Parallel Objects and Field Equations

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Abstract

This paper considers a generalization of the existing concept of parallel (with respect to a given connection) geometric objects and its possible usage as a suggesting rule in searching for adequate field equations in theoretical physics. The generalization tries to represent mathematically the two-sided nature of the physical objects, the change and the conservation.

The physical objects are presented mathematically by sections $\Psi$ of vector bundles, the admissible changes $D\Psi$ are described as a result of the action of appropriate differential operators $D$ on these sections, and the conservation properties are accounted for by the requirement that suitable projections of $D\Psi$ on $\Psi$ and on other appropriate sections must be zero. It is shown that the most important equations of theoretical physics obey this rule. Extended forms of Maxwell and Yang-Mills equations are also considered.

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1 Introduction

When we think about physical objects, e.g. classical particles, solid bodies, elementary particles, etc., we always keep in mind that, although we consider them as free, they can not in principle be absolutely free, otherwise they would be undetectable. What is really understood under ”free object” is, that some definite properties (e.g. mass, velocity) of the object under consideration do not change in time under the influence of the existing environment. The availability of such time-stable features of any physical object guarantees its identification during its existence in time. Without such an availability of constant in time properties, which are due to the object’s resistence abilities, we could not speak about objects and knowledge at all. So, a classical mass particle in external gravitational field is free with respect to its mass, and it is not free with respect to its behaviour as a whole, because in classical mechanics formalism its mass does not change during the influence of the external field on its accelereted way of motion.

In trying to formalize these views we have to give some initial explicit formulations of some most basic features (properties) of what we call physical object, which features would lead us to a, more or less, adequate theoretical notion of our intuitive notion of a physical object. Anyaway, the following properties of the theoretical concept ”physical object” we consider as necessary:

1. It can be created.
2. It can be destroyed.
3. It occupies finite 3-volumes at any moment of its existence, so it has structure.
4. It has a definite stability to withstand some external disturbances.
5. It has definite conservation properties.
6. It necessarily carries energy-momentum, and, possibly, other measurable (conservative or nonconservative) physical quantities.
7. It exists in an appropriate environment (called usually vacuum), which provides all necessary existence needs.
8. It can be detected by the rest of the world through allowed energy-momentum exchanges with the "rest of the world".
9. It may combine with other appropriate objects to form new objects of higher level structure.
10. Its death gives necessarily birth to new objects following definite rules of conservation.

Clearly, together with the purely qualitative features a physical object carry important physical properties which can be described quantitatively by corresponding quantities, and any interaction between two physical objects is, in fact, an exchange of such quantities provided both objects carry them. Hence, the more universal is a physical quantity the more useful for us it is, and this moment determines the exclusively important role of energy-momentum, which modern physics considers as the most universal one, i.e. no physical objects are known that carry no energy-momentum.

If we can identify a given physical object, represented locally in space-time by the mathematical object $\Psi$, at different moments of its existence, this means that the changes $D\Psi$ of its time-changing properties vanish when are "projected" upon the same object $\Psi$, making use of other appropriate objects $Q$. From formal point of view this means that some mathematical expression of the kind $F(\Psi, D\Psi; Q) = 0$, specifying what and how changes, and specifying also what is projected and how it is projected, should exist. Hence, specifying differentially some conservation properties of the system under consideration, we obtain equations of motion being consistent with these conservation properties. We recall that this idea has been used firstly by Newton in his momentum conservation equation $\dot{p} = F$, which is the restriction of the partial differential system $\nabla p = F$ on some trajectory. This Newton's system of equations just says that there are physical objects in Nature which admit the "point-like" approximation, and which can exchange energy-momentum with "the rest of the world" but keep unchanged some other intrinsic properties which allows their identification in time.

These two aspects of any physical object (or a system of objects) - change and conservation, - have been very successfully unified and presented as a working tool (computational prescription) by the variational procedure (Lagrange-Euler-Hamilton action principle). The central idea of this approach is that if something happens, i.e. some real process develops, in some 4-dimensional region in Nature, there is an optimization quantity characterizing its optimal way of development. This integral quantity has been called action, and its local representative is usually called lagrangean (or lagrangean density). If the lagrangean is known the procedure works perfectly in almost all theoretical physics, and gives explicit "equations of motion" and local "conserved quantities". The need of such a powerful tool is out of doubt, especially in microphysics where the system studied changes considerably during observation, and moreover, we have to get knowledge of it in a very indirect way. However, this approach has a formal nature, it does NOT prescribe the lagrangeans, moreover, many lagrangeans give the same equations of motion and integral conserved quantities. One needs initial knowledge of the system (like symmetry and stability properties, dynamical behaviour features, etc.) in order to guess the corresponding lagrangean. In field theory this situation frequently leads to studies of "model lagrangeans": scalar field, vector field, spinor field, etc., and to separation of "free field" terms from "interaction" terms in a lagrangean. The free field terms are meant to give the system's intrinsic dynamics, and the interaction terms describe some external influences. While in macrophysics, as a rule,
the external influences are such that they do NOT destroy the system, in microphysics a full restructuring is allowed: the old ingredients of the system may fully transform to new ones, (e.g. the electron-positron annihilation) provided the energy-momentum conservation holds. The essential point is that whatever the interaction is, it always results in appearing of relatively stable objects, carrying energy-momentum and some other particular physically measurable quantities.

This conclusion emphasizes once again the importance of having an adequate notion of what is called a physical object, and of its mathematical representation.

From the point of view of spatial extension the physical objects may be point-like, finite and infinite, but realistic seems to be just the second option (recall p.3 above), although the classical approximations of point-like (i.e. structureless) objects and the infinite plane waves have served as good approximations wherever they have been uncontradictingly introduced and used. However, modern science requires a better adequacy between the real objects and the corresponding mathematical model objects. So, the mathematical model objects Ψ must necessarily be spatially finite, and even temporally finite if the physical object considered has by its intrinsic nature finite life-time. This most probably means that Ψ must satisfy nonlinear partial differential equation(s), which should define in a consistent way the admissible changes and the conservation properties of the object under consideration. Hence, talking about physical objects we mean a spatially finite entities which have a well established balance between change and conservation, and this balance is kept by a permanent and strictly fixed interaction with the environment.

The conservation properties of an object manifest themselves through corresponding symmetry properties, and these physical symmetry properties appear as mathematical symmetries of the corresponding equations \( F(\Psi, D\Psi; Q) = 0 \) in the theory. Usually, responsible for these symmetries are some new (additional) mathematical objects defining the explicit form of the equation(s), e.g. the Minkowski pseudometric tensor \( \eta \) in the relativistic mechanics and relativistic field theory, the symplectic 2-form \( \omega \) in the Hamilton mechanics, etc. Knowing such symmetries we are able to find new solutions from the available ones, and in some cases to describe even the whole set of solutions. That’s why the Lie derivative operator (together with its generalizations and prolongations [1]) and the integrability conditions for the corresponding equation(s) \( F(\Psi, D\Psi; Q) = 0 \) play a very essential and hardly overestimated role in theoretical physics. Of course, before to start searching the symmetries of an equation, or of a mathematical object \( \Psi \) which is considered as a model of some physical object, we must have done the preliminary work of specifying the mathematical nature of \( \Psi \), and this information may come only from an initial data analysis of appropriately set and carried out experiments.

