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Improved Sliding Mode Finite-Time Synchronization of Chaotic Systems with Unknown Parameters

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Abstract: This work uses the sliding mode control method to conduct the finite-time synchronization of chaotic systems. The utilized parameter selection principle differs from conventional methods. The designed controller selects the unknown parameters independently from the system model. These parameters enable tracking and prediction of the additional variables that affect the chaotic motion but are difficult to measure. Consequently, the proposed approach avoids the limitations of selecting the unknown parameters that are challenging to measure or modeling the parameters solely within the relevant system. This paper proposes a novel nonsingular terminal sliding surface and demonstrates its finite-time convergence. Then, the adaptive law of unknown parameters is presented. Next, the adaptive sliding mode controller based on the finite-time control idea is proposed, and its finite-time convergence and stability are discussed. Finally, the paper presents numerical simulations of chaotic systems with either the same or different structures, thus verifying the proposed method’s applicability and effectiveness.

Keywords: chaotic synchronization; finite-time control; nonsingular terminal sliding mode surface; unknown parameter selection

1. Introduction

The chaos system is a special kind of nonlinear system, and chaos will occur in many aspects. The chaos phenomenon is generally caused by disturbances and the system parameters satisfying certain conditions. Thus, to improve the operation reliability of the system, sometimes this chaos phenomenon should be avoided. Discussions on the relation between the chaos and bifurcation phenomenon, and the non-linear oscillation of the system are very popular. Most scholars study how to ensure the stable operation of the system. This research is also of significance for the above-mentioned chaos control of one class of system in practice.

In most cases, the control of chaotic system refers to the synchronization control of the system. Therefore, in recent years, chaotic system synchronization control has attracted significant attention within the fields of mathematics, physics and, engineering science. The research findings on chaotic system synchronization are widely applied in securing communication, power conversion, biological systems, information processing, and chemical reactions [1]. The primary approach to two chaotic systems’ synchronization relies on designing an appropriate controller that controls the slave system, making the slave system state asymptotically track the main system state.

Numerous control techniques (e.g., in differential dynamics and bifurcation theory) of chaotic systems synchronization have emerged [2]. Specific methods include classical Proportional–Integral–Derivative (PID) control [3], adaptive fuzzy control, adaptive pulse...
perturbation [4], random Melnikov method [5], negative feedback algorithm [6], and chaotic
synchronization based on the ant colony algorithm [7]. Alzarano et al. [8] used the Melnikov
method to study the bifurcation phenomenon of chaotic motion, while Bikdash et al. [9] utilized it
in researching the fractal state of rolling chaotic motion. Path tracking technique was used in [10],
where the authors found a nonlinear bifurcation set with six degrees of freedom and obtained the chaotic
attractors generated by periodical system multiplication. Acanfora et al. [11] used the optimization
algorithm to explain the phase spectrum’s influence on resonance phenomenon development, given
the mixed nonlinear model of six degrees of freedom.

In early research works, the chaotic system’s parameters were commonly assumed to be known
in advance. However, in the majority of real-world systems, external natural factors inevitably disturb
the chaotic system parameters, thus changing the parameter values. Consequently, the synchronization
of chaotic systems’ unknown parameters has become an important research direction in recent
years. The chaotic systems unknown parameters’ synchronization methods include a sliding mode
control [12,13], backstepping design [14], optimum control [15], adaptive control [16], linear balanced
feedback control [17], active control [18], pulse control, and fuzzy control [19]. In most cases,
the parameters inside the system model are directly selected as unknown parameters, which has
certain limitations, because of the parameters outside the system model cannot be identified. The listed
studies on the synchronization of unknown parameters in chaotic systems pose no time requirements
and guarantee asymptotic stability. In other words, the time approaches infinity until the system
synchronization is achieved. However, practical applications require achieving system synchronization
within finite time. In addition, the finite-time control approaches were shown to result in better
robustness and anti-interference performance. Several researchers realized the chaotic synchronization
control using the finite-time control techniques, such as finite-time stochastic method [20], CLF-based
method [21], sliding mode control [22], and pure finite-time control method [23].

As mentioned above, the unknown parameters of a chaotic system are usually selected in a system
model. For example, in works such as [24,25], relevant model parameters are directly set and regarded
as unknown. Since system models contain multiple parameters, careful analysis and argumentation
are needed when choosing the unknown parameters. Additionally, in general, several parameters
cannot be regarded as unknown.

