Radius of Starlikeness for Bloch Functions

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Abstract. For normalised analytic functions \( f \) defined on the open unit disc \( D \) satisfying the condition \( \sup_{z \in D} (1 - |z|^2)|f'(z)| \leq 1 \), known as Bloch functions, we determine various starlikeness radii.

1. Introduction

The class \( A \) consists of all analytic functions \( f \) on the disc \( D := \{ z \in \mathbb{C} : |z| < 1 \} \) and normalized by the conditions \( f(0) = 0 \) and \( f'(0) = 1 \). The class \( S \) consists of all univalent functions \( f \in A \). The class \( B \) of Bloch functions consists of all functions \( f \in A \) satisfying \( \sup_{z \in D} (1 - |z|^2)|f'(z)| \leq 1 \) (see [1], [8]). Bonk [1] has shown that the radius of starlikeness of the class \( B \) is the same as radius of univalence and it equals \( 1/\sqrt{3} \approx 0.57735 \). It also follows from his distortion inequalities that the radius of close-to-convexity (with respect to \( z \)) is also \( 1/\sqrt{3} \). We extend the radius of starlikeness result by find various other starlikeness radii. An analytic function \( f \) is subordinate to the analytic function \( g \), denoted as \( f \prec g \), if there exists a function \( w : D \to D \) with \( w(0) = 0 \) satisfying \( f(z) = g(w(z)) \). If \( g \) is univalent, then \( f \prec g \) if and only if \( f(0) = g(0) \) and \( f(D) \subseteq g(D) \). The subclass of \( S \) of starlike functions \( S^* \) is the collection of functions \( f \in S \) with the condition \( \text{Re}(zf'(z)/f(z)) > 0 \) where \( z \in D \). The subclass \( K \) of convex functions consists of the functions in \( S \) with \( \text{Re}(1 + zf''(z)/f'(z)) > 0 \) for \( z \in D \). This characterisation gives a characterization of these classes in terms of the class \( P \) of Carathéodory functions or the functions with positive real part, comprising of analytic functions \( p \) with \( p(0) = 1 \) satisfying \( \text{Re}(p(z)) > 0 \) or equivalently the subordination \( p(z) \prec (1+z)/(1-z) \). Thus, the classes of starlike and convex functions are consist of \( f \in A \) with \( zf'(z)/f(z) \in P \) and \( 1 + zf''(z)/f'(z) \in P \) respectively. Several subclasses of starlike and convex functions were defined using subordination of \( zf'(z)/f(z) \) and \( 1 + zf''(z)/f'(z) \) to some function in \( P \). Ma and Minda [7] gave a unified treatment of growth, distortion, rotation and coefficient inequalities for function in classes \( S^*(\varphi) = \{ f \in A : zf'(z)/f(z) \prec \varphi(z) \} \) and \( K(\varphi) = \{ f \in A : 1 + zf''(z)/f'(z) \prec \varphi(z) \} \), where \( \varphi \in P \), starlike with respect to 1, symmetric about the x-axis and \( \varphi'(0) > 0 \). Numerous classes were defined for various choices of the function \( \varphi \) such as \( (1 + Az)/(1 + Bz) \), \( e^z \), \( z + \sqrt{1 + z^2} \) and so on.

For any two subclasses \( F \) and \( G \) of \( A \), the \( G \)-radius of the class \( F \), denoted as \( R_G(F) \) is the largest number \( R_G \in (0, 1) \) such that \( r^{-1}f(rz) \in G \) for all \( f \in F \) and \( 0 < r < R_G \). We determine the radius of various subclasses of starlike functions such as starlikeness

2010 Mathematics Subject Classification. 30C80, 30C45.

The first author is supported by the UGC-JRF Scholarship.
associated with the exponential function, lune, cardioid and a particular rational function for the class of Bloch Functions.

2. Radius Problems

In 2015, Mendaratta et al. [2] introduced a subclass \( S^*_e \) of starlike functions associated with the exponential function. This class \( S^*_e \) consists of all functions \( f \in \mathcal{A} \) satisfying the subordination \( zf'(z)/f(z) \prec e^z \). This subordination is equivalent to the inequality \( |\log(zf'(z)/f(z))| < 1 \). Our first theorem gives the \( S^*_e \)-radius of Bloch functions.