The mathematical concept of symmetry has many faces and admits various formulations and generalizations. The simplest case is a symmetry of a real valued function \( f : M \to \mathbb{R} \), where \( M \) is a manifold, with respect to a map \( \varphi : M \to M \): \( \varphi \) is a symmetry (or a symmetry transformation) of \( f \) if \( f(\varphi(x)) = f(x) \), \( x \in M \). If \( \varphi_t, t \in (0,1) \subset \mathbb{R} \) is 1-parameter group of diffeomorphisms of \( M \), then the symmetry \( f(\varphi_t(x)) = f(x) \) may be locally expressed through the Lie derivative \( L_X(f) = 0 \), where the vector field \( X \) on \( M \) generates \( \varphi \). If \( T \) is an arbitrary tensor field on \( M \) then the Lie derivative is naturally extended to act on \( T \) and we call \( T \) symmetric, or invariant with respect to \( X \), or with respect to the corresponding (local, in general) 1-parameter group of diffeomorphisms of \( M \), if \( L_XT = 0 \). In this way the Lie derivative represents an universal tool to search symmetries of tensor fields on \( M \) with respect to the diffeomorphisms of \( M \). Unfortunately, this universality of \( L_X \) does not naturally extend to sections of arbitrary vector bundles on \( M \), where we need additional structures in order to introduce some notion of symmetry or invariance.

We may find a suggestion how to approach this problem by slightly changing the point of view, namely, to look for those tensor fields \( T \) on \( M \) which satisfy the equation \( L_XT = 0 \), where the vector field \( X \) is given. We obtain in this way a system of differential equations for \( X \), i.e. we
search for the kernel $\text{Ker}(L_X)$ of the differential operator $L_X$, and call the solutions symmetric, or invariant, with respect to $X$. So, we may generalize the situation to any (physically sensible) differential operator $D : \Psi(x) \rightarrow (D\Psi)(x)$, where $\Psi(x)$ are the fields of interest, (sections of appropriate vector bundles), and to call the solutions of $D(\Psi) = 0$ symmetric, (invariant), with respect to $D$. As a rule, in such cases the solutions carry appropriate names, for example, if $D = \nabla$, where $\nabla$ is a linear connection in a vector bundle [2], then a section $\sigma$ of this bundle is called parallel with respect to $\nabla$ if $\nabla(\sigma) = 0$.

We’d like to note that, in general, the symmetry of an object is always with respect to something (group of transformations, differential operator, sections of some vector bundles, etc.) preliminary fixed. And if the symmetry we are looking for will be given some physical interpretation, the preliminary work needed to fix the symmetry operator should be done by theoretical physics.

Following the above stated views we are going to consider in this paper a more general view on the geometrical concept of parallel transport, more or less already used in some physical theories. The parallel transport concept appropriately unifies the two above mentioned features: change and a suitable projection. In some cases this concept may be given a physical interpretation of conservation (balance) equation (mainly energy-momentum balance). The examples presented show how it has been used and how it could be used as a field equations generating tool. An important feature, that deserves to be noted even at this moment, is that the corresponding equations may become nonlinear in a natural way, so we might be fortunately surprised by appearing of spatially finite (or soliton-like) solutions (we shall recall such examples).

## 2 The general Rule

We begin with the algebraic structure to be used further in the bundle picture. The basic concepts used are the tensor product $\otimes$ of two linear spaces (we shall use the same term linear space for a vector space over a field, and for a module over a ring, and from the context it will be clear which case is considered) and bilinear maps. Let $(U_1, V_1), (U_2, V_2)$ and $(U_3, V_3)$ be three couples of linear spaces. Let $\Phi : U_1 \times U_2 \rightarrow U_3$ and $\varphi : V_1 \times V_2 \rightarrow V_3$ be two bilinear maps. Then we can form the elements $(u_1 \otimes v_1) \in U_1 \otimes V_1$ and $(u_2 \otimes v_2) \in U_2 \otimes V_2$, and apply the given bilinear maps as follows: $(\Phi, \varphi)(u_1 \otimes v_1, u_2 \otimes v_2) = \Phi(u_1, u_2) \otimes \varphi(v_1, v_2)$. The obtained element is in $U_3 \otimes V_3$.

We give now the corresponding bundle picture. Let $M$ be a smooth $n$-dimensional real manifold. We assume that the following vector bundles over $M$ are constructed: $\xi_i, \eta_i$, with standard fibers $U_i, V_i$ and sets of sections $\text{Sec}(\xi_i), \text{Sec}(\eta_i), i = 1, 2, 3$.

Assume the two bundle maps are given: $(\Phi, id_M) : \xi_1 \times \xi_2 \rightarrow \xi_3$ and $(\varphi, id_M) : \eta_1 \times \eta_2 \rightarrow \eta_3$. Then if $\sigma_1$ and $\sigma_2$ are sections of $\xi_1$ and $\xi_2$ respectively, and $\tau_1$ and $\tau_2$ are sections of $\eta_1$ and $\eta_2$ respectively, we can form an element of $\text{Sec}(\xi_3 \otimes \eta_3)$:

$$\Phi(\varphi)(\sigma_1 \otimes \tau_1, \sigma_2 \otimes \tau_2) = \Phi(\sigma_1, \sigma_2) \otimes \varphi(\tau_1, \tau_2).$$

(1)

Let now $\tilde{\xi}$ be a new vector bundle on $M$ and $\sigma_2 \in \text{Sec}(\xi_2)$ is obtained by the action of the differential operator $D : \text{Sec}(\tilde{\xi}) \rightarrow \text{Sec}(\xi_2)$ on a section $\tilde{\sigma}$ of $\tilde{\xi}$, so we can form the section (instead of $\sigma_1$ we write just $\sigma$) $\Phi(\sigma, D\tilde{\sigma}) \otimes \varphi(\tau_1, \tau_2) \in \text{Sec}(\xi_3 \otimes \eta_3)$. We give now the following

**Definition:** The section $\tilde{\sigma}$ will be called $(\Phi, \varphi; D)$-parallel with respect to $\sigma$ if

$$\Phi(\varphi)(\sigma \otimes \tau_1, \tilde{\sigma} \otimes \tau_2) = \Phi(\sigma, D\tilde{\sigma}) \otimes \varphi(\tau_1, \tau_2) = 0.$$  

(2)

