Recursive Weak- and Strong Coupling
Expansions in a Cosine Potential.

March 31, 2022

Abstract
For the \( \text{Cos}(2x) \)-Potential the coefficients of the weak- and strong
coupling perturbation series of the ground state energy are constructed recursi
vely. They match the well-known expansion coefficients of the Mathieu
equation’s characteristic values. However presently there is no physically
intuitive method to extract the coefficients of the strong coupling series
from those of the weak one. The standard rule while giving excellent results
for the anharmonic oscillator fails completely in this case.

1 Introduction
The perturbative treatment of quantum-mechanical or field-theoretical
problems renders in general results in the form of divergent infinite power
series in some coupling constant \( g \). Various resummation schemes have
been applied to them, aiming at finite results for all values of the coupling
constant \( g \). For the anharmonic oscillator excellent results are known.(See
e.g. Chap. 16 of the book of Kleinert and Frohlinde\[1\] and the references
therein). Since the real value of the resummation schemes lies in their
generalization and application to field theories, it is worth while to study,
how generally applicable they are. But this requires models for which
higher order coefficients of the perturbation expansion are available. This is
practically only the case, if the coefficients can be constructed recursively.
Known models with this property are all closely related to the harmonic
oscillator. They permit weak coupling expansions only. Here we present
a model, which is based instead on the infinite square well potential. It
permits the recursive construction of a weak- as well as a strong coupling
series. Both series seem to have a finite radius of convergence and agree
with numerical data. A mapping of one series to the other, based on
physical intuition, is not known at present.

2 The Model
Consider the one dimensional Schrödinger equation:
\[
-\frac{1}{2} \Psi'' + (V_0 + g V_1) \Psi = E(g) \Psi,
\] (1)
where the coupling constant $g$ may take on positive or negative values. Its unperturbed potential is chosen to be the infinite square well:

$$V_0(x) = \begin{cases} 
0 & \text{if } |x| < \frac{\pi}{2} \\
\infty & \text{otherwise}
\end{cases}$$

As perturbing potential we take:

$$V_1(x) = \cos(2x)$$

The ground state energy $E_0(g)$ is known to be the characteristic value $b_1(g)$ of the Mathieu equation. We show now, how the expansion coefficients of $E_0(g) = b_1(g)$ can be obtained recursively in a quantum mechanical context.

### 3 The Weak Coupling Series

The recursion relation for the weak coupling coefficients $\epsilon_i$ in the series

$$E_0(x) = \sum_{i=0}^{\infty} \epsilon_i g^i$$

for the ground state energy will be obtained by the method of Hamprecht and Pelster [2]. To start off, we need to know the matrix elements of the perturbing potential $V_1(x)$ in the basis of unperturbed eigenfunctions:

$$\Psi_n(x) = \frac{\sqrt{2}}{\pi} \begin{cases} 
\cos nx & \text{for } n = 1, 3, 5, \ldots \\
\sin nx & \text{for } n = 2, 4, 6, \ldots
\end{cases}$$

They can be obtained by simple integration:

$$V_{n,m} := \langle \Psi_n \mid \cos(2x) \mid \Psi_m \rangle = \frac{1}{2} \begin{cases} 
1 & \text{if } m = n = 1 \\
1 & \text{if } |n - m| = 2 \text{ and } n \text{ odd} \\
1 & \text{if } |n - m| = 2 \text{ and } n \text{ even} \\
0 & \text{otherwise}
\end{cases}$$

Since the matrix $V_{n,m}$ is band-diagonal, the method of Hamprecht and Pelster applies and produces the well known weak coupling coefficients as shown in table 1. They are of Borell type in the sense, that triplets of consecutive coefficients alternate in sign. Also they decrease with an average ratio of about 4, so that the weak coupling series will have a radius of convergence of $|g| \simeq 4$. The recurrence relation for the $\epsilon_i$ can be found in appendix A.

### 4 The Strong Coupling Series

In this section we investigate the limit $g \to -\infty$. Scaling the x-axis with $x \to \alpha x$ with $\alpha = 1/\sqrt{|g|}$ transforms the Schrödinger equation (1) into:

$$-\frac{1}{2} \Psi'' + 2 \sin(2\alpha x) \Psi = e \Psi$$

(2)
Table 1: The first few coefficients of the weak coupling expansion for the ground state of the \(\cos(2x)\)-potential inside an infinite square well.

| Square Well with \(\cos(2x)\) Weak Coupling |
|--------|---|
| \(i\) | \(\epsilon_i\) |
| 0 | 1/2 |
| 1 | 1/2 |
| 2 | \(-1/16\) |
| 3 | \(-1/128\) |
| 4 | \(-1/3072\) |
| 5 | \(11/73728\) |
| 6 | \(49/1179648\) |
| 7 | \(55/18874368\) |
| 8 | \(-83/70778880\) |
| 9 | \(-124121/30198988800\) |
| 10 | \(-114299/3261490790400\) |
| 11 | \(192151/15655155793920\) |
| 12 | \(83513957/17533774489190400\) |

where \(E_0(g) = |g|(e - 1)\). A solution to this equation will be found by an expansion of \(\log(\psi)\) in powers of \(\alpha\). The square well barriers are withdrawn to \(|x| = \pi/(2\sqrt{|g|})\), so that they fall into a region where for small \(|g|\) the wavefunction \(\psi\) is exponentially small. Therefore these barriers will have no influence on the power expansion. With

\[
\psi = \exp(-\Phi), \quad \Phi = \sum_{k=1,j=1} c_{k,j} \alpha^k x^{2j} \quad \text{and} \quad e = \sum_{i=1} e_i \alpha^i
\]

inserted into equation (3), a recurrence relation for the \(c_{k,j}\) is obtained, which has a unique solution, if one takes the following initial conditions into account:

- Working up to order \(2n\) in \(\alpha\) and \(x\), we put \(c_{k,j} = 0\) for \(k > 2n + 1\).
- The power of \(x\) is to be restricted by \(c_{k,j} = 0\) for \(2j > k + 1\).

The results for the coefficients \(c_{k,j}\) are listed in table 3. They agree with the literature values [3]. The recurrence relation for the \(e_i\) can be found in appendix B.

This series is not of Borell type; again it has a finite radius of convergence of \(1/\sqrt{|g|} \approx 4\). The agreement of both expansions with numerical results is shown in figure 1 and in figure 2.

5 Appendix

- A: The Recurrence Relation for the Weak Coupling Coefficients
Figure 1: The first few approximations of order $n = 3, 5, 7, 9$ to the rescaled ground state energy $E_0(g)/g$, which tends to the constant value 1 for $g \to \infty$ are shown in comparison to numerical results (big dots). For small $|g|$ the weak coupling series (dotted lines) give good agreement and for large $|g|$ the strong coupling approximations (solid lines) fit very well.
Table 2: The first few coefficients of the strong coupling expansion for the ground state of the $\cos(2x)$-potential inside an infinite square well.

| $i$ | $e_i$ |
|-----|-------|
| 0   | 0     |
| 1   | 1     |
| 2   | $-1/8$ |
| 3   | $-1/64$ |
| 4   | $-3/512$ |
| 5   | $-53/16384$ |
| 6   | $-297/131072$ |
| 7   | $-3961/2097152$ |
| 8   | $-30363/16777216$ |
| 9   | $-2095501/1073741824$ |
| 10  | $-20057205/8589934592$ |
| 11  | $-421644859/137438953472$ |
| 12  | $-4828704237/109951162776$ |

Figure 2: The same data as in figure 1, but here for the unscaled ground state energy $E_0(g)$ and for an extended interval of the coupling constant $g$, including positive values as well.

$E_HgL$, to order 3, 5, 7, 9

$\rightarrow g$
\begin{equation*}
\gamma_{k,i} = \begin{cases}
1 & \text{for } k = i = 0 \\
0 & \text{for } k = 0 \text{ and } i \neq 0 \text{ or } i = 0 \text{ and } k \neq 0 \\
\frac{\gamma_{k-1,i} - \gamma_{k-2,i-1} - \gamma_{k+2,i-1} + \sum_{j=2}^{k-1} \gamma_{2,j-1} \gamma_{k,i-j}}{k^2 + 2k} & \text{else}
\end{cases}
\end{equation*}

\begin{equation*}
\epsilon_i = \frac{1}{2} \begin{cases}
1 & \text{for } i = 0, 1 \\
\gamma_{2,i-1} & \text{else}
\end{cases}
\end{equation*}

- **B: The Recurrence Relation for the Strong Coupling Coefficients**

To evaluate the strong coupling coefficients \( e_i \) up to order \( n \) in the expansion parameter \( \alpha = 1/\sqrt{|g|} \) the following relation may be used:

\begin{equation*}
\gamma_{k,j} = \begin{cases}
1 & \text{for } k = j = 1 \\
0 & \text{for } j < 1 \text{ or } k > 2n - 1 \text{ or } 2j > k + 1 \\
\beta_{k,j} + \frac{(-4)^{j-1}}{j(2j)!} & \text{for } 2j = k + 1 \\
\beta_{k,j} & \text{else}
\end{cases}
\end{equation*}

where

\begin{equation*}
(2j+1)(2j+2)\gamma_{k+1,j+1} + \sum_{l=2}^{\min(j, \frac{j+1}{2})} \sum_{i=\left[\frac{j-l-1}{4}\right]}^{\left[\frac{j-l+1}{4}\right]} i(j-i+1) \gamma_{l,i} \gamma_{k+1-l,j+1-i} = 2 \sum_{l=2}^{\min(j, \frac{j+1}{2})} \sum_{i=\left[\frac{j-l-1}{4}\right]}^{\left[\frac{j-l+1}{4}\right]} i(j-i+1) \gamma_{l,i} \gamma_{k+1-l,j+1-i} 
\end{equation*}

\begin{equation*}
\beta_{k,j} = \frac{1}{4j}
\end{equation*}

and

\begin{equation*}
\epsilon_i = \gamma_{i,1}
\end{equation*}
References

[1] Critical Properties of $\phi^4$-Theories (World Scientific, Singapore, 2001).

[2] Fluctuating Paths and Fields, (World Scientific, Singapore, 2001), pg. 347

[3] M. Abramowitz and I. A. Stegun, Handbook of Mathematical Functions, (Dover, 1965)