Exploring integrability-chaos transition with a sequence of independent perturbations

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A gas of interacting particles is a paradigmatic example of chaotic systems. It is shown here that even if all but one particle are fixed in generic positions, the excited states of the moving particle are chaotic. They are characterized by the number of principal components (NPC) — the number of integrable system eigenstates involved into the non-integrable one, which increases linearly with the number of strong scatterers. This rule is a particular case of the general effect of an additional perturbation on the system chaotic properties. The perturbation independence criteria supposing the system chaoticity increase are derived here as well. The effect can be observed in experiments with photons or cold atoms as the decay of observable fluctuation variance, which is inversely proportional to NPC, and, therefore, to the number of scatterers. This decay indicates that the eigenstate thermalization is approached. The results are confirmed by numerical calculations for a harmonic waveguide with zero-range scatterers along its axis.

Evolution of integrable systems is completely predictable and, according to the Kolmogorov-Arnold-Moser theorem (see [1]), weak perturbations do not affect this property. There are numerous examples of such classical and quantum systems, including stellar mechanics, hydrogen atoms, and many-body systems, both realized in experiments, such as quantum Newton cradle [2, 3] and cold-atom breathers [4, 5] (see [6–9] for the theoretical description) and those waiting for the realization [10].

When the integrability-lifting perturbation is sufficiently strong, the system evolution becomes unpredictable. Nevertheless, a completely-chaotic system relaxes to a state described by the Gibbs ensemble, thanks to the eigenstate thermalization mechanism — eigenstate expectation values are equal to microcanonical averages at the eigenstate energy — introduced in [11, 12] (see also [13, 14], the experimental work [15], the review [16] and the references therein). In contrast, integrable systems relax to states described by the generalized Gibbs ensemble [17–21].

However, a generic system lies between integrable and completely-chaotic systems [22–49]. The incomplete chaos can be related to weak integrability-lifting perturbations [29, 31, 32, 41, 47, 48] or phase-space separation in both classical and quantum systems. It can have also a specific quantum nature, such as many-body localization [33, 34, 49] or zero-range interactions [22, 23, 26, 28, 30]. Chaoticity of such incompletely-chaotic systems can be characterized by the inverse participation ratio (IPR) [28, 31, 33, 50]. Its inverse — the number of principal components (NPC) — estimates the number of integrable system eigenstates comprising the non-integrable one. NPC ranges from 1 for integrable systems to 0 for completely-chaotic ones. It governs the expectation values after relaxation in incompletely-chaotic systems with no selection rules [25, 31]. This regularity has been confirmed in different systems [41]. The fluctuations of expectation values over eigenstates are strong for integrable systems and vanish in completely-chaotic ones, according to the eigenstate thermalization hypothesis. These fluctuations, too, are governed by IPR [32].

Exploration of the integrability – eigenstate thermalization crossover is of special interest, and the means of the system chaoticity prediction based on its Hamiltonian, with no numerical calculations, would be very useful. The exponential decay of fluctuations in many-body systems with the number of particles was predicted in [12]. However, this sharp decay complicates the exploration due to the high sensitivity of the system chaoticity to the number of particles. Analytical predictions for single particles with random-matrix perturbations (see [31] and the references therein) are applicable to strong perturbations (and chaos) only. The system chaoticity can be modified by an additional perturbation, e.g., it can transform an integrable system into a chaotic one. However, the effect of an additional perturbation on a chaotic system is ambiguous, e.g., the system integrability can be restored if the additional perturbation is equal to the integrability-lifting one with opposite sign. The system chaoticity increases when the additional perturbation obeys the independence criteria determined here. Then, a linear increase of NPC with the number of independent perturbations of the same shape is predicted. Simultaneous decay of expectation value fluctuations indicates that the eigenstate thermalization is approached. These predictions are confirmed by numerical calculations for a harmonic waveguide with zero-range scatterers along its axis. Both NPC and expectation value fluctuations are specific characteristics of quantum chaos and it is unclear if they have classical counterparts.

Consider a sequence of Hamiltonians \( \hat{H}_s = \hat{H}_0 + \sum_{s'=1}^{S} \hat{V}_{s'} \). Here, the integrable one \( \hat{H}_0 \) has the eigenstates \( |n\rangle \) and eigenenergies \( E_n \) labeled by a proper set of integrals of motion \( \mathbf{n} \). The integrability is lifted by the perturbations \( \hat{V}_{s'} \). The eigenstates \( |\alpha_s\rangle \) of the non-integrable Hamiltonians \( \hat{H}_s \) are labeled in the increasing order of their eigenenergies \( E_{\alpha_s} \). If \( \hat{H}_s \) is invariant under some transformations, sets of \( |\alpha_s\rangle \) with certain symmetry have to be considered separately. For example, there may be eigenstates with certain angular momentum for
rotational symmetry of $\hat{H}_s$, or certain parity for inversion invariance, or certain quasimomentum for spatial periodicity. Such sets of eigenstates can be described by different Hamiltonians, e.g., with separated angular momentum for rotational symmetry, or restricted to the unit cell for spatial periodicity. Such Hamiltonians can contain fewer perturbations than the original one $\hat{H}_s$. Only the case when all $\hat{H}_s$ have the same invariance group is considered.

Each eigenstate $|\alpha_s\rangle$ can be expanded in terms of $|n\rangle$, and the strength function

$$W_s(E_n, E_{\alpha_s}) = \left\langle |n|\alpha_s\rangle |^2 \right\rangle$$

is the probability averaged over states with a fixed energy difference (see [51]). The relation between the strength functions for $|\alpha_s\rangle$ and $|\alpha_{s-1}\rangle$

$$W_s(E_n, E_{\alpha_s}) \approx \sum_{E_{\alpha_{s-1}}} W_{s-1}(E_n, E_{\alpha_{s-1}}) \left\langle |\alpha_{s-1}|\alpha_s\rangle |^2 \right\rangle$$

is obtained (see [51]) neglecting the interference terms. This approximation is applicable whenever the perturbation $\hat{V}_s$ is independent of $\hat{V}_{s'}$ with $s' < s$, i.e.,

$$\sum_{n, n'} \left\langle n|\hat{V}_s|n'\right\rangle \left\langle n'|\hat{V}_{s'}|n\right\rangle \ll \sum_{s', n'} \sum_{n, n'} \left\langle n|\hat{V}_{s'}|n'\right\rangle^2 .$$

(3)

The summations over $n$ and $n'$ in the microcanonical interval (see [51]) lead to the Berry autocorrelation function [52]. It is localized within the characteristic de Broglie wavelength determined by the characteristic eigenstate energy. Condition [53] is satisfied, for example, if the spatial separation between local potentials exceeds the characteristic de Broglie wavelength (see [51]). Effect of spatially-separated perturbations on certain characteristics of energy spectra was analyzed in [53]. Other examples are the angular-dependent potentials with no common spherical harmonics in their expansions, such as different terms in multipole expansion, and the potentials of different parity (see [51]).

