A generating polynomial for the pretzel knot

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Abstract

We collect statistics which consist of the coefficients in the expansion of the generating polynomials that count the Kauffman states associated with certain classes of pretzel knots having \( n \) tangles, of \( r \) half-twists respectively.

Keywords: generating polynomial, shadow diagram, Kauffman state.

1 Introduction

The generating polynomial for the shadow diagram of the knot \( K \) provides a refinement of counting the corresponding Kauffman states [1]. By state is meant the diagram obtained by splitting each vertex representing the initial diagram crossings, i.e., each \( \bigotimes \) to \( \bigotimes \), and repasting the edges as either \( \bigotimes \) or \( \bigotimes \). The generating polynomial for the knot \( K \) is then defined as the summation which is taken over all its states, namely

\[
K(x) = \sum_S x^{|S|},
\]

with \( |S| \) denoting the number of Jordan curves in the state diagram \( S \). For instance, the generating polynomial for the Hopf link is \( L(x) = 2x^2 + 2x \) (see Figure 1).

If \( K \) and \( K' \) are two arbitrary diagrams, and \( \bigodot \) denotes the unknot diagram, then we have the following properties and notations [2]:

(i) \( \bigodot(x) = x \); 
(ii) \( (\bigodot \uplus \bigodot \uplus \cdots \uplus \bigodot) (x) := (\bigodot \bigodot \cdots \bigodot) (x) = x^n \); 
(iii) \( (K \uplus \bigodot) (x) = xK(x) \);
Figure 1: The states of the Hopf link.

(iv) \((K \sqcup K')(x) = K(x).K'(x)\);

(v) \((K \# \bigcirc)(x) = K(x)\);

(vi) \(K_n(x) := (K \# K \# \cdots \# K) (x) = x \left(x^{-1} K(x)\right)^n\), with \(K_0 = \bigcirc\);

(vii) \((K \# K')(x) = x^{-1} K(x).K'(x)\),

where \# and \(\sqcup\) are respectively the connected sum and the disjoint union. Moreover, if we let \(\overline{K}\) denote the closure of \(K\), i.e., the connected sum with itself, then there exist two polynomials \(\alpha, \beta \in \mathbb{N}[x]\) such that

\[
\overline{K}(x) = x^2 \alpha(x) + x \beta(x), \quad \text{with} \quad K(x) = x^2 \beta(x) + x \alpha(x). \quad (2)
\]

With the notation and property in (vi) we obtain \(\overline{K}_0(x) = (\bigcirc \bigcirc)(x) = x^2\) and

\[
\overline{K}_n(x) = \alpha(x) \overline{K}_{n-1}(x) + \beta(x) K_{n-1}(x) = (\alpha(x) + x \beta(x))^n + (x^2 - 1) \alpha(x)^n. \quad (3)
\]

We can interpret (2) as follows: given the closure \(\begin{array}{c} K \end{array}\) of a knot diagram \(\begin{array}{c} K \end{array}\), its state diagrams can be divided into two subsets that are respectively counted by \(x^2 \alpha(x)\) and \(x \beta(x)\) as represented in Figure 2.

Figure 2: The two subsets of states associated with the closure of a knot.

(a) States counted by \(x^2 \alpha(x)\).

(b) States counted by \(x \beta(x)\).

In this note, we shall take advantage of these properties and establish the generating polynomial for a particular class of the pretzel knots.
2 Pretzel knot

A pretzel knot $P_{n,r} := P(r, r, \ldots, r)$ [4] is a knot composed of $n$ pairs of strands twisted $r$ times and attached along the tops and bottoms as in Figure 3(a).

Figure 3: The shadow diagram for the pretzel knot and the corresponding connected sums for constructing it.

If $F_r$ denotes the $r$-foil as pictured in Figure 3(b), then we have $P_{n,r} := (F_r)_n$. For the convenience, we set $F_0 = \bigcirc$ and $(F_r)_0 = \bigcirc$ so that $(F_0)_0 = \bigcirc$ and $(F_0)_n = \bigcirc\bigcirc\bigcirc\cdots\bigcirc$ for $n \geq 1$. Figure 4 displays some pretzel knots for small values of $n$ and $r$.

3 Generating polynomial

We begin with the generating polynomial for the closure of the $r$-foil (see Figure 3(b)) which yields the $r$-twist loop (see Figure 4(e)).

