HEAT KERNEL OF FRACTIONAL LAPLACIAN WITH HARDY DRIFT VIA DESINGULARIZING WEIGHTS

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Abstract. We establish sharp two-sided bounds on the heat kernel of the fractional Laplacian, perturbed by a drift having critical-order singularity, using the method of desingularizing weights.

1. In 1998, Milman and Semenov [MS0] introduced the method of desingularizing weights to establish two-sided weighted bounds on the heat kernel of the Schrödinger operator $-\Delta - V$, $V(x) = \delta \left( \frac{d-2}{2} \right)^2 |x|^{-2}$, $0 < \delta \leq 1$ in $L^2(\mathbb{R}^d, dx)$, $d \geq 3$ [MS1, MS2]. The corresponding $C_0$ semigroup is not ultra-contractive, but becomes one after transferring it to an appropriate weighted space.

In this paper we use the desingularization method to obtain sharp two-sided weighted bounds on the heat kernel of the operator $\left( -\Delta \right)^{\frac{\alpha}{2}} + b \cdot \nabla$, $b(x) = c|x|^{-\alpha}x$, $c > 0$, $1 < \alpha < 2$.

The vector field $b$ has a model critical-order singularity at $x = 0$. The standard upper bound in terms of the heat kernel of $\left( -\Delta \right)^{\frac{\alpha}{2}}$ does not hold.

The desingularization method rests on two assumptions: the Sobolev embedding property, and a "desingularizing" $(L^1, L^1)$ bound on the weighted semigroup. Namely, let $X$ be a locally compact space and $\mu$ a $\sigma$-finite Borel measure on $X$. Set

$$\langle u, v \rangle = \langle u\bar{v} \rangle := \int_X u\bar{v}d\mu.$$ 

Let $-\Lambda$ be the generator of a $C_0$ contraction semigroup $e^{-t\Lambda}$, $t > 0$, in the (complex) Banach space $L^p = L^p(X, \mu)$ for any $p \in [2, \infty]$. Assume that $\Lambda, \Lambda^*$ possess the Sobolev-type embedding property: There are constants $j > 1$ and $c_S > 0$ such that

$$\text{Re} \langle \Lambda f, f \rangle \geq c_S \|f\|_{2j}^2, \quad f \in D(\Lambda), \quad (N_1)$$

$$\text{Re} \langle \Lambda^* g, g \rangle \geq c_S \|g\|_{2j}^2, \quad g \in D(\Lambda^*), \quad (N_1^*)$$

where $\| \cdot \|_p = \| \cdot \|_{L^p}$, but $e^{-t\Lambda} \upharpoonright L^1 \cap L^p$ cannot be extended by continuity to a bounded map on $L^1$ and the ultra-contraction estimate

$$\|e^{-t\Lambda}f\|_\infty \leq c(t)\|f\|_1, \quad f \in L^1 \cap L^\infty, \quad t > 0$$

is not valid.

In this case we will be assuming that there exists a family of real valued weights $\varphi = \{\varphi_s\}_{s>0}$ on $X$ such that, for all $s > 0$,

$$\varphi_s, \quad 1/\varphi_s \in L^2_{\text{loc}}(X, \mu), \quad (N_2)$$

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and there exists constant $c_1$, independent of $s$ such that, for all $0 < t \leq s$
\[ \| \varphi_s e^{-t\Lambda} \varphi_s^{-1} f \|_1 \leq c_1 \| f \|_1, \quad f \in \mathcal{D} := \varphi_s L^\infty_{\text{com}}(X, \mu). \]  

(N3)

The following general theorem is the point of departure for the desingularization method in the non-selfadjoint setting:

**Theorem A.** In addition to (N1)-(N3) assume that
\[ \inf_{s>0, x \in X} |\varphi_s(x)| \geq c_0 > 0. \]  

(N4)

Then, for each $t > 0$, $e^{-t\Lambda}$ is integral operator, and there is a constant $C = C(j, c_s, c_1, c_0)$ such that the weighted Nash initial estimate
\[ |e^{-t\Lambda}(x, y)| \leq Ct^{-j'}|\varphi_t(y)|, \quad j' = j/(j - 1). \]  

(NIEw)

is valid for $\mu$ a.e. $x, y \in X$.

**Proof.** 1. There exists a constant $c_2$ such that the inequality
\[ \| e^{-t\Lambda} \varphi f \|_2 \leq c_2 t^{-\frac{j'}{2}} \| \varphi^2 f \|_1 \quad (\varphi \equiv \varphi_s) \]  

(*)

is valid for all $f \in \varphi^{-1}L^\infty_{\text{com}}$ and $0 < t \leq s$.

Indeed, set $L^2_\varphi = L^2(X, \varphi^2 d\mu)$, define a unitary map $\Phi : L^2_\varphi \to L^2 \varphi f = \varphi f$. Set $\Lambda_\varphi = \Phi^{-1} \Lambda \Phi$ of domain $D(\Lambda_\varphi) = \Phi^{-1} D(\Lambda)$. Then $\| e^{-t\Lambda_\varphi} \|_{L^p \to L^q} = \| e^{-t\Lambda} \|_{L^p \to L^q}$ for all $t \geq 0$. Here and below $\| \cdot \|_{p \to q} = \| \cdot \|_{L^p \to L^q}$, and the subscript $\varphi$ indicates that the corresponding quantities are related to the measure $\varphi^2 d\mu$.

Let $f = \varphi^{-1} h$, $h \in L^\infty_{\text{com}}$, and so $f \in L^2_\varphi \cap L^1_\varphi$ by (N2). Let $u_t = e^{-t\Lambda_\varphi} f$. Then $\varphi u_t = e^{-t\Lambda} \varphi f$ and

\[ \text{Re}(\Lambda_\varphi u_t, u_t)_\varphi \geq c_S \| \varphi u_t \|_{L^2_\varphi}^2 \\
\geq c_S \| \varphi u_t \|_2^{2 + \frac{j}{2}} \| \varphi u_t \|_1^{-\frac{j}{2}} \\
= c_S \langle u_t, u_t \rangle_\varphi^{1 + \frac{j}{2}} \| \varphi^{-1} \varphi e^{-t\Lambda} \varphi^{-1} \varphi^2 f \|_1^{-\frac{j}{2}}, \]

where (N1) and Hölder’s inequality have been used.

Clearly, $-\frac{1}{2} \varphi \partial_t \langle u_t, u_t \rangle_\varphi = \text{Re}(\Lambda_\varphi u_t, u_t)_\varphi$. Setting $w := \langle u_t, u_t \rangle_\varphi$ and using (N4) we have

\[ \frac{d}{dt} w^{-\frac{j}{2}} \geq \frac{2}{j} c_S (c_0^{-1} \| \varphi e^{-t\Lambda_\varphi} \varphi^{-1} \varphi^2 f \|_1)^{-\frac{j}{2}}. \]

By our choice of $f$, $\varphi^2 f = \varphi h \in \mathcal{D}$. Therefore we can apply (N3) and obtain

\[ \frac{d}{dt} w^{-\frac{j}{2}} \geq \frac{2}{j} c_S (c_1 c_0^{-1} \| f \|_{1, \varphi})^{-\frac{j}{2}}, \quad t \leq s. \]

Integrating this inequality over $[0, t]$ gives

\[ \| e^{-t\Lambda_\varphi} f \|_{2, \varphi} \leq c_2 t^{-\frac{j'}{2}} \| f \|_{1, \varphi}, \quad t \leq s, \]

or

\[ \| e^{-t\Lambda_\varphi} f \|_2 \leq c_2 t^{-\frac{j'}{2}} \| f \|_{1, \varphi}, \]

i.e. (\(\blacksquare\)).
2. Next, we claim that there is a constant $c_3 > 0$ such that
\[
\|e^{-t\Lambda}\|_{2 \to \infty} \leq c_3 t^{-\frac{d}{2}}.
\]
Indeed, since $\Lambda$ is accretive, $\Lambda^*$ is accretive as well. Since $e^{-t\Lambda}$ is a contraction on all $L^p$, $2 \leq p < \infty$, we have
\[
\|e^{-t\Lambda^*}g\|_1 \leq \|g\|_1, \quad g \in L^2 \cap L^1.
\]
Thus, arguing as above (with $\varphi \equiv 1$) and using $(N_1^*)$, we have $\|e^{-t\Lambda^*}\|_{1 \to 2} \leq c_3 t^{-\frac{d}{2}}$, and so via duality $(\ast \ast)$.

3. Combining $(\heartsuit)$ and $(\ast \ast)$, we obtain, for all $f \in \varphi^{-1}L^1_{\text{com}},$
\[
\|e^{-2t\Lambda}\varphi f\|_\infty \leq c_3 t^{-\frac{d}{2}}\|e^{-t\Lambda}\varphi f\|_2 \leq c_3 c_2 t^{-\frac{3}{2}}\|\varphi^2 f\|_1.
\]
The latter yields (after redefinition on a null set) $(NIE_m)$. The proof of Theorem A is completed. \[\square\]

**Remark.** $(N_1^*)$ provides the bound $\|e^{-t\Lambda}\|_{2 \to \infty} \leq ct^{-\frac{d}{2}}$, needed to prove $(NIE_m)$. There are other means to obtain the $(L^2, L^\infty)$ bound, e.g. replacing $(N_1^*)$ by $\text{Re} \langle \Lambda f, |f|^{p-1}\text{sgn} f \rangle \geq c_S \|f\|^p_{p_j}$, $f \in D(\Lambda)$, for all $p \geq 2$, and then arguing as in [KiS1] proof of Theorem 4.3].

In applications of Theorem A to concrete operators the main difficulty consists in verification of the $(L^1, L^1)$ bound (N3). In this paper we develop a new approach to the proof of $(N_3)$ for $\Lambda \supset (-\Delta)^{\frac{\alpha}{2}} + c|x|^{-\alpha}x \cdot \nabla$, $c > 0$, by verifying the hypotheses of the Lumer-Phillips Theorem for specially constructed $C_0$ semigroups approximating $\varphi_\epsilon e^{-t\Lambda} \varphi_\epsilon^{-1}$ in $L^1$. This construction of the approximating semigroups is a key observation.