**Theorem 2.1.** The \( S^*_e \)-radius of the class \( \mathcal{B} \) of Bloch functions is

\[
R_{S^*_e}(\mathcal{B}) = \frac{1}{4} \sqrt{3} \left( 3 - 3e + \sqrt{1 - 10e + 9e^2} \right) \approx 0.517387.
\]

The obtained radius is sharp.

**Proof.** For functions \( f \in \mathcal{B} \), Bonk [1] proved the following inequality

\[
\left| \frac{zf'(z)}{f(z)} - \frac{\sqrt{3}}{\sqrt{3} - r} \right| \leq \frac{\sqrt{3}r}{(\sqrt{3} - r)(\sqrt{3} - 2r)}, \quad |z| = r < \frac{1}{\sqrt{3}}.
\]

The function

\[
h(r) := \frac{\sqrt{3}}{\sqrt{3} - r} - \frac{\sqrt{3}r}{(\sqrt{3} - r)(\sqrt{3} - 2r)} = \frac{3 - 3\sqrt{3}r}{(\sqrt{3} - r)(\sqrt{3} - 2r)}
\]

is a decreasing function of \( r \) for \( 0 \leq r < 1/\sqrt{3} = R_{S^*_e}(\mathcal{B}) \). The number \( R = R_{S^*_e}(\mathcal{B}) < 1/\sqrt{3} = R_{S^*}(\mathcal{B}) \) is the smallest positive root of the polynomial

\[
2R^2 + 3\sqrt{3}(e - 1)R + 3(1 - e) = 0
\]

or \( h(R) = 1/e \). Therefore, for \( 0 \leq r < R \), it follows that \( 1/e = h(R) < h(r) \) and hence

\[
\frac{\sqrt{3}r}{(\sqrt{3} - r)(\sqrt{3} - 2r)} < \frac{\sqrt{3}}{\sqrt{3} - r} - \frac{1}{e}.
\]

Thus (2.1) and (2.3) give

\[
\left| \frac{zf'(z)}{f(z)} - \frac{\sqrt{3}}{\sqrt{3} - r} \right| < \frac{\sqrt{3}}{\sqrt{3} - r} - \frac{1}{e}, \quad |z| = r < R.
\]

The function \( C(r) = \sqrt{3}/(\sqrt{3} - r) \) is an increasing function of \( r \), so for \( r \in [0, R) \), it follows that \( C(r) \in [1, C(R)) \subseteq [1, C(0.6)) \approx [1, 1.53001] \subseteq (0.367879, 1.54308) \approx (1/e, (e + e^{-1})/2) \). By [2, Lemma 2.2], for \( 1/e < c < e \), we have \( \{ w : |w - c| < r_c \} \subseteq \{ w : |\log(w)| < 1 \} \) when \( r_c \) is given by

\[
r_c = \begin{cases} 
  c - e^{-1} & \text{if } e^{-1} < c \leq \frac{e + e^{-1}}{2}, \\
  e - c & \text{if } \frac{e + e^{-1}}{2} < c < e.
\end{cases}
\]

By (2.4), we see that \( w = zf'(z)/f(z), |z| < R \), satisfies \( |w - c| < c - e^{-1} \) and hence it follows that \( |\log(w)| < 1 \). This shows that \( S^*_e \)-radius of the class \( \mathcal{B} \) is at least \( R \).
We now show that \( R \) is the exact \( S_c^* \)-radius of the class \( \mathcal{B} \). The function \( f_0 : \mathbb{D} \to \mathbb{C} \) defined by

\[
f(z) = \frac{\sqrt{3}}{4} \left\{ 1 - 3 \left( \frac{z - \sqrt{1/3}}{1 - \sqrt{1/3}z} \right)^2 \right\} = \frac{3z(3 - 2\sqrt{3}z)}{(3 - \sqrt{3}z)^2}
\]
is an example of function in the class \( \mathcal{B} \) and it serves as an extremal function for the various problems. For this function, we have

\[
\frac{zf'(z)}{f(z)} = \frac{3\sqrt{3} - 9z}{2\sqrt{3}z^2 - 9z + 3\sqrt{3}}.
\]

Using the equation (2.2), we get

\[
\left| \log \left( \frac{zf'(z)}{f(z)} \right) \right| = \log \left( \frac{3\sqrt{3} - 9z}{2\sqrt{3}z^2 - 9z + 3\sqrt{3}} \right) = \left| \log \left( \frac{1}{e} \right) \right| = 1.
\]

This proves that \( R \) is the exact \( S_c^* \)-radius of the class \( \mathcal{B} \).

