SCHRÖDINGER PARTICLE IN MAGNETIC AND ELECTRIC FIELDS IN LOBACHEVSKY AND RIEMANN SPACES

Schrödinger equation in Lobachevsky and Riemann 4-spaces has been solved in the presence of external magnetic field that is an analog of a uniform magnetic field in the flat space. Generalized Landau levels have been found, modified by the presence of the space curvature. In Lobachevsky 4-model the energy spectrum contains discrete and continuous parts, the number of bound states is finite; in Riemann 4-model all energy spectrum is discrete. Generalized Landau levels are determined by three parameters, the magnitude of the magnetic field $B$, the curvature radius $\rho$ and the magnetic quantum number $m$. It has been shown that in presence of an additional external electric field the energy spectrum in the Riemann model can be also obtained analytically.

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1 Introduction. Spaces of a constant curvature and uniform magnetic field

Quantum-mechanical Kepler problems in spaces of constant positive and negative curvature were formulated and solved many years ago by Schrödinger, Infeld, Schild, and Stefenson [1, 2, 3, 4]. Dynamical symmetry for quantum Kepler problems in spaces of constant curvature was investigated in [5, 6, 7, 8]. These models were used to describe bound state in nuclear physics [10, 11, 12] and nano-physics [13, 14].

In the present paper the non-relativistic quantum mechanical problems for a particle in a magnetic field on the background of hyperbolical Lobachevsky and spherical Riemann spaces are solved. We start with Olevsky paper [15] where all 3-orthogonal coordinate systems \((x^1, x^2, x^3)\) in 3-space of constant curvature, negative and positive, permitting the full separation of variables in the extended Helmholtz equation \((\Delta_2 + \lambda)f(x) = 0\) have been found. An idea consists in searching for an analog of cylindrical coordinate of Euclid space \(E_3\), in which one could easily define the concept of a uniform magnetic field in spaces \(H_3\) and \(S_3\). Such uniform magnetic fields should be solutions of the corresponding Maxwell equations. Then we will try to examine the Landau problem [16, 18] of a Schrödinger’s particle in the external magnetic field on the background of curved space models, \(H_3\) and \(S_3\). Also, in present time we can note growing interest to Landau problem in the context of development in quantum mechanics in noncommutative space [24, 25, 26]. In the literature, the Landau problem was examined for Lobachevsky and spherical planes [19, 20, 21, 22, 23]; present paper agrees with the known results and adds to them.

In [15] in the list of coordinate systems, one can see the system number XI defined in space \(S_3\) by the relations

\[
\begin{align*}
  dS^2 &= c^2 dt^2 - [\cos^2 z (dr^2 + \sin^2 r d\phi^2)] - dz^2], \\
  z &\in [-\pi/2, +\pi/2], \quad r \in [0, \pi], \quad \phi \in [0, 2\pi], \\
  u_1 &= \cos z \sin r \cos \phi, \quad u_2 = \cos z \sin r \sin \phi, \\
  u_3 &= \sin z, \quad u_0 = \cos z \cos r, \quad u_1^2 + u_2^2 + u_3^2 + u_0^2 = 1;
\end{align*}
\]

and in space \(H_3\) as

\[
\begin{align*}
  dS^2 &= c^2 dt^2 - [\cosh^2 z (dr^2 + \sinh^2 r d\phi^2)] - dz^2], \\
  z &\in (-\infty, +\infty), \quad r \in [0, +\infty], \quad \phi \in [0, 2\pi], \\
  u_1 &= \cosh z \sinh r \cos \phi, \quad u_2 = \cosh z \sinh r \sin \phi, \\
  u_3 &= \sinh z, \quad u_0 = \cosh z \cosh r, \quad u_1^2 + u_2^2 + u_3^2 - u_0^2 = -1.
\end{align*}
\]

All coordinate \(u_a\) are dimensionless, what is achieved by dividing on the curvature radius \(\rho\). When \(\rho \to \infty\), these coordinates reduce to usual cylindrical ones in the space \(E_3\)

\[
dS^2 = c^2 dt^2 - (dr^2 + r^2 d\phi^2 + dz^2)
\]

Let us use the known form of the vector potential of the uniform constant magnetic field [2] in the flat space:

\[
A = \frac{1}{2} B \times r, \quad B = (0, 0, B), \quad A^a = \frac{Br}{2}(0, -\sin \phi, \cos \phi, 0),
\]

after translating to cylindrical coordinates

\[
A_t = 0, \ A_r = 0, \ A_z = 0, \ A_\phi = -\frac{Br^2}{2}.
\]
Correspondingly we have a single non-vanishing component of the strength tensor which obeys the Maxwell equations:

\[ F_{\phi r} = \partial_\phi A_r - \partial_r A_\phi = B r, \quad \frac{1}{r} \frac{\partial}{\partial r} r F^{\phi r} = \frac{1}{r} \frac{\partial}{\partial r} r \left( -\frac{1}{r^2} \right) B r \equiv 0. \]  

(3)

Such description of a uniform magnetic field in cylindrical coordinates may be extended quite easily to curved space models. Let us start with the following potential in the spherical space \( S_3 \)

\[ A_\phi = 2B (\cos r - 1), \quad F_{\phi r} = B \sin r, \]  

(4)

Analogously, in Lobachevsky model we start with

\[ A_\phi = -2B (\cosh r - 1), \quad F_{\phi r} = B \sinh r. \]  

(5)

As one can see both potentials are solutions of the respective Maxwell equations, and in the vanishing curvature limit they are reduced to (3).

