Picard-Fuchs Ordinary Differential Systems in $N = 2$ Supersymmetric Yang-Mills Theories

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Abstract

In general, Picard-Fuchs systems in $N = 2$ supersymmetric Yang-Mills theories are realized as a set of simultaneous partial differential equations. However, if the QCD scale parameter is used as unique independent variable instead of moduli, the resulting Picard-Fuchs systems are represented by a single ordinary differential equation (ODE) whose order coincides with the total number of independent periods. This paper discusses some properties of these Picard-Fuchs ODEs. In contrast with the usual Picard-Fuchs systems written in terms of moduli derivatives, there exists a Wronskian for this ordinary differential system and this Wronskian produces a new relation among periods, moduli and QCD scale parameter, which in the case of SU(2) is reminiscent of scaling relation of prepotential. On the other hand, in the case of the SU(3) theory, there are two kinds of ordinary differential equations, one of which is the equation directly constructed from periods and the other is derived from the SU(3) Picard-Fuchs equations in moduli derivatives identified with Appell’s $F_4$ hypergeometric system, i.e., Burchnall’s fifth order ordinary differential equation published in 1942. It is shown that four of the five independent solutions to the latter equation actually correspond to the four periods in the SU(3) gauge theory and the closed form of the remaining one is established by the SU(3) Picard-Fuchs ODE. The formula for this fifth solution is a new one.

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I. INTRODUCTION

It has been recognized that the low energy effective action of \( N = 2 \) supersymmetric Yang-Mills theory for any Lie gauge group including at most two derivatives and four fermions is dominated by a holomorphic function called prepotential \( \mathcal{F} \). Perturbatively, this prepotential is a sum of classical part and one-loop contribution, and further contributions from higher loop diagrams are excluded by the non-renormalization theorem in \( N = 2 \) theory. However, \( \mathcal{F} \) was expected to be affected by instantons and hence its non-perturbative determination was a longstanding problem.

In the case of SU(2) gauge theory, i.e., without any quark hypermultiplet, Seiberg and Witten showed that the vacuum configuration of the \( N = 2 \) action was parameterized by the moduli \( u = \langle \text{tr} \phi^2 \rangle \) (\( \phi \) is a complex scalar field in the adjoint representation of the gauge group) and singularities on this parameter space (moduli space) could split into pieces by instanton effect. This instanton corrected moduli space is often called quantum moduli space and Seiberg and Witten identified the quantum moduli space with the moduli space of a certain elliptic curve of genus one. According to their ansatz, since the vacuum expectation value \( a = \langle \phi \rangle \) and its magnetic dual can be also regarded as periods of a meromorphic 1-form on the elliptic Riemann surface, these periods can be calculated as a linear combination of solutions to Picard-Fuchs equations (for a historical review and introduction of Picard-Fuchs equation in mathematics, see Gray). Once the periods are calculated, it is immediate to obtain the prepotential because of rigid special geometry. In this way, Klemm et al. determined the SU(2) prepotential.

Seiberg and Witten’s approach to \( N = 2 \) supersymmetric SU(2) Yang-Mills theory was extended to other higher rank gauge group cases coupled with or without quark hypermultiplets and it was found that the quantum moduli spaces of those gauge theories could be identified with those of Riemann surfaces with certain genus. In these studies, general algorithms to get Picard-Fuchs equations were developed, but these equations are in general realized as a set of simultaneous partial differential equations (PDEs) in terms of moduli derivatives. For this reason, it is not easy to solve Picard-Fuchs equations, especially, in higher rank gauge group cases.

However, if the QCD scale parameter instead of moduli is used as unique independent variable, the resulting Picard-Fuchs systems will be represented by a single ordinary differential equation (ODE) whose order coincides with the total number of independent periods. Then the problem
solving Picard-Fuchs equations can be encoded into the language of ODE and therefore the study of periods are simplified. As another feature of this formalism, we remark that in contrast with the usual Picard-Fuchs systems written in terms of moduli derivatives there exists a Wronskian for these ordinary differential systems and the Wronskian produces a new relation among periods, moduli and QCD scale parameter. Especially, in the case of SU(2) it is quite reminiscent of scaling relation of the prepotential. This relation is highly non-linear in the case of higher rank gauge group, but it reflects the structure of the Picard-Fuchs ODE. Sec. II discusses these Picard-Fuchs ordinary differential systems. This realization of Picard-Fuchs systems via ODE becomes interesting when we consider a relation to hypergeometric differential equations in multiple variables. For example, in the case of SU(3) gauge theory, we can find another ODE which gives equivalent periods. That is Burchnall’s fifth order equation directly constructed from Appell’s $F_4$ hypergeometric differential equations. In general, the dimension of the solution space of a single ODE constructed from simultaneous PDEs can exceed that of the original PDEs (see also Srivastava and Karlsson and references therein), and Burchnall’s equation is the case. In addition, the extra solution which is not a solution to the original PDEs is known to have a very characteristic form. In the case of Burchnall’s equation for the SU(3) gauge theory, since four of the five independent solutions are found to correspond to the four periods of the SU(3) gauge theory (this identification is explicitly checked at the semi-classical regime) and these four periods are also solutions to the SU(3) Picard-Fuchs ODE, it is possible to extract a differential equation only for the fifth solution (although the fifth solution is irrelevant to the underlying physics). The formula for this fifth solution obtained in this way is a new one and takes a very different form compared with other fifth order ODEs constructed from a set of PDEs, e.g., a product of two Bessel functions or Whittaker functions (the “fifth solution” to these two cases are quite reminiscent each other). In our derivation, it is crucial to notice that the fifth solution of the fifth order ODE (that is, Burchnall’s equation) associated with Appell’s $F_4$ is constructed by “subtracting” fourth order equation satisfied by only periods of the SU(3) Seiberg-Witten curve which are part of the solutions to the fifth order ODE. In Sec. III, we discuss these aspects of Burchnall’s equation as application of the SU(3) Picard-Fuchs ODE to a theory of hypergeometric equations. Sec. IV is a brief summary.

