DYNAMICS IN A PARABOLIC-ELLIP틱 CHEMOTAXIS SYSTEM WITH GROWTH SOURCE AND NONLINEAR SECRETION

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ABSTRACT. In this work, we are concerned with a class of parabolic-elliptic chemotaxis systems with the prototype given by

\[
\begin{align*}
    u_t &= \nabla \cdot (\nabla u - \chi u \nabla v) + au - bu^\theta, \\
    0 &= \Delta v - v + u^\kappa,
\end{align*}
\]

with nonnegative initial condition for \( u \) and homogeneous Neumann boundary conditions in a smooth bounded domain \( \Omega \subset \mathbb{R}^n \) (\( n \geq 2 \)), where \( \chi, b, \kappa > 0 \), \( a \in \mathbb{R} \) and \( \theta > 1 \).

First, using different ideas from [9, 11], we re-obtain the boundedness and global existence for the corresponding initial-boundary value problem under, either

\[
\kappa + 1 < \max\{\theta, 1 + \frac{2}{n}\}
\]

or

\[
\theta = \kappa + 1, \quad b \geq \frac{(\kappa n - 2)}{\kappa n} \chi.
\]

Next, carrying out bifurcation from "old multiplicity", we show that the corresponding stationary system exhibits pattern formation for an unbounded range of chemosensitivity \( \chi \) and the emerging patterns converge weakly in \( L^\theta(\Omega) \) to some constants as \( \chi \to \infty \). This provides more details and also fills up a gap left in Kuto et al. [13] for the particular case that \( \theta = 2 \) and \( \kappa = 1 \). Finally, for \( \theta = \kappa + 1 \), the global stabilities of the equilibria \( ((a/b)^{2}, a/b) \) and \( (0, 0) \) are comprehensively studied and explicit convergence rates are computed out, which exhibits chemotaxis effects and logistic damping on long time dynamics of solutions. These stabilization results indicate that no pattern formation arises for small \( \chi \) or large damping rate \( b \); on the other hand, they cover and extend He and Zheng’s [6, Theorems 1 and 2] for logistic source and linear secretion \( (\theta = 2 \text{ and } \kappa = 1) \) (where convergence rate estimates were shown) to generalized logistic source and nonlinear secretion.

1. Introduction. Following the first classical chemotaxis model proposed by Keller and Segel in [12] to describe the aggregation phase of cellular slime mold, the mathematical modeling and analysis of chemotaxis have been rapidly developed in various

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ways (see the review articles \[2, 7, 8\] for instance). Due to its important applications in biological and medical sciences, chemotaxis research has become one of the hottest topics in applied mathematics nowadays and tremendous theoretical progress has been made in the past few decades. This work is devoted to the global dynamics including boundedness, pattern formation and long time behavior for the following parabolic-elliptic chemotaxis system with nonlinear production of signal and growth source:

\[
\begin{align*}
  u_t &= \nabla \cdot (\nabla u - \chi u \nabla v) + f(u), \quad x \in \Omega, t > 0, \\
  0 &= \Delta v - v + g(u), \quad x \in \Omega, t > 0, \\
  \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, \quad x \in \partial \Omega, t > 0, \\
  u(x, 0) &= u_0(x), \quad x \in \Omega,
\end{align*}
\]

(1.1)

where \( \Omega \subset \mathbb{R}^n (n \geq 1) \) is a bounded domain with smooth boundary \( \partial \Omega \) and \( \nu \) denotes the outward normal vector of \( \partial \Omega \), \( u(x, t) \) and \( v(x, t) \) denote the cell density and chemical concentration, respectively. The chemotactic sensitivity coefficient \( \chi (> 0) \) measures the strength of chemotaxis and the kinetic term \( f(u) \) describes cell proliferation and death (simply referred to as growth source) and \( g(u) \) accounts for the chemical secretion by cells. The parabolic-elliptic chemotaxis system (1.1) could be physically justified when the chemicals diffuse much faster than cells do; indeed, this simplified system was first introduced for the case \( f(u) = 0 \) and \( g(u) = u \) (minimal model) in [10] and thereafter various variants of (1.1) have been studied by many other authors (e.g. see [5, 9, 10, 21, 22, 30] and the references therein).

Based on the commonly used choices for \( f \) and \( g \) in the literature [5, 9, 21, 24, 25], throughout this paper, we assume that \( f \) is smooth in \([0, \infty)\) satisfying \( f(0) \geq 0 \) and there are \( a \geq 0, b > 0 \) and \( \theta > 1 \) such that

\[
f(u) \leq a - bu^\theta \quad \text{for all} \quad u \geq 0
\]

(1.2)

and, the secretion function \( g \) is continuous in \([0, \infty)\) and there are \( \beta > 0 \) and \( \kappa > 0 \) such that

\[
g(u) \leq \beta u^\kappa \quad \text{for all} \quad u \geq 0.
\]

(1.3)

This project originates from our two years ago’s preprint [23], which aims at extending the fundamental boundedness of Tello and Winkler [21] for logistic source and linear secretion to more general growth source and nonlinear secretion term. During the last two years, new progresses on variants and extensions of (1.1) have been obtained in [5, 9, 11]. In the starting work [21], for linear secretion \( g(u) = \beta u \) and logistic source \( \theta = 2 \), Neumann heat semigroup type arguments are used to obtain the global boundedness under

\[
b > \frac{(n-2)}{n} \chi.
\]

(1.4)

Moreover, for \( f(u) = bu(1 - u) \) with \( b > 2\chi \), the global bounded solution \((u, v)\) stabilizes according to

\[
\lim_{t \to \infty} \left( \|u(\cdot, t) - 1\|_{L^\infty(\Omega)} + \|v(\cdot, t) - 1\|_{L^\infty(\Omega)} \right) = 0.
\]

(1.5)

This fundamental global boundedness has been extended extensively in a sequel of works, cf. [3, 5, 9, 22, 30], for a system with nonlinear diffusion, nonlinear chemosensitivity, generalized logistic source or nonlinear production. Here, in this work, beyond boundedness, we wish to provide a full picture about other dynamical behaviors of solutions on the interactions between nonlinear cross-diffusion, generalized logistic source and signal production for (1.1) such as the ability of pattern
formations, the asymptotical behavior for large $\chi$ and the large time behavior of bounded solutions. For these purposes, we will stick to the parabolic-elliptic chemotaxis system (1.1) and, we will not go into further generality as done in [5, 9] instead. Therefore, we only mention the following direction of extensions for comparison: for some $\theta > 1$, $\kappa \geq 1$, $a \in \mathbb{R}$, $b > 0$, $\chi > 0$,

$$
\begin{align*}
\begin{cases}
u_t = \nabla \cdot (\nabla u - \chi u \nabla v) + au - bu^\theta, & x \in \Omega, t > 0, \\
0 = \Delta v - v + u^\kappa, & x \in \Omega, t > 0,
\end{cases}
\end{align*}
$$

its boundedness and global existence for (1.6) are guaranteed in the non-borderline cases that

$$
\begin{align*}
&\kappa = 1, \text{ either } 2 < \max\{\theta, 1 + \frac{2}{n}\} \text{ or } b > b_*= \frac{(n-2)}{n}\chi \text{ if } \theta = 2, \text{ see [30]}; \\
&\kappa \geq 1, \text{ either } \kappa + 1 < \theta \text{ or } b > b_* = \frac{n\kappa - 2}{n\kappa}\chi \text{ if } \theta = \kappa + 1, \text{ see [5].}
\end{align*}
$$

Under further conditions like $a = b$, $\theta \geq \kappa + 1$ and $b > 2\chi$, the latter work extends the comparison argument in [21] to show that the constant equilibrium (1,1) is globally stable and obeys (1.5).

The boundedness and global existence of Tello-Winkler were first extended for (1.6) to the borderline case by Kang and Stevens [11] under

$$
\chi = 1, \quad \theta = 2, \quad \kappa = 1, \quad n \geq 3, \quad b \geq b_* := \frac{(n-2)}{n}.
$$

In the same year as the work [11], Hu and Tao [9] extended the boundedness and global existence for (1.6) in [5] to the borderline case that

$$
\kappa \geq 1, \quad \theta = \kappa + 1, \quad n \geq 3, \quad b = b_* := \frac{(n\kappa - 2)}{n\kappa}\chi.
$$

Here, we notice that (1.8) and (1.9) impose restrictions like $\kappa \geq 1$ and $n \geq 3$, and that the methods in [11, 9] are relatively implicit or indirect. Here, for completeness, we wish to employ a simpler argument to show the borderline boundedness of solutions to (1.1). We now sketch our main results and give some comments for the motivation of our study:

(C1) **Bounded classical solutions.** In Section 4, by fully making use of the $L^{\kappa n/2+\epsilon}$-boundedness criterion obtained in Lemma 2.3, we establish the boundedness and global existence of classical solutions to (1.1) with $f$ and $g$ satisfying (1.2)-(1.3) if either

$$
\kappa + 1 < \max\{\theta, 1 + \frac{2}{n}\}
$$

or

$$
\theta = \kappa + 1, \quad b \geq \frac{(\kappa n - 2)}{\kappa n}\chi.
$$

The case of (1.10) is quite simple and its proof is short and can be readily adapted from existing approaches in literature, cf. [3, 5, 22, 30]. The idea used to prove boundedness under (1.11) is first to prove the $L^{\kappa n/2}$-boundedness of $u$ and then to use G-N interpolation inequality to prove its $L^{\kappa n/2+\epsilon}$-boundedness, which is different from (more direct) the existing methods in [9, 11]. For completeness and consistency, we include it to make the flow of the proof of (1.11) more smooth. The precise results are provided in Theorems 3.1 and 3.2, Corollary 1 and Remark 2. Here, we especially note that the logistic damping effect is always kept in force even when $\kappa n - 2 = 0$, that is, the premise $b > 0$ is always required. For instance, when $\kappa n - 2 = 0,$
our result ensures that the following chemotaxis-growth system with Neumann boundary condition

\begin{align}
  u_t &= \nabla \cdot (\nabla u - \chi u \nabla v) + au - bu^{1+\frac{2}{n}} ,
  
  0 &= \Delta v - v + u^{\frac{2}{n}} \\
  &\text{in } \Omega \subset \mathbb{R}^n
\end{align} (1.12)

has no blowup solutions for any \( n \geq 2, \chi > 0, a \in \mathbb{R} \) and \( b > 0 \); while, it is well-known that (1.12) does possess blowup solutions for \( a = b = 0, n = 2 \), cf. [8, 10].

(C2) **Pattern formations and their limiting behavior for large chemosensitivity.** In Section 5, we first use energy method to study the regularity and then perform Leray-Schauder index formula and bifurcation from "old multiplicity" [17, 18] to show the existence of non-constant steady states of (1.1) for an unbounded range of \( \chi \), which not only covers the results of Tello and Winkler [21] with logistic source \( (\theta = 2) \) and linear secretion \( (\kappa = 1) \), but also provide more verifiable conditions for the existence of pattern formations (Theorem 4.3). Furthermore, we investigate the asymptotic behavior of stationary solutions as \( \chi \to \infty \) in certain parameter regimes, which demonstrates that the emerging patterns will converge weakly in \( L^\theta(\Omega) \) to some constants as \( \chi \to \infty \), cf. Theorem 4.4. This part provides more details and clarifies a vague point made in Kuto et al [13] for the special cases \( f(u) = au - bu^2 \) and \( g(u) = \beta u \), see Remark 4.

(C3) **Large time behavior of bounded-in-time solutions.** In Section 6, instead of using the comparison arguments [21, 5], we apply the energy functional method from [1, 6] to undertake a comprehensive analysis for the global asymptotic stabilities of the parabolic-elliptic system (1.6) with \( \theta = \kappa + 1 \). Under explicit conditions, the global stabilities of the equilibria \( (\frac{a}{b})^{\frac{1}{\kappa}}, \frac{a}{b} \) and \( (0, 0) \) are obtained, which implies no pattern formations could arise for small chemosensitivity \( \chi \) or large damping rate \( b \). Moreover, we calculate their respective exponential and algebraic convergence rates explicitly in terms of the model parameters, which exhibits chemotaxis effects and logistic damping on long time dynamics of solutions, cf. Theorem 5.1. For logistic source \( (\theta = 2) \) and linear secretion \( (\kappa = 1) \), convergence rate estimates were derived but not explicitly stated in [6, Theorems 1 and 2]. In a word, our stability results extend the result of [6, Theorems 1 and 2] for logistic source and linear secretion (where convergence rate estimates were shown) to generalized logistic source and nonlinear nonlinear secretion \( (\theta > 1 \) and \( \kappa > 0 \) and, refine the uniform convergence in [21] and [5, \( m = 1 \)] to exponential convergence under different set of conditions.

Finally, we mention that various variants of (1.1) or its fully parabolic version have been investigated to understand the interplay of (nonlinear) diffusion, the chemotactic sensitivity and the cell kinetic in enforcing boundedness and stabilization toward constant equilibria, as well as more unexpected behavior witnessing a certain strength of chemotactic destabilization etc (e.g. see [20, 24, 25, 27, 28] and the references therein).

2. **Preliminaries and a boundedness criterion for the chemotaxis system.**

For convenience, we quote the well-known Gagliardo-Nirenberg interpolation inequality below and state the local well-posedness of the chemotaxis-growth system (1.1).
Lemma 2.1 (Gagliardo-Nirenberg interpolation inequality [4]). Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \) with a smooth boundary and let \( 1 \leq r \leq \infty \) and \( 0 < q < \infty \).

(i) Version 1: For any number \( \delta \in (0,1) \), set

\[
\frac{1}{p} = \delta \left( \frac{1}{r} - \frac{1}{n} \right) + (1 - \delta) \frac{1}{q}.
\]

Then

\[
\|w\|_{L^p(\Omega)} \leq C \|w\|_{W^{1,r}(\Omega)}^{\frac{\delta}{r}} \|w\|_{L^q(\Omega)}^{\frac{1-\delta}{r}}, \quad \forall \ w \in W^{1,r}(\Omega) \cap L^q(\Omega). \tag{2.1}
\]

(ii) Version 2: For any number \( \delta \in (0,1) \), set

\[
\frac{1}{p} = \delta \left( \frac{1}{r} - \frac{1}{n} \right) + (1 - \delta) \frac{1}{q}.
\]

Then, for any \( s > 0 \),

\[
\|w\|_{L^p(\Omega)} \leq C \|\nabla w\|_{L^r(\Omega)}^{\frac{\delta}{r}} \|w\|_{L^q(\Omega)}^{1-\delta} + C \|w\|_{L^q(\Omega)}^r, \quad \forall \ w \in W^{1,r}(\Omega) \cap L^q(\Omega). \tag{2.2}
\]

The constant \( C \) depends only on \( \Omega, q, r, \delta, n, s \).

The local-in-time existence of classical solutions to the chemotaxis-growth system (1.1) is quite standard; see similar discussions in [3, 21, 22, 30].

