SOLUTIONS OF THE YANG-BAXTER EQUATION
AND QUANTUM $\mathfrak{sl}(2)$

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Abstract. We construct a quantum deformation of a family of the Yang-Baxter equation solutions naturally arising from a Lie algebra $\mathfrak{sl}(2,\mathbb{C})$.

Introduction

Compared to the well established theory of quantum groups, the theory of quantum Lie algebras is much less developed. Quantum Lie algebras and related topics of bicovariant differential calculus are studied in [Be1], [Be2], [CSWW], [DF], [DH], [LS], and [Ma1], to mention just a few papers. However, many problems still seem to be open.

It is known that a solution $\mathcal{R}$ of the Quantum Yang-Baxter Equation (QYBE) can be naturally associated to any Lie algebra. Moreover, in some sense the Yang-Baxter relation is equivalent to the fact that the Lie bracket satisfies Jacobi identity and the invertibility of $\mathcal{R}$ is equivalent to the fact that the Lie bracket is antisymmetric. A non-trivial quantum deformation $\mathcal{R}^q$ of $\mathcal{R}$ can be regarded therefore as a version of quantum Lie algebra with the Yang-Baxter relation serving as quantum Jacobi identity and the invertibility serving as quantum antisymmetry.

In this paper we introduce a construction of a quantum deformation of the classical (family of) QYBE solutions in the case of $\mathfrak{sl}(2,\mathbb{C})$. The construction is presented in the framework of the Penrose-Kauffman graphical calculus. This example motivates an abstract definition of a (generalized) quantum Lie algebra and its module. The category of such modules is shown to have a natural tensor structure.

The paper is organized as follows. In 1.1 we discuss two families of QYBE solutions naturally arising from a Lie algebra (cf. [Ma2]). In 1.2 we discuss a family of QYBE solutions naturally arising from a module over a Lie algebra. In 1.3 we recall some basic facts about $\mathfrak{sl}(2,\mathbb{C})$ and the classical version of the Penrose-Kauffman graphical calculus, while in 2.1 we recall similar facts about $U_q(\mathfrak{sl}(2,\mathbb{C}))$ and the quantum graphical calculus. In 2.2 we introduce two families of QYBE solutions deforming the classical families for $\mathfrak{sl}(2,\mathbb{C})$ and in 2.3 we discuss quantum $\mathfrak{sl}_q(2,\mathbb{C})$-modules.

The proofs of all the lemmas are straightforward calculations or graphical calculus exercises; they are left to the reader.
1. Classical families of the QYBE solutions

1.1. Lie algebras.

Let $\mathbb{C}$ be the field of complex numbers, and let $\mathfrak{g}$ be a $\mathbb{C}$-vector space. Suppose that $\mathfrak{g}$ is an algebra, i.e. there is a multiplication map $[,] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$, and let $\tilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathbb{C}$. We define two families of linear maps

$$R(\lambda), R(\lambda)' : \tilde{\mathfrak{g}} \otimes \tilde{\mathfrak{g}} \to \tilde{\mathfrak{g}} \otimes \tilde{\mathfrak{g}}$$

$$R(\lambda) : (x + \alpha) \otimes (y + \beta) \mapsto (y + \beta) \otimes (x + \alpha) + [x, y] \otimes \lambda$$

$$R(\lambda)' : (x + \alpha) \otimes (y + \beta) \mapsto (y + \beta) \otimes (x + \alpha) + \lambda \otimes [x, y]$$

where $x, y \in \mathfrak{g}, \alpha, \beta, \lambda \in \mathbb{C}$. We set $R_{12}(\lambda) = R(\lambda) \otimes 1_{\tilde{\mathfrak{g}}}$ and $R_{23}(\lambda) = 1_{\tilde{\mathfrak{g}}} \otimes R(\lambda)$.

**Lemma 1.1.1.** a) $R(\lambda)R(\lambda)' = R(\lambda)'R(\lambda) = 1$ for any $\lambda$, if and only if $[,]$ is antisymmetric;

b) $R_{12}(\lambda)R_{23}(\lambda)R_{12}(\lambda) = R_{23}(\lambda)R_{12}(\lambda)R_{23}(\lambda)$ for any $\lambda$, if and only if $[,]$ satisfies Jacobi identity;

c) $R_{12}(\lambda)'R_{23}(\lambda)'R_{12}(\lambda)' = R_{23}(\lambda)'R_{12}(\lambda)'R_{23}(\lambda)'$ for any $\lambda$, if and only if $[,]$ satisfies Jacobi identity.