Remark: When we simply say as “Picard-Fuchs ODE”, it always means a single ordinary differential equation in terms of QCD scale parameter derivatives.
II. PICARD-FUCHS ORDINARY DIFFERENTIAL SYSTEMS

A. The hyperelliptic curve

Firstly, let us recall that the exact solution to the $SU(n + 1)$ ($n \in \mathbb{N}$) gauge theory as an example. On the affine local coordinates $x, y \in \mathbb{C}$, the hyperelliptic curve and the Seiberg-Witten differential are given by

$$y^2 = \tilde{W}_{SU(n+1)}^2 - z, \quad \lambda_{SW} = \frac{x \partial_x \tilde{W}_{SU(n+1)}}{y} dx,$$  \hspace{1cm} (2.1)

where $z = \Lambda_{SU(n+1)}^{2(n+1)}$ and

$$\tilde{W}_{SU(n+1)} = x^{n+1} - \sum_{i=2}^{n+1} s_i x^{n+1-i}.$$  \hspace{1cm} (2.2)

Eq.(2.2) shows the simple singularity with moduli $s_i$. This hyperelliptic curve can be compactified to a Riemann surface of genus $n$ after addition of infinity, hence there must be non-contractible 2n-cycles on this surface and these cycles can be chosen as the canonical bases, i.e., $\alpha_i \cap \alpha_j = \beta_i \cap \beta_j = 0, \alpha_i \cap \beta_j = -\beta_j \cap \alpha_i = +\delta_{i,j}$. Then we introduce the period vector

$$\Pi = \begin{pmatrix} a_{D_i} \\ a_i \end{pmatrix},$$  \hspace{1cm} (2.3)

where

$$a_i = \oint_{\alpha_i} \lambda_{SW}, \quad a_{D_i} = \oint_{\beta_i} \lambda_{SW}.$$  \hspace{1cm} (2.4)

Below, we often denote moduli as $u \equiv s_2$ and $v \equiv s_3$.

B. Derivation of Picard-Fuchs ODE

One of physically interesting behavior of periods is in the weak coupling region. Of course, the study of periods can be proceeded by using Picard-Fuchs equations, and in general periods are represented by series in moduli. However, by some rearrangement they are found to have a striking feature in their form. Namely, they can be summarized as

$$\text{periods} = \text{classical part} + \text{instanton corrections}.$$  \hspace{1cm} (2.5)
Eq. (2.5) suggests that it is more convenient to construct periods as a series in QCD scale parameter \( \Lambda \) rather than moduli. In a sense, (2.5) might be implied by Ito and Sasakura\(^27\) in their observation of general form of (a certain type of) Picard-Fuchs operators, which is summarized schematically as

\[
L = L_{cl} + \Lambda L_{\Lambda}.
\]

This equality means that the Picard-Fuchs operator \( L \) is a sum of operator \( L_{cl} \) whose kernel is classical periods and some operator \( L_{\Lambda} \). Once the classical periods are known, we can calculate instanton corrected periods, that is, the kernel of \( L \), by assuming \( L_{\Lambda} \) as a perturbation term for small \( \Lambda \) (at semi-classical regime). However, since in the case of other gauge theories with higher rank gauge groups Picard-Fuchs system is represented by a set of PDEs, such perturbative calculation involves technical problems, therefore another method to obtain instanton corrected periods should be developed. One of the candidates in view of differential equation is to construct Picard-Fuchs equations by regarding \( \Lambda \) as unique independent variable instead of moduli. Then the resulting Picard-Fuchs equations will be expressed by an ordinary differential equation. Since the QCD scale parameter always appears in any gauge theory with or without (massive) hypermultiplets, this formulation is convenient when we generalize the method to various gauge theories with any rank gauge group.

Now, let us consider the derivation of this ordinary differential equation. In general, \( k \)-times differentiation of \( \lambda_{SW} \) over \( z \) gives

\[
\frac{d^k\lambda_{SW}}{dz^k} = \text{polynomial in } x \frac{1}{y^{2k+1}} dx,
\]

but the right hand side can be decomposed into a sum of Abelian differentials and a total derivative term, if the well-known reduction algorithm is used.\(^17\),\(^18\),\(^19\),\(^20\),\(^21\) Accordingly, collecting (2.7) for various \( k \) can generate a differential equation. In addition, since the all independent periods should be solutions to this equation, the order of the equation must coincide with the total number of them and in fact it is determined as \( 2n \). This reduction method is easily confirmed, if the Seiberg-Witten curves are hyperelliptic type. In this way, we get the Picard-Fuchs ODE in the form

\[
\left[ \frac{d^{2n}}{dz^{2n}} + c_{2n-1} \frac{d^{2n-1}}{dz^{2n-1}} + \cdots + c_0 \right] \Pi = 0,
\]

(2.8)

where \( c_i \) are functions in moduli.
Also when massive quarks are included, we can obtain a similar ordinary differential equation, but in this case mass dependent polynomial appears in the denominator of the right hand side of (2.7). Reduction of such massive differential was also recognized by Marshakov et al.\textsuperscript{28} in their construction of massive WDVV equations.

C. Examples of Picard-Fuchs ODE

Let us see examples of Picard-Fuchs ODE. The first one is the SU(2) case and then the coefficients in (2.8) are given by

\[
c_1 = \frac{1}{z}, \quad c_0 = \frac{1}{16z(u^2 - z)}. \tag{2.9}
\]

Next, let us consider the SU(3) case. In this case, the coefficients are

\[
c_0 = \frac{-45(3z - 4u^3 + 27v^2)}{2z^2\Delta_{SU(3)}},
\]

\[
c_1 = \frac{45 (1053z^2 - 538zu^3 + 40u^6 + 3267zu^2v^2 - 54u^3v^2 - 1458v^4)}{2z^2\Delta_{SU(3)}},
\]

\[
c_2 = \frac{1}{4z^2\Delta_{SU(3)}} \left[ 445905z^3 - 8 \left( 4u^3 - 27v^2 \right)^3 + z^2 \left( -217368u^3 + 734589v^2 \right) \
+ 36z \left( 676u^6 - 135u^3v^2 - 29889v^4 \right) \right],
\]

\[
c_3 = \frac{1}{z\Delta_{SU(3)}} \left[ 76545z^3 - 162z^2 \left( 244u^3 - 297v^2 \right) - 4 \left( 4u^3 - 27v^2 \right)^3 \
+ 9z \left( 656u^6 - 1080u^3v^2 - 22599v^4 \right) \right], \tag{2.10}
\]

where \( \tilde{\Delta}_{SU(3)} \) is the product

\[
\tilde{\Delta}_{SU(3)} = (15z - 4u^3 + 27v^2)\Delta_{SU(3)} \tag{2.11}
\]

with the discriminant

\[
\Delta_{SU(3)} = \left[ 729z^2 + \left( 4u^3 - 27v^2 \right)^2 - 54z \left( 4u^3 + 27v^2 \right) \right] \tag{2.12}
\]

of the SU(3) hyperelliptic curve.

We can easily obtain Picard-Fuchs ODE for SO(5) or Sp(4) gauge group in a similar manner. Though the Sp(4) hyperelliptic curve may be seen to be different from that of the SO(5) theory, the isomorphism of Picard-Fuchs equations between these two theories could be observed by Ito and Sasakura.\textsuperscript{27} Also in the case of Picard-Fuchs ODE, this isomorphism can be easily established by the same transformation\textsuperscript{27}. 

