ON THE POLYNOMIAL MOMENT PROBLEM.

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1. Introduction.

In this paper we treat the following "polynomial moment problem": for a complex polynomial \( P(z) \) and distinct \( a, b \in \mathbb{C} \) such that \( P(a) = P(b) \) to describe polynomials \( q(z) \) such that

\[
\int_a^b P^i(z)q(z)\,dz = 0
\]

for all integer non-negative \( i \).

The polynomial moment problem was proposed in the series of papers of M. Briskin, J.-P. Francoise and Y. Yomdin [1]-[5] as an infinitesimal version of the center problem for the polynomial Abel equation in the complex domain in the frame of a programme concerning the classical Poincare center-focus problem for the polynomial vector field on the plane. It was suggested that the following "composition condition" imposed on \( P(z) \) and \( Q(z) = \int q(z)\,dz \) is necessary and sufficient for the pair \( P(z), q(z) \) to satisfy (*)

\[
P(z) = \tilde{P}(W(z)), \quad Q(z) = \tilde{Q}(W(z)), \quad \text{and} \quad W(a) = W(b).
\]

(**) It is easy to see that the composition condition is sufficient: since after the change of variable \( z \to W(z) \) the way of integration becomes closed, the sufficientness follows from the Cauchy theorem. The necessity of the composition condition in the case when \( a, b \) are not critical points of \( P(z) \) was proved by C. Christopher in [6] (see also the paper of N. Roytvarf [10] for a similar result) and in some other special cases by M. Briskin, J.-P. Francoise and Y. Yomdin in the papers cited above. Nevertheless, in general the composition conjecture fails to be true as it was shown by the author in [3].

In this paper we give a solution of the polynomial moment problem in the case when \( P(z) \) is indecomposable that is when \( P(z) \) can not be represented as a composition \( P(z) = P_1(P_2(z)) \) with non-linear polynomials \( P_1(z), P_2(z) \). We show that in this case the composition conjecture is true without any restrictions on points \( a, b \).
Theorem 1. Let $P(z), q(z)$ be complex polynomials and let $a, b$ be distinct complex numbers such that $P(a) = P(b)$ and

$$\int_a^b P^j(z)q(z)dz = 0$$

for $i \geq 0$. Suppose that $P(z)$ is indecomposable. Then there exists a polynomial $Q(z)$ such that $Q(z) = \int q(z)dz = \hat{Q}(P(z))$.

We also examine the following condition which is stronger than (*):

$$\int_a^b P^i(z)Q^j(z)Q'(z)dz = 0$$

for $i \geq 0, j \geq 0$. If $\gamma$ is a curve which is the image of the segment $[a, b]$ in $\mathbb{C}^2$ under the map $z \to (P(z), Q(z))$ then this condition is equivalent to the condition that $\int_\gamma \omega = 0$ for all global holomorphic 1-forms $\omega$ in $\mathbb{C}^2$ ("the moment condition").

Proofs.

2.1. Lemmata about branches of $Q(P^{-1}(z))$. Let $P(z)$ and $Q(z)$ be rational functions and let $U \subset \mathbb{C}$ be a domain in which there exists a single-valued branch $p^{-1}(z)$ of the algebraic function $P^{-1}(z)$. Denote by $Q(P^{-1}(z))$ the complete algebraic function obtained by the analytic continuation of the functional element $(U, Q(P^{-1}(z)))$. Since the monodromy group $G(P^{-1})$ of the algebraic function $P^{-1}(z)$ is transitive this definition does not depend of the choice of $p^{-1}(z)$. Denote by $d(Q(P^{-1}(z)))$ the degree of the algebraic function $Q(P^{-1}(z))$ that is the number of its branches.
Lemma 1. Let $P(z), Q(z)$ be rational functions. Then
\[ d(Q(P^{-1}(z))) = \deg P(z)/[C(z) : C(P,Q)]. \]

Proof. Since any algebraic relation over $C$ between $Q(p^{-1}(z))$ and $z$ supplies an algebraic relation between $Q(z)$ and $P(z)$ and vice versa we see that $d(Q(P^{-1}(z))) = [C(P,Q) : C(P)]$. As $[C(P,Q) : C(P)] = [C(z) : C(Q)]/[C(z) : C(P,Q)]$ the lemma follows now from the observation that $[C(z) : C(P)] = \deg P(z)$.