Building on the presented discussion, this paper proposes a novel finite-time control method for
chaotic systems with unknown parameters. The control strategy is based on sliding mode control
technology. The controller is model-based, but the controller unknown parameter adaptation only
depends on the sliding surface instead of directly selecting the relevant unknown parameters in the
system model, and the sliding surface has new forms. In this way, the selected parameters might be
the outside unknown variables of the system model that affect chaotic motion, which are challenging
to measure. The proposed approach avoids the difficulties of the unknown parameters selection
and mitigates the limitations of choosing the relevant parameters within the system model only.
The paper demonstrates the proposed method’s finite-time convergence. The adaptive law of unknown
parameters is then presented to make the synchronization error system reach the sliding mode surface
in finite-time. Simultaneously, simulation verification is conducted using chaotic systems with the
same, as well as different, structures. The results show a good control effect. This paper discusses the
mathematical theory enabling the realization of the proposed approach. However, further analysis is
required to determine the specific parameters’ value for different chaotic phenomena applications.

The rest of the paper is structured as follows. Section 1 presents the fundamental lemmas used in
the research. In Section 2, a new sliding surface is proposed for a general nonlinear chaotic system.
Additionally, the sliding mode controller that deals with unknown parameters independent of the
controlled system model is designed within this section. The adaptive law of unknown parameters is
described, and the finite-time stability is proved. In Section 3, simulation verification is performed,
demonstrating the same-structure synchronization of the parametric excitation rolling chaotic system
of a ship, as well as the different structures’ synchronization of the Liu system and the Lorenz system. Section 4 concludes the paper and discusses the directions for further research.

2. Fundamental Lemmas

In this section, the system description is presented, the synchronization problem is explained, and the relevant definitions and several necessary lemmas on finite-time synchronization are introduced. The following two $n$-dimensional chaotic systems are considered:

Master system:

$$\begin{cases} x_1(t) = f_1(x_1, x_2, \ldots, x_n) \\ x_2(t) = f_2(x_1, x_2, \ldots, x_n) \\ \vdots \\ x_n(t) = f_n(x_1, x_2, \ldots, x_n) \end{cases}$$ (1)

Slave system:

$$\begin{cases} y_1(t) = g_1(y_1, y_2, \ldots, y_n) + u_1(t) \\ y_2(t) = g_2(y_1, y_2, \ldots, y_n) + u_2(t) \\ \vdots \\ y_n(t) = g_n(y_1, y_2, \ldots, y_n) + u_n(t) \end{cases}$$ (2)

In the above equation, $f_i(x_1, x_2, \ldots, x_n), i = 1, 2, \ldots, n$ and $g_i(y_1, y_2, \ldots, y_n), i = 1, 2, \ldots, n$ are nonlinear functions, and $u_i(t), i = 1, 2, \ldots, n$ is a controller with unknown parameters. Equations (1) and (2) both represent chaotic systems.

This work aims at designing a suitable controller with particular unknown parameters independent of the system model to synchronize chaotic systems (described by Equations (1) and (2)) in a finite time while the systems remain chaotic. To achieve finite-time synchronization, a system error is defined as follows:

$$e_i(t) = x_i(t) - y_i(t)$$ (3)

The error dynamics equation is obtained as a derivative of the equation resulting from subtracting Equation (3) from Equation (2). Thus:

$$\begin{cases} e_1(t) = f_1(x_i) - g_1(y_i) - u_1(t) \\ e_2(t) = f_2(x_i) - g_2(y_i) - u_2(t) \\ \vdots \\ e_i(t) = f_i(x_i) - g_i(y_i) - u_i(t) \\ \vdots \\ e_n(t) = f_n(x_i) - g_n(y_i) - u_n(t) \end{cases}$$ (4)
**Definition 1.** Let the master system be defined with Equation (1) and the slave system with Equation (2). Suppose there is a constant $T > 0$ for which $\lim_{t \to T} |e_i(t)| = 0, i = 1, 2, \ldots, n$ and $|e_i(t)| \equiv 0, i = 1, 2, \ldots, n$ when $t > T$. Then, the chaotic synchronization of systems (1) and (2) is achieved in finite time $T$.

Achieving the finite-time stabilization of the error system (4) is equivalent to realizing the finite-time synchronization of chaotic systems (1) and (2).

