Small signal gain analysis for a wiggler with noncollinear laser wave and electron beam

A.I. Artemiev$^{a,b}$, D.N. Klochkov$^{a,b}$, G. Kurizki$^{b}$, N.P. Poluektov$^{a}$, N.Yu. Shubin$^{a,c}$

$^a$General Physics Institute RAS, 38 Vavilov street, Moscow, 119991 Russia
$^b$Chemical Physics Department, Weizmann Institute of Science, Rehovot 76100, Israel
$^c$Institute of Microprocessor Computer Systems RAS, 36/1 Nakhimovsky prospect, Moscow 117997 Russia

Abstract

The collective and single-electron amplification regimes of a non-collinear free electron laser are analyzed within the framework of dispersion equations. The small-signal gain and the conditions for self-amplified excitations are found. The collective excitations in a free electron laser are shown to be favored by the non-collinear arrangement of the relativistic electrons and the laser wave. Implications for free-electron lasing without inversion are discussed.

1 Introduction

In a free-electron laser (FEL) [1, 2], the accelerated motion of electrons in the pondermotive potential of the combined field of the wiggler and the amplified electromagnetic wave produces coherent stimulated radiation. The influence of the pondermotive potential induces structuring of the spatial density of electrons (bunching) on the scale of the laser wavelength. As a result net emission is enhanced. The electron beam in FELs is usually aligned along the amplified electromagnetic wave and the wiggler. The non-collinear geometry of the electron beam and laser waves structures the spacial and momentum distributions of electrons in a way that gives rise to a new amplification mechanism called free-electron laser without inversion (FELWI) [3]-[8]. These FELs use the advanced phase control to enhance the gain via interference of radiation produced in the two wigglers. To extend the analysis of FELWIs from the single-electron (Thompson) regime to the collective (Raman) regime of electromagnetic excitation a detailed study should be made of the amplification in a single wiggler for a non-collinear arrangement of the electron beam and the amplified wave. The goal of the current paper is to perform this analysis. Investigation of the regimes and conditions for the amplifications in a non-collinear FEL geometry is also interesting from academic point of view. This investigation generalizes the results of the book [10] and allows one to obtain the applicability limits of a single-electron approach used in papers [3]-[8]. It can have implications for other types of free-electron lasers as well.

An FELWI is composed of two magnetic wigglers being spatially separated by a drift region with magnetic field. There is a small angle between the axes of wigglers. The laser wave propagates in the direction having angles $\theta_1$ and $\theta_2$ with the axes of the first and the second wigglers, respectively. The electron beam is directed at the angles $\alpha_1$ and $\alpha_2$ to the axes of the wigglers. In the drift region the electrons are turned by the magnetic field. In this device the electrons having different acceleration in the first wiggler enter the
drift region at different directions. A magneto-optics set then separates these electrons and introduces delays in their entrance phases (times of entrance) for the second wiggler. It was shown [3]-[8] that one can control the time of electron entrance to the second wiggler so that the gain $G$ of FELWI as a function of the detuning from the resonance condition $\Omega = \omega (v_0 - v_{res}) / c$ is mostly positive and thus $\int G(\Omega) d\Omega > 0$. These results were obtained using the single-electron approximation (Thompson regime): the propagation of a single electron through the FEL system was considered and the resulting gain was averaged over the electron distribution. But it is known that the change of the system geometry may influence the type of excitation regime, i.e. it can lead to a change from the single-electron amplification regime to the collective one and vice versa. For example, in the paper [9] the collective regime was considered in a non-collinear wiggler filled with an overdense homogeneous plasma.

2 Basic equations

In order to find an analytical solution of the problem we assume without a loss of generality that all the electrons have equal velocities at the entrance of the first wiggler. We use the approach developed in plasma electronics [10], which naturally describes the beam oscillations.