For any two linear spaces $V$ and $W$ let $\sigma(V \otimes W)$ be the transposition of the tensor factors $\sigma(V \otimes W) : V \otimes W \to W \otimes V$. Let $R_{21}(\lambda) = \sigma(\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}})R(\lambda)\sigma(\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}})$.

**Lemma 1.1.2.** For any $\lambda$, $R(\lambda)R_{21}(-\lambda) = R_{21}(\lambda)R(-\lambda) = 1$.

Therefore, a classical Lie algebra $\mathfrak{g}$ provides a one-parameter family of the QYBE solutions in End $\tilde{\mathfrak{g}} \otimes \tilde{\mathfrak{g}}$.

1.2. Modules over Lie algebras.

Let $\mathfrak{g}$ be a Lie algebra. Let $V$ be a $\mathbb{C}$-vector space. We consider a linear map $A : \mathfrak{g} \otimes V \to V$, and define a family of intertwiners

$$R_V(\lambda) : \tilde{\mathfrak{g}} \otimes V \to V \otimes \tilde{\mathfrak{g}}$$

$$R_V(\lambda) : (x + \alpha) \otimes v \mapsto v \otimes (x + \alpha) + A(x, v) \otimes \lambda$$

where $x \in \mathfrak{g}, v \in V, \alpha, \lambda \in \mathbb{C}$.

**Lemma 1.2.1.** $(R_V(\lambda) \otimes 1_{\tilde{\mathfrak{g}}}) \circ (1_{\tilde{\mathfrak{g}}} \otimes R_V(\lambda)) \circ (R_V(\lambda) \otimes 1_V) = (1_V \otimes R_V(\lambda)) \circ (R_V(\lambda) \otimes 1_{\tilde{\mathfrak{g}}}) \circ (1_{\tilde{\mathfrak{g}}} \otimes R_V(\lambda))$ for any $\lambda$, if and only if $A$ is a $\mathfrak{g}$-action.

**Lemma 1.2.2.** Let $V, W$ be two $\mathfrak{g}$-modules. Then for any $\lambda$, $(\sigma(V \otimes W) \otimes 1_{\tilde{\mathfrak{g}}}) \circ (1_V \otimes R_W(\lambda)) \circ (R_V(\lambda) \otimes 1_W) = (1_W \otimes R_V(\lambda)) \circ (R_W(\lambda) \otimes 1_V) \circ (1_{\tilde{\mathfrak{g}}} \otimes \sigma(V \otimes W))$.

1.3. $\mathfrak{sl}(2, \mathbb{C})$ and Penrose-Kauffman graphical calculus.

Let us consider a special case $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$. Let $V_n$ denote the $(n + 1)$-dimensional irreducible $\mathfrak{sl}(2, \mathbb{C})$-module. In this notation the fundamental representation is $V_1$ and the adjoint representation is $V_2$. An intertwiner is a morphism in the category.
of \( \mathfrak{sl}(2, \mathbb{C}) \)-modules. The Lie bracket is the unique (up to a constant) intertwiner \( [ , ] : V_2 \otimes V_2 \to V_2 \).

Penrose-Kauffman graphical calculus is a way to represent the \( \mathfrak{sl}(2, \mathbb{C}) \)-intertwiners by planar diagrams. We will now recall the very basic conventions of graphical calculus; for details we refer the reader to [CFM], [KL], [FK] and references thereof (see, however, the warning below). The module \( V_1 \) (or rather \( \text{Id}_{V_1} \)) is depicted by a solid vertical strand \( | \). An intertwiner \( a : (V_1)^{\otimes m} \to (V_1)^{\otimes n} \) is depicted by certain curves connecting \( m \) distinct points on one horizontal line and \( n \) distinct points on another horizontal line lying below the first one. Only simple intersections are allowed. The module \( V_n \) (or rather the Jones-Wenzl projector \( p_n \)) is depicted by a box marked \( n \) with \( n \) strands attached to its top and \( n \) strands attached to its bottom.