Lemma 2.2. Let \( \Omega \subset \mathbb{R}^n \) be a bounded and smooth domain, the nonnegative initial data \((u_0, v_0) \in (C_0(\Omega), W^{1,q}(\Omega))\) for some \( q > n \), the growth source \( f \in W^{1,\infty}_{loc}(0, \infty)) \) with \( f(0) \geq 0 \) and the secretion function \( g \in C^0([0, \infty)) \). Then there is a maximal existence time \( T_m \in (0, \infty) \) and a unique pair of nonnegative functions \((u, v) \in C(\Omega \times [0, T_m)) \cap C^2(\Omega \times (0, T_m)) \) solving (1.1) classically in \( \Omega \times [0, T_m) \). In particular, if \( T_m < \infty \), then

\[
\|u(\cdot, t)\|_{L^\infty(\Omega)} \to \infty \quad \text{as} \ t \searrow T_m. \tag{2.3}
\]

Moreover, there exists a constant \( M_0 > 0 \) such that \( \|u(t)\|_{L^1} \leq M_0 \). Here and below, we will adopt the widely used short notations like \( \|u(t)\|_{L^r} = \|u(\cdot, t)\|_{L^r} = \|u(\cdot, t)\|_{L^r(\Omega)} = (\int_{\Omega} |u(x, t)|^r dx)^{\frac{1}{r}} \).

Proof. As mentioned above, the assertions concerning the local-in-time existence of classical solutions to the initial-boundary value problem (1.1) and the criterion (2.3) are well-studied. Since \( f(0) \geq 0 \), the maximum principle asserts that both \( u \) and \( v \) are nonnegative, as shown in [21, 28]. Integrating the \( u \)-equation in (1.1) and using (1.2), one can easily deduce that

\[
\frac{d}{dt} \int_{\Omega} u = \int_{\Omega} f(u) - \int_{\Omega} (a_0 - bu^q) \leq -\int_{\Omega} u + c|\Omega|,
\]

where \( c = \max\{a - bu^q + u : u \geq 0\} < \infty \) thanks to the fact that \( \theta > 1 \). Solving this standard Gronwall’s inequality shows that \( L^1 \)-norm of \( u \) is uniformly bounded. \( \square \)

For the chemotaxis model without growth, we know that the total cell mass \( \|u(t)\|_{L^1} \) is conservative. This is no long true for the chemotaxis model with growth, but \( \|u(t)\|_{L^1} \) is still uniformly bounded (cf. Lemma 2.2). However, the uniform boundedness of \( \|u(t)\|_{L^1} \) is not sufficient to prevent the blow-up of solutions in finite/infinite time (see [8, 26]). By [2, 19, 28], it is quite known that the hard task of proving the \( (L^\infty, W^{1,\infty}) \)-boundedness of \((u, v)\) can be reduced to proving only the \( L^p \)-boundedness of the \( u \)-component for suitably finite \( p \). Since the existing results (cf. [2, 19, 28]) don’t give us the precise information that we need in the
Lemma 2.3 (Criterion for boundedness). Assume that the hypotheses (1.2) and (1.3) hold, and \( \Omega \subset \mathbb{R}^n (n \geq 2) \) is a bounded domain with smooth boundary. Let \((u_0, v_0)\) be as in Lemma 2.2 and \((u, v)\) be the unique maximal solution of (1.1) defined on \([0, T_m]\). If there exist \( \varepsilon \in (0, \frac{\kappa n}{2}) \) and \( M > 0 \) such that
\[
\|u(\cdot,t)\|_{L^{\frac{n}{n+\varepsilon}}(\Omega)} \leq M, \quad \forall t \in (0, T_m),
\]
then \((u(\cdot,t), v(\cdot,t))\) is uniformly bounded in \( L^{\infty}(\Omega) \times W^{1,\infty}(\Omega) \) for all \( t \in (0, T_m) \), and so \( T_m = \infty \); that is, the solution \((u, v)\) exists globally with uniform-in-time bound.

Proof. For any \( p \geq 2 \), multiplying the \( u \)-equation in (1.1) by \( u^{p-1} \) and integrating over \( \Omega \) by parts, using Young’s inequality and the growth condition (1.2), we conclude that
\[
\frac{1}{p} \frac{d}{dt} \int_{\Omega} u^p = - \int_{\Omega} \nabla u \nabla (u^{p-1}) + \chi \int_{\Omega} u \nabla (u^{p-1}) \nabla v + \int_{\Omega} f(u) u^{p-1} 
\leq - \frac{4(p-1)}{p^2} \int_{\Omega} |\nabla u|^2 + \frac{2(p-1)\chi}{p} \int_{\Omega} u^2 |\nabla (u^{\frac{p}{2}})| |\nabla v| + \int_{\Omega} f(u) u^{p-1} 
\leq - \frac{2(p-1)}{p^2} \int_{\Omega} |\nabla (u^{\frac{p}{2}})|^2 + \frac{(p-1)\chi^2}{2} \int_{\Omega} u^p |\nabla v|^2 + \int_{\Omega} u^{p-1} (a - bw^\theta),
\]
which, upon the substitution \( w = u^{\frac{p}{2}} \), reads as
\[
\frac{1}{p} \frac{d}{dt} \int_{\Omega} w^2 \leq - \frac{2(p-1)}{p^2} \int_{\Omega} |\nabla w|^2 
+ \frac{(p-1)\chi^2}{2} \int_{\Omega} w^2 |\nabla v|^2 + \int_{\Omega} (aw^{2(p-1)} - bw^{2(p+\theta)}) \tag{2.4},
\]
where and hereafter \( \theta = \theta - 1 > 0 \).

Below, we shall apply the Gagliardo-Nirenberg interpolation inequality (2.1) to control the second integral on the right-hand side of (2.4).

Since \( \varepsilon \in (0, \frac{\kappa n}{2}) \) and \( n \geq 2 \), one can easily see that
\[
\frac{2\kappa n}{n + 2} < r := \frac{\kappa}{2} n + \varepsilon < \kappa n. \tag{2.5}
\]
Now, by assumption \( \|u(t)\|_{L^r} \) is bounded, it follows that \( \|g(u(t))\|_{L^{r/\kappa}} \) is bounded due to the fact that \( g(u) \leq \beta u^\alpha \). Then a simple application of the elliptic \( W^{2,q} \) estimate to the \( v \)-equation in (1.1) shows that \( \|v(t)\|_{W^{2,r/\kappa}} \) is bounded. This in turn entails by Sobolev embedding that \( \|v(t)\|_{W^{1,q'}} \) is bounded with
\[
q' = \frac{nr}{\kappa n - r} > 2
\]
by the choice of \( r \) in (2.5). Then we obtain from Hölder inequality that
\[
\|w^2 |\nabla v|^2\|_{L^r} \leq \|w^2\|_{L^q} \|\nabla v|^2\|_{L^{q'}} \leq \|w\|_{L^2}^2 \|\nabla v\|_{L^{q'}}^2 \leq C \|w\|_{L^2}^2 \tag{2.6}
\]
with
\[
q = \frac{q'}{2} = \frac{nr}{(n+2)r - 2\kappa n} > 1.
\]
A use of the Gagliardo-Nirenberg inequality (2.1) to (2.6) gives
\[
\|u^2\|L^1 \leq C\|w\|_{L^{2p}}^2 \leq C\|w\|_{W^{1,2}}^{2\delta} \|w\|^2_2 \leq C^{(1-\delta)}\|w\|^2_2 \tag{2.7}
\]
with
\[
\delta = \frac{np}{2(p+\vartheta)} - \frac{n}{2p} = \frac{n[p(q-1) - \vartheta]}{2q[2p - (n-2)\vartheta]} = \frac{2(\kappa n - r)p - [(n + 2)r - 2\kappa n]\vartheta}{[2p - (n-2)\vartheta]r}. \tag{2.8}
\]
Notice that \( r > \kappa n/2 \), a simple calculation from (2.8) shows that \( \delta \in (0, 1) \) as long as
\[
p > \max\left\{ \frac{(n-2)\vartheta}{2}, \frac{[(n + 2)r - 2\kappa n]\vartheta}{2(\kappa n - r)} \right\}. \tag{2.9}
\]
Hence, for any \( p \geq 2 \) fulfilling (4.22), the estimate (2.7) holds. Then applying Young’s inequality, we conclude from (2.7) that
\[
\|u^2\|L^1 \leq C\|w\|_{W^{1,2}}^{2\delta} \|w\|^2_2 \leq C\|w\|^2_2(1-\delta) + C_\epsilon \|w\|^2_2 + C(\epsilon_1, \epsilon_2) \tag{2.10}
\]
for any \( \epsilon_1, \epsilon_2 > 0 \) and some constant \( C \) depending on \( \epsilon_1, \epsilon_2 \). By Young’s inequality, one has
\[
\|w\|^2_{W^{1,2}} = \|w\|^2_{L^2} + \|\nabla w\|^2_{L^2} \leq \|w\|^2_{L^2} + \|\nabla w\|^2_{L^2} + C(\|w\|_2^2 + C(|\Omega|)). \tag{2.11}
\]
Then substituting (2.11) into (2.10), we have, for any \( \epsilon_1, \epsilon_2 > 0 \),
\[
\|u^2\|L^1 \leq (\epsilon_1 + \epsilon_2)\|w\|^2_{L^2} + \epsilon_2\|\nabla w\|^2_{L^2} + C(\epsilon_1, \epsilon_2, |\Omega|). \tag{2.12}
\]
Thus, for \( p \) satisfying (4.22), by taking \( \epsilon_1, \epsilon_2 > 0 \) in (2.12) such that
\[
(2.12)
\]
we deduce from (2.4) and (2.10) that
\[
\frac{1}{p} \int_{\Omega} w^2 \leq \int_{\Omega} \left( \alpha w^{2(p-1)\vartheta} + \frac{b}{2} w^{2(p+\vartheta)} \right) + C(p),
\]
which, together with the fact
\[
\max\left\{ \alpha w^{2(p-1)\vartheta} + \frac{b}{2} w^{2(p+\vartheta)} + w^2 : w \geq 0 \right\} < \infty,
\]
immediately gives that
\[
\frac{1}{p} \int_{\Omega} w^2 \leq - \int_{\Omega} w^2 + C(p)
\]
for some possibly large constant \( C \). The substitution of \( w = u^2 \) then yields
\[
\frac{1}{p} \int_{\Omega} u^p \leq - \int_{\Omega} u^p + C(p, r).
\]
Solving this Gronwall’s inequality, we deduce that \( \|u(t)\|_{L^p} \) is bounded for \( p \) satisfying (4.22) and our stipulation \( p \geq 2 \).

Now, the point-wise elliptic \( W^{2,q} \)-estimate applied to the \( v \)-equation in (1.1) shows that \( \|v(t)\|_{W^{2,p/\kappa}} \) is bounded, which is embedded in \( C^1(\overline{\Omega}) \) by choosing \( p \) such
that $p/k > n$. This shows that $\|v(t)\|_{W^{1, \infty}}$ are uniformly bounded with respect to $t \in (0, T_m)$. As such, we can perform the well-known Moser iteration technique to show that the $\|u(t)\|_{L^\infty}$ is bounded uniformly in time $t$; see details in [28, p. 4290-4292]. Accordingly, the extension criterion (2.3) implies $T_m = \infty$ and hence global existence follows. Moreover, $\|u(t)\|_{L^\infty}$ and $\|v(t)\|_{W^{1, \infty}}$ are uniformly bounded with respect to $t \in (0, \infty)$.

\textbf{Remark 1.} The boundedness criterion obtained in Lemma 2.3 holds also for the fully parabolic version of (1.1).

$$
\begin{aligned}
\left\{
\begin{array}{ll}
  u_t = \nabla \cdot (\nabla u - \chi u \nabla v) + f(u), & x \in \Omega, t > 0, \\
  v_t = \Delta v - v + g(u), & x \in \Omega, t > 0, \\
  \frac{\partial u}{\partial \nu} = 0, & x \in \partial \Omega, t > 0, \\
  u(x, 0) = u_0(x), v(x, 0) = v_0(x), & x \in \Omega.
\end{array}
\right.
\end{aligned}
$$

(2.13)

In such case, instead of using elliptic estimates, one uses the method of heat Neumann semigroup (see, for instance, [2, Lemma 3.2] and [28, Lemma 3.5]). Precisely, corresponding to (2.13), we have the following "reciprocal" lemma:

\textbf{Lemma 2.4.} Let $(u, v)$ be a maximal solution of (2.13) defined on its maximal existence interval $[0, T_m)$. If there exist $r \in [1, \kappa n)$ and $k_1 > 0$ such that

$$
\|u(\cdot, t)\|_{L^r(\Omega)} \leq k_1, \quad \forall t \in (0, T_m),
$$

then

$$
\|v(\cdot, t)\|_{W^{1, \infty}(\Omega)} \leq C(q, r, v_0)(1 + k_1), \quad \forall t \in (0, T_m)
$$

holds for all

$$
1 < q < \frac{nr}{\kappa n - r} = \frac{1}{\frac{r}{n} - 1}.
$$

Then, with Remark 1 at hand and noticing that $\|u(t)\|_{L^1}$ is uniformly bounded (cf. Lemma 2.2), uniform boundedness and global existence of solutions to (2.13) with sub-linear secretion follow directly from Lemma 2.3.

\textbf{Corollary 1.} Let the assumptions (1.2) and (1.3) with $\kappa < \frac{2}{n}$ hold. Then the solution $(u(\cdot, t), v(\cdot, t))$ of (2.13) or (1.1) is uniformly bounded in $L^\infty(\Omega) \times W^{1, \infty}(\Omega)$ for all $t \in (0, \infty)$.

\textbf{Remark 2.} Even in the absence of growth source, the assumption $g(u) \leq \beta u^\kappa$ with $\kappa < \frac{2}{n}$ induces that $(u(\cdot, t), v(\cdot, t))$ is bounded in $L^\infty(\Omega) \times W^{1/2}(\Omega)$ for some $q > n$, cf. [2, 15]. The point here is that the uniform spatial $L^1$-boundedness of $u$ is sufficient to prevent blowup of solutions. This is not usually the case as noted in the beginning of this section.

It is known from [26] that, even for a simpler chemotaxis-growth model than (1.1) with $\kappa = 1$, blow-up is still possible despite logistic dampening. Hence, there is a need to give an equivalent characterization of Lemma 2.3 in terms of blowup solutions.

