Lepton masses and mixings, and muon anomalous magnetic moment in an extended $B-L$ model with type I seesaw mechanism

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We propose a $B-L$ model combined with the $S_3 \times Z_3 \times Z_4$ discrete symmetry which successfully explains the recent 3 + 1 sterile - active neutrino data. The smallness of neutrino mass is obtained through the type-I seesaw mechanism. The active-active and sterile-active neutrino mixing angles are predicted to be consistent with the recent constraints in which $0.3401 (0.3402) \leq \sin^2 \theta_{12} \leq 0.3415 (0.3416), 0.456 (0.433) \leq \sin^2 \theta_{23} \leq 0.544 (0.545), 2.00 (2.018) \leq 10^2 \times \sin^2 \theta_{13} \leq 2.405 (2.424), 156 (140.8) \leq \delta^{(o)}_{CP} \leq 172 (167.2)$ for normal (inverted) ordering of the three neutrino scenario, and $0.015 (0.022) \leq s_{14}^2 \leq 0.045 (0.029), 0.005 (0.0095) \leq s_{24}^2 \leq 0.012 (0.012), 0.003 (0.009) \leq s_{34}^2 \leq 0.011$ for normal (inverted) ordering of the 3 + 1 neutrino scenario. Our model predicts flavour conserving leptonic neutral scalar interactions and successfully explains the muon $g-2$ anomaly.

I. INTRODUCTION

Recently, there have been experimental observations that cannot be explained by the three-neutrino oscillation framework [1-13]. However, these observations could be explained by adding at least one additional neutrino (called sterile neutrino) with mass in the eV range having non-trivial mixing with active neutrinos. The mentioned sterile neutrinos are singlets under $SU(2)_L$ which do not take part in the weak interaction but mix with the active ones that can be verified in the oscillation experiments. Nowadays, there are a number of schemes favouring the existence of sterile neutrinos, including the (3+1) scheme [13-29] in which one sterile neutrino with mass in eV scale is heavier than the three active ones; The (3+1+1) scheme [30-34] in which one sterile neutrino with mass in eV scale and the other is much heavier than 1 eV; The (1+3+1) scheme [35-37] in which one sterile neutrino is lighter than the three active ones and the other is heavier; The (3+2) scheme [38-44] in which the two sterile neutrinos are lighter than the three active ones are added to the standard three-neutrino oscillation framework. Among the schemes with sterile neutrinos, the one with one additional sterile neutrino with mass in the eV range (called four neutrino scheme) is the simplest extension of standard three neutrino mixing that can accommodate the anomalous results of short-baseline neutrino oscillations. Among four neutrino schemes, the 3+1 scheme is preferred because the 1+3 scheme whose one sterile neutrino is lighter than the active ones and the three active neutrinos are in eV scale is ruled out by Cosmology while the 2+2 scheme is not suitable with the atmospheric and the solar neutrino oscillation data. Currently, the neutrino mass squared differences and the mixing angles in the three-neutrino scheme [35] and 3+1 neutrino mixing angles [40], at the best-fit points and 3σ range, are shown in Table I. Besides, the 3σ CL ranges on the magnitude of the elements of the leptonic mixing matrix, for three neutrino scheme, are [47]:

\[
|U_{3\nu}\text{ with } SK\text{-atm}| = \begin{pmatrix}
0.801 & 0.845 & 0.513 & 0.579 & 0.143 & 0.155 \\
0.234 & 0.500 & 0.417 & 0.689 & 0.637 & 0.776 \\
0.271 & 0.535 & 0.477 & 0.694 & 0.613 & 0.756
\end{pmatrix}.
\]

One notable attribute of discrete symmetries is that they provide an explanation of the neutrino oscillation data. For the 3 + 1 scheme, the $U(1)_{B-L}$ extension with $S_3$ symmetry was presented in Ref. [48] without mentioning

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Table I: The neutrino oscillation data for three-neutrino scheme at 3\(\sigma\) range and the best fit points taken from Ref. [36], and 3+1 neutrino mixing constraints taken from Ref. [37]

| Parameters | Normal hierarchy (NH) | Inverted hierarchy (IH) |
|------------|----------------------|------------------------|
| \(\Delta m_{31}^2(10^{-5}\text{eV}^2)\) | 6.94 \(\rightarrow\) 8.14 (7.50) | 6.94 \(\rightarrow\) 8.14 (7.50) |
| \(\Delta m_{21}^2(10^{-3}\text{eV}^2)\) | 2.47 \(\rightarrow\) 2.63 (2.55) | 2.37 \(\rightarrow\) 2.53 (2.45) |
| \(s_{12}^2/10^{-1}\) | 2.71 \(\rightarrow\) 3.69 (3.38) | 2.71 \(\rightarrow\) 3.69 (3.38) |
| \(s_{23}^2/10^{-1}\) | 4.34 \(\rightarrow\) 6.10 (5.74) | 4.33 \(\rightarrow\) 6.08 (5.78) |
| \(s_{13}^2/10^{-2}\) | 2.000 \(\rightarrow\) 2.405 (2.200) | 2.018 \(\rightarrow\) 2.424 (2.225) |
| \(\delta/\pi\) | 0.71 \(\rightarrow\) 1.99 (1.08) | 1.11 \(\rightarrow\) 1.96 (1.58) |
| \(s_{14}\) | 0.0098 \(\rightarrow\) 0.0310 | 0.0098 \(\rightarrow\) 0.0310 |
| \(s_{24}\) | 0.0059 \(\rightarrow\) 0.0262 | 0.0059 \(\rightarrow\) 0.0262 |
| \(s_{34}\) | 0 \(\rightarrow\) 0.0369 | 0 \(\rightarrow\) 0.0369 |

The sterile-active neutrino mass and mixing which has been addressed in Ref. [39] with the results achieved at the first-order approximation, and only the normal mass spectrum was mentioned. The sterile neutrino issue has also been considered in Refs. [50-57] with the \(A_4\) symmetry and a very large amount of scalar fields, which is substantially different than the model considered in this paper whose scalar sector has a moderate amount of particle content. Namely, in Refs. [50, 51] the Standard Model (SM) symmetry is enlarged by the inclusion of the \(A_4 \times Z_3 \times U(1)_R\) discrete group. In Ref. [52], the SM symmetry is enlarged by the \(A_4 \times Z_4\) symmetry in which only \(|U_{e4}|\) has been predicted without \(|U_{\mu4}|\) and \(|U_{\tau4}|\); in Ref. [53], the SM symmetry is enlarged by the symmetry \(A_4 \times Z_3 \times Z_4 \times U(1)_R\) where two doublets and seventeen singlets are used; in Ref. [54], the symmetry \(A_4 \times Z_3 \times Z_4\) is added to the SM in which up to three Higgs doublets and twelve singlets are considered in the scalar sector; in Ref. [55] the symmetry \(A_4 \times Z_4\) is added to the SM in which one doublets and up to twelve singlet scalars are introduced; in Ref. [56], the symmetry \(A_4 \times C_4 \times C_6 \times C_2 \times U(1)\), is added to the SM in which one doublets and up to twelve singlets are used, and only the normal mass spectrum is satisfied, and in Ref. [57], the symmetry \(A_4 \times Z_3 \times Z_4\) is added to the \(B-L\) in which one Higgs doublet and up to eleven singlets are considered in the scalar sector and without mention in the muon anomalous magnetic moment. Thus, it would be useful to build a \(S_4\) flavoured model with a much more economical scalar content. To our knowledge, \(A_4\) has not been considered before in the \(3+1\) scheme with \(B-L\) extension.

This paper is organized as follows. The description of the model is presented in section II. Its implications in lepton masses and mixings as well as the corresponding numerical analysis is presented in section III. Section IV is devoted to the muon anomalous magnetic moment. Finally, some conclusions are given in section V. Appendix B provides the scalar potential of the model.

II. THE MODEL

The \(B-L\) gauge model [53, 59] is supplemented by one discrete symmetry \(S_4\) along with two Abelian symmetries \(Z_3\) and \(Z_4\). Further, three right-handed neutrinos \(\nu_R\), one sterile neutrino \(\nu_s\), one \(SU(2)_L\) doublet \(H^0\) and four singlet scalars \((\phi, \varphi, \rho, \phi_s)\) are additional introduction to the \(B-L\) model. Three right-handed neutrinos and three left-handed leptons \(\psi_L\) are grouped in 3 under \(S_4\) symmetry whereas the first right-handed charged leptons \(l_{1R}\) is assigned as 1 and the two others are grouped in 2 under the \(S_4\) symmetry. The sterile neutrino \(\nu_s\) is assigned as 1 under the \(S_4\) symmetry. The particle content of the model under consideration\(^1\) and its corresponding assignments under the symmetry \(SU(2)_L \times U(1)_Y \times U(1)_{B-L} \times S_4 \times Z_3 \times Z_4 \equiv \Gamma\) are shown in Table II.

\(^1\) Under \(SU(3)_C\) symmetry, all leptons and scalars are singlets.
With the above specified particle content, the following five dimensional Yukawa interactions invariant under the symmetries of the model arise:

\[- \mathcal{L}_{\text{cl}} = \frac{x_1}{\Lambda} (\bar{\psi}_L l_{1R})_3 (H \phi)_3 + \frac{x_2}{\Lambda} (\bar{\psi}_L l_{\alpha R})_3 (H \phi)_3 + \frac{x_3}{\Lambda} (\bar{\psi}_L l_{\alpha R})_3 (H' \phi)_3 + \frac{x_1}{\Lambda} (\bar{\psi}_L \nu_R)_4 (H \phi)_3 + \frac{x_2}{\Lambda} (\bar{\psi}_L \nu_R)_4 (H \phi')_3 + \frac{x_3}{\Lambda} (\bar{\psi}_L \nu_R)_4 (H' \phi')_3 + \frac{y_v}{2} (\bar{\nu}_R \nu_R)_3 \chi + \frac{y_s}{\Lambda} (\bar{\nu}_R \nu_R)_3 (\chi \phi)_3 + H.c.\]

In Eq. (2), \( \Lambda \) is the cut-off scale of the theory, \( y_v, y_s, x_k \) and \( x_{kp} \) are dimensionless constants. The VEVs of scalar fields satisfying the scalar potential minimum condition, presented in section B, are given as follows

\[ \langle H \rangle = (0 \ v^T), \quad \langle H' \rangle = (0 \ v'^T), \quad \langle \phi \rangle = (\langle \phi_1 \rangle, \langle \phi_2 \rangle, \langle \phi_3 \rangle), \quad \langle \phi_1 \rangle = \langle \phi_2 \rangle = \langle \phi_3 \rangle = v_{\phi}, \]

\[ \langle \phi \rangle = (0 \ 0 \ v_{\phi}), \langle \phi_2 \rangle = v_{\phi}, \langle \rho \rangle = v_{\rho}, \langle \chi \rangle = v_{\chi}, \langle \phi_s \rangle = (\langle \phi_{1s} \rangle, \langle \phi_{2s} \rangle, \langle \phi_{3s} \rangle), \quad \langle \phi_{1s} \rangle = \langle \phi_{2s} \rangle = \langle \phi_{3s} \rangle = v_{\phi_s}. \]

