STABILITY THEORY FOR NULLITY AND DEFICIENCY OF LINEAR RELATIONS

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Abstract. Let \( A \) and \( B \) be two closed linear relation acting between two Banach spaces \( X \) and \( Y \) and let \( \lambda \) be a complex number. We study the stability of the nullity and deficiency of \( A \) when it is perturbed by \( \lambda B \). In particular, we show the existence of a constant \( \rho > 0 \) for which both the nullity and deficiency of \( A \) remain stable under perturbation by \( \lambda B \) for all \( \lambda \) inside the disk \( |\lambda| < \rho \).

1. Introduction

For purposes of introduction we shall consider bounded linear operators \( A \) and \( B \) with domain \( X \) and range in \( Y \). As usual, let \( N(A) \) and \( R(A) \) denote the null space and range of \( A \) respectively. The dimensions of \( N(A) \) and \( Y/R(A) \) are called the nullity and the deficiency of \( A \) respectively and denoted by \( \alpha(A) \) and \( \beta(A) \). It is well known that \( \alpha(A) \) and \( \beta(A) \) have some kind of stability when \( A \) is subjected to some kind of perturbation (see for example [7]). More precisely, \( \alpha(A) \) and \( \beta(A) \) are unchanged when \( A \) is perturbed by some bounded linear operator \( B \) under certain prescribed conditions. This stability can be described in the form

\[
\alpha(A - B) - \beta(A - B) = \alpha(A) - \beta(A).
\]

Another convenient way of describing this stability is to put it in the form

\[
\alpha(A - B) = \alpha(A) \quad \text{and} \quad \beta(A - B) = \beta(A).
\]

The stability concept described here is very useful in studying eigenvalue problems of the form \( Ax = \lambda Bx \) and \( A^*y = \lambda B^*y \), where \( A^* \) denotes the adjoint operator.

This paper deals with the stability theory for nullity and deficiency of linear relations and it can be seen as a generalization of the classical theory for the corresponding quantities for linear operators. The theory and exposition developed here goes along the lines of the classical texts on the perturbation theory for linear operators (see for example [7] and [9]), but in a more general setting. Some stability theorems for multivalued linear operators or what we refer to here as linear relations, have been considered in [5] and more recently in [8]. In either of these cases, the perturbing multivalued linear operator \( B \) does not vary with the varying \( \lambda \) as the case we consider here.

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2. Preliminaries

2.1. Relations on sets. In this section we introduce some notation and consider some basic concepts concerning relations on sets. Let $U$ and $V$ be two nonempty sets. By a relation $\mathcal{T}$ from $U$ to $V$ we mean a mapping whose domain $D(\mathcal{T})$ is a nonempty subset of $U$, and taking values in $2^V \setminus \emptyset$, the collection of all nonempty subsets of $V$. Such a mapping $\mathcal{T}$ is also referred to as a multi-valued operator or at times as a set valued function. If $\mathcal{T}$ maps the elements of its domain to singletons, then $\mathcal{T}$ is said to be a single valued mapping or operator. Let $\mathcal{T}$ be a relation from $U$ to $V$ and let $\mathcal{T}(u)$ denote the image of an element $u \in U$ under $\mathcal{T}$. If we define $\mathcal{T}(u) = \emptyset$ for $u \in U$ and $u \notin D(\mathcal{T})$ then the domain $D(\mathcal{T})$ of $\mathcal{T}$ is given by

$$D(\mathcal{T}) = \{u \in U : \mathcal{T}(u) \neq \emptyset\}.$$ 

Denote by $R(U, V)$ the class of all relations from $U$ to $V$. If $\mathcal{T}$ belongs to $R(U, V)$, the graph of $\mathcal{T}$, which we denote by $G(\mathcal{T})$ is the subset of $U \times V$ defined by

$$G(\mathcal{T}) = \{(u, v) \in U \times V : u \in D(\mathcal{T}), v \in \mathcal{T}(u)\}.$$ 

A relation $\mathcal{T} \in R(U, V)$ is uniquely determined by its graph, and conversely any nonempty subset of $U \times V$ uniquely determines a relation $\mathcal{T} \in R(U, V)$.

For a relation $\mathcal{T} \in R(U, V)$ we define its inverse $\mathcal{T}^{-1}$ as the relation from $V$ to $U$ whose graph $G(\mathcal{T}^{-1})$ is given by

$$G(\mathcal{T}^{-1}) = \{(v, u) \in V \times U : (u, v) \in G(\mathcal{T})\}.$$ 

Let $\mathcal{T} \in R(U, V)$. Given a subset $M$ of $U$, we define the image of $M, T(M)$ to be

$$\mathcal{T}(M) = \bigcup \{\mathcal{T}(m) : m \in M \cap D(\mathcal{T})\}.$$ 

With this notation we define the range of $\mathcal{T}$ by

$$R(\mathcal{T}) := \mathcal{T}(U)$$

Let $N$ be a nonempty subset of $V$. The definition of $\mathcal{T}^{-1}$ given in (2.1) above implies that

$$\mathcal{T}^{-1}(N) = \{u \in D(\mathcal{T}) : N \cap \mathcal{T}(u) \neq \emptyset\}.$$ 

If in particular $v \in R(\mathcal{T})$, then

$$\mathcal{T}^{-1}(v) = \{u \in D(\mathcal{T}) : v \in \mathcal{T}(u)\}.$$ 

For a detailed study of relations, we refer to [1], [4], [3], [2], [5], and [9].

2.2. Linear Relations. Let $X$ and $Y$ be Linear spaces over a field $\mathbb{K} = \mathbb{R}$ (or $\mathbb{C}$) and let $\mathcal{T} \in R(X, Y)$. We say that $\mathcal{T}$ is a linear relation or a multi-valued linear operator if for all $x, z \in D(\mathcal{T})$ and any nonzero scalar $\alpha$ we have

1. $\mathcal{T}(x) + \mathcal{T}(z) = \mathcal{T}(x + z),$
2. $\alpha \mathcal{T}(x) = \mathcal{T}(\alpha x).$

The equalities in (1) and (2) above are understood to be set equalities. These two conditions indirectly imply that the domain of a linear relation is a linear subspace. The class of linear relations in $R(X, Y)$ will be denoted by $LR(X, Y)$. If $X = Y$ then we denote $LR(X, X)$ by $LR(X)$. We say that $\mathcal{T}$ is a linear relation in $X$ if $\mathcal{T} \in LR(X)$. We shall use the term operator to refer to a single valued linear operator while a multi-value linear operator will be generally referred to as a linear relation.
If $X$ and $Y$ are normed linear spaces, we say that $\mathcal{T} \in LR(X,Y)$ is closed if its graph $G(\mathcal{T})$ is a closed subspace of $X \times Y$. The collection of all such $\mathcal{T}$ will be denoted by CLR$(X,Y)$.

We conclude this section with the following theorems which are taken from [5].

**Theorem 2.1.** Let $\mathcal{T} \in R(X,Y)$. The following properties are equivalent.

(i) $\mathcal{T}$ is a linear relation.

(ii) $G(\mathcal{T})$ is a linear subspace of $X \times Y$.

(iii) $\mathcal{T}^{-1}$ is a linear relation.

(iv) $G(\mathcal{T}^{-1})$ is a linear subspace of $Y \times X$.

**Corollary 2.2.** Let $\mathcal{T} \in R(X,Y)$.

(i) Then $\mathcal{T}$ is a linear relation if and only if

$$\mathcal{T}(\alpha x_1 + \beta x_2) = \alpha \mathcal{T}(x_1) + \beta \mathcal{T}(x_2)$$

holds for all $x_1, x_2 \in D(\mathcal{T})$ and some nonzero scalars $\alpha$ and $\beta$.

(ii) If $\mathcal{T}$ is a linear relation then $\mathcal{T}(0)$ and $\mathcal{T}^{-1}(0)$ are linear subspaces.

For a linear relation $\mathcal{T}$, the subspace $\mathcal{T}^{-1}(0)$ is called the null space (or kernel) of $\mathcal{T}$ and is denoted by $N(\mathcal{T})$.

**Theorem 2.3.** Let $\mathcal{T}$ be a linear relation in a linear space $X$ and let $x \in D(\mathcal{T})$. Then $y \in T(x)$ if and only if

$$\mathcal{T}(x) = \mathcal{T}(0) + y.$$  

Theorem 2.3 shows that $\mathcal{T}$ is single valued if and only if $T(0) = \{0\}$.

**Theorem 2.4.** Let $\mathcal{T} \in R(X,Y)$. Then $\mathcal{T}$ is a linear relation if and only if for all $x_1, x_2 \in D(\mathcal{T})$ and all scalars $\alpha$ and $\beta$,

$$\alpha \mathcal{T}(x_1) + \beta \mathcal{T}(x_2) \subset \mathcal{T}(\alpha x_1 + \beta x_2).$$

**Theorem 2.5.** Let $\mathcal{T} \in LT(X,Y)$. Then

(a) $\mathcal{T}(M + N) = \mathcal{T}M + \mathcal{T}N$ for $M \subset X$ and $N \subset D(\mathcal{T})$.

(b) $\mathcal{T}T^{-1}(M) = M \cap R(\mathcal{T}) + \mathcal{T}(0)$ for $M \subset Y$.

(c) $\mathcal{T}^{-1}T(M) = M \cap D(\mathcal{T}) + \mathcal{T}^{-1}(0)$ for $M \subset X$.

2.3. **Normed linear relations.** Let $X$ be a normed linear space. By $B_X$ we shall mean the set

$$B_X := \{x \in X : |x| \leq 1\}.$$  

For a closed linear subspace $E$ of $X$, we denote by $Q_E$ the natural quotient map with domain $X$ and null space $E$. For $\mathcal{T} \in LR(X,Y)$, we shall denote $Q_{\mathcal{T}(0)}$ by $Q_{\mathcal{T}}$. It is well known that for $\mathcal{T} \in LR(X,Y)$, the operator $Q_{\mathcal{T}}\mathcal{T}$ is single valued (see [5]).

For $\mathcal{T} \in LR(X,Y)$, we set $\|Tx\| = \|Q_{\mathcal{T}}Tx\|$ for $x \in D(\mathcal{T})$ and $\|\mathcal{T}\| = \|Q_{\mathcal{T}}\mathcal{T}\|$. Note that these notions do not define a norm since nonzero relations can have zero norm.

