Non-contractible loops in the dense $O(n)$ loop model on the cylinder

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A lattice model of critical dense polymers $O(0)$ is considered for the finite cylinder geometry. Due to the presence of non-contractible loops with a fixed fugacity $\xi$, the model is a generalization of the critical dense polymers solved by Pearce, Rasmussen and Villani. We found the free energy for any height $N$ and circumference $L$ of the cylinder. The density $\rho$ of non-contractible loops is found for $N \to \infty$ and large $L$. The results are compared with those obtained for the anisotropic quantum chain with twisted boundary conditions. Using the latter method we obtained $\rho$ for any $O(n)$ model and an arbitrary fugacity.

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I. INTRODUCTION

The dense $O(n)$ loop model $\square$ is defined by drawing two lines in each elementary cell of the square lattice. Two possible states of the cell are shown in Fig$\square$. The lines on the whole lattice with appropriate boundary conditions form a system of closed loops with the Boltzmann weight $n$ ascribed to every loop. Two particular cases, $n = 1$ and $n = 0$ are of especial interest. For $n = 1$, it is due to remarkable observations concerning the ground state of the antiferromagnetic XXZ quantum chain and, equivalently, to properties
FIG. 1: Elementary cells.

FIG. 2: Loops on the horizontally periodic lattice. One non-contractible loop joins the left and the right sides of figure.

of the transfer matrix of dense loop model taken in the Hamiltonian limit of extreme spatial anisotropy [2–6]. It was noted that the elements of the ground state of the loop Hamiltonian are equal to the cardinality of different subsets of configurations of the Fully Packed Loop (FPL) model which in their turn are connected with classes of Alternating Sign Matrices (ASM).

In the case $n = 0$, the bulk loops disappear and the system of lines is converted into the model of critical dense polymers [7]. This model is a recent representative of the more general two dimensional polymer theory initiated by Saleur and Duplantier in the context of a conformal field theory [8–10]. The dense polymer model is the first member $\mathcal{LM}(1, 2)$ of the Yang-Baxter integrable series of logarithmic minimal models. In the scaling limit, the
central charge is \( c = -2 \) and the conformal weights yield the Kac formula for the infinitely extended Kac table [7].

The states of the \( O(n) \) model can be defined in terms of the connectivity condition for points of intersection between loops and a horizontal line cutting the loops at these points. Two points are connected by a link if there exists a line between them via the half space above the cut. For instance, imposing periodic boundary conditions in horizontal direction for the lattice in Fig.2 and specifying the boundary conditions at the upper edge, we obtain three minimal links between points 1 and 6, 2 and 3, 4 and 5 which are points of intersection between loops and the bottom line of lattice belonging to upper half space.

Pearce, Rasmussen and Villani [11] have solved the dense polymer model on the cylinder using the cylinder Temperley-Lieb algebra, the single-row transfer matrix and the inverse identity for the transfer matrix. They solved the inverse identity for the eigenvalues for odd and even numbers of defects including the case of zero number of defects when the non-contractible loops are allowed. The structure of the inversion identity to be solvable dictates the fugacity 2 for non-contractible loops preventing the evaluation of the grand canonical partition function and the average number of loops for a fixed height of the cylinder with a given circumference.

In this paper, we solve the model of dense polymers on the cylinder by calculating the partition function of a spanning web model on a finite cylinder in the presence of cycles winding around the cylinder. Our aim is to evaluate the grand partition function of dense polymer with an arbitrary fugacity of non-contractible loops on the cylinder and to find the density of non-contractible loops per unit of the height.

For an infinite cylinder of perimeter \( L \), there is an alternative, albeit simpler, method to compute the free energy and the density of non-contractible for any \( O(n) \) model and arbitrary fugacity. This method is based on the observation that the periodic Temperley-Lieb algebra [12, 13] has a quotient with a free parameter which can be identified with the fugacity of non-contractible loops when one uses the link representation of the algebra. The same algebra has a matrix representation which takes us to the XXZ spin 1/2 quantum chain with a twist depending on the fugacity. The ground state energy is known analytically [15] for large values of \( L \) for any anisotropy related to \( n \) of \( O(n) \) of the model. This allows us to compute the density of non contractible loops. In the special case of \( n = 0 \), one can use a Jordan Wigner transformation and obtain the ground state energy for any finite \( L \).
The paper is organized as follows. In Section 2, we present the calculation of the density of non-contractible loops using the XXZ quantum chain. We start with this presentation since, unlike the lattice case, the calculation is almost trivial. We also show that the probability distribution of non-contractible loops is not Gaussian.

In Section 3 we show that for the spacial case \( n = 0 \), due to the absence of contractible loops. The \( O(n) \) loop model can be mapped on the spanning webs model. In the latter model one can generalize the Kirchhoff theorem and bring our calculation to that of determinants. The application of this technique is presented in the next sections.

The calculation of the partition function on a \( L \times N \) torus is shown in Section 4 and the result of the calculation is given in Eq. (35).

The finite cylinder (height \( N \) and perimeter \( L \)) and different boundary conditions on the top and bottom of the cylinder is considered in Section 5.

Finally, in Section 6, we consider the case of the infinite cylinder taking \( N \) to infinity and compare the results thus obtained with the those of Section 2.

II. THE DENSITY OF NON-CONTRACTIBLE LOOPS OBTAINED FROM THE XXZ QUANTUM CHAIN

We remind the reader a few facts about the periodic Temperley-Lieb algebra (PTL) and some of its representations [13]. This algebra provides the key to the calculation of the density of non-contractible loops. The PTL has \( L \) generators \( e_i (i = 1, 2, \ldots, L) \) satisfying the relations:

\[
e_i^2 = xe_i, \quad e_i e_{i \pm 1} e_i = e_i, \quad e_i e_j = e_j e_i (|i - j| > 1), \quad e_{i+L} = e_i, \quad (1)
\]

where \( x \) is a parameter.

We are going to consider the case \( L \) even only. The PTL is infinite dimensional therefore we take a quotient which makes it finite dimensional:

\[
ABA = \alpha A \quad (2)
\]

where

\[
A = \prod_{i=1}^{L/2} e_{2i}, \quad B = \prod_{i=0}^{L/2-1} e_{1+2i}. \quad (3)
\]
FIG. 3: The six link patterns configurations for $L = 4$ sites on a cylinder and two circles without sites (noncontractible loops). The open arches and circles meet behind the cylinder.

We are interested in two representations of the PTL with the quotient (2). The first one is the spin representation in which

$$e_i = \sigma_i^+ \sigma_{i+1}^- e^{i \phi/L} + \sigma_i^- \sigma_{i+1}^+ e^{-i \phi/L} - \frac{\cos(\gamma)}{2} \sigma_i^z \sigma_{i+1}^z + \frac{i}{2} \sin(\gamma)(\sigma_{i+1}^z - \sigma_i^z),$$

where

$$x = 2 \cos(\gamma), \quad \alpha = 2 \cos(\frac{\phi}{2}),$$

and $\sigma^{\pm,z}$ are Pauli matrices [12].

