Magnetic Trajectories on 2-Step Nilmanifolds

Gabriela P. Ovando1 · Mauro Subils1

Received: 13 May 2022 / Accepted: 11 February 2023 / Published online: 7 April 2023
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Abstract
The aim of this work is the study of magnetic trajectories on 2-step nilmanifolds. The magnetic equation is written and the corresponding solutions are found for a family of invariant Lorentz forces on a 2-step nilpotent Lie group equipped with a left-invariant metric. Some explicit examples are computed in the Heisenberg Lie groups $H_n$ for $n = 3, 5$, showing differences with the case of exact forms. Interesting magnetic trajectories related to elliptic integrals appear in $H_3$. The question of existence of closed or periodic magnetic trajectories for every energy level on Lie groups or on compact quotients is treated. Conditions for the closeness of magnetic trajectories are proved.

Keywords Magnetic trajectories · 2-step nilmanifolds · Heisenberg Lie groups

Mathematics Subject Classification 70G45 · 22E25 · 53C99 · 70G65

1 Introduction

From the mechanical perspective, the behavior of a charged particle in presence of a force, known as Lorentz force, is described by an equation of the form:

$$\nabla_{\gamma'}\gamma' = q F \gamma'$$  \hspace{1cm} (1)

where $\gamma$ is a curve on a Riemannian manifold $(M, g)$, $\nabla$ is the corresponding Levi-Civita connection and $F$ is a skew-symmetric $(1, 1)$-tensor such that the corresponding 2-form $g(F \cdot, \cdot)$ is closed. So, geodesics describe the motion of particles that are not
experiencing any force, that is, for $F \equiv 0$. This situation appears in electromagnetism theory, and other examples arise associated to several geometries, for instance the potential gauge on a principal circle bundle $P(M^n, S^1)$, the Kähler (uniform) magnetic field on a Kähler manifold (see for instance [3]), or the contact magnetic field on a Sasakian manifold [7]. The main purpose is to solve the equation and to study the underlying geometries. From the dynamical perspective, many authors consider the topological entropy or the Anosov property of magnetic flows (see for instance [6]).

The magnetic trajectories are quite different from geodesics. In spaces of dimension two [8, 23] where the magnetic field is given by $q dA$, being $dA$ the area element, the magnetic trajectories follow: on the Euclidean plane $\mathbb{R}^2$ they are circles, on the sphere $S^2$ they are small circles, and on the hyperbolic plane the trajectories are either closed when $|q| \geq 1$, or open in the rest. These results were extended in different directions. For instance, trajectories corresponding to magnetic fields defined as scalar multiples of the Kähler form on a Kähler manifold were studied in [1, 2], on an almost Kähler manifold in [13]. Other spaces were considered, such as the sphere [22], the torus [24], and those related to Lie groups and their actions, for instance in [4, 5].

For 2-step nilmanifolds, the topic was recently considered by Epstein et al. [12] and Munteanu and Nistor in [20]. Epstein, Gornet and Mast concentrate in periodic magnetic geodesics on Heisenberg manifolds, obtaining an analysis of left-invariant exact magnetic flows. They search for closed trajectories and the corresponding energy level where they occur, and they determine the Mañé critical value. Indeed there are previous and important works describing closed geodesics on 2-step nilmanifolds such as [9, 10, 17, 18]. On the other hand, Munteanu and Nistor in [20] describe a magnetic trajectory associated to a quasi Sasakian structure as a Frenet helix of maximum order five.

In this paper, we concentrate the attention to magnetic trajectories associated to a Lorentz force on a given 2-step nilpotent Lie group equipped with a left-invariant metric $(N, \langle \cdot, \cdot \rangle)$ and associated compact quotients. We have two main purposes:

(i) To describe the solutions of the magnetic equation;
(ii) To determine closeness conditions of magnetic trajectories on compact quotients.

Indeed the first goal ask for solving the corresponding differential equation and a geometrical analysis of the solution curve, the second one connects with dynamical questions.

After the Euclidean spaces, the 2-step nilpotent Lie groups, have a simple algebraic structure but develop a very interesting geometry to study. They are the counterpart of 2-step Lie algebras, that satisfy $[[U, V], W] = 0$ for all $U, V, W$ in the Lie algebra, namely $\mathfrak{n}$. When equipped with a metric, such Lie algebra $\mathfrak{n}$ admits an orthogonal decomposition

$$\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}, \quad \text{where } \mathfrak{v} = \mathfrak{z}^\perp,$$

and $\mathfrak{z}$ denotes the center of the Lie algebra. This decomposition is deeply related to the geometry of the corresponding 2-step nilpotent Lie group $(N, \langle \cdot, \cdot \rangle)$ whenever the metric is left-invariant [11].
We focus on Lorentz forces on the Lie group equipped with a left-invariant metric \((N, \langle , \rangle)\) which are invariant by translations on the left. In this way they are determined by their values at the Lie algebra level. So, one has a Lorentz force \(F : n \rightarrow n\) and there is a magnetic equation for curves on \(n\) (Lemma 3.2). We say that \(F\) is of type I if it preserves the decomposition (2), while it is of type II if \(F(v) \subseteq z\) and \(F(z) \subseteq v\). The first observation is that every \(F : n \rightarrow n\) uniquely decomposes as \(F = F_1 + F_2\) where \(F_1\) is of type I, while \(F_2\) is of type II.

The study at the Lie algebra level enables to prove a geometrical property of magnetic trajectories \(\gamma\) corresponding to Lorentz forces of type I: if \((N, \langle , \rangle)\) has no Euclidean factor, there exists a vector field \(\xi \in \chi(M)\) such that the angle between \(\gamma'\) and \(\xi\) is constant along \(\gamma\) (Proposition 3.3). This is a weaker condition to that one of Frenet helix in [20].

We obtain a description of the magnetic trajectories on the Lie algebra \(n\) for Lorentz forces corresponding to exact forms (Theorem 3.4), by identifying them in the set of all differentiable curves on \(n\).

On the Lie group equipped with a left-invariant metric \((N, \langle , \rangle)\), the first step is to write clearly the magnetic equation. This is done by making use of the exponential map. We concentrate on this equation for Lorentz forces of type I, by getting the explicit formula for the magnetic trajectories (Theorem 3.8). The proof makes use of a similar reasoning to that one used by Eberlein in [11] to get the formulas for geodesics. In Ovando and Subils [21], the authors study left-invariant 2-forms on 2-step nilpotent nilmanifolds obtaining the next obstruction: If the 2-step nilpotent Lie algebra \((n, \langle , \rangle)\) is non-singular and its dimension satisfies: \(\dim n > 3 \dim z\), then any skew-symmetric map \(F : n \rightarrow n\) giving rise to a magnetic field, preserves the decomposition (2). So, for Lie groups satisfying this, all magnetic trajectories are described by Theorem 3.8.

Next, we work out some examples. We consider a left-invariant Lorentz force in the Heisenberg Lie group of dimension five that preserves the decomposition \(h_5 = v \oplus z\). This Lorentz force corresponds to a skew-symmetric derivation of \(h_5\). It is important to notice that such Lorentz force belongs to a family which strictly contains the Lorentz forces associated to quasi-Sasakian structures, among which one has Sasakian structures, cosymplectic structures, etc. as explained in [20]. See also [7] for magnetic trajectories on Sasakian manifolds of dimension 3D, where in addition the existence of periodic magnetic flowlines is investigated.

We study the existence of periodic magnetic trajectories and we find out differences to the results in [12]. In our case, there exist closed magnetic geodesics for every energy level, whenever the magnetic field is non-exact.

Finally, we study magnetic trajectories for Lorentz forces of type II on the Heisenberg Lie group of dimension three. By making use of geometrical properties of the magnetic equation and trajectories, the general case reduces to the study for a particular Lorentz force, \(F = -\text{ad}(e_2) + \text{ad}(e_2)^*\). In this situation, we prove that solutions of the magnetic equation are related to elliptic integrals. Indeed, those solutions are very different from the ones obtained in Theorem 3.8, see also Example 3.10. But according to Greenhill in [14], elliptic integrals were applied in electromagnetism theory in the 19th century, a fact known by Legendre and other mathematicians at that time.
On the induced compact nilmanifolds $\Gamma \backslash H_3$, we show conditions for periodicity of magnetic trajectories. The key is the following result obtained for any 2-step nilpotent Lie group

Lemma 4.5: Let $\lambda = \exp(W_1 + Z_1)$ be any element in the 2-step nilpotent Lie group $(N, \langle , \rangle)$. If a left-invariant Lorentz force $F$ of type II admits a $\lambda$-periodic trajectory then $W_1 \in \text{Ker} F$.

In a forthcoming paper, the authors will study the magnetic trajectories for any left-invariant Lorentz force $F$ on the Heisenberg Lie group $H_3$. The study is much more complicated but it seems to be close to the situation of Lorentz forces of type II. On the other hand, in [21] the authors proved that the only $H$-type Lie groups admitting Lorentz forces of type II are $H_3$, the complex Heisenberg Lie group of dimension six, and the quaternionic Heisenberg Lie group of dimension seven. Thus there are no much more examples of non-singular Lie algebras admitting Lorentz forces of type II. They constitute very interesting examples to study in the future.

In the next section, we recall main features of 2-step nilpotent Lie groups endowed with a left-invariant metric. In the following sections, we proceed with the study of magnetic trajectories by proving main results and working out the examples.

2 Lie Groups of Step Two with a Left-Invariant Metric

A Lie group is called 2-step nilpotent if its Lie algebra is 2-step nilpotent, that is, the Lie bracket satisfies $[[U, V], W] = 0$ for all $U, V, W \in n$. Throughout this paper Lie groups, so as their Lie algebras are considered over $\mathbb{R}$.

Whenever $N$ is simply connected, the Lie group $N$ is diffeomorphic to $\mathbb{R}^n$ and the exponential map is a diffeomorphism so that the multiplication map on $N$ obeys the following relation

$$ \exp(V) \exp(W) = \exp(V + W + \frac{1}{2}[V, W]), \quad \text{for all } U, V, W \in n. $$

Example 2.1 The smallest dimensional 2-step nilpotent Lie group is the Heisenberg Lie group $H_3$. It has dimension three and its Lie algebra is spanned by vectors $e_1, e_2, e_3$ satisfying the non-trivial Lie bracket relation $[e_1, e_2] = e_3$.

The Lie group $H_3$ can be modeled on $\mathbb{R}^3$ equipped with the product operation given by

$$(v_1, z_1)(v_2, z_2) = (v_1 + v_2, z_1 + z_2 + \frac{1}{2}v_1^t J v_2),$$

where $v_i = (x_i, y_i)$, $i=1,2$ and $J : \mathbb{R}^2 \to \mathbb{R}^2$ is the linear map $J(x, y) = (y, -x)$. By using this, usual computations show that a basis of left-invariant vector fields is given at $p = (x, y, z)$ by

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\[ e_1(p) = \partial_x - \frac{1}{2} y \partial_z, \quad e_2(p) = \partial_y + \frac{1}{2} x \partial_z, \quad e_3(p) = \partial_z. \]

Another presentation of the Heisenberg Lie group is given by $3 \times 3$-triangular real matrices with 1’s on the diagonal with the usual multiplication of matrices.

