PATTERN RECOGNITION ON ORIENTED MATROIDS:
HALFSPACES, CONVEX SETS AND TOPE COMMITTEES

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Abstract. The principle of inclusion-exclusion is applied to subsets of
maximal covectors contained in halfspaces of a simple oriented matroid
and to convex subsets of its ground set for enumerating tope committees.

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1. INTRODUCTION

Let \( \mathcal{M} = (E_t, \mathcal{T}) \) be a simple oriented matroid (it has no loops, parallel
or antiparallel elements) on the ground set \( E_t := \{1, \ldots, t\} \), with set of
topes \( \mathcal{T} \). Throughout we will suppose that \( \mathcal{M} \) is not acyclic.

The family \( \mathbf{K}_k^*(\mathcal{M}) \) of tope committees, of cardinality \( k \), \( 3 \leq k \leq |\mathcal{T}| - 3 \),
for the oriented matroid \( \mathcal{M} \) is defined as the collection

\[
\mathbf{K}_k^*(\mathcal{M}) := \{ \mathbf{K}^* \subset \mathcal{T} : |\mathbf{K}^*| = k, |\mathbf{K}^* \cap \mathcal{T}_e^+| > \frac{k}{2} \quad \forall e \in E_t \},
\]

where \( \mathcal{T}_e^+ := \{ T \in \mathcal{T} : T(e) = + \} \) is the positive halfspace of \( \mathcal{M} \)
that corresponds to the element \( e \), see [6, 7, 8, 9]. The family of tope anti-
committees, of cardinality \( k \), for \( \mathcal{M} \) is denoted by \( \mathbf{A}_k^*(\mathcal{M}) \); by definition,
\( \mathbf{A}^* \in \mathbf{A}_k^*(\mathcal{M}) \) iff \( -\mathbf{A}^* \in \mathbf{K}_k^*(\mathcal{M}) \), where \( -\mathbf{A}^* := \{-T : T \in \mathbf{A}^* \} \).

Key words and phrases. Blocker, blocking set, binomial poset, Boolean lattice, convex
set, committee, face lattice of a crosspolytope, halfspace, inclusion-exclusion, oriented
matroid, relative blocking, tope.

2010 Mathematics Subject Classification: 05E45, 52C40, 90C27.
Denote by \( ({\mathcal T}_k) \) the family of all \( k \)-subsets of the tope set \( \mathcal T \), and consider the families of tope subsets \( N^*_k(\mathcal M):=({\mathcal T}_k) - (K^*_k \cup A^*_k) \), \( 3 \leq k \leq |\mathcal T| - 3 \), that is the families

\[
N^*_k(\mathcal M) := \{ N^* \subset \mathcal T : |N^*| = k, \quad N^* \text{ neither a committee nor an anti-committee} \} ;
\]

we have

\[
\#K^*_k(\mathcal M) = \#A^*_k(\mathcal M) = \frac{1}{2}( |\mathcal T| - \#N^*_k(\mathcal M) ) , \quad 3 \leq k \leq |\mathcal T| - 3 .
\]

For an element \( e \in E_t \), we let \( \mathcal T^-_e := \{ T \in \mathcal T : T(e) = - \} \) denote the negative halfspace of \( \mathcal M \) that corresponds to the element \( e \). The family of all subsets, of cardinality \( j \), of the positive halfspace \( \mathcal T^+_e \) is denoted by \( (\mathcal T^+_e)^j \) and, similarly, \( (\mathcal T^-_e)^j \) denotes the family of \( j \)-subsets of the negative halfspace \( \mathcal T^-_e \).

The family of \( (i+j) \)-sets \( (\mathcal T^+_i)^j \oplus (\mathcal T^-_j) \) is defined as the family \( \{ A \cup B : A \in (\mathcal T^+_i)^j, B \in (\mathcal T^-_j) \} \).

On the one hand, \( K^*_k(\mathcal M) = \cap_{e \in E_t} \cup_{j=\lfloor (k+1)/2 \rfloor}^{\lfloor (k-1)/2 \rfloor} \left( \left( \mathcal T^+_e \right)^j \right) \), \( 3 \leq k \leq |\mathcal T| - 3 \). On the other hand, a \( k \)-subset \( K^* \subset \mathcal T \) is a committee for \( \mathcal M \) iff

\[
\begin{align*}
&\text{the set } K^* \text{ contains no set from the family } \cap_{e \in E_t} \left( \left( \mathcal T^+_e \right)^j \right) ; \\
&\text{the set } K^* \text{ contains at least one set from each family } \left( \left( \mathcal T^+_e \right)^j \right), \\
&\quad e \in E_t \quad \text{— in other words, the collection } \left( \left( \mathcal T^+_e \right)^j \right) \text{ is a blocking family for the family } \{ \left( \mathcal T^+_1 \right), \ldots, \left( \mathcal T^+_m \right) \}, \text{ that is, } \#\left( \left( \mathcal T^+_e \right) \right) \cap \left( \left( \mathcal T^+_e \right)^j \right) > 0 , e \in E_t .
\end{align*}
\]

As a consequence, the collection \( K^*_k(\mathcal M) \) is the family of all blocking \( k \)-sets of topes for the family \( \cup_{e \in E_t} \left( \left( \mathcal T^+_e \right)^j \right), \) and a committee \( K^* \in K^*_k(\mathcal M) \) is minimal if any its proper \( i \)-subset \( \mathcal T^i \subset K^* \) is not a blocking set for the family \( \cup_{e \in E_t} \left( \left( \mathcal T^+_e \right)^j \right) \).

Based on these remarks, we calculate in Sections 3 and 4 the numbers \( \#K^*_k(\mathcal M) \) of general committees of cardinality \( k \), in several possible ways, by applying the principle of inclusion-exclusion [1] [1] to subsets of maximal covectors contained in halfspaces of the oriented matroid \( \mathcal M \); in Section 4, these calculations involve the convex subsets of the ground set of \( \mathcal M \). In Section 4 we find the numbers \( \#K^*_k(\mathcal M) \) of tope committees, of cardinality \( k \), which contain no pairs of opposites. Sections 2, 5 and 7 list auxiliary results.

