1. Introduction

One of Franco Tricerri’s interests was in Hermitian manifolds. In [21] various types of Hermitian structures are discussed and conditions on the Lee form are of paramount importance. That Hermitian structures are closely connected with harmonic morphisms is shown in [2] and [25]. In this paper we study this connection for more general almost Hermitian manifolds. We obtain conditions involving the Lee form under which holomorphic maps between almost Hermitian manifolds are harmonic maps or morphisms. We show that the image of certain holomorphic maps from a cosymplectic manifold is cosymplectic if and only if the map is a harmonic morphism, generalizing a result of Watson [23]. Finally, in Theorem 5.1 we give conditions under which...
a harmonic morphism into a Hermitian manifold defines an integrable Hermitian structure on its domain.

2. Harmonic morphisms

For a smooth map \( \phi : (M, g) \to (N, h) \) between Riemannian manifolds its tension field \( \tau(\phi) \) is the trace of the second fundamental form \( \nabla d\phi \) of \( \phi \):

\[
\tau(\phi) = \sum_j \{ \nabla^\phi - H_x \} e_j d\phi(e_j) - d\phi(\nabla e_j e_j)
\]

where \( \{ e_j \} \) is a local orthonormal frame for \( TM \), \( \nabla^\phi \) denotes the pull-back of the Levi-Civita connection \( \nabla^N \) on \( N \) to the pull-back bundle \( \phi^{-1}TN \to M \) and \( d\phi : TM \to \phi^{-1}TN \) is the pull-back of the differential of \( \phi \). The map \( \phi \) is said to be harmonic if its tension field vanishes i.e. \( \tau(\phi) = 0 \). J. Eells and J. H. Sampson proved in \([7]\) that any holomorphic map between Kähler manifolds is harmonic and this was later generalized by A. Lichnerowicz in \([17]\). For information on harmonic maps, see \([4]\), \([5]\), \([6]\) and the references therein.

A harmonic morphism is a smooth map \( \phi : (M, g) \to (N, h) \) between Riemannian manifolds which pulls back germs of real-valued harmonic functions on \( N \) to germs of harmonic functions on \( M \). A smooth map \( \phi : M \to N \) is called horizontally (weakly) conformal if for each \( x \in M \) either

(i) the rank of the differential \( d\phi_x \) is 0, or

(ii) for the orthogonal decomposition \( T_xM = H_x \oplus V_x \) with \( V_x = \ker d\phi_x \) the restriction \( d\phi_x|_{H_x} \) is a conformal linear map onto \( T_{\phi(x)}N \).

Points of type (i) are called critical points of \( \phi \) and those of type (ii) regular points. The conformal factor \( \lambda(x) \) is called the dilation of \( \phi \) at \( x \). Setting \( \lambda = 0 \) at the critical points gives a continuous function \( \lambda : M \to \)
[0, ∞) which is smooth at regular points, but whose square \( \lambda^2 \) is smooth on the whole of \( M \). Note that at a regular point \( \phi \) is a submersion. A horizontally weakly conformal map is called horizontally homothetic if \( d\phi(\text{grad}(\lambda^2)) = 0 \). B. Fuglede showed in \[9\] that a horizontally homothetic harmonic morphism has no critical points.

The following characterization of harmonic morphisms is due to Fuglede and T. Ishihara, see \[8\], \[15\]: A smooth map \( \phi \) is a harmonic morphism if and only if it is a horizontally weakly conformal harmonic map. More geometrically we have the following result due to P. Baird and Eells, see \[1\]:

**Theorem 2.1.** Let \( \phi : (M, g) \to (N, h) \) be a non-constant horizontally weakly conformal map. Then

(i) if \( N \) is a surface, i.e. of real dimension 2, then \( \phi \) is a harmonic morphism if and only if its fibres are minimal at regular points,

(ii) if the real dimension of \( N \) is greater than 2 then any two of the following conditions imply the third:

(a) \( \phi \) is a harmonic morphism,

(b) the fibres of \( \phi \) are minimal at regular points,

(c) \( \phi \) is horizontally homothetic.