D. Wronskian

As is well-known, the scaling relation of SU(2) prepotential can be generated from Wronskian of Picard-Fuchs equation, but this is valid only for this SU(2) theory because Picard-Fuchs equations in other gauge theories consist of partial differential equations. In such theories, “Wronskian” does not generally exist. However, our Picard-Fuchs ODE (2.8) admits a Wronskian given by

\[ W_{SU(n+1)} = \begin{vmatrix}
    a_1 & \cdots & a_n & a_{D_1} & \cdots & a_{D_n} \\
    a'_1 & \cdots & a'_n & a'_{D_1} & \cdots & a'_{D_n} \\
    \vdots & \vdots & \vdots & \vdots & \vdots \\
    a^{(2n-1)}_1 & \cdots & a^{(2n-1)}_n & a^{(2n-1)}_{D_1} & \cdots & a^{(2n-1)}_{D_n}
\end{vmatrix}, \tag{2.13} \]

where \( ' = d/dz \). Substituting (2.13) into (2.8) shows that \( W_{SU(n+1)} \) satisfies

\[ W'_{SU(n+1)} + c_{2n-1} W_{SU(n+1)} = 0, \tag{2.14} \]

which is integrated to give

\[ W_{SU(n+1)} = \text{const.} \ e^{-\int c_{2n-1} dz}, \tag{2.15} \]

where const. is the integration constant to be determined from the comparison of left and right hand sides of (2.15) by asymptotic behavior of periods, but may depend on moduli because in our formulation moduli are regarded as constant.

Note that (2.15) produces a new non-linear relation between periods and other parameters. For example, in the case of the SU(2) theory with the normalization used by Klemm et al., (2.15) gives

\[ W_{SU(2)} = -\frac{iu}{2\pi z}, \tag{2.16} \]

which is a quite reminiscent expression with the homogeneity relation of prepotential. Similarly, for the SU(3) theory, we have

\[ W_{SU(3)} = \frac{15z - (4u^3 - 27v^2)}{z^4 \Delta_{SU(3)}^2}, \tag{2.17} \]

where the integration constant is normalized to 1 for convenience in the next section. Note that the regular singularities of (2.17) are the same with those of SU(3) Picard-Fuchs ODE.
III. BURCHNALL’S EQUATION

A. Multi-term differential equation

Picard-Fuchs equations obtained in the previous section can be shown to be classified in terms of multi-term ordinary differential equation discussed by Burchnall in his study of the relationship among hypergeometric differential equations in multiple variables and certain type of ordinary differential equations. Here, multi-term ordinary differential equation is defined by:

**Definition**: \(k\)-term ordinary differential equation is the differential equation taking the form

\[
\left[f(\theta_z) + \sum_{i=1}^{l} z^i g_i(\theta_z)\right] \Pi = 0 \quad (3.1)
\]

for some \(l\), where \(f\) and \(g_i\) are polynomial differential operators in the Euler derivative \(\theta_z = zd/dz\). \(k\) is the total number of \(f\) and non-zero \(g_i\).

For example, in terms of the Euler derivative, our SU(3) Picard-Fuchs ODE can be rewritten as the four-term ODE with

\[
f = 2916(x - y)^3(-1 + \theta_z)^2 \theta_z^2, \\
g_1 = -81(x - y) \theta_z^2 \left[43x - 52y - 60(x - y) \theta_z + 4(23x + 13y) \theta_z^2\right], \\
g_2 = 9 \left[(386x - 251y) \theta_z^2 + 144(2x - 7y) \theta_z^3 + 36(19x + y) \theta_z^4 - (x - y)(10 + 13\theta_z)\right], \\
g_3 = -5(1 + 3\theta_z)(2 + 3\theta_z)(-1 + 6\theta_z)(1 + 6\theta_z),
\]

where \(x = 4u^3/27\) and \(y = v^2\).

According to Srivastava and Saran, who extended the work of Burchnall to four-term ODE, our SU(3) Picard-Fuchs ODE seems to be representable by a hypergeometric function in the homogeneity form \(F(pz, qz, rz)\), where \(p, q\) and \(r\) are parameters. In this paper, we could not specify this function, but since the kernel of the SU(3) Picard-Fuchs ODE is essentially written by Appell’s \(F_4\) function, some property of \(F_4\) may appear in our SU(3) Picard-Fuchs ODE as a four-term equation. Furthermore, more detailed study indicates that Picard-Fuchs ODE in any rank gauge group can be classified as \(k\)-term equation, but \(k\) seems to correspond to \(2 \times (\text{rank of the gauge group})\).

Finally, note that the Picard-Fuchs ODE in SU(3) gauge theory has a factor \((-1 + \theta_z)^2 \theta_z^2\) in \(f\)-polynomial. Therefore, the indicial indices at semi-classical regime are degenerated to \(-1\) and 0.
This indicates that there are logarithmic solutions at this regime. Of course, similar observation holds also for SO(5) and Sp(4) Picard-Fuchs ODEs.

**B. Appell’s equations and Burchnall’s equation**

As is well-known, a hypergeometric function admits a lot of transformations and reducibilities. For example, Gaussian $\text{}_2F_1$ system has 24 solutions and Appell’s $F_1$ has 60 solutions. However, these solutions can be more systematically constructed, if we consider an equivalent ODE. In fact, Srivastava and Saran succeeded to find 120 solutions to the ODE for Lauricella’s $F_{D}^{(3)}$ function. Also in this sense, study of ODE for hypergeometric partial differential system is interesting. The method used in these studies followed to Burchnall’s work. In this paper, we do not attempt to obtain all solutions solutions to Burchnall’s equation (see below) like Kummer’s 24 solutions, but we can show that the basic five solutions to Burchnall’s equation, especially, the extra solution which is not a solution to $F_4$ system, can be derived by using SU(3) Picard-Fuchs ODE. Of course, it may be interesting if also this extra solution can be represented by Appell’s $F_4$, but we do not know whether it is possible or not. Nevertheless, we can establish the fifth solution as a simple formula. The reader should notice that our method presented in this paper is to use a fourth order ODE (SU(3) Picard-Fuchs ODE) satisfied by periods of Riemann surface in genus two (SU(3) Seiberg-Witten curve) and therefore our method is quite different from those mentioned above.

