Homotopy theory of equivariant operads with fixed colors

Peter Bonventre, Luís A. Pereira

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Abstract
We build model structures on the category of equivariant simplicial operads with a fixed set of colors, with weak equivalences determined by families of subgroups. In particular, by specifying to the family of graph subgroups (or, more generally, one of the indexing systems of Blumberg-Hill), we obtain model structures on the category of equivariant simplicial operads with a fixed set of colors, with weak equivalences determined by norm map data.

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1 Introduction

This paper follows [Per18], [BP21], [BP0] as part of a larger project studying \textit{equivariant operads with norm maps}. Here, norm maps are an extra piece of data (not present non-equivariantly) which must be considered when studying equivariant operads, the importance of which was made clear by Hill-Hopkins-Ravenel in their solution to the Kervaire invariant one problem [HHR16].

For concreteness, let us fix a finite group $G$ and consider the category $\mathsf{sOp}^G_\ast = \mathsf{Op}_\ast(\mathsf{sSet}^G)$ or, in words, the category of single-colored operads in the category $\mathsf{sSet}^G$ of simplicial sets with a $G$-action. We note that, for $\mathcal{O} \in \mathsf{sOp}^G_\ast$, the $n$-th operadic level $\mathcal{O}(n)$ has both a $\Sigma_n$-action and a $G$-action, commuting with each other, or, equivalently, a $G \times \Sigma_n$-action. One of the key upshots of Blumberg and Hill’s work [BH15] is that the preferred notion of weak equivalence in $\mathsf{sOp}^G_\ast$ is that of \textit{graph equivalence}, i.e. those maps $\mathcal{O} \rightarrow \mathcal{P}$ such that the fixed point maps

$$\mathcal{O}(n) \rightarrow \mathcal{P}(n) \quad \text{for } \Gamma \leq G \times \Sigma_n \text{ such that } \Gamma \cap \Sigma_n = \ast$$

(1.1)

are Kan equivalences in $\mathsf{sSet}$. Here, the term “graph” comes from a neat characterization of the $\Gamma$ as in (1.1): such a $\Gamma$ is necessarily the graph of a partial homomorphism $\phi : H \rightarrow \Sigma_n$ for some subgroup $H \leq G$, i.e. $\Gamma = \{(h, \phi(h)) \mid h \in H\}$. Note that one hence has a canonical isomorphism $\Gamma \simeq H$. Briefly, the need to consider such \textit{graph subgroups} $\Gamma$ comes from the study of algebras. Suppose $X \in \mathsf{sSet}^G_\ast$ is an algebra over $\mathcal{O}$, so that one has algebra multiplication maps as on the left below

$$\mathcal{O}(n) \times X^n \rightarrow X \quad \mathcal{O}(n) \times N_\Gamma X \rightarrow X$$

(1.2)

which are required to be $G \times \Sigma_n$-equivariant (where the target $X$ is given the trivial $\Sigma_n$-action). One then has induced $H$-equivariant maps on the right in (1.2), where the \textit{norm object} $N_\Gamma X$ denotes $X^n$ with the $H$-action determined by $\Gamma \simeq H$. In particular, each point $\rho \in \mathcal{O}(n)_{/\Gamma}$ determines a $H$-equivariant \textit{norm map} $\rho : N_\Gamma X \rightarrow X$, and such maps turn out to be a key piece of data for algebras in the equivariant context. The reason to prefer the graph equivalences in (1.1) is then to ensure that weakly equivalent operads have equivalent “spaces of norm maps” $\mathcal{O}(n)_{/\Gamma}^\Gamma$.

The existence of a model structure on $\mathsf{sOp}^G_\ast$ with weak equivalences given by the graph equivalences in (1.1) was established independently as a particular case of either [BP21, Thm. 1] or [GW18, Thm. 3.1]. It seems worth noting that these results are somewhat non formal. On the one hand, using the description $\mathsf{sOp}^G_\ast = \mathsf{Op}_\ast(\mathsf{sSet}^G)$ as single-colored operads enriched in $\mathsf{sSet}^G$, one could obtain a model structure on $\mathsf{sOp}^G_\ast$ by applying [BM03, Thm. 3.2] to the genuine/strong model structure on $\mathsf{sSet}^G$. Alternatively, one also has an identification $\mathsf{sOp}^G_\ast \simeq (\mathsf{Op}_\ast(\mathsf{sSet}))^G$ as $G$-objects on the category $\mathsf{Op}_\ast(\mathsf{sSet})$ of single-colored simplicial operads, so one could likewise build a model structure on $\mathsf{sOp}^G_\ast$ by applying [Ste16, Prop. 2.6] to the category $\mathsf{Op}_\ast(\mathsf{sSet})$ with its usual model structure. However, neither of these approaches recovers the graph equivalences in (1.1), instead leading to a weaker notion of equivalence for which the fixed points in (1.1) only need to be weak equivalences for $\Gamma \leq G \leq G \times \Sigma_n$.

The main result in this paper, Theorem 1, extends the graph equivalence model structures of [BP21, Thm. 1], [GW18, Thm. 3.1] from the context of single-colored operads to the context of $G$-operads $\mathsf{sOp}^G_\ast$ with any fixed $G$-set of colors $\mathcal{E}$. In the direct sequel [BP0], we will then use the model structures in Theorem 1 to obtain a Dwyer-Kan style model structure [BP0, Thm. A] on the larger category $\mathsf{sOp}^G_\ast = (\mathsf{Op}_\ast(\mathsf{sSet}))^G$ of $G$-objects on the category $\mathsf{Op}_\ast(\mathsf{sSet})$ of colored operads with varying sets of colors (here, and throughout, we use $\bullet$ to indicate that colors are allowed to change; note that in $\mathsf{Op}^G_\ast$ the color sets thus have $G$-actions). More precisely, one has inclusions $\mathsf{Op}^G \subset \mathsf{sOp}^G_\ast$, so that the model structures in Theorem 1 are almost (but not quite) the restrictions of the model structure in [BP0, Thm. A], cf. [BP0, Prop. 3.11], extending the analogue stories for categories $\mathsf{sCat}_\ast$ [Ber07, BM13] and operads $\mathsf{sOp}_\ast$ [Rob, CM13b, Cav].
The motivation for [BPb, Thm. A], and thus also for Theorem I, is that it allows us to formulate a Quillen equivalence [BPa]

\[ dSet^G \simeq sOp^{G}_c \] (1.3)

(where \(dSet^G\) is the category of equivariant dendroidal sets of [Per18]), thereby completing the generalization of the Cisinski-Moerdijk project of [CM11, CM13a, CM13b] to the equivariant setting. Further discussion on (1.3) can be found in the introduction to [BPa].

We end this introduction by noting that, as in the single-colored case, building the desired model structures on \(sOp^G_c\), \(sOp^G\) is again non formal, and the colored case brings extra nuances.

Focusing first on \(sOp^G_c\), it is tempting to try to identify this as either the category \(Op_c(sSet)^G\) of \(\mathcal{C}\)-colored operads on \(sSet^G\), or as the \(G\)-object category \((Op_c(sSet))^G\). However, both of these alternatives forget the \(G\)-action on \(\mathcal{C}\), so that the identifications only hold if \(G\)-acts trivially on \(\mathcal{C}\). As such, Theorem I cannot be derived from neither [BM03, Thm. 3.2] nor [Ste16, Prop. 2.6].

As for the single category \(sOp^G\), formal approaches again run into issues, albeit different ones. First, since one only has an inclusion \(Op_c(sSet)^G \subseteq sOp^G\), rather than an equivalence, the model structure on \(sOp^G\) can not be built using [Cav]. Second, since, by definition, \(sOp^G\) is a category of \(G\)-objects, one could try to build a model structure on \(sOp^G\) by applying [Ste16, Prop. 2.6] to the model structure on \(sOp_c\) from [Cav]. However, and as in the single-colored case, this approach would not produce the desired notion of equivalences suggested by graph subgroups.

1.1 Main Results

As noted in (1.1), our preferred notion of equivalence of equivariant operads is determined by the graph subgroups. However, throughout this paper we will find it technically no harder to work with a largely arbitrary collection of subgroups, defined as follows.

**Definition 1.4.** A \((G, \Sigma)\)-family is a a collection \(\mathcal{F} = \{\mathcal{F}_n\}_{n \in \mathbb{N}}\), where each \(\mathcal{F}_n\) is a family of subgroups of \(G \times \Sigma_n^op\).

The use of \(\Sigma_n^op\) rather than \(\Sigma_n\) in Definition 1.4 (and throughout the paper) is motivated by regarding \(\Sigma\) as the category of corollas (trees with a single node; see (1.5),(3.2),(3.3)), and the fact that the dendroidal nerve [MW07, §1] of an operad is contravariant on the category of trees.

Recall that a colored operad \(\mathcal{O}\) with color set \(\mathcal{C}\) has levels \(\mathcal{O}(\tilde{C}) = \mathcal{O}(\epsilon_1, \ldots, \epsilon_n; c_0)\) indexed by tuples \(\tilde{C} = (\epsilon_1, \ldots, \epsilon_n; c_0) = (\epsilon_i)_{0 \leq i \leq n}\) of elements in \(\mathcal{C}\), called \(\mathcal{C}\)-profiles. Our operads will always be symmetric, i.e. equipped with associative and unital isomorphisms \(\mathcal{O}(\epsilon_1, \ldots, \epsilon_n; c_0) \rightarrow \mathcal{O}(\epsilon_{\sigma(1)}, \ldots, \epsilon_{\sigma(n)}; c_0)\) for each permutation \(\sigma \in \Sigma_n\). Moreover, if \(\mathcal{O} \in sOp^G_c\) is a \(G\)-equivariant operad, the color set \(\mathcal{C}\) is itself a \(G\)-set, and one has additional associative and unital isomorphisms \(\mathcal{O}(\epsilon_1, \ldots, \epsilon_n; c_0) \rightarrow \mathcal{O}(g\epsilon_1, \ldots, g\epsilon_n; gc_0)\) for \(g \in G\). All together, one thus has isomorphisms

\[ \mathcal{O}(\epsilon_1, \ldots, \epsilon_n; c_0) \rightarrow \mathcal{O}(g\epsilon_{\sigma(1)}, \ldots, g\epsilon_{\sigma(n)}; gc_0) \] (1.5)

for \((g, \sigma) \in G \times \Sigma_n^op\). Note that these isomorphisms are associated with an action of \(G \times \Sigma_n^op\) on the set \(\mathcal{C}^{n+1}\) of \(n\)-ary profiles via \((g, \sigma)(\epsilon_i)_{0 \leq i \leq n} = (g\epsilon_{\sigma(i)})_{0 \leq i \leq n}\), where we implicitly write \(\sigma(0) = 0\).

As such, we say that a subgroup \(\Lambda \leq G \times \Sigma_n^op\) stabilizes a profile \(\tilde{C} = (\epsilon_i)_{0 \leq i \leq n}\) if, for any \((g, \sigma) \in \Lambda\), it is \(\epsilon_i = g\epsilon_{\sigma(i)}\) for all \(0 \leq i \leq n\). Note that, for \(\mathcal{O} \in sOp^G_c\), the level \(\mathcal{O}(\tilde{C})\) has a \(\Lambda\)-action.

**Theorem I.** Let \(G\) be a finite group. Fix a \(G\)-set of colors \(\mathcal{C}\) and a \((G, \Sigma)\)-family \(\mathcal{F} = \{\mathcal{F}_n\}_{n \geq 0}\).

Then there exists a model structure on \(sOp^G_c = Op^G_c(sSet)\), which we call the \(\mathcal{F}\)-model structure, such that a map \(\mathcal{O} \rightarrow \mathcal{P}\) is a weak equivalence (resp. fibration) if the maps

\[ \mathcal{O}(\tilde{C})^\Lambda \rightarrow \mathcal{P}(\tilde{C})^\Lambda \] (1.6)
are Kan equivalences (Kan fibrations) in \( \mathbf{sSet} \) for all \( \mathcal{C} \)-profiles \( \mathcal{C} \) and \( \Lambda \in \mathcal{F} \) which stabilize \( \mathcal{C} \).

More generally, a model structure on \( \text{Op}_\mathcal{F}^\mathcal{C}(\mathcal{V}) \) with weak equivalences and fibrations determined as in (1.6) exists provided that:

(i) \( \mathcal{V} \) is a cofibrantly generated model category such that the domains of the generating (trivial) cofibrations are small;

(ii) for any finite group \( G \), the \( G \)-object category \( \mathcal{V}^G \) admits the genuine model structure (Definition 4.1);

(iii) \( (\mathcal{V}, \otimes) \) is a closed symmetric monoidal model category with cofibrant unit;

(iv) \( (\mathcal{V}, \otimes) \) satisfies the global monoid axiom (Definition 4.6);

(v) \( (\mathcal{V}, \otimes) \) has cofibrant symmetric pushout powers (Definition 4.26).

The proof of Theorem I is given in §5.2. As usual, the \( \mathcal{F} \)-model structure on \( \text{Op}_\mathcal{F}^\mathcal{C}(\mathcal{V}) \) is lifted from a similar \( \mathcal{F} \)-model structure on the simpler category \( \text{Sym}_\mathcal{C}(\mathcal{V}) \) of \( \mathcal{C} \)-colored symmetric sequences (Definition 3.7), which are objects with the isomorphism data as in (1.5), but lacking the operadic composition maps.

For certain special \( (G, \Sigma) \)-families (motivated by the indexing systems of Blumberg-Hill [BH15]) we will also show that cofibrant objects in \( \text{Op}_\mathcal{F}^\mathcal{C}(\mathcal{V}) \) forget to cofibrant objects in \( \text{Sym}_\mathcal{C}(\mathcal{V}) \).

**Theorem II.** Suppose \( \mathcal{V} \) satisfies the hypotheses in Theorem I, and let \( \mathcal{F} \) be a pseudo indexing system (Definition 5.22). Then, if \( \mathcal{O} \rightarrow \mathcal{P} \) in \( \text{Op}_\mathcal{F}^\mathcal{C}(\mathcal{V}) \) is a cofibration between cofibrant objects for the \( \mathcal{F} \)-model structure, so is the underlying map of symmetric sequences in \( \text{Sym}_\mathcal{C}(\mathcal{V}) \).

Theorem II is proven in §5.3 as a particular case of Proposition 5.26. The notion of pseudo indexing systems extends that of weak indexing systems [BP21, Def. 4.58] (or, equivalently, realizable sequences [GW18, Def. 4.6]), which themselves extend the original indexing systems of Blumberg-Hill [BH15]. Notably, the graph subgroups in (1.1) form a (pseudo) indexing system.

**Remark 1.7.** There are identifications \( \text{Cat}_\mathcal{F}^G(\mathcal{V}) \cong \text{Op}_\mathcal{F}^\mathcal{C}(\mathcal{V}) \downarrow \ast \) and \( \text{Cat}_\mathcal{F}^G(\mathcal{V}) \cong \text{Op}_\mathcal{F}^\mathcal{C}(\mathcal{V}) \downarrow \ast_\mathcal{C} \), where \( \ast \) (resp. \( \ast_\mathcal{C} \)) denotes the terminal \( \mathcal{V} \)-category (with color set \( \mathcal{C} \)), so the \( \mathcal{F} \)-model structures on \( \text{Op}_\mathcal{F}^\mathcal{C}(\mathcal{V}) \), \( \text{Op}_\mathcal{F}^\mathcal{C}(\mathcal{V}) \) induce model structures on \( \text{Cat}_\mathcal{F}^G(\mathcal{V}) \), \( \text{Cat}_\mathcal{F}^G(\mathcal{V}) \). Since categories contain only unary operations, these latter model structures depend only on \( \mathcal{F}_1 \), which is identified with a family of subgroups of \( \mathcal{G} \) itself.

Moreover, we note that the analogues for \( \text{Cat}_\mathcal{F}(\mathcal{V}) \) of both Theorems I and II follow from our proofs without using the cofibrant pushout power condition (v) in Theorem I, and without additional restrictions on \( \mathcal{F}_1 \) (i.e. no analogue of the pseudo indexing system condition is needed). For details, see Remark 5.16. For a similar discussion concerning \( \text{Cat}_\mathcal{F}(\mathcal{V}) \) and [BPb, Thm. A], see [BPb, Rem. 1.14].

**Remark 1.8.** When working with operads, some authors (e.g. [Spi, Whi17, WY18]) discuss semi-model structures. Briefly, these are a weakening of Quillen’s original definition, where those factorization and lifting axioms that involve trivial cofibrations are only required to hold if the trivial cofibration has cofibrant source [WY18, §2.2]. We note that, in particular, semi-model structures suffice for performing bifibrant replacements.

The semi-model structure analogues of Theorems I and II can be obtained by slight variants of our proofs without using the global monoid axiom (iv). For details, see Remark 5.17.
Remark 1.9. It may be tempting to think that if the group $G = \ast$ is trivial one can omit the existence of genuine model structures assumption in (ii) of Theorem I. However, that is not the case since, even when $\mathcal{F}$ is the $(G, \Sigma)$-family of trivial subgroups (i.e. the projective case usually discussed in the literature), our arguments in the proofs of Theorems I, II are still using the rather strong cofibrant pushout powers assumption (v). However, in this specific case there are less restrictive sufficient conditions available in the literature, such as those in [PS18, Thm. 1.1].

1.2 Examples

The examples of model categories satisfying all of conditions (i) through (v) in Theorem I are fairly limited, mostly due to the cofibrant pushout powers axiom (v), which is rather restrictive. We further discuss the role of this condition in Remarks 1.11 and 1.12 below.

Below we list all known examples of categories satisfying all the above conditions.

(a) $(\text{sSet},\times)$ or $(\text{sSet}_*,\wedge)$ with the Kan model structure.

(b) $(\text{Top},\times)$ or $(\text{Top}_*,\wedge)$ with the usual Serre model structure.

(c) $(\text{Set},\times)$ the category of sets with its canonical model structure, where weak equivalences are the bijections and all maps are both cofibrations and fibrations.

(d) $(\text{Cat},\times)$ the category of usual categories with the “folk” or canonical model structure (e.g. [Rez]) where weak equivalences are the equivalences of categories, cofibrations are the functors which are injective on objects, and fibrations are the isofibrations.

In all these cases, conditions (i) and (iii) are well known. Moreover, the existence of the genuine model structures on $\mathcal{V}_G$ for condition (ii) and the global monoid axiom for condition (iv) all follow from Proposition 4.10, since these $(\mathcal{V}, \otimes)$ satisfy the usual monoid axiom and are readily seen to satisfy the conditions in Definition 4.9.

Lastly, we discuss the cofibrant symmetric pushout powers condition in (v). In case (a) this was shown in [BP21, Ex. 6.19]. Case (b) is a consequence of case (a), since the generating (trivial) cofibrations of (b) are geometric realizations of those in (a), and by [BP21, Rem. 6.17] it suffices to verify the cofibrant pushout power condition on generating sets. Case (c) is straightforward and left as an exercise (the “hardest” step is the observation that fixed points of isomorphisms are isomorphisms). For case (d), the only non obvious claim is that if $u$ is a trivial cofibration in $\text{Cat}$ then the pushout product map $u^\Sigma_n$ is a $\Sigma_n$-genuine trivial cofibration in $\text{Cat}^{\Sigma_n}$. By [BP21, Rem. 6.17] we can assume that $u = (\{0\} \to (0 \neq 1))$, i.e. that $u$ is the map from a singleton to the walking isomorphism category, as this is the only generating trivial cofibration of $\text{Cat}$. One can then either repeat the argument in [BP21, Ex. 6.19] or compute $u^\Sigma_n$ directly. The target of $u^{\Sigma_n}$ is the contractible groupoid on the set $\{0, 1\}^\times n$, so that $u^{\Sigma_n}$ is the inclusion of the full subcategory of $\{0, 1\}^\times n$ with $(1, 1, \cdots, 1)$ removed. It then follows that $u^{\Sigma_n}$ is a pushout of the map $\Sigma_n/\Sigma_n \cdot (0 \neq 1)$.

Remark 1.10. These examples also satisfy the axioms necessary for [BPb, Thm. A], yielding a Dwyer-Kan model structure on $\text{Op}^G(\mathcal{V})$, cf [BPb, §1.2].

Remark 1.11. As noted above, the cofibrant pushout powers condition (v) is the most restrictive out of all the conditions in Theorem I. Nonetheless, we chose to use this property since it has two very convenient bootstrapping properties. Firstly, as noted in the previous discussion, [BP21, Rem. 6.17] says that the cofibrant pushout powers condition needs only be checked on generating (trivial cofibrations). Secondly, the cofibrant pushout powers condition can be used...
to deduce more complex \(G\)-cofibrant pushout powers’’ analogue conditions for any groupoid \(G\), as formulated in Proposition 4.34.

In our view, the main obstacle to generalizing our results is the identification of a substitute for the cofibrant pushout powers condition (v) which has similar bootstrapping properties.

We end this section by discussing a noteworthy example for which our results do not apply.

**Remark 1.12.** The category \((\text{Sp}^\Sigma(s\text{Set}),\wedge)\) of symmetric spectra (on simplicial sets), with the positive \(S\) model structure, satisfies most of the axioms in Theorem 1 (and [BPb, Thm. A]), with the exceptions being the cofibrant unit requirement in (iii) and the cofibrant pushout powers axiom in (v). However, we believe that with some care both of these problems could be sidestepped if necessary. On the one hand, [GV12] showed that the non-cofibrancy of the unit does not prevent the existence of fixed color model structures and, on the other hand, the second author showed in [Per16] that pushout powers \(u^\Sigma n\) of positive \(S\) cofibrations \(u\) in \((\text{Sp}^\Sigma(s\text{Set}),\wedge)\) satisfy a “lax-\(\Sigma\)-cofibrancy” condition, which enjoys analogues of the bootstrapping conditions in Remark 1.11 (in fact, our key result concerning cofibrant pushout powers, Proposition 4.34, and its direct precursor [BP21, Prop. 6.25] were originally inspired by [Per16, Thm. 1.2]).

As such, we believe that symmetric spectra \(\text{Sp}^\Sigma(s\text{Set})\) likely satisfy a close analogue of Theorems I, II (and [BPb, Thm. A]). However, such results would be fundamentally unsatisfying, since the resulting notion of weak equivalence on \(G\)-symmetric spectra \((\text{Sp}^\Sigma(s\text{Set}))^G\) does not match the correct notion of genuine equivalences of \(G\)-spectra. More precisely, the latter equivalences are a localization of the former, so that the direct analogues of Theorems I, II (and [BPb, Thm. A]) would at best represent only an intermediate step towards the “genuine” results for equivariant colored spectral operads.

More generally, for an arbitrary model category \(\mathcal{V}\), the initial choice of the \((G,\Sigma)\)-family \(\mathcal{F}\) in Definition 1.4 should be replaced with a choice of model structures on \(\mathcal{V}^{G\times\Sigma_n}\) for each \(n \geq 0\).

### 1.3 Outline

We start in §2 by discussing some preliminary notions that will be needed throughout. As noted in the introduction to §2, our treatment will be simplified by using an “all colors” approach, working with the category \(\text{Op}_\ast(\mathcal{V})\) of all colored operads, regarded as suitably “fibered” over the category \(\text{Set}\) of sets. To that end, §2.1 recalls the necessary notion of Grothendieck fibration, while §2.2 discusses how the notions of adjunction and monad interact with such fibrations.

§3 then applies the abstract setup in §2 to discuss equivariant colored symmetric sequences and operads. §3.1 explores the category \(\text{Sym}_\ast(\mathcal{V})\) of all colored symmetric sequences, with a highlight being Proposition 3.17, which shows that the category \(\text{Sym}_\ast^G(\mathcal{V})\) of 

\[ G \]-symmetric sequences with a fixed \(G\)-set of objects \(\mathcal{C}\) can be described as a presheaf category. In §3.2 we prove Proposition 3.35, which provides a convenient description of the representable functors in \(\text{Sym}_\ast^G = \text{Sym}_\ast^G(\text{Set})\). Lastly, §3.3 briefly describes the category \(\text{Op}_\ast(\mathcal{V})\) of all colored operads as the algebras over a “fibered monad” on \(\text{Sym}_\ast(\mathcal{V})\), and unpacks the abstract discussion in §2.2 so as to likewise describe the category \(\text{Op}_\ast^G(\mathcal{V})\) of all equivariant colored operads as algebras on \(\text{Sym}_\ast^G(\mathcal{V})\).

§4 develops the equivariant homotopy theory needed to prove of Theorems I and II. First, §4.1 introduces the global monoid axiom featured in Theorem I. Then, in §4.2, we extend the work about equivariant model structures determined by families of subgroups in [BP21, §6] from the context of groups to that of groupoids. This culminates in Proposition 4.34, which concerns the properties of pushout powers \(f^\Sigma n\), and is one of the key technical results in the paper.

Lastly, §5 is dedicated to proving our two main results, Theorems I and II. §5.1 first specifies the theory in §4.2 to obtain model structures on the categories \(\text{Sym}_\ast^G(\mathcal{V})\) of symmetric sequences with fixed colors. These are then used in §5.2 to obtain transferred model structures on the
categories $\text{Op}_G^\bullet(V)$ of operads with fixed colors, establishing Theorem I. To finish, §5.3 proves Theorem II via a more careful analysis of the argument in the proof of Theorem I.

In Appendix A, we fill in some technical work that was postponed in §3 and §5, namely the full description of the “free operad monad” from §3.1, and the proof of Lemma 5.8, which provides the key filtrations used in the proofs of Theorems I and II.

2 Preliminaries

Much as in the non-equivariant case, the first step towards the construction of a model structure on the category $\text{Op}_G^\bullet(V)$ of operads on all sets of colors is to build model structures on each fixed color subcategory $\text{Op}_G^\bullet(V)$. However, the equivariant setting presents some technical challenges that will require us to somewhat repackage the non-equivariant narrative of [CM13b],[Cav].

To see why, recall that [CM13b],[Cav] follow a “work color by color and then assemble” strategy. More precisely, first the model structures on fixed color operads $\text{Op}_G^\bullet(V)$ are built by identifying these as algebras over a monad $F_\epsilon$ on the category $\text{Sym}_G^\bullet(V)$ of fixed color symmetric sequences. As such, maps of operads that do not fix colors only appear afterward when assembling the model structure on the full category $\text{Op}_G^\bullet(V)$.

However, when working equivariantly, while the maps in the fixed $G$-set of objects categories $\text{Op}_G^\bullet(V)$ do fix colors, the $G$-action on objects $\mathcal{O} \in \text{Op}_G^\bullet(V)$ involves maps of operads $\mathcal{O} \to \mathcal{O}$ that need not fix colors (unless $\mathcal{C}$ is a trivial $G$-set). In other words, when discussing equivariant operads, even the fixed color categories $\text{Op}_G^\bullet(V)$ require color change data. Due to this issue, our approach will be that the transition from the non-equivariant to the equivariant case is easier to describe if we regard the “all colors” framework as the primary framework, and then restrict to color fixed operads only when needed.