In the second representation, the generators act in the vector space of periodic link patterns. Each link pattern is one of the $\binom{L}{L/2}$ configurations of nonintersecting arches joining $L$ sites on a circle. One can visualize the circle on a cylinder. Besides the link patterns on the same cylinder, one takes $n$ circles with no sites on them (the index $n$ is unrelated to the $O(n)$ models). They represent $n$ non-contractible loops. In Fig. 3 we show the six configurations for $L = 4$ and $n = 2$. With few exceptions, the generators $e_i$ act on the configurations in the standard way (the non-periodic Temperley-Lieb algebra) [14]. In Fig. 4 one sees the action of the generator $e_2$ on one of the configurations of Fig. 3. The factor $x$ appears due to a contractible loop. The exceptions occur if one consider configurations having an arch of the size $L$ of the system and if the generator acts on the bond between the
FIG. 4: The action of the $e_2$ generator acting on the bond between the sites 2 and 3 in one of the configurations appearing in Fig. 3. The factor $x$ is due to the contractible loop.

FIG. 5: a) The action of the $e_2$ generator acting on the bond between the sites 2 and 3 which are the end-points of an arch of the size of the system. b) The action of $e_2$ gives a factor $\alpha$ to the new configuration.

ends of the arch (see Fig. 3k). The action of $e_2$ on the third configuration of Fig. 3 produces a new circle and one gets a configuration with $n = 3$. Instead of considering configurations with various numbers of non-contractible loops, we are going to consider configurations with no non-contractible loops but as the result of the action of $e_2$ we multiply with a fugacity $\alpha$ instead of adding a non-contractible loop (see Fig. 5b). With this rule one obtains a representation of the PTL with the quotient $[2]$.

We consider the Hamiltonian:

$$ H = -\sum_{i=1}^{L} e_i $$

(6)
Using the representation in terms of link paths of the PTL, this Hamiltonian is equal up to a factor to the Hamiltonian $H'$ obtained from the transfer matrix of the $O(n)$ models [11]. This factor is equal to the sound velocity $v_s = \frac{\pi}{\gamma} \sin(\gamma)$$
H' = \frac{H}{v_s}. \tag{7}$

In the spin representation of the PTL, using a similarity transformation, the Hamiltonian $H'$ can be written as:

$$H' = -\frac{1}{v_s} \sum_{i=1}^{L-1} \left[ (\sigma_i^+ \sigma_{i+1}^- + \sigma_i^- \sigma_{i+1}^+) \frac{\cos(\gamma)}{2} \sigma_i^z \sigma_{i+1}^z + (\sigma_L^+ \sigma_1^- e^{i\phi} + \sigma_1^+ \sigma_1^- e^{-i\phi}) - \frac{\cos(\gamma)}{2} \sigma_L^z \sigma_1^z \right]. \tag{8}$$

This Hamiltonian which is the XXZ quantum chain with a twist $\phi$ is integrable and its ground-state and energy spectrum is known [15]. In particular the ground-state energy is:

$$E'(\phi, \gamma, L) = \epsilon'_\infty L + \left( \frac{\phi^2}{4(\pi - \gamma)} - \frac{\pi}{6} \right) \frac{1}{L} + o(1/L), \tag{9}$$

where $\epsilon'_\infty$ is the bulk energy density. Notice that the choice $\alpha = 2$ and $\gamma = \pi/2$ used in [11] corresponds to the XX model. From (9) one can get two quantities of interest. Firstly, taking into account that the density of spin current is:

$$J_i = i(\sigma_i^+ \sigma_{i+1}^- - \sigma_i^- \sigma_{i+1}^+) \tag{10}$$

its average value at large values of $L$ is (see (4) and (6)):

$$J^z = -\frac{\partial E'}{\partial \phi} = -\frac{\phi}{2(\pi - \gamma)} \frac{1}{L}. \tag{11}$$

Since the coefficient in (11) is dimensionless, we expect it to be universal. A second quantity of interest is the density of non-contractible loops:

$$\rho_L(\gamma, \alpha) = -\alpha \frac{\partial E'}{\partial \alpha} = -2 \cot\left( \frac{\phi}{2} \right) J^z = \phi \cot(\phi/2) \frac{1}{\pi - \gamma} \frac{1}{L}. \tag{12}$$

Notice that the density of non-contractible loops is proportional to the current density (a physical explanation of this observation is still missing). We observe also that the dependence on $n$ in $O(n)$ has a very simple form.

There is a simple way to check if the probability distribution of non-contractible loops is Gaussian or not. Using Eqs. (5) and (9) one can compute all the moments $M_n$ of the probability distribution. They are all of order $L^{-1}$. One can check that, for example, the identity

$$M_3 = 2M_1M_2 - M_1^3 \tag{13}$$
valid for a Gauss distribution, is not satisfied.

Using the relation \( \gamma = \pi/2 \), no contractible loops

\[
\rho_L(\alpha) = \frac{4\alpha \arccos(\frac{\alpha}{2})}{\pi L \sqrt{4 - \alpha^2}} \quad \text{for } \alpha \leq 2,
\]  

and

\[
\rho_L(\alpha) = \frac{4\alpha \text{Arch}(\frac{\alpha}{2})}{\pi L \sqrt{\alpha^2 - 4}} \quad \text{for } \alpha > 2.
\]

The expressions (14) (15) are going to be compared with those obtained from spanning webs model presented in the next sections.

For the same case (\( x = 0 \)) only, one has a simple expression for the ground state energy valid for any value of \( L \):

\[
E'(\phi, \pi/2, L) = -\cos(\phi/L)/\sin(\pi/L)
\]

Using this relation one can compute the density of non-contractible loops for any fugacity and any size of the system \( L \).

III. THE SPANNING WEBS MODEL ON THE ROTATED SQUARE LATTICE

Another representation of the \( O(n) \) model relates loop configurations to clusters of bonds on sublattices of the original lattice. The square lattice of sites with integer coordinates can be divided into two sublattices, black and white. For sites of the black sublattice the sum of coordinates is even, for sites of the white one it is odd. The bijection between loop and bond configurations is shown in Fig.6. The neighboring sites of each sublattice are connected by a bond if it does not intersect the lines of elementary cell. Each connected cluster of bonds in the bulk of lattice is situated inside a loop. Each bulk cluster on the black sublattice is surrounded by a connected cluster of bonds on the white sublattice and vice versa.

The clusters of bonds corresponding to the loop configuration in Fig.2 are shown in Fig.7.

For \( n = 0 \), the absence of bulk loops contractible to a point implies the absence of isolated clusters of bonds on the black or white sublattice. The allowed bond configurations are so called spanning webs i.e. the graphs containing all vertices of a sublattice, non-contractible loops arising from the periodic boundary conditions and spanning trees connected either to open boundaries or to the non-contractible loops. Clearly, due to the bijection between bonds and elementary cells, the presence of every non-contractible loop in the bulk of the
FIG. 6: The bijection between bonds and elementary cells.