In fact the electroweak symmetry breaking scale is one order of magnitude lower than TeV scale and the \( B - L \) scale is assumed to be at the TeV scale, i.e.,

\[ v' \sim v = 1.23 \times 10^2 \text{GeV}, \quad v_{\chi} = 10^3 \text{GeV}. \]

Furthermore, in order to have very heavy right handed Majorana neutrino masses, thus allowing the implementation of the type I seesaw mechanism that produces the tiny masses of the active neutrinos, the VEVs of flavons \( \phi, \varphi, \rho \) and \( \phi_s \) are assumed to be at a very high scale:

\[ v_{\phi} \sim v_{\varphi} \sim v_{\rho} \sim v_{\phi_s} = 10^{11} \text{GeV}. \]

It is interesting to note that each of additional symmetry \( U(1)_{B-L}, S_4, Z_3 \) and \( Z_4 \) plays a crucial role in forbidding the unwanted terms, which are listed in Table V of Appendix A to obtain the lepton mass matrices in Eqs. (7) and (19). Furthermore, there exist invariant terms via Weinberg operators whose dimensions greater than or equal to six, \( \frac{1}{\Lambda^{2k+1}} \langle \bar{\psi}_L \nu_R^k \rangle (H \chi)^k (H' \phi)^k (P^c P) \) with \( k = 0, 1, 2, \ldots ; \quad H = H, H', \) and \( P = \phi, \varphi, \rho, \chi, \phi_s. \) The fact that \( v_H \ll v_P \ll \Lambda. \) Hence, the neutrino mass obtained by these Weinberg operators, \( v_H \left( \frac{\mu}{M} \right)^{2k+1} \left( \frac{v_P}{\Lambda} \right)^{2l+1}, \) is very small compared to the one obtained by the type-I seesaw mechanism as in Eq. (19) and thus have been ignored.

### III. NEUTRINO MASS AND MIXING

**A. Lepton mass and mixing in the three neutrino framework**

From the Yukawa interactions in Eq. (2), and using the tensor product rules of the \( S_4 \) discrete group (60), together with the VEVs of scalar fields in Eq. (3), the charged lepton mass Lagrangian is written in the form:

\[ \mathcal{L}_m = - (\bar{l}_{1L} \ l_{1R}) M_l (l_{1R} \ l_{2R})^T + H.c., \]

where

\[ M_l = \begin{pmatrix} x_{11} \left( \frac{v}{\Lambda} \right) & \frac{v}{\Lambda} \left( x_{21} v + x_{31} v' \right) & \frac{v}{\Lambda} \left( x_{22} v - x_{32} v' \right) \\ x_{12} \left( \frac{v}{\Lambda} \right) & x_{22} \left( \frac{v}{\Lambda} \right) & \frac{v}{\Lambda} \left( x_{22} v + x_{32} v' \right) \\ x_{13} \left( \frac{v}{\Lambda} \right) & x_{23} \left( \frac{v}{\Lambda} \right) & x_{33} \left( \frac{v}{\Lambda} \right) \end{pmatrix}. \]
which is diagonalized as, \( U^T_l M U_r = \text{diag}(m_e, m_\mu, m_\tau) \), with

\[
U^T_l = \frac{1}{\sqrt{3}} \begin{pmatrix}
1 & 1 & 1 \\
1 & \omega & \omega^2 \\
1 & \omega^2 & \omega
\end{pmatrix}, \quad U_r = \mathbf{I}_{3 \times 3} \quad (\omega = e^{i \frac{2\pi}{3}}),
\]

\[m_e = \sqrt{3} x_1 v_\phi \Lambda, \quad m_{\mu, \tau} = \sqrt{3} v_\phi \Lambda (x_2 v \pm x_3 v').\]

Equation (8) tells us that \( U_l \) is non trivial and hence it will affect on the lepton mixing matrix. Furthermore, Eq. (9) shows that \( m_\mu \) and \( m_\tau \) are distinguished by \( v' \). This is the reason why \( H' \) is additionally introduced to \( H \).

Furthermore, assuming the gauge singlet scalar fields adjoin VEVs much larger than the electroweak symmetry breaking scale, the scalar spectrum at low energies is the same as in the 2HDM theory. Then, the physical CP even and CP odd neutral scalar fields are given by:

\[
H^0_R = \sin \alpha h - \cos \alpha H, \quad H^0_R = - \cos \alpha h - \sin \alpha H,
\]

\[
H^0_I = \cos \beta G_Z + \sin \beta A^0, \quad H^0_I = \sin \beta G_Z - \cos \beta A^0,
\]

where \( G_Z \) is the Goldstone boson associated with the longitudinal component of the gauge boson \( Z \). \( h \) is the 126 GeV SM like Higgs, \( H \) is the heavy CP even Higgs, \( A \) is the heavy CP odd Higgs boson, \( \tan \beta = \frac{v'}{v} \) and \( \alpha \) is the mixing angle in the neutral scalar sector. Then, the leptonic Yukawa interactions involving neutral scalar fields are given by:

\[
\mathcal{L}^\text{y} = \frac{\sqrt{3}}{2} y_1 \sin \alpha \sigma_L h e_R + \frac{\sqrt{3}}{2} (y_2 \sin \alpha - y_3 \cos \alpha) \bar{\nu}_L h \mu_R
\]

\[+ \frac{\sqrt{3}}{2} (y_2 \sin \alpha + y_3 \cos \alpha) \tau_L h \tau_R + h.c,\]

\[
\mathcal{L}^\text{y} = - \frac{\sqrt{3}}{2} y_1 \cos \alpha \sigma_L H^0 e_R + \frac{\sqrt{3}}{2} (y_2 \cos \alpha + y_3 \sin \alpha) \bar{\nu}_L H^0 \mu_R
\]

\[+ \frac{\sqrt{3}}{2} (y_2 \cos \alpha - y_3 \sin \alpha) \tau_L H^0 \tau_R + h.c,\]

\[
\mathcal{L}^\text{y} = i \frac{\sqrt{3}}{2} y_1 \sin \beta \sigma_L A^0 e_R + i \frac{\sqrt{3}}{2} (y_2 \sin \beta - y_3 \cos \beta) \bar{\nu}_L A^0 \mu_R
\]

\[+ i \frac{\sqrt{3}}{2} (y_2 \sin \beta + y_3 \cos \beta) \tau_L A^0 \tau_R + h.c,\]

where

\[y_k = x_k \frac{v_\phi}{\Lambda} (k = 1, 2, 3).\]

Equations (12), (13) and (14) imply that the flavour changing leptonic neutral scalar interactions are absent in our model. Consequently, there are no scalar contributions to the charged lepton flavor violating decays \( \tau \rightarrow \mu \gamma \), \( \tau \rightarrow e \gamma \) and \( \mu \rightarrow e \gamma \). The charged lepton flavor violating decays \( \tau \rightarrow \mu \gamma \), \( \tau \rightarrow e \gamma \) and \( \mu \rightarrow e \gamma \) will receive one loop level contributions due to the electrically charged scalars and right handed Majorana neutrinos, however these contributions will scale with the inverse of fourth power of the model cutoff as well as with the fourth power of the very small neutrino Yukawa coupling, thus making their branching ratios very tiny. Another contributions to the charged lepton flavor violating decays will arise from the virtual exchange of a \( W \) gauge boson and sterile neutrinos however those contributions will have very tiny branching ratios as in the SM.

As we will see in the following, our model can successfully accommodate the SM charged lepton masses. Now, comparing the obtained result in Eq. (9) with the experimental values of \( m_e, m_\mu, m_\tau \), \( m_\tau = 1776.86 \text{ MeV} \), \( m_\mu = 105.65837 \text{ MeV} \), and considering the benchmark point \( v = v' = 123 \text{ GeV} \), \( v_\phi = 10^{11} \text{ GeV} \) and \( \Lambda = 10^{13} \text{ GeV} \), we get:

\[|x_1| \sim 10^{-4}, \quad |x_2| \sim |x_3| \sim 10^{-1}.\]

Furthermore, from the above given leptonic Yukawa interactions and using the benchmark point given above, it follows that the coupling of the 126 GeV SM like Higgs boson with the SM lepton-antilepton pair is about 0.7 the SM expectation.
Combining Eqs. (19) and (20) yields the and \( \alpha \) of ten real parameters. Considering the case of real VEVs for the scalars \( \phi \), we note that \( \nu_\tau \) is complex (\( \nu_\tau = \nu_\tau^* \)), where \( \nu_\tau \), \( \nu_R \), and \( \nu_s \) are defined as follows

\[ \nu_\tau = \nu_\tau^* \]

In the case of \( \nu_\tau, \nu_R, \nu_s \) are defined as follows

\[ (\bar{\nu}_R \nu_R, \nu_s) \]

With the VEV configuration given in Eq. (3), after symmetry breaking, the 7 × 7 neutrino mass matrix in the \( \nu_L, \nu_R, \nu_s \) basis takes the form

\[ M_7^{7 \times 7} = \begin{pmatrix} 0 & M_D & 0 \\ M_D^T & M_R & M_S^T \\ 0 & M_S & 0 \end{pmatrix}, \tag{18} \]

where \( M_D, M_R \) and \( M_S \) are the Dirac, Majorana and sterile neutrino mass matrices, respectively. They are given by:

\[ M_D = \begin{pmatrix} 0 & v_{x1\nu} & v_{x2\nu} & v_{x3\nu} \\ 0 & v_{x1\nu} & v_{x2\nu} & v_{x3\nu} \\ 0 & v_{x1\nu} & v_{x2\nu} & v_{x3\nu} \end{pmatrix}, \]

\[ M_R = y_\nu v_\nu^2 \begin{pmatrix} 1 & 0 \\ 0 & y_\nu v_\nu^2 \end{pmatrix}. \tag{19} \]

In the case of \( M_D < M_S \ll M_R \), the effective 4 × 4 light neutrino mass matrix is determined by 50, 52, 54, 56:

\[ M_\nu = -\begin{pmatrix} M_D M_R^{-1} M_D^T & M_D M_R^{-1} M_S^T \\ M_S M_R^{-1} M_D^T & M_S M_R^{-1} M_S^T \end{pmatrix}. \tag{20} \]