**Lemma 2.6.** Let $A, B \in CLR(X,Y)$ be such that $D(A) \supset D(B)$ and $B(0) \subset A(0)$. If $x_1, x_2 \in D(A)$ are such that $A(x_1) \cap B(x_2) \neq \emptyset$ then $A(x_1) - B(x_2) \subset A(0)$.

**Proof.** Let $z \in A(x_1) \cap B(x_2)$. Since $Q_A$ and $Q_B$ are single valued, we see that

$$Q_A(A(x_1) - B(x_2)) = Q_A(x_1) - Q_B(x_2) = \tilde{z} - \tilde{z} = 0.$$  

Hence $A(x_1) - B(x_2) \in A(0)$. \qed
Lemma 2.7. The following properties are equivalent for a linear relation $A$.

(i) $A$ is closed.
(ii) $QAQ$ is closed and $A(0)$ is closed.

Lemma 2.8. (a) Let $T \in LR(X, Y)$ be bounded. Then $\|Tx\| \leq \|T\|\|x\|$.

(b) For $S$, $T \in LR(X, Y)$ with $D(S) \subset D(T)$ and $T(0) \subset S(0)$ we have

$$\|Sx + Tx\| \geq \|Sx\| - \|Tx\|.$$  

Proof. (a) From [5, II.1.6] we have $\|T\| = \sup_{x \in B_D(T)} \|Tx\|$ so that

$$\|T\| = \sup_{x \in D(T)} \left\| \frac{1}{\|x\|}Tx \right\|$$ and

$$\|T\| \geq \left\| \frac{1}{\|x\|}Tx \right\|, \quad x \in D(T).$$

The inequality $\|T\|\|x\| \geq \|Tx\|$ for all $x \in D(T)$ then follows from [5, II.1.5].

(b) Since $T(0) \subset S(0)$, we see that $(S + T)(0) = S(0) + T(0) = S(0)$ since $S(0)$ is a subspace (linear subset). For $x \in D(S)$, let $s \in S(x)$ and let $t \in T(x)$. Then $s + t \in (S + T)(x) = S(x) + T(x)$ and so by [5, II.1.4] we get

$$\|Sx + Tx\| = \text{dist} (s + t, (S + T)(0))$$

$$= \text{dist} (s + t, S(0))$$

$$\geq \text{dist} (s, S(0)) - \text{dist} (t, (S(0))$$

$$\geq \text{dist} (s, S(0)) - \text{dist} (t, (T(0))$$

$$= \|Sx\| - \|Tx\|.$$  

Let $X$ be a normed space. By $X'$ we denote the norm dual of $X$, that is, the space of all continuous linear functionals $x'$ defined on $X$, with norm

$$\|x'\| = \inf \{ \lambda : |[x, x']| \leq \lambda \|x\| \text{ for all } x \in X\}$$

where $[x, x'] := x'(x)$ denotes the action of $x' \in X'$ on $x \in X$. If $M \subset X$ and $N \subset X'$, we write $M^\perp$ and $N^\perp$ to mean

$$M^\perp := \{x' \in X' : [x, x'] = 0 \text{ for all } x \in M\}$$ and

$$N^\perp := \{x \in X : [x, x'] = 0 \text{ for all } x' \in N\}.$$  

Let $T$ be a linear relation with $D(T) \subset X$ and $R(T) \subset Y$. We define the adjoint $T'$ of $T$ by

$$G(T') := G(-T^{-1})^\perp \subset Y' \times X'$$

where

$$[(y, x), (y', x')] = [x, x'] + [y, y'].$$

This means that

$$\text{(2.3) } (y', x') \in G(T') \text{ if and only if } [y, y'] - [x, x'] = 0 \text{ for all } (x, y) \in G(T).$$

From (2.3) we see that $y'(y) = x'(x)$ for all $y \in T(x), x \in D(T)$. Hence

$$\text{(2.4) } x' \in T'(y') \text{ if and only if } y'T(x) = x'(x) \text{ for all } x \in D(T).$$
This means that \( x' \) is an extension of \( y'T(x) \) and therefore the adjoint \( T' \) can be characterized as follows:

\[
G(T') = \{(y', x') \in Y' \times X' \mid \text{such that } x' \text{ is an extension of } y'T\}.
\]

Please note that \( T' \in CLR(Y', X') \) (see [5 III.1.2]).

**Lemma 2.9.** [5 III.1.4]

Let \( T \) be a closed linear relation. Then

(a) \( N(T') = R(T)^\perp \).

(b) \( T'(0) = D(T)^\perp \).

(c) \( N(T) = R(T')^\top \).

(d) \( T(0) = D(T')^\top \).

**Remark 2.10.** If \( T \) and \( S \) are closed linear relations with \( D(T) \subset D(S) \) and \( S(0) \subset T(0) \) then \( S'(0) \subset T'(0) \) by Lemma 2.9 (b).

## 3. LOWER BOUND OF A CLOSED LINEAR RELATION

Consider a closed linear relation \( A \) on a Banach space \( X \) and let \( N(A) \) denote the null space of \( A \) which is closed since \( A \) is closed. Since \( N(A) \subset D(A) \), a coset \( \bar{x} \in X = X/N(A) \) which contains a point of \( x \in D(A) \) consists entirely of points of \( D(A) \). To see that this is the case, let \( \bar{x} \in X \) and let \( x, y \in \bar{x} \) with \( x \in D(A) \). Then \( y-x \in N(A) \subset D(A) \) and the linearity of \( D(A) \) implies that \( y = x + (y-x) \in D(A) \).

Let \( D \) denote the collection of all such cosets \( \bar{x} \). On setting

\[
A(\bar{x}) := Q_A A(x) \quad \text{for } \bar{x} \in D,
\]

we define a linear operator \( \tilde{A} : \tilde{X} \to \tilde{X} \) where \( \tilde{X} := X/A(0) \). To see that (3.1) is well defined, let \( x, y \in \tilde{x} \). Then \( x-y \in N(A) \) and therefore

\[
0 \in A(0) = A(x-y) = A(x) - A(y).
\]

We see from (3.2) that \( A(x) \cap A(y) \neq \emptyset \). So, let \( u \in A(x) \cap A(y) \). Then

\[
A(x) = A(0) + u = A(y)
\]

so that \( Q_A Ax = Q_A Ay \). We have

\[
D(A) = \tilde{D}, \quad R(A) = R(Q_A A), \quad N(A) = \{0\}.
\]

**Remark 3.1.** Since \( A(0) \subset R(A) \), we also have that a coset \( \tilde{x} \in \tilde{X} \) that contains a point of \( R(A) \) consists entirely of element of \( R(A) \). To see that this is the case, let \( \tilde{x} \) be a coset in \( \tilde{X} \) and let \( u, v \in \tilde{x} \) with \( u \in R(A) \). Then \( v-u \in A(0) \subset R(A) \).

The linearity of \( R(A) \) implies that \( v = u + (v-u) \in R(A) \).

**Lemma 3.2.** The linear operator \( A \) defined by (3.1) is closed.

**Proof.** Let \( \{\tilde{x}_n\} \) be a sequence in \( \tilde{D} \) such that \( \tilde{x}_n \to \tilde{x} \in \tilde{X} \) and let \( \{Q_A Ax_n\} \) be a sequence in \( R(A) \) such that \( Q_A Ax_n \to \tilde{y} \in \tilde{X} \). Let \( x_n \in \tilde{x}_n \) and \( x \in \tilde{x} \). Since \( x_n \to x \), we see that \( \text{dist}(x_n - x, N(A)) \to 0 \). This means that \( x_n - x \) converges to some element of \( N(A) \), say,

\[
x_n - x \to u \in N(A).
\]

From (3.4) we see that \( x_n \to x + u = w \in \tilde{x} \).

Since \( Q_A Ax_n \to \tilde{y} \in \tilde{X} \), that is, \( \tilde{x}_n \to \tilde{y} \), we see that \( \text{dist}(z_n - y, A(0)) \to 0 \) as \( n \to \infty \) and so \( z_n \to y + v = z \in \tilde{y} \) for some \( v \in A(0) \) (where \( z_n \in A(x_n) \) for each
\(n \in \mathbb{N}\). The closedness of \(\mathcal{A}\) implies that \(w \in D(\mathcal{A})\) and \(z \in \mathcal{A}(w)\). Hence \(\tilde{x} \in \tilde{D}\) and \(\mathcal{A}(\tilde{x}) = Q_{\mathcal{A}}\mathcal{A}(x) = \tilde{g}\), showing that \(\mathcal{A}\) is closed. \(\square\)

We see that \(A^{-1}\) is single valued since \(A^{-1}(\{0\}) = \{0\}\). We now introduce the quantity \(\gamma(\mathcal{A})\) called the lower bound of the linear relation \(\mathcal{A}\). By definition,

\[
(3.5) \quad \gamma(\mathcal{A}) = \frac{1}{\|A^{-1}\|}
\]

with the understanding that \(\gamma(\mathcal{A}) = 0\) if \(A^{-1}\) is unbounded and that \(\gamma(\mathcal{A}) = \infty\) if \(A^{-1} = 0\). It follows from (3.5) that

\[
(3.6) \quad \gamma(\mathcal{A}) = \sup \{ \gamma \in \mathbb{R} : \|A(x)\| \geq \gamma \|x\| = \text{dist}(x, N(\mathcal{A})) \forall x \in D(\mathcal{A}) \}.
\]

Note that \(\gamma(\mathcal{A}) = \infty\) if and only if \(\mathcal{A}x = \mathcal{A}(0)\) for all \(x \in D(\mathcal{A})\). In order for (3.6) to hold even for this case, one should stipulate that \(\infty \times 0 = 0\). Obviously \(\gamma(\mathcal{A}) = \gamma(\mathcal{A})\).

Please note that characterization (3.6) implies that if \(\gamma(\mathcal{A}) = 0\) then the domain of \(\mathcal{A}\) cannot consist of the zero element alone.

The fact that \(\gamma(\mathcal{A}) = \infty\) if and only if \(\mathcal{A}(x) = \mathcal{A}(0)\) for all \(x \in D(\mathcal{A})\) leads to the following lemma (see also [5, Proposition II.2.2]).