See [3] and references therein on acyclic, convex and free sets of oriented matroids.

One can associate to the oriented matroid \( \mathcal M \) various “\( \kappa^* \)-vectors” (and their flag generalizations) whose components are the numbers of its tope committees of the corresponding cardinality, for example:
Example 1.1. Let \( \mathcal{M} := (E_6, T) \) be the simple oriented matroid represented by its third positive halfspace

\[
\tau^+_5 := \{ - - + + + +, - + + - + +, + - + - + -, + + + + - -, + + + - - -, - + + - - -, - + + + - - \};
\]

a realization of its reorientation \(-\{1, 2\}\mathcal{M}, \) by a hyperplane arrangement in \( \mathbb{R}^3, \) is shown in [7, Figure 3.1].

The oriented matroid \( \mathcal{M} \) has 28 maximal covectors and 238012 tope committees —

\[\kappa^*(\mathcal{M}) = (0, 0, 3, 0, 144, 1, 1942, 22, 11872, 136, 37775, 386, 66454, 542)\]

— among which 4496 committees are free of opposites:

\[\hat{\kappa}^*(\mathcal{M}) = (0, 0, 3, 0, 111, 1, 778, 14, 1935, 24, 1448, 24, 158, 0)\, .\]

2. Relative Blocking in Boolean Lattices

Let \( A \) be a nontrivial antichain in the Boolean lattice \( \mathbb{B}(n) \) of rank \( n, \) and \( \Lambda^\perp \) the set of lattice complements of the elements of \( A \) in \( \mathbb{B}(n); \) \( \rho(\cdot) \) denotes the rank function, \( \mathbb{B}(T)^{(i)} := \{ b \in \mathbb{B}(n) : \rho(b) = i \} \) denotes the \( i \)th
layer of $\mathcal{B}(n)$, and $\mathcal{I}(C)$ stands for the order ideal of the lattice $\mathcal{B}(n)$ generated by its antichain $C$.

For a rational number $r$, $0 \leq r < 1$, and for a positive integer number $k$, consider the subset

$$I_{r,k}(\mathcal{B}(n), A) := \{ b \in \mathcal{B}(n) : \rho(b) = k, \rho(b \wedge \lambda) > r \cdot k \quad \forall \lambda \in A \} \subset \mathcal{B}(n)^{(k)}$$

(2.1)

that consists of the relatively $r$-blocking elements, of rank $k$, for the antichain $A$.

Set $\nu(r \cdot k) := \lfloor r \cdot k \rfloor + 1$ and consider an antichain $A \subset \mathcal{B}(n)$ such that $\rho(\lambda) \geq \nu(r \cdot k)$ and $n - \rho(\lambda) \geq k - \nu(r \cdot k) + 1$, for each element $\lambda \in A$, that is,

$$\lfloor r \cdot k \rfloor + 1 \leq \min_{\lambda \in A} \rho(\lambda) \quad \text{and} \quad \max_{\lambda \in A} \rho(\lambda) \leq n + \lfloor r \cdot k \rfloor - k . \quad (2.2)$$

If the antichain $A$ satisfies constraints (2.2) then for an element $b' \in \mathcal{B}(n)^{(k)}$ we have $b' \not\in I_{r,k}(\mathcal{B}(n), A)$ iff $b' > d'$ for at least one element $d'$ of rank $k - \nu(r \cdot k) + 1 = k - \lfloor r \cdot k \rfloor$ such that $d' \in \mathcal{I}(A^\perp)$; therefore, on the one hand,

$$|I_{r,k}(\mathcal{B}(n), A)| = \binom{n}{k} + \sum_{D' \subseteq \mathcal{B}(n)^{(k-\lfloor r \cdot k \rfloor) \cap \mathcal{I}(A^\perp)}} (-1)^{|D'|} \cdot \binom{n - \rho(\bigvee_{d' \in D'} d')}{n - k} . \quad (2.3)$$

On the other hand, for an element $b \in \mathcal{B}(n)$ the inclusion $b \in I_{r,k}(\mathcal{B}(n), A)$ holds iff for each element $\lambda \in A$ we have $\rho(b \wedge \theta_\lambda) > 0$, for any element $\theta_\lambda \in \mathcal{B}(n)^{(\rho(\lambda) - \nu(r \cdot k) + 1) \cap \mathcal{I}(\lambda)}$, that is,

$$b \in I_{r,k}(\mathcal{B}(n), A) \iff \rho(b \wedge \theta_\lambda) > 0 \quad \forall \theta_\lambda \in \mathcal{B}(n)^{(\rho(\lambda) - \lfloor r \cdot k \rfloor) \cap \mathcal{I}(\lambda)} \forall \lambda \in A ,$$

and we have

$$|I_{r,k}(\mathcal{B}(n), A)| = \binom{n}{k} + \sum_{D \subseteq \min \bigcup_{\lambda \in A(\mathcal{B}(n)^{(\rho(\lambda) - \lfloor r \cdot k \rfloor) \cap \mathcal{I}(\lambda)}) : |D|>0}} (-1)^{|D|} \cdot \binom{n - \rho(\bigvee_{d \in D} d)}{k} . \quad (2.4)$$

(2.3)