More explicitly, the basis of our approach is to combine the fixed color symmetric sequence categories $\text{Sym}_G^\bullet(V)$ for all color sets $\mathcal{C} \in \text{Set}$ (and change of color data between them) into a single category $\text{Sym}_G^\bullet(V)$ (Definition 3.7). There is then a Grothendieck fibration $\text{Sym}_G^\bullet(V) \to \text{Set}$ which records the underlying set of colors, and whose fiber over $\mathcal{C} \in \text{Set}$ is $\text{Sym}_G^\bullet(V)$. Similarly, the monads $F_\epsilon$ on $\text{Sym}_G^\bullet(V)$ (and change of color data between them) assemble into a single monad $F$ on $\text{Sym}_G^\bullet(V)$ (Definition 3.44) which is suitably compatible with the Grothendieck fibration $\text{Sym}_G^\bullet(V) \to \text{Set}$, and, by considering (a suitable subcategory of) algebras over $F$, one obtains the category $\text{Op}_G^\bullet(V)$ of colored operads for all colors. These fit together as on the left below.

$$
\text{Sym}_G^\bullet(V) \xrightarrow{F} \text{Op}_G^\bullet(V) \quad \text{Sym}_G^\bullet(V) \xrightarrow{p^G} \text{Op}_G^\bullet(V)
$$

(2.1)

Within this “all colors” framework, passing to the equivariant case is simply a matter of applying $G$-objects throughout as on the right above, so that one has a Grothendieck fibration $\text{Sym}_G^\bullet(V) \to \text{Set}^G$ with a compatible monad $F^G$ from which one obtains the category $\text{Op}_G^\bullet(V)$ of $G$-equivariant colored operads for all colors.

The plan for this preliminary section is then as follows. §2.1 recalls the notion of Grothendieck fibration and introduces some related constructions that are used throughout. §2.2 discusses how the notions of adjunction and monad interact with Grothendieck fibrations, which will allow us in §3 to regard the monads (and associated adjunctions) in (2.1) as being suitably fibered over $\text{Set}$ and $\text{Set}^G$.

Remark 2.2. As an aside, we note that our discussion of fibered category theory will also streamline our work in the sequel [BPa]. Therein, we will need to establish a Quillen equivalence
$\text{PreOp}^G \simeq \text{sOp}^G$ between the category of (simplicial) preoperads and the category of simplicial operads. In that work, the claim that the adjunction is Quillen will be greatly simplified by noting that in both categories the generating (trivial) cofibrations can be described using a “fibered simplicial cotensoring”.

### 2.1 Grothendieck fibrations

Recall that a functor $\pi:\mathcal{C} \to \mathcal{B}$ is called a Grothendieck fibration if, for all arrows $\varphi:b' \to b$ in $\mathcal{B}$ and $c \in \mathcal{C}$ such that $\pi(c) = b$, there exists a cartesian arrow $\varphi^*c \to c$ lifting $\varphi$, meaning that for any choice of solid arrows

\[
\begin{array}{ccc}
c'' & \xrightarrow{\varphi^*c} & c \\
\downarrow & \searrow & \downarrow \\
b'' & \xrightarrow{\varphi} & b
\end{array}
\]

such that the right diagram commutes and $c'' \to c$ lifts $b'' \to b$, there exists a unique dashed arrow $c'' \to \varphi^*c$ lifting $b'' \to b'$ and making the left diagram commute.

A cleavage of $\pi$ is a fixed choice of cartesian arrows $\varphi^*c \to c$ for each $\varphi:b' \to b$ and $c$ with $\pi(c) = b$. Note that, writing $\mathcal{C}_b$ for the fiber over $b \in \mathcal{B}$, a cleavage determines functors $\varphi^*:\mathcal{C}_b \to \mathcal{C}_{b'}$ for each $\varphi:b' \to b$.

Dually, if $\pi^\text{op}:\mathcal{C}^\text{op} \to \mathcal{B}^\text{op}$ is a Grothendieck fibration, we say that $\pi$ is a Grothendieck opfibration. More explicitly, this means that, for any arrow $\varphi:b \to b'$ in $\mathcal{B}$ and $c \in \mathcal{C}$ such that $\pi(c) = b$, there exists a cocartesian arrow $c \to \varphi^*c$ lifting $\varphi$ and satisfying the dual of the universal property in (2.3). A cleavage of an opfibration is similarly defined as a choice of cocartesian arrows $c \to \varphi^*c$.

**Notation 2.4.** Given any functor of categories $\pi:\mathcal{C} \to \mathcal{B}$, one has a natural decomposition of mapping sets

\[
\mathcal{C}(c',c) = \coprod_{f \in \mathcal{B}(\pi(c'),\pi(c))} \mathcal{C}_\varphi(c',c)
\]

where $\mathcal{C}_\varphi(c',c)$ consists of the arrows projecting to $\varphi$.

**Remark 2.5.** Specifying to the case $b'' = b'$ in (2.3), one has that, when $\pi$ is a Grothendieck fibration, the contravariant functors

\[
\mathcal{C}_\varphi(-,c) : \mathcal{C}_{b'} \to \text{Set}
\]

are represented by (some choice of) $\varphi^*c$. Moreover, note that under the representing isomorphism $\mathcal{C}_b(\varphi^*c,\varphi^*c) \cong \mathcal{C}_\varphi(\varphi^*c,c)$ the identity $\varphi^*c \xrightarrow{\sim} \varphi^*c$ yields the canonical map $\varphi^*c \to c$ over $\varphi$.

**Remark 2.7.** The condition in (2.3) is stronger than the representability of (2.6). More precisely, let us say an arrow $\varphi^*c \to c$ is weakly cartesian if it represents (2.6), and that $\pi:\mathcal{C} \to \mathcal{B}$ is a weak Grothendieck fibration if it admits all weakly cartesian arrows. Then $\pi$ is in fact a Grothendieck fibration, i.e. the weakly cartesian arrows $\varphi^*c \to c$ satisfy the stronger condition in (2.3), iff the composites of weakly cartesian arrows are again weakly cartesian, i.e. if for any composable arrows $b'' \xrightarrow{\psi} b' \xrightarrow{\varphi} b$ and $c \in \mathcal{C}_b$ it is $\psi^*\varphi^*c \xrightarrow{\sim} (\varphi\psi)^*c$, where the isomorphism is in $\mathcal{C}_b$.

**Remark 2.8.** As previously noted, a cleavage of a Grothendieck fibration $\pi:\mathcal{C} \to \mathcal{B}$ determines functors $\varphi^*:\mathcal{C}_b \to \mathcal{C}_{b'}$ for each arrow $\varphi:b' \to b$. In addition, the claim that $\pi$ is also an opfibration is then equivalent to the existence of left adjoints

\[
\varphi^* : \mathcal{C}_{b'} \rightleftarrows \mathcal{C}_b : \varphi^*
\]
for all arrows φ: b' → b in B. Note that the required condition that the functors ϕψ and (ϕψ);) are naturally isomorphic (cf. Remark 2.7) is automatic, as these are left adjoints to ψ∗ϕ∗ = (ψϕ)∗.

**Remark 2.9.** Let π: C → B be a Grothendieck fibration and I a small category. Writing C I, B I for the categories of functors I → C, I → B, the functor π I: C I → B I is again a Grothendieck fibration, with cartesian arrows in C I the natural transformations built of cartesian arrows in C.

**Notation 2.10.** Given a functor B → Cat let us write C B ∈ Cat for the image of b ∈ B and ϕ: C b → C b for the functor induced by ϕ: b → b'. We then write B × C, for the associated (covariant) Grothendieck construction.

More explicitly, B × C is the category whose objects are pairs (b, c) with b ∈ B and c ∈ C b, and with an arrow (b, c) → (b', c') given by an arrow ϕ: b → b' in B together with an arrow f: ϕ c → c' in C. Note that the composite of f: ϕ c → c' and f': ϕ c' → c'' is given by

\[ \psi \psi f \xrightarrow{\psi f} \psi c' \xrightarrow{f'} c''. \]

Lastly, note that the natural projection π: B × C → B is naturally a Grothendieck opfibration.

**Remark 2.11.** It can be helpful to simplify notation and write elements (b, c) of B × C, simply as c. Under this convention, we depict arrows in B × C as composites c → ϕ c → c' where c ∼ ϕ c denotes the cocartesian arrow from c to ϕ c and f is the fiber arrow. Composites of either two cocartesian or two fiber arrows work as obvious: c → ϕ c → ϕ c' equals c → (ϕ c') c while c → c' → c'' equals f f c c'. The only non-obvious composites are then those of the form c → c' → ϕ c', which are determined by the commutativity of the square

\[
\begin{array}{ccc}
  c & \xrightarrow{f} & \phi c \\
  \downarrow & & \downarrow \phi f \\
  c' & \xrightarrow{\phi c'} & \\
\end{array}
\]  

(2.12)

**Remark 2.13.** If π: C → B is an opfibration then, by (the dual of) Remark 2.7, for any cleavage one must have associativity isomorphisms ϕ ψ ϕ = (ϕ ψ), but these need not be equalities. Should a cleavage be strictly associative, i.e. ϕ ψ ϕ = (ϕ ψ), for all composable ϕ, ψ and unital, i.e. (id b): id C b, then the Grothendieck opfibration is called split. One can show that an opfibration is split iff it is isomorphic to a Grothendieck construction in the sense of Notation 2.10.

The following are the two main instances of Notation 2.10 we will make use of.

**Example 2.14.** When G is a group regarded as a category with a single object, a functor G → Cat consists of a single category C with a G-action. In this case, G × C can be thought of as the category obtained from C by formally adding “action arrows” c → gc for each g ∈ G, c ∈ C.

More explicitly, an arrow from c to c' in G × C is uniquely described as an arrow ϕ: gc → c' in C for some g ∈ G, with the composite of ϕ: gc → c' and ϕ: gc' → c'' given by the composite g g c g c' g c' c'' in C.

**Remark 2.15.** Any group G admits an inversion isomorphism G = G op given by g → g⁻¹. If C is a G-category, so is C op, and the inversion isomorphism lifts to an isomorphism G × C op = (G × C) op which is the identity on objects. Indeed, an arrow from c to c in G × C op is an arrow f: c → gc in C while an arrow in (G × C) op is an arrow f': c' → gc in C. The isomorphism then identifies the arrow of G × C op represented by f: c → gc with the arrow of (G × C) op represented by g⁻¹ f: g⁻¹ c → c.
Notation 2.16. Let $\Sigma = \prod_{n \geq 0} \Sigma_n$ be the symmetric category, whose objects are the non-negative integers $n \geq 0$, and whose arrows are all automorphisms, with $n$ having automorphism group $\Sigma_n$.

For any category $\mathcal{C}$ there is then a functor

$$\Sigma^{op} \longrightarrow \text{Cat}, \quad n \mapsto \mathcal{C}^{\times n}$$

and we will abbreviate $\Sigma : \mathcal{C} = \left( \left( \Sigma^{op} \times \mathcal{C}^{\times (-)} \right)^{op} \right)$. Unpacking notation, the elements of $\Sigma : \mathcal{C}$ are tuples $(c_i)_{1 \leq i \leq n}$ of elements $c_i \in \mathcal{C}$ for some $n \geq 0$, and a map of tuples $(c_i) \to (d_j)$, necessarily of the same size $n$, consists of a permutation $\sigma \in \Sigma_n$ together with maps $c_i \to d_{\sigma(i)}$ for $1 \leq i \leq n$.

Remark 2.17. Writing $F$ for the skeleton of the category of finite sets consisting of the sets $\mathbb{N} = \{1, \ldots, n\}$ for $n \geq 0$, we can regard $\Sigma \in F$ as the maximal subgroupoid.

The tuple category $\Sigma \times \mathcal{C}$ in Notation 2.16 is then a subcategory of an analogous category $F \times \mathcal{C}$.

Remark 2.18. It is clear from the description of $\Sigma : (-)$ as a category of tuples that, should $(\mathcal{V}, \otimes)$ be a symmetric monoidal category, then $\otimes$ induces a functor $\Sigma : \mathcal{V} \to \mathcal{V}$ via $(v_i) \mapsto \otimes_i v_i$.

In fact, one can reverse this process: a symmetric monoidal structure on $\mathcal{V}$ can be described as a functor $\Sigma : \mathcal{V} \to \mathcal{V}$ satisfying suitable associativity and unitality conditions, and we will make use of this alternative description when defining operads.

However, there is a slight caveat. We will in fact prefer to describe a symmetric monoidal structure as a functor $(\Sigma : \mathcal{V})^{op} \to \mathcal{V}$ or, equivalently, $\Sigma : \mathcal{V}^{op} \to \mathcal{V}^{op}$. The equivalence between this description and the one above follows since a (symmetric) monoidal structure $\otimes$ on $\mathcal{V}$ is also a monoidal structure on $\mathcal{V}^{op}$ (or, using the isomorphism $\Sigma : \mathcal{V}^{op} \simeq (\Sigma : \mathcal{V})^{op}$). The reason for us to prefer this seemingly more cumbersome setup is because it actually seems to be more natural in practice. For example, if $\otimes = \Pi$ is the cartesian product then, in addition to symmetry isomorphisms, $\Pi$ also admits projection maps and diagonals, and to encode these one must use a map $(F : \mathcal{V}^{op})^{op} \to \mathcal{V}$ rather than $F : \mathcal{V} \to \mathcal{V}$. For further discussion, see [BP21, §2.2].

Remark 2.19. If $\mathcal{C}$ is a category with a $G$-action, the constructions in Example 2.14 and Notation 2.16 can be applied in either order to obtain categories $G \times (\Sigma : \mathcal{C})$ and $\Sigma : (G \times \mathcal{C})$.

In either of these categories, the objects are the tuples $(c_i)$ with $c_i \in \mathcal{C}$. As for the arrows, in $G \times (\Sigma : \mathcal{C})$ an arrow $(c_i) \to (d_j)$ between tuples of size $n$ is encoded by arrows $g (c_i) \to d_{\sigma(i)}$ in $\mathcal{C}$ for some $g \in G$, $\sigma \in \Sigma_n$ while in $(\Sigma : (G \times \mathcal{C}))$ an arrow is encoded by arrows $g (c_i) \to d_{g \sigma(i)}$ in $\mathcal{C}$ for some $(g_i) \in G^{\times n}$, $\sigma \in \Sigma_n$. Hence, we see that there is an inclusion of categories

$$G \times (\Sigma : \mathcal{C}) \rightarrow \Sigma : (G \times \mathcal{C})$$

Informally, $G \times (\Sigma : \mathcal{C})$ is the subcategory where the only $G$-action arrows are diagonal action arrows (i.e. those corresponding to constant tuples $(g_i)_{1 \leq i \leq n} \in G^{\times n}$).

Remark 2.20. Let $\pi : \mathcal{C} \to \mathcal{B}$ be a Grothendieck fibration. Then, if the base $\mathcal{B}$ and fibers $\mathcal{C}_b$ are all complete, so is the total category $\mathcal{C}$. Indeed, given a diagram $I \rightarrow \mathcal{C}$, and writing $b = \lim_{i \in I} \pi(c_i)$ and $\varphi_i : b \to \pi(c_i)$ for the canonical maps, one has

$$\lim_{i \in I} c_i = \lim_{i \in I} \varphi_i^* c_i$$

where the second limit formula is computed in $\mathcal{C}_b$.

Moreover, keeping the setup above, let $\varphi : b \to \pi(c_i)$ be an arbitrary cone in $\mathcal{B}$ and $\varphi : b \to b$ the induced map. Then (2.21) implies that one further has

$$\varphi^* \left( \lim_{i \in I} c_i \right) = \varphi^* \left( \lim_{i \in I} \varphi_i^* c_i \right) = \lim_{i \in I} \varphi_i^* \varphi_i^* c_i = \lim_{i \in I} \varphi_i^* c_i.$$
2.2 Fibered adjunctions and fibered monads

**Definition 2.22.** Let \( \pi: C \to B, \pi: D \to B \) be functors with a common target. A fibered adjunction is an adjunction

\[
L: C \rightleftarrows D: R
\]

where the functors \( L, R \), unit \( \eta: id_C \Rightarrow RL \) and counit \( \epsilon: LR \Rightarrow id_D \) are all fibered, i.e.

\[
\pi L = \pi, \quad \pi R = \pi, \quad \pi \eta = id, \quad \pi \epsilon = id.
\]

**Remark 2.23.** A fibered adjunction induces natural isomorphisms (cf. Notation 2.4)

\[
\mathcal{D}_\varphi (Lc, d) \cong \mathcal{C}_\varphi (c, Rd)
\]

for each \( c \in C, d \in D, \varphi: \pi(c) \to \pi(d) \).

**Proposition 2.24.** Let \( L: C \rightleftarrows D: R \) be an adjunction between Grothendieck fibrations over \( B \).

If the adjunction is fibered then \( R \) is a fibered functor which preserves cartesian arrows.

Conversely, if the right adjoint \( R \) is a fibered functor which preserves cartesian arrows, then one can modify the left adjoint (and unit, counit) so that the adjunction becomes a fibered adjunction.

**Proof.** For the first claim, letting \( \Phi: \bar{d} \to d \) be a cartesian arrow in \( D \), the fact that \( R(\Phi) \) is again cartesian follows from Remark 2.5 applied to the composite isomorphism

\[
\mathcal{C}_{\pi(d)} (-, R\bar{d}) \cong \mathcal{D}_{\pi(d)} (L(-), \bar{d}) \cong \mathcal{D}_{\pi(\Phi)} (L(-), d) \cong \mathcal{C}_{\pi(\Phi)} (-, Rd).
\]

For the “conversely” claim, noting that by assumption \( \pi RL = \pi L \), one can choose a cartesian natural transformation \( \bar{L} \Rightarrow L \) (i.e. a cartesian arrow in \( \mathcal{D}^C \)) over the projection of the adjunction unit \( \pi \eta \) (which is an arrow in \( \mathcal{E}^C \)). Moreover, noting that by assumption \( RL \Rightarrow RL \) is again cartesian, we write \( id_C \Rightarrow R\bar{L} \Rightarrow RL \) for the natural factorization with fibered \( \bar{\eta} \), as well as \( \epsilon \) for the composite \( LR \Rightarrow LR \Rightarrow id_D \). We claim that \( \bar{L}, R, \bar{\eta}, \epsilon \) now form a fibered adjunction, with the non obvious claim being that this is in fact still an adjunction. That the composite \( \bar{R} \Rightarrow \bar{R}L \Rightarrow R \) is the identity follows since we can rewrite this composite as \( \bar{R} \Rightarrow \bar{R}L \Rightarrow R \Rightarrow id \) and thus also as \( R \Rightarrow \bar{R}L \Rightarrow id \). The remaining claim is that the top horizontal composite in the diagram below is the identity,

\[
\begin{array}{ccccccc}
L & \xrightarrow{L\eta} & L\bar{L} & \xrightarrow{\epsilon} & \bar{L} \\
\downarrow & & \downarrow & & \downarrow \\
\bar{L} & \xrightarrow{\bar{L}\eta} & \bar{L}\bar{L} & \xrightarrow{\epsilon} & \bar{L}
\end{array}
\]

and, since \( \gamma: \bar{L} \Rightarrow L \) is cartesian, it in fact suffices to show that the overall composite \( \bar{L} \Rightarrow L \) in the diagram above is the map \( \gamma \), which is clear. \( \square \)
Remark 2.25. Let $\pi: C \to B$ be a functor that preserves coproducts and $G$ be a group. Then the free-forget adjunctions

\[
\begin{array}{ccc}
C & \xrightarrow{G(-)} & C^G \\
\downarrow & & \downarrow \\
B & \xrightarrow{G(-)} & B^G
\end{array}
\]

are compatible in the sense that both the left and right adjoints commute with the projections $\pi$. In addition, given $b \in B$, $b' \in B^G$ and $\varphi: b \to b'$ a map in $B$, write $G \cdot \varphi: G \cdot b \to G \cdot b'$ for the adjoint map. Then, for $c \in C_b$ and $c' \in C^G_b$, one has that the left isomorphism below decomposes into the isomorphisms on the right

\[
\text{C}(c, c') \simeq C^G(G \cdot c, c') \quad \text{C}_G(c, c') \simeq C^G_G(G \cdot c, c') \tag{2.26}
\]

Definition 2.27. Given a functor $\pi: C \to B$, a fibered monad is a monad $T: C \to C$ such that the functor $T$, multiplication $\mu: TT \Rightarrow T$ and unit $\eta: I \Rightarrow T$ are all fibered, i.e.

$\pi T = \pi, \quad \pi \mu = \pi \eta = \text{id}_\pi$.

Moreover, a fiber algebra is a $T$-algebra $c \in C$ such that the multiplication map $Tc \xrightarrow{\text{m}} c$ satisfies $\pi(m) = \text{id}_{\pi(c)}$. Lastly, we write $\text{Alg}_T(C) \subseteq \text{Alg}_T(C)$ for the full subcategory of fiber algebras.

Remark 2.28. For each $b \in B$, a fibered monad $T$ restricts to a monad on each fiber $C_b$, and we write $T_b$ to denote this restricted monad.

Remark 2.29. If $T$ is a fibered monad then any free algebra $Tc$ is automatically a fiber algebra, so that the free $T$-algebra functor factors as $C \to \text{Alg}_T(C) \subseteq \text{Alg}_T(C)$.

Proposition 2.30. Given a fibered monad on a Grothendieck fibration $\pi: C \to B$, the projection $\text{Alg}_T(C) \to B$ is again a Grothendieck fibration, with fibers $(\text{Alg}_T(C))_b = \text{Alg}_{T_b}(C_b)$.

Moreover, the free-algebra and forgetful functors then form a fibered adjunction $C \leftrightarrow \text{Alg}_T(C)$.

The key to this proof is that fiber algebra structures can be “pulled back” along cartesian arrows (which, by Proposition 2.24, must be the case if $C \leftrightarrow \text{Alg}_T(C)$ is to be a fibered adjunction).

Proof. The identification of the fibers is straightforward. Given a cartesian arrow $\Phi: \bar{c} \to c$ on $C$ and a fiber algebra structure on $c$, we claim there is a unique fiber algebra structure on $\bar{c}$ making $\Phi$ into an algebra map. Indeed, the properties of cartesian arrows imply that there is a unique way to choose a dashed fiber arrow in the diagram

\[
\begin{array}{ccc}
T \bar{c} & \xrightarrow{T \Phi} & Tc \\
\downarrow & & \downarrow \\
\bar{c} & \xrightarrow{\Phi} & c
\end{array}
\]

which provides the multiplication on $\bar{c}$. The claims that $T \bar{c} \to \bar{c}$ is an algebra structure and that $\Phi$ is again cartesian when viewed as an algebra map likewise follow from $\Phi$ being cartesian in $C$. The “moreover” claim that $C \leftrightarrow \text{Alg}_T(C)$ is a fibered adjunction follows by noting that the adjunction unit is the unit $I \Rightarrow T$ of the monad $T$, which is fibered by assumption, while the counit, evaluated on a fiber algebra $c$, is the multiplication $Tc \to c$, and hence fibered by the definition of fiber algebras. \qed
Remark 2.31. Suppose a cleavage of a Grothendieck fibration \( \pi : \mathcal{C} \to \mathcal{B} \) has been chosen.

A fibered monad \( T \) is then equivalent to the data of the fiber monads \( T_b \) on the fibers \( \mathcal{C}_b \) together with, for each arrow \( \varphi : b' \to b \) in \( \mathcal{B} \), natural transformations \( T_{b'} \varphi^* \Rightarrow \varphi^* T_b \), such that

(a) for composites \( b'' \xrightarrow{\psi} b' \xrightarrow{\varphi} b \) and identities \( b \xrightarrow{id_b} b \), the induced diagrams below commute

\[
\[
\begin{array}{ccc}
T_{b''} \psi^* \varphi^* & \xrightarrow{\cong} & T_{b'} \psi^* \\
\downarrow & & \downarrow \\
(\varphi \psi)^* T_b & \xrightarrow{\cong} & (\varphi \psi)^* T_b
\end{array}
\]

(b) the natural squares below, with vertical maps induced by the monads \( T_{b'}, T_b \), commute

\[
\[
\begin{array}{ccc}
T_{b'} T_b \varphi^* & \xrightarrow{=} & \varphi^* T_b T_b \\
\downarrow & & \downarrow \\
\varphi^* T_b & \xrightarrow{=} & \varphi^* T_b
\end{array}
\]

Remark 2.33. Suppose the Grothendieck fibration \( \pi : \mathcal{C} \to \mathcal{B} \) in Remark 2.31 is also an opfibration so that, by Remark 2.8, the cleavage functors \( \varphi^* \) for \( \varphi : b' \to b \) admit left adjoints \( \varphi_1 \).

Then \( \varphi^* T_b \varphi_1 \) has a monad structure (which combines the multiplication and unit of \( T_b \) with the unit and counit of the \( (\varphi_1, \varphi^*) \) adjunction) and commutativity of the diagrams in (2.32) is equivalent to the claim that the induced natural transformation \( T_{b'} \Rightarrow \varphi^* T_b \varphi_1 \) is a map of monads.

Remark 2.34. Suppose that both \( \mathcal{C} \) and \( \text{Alg}_{\mathcal{F}}(\mathcal{C}) \) admit adjunctions as in Remark 2.33 for each map \( \varphi : b' \to b \). We denote these two adjunctions by

\[
\varphi : \mathcal{C}_{b'} \rightleftarrows \mathcal{C}_b : \varphi^* \quad \text{and} \quad \varphi : \text{Alg}_{T_{b'}}(\mathcal{C}_{b'}) \rightleftarrows \text{Alg}_{T_b}(\mathcal{C}_b) : \varphi^*
\]

to emphasize the fact that the algebraic cleavage functor \( \varphi^* \) lifts the underlying \( \varphi^* \) (cf. the proof of Proposition 2.30).

On the other hand, the algebraic \( \varphi_1 \) functor is not a lift of the underlying \( \varphi_1 \). Rather, by the dual of Proposition 2.24, one has that \( T : \mathcal{C} \to \text{Alg}_{T_{b'}}(\mathcal{C}) \) preserves cocartesian arrows, so that the map \( T_{b'} \Rightarrow \varphi_1 T_b \) consisting of algebraic cocartesian arrows is identified with \( T_{b'} \Rightarrow T_b \varphi_1 \), i.e. the image under \( T \) of the natural map \( id_{\mathcal{C}_{b'}} \Rightarrow \varphi_1 \) consisting of underlying cocartesian arrows in \( \mathcal{C} \).

Thus, since any \( c \in \text{Alg}_{T_{b'}}(\mathcal{C}_{b'}) \) is given by a coequalizer of free algebras \( c \cong \text{coeq}(T_{b'} T_{b'} c \cong T_{b'} c), \) one has the formula \( \hat{\varphi}_1 c \cong \text{coeq}(T_{b'} \varphi_1 T_{b'} c \cong T_{b'} \varphi_1 c). \)

Proposition 2.35. Let \( I \) be a fixed small category, and \( T \) a fibered monad with respect to a Grothendieck fibration \( \pi : \mathcal{C} \to \mathcal{B} \). Then:

(i) \( T^I \) is fibered monad with respect to \( \pi^I : \mathcal{C}^I \to \mathcal{B}^I; \)

(ii) there is a natural identification \( \text{Alg}^I_{\pi^I}(\mathcal{C}^I) \cong (\text{Alg}_{\pi}(\mathcal{C}))^I. \)

Proof. Both parts follow readily from the definitions. \( \square \)
3 Equivariant colored operads

This section discusses the categories of equivariant colored symmetric sequences and operads. First, §3.1 applies the abstract theory from §2 to the categories \( \text{Sym}_*(\mathcal{V}) \), \( \text{Sym}_c^G(\mathcal{V}) \) of symmetric sequences with varying sets of colors (Definition 3.7), with most of the work being devoted to providing explicit descriptions of the equivariant fibers \( \text{Sym}_c^G(\mathcal{V}) \) for \( \mathcal{C} \in \text{Set}^G \) as a presheaf category, given in Proposition 3.17. Then, §3.2 provides a convenient description of the representable functors in these fibers in Proposition 3.35. Lastly, in §3.3, we describe the fibered monad \( F \) on \( \text{Sym}_*(\mathcal{V}) \) which defines operads \( \text{Op}_*(\mathcal{V}) \), cf. (2.1), as given in Definition 3.44 (and elaborated on in Appendix A). Moreover, in (3.48) we determine the fibers of the equivariant monad \( F^G \) on \( \text{Sym}_c^G(\mathcal{V}) \).

