Abstract

Ten years ago, Glaßer, Pavan, Selman, and Zhang [GPSZ08] proved that if $P \neq NP$, then all NP-complete sets can be simply split into two NP-complete sets.

That advance might naturally make one wonder about a quite different potential consequence of NP-completeness: Can the union of easy NP sets ever be hard? In particular, can the union of two non-NP-complete NP sets ever be NP-complete?

Amazingly, Ladner [Lad75] resolved this more than forty years ago: If $P \neq NP$, then all NP-complete sets can be simply split into two non-NP-complete NP sets. Indeed, this holds even when one requires the two non-NP-complete NP sets to be disjoint.

We present this result as a mini-tutorial. We give a relatively detailed proof of this result, using the same technique and idea Ladner [Lad75] invented and used in proving a rich collection of results that include many that are more general than this result: delayed diagonalization. In particular, the proof presented is based on what one can call team diagonalization (or if one is being playful, perhaps even tag-team diagonalization): Multiple sets are formed separately by delayed diagonalization, yet those diagonalizations are mutually aware and delay some of their actions until their partner(s) have also succeeded in some coordinated action.

We relatedly note that, as a consequence of Ladner’s result, if $P \neq NP$, there exist OptP functions $f$ and $g$ whose composition is NP-hard yet neither $f$ nor $g$ is NP-hard.

1 Introduction

This paper is about the fact that if $P \neq NP$, then NP-hardness can be built by in a simple way by composing/combining non-NP-hard subparts.
Our initial interest in this came from a question not about sets but about functions. At a computational biology talk one of us attended, two actions were sequentially taken on the input and the overall transformation was clearly NP-hard. During a discussion of the talk, the question came up of whether that meant that at least one of the two composed transformations must itself be NP-hard. Although for the example of that talk it probably was the case that one of the constituent transformations could be directly proved NP-hard, what we eventually realized (not nearly as quickly as we should have) is that (assuming \( P \neq NP \)) this is not a general behavior. In particular, if \( P \neq NP \), then there exist NP optimization functions (i.e., \( \text{OptP} \) functions) \( f \) and \( g \) such that neither \( f \) nor \( g \) is even NP-Turing-hard, yet \( g(f(x)) \) is (functional) NP-many-one-hard. That is, non-NP-hard functions can via composition achieve NP-hardness.

The natural way to show this is simply to observe that the function case follows immediately from the language case, and the language case already put to bed in the 1970s by Ladner [Lad75]! So this paper presents, as a mini-tutorial on delayed diagonalization, and in honor of Professor Ladner’s career on the occasion of his retirement, a proof of the language-case result, which is in fact a special case of one of his 1975 results. (The paper also provides the observation that the (\( \text{OptP} \)) function case follows immediately from that.)

Credit Where Credit Is Due and Blame Where Blame Is Due It is important to mention up front that, as this is a tutorial, the credit for the results here belongs not to the authors of this tutorial, but to the author of the underlying paper, namely Richard Ladner. All of the results of this paper are either explicitly in his seminal paper on delayed diagonalization or are implicit from or (for the case of Section 5) follow easily from its results and techniques. Similarly, though the particular proof write-up of Theorem 4.1 and the framing of the theorem are our attempts to frame a clear yet very accessible proof for a tutorial article, it is very important to stress that all we are doing is employing Ladner’s breakthrough technique, delayed diagonalization, that he developed to prove results of this sort and of many related sorts, and indeed the proof is his in every sense other than that any errors our version might have, which of course are ours. In fact, in some sense we are giving a rather long proof to get a result weaker than ones that his original paper gets; the reason for this is that our goal here is to be an expository paper, and to make as clear as possible, a particular flavor of delayed diagonalization. So in brief, all the ideas here and much of the detail is due to the seminal paper of Ladner—except for any errors here, as those will be due to flaws on our part in writing this tutorial.\(^1\)