Firstly, let us recall that the SU(3) Picard-Fuchs system

\[
\begin{align*}
\theta_x\left(\theta_x - \frac{1}{3}\right) - \bar{x} \left(\theta_x + \theta_y - \frac{1}{6}\right) \left(\theta_x + \theta_y - \frac{1}{6}\right) \Pi = 0, \\
\theta_y\left(\theta_y - \frac{1}{2}\right) - \bar{y} \left(\theta_x + \theta_y - \frac{1}{6}\right) \left(\theta_x + \theta_y - \frac{1}{6}\right) \Pi = 0,
\end{align*}
\]

where we have introduced $\bar{x} = 4u^3/(27\Lambda_{SU(3)}^6)$ and $\bar{y} = v^2/\Lambda_{SU(3)}^6$, and $\theta_x = x\partial/\partial x$ and $\theta_y = y\partial/\partial y$ are Euler partial derivatives. In the terminology of previous subsection, (3.3) consists of two two-term equations and is nothing but the Appell’s differential system for the type $F_4$ hypergeometric double series

\[
F_4(\alpha, \beta; \gamma, \gamma'; x, y) = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_{m+n} x^m y^n}{(\gamma)_m(\gamma')_n m! n!},
\]

where $\alpha = \beta = -1/6, \gamma = 2/3$ and $\gamma' = 1/2$. However, by the scaling transformation

\[
\bar{x} = x\bar{z}, \quad \bar{y} = y\bar{z}, \quad x = \frac{4u^3}{27}, \quad y = v^2, \quad \bar{z} = \frac{1}{\Lambda_{A_2}^6},
\]

(3.5)
we see that (3.3) turns to
\[
\theta_x(\theta_x + \gamma - 1) - x\tilde{z}(\theta_x + \theta_y + \alpha)(\theta_x + \theta_y + \beta) F = 0,
\]
\[
\theta_y(\theta_y + \gamma' - 1) - y\tilde{z}(\theta_x + \theta_y + \alpha)(\theta_x + \theta_y + \beta) F = 0,
\]
where one of analytic solutions near \((x, y) = (0, 0)\) is given by
\[
F(x\tilde{z}, y\tilde{z}) = \sum_{m,n=0}^{\infty} \frac{(\alpha)^m(\beta)^n}{\gamma^n(n!)^2} \frac{(x\tilde{z})^m(y\tilde{z})^n}{m!n!}. \tag{3.7}
\]
At first sight, since \(F(x\tilde{z}, y\tilde{z})\) reduces to \(F_4\) for \(\tilde{z} \to 1\), this scale transformation may be trivial, but (3.6) was used as a starting point in the Burchnall’s work on a set of partial differential equations.\(^{25}\)

In fact, Burchnall noticed on the homogeneity relation of \(F\)
\[
(\theta_x + \theta_y - \theta_{\tilde{z}}) F = 0, \tag{3.8}
\]
where \(\theta_{\tilde{z}}\) is the ordinary differential operator \(\tilde{z}d/d\tilde{z}\), and finally arrived at the ordinary differential equation of fifth order (see also appendix A)
\[
\left[ f_0 - 2(x + y)\tilde{z} f_1(\theta_{\tilde{z}} + \alpha)(\theta_{\tilde{z}} + \beta) + \frac{1}{2}(x - y)\tilde{z} f_2(\theta_{\tilde{z}} + \alpha)(\theta_{\tilde{z}} + \beta) \\
+ (x - y)^2\tilde{z}^2 f_3(\theta_{\tilde{z}} + \alpha)(\theta_{\tilde{z}} + \alpha + 1)(\theta_{\tilde{z}} + \beta)(\theta_{\tilde{z}} + \beta + 1) \right] F = 0, \tag{3.9}
\]
where
\[
f_0 = \theta_{\tilde{z}}(\theta_{\tilde{z}} + \gamma - 1) (\theta_{\tilde{z}} + \gamma' - 1) (\theta_{\tilde{z}} + \gamma + \gamma' - 2) \left( \theta_{\tilde{z}} + \frac{\gamma}{2} + \frac{\gamma'}{2} - 2 \right),
\]
\[
f_1 = \left( \theta_{\tilde{z}} + \frac{\gamma}{2} + \frac{\gamma'}{2} \right) \left( \theta_{\tilde{z}} + \frac{\gamma}{2} + \frac{\gamma'}{2} - 1 \right) \left( \theta_{\tilde{z}} + \frac{\gamma}{2} + \frac{\gamma'}{2} - 1 \right),
\]
\[
f_2 = (\gamma - \gamma') (\gamma + \gamma' - 2) \left( \theta_{\tilde{z}} + \frac{\gamma}{2} + \frac{\gamma'}{2} - 1 \right),
\]
\[
f_3 = \left( \theta_{\tilde{z}} + \frac{\gamma}{2} + \frac{\gamma'}{2} + 1 \right). \tag{3.10}
\]
In contrast with the SU(3) Picard-Fuchs ODE, since Burchnall’s equation (3.9) is classified as a three-term ODE, it reflects a consequence of the nature of two variables hypergeometric function in the form \(F(pz, qz)\), where \(p\) and \(q\) are parameters.\(^{25}\)

As a direct calculation shows, (3.9) can not be expressed in the form \([\theta_{\tilde{z}} + K(x, y, \tilde{z})]LF = 0\), where \(K(x, y, \tilde{z})\) is some function of \(x, y\) and \(\tilde{z}\), and \(L\) is some differential operator of fourth order. Accordingly, such \(L\) does not exist and therefore the relation between our SU(3) Picard-Fuchs
ODE and Burchnall’s equation is non-trivial. However, note that the coefficient of highest power in the Euler derivative corresponds to the discriminant of the SU(3) curve

\[
\Delta \equiv \left[ 1 - 2(x + y)\tilde{z} + (x - y)^2\tilde{z}^2 \right] = \frac{1}{729\Lambda_{SU(3)}^{12}} \left[ 4u^3 - 27(v + \Lambda_{SU(3)}^3)^2 \right] \left[ 4u^3 - 27(v - \Lambda_{A_2}^3)^2 \right].
\]

Therefore, in a sense Burchnall’s equation is reminiscent of the SU(3) Picard-Fuchs ODE, but these two are not completely equivalent. Clarifying the relation between these two equations is the subject in the rest of the paper.

C. Four solutions at semi-classical regime

It would be instructive to get solutions explicitly around \( \Lambda_{SU(3)} = 0 \) (\( \tilde{z} = \infty \)) which is a regular singular point of the equation. In the work of Burchnall, solutions of (3.9) were not calculated at any singularities, but since (3.9) is a linear ordinary differential equation, it is easy to solve it by traditional Frobenius’s method under the assumption 

\[
F = \tilde{z}^{-\nu} \sum_{n=0}^{\infty} A_n \tilde{z}^{-n}
\]

for some \( \nu \) and \( A_n \).

Then the indicial indices are determined as

\[
\nu = \alpha, \ \beta, \ \alpha + 1, \ \beta + 1, \ \frac{1}{2}(\gamma + \gamma') + 2
\]

or equivalently,

\[
\nu_1 = -\frac{1}{6}, \ \nu_2 = \frac{5}{6}, \ \nu_3 = \frac{19}{12},
\]

where \( \nu_1 \) and \( \nu_2 \) are actually double roots. The solution for \( \nu_3 \) is the subject in the next subsection.