(where $\min \cdot$ stands for the set of minimal elements of a subposet) or, via Vandermonde’s convolution,

$$|I_{r,k}(\mathcal{B}(n), A)| = -\sum_{D \subseteq \min \bigcup_{\lambda \in A(\mathcal{B}(n)^{(\rho(\lambda) - \lfloor r \cdot k \rfloor) \cap \mathcal{I}(\lambda)}) : |D|>0}} (-1)^{|D|} \cdot \sum_{1 \leq h \leq k} \binom{\rho(\bigvee_{d \in D} d)}{h} \binom{n - \rho(\bigvee_{d \in D} d)}{k - h} . \quad (2.5)$$
One more inclusion-exclusion type formula for the cardinality of the set \(I_{r,k}(\mathbb{B}(n), A)\), for an antichain \(A\) such that \(\rho(\lambda) \geq \nu(r \cdot k)\), for all \(\lambda \in A\), is given in [10, (5.4)]: if
\[
[r \cdot k] + 1 \leq \min_{\lambda \in A} \rho(\lambda)
\] (2.6)
then
\[
|I_{r,k}(\mathbb{B}(n), A)| = \sum_{D \subseteq \mathbb{B}(n) \setminus (\{r \cdot k\} \cap \mathfrak{A}(A); |D| > 0)} (-1)^{|D|} \cdot \left( \sum_{C \subseteq A; D \subseteq \mathfrak{A}(C)} (-1)^{|C|} \left( n - \rho(\nu_{d \in D} d) \right) \right) .
\] (2.7)

We now refine formulas (2.3), (2.4) and (2.5) with the help of the Möbius function [11, 11], see below expressions (2.8), (2.9) and (2.10), respectively. Let \(X\) be a non-trivial antichain in the Boolean lattice \(\mathbb{B}(n)\). Denote by \(\mathcal{E}(\mathbb{B}(n), X)\) the sub-join-semilattice of \(\mathbb{B}(n)\) generated by the set \(X\) and augmented by a new least element \(\hat{0}\); the greatest element \(\hat{1}\) of the lattice \(\mathcal{E}(\mathbb{B}(n), X)\) is the join \(\bigvee_{x \in X} x\) in \(\mathbb{B}(n)\). The Möbius function of the lattice \(\mathcal{E}(\mathbb{B}(n), X)\) is denoted by \(\mu_{\mathcal{E}}(\cdot, \cdot)\).

Let \(A\) be a non-trivial antichain in the Boolean lattice \(\mathbb{B}(n)\) that complies with constraints (2.2). We have
\[
|I_{r,k}(\mathbb{B}(n), A)| = \binom{n}{k} + \sum_{z \in \mathcal{E}(\mathbb{B}(n) \setminus (\{r \cdot k\} \cap \mathfrak{A}(A^+)); z > \hat{0}} \mu_{\mathcal{E}}(\hat{0}, z) \cdot \left( n - \rho(z) \right) \right) .
\] (2.8)

\[
|I_{r,k}(\mathbb{B}(n), A)| = \binom{n}{k} + \sum_{z \in \mathcal{E}(\min \bigcup_{\lambda \in \mathfrak{A}(\mathbb{B}(n) \setminus (\{r \cdot k\} \cap \mathfrak{A}(\lambda)))); z > \hat{0}} \mu_{\mathcal{E}}(\hat{0}, z) \cdot \left( n - \rho(z) \right) \right) .
\] (2.9)

\[
|I_{r,k}(\mathbb{B}(n), A)| = -\sum_{z \in \mathcal{E}(\min \bigcup_{\lambda \in \mathfrak{A}(\mathbb{B}(n) \setminus (\{r \cdot k\} \cap \mathfrak{A}(\lambda)))); z > \hat{0}} \mu_{\mathcal{E}}(\hat{0}, z)
\]
\[
\cdot \sum_{1 \leq h \leq k} \left( \frac{\rho(z)}{h} \right) \left( \frac{n - \rho(z)}{k - h} \right) .
\] (2.10)

A companion formula to (2.7) is given in [10, (5.6)]: let \(A \subseteq \mathbb{B}(n)\) be an antichain that obeys constraint (2.6), and let \(C_{r,k}(\mathbb{B}(n), A)\) be the join-semilattice of all sets from the family \(\{\mathbb{B}(n) \setminus (\{r \cdot k\} + 1) \cap \mathfrak{A}(C) : C \subseteq A, |C| > 0\}\).
ordered by inclusion and augmented by a new least element \( \hat{0} \); the greatest element \( \hat{1} \) of the lattice \( C_{r,k}(B(n), \Lambda) \) is the set \( B(n)^{-} \cap \mathcal{I}(\Lambda) \). We denote the Möbius function of \( C_{r,k}(B(n), \Lambda) \) by \( \mu_C(\cdot, \cdot) \). We have

\[
|I_{r,k}(B(n), \Lambda)| = \sum_{X \in C_{r,k}(B(n), \Lambda): X > \hat{0}} \mu_C(\hat{0}, X) \cdot \sum_{z \in E(B(n), X): z > \hat{0}} \mu_C(\hat{0}, z) \cdot \left( \frac{n - \rho(z)}{n - k} \right). \tag{2.11}
\]

### 3. Halfspaces and Tope Committees

Let \( B(T) \) be the Boolean lattice of all subsets of the tope set \( T \), and \( \Upsilon := \{v_1, \ldots, v_k\} \subset B(T)^{(|T|/2)} \) its antichain whose element \( v_e \) represents in \( B(T) \) the \( e \)th positive halfspace \( T^+_e \) of the oriented matroid \( M \). The family \( K^+_k(M) \) of tope committees, of cardinality \( k, 3 \leq k \leq |T| - 3 \), for \( M \) is represented in the lattice \( B(T) \) by the antichain

\[
I_{\frac{k}{2}, k}(B(T), \Upsilon) := \{b \in B(T) : \rho(b) = k, \rho(b \land v_e) > \frac{k}{2}, \forall e \in E_t \} \subset B(T)^{(k)};
\]

thanks to axiomatic symmetry \( T = -T \), see \([1\] §4.1.1, (L1)]\), the cardinality of this set is

\[
|I_{\frac{k}{2}, k}(B(T), \Upsilon)| = \binom{|T|}{k} + \sum_{D \subseteq B(T)^{(|T|/2)} \cap \Upsilon(\Lambda) : |D| > 0} (-1)^{|D|} \cdot \left( \binom{|T| - \rho(\bigvee_{d \in D} d)}{|T| - k} \right), \tag{3.1}
\]

by \([23]\). Note that for an integer \( j, 1 \leq j \leq |T|/2 \), we have

\[
|B(T)^{(j)} \cap \Upsilon(\Lambda)| = -\sum_{A \in L_{\text{conv}}(M) - \{\hat{0}\}: A \text{ free}} (-1)^{|A|} \cdot \binom{|T^+_A|}{j},
\]

