the approximations used to derive the parallel distribution function in Fourier space. In Equation (70), we can clearly see that compared to previous descriptions of perpendicular transport, there is a new term containing explicitly the field line diffusion coefficient \( \kappa_{\perp L} \). In the next section, we explore the physical meaning of this term.

6. Detailed Analytical Discussion for 2D Turbulence

In applications analytical forms of the perpendicular diffusion coefficient based on two-component turbulence with a dominant 2D contribution are frequently used. This concerns studies of solar modulation (see, e.g., Moloto & Engelbrecht 2020; Engelbrecht & Wolmarans 2020; Engelbrecht & Moloto 2021) as well diffusive shock acceleration (see, e.g., Zank et al. 2000; Li et al. 2003, 2005; Zank et al. 2006; Li et al. 2012; Ferrand et al. 2014). Therefore, we now focus on the new result obtained for two-dimensional turbulence after we have employed the diffusion approximation as given by Equation (70).

6.1. Detailed Investigation of the New Term

In order to better understand the physics of the new term in Equation (70), we consider the case of short parallel mean-free paths but assume that the term containing the field line diffusion coefficient is still dominant, so we approximate

\[ \frac{\nu_{\perp}L_{\perp}}{3\kappa_\parallel} k_\parallel \sqrt{\frac{v}{\lambda_\|}} + \kappa_\perp k_\parallel^2 + \frac{\nu}{\lambda_\|} + \kappa_\parallel k_\perp^2 \approx \frac{\nu_{\perp}L_{\perp}}{3\kappa_\parallel} k_\parallel \sqrt{\frac{v}{\lambda_\|}}. \tag{71} \]

Using this approximation in Equation (70) yields

\[ \kappa_\parallel = \frac{\pi v^2}{3B_0^2} \int_0^\infty dk_\perp g^{2D}(k_\perp) \times \frac{1}{\lambda_\parallel v} \int_0^\infty dk_\| g^{2D}(k_\|) k_\parallel^{-1}. \tag{72} \]

The remaining integral could easily be evaluated for a certain spectrum. In the following we want to keep our investigations as general as possible. Therefore, we replace the \( k_\parallel \) integral by the so-called perpendicular integral scale of the turbulence \( L_{\perp} \), which is defined via (see, e.g., Shalchi 2020a)

\[ \delta B^2_\perp L_{\perp} = 2\pi \int_0^\infty dk_\| g^{2D}(k_\|) k_\|^{-1}. \tag{73} \]

Therefor, Equation (72) can be written as

\[ \kappa_\parallel = \frac{v}{6B_0^2} \frac{\sqrt{3\kappa_\parallel}}{\nu_{\perp}L_{\perp}} \frac{\lambda_\parallel}{\nu} L_{\perp} \delta B^2_\perp. \tag{74} \]

To simplify this further, we replace the parallel mean-free path by the parallel spatial diffusion coefficient via \( \lambda_\parallel = 3\kappa_\parallel/v \) to find

\[ \frac{\kappa_\parallel}{\kappa_\|} = \frac{1}{2} \frac{L_{\perp}}{\kappa_{\perp L} B_0^2}. \tag{75} \]

The latter result can be rewritten in different ways. First, we can replace the field line diffusion coefficient by the two-dimensional result given by Equation (7). Using this and taking the square of Equation (75) gives us

\[ \frac{\kappa_\parallel}{\kappa_\|} = \frac{L_{\perp}^2}{4\kappa_{\perp L}^2 B_0^2} \frac{\delta B^2_\perp}{B_0^2}. \tag{76} \]

where we again have used the ultra-scale \( L_{\perp} \). In order to evaluate this in more detail, we now need to specify the turbulence spectrum. In what follows we use a general spectrum with energy and inertial range. Those two parts of the spectrum are separated via the characteristic scale \( \ell_{\perp} \). Such a spectrum was developed in Shalchi & Weinhorst (2009), and has the following form:

\[ g^{2D}(k_\perp) = \frac{4D(s, q)}{\pi} \delta B^2_\perp \ell_{\perp}^{-s} (k_{\perp} \ell_{\perp})^q \left[ 1 + (k_{\perp} \ell_{\perp})^2 (s+q)/2 \right]. \tag{77} \]

In the inertial range the spectrum scales like \( k_{\perp}^{-s} \), whereas in the energy range it scales like \( k_{\perp}^q \). In Equation (77) we have also used the normalization function

\[ D(s, q) = \frac{\Gamma\left(\frac{s+q}{2}\right)}{2\Gamma\left(\frac{s-1}{2}\right)} \frac{\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{q-1}{2}\right)}. \tag{78} \]
According to the formulas listed above, this becomes

\[ \frac{\kappa_\perp}{\kappa_\parallel} = \frac{L_\perp^2}{4L_\parallel^3}. \tag{84} \]