Eq.(3.9) produces the recursion relations

\[
(x - y)^2 \rho_1 A_1 - (\nu_i - \alpha)(\nu_i - \beta)\phi_1 A_0 = 0,
\]

\[
(x - y)^2 \rho_n A_n - (\nu_i + n - \alpha - 1)(\nu_i + n - \beta - 1)\sigma_n A_{n-1} + \chi_n A_{n-2} = 0, \ n > 1,
\]

where

\[
\rho_n = (2\nu_i + 2n - \gamma - \gamma' - 2)(\nu_i + n - \alpha)(\nu_i + n - \beta)(\nu_i + n - \alpha - 1)(\nu_i + n - \beta - 1),
\]

\[
\sigma_n = (\nu_i + n - \gamma - \gamma')(2\nu_i + 2n - \gamma - \gamma')[x(\nu_i + n - \gamma' - 1) + y(\nu_i + n - \gamma - 1)]
\]

\[+(\nu_i + n - 1)(2\nu_i + 2n - \gamma - \gamma' - 2)[x(\nu_i + n - \gamma) + y(\nu_i + n - \gamma')],
\]

\[
\chi_n = (\nu_i + n - 2)(\nu_i + n - \gamma - 1)(\nu_i + n - \gamma' - 1)(\nu_i + n - \gamma - \gamma')(2\nu_i + 2n - \gamma - \gamma')
\]
with $A_0 = 1$. Here, repeated indices are assumed not to be summed. If these recursion relations are used, the solutions corresponding to respective indicial indices will be obtained, but we must be careful, because there are indicial indices which differ by unit among them, i.e., \( \alpha \) and \( \alpha + 1 \), and \( \beta \) and \( \beta + 1 \). For example, let \( \nu = \alpha \). Then the recursion relations produce the \( n \)-th coefficient as a linear combination of \( A_0 \) and \( A_1 \). Thus the solution is given by a linear combination of \( \tilde{z}^{-\alpha}(A_0 + \cdots) \) and \( \tilde{z}^{-\alpha-1}(A_1 + \cdots) \), but the “indicial index” of the last series can be seen as \( \alpha + 1 \). Therefore, the last series can be also regarded as a solution corresponding to this index. In fact, it is easy to see that explicit construction of the solution supports this observation. Of course, in this case, since we would like to get a solution corresponding to the index \( \alpha \), \( A_1 \) can be set to zero without loss of generality. For this reason, \( A_1 \) is chosen as zero for the indices \( \alpha \) and \( \beta \), while that for \( \alpha + 1, \beta + 1 \) and \( (\gamma + \gamma' - 2)/2 \) should be determined from the first equation in (3.14).

In this way, we get the regular series solutions \((i = 1, 2)\)

\[
\varphi_i = \tilde{z}^{-\nu_i} \sum_{n=1}^{\infty} A_{i,n} \tilde{z}^{-n},
\]

(3.16)

where the first few coefficients are given by

\[
\begin{align*}
A_{i,0} &= 1, \\
A_{1,1} &= 0, \\
A_{1,2} &= \frac{5}{648(x-y)^2}, \\
A_{1,3} &= \frac{35(41x+40y)}{209952(x-y)^4}, \\
A_{2,1} &= \frac{5(5x+4y)}{48(x-y)^2}, \\
A_{2,2} &= \frac{35(157x^2+460xy+112y^2)}{15552(x-y)^4}, \\
A_{2,3} &= \frac{385(18671x^3+119352x^2y+105504xy^2+12352y^3)}{26873856(x-y)^6}.
\end{align*}
\]

(3.17)

On the other hand, the degeneracy of \( \nu_1 \) and \( \nu_2 \) produce the logarithmic solutions \((j = 1, 2)\)

\[
\tilde{\varphi}_j = \varphi_j \ln \frac{1}{\tilde{z}} + \tilde{z}^{-\nu_j} \sum_{n=1}^{\infty} B_{j,n} \tilde{z}^{-n},
\]

(3.18)

where some of \( B_{j,n} \) are

\[
B_{1,1} = 0,
\]
\[ B_{1,2} = -\frac{17}{1296(x-y)^2}, \]
\[ B_{1,3} = -\frac{(11761x + 11216y)}{1259712(x-y)^4}, \]
\[ B_{2,1} = \frac{49x + 104y}{144(x-y)^2}, \]
\[ B_{2,2} = \frac{14273x^2 + 70940xy + 28268y^2}{46656(x-y)^4}, \]
\[ B_{2,3} = \frac{41936917x^3 + 383568144x^2y + 472854144xy^2 + 79210112y^3}{161243136(x-y)^6}. \] (3.19)

It is interesting to notice that these series are composed by a series in powers of \( 1/(x-y) = 27/(4u^3 - 27v^2) \) which detects the discriminant of the semi-classical SU(3) curve. This feature is useful when we consider the structure of the quantum moduli space of the SU(3) gauge theory.

For this purpose, let us recall the work of Klemm et al.\(^5\) In the course of the analysis of the quantum moduli space of the SU(3) gauge theory, Klemm et al.\(^5\) found that the quantum moduli space could be better understood as the complex projective space \( \mathbb{CP}^2 \) with singularities which correspond to the strong coupling regime. Then this space can be covered by the three local (inhomogeneous) coordinates

\[ P_1 : \left( \frac{4u^3}{27 \Lambda^6} : \frac{v^2}{\Lambda^6} : 1 \right), \quad P_2 : \left( \frac{4u^3}{27 v^2} : 1 : \frac{\Lambda^6}{v^2} \right), \quad P_3 : \left( 1 : \frac{27v^2}{4u^3} : \frac{27 \Lambda^6}{4u^3} \right). \] (3.20)

For this reason, periods should be obtained at each coordinate patch, hence the periods derived in this way are locally valid. However, our solutions have a (slightly) nice property, because the basis of solution space are common both on \( P_2 \) and \( P_3 \). That is, to get periods on \( P_2 \), it is enough to further expand \( \varphi_i \) by \( v \), while on \( P_3 \) by large \( u \).

In fact, we can see that the four periods are expressed by linear combinations of (3.16) and (3.18). For example, to match periods on the patch \( P_3 \), let us define \( \omega_i \) and \( \Omega_j \) by linear combinations of \( \varphi_i \)

\[ \tilde{\omega}_1 = c_1 \varphi_1 + c_2 \varphi_2, \quad \tilde{\omega}_2 = c_3 \varphi_1 + c_4 \varphi_2, \]
\[ \tilde{\Omega}_1 = \tilde{\omega}_1 \ln \frac{27}{4u^3 \tilde{z}} + \sum_{i=1}^{\infty} \sum_{n=1}^{\infty} c_i B_{i,n} \tilde{z}^{-\nu_i-n} + c_5 \varphi_1 + c_6 \varphi_2, \]
\[ \tilde{\Omega}_2 = \tilde{\omega}_2 \ln \frac{27}{4u^3 \tilde{z}} + \sum_{i=1}^{\infty} \sum_{n=1}^{\infty} c_{i+2} B_{i,n} \tilde{z}^{-\nu_i-n} + c_7 \varphi_1 + c_8 \varphi_2, \] (3.21)