Let us consider the induced radiation by an electron beam in the first wiggler. We choose the coordinate system so that the axis $0z$ coincides with the axis of the wiggler while the wiggler vector-potential is parallel to the axis $0y$. By assuming that the static magnetic field of a plane wiggler $A_w$ is independent of the transverse coordinates $x$ and $y$, we can approximate it by harmonic function

$$A_w = A_w e_y,$$

where $A_w = A_0 e^{-ik_w r} + c.c.$, (1)

where $k_w = (0, 0, k_w)$ is the wiggler wave vector, “c.c.” denotes complex conjugation, and $e_y$ is a unit vector along $y$ axis. The wiggler field causes the electron oscillations along the $y$-axis, therefore such an electron interacts most efficiently with a linearly polarized light wave. So we assume that vector potential of laser wave has linear polarization $A_L = A_L(t, x, z)e_y$. In this case the vector potential $A_L$ defines the purely vortex part of the field $\text{div} A_L = 0$, while the scalar potential $\phi = \phi(t, x, z)$ defines longitudinal beam waves in the system. The Maxwell equations can be written in the form:

$$\Delta_l \phi \equiv (\partial_x^2 + \partial_z^2)\phi = -4\pi \rho,$$

$$\left( c^2 \partial_t^2 + \partial_x^2 - \partial_y^2 \right) A_L = -4\pi c j_y. $$

The electron beam entering the wiggler is assumed to have uniform density $n_b$ and uniform electron velocity $u = (-u \sin \alpha; 0; u \cos \alpha)$. Then the initial distribution function can be written in the form $f_0 = n_b \delta(p_0 - m\gamma_0 u)$. Here $e$ and $m$ are the electron charge and mass, $\gamma$ is the Lorentz factor. The integral over the initial coordinates with this initial distribution function gives the charge and current densities for beam with charge compensated

$$\rho = en_b \left\{ \int \delta[x - x(t, x_0, z_0)] \delta[z - z(t, x_0, z_0)] dx_0 dz_0 - 1 \right\},$$

$$j_y = en_b \int v_y(t, x_0, z_0) \delta[x - x(t, x_0, z_0)] \delta[z - z(t, x_0, z_0)] dx_0 dz_0. $$

2
Here \(x(t, x_0, z_0)\) and \(z(t, x_0, z_0)\) are the solutions of Hamilton equations

\[
\dot{r} = \frac{\partial H}{\partial P}, \quad \dot{P} = -\frac{\partial H}{\partial r}
\]

(4)

with initial conditions \(r_\parallel(0) = r_\parallel(0), P_\parallel(0) = m\gamma_0 u\). \(P = p + \frac{e}{c}A\) is the canonical momentum; \(A = A_w + A_L\) is a sum of vector potentials. The Hamiltonian of the electron in the field

\[
H = \sqrt{m^2 c^4 + c^2 (P - \frac{e}{c}A)^2 + e\phi} = mc^2 \gamma + e\phi
\]

(5)

does not depend on \(y\): \(\partial H/\partial y = 0\), and so we obtain the first integral

\[
v_y = \left. -\frac{e}{mc} \frac{A(t, x, z)}{\gamma} \right|_{x=x(t, x_0, z_0)}^{z=z(t, x_0, z_0)}.
\]

(6)

We represent all vectors as sums of two components: the first component (designated as \(f_\parallel = (f_x, 0, f_z)\)) being in the plane \(xz\), and the second component (designated as \(f_y = e_y\)) being parallel to the vector-potential or vector \(e_y\). The Hamilton equations (4) determine the electron coordinate and velocity

\[
\dot{r}_\parallel = \frac{\partial H}{\partial P_\parallel} = v_\parallel,
\]

\[
\dot{v}_\parallel = -\frac{e}{m\gamma} \left[ \nabla_\parallel - \frac{1}{c^2}v_\parallel (v_\parallel \cdot \nabla_\parallel) \right] \phi - \frac{1}{2} \left( \frac{e}{mc} \right)^2 \frac{1}{\gamma^2} \left[ \nabla_\parallel + \frac{v_\parallel}{c^2} \frac{\partial}{\partial t} \right] A^2 .
\]

(7)