Sharma et al. studied the class \( S_c^* = S^*(\phi_c) = S^*(1 + (4/3)z + (2/3)z^2) \) and gave Lemma 2.5

For \( 1/3 < c < 3 \),

\[
r_c = \begin{cases} 
\frac{3c - 1}{3} & \text{if } \frac{1}{3} < c \leq \frac{5}{3} \\
3 - c & \text{if } \frac{5}{3} < c < 3 
\end{cases}
\]

then \( \{ w : |w - c| < r_c \} \subseteq \Omega_c \). Here \( \Omega_c \) is the region bounded by the cardioid \( \{ x + iy : (9x^2 + 9y^2 - 18x + 5)^2 - 16(9x^2 + 9y^2 - 6x + 1) = 0 \} \).

**Theorem 2.2.** The \( S_c^* \)-radius \( \mathcal{R}_{S_c^*} \approx 0.524423 \). This radius is sharp.

**Proof.** \( R = \mathcal{R}_{S_c^*} \) is the smallest positive root of the equation

\[
R^2 + 3\sqrt{3}R - 3 = 0.
\]

The function

\[
h(r) := \frac{\sqrt{3}}{\sqrt{3} - r} - \frac{\sqrt{3}r}{(\sqrt{3} - r)(\sqrt{3} - 2r)} = \frac{3 - 3\sqrt{3}r}{(\sqrt{3} - r)(\sqrt{3} - 2r)}
\]
is a decreasing function of \( r \) for \( 0 \leq r < 1/\sqrt{3} = \mathcal{R}_{S_c^*} \). Note that the class \( S_c^* \) is a subclass of the parabolic starlike class \( S^* \). Also since, \( R = \mathcal{R}_{S_c^*} \) is the smallest positive root of the equation \( h(r) = 1/3 \). For \( 0 \leq r < R \), we have

\[
\frac{\sqrt{3}r}{(\sqrt{3} - r)(\sqrt{3} - 2r)} < \frac{\sqrt{3}}{\sqrt{3} - r} - \frac{1}{3}
\]

Thus (2.1) and (2.7) give

\[
\left| \frac{zf'(z)}{f(z)} - \frac{1}{1 - ar} \right| < \frac{1}{1 - ar} - \frac{1}{3}; \ |z| \leq r, \ a = \frac{1}{\sqrt{3}}.
\]

The center \( C(r) \) of (2.1) is an increasing function of \( r \), so for \( r \in [0, R) \), \( C(r) \in [1, C(R)] \subseteq [1, c(0.6)) \approx [1, 1.53001] \subseteq (1/3, 5/3) \). Now, by (2.6) we get that the disc \( \{ w : |w - c| <
Thus (2.1) and (2.9) give the smallest positive root of the equation
\[ h_{r} = \text{a decreasing function of } r. \]
So, \( R \) for this function, \( zf(z) = \frac{3\sqrt{3} - 9z}{2\sqrt{3}z^2 - 9z + 3\sqrt{3}} \), and using the equation for \( R \), we get
\[ 2\sqrt{3}r^2 - 9r + 3\sqrt{3} = 3(3\sqrt{3} - 9r), \]
thus for \( z = R \)
\[ \frac{zf'(z)}{f(z)} = \frac{1}{3} = \phi_{c}(-1). \]

The class \( S_{\sqrt{3}}^{*} = \mathcal{S}^{*}(z + \sqrt{1 + z^2}) \) was introduced in 2015 by Rain and Sokól [4] in 2015 and proved that \( f \in S_{\sqrt{3}}^{*} \iff zf'(z)/f(z) \) lies in the lune region \( \{ w : |w^2 - 1| < 2|w| \} \). Gandhi and Ravichandran [5] Lemma 2.1] proved that for \( \sqrt{2} - 1 < c \leq \sqrt{2} + 1, \)
\[ \{ w : |w - c| < 1 - |\sqrt{2} - c| \} \subseteq \{ w : |w^2 - 1| < 2|w| \} \tag{2.8} \]