where
\[ c_1 = 2F_1 \left( \frac{-1}{6} : \frac{1}{3} : \frac{1}{2} : \frac{27v^2}{4w^3} \right), \quad c_2 = -\frac{3}{16u^3} 2F_1 \left( \frac{5}{6} : \frac{1}{3} : \frac{1}{2} : \frac{27v^2}{4w^3} \right), \]
\[ c_3 = 2F_1 \left( \frac{2}{3} : \frac{2}{3} : \frac{27v^2}{4w^3} \right), \quad c_4 = \frac{3}{2u^3} 2F_1 \left( \frac{4}{3} : \frac{5}{3} : \frac{27v^2}{4w^3} \right), \]
\[ c_5 = c_1 + \sum_{n=0}^{\infty} \frac{(-1/6)_n (1/6)_n}{(1/2)_n n!} \times \left[ \psi \left( n - \frac{1}{6} \right) - \psi \left( -\frac{1}{6} \right) + \psi \left( n + \frac{1}{6} \right) - \psi \left( \frac{1}{6} \right) \right] \left( \frac{27v^2}{4w^3} \right)^n, \]
\[ c_6 = c_2 - \frac{3}{16u^3} \sum_{n=0}^{\infty} \frac{(5/6)_n (7/6)_n}{(1/2)_n n!} \times \left[ 2\psi(1) - 2\psi(2) + \psi \left( n + \frac{5}{6} \right) - \psi \left( -\frac{1}{6} \right) + \psi \left( n + \frac{7}{6} \right) - \psi \left( \frac{1}{6} \right) \right] \left( \frac{27v^2}{4w^3} \right)^n, \]
\[ c_7 = c_3 + \sum_{n=1}^{\infty} \frac{(1/3)_n (2/3)_n}{(3/2)_n n!} \times \left[ \psi \left( n + \frac{1}{3} \right) - \psi \left( \frac{1}{3} \right) + \psi \left( n + \frac{2}{3} \right) - \psi \left( \frac{2}{3} \right) \right] \left( \frac{27v^2}{4w^3} \right)^n, \]
\[ c_8 = c_4 + \frac{3}{2u^3} \sum_{n=0}^{\infty} \frac{(4/3)_n (5/3)_n}{(3/2)_n n!} \left[ 2\psi(1) - 2\psi(2) + \psi \left( n + \frac{4}{3} \right) - \psi \left( \frac{1}{3} \right) + \psi \left( n + \frac{5}{3} \right) - \psi \left( \frac{2}{3} \right) \right] \left( \frac{27v^2}{4w^3} \right)^n. \]  

(3.22)

Here, \( \psi(x) = d\ln \Gamma(x)/dx \) is the digamma function and \( 2F_1 \) is the hypergeometric function whose series representation is given by

\[ 2F_1(a, b; c; x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} x^n, \]  

(3.23)

where \((*)_n = \Gamma(* + n)/\Gamma(*)\) is the Pochhammer symbol. In order to make a contact with the normalization used by Klemm et al., we rescale as

\[ \omega_1 = 2\sqrt{u\Lambda}\tilde{\omega}_1, \quad \omega_2 = \frac{v\Lambda}{u}\tilde{\omega}_2, \]
\[ \Omega_1 = 2\sqrt{u\Lambda}\tilde{\Omega}_1, \quad \Omega_2 = \frac{v\Lambda}{u}\tilde{\Omega}_2. \]  

(3.24)

Then the periods \( a_j \) and \( a_{D_j} \) can be given by

\[ a_1 = \frac{1}{2} (\omega_1 + \omega_2), \quad a_2 = \frac{1}{2} (\omega_1 - \omega_2), \]
\[ a_{D_1} = -\frac{i}{4} (\Omega_1 + 3\Omega_2) - \frac{i}{\pi} (\delta_1 \omega_1 - \delta_2 \omega_2), \]
\[ a_{D_2} = -\frac{i}{4} (\Omega_1 - 3\Omega_2) - \frac{i}{\pi} (\delta_1 \omega_1 + \delta_2 \omega_2), \]  

(3.25)

where \( \delta_1 = i(5 - 3\ln 3 - 4\ln 2)/4 \) and \( \delta_2 = 3i(1 + 3\ln 3)/4 \) are constants determined from asymptotic expansion of periods. The identification of periods by our solutions can be easily established by...
expanding (3.25) at \( u = \infty \). On the other hand, for large \( v \), i.e., on the patch \( P_2 \), \( \varphi_i \) are expanded at \( v = \infty \) and then consider similar linear combinations.

In this way, we can check that the series solutions for \( \nu = \nu_1 \) and \( \nu_2 \) comprise in fact the four periods.

\( \text{D. The fifth solution} \)

We have seen that the four solutions with indicial indices \( \nu_1 \) and \( \nu_2 \) of Burchnall’s fifth order equation in fact yield the four periods of the SU(3) gauge theory. However, there exists an extra solution in (3.9). Therefore, the appearance can be regarded to be characteristic in the ordinary differential form of the partial differential system.

In this subsection, we show that it is possible to derive a fourth order equation satisfied by this fifth solution with aid of the SU(3) Picard-Fuchs ODE and we derive the closed formula for this fifth solution as an application of the SU(3) Picard-Fuchs ODE.

Firstly, let us rewrite (3.9) in the form

\[
\frac{d^5}{dz^5} + c_{B,A} \frac{d^4}{dz^4} + \cdots + c_{B,0} F = 0,
\]

(3.26)

where \( z = 1/\tilde{z} \) and \( c_{B,i} \) are some functions in \( x, y \) and \( z \). As we have already seen in the previous section, since the four of the five independent solutions to (3.9) are regarded as the four periods of the SU(3) gauge theory, we can write the Wronskian for (3.26) as

\[
W_B = \begin{vmatrix}
  a_1 & a_2 & a_{D_1} & a_{D_2} & h \\
  a_1' & a_2' & a_{D_1}' & a_{D_2}' & h' \\
  \vdots & \vdots & \vdots & \vdots & \vdots \\
  a_1^{(4)} & a_2^{(4)} & a_{D_1}^{(4)} & a_{D_2}^{(4)} & h^{(4)}
\end{vmatrix},
\]

(3.27)

where \( ' = d/dz \) and the fifth solution is denoted by \( h \). Again from basic differential calculation, \( W_B \) is generated from \( c_{B,A} \) and is found to be (c.f (2.15))

\[
W_B = \frac{1}{z^{85/12}[x^2 + (y - z)^2 - 2x(y + z)]^{5/2}}
\]

(3.28)

up to the normalization of integration constant, which is irrelevant to the following discussion.