The expression [19] shows that, in the neutrino sector, there are five complex parameters, thus implying the existence of ten real parameters. Considering the case of real VEVs for the scalars \( H, H', \phi, \chi \) and \( \eta \) while taking the VEV of \( \varphi \) to be complex (\( \varphi = \nu_0 e^{i\alpha} \)), the phase redefinition of \( \psi_L \) and \( \nu_R \) allows to rotate away the phases of three Yukawa couplings \( x_{1\nu}, y_\nu \) and \( y_\nu \), reducing to seven parameters. Furthermore, two parameters are absorbed during the formation of the neutrino mass matrix. Thus, there are left five real and dimensionless parameters \( k_{2,3}, m, m_s \) and \( \alpha \), where \( k_{2,3}, m, m_s \) are defined as follows

\[ k_2 = \frac{v_0 x_{2\nu}}{v_{\nu} x_{1\nu}}, \quad k_3 = \frac{v_0 v' x_{3\nu}}{v_{\nu} x_{1\nu}}, \quad m = \frac{v_0^2 v_{x1\nu}^2}{\Lambda^2 v_\nu y_\nu}, \quad m_s = \frac{v_\nu v_{x1\nu}^2 y_s}{\Lambda^2 y_\nu}. \tag{21} \]

Combining Eqs. [19] and [20] yields the 4 × 4 active-sterile mass matrix in the explicit form

\[ M_\nu^{4 \times 4} = \begin{pmatrix} 1 + (k_2 - k_3)^2 e^{2i\alpha} & 0 & 2k_2 e^{i\alpha} & \sqrt{\frac{m}{m_s}} (1 + (k_2 - k_3) e^{i\alpha}) \\ 0 & 1 & 0 & \sqrt{\frac{m}{m_s}} (1 + (k_2 - k_3) e^{i\alpha}) \\ 2k_2 e^{i\alpha} & 0 & 1 + (k_2 + k_3)^2 e^{2i\alpha} & 1 + (k_2 + k_3) e^{i\alpha} \\ \sqrt{\frac{m}{m_s}} (1 + (k_2 - k_3) e^{i\alpha}) & 0 & 1 + (k_2 + k_3) e^{i\alpha} & m_s \end{pmatrix}. \tag{22} \]

In the case of \( M_D < M_S \), one can apply the type-I seesaw mechanism on Eq. [20] to get the active neutrino mass matrix as 50, 52, 54, 56

\[ M_\nu = M_D M_R^{-1} M_D^T (M_S M_R^{-1} M_S^T)^{-1} M_S M_R^{-1} M_D^T - M_D M_R^{-1} M_D^T \]

\[ = m \begin{pmatrix} 0 & 0 & 2k_3 - k_0 - i2k_3 \sin \alpha \\ 0 & 0 & 0 \\ -2 & -2 & 0 \\ 0 & 0 & k_+ \end{pmatrix}. \tag{23} \]

We note that \( M_\nu \) is complex because \( m, k_{2,3} \) and \( \alpha \) are real parameters. Hence, to determine the active neutrino masses, we define a Hermitian matrix \( m_\nu^2 \) given by

\[ m_\nu^2 = M_\nu M_\nu^\dagger = \frac{m^2 k_0}{2} \begin{pmatrix} k_- & 0 & 2k_3 - k_0 - i2k_3 \sin \alpha \\ 0 & 0 & 0 \\ 2k_3 - k_0 + i2k_3 \sin \alpha & 0 & k_+ \end{pmatrix}. \tag{24} \]
where

\[ k_0 = 1 + k_2^2 + k_3^2 - 2k_2 \cos \alpha, \quad k_\mp = 1 + (k_2 \mp k_3)^2 - 2(k_2 \mp k_3) \cos \alpha. \]  

(25)

The squared matrix \( m^2_\nu \) in Eq. (24) is diagonalized by the rotation matrix \( U_\nu \) satisfying

\[
U^\dagger_{\nu} m^2_{\nu} U_{\nu} = \begin{cases}
\left( \begin{array}{ccc}
0 & 0 & 0 \\
0 & m^2 & 0 \\
m^2 & 0 & 0
\end{array} \right) , & U_{\nu} = \begin{pmatrix}
g_0(g_1 + ig_2) & 0 & -r_0(g_1 + ig_2) \\
0 & 1 & 0 \\
r_0(g_1 + ig_2) & 0 & 1
\end{pmatrix}
\end{cases}
\]

for NH, \( \alpha \),

\[
\begin{pmatrix}
g_0(g_1 + ig_2) & 0 & -r_0(g_1 + ig_2) \\
0 & 1 & 0 \\
r_0(g_1 + ig_2) & 0 & 1
\end{pmatrix}
\]

for IH, \( \alpha \).

(26)

where

\[
g_0 = \sqrt{\frac{k_0}{k_+}}, \quad r_0 = \sqrt{\frac{k_0}{k_-}}, \quad g_1 = \frac{1}{\sqrt{2}} - \sqrt{\frac{2}{\sqrt{2} k_3}}, \quad g_2 = \frac{\sqrt{2} k_3 \sin \alpha}{k_0}.
\]

(27)

The corresponding leptonic mixing matrix is

\[
U_{\text{lep}} = U_{\nu}^\dagger U_{\nu} = \begin{cases}
\left( \begin{array}{ccc}
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\
0 & 0 & 1
\end{array} \right), & \text{for NH,} \\
\left( \begin{array}{ccc}
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\
0 & 0 & 1
\end{array} \right), & \text{for IH.}
\end{cases}
\]

(28)

Three light neutrino masses are given by

\[
\begin{align*}
m^2_1 &= 0, & m^2_2 &= m^2, & m^2_3 &= k_3^2 m^2 & \text{for NH}, \\
m^2_1 &= k_3^2 m^2, & m^2_2 &= m^2, & m^2_3 &= 0 & \text{for IH},
\end{align*}
\]

(29)

which implies the neutrino mass ordering should be either \((0, m, m k_0)\) or \((m k_0, m, 0)\). It is important to note that in the considered model both NH \((m_1 < m_2 < m_3)\) and IH \((m_3 < m_1 < m_2)\) are satisfied due to the fact that \(k_0\) in Eq. 25 can be greater or less than the one in the case of \(k_2\) and \(k_3\) are real parameters. Hence, the model can predict both NH and IH spectrum which are consistent with the recent experimental data [15] and different from that of Ref. 50 in which only the NH is allowed.

In the three-neutrino scheme [61], the lepton mixing angles are determined from Eq. 28,

\[
\begin{align*}
s^2_{13} &= \left| U_{e3} \right|^2 = \begin{cases}
\frac{2 k_3^2}{\sqrt{3} k_0} & \text{for NH}, \\
\frac{2 k_3^2}{k_0} & \text{for IH},
\end{cases}
\end{align*}
\]

(30)

\[
\begin{align*}
s^2_{12} &= \left| U_{e2} \right|^2 = \frac{1}{1 - \left| U_{e3} \right|^2}, & s^2_{23} &= \left| U_{e3} \right|^2 = \begin{cases}
\frac{1}{2} + \frac{\sqrt{3} k_3 \sin \alpha}{\sqrt{k_0^2 - 2 k_3^2}} & \text{for NH}, \\
\frac{1}{2} - \frac{\sqrt{3} k_3 \sin \alpha}{\sqrt{k_0^2 + 2 k_3^2}} & \text{for IH},
\end{cases}
\end{align*}
\]

(31)

(32)

where \(s_{ij} = \sin \theta_{ij}, \ c_{ij} = \cos \theta_{ij}, \ t_{12} = \frac{a_{12}}{c_{12}}, \text{ and } t_{23} = \frac{a_{23}}{c_{23}}\) with \(\theta_{ij}\) are neutrino mixing angles.

Furthermore, from Eqs. 29, we can express \(m\) and \(k_0\) in terms of two observables \(\Delta m^2_{21}\) and \(\Delta m^2_{31}\) as follows:

\[
m = \begin{cases}
\frac{\sqrt{\Delta m^2_{21}}}{\sqrt{\Delta m^2_{21} - \Delta m^2_{31}}} & \text{for NH}, \\
\frac{\sqrt{\Delta m^2_{31}}}{\sqrt{\Delta m^2_{31} - \Delta m^2_{21}}} & \text{for IH},
\end{cases}
\]

(33)

\[
k_0 = \begin{cases}
\frac{\sqrt{m}}{m - \Delta m^2_{31}} & \text{for NH}, \\
\frac{\sqrt{m}}{m - \Delta m^2_{21}} & \text{for IH}.
\end{cases}
\]

(34)

The fact that the neutrino mass ordering can be normal \((m_1 < m_2 < m_3)\) or inverted \((m_3 < m_1 < m_2)\) depending on the sign of \(\Delta m^2_{31}\) [16, 22]. We will show that the considered model can provide the satisfied explanation on neutrino masses and mixings data, for both three-neutrino scheme and \(3 + 1\) scheme, given in Table 4.
B. 3+1 sterile-active neutrino mixing

In the case of $M_D < M_S$, the mass of the $4^{th}$ mass eigenstate is given by \[ m_4 = M_S M_R^{-1} M_S^T = m_s. \] (35)

Combining Eqs. (29) and (35) yields:

\[
m_s = m_4 = \begin{cases} \sqrt{\Delta m_{41}^2} & \text{for NH}, \\ \sqrt{m^2 k_0^2 + \Delta m_{41}^2} & \text{for IH}. \end{cases}
\] (36)

The $4 \times 4$ neutrino mixing matrix is given by \[ U = U_L^\dagger U_\nu = \begin{pmatrix} U_\nu^T (1 - \frac{1}{2} R R^T) & U_\nu^T R \\ -R^T U_\nu & 1 - \frac{1}{2} R^T R \end{pmatrix}, \] (37)

where $R$ is given by \[ R = M_D M_R^{-1} M_S^T (M_S M_R^{-1} M_S^T)^{-1} = \sqrt{\frac{m}{2 m_s}} \begin{pmatrix} 1 + (k_2 - k_3) e^{i \alpha} \\ 0 \\ 1 + (k_2 + k_3) e^{i \alpha} \end{pmatrix}. \] (38)