2 Separation of variables in the Schrödinger equation

Non-relativistic Schrödinger equation in a space-time with the metrics

\[ dS^2 = (dx^0)^2 + g_{kl}(x) dx^k dx^l, \quad g = \det (g_{kl}(x)) \]

has the form

\[ (i \hbar \partial_t + e A_0) \Psi = H \Psi, \]

\[ H = -\frac{1}{2M} \left[ \left( i \hbar \sqrt{-g} \partial_k \sqrt{-g} + \frac{e}{c} A_k \right) g^{kl}(i \hbar \partial_l + \frac{e}{c} A_l) \right] \Psi. \]

In the space \( S_3 \), parameterized by cylindric coordinates and in the presence of the external uniform magnetic field, the Hamiltonian \( H \) takes the form

\[ H = \frac{\hbar^2}{2MR^2} \left[ -\frac{1}{\cos^2 z \sin r} \frac{\partial}{\partial r} \sin r \frac{\partial}{\partial r} + \frac{1}{\cos^2 z \sin^2 r} \left( i \frac{\partial}{\partial \phi} + \frac{e}{\hbar c} A_\phi \right)^2 - \frac{1}{\cos^2 z \sin^2 \phi} \frac{\partial}{\partial z} \right] \]

and the variables may be separated by the substitution:

\[ \Psi = e^{-iEt/\hbar} e^{im\phi} Z(\phi) R(\phi), \quad \epsilon = E/(\hbar^2/M_N^2), \]

\[ \frac{1}{Z} \left( 2\epsilon \cos^2 z \frac{Z}{Z} + \frac{d}{dz} \cos^2 z \frac{dZ}{dz} \right) \]

\[ = \frac{1}{R} \left( -\frac{1}{\sin r} \frac{d}{dr} \sin r \frac{dR}{dr} + \frac{1}{\sin^2 r} \left( m + \frac{eB}{\hbar c} \frac{1}{2} \right)^2 R \right). \]

Introducing the separation constant \( \lambda \), we arrive at

\[ \frac{d}{dz} \cos^2 z \frac{dZ}{dz} + 2\epsilon \cos^2 z Z = \lambda Z, \]

(7)

\[ -\frac{1}{\sin r} \frac{d}{dr} \sin r \frac{dR}{dr} + \frac{1}{\sin^2 r} (m + \frac{eB}{\hbar c} 2\rho^2 \sin^2 \frac{r}{2})^2 R = \lambda R. \]

(8)
In the same manner, in the Lobachevsky space \( H_3 \) we arrive at

\[
\frac{d}{dz} \cosh^2 z \frac{dZ}{dz} + 2 \epsilon \cosh^2 z Z = \lambda Z, \tag{9}
\]

\[
-\frac{1}{\sinh r} \frac{d}{dr} \sinh r \frac{dR}{dr} + \frac{1}{\sinh^2 r} \left( m + \frac{eB}{\hbar c} \frac{2 \rho^2 \sinh^2 r}{2} \right)^2 R = \lambda R. \tag{10}
\]

In the limit of flat space these equations become more simple:

\[
\frac{d^2 Z}{dz^2} + \frac{2EM}{\hbar^2} Z = \Lambda Z, \quad Z = e^{\pm iPz/\hbar}, \quad \Lambda = \frac{2M}{\hbar^2} \left( E - \frac{P^2}{2M} \right),
\]

\[
-\frac{1}{r} \frac{d}{dr} r \frac{dR}{dr} + \frac{1}{r^2} \left( m + \frac{eB}{\hbar c} \right)^2 R = \frac{2M}{\hbar^2} \left( E - \frac{P^2}{2M} \right) R \tag{11}
\]

which coincide with the known one [18]:

\[
\frac{\hbar^2}{2M} \left( \frac{d^2 R}{dr^2} - \frac{1}{r} \frac{dR}{dr} - \frac{m^2}{r^2} R \right) + \left[ E - \frac{P^2}{2M} - \frac{\omega^2}{8} r^2 - \frac{\hbar \omega m}{2} \right] R = 0
\]

with \( \omega = eB/Mc \).

### 3 Radial solutions in space \( H_3 \)

Below the notion \( \frac{eB}{\hbar c} \rho^2 \Rightarrow B \) will be used, then eq. (10) reads

\[
\frac{d^2 R}{dr^2} + \frac{1}{r} \frac{dR}{dr} - \frac{m^2}{r^2} R = \frac{2M}{\hbar^2} \left( E - \frac{P^2}{2M} \right) R + \lambda R = 0. \tag{12}
\]

In new variables

\[
\cosh r - 1 = -2z, \quad z = -\sinh^2 \frac{r}{2} \in (-\infty, 0],
\]

for the radial equation we get

\[
\left[ z(1 - z) \frac{d^2}{dz^2} + (1 - 2z) \frac{d}{dz} - \frac{1}{4} \left( \frac{m^2}{z} - 4B^2 + \frac{(m - 2B)^2}{1 - z} \right) - \lambda \right] R = 0. \tag{13}
\]

Let us specify the behavior of the function \( R \) near to the points 0 and 1 (take note that the value 1 does not belong to the physical domain of the coordinate \( r \)):

- if \( z \to 0 \), then \( R(z) \sim z^a \), \( a = \pm \frac{m}{2} \);
- if \( z \to 1 \), then \( R(z) \sim (1 - z)^b \), \( b = \pm \frac{m - 2B}{2} \).

Making substitution \( R = z^a (1 - z)^b F \), from eq. (13) we get

\[
+ z(1 - z) F'' + \left[ a(1 - z) - bz + a(1 - z) - bz + (1 - 2z) \right] F' + \frac{1}{z} \left[ a(a - 1) + a - \frac{m^2}{4} \right] F + \frac{1}{1 - z} \left[ b(b - 1) + b - \frac{(m - 2B)^2}{4} \right] F - \left[ a(a + 1) + 2ab + b(b + 1) - B^2 + \lambda \right] F = 0; \tag{14}
\]
Imposing conditions $a = \pm m/2$, $b = \pm(m - 2B)/2$ we obtain

\[ R = z^n(1 - z)^b F, \]

\[ z(1 - z)F'' + [(2a + 1) - 2(a + b + 1)z]F' - [a(a + 1) + 2ab + b(b + 1) - B^2 + \lambda]F = 0 \]

which is the equation of hypergeometric type: $z(1 - z)F + [\gamma - (\alpha + \beta + 1)z]F' - \alpha\beta F = 0$. Thus, the radial function is constructed in hypergeometric functions as follows: (to obtain the bound states we must take the parameters $a$ and $b$ positive and negative respectively)

\[ z = -\sinh^2 \frac{r}{2}, z \in (-\infty, +0], \]

\[ R = (-\sinh \frac{r}{2})^{m|} (\cosh \frac{r}{2})^{-|m-2B|} F_1(\alpha, \beta; \gamma; -\sinh^2 \frac{r}{2}) \quad (15) \]

where $(\alpha, \beta)$ are defined by

\[ \alpha + \beta = 2a + 2b + 1, \quad \alpha\beta = (a + b)(a + b + 1) - B^2 + \lambda \]

that is

\[ a = +\frac{|m|}{2}, \quad b = -\frac{|m - 2B|}{2}, \]

\[ \alpha = a + b + \frac{1}{2} - \sqrt{B^2 + \frac{1}{4} - \lambda}, \]

\[ \beta = a + b + \frac{1}{2} + \sqrt{B^2 + \frac{1}{4} - \lambda}, \]

\[ \gamma = 2a + 1 = + |m| + 1. \quad (16) \]