The relation [53] means that an addition of an independent perturbation increases the number of the integrable system eigenstates involved to the non-integrable one (see Fig. 1). This intuitive picture illustrates the quantitative relation presented below.

Since $W_s(E_n, E_{\alpha_s})$ should decay as $(E_n - E_{\alpha_s})^{-2}$ in the limit $|E_n - E_{\alpha_s}| \to \infty$, (see [51]) but has no singularities as $E_n$ and $E_{\alpha_s}$ never coincide, the Lorentzian profile

$$W_L(E, \Gamma) = \frac{1}{\pi} \frac{\Gamma}{E^2 + \Gamma^2}$$

(4)

is a natural choice for the continuous strength function $W_s(E_n, E_{\alpha_s}) \approx W_L(E_{\alpha_s} - E_n, \Gamma_s) \Delta E$. Here $\Delta E$ is the average difference between eigenenergies in the vicinity of $E_{\alpha_s}$. Such a strength function has been applied to systems with strong random-matrix perturbations [31, 54–57], when the profile contains many energy levels and $\Gamma$ can be evaluated using the Fermi golden rule. On the integrability-chaos crossover, explored here, the profile may contain only a few levels and the Fermi golden rule can be inapplicable, but the strength function with some $\Gamma$ retains the necessary properties. The averaged IPR $\eta_s \equiv \sum_n |\langle \alpha_s|n\rangle|^2 \nu_s$ (where the overbar means the microcanonical average over the states $\alpha_s$) is related to the Lorentzian width $\Gamma_s$ as (see [51])

$$\eta_s = \frac{2}{3} \frac{\Delta E}{2 \pi \Gamma_s} ,$$

(5)

where the factor 3 is chosen for the time reversal (T) invariant and PT invariant systems (where P is the inversion) and 2 is chosen otherwise. For the fixed integrable Hamiltonian $\hat{H}_0$ determining the energy difference $\Delta E$, Eq. [5] together with the relation $\Gamma_s = \Gamma_{s-1} + \nu$ (see [51]), leads to the recurrence relation for NPC $\eta_s^{-1}$

$$\eta_s^{-1} = \eta_s^{-1} + \nu .$$

(6)

The parameter $\nu$ is approximately independent of $s$ if the chaotic properties of $|\alpha_s\rangle$ weakly depend on $s$ and the shape of $\hat{V}_s$ is independent of $s$ (as for strong scatterers with the same strength). Then [6] provides a linear dependence $\eta_s^{-1} = \eta_2^{-1} + (s - 2)\nu$ of NPC on the number of scatterers. An additional independent perturbation increases NPC even if its value is so high that the system can be considered as a completely chaotic one.

The prediction [6] is tested for models where the integrability of a particle in a potential with separable coordinates is lifted by $s$ fixed zero-range scatterers. Other examples of such models are flat orthogonal billiards — multiscatterer generalizations of the Seba billiard [22]. Their energy spectrum properties were analyzed for up to 6 scatterers [23, 24, 43]. Scattering in a harmonic potential was analyzed in [53]. The set of scatterers is a particular case of the rank-$s$ separable potential $\hat{V}_s = V_s \left| \mathcal{F}_{s'} \right\rangle \langle \mathcal{F}_{s'} |$ with the formfactors $|\mathcal{F}_{s'}\rangle$. For such potentials the eigenstate expansion coefficients can be expressed as (see [51])

$$\langle n|\alpha_s\rangle = \sum_{s'=1}^s V_{s'} \frac{\langle n|\mathcal{F}_{s'}\rangle \langle \mathcal{F}_{s'}|\alpha_s\rangle}{E_{\alpha_s} - E_n}$$

(7)
in terms of $s$ overlaps $\langle F_{s'}|\alpha_s \rangle$ which obey the set of linear equations

$$\sum_{s''=1}^{s} \left( V_{s''} \sum_{n} \frac{\langle F_{s'}|n\rangle \langle n|F_{s''}\rangle}{E_{\alpha_s} - E_n} - \delta_{s''s'} \right) \langle F_{s''}|\alpha_s \rangle = 0. \tag{8}$$

This system has non-trivial solutions if the determinant of its matrix is equal to zero. Then the eigenenergies $E_{\alpha_s}$ are roots of the determinant. High rank separable potentials can also approximate long-range, e.g., dipole-dipole, ones [59]. This approximation was used for energy spectra calculation [60].

The present models are generalizations of the single-scatterer model [26,28,90]. The integrable Hamiltonian contains the kinetic energy and the radial harmonic potential with the frequency $\omega_\perp$,

$$\hat{H}_0 = \frac{\hbar^2}{2m} \left( \frac{1}{i} \frac{\partial}{\partial z} - A \right)^2 - \Delta_\rho + \frac{m\omega_\perp^2 \rho^2}{2}, \tag{9}$$

where $z$ and $\rho$ are the axial and radial coordinates, respectively, $m$ is the particle mass, and $A$ is a vector potential. The discrete energy spectrum is provided either by the periodic boundary conditions (PBC), $\langle z + L|\alpha_s \rangle = \langle z|\alpha_s \rangle$, or by a hard-wall box, $\langle z = 0|\alpha_s \rangle = \langle z = L|\alpha_s \rangle = 0$. The formfactors of the separable perturbation are $\langle r|F_{s'} \rangle = \delta_{\text{reg}}(r - R_{s'})$, where $\delta_{\text{reg}}(r)$ is the Fermi-Huang pseudopotential, and the scatterer position $R_{s'} = (0, 0, z_{s'})$ has the zero radial component (see Fig. 3(a)). The Hamiltonian $\hat{H}$ is rotationally symmetric along the waveguide axis and the perturbation affects only the states with zero angular momentum. Then, only products $|nl\rangle$ of two-dimensional harmonic oscillator and (for PBC) a plane wave with the momentum $2\pi \hbar/L$ are considered here. For hard-wall box, the standing waves with the momentum $\pi \hbar/L$ replace the plane waves (see [51]).