2. We now state our main result concerning $(-\Delta)^{\frac{\alpha}{2}} + c|x|^{-\alpha}x \cdot \nabla$, $1 < \alpha < 2$, $c > 0$, in detail. Let $d \geq 3$. Set
\[
c(\alpha, p, d) := \frac{\gamma(d - \frac{\alpha}{2})}{\gamma(d - \frac{\alpha}{2})}, \quad \gamma(\alpha) := \frac{2\pi^{\frac{d}{2}} \Gamma(\frac{d}{2})}{\Gamma(\frac{d - \alpha}{2})}, \quad 1 < p < \frac{d}{\alpha}.
\]
Set
\[
b(x) := \kappa|x|^{-\alpha}x, \quad \kappa := \delta(d - \alpha)^{-1}2e^{-2 \left(\frac{\alpha}{2}, 2, d\right)}, \quad 0 < \delta < 1.
\]

**Proposition 1.** $\Lambda := (-\Delta)^{\frac{\alpha}{2}} + b \cdot \nabla$, $D(\Lambda) = D((-\Delta)^{\frac{\alpha}{2}}) = W^{\alpha,2}$, is the (minus) generator of a holomorphic semigroup in $L^2$.

We prove Proposition 1 below by showing that $b \cdot \nabla$ is Rellich’s perturbation of $(-\Delta)^{\frac{\alpha}{2}}$.

Define $\beta$ by $\frac{\gamma(\beta)}{(\beta - \alpha)(\beta - \alpha)} = \kappa$. This choice of $\beta$ entails that $|x|^{-\alpha+\beta}$ is a Lyapunov function to the formal operator $\Lambda^* = (-\Delta)^{\frac{\alpha}{2}} - b \cdot \nabla$, i.e. $\Lambda^* |x|^{-\alpha+\beta} = 0$, cf. Appendix [A].

Let $\eta$ be a $C^2([0, \infty[)$ function such that
\[
\eta(r) = \begin{cases} r^{-d+\beta}, & 0 < r < 1, \\ \frac{1}{2}, & r \geq 2. \end{cases}
\]

**Theorem 1.** $e^{-t\Lambda}$ is an integral operator for each $t > 0$; there exists a constant $C$ such that the weighted Nash initial estimate
\[
e^{-t\Lambda}(x, y) \leq Ct^{-j\prime}\varphi_t(y), \quad j' = \frac{d}{\alpha}, \quad \varphi_t(y) = \eta(t^{-\frac{\alpha}{2}}|y|)
is valid for all \( x, y \in \mathbb{R}^d, y \neq 0 \) and \( t > 0 \).

Having at hand Theorem \([\text{I}]\) we obtain below the following.

**Theorem 2.** \( e^{-t\Lambda}(x,y) \approx e^{-t(-\Delta)^{\frac{\alpha}{2}}}(x,y)\varphi_t(y), \quad x, y \in \mathbb{R}^d, y \neq 0, \quad t > 0. \)

Here \( e^{-t(-\Delta)^{\frac{\alpha}{2}}}(x,y) \approx t^{-\frac{\alpha}{4}} \wedge \frac{t}{|x-y|^{d+\alpha}} \). \( a(z) \approx b(z) \) means that \( c^{-1}b(z) \leq a(z) \leq cb(z) \) for some constant \( c > 1 \) and all admissible \( z \).

Sharp two-sided weighted bounds for the heat kernel of \((-\Delta)^{\frac{\alpha}{2}} - \delta c_\alpha^{-2} |x|^{-\alpha}, 0 < \alpha < 2, 0 < \delta \leq 1\) is the subject of \([\text{BGJP}]\). Our method gives a short and transparent operator-theoretic proof of these bounds for \( 0 < \delta < 1 \) \([\text{KiS2}]\). Concerning \((-\Delta)^{\frac{\alpha}{2}} + c|x|^{-\alpha}, c > 0, \) see \([\text{CKSV}]\) and \([\text{JW}]\).

1. **Proof of Proposition \([\text{I}]\)**

For brevity, write \( \| \cdot \| \equiv \| \cdot \|_{2\to 2} \) and \( A \equiv (-\Delta)^{\frac{\alpha}{2}} \) in \( L^2 \).

Define \( T = b \cdot \nabla(\zeta + A)^{-1}, \) \( \Re \zeta > 0, \) and note that

\[
\|T\| \leq \|b\| (\zeta + A)^{-1+\frac{\alpha}{2}} \|\nabla(\zeta + A)^{-\frac{\alpha}{2}}
\]

(we are using \( \|\nabla g\|_2 = \|(-\Delta)^{\frac{\alpha}{4}} g\|_2 \))

\[
\leq \|b\| (\Re \zeta + A)^{-1+\frac{\alpha}{2}} \|A^{\frac{\alpha}{2}} (\zeta + A)^{-\frac{\alpha}{2}}
\]

(by the Spectral Theorem, \( \|A^{\frac{\alpha}{2}} (\zeta + A)^{-\frac{\alpha}{2}}\| \leq 1 \))

\[
\leq \|b\| (-\Delta)^{-\frac{\alpha-1}{2}}
\]

(we are using \([\text{KPS}]\) Lemma 2.7)

\[
= \kappa c(\alpha - 1, 2, d) < \delta \quad (\delta < 1)
\]

because \( c(\alpha - 1, 2, d) < (d - \alpha)^{-1} c^2(\frac{\alpha}{2}, 2, d) \) or, equivalently,

\[
F(\alpha) \equiv (d - \alpha) \Gamma\left(\frac{d - 2 + 2\alpha}{4}\right) \left[\Gamma\left(\frac{d - \alpha}{4}\right)\right]^2 - 4\Gamma\left(\frac{d + 2 - 2\alpha}{4}\right) \left[\Gamma\left(\frac{d + \alpha}{4}\right)\right]^2 > 0
\]

(the latter is due to \( \frac{d^2}{dt^2} \log \Gamma(t) \geq 0 \) and \( F(2) = 0 \) \( (d - 2) \Gamma(d-2) = 4 \Gamma(d+2) \)).

Thus, the Neumann series for \((\zeta + A)^{-1} = (\zeta + A)^{-1}(1 + T)^{-1}\) converges, and

\[
\| (\zeta + A)^{-1} \| \leq (1 - \delta)^{-1} |\zeta|^{-1}, \quad \Re \zeta > 0,
\]

i.e. \(-\Lambda\) is the generator of a holomorphic semigroup. \hfill \Box

2. **Proof of Theorem \([\text{I}]\)**

First, we are going to verify the assumptions of Theorem A for the operators

\[
P^\varepsilon := (-\Delta)^{\frac{\alpha}{2}} + b_\varepsilon \cdot \nabla + U_\varepsilon \quad \text{in} \ L^2, \quad D(P^\varepsilon) = D((-\Delta)^{\frac{\alpha}{2}}) \equiv \mathcal{W}^{\alpha,2} \quad \text{(Bessel potential space)},
\]

where \( \varepsilon > 0, \)

\[
b_\varepsilon(x) = \kappa |x|^{-\alpha} , \quad |x|_\varepsilon := \sqrt{|x|^2 + \varepsilon}, \quad U_\varepsilon(x) := \alpha \kappa \varepsilon |x|^{-\alpha - 2} (\varepsilon > 0),
\]

and for the weights \( \varphi_\varepsilon \) defined in Theorem \([\text{I}]\)

\( P_{\varepsilon}, \varepsilon > 0, \) is the generator of a \( C_0 \) semigroup in \( L^2 \) (for example, by the Hille Perturbation Theorem \([\text{Ka}]\) Ch. IX, sect. 2.2). Similarly, \( \Lambda^\varepsilon := (-\Delta)^{\frac{\alpha}{2}} + b_\varepsilon \cdot \nabla \) generates a \( C_0 \) semigroup in
Moreover, it is well known that $e^{-t\Delta}L^2_+ \subset L^2_+$ and $\|e^{-tP_\varepsilon}f\|_\infty \leq \|e^{-t\Lambda}\|_\infty \leq \|f\|_\infty$, $f \in L^2 \cap L^\infty$. It follows from (N1) (see below) that $e^{-tP_\varepsilon}$ is a contraction in $L^2$. In particular, $e^{-tP_\varepsilon}$ is a $C_0$ contraction semigroup in all $L^p$, $2 \leq p < \infty$.

(N1): There is a constant $c > 0$ such that, for all $f \in D(P_\varepsilon)$ and $\varepsilon > 0$,

$$\Re\langle P_\varepsilon f, f \rangle \geq c\|f\|_{2j}^2, \quad j = \frac{d}{d-\alpha}.$$  

Proof. Indeed, $\Re\langle P_\varepsilon^* f, f \rangle = \|(-\Delta)^{\frac{\alpha}{2}}f\|_2^2 + \kappa \Re\|x|^{-\alpha}x \cdot \nabla f \rangle + \langle U_{\varepsilon} f, f \rangle$ and

$$\Re\|x|^{-\alpha}x \cdot \nabla f \rangle = -\kappa \frac{d-\alpha}{2}\| x |^{-\frac{\alpha}{2}}f \|_2^2 - \frac{1}{2}\langle U_{\varepsilon} f, f \rangle.$$  

Now applying the Hardy-Relliich inequality $\|(-\Delta)^{\frac{\alpha}{2}}f\|_2^2 \geq c^{-2}(\frac{\alpha}{2}, 2, d)\|x|^{-\frac{\alpha}{2}}f\|_2^2$ (see [KPS] Lemma 2.7) and the uniform Sobolev inequality $\|(-\Delta)^{\frac{\alpha}{2}}f\|_2^2 \geq c_S\|f\|_{2j}^2$, we obtain (N1) with $c = (1 - \delta)c_S$. \hfill \Box

(N1*): There is a constant $c > 0$ such that, for all $g \in D((P_\varepsilon)^*)$ and $\varepsilon > 0$,

$$\Re\langle (P_\varepsilon)^* g, g \rangle \geq c\|g\|_{2j}^2.$$  

Proof. Since $D((P_\varepsilon)^*) = D((-\Delta)^{\frac{\alpha}{2}}) \equiv D(P_\varepsilon)$, (N1*) is a consequence of (N1). \hfill \Box

(N2), (N4): $\varphi_{\pm1} \in L^2_{\text{loc}}$ and $\inf_{s > 0, x \in \mathbb{R}^d} \varphi_s(x) \geq \frac{1}{2}$. By the construction of $\varphi$, (N2), (N4) are valid.

(N3): There exists a constant $\omega > 0$ such that, for all $0 < t \leq s$,

$$\|\varphi_s e^{-tP_\varepsilon} \varphi_{-1}^* h\|_1 \leq e^{\omega s\frac{t}{2}}\|h\|_1, \quad h \in L^1 \cap L^2, \quad \omega \neq \omega(\varepsilon).$$

See the proof of (N3) below.