Harmonic morphisms exhibit many properties which are “dual” to those of harmonic maps. For example, whereas harmonic maps exhibit conformal invariance in a 2-dimensional domain (cf. \[7\], Proposition p.126), harmonic morphisms exhibit conformal invariance in a 2-dimensional codomain: If \( \phi : (M, g) \to (N, h) \) is a harmonic morphism to a 2-dimensional Riemannian manifold and \( \psi : (N, h) \to (\tilde{N}, \tilde{h}) \) is a weakly conformal map to another 2-dimensional Riemannian manifold, then the composition \( \psi \circ \phi \) is a harmonic morphism. In particular the concept of a harmonic morphism to a Riemann surface is well-defined. For information on harmonic morphisms see \[2\] and \[26\].
3. Almost Hermitian manifolds

Let \((M^m, g, J)\) be an almost Hermitian manifold i.e. a Riemannian manifold \((M, g)\) of even real dimension \(2m\) together with an almost complex structure \(J : TM \to TM\) which is isometric on each tangent space and satisfies \(J^2 = -I\). Let \(T^CM\) be the complexification of the tangent bundle. We then have an orthogonal decomposition

\[ T^CM = T^{1,0}M \oplus T^{0,1}M \]

of \(T^CM\) into the \(\pm i\)-eigenspaces of \(J\), respectively. Each vector \(X \in T^CM\) can be written as \(X = X^{1,0} + X^{0,1}\) with

\[ X^{1,0} = \frac{1}{2}(X - iJX) \in T^{1,0}M \quad \text{and} \quad X^{0,1} = \frac{1}{2}(X + iJX) \in T^{0,1}M, \]

and locally one can always choose an orthonormal frame \(\{e_1, \ldots, e_m, Je_1, \ldots, Je_m\}\) for \(TM\) such that

\[ T^{1,0}M = \text{span}_\mathbb{C}\{Z_1 = \frac{e_1 - iJe_1}{\sqrt{2}}, \ldots, Z_m = \frac{e_m - iJe_m}{\sqrt{2}}\}, \]

\[ T^{0,1}M = \text{span}_\mathbb{C}\{\bar{Z}_1 = \frac{e_1 + iJe_1}{\sqrt{2}}, \ldots, \bar{Z}_m = \frac{e_m + iJe_m}{\sqrt{2}}\}. \]

The set \(\{Z_k| k = 1, \ldots, m\}\) is called a local Hermitian frame on \(M\).

As for any other \((1,1)\)-tensor the divergence of \(J\) is given by

\[ \delta J = \text{div}(J) = \sum_{k=1}^{m} (\nabla_{e_k} J)(e_k) + (\nabla_{Je_k} J)(Je_k) \]

\[ = \sum_{k=1}^{m} (\nabla_{Z_k} J)(Z_k) + (\nabla_{\bar{Z}_k} J)(\bar{Z}_k). \]

**Remark 3.1.** Modulo a constant, the vector field \(J\delta J\) is the dual to the Lee form, see \([21]\). It is called the Lee vector field.

Following Kotô \([19]\) and Gray \([12]\) with alternative terminology due to Salamon \([20]\) we call an almost Hermitian manifold \((M, g, J)\)

(i) quasi-Kähler or \((1,2)\)-symplectic if

\[ (\nabla_X J)Y + (\nabla_J X)JY = 0 \quad \text{for all} \quad X, Y \in C^\infty(TM), \quad \text{and} \]
(ii) semi-Kähler or cosymplectic if $\delta J = 0$.

Note that a $(1, 2)$-symplectic manifold $(M, g, J)$ is automatically cosymplectic. It is an easy exercise to prove the following two well-known results:

**Lemma 3.2.** Let $(M, g, J)$ be an almost Hermitian manifold. Then the following conditions are equivalent:

(i) $M$ is $(1, 2)$-symplectic,
(ii) $\nabla_Z W \in C^\infty(T^{1,0}M)$ for all $Z, W \in C^\infty(T^{1,0}M)$.

**Lemma 3.3.** Let $(M, g, J)$ be an almost Hermitian manifold. Then the following conditions are equivalent:

(i) $M$ is cosymplectic,
(ii) $\sum_{k=1}^m \nabla_{Z_k} Z_k \in C^\infty(T^{1,0}M)$ for any local Hermitian frame $\{Z_k\}$.