\textbf{Corollary 2.} Suppose that $(u, v)$ is a solution of (1.1) which blows up at time $t = T_m$. Then $u$ and $v$ blow up simultaneously at $t = T_m$ in the following manner:

$$
\limsup_{t \to T_m^-} |u(\cdot, t)|_{L^p(\Omega)} = \infty \quad \text{for all} \quad p > \kappa n/2 \quad \text{and} \quad \limsup_{t \to T_m^-} |v(\cdot, t)|_{W^{1, \infty}(\Omega)} = \infty.
$$

This means, for any blowup solution $(u, v)$ of (1.1), $u$ blows up not only in $L^\infty$-topology but also in $L^p$-topology for any $p > \kappa n/2$, and $v$ blows up in $W^{1, \infty}$-topology.
Proof. If \( \|v(t)\|_{W^{1,\infty}} \) is bounded, then the crucial inequality (2.6) is valid. Then one can readily see from the proof of the Lemma that \( \|u(\cdot,t)\|_{L^p} \) is bounded. \( \square \)

3. The \( L^{kn/2+\varepsilon} \)-boundedness of \( u \) and global existence. In this section, we use the criterion established in Lemma 2.3 to study the boundeness and global existence for (1.1). This idea is different from \([9, 11]\). To make our presentation self-contained, we would like to provide necessary details to make the flow of the proof of (1.11) more smooth. Also, we would like to rewrite (1.1) here for purpose of reference.

\[
\begin{aligned}
&\left\{ \begin{array}{l}
\partial_t u = \nabla \cdot (\nabla u - \chi u \nabla v) + f(u), \quad x \in \Omega, t > 0, \\
0 = \Delta v - v + g(u), \quad x \in \Omega, t > 0, \\
\frac{\partial v}{\partial \nu} = 0, \quad x \in \partial \Omega, t > 0, \\
u(x,0) = u_0(x) \geq 0, \quad x \in \Omega.
\end{array} \right.
\tag{3.1}
\end{aligned}
\]

The growth source \( f \in W^{1,\infty}_{\text{loc}}([0,\infty)) \) satisfies \( f(0) \geq 0 \) and

\[
f(u) \leq a - bu^\theta, \quad \forall \ u \geq 0
\tag{3.2}
\]

for some \( a \geq 0, \ b > 0 \) and \( \theta > 1 \), and the production term \( g \in C^1((0,\infty)) \) and satisfies

\[
g(u) \leq \beta u^\kappa, \quad \forall \ u \geq 0
\tag{3.3}
\]

for some \( \beta > 0 \) and \( \kappa > 0 \).

If \( \kappa < \frac{2}{n} \), then the boundedness for (3.1) is ensured by Corollary 1 and Remark 2. Therefore, we will consider the case \( \kappa \geq \frac{2}{n} \) only in the rest of this section.

**Theorem 3.1.** Let \( u_0 \in C(\bar{\Omega}) \) and let \( f \) and \( g \) fulfill (3.2) and (3.3) with

\[
\theta - \kappa > 1.
\tag{3.4}
\]

Then the unique classical global solution \((u(\cdot,t), v(\cdot,t))\) of the minimal chemotaxis-growth system (3.1) is uniformly bounded in \( L^\infty(\Omega) \times W^{1,\infty}(\Omega) \) for all \( t \in (0,\infty) \).

**Proof.** For any \( p > 1 \), we multiply the \( u \)-equation in (3.1) by \( pu^{p-1} \) and integrate the result over \( \Omega \) by parts to deduce that

\[
\frac{d}{dt} \int_{\Omega} u^p + p(p-1) \int_{\Omega} u^{p-2} |\nabla u|^2 = (p-1) \chi \int_{\Omega} u^{p-1} \nabla u \nabla v + p \int_{\Omega} f(u) u^{p-1}.
\tag{3.5}
\]

Testing the \( v \)-equation in (3.1) against \( u^p \), we end up with

\[
p \int_{\Omega} u^{p-1} \nabla u \nabla v = - \int_{\Omega} u^p v + \int_{\Omega} u^p g(u).
\tag{3.6}
\]

Substituting (3.6) into (3.5) and using (3.2) and (3.3) yield

\[
\frac{d}{dt} \int_{\Omega} u^p + p(p-1) \int_{\Omega} u^{p-2} |\nabla u|^2 \leq -(p-1) \chi \int_{\Omega} u^p v + (p-1) \beta \chi \int_{\Omega} u^{p+\kappa} + ap \int_{\Omega} u^{p-1} - bp \int_{\Omega} u^{p+\theta-1}.
\tag{3.7}
\]

Thanks to the relation (3.4), we conclude

\[
C_p := \max \left\{ (p-1) \beta \chi z^{p+\kappa} + ap z^{p-1} - bp z^{p+\theta-1} + z^\theta : z \geq 0 \right\} < \infty.
\]

Then it follows from (3.7) that

\[
\frac{d}{dt} \int_{\Omega} u^p \leq - \int_{\Omega} u^p + C_p.
\]
which, upon a use of Gronwall’s inequality, yields that
\[
\int_\Omega u^p \leq \int_\Omega u_0^p + C_p
\]
for any \( p > 1 \) and for any \( t \in (0, T_m) \). As a consequence, the \( L^{\kappa + 1/2} \)-boundedness criterion provided by Lemma 2.3 immediately guarantees that \( T_m = \infty \) and, furthermore, \( \|u(t)\|_{L^\infty} \) and \( \|v(t)\|_{W^{1, \infty}} \) are uniformly bounded for \( t \in (0, \infty) \).

Next, we explore the borderline case \( \theta - \kappa = 1 \). In this case, we will see that the \( L^{\kappa + 1/2} \)-boundedness criterion in Lemma 2.3 plays a crucial role.

**Theorem 3.2.** Let \( u_0 \in C(\Omega) \) and let \( f \) and \( g \) fulfill (3.2) and (3.3) with \( \theta - \kappa = 1 \).

If either
\[
b > \left( \frac{\kappa n - 2}{\kappa n} \right) \beta \chi
\]
(3.9)
or
\[
b = \left( \frac{\kappa n - 2}{\kappa n} \right) \beta \chi,
\]
(3.10)
then the unique classical global solution \( (u(\cdot, t), v(\cdot, t)) \) of the chemotaxis-growth system (3.1) is uniformly bounded in \( L^\infty(\Omega) \times W^{1, \infty}(\Omega) \) for all \( t \in (0, \infty) \).

**Proof.** Due to Lemma 2.3, it suffices to prove that \( \|u(t)\|_{L^{\kappa + 1/2}} \) is uniformly bounded for some sufficiently small \( \epsilon > 0 \). To this end, for any \( p \geq 2 \), we apply (3.8) to (3.7) to deduce
\[
\frac{d}{dt} \int_\Omega u^p + 4(p - 1) \int_\Omega |\nabla u|^2 \leq - (p - 1) \int_\Omega u^p v - [bp - (p - 1)\beta \chi] \int_\Omega u^{p+\kappa} + ap \int_\Omega u^{p-1}.
\]
(3.11)
Let us first treat the strict inequality case of (3.9); that is
\[
b > \left( \frac{\kappa n - 2}{\kappa n} \right) \beta \chi = \left( \frac{\kappa n}{2} - 1 \right) \beta \chi.
\]
This allows us to fix a small \( \epsilon > 0 \) in such a way that
\[
b > \left( \frac{\kappa n}{2} + \epsilon \right) \beta \chi.
\]
(3.12)
Setting \( p = \kappa n/2 + \epsilon \) and using (3.12), one has \( [bp - (p - 1)\beta \chi] > 0 \). The fact \( \kappa > 0 \) then ensures
\[
C = \max \left\{ - [bp - (p - 1)\beta \chi] z^{p+\kappa} + ap z^{p-1} + z^p : z \geq 0 \right\} < \infty,
\]
which combined with (3.11) leads us to
\[
\frac{d}{dt} \int_\Omega u^{\kappa n/2+\epsilon} \leq - \int_\Omega u^{\kappa n/2+\epsilon} + C(n, \epsilon, \beta, \chi, a, b, \kappa, \Omega).
\]
This immediately shows that \( \|u(t)\|_{L^{\kappa n/2+\epsilon}} \) is uniformly bounded for \( t \in (0, T_m) \).

Let us now examine the borderline case of (3.10). In this case, the premise \( b > 0 \), cf. (3.2), entails \( \kappa n/2 > 1 \); then, for any \( p \in (1, \kappa n/2) \) (nonempty), we have
\[
b p - (p - 1)\beta \chi = \beta \chi p \left( \frac{1}{p} - \frac{2}{\kappa n} \right) \geq 0.
\]
(3.13)
Accordingly, we infer from (3.11) and (3.13) that
\[ \frac{d}{dt} \int_{\Omega} u^p + \frac{4(p-1)}{p} \int_{\Omega} |\nabla u|^2 \leq a_p \int_{\Omega} u^{p-1} \leq \int_{\Omega} u^p + a_p (p-1)^{p-1} |\Omega|. \tag{3.14} \]

Now, we apply the Gagliardo-Nirenberg interpolation inequality (2.2), $L^1$-boundedness of $u$ and Young’s equality with epsilon to derive that
\[ \|u\|_{L^p}^p = \|u_0\|_{L^p}^p \leq C \|\nabla u_0\|_{L^2}^{\frac{2p}{p-2}} \|u_0\|_{L^{\frac{p(p-1)}{2}}}^{2(1-\delta)} + C \|u_0\|_{L^{\frac{p(p-1)}{2}}}^2 \]
\[ \leq C \|\nabla u_0\|_{L^2}^2 + C \leq \eta \|\nabla u_0\|_{L^2}^2 + C_\eta \] \tag{3.15}
for any $\eta > 0$, where
\[ \delta = \frac{p}{2} - \frac{1}{2} + \frac{1}{n} \in (0, 1). \]

A combination of (3.14) and (3.15) gives rise to
\[ \frac{d}{dt} \int_{\Omega} u^p + \frac{2(p-1)}{p} \int_{\Omega} |\nabla u|^2 \leq C(p, a, |\Omega|), \]
which coupled with (3.15) once more implies
\[ \frac{d}{dt} \int_{\Omega} u^p + \int_{\Omega} u^p \leq C(p, a, |\Omega|). \tag{3.16} \]

Solving this differential inequality immediately yields
\[ \|u\|_{L^p} \leq \|u_0\|_{L^p} + C(p, a, |\Omega|), \quad \forall p \in (1, \frac{\kappa n}{2}). \tag{3.17} \]

In the sequel, we shall prove that $\|u\|_{L^{p_0}}$ is also uniformly bounded for some $p_0 > \kappa n/2$. To this end, for any $p \in (\frac{\kappa n}{2}, \frac{\kappa n}{4} + 1)$, the Hölder inequality along with (3.17) yields that
\[ a_p \int_{\Omega} u^{p-1} \leq a_p \left( \int_{\Omega} u_0^{\frac{n}{n-2}} \right)^{\frac{2(p-1)}{n}} |\Omega| \left( \int_{\Omega} u_0^{\frac{n}{n-2}} \right)^{\frac{n-2}{n}} \leq C_p, \]
which in conjunction with (3.11) allows us to conclude that
\[ \frac{d}{dt} \int_{\Omega} u^p + \frac{4(\kappa n - 2)}{\kappa n} \int_{\Omega} |\nabla u|^2 \]
\[ \leq \beta \chi (\frac{\kappa n}{2} + 1)(\frac{2}{\kappa n} - \frac{1}{p}) \int_{\Omega} u^{p+\kappa} + C_p, \quad \forall p \in (\frac{\kappa n}{2}, \frac{\kappa n}{4} + 1), \tag{3.18} \]
where we have substituted the value of $b$ in (3.10).

In the sequel, we wish to bound the first term on the right-hand side of (3.18) in terms of the dissipation term on its left-hand side. Case I: $n > 2$. In this case, Hölder’s inequality shows
\[ \int_{\Omega} u^{p+\kappa} = \int_{\Omega} u^{\frac{n}{n-2} \cdot \frac{n}{n-2}^*} \cdot u^{\frac{n}{n-2}^*} \leq \left( \int_{\Omega} u^{\frac{n}{n-2}} \right)^{\frac{n}{n-2}^*} \left( \int_{\Omega} u^{\frac{n}{n-2}^*} \right)^2 = \|u_{|\nabla u|^2}\|_{L^{\frac{n}{n-2}}}^2 \|u\|_{L^{\frac{n}{n-2}}}. \]

Then we infer from the Sobolev embedding $W^{1,2} \hookrightarrow L^{\frac{2n}{n-2}}$, (3.15) and (3.17) that
\[ \int_{\Omega} u^{p+\kappa} \leq \|u\|_{L^{\frac{2n}{n-2}}}^\kappa \|u\|_{L^{\frac{2n}{n-2}}}^2 \leq C \|u\|_{L^{\frac{2n}{n-2}}}^\kappa (\|\nabla u\|_{L^2}^2 + \|u\|_{L^2}^2) \leq C(\|\nabla u\|_{L^2}^2 + 1). \tag{3.19} \]
Here, we emphasize that the constant $C$ in (3.19) is independent of $p$ since we used only the Sobolev embedding $W^{1,2} \hookrightarrow L^{\frac{2n}{n-2}}$ and (3.15) with $\eta = 1.$
Case II: $n = 1, 2$. In this case, we set $q := 3p$, which implies $q > p + \kappa$ by the choice $p > \frac{\kappa n}{2}$; then we choose

$$\sigma = \frac{p + \kappa - \frac{\kappa n}{2}}{3p - \frac{\kappa n}{2}} \iff p + \kappa = 3p\sigma + \frac{\kappa n}{2}(1 - \sigma).$$

(3.20)

A use of the Hölder inequality leads to

$$\int_{\Omega} u^{p+\kappa} = \int_{\Omega} u^{3p\sigma} \cdot \frac{u^{\kappa n}}{3p} \cdot (1 - \sigma) \leq \left( \int_{\Omega} u^{3p} \right)^\sigma \left( \int_{\Omega} \frac{u^{\kappa n}}{3p} \right)^{1 - \sigma} = \|u\|_{L^\infty}^\sigma \|u\|_{L^{\frac{3p}{\kappa n}}}^{\frac{\kappa n}{2}(1 - \sigma)}.

Then we conclude from the G-N interpolation inequality (2.2) and (3.17) that

$$\int_{\Omega} u^{p+\kappa} \leq C\|u\|_{L^{\frac{3p}{\kappa n}}} \left( \|\nabla u\|_{L^2}^{\delta} \|u\|_{L^{\frac{3p}{\kappa n}}}^{1 - \delta} + \|u\|_{L^{\frac{3p}{\kappa n}}} \right)^{\delta \sigma}$$

(3.21)

where we have utilized the following facts

$$\delta = \frac{p - \frac{n}{2}}{p + \frac{n}{2}} \in (0, 1), \quad 6\delta \sigma = 2,$$

(3.22)

the latter equality is due to (3.20) and (3.22). Observe that

$$\frac{2\kappa n}{\kappa n + 2} < \frac{\kappa n}{p} < 2, \quad \forall p \in (\frac{\kappa n}{2}, \frac{\kappa n}{2} + 1),$$

which implies that the constant $C$ in (3.21) can be uniformly bounded in $p \in (\frac{\kappa n}{2}, \frac{\kappa n}{2} + 1)$ and then can be chosen independent of such $p$. That is, (3.19) is also valid in the case of II.