In order to understand the latter formula, we need to replace the scales \( L_\perp \) and \( L_\parallel \). As before, we employ Equations (79) and (79). Therewith, we can write Equation (84) as

\[ \frac{\kappa_\perp}{\kappa_\parallel} = b_2^5 \frac{\kappa_{FL}}{\ell_\perp^2}, \tag{85} \]

where we have used

\[ b_2 = \frac{(q - 1)^2}{(s - 1)^2} \left[ \frac{\Gamma(s/2)}{\Gamma(s-1/2)} \right]^2. \tag{86} \]

Equation (85) is similar to the heuristic result discussed in the introduction (see Equation (2)). Therefore, we can easily understand the physics of the new term found here. Particles follow diffusive field lines while their parallel motion is diffusive as well. In this case, we find compound sub-diffusion. As soon as the transverse complexity of the turbulence becomes important, particles leave the original field line they followed and diffusion is restored. The obtained diffusion coefficient is comparable to the CLRR limit.

6.2. The Limit of Short Parallel Mean-free Paths

So far we have explored the new term and neglected the previously obtained terms in Equation (70). However, there is a problem with these considerations and that is that there is a competition between two terms in the denominator of Equation (70). In the formal limit \( \lambda_\parallel \to 0 \) the latter formula can be simplified to

\[ \frac{\kappa_\perp}{\kappa_\parallel} = \frac{\pi}{B_0^2} \int_0^{\infty} dk \frac{\Gamma(2q)}{1 + \kappa_{FL}L_\perp \sqrt{\kappa_\perp/\kappa_\parallel}}. \tag{87} \]

To be more specific we use spectrum (77) and employ the transformation \( x = k_\perp \ell_\perp \) so that Equation (87) becomes

\[ \frac{\kappa_\perp}{\kappa_\parallel} = 4D(s, q) \frac{\delta B^2}{B_0^2} \int_0^{\infty} dx \frac{h(x)}{1 + \alpha x}, \tag{88} \]

where we have used the dimensionless spectrum

\[ h(x) = \frac{x^q}{(1 + x^2)^{\alpha+q/2}}. \tag{89} \]

and the parameter

\[ \alpha \equiv \frac{\kappa_{FL}}{\ell_\perp} \sqrt{\frac{\kappa_\perp}{\ell_\perp}} \equiv \frac{\kappa_{FL}}{\ell_\perp} \sqrt{\frac{\kappa_\parallel}{\ell_\perp}}. \tag{90} \]

It follows from Equation (88) that we find the fluid limit (see Equation (4)) if \( \alpha \ll 1 \) and CLRR diffusion for \( \alpha \gg 1 \). Therefore, the critical value, where the turnover from the fluid limit to CLRR diffusion occurs, is \( \alpha \approx 1 \). According to Equation (90), this condition can be written as

\[ \frac{\kappa_\perp}{\kappa_\parallel} = \left( \frac{\kappa_{FL}}{\ell_\perp} \right)^2 \quad \text{for} \quad \alpha = 1. \tag{91} \]
which actually corresponds to CLRR diffusion (see Equation (2)). However, both sides of the latter equation are always directly proportional to $\delta B_0^2 / B_0^2$, and therefore, changing the magnetic field ratio does not restore the fluid limit. We conclude that real perpendicular diffusion, for the case of very short parallel mean-free paths and large Kubo numbers, is always a combination of the fluid limit and CLRR diffusion. The physics behind this is not too difficult to understand. The fluid limit is valid in cases where parallel transport becomes diffusive almost instantaneously and particles interact with ballistic field lines before transverse complexity becomes important. CLRR diffusion, on the other hand, is obtained if parallel transport becomes diffusive almost instantaneously, then the field lines become diffusive and thereafter transverse complexity becomes significant. However, for large Kubo number turbulence such as quasi-2D turbulence, field lines become diffusive exactly for the scales where transverse complexity becomes relevant, namely, $\langle (\Delta x)^2 \rangle \approx 2 \kappa^2_{\perp}$. Equation (87) takes into account all those effects and weights them appropriately. Therefore, for three-dimensional turbulence with large Kubo numbers, including two-dimensional turbulence and short parallel mean-free paths, we are in a hybrid regime. Figure 3 shows the different regimes of perpendicular diffusion.

