EXTREME KOHONOV SPECTRA

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Abstract. We prove that the spectrum constructed by González-Meneses, Manchón and the second author is stably homotopy equivalent to the Khovanov spectrum of Lipshitz and Sarkar at its extreme quantum grading.

1. Introduction

Khovanov homology is a powerful link invariant introduced by Mikhail Khovanov in [Kho00] as a categorification of the Jones polynomial. More precisely, given an oriented diagram \( D \) representing a link \( L \), he constructed a finite \( \mathbb{Z} \)-graded family of chain complexes

\[
\cdots \rightarrow C^{i,j}(D) \xrightarrow{d_i} C^{i+1,j}(D) \xrightarrow{d_{i+1}} C^{i+2,j}(D) \rightarrow \cdots
\]

whose bigraded homology groups, \( Kh^{i,j}(D) \), are link invariants satisfying

\[
J(L)(q) = \sum_{ij} q^i (-1)^j \text{rank}(Kh^{i,j}(L)),
\]

where \( J(L) \) is the Jones polynomial of \( L \). The groups \( Kh^{i,j}(L) \) are known as Khovanov homology groups of \( L \), and the indexes \( i \) and \( j \) as homological and quantum gradings, respectively.

A decade later, Lipshitz and Sarkar [LS14] constructed a \( \mathbb{Z} \)-graded family of spectra \( X^j(D) \) associated to a link diagram \( D \), and they proved that

For each \( j \in \mathbb{Z} \), the spectrum \( X^j(D) \) is a link invariant up to homotopy and there is a canonical isomorphism \( H^*(X^j(D)) \cong Kh^{*,j}(D) \).

The construction of these spectra was later simplified in [LLS15] and [LLS17], where it was shown that each spectrum \( X^j \) can be obtained as the suspension spectrum of the realisation of a certain cubical functor on pointed topological spaces.

For a given link diagram \( D \), the Khovanov chain complex is trivial for all but finitely many \( j \)'s. Let \( j_{min}(D) \) be the minimal quantum grading such that the complex \( \{C^{i,j}(D), d_i\} \) is non-trivial. In [CMS17] González-Meneses, Manchón and Silvero introduced a simplicial complex \( X_D \) satisfying the following

The simplicial complex \( X_D \), if not contractible, is a link invariant up to stable homotopy and there is a canonical isomorphism \( H^{*+n-1}(X_D) \cong Kh^{*,j_{min}}(D) \), with \( n_- \) the number of negative crossings of \( D \).

In this paper we show that, for the minimal quantum grading, both constructions are stably homotopy equivalent.

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2. A stable homotopy equivalence

2.1. States and enhancements. Let $2^n$ be the poset $\{1 \to 0\}^n$, which has an initial element $\vec{0} = (1,1,\ldots,1)$ and a terminal element $\vec{0} = (0,0,\ldots,0)$, and write $|v| = \sum_{i=1}^{n} v_i$.

Let $D$ be an oriented link diagram with $n$ ordered crossings, where $n_+ (n_-)$ of them are positive (negative). A state is an assignment of a label, 0 or 1, to each crossing in $D$. There is a bijection between the set $S$ of states of $D$ and the elements of $2^n$ by considering $v \in 2^n$ as the state that assigns the $i^{th}$ coordinate of $v$ to the $i^{th}$ crossing of $D$. Write $D(v)$ for the set of topological circles and chords obtained after smoothing each crossing of $D$ according to its label by following Figure 1.

An enhancement of a state $v$ is a map $x$ assigning a sign $\pm 1$ to each of the $|D(v)|$ circles in $D(v)$. Write $\tau(v,x) = \sum x(c)$ where $c$ ranges over all circles in $D(v)$, and define, for the enhanced state $(v,x)$, the integers

$$h(v,x) = h(v) = -n_+ + |v|, \quad q(v,x) = n_+ - 2n_- + |v| + \tau(v,x).$$

Let $j_{\min} = \min \{ q(v,x) \mid (v,x) \text{ is an enhanced state of } D \}$, and for any state $v$ write $v^\sim = (v,x_-)$ with $x_-$ the constant function with value $-1$.

**Proposition 1.** [GMS17, Proposition 4.1] In this setting, $j_{\min} = q(\vec{0}^-)$ and $q(v,x) = j_{\min}$ if and only if $(v,x) \in S_{\min}$, where

$$S_{\min} = \{ \text{enhanced states } (v,x) \text{ such that } |D(v)| = |D(\vec{0})| + |v| \text{ and } x = x_- \}.$$  

In particular, $j_{\min} = n_+ - 2n_ - - |D(\vec{0})|$. 

