A Mixed Finite Element Method for a Class of Evolution Differential Equations with $p$-Laplacian and Memory

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Abstract

We present a new mixed finite element method for a class of parabolic equations with $p$-Laplacian and nonlinear memory. The applicability, stability and convergence of the method are studied. First, the problem is written in a mixed formulation as a system of one parabolic equation and a Volterra equation. Then, the system is discretized in the space variable using the finite element method with Lagrangian basis of degree $r \geq 1$. Finally, the Crank-Nicolson method with the trapezoidal quadrature is applied to discretize the time variable. For each method, we establish existence, uniqueness and regularity of the solutions. The convergence order is found to be dependent on the parameter $p$ on the $p$-Laplacian in the sense that it decreases as $p$ increases.

Keywords: Finite elements, integrodifferential equation, $p$-Laplacian, memory term, Lagrange polynomials.

1. Introduction

In this work, we study the evolutionary integro-differential equation with $p$-Laplacian and memory,

$$u_t(x,t) - \Delta_p u(x,t) = \int_0^t g(t-s) \Delta_p u(x,s) \, ds + f(x,t),$$

where $f$ and $g$ are given functions. This type of equation appears in the mathematical description of heat propagation in materials with memory, where the heat flux can depend on the process history. In addition to the common problem of the memory term, whose time discretization forces a large volume of calculations and memory consumption, problem (1) also presents a new difficulty when containing the $p$-Laplacian in that it makes the memory term nonmonotonous (see [2]).
Since the 70’s, evolution equations with memory terms have attracted the attention of researchers. Issues related to the existence and properties of solutions to partial integro-differential equations (PIDEs) such as

\[ u_t - Au = \int_0^t g(t-s)Bu(s)ds + f, \]  

(2)

where \( A \) and \( B \) are symmetric positive definite operators of at most second order and \( g \) is a memory kernel, were addressed, for example, in [4, 11, 17, 19]. Concerning numerical approximations to the solutions of (2), several methods have already been investigated. For a review of the finite element method applied to PIDEs, we refer, for instance, to [6] and the references therein. The finite volume method and the collocation method with splines were studied in [22] and [20], respectively. The mixed finite element method was considered in [23], while the discontinuous Galerking method was studied in [18]. Subsequently, an error analysis for the Crank-Nicolson finite element method was made in [21]. A two grid finite element method was proposed in [25] and, more recently, a pseudospectral method was investigated in [24].

Partial differential equations involving the \( p \)-Laplacian operator have been extensively studied in the last decades. For a survey of the theory, we refer to the monographs [1, 12, 16]. Concerning the numerical simulations of the \( p \)-Laplacian with the finite element method, it was found that the regularity of the solutions limits the convergence rates. In [14], Glowinski and Marroco proved a convergence of \( O(h^{1/p}) \) in the \( W^{1,p} \) norm. Later, Chow [9] improved this convergence order to \( O(h^{2/p}) \). In 1993, assuming a stronger regularity for the weak solution, Barret and Liu, in [5], proved optimal error bounds of order \( O(h) \) in \( W^{1,p} \).

The lack of monotonicity in the memory term of equation (1) makes it unfeasible to use most of the well-developed techniques available. In [2], Antontsev and his coauthors studied equation (1) with a nonlinear source term \( \Theta(x, t, u) \), substituting the equation with a system composed of a diffusion-reaction equation and an integral equation. They proved that for \( \max\{1, \frac{2n}{n+2}\} < p < \infty \), \( u_0 \in W^{1,p}_0(\Omega) \), \( f \in L^2(Q) \) and \( g, g' \in L^2(0, T) \), the problem has a weak solution that is local or global in time depending on the growth rate of \( \Theta(x, t, s) \), when \( |s| \to \infty \). Uniqueness conditions were established and they also proved that for \( p > 2 \) and \( s\Theta(x, t, s) \leq 0 \), the data disturbances propagate with finite speed and that the waiting time effect is possible.

Nowadays, problem (1), with \( p \) depending on \( x \), is attracting considerable attention, perhaps because of its various physical applications. We refer to [3, 24, 13], where questions on the solvability and properties of the solutions are addressed.

In this paper, we present a new mixed finite element method for equation (1). The existence, uniqueness and regularity of the discrete solutions are established. Error bounds depending on the parameter \( p \) are also obtained. An auxiliary problem and its variational formulation is presented in Section 2. The discretization of the space variable is developed in Section 3. The discretization
of the time variable is studied in Section 4. Finally, in Section 5, we draw some final conclusions.

2. Parabolic equation with \( p \)-Laplacian

Let us consider the evolutionary integro-differential equation with the homogeneous Dirichlet condition,

\[
\begin{align*}
\frac{du}{dt} - \Delta_p u &= \int_0^t g(t-s) \Delta_p u(x,s) \, ds + f(x,t), \quad \forall (x,t) \in Q = \Omega \times [0,T], \\
u(x,t) &= 0, \quad \forall (x,t) \in \partial \Omega \times [0,T], \\
u(x,0) &= u_0(x), \quad \forall x \in \Omega,
\end{align*}
\]

(3)

where \( u_0, g \) and \( f \) are given functions, \( \Omega \subset \mathbb{R}^n \) is a bounded domain with Lipschitz-continuous boundary. The \( p \)-Laplacian \( \Delta_p u \) is given by

\[
\Delta_p u = \text{div} \left( |\nabla u|^{p-2} \nabla u \right), \quad 2 < p < \infty
\]

and

\[
y(x,t) = \int_0^t g(t-s) \Delta_p u(x,s) \, ds
\]

is the memory term of the evolutionary integro-differential equation.

Assuming

\[
g, g' \in L^2(0,T), \quad f \in L^2(Q), \quad u_0 \in L^2(\Omega) \cap W^{1,p}_0(\Omega),
\]

it is proved in [2] that problem (3) has a unique weak solution. For notions on Sobolev spaces, we refer to [7, 13, 16]. Below, we present an important lemma which can be found in [5, 8, 9].

**Lemma 1.** For every \( p > 1 \) and \( \delta \geq 0 \), there are three positive constants \( C_1, C_2 \) and \( C_3 \) such that for every \( \zeta, \gamma \in \mathbb{R}^n \), \( \zeta \neq \gamma \), we have

1. \[
\left| |\zeta|^{p-2} \zeta - |\gamma|^{p-2} \gamma \right| \leq C_1 |\zeta - \gamma|^{1-\delta} (|\zeta| + |\gamma|)^{p-2+\delta};
\]

2. \[
\left( |\zeta|^{p-2} \zeta - |\gamma|^{p-2} \gamma, \zeta - \gamma \right)_{\mathbb{R}^n} \geq C_2 |\zeta - \gamma|^{2+\delta} (|\zeta| + |\gamma|)^{p-2-\delta};
\]

3. if \( p > 2 \),

\[
\left( |\zeta|^{p-2} \zeta - |\gamma|^{p-2} \gamma, \zeta - \gamma \right)_{\mathbb{R}^n} \geq C_3 |\zeta - \gamma|^p.
\]
2.1. Auxiliary problem

It is easy to prove that the memory term satisfies the integral equation

\[ y(x,t) = - \int_0^t g(t-s) y(x,s) ds + f_2(x,t,u), \quad (5) \]

where \( f_2 \) is the nonlinear nonlocal operator

\[ f_2(x,t,u(x,t)) = u(x,t)g(0) - u_0(x)g(t) + \int_0^t g'(t-s)u(x,s)ds - \int_0^t g(t-s)f(x,s)ds. \quad (6) \]

In fact, taking equation (3) and convoluting with \( g \), we obtain

\[ \int_0^t g(t-s) \Delta_p u(x,s)ds = - \int_0^t g(t-s) \int_0^s g(s-\tau) \Delta_p u(x,\tau) d\tau ds + \int_0^t g(t-s)u_s(x,s)ds - \int_0^t g(t-s)f(x,s)ds. \quad (7) \]

Integrating by parts the second term on the right side of equation (7), we obtain

\[ \int_0^t g(t-s) \Delta_p u(x,s)ds = - \int_0^t g(t-s) \int_0^s g(s-\tau) \Delta_p u(x,\tau) d\tau ds + g(0)u(x,t) - g(t)u_0(x) + \int_0^t g'(t-s)u(x,s)ds - \int_0^t g(t-s)f(x,s)ds, \]

which means that \( y(x,t) \), defined in (4), satisfies equation (5).