This relation (2) we call the **GENERAL RULE (GR)**, the map $\Phi$ ”projects” the ”changes” $D\tilde{\sigma}$ of the section $\tilde{\sigma}$ on the section $\sigma$ ($\sigma$ may depend on $\tilde{\sigma}$), and $\varphi$ ”works” usually on the (local)
bases of the bundles where $\sigma$ and $D\tilde{\sigma}$ take values. As an example of a differential operator we note the particular case when $\xi$ is the bundle of exterior $p$-forms on $M$ with the available differential operator $\text{exterior derivative } d : \Lambda^p(M) \rightarrow \Lambda^{p+1}(M)$. In the case of the physically important example of Lie algebra $g$-valued differential forms, with "$\Phi =$ exterior product" and "$\varphi =$ Lie bracket $[,]$", $\xi_1 = \Lambda^p(M) = \xi$, $\xi_2 = \Lambda^{p+1}(M)$, $\eta_1 = \eta_2 = M \times g$, the GR (2) looks as follows:

$$\langle \Lambda, [,] ; d \rangle (\alpha^i \otimes E_i, \beta^j \otimes E_j) = \langle \Lambda, [,] \rangle (\alpha^i \otimes E_i, d\beta^j \otimes E_j) = \alpha^i \wedge d\beta^j \otimes [E_i, E_j] = 0,$$

where $\{E_i\}$ is a basis of $g$, and a summation over the repeated indexes is understood. Further we are going to consider particular cases of the (GR) (2) with explicitly defined differential operators whenever they participate in the definition of the section of interest.

### 3 The General Rule in Action

#### 3.1 Classical mechanics

We begin studying the potential strength of the GR in the frame of classical mechanics.

1. **Integral invariance relations**

   These relations have been introduced and studied from the point of view of applications in mechanics by Lichnerowicz [3].

   We specify the bundles over the real finite dimensional manifold $M$ introduced in sec.2:
   
   $\xi_1 = TM$; $\xi_2 = T^*(M)$; $\eta_1 = \eta_2 = \xi_3 = \eta_3 = M \times \mathbb{R}$, denote $\text{Sec}(M \times \mathbb{R}) \equiv C^\infty(M)$

   $\Phi=$substitution operator, denoted by $i(X), X \in \text{Sec}(TM)$;

   $\varphi=$point-wise product of functions.

   We denote by 1 the function $f(x) = 1, x \in M$. Consider the sections $X \otimes 1 \in \text{Sec}(TM \otimes (M \times \mathbb{R})); \alpha \otimes 1 \in \text{Sec}(T^*M \otimes (M \times \mathbb{R})$. Then the GR leads to

   $$\langle \Phi, \varphi \rangle (X \otimes 1, \alpha \otimes 1) = i(X)\alpha \otimes 1 = i(X)\alpha = 0. \quad (3)$$

   We introduce now the differential operator $d$: if $\alpha$ is an exact 1-form, $\alpha = df$, so that $\xi = M \times \mathbb{R}$, the relation (3) becomes

   $$i(X)\alpha = i(X)df = X(f) = 0,$$

   i.e. the derivative of $f$ along the vector field $X$ is equal to zero. So, we obtain the well known relation, defining the first integrals $f$ of the dynamical system determined by the vector field $X$. In this sense $f$ may be called $(\Phi, \varphi, d)$-parallel with respect to $X$, where $\Phi$ and $\varphi$ are defined above. In [3] $\alpha$ is a p-form, $\alpha \in \text{Sec}(\Lambda^p(T^*M))$, but this does not change the validity of the above relation (3).

2. **Absolute and relative integral invariants**

   These quantities have been introduced and studied in mechanics by Cartan [4]. By definition, a p-form $\alpha$ is called an absolute integral invariant of the vector field $X$ if $i(X)\alpha = 0$ and $i(X)df = 0$. And $\alpha$ is called a relative integral invariant of the field $X$ if $i(X)df = 0$. So, in our terminology (the same bundle picture as above), we can call the relative integral invariants of $X$ $(\Phi, \varphi, d)$-parallel with respect to $X$, and the absolute integral invariants of $X$ have additionally $(\Phi, \varphi)$-parallelism with respect to $X$, with $(\Phi, \varphi)$ as defined above. A special case is when $p = n$, and $\omega \in \Lambda^n(M)$ is a volume form on $M$. 

3. Symplectic mechanics

Symplectic manifolds are even dimensional and have a distinguished nondegenerate closed 2-form $\omega$, $d\omega = 0$. This structure may be defined in terms of the GR in the following way. Choose $\xi_1 = \eta_1 = \eta_2 = M \times \mathbb{R}$, $\xi_2 = \Lambda^2(T^*M)$, and $d$ as a differential operator. Consider now the section $1 \in Sec(M \times \mathbb{R})$ and the section $\omega \otimes 1 \in Sec(\Lambda^2(T^*M)) \otimes Sec(M \times \mathbb{R})$, with $\omega$ nondegenerate. The map $\Phi$ is the product $f, \omega$ and the map $\varphi$ is the product of functions. So, we have

$$(\Phi, \varphi; d)(1 \otimes 1, \omega \otimes 1) = 1.d\omega \otimes 1 = d\omega = 0.$$ 

Hence, the relation $d\omega = 0$ is equivalent to the requirement $\omega$ to be $(\Phi, \varphi; d)$-parallel with respect to the section $1 \in Sec(M \times \mathbb{R})$.

The hamiltonian vector fields $X$ are defined by the condition $L_X \omega = di(X)\omega = 0$. If $\Phi = \varphi$ is the point-wise product of functions we have

$$(\Phi, \varphi; d)(1 \otimes 1, i(X)\omega \otimes 1) = (\Phi, \varphi)(1 \otimes 1, di(X)(\omega) \otimes 1) = L_X\omega \otimes 1 = L_X\omega = 0.$$ 

In terms of the GR we can say that $X$ is hamiltonian if $i(X)\omega$ is $(\Phi, \varphi; d)$-parallel.

The induced Poisson structure $\{f, g\}$, is given in terms of the GR by setting $\Phi = \omega^{-1}$, where $\omega^{-1}.\omega = id_{TM}$, $\varphi$ = point-wise product of functions, and $1 \in Sec(M \times \mathbb{R})$. We get

$$(\Phi, \varphi)(df \otimes 1, dg \otimes 1) = \omega^{-1}(df, dg) \otimes 1.$$ 

A closed 1-form $\alpha$, $d\alpha = 0$, is a first integral of the hamiltonian system $Z$, $d\alpha(Z)\omega = 0$, if $i(Z)\alpha = 0$. In terms of the GR we can say that the first integrals $\alpha$ are $(i, \varphi)$-parallel with respect to $Z$: $(i, \varphi)(Z \otimes 1, \alpha \otimes 1) = i(Z)\alpha \otimes 1 = 0$. From $L_Z\omega = 0$ it follows $L_Z\omega^{-1} = 0$. The Poisson bracket $(\alpha, \beta)$ of two first integrals $\alpha$ and $\beta$ is equal to $-d(\omega^{-1}(\alpha, \beta))$ [5]. The well known property that the Poisson bracket of two first integrals of $Z$ is again a first integral of $Z$ may be formulated as: the function $\omega^{-1}(\alpha, \beta)$ is $(i, \varphi; d)$-parallel with respect to $Z$,

$$(i, \varphi; d)(Z \otimes 1, \omega^{-1}(\alpha, \beta) \otimes 1) = i(Z)d\omega^{-1}(\alpha, \beta) \otimes 1 = 0.$$ 