The first possibility to obtain polynomial is

\[ \alpha = a + b + \frac{1}{2} - \sqrt{B^2 + \frac{1}{4} - \lambda} = -n = 0, -1, \ldots \]

from whence it follows the quantization rule

\[ a + b + \frac{1}{2} + n \geq 0, \quad \lambda = \frac{1}{4} + B^2 - \left( a + b + n + \frac{1}{2} \right)^2 \quad (17) \]

or

\[ \frac{|m|}{2} - \frac{|m - 2B|}{2} + n + \frac{1}{2} \geq 0, \]

\[ \lambda = \frac{1}{4} + B^2 - \left( \frac{|m|}{2} - \frac{|m - 2B|}{2} + n + \frac{1}{2} \right)^2, \quad (18) \]

\[ R = (-\sinh \frac{r}{2})^{m|} (\cosh \frac{r}{2})^{-|m-2B|} F_1(-n, |m| - |m - 2B| +1 + n; |m| +1, -\sinh^2 \frac{r}{2}). \quad (19) \]

The second possibility to obtain the polynomial is

\[ \beta = a + b + \frac{1}{2} + \sqrt{B^2 + \frac{1}{4} - \lambda} = -n \]
that is
\[ a + b + \frac{1}{2} + n \leq 0, \quad \lambda = \frac{1}{4} + B^2 - (a + b + \frac{1}{2} + n)^2 \]  
(20)
or
\[
\lambda = \frac{1}{4} + B^2 - \left( \frac{|m|}{2} - \frac{|m-2B|}{2} + n + \frac{1}{2} \right)^2, 
\]
(21)
\[
R = (-\sinh \frac{r}{2})^{|m|}(\cosh \frac{r}{2})^{-|m-2B|} \binom{2}{1}([m]| - |m - 2B| + 1 + n, -n; |m| + 1; -\sinh^2 \frac{r}{2}). 
\]
(22)

Now, let we assume that \( B > 0 \). There exists five possibilities:
\[
m = 0, m < 0, 0 < m < 2B, m > 2B, m = 2B. 
\]

For the case \( m = 0 \) we have \( a = 0, b = -B \). Applying the quantization condition \( (18) \) we get a very special situation: namely, one separate bound state may exist or not depending on the value of a magnetic field:
\[
0 < B - n \leq \frac{1}{2}, \quad \lambda - \frac{1}{4} = 2B(n + 1/2) - (n + 1/2)^2, 
\]
\[
R = (\cosh \frac{r}{2})^{-2B} \binom{2}{1}(-n, -2B + 1 + n; +1; -\sinh^2 \frac{r}{2}), 
\]
\[
R_{r \to +\infty} \sim (\cosh \frac{r}{2})^{-2B}(\sinh^2 \frac{r}{2})^n \sim e^{-(B-n)r}. 
\]
(23)

Applying the quantization rule \( (21) \) we get a finite series of bound states:
\[
n + \frac{1}{2} \leq B, n = 0, 1, 2, \ldots, N_B, \quad \lambda - 1/4 = 2B(n + 1/2) - (N + 1/2)^2, 
\]
\[
R = (\cosh \frac{r}{2})^{-2B} \binom{2}{1}(-2B + 1 + n, -n; |m| + 1; -\sinh^2 \frac{r}{2}), 
\]
\[
R_{r \to +\infty} \sim (\cosh \frac{r}{2})^{-2B}(\sinh^2 \frac{r}{2})^n \sim e^{-(B-n)r}. 
\]
(24)

Now let us consider the case
\[
m < 0, a = -\frac{m}{2} > 0, \quad b = \frac{m}{2} - B < 0. 
\]

Applying the quantization condition \( (18) \) we again get a special situation: one separate bound state may exist or not depending on the value of magnetic field:
\[
0 < B - n \leq \frac{1}{2}, \quad \lambda - \frac{1}{4} = 2B(n + 1/2) - (n + 1/2)^2, 
\]
\[
R = (-\sinh \frac{r}{2})^{-m}(\cosh \frac{r}{2})^{m-2B} \binom{2}{1}(-n, -2B + 1 + n; -m + 1; -\sinh^2 \frac{r}{2}), 
\]
\[
R_{r \to +\infty} \sim e^{-(B-n)r}. 
\]
(25)

Applying the quantization rule \( (21) \) we get a finite series of bound states:
\[
n + \frac{1}{2} \leq B, n = 0, 1, 2, \ldots, N_B, 
\]
\[
\lambda - 1/4 = 2B(n + 1/2) - (n + 1/2)^2, 
\]
\[
R = (-\sinh \frac{r}{2})^{-m}(\cosh \frac{r}{2})^{m-2B} \binom{2}{1}(-2B + 1 + n, -n; -m + 1; -\sinh^2 \frac{r}{2}), 
\]
\[
R_{r \to +\infty} \sim e^{-(B-n)r}. 
\]
(26)
Now let us consider the case

\[ 0 < m < 2B, \ a = \frac{m}{2} > 0, \ b = \frac{m}{2} - B < 0. \]

Applying the quantization condition (18) we get a very special situation: one separate bound state may exist or not depending on the value of magnetic field and magnetic quantum number:

\[ 0 \leq B - m - n \leq \frac{1}{2}, \quad \lambda - \frac{1}{4} = 2B(m + n + 1/2) - (m + n + 1/2)^2, \]

\[ R = \left( - \sinh \frac{r}{2} \right)^m (\cosh \frac{r}{2})^{m-2B} \mathbf{2}_1 F_1 (-m, 2m - 2B + 1 + n; m + 1; - \sinh^2 \frac{r}{2}), \]

\[ R_{r \to +\infty} \sim e^{-(B-m-n)r}. \tag{27} \]