This allows us to consider the equivalent auxiliary problem of finding the pair \((u,y)\) that satisfies the conditions

\[
\begin{align*}
  & u_t - \Delta_p u = y(x,t) + f(x,t), \quad \forall (x,t) \in Q, \\
  & y(x,t) = - \int_0^t g(t-s) y(x,s) ds + f_2(x,t,u), \quad \forall (x,t) \in Q, \\
  & u(x,t) = 0, \quad \forall (x,t) \in \partial \Omega \times [0,T], \\
  & u(x,0) = u_0(x), \quad \forall x \in \Omega, \\
  & y(x,0) = 0, \quad \forall x \in \Omega.
\end{align*}
\]

2.2. Variational formulation

If we multiply the first equation of problem (8) by \( w \in H^1_0(\Omega) \) and integrate in \( \Omega \), we get

\[ \int_\Omega u_t w dx - \int_\Omega \text{div} \left( |\nabla u|^{p-2} \nabla u \right) w dx = \int_\Omega y w dx + \int_\Omega f w dx. \quad (9) \]
Applying Green’s theorem to the second term on the right side of equation (10) and using the definition of the space $H^1_0(\Omega)$, we have

$$\int_\Omega u_twdx + \int_\Omega |\nabla u|^{p-2} \nabla u \nabla wdx = \int_\Omega ywdx + \int_\Omega fwdx.$$  

Multiplying the second equation of the problem (8) by $v \in H^1_0(\Omega)$ and integrating in $\Omega$ we obtain

$$\int_\Omega ywdx = -\int_\Omega v \int_0^t g(t-s)y(x,s)dsdx + \int_\Omega f_2(x,t,u)vdx.$$  

So, we are left with the system

$$\int_\Omega u_twdx + \int_\Omega |\nabla u|^{p-2} \nabla u \nabla wdx = \int_\Omega ywdx + \int_\Omega fwdx, \forall w \in H^1_0(\Omega),$$  

$$\int_\Omega ywdx = -\int_\Omega v \int_0^t g(t-s)y(x,s)dsdx + \int_\Omega f_2vdx, \forall v \in H^1_0(\Omega).$$  

(10)

A pair of bounded and measurable functions $(u(x,t), y(x,t))$, is said to be a weak solution of the initial value problem (8), with initial data $u_0$ limited and measurable, if (10) is valid for all $(w,v) \in (H^1_0(\Omega))^2$. Henceforth, we assume that problem (8) has a unique weak solution with sufficient regularity in order to perform the calculations needed in next sections.

3. Discretization in space

3.1. Lagrangian Bases

Let us consider a regular partition $T_h = \{T_0, \ldots, T_m\}$ of $\Omega$ in simplexes with parameter $h$ and the space $S^h \subset H^1_0(\Omega)$ defined by

$$S^h = \{w \in C^0(\Omega) : w(x) = 0, x \in \partial \Omega, w(x)|_{T_k} \in \mathcal{P}_r(T_k), k = 0, \ldots, m\},$$

where $\mathcal{P}_r(T_k)$ is the set of polynomials of degree less than or equal to $r$ defined in $T_k$. We denote the interpolation operator into $S^h$ by $\Pi_h$. An estimate of the interpolation error is given in the next Lemma, which may be found in [10].

**Lemma 2.** If $\Pi_h : H^{r+1}(\Omega) \cap H^1_0(\Omega) \to S^h$ is the interpolation operator, then

$$\|u - \Pi_h u\|_{L^2(\Omega)} + h\|\nabla(u - \Pi_h u)\|_{L^2(\Omega)} \leq C h^s\|u\|_{H^s(\Omega)}, \quad 1 \leq s \leq r + 1,$$

with $u \in H^s(\Omega) \cap H^1_0(\Omega)$ and $C$ is a positive constant.

The semi-discrete problem is to find $(u^h, y^h) \in (S^h)^2$ such that

$$\int_\Omega u^h w^h dx + \int_\Omega |\nabla u^h|^{p-2} \nabla u^h \nabla w^h dx = \int_\Omega y^h w^h dx + \int_\Omega f w^h dx, \forall w^h \in S^h,$$

$$\int_\Omega y^h v^h dx = -\int_\Omega v^h \int_0^t g(t-s)y^h(x,s)dsdx + \int_\Omega f_2 v^h dx, \forall v^h \in S^h,$$

(11)

and

$$u^h(x,0) = u^h_0 = \Pi_h u_0, \quad y^h(x,0) = 0, \quad \forall x \in \Omega.$$
We note that problem (11) has a solution. In fact, from the second equation we obtain a solution \( y^h(u^h) \) which is substituted in the first equation to give a solution \( u^h(x, t) \). Substituting the latter solution in \( y^h(u^h) \) gives the solution \( y^h(x, t) \) (see [2] for more details).

**Theorem 3** (Uniqueness). If \( g, g' \in L^\infty(0, T) \), then the solution of the semi-discrete problem (11) is unique.

**Proof.** Suppose that \((u^h_1, y^h_1)\) and \((u^h_2, y^h_2)\) are two solutions of the semi-discrete problem (11). Subtracting the equation for \( y^h_2 \) from the equation for \( y^h_1 \), we obtain

\[
\int_{\Omega} (y^h_1 - y^h_2)v^h \, dx = -\int_{\Omega} v^h \int_{0}^{t} g(t - s)(y^h_1 - y^h_2) \, ds \, dx + g(0) \int_{\Omega} (u^h_1 - u^h_2)v^h \, dx
\]

\[+ \int_{\Omega} v^h \int_{0}^{t} g'(t - s)(u^h_1 - u^h_2) \, ds \, dx. \tag{12}
\]

Let \( v^h = y^h_1 - y^h_2 \in S^h \). As \( g \) and \( g' \) are bounded, we may apply Young’s inequality to (12) and thus obtain

\[
\int_{\Omega} (y^h_1 - y^h_2)^2 \, dx \leq C \int_{0}^{t} (y^h_1 - y^h_2)^2 \, dx + C \int_{0}^{t} (u^h_1 - u^h_2)^2 \, dx.
\tag{13}
\]

Applying Gronwall’s lemma to (13), we have

\[
\int_{\Omega} (y^h_1 - y^h_2)^2 \, dx \leq C \int_{0}^{t} (u^h_1 - u^h_2)^2 \, dx + C \int_{0}^{t} (u^h_1 - u^h_2)^2 \, dx.
\tag{14}
\]

Using a similar argument with \( u^h_1 \) and \( u^h_2 \), we get

\[
\int_{\Omega} |(u^h_1)_t - (u^h_2)_t|w^h \, dx + \int_{\Omega} \left( |\nabla u^h_1|^{p-2} \nabla u^h_1 - |\nabla u^h_2|^{p-2} \nabla u^h_2 \right) \nabla w^h \, dx
\]

\[= \int_{\Omega} (y^h_1 - y^h_2)w^h \, dx. \tag{15}
\]

Let \( w^h = u^h_1 - u^h_2 \in S^h \). Applying Young’s inequality in (15), using (14) and Lemma 1, we have

\[
\frac{d}{dt} \int_{\Omega} (u^h_1 - u^h_2)^2 \, dx \\
\leq \int_{\Omega} (u^h_1 - u^h_2)^2 \, dx + C \int_{0}^{t} (u^h_1 - u^h_2)^2 \, dx + C \int_{0}^{t} (u^h_1 - u^h_2)^2 \, dx. \tag{16}
\]

Integrating equation (16) from 0 to \( t \) and noting that \( u^h_1(x, 0) - u^h_2(x, 0) = 0 \), we obtain

\[
\int_{\Omega} (u^h_1 - u^h_2)^2 \, dx \leq C(1 + t) \int_{0}^{t} (u^h_1 - u^h_2)^2 \, dxds.
\]
Now, let $\zeta(t) = \int_{\Omega} (u^h_t - u^h_s)^2 dx \geq 0$. This implies that $\zeta(t) \leq C \int_0^t \zeta(s) ds$ and $\zeta(0) = 0$. By Gronwall’s lemma, $\zeta(t) = 0$ for every $t \in [0, T]$, that is,

$$\int_{\Omega} (u^h_t - u^h_s)^2 dx = 0, \quad u^h_t = u^h_s \text{ in } L^2(\Omega).$$

Returning to $y$, from equation (14), we have

$$\int_{\Omega} (y^h_t - y^h_s)^2 dx = 0, \quad \text{for every } t \in [0, T],$$

which proves the desired result. \hfill \Box

**Theorem 4** (Regularity). Let $(u^h, y^h) \in S^h \times S^h$ be a solution to problem (11) and $g, g' \in L^\infty(0, T)$. Then, for every $t \in [0, T]$,

$$\|u^h\|^2_{L^2(\Omega)} + \|\nabla u^h\|_{L^p(Q)}^p \leq C \|u_0\|^2_{L^2(\Omega)} + C\|f\|^2_{L^2(Q)(20)}$$

and

$$\|y^h\|^2_{L^2(\Omega)} \leq C \|u_0\|^2_{L^2(\Omega)} + C\|f\|^2_{L^2(Q),}$$

where

$$C = C(T, \|g\|_{L^\infty(0, T)}, \|g'\|_{L^\infty(0, T)}).$$

Proof. Since $y^h$ is a solution to problem (11),

$$\int_{\Omega} y^h v^h dx = - \int_{\Omega} v^h \int_0^t g(t - s)y^h(x, s)dsdx + \int_{\Omega} f^h v^h dx. \quad (17)$$

For $v^h = y^h \in S^h$,

$$\int_{\Omega} (y^h)^2 dx = - \int_{\Omega} y^h(x, t) \int_0^t g(t - s)y^h(x, s)dsdx + g(0) \int_{\Omega} u^h y^h dx \quad (18)$$

Applying Young’s inequality to equation (18) and using the fact that $g$ and $g'$ are limited, we have

$$\int_{\Omega} (y^h)^2 dx \leq C \int_0^t \int_{\Omega} (y^h(x, s))^2 dxds + C \int_{\Omega} (u^h)^2 dx + C \int_{\Omega} (u^h_s)^2 dx \quad \text{for every } t \in [0, T].$$

