More on trees and Cohen reals

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In this paper we analyse some questions concerning trees on $\kappa$, both for the countable and the uncountable case, and the connections with Cohen reals. In particular, we provide a proof for one of the implications left open in [6, Question 5.2] about the diagram for regularity properties.

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1 Introduction

Throughout the paper we deal with trees on $\eta^{<\kappa}$, with $\kappa \geq \omega$ being any regular cardinal and $\eta \geq 2$ or if $\eta$ is infinite then $\eta$ regular too. A tree forcing $\mathbb{P}$ is a poset whose conditions are perfect trees $p \subseteq \eta^{<\kappa}$ with the property that for every $p \in \mathbb{P}$ and every $t \in p$ one has $p[t] := \{ t' \in p : t' \subseteq t \cap t' \} \in \mathbb{P}$; the ordering is $q \leq p \iff q \subseteq p$. In case $\kappa = \omega$ and $\eta \in \{2, \omega\}$ some of the most popular tree forcings are, e.g., Hechler forcing $\mathbb{D}$ [1, Definition 3.1.9], eventually different forcing $\mathbb{E}$ [1, Definition 7.4.8], Sacks forcing [2, p. 3], Silver forcing $\mathbb{V}$ [2, p. 4], Miller forcing $\mathbb{M}$ [2, p. 3], Laver forcing [2, p. 3], Mathias forcing $\mathbb{R}$ [2, p. 4], random forcing $\mathbb{B}$ [1, p. 99]. The relation between tree forcings and Cohen reals has been rather extensively developed in the literature. The reason to study such connections for different types of tree forcing notions was mainly to “separate” different kinds of cardinal characteristics, in particular from cov($\mathcal{M}$). We can associate a tree forcing $\mathbb{P}$ in a standard way with a notion of $\mathbb{P}$-nowhere dense sets, $\mathbb{P}$-meagre sets and $\mathbb{P}$-measurable sets.

Definition 1.1 Given $\mathbb{P}$ a tree forcing notion and $X \subseteq \eta^\kappa$ a set of $\kappa$-reals, we say that $X$ is $\mathbb{P}$-nowhere dense if

$$\forall p \in \mathbb{P} \exists q \leq p([q] \cap X = \emptyset),$$

and we put $\mathcal{N}_\mathbb{P} := \{ X : X \text{ is } \mathbb{P}\text{-nowhere dense} \}$. We say that $X$ is $\mathbb{P}$-meagre if there are $A_i \in \mathcal{N}_\mathbb{P}$ such that $X \subseteq \bigcup_{i \in \omega} A_i$, and we put $\mathcal{I}_\mathbb{P} := \{ X : X \text{ is } \mathbb{P}\text{-meagre} \}$. We say that $X$ is $\mathbb{P}$-measurable if

$$\forall p \in \mathbb{P} \exists q \leq p([q] \cap X \in \mathcal{I}_\mathbb{P} \lor [q] \setminus X \in \mathcal{I}_\mathbb{P}).$$

A family $\Gamma$ of subsets of $\kappa$-reals is called well-sorted if it is closed under continuous pre-images. We abbreviate the sentence “every set in $\Gamma$ is $\mathbb{P}$-measurable” by $\Gamma(\mathbb{P})$.

E.g., when $\mathbb{P}$ is Cohen forcing $\mathbb{C}$, then $\mathbb{C}$-meagreness coincides with topological meagreness and $\mathbb{C}$-measurability coincides with the Baire Property. When $\mathbb{P}$ is the Random forcing $\mathbb{R}$, then $\mathbb{R}$-meagreness coincides with Lebesgue measure zero and $\mathbb{R}$-measurability coincides with Lebesgue measurability.

The presence of Cohen reals added by a tree forcing $\mathbb{P}$ has an impact both on the structure of $\mathcal{I}_\mathbb{P}$ and on the corresponding notion of $\mathbb{P}$-measurability, as specified in the tables introduced below. More specifically, if $\mathbb{P}$ adds a Cohen real then the way of coding the $\mathbb{P}$-generic into a Cohen real often induces a construction providing $\Gamma(\mathbb{P}) \Rightarrow \Gamma(\mathbb{C})$ (cf., e.g., [5, Theorem 3.1] where such a connection is shown in case of $\mathbb{P} = \mathbb{D}$). Moreover the presence of a coded Cohen real often implies that $\mathcal{N}_\mathbb{P}$ and $\mathcal{I}_\mathbb{P}$ do not coincide. E.g., this holds for the Hechler forcing $\mathbb{D}$ and for the eventually different forcing $\mathbb{E}$. Both these forcings are c.c.c., and indeed $\sigma$-centered. So, a natural question that arises is whether one can find a non-c.c.c. tree forcing notion $\mathbb{P}$ for which $\Gamma(\mathbb{P}) \Rightarrow \Gamma(\mathbb{C})$.

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and \( \mathcal{I}_d \neq \mathcal{N}_n \). In this paper we give a positive answer, by defining and analysing a variant of Mathias forcing in the space \( 3^\omega \) instead of \( 2^\omega \).

As a more general question, for a tree forcing \( \mathbb{P} \), one can consider the four properties mentioned so far, namely: 1) \( \mathbb{P} \) adds Cohen reals; 2) \( \Gamma(\mathbb{P}) \Rightarrow \Gamma(\mathbb{C}) \); 3) \( \mathcal{I}_d \neq \mathcal{N}_n \); 4) \( \mathbb{P} \) is c.c.c. So, e.g., if we consider the most popular tree forcings we get the following table, where \( T \) stands for the variant of Mathias forcing defined in § 2, and \( M^\text{full} \) is the variant of Miller forcing where we require that every splitting node splits into the whole \( \omega \). The results in Table 1 without an explicit reference are deemed as folklore.

Note that the table above refers to the tree forcings in the \( \omega \)-case, and so defined on spaces like \( 2^\omega \), \( \omega^\omega \) or \( [\omega]^{\omega} \).

For \( \kappa > \omega \) we could consider the same table, but then the situation changes and we can get several different developments. We always assume \( \kappa^{<\kappa} = \kappa \).