Lemma 1 ([2]). The generating polynomials for the $r$-twist loop and the $r$-foil knot are respectively given by

$$T_r(x) = x(x + 1)^r$$

and

$$F_r(x) = \overline{T_r(x)} = (x + 1)^r + x^2 - 1.$$  \hspace{1cm} (6)

We shall deduce the two polynomials $\alpha_r$, $\beta_r$ associated with closure of the $r$-foil with the help of formula (5).

Lemma 2. The following expression holds for $\overline{T_r(x)}$

$$\overline{T_r(x)} = x^2\alpha_r(x) + x\beta_r(x),$$

where $\alpha_r(x) := \frac{(x + 1)^r - 1}{x}$ and $\beta_r(x) = 1$.  \hspace{1cm} (7)
Figure 4: For some values of \( n \) and \( r \) we have: (a) a disjoint union of \( n \) unknots \((r = 0, n \geq 1)\); (b) an \( n \)-foil \((r = 1)\); (b) an \( n \)-chain link \((r = 2)\); (c) an \( n \)-sinnet of square knotting \((r = 3)\); (e) an \( r \)-twist loop \((n = 1)\) and (f) a \( 2r \)-foil \((n = 2)\).

**Proof.** First, note that \( \overline{F_r} = (\overline{F_r})_1 = P_{1,r} = T_r \). Among the states of the \( r \)-twist loop, there is exactly one which is made up of one component as shown in Figure 5. Hence \( \beta_r(x) = 1 \). Then by (5), we get

\[
T_r(x) = x^2 \left( \frac{(x+1)^r - 1}{x} \right) + x.
\]

In fact if we let \( \alpha_r(x) = \frac{(x+1)^r - 1}{x} \), then the expansion of \( x^2 \alpha_r(x) \), namely

\[
x^2 \alpha_r(x) = x \left( x(x+1)^0 + x(x+1)^1 + x(x+1)^2 + \cdots + x(x+1)^{r-1} \right)
= (\bigcirc \cup \bigcirc) (x) + (\bigcirc \cup \bigcirc \bigcirc) (x) + (\bigcirc \cup \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc) (x) + \cdots + (\bigcirc \cup \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc) (x),
\]

counts as expected the states which might result to that in Figure 5(a).

**Proposition 3.** The generating polynomial for the Pretzel knot \( P_{n,r} \) is given by

\[
P_{n,r}(x) = (\alpha_r(x) + x)^n + (x^2 - 1) \alpha_r(x)^n.
\]

**Proof.** We write

\[
P_{n,r}(x) = \overline{(F_r)}_n(x) = \alpha_r(x)P_{n-1,r}(x) + (F_r)_{n-1}(x).
\]

and we conclude by (4).
Remark 4. Since \((F_r)_n(x) = x(\alpha_r(x) + x)^n\), and for some values of \(r\), we obtain the generating polynomials for the following knots [2]:

- \((F_1)_n = x(x + 1)^n\), \(n\)-twist loop [3, A097805, A007318];
- \((F_2)_n = x(2x + 2)^n\), \(n\)-link [3, A038208];
- \((F_3)_n = x(x^2 + 4x + 3)\), \(n\)-overhand knot [3, A299989].

4 Results

In this section, we retrieve some of our previous results (case \(r = 1, 2, 3\)) [2] which confirm that the generating polynomial agrees with the construction in section 2.

1. Case \(r = 0\).

(a) Generating polynomial:

\[
P_{n,0}(x) = \begin{cases} 
x^2 & \text{if } n = 0; 
x^n & \text{if } n \geq 1.
\end{cases}
\]

(b) Coefficients table: [3, A010054, A023531, A073424 \((n \geq 1, \text{all read as triangle})\)]

| \(n \setminus k\) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|------------------|---|---|---|---|---|---|---|---|
| 0                | 0 | 0 | 1 |
| 1                |   | 1 |
| 2                |   | 0 | 1 |
| 3                |   | 0 | 0 | 1 |
| 4                |   | 0 | 0 | 0 | 1 |
| 5                |   | 0 | 0 | 0 | 0 | 1 |
| 6                |   | 0 | 0 | 0 | 0 | 0 | 1 |
| 7                |   | 0 | 0 | 0 | 0 | 0 | 0 | 1 |

Table 1: Values of \(p_0(n,k)\) for \(0 \leq n \leq 8\) and \(0 \leq k \leq 8\).
2. Case $r = 1$.

