Two Models Relevant to the Interaction of a Point Charge and a Magnetic Moment

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Abstract

An understanding of the interaction of a point charge and a magnetic moment is crucial for understanding the experiments involving electromagnetic momentum carried by permeable materials as well as the experimentally-observed Aharonov-Bohm and Aharonov-Casher phase shifts. Here we present two simple models for a magnetic moment which have vastly different interactions with a distant point charge. It is suggested that a satisfactory theoretical understanding of the interaction is still lacking and that the “hidden momentum” interpretation has been introduced into the textbook literature prematurely.
I. INTRODUCTION

The interaction of a point charge and a magnetic moment represents one of the persistent problems of classical electromagnetism. The interaction is intriguing partly because the combination of the electric field of the charge and the magnetic field of the magnetic moment introduces electromagnetic field linear momentum. One mystery of the charge-magnetic-moment interaction is just how Nature incorporates this electromagnetic field momentum into the relativistic conservation law for linear momentum. The interaction of a point charge and a magnetic moment forms the basis for controversies involving "hidden momentum," the Aharonov-Bohm effect, and the Aharonov-Casher effect. However, a certain version of the interaction involving hidden momentum has made its way into the textbook literature despite some objections. Indeed some research journals have rejected a competing description of the interaction, even refusing to send out for review manuscripts which explore the competing description. Most recently, the interaction has been used as the basis for the astonishing claim that the Lorentz force law is incompatible with relativity. Given the controversies which still exist, it seems wise to review the varying suggestions for the interaction of a point charge and a magnetic moment. Here we will carry out calculations involving two different models for a magnetic moment which interact with a point charge in strikingly different ways. The contrasting behaviors suggest that there is yet more to be understood regarding the interaction as it exists in Nature. The contrast also suggests the possibility that the current "hidden momentum" interpretation in the classical electromagnetism textbooks may be suspect and that the Aharonov-Bohm effect and Aharonov-Casher effect may be wrongly presented in the quantum mechanics texts.

II. TWO MODELS FOR A MAGNETIC MOMENT

In the present article, two different models for a magnetic moment are considered. The unperturbed magnetic-moment models both have a point charge $e$ of mass $m$ moving in a circular orbit $r(t)$ of radius $r_0$ and angular frequency $\omega_0$ about a charge $-e$ of large mass $M$ locate at the center of the orbit. Thus for the unperturbed magnetic moment orbit, we have the displacement, velocity, and acceleration of the charge $e$ located at angle $\phi = \omega_0 t + \phi_0$. 
given by

\[ \mathbf{r}(t) = r_0[\hat{i}\cos(\omega_0 t + \phi_0) + \hat{j}\sin(\omega_0 t + \phi_0)] \] (1)

\[ \mathbf{v}(t) = \omega_0 r_0[-\hat{i}\sin(\omega_0 t + \phi_0) + \hat{j}\cos(\omega_0 t + \phi_0)] \] (2)

\[ \mathbf{a}(t) = -\omega_0^2 r_0[\hat{i}\cos(\omega_0 t + \phi_0) + \hat{j}\sin(\omega_0 t + \phi_0)] \] (3)

The system has a magnetic moment

\[ \mathbf{\mu} = \frac{\hbar c \omega_0 r_0^2}{2c} \] (4)

obtained from the ensemble-averaged (averaging over the initial phase \( \phi_0 \)) or time-averaged current density, and the steady-state current formula \( \mathbf{\mu} = \int d^3 r \times \mathbf{J}/(2c) \). The crucial difference between the models involves the binding which leads to the circular orbit. The magnetic-moment model favored by the proponents of hidden momentum involves a “fixed-path” constraint such that during any interaction the point charge \( e \) may accelerate along the circular orbital path but cannot depart from the circular orbital path of radius \( r_0 \). The magnetic-moment model favored by the present author involves a Coulomb-potential interaction between the charge \( e \) and the central charge \( -e \), corresponding to the behavior of a hydrogen-like atom. Both these models have been presented previously in the literature, but never before in direct comparison.