To sum up our discussion, we have shown that

$$\beta \chi \left( \frac{\kappa n}{2} + 1 \right) \left( \frac{2}{\kappa n} - \frac{1}{p} \right) \int_{\Omega} u^{p+\kappa} \leq C\beta \chi \left( \frac{\kappa n}{2} + 1 \right) \left( \frac{2}{\kappa n} - \frac{1}{p} \right) \left( \int_{\Omega} |\nabla u|^2 + 1 \right)$$

(3.23)

for any $p \in (\frac{\kappa n}{2}, \frac{\kappa n}{2} + 1)$, where the constant $C$ is independent of such $p$.

Now, we fix a

$$p_0 \in \left( \frac{\kappa n}{2}, \frac{\kappa n}{2} + 1 \right) \cap \left( 0, \frac{1}{\left( \frac{2}{\kappa n} - \frac{4(\kappa n - 2)}{C\beta \chi \kappa n (\kappa n + 2)} \right)^+} \right)$$

(3.24)

so that

$$C\beta \chi \left( \frac{\kappa n}{2} + 1 \right) \left( \frac{2}{\kappa n} - \frac{1}{p_0} \right) \leq 2\left( \frac{\kappa n - 2}{\kappa n} \right).$$

(3.25)

Finally, we take $\eta = \frac{2(\kappa n - 2)}{\kappa n}$ in (3.19), and then we deduce from (3.23), (3.24), (3.25) and (3.18) a Gronwall differential inequality for $\|u\|_{L^p_0}$:

$$\frac{d}{dt} \int_{\Omega} u^{p_0} + \int_{\Omega} u^{p_0} \leq C(p_0, a, |\Omega|),$$

trivially yielding that $\|u(t)\|_{L^p_0}$ is uniformly bounded. Thanks to the fact $p_0 > \kappa n/2$ by (3.24), the assertions of Theorem 3.2 follow as a consequence of Lemma 2.3. \qed

Remark 3. From the discussion in Section 3 and the work of [24] on sub-quadratic dampening enforcing the existence of global “very weak” solutions, we are led to speculate that no blow-up would occur for the minimal-chemotaxis-growth model (3.1) whenever

$$\theta - \kappa > 1 - \frac{1}{n}.$$
If this turned out to be true, then it would be a significant improvement of Theorems 3.1 and 3.2 and hence of existing results (cf. [3, 21, 22, 30]). In particular, under additional smallness assumptions, this has been verified in [24] for the KS system (3.1) with \( g(u) = u \) (or \( \kappa = 1 \)) and \( f \) satisfying \( f(u) \leq a - bu^\theta \) for all \( u \geq 0 \) and for some \( a \geq 0, b > 0 \) and 
\[
\theta > 2 - \frac{1}{n}.
\]
We are unable to obtain such a sharp conjectured result via the approach described above. Innovative ways should be found to explore this speculation.

4. Steady states for the K-S model. In this section, we study the steady states of the minimal chemotaxis-growth model (3.1):

\[
\begin{aligned}
0 &= \nabla \cdot (\nabla u - \chi u \nabla v) + f(u), \quad x \in \Omega, \\
0 &= \Delta v - v + g(u), \quad x \in \Omega, \\
\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, \quad x \in \partial \Omega.
\end{aligned}
\]

(4.1)

First of all, some \textit{a priori} estimates and regularity results for the solution in integral sense (cf. [21, Definition 4.1]) of (4.1) are needed in the subsequent discussions.

**Lemma 4.1.** Let \( f \) and \( g \) satisfy (3.2) and (3.3) with \( \theta > \kappa \), and \((u, v)\) be a positive solution of (4.1). Then

\[
\int_{\Omega} u^\theta \leq \frac{a}{b} |\Omega|, \quad \min u \leq K, \quad \int_{\Omega} v \leq \beta \left( \frac{a}{b} \right)^{\frac{\theta}{\kappa}} |\Omega|,
\]

(4.2)

where \( K \) is the largest zero point of \( f \). Furthermore, the \( W^{2, \frac{\theta}{\kappa}} \)-norm of \( v \) is uniformly bounded in \( \chi \). In particular, if \( f(u) = cu - bu^\theta \), then \( \max_{\Omega} u \geq K = (c/b)^{\theta - 1} - 1 \).

**Proof.** Integrating the \( u \)-equation and using the fact \( f(u) \leq a - bu^\theta \), we have

\[
0 = \int_{\Omega} f(u) \leq \int_{\Omega} (a - bu^\theta),
\]

which directly gives the first two inequalities in (4.2). Then integrating the \( v \)-equation, using \( g(u) \leq \beta u^\kappa \) and Hölder inequality, we arrive at the last desired inequality in (4.2).

Notice that

\[
\|g(u)\|_{L^\frac{\theta}{\kappa}} \leq \beta \|u^\kappa\|_{L^\frac{\theta}{\kappa}} = \beta \left( \int_{\Omega} u^\theta \right)^{\frac{\theta}{\kappa}} \leq \beta \left( \frac{a}{b} |\Omega| \right)^{\frac{\theta}{\kappa}},
\]

then the elliptic regularity applied to the \( v \)-equation in (4.1) yields the stated \( W^{2, \frac{\theta}{\kappa}} \)-estimate for \( v \). Especially, for \( f(u) = cu - bu^\theta \), if \( \max_{\Omega} u < K \), then \( f(u) > 0 \) on \( \Omega \) and so \( \int_{\Omega} f(u) > 0 \), which is a contradiction.

Performing the similar (test function) argument as done in [21], we derive some regularity results for (4.1), showing the solution typically will never become singular (i.e., unbounded).

**Lemma 4.2.** Let \( f \) and \( g \) satisfy (3.2) and (3.3) with \( \theta - 1 \geq \kappa \) and let \((u, v)\) be a nonnegative solution of the elliptic chemotaxis system (4.1). Then the following statements are true.

(i) \( u \in L^{p + \kappa}(\Omega) \) and \( v \in L^{q + 1}(\Omega) \) for any \( p < \frac{\beta \chi}{(\beta \chi - b)^\theta} \) and \( q < \frac{\beta \chi}{\kappa(\beta \chi - b)^\theta} \);
(ii) for \( \theta - 1 > \kappa \), \((u, v)\) is bounded in \( L^\infty(\Omega) \) and \( u, v \in C^{1+\gamma}(\overline{\Omega}) \) for all \( \gamma \in (0, 1) \);
(iii) for \( \theta - 1 = \kappa \), if
\[
\begin{align*}
n \leq \frac{2}{\kappa} + 2 \text{ or } \left\{ n > \frac{2}{\kappa} + 2, \quad b > \frac{[(n - 2)\kappa - 2]}{(n - 2)\kappa} \beta \chi \right\},
\end{align*}
\]
then \( u, v \) is bounded in \( L^\infty(\Omega) \) and \( u, v \in C^{1+\gamma}(\overline{\Omega}) \) for all \( \gamma \in (0, 1) \);
(iv) if \( f(u) > 0 \) on \( (0, (\frac{p}{\beta})^+) \), then any solution of \((4.1)\) satisfies
\[
(\frac{a}{\beta})^+ e^{\chi(\min_{\Omega} v - \max_{\Omega} v)} \leq u \leq (\frac{a}{\beta})^+ e^{\chi(\max_{\Omega} v - \min_{\Omega} v)}.
\]
In particular, if \( v \) is bounded, then \( u, v \in C^{1+\gamma}(\overline{\Omega}) \) for all \( \gamma \in (0, 1) \).

**Proof.** (i) The elliptic counterpart of \((3.7)\) is
\[
p(p - 1) \int_\Omega u^{p - 2} |\nabla u|^2 + (p - 1) \chi \int_\Omega u^p v - (p - 1) \beta \chi \int_\Omega u^{\nu + \kappa} + bp \int_\Omega u^{p + \theta - 1}
\leq ap \int_\Omega u^{p - 1},
\]
In the case \( \theta - 1 > \kappa \), a simple application of Young’s inequality with \( \epsilon \) to \((4.5)\) shows that \( \int_\Omega u^{p + \theta - 1} \) is bounded for any \( p > 1 \); while, in the case \( \theta - 1 = \kappa \), it follows from \((4.5)\) that
\[
[bp - (p - 1) \beta \chi] \int_\Omega u^{\nu + \kappa} \leq ap \int_\Omega u^{p - 1},
\]
which immediately implies \( u \in L^{p + \kappa}(\Omega) \) for any \( p < \frac{\beta \chi}{(\beta \chi - b)^+} \). Then multiplying the \( v \)-equation by \( v^q \), integrating by parts and using \((3.3)\) and Young’s inequality, we deduce
\[
q \int_\Omega v^{q - 1} |\nabla v|^2 + \int_\Omega v^{q + 1} \beta \int_\Omega u^\kappa v^q \leq \frac{1}{2} \int_\Omega v^{q + 1} + \frac{\beta q}{q + 1} (\frac{2 \beta q}{q + 1})^q \int_\Omega u^{(q + 1)\kappa},
\]
which, coupled with the integrability of \( u \), yields that \( v \in L^{q + 1}(\Omega) \) for any \( q < \frac{\beta \chi}{\kappa (\beta \chi - b)^+} \).

(ii) Let us use the \( v \)-equation in \((4.1)\) to rewrite the system \((4.1)\) as
\[
\begin{align*}
\Delta u - \chi \nabla u \nabla v &= \chi u(v - g(u)) - f(u), \quad x \in \Omega, \\
\Delta v &= v - g(u), \quad x \in \Omega, \\
\frac{\partial u}{\partial \nu} &= \frac{\partial v}{\partial \nu} = 0, \quad x \in \partial \Omega.
\end{align*}
\]
Let \( x_1 \) and \( x_2 \) be the maximum points of \( u \) and \( v \) in \( \overline{\Omega} \), respectively. We then apply the Hopf lemma and maximum principle to \((4.6)\), cf. [16, Lemma 2.1] and use \((3.2)\) and \((3.3)\) to get
\[
\begin{align*}
\chi u(x_1)(v(x_1) - \beta u^\kappa(x_1)) \leq \chi u(x_1)(v(x_1) - g(u(x_1))) &\leq f(u(x_1)) \leq a - bu^\theta(x_1), \\
v(x_2) &\leq g(u(x_2)) \leq \beta u^\kappa(x_2),
\end{align*}
\]
from which it follows that
\[
\begin{align*}
\chi u(x_1)(v(x_1) - \beta u^\kappa(x_1) + \frac{b}{\beta} u^{\theta - 1}(x_1)) &\leq a, \\
v(x_2) \leq g(u(x_2)) \leq \beta u^\kappa(x_2) &\leq \beta u^\kappa(x_1).
\end{align*}
\]
Because \( \theta - 1 > \kappa \), the boundedness of \( u \) and \( v \) follows from \((4.7)\). Then the regularity \( u, v \in C^{1+\gamma}(\Omega) \) for all \( \gamma \in (0, 1) \) follows from the standard elliptic regularity, cf. [21, P. 868, Step 4].
(iii) The $W^{2,p}$-elliptic regularity applied to
\[ -\Delta v + v = g(u) \in L^{\frac{p+1}{p}}(\Omega), \quad \forall p < \frac{\beta \chi}{(\beta \chi - b)^\tau} \]
shows that $v \in W^{2,\frac{p+1}{p}}(\Omega)$. Notice that (4.3) implies
\[ \frac{2\beta \chi}{\kappa(\beta \chi - b)^\tau} + 2 > n. \]
Then the Sobolev embedding says $v \in L^\infty(\Omega)$. To show the boundedness of $u$, we rewrite the $u$-equation as
\[ -\nabla \cdot (e^{\chi v} \nabla (ue^{-\chi v})) = f(u). \quad (4.8) \]
Testing it against $(ue^{-\chi v} - (\frac{a}{b})^{\frac{1}{\tau}} e^{-\chi v})$ and using (3.2), we end up with
\[ \int_{\Omega} e^{\chi v} \nabla (ue^{-\chi v} - (\frac{a}{b})^{\frac{1}{\tau}} e^{-\chi v})^2 = \int_{\Omega} f(u)(ue^{-\chi v} - (\frac{a}{b})^{\frac{1}{\tau}} e^{-\chi v})_+ \]
\[ = \int_{\{u > (\frac{a}{b})^{\frac{1}{\tau}} e^{-\chi v}\}} f(u)(ue^{-\chi v} - (\frac{a}{b})^{\frac{1}{\tau}} e^{-\chi v})_+ \leq 0, \]
which implies
\[ u \leq (\frac{a}{b})^{\frac{1}{\tau}} e^{\chi v}, \]
leading to the desired upper bound for $u$.
(iv) Let $(u, v)$ be a solution of (4.1). Then we test (4.8) by
\[ (ue^{-\chi v} - (\frac{a}{b})^{\frac{1}{\tau}} e^{-\chi v})_+ \]
to derive
\[ \int_{\Omega} f(u)(ue^{-\chi v} - (\frac{a}{b})^{\frac{1}{\tau}} e^{-\chi v})_+ \]
\[ = \int_{\Omega} e^{\chi v} \nabla (ue^{-\chi v}) \nabla (ue^{-\chi v} - (\frac{a}{b})^{\frac{1}{\tau}} e^{-\chi v})_+ \leq 0, \]
from which it follows
\[ \int_{\{u < (\frac{a}{b})^{\frac{1}{\tau}} e^{-\chi v}\}} f(u)e^{-\chi v} \left(u - (\frac{a}{b})^{\frac{1}{\tau}} e^{\chi(v-\max v)}\right)_+ \]
\[ = \int_{\Omega} f(u)e^{-\chi v} \left(u - (\frac{a}{b})^{\frac{1}{\tau}} e^{\chi(v-\max v)}\right)_+ \leq 0. \]
The positivity of $f$ on $(0, (\frac{a}{b})^{\frac{1}{\tau}})$ directly gives
\[ \left(u - (\frac{a}{b})^{\frac{1}{\tau}} e^{\chi(v-\max v)}\right)_+ \equiv 0 \implies u \geq (\frac{a}{b})^{\frac{1}{\tau}} e^{\chi(v-\max v)}. \]
In a similar way, testing (4.8) by
\[ (ue^{-\chi v} - (\frac{a}{b})^{\frac{1}{\tau}} e^{-\chi v})_+ \]
yields the upper bound for $u$ in (4.4).

In what follows, we study the capability of the system (4.1) to form patterns. We perform Leray-Schauder index formula (The possibility of realization of such method was mentioned in [13] but not carried out even for a simpler model than (4.1)) to show that, for each equilibrium state, the stationary system (4.1) admits an increasing sequence of $\{\chi_k\}_{k=1}^\infty$ such that it has at least one nonconstant solution.
whenever \( \chi \in (\chi_{2k-1}, \chi_{2k}), k = 1, 2, \ldots \). More precisely, we have the following existence result for pattern formations.