Unlike a flat billiard with a constant energy density of states, in the present model $E_\alpha \propto \alpha^{2/3}$, as for a three-dimensional free particle, and the energy density of states $\partial \alpha^2 / E_\alpha \propto E_\alpha^{1/2}$ increases with the energy. The logarithmic asymptotic freedom [23] found for flat billiards is related to decreasing effective coupling $V_{\text{eff}} \propto 1/\log E$. However, it is a specific property of the systems with $E_\alpha \propto \alpha$. If $E_\alpha \propto \alpha^\gamma (\gamma \neq 1)$, one can see from the derivation [23] that $V_{\text{eff}} \propto E^{\gamma - 1/\gamma}$ has the same energy dependence as the energy difference between the states $\partial E_\alpha / \partial \alpha \propto E^{-1/\gamma}$. Then, the present model, as well as a generic system with $\gamma \neq 1$, does not show the asymptotic freedom.

In the absence of the vector potential, $A = 0$, the energy spectrum of the integrable Hamiltonian is degenerate, $E_{nl} = E_{n-1}$. The degeneracy will be lifted by any potential with undefined parity. The vector potential lifts it as well, with no complication of the wavefunctions. However, the Hamiltonian loses the T invariance.

Four models are considered here. The non-symmetric model is T-noninvariant and the scatterer positions $z_1 = 0, z_2 = (s' - 1 + \delta_s)/L/s$ (with $s > 1$) have no symmetry due to random shifts $-0.25 \leq \delta_s < 0.25$ chosen once for each $s$. The symmetric model with $z_{s-s'+1} = z_s - z_{s'}$ and equal $V_{s'}$ is PT-invariant, where P is the inversion over $z_\perp/2$ (this model is not P-invariant, as $\hat{H}_0$ is not P-invariant if $A \neq 0$). The T-invariant model has $A = 0$ and the same scatterer positions as the non-symmetric one. Only this model has degenerate $E_{nl}$. The three previous models correspond to PBC, while the fourth, box, model, corresponds to the hard-wall box, has $A = 0$, and $z_{s'} = (s' + \delta_{s'})/L/(s + 1)$.

Summation over $l$ in [51] can be done analytically (see [51]), leaving a sum over $\sim E_\alpha/(\hbar \omega_\perp)$ values of $n$ (the closed-channel contributions with $n\hbar \omega_\perp > E_\alpha$ decay exponentially with $n$). As $E_\alpha \propto \alpha^{2/3}$, calculation of the system [3] matrix and its solution require $\sim s^2 \alpha^{2/3}$ and $\sim s^3$ operations, respectively. Then $\alpha$ eigenstates are calculated for $\alpha \gg s^{3/2}$ with $\sim s^{4/3}$ operations (cf. with $\sim s^3$ operations required by the direct diagonalization method).

The integrable system is described by two dimensionless parameters: $\lambda = mL^2 \omega_\perp / \hbar$, characterizing the aspect ratio, and the scaled vector potential $l_0 = L \lambda / (2\pi)$. In the calculations, $\lambda = \pi^3(1 + \sqrt{5}) \approx 100$ and $l_0 = 0.25 - e^{-4} \approx 0.232$ are expressed in terms of transcendent numbers. Approximately the same results are obtained for any $l_0 > 0.01$. In Fig. 3(a) and (c), $V_{s'} = 10^6 V_0$ for all scatterers is in the unitary regime ($V_0 = 2\pi \hbar^2 m^{-3/2} \omega^{-1/2}$ is the scale of the interaction strength). Approximately the same results are obtained for any $V_{s'} > V_0$. In Fig. 3(b), $V_{s'}/V_0 = 10^6, 10^{-1}, 10^{-2}, 2 \times 10^{-3}, 10^{-3}, 10^{-4}, 0$ for the 7 IPR values from left to right (the points for $V_{s'}/V_0 = 10^6$ and $10^{-1}$ are almost indistinguishable). Each point in Fig. 3 represents an average over the states $101 \leq \alpha \leq 10^6$ of the non-integrable system.

The plots of NPC [see Fig. 3(a)] as a function of the number of scatterers for the four models confirm the
linear dependence (6). The linear fits have two model-dependent parameters: $\eta_2$, which cannot be predicted by (6) since the system with a single scatterer is not chaotic enough, and $\nu$. In Fig. 3(a), $\nu = 1.07, 0.7, 0.68,$ and 0.53 for the non-symmetric, symmetric, box, and $T$-invariant models, respectively. Taking into account the symmetry-dependent factors in Eq. (5), we can see that $\Gamma'$ is approximately the same for the first three models.

The Lorentzian width is related to IPR by Eq. (5). Figure 3(a) demonstrates the linear dependence even when $\Gamma_s \sim \Delta E$. Therefore, even when the Fermi golden rule is inapplicable to $\Gamma_s$ as the profile contains only few energy levels, the strength function can be approximated by the Lorentzian profile.