In particular, \( V_2 \) is represented by the following diagram

We omit the marking of the box if it does not lead to a confusion. A graphical version of the relation \( V_1 \otimes V_1 \simeq V_2 \oplus V_0 \) is depicted as follows:

\[(1.3.1)\]

Using the graphical calculus we can present the Lie bracket \( [ , ] \), the transposition \( \sigma(V_1 \otimes V_1) \), and our families \( \mathcal{R}(\lambda), \mathcal{R}(\lambda)', \mathcal{R}_{V_n}(\lambda) \) as follows:

\[
[ , ] = \quad \sigma(V_1 \otimes V_1) = \\
\mathcal{R}(\lambda) = \\
\mathcal{R}(\lambda)' = \\
\mathcal{R}_{V_n}(\lambda) =
\]

\[+ \lambda \]

\[+ \frac{1}{2} \]
2. Families of the QYBE solutions arising from quantum $\mathfrak{sl}(2, \mathbb{C})$

2.1. $U_q(\mathfrak{sl}(2, \mathbb{C}))$ and Penrose-Kauffman graphical calculus.

The quantum group $U = U_q(\mathfrak{sl}(2, \mathbb{C}))$ is a certain Hopf algebra over the field $\mathbb{C}(q)$, defined by generators and relations. For details on the definition of $U$ we refer the reader to any of the numerous books on quantum groups (cf. [Dr], [J1], [J2]). We follow the notation and conventions of [FK]. For $n \in \mathbb{Z}$ there is a $U$-module $V^q_n$ deforming the usual $(n + 1)$-dimensional module over $\mathfrak{sl}(2, \mathbb{C})$. A linear map between two $U$-modules is called an intertwiner if it is a morphism in the category of $U$-modules. We are especially interested in two $U$-modules: the deformation of the natural representation $V^q_1$ and the deformation of the adjoint representation $V^q_2$.

There are certain basic intertwining operators which we would like to mention:

\[
\begin{align*}
\epsilon_1 & : V^q_1 \otimes V^q_1 \rightarrow V^q_0 \\
\epsilon_1(v^1 \otimes v^1) &= \epsilon_1(v^{-1} \otimes v^{-1}) = 0 \\
\epsilon_1(v^{-1} \otimes v^1) &= 1, \epsilon_1(v^1 \otimes v^{-1}) = -q \\
\delta_1 & : V^q_0 \rightarrow V^q_1 \otimes V^q_1 \\
\delta_1 &= v^1 \otimes v^{-1} - q^{-1}v^{-1} \otimes v^1 \\
\tilde{R}_{11} & : V^q_1 \otimes V^q_1 \rightarrow V^q_1 \otimes V^q_1 \\
\tilde{R}_{11} &= \sigma R
\end{align*}
\]

where $\sigma$ is the transposition of the tensor factors, and $R(V^q_1 \otimes V^q_2)$ is the representation in $V^q_1 \otimes V^q_2$ of a certain distinguished element $R \in U \otimes U$, called the universal $R$-matrix. We have the following relation

\[(2.1.1) \quad \tilde{R}_{11} = q^{\frac{1}{2}} \delta_1 \circ \epsilon_1 + q^{-\frac{1}{2}} I\]

Remark. For the explanation of the appearance of the square root $q^{\frac{1}{2}}$ see e.g. [FK, 1.3].

Clearly, there is a unique (up to a constant) intertwiner $[\ , \ ]_q : V^q_2 \otimes V^q_2 \rightarrow V^q_2$ which provides a structure of an algebra for the $\mathbb{C}(q)$-vector space $V^q_2$. We call this intertwiner the quantum Lie bracket. We also denote $V^q_2$ by $\mathfrak{g}_q$ or $\mathfrak{sl}_q(2, \mathbb{C})$, and $V^q_1 \otimes V^q_1 \simeq V^q_2 \oplus V^q_0 \simeq \mathfrak{g}_q \oplus \mathbb{C}(q)$ by $\tilde{\mathfrak{g}}_q$.