Applying the quantization rule (21) we get a finite series of bound states:

\[ m + \frac{1}{2} + n \leq B, \quad n = 0, 1, ..., N_{B,m}, \]

\[ \lambda - \frac{1}{4} = 2B(m + n + 1/2) - (m + n + 1/2)^2, \]

\[ R = \left( - \sinh \frac{r}{2} \right)^m (\cosh \frac{r}{2})^{m-2B} \mathbf{2}_1 F_1 (2m - 2B + 1 + n, -n; m + 1; - \sinh^2 \frac{r}{2}), \]

\[ R_{r \to +\infty} \sim e^{(B-n)r} \to \infty; \tag{28} \]

For the case

\[ m > 2B \quad a = \frac{m}{2}, \quad b = -\frac{m - 2B}{2} < 0, \]

with the rule (18) we get

\[ B + \frac{1}{2} + n \geq 0, \quad \lambda - \frac{1}{4} = -2B(n + \frac{1}{2}) - (n + \frac{1}{2})^2, \]

\[ R = \left( - \sinh \frac{r}{2} \right)^m (\cosh \frac{r}{2})^{-m+2B} \mathbf{2}_1 F_1 (-n, 2B + 1 + n; m + 1; - \sinh^2 \frac{r}{2}), \]

\[ R_{r \to +\infty} \sim e^{(B+n)r} \to \infty; \tag{29} \]

this is not the case of bound states. With the rule (21) we arrive at

\[ B + \frac{1}{2} + n \leq 0 \tag{30} \]

that can not be fulfilled together with the quantization conditions whereas \( n \geq 0 \).

And finally for the last variant:

\[ m = 2B > 0, \quad a = m = 2B, \quad b = 0, \]

we get

\[ B + n + \frac{1}{2} \geq 0, \quad \lambda - \frac{1}{4} = 2B(n + \frac{1}{2}) - (n + \frac{1}{2})^2, \]

\[ R = \left( - \sinh \frac{r}{2} \right)^{2B} \mathbf{2}_1 F_1 (-n, |m| - |m - 2B| + 1 + n; m |+1; - \sinh^2 \frac{r}{2}), \]

\[ R_{r \to +\infty} \sim e^{(B+n)r} \to \infty \]
which is not a bound state. Quantization according to (21) cannot provide us with bound states because of relationship \( B + n + 1/2 > 0 \).

Let us collect the results together:

**separate states:** that may be presented by the single formula:

\[
\lambda - \frac{1}{4} = 2B \left( \frac{m+|m|}{2} + n + 1/2 \right) - \left( \frac{m+|m|}{2} + n + 1 \right)^2 ;
\]

(31)

**series of states:** given by a formula

\[
\lambda - 1/4 = 2B \left( \frac{m+|m|}{2} + n + 1/2 \right) - \left( \frac{m+|m|}{2} + n + 1/2 \right)^2 , \; n = 0, 1, \ldots , N_B.
\]

(32)

In usual unites, the last relation reads

\[
\lambda - \frac{1}{4} = \rho^2 \lambda_0 - \frac{1}{4}, \; \lim_{\rho \to \infty} \lambda_0 = \frac{2M}{\hbar^2} (E - \frac{P^2}{2M}), \; m < 2B, \; m + n + 1/2 \leq \frac{eB}{\hbar c} \rho^2,
\]

\[
\rho^2 \lambda_0 - \frac{1}{4} = 2 \frac{eB}{\hbar c} \rho^2 \left( \frac{m+|m|}{2} + n + 1/2 \right) - \left( \frac{m+|m|}{2} + n + 1/2 \right)^2 , \; n = 0, 1, \ldots , N_B.
\]

(33)

In the limit of a flat space, from (33) the known result follows:

\[
E - \frac{P^2}{2M} = \frac{eB \hbar}{Mc} \left( \frac{m+|m|}{2} + n + 1/2 \right) .
\]

For the case of the positive orientation of magnetic field \( B < 0 \), we should take into account the symmetry provided by initial differential equation: \( m \to m' = -m, \; B \to B' = -B \).

### 4 Differential equation for \( Z(z) \) in \( H_3 \)

Now, let us examine the differential equation for \( Z(z) \) in space \( H_3 \):

\[
\frac{d^2Z}{dz^2} + 2 \frac{\sinh z}{\cosh z} \frac{dZ}{dz} + 2eZ - \frac{\lambda}{\cosh^2 z} Z = 0.
\]

(34)

Changing the variable as \( \sinh z = i(2x - 1) \), we get

\[
x(1-x) \frac{d^2Z}{dx^2} - 3(2x-1) \frac{1}{2} \frac{dZ}{dx} - \left( 2e - \frac{\lambda}{4x(1-x)} \right) Z = 0.
\]

Taking the function in the form \( Z = x^a(1-x)^bF \), we obtain

\[
+ x(1-x)F'' + [2a(1-x) - 2bx - 3x + 3/2]F' + \left[ a(a-1) + \frac{3a}{2} + \frac{\lambda}{4} \right] \frac{1}{x} F
\]

\[
+ \left[ b(b-1) + 3b - \frac{3b}{2} + \frac{\lambda}{4} \right] \frac{1}{1-x} F + [-a(a-1) - ab - ab - b(b-1) - 3a - 3b - 2e]F = 0.
\]