Recall that by L"uroth theorem each field $k$ such that $C \subset k \subset C(z)$ and $k \neq C$ is of the form $k = C(R), R \in C(z) \setminus C$. Therefore, the field $C(P,Q)$ is a proper subfield of $C(z)$ if and only if $P(z) = \tilde{P}(W(z)), Q(z) = \tilde{Q}(W(z))$ for some rational functions $\tilde{P}(z), \tilde{Q}(z), W(z)$ with $\deg W(z) > 1$; in this case we say that $P(z)$ and $Q(z)$ have a common right divisor in the composition algebra. The lemma 1 implies the following explicit criterion which essentially due to Ritt (cf. also [3], [11]).

Corollary 1. Let $P(z), Q(z)$ be rational functions. Then $P(z)$ and $Q(z)$ have a common right divisor in the composition algebra if and only if
\[ Q(p^{-1}(z)) = \tilde{Q}(\tilde{p}^{-1}(z)) \tag{1} \]
for two different branches $p^{-1}(z), \tilde{p}^{-1}(z)$ of $P^{-1}(z)$.

Proof. Indeed, by lemma 1 the field $C(P,Q)$ is a proper subfield of $C(z)$ if and only if $d(Q(P^{-1}(z))) < \deg P(z)$. On the other hand, the last inequality is clearly equivalent to condition (1).

Lemma 2. Let $P(z), Q(z)$ be rational functions, $\deg P(z) = n$. Suppose that there exist $a_i \in C, 1 \leq i \leq n$, not all equal between themselves such that
\[ \sum_{i=1}^{n} a_i Q(p_i^{-1}(z)) = 0. \tag{2} \]

If, in addition, the group $G(P^{-1})$ is doubly transitive then $Q(z) = \tilde{Q}(P(z))$ for some rational function $\tilde{Q}(z)$.

Proof. Let $G \subset S_n$ be a permutation group and let $\rho_G : G \to GL(C^n)$ be the permutation representation of $G$ that is $\rho_G(g), g \in G$ is the linear map which sends a vector $\vec{a} = (a_1, a_2, ..., a_n)$ to the vector $\vec{a}_g = (a_{g(1)}, a_{g(2)}, ..., a_{g(n)})$. It is well known (see e.g. [2], Th. 29.9) that $G$ is doubly transitive if and only if $\rho_G$ is the sum of the identical representation and an absolutely irreducible representation. Clearly, the one-dimensional $\rho_G$-invariant subspace $E \subset C^n$ corresponding to the identity representation is generated by the vector $(1,1,...,1)$. Therefore, since the Hermitian inner product $(\vec{a}, \vec{b}) = a_1 b_1 + a_2 b_2 + ... + a_n b_n$ is invariant with respect to $\rho_G$, the group $G$ is doubly transitive if and only if the subspace $E$ and its orthogonal complement $E^\perp$ are the only $\rho_G$-invariant subspaces of $C^n$.