3.2 Frobenius integrability theorems and linear connections

1. Frobenius integrability theorems

Let $\Delta = (X_1, \ldots, X_r)$ be a differential system on $M$, i.e. the vector fields $X_i, i = 1, \ldots, r$ define a locally stable submodule of $Sec(TM)$ and at every point $p \in M$ the subspace $\Delta^r_p \subset T_p(M)$ has dimension $r$. Then $\Delta^r$ is called integrable if $[X_i, X_j] \in \Delta^r, i, j = 1, \ldots, r$. Denote by $\Delta^r_{p-r} \subset T_p(M)$ the complimentary subspace: $\Delta^r_p \oplus \Delta^r_{p-r} = T_p(M)$, and let $\pi : T_p(M) \rightarrow \Delta^r_{p-r}$ be the corresponding projection. So, the corresponding Frobenius integrability condition means $\pi([X_i, X_j]) = 0, i, j = 1, \ldots, r$.

In terms of the GR we set $D(X_i) = \pi \circ L_{X_i}, \Phi =$ “product of functions and vector fields”, and $\varphi$ again the product of functions. The integrability condition now is

$$(\Phi, \varphi; D(X_i))(1 \otimes 1, X_j \otimes 1) = (\Phi, \varphi)(1 \otimes 1, \pi([X_i, X_j] \otimes 1)) = 1.\pi([X_i, X_j]) \otimes 1.1 = 0, \quad i, j = 1, \ldots, r.$$ 

In the dual formulation we have the Pfaff system $\Delta^s_{n-r}$, generated by the linearly independent 1-forms $(\alpha_1, \ldots, \alpha_{n-r})$, such that $\alpha_m(X_i) = 0, i = 1, \ldots, r; m = 1, \ldots, n - r$. Then $\Delta^s_{n-r}$ is
integrable if \( d\alpha \wedge \alpha_1 \wedge \cdots \wedge \alpha_{n-r} = 0, \alpha \in \Delta^r_{n-r} \). In terms of GR we set \( \varphi \) the same as above, \( \Phi = \wedge \) and \( d \) as differential operator.

\[
(\Phi, \varphi; d)(\alpha_1 \wedge \cdots \wedge \alpha_{n-r} \otimes 1, \alpha \otimes 1) = d\alpha \wedge \alpha_1 \wedge \cdots \wedge \alpha_{n-r} \otimes 1 = 0.
\]

2. Linear connections

The concept of a linear connection in a vector bundle has proved to be of great importance in geometry and physics. In fact, it allows to differentiate sections of vector bundles along vector fields, which is a basic operation in differential geometry, and in theoretical physics the physical fields are represented mainly by sections of vector bundles. We recall now how one comes to it.

Let \( f : \mathbb{R}^n \to \mathbb{R} \) be a differentiable function. Then we can find its differential \( df \). The map \( f \to df \) is \( \mathbb{R} \)-linear: \( df(\kappa f) = \kappa df, \kappa \in \mathbb{R} \), and it has the derivative property \( df(g) = f dg + g df \). These two properties are characteristic ones, and they are carried to the bundle situation as follows.

Let \( \xi \) be a vector bundle over \( M \). We always have the trivial bundle \( \xi_0 = M \times \mathbb{R} \). Consider now \( f \in C^\infty(M) \) as a section of \( \xi_0 \). We note that \( \text{Sec}(\xi_0) = C^\infty(M) \) is a module over itself, so we can form \( df \) with the above two characteristic properties. The new object \( df \) lives in the space \( \Lambda^1(M) \) of 1-forms on \( M \), so it defines a linear map \( df : \text{Sec}(TM) \to \text{Sec}(\xi_0), df(X) = X(f) \).

Hence, we have a map \( \nabla \) from \( \text{Sec}(\xi_0) \) to the 1-forms with values in \( \text{Sec}(\xi_0) \), and this map has the above two characteristic properties. We say that \( \nabla \) defines a linear connection in the vector bundle \( \xi_0 \).

In the general case the sections \( \text{Sec}(\xi) \) of the vector bundle \( \xi \) form a module over \( C^\infty(M) \). So, a linear connection \( \nabla \) in \( \xi \) is a \( \mathbb{R} \)-linear map \( \nabla : \text{Sec}(\xi) \to \Lambda^1(M, \xi) \). In other words, \( \nabla \) sends a section \( \sigma \in \text{Sec}(\xi) \) to a 1-form \( \nabla \sigma \) valued in \( \text{Sec}(\xi) \) in such a way, that

\[
\nabla(k \sigma) = k \nabla(\sigma), \quad \nabla(f \sigma) = df \otimes \sigma + f \nabla(\sigma),
\]

where \( k \in \mathbb{R} \) and \( f \in C^\infty(M) \). If \( X \in \text{Sec}(TM) \) then we have the composition \( i(X) \circ \nabla \), so that

\[
i(X) \circ \nabla(f \sigma) = X(f) \sigma + f \nabla_X(\sigma),
\]

where \( \nabla_X(\sigma) \in \text{Sec}(\xi) \).

In terms of the GR we put \( \xi_1 = TM = \xi \) and \( \xi_2 = \Lambda^1(M) \otimes \xi \), and \( \eta_1 = \eta_2 = \xi_0 \). Also, \( \Phi(X, \nabla \sigma) = \nabla_X \sigma \) and \( \varphi(f, g) = f.g \). Hence, we obtain

\[
(\Phi, \varphi; \nabla)(X \otimes 1, \sigma \otimes 1)(\Phi, \varphi)(X \otimes 1, (\nabla \sigma) \otimes 1) = \nabla_X \sigma \otimes 1 = \nabla_X \sigma,
\]

and the section \( \sigma \) is called \( \nabla \)-parallel with respect to \( X \) if \( \nabla_X \sigma = 0 \).

3. Covariant exterior derivative

The space of \( \xi \)-valued \( p \)-forms \( \Lambda^p(M, \xi) \) on \( M \) is isomorphic to \( \Lambda^p(M) \otimes \text{Sec}(\xi) \). So, if \( (\sigma_1, \ldots, \sigma_r) \) is a local basis of \( \text{Sec}(\xi) \), every \( \Psi \in \Lambda^p(M, \xi) \) is represented by \( \psi^i \otimes \sigma_i, i = 1, \ldots, r \), where \( \psi^i \in \Lambda^p(M) \). Clearly the space \( \Lambda(M, \xi) = \bigoplus_{p=0}^n \Lambda^p(M, \xi) \), where \( \Lambda^p(M, \xi) = \text{Sec}(\xi) \), is a \( \Lambda(M) = \bigoplus_{p=0}^n \Lambda^p(M) \)-module: \( \alpha \Psi = \alpha \Psi = (\alpha \wedge \psi^i) \otimes \sigma_i \).