III. ELECTROMAGNETIC MOMENTUM OF INTERACTION

We assume that the external charge \( q \) (with which the magnetic moment is interacting) is located on the \( x \)-axis at coordinate \( x_q \), \( \mathbf{r}_q = \hat{i}x_q \), where \( x_q >> r_0 \), so that we may think of the electric field \( \mathbf{E}_q(\mathbf{r}) = q(\mathbf{r} - \hat{i}x_q)/|\mathbf{r} - \hat{i}x_q|^3 \) due to the charge \( q \) as approximately constant over the magnetic moment at the value \( \mathbf{E}_q = -\hat{i}q/x_q^2 \) with small corrections of order \( r_0/x_q \).

The location of the magnetic moment in the electric field of the charge \( q \) leads to linear momentum \( \int d^3 r \mathbf{E} \times \mathbf{B}/(4\pi c) \) in the electromagnetic fields in vacuum arising from the electric field \( \mathbf{E}_q \) of the charge \( q \) and the magnetic field \( \mathbf{B}_e \) of the magnetic moment. The electromagnetic field momentum appears in order \( v^2/c^2 \), and therefore we can work with the Darwin Lagrangian and calculate perturbations using nonrelativistic physics. Since the electromagnetic field linear momentum \( \mathbf{p}_{em} \) is already first order in the perturbing charge \( q \), the momentum can be calculated through first order in \( q \) by using the unperturbed motion of the charge \( e \) as
\[ \mathbf{p}_{em} = \frac{eq}{2c^2} \left( \frac{\mathbf{v}(t)}{|\mathbf{r}(t) - \hat{i}x_q|} + \frac{\mathbf{v}(t) \cdot (\mathbf{r}(t) - \hat{i}x_q)(\mathbf{r}(t) - \hat{i}x_q)}{|\mathbf{r}(t) - \hat{i}x_q|^3} \right) \]

\[ = \frac{eq}{2c^2} \left[ \frac{\mathbf{v}(t)}{x_q} \left( 1 + \frac{\hat{i} \cdot \mathbf{r}(t)}{x_q} \right) + \frac{\mathbf{v}(t) \cdot [\mathbf{r}(t) - \hat{i}x_q][\mathbf{r}(t) - \hat{i}x_q]}{x_q^3} \left( 1 + 3 \frac{\hat{i} \cdot \mathbf{r}(t)}{x_q} \right) \right] \quad (5) \]

where we have expanded the denominators assuming that \( r_0 \ll x_q \) to obtain the expressions

\[ |\mathbf{r}(t) - \hat{i}x_q|^{-1} = x_q^{-1}(1 + \hat{i} \cdot \mathbf{r}(t)/x_q + ...), \quad |\mathbf{r}(t) - \hat{i}x_q|^{-3} = x_q^{-3}(1 + 3 \hat{i} \cdot \mathbf{r}(t)/x_q + ...). \]

Then the average linear momentum \( \langle \mathbf{p}_{em} \rangle \) in the electromagnetic field is found from Eqs. (1) and (2) by either ensemble averaging or averaging over time \( t \) with

\[ \langle \cos^2 \theta \rangle = \langle \sin^2 \theta \rangle = \frac{1}{2}, \quad \langle \cos \theta \sin \theta \rangle = 0, \]

\[ \langle \mathbf{p}_{em} \rangle = \frac{eq}{2c^2} \left( j \frac{\omega_0 r_0^2}{x_q^2} \right) = -\frac{1}{c} \mathbf{\mu} \times \mathbf{E} \quad (6) \]

This is the electromagnetic field momentum which must be consistent with the conservation of linear momentum.

IV. NONRELATIVISTIC PERTURBATION CALCULATION – FIXED-PATH MODEL

In the fixed-path model\(^1\)\(^4\) for a magnetic moment, the current-carrying charge \( e \) remains on the circular path of radius \( r_0 \) but may change its velocity along this path. In the nonrelativistic perturbation calculation, the angular acceleration \( \frac{d^2 \phi}{dt^2} \) of the charge \( e \) in its circular orbit is produced by the component of the electric field \( \mathbf{E}_q \) tangential to the orbit

\[ \frac{d^2 \phi}{dt^2} = a_{\phi}/r_0 = \frac{e \mathbf{E}_q \sin(\phi)}{r_0 m} = \frac{eq \sin(\phi)}{mr_0 x_q^2} \quad (7) \]