**Lemma 1** [26]. Consider a system (5)

$$\dot{x} = f(x), f(0) = 0, x \in \mathbb{R}^n,$$

where $f : D \to \mathbb{R}^n$ is continuous on an open neighborhood and $D \subset \mathbb{R}^n$. Assuming that there is a continuous differential positive-definite function $V(x) : D \to \mathbb{R}$, and the real numbers $p > 0$ and $0 < \eta < 1$ satisfy:

$$V(x) + pV'(x) \leq 0, \forall x \in D, \quad (6)$$

then, the system (5) is a finite-time stable system whose stabilization time depends on the initial state $x(0) = x_0, \text{(here } x(0) \text{ represent vectors, for example } x = (x_1, x_2, \ldots, x_n), \text{ so } x(0) = x_0 = (x_1(0), x_2(0), \ldots, x_i(0)), \text{ the initial conditions are simple initial conditions)}$ and satisfies:

$$T(x_0) \leq \frac{V^{1-\eta}(x_0)}{p(1-\eta)}. \quad (7)$$

**Note 1.** Throughout the paper, symbols written in boldface represent vectors.

**Lemma 2** [27]. For every $a_1, a_2, \ldots, a_n \in \mathbb{R}$, the following equation is satisfied:

$$|a_1| + |a_2| + \ldots + |a_n| \geq \sqrt{a_1^2 + a_2^2 + \ldots + a_n^2}. \quad (8)$$

**3. Design of Sliding Mode Surface and Controller**

This section introduces a new sliding mode controller that realizes the chaotic synchronization of two different chaotic systems in a finite time. In doing so, the two main procedures include designing a sliding surface and guaranteeing the finite-time stability of the system on a sliding surface. First, an adaptive finite-time controller with unknown parameters independent of the system model is designed. Then, convergence to sliding surface within finite time is guaranteed, and the adaptive law of unknown parameters is designed.

The designed novel sliding surface is:

$$s_i(t) = c_i e_i(t) + \int_0^t \text{sgn}(e_i(\tau)) |e_i(\tau)|^\alpha d\tau + e_i(t) \text{sgn}(e_i(t)), i = 1, 2, \ldots, n, \quad (9)$$

where $s_i(t) \in \mathbb{R}$, and $c_i$ and $\alpha$ are constants for which $c_i > 0$ and $0 < \alpha < 1$. Taking the first-order derivative of the sliding surface, one obtains:

$$\dot{s}_i(t) = c_i \dot{e}_i(t) + \text{sgn}(e_i(t)) |e_i(t)|^\alpha + e_i(t) \text{sgn}(e_i(t)), i = 1, 2, \ldots, n. \quad (10)$$

Based on the sliding mode control principle [28],

$$\dot{e}_i(t) = -\frac{\text{sgn}(e_i(t))}{c_i + \text{sgn}(e_i(t))} |e_i(t)|^\alpha. \quad (11)$$
Theorem 1. Consider the error dynamics Equation (13). Under the introduced sliding surface, the system is finite-time stable, and the time $T_1$ required to reach the equilibrium $e_i(t) = 0$ is determined by:

$$T_1 \leq \frac{(1/2 \sum_{i=1}^{n} e_i^2(0))}{(1 - \alpha)\rho 2^{\frac{\alpha+1}{2}}}.$$

where $\rho$ is the minimum value among $\frac{1}{d_i}$, $\forall i = 1, 2, \ldots, n$.

**Proof of Theorem 1.** Consider the Lyapunov function: $V(t) = \frac{1}{2} \sum_{i=1}^{n} e_i^2(t)$.

The function’s first-order derivative relative to time equals: $V(t) = \sum_{i=1}^{n} e_i(t)e_i(t)\dot{e}_i(t)$.

Then, using (13), one obtains: $V(t) = \sum_{i=1}^{n} e_i(t)(-\frac{1}{d_i}\text{sgn}(e_i(t))|e_i(t)|^\alpha)$,

$$\dot{V}(t) = -\sum_{i=1}^{n} \frac{1}{d_i}|e_i(t)|^{\alpha+1} \leq -\rho \sum_{i=1}^{n} |e_i(t)|^{\alpha+1},$$

Utilizing Lemma 2, it follows: $V(t) \leq -\rho \sum_{i=1}^{n} |e_i(t)|^{\alpha+1} \leq -2^{\frac{\alpha+1}{2}}\rho (\frac{1}{2} \sum_{i=1}^{n} e_i^2(t))^{\frac{\alpha+1}{2}} = -2^{\frac{\alpha+1}{2}}\rho V^{\frac{\alpha+1}{2}}(t)$.