**Theorem 2.3.** The \( S_{\sqrt{3}}^{*} \) radius, \( R_{S_{\sqrt{3}}^{*}} \approx 0.507306 \). The radius is sharp.

**Proof.** For \( R = R_{S_{\sqrt{3}}^{*}} \), the center of (2.1), \( C(R) = \sqrt{2} \); since \( C(r) \) is an increasing function of \( r \), thus for \( 0 \leq r < R, 1 \leq C(r) < \sqrt{2} \), or for \( 0 \leq r < R, \sqrt{2} - C(r) \geq 0. \)
So, \( R = R_{S_{\sqrt{3}}^{*}} \) is the smallest positive root of the equation
\[ (2 - 2\sqrt{2})R^2 + \sqrt{3}(3\sqrt{2} - 6)R + 3(2 - \sqrt{2}) = 0. \]
The function
\[ h(r) := \frac{\sqrt{3}}{\sqrt{3} - r} - \frac{\sqrt{3}r}{(\sqrt{3} - r)(\sqrt{3} - 2r)} = \frac{3 - 3\sqrt{3}r}{(\sqrt{3} - r)(\sqrt{3} - 2r)} \]
is a decreasing function of \( r \) for \( 0 \leq r < 1/\sqrt{3} = R_{S_{\sqrt{3}}^{*}}. \) [1] Corollary, P.455] Note that the class \( S_{\sqrt{3}}^{*} \) is a subclass of the parabolic starlike class \( S^{*} \). Also since, \( R = R_{S_{\sqrt{3}}^{*}} \) is the smallest positive root of the equation \( h(r) = \sqrt{2} - 1 \). For \( 0 \leq r < R, \) we have
\[ \frac{\sqrt{3}r}{(\sqrt{3} - r)(\sqrt{3} - 2r)} < 1 - \sqrt{2} + \frac{\sqrt{3}}{\sqrt{3} - r} = 1 - \left| \sqrt{2} - \frac{\sqrt{3}}{\sqrt{3} - r} \right|. \tag{2.9} \]
Thus (2.1) and (2.9) give
\[ \left| \frac{zf'(z)}{f(z)} - \frac{1}{1 - ar} \right| < 1 - \left| \sqrt{2} - \frac{1}{1 - ar} \right| ; |z| \leq r, a = \frac{1}{\sqrt{3}}. \]
The center $C(r)$ of (2.1) is an increasing function of $r$, so for $r \in [0, R)$, $C(r) \in [1, C(R)) \subseteq [1, c(0.6)) \approx [1, 1.53001) \subseteq (\sqrt{2} - 1, \sqrt{2} + 1)$. Now, by (2.8) we get that the $R$ is the required radius.

Consider the

$$ f(z) = \frac{\sqrt{3}}{4} \left\{ 1 - 3 \left( \frac{z - \sqrt{1/3}}{1 - \sqrt{1/3}} \right)^2 \right\} $$

for this function, $zf'(z) = \frac{3\sqrt{3} - 9z}{2\sqrt{3}z^2 - 9z + 3\sqrt{3}}$, and we can easily see that for $z = \frac{1}{2}[2\sqrt{3} - \sqrt{6}]$, $\left| \left( \frac{zf'(z)}{f(z)} \right)^2 - 1 \right| = 2 \left( \frac{zf'(z)}{f(z)} \right) = 2(\sqrt{2} - 1)$.

Thus, the result is sharp.

The next class that we consider is the class of starlike functions associated with a rational function. Kumar and Ravichandran [6] introduced the class of starlike functions associated with the rational function $\psi(z) = 1 + ((z^2 + k)/(k^2 - kz))$ where $k = \sqrt{2} + 1$, denoted by $S^*_R = S^*(\psi(z))$. They proved [6] Lemma 2.2] that for $2(\sqrt{2} - 1) < c < 2$,

$$ r_c = \begin{cases} c - 2(\sqrt{2} - 1) & \text{if } 2(\sqrt{2} - 1) < c \leq \sqrt{2} \\ 2 - c & \text{if } \sqrt{2} \leq c < 2 \end{cases} \quad (2.10) $$

then $\{w : |w - c| < r_c \} \subseteq \psi(\mathbb{D})$.

**Theorem 2.4.** The $S^*_R$ radius is the smallest positive root of the polynomial $4(1 - \sqrt{2})r^2 + 3\sqrt{3}(2\sqrt{2} - 3)r + 3(3 - 2\sqrt{2}) = 0$. The result is sharp.

**Proof.** $R = S^*_R$ is the smallest positive root of the equation $4(1 - \sqrt{2})R^2 + 3\sqrt{3}(2\sqrt{2} - 3)R + 3(3 - 2\sqrt{2}) = 0$.