3.1 Equivariant colored symmetric sequences

We now discuss the categories \( \text{Sym}_*(\mathcal{V}) \), \( \text{Sym}_c^G(\mathcal{V}) \) of symmetric sequences appearing in (2.1).

Definition 3.1. Let \( \mathcal{C} \in \text{Set} \) be a fixed set of colors (or objects). A tuple \( \mathcal{C} = (c_1, \ldots, c_n; c_0) \in \mathcal{C}^{n+1} \) is called a \( \mathcal{C} \)-profile of arity \( n \). The \( \mathcal{C} \)-symmetric category \( \Sigma \mathcal{C} \) is the category whose objects are the \( \mathcal{C} \)-profiles and whose morphisms are action maps

\[
\mathcal{C} = (c_1, \ldots, c_n; c_0) \xrightarrow{\sigma} (c_{\sigma^{-1}(1)}, \ldots, c_{\sigma^{-1}(n)}; c_0) = \mathcal{C}^{\sigma^{-1}}
\]

(3.2)

for each permutation \( \sigma \in \Sigma_n \), with the natural notion of composition.

Alternatively, we will find it useful to visualize profiles as corollas (i.e. trees with a single node) with edges decorated by colors in \( \mathcal{C} \), as depicted below, so that the map labeled \( \sigma \) is the unique map of trees indicated such that the coloring of an edge equals the coloring of its image.

\[
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{\sigma} & \mathcal{C}^{\sigma^{-1}} \\
\cdots & & \cdots \\
\overset{c_0}{\bullet} & \overset{c_1}{\bullet} & \cdots \overset{c_n}{\bullet}
\end{array}
\]

(3.3)

Given any map of color sets \( \varphi: \mathcal{C} \to \mathcal{D} \), there is a functor (abusively written) \( \varphi: \Sigma \mathcal{C} \to \Sigma \mathcal{D} \), given by \( \varphi(c_1, \ldots, c_n; c_0) = (\varphi(c_1), \ldots, \varphi(c_n); \varphi(c_0)) \).

Remark 3.4. The notation \( \mathcal{C}^{\sigma^{-1}} \) in (3.2),(3.3) reflects the fact that \( \Sigma_n \) acts on the right on \( \mathcal{C} \)-profiles of arity \( n \) via \( \mathcal{C}^{\sigma} = (c_\cdot; c_0) \xrightarrow{\sigma} \mathcal{C}^{\sigma^{-1}} \), where we make the convention that \( \sigma(0) = 0 \).

Remark 3.5. If one regards \( \mathcal{C} \in \text{Set} \) as a discrete category, following Notation 2.16 one has an identification of groupoids \( \Sigma \mathcal{C} = (\Sigma; \mathcal{C}) \times \mathcal{C} \), where the \( \Sigma; \mathcal{C} \) factor accounts for the sources \( c_1, \ldots, c_n \) of a profile and the \( \mathcal{C} \) factor accounts for the target \( c_0 \).

Notation 3.6. We will (slightly abusively) write \( \Sigma \to \text{Set} \) for the Grothendieck construction (see Notation 2.10) of the functor \( \mathcal{C} \to \Sigma \mathcal{C} \).

More explicitly, the objects of \( \Sigma \) are the \( \mathcal{C} \in \Sigma \mathcal{C} \) for some set of colors \( \mathcal{C} \) and an arrow from \( \overset{\mathcal{C}}{\mathcal{D}} \in \Sigma \mathcal{C} \) to \( \overset{\mathcal{D}}{\mathcal{C}} \in \Sigma \mathcal{D} \) over \( \mathcal{C}: \mathcal{C} \to \mathcal{D} \) is an arrow \( \varphi: \mathcal{D} \to \mathcal{C} \) in \( \Sigma \).

Definition 3.7. Let \( \mathcal{V} \) be a category. The category \( \text{Sym}_*(\mathcal{V}) \) of symmetric sequences on \( \mathcal{V} \) (on all colors) is the category with:

- objects given by pairs \( (\mathcal{C}, X) \) with \( \mathcal{C} \in \text{Set} \) a set of colors and \( X: \Sigma \mathcal{C} \to \mathcal{V} \) a functor;
• arrows \((\mathcal{C}, X) \rightarrow (\mathcal{D}, Y)\) given by a map \(\varphi: \mathcal{C} \rightarrow \mathcal{D}\) of colors and a natural transformation \(X \Rightarrow Y \varphi\) as below.

\[
\begin{array}{ccc}
\Sigma_{\mathcal{C}}^{\text{op}} & \xrightarrow{\varphi} & \mathcal{V} \\
\downarrow & & \downarrow \\
\Sigma_{\mathcal{D}}^{\text{op}} & \xrightarrow{\varphi^*} & \mathcal{V}
\end{array}
\]

**Remark 3.8.** Given a category \(\mathcal{V}\), let \(\text{Cat}^{/\mathcal{V}}\) denote the category with objects given by functors \(\mathcal{C} \rightarrow \mathcal{V}\), and arrows from \(\mathcal{C} \rightarrow \mathcal{V}\) to \(\mathcal{D} \rightarrow \mathcal{V}\) given by pairs \((\varphi, \phi)\) with \(\varphi: \mathcal{C} \rightarrow \mathcal{D}\) a functor and \(\phi\) a natural transformation

\[
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{\varphi} & \mathcal{V} \\
\downarrow & & \downarrow \\
\mathcal{D} & \xrightarrow{\phi} & \mathcal{V}
\end{array}
\] \hspace{1cm} (3.9)

Then \(\text{Sym}_\bullet(\mathcal{V})\) is naturally a (neither wide nor full) subcategory of \(\text{Cat}^{/\mathcal{V}}\).

**Remark 3.10.** We caution that \(\text{Sym}_\bullet(\mathcal{V})\) is quite different from the presheaf category \(\text{Fun}(\Sigma_{\mathcal{V}}^{\text{op}}, \mathcal{V})\). Instead, one should think of \(\text{Sym}_\bullet(\mathcal{V})\) as a form of “fibered presheaves”. More precisely, the obvious functor \(\text{Sym}_\bullet(\mathcal{V}) \rightarrow \text{Set}\) is a (split) Grothendieck fibration, with fiber over \(\mathcal{C} \in \text{Set}\) the presheaf category \(\text{Sym}_\bullet(\mathcal{V}) = \text{Fun}(\Sigma_{\mathcal{V}}^{\text{op}}, \mathcal{V})\).

Note that, for any map \(\varphi: \mathcal{C} \rightarrow \mathcal{D}\), one has adjunctions

\[
\varphi!: \text{Sym}_\mathcal{C}(\mathcal{V}) \cong \text{Sym}_\mathcal{D}(\mathcal{V}): \varphi^*
\] \hspace{1cm} (3.11)

where \(\varphi^*\) (resp. \(\varphi_\ast\)) is precomposition with (resp. left Kan extension along) \(\varphi: \Sigma_{\mathcal{V}}^{\text{op}} \rightarrow \Sigma_{\mathcal{D}}^{\text{op}}\), so that the Grothendieck fibration \(\text{Sym}_\bullet(\mathcal{V}) \rightarrow \text{Set}\) is also an opfibration.

**Remark 3.12.** The forgetful functor \(\text{Cat}^{/\mathcal{V}} \rightarrow \text{Cat}\) remembering only the source category is likewise both a fibration and an opfibration, with a diagram (3.9) being a cartesian arrow if \(\phi\) an isomorphism and cocartesian if it is a left Kan extension.

Building on Remark 3.10, one can define a fibered Yoneda embedding, as in the following, where we abbreviate \(\text{Sym}_\bullet = \text{Sym}_\bullet(\text{Set})\).

**Notation 3.13.** Let \(\mathcal{C} \in \text{Set}\), \(\tilde{\mathcal{C}} \in \Sigma_{\mathcal{C}}\). We write \(\Sigma_{\mathcal{C}}[\tilde{\mathcal{C}}] \in \text{Sym}_\mathcal{C} = \text{Set}^{\Sigma_{\mathcal{C}}^{\text{op}}}\) for the representable presheaf

\[
\Sigma_{\mathcal{C}}[\tilde{\mathcal{C}}](-) = \text{Set}_{\mathcal{C}}(-, \tilde{\mathcal{C}}).
\]

Moreover, we define the fibered Yoneda embedding

\[
\Sigma_{\mathcal{C}} \xrightarrow{\Sigma_{\mathcal{C}}[\cdot]} \text{Sym}_\bullet
\] \hspace{1cm} (3.14)

to be \(\Sigma_{\mathcal{C}}[\tilde{\mathcal{C}}]\) when evaluated on an object \(\tilde{\mathcal{C}} \in \Sigma_{\mathcal{C}}\), and on an arrow \(\varphi: \tilde{\mathcal{C}} \rightarrow \tilde{\mathcal{D}}\) over \(\varphi: \mathcal{C} \rightarrow \mathcal{D}\) to be the natural transformation \(\Sigma_{\mathcal{C}}[\tilde{\mathcal{C}}] \Rightarrow \varphi^*\Sigma_{\mathcal{D}}[\tilde{\mathcal{D}}]\) given by the natural composites

\[
\Sigma_{\mathcal{C}}[\tilde{\mathcal{C}}](-) = \Sigma_{\mathcal{C}}(-, \tilde{\mathcal{C}}) \rightarrow \Sigma_{\mathcal{D}}(\varphi(-), \tilde{\mathcal{C}}) \rightarrow \Sigma_{\mathcal{D}}(\varphi(-), \tilde{\mathcal{D}}) = \varphi^*\Sigma_{\mathcal{D}}[\tilde{\mathcal{D}}](-).
\]

**Proposition 3.15.** Let \(\tilde{\mathcal{C}} \in \Sigma_{\mathcal{C}}\), \(\varphi: \mathcal{C} \rightarrow \mathcal{D}\) be a map of colors and consider the adjunction (3.11).

Then the adjoint of the canonical map \(\Sigma_{\mathcal{C}}[\tilde{\mathcal{C}}] \rightarrow \varphi^*\Sigma_{\mathcal{D}}[\varphi(\tilde{\mathcal{C}})]\) in \(\text{Sym}_\mathcal{C}\) is an isomorphism \(\varphi_!\Sigma_{\mathcal{C}}[\tilde{\mathcal{C}}] \cong \Sigma_{\mathcal{D}}[\varphi(\tilde{\mathcal{C}})]\) in \(\text{Sym}_\mathcal{D}\).
Alternatively, this result states that the fibered Yoneda \( \Sigma_\ast[-] \) preserves cocartesian arrows.

\begin{proof}
For \( Y \in \text{Sym}_B \), by the (usual) Yoneda lemma one has isomorphisms
\[
\text{Sym}_B(\Sigma_B[\varphi \tilde{C}], Y) \cong Y(\varphi \tilde{C}) \cong \text{Sym}_C(\Sigma_C[\tilde{C}], \varphi^* Y)
\]
proving \( \Sigma_B[\varphi \tilde{C}] \cong \varphi_! \Sigma_C[\tilde{C}] \). That this identification is adjoint to the canonical map \( \Sigma_C[\tilde{C}] \to \varphi^* \Sigma_B[\varphi \tilde{C}] \) in \( \text{Sym}_C \) follows since this sends \( id_{\tilde{C}} \) to \( id_{\varphi \tilde{C}} \) (and since these identities determine the isomorphisms in (3.16)).
\end{proof}

Now let \( G \) be a group. Writing \( \text{Sym}^G_C(V) = (\text{Sym}_C(V))^G \) for the category of \( G \)-objects on \( \text{Sym}_C(V) \), the argument in Remark 2.9 yields that \( \text{Sym}^G_C(V) \to \text{Set}^G \) is again a Grothendieck fibration. For each \( C \in \text{Set}^G \), we will then write \( \text{Sym}^G_C(V) \) to denote the fiber of \( \text{Sym}^G_C(V) \to \text{Set}^G \) over \( C \). We caution that \( \text{Sym}^G_C(V) \) is not the category \( (\text{Sym}_C(V))^G \) of \( G \)-objects on \( \text{Sym}_C(V) \) unless the \( G \)-action on \( C \) happens to be trivial.

We will thus find it convenient to have a more explicit description of \( \text{Sym}^G_C(V) \), provided by the following result, which adapts [BF21, Lemma A.6]. In fact, we prove a more general description for the categories \( \text{Cat} \^{!} | V \) in Remark 3.8, for which the natural “source category” functor \( \text{Cat} \^{!} | V \to \text{Cat} \) is likewise a split Grothendieck fibration, with fiber over \( C \in \text{Cat} \) given by \( \text{Fun}(C, V) \).

\begin{proposition}
Let \( G \) be a group and \( C \in \text{Set}^G \) be a \( G \)-set. There is then a natural identification
\[
\text{Sym}^G_C(V) \simeq \text{Fun}(G \times \Sigma_C^G, V).
\]

More generally, for a category \( B \), the fiber of \( (\text{Cat} \^{!} | V)^B \) over \( \text{Fun}(B \times C, V) \) is given by
\[
\text{Fun}(B \times C, V).
\]
\end{proposition}

\begin{proof}
It is immediate from the definitions that \( \text{Sym}^G_C(C) \) matches the fiber of \( (\text{Cat} \^{!} | V)^G \to \text{Set}^G \) over \( \Sigma_C^G \in \text{Cat}^G \), so we need only address the general case.

As in Notation 2.10, let us write \( \phi_! : C \to C' \) for the functor induced by the arrow \( \varphi : b \to b' \) in \( B \). Unpacking definitions, an object of \( (\text{Cat} \^{!} | V)^B \) over \( C \times B \to \text{Cat} \) corresponds to the data of functors \( \gamma_b : C_b \to V \) for each \( b \in B \) and natural transformations \( \phi_!: \gamma_b \Rightarrow \gamma_{b'} \) for each \( \varphi : b \to b' \) in \( B \)
\[
\begin{diagram}
C_b & \xrightarrow{\phi} & C_{b'}
\end{diagram}
\xrightarrow{\gamma_b} V
\]
subject to the requirements that \( \gamma_{b'} \circ \phi_! = \gamma_{b'} \circ \phi_! \circ \gamma_b \) equals \( \gamma_b \circ \phi_{b, b'} \) for composable \( b \xrightarrow{c} b' \xrightarrow{c'} b'' \) and that \( \phi_{b, b} = id_{\gamma_b} \).

We now claim that the data above is exactly the data of a functor \( F : B \times C \to V \). To ease notation, we follow Remark 2.11 and describe arrows in \( B \times C \) as composites \( c \twoheadrightarrow c' \). On fiber arrows \( f : c \to c' \) in \( C_b \) set \( F(f) = \gamma_b(f) \), and on cocartesian arrows set \( F(c \twoheadrightarrow c') = (\phi_{c'})_\ast \).

Then: (i) associativity and unitality of \( F \) with respect to fiber arrows is equivalent to associativity and unitality of the \( \gamma_b \); (ii) associativity and unitality of \( F \) with respect to the cocartesian arrows is equivalent to the conditions following (3.19); (iii) \( F \) respecting the commutative squares (2.12) is equivalent to naturality of the \( \phi_{\varphi} \).

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This finishes the proof that (3.18) has the correct set of objects. The claim that it also has the correct arrows is similar.

Following the result above, we will represent $G$-equivariant symmetric sequences $X \in \text{Sym}^G_\mathcal{C}(\mathcal{V})$ for some $\mathcal{C} \in \text{Set}^G$ as functors $X: G \times \Sigma_\mathcal{C}^G \to \mathcal{V}$, and objects of $(\text{Cat} \downarrow \mathcal{V})^G$ as functors $G \times \mathcal{C} \to \mathcal{V}$.

**Remark 3.20.** Recall that, if $\mathcal{C} \in \text{Cat}^G$ is a category with a $G$-action, then $\Sigma \downarrow \mathcal{C}$ has a $G$-action where $G$ acts diagonally on tuples $(c_i)$ with $c_i \in \mathcal{C}$. Therefore, by applying the $\Sigma(-)$ construction to the categories $\text{Cat} \downarrow \mathcal{V}$ in Remark 3.8 (note that that one can apply $\Sigma(-)$ to the entirety of diagram (3.8)), one obtains a description of the functor

$$
\begin{array}{ccc}
(\text{Cat} \downarrow \mathcal{V})^G & \xrightarrow{\Sigma(-)} & (\text{Cat} \downarrow \Sigma \downarrow \mathcal{V})^G \\
G \times \mathcal{C} \to \mathcal{V} & \longmapsto & (G \times (\Sigma \downarrow \mathcal{C}) \to \Sigma \downarrow (G \times \mathcal{C}) \to \Sigma \downarrow \mathcal{V})
\end{array}
$$

where the functor $G \times (\Sigma \downarrow \mathcal{C}) \to \Sigma \downarrow (G \times \mathcal{C})$ is the natural inclusion described in Remark 2.19.

### 3.2 Representable functors

Since $\text{Sym}^G_\mathcal{C}(\mathcal{V}) \cong \mathcal{V}^{G \times \Sigma_\mathcal{C}^G}$ is again a functor category, we will find it useful to discuss its representable functors. In the following discussion we set $\mathcal{V} = \text{Set}$ and abbreviate $\text{Sym}^G_\mathcal{C} = \text{Sym}^G_\mathcal{C}(\text{Set})$.

As the objects of $G \times \Sigma_\mathcal{C}^G$ are simply the profiles $\bar{C} \in \Sigma_\mathcal{C}$, each such profile induces a representable functor $(G \times \Sigma_\mathcal{C}^G)(\bar{C}, -)$ in $\text{Sym}^G_\mathcal{C}$. However, caution is needed. If one forgets the $G$-action on $\mathcal{C}$, then $\bar{C} \in \Sigma_\mathcal{C}$ likewise induces a representable functor $\text{Sym}^G_\mathcal{C}(\bar{C}) = \Sigma_\mathcal{C}^G(\bar{C}, -)$ in $\text{Sym}^G_\mathcal{C}$, as discussed in Notation 3.13. As such, our next task is to understand the relation between $(G \times \Sigma_\mathcal{C}^G)(\bar{C}, -)$ and $\text{Sym}^G_\mathcal{C}(\bar{C})$ in such a way that we have analogues of the fibered Yoneda (3.14) and Proposition 3.15.

As in our previous work in [Per18, Not. 5.56], [BP20, §2.3], the key to achieve this will be to extend the $\Sigma_\mathcal{C}(\bar{C})$ notation to be defined not only for corollas $\bar{C}$ but also for “colored forests of corollas”. In fact, we will do a little more. In anticipation of our discussion of operads, we will extend this notation by defining $\Sigma_\mathcal{C}(\bar{F})$ for $\bar{F}$ a general colored forest (of trees). In the following, $\Phi$ denotes the category of forests (i.e. formal coproducts of trees; see [Per18, §5.1]).

**Definition 3.21.** Let $\mathcal{C}$ be a set of colors. The category $\Phi_\mathcal{C}$ of $\mathcal{C}$-colored forests has

- objects pairs $\bar{F} = (F, c)$ where $F \in \Phi$ is a forest and $c: E(F) \to \mathcal{C}$ is a coloring of its edges;
- a map $\bar{F} = (F, c) \to (F', c') = \bar{F}'$ is a map $\varphi: F \to F'$ in $\Phi$ such that $c = c' \varphi$.

If $\varphi: \mathcal{C} \to \mathcal{D}$ is a map of colors, we write $\varphi: \Phi_\mathcal{C} \to \Phi_\mathcal{D}$ for the functor sending $\bar{F} = (F, c)$ to $\varphi \bar{F} = (F, \varphi c)$. Note that this defines a Grothendieck opfibration

$$
\Phi_\mathcal{C} \to \text{Set}
$$

whose objects are the $\bar{F} \in \Phi_\mathcal{C}$ for some $\mathcal{C} \in \text{Set}$, and with an arrow from $\bar{F}$ to $\bar{F}'$ over $\varphi: \mathcal{C} \to \mathcal{D}$ given by an arrow $\varphi \bar{F} \to \varphi \bar{F}'$ in $\Phi_\mathcal{D}$.

For each vertex $v \in \mathcal{V}(F)$ in a forest, we write $F_v$ for the associated corolla. Note that, given a $\mathcal{C}$-coloring $\bar{F}$ on $F$, one one likewise obtains colorings $\bar{F}_v$ on $F_v$. 


Notation 3.22. Given $\vec{F} \in \Phi_{C}$ we define
\[ \Sigma_{C}[\vec{F}] = \bigsqcup_{c \in V(F)} \Sigma_{C}[\vec{F}_{c}] \] (3.23)
where we highlight that the coproduct $u^{C}$ is fibered, i.e. it takes place in $\text{Sym}_{C}$ rather than $\text{Sym}$. 

Example 3.24. Let $C = \{a, b, c\}$. On the left below we depict a $\mathcal{C}$-colored forest $\vec{F} = \vec{T} \cup \vec{S}$ with tree components $\vec{T}, \vec{S}$.

Moreover, on the right we depict the $\mathcal{C}$-profiles/corollas $\vec{T}_{i}$ and $\vec{S}_{j}$ corresponding to the vertices of $T, S$, so that
\[ \Sigma_{C}[\vec{F}] = \sum_{1 \leq i \leq 4} \Sigma_{C}[\vec{T}_{i}] = \sum_{1 \leq j \leq 3} \Sigma_{C}[\vec{S}_{j}] \]

Remark 3.25. The representables $\Sigma_{C}[-]$ in (3.23) do not quite define a functor on the entire category $\Phi_{\bullet}$, due to the fact that the only maps of forests sending vertices to vertices are the outer maps $BP_{21}, \Sect{3.2}$. Writing $\Phi_{o} \hookrightarrow \Phi_{\bullet}$ for the wide subcategory of those arrows whose underlying maps of uncolored forests are outer (on each tree component), (3.14) extends to give generalized fibered Yoneda embeddings (where the right functor is obtained from the left functor by taking $G$-objects)
\[ \Phi_{o} \xrightarrow{\Sigma_{C}[-]} \text{Sym}_{\bullet}, \quad \Phi_{o,G} \xrightarrow{\Sigma_{C}[-]} \text{Sym}_{G} \]
which are fibered over $\text{Set}$ and $\text{Set}^{G}$, respectively. Note that, thanks to formula (3.23), Proposition 3.15 automatically generalizes, i.e. one has natural identifications
\[ \varphi_{!}\Sigma_{C}[\vec{F}] = \Sigma_{D}[\varphi\vec{F}] \] (3.26)
for each map of colors $\varphi : \mathcal{C} \to \mathcal{D}$ (which is an equivariant map in the equivariant case).

Notation 3.27. We write $(-)^{\gamma} : \Phi \to \Phi_{\bullet}$ for the tautological coloring functor which sends $F \in \Phi$ to $F^{\gamma} \in \Phi_{E(T)}$ where $F^{\gamma} = (F, t)$ is the underlying forest $F$ together with the identity coloring $t : E(T) \to E(T)$. Moreover, we then abbreviate $\Sigma_{C}[F] = \Sigma_{E(F)}[F^{\gamma}]$.

Remark 3.28. For any colored forest $\vec{F} = (F, c)$, regarding $c : E(T) \to \mathcal{C}$ as a change of color map, one has $\vec{F} = cF^{\gamma}$, so that (3.26) then yields
\[ \Sigma_{C}[\vec{F}] = c!\Sigma_{C}[F] \] (3.29)
**Definition 3.30.** Let $G$ be a group, $\mathcal{C} \in \text{Set}^G$ be a $G$-set of colors, and $\mathcal{C} \in \Sigma_{\mathcal{C}}$ be a $\mathcal{C}$-profile/corolla. Write $\mathcal{C} = (C, c)$ with $C \in \Sigma$ the underlying corolla and $c : E(T) \to \mathcal{C}$.

Writing $G \cdot c : G \cdot E(T) \to \mathcal{C}$ for the adjoint map, $G \cdot C \in \Phi^G$ for the $G$-free forest determined by $C$, and noting that $E(G \cdot C) \simeq G \cdot E(C)$, we define $G \cdot \mathcal{C} \in \Phi^G$ by

$$G \cdot \mathcal{C} = (G \cdot c)(G \cdot C)^\circ.$$

(3.31)

**Remark 3.32.** Writing $g \cdot \mathcal{C} \to \mathcal{C}$ for the $G$-action maps, one has the more explicit formula (see Example 3.34)

$$G \cdot \mathcal{C} = \bigsqcup_{g \in G} g \mathcal{C}$$

However, in practice we will prefer to use (3.31) for technical purposes.

**Remark 3.33.** One can further extend (3.31) to define a functor $G \cdot \mathcal{C} (-) : \Phi_{\mathcal{C}} \to \Phi^G_{\mathcal{C}}$, which is the left adjoint to the forgetful functor $\Phi^G_{\mathcal{C}} \to \Phi_{\mathcal{C}}$.

**Example 3.34.** Let $G = \{1, i, -1, -i\} \simeq \mathbb{Z}_4$ be the group of quartic roots of unit and $\mathcal{C} = \{a, -a, a, -i, b, i, b\}$ where we implicitly have $-b = b$. The following depicts the forest (of corollas) $G \cdot \mathcal{C}$ in $\Phi^G_{\mathcal{C}}$ for $\mathcal{C}$ in $\Sigma_{\mathcal{C}}$ the leftmost corolla.

Note that the pairs $\mathcal{C}, -\mathcal{C}$ and $i\mathcal{C}, -i\mathcal{C}$ are isomorphic in $\Sigma_{\mathcal{C}}$ while any other pair, such as $\mathcal{C}, i\mathcal{C}$, is not. In general, it is moreover possible for two or more tree components of $G \cdot \mathcal{C}$ to be equal.

**Proposition 3.35.** For any $G$-set of colors $\mathcal{C}$ and $\mathcal{C}$-profile $\mathcal{C} \in \Sigma_{\mathcal{C}}$ one has a natural identification

$$(G \ltimes \Sigma^\text{op}) (\mathcal{C}, -) \simeq \Sigma_{\mathcal{C}} [G \cdot \mathcal{C}].$$

**Proof.** Recalling the $C_{\varphi}(-, -)$ notation (cf. Notation 2.4) for maps over $\varphi : b \to b'$ we likewise write $\text{Sym}^G_{\varphi}(-, -)$, $\text{Sym}_{\varphi}(-, -)$ for maps over the map of colors $\varphi$. The result now follows from the string of isomorphisms (which show that $\mathcal{C}$ represents $\Sigma_{\mathcal{C}} [G \cdot \mathcal{C}] \in \text{Set}^{G \ltimes \Sigma^\text{op}_{\mathcal{C}}}$)

$$\text{Sym}^G_{\Sigma_{\mathcal{C}}}[G \cdot \mathcal{C}, X] \simeq \text{Sym}^G_{\Sigma_{\mathcal{C}}}[G \cdot C, X] \simeq \text{Sym}_{\Sigma_{\mathcal{C}}}[C, X] \simeq \text{Sym}_{\Sigma_{\mathcal{C}}}[\mathcal{C}, X] = X(\mathcal{C})$$

where: the first and third steps use the canonical pushforwards (3.29) and (the dual of) Remark 2.5, the second step uses (2.26) and the observation that $\Sigma_{\mathcal{C}}[G \cdot C] \simeq G \cdot \Sigma_{\mathcal{C}}[C]$, and the last step is the Yoneda lemma in $\text{Sym}_{\mathcal{C}}$. $\blacksquare$

We end this section by discussing convenient notation for subgroups $\Lambda \leq \text{Aut}_{G \ltimes \Sigma^\text{op}_{\mathcal{C}}}(\mathcal{C})$, where the $\mathcal{C}$-profile $\mathcal{C}$ is regarded as an object of $G \ltimes \Sigma^\text{op}_{\mathcal{C}}$. 