Thus, Theorem A applies and yields

$$\|e^{-tP_\varepsilon} \varphi_\varepsilon f\|_\infty \leq C t^{-\frac{j}{2}}\| \varphi_\varepsilon^2 f\|_1, \quad C \neq C(\varepsilon), \quad f \in L^1_\varphi.$$  

It remains to take $\varepsilon \downarrow 0$ in (*). In Remark [1] we prove that $e^{-tP_\varepsilon} \to e^{-t\Lambda}$ strongly in $L^2$. The latter and (*) clearly yield $\|e^{-t\Lambda} \varphi_\varepsilon f\|_\infty \leq C t^{-\frac{j}{2}}\| \varphi_\varepsilon^2 f\|_1$ and hence Theorem [1]

**Proof of (N3).** In $L^1$ define operators

$$P_\varepsilon := (-\Delta)^{\frac{\alpha}{2}} + b_\varepsilon \cdot \nabla + U_\varepsilon, \quad D(P_\varepsilon) = D((-\Delta)^{\frac{\alpha}{2}}) \equiv W^{\alpha/2, 1},$$

$$(P_\varepsilon)^* := (-\Delta)^{\frac{\alpha}{2}} - \nabla \cdot b_\varepsilon + U_\varepsilon = (-\Delta)^{\frac{\alpha}{2}} - b_\varepsilon \cdot \nabla - W_\varepsilon, \quad D((P_\varepsilon)^*) = D((-\Delta)^{\frac{\alpha}{2}}),$$

where $W_\varepsilon(x) = (d-\alpha)\kappa|x|^{-\alpha}$. Note that for each $\varepsilon > 0$ $e^{-tP_\varepsilon}$, $e^{-t(P_\varepsilon)^*}$ can be viewed as $C_0$ semigroups in $L^1$ and $C_0 = \{ f \in C(\mathbb{R}^d) \mid f \text{ are uniformly continuous and bounded} \}$ with the sup-norm (e.g. by the Hille Perturbation Theorem).

Set

$$\phi_n = \left( e^{-t(P_\varepsilon)^*} \right) \varphi \Rightarrow \varphi s_n, \quad n = 1, 2, \ldots$$

Since $\varphi = \varphi_{(1)} + \varphi_{(u)}$, $\varphi_{(1)} \in D((-\Delta)^{\frac{\alpha}{2}})$, $\varphi_{(u)} \in D((-\Delta)^{\frac{\alpha}{2}}C_0)$, the weights $\phi_n$ are well defined.

**Remark.** We emphasize that this choice of $\phi_n$, the regularization of $\varphi$, is the key observation that allows to carry out the method in the case $\alpha < 2$.  

Put \( Q = \varphi_n P^c \varphi_n^{-1}, \) \( D(Q) = \varphi_n D(P^c) = \varphi_n D((-\Delta)^{1/2}), \) \( P_{\varepsilon,n} = \varphi_n e^{-tP^c} \varphi_n^{-1}. \)

Here \( \varphi_n D(P^c) := \{ \varphi_n u \mid u \in D(P^c) \}. \) Since \( \varphi_n \geq \frac{1}{2} \) and \( \varphi_n, \varphi_n^{-1} \in L^\infty, \) these operators are well defined. In particular, \( P_{\varepsilon,n} \) is a quasi bounded \( C_0 \) semigroup in \( L^1, \) say \( e^{-tG}. \) Set

\[
M := \varphi_n (1 + (-\Delta)^{1/2})^{-1}[L^1 \cap C_u] = \varphi_n (\lambda_c + P^c)^{-1}[L^1 \cap C_u], \quad 0 < \lambda_c \in \rho(-P^c).
\]

Clearly, \( M \) is a dense subspace of \( L^1, M \subset D(Q) \) and \( M \subset D(G). \) Moreover, \( Q \upharpoonright M \subset G. \) Indeed, for \( f = \varphi_n u \in M, \)

\[
Gf = s-L^1\lim_{t \downarrow 0} t^{-1}(1 - e^{-tG})f = \varphi_n s-L^1\lim_{t \downarrow 0} t^{-1}(1 - e^{-tP^c})u = \varphi_n P^c u = Qf.
\]

Thus \( Q \upharpoonright M \) is closable and \( \tilde{Q} := (Q \upharpoonright M)^{\text{clos}} \subset G. \)

Next, let us show that \( R(\lambda_c + \tilde{Q}) \) is dense in \( L^1. \) If \( \langle (\lambda_c + \tilde{Q})h, v \rangle = 0 \) for all \( h \in D(\tilde{Q}) \) and some \( v \in L^\infty, \|v\|_\infty = 1, \) then taking \( h \in M \) we would have \( \langle (\lambda_c + \tilde{Q}) \varphi_n (\lambda_c + P^c)^{-1} g, v \rangle = 0, \) \( g \in L^1 \cap C_u, \) or \( \langle \varphi_n g, v \rangle = 0. \) Choosing \( g = e^{\tilde{Q}t} (\chi_n v), \) where \( \chi_n \in C_\infty^c \) with \( \chi_n(x) = 1 \) when \( x \in B(0, n), \) we would have \( \lim_{t \uparrow \infty} \langle \varphi_n (\lambda_c + \tilde{Q}) h, v \rangle = \langle \varphi_n \chi_n, |v|^2 \rangle = 0, \) and so \( v \equiv 0. \) Thus, \( R(\lambda_c + \tilde{Q}) \) is dense in \( L^1. \)

**Proposition 2** (The main step). There is a constant \( \hat{c} = \hat{c}(d, \alpha, \delta) \) such that

\[
\lambda + \tilde{Q} \text{ is accretive whenever } \lambda \geq \hat{c}s^{-1}.
\]

Taking Proposition 2 for granted, we immediately establish the bound

\[

\|e^{-tG}\|_{1 \to 1} \equiv \|\varphi_n e^{-tP^c} \varphi_n^{-1}\|_{1 \to 1} \leq e^{\omega t}, \quad \omega = \hat{c}s^{-1}. \tag{**}
\]

Indeed, the facts: \( \tilde{Q} \) is closed and \( R(\lambda_c + \tilde{Q}) \) is dense in \( L^1 \) together with Proposition 2 imply \( R(\lambda_c + \tilde{Q}) = L^1 \) (Appendix B). But then, by the Lumer-Phillips Theorem, \( \lambda + \tilde{Q} \) is the (minus) contraction generator of a contraction semigroup, and \( \tilde{Q} = G \) due to \( \tilde{Q} \subset G. \)

In turn, \( (**\)\) easily yields \((N_3).\) Indeed, \((**)\) implies that \( \lim_{n \uparrow \infty} \|\varphi_n e^{-tP^c} v\|_1 \leq e^{\omega t} \lim_{n \uparrow \infty} \|\varphi_n v\|_1 \)

for all \( v \in L^1 \cap L^2. \) But

\[

\lim_{n \uparrow \infty} \|\varphi_n v\|_1 = \lim_{n \uparrow \infty} \langle \varphi, e^{-\frac{P^c}{n}} |v| \rangle = \langle \varphi, |v| \rangle < \infty;
\]

\[

\lim_{n \uparrow \infty} \|\varphi_n e^{-tP^c} v\|_1 = \lim_{n \uparrow \infty} \langle \varphi, e^{-\frac{P^c}{n}} |e^{-tP^c} v| \rangle = \langle \varphi, |e^{-tP^c} v| \rangle < \infty.
\]

Therefore, taking \( v = \varphi^{-1} h \) we arrive at \((N_3).\)

**Proof of Proposition 2** First we note that, for \( f = \varphi_n u \in M, \)

\[

\langle Qf, \frac{f}{|f|} \rangle = \langle \varphi_n P^c u, \frac{f}{|f|} \rangle = \lim_{t \downarrow 0} t^{-1} \langle \varphi_n (1 - e^{-tP^c}) u, \frac{f}{|f|} \rangle,
\]

\[

\text{Re}(Qf, \frac{f}{|f|}) \geq \lim_{t \downarrow 0} t^{-1} \langle (1 - e^{-tP^c}) |u|, \varphi_n \rangle = \langle P^c e^{-\frac{P^c}{n}} |u|, \varphi \rangle.
\]

We emphasize that \( e^{-tP^c} \) is holomorphic due to Hille's Perturbation Theorem.
We are going to estimate $J := \langle P^c e^{-\frac{|u|^2}{n}} | \varphi \rangle$ from below using the representation
\[ (-\Delta)^\frac{\alpha}{2} \varphi = -I_{2-\alpha} \Delta \varphi, \]
where $I_\nu \equiv (-\Delta)^{-\frac{\nu}{2}}$.

Since $e^{-t(P^c)^*}$ is a $C_0$ semigroup in $L^1$ and $C_\alpha$, and $\varphi = \varphi(1) + \varphi(u)$, $\varphi(1) \in D((-\Delta)^\frac{\alpha}{2})$, $\varphi(u) \in D((-\Delta)^\frac{\alpha}{2})_{C_\alpha}$, $(P^c)^* \varphi$ is well defined and belongs to $L^1 + C_\alpha = \{ w + v \mid w \in L^1, v \in C_\alpha \}$.

Define $\tilde{\varphi}_1(x) = |x|^{-d+\beta}$, $V(x) := (\beta - \alpha)\kappa|x|^{-\alpha} (= \frac{\kappa(\beta)}{\gamma(\beta-\alpha)}|x|^{-\alpha}$ by the choice of $\beta$). Using the identity $(-\Delta)^\frac{\alpha}{2} \tilde{\varphi}_1 = V\tilde{\varphi}_1$ (see Appendix A), we obtain
\[ (-\Delta)^\frac{\alpha}{2} \varphi_1 = -I_{2-\alpha} 1_{B(0,1)} \Delta \tilde{\varphi}_1 - I_{2-\alpha} 1_{B^c(0,1)} \Delta \varphi_1 \]
\[ = V \tilde{\varphi}_1 - I_{2-\alpha} 1_{B^c(0,1)} \Delta (\varphi_1 - \tilde{\varphi}_1). \]

Routine calculation shows that $-I_{2-\alpha} 1_{B^c(0,1)} \Delta (\varphi_1 - \tilde{\varphi}_1) \geq -c_0$ for a constant $c_0$.

Also, by straightforward calculation, $-\left( b_\epsilon \cdot \nabla + W_\epsilon \right) \varphi_1 \geq -V \tilde{\varphi}_1 - c_1$ for a constant $c_1$.