**Example 3.4.** For $r, s \geq 0$ let $(M, g)$ be the product $S^{2r+1} \times S^{2s+1}$ of the two unit spheres in $\mathbb{C}^{r+1}$ and $\mathbb{C}^{s+1}$ equipped with their standard Euclidean metrics. The manifold $(M, g)$ has a standard almost Hermitian structure $J$ which can be described as follows (cf. [14] and [22]): Let $n_1, n_2$ be the unit normals to $S^{2r+1}, S^{2s+1}$ in $\mathbb{C}^{r+1}, \mathbb{C}^{s+1}$ and let $\mathcal{H}_1, \mathcal{H}_2$ be the horizontal spaces of the Hopf maps $S^{2r+1} \to \mathbb{C}P^r$, $S^{2s+1} \to \mathbb{C}P^s$, respectively. Then any vector tangent to $M$ has the form

$$X = X_1 + aJ_1 n_1 + X_2 + bJ_2 n_2$$

where $a, b \in \mathbb{R}$, $X_1 \in \mathcal{H}_1$, $X_2 \in \mathcal{H}_2$, and $J_1, J_2$ are the standard Kähler structures on $\mathbb{C}^{r+1}$ and $\mathbb{C}^{s+1}$, respectively. Then the almost complex structure $J$ on $M$ is given by

$$J : X \mapsto J_1 X_1 - bJ_1 n_1 + J_2 X_2 + aJ_2 n_2.$$
We calculate that (cf. [24]):

$$\delta J = -2(rJ_1n_1 + sJ_2n_2).$$

The almost Hermitian manifold $$(M, g, J)$$ is called the Calabi-Eckmann manifold. It is cosymplectic if and only if $$s = r = 0$$ i.e. when $$M$$ is the real 2-dimensional torus in $$\mathbb{C}^2$$.

**Example 3.5.** Any invariant metric on a 3-symmetric space gives it a $$(1, 2)$$-symplectic structure (cf. Proposition 3.2 of [13]). Such 3-symmetric spaces occur as twistor spaces of symmetric spaces. One interesting example is the complex Grassmannian $$G_n(\mathbb{C}^{m+n}) = SU(m+n)/S(U(m) \times U(n))$$ with twistor bundle the flag manifold $$N = SU(m+n)/S(U(m) \times U(k) \times U(n-k))$$ and projection $$\pi : N \rightarrow G_n(\mathbb{C}^{m+n})$$ induced by the inclusion map $$U(k) \times U(n-k) \hookrightarrow U(n)$$. The manifold $$N$$ has an almost Hermitian structure usually denoted by $$J^2$$ such that $$(N, g, J^2)$$ is $$(1, 2)$$-symplectic for any $$SU(m+n)$$-invariant metric $$g$$ on $$N$$. For further details see [20].

Finally, recall that an almost Hermitian manifold is called Hermitian if its almost complex structure is integrable. A necessary and sufficient condition for this is the vanishing of the Nijenhuis tensor (cf. [18]), or equivalently, that $$T^{1,0}M$$ is closed under the Lie bracket i.e. $$[T^{1,0}M, T^{1,0}M] \subset T^{1,0}M$$.

4. **The harmonicity of holomorphic maps**

Throughout this section we shall assume that $$(M^m, g, J)$$ and $$(N^n, h, J^N)$$ are almost Hermitian manifolds of complex dimensions $$m$$ and $$n$$ with Levi-Civita connections $$\nabla$$ and $$\nabla^N$$, respectively. Furthermore we suppose that the map $$\phi : M \rightarrow N$$ is holomorphic i.e. its differential $$d\phi$$ satisfies $$d\phi \circ J = J^N \circ d\phi$$. We are interested in studying under what additional assumptions the map $$\phi$$ is a harmonic map or morphism.
For a local Hermitian frame \( \{ Z_k \} \) on \( M \) we define

\[
A = \sum_{k=1}^{m} \nabla_{\phi^{-1}TN} Z_k d\phi(Z_k) \quad \text{and} \quad B = -\sum_{k=1}^{m} d\phi(\nabla_{\bar{Z}_k} Z_k).
\]

**Lemma 4.1.** Let \( \phi : (M, g, J) \to (N, h, J^N) \) be a map between almost Hermitian manifolds. If \( N \) is \((1, 2)\)-symplectic then the tension field \( \tau(\phi) \) of \( \phi \) is given by

\[
\tau(\phi) = -d\phi(J\delta J).
\]