Suppose that (2) holds. In this case also
\[ \sum_{i=1}^{n} a_i Q(p^{-1}_{\sigma(i)}(z)) = 0 \tag{3} \]
for all $\sigma \in G(P^{-1})$ by the analytic continuation. To prove the lemma it is enough to show that $Q(p_{i}^{-1}(z)) = Q(p_{j}^{-1}(z))$ for all $i, j, 1 \leq i, j \leq n$; then by lemma 1 $[C(z) : C(P,Q)] = \deg P(z) = [C(z) : C(P)]$ and therefore $Q(z) = \tilde{Q}(P(z))$ for some rational function $\tilde{Q}(z)$. Assume the converse i.e. that there exists $z_0 \in U$
such that not all \(Q(p_{i}^{-1}(z_0))\), \(1 \leq i \leq n\), are equal between themselves. Without loss of generality we can suppose that all \(Q(p_{i}^{-1}(z_0))\), \(1 \leq i \leq n\), are finite. Consider the subspace \(V \subset \mathbb{C}^n\) generated by the vectors \(\vec{v}_\sigma\), \(\sigma \in G(P^{-1})\), where \(\vec{v}_\sigma = (Q(p_{\sigma(1)}^{-1}(z_0))), Q(p_{\sigma(2)}^{-1}(z_0)), ..., Q(p_{\sigma(n)}^{-1}(z_0))\). Clearly, \(V\) is \(\rho_G(P^{-1})\)-invariant and \(V \neq E\). Moreover, it follows from (3) that \(V\) is contained in the orthogonal complement \(A^\perp\) of the subspace \(A \subset \mathbb{C}^n\) generated by the vector \((\vec{a}_1, \vec{a}_2, ..., \vec{a}_n)\). Since \(A \neq E\) we see that \(V\) is a proper \(\rho_G\)-invariant subspace of \(\mathbb{C}^n\) distinct from \(E\) and \(E^\perp\) that contradicts the assumption that the group \(G(P^{-1})\) is doubly transitive.

2.2. Lemma about preimages of domains. For a polynomial \(P(z)\) denote by \(c(P)\) the set of finite critical values of \(P(z)\).

**Lemma 3.** Let \(P(z)\) be a polynomial and let \(U \subset \mathbb{C}P^1\) be an unbounded simply connected domain such that \(c(P) \cap U = \emptyset\). Then \(P^{-1}\{U\}\) is conformally equivalent to the unit disk and \(P^{-1}\{\partial U\}\) is connected.

**Proof.** Indeed, by the Riemann theorem \(U\) is conformally equivalent to the unit disk \(D\) whenever \(\partial U\) contains more than one point. It follows from \(c(P) \cap U = \emptyset\) that \(\partial U\) contains a unique point if and only if \(P(z)\) has a unique finite critical value \(c\) and \(\partial U = c\); in this case there exist linear functions \(\sigma_1, \sigma_2\) such that \(\sigma_1(P(\sigma_2(z))) = z^n\), \(n \in \mathbb{N}\) and the lemma is obvious. Therefore, we can suppose that \(U \cong \mathbb{D}\). Since \(c(P) \cap U = \emptyset\) the restriction of the map \(P(z) : \mathbb{C}P^1 \rightarrow \mathbb{C}P^1\) on \(P^{-1}\{U\}\setminus P^{-1}\{\infty\}\) is a covering map. As \(U \setminus \infty\) is conformally equivalent to the punctured unit disc \(\mathbb{D}^*\) it follows from covering spaces theory that \(P^{-1}\{U\}\setminus P^{-1}\{\infty\}\) is a disjoint union of domains \(\cup U_i\) conformally equivalent to \(\mathbb{D}^*\) such that all induced maps \(f_i : \mathbb{D}^* \rightarrow \mathbb{D}^*\) are of the form \(z \rightarrow z^{l_i}\), \(l_i \in \mathbb{N}\). But, as \(P^{-1}\{\infty\} = \{\infty\}\), there may be only one such a domain. Therefore, the preimage \(P^{-1}\{U\}\) is conformally equivalent to the unit disk. In particular, since \(P^{-1}\{\partial U\} = \partial P^{-1}\{U\}\) we see that \(P^{-1}\{\partial U\}\) is connected.

2.3. Proof of theorem 2: the case of a regular value. In this section we investigate the case when \(t_0 = P(a) = P(b)\) is not a critical value of the polynomial \(P(z)\). For a simple closed curve \(M \subset \mathbb{C}\) denote by \(D_M^+\) (resp. \(D_M^-\)) the domain that is interior (resp. exterior) with respect to \(M\).