Therefore,
\[ (P^c)^* \varphi_1 = (-\Delta)^\frac{\alpha}{2} \varphi_1 - (b_\epsilon \cdot \nabla + W_\epsilon) \varphi_1 \geq -C, \quad C := c_0 + c_1, \]
so, by scaling,
\[ J = \langle e^{-\frac{|u|^2}{n}} | \varphi \rangle (P^c)^* \varphi \geq -CS^{-1} \| e^{-\frac{|u|^2}{n}} \| u \|_1 \geq -CS^{-1} \| e^{-\frac{|u|^2}{n}} \|_1 \| \phi_n^{-1} f \|_1, \]
or due to $\phi_n \geq \frac{1}{2}$,
\[ J \geq -2CS^{-1} \| e^{-\frac{|u|^2}{n}} \|_1 \| f \|_1. \]

Noticing that $\| W_\epsilon \|_\infty \leq c_2 \epsilon^{-\frac{\alpha}{2}}$, $c := \kappa(d-\alpha)$, we have $\| e^{-\frac{|u|^2}{n}} \|_1 \leq c \epsilon^{-\frac{\alpha}{2}} n^{-1} = 1 + o(n)$. Taking $\lambda = 3Cs^{-1}$ we obtain that
\[ \text{Re} \langle (\lambda + Q) f, \frac{f}{|f|} \rangle \geq 0 \quad f \in M. \]

The latter holds for all $f \in D(\tilde{Q})$. The proof of Proposition 2 is completed.

The proof of (N3) is completed. The proof of Theorem 1 is completed.

**Remark 1** (Proof of $e^{-tP^c} \overset{\delta}{\to} e^{-t\Lambda}$). It suffices to show that $(\mu + P^c)^{-1} \overset{\delta}{\to} (\mu + \Lambda)^{-1}$ for a $\mu > 0$.

First, we show that $(\mu + \Lambda^c)^{-1} \overset{\delta}{\to} (\mu + \Lambda)^{-1}$. We will use notation introduced in the proof of Proposition 1 above. Recall: $(\mu + \Lambda)^{-1} = (\mu + A)^{-1}(1 + T)^{-1}$, $\| (\mu + \Lambda)^{-1} \| \leq (1 - \delta)^{-1} \mu^{-1}$. Since $\| (T - T_\epsilon)f \|_2 \leq \| b - b_\epsilon \| (\mu + A)^{-1} \| \nabla f \|_2 \to 0$ for every $f \in C^\infty_c$ by the Domminated Convergence Theorem, we have $T_\epsilon \overset{\delta}{\to} T$. Therefore, $(\mu + \Lambda^c)^{-1} \overset{\delta}{\to} (\mu + \Lambda)^{-1}$.

We show that $(\mu + P^c)^{-1} - (\mu + \Lambda^c)^{-1} \overset{\delta}{\to} 0$. Set $S = (\mu + A)^{-1} + \frac{\delta}{\mu} b \cdot \nabla (\mu + A)^{-1}$ and $S_\epsilon = (\mu + A)^{-1} + \frac{\delta}{\mu} b_\epsilon \cdot \nabla (\mu + A)^{-1}$. Then $\sup \| S_\epsilon \|, \| S \| < 1$ and
\[ (\mu + \Lambda^c)^{-1} = (\mu + A)^{-1} + \frac{\delta}{\mu} (1 + S_\epsilon)^{-1}(\mu + A)^{-1} + \frac{\delta}{\mu} S_\epsilon, \quad \mu > 0. \]

Now, let $h \in L^2 \cap L^\infty$. Then
\[ \| (\mu + P^c)^{-1}h - (\mu + \Lambda^c)^{-1}h \|_2 = \| (\mu + \Lambda^c)^{-1}U_\epsilon (\mu + P^c)^{-1}h \|_2 \leq K_1 + K_2, \]
Remark 2. N contractions (due to (Theorem 1) yields \(\Gamma(\alpha, 0 < \delta < 1, b(x) := \delta_2 c e^{-2(\alpha-1, 2, d)}|x|^{-\alpha}, 0 < \delta_2 < 1, by following the arguments in [KiS1, Section 4]). Note that

\[ c^{-1}(\alpha - 1, 2, d) < c^{-2}(\alpha - 1, 2, d) \]

(\text{indeed, } \Gamma(\frac{d+2-2\alpha}{4})[\Gamma(\frac{d-1+\alpha}{4})]^2 - \Gamma(\frac{d-2+2\alpha}{4})[\Gamma(\frac{d-1-\alpha}{4})]^2 > 0), \text{i.e. these assumptions are less restrictive than the ones needed in the proof of Proposition [1].}

Then, in particular,

\[ \|e^{-t\Lambda} f\|_q \leq c r^{-\frac{\gamma}{2}} \|f\|_r, \quad f \in L^r \cap L^q, \quad 2 \leq r < q \leq \infty \]

(arguing as in the proof of [KiS1, Theorem 4.3]).

The following inequalities, which will be needed in the proof of Theorem 2 below, are simple consequences of (N3) and (3):

Corollary 1.

\[ e^{-t(P^c)^*} \varphi(x) \leq c \varphi(x), \quad \langle e^{-t(P^c)^*} (x, \cdot) \rangle \leq 2c \varphi(x) \quad x \neq 0, \quad s \geq t > 0. \]

3. Proof of Theorem 2. The upper bound \(e^{-t\Lambda}(x, y) \leq C e^{-t(-\Delta)^{\frac{\gamma}{4}}}(x, y) \varphi(t) \quad (y \neq 0).\)

For brevity, everywhere below \((-\Delta)^{\frac{\gamma}{4}} =: A.\)

By scaling, it suffices to consider \(t = 1.\) It suffices to prove the bound \((\varepsilon > 0)\)

\[ e^{-t(P^c)^*}(x, y) \leq C e^{-A}(x, y) \varphi(x), \quad C \neq C(\varepsilon), \quad \varphi \equiv \varphi_1. \]

Let \(R > 1\) to be chosen later.

The case \(|x|, |y| \leq 2R.\)

Since \(e^{-A}(x, y) \approx 1 \land |x - y|^{-d-\alpha} \quad (x \neq y),\) the Nash initial estimate \(e^{-t(P^c)^*}(x, y) \leq C t^{-\frac{\gamma}{2}} \varphi(x)\)

(Theorem [1]) yields

\[ e^{-t(P^c)^*}(x, y) \leq C_R e^{-A}(x, y) \varphi(x), \quad C_R \neq C_R(\varepsilon). \]

To consider the other cases we will be using the Duhamel formula,

\[ e^{-t(P^c)^*} = e^{-A} + \int_0^1 e^{-t(P^c)^*}(B_{\varepsilon,R} + B_{\varepsilon,R}^c) e^{-t(1-\tau)^A} d\tau \]

\[ =: e^{-A} + K_R + K_R^c. \]
where \( B_{\varepsilon,R} := 1_{B(0,R)}B_{\varepsilon} \), \( B_{\varepsilon,R}^c := 1_{B^c(0,R)}B_{\varepsilon} \) and \( B_{\varepsilon} := b_{\varepsilon} \cdot \nabla + W_{\varepsilon} \) (recall, \( W_{\varepsilon}(x) = \kappa(d-\alpha)|x|^{-\alpha} \), \( b_{\varepsilon}(x) = \kappa|x|^{-\alpha}x \).

Below we prove that \( K_R(x,y), K_R^c(x,y) \leq C\tau e^{-A(x,y)}\varphi(x) \), which would yield the upper bound. We will need the following.

**Lemma 1.** Set \( E^t(x,y) = t(|x-y|^{-d-\alpha-1} \wedge t^{-\frac{d+1}{\alpha}}) \), \( E^t f(x) := \langle E^t(x,\cdot)f(\cdot) \rangle \).

Let \( 0 < t \leq 1 \). Then

1. \(|\nabla_x e^{-tA}(x,y)| \leq c_0E^t(x,y)\);
2. \( \int_0^t (e^{-(t-\tau)A}(x,\cdot)E^\tau(\cdot,y))d\tau \leq c_1 e^{-tA}(x,y)\);
3. \( \int_0^t (E^{t-\tau}(x,\cdot)E^\tau(\cdot,y))d\tau \leq c_2 E^t(x,y) \).

**Proof.** For the proof of (i), (ii) see e.g. [BJ]. Essentially the same argument yields (iii). For the sake of completeness, we provide the details:

\[
E^t(x,z) \wedge E^\tau(z,y) = (t|x-z|^{-d-\alpha-1} \wedge t^{-\frac{d+1}{\alpha}}) \wedge (\tau|z-y|^{-d-\alpha-1} \wedge \tau^{-\frac{d+1}{\alpha}})
\]

\[
\leq C_0 \left( \frac{t + \tau}{2} \right)^{-\frac{d+1}{\alpha}} \wedge \left( (t + \tau) \left( \frac{|x-z| + |z-y|}{2} \right)^{-d-\alpha-1} \right) \quad (C_0 > 1)
\]

\[
\leq C (t + \tau)^{-\frac{d+1}{\alpha}} \wedge \left( |t + \tau|(|x-y|)^{-d-\alpha-1} \right) = CE^{t+\tau}(x,y),
\]

so (iii) follows from the inequality \( ac = (a \wedge c)(a \lor c) \leq (a \wedge c)(a + c) \quad (a,c \geq 0) \):

\[
\int_0^t (E^{t-\tau}(x,\cdot)E^\tau(\cdot,y))d\tau \leq E^{t+\tau}(x,y) \int_0^t (E^{t-\tau}(x,\cdot) + E^\tau(\cdot,y))d\tau,
\]

where, routine calculation shows, \( \int_0^t (E^{t-\tau}(x,\cdot) + E^\tau(\cdot,y))d\tau \leq c_2 < \infty \) (we use that \( t \leq 1 \)).

\[\square\]

The case \(|y| > 2R, 0 < |x| \leq |y|\).