**Lemma 3.3.** For \(\mathcal{A} \in \text{CLR}(X, Y)\) we have

\[
\gamma(\mathcal{A}) = \begin{cases} 
\infty & \text{if } D(\mathcal{A}) \subset N(\mathcal{A}) \text{ and } \\
\inf \left\{ \frac{\|A(x)\|}{\|x\|} : x \in D(\mathcal{A}) \& x \notin N(\mathcal{A}) \right\} & \text{otherwise}.
\end{cases}
\]

**Remark 3.4.** A bounded linear operator \(T\) is closed if and only if \(D(T)\) is closed.

**Proof.** Suppose that \(u_n \to u\) with \(u_n \in D(T)\). The boundedness of \(T\) implies that \(T(u_n)\) is a Cauchy sequence and therefore converges, say \(T(u_n) \to v\). The closedness of \(T\) implies that \(u \in D(T)\) and \(T(u) = v\). This shows that \(D(T)\) is closed. \(\square\)

If \(\mathcal{S}\) is a closed linear relation from \(X\) to \(Y\), the graph of \(\mathcal{S}\), \(G(\mathcal{S})\) is a closed subset of \(X \times Y\). Sometimes it is convenient to regard it as a subset of \(Y \times X\). More precisely, let \(G'(\mathcal{S})\) be the linear subset of \(Y \times X\) consisting of all pairs of the form \((v, u)\) where \(u \in D(\mathcal{S})\) and \(v \in \mathcal{S}(u)\). We shall call \(G'(\mathcal{S})\) the inverse graph of \(\mathcal{S}\). As in the case of the graph \(G(\mathcal{S})\), \(G'(\mathcal{S})\) is closed if and only \(\mathcal{S}^{-1}\) is closed. Clearly, \(G(\mathcal{S}) = G'(\mathcal{S}^{-1})\). Thus \(\mathcal{S}^{-1}\) is closed if and only \(\mathcal{S}\) is closed.

**Lemma 3.5.** If \(\mathcal{A}\) is a closed linear relation then \(R(\mathcal{A})\) is closed if and only if \(\gamma(\mathcal{A}) > 0\).

**Proof.** By definition \(\gamma(\mathcal{A}) > 0\) if and only if \(A^{-1}\) is bounded (where \(A\) is the operator defined in (3.1)), and this is true if and only if \(D\left(A^{-1}\right) = R\left(A\right) = R(Q_{\mathcal{A}}\mathcal{A})\) is closed (we use the fact that \(A^{-1}\) is closed because \(A\) is closed, and then apply Remark (3.3)).

Now assume that \(\gamma(\mathcal{A}) > 0\) and set \(\{y_n\}\) be a convergent sequence in \(R(\mathcal{A})\) with

\[
(3.7) \quad y_n \to y.
\]

Since \(Q_{\mathcal{A}}\) is a bounded linear operator, the sequence \(\{Q_{\mathcal{A}}y_n\}\) is a Cauchy sequence in \(R(Q_{\mathcal{A}}\mathcal{A})\) and therefore converges to a point \(\hat{z} \in R(Q_{\mathcal{A}}\mathcal{A}) \subset \hat{H}\) since \(R(Q_{\mathcal{A}}\mathcal{A})\) is
closed. We see that dist\((y_n - z, A(0))\) → 0 as \(n \to \infty\) so that \(y_n - z \to v\) for some \(v \in A(0)\), that is,
\[
y_n \to z + v \in \hat{z}.
\]
(3.8)

Since \(A(0) \subset R(A)\), a coset \(\hat{x} \in \hat{H}\) that contains a point of \(R(A)\) consists entirely of element of \(R(A)\). To see that this is the case, let \(\hat{x}\) be a coset in \(\hat{H}\) and let \(u, v \in \hat{x}\) with \(u \in R(A)\). Then \(v - u \in A(0) \subset R(A)\). The linearity of \(R(A)\) implies that \(v = u + (v - u) \in R(A)\).

We see from (3.7) and (3.8) that \(y \in \hat{z}\) and that \(y \in R(A)\) since \(z \in R(A)\) and \(y \in \hat{z}\). This shows that \(R(A)\) is closed.

On the other hand, assume that \(R(A)\) is closed. Since \(A^{-1}\) is closed (since \(A\) is closed), it is enough, by the closed graph theorem, to show that \(D(A^{-1}) = R(A) = R(Q_{\mathcal{A}}A)\) is closed. So, assume that \(\{\hat{z}_n\}\) is a sequence in \(R(Q_{\mathcal{A}}A)\) such that \(\hat{z}_n \to \hat{z} \in \hat{H}\). Then dist\((z_n - z, A(0))\) → 0 as \(n \to \infty\). Hence, there exists an element \(w \in A(0)\) such that \(z_n \to z + w \in \hat{z}\). The closedness of \(R(A)\) implies that \(z + w \in R(A)\) so that \(\hat{z} \in R(Q_{\mathcal{A}}A)\).

Please see [5, III.5.3] for another proof of Lemma 3.5.

For the definition of continuity and openness of a linear relation \(T\) mentioned in the following two lemmas, please refer to [5].

**Lemma 3.6.** [5, II.3.2, III.1.3, III.1.5, III.4.6]

Let \(S, T \in LR(X,Y)\). Then

(a) \(T\) is continuous if and only if \(\|T\| < \infty\).
(b) \((\lambda T)' = \lambda T'\) (for \(\lambda \neq 0\)).
(c) \(T\) is open if and only if \(\gamma(T) > 0\).
(d) If \(D(S) \supset D(T)\) and \(\|S\| < \infty\) then \((T + S)' = T' + S'\).

**Lemma 3.7.** [5, III.4.6]

(a) \(T\) is continuous if and only if \(D(T') = T(0)\).
(b) \(T\) is open if and only if \(R(T') = N(T)^{\perp}\).
(c) If \(T\) is continuous, then \(\|T'\| = \|T\|\).
(d) If \(T\) is open, then \(\gamma(T) = \gamma(T')\).

4. **The gap between closed linear manifolds and their dimensions**

Let \(Z\) be a Banach space and let \(L\) be a closed subspaces of \(Z\). We denote by \(S_L\) the unit sphere of \(L\), that is, \(S_L := \{u \in L : \|u\| = 1\}\). For any two closed linear manifolds \(M\) and \(N\) of \(Z\) with \(M \neq \{0\}\), define the gap between \(M\) and \(N\), denoted by \(\delta(M,N)\) to be

\[
\delta(M,N) := \sup_{u \in S_M} \text{dist} (u,N)
\]

and set \(\delta(M,N) = 0\) if \(M = \{0\}\). \(\delta(M,N)\) can also be characterized as the smallest number \(\delta\) for which

\[
\text{dist} (u,N) \leq \delta \|u\| \text{ for all } u \in M.
\]

It can be seen from the definition that \(0 \leq \delta(M,N) \leq 1\).

See [7] for the following lemma.
Lemma 4.1. Let $M$ and $N$ be linear manifolds in a Banach space $Z$. If $\dim M > \dim N$ then there exists an $x \in M$ such that
\[
\operatorname{dist}(x, N) = \|x\| > 0.
\]

The above lemma can be expressed in the language of the quotient space as follows.

Lemma 4.2. Let $M$ and $N$ be linear manifolds in a Banach space $Z$. If $\dim M > \dim N$ then there exists an $x \in M$ such that
\[
\|\tilde{x}\| = \|x\| > 0, \quad \text{where } \tilde{x} \in \tilde{X} := X/N. \quad (N \text{ is closed since } \dim N < \infty)
\]

The following lemma is a direct consequence of the preceding one.

Lemma 4.3. If $\|\tilde{x}\| < \|x\|$ for every non-zero $x \in M$ where $\tilde{x} \in \tilde{X} = X/N$ then $\dim M \leq \dim N$.

See [7, Page 200] and [6] for Lemma 4.4 and Lemma 4.5 respectively.

Lemma 4.4. Let $M$ and $N$ be closed linear manifolds of a Banach space $Z$. If $\delta(M, N) < 1$ then $\dim M \leq \dim N$.

Lemma 4.5. Let $x$ be an element of a normed linear space $X$ and let $M$ and $N$ be closed linear subspaces of $X$. Consider the quotient space $\tilde{X} := X/N$ and let $\tilde{x}$ denote the quotient class of $x$. For any $\varepsilon > 0$ there exists $x_0 \in \tilde{x}$ such that
\[
\operatorname{dist}(x_0, M) \geq (1 - \varepsilon) \left(1 - \frac{\delta(M, N)}{1 + \delta(M, N)}\right) \|x_0\|.
\]

5. The quantity $\nu(A : B)$

Let $X$ and $Y$ be two linear spaces and let $A, B \in LR(X, Y)$ with $B(0) \subset A(0)$. For $n \in \mathbb{N}$, let $M_n$ and $N_n$ be the linear manifolds of $X$ and $M'_n$ and $N'_n$ be the linear manifolds of $Y'$ defined inductively as follows:
\[
M_0 = X, \quad M_n = B^{-1}(A(M_{n-1})) \quad \text{for } n = 1, 2, \ldots,
\]
\[
N_1 = A^{-1}(0), \quad N_n = A^{-1}(B(N_{n-1})) \quad \text{for } n = 2, 3, \ldots,
\]
\[
M'_0 = Y', \quad M'_n = B'^{-1}(A'(M'_{n-1})) \quad \text{for } n = 1, 2, \ldots,
\]
\[
N'_1 = A'^{-1}(0), \quad N'_n = A'^{-1}(B'(N'_{n-1})) \quad \text{for } n = 2, 3, \ldots.
\]

If $M_k \supset M_{k+1}$ then $A(M_k) \supset A(M_{k+1})$ and therefore
\[
M_{k+1} = B^{-1}(A(M_k)) \supset B^{-1}(A(M_{k+1})) = M_{k+2}.
\]

Since $M_0 = X \supset D(B) \supset M_1$, we conclude by induction that
\[
M_0 \supset M_1 \supset M_2 \supset \cdots \supset N(B).
\]

Similarly,
\[
N_1 \subset N_2 \subset N_3 \subset \cdots \subset D(A).
\]

Note that
\[
N_1 = N(A).
\]
Lemma 5.1. Let \( n \) be a positive integer. The following first \( n \) conditions are equivalent to one another and they in turn imply that condition \((\kappa)\) holds.

1. \( N_1 \subset M_n \),
2. \( N_2 \subset M_{n-1} \),
3. \( \vdots \),
4. \( N_n \subset M_1 \),
5. \( A(N_{k+1}) \cap B(N_k) \neq \emptyset \), \( N_k \subset D(B) \), for \( k = 1, 2, \ldots, n \).