Since the angular acceleration is regarded as small, we may introduce the unperturbed expression \( \phi_{\text{unperturbed}}(t) = \omega_0 t + \phi_0 \) on the right-hand side of Eq. (7) to obtain

\[ \frac{d^2 \phi}{dt^2} = \frac{eq \sin(\omega_0 t + \phi_0)}{mr_0 x_q^2} \quad (8) \]

Assuming that the unperturbed motion holds at time \( t = 0 \), this equation can be integrated with respect to time to obtain

\[ \phi(t) = \omega_0 t + \phi_0 - \frac{eq \sin(\omega_0 t + \phi_0)}{\omega_0^2 mr_0 x_q^2} \quad (9) \]
Thus when interacting with the charge \( q \), the orbiting particle \( e \) of the magnetic moment has the position

\[
\mathbf{r}(t) = r_0 [\hat{i} \cos(\phi) + \hat{j} \sin(\phi)]
\]

\[
= r_0 \left[ \hat{i} \cos \left( \omega_0 t + \phi_0 - \frac{eq \sin(\omega_0 t + \phi_0)}{\omega_0^2 m_0 x_q^2} \right) + \hat{j} \sin \left( \omega_0 t + \phi_0 - \frac{eq \sin(\omega_0 t + \phi_0)}{\omega_0^2 m_0 x_q^2} \right) \right]
\]

\[
= r_0 \hat{i} \left[ \cos (\omega_0 t + \phi_0) + \frac{eq \sin(\omega_0 t + \phi_0)}{\omega_0^2 m_0 x_q^2} \sin(\omega_0 t + \phi_0) \right]
\]

\[
+ r_0 \hat{j} \left[ \sin(\omega_0 t + \phi_0) - \frac{eq \sin(\omega_0 t + \phi_0)}{\omega_0^2 m_0 x_q^2} \cos (\omega_0 t + \phi_0) \right]
\]

(10)

where we have used the approximations for small \( \delta \), \( \sin(\theta + \delta) \approx \sin \theta + \delta \cos \theta \) and \( \cos(\theta + \delta) \approx \cos \theta - \delta \sin \theta \).

The first thing which we notice is that the interaction of the magnetic moment (fixed-path model) has led to a nonrelativistic (zero-order in \( v/c \)) average electric dipole moment \( \langle \mathbf{p} \rangle \). Thus the average electric dipole moment follows from Eq. (10) as

\[
\langle \mathbf{p} \rangle = \langle \mathbf{e} \mathbf{r}(t) \rangle = \hat{i} \frac{e^2 q}{2m \omega_0 x_q^2} = -\frac{e^2}{2m \omega_0^2} \mathbf{E}_q
\]

(11)

This looks rather like the polarization found for a charged harmonic oscillator in the electrostatic field \( \mathbf{E}_q \) of a charge \( q \) except that the polarization is in the opposite direction from the polarizing electric field. Rather than the usual attraction between a charge and a polarizable material, we find here a repulsion.

There is no average nonrelativistic linear momentum in the circular orbit of the charge \( e \) since \( \langle \mathbf{p}_{\text{mech nonrel}} \rangle = m \langle \mathbf{v} \rangle = m \langle \mathbf{d} \mathbf{r}/dt \rangle = 0 \) using \( \mathbf{d} \mathbf{r}/dt \) from Eq. (10). However, there is a net linear momentum at the relativistic order \( v^2/c^2 \). The relativistic expression for mechanical linear momentum of a particle is \( \mathbf{p}_{\text{mech}} = m\gamma \mathbf{v} = m\gamma c^2 \mathbf{v}/c^2 \) where we recognize \( m\gamma c^2 \) as the mechanical energy (rest energy plus kinetic energy) of the particle. The energy conservation law which goes along with the fixed-path perturbation analysis gives energy balance for mechanical plus electrostatic potential energy as

\[
m\gamma_0 c^2 + \frac{eq_0 \cos \phi_0}{x_q^2} = m\gamma c^2 + \frac{eq_0 \cos \phi}{x_q^2}
\]

(12)

Thus we have

\[
\mathbf{p}_{\text{mech}} = m\gamma \mathbf{v} = \frac{m\gamma c^2 \mathbf{v}}{c^2} = \left( m\gamma_0 c^2 + \frac{eq_0 \cos \phi_0}{x_q^2} - \frac{eq_0 \cos \phi}{x_q^2} \right) \frac{1}{c^2} \frac{d \mathbf{r}}{dt}
\]