A Riemannian metric $\langle \cdot, \cdot \rangle$ on the Lie group $N$ is called left-invariant if translations on the left by elements of the group are isometries. Thus, a left-invariant metric is determined at the Lie algebra level $\mathfrak{n}$, usually identified with the tangent space at the identity element $T_e N$. The metric on the Lie algebra, also denoted $\langle \cdot, \cdot \rangle$, determines an orthogonal decomposition as vector spaces on the Lie algebra as in Eq. (2):

\[ \mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}, \quad \text{where} \quad \mathfrak{v} = \mathfrak{z}^\perp \]

and $\mathfrak{z}$ denotes the center of $\mathfrak{n}$. The subspaces $\mathfrak{v}$ and $\mathfrak{z}$ induce left-invariant distributions on $N$, denoted by $\mathcal{V}$ and $\mathcal{Z}$.

The decomposition in Eq. (2) induces the skew-symmetric maps $j(Z) : \mathfrak{v} \rightarrow \mathfrak{v}$, for every $Z \in \mathfrak{z}$, implicitly defined by

\[ \langle Z, [V, W] \rangle = \langle j(Z)V, W \rangle \quad \text{for all} \quad Z \in \mathfrak{z}, \ V, W \in \mathfrak{v}. \quad (3) \]

Note that $j : \mathfrak{z} \rightarrow \mathfrak{so}(\mathfrak{v})$ is a linear map. Let $C(\mathfrak{n})$ denote the commutator of the Lie algebra $\mathfrak{n}$. One has the splitting

\[ \mathfrak{z} = C(\mathfrak{n}) \oplus \ker(j) \]

as orthogonal direct sum of vector spaces. In fact,

- Since $\langle Z, [U, V] \rangle = 0$ for all $U, V \in \mathfrak{v}$ and $Z \in \ker(j)$, then $\ker(j) \perp C(\mathfrak{n})$.
- $\dim \mathfrak{z} = \dim \ker(j) + \dim C(\mathfrak{n})$.
- The restriction $j : C(\mathfrak{n}) \mapsto \mathfrak{so}(\mathfrak{v})$ is injective.

See the proof of the next result in Proposition 2.7 in [11].

**Proposition 2.2** [11] Let $(N, \langle \cdot, \cdot \rangle)$ denote a 2-step nilpotent Lie group with a left-invariant metric. Then

- The subspaces $\ker j$ and $C(\mathfrak{n})$ are commuting ideals in $\mathfrak{n}$.
- Let $E = \exp(\ker(j))$. Then $E$ is the Euclidean de Rham factor of $N$ and $N$ is isometric to the Riemannian product of the totally geodesic submanifolds $E$ and $\bar{N}$ where $\bar{N} = \exp(\mathfrak{v} \oplus C(\mathfrak{n}))$.

**Example 2.3** Let $\mathfrak{h}_3$ denote the Heisenberg Lie algebra of dimension three with basis $e_1, e_2, e_3$ as in Example 2.1. Take the metric so that this basis is orthonormal. It is not hard to see that the center is the subspace $\mathfrak{z} = \text{span}(e_3)$, while its orthogonal complement is the subspace $\mathfrak{v} = \text{span}(e_1, e_2)$ and moreover the map $j : \mathfrak{z} \rightarrow \mathfrak{so}(\mathfrak{v})$ is generated by

\[ j(e_3) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \]
in the basis $e_1, e_2$ of $\mathfrak{v}$.

Let $\nabla$ denote the Levi-Civita connection on the 2-step Lie group $(N, \langle , \rangle)$. Since the metric is invariant by left-translations, for $X, Y$ left-invariant vector fields one has the following formula for the covariant derivative:

$$\nabla_X Y = \frac{1}{2} \left( [X, Y] - \text{ad}(X)^*(Y) - \text{ad}(Y)^*(X) \right)$$

where $\text{ad}(X)^*, \text{ad}(Y)^*$ denote the adjoints of $\text{ad}(X), \text{ad}(Y)$ respectively. Thus, one obtains

$$\begin{cases}
(i) \quad \nabla_Z \tilde{Z} = 0 & \text{for all } Z, \tilde{Z} \in \mathfrak{z}, \\
(ii) \quad \nabla_Z X = \nabla_X Z = -\frac{1}{2} j(Z)X & \text{for all } Z \in \mathfrak{z}, X \in \mathfrak{v}, \\
(iii) \quad \nabla_X \tilde{X} = \frac{1}{2} [X, \tilde{X}] & \text{for all } X, \tilde{X} \in \mathfrak{v}.
\end{cases}$$

Furthermore the isometry group of the nilpotent Lie group $(N, \langle , \rangle)$ is the semidirect product

$$\text{Iso}(N) = H \ltimes N,$$

where $N$ is the nilradical of $\text{Iso}(N)$ and $H$ is the group of orthogonal automorphisms,

result that was proved in 1963 by Wolf [26] (see also [25]). The action of the subgroup $H$ on the ideal $N$ is given by $f \cdot L_n = L_{f(n)}$ for every left translation $L_n$ with $n \in N$, and every $f \in H$.

Whenever $N$ is simply connected, the set of orthogonal automorphisms on $H$ is in correspondence with the set of orthogonal authomorphisms of $\mathfrak{n}$. Thus, the Lie algebra of the isometry group, denoted by $\text{iso}(N)$, is the direct sum as vector spaces $\text{iso}(N) = \mathfrak{h} \oplus \mathfrak{n}$, where $\mathfrak{h}$ is the Lie subalgebra of $H$ and $\mathfrak{n}$ an ideal being the Lie algebra of $N$. The Lie subalgebra $\mathfrak{h}$ consists of skew-symmetric derivations,

$$\mathfrak{h} = \{ d : \mathfrak{n} \to \mathfrak{n} : d[X, Y] = [dX, Y] + [X, dY] \text{ and } \langle dX, Y \rangle + \langle X, dY \rangle = 0 \}.$$

**Example 2.4** In the case of the Heisenberg Lie algebra equipped with its canonical metric as above, any skew-symmetric derivation has a matrix in the basis $e_1, e_2, e_3$ of the form

$$\begin{pmatrix}
0 & -b & 0 \\
b & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad b \in \mathbb{R}.$$

Thus the isometry Lie algebra $\text{iso}(\mathfrak{h}_3) = \mathbb{R} \oplus \mathfrak{h}_3$ where any $b \in \mathbb{R}$ corresponds to a derivation as above. Furthermore, the Lie algebra $\text{iso}(\mathfrak{h}_3)$ is isomorphic to the oscillator Lie algebra of dimension four. The isometry group of the Heisenberg Lie group $(H_3, \langle , \rangle)$ is $\text{Iso}(H_3) = O(2) \ltimes H_3$, where the action is explicitly given by

$$A \cdot (V, Z) = (A(V), \det(A)Z), \quad \text{for } A \in O(2), V \in \mathfrak{v}, Z \in \mathfrak{z}.$$
Definition 2.5 A 2-step nilpotent real Lie algebra $\mathfrak{n}$ with center $\mathfrak{z}$ is called non-singular if $\text{ad}(X) : \mathfrak{n} \to \mathfrak{z}$ is onto for any $X \notin \mathfrak{z}$ [11]. The corresponding 2-step nilpotent Lie group will be called non-singular.

See the next examples of non-singular Lie algebras.

Example 2.6 Heisenberg Lie algebras. Let $n \geq 1$ be any integer and let $X_1, Y_1, X_2, Y_2, \ldots, X_n, Y_n$ be any basis of a real vector space $\mathfrak{v}$ isomorphic to $\mathbb{R}^{2n}$. Let $Z$ be an element generating a one-dimensional space $\mathfrak{z}$. Define a Lie bracket by $[X_i, Y_i] = -[Y_i, X_i] = Z$ and the other Lie brackets by zero. The Lie algebra $\mathfrak{h}_{2n+1} = \mathfrak{v} \oplus \mathfrak{z}$ is the $(2n+1)$-dimensional Heisenberg Lie algebra.

Example 2.7 Quaternionic Heisenberg Lie algebras. Let $n \geq 1$ be any integer. For each integer $1 \leq i \leq n$, let $\mathbb{H}^i$ be a four-dimensional real vector spaces with basis $X_i, Y_i, V_i, W_i$. Let $\mathfrak{z}$ denote a three-dimensional real vector space with basis $Z_1, Z_2, Z_3$. Consider the vector space direct sum $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$, where $\mathfrak{v} = \bigoplus_i \mathbb{H}^i$. Define a Lie bracket on the Lie algebra $\mathfrak{n}, [\cdot, \cdot]$, that is $\mathbb{R}$-bilinear and skew-symmetric with non-trivial relations as follows:

$$
[X_i, Y_i] = Z_1 \quad [X_i, V_i] = Z_2 \quad [X_i, W_i] = Z_3 \quad \text{for } 1 \leq i \leq n, \\
[V_i, W_i] = Z_1 \quad [Y_i, W_i] = -Z_2 \quad [Y_i, V_i] = Z_3 \quad \text{for } 1 \leq i \leq n.
$$

The resulting Lie algebra is called the quaternionic Heisenberg Lie algebra of dimension $4n + 3$.

Once the 2-step nilpotent Lie algebra $\mathfrak{n}$ is equipped with a metric, one has the corresponding maps $j(Z) : \mathfrak{v} \to \mathfrak{v}$ defined in Equation (3). The non-singularity property is equivalent to the condition that any map $j(Z) : \mathfrak{v} \to \mathfrak{v}$ is non-singular for every $Z \in \mathfrak{z}$. And this condition is independent of the metric.

Non-singular Lie algebras are also known as fat algebras because they are the symbol of fat distributions (see [19]).

More generally, a 2-step nilpotent Lie algebra $\mathfrak{n}$ is said

- **Almost non-singular** if there are elements $Z, \tilde{Z} \in \mathfrak{z}$ such that $j(Z)$ is non-singular but $j(\tilde{Z})$ is singular.
- **Singular** if any map $j(Z)$ is singular for every $Z \in \mathfrak{z}$.

A family of non-singular Lie algebras is provided by H-type Lie algebras, which are defined as follows.

Let $(\mathfrak{n}, \langle \cdot, \cdot \rangle)$ denote a 2-step nilpotent Lie algebra equipped with a metric. If the map $j(Z) : \mathfrak{v} \to \mathfrak{v}$ is orthogonal for every $Z \in \mathfrak{z}$ with $\langle Z, Z \rangle = 1$, then the Lie algebra $\mathfrak{n}$ is a Lie algebra of type $H$ [16] (also known as $H$-type Lie algebras). Equivalently, the 2-step nilpotent Lie algebra $\mathfrak{n}$ is of type $H$ if and only if

$$
j(Z)^2 = -(Z, Z)Id, \quad \text{for every } Z \in \mathfrak{z},
$$

which is equivalent to $j(Z)j(\tilde{Z}) + j(\tilde{Z})j(Z) = -2\langle Z, \tilde{Z} \rangle Id$, for $Z, \tilde{Z} \in \mathfrak{z}$. By making use of this identity one can also prove that

$$
[X, j(Z)X] = \langle X, X \rangle Z
$$

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for every \( X \in \mathfrak{v} \) and \( Z \in \mathfrak{z} \). The real, complex and quaternionic Heisenberg algebras are examples of Lie algebras of type \( H \).

### 3 Trajectories for Left-Invariant Magnetic Fields

In this section, we write explicitly the magnetic equation associated to a left-invariant Lorentz force on a 2-step nilpotent Lie group endowed with a left-invariant metric. We derive the solution for invariant Lorentz forces preserving the decomposition in Equation (2).