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Alternatively, in profile notation we have 
leaf-root functor which happen to be trees, as well as \( \Omega \) as the category of fiber algebras over a certain fibered monad \( c \). The colored leaf-root functor is in fact fibered over \( c \) \( \Rightarrow \) any map of colors \( \phi \). More explicitly, for \( g, \sigma \) \( \in \Omega \), \( g, \sigma \) isomorphisms. Next, following \([3.38]\) restrictions \( G \times \Sigma^{op} \) for the wide subcategory whose arrows are the isomorphisms. Next, following \([3.38]\), there is a “colored arity functor”

\[
\Omega^0_{\mathcal{C}} \xrightarrow{lr} \Sigma_{\mathcal{C}} \tag{3.38}
\]

which we call the leaf-root functor, described as follows. Given \( \bar{T} \in \Omega^0_{\mathcal{C}} \), its leaf-root \( lr(\bar{T}) \in \Sigma_{\mathcal{C}} \) is the only \( \mathcal{C} \)-corolla admitting a planar tall map \( lr(\bar{T}) \rightarrow T \) (see \([3.38]\)), where by tall map we mean a map which sends leaves to leaves and the root to the root.

**Example 3.39.** For \( \bar{T}, \bar{S} \) the trees in Example 3.24, we depict \( lr(\bar{T}) \), \( lr(\bar{S}) \) below, which, informally, are obtained by keeping only the leaves and roots of the trees \( \bar{T}, \bar{S} \).

![Diagram](image)

Alternatively, in profile notation we have \( lr(\bar{T}) = (b, c; a) \) and \( lr(\bar{S}) = (; a) \).

**Remark 3.40.** Note that for a stick tree \( \eta_\epsilon \), consisting of a single edge decorated by the color \( \epsilon \in \mathcal{C} \), it is \( lr(\eta_\epsilon) = (; \epsilon) \), which is the corolla with two edges (a leaf and a root), both labeled by \( \epsilon \).

**Remark 3.41.** The colored leaf-root functor is in fact fibered over \( \text{Set}^G \). More explicitly, for any map of colors \( \varphi : \mathcal{C} \rightarrow \mathcal{D} \), we have the following commuting diagram.

![Diagram](image)
For each \( \mathcal{C} \)-profile \( \mathcal{C} \), we write \( \mathcal{C} \downarrow \Omega^\mathcal{C}_X \) for the undercategory with respect to \( \mathcal{C} \), whose objects consist of a tree \( \tilde{T} \in \Omega^\mathcal{C}_X \) together with a choice of isomorphism \( \mathcal{C} \to \mathcal{C} \). Morally, \( \mathcal{C} \downarrow \Omega^\mathcal{C}_X \) is the “groupoid of trees with arity \( \mathcal{C} \).

We can now provide the “usual” formula for the “free operad monad” (see [BM07, page 816] for the non-colored case). Letting \( X \in \text{Sym}_\mathcal{C}(\mathcal{V}) \), then for each \( \mathcal{C} \)-profile \( \mathcal{C} \) we have

\[
F_\mathcal{C}X(\mathcal{C}) = \prod_{[T]_{\text{iso}(\mathcal{C})}} \left( \bigotimes_{v \in \mathcal{V}(T)} X(\tilde{T}_v) \right) \cdot_{\text{Aut}^\mathcal{C}(\mathcal{T})} \text{Aut}_{\Omega^\mathcal{C}}(\mathcal{C})
\]

where \( \text{Iso}(\mathcal{C}) \) denotes isomorphism classes of objects.

However, one drawback of the formula (3.42) is that it is not immediately clear how it should be modified in the equivariant case, where \( \mathcal{C} \) is a \( G \)-set and \( X \in \text{Sym}_G^\mathcal{C} \) is a \( G \times \Sigma^G \to \mathcal{V} \). To address this, we first repackage (3.42), following our approach in [BP21, §4]. We first need to define another functor, which we call the \textit{vertex functor}. As motivation, we note that in (3.42) the \( \text{Aut}_{\Omega^G} \mathcal{T} \)-action on the term \( \bigotimes_{v \in \mathcal{V}(T)} X(\tilde{T}_v) \) depends on both permutations of the set \( \mathcal{V}(T) \) and on automorphisms of the corollas \( \tilde{T}_v \). As such, rather than regard the vertices of \( \tilde{T} \) as merely a set we define

\[
\Omega^\mathcal{C}_X (V) \xrightarrow{V \cdot} \Sigma ; \Omega^\mathcal{C} \quad \tilde{T} \mapsto V(\tilde{T}) = (\tilde{T}_v)_{v \in \mathcal{V}(T)}
\]

In words, \( V(\tilde{T}) \) is the tuple of corollas indexed by the vertices of \( T \). Note that, by regarding \( V(\tilde{T}) \) as an object in \( \Sigma ; \Omega^\mathcal{C} \) rather than just a set, we keep track of extra automorphism data. Noting that both the leaf root and vertex functors are naturally compatible with change of colors \( \varphi : \mathcal{C} \to \mathcal{D} \), we can now provide the following alternative description of (3.42).

**Definition 3.44.** Let \( \mathcal{V} \) be a closed symmetric monoidal category.

The \textit{fibered free operad monad} \( F \) on \( \text{Sym}_\mathcal{C}(\mathcal{V}) \) assigns to \( \Sigma^G \to \mathcal{V} \) the left Kan extension

\[
\begin{array}{ccc}
\Omega^G \xrightarrow{V \cdot} & (\Sigma ; \Omega^\mathcal{C})^G & (\Sigma \times V)^G \xrightarrow{\smash{\otimes \atop \circ \atop \circ}} & \mathcal{V} \\
\downarrow \smash{\atop \circ \atop \circ} & \downarrow \smash{\atop \circ \atop \circ} & \downarrow \smash{\atop \circ \atop \circ} & \\
\mathcal{V} \xrightarrow{\smash{\atop \circ \atop \circ} \circ} & \mathcal{V} \xrightarrow{\smash{\atop \circ \atop \circ} \circ} & \mathcal{V} \xrightarrow{\smash{\atop \circ \atop \circ} \circ}
\end{array}
\]

Then \( \mathcal{O}_\mathcal{C}(\mathcal{V}) = \text{Alg}_{\mathcal{C}}(\text{Sym}_\mathcal{C}(\mathcal{V})) \), with fibers \( \mathcal{O}_\mathcal{C}(\mathcal{V}) = \text{Alg}_{\mathcal{C}}(\text{Sym}_\mathcal{C}(\mathcal{V})) \) over each \( \mathcal{C} \in \text{Set} \).

The complete discussion of the monad structure \( FF \Rightarrow F \), \( id \Rightarrow F \) is postponed to Appendix A, culminating in Definition A.32.

**Remark 3.46.** To relate (3.42) and (3.45), recall that, for any span \( \mathcal{G} \xleftarrow{k} \mathcal{G} \to \mathcal{V} \) with \( \mathcal{G}, \tilde{\mathcal{G}} \) groupoids, one has the formula

\[
\text{Lan} X(\mathcal{G}) \simeq \text{colim}(k(g) \to g) X(g) \simeq \prod_{[k(g) \to g]_{\text{iso}(\mathcal{G})}} \text{Aut}_{\mathcal{G}}(g) \cdot_{\text{Aut}_0(\mathcal{G})} X(g)
\]

where the first identification is (the dual of) [Rie14, Thm. 1.3.5] and the second identification uses the observation that \( \mathcal{G} \downarrow \tilde{\mathcal{G}} \) is also a groupoid.

By the categorical argument in Proposition 2.35, we have that, by taking \( G \)-objects, \( F \) also induces a fibered monad \( F^G \) on \( \text{Sym}_G^\mathcal{C}(\mathcal{V}) \).

To describe \( F^G \), note that (3.45) can be regarded as an arrow in the category \( \text{Cat} \xleftarrow{id} \mathcal{V} \) from Remark 3.8 which, being a left Kan extension, is cocartesian over \( \text{Cat} \) (cf. Remark 3.12).
Hence, if \( X \in \text{Sym}_G^G(\mathcal{V}) \) is \( G \)-equivariant, (3.45) is then a cocartesian arrow in \((\text{Cat} i^! \mathcal{V})^G\) over \( \text{Cat}^G \). By Proposition 3.17, we can hence rewrite such a \( G \)-equivariant (3.45) as the left Kan extension for a span \( G \times \Sigma^\varepsilon \rightarrow G \times \Omega^\varepsilon \rightarrow \mathcal{V} \). To fully describe this span, we need to understand how equivariance affects the top composite in (3.45), with the non-obvious issue being that of understanding what happens to the middle map therein, which can be described using Remark 3.20. Putting all of this together (and using the isomorphisms \( G \times \mathcal{C}^\varepsilon \cong (G \times \mathcal{C})^\varepsilon \) from Remark 2.15), we get the following.

**Proposition 3.47.** The monad \( \mathbb{F}^G \) on \( \text{Sym}_G^G(\mathcal{V}) \) assigns to \( G \times \Sigma^\varepsilon \rightarrow \mathcal{V} \) the left Kan extension below.

\[
\begin{array}{ccc}
G \times \Omega^\varepsilon & \xrightarrow{G \times !} & G \times (\Sigma \varepsilon)^\varepsilon \\
\downarrow \text{Gwh}^\varepsilon & & \downarrow \text{Lan}_{D^G \times \mathcal{V}} \varepsilon^\varepsilon \\
G \times \Sigma^\varepsilon & \xrightarrow{\text{Str}^G (3.50)} & \mathcal{V}
\end{array}
\]

Then \( \text{Op}^G_\varepsilon (\mathcal{V}) = \text{Alg}^\varepsilon_\varepsilon (\text{Sym}_G^G(\mathcal{V})) \), with fibers \( \text{Op}^G_\varepsilon (\mathcal{V}) = \text{Alg}^\varepsilon_\varepsilon (\text{Sym}_G^G(\mathcal{V})) \) over each \( \varepsilon \in \text{Set}^G \).

**Remark 3.49.** By Remark 3.46 we now have the following analogue of (3.42).

\[
\mathbb{F}^G_\varepsilon X(\tilde{C}) = \coprod_{\{T\} \in \text{Aut}_{G \times \Omega^\varepsilon}(\tilde{C})} \left( \bigotimes_{i \in \mathcal{V}(T)} X(\tilde{T}_i) \right) \text{Aut}_{G \times \Sigma^\varepsilon}(\tilde{C})
\]

In comparing (3.50) with (3.42), note that, since \( G \times \Omega^\varepsilon \) has more morphisms than \( \Omega^\varepsilon \), equation (3.50) has fewer coproduct summands than (3.42), though this is compensated by the fact that the inductions \((-) \text{Aut}_{G \times \Omega^\varepsilon}(\tilde{C}) \text{Aut}_{G \times \Sigma^\varepsilon}(\tilde{C}) \) are correspondingly larger than the inductions \((-) \text{Aut}_{\Sigma^\varepsilon}(\tilde{C}) \text{Aut}_{\Sigma^\varepsilon}(\tilde{C}) \).

**Remark 3.51.** Following Remark 2.34, for any map of \( G \)-sets \( \varphi : \varepsilon \rightarrow \mathcal{D} \) and \( \mathcal{D} \)-symmetric sequence \( X \), one has a pullback \( \varepsilon \)-symmetric sequence \( \varphi^* X \) given by \( \varphi^* X(\tilde{D}) = X(\varphi(\tilde{D})) \), which is an operad if \( X \) itself is an operad. Moreover, one then has a pair of adjunctions

\[
\begin{array}{ccc}
\text{Op}^G_\varepsilon (\mathcal{V}) & \xleftarrow{\varphi_1} & \text{Op}^\mathcal{D}_\varepsilon (\mathcal{V}) \\
\downarrow \text{fgt} & & \downarrow \text{fgt} \\
\text{Sym}^G_\varepsilon (\mathcal{V}) & \xleftarrow{\varphi_1} & \text{Sym}^\mathcal{D}_\varepsilon (\mathcal{V})
\end{array}
\]

where we highlight that the right adjoints are compatible with the forgetful functors, in the sense that \( \varphi^* \circ \text{fgt} = \text{fgt} \circ \varphi^* \), but the left adjoints are not: \( \varphi_1 \) is simply a left Kan extension, while \( \varphi_1 \) is given by the coequalizer

\[
\varphi_1 \mathcal{O} = \text{coeq}(\mathbb{F}_\mathcal{D} \varphi_1 \mathbb{F}_{\varepsilon} \mathcal{O} \Rightarrow \mathbb{F}_\mathcal{D} \varphi_1 \mathcal{O}).
\]

In general, we do not have a more explicit description of \( \varphi_1 \). However, when \( \varphi \) is injective, \( \varphi_1 X \) is the extension by \( \emptyset \), from which it follows that \( \mathbb{F}_\mathcal{D} \varphi_1 = \varphi_1 \mathbb{F}_{\varepsilon} \), and (3.52) then says that \( \varphi_1 \mathcal{O} = \text{coeq} (\varphi_1 \mathbb{F}_\varepsilon \mathbb{F}_{\varepsilon} \mathcal{O} \Rightarrow \varphi_1 \mathbb{F}_{\varepsilon} \mathcal{O}) \cong \varphi_1 (\text{coeq} (\mathbb{F}_\varepsilon \mathbb{F}_\varepsilon \mathcal{O} \Rightarrow \mathbb{F}_\varepsilon \mathcal{O})) \cong \varphi_1 \mathcal{O} \), so that \( \varphi_1 \circ \text{fgt} \cong \text{fgt} \circ \varphi_1 \).
4 Equivariant homotopy theory

This section develops the equivariant homotopy theory needed for our main proofs. §4.1 introduces the global monoid axiom in Definition 4.6, with sufficient conditions for this axiom given in Proposition 4.10. In §4.2, motivated by the identification $\text{Sym}_G^G(V) \simeq V^{G \times \Sigma}^G$ in Proposition 3.17, we extend our discussion from [BP21, §6] regarding model structures determined by families from the context of $V^G$ with $G$ a group to that of $V^G$ with $G$ a groupoid. This culminates in Proposition 4.34, an extension of [BP21, Prop. 6.25] which will greatly simplify the proof of Theorem II, cf. Remark 5.27.

4.1 Global monoid axiom

Definition 4.1. Let $V$ be a model category and $G$ a group.

The genuine (or fine) model structure on $G$-objects $V^G$ is the model structure (if it exists) such that $f:X \to Y$ is a weak equivalence (resp. fibration) if the fixed point maps $f^H:X^H \to Y^H$ are weak equivalences (fibrations) in $V$ for all $H \leq G$.

Notation 4.2. We write $W_G$ to denote the class of genuine weak equivalences in $V^G$.

Remark 4.3. If $V$ is cofibrantly generated with $I$ (resp. $J$) the sets of generating (trivial) cofibrations, then the genuine model structure on $V^G$, should it exist, is again cofibrantly generated with generating sets

$$I_G = \{G/H \cdot i \mid H \leq G, i \in I\} \quad J_G = \{G/H \cdot j \mid H \leq G, j \in J\}. \quad (4.4)$$

Definition 4.6. Let $(V, \otimes)$ be a cofibrantly generated monoidal model category.

For a finite group $G$, let $J^G_\otimes$ be the set of maps in $V^G$ given by

$$J^G_\otimes = \{j \otimes X \mid j \in J, X \in V^G\}$$

We refer to the class of maps $J^G_\otimes$-cof in $V^G$ as the $G$-genuine $\otimes$-trivial cofibrations.

Moreover, we say $V$ satisfies the global monoid axiom if $G$-genuine $\otimes$-trivial cofibrations are $G$-genuine weak equivalences, i.e. if $J^G_\otimes$-cof $\subseteq W_G$, for all finite groups $G$.

Remark 4.7. The global monoid axiom holds provided $J^G_\otimes$-cell $\subseteq W_G$ for all finite groups $G$.

Remark 4.8. Restricting to $G = \ast$, the global monoid axiom recovers the monoid axiom of Schwede-Shipley [SS00].

We now discuss convenient sufficient conditions for the existence of the genuine model structures in Definition 4.1 and for the global monoid axiom in Definition 4.6. These are given by the following definition, which is motivated by [Ste16, Remark 2.7], and gives two variants of the cellular fixed points conditions of [Ste16, Prop. 2.6].

Definition 4.9. Let $V$ (resp. $(V, \otimes)$) be a cofibrantly generated (monoidal) model category.

We say $V$ has weak acyclic cellular fixed points (resp. $(V, \otimes)$ has monoidal weak acyclic cellular fixed points) if, for all finite groups $G$ and subgroups $H \leq G$, the fixed point functor $(-)^H:V^G \to V$...
(i) preserves direct colimits of maps in $\mathcal{J}_G$-cof (resp. $\mathcal{J}_G^\oplus$-cof);

(ii) preserves pushout diagrams where one leg is in $\mathcal{J}_G$ (resp. in $\mathcal{J}_G^\oplus$);

(iii) sends maps in $\mathcal{J}_G$ to maps in $\mathcal{J}$-cof (resp. maps in $\mathcal{J}_G^\oplus$ to maps in $\mathcal{J}^\circ$-cof).

**Proposition 4.10.** Let $\mathcal{V}$ (resp. $(\mathcal{V}, \otimes)$) be a cofibrantly generated (monoidal) model category.

(i) If $\mathcal{V}$ has weak acyclic cellular fixed points, the genuine model structure on $\mathcal{V}^G$ exists for any finite group $G$.

Moreover, for any $H \leq G$, fixed points $(-)^H: \mathcal{V}^G \rightarrow \mathcal{V}$ send genuine trivial cofibrations to trivial cofibrations.

(ii) If $(\mathcal{V}, \otimes)$ has monoidal weak acyclic cellular fixed points and satisfies the usual monoid axiom (cf. Remark 4.8), then $\mathcal{V}$ satisfies the global monoid axiom.

**Proof.** For (i), we apply [Hov99, Theorem 2.1.19] to the generating sets in (4.4). All conditions therein are immediate except for the requirement that $\mathcal{J}_G$-cell $\subseteq \mathcal{W}_G$, which holds since the conditions in Definition 4.9 guarantee that $H$-fixed points of maps in $\mathcal{J}_G$-cell are in $\mathcal{J}$-cell.

(ii) is similar. Definition 4.9 implies that $H$-fixed points of maps in $\mathcal{J}_G^\oplus$-cell are in $\mathcal{J}^\circ$-cell. $\blacksquare$

### 4.2 Families in groupoids and pushout powers

In addition to the genuine model structures from Definition 4.1 in §4.1, we will need to consider variants where the subgroups $H \leq G$ therein are restricted to a family of subgroups. In fact, motivated by the identification $\text{Sym}_c^G(\mathcal{V}) = \text{Fun}(G \times \Sigma_c^G, \mathcal{V})$ in Proposition 3.17, we will further consider a groupoid variant. We first extend the notion of family of subgroups to the context of groupoids.

**Definition 4.11.** Let $\mathcal{G}$ be a groupoid. A family of subgroups of $\mathcal{G}$ is a collection $\mathcal{F} = \{ \mathcal{F}_x | x \in \mathcal{G} \}$ where each $\mathcal{F}_x$ is itself a collection of subgroups $H \leq \text{Aut}_\mathcal{G}(x)$ and such that:

(i) if $H \in \mathcal{F}_x$ and $K \leq H$ then $K \in \mathcal{F}_x$;

(ii) if $H \in \mathcal{F}_x$ then for any arrow $g: x \rightarrow x'$ it is $gHg^{-1} \in \mathcal{F}_{x'}$.

**Remark 4.12.** If $\mathcal{G}$ has a single object, i.e. if $\mathcal{G}$ is a group $G$ regarded as a category with a single object, Definition 4.11 recovers the usual definition of a family of subgroups of $G$ as a collection of subgroups $H \leq G$ closed under inclusion and conjugation.

Moreover, if $\mathcal{F}$ is a family of subgroups of $\mathcal{G}$, each $\mathcal{F}_x$ is a family of subgroups of $\text{Aut}(x)$ in the usual sense. In fact, $\mathcal{F}$ is completely determined by a choice of families $\mathcal{F}_x$ for $x$ ranging over a set of representatives of the isomorphism classes/components of $\mathcal{G}$.

**Definition 4.13.** Let $\mathcal{V}$ be a model category, $\mathcal{G}$ a groupoid and $\mathcal{F}$ a family of subgroups of $\mathcal{G}$.

The $\mathcal{F}$-model structure on $\mathcal{V}_\mathcal{F}^\mathcal{G} = \text{Fun}(\mathcal{G}, \mathcal{V})$, which we denote by $\mathcal{V}_\mathcal{F}^\mathcal{G}$, is the model structure (if it exists) such that $f: X \rightarrow Y$ is a weak equivalence (resp. fibration) if the maps $f(x)^H: X(x)^H \rightarrow Y(x)^H$ are weak equivalences (fibrations) in $\mathcal{V}$ for all $x \in \mathcal{G}$ and $H \in \mathcal{F}_x$.

**Remark 4.14.** Generalizing (4.4), one has that, if $\mathcal{V}$ is cofibrantly generated and the model structure $\mathcal{V}_\mathcal{F}^\mathcal{G}$ exists, then the latter is likewise cofibrantly generated with generating sets

$$\mathcal{I}_\mathcal{F} := \{ \mathcal{G}(x,-)/H \cdot i | x \in \mathcal{G}, H \in \mathcal{F}_x, i \in \mathcal{I} \} \quad \mathcal{J}_\mathcal{F} := \{ \mathcal{G}(x,-)/H \cdot j | x \in \mathcal{G}, H \in \mathcal{F}_x, j \in \mathcal{J} \}.$$

(4.15)
Remark 4.16. Since, for any groupoid, one has an equivalence of categories $\mathcal{G} \cong \coprod_{x \in \text{ob}(\mathcal{G})} \text{Aut}(x)$, (trivial) cofibrations in $\mathcal{V}_F^\mathcal{G}$ also admit a pointwise description: a map $f$ in $\mathcal{V}_F^\mathcal{G}$ is a (trivial) cofibration iff $f(x)$ is a (trivial) cofibration in $\mathcal{V}_{F_x}^{\text{Aut}(x)}$ for all $x \in \mathcal{G}$.

Proposition 4.17. Suppose $\mathcal{V}$ is cofibrantly generated. Then the $\mathcal{V}_F^\mathcal{G}$ model structures exist for all groupoids $\mathcal{G}$ and families $\mathcal{F}$ if and only if the genuine model structures on $\mathcal{V}_{\text{Aut}}^\mathcal{G}$ exist for all groups $\mathcal{G}$.

In particular, the $\mathcal{V}_F^\mathcal{G}$ model structures exist whenever $\mathcal{V}$ has weak acyclic cellular fixed points.

Proof. For the main claim, only the “if” direction requires proof. Much as in Proposition 4.10(i), we consider $[\mathcal{F}]$, where the middle inclusion follows by the existence of the genuine model structure on $\mathcal{V}_{\text{Aut}}^\mathcal{G}$. This now follows since

$$\mathcal{F}_x\text{-cell} \subseteq \mathcal{F}_{\text{Aut}(x)}\text{-cell} \subseteq \mathcal{W}_{\text{Aut}(x)} \subseteq \mathcal{W}_x$$

where the middle inclusion follows by the existence of the genuine model structure on $\mathcal{V}_{\text{Aut}(x)}$.

The “in particular” claim now reduces to Proposition 4.10(i).

Remark 4.19. The families of subgroups $\mathcal{F}$ of a groupoid $\mathcal{G}$ form a poset (in fact lattice) under inclusion. Given a map of groupoids $\phi: \mathcal{G} \to \bar{\mathcal{G}}$ and family $\bar{\mathcal{F}}$ of subgroups of $\bar{\mathcal{G}}$ we have a “pullback family” $\phi^* \bar{\mathcal{F}}$ of subgroups of $\mathcal{G}$ given by

$$\left(\phi^* \bar{\mathcal{F}}\right)_x = \left\{ H \subseteq \text{Aut}(x) \mid \phi(H) \in \bar{\mathcal{F}}_{\phi(x)} \right\}.$$

The purpose of the $\phi^* \bar{\mathcal{F}}$ families is given by the following, which adapts [BP21, Props. 6.5, 6.6] in light of Remark 4.16.

Proposition 4.21. Suppose all the model structures appearing in (4.22) exist.

Let $\phi: \mathcal{G} \to \bar{\mathcal{G}}$ be a map of groupoids and $\mathcal{F}, \bar{\mathcal{F}}$ families of subgroups of $\mathcal{G}, \bar{\mathcal{G}}$. Then the adjunctions

$$\phi_\ast: \mathcal{V}_F^\mathcal{G} \rightleftarrows \mathcal{V}_F^{\bar{\mathcal{G}}}: \phi^* \quad \phi^*: \mathcal{V}_{\bar{\mathcal{F}}}^{\bar{\mathcal{G}}} \rightleftarrows \mathcal{V}_{\mathcal{F}}^{\mathcal{G}}: \phi_\ast$$

are Quillen provided $\mathcal{F} \subseteq \phi^* \bar{\mathcal{F}}$ for the left adjunction and $\phi^* \bar{\mathcal{F}} \subseteq \mathcal{F}$ for the right adjunction.

We next discuss the interactions of equivariant model structures on $\mathcal{V}$ with the monoidal structure $\otimes$ on $\mathcal{V}$.

Definition 4.23. Extending Definition 4.6, we write $\mathcal{F}_\otimes^\mathcal{G} = \mathcal{F} \otimes \mathcal{V}^\mathcal{G} = \{ j \otimes X \mid j \in \mathcal{F}, X \in \mathcal{V}^\mathcal{G}\}$ and refer to the class of maps $\mathcal{J}_\otimes^\mathcal{G}$-cof in $\mathcal{V}^\mathcal{G}$ as the $\mathcal{G}$-genuine $\otimes$-trivial cofibrations.

Note that, as in Remark 4.16, $f$ is a $\mathcal{G}$-genuine $\otimes$-trivial cofibration if and only if $f(x)$ is a $\text{Aut}(x)$-genuine $\otimes$-trivial cofibration for all $x \in \mathcal{G}$.

Proposition 4.24. Both left adjoints in (4.22) send genuine $\otimes$-trivial cofibrations to genuine $\otimes$-trivial cofibrations.

Proof. This follows from the identifications $\phi_\ast (j \otimes X) \simeq j \otimes \phi_\ast (X)$ and $\phi^* (j \otimes X) \simeq j \otimes \phi^* (X)$.