The physical implication of the rule (6) is related to the fluctuations of expectation values $\langle \alpha | \hat{O} | \alpha \rangle$ of an observable $\hat{O}$, characterized by their variance $\text{Var}_\alpha(\hat{O}) = \langle \alpha | \hat{O} | \alpha \rangle^2 - \langle \alpha | \hat{O} | \alpha \rangle^2$ over the eigenstates $|\alpha\rangle$. It is proportional to IPR and the variance for the underlying integrable system

$$\text{Var}_\alpha(\hat{O}) = \eta \text{Var}_n(\hat{O}) \tag{10}$$

as was derived, in slightly different form, in [32]. The applicability criteria of this relation can be determined using the fact that it can also be derived in the same way as the relation between the initial, thermal, and relaxed expectation values (7) in [28] replacing the density matrix by the observable. Then (10), like the relation (7) in [28], is applicable to perturbations with no selection rules. The relation (10) was compared with numerical results [32] for a many-body system with two-body interactions. However, such systems do have selection rules, as each two-body interaction conserves quantum numbers of other particles. It is probably the reason why the numerical results [32] were described by (10) only up to some energy-dependent factor. Systems with separable perturbations have no selection rules, and the relation (10) describes the dependence of variances on the number of scatterers and interaction strength for four observables [see Fig. 3 (b) and (c)]. The observables are: the axial momentum (crosses), the part of the transverse potential energy in the total energy (pluses), and the occupations of the positive momenta (triangles) and of the odd axial modes (circles). The lines show IPR.
acting particles are studied in numerous experimental and theoretical researches. However, due to computational difficulties, a direct numerical simulation is performed for lattice systems (e.g., \[14, 17, 18, 31-33, 37, 38, 40, 41, 44, 47, 48\]) with a finite Hilbert space, while the problem complexity increases as a high power of the lattice site number and exponentially with the number of particles. A single particle in an external potential allows us to explore an infinite Hilbert space. Eigenstate thermalization has been analyzed \([62]\) for \(3 \times 10^4\) states of a Sinai-type billiard. However, the chaoticity of that system cannot be tuned and the calculation of highly excited states is obstructed by the increase of the coordinate grid size. The properties of systems with several independent perturbations (particularly, scatterers) can be tuned by the number of perturbations and their strengths. Although the general results are confirmed by numerical calculations for a specific model, they are applicable to any integrable system, perturbed by several scatterers.

The predictions should be testable experimentally in several physical systems. Tightly-trapped cold atoms of one kind can play the role of scatterers for an atom of a second kind in a wide trap. Moreover, several atoms of the second kind — with interactions between them turned off by a broad Feshbach resonance — can be used to get averages. In optics, photons in a cavity can be scattered by optical defects \([63]\).

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**SUPPLEMENTAL MATERIAL FOR:**

**EXPLORING INTEGRABILITY-CHAOS TRANSITION WITH A SEQUENCE OF INDEPENDENT PERTURBATIONS**

Numbers of equations in the Supplemental material start with S. References to equations in the Letter do not contain S.

I. AVERAGES

The microcanonical window \([E_{\text{min}}, E_{\text{max}}]\), containing a large number of states \(N_{\text{MC}}\), is small compared to the system energy scales. The microcanonical average is defined as

\[
\bar{F}(\alpha) = \frac{1}{N_{\text{MC}}} \sum_{\alpha \in \text{MC}} F(\alpha),
\]

(S-1)

where \(\alpha \in \text{MC}\) means that \(E_{\text{min}} < E_\alpha < E_{\text{max}}\).

The average over states with a fixed energy difference in the strength function (1) is defined as

\[
\langle F(\alpha, \beta) \rangle = \frac{1}{N_{\text{MC}}} \sum_{\alpha', \beta' \in \text{MC}} F(\alpha', \beta') \delta_{E_\beta - E_\alpha + E_{\alpha'} + E_{\beta'}},
\]

(S-2)

where \(\delta_{E_\beta - E_\alpha + E_{\alpha'} + E_{\beta'}}\) selects the state \(\beta'\) with the energy \(E_{\beta'}\) closest to \(E_\beta - E_\alpha + E_{\alpha'}\).

II. DERIVATION OF THE RELATION (2)

Consider at first the Hamiltonian \(\hat{H}_2 = \hat{H}_0 + \hat{V}_1 + \hat{V}_2\). The Schrödinger equations lead to the following relations between the eigenstates \(|\alpha_2\rangle, |\alpha_1\rangle,\) and \(|n\rangle\) of the Hamiltonians \(\hat{H}_2, \hat{H}_1,\) and \(\hat{H}_0,\) respectively,

\[
\langle \alpha_1 | \alpha_2 \rangle = \frac{\langle \alpha_1 | \hat{V}_2 | \alpha_2 \rangle}{E_{\alpha_2} - E_{\alpha_1}}, \quad \langle n | \alpha_1 \rangle = \frac{\langle n | \hat{V}_1 | \alpha_1 \rangle}{E_{\alpha_1} - E_n}
\]

(S-3)

Let us represent the expansion coefficients as \(\langle n | \alpha_2 \rangle = \sum_{\alpha_1} \langle n | \alpha_1 \rangle \langle \alpha_1 | \alpha_2 \rangle\) and separate the diagonal part of the strength function (1)

\[
W_{\text{diag}}^2(E_n, E_{\alpha_2}) = \left\langle \sum_{\alpha_1} |\langle n | \alpha_1 \rangle|^2 |\langle \alpha_1 | \alpha_2 \rangle|^2 \right\rangle
= \frac{1}{N_{\text{MC}}} \sum_{E_{\alpha_1}, n, \alpha_1'} \sum_{\text{MC}} |\langle n | \alpha_1' \rangle|^2 \delta_{E_{\alpha_1} - E_{\alpha_1'} - E_{\alpha_1'} + E_{\alpha_1'}} \left(\frac{1}{E_{\alpha_2} - E_{\alpha_1'}}\right)^2 \sum_{\alpha_2'} |\langle \alpha_1' | \hat{V}_2 | \alpha_2' \rangle|^2 \delta_{E_{\alpha_2} - E_{\alpha_1} + E_{\alpha_1'}, E_{\alpha_2'}}.
\]

(S-4)

The eigenstates \(|\alpha_1'\rangle\) are independent of the potential \(\hat{V}_2\) as the Hamiltonian \(\hat{H}_1\) does not include \(\hat{V}_2\). Besides, \(E_{\alpha_1'} - E_{\alpha_2'}\) is independent of \(\alpha_1'\) since \(E_{\alpha_1'}\) and \(E_{\alpha_2'}\) are independent being eigenenergies of different non-integrable Hamiltonians. This means that the value of the last sum in Eq. (S-4) appears as the states \(|\alpha_1'\rangle\) were indiscriminately chosen from a microcanonical interval. Then the sum can be approximated by its microcanonical average over \(\alpha_1'\), leading to

\[
W_{\text{diag}}^2(E_n, E_{\alpha_2}) \approx \sum_{E_{\alpha_1}} W_1(E_n, E_{\alpha_1}) \left\langle |\langle \alpha_1 | \alpha_2 \rangle|^2 \right\rangle.
\]

(S-5)

If the non-diagonal part of the strength function can be neglected, replacing \(\alpha_2, \alpha_1, \hat{V}_2,\) and \(\hat{V}_1\) by \(\alpha_s, \alpha_{s-1}, \hat{V}_s,\) and \(\sum_{s'=1}^{s-1} \hat{V}_{s'}\), respectively, we get the relation (2).