A Lorentz force on a Riemannian manifold \((M, g)\) is a \((1, 1)\)-tensor \( F : TM \to TM \), which is skew-symmetric and such that the associated 2-form, \( \Omega_F \), is closed:

\[
\Omega_F(U, V) = g(FU, V), \quad \text{for all} \quad U, V \in \chi(M).
\]

The closeness condition will impose restrictions to the 2-form as seen in [21].

On any Riemannian manifold \((M, g)\) with Levi-Civita connection \( \nabla \), a solution curve of the magnetic equation in (1), namely \( \gamma \), has constant velocity. This follows easily from the next computation

\[
\frac{d}{dt} g(\gamma'(t), \gamma'(t)) = 2g(\nabla_{\gamma'(t)}\gamma'(t), \gamma'(t)) = 2g(F\gamma'(t), \gamma'(t)) = 0.
\]

However a reparametrization could not be a solution. In fact, assume \( g(\gamma'(t), \gamma'(t)) = E \neq 0 \) and take \( \tau(t) = \gamma(t/E) \), then \( \tau'(t) = 1/E \gamma'(t/E) \) so that one has \( \nabla_{\tau'(t)}\tau'(t) = 1/E^2 F\gamma'(t/E) \). On the other side \( F\tau'(t) = 1/E F\gamma'(t/E) \).

Now, let \( \psi : M \to M \) denote an isometry and let \( F \) denote a Lorentz force on the Riemannian manifold \( M \). If \( F \) commutes with the differential \( d\psi \), then \( \psi \circ \gamma \) is a magnetic trajectory whenever \( \gamma \) is. In fact

\[
d\psi \nabla_{\gamma'}\gamma' = \nabla_{d\psi\gamma'}d\psi\gamma' = F \circ d\psi\gamma' = d\psi \circ F\gamma'.
\]

Moreover take a non-trivial number \( r \in \mathbb{R} \) and let \( \gamma \) denote a magnetic solution for the Lorentz force \( F \). Write \( \sigma(t) = \gamma(rt) \). Then \( \sigma \) is magnetic solution for the Lorentz force \( rF \):

\[
\nabla_{\sigma'(t)}\sigma'(t) = r^2\nabla_{\gamma'(rt)}\gamma'(rt) = r^2 F\gamma'(rt) = rF\sigma'(t).
\]

Analogous computations prove the next lemma.

**Lemma 3.1** Let \((M, g)\) denote a Riemannian manifold with Levi-Civita connection \( \nabla \). Let \( F \) denote a Lorentz force on \( M \).

(i) If \( \psi : M \to M \) is an isometry that preserves \( F \): \( d\psi \circ F = F \circ d\psi \), then \( \psi \circ \gamma \) is a magnetic trajectory (for \( F \)).

(ii) If \( \gamma \) is a magnetic trajectory for the Lorentz force \( F \), then \( (\psi \circ \gamma)(rt) \) is a magnetic trajectory for the Lorentz force \( d\psi \circ rF \circ d\psi^{-1} \), for any \( r \in \mathbb{R} \) and for any isometry \( \psi : M \to M \).
Let $(N, \langle , \rangle)$ denote a Lie group equipped with a left-invariant metric and let $\nabla$ be the corresponding Levi-Civita connection for the metric. Let $F$ be a skew-symmetric endomorphism on $TN$ which is left-invariant, that is $F \circ dL_p = dL_p \circ F$. In this situation the map $F$ is determined by its values on $T_eN \equiv \mathfrak{n}$. Assume that $F$ is a left-invariant Lorentz force, that is, it gives rise to a left-invariant closed 2-form. This says:

$$\langle [X, Y], FU \rangle + \langle [Y, U], FX \rangle + \langle [U, X], FY \rangle = 0 \quad \text{for all } X, Y, U \in \mathfrak{n}.$$ 

For any magnetic trajectory $\gamma : I \to N$, write $\gamma' = dL_{\gamma(t)}x(t)$ where $x(t)$ is a curve at the Lie algebra. Since the metric and $F$ are invariant by translations on the left, we have

$$\nabla_{\gamma'(t)}(\gamma'(t)) = dL_{\gamma(t)}(x'(t) + \nabla x(t)) = qF \circ dL_{\gamma(t)}x(t) = qdL_{\gamma(t)}x(t),$$

for $q \in \mathbb{R}$, and where we denote also by $\nabla$ the $\mathbb{R}$-bilinear map on $\mathfrak{n}$ determined by the Levi-Civita connection, sometimes called the Levi-Civita connection on the Lie algebra $\mathfrak{n}$. By making use of the Koszul formula, one proves that $\nabla x(t) = - ad^*(x(t))x(t)$, where $ad^*(x)$ denotes the adjoint of $ad(x) : \mathfrak{n} \to \mathfrak{n}$ with respect to the metric. Thus, it is clear that magnetic trajectories are determined at the identity, furthermore, by curves $x : I \to \mathfrak{n}$ satisfying the equation

$$x'(t) = ad^*(x(t))(x(t)) + qFx(t), \quad \text{for } q \in \mathbb{R} - \{0\}.$$ 

Clearly, for $F \equiv 0$ one obtains the Euler equation.

**Lemma 3.2** Let $(N, \langle , \rangle)$ denote a Lie group equipped with a left-invariant metric and left-invariant Lorentz force $F$. The magnetic equation at the Lie algebra level is given by:

$$x'(t) = ad^*(x(t))(x(t)) + qFx(t), \quad \text{ad}^*(x) \text{is the adjoint of ad}(x) \text{ w.r.t. } \langle , \rangle. \quad (4)$$

Assume now that $(N, \langle , \rangle)$ denotes a 2-step nilpotent Lie group with a left-invariant metric. Let $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$ as in Eq. (2), denote the corresponding Lie algebra and let $F : \mathfrak{n} \to \mathfrak{n}$ be a skew-symmetric linear map giving rise to a closed left-invariant 2-form on $N$.

We say that $F$ is of

(i) **Type I** if it preserves the decomposition $\mathfrak{v} \oplus \mathfrak{z}$, while it is of

(ii) **Type II** if $F(\mathfrak{v}) \subseteq \mathfrak{z}$ and $F(\mathfrak{z}) \subseteq \mathfrak{v}$.

Any Lorentz force $F$ on the 2-step nilpotent Lie algebra $\mathfrak{n}$, admits a decomposition $F = F_1 + F_2$, where $F_1$ is a Lorentz force of type I, while $F_2$ is of type II, see more details in [21].

The closeness condition for $F_1$ follows

$$F_1(\mathfrak{z}) \subseteq \ker(j),$$
while for $F_2$ follows

$$\langle [U, V], FW \rangle + \langle [V, W], FU \rangle + \langle [W, U], FV \rangle = 0,$$

for all $U, V, W \in \mathfrak{v}$.

In particular, whenever $F$ is of type I and the Lie group $N$ has no Euclidean factor, the closeness condition says that $F_{\mathfrak{h}} \equiv 0$.

Let $x : I \to \mathbb{R}$ denote a curve on the 2-step nilpotent Lie algebra $\mathfrak{n}$. Write $x(t) = v(t) + z(t) \in \mathfrak{v} \oplus \mathfrak{z}$ with respect to the orthogonal decomposition considered in (2).

For the Lorentz force $F : \mathfrak{n} \to \mathfrak{n}$, the curve $x$ satisfies the magnetic equation (4) if and only if for the curves $v : I \to \mathfrak{v}$ and $z : I \to \mathfrak{z}$, where $I$ is a real interval, $I \subseteq \mathbb{R}$, it holds

$$v'(t) + z'(t) = j(z(t))v(t) + qFv(t). \quad (5)$$

Recall that a curve $\gamma$ on an almost contact Riemannian manifold $M$ is called a slant curve whenever the angle between $\gamma'(t)$ and the Reeb vector is constant along $\gamma$ (see [15]). By generalizing this notion, we shall say that the curve $\gamma$ on a Riemannian manifold is a slant curve if there exists a field $\xi \in \chi(M)$ such that the angle between $\gamma'(t)$ and $\xi$ is constant along $\gamma$.

**Proposition 3.3** Let $(N, \langle, \rangle)$ denote a 2-step nilpotent Lie group without Euclidean factor, equipped with a left-invariant metric and left-invariant Lorentz force $F$ of type I, that is $F(\mathfrak{v}) \subseteq \mathfrak{v}$ and $F(\mathfrak{z}) \subseteq \mathfrak{z}$. Then magnetic trajectories are slant curves.

**Proof** The fact that $F$ induces a closed 2-form, implies that $F(Z) = 0$ for all $Z \in \mathfrak{z}$. Thus the Eq. (5) written in terms of the decomposition $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$ says

$$\begin{cases} v'(t) = j(Z_0)v(t) + qFv(t) \\ z'(t) = 0, \end{cases}$$

where $v(0) = V_0 \in \mathfrak{v}$ and $z(0) = Z_0 \in \mathfrak{z}$. Thus, taking any $\xi \in \mathfrak{z}$ we have that $c = \langle Z_0, \xi \rangle = \langle \gamma'(t), \xi \rangle$ for some $c \in \mathbb{R}$.

By abuse, we shall say that a magnetic trajectory is exact if it is a solution of the magnetic equation for an “exact” magnetic field, that is, an exact 2-form. Exact 2-forms are in correspondence with maps $F : \mathfrak{n} \to \mathfrak{n}$ of the form $F = j(\tilde{Z})$ for some $\tilde{Z} \in C(\mathfrak{n})$. In fact, by definition $\Omega$ is exact if and only if $\Omega = d\theta$ for some 1-form $\theta$. Thus, there exist $V + Z \in \mathfrak{v} \oplus \mathfrak{z}$ such that $\theta(\cdot) = \langle V + Z, \cdot \rangle$. And it holds that $d\theta(\cdot, \cdot) = \langle Z, [\cdot, \cdot] \rangle$ which gives the maps $j(Z) : \mathfrak{v} \to \mathfrak{v}$ defined in Eq. (3). Notice that any $j(Z)$ is a Lorentz force of type I, trivially extended to the center.

The following theorem characterizes the magnetic trajectories corresponding to exact left-invariant magnetic fields among the set of differentiable curves on $\mathfrak{n}$.

**Theorem 3.4** Let $(N, \langle, \rangle)$ denote a 2-step nilpotent Lie group equipped with a left-invariant metric. Let $y(t) = v(t) + z(t)$ denote a curve on the Lie algebra $\mathfrak{n}$. The following statements are equivalent:

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(i) \( y(t) \) is a magnetic trajectory for \( F = j(\tilde{Z}) \);
(ii) \( y(t) = x(t) - q\tilde{Z} \) where \( x(t) \) is the geodesic with initial condition \( X_0 + Z_0 + q\tilde{Z} \) for some \( q \in \mathbb{R} - \{0\} \);
(iii) The curve \( y(t) = v(t) + z(t) \) in \( n \) satisfies:
   
   (a) \( z(t) \) is constant, namely \( z(t) = Z_0 \),
   (b) \( v'(t) = Rv(t) \), with \( R \in \mathfrak{so}(v) \), and
   (a) \( R \) belongs to the image of \( j : R \in \text{Image}(j) \).

**Proof** Let \( \gamma : I \rightarrow N \) be a curve on the Lie group \( N \) such that \( \gamma'(t) = dL_{\gamma(t)}y(t) \), for the curve \( y(t) \) on the Lie algebra \( n \). By using the orthogonal decomposition of \( n \) in (2), any magnetic trajectory on \( n \) must satisfy an equation of the form

\[
v'(t) = j(z(t))v(t) + qFv(t), \quad z'(t) = F_3v(t),
\]

where \( F : n \rightarrow n \) is skew-symmetric with respect to \( \langle , \rangle \) and it gives rise to a closed two form \( (F, \cdot) \).