1. For \( D_\kappa \) (and similarly for \( E_\kappa \)), the constructions done for the \( \omega \)-case (e.g., the proof of [5, Theorem 3.1]) easily generalise;
2. for the \( \kappa \)-Silver forcing, the situation seems to depend on whether \( \kappa \) is inaccessible or not; but it is rather independent of whether we consider club splitting or other version of \( < \kappa \)-closure;
3. for \( \kappa \)-Mathias forcing, the situation is drastically different from the \( \omega \)-case, as we can prove a strict connection with the Baire property and Cohen reals;

The table for \( \kappa \) uncountable then appears as follows, where \( \kappa \) denotes any cardinal, \( \lambda \) any inaccessible cardinal and \( \gamma \) any not inaccessible cardinal:

| Adding Cohen | \( \mathcal{I}_d \neq \mathcal{N}_n \) | \( \Gamma(\mathbb{P}) \Rightarrow \Gamma(\mathbb{C}) \) | \( \kappa^{+\text{-c.c}} \) |
|-------------|-------------------------------|---------------------------------|----------------|
| \( D_\kappa \) \( \times \) | \( \times \) | \( \times \) | \( \times \) |
| \( E_\kappa \) \( \times \) | \( \times \) | \( \times \) | \( \times \) |
| \( M^\text{club} \) \( \times \) | \( \times \) | \( \times \) | \( \times \) |
| \( M^\text{club} \) \( ? \) | \( ? \) | \( ? \) | \( ? \) |
| \( M^\text{club} \) \( \times \) | \( \times \) | \( \times \) | \( \times \) |
| \( E_\kappa \) \( \times \) | \( \times \) | \( \times \) | \( \times \) |

Table 2 Properties of tree forcings on \( \kappa \).

Basic notions and definitions. The elements in \( \eta^\kappa \) are called \( \kappa \)-reals or \( \kappa \)-sequences, where \( \eta \) is also a regular cardinal, usually \( \eta = 2 \) or \( \eta = \kappa \). Given \( s, t \in \eta^{<\kappa} \) we write \( s \perp t \) if and only if neither \( s \subseteq t \) nor \( t \subseteq s \) (and we say \( s \) and \( t \) are incompatible). The following notations are also used.

1. A tree \( p \subseteq \eta^{<\kappa} \) is a subset closed under initial segments and its elements are called nodes. We consider \( < \kappa \)-closed trees \( p \), i.e., for every \( \subseteq \)-increasing sequence of length \( < \kappa \) of nodes in \( p \), the supremum (i.e., union) of these nodes is still in \( p \). Moreover, we abuse of notation denoting by \( |t| \) the ordinal \( \text{dom}(t) \).
2. We say that a \( < \kappa \)-closed tree \( p \) is perfect if and only if for every \( s \in p \) there exists \( t \supseteq s \) and \( \alpha, \beta \in \eta \), \( \alpha \neq \beta \), such that \( t \uparrow \alpha \in p \) and \( t \uparrow \beta \in p \); we call such \( t \) a splitting node (or splitnode) and set \( \text{Split}(p) := \{ t \in p : t \text{ is splitting} \} \).
3. We say that a splitnode \( t \in p \) has order type \( \alpha \) (and we write \( t \in \text{Split}_\alpha(p) \)) if and only if \( \text{ot}(\{ s \in p : s \subseteq t \land s \in \text{Split}(p) \}, \subseteq) = \alpha \).

\( \{ \alpha \} \) denote any not inaccessible cardinal.

Table 1 Properties of classical tree forcings.
4. stem$(p)$ is the longest node in $p$ which is compatible with every node in $p$;
\[ p[t := \{s \in p : s \text{ is compatible with } t\}]. \]
5. $|p| \defeq \{x \in n^\omega : \alpha < x(\alpha \in p)\}$ is called the set of branches (or body) of $p$.
6. $\text{succ}(t, p) \defeq \{\alpha \in n : t^\alpha \in p\}$, for $t \in p$.
7. A poset $\mathbb{P}$ is called tree forcing if its conditions are perfect trees and for every $p \in \mathbb{P}$, and every $t \in p$, one has $p[t \in \mathbb{P}$ too.

**Remark 1.2** When comparing different notions of $\mathbb{P}$-measurability, i.e., investigating the relationship between $\Gamma(\mathbb{P})$ and $\Gamma(\mathbb{Q})$ for different tree forcings $\mathbb{P}$ and $\mathbb{Q}$, we often refer to different topological spaces. As Brendle pointed out explicitly in [2] the idea is to consider the analogous versions in the space of strictly increasing sequences $\omega^{\omega\omega}$ which can be seen to be almost isomorphic to the spaces we deal with (for the details, cf. [2, §1.2]). The only case that is not covered in [2] is $\omega^{\omega\omega}$. In this paper we need to implement this case as well, as we are going to work with it in the coming section. Actually in trying to describe a suitable isomorphism, we need to consider a special subspace, in the same fashion as we do when we consider only the subspace of $2^\omega$ consisting of binary sequences that are not eventually 0. Analogously we consider $H \defeq \{x \in \omega^\omega : \exists^\infty n(x(n) = 2)\}$ and we define the appropriate map $\varphi : H \to \omega^{\omega\omega}$ as follows: we fix the lexicographic enumeration $b : 2^{\omega\omega} \to \omega$. So $b(s) \leq b(t)$, whenever $s \leq t$ and in particular $b(\emptyset) = 0$. For every $x \in H$ let $\{n_k : k \in \omega\}$ enumerate the set of all inputs $n$ such that $x(n) = 2$. Then define $\sigma^0_n : = (x(i) : 0 \leq i < n_0)$ and for every $j \in \omega$, $\sigma^{j+1}_n : = (x(i) : n_j < i < n_{j+1})$. Finally put
\[ \varphi(x) : = (b(\sigma^0_n), b(\sigma^0_n) + b(\sigma^1_n) + 1, b(\sigma^0_n) + b(\sigma^1_n) + b(\sigma^2_n) + 2, \ldots) = (\sum_{i \leq n} b(\sigma^i_n) + n : n \in \omega). \]
One can easily check that $\varphi$ is an isomorphism.