The quantum version of Penrose-Kauffman graphical calculus is a way to represent the $U$-intertwiners by planar diagrams. Again, we recall the very basic conventions of graphical calculus; for details we refer the reader to [KL] or [FK] (see, however, the warning below). The deformation of the natural representation $V^q_1$ is depicted by a solid vertical strand $\cdot$. An intertwiner $a : (V^q_1)^{\otimes m} \rightarrow (V^q_1)^{\otimes n}$ is depicted by certain curves connecting $m$ distinct points on one horizontal line and $n$ distinct points on another horizontal line lying below the first one. Only simple intersections are allowed. At each intersection we specify the type of intersection: overcrossing or undercrossing. Thus, any diagram can be viewed as a projection of a system of curves in three dimensions.
The quantum adjoint representation $V_2^q$ is depicted by a box with two strands attached to its top and two strands attached to its bottom. We need the quantum version of the relation (1.3.1):

$$= \frac{1}{[2]}$$

where $[2] = q + q^{-1}$ (in general, $[n] = \frac{q^n - q^{-n}}{q - q^{-1}}$), and the graphical version of the relation (2.1.1):

$$= q^\frac{1}{2} + q^{-\frac{1}{2}}$$

The quantum Lie bracket $[,]_q$ is depicted as follows:

Roughly speaking, to recover the classical graphical calculus from its quantum version one has to stop distinguishing between overcrossings and undercrossings and set $q = 1$.

**Warning.** In our conventions the intertwiners “go downward” while in [FK] the intertwiners “go upward”. Therefore, all our pictures are upside down compared to those of [FK].

**2.2. $\mathfrak{sl}_q(2, \mathbb{C})$ in the framework of graphical calculus.**

We consider the following eight intertwiners $r, r', a, a', b, b', c, c'$: $\tilde{\mathfrak{g}}_q \otimes \tilde{\mathfrak{g}}_q \rightarrow \tilde{\mathfrak{g}}_q \otimes \tilde{\mathfrak{g}}_q$:

$$r = \quad a = \quad b = \quad c =$$

and

$$r' = \quad a' = \quad b' = \quad c' =$$

It is clear how to define the above intertwiners algebraically, using the projectors and injectors (see [FK, 1.4]) and the basic intertwiners $\epsilon_1, \delta_1, \tilde{R}_{11}, \tilde{R}_{11}^{-1}$. We leave this exercise to the reader. We define two families of intertwiners

$\mathfrak{R}^q(\lambda), \mathfrak{R}^q(\lambda)': \tilde{\mathfrak{g}}_q \otimes \tilde{\mathfrak{g}}_q \rightarrow \tilde{\mathfrak{g}}_q \otimes \tilde{\mathfrak{g}}_q$

$$\mathfrak{R}^q(\lambda) = r + (\lambda - \frac{1}{[2]}a + \frac{1}{[2][2\lambda - 1]}b + \frac{1}{[2]^2}c$$

$$\mathfrak{R}^q(\lambda)' = r' + (\lambda - \frac{1}{[2]}a' + \frac{1}{[2][2\lambda - 1]}b' + \frac{1}{[2]^2}c'$$
where \( \lambda \in \mathbb{C}(q) \), \( \lambda \neq \frac{1}{[2]} \). It is easy to see that

\[
\mathcal{R}^q(\lambda) = \begin{array}{c}
\begin{array}{c}
\quad + \lambda \\
\quad + \frac{\lambda}{[2]\lambda - 1}
\end{array}
\end{array}
\]

and

\[
\mathcal{R}^q(\lambda)' = \begin{array}{c}
\begin{array}{c}
\quad + \lambda \\
\quad + \frac{\lambda}{[2]\lambda - 1}
\end{array}
\end{array}
\]

**Lemma 2.2.1.** a) For any \( \lambda \neq \frac{1}{[2]} \), \( \mathcal{R}^q(\lambda)\mathcal{R}^q(\lambda)' = \mathcal{R}^q(\lambda)'\mathcal{R}^q(\lambda) = 1 \);

b) For any \( \lambda \neq \frac{1}{[2]} \), \( \mathcal{R}^q_{12}(\lambda)\mathcal{R}^q_{23}(\lambda)\mathcal{R}^q_{12}(\lambda) = \mathcal{R}^q_{23}(\lambda)\mathcal{R}^q_{12}(\lambda)\mathcal{R}^q_{23}(\lambda) \);

c) For any \( \lambda \neq \frac{1}{[2]} \), \( \mathcal{R}^q_{12}(\lambda)'\mathcal{R}^q_{23}(\lambda)'\mathcal{R}^q_{23}(\lambda)' = \mathcal{R}^q_{23}(\lambda)\mathcal{R}^q_{12}(\lambda)'\mathcal{R}^q_{23}(\lambda)' \).