(i) \( \implies \) (ii) Assume that the Lorentz force \( F \) corresponds to an exact 1-form, that is, \( F = j(\tilde{Z}) \). Let \( y(t) \) denote the magnetic trajectory on \( n \) with initial condition \( v(0) + z(0) = X_0 + Z_0 \). Then it satisfies the corresponding magnetic equation in (5).

Let \( x(t) \) denote the unique solution of the geodesic equation on \( n \), with initial condition \( X_0 + Z_0 + q\tilde{Z} \):

\[
v'(t) = j(z(t))v(t), \quad z'(t) = 0,
\]

Then \( x(t) - q\tilde{Z} \) coincides with \( y(t) \). In fact, let \( \beta(t) = x(t) - q\tilde{Z} \). The curve \( \beta \) satisfies

- \( \beta(0) = X_0 + Z_0 \)
- \( \beta'(t) = x'(t) \) with \( x(t) = v(t) + z(t) \) satisfying (7), which is
  \[
v'(t) = j(Z_0 + q\tilde{Z})v(t), \quad z'(t) = 0. \quad \text{Thus } \beta \text{ also satisfies } v'(t) = j(Z_0 + q\tilde{Z})v(t).
\]

Since \( F = j(\tilde{Z}) \), the magnetic trajectory with initial condition \( X_0 + Z_0 \) satisfies the Eq. (6), which takes the form \( v'(t) = j(Z_0 + q\tilde{Z})v(t), \quad z(t) = 0 \).

Thus by existence and uniqueness of the solution, it follows that \( \beta(t) \) coincides with \( y(t) \).

(ii) \( \implies \) (iii) Assume now that we have the curve on \( n \) given by \( y(t) = x(t) - q\tilde{Z} \) where \( x(t) \) is the geodesic with initial condition \( X_0 + Z_0 + q\tilde{Z} \). Thus,

\[
y'(t) = v'(t) + z'(t) = j(Z_0 + q\tilde{Z})v(t). \quad \text{Then } y(t) \text{ satisfies (iii). In fact, since } x(t) \text{ satisfies the geodesic equation on } n, \text{ one has } y'(t) = j(Z_0 - \tilde{Z})v(t) \text{ and } z'(t) = 0.
\]

Clearly, if we denote by \( R = j(Z_0 - \tilde{Z}) \), then \( R \in \text{Image}(j) \).

(iii) \( \implies \) (i) Let \( y(t) \) denote a curve satisfying the conditions in (iii). We shall see that \( y(t) \) is a magnetic trajectory.

Since \( z(t) = 0 \), then the curve satisfies \( y(t) = v(t) + Z_0 \) where \( Z_0 \in Z \). On the other hand, since \( R \in \text{Image}(j) \), take a non-trivial element in the commutator, \( \tilde{Z} \), such that \( R - j(Z_0) = j(\tilde{Z}) \). Then \( v \) and \( z \) satisfy the equations

\[
v'(t) = j(Z_0)v(t) + j(\tilde{Z})v(t), \quad z(t) = Z_0,
\]

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which is the magnetic equation with \( q = 1 \), and \( F = j(\tilde{Z}) \). This implies that \( F \) is exact. And this finishes the proof. \( \square \)

**Example 3.5** Consider the Heisenberg Lie algebra of dimension three, \( \mathfrak{h}_3 \), let \( x(t) \) denote a curve on \( \mathfrak{n} \) such that \( x(t) = v(t) + z(t) \in v \oplus \mathfrak{j} \). Take the Lorentz force given by \( j(\rho e_3) \) for some \( \rho \in \mathbb{R} \). Assume \( x(t) \) is a magnetic trajectory associated to \( j(\rho e_3) \) with \( x(0) = a e_1 + b e_2 + c e_3 \). As above, the curves \( v(t) \) and \( z(t) \) satisfy \( v'(t) = (c + q \rho) j(e_3) v(t) \), \( z(t) = ce_3 \), and this implies that the curve \( x(t) \) is given by:

\[
x(t) = e^{(c+q\rho)tj(e_3)}v_0 + ce_3, \quad \text{where } v_0 = u_1 e_1 + u_2 e_2.
\]

Notice that we assume \( \rho \neq 0 \) for the Lorentz force to be non-trivial.

As above, let \( F \) be a Lorentz force, that is, a \((1,1)\)-tensor which is skew-symmetric on \( TN \), left-invariant and it gives rise to a closed 2-form. Let \( F_v \) and \( F_3 \) denote the projections of \( F \) to \( v \) and \( \mathfrak{j} \) respectively, that is \( F(W) = F_v(W) + F_3(W) \in v \oplus \mathfrak{j} \).

Let \( \gamma : I \to N \) be a curve on the 2-step Lie group \( N \) and write \( \gamma(t) = \exp(X(t) + Z(t)) \) where \( X(t) \in v \) and \( Z(t) \in \mathfrak{j} \). We shall derive the conditions for \( \gamma \) to be a magnetic trajectory with \( \gamma(0) = e \) and \( \gamma'(0) = X_0 + Z_0 \).

Recall that \( d \exp_{\xi}(A\xi) = \frac{d}{dt}|_{t=0} \exp(\xi + tA) = d\exp_{\xi}(A + \frac{1}{2}[A, \xi]) \) to obtain the formulas in the next lemma (analogous to the geodesic Equation, see [11]).

**Lemma 3.6** Let \( \gamma : I \to \mathbb{R} \) be a curve on \((N, \langle, \rangle)\) given as \( \gamma(t) = \exp(X(t) + Z(t)) \), where \( \exp : \mathfrak{n} \to N \) denote the usual exponential. Then \( \gamma \) is a magnetic trajectory for the Lorentz force \( F \) if and only if the curves on \( \mathfrak{n} \) given by \( X(t) \) and \( Z(t) \) satisfy the following equations:

\[
\begin{align*}
X'' - j(Z' + \frac{1}{2}[X', X])X' &= q F_v(X' + Z' + \frac{1}{2}[X', X]) \\
Z'' + \frac{1}{2}[X'', X] &= q F_3(X' + Z' + \frac{1}{2}[X', X]).
\end{align*}
\]

(8)

A particular case of Eq. (8) occurs if \( F \) satisfies that \( F_3 \equiv 0 \), since \( F \) is skew-symmetric, one also has \( F_v(\mathfrak{j}) \equiv 0 \). In this case Eq. (8) above reduces to

\[
\begin{align*}
X'' - j(Z_0)X' &= q F_v(X') \\
Z' + \frac{1}{2}[X', X] &= Z_0
\end{align*}
\]

(9)

**Proposition 3.7** Let \( F \) denote a left-invariant Lorentz force on \( \mathfrak{n} \) and assume \( F_3 \equiv 0 \). Let \( \gamma(t) \) denote a magnetic trajectory with \( \gamma(0) = e \), \( \gamma'(0) = X_0 + Z_0 \in v \oplus \mathfrak{j} \). Then

\[
\gamma'(t) = dL_{\gamma(t)} e^{t(j(Z_0)+qF_v)} X_0 + Z_0, \quad \text{where } \gamma'(0) = X_0 + Z_0 \text{ for all } t \in \mathbb{R}, (10)
\]

and \( e^{t(j(Z_0)+qF_v)} = \sum_{n=0}^{\infty} \frac{t^n}{n!}(j(Z_0) + qF_v)^n \).

**Proof** Write \( \gamma(t) = \exp(X(t) + Z(t)) \) where \( X(t) \subset v \) and \( Z(t) \subset \mathfrak{j} \) for all \( t \in \mathbb{R} \). Then

\[
\gamma'(t) = d\exp_{\gamma(t)}(X'(t) + Z'(t))_{X(t)+Z(t)} = dL_{\gamma(t)}(X' + Z' + \frac{1}{2}[X', X]) = dL_{\gamma(t)}(X' + Z_0).
\]
By integrating the first equation of (9) we obtain $X'(t) = e^{t(j(Z_0) + qF_v)}X_0$, which finishes the proof. □

Assume now that the Lorentz force satisfies $F(v) \subset v$ and $F(\zeta) \subset \zeta$. In this case Eq. (8) reduces to

$$
\begin{align*}
\{ X'' - j(Z' + \frac{1}{2}[X', X])X' &= q F(X') \\
Z'' + \frac{1}{2}[X'', X] &= q F(Z' + \frac{1}{2}[X', X])
\}
\tag{11}
\end{align*}
$$

To solve the above equations, it suffices to consider the solutions passing through the identity, that is $\gamma(0) = e$. In fact, this is possible since the metric is invariant by translations on the left, so as the Lorentz force $F$.

Write $\gamma'(0) = X_0 + Z_0 \in v \oplus \zeta$, and suppose that $Z_0 = Z_0^\xi + Z_0^\eta$ where $Z_0^\xi \in C(n)$ and $Z_0^\eta \in \ker(j)$, with respect to the orthogonal splitting $\zeta = C(n) \oplus \ker(j)$.

The closeness condition for the Lorentz force $F$ implies that $F(\zeta) \subseteq \ker(j)$. Since $F$ is a skew-symmetric map one has

$$(F[U, V], Z) = -(F(Z), [U, V]) = 0, \quad \text{for all } U, V \in v, Z \in \zeta,$$

gives $F(C(n)) \equiv 0$. So, in particular $F(Z_0^\xi) = 0$ and $F(Z_0^\eta) \in \ker(j)$.

By writing $Z(t) = Z_\xi(t) + Z_\eta(t) \in C(n) \oplus \ker(j)$, the second equation in (11) is equivalent to

$$
\begin{align*}
Z_\xi'' + \frac{1}{2}[X'', X] &= 0 \quad \text{equivalently} \quad Z_\xi'' + \frac{1}{2}[X', X] = Z_\xi^\xi, \\
Z_\eta'' &= q F Z_\eta.'
\end{align*}
$$

Since $Z'(t) = Z_\xi'(t) + Z_\eta'(t)$ and $Z_\eta' \in \ker(j)$, the first equation in (11) reduces to

$$X'' - j(Z_0^\xi)X' = q F(X'). \tag{13}$$

Let $J : v \rightarrow v$ be the skew-symmetric map given by $J = j(Z_0^\xi) + q F_v$. And write $v = v_1 \oplus v_2$ where $v_1$ is the kernel of $J$ and $v_2$ is the orthogonal complement of $v_1$ in $v$. Note that $v_2$ is invariant by $J$ and the map $J$ is non-singular on $v_2$.