The function

$$ h(r) := \frac{\sqrt{3}}{\sqrt{3} - r} - \frac{\sqrt{3}r}{(\sqrt{3} - r)(\sqrt{3} - 2r)} = \frac{3 - 3\sqrt{3}r}{(\sqrt{3} - r)(\sqrt{3} - 2r)} $$

is a decreasing function of $r$ for $0 \leq r < 1/\sqrt{3} = R_{S^*_R}$. [11] Corollary, P.455] Note that the class $S^*_R$ is a subclass of the parabolic starlike class $S^*$. Also since, $R = S^*_R$ is the smallest positive root of the equation $h(r) = 2(\sqrt{2} - 1)$. For $0 \leq r < R$, we have

$$ \frac{\sqrt{3}r}{(\sqrt{3} - r)(\sqrt{3} - 2r)} < \frac{\sqrt{3}}{\sqrt{3} - r} - 2(\sqrt{2} - 1) \quad (2.11) $$

Thus (2.1) and (2.11) give

$$ \left| \frac{zf'(z)}{f(z)} - \frac{1}{1 - ar} \right| < \frac{1}{1 - ar} - 2(\sqrt{2} - 1); \ |z| \leq r, \ a = \frac{1}{\sqrt{3}}. $$
The center $C(r)$ of (2.1) is an increasing function of $r$, so for $r \in [0, R)$, $C(r) \in [1, C(R)) \subseteq [1, c(0.4)) \approx [1, 1.30029) \subseteq (2(\sqrt{2} - 1), \sqrt{2})$. Now, by (2.10) we get that the disc \{ $w : |w - c| < x - 2(\sqrt{2} - 1)$ \} $\subseteq \psi(\mathbb{D})$.

To show that the result is sharp, consider the function

$$ f(z) = \frac{\sqrt{3}}{4} \left\{ 1 - 3 \left( \frac{z - \sqrt{1/3}}{1 - \sqrt{1/3}z} \right)^2 \right\} $$

for this function, $zf'(z) = \frac{3\sqrt{3} - 9z}{2\sqrt{3}z^2 - 9z + 3\sqrt{3}}$, and using the equation for $R$ we get $3\sqrt{3} - 9r = (2\sqrt{2} - 2)(2\sqrt{3}r^2 - 9r + 3\sqrt{3})$ thus for $z = R$

$$ \frac{zf'(z)}{f(z)} = 2\sqrt{2} - 2 = \psi(-1). $$

Theorem 2.5. For the class $\mathcal{B}$ the following results hold:

1. The Lemniscate starlike radius, $R_{S_L} = \frac{2\sqrt{\frac{3}{5} - \sqrt{6}}}{4} \approx 0.253653$.
2. The starlike radius associated with the sine function, $R_{S_{\sin}} = \frac{\sqrt{3}\sin \frac{1}{2} + 2\sin 1}{2 + 2\sin 1} \approx 0.395735$.
3. The nephroid radius, $R_{S_{Ne}} = \frac{\sqrt{3}}{5} \approx 0.34641$.
4. The sigmoid radius, $R_{S_{SG}} = \frac{\sqrt{3}(e - 1)}{4e} \approx 0.273716$.

References

[1] M. Bonk, Distortion estimates for Bloch functions, Bull. London Math. Soc. 23 (1991), no. 5, 454–456.
[2] R. Mendiratta, S. Nagpal and V. Ravichandran, On a subclass of strongly starlike functions associated with exponential function, Bull. Malays. Math. Sci. Soc. 38 (2015), no. 1, 365–386.
[3] K. Sharma, N. K. Jain and V. Ravichandran, Starlike functions associated with a cardioid, Afr. Mat. 27 (2016), no. 5-6, 923–939.
[4] R. K. Raina and J. Soköl, Some properties related to a certain class of starlike functions, C. R. Math. Acad. Sci. Paris 353 (2015), no. 11, 973–978.
[5] S. Gandhi and V. Ravichandran, Starlike functions associated with a lune, Asian-Eur. J. Math. 10 (2017), no. 4, 1750064, 12 pp.
[6] S. Kumar and V. Ravichandran, A subclass of starlike functions associated with a rational function, Southeast Asian Bull. Math. 40 (2016), no. 2, 199–212.
[7] W. C. Ma and D. Minda, A unified treatment of some special classes of univalent functions, in Proceedings of the Conference on Complex Analysis (Tianjin, 1992), 157–169, Conf. Proc. Lecture Notes Anal., I, Int. Press, Cambridge, MA.
[8] Ch. Pommerenke, On Bloch functions, J. London Math. Soc. (2) 2 (1970), 689–695.