**Proof:** Let \( \{ Z_k \} \) be a local Hermitian frame, then a simple calculation shows that

\[
J\delta J = \sum_{k=1}^{m} \{(1 + iJ)\nabla_{Z_k} Z_k + (1 - iJ)\nabla_{\bar{Z}_k} \bar{Z}_k\},
\]

so that the \((0, 1)\)-part of \( J\delta J \) is given by

\[
(J\delta J)^{0,1} = (1 + iJ) \sum_{k=1}^{m} \nabla_{\bar{Z}_k} Z_k.
\]

The holomorphy of \( \phi \) implies that \( d\phi(Z_k) \) belongs to \( C^\infty(\phi^{-1}T^{1,0}N) \) and the \((1, 2)\)-symplecticity on \( N \) that \( \nabla_{\phi^{-1}TN} Z_k d\phi(Z_k) \in C^\infty(\phi^{-1}T^{1,0}N) \).

This means that \( A^{0,1} = 0 \). From Equation (1) and the symmetry of the second fundamental form \( \nabla d\phi \) we deduce that \( \tau(\phi) = 2(A + B) \). Taking the \((0, 1)\)-part and using the fact that \( \phi \) is holomorphic we obtain

\[
\tau(\phi)^{0,1} = 2(A^{0,1} + B^{0,1}) = 2B^{0,1} = -d\phi(J\delta J)^{0,1}.
\]

Since \( \tau(\phi) \) and \( d\phi(J\delta J) \) are both real, we deduce the result.

The next proposition gives a criterion for harmonicity in terms of the Lee vector field.

**Proposition 4.2.** Let \( \phi : (M, g, J) \to (N, h, J^N) \) be a holomorphic map from an almost Hermitian manifold to a \((1, 2)\)-symplectic manifold. Then \( \phi \) is harmonic if and only if \( d\phi(J\delta J) = 0 \).
Note that since we are assuming that $\phi$ is holomorphic $d\phi(J\delta J) = 0$ is equivalent to $d\phi(\delta J) = 0$. As a direct consequence of Proposition 4.2 we have the following result of Lichnerowicz, see [17]:

**Corollary 4.3.** Let $\phi : (M, g, J) \rightarrow (N, h, J^N)$ be a holomorphic map from a cosymplectic manifold to a $(1,2)$-symplectic one. Then $\phi$ is harmonic.

To deduce that $\phi$ is a harmonic morphism we must assume that $\phi$ is horizontally weakly conformal. In that situation we can say more:

**Proposition 4.4.** Let $\phi : (M^m, g, J) \rightarrow (N^n, h, J^N)$ be a surjective horizontally weakly conformal holomorphic map between almost Hermitian manifolds. Then any two of the following conditions imply the third:

(i) $\phi$ is harmonic and so a harmonic morphism,
(ii) $d\phi(J\delta J) = 0$.
(iii) $N$ is cosymplectic,

**Proof:** By taking the $(0,1)$-part of equation (1) we obtain

$$\tau(\phi)^{0,1} = 2(A^{0,1} + B^{0,1}).$$

The tension field $\tau(\phi)$ is real so that the map $\phi$ is harmonic if and only if $\tau(\phi)^{0,1} = 0$. Since $2B^{0,1} = -d\phi(J\delta J)^{0,1}$ and the vector field $d\phi(J\delta J)$ is real the condition $d\phi(J\delta J) = 0$ is equivalent to $B^{0,1} = 0$. To complete the proof we shall now show that $A^{0,1} = 0$ on $M$ if and only if $N$ is cosymplectic.

Let $R$ be the open subset of regular points of $\phi$. Let $p \in R$ and \{Z'$_1$, \ldots, Z'$_n$\} a local Hermitian frame on an open neighbourhood $V$ of $\phi(p) \in N$. Let Z'$_1$, \ldots, Z'$_n$ be the unique horizontal lifts of Z'$_1$, \ldots, Z'$_n$ to $\phi^{-1}(V)$ and normalize by setting $Z_k = \lambda Z'_k$ for $k = 1, 2, \ldots, n$, where $\lambda$ is the dilation of $\phi$ defined in Section 2. Then we can, on an open
neighbourhood of $p$, extend $\{Z_1, \ldots, Z_n\}$ to a local Hermitian frame $\{Z_1, \ldots, Z_m\}$ for $M$. We then have

$$A = \sum_{k=1}^{n} \nabla \phi^{-1} T N Z_k^\prime (\lambda Z_k^\prime) = \sum_{k=1}^{n} \bar{Z}_k(\lambda) Z_k^\prime + \lambda^2 \sum_{k=1}^{n} \nabla N \bar{Z}_k Z_k^\prime.$$  

The vector field $\sum_{k=1}^{n} \bar{Z}_k(\lambda) Z_k^\prime$ belongs to $\phi^{-1} T^{1,0} N$, so by Lemma 3.3, $A^{0,1}$ vanishes on $R$ if and only if $N$ is cosymplectic at each point of $\phi(R)$.