The following is immediate from [BP21, Rem. 6.14] and Remark 4.16.
Proposition 4.25. Suppose \((\mathbb{V}, \otimes)\) is a closed monoidal model category which is cofibrantly generated, and also that the model structures appearing below exist. Further, let \(\mathcal{G}, \mathcal{G}'\) be groupoids and \(\mathcal{F}, \mathcal{F}'\) families of subgroups of \(\mathcal{G}, \mathcal{G}'\). Then \(\otimes\) includes a left Quillen bifunctor
\[
\mathbb{V}^\mathcal{G}_\mathcal{F} \times \mathbb{V}^\mathcal{G}'_{\mathcal{F}'} \to \mathbb{V}^{\mathcal{G} \times \mathcal{G}'}_{\mathcal{F} \cap \mathcal{F}'}
\]
with the family \(\mathcal{F} \cap \mathcal{F}'\) of \(\mathcal{G} \times \mathcal{G}'\) is defined as follows (for \(\pi_i^G, \mathcal{G} \times \mathcal{G}' \to \mathcal{G}, \pi_i^G, \mathcal{G} \times \mathcal{G}' \to \mathcal{G}\) the projections)
\[
(\mathcal{F} \cap \mathcal{F}')_{(x, x')} = \pi_i^G(\mathcal{F}_x) \cap \pi_i^{G'}(\mathcal{F}'_{x'}) = \{K \leq \text{Aut}(x, x') \mid \pi_i(\mathcal{G})(K) \in \mathcal{F}_x, \pi_i^{G'}(K) \in \mathcal{F}'_{x'}\}.
\]

Unpacking the definition of left Quillen bifunctor, Proposition 4.25 says that, if maps \(f, \bar{f}\) in \(\mathbb{V}^\mathcal{G}_\mathcal{F}\), \(\mathbb{V}^\mathcal{G}'_{\mathcal{F}'}\) are \(\mathcal{F}, \mathcal{F}'\) cofibrations, then the pushout product map \(f \Box \bar{f}\) in \(\mathbb{V}^{\mathcal{G} \times \mathcal{G}'}_{\mathcal{F} \cap \mathcal{F}'}\) (defined in e.g. [Rie14, 11.1.7]) is a \(\mathcal{F} \cap \mathcal{F}'\)-cofibration, which is trivial if either \(f\) or \(\bar{f}\) are.

Note, however, that should \(\otimes\) be a symmetric monoidal structure, then when \(\mathcal{G} = \mathcal{G}'\) and \(\bar{f} = f\) the map \(f \Box \bar{f}\) admits an additional \(\Sigma_2\)-action (and, more generally, \(f^{\mathcal{G}}\) admits a \(\Sigma_n\)-action) which is ignored by Proposition 4.25. To discuss such “actions on powers” we need a few preliminaries, starting with the following additional hypothesis on \(\mathcal{V}\).

Definition 4.26 ([BP21, Def. 6.16]). We say a symmetric monoidal model category \(\mathcal{V}\) has cofibrant symmetric pushout powers if, for all (trivial) cofibrations \(f\), the pushout product power \(f^{\mathcal{G}_n}\) is a \(\Sigma_n\)-genuine (trivial) cofibration in \(\mathbb{V}^\mathcal{G}_n\) for all \(n \geq 1\).

Remark 4.27. We purposely excluded the \(n = 0\) case in Definition 4.26.

Unwinding definitions, one sees that for any map \(f\), \(f^{\mathcal{G}_0}\) is always the map \(\varnothing \to 1_\mathcal{V}\) from the initial object to the monoidal unit (indeed, \(\Box\) determines a monoidal structure on the category of arrows, with \(\varnothing \to 1\) being the unit). However, while we assume that \(1_\mathcal{V}\) is cofibrant at certain points in this work, we never wish to assume that \(\varnothing \to 1_\mathcal{V}\) is a trivial cofibration.

Given a map \(f\) in \(\mathbb{V}^\mathcal{G}\) we write \(f^{\mathcal{G}_n}\) and \(f^{\mathcal{G}_n}\) for the maps in \(\mathbb{V}^{\mathcal{G}_n\mathcal{G}}\) given by
\[
f^{\mathcal{G}_n}((x_i)_{1 \leq i \leq n}) = \Box_{1 \leq i \leq n} f(x_i), \quad f^{\mathcal{G}_n}((x_i)_{1 \leq i \leq n}) = \bigotimes_{1 \leq i \leq n} f(x_i).
\]

Next, we write
\[
\pi_{\mathcal{G}_n} : \Sigma_n \times \mathcal{G}^{\times n} \to \mathcal{G}_n \quad \pi_i^G : \Sigma_n \times \mathcal{G}^{\times n} \to \mathcal{G}, 1 \leq i \leq n
\]
for the projections onto each coordinate. We warn that, while \(\pi_{\mathcal{G}_n}\) is a map of groupoids, the \(\pi_i^G\) are not. Nonetheless, writing \(\Sigma_n \leq \Sigma_n\) for the subgroup of permutations that fix \(i\), one has that \(\pi_i^G\) is a homomorphism when restricted to \(\pi_i^{\Sigma_n} (\Sigma_n)\) (indeed, one has an isomorphism \(\pi_i^{\Sigma_n} (\Sigma_n) \cong (\Sigma_1 \times \mathcal{G}) \times (\Sigma_{n-1} \times \mathcal{G})\), which identifies \(\pi_i^G\) with the projection to \((\Sigma_1 \times \mathcal{G}) \cong \mathcal{G}\). Given \((x_i) \in \Sigma_n \times \mathcal{G}\) and a subgroup \(H \leq \text{Aut}(\Sigma_n)\), we then write \(H_i = H \cap \pi_i^{\Sigma_n} (\Sigma_n)\) for the subgroup of \(H\) whose projection to \(\Sigma\) fixes \(i\).

Given a family \(\mathcal{F}\) of subgroups of \(\mathcal{G}\), we can now finally define the family \(\mathcal{F}^{\times n}\) of subgroups of \(\Sigma_n \times \mathcal{G}\) for \(n \geq 1\) by
\[
(\mathcal{F}^{\times n})_{(x_i)} = \{H \leq \text{Aut}(\Sigma_n) \mid \pi_i^G (H_i) \in \mathcal{F}_x, 1 \leq i \leq n\}.
\]

Note that this construction is compatible with pullbacks along a functor \(\phi : \mathcal{G} \to \mathcal{G}'\), in the sense that
\[
(\Sigma_n \times \phi)^* \mathcal{F}^{\times n} = (\phi^* \mathcal{F})^{\times n}.
\]
Lemma 4.31 (cf. [BP21, Prop. 6.23]). Let $\mathcal{F}$ be a family of subgroups in the groupoid $\mathcal{G}$ and write $\iota: (\Sigma_n \triangleleft \mathcal{G}) \times (\Sigma_m \triangleleft \mathcal{G}) \to \Sigma_{n+m} \triangleleft \mathcal{G}$ for the standard inclusion. Then

$$\mathcal{F}^{\Sigma_n} \cap \mathcal{F}^{\Sigma_m} \subseteq \iota^* \mathcal{F}^{\Sigma_{n+m}}$$

(4.32)

so that the composite

$$\mathcal{V}^{\Sigma_n \triangleleft \mathcal{G}} \times \mathcal{V}^{\Sigma_m \triangleleft \mathcal{G}} \xrightarrow{\otimes} \mathcal{V}^{(\Sigma_n \triangleleft \mathcal{G}) \times (\Sigma_m \triangleleft \mathcal{G})} \xrightarrow{\iota^*} \mathcal{V}^{\Sigma_{n+m} \triangleleft \mathcal{G}}$$

(4.33)

is a left Quillen bifunctor.

Proof. To ease notation, we write $\{1, \ldots, n + m\} = \{1, \ldots, n\} \cup \{1, \ldots, m\}$, and allow $1 \leq i \leq n$ to range over the first summand and $1 \leq j \leq m$ to range over the second summand.

Note now that, if $H$ is a group of automorphisms in $(\Sigma_n \triangleleft \mathcal{G}) \times (\Sigma_m \triangleleft \mathcal{G})$, then $\pi_n^i(H) = \pi_n^j((\pi_{\Sigma_n}(g(H)))$ and $\pi_m^i(H) = \pi_m^j((\pi_{\Sigma_m}(g(H)))$, so that (4.32) is immediate from definition of $\mathcal{F} \cap \mathcal{F}$.

That (4.33) is a Quillen bifunctor simply combines Propositions 4.21 and 4.25.

We now have the following, which is a strengthening of [BP21, Prop. 6.25].

Proposition 4.34. Suppose $(\mathcal{V}, \otimes)$ is as in Proposition 4.25 and, in addition, that it has cofibrant symmetric pushout powers. Further, let $\mathcal{G}$ be a groupoid and $\mathcal{F}$ a family of subgroups of $\mathcal{G}$. Then:

(i) if $f$ is a (trivial) $\mathcal{F}$-cofibration in $\mathcal{V}$ then $f^{\otimes n}$ is a (trivial) $\mathcal{F}^{\otimes n}$-cofibration in $\mathcal{V}^{\otimes n \triangleleft \mathcal{G}}$;

(ii) if $f$ is a (trivial) $\mathcal{F}$-cofibration between $\mathcal{F}$-cofibrant objects in $\mathcal{V}$ then $f^{\otimes n}$ is a (trivial) $\mathcal{F}^{\otimes n}$-cofibration between $\mathcal{F}^{\otimes n}$-cofibrant objects in $\mathcal{V}^{\otimes n \triangleleft \mathcal{G}}$.

Proof. We first prove the result when $\mathcal{G} = \mathcal{G}$ is in fact a group. In this case, (i) is almost exactly [BP21, Prop. 6.25], except for the fact that throughout [BP21] we assume $\mathcal{V}$ has cellular fixed points (i.e. that it satisfies the conditions in [Ste16, Prop. 2.6], which are a stronger version of Definition 4.9), so we must check that this assumption is not needed in the proof of [BP21, Prop. 6.25]. Analyzing the proof therein, one sees that the only model structure requirements are the cofibrancy of pushout powers and that the functors in [BP21, Props 6.5 and 6.23] are left Quillen (bi)functors. Noting that the latter results are generalized by Proposition 4.21 and Lemma 4.31 yields (i).

For (ii) when $\mathcal{G} = \mathcal{G}$ is a group, we need to recall an argument in the proof of [BP21, Prop. 6.25]. Given composable arrows $Z_0 \xrightarrow{g} Z_1 \xrightarrow{f} Z_2$ in $\mathcal{V}$, denote by $Q^n(g), Q^n(f)$ the domains of $g^{\otimes n}, f^{\otimes n}$. There is a filtration of the box product of the composite $(fg)^{\otimes n}$ as

$$Q^n(fg) \xrightarrow{k_0} \cdots \xrightarrow{k_1} \cdots \xrightarrow{k_{n-1}} \cdots \xrightarrow{k_n} Z_2^{\otimes n}$$

where each $k_r, 0 \leq r \leq n$ is given by a pushout as on the left below, with the right diagram specifying the $k_0$ case (this filtration is induced by a $\Sigma_n$-equivariant filtration $P_0 \subseteq P_1 \subseteq \cdots \subseteq P_n$ of the poset $P_n = (0 \to 1 \to 2)^{\otimes n}$, where $P_0$ consists of the tuples with at least one 0-coordinate and tuples in $P_r$ have at most $r$ 2-coordinates; for more details, see the proof of [Per16, Lemma 4.8])

Specifying to the case $Z_0 = \emptyset$, one has $Q^n(g) = Q^n(fg) = \emptyset$ so that $f^{\otimes n}: Z_1^{\otimes n} \to Z_2^{\otimes n}$ is

$$Z_1^{\otimes n} \xrightarrow{k_1} \cdots \xrightarrow{k_2} \cdots \xrightarrow{k_{n-1}} \cdots \xrightarrow{k_n} Z_2^{\otimes n}.$$
Part (i) of the result now implies that \( g^{\le r} \) is a \( \mathcal{F}^{\le r}, \text{cofibration} \) and \( f^{\le r} \) is a \( \mathcal{F}^{\le r}, \text{-(trivial) cofibration} \) (the trivial case uses \( r \ge 1 \)), so (ii) for \( \mathcal{G} = \mathcal{G} \) a group follows from Lemma 4.31.

We now prove the result for general groupoids. Since any groupoid is equivalent to a disjoint union of groups, we may reduce to that case, i.e., we may assume that two objects of \( \mathcal{G} \) are isomorphic iff they are equal.

Given \( (x_i) \in \Sigma_k \times \mathcal{G}^{\lambda} \), we now need to check that the maps in (4.28) are (trivial) \( (\mathcal{F}^{\le n})_{(x_i)} \)-cofibrations. Form the partition \( \{1, \cdots, n\} = \lambda_1 \cdots \lambda_k \) such that \( i, j \) are in the same piece iff \( x_i = x_j \), and write \( n_l = |\lambda_l| \). Writing \( x_{\lambda_l} \) for the common value of the \( x_i \) with \( i \in \lambda_l \), we then have

\[
\begin{align*}
    f^{\le n}((x_i)_{1 \le i \le n}) &= \bigotimes_{1 \le i \le k} f(x_{\lambda_i})^{\le n_{\lambda_i}}, \\
    f^{\le n}((x_i)_{1 \le i \le n}) &= \bigotimes_{1 \le i \le k} f(x_{\lambda_i})^{\le n_{\lambda_i}}.
\end{align*}
\]

Writing \( G_l \) for the automorphism group of \( x_{\lambda_l} \), the group case shown above shows that the \( f(x_{\lambda_i})^{\le n_{\lambda_i}}, f(x_{\lambda_l})^{\le n_{\lambda_l}} \) are (trivial) \( \mathcal{F}^{\le n_{\lambda_l}} \)-cofibrations in \( \mathcal{V}^{\le n_{\lambda_l}} \), the latter with \( \mathcal{F}^{\le n_{\lambda_l}} \)-cofibrant domain. Next, note that for any subgroup \( H \le \text{Aut}_{\Sigma_n}((x_i)) \) the projection \( \pi_{\Sigma}(H) \) must preserve the partition (or else there would be distinct \( x_i \) which are isomorphic) so that, writing \( \pi_{\Sigma} : \Sigma_n \times \mathcal{G}^{\lambda_1} \rightarrow \mathcal{G}^{\lambda_1} \) for the projections, one has that the \( \pi_{\mathcal{G}^{\lambda_1}} \) are homomorphisms when restricted to \( H \). The result now follows by Lemma 4.31 and the observation that \( H \in (\mathcal{F}^{\le n})_{(x_i)} \) iff \( \pi_{\mathcal{G}^{\lambda_1}}(H) \in (\mathcal{F}^{\le n})_{(x_i)} \) for all \( l \).

**Remark 4.35.** If one focuses exclusively on cofibrations and ignores statements concerning trivial cofibrations, Proposition 4.34 actually subsumes Proposition 4.25. Indeed, by considering the disjoint union groupoid \( \mathcal{G} \cup \mathcal{G} \) with the disjoint union family \( \mathcal{F} \cup \mathcal{F} \), one can check that, for \( x \in \mathcal{G}, x \in \mathcal{G} \), one has that for the tuple \( (x, x) \in \Sigma_2 i (\mathcal{G} \cup \mathcal{G}) \) it is \( \left( (\mathcal{F} \cup \mathcal{F})^{\times 2} \right)_{(x, x)} = \mathcal{F} \cap \mathcal{F} \).

## 5 Model structures on equivariant operads with fixed set of colors

The goal of this section is to prove our main results, Theorems I and II.

As discussed in the introduction to §2, in the non-equivariant context the model structures on color fixed operads \( \mathcal{O}_{\mathcal{C}}(\mathcal{V}) \) are obtained by via transfer from model structures on the categories \( \mathcal{S}_{\mathcal{C}}(\mathcal{V}) \) of color fixed symmetric sequences. Likewise, our model structures on color fixed equivariant operads \( \mathcal{O}_{\mathcal{C}}(\mathcal{V}) \) will be transferred from model structures on color fixed equivariant symmetric sequences \( \mathcal{S}_{\mathcal{C}}(\mathcal{V}) \) where, just as in (1.6), the weak equivalences in \( \mathcal{S}_{\mathcal{C}}(\mathcal{V}) \) are determined by a \( (G, \Sigma) \)-family (cf. Definition 1.4). This section is then organized as follows.

In §5.1 we simply translate the work in §4.2 to the categories \( \mathcal{S}_{\mathcal{C}}(\mathcal{V}) \cong \mathcal{O}_{\mathcal{C}}(\mathcal{V}) \).

Then in §5.2 we prove Theorem I by transferring the model structures on \( \mathcal{S}_{\mathcal{C}}(\mathcal{V}) \) to obtain model structures on \( \mathcal{O}_{\mathcal{C}}(\mathcal{V}) \). The key ingredients to this proof are Proposition 4.34 in §4.2 and the filtrations of free operad extensions in Lemma 5.8 (whose proof is deferred to Appendix A).

Lastly, §5.3 proves Theorem II via a more careful analysis of the filtrations in the proof of Theorem I.

### 5.1 Homotopy theory of symmetric sequences with a fixed color \( G \)-set

Following the identification \( \mathcal{S}_{\mathcal{C}}(\mathcal{V}) \cong \mathcal{O}_{\mathcal{C}}(\mathcal{V}) \) in Proposition 3.17, we now apply the framework from §4.2 to the groupoids \( \mathcal{G} \cong \mathcal{G} \times \mathcal{S}_{\mathcal{C}} \) to build model structures on the categories \( \mathcal{S}_{\mathcal{C}}(\mathcal{V}) \).

We now recall and elaborate on the \( (G, \Sigma) \)-families in Definition 1.4. Notably, the families \( \mathcal{F}_{\mathcal{C}} \) below are such that change of colors \( \varphi : \mathcal{C} \rightarrow \mathcal{D} \) induce Quillen adjunctions, cf. Corollary 5.6(i).
**Definition 5.1.** A \((G, \Sigma)\)-family is a family of subgroups \(\mathcal{F}\) of the groupoid \(G \times \Sigma\); i.e. a collection of families \(\mathcal{F}_n\) of the groups \(G \times \Sigma\) for each \(n \geq 0\).

Moreover, given a \((G, \Sigma)\)-family \(\mathcal{F}\) and a color \(G\)-set \(\mathcal{C} \in \text{Set}^G\), we define the family \(\mathcal{F}_\mathcal{C}\) in \(G \times \Sigma\) by \(\mathcal{F}_\mathcal{C} = \pi \mathcal{C}(\mathcal{F})\), with \(\pi \mathcal{C}: G \times \Sigma \to G \times \Sigma\) the canonical forgetful functor.

**Remark 5.2.** The functors \(\pi \mathcal{C}: G \times \Sigma \to G \times \Sigma\) are faithful and can thus be regarded as inclusions on hom sets. Thus, letting \(\mathcal{C} \in \Sigma\) be a \(\mathcal{C}\)-colored corolla with \(n\) leaves, one has that \(\mathcal{F}_\mathcal{C} = \{\mathcal{F}_n\}\) where

\[
\mathcal{F}_\mathcal{C} = \mathcal{F}_n \cap \text{Aut}_{G \times \Sigma}(\mathcal{C}).
\]

Alternatively, following Definition 3.37, \(\mathcal{F}_\mathcal{C}\) consists of the \(\Lambda \in \mathcal{F}_n\) such that \(\Lambda\) stabilizes \(\mathcal{C}\).

In this paper and the sequel [BPa], we will be interested in three main examples of \((G, \Sigma)\)-families:

(a) First, there is the family \(\mathcal{F}_{all}\) of all the subgroups of \(G \times \Sigma\) (in which case the \(\mathcal{F}_{all, \mathcal{C}}\) are also the families of all subgroups), which is useful mainly for technical purposes.

(b) Secondly, there is the family of \(\mathcal{F}_G^\mathcal{T}\) of G-graph subgroups (e.g. [BP21, Def. 6.36]), where \(\mathcal{F}_G^\mathcal{T}\) consists of the subgroups \(\Gamma \leq G \times \Sigma\) such that \(\Gamma \cap \Sigma = \{\ast\}\). We note that the elements of such \(\Gamma\) have the form \((h, \phi(h)^{-1})\) for \(h\) ranging over some subgroup \(H \leq G\) and \(\phi: H \to \Sigma\), a homomorphism, motivating the “graph subgroup” terminology.

Although secondary for our work in the current paper, we regard \(\mathcal{F}_G^\mathcal{T}\) as the “canonical choice” of \((G, \Sigma)\)-family, as this is the family featured in the Quillen equivalence \(W: \text{dSet}^G \cong \text{dSet}^G: heN\) in [BPa, Thm. 1].

(c) Lastly, there are the indexing systems of Blumberg and Hill, which are special subfamilies of \(\mathcal{F}_G^\mathcal{T}\) which share the key technical properties of \(\mathcal{F}_G^\mathcal{T}\) itself, and are discussed in §5.3.

**Example 5.3.** Let \(G = \mathbb{Z}/2 = \{\pm 1\}\) and \(\mathcal{C} = \{a, -a, b\}\) where we implicitly have \(-b = b\). Consider the two \(\mathcal{C}\)-corollas \(\mathcal{C}, \mathcal{D} \in \Sigma\) below.

The non-trivial \(G\)-graph subgroups of \(\mathcal{F}_G^\mathcal{T}, \mathcal{F}_G^\mathcal{T}\) then correspond to the possible \(\mathbb{Z}/2\)-actions on the underlying trees \(C, D\) which are compatible with the action on labels (in the sense that the composites \(E(C) \to E(C) \to \mathcal{C}\) and \(E(C) \to \mathcal{C}\) coincide). In this particular case, both \(\mathcal{F}_G^\mathcal{T}, \mathcal{F}_G^\mathcal{T}\) have exactly two non-trivial groups, which correspond to the \(\mathbb{Z}/2\)-actions on the underlying corollas that are depicted below.
Definition 5.4. Let $\mathcal{C}$ be a $G$-set and $\mathcal{F}$ a $(G, \Sigma)$-family. Then the $\mathcal{F}$-model structure (if it exists) on the fiber $\text{Sym}^G_{\mathcal{F}}(V)$ of $\text{Sym}(V)$ over $\mathcal{C}$ is the $\mathcal{F}_\mathcal{C}$-model structure

$$\text{Sym}^G_{\mathcal{F}, \mathcal{C}}(V) = \mathcal{V}^{G\times\Sigma\mathbb{C}}_{\mathcal{F}, \mathcal{C}}$$

where $\mathcal{F}_\mathcal{C}$ is as in Definition 5.1. Explicitly, a map $X \to Y$ in $\text{Sym}^G_{\mathcal{F}, \mathcal{C}}(V)$ is a weak equivalence (resp. fibration) if $X(\mathcal{C})^A \to Y(\mathcal{C})^A$ is so in $\mathcal{V}$ for all $\mathcal{C}$-profiles $\mathcal{C}$ and $A \in \mathcal{F}_\mathcal{C}$.

When $\mathcal{F} = \mathcal{F}_{\text{all}}$ is the family of all subgroups, we refer to this model structure simply as the genuine model structure on $\text{Sym}^G_{\mathcal{F}}(V)$.

Remark 5.5. If $\mathcal{V}$ is cofibrantly generated, combining (4.15) with the $\Sigma_{\mathcal{E}}[G \cdot \mathcal{C}]$ notation for the representable functors in Proposition 3.35, one has that the generating (trivial) cofibrations of $\text{Sym}^G_{\mathcal{F}, \mathcal{C}}(V)$ are the sets of maps

$$\mathcal{I}_{\mathcal{E}, \mathcal{F}} = \left\{ \Sigma_{\mathcal{E}}[G \cdot \mathcal{C}] / \Lambda \cdot i \right\} \quad \mathcal{J}_{\mathcal{E}, \mathcal{F}} = \left\{ \Sigma_{\mathcal{E}}[G \cdot \mathcal{C}] / \Lambda \cdot j \right\}$$

where $\mathcal{C}$ ranges over $\Sigma_{\mathcal{E}}$, $\Lambda$ ranges over $\mathcal{F}_\mathcal{C}$, $i$ ranges over $\mathcal{I}$ and $j$ ranges over $\mathcal{J}$.

Corollary 5.6. Suppose the model structures $\text{Sym}^G_{\mathcal{F}, \mathcal{C}}(V)$ exist for all $(G, \Sigma)$-families $\mathcal{F}$ and $G$-sets $\mathcal{C}$.

(i) For any $(G, \Sigma)$-family $\mathcal{F}$ and map of colors $\phi: \mathcal{C} \to \mathcal{D}$ the induced adjunction

$$\phi_*: \text{Sym}^G_{\mathcal{F}, \mathcal{C}}(V) \rightleftarrows \text{Sym}^G_{\mathcal{F}, \mathcal{D}}(V): \phi^*$$

is a Quillen adjunction.

(ii) For any homomorphism $\phi: G \to \bar{G}$ (and writing $\phi: G \times \Sigma^{\text{op}} \to \bar{G} \times \Sigma^{\text{op}}$ for the induced homomorphism), $(G, \Sigma)$-family $\mathcal{F}$ and $(\bar{G}, \Sigma)$-family $\bar{\mathcal{F}}$, and $G$-set of colors $\mathcal{C}$, the adjunction

$$\phi_*: \text{Sym}^G_{\mathcal{F}, \mathcal{C}}(V) \rightleftarrows \text{Sym}^G_{\bar{\mathcal{F}}, \mathcal{C}}(V): \phi^*$$

is a Quillen adjunction whenever $\mathcal{F} \subseteq \phi^* \bar{\mathcal{F}}$.

(iii) For any homomorphism $\phi: G \to \bar{G}$, $(G, \Sigma)$-family $\mathcal{F}$ and $(\bar{G}, \Sigma)$-family $\bar{\mathcal{F}}$, and $G$-set of colors $\mathcal{C}$, the adjunction

$$\bar{G} \cdot (-): \text{Sym}^G_{\mathcal{F}, \mathcal{C}}(V) \rightleftarrows \text{Sym}^G_{\bar{G} \cdot \mathcal{C}, \mathcal{F}}(V): \text{fgt}$$

is a Quillen adjunction whenever $\mathcal{F} \subseteq \phi^* \bar{\mathcal{F}}$.

Proof. Parts (i) and (ii) are immediate from Proposition 4.21. Part (iii) follows by combining (i) applied to the map of $G$-sets $\mathcal{C} \to \bar{G} \cdot \mathcal{C}$ with (ii) applied to the $G$-set $\bar{G} \cdot \mathcal{C}$. 

5.2 Homotopy theory of operads with a fixed color $G$-set

Recall that $\mathbb{P}^G_{\mathcal{C}}$ denotes the fiber of the monad $\mathbb{P}^G$ for $\mathcal{C} \in \text{Set}^G$ (see Proposition 3.47). Our goal in this section is to prove Theorem 1 by transferring the $\mathcal{F}$-model structures on $\text{Sym}^G_{\mathcal{F}}(V)$ from Definition 5.4 to $\text{Op}^G_{\mathcal{F}}(V)$ along the free-forgetful adjunction

$$\mathbb{P}^G_{\mathcal{C}}: \text{Sym}^G_{\mathcal{F}}(V) \rightleftarrows \text{Op}^G_{\mathcal{F}}(V): \text{fgt}$$
so that a map in \( \text{Op}_{G}^{C}(V) \) is a weak equivalence/fibration iff so is the underlying map in \( \text{Sym}_{G}^{C}(V) \).

The key to proving Theorem I will be a suitable filtration, given by Lemma 5.8, of free operad extensions in \( \text{Op}_{G}^{C}(V) \), i.e. of pushouts as in (5.9) below. The following discussion and lemma summarizes the key properties of the filtration we will need.