(13)
Then time-averaging using $dr/dt$ from Eq. (10) (or indeed from the unperturbed $v(t)$ of Eq. (2)), we find

$$
\langle P_{\text{mech}} \rangle = -\hat{j}eq\omega_0r_0^2
$$

(14)

This relativistic mechanical linear momentum is exactly equal in magnitude and opposite in direction from the electromagnetic linear momentum found in Eq. (13). Thus it is claimed that the mechanical hidden momentum balances the electromagnetic field momentum giving a self-consistent, self-contained system of zero linear momentum with no mystery regarding conservation of linear momentum. Of course, this description of the charge-magnetic-moment interaction says nothing about the source of the crucial nonrelativistic (zero-order in $v/c$) forces which constrain the motion of the current-carrying charge $e$ so that it moves only in a circular path. Also, there is no mention of the unusual electrostatic force between the magnetic moment and the charge $q$ associated with the nonrelativistic electric dipole moment in Eq. (11). In the opinion of the present writer, this fixed-path charge-magnetic-moment interaction description has no more validity than the interaction description of two point charges $e$ and $q$ which are at rest at a separation $x_m$ and are noted with triumph to have zero linear momentum. The interaction description remains worthlessly incomplete unless one accounts for the external nonelectromagnetic forces which are required for equilibrium or else (if there are no non-electromagnetic forces) one notes the time evolution of the system under the electromagnetic forces.

V. NONRELATIVISTIC PERTURBATION CALCULATION – PURELY ELECTROMAGNETIC MODEL

Our second model for a magnetic moment involves the charge $e$ attracted to the massive opposite charge $-e$ by Coulomb attraction. In this case, all the forces are electromagnetic, and there are no additional nonelectromagnetic forces of constraint which keep the charge $e$ in a circular orbit. Indeed, as pointed out by Solem in his article, “The Strange Polarization of the Classical Atom,” the interaction of the nonrelativistic hydrogen atom with an external electric field $E_q$ produces an electric dipole moment for the magnetic moment which is perpendicular to the electric field $E_q$. Qualitatively, the situation is easy to understand. A charge $e$ in a circular Coulomb orbit will be slowed down when moving toward the external charge $q$, and therefore the charge $e$ will tend to fall in closer to the
central charge \(-e\); on the other hand, the charge \(e\) will be speeded up when moving away from the external charge \(q\), and therefore the charge \(e\) will tend to move further away from the central charge \(-e\). Since the external charge \(q\) produces only a small orbital perturbation of the charge \(e\), we expect that the orbit of the charge \(e\) will be an elliptical Coulomb orbit. Furthermore, the semi-major axis of the elliptical orbit will remain essentially unchanged since the charge \(e\) is in approximately periodic motion, moving repeatedly towards and away from the external charge \(q\), successively losing and gaining kinetic energy from the field \(E_q\). The behavior involving a changing elliptical Coulomb orbit of fixed semi-major axis is totally different from the tangential-velocity-changing behavior portrayed in the fixed-path analysis discussed above. The changing ellipticity of the Coulomb orbit leads to a changing average electric dipole field back at the external charge \(q\) which produces forces on \(q\) which are completely different from those predicted by the fixed-path analysis.

The perturbation analysis of the Coulomb orbit by an external electric field has been given in other publications\(^{18}\). Here we will sketch the analysis. The displacement \(\mathbf{r}\) of the charge \(e\) with mass \(m\) in a Coulomb orbit with a charge \(-e\) at the origin is given by\(^{19}\)

\[
\mathbf{r} = \frac{3}{2} \frac{\mathbf{K}}{(-2mH_0)^{1/2}} + \frac{1}{4H_0} \frac{d}{dt} \left[ m(\mathbf{r} \times \mathbf{v} \times \mathbf{r} + m\mathbf{v}r^2) \right]
\]

where \(\mathbf{K}\) is the Laplace-Runge-Lenz vector\(^{20}\)

\[
\mathbf{K} = \frac{1}{(-2mH_0)^{1/2}} \left( [\mathbf{r} \times (m\mathbf{v})] \times (m\mathbf{v}) + me^2 \frac{\mathbf{r}}{r} \right)
\]

and \(H_0\) is the energy of the particle

\[
H_0 = \frac{1}{2} mv^2 - \frac{e^2}{r}
\]