Combining Eqs. (8) and (38), we find that the strength of the active-sterile mixing is given by

\[
U_L^\dagger R = \sqrt{\frac{m}{6 m_s}} \begin{pmatrix} 2 + 2k_2 e^{i \alpha} \\ \frac{\omega^2 (k_2 + k_3) + k_2 - k_3}{\omega (k_2 + k_3) + k_2 - k_3} e^{i \alpha} - \omega \omega^2 \end{pmatrix} \equiv \begin{pmatrix} U_{e4} \\ U_{\mu 4} \\ U_{\tau 4} \end{pmatrix},
\] (39)

which leads to

\[
\begin{align*}
&s_{14}^2 = |U_{e4}|^2 = \frac{2 m}{3 m_s} (1 + k_2^2 + 2 k_2 \cos \alpha), \\
&s_{24}^2 = \frac{|U_{\mu 4}|^2}{1 - |U_{e4}|^2} = \frac{1 + k_2^2 + 3 k_3^2 + 2 k_2 \cos \alpha + 2 \sqrt{3} k_3 \sin \alpha}{6 m_s - (1 + k_2^2 + 2 k_2 \cos \alpha)}, \\
&s_{34}^2 = \frac{|U_{\tau 4}|^2}{1 - |U_{e4}|^2 - |U_{\mu 4}|^2} = \frac{1 + k_2^2 + 3 k_3^2 + 2 k_2 \cos \alpha - 2 \sqrt{3} k_3 \sin \alpha}{6 m_s - (5 + 5 k_2^2 + 3 k_3^2 + 10 k_2 \cos \alpha + 2 \sqrt{3} k_3 \sin \alpha)}. \end{align*}
\] (40)

C. Effective neutrino mass parameter and Jarlskog invariant

The Jarlskog invariant, obtained from Eq. (28), has the form \[ J_{C P} = \text{Im}(U_{12} U_{23} U_{13}^* U_{22}^*) = \begin{cases} \frac{k_3 (k_2 - \cos \alpha)}{3 \sqrt{3} k_0} & \text{for NH}, \\ -\frac{k_3 (k_2 - \cos \alpha)}{3 \sqrt{3} k_0} & \text{for IH}. \end{cases} \] (41)
The effective neutrino masses get the following forms [66–68],

\[
m^{(3)}_\beta = \sqrt{\sum_{i=1}^{3} |U_{ei}|^2 m_i^2} = \sqrt{\frac{m^2}{3} + \frac{m^2 k_0 \left( (2k_3^2 - k_0 + k_+)^2 + 4k_3^2 \sin^2 \alpha \right)}{6k_+}},
\]

(42)

\[
\langle m^{(3)}_{ee} \rangle = \sum_{i=1}^{3} U_{ei}^2 m_i = \frac{m}{6} \sqrt{\frac{16k_3^2 \sin^2 \alpha (2k_3^2 - k_0 + k_+)^2 + \left[ (k_+ - k_0 + 2k_3^2)^2 - 4k_3^2 \sin^2 \alpha + 2k_+ \right]^2}{k_+^2}},
\]

(43)

\[
m_{ee} = \sqrt{\sum_{i=1}^{4} |U_{ei}|^2 m_i^2} = \left\{ \frac{m^2}{3} + \frac{m^2 k_0 \left( (2k_3^2 - k_0 + k_+)^2 + 4k_3^2 \sin^2 \alpha \right)}{6k_+} \right. \\
+ \frac{2mm_\alpha}{3} \left[ (k_2 \cos \alpha + 1)^2 + k_2^2 \sin^2 \alpha \right] \right\}^{\frac{1}{2}},
\]

(44)

\[
\langle m_{ee} \rangle = \sum_{i=1}^{4} U_{ei}^2 m_i = \frac{m}{3} \left\{ 16 \sin^2 \alpha \left[ 2k_2 k_+ (k_2 \cos \alpha + 1) + k_0 k_3 - 2k_3^2 - k_3 k_+ \right]^2 \right. \\
+ \left[ k_0^2 + 6k_+ + 4k_2 k_+ (k_2 \cos \alpha + 2) \cos \alpha - 2k_0 (2k_3^2 + k_+) \right. \\
- 4 \left( k_2 k_+ + k_3^2 \right) \sin^2 \alpha + \left( 2k_3^2 + k_+ \right)^2 \right\}^{\frac{1}{2}}.
\]

(45)

D. Numerical analysis

Firstly, provided that \( s_{13} \) is located inside the \( 3\sigma \) experimentally allowed range\(^2\) taken from Ref. [45], i.e., \( s_{13}^2 \in (2.000, 2.405)10^{-2} \) for NH and \( s_{13}^2 \in (2.018, 2.424)10^{-2} \) for IH, from Eq. (31), we predict solar mixing angle

\[
s_{12}^2 \in \left\{ \begin{array}{ll} (0.3401, 0.3415) & \text{for NH,} \\
(0.3402, 0.3416) & \text{for IH,} \end{array} \right.
\]

(46)

which is consistent with the \( 2\sigma \) experimental range given in Ref. [45]. Furthermore, Eqs. (33) and (34) imply that

\[
m \in \left\{ \begin{array}{ll} (8.331, 9.022) \text{meV for NH,} \\
(49.39, 51.10) \text{meV for IH,} \end{array} \right.
\]

(47)

\[
\left\{ \begin{array}{ll} m_1 = 0, & m_2 \in (8.331, 9.022) \text{meV,} \\
m_3 \in (49.70, 51.28) \text{meV for NH,} \\
m_1 \in (48.68, 50.30) \text{meV,} & m_2 \in (49.39, 50.10) \text{meV,} \\
m_3 = 0 & \text{for IH,} \end{array} \right.
\]

(48)

\[
\sum_{i=1}^{3} m_i \in \left\{ \begin{array}{ll} (58.03, 60.31) \text{meV for NH,} \\
(98.07, 101.30) \text{meV for IH,} \end{array} \right.
\]

(49)

provided that the light neutrino mass-squared differences \( \Delta m_{21}^2 \) and \( \Delta m_{31}^2 \) to be varied in \( 3\sigma \) range of Ref. [45] as shown in Table II.

To find the allowed ranges of \( k_2, k_3, m, m_\alpha \) and \( \alpha \), and predictive ranges of the experimental parameters \( \sin^2 \theta_{23}, \sin \delta, \langle m_{ee} \rangle \) and \( s_{k4}^2 (k = 1, 2, 3) \), we utilize the observables \( \Delta m_{21}^2, \Delta m_{31}^2, \Delta m_{12}^2, \sin^2 \theta_{13} \) and \( \sin^2 \theta_{23} \) with values given in Table III.

\(^2\) In present work, numbers are displayed with \( 4 \) significant digits to the right of the decimal point.

\(^3\) As will be mentioned below, at present, there are various experimental bounds on \( \Delta m_{21}^2 \). In this work, \( \Delta m_{21}^2 \) is assumed in the range of \( \Delta m_{21}^2 \in (5.0, 10) \text{eV}^2 \) for NH while \( \Delta m_{21}^2 \in (30.0, 50.0) \text{eV}^2 \) for IH.
From Eqs. (25), (30)-(32) and (41), we get:

\[
\begin{align*}
k_2 &= \left\{ \begin{array}{ll}
\frac{-1}{s_{13}} \sqrt{\frac{t_N k_2}{2} + \Phi_N + s_{13}^2} & \text{for NH}, \\
\sqrt{1 + \frac{1}{2s_{13}} \left( \frac{3t_N k_2}{2} + \sqrt{3} \Phi_I \right)} & \text{for IH},
\end{array} \right. \\
k_3 &= \left\{ \begin{array}{ll}
\sqrt{\frac{3t_N}{2} s_{13}} & \text{for NH}, \\
\left( 1 - \frac{3}{2} s_{13}^2 \right) t_I & \text{for IH},
\end{array} \right. \\
\cos \alpha &= \left\{ \begin{array}{ll}
\sqrt{\frac{t_N k_2 + \Phi_N + s_{13}^2}{\sin^2 \theta_{23} + \cos^2 \theta_{23} + \Phi_N - 2s_{13}^2}} & \text{for NH}, \\
\frac{t_N (3s_{13}^2 - 2) + 2s_{13}^3}{\sqrt{2 - 3s_{13}^2 + \frac{3t_N k_2}{2} + \sqrt{3} \Phi_I \left( 3t_I c_{13}^2 \cos^2 \theta_{23} + \sqrt{3} \Phi_I + 6s_{13}^2 - 4 \right)}} & \text{for IH},
\end{array} \right. \\
\sin \delta_{CP} &= \left\{ \begin{array}{ll}
- \frac{\sqrt{\sin^2 \theta_{23} \cos^2 \theta_{23} + \Phi_N - 2s_{13}^2}}{\left( s_{13}^3 + \frac{t_N c_{13}^2 \cos^2 \theta_{23} + \Phi_N - 2s_{13}^2}{4s_{13}^3 \left( 3s_{13}^2 - 2 \right) + 2s_{13}^3} \right)} & \text{for NH}, \\
\frac{1}{s_{13} \sqrt{2 - 3s_{13}^2 + \frac{3t_N k_2}{2} + \sqrt{3} \Phi_I \left( \left( 3t_I c_{13}^2 \cos^2 \theta_{23} + \sqrt{3} \Phi_I + 6s_{13}^2 - 4 \right) + 1 \right)}} & \text{for IH},
\end{array} \right. \\
\end{align*}
\]

with

\[
\Phi_N = \sqrt{(\cos^2 \theta_{13} - c_{13}^2 \sin^2 \theta_{23}) \left( t_N c_{13}^2 \cos^2 \theta_{23} - 2s_{13}^2 \right)}, \\
\Phi_I = \sqrt{(\cos^2 \theta_{13} - c_{13}^2 \sin^2 \theta_{23}) \left( 3t_I c_{13}^2 \cos^2 \theta_{23} + 6s_{13}^2 - 4 \right) t_I}, \\
k_\Phi = (6 - 5s_{13}^2) s_{13}^2 + 2c_{13}^4 \sin^2 \theta_{23} - 1, \\
t_N = \sqrt{\frac{\Delta m_{21}^2}{\Delta m_{31}^2}}, \quad t_I = \sqrt{-\frac{\Delta m_{21}^2}{\Delta m_{31}^2} - \Delta m_{31}^2}.
\]

Equations (50)–(57) imply that \(k_3\) depends on three parameters \(\Delta m_{21}^2, \Delta m_{31}^2,\) and \(s_{13}\) while \(k_2\) and \(\cos \alpha\) depend on four parameters \(\Delta m_{21}^2, \Delta m_{31}^2, s_{13},\) and \(s_{23}.\) As a consequence, Eqs. (40)–(57) imply that \(J_{CP}, \sin \delta_{CP}, \langle m_{ee}^{(3)} \rangle,\) \(\langle m_{ee} \rangle\) and \(m_{ee}^{(3)}\) as well as \(|U_{ij}| (i = 1, 2, 3; j = 1, 3)\) depend on four parameters \(\Delta m_{21}^2, \Delta m_{31}^2, s_{13}\) and \(s_{23}\) while \(s_{24}^2 (k = 1, 2, 3)\) and \(m_\beta\) depend on five parameters \(\Delta m_{21}^2, \Delta m_{31}^2, s_{13}, s_{23}\) and \(m_\ell \equiv m_4 \equiv \sqrt{\Delta m_{31}^2}\) in which three parameters \(\Delta m_{21}^2, \Delta m_{31}^2,\) and \(s_{13}\) are measured with more accuracy that can be used to constrain the others.