6.3. The Limit of Long Parallel Mean-free Paths

So far, we have focused on short parallel mean-free paths. In the formal limit $\lambda_0 \rightarrow \infty$, on the other hand, Equation (70) can be simplified to

$$\kappa_\perp = \frac{v^2}{3} v_{\text{KFL}}^2 + \frac{1}{B_0^2} \int_0^\infty dk_\perp g_{\text{2D}}(k_\perp) k_\perp^{-2}.$$  (92)

The field line diffusion coefficient for quasi-2D turbulence is given by (see Matthaeus et al. 1995)

$$\kappa_{\text{FL}}^2 = \frac{\pi}{B_0^2} \int_0^\infty dk_\perp g_{\text{2D}}(k_\perp) k_\perp^{-2},$$  (93)

and therefore, Equation (92) can simply be written as

$$\kappa_\perp = \frac{v^2}{3} \frac{\kappa_{\text{FL}}^2}{v^2_{\text{KFL}} / \sqrt{3} + \kappa_\perp}.$$  (94)

The latter formula corresponds to the quadratic equation

$$\kappa_\perp^2 + \frac{v_{\text{KFL}}}{\sqrt{3}} \kappa_\perp - \frac{v^2}{3} \kappa_{\text{FL}}^2 = 0.$$  (95)

The only physical solution of the latter equation is

$$\kappa_\perp = \frac{\sqrt{5} - 1}{2} v_{\text{KFL}} / 2 \approx 0.7136 \frac{v}{2} \kappa_{\text{FL}}.$$

Compared to the real FLRW limit, which is given by $\kappa_\perp = v_{\text{KFL}} / 2$, our new result is reduced by a factor of 1.5. This can also be understood. The real FLRW limit is obtained only if parallel diffusion is suppressed forever and if particles follow field lines forever. Both assumptions are unrealistic. In reality, particle transport along the mean field always becomes diffusive if time passes and particles always need to leave the original field line they were tied to. However, in the limit of very long parallel mean-free paths, perpendicular transport is still predominantly controlled by the FLRW because pitch-angle scattering is in this case a weak process. Figure 3 depicts the different transport regimes contained in FLPD theory.

7. Comparison with Test-particle Simulations

A significant amount of test-particle simulations have been performed in the past for a slab+2D turbulence model also known as two-component turbulence (see, e.g., Qin et al. 2002a, 2002b; Qin & Shalchi 2012; Hussein et al. 2015; Arendt & Shalchi 2020). The two-component model is strongly supported by solar wind observations where one can find a cross distribution of magnetic fluctuations (see Matthaeus et al. 1990), indicating that real turbulence can be approximated by a superposition of slab and two-dimensional modes. Furthermore, as demonstrated by Bieber et al. (1996), solar wind turbulence possesses a dominant (roughly 80%) by magnetic energy) two-dimensional component and a smaller (roughly 20%) slab component. Theoretically, the strong two-dimensional component was explained based on the theory of nearly incompressible magnetohydrodynamics (see Zank & Matthaeus 1993). In recent years, those conclusions have been confirmed (see, e.g., Zank et al. 2017; 2020). In the following we employ the two-component turbulence model and compute perpendicular diffusion coefficients based on the theory developed in this paper. We then compare our analytical findings with previously performed test-particle simulations.

The explicit slab contribution to the perpendicular diffusion coefficient is zero. For two-dimensional turbulence we derived Equation (70). In the following we rewrite this to make it more...
we have also shown the NLGC mean-free path. The black solid lines correspond to the diffusive FLPD results as given by Equation following parameter values have been used: $\delta B^2/B_0^2 = 1$, $\ell_\perp = \ell_p$, $q = 2$, and a slab fraction of 20%.

Figure 4. The left panel shows the perpendicular mean-free path vs. the parallel mean-free path and the right panel shows the ratio $\lambda_\perp/\lambda_\parallel = \kappa_\perp/\kappa_\parallel$ vs. the parallel mean-free path. The black solid lines correspond to the diffusive FLPD results as given by Equation (98) and the circles represent the different test-particle simulations published in Hussein et al. (2015). The crosses represent time-dependent FLPD theory as given by Equation (53), or in code units, by Equation (B18). For comparison we have also shown the NLGC/UNLT results without a correction factor, meaning that we have set $\alpha^2 = 1$ (gray solid line). To obtain the visualized results the following parameter values have been used: $\delta B^2/B_0^2 = 1$, $\ell_\perp = \ell_p$, $q = 2$, and a slab fraction of 20%.

convenient for numerical investigations. Replacing the perpendicular diffusion coefficient by the perpendicular mean-free path via $\kappa_\perp = v\lambda_\perp/3$ allows us to write Equation (70) as

$$\frac{\lambda_\perp}{\lambda_\parallel} = \frac{2}{B_0^2} \int_0^\infty dk_\perp g^{2D}(k_\perp)$$

$$\times \left. \frac{1}{\kappa_{FL} k_\perp \Gamma \left( \frac{s}{q} \right) \left[ 1 + \frac{\lambda_\parallel}{\lambda_\perp} \right]^2 + 1 + \frac{\lambda_\parallel}{\lambda_\perp} \right]^{s/2},$$