We introduce two relativistic factors

\[
\gamma_\parallel = \left(1 - \frac{v_\parallel^2}{c^2}\right)^{-1/2}, \quad \gamma = \gamma_\parallel \left[1 + \frac{1}{c^2} \left(\frac{e}{mc}\right)^2 A^2 \right]^{1/2}.
\]

(8)

The field equations (2) take the form

\[
\Delta_\parallel \phi = -\frac{m}{e} \omega_b^2 \left\{ \int \delta [r_\parallel - r_\parallel(t, r_\parallel(0))] dr_\parallel + 1 \right\},
\]

(9)

\[
(c^2 \Delta_\parallel - \partial_t^2)A_L - \omega_b^2 \int \frac{A_L}{\gamma} \delta [r_\parallel - r_\parallel(t, r_\parallel(0))] dr_\parallel = \omega_b^2 \int \frac{A_w}{\gamma} \delta [r_\parallel - r_\parallel(t, r_\parallel(0))] dr_\parallel.
\]

(9)'

Here \(\omega_b^2 = 4\pi e^2 n_b/m\) is square of Langmuir frequency of the electron beam; here and below \(\gamma = \gamma(t, r_\parallel(0))\).

We look for the solutions for the field in the forms

\[
\phi = \frac{1}{2} [\psi e^{i k_0 r_\parallel} + c.c.],
\]

\[
A_L = A_+ e^{i(k_0 - k_w) r_\parallel} + A_- e^{-i(k_0 + k_w) r_\parallel}.
\]

(10)
Here vector $k_0 = k_0 \sin \theta, 0, \cos \theta$ lies in plane $xz$. We denote the dimensionless coordinates as $\xi = k_0 r_\parallel, \xi_0 = k_0 r_\parallel 0$ and introduce the dimensionless spatial Fourier-components of the electron charge and current density, $\sigma$ and $\hat{\sigma}$, respectively:

$$\sigma = \frac{1}{\pi} \int_0^{2\pi} e^{-i\xi} d\xi_0, \quad \hat{\sigma} = \frac{1}{\pi} \int_0^{2\pi} e^{-i\xi} \frac{\dot{\gamma}}{\gamma} d\xi_0,$$

(11)

Note that the integration is performed over the laser wavelength. Here and below $\xi = \xi(t, \xi_0)$. Substituting the solutions (10) in Eqs. (9) and averaging these equations over wavelength we get

$$\phi = \frac{1}{2} m e \omega^2 b k_0^2 \left[ \sigma e^{i\xi} + \text{c.c.} \right],$$

(12)

$$\frac{d^2 A_+}{dt^2} + \omega_+^2 A_+ + \omega_0^2 I_0 A_+ = -\frac{1}{2} \omega_0^2 \hat{\sigma} A_0,$$

(12')

$$\frac{d^2 A_-}{dt^2} + \omega_-^2 A_- + \omega_0^2 I_0^* A_- = -\frac{1}{2} \omega_0^2 \hat{\sigma}^* A_0.$$

(12'')

where

$$\omega_+^2 = (k_0 \pm k_w)^2 c^2 + \omega_0^2 \langle \gamma^{-1} \rangle,$$

$$I_0 = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{-2i\xi(t, \xi_0)}}{\gamma(t, \xi_0)} d\xi_0,$$

(13)

$$\langle \gamma^{-1} \rangle = \frac{1}{2\pi} \int_0^{2\pi} \frac{d\xi_0}{\gamma(t, \xi_0)}.$$

Eqs. (12), (12') are the equations of stimulated oscillations of two coupled systems: the electron beam and the amplified electromagnetic field.