Therefore, $\dot{V}(t) + 2^{\frac{\alpha+1}{2}}\rho V^{\frac{\alpha+1}{2}}(t) \leq 0$, where $2^{\frac{\alpha+1}{2}}\rho > 0$ and $0 < \frac{\alpha+1}{2} < 1$. Since Equation (6) is satisfied, following the Lemma 1 one obtains: $T_1 \leq \frac{(1/2 \sum_{i=1}^{n} e_i^2(0))}{(1 - \alpha)\rho 2^{\frac{\alpha+1}{2}}}$. Hence, the error $e_i(t), i = 1, 2, \ldots, n$ converges to zero in finite time $T_1$. In other words, systems (1) and (2) synchronize in a finite time. The proof is finished. $\square$

Once the sliding surface is designed, one needs to design a controller with unknown parameters independent of the system model and ensure the error system converges to the sliding surface within finite time. Therefore, the existence of a sliding surface, as well as the convergence of the error trajectory $e_i(t), i = 1, 2, \ldots, n$ to sliding surface $s_i(t) = 0, i = 1, 2, \ldots, n$ should be guaranteed. Then, Systems (1) and (2) reach chaotic synchronization.

The designed controller is as follows:

$$u_i(t) = f_i(x_i) - g_i(y_i) + \frac{1}{d_i}[\text{sgn}(e_i(t))|e_i(t)|^\alpha + k_i \frac{|s_i(t)|}{s_i(t)}],$$

where $i = 1, 2, \ldots, n$, $\hat{d}_i = c_i + \text{sgn}(e_i(t)) > 0$, and $k_i > 0$. Here $k_i$ denotes the unknown parameters that are greater than zero and independent of the synchronized system model. Symbol $\hat{k}_i$ denotes the estimated value of unknown parameter $k_i$. Note that the unknown parameter $k_i$ is not within the system model (1) and only depends on the sliding surface. For the conventional approach, the unknown parameters are selected in the
system model (e.g., [22] and [23]). Therefore, these parameters can be seen as the arbitrary factors that affect the system and are difficult to predict or measure. The adaptive law of unknown parameter $k_i$ is defined as:

\[
\dot{k}_i = |s_i(t)| - k_i \frac{[\hat{k}_i - k_i]}{(|\hat{k}_i - k_i|)}.
\] (16)

The control law proposed in Equation (15) and the adaptive law in Equation (16) ensure the sliding motion occurs in a finite time. The following theorem demonstrates such a claim.

**Theorem 2.** When the control law (15) and the adaptive law (16) are used to control the error system (4), the system state moves towards the sliding surface and approaches the sliding surface $s_i(t) = 0, i = 1, 2, \ldots, n$ in a finite time, denoted $T_2$. Time $T_2$ is determined by:

\[
T_2 \leq \frac{(\frac{1}{2} \sum_{i=1}^{n} |s_i(0)|^2 + \frac{1}{2} \sum_{i=1}^{n} \overline{k}_i^2(0))^2}{2^{-\frac{1}{4}} \rho}.
\] (17)

where $\overline{k}_i$ denotes the errors of unknown parameters ($\overline{k}_i = \hat{k}_i - k_i$, $\forall i = 1, 2, \ldots, n$).

**Proof of Theorem 2.** Consider the Lyapunov function: $V(t) = \frac{1}{2} \sum_{i=1}^{n} |s_i(t)|^2 + \frac{1}{2} \sum_{i=1}^{n} \overline{k}_i^2$, where $\overline{k}_i$ is the error of unknown parameter. Clearly, $\dot{k}_i = \dot{\hat{k}}_i$. Calculating the first-order derivative of $V(t)$, one obtains:

\[
\dot{V}(t) = \sum_{i=1}^{n} [s_i(t)\dot{s}_i(t)] + \sum_{i=1}^{n} [\dot{k}_i \dot{k}_i].
\]

Utilizing Equation (12), it follows: $\dot{V}(t) = \sum_{i=1}^{n} [s_i(t)(d_i f_i(x_i) - g_i(y_i) - u_i(t)) + \text{sgn}(e_i(t))[e_i(t)]^2] + \sum_{i=1}^{n} [\dot{k}_i \dot{k}_i]$. Next, using Equation (4), one derives:

\[
\dot{V}(t) = \sum_{i=1}^{n} [s_i(t)(d_i f_i(x_i) - g_i(y_i) - u_i(t)) + \text{sgn}(e_i(t))[e_i(t)]^2] + \sum_{i=1}^{n} [\dot{k}_i \dot{k}_i].
\]

