ON C.T.C. WALL’S SUSPENSION THEOREM

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Abstract. The concept of thickening was systematically studied by C.T.C. Wall in [Wa1]. The suspension theorem of that paper is an exact sequence relating the n-dimensional thickenings of a finite complex to its (n+1)-dimensional ones. The object of this note is to fill in what we believe is a missing argument in the proof of that theorem.

1. Introduction

One of the classical industries in differential topology is to list up to diffeomorphism all manifolds within a given homotopy type. In dimensions \( \geq 6 \), assuming that the homotopy type of the boundary is held fixed, the surgery exact sequence [Wa2] reduces the problem to bundle theory and the stable algebra of bilinear forms over the integral group ring of the fundamental group.

However, if the homotopy type of the boundary is not held fixed, the methods of surgery theory do not apply. Nearly forty years ago, C.T.C. Wall [Wa1] developed tools to study this version of the problem. Of these, probably the most computationally useful is the suspension theorem (cf. [Wa1, §5]), which is a variant of the classical EHP sequence of I.M. James [Ja]. This note is a comment on the suspension theorem.

Thickenings. Let \( K \) be a connected finite CW complex of dimension \( \leq k \). An \( n \)-thickening of \( K \) consists of a simple homotopy equivalence

\[ f : K \to M \]

in which \( M \) is a compact smooth manifold of dimension \( n \) such that the inclusion \( \partial M \to M \) is 1-connected and induces an isomorphism of fundamental groups (such manifolds are said to satisfy the \( \pi-\pi \) condition). We will assume that \( k \leq n-3 \) and \( n \geq 6 \).

Two \( n \)-thickenings \( f_0 : K \to M_0 \) and \( f_1 : K \to M_1 \) are said to be equivalent if there is a diffeomorphism \( h : M_0 \to M_1 \) such that \( hf_0 \)…

\[\text{Date: July 5, 2018.}\]
\[\text{The second author is partially supported by NSF Grant DMS-0201695.}\]
\[\text{2000 MSC. Primary: 55R19. Secondary: 55N65.}\]
is homotopic to \( f_1 \). This gives an equivalence relation on the set of \( n \)-thickenings; let

\[ T_n(K) \]

denote the associated set of equivalence classes.

**Suspension.** The operation obtained by taking the cartesian product with the unit interval gives a map

\[ E: T_n(K) \to T_{n+1}(K) \]

called suspension. Let us say that an \((n+1)\)-thickening compresses if its associated equivalence class is in the image of \( E \).

Wall’s suspension theorem provides criteria for deciding when a thickening compresses. Wall assumes

- \( K \) is \((2k-n)\)-connected,
- \( 3k+2 \leq 2n \).

He then constructs a function

\[ H: T_{n+1}(K) \to [K \times K, S^{n+1}] \]

where the target is the homotopy classes of maps from \( K \times K \) to \( S^{n+1} \) (we will review the definition of \( H \) below). Wall then shows that a thickening compresses if and only if it vanishes under \( H \). This is the content of one half of the suspension theorem. The other half gives criteria for deciding the extent to which \( E \) is injective (we will not be concerned with the latter in this paper).

**Definition of \( H \).** Let \( f: K \to M \) be an \((n+1)\)-thickening representing an element \( x \in T_{n+1}(K) \). Let \( C \) be the complement of a tubular neighborhood of the diagonal embedding \( M \to M \times M \). Then we have an identification

\[ (M \times M)/C \cong M^\tau, \]

where \( M^\tau \) denotes the Thom space tangent bundle \( \tau \) of \( M \). Choose a basepoint for \( K \). Applying \( f \), we obtain a basepoint for \( M \). Identify \( S^{n+1} \) with the one point compactification of the fiber of \( \tau \) at the basepoint. This gives an inclusion

\[ S^{n+1} \subset M^\tau. \]

Wall observes that the range assumptions imply that this inclusion is \((2k+1)\)-connected.

The composite map

\[ K \times K \xrightarrow{f \times f} M \times M \xrightarrow{\text{collapse} C} (M \times M)/C = M^\tau \]
ON C.T.C. WALL’S SUSPENSION THEOREM

Gives rise to a homotopy class in \([K \times K, M^\tau]\). Since \(K \times K\) has dimension \(\leq 2k\), the induced map of homotopy sets

\([K \times K, S^{n+1}] \to [K \times K, M^\tau]\)

is a bijection. Wall defines

\(H(x) \in [K \times K, S^{n+1}]\)

to be the unique element that pushes forward to the above homotopy class.