Let $\{i\theta_1, -i\theta_1, \ldots, i\theta_m, -i\theta_m\}$ be the distinct non-null eigenvalues of $J$ and decompose $v_2$ as an orthogonal direct sum $v = v_1 \oplus \ldots \oplus v_m$ where every subspace $v_l$ is invariant by $J$ and $J^2 = -\theta_l 1d$ on $v_l$. Write

$$X_0 = X_1 + X_2, \quad \text{where } X_1 \in v_1, X_2 \in v_2.$$

$$X_2 = \sum_{l=1}^m \xi_l, \quad \text{where } \xi_l \in v_l, \text{ for all } l = 1, \ldots, m.$$

Let $\ker(j) = \zeta_1 \oplus \zeta_2$ be a orthogonal splitting as vector spaces, where $\zeta_1 = \ker(j) \cap \ker(F_\zeta)$ and $\zeta_2 = \zeta_1^\perp$ is its orthogonal complement. Thus the map $F$ leaves the subspace $\zeta_2$ invariant and the map $F$ is non-singular on $\zeta_2$. Write

$$Z_0^\eta = Z_1^\eta + Z_2^\eta, \quad \text{where } Z_l^\eta \in \zeta_l, \quad l = 1, 2.$$
Theorem 3.8  Let \((N, \langle, \rangle)\) denote a 2-step nilpotent Lie group with a left-invariant metric. Let \(\gamma(t) = \exp(X(t) + Z(t))\) denote a magnetic trajectory through the identity with initial condition \(\gamma'(0) = X_0 + Z_0 \in \mathfrak{u} \oplus Z^1_0\), where \(Z_0 = Z_0^\mathfrak{n} + Z_0^\mathfrak{n} \in C(\mathfrak{n}) \oplus \text{ker}(j)\). Then, with respect to the notations above, one has

(i) \[X(t) = tX_1 + (e^{tJ} - \text{Id})J^{-1}X_2, \text{ with } J = j(Z_0^\mathfrak{n}) + qF_0 \text{ and}\]

(ii) \[Z(t) = tZ_1 + Z_2(t) \text{ where}\]

(a) \[Z_1(t) = Z_0^\mathfrak{n} + Z_1^\mathfrak{n} + \frac{1}{2}[X_1, (e^{tJ} + \text{Id})J^{-1}X_2] + \frac{1}{2} \sum_{i=1}^m [J^{-1}x_i, \xi_i],\]

(b) \[Z_2(t) \text{ is a function of uniformly bounded absolute value:}\]

\[
Z_2(t) = \frac{1}{q}(e^{tqF} - \text{Id})F^{-1}Z_2^\mathfrak{n} + [X_1, (e^{tJ} - \text{Id})J^{-1}X_2] + \frac{1}{2} e^{tJ}J^{-1}X_2, J^{-1}X_2 \]

\[- \frac{1}{2} \sum_{i \neq j}^m \frac{1}{\theta_i^{-1} - \theta_j^{-1}} \left(\left[e^{tJ}J\xi_i, e^{tJ}J^{-1}\xi_j\right] - \left[e^{tJ}\xi_i, e^{tJ}\xi_j\right]\right)\]

\[+ \frac{1}{2} \sum_{i \neq j}^m \frac{1}{\theta_i^{-1} - \theta_j^{-1}} \left(\left[J\xi_i, J^{-1}\xi_j\right] - \left[J\xi_i, \xi_j\right]\right)\]

Proof  Clearly \(X(t)\) is the solution of \(X''(t) = JX'(t)\), which is equivalent to Eq. (13). And in terms of \(\mathfrak{u}_t\) we shall have the solution \(X(t) = tX_1 + (e^{tJ} - \text{Id})J^{-1}X_2\). Now, the equation on the center gives, on the one hand:

\[Z_\mathfrak{n}(t) = tZ_\mathfrak{n} + \frac{1}{q}(e^{tqF} - \text{Id})F^{-1}Z_\mathfrak{n}^2.\]

And on the other hand,

\[Z_\mathfrak{n}'(t) = Z_0^\mathfrak{n} - \frac{1}{2} \left|[X_1, (e^{tJ} - \text{Id})J^{-1}X_2] + t[e^{tJ}X_2, X_1] + [e^{tJ}X_2, (e^{tJ} - \text{Id})J^{-1}X_2]\right|.

So, by computing we have that:

- The integral of \(Z_0^\mathfrak{n}\) is \(tZ_0^\mathfrak{n} + \text{constant}\),
- The integral of \([X_1, (e^{tJ} - \text{Id})J^{-1}X_2]\) is \([X_1, J^{-1}e^{tJ}J^{-1}X_2 - tJ^{-1}X_2]\) + constant,
- The integral of \(t[e^{tJ}X_2, X_1]\) is \(t[e^{tJ}J^{-1}X_2, X_1] - [J^{-1}e^{tJ}J^{-1}X_2, X_1]\) + constant.
- The integral of \([e^{tJ}X_2, J^{-1}X_2]\) is \([e^{tJ}J^{-1}X_2, J^{-1}X_2]\) + constant.

Finally we need information about the integral of \([e^{tJ}X_2, e^{tJ}J^{-1}X_2]\).

Notice that

\[ [e^{tJ}X_2, e^{tJ}J^{-1}X_2] = \sum_{i \neq j=1}^m [e^{tJ}\xi_i, e^{tJ}J^{-1}\xi_j] + \sum_{i=1}^m [\xi_i, J^{-1}\xi_i]. \]

The last terms come from the equality \([\xi_i, J^{-1}\xi_i] = [e^{tJ}\xi_i, e^{tJ}J^{-1}\xi_i]\) for all \(i\). In fact, let \(f_i(t) = [e^{tJ}\xi_i, e^{tJ}J^{-1}\xi_i]\) and derive with respect to \(t\) to obtain that \(f_i'(t) = 0\) for all \(t\). So that \(f_i\) is a constant function, \(f_i(t) = f_i(0)\).

Recall that \(e^{tJ}J = Je^{tJ}\) for all \(t \in \mathbb{R}\). Notice that \(J = -\theta_i^2 J^{-1}\) on \(\mathfrak{u}_t\). And make use of this information to prove that the functions \(X(t)\) and \(Z(t)\) are solutions of the magnetic equation (11) with the required initial conditions. \(\square\)

Remark 3.9 Let \((N, \langle, \rangle)\) denote a 2-step nilpotent Lie group equipped with a left-invariant metric. Let \(F = j(Z)\) be a left-invariant Lorentz force on \(N\). Indeed, the
magnetic equation for $F$ has the form (9) with $F_\nu = j(\tilde{Z})$. Thus the solution is given in theorem above with $J = j(Z_0^\nu + q\tilde{Z})$. These solutions correspond to exact magnetic fields.

**Example 3.10** Let $H_3$ denote the Heisenberg Lie group of dimension three endowed with the canonical left-invariant metric, that is, it makes the basis of left-invariant vectors $e_1, e_2, e_3$ to an orthonormal basis.

Let $\tilde{Z} = \rho e_3$ such that one gets the Lorentz force $\rho j(e_3)$ giving rise to an exact magnetic field.

Let $\gamma(t) = \exp(x(t)e_1 + y(t)e_2 + z(t)e_3)$ denote a magnetic trajectory passing through the identity, for the Lorentz force $\rho j(e_3)$. Assume $\gamma'(0) = x_0 e_1 + y_0 e_2 + z_0 e_3$.

In Epstein et al. [12] the authors obtain the explicit solutions as:

- If $z_0 - \rho \neq 0$ then the solution is
  \[
  \begin{pmatrix}
  x(t) \\
  y(t)
  \end{pmatrix} = \frac{1}{z_0 - \rho} \begin{pmatrix}
  \sin(t(z_0 - \rho)) & -1 + \cos(t(z_0 - \rho)) \\
  1 - \cos(t(z_0 - \rho)) & \sin(t(z_0 - \rho))
  \end{pmatrix}
  \]
  and for $v_0 = x_0 e_1 + y_0 e_2$ set
  \[
  z(t) = z_0 + \frac{||v_0||^2}{2(z_0 - \rho)} t - \frac{||v_0||^2}{2(z_0 - \rho^2)} \sin(t(z_0 - \rho)).
  \]

- If $z_0 = \rho$ the solution is $\gamma(t) = \exp(t(x_0 e_1 + y_0 e_2 + z_0 e_3))$.

In Inoguchi et al. [15] the authors prove that for a Lorentz force $F$ associated to a quasi-Sasakian structure (in particular as above), a magnetic trajectory is a geodesic for an Okumura type connection (see Theorem 2).

**Remark 3.11** Any orthogonal automorphism on the 2-step nilpotent Lie group $(N, \langle \cdot, \cdot \rangle)$ is determined at the Lie algebra level.

Let $\phi : n \to n$ be an orthogonal automorphism of $n$ and $0 \neq r \in \mathbb{R}$. If the curve $\gamma : I \to N$, given as $\gamma(t) = \exp(X(t) + Z(t))$ is a magnetic trajectory through the identity for the Lorentz force $F$, then by Lemma 3.1 the curve $\gamma_{\phi,r}(t) = \exp(\phi(X(rt) + Z(rt)))$ is a magnetic trajectory through the identity for the Lorentz force $F_{\phi,r} = r \phi \circ F \circ \phi^{-1}$. Furthermore, whenever $F$ is of type I or type II then $F_{\phi,r}$ is of type I or type II, respectively (see Sect. 4).

Observe that an orthogonal automorphism $\phi : n \to n$ verifies $\phi(\tilde{z}) = \tilde{z}$, $\phi(v) = v$ and

$$\phi \circ J(Z) \circ \phi^{-1} = J(\phi(Z)) \quad \text{for every} \ Z \in \tilde{z}.$$

**Remark 3.12** In Ovando and Subils [21] the authors prove the following lemma that imposes obstructions to the existence of Lorentz forces of type II. Let $n$ denote a non-singular 2-step nilpotent Lie algebra such that $\dim n > 3 \dim \tilde{z}$. Then any closed 2-form on $n$ satisfies

$$\Omega(Z, U) = 0, \quad \text{for all} \ Z \in \tilde{z}, U \in n.$$
In particular, under these conditions, there only exist Lorentz forces of type I. Therefore, in those cases all magnetic trajectories are described by Theorem 3.8.

### 3.1 Magnetic Trajectories for Lorentz Forces of Type I

Here we shall study some magnetic trajectories in the Heisenberg Lie group of dimension five, $H_5$ (see Example 2.6), where we take a Lorentz force of type I associated to skew-symmetric derivations.

As seen in Proposition 3.3, any magnetic trajectory is a curve of constant slant.

**Proposition 3.13** Let $(H_{2n+1}, \langle , \rangle)$ denote a Heisenberg Lie group of dimension $2n+1$ with a left-invariant metric and a left-invariant Lorentz force of type I. The following conditions are equivalent

(i) $Fj(Z) - j(Z)F = 0$ for all $Z \in \mathfrak{z}$,

(ii) $F$ is a skew-symmetric derivation of $h_{2n+1}$.

(iii) $\nabla_Z F = 0$ for any $Z \in \mathfrak{z}$.

**Proof** Recall that a linear map $d : h_{2n+1} \to h_{2n+1}$ is a derivation whenever

$$d[X, Y] = [dX, Y] + [X, dY] \quad \text{for all } X, Y \in h_{2n+1}.$$ 

Moreover, since $d$ is skew-symmetric $dZ = 0$ for any $Z \in \mathfrak{z}$, thus

$$0 = \langle [dV_1, V_2] + [V_1, dV_2], Z \rangle = \langle j(Z)dV_1, V_2 \rangle - \langle d j(Z) V_1, V_2 \rangle,$$

for any $Z \in \mathfrak{z}$, which says, that $d : h_{2n+1} \to h_{2n+1}$ is a skew-symmetric derivation if and only if $d(z) = 0$, $d(v) \subseteq v$ and $j(Z)d - d j(Z) = 0$. This gives (i) $\Leftrightarrow$ (ii).

Notice that for any $V \in v, \tilde{Z} \in \mathfrak{z}$ one has

$$(\nabla_Z F)(V + \tilde{Z}) = \nabla_Z(FV) - F\nabla_Z(V) = -\frac{1}{2} j(Z) F V + \frac{1}{2} F j(Z) V,$$

so that (i) $\Leftrightarrow$ (iii).