## 2 A variant of Mathias forcing

**Definition 2.1** We define $\mathbb{T}$ as the tree forcing consisting of perfect trees $p \subseteq 3^{<\omega}$ with $A_p \subseteq \omega$ such that
1. for every $t \in p$ ($|t| \in A_p \iff t \in \text{Split}(p)$), we refer to $A_p$ as the set of splitting levels of $p$;
2. if $t \in \text{Split}(p)$, then $t$ is fully splitting (i.e., for every $i \in 3, t^i \in p$);
3. for every $s \subseteq \text{stem}(p)$, if $s \notin \text{Split}(p)$ then $s^2 \notin p$;
4. for every $s, t \in p$, $|s| = |t|$, $s, t \notin \text{Split}(p)$, one has
\[ \forall i \in 2(x^i \in p \iff t^i \in p). \]

Intuitively, any condition $p \in \mathbb{T}$ is a perfect tree in $3^{<\omega}$ such that at any level $n \in \omega$ either $p$ uniformly splits, or uniformly takes the same value.

Note that $\mathbb{T}$ is not c.c.c. To show that let $E \subseteq \omega$ be the set of even numbers and $O = \omega \setminus E$. For each $a \subseteq O$ we define a condition $p_a \in \mathbb{T}$ in the following way: on even levels we uniformly split and on odd levels $n$ we uniformly choose the value 1 whenever $n \in a$ and 0 otherwise, so
\[ p_a : = \{t \in 3^{<\omega} : \forall n \in O \cap |t| ((n \in a \to t(n) = 1) \land (n \notin a \to t(n) = 0))\}. \]

We claim that $\{p_a : a \subseteq O\}$ is an antichain. In fact, let $a, b \subseteq O$ be two different subsets and fix $n \in O$ such that $n \in a \setminus b$ or $n \in b \setminus a$. Without loss of generality, assume $n \in a \setminus b$. Then each branch $x$ through $p_a$ must satisfy $x(n) = 1$, whereas each branch $y$ through $b$ satisfies $y(n) = 0$. Thus $[p_a] \cap [p_b] = \emptyset$ and in particular $p_a \perp p_b$.

Under a certain point of view $\mathbb{T}$ seems to behave like the original Mathias forcing $\mathbb{R}$. E.g., the following proof showing that $\mathbb{T}$ satisfies Axiom A follows the same line as for $\mathbb{R}$. However, going more deeply one has to be careful, as even if $\mathbb{T}$ still satisfies quasi pure decision (Lemma 2.3), it fails to satisfy pure decision (Lemma 2.4). Thus, we examine these proofs in closer detail to better understand the main differences between $\mathbb{T}$ and $\mathbb{R}$.

**Proposition 2.2** The forcing $\mathbb{T}$ satisfies Axiom A.

**Proof.** We define the partial orderings $\langle \leq_n : n \in \omega \rangle$ in the expected way: For $p, q \in \mathbb{T}$ we put $q \leq_n p$ if and only if $q \leq p$ and the two sets of splitting levels $A_q$ and $A_p$ coincide on the first $n + 1$ elements. So, in
particular \( q \leq_0 p \) implies \( \text{stem}(q) = \text{stem}(p) \). It is easy to check that fusion sequences exist. Let \( p \in \mathbb{T} \), \( k \in \omega \) and \( D \subseteq \mathbb{T} \) a dense subset be given. We show that there is a stronger condition \( q \leq_k p \) and a finite set \( E \subseteq D \) pre dense below \( q \). This proves that \( \mathbb{T} \) satisfies Axiom A. Let \( A_p = \{ n_i : i < \omega \} \) be an increasing enumeration of the splitting levels of \( p \). Observe that there are exactly \( 3^k \) nodes \( t \in p \) of length \( n_k \). Each of those nodes is splitting, so that there are exactly \( 3^{k+1} \) immediate successor-nodes. Let \( \{ t_i : i < 3^{k+1} \} \) enumerate all nodes \( t \in p \) of length \( n_k + 1 \). We construct \( q \leq_k p \) together with a decreasing sequence \( p = q_0 \geq q_1 \geq \ldots \geq q_{3^{k+1}} = q \). Assume we want to construct \( q_{j+1} \). Find \( p_j \in D \) so that \( p_j \leq q_j | t_j \) (this is always possible since \( D \) is dense). We define \( q_{j+1} \) to be the condition which is obtained from \( q_j \), by copying \( p_j \) above each node in \( q_j \) of length \( n_k + 1 \). More precisely:

\[
q_{j+1} := \{ t \in q_j : \{ 1 \wedge \exists x \in p_j \forall n \in \omega \\
\quad (n_k < n < |t| \rightarrow s(n) = t(n)) \} 
\]

It follows from the construction that for \( q := q_{3^{k+1}} \) and \( j < 3^{k+1} \) we must have \( q | t_j \leq p_j \). In particular, we have that \( q \leq_k p \). Put \( E := \{ p_j : j < 3^{k+1} \} \). We want to check that \( E \) is pre dense below \( q \). Therefore, let \( r \leq q \) be given. Then there is \( j < 3^{k+1} \) such that \( r | t_j \leq q | t_j \). But also \( q | t_j \leq p_j \in E \) and so \( r \) and \( p_j \) are compatible via \( r | t_j \).

**Lemma 2.3** \( \mathbb{T} \) satisfies quasi pure decision, i.e., for every open dense \( D \subseteq \mathbb{T} \), \( p \in \mathbb{T} \), there is \( q \leq_0 p \) satisfying what follows: if there exists \( q' \leq q \) such that \( q' \in D \), then \( q | \text{stem}(q') \in D \) as well.

**Proof.** Let \( p \in \mathbb{T} \) and \( D \subseteq \mathbb{T} \) open dense be given. We construct a fusion sequence \( p = q_0 \geq q_1 \geq \ldots \geq q_{3^{k+1}} = q \) such that the fusion \( \mathbb{T} = \bigcap q_k \) witnesses quasi pure decision. Assume we are at step \( k+1 \) of the construction, i.e., we have already constructed \( q_k \). Let \( A_q = \{ n_i : i < \omega \} \) be the corresponding set of splitting levels. Let \( \{ t_j \in q_k : j < 3^k \} \) denote all nodes in \( q_k \) of length \( n_k \). Similar to above we construct a decreasing sequence \( q_k = q_0^k \geq q_1^k \geq \ldots \geq q_{3^k}^k \). Assume we are at step \( j < 3^k \). There are two cases:

- **Case 1:** There is no stronger condition \( p' < q_j^k \) in \( D \) with \( \text{stem}(p') = t_j \). Then do nothing and put \( q_{j+1}^k := q_j^k \).
- **Case 2:** Otherwise there is a \( p' \leq q_j^k \) in \( D \) with \( \text{stem}(p') = t_j \). As in the proof above we define:

\[
q_{j+1}^k := \{ t \in q_j^k : \{ 1 \wedge \exists x \in p' \forall n \in \omega \\
\quad (n_k < n < |t| \rightarrow s(n) = t(n)) \} 
\]

specifically \( q_{j+1}^k | t_j = p' \). Finally defining \( q_{j+1} := q_{3^k}^k \), we get that the corresponding two sets of splitting levels \( A_{q_j} \) and \( A_{q_{j+1}} \) coincide on the first \( k+1 \) elements and therefore \( q_{j+1} \leq_k q_k \). This completes the construction.