For the spectral function associated with the two-dimensional modes, we employ again Equation (77). After using this in Equation (97) and performing the integral transformation $x = k_\perp \ell_\perp$, we find

$$\frac{\lambda_\perp}{\lambda_\parallel} = 4D(s, q) \frac{\delta B_\perp^2}{B_0^2}$$

$$\times \int_0^\infty dx \frac{\Gamma(s)}{\Gamma \left( \frac{s}{q} \right)} \left[ \frac{\lambda_\parallel}{\lambda_\perp} \right]^2 + 1 + \frac{\lambda_\parallel}{\lambda_\perp} x^2,$$

where we have used the dimensionless spectrum given by Equation (89).

A peculiarity of FLPD theory is that it explicitly contains the field line diffusion coefficient $\kappa_{FL}$. The total field line diffusion coefficient in two-component turbulence is (see Matthaeus et al. 1995)

$$\kappa_{FL} = \frac{1}{4} \left\{ \kappa_{slab} + \sqrt{\kappa_{slab}^2 + 4 \kappa_{2D}^2} \right\},$$

which depends on the slab as well as the two-dimensional diffusion coefficients. However, a problem with FLPD theory is that it is not clear whether the parameter $\kappa_{FL}$ in Equation (98) is the total field line diffusion coefficient or the field line diffusion coefficient associated with the two-dimensional modes $\kappa_{2D}$. In the following we assume that it is the total diffusion coefficient but further investigations need to explore whether this is true or not.

In order to determine the individual coefficients $\kappa_{slab}$ and $\kappa_{2D}$, we need to specify the two spectra. For the slab modes we use the Bieber et al. (1994) model, which is given by

$$g_{slab}(k_\parallel) = \frac{1}{2\pi} C(s) \delta B_{slab}^2 \ell_\parallel \left[ 1 + (k_\parallel \ell_\parallel)^2 \right]^{-s/2},$$

where we have used the normalization function

$$C(s) = \frac{\Gamma \left( \frac{s}{2} \right)}{2\sqrt{\pi} \Gamma \left( \frac{s}{q} - 1 \right)},$$

where $s$ is the inertial range spectral index as before and $\ell_\parallel$ is the parallel or slab bendover scale. For the two-dimensional modes we employ the form given by Equation (77). For those spectra, the slab field line diffusion coefficient is given by

$$\kappa_{slab} = L_U \frac{\delta B_{slab}^2}{B_0^2} = \pi C(s) \ell_\parallel \delta B_{slab}^2 B_0^2$$

and the two-dimensional field line diffusion coefficient is given by (see, e.g., Shalchi 2020a for a systematic derivation of such results)

$$\kappa_{2D} = L_U \frac{\delta B_{2D}}{B_0^2} = \sqrt{\frac{s - 1}{2(q - 1)}} \delta B_{2D} B_0^2.$$

Equation (98) can easily be solved numerically. In fact, it is not more difficult to solve this equation than the previous equations derived in Matthaeus et al. (2003) or Shalchi (2010) (see, e.g., Equations (22) and (24)).

In Figures 4–9 we compare the FLPD theory developed in this paper with previously performed test-particle simulations. Those simulations were performed for a two-component turbulence model and were taken from Qin & Shalchi (2012), Hussein et al. (2015), and Arendt & Shalchi (2020). The simulations are compared with the diffusive version of FLPD theory, as given by Equation (66), as well as the time-dependent version given by Equation (53). For the latter case the reader can find the corresponding mathematical details in Appendix B (see Equation (B18)). The parameter values used can be found in the caption of the corresponding figure.
Figure 5. The left panel shows the perpendicular mean-free path vs. the parallel mean-free path and the right panel shows the ratio $\lambda_\perp/\lambda_\parallel \equiv \kappa_\perp/\kappa_\parallel$ vs. the parallel mean-free path. The black solid lines correspond to the diffusive FLPD results as given by Equation (98) and the circles represent the test-particle simulations published in Arendt & Shalchi (2020). The crosses represent time-dependent FLPD theory as given by Equation (53), or in code units, by Equation (B18). For comparison we have also shown the NLGC/UNLT results without a correction factor, meaning that we have set $a^2 = 1$ (gray solid line). To obtain the visualized results, the following parameter values have been used: $\delta B^2/B_0^2 = 1$, $\ell_\perp = \ell_\parallel$, $q = 3$, and a slab fraction of 20%.

Figure 6. Caption is as in Figure 5 but the following parameter values have been used: $\delta B^2/B_0^2 = 1$, $\ell_\perp = \ell_\parallel$, $q = 3$, and a slab fraction of 50%.