FIG. 7: The bond configuration corresponding to the loop configuration from Fig.2.

cylinder is equivalent to the presence of a pair of non-contractible polymers in the dense polymer representation. If there are no non-contractible loops in the system, polymers can propagate from the top to the bottom of the cylinder. These polymers are treated as defect lines which separate clusters of bonds one from another.

The reformulation of the dense polymer model in terms of bond configurations leads us to the standard problem of spanning graphs on the square lattice. The cycle-free spanning graphs are called spanning trees; the graphs containing a number of cycles are called spanning webs. The enumeration of spanning trees is traced back to the classical Kirchhoff theorem [16, 21]. The spanning web model appears in the statistical mechanics as the Temperley representation [17] of the dimer model solved by Kasteleyn [18] and by Temperley and Fisher.
A particular case we consider here is the spanning web model on the cylinder where non-contractible cycles are supplied by fugacity $\xi$. This decoration needs a generalization of the Kirchhoff theorem. A similar model, considered as the (1,2) logarithmic minimal model has been solved in [20] where the fugacity of non-contractible loops has been first introduced. A basic accent of the present work is the calculation of density of non-contractible cycles in a finite geometry.

We consider an oriented labeled graph $G = (V, E)$ with vertex set $V$ and set of bonds $E$. Vertices are the sites of a finite square lattice rotated by $\pi/4$ and wrapped on a cylinder. The graph $G = (V, E)$ can be considered as a sublattice of square superlattice $L$ with standard orientation, containing $N$ rows and $L$ columns of cells. Vertices of the superlattice are shown in Fig. 8 as open and filled circles. The vertex set $V$ is the sublattice of filled circles. For convenience, in the remainder we call $N$ “height” and $L$ “perimeter” of the cylinder. Let the cells $r_{ij} = (i, j)$ of $L$ be labeled by the integer coordinates $i = 1, \ldots, L$ and $j = 1, \ldots, N$, so that the row $\{r_{i1} = (i, 1) : i = 1, \ldots, L\}$ is the bottom boundary of the cylinder and the row $\{r_{iN} = (i, N) : i = 1, \ldots, L\}$ is its top boundary. Due to the periodicity in the horizontal direction, the first column $\{r_{1j} = (1, j) : j = 1, \ldots, N\}$ is arbitrarily fixed. For the sake of convenience, both $L$ and $N$ are chosen to be even. The vertex set $V$ of the rotated square lattice $G$ then consists of the vertices of the sublattice of $L$ with, say, even sum of the horizontal and vertical coordinates, i.e. $V = \{r_{ij} = (i, j) : i + j = \text{even}\}$. Explicitly, we have

$$V = \bigcup_{i=1}^{L/2} \bigcup_{j=1}^{N/2} \{(2i - 1, 2j - 1), (2i, 2j)\}. \quad (17)$$
The edges in $E$ we take oriented from a site $(i, j) \in V$ to its nearest neighbors on the right-hand side, $(i+1, j+1)$ and $(i+1, j-1)$. This direction we call “positive”, and the opposite one, from a site $(i, j) \in V$ to its nearest neighbors on the left-hand side, $(i-1, j+1)$ and $(i-1, j-1)$, we call “negative”.

We find it convenient to analyze the construction of spanning web configurations on the above oriented graph by using the arrow representation, see e.g. [21]. Accordingly, to each vertex $r \in V$ we attach an arrow directed along one of the bonds $(r, r')$ incident to it. Each arrow defines a directed bond $(r \rightarrow r')$ and each configuration of arrows $A$ on $G$ defines a spanning directed graph (digraph) $G_{sd}(A)$ with set of bonds $E_{sd}(A) = \{(r \rightarrow r') : r, r' \in V\}$ depending on $A$.

A cycle of length $k$ is a sequence of directed bonds $(r_1, r_2), (r_2, r_3), (r_3, r_4), \ldots, (r_k, r_1)$, where all $r_j$, $1 \leq j \leq k$ are distinct. If both $(r \rightarrow r')$ and $(r' \rightarrow r)$ belong to the same spanning web we say that it contains a cycle of length 2. Our aim is to study sets of spanning digraphs with no other cycles than those which wrap the cylinder. The relevant configurations will be enumerated with the aid of a generating function defined as the determinant of an appropriately constructed weight matrix. In this respect, the derivation below is nothing more than a generalization of the matrix Kirchhoff theorem [16].

We begin with the examination of the determinant expansion of the usual Laplace matrix $\Delta$ for the graph $G$. Let the vertices $r \in V$ be labeled in arbitrary order from 1 to $n = |V| = LN/2$. Then $\Delta$ has the following elements ($\alpha, \beta \in \{1, \ldots, n\}$)

$$
\Delta_{\alpha, \beta} = \begin{cases} 
  z_\alpha, & \text{if } \alpha = \beta, \\
  -1, & \text{if } \alpha \text{ and } \beta \text{ are adjacent,} \\
  0, & \text{otherwise.}
\end{cases}
$$

(18)

where $z_\alpha$ is the order of vertex $r_\alpha$ in the rotated square lattice $G$. Since the matrix $\Delta$ has a zero eigenvalue, its determinant vanishes. On the other hand, the Leibniz formula expresses the determinant of $\Delta$ as a sum over all permutations $\sigma$ of the set $\{1, 2, \ldots, n\}$:

$$
\det \Delta = \sum_{\sigma \in S_n} \sgn(\sigma) \Delta_{1, \sigma(1)} \Delta_{2, \sigma(2)} \cdots \Delta_{n, \sigma(n)} = 0,
$$

(19)

where $S_n$ is the symmetric group, $\sgn(\sigma) = \pm 1$ is the signature of the permutation $\sigma$. The identity permutation $\sigma = \sigma_{id}$ in [19] yields the term $z_1 z_2 \cdots z_n$ equal to the number of all possible arrow configurations on $G$. 