The key step in Wall’s proof. Wall wishes to show that the triviality of

\(H(x) \in [K \times K, S^{n+1}]\)

implies that the inclusion \(\partial M \to M\) admits a section up to homotopy.

If there is such a homotopy section \(s: M \to \partial M\), then by the numerical assumptions and the Stallings-Wall embedding theorem ([St], [Wa1]), the map

\[
\begin{align*}
K \xrightarrow{f} M \xrightarrow{s} \partial M
\end{align*}
\]

admits an embedding up to homotopy, i.e., there is a compact codimension zero submanifold \(N \subset \partial M\) and a simple homotopy equivalence \(g: K \to N\) such that the composite

\[
\begin{align*}
K \xrightarrow{g} N \xrightarrow{c} M
\end{align*}
\]

is homotopic to \(f\). Then, using the s-cobordism theorem, one sees that \(g: K \to N\) is a representative of a compression of \(x\).

In summary, the proof hinges on the following:

Assertion. (Wall). In the given range, \(H(x)\) vanishes if and only if \(\partial M \to M\) admits a homotopy section.

On page 87 of [Wa1], a proof of this assertion is purportedly given. We cite the relevant paragraph here:

Now let \(M'\) be the complement of a collar neighborhood of \(\partial M\) in \(M\), \(i: M' \to M\) the inclusion. Since \(K, M,\) and \(M'\) all have the same homotopy type, the assertion above shows that \(f \times i: K \times M' \to M \times M\) is homotopic to a map into \(C\), and hence to a map into \(\text{Int}(M \times M - \Delta M)\). But the projection onto the second factor \(\text{Int}(M)\) is a fibration (it is well known to be locally trivial); the above homotopy projects to one in \(\text{Int}(M)\), whose inverse lifts to a homotopy of the constructed map \(K \times M' \to C\) to map of the form \(g' \times i\).
Thus \( f \simeq g' \), and the image of \( g' \) lies in a collar neighborhood of \( \partial M \); a further evident homotopy sends \( g' \) to a map \( g: K \to \partial M \).

(note: we have changed Wall’s notation to conform with ours, but with this exception, what is displayed here is verbatim). In line 3 of the citation, the expression ‘the assertion above’ amounts to the statement that the cofibration sequence

\[ C \to M \times M \to M^\tau \]

is a homotopy fiber sequence in dimensions \( \leq 2k \). This follows from the Blakers-Massey theorem and the range assumption. Hence the first part of the argument is clear to us.

However, the latter half of the paragraph eludes us. Specifically, we do not understand how the map \( g' \times i \) is produced having the required property: namely, the map \( g': K \to M \) is supposed to satisfy \( g'(x) \neq i(y) \) for all \( x \in K \) and \( y \in M' \). It seems to us that the map \( g' \) obtained from Wall’s argument really depends on both factors, and so is a map of the form \( g': K \times M' \to M \). Unfortunately a map of the latter kind is insufficient for producing the desired map \( g \).

Furthermore, once \( f \times i \) is homotoped to a map into \( C \), the range assumption is not used anymore in the argument. Since \( f \) and \( i \) are homotopy equivalences, one can reformulate what Wall writes as saying that, without any range restrictions, the only obstruction to finding a homotopy section of the inclusion \( \partial M \to M \) is to find a homotopy factorization of the identity map of \( M \times M \) through \( M \times M - \Delta M \). We think this is unlikely in general to be true, but we do not have a counterexample.

Rather than trying to sort out Wall’s argument, we will provide one of our own.

**Reformulation.** The above description of \( H \) used the collapse map

\[ M \times M \to M^\tau \]

which make into a based map

\[ c_0: M_+ \wedge M_+ \to M^\tau, \]

where \( M_+ \) is the effect of adjoining a disjoint basepoint to \( M \) and we have used the identification \( (M \times M)_+ = M_+ \wedge M_+ \).

It is clear from the range assumption and the definition of \( H \) that \( H(x) \) vanishes if and only if \( c \) is stably null homotopic.

We are ready to formulate the main result of this paper.
Theorem A. Let $M$ be a compact manifold of dimension $n+1$ satisfying the $\pi$-$\pi$ condition. Assume that $M$ has the homotopy type of a CW complex of dimension $\leq k$. Furthermore, assume

- $M$ is $(2k-n)$-connected, and
- $3k+1 \leq 2n$.

Then the inclusion $\partial M \to M$ admits a section up to homotopy if and only $c_0: M_+ \wedge M_+ \to M^\tau$ is stably null homotopic.

Note that the range of our theorem is one dimension better than the one used by Wall.

