Winkler’s Hat Guessing Game: Better Results for Imbalanced Hat Distributions*

Benjamin Doerr
Max-Planck-Institute for Informatics
66123 Saarbrücken
Germany

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Abstract

In this note, we give an explicit polynomial-time executable strategy for Peter Winkler’s hat guessing game that gives superior results if the distribution of hats is imbalanced. While Winkler’s strategy guarantees in any case that \( \lfloor n/2 \rfloor \) of the \( n \) players guess their hat color correct, our strategy ensures that the players produce \( \max\{r, b\} - \frac{1.2n^{2/3}}{2} \) correct guesses for any distribution of \( r \) red and \( b = n - r \) blue hats. We also show that any strategy ensuring \( \max\{r, b\} - f(n) \) correct guesses necessarily has \( f(n) = \Omega(\sqrt{n}) \).

1 The Hat Color Guessing Game

In this note, we deal with the following game suggested by Peter Winkler [Win02]. In the *simultaneous hat guessing game*, there are \( n \) players each wearing a red or blue hat. Each player can see all hats except his own. Simultaneously, the players have to guess the color of their own hat. No communication is allowed during the game. The players may, however, discuss their strategy before they get to see the hats.

It is easy to see from the rules that no player can make sure that he guesses his hat color, no matter what strategy the players agree on. Thus the following result in [Win02] is quite surprising. There is a strategy that *guarantees* that \( \lfloor n/2 \rfloor \) players guess their hat color

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*This is an old result of mine never published except in my Habilitation thesis.*
correctly. In fact, the strategy is not too difficult, and the interested reader is encouraged to stop reading now and try to find such a strategy on his own.

A drawback of this strategy is that it also ensures that not more than $\lceil n/2 \rceil$ players guess correctly. This is particularly annoying since there is a strategy (and it is probably the first one most people think of) that seems to gain an advantage from the fact that there are more hats in one color than the other.

Assume for simplicity that $n$ is even and that there are $r$ red hats and $b = n - r$ blue ones. The majority strategy is for each player to guess that color which he can see more hats in. If there are more red hats than blue ones, the assumption that $n$ is even ensures that the difference is at least 2. Thus all players can see more red hats than blue ones. Using the majority strategy, all players wearing a red hat guess right. Hence this strategy is superior, leading to $\max\{r, b\}$ correct guesses, if the distribution of hats is imbalanced. Unfortunately, the majority strategy fails badly if there are as many red as blue hats. In this case, all player can see more hats in the color they are not wearing. Hence all players guess wrong.

In this note, we are looking for strategies that combine advantages of the 50%-strategy and the majority strategy. We present an explicit strategy that produces more correct guesses if the distribution of hats is imbalanced, but ensures that at least nearly 50% of the players guess right in any case. A probabilistic argument shows that no strategy can ensure the better outcome of the majority and the 50%-strategy in all cases. Thus to exploit imbalanced distributions, one has to pay a price in the sense that less than half of the players guess right for balanced partitions. But this price can be kept small: Our strategy produces $\max\{b, r\} - o(n)$ correct guesses on any distribution of $r$ red and $b$ blue hats. More precisely, we show the following.

**Theorem 1.** There is an explicit strategy such that $n$ players surely produce $\max\{r, b\} - 1.2n^{2/3} - 2$ correct guesses for any distribution of $r$ red and $b = n - r$ blue hats. This strategy requires the players to do only elementary, polynomial-time computations.

We also show that no strategy can provide a guarantee of better than $\max\{r, b\} - \Omega(\sqrt{n})$.

Subsequent to the first version of this work, Uriel Feige [Fei04] gave an existential proof for a strategy producing $\max\{r, b\} - O(\sqrt{n})$ correct guesses, but left open the problem whether there is a strategy of this quality such that all computations done by the players can be performed in polynomial time.
2 Notation and the Pairing Strategy

Let us assume from now on that \( n \) is even unless otherwise stated. We shall show that in this case there is a strategy ensuring \( \max\{r, b\} - 1.2n^{2/3} - 1 \) correct guesses for any distribution of \( r \) red and \( b \) blue hats. This yields Theorem 1.