Let \(L \subset \mathbb{C}\) be a simple closed curve such that \(t_0 \in L\) and \(c(P) \subset D_L^+\). Denote by \(\tilde{L}\) the same curve considered as an embedded into the complex plane oriented graph. By definition, the graph \(\tilde{L}\) has one vertex \(t_0\) and one counter-clockwise oriented edge \(l\). Let \(\tilde{\Omega} = P^{-1}\{\tilde{L}\}\) be an oriented graph which is the preimage of the graph \(\tilde{L}\) under the mapping \(P(z) : \mathbb{C} \rightarrow \mathbb{C}\), i.e. vertices of \(\tilde{\Omega}\) are preimages of \(t_0\) and oriented edges of \(\tilde{\Omega}\) are preimages of \(l\). As \(L \cap c(P) = \emptyset\) the graph \(\tilde{\Omega}\) has \(n = \deg P(z)\) vertices and \(n\) edges. Furthermore, by lemma 3 the graph \(\tilde{\Omega} = P^{-1}\{\partial D_L^+\}\) is connected.

Therefore, as a point set in \(\mathbb{C}\) the graph \(\tilde{\Omega}\) is a simple closed curve. Let \(l_j, 1 \leq j \leq n\), be oriented edges of \(\tilde{\Omega}\) and let \(a_j\) (resp. \(b_j\)) be the starting (resp. ending) point of \(l_j\). We will suppose that edges of \(\tilde{\Omega}\) are numerated by such a way that \(a_1 = a\) and that under a moving around the domain \(P^{-1}\{D_L^+\}\) along its boundary \(\tilde{\Omega}\) the edge \(l_i\), \(1 \leq i \leq n - 1\), is followed by the edge \(l_{i+1}\) (see fig. 1).

Let \(U \subset \mathbb{C}\) be a simply connected domain such that \(U \cap c(P) = \emptyset\) and \(L \setminus \{t_0\} \subset U\). By monodromy theorem, in such a domain there exist \(n\) single-valued branches of \(P^{-1}(t)\). Denote by \(p_j^{-1}(t)\), \(1 \leq j \leq n\), the single-valued branch of \(P^{-1}(t)\) defined
in $U$ by the condition $p_j^{-1}\{t \setminus t_0\} = l_j \setminus \{a_j, b_j\}$; such a numeration of branches of $P^{-1}(t)$ means that the analytic continuation of the functional element $\{U, p_j^{-1}(t)\}$, $1 \leq j \leq n-1$, along $L$ is the functional element $\{U, p_j^{-1+1}(t)\}$. Let $l_k, k < n$, be the edge of $\Omega$ such that $b_k = b$ and let $\Gamma = \{l_1, l_2, ..., l_k\}$ be the oriented path in the graph $\Omega$ joining the vertices $a_1 = a$ to $b_k = b$. For $t \in U$ set $\varphi(t) = \sum_{j=1}^k Q(p_j^{-1}(t))$. Clearly, $\varphi(t)$ is analytic on $U$ and extends to a continuous on $U \cup t_0$ function since $\sum_{j=1}^k Q(a_j) - \sum_{j=1}^k Q(b_j) = Q(a) - Q(b) = 0$

Consider an analytic on $\mathbb{C}P^1 \setminus L$ function

$$I(\lambda) = \oint_{\Gamma} \frac{\varphi(t)}{t-\lambda} dt = \int_{\Gamma} \frac{Q(z)P'(z)dz}{P(z) - \lambda}.$$ 