Acknowledgements. This paper arose out of discussions related to the first author’s Ph.D. thesis. We are indebted to Bill Richter for suggesting that a proof of the main result ought to exist using some form of duality. We are also grateful to Bruce Williams and Michael Weiss for help in tracking down references to theorem 3.3.

2. The proof of Theorem A

We first observe that the inclusion

$$M \times \partial M \subset M \times M$$

is homotopic to a map into $C$ (to see this, let $M_0$ be the complement of a collar neighborhood of $\partial M$. Then $\partial M \times M_0$ is a subspace of $C$ and $M_0 \subset M$ is a homotopy equivalence). This means that the map $c_0$ factors up to homotopy as

$$M_+ \wedge M_+ \xrightarrow{\text{id} \wedge p} M_+ \wedge M/\partial M \xrightarrow{c} M^\tau$$

in which $p: M_+ \to M/\partial M$ is the evident quotient map and the map $c$ is the map on quotients induced by the map of pairs

$$(M \times M, M_0 \times \partial M) \to (M \times M, C)$$

(here we are implicitly identifying the quotient $(M \times M)/(M_0 \times \partial M)$ with the quotient $M_+ \wedge M/\partial M = (M \times M)/(M \times \partial M)$).

The map $c$ has an adjoint (cf. the exponential law)

$$c^\#: M/\partial M \to \text{maps}(M_+, M^\tau).$$

Lemma 2.1. The map $c^\#$ is $(k+1)$-connected.

We will sketch a proof of 2.1 in §3.

Proof of Theorem A, assuming 2.1 As $M$ has the homotopy type of a CW complex of dimension $\leq k$, it follows from 2.1 that the map $p: M_+ \to M/\partial M$ is null homotopic if and only if the map

$$c^\# \circ p: M_+ \to \text{maps}(M_+, M^\tau)$$
is null homotopic. The $\pi_\bullet \pi_\bullet$ condition, the relative Hurewicz theorem and Poincaré duality, imply that the map $\partial M \to M$ is $(n-k)$-connected. From this it follows that $\partial M$ is $\min(n-k-1, 2k-n)$-connected since $M$ is $(2k-n)$-connected. As $2k-n \leq n-k-1$ if and only if $3k+1 \leq 2n$ (which is part of our assumptions), the minimum coincides with $2k-n$. Consequently, $\partial M$ is also $(2k-n)$-connected.

By the Blakers-Massey theorem, the cofiber sequence

$$\partial M \to M \to M/\partial M$$

is a fibration up to homotopy up through dimension $k$. This means that the map

$$\partial M \to \text{hofiber}(M \to M/\partial M)$$

is $k$-connected. As $M$ has the homotopy type of a CW complex of dimension $\leq k$, we infer from this that $\partial M \to M$ has a homotopy section if and only if the map $p: M_+ \to M/\partial M$ is null homotopic.

By Lemma 2.1, $p$ is null homotopic if and only if $c^\# \circ p$ is. But the adjoint of the latter is a map $M_+ \land M_+ \to M^\tau$, which is easily checked to coincide with Wall’s map $c_0$. $\square$

3. The proof of Lemma 2.1

The proof will involve a generalized version of Poincaré duality. Let

$$E(\tau) \to M$$

be the spherical fibration whose fiber $E(\tau)_x$ over $x \in M$ is the one point compactified tangent space at $x$ (after a Riemannian metric is chosen, we have an identification $E(\tau)_x \cong D(\tau)_x/S(\tau)_x$, where $D(\tau)_x$ is the unit tangent disk and $S(\tau)_x$ is the unit tangent sphere at $x$). This sphere bundle comes equipped with a section “at infinity,” which we regard as a basepoint for its space of sections $\text{sec}(E(\tau) \to M)$.

One defines a based map

$$d: M/\partial M \to \text{sec}(E(\tau) \to M),$$

in the following way: the tubular neighborhood theorem gives a homotopy pushout square

$$\begin{array}{ccc}
S(\tau) & \to & M \times M - \Delta M \\
\downarrow & & \downarrow \\
D(\tau) & \to & M \times M.
\end{array}$$
We take the fiber at $x \in M$ of this diagram along the second factor projection $M \times M \to M$. This gives a homotopy pushout
\[
\begin{array}{ccc}
S(\tau)_x & \longrightarrow & M - x \\
\downarrow & & \downarrow \\
D(\tau)_x & \longrightarrow & M \times x.
\end{array}
\]

We therefore have a preferred identification
\[E(\tau)_x \simeq \text{cone}(M - x \to M \times x)\]
where the right side denotes the mapping cone of $M - x \to M \times x$. So we get a (collapse) map
\[M \times x \to \text{cone}(M - x \to M \times x) \simeq E(\tau)_x.\]