Now note that if $p$ is a critical point of $\phi$ then either $p$ is a limit point of a sequence of regular points or $p$ is contained in an open subset $W$ of critical points. In the first case, if $A^{0,1}$ vanishes on $R$ then it vanishes also at $p$ by continuity. In the second case $d\phi = 0$ on $W$ so that $A^{0,1} = 0$ at $p$. This means that $A^{0,1}$ vanishes on $R$ if and only if it vanishes on $M$.

On the other hand, since $\phi$ is surjective it follows from Sard’s theorem that $\phi(R)$ is dense in $N$. This implies that $N$ is cosymplectic at points of $\phi(R)$ if and only if $N$ is cosymplectic everywhere. Putting the above remarks together yields the proof.

As a direct consequence of Proposition 4.4 we have the following:

**Theorem 4.5.** Let $\phi : (M, g, J) \rightarrow (N, h, J_N)$ be a surjective horizontally weakly conformal holomorphic map from a cosymplectic manifold to an almost Hermitian manifold. Then $N$ is cosymplectic if and only if $\phi$ is a harmonic morphism.

Combining Theorems 4.5 and 2.1 we then obtain:

**Corollary 4.6.** Let $\phi : (M, g, J) \rightarrow (N, h, J_N)$ be a surjective horizontally homothetic holomorphic map from a cosymplectic manifold to an almost Hermitian manifold. Then $N$ is cosymplectic if and only if $\phi$ has minimal fibres.
Corollary 4.6 generalizes a result of B. Watson in [23] where it is assumed that the map $\phi$ is a Riemannian submersion. If the manifold $(M, g, J)$ is $(1, 2)$-symplectic we have the following version of Theorem 4.5:

**Proposition 4.7.** Let $\phi : (M, g, J) \to (N^n, h, J^N)$ be a horizontally weakly conformal holomorphic map from a $(1, 2)$-symplectic manifold to a cosymplectic one. Then $\phi$ is a harmonic morphism whose fibres are minimal at regular points. If $n > 1$ then $\phi$ is horizontally homothetic.

**Proof:** The inclusion maps of the fibres of $\phi$ are holomorphic maps between $(1, 2)$-symplectic manifolds. They are isometric immersions and, by Corollary 4.3, harmonic so the fibres are minimal. For an alternative argument see [11]. If $n > 1$ then Theorem 2.1 implies that $\phi$ is horizontally homothetic.

Now assume that $n = 1$, then $N$ is automatically Kähler and therefore $(1, 2)$-symplectic. Further, any holomorphic map from an almost Hermitian manifold $(M, g, J)$ to $N$ is horizontally weakly conformal. Hence Proposition 4.2 implies the following results:

**Corollary 4.8.** Let $\phi : (M, g, J) \to N$ be a holomorphic map from an almost Hermitian manifold to a Riemann surface. Then $\phi$ is a harmonic morphism if and only if $d\phi(J\delta J) = 0$.

**Corollary 4.9.** Let $\phi : (M, g, J) \to N$ be a holomorphic map from a cosymplectic manifold to a Riemann surface. Then $\phi$ is a harmonic morphism.

**Example 4.10.** For two integers $r, s \geq 0$ let $M$ be the Calabi-Eckmann manifold $(S^{2r+1} \times S^{2s+1}, g, J)$ and $\phi : M \to \mathbb{C}P^r \times \mathbb{C}P^s$ be the product of the Hopf maps $S^{2r+1} \to \mathbb{C}P^r$, $S^{2s+1} \to \mathbb{C}P^s$. Then it is
not difficult to see that $\phi$ is holomorphic. Further the kernel of $d\phi$ is given by $\ker d\phi = \text{span}\{J_1n_1, J_2n_2\}$. From Example 3.4 we get $d\phi(\delta J) = -2d\phi(rJ_1n_1 + sJ_2n_2) = 0$. Since the map $\phi$ is a Riemannian submersion we deduce by Proposition 4.4 that $\phi$ is a harmonic morphism.