\(^1\)Some Fine Print and Disclaimers: Ladner’s article is in part focused on Turing reductions, as was common back then. Yet it also weaves in explicitly, and implies, a wide range of results about many-one reductions. In particular, the reader wanting to see Ladner’s far more general treatment of the type of thing we cover here is enthusiastically pointed to Ladner’s seminal paper [Lad75], and in particular to, on his page 160, his Theorem 2 which is a quite general version of the central result that we are presenting in this article, i.e., his Theorem 2 for the case of \( A \) being any \( P \) set and \( B \) being an NP-complete (i.e., NP-many-one-complete) set is, in effect, the special case being presented in Section 4 as Theorem 4.1, give or take the fact that in Theorem 4.1 here we have put right into the theorem statement some details of the simple splitting’s framework, namely, the polynomial-time function \( r \) and how it controls the splitting.
**Organization** The rest of this article is organized as follows. Section 2 mentions some related or contrasting results. Section 3 provides some preliminaries and definitions. Section 4 contains our write-up, but using Ladner’s delayed diagonalization, proving the result, due to Ladner, that if $P$ and $NP$ differ, then there exists a pair of disjoint, non-NP-complete sets in $NP − P$ whose union is NP-complete. The case of function composition described above will then follow easily, and is covered in Section 5, and Section 6 provides a brief teaser for some of the rest of the world of results that Ladner’s work provides and/or underpins.

2 Related and Contrasting Work

The fascinating related work of Glaßer et al. [GSTW08] has a different focus. That paper on the complexity of unions of disjoint sets primarily focuses on whether unions of disjoint NP-complete sets remain hard. That paper does have a section—Section 4.2 in that paper’s numbering—where the union of two disjoint sets is harder than its components, but the results of that section do not overlap at all with Ladner’s [Lad75] work, due to Glaßer et al.’s focus on equivalent (to each other) constituent sets.

Another related paper by Glaßer et al. [GPSZ08] shows that every nontrivial (in the sense of the set and its complement each containing at least two elements) NP-complete set is (so-called) m-mitotic. This result is interesting for us here because it implies that every nontrivial NP-complete set can be partitioned into two P-separable sets that are NP-complete. However, while Ladner’s work splits NP-complete sets into NP-noncomplete NP sets, Glaßer et al. are interested in splitting NP-complete sets into NP-complete sets.

We also mention the work of Hemaspaandra et al. [HJRW98] that shows that the join operator (the marked union operator) can yield a set of lower complexity (in the extended low hierarchy) than either of its constituents. This regards a focus opposite that of Ladner’s work presented in this paper. In particular, that work is about using combinations to lower complexity; the focus of the Ladner work that we are presenting is on using combinations to rise from non-NP-completeness to NP-completeness.

We prove Ladner’s key result using team diagonalization. Delayed diagonalization is the powerful technique first used by Ladner [Lad75] (see also, e.g., [Koz80, Sch82, Reg92, For00, RV97]) to show that if $P \neq NP$, then there are incomplete sets in $NP − P$. Many people, rather naturally given which example one tends to see in courses and textbooks, think of delayed diagonalization as the technique of having one set handle a list (actually two lists) of requirements by looking deeply into its own history. But in fact, delayed diagonalization is far more flexible and powerful than merely being able to do that. In (what we here will call) team diagonalization, though, Ladner in effect has two sets, each with its own list of requirements to satisfy, but the two sets will take long turns as to which of them is working on its requirements, and while one is doing that, the other will politely remain simple and boring. Loosely put, the sets will each respect the goals of the other set, and will take on burdens in a completely coordinated “lock-step” fashion.

Ladner’s Corollary 2.1 on page 160 of his paper gives the many-one analogue of his Theorem 2.
3 Preliminaries

For each string $x$, the number of characters in $x$ will be denoted $|x|$. For each set $A$ and each natural number $k$, $A^=k$ will denote $\{x \mid x \in A \land |x| = k\}$. We take all sets and classes to be with respect to the alphabet $\Sigma = \{0, 1\}$. The symmetric difference operation for sets, $(A - B) \cup (B - A)$, will be denoted $A \triangle B$.

All logarithms in this paper are base two, e.g., $\log \log i$ means $\log_2(\log_2(i))$.

We say $A \leq^p_m B$ exactly if $A$ is polynomial-time many-one reducible to $B$ (i.e., there is a polynomial-time function $f$ such that, for each $x$, it holds that $x \in A \iff f(x) \in B$). A set $B$ is NP-hard (with respect to $\leq^p_m$ reductions) exactly if for all $A \in$ NP, $A \leq^p_m B$. A set $B$ is NP-complete (with respect to $\leq^p_m$ reductions) exactly if $B \in$ NP and $B$ is NP-hard. We say $A \leq^p_T B$ if $A \in \text{P}^B$. A set $B$ is NP-hard with respect to $\leq^p_T$ reductions exactly if for all $A \in$ NP, $A \leq^p_T B$. A set $B$ is NP-complete with respect to $\leq^p_T$ reductions exactly if $B \in$ NP and $B$ is NP-hard with respect to $\leq^p_T$ reductions.