Adjointly, what we have so far produced is a map
\[M \to \text{sec}(E(\tau) \to M).\]

Let $M_0$ denote the complement of a collar neighborhood of the inclusion $\partial M \subset M$. By construction, the composite
\[M \longrightarrow \text{sec}(E(\tau) \to M) \xrightarrow{\text{restrict}} \text{sec}(E(\tau)|_{M_0} \to M_0)\]
has the property that it maps $\partial M$ to the basepoint of the section space (this is because $\partial M$ and $M_0$ are disjoint). So, up to the above identifications, what we have really produced is a map
\[d: M/\partial M \to \text{sec}(E(\tau) \to M).\]

Let us next think of the function space maps$(M_+, M^{\tau})$ as a section space of the trivial fibration
\[M^{\tau} \times M \to M.\]

Change the notation of the latter to $\tilde{E}(\tau) \to M$. By construction, the map $\tilde{E}^#$ factors up to homotopy as
\[M/\partial M \xrightarrow{d} \text{sec}(E(\tau) \to M) \xrightarrow{q} \text{sec}(\tilde{E}(\tau) \to M)\]
in which $q$ is the map coming from the map of fibrations $E(\tau) \to \tilde{E}(\tau)$ defined using the evident inclusion of each fiber $E(\tau)_x \to M^{\tau} \times x$.

**Lemma 3.1.** The map $q$ is $(k+1)$-connected.

**Proof.** As noted in the introduction, the map of fibers
\[E(\tau)_x \to M^{\tau} \times x\]
is $(2k+1)$-connected. Obstruction theory then shows the connectivity of $q$ is this number minus the dimension of $M$. $\square$
Lemma 3.2. The map $d$ is $(2n - 2k + 1)$-connected.

Before proving Lemma 3.2, we show why these results imply Theorem 2.1. Observe $k + 1 \leq 2n - 2k + 1$ is equivalent to $3k \leq 2n$, so Lemma 3.2 and our range assumption will imply that $d$ is $(k+1)$-connected. This, together with Theorem 3.1 and the factorization $c^\sharp = q \circ d$ implies that $c^\sharp$ is $(k+1)$-connected, yielding Theorem 2.1.

Proof of Lemma 3.2. There is a commutative diagram

$$
\begin{array}{ccc}
M/\partial M & \xrightarrow{d} & \text{sec}(E(\tau) \to M) \\
\downarrow & & \downarrow \\
Q(M/\partial M) & \xrightarrow{d^\mathrm{st}} & \text{sec}^\mathrm{st}(E(\tau) \to M)
\end{array}
$$

where $Q$ is the stable homotopy functor, $\text{sec}^\mathrm{st}$ means stable sections (= sections of the fibration whose fiber at $x \in M$ is $Q(E(\tau)_x)$), and the vertical maps are the natural inclusions. The bottom map $d^\mathrm{st}$ is a stable version of $d$; one way to see this is to think of $Q(M/\partial M)$ as a limit over $j$ of $\Omega^j(M \times D^j)/\partial(M \times D^j)$, so we can take a version of the top horizontal map using $M \times D^j$ instead of $M$ and then loop it $j$ times. The details will be left to the reader.

Since $M/\partial M$ is $(n-k)$-connected, the Freudenthal theorem shows that the left vertical map is $(2(n-k) + 1)$-connected. Since $E(\tau)_x$ is $n$-connected, the map $E(\tau)_x \to Q(E(\tau)_x)$ is $(2n+1)$-connected. By elementary obstruction theory, we see that the right vertical map is $(2n-k+1)$-connected. Using these observations, Lemma 3.2 is then a direct consequence of the following:

Theorem 3.3 (Poincaré Duality). For any compact manifold smooth manifold $M$, The map $d^\mathrm{st}$ is a weak equivalence (of infinite loop spaces).

Remark 3.4. The earliest reference we could find for Theorem 3.3 is in a paper of Dax [Da, Prop. 5.1], who formulates it in terms of normal bordism theory, and proves it using transversality. Our map $d^\mathrm{st}$ corresponds, via the Thom-Pontryagin map, to the isomorphism denoted by $U$ in his proof.

In a paper of Weiss and Williams [W-W, Prop. 2.4], a statement of the above kind is proved using a map going in the other direction (their statement allows for any spectrum of coefficients, not just the sphere spectrum). Williams has informed us that our map is not a homotopy
inverse to the Weiss-Williams map; one must first twist by a certain involution.

Another proof of 3.3 which is more generally valid for Poincaré duality spaces (using a an alternative description of $d^s$ as a ‘norm map’) can be found in [Kl, Thms. A,D].

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