Let us specialize the family \( \mathcal{R}^q(\lambda) \) to the case when \( \lambda = \frac{\mu}{[2](1-q^{-2})} \), \( \mu \in \mathbb{C} \), \( \mu \neq 0 \). Then it is easy to see that if \( q = 1 \), the family \( \mathcal{R}^q(\lambda) \) degenerates to our classical family \( \mathcal{R}(\mu) \), \( \mu \neq 0 \). A similar statement is true for \( \mathcal{R}^q(\lambda)' \) and \( \mathcal{R}(\mu)' \). Thus, we can regard the \( \mathbb{C}(q) \)-vector space \( \tilde{g}_q \) equipped with \( \mathcal{R}^q(\lambda) \) as a quantum Lie algebra. The relation of Lemma 2.2.1.a) should be regarded as quantum antisymmetry and the relation of Lemma 2.2.1.b) or Lemma 2.2.1.c) as quantum Jacobi identity.

**Remarks.** 1) We would obtain two more families of QYBE solutions if we turn all the diagrams in the definition of \( \mathcal{R}^q(\lambda) \), \( \mathcal{R}^q(\lambda)' \) upside down;

2) Along the lines of Majid [Ma2], a solution of the QYBE can be constructed in the framework of [LS]. It is easy to see that such a solution arising in the case of \( \mathfrak{sl}(2, \mathbb{C}) \) is a member of our family. A similar technique can be applied in the case of \( \mathfrak{sl}(n, \mathbb{C}) \); we do not know, however, how to extend it to the case of an arbitrary semisimple Lie algebra.

### 2.3. \( \mathfrak{sl}_q(2, \mathbb{C}) \)-modules in the framework of graphical calculus.

Let \( V \) be an irreducible finite dimensional \( \mathbf{U} \)-module, i.e. \( V = V_n^q \) for some \( n \). Let us consider the following four intertwiners \( \tilde{g}_q \otimes V \rightarrow V \otimes \tilde{g}_q \):

\[
\begin{array}{c}
r(V) = \begin{array}{c}
\begin{array}{c}
\quad + \lambda \\
\quad + \frac{\lambda}{[2]\lambda - 1}
\end{array}
\end{array}
\end{array}
\quad a(V) = \begin{array}{c}
\begin{array}{c}
\quad + \lambda \\
\quad + \frac{\lambda}{[2]\lambda - 1}
\end{array}
\end{array}
\quad b(V) = \begin{array}{c}
\begin{array}{c}
\quad + \lambda \\
\quad + \frac{\lambda}{[2]\lambda - 1}
\end{array}
\end{array}
\quad c(V) = \begin{array}{c}
\begin{array}{c}
\quad + \lambda \\
\quad + \frac{\lambda}{[2]\lambda - 1}
\end{array}
\end{array}
\end{array}
\]

It is clear how to define the above intertwiners algebraically. We leave this exercise to the reader. We define a family of intertwiners

\[
\mathcal{R}^q_V(\lambda) : \tilde{g}_q \otimes V \rightarrow V \otimes \tilde{g}_q
\]

\[
\mathcal{R}^q_V(\lambda) = r(V) + (\lambda - \frac{1}{[2]})a(V) + \frac{1}{[2][2]\lambda - 1}b(V) + \frac{1}{[2]^2}c(V)
\]
where \( \lambda \in \mathbb{C}(q), \lambda \neq -\frac{1}{2q} \). It is easy to see that

\[
\mathcal{R}^q_{\lambda}(\lambda) = \begin{array}{c}
\begin{array}{c}
\lambda
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\lambda
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\lambda
\end{array}
\end{array}
\]