Applying Gronwall’s lemma to (19), we obtain

$$\|y^h\|^2_{L^2(\Omega)} \leq C\|u^h\|^2_{L^2(\Omega)} + C\|u_0^h\|^2_{L^2(\Omega)} + C \int_0^t \|u^h\|^2_{L^2(\Omega)} ds + C\|f\|^2_{L^2(Q)}. \quad (20)$$
Now, considering the first equation of problem (11) with \( w^h = w^h \), we have

\[
\int_{\Omega} u^h u^h \, dx + C \int_{\Omega} |\nabla u^h|^p \, dx = \int_{\Omega} y^h u^h \, dx + \int_{\Omega} f u^h \, dx.
\]  

(21)

Applying Young’s inequality to (21) and equation (20) we have

\[
dt \| u^h \|_{L^2(\Omega)}^2 + C \| \nabla u^h \|_{L^p(\Omega)}^p \leq C \| u^h \|_{L^2(\Omega)}^2 + C \| u_0^h \|_{L^2(\Omega)}^2 + \int_0^t \| u^h \|_{L^2(\Omega)}^2 \, ds
\]

\[
\qquad + C \| f \|_{L^2(0,T;L^2(\Omega))}^2 + \frac{1}{2} \| f \|_{L^2(\Omega)}^2.
\]

(22)

Integrating equation (22) from 0 to \( t \) gives

\[
\| u^h \|_{L^2(\Omega)}^2 + C \| \nabla u^h \|_{L^p(\Omega)} = C \| u_0^h \|_{L^2(\Omega)}^2 + C \| f \|_{L^2(0,T;L^2(\Omega))}^2.
\]

(23)

Ignoring the second term on the left side, since it is nonnegative, and applying Gronwall’s lemma to (23), we obtain

\[
\| u^h \|_{L^2(\Omega)}^2 \leq C \| u_0^h \|_{L^2(\Omega)}^2 + C \| f \|_{L^2(0,T;L^2(\Omega))}^2.
\]

Using this estimate in (23) completes the proof of (17). Finally, from equation (20), we obtain

\[
\| y^h \|_{L^2(\Omega)}^2 \leq C \| u_0^h \|_{L^2(\Omega)}^2 + C \| f \|_{L^2(0,T;L^2(\Omega))}^2,
\]

where

\[
C = C(T, \| g \|_{L^\infty(0,T)}, \| g' \|_{L^\infty(0,T)}),
\]

as required.

\[\square\]

**Theorem 5** (Convergence). Let \((u, y)\) and \((u^h, y^h)\) be solutions to problems (10) and (11), respectively. If \( g, g' \in L^\infty(0,T), u_0 \in H^{r+1}(\Omega) \) and \( f \in L^2(Q) \), then, for every \( t \in [0,T] \),

\[
\| u - u^h \|_{L^2(\Omega)} \leq Ch^{\frac{r}{2p-1}}
\]

(24)

and

\[
\| y - y^h \|_{L^2(\Omega)} \leq C h^{\frac{r}{2p-1}},
\]

(25)

where the constant \( C \) does not depend on \( h \) but may depend on \( g, u, y \) and their derivatives.
Proof. Noting that \((u, y)\) and \((u^h, y^h)\) are solutions to problems \((10)\) and \((11)\), respectively, and subtracting the second equation of \((11)\) from the second equation of \((10)\), with \(v = v^h \in S^h \subset H^1_0\), we get

\[
\int_\Omega (y - y^h)v^h \, dx = -\int_\Omega v^h \int_0^t g(t-s)(y(x,s) - y^h(x,s)) \, ds \, dx + \int_\Omega g(0)(u - u^h)v^h \, dx \\
- \int_\Omega g(t)(u_0 - u_0^h)v^h \, dx + \int_\Omega v^h \int_0^t g'(t-s)(u(x,s) - u^h(x,s)) \, ds \, dx.
\]

By writing \((y - y^h) = (y - \Pi_h y) + (\Pi_h y - y^h) = \varphi + \psi\) and \((u - u^h) = (u - \Pi_h u) + (\Pi_h u - u^h) = \rho + \theta\), with \(\Pi_h u\) the interpolation of \(u\) in \(S^h\), we have

\[
\int_\Omega \varphi \psi^h \, dx = -\int_\Omega v^h \int_0^t g(t-s)\varphi(x,s) \, ds \, dx + \int_\Omega g(0)\int_\Omega \theta v^h \, dx \\
+ \int_\Omega g(0)\int_\Omega \rho v^h \, dx - g(t)\int_\Omega (u_0 - u_0^h)v^h \, dx - \int_\Omega v^h \int_0^t g(t-s)\psi(x,s) \, ds \, dx \\
+ \int_\Omega \int_0^t g'(t-s)\rho(x,s) \, ds \, dx + \int_\Omega v^h \int_0^t g'(t-s)\theta(x,s) \, ds \, dx. \tag{26}
\]

Making \(v^h = \psi \in S^h\) and applying Young’s inequality to \((26)\), we have

\[
\int_\Omega \psi^2 \, dx \\
\leq C(\epsilon) \int_\Omega \varphi^2 \, dx + \epsilon \int_\Omega \psi^2 \, dx + C(\epsilon) \int_\Omega \theta^2 \, dx + \epsilon \int_\Omega \psi^2 \, dx + C(\epsilon) \int_\Omega \rho^2 \, dx \\
+ \epsilon \int_\Omega \psi^2 \, dx + C(\epsilon) \int_\Omega (u_0 - u_0^h)^2 \, dx + \epsilon \int_\Omega \psi^2 \, dx + \epsilon \int_\Omega \psi^2 \, dx \\
+ C(\epsilon) \int_0^t \varphi^2 ds + \epsilon \int_\Omega \psi^2 \, dx + C(\epsilon) \int_\Omega \int_0^t \rho^2 \, ds \, dx + \epsilon \int_\Omega \psi^2 \, dx \\
+ C(\epsilon) \int_0^t \theta^2 ds + \epsilon \int_\Omega \psi^2 \, dx + C(\epsilon) \int_\Omega \int_0^t \psi^2(x,s) \, ds \, dx.
\]

Choosing \(\epsilon\) appropriately and using Lemma 2, we obtain the inequality

\[
\int_\Omega \psi^2 \, dx \leq Ch^{2(r+1)} + C \int_\Omega \vartheta^2 \, dx + C \int_0^t \int_\Omega \vartheta^2(x,s) \, ds \, dx \\
+ C \int_0^t \int_\Omega \psi^2(x,s) \, ds \, dx. \tag{27}
\]

Applying Gronwall’s lemma to \((27)\), we have

\[
\int_\Omega \psi^2 \, dx \leq Ch^{2(r+1)} + C \int_\Omega \vartheta^2 \, dx + C(1+t) \int_0^t \int_\Omega \vartheta^2(x,s) \, ds \, dx. \tag{28}
\]

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Now, subtracting the first equation in (11) from the first equation in (10), with 
\( w = w^h \in S^h \), we get
\[
\int_{\Omega} \theta w^h \, dx + \int_{\Omega} \left( |\Pi_h \nabla u|^{p-2} \Pi_h \nabla u - |\nabla u^h|^{p-2} \nabla u^h \right) \nabla w^h \, dx = \int_{\Omega} \varphi w^h \, dx \\
+ \int_{\Omega} \psi w^h \, dx - \int_{\Omega} \rho t w^h \, dx + \int_{\Omega} \left( |\Pi_h \nabla u|^{p-2} \Pi_h \nabla u - |\nabla u^h|^{p-2} \nabla u^h \right) \nabla w^h \, dx.
\]

Making \( w^h = \theta \in S^h \), applying Young’s inequality and Lemma 1, we obtain
\[
\frac{1}{2} d \int_{\Omega} \theta^2 \, dx + \frac{1}{2} \int_{\Omega} |\nabla \theta|^p \, dx \leq \frac{1}{2} \int_{\Omega} \varphi^2 + \frac{1}{2} \int_{\Omega} \psi^2 + \frac{1}{2} \int_{\Omega} \theta^2 + \frac{1}{2} \int_{\Omega} \rho^2 + \frac{1}{2} \int_{\Omega} \rho^2 \nabla \theta \, dx.
\]
Choosing \( \varepsilon \) appropriately and using estimate (27) and Lemma 2, we have
\[
\int_{\Omega} \theta^2 \, dx \leq C h^{2(r+1)} + C \int_{\Omega} \theta^2 + \int_{0}^{t} \int_{\Omega} \theta^2 (x,s) \, dx \, ds. \quad (29)
\]
Integrating (29) with respect to \( t \), we get
\[
\int_{\Omega} \theta^2 \, dx \leq C h^{2(r+1)} + C \int_{0}^{t} \int_{\Omega} \theta^2 (x,s) \, dx \, ds. \quad (30)
\]
Applying Gronwall’s lemma to (30), we obtain the inequality
\[
\int_{\Omega} \theta^2 \, dx \leq C h^{2(r+1)} + C h^{\frac{2r}{r+1}}. \quad (31)
\]
Equation (31) with the estimate for \( \rho \) given in Lemma 2 proves (24). Substituting equation (31) in equation (28) and adding the estimate for \( \varphi \), we obtain (25), as required.