**Theorem 4.3.** Let \( Z = \{ \tilde{u} > 0 \} \). Then, for each \( \tilde{u} \in Z \), there exists a positive increasing sequence \( \{ \chi_k = \chi_k(\tilde{u}) \} \) with the property

\[ 0 < \chi_1(\tilde{u}) < \chi_2(\tilde{u}) < \cdots < \chi_k(\tilde{u}) < \chi_{k+1}(\tilde{u}) \to \infty \]

such that, for every

\[ \chi \in \bigcup_{\tilde{u} \in Z} \bigcup_{k=1}^{\infty} (\chi_{2k-1}(\tilde{u}), \chi_{2k}(\tilde{u})) := P_\chi, \]

the stationary chemotaxis-growth system (4.1) has at least one nonconstant solution.

Before presenting the proof, we want to remark that Theorem 4.3 not only gives the existence of non-constant solutions for (4.1) which is a generalization of the model considered in [21] where \( f \) is of logistic type, but also provides more explicit conditions which are cleaner and easier to verify. Our proof is the consequence of bifurcation from “eigenvalues” of odd multiplicity.

**Proof of Theorem 4.2.** By the \( v \)-equation, (4.1) is equivalent to

\[
\begin{cases}
-\Delta u + \chi \nabla u \nabla v = -\chi u[v - g(u)] + f(u) & \text{in } \Omega, \\
-\Delta v + v = g(u) & \text{in } \Omega, \\
\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 & \text{on } \partial \Omega.
\end{cases}
\]

Linearizing the system (4.9) about a generic equilibrium state \((\tilde{u}, \tilde{v}) := (\tilde{u}, g(\tilde{u}))\) with \( \tilde{u} \in Z \), we arrive at the linearized system

\[
\begin{cases}
-\Delta u + u = [\chi g'(\tilde{u})\tilde{u} + f'(\tilde{u}) + 1]u - \chi \tilde{u}v & \text{in } \Omega, \\
-\Delta v + v = g'(\tilde{u})u & \text{in } \Omega, \\
\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 & \text{on } \partial \Omega.
\end{cases}
\]

Let

\[ A(\chi) = \begin{pmatrix}
g'(\tilde{u})\tilde{u} \chi + f'(\tilde{u}) + 1 & -\chi \tilde{u} \\
g'(\tilde{u}) & 0
\end{pmatrix}.\]

By direct computations, the eigenvalues of the matrix \( A \) are

\[ \lambda^\pm(\chi) = \frac{1}{2} \left[ g'(\tilde{u})\tilde{u} \chi + f'(\tilde{u}) + 1 \pm \sqrt{[g'(\tilde{u})\tilde{u} \chi + f'(\tilde{u}) - 1]^2 + 4f'(\tilde{u})} \right]. \]

Case 1: \( f'(\tilde{u}) \geq 0 \). In this case, \( \lambda^\pm(\chi) \) are defined for all \( \chi > 0 \) and are increasing with

\[ 0 < \lambda^-(\chi) < \lambda^-(\infty) = 1 \leq f'(\tilde{u}) + 1 < \lambda^+(\chi) < \lambda^+(\infty) = \infty.\]

Case 2: \( f'(\tilde{u}) < 0 \). In this case, \( \lambda^\pm(\chi) \) are defined for all

\[ \chi \geq 1 + 2\sqrt{-f'(\tilde{u}) - f'(\tilde{u})} \quad : = \hat{\chi}_0, \]

where \( \lambda^+(\chi) \) is increasing and \( \lambda^-(\chi) \) is decreasing for all \( \chi \geq \hat{\chi}_0 \) with

\[ \lambda^-(\infty) < \lambda^-(\chi) < \lambda^-(\hat{\chi}_0) = 1 + \sqrt{-f'(\tilde{u})} < \lambda^+(\chi) < \lambda^+(\infty) = \infty.\]

It is well-known that the eigenvalue problem

\[ -\Delta w + w = \sigma w \quad \text{in } \Omega, \quad \frac{\partial w}{\partial \nu} = 0 \quad \text{on } \partial \Omega \]
has a sequence of eigenvalues \( \{ \sigma_k \}_{k=0}^{\infty} \) with \( 1 = \sigma_0 < \sigma_1 < \cdots < \sigma_k < \cdots \to \infty \) and the collection of their corresponding eigenfunctions \( \{ e_k(x) \}_{k=0}^{\infty} \) forms a complete orthogonal basis for \( L^2(\Omega) \).

In case of \( f'(\tilde{u}) \geq 0 \), we have \( \{ \lambda^- : \chi > 0 \} \cap \Sigma = \emptyset \), hereafter \( \Sigma = \{ \sigma_j : j = 0, 1, 2, \cdots \} \). In the case of \( f'(\tilde{u}) < 0 \), we fix \( \tilde{\chi}_1 \) according to

\[
\tilde{\chi}_1 = \begin{cases} 
\frac{1}{\lambda^-(\sigma_1)}, & \text{if } \sigma_1 < 1 + \sqrt{-f'(\tilde{u})}, \\
\text{any number } \geq \tilde{\chi}_0, & \text{if } \sigma_1 \geq 1 + \sqrt{-f'(\tilde{u})}.
\end{cases}
\]

Since \( \lambda^-(\chi) \) is decreasing and \( \lambda^-(\chi) > 1 = \sigma_0 \), we have \( \{ \lambda^- : \chi > \tilde{\chi}_1 \} \cap \Sigma = \emptyset \).

Let \( \tilde{\chi}_1 = 0 \) if \( f'(\tilde{u}) \geq 0 \) and \( \tilde{\chi}_1 = \frac{1}{\lambda^-(\sigma_1)} \) if \( f'(\tilde{u}) < 0 \). For any eigenvalue \( \sigma_k \in \Sigma \cap \{ \lambda^+ : \chi > \tilde{\chi}_1 \} \), we set \( \tilde{\chi}_k = (\lambda^+)^{-1}(\sigma_k) \), which is well-defined by the properties of \( \lambda^+ \). Then it follows readily from the properties of \( \lambda^\pm \) that

\[
\lambda^- (\tilde{\chi}_k) \in \Sigma, \quad \lambda^- (\tilde{\chi}_k) \notin \Sigma.
\]

Choosing an open neighborhood \( O_k = (\lambda^+)^{-1}(\sigma_k) \cap (\tilde{\chi}_k, \infty) \) of \( \tilde{\chi}_k \) such that for any \( \sigma \in \Sigma \) with \( \sigma \neq \lambda^\pm (\tilde{\chi}_k) \), we have \( \sigma \neq \lambda^\pm (\chi) \) for any \( \chi \in O_k \). That is, \( (\lambda^+ (O_k) \cup \lambda^- (O_k)) \cap \Sigma = \{ \lambda^\pm (\tilde{\chi}_k) \} \). We now consider the subsets

\[
\begin{align*}
O_k^+ &= \{ \chi \in O_k : \lambda^+ (\chi) > \lambda^+ (\tilde{\chi}_k) \} = (\lambda^+)^{-1}(\sigma_k) \cap (\tilde{\chi}_k, \infty) = (\tilde{\chi}_k, \tilde{\chi}_{k+1}), \\
O_k^- &= \{ \chi \in O_k : \lambda^- (\chi) < \lambda^- (\tilde{\chi}_k) \} = (\lambda^-)^{-1}(\sigma_k) \cap (\tilde{\chi}_k, \infty) = (\tilde{\chi}_{k-1}, \tilde{\chi}_k),
\end{align*}
\]

and the space \( X = \{ u \in C^1(\Omega) : \frac{\partial u}{\partial v} = 0 \text{ on } \partial \Omega \} \). Let \( L_k^\pm : X^2 \to X^2 \) be defined by

\[
L_k^\pm = I - (-\Delta + I)^{-1}A(\chi), \quad \chi \in O_k^\pm,
\]

where the compact operator \( (-\Delta + I)^{-1} : X^2 \to X^2 \) is the inverse of \( -\Delta + I \) in \( X^2 \).

In the sequel, we will show that \( (0, 0) \not\in L_k^\pm (\partial B((0, 0), r_k^\pm)) \) for any \( r_k^\pm > 0 \), where \( B((\tilde{u}, \tilde{v}), r) \) denotes the open ball in \( X^2 \) centered at \( (\tilde{u}, \tilde{v}) \) with radius \( r \). Indeed, suppose not, then (4.10) has a nontrivial solution \( (u, v) \). Let

\[
U_j = \int_{\Omega_j} u \, e_j, \quad V_j = \int_{\Omega_j} v \, e_j.
\]

Then multiplying (4.10) by \( e_j \) and integrating over \( \Omega \), we get an algebraic system for \( U_j \) and \( V_j \):

\[
\begin{align*}
\{ \sigma_j - g^\prime(\tilde{u})\tilde{u}\chi - f'(\tilde{u}) - 1 \} U_j + \chi \tilde{u} V_j &= 0, \\
g^\prime(\tilde{u}) U_j + \sigma_j V_j &= 0,
\end{align*}
\]

which has a nonzero solution \( (U_j, V_j) \) for some \( j \) if and only if

\[
\sigma_j^2 - [g^\prime(\tilde{u}) \tilde{u}\chi + f'(\tilde{u}) + 1] \sigma_j + g^\prime(\tilde{u}) \tilde{u}\chi = 0. \tag{4.12}
\]

Solving (4.12) in \( \sigma_j \) and comparing with (4.11), we discover that \( \sigma_j = \lambda^\pm (\chi) \), which contradicts the fact that \( (\lambda^+ (O_k^+) \cup \lambda^- (O_k^-)) \cap \Sigma = \emptyset \). Hence, \( (U_j, V_j) = (0, 0) \) for all \( j \geq 0 \), and so (4.10) has only the zero solution by the completeness of eigenfunctions. This tells us that \( (0, 0) \not\in L_k^\pm (\partial B((0, 0), r_k^\pm)) \) or, equivalently, 1 is not an eigenvalue of \( (-\Delta + I)^{-1}A(\chi) \) for \( \chi \in O_k^\pm \).

Because \( L_k^\pm \) is a compact perturbation of identity, the Leray-Schauder degree, \( \deg(L_k^\pm, B((0, 0), r_k^\pm)), \cdot, \cdot \), is well-defined, and

\[
\deg(L_k^+, B((0, 0), r_k^+), (0, 0)) = -\deg(L_k^-, B((0, 0), r_k^-), (0, 0)),
\]

cf Nirenberg [17]. In light of (4.9), we consider the nonlinear operator defined by

\[
h(u, v; \chi) = \left( -\chi \nabla u \nabla v - \chi [v - g(u)] + f(u) + u \right).
\]
Notice that, for $\chi \in O_k^+ \cap r_k^+$ small enough, the operator
\[
H_k^+(u,v;\chi) = I - (-\Delta + I)^{-1}(h(u,v;\chi))
\]
is a continuous and compact perturbation of the identity in $B((\tilde{u},\tilde{v}), r_k)$. Moreover, for such small $r_k$, we have
\[
deg(L_k^+, B((0,0), r_k^+), (0,0)) = \deg(H_k^+, B((\tilde{u},\tilde{v}), r_k), (0,0)).
\] (4.13)

To calculate $\deg(L_k^+, B((0,0), r_k^+), 0)$, we shall employ the Leray-Schauder index formula [17, Theorem 2.8]. To this end, we need to ensure that $\mu = 1$ is not an eigenvalue of $(-\Delta + I)^{-1}A(\chi)$ for $\chi \in O_k^+$. This indeed has been shown above; while, for our later usage, we give its proof via method of eigen-expansion.

By definition, $((u,v), \mu)$ is an eigen-pair of $(-\Delta + I)^{-1}A(\chi)$ if and only if
\[
\begin{cases}
-\mu \Delta u + \mu u = [g'(\tilde{u})\tilde{u} + f'(\tilde{u}) - 1]u - \chi uv & \text{in } \Omega, \\
-\mu \Delta v + \mu v = g'(\tilde{u})u & \text{in } \Omega, \\
\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 & \text{on } \partial \Omega.
\end{cases}
\] (4.14)

By the idea of eigen-expansion, we let
\[
u(x) = \sum_{j=0}^{\infty} \nu_j e_j(x), \quad v(x) = \sum_{j=0}^{\infty} \nu_j e_j(x).
\] (4.15)

Substituting (4.15) into (4.14) and using the completeness of eigenfunctions $\{e_j\}$, we obtain an algebraic system in $u_j$ and $v_j$ as follows.
\[
\begin{cases}
\sigma_j \mu - g'(\tilde{u})\tilde{u} + f'(\tilde{u}) - 1\sigma_j u_j + \chi v_j = 0, \\
-\sigma_j \mu u_j + g'(\tilde{u})\tilde{u} v_j = 0,
\end{cases}
\]
which has a nonzero solution $(u_j, v_j)$ for some $j$ if and only if
\[
\sigma_j^2 \mu^2 - [g'(\tilde{u})\tilde{u} + f'(\tilde{u}) - 1]\sigma_j \mu + g'(\tilde{u})\tilde{u} = 0.
\] (4.16)

Solving (4.16) and comparing (4.11), we find that the eigenvalues of $(-\Delta + I)^{-1}A(\chi)$ are
\[
\mu_j^\pm(\chi) = \frac{\lambda_j^\pm(\chi)}{\sigma_j}, \quad j = 0, 1, 2 \ldots.
\] (4.17)

Recall that $(\lambda^+(O_k^+ \cup \lambda^-(O_k^+)) \cap \Sigma = \emptyset$, and so 1 is not an eigenvalue of $(-\Delta + I)^{-1}A(\chi)$ for $\chi \in O_k^+$. Then the Leray-Schauder index formula gives
\[
de(\mu, B((0,0), r_k^+), (0,0)) = (-1)^{\gamma_k^+},
\] (4.18)
where $\gamma_k^+$ is the sum of the algebraic multiplicities of the real eigenvalues of $(-\Delta + I)^{-1}A(\chi), \chi \in O_k^+$ which are greater than 1. In the case of $f'(\tilde{u}) < 0$, since $\lambda^-(\chi) < \sigma_j$ for any $j \geq 1$ and $\chi > \bar{\chi}_1$, we conclude from (4.17) and the properties of $\lambda^+$ that
\[
\gamma_k^+ = \sharp(\sigma_{0}) + \sum_{\sigma_j < \lambda^+(\bar{\chi}_{k+1})} \sharp(\sigma_j), \quad \gamma_k^- = \sharp(\sigma_{0}) + \sum_{\sigma_j < \lambda^+(\bar{\chi}_{k})} \sharp(\sigma_j).
\]
Here the notation $\sharp(\sigma_k)$ denotes the finite algebraic multiplicity of $\sigma_k$. While, in the case of $f'(\tilde{u}) \geq 0$, since $\lambda^-(\chi) < \sigma_j$ for any $j \geq 0$ and $\chi > \bar{\chi}_1$, we conclude from (4.17) and the properties of $\lambda^+$ that
\[
\gamma_k^+ = \sum_{\sigma_j < \lambda^+(\bar{\chi}_{k+1})} \sharp(\sigma_j), \quad \gamma_k^- = \sum_{\sigma_j < \lambda^+(\bar{\chi}_{k})} \sharp(\sigma_j).
\]

Hence, in either case, we obtain
\[
\gamma_k^+ - \gamma_k^- = \sharp(\lambda^+(\bar{\chi}_{k})) = \sharp(\sigma_k).
\] (4.19)
From (4.18) and (4.19) we deduce that
\[
\deg(L_+^k, B((0,0), r_+^k), (0,0)) - \deg(L_-^k, B((0,0), r_-^k), (0,0)) = (-1)^{\gamma_k} \left((-1)^{2(\sigma_k)} - 1\right).
\]
(4.20)

Now, if \( \mathbb{g}(\sigma_k) \) is an odd number, then by (4.13) and (4.20) the topological structures of \( L_+^k \) and hence of \( H_+^k \) change when \( \chi \) crosses \( \tilde{\chi}_k \). Indeed, by the well-known bifurcation from “eigenvalues” of odd multiplicity (cf. [17, 18]), it follows that \( \tilde{\chi}_k \) is a bifurcation value. Consequently, there exists a bifurcation branch \( C_k \) containing \( (\tilde{u}, \tilde{v}, \chi_k) \) such that either \( C_k \) is not compact in \( X \times X \times \mathbb{R} \) or \( C_k \) contains \( (\tilde{u}, \tilde{v}, \sigma_j) \) with \( \sigma_j \neq \sigma_k \).