Lemma 2.3.1. For any \( \lambda \neq -\frac{1}{2q} \), \( (\mathcal{R}^q_{V}(\lambda) \otimes 1_{\tilde{g}}) \circ (1_{\tilde{g}} \otimes \mathcal{R}^q_{V}(\lambda)) \circ (\mathcal{R}^q_{\lambda}(\lambda) \otimes 1_{V}) = (1_{V} \otimes \mathcal{R}^q_{\lambda}(\lambda)) \circ (\mathcal{R}^q_{V}(\lambda) \otimes 1_{\tilde{g}}) \circ (1_{\tilde{g}} \otimes \mathcal{R}^q_{V}(\lambda)). \)

For two \( U \)-modules \( V, W \), we define \( R(V \otimes W) \) as the representation of the universal \( R \)-matrix in \( V \otimes W \). We set \( \tilde{R}(V \otimes W) = \sigma R(V \otimes W) \), where \( \sigma \) is the transposition of the tensor factors. The following Lemma is a \( q \)-deformation of Lemma 1.2.2.

Lemma 2.3.2. Let \( V, W \) be two irreducible finite dimensional \( U \)-modules. Then for any \( \lambda \neq -\frac{1}{2q} \), \( (\tilde{R}(V \otimes W) \otimes 1_{\tilde{g}}) \circ (1_{\tilde{g}} \otimes \mathcal{R}^q_{V}(\lambda)) \circ (\mathcal{R}^q_{V}(\lambda) \otimes 1_{W}) = (1_{W} \otimes \mathcal{R}^q_{V}(\lambda)) \circ (\mathcal{R}^q_{W}(\lambda) \otimes 1_{V}) \circ (1_{\tilde{g}} \otimes \tilde{R}(V \otimes W)). \)

Let us consider the \( U \)-module \( V^n_q \). Let us specialize the family \( \mathcal{R}^q_{V^n_q}(\lambda) \) to the case when \( \lambda = \frac{\mu}{m(1-q^{-2})}, \mu \in \mathbb{C}, \mu \neq 0 \). Then it is easy to see that if \( q = 1 \), the family \( \mathcal{R}^q_{V^n_q}(\lambda) \) degenerates to our classical family \( \mathcal{R}^q_{V_n}(\mu), \mu \neq 0 \).

Remarks. 1) We would obtain another family of intertwiners if we turn all the diagrams in the definition of \( \mathcal{R}^q_{V}(\lambda) \) upside down.

2) Let us consider the case when \( V = V_2^q g_q \). We can regard \( a(g_q) \) as a map \( g_q \otimes g_q \rightarrow g_q \) and \( c(g_q) \) as a map \( g_q \rightarrow g_q \). Then \( a(g_q) = [2](1-q^{-2})[\ , \ ]_q \), \( c(g_q) = -(q^3 + q^{-3}) \text{Id}_{g_q} \). Let us consider the components of \( \mathcal{R}^q_{12}(\lambda)\mathcal{R}^q_{23}(\lambda)\mathcal{R}^q_{12}(\lambda) \) and \( \mathcal{R}^q_{23}(\lambda)\mathcal{R}^q_{12}(\lambda)\mathcal{R}^q_{23}(\lambda) \) mapping \( g_q \otimes g_q \otimes g_q \) to \( g_q \otimes \mathbb{C}(q) \otimes \mathbb{C}(q) \). These components may be regarded as maps from \( g_q \otimes g_q \otimes g_q \) to \( g_q \). Then the Yang-Baxter equation implies

\[
(2.3.1) \quad (q^2 + q^{-2} - 1)[\ , \ ]_q \circ ([\ , \ ]_q \otimes 1) = [\ , \ ]_q \circ (1 \otimes [\ , \ ]_q) \circ (1 \otimes -r(g_q) \otimes 1)
\]

It is clear that the maps \( T^h \) and \( \sigma^h \) introduced in [V] may be considered as \( \mathbb{C}(q) \)-linear maps between \( \mathbb{C}(q) \)-linear spaces. Then \( T^h \) is equal to \( [\ , \ ]_q \) up to multiplication by a constant and \( 1 \otimes -r(g_q) = \frac{a^2 + q^{-2}}{2} (1 \otimes -\sigma^h) \). Therefore, \( (2.3.1) \) is equivalent to the second of the relations (6.1) of [V]. The first relation (6.1) can be obtained in a similar way.