Then, one can utilize Equation (15) to obtain:

\[
\dot{V}(t) = -\sum_{i=1}^{n} [k_i s_i(t)] + \sum_{i=1}^{n} [\dot{k}_i s_i(t)] = -\sum_{i=1}^{n} [k_i s_i(t)] + \sum_{i=1}^{n} [\dot{k}_i s_i(t)] - \sum_{i=1}^{n} [\dot{k}_i s_i(t)] - k_i [\dot{k}_i k_i (\dot{k}_i - k_i)]
\]

From Equation (16) follows:

\[
\dot{V}(t) = -\sum_{i=1}^{n} [k_i s_i(t)] - \sum_{i=1}^{n} k_i [\dot{k}_i k_i - k_i]
\]
Thus,
\[ V(t) = -\gamma(\sum_{i=1}^{n} |s_i(t)|^2 + \sum_{i=1}^{n} k_i^2) \leq -\gamma(\sum_{i=1}^{n} |s_i(t)|^2 + \sum_{i=1}^{n} k_i^2), \]
where \( \gamma \) denotes the minimum value among \( k_i \). Using Equation (8) in Lemma 2, it follows:
\[
-\gamma(\sum_{i=1}^{n} |s_i(t)|^2 + \sum_{i=1}^{n} k_i^2) \leq -\gamma(\sum_{i=1}^{n} |s_i(t)|^2 + \sum_{i=1}^{n} k_i^2) \leq -2^{1.5} \gamma(\sum_{i=1}^{n} |s_i(t)|^2 + \sum_{i=1}^{n} k_i^2) \leq -2^{1.5} \gamma V^1(t) \leq 0
\]
where \( 2^{1.5} \gamma > 0, 0 < \frac{1}{2} < 1 \). Since Equation (6) is satisfied, Lemma 1 gives:
\[
T_2 \leq \frac{1}{2^{1.5} \gamma} \left( \sum_{i=1}^{n} |s_i(0)|^2 + \frac{1}{2} \sum_{i=1}^{n} k_i^2(0) \right)^{1.5}
\]
Therefore, the error trajectory \( e_1(t) \) converges to sliding mode surface \( s_1(t) = 0 \) in finite time \( T_2 \). The proof is finished. □

In summary, the block diagram of synchronization and control scheme is as follows:

**Note 2.** Following Theorems 1 and 2, the sliding mode controller (15) with the adaptive law (16) and sliding surface (9) enables synchronization of systems (1) and (2) within finite time. The synchronization time equals \( T = T_1 + T_2 \).

**Note 3.** Constant \( c_i \) is directly proportional to convergence times \( T_1 \) and \( T_2 \). Thus, a smaller \( c_i \) results in shorter \( T_1 \) and \( T_2 \). On the other hand, from Equation (15), it follows that the controller input \( u_i(t) \) is inversely proportional to \( c_i \), meaning that smaller \( c_i \) leads to greater \( u_i(t) \).

4. Simulation Verification

In this section, MATLAB software is used to perform numerical simulations of chaotic systems with either the same or with different structures [29,30]. The chaotic systems’ synchronization is realized utilizing the novel sliding surface and controller designed in the previous section. Additionally, the introduction of unknown parameters independent of the system model is realized, and the adaptive law is presented.

4.1. Simulation Verification of Parametric Excitation Rolling Chaotic System of Ship with the Same Structure

The nonlinear mathematical model for the roll system of a ship with parametric and forced excitation in a regular longitudinal wave is as follows:
\[
\dot{\phi} + \mu_1 \dot{\phi} + \mu_3 \phi + \omega_0^2 (\phi + \alpha_3 \phi^3 + \alpha_5 \phi^5 + \omega_0 \phi \cos(\omega_1 t)) = K_c \sin(\omega_1 t + \delta_0),
\]
where \( \phi(t) \) is the roll angle; \( \omega_0 \) is the amplitude of parametric excitation; \( \omega_1 \) is the frequency of metric excitation, which is usually twice that of the natural frequency when chaos occurs in the system; \( \mu_1 \) and \( \mu_3 \) are the damping factors of the roll; \( \alpha_3 \) and \( \alpha_5 \) are the dimensionless righting moment coefficients; \( \omega_0 \) is the natural frequency of the ship roll; and \( K_c \sin(\omega_1 t + \delta_0) \) is the forced roll moment in regular waves, where \( K_c \) is the amplitude of the forced excitation and \( \delta_0 \) is encounter location of the ship and regular wave, assigned as 0.
After the above parameters are determined, the nonlinear mathematical model for the roll system of a ship with parametric and forced excitation is changed to the following:

\[ \ddot{\phi} + 0.069\dot{\phi} + 0.5367\phi^3 + 1.25\phi - 0.9926\phi^3 + 0.0998\phi^5 + 2.875\phi \cos 2.236t = 0.16 \sin 2.236t \]

