Abstract

The extended RC association models introduced in this paper allow the user to select the type of logit (local, global, continuation, reverse continuation) suitable for the row and column variables and the form of the divergence measure which, as in Kateri and Papaioannou (1994) may be used to define an extended class of bivariate interactions. An algorithm is described which fits these models; it exploits a reduced rank property without the need for identifiability constraints. Linear constraints that define parsimonious models are allowed on marginal logits and generalized interactions. An application to social mobility data is presented and discussed.

Keywords: RC association models, Divergence measures, reduced rank matrices, logit types, marginal models

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1. Introduction

The RC association models introduced by Goodman (1979) as a flexible class of models for investigating association in a two-way contingency table, were, later, extended by Goodman (1981) and Gilula et al. (1988) to include models of Correspondence Analysis which had been developed, initially, with partially different objectives. Later, Kateri and Papaioannou (1994) formulated a general class of RC association models that includes the original association models and the correspondence analysis model as special cases. They also showed that each model in their class is optimal with respect to a divergence measure which determines their generalized interactions.

An extension in a different direction was proposed by Bartolucci and Forcina (2002) who noted that the original RC association models could be seen as defined by a rank constraint on the matrix of adjacent log-odds ratios (LOR for short in the following); they suggested that a similar rank constraint could be imposed to the matrix of LOR defined by choosing logits of a general type (local, global, continuation) for the row and column variable. The motivation for this...
extension being that logits of a certain type may be more appropriate than the
adjacent to the row and columns variable of a two way table; these choices may
be combined (Douglas et al., 1990) leading to a variety of interaction parameters
which can be subjected to a rank constraint. Thus, for each chosen pair of logits,
a corresponding RC association model may be defined and fitted by imposing,
if suitable, additional linear constraints on marginal logits and LOR.

More recently, Espendiller (2017), Chapter 5, has proposed an extension of
RC association models by combining log-odds rations of four different types
with the function used by Kateri and Papaioannou (1994) to define generalized
interactions. In the same dissertation, methods for the estimation of U, R or C
association model of the new type are presented.

This paper defines a new class of interaction parameters for two-way ta-
tables obtained by combining the work of Kateri and Papaioannou (1994) and
Bartolucci and Forcina (2002) and may be seen as a generalization of Espendiller
(2017). In particular, if we restrict, for convenience, to Cressie and Read (1984)
power divergence measures, an RC association model may be defined by specifying
the value of the power parameter in the divergence measure, the type of
logit for the row and column variable and the rank of the corresponding matrix
of interactions. An algorithm for maximum likelihood estimation (ML) which
implements the results on rank constraints described by Bartolucci and Forcina
(2002) is presented. The idea behind this approach is that the set of identifiabil-
ity constraints used in the tradition of RC association models are unnecessary for
fitting the model, though, once the model has been fitted, they are useful tools
for interpreting and presenting the results. An adaptation of the regression
version of the Aitchison and Silvey (1958) approach to constrained ML estimation
described by Evans and Forcina (2013) is implemented together with a suitable
line search for improving stability.

The models are defined in section 2 and an algorithm for ML estimation is
presented in section 3. An application to social mobility is discussed in section
4.

2. A new class of extended RC association models

2.1. Notations

Consider an $I_1 \times I_2$ contingency table containing the observed frequencies for
$X_1, X_2$, two categorical variables; let $y$ denote the vector of observed frequencies
with the columns categories running faster; let also $\pi$ denote the corresponding
vector of cell probabilities. Let $l_i, i = 1, 2$, denote the logit type chosen for
$X_i, i = 1, 2$ respectively. There exist (see Bartolucci et al., Appendix) suitable matrices $C_h(l_i), h = 0, 1, i = 1, 2$ such that marginal logits of the given
type may be computed as

$$\eta_i = \log[C_1(l_i)\pi] - \log[C_0(l_i)\pi].$$

In order to define interactions, let $E(x_i, b_i, l_i)$ denote a class of events determined
by the cut point $x_i < I_i$, the binary indicator $b_i$ and the logit type $l_i$ according
to the following rules

\[ E(x_i, 0, l_i) = \begin{cases} x_i & \text{if } l_i \text{ is local or continuation} \\ 1, \ldots, x_i & \text{if } l_i \text{ is global or reverse continuation} \end{cases} \]

\[ E(x_i, 1, l_i) = \begin{cases} x_i + 1 & \text{if } l_i \text{ is local or reverse continuation} \\ x_i + 1, \ldots, I_i & \text{if } l_i \text{ is global or continuation} \end{cases} \]