Let $F$ be a Lorentz force of type I on the Heisenberg Lie group of dimensions five $H_5$, satisfying one the of the equivalent conditions in the proposition above. Then $F$ can be identified with a skew-hermitian matrix on $\mathbb{R}^4 \simeq v$ which has purely imaginary eigenvalues $i\mu_1, i\mu_2$ and it is diagonalizable by $S \in U(2)$. This means that there is $S \in O(4, \mathbb{R})$ such that $S j(Z) S^{-1} = j(Z)$ and there is a basis of $v$ of the form $U_1, V_1, U_2, V_2$ giving the following matricial presentation

$$S F S^{-1} = \begin{pmatrix}
0 & -\mu_1 \\
\mu_1 & 0 \\
0 & -\mu_2 \\
\mu_2 & 0
\end{pmatrix}.$$ 

Observe that $F$ is exact if and only if $\mu_1 = \mu_2$, while it is harmonic if $\mu_1 = -\mu_2$. 

\(\square\)
Since $Sj(Z)S^{-1} = j(Z)$ we have that

$$S(z_0 j(Z) + F)S^{-1} = \begin{pmatrix} 0 & -z_0 - \mu_1 \\ z_0 + \mu_1 & 0 \\ 0 & -z_0 - \mu_2 \\ z_0 + \mu_2 \end{pmatrix}. $$

Now, take the curve $\tilde{\sigma}(t) = \exp(SV(t) + Z(t))$, for $S \in U(2)$ as above. Then $||\sigma'(t)|| = ||\tilde{\sigma}'(t)||$ for all $t$, and from the equations above we have that the curves $SV(t)$ and $Z(t)$ in the Lie algebra $\mathfrak{n}$, satisfy:

$$SV'' = S(F + z_0 j(Z))S^{-1}SV' = \tilde{J}SV', \quad Z' + \frac{1}{2}[SV', SV] = z_0 Z.$$

This means that finding solutions to these last equations gives magnetic trajectories for the original magnetic equation. Moreover, the curve $\tilde{\sigma}$ is closed or periodic if equivalently the curve $\sigma$ satisfies this.

Solutions to the magnetic equations above for the Lorentz force $\tilde{J} = S(z_0 j(Z) + F)S^{-1}$ have the form $\tilde{\sigma}(t) = \exp(SV(t) + z(t)Z)$ for $SV(t) \in \mathfrak{v}$, explicitly:

- For $-z_0 = \mu_1 = \mu_2$:
  $$SV(t) = SV_0 t, \quad V_0 = x_1^0 X_1 + y_1^0 Y_1 + x_2^0 X_2 + y_2^0 Y_2$$
  $$z(t) = z_0 t.$$

- For $z_0 + \mu_1 \neq 0$ and $z_0 + \mu_2 \neq 0$:
  $$SV(t) = (u_1(t), v_1(t), u_2(t), v_2(t)) \quad \text{with}$$
  $$u_1(t) = \frac{x_1^0}{z_0 + \mu_1} \sin((z_0 + \mu_1)t) - \frac{y_1^0}{z_0 + \mu_1} (1 - \cos((z_0 + \mu_1)t)), $$
  $$v_1(t) = \frac{x_1^0}{z_0 + \mu_1} (1 - \cos((z_0 + \mu_1)t)) + \frac{y_1^0}{z_0 + \mu_1} \sin((z_0 + \mu_1)t), $$
  $$u_2(t) = \frac{x_2^0}{z_0 + \mu_2} \sin((z_0 + \mu_2)t) - \frac{y_2^0}{z_0 + \mu_2} (1 - \cos((z_0 + \mu_2)t)), $$
  $$v_2(t) = \frac{x_2^0}{z_0 + \mu_2} (1 - \cos((z_0 + \mu_2)t)) + \frac{y_2^0}{z_0 + \mu_2} \sin((z_0 + \mu_2)t), $$
  $$z(t) = (z_0 + \frac{1}{2} \sum_{i=1}^{2} (x_i^0)^2 \sin((z_0 + \mu_i)t) + \frac{1}{2} \sum_{i=1}^{2} (y_i^0)^2 \sin((z_0 + \mu_i)t),$$

where $\tilde{V}_i^0 = (x_i^0, y_i^0)$.

- For $-z_0 = \mu_2 \neq \mu_1$:
  $$SV(t) = (u_1(t), v_1(t), u_2(t), v_2(t)) \quad \text{with}$$
  $$u_1(t) = \frac{x_1^0}{z_0 + \mu_1} \sin((z_0 + \mu_1)t) - \frac{y_1^0}{z_0 + \mu_1} (1 - \cos((z_0 + \mu_1)t)), $$
  $$v_1(t) = \frac{x_1^0}{z_0 + \mu_1} (1 - \cos((z_0 + \mu_1)t)) + \frac{y_1^0}{z_0 + \mu_1} \sin((z_0 + \mu_1)t), $$
  $$u_2(t) = x_2^0 t, $$
  $$v_2(t) = y_2^0 t,$$
  $$z(t) = (z_0 + \frac{1}{2} (x_2^0)^2 t + \frac{1}{2} (y_2^0)^2 t).$$
where $\hat{V}_1^0 = (\hat{x}_1^0, \hat{y}_1^0)$. 

The values $\hat{x}_1^0, \hat{y}_1^0, \hat{x}_2^0, \hat{y}_2^0$ are the transformed by $S$ of the initial conditions for $V(t)$.

**Remark 3.14** In Munteanu and Nistor [20] the authors study magnetic trajectories on Heisenberg Lie groups $H_{2n+1}$ associated to a quasi-Sasakian structure. This is a stronger condition than the asked in Proposition 3.13. They prove that a non-geodesic magnetic trajectory is a Frenet helix of maximum order five.

Recall that the **energy** is $E(X) = \frac{1}{2} \langle X, X \rangle$, for any $X \in \mathfrak{n}$. In particular, the **energy of a curve** $\sigma$ is the energy of $\sigma'(0)$.

**Proposition 3.15** If the map $F : \mathfrak{n} \to \mathfrak{n}$ does not correspond to an exact form, there is a periodic magnetic trajectory with energy $E$ for every $E \geq 0$.

**Proof** The condition of the map $F$ giving rise to a non-exact closed 2-form, says that the eigenvalues satisfy $\mu_1 \neq \mu_2$ and we can assume without losing generality that $\mu_1 < \mu_2$. Note that to have periodic trajectories, one needs to consider the linear part of the coordinate $z(t)$ of the solution above.

Take $z_0$ such that $-\mu_2 < z_0 < -\mu_1$ then the line

$$
\frac{1}{2} \frac{x}{z_0 + \mu_1} + \frac{1}{2} \frac{y}{z_0 + \mu_2} = -z_0
$$

has positive slope and the set of points on the line with non-negative coordinates is not bounded. More precisely, for $E \geq \max \{ \frac{|\mu_1|^2}{2}, \frac{|\mu_2|^2}{2} \}$ we have that $2E - z_0^2 > 0$ and

$$
2E - z_0^2 \geq \mu_i^2 - z_0^2 > \mu_i^2 - z_0^2 - (\mu_i + z_0)^2 = -2z_0(z_0 + \mu_i)
$$

for $i = 1, 2$. So the system

$$
\begin{cases}
\frac{1}{2} \frac{x}{z_0 + \mu_1} + \frac{1}{2} \frac{y}{z_0 + \mu_2} = -z_0 \\
x + y = 2E - z_0^2.
\end{cases}
$$

has a solution with $x \geq 0$ and $y \geq 0$. Take $\hat{V}_1^0, \hat{V}_2^0 \in \mathfrak{v}$ such that $||\hat{V}_1^0||^2 = x, ||\hat{V}_2^0||^2 = y$ is the solution of the system (14). Then every magnetic trajectory with initial condition $(\hat{V}_1^0, \hat{V}_2^0, z_0)$ is periodic and has energy $E = \frac{1}{2}(z_0^2 + ||\hat{V}_1^0||^2 + ||\hat{V}_2^0||^2)$.

Now suppose $E < \max \{ \frac{|\mu_1|^2}{2}, \frac{|\mu_2|^2}{2} \} = \frac{|\mu_1|^2}{2}$ (the case $\max \{ \frac{|\mu_2|^2}{2}, \frac{|\mu_3|^2}{2} \} = \frac{|\mu_3|^2}{2}$ is analogous). If there is a periodic magnetic trajectory with $\hat{V}_2^0 = 0$ and energy $E$ then

$$
z_0 + \frac{1}{2} \frac{||\hat{V}_1^0||^2}{z_0 + \mu_1} + \frac{1}{2} \frac{||\hat{V}_2^0||^2}{z_0 + \mu_2} = 0 \iff (z_0 + \mu_1)^2 + 2E
$$

$$
= \mu_1^2 \iff z_0 = -\mu_1 \pm \sqrt{\mu_1^2 - 2E}.
$$

Observe that $(-\mu_1 - \sqrt{\mu_1^2 - 2E})(-\mu_1 + \sqrt{\mu_1^2 - 2E}) = 2E$. Since $z_0^2 \leq 2E$, the only possibility is $z_0 = -\mu_1 + sgn(\mu_1)\sqrt{\mu_1^2 - 2E}$. Then taking $\hat{V}_1^0$ such that

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\[ ||V_1^0||^2 = 2E - z_0^2 \] we have that the corresponding magnetic trajectory is periodic and has energy \( E \).

Note that in the special case that \( z_0 = -\mu_2 \) (when \( 2E = \mu_1^2 - (\mu_1 - \mu_2)^2 \)) this works as well since \( \tilde{V}_2^0 = 0 \) and \( u_2(t) = v_2(t) = 0 \) for all \( t \).

\[ \square \]

4 Magnetic Trajectories on the Heisenberg Lie Group \( H_3 \).

In this section, we study magnetic trajectories on the Heisenberg Lie group of dimension three. We start by calculating the magnetic trajectories for Lorentz forces of type II on \( H_3 \). We also study closed magnetic trajectories on compact spaces \( \Lambda \backslash H_3 \), where \( \Lambda \) is a cocompact lattice.

4.1 Magnetic Trajectories on the Three-Dimensional Heisenberg Lie Group

Let \( h_3 \) denote the three-dimensional Heisenberg Lie algebra. Let \( \langle , \rangle \) denote the canonical metric on \( h_3 \) (hence in \( H_3 \)). Any skew-symmetric map of type II, \( F : h_3 \to h_3 \) has the form \( FU = -\text{ad}(U) + \text{ad}(U)^T \) for some \( U \in \mathfrak{v} \), i.e.

\[
F_U(V + Z) = [V, U] + j(Z)U, \quad \text{for all } V + Z \in \mathfrak{v} \oplus \mathfrak{j} = h_3.
\]

In fact, assume that such linear map \( F \) has a matricial presentation in the basis \( e_1, e_2, e_3 \) as

\[
F = \begin{pmatrix}
0 & 0 & -\beta \\
0 & 0 & -\alpha \\
\beta & \alpha & 0
\end{pmatrix}, \quad \alpha, \beta \in \mathbb{R}.
\]

Define the element \( U \in \mathfrak{v} \) by \( U = -\alpha e_1 + \beta e_2 \). Thus, one verifies

- \( [e_1, U] = \beta e_3, [e_2, U] = \alpha e_3 \) and
- \( \text{ad}(U)'(Z) = j(Z)U \), for \( Z = ze_3 \).