Let \( x_1(t) = \phi(t) \), \( x_2(t) = \dot{\phi}(t) \), and, by conversion to state equations, it is obtained as follows:

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -0.069x_2 - 0.5367x_2^3 - 1.25x_1 + 0.9926x_1^3 - 0.0998x_1^5 - 2.875x_1 \cos 2.236t + 0.16 \sin 2.236t
\end{align*}
\]  

(18)

The parametric excitation roll system of a ship is a chaos system that can be proved by MATLAB simulation results. The history charts of \( x_1 \) and \( x_2 \) of the parametric excitation roll system of the ship are shown in Figures 1 and 2, respectively, in which it is clear that the system is in chaos. The phase diagram of the parametric excitation roll system of the ship is shown in Figure 3, which further proves that the system is a chaos system and similar to a Duffing chaos system.

**Figure 1.** The time response of \( x_1 \) about ship parametric excitation rolling system Equation (18).

**Figure 2.** The time response of \( x_2 \) about ship parametric excitation rolling system Equation (18).
This method can be seen in references [24] and [25].

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parametric excitation rolling system with the same fundamental structure. In other words, the chaotic control is achieved over the same structure systems.

In the method discussed in this paper, a ship’s parametric excitation rolling system is regarded as a master system. The controller’s chaotic system with unknown parameters independent of the system model is regarded as a slave system. The master system and slave system are based on the ship parametric excitation rolling system with the same fundamental structure. In other words, the chaotic control is achieved over the same structure systems.

Master system:

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -0.069x_2 - 0.5367x_2^3 - 1.25x_3 + 0.9926x_1^3 - 0.0998x_1^5 , \\
&\quad -2.875x_1 \cos 2.236t + 0.16 \sin 2.236t
\end{align*}
\]  \hspace{1cm} (19)

Slave system:

\[
\begin{align*}
\dot{y}_1 &= y_2 + u_1(t) \\
\dot{y}_2 &= -0.069y_2 - 0.5367y_2^3 - 1.25y_1 + 0.9926y_1^3 - 0.0998y_1^5 , \\
&\quad -2.875y_1 \cos 2.236t + 0.16 \sin 2.236t + u_2(t)
\end{align*}
\]  \hspace{1cm} (20)

Error dynamics equation:

\[
\begin{align*}
\dot{e}_1 &= x_1 - y_1 \\
\dot{e}_2 &= x_2 - y_2
\end{align*}
\]

Error system:

\[
\begin{align*}
\dot{e}_1 &= x_2 - y_2 - u_1(t) \\
\dot{e}_2 &= -0.069x_2 - 0.5367x_2^3 - 1.25x_1 + 0.9926x_1^3 - 0.0998x_1^5 \\
&\quad -2.875x_1 \cos 2.236t + 0.16 \sin 2.236t + 0.069y_2 + \\
&\quad 0.5367y_2^3 - 1.25y_1 - 0.9926y_1^3 + 0.0998y_1^5 \\
&\quad +2.875y_1 \cos 2.236t - 0.16 \sin 2.236t - u_2(t)
\end{align*}
\]  \hspace{1cm} (21)

Using Equation (9), the sliding mode surface is designed as follows:

\[
\begin{align*}
s_1 &= c_1e_1(t) + \int_0^t \text{sgn}(e_1(\tau))|e_1(\tau)|^{\alpha} d\tau + e_1(t)\text{sgn}(e_1(t)) \\
s_2 &= c_2e_2(t) + \int_0^t \text{sgn}(e_2(\tau))|e_2(\tau)|^{\alpha} d\tau + e_2(t)\text{sgn}(e_2(t))
\end{align*}
\]  \hspace{1cm} (22)
According to Equation (15), the controller with unknown parameters independent of the system is designed as:

\[
\begin{align*}
  u_1(t) &= x_2 - y_2 + \frac{1}{c_1 + \|\text{sgn}(e_1(t))\|} |\text{sgn}(e_1(\tau))| e_1(\tau)|^{\alpha} + \hat{k}_1 \frac{|\dot{e}_1|}{s_1} \\
  u_2(t) &= -0.069x_2 - 0.5367x_2^3 - 1.25x_1 + 0.9926x_1^3 - 0.0998x_1^5 \\
  &\quad -2.875x_1 \cos 2.236t + 0.16 \sin 2.236t + 0.069y_2 + 0.5367y_2^3 \\
  &\quad + 1.25y_1 - 0.9926y_1^3 + 0.0998y_1^5 + 2.875y_1 \cos 2.236t \\
  &\quad -0.16 \sin 2.236t + \frac{1}{c_2 + \|\text{sgn}(e_2(t))\|} |\text{sgn}(e_2(\tau))| e_2(\tau)|^{\alpha} + \hat{k}_2 \frac{|\dot{e}_2|}{s_2} 
\end{align*}
\]

From Equation (16), the adaptive law of unknown parameters is:

\[
\begin{align*}
  \dot{\hat{k}}_1 &= |s_1| - k_1 \frac{|\dot{e}_1|}{|\hat{k}_1 - k_1|} \\
  \dot{\hat{k}}_2 &= |s_2| - k_2 \frac{|\dot{e}_2|}{|\hat{k}_2 - k_2|}. 
\end{align*}
\]

In the simulation, the step size was set to 0.01 s, and the initial values were \(x_1(0) = 3, x_2(0) = 3, y_1(0) = 1, y_2(0) = 1, \) and \(c_1 = 8, c_2 = 9.\) In addition, the initial values of unknown parameters were set to \(k_1 = 0.1\) and \(k_2 = 0.2.\)

Figures 4 and 5 reflect the synchronization process of Systems (19) and (20), and synchronization error is shown in Figure 6, the identification processes of unknown parameters \(k_1\) and \(k_2\) are shown in Figure 7. Figures 4 and 5 demonstrate that, when \(t\) approaches 2 s, the drive system parameters \((x_1\) and \(y_1)\) synchronize with the response system parameters \((x_2\) and \(y_2)\). The error effect in Figure 6 shows that, when \(t\) approaches 2 s, errors \(e_1\) and \(e_2\) stabilize around zero. Finally, Figure 7 demonstrates that, when \(t\) approaches 2 s, the parameters \(k_1\) and \(k_2\) reach values 6 and 7, respectively.
Figure 4. Synchronization process of $y_1$ arrives $x_1$.

Figure 5. Synchronization process of $y_2$ arrives $x_2$.

Figure 6. Synchronization errors.

Figure 7. Identification process of $k_1$ and $k_2$.

By using the parameters to update the rule (24), one can identify the special unknown parameters in the controller (23). In other words, the drive system and the response system reach a finite-time synchronization. After the controller facilitates the ship parametric excitation rolling system with unknown parameters independent of the system model, the response system successfully tracks the original system in a finite time. The adaptive parameters converge to bounded values within the finite time.

4.2. Simulation Verification of Chaotic System with Different Structures

Both the Liu system and the Lorenz system [31,32] are well-known chaotic systems that do not require a detailed introduction. In addition, their structures are different. In the traditional methods, the coefficients of the equations are usually chosen as unknown parameters. The coefficients of
these equations can affect the chaotic state of the system. This treatment method can be seen in references [24,25].

Here, the method mentioned in the previous section are used to verify the proposed method’s performance in the chaotic synchronization of systems with different structures.

Master system:

\[
\text{Liu : } \begin{cases}
    \dot{x}_1 = -\frac{10}{3}x_1 - x_2x_3 + 10 \\
    \dot{x}_2 = -10x_2 + x_1x_3 \\
    \dot{x}_3 = -4x_3 + x_1x_2
\end{cases}
\]

(25)

Slave system:

\[
\text{Lorenz : } \begin{cases}
    \dot{y}_1 = 10(y_2 - y_1) \\
    \dot{y}_2 = 28y_1 - y_2 - y_1y_3 \\
    \dot{y}_3 = y_1y_2 - \frac{8}{3}y_3
\end{cases}
\]

(26)