4. Discretization in time

The memory term will be discretized using a numerical quadrature and, in order to maintain a good convergence order, we use the Crank-Nicolson method along with the trapezoidal quadrature.

4.1. Crank-Nicolson method

Consider the partition \( 0 = t_0 < t_1 < \cdots < t_N = T \), with step \( \delta = \frac{T}{N} \), of \([0,T]\). Evaluating (11), at \( t = t_{k+\frac{1}{2}} = \frac{t_{k+1} + t_k}{2} \), we obtain
\[
\int_{\Omega} u^h \left( x, t_{k+\frac{1}{2}} \right) w^h \, dx + \int_{\Omega} |\nabla u^h \left( x, t_{k+\frac{1}{2}} \right)|^{p-2} \nabla u^h \left( x, t_{k+\frac{1}{2}} \right) \nabla w^h \, dx = \int_{\Omega} y^h \left( x, t_{k+\frac{1}{2}} \right) w^h \, dx + \int_{\Omega} f \left( x, t_{k+\frac{1}{2}} \right) w^h \, dx
\]
and
\[
\begin{align*}
\int_{\Omega} y^h(x, t_{k+1}) v^h(x) dx &= -\int_{0}^{t_{k+1}} g(t_{k+\frac{1}{2}} - s) \int_{\Omega} y^h(x, s) v^h(x) dxds \\
+ g(0) \int_{\Omega} u^h(x, t_{k+1}) v^h(x) dx - g(t_{k+1}) \int_{\Omega} u_0(x) v^h(x) dx \\
+ \int_{0}^{t_{k+1}} g'(t_{k+\frac{1}{2}} - s) \int_{\Omega} u^h(x, s) v^h(x) dxds \\
&- \int_{0}^{t_{k+1}} g(t_{k+\frac{1}{2}} - s) \int_{\Omega} f(x, s) v^h(x) dxds.
\end{align*}
\]

Let us consider the approximations
\[
\begin{align*}
u^h_k(x, t_{k+\frac{1}{2}}) &\approx \frac{u^h(x, t_{k+1}) - u^h(x, t_k)}{\delta} = \tilde{u}(k+\frac{1}{2}) \\
u^h_k(x, t_{k+\frac{1}{2}}) &\approx \frac{u^h(x, t_{k+1}) + u^h(x, t_k)}{2} = \bar{u}(k+\frac{1}{2})
\end{align*}
\]
and
\[
\begin{align*}
y^h_k(x, t_{k+\frac{1}{2}}) &\approx \frac{y^h(x, t_{k+1}) + y^h(x, t_k)}{2} = \bar{y}(k+\frac{1}{2}).
\end{align*}
\]

We will now approximate the integrals in time using the trapezoidal quadrature so that the order of precision is maintained.
\[
\begin{align*}
\int_{0}^{t_{k+1}} g'(t_{k+\frac{1}{2}} - s) \int_{\Omega} u^h(x, s) v^h(x) dxds \\
\approx \frac{\delta}{2} g'(t_{k+\frac{1}{2}}) \int_{\Omega} u^h_0(x) v^h(x) dx + \delta \sum_{j=1}^{k-1} g'(t_{k+\frac{1}{2}} - t_j) \int_{\Omega} u^h(x, t_j) v^h(x) dx \\
+ \frac{3\delta}{4} g'(t_{k+\frac{1}{2}} - t_k) \int_{\Omega} u^h(x, t_k) v^h(x) dx + \frac{\delta}{8} g'(0) \int_{\Omega} u^h(x, t_k) v^h(x) dx \\
+ \frac{\delta}{8} g'(0) \int_{\Omega} u^h(x, t_{k+1}) v^h(x) dx = Q_g'(u^h)
\end{align*}
\]
and
\[
\begin{align*}
\int_{0}^{t_{k+1}} g(t_{k+\frac{1}{2}} - s) \int_{\Omega} y^h(x, s) v^h(x) dxds \\
\approx \frac{\delta}{2} g(t_{k+\frac{1}{2}}) \int_{\Omega} y^h_0(x) v^h(x) dx + \delta \sum_{j=1}^{k-1} g(t_{k+\frac{1}{2}} - t_j) \int_{\Omega} y^h(x, t_j) v^h(x) dx \\
+ \frac{3\delta}{4} g(t_{k+\frac{1}{2}} - t_k) \int_{\Omega} y^h(x, t_k) v^h(x) dx + \frac{\delta}{8} g(0) \int_{\Omega} y^h(x, t_k) v^h(x) dx \\
+ \frac{\delta}{8} g(0) \int_{\Omega} y^h(x, t_{k+1}) v^h(x) dx = Q_g(y^h).
\end{align*}
\]
4.2. Totally discrete formulation

To simplify the notation, whenever there is no danger of confusion, we will consider a function with the superscript \((j)\) to represent this function evaluated at instant \(t = t_j\).

The totally discrete problem is to find \((U^{(k+1)}, Y^{(k+1)})\), the solution to

\[
\int_{\Omega} \partial U^{(k+\frac{1}{2})} w^h dx + \int_{\Omega} \left| \nabla U^{(k+\frac{1}{2})} \right|^{p-2} \nabla U^{(k+\frac{1}{2})} \nabla w^h dx = \int_{\Omega} Y^{(k+\frac{1}{2})} w^h dx \\
+ \int_{\Omega} f^{(k+\frac{1}{2})} w^h dx \tag{32}
\]

and

\[
\int_{\Omega} Y^{(k+\frac{1}{2})} u^h dx = g(0) \int_{\Omega} U^{(k+\frac{1}{2})} u^h dx - g(t_{k+\frac{1}{2}}) \int_{\Omega} u^h_0(x) v^h dx \\
- Q_g(Y) + Q_{g'}(U) - I(f), \tag{33}
\]

where

\[
I(f) = \int_0^{t_{k+\frac{1}{2}}} g(t_{k+\frac{1}{2}} - s) \int_{\Omega} f(x, s) v^h(x) dx ds.
\]

Equation (33) is a linear system for \(Y^{(k+1)}\). Obtaining a solution \(Y^{(k+1)}\) depending on \(U^{(k+1)}\) from (33) and substituting in (32), one obtains a non linear equation for \(U^{(k+1)}\). Then, using the fixed point theorem, it follows easily that (32) has a solution.

**Theorem 6 (Uniqueness).** If \(g, g' \in L^\infty(0, T)\), the solution to the discrete problem is unique.

**Proof.** The proof is similar to that of Theorem 5 but it is more technical. Suppose that \((U_1^{(k+1)}, Y_1^{(k+1)})\) and \((U_2^{(k+1)}, Y_2^{(k+1)})\) are two solutions to problem (32)-(33). Subtracting the equation for \(Y_2^{(k+1)}\) from the equation for \(Y_1^{(k+1)}\),
we obtain

\[
\left( \frac{1}{2} - \frac{\delta}{8} g(0) \right) \int_{\Omega} (Y_1^{(k+1)} - Y_2^{(k+1)}) v^h \, dx
= \left( \frac{3\delta}{4} g(t_{k+\frac{1}{2}} - t_k) + \frac{\delta}{8} g(0) - \frac{1}{2} \right) \int_{\Omega} (Y_1^{(k)} - Y_2^{(k)}) v^h \, dx
+ \delta \sum_{j=1}^{k-1} g(t_{k+\frac{j}{2}} - t_j) \int_{\Omega} (Y_1^{(j)} - Y_2^{(j)}) v^h \, dx + \frac{\delta}{2} g(t_{k+\frac{j}{2}}) \int_{\Omega} (Y_1^{(0)} - Y_2^{(0)}) v^h \, dx
\]

\[
+ \left( -\frac{1}{2} g(0) - \frac{\delta}{8} g'(0) \right) \int_{\Omega} (U_1^{(k+1)} - U_2^{(k+1)}) v^h \, dx
+ \left( \frac{1}{2} g(0) - \frac{3\delta}{8} g'(t_{k+\frac{j}{2}} - t_k) - \frac{\delta}{8} g'(0) \right) \int_{\Omega} (U_1^{(k)} - U_2^{(k)}) v^h \, dx
\]

\[
+ \delta \sum_{j=1}^{k-1} g'(t_{k+\frac{j}{2}} - t_j) \int_{\Omega} (U_1^{(j)} - U_2^{(j)}) v^h \, dx
+ \left( -g(t_{k+\frac{j}{2}}) + \frac{\delta}{8} g'(t_{k+\frac{j}{2}}) \right) \int_{\Omega} (U_1^{(0)} - U_2^{(0)}) v^h \, dx.
\] (34)

Let \( v^h = Y_1^{(k+1)} - Y_2^{(k+1)} \). As \( g \) and \( g' \) are bounded, we may apply Young's inequality in (34), so

\[
\int_{\Omega} (Y_1^{(k+1)} - Y_2^{(k+1)})^2 \, dx \leq C \int_{\Omega} (Y_1^{(k)} - Y_2^{(k)})^2 \, dx
+ C \delta \sum_{j=1}^{k-1} \int_{\Omega} (Y_1^{(j)} - Y_2^{(j)})^2 \, dx
+ C \int_{\Omega} (U_1^{(k)} - U_2^{(k)})^2 \, dx + C \delta \sum_{j=1}^{k-1} \int_{\Omega} (U_1^{(j)} - U_2^{(j)})^2 \, dx.
\] (35)