Sets $A$ and $B$ are said to be P-separable exactly if there exists a set $L \in$ P such that $A \subseteq L \subseteq \overline{B}$, i.e., there is a polynomial-time set that is a (possibly nonstrict) superset of $A$ yet has no intersection with $B$.

Given functions $f$ and $g$, we say that $f \leq^p_m g$ (f functional many-one reduces to $g$) if there are polynomial-time functions $h_1$ and $h_2$ such that, for all $x$, it holds that $f(x) = h_2(g(h_1(x)))$ [Zan91]. Functional many-one reductions are even more restrictive than metric reductions [Kre88] (which have almost the same definition except they allow $h_2$ to have direct access to $x$, i.e., the definition’s key part is $f(x) = h_2(x, g(h_1(x)))$) and are most commonly studied in the context of #P-completeness in order to prove stronger completeness results than mere #P-metric-completeness or #P-Turing completeness. For example, Valiant’s notion of #P-completeness in his seminal paper [Val79] is the Turing-reduction notion, and for the permanent of $(0, 1)$ matrices the reduction he builds is not a many-one reduction (although for the permanent of $(-1, 0, 1, 2, 3)$ matrices Valiant does establish #P-many-one-completeness); #P-many-one-completeness for the permanent of $(0, 1)$-matrices was obtained only more than a decade later, by Zankó [Zan91]. As another example, Deng and Papadimitriou’s [DP94] proof that the Shapley–Shubik power index is #P-metric-complete was later strengthened to a #P-many-one-completeness result [FI09].

Let $\chi_A$ denote the characteristic function of $A$, that is, $\chi_A(x)$ equals 0 if $x \notin A$ and equals 1 if $x \in A$. We say a function $g$ is NP-hard exactly if, for every $A \in$ NP, $\chi_A \leq^p_m g$. (Note that this is equivalent to $\chi_{\text{SAT}} \leq^p_m g$.) We say that a function $g$ is NP-Turing-hard exactly if, for all $A \in$ NP, $A \in \text{P}^g$ (equivalently, SAT $\in$ P$^g$).

We remark that in the literature when “NP-hard functions” are mentioned, the term often means what we here call NP-Turing-hard. However, for clarity, in this paper we always use, for both sets and functions, the terms NP-hard (for the many-one case) and NP-Turing-hard (for the Turing case). The key theorems statements here—in particular, Theorems 4.1, 5.1, and 6.1—are stated for the most demanding choices regarding many-one versus Turing reductions (even when this requires mixing and matching within the theorem statements, as indeed happens in each of these three theorem statements), and so they
imply the weaker choices.

4 Splitting NP-Complete Sets into NP-Incomplete Sets

The following captures Ladner’s beautiful insight that if P ≠ NP, then each NP-complete set can be built from (or looked at from the other direction, partitioned into) two non-NP-complete NP sets, in a very simple way.

**Theorem 4.1.** If P ≠ NP and S is an NP-complete set, then there is a function r : N → N such that, for all n, r(n) can be computed in time polynomial in n and the following disjoint sets A and B belong to NP − P, satisfy A ∪ B = S, and are not NP-hard even under polynomial-time Turing reductions:

\[ A = \{ x \mid x \in S \text{ and } r(|x|) \text{ is even} \} \]  

(1) and

\[ B = \{ x \mid x \in S \text{ and } r(|x|) \text{ is odd} \}. \]  

(2)

**Proof.** Suppose P ≠ NP. Let M₁, M₂, . . . be a standard enumeration of deterministic oracle Turing machines, each of which is explicitly polynomially clocked (upper-bounded) independently of its oracle. We assume, w.l.o.g., that the enumeration and the clocking are such that there is a universal oracle Turing machine \( U \) such that the following is true:

1. For each \( j \geq 1 \), each \( x \), and each \( A \), \( U^A(x, j) \) simulates \( M^A_j(x) \) (in the sense that \( x \in L(M^A_j) \iff x \in L(U^A(x, j)) \)), and