where \( L_{\text{conv}}(M) \) denotes the meet-semilattice of \textit{convex subsets} of the ground set \( E_t \), and \( T^+_A := \bigcap_{a \in A} T^+_a \); \( \hat{0} \) denotes the least element of \( L_{\text{conv}}(M) \). Recall that from the algebraic combinatorial point of view \([2]\), the set \( B(T)^{(j)} \cap \Upsilon(\Lambda) \) is a \textit{subset} in the \textit{Johnson association scheme} \( J(|T|, j) := (X, \mathcal{R}) \) on the set \( X := B(T)^{(j)} \), with the partition \( \mathcal{R} := (R_0, R_1, \ldots, R_j) \) of \( X \times X \), defined by \( R_i := \{(x, y) : j - \rho(x \land y) = i\} \), for all \( 0 \leq i \leq j \).
Reformulate observation (3.1) in the following way:

\[
\#K^*_k(M) = \#K^*_{|T|-k}(M) = \binom{|T|}{|T| - \ell} + \sum_{\mathcal{G} \subseteq \bigcup_{e \in E_t} (\mathcal{T} + e \cdot \left\lfloor \frac{|T|}{2} + 1 \right\rfloor / 2)} (-1)^{|\mathcal{G}|} \cdot \left( \frac{|T| - \sum_{G \in \mathcal{G}} |G|}{|T| - \ell} \right),
\]

(3.2)

where \( \ell \in \{k, |T| - k\} \); this formula counts the number of all blocking \( k \)-sets of topes for the family \( \bigcup_{e \in E_t} (\mathcal{T} + e \cdot \left\lfloor \frac{|T|}{2} + 1 \right\rfloor / 2) \), cf. (2.4), and it counts the number of all blocking \((|T| - k)\)-sets of topes for the family \( \bigcup_{e \in E_t} (\mathcal{T} + e \cdot \left\lfloor \frac{k}{2} + 1 \right\rfloor / 2) \).

We can also rewrite (3.1) by means of Vandermonde’s convolution in the form:

\[
\left| I_{1,k} (B(T), T) \right| = - \sum_{D \subseteq B(T) \cap \bigcup_{e \in E_t} \mathcal{T} + e \cdot \left\lfloor \frac{|T|}{2} + 1 \right\rfloor / 2} (-1)^{|D|} \sum_{1 \leq h \leq k} \rho(d \cdot D) \left( \frac{|T| - \sum_{d \in D} |d|}{h} \right)^{k - h - \rho(d \cdot D) / k},
\]

cf. (2.5), that is,

\[
\#K^*_k(M) = \#K^*_{|T|-k}(M) = - \sum_{\mathcal{G} \subseteq \bigcup_{e \in E_t} (\mathcal{T} + e \cdot \left\lfloor \frac{|T|}{2} + 1 \right\rfloor / 2)} (-1)^{|\mathcal{G}|} \rho(d \cdot \mathcal{G}) \sum_{\max\{1, \ell - |\mathcal{G}|, \min\{\ell, \sum_{G \in \mathcal{G}} |G|\} \leq h \leq \min\{\ell, \sum_{G \in \mathcal{G}} |G|\}} \left( \frac{|\mathcal{G}|}{h} \right)^{k - h - \rho(d \cdot \mathcal{G}) / k},
\]

(3.3)

where \( \ell \in \{k, |T| - k\} \).

If \( \mathcal{G} \) is a family of tope subsets then we denote by \( \mathcal{E}(\mathcal{G}) \) the join-semilattice \( \{ \bigcup_{F \in \mathcal{F}} F : \mathcal{F} \subseteq \mathcal{G}, \#\mathcal{F} > 0 \} \) composed of the unions of the sets from the family \( \mathcal{G} \) ordered by inclusion and augmented by a new least element \( \hat{0} \); the greatest element \( \hat{1} \) of the lattice \( \mathcal{E}(\mathcal{G}) \) is the set \( \bigcup_{G \in \mathcal{G}} G \). The Möbius function of the poset \( \mathcal{E}(\mathcal{G}) \) is denoted by \( \mu_{\mathcal{G}} (\cdot, \cdot) \).

Expressions (3.4) and (3.5) below refine formulas (3.2) and (3.3), respectively.

**Proposition 3.1.** The number \( \#K^*_k(M) \) of tope committees, of cardinality \( k \), \( 3 \leq k \leq |T| - 3 \), for the oriented matroid \( M := (E_t, T) \), is:
\( \#K^*_k(M) = \#K^*_|T| - k(M) \)

\[
(3.4)
\]

where \( \ell \in \{ k, |T| - k \}. \)

(ii)

\[
\#K^*_k(M) = \#K^*_|T| - k(M) = - \sum_{\substack{G \in \mathcal{E}(U_{k \in E_1 \left( \frac{\ell}{|T| + 1} \right)}): |G| > 0 \}} \mu_e(\hat{0}, G) \cdot \sum_{\max(1, \ell - |T| + |G|) \leq h \leq \min(\ell, |G|)} \binom{|G|}{h} \binom{|T| - |G|}{\ell - h},
\]

\[
(3.5)
\]

where \( \ell \in \{ k, |T| - k \}. \)

Let \( C_{1, k} (\mathbb{B}(T), Y) \) be the join-semilattice of all sets from the family \( \{ \mathbb{B}(T)^{\left( \frac{(k+1)/2} \right)} \cap \mathcal{Y}(C) : C \subseteq Y, |C| > 0 \} \) ordered by inclusion and augmented by a new least element \( \hat{0} \). The greatest element \( \hat{1} \) of the lattice \( C_{1, k} (\mathbb{B}(T), Y) \) is the set \( B(T)^{\left( \frac{(k+1)/2} \right)} \cap \mathcal{Y}(Y) \). Similarly, for an element \( X \in C_{1, k} (\mathbb{B}(T), Y) \) we denote by \( \mathcal{E}(\mathbb{B}(T), X) \) the sub-join-semilattice of \( \mathbb{B}(T) \) generated by the set \( X \subseteq \mathbb{B}(T) \) and augmented by a new least element \( 0 \). The Möbius functions of the posets \( C_{1, k} (\mathbb{B}(T), Y) \) and \( \mathcal{E}(\mathbb{B}(T), X) \) are denoted by \( \mu_C(\cdot, \cdot) \) and \( \mu_e(\cdot, \cdot) \), respectively.