Now, applying Gronwall's lemma to (35), we have

\[
\int_{\Omega} (Y_1^{(k+1)} - Y_2^{(k+1)})^2 \, dx \leq C \int_{\Omega} (U_1^{(k+1)} - U_2^{(k+1)})^2 \, dx + C \int_{\Omega} (U_1^{(k)} - U_2^{(k)})^2 \, dx
+ C \delta \sum_{j=1}^{k-1} \int_{\Omega} (U_1^{(j)} - U_2^{(j)})^2 \, dx.
\] (36)

Considering \( w^h = \tilde{U}^{(k+\frac{1}{2})} \) in equation (32) and repeating the procedure for
Applying Young’s inequality to (37) and (36), we have

\[ \int_{\Omega} (U_1^{(k+1)} - U_2^{(k+1)})^2 \, dx \leq C \int_{\Omega} (U_1^{(k)} - U_2^{(k)})^2 \, dx + C \int_{\Omega} (U_1^{(k-1)} - U_2^{(k-1)})^2 \, dx + C \delta \sum_{j=1}^{k-1} \int_{\Omega} (U_1^{(j)} - U_2^{(j)})^2 \, dx. \]

As \( \zeta^{(k+1)} = \int_{\Omega} (U_1^{(k+1)} - U_2^{(k+1)})^2 \, dx \geq 0 \) and \( \zeta^{(0)} = 0 \), we may apply the discrete version of Gronwall’s lemma. Therefore \( \zeta^{(k)} = 0 \) for every \( k \geq 0 \) and so

\[ \int_{\Omega} (U_1^{(k+1)} - U_2^{(k+1)})^2 \, dx = 0, \quad U_1^{(k+1)} = U_2^{(k+1)} \text{ in } L^2(\Omega). \] 

Going back to \( Y \) and substituting (38) in (36), we have

\[ \int_{\Omega} (Y_1^{(k+1)} - Y_2^{(k+1)})^2 \, dx = 0, \]

which proves the required result.

\[ \mathbf{Theorem 7} \text{ (Stability). Let } (U^{(k+1)}, Y^{(k+1)}) \in S^h \times S^h \text{ be solutions to problem (32), (33)}. \text{ If } g, g' \in L^\infty(0,T), \ u_0 \in L^2(\Omega) \text{ and } f \in L^2(Q), \text{ then, for every } k \geq 0, \]

\[ \|U^{(k+1)}\|^2_{L^2(\Omega)} \leq C \|u_0\|^2_{L^2(\Omega)} + C \|f\|^2_{L^2(0,T;L^2(\Omega))} + C \delta \sum_{j=0}^{k} \|f^{(j+\frac{1}{2})}\|^2_{L^2(\Omega)}, \]

and

\[ \|Y^{(k+1)}\|^2_{L^2(\Omega)} \leq C \|u_0\|^2_{L^2(\Omega)} + C \|f\|^2_{L^2(0,T;L^2(\Omega))} + C \delta \sum_{j=0}^{k} \|f^{(j+\frac{1}{2})}\|^2_{L^2(\Omega)}, \]

where

\[ C = C(T, \|g\|_{L^\infty(0,T)}), \|g'\|_{L^\infty(0,T)}). \]
Proof. We write equation (33) in the form

\[
\left( \frac{1}{2} - \frac{\delta}{8}g(0) \right) \int_{\Omega} Y^{(k+1)}v^h dx
= \left( \frac{3\delta}{4}g(t_{k+\frac{1}{2}} - t_k) + \frac{\delta}{8}g(0) - \frac{1}{2} \right) \int_{\Omega} Y^{(k)}v^h dx
+ \delta \sum_{j=1}^{k-1} g(t_{k+\frac{1}{2}} - t_j) \int_{\Omega} Y^{(j)}v^h dx + \left( -\frac{\delta}{8}g(0) - \frac{\delta}{8}g'(0) \right) \int_{\Omega} U^{(k)}v^h dx
+ \left( \frac{1}{2}g(0) - \frac{3\delta}{8}g'(t_{k+\frac{1}{2}} - t_k) - \frac{\delta}{8}g'(0) \right) \int_{\Omega} U^{(k)}v^h dx
+ \delta \sum_{j=1}^{k-1} g'(t_{k+\frac{1}{2}} - t_j) \int_{\Omega} U^{(j)}v^h dx + \left( -\delta g(t_{k+\frac{1}{2}}) + \frac{\delta}{8}g'(t_{k+\frac{1}{2}}) \right) \int_{\Omega} u_0^h v^h dx
- \int_0^{t_{k+\frac{1}{2}}} g(t_{k+\frac{1}{2}} - s) \int_{\Omega} f(x,s) v^h dx.
\]  
and consider \( v^h = Y^{(k+1)} \). Then, since \( g \) and \( g' \) are bounded, we may apply Young’s inequality to (39) and thus obtain

\[
\|Y^{(k+1)}\|^2_{L^2(\Omega)} \leq C\|Y^{(k)}\|^2_{L^2(\Omega)} + C\delta \sum_{j=1}^{k-1} \|Y^{(j)}\|^2_{L^2(\Omega)} + C\|U^{(k+1)}\|^2_{L^2(\Omega)}
+ C\|U^{(k)}\|^2_{L^2(\Omega)} + C\delta \sum_{j=1}^{k-1} \|U^{(j)}\|^2_{L^2(\Omega)} + C\|u_0^h\|^2_{L^2(\Omega)} + C\|f\|^2_{L^2(\Omega)}. \tag{40}
\]

Applying discrete version of Gronwall’s lemma to (40), we have

\[
\|Y^{(k+1)}\|^2_{L^2(\Omega)} \leq C\|U^{(k+1)}\|^2_{L^2(\Omega)} + C\|U^{(k)}\|^2_{L^2(\Omega)} + C\delta \sum_{j=1}^{k-1} \|U^{(j)}\|^2_{L^2(\Omega)}
+ C\|f\|^2_{L^2(0,T;L^2(\Omega))} + C\|u_0^h\|^2_{L^2(\Omega)}. \tag{41}
\]

Returning now to equation (32) and considering \( w^h = \bar{U}^{(k+\frac{1}{2})} \), we have

\[
\int_{\Omega} (U^{(k+1)})^2 dx - \int_{\Omega} (U^{(k)})^2 dx + 2C\delta \int_{\Omega} \sqrt{\frac{\nabla U^{(k+1)} + \nabla U^{(k)}}{2}} \left| \frac{\nabla U^{(k+1)} + \nabla U^{(k)}}{2} \right| dx
= 2\delta \int_{\Omega} (U^{(k+\frac{1}{2})} - \bar{U}^{(k+\frac{1}{2})}) dx + 2\delta \int_{\Omega} \bar{U}^{(k+\frac{1}{2})} dx. \tag{42}
\]

Applying Young’s inequality to (42) and (41), we get

\[
\|U^{(k+1)}\|^2_{L^2(\Omega)} \leq C\|U^{(k)}\|^2_{L^2(\Omega)} + C\|U^{(k-1)}\|^2_{L^2(\Omega)} + C\delta \sum_{j=1}^{k-1} \|U^{(j)}\|^2_{L^2(\Omega)}
+ C\|u_0^h\|^2_{L^2(\Omega)} + C\|f\|^2_{L^2(0,T;L^2(\Omega))} + C\delta \|f\|^2_{L^2(\Omega)}. \tag{43}
\]
Applying the discrete version of Gronwall’s lemma to (43), we have

$$\|U^{(k+1)}\|^2_{L^2(\Omega)} \leq C\|u^h_0\|^2_{L^2(\Omega)} + C\|f\|^2_{L^2([0,T];L^2(\Omega))} + C\delta \sum_{j=0}^k \|f^{(j+\frac{1}{2})}\|^2_{L^2(\Omega)}.$$ 

Then, from equation (40), we obtain

$$\|Y^{(k+1)}\|^2_{L^2(\Omega)} \leq C\|u^h_0\|^2_{L^2(\Omega)} + C\|f\|^2_{L^2([0,T];L^2(\Omega))} + C\delta \sum_{j=0}^k \|f^{(j+\frac{1}{2})}\|^2_{L^2(\Omega)},$$

where

$$C = C(T, \|g\|_{L^\infty(0,T)}, \|g'\|_{L^\infty(0,T)}),$$

which proves the theorem. \qed

**Theorem 8 (Convergence).** Let $(u, y)$ and $(U, Y)$ be solutions to problems (10) and (42)–(43), respectively. If $g, g' \in L^\infty(0,T)$ and $u_0 \in H^{r+1}(\Omega)$, then, for $\delta$ sufficiently small,

$$\|u(x, t_k) - U^{(k)}(x)\|_{L^2(\Omega)} \leq C\left(h^{r+1} + \delta^2 + h^{|\alpha|} + \delta^{\frac{|\alpha|}{2}}\right)$$

and

$$\|y(x, t_k) - Y^{(k)}(x)\|_{L^2(\Omega)} \leq C\left(h^{r+1} + \delta^2 + h^{|\alpha|} + \delta^{\frac{|\alpha|}{2}}\right),$$

where the constant $C$ does not depend on $h$ or $\delta$ but may depend on $g, u, y$ and their derivatives.