Next, recall that the fourth order derivatives of periods can be reduced to a linear combination of lower order derivatives by using the SU(3) Picard-Fuchs ODE. Therefore, from (3.27) with the SU(3) Picard-Fuchs ODE, we see that
\[ W_B = W_{SU(3)}(h^{(4)} + c_3 h^{\prime} + c_2 h^\prime + c_1 h + c_0 h), \]  

(3.29)

where \( c_i \) are given by (2.10). From (3.28) and (2.17), it is immediate to obtain the differential equation for \( h \)

\[ h^{(4)} + c_3 h^{\prime} + c_2 h^\prime + c_1 h + c_0 h = R, \]

(3.30)

where

\[ R = \frac{1}{z^{37/12}[5z - 9(x - y)][x^2 + (y - z)^2 - 2x(y + z)]}. \]

(3.31)

It is interesting to note that \( h \) satisfies the SU(3) Picard-Fuchs ODE with a source term. It is now easy to get a general solution to (3.30) (see also appendix B)

\[ h = \sum_{i=1}^{2} \rho_i a_i + \sum_{i=1}^{2} \epsilon_i a_{D_i}, \]

where \( \rho_i \) and \( \epsilon_i \) are integration constants, the integration symbol is the integration constant free integral and \( w_i \) are determinants defined by

\[ w_1 = \begin{vmatrix} a_2 & a_{D_1} & a_{D_2} \\ a_2' & a_{D_1}' & a_{D_2}' \\ a_2'' & a_{D_1}'' & a_{D_2}'' \end{vmatrix}, \quad w_2 = \begin{vmatrix} a_1 & a_{D_1} & a_{D_2} \\ a_1' & a_{D_1}' & a_{D_2}' \\ a_1'' & a_{D_1}'' & a_{D_2}'' \end{vmatrix}, \quad w_3 = \begin{vmatrix} a_1 & a_2 & a_{D_2} \\ a_1' & a_2' & a_{D_2}' \\ a_1'' & a_2'' & a_{D_2}'' \end{vmatrix}, \quad w_4 = \begin{vmatrix} a_1 & a_2 & a_{D_1} \\ a_1' & a_2' & a_{D_1}' \\ a_1'' & a_2'' & a_{D_1}'' \end{vmatrix}. \]

(3.33)

In (3.32), we have written as \( W_{SU(3)} \) for simplicity, but actually it should be substituted by (2.17).

To summarize, we have succeeded to find a closed representation of the fifth solution by using the SU(3) Picard-Fuchs ODE. It is interesting to compare (3.32) with the fifth solution of other three-term differential equation discussed by Burchnall.\(^3\) In the case of a product of two Bessel functions, for instance, the fifth solution is expressed by an integral of a product of two “Wronskians” each of which is written by other two independent solutions.\(^3\) However, our fifth solution does not admit such factorization, so (3.32) seems to imply the fact that the Appell function \( F_4 \) with parameters \( \alpha = \beta = -1/6, \gamma = 2/3 \) and \( \gamma' = 1/2 \) (or equivalently, the SU(3) periods) can not be factored into a product of two (non-trivial) functions.

**Remark:** In the semi-classical regime, series representation of this fifth solution corresponds to \( \nu_3 = 19/12 \) in (3.13), and this can be also seen from the terms not including \( \rho_i \) and \( \epsilon_i \) in (3.32).
IV. SUMMARY

In this paper, we have discussed the Picard-Fuchs equations appearing in $N = 2$ supersymmetric Yang-Mills theories in view of ordinary differential equation and realized Picard-Fuchs equations as a system of ODEs. This construction has given a new relation among periods, moduli and QCD parameter by using the Wronskian and this is a systematic way to get such non-linear relation among periods in higher rank gauge group cases.

In the case of SU(3), we have also found that Burchnall’s ordinary differential equation for Appell’s $F_4$ is a candidate of Picard-Fuchs ODE by identifying the SU(3) QCD mass scale parameter with the scaling variable used in Burchnall’s observation and confirmed that the four of five solutions of Burchnall’s equation in fact coincide with SU(3) periods. As for the fifth solution, it has been shown that it has a simple and closed form by using the SU(3) Picard-Fuchs ODE. Of course, even if we consider arbitrary Riemann surface of genus two and try to get a similar result for the fifth solution to Burchnall’s equation, the derivation will be failed because we can not always have an equation in fourth order like SU(3) Picard-Fuchs ODE. Note that the SU(3) Seiberg-Witten curve is a specific choice and the appearance of $\Lambda_{SU(3)}$ plays the central role in the discussion.

Generalization of Burchnall’s construction of ordinary differential equation from a set of partial differential equations to Picard-Fuchs equations in other gauge theories is straightforward, but we do not know whether Burchnall type equations exist or not for Picard-Fuchs equations in those gauge theories. Studying these cases will open further aspect of Picard-Fuchs equations and hypergeometric nature of the equations constructed from the homogeneous hypergeometric equations in multiple variables.

Finally, as another direction, since our equation is ODE in contrast with the usual Picard-Fuchs systems, it may be possible to consider a relation to classical $W$-algebras in view of Picard-Fuchs ODE.

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APPENDIX A. DERIVATION OF THE BURCHNALL’S EQUATION

In this appendix, we briefly review the derivation of the Burchnall’s equation. The reader is also recommended to refer to the original paper.

Firstly, notice that from (3.6) it is easy to obtain

\[(\theta \tilde{z} + \gamma + \gamma' - 2)\theta_x \theta_y F = (x\tilde{z} \theta_y + y\tilde{z} \theta_x)XF, \tag{A1}\]

where \(X = (\theta \tilde{z} + \alpha)(\theta \tilde{z} + \beta)\). We can also obtain

\[UF \equiv [\theta \tilde{z} (\theta \tilde{z} + \gamma - 1) - (x + y)\tilde{z} X] F = (2\theta_x + \gamma - \gamma') \theta_y F, \tag{A2}\]

and

\[U'F \equiv [\theta \tilde{z} (\theta \tilde{z} + \gamma' - 1) - (x + y)\tilde{z} X] F = (2\theta_y + \gamma' - \gamma) \theta_x F \tag{A3}\]

Addition of the two equations in (A3) provides

\[VF \equiv [\theta \tilde{z} (\theta \tilde{z} + \gamma - 1)(\theta \tilde{z} + \gamma' - 1)F = [(\theta \tilde{z} + \gamma' - 1)\theta_x \theta_y + x\tilde{z} (\theta \tilde{z} + \gamma')X] F, \]

\[\theta_y (\theta \tilde{z} + \gamma' - 1)(\theta \tilde{z} + \gamma - 1)F = [(\theta \tilde{z} + \gamma - 1)\theta_x \theta_y + y\tilde{z} (\theta \tilde{z} + \gamma)X] F. \tag{A3}\]