More precisely, the integral above defines two analytic functions: one of them $I^+(\lambda)$ is analytic in $D^+_L$ and the other one $I^-(\lambda)$ is analytic in $D^-_L$. Furthermore, calculating the Taylor expansion at infinity we see that condition (*) reduces to the condition that $I^- (\lambda) \equiv 0$ in $D^-_{L}$. Therefore, by a well-known result about integrals of the Cauchy type (see e.g. [1], p. 63) the function $\varphi(t)$ is the boundary value on $L$ of the analytic in $D^+_L$ function $I^+(\lambda)$. It follows from the uniqueness theorem for boundary values of analytic functions that the functional element $\{U, \varphi(t)\}$ can be analytically continued along any curve $M \subset D^+_L$. As $c(P) \subset D^+_L$ this fact implies that $\{U, \varphi(t)\}$ can be analytically continued along any curve $M \subset \mathbb{C}$. Therefore, by the monodromy theorem, the element $\{U, \varphi(t)\}$ extends to a single-valued analytic function in the whole complex plane. In particular, the analytic continuation of $\{U, \varphi(t)\}$ along any closed curve coincides with $\{U, \varphi(t)\}$. On the other hand, by construction the analytic continuation of $\{U, \varphi(t)\}$ along the curve $L$ is $\{U, \varphi_L(t)\}$, where $\varphi_L(t) = \sum_{j=1}^{k+1} Q(p_j^{-1}(t))$. It follows from $\varphi(t) = \varphi_L(t)$ that $Q(p_1^{-1}(t)) = Q(p_k^{-1}(t))$ and by corollary 1 we conclude that $P(z)$ and $Q(z)$ have a common right divisor in the composition algebra.

As the field $\mathbb{C}(P, Q)$ is a proper subfield of $\mathbb{C}(z)$ and $P(z), Q(z)$ are polynomials it is easy to prove that $\mathbb{C}(P, Q) = \mathbb{C}(W)$ for some polynomial $W(z)$, $\deg W(z) > 1$. It means that $P(z) = \tilde{P}(W(z)), Q(z) = \tilde{Q}(W(z))$ for some polynomials $\tilde{P}(z), \tilde{Q}(z)$ such that $\tilde{P}(z)$ and $\tilde{Q}(z)$ have no a common right divisor in the composition algebra. Let us show that $W(a) = W(b)$. Since $t_0$ is not a critical value of the polynomial $P(z) = \tilde{P}(W(z))$ the chain rule implies that $t_0$ is not a critical value of the polynomial $\tilde{P}(z)$. Therefore, if $W(a) \neq W(b)$ then after the change of variable $z \mapsto W(z)$ in the same way as above we find that $\tilde{P}(z) = \tilde{P}(U(z)), \tilde{Q}(z) = \tilde{Q}(U(z))$.
for some polynomials \( \tilde{P}(z), \tilde{Q}(z), U(z) \) with \( \deg U(z) > 1 \) that contradicts the fact that \( \tilde{P}(z), \tilde{Q}(z) \) have no a common right divisor in the composition algebra. This completes the proof in the case when \( z_0 \) is not a critical value of \( P(z) \).

2.4. Proof of theorem 2: the case of a critical value. Assume now that \( t_0 = P(a) = P(b) \) is a critical value of \( P(z) \). In this case let \( L \) be a simple closed curve such that \( t_0 \in L \) and \( c(P) \setminus t_0 \subset D^+_L \). Consider again a graph \( \bar{\Omega} = P^{-1}\{\bar{L}\} \). Since \( P^{-1}\{D^-_L\} \) is still conformally equivalent to the unit disk by lemma 3, we see that the graph \( \bar{\Omega} \) topologically is the boundary of a disc although it is not a simple closed curve any more. Let \( l_j, 1 \leq j \leq n, \) be oriented edges of \( \bar{\Omega} \) and let \( a_j \) (resp. \( b_j \)) be the starting (resp. the ending) point of \( l_j \). Let us fix again such a numeration of edges of \( \bar{\Omega} \) that \( a_1 = a \) and that under a moving around the domain \( P^{-1}\{D^-_L\} \) along its boundary \( \bar{\Omega} \) the edge \( l_i, 1 \leq i \leq n - 1, \) is followed by the edge \( l_{i+1} \). As above denote by \( U \) a domain in \( \mathbb{C} \) such that \( U \cap c(P) = \emptyset, \) \( L \setminus \{t_0\} \subset U \) and let \( p_j^{-1}(t), 1 \leq j \leq n, \) be the single-valued branch of \( P^{-1}(t) \) defined in \( U \) by the condition \( p_j^{-1}(l \setminus t_0) = l_j \setminus \{a_j, b_j\} \). If \( k < n \) is a number such that \( b_k = b \) then for the same reason as above the function \( \varphi(t) = \sum_{j=1}^{k} Q(p_j^{-1}(t)) \) extends to an analytic in \( U \cup D^+_L \) function but this fact does not imply now that \( \varphi(t) \) extends to an analytic in the whole complex plane function since \( D^+_L \) does not contain \( t_0 \in c(P) \). Nevertheless, if \( V \) is a simply connected domain such that \( U \subset V \) and \( t_0 \notin V \) then \( \varphi(t) \) still extends to a single-valued analytic function in \( V \).