The equations of the electron motion take the form

$$\dot{\mathbf{v}}_\parallel = -\frac{i \omega_0^2}{2 k_0^2} \left[ \mathbf{k}_0 - \frac{1}{c^2} \mathbf{v}_\parallel (k_0 \mathbf{v}_\parallel) \right] \sigma e^{i\xi} - \left( \frac{e}{mc} \right)^2 e^{2i\xi} \left( i \mathbf{k}_0 + \frac{\mathbf{v}_\parallel}{c^2} \frac{d}{dt} \right) (A_0^* A_+ + A_0 A_+^*) + \text{c.c.},$$

(14)

$$\dot{r}_\parallel = \mathbf{v}_\parallel,$$

(14')

with the initial conditions $r_\parallel(t = 0) = r_\parallel 0, \mathbf{v}_\parallel(t = 0) = \mathbf{u}$. The self-consistent system of Eqs. (11)–(14) determines the stimulated radiation in the wiggler and describes both linear and nonlinear regimes of the FEL instability.

3 Small signal gain

3.1 Dispersion equation

Further we consider the linear stage of instability (small signal gain). We linearize Eqs. (11)–(14) for small perturbations $\delta \mathbf{r}, \delta \mathbf{v}$, which are proportional to the amplitudes of the laser
waves $A_{\pm}$. All values are expanded in sums of non-disturbed and disturbed components: $r_{\parallel} = r_{\parallel 0} + u t + \delta r_{\parallel}$ or $\xi = \xi_0 + k_0 u t + k_0 \delta r_{\parallel}$, $v_{\parallel} = u + \delta v_{\parallel}$, $\omega = k_0 u + \Delta \omega$, $\gamma = \gamma_0 + \delta \gamma$ and $\gamma_{\parallel} = \gamma_{\parallel 0} + \delta \gamma_{\parallel}$. Here

$$\gamma_0 = \gamma_{\parallel 0} \sqrt{1 + \mu}, \quad \gamma_{\parallel 0} = (1 - \beta^2)^{-1/2},$$

where $\beta = u/c$. The wiggler parameter $\mu$, which will play a significant role, is defined as dimensionless square of the wiggler field amplitude

$$\mu = \frac{2}{c^2} \left( \frac{e}{mc} \right)^2 |A_0|^2$$

By linearizing equations over small perturbations, we obtain $I_0 = 0$ and

$$\sigma = \delta \sigma e^{-ik_0 u t}, \quad \delta \sigma = \frac{1}{\pi} \int_0^{2\pi} (-i k_0 \delta r_{\parallel}) e^{-i \xi_0} d\xi_0$$

$$\dot{\sigma} = \delta \dot{\sigma} e^{-ik_0 u t}, \quad \delta \dot{\sigma} = \frac{\delta \sigma}{\gamma_0} - \frac{1}{\pi \gamma_0} \int_0^{2\pi} \frac{\delta \gamma}{\gamma_0} e^{-i \xi_0} d\xi_0$$

For the small signal gain the vector-potential is a harmonic function of time

$$A_{\pm} = a_{\pm} e^{i \omega t}.$$  

The frequency $\omega$ is complex and its imaginary part defines the growth rate of the FEL instability.

The solution to the linearized equations of motion follows:

$$\delta v_{\parallel} = \left( \frac{e}{mc} \right)^2 \frac{e^{i \xi_0}}{D_b \gamma_0^3} \left( \beta_1 k_0 - \frac{\omega}{c^2} \beta_2 u \right) (A_0^* a_+ + A_0 a_+^*) e^{-i \Delta \omega t} + \text{c.c.}$$

$$\delta r_{\parallel} = i \left( \frac{e}{mc} \right)^2 \frac{e^{i \xi_0}}{D_b \gamma_0^3 \Delta \omega} \left( \beta_1 k_0 - \frac{\omega}{c^2} \beta_2 u \right) (A_0^* a_+ + A_0 a_+^*) e^{-i \Delta \omega t} + \text{c.c.}$$

Here

$$D_b = (\omega - k_0 u)^2 - \Omega_b^2$$

is the dispersion function of electron beam wave associated with the beam frequency $\Omega_b$, where

$$\Omega_b^2 = \frac{\omega_b^2}{\gamma_0} \left[ 1 - \frac{(k_0 u)^2}{\gamma_0^2 c^2} \right].$$