A linear connection \( \nabla \) in \( \xi \) generates covariant exterior derivative \( D : \Lambda^p(M, \xi) \to \Lambda^{p+1}(M, \xi) \) in \( \Lambda(M, \xi) \) according to the rule

\[
D\Psi = D(\psi^i \otimes \sigma_i) = d\psi^i \otimes \sigma_i + (-1)^p \psi^i \wedge \nabla(\sigma_i) = (d\psi^i + (-1)^p \psi^i \wedge \Gamma^i_{\mu\jmath} dx^\mu) \otimes \sigma_i = (D\Psi)^i \otimes \sigma_i.
\]
We may call now a $\xi$-valued $p$-form $\Psi$ $\nabla$-parallel if $D\Psi = 0$, and $(X, \nabla)$-parallel if $i(X)D\Psi = 0$. This definition extends in a natural way to $q$-vectors with $q \leq p$. Actually, the substitution operator $i(X)$ extends to (decomposable) $q$-vectors $X_1 \wedge X_2 \wedge \cdots \wedge X_q$ as follows:

$$i(X_1 \wedge X_2 \wedge \cdots \wedge X_q)\Psi = i(X_q) \circ i(X_{q-1}) \circ \cdots \circ i(X_1)\Psi,$$

and extends to nondecomposable $q$-vectors by linearity. Hence, if $\Theta$ is a section of $\Lambda^q(TM)$ we may call $\Psi (\Theta, \nabla)$-parallel if $i(\Theta)D\Psi = 0$.

Denote now by $L_\xi$ the vector bundle of (linear) homomorphisms $(\Pi, id) : \xi \to \xi$, and let $\Pi \in \text{Sec}(L_\xi)$. Let $\chi \in \text{Sec}(\Lambda^q(TM) \otimes L_\xi)$ be represented as $\Theta \otimes \Pi$. The map $\Phi$ will act as: $\Phi(\Theta, \Psi) = i(\Theta)\Psi$, and the map $\varphi$ will act as: $\varphi(\Pi, \sigma_i) = \Pi(\sigma_i)$. So, if $\nabla(\sigma_k) = \Gamma_{\mu k}^j dx^\mu \otimes \sigma_j$, we may call $\Psi (\nabla)$-parallel with respect to $\chi$ if

$$(\Phi, \varphi; D)(\Theta \otimes \Pi, \Psi = \psi^i \otimes \sigma_i) = (\Phi, \varphi)(\Theta \otimes \Pi, (D\Psi)^i \otimes \sigma_i) = i(\Theta)(D\Psi)^i \otimes \Pi(\sigma_i) = 0. \quad (6)$$

If we have isomorphisms $\otimes^p TM \sim \otimes^p T^* M, p = 1, 2, \ldots$, defined in some natural way (e.g. through a metric tensor field), then to any $p$-form $\alpha$ corresponds unique $p$-vector $\tilde{\alpha}$. In this case we may talk about $\sim$- autoparallel objects with respect a (point-wise) bilinear map $\varphi : (\xi \times \xi) \to \eta$, where $\eta$ is also a vector bundle over $M$. So, $\Psi = \alpha^k \otimes \sigma_k \in \Lambda^p(M, \xi)$ may be called $(i, \varphi; \nabla)$-autoparallel with respect to the isomorphism $\sim$ if

$$(i, \varphi; \nabla)(\tilde{\alpha}^k \otimes \sigma_k, \alpha^m \otimes \sigma_m) = i(\tilde{\alpha}^k) d\alpha^m \otimes \varphi(\sigma_k, \sigma_m) + (-1)^{p} i(\tilde{\alpha}^k)(\alpha_j^m \otimes \Gamma_{\mu k}^j dx^\mu) \otimes \varphi(\sigma_k, \sigma_m) = 0. \quad (7)$$

Although the above examples do not, of course, give a complete list of the possible applications of the GR (2), they will serve as a good basis for the physical applications we are going to consider further.

4 Physical applications of GR

1. Autoparallel vector fields and 1-forms

In nonrelativistic and relativistic mechanics the vector fields $X$ on a manifold $M$ are the local representatives (velocity vectors) of the evolution trajectories for point-like objects. The condition that a particle is free is mathematically represented by the requirement that the corresponding vector field $X$ is autoparallel with respect to a given connection $\nabla$ (covariant derivative) in $TM$:

$$i(X)\nabla X = 0, \quad \text{or in components,} \quad X^\sigma \nabla_\sigma X^\mu + \Gamma_{\sigma \nu}^\mu X^\sigma X^\nu = 0. \quad (8)$$

In view of the physical interpretation of $X$ as velocity vector field the usual latter used instead of $X$ is $u$. The above equation (8) presents a system of nonlinear partial differential equations for the components $X^\mu$, or $u^\mu$. When reduced to 1-dimensional submanifold which is parametrisated locally by the appropriately chosen parameter $s$, (8) gives a system of ordinary differential equations:

$$\frac{d^2 x^\mu}{ds^2} + \Gamma_{\sigma \nu}^\mu \frac{dx^\nu}{ds} \frac{dx^\nu}{ds} = 0, \quad (9)$$

and (9) are known as ODE defining the geodesic (with respect to $\Gamma$) lines in $M$. When $M$ is reinitialniam with metric tensor $g$ and $\Gamma$ the corresponding Levi-Civita connection, i.e. $\nabla g = 0$
and \( \Gamma^\mu_{\nu\sigma} = \Gamma^\nu_{\mu\sigma} \), then the solutions of (9) give the extreme (shortest or longest) distance \( \int_a^b ds \) between the two points \( a, b \in M \), so (9) are equivalent to

\[
\delta \left( \int_a^b ds \right) = \delta \left( \int_a^b \sqrt{g_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds}} \right) = 0.
\]

A system of particles that move along the solutions to (9) with \( g \)-the Minkowski metric and \( g_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} > 0 \), is said to form an inertial frame of reference.

It is interesting to note that the system (8) has (3+1)-soliton-like (even spatially finite) solutions on Minkowski space-time [6]. In fact, in canonical coordinates \((x^1, x^2, x^3, x^4) = (x, y, z, \xi = ct)\) let \( u^\mu = (0, 0, \pm \frac{v}{c}, f) \) be the components of \( u \), where \( 0 < v = const < c \), and \( c \) is the velocity of light, so \( \frac{\|^u}{c} < 1 \) and \( u^\sigma u_\sigma = \left(1 - \frac{v^2}{c^2}\right) f^2 > 0 \). Then every function \( f \) of the kind

\[
f(x, y, z, \xi) = f \left(x, y, z, \xi \pm \frac{v}{c} \right), \quad \alpha = const, \quad \text{for example} \quad \alpha = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}},
\]

defines a solution to (8). If \( u_\mu u^\mu = 0 \) then equations (8) \((X = u)\), are equivalent to \( u^\mu (du)_{\mu\nu} = 0 \), where \( du \) is the exterior derivative. In fact, since the connection used is riemannian, we have \( 0 = \nabla_{u^\alpha} u_{\beta} \wedge u_{\gamma} \wedge u_{\delta} \), so the relation \( u^\nu \nabla_{u_{\mu}} u_{\nu} - u^\nu \nabla_{u_{\nu}} u_{\mu} = 0 \) holds and is obviously equal to \( u^\nu (du)_{\mu\nu} = 0 \). The soliton-like solution is defined by \( u = (0, 0, \pm f, f) \) where the function \( f \) is of the form

\[
f(x, y, z, \xi) = f(x, y, z, \xi).
\]