**Claim 1.** If \(|y| > 2R, 0 < |x| \leq |y|\), then

\[
K_R(x,y) \equiv \int_0^{\tau} \langle e^{-r(P_\varepsilon)^{\ast}}(x,\cdot)B_{\varepsilon,R}(\cdot)e^{-(1-r)A}(\cdot,y) \rangle d\tau \leq \hat{C}e^{-A(x,y)}\varphi(x), \quad \hat{C} \neq \hat{C}(\varepsilon).
\]

**Proof.** Claim (ii) clearly follows from

\[j\int_0^t \langle e^{-(r(P_\varepsilon)^{\ast})}(x,\cdot)1_{B(0,R)}(\cdot)W_\varepsilon(\cdot)e^{-(1-r)A}(\cdot,y) \rangle d\tau \leq c_4 e^{-tA}(x,y)\varphi(x),\]

and, in view of Lemma (i), from

\[jj\int_0^t \langle e^{-(r(P_\varepsilon)^{\ast})}(x,\cdot)1_{B(0,R)}(\cdot)Z_\varepsilon(\cdot)E^{t-\tau}(\cdot,y) \rangle d\tau \leq c_3 e^{-tA}(x,y)\varphi(x), \quad \text{where} \ Z_\varepsilon(x) := |x|^{-\alpha}|x|,\]
Let us prove \((ij)\):

\[
\int_0^t \langle e^{-\tau(P_c)^*} (x, \cdot) \rangle 1_{B(0,R)}(\cdot) |Z_\varepsilon(\cdot)E^{t-\tau}(\cdot, y)| \, d\tau
\]

(we are using \(E^{t-\tau}(\cdot, y) \leq Ce^{-(t-\tau)A}(\cdot, y)| \cdot - y|^{-1}\))

\[
\leq C \int_0^t \langle e^{-\tau(P_c)^*} (x, \cdot) \rangle 1_{B(0,R)}(\cdot) |Z_\varepsilon(\cdot) e^{-(t-\tau)A}(\cdot, y)| \cdot - y|^{-1} \, d\tau
\]

(we are using \(1_{B(0,R)}(\cdot) | \cdot - y|^{-1} \leq | \cdot |^{-1}\))

\[
\leq C' \int_0^t \langle e^{-\tau(P_c)^*} (x, \cdot) \rangle 1_{B(0,R)}(\cdot) |W_\varepsilon(\cdot) e^{-(t-\tau)A}(\cdot, y)| \, d\tau
\]

(we are using \(1_{B(0,R)}(\cdot) e^{-(t-\tau)A}(\cdot, y) \leq e^{-tA}(x, y)\))

\[
\leq C'' e^{-tA}(x, y) \int_0^t \langle e^{-\tau(P_c)^*} (x, \cdot) \rangle 1_{B(0,R)}(\cdot) |W_\varepsilon(\cdot)| \, d\tau.
\]

According to the Duhamel formula \(e^{-t(P_c)^*} = e^{-tA} + \int_0^t e^{-\tau(P_c)^*} (b_\varepsilon \cdot \nabla + W_\varepsilon) e^{-(t-\tau)A} \, d\tau\),

\[
1 + \int_0^t \langle e^{-\tau(P_c)^*} (x, \cdot) W_\varepsilon(\cdot) \rangle \, d\tau = \langle e^{-t(P_c)^*} (x, \cdot) \rangle.
\]

Using the inequality \(\langle e^{-t(P_c)^*} (x, \cdot) \rangle \leq 2c\varphi(x)\) from Corollary \(1\) it is seen that

\[
\int_0^t \langle e^{-\tau(P_c)^*} (x, \cdot) W_\varepsilon(\cdot) \rangle \, d\tau \leq 2c\varphi(x).
\]

The latter and the previous estimate yield \((jj)\). Incidentally, we have also proved \((j)\).

\[\square\]

**Claim 2.** If \(|y| > 2R, |x| \leq |y|\), then

\[
K_R^c(x, y) \equiv \int_0^1 \langle e^{-\tau(P_c)^*} (x, \cdot) B_{\varepsilon,R}(\cdot) e^{-(1-\tau)A}(\cdot, y) \rangle \, d\tau \leq Ce^{-A}(x, y)\varphi(x).
\]

**Proof.** Lemma \((1) (i)\) yields

\[
|B_{\varepsilon,R}(\cdot) e^{-(\tau-\tau')A}(\cdot, y)| \leq C_0 \left( R^{-\alpha} e^{-(\tau-\tau')A}(\cdot, y) + R^{-\alpha+1} E^{\tau-\tau'}(\cdot, y) \right),
\]

\[
\tag{(*)}
\]

\[
K_R^c(x, y) \equiv \int_0^1 \langle e^{-\tau(P_c)^*} (x, \cdot) B_{\varepsilon,R}(\cdot) e^{-(1-\tau)A}(\cdot, y) \rangle \, d\tau
\]

\[
\leq C_0 R^{-\alpha} \int_0^1 \langle e^{-\tau(P_c)^*} (x, \cdot) e^{-(1-\tau)A}(\cdot, y) \rangle \, d\tau + C_0 R^{-\alpha+1} \int_0^1 \langle e^{-\tau(P_c)^*} (x, \cdot) E^{1-\tau}(\cdot, y) \rangle \, d\tau.
\]

\[
\tag{(***)}
\]

1. Let us estimate the first term in the RHS of \(\text{(***)}\). By the Duhamel formula,

\[
\int_0^1 e^{-\tau(P_c)^*} e^{-(1-\tau)A} \, d\tau
\]

\[
= \int_0^1 e^{-tA} e^{-(1-\tau)A} \, d\tau + \int_0^1 \int_0^\tau e^{-\tau'(P_c)^*} (B_{\varepsilon,R} + B_{\varepsilon,R}^c) e^{-(\tau-\tau')A} \, d\tau' e^{-(1-\tau)A} \, d\tau
\]

\[
\equiv e^{-A} + I_R + I_R^c.
\]
We have $I_R = \int_0^1 I_R^c e^{-(1-\tau)A} d\tau$, where $I_R^c := \int_0^\tau e^{-\tau'(P^c)^*} B_{\epsilon, R} e^{-(\tau-\tau')A} d\tau'$. By Claim \[\square\]

$$|I_R^c(x, y)| \leq \tilde{C} e^{-A}(x, y) \phi(x)$$

and so $|I_R(x, y)| \leq \tilde{C} e^{-A}(x, y) \phi(x)$.

In turn, $I_R^c = \int_0^1 (I_R^c)^\tau e^{-(1-\tau)A} d\tau$, where $(I_R^c)^\tau := \int_0^\tau e^{-\tau'(P^c)^*} B_{\epsilon, R} e^{-(\tau-\tau')A} d\tau'$, so

$$|(I_R^c)^\tau(x, y)| \leq C_0 R^{-\alpha} \int_0^\tau \langle e^{-\tau'(P^c)^*} e^{-(\tau-\tau')A} e^{-(1-\tau')A} \rangle(x, y) d\tau' + C_0 R^{-\alpha+1} \int_0^\tau \langle e^{-\tau'(P^c)^*} e^{-(\tau-\tau')A} E^{\tau-t'} e^{-(1-\tau')A} \rangle(x, y) d\tau'$$

Then

$$|I_R^c(x, y)| \leq C_0 R^{-\alpha} \int_0^1 \int_0^\tau \langle e^{-\tau'(P^c)^*} e^{-(1-\tau')A} \rangle(x, y) d\tau' d\tau$$

$$+ C_0 R^{-\alpha+1} \int_0^1 \int_0^\tau \langle e^{-\tau'(P^c)^*} E^{\tau-t'} e^{-(1-\tau')A} \rangle(x, y) d\tau' d\tau,$$

where we estimate the first and second integrals as follows.

$$\int_0^1 \int_0^\tau \langle e^{-\tau'(P^c)^*} e^{-(1-\tau')A} \rangle(x, y) d\tau' d\tau$$

$$\leq \int_0^1 \int_0^\tau \langle e^{-\tau'(P^c)^*} e^{-(1-\tau')A} \rangle(x, y) d\tau' d\tau = \int_0^1 \langle e^{-\tau'(P^c)^*} (x, \cdot) e^{-(1-\tau')A} \rangle(y) d\tau',$$

$$\int_0^1 \int_0^\tau \langle e^{-\tau'(P^c)^*} E^{\tau-t'} e^{-(1-\tau')A} \rangle(x, y) d\tau' d\tau$$

(we are changing the order of integration in $\tau$ and $\tau'$)

$$= \int_0^1 \int_{\tau'}^1 \langle e^{-\tau'(P^c)^*} E^{\tau-t'} e^{-(1-\tau')A} \rangle(x, y) d\tau d\tau'$$

(by Lemma \[\square\](ii), $\int_0^1 \langle E^{\tau-t'} e^{-(1-\tau')A} \rangle(\cdot, y) d\tau = c_1 e^{-(1-\tau')A}(\cdot, y)$)

$$\leq c_1 \int_0^1 \langle e^{-\tau'(P^c)^*} (x, \cdot) e^{-(1-\tau')A} \rangle(y) d\tau'.$$

Thus,

$$|I_R^c(x, y)| \leq C_0 (R^{-\alpha} + c_1 R^{-\alpha+1}) \int_0^1 \langle e^{-\tau'(P^c)^*} (x, \cdot) e^{-(1-\tau')A} \rangle(y) d\tau.$$

Therefore, for $R > 1$ such that $C_0(R^{-\alpha} + c_1 R^{-\alpha+1}) \leq \frac{1}{2}$,

$$\int_0^1 \langle e^{-\tau'(P^c)^*} (x, \cdot) e^{-(1-\tau')A} \rangle(y) d\tau$$

$$\leq e^{-A} (x, y) + \frac{1}{2} \int_0^1 \langle e^{-\tau'(P^c)^*} (x, \cdot) e^{-(1-\tau')A} \rangle(y) d\tau + \tilde{C} e^{-A} (x, y) \phi(x),$$

i.e.

$$\int_0^1 \langle e^{-\tau'(P^c)^*} (x, \cdot) e^{-(1-\tau')A} \rangle(y) d\tau \leq 2 (2 + \tilde{C}) e^{-A} (x, y) \phi(x).$$
2. Let us estimate the second term in the RHS of (3). By the Duhamel formula
\[
\int_0^1 e^{-\tau(P^*)^\alpha} E^{1-\tau} d\tau
\]
\[
= \int_0^1 e^{-\tau A} E^{1-\tau} d\tau + \int_0^1 \int_0^\tau e^{-\tau(t)} (B_{\varepsilon,R} + B_{\varepsilon,R}^c) e^{-(\tau-t)A} d\tau' E^{1-\tau} d\tau
\]
\[
\equiv \int_0^1 e^{-\tau A} E^{1-\tau} d\tau + J_R + J_R^c,
\]
where, by Lemma 1 (ii), \([0,1] e^{-\tau A} E^{1-\tau}(x,y) ds \leq c_1 e^{-A}(x,y).\) Let us estimate \(J_R\) and \(J_R^c.\)