Since \( \langle j(Z)U, e_1 \rangle = \langle Z, [U, e_1] \rangle = -\beta z \), and \( \langle j(Z)U, e_2 \rangle = \langle Z, [U, e_2] \rangle = -\alpha z \), it holds \( F \equiv F_U \).

Let \( \gamma(t) = \exp(V(t) + Z(t)) \) be a magnetic trajectory on the Heisenberg group through the identity element. Now, the magnetic equations in (8) for the map \( F_U \) can be written as

\[
\begin{align*}
V'' - j(Z' + \frac{1}{2}[V', V])V' &= j(Z' + \frac{1}{2}[V', V])U \\
Z'' + \frac{1}{2}[V'', V] &= [V', U].
\end{align*}
\]

(15)

In the above equation, we have reduced to the case \( q = 1 \) since \( U \) is an arbitrary element of the Lie algebra and \( q \) in the right hand side can be absorbed by \( U \).
One has the initial conditions \( V(0) = (0,0), Z(0) = 0 \) and \( V'(0) = V_0 = x_0 e_1 + y_0 e_2 \) and \( Z'(0) = z_0 e_3 \). From the second equation one gets
\[
Z' + \frac{1}{2} [V', V] = [V, U] + Z_0.
\]

Replace in the first equation to obtain
\[
V'' - j ([V, U] + Z_0) V' = j ([V, U] + Z_0) U,
\]
equivalently
\[
V'' - j ([V, U] + Z_0)(V' + U) = 0. \tag{16}
\]

Let \( \phi : \mathfrak{h}_3 \rightarrow \mathfrak{h}_3 \) be an orthogonal automorphism of \( \mathfrak{h}_3 \) and \( r \in \mathbb{R} \). Then by Remark (3.11), one gets
\[
r \phi \circ F_U \circ \phi^{-1}(V + Z) = r \phi([\phi^{-1}(V), U] + j(\phi^{-1}(Z))U)
= r ([V, \phi(U)] + j(Z)\phi(U)
= Fr\phi(U)(V + Z).
\]
We saw in Example 2.4 that for any orthogonal automorphism of \( \mathfrak{h}_3 \) there exists a transformation \( A \in O(v) = O(2) \) such that \( \phi(V + Z) = A(V) + det(A)Z \). Since the group \( O(2) \) acts transitively on the spheres of \( v \), for \( U \neq 0 \), there exist a real number \( r > 0 \) and an orthogonal automorphism \( \phi \) of the Heisenberg Lie algebra \( \mathfrak{h}_3 \) such that \( r\phi(U) = e_2 \). By Lemma 3.1 we may just compute the magnetic trajectory \( \sigma(t) \) corresponding to a Lorentz force \( F_U \), for \( U = e_2 \).

Write \( V(t) = x(t)e_1 + y(t)e_2 \) and \( U = e_2 \). Now, in usual coordinates, the system (16) reduces explicitly to the system:
\[
\begin{cases}
x''(t) + (x(t) + z_0)(y'(t) + 1) = 0 \\
y''(t) - (x(t) + z_0)x'(t) = 0
\end{cases} \tag{17}
\]

From the second equation, we have
\[
y''(t) = \frac{x(t)^2}{2} + z_0x(t) \Rightarrow y'(t) = \frac{x(t)^2}{2} + z_0x(t) + y_0.
\]
Again replace in the first equation to get
\[
x''(t) + (x(t) + z_0)\left(\frac{x(t)^2}{2} + z_0x(t) + y_0 + 1\right) = 0 \tag{18}
\]
This is a second-order autonomous differential equation that has the following important property:

If \( x(t) \) is a solution of Eq. (18) then \( x(t + c) \) and

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\( x(-t + c) \) are also solutions for all \( c \in \mathbb{R} \). \hspace{1cm} (19)

In particular, whenever \( x(t) \) is a solution of Eq. (18) with initial condition \( x'(0) = x_0 \), then \( x(-t) \) is a solution with initial condition \( x'(0) = -x_0 \). So, we may assume that \( x_0 \geq 0 \).

Let \( h: \mathbb{R} \to \mathbb{R} \) be the function given by \( h(x) = \frac{x^2}{2} + z_0 x + y_0 + 1 \) then Eq. (18) is rewritten as

\[
\begin{align*}
\frac{d^2 x}{dt^2} + h'(x(t))h(x(t)) &= 0 \\
\Rightarrow \quad \left( \frac{dx}{dt} \right)^2 + h(x(t))^2 &= \text{const.}
\end{align*}
\]

Observe that this implies that \( \left( \frac{dx}{dt} \right)^2 + \left( y'(t) + 1 \right)^2 = |V_1|^2 = x_0^2 + y_1^2 \) is constant, where \( V_1 = V_0 + e_2 \) and \( y_1 = y_0 + 1 \). Thus, one gets

\[
x'(t)^2 = |V_1|^2 - h(x(t))^2.
\]

If \( x_0 = 0, y_0 = -1 \) or \( x_0 = z_0 = 0 \), the curve \( x(t) \) must be constant and the solution curve passing through the identity with initial conditions \( V'(0) = V_0, Z'(0) = z_0 e_3 \) is given by:

\[
x(t) \equiv 0, \quad y(t) \equiv y_0 t \quad \text{and} \quad z(t) \equiv z_0 t.
\]

We fix \( |V_1| > 0 \) and \( x_0^2 + z_0^2 > 0 \). Suppose \( x'(t) > 0 \) in some neighborhood of 0 then

\[
\frac{x'(t)}{\sqrt{|V_1|^2 - h(x(t))^2}} = 1
\]

implying that

\[
\int_0^{x(t)} \frac{du}{\sqrt{|V_1|^2 - h(u)^2}} = t.
\]

Then we must study the elliptic integral

\[
E(x) = \int_0^x \frac{du}{\sqrt{|V_1|^2 - h(u)^2}}
\]

and its inverse.

By the substitution \( v = u + z_0 \) we get

\[
E(x) = \int_0^x \frac{du}{\sqrt{|V_1|^2 - \left( \frac{u^2}{2} + z_0 u + y_1 \right)^2}} = \int_{z_0}^{x+z_0} \frac{dv}{\sqrt{|V_1|^2 - \left( \frac{v^2}{2} + y_1 - \frac{z_0^2}{2} \right)^2}}
\]
\[
2 \int_{z_0}^{x+z_0} \frac{dv}{\sqrt{(2|V_1| + 2y_1 - z_0^2 + v^2)(2|V_1| - 2y_1 + z_0^2 - v^2)}}
\]

Observe that \(2|V_1| - 2y_1 + z_0^2 > 0\) since \(|V_1| \geq y_1\) and \(x_0^2 + z_0^2 > 0\). Then we have three cases to consider: \(2|V_1| + 2y_1 - z_0^2\) being positive, negative or zero.

If \(2|V_1| + 2y_1 - z_0^2 > 0\), using formula (7) on page 33 of [14] we have

\[
\mathcal{E}(x) = \frac{1}{2\sqrt{|V_1|}} \left( \text{cn}^{-1} \left( \frac{z_0}{\sqrt{2|V_1| - 2y_1 + z_0^2}}, k_1 \right) - \text{cn}^{-1} \left( \frac{x + z_0}{\sqrt{2|V_1| - 2y_1 + z_0^2}}, k_1 \right) \right)
\]

where \(\text{cn}\) is the cosine amplitude, one of Jacobi’s elliptic function, with elliptic modulus \(k_1 = \sqrt{\frac{2|V_1| - 2y_1 + z_0^2}{4|V_1|}}\). This function is defined on \([-z_0 - \sqrt{2|V_1| - 2y_1 - z_0^2}, -z_0 + \sqrt{2|V_1| - 2y_1 - z_0^2}]\).

Its inverse is

\[
\Phi(t) = \sqrt{2|V_1| - 2y_1 + z_0^2} \text{cn} \left( C_0 - 2\sqrt{|V_1|}t, k_1 \right) - z_0 \quad (20)
\]

where \(C_0\) is such that \(\Phi(0) = 0\). The map \(\Phi(t)\) is the solution of the differential equation (18), it is defined on \(\mathbb{R}\) and it is periodic with period \(2\frac{K(k_1)}{\sqrt{|V_1|}}\) where \(K(k)\) is the quarter period defined as

\[
K(k) = \int_0^{\pi/2} \frac{1}{\sqrt{1 - k^2 \sin^2 \theta}} d\theta.
\]

Analogously, if \(2|V_1| + 2y_1 - z_0^2 < 0\), we use formula (9) on page 33 of [14] to compute

\[
\mathcal{E}(x) = \frac{1}{\sqrt{2|V_1| - 2y_1 + z_0^2}} \left( \text{dn}^{-1} \left( \frac{z_0}{\sqrt{2|V_1| - 2y_1 + z_0^2}}, k_2 \right) - \text{dn}^{-1} \left( \frac{x + z_0}{\sqrt{2|V_1| - 2y_1 + z_0^2}}, k_2 \right) \right)
\]

where \(\text{dn}\) is the delta amplitude with modulus \(k_2 = \sqrt{\frac{4y_1 - 2z_0^2}{2|V_1| - 2y_1 + z_0^2}}\). In this case \(\mathcal{E}(x)\) is defined on \([-z_0 - \sqrt{z_0^2 - 2y_1 + 2|V_1|}, -z_0 - \sqrt{z_0^2 - 2y_1 - 2|V_1|}]\) if \(z_0 < 0\) or on \([-z_0 + \sqrt{z_0^2 - 2y_1 - 2|V_1|}, -z_0 + \sqrt{z_0^2 - 2y_1 + 2|V_1|}]\) if \(z_0 > 0\).
The corresponding solution of the differential equation is

\[ \Phi(t) = \sqrt{2|V_1| - 2y_1 + z_0^2} \, \text{dn} \left( \frac{C_1 - \sqrt{2|V_1| - 2y_1 + z_0^2}}{k_2}, k_2 \right) - z_0 \]  

where \( C_1 \) is also the constant such that \( \Phi(0) = 0 \). This solution is also periodic on \( \mathbb{R} \) and its period is \( 2 \sqrt{2|V_1| - 2y_1 + z_0^2} / k_2 \).

Now if \( 2|V_1| + 2y_1 - z_0^2 = 0 \), take the function

\[ E(x) = 2 \int_{z_0}^{x+z_0} \frac{dv}{|v|\sqrt{4|V_1| - v^2}}. \]

Thus, one has

\[ E(x) = \frac{\text{sgn}(z_0)}{\sqrt{|V_1|}} \left( \text{artanh} \left( \frac{\sqrt{|V_1| - z_0^2}}{4|V_1|} \right) - \text{artanh} \left( \frac{\sqrt{|V_1| - (x+z_0)^2}}{4|V_1|} \right) \right) \]

where the function \( \text{artanh} \) is the inverse hyperbolic tangent, i.e. \( \text{artanh}(x) = \frac{1}{2} \ln \left( \frac{1+x}{1-x} \right) \). Observe that the function \( E(x) \) is defined for \( -z_0 < x \leq 2\sqrt{|V_1|} - z_0 \) if \( z_0 > 0 \), and for \( -2\sqrt{|V_1|} - z_0 \leq x < -z_0 \) if \( z_0 < 0 \).