The coefficients $\beta_1$ and $\beta_2$ equal

$$\beta_1 = \gamma_0 (\omega - (k_0 u)) - \frac{\omega_b^2 (k_0 u)}{k_0^2 c^2}, \quad \beta_2 = \gamma_0 (\omega - (k_0 u)) - \frac{\omega_b^2}{\omega}.$$  

The perturbations of the dimensionless charge density $\sigma$ and the dimensionless current density $\dot{\sigma}$ follow:

$$\delta \sigma = 2 \left( \frac{e}{mc} \right)^2 \frac{1}{D_b \gamma_0^2} \left( \frac{k_0^2}{\gamma_0^2} - \frac{(k_0 u) \omega}{c^2} \right) (A_0^* a_+ + A_0 a_+^*) e^{-i \Delta \omega t},$$

$$\delta \dot{\sigma} = \frac{2}{c^2} \left( \frac{e}{mc} \right)^2 \frac{c^2 k_0^2 - \omega^2 + \omega_b^2 \gamma_0^{-1}}{D_b \gamma_0 \gamma_{\parallel 0} (1 + \mu)} (A_0^* a_+ + A_0 a_+^*) e^{-i \Delta \omega t}.$$
Substituting (23′) in field equation (12′), (12′′) we obtain the dispersion equation, which defines relation $\omega = \omega(k)$.

Let us consider the resonant case $\omega \approx \omega_+ = k_0 u - \Omega_b$, which corresponds to the maximal growth rate of the FEL instability. In this case $A_\perp = a_\perp = 0$. As a result the dispersion equation takes the simple form

$$D_b(\omega^2 - \omega_+^2) = \frac{1}{2} \frac{\omega_b^2}{1 + \mu} \frac{\omega_b^2}{\Omega_b^2 \gamma_0^2} \delta = |q|,$$  \hspace{1cm} (24)

Here

$$\omega_+^2 = (k_0 - k_\perp)^2 c^2 + \frac{\omega_b^2}{\gamma_0}$$ \hspace{1cm} (25)

The solution of the dispersion equation (24) under the resonant condition gives the frequency $\omega$

$$\omega = \omega_+ + \delta \omega = (k_0 u) - \Omega_b + \delta \omega.$$  \hspace{1cm} (26)

The presence of the beam leads to the complex shift of frequency $\delta \omega$ (where $|\delta \omega| \ll \omega_+$). Its imaginary part is the growth rate.

For the resonant conditions described above, the dispersion function of the beam and the detuning of the frequency from the resonance are equal to $D_b = \delta \omega^2 - 2 \delta \omega \Omega_b$, $\Delta_\omega = \delta \omega - \Omega_b$, respectively.

We introduce the complex dimensionless shift frequency $\delta = \delta \omega/\Omega_b$. The dispersive equation (24) can be written in terms of $\delta$ as

$$\delta^2(\delta - 2) + \frac{1}{2} \frac{\omega_b^2}{1 + \mu} \frac{\omega_b^2}{\Omega_b^2 \gamma_0^2} \delta = |q|.$$  \hspace{1cm} (27)

Where

$$|q| = \frac{1}{4} \frac{\mu}{1 + \mu} \frac{(1 + \nu)^2}{\nu} \left( \frac{k_0 c}{k_0 u} \right)^2 \frac{\omega_b^2}{\Omega_b^2 \gamma_0^2 \gamma_\parallel^2},$$  \hspace{1cm} (28)

and

$$\nu = \frac{\omega_b^2}{\omega_+ \Omega_b \gamma_0}.$$  \hspace{1cm} (29)

For a non-relativistic beam ($\beta \ll 1$) the parameter $\nu$ reduces to the ratio of the frequencies $\nu = \omega_b/\omega_+$, i.e. to dimensionless Langmuir frequency. It is shown below that the parameter $\nu$ defines the normal or anomalous behaviors of the growth rate, while the parameter $|q|$ defines the regime of instability (Raman or Thompson).