Using (2.11), we obtain the expression

\[
|I_{1, k} (\mathbb{B}(T), Y)| = \sum_{X \in C_{1, k} (\mathbb{B}(T), Y): X > 0} \mu_C(\hat{0}, X) \cdot \sum_{z \in \mathcal{E}(\mathbb{B}(T), X): z > 0} \mu_e(\hat{0}, z) \cdot \binom{|T| - \rho(z)}{|T| - k}.
\]

(3.6)

Restate (3.6) in the following way:

**Proposition 3.2.** The number \( \#K^*_k(M) \) of tope committees, of cardinality \( k, 3 \leq k \leq |T| - 3 \), for the oriented matroid \( M := (E_1, T) \), is

\[
\#K^*_k(M) = \#K^*_|T| - k(M) = \sum_{\substack{G \in \{ U_{k \in E_1 \left( \frac{\ell}{|T| + 1} \right)}: E \subseteq E_2, |E| > 0 \} \}} \mu_C(\hat{0}, G) \cdot \sum_{\substack{G \in \mathcal{E}(G): 0 < |G| \leq \ell \}} \mu_e(\hat{0}, G) \cdot \binom{|T| - |G|}{\ell - |G|},
\]

(3.5)
where $\ell \in \{k, |T| - k\}$; $\mu_C(\cdot, \cdot)$ denotes the Möbius function of the family $C$ defined as
$$
\mu_C(\cdot, \cdot) = \left\{ \begin{array}{ll}
0, & \text{if } 0 \in C \cap D, \\
(-1)^{|C-D|}, & \text{if } 0 \not\in C \cap D.
\end{array} \right.
$$

4. Convex Sets and Tope Committees

Let the antichain $\Upsilon := \{v_1, \ldots, v_t\} \subset B(T)^{\left\lfloor \frac{|T|}{2} \right\rfloor}$ again represent the family of positive halfspaces of the oriented matroid $M$ in the Boolean lattice $B(T)$ of all subsets of the tope set $T$. We have

$$
|I_{T,k}(B(T), \Upsilon)| = \sum_{D \subseteq B(T)^{\left\lceil \frac{(k+1)}{2} \right\rceil}} (-1)^{|D|} \sum_{C \subseteq \Upsilon: D \subseteq \Upsilon(C)} (-1)^{|C|} \binom{|T| - \rho(D)}{|T| - k}, \quad 3 \leq k \leq |T| - 3 , \quad (4.1)
$$
cf. [27].

Consider the mapping

$$
\gamma_k : B(T)^{\left\lceil \frac{(k+1)}{2} \right\rceil} \cap \Upsilon(\Upsilon) \to L_{\text{conv}}(M),
\quad d \mapsto \max \{ A \in L_{\text{conv}}(M) : d \subseteq T_A^+ \}, \quad (4.2)
$$

that sends a $\left\lfloor \frac{(k+1)}{2} \right\rfloor$-subset of topes $d \in \Upsilon(\Upsilon)$ to the inclusion-maximum convex subset $A \subset E_t$ with the property $d \subseteq T_A^+$; we are actually interested in such a mapping to the subposet $L_{\text{conv}}, \geq_{\lceil \frac{(k+1)}{2} \rceil}(M)$, the order ideal of the semilattice $L_{\text{conv}}(M)$ defined as $L_{\text{conv}}, \geq_{\lceil \frac{(k+1)}{2} \rceil}(M) := \{ A \in L_{\text{conv}}(M) : |T_A^+| \geq \left\lceil \frac{(k+1)}{2} \right\rceil \}$.

Fix a nonempty subset $D \subseteq B(T)^{\left\lceil \frac{(k+1)}{2} \right\rceil} \cap \Upsilon(\Upsilon)$ and consider the blocker $B(\gamma_k(D))$ of the image $\gamma_k(D)$; if we let $\min_\gamma(D)$ denote the subfamily of all inclusion-minimal sets from the family $\gamma_k(D)$ then $B(\gamma_k(D)) = B(\min_\gamma(D))$.

Let $\Delta^*(D)$ be the abstract simplicial complex whose facets are the complements $E_t - B$ of the sets $B \in B(\min_\gamma_k(D))$ from the blocker of the Sperner family $\min_\gamma_k(D)$, and let $\Delta(D)$ be the complex whose facets are the complements $E_t - G$ of the sets $G \in \min_\gamma_k(D)$; if the complexes $\Delta(D)$ and $\Delta^*(D)$ have the same vertex set then $\Delta^*(D)$ is the Alexander dual of $\Delta(D)$. The reduced Euler characteristics $\bar{\chi}(\cdot)$ of the complexes satisfy the equality $\bar{\chi}(\Delta^*(D)) = (-1)^{|T|-1} \bar{\chi}(\Delta(D))$.

For a subset $C := \{v_{i_1}, \ldots, v_{i_j}\} \subseteq \Upsilon$ we have $D \subseteq \Upsilon(C)$ iff the collection of indices $\{i_1, \ldots, i_j\}$ is a blocking set for the family $\min_\gamma_k(D)$; therefore

$$
\sum_{C \subseteq \Upsilon: D \subseteq \Upsilon(C)} (-1)^{|C|} = (-1)^{|T|-1} \bar{\chi}(\Delta^*(D)).
$$

If $\bigcup_{F \in \min_\gamma_k(D)} F \neq E_t$ then the complex $\Delta^*(D)$ is a cone and, as a consequence, $\bar{\chi}(\Delta^*(D)) = 0$. 