Proof. First we prove the equivalence of the conditions (1) to (5.9). For each \( r = 1, 2, \ldots, n - 1 \), (r) implies \((r + 1)\). In fact if \( N_r \subset M_{n-r+1} \), then (5.5), (5.6) and (5.8) imply that

\[
N_{r+1} = A^{-1}(B(M_r)) \subset A^{-1}(B(M_{n-r+1})) \subset A^{-1}(A(M_{n-r}) + B(0))
\]

so that each \( x \in N_r \) has the property that there exists a \( z \in B(x) \) such that \( z \in A(y) \) for some \( y \in M_{n-r-1} \). Then \( y \in A^{-1}(B(N_r)) = N_{r+1} \subset M_{n-r} \) and so \( x \in B^{-1}(A(M_{n-r})) = M_{n-r+1} \) This proves that \( N_r \subset M_{n-r+1} \).

Next we prove that (n) implies (\( \kappa \)). So, suppose that (n) is satisfied. Then \( N_k \subset N_n \subset M_1 = B^{-1}(AX) \subset D(B) \) for \( k < n \), so that for each \( x \in N_k \), there exists a \( z \in B(x) \) such that \( z \in A(y) \) for some \( y \in X \). Then \( y \in A^{-1}(B(N_k)) = N_{k+1} \) and so \( A(N_{k+1}) \cap B(N_k) \neq \emptyset \).

If \( N_1 \subset M_n \) then \( N_1 \subset M_{n'} \), for all \( n' < n \) since \( M_n \) is a non increasing sequence. We denote by \( \nu(A : B) \) the smallest number \( n \) for which the condition \( N_1 \subset M_n \) (or any one of the other equivalent conditions) is not satisfied. We set \( \nu(A : B) = \infty \) if there is no such \( n \). This is the case if for example \( A^{-1}(0) \subset B^{-1}(0) \).

Lemma 5.2. Let \( X \) and \( Y \) be Banach spaces and let \( A, B \in \text{CLR}(X,Y) \) with \( D(A) = D(B) = X \). Then

\[
M_n' \subset (B(N_n))^\perp \text{ and } N_n' \subset (A(M_{n-1}))^\perp \text{ for } n = 1, 2, \ldots.
\]

Proof. First we show that (5.9) holds for \( n = 1 \). To begin with, let \( y' \in M_1' \) and let \( x \in D(B) \cap N_1 \). Then by definition, \( y' \in B^{-1}[A'(Y')] \) and \( x \in A^{-1}(0) \cap D(B) \). Hence there exists an element \( x' \in A'(Y') \cap R(B') \) such that \( (y', x') \in G(B') \). Since \( x' \in A'(Y') \), there exists an element \( f' \in D(A') \subset Y' \) such that \( (f', x') \in G(A') \). Since \( (x, 0) \in G(A), (5.8) \) implies that \( f'(0) = x'(x) \) so that \( x'(x) = 0 \). So, for \( y \in B(x), y'(y) = x'(x) = 0 \), showing the \( y' \in [B(N_1)]^\perp \).

The second inclusion follows from

\[
N_1' = N(A') = R(A)^\perp = [A(M_0)]^\perp \text{ (see Lemma 2.2 (a)).}
\]

We shall therefore assume that (5.9) has been proved for \( n = k \) and try to prove it for \( n = k + 1 \). So, let \( g' \in M_{k+1}' \) and let \( z \in D(B) \cap N_{k+1} \). Then \( g' \in B^{-1}[A'(M_{k+1})^\perp] \) and \( z \in A^{-1}[B(N_k)] \cap D(B) \). Hence there exists an element \( h' \in A'(M_{k+1})^\perp \) such
that \((g',h') \in G(B')\). Since \(h' \in \mathcal{A}'(M'_k)\) it follows that there exists an element \(l' \in M'_k\) such that \((l',h') \in G(A')\). The fact that \(z \in N_{k+1}\) means that there is an element \(w \in B(N_k)\) such that \((z,w) \in G(A)\). This means that \(l'(w) = h'(z)\) and \(h'(z) = 0\) since \(l \in [B(N_k)]^\perp\). So, for \(u \in B(z)\), \(g'(u) = h'(z) = 0\) meaning that \(g' \in [B(N_{k+1})]^\perp\) and that \(M'_{k+1} \subset [B(N_{k+1})]^\perp\). This proves the first inclusion in (5.9). The second inclusion can be proved in a similar way.

**Lemma 5.3.** Let \(\mathcal{A} \in \text{CLR}(X,Y)\). For every \(f' \in N(\mathcal{A})^\perp\) there exists \(g' \in Y'\) such that \(g'(y) = f'(x)\) for all \(y \in \mathcal{A}(x)\) and all \(x \in D(\mathcal{A})\).

**Proof.** Define a linear functional \(g'\) on \(Y'\) by setting \(g'(y) = f'(x)\) for all \(y \in \mathcal{A}(x)\) and all \(x \in D(\mathcal{A})\). Then \(g'\) is defined on \(R(\mathcal{A})\) and is bounded. To show that \(g'\) is indeed bounded, we first note that for \(y \in \mathcal{A}(x)\),

\[
|g'(y)| = |f'(x)| \leq \|f\|\|x\|
\]

and consider the quotient space \(\tilde{X} := X/N(\mathcal{A})\). Let \(x_1 \in \tilde{x}\). Then \(x - x_1 = u\) for some \(u \in N(\mathcal{A})\) so that \(f(x) = f(x_1)\). This equality means that \(\|x\|\) in (5.10) can be replaced with \(\|x_1\|\) for any \(x_1 \in \tilde{x}\) without changing the inequality. This therefore means that

\[
|g'(y)| \leq \|f'\|\|\tilde{x}\| \leq \|f'\|\|\gamma(\mathcal{A})^{-1}\|\|\mathcal{A}x\| = \|f'\|\|\gamma(\mathcal{A})^{-1}\|Q_\mathcal{A}y\| \leq \|f'\|\|\gamma(\mathcal{A})^{-1}\|Q_\mathcal{A}\|\|y\|,
\]

that is, \(g'\) is bounded on \(R(\mathcal{A})\). The Hahn-Banach extension theorem implies that \(g'\) can be extended to the whole of \(Y'\) without changing its bound.

**Remark 5.4.** Lemma 5.3 above implies that \(N(\mathcal{A})^\perp \subset R(\mathcal{A}')\) and that \(N(\mathcal{A})^\perp = R(\mathcal{A}')\) by Lemma 2.9 (c).

**Lemma 5.5.** Let \(\mathcal{A}, \mathcal{B} \in \text{CLR}(X,Y)\) with \(D(\mathcal{A}) = D(\mathcal{B}) = X\), \(R(\mathcal{A})\) closed and \(\mathcal{B}\) bounded. If \(B(0) \subseteq A(0)\) then

\[
M'_1 = [B(N_1)]^\perp.
\]

(5.12)

\[
\nu(\mathcal{A}') : B' = \nu(\mathcal{A} : B).
\]

**Proof.** Let \(f' \in [B(N_1)]^\perp = (B(\mathcal{A}'(0)))^\perp\). Since \(B(0) \subseteq B(N_1)\), Lemma 3.10 (a) together with Lemma 3.7 (a) imply that \(f' \in D(B')\). So, let \(g' \in B'(f')\), that is, \((f',g') \in G(B')\). This means that for \(x \in N_1\) and \(y \in B(x)\), \(g'(x) = f'(y) = 0\), which shows that \(g' \in N_1^\perp = N(\mathcal{A})^\perp\) and therefore \(g' \in R(\mathcal{A}')\) and so \(g' \in R(\mathcal{A}')\) by Remark 5.4. It follows that \(f' \in B'^{-1}[A'(Y')] = M'_1\). This shows that \([B(N_1)]^\perp \subseteq M'_1\). Equality (5.11) then follows by (5.9). To prove the second equality, let \(v = \nu(\mathcal{A} : B)\). Then \(N_1 \subseteq M_n\) for all \(n < v\). Since \(M_n = B^{-1}[\mathcal{A}(M_{n-1})]\) we see that

\[
B(N_1) \subseteq B(M_n) \subseteq \mathcal{A}(M_{n-1}) + B(0) \subseteq \mathcal{A}(M_{n-1}) + A(0) = \mathcal{A}(M_{n-1}),
\]

where the last equality follows from the fact that \(A(0) \subseteq \mathcal{A}(M_{n-1})\) and \(\mathcal{A}(M_{n-1})\) is a linear space. We see from (5.13) that \([\mathcal{A}(M_{n-1})]^\perp \subset [B(N_1)]^\perp\). If then follows from (5.9) and (5.11) that \(N'_n \subseteq M'_1\). This means that \(v' = (\mathcal{A}' : B') > n\) and that \(v' \geq v\).
To prove the opposite inequality, let \( n < v' \). Then we have \( N'_n \subset M'_n \). If follows from Lemma 2.9(a), (3.3), and (6.9) that \( [A(X)]^\perp \subset [B(N_n)]^\perp \). Since \( R(A) = A(X) \) is closed, this implies that \( B(N_n) \subset A(X) \). Since \( D(B) = X \) we see that \( N_n \subset N_n + B(0) \subset B^{-1}[A(X)] = M_1 \). This shows that \( v > n \) and therefore \( v \geq v' \).

\[ \square \]

6. Nullity and Deficiency

In this section we study the behaviour of the nullity and deficiency for linear relations under some perturbations. For \( A \in LR(X, Y) \), the nullity \( \alpha(A) \) and the deficiency \( \beta(A) \) are defined by

\[ \alpha(A) := \dim N(A) \quad \text{and} \quad \beta(A) := \dim Y/R(A). \]

**Lemma 6.1.** [5, III.7.2]
Let \( T \) be a closed linear relation with \( \gamma(T) > 0 \). Then \( \alpha(T') = \beta(T) \).

Let \( X \) and \( Y \) be Banach spaces and let \( A \) be a closed linear relation with \( D(A) \subset X \) and \( R(A) \subset Y \). Let \( n \in \{ \mathbb{N} \cap 0 \} \) be such that for any \( \varepsilon > 0 \) there exists an \( n \)-dimensional closed linear subset \( N_\varepsilon \) of \( N(A) \) such that

\[ \|A(x)\| \leq \varepsilon \|x\| \quad \text{for all} \quad x \in N_\varepsilon \]

while this is not true if \( n \) is replaced by a larger number. In such a case we set \( \alpha'(A) := n \) and define \( \beta'(A) \) to be

\[ \beta'(A) := \alpha'(A'). \]

The following two lemmas show that \( \alpha'(A) \) is defined for every closed linear relation \( A \).