Note that for collinear FEL geometry, when $\alpha = \theta = 0$, and relativistic electron beams we get $|q| \approx 0.25 \mu/(1 + \mu) \cdot (1 + \nu)^2/\nu$, that is the parameter $|q|$ depends on $\gamma_\parallel$ only through the intermediary value $\nu$. To the contrary, for non-collinear FEL geometry, when $\alpha + \theta \neq 0$, and relativistic electrons the parameter $|q|$ will strongly depend on $\gamma_\parallel$. For $\gamma_\parallel \sin(\alpha + \theta) \gg 1$ we obtain the asymptotic

$$|q| \approx \frac{\mu}{1 + \mu} \frac{(1 + \nu)^2}{\nu} \frac{1}{\gamma_\parallel^2 \sin^2(2\alpha + 2\theta)}.$$  \hspace{1cm} (30)

In addition to, for collinear FEL geometry with $\gamma_\parallel$ increasing the parameter $\nu$ grows as a function $\nu = (\omega_b/\omega_+) \sqrt{\gamma_\parallel/(1 + \mu)} \propto \sqrt{\gamma_\parallel}$, while for non-collinear FEL geometry
under condition \( \gamma_{||0} \sin(\alpha + \theta) \gg 1 \) the parameter \( \nu \) drops as \( \gamma_{||0} \) increasing, namely \( \nu = \omega_b/(\omega + \sqrt{\gamma_{||0}^2 \sin(\alpha + \theta)}) \propto 1/\sqrt{\gamma_{||0}} \). This distinction leads to different dependance of the parameter \( |q| \) on \( \gamma_{||0} \): while for collinear laser geometry we have \( |q| \sim \sqrt{\gamma_{||0}} \) (for \( \nu \ll 1 \)) and \( |q| \sim 1/\sqrt{\gamma_{||0}} \) (for \( \nu \gg 1 \)), then for non-collinear laser geometry under ultra-relativistic conditions \( \gamma_{||0} \sin(\alpha + \theta) \gg 1 \) we have \( |q| \sim \gamma_{||0}^{-3/2} \). As was shown later, this mean that for relativistic electron beams \( \gamma_{||0} \sqrt{\nu} \sin(\alpha + \theta) \gg 1 \) for \( \nu \ll 1 \) and \( \gamma_{||0} \sin(\alpha + \theta)/\sqrt{\nu} \gg 1 \) for \( \nu \gg 1 \) propagating at a small angle to laser wave direction, the collective amplification is possible for any value of parameter \( \mu \) (as distinct from collinear wiggler geometry \[10\], for which Raman regime is absent for \( \mu > 1 \)), that is for any lateral relativistic velocity of electrons.

Let us consider different regimes of excitation.

### 3.2 Collective amplification

For the collective regime, when \( |\delta\omega| \ll \Omega_b \) or \( |\delta| \ll 1 \), and for relativistic beam \( \gamma_{||0} \sin(\alpha + \theta) \gg 1 \) the dispersion equation \((27)\) reduces to the quadratic form

\[
\delta^2 - \frac{1}{4} \frac{\mu}{1 + \mu \gamma_{||0}^2 \sin^2(\alpha + \theta)} \delta + \frac{1}{2} |q| = 0 \tag{31}
\]

leading to the growth rate for the collective regime: \( \text{Im}(\delta) = \sqrt{|q|/2} \) or

\[
\text{Im}(\delta\omega) = \frac{1}{2} \sqrt{\frac{\mu}{2(1 + \mu) k_0 c} \frac{\Omega_b \omega_b}{\gamma_{||0}}} \left( 1 + \frac{\omega_b^2}{\omega_b^2 + \Omega_b \gamma_{||0}} \right). \tag{32}
\]

The condition for Raman (collective) amplification can be rewritten as \( |q| \ll 1 \). Thus for non-collinear FEL geometry under relativistic condition \( \gamma_{||0} \sin(\alpha + \theta) \gg 1 \) the collective regime holds for any lateral relativistic velocity of electrons. The increasing of the longitudinal velocity (or relativistic factor \( \gamma_{||0} \)) for the non-collinear FEL geometry decreases the parameter \( |q| \) and thus leads to the collective regime of amplification, independently from the value of the wiggler parameter \( \mu \).