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Rewrite (4.1) in the following way:

\[
|I_{\mathcal{T}}^{\gamma}(\mathcal{B}(\mathcal{T}), \mathcal{Y})| = \sum_{D \subseteq \mathcal{B}(\mathcal{T})(\lfloor(k+1)/2\rfloor) \cap \mathcal{Y}} \left( -1 \right)^{|D|} \cdot \chi(\Delta(D)) \cdot \left( |T| - \rho(\vee_{d \in D} d) \right) \quad (4.3)
\]

note that singleton sets \( D \) := \{d\}, where \( d \in \mathcal{B}(\mathcal{T})(\lfloor(k+1)/2\rfloor) \cap \mathcal{Y} \), do not play a role in (4.3).

Given a subset \( D \subseteq \mathcal{B}(\mathcal{T})(\lfloor(k+1)/2\rfloor) \cap \mathcal{Y} \) such that \( \bigcup_{F \in \min \gamma_k(D)} F = E_t \), let \( \mathcal{S}(D) \) denote the family of the unions \( \{ \bigcup_{F \in F} : F \subseteq \min \gamma_k(D), \#F > 0 \} \) ordered by inclusion and augmented by a new least element \( \hat{0} \); the greatest element \( \hat{1} \) of the lattice \( \mathcal{S}(D) \) is the ground set \( E_t \). The reduced Euler characteristic \( \tilde{\chi}(\Delta(D)) = \sum_{F \subseteq \min \gamma_k(D)} (-1)^{\#F} \) of the complex \( \Delta(D) \) is equal to the Möbius number \( \mu_{\mathcal{S}(D)}(0, \hat{1}) \) and, in particular, to \( (-1)^{\# \min \gamma_k(D)} \) when the sets in the family \( \min \gamma_k(D) \) are pairwise disjoint. Restate observation (4.3):

**Proposition 4.1.** The number \( \#K^*_k(\mathcal{M}) \) of tope committees, of cardinality \( k \), \( 3 \leq k \leq |\mathcal{T}| - 3 \), for the oriented matroid \( \mathcal{M} := (\mathcal{E}_t, \mathcal{T}) \), is

\[
\#K^*_k(\mathcal{M}) = \#K^*_{|\mathcal{T}| - k}(\mathcal{M}) = \sum_{\mathcal{G} \subseteq \bigcup_{E_t \in E_t} \left( \lfloor(k+1)/2\rfloor \right)} \left( -1 \right)^{\#\mathcal{G}} \cdot \mu_{\mathcal{S}(\mathcal{G})}(0, \hat{1}) \cdot \left( |\mathcal{T}| - |\bigcup_{G \in \mathcal{G}} G| \right) \left( \ell - |\bigcup_{G \in \mathcal{G}} G| \right),
\]

where \( \ell \in \{k, |\mathcal{T}| - k\} \).

Consider the abstract simplicial complex whose facets are the positive halfspaces of the oriented matroid \( \mathcal{M} \). If some its relevant \( (\lfloor(k+1)/2\rfloor - 1) \)-dimensional faces, sets from the family \( \bigcup_{e \in E_t} \left( \lfloor(k+1)/2\rfloor \right) \), are free — each of them is contained in exactly one facet \( \mathcal{T}^+_e \), for some element \( e \in E_t \) — then the Möbius numbers \( \mu_{\mathcal{S}(\mathcal{G})}(0, \hat{1}) \) in (4.3), under \( \ell := k \), are all equal to \( (-1)^{\ell} \):

**Corollary 4.2.** Let \( k \) be an integer, \( 3 \leq k \leq |\mathcal{T}| - 3 \). If for any family \( \mathcal{G} \subseteq \bigcup_{E_t \in E_t} \left( \lfloor(k+1)/2\rfloor \right) \) such that \( \bigcup_{F \in \min \gamma_k(\mathcal{G})} F = E_t \) and \( \bigcup_{G \in \mathcal{G}} G \leq k \), it holds \( |\gamma_k(G)| = 1 \), for any set \( G \in \mathcal{G} \), then the number \( \#K^*_k(\mathcal{M}) \) of tope
committees, of cardinality \( k \), for the oriented matroid \( \mathcal{M} := (E_t, T) \), is

\[
\#K_k^r(\mathcal{M}) = \#K_{|T|-k}^r(\mathcal{M}) = (-1)^t \sum_{\Omega \subseteq \cup_{i \in E_t} \left( r_{(k+1)/2} \right)} (-1)^{\|\Omega\|} \binom{|T| - \sum_{\Gamma \subseteq \cup_{G \in \mathcal{G}}} |G|}{k - \sum_{\Gamma \subseteq \cup_{G \in \mathcal{G}}} |G|} . 
\]

5. Relative Blocking in Posets Isomorphic to the Face Lattices of Crosspolytopes

Consider a poset \( \mathcal{O}'(m) \), with the rank function \( \rho(\cdot) \), which is isomorphic to the graded face meet-semilattice of the boundary of a \( m \)-dimensional crosspolytope and is defined in the following way: the semilattice \( \mathcal{O}'(m) \) is composed of all subsets, free of opposites, of a set \( \{-m, \ldots, -1, 1, \ldots, m\} \), ordered by inclusion. We denote by \( \mathcal{O}(m) \) the lattice \( \mathcal{O}(m) := \mathcal{O}'(m) \cup \{1\} \), where \( 1 \) is a new greatest element. Let \( A \subseteq \mathcal{O}'(m) \) be a nontrivial antichain in the lattice \( \mathcal{O}(m) \).

For a rational number \( r \), \( 0 \leq r < 1 \), and for a positive integer number \( k \), we define the set \( I_r,k(\mathcal{O}'(m), A) \) of relatively \( r \)-blocking elements, of rank \( k \), for the antichain \( A \) in analogy with the sets \( I_r,k(\mathcal{B}(n), \cdot) \) for antichains in Boolean lattices, cf. \([2.1]\):

\[
I_r,k(\mathcal{O}'(m), A) := \{ b \in \mathcal{O}'(m) : \rho(b) = k, \rho(b \wedge \lambda) > r \cdot k \ \forall \lambda \in A \} \subseteq \mathcal{O}'(m)^{(k)},
\]

where \( \mathcal{O}'(m)^{(k)} \) is the \( k \)-th layer of the semilattice \( \mathcal{O}'(m) \).