Before showing that the fusion \( q := \bigcap q_k \) witnesses quasi pure decision we make the following observation: Since in the \( (k+1) \)st step in the construction of the fusion the \( k \)th splitting level is fixed, we know for each \( k \in \omega \) and \( l > k \) that \( q \leq_k q_l \). Therefore the two sets of splitting levels \( A_{q_j} \) and \( A_{q_{j+1}} \) coincide on the first \( k+1 \) elements.

Now let \( q' \leq q \) in \( D \) be given. Put \( t := \text{stem}(q') \). Again we denote the splitting levels of \( p \) by \( A_q = \{ n_k : k \in \omega \} \) and take \( n_k \) such that \( |t| = n_k \). We look at the construction of \( q_{j+1}^k \). Then there is \( j < 3^k \) with \( t_j = t \). Since \( q' \leq q_j^k \) and \( q' \in D \) we know that in the construction of \( q_{j+1}^k \) case 2 was applied, i.e., \( q_{j+1}^k | t = p' \) for some \( p' \in D \). Thus, using openness of \( D \) and \( q | t \leq q_{j+1}^k | t \), we also get \( q | t \in D \).

**Lemma 2.4** The forcing \( \mathbb{T} \) does not satisfy pure decision and adds Cohen reals.

**Proof.** We first show that \( \mathbb{T} \) does not satisfy pure decision: We have to find a condition \( p \in \mathbb{T} \) and a sentence \( \varphi \) such that no \( q \leq_0 p \) decides \( \varphi \). We prove something slightly stronger: Given any \( p \in \mathbb{T} \) we can find a sentence \( \varphi_p \) such that there is no \( q \leq_0 p \) deciding \( \varphi_p \).

So let \( p \in \mathbb{T} \) and \( q \leq_0 p \) be given (i.e., \( q \leq p \wedge \text{stem}(p) = \text{stem}(q) \)). Let \( \dot{z} \) be the \( \mathbb{T} \)-name for the generic real. It is clear that \( \mathbb{T} \models \exists n \in \mathbb{N} \dot{z}(n) = 2 \). We can define a name \( \dot{\sigma}_z \in \omega^\omega \cap V^\mathbb{T} \) such that:

\[
\mathbb{T} \models \dot{\sigma}_z(k) = k \text{th 2 occurring in } \dot{z}.
\]

This means that in any generic extension \( V[z] \) the evaluation of \( \dot{\sigma}_z \) enumerates the set \( \{ k \in \omega : z(k) = 2 \} \in V[z] \). For \( k \in \omega \) we define

\[
\varphi_k := \text{“there are even many 1’s occurring in } \dot{z} \text{ between } \dot{\sigma}_z(k) \text{ and } \dot{\sigma}_z(k+1)’’.
\]

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Put $k := \{|n < \text{stem}(q)| : \text{stem}(q)(n) = 2|\}$ and let $n_0^q < n_1^q$ denote the first two splitting levels of $q$. Take $q_0, q_1 \leq q$ such that

1. $\text{stem}(q_0)(n_0^q) = 0$ and $\text{stem}(q_0)(n_1^q) = 2$,
2. $\text{stem}(q_1)(n_0^q) = 1$ and $\text{stem}(q_1)(n_1^q) = 2$.

Then there are at least $k + 1$ many $2$'s occurring in $\text{stem}(q_i)$, therefore $\varphi_k$ is decided by $q_i$, $i \in 2$ and we get

$$q_0 \Vdash \varphi_k \iff q_1 \Vdash \neg \varphi_k.$$ 

This proves that $q$ does not decide $\varphi_k$.

We now show with a similar idea that $\mathbb{T}$ adds Cohen reals. Again let $\dot{z}$ be the $\mathbb{T}$-name for the generic real and let $\dot{\sigma}_z$ be as above. For every $k \in \omega$,

1. $c(k) = 0$ if and only if $\{|i \in \omega : \dot{\sigma}_z(k) \leq i < \dot{\sigma}_z(k + 1) \land \dot{z}(i) = 1|\}$ is even and
2. $c(k) = 1$ if and only if $\{|i \in \omega : \dot{\sigma}_z(k) \leq i < \dot{\sigma}_z(k + 1) \land \dot{z}(i) = 1|\}$ is odd.

Then $\models_{\mathbb{T}} c \in 2^\omega$. We want to show that $c$ is Cohen. So fix $p \in \mathbb{T}$, $\alpha \in 2^{<\omega}$ and let $c_\alpha \subseteq c$ be the part of $c$ decided by $p$. We aim to find $q \leq p$ such that $q \models c_\alpha^\bot \sigma \subseteq c$. This is sufficient to show that $c$ is Cohen.

Let $k = |c_\alpha|$, i.e., $k$ is minimal such that $c(k)$ is not decided by $p$. Define $p = q_0 \geq q_1 \geq \cdots \geq q_{|\alpha|}$ by recursion as follows.

Assume we have constructed $q_j, j < |\alpha|$. Let $n^j_1 < n^j_1$ be the first two splitting levels of $q_j$. For $i \in 2$ take $t_i \in q_j$ of length $n^j_1 + 1$ so that $t_i(n^j_1) = i$ and $t_i(n^j_1) = 2$. Put $q^j_1 := q_j | t_i$. Then we must have

$$\{|m \in \omega : n^j_1 \leq m < n^j_1 \land \text{stem}(q^j_1)(m) = 1|\} = \text{mod}_2 \sigma(j)$$

(1)

for exactly one $i \in 2$. Let $q_{j+1} = q^j_1$ such that (1) holds.