2. For each \( j \geq 1 \), each \( x \), and each \( A \), \( U^A(x, j) \) runs in time at most \( |x|^j + j \).

Technical aside (that we suggest ignoring during a first reading): The w.l.o.g.s in that sentence and a few lines above are the kind of somewhat painful groundwork that is often skipped over—as we also will mostly do here. But, to touch for those interested on what is under the hood: The natural way to build a nice clocked enumeration is to pair each machine (from a truly standard enumeration of all Turing machines) together with every possible clock drawn from some nice family of polynomials that for every polynomial has at least one member that majorizes that polynomial over all natural numbers. The typical family used for that is \( n^k + k \). One needs to then interlace and assign numbers in the enumeration so that even the earliest members of the enumeration meet their own claimed time bounds (such as that the \( k \)th machine will run in time \( n^k + k \)), and doing that often involves delaying (not at all in the same sense of the word as in delayed diagonalization) when machines come in (i.e., making sure their location in the enumeration has a high enough number) or even making some machines in our list dummy machines that ignore everything and in one step halt. And in doing what was just described, one has to account for the fact that clocking the time of a machine itself has overhead, since one puts a supervisor on top of the machine; but the overhead is mild—certainly at most polynomial. Beyond that, here we want to be able to uniformly simulate any given machine on the fly, and so one also has to take into account the cost of the universal machine’s own simulation of machines—itself also incurring a mild overhead—so that whatever claim one wants to make about the universal machine’s running times is correct. Despite that, all the w.l.o.g. claims above indeed can be routinely achieved. Important in that is that each (deeply) underlying machine is paired with polynomials (infinitely many) greater than particular given polynomial \( p \), and so basically each machine appears infinitely often on the listing, and indeed occurs with any particular needed polynomial “headroom for simulation”
Let $S$ be an arbitrary NP-complete set. Let $c \geq 1$ be a constant such that $S \in \text{DTIME}(2^{n^c})$. Let $A$ and $B$ be defined by Eqns. (1) and (2); of course, for those definitions to be meaningful, we will need to define $r$. In particular, we will now define $r$ such that $r$ is nondecreasing, and $r(n)$ can be computed in time polynomial in $n$. It follows, keeping in mind that $S \in \text{NP}$, that $A \in \text{NP}$ and $B \in \text{NP}$. It also follows, from the definition of $A$ and $B$, that $A$ and $B$ are disjoint and satisfy $A \cup B = S$. In addition, our definition of $r$ will be such that the other claims in the conclusion of Theorem 4.1 hold, namely, that neither $A$ nor $B$ belongs to $\text{P}$, and neither $A$ nor $B$ is NP-hard even under polynomial-time Turing reductions.

Let $r(0) = r(1) = r(2) = 2$. To define $r$, we describe a procedure that, for any $i \geq 2$, computes the value $r(i + 1)$ in time polynomial in $i$ based on the values the values $r(0), r(1), \ldots, r(i)$.

**Computation of $r(i + 1)$ based on $r(0), r(1), \ldots, r(i)$ ($i \geq 2$):**

When we determine $r(i + 1)$, we try to diagonalize against machine $M_{r(i)/2}$. If $r(i)$ is odd then we try to make sure that $M_{r(i)/2}$ does not decide SAT with the help of oracle $A$ and if $r(i)$ is even then we try to make sure that $M_{r(i)/2}$ does not decide SAT with the help of oracle $B$. If we succeed then we set $r(i + 1) = r(i) + 1$. Otherwise, we set $r(i + 1) = r(i)$. (Then the same machine and oracle will be tried to diagonalize against when $r(i + 2)$ is determined.)

If

$$2^{(\log \log i)^{r(i)/2} + |r(i)/2|^c} \geq i$$

(3)

then the diagonalization fails, and we set $r(i + 1) = r(i)$. Otherwise there are two cases, as follows. (As one reads the cases, the fact that $A$ and $B$ are being used in text that in part is also creating them may seem circular. But why this is not fatally naughty is explained in the discussion justifying the correctness of the construction.)

**Case 1: $r(i)$ is odd.** We try to diagonalize against $M_{r(i)/2}$ with oracle $A$. Determine if there exists a string $y$ of length at most $\log \log i$ satisfying

$$y \in \text{SAT} \iff y \notin L(M^A_{r(i)/2}).$$

(4)

If such a $y$ exists then SAT is not polynomial-time Turing reducible to $A$ via oracle machine $M_{r(i)/2}$. Hence the diagonalization is successful, and so we set $r(i + 1) = r(i) + 1$. Otherwise, we set $r(i + 1) = r(i)$.

**Case 2: $r(i)$ is even.** We try to diagonalize against $M_{r(i)/2}$ with oracle $B$. Determine if there exists a string $y$ of length at most $\log \log i$ satisfying

$$y \in \text{SAT} \iff y \notin L(M^B_{r(i)/2}).$$

(5)

even on top of the underlying time cost. So if an (deeply) underlying machine with a given oracle $A$ does happen to run in some polynomial time, one of the machines in our enumeration will capture that in such a way that even its simulation within the universal machine will duplicate, under oracle $A$, the action of the underlying machine in terms of acceptance and rejection, and will do so relative to our time claims without being cut off by time issues.