Addition of the two equations in (A3) provides

\[VF \equiv [\theta \tilde{z} (\theta \tilde{z} + \gamma - 1)\theta \tilde{z} + \gamma' - 1)F = [(\theta \tilde{z} + \gamma' - 1)\theta_x \theta_y + x\tilde{z} (\theta \tilde{z} + \gamma')X - y\tilde{z} (\theta \tilde{z} + \gamma)X] F = (2\theta \tilde{z} + \gamma + \gamma' - 2)\theta_x \theta_y F, \tag{A4}\]

thus

\[(\theta \tilde{z} + \gamma + \gamma' - 2)VF = [4xy\tilde{z}^2XX_{+1} + x\tilde{z}XU + y\tilde{z}XU' + 2(\theta \tilde{z} + \gamma + \gamma' - 2)\theta_x \theta_y] F, \tag{A5}\]

where \(X_{+1} = (\theta \tilde{z} + \alpha + 1)(\theta \tilde{z} + \beta + 1)\), and (A1) and (A2) have been used. Moreover, from (A5) we have

\[WF \equiv [\theta \tilde{z} (\theta \tilde{z} + \gamma + \gamma' - 2)(\theta \tilde{z} + \gamma - 1)(\theta \tilde{z} + \gamma' - 1)\]

\[-x\tilde{z} [(\theta \tilde{z} + \gamma + \gamma' - 1)(\theta \tilde{z} + \gamma') + \theta \tilde{z} (\theta \tilde{z} + \gamma - 1)] X\]

\[-y\tilde{z} [(\theta \tilde{z} + \gamma + \gamma' - 1)(\theta \tilde{z} + \gamma) + \theta \tilde{z} (\theta \tilde{z} + \gamma' - 1)] X + (x - y)^2\tilde{z}^2XX_{+1}] F = 2(\theta \tilde{z} + \gamma + \gamma' - 2)\theta_x \theta_y F. \tag{A6}\]
Therefore, the expected equation is given by rearrangement of

\[(2\theta_z + \gamma + \gamma' - 2)WF = 2(\theta_z + \gamma + \gamma' - 2)VF, \quad (A7)\]
i.e.,

\[\left[ Y_0 - x\bar{z}Y_1X - y\bar{z}Y_2X + (x - y)^2\bar{z}^2(2\theta_z + \gamma + \gamma' + 2)XX_1 \right] F = 0, \quad (A8)\]

where

\[Y_0 = \theta_z(\theta_z + \gamma - 1)(\theta_z + \gamma' - 1)(\theta_z + \gamma + \gamma' - 2)(2\theta_z + \gamma + \gamma' - 4),\]
\[Y_1 = (\theta_z + \gamma')(\theta_z + \gamma + \gamma' - 1)(2\theta_z + \gamma + \gamma' - 2) + \theta_z(\theta_z + \gamma - 1)(2\theta_z + \gamma + \gamma'),\]
\[Y_2 = (\theta_z + \gamma)(\theta_z + \gamma + \gamma' - 1)(2\theta_z + \gamma + \gamma' - 2) + \theta_z(\theta_z + \gamma' - 1)(2\theta_z + \gamma + \gamma'). \quad (A9)\]

Of course (A8) is equivalent to (3.9).

**APPENDIX B. GENERAL SOLUTION TO FOURTH ORDER ODE**

This appendix reviews a construction of a general solution to the fourth order linear ordinary differential equation

\[y^{(4)} + P(x)y''' + Q(x)y'' + R(x)y' + S(x)y = T(x), \quad (B1)\]

where \(\dot{} = d/dx\) and \(P, Q, R, S\) and \(T\) are some functions in \(x\).

Firstly, let \(T = 0\) and let \(y_i\) \((i = 1, \cdots, 4)\) be the fundamental solutions to

\[y^{(4)} + P(x)y''' + Q(x)y'' + R(x)y' + S(x)y = 0. \quad (B2)\]

Then we assume that the general solution to (B1) is represented in the form

\[y = \sum_{i=1}^{4} C_i(x)y_i, \quad (B3)\]

by using unknown coefficients \(C_i\). Differentiating (B3), we can obtain

\[y' = \sum_{i=1}^{4} C_i'y_i + \sum_{i=1}^{4} C_i y_i', \quad (B4)\]

but we further assume that the first term in the right hand side vanishes. Namely, we have

\[\sum_{i=1}^{4} C_i'y_i = 0 \quad (B5)\]
and
\[ y' = \sum_{i=1}^{4} C_i y'_i. \]  \hfill (B6)

Repeating differentiation and imposing the vanishing of terms including \( C'_i \), we get
\[ \sum_{i=1}^{4} C'_i y'_i = 0, \quad \sum_{i=1}^{4} C'_i y''_i = 0 \]  \hfill (B7)

and
\[ y'' = \sum_{i=1}^{4} C_i y''_i, \quad y''' = \sum_{i=1}^{4} C_i y'''_i. \]  \hfill (B8)

As for \( y^{(4)} \), we assume
\[ y^{(4)} = \sum_{i=1}^{4} C_i y'''_i + \sum_{i=1}^{4} C_i y^{(4)}_i. \]  \hfill (B9)

Then from (B1), (B9) and (B8), we get
\[ \sum_{i=1}^{4} C'_i y''''_i = T. \]  \hfill (B10)

In this way, we can arrive at the matrix equation determining all \( C_i \)
\[ YC = T(0, 0, 0, T), \]  \hfill (B11)

where
\[ Y = \begin{pmatrix} y_1 & \cdots & y_4 \\ y'_1 & \cdots & y'_4 \\ y''_1 & \cdots & y''_4 \\ y'''_1 & \cdots & y'''_4 \end{pmatrix}, \quad C = \begin{pmatrix} C'_1 \\ \vdots \\ C'_4 \end{pmatrix}. \]  \hfill (B12)

Consequently, \( C_i \) are given by
\[ C_1 = c_1 - \int_0^T \frac{w_1}{\det Y} dx, \quad C_2 = c_2 + \int_0^T \frac{w_2}{\det Y} dx, \quad C_3 = c_3 - \int_0^T \frac{w_3}{\det Y} dx, \quad C_4 = c_4 + \int_0^T \frac{w_4}{\det Y} dx, \]  \hfill (B13)

where \( c_i \) are integration constants, the integration symbol is the integration constant free integral and
\[ w_1 = \begin{vmatrix} y_2 & y_3 & y_4 \\ y'_2 & y'_3 & y'_4 \\ y''_2 & y''_3 & y''_4 \end{vmatrix}, \quad w_2 = \begin{vmatrix} y_1 & y_3 & y_4 \\ y'_1 & y'_3 & y'_4 \\ y''_1 & y''_3 & y''_4 \end{vmatrix}, \quad w_3 = \begin{vmatrix} y_1 & y_2 & y_4 \\ y'_1 & y'_2 & y'_4 \\ y''_1 & y''_2 & y''_4 \end{vmatrix}, \quad w_4 = \begin{vmatrix} y_1 & y_2 & y_3 \\ y'_1 & y'_2 & y'_3 \\ y''_1 & y''_2 & y''_3 \end{vmatrix}. \]  \hfill (B14)
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