Then by construction, for every $j < |\alpha|, q_j |\alpha| \models c(|c_\alpha| + j) = \sigma(j)$, i.e., $q_j |\alpha| \models c_\alpha^\bot \sigma \subseteq c$.

Before moving to the issue concerning the ideals $\mathcal{I}_\mathbb{T}$ and $\mathcal{N}_\mathbb{T}$, we have to clarify the space we are interesting in working with. To understand the point let us consider the standard Mathias forcing $\mathbb{R}$. If we work in the Cantor space $2^\omega$ literally, then we end up with a trivial example to show that $\mathcal{N}_{\mathbb{R}} \neq \mathcal{I}_{\mathbb{R}}$, namely the set of “rational numbers”, i.e., the set $Q := \{x \in 2^\omega : \exists n \forall m \geq n (x(m) = 0)\}$. In a similar fashion one can check that the sets $N_n := \{x \in 2^\omega : \exists n \forall i \geq n (x(i) \neq 1)\}$ are $\mathbb{T}$-nowhere dense, but the union $\bigcup_{n \in \omega} N_n$ is not. We leave the straightforward proof to the reader.

For the same argument we specified in Remark 1.2, indeed the space we really refer to when we work with the standard Mathias forcing is not literally $2^\omega$, but is the subspace obtained via the identification of $[\omega]^\omega$ and $2^\omega$, i.e., the set $\{x \in 2^\omega : \exists n x(n) = 1\}$. In such a space the counterexample disappears and indeed we get $\mathcal{I}_{\mathbb{R}} = \mathcal{N}_{\mathbb{R}}$.

The main difference we want to make is that $\mathbb{T}$ behaves completely differently. In fact even when we take the “proper” space $H := \{x \in 3^\omega : \exists n x(n) = 2\}$ we cannot show that $\mathcal{N}_{\mathbb{T}} = \mathcal{I}_{\mathbb{T}}$, as the following result highlights (where the ideals are considered in the space $H$).

**Lemma 2.5** $\mathcal{N}_{\mathbb{T}} \neq \mathcal{I}_{\mathbb{T}}$.

**Proof.** Given $z \in H$ consider $\sigma_z \in \omega^\omega$ as in the proof of the previous Lemma and also remind $c_z \in 2^\omega$ be as follows:

1. $c_z(k) = 0$ if and only if $\{|i \in \omega : \sigma_z(k) \leq i < \sigma_z(k + 1) \land z(i) = 1|\}$ is even
2. $c_z(k) = 1$ if and only if $\{|i \in \omega : \sigma_z(k) \leq i < \sigma_z(k + 1) \land z(i) = 1|\}$ is odd.

Then define $M_n := \{z \in H : \forall k \geq n (c_z(k) = 0)\}$. We claim each $M_n$ is $\mathbb{T}$-nowhere dense, but $\bigcup_{n \in \omega} M_n$ is not. In fact given $n \in \omega$ and $p \in \mathbb{T}$ we can lengthen the stem of $p$ to get a stronger condition $p^* \leq p$ such that $\{|k < |\text{stem}(p')| : p'(k) = 2\}$ has size $> n$. Let $A_p := \{n_i : i \in \omega\}$. Now we take $t \in \text{Split}_2(p')$ extending $\text{stem}(p')^\bot 2$, i.e., $t(n_0) = 2$ such that $t(n_1) \neq 2$ and the set of $\{k > |\text{stem}(p')| : t(k) = 1\}$ is odd. Then $q := p^* | t^\bot 2$ has no common branch with $M_n$. On the other hand there is always a branch $z \in [p] \cap H$ such that for all $k > \text{stem}(p), c_z(k) = 0$. □
3 \( \Gamma(\mathbb{P}) \Rightarrow \Gamma(\mathbb{C}) \)

We now prove a rather general result, showing how the “Cohen coding” allows us to prove a classwise connection between \( \mathbb{P} \)-measurability and Baire property. Beyond its own interest, the technique used will also permit us to apply it in other specific cases that we shall summarize along the paper, in particular to answer a question connected to the diagram of regularity properties at uncountable investigated in [6]. Recall that a family of sets \( \Gamma \) is well-sorted if it is closed under continuous pre-images and \( \Gamma(\mathbb{P}) \) stands for “every set in \( \Gamma \) is \( \mathbb{P} \)-measurable”.

**Proposition 3.1** Let \( \mathcal{X} \) be a set of size \( \leq \kappa \) endowed with the discrete topology, \( \mathcal{X}^\omega \) the topological product space equipped with the bounded topology (i.e., the topology generated by \( [t] := \{x \in \mathcal{X}^\omega : x \geq t\} \) with \( t \in \mathcal{X}^{-}\omega \)), \( \mathbb{P} \) be a \( \kappa \)-closed tree forcing notion defined on \( \mathcal{X}^{-}\omega \). Let \( \varphi^* : \mathcal{X}^{-}\omega \rightarrow 2^{-}\omega \) be order-preserving and \( \varphi : \mathcal{X}^\omega \rightarrow 2^\omega \) it’s natural expansion (i.e., \( \varphi(x) = \bigcup_{\sigma \leq x} \varphi^*(\sigma) \)). Assume

for every \( p \in \mathbb{P} \), \( \varphi([p]) \) is open dense in \([\varphi^*(\text{stem}(p))]. \)

Then \( \Gamma(\mathbb{P}) \) implies \( \Gamma(\mathbb{C}) \).

We note that \( \varphi \) is continuous. Since for each \( p \in \mathbb{P} \) we have \( \varphi([p]) \) open dense in \([\varphi^*(\text{stem}(p))]) \), we also get \( \forall q \in \mathbb{P} \forall s \in 2^{-}\omega \exists \sigma \in q \) such that \( \varphi^*(\sigma) \supseteq \varphi^*(\text{stem}(q)) \). The key step for the proof is the following lemma.