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If such a $y$ exists then SAT is not polynomial-time Turing reducible to $B$ via oracle machine $M_{[r(i)/2]}$. Hence the diagonalization is successful, and so we set $r(i + 1) = r(i) + 1$. Otherwise, we set $r(i + 1) = r(i)$.

For any given $n$, we compute $r(n)$ as follows: For $i = 2, 3, \ldots, n - 1$, we successively compute $r(i + 1)$ as described above based on the values $r(0), r(1), \ldots, r(i - 1), r(i)$. Note that if for each $i \in \{2, 3, \ldots, n - 1\}$ this is possible in time polynomial in $i$, then $r(n)$ can be computed in time polynomial in $n$.

It remains to show that this construction is correct.

1. The construction of $r(i + 1)$ in cases 1 and 2 above obviously depends on the oracle sets $A$ and $B$, which according to Eqn. (1) and Eqn. (2) depend on $r$ being odd or even. However, the construction is not circular: To determine whether $y \notin L(M_{[r(i)/2]}^{(i)})$, we only need to know the oracle up to length $(\log \log i)^{\lceil r(i)/2 \rceil} + \lfloor r(i)/2 \rfloor$, which is smaller than $i$ by Eqn. (3). Hence the only $r(k)$ values that $r(i + 1)$ may depend on are values $r(k)$ for $k$ less than than $i$.

2. We argue—and recall that, as noted above, showing this establishes that $r(n)$ can be computed (from scratch) in time polynomial in $n$—that for each $i \geq 2$, the procedure that determines $r(i + 1)$ based on $r(0), r(1), \ldots, r(i - 1), r(i)$ runs in time polynomial in $i$. To this end, we have to show that the conditions in Eqns. (4) and (5) can be checked for all $y$ with $|y| \leq \log \log i$ in time polynomial in $i$.

First, for each $y$, checking whether $y \in \text{SAT}$ can be done by brute force in time polynomial in $i$ since $y$ is of length at most $\log \log i$.

Second, the running time of $M_{[r(i)/2]}^{(i)}$ on inputs of length $\log \log i$ is at most $t = (\log \log i)^{\lceil r(i)/2 \rceil} + \lfloor r(i)/2 \rfloor$. Hence the length of each oracle query $q$ is at most $t$. Since $S \in \text{DTIME}(2^{n^c})$, there exists a constant $s > 0$ such that we can determine in time

\[ s \cdot 2^{c^e} \leq s \cdot 2^{((\log \log i)^{\lceil r(i)/2 \rceil} + \lfloor r(i)/2 \rfloor)^c} \]

whether $q \in S$. And since we got past the test of Eqn. (3), we thus know that the time used is at most $s \cdot i$.

Finally, note that no more than $(2\log i) - 1$ different strings $y$ have to be checked in whichever one of Eqns. (4) or (5) applies. Hence the whole procedure of determining $r(i + 1)$ based on $r(0), r(1), \ldots, r(i)$ takes time polynomial in $i$.

3. In the above construction, we try to diagonalize against machine $M_{[r(i)/2]}$ when we determine $r(i + 1)$. Hence to check that we eventually diagonalize against all deterministic polynomial-time oracle Turing machines $M_{j}^{(i)}$, we only have to show that $r(i)$—which, recall, is nondecreasing—grows indefinitely.

Suppose that there exist $k$ and $n_0$ such that $r(n) = k$ for all $n \geq n_0$. Then there are two cases:
Case 1: $k$ is odd.

Then for all $i \geq n_0$ it holds that, for all $y$ with $|y| \leq \log \log i$:

\[ y \in \text{SAT} \iff y \in L(M^A_{\lfloor r(i)/2 \rfloor}), \quad \text{and} \]
\[ y \in L(M^A_{\lfloor r(i)/2 \rfloor}) \iff y \in L(M^A_{k/2}). \]

So for every string $y$, it holds that $y \in \text{SAT} \iff y \in L(M^A_{k/2})$. Furthermore, by construction, $A$ contains in this case only finitely many strings. It follows that SAT can be decided in polynomial time, which contradicts our assumption that $P \neq \text{NP}$.

Case 2: $k$ is even.