(35)
Imposing additional restrictions
\[
a^2 + \frac{a}{2} + \frac{\lambda}{4} = 0, \quad a = \frac{-1 \pm \sqrt{1 - 4\lambda}}{4}, \quad b^2 + \frac{b}{2} + \frac{\lambda}{4} = 0, \quad b = \frac{-1 \pm \sqrt{1 - 4\lambda}}{4},
\]
we arrive at more simple equation
\[
x(1 - x)F'' + [(2a + 3/2) - x(2a + 2b + 3)]F' - [(a + b + 1)^2 + 2\epsilon - 1]F = 0
\]
which is of hypergeometric type
\[
x(1 - x)F + [C - (A + B + 1)x]F' - ABF = 0
\]
with parameters defined by
\[
A + B = 2(a + b + 1), \quad AB = (a + b + 1)^2 + 2\epsilon - 1.
\]
That is
\[
A = a + b + 1 + \sqrt{1 - 2\epsilon}, \quad B = a + b + 1 - \sqrt{1 - 2\epsilon}, \quad C = 2a + 3/2;
\]
\[
Z = x^a(1 - x)^b 2 F_1(A, B; C; x), \quad x = \frac{1 - \frac{1}{2} \sinh z}{2}, \quad (1 - x) = \frac{1 + \frac{1}{2} \sinh z}{2},
\]
\[
a = \frac{-1 \pm \sqrt{1 - 4\lambda}}{4}, \quad b = \frac{-1 \pm \sqrt{1 - 4\lambda}}{4},
\]
To find behavior of these solutions at infinity, one can use the Kummer’s relations [27]
\[
U_1 = 2 F_1(A, B; C; x), \quad U_3 = (-x)^{-A} 2 F_1(A, A - C + 1; A - B + 1; x^{-1}),
\]
\[
U_4 = (-x)^{-B} 2 F_1(B - C + 1, B; B - A + 1; x^{-1}), \quad U_1 = \frac{\Gamma(C)\Gamma(B - A)}{\Gamma(C - A)\Gamma(B)} U_3 + \frac{\Gamma(C)\Gamma(A - B)}{\Gamma(C - B)\Gamma(A)} U_4.
\]
Then
\[
Z = x^a(1 - x)^b U_1 = \frac{\Gamma(C)\Gamma(B - A)}{\Gamma(C - A)\Gamma(B)} x^a(1 - x)^b (-x)^{-a - b + 1 + i\sqrt{2\epsilon - 1}} 2 F_1(A, A - C + 1; A - B + 1; x^{-1}) + \frac{\Gamma(C)\Gamma(A - B)}{\Gamma(C - B)\Gamma(A)} x^a(1 - x)^b (-x)^{-a - b + 1 - i\sqrt{2\epsilon - 1}} 2 F_1(B - C + 1, B; B - A + 1; x^{-1}).
\]
Therefore, asymptotically all three solutions \(U_1, U_3, U_4\) vanish when \(x \to \infty\):
\[
Z = x^a(1 - x)^b U_1 \sim (-1)^a \left[ \frac{\Gamma(C)\Gamma(B - A)}{\Gamma(C - A)\Gamma(B)} (-x)^{-1 - i\sqrt{2\epsilon - 1}} + \frac{\Gamma(C)\Gamma(A - B)}{\Gamma(C - B)\Gamma(A)} (-x)^{-1 + i\sqrt{2\epsilon - 1}} \right] \to 0.
\]
Let $a = b$, then

$$a = b = \frac{-1 \pm 2i\sqrt{\lambda - 1/4}}{2}, \quad a + b = \frac{-1 \pm 2i\sqrt{\lambda - 1/4}}{4},$$

$$Z = x^a (1-x)^b \, {}_2F_1(A, B; C; x) = \left( \frac{\cosh^2 z}{4} \right)^a \, {}_2F_1(A, B; C; x),$$

$$A = \frac{1}{2} + i(\sqrt{2\epsilon - 1} \pm \sqrt{\lambda - \frac{1}{4}}), \quad B = \frac{1}{2} - i(\sqrt{2\epsilon - 1} \mp \sqrt{\lambda - \frac{1}{4}}),$$

$$C = 1 \pm i\sqrt{\lambda - 1/4}. \quad (42)$$

One may note that from (42) it follows

$$(A - \frac{1}{2})(B - \frac{1}{2}) = (2\epsilon - 1) - (\lambda - 1/4).$$

It has sense to introduce a new parameter

$$\lambda = (2\epsilon - 1) - (\lambda - 1/4) = \nu^2, \quad \text{or} \quad (\lambda - 1/4) = (2\epsilon - 1) - \nu^2 \quad (43)$$

which generalizes the known relation in the flat space model $\lambda = 2\epsilon - k^2$. It is the matter of simple calculation to determine the operator with eigenvalues $\nu^2 + 3/4$:

$$\nu^2 + \frac{3}{4} = 2\hat{H}(r, \phi, z) - \hat{\lambda}(r).$$

Indeed, with the notion

$$\hat{\lambda}(r, \phi) = -\frac{1}{\sinh r} \frac{\partial}{\partial r} \sinh r \frac{\partial}{\partial r} + \frac{1}{\sinh^2 r} \left[ \frac{i}{\partial \phi} - B(\cosh r - 1) \right]^2,$$

$$2\hat{H}(r, \phi, z) = \frac{1}{\cosh^2 z} \hat{\lambda}(r, \phi) - \frac{1}{\cosh^2 z \partial z} \cosh^2 z \frac{\partial}{\partial z}. \quad (44)$$

We get

$$\hat{\nu}^2(r, \phi, z) + \frac{3}{4} = \left( \frac{1}{\cosh^2 z} - 1 \right) \hat{\lambda}(r, \phi) - \frac{1}{\cosh^2 z \partial z} \cosh^2 z \frac{\partial}{\partial z}. \quad (45)$$

and the commutative relations $[2\hat{H}, \hat{\lambda}(r)] = 0$ holds.