**Lemma 3.2** Let \( \mathbb{P}, \varphi, \varphi^* \) be as in the Proposition and \( X \subseteq 2^\omega \). Define \( Y := \varphi^{-1}[X] \). Assume there is \( q \in \mathbb{P} \) such that \( Y \cap [q] \in \mathbb{P} \)-comeagre in \([q] \). Then \( X \cap [\varphi^*(\text{stem}(q))] \) is comeagre in \([\varphi^*(\text{stem}(q))]. \)

**Proof.** We are assuming \( Y \cap [q] \) is \( \mathbb{P} \)-comeagre, for some \( q \in \mathbb{P} \). This implies that there is a collection \( \{A_\alpha : \alpha < \kappa \land A_\alpha \in \mathbb{P}\} \) open dense in \([q] \) such that \( \bigcap_{\alpha} A_\alpha \subseteq [q] \cap Y \). Without loss of generality, assume \( A_\alpha \supseteq A_\beta \) whenever \( \alpha < \beta < \kappa \). Let \( t = \varphi^*(\text{stem}(q)) \). We want to show that \( \varphi[Y] \cap t = X \cap t \) is comeagre in \([t] \), i.e., we want to find \( \{B_\alpha : \alpha < \kappa \} \) open dense sets in \([t] \) such that \( \bigcap_\alpha B_\alpha \subseteq X \cap [t] \). Given \( \sigma \in \kappa^{-}\omega \) we recursively define on the length of \( \sigma \) a set \( \{q_\sigma : \sigma \in \kappa^{-}\omega \} \subseteq \mathbb{P} \) with the following properties:

1. \( q_\emptyset = q \).
2. \( \forall \sigma \in \kappa^{-}\omega \cup_i [\varphi^*(\text{stem}(q_\sigma^{-1}))] \) is open dense in \([\varphi^*(\text{stem}(q_\sigma))]. \)
3. \( \forall \sigma \in \kappa^{-}\omega \forall i \in \kappa (\{q_\sigma^{-1} : \sigma \leq q_\sigma^{-1} \subseteq A_\sigma \land q_\sigma^{-1} \leq q_\sigma \}) \).

Assume we are at step \( \alpha = |\sigma| \). Fix \( \sigma \in \kappa^{-}\omega \) arbitrarily and then put \( t_\sigma = \varphi^*(\text{stem}(q_\sigma)) \). We first make sure that \( 2 \). holds. Therefore let \( \{s_i : i < \kappa \} \) enumerate \( 2^{-}\omega \). As noted right below Proposition 3.1, we can find \( p_i \leq q_\sigma \) such that \( \varphi^*(\text{stem}(p_i)) \supseteq t_\sigma \cap s_i \). Since each \( A_\alpha \) is \( \mathbb{P} \)-open dense in \([q] \) we can find for each \( i < \kappa \) an extension \( q_i \leq p_i \) such that \( [q_i] \subseteq \bigcap_{i < \kappa} A_\alpha \). This ensures that also \( 3 \). holds and we put \( q_\sigma^{-1} := q_\sigma \). At limit steps \( \lambda \), we put for every \( \sigma \in \kappa^{-}\omega \), \( \lambda := \bigcap_{\sigma \leq q_\sigma^{-1}} q_\sigma \). Finally we put \( B_\alpha := \bigcup_{\sigma \leq q_\sigma^{-1}} [\varphi_\sigma : \sigma \in \kappa^{-}\omega \} \). We have to check that \( \bigcap_{\alpha} B_\alpha \subseteq X \cap [t] \). Since \( t = \varphi^*(\text{stem}(q)) \) and \( q_\alpha \leq q \) we have \( \varphi([q_\alpha]) \subseteq [\varphi^*(t)] \) and therefore \( B_\alpha \subseteq [t] \) for each \( \alpha \in \kappa \). On the other hand by construction of \( \alpha \) we know \( \varphi^{-1}[B_{\alpha+1}] \subseteq A_\alpha \) and hence \( \varphi^{-1}[\bigcap B_\alpha] \subseteq \bigcap B_\alpha \) which implies \( \bigcap B_\alpha \subseteq X \).

**Proof of Proposition 3.1.** Let \( X \in \Gamma \) be given and put \( Y := \varphi^{-1}[X] \). Then also \( Y \in \Gamma \), since \( \Gamma \) is well-sorted and \( \varphi \) is continuous. We now use the lemma to show that for every \( t \in 2^{-}\omega \) there exists \( t' \supseteq t \) such that \( X \cap [t'] \) is meagre or \( X \cap [t'] \) is comeagre.

Fix \( t \in 2^{-}\omega \) arbitrarily and pick \( p \in \mathbb{P} \) such that \( \varphi^*(\text{stem}(p)) \supseteq t \). By assumption \( Y \in \mathbb{P} \)-measurable, and so:

1. \( \text{in case there exists } q \leq p \text{ such that } Y \cap [q] \text{ is } \mathbb{P} \text{-comeagre; put } t' := \varphi^*(\text{stem}(q)) \). By the lemma above, \( X \cap [t'] \) is comeagre in \([t'] \);
2. \( \text{in case there exists } q \leq p \text{ such that } Y \cap [q] \text{ is } \mathbb{P} \text{-meagre}, \text{ then apply the lemma above to the complement of } Y \text{, in order to get } X \cap [t'] \text{ be meagre in } [t'], \text{ with } t' := \varphi^*(\text{stem}(q)). \)

By the remark directly after Definition 1.1 this suffices to complete the proof.

**Proposition 3.3** Let \( \Gamma \) be a well-sorted family of sets. Then \( \Gamma(\mathbb{T}) \Rightarrow \Gamma(\mathbb{C}) \).

**Proof.** Consider \( H := \{x \in 3^\omega : \exists^n n x(n) = 2\} \). As we remarked right above Lemma 2.5, \( H \) is \( \mathbb{T} \)-comeagre. Thus we have for each set \( X \subseteq 3^\omega \):

\[ X \text{ is } \mathbb{T} \text{-measurable} \iff X \cap H \text{ is } \mathbb{T} \text{-measurable}. \]
Since we are only concerned with \( \mathbb{T} \)-measurability we can work with the set \( H \) instead of the whole space \( 2^\omega \).