We denote by \( \Omega_{C}^{a} \) the \( C \)-colored variant of the alternating trees \( \Omega^{a} \) of [BP21, Def. 5.52] (formally, \( \Omega_{C}^{a} \) is defined by copying Definition 3.21 but with \( F, \rho : F \to F' \) therein now in \( \Omega^{a} \)). Its objects are trees \( T \) whose set of vertices is partitioned into “active” and “inert” vertices, \( V(T) = V^{ac}(T) \sqcup V^{in}(T) \), in such a way that adjacent vertices are in different sides of the partition and the “outer vertices” (i.e. those adjacent to a leaf or root) are active. Its maps are informally described as tall maps of trees that “send inert vertices to inert vertices”.

**Example 5.7.** Below we depict a planar alternating map \( f : T \to S \) between alternating trees \( S, T \in \Omega^{a} \). Active vertices are black \( \bullet \) and inert vertices are white \( \circ \). Pictorially, active vertices \( \bullet \) can be expanded to an alternating tree of the same arity, while inert vertices \( \circ \) are preserved.

![Diagram of alternating trees](#)

Furthermore, we write \( \Omega_{C}^{a}[k] \subseteq \Omega_{C}^{a} \) for the full subcategory of \( \bar{T} \) such that \( |V^{in}(\bar{T})| = k \), i.e. such that \( \bar{T} \) has exactly \( k \) inert vertices. Crucially, \( \Omega_{C}^{a}[k] \) turns out to be a groupoid (indeed, maps in \( \Omega_{C}^{a}[k] \) can only replanarize vertices, or else the number of \( \circ \) vertices would increase). Adapting (3.38) and (3.43), we have functors

\[
\text{lr}^{C} : \Omega_{C}^{a}[k] \to \Sigma \quad \text{V}^{in}: \Omega_{C}^{a}[k] \to \Sigma \wr \Sigma
\]

where \( \text{lr}^{C} \) is defined as before (one just ignores the vertex partition data) and \( V^{in} \) is the tuple containing only the inert vertices.

The proof of the following key Lemma is postponed to §A.4 in the Appendix.

**Lemma 5.8.** Fix a \( G \)-set of colors \( C \) and let \( u : X \to Y \) be a map in \( \text{Sym}_{C}^{G}(V) \) and \( FX \to O \) be a map in \( \text{Op}_{C}^{G}(V) \). Then, for the pushout

\[
\begin{array}{ccc}
FX & \longrightarrow & O \\
\downarrow \quad \text{F}\text{u} & & \downarrow \\
FY & \longrightarrow & O[u]
\end{array}
\]

in the category of operads \( \text{Op}_{C}^{G}(V) \) the map \( O \to O[u] \) admits an underlying filtration

\[
O = O_{0} \to O_{1} \to O_{2} \to \cdots \to O_{\infty} = O[u]
\] (5.10)
of maps in $\text{Sym}^{G}_{\mathcal{C}}(V)$ where each map $\mathcal{O}_{k-1} \to \mathcal{O}_k$ fits into a pushout

\[
\begin{array}{c}
\bullet \\
\downarrow
\end{array}
\begin{array}{c}
\mathcal{O}_{k-1} \\
\mathcal{O}_k
\end{array}
\]

(5.11)

for $(l_{\mathcal{C}}^{a,\mathcal{F}}) : \mathcal{V}^{G \ast \Sigma[1]}_{\mathcal{C}} \to \mathcal{V}^{G \ast \Sigma[1]}_{\mathcal{C}} = \text{Sym}^{G}_{\mathcal{C}}(V)$ is as in Proposition 4.21 and $n_k^{(\mathcal{O},X,Y)}$ the natural transformation in $\mathcal{V}^{G \ast \Omega[1]}_{\mathcal{C}}$ whose constituent arrows for $\bar{T} \in \Omega[1]_{\mathcal{C}}$ are

\[
n_k^{(\mathcal{O},X,Y)}(\bar{T}) = \left( \bigotimes_{v \in V^{in}(\bar{T})} \mathcal{O}(\bar{T}_v) \otimes \Box u(\bar{T}_v) \right) = \left( \bigotimes_{v \in V^{in}(\bar{T})} (\varnothing \to \mathcal{O}(\bar{T}_v)) \Box u(\bar{T}_v) \right)
\]

(5.12)

We can now prove our first main result, Theorem I.

Proof of Theorem I. Since we are assuming $\mathcal{V}$ is cofibrantly generated and that the genuine model structure on $\mathcal{V}^{G}$ exists, Propositions 3.17 and 4.17 imply that the model structure $\text{Sym}^{G}_{\mathcal{C}}(V)$ exists for all $(G, \Sigma)$-families $\mathcal{F}$ and $G$-sets $\mathcal{C}$. Unpacking Definition 5.4, the desired $\mathcal{F}$-model structure $\mathcal{O}_{\mathcal{E},\mathcal{F}}^{G}_{\mathcal{C}}(V)$ from Theorem I is the model structure transferred from $\text{Sym}^{G}_{\mathcal{C}}(V)$ along the free-forgetful adjunction. Thus, writing $\mathcal{I}_{\mathcal{E},\mathcal{F}}$, $\mathcal{J}_{\mathcal{E},\mathcal{F}}$ for the generating sets of $\text{Sym}^{G}_{\mathcal{C}}(V)$ (cf. Remark 5.5), the $\mathcal{F}$-model structure $\mathcal{O}_{\mathcal{E},\mathcal{F}}^{G}_{\mathcal{C}}(V)$ will have generating sets $\mathcal{F}^{G}_{\mathcal{E},\mathcal{F}}$, $\mathcal{F}^{G}_{\mathcal{E},\mathcal{F}}$.

By [Hir03, Thm. 11.3.2], we need only show that all maps in $(\mathcal{F}^{G}_{\mathcal{E},\mathcal{F}})$ are $\mathcal{F}$-weak equivalences in $\text{Sym}^{G}_{\mathcal{C}}(V)$. Moreover, since trivial $\mathcal{F}$-cofibrations are genuine trivial cofibrations and genuine weak equivalences are $\mathcal{F}$-weak equivalences, one needs only consider the genuine case, i.e. the case of $\mathcal{F}_{\text{aut}}$ the family of all subgroups (this repeats the argument in (4.18)).

Since we are assuming $\mathcal{V}$ is a symmetric monoidal model category that satisfies the global monoid axiom (Definition 4.6), it suffices to show that pushouts of maps in $(\mathcal{F}^{G}_{\mathcal{E},\mathcal{F}})$, i.e. maps $\mathcal{O} \to \mathcal{O}[u]$ in (5.9) for $u \in \mathcal{J}_{\mathcal{E},\mathcal{F},\text{aut}}$, are equivariant $\boxtimes$-trivial cofibrations in $\text{Sym}^{G}_{\mathcal{C}}(V) = \mathcal{V}^{G \ast \Sigma[1]}_{\mathcal{C}}$ (Definition 4.23).

We thus now let $u \in \mathcal{J}_{\mathcal{E},\mathcal{F},\text{aut}}$. Now note that, for $\bar{T} \in \Omega[1]_{\mathcal{C}}$, we have

\[
\Box u(\bar{T}_v) = u^{G_k}(V^{in}(\bar{T}))
\]

where $u^{G_k} \in \mathcal{V}^{G \ast \Omega[1](G \ast \Sigma[1])}$ is as defined in (4.28). Since we are assuming $\mathcal{V}$ has cofibrant symmetric pushout powers, Proposition 4.34 implies that $u^{G_k}$ is a genuine trivial cofibration in $\mathcal{V}^{G \ast \Omega[1](G \ast \Sigma[1])}$. In turn, Proposition 4.21 implies that $u^{G_k}(V^{in}(\bar{T}))$ is similarly a genuine trivial cofibration in $\mathcal{V}^{G \ast \Omega[1](G \ast \Sigma[1])}$.

It is now clear that the map $n_k^{(\mathcal{O},X,Y)}$ in (5.12) is an equivariant $\boxtimes$-trivial cofibration in $\mathcal{V}^{G \ast \Omega[1](G \ast \Sigma[1])}$, so that by Proposition 4.24 the maps $\mathcal{O}_{k-1} \to \mathcal{O}_k$ in the pushouts (5.11) are genuine $\boxtimes$-trivial cofibrations in $\text{Sym}^{G}_{\mathcal{C}}(V) = \mathcal{V}^{G \ast \Sigma[1]}_{\mathcal{C}}$. Thus the composite $\mathcal{O} \to \mathcal{O}[u]$ is also an equivariant $\boxtimes$-trivial cofibration, and the result follows.

For later reference, we highlight two consequences of the previous proof.

Remark 5.13. $\mathcal{F}$-trivial cofibrations in $\mathcal{O}_{\mathcal{E},\mathcal{F}}^{G}_{\mathcal{C}}(V)$ are underlying genuine $\boxtimes$-trivial cofibrations in $\text{Sym}^{G}_{\mathcal{C}}(V)$.
Remark 5.14. The generating (trivial) cofibrations in $\mathcal{O}_C^G(\mathcal{V})$ are the sets

\[ \mathcal{F}^G_C/I, \mathcal{F} = \left\{ \mathcal{F} \left( \Sigma \mathcal{E}[G \cdot e \tilde{C}] / \Lambda \cdot j \right) \right\} \]

where $\tilde{C}$ ranges over $\Sigma \mathcal{E}$, $\Lambda$ ranges over $\mathcal{F}_C$, $i$ ranges over $I$, and $j$ ranges over $J$.

Corollary 5.15. Suppose the assumptions of Theorem 1 hold, so the model structures discussed in the items below exist.

(i) For any $(G, \Sigma)$-family $\mathcal{F}$ and map of colors $\varphi: \mathcal{E} \to \mathcal{D}$, the induced adjunction

\[ \tilde{\varphi}: \mathcal{O}_C^G(\mathcal{V}) \rightleftarrows \mathcal{O}_D^G(\mathcal{V}): \varphi^* \]

is a Quillen adjunction.

(ii) For any homomorphism $\varphi: G \to \bar{G}$, $(G, \Sigma)$-family $\mathcal{F}$ and $(\bar{G}, \Sigma)$-family $\bar{\mathcal{F}}$, and $\bar{G}$-set of colors $\mathcal{E}$, the adjunction

\[ \tilde{\phi}: \mathcal{O}_C^G(\mathcal{V}) \rightleftarrows \mathcal{O}_{\bar{G}}^G(\mathcal{V}): \phi^* \]

is a Quillen adjunction whenever $\mathcal{F} \subseteq \phi^* \bar{\mathcal{F}}$.

(iii) For any homomorphism $\varphi: G \to \bar{G}$, $(G, \Sigma)$-family $\mathcal{F}$ and $(\bar{G}, \Sigma)$-family $\bar{\mathcal{F}}$, and $\bar{G}$-set of colors $\mathcal{E}$, the adjunction

\[ \bar{G} \cdot \mathcal{O} (\cdot): \mathcal{O}_C^G(\mathcal{V}) \rightleftarrows \mathcal{O}_{\bar{G} \cdot \mathcal{O}_C^G}(\mathcal{V}): \mathcal{F} \mathcal{G} \]

is a Quillen adjunction whenever $\mathcal{F} \subseteq \phi^* \bar{\mathcal{F}}$.

Proof. Since operadic weak equivalences, fibrations and the forgetful functors $\varphi^*, \phi^*, \mathcal{F} \mathcal{G}$ are all defined in terms of the underlying symmetric sequences, this follows from Corollary 5.6.

Remark 5.16. When (5.9) is a diagram in $\text{Cat}^G_C(\mathcal{V})$, the map $n^G_{k,(X,Y)}(\bar{T})$ in (5.12) is necessarily $\emptyset \to \emptyset$ unless $\bar{T}$ is a linear tree. But, if $\bar{T}$ is linear, its automorphism group in $G \times \Omega_{G, 0}^a[k]$ is simply a subgroup of $G$ and does not permute the factors in (5.12). Hence, the key claim in the proof of Theorem 1, i.e. that $n^G_{k,(X,Y)}$ is an equivariant $\otimes$-trivial cofibration in $V^{G \times \Omega_{G, 0}^a}[k]$, follows by replacing the use of Proposition 4.34 with the more elementary Proposition 4.25, and thus does not require the cofibrant pushout powers condition in Definition 4.26 and Theorem 1.

Remark 5.17. If $\mathcal{O}$ in (5.9) is known to be underlying genuine cofibrant (i.e. $\mathcal{F}_{alt}$-cofibrant) in $\text{Sym}^G_C(\mathcal{V})$ then, replacing the use of the global monoid axiom with Proposition 4.25, the argument in the proof of Theorem 1 shows that the maps $\mathcal{O}_{k-1} \to \mathcal{O}_k$ and their composite $\mathcal{O} \to \mathcal{O}_[a]$ are genuine trivial cofibrations (rather than just genuine $\otimes$-trivial cofibrations).

On the other hand, conditions (i),(ii),(iii),(iv) in Theorem 1 suffice to show that if $\mathcal{O}$ is $\mathcal{F}_{alt}$-cofibrant in $\mathcal{O}^G_C(\mathcal{V})$ then it is underlying $\mathcal{F}_{alt}$-cofibrant in $\text{Sym}^G_C(\mathcal{V})$ (this follows either by the proof of Theorem II in §5.3, or by repeating the argument in the proof of Theorem I but now with $u \in I_{C, \mathcal{F}_{alt}}$ a generating cofibration).

Therefore, by [WY18, Thm. 2.2.2], the semi-model structure analogue of Theorem 1 does not require the global monoid axiom (v).

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5.3 Pseudo indexing systems and underlying cofibrancy

Our goal in this subsection is to prove Theorem II, stating that for certain \((G, \Sigma)\)-families \(\mathcal{F}\) the forgetful functor \(\text{Op}_G^\mathcal{F}(V) \to \text{Sym}_G^\mathcal{F}(V)\) preserves cofibrations between cofibrant objects.

Definition 5.18. Let \(\mathcal{F}\) be a \((G, \Sigma)\)-family. We say \(\mathcal{O} \to \mathcal{P}\) in \(\text{Op}_G^\mathcal{F}(V)\) is a \(\Sigma_{\mathcal{F}}\)-cofibration if the underlying map is a cofibration in \(\text{Sym}_G^\mathcal{F}(V)\). Similarly, we say \(\mathcal{O}\) is \(\Sigma_{\mathcal{F}}\)-cofibrant if the map \(\varnothing \to \mathcal{O}\) in \(\text{Sym}_G^\mathcal{F}\) is an \(\mathcal{F}\)-cofibration.

Remark 5.19. Following Remark 4.16, note that \(\mathcal{O} \to \mathcal{P}\) is a \(\Sigma_{\mathcal{F}}\)-cofibration iff \(\mathcal{O}(\widehat{\mathcal{O}}) \to \mathcal{P}(\widehat{\mathcal{O}})\) is an \(\mathcal{F}_{\mathcal{O}}\)-cofibration in \(\text{V}^{\text{Aut}(\mathcal{O})}\) for all \(\mathcal{O}\)-profiles \(\widehat{\mathcal{O}}\).

We now introduce some notation needed to define the pseudo indexing systems in Theorem II.

Notation 5.20. Let \(\mathcal{F}\) be a family of subgroups of the groupoid \(\mathcal{G}\), cf. Definition 4.11. We write \(\mathcal{F}^\ast\) for the family in \(\Sigma : \mathcal{G} = \bigsqcup_{n \geq 0} \Sigma_n \mathcal{G}\) given by the union \(\bigsqcup_{n \geq 0} \mathcal{F}^{\ast n}\), where \(\mathcal{F}^{\ast n}\) for \(n \geq 1\) is as in (4.29), and \(\mathcal{F}^{\ast 0}\) is the non-empty family of \(\Sigma_0\).

Similarly, for a map \(f \in \mathcal{V}^{\mathcal{F}}\), we write \(f^{\mathcal{F}}\) for the map in \(\mathcal{V}^{\Sigma \mathcal{G}}\) which is given by the map \(f^{\mathcal{F}^\ast n}\) in (4.28) on each summand \(\Sigma_n : \mathcal{G}\) with \(n \geq 1\), and by \(f^{\mathcal{F}^\ast 0} = (\varnothing \to 1_\mathcal{V})\) on the summand \(\Sigma_0 : \mathcal{G} = \Sigma_0\).

Notation 5.21. For any \(G\)-set \(\mathcal{E}\), we let \(V_\mathcal{E}\) denote composite below, which is the (opposite of the) composite of the first two horizontal maps in (3.48).

\[
V_\mathcal{E} : G^{op} \times \Omega_\mathcal{E}^0 \xrightarrow{G^{op} \times V} G^{op} \times (\Sigma : \Omega_\mathcal{E}) \xrightarrow{\Delta} \Sigma : (G^{op} \times \Sigma_\mathcal{E})
\]

If \(\mathcal{E} = \ast\), we abuse notation slightly and simply write \(V = V_\ast\).

In the following we slightly abuse notation by conflating a \((G, \Sigma)\)-family \(\mathcal{F}\), i.e. a family in \(G \times \Sigma^{op}\), with its opposite family in \(G^{op} \times \Sigma\).

Definition 5.22. A \((G, \Sigma)\)-family \(\mathcal{F}\) is called a pseudo indexing system if

\[
V^{\ast \mathcal{F}^\ast} \subseteq \text{lr}^{\ast \mathcal{F}},
\]

for \(\mathcal{F}^\ast\) as in Notation 5.20, and \(V^{\ast \mathcal{F}^\ast}, \text{lr}^{\ast \mathcal{F}}\) as defined by (4.20).

Remark 5.23. Pseudo indexing systems generalize the weak indexing systems in our previous work [Per18, Def. 9.5], [BP21, Def. 4.58], [BP20, Def. 6.2] which are themselves a generalization of the indexing systems introduced by Blumberg-Hill [BH15, Def. 3.22].

More precisely, [BP21, Rem. 6.47] implies that a weak indexing system is precisely a pseudo indexing system consisting of \(G\)-graph subgroups (cf. (1.1), [BP21, Def. 6.36]).

Remark 5.24. Unpacking (4.29), one obtains a more explicit description for the family \(V^{\ast \mathcal{F}^\ast}\).

For a tree \(T \in \Omega\) and vertex \(v \in V(T)\), write \(\text{Aut}_v(T) \leq \text{Aut}(T)\) for the subgroup which fixes the root of \(T_v\). One then has a natural homomorphism \(\pi_{T_v} : \text{Aut}_v(T) \to \text{Aut}(T_v)\) (note that \(\pi_{T_v}\) can not be defined on the full group \(\text{Aut}(T)\)). Given a \((G, \Sigma)\)-family \(\mathcal{F}\) and a subgroup \(\Delta \leq G^{op} \times \text{Aut}(T)\), one then has \(\Delta \in (V^{\ast \mathcal{F}^\ast})_T\) iff \(\pi_{T_v}(\Delta \cap (G^{op} \times \text{Aut}_v(T))) \in \mathcal{F}_T\) for all \(v \in V(T)\).

Lastly, we note that the family \(V^{\ast \mathcal{F}^\ast}\) construction generalizes a construction in [BP21]. Following [BP21, Rem. 6.48], if \(\mathcal{F}\) consists of \(G\)-graph subgroups (so that it can be identified with a family of \(G\)-corollas in the sense of [BP21, Def. 4.53]) the family \((V^{\ast \mathcal{F}^\ast})_T\) agrees with the family \(\mathcal{F}_T\) from [BP21, Prop. 6.46].
Remark 5.25. Specifying (5.22) for the stick tree $\eta \in \Omega$, one has that $(V^*F^*)_c$ is the family of all subgroups of $G^{op} \cong G^{op} \times \text{Aut}(\eta)$, while $(\text{lr} F)_c$ is identified with the family $F_1$. Hence, in particular, if $F$ is a pseudo indexing system then $F_1$ must be the family of all subgroups of $G^{op} \cong G^{op} \times \Sigma_1$.

We can now prove the main result of this section, which is a refinement of Theorem II.

Proposition 5.26. Suppose $(V, \otimes)$ is as in Theorem I, and let $F$ be a pseudo indexing system.

If $F : \mathcal{O} \to \mathcal{P}$ is an $F$-cofibration in $\text{Op}_G^C(V)$ and $\mathcal{O}$ is $\Sigma_F$-cofibrant, then $F$ is a $\Sigma_F$-cofibration.

Remark 5.27. Theorem II and Proposition 5.26 are the colored analogues of [BP21, Lemma 6.64] and [BP21, Rem. 6.70], and it would be straightforward but tedious to adapt the proofs therein. However, as we have established Proposition 4.34 for all groupoids (while in [BP21] we only had access to [BP21, Prop. 6.25], which covers only groups), here we will be able to provide a significantly simpler proof.

In the following, we use the categories $\Omega^G_2[k] \in \Omega^G_2$ of alternating trees discussed in §5.2. Moreover, we write $V^*_G (G^{op} \times \Omega^G_2[k]) \to \Sigma : ((G^{op} \times \Sigma_1)^{\mathfrak{u}2})$ for the natural variant of the vertex functor where the summands $(G^{op} \times \Sigma_1)^{\mathfrak{u}2} \cong (G^{op} \times \Sigma_1)^{\mathfrak{u}(n,1)}$ separate active and inert vertices.

Proof of Proposition 5.26. It is enough to show that, if $\mathcal{O} \to \mathcal{O}[u]$ is a free operad extension as in (5.9), and for which $\mathcal{O}$ is $\Sigma_F$-cofibrant and $u : X \to Y$ is a $F$-cofibration in $\text{Sym}_G^C(V)$, then each of the filtration pieces $\mathcal{O}_{k-1} \to \mathcal{O}_k$ in (5.10) are $F$-cofibrations in $\text{Sym}_G^C(V)$.

We now consider the following commutative diagrams, where the composites of the top squares coincide. To ease notation, we have written $G^o$ as $G^{op}$ and $\Sigma : ((G^{op} \times \Sigma_1)^{\mathfrak{u}2})$ as $\Sigma : (G^o \times \Sigma_1)^{\mathfrak{u}2}$, while $\nabla$ denotes the fold map.

Recalling the family $F_c$ on $G^{op} \times \Sigma_1$ in Definition 5.1, we now have a sequence of identifications (of families in $G^{op} \times \Omega^G_2[k]$)

$$(V^*_c)^* ((\nabla^*F_c)^*) = (V^*_c)^* ((\nabla^*\pi^*_cF)^*) = (V^*_c)^* (\Sigma \nabla)^* (\Sigma \pi^*_c)^* F^* = (\pi^*_c)^* \nabla^* F^*$ (5.29)

where the first step is the definition $F_c = \pi^*_cF$, the second step follows from (4.30), and the third step follows since the composite top sections in (5.28) coincide. Moreover, there are identifications

$$(\pi^*_c)^* \text{lr}^*_c F_c = (\pi^*_c)^* \text{lr}^*_c \pi^*_c F = (\pi^*_c)^* \pi^*_c \text{lr}^* F (5.30)$$

where the first step again uses the definition of $F_c$ and the second step uses the commutativity of the left bottom square in (5.28).

It now follows from (5.29), (5.30) that, if $F$ is a pseudo indexing system, i.e. if $V^*F \subseteq \text{lr}^* F$, then

$$(V^*_c)^* ((\nabla^*F_c)^*) \subseteq (\pi^*_c)^* \text{lr}^*_c F_c. (5.31)$$
Abbreviating \( br_\omega^k = br_\xi \pi^k_\omega \) for the left vertical composite in (5.28), applying Proposition 4.21 to (5.31) yields that \( (br_{\xi,op}^k)^2 : \mathcal{V}^{G_k \Omega_{\xi}^k[k]_{op}} \to \mathcal{V}^{G_k \Sigma_{op}^k} \) sends \( (\mathcal{V}_\omega^k)^* ((\mathcal{V}_\xi^k)^*) \)-cofibrations to \( \mathcal{F}_\xi \)-cofibrations.

Next, notice that the map \((\mathcal{O}, u) = (\emptyset \to \mathcal{O}) u (X \xrightarrow{u} Y)\) is a \( \mathcal{V}_\xi \)-cofibration in \( \mathcal{V}^{G_k \Sigma_{op}^k} \) (this simply restates the assumption that \( \emptyset \to \mathcal{O}, X \xrightarrow{u} Y \) are \( \mathcal{F}_\xi \)-cofibrations in \( \mathcal{V}^{G_k \Sigma_{op}^k} = \text{Sym}_\xi^G(V) \)). And, since the functor \( n_{k}^{\mathcal{O},X,Y}^\mathcal{G} = G \times \Omega_{\xi}^k[k]_{op} \to \mathcal{V} \) from (5.12) can be described by

\[
\eta_{\mathcal{O},X,Y} = (\mathcal{V}_\xi^k)^* ((\mathcal{O}, u)^G,
\]

Propositions 4.34(i) and 4.21 imply that \( n_{k}^{\mathcal{O},X,Y}^\mathcal{G} \) is a \( (\mathcal{V}_\xi^k)^* ((\mathcal{V}_\xi^k)^*) \)-cofibration.

But it now follows that \( (br_{\xi,op}^k, n_{k}^{\mathcal{O},X,Y}^\mathcal{G}) \) is a \( \mathcal{F}_\xi \)-cofibration in \( \mathcal{V}^{G_k \Sigma_{op}^k} = \text{Sym}_\xi^G(V) \) and thus, using the pushout squares (5.11), so is \( \mathcal{O}_{k,-1} \to \mathcal{O}_k \), finishing the proof.

In the following, we write \( \eta_{\mathcal{G}} \) for the initial object of \( \text{Op}_{\mathcal{G}}(V) \). This notation is motivated by the underlying identification \( \eta_{\mathcal{G}} = \prod_{\mathcal{G}_k} \Sigma_{\mathcal{G}_k}[\eta_{\mathcal{G}_k}] \) in \( \text{Sym}_\xi^G(V) \), where \( \eta_{\mathcal{G}} \) is as in Remark 3.40.

**Proof of Theorem II.** Remark 5.25 guarantees that the initial operad \( \eta_{\mathcal{G}} \in \text{Op}_{\mathcal{G}}(V) \) is \( \Sigma_{\mathcal{F}} \)-cofibrant for any pseudo indexing system \( \mathcal{F} \). Hence, by first applying Proposition 5.26 to maps of the form \( \eta_{\mathcal{G}} \to \mathcal{O} \), one has that all \( \mathcal{F} \)-cofibrant operads \( \mathcal{O} \in \text{Op}_{\mathcal{G}}(V) \) are \( \Sigma_{\mathcal{F}} \)-cofibrant, so that Theorem II becomes a particular instance of Proposition 5.26.

## A The monad for free colored operads

This appendix is fairly technical. Its goal is to complete Definition 3.44 by fully describing the monad structure on the fibered free operad monad \( \mathcal{F} \) of (3.45), and to use this to prove some necessary technical results, most notably the key filtration result in Lemma 5.8.

Our approach builds off our previous work in [BP21], and is motivated by the fact that the left Kan extension in (3.45) makes the monad structure somewhat awkward to describe and work with directly. As such, our strategy is to note that there is an adjunction

\[
\text{Lan} : \text{WSpan}^\mathcal{I}(\Sigma^\mathcal{G}_{\mathcal{G}}, \mathcal{V}) \rightleftarrows \text{Sym}^\mathcal{G}(\mathcal{V}) : \nu
\]

(\text{A.1})

that identifies \( \text{Sym}^\mathcal{G}(\mathcal{V}) \) as a reflexive subcategory of a larger category \( \text{WSpan}^\mathcal{I}(\Sigma^\mathcal{G}_{\mathcal{G}}, \mathcal{V}) \) of spans (Definition A.18), with the left Kan extension being the reflection. We then build a monad \( \mathcal{N} \) on the larger category \( \text{WSpan}^\mathcal{I}(\Sigma^\mathcal{G}_{\mathcal{G}}, \mathcal{V}) \), and show that this monad can be transferred to \( \text{Sym}^\mathcal{G}(\mathcal{V}) \).