2.1 Notation

Let the set of players simply be \([n]\). Then a distribution of hats is an \( \omega \) from \( \Omega := \{R, B\}^n \). Put \( R_\omega := \{i \in [n] \mid \omega_i = R\} \) and \( B_\omega := [n] \setminus R_\omega \). Formally, a strategy for the players is a function \( S : \Omega \to \Omega \) such that for all \( i \in [n] \) the \( i \)-th player’s guess \( S(\omega)_i \) is independent of \( \omega_i \). For a strategy \( S \) and \( \omega \in \Omega \) put \( \text{cor}(S, \omega) = |\{i \in [n] \mid S(\omega)_i = \omega_i\}| \), the number of correct guesses produced by \( S \) on \( \omega \).

2.2 Pairing Strategy

We briefly review from [Win02] the strategy ensuring \( \frac{n}{2} \) correct guesses. Assume the set \([n]\) of players partitioned into ordered pairs, i.e., there are \( x_i, y_i \in [n], i \in [n/2] \), such that \( \{x_i, y_i \mid i \in [n/2]\} = [n] \). Assume further that this pairing is known to the players. The pairing strategy with respect to this pairing is as follows: For all \( i \in [n/2] \), player \( x_i \) calls the color of \( y_i \)'s hat, player \( y_i \) calls the opposite color of \( x_i \)'s hat. Thus if \( x_i \)'s and \( y_i \)'s hat have the same color, then \( x_i \) guesses right and \( y_i \) wrong. If their hat colors are different, \( y_i \)'s guess is right and \( x_i \)'s is wrong. In particular, this strategy ensures that exactly one player from each pair guesses right.

Note that the pairing is independent of the hat colors. Thus the player may agree on the pairing prior to the guessing as part of their agreement on a strategy. Doing so and playing the pairing strategy ensures \( \frac{n}{2} \) correct guesses.

3 Probabilistic Analysis and Lower Bounds

As indicated in the introduction, the pairing strategy not only guarantees that \( \frac{n}{2} \) players guess right, it also guarantees that that many players guess wrong. This cannot be helped as can be seen from the following elementary probabilistic argument, which was already sketched in [Win02].

Assume that we pick a distribution of hats uniformly at random from \( \Omega \), i.e., we view \( \Omega \) as a probability space with probability distribution \( \Pr : \Omega \to [0, 1] \) defined by \( \Pr(\omega) = \frac{1}{|\Omega|} = 2^{-n} \) for all \( \omega \in \Omega \). Then any strategy in expectation produces \( \frac{n}{2} \) correct guesses.
Lemma 2. Let $S : \Omega \to \Omega$ be any strategy. Then the expected number of correct guesses produced by $S$ on a random hat distribution is $\frac{n}{2}$.

Proof. Define the following random variables. Denote by $X$ the number of correct guesses produced by $S$. For $i \in [n]$ let $X_i$ be $1$, if player $i$ guesses correctly, and $0$ otherwise. Then $X = \sum_{i=1}^{n} X_i$ and $\Pr(X_i = 1) = \Pr(X_i = 0) = \frac{1}{2}$. Thus $EX = \sum_{i=1}^{n} EX_i = \frac{n}{2}$. □

Thus from the viewpoint of average case analysis, the game regarded is rather boring. Lemma 2 has a nice combinatorial corollary.

Corollary 3. For all even $n \in \mathbb{N}$,

$$\sum_{0 \leq i \leq n} \binom{n}{i} \max\{i, n-i\} = 2^n \frac{n}{2}.$$

Proof. The expected number of correct guesses produced by the maximum strategy just is $2^{-n} \sum_{0 \leq i \leq n, \ i \neq n/2} \binom{n}{i} \max\{i, n-i\}$. Hence the claim follows from Lemma 2. □

Another consequence of the lemma is that no strategy can ensure $\max\{|R_\omega|, |B_\omega|\}$ correct guesses for all $\omega \in \Omega$. More precisely, we obtain the following.

Lemma 4. For any $n$, there is no strategy that produces more than $\max\{|R_\omega|, |B_\omega|\} - \sqrt{n/(2\pi)} \exp(-1/(3n)) + 1$ correct guesses on all $\omega \in \Omega$.