**Lemma 6.2.** Assume that for every \( \varepsilon > 0 \) and any closed linear subset \( M \) of \( X \) of finite codimension, there is an \( x \in M \cap D(A) \) such that \( \|x\| = 1 \) and \( \|A(x)\| \leq \varepsilon \), then \( \alpha'(A) = \infty \).

**Proof.** We have to show that for each \( \varepsilon > 0 \), there exists an infinite dimensional closed linear subset \( N_\varepsilon \subset D(A) \) with the property (6.1). First we construct two sequences \( x_n \in D(A) \) and \( f_n \in X' \) such that

\[ \|x_n\| = 1, \|f_n\| = 1, f_n(x_n) = 1, \]

\[ f_k(x_n) = 0, \quad k = 1, 2, \ldots, n - 1, \]

\[ \|A(x_n)\| \leq 3^{-n} \varepsilon, \quad n \in \mathbb{N}. \]

For \( n = 1 \), the result holds by [7, III-Corollary 1.24]. Suppose that \( x_n, f_k \) have been constructed for \( k = 1, 2, \ldots, n - 1 \). Then \( x_n \) and \( f_n \) can be constructed in the following way. Let \( M \subset X \) be the collection of all \( x \in X \) such that \( f_k(x) = 0 \), \( k = 1, 2, \ldots, n - 1 \). Since \( M \) is a closed linear subset of \( X \) with finite codimension (dim \( M^\perp \leq n - 1 \) and use codim \( M = \dim M^\perp \)), there is an \( x_n \in M \cap D(A) \) such that \( \|x_n\| = 1 \) and \( \|A(x_n)\| \leq 3^{-n} \varepsilon \). For this \( x_n \), there exists an \( f_n \in X' \) such that \( \|f_n\| = 1 \) and \( f_n(x_n) = 1 \) (see [7, III-Corollary 1.24]). It follows from (6.3) that the \( x_n \) are linearly independent so that \( M'_n := \text{span} \{ x_1, x_2, \ldots \} \) is infinite dimensional. Each \( x \in M'_n \) has the form

\[ x = \xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_n x_n \]

for some positive integer \( n \). Hence for \( k = 1, 2, \cdots, n, \)

\[ f_k(x) = \xi_1 f_k x_1 + \xi_2 f_k(x_2) + \cdots + \xi_{k-1} f_k(x_{k-1}) + \xi_k. \]
We show that the coefficients \( \xi_k \) satisfy the inequality
\[
|\xi_k| \leq 2^{k-1}||x||, \quad k = 1, 2, \ldots, n.
\]
For \( k = 1 \), this is clear from (6.3) and (6.1). If we assume that (6.6) has been proved for \( k < j \), we see from (6.6) that
\[
|\xi_j| \leq |f_j(x)| + |\xi_1| |f_1(x)| + \cdots + |\xi_{j-1}| |f_{j-1}(x)|
\leq ||x|| + |\xi_1| + |\xi_2| + \cdots + |\xi_{j-1}|
\leq ||x|| + 2||x|| + \cdots + 2^{j-2}||x||
= ||x|| \left[ 2 + 2(1 + 2 + 2^2 + \cdots + 2^{j-1}) \right]
= 2^{j-1}||x||.
\]
It follows from (6.3), (6.1) and (6.1) that
\[
||A(x)|| \leq |\xi_1||Ax_1| + \cdots + |\xi_n||Ax_n|
\leq \left( \frac{1}{3} + \frac{2}{3^2} + \frac{2^2}{3^3} + \cdots + \frac{2^{n-1}}{3^n} \right) \varepsilon \|x\|
\leq \varepsilon \|x\|.
\]
Let \( u \in \overline{M}_\gamma \) and let \( \{u_n\} \) be a sequence in \( M'_\gamma \) such that \( u_n \to u \). The boundedness of \( Q_\gamma A \) on \( M'_\gamma \) implies that \( \{Q_\gamma A(x_n)\} \) is a Cauchy sequence in \( \overline{Y} := Y/A(0) \) and therefore converges, say \( Q_\gamma A(x_n) \to \tilde{v} \in \overline{Y} \). This means that dist \( (x_n - v, A(0)) \to 0 \) as \( n \to \infty \), that is, \( x_n - v \to z \in A(0) \) for some \( z \in A(0) \). In other words, \( x_n \to v + z = w \in \tilde{v} \). The closedness of \( A \) implies that \( x \in D(A) \) and \( w \in A(x) \). Hence \( Q_\gamma A \) is defined and bounded on the closure of \( M'_\gamma \) with the same bound. \( \square \\

**Lemma 6.3.** If \( A \) is a closed linear relation with closed range (that is, \( \gamma(A) > 0 \)) then \( \alpha'(A) = \alpha(A) \) and \( \beta'(A) = \beta(A) \).

**Proof.** By Lemma 3.3, \( \gamma(A) > 0 \) implies \( \gamma(A') > 0 \) while Lemma 6.1 implies that \( \alpha(A') = \beta(A) \). In view of (6.2), it is enough to show that \( \alpha'(A) = \alpha(A) \). It is clear that \( \alpha'(A) \geq \alpha(A) \). Now suppose that there exists a closed linear manifold \( N_\varepsilon \) with \( \dim N_\varepsilon > \alpha(A) = \dim N(A) \) and with property (6.1). Pick \( x \in N_\varepsilon \) such that \( ||\tilde{x}|| = ||x|| = 1 \) where \( \tilde{x} \in \overline{X} = X/N(A) \) (this is possible by Lemma 241). For this \( x \), \( ||A(x)|| \geq \gamma(A) \) on the one hand and \( ||A(x)|| \leq \varepsilon \) on the other hand, leading to the inequality \( \gamma(A) \leq \varepsilon \). In other words, there is no \( N_\varepsilon \) with \( \dim N_\varepsilon > \alpha(A) = \dim N(A) \) for \( \varepsilon < \gamma(A) \). This proves that \( \alpha'(A) \leq \alpha(A) \) and that \( \alpha'(A) = \alpha(A) \). The second equality follows from (6.2) and Lemma 6.1. \( \square \\

**Lemma 6.4.** Let \( T \in CLR(X) \) with non closed range (that is, \( \gamma(T) = 0 \)), then
\[
\alpha'(T) = \infty.
\]

**Proof.** Let \( M \) be any closed linear manifold of \( X \) with finite codimension and let \( QT \) be denoted by \( Q \). Consider the mapping \( T : X/M \to QT(X)/QT(M) \) defines by setting \( T(\tilde{x}) = \widehat{QT(x)} \). Then \( T \) is clearly well defined and linear. It is well defined since
\[
T(\tilde{x} + v) = \widehat{QT(x + v)} = QT(x) + QT(v) = \widehat{QT(x)} = T\tilde{x}
\]
for any \( v \in M \). If follows that \( QT(X)/QT(M) \) is a finite dimensional space since \( M \) has finite codimension. \([7] \) III-Lemma 1.9 implies that \( QT(X) \) is a closed subset of \( \overline{Y} := Y/T(0) \) if \( QT(M) \) is a closed subspace of the same space. This would mean
that $\mathcal{T}(X)$ is a closed subset of $Y$. To see why this is true, let $\{y_n\}$ be a convergent sequence in $\mathcal{T}(X)$ with $y_n \to y \in Y$. Then $\{Qy_n\}$ is a Cauchy sequence in $\tilde{Y}$ and therefore converges to some $\tilde{z} \in Q\mathcal{T}(X)$. In other words, $y_n - z \to w \in \mathcal{T}(0)$, so that $y_n \to z + w \in \tilde{z}$. The uniqueness of the limit implies that $y = z + w \in \tilde{z}$ and that $y \in R(T)$ since $z \in R(T)$ and every coset that contains element of $R(T)$ consists entirely of elements of $R(T)$. Next we show that if $\mathcal{T}(M)$ is closed then $Q\mathcal{T}(M)$ is closed. So, assume that $\mathcal{T}(M)$ is closed and let $\{\tilde{z}\}$ be a sequence in $Q\mathcal{T}(M)$ that converges to an element $\tilde{z} \in \tilde{Y}$. Then $z_n - z \to v \in \mathcal{T}(0)$ and so $z_n \to z + v \in \tilde{z}$. The closedness of $\mathcal{T}(M)$ implies that $z + v \in \mathcal{T}(M)$ and that $\tilde{z} \in Q\mathcal{T}(M)$.

The contradiction that $\mathcal{T}(X)$ is both open and closed means that $Q\mathcal{T}(M)$ is not closed and that $\mathcal{T}(M)$ is not closed and therefore $\gamma(\mathcal{T}_M) = 0$. Hence there exists, for any $\varepsilon > 0$, an $x \in M \cap D(T)$ such that $||x|| = 1$ and $||T(x)|| \leq \varepsilon ||\tilde{x}|| \leq \varepsilon ||x|| = \varepsilon$ where $\tilde{x} \in \tilde{X} = X/N(T)$. This shows that the conditions of Lemma 6.2 are satisfied and therefore $\alpha'(T) = \infty$.

**Theorem 6.5.** Let $X$ and $Y$ be Banach spaces and let $A$ be a closed linear relation with $D(A) \subset X$, having closed range $R(A) \subset Y$, and with $\alpha(A)$ finite. Let $B$ be a closed bounded linear relation such that $D(B) \supset D(A)$, $B(0) \subset A(0)$ and

$$
\|B\| < \gamma(A).
$$

Then the linear relation $A + B$ is closed and has closed range. Moreover,

$$
\alpha(A + B) \leq \alpha(A), \quad \beta(A + B) \leq \beta(A).
$$

**Proof.** Let $\{x_n\}$ be a sequence in $D(A)$ such that $x_n \to x \in X$ and let $\{y_n\}$ be a sequence in $R(A + B)$ such that $y_n \to y \in Y$, where $y_n = u_n + v_n$ with $u_n \in A(x_n)$ and $v_n \in B(x_n)$ for each $n \in \mathbb{N}$. In other words,

$$
u_n + v_n \to y.\tag{6.10}
$$

Note that (6.8) implies that $\{Q_B B(x_n)\}$ is a Cauchy sequence in $\tilde{Y} := Y/B(0)$ and therefore converges to a point of $\tilde{Y}$, say $Q_B B(x_n) \to \tilde{v} \in \tilde{Y}$. Hence $\text{dist}(v_n - B(x_n) \to 0$ as $n \to 0$, that is, $v_n \to z$ for some $z \in B(0)$. Hence $v_n \to v + z \in \tilde{v}$. The closedness of $B$ implies that $x \in D(B)$ and $v + z \in B(x)$. Hence $y = y - v - z + v + z \in A(x) + B(x)$ and so $A + B$ is closed.