**Proof.** Subtracting (43) from the second equation of (10) evaluated at $t = t_{k+\frac{1}{2}}$ with $v = v^h$, we have

$$\int_\Omega \left(y^{(k+\frac{1}{2})} - Y^{(k+\frac{1}{2})}\right)v^h dx$$

$$= g(0) \int_\Omega \left(u^{(k+\frac{1}{2})} - U^{(k+\frac{1}{2})}\right)v^h dx + g(t_{k+\frac{1}{2}}) \int_\Omega (u^h_0 - u_0) v^h dx$$

$$- \int_0^{t_{k+\frac{1}{2}}} g(t_{k+\frac{1}{2}} - s) \int_\Omega y(x, s)v^h(x) dx ds + Q_y(Y)$$

$$+ \int_0^{t_{k+\frac{1}{2}}} g'(t_{k+\frac{1}{2}} - s) \int_\Omega u(x, s)v^h(x) dx ds - Q_{y'}(U)$$

$$= g(0)I_1 + g\left(t_{k+\frac{1}{2}}\right)I_2 + I_3 - I_4.$$

Considering $y(x, t_j) - Y^{(j)} = \varphi(x, t_j) + \psi(x, t_j)$ as before, we have

$$\int_\Omega \left(y^{(k+\frac{1}{2})} - Y^{(k+\frac{1}{2})}\right)v^h dx$$

$$= \int_\Omega \left(y^{(k+\frac{1}{2})} - \bar{y}^{(k+\frac{1}{2})}\right)v^h dx$$

$$+ \int_\Omega \bar{\varphi}^{(k+\frac{1}{2})}v^h dx + \int_\Omega \bar{\psi}^{(k+\frac{1}{2})}v^h dx.$$
Likewise, Letting $u(x, t_j) - U^{(j)} = \rho(x, t_j) + \theta(x, t_j) = \rho^{(j)} + \theta^{(j)}$, we have

\[
I_1 = \int_{\Omega} \left( u^{(k+\frac{1}{2})} - \tilde{U}^{(k+\frac{1}{2})} \right) v^h dx = \int_{\Omega} \left( u^{(k+\frac{1}{2})} - \tilde{u}^{(k+\frac{1}{2})} \right) v^h dx + \int_{\Omega} \tilde{\rho}^{(k+\frac{1}{2})} v^h dx + \int_{\Omega} \tilde{\theta}^{(k+\frac{1}{2})} v^h dx.
\]

We can write

\[
I_3 = \int_0^{t+\frac{1}{2}} g(t+\frac{s}{2} - s) \int_{\Omega} g(x, s)v^h(x)dx ds - Q_g(y) + Q_g(y) - Q_g(Y),
\]

where

\[
Q_g(y) - Q_g(Y) = \frac{\delta}{2} g(t+\frac{s}{2}) \int_{\Omega} \left( y_0 - Y^{(0)} \right) v^h dx + \delta \sum_{j=1}^{k-1} g(t+\frac{s}{2} - t_j) \int_{\Omega} \varphi^{(j)} v^h dx + \frac{3\delta}{4} g(t+\frac{s}{2} - t_k) + \frac{\delta}{8} g(0) \int_{\Omega} \varphi^{(k)} v^h dx + \frac{3\delta}{4} g(t+\frac{s}{2} - t_k) \int_{\Omega} \psi^{(k)} v^h dx + \frac{\delta}{8} g(0) \int_{\Omega} \varphi^{(k+1)} v^h dx + \frac{3\delta}{4} g(t+\frac{s}{2} - t_k) \int_{\Omega} \theta^{(k)} v^h dx.
\]

Likewise,

\[
I_4 = \int_0^{t+\frac{1}{2}} \frac{g'}{2} \left( t+\frac{s}{2} - s \right) \int_{\Omega} g(x, s)v^h(x)dx ds - Q_{g'}(u) + Q_{g'}(u) - Q_{g'}(U),
\]

where

\[
Q_{g'}(u) - Q_{g'}(U) = \frac{\delta}{2} g'(t+\frac{s}{2}) \int_{\Omega} \left( u_0(x) - U^{(0)} \right) v^h dx + \frac{3\delta}{4} g'(t+\frac{s}{2} - t_k) \int_{\Omega} \rho^{(k)} v^h dx + \frac{3\delta}{4} g'(t+\frac{s}{2} - t_k) \int_{\Omega} \theta^{(k)} v^h dx + \frac{\delta}{8} g'(0) \int_{\Omega} \rho^{(k+1)} v^h dx + \frac{\delta}{8} g'(0) \int_{\Omega} \theta^{(k+\frac{1}{2})} v^h dx,
\]

\[\text{17}\]
and hence we can write

\[
\int_{\Omega} \frac{\psi(\kappa + \frac{1}{2})}{g} v^h dx = - \int_{\Omega} \left( g(\kappa + \frac{1}{2}) - \overline{g}(\kappa + \frac{1}{2}) \right) v^h dx - \frac{1}{2} \int_{\Omega} \varphi(\kappa + 1) v^h dx
\]

\[- \frac{1}{2} \int_{\Omega} \varphi(\kappa) v^h dx + g(0) \int_{\Omega} \left( h(\kappa + \frac{1}{2}) - \overline{h}(\kappa + \frac{1}{2}) \right) v^h dx + \frac{g(0)}{2} \int_{\Omega} \varphi(\kappa + 1) v^h dx
\]

\[+ \frac{g(0)}{2} \int_{\Omega} \varphi(\kappa) v^h dx + \int_{\Omega} \theta(\kappa + \frac{1}{2}) v^h dx + g(t_{\kappa + \frac{1}{2}}) \int_{\Omega} \left( U(0) - U_0 \right) v^h dx
\]

\[- \int_{0}^{t_{\kappa + \frac{1}{2}}} g(t_{\kappa + \frac{1}{2}} - s) \int_{\Omega} y(x, s) v^h dx ds + Q_g(y)
\]

\[- \frac{\delta}{2} g(t_{\kappa + \frac{1}{2}}) \int_{\Omega} (y_0 - Y(0)) v^h dx - \delta \sum_{j=1}^{k-1} g(t_{\kappa + \frac{1}{2}} - t_j) \int_{\Omega} \varphi(j) v^h dx
\]

\[- \delta \sum_{j=1}^{k-1} g(t_{\kappa + \frac{1}{2}} - t_j) \int_{\Omega} \psi(j) v^h dx - \frac{3\delta}{4} g(t_{\kappa + \frac{1}{2}} - t_k) \int_{\Omega} \varphi(k) v^h dx
\]

\[- \frac{3\delta}{4} g(t_{\kappa + \frac{1}{2}} - t_k) \int_{\Omega} \varphi(\kappa) v^h dx - \frac{\delta}{8} g(0) \int_{\Omega} \varphi(\kappa) v^h dx + \frac{\delta}{8} g(0) \int_{\Omega} \varphi(\kappa + 1) v^h dx
\]

\[- \frac{\delta}{4} g(0) \int_{\Omega} \psi(\kappa + \frac{1}{2}) v^h dx - \int_{0}^{t_{\kappa + \frac{1}{2}}} g'(t_{\kappa + \frac{1}{2}} - s) \int_{\Omega} u(x, s) v^h(x) dx ds - Q_{\theta'}(u)
\]

\[+ \frac{\delta}{2} g'(t_{\kappa + \frac{1}{2}}) \int_{\Omega} (u_0 - U(0)) v^h(x) dx + \delta \sum_{j=1}^{k-1} g'(t_{\kappa + \frac{1}{2}} - t_j) \int_{\Omega} \rho(j) v^h dx
\]

\[+ \delta \sum_{j=1}^{k-1} g'(t_{\kappa + \frac{1}{2}} - t_j) \int_{\Omega} \theta(j) v^h dx + \frac{3\delta}{4} g'(t_{\kappa + \frac{1}{2}} - t_k) \int_{\Omega} \rho(k) v^h dx
\]

\[+ \frac{3\delta}{4} g'(t_{\kappa + \frac{1}{2}} - t_k) \int_{\Omega} \varphi(k) v^h dx + \frac{\delta}{8} g'(0) \int_{\Omega} \rho(\kappa) v^h dx
\]

\[+ \frac{\delta}{4} g'(0) \int_{\Omega} \varphi(\kappa + 1) v^h dx + \frac{\delta}{4} g'(0) \int_{\Omega} \varphi(k + \frac{1}{2}) v^h dx.
\]