In particular, the analytic continuation of \( \{U, \varphi(t)\} \) along any simple closed curve \( M \) such that \( t_0 \subset D^+_M \) coincides with \( \{U, \varphi(t)\} \).

Let \( t_1 \in U \) be a point and let \( M_1 \) (resp. \( M_2 \)) be a simple closed curve such that \( t_1 \in M_1, \) \( M_1 \cap c(P) = \emptyset \) and \( D^+_M \cap c(P) = t_0 \) (resp. \( t_1 \in M_2, \) \( M_2 \cap c(P) = \emptyset \) and \( D^+_M \cap c(P) = c(P) \setminus t_0 \)). Define a permutation \( \rho_1 \in S_n \) (resp. \( \rho_2 \in S_n \)) by the condition that the functional element \( \{U, p_1^{-1}(t)\} \) (resp. \( \{U, p_2^{-1}(t)\} \)) is the result of the analytic continuation of the functional element \( \{U, p_j^{-1}(t)\}, 1 \leq j \leq n, \) from \( t_1 \) along the curve \( M_1 \) (resp. \( M_2 \)). Having in mind the identification of the set of elements \( \{U, p_j^{-1}(t)\}, 1 \leq j \leq n, \) with the set of oriented edges of the graph \( \bar{\Omega} \) the permutations \( \rho_1, \rho_2 \) can be described as follows: \( \rho_1 \) cyclically permutes the edges of \( \bar{\Omega} \) around the vertices from which they go while cycles \( (j_1, j_2, ..., j_k) \) of \( \rho_2 \)
correspond to simple cycles \((l_1, l_2, ..., l_n)\) of the graph \(\tilde{\Omega}\) and \(\rho_1 \rho_2 = (12...n)\) (see fig. 2).

To unload notation denote temporarily the element \(\{U, Q(p_i^{-1}(t))\}\), \(1 \leq i \leq n\), by \(s_i\). Since \(t_0 \subset D_{M_2}^{-}\), we have:

\[
0 = \sum_{j=1}^{k} s_{p_2(j)} - \sum_{j=1}^{k} s_j = s_{p_2(k)} + \sum_{j=1}^{k-1} [s_{p_2(j)} - s_{j+1}] - s_1. \quad (4)
\]

Using \(\rho_1 \rho_2 = (12...n)\) we can rewrite (4) as

\[
s_{\rho_1^{-1}(k+1)} - s_1 + \sum_{j=1}^{k-1} [s_{p_2(j)} - s_{\rho_1 \rho_2(j)}] = 0.
\]

Therefore, by the analytic continuation

\[
s_{\rho_1^{-1}(k+1)} - s_{\rho_1^{-1}(1)} + \sum_{j=1}^{k-1} [s_{\rho_1 \rho_2(j)} - s_{\rho_1^{-1} \rho_2(j)}] = 0 \quad (5)
\]

for \(f \geq 0\). Summing equalities (5) from \(f = 1\) to \(f = o(\rho_1)\), where \(o(\rho_1)\) is the order of the permutation \(\rho_1\), changing the order of summing, and observing that

\[
\sum_{f=0}^{o(\rho_1)-1} \left[ s_{\rho_1 \rho_2(j)} - s_{\rho_1^{-1} \rho_2(j)} \right] = s_{p_2(j)} - s_{\rho_1^{-1} \rho_2(j)} = 0
\]

we obtain:

\[
\sum_{f=0}^{o(\rho_1)-1} s_{\rho_1^{-1}(k+1)} = \sum_{f=0}^{o(\rho_1)-1} s_{\rho_1^{-1}(1)}. \quad (6)
\]