The next result can be extended to any of the twistor spaces considered by Salamon in [20], but for clarity we state it for a particular case.

Proposition 4.11. Let $\pi : N \to G_n(\mathbb{C}^{m+n})$ be the twistor fibration of Example 3.7 and $\phi : (M, g, J) \to N$ be a holomorphic map from an almost Hermitian manifold into the flag manifold $N$. Although $\psi = \pi \circ \phi : M \to G_n(\mathbb{C}^{m+n})$ is not, in general, a holomorphic map, we have

$$\tau(\psi) = -d\psi(J\delta J).$$

Proof: Let $\{Z_k\}$ be a local Hermitian frame on $M$. Then by using the Composition Law for the tension field and Lemmas 4.1 and 4.12 we obtain:

$$\tau(\psi) = d\pi(\tau(\phi)) + \sum_{k=1}^{m} \nabla d\pi(d\phi(\bar{Z}_k), d\phi(Z_k))$$

$$= -d\pi(d\phi(J\delta J)) + 0$$

$$= -d\psi(J\delta J).$$

Lemma 4.12. The twistor fibration $\pi : N \to G_n(\mathbb{C}^{m+n})$ is $(1,1)$-geodesic i.e.

$$\nabla d\pi(Z, W) = 0$$

for all $p \in N$, $Z \in T_p^{1,0}N$ and $W \in T_p^{0,1}N$.

Proof: Decompose $Z$ and $W$ into vertical and horizontal parts $Z = Z^V + Z^H$, $W = W^V + W^H$. Now since $\pi$ is a Riemannian submersion
\[ \nabla d\pi(Z^H, W^H) = 0 \] by Lemma 1.3 of \([19]\). Further
\[
\nabla d\pi(Z^V, W) = \nabla \phi^{-1} T_{G_n(C^{m+n})} Z \nabla d\pi(W) - d\pi(\nabla Z^V W).
\]
The first term is zero and the second term is of type \((0, 1)\) with respect to the almost Hermitian structure \(J_p\) on \(G_n(C^{m+n})\) defined by \(p\), since \(\nabla Z^V W\) is of type \((0, 1)\) and \(d\pi_p : T_p M \to T_{\pi(p)} G_n(C^{m+n})\) intertwines \(J\) and \(J_p\).

Similarly \(\nabla d\pi(W, Z^V)\) is of type \((1, 0)\), so by the symmetry of \(\nabla d\pi\), \(\nabla d\pi(Z^V, W) = 0\). Hence
\[
\nabla d\pi(Z, W) = \nabla d\pi(Z^V, W) + \nabla d\pi(Z^H, W^H) + \nabla d\pi(Z^H, W^V) = 0.
\]

5. **Superminimality**

Let \(\phi : (M, g) \to (N, h, J^N)\) be a horizontally conformal submersion from a Riemannian manifold to an almost Hermitian manifold. Assume that the fibres of \(\phi\) are orientable and of real dimension 2. Then we can construct an almost Hermitian structure \(J\) on \((M, g)\) such that \(\phi\) becomes holomorphic: make a smooth choice of an almost Hermitian structure on each fibre and lift \(J^N\) to the horizontal spaces \(H\) using \(d\phi \circ J = J^N \circ d\phi\).

For an almost Hermitian manifold \((M, g, J)\) we shall call an almost complex submanifold \(F\) of \(M\) **superminimal** if \(J\) is parallel along \(F\) i.e. \(\nabla_V J = 0\) for all vector fields \(V\) tangent to \(F\). It is not difficult to see that any superminimal \(F\) is minimal. Superminimality of surfaces in 4-dimensional manifolds has been discussed by several authors, see for example \([3]\).

**Theorem 5.1.** Let \(\phi : (M, g, J) \to (N, h, J^N)\) be a horizontally conformal holomorphic map from an almost Hermitian manifold to a Hermitian manifold with complex 1-dimensional fibres. If
(i) the fibres of \( \phi \) superminimal with respect to \( J \), and
(ii) the horizontal distribution \( \mathcal{H} \) satisfies \([\mathcal{H}^{1,0}, \mathcal{H}^{1,0}]^V \subset \mathcal{V}^{1,0}\),

then \( J \) is integrable.