Using the Equation (9), the sliding mode surface is designed as follows:

\[
\begin{align*}
    s_1 &= c_1e_1(t) + \int_0^t \text{sgn}(e_1(\tau))|e_1(\tau)|^\alpha d\tau + e_1(t)\text{sgn}(e_1(t)) \\
    s_2 &= c_2e_2(t) + \int_0^t \text{sgn}(e_2(\tau))|e_2(\tau)|^\alpha d\tau + e_2(t)\text{sgn}(e_2(t)) \\
    s_3 &= c_3e_3(t) + \int_0^t \text{sgn}(e_3(\tau))|e_3(\tau)|^\alpha d\tau + e_3(t)\text{sgn}(e_3(t))
\end{align*}
\]

(27)

According to Equation (15), the controller with unknown parameters independent of the system is designed as:

\[
\begin{align*}
    u_1(t) &= 10(x_2 - x_1) - 10(y_2 - y_1) + \frac{1}{c_1 + \text{sgn}(e_1(t))} \text{sgn}(e_1(t))|e_1(t)|^\alpha + \dot{k}_1|s_1| \\
    u_2(t) &= 40x_1 - x_1x_3 - (28y_1 - y_2 - y_1y_3) + \frac{1}{c_2 + \text{sgn}(e_2(t))} \text{sgn}(e_2(t))|e_2(t)|^\alpha + \dot{k}_2|s_2| \\
    u_3(t) &= -2.5x_3 + 4x_1^2 - (y_1y_2 - \frac{8}{3}y_3) + \frac{1}{c_3 + \text{sgn}(e_3(t))} \text{sgn}(e_3(t))|e_3(t)|^\alpha + \dot{k}_3|s_3|
\end{align*}
\]

(28)

From Equation (16), the adaptive law of unknown parameters is:

\[
\begin{align*}
    \dot{k}_1 &= |s_1| - k_1 \frac{|s_2|}{(s_3 - k_3)} \\
    \dot{k}_2 &= |s_2| - k_2 \frac{|s_3|}{(s_3 - k_3)} \\
    \dot{k}_3 &= |s_3| - k_3 \frac{|s_3|}{(s_3 - k_3)}
\end{align*}
\]

(29)

In the simulation, the step size was set to 0.01 s, and the initial values were \(x_1(0) = -1.5, x_2(0) = 2.25, x_3(0) = 2.1, y_1(0) = 1, y_2(0) = 1, y_3(0) = 1,\) and \(c_1 = 8, c_2 = 9, c_3 = 10.\) In addition, the initial values of unknown parameters were set to \(k_1 = 0.1, k_2 = 0.2, k_3 = 0.3.\)

Figures 8–10 reflect the synchronization process of Systems (25) and (26), and synchronization error is shown in Figure 11, the identification processes of unknown parameters \(k_1, k_2, k_3\) are shown in Figure 12. Figures 8–10 demonstrate that, when \(t\) approaches 1.7 s, the drive system parameters \((x_1, y_1, z_1)\) synchronize with the response system parameters \((x_2, y_2, z_2)\). The error effect in Figure 11 shows that, when \(t\) approaches 1.7 s, errors \(e_1, e_2,\) and \(e_3\) stabilize around zero. Finally, Figure 12 demonstrates that, when \(t\) approaches 1.7 s, the parameters \(k_1, k_2,\) and \(k_3\) reach values 5.2, 2.5, and 2, respectively.
synchronization error is shown in Figure 11. Figures 8–10 demonstrate that, when
and
addition, the initial values of unknown parameters were set to
,
,
, and

are shown in Figure 12, and

reach values 5.2, 2.5, and 2, respectively.

Figure 8. Synchronization of \( x_1 \) and \( y_1 \).

Figure 9. Synchronization of \( x_2 \) and \( y_2 \).

Figure 10. Synchronization of \( x_3 \) and \( y_3 \).

Figure 11. Synchronization error.
By using the parameters to update the rule (29), one can identify the special unknown parameters in the controller (28). In other words, the master system and the slave system reach a finite-time synchronization. The adaptive parameters converge to bounded values within the finite time. The effectiveness of the algorithm is further proved.

5. Conclusions

This paper uses a sliding mode control theory to study the finite-time synchronization problem of chaotic systems. A novel nonsingular terminal sliding surface was proposed. Additionally, a controller with unknown parameters only depends on the sliding surface, instead of directly selecting the relevant unknown parameters in the system model. The adaptive laws were designed, the convergence and the stability within finite time were proved. The simulation results demonstrate that synchronization of chaotic systems can be achieved in a finite time with high control speed, irrespective of whether the systems differ in structures or not. The practical meaning of the selected unknown parameters in specific chaotic systems will be further discussed in the subsequent studies.

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