Clearly, for every autoparallel vector field \( u \) (or one-form \( u \)) there exists a canonical coordinate system on the Minkowski space-time, in which \( u \) takes such a simple form: \( u^\mu = (0, 0, \alpha f, f), \alpha = const \). The dependence of \( f \) on the three spatial coordinates \((x, y, z)\) is arbitrary, so it is allowed to be chosen soliton-like and, even, finite. Let now \( \rho \) be the mass-energy density function, so that \( \nabla_{\sigma}(\rho u^\sigma) = 0 \) gives the mass-energy conservation, i.e. the function \( \rho \) defines those properties of our physical system which identify the system during its evolution. In this way the tensor conservation law

\[
\nabla_{\sigma}(\rho u^\sigma u^\mu) = (\nabla_{\sigma}\rho u^\sigma)u^\mu + \rho u^\sigma \nabla_{\sigma} u^\mu = 0
\]

describes the two aspects of the physical system: its dynamics through equations (8) and its mass-energy conservation properties.

The properties described give a connection between free point-like objects and (3+1) soliton-like autoparallel vector fields on Minkowski space-time. Moreover, they suggest that extended free objects with more complicated space-time dynamical structure may be described by some appropriately generalized concept of autoparallel mathematical objects.

### 2. Electrodynamics

#### 2.1 Maxwell equations

The Maxwell equations \( dF = 0, d * F = 0 \) in their 4-dimensional formulation on Minkowski space-time \((M, \eta)\), \( sign(\eta) = (-, -, -) \) and the Hodge * is defined by \( \eta \), make use of the exterior derivative as a differential operator. The field has, in general, 2 components \((F, *F)\), so the interesting bundle is \( \Lambda^2(M) \otimes V \), where \( V \) is a real 2-dimensional vector space. Hence the adequate mathematical field will look like \( \Omega = F \otimes e_1 + *F \otimes e_2 [7] \), where \((e_1, e_2)\) is a basis of \( V \). The exterior derivative acts on \( \Omega \) as: \( d\Omega = dF \otimes e_1 + d * F \otimes e_2 \), and the equation \( d\Omega = 0 \) gives the vacuum Maxwell equations.
In order to interpret in terms of the above given general view (GR) on parallel objects with respect to given sections of vector bundles and differential operators we consider the sections (see the above introduced notation) \((1 \times 1, \Omega \times 1)\) and the differential operator \(\mathbf{d}\). Hence, the GR acts as follows:

\[
(\Phi, \varphi; \mathbf{d})(1 \times 1, \Omega \times 1) = (\Phi, \varphi)(1 \times 1, \mathbf{d}\Omega \times 1) = (1 \mathbf{d}\Omega \otimes 1.1)
\]

The corresponding \((\Phi, \varphi; \mathbf{d})\)-parallelism leads to \(\mathbf{d}\Omega = 0\). In presence of electric \(j\) and magnetic \(m\) currents, considered as 3-forms, the parallelism condition does not hold and on the right-hand side we’ll have non-zero term, so the full condition is

\[
(\Phi, \varphi)(1 \times 1, (\mathbf{d}F \otimes e_1 + \mathbf{d} * F \otimes e_2) \times 1) = (\Phi, \varphi; \mathbf{d})(1 \times 1, (m \otimes e_1 + j \otimes e_2) \times 1)
\]

The case \(m = 0, F = \mathbf{d}A\) is, obviously a special case.

### 2.2 Extended Maxwell equations

The extended Maxwell equations (on Minkowski space-time) in vacuum read [8]:

\[
F \wedge * \mathbf{d}F = 0, \quad (\ast F) \wedge (\ast \mathbf{d} * F) = 0, \quad F \wedge (\ast \mathbf{d} * F) + (\ast F) \wedge (\ast \mathbf{d} F) = 0
\]

They may be expressed through the GR in the following way. On \((M, \eta)\) we have the bijection between \(\Lambda^2(TM)\) and \(\Lambda^2(T^*M)\) defined by \(\eta\), which we denote by \(\tilde{F} \leftrightarrow F\). So, equations (11) are equivalent to

\[
i(\tilde{F})\mathbf{d}F = 0, \quad i(\ast \tilde{F})\mathbf{d} * F = 0, \quad i(\tilde{F})\mathbf{d} * F + i(\ast \tilde{F})\mathbf{d} F = 0.
\]

We consider the sections \(\tilde{\Omega} = \tilde{F} \otimes e_1 + \ast \tilde{F} \otimes e_2\) and \(\Omega = F \otimes e_1 + \ast F \otimes e_2\) with the differential operator \(\mathbf{d}\). The maps \(\Phi\) and \(\varphi\) are defined as: \(\Phi\) is the substitution operator \(i\), and \(\varphi = \vee\) is the symmetrized tensor product in \(V\). So we obtain

\[
(\Phi, \varphi; \mathbf{d})(\tilde{F} \otimes e_1 + \ast \tilde{F} \otimes e_2, F \otimes e_1 + \ast F \otimes e_2) = i(\tilde{F})\mathbf{d}F \otimes e_1 \vee e_1 + i(\ast \tilde{F})\mathbf{d} * F \otimes e_2 \vee e_2 + (i(\tilde{F})\mathbf{d} * F + i(\ast \tilde{F})\mathbf{d} F) \otimes e_1 \vee e_2 = 0.
\]

Equations (12) may be written down also as \(((i, \vee)\tilde{\Omega})\mathbf{d}\Omega = 0\).

Equations (12) are physically interpreted as describing locally the intrinsic energy-momentum exchange between the two components \(F\) and \(\ast F\) of \(\Omega\): the first two equations \(i(\tilde{F})\mathbf{d}F = 0\) and \(i(\ast \tilde{F})\mathbf{d} * F = 0\) say that every component keeps locally its energy-momentum, and the third equation \(i(\tilde{F})\mathbf{d} * F + i(\ast \tilde{F})\mathbf{d} F = 0\) says (in accordance with the first two) that if \(F\) transfers energy-momentum to \(\ast F\), then \(\ast F\) transfers the same quantity energy-momentum to \(F\).