At the best-fit points of \(\Delta m_{31}^2\) [45], \(\Delta m_{31}^2 = 2.55 \times 10^3 \text{ meV}^2\) for NH and \(\Delta m_{31}^2 = -2.45 \times 10^3 \text{ meV}^2\) for IH, the parameter \(k_3\) depends on \(s_{13}^2\) and \(\Delta m_{21}^2\) which is depicted in Fig. 1. This figure implies

\[
k_3 \in \left\{ \begin{array}{ll}
(0.41, 0.46) & \text{for NH}, \\
(0.974, 0.9775) & \text{for IH},
\end{array} \right.
\]

At the best-fit points of two light neutrino mass squared differences taken from Ref. [45], \(\Delta m_{31}^2 = 2.55 \times 10^3 \text{ meV}^2\) for NH while \(\Delta m_{31}^2 = -2.45 \times 10^3 \text{ meV}^2\) for IH and \(\Delta m_{31}^2 = 75.0 \text{ meV}^2\), we get

\[
k_0 = \left\{ \begin{array}{ll}
5.831 & \text{for NH}, \\
0.985 & \text{for IH},
\end{array} \right.
\]

and the parameters \(k_2, \cos \alpha, \sin \delta_{CP}, \langle m_{ee}^{(3)} \rangle, \langle m_{ee} \rangle\) and \(m_{ee}^{(3)}\) depend on \(s_{13}\) and \(s_{23}\) which are, respectively, depicted in Figs. 2, 3, 6, 8 and 9.

Figure 2 implies that

\[
k_2 \in \left\{ \begin{array}{ll}
(-3.20, -2.40) & \text{for NH}, \\
(1.13, 1.18) & \text{for IH}.
\end{array} \right.
\]
Figure 1: $k_3$ as a function of $s_{13}^2$ and $\Delta m^2_{21}$ with $\Delta m^2_{21} \in (69.4, 81.4) \text{meV}^2$ and $s_{23}^2 \in (2.000, 2.405)10^{-2}$ for NH (in the left panel) while $s_{13}^2 \in (2.018, 2.424)10^{-2}$ for IH (in the right panel).

Figure 2: $k_2$ as a function of $s_{23}^2$ and $s_{13}^2$ with $s_{23}^2 \in (0.456, 0.544)$ and $s_{13}^2 \in (2.00, 2.405)10^{-2}$ for NH (in the left panel) while $s_{23}^2 \in (0.433, 0.545)$ and $s_{13}^2 \in (2.018, 2.424)10^{-2}$ for IH (in the right panel).

From figure 3 it follows that:

$$\cos \alpha \in \begin{cases} (0.90, 0.0) \text{ for NH}, \\ (0.994, 0.999) \text{ for IH}, \end{cases} \text{i.e., } \alpha^{(e)} \in \begin{cases} (90.0, 154.0) \text{ for NH}, \\ (2.563, 6.28) \text{ for IH}. \end{cases} \tag{61}$$

The dependence of $J_{CP}$ on $s_{23}^2$ and $s_{13}^2$ is depicted in Fig. 4.

Figure 4 implies that the Jarlskog invariant takes the values

$$J_{CP} \in \begin{cases} (-3.5, -3.0) \times 10^{-2} \text{ for NH}, \\ (-3.4, -2.6) \times 10^{-2} \text{ for IH}. \end{cases} \tag{62}$$
Figure 3: $\cos \alpha$ as a function of $s_{23}^2$ and $s_{13}^2$ with $s_{23}^2 \in (0.456, 0.544)$ and $s_{13}^2 \in (2.00, 2.405)10^{-2}$ for NH (in the left panel) while $s_{23}^2 \in (0.433, 0.545)$ and $s_{13}^2 \in (2.018, 2.424)10^{-2}$ for IH (in the right panel).

Figure 4: $J_{CP} \times 10^2$ as a function of $s_{23}^2$ and $s_{13}^2$ with $s_{23}^2 \in (0.456, 0.544)$ and $s_{13}^2 \in (2.00, 2.405)10^{-2}$ for NH (in the left panel) while $s_{23}^2 \in (0.433, 0.545)$ and $s_{13}^2 \in (2.018, 2.424)10^{-2}$ for IH (in the right panel).

Figure 5 implies that

$$\sin \delta_{CP} \in \begin{cases} (-0.99, -0.91) & \text{for NH}, \\ (-0.975, -0.775) & \text{for IH} \end{cases}$$

i.e., $\delta^{(\phi)}_{CP} \in \begin{cases} (156.0, 172.0) & \text{for NH}, \\ (140.8, 167.2) & \text{for IH} \end{cases}$.
Figure 5: $\sin \delta_{CP}$ as a function of $s_{23}^2$ and $s_{13}^2$ with $s_{23}^2 \in (0.456, 0.545)$ and $s_{13}^2 \in (2.00, 2.405) \times 10^{-2}$ for NH (in the left panel) while $s_{23}^2 \in (0.433, 0.545)$ and $s_{13}^2 \in (2.018, 2.424) \times 10^{-2}$ for IH (in the right panel).

Figure 6: $\langle m_{ee}^{(3)} \rangle$ (meV) as a function of $s_{23}^2$ and $s_{13}^2$ with $s_{23}^2 \in (0.456, 0.544)$ and $s_{13}^2 \in (2.00, 2.405) \times 10^{-2}$ for NH (in the left panel) while $s_{23}^2 \in (0.433, 0.545)$ and $s_{13}^2 \in (2.018, 2.424) \times 10^{-2}$ for IH (in the right panel).

Figures [6] and [7] imply that

$$\langle m_{ee}^{(3)} \rangle \in \begin{cases} (3.50, 4.10) \text{ meV} & \text{for NH,} \\ (48.20, 48.70) \text{ meV} & \text{for IH,} \end{cases} \quad (64)$$

$$m_{\beta}^{(3)} \in \begin{cases} (8.80, 9.20) \text{ meV} & \text{for NH,} \\ (49.16, 49.24) \text{ meV} & \text{for IH,} \end{cases} \quad (65)$$

$$\langle m_{ee} \rangle \in \begin{cases} (40.0, 110.0) \text{ meV} & \text{for NH,} \\ (198.0, 208.0) \text{ meV} & \text{for IH.} \end{cases} \quad (66)$$

We see that the resulting effective neutrino mass for three neutrino scheme in Eqs. (64)–(66), for both NH and IH, are
Figure 7: $m_{3}^{(s)}$ (meV) as a function of $s_{23}^2$ and $s_{13}^2$ with $s_{23}^2 \in (0.456, 0.544)$ and $s_{13}^2 \in (2.00, 2.405) \times 10^{-2}$ for NH (in the left panel) while $s_{23}^2 \in (0.433, 0.545)$ and $s_{13}^2 \in (2.018, 2.424) \times 10^{-2}$ for IH (in the right panel).

below all the upper limits taken from GERDA \cite{69} $\langle m_{ee} \rangle < (120 \div 260)$ meV, MAJORANA \cite{70} $\langle m_{ee} \rangle < (24 \div 53)$ meV, CUORE \cite{71} $\langle m_{ee} \rangle < (110 \div 500)$ meV, KamLAND-Zen \cite{72} $\langle m_{ee} \rangle < (61 \div 165)$ meV, GERDA \cite{73} $\langle m_{ee} \rangle < (104 \div 228)$ meV, CUORE \cite{74} $\langle m_{ee} \rangle < (75 \div 350)$ meV and CUPID-Mo Collaboration \cite{75} $\langle m_{ee} \rangle < (310 \div 540)$ meV.

The dependence of $\langle m_{ee} \rangle$ on $s_{23}^2$ and $\Delta m_{41}^2$ with $s_{23}^2 \in (0.433, 0.545)$ and $\Delta m_{41}^2 \in (10.0, 30.0) \times 10^6$ meV$^2$ are depicted in Fig. 8.

The dependences of the absolute values of the entries of the lepton mixing matrix in Eq. 28 on $s_{13}$ and $s_{23}$ are

Figure 8: $\langle m_{ee} \rangle$ (meV) as a function of $s_{23}^2$ and $s_{13}^2$ with $s_{23}^2 \in (0.456, 0.544)$ and $s_{13}^2 \in (2.00, 2.405) \times 10^{-2}$ for NH (in the left panel) while $s_{23}^2 \in (0.541, 0.598)$ and $s_{13}^2 \in (2.018, 2.424) \times 10^{-2}$ for IH (in the right panel).
presented in Figs. [15] and [16] which indicate that

$$|U_{\text{lep}}| \in \begin{cases} 
0.8020 \to 0.8040 & 0.142 \to 0.154 \\
0.370 \to 0.420 & 0.700 \to 0.730 \\
0.430 \to 0.470 & 0.670 \to 0.695 \\
0.8020 \to 0.8040 & 0.144 \to 0.154 \\
0.440 \to 0.490 & 0.655 \to 0.690 \\
0.340 \to 0.400 & 0.710 \to 0.745 
\end{cases} \quad \text{for NH},$$

$$|U_{\text{lep}}| \in \begin{cases} 
0.8020 \to 0.8040 & 0.142 \to 0.154 \\
0.370 \to 0.420 & 0.700 \to 0.730 \\
0.430 \to 0.470 & 0.670 \to 0.695 \\
0.8020 \to 0.8040 & 0.144 \to 0.154 \\
0.440 \to 0.490 & 0.655 \to 0.690 \\
0.340 \to 0.400 & 0.710 \to 0.745 
\end{cases} \quad \text{for IH}. \quad (67)$$

At the best-fit points of the two light neutrino mass squared differences and reactor neutrino mixing angle taken from Ref. [4], $\Delta m_{31}^2 = 75.0 \, \text{meV}^2$ and $\Delta m_{32}^2 = 2.55 \times 10^3 \, \text{meV}^2$, $s_{13}^2 = 2.200 \times 10^{-2}$ for NH while $\Delta m_{32}^2 = -2.45 \times 10^3 \, \text{meV}^2$, $s_{13}^2 = 2.225 \times 10^{-2}$ for IH, the parameter $m_\beta$ and $s_{2k}^2$ $(k = 1, 2, 3)$ depend on $s_{23}$ and $m_s = \sqrt{\Delta m_{12}^2}$. At present, there are various experimental bounds on $\Delta m_{32}^2$. Going by this assumption, the dependences of $m_\beta$ and $s_{2k}^2$ $(k = 1, 2, 3)$ depend on $s_{23}$ and $m_s = \sqrt{\Delta m_{12}^2}$.