In general, each permutation $\sigma \in S_n$ can be factored into a product (composition) of disjoint cyclic permutations, say, $\sigma = c_1 \circ c_2 \cdots \circ c_k$. This representation partitions the set of vertices $V$ into non-empty disjoint subsets - the orbits $O_i$ of the corresponding cycles $c_i$, $i = 1, \ldots, k$. More precisely, if $O_i = \{v_{i,1}, v_{i,2}, \ldots, v_{i,l_i}\} \subset V$ is the orbit of $c_i$, then $\cup_{i=1}^k O_i = V$ and $\sum_{i=1}^k l_i = n$, where $l_i$ is the cardinality of the orbit $O_i$, equivalently, the length of the cycle $c_i$. The orbits consisting of just one element, if any, constitute the set $S_{fp}(\sigma)$ of fixed points of the permutation: $S_{fp}(\sigma) = \{v = \sigma(v), v \in V\}$. In the case of the identity permutation $\sigma_{id} \in S_n$ all orbits consist of exactly one element, $O_i(\sigma_{id}) = \{v_i\} \subset V$, $i = 1, \ldots, n$, and $S_{fp}(\sigma_{id}) = V$. A cycle $c_i$ of length $|c_i| = l_i \geq 2$ will be called a proper cycle. A proper cycle of length 2 corresponds to two oppositely directed edges which connect a pair of adjacent vertices: $(v_{i,1} \to v_{i,2})$, $(v_{i,2} \to v_{i,1})$. Note that the vertices of an orbit $O_i$ of cardinality $l_i = |O_i(\sigma)| \geq 3$ are connected by a closed path on $\mathcal{G}$ which can be traversed in two opposite directions: if $c_i$ is the cycle defined by $v_{i,1} \to \sigma(v_{i,1}) = v_{i,2}, \to \ldots \to \sigma(v_{i,l_i}) = v_{i,1}$, then the reverse cycle $c_i'$ can be represented as $v_{i,l_i} \to \sigma(v_{i,l_i}) = v_{i,l_i-1}, \to \ldots \to \sigma(v_{i,1}) = v_{i,1}$.

Now we take into account that the proper cycles on $\mathcal{G}$ are of even length only, hence, the signature of every permutation in the expansion of the determinant depends on the number of proper cycles in its factorization, i.e., if $\sigma = c_1 \circ c_2 \cdots \circ c_p$, where $|c_i| \geq 2$, $i = 1, \ldots, p$, then $\text{sgn}(\sigma) = (-1)^p$. Thus, the terms in Eq. (19) can be rearranged according to the number $p$ of disjoint proper cycles as follows:

$$
\prod_{i=1}^n z_i + \sum_{p=1}^{[n/2]} (-1)^p \sum_{\sigma = c_1 \circ \cdots \circ c_p} \prod_{i=1}^p \Delta_{v_i, c_i(v_i)} \Delta_{c_i(v_i), c_i^2(v_i)} \cdots \Delta_{c_i^{l_i-1}(v_i), v_i} \prod_{j \in S_{fp}(\sigma)} z_j. \quad (20)
$$

Here $c_i^k$ is the $k$-fold composition of the cyclic permutation $c_i$ of even length $l_i$, $v_i \in O_i(\sigma)$, so that $c_i^{k-1}(v_i) \neq c_i^k(v_i)$ and $c_i^0(v_i) = v_i$. Note that all non-vanishing off-diagonal elements are equal to $-1$.

The above expansion reveals the following features: (i) As expected, all spanning digraphs on $\mathcal{G}$ have at least one proper cycle; (ii) Each term with $S_{fp}(\sigma) \neq \emptyset$ represents a set of $\prod_{j \in S_{fp}} z_j$ distinct spanning digraphs which have in common the specified cycles $c_1, \ldots, c_p$, and differ in the oriented edges outgoing from the vertices $j \in S_{fp}(\sigma)$. These oriented edges may form cycles on their own which do not enter into the list $c_1, \ldots, c_p$; (iii) Since the sets $\cup_{i=1}^p O_i$ and $S_{fp}(c_1, \ldots, c_p)$ are disjoint, the proper cycles formed by the oriented edges incident to the fixed points of a given permutation $\sigma = c_1 \circ c_2 \circ \cdots \circ c_p$ should enter into the
enlarged list of cycles $c_1, c_2, \ldots, c_p, \ldots, c_{p'}$, $p' > p$, corresponding to the cycle decomposition of another permutation $\sigma'$.

![FIG. 9: Spanning digraph generated by a single term in the determinant expansion of the Laplacian matrix (see text).](image)

For example, consider the determinant of the Laplacian matrix of a cylinder of height 3 and perimeter 6 shown in Fig. 9. The set of vertices $V$ consists of filled circles marked by $1, 2, \ldots, 9$. The set of oriented edges is a collection of 24 inclined vectors of type $1 \to 4, 1 \to 6, 4 \to 1, 6 \to 1, \ldots$. The corresponding Leibniz expansion contains the term

$$(-1)^2(\Delta_{1,4}\Delta_{4,7}\Delta_{7,6}\Delta_{6,1})(\Delta_{2,5}\Delta_{5,2})\Delta_{3,3}\Delta_{8,8}\Delta_{9,9} \quad (21)$$

which represents $z_3z_8z_9 = 8$ spanning digraphs on $G$ with 2 specified cycles and all possible oriented bonds outgoing from the vertices 3, 8 and 9.

As noticed first in [21], the expansion (20) parallels in form the inclusion-exclusion principle in combinatorial mathematics. Indeed, let $c_1, c_2, \ldots, c_m$ be the list of all possible proper cycles on $G$, labeled in an arbitrary order. Define $A_i, i = 1, 2, \ldots, m$ as the set of all spanning digraphs on $G$ containing the particular cycle $c_i$. Then, expansion (20) can be written in the form of the inclusion-exclusion principle:

$$|\bigcup_{i=1}^{m} A_i| - \sum_{i=1}^{m} |A_i| + \sum_{1 \leq i < j \leq m} |A_i \cap A_j| - \sum_{1 \leq i < j < k \leq m} |A_i \cap A_j \cap A_k| + \cdots - (-1)^{m+1}|A_1 \cap \cdots \cap A_m| \quad (22)$$

which holds for any finite sets $A_1, A_2, \ldots, A_m$, where $|A|$ is the cardinality of the set $A$. This sum equals zero, because all spanning digraphs on $G$ have at least one proper cycle $c_i$, 

The first term of the expansion originates from the term $\prod_{i=1}^{m} z_i$ in Eq. (20) and represents the set of all possible cycles formed by oriented edges incident to every vertex of $\mathcal{G}$. To obtain the number of spanning webs, one has to subtract all the digraphs having contractible cycles, and leave all those with non-contractible cycles wrapping the cylinder. In order to keep the number of non-contractible cycles we have to change the weights of the nondiagonal terms in such a way, that every cycle containing difference between the numbers of positive and negative steps equal to $+L$ or $-L$, depending on the orientation, enters the determinant expansion with the opposite sign and sums up with the corresponding non-contractible cycle, generated by the arrows representing the diagonal term $\prod_{i=1}^{m} z_i$. At that, all the proper contractible cycles should keep their sign in the expansion in order to cancel out. This can be readily achieved by using the fact that every contractible cycle contains equal numbers of positive and negative steps.