Figure 7. Caption is as in Figure 5 but the following parameter values have been used: $\delta B^2/B_0^2 = 0.25$, $\ell_\perp = \ell_\parallel$, $q = 3$, and a slab fraction of 20%.
We have covered a large parameter space meaning that we have changed turbulence strength $\delta B / B_0$, the scale ratio $\ell_\perp / \ell_\parallel$, the slab fraction, and even the energy range spectral index $q$. One can easily see that the agreement between simulations and FLPD theory is remarkable in all cases. A correction parameter $a^2$ is no longer needed in order to achieve this agreement nor does FLPD theory contain any free parameter (see Table 1 for a comparison of the different theories). Of course, FLPD theory without diffusion approximation is more accurate, but even if a diffusion approximation is used, the agreement is more than satisfactory. This is important because using the diffusion approximation leads to a strong mathematical simplification of our new theory.

Figure 10 shows as an example the time-dependent perpendicular diffusion coefficient for two different sets of particle and turbulence parameters. Very nicely we can see that the time-dependent version of FLPD theory provides the initial ballistic regime and thereafter the transport becomes

**Figure 8.** Caption is as in Figure 5 but the following parameter values have been used: $\delta B / B_0 = 2$, $\ell_\perp = \ell_\parallel$, $q = 3$, and a slab fraction of 20%.

**Figure 9.** The left panel shows the perpendicular mean-free path vs. the parallel mean-free path and the right panel shows the ratio $\lambda_\perp / \lambda_\parallel \equiv \kappa_\perp / \kappa_\parallel$ vs. the parallel mean-free path. The black solid lines correspond to the diffusive FLPD results as given by Equation (98). The circles represent the test-particle simulations published in Qin & Shalchi (2012). The crosses represent time-dependent FLPD theory as given by Equation (53), or in code units, by Equation (B18). For comparison we have also shown the NLGC/UNLT results without a correction factor, meaning that we have set $a^2 = 1$ (gray solid line). To obtain the visualized results the following parameter values have been used: $\delta B / B_0 = 1$, $\ell_\perp = 0.1 \ell_\parallel$, $q = 1.5$, and a slab fraction of 20%.

### Table 1

| Theory | Agreement with Heuristic Arguments | Agreement with Simulations for Small Kubo Number Turbulence | Agreement with Simulations for Large Kubo Number Turbulence |
|--------|-----------------------------------|-------------------------------------------------------------|-----------------------------------------------------------|
| NLGC   | No                                | No                                                          | Only with correction factor $a^2$                         |
| UNLT   | Only for small Kubo numbers       | Yes                                                         | Only with correction factor $a^2$                         |
| FLPD   | Yes                               | Yes                                                         | Yes                                                       |

**Note.** Compared are the NLGC theory of Matthaeus et al. (2003), the UNLT theory, as well as the FLPD theory developed in this paper. Simple concepts such as quasi-linear theory or hard-sphere scattering disagree with simulations and heuristic arguments.
weakly sub-diffusive. For later times, diffusion is restored because the transverse complexity of the turbulence influences the particle motion. The sequence ballistic transport $\rightarrow$ sub-diffusion $\rightarrow$ normal diffusion is more pronounced for shorter parallel mean-free paths corresponding to lower particle rigidities.

8. Summary and Conclusion

Particle transport and perpendicular diffusion in particular are fundamental problems in modern physics. To explore how particles move across a mean magnetic field has been explored for more than half a century and started with the work of Jokipii (1966). For decades it was unclear how perpendicular transport in space and astrophysical plasmas works. Significant progress has been made by the famous paper of Rechester & Rosenbluth (1978) but their work focused on laboratory plasmas where collisions play a major role. Over the years, several attempts have been made to solve this problem in space and astrophysics often based on simple concepts such as quasi-linear theory or hard-sphere scattering. However, all those approaches failed in almost all cases. A major breakthrough has been made by Matthaeus et al. (2003) where a nonlinear integral equation has been derived based on a set of approximations. The latter theory, commonly referred to as the NLGC theory, was the first approach in history that showed agreement with simulations and observations. However, this approach was not the final solution to the problem, mainly due to two reasons. First, NLGC theory provides a nonvanishing diffusion coefficient for slab turbulence but it is known for sure that perpendicular transport in turbulence without transverse structure has to be sub-diffusive since particles are tied to magnetic field lines forever. Second, for other turbulence configurations, the theory provides a result that is typically a factor of 3 too large. Therefore, Matthaeus et al. (2003) incorporated a correction factor $a^2$ to achieve agreement with test-particle simulations.