Thus in this work, $\Delta m_{31}^2$ is assumed in the range of $\Delta m_{31}^2 \in (5.0, 10) \, \text{eV}^2$ for NH while $\Delta m_{31}^2 \in (30.0, 50.0) \, \text{eV}^2$ for IH. Going by this assumption, the dependences of $m_\beta$ and $s_{2k}^2$ $(k = 1, 2, 3)$ depend on $s_{23}$ and $\Delta m_{12}^2$, with $s_{2k}^2 \in (0.456, 0.544)$ and $\Delta m_{12}^2 \in (5.0, 10.0) \times 10^6 \, \text{meV}^2$ for NH while $s_{23}^2 \in (0.433, 0.545)$ and $\Delta m_{12}^2 \in (30.0, 50.0) \times 10^6 \, \text{meV}^2$ for IH, are presented in Figs. [9] [10] and [11] respectively.

![Figure 9](image1.png)

Figure 9: $m_\beta$ (meV) as a function of $s_{2k}^2$ and $s_{13}^2$ with $s_{2k}^2 \in (0.456, 0.544)$ and $\Delta m_{12}^2 \in (5.0, 10.0) \times 10^6 \, \text{meV}^2$ for NH (in the left panel) while $s_{23}^2 \in (0.433, 0.545)$ and $\Delta m_{12}^2 \in (30.0, 50.0) \times 10^6 \, \text{meV}^2$ for IH (in the right panel).

Figures [10][11] show that the considered model predicts the range of $s_{14}^2$, $s_{24}^2$ and $s_{34}^2$ as follows

$$s_{14}^2 \in \begin{cases} 
(0.015, 0.045) & \text{for NH}, \\
(0.022, 0.029) & \text{for IH}, 
\end{cases} \quad (68)$$

$$s_{24}^2 \in \begin{cases} 
(0.005, 0.012) & \text{for NH}, \\
(0.0095, 0.012) & \text{for IH}, 
\end{cases} \quad (69)$$

$$s_{34}^2 \in \begin{cases} 
(0.003, 0.011) & \text{for NH}, \\
(0.009, 0.012) & \text{for IH}, 
\end{cases} \quad (70)$$
Finally, from the above analysis, the obtained parameters of the model are summarized in Table III. In the case $s_{23}^2 = 0.544 (\theta_{23} = 47.50^\circ)$ and $\Delta m_{31}^2 = 5.0 \text{eV}^2$ for NH while $s_{23}^2 = 0.545 (\theta_{23} = 47.58^\circ)$ and $\Delta m_{31}^2 = 30.0 \text{eV}^2 (m_s = 4477.23 \text{meV})$ for IH, we obtain the parameters as given in Tab. IV and the lepton mixing matrix gets the explicit form:

$$
U_{lep} = \begin{cases} 
\begin{pmatrix} 0.8012 + 0.05287i & 0.5774 & -0.1285 - 0.07406i \\
0.09636 - 0.354i & -0.2887 + 0.5i & -0.6317 - 0.3846i \\
0.09636 - 0.4598i & -0.2887 - 0.5i & -0.6317 + 0.2165i \\
0.8009 - 0.05405i & 0.5774 & -0.1285 + 0.07575i \\
0.09596 - 0.4611i & -0.2888 + 0.5i & -0.6315 - 0.2147i \\
0.09596 + 0.353i & -0.2888 - 0.5i & -0.6315 + 0.3662i 
\end{pmatrix} & \text{for NH,} \\
\begin{pmatrix} 0.8012 - 0.05287i & 0.5774 & -0.1285 + 0.07406i \\
0.09636 + 0.354i & -0.2887 + 0.5i & -0.6317 - 0.3846i \\
0.09636 + 0.4598i & -0.2887 - 0.5i & -0.6317 + 0.2165i \\
0.8009 + 0.05405i & 0.5774 & -0.1285 - 0.07575i \\
0.09596 + 0.4611i & -0.2888 + 0.5i & -0.6315 - 0.2147i \\
0.09596 - 0.353i & -0.2888 - 0.5i & -0.6315 + 0.3662i 
\end{pmatrix} & \text{for IH,} 
\end{cases}
$$

which are unitary and in good agreement with the constraint on the absolute values of the entries of the lepton mixing matrix given in Eq. (71).
Figure 11: $s_{14}^1$, $s_{24}^1$ and $s_{34}^1$ as functions of $s_{23}^2$ and $\Delta m_{21}^2$ with $s_{23}^2 \in (0.433, 0.545)$ and $\Delta m_{21}^2 \in (30.0, 50.0) \times 10^{6} \text{meV}^2$ for IH.

IV. MUON ANOMALOUS MAGNETIC MOMENT

The experimental data shows a significant deviation of the muon anomalous magnetic moment from its SM value

$$\Delta a_\mu = a_\mu^{\text{exp}} - a_\mu^{\text{SM}} = (2.51 \pm 0.59) \times 10^{-9}$$  \[82\, [89] \]

In this subsection, we will analyze the implications of our model in the muon anomalous magnetic moment. Muon anomalous magnetic moments mainly arises from one-loop diagrams involving the exchange of electrically neutral CP even and CP odd scalars and the muon running in the internal lines of the loop. It is worth mentioning that due to the symmetries in our model, there are no tree level flavor changing neutral scalar interaction in the leptonic Yukawa terms, thus implying that the muon is the only charged lepton contributing to the muon anomalous magnetic moment. There are also contributions arising from electrically charged scalar and right handed Majorana neutrinos but these contributions are strongly suppressed. Thus, the leading contributions to the muon anomalous magnetic
moment take the form:

\[
\Delta a_\mu \approx \left[ (y_{h\pi_RH})^2 - (y_{h^0\pi_RH})^2 \right] m_\mu \frac{\mu}{\nu} I_H^{(\mu)}(m_\mu, m_\nu) \\
+ \frac{m_\mu^2}{8\pi^2} \left[ \frac{3}{2} (y_{H^\pm\pi_RH})^2 I_H^{(\mu)}(m_\mu, m_{H^\pm}) + \frac{3}{2} (y_{H^0\pi_RH})^2 I_A^{(\mu)}(m_\mu, m_{A^0}) \right],
\]

(73)

where the Yukawa couplings appearing in Eq. (73) are given by:

\[
y_{h\pi_RH} = \sqrt{\frac{3}{2}} (y_2 \sin \alpha - y_3 \cos \alpha), \quad \langle SM \rangle \quad y_{h^0\pi_RH} = \frac{m_\mu}{\nu},
\]

(74)

\[
y_{H^\pm\pi_RH} = \sqrt{\frac{3}{2}} (y_2 \cos \alpha + y_3 \sin \alpha), \quad y_{A^0\pi_RH} = \sqrt{\frac{3}{2}} (y_2 \sin \beta - y_3 \cos \beta),
\]

(75)

whereas the loop function \(I_H^{(\mu)}(m_f, m_{H,A})\) has the form [93][94]:

\[
I_H^{(\mu)}(m_f, m_{H,A}) = \int_0^1 \frac{x^2 \left(1 - x \pm \frac{m_f}{m_\nu} \right)}{m_\mu^2 x^2 + \left(\frac{m_f^2 - m_\mu^2}{m_\nu^2} \right) x + m_{H,A}^2 (1 - x)} dx.
\]

(76)
Figure 12 shows the allowed parameter space in the $m_{H^0} - m_{A^0}$ plane consistent with the muon anomalous magnetic moment. We find that our model can successfully accommodate the experimental values of the muon anomalous magnetic moment.

V. CONCLUSIONS

We have proposed a $B-L$ model combined with the $S_4 \times Z_3 \times Z_4$ discrete group which successfully explains the recent 3+1 sterile-active neutrino data. The tiny masses of the light active neutrinos are obtained through the type-I seesaw mechanism. The active-active and sterile-active neutrino mixing angles are predicted to be consistent with the recent constraints where $0.3401 (0.3402) \leq \sin^2 \theta_{12} \leq 0.3415 (0.3416)$, $0.456 (0.433) \leq \sin^2 \theta_{23} \leq 0.544 (0.545)$, $2.00 (2.018) \leq 10^2 \times \sin^2 \theta_{13} \leq 2.405 (2.424)$, $156 (140.8) \leq s_{CP}^0 \leq 172 (167.2)$ for normal (inverted) ordering of the three neutrino scenario, and $0.015 (0.022) \leq s_{14}^2 \leq 0.045 (0.029)$, $0.005 (0.0095) \leq s_{24}^2 \leq 0.012 (0.012)$, $0.003 (0.009) \leq s_{34}^2 \leq 0.011$ for normal (inverted) ordering of the 3+1 neutrino scenario. The effective neutrino masses are predicted to be in the ranges $40.0 (185) \leq \langle m_{ee} \rangle [\text{meV}] \leq 110.0 (205.0)$ and $300 (700) \leq \langle m_\beta \rangle [\text{meV}] \leq 550 (900)$, for normal and inverted neutrino mass orderings, respectively, range of values consistent with the recent experimental data. Our model predicts flavour conserving leptonic neutral scalar interactions and successfully explains the muon $g-2$ anomaly.

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### Appendix B: Higgs potential invariant under $\Gamma$ symmetry

The total renormalizable scalar potential invariant under $\Gamma$ symmetry is given by\(^4\):

$$
V_{\text{scal}} = V(H) + V(H') + V(\phi) + V(\varphi) + V(\rho) + V(\chi) + V(\phi_s) + V(HH') + V(H\phi) \\
+ V(H\varphi) + V(H\rho) + V(H\chi) + V(H\phi_s) + V(H'\phi) + V(H'\varphi) + V(H'\rho) + V(H'\chi) \\
+ V(H'\phi_s) + V(H'\rho_\nu) + V(H'\rho) + V(\phi_s) + V(\rho) + V(\varphi) + V(\varphi_s) \\
+ V(\rho\chi) + V(\rho\phi_s) + V(\chi\phi_s) + V_{\text{trip}} + V_{\text{quart}},
$$

(B1)

where

$$
V(H) = \mu_H^2 H^2, \quad V(H') = V(H, H \rightarrow H'), \\
V(\phi) = \mu_\phi^2 \phi^2 + \lambda_\phi \phi^4, \\
V(\varphi) = \mu_\varphi^2 \varphi^2 + \lambda_\varphi \varphi^4, \\
V(\rho) = \mu_\rho^2 \rho^2 + \lambda_\rho \rho^4, \\
V(\chi) = \mu_\chi^2 \chi^2 + \lambda_\chi \chi^4.
$$

\(^4\) Here, $V(a_1 \rightarrow b_2, b_1 \rightarrow b_2, \cdots) \equiv V(a_1, b_1, \cdots)_{(a_1 \neq b_2, b_1 \neq b_2, \cdots)}$. 