Proof. Let first $n$ be even. Assume there is a strategy $S$ such that $\text{cor}(S, \omega) \geq \max\{|R_\omega|, |B_\omega|\} - \sqrt{n/(2\pi)} \exp(-1/(3n))$ for all $\omega \in \Omega$. From Lemma 2 and Corollary 3 we have

$$2^n \frac{n}{2} = \sum_{\omega \in \Omega} \text{cor}(S, \omega)$$
$$> \sum_{\omega \in \Omega} \left( \max\{|R_\omega|, |B_\omega|\} - \sqrt{n/(2\pi)} \exp(-1/(3n)) \right)$$
$$= \sum_{i=0}^{n} \binom{n}{i} \max\{i, n-i\} - 2^n \sqrt{n/(2\pi)} \exp(-1/(3n))$$
$$= 2^n \frac{n}{2} + \binom{n}{n/2} \frac{n}{2} - 2^n \sqrt{n/(2\pi)} \exp(-1/(3n)).$$

Estimating $\binom{n}{n/2} \geq 2^n \sqrt{2/(\pi n)} \exp(-1/(3n))$, cf. e.g. Robbins [Rob55], yields a contradiction.
Now let \( n \) be odd and \( S \) any strategy. Extend \( S \) to a strategy \( S' \) for \( n+1 \) players by letting the \((n+1)\)-st player always guess \( R \) and all other players ignore the \((n+1)\)-st player’s hat. By the above, there is an \( \omega' \in \{R,B\}^{n+1} \) such that \[ \text{cor}(S', \omega') \leq \max\{|R_{\omega'}|, |B_{\omega'}|\} - \sqrt{(n+1)/(2\pi)} \exp(-1/(3n+3)). \] For \( \omega = (\omega_1, \ldots, \omega_n) \) we have
\[
\text{cor}(S, \omega) \leq \text{cor}(S', \omega') \leq \max\{|R_{\omega'}|, |B_{\omega'}|\} - \sqrt{n/(2\pi)} \exp(-1/(3n)).
\]

\[ \square \]

## 4 An Explicit Strategy

It seems that an easy solution to our problem is to play the pairing strategy, if the distribution of hats is balanced, and the majority strategy otherwise. It turns out that this does not work. The problem is that the pairing strategy works well only if both players from each pair apply it. Thus we needed to ensure that either all players apply the majority strategy or all apply the pairing strategy. The problem with our initial idea is that depending on whether he is wearing a majority color hat or not, the player regards the distribution as more or less balanced. Thus it seems difficult to get the players organized. We solve this problem as follows.

### 4.1 Partial Strategies \( S(T, a, b) \)

For our strategy we also assume that the players have agreed on a pairing as above. We say that a subset \( T \subseteq [n] \) of the players respects the pairing, if no pair intersects \( T \) non-trivially, i.e., if \( \{x_i, y_i\} \subseteq T \) or \( \{x_i, y_i\} \cap T = \emptyset \) holds for all \( i \in [n/2] \).

For such a subset \( T \) and integers \( a, b \) such that \( a < \frac{1}{2}|T| \leq b \) and \( a + 2 \leq b \), we define the following strategy \( S(T, a, b) \) for the players in \( T \): If a player in \( T \) can see at least \( b \) red hats, he guesses ‘red’. If he can see at most \( a \) red hats, he guesses ‘blue’. Otherwise he guesses according to the pairing strategy.

**Lemma 5.** Let \( \omega \in \Omega \) be any distribution of hats and \( m = \max\{|B_{\omega} \cap T|, |R_{\omega} \cap T|\} \). Then the strategy \( S(T, a, b) \) produces at least \( \text{cor}(\omega, T, a, b) \) correct guesses, where

\[
\text{cor}(\omega, T, a, b) := \begin{cases}
  m = |R_{\omega} \cap T| & \text{if } |R_{\omega} \cap T| > b, \\
  b - \frac{1}{2}|T| = m - \frac{1}{2}|T| & \text{if } |R_{\omega} \cap T| = b, \\
  \frac{1}{2}|T| - a - 1 = m - \frac{1}{2}|T| & \text{if } a + 2 \leq |R_{\omega} \cap T| \leq b - 1, \\
  m = |B_{\omega} \cap T| & \text{if } |R_{\omega} \cap T| = a + 1, \\
  \frac{1}{2}|T| - a - 1 = m - \frac{1}{2}|T| & \text{if } |R_{\omega} \cap T| \leq a.
\end{cases}
\]
Proof. If \(|R_\omega \cap T| > b\), then all players can see at least \(b\) red hats. Thus they all guess ‘red’ and \(m = |R_\omega \cap T|\) of them naturally are right. If \(|R_\omega \cap T| = b\), then those players wearing a blue hat can see \(b\) red ones and (wrongly) guess ‘red’, whereas the players wearing a red hat guess according to the pairing strategy. Since there are only \(|T| - b\) blue hats, at least \(2b - |T|\) players wearing a red hat have a partner wearing a red hat as well. Hence from at least \(b - \frac{1}{2}|T|\) pairs both partners guess according to the pairing strategy, producing one correct guess (and one false one) per pair.