**Proof:** We will show that \( T^{1,0}M \) is closed under the Lie bracket i.e. \([T^{1,0}M, T^{1,0}M] \subset T^{1,0}M\), or equivalently:

\begin{align*}
(a) \ [V^{1,0}, V^{1,0}] & \subset T^{1,0}M, \\
(b) \ [\mathcal{H}^{1,0}, \mathcal{H}^{1,0}] & \subset T^{1,0}M, \\
(c) \ [\mathcal{H}^{1,0}, V^{1,0}] & \subset T^{1,0}M.
\end{align*}

The fibres are complex 1-dimensional so \([V^{1,0}, V^{1,0}] = 0\). This proves (a). Let \( Z, W \) be two vector fields on \( N \) of type \((1, 0)\) and let \( Z^*, W^* \) be their horizontal lifts to \( \mathcal{H}^{1,0} \). Then \( d\phi[Z^*, W^*] = [Z, W] \) is of type \((1, 0)\) since \( J^N \) is integrable. The holomorphy of \( \phi \) and assumption (ii) then imply (b).

Let \( \langle , \rangle \) be the complex bilinear extention of \( g \) to \( T^\mathbb{C}M \) and \( V \) be a vertical vector field of type \((1, 0)\). Then \( d\phi([V, Z^*]) = [d\phi(V), Z] = 0 \), so \([V, Z^*]\mathcal{H} = 0\). On the other hand,

\[
\langle [V, Z^*], V \rangle = \langle \nabla_V Z^*, V \rangle - \langle \nabla_{Z^*} V, V \rangle \\
= -\langle Z^*, \nabla_V V \rangle - \frac{1}{2} Z^*(\langle V, V \rangle).
\]

The subspace \( T^{1,0}M \) is isotropic w.r.t. \( \langle , \rangle \) so \( \langle V, V \rangle = 0 \). The superminimality of the fibres implies that \( J(\nabla_V V) = \nabla_V JV = i\nabla_V V \). Hence \( \nabla_V V \) is an element of \( T^{1,0}M \) so \( \langle Z^*, \nabla_V V \rangle = 0 \). This shows that \( \langle [V, Z^*], V \rangle = 0 \) so \([V, Z^*]^\mathcal{H} \) belongs to \( T^{1,0}M \). This completes the proof.

The reader should note that condition (ii) of Theorem 5.1 is satisfied when the horizontal distribution \( \mathcal{H} \) is integrable, or when \( N \) is complex 1-dimensional, since in both cases \([\mathcal{H}^{1,0}, \mathcal{H}^{1,0}]^V \) = 0. Another example where the Theorem 5.1 applies is the following:
Example 5.2. The Hopf map $\phi : \mathbb{C}^{n+1} - \{0\} \to \mathbb{C}P^n$ is a horizontally conformal submersion with complex 1-dimensional fibres. The horizontal distribution is non-integrable, but it is easily seen that condition (ii) is satisfied, in fact $[\mathcal{H}^{1,0}, \mathcal{H}^{1,0}]^\mathcal{V} = 0$. The Kähler structure on $\mathbb{C}P^n$ lifts to two almost Hermitian structures on $\mathbb{C}^{n+1} - \{0\}$, the fibres are superminimal with respect to both of these, so by Theorem 5.1 they are both Hermitian. In fact one is the standard Kähler structure; the other is not Kähler.

If $N$ is complex 1-dimensional and $M$ is real 4-dimensional then Theorem 5.1 reduces to a result of the second author given in Proposition 3.9 of [25].

Example 5.3. Let $(M^2, g, J)$ be an almost Hermitian manifold of complex dimension 2 and $N$ a Riemann surface. Then we have the identities

$$\nabla_{\delta J} J = \nabla_{J \delta J} J = 0.$$

Otherwise said, span$_\mathbb{R}\{\delta J, J \delta J\}$ is contained in ker $\nabla J$. The condition $d\phi(\delta J) = 0$ for a non-constant holomorphic map $\phi : M \to N$ is thus equivalent to the superminimality of the fibres (at regular points) so that Corollary 4.8 translates into the following result of the second author, see Proposition 1.3 of [25]. A holomorphic map from a Hermitian manifold of complex dimension 2 to a Riemann surface is a harmonic morphism if and only if its fibres are superminimal at the regular points of $\phi$.

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