So, if \( z_0 > 0 \) the solution of Eq. (18) is given by

\[ \Phi(t) = 2\sqrt{|V_1|} \, \text{sech}(C_2 - \sqrt{|V_1|} t) - z_0 \]  

and if \( z_0 < 0 \), it is given by

\[ \Phi(t) = -2\sqrt{|V_1|} \, \text{sech}(\sqrt{|V_1|} t - C_2) - z_0, \]

where \( \text{sech} \) is the hyperbolic secant and \( C_2 = \text{artanh} \left( \frac{\sqrt{|V_1| - z_0^2}}{4|V_1|} \right) \). These solutions are also defined on \( \mathbb{R} \) but they are not periodic.

All this is exposed in the next result.

**Theorem 4.1** The magnetic trajectory \( \sigma(t) = \exp(x(t)e_1 + y(t)e_2 + z(t))e_3 \) on the Heisenberg Lie group \( H_3 \) corresponding to a Lorentz force \( F_{e_2} \), with initial conditions \( \sigma(0) = e \in H_3 \) and \( \sigma'(0) = x_0e_1 + y_0e_2 + z_0e_3 \) is explained below:
Table 1 \(|V_1|^2 = x_0^2 + (y_0 + 1)^2\) and \(y_1 = y_0 + 1\)

| Condition | \(\Phi\) | Period | Image |
|-----------|----------|--------|-------|
| \(z_0 < -\sqrt{2(|V_1| + y_1)}\) | (21) | \(\frac{2K(k_2)}{\sqrt{2|V_1|-2y_1+z_0^2}}\) | \([-z_0 - \sqrt{\frac{2}{z_0^2} - 2A}, -z_0 - \sqrt{\frac{2}{z_0^2} - 2B}]\) |
| \(z_0 = -\sqrt{2(|V_1| + y_1)}\) | (23) | Non | \([-z_0 - \sqrt{\frac{2}{z_0^2} - 2A}, -z_0]\) |
| \(|z_0| < \sqrt{2(|V_1| + y_1)}\) | (20) | \(\frac{2K(k_1)}{\sqrt{|V_1|}}\) | \([-z_0 - \sqrt{\frac{2}{z_0^2} - 2A}, -z_0 + \sqrt{\frac{2}{z_0^2} - 2A}]\) |
| \(z_0 = \sqrt{2(|V_1| + y_1)}\) | (22) | Non | \((-z_0, -z_0 + \sqrt{\frac{2}{z_0^2} - 2A}]\) |
| \(z_0 > \sqrt{2(|V_1| + y_1)}\) | (21) | \(\frac{2K(k_2)}{\sqrt{2|V_1|-2y_1+z_0^2}}\) | \([-z_0 + \sqrt{\frac{2}{z_0^2} - 2B}, -z_0 + \sqrt{\frac{2}{z_0^2} - 2A}]\) |

where \(A = y_1 - |V_1|, B = y_1 + |V_1|\)

- If \(x_0^2 + z_0^2(y_0 + 1)^2 \neq 0\),

\[
\begin{cases}
  x(t) = \Phi(t) \\
  y(t) = \int_0^t \left( \frac{\Phi(s)^2}{2} + z_0 \Phi(s) + y_0 \right) ds \\
  z(t) = \frac{1}{2} \int_0^t \left( -\Phi'(s)y(s) + \Phi(s)(y'(s) + 2) \right) ds + z_0 t
\end{cases}
\]

where the function \(\Phi(t)\) is defined in Table 1 for \(x_0 \geq 0\) and we make use of the corresponding \(\Phi(-t)\) whenever \(x_0 < 0\).

- If \(x_0 = 0\) and \(z_0(y_0 + 1) = 0\), then

\[
\begin{cases}
  x(t) = 0 \\
  y(t) = y_0 t \\
  z(t) = z_0 t
\end{cases}
\]

Remark 4.2 See more details about elliptic functions and elliptic integrals in [14]. There, one can see some applications of these integrals in many situations.

Remark 4.3 Notice that any invariant Lorentz force on the Heisenberg Lie group \(H_3\) will be of the form \(F = j(\tilde{Z}) + F_U\) for some \(U \in \mathfrak{v}\), in such way that whenever \(\tilde{Z} = 0\), we are in the situation above. In the other hand, for \(U = 0\) we have an exact 2-form and \(F = j(\tilde{Z})\).
4.2 Closed Magnetic Trajectories on Compact Quotients

The aim now is the study of closed trajectories for the Lorentz forces of type II on Heisenberg nilmanifolds, that is $M = \Lambda \backslash H_3$, where $\Lambda$ denotes a cocompact lattice in $H_3$.

Let $(N, \langle , \rangle)$ denote a Lie group endowed with a left-invariant metric. Assume $\Lambda$ is a discrete subgroup of $N$ such that the quotient $\Lambda \backslash N$ is compact. A natural metric on the quotient is the one induced from $N$ and also a magnetic field on $\Lambda \backslash N$ is induced from the left-invariant magnetic field on $N$.

**Definition 4.4** Let $N$ be a simply connected nilpotent Lie group. For any element $\lambda \in N$ different from the identity, a curve $\sigma(t)$ is called $\lambda$-periodic with period $\omega$ if $\omega \neq 0$ and for all $t \in \mathbb{R}$ it holds:

$$\lambda \sigma(t) = \sigma(t + \omega).$$

It is clear that whenever the subgroup $\Lambda < N$ is a cocompact discrete subgroup, called lattice, for any element $\lambda \in \Lambda$, a $\lambda$-periodic magnetic trajectory will project to a smoothly closed magnetic trajectory under the mapping $N \to \Lambda \backslash N$. Conversely, every closed magnetic trajectory $\sigma$ on $\Lambda \backslash N$ lifts to a $\lambda$-periodic magnetic trajectory on the Lie group $N$, if $\sigma$ is non-contractible, or directly lifts to a closed magnetic trajectory on $N$.

Choose an element $\lambda = \exp(W_1 + Z_1) \in N$. A magnetic trajectory $\sigma(t) = \exp(V(t) + Z(t))$ is $\lambda$-periodic with period $\omega$, for $\omega \neq 0$, if and only if the following equations are verified:

$$\begin{align*}
W_1 + V(t) &= V(t + \omega) \\
Z_1 + Z(t) + \frac{1}{2}[W_1, V(t)] &= Z(t + \omega) + Z_0.
\end{align*} \quad (24)$$

for all $t \in \mathbb{R}$.

By assuming that the curve $\sigma(t) = \exp(V(t) + Z(t))$ is a magnetic trajectory for a left-invariant Lorentz force $F$ of type II on a 2-step nilpotent Lie group $(N, \langle , \rangle)$, then it satisfies the following system

$$\begin{align*}
V'' - j(Z' + \frac{1}{2}[V', V])V' &= q F_0(Z' + \frac{1}{2}[V', V]) \\
Z'' + \frac{1}{2}[V'', V] &= q F_3(V').
\end{align*}$$

From the second equation one gets:

$$Z' + \frac{1}{2}[V', V] = q F_3(V) + Z_0.$$

Evaluating in $t + \omega$ and using (24) we get

$$Z' + \frac{1}{2}[W_1, V'] + \frac{1}{2}[V', W_1 + V] = q F_3(W_1 + V) + Z_0.$$
By substracting the last two equations, one obtains the following condition for $W_1$:

$$F_3(W_1) = 0$$ (25)

**Lemma 4.5** Let $\lambda = \exp(W_1 + Z_1)$ be any element in the 2-step nilpotent Lie group $(N, \langle , \rangle)$. If a left-invariant Lorentz force $F$ of type II admits a $\lambda$-periodic trajectory then $W_1 \in \ker F$.

In the 3-dimensional Heisenberg, take the Lorentz force $F_{e_2}$. Clearly, the kernel is given by

$$F(W_1) = 0 \iff W_1 = ye_2,$$

and more generally, for a non-trivial Lorentz force $F_U$ the kernel is $\ker F_U = \text{span}\{U\}$.

Thus, the curve $\sigma(t) = (x(t), y(t), z(t))$ is $\lambda$-periodic for $\lambda = \exp(y_1 e_2 + z_1 e_3)$ if and only if

$$x(t) = x(t + \omega)$$
$$y(t) + y_1 = y(t + \omega)$$
$$z(t) + z_1 - \frac{1}{2}y_1 x(t) = z(t + \omega)$$

for all $t \in \mathbb{R}$. So the function $x$ must be periodic and $\omega$ must be a multiple of the period of $x$. By using the expression of $y(t)$ given in Theorem 4.1, we see that if $y_1 = y(\omega)$, the second equation above holds. From the expression of $z(t)$ in Theorem 4.1 one has

$$(z(t + w) - z(t) + \frac{1}{2}y_1 x(t))' =$$
$$\frac{1}{2}(-x'(t + \omega)y(t + \omega) + x(t + \omega)(y'(t + \omega) + 2)$$
$$- (-x'(t)y(t) + x(t)(y'(t) + 2)) + y_1 x'(t))$$
$$= \frac{1}{2}(-x'(t)y(t + \omega) + x(t)(y'(t) + 2) - (-x'(t)y(t) + x(t)(y'(t) + 2)) + y_1 x'(t))$$
$$= \frac{1}{2}x'(t)(-y(t + \omega) + y(t) + y_1) = 0$$

Then any magnetic trajectory $\sigma(t) = (x(t), y(t), z(t))$ where $x(t)$ is a periodic function with period $\omega$, is $\lambda$-periodic for $\lambda = \exp(y(\omega)e_2 + z(\omega)e_3)$.

The next result summarizes explicit conditions for $\lambda$-periodicity of magnetic trajectories.

**Proposition 4.6** Let $(H_3, \langle , \rangle)$ be the Heisenberg group equipped with its canonical metric, let $F = F_{e_2}$ denote a Lorentz force of type II and let $\sigma$ be the magnetic trajectory through the identity such that $\sigma'(0) = x_0 e_1 + y_0 e_2 + z_0 e_3$.

- If $z_0^2 \neq 2(\sqrt{x_0^2 + (y_0 + 1)^2} + y_0 + 1)$, and $x_0^2 + z_0^2(y_0 + 1)^2 \neq 0$ then $\sigma$ is periodic or $\lambda$-periodic for $\lambda = \sigma(\omega)$ where $\omega$ is the corresponding period given in Table 1.
• If \( x_0^2 + z_0^2(y_0 + 1)^2 = 0 \) then \( \sigma \) is \( \lambda \)-periodic for \( \lambda = \exp(y_0 e_2 + z_0 e_3) \).

• If \( z_0^2 = 2(\sqrt{x_0^2} + (y_0 + 1)^2 + y_0 + 1) \), and \( x_0^2 + z_0^2(y_0 + 1)^2 \neq 0 \) then \( \sigma \) is not \( \lambda \)-periodic for any \( \lambda \).

**Remark 4.7** Notice that for exact 2-forms in the Heisenberg Lie group there exist closed magnetic trajectories only for sufficiently small energy levels and the \( \lambda \)-periodic trajectories have energy bounded below by the Mañé critical value [12]. The results in Proposition 3.15 and Proposition 4.6 show that for some non-exact magnetic fields there are periodic and \( \lambda \)-periodic magnetic geodesics trajectories at every energy level.

**Remark 4.8** We see that different kind of magnetic fields, equivalently Lorentz forces, show different dynamical properties in \( H_3 \), but also in \( H_5 \) when we consider non-exact magnetic fields. In particular, when the magnetic fields are of type I, solutions correspond to slant curves.

**Acknowledgements** The authors kindly thank the reviewers for several comments and suggestions to improve the original version of the present work. **Muchas gracias!**

**Funding** Partially supported by SCyT (UNR).

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