### 5 Radial equation in spherical space

Now let us consider the radial equation arising in spherical space. With the use of dimensionless parameter $\frac{e^B}{h}\nu^2 \implies B$ the equation reads

$$\frac{d^2}{dr^2} R + \frac{1}{\tan r} \frac{dR}{dr} - \frac{1}{\sin^2 r} \left[ m + B(1 - \cos r) \right]^2 \frac{R}{R} + \lambda \frac{R}{R} = 0. \quad (46)$$
With variable change given by $1 - \cos r = 2z$ we have

$$[z(1-z)\frac{d^2}{dz^2} + (1-2z)\frac{d}{dz} - \frac{1}{4}(\frac{m^2}{z} - 4B^2 + \frac{(m+2B)^2}{1-z}) + \lambda]R = 0. \quad (47)$$

After substitution $R = z^a(1-z)^bF$ we produce

$$z(1-z)F'' + [a(1-z) - bz + a(1-z) - bz + (1-2z)]F' + \frac{1}{z}[a(a-1) + a - m^2/4]F$$

$$+ \frac{1}{1-z}\left[(b-1) + b - (m+2B)^2/\left[F - [a(a+1) + 2ab + b(b+1) - B^2 - \lambda]F = 0. \quad (48)\right.$$  

Imposing the conditions $a = \pm m/2, \ b = \pm(m + 2B)/2$ we arrive at

$$z(1-z)F'' + [(2a+1) - 2(a+b+1)z]F' - [a(a+1) + 2ab + b(b+1) - B^2 - \lambda]F = 0 \quad (49)$$

which is of hypergeometric type. Thus, the problem is solved as

$$z = \sin^{2\frac{r}{2}}, \ z \in [0, +1], \ r \in [0, +\pi];$$

$$R = (\sin^{r/2} + |m|) (\cos^{r/2} + |m+2B|) F_1(\alpha, \beta; \gamma; -\sin^{2\frac{r}{2}}) \quad (50)$$

where

$$a = +\left|\frac{m}{2}\right|, \ b = +\left|\frac{m+2B}{2}\right|, \quad \alpha = a + b + \frac{1}{2} - \sqrt{B^2 + \frac{1}{4} + \lambda},$$

$$\beta = a + b + \frac{1}{2} + \sqrt{B^2 + \frac{1}{4} + \lambda}, \quad \gamma = + |m| + 1. \quad (51)$$

The quantization condition is given by

$$\alpha = a + b + \frac{1}{2} - \sqrt{B^2 + \frac{1}{4} + \lambda} = -n, \quad n = 0, -1, \ldots$$

from whence it follows relations for possible bound states:

$$\lambda + \frac{1}{4} = -B^2 + (a + b + \frac{1}{2} + n)^2, \quad n = 0, -1, -2, \ldots$$

$$R = (\sin^{r/2} + |m|)(\cos^{r/2} + |m+2B|) F_1(-n, |m| + |m+2B| + 1 + n; |m| + 1; -\sin^{2\frac{r}{2}}). \quad (52)$$

While examining wave functions in spherical space one must take into account peculiarities in parameterization of the space $S_3$ by the coordinates $(r, \phi, z)$:

$$z \in [-\pi/2, +\pi/2], \ r \in [0, \pi], \ \phi \in [0, 2\pi],$$

$$u_1 = \cos z \sin r \cos \phi, \ u_2 = \cos z \sin r \sin \phi,$$

$$u_3 = \sin z, \ u_0 = \cos z \cos r, \ u_1^2 + u_2^2 + u_3^2 + u_0^2 = 1. \quad (53)$$

In particular, we have

$$r = 0, \ u_1 = 0, \ u_2 = 0, \ u_3 = \sin z, \ u_0 = +\cos z,$$

$$r = \pi, \ u_1 = 0, \ u_2 = 0, \ u_3 = \sin z, \ u_0 = -\cos z,$$

$$z \in [-\pi/2, +\pi/2] \quad (54)$$
which means that the full closed curve $u_0^2 + u_3^2 = 1$ in $S_3$ is parameterized by two pieces:

$$\{(r = 0, \phi - \text{mute}, z) + (r = \pi, \phi - \text{mute}, z)\}.$$

Correspondingly, when $m \neq 0$, the function $R(r)$ must vanish in $r = 0, \pi$:

$$m \neq 0, \quad R(0) = 0, \quad R(\pi) = 0. \quad (55)$$

First, let us examine the case $m = 0$, then

$$a = 0, \quad b = +B,$$

$$\lambda + \frac{1}{4} = 2B(n + 1/2) + (n + 1/2)^2,$$

$$R = (\cos \frac{r}{2})^{+2B} F_1(-n, 2B + n + 1; 1; -\sin^2 \frac{r}{2}),$$

$$R_{r \to 0} = 1, \quad R_{r \to \pi} = 0. \quad (56)$$

Because $m = 0$ and the wave function does not depend on $\phi$, this solution is acceptable by continuity reasons. Now, let us examine the case $m > 0$, then

$$a = +\frac{m}{2}, \quad b = +\frac{m + 2B}{2},$$

$$\lambda + \frac{1}{4} = 2B(n + m + 1/2) + (n + m + 1/2)^2,$$

$$R = (\sin \frac{r}{2})^{+m}(\cos \frac{r}{2})^{m+2B} \times_2 F_1(-n, 2B + 2m + n + 1; m + 1; -\sin^2 \frac{r}{2}),$$

$$R_{r \to 0} = 0, \quad R_{r \to \pi} = 0; \quad (57)$$

these are the correct single-valued solutions. Let us consider the variant $m < -2B$, then

$$a = -\frac{m}{2}, \quad b = -\frac{m + 2B}{2} > 0,$$

$$\lambda + \frac{1}{4} = -2B(n - m + 1/2) + (n - m + 1/2)^2$$

$$= (n - m + 1/2)((n - m + 1/2 - 2B) > 0,$$

$$R = (\sin \frac{r}{2})^{-m}(\cos \frac{r}{2})^{-(m+2B)} \times_2 F_1(-n, -2m - 2B + 1 + n; -m + 1; -\sin^2 \frac{r}{2}),$$

$$R_{r \to 0} = 0, \quad R_{r \to \pi} = 0. \quad (58)$$

Let us examine the case $-2B < m < 0$, then

$$a = -\frac{m}{2} > 0, \quad b = \frac{m + 2B}{2} > 0,$$

$$\lambda + \frac{1}{4} = 2B(n + 1/2) + (n + 1/2)^2,$$

$$R = (\sin \frac{r}{2})^{-m}(\cos \frac{r}{2})^{m+2B} \times_2 F_1(-n, 2B + n + 1; -m + 1; -\sin^2 \frac{r}{2}),$$

$$R_{r \to 0} = 0, \quad R_{r \to \pi} = 0. \quad (59)$$
And finally, the last variant is \( m = -2B \), then

\[
a = +B, \ b = 0,
\]
\[
\lambda + \frac{1}{4} = 2B(n + 1/2) + (n + 1/2)^2,
\]
\[
R = (-\sin\frac{r}{2})^{+2B} \times _2 F_1(-n, 2B + n + 1; -B + 1; -\sin^2\frac{r}{2}),
\]
\[
R_{r \to 0} = 0, \ R_{r \to \pi} = 1. \quad (60)
\]

We faced here with the case of discontinuities of wave functions in \( S_3 \), because the wave function preserves dependence on \( \phi \) at \( r \to \pi \).

