Uniqueness of holomorphic Abel functions at a complex fixed point pair

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Abstract. We give a simple uniqueness criterion (and some derived criteria) for holomorphic Abel functions and show that Kneser’s real analytic Abel function of the exponential is subject to this criterion.

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1. Introduction

There is a lot of discussion about the “true” or “best” fractional iterates of the function $e^x$ in the (lay-)mathematical community. In 1949 Kneser [4] proved the existence of real analytic fractional iterates. However, Szekeres (a pioneer in developing the theory of fractional iteration [9]) states in 1961 [10]:

“The solution of Kneser does not really solve the problem of ‘best’ fractional iterates of $e^x$. Quite apart from practical difficulties involved in the calculation of Kneser’s function on the real axis, there is no indication whatsoever that the function will grow more regularly to infinity than any other solution. There is certainly no uniqueness attached to the solution; in fact if $g(x)$ is a real analytic function with period 1 and $g'(x) + 1 > 0$ (e.g. $g(x) = \frac{1}{4\pi} \sin(2\pi x)$ then $B^*(x) = B(x) + g(B(x))$ is also an analytic Abel function of $e^x$ which in general yields a different solution of the equation”.

A recent discussion with Prof. Jean Écalle supports the impression that no uniqueness criterion was found until today and that there is even evidence against the existence of a criterion concerned with the growth-scale or asymptotic behavior at infinity.
By turning our attention from the purely real analytic behavior of the Abel function to the behavior in the complex plane we can succeed in giving a simple uniqueness criterion for the Abel functions of a whole class of real analytic (or arbitrary holomorphic) functions with two complex fixed points.

We show the usefulness of the criterion by providing an Abel function that satisfies the criterion. This is the above mentioned one, constructed by Kneser, the Abel function of $e^x$ (which can be easily generalized to functions $b^x$ with $b > e^{1/e}$).

We also have a suggestion for the numerical computation of this Abel function and the corresponding fractional iterates of $e^x$ (also of $b^x$ for $b > e^{1/e}$ in generalizations) by a method developed in [5]. Several other methods for the numerical computation of holomorphic fractional iterates of $e^x$ or the holomorphic Abel function have emerged in the past years (for example one is given in [12]). A future research goal would be to put them on a thorough theoretic base (proving convergence and holomorphy) and to verify the here given uniqueness criterion.

2. Motivation

Our original motivation was the investigation of a fourth stage of operations after the third stage containing power, exponential and logarithm.

Various terms for such operations were used in the past like: “generalized exponential” and “generalized logarithm” by Walker [11], “ultra exponential” and “infra logarithm” by Hooshmand [3], “super-exponential” by Bromer [1], tetration and superlogarithm [5]. In this paper we give them the more succinct names “4-exponential” and “4-logarithm”.

Definition 1. (4-exponential) A 4-exponential to base $b > 0$ is a function $f$ that satisfies

$$f(0) = 1$$
(1)

$$f(z + 1) = \exp_b(f(z))$$
(2)

for all applicable $z$.

For any $\tilde{f}$ that only satisfies (2) and contains 1 in its codomain: $\tilde{f}(z_0) = 1$, the function $f(z) = \tilde{f}(z + z_0)$ is a 4-exponential.

Definition 2. (4-logarithm) A 4-logarithm to base $b > 0$ is a function $g$ that satisfies the Abel equation (4) (see [7]) (for all applicable $z$) with the following initial condition:

$$g(1) = 0$$
(3)

$$g(\exp_b(z)) = g(z) + 1.$$  (4)