2.2. A new class of interaction parameters

Define

\[ p_M(x_M; b_M; l_M) = P(X_i \in E(x_i, b_i, l_i), \forall i \in M), \]

let also

\[ \varphi_{ij}(b_1, b_2; l_1, l_2) = \frac{p_{1,2}(i, j; b_1, b_2; l_1, l_2)}{p_1(i; b_1; l_1) p_2(j; b_2; l_2)}. \]

Denote with \( F(v) \) the function defined in Kateri (2018), section 3; then an extended class of interaction parameters for two-way contingency tables may be defined as

\[ \gamma_{ij}(F; l_1, l_2) = F[\varphi_{ij}(1, 1; l_1, l_2)] - F[\varphi_{ij}(1, 0; l_1, l_2)] - F[\varphi_{ij}(0, 1; l_1, l_2)] + F[\varphi_{ij}(0, 0; l_1, l_2)]. \]

To be specific, let \( \pi_0 \) denote a reference distribution, than the Cressie and Read (1984) power measure of divergence between \( \pi \) and \( \pi_0 \), may be written as

\[ I(\pi, \pi_0; \lambda) = \frac{1}{\lambda(\lambda + 1)} \sum_v \pi_v \left( \frac{\pi_v}{\pi_0 v} \right)^\lambda - 1, \]

where \( \lambda \) is a real number and \( I(\pi, \pi_0; \lambda) \) convergence to the Kullback-Leibler divergence measure when \( \lambda \) tends to 0. Then, Theorem 2.1 in Kateri and Papaioannou (1994) implies that by setting \( F(v) = (v^\lambda - 1)/\lambda \), the resulting joint distribution has some desirable properties.

2.3. Defining RC association models by a rank constraint

The original formulation by Goodman, followed in most treatments of association models, assumes that

\[ \log \pi_{ij} = \zeta + \alpha_i + \beta_j + \sum_k \psi_k \mu_{ik} \nu_{jk}, \tag{1} \]

where \( K < (I_1 - 1)(I_2 - 1) \) and, for fixed \( k \), \( \mu_{ik} \), \( \nu_{jk} \) are, respectively, row and column scores and the \( \alpha \)s, \( \beta \)s, \( \mu \)s and \( \nu \)s are subject to suitable identifiability constraints. This formulation, which is closely related to that used for log-linear models, is less convenient when when modelling also the marginal distributions; this may be the case when, for instance, the \( X_i \) variables are homologous and marginal homogeneity is of interest.
It is well known that the marginal distributions of the row and column variables are determined by $I_1 - 1$ and $I_2 - 1$ marginal logits respectively, while the association depends on an $(I_1 - 1) \times (I_2 - 1)$ matrix of interactions. Let $M$ be such a matrix; it can be easily shown that (1) is equivalent to assume that $M$ has rank $K$. The idea proposed by Bartolucci and Forcina (2002) was to impose directly the rank constraint without inflating the parameter space and adding suitable identifiability constraints. In particular, Bartolucci and Forcina (2002) proved the following simple result: suppose that $m_{ij} \neq 0$, let $m_{1i}$ be the column vector containing the $i$th row of $M$ and $m_{2j}$ its jth column; let also $H_{hv}$ be the identity matrix of size $I_h - 1$ without the $v$th row.

**Lemma 1.** Let $F(M) = H_{1i}(M - m_{1i}m_{2j}'/m_{ij})H_{2j}';$ if $M$ has rank $K$, then $F(M)$ has rank $K - 1$.

**Proof** See Bartolucci and Forcina (2002), Appendix.

Because the transformation in Lemma 1 may be applied recursively, let $F^r(M)$ denote $r$ applications of the same transformation; if rank $M = K$, then Lemma 1 implies that $F^K(M)$ is a matrix of zeros. It follows that, under the assumed model, the vectorized version of the matrix of interactions may be transformed into a vector whose elements, under model (1) is a vector of 0s.

This approach has two convenient features:

1. the result in Lemma 1 is independent of the type of interactions adopted and thus it can be applied to the extended interactions defined above;
2. the derivative of $F^r(M)$ with respect to $M$, in its vectorised form, may be computed by a simple recursive formula.

### 3. Maximum likelihood estimation

Given a table of observed frequencies $y$, a model $\mathcal{M}$ is defined by choosing the logits type for the row and column variables, the value of the power divergence $\lambda$, the rank $K$ of the RC model and, possibly, additional linear constraints on the marginal logits and the interactions. Maximum likelihood may be computed by the Aitchison and Silvey (1958) algorithm where, in each step, a quadratic approximation of the log-likelihood is maximized subject to a linear approximation of the constraints. Below, a special version of this algorithm suitable in the present context is recalled briefly.