Case 1: If, for some \( k \), the bifurcation branch \( C_k \) is not compact in \( X \times X \times \mathbb{R} \), then \( C_k \) extends to infinity in \( \chi \) due to the elliptic regularity that any closed and bounded subset of the solution triple \( (u, v, \chi) \) of our chemotaxis system (4.1) in \( X \times X \times \mathbb{R} \) is compact; this can be easily shown by the Sobolev embeddings and results from [14, Chapter 3], see similar discussions in [29, Proposition 4.1]. Clearly, in this case, we can find a sequence \( \{\chi_k(\tilde{u})\}_{k=1}^{\infty} \) fulfilling the statement of the theorem.

Case 2: If, for any \( k \), the branch \( C_k \) contains \( (\tilde{u}, \tilde{v}, \tilde{\chi}_j) \) with \( \tilde{\chi}_j \neq \tilde{\chi}_k \), then we define
\[
\chi^-_k = \inf \{\chi \in C_k \}, \quad \chi^+_k = \sup \{\chi \in C_k \} < \infty.
\]
Then, for any \( \chi \in \cap_{k=0}^{\infty} (\chi^-_k , \chi^+_k) \), the system (4.1) has at least one non-constant solution. From this and the fact that \( \sigma_k \to \infty \) and \( \tilde{\chi}_k = (\lambda^+)^{-1}(\sigma_k) \to \infty \) as \( k \to \infty \), a sequence \( \{\chi_k(\tilde{u})\}_{k=1}^{\infty} \) satisfying the description of the theorem can be readily constructed. Finally, the theorem follows by unifying all \( \tilde{u} \in Z \).

For the constant steady state \( (\tilde{u}, \tilde{v}) \), the length of the associated interval \( (\chi_{2k-1}, \chi_{2k}) \) of existence of nonconstant solutions is positive:
\[
|\chi_{2k} - \chi_{2k-1}| \geq \inf \{|(\lambda^+)^{-1}(\sigma_{2k-1}) - (\lambda^+)^{-1}(\sigma_j)| : j \neq 2k - 1\} > 0.
\]
This, joined with \( \chi_k \to \infty \), illustrates that the set \( P_\chi \) specified in the theorem is unbounded. However, it is yet unknown whether or not (4.1) has a nonconstant solution for \( \chi \) in the complement of the unbounded set \( P_\chi \).

Based on Theorem 4.3, we naturally wish to explore the asymptotic behavior of the nontrivial solutions \( (u, v) \) of (4.1) as \( \chi \to \infty \). By using the \( a \) priori estimates in Lemma 4.1, we obtain the following result on their asymptotic behavior as \( \chi \to \infty \).

**Theorem 4.4.** Let \( f(u) = au - bu^\theta \) with \( a \geq 0, b > 0, \theta > 1 \) and \( g(u) = \beta u^\kappa \) with
\[
\frac{\theta}{\kappa} > \max \left\{ 1, \frac{n}{2} \right\}, \quad \frac{\theta}{\kappa + 1} > \frac{n}{n + 1}
\]
and let \( \{u_\chi, v_\chi\} \) be any positive solution of (4.1). Then there is a subsequence \( \{\chi_j\} \) with \( \lim_{j \to \infty} \chi_j = \infty \) such that \( (u_j, v_j) = (u_{\chi_j}, v_{\chi_j}) \) fulfills
\[
\lim_{j \to \infty} u_j = M \quad \text{weakly in } L^\theta(\Omega),
\lim_{j \to \infty} \int_\Omega u_j^\theta = bM|\Omega|/a,
\lim_{j \to \infty} v_j = \beta M^\kappa \quad \text{weakly in } W^{2,\frac{\theta}{\kappa}}(\Omega),
\lim_{j \to \infty} v_j = \beta M^\kappa \quad \text{strongly in } W^{1,p}(\Omega),
\lim_{j \to \infty} v_j = \beta M^\kappa \quad \text{uniformly in } \Omega.
\]
for some nonnegative constant \( M \), where
\[
p < \left\{ \begin{array}{ll}
\frac{\theta}{n\kappa - \theta}, & \text{if } \frac{\theta}{\kappa} < n, \\
n, & \text{if } \frac{\theta}{\kappa} \geq n.
\end{array} \right.
\]
Proof. By Lemma 4.1, we see that \( \|u_\chi\|_{L^p(\Omega)} \) and \( \|v_\chi\|_{W^{2,\frac{\alpha}{\alpha-\theta}}(\Omega)} \) are uniformly bounded with respect to \( \chi \). Hence, the reflexivity and Sobolev embedding allow us to find a subsequence \( \{\chi_j\} \) with \( \lim_{j \to \infty} \chi_j = \infty \) such that \( (u_j, v_j) = (u_{\chi_j}, v_{\chi_j}) \) satisfies

\[
\begin{aligned}
\lim_{j \to \infty} u_j &= u_\infty \text{ weakly in } L^\theta(\Omega), \\
\lim_{j \to \infty} v_j &= v_\infty \text{ weakly in } W^{2,\frac{\alpha}{\alpha-\theta}}(\Omega), \\
\lim_{j \to \infty} v_j &= v_\infty \text{ strongly in } W^{1,p}(\Omega), \\
\lim_{j \to \infty} v_j &= v_\infty \text{ uniformly in } \Omega.
\end{aligned}
\tag{4.23}
\]

for some \( (u_\infty, v_\infty) \in L^\theta(\Omega) \times W^{2,\frac{\alpha}{\alpha-\theta}}(\Omega) \), \( p \) is defined in (4.22) and \( v_\infty \) is a (weak) solution of

\[-\Delta v_\infty + v_\infty = \beta u_\infty^\kappa \text{ in } \Omega, \quad \frac{\partial v_\infty}{\partial \nu} = 0 \text{ on } \partial \Omega. \tag{4.24}\]

The last convergence in (4.23) follows from the compact Sobolev embedding \( W^{2,\frac{\alpha}{\alpha-\theta}}(\Omega) \hookrightarrow C^0(\Omega) \) since \( \theta/\kappa > n/2 \). One can easily infer from (4.2) and (4.23) that

\[\|u_\infty\|_{L^p(\Omega)} \leq \liminf_{j \to \infty} \|u_j\|_{L^p(\Omega)} \leq \left( \frac{a}{b} \right)^{\frac{p}{\theta}}.\]

On the other hand, multiplying the first equation in (4.1) by \( w \in W^{2,\frac{\alpha}{\alpha-\theta}}_N(\Omega) \) and dividing by \( \chi = \chi_j \) with \( W^{2,\frac{\alpha}{\alpha-\theta}}(\Omega) = \{u \in W^{2,\frac{\alpha}{\alpha-\theta}}(\Omega) : \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial \Omega\} \), we obtain

\[
\frac{1}{\chi_j} \int_{\Omega} u_j \Delta w + \int_{\Omega} u_j \nabla v_j \nabla w + \int_{\Omega} (au_j - bu_j^\theta) w = 0. \tag{4.25}
\]

Noticing that

\[
\int_{\Omega} u_j \nabla v_j \nabla w = \int_{\Omega} u_j \nabla v_\infty \nabla w
\]

in the case of \( n\kappa > \theta \) and \( \theta > n/(n-1) \), we use the conditions in (4.21) to derive

\[
\int_{\Omega} |u_j \nabla w|^{\frac{\alpha}{\alpha-\theta}} \leq C \int_{\Omega} |u_j \nabla w|^{\frac{\alpha}{\alpha-\theta}} \leq C \int_{\Omega} |\nabla w|^{\frac{\alpha}{\alpha-\theta}} \leq C \int_{\Omega} |\nabla w|^{\frac{\alpha}{\alpha-\theta}} \leq C \int_{\Omega} |\nabla w|^{\frac{\alpha}{\alpha-\theta}};
\]

and

\[
\int_{\Omega} |\nabla v_\infty \nabla w|^{\frac{\alpha}{\alpha-\theta}} \leq \int_{\Omega} |\nabla v_\infty \nabla w|^{\frac{\alpha}{\alpha-\theta}} \leq C \|v_\infty\|_{W^{2,\frac{\alpha}{\alpha-\theta}}} \leq C \|v_\infty\|_{W^{2,\frac{\alpha}{\alpha-\theta}}};
\]

then sending \( j \to \infty \) in (4.25), we conclude from the (weak) convergence in (4.23) that

\[
\int_{\Omega} u_\infty \nabla v_\infty \nabla w = 0, \quad \forall w \in W^{2,\frac{\alpha}{\alpha-\theta}}_N(\Omega). \tag{4.26}
\]

Taking the test function \( w = v_\infty \) in (4.26) yields

\[
\int_{\Omega} u_\infty \nabla v_\infty \leq 0. \tag{4.27}
\]

Let \( \Omega_\infty = \{x \in \Omega : |\nabla v_\infty(x)| > 0\} \). If \( |\Omega_\infty| > 0 \) then \( u_\infty = 0 \text{ a.e. in } \Omega_\infty \) by (4.27); observe that \( \nabla v_\infty = 0 \) on \( \partial \Omega_\infty \), and so an integration of (4.24) with \( \Omega \) replaced with \( \Omega_\infty \) says that \( \int_{\Omega_\infty} v_\infty = 0 \), and thus the nonnegativity and the
inclusion \( v_\infty \in W^{2,\frac{\theta}{\kappa}}(\Omega) \to C(\overline{\Omega}) \) imply \( v_\infty \equiv 0 \) in \( \Omega_\infty \). This is a contradiction to the definition of \( \Omega_\infty \). Whence, \( |\Omega_\infty| = 0 \), which along with the continuity of \( v_\infty \), concludes that \( v_\infty \) is a non-negative constant, say, \( \beta M^\kappa \). Then (4.24) again shows that \( u_\infty = M \) is a nonnegative constant almost everywhere in \( \Omega \). Now, an integration of the \( u \)-equation in (4.1) directly deduces
\[
\int_\Omega u_\theta^\beta = \frac{a}{b} \int_\Omega u_j \to \frac{a}{b} M |\Omega|,
\]
where the weak convergence of \( u_j \) was used. This completes the proof of the theorem.

**Remark 4.** Based on the merely weak convergence of \( \{u_j\} \) in \( L^\theta(\Omega) \), we are unfortunately unable to determine the precise values of \( M \). The natural candidate for \( M \) is 0 or \( \left( \frac{a}{b} \right)^{1/\theta - 1} \) because of (4.28). Indeed, Kuto et al [13] claimed either \( M = 0 \) or \( a/b \) for the specific choices \( \theta = 2 \) and \( \kappa = 1 \). We underline that their claim is in general incorrect as to be discussed below. Indeed, they claimed from (4.28) that \( \{u_j\} \) contains a subsequence, still denoted by \( \{u_j\} \), which converges to \( u_\infty \) almost everywhere in \( \Omega \) as \( j \to \infty \). However, the equality (4.28) does not exclude oscillating functions (a priori, we do not know whether or not the the solution \( u_j \) will behave like this), and hence the claim is not guaranteed in general. For example, if we take
\[
u_j(x) = 1 + \sin(jx),
\]
then it follows that
\[
u_j \to 1 \text{ weakly in } L^2(0, 2\pi), \quad \int_0^{2\pi} u_j^2 = \frac{3}{2} \int_0^{2\pi} u_j = 3\pi. \tag{4.29}
\]
This says that \( u_j \) satisfies (4.28) with \( a = 3, b = 2 \) and \( \theta = 2 \). While, if there is a subsequence \( \{j\}' \) of \( \{j\} \) such that \( u_j \to 1 \) a.e. in \( (0, 2\pi) \), then the Lebesgue dominated convergence theorem (0 \( \leq u_j^2 \leq 4 \)) gives
\[
\lim_{j' \to \infty} \int_0^{2\pi} u_j'^2 = \int_0^{2\pi} 1^2 = 2\pi,
\]
which contradicts (4.29). Therefore, \( u_j \) has no subsequence that converges a.e. to 1 in \( (0, 2\pi) \).

The other gap of their proof lies in the application of Lebesgue dominated convergence theorem without finding the dominating function for \( u_j \). Typically, there is no dominating function for \( u_j \), since, on the one hand, the cells will aggregate when chemotactic effect is strong, and, on the other hand, we would get a stronger convergence if a dominating function was found. However, a stronger convergence than that of Theorem 4.4 seems unavailable, since boundedness results in Lemma 4.2 are not uniform with respect to \( \chi \), even in \( L^p \)-topology.

5. **Large time behavior for the K-S model.** In this section, we shall study the large time behavior for a specific chemotaxis-growth model with nonlinear production in the chemical equation as follows:
\[
\begin{align*}
    u_t &= \nabla \cdot (\nabla u - \chi u \nabla v) + u(a - bu^\kappa), & x \in \Omega, t > 0, \\
    0 &= \Delta v - v + u^\kappa, & x \in \Omega, t > 0, \\
    \frac{\partial u}{\partial v} = \frac{\partial v}{\partial u} = 0, & \quad x \in \partial \Omega, t > 0, \\
    u(x, 0) &= u_0(x), & x \in \Omega, \\
\end{align*}
\tag{5.1}
\]
where \( a \in \mathbb{R}, b > 0, \chi > 0, \kappa > 0 \) and \( \Omega \subset \mathbb{R}^n \) is a bounded smooth domain with \( n \geq 1 \).