In \$\text{A.1} \text{, A.2} \text{, A.3} \text{, and A.4} \text{, we adapt (and improve on) some constructions in [BP21] to the colored context. A.3 then describes the monads on } \text{Sym}^\mathcal{G}(\mathcal{V}), \text{WSpan}^\mathcal{I}(\Sigma^\mathcal{G}_{\mathcal{G}}, \mathcal{V}). \text{A.4 is dedicated to proving Lemma 5.8. Lastly, A.5 proves some quick results that will be needed in the sequel [BP2b].}

### A.1 Colored strings

We now recall some key notions in [BP21] regarding strings of trees.

Firstly, and as in \$3.3 \text{, we write } \Omega_{\mathcal{G}} \text{ for the category of } \mathcal{G} \text{-colored trees } \mathcal{T} = (T, c : E(T) \to \mathcal{G}) \text{ and color preserving maps.}

We will make use of certain special types of maps in \( \Omega_{\mathcal{G}} \), all of which are defined in terms of the underlying map of trees. A map \( f : \mathcal{T} \to \mathcal{S} \) is called: planar if the underlying map of trees is planar; tall if it sends leaves to leaves and the root to the root; an outer face if, for any factorization \( f = f't \) with \( t \) a tall map, one has that \( t \) is an isomorphism.
Any map $T \xrightarrow{f} S$ in $\Omega_C$ the has a factorization $T \xrightarrow{f'} R \xrightarrow{f''} S$, unique up to unique isomorphism, with $f'$ a tall map and $f''$ an outer map [BP21, Prop. 3.36]. Moreover, this factorization is strictly unique if $f, f''$ are required to be planar.

In the following, we extend the notation $T_v$ for the corolla associated to a vertex $v \in V(T)$.

**Notation A.2.** Given a planar map $\hat{T} \to \hat{S}$ and $v \in V(T)$, we write $\hat{T}_v \to \hat{S}_v \to \hat{S}$ for the “tall map followed by outer face” factorization of the composite $\hat{T}_v \to \hat{T} \to \hat{S}$.

**Example A.3.** Consider the planar map $f: T \to S$ on the left below sending each edge of $T$ to the edge of $S$ with the same name. For each of the four vertices $v_1, v_2, v_3, v_4$ of $T$ (these are ordered according to the planarization of $T$, following the convention in [BP21, §3.1]) we present the corresponding planar outer face $S_{v_i}$ on the right.

![Diagram](image.png)

We next recall some notation concerning the $\Sigma i (-) \iota$ construction in Notation 2.16. For any category $C$, there is a natural “unary tuple functor" $\delta^i: C \to \Sigma i C$ sending $c \in C$ to $(c) \in \Sigma i C$, as well as a natural “concatenation functor" $\sigma^i: \Sigma \iota C \to \Sigma i C$ given by

$$(\{c_1, \ldots, c_{1,n_1}\}, (c_{2,1}, \ldots, c_{2,n_2}), \ldots, (c_{k,1}, \ldots, c_{k,n_k}) \mapsto (c_{1,1}, \ldots, c_{1,n_1}, c_{2,1}, \ldots, c_{2,n_2}, \ldots, c_{k,1}, \ldots, c_{k,n_k})$$

More generally, these functors are part of an augmented cosimplicial object $\Sigma^{n+1} i C, n \geq -1$ in $\text{Cat}$, meaning that one has maps

$$\Sigma^{i+1} i C \xrightarrow{\sigma^i} \Sigma^i i C, \quad 0 \leq i \leq n - 1 \quad \Sigma^{i+1} i C \xrightarrow{\delta^i} \Sigma^{i+2} i C, \quad 0 \leq j \leq n + 1$$

satisfying the cosimplicial identities. Explicitly, $\sigma^i$ concatenates the $(i + 1)$-th and $(i + 2)$-th $\Sigma$ coordinates and $\delta^i$ inserts a unary $\Sigma$ coordinate in the $(j + 1)$-th position.

**Definition A.4.** Let $\mathcal{C}$ be a set of colors. For $n \geq 0$, the category $\Omega^i_\mathcal{C}$ of $n$-strings has as objects strings $T_0 \to T_1 \to \cdots \to T_n$ of maps that are planar and tall, and arrows given by tuples $\mathcal{T}_i \xrightarrow{\cong} \mathcal{T}_i'$ of compatible isomorphisms. In addition, we set $\Omega_\mathcal{C}^1 = \Sigma_\mathcal{C}$.

**Remark A.5.** By definition of tall planar map, all trees $T_i$ in a string have the same leaf-root $\text{lr}(\mathcal{T}_i)$. The convention $\Omega^i_\mathcal{C} = \Sigma_\mathcal{C}$ is motivated by the fact that one can canonically extend an $n$-string as

$$\text{lr}(\mathcal{T}_0) = T_{-1} \to \mathcal{T}_0 \to \mathcal{T}_1 \to \cdots \to \mathcal{T}_n.$$ 

The key functors in [BP21, §3.4] now extend to the strings $\Omega^n_\mathcal{C}$. Firstly, one has simplicial operators

$$d_i: \Omega^n_\mathcal{C} \to \Omega^{n-1}_\mathcal{C}, \quad 0 \leq i \leq n; \quad s_j: \Omega^n_\mathcal{C} \to \Omega^{n+1}_\mathcal{C}, \quad -1 \leq j \leq n$$

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which remove (resp. repeat) the \( i \)-th (resp. \( j \)-th) tree in the string (where the boundary cases must be interpreted in light of Remark \ref{rem:A.5}). Secondly, there are vertex operators

\[
\Omega^n_{\xi} \xrightarrow{V} \Sigma : \Omega^{n-1}_{\xi} \xrightarrow{\Sigma V^{k-1}} \Sigma^2 : \Omega^{n-k-1}_{\xi} \xrightarrow{\sigma^0} \Sigma : \Omega^{n-k-1}_{\xi}.
\]

(A.6)

where we note that the indexing set \( V(T_0) \) of the tuple is ordered according to the planarization of \( T_0 \). Moreover, one iteratively defines higher order vertex functors by setting \( V^k : \Omega^n_{\xi} \to \Sigma : \Omega^{n-k-1}_{\xi} \) to be the composite

\[
\Omega^n_{\xi} \xrightarrow{V} \Sigma : \Omega^{n-1}_{\xi} \xrightarrow{\Sigma V^{k-1}} \Sigma^2 : \Omega^{n-k-1}_{\xi} \xrightarrow{\sigma^0} \Sigma : \Omega^{n-k-1}_{\xi}.
\]

One has an identification \( V^k(\tilde{T}_0 \to \cdots \to \tilde{T}_n) \cong (\tilde{T}_{k+1,0} \to \cdots \to \tilde{T}_{n,v})_{v \in V(T_0)} \), though some care is needed, as the order on \( V(T_k) \) in this formula does not depend only on the planarization of \( T_k \). Rather, cf. \cite[(3.99)]{BP21}, one has natural maps \( V(T_k) \to V(T_{k-1}) \to \cdots \to V(T_0) \) and, for \( v, w \in V(T_k) \), the condition \( v < w \) is determined by the lowest \( i \) such that \( v, w \) have distinct images in \( V(T_i) \).

Lastly, for a map of colors \( \varphi : \mathcal{C} \to \mathcal{D} \), there are change of color functors \( \varphi : \Omega^n_{\xi} \to \Omega^m_{\eta} \) defined by \( (\varphi \tilde{T}_0 \to \cdots \to \tilde{T}_n) \mapsto (\varphi \tilde{T}_0 \to \cdots \to \varphi \tilde{T}_n) \), where \( \varphi \tilde{T}_i \) applies \( \varphi \) to the colors (cf. Definition \ref{def:3.2}). These operators satisfy a number of compatibilities (cf. \cite[Prop. 3.102]{BP21}). Firstly, the \( d_i, s_j \) operators satisfy the simplicial identities, and the \( V^k \) operators are “additive” in the sense that the composite below is \( V^{k+l+1} \).

\[
\Omega^n_{\xi} \xrightarrow{V^l} \Sigma : \Omega^{n-l-1}_{\xi} \xrightarrow{\Sigma V^k} \Sigma^2 : \Omega^{n-k-l-2}_{\xi} \xrightarrow{\sigma^0} \Sigma : \Omega^{n-k-l-2}_{\xi}.
\]

(A.7)

The next results list the compatibilities between \( d_i, s_j \) and the \( V^k \) operators. We note that the natural isomorphisms \( \pi_{i,k} \) are needed to account for different orderings on \( V(T_k) \), cf. the comment following (A.6).

**Proposition A.8.** One has the following diagrams in the 2-category \( \text{Cat} \).

(i) For \( 0 \leq i \leq k \leq n \) there are 2-isomorphisms \( \pi_{i,k} \), and for \( 0 \leq l \leq k \leq n \) there are commutative diagrams

\[
\begin{array}{ccc}
\Omega^n_{\xi} & \xrightarrow{V^k} & \Sigma : \Omega^{n-k-1}_{\xi} \\
\downarrow d_i & & \downarrow s_j \\
\Omega^{n-1}_{\xi} & \xrightarrow{V^{k-1}} & \Sigma : \Omega^{n-k-1}_{\xi} \\
\end{array}
\]

(ii) For \( -1 \leq k < i \leq n \) and for \( -1 \leq l \leq k \leq j \leq n \) there are commutative diagrams

\[
\begin{array}{ccc}
\Omega^n_{\xi} & \xrightarrow{V^k} & \Sigma : \Omega^{n-k-1}_{\xi} \\
\downarrow d_{i-k-1} & & \downarrow s_{j-k-1} \\
\Omega^{n-1}_{\xi} & \xrightarrow{V^{k-1}} & \Sigma : \Omega^{n-k-2}_{\xi} \\
\end{array}
\]

Furthermore, these diagrams are pullback squares in \( \text{Cat} \).
(iii) all $d_i, s_j, V^k$, and $\pi_{i,k}$ are natural in $\mathcal{C}$, i.e. for each map of colors $\varphi : \mathcal{C} \to \mathcal{D}$ one has commutative diagrams (for the prism, commutativity means that $\varphi \pi = \pi \varphi$)

\[ \begin{array}{cccc}
\Omega^n_{\mathcal{C}} & \xrightarrow{d_i} & \Omega^{n-1}_{\mathcal{C}} & \\
\varphi & \downarrow & \varphi & \\
\Omega^n_{\mathcal{D}} & \xrightarrow{s_j} & \Omega^{n+1}_{\mathcal{D}} & \\
\end{array} \]

\[ \begin{array}{cccc}
\Omega^n_{\mathcal{C}} & \xrightarrow{V^k} & \Omega^{n-k-1}_{\mathcal{C}} & \\
\varphi & \downarrow & \varphi & \\
\Omega^n_{\mathcal{D}} & \xrightarrow{\Sigma_1} & \Omega^{n-k-1}_{\mathcal{D}} & \\
\end{array} \]

\[ \begin{array}{cccc}
\Omega^n_{\mathcal{C}} & \xrightarrow{V^k} & \Sigma_1 \Omega^{n-k-1}_{\mathcal{C}} & \\
\varphi & \downarrow & \varphi & \\
\Omega^n_{\mathcal{D}} & \xrightarrow{\Sigma_1} & \Sigma_1 \Omega^{n-k-1}_{\mathcal{D}} & \\
\end{array} \]

\[ \begin{array}{cccc}
\Omega^n_{\mathcal{C}} & \xrightarrow{V^k} & \Sigma_1 \Omega^{n-k-1}_{\mathcal{C}} & \\
\varphi & \downarrow & \varphi & \\
\Omega^n_{\mathcal{D}} & \xrightarrow{\Sigma_1} & \Sigma_1 \Omega^{n-k-1}_{\mathcal{D}} & \\
\end{array} \]

The following lists the compatibilities between the $\pi_{i,k}$ isomorphisms, which are extensions of the additivity of $V^k$ in (A.7) and of the simplicial identities between the $d_i, s_j$ operators.

**Proposition A.9.** In each item, the two composite natural transformations coincide.

(IT1) For $0 \leq i < k$ and $-1 \leq l \leq n - k - 1$

\[ \begin{array}{cccc}
\Omega^n_{\mathcal{C}} & \xrightarrow{V^k} & \Sigma_1 \Omega^{n-k-1}_{\mathcal{C}} & \\
\varphi & \downarrow & \varphi & \\
\Omega^n_{\mathcal{D}} & \xrightarrow{d_i} & \Sigma_1 \Omega^{n-k-1}_{\mathcal{D}} & \\
\end{array} \]

(IT2) For $-1 \leq k < i < k + l + 1 \leq n$

\[ \begin{array}{cccc}
\Omega^n_{\mathcal{C}} & \xrightarrow{V^k} & \Sigma_1 \Omega^{n-k-1}_{\mathcal{C}} & \\
\varphi & \downarrow & \varphi & \\
\Omega^n_{\mathcal{D}} & \xrightarrow{V^l} & \Sigma_1 \Omega^{n-k-1}_{\mathcal{D}} & \\
\end{array} \]

(FF1) For $0 \leq i < i' < k \leq n$

\[ \begin{array}{cccc}
\Omega^n_{\mathcal{C}} & \xrightarrow{V^k} & \Sigma_1 \Omega^{n-k-1}_{\mathcal{C}} & \\
\varphi & \downarrow & \varphi & \\
\Omega^n_{\mathcal{D}} & \xrightarrow{V^k} & \Sigma_1 \Omega^{n-k-1}_{\mathcal{D}} & \\
\end{array} \]

(FF2) For $0 \leq i < k < i' \leq n$

\[ \begin{array}{cccc}
\Omega^n_{\mathcal{C}} & \xrightarrow{V^k} & \Sigma_1 \Omega^{n-k-1}_{\mathcal{C}} & \\
\varphi & \downarrow & \varphi & \\
\Omega^n_{\mathcal{D}} & \xrightarrow{V^k} & \Sigma_1 \Omega^{n-k-1}_{\mathcal{D}} & \\
\end{array} \]
A.2 The $(-) : A$ construction

One of the key ingredients used in [BP21, §4.2] when describing the monad on spans (cf. (A.1)) is the use of categories $\Omega^n : A$ defined by pullbacks diagrams of the form

\[
\Omega^n : A \xrightarrow{V^n} \Sigma : A \\
\Omega^n \xrightarrow{\pi} \Sigma : \Sigma
\]

Moreover, these categories are related by analogues of the operators $d_i, s_j, V^k, \pi_{i,k}$ which satisfy all the analogues of the compatibilities listed in Propositions A.8 and A.9.
In [BP21] we built these analogue operators in a somewhat ad-hoc manner. Here we use a more systematic approach, which views the \((-\) : \mathcal{A}\) construction as a 2-categorical extension of the pullback operation in \textbf{Cat}. We now define the required 2-categories, which are a variant of the \textbf{Cat} \iota \mathcal{V} categories in Remark 3.8 with regard to a split Grothendieck fibration \(\mathcal{E} \rightarrow \mathcal{B}\) (Remark 2.13). In what follows, arrows \(\varphi^*e \rightarrow e\) in the chosen cleavage of \(\mathcal{E}\) are called pullback arrows.

**Definition A.11.** Let \(\mathcal{E} \rightarrow \mathcal{B}\) be a split Grothendieck fibration. We write \(\textbf{Cat} \downarrow^r \mathcal{B} \mathcal{E}\) for the 2-category such that:

- objects are functors \(F : \mathcal{C} \rightarrow \mathcal{E}\);
- an 1-arrow from \(F : \mathcal{C} \rightarrow \mathcal{E}\) to \(F' : \mathcal{C}' \rightarrow \mathcal{E}\) is a pair \((\varphi, \phi)\) formed by a functor \(\varphi : \mathcal{C} \rightarrow \mathcal{C}'\) and a natural transformation \(\phi : F' \varphi \Rightarrow F\) consisting of pullback arrows over \(\mathcal{B}\).

\[
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{\varphi} & \mathcal{E} \\
\downarrow & \nearrow \phi & \\
\mathcal{C}' & \xrightarrow{F'} & \mathcal{E}
\end{array}
\]

- a 2-arrow from \((\varphi, \phi)\) to \((\varphi', \phi')\) is a 2-arrow \(\Phi : \varphi \Rightarrow \varphi'\) such that \(\phi' \circ F' \Phi = \phi\).

\[
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{\varphi} & \mathcal{E} \\
\downarrow & \nearrow \phi & \\
\mathcal{C}' & \xrightarrow{F'} & \mathcal{E}
\end{array}
\]

Given a map \(\rho : \mathcal{E} \rightarrow \mathcal{F}\) of split Grothendieck fibrations over \(\mathcal{B}\) (i.e. \(\rho\) sends pullback arrows to pullback arrows), we now define a pullback 2-functor

\[
\rho^* : \textbf{Cat} \downarrow^r \mathcal{B} \mathcal{F} \rightarrow \textbf{Cat} \downarrow^r \mathcal{B} \mathcal{E}.
\]

On objects, i.e. functors \(F : \mathcal{C} \rightarrow \mathcal{F}\), one sets \(\rho^* (C \rightarrow \mathcal{E}) = (C \times \mathcal{F} \mathcal{E} \rightarrow \mathcal{E})\).

On 1-arrows, i.e. pairs \((\varphi, \phi) : F_2 \circ \varphi \Rightarrow F_1\) as in the bottom of the diagram below

\[
\begin{array}{ccc}
\mathcal{C}_1 \times \mathcal{F} \mathcal{E} & \xrightarrow{\tilde{\varphi} \tilde{\phi}} & \mathcal{E} \\
\downarrow & \nearrow \phi & \\
\mathcal{C}_2 \times \mathcal{F} \mathcal{E} & \xrightarrow{\tilde{\varphi} \tilde{\phi}} & \mathcal{E}
\end{array}
\]

we define \(\rho^* (\varphi, \phi)\) as the only possible choice of dashed data \((\tilde{\varphi}, \tilde{\phi})\) such that: (i) \(\tilde{\phi}\) consists of pullback arrows over \(\mathcal{B}\) and; (ii) the diagram commutes in the sense that \(p\tilde{\varphi} = \varphi p\) and \(p\tilde{\phi} = \phi p\).

Alternatively, one has the explicit formula

\[
\rho^* (\varphi, \phi) = \left( (\varphi p, (\phi p)^* E_1), (\phi p)^* E_1 \Rightarrow E_1 \right).
\]
Lastly, on a 2-arrow $\Phi : (\varphi, \phi) \Rightarrow (\varphi', \phi')$ as on the bottom of the leftmost diagram below

\[
\begin{array}{ccc}
C_1 \times_\Sigma \mathcal{E} & \xrightarrow{\Phi} & \mathcal{E} \\
\downarrow p & & \downarrow E_1^{-1} \\
\mathcal{E} & \xrightarrow{\Phi'} & \mathcal{E} \\
\downarrow p & & \downarrow E_2^{-1} \\
C_2 \times_\Sigma \mathcal{E} & \xrightarrow{\Phi} & \mathcal{E} \\
\end{array}
\]

we define $\rho^* (\Phi)$ as the only choice (recall that $\varphi'$ is a pullback arrow) of dashed $\Phi$ such that $\varphi' \circ E_2 \Phi = \phi$ and $p \Phi = \Phi p$.

We are now ready to extend the $(-) : A$ construction from (A.10).

First, note that, using the functor $V^n : \Omega^n \rightarrow \Sigma : \Sigma \mathcal{E}$, the categories $\Omega^n$ may be regarded as objects in $\text{Cat}(\Sigma : \Sigma \mathcal{E})$. Hence, given a functor $A \rightarrow \Sigma \mathcal{E}$ we define

\[
(-) : A : \text{Cat}(\Sigma : \Sigma \mathcal{E}) \rightarrow \text{Cat}(\Sigma : \Sigma \mathcal{E})
\]

as the pullback 2-functor (A.12) for the map $\Sigma : A \rightarrow \Sigma : \Sigma \mathcal{E}$.

Focusing on objects, (A.13) then defines $\Omega^n : A$ together with maps $\Omega^n : A \rightarrow \Sigma : A$.

Next, by specifying Proposition A.8(i) to $k = n$, one obtains 1-arrows $(d_i, \pi_{n_i}), 0 \leq i < n$ and $(s_j, \delta_{V^n}), -1 \leq j \leq n$ in $\text{Cat}(\Sigma : \Sigma \mathcal{E})$. Thus, by specifying (A.13) to 1-arrows one also obtains 1-arrows $(d_i, \pi_{n_i}), 0 \leq i < n$ and $(s_j, \delta_{V^n}), -1 \leq j \leq n$ in $\text{Cat}(\Sigma : \Sigma \mathcal{E})$ between the $\Omega^n : A \rightarrow \Sigma : A$.

We next describe the functors $V^k : \Omega^n : A \rightarrow \text{Cat}(\Omega^k : A)$. Note that the $\Sigma : (-)$ operation can be extended to a 2-endofunctor

\[
\text{Cat}(\Sigma : A) \xrightarrow{\Sigma(-)} \text{Cat}(\Sigma : A)
\]

so that, noting that the diagram below consists of pullback squares,

\[
\begin{array}{ccc}
\Sigma : (\Omega^n : A) & \xrightarrow{\Sigma V^n} & \Sigma : (\Sigma : A) \\
\downarrow & & \downarrow \\
\Sigma : \Omega^n & \xrightarrow{\Sigma V^n} & \Sigma : (\Sigma : \Sigma \mathcal{E})
\end{array}
\]

one has identifications $\Sigma : (\Omega^n : A) \cong (\Sigma : \Omega^n : A)$, which are seen to agree with the $\sigma^i$ operators on $\Sigma A$. Thus, we henceforth omit parenthesis and write $\Sigma : \Omega^n : A$ to denote $\Sigma : (\Omega^n : A)$.

Specifying (A.14) to $A = \Sigma \mathcal{E}$, by (A.7) when $k = n - l - 1$ one has that $(V^l, id V^n)$ defines a 1-arrow from $\Omega^l : \Sigma : \Sigma \mathcal{E}$ to $\Sigma : \Omega^{n-l} : \Sigma : \Sigma \mathcal{E}$ in $\text{Cat}(\Sigma \mathcal{E})$. Thus, applying (A.13) yields a 1-arrow $(V^l, id V^n)$ from $\Omega^l : A \rightarrow \Sigma : A$ to $\Sigma : \Omega^{n-l} : A \rightarrow \Sigma : A$ in $\text{Cat}(\Sigma : A)$.

It is now immediate from 2-functoriality of (A.13) that one has natural transformations $\pi_{i,k}$ between the $\Omega^n : A$ satisfying the analogues of Proposition A.8(i)(ii) and Proposition A.9.

In what follows, we will find it convenient to abbreviate $\Omega^n : A \rightarrow \Sigma : A$ as $\Omega^n : A$ and $(d_i, \pi_{n_i}), (s_j, \delta_{V^n}), (V^k, id V^n)$ as $d_i, s_j, V^k$, i.e. we will leave the $\Sigma : A$ data implicit.

The analogue of Proposition A.8(iii) requires an extra argument, and is stated in the following.
Proposition A.15. A commutative square

\[
\begin{array}{ccc}
A & \xrightarrow{\varphi} & B \\
\downarrow & & \downarrow \\
\Sigma \varphi & \xrightarrow{\varphi} & \Sigma D
\end{array}
\]  
(A.16)

induces natural maps \( \varphi : \Omega^n \circ \varphi A \to \Omega^n \circ \varphi B \) such that the diagrams below coincide.

\[
\begin{array}{cccc}
\Omega^n \circ \varphi A & \xrightarrow{d_i} & \Omega^{n-1} \circ \varphi A & \xrightarrow{s_j} & \Omega^{n+1} \circ \varphi A \\
\downarrow & & \downarrow & & \downarrow \\
\Omega^n \circ \varphi B & \xrightarrow{d_i} & \Omega^{n-1} \circ \varphi B & \xrightarrow{s_j} & \Omega^{n+1} \circ \varphi B
\end{array}
\]

Proof. The desired maps \( \varphi : \Omega^n \circ \varphi A \to \Omega^n \circ \varphi B \) are obtained by just drawing the pullback diagrams defining each term, so we focus on the more interesting claim that the given diagrams commute. To see this, we first factor (A.16) as

\[
\begin{array}{ccc}
A & \xrightarrow{A} & B \times \Sigma \varphi \\
\downarrow & & \downarrow \\
\Sigma \varphi & \xrightarrow{\Sigma \varphi} & \Sigma D
\end{array}
\]

and note that it suffices to prove the result separately for each half. For the left half, the desired commutativity claims follow from naturality of \( (-) \circ \varphi A \) with respect to \( A \). On the other hand, for the right square the commutativity claims follow by instead noting that all diagrams in Proposition A.8(iii) can be regarded as diagrams in the 2-category \( \text{Cat} \), \( \Sigma \circ \Sigma D \) (by using the composites \( \varphi \circ \Omega^n \circ \varphi A \xrightarrow{V_n} \Sigma \circ \Sigma D \)) and then applying the pullback functor \( (-) \circ \varphi B \).

\[\square\]

In what follows, for an increasing sequence \( i_1 < i_2 < \cdots < i_k \), we write \( d_{i_1, i_2, \ldots, i_k} = d_{i_1} \circ d_{i_2} \circ \cdots \circ d_{i_k} \).

Next, note that, using the composite functors \( \Omega^n \circ \varphi A \to \Omega^n \circ \varphi B \xrightarrow{d_{0, \ldots, n}} \Sigma \varphi \), one can regard the \( \Omega^n \circ (-) \) constructions as endofunctors on the usual 1-overcategory \( \text{Cat} \).\( \Sigma \varphi \).

Proposition A.17. Let \( k, l \geq -1 \). One has canonical natural identifications \( \Omega^k \circ \Omega^l A = \Omega^{k+l+1} A \).

Moreover, these identifications are associative in the sense that, for any \( k, l, m \leq -1 \), the iterated composite identifications below coincide.

\[
\begin{align*}
\Omega^k \circ \Omega^l A &= \Omega^{k+l+1} A \\
\Omega^k \circ \Omega^m A &= \Omega^{k+m+1} A \\
\Omega^k \circ \Omega^l A &= \Omega^{k+l+1} A
\end{align*}
\]

Furthermore, the identifications above also induce the following identifications

\[
\begin{align*}
d_i \circ d_l &\simeq d_{i+l} \quad \pi_{i,k} \circ d_l &\simeq \pi_{i+k,l} \\
p_i \circ d_l &\simeq p_{i+k,l} \quad d_i \circ s_j &\simeq s_{i+j} \quad O^k \circ d_i &\simeq d_{k+i+1} \\
p_i \circ s_j &\simeq s_{i+j} \quad O^k \circ s_j &\simeq s_{k+j+1}
\end{align*}
\]
Proof. The first claim follows by noting that all squares in the diagram below are pullback squares

\[
\begin{array}{ccc}
\Omega_k^{k+l+1} \times A & \xrightarrow{V} & \Sigma \times \Omega_k^{l} \times A \\
\downarrow & & \downarrow \\
\Omega_k^{k+l+1} & \xrightarrow{V} & \Sigma \times \Omega_k^{l} \\
\downarrow & & \downarrow \\
ds_{k+1,\ldots,k+l+1} & \xrightarrow{V} & \Sigma \times \Omega_k^{l} \\
\end{array}
\]

while associativity follows from the obvious “3 level analogue” of the diagram above.