On the one hand, we have

\[
|I_r,k(\mathcal{O}'(m), A)| = \sum_{X \in \mathcal{E}_r,k(\mathcal{O}'(m), A): X > \hat{0}} \mu_{\mathcal{E}}(\hat{0}, X) \cdot \sum_{z \in \mathcal{E}(\mathcal{O}'(m), X): z > \hat{0}} \mu_{\mathcal{E}}(\hat{0}, z) \cdot 2^{k - \rho(z)} \cdot 2^{m - \rho(z)} / (m - k) ,
\]

cf. \([2.1]\), where \( \mathcal{E}_r,k(\mathcal{O}'(m), A) \) denotes the join-semilattice of all sets from the family \( \{ \mathcal{O}'(m)^{(r-k+1)} \cap \mathcal{F}(C) : C \subseteq \Lambda, \ |C| > 0 \} \) ordered by inclusion and augmented by a new least element \( \hat{0} \); the greatest element \( \hat{1} \) of the lattice \( \mathcal{E}_r,k(\mathcal{O}'(m), A) \) is the set \( \mathcal{O}'(m)^{(r-k+1)} \cap \mathcal{F}(A) \). For an element \( X \in \mathcal{E}_r,k(\mathcal{O}'(m), A) \), the notation \( \mathcal{E}(\mathcal{O}'(m), X) \) is used to denote the sub-join-semilattice of the lattice \( \mathcal{O}(m) \) generated by the set \( X \subseteq \mathcal{O}'(m) \), with the greatest element of \( \mathcal{O}(m) \) deleted from it, and augmented by a new least element \( \hat{0} \). The Möbius functions of the posets \( \mathcal{E}_r,k(\mathcal{O}'(m), A) \),
and \( \hat{\mathcal{E}}(O'(m), X) \) are denoted by \( \mu_{\varepsilon}(\cdot, \cdot) \) and \( \mu_{\varepsilon}(\cdot, \cdot) \), respectively; \( \rho(\cdot) \) denotes the rank of an element in the poset \( O'(m) \).

On the other hand, we have

\[
|I_{r,k}(O'(m), A)| = \sum_{D \subseteq O'(m)^{(r-k)+1}} (-1)^{|D|} \sum_{C \subseteq A: D \subseteq \mathcal{D}(C)} (-1)^{|C|} \cdot 2^{k - \rho(\lor_{d \in D} d)} \cdot \left( \frac{m - \rho(\lor_{d \in D} d)}{m - k} \right),
\]

where \( G = (\mathcal{D}(\varepsilon), \mathcal{D}(\eta)) \) and \( \mathcal{D}(\varepsilon) \) is a family of tope subsets that are free of opposites then we denote \( \mathcal{D}(\varepsilon) \) by \( \mathcal{D}(\varepsilon) \). The lattice \( O(\mathcal{T}) := O'(\mathcal{T}) \cup \{ \hat{1} \} \) is the semilattice \( O'(\mathcal{T}) \) augmented by a new greatest element 1. We again turn to the mapping \( \gamma_k: \mathbb{B}(\mathcal{T})^{(\lceil (k+1)/2 \rceil)} \cap \mathcal{J}(\mathcal{Y}) \rightarrow L_{\text{conv}}(\mathcal{M}) \) defined in (4.1), and to the lattices \( \mathcal{S}(\cdot) \) considered in Section 4.

If \( G \) is a family of tope subsets which are free of opposites we then denote by \( \mathring{\mathcal{E}}(G) \) the join-semilattice \( \{ \bigcup_{F \in \mathcal{F}} F : \mathcal{F} \subseteq G, \# \mathcal{F} > 0, \bigcup_{F \in \mathcal{F}} F \text{ free of opposites} \} \) composed of the unions, free of opposites, of the sets from the family \( G \) ordered by inclusion and augmented by a new least element \( \hat{0} \); the Möbius function of the poset \( \mathring{\mathcal{E}}(G) \) is denoted by \( \mu_{\mathring{\mathcal{E}}}(\cdot, \cdot) \).

Formula (6.1) below is deduced from (5.1). Formulas (6.2) and (6.3) are deduced from (5.2); they are direct analogues of formulas (4.4) and (4.5), respectively. See also [9, Section 3].

**Theorem 6.1.** The number \( \#K_k^+(\mathcal{M}) \) of tope committees which are free of opposites, of cardinality \( k, 3 \leq k \leq |\mathcal{T}|/2 \), for the oriented matroid \( \mathcal{M} := (E_t, \mathcal{T}) \), is:

\[
\#K_k^+(\mathcal{M}) = \sum_{G \in \mathring{\mathcal{E}}(G)} \mu_{\mathring{\mathcal{E}}}(\hat{0}, G) \cdot \sum_{G \in \mathring{\mathcal{E}}(G)} \mu_{\mathring{\mathcal{E}}}(\hat{0}, G) \cdot 2^{k - |G|} \cdot \left( \frac{\frac{1}{2}|\mathcal{T}| - |G|}{k - |G|} \right), \quad (6.1)
\]
where \( \mu_C(\cdot, \cdot) \) denotes the Möbius function of the family \( \hat{C} := \{\hat{0}\} \)
\( \cup \{ \bigcup_{E \in E_t} \left( \frac{T_k^+}{(k+1)/2} \right) : E \subseteq E_t, |E| > 0 \} \) ordered by inclusion.