If the field exchanges (loses or gains) energy-momentum with some external systems, Extended Electrodynamics describes the potential abilities of the external systems to gain or lose energy-momentum from the field by means of 4 one-forms (currents) \(J_a, a = 1, 2, 3, 4\), and explicitly the exchange is given by [8]

\[
i(\tilde{F})\mathbf{d}F = i(\tilde{J}_1)F, \quad i(\ast \tilde{F})\mathbf{d} * F = i(\tilde{J}_2)F, \quad i(\tilde{F})\mathbf{d} * F + i(\ast \tilde{F})\mathbf{d} F = i(\tilde{J}_3)F + i(\tilde{J}_4) * F.
\]

It is additionally assumed that every couple \((J_a, J_b)\) defines a completely integrable Pfaff system, i.e. the following equations hold:

\[
\mathbf{d}J_a \wedge J_a \wedge J_b = 0, \quad a, b = 1, \ldots, 4.
\]

The system (12) has \((3+1)\)-localised photon-like (massless) solutions [7], and the system (13)-(14) admits a large family of \((3+1)\)-soliton solutions [8].
3. Yang-Mills theory

3.1 Yang-Mills equations

In this case the field is a connection, represented locally by its connection form \( \omega \in \Lambda^1(M) \otimes \mathfrak{g} \), where \( \mathfrak{g} \) is the Lie algebra of the corresponding Lie group \( G \). If \( D \) is the corresponding covariant derivative, and \( \Omega = D\omega \) is the curvature, then Yang-Mills equations read \( D \ast \Omega = 0 \). The formal difference with the Maxwell case is that \( G \) may NOT be commutative, and may have, in general, arbitrary finite dimension. So, the two sections are \( 1 \otimes 1 \) and \( \ast \Omega \otimes 1 \), the maps \( \Phi \) and \( \varphi \) are product of functions and the differential operator is \( D \). So, we may write

\[
(\Phi, \varphi; D)(1 \otimes 1, \ast \Omega \otimes 1) = D \ast \Omega \otimes 1 = 0.
\]

Of course, equations (13) are always coupled to the Bianchi identity \( D\Omega = 0 \).

3.2 Extended Yang-Mills equations

The extended Yang-Mills equations are written down in analogy with the extended Maxwell equations. The field of interest is an arbitrary 2-form \( \Psi \) on \( (M, \eta) \) with values in a Lie algebra \( \mathfrak{g} \), \( \dim(\mathfrak{g}) = r \). If \( \{E_i\}, i = 1, 2, \ldots, r \) is a basis of \( \mathfrak{g} \) we have \( \Psi = \tilde{\psi}^i \otimes E_i \) and \( \bar{\Psi} = \tilde{\psi}^i \otimes E_i \). The map \( \Phi \) is the substitution operator, the map \( \varphi \) is the corresponding Lie product \([,] \), and the differential operator is the exterior covariant derivative with respect to a given connection \( \omega \): \( D\Psi = d\Psi + [\omega, \Psi] \). We obtain

\[
(\Phi, \varphi; D)(\tilde{\psi}^i \otimes E_i, \psi^j \otimes E_j) = i(\tilde{\psi}^i)(d\psi^m + \omega^j \wedge \psi^k C_{jk}^m) \otimes [E_m, E_i] = 0,
\]

where \( C_{ij}^k \) are the corresponding structure constants. If the connection is the trivial one, then \( \omega = 0 \) and \( D \to d \), so, this equation reduces to

\[
i(\tilde{\psi}^i)d\psi^j C_{ij}^k \otimes E_k = 0 \tag{17}
\]

If, in addition, instead of \([,] \) we assume for \( \varphi \) some bilinear map \( f : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g} \), such that in the basis \( \{E_i\} \) \( f \) is given by \( f(E_i, E_i) = E_i \), and \( f(E_i, E_j) = 0 \) for \( i \neq j \) the last relation reads

\[
i(\tilde{\psi}^i)d\psi^j \otimes E_i = 0, \quad i = 1, 2, \ldots, r. \tag{18}
\]

The last equations (18) define the components \( \psi^j \) as independent 2-forms (of course \( \psi^j \) may be arbitrary \( p \)-forms). If the bilinear map \( \varphi \) is chosen to be the symmetrized tensor product \( \vee : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g} \vee \mathfrak{g} \), we obtain

\[
i(\psi^i)d\psi^j \otimes E_i \vee E_j = 0, \quad i \leq j = 1, \ldots, r. \tag{19}
\]

Equations (17) and (19) may be used to model bilinear interaction among the components of \( \Psi \). If the terms \( i(\tilde{\psi}^i)d\psi^j \otimes E_i \vee E_j \) have the physical sense of energy-momentum exchange we may say that every component \( \psi^j \) gets locally as much energy-momentum from \( \tilde{\psi}^i \) as it gives to it. Since \( C^k_{ij} = -C^k_{ji} \), equations (17) consider only the case \( i < j \), while equations (19) consider \( i \leq j \), in fact, for every \( i, j = 1, 2, \ldots, r \) we obtain from (19)

\[
i(\tilde{\psi}^i)d\psi^j = 0, \quad \text{and} \quad i(\tilde{\psi}^i)d\psi^j + i(\tilde{\psi}^j)d\psi^i = 0.
\]

Clearly, these last equations may be considered as a natural generalization of equations (12), so spatial soliton-like solutions are expectable.

4. General Relativity

In General Relativity the field function of interest is in a definite sense identified with a pseudometric \( g \) on a 4-dimensional manifold, and only those \( g \) are considered as appropriate
to describe the real gravitational fields which satisfy the equations \( R_{\mu\nu} = 0 \), where \( R_{\mu\nu} \) are the components of the Ricci tensor. The main mathematical object which detects possible gravity is the Riemann curvature tensor \( R_{\alpha\mu,\beta\nu} \), which is a second order nonlinear differential operator \( R : g \rightarrow R(g) \). We define the map \( \Phi \) to be the contraction, or taking a trace:

\[
\Phi : (g_{\alpha\beta}, R_{\alpha\mu,\beta\nu}) = g^{\alpha\beta} R_{\alpha\mu,\beta\nu} = R_{\mu\nu},
\]

so it is obviously bilinear. The map \( \varphi \) is a product of functions, so the GR gives

\[
(\Phi, \varphi; R)(g \otimes 1, g \otimes 1) = \Phi(g, R(g)) \otimes 1 = \text{Ric}(R(g)) \otimes 1 = 0. \tag{20}
\]

5. Schrödinger equation

The object of interest in this case is a map \( \Psi : \mathbb{R}^4 \rightarrow \mathbb{C} \), and \( \mathbb{R}^4 = \mathbb{R}^3 \times \mathbb{R} \) is parametrized by the canonical coordinates \((x, y, z; t)\), where \( t \) is the (absolute) time "coordinate". The operator \( D \) used here is

\[
D = i\hbar \frac{\partial}{\partial t} - H,
\]

where \( H \) is the corresponding hamiltonian. The maps \( \Phi \) and \( \varphi \) are products of functions, so the GR gives

\[
(\Phi, \varphi; D)(1 \otimes 1, \Psi \otimes 1) = \left( 1 \otimes \left( i\hbar \frac{\partial \Psi}{\partial t} - H\Psi \right) \right) \otimes 1 = 0. \tag{21}
\]