Now we define a matrix $D$, associated with the graph $\mathcal{G}$, such that $\det D$ is the generating function of all spanning digraphs on $\mathcal{G}$ which have no contractible cycles. The elements $D_{\alpha,\beta}$, $\alpha, \beta = 1, \ldots, n = LN/2$ of $D$ are explicitly given as:

$$
(D_{L,N}(\omega))_{\alpha,\beta} = \begin{cases} 
z_{\alpha}, & \text{if } \alpha = \beta, 
-b, & \text{if } r_\beta \text{ is right neighbor of } r_\alpha, 
-b^{-1}, & \text{if } r_\beta \text{ is left neighbor of } r_\alpha, 
0, & \text{otherwise.}
\end{cases}
$$

(23)

Here $b = \omega^{1/L} e^{-i\pi/L}$, the condition “$r_\beta$ is right (left) neighbor of $r_\alpha$” means that if $r_\alpha = (i, j)$ then $r_\beta = (i+1, j \pm 1)$ ($r_\beta = (i-1, j \pm 1)$). Note that all closed paths which do not wrap the cylinder contain an equal number of edges with either orientations, hence, their weight in $\det D$ remains the same as in $\det \Delta$. Therefore, all the configurations which contain such closed paths (contractible cycles) cancel out in the expansion of $\det D$. On the other hand, cycles generated by off-diagonal elements that wrap the cylinder change their sign, because they contain edges oriented in one direction exceeding by $L$ the number of edges in the opposite direction. This amounts to the total factor of $b^L = -\omega$ or $b^{-L} = -\omega^{-1}$ depending on the orientation. Therefore, each non-contractible cycle with a given orientation is counted twice, however, with different weight - once it enters into the determinant expansion with unit weight, being generated by diagonal elements of the matrix $D$, and second time it enters with a factor $\omega$ or $\omega^{-1}$ (depending on the orientation) as generated by off-diagonal
elements of that matrix. Thus, the total number of non-contractible cycles, irrespective of their origin and orientation, is given by the coefficient in front of the corresponding power of $\omega + \omega^{-1} + 2 \equiv \xi$ in the series expansion of the partition function. In general, besides the non-contractible cycles, the average number of which is controlled by fugacity $\xi$, the spanning digraph contains tree subgraphs connected to them. All branches of the trees can be generated only by the diagonal elements of $D$ and, hence, carry unit weight.

**IV. THE PARTITION FUNCTION ON A TORUS**

The average density (per unit height of the cylinder) of the noncontractible cycles as, a function of the fugacity $\xi$ is defined as

$$\rho_{L,N}(\xi) = \frac{1}{N} \frac{\partial}{\partial \xi} \ln \det D_{L,N}(\omega(\xi)).$$

(24)

To calculate the determinant in the above expression we make some transformations which allow for an easy diagonalization of the matrix $D_{L,N}(\omega)$. First of all, the decomposition of the vertex set $V$ of the rotated square lattice $G$ suggests its rearrangement by combining all pairs of nearest neighbors $(2i - 1, 2j - 1)$ and $(2i, 2j)$, $i = 1, 2, \ldots, L' = L/2$, $j = 1, 2, \ldots, N' = N/2$ into two-site unit cells. Thus we obtain a square $L' \times N'$ array of $LN/4$ unit cells with the connectivity of a triangular lattice. Under neglect of the boundary effects at the top and bottom of the cylinder, we describe the weighted connectivity of the sites in an unit cell with their neighbors in $G$, taking into account the bond orientation, by introducing the following $2 \times 2$ matrices:

$$a(0,0) = \begin{pmatrix} 0 & b^{-1} \\ b & 0 \end{pmatrix}, \quad a(1,0) = \begin{pmatrix} 0 & 0 \\ b^{-1} & 0 \end{pmatrix}, \quad a(0,1) = \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix}, \quad a(1,1) = \begin{pmatrix} 0 & 0 \\ b^{-1} & 0 \end{pmatrix},$$

$$a(-1,0) = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}, \quad a(0,-1) = \begin{pmatrix} 0 & b^{-1} \\ 0 & 0 \end{pmatrix}, \quad a(-1,-1) = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}.$$

Now the matrix $D_{L,N}(\omega)$, see Eq. (23), can be written as

$$[4I_2 - a(0,0)] \otimes I_{L'} \otimes I_{N'} - a(1,0) \otimes R_{L'} \otimes I_{N'} - a(-1,0) \otimes R_{L'}^T \otimes I_{N'} - a(0,1) \otimes I_{L'} \otimes R_{N'},$$

$$-a(0,-1) \otimes I_{L'} \otimes R_{N'}^T - a(1,1) \otimes R_{L'} \otimes R_{N'} - a(-1,-1) \otimes R_{L'}^T \otimes R_{N'}^T.$$
Here $R_M$ is the $M \times M$ matrix

$$R_M = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \ldots & 0 & 1 \\ 1 & 0 & \ldots & 0 & 0 \end{pmatrix}, \quad (26)$$

and $R_M^T$ is the matrix transposed of $R_M$. Now we note that both $R_M$ and $R_M^T$ are diagonalized by the similarity transformation

$$S_M^{-1} R_M S_M = \text{diag}\{e^{i2\pi m/M}, m = 1, 2, \ldots, M\},$$

where the $S_M$ is the matrix with elements

$$(S_M)_{n,m} = M^{-1/2} e^{i2\pi mn/M}, \quad m, n = 1, 2, \ldots, M. \quad (27)$$

Since $R_M R_M^T = I_M$, we have

$$S_M^{-1} R_M^T S_M = \text{diag}\{e^{-i2\pi m/M}, m = 1, 2, \ldots, M\}.$$

Therefore, with the similarity transformation generated by the matrix $I_2 \otimes S_{L'} \otimes S_{N'}$ we can diagonalize the matrix $D_{L,N}(\omega)$ in the $L' = L/2$ and $N' = N/2$ dimensional subspaces. Then for the determinant we readily obtain

$$\det D_{L,N}(\omega) = \prod_{m=1}^{L'} \prod_{n=1}^{N'} \det Q(2\pi m/L', 2\pi n/N') \quad (28)$$

where $Q(\theta_1, \theta_2)$ is the $2 \times 2$ matrix

$$Q(\theta_1, \theta_2) = 4I_2 - a(0, 0) - a(1, 0)e^{i\theta_1} - a(-1, 0)e^{-i\theta_1} - a(0, 1)e^{i\theta_2} - a(1, 1)e^{i\theta_1+i\theta_2} - a(-1, -1)e^{-i\theta_1-i\theta_2}. \quad (29)$$