For \( \kappa = 1 \) and \( b = a \), under the assumption \( b > 2\chi \), Tello and Winkler in [21] used comparison arguments to show that the solution of (5.1) converges in \( L^\infty \)-topology to its constant steady state \((1,1)\). Recently, such methods were extended for a model with nonlinear chemosensitivity and secretion [5]. On the other hand, for \( \kappa = 1 \), He and Zheng [6] modified the energy functional method from [1] to obtain the stabilities of the constant equilibria \((0,0)\) and \((a/b, a/b)\) for \( \kappa = 1 \) with convergence rate estimates. Here, we extend the energy functional method to undergo a comprehensive analysis for the global stabilities with explicit convergence rates of the constant steady states \( ((a/b)^{\frac{1}{\kappa}}, a/b)\) and \((0,0)\). Our precise long time behaviors for (5.1) as \( t \) tends to infinity go as follows.

**Theorem 5.1.** Let \( u_0 \in C(\Omega) \) with \( u_0 \geq 0 \) and let \( b, \chi > 0, b \geq \frac{(\kappa n - 2)}{\kappa n} \chi \).

(i) In the case of \( a > 0 \), assume additionally that

\[
\begin{align*}
& b > \frac{\sqrt{a}}{\kappa} \chi, \\
& a > (1 - \frac{1}{\kappa})^2, \quad b > \frac{(1 - \frac{1}{\kappa} + \sqrt{a})}{4[a - (1 - \frac{1}{\kappa})]} \chi, \\
& \text{if } 0 < \kappa \leq 1, \\
& \text{if } \kappa > 1.
\end{align*}
\]

Then the global bounded solution \((u, v)\) of (5.1) converges exponentially:

\[
\|u(\cdot, t) - \frac{a}{b} \|^2_{L^\infty(\Omega)} + \|v(\cdot, t) - \frac{a}{b} \|^2_{L^\infty(\Omega)} \leq C_\kappa \begin{cases} 
\mu_1 t, & \text{if } 0 < \kappa \leq 1, \\
\mu_2 t, & \text{if } \kappa > 1
\end{cases}
\]

for all \( t \geq 0 \) and some large constant \( C_\kappa \) independent of \( t \). Here

\[ \mu_1 = \frac{a_\kappa}{(n+2)b^2}(b^2 - \frac{a_\chi^2}{16}) > 0 \]

and

\[ \mu_2 = \frac{\kappa}{16ab(n+2)} \left( \frac{a}{b} \right)^{\frac{2-\kappa}{2}} \left( 16[a - (1 - \frac{1}{\kappa})^2]b^2 - 8a_\chi(1 - \frac{1}{\kappa})b - a_\chi^2 \right) > 0. \]

(ii) In the case of \( a = 0 \), the global solution \((u, v)\) of (5.1) converges algebraically:

\[
\|u(\cdot, t)\|_{L^\infty(\Omega)} \leq C_\kappa (t + 1)^{-\frac{1}{n+1}}
\]

and

\[
\|v(\cdot, t)\|_{L^\infty(\Omega)} \leq C_\kappa \begin{cases} 
(t + 1)^{-\frac{1}{n+1}}, & \text{if } 0 < \kappa \leq 1, \\
(1 + \frac{a_\chi}{(n+1)a}) t^{-\frac{1}{n+1}}, & \text{if } \kappa > 1
\end{cases}
\]

for all \( t \geq 0 \) and some large constant \( C_\kappa \) independent of \( t \).

(iii) In the case of \( a < 0 \), the global solution \((u, v)\) of (5.1) converges exponentially:

\[
\|u(\cdot, t)\|_{L^\infty(\Omega)} \leq C_\kappa e^{\frac{a_\kappa}{(n+1)a} t^-^1}, \quad \|v(\cdot, t)\|_{L^\infty(\Omega)} \leq C_\kappa \begin{cases} 
e \frac{\kappa a_\chi}{a_\chi^2} t, & \text{if } 0 < \kappa \leq 1, \\
e \frac{(n+1)}{a_\chi^2} t, & \text{if } \kappa > 1
\end{cases}
\]

for all \( t \geq 0 \) and some large constant \( C_\kappa \) independent of \( t \).

**Corollary 3.** In the case of (i), the equilibrium \(((a/b)^{\frac{1}{\kappa}}, a/b)\) is globally asymptotically stable; in the case of (ii) or (iii), \((0,0)\) is globally asymptotically stable. Thus, under the conditions of the theorem, the chemotaxis system (5.1) has no nonconstant steady state.
Remark 5. Theorem 5.1 gives explicit convergence rates for \((u, v)\), which were not explicitly stated in [6, Theorems 1 and 2] for \(\kappa = 1\); besides, it extends their linear secretion case \((\kappa = 1)\) to nonlinear secretion case \((\kappa \neq 1)\). As can be easily seen from the proof below, the condition \(b \geq \frac{a\kappa - 2}{\kappa b} \chi\) is merely used to ensure uniform boundedness and hence global existence. While, if we have only \(b > 0\), then we can adapt the arguments in [21, 26] to infer that the chemotaxis system (5.1) has a global weak solution which will become eventual smooth and bounded. Therefore, the decay estimates (5.4), (5.5) and (5.6) will continue to hold for \(t \geq T_0\) with some \(T_0 > 0\).

The key of the proof of Theorem 5.1 relies on finding so-called Lyapunov functionals, which are inspired from [1, 6]. Here, we will present all the necessary details for the clarity of obtaining the explicit convergence rates.

**Lemma 5.2.** In the case of (i) of Theorem 5.1, the solution \((u, v)\) of (5.1)\n\[
\int_{\Omega} \left( u - \left( \frac{a}{b} \right)^{\frac{1}{2}} \right)^2 \to 0, \quad \int_{\Omega} (v - \frac{a}{b})^2 \leq \int_{\Omega} (u^\kappa - \frac{a}{b})^2 \to 0 \quad \text{as} \quad t \to \infty. \quad (5.7)
\]

**Proof.** In the case of \(\kappa \in (0, 1]\), motivated by [1, 6], we define the functional\n\[
W(t) = \int_{\Omega} \left( u - c - c \ln \left( \frac{u}{c} \right) \right), \quad c = \left( \frac{a}{b} \right)^{\frac{1}{2}}, \quad t > 0. \quad (5.8)
\]
By taking derivative, one sees easily that \(w(s) = s - c - c \ln \left( \frac{s}{c} \right), s > 0\) achieves its global minimum zero at \(s = c\). Hence, \(W(t) = \int_{\Omega} w(u) \geq 0\) for all \(t \geq 0\).

Using the first equation in (5.1), we deduce from Cauchy-Schwarz inequality that\n\[
\frac{d}{dt} W(t) = - \int_{\Omega} \frac{u - c}{u} u_t \leq \int_{\Omega} \frac{u - c}{u} \left( \nabla \cdot (\nabla u - \chi u \nabla v) + u(a - bu^\kappa) \right) \quad (5.9)
\]
\[
= -c \int_{\Omega} \frac{\nabla u}{u} \cdot \nabla v - c \chi \int_{\Omega} \nabla v - b \int_{\Omega} (u - c)(u^\kappa - c^\kappa) \\
\leq \frac{c \chi^2}{4} \int_{\Omega} |\nabla v|^2 - b \int_{\Omega} (u - c)(u^\kappa - c^\kappa).
\]

Multiplying the second equation in (5.1) by \((v - c^\kappa)\), we find\n\[
\int_{\Omega} |\nabla v|^2 = - \int_{\Omega} (v - c^\kappa)^2 + \int_{\Omega} (u^\kappa - c^\kappa)(v - c^\kappa). \quad (5.10)
\]
We then substitute (5.10) into (5.9) to derive\n\[
\frac{d}{dt} W(t) \leq \frac{c \chi^2}{4} \int_{\Omega} (v - c^\kappa)^2 + \frac{c \chi^2}{4} \int_{\Omega} (u^\kappa - c^\kappa)(v - c^\kappa) \\
- b \int_{\Omega} (u - c)(u^\kappa - c^\kappa) \\
\leq \frac{c \chi^2}{16} \int_{\Omega} (u^\kappa - c^\kappa)^2 - b \int_{\Omega} (u - c)(u^\kappa - c^\kappa) \quad (5.11)
\]
\[
\leq \frac{(c \chi^2}{16} - b) \int_{\Omega} (u - c)(u^\kappa - c^\kappa) = -(b - \frac{a \chi^2}{16b}) \int_{\Omega} (u - c)(u^\kappa - c^\kappa) \\
:= - \delta \int_{\Omega} (u - c)(u^\kappa - c^\kappa),
\]
where we used the fact that
\[(u^\kappa - c^\kappa)^2 \leq c^{\kappa-1}(u - c)(u^\kappa - c^\kappa)\] (5.12)
due to \(\kappa \in (0, 1]\). Then, for any \(t_0 \geq 0\), an integration of (5.11) on \((t_0, t)\) yields
\[W(t) - W(t_0) \leq -\delta \int_{t_0}^{t} \int_\Omega (u - c)(u^\kappa - c^\kappa),\]
and then the nonnegativity of \(G\) and the nonnegativity of \(\delta\) ensured by \(b > \frac{\sqrt{\alpha}}{4}\) give
\[\int_{t_0}^{\infty} \int_\Omega (u - c)(u^\kappa - c^\kappa) \leq \frac{W(t_0)}{\delta} < \infty.\]
Since \(b \geq \frac{\kappa - 2}{\kappa} \chi\), it follows from Theorem 3.2 that \((u, v)\) is a global bounded classical solution. Hence, the standard parabolic regularity for parabolic equations (cf. [14]) shows the existence of \(\sigma \in (0, 1)\) and \(C > 0\) such that
\[\|u\|_{L^2(\Omega \times [t, t+1])}^{\kappa-1} + \|v\|_{L^2(\Omega \times [t, t+1])} \leq C, \quad \forall t \geq 1.\] (5.13)
This clearly shows that \(\int_\Omega (u - c)(u^\kappa - c^\kappa)\) is globally bounded and uniformly continuous with respect to \(t\). Therefore, upon a use of (5.12), one has
\[\frac{1}{c^{\kappa-1}} \int_\Omega (u^\kappa - c^\kappa)^2 \leq \int_\Omega (u - c)(u^\kappa - c^\kappa) \to 0 \text{ as } t \to \infty.\] (5.15)
On the other hand, we apply Hölder inequality to (5.10) to obtain
\[\int_\Omega |\nabla v|^2 \leq -\frac{1}{2} \int_\Omega (v - c^\kappa)^2 + \frac{1}{2} \int_\Omega (u^\kappa - c^\kappa)^2\]
and so
\[\int_\Omega (v - c^\kappa)^2 \leq \int_\Omega (u^\kappa - c^\kappa)^2 \to 0 \text{ as } t \to \infty.\] (5.14)
Since \(u\) is bounded, say \(u(x, t) \leq R\) on \(\bar{\Omega} \times [0, \infty)\), then, for \(z(x) = x^\frac{\kappa}{2}\), the mean value theorem gives \(u - c = z(u^\kappa) - z(c^\kappa) = \frac{1}{\kappa} \frac{a}{b} \ln(\frac{b}{a} u^\kappa)\) for some \(\xi\) between \(R^\kappa\) and \(c^\kappa\). So
\[\int_\Omega (u - c)^2 \leq \frac{R^{2(\kappa-1)}}{\kappa^2} \int_\Omega (u^\kappa - c^\kappa)^2 \to 0 \text{ as } t \to \infty.\] (5.15)
This proves the lemma for the case that \(0 < \kappa \leq 1\). Similarly, in the case of \(\kappa > 1\), we consider the functional
\[H(t) = \frac{1}{\kappa} \int_\Omega \left( u^\kappa - \frac{a}{b} - \frac{a}{b} \ln(\frac{b}{a} u^\kappa) \right), \quad t > 0.\] (5.16)
Since the function \(h(s) = s - \frac{a}{b} - \frac{a}{b} \ln(\frac{b}{a} s)\) achieves its global minimum zero over \((0, \infty)\) at \(s = \frac{b}{a}\), it then follows that \(H(t) = \frac{1}{\kappa} \int_\Omega h(u^\kappa) \geq 0\) for all \(t \geq 0\).
Using the first equation in (5.1), we deduce from Cauchy-Schwarz inequality that
\[
\frac{d}{dt} H(t) = \int_{\Omega} \frac{u^\kappa - a}{u} u_t
\]
\[
= \int_{\Omega} \frac{u^\kappa - a}{u} \left( \nabla \cdot (\nabla u - \chi u \nabla v) + u(a - bu^\kappa) \right)
\]
\[
= -\frac{a}{b} \int_{\Omega} \frac{\nabla u^2}{u^2} + \frac{a \chi}{b} \int_{\Omega} \frac{\nabla u}{u} \cdot \nabla v - (\kappa - 1) \int_{\Omega} u^{\kappa - 2} |\nabla u|^2
\]
\[
+ \chi(\kappa - 1) \int_{\Omega} u^{\kappa - 1} \nabla u \nabla v - b \int_{\Omega} (u^\kappa - \frac{a}{b})^2
\]
\[
\leq \frac{a \chi^2}{4b} \int_{\Omega} |\nabla v|^2 - (\kappa - 1) \int_{\Omega} u^{\kappa - 2} |\nabla u|^2
\]
\[
- [b - (1 - \frac{1}{\kappa}) \chi] \int_{\Omega} (u^\kappa - \frac{a}{b})^2 - (1 - \frac{1}{\kappa}) \chi \int_{\Omega} (u^\kappa - \frac{a}{b})(v - \frac{a}{b}).
\]
Testing the second equation in (5.1) by \( (u^\kappa - \frac{a}{b}) \), we have
\[
\kappa \int_{\Omega} u^{\kappa - 1} \nabla u \nabla v = - \int_{\Omega} (u^\kappa - \frac{a}{b})(v - \frac{a}{b}) + \int_{\Omega} (u^\kappa - \frac{a}{b})^2. \tag{5.18}
\]
A substitution of (5.18) into (5.17) gives rise to
\[
\frac{d}{dt} H(t) \leq \frac{a \chi^2}{4b} \int_{\Omega} |\nabla v|^2 - (\kappa - 1) \int_{\Omega} u^{\kappa - 2} |\nabla u|^2
\]
\[
- [b - (1 - \frac{1}{\kappa}) \chi] \int_{\Omega} (u^\kappa - \frac{a}{b})^2 - (1 - \frac{1}{\kappa}) \chi \int_{\Omega} (u^\kappa - \frac{a}{b})(v - \frac{a}{b}). \tag{5.19}
\]
Multiplying the second equation in (5.1) by \( (v - \frac{a}{b}) \), we get
\[
\int_{\Omega} |\nabla v|^2 = - \int_{\Omega} (v - \frac{a}{b})^2 + \int_{\Omega} (u^\kappa - \frac{a}{b})(v - \frac{a}{b}). \tag{5.20}
\]
Combining (5.19) and (5.20) and using Cauchy-Schwarz inequality, we get
\[
\frac{d}{dt} H(t) + (\kappa - 1) \int_{\Omega} u^{\kappa - 2} |\nabla u|^2
\]
\[
\leq -[b - (1 - \frac{1}{\kappa}) \chi] \int_{\Omega} (u^\kappa - \frac{a}{b})^2 - \frac{a \chi^2}{4b} \int_{\Omega} (v - \frac{a}{b})^2
\]
\[
+ (\frac{a \chi^2}{4b} - (1 - \frac{1}{\kappa}) \chi) \int_{\Omega} (u^\kappa - \frac{a}{b})(v - \frac{a}{b})
\]
\[
\leq -\left\{ [b - (1 - \frac{1}{\kappa}) \chi] - \frac{b \chi}{a} \frac{a \chi}{4b} - (1 - \frac{1}{\kappa})^2 \right\} \int_{\Omega} (u^\kappa - \frac{a}{b})^2
\]
\[
:= -\epsilon \int_{\Omega} (u^\kappa - \frac{a}{b})^2.
\]
A simple calculation from the second assumption in (5.2) shows that \( \epsilon > 0 \) and then
an integration of the above inequality from any fixed \( t_0 \geq 0 \) to \( t \) entails
\[
H(t) - H(t_0) \leq -\epsilon \int_{t_0}^{t} \int_{\Omega} (u^\kappa - \frac{a}{b})^2,
\]
and thus the nonnegativity of \( H \) yields
\[
\int_{t_0}^{\infty} \int_{\Omega} (u^\kappa - \frac{a}{b})^2 \leq \frac{H(t_0)}{\epsilon} < \infty.
\]
Again, the global boundedness and uniform continuity of $\int_\Omega (u^\kappa - \frac{a}{b})^2$ in $t$ entails
\[
\int_\Omega (u^\kappa - \frac{a}{b})^2 \to 0 \text{ as } t \to \infty.
\]