In order to evaluate the non-diagonal part, let us represent the expansion coefficients as

\[
\langle n | \alpha_2 \rangle = \frac{1}{E_{\alpha_2} - E_n} \sum_{\alpha_1} \left(\langle n | \hat{V}_1 | \alpha_1 \rangle \langle \alpha_1 | \alpha_2 \rangle + \langle n | \alpha_1 \rangle \langle \alpha_1 | \hat{V}_2 | \alpha_2 \rangle\right)
\]
using Eqs. (S-3) and the completeness of the set $|\alpha_1\rangle$ $\alpha_1$ here is the same as in [S-4]. The corresponding probabilities can be expressed

$$
|\langle n|\alpha_2\rangle|^2 = \frac{1}{(E_{\alpha_2} - E_n)^2} \sum_{\alpha_1,\alpha'_1} \left( B_{\alpha_1,\alpha'_1} (\hat{V}_1, \hat{V}_1, n) B_{\alpha'_1,\alpha_1} (\hat{I}, \hat{I}, \alpha_2) + B_{\alpha_1,\alpha'_1} (\hat{I}, \hat{I}, \alpha_2) B_{\alpha'_1,\alpha_1} (\hat{V}_2, \hat{V}_2, \alpha_2) 
+ B_{\alpha_1,\alpha'_1} (\hat{I}, \hat{V}_1, n) B_{\alpha'_1,\alpha_1} (\hat{I}, \hat{V}_2, \alpha_2) + B_{\alpha_1,\alpha'_1} (\hat{V}_1, \hat{I}, n) B_{\alpha'_1,\alpha_1} (\hat{V}_2, \hat{I}, \alpha_2) \right)
$$

(S-6)
in terms of the products of the matrix elements

$$
B_{\alpha_1,\alpha'_1} (\hat{O}_1, \hat{O}_2, \beta) = \langle \alpha_1|\hat{O}_1|\beta\rangle \langle \beta|\hat{O}_2|\alpha'_1\rangle.
$$

In the generic case, different degrees of freedom of $\hat{H}_0$ have incommensurate frequencies, and the quantum numbers $n$ and $n'$ of the energy-neighboring states will be mutually uncorrelated. Then, substituting (S-6) into the strength function (1), we can replace $B_{\alpha_1,\alpha'_1} (\hat{O}_1, \hat{O}_2, n)$ by its microcanonical average $B_{\alpha_1,\alpha'_1} (\hat{O}_1, \hat{O}_2, n)$ [see (S-1)] and obtain the following approximation for the fixed energy difference average [see (S-2)]

$$
\left\langle \frac{1}{(E_{\alpha_2} - E_n)^2} B_{\alpha_1,\alpha'_1} (\hat{O}_1, \hat{O}_2, n) B_{\alpha'_1,\alpha_1} (\hat{O}_3, \hat{O}_4, \alpha_2) \right\rangle \approx \frac{1}{(E_{\alpha_2} - E_n)^2} B_{\alpha_1,\alpha'_1} (\hat{O}_1, \hat{O}_2, n) B_{\alpha'_1,\alpha_1} (\hat{O}_3, \hat{O}_4, \alpha_2)^{n_2}\eta_{\alpha_1,\alpha'_1} (\hat{O}_1, \hat{O}_2, n)
$$

(S-7)
The independence of the random functions $\langle n|\alpha_s\rangle$ and $\langle n|\alpha'_s\rangle$ for $\alpha_s \neq \alpha'_s$ [12, 52] leads to the approximate equality of the microcanonical averages

$$
|\alpha_s\rangle|\alpha_s\rangle^\alpha_s \approx \langle n|\alpha_s\rangle^n
$$

(S-8)
of the product of two eigenfunctions. This results in

$$
B_{\alpha_1,\alpha'_1} (\hat{I}, \hat{I}, \beta) \approx \frac{1}{N_{\text{MC}}} \delta_{\alpha_1,\alpha'_1}, \quad B_{\alpha_1,\alpha'_1} (\hat{I}, \hat{V}, \beta) \approx \frac{1}{N_{\text{MC}}} \langle \alpha_1|\hat{V}|\alpha'_1\rangle
$$

(S-9)
if $\alpha_1$ and $\alpha'_1$ belong to the microcanonical interval (otherwise, the averages vanish). Then, using Eqs. (S-6), (S-7), (S-8), and (S-9) one can approximate the diagonal and non-diagonal parts of the strength function (1) as

$$
W_2^{\text{diag}}(E_n, E_{\alpha_2}) \approx \frac{1}{N_{\text{MC}}^2(E_{\alpha_2} - E_n)^2} \left[ \sum_{n,n' \in \text{MC}} \left( |\langle n|\hat{V}_1|n'\rangle|^2 + |\langle n|\hat{V}_2|n'\rangle|^2 \right) + 2 \sum_{\alpha_1 \in \text{MC}} \langle \alpha_1|\hat{V}_1|\alpha_1\rangle \langle \alpha_1|\hat{V}_2|\alpha_1\rangle \right]
$$

(S-10)

$$
W_2^{\text{non}}(E_n, E_{\alpha_2}) \approx \frac{2}{N_{\text{MC}}^2(E_{\alpha_2} - E_n)^2} \left[ \sum_{n,n' \in \text{MC}} \langle n|\hat{V}_1|n'\rangle \langle n'|\hat{V}_2|n\rangle - \sum_{\alpha_1 \in \text{MC}} \langle \alpha_1|\hat{V}_1|\alpha_1\rangle \langle \alpha_1|\hat{V}_2|\alpha_1\rangle \right].
$$

The last, single, sums here can be neglected compared to the double sums. Then, the non-diagonal part is negligible when the condition (3) for $s = 2$ is satisfied. The recurrent application of this condition provides the condition (3) for any $s$.