To complete the proof, it is enough to show that

$$
\alpha'(A + B) \leq \alpha(A) \quad \text{and that} \quad \beta'(A + B) \leq \beta'(A)
$$

and then apply Lemma 6.3 to conclude that $A + B$ has closed range and Lemma 6.3 to establish the inequalities in the theorem since $\alpha'(A + B) \geq \alpha(A + B)$ by definition and $\beta'(A + B) \geq \alpha(A + B)$ by 6.2 and Lemma 6.3.

To prove (6.11), suppose that for a given $\varepsilon > 0$ there exists a closed linear manifold $N_\varepsilon \subset D(A + B) = D(A)$ such that

$$
\|(A + B)(x)\| \leq \varepsilon \|x\| \quad \text{for every} \ x \in N_\varepsilon.
$$

It then follows form (6.12) and Lemma 2.8 that

$$
\|B\| + \varepsilon \|x\| \geq \|B(x)\| + \|(A + B)(x)\| \geq \|B(x)\| + \|(A + B)(x)\| = \|A(x)\| \geq \gamma(A) \|\tilde{x}\| \tag{6.13}
$$
where \( \tilde{x} \in \tilde{X} := X/N(A) \). If we pick \( \varepsilon \) such that \( 0 < \varepsilon < \gamma(A) - \|B\| \) we see from (6.13) that \( \|\tilde{x}\| < \|x\| \) for all non-zero \( x \in D(A) \). It therefore follows from Lemma 6.3 that

\[
\dim N_\varepsilon \leq \dim N(A) = \alpha(A),
\]

which means that \( \alpha'(A + B) \leq \alpha(A) \).

To prove the second inequality, we note that Lemma 6.6 together with Lemma 6.7 imply that \( \|B'\| = \|B\|, \gamma(A') = \gamma(A) \), and \( (A + B)' = A' + B' \). It therefore follows that \( \|B'\| \leq \gamma(A') \). Applying what has been proved above to the pair \( A', B' \), we see that

\[
\beta'(A + B) = \alpha'((A + B}') = \alpha'(A' + B') \leq \alpha(A') = \beta(A),
\]

where the last equality follows from Lemma 6.1.

**Lemma 6.6.** Let \( X \) and \( Y \) be Banach spaces and let \( T \) be a closed linear relation with \( D(T) \subset X \) and \( R(T) \subset Y \). Set

\[
(6.14) \quad \|x\|_{D(T)} := \|x\| + \|T(x)\|, \quad x \in D(T).
\]

Then \( D(T) \) becomes a Banach space if \( \|\cdot\|_{D(T)} \) is chosen as the norm.

**Proof.** That \( \|\cdot\|_{D(T)} \) defines a norm on \( D(T) \) is clear. To prove completeness, assume that \( \{x_n\} \) is a Cauchy sequence in \( D(T) \). Then \( \{x_n\} \) and \( \{Q_T T(x_n)\} \) are Cauchy sequences in \( X \) and \( \tilde{Y} = Y/T(0) \) respectively and therefore converge, say, \( x_n \to x \in X \) and \( Q_T T(x_n) \to \tilde{u} \in \tilde{Y} \). Let \( u_n \in T(x_n) \) for each \( n \in \mathbb{N} \). Then \( \tilde{u}_n \to \tilde{u} \) and so dist \((u_n - u, T(0))\) \( \to 0 \) as \( n \to \infty \), that is, \( u_n - u \to v \in T(0) \). We therefore see that \( u_n \to u + v = s \in \tilde{u} \). The closedness of \( T \) implies that \( x \in D(T) \) and that \( s \in T(x) \). Now,

\[
\|x_n - x\|_{D(T)} = \|x_n - x\| + \|Q_T T(x_n - x)\| = \|x_n - x\| + \|Q_T u_n - Q_T s\|
\]

\[
= \|x_n - x\| + \|\tilde{u}_n - \tilde{u}\| \to 0 \text{ as } n \to \infty.
\]

This shows that \( D(T) \) is complete. \( \square \)

Let \( X \) and \( Y \) be Banach spaces and let \( A, B \in CLR(X,Y) \) be such that \( D(A) \subset D(B) \) and \( B(0) \subset A(0) \). In the following theorem we write \( \|B(x)\|_A \) to mean the quantity \( \|Q_A B(x)\| \). The quantities \( \|A(x)\|_A \) and \( \|B(x)\|_B \) are defined in a similar way

**Theorem 6.7.** Let \( X \) and \( Y \) be Banach spaces and let \( A \) be a closed linear relation with \( D(A) \subset X \) and with closed range \( R(A) \subset Y \). Let \( B \) be a closed linear relation such that \( D(A) \subset D(B) \subset X \), \( R(B) \subset Y \), \( B(0) \subset A(0) \), and

\[
(6.15) \quad \|B(x)\|_B \leq \sigma \|x\| + \tau \|A(x)\|_A, \quad \forall x \in D(A),
\]

where \( \sigma \) and \( \tau \) are non-negative constants such that

\[
(6.16) \quad \sigma + \tau \gamma(A) < \gamma(\mathcal{A}).
\]

Then the linear relation \( A + B \) is closed and has closed range. If \( \alpha(A) < \infty \) then

\[
(6.17) \quad \alpha(A + B) \leq \alpha(A), \text{ and } \beta(A + B) \leq \beta(A).
\]

**Proof.** Let \( \{x_n\} \) be a sequence in \( D(A) \) such that \( x_n \to x \in X \) and let \( \{y_n\} \) be a sequence in \( R(A + B) \) such that \( y_n \to y \in Y \), where \( y_n = u_n + v_n \) with \( u_n \in A(x_n) \) and \( v_n \in B(x_n) \) for each \( n \in \mathbb{N} \). Note that (6.15) implies that

\[
(6.18) \quad \|A(x)\|_A - \|B(x)\|_B \geq (1 - \tau) \|A(x)\|_A - \sigma \|x\|.
\]
Since \( \| B(x) \|_B = \| Q_A B(x) \|_B \geq \| Q_A Bx \|_A \) we see that
\[
(6.19) \quad \| Q_A A(x) \|_A - \| Q_A B(x) \|_A \geq (1 - \tau)\| A(x) \| - \sigma \| x \|
\]
and that
\[
(6.20) \quad \| Q_A A x + Q_A B x \|_A \geq (1 - \tau)\| Q_A A x \| - \sigma \| x \|.
\]
Inequality (6.20) and the linearity of \( Q_A \) implies that
\[
(6.21) \quad \| Q_A (u_n + v_n) \|_A \geq (1 - \tau)\| Q_A u_n \| - \sigma \| x_n \|
\]
so that
\[
(6.22) \quad \| y_n \| = \| u_n + v_n \| \geq (1 - \tau)\| Q_A u_n \| - \sigma \| x_n \|.
\]
It therefore follows that for \( m, n \in \mathbb{N} \),
\[
(6.23) \quad \| y_n - y_m \| \geq (1 - \tau)\| Q_A u_n - Q_A u_m \| - \sigma \| x_n - x_m \|.
\]
Since \( 1 - \tau > 0 \) by (6.10) and both \( \{ x_n \} \) and \( \{ y_n \} \) are Cauchy sequences, it follows by (6.23) that \( \{ Q_A u_n \} \) is a Cauchy sequence and therefore converges, say,
\[
(6.24) \quad \tilde{u}_n \to \tilde{u},
\]
where we denote \( Q_A u_n \) by \( \tilde{u}_n \) in \( Y/\mathcal{A}(0) \). The convergence in (6.24) implies that \( \| u_n - u, A(0) \| \to 0 \) as \( n \to \infty \). This means that \( u_n - u \) converges to an element of \( A(0) = A(0) \), say \( u_n - u \to z \in \mathcal{A}(0) \). This means that \( u_n \to z - u = s \). The closedness of \( \mathcal{A} \) implies that \( x \in D(\mathcal{A}) \) and \( s \in \mathcal{A}(x) \). Since \( u_n \to s \), we see that \( Q_A u_n = Q_A s \). Applying (6.15) to \( x_n - x \) we see that \( Q_B B(x_n) \to Q_B B(x) \), that is, \( \text{dist} (v_n - v, B(0)) \to 0 \) as \( n \to \infty \), \( v \in Bx \). This shows that \( v_n - v \) converges to an element say \( w \) of \( B(0) \), that is, \( v_n \to w - v = r \in Bx \) since \( B(x) = B(0) + v \). Hence \( y = s + r \in (A + B)(x) \), showing that \( A + B \) is closed.

We introduce a norm on \( D(\mathcal{A}) \) by
\[
(6.25) \quad \| x \|_D := (\sigma + \varepsilon)\| x \| + (\tau + \varepsilon\| A(x) \| \geq \varepsilon\| x \|,
\]
for some arbitrary but fixed positive constant \( \varepsilon \). Note that the space \( D(\mathcal{A}) \) becomes a Banach space by Lemma 6.6 which we denote by \( \tilde{D} \). We now regard \( \mathcal{A} \) and \( \mathcal{B} \) as linear relations with \( D(\mathcal{A}) = D(\mathcal{B}) = \tilde{D} \) and denote them by \( \tilde{\mathcal{A}} \) and \( \tilde{\mathcal{B}} \) respectively. Since \( \| x \|_D = (\sigma + \varepsilon)\| x \| + (\tau + \varepsilon\| A(x) \| \geq \sigma\| x \| + \tau\| A(x) \| \geq \| Bx \| \) for every \( x \in \tilde{D} \) and \( \| \tilde{B} \| := \sup_{x \in \tilde{D}} \| \tilde{B}x \| \), we see that \( \| \tilde{B} \| \leq 1 \). From \( \| A(x) \| \leq (\tau + \varepsilon)^{-1}\| x \|_D \) and the definition of \( \| \tilde{\mathcal{A}} \| \) we also see that \( \| \tilde{\mathcal{A}} \| \leq (\tau + \varepsilon)^{-1} \).