In Shalchi (2010), the so-called UNLT theory was derived systematically (see Equation (22)). Diffusive UNLT theory provides $\kappa_{\perp} = 0$ for slab turbulence, resolving the first of the two problems listed above. Furthermore, the theory contains the FLRW theory developed by Matthaeus et al. (1995) as a special limit and can, therefore, be seen as a unified theory for particles and magnetic field lines. A time-dependent version of UNLT theory (see Equation (31)) was developed by Shalchi (2017) and Lasuik & Shalchi (2017). The latter theory allows for a description of perpendicular transport in all regimes, including the initial ballistic regime, the sub-diffusive regime, and the normal diffusive regime. The latter case is obtained due to the impact the transverse complexity of the turbulence has. However, diffusive as well as time-dependent UNLT theories did not solve the second problem listed above. Both theories require using the correction parameter $a^2$ in the case of large Kubo number turbulence, including quasi-2D turbulence and two-component turbulence.

Independent of the development of analytical theories and numerical tools, there was always a desire to understand perpendicular transport in space plasmas in a more heuristic way. By following the spirit of Rechester & Rosenbluth (1978), a heuristic approach was finally developed by Shalchi (2019a, 2020b). This heuristic description provides simple analytical forms of the perpendicular diffusion coefficient in certain limits. Those forms include the traditional FLRW limit (see Equation (1)), the so-called fluid limit (see Equation (4)), as well as a collisionless version of the CLRR limit (see Equation (2)). Although it was not clear from this heuristic description whether one finds the fluid or the CLRR limit for short parallel mean-free paths and large Kubo number turbulence, the heuristic approach provided a possible explanation as to why $a^2$ is needed and why one needs to set $a^2 = 1/3$.

Heuristic arguments are important in order to understand the underlying physics of particle transport. However, one still needs an analytical theory in order to describe perpendicular transport systematically. Such a systematic theory has finally developed in this paper. The new theory can be understood as a generalized compound sub-diffusion approach where a transverse complexity factor is included allowing the particles to

![Figure 10. Shown are time-dependent perpendicular diffusion coefficients. Here, we have used the dimensionless time $T = \Omega t$ and the gyro-frequency $\Omega$. Compared are the test-particle simulations published in Arendt & Shalchi (2020) represented by the black solid lines and the time-dependent FLPD theory represented by the gray solid lines. The latter results were obtained by solving Equation (B18) numerically. The parameter values used are $q = 3$, $t_\perp = t_{\perp,0}$, $\delta B_{\perp 0}/B_\parallel^0 = 0.20$, and $\delta B_{\parallel 0}/B_\parallel^0 = 0.80$. For comparison we have also shown time-dependent FLPD theory without a transverse complexity factor corresponding to compound sub-diffusion (gray dotted lines). The left panel shows the perpendicular diffusion coefficient for lower particle energies ($R = \sqrt{t}/(\Omega t_{\perp,0}) = 0.01$) corresponding to a parallel mean-free path of $\lambda_\parallel/\ell_\parallel = 0.954$. The right panel shows the results for intermediate particle energies ($R = \sqrt{t}/(\Omega t_{\perp,0}) = 1$) corresponding to a parallel mean-free path of $\lambda_\parallel/\ell_\parallel = 17.1$.](image-url)
decorrelate from the original magnetic field line they were tied to. Thus, we call the new approach FLPD theory. We derived a time-dependent version (see Equation (53)) as well as a diffusive version (see Equation (66)) of this theory. FLPD theory has several important features. For small Kubo number turbulence the new theory becomes identical to UNLT theory (see Figure 1 for a detailed explanation of the relations between all those different approaches). For slab turbulence, we find compound sub-diffusion as the final state of the transport. For non-slab turbulence, we find compound sub-diffusion after the initial ballistic regime as long as transverse complexity is not yet relevant. Thereafter, diffusion is restored and this happens as soon as transverse complexity becomes relevant (see, e.g., Figure 10). Both the fluid limit as well as the CLRR limit are contained in the new theory. For large Kubo number and quasi-2D turbulence, we find in the diffusion limit Equation (70). Compared to the theories of Matthaeus et al. (2003) and Shalchi (2010), we find an extra term in the denominator of the fraction in the wave number integral. This extra term corresponds to a CLRR diffusion term and provides the final piece of the puzzle that has been missing for almost two decades. Figure 3 explains our current understanding of perpendicular transport in three-dimensional turbulence. Figure 1 compares different analytical theories developed previously and Table 1 explains when they agree with heuristic arguments and simulations.