If \(|R_\omega \cap T| \in \{a + 2, \ldots, b - 1\}\), then all players guess according to the pairing strategy, which yields \(\frac{1}{2}|T|\) correct guesses. If \(|R_\omega \cap T| = a + 1\), the players wearing a red hat can see only \(a\) red hats and thus (wrongly) guess ‘blue’. The \(m\) players wearing a blue hat can see \(a + 1\) red hats and hence guess according to the pair strategy. As above, this produces at least \(\frac{1}{2}|T| - a - 1\) correct guesses. If \(|R_\omega \cap T| \leq a\), all players guess ‘blue’, \(m = |B_\omega \cap T|\) of them being correct.

4.2 A Strategy for all Players

The strategy \(S(T, a, b)\) is not bad unless there are exactly \(a + 1\) or \(b\) red hats in \(T\). In this case we say that \(S(T, a, b)\) fails. Our plan is to partition the set of all players \([n]\) into \(k \geq 2\) subsets \(T_1, \ldots, T_k\) respecting the pairing and choose integers \(a_i, b_i\) for all \(i \in [k]\) in such a way that at most one strategy \(S(T_i, a_i, b_i)\) fails.

Assume the partition \([n] = T_1 \cup \ldots \cup T_k\) be given and known to the players. For all \(i \in [k]\) put \(T_i = [n] \setminus T_i\). Let \(b_i\) be minimal subject to \(b_i \geq \frac{1}{2}|T_i|\) and

\[|R_\omega \cap T_i| + b_i \equiv i \pmod{k}.\]

Note that each player in \(T_i\) can compute this number as he only needs to know the number of red hats in \([n] \setminus T_i\). Put \(a_i = b_i - k - 1\).

Lemma 6. Let \(\omega \in \Omega\) be any distribution of hats and \(T_i, a_i, b_i\) as above. Let \(i \in [k]\) such that \(i \equiv |R_\omega| \pmod{k}\). Then no strategy \(S(T_j, a_j, b_j)\), \(j \neq i\), fails. \(S(T_i, a_i, b_i)\) fails if and only if \(|R_\omega \cap T_i| \in \{a_i + 1, b_i\}\).

Proof. Let \(j \in [k]\) such that \(S(T_j, a_j, b_j)\) fails. Then \(|R_\omega \cap T_j| \in \{a_j + 1, b_j\}\). Note that \(a_j + 1 \equiv b_j \pmod{k}\) by definition. Hence

\[|R_\omega| = |R_\omega \cap T_j| + |R_\omega \cap T_j| \equiv |R_\omega \cap T_j| + b_j \equiv j \pmod{k}.\]

Thus at most one strategy may fail, namely the strategy \(S(T_i, a_i, b_i)\). This happens if and only if \(|R_\omega \cap T_i| \in \{a_i + 1, b_i\}\). \(\square\)

Let \(S\) be the union of the strategies \(S(T_i, a_i, b_i)\), \(i \in [k]\), i.e., the strategy such that a player contained in \(T_i\) follows the strategy \(S(T_i, a_i, b_i)\).
Lemma 7. For all $\omega \in \Omega$,

$$\text{cor}(S, \omega) \geq \max\{|R_\omega|, |B_\omega|\} - \frac{1}{2} \max_{i \in [k]} |T_i| - (k - 1)^2.$$  

Proof. Let $i \in [k]$ such that $|R_\omega| \equiv i \pmod{k}$. Assume that $\max\{|R_\omega|, |B_\omega|\} = |R_\omega|$. Let $j \neq i$. Then $S(T_j, a_j, b_j)$ does not fail by Lemma 6. From Lemma 5 we conclude that if $|R_\omega \cap T_j| \in \{a_j + 2, \ldots, b_j - 1\}$, then

$$\text{cor}(\omega, T_j, a_j, b_j) = \frac{1}{2}|T_j| = |R_\omega \cap T_j| - (|R_\omega \cap T_j| - \frac{1}{2}|T_j|) \geq |R_\omega \cap T_j| - (b_j - 1 - \frac{1}{2}|T_j|) \geq |R_\omega \cap T_j| - (k - 2).$$

If $|R_\omega \cap T_j| > b_j$, then $\text{cor}(\omega, T_j, a_j, b_j) = |R_\omega \cap T_j|$, and if $|R_\omega \cap T_j| \leq a_j$, then $\text{cor}(\omega, T_j, a_j, b_j) = |B_\omega \cap T_j| \geq |R_\omega \cap T_j|$. Hence in all cases we have $\text{cor}(\omega, T_j, a_j, b_j) \geq |R_\omega \cap T_j| - (k - 2)$. The possibly failing strategy $S(T_i, a_i, b_i)$ yields $\text{cor}(\omega, T_i, a_i, b_i) = \max\{|B_\omega \cap T_i|, |R_\omega \cap T_i|\} - \frac{1}{2}|T_i| \geq |R_\omega \cap T_i| - \frac{1}{2}|T_i|$ correct guesses.