(a) Generating polynomial:

$$P_{n,1}(x) = (x + 1)^n + x^2 - 1.$$  

(b) Coefficients table: $[2, 3, \text{A007318} \ (3 \leq k \leq n)]$

| $n \ \backslash \ k$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---------------------|---|---|---|---|---|---|---|---|---|---|----|
| 0                   | 0 | 0 | 1 |   |   |   |   |   |   |   |    |
| 1                   | 0 | 1 | 1 |   |   |   |   |   |   |   |    |
| 2                   | 0 | 2 | 2 |   |   |   |   |   |   |   |    |
| 3                   | 0 | 3 | 4 | 1 |   |   |   |   |   |   |    |
| 4                   | 0 | 4 | 7 | 4 | 1 |   |   |   |   |   |    |
| 5                   | 0 | 5 | 11| 10| 5 | 1 |   |   |   |   |    |
| 6                   | 0 | 6 | 16| 20| 15| 6 | 1 |   |   |   |    |
| 7                   | 0 | 7 | 22| 35| 21| 7 | 1 |   |   |   |    |
| 8                   | 0 | 8 | 29| 56| 28| 8 | 1 |   |   |   |    |
| 9                   | 0 | 9 | 37| 84| 70| 36| 9 | 1 |   |   |    |
| 10                  | 0| 10| 46|120|210|252|210|120|45|10|  1 |

Table 2: Values of $p_1(n, k)$ for $0 \leq n \leq 10$ and $0 \leq k \leq 10$.

3. Case $r = 2$.

(a) Generating polynomial:

$$P_{n,2}(x) = (2x + 2)^n + (x^2 - 1) (x + 2)^n.$$  

(b) Coefficients table: $[3, \text{A300184}]$

| $n \ \backslash \ k$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---------------------|---|---|---|---|---|---|---|---|---|---|----|----|
| 0                   | 0 | 0 | 1 |   |   |   |   |   |   |   |    |    |
| 1                   | 0 | 1 | 1 | 2 | 1 |   |   |   |   |   |    |    |
| 2                   | 0 | 4 | 7 | 4 | 1 |   |   |   |   |   |    |    |
| 3                   | 0 |12 |26 |19 |6 | 1 |   |   |   |   |    |    |
| 4                   | 0 |32 |88 |88 |39| 8 | 1 |   |   |   |    |    |
| 5                   | 0 |80 |272|360|1230|71 |10 | 1 |   |   |    |    |
| 6                   | 0 |192|784|1312|1140|532|123 |12 | 1 |   |    |    |
| 7                   | 0 |448|2144|4368|4872|3164|1162|211|14 | 1 |    |    |
| 8                   | 0 |1024|5632|13568|18592|15680|8176|2480|367|16 | 1 |    |
| 9                   | 0 |2304|14336|39936|65088|67872|46368|20304|5262|655|18 | 1 |    |

Table 3: Values of $p_2(n, k)$ for $0 \leq n \leq 9$ and $0 \leq k \leq 11$.  

6
4. Case $r = 3$.

(a) Generating polynomial:

$$P_{n,3}(x) = (x^2 + 4x + 3)^n + (x^2 - 1) (x^2 + 3x + 3)^n.$$

(b) Coefficients table: [2, Table 14]

| $n \setminus k$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|-----------------|---|---|---|---|---|---|---|---|---|---|----|----|----|
| 0               | 0 | 0 | 1 |
| 1               | 0 | 1 | 3 | 3 | 1 |
| 2               | 0 | 6 | 16| 20| 15| 6 | 1 |
| 3               | 0 | 27| 90 |136| 129| 84 |36 |9 | 1 |
| 4               | 0 | 108| 459| 876| 1021| 832| 501| 220| 66| 12 | 1  |
| 5               | 0 | 405| 2133| 5085| 7350| 7321| 5420| 3103| 1375| 455| 105 | 15 | 1 |

Table 4: Values of $p_3(n,k)$ for $0 \leq n \leq 5$ and $0 \leq k \leq 12$.

5. Case $r = n$.

(a) Generating polynomial:

$$P_{n,n}(x) = \left(\frac{(x+1)^n - 1}{x} + x\right)^n + (x^2 - 1) \left(\frac{(x+1)^n - 1}{x}\right)^n.$$

(b) Coefficients table:

| $n \setminus k$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
|-----------------|---|---|---|---|---|---|---|---|---|---|----|----|----|----|----|
| 0               | 0 | 0 | 1 |
| 1               | 0 | 1 | 1 |
| 2               | 0 | 4 | 7 | 4 | 1 |
| 3               | 0 | 27| 90 |136| 129| 84 |36 |9 | 1 |
| 4               | 0 | 256| 1504| 4336| 8273| 11744| 13036| 11488| 8014| 4368| 1820| 560| 120| 16 | 1 |

Table 5: Values of $p_n(n,k)$ for $0 \leq n \leq 4$ and $0 \leq k \leq 14$.