It is clear that \( R(\tilde{\mathcal{A}}) = R(\mathcal{A}) \) is closed and that
\[
(6.26) \quad \alpha(\tilde{\mathcal{A}}) = \alpha(\mathcal{A}), \quad \beta(\tilde{\mathcal{A}}) = \beta(\mathcal{A}), \quad \alpha(\tilde{\mathcal{A}} + \tilde{\mathcal{B}}) = \alpha(\mathcal{A} + \mathcal{B}), \quad \beta(\tilde{\mathcal{A}} + \tilde{\mathcal{B}}) = \beta(\mathcal{A} + \mathcal{B})
\]
Please note that \( \gamma(\tilde{\mathcal{A}}) = \gamma(\mathcal{A}) \) if \( \gamma(\mathcal{A}) = \infty \). In order to relate \( \gamma(\tilde{\mathcal{A}}) \) to \( \gamma(\mathcal{A}) \) in the other case, we recall that in this case,
\[
\gamma(\tilde{\mathcal{A}}) = \inf \left\{ \frac{\| \tilde{\mathcal{A}}(x) \|}{\| \tilde{x} \|_D} : x \in \tilde{D}, \tilde{x} \notin N(\tilde{\mathcal{A}}) \right\} = \inf \left\{ \frac{\| \mathcal{A}(x) \|}{\| \tilde{x} \|_D} : x \in \tilde{D}, \tilde{x} \notin N(\tilde{\mathcal{A}}) \right\}
\]
where \( \tilde{x} \in \tilde{X} := X/N(\mathcal{A}) \).
But
\[
\|\tilde{x}\|_D = \inf_{z \in N(A)} \|x - z\|_D = \inf_{z \in N(A)} \|\sigma + \varepsilon\|_D + (\tau + \varepsilon)\|A(x - z)\|
\]
where we have used the linearity of the natural quotient map and the fact that 
\(A(z) = A(0)\).

Hence
\[
\gamma(\tilde{A}) = \inf \left\{ \frac{\|A(x)\|}{(\sigma + \varepsilon)\|x\| + (\tau + \varepsilon)\|Ax\|} : x \in D(A), x \notin N(A) \right\}
\]
\[
= \frac{\gamma(A)}{\sigma + \varepsilon + (\tau + \varepsilon)\gamma(A)},
\]
where we have used the fact that \(f(t) = \frac{t}{\sigma + t}\) is an increasing function for any constant \(\alpha\).

In view of (6.16), we can make \(\gamma(\tilde{A}) > 1\) by choosing \(\varepsilon\) small enough. Since 
\(\|\tilde{B}\| \leq 1\), we can apply Theorem 6.5 to the pair \(\tilde{A}, \tilde{B}\) with the result that
\(R(\tilde{A} + \tilde{B}) = R(A + B)\) is closed and (6.9) holds with \(A, B\) replaced with \(\tilde{A}, \tilde{B}\). The result then
follows by (6.26).

\[\Box\]

7. Stability Theorems

Consider an eigenvalue problem of the form
\[
(7.1) \quad Ax = \lambda B
\]
where \(A\) and \(B\) are linear operators from \(X\) to \(Y\) and the associated
\[
(7.2) \quad A^* f' = \lambda B^* f'
\]
where the adjoints \(A^*\) and \(B^*\) exist. The null space \(N(A - \lambda B)\) of the linear
operator \(A - \lambda B\) is the solution set of the eigenvalue problem (7.1). Similarly,
\(N(A^* - \lambda B^*) = R(\tilde{A} + \tilde{B})\) is the solution set of the eigenvalue problem (7.2).

In the setting of linear relations, the eigenvalue problems (7.1) and (7.2) can be
formulated as
\[
(7.3) \quad \mathcal{A}(x) \cap \lambda \mathcal{B}(x) \neq \emptyset
\]
and
\[
(7.4) \quad \mathcal{A}'(x') \cap \lambda \mathcal{B}'(x') \neq \emptyset
\]
where \(\mathcal{A}, \mathcal{B} \in \text{LR}(X, Y)\). Conditions (7.3) and (7.4) are equivalent to
\[
(7.5) \quad (\mathcal{A} - \lambda \mathcal{B})(x) = (\mathcal{A} - \lambda \mathcal{B})(0)
\]
and
\[
(7.6) \quad (\mathcal{A}' - \lambda \mathcal{B}')(x') = (\mathcal{A}' - \lambda \mathcal{B}')(0)
\]
respectively.

As before, the solution sets of (7.5) and (7.6) are \(N(\mathcal{A} - \lambda \mathcal{B})\) and \(N(\mathcal{A}' - \lambda \mathcal{B}') = \)
\(R(\mathcal{A} - \lambda \mathcal{B})\) respectively. In this last section we study the stability of the dimensions
of the null spaces of $A - \lambda B$ and $A' - \lambda B'$ as $\lambda$ varies in some specified subset of the complex plane. This is considered in the following theorems.

**Theorem 7.1.** Let $X$ and $Y$ be Banach spaces and let $A, B \in \text{CLR}(X, Y)$ be such that $A$ has closed range, $D(B) \supset D(A)$, $B(0) \subset A(0)$, and

$$\|B(x)\| \leq \sigma \|x\| + \tau \|A(x)\| \quad \text{for every } x \in D(A)$$

where $\sigma$ and $\tau$ are non-negative constants. Then $A - \lambda B$ is closed for $|\lambda| < \frac{\gamma(A)}{\sigma + \gamma(A)}$ and if $R(A) \setminus A(0) \neq \emptyset$, then $\gamma(A - \lambda B) < \infty$ for $|\lambda| \geq \frac{\gamma(A)}{\sigma + \gamma(A)}$.

**Proof.** If follows from Theorem 6.7 that $A - \lambda B$ is closed if $|\lambda| < \frac{\gamma(A)}{\sigma + \gamma(A)}$.

If $\gamma(A - \lambda B) = \infty$ then $(A - \lambda B)(x) = (A - \lambda B)(0) = A(0)$. The fact that $(A - \lambda B)(x) = (A - \lambda B)(0)$ for every $x \in D(A - \lambda B) = D(A)$ implies that $A(x) \cap \lambda B(x) \neq \emptyset$ for every $x \in D(A)$. Since $B(0) \subset A(0)$, it follows that $\|A(x)\| \leq \|\lambda B(x)\|$ for every $x \in D(A)$ and therefore

$$\|A(x)\| \leq |\lambda|\|B(x)\| \leq |\lambda|(|\sigma\|x\| + \tau\|A(x)\|)$$

so that

$$|\lambda|\|B(x)\| \leq |\lambda|\|x\|/(1 - |\lambda|\tau).$$

Since $x$ can vary freely in $\tilde{x}$, we conclude that $\gamma(A) \leq |\lambda|/(1 - |\lambda|\tau)$. Since $|\lambda| \geq \frac{\gamma(A)}{\sigma + \gamma(A)}$. \hfill \Box

**Theorem 7.2.** Let $X$ and $Y$ be Banach spaces and let $A, B \in \text{CLR}(X, Y)$ be such that $A$ has closed range, $D(B) \supset D(A)$, $B(0) \subset T(0)$, and

$$\|B(x)\| \leq \sigma \|x\| + \tau \|A(x)\| \quad \text{for every } x \in D(A),$$

where $\sigma$ and $\tau$ are non-negative constants. If $\nu(A : B) = \infty$ then

$$\delta(N(A), N(A - \lambda B)) \leq \frac{\sigma |\lambda|}{\gamma(A) - |\lambda|(|\sigma + \tau\gamma(A)|)}.$$

**Proof.** Let $N_k$ be as defined in 5.2 and consider a sequence $z_k$ with the following properties:

$$z_k \in N_k, \quad A(z_{k+1}) \cap B(z_k) \neq \emptyset$$

$$\xi \|z_{k+1}\| \leq \|A(z_{k+1})\|, \quad k = 1, 2, \ldots$$

where $\xi$ is a positive constant. We show that for each $z \in N(A)$ and $\xi < \gamma(A)$, there is a sequence $z_k$ that satisfies (7.12) such that $z = z_1$. We set $z = z_1$ and construct $z_k$ by induction. Suppose $z_1, z_2, \ldots z_k$ have been constructed with properties (7.12). Since $z_k \in N_k \subset M_j = B^{-1}(A(X))$, there exists a $z_{k+1} \in D(A)$ such that $A(z_{k+1}) \cap B(z_k) \neq \emptyset$. Since $\gamma(A)||z_{k+1}\| \leq \|A(z_{k+1})\|$ and $z_{k+1}$ can be replaced by any other element of $z_{k+1}$, we can choose $z_{k+1}$ such that $\xi ||z_{k+1}\| \leq \|A(z_{k+1})\|$. Since $A(z_{k+1}) \cap B(z_k) \neq \emptyset$, we see that $z_{k+1} \in A^{-1}(B(N_k)) = N_{k+1}$. This completes the induction process.

Since $A(z_{k+1}) \cap B(z_k) \neq \emptyset$ and $A(0) \subset B(0)$, we see that

$$\|A(z_{k+1})\| \leq \|B(z_k)\| \leq \sigma \|z_k\| + \tau \|A(z_k)\|.$$
For \( k = 1 \), (7.13) gives \( \|A(z_2)\| \leq \|B(z_1)\| \leq \sigma \|z_1\| \) since \( z_1 \in N(A) \). For \( k \geq 2 \), (7.12) implies that
\[
(7.14) \|A(z_{k+1})\| \leq \|B(z_k)\| \leq \sigma \|z_k\| + \tau \|A(z_k)\| \leq (\sigma \xi^{-1} + \tau) \|A(z_k)\| \\
\leq (\sigma \xi^{-1} + \tau) \|A(z_{k-1})\| \leq \cdots \leq (\sigma \xi^{-1} + \tau)^{k-1} \|A(z_2)\| \\
= \xi^{-(k-1)}(\sigma + \xi \tau)^{k-1} \|A(z_2)\| \leq \sigma \xi^{-(k-1)}(\sigma + \xi \tau)^{k-1} \|z_1\|.
\]

We also see from (7.12) and (7.14) that
\[
\|z_{k+1}\| \leq \sigma \xi^{-k}(\sigma + \xi \tau)^{k-1} \|z_1\|, \quad k = 1, 2, \ldots.
\]

The bounds in (7.14) and (7.15) imply that the series
\[
u(\lambda) = \sum_{k=1}^{\infty} \lambda^{k-1} z_k, \quad \lambda(A) = \sum_{k=1}^{\infty} \lambda^k Q_A A(z_{k+1}), \quad \lambda(B) = \sum_{k=1}^{\infty} \lambda^{k-1} Q_B B(z_k)
\]
are absolutely convergent for \( |\lambda| < \frac{\xi}{\sigma + \xi \tau} \). The convergence of the last series follows from the fact that \( \|Q_A B(z_k)\| \leq \|Q_B B(z_k)\| \) since \( B(0) \subset A(0) \).