The microcanonical average of the product of two eigenfunctions

$$
\langle r_1|n\rangle \langle n|r_2\rangle^n \approx C_{\text{Berry}}(|r_1 - r_2|, (r_1 + r_2)/2)
$$

is the Berry autocorrelation function [52]

$$
C_{\text{Berry}}(r, R) = \Gamma(D/2) \frac{J_{D/2-1}(K(r)R)}{(K(r)R)^{D/2-1}}
$$

where $D$ is the number of degrees of freedom (the dimension of the vectors $n$, $r$, and $R$), $\Gamma$ is the $\Gamma$-function, $J$ is the Bessel function (see [64]), and $K(R)$ is the absolute value of the local wavevector. The autocorrelation function decays
when $r$ exceeds the characteristic de Broglie wavelength $K^{-1}$, determined by the mean energy of the microcanonical interval.

Consider several important cases when $W_{2}^{\text{bond}}$ becomes negligible and, therefore, the condition (3) is satisfied. For local coordinate-dependent potentials $\hat{V}_1$ and $\hat{V}_2$, the sum in $W_{2}^{\text{bond}}$ can be transformed as

$$\sum_{\mathbf{n}, \mathbf{n}' \in \text{MC}} \langle \mathbf{n} | \hat{V}_1 | \mathbf{n}' \rangle \langle \mathbf{n}' | \hat{V}_2 | \mathbf{n} \rangle \approx \int d^D r d^D r' \hat{V}_1(\mathbf{R} + r/2) \hat{V}_2(\mathbf{R} - r/2) C_{\text{Berry}}^2(r, \mathbf{R}).$$  \hspace{1cm} (S-11)

Then, $W_{2}^{\text{bond}}$ becomes negligible when the distance between $V_1$ and $V_2$ exceeds the characteristic de Broglie wavelength.

If the ranges of $V_1$ and $V_2$ are substantially less than the characteristic scale of $\mathcal{H}_0$, we can neglect the $\mathbf{R}$-dependence of the Berry autocorrelation function, substituting $C_{\text{Berry}}(r, \mathbf{R}) \approx C_{\text{Berry}}(r, \mathbf{R}_0)$, where $\mathbf{R}_0$ is any point in the vicinity of $V_1$ and $V_2$. Then Fourier transform leads to

$$\sum_{\mathbf{n}, \mathbf{n}' \in \text{MC}} \langle \mathbf{n} | \hat{V}_1 | \mathbf{n}' \rangle \langle \mathbf{n}' | \hat{V}_2 | \mathbf{n} \rangle \approx \int d^D p \hat{V}_1(p) \hat{V}_2(-p) C_{\text{Berry}}^2(p),$$ \hspace{1cm} (S-12)

where

$$\hat{V}_i(\mathbf{p}) = (2\pi)^{-D/2} \int d^D r \hat{V}_i(\mathbf{r}) e^{-i \mathbf{p} \cdot \mathbf{r}}.$$  \hspace{1cm} (S-12)

In any dimension, if $V_1$ and $V_2$ can be expanded in terms of hyperspherical harmonics, $V_i(\mathbf{r})$ and its Fourier transform $\hat{V}_i(p)$ are expanded in terms of the same hyperspherical harmonics (see [55]). For example, in the 3D case ($D = 3$), when $V_i(\mathbf{r}) = \sum_{l,m} v_{i lm}(r) Y_{i lm}(\mathbf{r}/r)$, its Fourier transforms can be expanded in terms of the same spherical harmonics, $\hat{V}_i(p) = \sum_{l,m} \hat{v}_{i lm}(p) Y_{i lm}(\mathbf{p}/p)$, where $\hat{v}_{i lm}(p) = (2/\pi)^{1/2} \int_0^{\infty} r^2 dr j_l(pr) v_{i lm}(r)$ and $j_l$ is the spherical Bessel function. Substituting these expansions into Eq. (S-12) we see that, due to the orthogonality of the spherical (or hyperspherical) harmonics, $W_{2}^{\text{bond}}$ vanishes when $V_1$ and $V_2$ do not contain the same harmonics. For example, it is the case when $V_1$ and $V_2$ are different terms in multipole expansion of a potential. The odd (even) $V_i(\mathbf{r})$ includes only spherical harmonics with odd (even) $l$. Then for the perturbations of different parity the condition (3) is satisfied too.

The pseudopotential expansion of multipolar interactions [59] has the form

$$\hat{V}_i = \sum_{l, m', m} |v'_{i lm'}(r) Y_{i lm'}(\mathbf{r}/r)| v_{i lm}(r) Y_{i lm}(\mathbf{r}/r).$$ \hspace{1cm} (S-13)

The sum in $W_{2}^{\text{bond}}$ contains the matrix elements $\langle v_{i lm}(r_1) Y_{i lm}(r_1/r_1) | C_{\text{Berry}}(r_1 - r_2, \mathbf{R}_0) | v_{i lm'}(r_2) Y_{i lm'}(r_2/r_2) \rangle \propto \delta_{l l'} \delta_{m m'}$ [since the matrix elements have the same form as Eq. (S-11), the derivation above is applicable]. Then different terms in the expansion (S-11) can be considered as independent perturbations.

Equation (S-10) also shows that $W_{s}(E_n, E_{\alpha_s})$ should decay as $(E_n - E_{\alpha_s})^{-2}$ in the limit $|E_n - E_{\alpha_s}| \to \infty$.