**Proposition 2.** Let $v \in S_{\min}$. If $u < v$ then $u^- \in S_{\min}$.

**Proof.** Note that the Khovanov differential $d$ either splits one circle into two or merges two circles into one. As $|D(v)| = |D(\vec{0})| + |v|$, necessarily $v$ is obtained from $\vec{0}$ by performing $|v|$ splittings in the crossings corresponding to non-zero coordinates of $v$. Hence, if $u$ and $v$ differ on $k$ coordinates, $v$ is obtained from $u$ by performing $k$ splittings, that is, $|D(u)| = |D(v)| - k = |D(\vec{0})| + |u|$. \hfill $\square$

2.2. The simplicial complex for extreme Khovanov homology. Let $D$ be an oriented link diagram. In [GMS17], a simplicial complex $X_D$ was constructed, whose simplicial cochain complex is canonically isomorphic to the extreme Khovanov complex $\{C^{i,j_{\min}}(D), d_i\}$ shifted by $n_+ - 1$. Next, we review the construction of $X_D$ (cf. Figure 2).

The Lando graph $G_D$ associated to $D$ consists of a vertex for each chord in $D(\vec{0})$ having both endpoints in the same circle, and an edge between two vertices if the endpoints of the corresponding chords alternate in the same circle. The simplicial complex $X_D$ is defined as the independence complex of the graph $G_D$; in other words, the simplices of $X_D$ are the subsets of pairwise non-adjacent vertices of $G_D$. Alternatively, it is the clique complex of the complement graph of $G_D$.
2.3. Functors to the Burnside category. Let $\mathbf{Top}_*$ be the category of pointed topological spaces with basepoint $\ast$. Let $\mathbf{Set}_*$ be the category of pointed sets, which we view as a subcategory of $\mathbf{Top}_*$ as the subcategory of discrete spaces. Let $\mathcal{B}$ be the Burnside 2-category for the trivial group, whose objects are finite sets, morphisms are spans and 2-morphisms are correspondences. We will freely refer to the results and notation of [LLS15] and [LLS17] in what follows (see also the survey [LS17]).

The category $\mathbf{Set}_*$ sits inside $\mathcal{B}$ by sending a pointed set $A$ to $A \setminus \{\ast\}$, and a morphism $f: A \to B$ to the span $A \setminus \{\ast\} \leftarrow A \setminus f^{-1}(\ast) \rightarrow B \setminus \{\ast\}$.

Recall from [LLS15, Definition 5.1] that an $N$-dimensional spatial refinement of a functor $F: 2^n \to \mathcal{B}$ is another functor $\tilde{F}: 2^n \to \mathbf{Top}_*$ with values in wedges of spheres of dimension $N$ satisfying certain properties. The following observation is straightforward from that definition:

**Lemma 3.** A functor $F: 2^n \to \mathcal{B}$ has a $0$-dimensional spatial refinement $\tilde{F}$ if and only if $F$ factors as $2^n \to \mathbf{Set}_* \hookrightarrow \mathcal{B}$. If this is the case, the refinement is $\tilde{F}: 2^n \to \mathbf{Set}_* \subset \mathbf{Top}_*$.

Let $2^n_+$ be the poset obtained as follows: Take a second copy of $2^n$, and rename its terminal object $\vec{0}$ as $\circ$. The poset $2^n_+$ is the union of both copies along the subposet $2^n \setminus \{\vec{0}\}$. Alternatively, it is the result of adding two cones to $2^n \setminus \{\vec{0}\}$ with apices $\vec{0}$ and $\circ$. If $\tilde{F}: 2^n \to \mathbf{Top}_*$ is an $N$-dimensional spatial refinement, then its totalisation is defined as follows: extend $\tilde{F}$ to a functor $\tilde{F}_+: 2^n_+ \to \mathbf{Top}_*$ by declaring $\tilde{F}_+(\circ) = \ast$ and define:

$$\text{Tot} \tilde{F} = \operatorname{hocolim} \tilde{F}_+ \in \mathbf{Top}_*.$$