Let \( u_n(\lambda), \lambda_n(A), \lambda_n(B) \) and \( \lambda_n(B_A) \) denote the sequences of the partial sums of the above series in that order. Then for each \( n \), \( u_n(\lambda) \in D() \) and \( \lambda_n(A) \in \tilde{Y} := Y / A(0) \). Furthermore, \( u_n(\lambda) \to u(\lambda) \) and \( \lambda_n(A) \to \lambda(A) \). Since \( Q_A A \) is closed by Lemma 2.7, we see that \( u(\lambda) \in D(Q_A A) = D(A) \) and that
\[
(7.16) \quad Q_A A(u(\lambda)) = \lambda(A) = \sum_{k=1}^{\infty} \lambda^k Q_A A(z_{k+1}).
\]

Since \( A(z_{k+1}) \cap B(z_k) \neq \emptyset \), a similar argument shows that
\[
(7.17) \quad Q_A B(u(\lambda)) = \lambda(B_A) = \sum_{k=1}^{\infty} \lambda^{k-1} Q_A B(z_k) = \sum_{k=1}^{\infty} \lambda^k Q_A A(z_{k+1}) = \lambda(A).
\]

One also obtains the equality \( Q_B B(u(\lambda)) = \lambda(B) = \sum_{k=1}^{\infty} \lambda^{k-1} Q_B B(z_k) \) using the closedness of \( B \).

From (7.16) and (7.17) we see that
\[
\lim_{\lambda \to 0} \frac{Q_A A(u(\lambda)) - \lambda B(u(\lambda))}{\lambda} = 0
\]
and so \( u(\lambda) \in N(A - \lambda B) \).

Furthermore,
\[
\|u(\lambda) - z_1\| \leq \sum_{k=1}^{\infty} |\lambda|^{k-1} \|z_k\| \leq \left( \frac{\sigma |\lambda|}{\xi - |\lambda| (\sigma + \xi \tau)} \right) \|z_1\|.
\]

Since there is such a \( u(\lambda) \in N(A - \lambda B) \) for every \( z - z_1 \in N(A) \), we conclude that
\[
(7.18) \quad \delta(N(A), N(A - \lambda B)) \leq \frac{\sigma |\lambda|}{\gamma(A) - |\lambda| (\sigma + \xi \tau(A))}.
\]

We observe that if \( \alpha(A) < \infty \) then Theorem 6.7 can be used to conclude that \( A - \lambda B \) has closed range if \( |\lambda| < \frac{\gamma(A)}{\sigma+\xi \gamma(A)} \). However, this conclusion is not possible if no restriction is imposed on \( \alpha(A) \). This case is considered in the next lemma.
Lemma 7.3. Let $A$ and $B$ be as in Theorem 7.2 with $\nu(A : B) = \infty$. Then $A - \lambda B$ has closed range for $|\lambda| < \frac{\gamma(A)}{3\sigma + \tau\gamma(A)}$.

Proof. In the present case, let $x \in X$ and set $y = x - u$ for any $u \in N(A - \lambda B)$. Lemma 4.5 implies that for any $\varepsilon > 0$,

$$
(7.19) \quad \|\tilde{y}\| = \text{dist} (y, N(A)) \geq 1 - \delta(N(A), N(A - \lambda B)) \frac{1 - \varepsilon}{1 + \delta} \|y\|.
$$

Suppose that $x \in D(A) = D(A - \lambda B)$ and let $\delta := \delta(N(A) : N(A - \lambda B))$. Since $(A - \lambda B)(u) = (A - \lambda B)(0) = A(0)$, we see that

$$
(7.20) \quad \|y\| = \|A(y)\| - |\lambda| \|B(y)\| \ (\text{by Lemma 2.82})
$$

$$
\geq \|A(y)\| - |\lambda| (|\sigma| \|y\| + \tau \|A(y)\|)
$$

$$
\geq (1 - \tau |\lambda|) \|A(y)\| - |\lambda| \|y\|
$$

$$
\geq (1 - \tau |\lambda|) \|A\| \|\tilde{y}\| - |\lambda| \|y\| \ (\text{by (7.19)})
$$

$$
\geq |\gamma(A) - (2\sigma + \tau\gamma(A))| |\lambda| (1 - \varepsilon) \|y\| - |\lambda| \|y\| \ (\text{by (7.11)})
$$

$$
= |\gamma(A) - (2\sigma + \tau\gamma(A))| |\lambda| (1 - \varepsilon) - |\lambda| \|y\|.
$$

Let $\tilde{X}$ denote the quotient space $X/N(A - \lambda B)$. Since $x - y = u \in N(A - \lambda B)$, we see that $\|y\| \geq \|\tilde{y}\| = \|\tilde{x}\|$ and therefore (7.20) implies that

$$
(7.21) \quad \|A - \lambda B\| (x) \geq |(\gamma(A) - (2\sigma + \tau\gamma(A))| |\lambda| (1 - \varepsilon) - |\lambda| \|\tilde{x}\|,
$$

from which we conclude that

$$
\gamma(A - \lambda B) \geq |(\gamma(A) - (3\sigma + \tau\gamma(A))| |\lambda|.
$$

It therefore follows that $\gamma(A - \lambda B) > 0$ and therefore $R(A - \lambda B)$ is closed if $|\lambda| < \frac{\gamma(A)}{3\sigma + \tau\gamma(A)}$.

Finally, we establish the stability of both the nullity and deficiency of $A - \lambda B$ for $\lambda$ inside the disk $|\lambda| < \rho$ for some constant $\rho$.

Theorem 7.4. Let $X$ and $Y$ be Banach spaces and let $A, B \in \text{CLR}(X, Y)$ be such that $A$ has closed range, $D(B) \supset D(A)$, $B(0) \subset T(0)$, and

$$
(7.23) \quad \|B(x)\| \leq \sigma \|x\| + \tau \|A(x)\| \quad \text{for every} \ x \in D(A),
$$

where $\sigma$ and $\tau$ are non-negative constants. If $\nu(A : B) = \infty$ then $\alpha(A - \lambda B)$ and $\beta(A - \lambda B)$ are constants for all $\lambda$ for which $|\lambda| < \frac{\gamma(A)}{3\sigma + \tau\gamma(A)}$.

Proof. Let $u \in N(A - \lambda B)$. Then $A(u) \cap \lambda B(u) \neq \emptyset$ and we see from (7.9) that

$$
\|\tilde{u}\| \leq |\lambda| \|u\|/(1 - |\lambda| \tau \gamma(A)).
$$

Since $\|\tilde{u}\| = \text{dist} (u, N(A))$, we see from characterization (4.11) that

$$
\delta(N(A - \lambda B), N(A)) \leq \frac{\sigma |\lambda|}{(1 - |\lambda| \tau \gamma(A))}.
$$
Since \( \frac{\sigma |\lambda|}{(1-|\lambda| \tau)(\lambda)} \) if \(|\lambda| < \frac{\gamma(A)}{\sigma + \tau \gamma(A)}\), Lemma 4.4 implies that

\[
\alpha(A - \lambda B) \leq \alpha(A) \quad \text{for} \quad |\lambda| < \frac{\gamma(A)}{\sigma + \tau \gamma(A)}. \tag{7.24}
\]

The reverse inequality follows from Theorem 7.2 by noting that the righthand side of (7.11) is less than one if \(|\lambda| < \frac{\gamma(A)}{2\sigma + \tau \gamma(A)}\). We therefore conclude by Lemma 4.4 that \(\alpha(A) \leq \alpha(A - \lambda B)\) if \(|\lambda| < \frac{\gamma(A)}{2\sigma + \tau \gamma(A)}\). Combined with (7.24) we conclude that

\[
\alpha(A) = \alpha(A - \lambda B) \quad \text{for} \quad |\lambda| < \frac{\gamma(A)}{2\sigma + \tau \gamma(A)}. \tag{7.25}
\]

To show that \(\beta(A - \lambda) = \beta(A)\), we make use of the linear relations \(\tilde{A}\) and \(\tilde{B}\) as defined in the proof of Theorem 6.7. Since \(\tilde{A}\) is bounded, Lemmas 3.6(c), 3.7(d), and 3.5 imply that \(R(\tilde{A}')\) has closed range. Since \(\tilde{B}(0)' \subset \tilde{A}(0)'\) by Remark 2.10 and \(\nu(\tilde{A}' : \tilde{B}') = \infty\) by Lemma 5.5 all the assumptions of Theorem 7.4 are satisfied by the pair \(\tilde{A}'\) and \(\tilde{B}'\). Since \(\|\tilde{B}'\| < 1\) by Lemmas 3.6(a) and 3.7(c), it follows from (7.25) that

\[
\alpha(\tilde{A}' - \lambda \tilde{B}') = \alpha(\tilde{A}') \quad \text{for} \quad |\lambda| < \frac{\gamma(\tilde{A}')}{2\|\tilde{B}'\|}. \tag{7.26}
\]

Since \((\tilde{A} - \lambda \tilde{B})' = (\tilde{A}' - \lambda \tilde{B}')\) by Lemma 3.6(b) and (d) and \((\tilde{A} - \lambda \tilde{B})\) has closed range (since \(A - \lambda B\) has closed range), it follows from (6.26), Lemma 6.1 and (7.26) that

\[
\beta(A - \lambda B) = \beta(\tilde{A} - \lambda \tilde{B}) = \alpha(\tilde{A}' - \lambda \tilde{B}') = \alpha(\tilde{A}') = \beta(\tilde{B}) = \beta(B). \tag{7.27}
\]

Theorem 7.4 remains true if we replace the requirement \(\nu(A : B) = \infty\) with \(B^{-1}(0) \subset A^{-1}(0)\).

References

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