A simple use of Hölder inequality to (5.20) immediately shows
\[
\int_\Omega |\nabla u|^2 \leq -\frac{1}{2} \int_\Omega (v - \frac{a}{b})^2 + \frac{1}{2} \int_\Omega (u^\kappa - \frac{a}{b})^2
\]
and so
\[
\int_\Omega (v - \frac{a}{b})^2 \leq \int_\Omega (u^\kappa - \frac{a}{b})^2 \to 0 \text{ as } t \to \infty.
\]

The fact $\kappa > 1$ leads to
\[
M = \sup_{z \in (0, \infty)} \frac{(z - \frac{a}{b})^\frac{1}{\kappa}}{(z^\kappa - \frac{a}{b})^2} < \infty.
\]

and thus
\[
\int_\Omega (u - \frac{a}{b})^\frac{1}{\kappa} \leq M \int_\Omega (u^\kappa - \frac{a}{b})^2 \to 0 \text{ as } t \to \infty,
\]

Finally, the $L^2$-convergence in (5.7) follows from (5.23) and (5.22).

**Proof of (i) of Theorem 5.1.** We conclude from the Gagliardo-Nirenberg inequality (2.1), (5.7), (5.15), (5.13) and (5.23) of Lemma 5.2 that
\[
\|u(\cdot, t) - \frac{a}{b}\|_{L^\infty(\Omega)} \leq C_{GN}\|u(\cdot, t) - \frac{a}{b}\|_{W^{1, \infty}(\Omega)} \leq C\|u(\cdot, t) - \frac{a}{b}\|_{L^2(\Omega)} \leq C\|u^\kappa(\cdot, t) - \frac{a}{b}\|_{L^2(\Omega)} \to 0 \text{ as } t \to \infty.
\]

Based on the definition of $W$ in (5.8) and (5.11), we compute via the L’Hospital rule that
\[
\lim_{u \to c} \frac{w(u)}{(u - c)(u^\kappa - c^\kappa)} = \lim_{u \to c} \frac{u - c - c \ln \frac{u}{c}}{(u - c)(u^\kappa - c^\kappa)} = \frac{1}{2c\kappa c^\kappa}, \quad c = \frac{a}{b}.
\]

This together with (5.24) allows one to find $t_1 \geq 0$ such that
\[
\frac{1}{4\kappa c^\kappa}(u(t) - c)(u^\kappa(t) - c^\kappa) \leq w(u(t)) \leq \frac{1}{\kappa c^\kappa}(u(t) - c)(u^\kappa(t) - c^\kappa), \quad t \geq t_1,
\]

and so
\[
\frac{1}{4\kappa c^\kappa} \int_\Omega (u(t) - c)(u^\kappa(t) - c^\kappa) \leq W(t) = \int_\Omega w(u(t)) \leq \frac{1}{\kappa c^\kappa} \int_\Omega (u(t) - c)(u^\kappa(t) - c^\kappa), \quad t \geq t_1.
\]

By (5.11) and (5.25), we establish an ordinary differential inequality (ODI) for $W$:
\[
\frac{d}{dt} W(t) \leq -\delta c\kappa W(t), \quad t \geq t_1,
\]

directly implying
\[
W(t) \leq W(t_1)e^{-\delta c\kappa(t-t_1)}, \quad t \geq t_1.
\]
Accordingly, we deduce from (5.12), (5.24), (5.25) and (5.26) that
\[
\|u(\cdot, t) - c\|_{L^\infty(\Omega)} \leq C_\kappa \|u^\kappa(\cdot, t) - c^\kappa\|_{L^2(\Omega)} e^{\frac{\kappa}{4} h(t)} e^{-\frac{c}{2\kappa} h(t)}, \quad t \geq t_1. 
\] (5.27)

Keeping (5.14) and (5.27) in mind and repeating the similar arguments for \( v \), we arrive at
\[
\|v(\cdot, t) - c^\kappa\|_{L^\infty(\Omega)} \leq C_\kappa \left(4\kappa c^{2\kappa-1}W(t_1)\right)^{\frac{1}{4\kappa}} e^{-\frac{c}{2\kappa} h(t)}, \quad t \geq t_1. 
\] (5.28)

Similarly, in the case of \( \kappa > 1 \), by the definition of \( H \) in (5.16) and (5.21), we calculate
\[
\lim_{u \to c} \frac{h(u^\kappa)}{(u^\kappa - c^\kappa)^2} = \lim_{z \to c^\kappa} \frac{z - c^\kappa - c^\kappa \ln\left(\frac{z}{c^\kappa}\right)}{(z - c^\kappa)^2} = \frac{c^{\kappa-2}}{2\kappa}, \quad c = \left(\frac{a}{b}\right)^{\frac{1}{\kappa}}. 
\]

This in conjunction with (5.24) gives the existence of \( t_2 \geq 0 \) such that
\[
\frac{c^{\kappa-2}}{4\kappa} (u^\kappa(t) - c^\kappa)^2 \leq h(u^\kappa(t)) \leq \frac{c^{\kappa-2}}{\kappa} (u^\kappa(t) - c^\kappa)^2, \quad t \geq t_2, 
\]
and so
\[
\frac{c^{\kappa-2}}{4\kappa} \int_\Omega (u^\kappa(t) - c^\kappa)^2 \leq H(t) = \int_\Omega \frac{h(u^\kappa(t))}{\kappa} \leq \frac{c^{\kappa-2}}{\kappa} \int_\Omega (u^\kappa(t) - c^\kappa)^2, \quad t \geq t_2. 
\] (5.29)

By (5.21) and (5.29), we establish an ODE for \( H \):
\[
\frac{d}{dt} H(t) \leq -\epsilon c^{2-\kappa} H(t), \quad t \geq t_2, 
\]
which quickly gives
\[
H(t) \leq H(t_2) e^{-\epsilon c^{2-\kappa} (t-t_2)}, \quad t \geq t_2. 
\] (5.30)

Finally, we infer from (5.24), (5.29) and (5.30) that
\[
\|u(\cdot, t) - c\|_{L^\infty(\Omega)} \leq C_\kappa \|u^\kappa(\cdot, t) - c^\kappa\|_{L^2(\Omega)} e^{\frac{\kappa}{4} h(t)} e^{-\frac{c}{2\kappa} h(t)}, \quad t \geq t_2. 
\] (5.31)

Analogously, taking (5.14) and (5.27) into account, one can easily get
\[
\|v(\cdot, t) - c^\kappa\|_{L^\infty(\Omega)} \leq C_\kappa \left(4\kappa c^{2\kappa-1}H(t_2)\right)^{\frac{1}{4\kappa}} e^{-\frac{c}{2\kappa} h(t)}, \quad t \geq t_2. 
\] (5.32)

Finally, plugging the definitions of \( \delta \) and \( \epsilon \) into (5.27), (5.28), (5.31) and (5.32) and taking large \( C_\kappa \), we end up with the decay estimate (5.3).

\[\square\]

**Remark 6.** In the absence of chemotaxis, i.e., \( \chi = 0 \), we get from (5.17) that
\[
\frac{d}{dt} H(t) \leq -b \int_\Omega (u - \frac{a}{b})^2. 
\]

Consequently, the estimates (5.31) and (5.32) imply the exponential convergence:
\[
\|u(\cdot, t) - \left(\frac{a}{b}\right)^{\frac{1}{\kappa}}\|_{L^\infty(\Omega)} + \|v(\cdot, t) - \frac{a}{b}\|_{L^\infty(\Omega)} \leq C_\kappa e^{-\frac{\kappa}{4\kappa} (t-t_2)^{\frac{2}{\kappa}}}, \quad t \geq 0. 
\]

This holds true for all \( a > 0 \). While, in the presence of chemotaxis, especially, with super-linear secretion, i.e., \( \kappa > 1 \), we need further restrict \( a \) to satisfy \( a > (1 - \frac{1}{\kappa})^2 \) as stated in (5.2) in order to have such exponential convergence. Hence, there is a
gap left as to whether or not the exponential stabilization of solution still occurs when \(0 < a \leq (1 - \frac{1}{\kappa})^2\).

**Proof of (ii) and (iii) of Theorem 5.1.** It is straightforward to check from the proofs in sections 2-4 that the sign of \(a\) does not play any role in the boundedness and global existence. Thus, \((u, v)\) is still a global bounded classical solution under the condition of Theorem 5.1. In the case of \(a = 0\), we integrate the first equation in (5.1) and use Hölder inequality to obtain

\[
\frac{d}{dt} \int_{\Omega} u = -b \int_{\Omega} u^{\kappa+1} \leq -b|\Omega|^{-\kappa} \left( \int_{\Omega} u \right)^{\kappa+1}, \quad t > 0,
\]

which entails

\[
\int_{\Omega} u \leq \left( \int_{\Omega} u_0 \right)^{-\kappa} + b|\Omega|^{-\kappa} t^{-\frac{1}{\kappa}}, \quad t > 0.
\]

(5.33)

Hence, the Gagliardo-Nirenberg inequality coupled with the boundedness of \(u\) shows that

\[
\|u(\cdot, t)\|_{L^\infty(\Omega)} \leq C_{GN} \left( \|u(\cdot, t)\|_{W^{1,1}(\Omega)} \right)^\kappa \|u(\cdot, t)\|_{L^\infty(\Omega)}^{1-\kappa} \leq \left( \int_{\Omega} u_0 \right)^{-\kappa} + b|\Omega|^{-\kappa} t^{-\frac{1}{\kappa}}, \quad t > 0.
\]

(5.34)

An integration of the second equation in (5.1) shows

\[
\int_{\Omega} v = \int_{\Omega} u^{\kappa} \leq \left\{ \begin{array}{ll}
|\Omega|^{1-\kappa} \left( \int_{\Omega} u_0 \right)^{-\kappa} + b|\Omega|^{-\kappa} t & \text{if } 0 < \kappa \leq 1, \\
\|u(\cdot, t)\|_{L^\infty(\Omega)}^{\kappa} \int_{\Omega} u_0 & \text{if } \kappa > 1.
\end{array} \right.
\]

This combined with (5.33) and (5.34) yields

\[
\int_{\Omega} v \leq \left\{ \begin{array}{ll}
|\Omega|^{1-\kappa} \left( \int_{\Omega} u_0 \right)^{-\kappa} + b|\Omega|^{-\kappa} t & \text{if } 0 < \kappa \leq 1, \\
\left( \int_{\Omega} u_0 \right)^{-\kappa} + b|\Omega|^{-\kappa} t & \text{if } \kappa > 1.
\end{array} \right.
\]

Then we conclude from (5.34) with \(u\) replaced by \(v\) that, \(t > 0,\)

\[
\|v(\cdot, t)\|_{L^\infty(\Omega)} \leq \left\{ \begin{array}{ll}
C_\kappa \left( \int_{\Omega} u_0 \right)^{-\kappa} + b|\Omega|^{-\kappa} t^{-\frac{1}{\kappa}} & \text{if } 0 < \kappa \leq 1, \\
C_\kappa \left( \int_{\Omega} u_0 \right)^{-\kappa} + b|\Omega|^{-\kappa} t^{-\frac{1}{\kappa}} & \text{if } \kappa > 1.
\end{array} \right.
\]

(5.35)

In the case of \(a < 0\), we integrate the first equation in (5.1) to get

\[
\frac{d}{dt} \int_{\Omega} u = a \int_{\Omega} u - b \int_{\Omega} u^{\kappa+1} \leq -a \int_{\Omega} u, \quad t > 0,
\]

and thus

\[
\int_{\Omega} u \leq e^{at} \int_{\Omega} u_0, \quad t > 0.
\]

(5.36)

Then the GN inequality (5.34) implies

\[
\|u(\cdot, t)\|_{L^\infty(\Omega)} \leq C_\kappa e^{\frac{\kappa-1}{\kappa+1} t} \left( \int_{\Omega} u_0 \right)^{\frac{1}{\kappa+1}}, \quad t > 0.
\]

(5.37)

With this decay estimate at hand, using the discussions leading to (5.35), we derive

\[
\|v(\cdot, t)\|_{L^\infty(\Omega)} \leq \left\{ \begin{array}{ll}
C_\kappa e^{\frac{\kappa-1}{\kappa+1} t} (\int_{\Omega} u_0)^{-\frac{1}{\kappa+1}} & \text{if } 0 < \kappa \leq 1, \\
C_\kappa e^{\frac{\kappa-1}{\kappa+1} t} (\int_{\Omega} u_0)^{-\frac{1}{\kappa+1}} & \text{if } \kappa > 1.
\end{array} \right.
\]

(5.38)

Extracting the essential ingredients of the estimates (5.34), (5.35), (5.37) and (5.38), we readily conclude the decay estimates (5.4), (5.5) and (5.6).
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