We have \(J_R = \int_0^1 J_R^t E^{1-\tau} d\tau,\) where \(J_R^t := \int_0^\tau e^{-\tau(t)} (B_{\varepsilon,R}^c) e^{-(\tau-t)A} d\tau'.\) By Claim 1 (ii),
\[
|J_R^t(x,y)| \leq \tilde{C} e^{-A}(x,y) \varphi(x), \quad \text{and so by Lemma 1 (ii),}
\]
\[
|J_R(x,y)| \leq C_1 e^{-A}(x,y) \varphi(x).
\]

In turn, \(J_R^c = \int_0^1 (J_R^c)^\tau E^{1-\tau} d\tau,\) where \((J_R^c)^\tau := \int_0^\tau e^{-\tau(t)} (B_{\varepsilon,R}^c) e^{-(\tau-t)A} d\tau'.\) By (3) and Lemma 1 (ii), \(|(J_R^c)^\tau(x,y)| \leq C_0 R^{-\alpha} \int_0^\tau (e^{-\tau(t)} e^{-(\tau-t)A}) (x,y) d\tau' + C_0 R^{-\alpha+1} \int_0^\tau (e^{-\tau(t)} E^{1-\tau}) (x,y) d\tau'.\)

Due to Lemma 1 (ii),(iii),
\[
|J_R^c(x,y)| \leq C_0 c_1 R^{-\alpha} \int_0^1 (e^{-\tau(P^*)^\alpha} (x,\cdot) e^{-(1-\tau)A}(\cdot,y)) d\tau'
\]
\[
+ C_0 c_2 R^{-\alpha+1} \int_0^1 (e^{-\tau(P^*)^\alpha} (x,\cdot) E^{1-\tau}(\cdot,y)) d\tau'.
\]

Thus, for \(R > 1\) such that \(C_0 c_1 R^{-\alpha}, C_0 c_2 R^{-\alpha+1} \leq \frac{1}{2},\)
\[
\int_0^1 (e^{-\tau(P^*)^\alpha} (x,\cdot) E^{1-\tau}(\cdot,y)) d\tau \leq c_1 e^{-A}(x,y) + \frac{1}{2} \int_0^1 (e^{-\tau(P^*)^\alpha} (x,\cdot) e^{-(1-\tau)A}(\cdot,y)) d\tau
\]
\[
+ \frac{1}{2} \int_0^1 (e^{-\tau(P^*)^\alpha} (x,\cdot) E^{1-\tau}(\cdot,y)) d\tau + C_1 e^{-A}(x,y) \varphi(x).
\]

Using 1 we arrive at \(\int_0^1 (e^{-\tau(P^*)^\alpha} (x,\cdot) E^{1-\tau}(\cdot,y)) d\tau \leq 2(2c_1 + 2 + \hat{C} + C_1) e^{-A}(x,y) \varphi(x).\)

Now 1 and 2 applied in (3) yield Claim 2.

\[\square\]

The case \(|x| > 2R, |y| \leq |x|\) is treated similarly, so we omit the details.

The proof of the upper bound is completed.

4. Proof of Theorem 2. The lower bound \(e^{-t\Lambda}(x,y) \geq C e^{-t(-\Delta)^\frac{\theta}{2}}(x,y) \varphi_t(y)\)
\((C > 0, x, y \neq 0).\)

Proposition 3. Define \(g = \varphi h, \varphi \equiv \varphi_s, 0 \leq h \in S\text{-the L. Schwartz space of test functions. There is a constant } 0 < \mu \text{ such that, for all } 0 < t \leq s,\)
\(e^{-\tilde{\Theta}t}(g) \leq \langle \varphi e^{-t\Lambda} \varphi^{-1} g \rangle.\)

Proof. Set \(g_n = \phi_n h, \phi_n(x) = (e^{-\tilde{\Theta}t}) \varphi(x).\) Then
\[
\langle g_n \rangle - \langle \phi_n e^{-t(P^* - \mu)} h \rangle = -\mu \int_0^t \langle \varphi, e^{-t(P^* - \mu)} e^{-\tilde{\Theta} t} h \rangle d\tau + \int_0^t \langle \varphi, P^* e^{-t(P^* - \mu)} e^{-\tilde{\Theta} t} h \rangle d\tau,
\]
where $\mu = \frac{\hat{\mu}}{s} > 0$ is to be chosen. Let $\tilde{\varphi}(x) = (s^{-\frac{1}{2}}|x|)^{-d+\beta}$. Write $(P^s)\varphi = (P^s)^*\phi + (P^s)^*(\varphi - \phi) = 1_{B(0,1)}(V-V_\varepsilon)\varphi + v_\varepsilon V_\varepsilon, V(x) = V(|x|) = \kappa(\beta-\alpha)|x|^{-\alpha}, V_\varepsilon(x) = V_\varepsilon(|x|) = V(|x|\varepsilon)$. Routine calculation shows that $\|v_\varepsilon\|_\infty \leq \frac{\mu_1}{s}$ for a $\mu_1 \neq \mu_1(\varepsilon)$ (cf. the proof of Proposition 2). Thus

$$\int_0^t \langle v_\varepsilon, e^{-\tau(P^s-\mu)}e^{-\frac{P^s}{\varepsilon} h}\rangle d\tau \leq \frac{\mu_1}{s} \int_0^t \langle e^{-\tau(P^s-\mu)}e^{-\frac{P^s}{\varepsilon} h}\rangle d\tau \leq \frac{2\mu_1}{s} \int_0^t \langle \varphi, e^{-\tau(P^s-\mu)}e^{-\frac{P^s}{\varepsilon} h}\rangle d\tau.$$

Taking $\hat{\mu} = 2\mu_1$, we have

$$\langle g_\mu \rangle - \langle \phi_n e^{-t(P^s-\mu)h}\rangle \leq \int_0^t \langle 1_{B(0,1)}(V-V_\varepsilon)\varphi, e^{-\tau(P^s-\mu)}e^{-\frac{P^s}{\varepsilon} h}\rangle d\tau,$$

or, sending $n \to \infty$,

$$\langle g \rangle - e^{t\hat{\mu}} \langle \varphi e^{-tP^s h}\rangle \leq e^{\hat{\mu}} \int_0^t \langle 1_{B(0,1)}(V-V_\varepsilon)\varphi, e^{-\tau P^s h}\rangle d\tau. \quad (\hat{\circ})$$

It remains to take $\varepsilon \downarrow 0$ in $(\hat{\circ})$. Since $\|e^{-\tau P^s h}\|_\infty \leq \|h\|_\infty$ and

$$1_{B(0,1)}|V-V_\varepsilon|\varphi \leq 2\varphi 1_{B(0,1)} V \leq C1_{B(0,1)}|x|^{-d+\beta-\alpha}, \quad d - \beta + \alpha < d,$$

the RHS of $(\hat{\circ})$ tends to 0 as $\varepsilon \downarrow 0$ due to the Dominated Convergence Theorem. The latter, $e^{-tP^s h} \to e^{-t\Lambda h}$ strongly in $L^2$ (see Remark 1) and $(N_3)$ yield Proposition 3.

We also need the following consequence of the upper bound and Proposition 3.

**Proposition 4.** Fix $t > 0$. Set $g := \varphi h, \varphi = \varphi_t, 0 \leq h \in S$ with $\text{sprt} h \subset B(0,R_0)$ for some $R_0 \geq 1$. Then there are $0 < r_t < R_0 \lor \frac{t}{2} \lor R_t, R_0$ such that, for all $r \in [0, r_t]$ and $R \geq 2R_t, R_0\lor \infty$,

$$e^{-\hat{\mu}^{-1}}(g) \leq \langle 1_{B(0,1)} \varphi e^{-t\Lambda \varphi^{-1}} \hat{g}, 1_{B(1,0)} \rangle, \quad 1_{B(0,1)} := 1_{B(0,1)} - 1_{B(0,R)}.$$

In particular,

$$e^{-\hat{\mu}^{-1}}(\varphi(x)) \leq e^{-t\Lambda^* \varphi} 1_{B(0,1)}(x) \quad \text{for all } x \in B(0, R_0).$$

**Proof.** By the upper bound,

$$\langle 1_{B(0,r)} \varphi e^{-t\Lambda \varphi^{-1}} \rangle \leq C\langle 1_{B(0,r)} \varphi t, e^{-t\Lambda} \rangle \leq CC_t^{-\frac{\alpha}{2}} \|1_{B(0,r)} \varphi t\|_1\|g\|_1 = o(r_t)\|g\|_1, \quad o(r_t) \to 0 \text{ as } r_t \downarrow 0;$$

$$\langle 1_{B(0,r)} \varphi e^{-t\Lambda \varphi^{-1}} \rangle \leq C\langle 1_{B(0,r)} \varphi, e^{-t\Lambda} \rangle \leq C\langle e^{-t\Lambda} 1_{B(0,r)} \varphi, e^{-t\Lambda} \rangle, \quad \text{where } R \geq 2R_t, R_0 \lor 2(R_0 \lor \frac{t}{2}) \leq C \sup_{x \in B(0,R_0)} e^{-t\Lambda} 1_{B(0,r)}(x)\|g\|_1 \leq C \hat{C}CC_t^{-\alpha} R_t, R_0\|g\|_1 = o(R_t, R_0)\|g\|_1, \quad o(R_t, R_0) \to 0 \text{ as } R_t, R_0 \uparrow \infty$$

due to $e^{-t\varphi}(x, y) \leq \hat{C}(t|x - y|^{-d-\alpha} \lor t^{-\frac{\alpha}{2}}) \leq \hat{C}2^{d+\frac{\alpha}{2}} |y|^{-d-\frac{\alpha}{2}}$ if $|x| \leq R_0$ and $|y| \geq R$.

It remains to apply Proposition 3.