---

### Table V: Forbidden terms

| Couplings | Forbidden by |
|-----------|-------------|
| $(\bar{\psi}_L \psi_R^c)_{1} H^2$, $(\bar{\psi}_L \psi_R^c)_{1} H^4$, $(\bar{\psi}_R \nu R)_{1} (\nu^* \nu)_{1}$, $(\bar{\psi}_R \nu R)_{1} (\phi^* \phi)_{1}$, $(\bar{\psi}_R \nu R)_{2} (\phi^* \phi)_{2}$ | $U(1)_{B-L}$ |
| $(\bar{\psi}_R \nu R)_{1} (\phi^* \phi)_{1}$, $(\bar{\psi}_R \nu R)_{2} (\phi^* \phi)_{2}$, $(\bar{\psi}_R \nu R)_{3} (\phi^* \phi)_{3}$, $(\bar{\psi}_R \nu R)_{4} (\phi^* \phi)_{4}$, $(\bar{\psi}_R \nu R)_{5} (\phi^* \phi)_{5}$, $(\bar{\psi}_R \nu R)_{6} (\phi^* \phi)_{6}$ | $S_3$ |
| $(\bar{\psi}_L\nu L)_{1} (H' \phi^* \phi)_{1}$, $(\bar{\psi}_L\nu L)_{1} (H' \phi^* \phi)_{2}$, $(\bar{\psi}_L\nu L)_{1} (H' \phi^* \phi)_{3}$, $(\bar{\psi}_L\nu L)_{1} (H' \phi^* \phi)_{4}$, $(\bar{\psi}_L\nu L)_{1} (H' \phi^* \phi)_{5}$, $(\bar{\psi}_L\nu L)_{1} (H' \phi^* \phi)_{6}$ | $Z_3$ |
| $(\bar{\psi}_L\nu L)_{1} (H' \phi^* \phi)_{1}$, $(\bar{\psi}_L\nu L)_{1} (H' \phi^* \phi)_{2}$, $(\bar{\psi}_L\nu L)_{1} (H' \phi^* \phi)_{3}$, $(\bar{\psi}_L\nu L)_{1} (H' \phi^* \phi)_{4}$, $(\bar{\psi}_L\nu L)_{1} (H' \phi^* \phi)_{5}$, $(\bar{\psi}_L\nu L)_{1} (H' \phi^* \phi)_{6}$ | $Z_4$ |
\[ V(H, H') = \lambda_1^{HH'} (H^\dagger H)_{1} (H^\dagger H')_{1} + \lambda_2^{HH'} (H^\dagger H')_{1} (H^\dagger H')_{2} , \quad V(H, \phi) = \lambda_{1}^{H\phi} (H^\dagger H)_{1} (\phi^* \phi)_{1} + \lambda_{2}^{H\phi} (H^\dagger H')_{3} (\phi^* \phi)_{3} , \quad V(H, \varphi) = V(H, \phi \to \varphi) \]

\[ V(H, \rho) = \lambda_1^{H\rho} (H^\dagger H)_{1} (\rho^* \rho)_{1} + \lambda_2^{H\rho} (H^\dagger H')_{1} (\rho^* \rho)_{2} \]

\[ V(H', \phi) = \lambda_1^{H'\phi} (H'^\dagger H')_{1} (\phi^* \phi)_{1} + \lambda_2^{H'\phi} (H'^\dagger H')_{3} (\phi^* \phi)_{3} , \quad V(H', \varphi) = V(H', \phi \to \varphi) \]

\[ V(H', \rho) = \lambda_1^{H'\rho} (H'^\dagger H')_{1} (\rho^* \rho)_{1} + \lambda_2^{H'\rho} (H'^\dagger H')_{1} (\rho^* \rho)_{2} \]

\[ (B2) \]

Now we show that the VEV alignments in Eq. \([B1]\) satisfy the minimization condition of the scalar potential \( V_{\text{scal}} \) of the model whose explicit expression is given in Eqs. \([B1],[B2]\). For this purpose we suppose that the VEVs of the scalars \( H, H', \phi, \rho, \chi \) and \( \varphi \) are real while that of \( \varphi \) is complex, i.e., \( v^* = v, \phi^* = \phi, v^*_x = v_x, v^*_\phi = v_\phi, v^*_\chi = v_\chi, v^*_\rho = v_\rho, v^*_\varphi = v_\varphi \)

\[ \lambda^i_{\phi} = \lambda^i_{\phi} \]

\[ \lambda^i_{\rho} = \lambda^i_{\rho} \]

\[ \lambda^i_{\varphi} = \lambda^i_{\varphi} \]

\[ (B4) \]

For simplicity we will work with the following benchmark points:

\[ \lambda_1^H = \lambda_2^H = \lambda_1^{H'} = \lambda_2^{H'} = \lambda_1^H = \lambda_2^H = \lambda_1^{H'} = \lambda_2^{H'} = \lambda_1^{H\phi} = \lambda_2^{H\phi} = \lambda_1^{H\rho} = \lambda_2^{H\rho} = \lambda_1^{H\varphi} = \lambda_2^{H\varphi} \]

\[ \lambda_1^{H'\phi} = \lambda_2^{H'\phi} = \lambda_1^{H'\rho} = \lambda_2^{H'\rho} = \lambda_1^{H'\varphi} = \lambda_2^{H'\varphi} \]

\[ \lambda_1^{H\phi} = \lambda_2^{H\phi} = \lambda_1^{H\rho} = \lambda_2^{H\rho} = \lambda_1^{H\varphi} = \lambda_2^{H\varphi} \]

\[ \lambda_1^{H'\phi} = \lambda_2^{H'\phi} = \lambda_1^{H'\rho} = \lambda_2^{H'\rho} = \lambda_1^{H'\varphi} = \lambda_2^{H'\varphi} \]

\[ \lambda_1^{H\phi} = \lambda_2^{H\phi} = \lambda_1^{H\rho} = \lambda_2^{H\rho} = \lambda_1^{H\varphi} = \lambda_2^{H\varphi} \]

\[ \lambda_1^{H'\phi} = \lambda_2^{H'\phi} = \lambda_1^{H'\rho} = \lambda_2^{H'\rho} = \lambda_1^{H'\varphi} = \lambda_2^{H'\varphi} \]

\[ \lambda_1^{H\phi} = \lambda_2^{H\phi} = \lambda_1^{H\rho} = \lambda_2^{H\rho} = \lambda_1^{H\varphi} = \lambda_2^{H\varphi} \]

\[ \lambda_1^{H'\phi} = \lambda_2^{H'\phi} = \lambda_1^{H'\rho} = \lambda_2^{H'\rho} = \lambda_1^{H'\varphi} = \lambda_2^{H'\varphi} \]

\[ \lambda_1^{H\phi} = \lambda_2^{H\phi} = \lambda_1^{H\rho} = \lambda_2^{H\rho} = \lambda_1^{H\varphi} = \lambda_2^{H\varphi} \]

\[ \lambda_1^{H'\phi} = \lambda_2^{H'\phi} = \lambda_1^{H'\rho} = \lambda_2^{H'\rho} = \lambda_1^{H'\varphi} = \lambda_2^{H'\varphi} \]

\[ (B5) \]

\[ \lambda_1^{H\phi} = \lambda_2^{H\phi} = \lambda_1^{H\rho} = \lambda_2^{H\rho} = \lambda_1^{H\varphi} = \lambda_2^{H\varphi} \]

\[ \lambda_1^{H'\phi} = \lambda_2^{H'\phi} = \lambda_1^{H'\rho} = \lambda_2^{H'\rho} = \lambda_1^{H'\varphi} = \lambda_2^{H'\varphi} \]

\[ (B6) \]
As a consequence, the condition \( B3 \) becomes

\[
\begin{align*}
\mu_H^2 &+ 2\lambda_H v^2 + 2\lambda^\varphi (v_\varphi^2 + v_\chi^2 + v_\rho^2 + v_\rho^2 + 3v_\phi^2 + v_\phi^2 + 2v_\chi^2) = 0, \\
\mu_{H'}^2 &+ 2\lambda_{H'} v^2 + 2\lambda^\varphi (v_\varphi^2 + v_\chi^2 + v_\rho^2 + v_\rho^2 + 3v_\phi^2 + v_\phi^2 + 2v_\chi^2) = 0, \\
6\mu_{\varphi}^2 v_\varphi &+ 84\lambda^\varphi v_\varphi^3 + 2\lambda^\varphi [6v_\varphi(v_\varphi^2 + v_\chi^2 + v_\rho^2 + v_\rho^2) + 20v_\rho v_\rho^2 + e^{i\alpha} v_\varphi v_\rho (4av_\rho + 3v_\chi)] + e^{-i\alpha} v_\varphi v_\rho (4v_\phi + 3av_\rho) = 0, \\
\mu_\varphi^2 v_\varphi &+ 6\lambda^\varphi v_\varphi^3 + \lambda^\varphi [2v_\varphi(v_\varphi^2 + v_\chi^2 + v_\rho^2 + 3v_\phi^2 + 2v_\chi^2) + e^{i\alpha} \lambda^\varphi v_\varphi (2av_\rho + 3v_\chi)] + e^{-i\alpha} \lambda^\varphi v_\varphi (2v_\phi + 3av_\rho) = 0, \\
v_\rho \left[ \mu_\rho^2 + 2\lambda_\rho v_\rho^2 + 2\lambda^\varphi (v_\varphi^2 + v_\chi^2 + v_\rho^2 + 3v_\phi^2 + 2v_\chi^2) \right] + e^{i\alpha} \lambda^\varphi v_\varphi (2av_\rho + 3v_\chi) + e^{-i\alpha} \lambda^\varphi v_\varphi (2v_\phi + 3av_\rho) = 0, \\
\mu_\chi^2 &+ 2\lambda_\chi v_\chi^2 + 2\lambda^\varphi (v_\varphi^2 + v_\chi^2 + v_\rho^2 + 3v_\phi^2 + 2v_\chi^2) = 0, \\
3\lambda^\varphi v_\varphi v_\rho (e^{-i\alpha} + e^{i\alpha}) &+ 2v_\chi \left[ \mu_{\chi}^2 + 2\lambda^\varphi (v_\varphi^2 + v_\chi^2 + v_\rho^2 + 3v_\phi^2 + v_\rho^2) + 10\lambda^\varphi v_\phi^2 \right] = 0, \\
\mu_{\varphi}^2 + 6\lambda_H v^2 &+ 2\lambda^\varphi (v_\varphi^2 + v_\chi^2 + v_\rho^2 + 3v_\phi^2 + v_\rho^2 + 2v_\chi^2) > 0, \\
\mu_{H'}^2 &+ 6\lambda_{H'} v^2 + 2\lambda^\varphi (v_\varphi^2 + v_\chi^2 + v_\rho^2 + 3v_\phi^2 + v_\rho^2 + 2v_\chi^2) > 0, \\
3\mu_\phi^2 + 126\lambda^\varphi v_\phi^2 &+ 4\lambda^\varphi v_\varphi v_\rho (e^{-i\alpha} + e^{i\alpha}) + 6\lambda^\varphi (v_\varphi^2 + v_\chi^2 + v_\rho^2 + 3v_\phi^2 + v_\rho^2) + 20\lambda^\varphi v_\phi^2 > 0, \\
\mu_\rho^2 &+ 18\lambda^\varphi v_\rho^2 + 2\lambda^\varphi (v_\varphi^2 + v_\chi^2 + v_\rho^2 + 3v_\phi^2 + v_\rho^2) > 0, \\
\mu_{\chi}^2 &+ 6\lambda_\chi v_\chi^2 + 2\lambda^\varphi (v_\varphi^2 + v_\chi^2 + v_\rho^2 + 3v_\phi^2 + v_\rho^2) > 0, \\
\mu_{\phi}^2 &+ 2\lambda^\varphi (v_\varphi^2 + v_\chi^2 + v_\rho^2 + 3v_\phi^2 + v_\rho^2) + 30\lambda^\varphi v_\phi^2 > 0, \\
-2v_\chi - 3av_\phi - e^{2i\alpha} (2av_\rho + 3v_\chi) > 0. 
\end{align*}
\]