If \(a, b\) are not critical points of \(P(z)\) then \(p_{\rho_1^{-1}}^{-1}(t) = p_1^{-1}(t)\), \(p_{\rho_1^{-1}(k+1)}^{-1}(t) = p_{k+1}^{-1}(t)\) and (6) reduces to the equality \(Q(p_{k+1}^{-1}(t)) = Q(p_1^{-1}(t))\).

Suppose now that at least one of the points \(a, b\) is a critical point of \(P(z)\). Observe that (6) is hold for any polynomial \(Q(z)\) such that \(q(z) = Q'(z)\) satisfies (*). Therefore, substituting in (6) \(Q^j(z)\), \(2 \leq j \leq d_a + d_b - 1\), instead of \(Q(z)\) we conclude that

\[
\sum_{s=0}^{o(\rho_1)-1} Q^j(p_{\rho_1^{-1}(k+1)}^{-1}(t)) = \sum_{s=0}^{o(\rho_1)-1} Q^j(p_{\rho_1^{-1}(1)}^{-1}(t)) \quad (7)
\]

for all \(j, 1 \leq j \leq d_a + d_b - 1\). Consider a Vandermonde determinant \(D = \| d_{j,i} \|\), where \(d_{j,i} = Q^j(p_i^{-1}(t))\), \(0 \leq j \leq d_a + d_b - 1\) and \(i\) ranges the set of different indices from the cycles of \(\rho_1\) containing 1 and \(k+1\). Since (7) implies that \(D = 0\) we conclude again that \(Q(p_{\rho_1^{-1}}^{-1}(t)) = Q(p_{\rho_1^{-1}}^{-1}(t))\) for some \(i \neq j, 1 \leq i, j \leq n\). Therefore, \(P(z)\) and \(Q(z)\) have a common right divisor in the composition algebra and we can finish the proof by the same argument as in section 2.3 taking into account that the multiplicity of a point \(c \in \mathbb{C}\) with respect to \(P(z) = \tilde{P}(W(z))\) is greater or equal then the multiplicity of the point \(W(c)\) with respect to \(\tilde{P}(z)\).
2.5. **Proof of theorem 1.** Suppose at first that $n = \deg P(z)$ is a prime number. In this case the degree of the algebraic function $Q(P^{-1}(t))$ equals either $n$ or 1 since $d(Q(P^{-1}(t)))$ divides $\deg P(z)$. If $d(Q(P^{-1}(t))) = n$ then Puiseux expansions at infinity

$$Q(p^{-1}_n(t)) = \sum_{k \geq k_0} a_k \varepsilon^k t^k,$$

(8)

$satisfies an algebraic polynomial with integer coefficients distinct from the $n$-th cyclotomic polynomial $\Phi_n(z) = 1 + z + \ldots + z^{n-1}$. Since $\varepsilon^k$ is a primitive $n$-th root of unity it is a contradiction. Therefore, $d(Q(p^{-1}(t))) = 1$ and $Q(z) = \tilde{Q}(P(z))$ for some polynomial $\tilde{Q}(z)$.

Suppose now that $n$ is composite. Since $P(z)$ is indecomposable the group $G(P^{-1})$ is primitive by the Ritt theorem \[9\]. By the Schur theorem (see e.g. \[13\], Th. 25.3) a primitive permutation group of composite degree $n$ which contains an $n$-cycle is doubly transitive. Therefore, by lemma 2 equality (6) implies that $Q(z) = \tilde{Q}(P(z))$ for some polynomial $\tilde{Q}(z)$.

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