**Proposition 5.** $\langle h \rangle = \langle e^{-t\Lambda^* h}\rangle$ for every $h \in L^1$, $t > 0$. 

\[\Box\]
Proof. We have, for $h \in S$,
\[
\langle h \rangle - \langle e^{-t(P^c)^r} h \rangle = \int_0^t \langle 1,(P^c)^r e^{-\tau(P^c)^r} h \rangle d\tau = \int_0^t \langle U_\varepsilon e^{-\tau(P^c)^r} h \rangle d\tau
\]
\[
= \int_0^t \langle B_{(0,1)} U_\varepsilon e^{-\tau(P^c)^r} h \rangle d\tau + \int_0^t \langle B_{(0,1)} U_\varepsilon e^{-\tau(P^c)^r} h \rangle d\tau.
\]
It is clear that $\langle B_{(0,1)} U_\varepsilon e^{-\tau(P^c)^r} h \rangle \leq \|B_{(0,1)} U_\varepsilon\|_\infty \|h\|_1 \to 0$ as $\varepsilon \downarrow 0$, and so the first integral converges to 0. Let us estimate the second integral:
\[
\int_0^t \langle B_{(0,1)} U_\varepsilon e^{-\tau(P^c)^r} h \rangle d\tau = \int_0^t \langle e^{-\tau(P^c)^r} B_{(0,1)} U_\varepsilon, h \rangle d\tau
\]
(we are using the upper bound $e^{-tP^c} (x,y) \leq Ce^{-tA}(x,y)\varphi_t(y)$)
\[
\leq C \int_0^t \langle e^{-\tau A} \varphi B_{(0,1)} U_\varepsilon, |h| \rangle d\tau
\]
\[
\leq C \|h\|_\infty \|\varphi B_{(0,1)} U_\varepsilon\|_1 \to 0 \text{ as } \varepsilon \downarrow 0 \text{ due to } d - \beta + \alpha < d.
\]
Thus, $\langle h \rangle = \lim_\varepsilon \langle e^{-t(P^c)^r} h \rangle$. Next, since $e^{-t(P^c)^r} h \to e^{-tA} h$ strongly in $L^2$ (see Remark [1], we may suppose that $e^{-t(P^c)^r} h \to e^{-tA} h$ a.e. The upper bound $e^{-t(P^c)^r} (x,y) \leq Ce^{-tA(x,y)\varphi_t(x)}$, yields $|e^{-t(P^c)^r} h| \leq C \varphi_t e^{-tA}|h| \in L^1$, and so $\lim_\varepsilon (e^{-t(P^c)^r} h) = (e^{-tA} h)$ by the Dominated Convergence Theorem. Thus, equality $\langle h \rangle = \langle e^{-tA} h \rangle$ holds for every $h \in S$ and hence for every $h \in L^1$.

**Proposition 6.** Fix $t > 0$. Let $0 \leq h \in S$ with $\text{sprt} h \subset B(0,R_0)$ for some $R_0 \geq 1$. Then there are $0 < r_1 < R_0 \vee t^\frac{\alpha}{2} < R_{t,R_0}$ such that, for all $r \in [0,r_1]$ and $R \in [2R_{t,R_0}, \infty[,$
\[
\frac{1}{2} \langle h \rangle \leq \langle 1_{R,r} e^{-tA} h \rangle.
\]
In particular,
\[
\frac{1}{2} \leq e^{-tA} 1_{R,r}(x) \text{ for all } x \in B(0,R_0).
\]

**Proof.** We follow the argument in the proof of Proposition [1]. By the upper bound,
\[
\langle 1_{B(0,r)} e^{-tA} h \rangle \leq C \langle 1_{B(0,r)} \varphi_t, e^{-tA} h \rangle
\]
\[
\leq C \|1_{B(0,r)} \varphi_t\|_1 \|h\|_1
\]
\[
= o(r_t) \|h\|_1, \quad o(r_t) \to 0 \text{ as } r_t \downarrow 0;
\]
\[
\langle 1_{B^c(0,R)} e^{-tA} h \rangle \leq C \langle 1_{B^c(0,R)} \varphi_t, e^{-tA} h \rangle
\]
\[
\leq C \langle e^{-tA} 1_{B^c(0,R)}, h 1_{B(0,R_0)} \rangle, \text{ where } R \geq 2R_{t,R_0} \geq 2(R_0 \vee t^\frac{\alpha}{2})
\]
\[
\leq C \sup_{x \in B(0,R_0)} e^{-tA} 1_{B^c(0,R)}(x) \|h\|_1
\]
\[
\leq C \tilde{C} d R \|h\|_1
\]
\[
= o(R_t) \|h\|_1, \quad o(R_t) \to 0 \text{ as } R_t \uparrow \infty
\]
due to $e^{-tA}(x,y) \leq \tilde{C} (t|x-y|^{-d-\alpha} \wedge t^{-\frac{\alpha}{2}}) \leq \tilde{C} 2^{d+\frac{\alpha}{2}} |y|^{-d-\frac{\alpha}{2}}$ if $|x| \leq R_0$ and $|y| \geq R$.

The last two estimates and Proposition [3] yield $\frac{1}{2} \langle h \rangle \leq \langle 1_{R,r} e^{-tA} h \rangle$. \qed
Claim 3. For every \( r > 0 \) there exist a constant \( t(r) > 0 \) such that
\[
e^{-tA} (x, y) \geq \frac{1}{2} e^{-tA} (x, y) \quad \text{for all } |x| \geq r, \ |y| \geq r, \quad 0 < t \leq t(r).
\]

Proof. By the Duhamel formula,
\[
e^{-(t^p)^+} (x, y) \geq e^{-tA} (x, y) + M_t (x, y), \quad M_t (x, y) \equiv \int_0^t \left\langle e^{-(r^p)(P^p)^+} (x, \cdot) b_t (\cdot) \cdot \nabla, e^{-rA} (\cdot, y) \right\rangle d\tau.
\]

By Lemma [ii],
\[
|M_t (x, y)| \leq c_1 \int_0^t \left\langle e^{-(t-r)(P^p)^+} (x, \cdot) \right| \cdot |1-\alpha E^r (\cdot, y) \right\rangle d\tau
\]
(we apply the upper bound)
\[
\leq c_1 C \int_0^t \varphi_{t-r} (x) \left\langle e^{-(t-r)A} (x, \cdot) \right| \cdot |1-\alpha E^r (\cdot, y) \right\rangle d\tau
\]
(since \( |x| \geq r \), we may select \( t = t(r) > 0 \) sufficiently small so that \( \varphi_{t-r} (x) = \frac{1}{2} \))
\[
\leq \frac{c_1 C}{2} \int_0^t \left\langle e^{-(t-r)A} (x, \cdot) \right| \cdot |1-\alpha E^r (\cdot, y) \right\rangle d\tau =: J(|1-\alpha|).
\]

Next, select \( \gamma > 0 \) sufficiently small (\( \gamma \ll r \)) so that, for all \( 0 < \tau < t, \ |x|, \ |y| \geq r, \)
\[
1_{B(0, \gamma)} (\cdot) e^{-(t-r)A} (x, \cdot) \leq C_5 e^{-tA} (x, 0),
\]
\[
1_{B(0, \gamma)} (\cdot) e^{-\gamma A} (\cdot, y) \leq C_6 e^{-tA} (0, y),
\]
\[
1_{B(0, \gamma)} (\cdot) E^r (\cdot, y) \leq C_7 e^{-tA} (0, y).
\]

Using the inequality
\[
e^{-tA} (x, z) e^{-\gamma A} (z, y) \leq K e^{-(t+\gamma)A} (x, y) \left( e^{-tA} (x, z) + e^{-\gamma A} (z, y) \right), \quad (*)
\]
which holds for a constant \( K = K(d, \alpha) \), all \( x, z, y \in \mathbb{R}^d \) and \( t, \tau > 0 \) (see e.g. [3.1]), we have
\[
J(1_{B(0, \gamma)} |1-\alpha|) \leq c \int_0^t \left( 1_{B(0, \gamma)} (\cdot) \right| \cdot |1-\alpha| d\tau e^{-tA} (x, 0) + e^{-tA} (0, y)) e^{-2A} (x, y)
\]
\[
\leq c C(r) \gamma^{d-\alpha+1} e^{-tA} (x, y).
\]

In turn,
\[
J(1_{B(0, \gamma)} |1-\alpha|) \leq \frac{c_1 C}{2} C_0 \gamma^{1-\alpha} t^{1-\alpha} e^{-tA} (x, y), \quad (**) \]
follows immediately from
\[
\int_0^t \left\langle e^{-(t-r)A} (x, \cdot) E^r (\cdot, y) \right\rangle d\tau \leq C_0 t^{1-\frac{1}{\alpha}} e^{-tA} (x, y)
\]
proved in Appendix [C].

Thus, putting \( t = \gamma^{2\alpha} \) and selecting \( \gamma > 0 \) sufficiently small in (**) and (***) we have
\[
|M_t (x, y)| \leq \frac{1}{2} e^{-tA} (x, y).
\]

Thus,
\[
e^{-t(P^p)^+} (x, y) \geq \frac{1}{2} e^{-tA} (x, y), \quad |x| \geq r, \ |y| \geq r, \quad 0 < t \leq t(r).
\]
Finally, using $L^2$-strong convergence $e^{-t(Px)^*} \to e^{-t\Lambda^*}$ (see Remark [1]), we complete the proof of the claim.

Claim 4. For every $r > 0$ there exists a constant $c(r) > 0$ such that
\[
e^{-\Lambda^*}(x, y) \geq c(r) e^{-A}(x, y) \quad \text{for all } |x| \geq r, |y| \geq r, \quad x \neq y.
\]

Proof. By the reproduction property,
\[
e^{-2t_0\Lambda^*}(x, y) \geq (e^{-t_0\Lambda^*}(x, \cdot) 1_{B^c(0, r)}(\cdot)) e^{-t_0\Lambda^*}(\cdot, y)
\]
(w we are applying Claim [3])
\[
\geq c_1^2 (e^{-t_0A}(x, \cdot) 1_{B^c(0, r)}(\cdot) e^{-t_0A}(\cdot, y)), \quad c_1 := \frac{1}{2}, \quad t_0 = t(r).
\]

Consider the following cases:

1) If $(r \leq |x|, |y| \leq r_m$, where $r_m (> r)$ is to be chosen, then the above inequality yields
\[
e^{-2t_0\Lambda^*}(x, y) \geq C_{1, r_m} e^{-2t_0A}(x, y), \quad C_{1, r_m} > 0.
\]

2) If $|x|, |y| > r_m$, then
\[
e^{-2t_0\Lambda^*}(x, y) \geq c_1^2 (e^{-t_0A}(x, \cdot) 1_{B(0, r)}(\cdot) e^{-t_0A}(\cdot, y))
\]
(w we are applying [3])
\[
\geq c_1^2 e^{-2t_0A}(x, y)(1 - K(1_{B(0, r)}(\cdot))(e^{-t_0A}(x, \cdot) + e^{-t_0A}(\cdot, y))))
\]
\[
\geq c_1^2 e^{-2t_0A}(x, y)(1 - K_1(1_{B(0, r)})(r_m - r)^{-d-\alpha})
\]
(w select $r_m$ sufficiently large)
\[
\geq C_{2, r_m} e^{-2t_0A}(x, y) \quad C_{2, r_m} > 0.
\]

3) If $r \leq |x| \leq r_m, |y| > r_m$, then
\[
e^{-2t_0\Lambda^*}(x, y) \geq c_1^2 (e^{-t_0A}(x, \cdot) 1_{B^c(0, r)}(\cdot) e^{-t_0A}(\cdot, y))
\]
\[
\geq C_{3, r_m} e^{-t_0A}(x, \cdot) 1_{B^c(0, r)}(\cdot)(r + |y|)^{-d-\alpha}
\]
\[
\geq C_{4, r_m} e^{-2t_0A}(0, y) \geq C_{5, r_m} e^{-2t_0A}(x, y), \quad C_{i, r_m} > 0 \quad (i = 3, 4, 5).
\]

4) If $r \leq |y| \leq r_m, |x| > r_m$, then, by the symmetry of $e^{-t_0A}$, $e^{-2t_0\Lambda^*}(x, y) \geq C_{5, r_m} e^{-2t_0A}(x, y)$.
Thus, we have proved that $e^{-2t_0\Lambda^*}(x, y) \geq c_2 e^{-2t_0A}(x, y), c_2 > 0$, for all $|x|, |y| \geq r$. Continuing this process, we obtain the assertion of the claim.