It is convenient to cast its determinant in the form

$$\det Q(\theta_1, \theta_2) = 4 \cos^2(\theta_2/2) \left[ \frac{4}{\cos^2(\theta_2/2)} - 2 - 2 \cos(\theta_1 + 2i \ln b) \right], \quad (30)$$

where $2i \ln b = (i \ln \omega + \pi)/L'$. Thus, from Eq. (28) we obtain the $\omega$-dependent part of the partition function

$$Z_{L',N'}(\omega) = \prod_{m=1}^{L'} \prod_{n=1}^{N'} \left[ \frac{4}{\cos^2(\pi n/N')} - 2 - 2 \cos(2\pi m/L' + (i \ln \omega + \pi)/L') \right]. \quad (31)$$
Next we set
\[
\frac{4}{\cos^2(\pi n/N')} - 2 = A^2(\pi n/N') + A^{-2}(\pi n/N'),
\]
where
\[
A^2(\pi n/N') = \frac{(1 + |\sin(\pi n/N')|)^2}{\cos^2(\pi n/N')}.
\]
Now, making use of the identity
\[
\prod_{m=1}^{L'} [A^2 + A^{-2} - 2 \cos (2\pi m/L' + \alpha)] = A^{2L'} + A^{-2L'} - 2 \cos (L' \alpha),
\]
with \(\alpha = (i \ln \omega + \pi)/L'\), we perform exactly the product over \(m\):
\[
Z_{L',N'}(\omega) = \prod_{n=1}^{N'} \left[ \frac{(1 + \sin(\pi n/N'))^L}{\cos^L(\pi n/N')} + \frac{(1 - \sin(\pi n/N'))^L}{\cos^L(\pi n/N')} + \omega + \omega^{-1} \right].
\]
Strictly speaking, this expression is valid for spanning webs on a torus, since we have not considered boundary conditions at edges of the cylinder properly. In the next section, we derive the exact partition function taking into account the closed or open boundary at the top and bottom of the cylinder.

V. CYLINDRICAL BOUNDARY CONDITIONS

In the preceding section, we computed the partition function of spanning webs on the torus. Here, we present a technique, which allows consideration of different boundary conditions shown in Fig. 10. Vertical lines correspond to a single line on the cylinder, due to periodic boundary conditions in the horizontal direction. On the top and the bottom of cylinder, boundaries may be closed or open.

As before, \(L\) is the perimeter of cylinder and \(N\) is its height. The total number of vertices in the graph is \(|V| = LN/2\). For instance, the top left lattice in Fig. 10 has \(L = 10, N = 7\). It can be seen from the figure that there is a difference between the cases of odd and even \(N\) (in the latter case the lower edge is shifted with respect to the top one). In our computations \(L = 2L'\) is even, whereas \(N\) is of any parity.

A. Partition function of the model with open boundaries

In the case of both open boundaries, the diagonal elements of the matrix (23) are constant: \(z_\alpha = 4\), because every vertex has four outgoing edges. The partition function is \(\det D\).
Operator $D$ acts on the linear space $V$ of complex numbers assigned to vertices of the graph. Consider $N$-dimensional subspace of $V$ with elements of the form shown on Fig. (11) where $x_1, x_2, \ldots x_n$ are arbitrary, $\alpha$ is fixed. Due to periodicity in the horizontal direction, $\alpha^{2L'} = 1$, so the possible values of $\alpha$ are $\alpha_l = e^{i\pi l/L'}, \ l = 0, 1, \ldots, L' - 1$. The subspace, corresponding to the $\alpha_l$ is denoted by $V_l$, $l = 0, 1, \ldots, L' - 1$. Then, we can represent $V$ as the direct sum of subspaces $V_l$: $V = \bigoplus_{l=0}^{L'-1} V_l$.

The key point is $D$-invariance of each subspace $V_l$ what allows to express the determinant of $D$ as the product:

$$\det D = \prod_{l=0}^{L'-1} \det D \bigg|_{V_l}.$$
The restriction of $D$ on subspace $V_l$ is

$$D \bigg|_{V_l} = A_{l,N} = \begin{bmatrix} 4 & -c_l & & & & & \\ -c_l & 4 & -c_l & & & & \\ & -c_l & 4 & -c_l & & & \\ & & \vdots & \ddots & \ddots & \ddots & \\ & & & & -c_l & 4 & \\ & & & & & & 4 \end{bmatrix},$$

where $c_l = b\alpha_l + \frac{1}{b\alpha_l}$ and, as above, $b = \omega^{1/L} e^{-i\pi/L}.$

The partition function $Z_{L,N}^{op,op}$ (the upper indices mean the pair of open boundaries) is

$$Z_{L,N}^{op,op} = \det D = \prod_{l=0}^{L'-1} \det A_{l,N}.$$  \hspace{1cm} (36)

The determinant of three-diagonal matrix $A_{l,N}$ is computed by using the recurrence

$$\det A_{l,N} = 4 \det A_{l,N-1} - c_l^2 \det A_{l,N-2},$$

with the solution

$$\det A_{l,N} = \frac{q_{l+1}^{N+1}(l) - q_{l+2}^{N+1}(l)}{q_1(l) - q_2(l)}, \quad q_{1,2}(l) = 2 \pm \sqrt{4 - c_l^2}.$$  

We represent $b$ as $b = e^{i\varphi}$ and write

$$c_l = \exp\left[i(\varphi + \frac{\pi l}{L'})\right] + \exp\left[-i(\varphi + \frac{\pi l}{L'})\right] = 2 \cos(\varphi + \frac{\pi l}{L'}).$$
$q_{1,2}(l) = 2 \pm i \left( b_{\alpha l} - \frac{1}{b_{\alpha l}} \right) = 2 \pm 2 \cos(\varphi + \frac{\pi l}{L} - \frac{\pi}{2}) = 2 \pm 2 \cos(\psi + \frac{\pi l}{L})$. where we denoted

$$\psi = \varphi - \frac{\pi}{2}$$

(37)

Now, the partition function (36) is

$$Z_{L,N}^{op,op} = \prod_{l=0}^{L'-1} \det A_{l,N} = \prod_{l=0}^{L'-1} \frac{q_1^{N+1}(l) - q_2^{N+1}(l)}{q_1(l) - q_2(l)} =$$

$$= \prod_{l=0}^{L'-1} 4^N \frac{\cos^{2N+2}(\frac{\psi}{2} + \frac{\pi l}{2L}) - \sin^{2N+2}(\frac{\psi}{2} + \frac{\pi l}{2L})}{\cos^2(\frac{\psi}{2} + \frac{\pi l}{2L}) - \sin^2(\frac{\psi}{2} + \frac{\pi l}{2L})}.$$  

We use the equality (3.19) from [7]:

$$\frac{\cos^{2N+2} u - \sin^{2N+2} u}{\cos^2 u - \sin^2 u} = \begin{cases} 
\frac{N+1}{2^N} \prod_{j=1}^{(N-1)/2} \left( \frac{1}{\sin^2 \frac{\pi j}{N+1}} - \sin^2 2u \right), & \text{odd } N \\
\frac{1}{2^N} \prod_{j=1}^{N/2} \left( \frac{1}{\sin^2 \frac{\pi (j-1/2)}{N+1}} - \sin^2 2u \right), & \text{even } N 
\end{cases}$$

to obtain

$$Z_{L,N}^{op,op} = \begin{cases} 
\prod_{l=0}^{L'-1} 2^N (N+1) \prod_{j=1}^{(N-1)/2} \left( \frac{1}{\sin^2 \frac{\pi j}{N+1}} - \sin^2 (\psi + \frac{\pi l}{L}) \right), & \text{odd } N \\
\prod_{l=0}^{L'-1} 2^N \prod_{j=1}^{N/2} \left( \frac{1}{\sin^2 \frac{\pi (j-1/2)}{N+1}} - \sin^2 (\psi + \frac{\pi l}{L}) \right), & \text{even } N.
\end{cases}$$