Collecting results together, we have

\[
m > 0, \quad \lambda + \frac{1}{4} = (n + 1/2 + m)(n + 1/2 + m + 2B);
\]
\[
m < -2B, \quad \lambda + \frac{1}{4} = (n + 1/2 - m)(n + 1/2 - m - 2B);
\]
\[
-2B < m \leq 0, \quad \lambda + \frac{1}{4} = (n + 1/2)(n + 1/2 - 2B). \quad (61)
\]

In usual units, these formulas look

\[
m > 0 \quad \rho^2 \lambda_0 + \frac{1}{4} = +2\frac{eB}{\hbar c} \rho^2(n + m + 1/2) + (n + m + 1/2)^2 ;
\]
\[
m < -2\frac{eB}{\hbar c} \rho^2 \quad \rho^2 \lambda_0 + \frac{1}{4} = -2\frac{eB}{\hbar c} \rho^2(n - m + 1/2) + (n - m + 1/2)^2 ;
\]
\[
-2\frac{eB}{\hbar c} \rho^2 < m \leq 0, \quad \rho^2 \lambda_0 + \frac{1}{4} = 2\frac{eB}{\hbar c} \rho^2(n + 1/2) + (n + 1/2)^2 . \quad (62)
\]

When \( \rho \to 0 \), the case \( m < -\infty \) turns to be vanishing, and two other cases give

\[
m < 0, \quad \lambda_0 = 2\frac{eB}{\hbar c}(n + 1/2),
\]
\[
m \geq 0, \quad \lambda_0 = +2\frac{eB}{\hbar c}(n + m + 1/2). \quad (63)
\]

Thus, we arrive at the known result in the flat space model:

\[
\lim_{\rho \to \infty} \lambda_0 = \frac{2M}{\hbar^2} \left( E - \frac{P^2}{2M} \right), \quad E - \frac{P^2}{2M} = \frac{eB \hbar}{M c} \left( \frac{m + |m|}{2} + n + 1/2 \right). \quad (64)
\]

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6 Solutions of the equation for \( Z(z) \) in the space \( H_3 \)

Now, let us consider the differential equation for \( Z(z) \) in space \( S_3 \):

\[
\frac{d^2 Z}{dz^2} - 2 \frac{\sin z}{\cos z} \frac{dZ}{dz} + 2\epsilon Z - \frac{\lambda}{\cos^2 z} Z = 0 . \tag{65}
\]

Changing the variable \( \sin z = y \), we get

\[
(1 - y^2) \frac{d^2 Z}{dy^2} - 3y \frac{dZ}{dy} + \left( 2\epsilon - \frac{\lambda}{1 - y^2} \right) Z = 0. \tag{66}
\]

In the variable \( y = (2x - 1) \) it takes the form

\[
x(1-x) \frac{d^2 Z}{dx^2} - \frac{3}{2} (2x - 1) \frac{dZ}{dx} + \left( 2\epsilon - \frac{\lambda}{4x(1-x)} \right) Z = 0. \tag{67}
\]

With the use of the substitution \( Z = x^a(1-x)^b F \), eq. (67) leads to

\[
x(1-x) F'' + \left[ (2a + 3/2) - x(2a + 2b + 3) \right] F' - \left[ (a + b + 1)^2 - 2\epsilon - 1 \right] F = 0, \tag{68}
\]

Imposing the following restrictions

\[
a^2 + \frac{a}{2} - \frac{\lambda}{4} = 0 , \quad a = \frac{-1 \pm \sqrt{1 + 4\lambda}}{4} ,
\]

\[
b^2 + \frac{b}{2} - \frac{\lambda}{4} = 0 , \quad b = \frac{-1 \pm \sqrt{1 + 4\lambda}}{4} \tag{69}
\]

we reduce eq. (68) to

\[
x(1-x) F'' + \frac{[(2a + 3/2) - x(2a + 2b + 3)] F'}{-[(a + b + 1)^2 - 2\epsilon - 1]} F = 0, \tag{70}
\]

which is of hypergeometric type

\[
x(1-x) F + [C - (A + B + 1)x] F' - AB F = 0 ,
\]

with parameters given by (to obtain the single-valued wave functions we must take \( a \) and \( b \) positive)

\[
Z = x^a(1-x)^b \binom{2}{1} F_1(A, B; C; x) , \quad a = \frac{-1 + \sqrt{1 + 4\lambda}}{4}, \quad b = \frac{-1 + \sqrt{1 + 4\lambda}}{4};
\]

\[
C = 1 + \sqrt{\lambda + 1/4}, \quad A = \frac{1}{2} + \sqrt{\lambda + \frac{1}{4} + \sqrt{2\epsilon + 1}}, \quad B = \frac{1}{2} + \sqrt{\lambda + \frac{1}{4} - \sqrt{2\epsilon + 1}}. \tag{71}
\]

To obtain polynomials we must impose the following conditions:

\[
\lambda + \frac{1}{4} > 0, \quad 2\epsilon + 1 > 0, \quad B = \frac{1}{2},
\]

\[
+ \sqrt{\lambda + \frac{1}{4} - \sqrt{2\epsilon + 1}} = -N, \quad N = 0, 1, 2, \ldots
\]
from whence it follows the quantization rule

\[ 2\epsilon + 1 = \left( \frac{1}{2} + N + \sqrt{\lambda + \frac{1}{4}} \right)^2. \tag{72} \]