We are in position to complete the proof of the lower bound using the so-called $3q$ argument.

Set $q_i(x, y) := \varphi^{-1}(x) e^{-t\Lambda^*}(x, y)$ ($\varphi \equiv \varphi_1$).

(a) Let $x, y \in B^c(0, 1), x \neq y$. Then by Claim [3]
\[
q_3(x, y) \geq \varphi^{-1}(x) e^{-3\Lambda^*}(x, y) \geq e^{-3\Lambda^*}(x, y) \geq ce^{-3A}(x, y).
\]

Now, fix $R_0 = 1$. 

(b) Let \( x \in B(0, 1) \), \( |y| \geq r \), \( x \neq y \). By the reproduction property,
\[
q_2(x, y) \geq \varphi^{-1}(x)(e^{-\Lambda^*(x, \cdot)}\varphi^{-1}(\cdot)e^{-\Lambda^*(\cdot, y)}1_{R, r}(\cdot))
\]
\[
\geq \varphi^{-1}(x)(e^{-\Lambda^*(x, \cdot)}\varphi(\cdot)e^{-\Lambda^*(\cdot, y)}1_{R, r}(\cdot))
\]
(we are applying Proposition 4)
\[
\geq e^{-\hat{\mu}-1}\varphi^{-1}(y)\inf_{r \leq |z| \leq R}e^{-\Lambda^*(z, y)}
\]
(we are applying Claim 4)
\[
\geq e^{-\hat{\mu}-1}\varphi^{-1}(y)c(r)e^{-\Lambda}(x, y)
\]
\[
\geq C_1(r)e^{-\Lambda}(x, y).
\]

(b') Let \( x \in B(0, 1) \), \( |y| \geq 1 \) (\( > r \)), \( x \neq y \). Arguing as in (b), we obtain
\[
q_3(x, y) \geq C_2e^{-3\Lambda}(x, y).
\]

(c) Let \( |x| \geq r \), \( y \in B(0, 1) \), \( x \neq y \). We have
\[
q_2(x, y) \geq \varphi^{-1}(x)(e^{-\Lambda^*(x, \cdot)}e^{-\Lambda^*(y, \cdot)}1_{R, r}(\cdot))
\]
\[
\geq \varphi^{-1}(x)(e^{-\Lambda^*(x, \cdot)}e^{-\Lambda(y, \cdot)}1_{R, r}(\cdot))
\]
(we are applying Claim 4)
\[
\geq \varphi^{-1}(x)c(r)(e^{-\Lambda(x, \cdot)}1_{R, r}(\cdot))
\]
\[
\geq C_3(r)(R + |x|)^{1-d-\alpha}(e^{-\Lambda(y, \cdot)}1_{R, r}(\cdot))
\]
(we are applying Proposition 6)
\[
\geq C_3(r)2^{-1}(R + |x|)^{-1} \geq C_4(r)e^{-2\Lambda}(x, y).
\]

(c') Let \( |x| \geq 1 \) (\( > r \)), \( y \in B(0, 1) \), \( x \neq y \). Arguing as in (c), we obtain
\[
q_3(x, y) \geq C_5(r)e^{-3\Lambda}(x, y).
\]

(d) Let \( x, y \in B(0, 1) \), \( x \neq y \). By the reproduction property,
\[
q_3(x, y) \geq \varphi^{-1}(x)(e^{-\Lambda^*(x, \cdot)}e^{-2\Lambda^*(\cdot, y)}1_{R, r}(\cdot))
\]
(we are using (c))
\[
\geq \varphi^{-1}(x)C_4(r)(e^{-\Lambda^*(x, \cdot)}\varphi(\cdot)e^{-2\Lambda(\cdot, y)}1_{R, r}(\cdot))
\]
(we are using \( e^{-2\Lambda}(z, y) \geq c_{r, R} > 0 \) for \( r \leq |z| \leq R, |y| \leq 1 \))
\[
\geq C_4c_{r, R}\varphi^{-1}(x)(e^{-\Lambda^*(x, \cdot)}1_{R, r}(\cdot)\varphi(\cdot))
\]
(we are applying Proposition 4)
\[
\geq C_4c_{r, R}e^{-\hat{\mu}-1} \geq C_5(r, R)e^{-3\Lambda}(x, y).
\]

By (a), (b'), (c'), (d), \( q^2(x, y) \geq Ce^{-3\Lambda}(x, y) \) for all \( x, y \in \mathbb{R}^d \), \( x \neq y \), and so \( e^{-3\Lambda^*(x, y)} \geq Ce^{-3\Lambda}(x, y)\varphi(x) \). Now the scaling argument yields the lower bound.
Appendix A.

Set \(I_\alpha = (-\Delta)^{-\frac{\alpha}{2}}\), the Riesz potential defined by the formula
\[
I_\alpha f(x) := \frac{1}{\gamma(\alpha)} \langle |x - \cdot|^{-d + \alpha} f(\cdot) \rangle, \quad \gamma(\alpha) := \frac{2^n \pi^{\frac{n}{2}} \Gamma(\frac{n}{2})}{\Gamma\left(\frac{n}{2} - \frac{\alpha}{2}\right)}.
\]

The identity
\[
\frac{\gamma(\beta - \alpha)}{\gamma(\beta)} |x|^{-d + \beta} = I_\alpha |x|^{-d + \beta - \alpha}, \quad 0 < \alpha < \beta < d,
\]
follows e.g. from \(I_\beta = I_\alpha I_{\beta - \alpha}\).

In the proof of Theorem \(\|\) we use a consequence of (2):
\[
(-\Delta)^\frac{\alpha}{2} |x|^{-d + \beta} = V(x)|x|^{-d + \beta}, \quad V(x) = \frac{\gamma(\beta)}{\gamma(\beta - \alpha)} |x|^{-\alpha},
\]
(i.e. \(\tilde{\varphi}_1(x) = |x|^{-d + \beta}\) is a Lyapunov’s function to the formal operator \((-\Delta)^\frac{\alpha}{2} - V\)).

Appendix B.

Let \(P\) be a closed operator on \(L^1\) such that \(\text{Re}((\lambda + P)f, \frac{f}{|f|}) \geq 0\) for all \(f \in D(P)\), and \(R(\mu + P)\) is dense in \(L^1\) for a \(\mu > \lambda\).

Then \(R(\mu + P) = L^1\).

Indeed, let \(y_n \in R(\mu + P)\), \(n = 1, 2, \ldots\), be a Cauchy sequence in \(L^1\); \(y_n = (\mu + P)x_n, x_n \in D(P)\). Write \([f, g] := \langle f, \frac{g}{|g|} \rangle\). Then
\[
(\mu - \lambda)\|x_n - x_m\|_1 = (\mu - \lambda)[x_n - x_m, x_n - x_m]
\leq (\mu - \lambda)[x_n - x_m, x_n - x_m] + [(\lambda + P)(x_n - x_m), x_n - x_m]
= [(\mu + P)(x_n - x_m), x_n - x_m] \leq \|y_n - y_m\|_1.
\]

Thus, \(\{x_n\}\) is itself a Cauchy sequence in \(L^1\). Since \(P\) is closed, the result follows.

Appendix C.

Let us show that
\[
\int_0^t \langle e^{-(t-\tau)A}(x, \cdot)E^\tau(\cdot, y) \rangle d\tau \lesssim t^{1 - \frac{\alpha}{d}} e^{-tA}(x, y) \quad \text{for all } x, y \in \mathbb{R}^d, \quad t > 0.
\]

Indeed,
\[
e^{-(t-\tau)A}(x, z)E^\tau(z, y) \approx e^{-(t-\tau)A}(x, z)e^{-\tau A}(z, y)(|z - y|^{-1} \wedge \tau^{-\frac{\alpha}{d}})
\]
(we are applying (11))
\[
\lesssim e^{-tA}(x, y)(e^{-(t-\tau)A}(x, z) + e^{-\tau A}(z, y))(|z - y|^{-1} \wedge \tau^{-\frac{\alpha}{d}}).
\]

Therefore, using \(e^{-tA}(x, z) \lesssim (t|z - x|^{-d - \alpha}) \wedge t^{-\frac{\alpha}{d}} \lesssim |z - x|^{-d} \wedge t^{-\frac{\alpha}{d}}\), we obtain
\[
e^{-(t-\tau)A}(x, z)E^\tau(z, y) \lesssim e^{-tA}(x, y)[(|z - x|^{-d} \wedge (t - \tau)^{-\frac{\alpha}{d}}) + (|z - y|^{-d} \wedge \tau^{-\frac{\alpha}{d}})](|z - y|^{-1} \wedge \tau^{-\frac{\alpha}{d}})
= e^{-tA}(x, y) I,
\]
where, it is easily seen using Young’s inequality,
\[
I \lesssim |x - z|^{-d-1} \wedge (t - \tau)^{-\frac{d+1}{\alpha}} + |z - y|^{-d-1} \wedge \tau^{-\frac{d+1}{\alpha}},
\]
and so
\[
\int_0^t \langle I \rangle_z d\tau \lesssim t^{1-\frac{d}{\alpha}}.
\]

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