2.4. Khovanov spectra. Fix a link diagram $D$ and let $\mathcal{F}: 2^n \to \mathcal{B}$ be the functor constructed in [LLS17, Proposition 6.1] whose value at a vertex $v$ is the set of all possible enhancements associated to the state $v$. Let $\mathcal{F}^j$ be the subfunctor whose values are those enhancements with quantum grading $j$. If $\mathcal{F}^j$ is an $N$-dimensional spatial refinement of $\mathcal{F}^j$, then the Khovanov spectrum of Lipshitz and Sarkar in quantum grading $j$ is [LLS15, Theorem 3]

$$X^j \simeq \Sigma^{-N-n} - \Sigma^\infty \text{Tot} \mathcal{F}^j. \quad (1)$$

When $j = j_{\text{min}}$, we can restate Propositions 1 and 2 in the following way:
Proposition 4. The value of $\mathcal{F}_{\text{min}}^j$ at a vertex $v \in 2^n$ is either the singleton $x_-$ for the case when $(v, x_-) \in S_{\text{min}}$, or empty otherwise. Moreover, the value of $\mathcal{F}_{\text{min}}^j$ at an arrow $v > u$ is, depending on the values of $\mathcal{F}_{\text{min}}^j(u)$ and $\mathcal{F}_{\text{min}}^j(v)$,

| $\mathcal{F}_{\text{min}}^j(u)$ | $\mathcal{F}_{\text{min}}^j(v)$ |
|-------------------------------|-------------------|
| $\emptyset$                  | $\emptyset$       |
| $x_-$                        | $x_-$             |
| $\emptyset$                  | $\emptyset$       |

In particular, we obtain the following corollary:

Corollary 5. $\mathcal{F}_{\text{min}}^j$ factors through $\text{Set}_\ast$, and therefore the factorisation $\tilde{\mathcal{F}}_{\text{min}}^j$ is the 0-dimensional spatial refinement of $\mathcal{F}_{\text{min}}^j$. In fact, it further factors through the inclusion $\text{Set} \subset \text{Set}_\ast$ sending a set $A$ to the pointed set $A \cup \{\ast\}$. If we write $\hat{\mathcal{F}}_{\text{min}}^j$ for the latter factorisation, we get

$$\begin{array}{c}
2^n \xrightarrow{\mathcal{F}_{\text{min}}^j} B \\
\text{Set} \xrightarrow{\mathcal{F}_{\text{min}}^j} \text{Set}_\ast
\end{array}$$

2.5. A homotopy equivalence. Let $\text{Poset}$ be the category of posets, and let $\text{SComp}$ be the category of simplicial complexes. There are functors

$$\begin{array}{c}
\text{Poset} \xrightarrow{\kappa} \text{SComp} \xrightarrow{|\cdot|} \text{Top},
\end{array}$$

where $\kappa$ takes a simplicial complex to its poset of non-empty faces, $\kappa$ takes a poset $P$ to the simplicial complex whose 0-simplices are the elements of $P$, and whose $i$-simplices are ascending chains of $i+1$ elements in $P$. The functor $|\cdot|$ takes a simplicial complex to its realisation. The composition $\kappa \circ |\cdot|$ takes a simplicial complex $Y$ to its barycentric subdivision $\text{sd}(Y)$. We will denote the composition $|K(\cdot)|$ by $\|\cdot\|$. If $P$ is a poset and $F : P \to \text{Top}$ is a functor taking every element of $P$ to a singleton, then $\|P\|$ is a model for the homotopy colimit of $F$.

Let $S'_{\text{min}} \subset 2^n$ be the subposet of those states $v$ such that $(v, x_-) \in S_{\text{min}}$. The poset $2^n$ can be identified with the poset of faces of the $(n-1)$-dimensional simplex with the arrows reversed, where we identify $\vec{0}$ with the empty face and $\vec{1}$ with the top-dimensional face. Under this identification, the poset of faces of $X_D$ becomes precisely $S'_{\text{min}}$ [GMS17, Proposition 4.3]. Therefore, if $F : S'_{\text{min}} \to \text{Top}$ is a functor with values on singletons, then

$$\text{hocolim} F \simeq \|S'_{\text{min}} \setminus \{\vec{0}\}\| = |\text{sd}(X_D)| \cong |X_D|.$$  

(2) hocolim $F \simeq \|S'_{\text{min}} \setminus \{\vec{0}\}\| = |\text{sd}(X_D)| \cong |X_D|.$