We want to apply Proposition 3.1. For an element \( x \in H \) let \( A_x = \{ n_i : i < \omega \} \) be an increasing enumeration of all \( n \in \omega \) such that \( x(n) = 2 \). This is by definition of \( H \) an infinite set. Using this notation we define a function \( \varphi : H \rightarrow 2^\omega \) via:

\[
\varphi(x)(i) = \begin{cases} 
0 & \text{if } \{ j < \omega : n_i < j < n_{i+1} \land x(j) = 1 \} \text{ is even,} \\
1 & \text{else.}
\end{cases}
\]

Note that \( \varphi \) is surjective but not injective and observe that \( \varphi \) induces a map \( \varphi^* : 3^{<\omega} \rightarrow 2^{<\omega} \) such that for each \( x \in H \) and \( i < \omega \) we have \( \varphi(x)[i] = \varphi^*(x[n_i]) \). We have to check that the requirement from Proposition 3.1 is satisfied. Therefore fix \( p \in \mathbb{T} \) and \( s \in 2^{<\omega} \). Let \( A_p = \{ n_i : i < \omega \} \) be the corresponding set of splitting levels and \( s = (i_1, \ldots, i_k) \). Then we can lengthen \( \text{stem}(p) \) in order to have the parity of 1s between two subsequent 2 according to the corresponding \( i_j \), that means we find \( t \in p \) such that \( \varphi^*(t) = \varphi^*(\text{stem}(p)) \downarrow s \). Thus, we even get equality \( \varphi([p]) = [\varphi^*(\text{stem}(p))] \).

So we are able to apply Proposition 3.1 and get \( \Gamma(\mathbb{T}) \Rightarrow \Gamma(\mathbb{C}) \).

\[\square\]

4 Some results for the uncountable case

In this section, we investigate some issues concerning Table 2. We shall always assume that \( \kappa \) is an uncountable regular cardinal such that \( \kappa = 2^{<\kappa} \).

**Definition 4.1** (Club \( \kappa \)-Miller forcing \( \mathbb{M}_{\kappa}^\text{Club} \)) A tree \( p \subseteq \kappa \) is called \( \kappa \)-Miller tree if it is pruned, \( \kappa \)-closed and

(a) for every \( s \in p \) there is an extension \( t \supseteq s \) in \( p \) such that \( \text{succ}(t, p) \subseteq \kappa \) is club. Such a splitting node \( t \) is called **club-splitting**.

(b) for every \( x \in [p] \) the set \( \{ \alpha < \kappa : x [\alpha] \text{ is club-splitting} \} \) is club.

We note that both (a) and (b) ensure that \( \mathbb{M}_{\kappa}^\text{Club} \) is a \( \kappa \)-closed forcing. The set of trees that consist of nodes that are either club-splitting or not splitting is a dense subset of \( \mathbb{M}_{\kappa}^\text{Club} \).

The following remark highlights the connection with \( \kappa \)-Cohen reals. We remark that a similar result (though in a different context, dealing with a version of \( \mathbb{M}_{\kappa} \) satisfying (a) but not (b)) has been proven by Mildenberger and Shelah in [12].

**Remark 4.2** We introduce a coding function \( \varphi^* : \kappa^{<\kappa} \rightarrow 2^{<\kappa} \) that allows us to read off a Cohen \( \kappa \)-real from the \( \mathbb{M}_{\kappa}^\text{Club} \)-generic. Therefore fix a \( \kappa \) sized family \( \{ S_i \subseteq \kappa : i \in 2^{<\kappa} \} \) of pairwise disjoint stationary sets such that the union of all \( S_i \)'s covers \( \kappa \) (this is possible since we assume \( \kappa = 2^{<\kappa} \)). Let \( \sigma \in \kappa^{<\kappa} \). We define \( \varphi^*(\sigma) = \{ \tau \in \kappa^{<\kappa} : \text{card}(\sigma(\tau)) = \text{card}(\sigma) \} \). Let \( \hat{\varphi} \) be the \( \mathbb{M}_{\kappa}^\text{Club} \)-name for the generic \( \kappa \)-real and \( \check{c} \) the \( \mathbb{M}_{\kappa}^\text{Club} \)-name such that \( \check{c} = \varphi^*(\hat{\varphi}) \). We claim that \( \check{c} \) is \( \kappa \)-Cohen in every generic extension. Therefore fix \( p \in \mathbb{M}_{\kappa}^\text{Club} \) and let \( c_p \in 2^{<\kappa} \) be the initial part of \( \check{c} \) decided by \( p \) so \( c_p = \varphi^*(\text{stem}(p)) \). Let \( t \in 2^{<\kappa} \) be given. We want to find \( q \leq p \) such that \( q \forces \check{c} \rightarrow t \subseteq \check{c} \). Since \( \text{stem}(p) \) is club-splitting we can find an \( \alpha_0 \in S_t \cap \{ \alpha < \kappa : \text{stem}(p) \cup \alpha \in p \} \) and take \( q \) to be \( p \upharpoonright \text{stem}(p) \cup \alpha_0 \), i.e., \( \text{stem}(q) \) extends \( \text{stem}(p) \cup \alpha_0 \). This implies that \( \varphi^*(\text{stem}(q)) \supseteq c_p \downarrow t \) and therefore \( q \forces c_p \rightarrow t \subseteq \check{c} \) as demanded.

We also remark that the fact that \( \mathbb{M}_{\kappa}^\text{Club} \) adds Cohen \( \kappa \)-reals is not new and it was proven in [4], even if the authors use a different coding map.

Differently from \( \mathbb{T} \), the Cohen-like behaviour of the \( \mathbb{M}_{\kappa}^\text{Club} \)-generic does not have an impact on the ideals, as shown in the next result.

**Lemma 4.3** \( \mathbb{N}_{\mathbb{M}_{\kappa}^\text{Club}} = \mathbb{I}_{\mathbb{M}_{\kappa}^\text{Club}} \)

**Proof.** The proof is rather standard. We report a sketch of it here just for completeness. Given \( \{ D_i : i < \kappa \} \) a family of \( \mathbb{M}_{\kappa}^\text{Club} \)-open dense sets and \( p \in \mathbb{M}_{\kappa}^\text{Club} \) we simply construct a fusion sequence \( \{ q_i : i < \kappa \} \) so that \( q := \bigcup_{i < \kappa} q_i \leq p \), for every \( i < \kappa \), \( [q_i] \subseteq D_i \), and for every \( j < i \), \( q_i \leq j \), \( q_i \leq j \), i.e., \( q_i \leq j \), and for every \( j \leq i \), \( \text{Split}_j(q_i) = \text{Split}_j(q_j) \). This can be done via an easy recursive construction: at limit steps \( i \), simply put \( q_i := \)
Proposition 4.4 Let $\Gamma$ be a well-sort family of subsets of $\kappa$-reals. Then $\Gamma(M^\omega_{\text{Club}}) \Rightarrow \Gamma(C)$.