(38)

Changing the order of products, we rewrite the partition function as

$$Z_{L,N}^{op,op} = \begin{cases} 
2^{L'} (N+1)^{L'} \prod_{j=1}^{(N-1)/2} \prod_{l=0}^{L'-1} \left( \frac{4}{\sin^2 \frac{\pi j}{N+1}} - 2 + 2 \cos(2\psi + \frac{\pi l}{L'}) \right), & \text{odd } N \\
\prod_{j=1}^{N/2} \prod_{l=0}^{L'-1-1} \left( \frac{4}{\sin^2 \frac{\pi (j-1/2)}{N+1}} - 2 + 2 \cos(2\psi + \frac{\pi l}{L'}) \right), & \text{even } N.
\end{cases}$$

(39)

or, introducing $p_j$:

$$p_j = \begin{cases} 
\left( 1 + \cos \frac{\pi j}{N+1} \right)^2, & \text{odd } N \\
\left( 1 + \cos \frac{\pi (j-1/2)}{N+1} \right)^2, & \text{even } N.
\end{cases}$$

(40)

as

$$Z_{L,N}^{op,op} = \begin{cases} 
2^{L'} (N+1)^{L'} \prod_{j=1}^{(N-1)/2} \prod_{l=0}^{L'-1} \left( p_j + p_j^{-1} - 2 \cos(2\varphi + \frac{\pi l}{L'}) \right), & \text{odd } N \\
\prod_{j=1}^{N/2} \prod_{l=0}^{L'-1-1} \left( p_j + p_j^{-1} - 2 \cos(2\varphi + \frac{\pi l}{L'}) \right), & \text{even } N.
\end{cases}$$

(41)
Due to \( [34] \), we get

\[
Z_{L,N}^{\text{op,op}} = \begin{cases} 
2^{L'}(N+1)^L \prod_{j=1}^{(N-1)/2} \left[ p_j^{L'} + p_{j-L'} - 2 \cos(2L' \varphi) \right], & \text{odd } N \\
\prod_{j=1}^{N/2} \left[ p_j^{L'} + p_{j-L'} - 2 \cos(2L' \varphi) \right], & \text{even } N.
\end{cases}
\tag{42}
\]

and obtain finally the exact expression for partition function for the open boundaries:

\[
Z_{L,N}^{\text{op,op}} = \begin{cases} 
2^{L'}(N+1)^L \prod_{j=1}^{(N-1)/2} \left[ \left( 1 + \cos \frac{\pi j}{N+1} \right) \right]^{L/2} \left( 1 - \cos \frac{\pi j}{N+1} \right)^L + \omega + \omega^{-1}, & \text{odd } N \\
\prod_{j=1}^{N/2} \left[ \left( 1 + \cos \frac{\pi j}{N+1} \right) \right]^{L/2} \left( 1 - \cos \frac{\pi j}{N+1} \right)^L + \omega + \omega^{-1}, & \text{even } N.
\end{cases}
\tag{43}
\]

**B. Partition function for closed boundaries**

If the top boundary is closed and the bottom one is closed or open, the matrices \( A_{l,N} \) are transformed into \( C_{l,N} \) or \( B_{l,N} \):

\[
C_{l,N} = \begin{bmatrix} 2 & -c_l \\ -c_l & 4 & -c_l \\ & -c_l & 4 & -c_l \\ \vdots \\ & & & & -c_l & 2 \end{bmatrix}, \quad B_{l,N} = \begin{bmatrix} 2 & -c_l \\ -c_l & 4 & -c_l \\ & -c_l & 4 & -c_l \\ \vdots \\ & & & & -c_l & 4 \end{bmatrix}.
\]

It is easy to find the identities

\[
det B_{l,N} = det A_{l,N} - 2 det A_{l,N-1} = \frac{q_1^N(q_1 - 2) - q_2^N(q_2 - 2)}{q_1 - q_2} = \frac{q_1^N(q_1 - q_2) - q_2^N(q_2 - q_1)}{2(q_1 - q_2)} = \frac{q_1^N(l) + q_2^N(l)}{2},
\]

and

\[
det C_{l,N} = det B_{l,N} - 2 det B_{l,N-1} = \frac{q_1^{N-1}(q_1 - 2) + q_2^{N-1}(q_2 - 2)}{2} = \frac{(q_1 - q_2)(q_1^{N-1} - q_2^{N-1})}{4} = \frac{(q_1(l) - q_2(l))^2}{4} det A_{l,N-2}.
\]

Then the partition function \( Z_{L,N}^{\text{cl,cl}} \) for the closed-closed boundary conditions can be expressed via \( Z_{L,N}^{\text{op,op}} \).
where \( Z_{L,N}^{\text{cl,cl}} = \prod_{l=0}^{L-1} \det C_{l,N} = \prod_{l=0}^{L-1} \frac{(q_1(l) - q_2(l))^2}{4} \prod_{l=0}^{L-1} \det A_{l,N-2} = \xi Z_{L,N-2}^{\text{op,op}} \). Because
\[
\prod_{l=0}^{L-1} \frac{(q_1(l) - q_2(l))^2}{4} = \prod_{l=0}^{L-1} 4 \cos^2(\psi + \frac{\pi l}{L}) =
\]
\[
= (-1)^L \prod_{l=0}^{2L-1} 2 \cos \left( \psi + \frac{2\pi l}{2L} \right) = (-1)^L \prod_{l=0}^{2L-1} \left[ i + i^{-1} - 2 \cos \left( \psi + \frac{2\pi l}{2L} \right) \right] =
\]
\[
= (-1)^L \left[ i^{2L'} + i^{-2L'} - 2 \cos 2L' \psi \right] = 2 + (-1)^{L'+1} 2 \cos 2L' \psi = \xi.
\]
In the same way, we calculate \( Z_{L,N}^{\text{op,cl}} \) for the open-closed conditions:
\[
Z_{L,N}^{\text{op,cl}} = \prod_{l=0}^{L-1} \det B_{l,N} = \prod_{l=0}^{L-1} \frac{q_1^N(l) + q_2^N(l)}{2} =
\]
\[
= 2^{-L'} \prod_{l=0}^{L-1} \frac{q_1^N(l) - q_2^N(l)}{q_1(l) - q_2(l)} \frac{q_1(l) - q_2(l)}{q_1^N(l) - q_2^N(l)} = 2^{-L'} Z_{L,N/2-1}^{\text{op,op}} Z_{L,N}^{\text{op,op}},
\]
and \( Z_{L,N} \) for the torus:
\[
Z_{L,N}^{\text{torus}} = \prod_{l=0}^{L-1} \left( q_1^{N/2}(l) - q_2^{N/2}(l) \right)^2 = \prod_{l=0}^{L-1} \left( \frac{q_1^{N/2}(l) - q_2^{N/2}(l)}{q_1(l) - q_2(l)} \right)^2 (q_1(l) - q_2(l))^2 =
\]
\[
= 4^{L'} \left( Z_{L,N/2-1}^{\text{op,op}} \right)^2 \prod_{l=0}^{L-1} \frac{(q_1(l) - q_2(l))^2}{4} = 4^{L'} \xi \left( Z_{L,N/2-1}^{\text{op,op}} \right)^2.
\]
From these expressions, it can be easily seen that the density of non-contractible loops is equal for different boundary conditions in the limit \( N \to \infty \).