Then for all $i \geq n_0$ it holds that, for all $y$ with $|y| \leq \log \log i$:

\[ y \in \text{SAT} \iff y \in L(M^B_{\lfloor r(i)/2 \rfloor}) \quad \text{and} \]
\[ y \in L(M^B_{\lfloor r(i)/2 \rfloor}) \iff y \in L(M^B_{k/2}). \]

So for every string $y$, it holds that $y \in \text{SAT} \iff y \in L(M^B_{k/2})$. Furthermore, by construction, $B$ contains in this case only finitely many strings. It follows that SAT can be decided in polynomial time, which contradicts our assumption that $P \neq \text{NP}$.

4. Finally, we must argue that $A \not\in P$ and $B \not\in P$. (Recall that, at the start of the proof, we pointed out that both $A$ and $B$ belong to NP, that $A$ and $B$ are disjoint, and that $A \cup B = S$. So we do not need to re-argue those points here.) Suppose for example that $A \in P$. That implies, since $S = A \cup B$, that $P^S \subseteq P^B$. So the fact that $S$ is NP-complete certainly yields that $B$ is NP-Turing-hard. But that contradicts the fact that, as argued above, $B$ is not NP-Turing-hard. So $A \not\in P$.

By the analogous argument, it also holds that $B \not\in P$.

One has the following easy corollaries. (For disjoint sets $A_1$ and $A_2$, note that $A_1 \cup A_2 = A_1 \Delta A_2$, so one could equally well in the theorems below make the theorems be stated not about a union ($A_1 \cup A_2$) but about a symmetric difference ($A_1 \Delta A_2$).)

**Corollary 4.2.** $P \neq \text{NP}$ if and only if for each NP-complete set $S$ there exist disjoint NP sets $A_1 \subseteq S$ and $A_2 \subseteq S$ such that $S = A_1 \cup A_2$ and neither $A_1$ nor $A_2$ is NP-complete.

**Proof.** The direction from left to right follows directly from Theorem 4.1.

Now suppose that $P = \text{NP}$. Let $S$ be any set such that $S \neq \Sigma^*$ and $S \neq \emptyset$. Then $S$ is NP-complete. The only subset of $S$ that is not NP-complete is the empty set. Hence there are no NP-incomplete sets $A_1 \subseteq S$ and $A_2 \subseteq S$ such that $A_1 \cup A_2 = S$. 

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If $P \neq NP$ then the sets $A_1$ and $A_2$ in Corollary 4.2 are both not in $P$.

**Corollary 4.3.** If $P \neq NP$, then every $NP$-complete set $S$ has the property that there is a $P$ set $D$ such that both $S \cap D$ and $S \cap \overline{D}$ are $NP$-incomplete (i.e., are in $NP$ yet are not $NP$-complete).

That is, for every $NP$-complete set $S$ there exist $P$-separable sets $A_1$ and $A_2$ such that $S = A_1 \cup A_2$ and neither $A_1$ nor $A_2$ is $NP$-hard.

**Proof.** Let

$$D = \{x \in \Sigma^* \mid r(|x|) \text{ is even}\}.$$ 

Note that $A = S \cap D$ and $B = S \cap \overline{D}$ are the sets $A$ and $B$ in Theorem 4.1. 

We mention a result by Glaßer et al. [GPSZ08] that is similar in spirit to Corollary 4.3 even though it is trying to achieve the opposite. Glaßer et al. [GPSZ08] call a set $A$ nontrivial exactly if both $A$ and $\overline{A}$ contain at least two elements. Glaßer et al. showed (among other things) that every nontrivial set $A$ that is $NP$-complete is also $m$-mitotic. This easily implies the following theorem.

**Theorem 4.4 ([GPSZ08]).** If $P \neq NP$, then every $NP$-complete set $S$ has the property that there is a $P$ set $D$ such that both $S \cap D$ and $S \cap \overline{D}$ are $NP$-complete.

While Corollary 4.3 shows (if $P \neq NP$) that every $NP$-complete set $S$ can be split into $P$-separable sets that are not $NP$-complete, Glaßer et al.'s theorem implies that (regardless of whether or not $P \neq NP$) every nontrivial $NP$-complete set $S$ can be split into $P$-separable sets such that both sets are indeed $NP$-complete.

Note that regarding $P = NP$ we have the following.

**Theorem 4.5.** If $P = NP$, then no $NP$-complete set $S$ has the property that there is a $P$ set $D$ such that both $S \cap D$ and $S \cap \overline{D}$ are $NP$-incomplete (i.e., are in $NP$ yet are not $NP$-complete).

**Proof.** Suppose $P = NP$. Then the only $NP$-incomplete sets are $\emptyset$ and $\Sigma^*$. But the only sets that can be formed by the union of two sets chosen from $\{\emptyset, \Sigma^*\}$ are $\emptyset$ and $\Sigma^*$, which as just mentioned are $NP$-incomplete, yet the theorem’s claim is that $S$ is $NP$-complete.