6. Dirac equation

The original free Dirac equation on the Minkowski space-time \((M, \eta)\) makes use of the following objects: \( \mathbb{C}^4 \) - the canonical 4-dimensional complex vector space, \( L_{\mathbb{C}^4} \) - the space of \( \mathbb{C} \)-linear maps \( \mathbb{C}^4 \rightarrow \mathbb{C}^4 \), \( \Psi \in \text{Sec}(M \times \mathbb{C}^4) \), \( \gamma \in \text{Sec}(T^*M \otimes L_{\mathbb{C}^4}) \), and the usual differential \( d : \psi^i \otimes e_i \rightarrow d\psi^i \otimes e_i \), where \( \{e_i\}, i = 1, 2, 3, 4 \), is a basis of \( \mathbb{C}^4 \). We identify further \( L_{\mathbb{C}^4} \) with \( (\mathbb{C}^4)^* \otimes \mathbb{C}^4 \) and if \( \{\varepsilon^i\} \) is a basis of \( (\mathbb{C}^4)^* \), dual to \( \{e_i\} \), we have the basis \( \varepsilon^i \otimes e_j \) of \( L_{\mathbb{C}^4} \). Hence, we may write

\[
\gamma = \gamma^j_{\mu\lambda} dx^\mu \otimes (\varepsilon^i \otimes e_j),
\]

and

\[
\gamma(\Psi) = \gamma^j_{\mu\lambda} dx^\mu \otimes (\varepsilon^i \otimes e_j)(\psi^k \otimes e_k) = \gamma^j_{\mu\lambda} dx^\mu \otimes \psi^k < \varepsilon^i, e_k > e_j = \gamma^j_{\mu\lambda} dx^\mu \otimes \psi^k \delta^i_k e_j = \gamma^j_{\mu\lambda} \psi^i dx^\mu \otimes e_j.
\]

The 4 matrices \( \gamma_{\mu} \) satisfy \( \gamma_{\mu} \gamma_{\nu} + \gamma_{\nu} \gamma_{\mu} = \eta_{\mu\nu} \text{id}_{\mathbb{C}^4} \), so they are nondegenerate: \( \text{det}(\gamma_{\mu}) \neq 0 \), \( \mu = 1, 2, 3, 4 \), and we can find \( (\gamma_{\mu})^{-1} \) and introduce \( \gamma^{-1} \) by

\[
\gamma^{-1} = (\gamma_{\mu})^{-1} \frac{1}{i} dx^\mu \otimes (\varepsilon^i \otimes e_j).
\]

We introduce now the differential operators \( D^\pm : \text{Sec}(M \times \mathbb{C}^4) \rightarrow \text{Sec}(T^*M \otimes \mathbb{C}^4) \) through the formula:

\[
D^\pm = i\text{id} \pm \frac{1}{2}m\gamma^{-1}, i = \sqrt{-1}, m \in \mathbb{R}. \]

The corresponding maps are: \( \Phi = \eta \), \( \varphi : L_{\mathbb{C}^4} \times \mathbb{C}^4 \rightarrow \mathbb{C}^4 \) given by \( \varphi(\alpha^* \otimes \beta, \rho) =< \alpha^*, \rho > \beta \). We obtain
(Φ, φ; D±)(γ, Ψ) = (Φ, φ)(γ^j_µdx^µ ⊗ (ε^i ⊗ e_j), i∂ψ^k_µdx^ν ⊗ e_k ≡ 1/2m(γ^j_µ)^ν_ρdx^ν ⊗ (ε^r ⊗ e_s)ψ^m_νe_m)

= iγ^j_µ∂ψ^k_νη(dx^µ, dx^ν) < ε^i, ε_k > e_j + 1/2mγ^j_µ(γ^k_ρ)^ε_δe_j

= iγ^j_µ∂ψ^k_νδ^i_µε_j + 1/2mη^µνγ^j_µ(γ^k_ρ)^ε_δe_j

= iγ^j_µ∂ψ^k_νε_j ± 1/2m(−2δ^i_µε_j) e_j = (iγ^j_µ∂ψ^k_νε_j ± mψ^j_µ) e_j = 0

(22)

In terms of parallelism we can say that the Dirac equation is equivalent to the requirement the section Ψ ∈ Sec(M × C^4) to be (η, φ; D±)-parallel with respect to the given γ ∈ Sec(M × L_C^4).

Finally, in presence of external electromagnetic field A = A_µdx^µ the differential operators D± modify to D± = id − eA ⊗ id_C^4 ± 1/2mγ^−1, where e is the electron charge.

5 Conclusion

It was shown that the GR, defined by relation (2), naturally generalizes the geometrical concept of parallel transport, and that it may be successfully used as a unified tool to represent formally important equations in theoretical physics. If Ψ is the object of interest then the GR specifies mainly the following things: the change DΨ of Ψ, the object Ψ_1 with respect to which we consider the change, the projection of the change DΨ on Ψ_1, and the bilinear amp φ determines the space where the final object lives. When Ψ = Ψ_1 we may speak about autoparallel objects, and in this case, as well as when the differential operator D depends on Ψ and its derivatives, we obtain nonlinear equation(s). In most of the examples considered the main differential operator used was the usual differential d and its covariant generalization.

In the case of vector fields and one-forms on the Minkowski space-time we recalled our previous result that among the corresponding autoparallel vector fields there are finite (3+1) soliton-like ones, time-like, as well as isotropic. This is due to the fact that the trajectories of these fields define straight lines, so their “transverse” components should be zero if the spatially finite (or localized) configuration must move along these trajectories. Moreover, the corresponding equation can be easily modified to be interpreted as local energy-momentum conservation relation.

It was further shown that Maxwell vacuum equations appear as d-parallel, i.e. without specifying any projection procedure. This determines their linear nature and leads to the lack of spatial soliton-like solutions. The vacuum extended Maxwell equations (11) are naturally cast in the form of autoparallel (nonlinear) equations, and, as it was shown in our former works, they admit photon-like (3+1) spatially finite and spatially localized solutions, and some of them admit naturally defined spin properties [9]. The general extended Maxwell equations (13)-(14) may also be given such a form if we replace the operator d with d − i(J_µ) in (13), and (3+1) soliton solutions of this system were also found [8].

The Yang-Mills equations were also described in this way. The introduced in this paper extended Yang-Mills equations (16)-(19) are expected to give spatial soliton solutions. The vacuum Einstein equations of General Relativity also admit such a formulation.

In quantum physics the Schrödinger equation admits the "parallel" formulation without any projection. A bit more complicated was to put the Dirac equation in this formulation, and this is due to the bit more complicated mathematical structure of γ = γ^j_µdx^µ ⊗ (ε^i ⊗ e_j).

These important examples make us think that the introduced in this paper extended concept for (Φ, φ; D)-parallel objects as a natural generalization of the existing geometrical concept for
∇-parallel objects, may be successfully used in various directions, in particular, in searching for appropriate nonlinearizations of the existing linear equations in theoretical and mathematical physics. It may also turn out to find it useful in looking for appropriate lagrangeans in some cases. In our view, as we pointed out in the Introduction, this is due to the fact that it expresses in a unified manner the dual change-conservation nature of the physical objects.

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