Consider asymptotic formulas for growth rates of the undulator radiation in the case of ultra-relativistic electron beams, \( \gamma_{||0} \sin(\alpha + \theta) \gg 1 \):

\[
\text{Im}(\delta\omega) = \begin{cases} 
\frac{1}{2} \sqrt{2(1 + \mu)} \frac{\sqrt{\omega_b \sin(\alpha + \theta)}}{\gamma_{||0} \cos(\alpha + \theta)}, & \frac{\mu}{1 + \mu \gamma_{||0}^2 \sin^2(\alpha + \theta)} \ll \nu \ll 1 \\
\frac{1}{2} \frac{\sqrt{\pi}}{2(1 + \mu)^{3/4} \gamma_{||0}^{3/4} \cos(\alpha + \theta) \sqrt{\omega_b \sin(\alpha + \theta)}}, & 1 \ll \nu \ll \frac{1 + \mu \gamma_{||0}^2 \sin^2(\alpha + \theta)}{\mu}. \tag{33}
\end{cases}
\]

The first growth rate \((33)\) is the usual one \([10]\) for collective regimes, since its dependence on Langmuir beam frequency is \( \omega_b^{1/2} \). The second growth rate is described by dependence \( \omega_b^{3/2} \). This anomalous behavior is a result of energy phase equalizing, which takes place both in collinear \([10]\) and non-collinear wiggler geometry. For a non-collinear FEL geometry the growth rate depends on the geometric parameter \( \sin(\alpha + \theta) \) yet. Note that the condition \( \nu \gg 1 \) can hold for overdense ultra-relativistic beam, when \( (\omega_b/\omega_+)^2 \gg \sin(\alpha + \theta)/\sqrt{1 + \mu} \).

The condition of the amplification with the second growth rate of Eq.(33) can be written in the form

\[
\max \left\{ \frac{1}{2}, \left( \frac{\mu}{1 + \mu} \frac{\omega_b}{\sqrt{\sin(\alpha + \theta)}} \right)^{3/4} \right\} \ll \gamma_{||0} \sin(\alpha + \theta) \ll \frac{\omega_b (1 + \mu)^{1/4}}{\omega_+ \sqrt{\sin(\alpha + \theta)}}. \tag{34}
\]
The formula (33) shows that the increasing of longitudinal velocity (or relativistic factor $\gamma_{||0}$) for non-collinear wiggler geometry leads to excitation of collective regime independently from the values $\mu$ and $\nu$.

### 3.3 Single-electron amplification

For the single-electron amplification (Thompson regime) the frequency shift $|\delta \omega|$ is larger than beam frequency, namely $|\delta \omega| \gg \Omega_b$ or $|\delta| \gg 1$, and the dispersion equation (27) is cubic

$$\delta^3 + \frac{1}{2} \frac{\mu}{1 + \mu} \frac{\omega_b^2}{\Omega_b^2 \gamma_{||0}^2} \delta - |q| = 0. \quad (35)$$

The solution Eq.(35), being written for image part of $\delta$, is

$$\text{Im}(\delta) = \frac{\sqrt{3}}{2} |q|^{1/3}. \quad (36)$$

The above definition of Thompson type of amplification ($|\delta| \gg 1$) can be rewritten as $|q| \gg 1$.