III. DERIVATION OF THE RECURRENCE RELATION (6) FOR NPC

Choosing the Lorentzian profile (4) for the continuous strength function $W_s(E_n, E_{\alpha_s}) \approx W_L(\alpha_{s-1} - E_n, \Gamma_s) \Delta E$, replacing the sum over $E_{\alpha_{s-1}}$ by the integral, and substituting $\left\langle |(\alpha_s | \alpha_{s-1})|^2 \right\rangle = (\alpha_{s} - E_{\alpha_{s-1}})^{\Delta E}$, we reduce (2) to the integral equation:

$$W_L(E_{\alpha_s} - E_n, \Gamma_s) = C_{SI} \int dE_{\alpha_{s-1}} W'(E_{\alpha_s}, E_{\alpha_{s-1}}) W_L(E_{\alpha_{s-1}} - E_n, \Gamma_{s-1}),$$ \hspace{1cm} (S-14)

where the factor $C_{SI}$ compensates for inaccuracy of the sum-integral replacement. The solution of $W'(E_{\alpha_s}, E_{\alpha_{s-1}}) = W_L(\alpha_{s-1} - E_{\alpha_{s-1}}, \Gamma')/C_{SI}$ with $\Gamma' = \Gamma_s - \Gamma_{s-1}$ is obtained using the Fourier transform.

Whenever it is applicable, the Fermi golden rule provides [31] [54] [57]

$$\Gamma_s \approx \frac{\pi}{\mathcal{N}_{MC} \Delta E} \sum_{\mathbf{n}, \mathbf{n}' \in \text{MC}} \left\langle \mathbf{n} | \sum_{s'=1}^{s} \hat{V}_{s'} \mathbf{n}' \right\rangle^2.$$
For independent perturbations, obeying the condition (3), the sum over \(n\) and \(n'\) above can be split as

\[
\sum_{n, n' \in \mc} \left| \langle n | \sum_{s'=1}^{s} \hat{V}_{s'} | n' \rangle \right|^2 = \sum_{n, n' \in \mc} \left| \langle n | \sum_{s'=1}^{s-1} \hat{V}_{s'} | n' \rangle \right|^2 + \sum_{n, n' \in \mc} \left| \langle n | \hat{V}_{s} | n' \rangle \right|^2.
\]

Then \(\Gamma_s = \Gamma_{s-1} + \Gamma'_s\), in the agreement with the general relation derived above.

Using Eq. (S-2), we can represent IPR as the average over states with a fixed energy difference

\[
\eta_s \equiv \sum_n \langle |\alpha_s |n\rangle|^2 = \sum_n \left\langle \langle |\alpha_s |n\rangle|^4 \right\rangle.
\]

According to Berry’s conjecture \[52\], non-integrable system eigenfunctions are Gaussian random functions. The Gaussian distribution of the \(|\alpha_s\rangle\) values was confirmed even for single-scatterer models \[22\, 27\] which have IPR \(\sim 0.5\) and, therefore, are rather far from the eigenstate thermalization regime. As a consequence, \(|n|\alpha_s\rangle = \int d^n\mathbf{r} \langle n|\mathbf{r}\rangle \langle \mathbf{r}|\alpha_s\rangle\) can be considered as random Gaussian variables. In T-invariant systems, \(\langle \mathbf{r}|\alpha_s\rangle\) and \(\langle \mathbf{n}|\alpha_s\rangle\) can always be chosen to be everywhere real. Therefore, \(|\mathbf{n}|\alpha_s\rangle\) are real random Gaussian variables. In PT-invariant systems, \(\langle \mathbf{r}|\alpha_s\rangle\) and \(\langle \mathbf{n}|\alpha_s\rangle\) can always be chosen to be everywhere real. Therefore, \(|\mathbf{n}|\alpha_s\rangle\) are real random Gaussian variables. In PT-invariant systems, \(\langle \mathbf{r}|\alpha_s\rangle\) and \(\langle \mathbf{n}|\alpha_s\rangle\) being a real random Gaussian variable, or \(\langle \mathbf{n}|\alpha_s\rangle\) being a real random Gaussian variable multiplied by \(\mathbf{i}\). If the system is not T-invariant nor PT-invariant \(|\mathbf{n}|\alpha_s\rangle\) should be considered as complex random Gaussian variables.

The fourth moment of a Gaussian variable is the second one squared and multiplied by 2 or 3 for complex or real variables, respectively (see \[31\]). Then averaged IPR \(\langle\rangle\) can be transformed as

\[
\eta_s = \left\{ \frac{2}{3} \right\} \sum_n \left\langle \langle |\alpha_s |n\rangle|^2 \right\rangle^2 \approx \left\{ \frac{2}{3} \right\} \int dE_n W^2_n |E_{\alpha_s} - E_n| \Delta E = \left\{ \frac{2}{3} \right\} \frac{\Delta E}{2\pi \Gamma_s} \]

This leads to the relation for NPC (6) with \(\nu = 2\pi \Gamma'/(2\Delta E)\) or \(\nu = 2\pi \Gamma'/(3\Delta E)\), respectively. For large \(s\) the chaotic properties of \(|\alpha_s\rangle\) weakly depend on \(s\) and if the shape of \(\hat{V}_s\) is independent of \(s\) (as for scatterers with the same strength), \(\nu\) will be approximately independent of \(s\).

### IV. HARMONIC WAVEGUIDE WITH ZERO-RANGE SCATTERERS ALONG ITS AXIS

Projecting the Schrödinger equation with the rank-\(s\) separable potential

\[
\left( \hat{H}_0 + \sum_{s'=1}^{s} V_{s'} |\mathcal{F}_{s'}\rangle \langle \mathcal{F}_{s'}| \right) |\alpha_s\rangle = E_{\alpha_s} |\alpha_s\rangle
\]

onto eigenstates \(|n\rangle\) of \(\hat{H}_0\), we get the expansion coefficients (7). Then we can express \(|\alpha_s\rangle\) in terms of the overlaps \(|\mathcal{F}_{s'}|\alpha_s\rangle\),

\[
|\alpha_s\rangle = \sum_n \sum_{s'=1}^{s} \frac{|n| \langle n|\mathcal{F}_{s'}\rangle}{E_{\alpha_s} - E_n} V_{s'} \langle \mathcal{F}_{s'}|\alpha_s\rangle. \tag{S-17}
\]

Finally, projection of this equation onto the formfactors \(|\mathcal{F}_{s'}\rangle\) leads to the set of linear equations (8) for the overlaps.