The Laplace-Runge-Lenz vector \(\mathbf{K}\) is constant in time for a Coulomb orbit. Thus the time-average value of \(\mathbf{r}\) averaged over the orbit follows from Eq. \((15)\) as

\[
\langle \mathbf{r} \rangle = \frac{3}{2} \frac{\mathbf{K}}{(-2mH_0)^{1/2}}
\]

The nonrelativistic equation of motion for the charge \(e\) in the presence of the perturbing electric field \(E_q\) is

\[
m \frac{d^2 \mathbf{r}}{dt^2} = -\frac{e^2 \mathbf{r}}{r^3} + e\mathbf{E}_q
\]

Then the time-derivative of \(\mathbf{K}\) follows from Eqs. \((16)\) and \((19)\) as

\[
\frac{d\mathbf{K}}{dt} = me[-2\mathbf{r}(\mathbf{E}_q) + \mathbf{E}_q(\mathbf{r} \cdot \mathbf{v}) + \mathbf{v} (\mathbf{r} \cdot \mathbf{E}_q)]
\]
Since the right-hand side of Eq. (20) is already first order in the perturbing field $\mathbf{E}_q$, we may evaluate the time derivative through first order by averaging the right-hand side over an unperturbed Coulomb orbit to obtain

$$\frac{d\mathbf{K}}{dt} = \frac{3}{2}m_e(\mathbf{r} \times \mathbf{v}) \times \mathbf{E}_q = \frac{3}{2}e\mathbf{L} \times \mathbf{E}_q = 3cm\mu \times \mathbf{E}_q$$

(21)

where $\mathbf{L} = mr \times \mathbf{v}$ is the orbital angular momentum of the charge $e$. Thus in the Coulomb-orbit model for a magnetic moment, the current-carrying charge $e$ may start out in a circular orbit where the Laplace-Runge-Lenz vector $\mathbf{K}$ is zero, but the orbit changes in time becoming increasingly elliptical. The magnetic moment $\mu$ changes in time along with the orbital angular momentum $\mathbf{L}$

$$\langle \frac{d\mu}{dt} \rangle = \frac{e}{2mc} \langle \frac{d\mathbf{L}}{dt} \rangle = \frac{e}{2mc} \langle \mathbf{I} \rangle = \frac{e}{2mc} \langle e\mathbf{r} \rangle \times \mathbf{E}_q$$

(22)

The changing orbit of the charge $e$ in the magnetic moment will lead to electrical forces back on the external charge $q$, both nonrelativistic electrostatic forces associated with the presence of the electric dipole moment and also relativistic forces associated with the electric field induced by the changing magnetic moment. It has been pointed out that the forces associated with this changing magnetic moment are qualitatively appropriate to account for the Aharonov-Bohm phase shift as a lag effect associated with classical electromagnetic forces.\textsuperscript{21}

A. Closing Summary

The problem of the interaction of a point charge and a magnetic moment is an old problem surrounded by controversy. In recent years, one version (involving hidden momentum) of this controversy has been introduced into the textbook literature of electromagnetism. In this article, we present two models for magnetic moments and note their contrasting behaviors in the presence of an external charged particle. On the one hand, the fixed-path model for the magnetic moment indeed exhibits hidden momentum, but it involves unmentioned nonelectromagnetic forces which are of nonrelativistic order and are vastly greater than the relativistic mechanical effects which are touted in the textbook literature. The fixed-path model also involves an unusual nonrelativistic electric dipole moment. On the other hand,
the Coulomb-orbit model involves only electromagnetic interactions. This electromagnetic
type shows a changing magnetic moment which introduces both electrostatic fields and
induced electric fields. We believe that the interaction of a charged particle and a magnetic
moment (appropriate for describing Nature) remains a poorly-understood aspect of electro-
magnetic theory and that it is premature to accept the hidden-momentum description of
the interaction.

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8 See, for example, Ref. 1, pp. 357, 361, 520-521.
9 See, for example, Ref. 2, pp. 189, 618.
10 See, for example, T. H. Boyer, “Relativity, energy flow, and hidden momentum,” Am. J. Phys.
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13 See, for example, Ref. 2, p. 186.

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15 See ref. 2, pp. 596-598.

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19 Equation (15) can be check by carrying out the time derivative and inserting the equation of motion \( a = -\frac{e^2 r}{mr^3} \).

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