Note that Corollary 4.3 and Theorem 4.5 are not converses of each other. They are actually stronger than just giving an “if and only” statement.

The following notes that the sets being used are not merely disjoint in the sense that no string participates in both of the sets, but also differ so strongly that no length participates in both of the sets.

**Definition 4.6.** Two sets $A_1$ and $A_2$ are strongly disjoint exactly if, for every $k$, $A_1^k = \emptyset$ or $A_2^k = \emptyset$.

**Corollary 4.7.** $P \neq NP$ if and only if for each $NP$-complete set $S$ there exist strongly disjoint, $P$-separable, $NP$ sets $A_1 \subseteq S$ and $A_2 \subseteq S$ such that $A_1 \cup A_2 = S$ and neither $A_1$ nor $A_2$ is $NP$-complete.
Proof. The direction from left to right follows easily from Theorem 4.1 because it is easy to see that the sets \( A \) and \( B \) in that theorem are strongly disjoint P-separable NP-sets.

The direction from right to left follows directly from Corollary 4.2. \( \square \)

5 Composition of Functions: Hard Functions Composed from Nonhard Ones

Consider NPTMs where each nondeterministic path outputs one value (say, a string over the alphabet \( \Sigma^* \)). Paths that do not explicitly output a value are by convention viewed as outputting \( \varepsilon \), the empty string. A function \( f \) is said to be in \( \text{OptP} \) \([\text{Kre88}]\) exactly if there is a thus-viewed NPTM \( N \) such that, for each \( x \), \( f(x) \) is the lexicographically maximum value among all values output by paths of \( N \) on input \( x \). (By lexicographical order, we mean as is standard the order in which \( \varepsilon < 0 < 1 < 00 < 01 < 10 < 11 < 111 < \ldots \). \( \text{OptP} \) is often viewed as having codomain \( \mathbb{N} \) rather than \( \Sigma^* \), but by the natural bijection, the views are the same.)

We now observe, as an easy consequence of the main result of the previous section, that the composition of non-NP-hard NP optimization functions can achieve NP-hardness. Let \( g \circ f \) denote the composition of the functions, i.e., the function defined by, for each \( x \),

\[
(g \circ f)(x) = g(f(x)).
\]

Theorem 5.1. If \( P \neq \text{NP} \), then there exist \( \text{OptP} \) functions \( f \) and \( g \) such that neither \( f \) nor \( g \) is \( \text{NP} \)-hard (or even \( \text{NP} \)-Turing-hard), yet \( g \circ f \) is an \( \text{NP} \)-hard \( \text{OptP} \) function.

Proof. Let \( S = \text{SAT} \) and let \( A \) and \( B \) be disjoint NP sets such that neither \( A \) nor \( B \) is \( \text{NP} \)-Turing-complete and \( A \cup B = S \). Such sets exist according to Theorem 4.1.

We define \( f \) and \( g \) as follows:

For each \( x \in \Sigma^* \),

\[
f(x) = \begin{cases} 
|\{ \text{\varepsilon} \} | + 1 & \text{if } x \in A \\
0 & \text{otherwise,}
\end{cases}
\]

and for each \( z \in \Sigma^* \),

\[
g(z) = \begin{cases} 
1 & \text{if } (z's \text{ first bit is a } 1) \text{ or } (z = 0x \text{ and } x \in B) \\
0 & \text{otherwise.}
\end{cases}
\]

It is easy to see that for each \( x \in \Sigma^* \),

\[
g(f(x)) = \begin{cases} 
1 & \text{if } x \in A \cup B \\
0 & \text{otherwise.}
\end{cases}
\]

Hence \( \chi_{\text{SAT}} \leq^p_m g \circ f \), and therefore \( g \circ f \) is \( \text{NP} \)-hard under polynomial-time many-one reductions. Furthermore, \( f \), \( g \), and \( g \circ f \) are easily seen to be \( \text{OptP} \) functions.

It remains to show that \( f \) and \( g \) are not \( \text{NP} \)-Turing-hard.
Suppose \( f \) is NP-Turing-hard. Then there exists a polynomial-time oracle machine \( M \) such that \( \text{SAT} = L(M^f) \). Since given \( A \) as an oracle it is easy to compute \( f \), it follows that there is a polynomial-time oracle Turing machine \( M' \) such that \( \text{SAT} = L(M'^A) \). But this is a contradiction because we assumed that \( A \) is not NP-Turing-hard.