6. Case $k = 1$: $p_r(n,1) = nr^{n-1}$, $r \geq 1$ [3, A104002 ($1 \leq r \leq n$)], see Table 6.

7. Case $k = 2$: $p_r(n,2) = \binom{n}{2} r^{n-2} \left(2 \binom{r}{2} + 1\right) + r^n$, $r \geq 1$, see Table 7.

We observe the following formulas:

- First column in Table 7 is 1, 0, 1 followed by 0, 0, 0, ... [3, A154272];
| $n \setminus r$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|-----------------|---|---|---|---|---|---|---|---|---|---|
| 0               | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1               | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2               | 0 | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 |
| 3               | 0 | 3 | 12 | 27 | 48 | 75 | 108 | 147 | 192 | 243 |
| 4               | 0 | 4 | 32 | 108 | 256 | 500 | 864 | 1372 | 2048 | 2916 |
| 5               | 0 | 5 | 80 | 405 | 1280 | 3125 | 6480 | 12005 | 20480 | 32805 |
| 6               | 0 | 6 | 192 | 1458 | 6144 | 18750 | 46656 | 100842 | 196608 | 354294 |
| 7               | 0 | 7 | 448 | 5103 | 28672 | 109375 | 326592 | 823543 | 1835008 | 3720087 |
| 8               | 0 | 8 | 1024 | 17496 | 131072 | 625000 | 2239488 | 6588344 | 16777216 | 38263752 |

Table 6: Values of $p_r(n, 1)$ for $0 \leq n \leq 8$ and $0 \leq r \leq 9$.

| $n \setminus r$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|-----------------|---|---|---|---|---|---|---|---|---|
| 0               | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 1               | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| 2               | 1 | 2 | 7 | 16 | 29 | 46 | 67 | 92 | 121 |
| 3               | 0 | 4 | 26 | 90 | 220 | 440 | 774 | 1246 | 1880 |
| 4               | 0 | 7 | 88 | 459 | 1504 | 3775 | 7992 | 15043 | 25984 |
| 5               | 0 | 11 | 272 | 2133 | 9344 | 229375 | 47436 | 164297 | 324608 |
| 6               | 0 | 16 | 784 | 9234 | 54016 | 212500 | 649296 | 1666294 | 3764224 |
| 7               | 0 | 22 | 2144 | 37908 | 295936 | 1456250 | 5342112 | 16000264 | 41320448 |
| 8               | 0 | 29 | 5632 | 149445 | 1556480 | 9578125 | 42177024 | 147414197 | 435159040 |
| 9               | 0 | 37 | 14336 | 570807 | 7929856 | 61015625 | 322486272 | 1315198171 | 4437573632 |

Table 7: Values of $p_r(n, 2)$ for $0 \leq n \leq 9$ and $0 \leq r \leq 8$.

- $p_1(n, 2) = \binom{n}{2} + 1$ [3, A152947];
- $p_0(2, 0) = 0$, $p_1(2, 1) = 2$, $p_2(2, 2) = 7$ and $p_n(2, n) = \binom{2n}{n}$ [3, A000984];
- $p_2(n, 2) = (3n^2 - 3n + 8)2^{n-3}$ [3, A300451];
- $p_2(n, n) = 2(n(n - 1) + 2^n - 1$ [3, A295077];
- $p_3(n, 2n - 1) = \binom{3n}{3}$ [3, A006566];
- $p_3(n, 2n) = \binom{3n}{2}$ [3, A062741];
- $p_0(0, 1) = 0$ and $p_n(n, 1) = n^n$ [3, A000312];
• \( p_n(2, 2) = 2n^2 - n + 1 \) [3, A130883];

• \( p_n(n, n^2 - n + 1) = n^2 \) [3, A000290];

• \( p_0(0, 0) = p_1(1, 0) = 0, p_2(2, 2) = 7 \) and \( p_n(n, n^2 - n) = \binom{n^2}{2} \) [3, A083374];

• \( p_n(n, n^2 - n - 1) = \binom{n^2}{3} \) [3, A178208].

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