Thus the total number of correct guesses is

$$\text{cor}(S, \omega) \geq |R_\omega \cap T_i| - \frac{1}{2}|T_i| + \sum_{j \in [k] \setminus \{i\}} (|R_\omega \cap T_j| - (k - 2)) \geq |R_\omega| - \frac{1}{2}|T_i| - (k - 1)(k - 2).$$

Assume now that $|R_\omega| < |B_\omega|$. For $j \neq i$, $S(T_j, a_j, b_j)$ does not fail. We have $\text{cor}(\omega, T_j, a_j, b_j) = |R_\omega \cap T_j| > |B_\omega \cap T_j|$, if $|R_\omega \cap T_j| > b_j$, and $\text{cor}(\omega, T_j, a_j, b_j) = |B_\omega \cap T_j|$, if $|R_\omega \cap T_j| \leq a_j$. If $|R_\omega \cap T_j| \in \{a_j + 2, \ldots, b_j - 1\}$, then

$$\text{cor}(\omega, T_j, a_j, b_j) = \frac{1}{2}|T_j| = |B_\omega \cap T_j| - (|B_\omega \cap T_j| - \frac{1}{2}|T_j|) \geq |B_\omega \cap T_j| - (|T_j| - (a_j + 2) - \frac{1}{2}|T_j|) \geq |B_\omega \cap T_j| - (k - 1).$$

Hence $\text{cor}(\omega, T_j, a_j, b_j) \geq |B_\omega \cap T_j| - (k - 1)$ for all $j \neq i$. Together with $\text{cor}(\omega, T_i, a_i, b_i) \geq |B_\omega \cap T_i| - \frac{1}{2}|T_i|$, we conclude

$$\text{cor}(S, \omega) \geq |B_\omega \cap T_i| - \frac{1}{2}|T_i| + \sum_{j \in [k] \setminus \{i\}} (|B_\omega \cap T_j| - (k - 1)) \geq |B_\omega| - \frac{1}{2}|T_i| - (k - 1)^2.$$

This proves the claim. \qed
4.3 Optimizing the Partition

It remains to choose a suitable partition $[n] = T_1 \cup \ldots \cup T_k$. Let $k = \left\lceil \sqrt[n/4] \right\rceil$. For any number $r \in \mathbb{R}$ denote by $\lceil r \rceil_2$ the smallest even integer not smaller than $r$, and by $\lfloor r \rfloor_2$ the largest even integer not exceeding $r$. Choose $\ell \in [k]$ such that $n = \ell \lceil n/k \rceil_2 + (k - \ell) \lfloor n/k \rfloor_2$ — recall that we assumed $n$ to be even. Let $[n] = T_1 \cup \ldots \cup T_k$ be such that $|T_i| = \lceil n/k \rceil_2$ for $i \in [\ell]$ and $|T_i| = \lfloor n/k \rfloor_2$ for $i \in [\ell + 1..k]$ and such that all $T_i$ respect our initially chosen pairing. Then the loss compared to $\max\{|R_\omega|, |B_\omega|\}$ as given by the previous lemma is at most

$$\frac{1}{2} \lceil n/k \rceil_2 + (k - 1)^2 \leq 1 + \frac{1}{2} 3\sqrt[3]{4n^{2/3}} + \frac{1}{\sqrt[3]{10}} n^{2/3} \leq 1 + 1.2n^{2/3}.$$

This proves Theorem \[\Box\]

References

[Fei04] U. Feige. You Can Leave Your Hat On (If You Guess Its Color). Technical report MCS04-03 of the Weizmann Institute, 2004. Available at www.wisdom.weizmann.ac.il/~feige/TechnicalReports/hats.ps

[Rob55] H. Robbins. A remark on Stirling’s formula. Amer. Math. Monthly, 62:26–29, 1955.

[Win02] P. Winkler. Games people don’t play. In D. Wolfe and T. Rodgers, editors, Puzzlers’ tribute. A feast for the mind. A K Peters, 2001.