Consider the asymptotic of the growth rate $\text{Im}(\delta \omega)$ for $\gamma_{||0} \sin(\alpha + \theta) \gg 1$, the case of interest for FELWI applications. Under conditions

$$\nu \ll \min \left\{ \frac{\mu}{1 + \mu} \frac{1}{\gamma_{||0}^2 \sin^2(\alpha + \theta)}, 1 \right\}, \quad (37)$$

the asymptotic behavior of the growth rate is

$$\text{Im}(\delta \omega) = \frac{\sqrt{3}}{2^{5/3}} \left[ \frac{\mu}{(1 + \mu)^2} \frac{\omega_b^2 \omega^+}{\gamma_{||0}^3 \tan^2(\alpha + \theta)} \right]^{1/3}. \quad (38)$$

For very large $\nu$, namely $\nu \gg \gamma_{||0}^2 \sin^2(\alpha + \theta)(1 + \mu)/\mu$, the growth rate of single-electron amplification has the anomalous behavior

$$\text{Im}(\delta \omega) = \frac{\sqrt{3}}{2^{5/3}} \frac{\mu^{1/3}}{\sqrt{1 + \mu}} \left( \frac{\omega_b}{\omega^+} \right)^{1/3} \frac{\omega_b}{\gamma_{||0}^{4/3} \cos^{2/3}(\alpha + \theta)}. \quad (39)$$

As for collinear FEL geometry, here the growth rate depends on Langmuir frequency of the electron beam as $\omega_b^{4/3}$ and is almost independent from the angle between the electron beam and the laser wave.

However, the realization of this amplification regime using ultra-relativistic beam is almost impossible because of the large required charge of beam $(\omega_b/\omega^+) \gg \gamma_{||0}^{5/2} \sin^3(\alpha + \theta)(1 + \mu)^{3/4}/\mu$, and as a consequence it is necessary very big current, which is limited by vacuum current for vacuum devices.

The above calculations indicate that if the wiggler is loaded with the non-collinear electron beam and the laser wave, then the Raman-type amplification is feasible for relatively small densities of the electron beam (the first growth rate in Eq.(33)). We find that the electron current density required for Raman-type amplification drops with increasing the relativistic factor $\gamma_{||0}$ of beam. This means that collective amplification can be realized in optical wigglers, in particular, in FELWI, in which the ultra-relativistic non-collinear beams are used.
4  Conclusions

Summarizing, we consider Thompson and Raman regimes of FEL amplification for the non-collinear geometry of the electron and laser beams. It was found that the non-collinear geometry shifts the conditions for the amplification toward collective (Raman) regime. It was found that if the wiggler is loaded with the non-collinear electron beam and the laser wave, then the Raman-type amplification is feasible for relatively small densities of the electron beam (the first growth rate in Eq. (33)). We find that the electron current density required for Raman-type amplification drops with increasing the relativistic factor $\gamma_{0}$ of beam. This means that collective amplification can be realized exactly in optical wigglers, in particular, in an FELWI, which employs ultra-relativistic electron beams non-collinear laser wave.

5  Acknowledgements

N.Yu.Sh., D.N.K. and A.I.A. gratefully acknowledge support by the RFBR grant 02-02-17135 and support by the International Science and Technology Center, Moscow, through the Project A-820.

References

[1] J.M.J. Madey, Nuovo Cimento Soc. Ital. Fis. 50B, 64 (1979).
[2] C. A. Brau, Free-Electron Lasers (Academic, Boston, 1990).
[3] G. Kurizki, M.O. Scully, C. Keitel, Phys. Rev. Lett. 70, 1433 (1993).
[4] B. Sherman, G. Kurizki, Phys. Rev. Lett. 75, 4602 (1995).
[5] D.E. Nikonov, B. Scherman, G. Kurizki, M.O. Scully, Opt. Commun. 123, 363 (1996).
[6] D.E. Nikonov, M.O. Scully, G. Kurizki, Phys. Rev. E 54, 6780 (1996).
[7] D.E. Nikonov, Yu.V. Rostovtsev, G. Sussmann, Phys. Rev. E 57, 3444 (1998).
[8] Yu. V. Rostovtsev, S. Trendafilov, A. I. Artemyev, K. T. Kapale, G. Kurizki, and M. O. Scully, Phys. Rev. Lett. 90, 214802 (2003).
[9] A.B. Matsko, Yu.V. Rostovtsev, Phys. Rev. E 58, 7846, (1998).
[10] M.V. Kuzelev, A.A. Rukhadze Plasma Free Electron Lasers. Edition Frontier, Paris, 1995.