In the same way, we can show that \( g \) is not NP-Turing-hard. \( \square \)

6 Generalizations and Variants

Theorem 4.1 can be generalized/varied in many ways. Please see Ladner’s original paper [Lad75] for results in such a general form that this can be done very broadly. But for now let us mention a few examples. First, the theorem holds for many classes other than NP (because Ladner proves it in a very general setting, see especially p. 160 of his paper), for instance for PSPACE, EXP, EXPSPACE, EEXP, \( \Sigma^p_i \), \( \Pi^p_i \), etc. Second, Theorem 4.1 says that we can split \( S \) into two sets that are NP-incomplete. Clearly, a straightforward adaptation of the proof shows that for any integer \( k > 1 \), \( S \) can be split into \( k \) incomplete sets. As an example, we mention a variation of Theorem 4.1 for the case of EEXP, with things split into three incomplete sets. Of course, here one does not need any assumption of the form \( P \neq \text{NP} \).

**Theorem 6.1.** Let \( S \) be any EEXP-complete set. Then there is a function \( r : \mathbb{N} \to \mathbb{N} \) such that for all \( n \), \( r(n) \) can be computed in time polynomial in \( n \) and the following pairwise-disjoint EEXP sets \( A, B, \) and \( C \) satisfy \( S = A \cup B \cup C \) and are not EEXP-hard under polynomial-time Turing reductions:

\[
A = \{ x \mid x \in S \land (r(|x|) \equiv 0 \text{ (mod 3)}) \},
\]

\[
B = \{ x \mid x \in S \land (r(|x|) \equiv 1 \text{ (mod 3)}) \},
\]

and

\[
C = \{ x \mid x \in S \land (r(|x|) \equiv 2 \text{ (mod 3)}) \}.
\]

The proof differs from the proof of Theorem 4.1 in that one now has to consider three different cases: \( r(|x|) \equiv 0 \text{ (mod 3)} \), \( r(|x|) \equiv 1 \text{ (mod 3)} \), and \( r(|x|) \equiv 2 \text{ (mod 3)} \), and further, the length of the string \( y \) that one uses in one’s diagonalization must be of length roughly triple-logarithmic in \( i \), that is, computations have to look even more deeply back within the history.

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References

[DP94] X. Deng and C. Papadimitriou. On the complexity of comparative solution concepts. *Mathematics of Operations Research*, 19(2):257–266, 1994.

[FH09] P. Faliszewski and L. Hemaspaandra. The complexity of power-index comparison. *Theoretical Computer Science*, 410(1):101–107, 2009.

[For00] L. Fortnow. Diagonalization. *Bulletin of the EATCS*, 71:102–112, 2000.

[GPSZ08] C. Glaßer, A. Pavan, A. Selman, and L. Zhang. Splitting NP-complete sets. *SIAM Journal on Computing*, 37:1517–1535, 2008.

[GSTW08] C. Glaßer, A. Selman, S. Travers, and K. Wagner. The complexity of unions of disjoint sets. *Journal of Computer and System Sciences*, 74:1173–1187, 2008.

[HJR98] L. Hemaspaandra, Z. Jiang, J. Rothe, and O. Watanabe. Boolean operations, joins, and the extended low hierarchy. *Theoretical Computer Science*, 205(1–2):317–327, 1998.

[HS18] L. Hemaspaandra and H. Spakowski. Team diagonalization. *SIGACT News*, 49(3), 2018. To appear.

[Koz80] D. Kozen. Indexings of subrecursive classes. *Theoretical Computer Science*, 11(3):277–301, 1980.

[Kre88] M. Krentel. The complexity of optimization problems. *Journal of Computer and System Sciences*, 36(3):490–509, 1988.

[Lad75] R. Ladner. On the structure of polynomial time reducibility. *Journal of the ACM*, 22(1):155–171, 1975.

[Reg92] K. Regan. Diagonalization, uniformity, and fixed-point theorems. *Information and Computation*, 98:1–40, 1992.

[RV97] K. Regan and H. Vollmer. Gap-languages and log-time complexity classes. *Theoretical Computer Science*, 188(1–2):101–116, 1997.

[Sch82] U. Schöning. A uniform approach to obtain diagonal sets in complexity classes. *Theoretical Computer Science*, 18:95–103, 1982.

[Val79] L. Valiant. The complexity of computing the permanent. *Theoretical Computer Science*, 8(2):189–201, 1979.

[Zan91] V. Zankó. #P-completeness via many-one reductions. *International Journal of Foundations of Computer Science*, 2(1):76–82, 1991.