Proof. Let $\varphi^*$ and $\varphi$ as in Remark 4.2. We cannot apply Proposition 3.1 directly, since $\varphi(\{p\})$ might not be open dense in $[\varphi^*(\text{stem}(p))]$. But a similar argument can be made and in fact makes the proof easier. The key step is to adjust Lemma 3.2. Fix $X \subseteq 2^\kappa$ and $q \in M^\omega_{\text{Club}}$ and put $Y := \varphi^{-1}[X]$. Assume $Y \cap [q]$ is $M^\omega_{\text{Club}}$-comeagre in $[q]$. We note that Lemma 4.3 implies there is $q' \leq q$ such that $[q'] \subseteq Y$. It is enough to show that $\varphi([q']) \subseteq X$ is comeagre in $[\varphi^*(\text{stem}(q'))]$. To this end we recursively define a decreasing sequence $\{D_\alpha : \alpha < \kappa\}$ of open dense sets in $[\varphi^*(\text{stem}(q'))]$ as follows: $D_0 := \bigcup\{\varphi^*(t) : t \in \text{Split}_1(q')\}$, $D_{\alpha+1} := \bigcup\{\varphi^*(t) : t \in \text{Split}_{\alpha+2}(q')\}$ and for $\lambda$ limit $D_\lambda := \bigcap_{\alpha < \lambda} D_\alpha$. The same argument used in Remark 4.2 shows that the $D_\alpha$’s are open dense and the construction clearly implies $\bigcap_{\alpha < \kappa} D_\alpha = [\varphi([q'])].$

The proof is completed in the same manner as Proposition 3.1.

Lemma 4.6 $\mathbb{N}_{\mathbb{R}_k} \neq \mathbb{I}_{\mathbb{R}_k}$

Proof 18 We first clarify what is meant with $\mathbb{N}_{\mathbb{R}_k} : X \subseteq [\kappa]^\kappa$ is called $\mathbb{R}_k$-nowhere dense if for each $(s, A) \in \mathbb{R}_k$ there is a stronger condition $(t, B) \leq (s, A)$ such that

$$\forall x \in X \forall y \in [B]^\kappa (x \notin t \cup y).$$

(2)

We define an equivalence relation on the set of countably infinite subsets of $\kappa$. For $a, b \in [\kappa]^\omega$ let $a \sim b : \iff |a \setminus b| < \omega$. We fix a system of representatives. For $a \in [\kappa]^\omega$ we denote the representative of $\{b \in [\kappa]^\omega : b \sim a\}$ with $\bar{a}$. Then we define a colouring function $C : [\kappa]^\omega \to \{0, 1\}$ as follows:

$$C(a) = \begin{cases} 0 & \text{if } |a \setminus \bar{a}| \text{ is even}, \\ 1 & \text{else}. \end{cases}$$

We can identify $x \in [\kappa]^\kappa$ with it’s increasing enumeration $\chi : \kappa \to \kappa$ given by $\chi(\xi) := \min(x \setminus \bigcup_{\xi < \xi} \chi(\alpha))$. Let $\{\alpha_i : i < \kappa\}$ enumerate the limit ordinals $< \kappa$. For $x \in [\kappa]^\kappa$ and $i < \kappa$ we define the countable set $b^i_j := \{x(\xi) : \alpha_i < \xi < \alpha_{i+1}\} \subseteq \kappa$.

Claim 4.7 The set $X_i := \{x \in [\kappa]^\kappa : \forall j > i \ C(b^i_j) = 0\}$ is $\mathbb{R}_k$-nowhere dense for all $i < \kappa$, but their union is not.

Proof. Let $(s, A)$ be a $\kappa$-Mathias condition and $i < \kappa$ be given. Fix $j > i$. Then $A \subseteq \kappa$ is of size $\kappa$. By removing at most one element of $A$, we find $A' \subseteq \kappa$ such that $C(b^j_i) = 1$. We extend $s$ with the first $\alpha_{j+1}$ elements of $A'$ to get $t := s \cup (A'(\xi) : \xi \leq \alpha_{j+1}) \in [\kappa^\omega]$. Now we can shrink $A'$ to $B := A' \setminus (A'(\alpha_{j+1}) + 1)$ in order to obtain a $\kappa$-Mathias condition $(t, B) \leq (s, A)$ fulfilling the requirement (2).

However the union $X := \bigcup_{i < \kappa} X_i$ can not be $\mathbb{R}_k$-nowhere dense. In fact, let $(s, A)$ be a $\kappa$-Mathias condition. We can always find for $i > j$ a subset $B \subseteq A$ of size $\kappa$ such that $C(b^j_i) = 0$, for all $j > i$ and hence (2) is false for $X_i$ and $(s, B)$.

Proposition 4.8 $\Gamma(\mathbb{R_k}) \Rightarrow \Gamma(C)$.

Proof. We use the colouring function $C$ introduced above to define $\varphi : \kappa^\omega \to 2^\kappa$. Therefore let $\{\alpha_i : i < \kappa\}$ enumerate the limit ordinals $< \kappa$ and put $\varphi(x) := \{C((x(\xi) : \alpha_i < \xi < \alpha_{i+1}) : i < \kappa\}$. Let $\varphi^*$ be the natural corresponding function, then $\varphi^*[\{x(\xi) : \alpha_i < \xi < \alpha_{i+1}\}] = \varphi^*[\varphi(\hat{x})]$ holds and we can apply Proposition 3.1.

The colouring introduced above requires AC. However the result does not need AC, as we can also consider another kind of colouring, as noted by Wohofsky and Koelbing during the writing of [7]: fix $S \in [\kappa]^\kappa$ stationary and co-stationary and define the colouring $C : [\kappa]^\omega \to \{0, 1\}$ by $C(a) := 0$ if and only if $\sup a \in S$. 

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Remark 4.9 Proposition 3.1 also applies for $P \in \{D_\kappa, E_\kappa\}$. The coding function $\varphi : \kappa^\kappa \to 2^\kappa$ we need in this case is given by $\varphi(x)(i) = x(i) \mod 2$, similarly to the $\omega$-case. It is straightforward to prove that such a $\varphi$ (and the natural corresponding $\varphi^*$) satisfies the required properties of Proposition 3.1.

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