Time-dependent and diffusive versions of the new theory can be combined with a two-component turbulence model (see, e.g., Equation (B18)). This is an important example because solar wind turbulence can be well approximated by this type of model (see, e.g., Matthaeus et al. 1990). For this specific turbulence approximation, we compared the new theory with previously performed test-particle simulations published in Qin & Shalchi (2012), Hussein et al. (2015), and Arendt & Shalchi (2020). These sets of simulations were performed over the past few years by using four different test-particle codes. The comparison can be seen in Figures 4–9 as well as Figure 10. Very clearly one finds a remarkable agreement between the new theory and the simulations regardless of whether the diffusion approximation is used or not. It needs to be emphasized that the new theory does not contain any correction parameter $a^2$ or any other type of free parameter.

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Appendix A

The Particle Velocity Correlation Function

Our aim is to develop a systematic theory for the perpendicular particle diffusion coefficient, and more generally, mean square displacements of particle orbits. In the following, we focus on the perpendicular velocity correlation function $\langle V_x(t)V_x(0) \rangle$. Instead of using Equation (35), we start our derivation with

$$\langle V_x(t)V_x(0) \rangle = \frac{v^2}{3B_0^2} \int_{-\infty}^{+\infty} dz \langle \delta B_x[x(z)] \delta B_x[x(0)] \rangle f_\parallel(z; t).$$  \hspace{1cm} (A1)

The factor $v^2/3$ therein is needed to satisfy the assumption of isotropic initial conditions. Note, the magnetic field correlations have to be computed along the magnetic field line $x(z, t)$. It follows from the TGK formula (15) that velocity correlations and means square displacement are related via Equation (30). The same can be assumed for magnetic field lines, where we have

$$\frac{d^2}{dz^2} \langle \delta B_x^2 \rangle_{FL} = \frac{2}{B_0^2} \langle \delta B_x[x(z)] \delta B_x[x(0)] \rangle.$$  \hspace{1cm} (A2)

Therefore, Equation (A1) can be written as

$$\frac{d^2}{dz^2} \langle \delta B_x^2 \rangle_{FL} = \int_{-\infty}^{+\infty} dz \left[ \frac{d^2}{dz^2} \langle \delta B_x^2 \rangle_{FL} \right] f_\parallel(z; t).$$  \hspace{1cm} (A3)

Shalchi & Kourakis (2007b) derived Equation (41) for the field line mean square displacement. Using this in Equation (A3) yields

$$\frac{d^2}{dz^2} \langle \delta B_x^2 \rangle_{FL} = \int_{-\infty}^{+\infty} dz f_\parallel(z; t) \cos(k_\parallel z) e^{-\frac{k_\parallel^2}{2}d_{FL}(z)}.$$  \hspace{1cm} (A4)

For the magnetic field lines, we now employ a diffusion approximation of the form $\langle \delta B_x^2 \rangle_{FL} = 2\kappa_{FL}[z]$, and we find

$$\frac{d^2}{dz^2} \langle \delta B_x^2 \rangle_{FL} = \int_{-\infty}^{+\infty} dz f_\parallel(z; t) \cos(k_\parallel z) e^{-\frac{k_\parallel^2}{2}d_{FL}(z)}.$$  \hspace{1cm} (A5)

The next step is to replace the parallel distribution function of the particle $f_\parallel(z; t)$. First, we use the Fourier representation given by Equation (37). Using this in Equation (A5) yields

$$\frac{d^2}{dz^2} \langle \delta B_x^2 \rangle_{FL} = \int_{-\infty}^{+\infty} dk f(k_\parallel t)$$

$$\int_{-\infty}^{+\infty} dz \cos(k_\parallel z) e^{-\frac{k_\parallel^2}{2}d_{FL}(z)}.$$  \hspace{1cm} (A6)

The z-integral therein can be solved via Equation (43). Furthermore, we use the resonance function defined via Equation (44). Therewith, we can write Equation (A6) as

$$\frac{d^2}{dz^2} \langle \delta B_x^2 \rangle_{FL} = \int_{-\infty}^{+\infty} dk f(k_\parallel t)$$

$$\int_{-\infty}^{+\infty} dz \cos(k_\parallel z) e^{-\frac{k_\parallel^2}{2}d_{FL}(z)}.$$  \hspace{1cm} (A7)

For the parallel distribution function in Fourier space, we can employ Equation (48), where the two parameter $\omega_+$ and $\omega_-$ are given by Equation (33). Comparing this with Equation (32) allows us to write this as

$$F(k_\parallel t) = \frac{1}{2\pi v^2} 3 \xi(k_\parallel t).$$  \hspace{1cm} (A8)
Using this in Equation (A7) yields
\[
\frac{d^2}{dt^2} \langle (\Delta x)^2 \rangle_p = \frac{1}{\pi B_0^2} \int_{-\infty}^{\infty} dk_{\parallel} \frac{\omega^2}{\omega^2 - \omega^2} \xi(k_{\parallel}, t)
\]
\[
\int d^2k_{\perp} P_{kk} \{R_+(k, k_{\parallel}) + R_-(k, k_{\parallel})\} e^{-\frac{1}{2} (\Delta x)^2} k_{\perp}^2.
\]
(A9)
in perfect agreement with Equation (51).