The system of Eqs. \( B7 \) to \( B14 \) yields the following solution:

\[
\begin{align*}
\lambda^H &= -\frac{\mu_H^2 + 2\lambda^\varphi (v_\varphi^2 + v_\chi^2 + v_\rho^2 + 3v_\phi^2 + v_\rho^2 + 2v_\chi^2)}{2v_\rho^2}, \\
\lambda^{H'} &= -\frac{\mu_{H'}^2 + 2\lambda^\varphi (v_\varphi^2 + v_\chi^2 + v_\rho^2 + 3v_\phi^2 + v_\rho^2 + 2v_\chi^2)}{2v_\rho^2}, \\
\lambda^\phi &= -\frac{1}{42v_\varphi^3} \left[ (3\mu_\phi^2 + 6\lambda^\varphi v_\phi^2 + 6\lambda^\varphi v_\phi^2 + 6\lambda^\varphi v_\phi^2 + 6\lambda^\varphi v_\phi^2 + 20\lambda^\varphi v_\phi^2) v_\phi \\
+ \lambda^\varphi v_\varphi v_\rho \left( \frac{(4av_\rho + 3v_\chi) \sqrt{2v_\rho + 3av_\chi}}{2av_\rho + 3v_\chi} + \frac{\sqrt{2v_\rho + 3av_\chi}}{\sqrt{2v_\rho + 3av_\chi}} \right) \right], \\
\lambda^\varphi &= -\frac{\mu_\varphi^2 v_\varphi + 2\lambda^\varphi \left[ v_\varphi (v_\varphi^2 + v_\chi^2 + v_\rho^2 + 3v_\phi^2 + v_\rho^2) + v_\chi v_\rho \sqrt{(2av_\rho + 3v_\chi)(2v_\rho + 3av_\chi)} \right]}{6v_\varphi^3}, \\
\lambda^\rho &= -\frac{\mu_\rho^2 v_\rho + 2\lambda^\varphi \left[ v_\rho (v_\varphi^2 + v_\chi^2 + v_\rho^2 + 3v_\phi^2 + 2v_\chi^2) + v_\chi v_\rho \sqrt{(2av_\rho + 3v_\chi)(2v_\rho + 3av_\chi)} \right]}{2v_\rho^3}, \\
\lambda^\chi &= -\frac{\mu_\chi^2 v_\chi + 2\lambda^\varphi \left[ v_\chi (v_\varphi^2 + v_\chi^2 + v_\rho^2 + 3v_\phi^2 + 2v_\chi^2) \right]}{2v_\chi^2}, \\
\lambda^\phi &= \frac{\mu_{\phi}^2 v_\phi + 2\lambda^\varphi \left[ v_\phi (v_\varphi^2 + v_\chi^2 + v_\rho^2 + 5v_\phi^2 + v_\rho^2) \right]}{10v_\phi^3} - \frac{3\lambda^\varphi v_\varphi v_\rho (v_\phi + 2v_\phi^2 + 3av_\phi + 3av_\chi)}{10v_\phi^3 \sqrt{(2av_\rho + 3v_\chi)(2v_\rho + 3av_\chi)}}.
\end{align*}
\]
With the aid of the solution (B23), expressions (B15)-(B18) become

\[ \delta^2_v \sim -\mu^2_H - 2\lambda^x (v^2 + v_\chi^2 + v'^2 + 3v_\phi^2 + v_\rho^2 + 2v^2) > 0, \]  
(B24)

\[ \delta^2_v' \sim -\mu^2_H' - 2\lambda^x (v^2 + v_\chi^2 + v'^2 + 3v_\phi^2 + v_\rho^2 + 2v^2) > 0, \]  
(B25)

\[ \delta^2_\phi \sim -3\mu^2_\phi - \lambda^x [6v^2 + 6v_\phi^2 + 6(v_\rho^2 + v^2) + 20v_\rho^2 \] 
\[ + v_\phi v_\rho [a v_\phi (21av_\phi + 16v_\rho) + 3v_s (9av_\phi + 7v_\rho)] \] 
\[ \frac{v_\rho}{v_\phi} \sqrt{3av_\phi + 2v_\rho} \sqrt{2av_\phi + 3v_s} \] 
\[ > 0, \]  
(B26)

\[ \delta^2_\rho \sim -\mu^2_\rho - 2\lambda^x (v^2 + v_\chi^2 + v'^2 + 3v_\phi^2 + v_\rho^2) - \frac{3\lambda v_\phi v_\rho \sqrt{(2av_\phi + 3v_s)(2v_\phi + 3av_s)}}{v_\phi} > 0, \]  
(B27)

\[ \delta^2_\rho \sim -\mu^2_\rho - 2\lambda^x (v^2 + v_\chi^2 + v'^2 + 3v_\phi^2 + v_\rho^2) - \frac{3\lambda v_\phi v_\rho \sqrt{(2av_\phi + 3v_s)(2v_\phi + 3av_s)}}{v_\rho} > 0, \]  
(B28)

\[ \delta^2_\chi \sim -\mu^2_\chi - 2\lambda^x (v^2 + v_\phi^2 + v'^2 + 3v_\phi^2 + v_\rho^2 + 2v^2) > 0, \]  
(B29)

\[ \delta^2_\phi_s \sim -2\mu^2_\phi_s - 4\lambda^x (v^2 + v_\chi^2 + v'^2 + 5v_\phi^2 + v_\rho^2) - \frac{9\lambda v_\phi^2 v_\rho^2 (v_\phi + a^2 v_\phi + 3av_s)}{v_\phi \sqrt{(2av_\phi + 3v_s)(2v_\phi + 3av_s)}} > 0, \]

\[ \delta^2_\alpha \sim -\lambda^x v_\phi v_\rho v_\phi \sqrt{3av_\phi + 2v_\rho} \sqrt{2av_\phi + 3v_s} > 0. \]  
(B30)

Assuming that \( \mu^2_H, \mu^2_H', \mu^2_\phi, \mu^2_\rho, \mu^2_\phi_s, \mu^2_\rho_s, \mu^2_\chi \) and \( \mu^2_\phi_s \) are negative and of the same order of magnitude and the same as that of the SM \([6, 7]\),

\[ \mu^2_H \sim \mu^2_H' \sim \mu^2_\phi \sim \mu^2_\rho \sim \mu^2_\phi_s \sim \mu^2_\rho_s \sim \mu^2_\chi \sim -10^4 \text{ GeV}. \]
(B31)

Expressions Eqs. (4), (5), (B25)-(B30) and (B31) tell us that \( \delta^2_v = \delta^2_v' \) and \( \delta^2_\phi \) depend on one parameter \( \lambda^x \) while \( \delta^2_\phi_s, \delta^2_\phi, \delta^2_\rho \) and \( \delta^2_\chi \) depend on two parameters \( \lambda^x \) and \( a \) which are respectively plotted in Figs. 13 and 14 with \( \lambda^x \in (-10^{-2}, -10^{-4}) \) and \( a \in (1.0, 5.0) \). These figures imply that the expressions (B25)-(B30) are always satisfied by the VEV alignments in Eq. (5).

Figure 13: \( \delta^2_v = \delta^2_v' \) versus \( \lambda^x \) (in the left panel), and \( \delta^2_\phi \) versus \( \lambda^x \) and \( a \) (in the right panel) with \( \lambda^x \in (-10^{-2}, -10^{-4}) \) and \( a \in (1.0, 5.0) \)

---

5 In the SM \([61]\), \( |\mu| = 88.4 \) GeV. Here, we use \( |\mu_H| \sim |\mu_\phi| \sim |\mu_\rho| \sim |\mu_\phi_s| \sim |\mu_\rho_s| \sim |\mu_\chi| = 10^2 \) GeV for their scales.
Figure 14: $\delta^2_\phi$ (upper left), $\delta^2_\rho$ (upper right), $\delta^2_{\phi s}$ (bottom left) and $\delta^2_\alpha$ (bottom right) versus $\lambda^x$ and $a$ with $\lambda^x \in (-10^{-2}, -10^{-4})$ and $a \in (1.0, 5.0)$

Appendix C: The dependence of $|U_{ij}|$ ($i = 1, 2, 3; j = 1, 3$) on $s_{13}$ and $s_{23}$ for normal hierarchy
Appendix D: The dependence of $|U_{ij}| (i = 1, 2, 3; j = 1, 3)$ on $s_{13}$ and $s_{23}$ for inverted hierarchy
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Figure 15: $|U_{ij}| (i = 1, 2, 3; j = 1, 3)$ as functions of $s_{23}^2$ and $s_{13}^2$ with $s_{23}^2 \in (0.456, 0.544)\, \text{and}\, s_{13}^2 \in (2.00, 2.405) \times 10^{-2}$ for NH
Figure 16: $|U_{ij}|$ ($i = 1, 2, 3; j = 1, 3$) as functions of $s_{23}^2$ and $s_{13}^2$ with $s_{23}^2 \in (0.433, 0.545)$ and $s_{13}^2 \in (2.018, 2.242)10^{-2}$ for IH