Using the estimates from the error term of the trapezoidal rule with $v^h =$
\( \psi^{(k+\frac{1}{2})} \) and applying Young’s inequality, we have

\[
\left( 1 - 25\varepsilon + \frac{\delta g(0)}{4} \right) \| \psi^{(k+\frac{1}{2})} \|_{L^2(\Omega)}^2 \leq C(\varepsilon) \| \psi^{(k+\frac{1}{2})} - \bar{y}^{(k+\frac{1}{2})} \|_{L^2(\Omega)}^2 \\
+ C(\varepsilon)(1 + \delta^2) \| \varphi^{(k+1)} \|_{L^2(\Omega)}^2 + C(\varepsilon)(1 + 2\delta^2) \| \varphi^{(k)} \|_{L^2(\Omega)}^2 \\
+ C(\varepsilon)\| u^{(k+\frac{1}{2})} - \bar{u}^{(k+\frac{1}{2})} \|_{L^2(\Omega)}^2 + C(\varepsilon)(1 + \delta^2) \| \rho^{(k+1)} \|_{L^2(\Omega)}^2 \\
+ C(\varepsilon)(1 + 2\delta^2) \| \rho^{(k)} \|_{L^2(\Omega)}^2 + C(\varepsilon)(1 + \delta^2) \| u_0(x) - U(0) \|_{L^2(\Omega)}^2 \\
+ C(\varepsilon)\delta^4 \left( \| y \|_{L^\infty(0,T;L^2(\Omega))} + \| y_t \|_{L^\infty(0,T;L^2(\Omega))} + \| y_{tt} \|_{L^\infty(0,T;L^2(\Omega))} \right) \\
+ C(\varepsilon)\delta^2 \| y_0 - Y(0) \|_{L^2(\Omega)}^2 + C(\varepsilon)\delta^2 \sum_{j=1}^{k-1} \| \theta^{(j)} \|_{L^2(\Omega)}^2 \\
+ C(\varepsilon)\delta^2 \sum_{j=1}^{k-1} \| \psi^{(j)} \|_{L^2(\Omega)}^2 + C(\varepsilon)\delta^2 \sum_{j=1}^{k-1} \| \varphi^{(j)} \|_{L^2(\Omega)}^2 + C(\varepsilon)\delta^4 \| \psi^{(k)} \|_{L^2(\Omega)}^2 \\
+ C(\varepsilon)\delta^4 \left( \| u \|_{L^\infty(0,T;L^2(\Omega))} + \| u_t \|_{L^\infty(0,T;L^2(\Omega))} + \| u_{tt} \|_{L^\infty(0,T;L^2(\Omega))} \right) \\
+ C(\varepsilon)\delta^2 \sum_{j=1}^{k-1} \| \rho^{(j)} \|_{L^2(\Omega)}^2 + C(\varepsilon)\delta^2 \| \theta^{(k)} \|_{L^2(\Omega)}^2 \\
+ C(\varepsilon)(1 + \delta^2) \| \bar{\theta}^{(k+\frac{1}{2})} \|_{L^2(\Omega)}^2. \quad (44)
\]

By Lemma 2, some interpolation error bounds and choosing appropriate \( \varepsilon \) and \( \delta \), we can write

\[
\| \psi^{(k+\frac{1}{2})} \|_{L^2(\Omega)}^2 \leq C(h^{2(r+1)} + \delta^4) + C\| \bar{\theta}^{(k+\frac{1}{2})} \|_{L^2(\Omega)}^2 + C\delta^2 \sum_{j=0}^{k} \| \theta^{(j)} \|_{L^2(\Omega)}^2 \\
+ C\delta^2 \sum_{j=0}^{k} \| \psi^{(j)} \|_{L^2(\Omega)}^2,
\]

Then

\[
\| \psi^{(k+1)} \|_{L^2(\Omega)}^2 \leq C\| \psi^{(k)} \|_{L^2(\Omega)}^2 + C(h^{2(r+1)} + \delta^4) + C\| \bar{\theta}^{(k+\frac{1}{2})} \|_{L^2(\Omega)}^2 \\
+ C\delta^2 \sum_{j=0}^{k} \| \theta^{(j)} \|_{L^2(\Omega)}^2 + C\delta^2 \sum_{j=0}^{k} \| \psi^{(j)} \|_{L^2(\Omega)}^2.
\]

By the discrete version of Gronwall’s lemma,

\[
\| \psi^{(k+1)} \|_{L^2(\Omega)}^2 \leq C(1 + \delta)(h^{2(r+1)} + \delta^4) + C\| \bar{\theta}^{(k+1)} \|_{L^2(\Omega)}^2 + C\| \theta^{(k)} \|_{L^2(\Omega)}^2 \\
+ C\| \theta^{(k-1)} \|_{L^2(\Omega)}^2 + C(1 + \delta)\delta^2 \sum_{j=0}^{k-1} \| \theta^{(j)} \|_{L^2(\Omega)}^2.
\]
Now we go back to the equation for $u$. Subtracting equation (32) from the first equation of (10), evaluated at $t = \frac{k}{2}$, and considering $w = w^h \in S^h$, we obtain

$$
\int_{\Omega} \left( u_i^{(k+\frac{1}{2})} - \partial U^{(k+\frac{1}{2})} \right) w^h dx \\
+ \int_{\Omega} \left( \nabla u^{(k+\frac{1}{2})} \right)^{p-2} \nabla u^{(k+\frac{1}{2})} - \left| \nabla U^{(k+\frac{1}{2})} \right|^{p-2} \nabla U^{(k+\frac{1}{2})} \right) \nabla w^h dx \\
= \int_{\Omega} \left( y^{(k+\frac{1}{2})} - \bar{y}^{(k+\frac{1}{2})} \right) w^h dx.
$$

In this case,

$$
\int_{\Omega} \left( u_i^{(k+\frac{1}{2})} - \partial U^{(k+\frac{1}{2})} \right) w^h dx + \int_{\Omega} \left( \partial u^{(k+\frac{1}{2})} - \partial U^{(k+\frac{1}{2})} \right) w^h dx \\
+ \int_{\Omega} \left( \left| \nabla u^{(k+\frac{1}{2})} \right|^{p-2} \nabla u^{(k+\frac{1}{2})} - \left| \nabla U^{(k+\frac{1}{2})} \right|^{p-2} \nabla U^{(k+\frac{1}{2})} \right) \nabla w^h dx \\
= \int_{\Omega} \left( y^{(k+\frac{1}{2})} - \bar{y}^{(k+\frac{1}{2})} \right) w^h dx + \int_{\Omega} \left( \bar{y}^{(k+\frac{1}{2})} - \bar{y}^{(k+\frac{1}{2})} \right) w^h dx,
$$

and therefore

$$
\int_{\Omega} \partial \theta^{(k+\frac{1}{2})} w^h dx \\
+ \int_{\Omega} \left( \left| \nabla \Pi_h u^{(k+\frac{1}{2})} \right|^{p-2} \nabla \Pi_h u^{(k+\frac{1}{2})} - \left| \nabla U^{(k+\frac{1}{2})} \right|^{p-2} \nabla U^{(k+\frac{1}{2})} \right) \nabla w^h dx \\
= - \int_{\Omega} \left( u_i^{(k+\frac{1}{2})} - \bar{u}^{(k+\frac{1}{2})} \right) w^h dx - \int_{\Omega} \partial \rho^{(k+\frac{1}{2})} w^h dx \\
+ \int_{\Omega} \left( \left| \nabla u^{(k+\frac{1}{2})} \right|^{p-2} \nabla u^{(k+\frac{1}{2})} - \left| \nabla \Pi_h u^{(k+\frac{1}{2})} \right|^{p-2} \nabla \Pi_h u^{(k+\frac{1}{2})} \right) \nabla w^h dx \\
+ \int_{\Omega} \left( y^{(k+\frac{1}{2})} - \bar{y}^{(k+\frac{1}{2})} \right) w^h dx + \int_{\Omega} \bar{y}^{(k+\frac{1}{2})} w^h dx + \int_{\Omega} \bar{y}^{(k+\frac{1}{2})} w^h dx.
$$
Taking $w^h = \bar{\theta}^{(k+\frac{1}{2})}$ and applying Young’s inequality, we obtain

$$
\begin{align*}
\int_{\Omega} \bar{\theta}^{(k+\frac{1}{2})} \bar{\theta}^{(k+\frac{1}{2})} \, dx + C_2 \| \nabla \bar{\theta}^{(k+\frac{1}{2})} \|_{L^p(\Omega)}^p \\
\leq C_{c_1} \| u_t \|_{L^2(\Omega)}^2 + C_{c_1} \| \partial \bar{\theta}^{(k+\frac{1}{2})} \|_{L^2(\Omega)}^2 + \epsilon_1 \| \bar{\theta}^{(k+\frac{1}{2})} \|_{L^2(\Omega)}^2 \\
+ C_{c_1} \| \partial \theta \|_{L^2(\Omega)}^2 \leq \bar{C} - \epsilon_1 \| \bar{\theta}^{(k+\frac{1}{2})} \|_{L^2(\Omega)}^2 + C_{c_1} \| \bar{\theta}^{(k+\frac{1}{2})} \|_{L^2(\Omega)}^2 \\
+ C_{c_1} \| \psi \|_{L^2(\Omega)}^2 + \epsilon_1 \| \bar{\theta}^{(k+\frac{1}{2})} \|_{L^2(\Omega)}^2 + C_{c_1} \| \psi \|_{L^2(\Omega)}^2 + \epsilon_1 \| \bar{\theta}^{(k+\frac{1}{2})} \|_{L^2(\Omega)}^2.
\end{align*}
$$