Theorem. There is a homotopy equivalence

$$\mathcal{X}_{\text{min}}^j \simeq \Sigma^{1-n} - \Sigma^\infty |X_D|.$$  

Proof. From (1) and Corollary 5 we have that $\mathcal{X}_{\text{min}}^j \simeq \Sigma^{-n} - \Sigma^\infty \text{hocolim} \mathcal{F}_{\text{min}}^j$. We will prove that $\text{hocolim} \mathcal{F}_{\text{min}}^j \simeq \Sigma^{\|S'_{\text{min}} \setminus \{\vec{0}\}\|} |X_D|$, and the result will follow from the homeomorphism $\|S'_{\text{min}} \setminus \{\vec{0}\}\| \cong |X_D|.$
As $2^n_+$ is constructed as the pushout of two cubes, there is a pushout diagram

$$
\begin{align*}
\text{hocolim } \tilde{F}^{j_{\text{min}}} |_{2^n \setminus \{\vec{0}\}} & \longrightarrow \text{hocolim } \tilde{F}^{j_{\text{min}}} |_{2^n} \\
\downarrow & \quad \downarrow \\
\text{hocolim } \tilde{F}^{j_{\text{min}}} |_{2^n \setminus \{\vec{0}\}} & \longrightarrow \text{hocolim } \tilde{F}^{j_{\text{min}}},
\end{align*}
$$

and as the two cubes have final elements $\vec{0}$ and $\circ$, we have

$$
\text{hocolim } \tilde{F}^{j_{\text{min}}} |_{2^n} \simeq \tilde{F}^{j_{\text{min}}} (\vec{0}) = \{x_-, *\}, \quad \text{hocolim } \tilde{F}^{j_{\text{min}}} |_{2^n \setminus \{\vec{0}\}} \simeq \tilde{F}^{j_{\text{min}}} (\circ) = \ast.
$$

We now proceed to the computation of the upper left term in the diagram. Recall from the second part of Corollary 5 that $\tilde{F}^{j_{\text{min}}}$ factors as $\hat{F}^{j_{\text{min}}}$:

$$
2^n \rightarrow \text{Set} \subset \text{Top} \subset \text{Set}^\ast.
$$

Since the inclusion $\text{Top} \subset \text{Top}^\ast$ is a left adjoint, it preserves colimits, and therefore

$$
\text{hocolim } \hat{F}^{j_{\text{min}}} |_{2^n \setminus \{\vec{0}\}} = \text{hocolim } \hat{F}^{j_{\text{min}}} |_{2^n \setminus \{\vec{0}\} \cup \{*\}}.
$$

Now, from Proposition 4, it follows that $\hat{F}^{j_{\text{min}}}(u)$ is either a point or empty depending on whether $u$ belongs to $S'_{\text{min}}$ or not; therefore

$$
\text{hocolim } \hat{F}^{j_{\text{min}}} |_{2^n \setminus \{\vec{0}\}} = \text{hocolim } \hat{F}^{j_{\text{min}}} |_{S'_{\text{min}} \setminus \{\vec{0}\}}
$$

and since the latter functor is constant with values on singletons, (2) leads to

$$
\text{hocolim } \hat{F}^{j_{\text{min}}} |_{S'_{\text{min}} \setminus \{\vec{0}\}} \simeq \|S'_{\text{min}} \setminus \{\vec{0}\}||.
$$

Finally, we face again the original pushout diagram in $\text{Top}^\ast$:

$$
\|S'_{\text{min}} \setminus \{\vec{0}\}|| \cup \{*\} \longrightarrow \ast
$$

$$
\begin{align*}
\{x_-, *\} \longrightarrow & \text{hocolim } \hat{F}^{j_{\text{min}}} \\
\downarrow & \quad \downarrow \\
& \{x_-, *\} \longrightarrow \text{hocolim } \hat{F}^{j_{\text{min}}},
\end{align*}
$$

where the left vertical map collapses $\|S'_{\text{min}} \setminus \{\vec{0}\}||$ to $\{x_-, *\}$. Replacing $\{x_-, *\}$ by $\text{Cone}(\|S'_{\text{min}} \setminus \{\vec{0}\}||) \cup \{*\}$ and the $\ast$ in the upper right corner by $\text{Cone}(\|S'_{\text{min}} \setminus \{\vec{0}\}||)$ with basepoint the apex of the cone, we obtain a homotopy equivalent cofibrant pushout diagram, whose colimit is the (unreduced) suspension of $\|S'_{\text{min}} \setminus \{\vec{0}\}||$. □

**Remark 6.** One can similarly define a maximal quantum grading $j_{\text{max}}$ and define a simplicial complex $Y_D$ as the Alexander dual of $X_D$ where $D^*$ is the mirror image of $D$ (cf. [PS17, Theorem 7.4]). The fact that the Khovanov spectrum of a link diagram is the Spanier-Whitehead dual of the Khovanov spectrum of its mirror image, immediately implies that $X^{j_{\text{max}}} \simeq \Sigma^{n+1} \Sigma^\infty Y_D$.

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