**VI. DENSITY OF NON-CONTRACTIBLE LOOPS**

Since we are interested in the case of the density on an infinitely long cylinder, we expect the boundary corrections to vanish in the limit \( N' \to \infty \). To take this limit, we factor out another \( \omega \)-independent term from \( Z_{L',N'}(\omega) \),
\[
Z_{L',N'}(\omega) = \left[ \prod_{n=1}^{N'} \frac{[1 + \sin(\pi n/N')]^L}{\cos^L(\pi n/N')} \right]
\times \prod_{n=1}^{N'} \left[ 1 + (\omega + \omega^{-1}) \frac{\cos^L(\pi n/N')}{[1 + \sin(\pi n/N')]^L} + \frac{\cos^{2L}(\pi n/N')}{[1 + \sin(\pi n/N')]^{2L}} \right],
\]
and turn back to the density definition (24):

\[
\rho_{L,N}(\xi) = \frac{\xi}{N} \sum_{n=1}^{N'} \ln \left[ 1 + (\xi - 2) \frac{\cos^L(\pi n/N')}{[1 + \sin(\pi n/N')]^L} + \frac{\cos^{2L}(\pi n/N')}{[1 + \sin(\pi n/N')]^{2L}} \right].
\]

(46)

Hence, in the limit \(N' \to \infty\)

\[
\rho_{L,\infty}(\xi) = \frac{\xi}{2\pi} \int_{0}^{\pi} \frac{g^L(\phi) \, d\phi}{1 + (\xi - 2)g^L(\phi) + g^{2L}(\phi)},
\]

(47)

where \((L = 2L' \text{ is even})\)

\[
g(\phi \in [0, \pi]) = \frac{|\cos(\phi)|}{[1 + \sin(\phi)]} \in [0, 1], \quad \max_{\phi \in [0, \pi]} g(\phi) = g(0) = g(\pi) = 1.
\]

(48)

Therefore, when \(L \gg 1\) the essential contribution in \(\rho_{L,\infty}(\xi)\) comes from the integration over the two small intervals \(0 \leq \phi \leq \varepsilon\) and \(\pi - \varepsilon \leq \phi \leq \pi\) with \(\varepsilon \ll 1\). To the leading-order in \(L \gg 1\) it suffices to take the linear term in the expansion

\[
\ln g(\phi) = -\phi + O(\phi^2), \quad \phi \to 0^+, \quad \text{or} \quad \phi \to \pi,
\]

(49)

which yields

\[
\rho_{L,\infty}(\xi) \approx \frac{\xi}{\pi} \int_{0}^{\varepsilon} \frac{e^{-L\phi} \, d\phi}{1 + (\xi - 2)e^{-L\phi} + e^{-2L\phi}} = \frac{\xi}{\pi L} \int_{e^{-L\varepsilon}}^{1} \frac{dy}{1 + (\xi - 2)y + y^2}.
\]

(50)

Extending the lower limit in the latter integral to \(y = 0\), we finally obtain the general expression

\[
\rho_{L,\infty}(\xi) = \begin{cases} 
\frac{\xi}{\pi L \sqrt{\xi(\xi-4)}} \ln \frac{\xi + \sqrt{\xi(\xi-4)}}{\xi - \sqrt{\xi(\xi-4)}}, & \text{if } (\xi - 4) > 0, \\
\frac{2\varepsilon}{\pi L} & \text{if } (\xi - 4) = 0, \\
\frac{2\varepsilon}{\pi L \sqrt{\xi(4-\xi)}} \left[\tan^{-1} \frac{\xi}{\sqrt{\xi(4-\xi)}} - \tan^{-1} \frac{\xi - 2}{\sqrt{\xi(4-\xi)}}\right], & \text{if } (\xi - 4) < 0.
\end{cases}
\]

(51)

We are going to compare the obtained result with Eq.(14) in the interval of the loop fugacity \(0 \leq \alpha \leq 2\) corresponding to real values of the twist parameter \(\phi\). To this end, we transform the bottom expression in Eq.(51)

\[
\frac{2\varepsilon}{\pi L \sqrt{\xi(4-\xi)}} \left[\tan^{-1} \frac{\xi}{\sqrt{\xi(4-\xi)}} - \tan^{-1} \frac{\xi - 2}{\sqrt{\xi(4-\xi)}}\right] = \frac{2\xi}{\pi L \sqrt{\xi(4-\xi)}} \arccos \left(\frac{\sqrt{\xi}}{2}\right)
\]

(52)
and remember that each cycle in the spanning web model corresponds to a pair of non-contractible loops with fugacity $\xi = \alpha^2$. Then, the density of non-contractible loops in the dense polymer model is

$$\rho_L(\alpha) = \frac{4\alpha \arccos(\alpha/2)}{\pi L \sqrt{(4 - \alpha^2)}}$$

in full agreement with (14).

For $\xi > 4$ and $\alpha > 2$, we use the formula

$$\ln\left(\frac{1 + x}{1 - x}\right) = 2\text{Arth}(x)$$

and get

$$\rho_L(\alpha) = \frac{4\alpha \text{Arch}(\alpha/2)}{\pi L \sqrt{(\alpha^2 - 4)}}$$

We see that the formula for $\alpha > 2$ corresponds to the quantum chain result (15) with the complex twist

$$\phi = i\Omega, \quad \alpha = 2 \cosh(\Omega/2).$$

The crucial check of Eq.(53) is the value of density $\rho_L(\alpha)$ for $\alpha = \sqrt{2}$, when the fugacity of non-contractible cycles in the spanning web model is $\xi = 2$. In this case, the non-contractible cycles enter into the partition function as two non-weighted sequences of bonds oriented clockwise and anticlockwise. Due to symmetry of the $O(n)$ model, the spread of each sequence in horizontal direction (that is $L$ by the definition of loops) coincides in average with that in the direction of the cylinder axis. Each two loops on the rotated lattice (Fig.8) are separated by a loop on the dual lattice. Thus, the spanning web configuration in the vertical direction is a sequence of sandwiches of loops and dual loops of average thickness $2L$. Then, the density of web cycles is $1/(2L)$ and the density of non-contractible loops $\rho_L(\sqrt{2}) = 1/L$.

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