Using Lemma 1 and Lemma 2, some numerical differentiation and interpolation error bounds and choosing $\epsilon_2 < \frac{C}{2}$, we can write

$$
\begin{align*}
\frac{1}{2\delta} \left( \| \bar{\theta}^{(k+1)} \|_{L^2(\Omega)}^2 + \| \bar{\theta}^{(k)} \|_{L^2(\Omega)}^2 \right) \\
\leq C \delta^4 + C \| \bar{\theta}^{(k+1)} \|_{L^2(\Omega)}^2 + C \| \bar{\theta}^{(k)} \|_{L^2(\Omega)}^2 + C \| \psi^{(k+1)} \|_{L^2(\Omega)}^2 + C \| \psi^{(k)} \|_{L^2(\Omega)}^2 \\
+ C \| \psi \|_{L^2(\Omega)}^2 + C \| \psi \|_{L^2(\Omega)}^2 + C \| \psi \|_{L^2(\Omega)}^2 + C \| \psi \|_{L^2(\Omega)}^2 \\
\leq (1 - C \delta) \| \bar{\theta}^{(k+1)} \|_{L^2(\Omega)}^2 + C \delta \left( h^{2(r+1)} + \delta \right) + C \delta \left( \delta h^{2(r+1)} + h \delta \right) \\
+ (C \delta - 1) \| \bar{\theta}^{(k)} \|_{L^2(\Omega)}^2 + C \delta \| \bar{\theta}^{(k-1)} \|_{L^2(\Omega)}^2 + C \delta^3 \sum_{j=0}^{k-1} \| \bar{\theta}^{(j)} \|_{L^2(\Omega)}^2.
\end{align*}
$$

For $\delta$ sufficiently small, we have

$$
\| \bar{\theta}^{(k+1)} \|_{L^2(\Omega)}^2 \leq C \delta \left( h^{2(r+1)} + \delta^4 \right) + C \delta \left( \delta h^{2(r+1)} + h \delta \right) - C \| \bar{\theta}^{(k)} \|_{L^2(\Omega)}^2 \\
+ C \delta \| \bar{\theta}^{(k-1)} \|_{L^2(\Omega)}^2 + C \delta^3 \sum_{j=0}^{k-1} \| \bar{\theta}^{(j)} \|_{L^2(\Omega)}^2.
$$

From the discrete version of Gronwall’s lemma,

$$
\| \bar{\theta}^{(k+1)} \|_{L^2(\Omega)}^2 \leq C \delta \left( h^{2(r+1)} + \delta^4 + h \delta \right).
$$
Returning to the equation of $\|\psi^{(k+1)}\|_{L^2(\Omega)}^2$,

$$
\|\psi^{(k+1)}\|_{L^2(\Omega)}^2 \leq C \left( h^{2(r+1)} + \delta^4 \right) + C \delta \left( h^{2(r+1)} + \delta^4 + h^{\frac{rp}{r-1}} + \delta^{\frac{2p}{r}} \right) 
+ C(1 + \delta)\delta^2 \left( h^{2(r+1)} + \delta^4 + h^{\frac{rp}{r-1}} + \delta^{\frac{2p}{r}} \right).
$$

Finally, adding the estimates of $\rho^{(k+1)}$ and $\varphi^{(k+1)}$ given by Lemma 2 the required result is obtained.

We notice that in (44) if $g(0) \geq 0$ then $\delta$ can be any positive value, otherwise should be $\delta < -\frac{1}{g(0)}$ sufficiently small, for example $\delta = -\frac{1}{g(0)}$ for $\epsilon = \frac{1}{100}$.

5. Final comments

In this paper, we applied the finite element method with a polynomial basis of degree $r$ complemented with the Crank-Nicolson method and the trapezoid quadrature to a class of evolution differential equations with $p$-Laplacian and memory. The memory term was separated from the $p$-Laplacian using a mixed formulation. We demonstrated the existence, uniqueness and regularity of the discrete solutions under mild conditions on the data. We also obtained the convergence order depending on $p$ in the classical norms. It was found that the convergence order decreases as $p \to \infty$, but it is always bigger than $r^2$ for $h$ and bigger than 1 for $\delta$.

As future work, we intend to find an efficient method to solve the nonlinear system of algebraic equations and to implement the method in a computational system, such as in a Matlab environment, to illustrate the theoretical results. An interesting challenge is to do a similar study for equation (1) with $p$ depending on $x$. The fact that the $p(x)$-Laplacian is not homogeneous makes the problem more complicated than the problem with constant $p$.

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References

[1] S. Antontsev, S. Shmarev, Evolution PDEs with nonstandard growth conditions, volume 4 of Atlantis Studies in Differential Equations, Atlantis Press, Paris, 2015. Existence, uniqueness, localization, blow-up.

[2] S. Antontsev, S. Shmarev, J. Simsen, M.S. Simsen, On the evolution $p$-Laplacian with nonlocal memory, Nonlinear Anal. 134 (2016) 31–54.
[3] S. Antontsev, S. Shmarev, J. Simsen, M. Stefanello Simsen, Differential inclusion for the evolution $p(x)$-Laplacian with memory, Electron. J. Differential Equations (2019) Paper No. 26, 28.

[4] V. Barbu, M.A. Malik, Semilinear integro-differential equations in Hilbert space, J. Math. Anal. Appl. 67 (1979) 452–475.

[5] J.W. Barrett, W.B. Liu, Finite element approximation of the parabolic $p$-Laplacian, SIAM J. Numer. Anal. 31 (1994) 413–428.

[6] C. Chen, T. Shih, Finite element methods for integrodifferential equations, volume 9 of Series on Applied Mathematics, World Scientific Publishing Co., Inc., River Edge, NJ, 1998.

[7] M. Chipot, Elements of nonlinear analysis, Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks], Birkhäuser Verlag, Basel, 2000.

[8] M. Chipot, T. Savitska, Nonlocal $p$-Laplace equations depending on the $L^p$ norm of the gradient, Adv. Differential Equations 19 (2014) 997–1020.

[9] S.S. Chow, Finite element error estimates for nonlinear elliptic equations of monotone type, Numer. Math. 54 (1989) 373–393.

[10] P.G. Ciarlet, The finite element method for elliptic problems, volume 40 of Classics in Applied Mathematics, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2002. Reprint of the 1978 original [North-Holland, Amsterdam; MR0520174 (58 #25001)].

[11] M.G. Crandall, S.O. Londen, J.A. Nohel, An abstract nonlinear Volterra integrodifferential equation, J. Math. Anal. Appl. 64 (1978) 701–735.

[12] E. DiBenedetto, Degenerate parabolic equations, Universitext, Springer-Verlag, New York, 1993.

[13] L.C. Evans, Partial differential equations, volume 19 of Graduate Studies in Mathematics, American Mathematical Society, Providence, RI, 1998.

[14] R. Glowinski, A. Marrocco, Sur l’approximation, par éléments finis d’ordre un, et la résolution, par pénalisation-dualité, d’une classe de problèmes de Dirichlet non linéaires, Rev. Française Automat. Informat. Recherche Opérationnelle Sér. Rouge Anal. Numér. 9 (1975) 41–76.

[15] A. Himadan, Well defined extinction time of solutions for a class of weak-viscoelastic parabolic equation with positive initial energy, AIMS Math. 6 (2021) 4331–4344.

[16] J.L. Lions, Quelques méthodes de résolution des problèmes aux limites non linéaires, Dunod; Gauthier-Villars, Paris, 1969.
[17] R.C. MacCamy, Stability theorems for a class of functional differential equations, SIAM J. Appl. Math. 30 (1976) 557–576.

[18] K. Mustapha, H. Brunner, H. Mustapha, D. Schötzau, An $hp$-version discontinuous Galerkin method for integro-differential equations of parabolic type, SIAM J. Numer. Anal. 49 (2011) 1369–1396.

[19] J.A. Nohel, Nonlinear Volterra equations for heat flow in materials with memory, in: Integral and functional differential equations (Proc. Conf., West Virginia Univ., Morgantown, W. Va., 1979), volume 67 of Lecture Notes in Pure and Appl. Math., Dekker, New York, 1981, pp. 3–82.

[20] A.K. Pani, G. Fairweather, R.I. Fernandes, Alternating direction implicit orthogonal spline collocation methods for an evolution equation with a positive-type memory term, SIAM J. Numer. Anal. 46 (2007/08) 344–364.

[21] G.M.M. Reddy, R.K. Sinha, J.A. Cuminato, A posteriori error analysis of the Crank-Nicolson finite element method for parabolic integro-differential equations, J. Sci. Comput. 79 (2019) 414–441.

[22] R.K. Sinha, R.E. Ewing, R.D. Lazarov, Some new error estimates of a semidiscrete finite volume element method for a parabolic integro-differential equation with nonsmooth initial data, SIAM J. Numer. Anal. 43 (2006) 2320–2343.

[23] R.K. Sinha, R.E. Ewing, R.D. Lazarov, Mixed finite element approximations of parabolic integro-differential equations with nonsmooth initial data, SIAM J. Numer. Anal. 47 (2009) 3269–3292.

[24] F. Tchier, I. Dassios, F. Tawfiq, L. Ragoub, On the approximate solution of partial integro-differential equations using the pseudospectral method based on Chebyshev cardinal functions, Mathematics 9 (2021) 286.

[25] W. Wang, Q. Hong, Two-grid economical algorithms for parabolic integro-differential equations with nonlinear memory, Appl. Numer. Math. 142 (2019) 28–46.

[26] K. Zennir, T. Miyasita, Lifespan of solutions for a class of pseudo-parabolic equation with weak-memory, Alexandria Engineering Journal 59 (2020) 957–964.

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