#### 3.1. The regression algorithm

Let $\theta$ denote a vector of canonical parameters for the multinomial distribution; this may be defined by a full rank matrix $G$ of size $I_1 I_2 \times (I_1 I_2 - 1)$ whose columns do not span the unitary vector, then $\log \pi = G\theta - 1 \log(\sum \exp(G\theta))$. Suppose that the model $\mathcal{M}$ is defined by the vector of constraints $h(\theta) = 0$; let $H$ be the derivative of $h'$ with respect to $\theta$; let also $H^- = H(H'H)^{-1}$ denote a right inverse of $H'$. A first order approximation of the constraints may be written as

$$h = h_0 + H'(\theta - \theta_0) = H'(H^- h_0 + \theta - \theta_0) = H'v = 0.$$  

(2)
Let $X$ be a matrix that spans the space orthogonal to the columns of $H$, then (2) implies $v = X\beta$, say. Let $Q$ denote the quadratic approximation of the log-likelihood at $\theta_0$ which has the same score and the same expected information, then the least square solution that maximizes $Q$ subject to the linear approximation of the constraints in (2) has the form

$$
\hat{v} = X(X'F_0X)^{-1}X'F_0(H^{-1}h_0 + F_0^{-1}s_0),
$$

where $s = G'(y - n\pi)$ and $F = nG'(\text{diag}(\pi) - \pi\pi')G$ denote, respectively, the score and information matrix. From this, an updated estimate along the line determined by the least square solution may be computed as $\theta(t) = \theta_0 + t(\hat{v} - H^{-1}h_0)$, where $t$ denotes the step length. Clearly the value of $t$ should be such that the likelihood increases and $h(\theta(t))$ is possibly smaller than $h(\theta_0)$.

### 3.2. Line search

Though one could choose the step length by trial, the approach described here can improve efficiency. It consists in computing the value of the function

$$
f(t) = y'\log(\pi(t))/n - h(t)'h(t)/2
$$

computed at $t = 0, 1/4, 1/2$ and maximizing the cubic approximation of $f(t)$ which uses also its first derivative at $t = 0$.

### 4. Application

The data in the left side of Table 1 is taken from Mosteller (1968) and classifies 3,500 British adults according to their social class and that of their fathers.

|       | $S_1$ | $S_2$ | $S_3$ | $S_4$ | $S_5$ | $S_1$ | $S_2$ | $S_3$ | $S_4$ |
|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| $F_1$ | 50    | 45    | 8     | 18    | 8     | 0.3932| 0.7188| 0.7859| 0.9457|
| $F_1$ | 28    | 174   | 84    | 154   | 55    | 0.0560| 0.3852| 0.5670| 0.8906|
| $F_1$ | 11    | 78    | 110   | 223   | 96    | 0.0159| 0.1747| 0.3683| 0.8214|
| $F_1$ | 14    | 150   | 185   | 714   | 447   | 0.0106| 0.0970| 0.2311| 0.7128|
| $F_1$ | 3     | 42    | 72    | 320   | 411   | 0.0058| 0.0516| 0.1314| 0.5298|

As a preliminary step, a quick search among possible model specifications with $K = 1$ was performed; because the row and column variables have the same nature, it seems reasonable to restrict attention to models with the same logit type for $X_1$, $X_2$. The deviance of models with logit type local, global and continuation as a function of $\lambda$ are plotted in Figure 1. The model with logits of type global with $\lambda = -0.04$ seems to do best; however, for a range
of $\lambda$s close to 0, difference in model fit are not substantial. Both the models with equally spaced row and column scores are rejected as well as the model of marginbal homogeneity. However, the model which assumes a constant shift between the corresponding marginal logits is not rejected ($P$ value 14.30%); it implies that the distribution of occupational status of the sons is stochastically larger relative to that of the fathers. Taking this as the final model, $\hat{\psi} = 1.98$ and the correlation between $X, Y$ based on the estimated scores equals 0.46. This seems to be in accordance with the fact that, being the estimated row and column scores increasing, all elements of the estimated matrix of extended interactions are positive. The estimated values of the cumulative conditional distributions given in Table indicate clearly that, the estimated model, satisfies positive quadrant dependence.

Table 2: Left side: deviance and degrees of freedom (dof) for a selection of models with $K = 1$, $\lambda = -0.04$ and logit type global for row and column variable; Right panel: estimated row and column scores for the RC model with shift on marginal logits

| Deviance | dof | row scores | column scores |
|----------|-----|------------|---------------|
| RC       | 7.60| -2.8343    | -2.9825       |
| R        | 50.15| -1.5076   | -1.6513       |
| C        | 55.88| -0.5235   | -0.6150       |
| RC+m.h.  | 40.47| 0.3199    | 0.2474        |
| RC+m.s  | 17.19| 1.0738    | 1.0162        |

m.h. = marginal homogeneity, m.s. = constant shift on marginal logits
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