(ii)
\[
\#K_k^*(\mathcal{M}) = \sum_{\mathcal{G} \subseteq \bigcup_{E \in E_t} \left( \frac{T_k^+}{(k+1)/2} \right)} (-1)^{\#\mathcal{G}} \cdot \mu_S(\mathcal{G})(\hat{0}, \hat{1})
\]
\[
= \sum_{\mathcal{G} \subseteq \bigcup_{E \in E_t} \left( \frac{T_k^+}{(k+1)/2} \right)} (-1)^{\#\mathcal{G}} \cdot 2^{k-|\bigcup_{G \subseteq E} G|} \cdot \left( \frac{1}{2} |T| - \left| \bigcup_{G \subseteq E} G \right| \right) \cdot \left( k - \left| \bigcup_{G \subseteq E} G \right| \right). \tag{6.2}
\]

In particular, if for any family \( \mathcal{G} \subseteq \bigcup_{E \in E_t} \left( \frac{T_k^+}{(k+1)/2} \right) \) such that \( \bigcup_{G \subseteq E} G \) is free of opposites, \( \bigcup_{F \in \min \gamma_k(\mathcal{G})} F = E_t \) and \( |\bigcup_{G \subseteq E} G| \leq k \), it holds \( |\gamma_k(G)| = 1 \), for any set \( G \in \mathcal{G} \), then
\[
K_k^*(\mathcal{M}) = (-1)^t
\]
\[
\sum_{\mathcal{G} \subseteq \bigcup_{E \in E_t} \left( \frac{T_k^+}{(k+1)/2} \right)} (-1)^{\#\mathcal{G}} \cdot 2^{k-|\bigcup_{G \subseteq E} G|} \cdot \left( \frac{1}{2} |T| - \left| \bigcup_{G \subseteq E} G \right| \right) \cdot \left( k - \left| \bigcup_{G \subseteq E} G \right| \right). \tag{6.3}
\]

7. Relative Blocking in Principal Order Ideals of Binomial Posets

In this section we mention an analogue of formula (4.3) in the more general context of binomial posets.

Let \( P \) be a graded lattice of rank \( n \) which is a principal order ideal of some binomial poset. The factorial function \( B(k) \) of \( P \) counts the number of maximal chains in any interval of length \( k \) in \( P \). The number \( \left[ \frac{j}{2} \right] \) of elements of rank \( i \) in an interval of length \( j \) in \( P \) is equal to \( \frac{B(j)}{B(2j)B(j-i)} \), see [11] §3.15.

Let \( \Lambda \) be a nontrivial antichain in the lattice \( P \). If \( r \) is a rational number, \( 0 \leq r < 1 \), and \( k \) is a positive integer, then the set \( I_{r,k}(P, \Lambda) \) of relatively \( r \)-blocking elements, of rank \( k \), for the antichain \( \Lambda \) in \( P \), is defined as follows:
\[
I_{r,k}(P, \Lambda) := \{ b \in P : \rho(b) = k, \rho(b \wedge \lambda) > r \cdot k \quad \forall \lambda \in \Lambda \} \subseteq P^{(k)},
\]
where \( \rho(\cdot) \) is the rank function of \( P \), and \( P^{(k)} \) is the \( k \)th layer of \( P \).
Let $\mathcal{N}(A)$ be the abstract simplicial complex whose facets are the inclusion-maximal sets of indices $\{i_1, \ldots, i_j\}$ such that for the corresponding antichains $\{\lambda_{i_1}, \ldots, \lambda_{i_j}\} \subseteq A$ it holds $\lambda_{i_1} \land \cdots \land \lambda_{i_j} > 0$, where $0$ is the least element of $P$. If the poset $P$ is the Boolean lattice $\mathbb{B}(n)$ then the complex $\mathcal{N}(A)$ is the nerve of the corresponding Sperner family; see, e.g., [3, §10] on the topological combinatorics of the nerve.

Set $\nu(r \cdot k) := [r \cdot k] + 1$. Let
\[
\nu(r \cdot k) \colon P^{(\nu(r \cdot k))} \cap \mathcal{J}(A) \rightarrow \mathcal{N}(A),
\]
\[
d \mapsto \max \{ N \in \mathcal{N}(A) : d \leq \bigwedge_{i \in N} \lambda_i \}
\]
be the mapping that reflects an element $d$, of rank $\nu(r \cdot k)$, of the order ideal $\mathcal{J}(A)$ generated by the antichain $A$ to the inclusion-maximum face of the complex $\mathcal{N}(A)$ with the property $d \leq \bigwedge_{i \in N} \lambda_i$.

Associate to a subset $D \subseteq P^{(\nu(r \cdot k))} \cap \mathcal{J}(A)$, such that $|\bigcup_{F \in \min c_{r,k}(D)} F| = |A|$, a poset $S(D)$ which is the family $\{\bigcup_{F \in \mathcal{F}} F : F \leq \min c_{r,k}(D), |\mathcal{F}| > 0\}$ ordered by inclusion, with a new least element $0$ adjoined; here $\min c_{r,k}(D)$ denotes the subfamily of all inclusion-minimal sets from the image $c_{r,k}(D)$. Let $\mu_{S(D)}(0,1)$ denote the corresponding Möbius number, where $1$ is the greatest element of $S(D)$.

Suppose that $\nu(r \cdot k) \leq \min_{\lambda \in A} \rho(\lambda)$. Since
\[
|I_{r,k}(P,A)| = \sum_{D \subseteq P^{(\nu(r \cdot k))} \cap \mathcal{J}(A) : |D| > 0} (-1)^{|D|}
\]
\[
\cdot \left( \sum_{C \subseteq A : \bigwedge_{D \subseteq \mathcal{J}(C)} \mu_{S(D)}(0,1) \cdot \left( \frac{n - \rho(\bigvee_{d \in D} d)}{n - k} \right) \right),
\]
by [10] (5.4), we have
\[
|I_{r,k}(P,A)| = \sum_{D \subseteq P^{(\nu(r \cdot k))} \cap \mathcal{J}(A) : 1 \leq |D| \leq \nu(r \cdot k), |\bigcup_{F \in \min c_{r,k}(D)} F| = |A|, \rho(\bigvee_{d \in D} d) \leq k} (-1)^{|D|} \cdot \mu_{S(D)}(0,1) \cdot \left( \frac{n - \rho(\bigvee_{d \in D} d)}{k - \rho(\bigvee_{d \in D} d)} \right).
\]

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