Appendix B
Time-dependent Results for Two-component Turbulence

In order to achieve an analytical description of perpendicular transport with high accuracy, we need to employ the time-dependent version of FLPD theory, meaning that we do not use a diffusion approximation. The corresponding equation is given by (see Equation (53) in the main part of this paper)
\[
\frac{d^2}{dt^2} \langle (\Delta x)^2 \rangle = \frac{1}{\pi B_0^2} \int_{-\infty}^{\infty} dk_{\parallel} \xi(k_{\parallel}, t)
\]
\[
\int d^2k_{\perp} P_{kk} \{R_+(k, k_{\parallel}) + R_-(k, k_{\parallel})\} e^{-\frac{1}{2} (\Delta x)^2} k_{\perp}^2.
\]
(B10)

In the following we combine Equation (B10) with the two-component turbulence model. Therefore, there will be two contributions to the spectral tensor, and Equation (B10) will consist of two contributions as well. As pointed out in the main part of this paper, the slab contribution provided by the new theory is in perfect coincidence with the time-dependent UNLT result. For the two-dimensional contribution, on the other hand, we can employ Equations (58) and (59). Therefore, Equation (B10) becomes for two-component turbulence
\[
\frac{d^2}{dt^2} \langle (\Delta x)^2 \rangle = \frac{8 \pi}{B_0^2} \int_{0}^{\infty} dk_{||} g^{\text{slab}}(k_{||}) \xi(k_{||}, t)
\]
\[
\int_{0}^{\infty} dk_{\perp} g^{\text{2D}}(k_{\perp}) \frac{\omega^2 k_{\perp}^2}{k_{||}^2 + (\omega \ell_{2D})^2} e^{-\frac{1}{2} (\Delta x)^2} k_{\perp}^2.
\]
(B11)

The integral transformation \(x = k_{||} l_{\parallel}\) in the remaining \(k_{||}\) integral allows us to write Equation (B11) as
\[
\frac{d^2}{dt^2} \langle (\Delta x)^2 \rangle = 4 C(s) \frac{h^2 \ell_{2D}}{B_0^2} \int_{0}^{\infty} dx \frac{1}{(1 + x^2/\tau)^2} \omega_+ e^{\omega_+ x} - \omega_- e^{\omega_- x}.
\]
(B12)

The latter formula is a second-order differential equation for the function \(\sigma\). Since all quantities therein are dimensionless, this differential equation can be solved numerically. This was done in this paper to produce the crosses in Figures 4–9, as well as the time-dependent plots in Figure 10. It has to be noted that the first term in Equation (B12), which comes from the slab modes, is sub-diffusive. We still kept it to obtain a more accurate result, especially for earlier times.

In the following we combine Equation (B10) with the two-component turbulence model. Therefore, there will be two contributions to the spectral tensor, and Equation (B10) will consist of two contributions as well. As pointed out in the main part of this paper, the slab contribution provided by the new theory is in perfect coincidence with the time-dependent UNLT result. For the two-dimensional contribution, on the other hand, we can employ Equations (58) and (59). Therefore, Equation (B10) becomes for two-component turbulence
\[
\frac{d^2}{dt^2} \langle (\Delta x)^2 \rangle = \frac{8 \pi}{B_0^2} \int_{0}^{\infty} dk_{\parallel} g^{\text{slab}}(k_{\parallel}) \xi(k_{\parallel}, t)
\]
\[
\int_{0}^{\infty} dk_{\perp} g^{\text{2D}}(k_{\perp}) \frac{\omega^2 k_{\perp}^2}{k_{||}^2 + (\omega \ell_{2D})^2} e^{-\frac{1}{2} (\Delta x)^2} k_{\perp}^2.
\]
(B11)

Therein, we need to replace three functions, namely, the slab spectrum \(g^{\text{slab}}(k_{||})\), the 2D spectrum \(g^{\text{2D}}(k_{\perp})\), as well as the parallel correlation function \(\xi(k_{\parallel}, t)\). For the slab spectrum, we employ Equation (100) and the 2D spectrum and the parallel correlation function are given by Equations (77) and (32), respectively. Therewith, we can write
\[
\frac{d^2}{dt^2} \langle (\Delta x)^2 \rangle = 4 C(s) \frac{h^2 \ell_{2D}}{B_0^2} \int_{0}^{\infty} dx \frac{1}{(1 + x^2/\tau)^2} \omega_+ e^{\omega_+ x} - \omega_- e^{\omega_- x}.
\]
(B12)
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