In particular, this formula gives

\[ (2\epsilon + 1) - \left( \lambda + \frac{1}{4} \right) = \left( \frac{1}{2} + N \right)^2 + 2 \left( \frac{1}{2} + N \right) \sqrt{\lambda + \frac{1}{4}}, \tag{73} \]

with the notion

\[ \left( \frac{1}{2} + N \right)^2 + 2\left( \frac{1}{2} + N \right) \sqrt{\lambda + \frac{1}{4}} = \nu^2 > 0 \]

it reads

\[ (2\epsilon + 1) = \nu^2 + \left( \lambda + \frac{1}{4} \right) \tag{74} \]

which generalizes the known relation in flat space model: \( 2\epsilon = k^2 + \lambda \). The identity \( \tag{74} \) permits us to express all parameters through \( \epsilon, \nu^2 \):

\[
\begin{align*}
C &= 1 + \sqrt{(2\epsilon + 1) - \nu^2}, \\
A &= \frac{1}{2} + \sqrt{(2\epsilon + 1) - \nu^2 + \sqrt{2\epsilon + 1}}, \\
B &= \frac{1}{2} + \sqrt{(2\epsilon + 1) - \nu^2} - \sqrt{2\epsilon + 1}. \tag{75}
\end{align*}
\]

However, it should be noted that such a form does not provide us with any advantage because quantization for the quantity \( (2\epsilon + 1) - \nu^2 \) is given by the differential equation for radial function \( R(r) \). It is easily to obtain an operator with eigenvalues \( \nu^2 + 3/4 \):

\[
\begin{align*}
\hat{\nu}^2 - \frac{3}{4} &= 2\hat{H}(r, \phi, z) - \hat{\lambda}(r), \\
\hat{\lambda}(r, \phi) &= -\frac{1}{\sin r} \frac{\partial}{\partial r} \sin r \frac{\partial}{\partial r} + \frac{1}{\sin^2 r} \left[ -i \frac{\partial}{\partial \phi} + B(1 - \cos r) \right]^2,
\end{align*}
\]

so that the operator is given by

\[ \hat{\nu}^2(r, \phi, z) - \frac{3}{4} = \left( \frac{1}{\cos^2 z} - 1 \right) \hat{\lambda}(r, \phi) - \frac{1}{\cos^2 z} \]

\[ \times \frac{\partial}{\partial z} \cos^2 z \frac{\partial}{\partial z} \tag{77} \]

It commutes with Hamiltonian.

7 Presence of an electric field, the case of space \( S_3 \)

Let us introduce into consideration an additional external electric field along the \( z \) axis

\[ A_0 = E_0 \frac{\sin z}{\cos^2 z} \tag{78} \]
in the limit of flat space it becomes the uniform constant field \( E \) along \( z \). As we will see below, this additional field does not destroy hypergeometric structure of the solution \( Z(z) \). To take into account the presence of this field, it is sufficient to make one formal change in the previous calculations (in dimensionless notation)

\[
\epsilon \mapsto \epsilon + eE_0 \frac{M\rho^2}{\hbar^2} \sin \frac{z}{\cos^2 z} = \epsilon + \mu \frac{\sin z}{\cos^2 z}.
\] (79)

Also, one should use the identity

\[
(2\epsilon + 2\mu \frac{\sin z}{\cos^2 z}) = 2\epsilon + 2\mu \frac{y}{1-y^2} = 2\epsilon + 2\mu \frac{2x-1}{4x(1-x)} = 2\epsilon + \frac{2\mu}{4} (-\frac{1}{x} + \frac{1}{1-x}).
\]

Correspondingly, instead of eq. (68), we have

\[
x(1-x)F'' + [2a(1-x) - 2bx - 3x + 3]F' + \left[ a(a-1) + \frac{3a}{2} - \frac{\lambda}{4} - \frac{2\mu}{4} \right] \frac{1}{x} F + \left[ b(b-1) + \frac{3b}{2} - \frac{\lambda}{4} + \frac{2\mu}{4} \right] \frac{1}{1-x} F + \left[ -(a-1) - ab - ab(b-1) - 3a - 3b + 2\epsilon \right] F = 0.
\] (80)

By imposing two conditions

\[
a^2 + \frac{a}{2} - \frac{\lambda + 2\mu}{4} = 0, \quad b^2 + \frac{b}{2} - \frac{\lambda - 2\mu}{4} = 0
\] (81)

eq. (80) reduces to

\[
x(1-x)F'' + [(2a + 3/2) - x(2a + 2b + 3)]F' - [(a + b + 1)^2 - 2\epsilon - 1] F = 0
\] (82)

which is of hypergeometric type. Thus the solutions are given by

\[
Z = x^a(1-x)^b \binom{A}{B} (A; B; C; x), \quad x = \frac{1 + \sin z}{2}, \quad (1-x) = \frac{1 - \sin z}{2},
\]

\[
a = -1 + \sqrt{1 + 4\lambda + 4\mu}, \quad b = -1 + \sqrt{1 + 4\lambda - 4\mu},
\]

\[
A = \frac{1}{2} + \frac{1}{2} \sqrt{\lambda + 2\mu + \frac{1}{4} + \frac{1}{2} \sqrt{\lambda - 2\mu + \frac{1}{4} + \sqrt{2\epsilon + 1}}},
\]

\[
B = \frac{1}{2} + \frac{1}{2} \sqrt{\lambda + 2\mu + \frac{1}{4} + \frac{1}{2} \sqrt{\lambda - 2\mu + \frac{1}{4} - \sqrt{2\epsilon + 1}}},
\]

\[
C = 1 + \sqrt{\lambda - 2\mu + 1/4}.
\] (83)

To obtain polynomials, we must impose the following restriction

\[
(\lambda + 2\mu) + \frac{1}{4} > 0, \quad 2\epsilon + 1 > 0,
\]

\[
B = \frac{1}{2} + \sqrt{\lambda + 2\mu + \frac{1}{4} + \sqrt{\lambda - 2\mu + \frac{1}{4} - \sqrt{2\epsilon + 1}}} = -N, \quad N = 0, 1, 2, \ldots
\]

from whence it follows the quantization rule

\[
2\epsilon + 1 = \left( \frac{1}{2} + N + \frac{1}{2} \sqrt{\lambda + 2\mu + \frac{1}{4} + \sqrt{\lambda - 2\mu + \frac{1}{4}}} \right)^2
\] (84)

where the quantization of \( \lambda \) is given by (61)). Thus, the energy levels are determined by two quantum numbers, \((n, N)\), and depends upon three parameters, curvature radius \( \rho \), and magnitudes of fields: \( B \) and \( E \).
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