Motivated by the above examples we give the following abstract definitions.

Definition 2.3.3. A \( \mathbb{C}(q) \)-vector space \( g_q \) is called a (generalized) quantum Lie algebra if it is equipped with a \( \mathbb{C}(q) \)-linear family \( \mathcal{R}(\lambda) : \tilde{g}_q \otimes \tilde{g}_q \rightarrow \tilde{g}_q \otimes \tilde{g}_q \), where \( \tilde{g}_q = g_q \oplus \mathbb{C}(q) \), of the invertible solutions of the Yang-Baxter equation. A \( \mathbb{C}(q) \)-vector space \( V \) is called a (generalized) \( g_q \)-module if there is a \( \mathbb{C}(q) \)-linear family \( \mathcal{R}_V : \tilde{g}_q \otimes V \rightarrow V \otimes \tilde{g}_q \) satisfying the relation of Lemma 2.3.1.

For example, any \( V^q_n \) is an \( \mathfrak{sl}_2(q, \mathbb{C}) \)-module.
**Lemma 2.3.4.** Let $V$, $W$ be two $\mathfrak{g}_q$-modules. Then the vector space $V \otimes W$ has a natural structure of a $\mathfrak{g}_q$-module: $\mathfrak{g}_V \otimes \mathfrak{g}_W = (1_V \otimes \mathfrak{g}_W)(\mathfrak{g}_V \otimes 1_W)$.

Such a tensor product is clearly associative. Therefore, the category of $\mathfrak{g}_q$-modules is a monoidal category.

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**References**

[Be1] D. Bernard, *Quantum Lie algebras and differential calculus on quantum Lie groups*, Progr. Theoret. Phys. Suppl. 102 (1990), 49-66.

[Be2] D. Bernard, *A remark on quasi-triangular quantum Lie algebras*, Phys. Lett. B 260 (1991), 389-393.

[CFM] J.C. Carter, D.E. Flath, and M. Saito, *The classical and quantum $\ell$-j-symbols*, Princeton Univ. Press, Princeton, 1995.

[CSWW] U. Carow-Watamura, M. Schlieker, S. Watamura, and W. Weich, *Bicovariant Differential Calculus on quantum groups $SU_q(N)$ and $SO_q(N)$*, Comm. Math. Phys. 142 (1991), 605-641.

[DF] J. Ding and I. Frenkel, *Quantum Lie algebra $\mathfrak{gl}_q(n)$*, preprint, 1996.

[DH] G. W. Delius and A. Hueffman, *On quantum Lie algebras and quantum root systems*, J. Phys. A: Math. Gen. 29 (1996), 1703-1722.

[Dr] V.G. Drinfeld, *Quantum groups*, Amer. Math. Soc., Providence, RI, Proc. Internat. Congr. Math. (Berkeley, 1986) (1987), 798-820.

[FK] I. Frenkel and M. Khovanov, *Canonical bases in tensor products and graphical calculus for $U_q(sl_2)$*, Duke Math. J. 87 (1997), 409-480.

[J1] M. Jimbo, *A q-difference analogue of $U(g)$ and the Yang-Baxter equation*, Lett. Math. Phys. 10 (1985), 63-69.

[J2] M. Jimbo, *Quantum R-matrix for the generalized Toda system*, Comm. Math. Phys. 102 (1986), 537-547.

[KL] L. H. Kauffman and S. L. Lins, *Temperley-Lieb Recoupling Theory and Invariants of 3-Manifolds*, Ann. of Math. Stud., vol. 134, Princeton Univ. Press, Princeton, 1994.

[LS] V. Lyubashenko and A. Sudbery, *Quantum Lie algebras of type $A_n$*, q-alg/9510004, J. Math. Phys. (to appear).

[Ma1] S. Majid, *Quantum and braided Lie algebras*, J. Geom. and Phys. 13 (1994), 307-356.

[Ma2] S. Majid, *Solutions of the Yang-Baxter equations from braided-Lie algebras and braided groups*, J. Knot Theory and Its Ramifications 4 (1995), 673-697.

[V] M. Vybornov, *Drinfeld-Kohno correspondence and quantum Jacobi identity*, Funct. Anal. Appl. (to appear).

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