Eigenfunctions and eigenenergies of \(\hat{H}_0\) (9) are expressed as

\[
\langle \rho, z|nl\rangle = L^{-1/2} e^{2\pi i l z/L} \langle \rho|n\rangle, \quad E_{nl} = \hbar \omega (2n + 1) + \hbar^2 (2\pi l/L - A)^2/(2m) \tag{S-18}
\]

with \(-\infty < l < \infty\) for PBC and

\[
\langle \rho, z|nl\rangle = (2/L)^{1/2} \langle \rho|n\rangle \sin \pi l z/L, \quad E_{nl} = \hbar \omega (2n + 1) + (\pi \hbar l/L)^2/(2m) \tag{S-19}
\]

with \(1 \leq l < \infty\) for the hard-wall box. Here

\[
\langle \rho|n\rangle = \frac{1}{\sqrt{\pi a_\perp}} L_n^{(0)} \left(\rho/a_\perp\right)^2 \exp(-\rho/a_\perp^2/2) \tag{S-20}
\]
is the radial wavefunction, $a_\perp = (\hbar/m\omega_\perp)^{1/2}$ is the transverse oscillator range, and $P^{(0)}_n$ are the Laguerre polynomials (see [64]). Now $\textbf{n} = (n, l)$ and, like in [26], the sum over $l$ in (8) with the eigenfunctions and eigenenergies (S-18) can be calculated analytically using the formula

$$
\sum_{l=\infty}^{\infty} \frac{\exp(\text{i} l \pi \zeta)}{l + a} = \frac{\pi}{\sin \pi a} \exp(-2\pi a(\zeta - \lfloor \zeta \rfloor - 1/2))
$$

(S-21)

(see Eq. (5.4.3.4) in [66]). Then the system (8) attains the form

$$
\sum_{s''=1}^{s} S_{s',s''}(\varepsilon) \langle F_{s''}|\alpha_s \rangle = 0
$$

(S-22)

with $S_{s',s''}(\varepsilon) = \langle F_{s'}|S(z - z_{s''}, \varepsilon)|V_{s''}/V_0 - \delta_{s's''} \rangle$. Here $S(z, \varepsilon)$ is the Green function of the integrable system on the waveguide axis. For $z \geq 0$ it is expressed as

$$
S(z, \varepsilon) = \sqrt{\lambda} \sum_{n=0}^{\infty} \frac{1}{2p_n} e^{2\text{i}p_n z/L} \left[ e^{2\text{i}p_n z/L} (\cot(\pi l_0 + p_n) - i) - e^{-2\text{i}p_n z/L} (\cot(\pi l_0 - p_n) - i) \right],
$$

(S-23)

where $\varepsilon = mL^2(E - \hbar \omega_\perp)/(2\hbar^2)$ and $p_n = \sqrt{\varepsilon - \lambda n}$. For $z < 0$ we have $S(-z, \varepsilon) = S^*(z, \varepsilon)$. The formfactors $\langle \mathbf{r}|F_{s'} \rangle = \delta_{\text{reg}}(\mathbf{r} - \mathbf{R}_{s'})$ are expressed in terms of the Fermi-Huang pseudopotential $\delta_{\text{reg}}(\mathbf{r})f(\mathbf{r}) = \theta(\mathbf{r})|f(\mathbf{r})|_{r=0}$ which extracts the regular part of the function $f(\mathbf{r})$ at $r \to 0$. Then for $s' \neq s''$ we have $S_{s',s''}(\varepsilon) = S(z_{s'} - z_{s''}, \varepsilon)$ since $S(z_{s'} - z_{s''}, \varepsilon)$ is regular in this case. However, $S(z, \varepsilon)$ has a pole at $z \to 0$. The asymptotic of the irregular part at $z \to 0$

$$
-\sqrt{\lambda} \sum_{n=|\varepsilon/\lambda|+1}^{\infty} \frac{e^{-2|p_n| z/L}}{|p_n|} \sim -\frac{L}{\sqrt{\lambda \varepsilon} - \lambda \left( \frac{1}{2} \left[ \frac{\varepsilon}{\lambda} \right] + 1 - \frac{\varepsilon}{\lambda} \right)}
$$

is expressed, as in [67] in terms of the Hurwitz zeta function (see [64]). The term $\sim 1/z$ is eliminated by the Fermi-Huang pseudopotential. Then we can represent the diagonal matrix elements as

$$
S_{s',s''}(\varepsilon) = \frac{V_{s'}}{V_0} \left[ \sqrt{\lambda} \sum_{n=0}^{\infty} \left( \frac{\sin 2p_n}{p_n(\cos 2\pi l_0 - 2p_n \lambda)} + \frac{\theta(n - \varepsilon/\lambda)}{|p_n|} \right) - \zeta \left( \frac{1}{2} \left[ \frac{\varepsilon}{\lambda} \right] + 1 - \frac{\varepsilon}{\lambda} \right) \right] - 1.
$$

In the T-invariant model, $A = l_0 = 0$ and the function (S-23) is real.

The case of a hard-wall box is treated in a similar way. For $s' \geq s''$ we have

$$
S_{s',s''}(\varepsilon) = -\frac{V_{s'}}{V_0} \left[ \sqrt{\lambda} \sum_{n=0}^{\infty} \left( \frac{\sin 2p_n}{p_n(1 - z_{s''}/L)} \frac{\sin 2p_n z_{s''}/L}{p_n \sin 2p_n} - \frac{\theta(n - \varepsilon/\lambda)}{|p_n|} \delta_{s',s''} \right) + \zeta \left( \frac{1}{2} \left[ \frac{\varepsilon}{\lambda} \right] + 1 - \frac{\varepsilon}{\lambda} \right) \delta_{s',s''} \right] - \delta_{s',s''}
$$

and $S_{s'',s'}(\varepsilon) = S_{s',s''}(\varepsilon)$.

The homogeneous system of linear equations (S-22) has non-trivial solutions if the determinant of its matrix is equal to zero. Then the roots $\varepsilon_n$ of the determinant, $\det(S_{s',s''}(\varepsilon_n)) = 0$, give us the non-integrable system eigenenergies $E_\alpha = 2\hbar^2 \varepsilon_\alpha/(mL^2) + \hbar \omega_\perp$ and solutions of the system (S-22), being substituted to Eq. (S-17), provide the eigenfunctions.