For the additional identifications, those identifications concerning \(d_i\) and \(\pi_{i,k}\) follow from the left diagram below (the bottom section of which commutes by Proposition A.9(FF2)), the identification concerning \(d_{k+i}\) follows from the rightmost diagram, and the identifications concerning \(s_j\) and \(s_{k+j+1}\) follow from obvious analogues of these diagrams.

\[
\begin{array}{ccc}
\Omega_k^{k+l+1} \times A & \xrightarrow{V} & \Sigma \times \Omega_k^{l} \\
\downarrow & & \downarrow \\
\Omega_k^{k+l+1} & \xrightarrow{V} & \Sigma \times \Omega_k^{l} \\
\downarrow & & \downarrow \\
ds_i & \xrightarrow{V} & \Sigma \times \Omega_k^{l} \\
\end{array}
\]

A.3 The fibered monads on spans and symmetric sequences

In this section we finally complete the definition of the fibered free operad monad \(\mathcal{F}\) in Definition 3.44, starting with the promised monad \(N\) on the category \(\text{WSpan}^l(\Sigma^{\text{op}}, V)\) of spans (cf. (A.1)).

The fibered monad on colored spans

Definition A.18. The category \(\text{WSpan}^l(\Sigma^{\text{op}}, V)\) has

- objects given by a choice of a set of colors \(\mathcal{C}\) and a span \(\Sigma^{\text{op}} \leftarrow A^{\text{op}} \rightarrow V\)

- morphisms given by a choice of a map of colors \(\varphi: \mathcal{C} \rightarrow \mathcal{D}\), together with a commutative square and natural transformation as given below.

\[
\begin{array}{ccc}
\Sigma^{\text{op}} & \xleftarrow{\varphi^{\text{op}}} & A^{\text{op}} \\
\downarrow & \downarrow & \downarrow \\
\Sigma^{\text{op}} & \xleftarrow{B^{\text{op}}} & V \\
\end{array}
\]
Remark A.20. By definition, there is a forgetful functor $\text{WSpan}^l(\Sigma^{op}, V) \to \text{Set}$ which remembers the set of colors. Moreover, this is a Grothendieck fibration, with the cartesian arrows the diagrams (A.19) where the square is a pullback square and the natural transformation is an isomorphism.

Remark A.21. Given a span $\Sigma^{op} \leftarrow A^{op} \rightarrow V$ one can the form the left Kan extension $\text{Lan}F \colon \text{WSpan}^l(\Sigma^{op}, V) \rightleftarrows \text{Sym}(V) : \upsilon$ where the inclusion $\upsilon$ sends $\Sigma^{op} \rightarrow V$ to the span $\Sigma^{op} \leftarrow \Sigma^{op} \rightarrow V$.

Remark A.22. One can also define a larger category $\text{WSpan}^l(-, V)$ where the categories $\Sigma^{op}$ in the spans (and functors between them) are allowed to be any category (any functor), in which case left Kan extension defines a fibered adjunction over $\text{Cat}$ (cf. Remark 3.8)

Definition A.23 (cf. [BP21], Def. 4.15). The monad $N$ on $\text{WSpan}^l(\Sigma^{op}, V)$ sends the span $\Sigma^{op} \leftarrow A^{op} \rightarrow V$ to the (opposite of the) composite span in

$$
\array{
\Omega_0 \uplus A & \xymatrix{\Sigma \downarrow V \ar[r] & \Sigma \downarrow V^{op} & \Sigma \downarrow V^{op} \\
\Omega_0 \uplus A & \Sigma \downarrow V \ar[r] & \Sigma \downarrow V^{op} & \Sigma \downarrow V^{op} \\
\Sigma \downarrow V & \Sigma \downarrow V^{op} & \Sigma \downarrow V^{op} & \Sigma \downarrow V^{op} \\
\Sigma \downarrow V & \Sigma \downarrow V^{op} & \Sigma \downarrow V^{op} & \Sigma \downarrow V^{op} \\
\Sigma \downarrow V & \Sigma \downarrow V^{op} & \Sigma \downarrow V^{op} & \Sigma \downarrow V^{op} \\
}\}
$$

where the inclusion $\upsilon$ sends $\Sigma^{op} \rightarrow V$ to the span $\Sigma^{op} \leftarrow \Sigma^{op} \rightarrow V$.

Functoriality of $N$ with respect to maps that change colors follows from Proposition A.15.
Proposition A.26 (cf. [BP21, Prop. 4.18]). \(N\) is a monad on \(\text{WSpan}^l(\Sigma^{op}, \mathcal{V})\), fibered over \(\text{Set}\).

**Proof.** To check associativity, the functor \(\mu_N: \text{NNN} \Rightarrow \text{NN}\) is encoded by the diagram

\[
\begin{array}{cccccccccc}
\Omega^2 \circ A & \rightarrow & \Sigma \circ \Omega^1 \circ A & \rightarrow & \Sigma^2 \circ \Omega^0 \circ A & \rightarrow & \Sigma^3 \circ A & \rightarrow & \Sigma^3 \circ \mathcal{V}^{op} & \oplus & \Sigma^2 \circ \mathcal{V}^{op} & \oplus & \Sigma \circ \mathcal{V}^{op} & \oplus & \mathcal{V}^{op} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Omega^2 \circ A & \rightarrow & \Sigma \circ \Omega^0 \circ A & \rightarrow & \Sigma^2 \circ A & \rightarrow & \Sigma^2 \circ \mathcal{V}^{op} & \oplus & \Sigma \circ \mathcal{V}^{op} & \oplus & \mathcal{V}^{op} & \oplus & \mathcal{V}^{op} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Omega^0 \circ A & & & & \Sigma \circ A & & & & \Sigma \circ \mathcal{V}^{op} & & & & \mathcal{V}^{op} \\
\end{array}
\]

while the functor \(N\mu: \text{NNN} \Rightarrow \text{NN}\) is encoded by

\[
\begin{array}{cccccccccc}
\Omega^2 \circ A & \rightarrow & \Sigma \circ \Omega^1 \circ A & \rightarrow & \Sigma^2 \circ \Omega^0 \circ A & \rightarrow & \Sigma^3 \circ A & \rightarrow & \Sigma^3 \circ \mathcal{V}^{op} & \oplus & \Sigma^2 \circ \mathcal{V}^{op} & \oplus & \Sigma \circ \mathcal{V}^{op} & \oplus & \mathcal{V}^{op} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Omega^2 \circ A & \rightarrow & \Sigma \circ \Omega^0 \circ A & \rightarrow & \Sigma^2 \circ A & \rightarrow & \Sigma^2 \circ \mathcal{V}^{op} & \oplus & \Sigma \circ \mathcal{V}^{op} & \oplus & \mathcal{V}^{op} & \oplus & \mathcal{V}^{op} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Omega^0 \circ A & & & & \Sigma \circ A & & & & \Sigma \circ \mathcal{V}^{op} & & & & \mathcal{V}^{op} \\
\end{array}
\]

That the leftmost sections of these diagrams match follows by parts (IT1) and (FF1) of Proposition A.9, while the fact that the rightmost sections match follows since \(\mathcal{V}\) is a monoidal category.

Unitality of \(N\) follows by a simpler version of the argument above, cf. [BP21, (4.21)(4.22)].

The fibered monad on colored symmetric sequences

We will now use the fibered adjunction

\[
\text{Lan}: \text{WSpan}^l(\Sigma^{op}, \mathcal{V}) \rightleftarrows \text{Sym}(\mathcal{V}): \upsilon
\]

from Remark A.21 to induce a fibered monad on \(\text{Sym}(\mathcal{V})\). To do so, we will verify the conditions in [BP21, Prop. 2.27], requiring that the natural transformations

\[
\text{Lan} \upsilon \Rightarrow \text{id} \quad \text{Lan} \upsilon \eta \Rightarrow \text{Lan} \upsilon \nu \text{Lan}
\]

are natural isomorphisms. This is clear for \(\epsilon\) while for \(\eta\) it follows from the following two lemmas, the first of which is proven exactly as in [BP21, Lemma 2.21].

**Lemma A.27** (cf. [BP21, Lemma 2.21]). If in \(\mathcal{V}\) the monoidal product commutes with colimits in each variable, and the leftmost diagram

\[
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{F} & \mathcal{V}^{op} \\
\downarrow{k} & & \downarrow{H} \\
\mathcal{D} & \xrightarrow{H} & \mathcal{V}^{op}
\end{array}
\]

is a right Kan extension diagram, then so is the composite of the rightmost diagram.
Lemma A.28 (cf. [BP21, Lemma 4.27]). Suppose that $\mathcal{V}$ is complete. If the rightmost triangle in

$$
\begin{align*}
\Omega^0_\mathcal{E} \rightarrow A & \xrightarrow{\mathcal{V}} \Sigma \downarrow A \xrightarrow{\mathcal{V}^{op}} \\
\Omega^0_\mathcal{E} & \xrightarrow{\mathcal{V}} \Sigma \downarrow \Sigma_\mathcal{E}
\end{align*}
$$

is a right Kan extension diagram then so is the composite diagram.

Our proof Lemma A.28 will be a more formalized version of the proof in [BP21, Lemma 4.27]. For $\pi: \mathcal{E} \to \mathcal{B}$ a Grothendieck fibration and $e \in \mathcal{E}$, we write $e \downarrow_\mathcal{B} \mathcal{E}$ for the subcategory of the undercategory $e \downarrow \mathcal{E}$ consisting of the objects and maps over $id_{\pi(e)}$. Pullbacks over $\mathcal{B}$ then provide a retraction $r: e \downarrow \mathcal{E} \to e \downarrow_\mathcal{B} \mathcal{E}$, which is a right adjoint to the inclusion $e \downarrow_\mathcal{B} \mathcal{E} \hookrightarrow e \downarrow \mathcal{E}$. In other words, $e \downarrow_\mathcal{B} \mathcal{E}$ is a coreflexive subcategory of $e \downarrow \mathcal{E}$, so that the inclusion $e \downarrow_\mathcal{B} \mathcal{E} \hookrightarrow e \downarrow \mathcal{E}$ is final.

Proof. Firstly, note that the composite

$$
\begin{align*}
\tilde{T}_e & \downarrow \Omega^0_\mathcal{E} \longrightarrow (\tilde{T}_e)_{V(T)} \downarrow \Sigma \downarrow \Sigma_\mathcal{E} \xrightarrow{\pi} (\tilde{T}_e)_{V(T)} \downarrow \Sigma \downarrow \Sigma_\mathcal{E} 
\end{align*}
$$

is an isomorphism. Indeed, the objects of $\tilde{T}_e \downarrow \Omega^0_\mathcal{E}$ are determined by underlying isomorphisms $f: T \xrightarrow{\cong} T'$ in $\Omega$, which are in turn determined by a tuple of isomorphisms of vertices $f_v: T_v \xrightarrow{\cong} T'_v$ in $\Sigma$ for each $v \in V(T)$, cf. [BP21, Prop. 3.12]. We now claim that the maps

$$
\begin{align*}
(\tilde{T}_e, (a_v)_{V(T)}) \downarrow \Omega^0_\mathcal{E} \rightarrow (a_v)_{V(T)} \downarrow \Sigma \downarrow A \xrightarrow{\pi} (a_v)_{V(T)} \downarrow \Sigma \downarrow A 
\end{align*}
$$

are likewise isomorphisms. To see this, we first write $D, \tilde{D}$ for the composite functors in (A.29),(A.30), and $\rho: A \to \Sigma_\mathcal{E}$ for the given map. An object in the target of (A.30) is a tuple $f_v: a_v \to b_v$ of maps in $A$ for $v \in V(T)$. Writing $\rho(f_v): \rho(a_v) \to \rho(b_v)$ as $f_v: \tilde{T}_v \to \tilde{T}'_v$ and $D^{-1}(\rho(f_v))$ as $f: \tilde{T} \to \tilde{T}'$ one then has

$$
\tilde{D}^{-1}\left((a_v \overset{f_v}{\rightarrow} b_v)_{v \in V(T)}\right) = (\tilde{T} \xrightarrow{f} \tilde{T}', (a_v)_{v \in V(T)} \rightarrow (b_v)_{v \in V(T)})
$$

where we note that the map $(a_v)_{v \in V(T)} \rightarrow (b_v)_{v \in V(T)}$ involves a permutation of tuples induced by the isomorphism $V(T) \xrightarrow{\cong} V(T')$. Now consider the diagram

$$
\begin{align*}
(\tilde{T}_e, (a_v)) \downarrow \Sigma \downarrow A & \xrightarrow{\pi} (a_v) \downarrow \Sigma \downarrow A \\
(\tilde{T}_e, (a_v)) \downarrow \Sigma \downarrow A & \xrightarrow{\pi} (a_v) \downarrow \Sigma \downarrow A
\end{align*}
$$

To finish the proof, we show that the first map in (A.31) is final. Since this first map is naturally isomorphic to the full composite in (A.31), we need only show that the latter is final. But this follows since (A.30) is an isomorphism and the last map in (A.31) is known to be final.

Since Lemmas A.27,A.28 verify the hypotheses of [BP21, Prop. 2.27], we can finally complete Definition 3.44 by describing the monad structure on $\mathbb{F}$.

Definition A.32. The fibered free operad monad $\mathbb{F}$ on $\mathbf{Sym}_\bullet(\mathcal{V})$ has underlying functor $\mathbb{F} = \operatorname{Lan}N\mathcal{V}$ and multiplication and unit given by

$$
\begin{align*}
\operatorname{Lan}N\mathcal{V}\operatorname{Lan}N\mathcal{V} & \xrightarrow{\delta} \operatorname{Lan}N\mathcal{V} \rightarrow \operatorname{Lan}N\mathcal{V} \\
id & \xrightarrow{\epsilon} \operatorname{Lan} \rightarrow \operatorname{Lan}N\mathcal{V}
\end{align*}
$$
A.4 Free extensions of operads

Our overall goal in this section is to prove Lemma 5.8, which allows us to understand free operad extensions, i.e. pushouts of the form

\[
\begin{array}{ccc}
FX & \longrightarrow & O \\
\text{Fu} & & \downarrow \\
FY & \longrightarrow & O[u].
\end{array}
\]  
\tag{A.33}

where \( u: X \to Y \) is a map of symmetric sequences, and where we moreover require that (A.33) is a fibered diagram over \( \text{Set} \), i.e. that all maps therein are the identity on color sets.

Moreover, in order to understand the equivariant case, we will not fix the set of colors, but rather consider all colors simultaneously, and note that our constructions are natural on the diagrams (A.33) with respect to change of colors. More explicitly, this means that the work in this section will be natural with regard to commutative diagrams

\[
\begin{array}{ccc}
X & \xrightarrow{u} & Y \\
\downarrow & & \downarrow \\
X' & \xrightarrow{u} & Y'.
\end{array}
\]

where all vertical maps induce the same map \( \varphi: \mathcal{C} \to \mathcal{D} \) on objects.

To understand the pushouts (A.33), we will produce a filtration

\[
\mathcal{O} = \mathcal{O}_0 \hookrightarrow \mathcal{O}_1 \hookrightarrow \mathcal{O}_2 \hookrightarrow \ldots \hookrightarrow \text{colim}_k \mathcal{O}_k = \mathcal{O}[u]
\]  
\tag{A.34}

of the underlying symmetric sequences, i.e. with \( \mathcal{O}_i \in \text{Sym}_*(\mathcal{V}) \) (moreover, all maps in (A.34) will, again, be the identity on colors).

Writing \( \mathcal{U}_\text{Set} \) and \( \mathcal{U}_\text{set} \) for the fibered coproducts in \( \text{Sym}_*(\mathcal{V}) \) and \( \mathcal{O}_\text{Op}(\mathcal{V}) \) (i.e. these are the coproducts within each fixed color fiber over \( \text{Set} \), rather than the coproducts in the overall categories), the discussion in (5.3) through (5.7) of [BP21] yields that

\[
\mathcal{O}[u] \simeq \text{coeq}(\mathcal{U}_\text{set} FX \mathcal{U}_\text{set} FY \Rightarrow \mathcal{O} \mathcal{U}_\text{set} FY) \\
= \text{colim}_{[\pi] \Delta^{op}} B_\pi(\mathcal{O}, FX, FX, FX, FY) \\
= \text{colim}_{[\pi] \Delta^{op}, [\nu] \Delta^{op}} B_{\pi \nu} \left( (\mathcal{U}_\text{set})^{op} \mathcal{O}, FX, FX, FX, FY \right) \\
= \text{colim}_{[\pi] \Delta^{op}, [\nu] \Delta^{op}} \text{Lan}_{\mathcal{O}} \circ \left( N^{\text{op}} \mathcal{U}_\text{set} v X^{\text{set}} v Y \cap \mathcal{U}_\text{set} v Y \right),
\]  
\tag{A.35}

where \( B_* \) denotes the double bar construction with respect to \( \mathcal{U}_\text{set} \), \( \mathcal{F}^{op+1} \mathcal{O} \) denotes the simplicial resolution of \( \mathcal{O} \), and \( N \) is the monad on spans in Definition A.23. Crucially, we note that colimits over \( \Delta^{op} \) are computed by the reflexive coequalizer determined by levels 0 and 1, so that the colimits in (A.35) can be computed in \( \text{Sym}_*(\mathcal{V}) \) rather than in \( \mathcal{O}_\text{Op}(\mathcal{V}) \).

By construction, \( N \left( N^{\text{op}} \mathcal{O} \mathcal{U}_\text{set} v X^{\text{set}} v Y \cap \mathcal{U}_\text{set} v Y \right) \) denotes a certain span \( \Sigma^{\text{op}} \mapsto (\Omega^{\text{op}}_{\mathcal{V}})^{\text{op}} \to \mathcal{V} \) which we will explicitly identify in §A.4. This will then allows us to apply (the natural analogue of) [BP21, Prop. 5.42] along each simplicial direction to convert the last line of (A.35) into a \( \text{Lan} \) over a single span \( \Sigma^{\text{op}} \mapsto (\Omega^{\text{op}}_{\mathcal{V}})^{\text{op}} \to \mathcal{V} \).

The task of describing \( \Omega^{\text{op}}_{\mathcal{V}} \) is similar to the spirit of Proposition A.17, which shows that \( N^{\text{op}+1} \) is naturally calculated using the \( \Omega^{\text{op}}_{\mathcal{V}} (-) \) construction.
In practice, we will prefer to describe a slightly more general variant of the $\Omega_{n,s,\lambda}$ categories. For $\lambda = \lambda_a \cup \lambda_i$, a partition of $\{1, 2, \ldots, l\}$, we write $N^{x, \lambda}$ for the monad (cf. [BP21, §2.3]) on $(\text{WSpan}(\Sigma^*, \mathcal{V}))^{s,t}$ given by

$$\left( N^{x, \lambda}(A_j) \right)_k = \begin{cases} N(A_k) & \text{if } k \in \lambda_a \\ A_k & \text{if } k \in \lambda_i \end{cases}$$

(A.36)

Note that, writing $\{(l)\} = \{-\infty, -l, \ldots, 0, \ldots, l, \infty\}$ and $\lambda_i$ for the partition on $\{(l)\}$ with $(\lambda_i)_a = \{-\infty\}$, the last term in (A.35) is then $\text{Lan}_N \Sigma_{\text{Set}}(N^{\lambda_i})^\circ \circ \iota_0 \circ \iota_1 \circ \phi_1 \circ \phi_2 \circ \ldots \circ \phi_N$.

We note that $N^{x, \lambda}$ preserves the fibered product $(\text{WSpan}(\Sigma^*, \mathcal{V}))^{s,t}$, i.e. the subcategory of those tuples $(A_j)$ where all $A_j$ have the same color set $\mathcal{C} \in \text{Set}$ (and likewise for maps), and we abuse notation by also writing $N^{x, \lambda}$ for the monad restricted to this subcategory.

Out next task is to understand the composites $N \amalg \Sigma_{\text{Set}}(N^{\lambda_i})^\circ$.

**Labeled colored strings**

Let $l \geq 1$. A $l$-labeling of a tree $\mathcal{T} \in \mathcal{C}$ is a map $V(T) \to \{1, \ldots, l\}$. Further, a map $T \to \mathcal{S}$ of labeled trees is called a label map if, for all $v \in V(T)$, all the vertices in $\mathcal{S}_v$ (Notation A.2) have the same label as $v$. Lastly, given a subset $\lambda_i \subseteq \{1, \ldots, l\}$, a label map $\mathcal{T} \to \mathcal{S}$ is called $\lambda_i$-inert if $\mathcal{S}_v$ is a corolla whenever the label of $v \in V(T)$ is in $\lambda_i$.

The categories $\Omega_{\mathcal{C}}^{n,s,\lambda}$ defined below will represent the functors $N^{x, \lambda} \circ \amalg_{\text{Set}} \circ (N^{x, \lambda})^{s,t}$.

**Definition A.37** (cf. [BP21, Def. 5.11]). Given $-1 \leq s \leq n$, $l \geq 0$, and a partition $\lambda = \lambda_a \cup \lambda_i$ of $\{1, 2, \ldots, l\}$, define $\Omega_{\mathcal{C}}^{n,s,\lambda}$ to have as objects $n$-planar strings

$$\text{lr}(\mathcal{T}_0) = \mathcal{T}_0 \xrightarrow{f_0} \mathcal{T}_1 \xrightarrow{f_2} \mathcal{T}_2 \xrightarrow{f_3} \cdots \xrightarrow{f_{s+1}} \mathcal{T}_{s+1} \xrightarrow{f_{s+2}} \cdots \xrightarrow{f_{n}} \mathcal{T}_n$$

together with $l$-labelings of $\mathcal{T}_s, \mathcal{T}_{s+1}, \cdots, \mathcal{T}_n$, such that $f_{s+1}, r > s$ are $\lambda_i$-inert label maps.

Arrows in $\Omega_{\mathcal{C}}^{n,s,\lambda}$ are tuples of isomorphisms $\left( \rho_r : \mathcal{T}_r \to \mathcal{T}_r' \right)$ such that $\rho_r, r \geq s$ are label maps. Further, for any $s < 0$ or $n < s'$, we write

$$\Omega_{\mathcal{C}}^{n,s,\lambda} = \Omega_{\mathcal{C}}^{n,-1,\lambda}, \quad \Omega_{\mathcal{C}}^{n,s',\lambda} = \Omega_{\mathcal{C}}^{n,0}.$$

We now discuss the functors relating the $\Omega_{\mathcal{C}}^{n,s,\lambda}$ categories. Firstly, for $s \leq s'$ and map of labels $\gamma : \{1, \ldots, l'\} \to \{1, \ldots, l\}$ such that $\lambda_i' \subseteq \gamma^{-1}(\lambda_i)$ there are natural functors

$$\Omega_{\mathcal{C}}^{n,s,\lambda} \to \Omega_{\mathcal{C}}^{n,s',\lambda}, \quad \Omega_{\mathcal{C}}^{n,s,\lambda} \to \Omega_{\mathcal{C}}^{n,s,\lambda}.$$

Second, by keeping track of labels on vertices, the functors from §A.1 relating the categories $\Omega_{\mathcal{C}}^{n,s,\lambda}$ extend to the categories $\Omega_{\mathcal{C}}^{n,s,\lambda}$. Indeed, for $k \leq n$ and $\varphi : \mathcal{C} \to \mathcal{D}$ a map of colors one has functors

$$\Omega_{\mathcal{C}}^{n,s,\lambda} \xrightarrow{\varphi^k} \Omega_{\mathcal{C}}^{n-k,1,s-k,1,\lambda}, \quad \Omega_{\mathcal{C}}^{n,s,\lambda} \xrightarrow{\varphi} \Omega_{\mathcal{C}}^{n,s,\lambda}.$$

(A.38)

Lastly, one also has simplicial operators $d_i, s_j$, but some care is needed with the way these interact with the index $s$. To do so, defining functions $d_i, s_j : \mathbb{Z} \to \mathbb{Z}$ by

$$d_i(s) = \begin{cases} s - 1, & i < s \\ s, & s \leq i \end{cases} \quad \text{and} \quad s_j(s) = \begin{cases} s + 1, & j < s \\ s, & s \leq j \end{cases}$$

(A.39)
one has simplicial operators
\[
\Omega_{c}^{n,s,\lambda} \xrightarrow{d_{i}} \Sigma \Omega_{c}^{n,d_{i}(s),\lambda}, \quad \Omega_{c}^{n,s,\lambda} \xrightarrow{s_{j}} \Sigma \Omega_{c}^{n,s_{j}(s),\lambda},
\]
for \(0 \leq i \leq n\) and \(-1 \leq j \leq n\). In practice, we will prefer to suppress \(s\) from the notation, and write \(\Omega_{c}^{n,s,\lambda}\) to denote the string of categories \(\Omega_{c}^{n,s,\lambda}\) as a whole. Lastly, the \(\pi_{i,k}\) natural isomorphisms for \(i < k\) from Proposition A.8 generalize to natural isomorphisms
\[
\Omega_{c}^{n,s,\lambda} \xrightarrow{V^{k}} \Sigma \Omega_{c}^{n-k-1,s-k-1,\lambda}
\]
\[
\Omega_{c}^{n-1,d_{i}(s),\lambda} \xrightarrow{\pi_{i,k}} V_{k-1} \rightarrow \Sigma \Omega_{c}^{n-k-1,1,s-k,\lambda}
\]
(not note that the right vertical map is an identity even if \(s - k - 1 \neq d_{i}(s) - k\), since that can only occur if \(s \leq i \leq k\), implying that the rightmost terms are both \(\Sigma \Omega_{c}^{n-k-1,1,s-k,\lambda}\)).

**Remark A.40.** We now discuss the naturality of the given functors on the categories \(\Omega_{c}^{n,s,\lambda}\) just described.

(i) by keeping track of vertex labels, all the analogues of the properties in Propositions A.8 and A.9 extend (note that this includes the pullback claims in Proposition A.8(ii)).
(ii) the change of color functors \(\varphi\), change of label functors \(\gamma\), and the forgetful functors in (A.38) are all natural with respect to each other.
(iii) \(d_{i}\), \(s_{j}\), \(V^{k}\) and \(\pi_{i,k}\), are natural with respect to the change of color functors \(\varphi\), change of label functors \(\gamma\), and the forgetful functors in (A.38), in the sense that they satisfy the analogues of Proposition A.8(iii) with the role of \(\varphi\) replaced with the latter functors.
(iv) For \(k \leq s \leq s'\) the following squares are pullback squares
\[
\Omega_{c}^{n,s,\lambda} \xrightarrow{V^{k}} \Sigma \Omega_{c}^{n-k-1,s-k-1,\lambda}
\]
\[
\Omega_{c}^{n,s',\lambda} \xrightarrow{V^{s'}} \Sigma \Omega_{c}^{n-k-1,s'-k-1,\lambda}
\]

The following is the main purpose of the \(\Omega_{c}^{n,s,\lambda}\) categories, adapting the work in §A.2.

**Definition A.41** (cf. [BP21, Not. 5.25]). Given a \(l\)-tuple of functors \((A_{j} \rightarrow \Sigma c)_{1 \leq j \leq l}\), we write
\[
(-) \circ (A_{j}) : \text{Cat} \downarrow \Sigma c \rightarrow \text{Cat} \downarrow \Sigma c_{j} \rightarrow \text{Cat} \downarrow \Sigma \cup A_{j}
\]
for the pullback (A.12) for the map \(\Sigma : \cup A_{j} \rightarrow \Sigma : \Sigma c_{j}\).

In particular, for all \(-1 \leq s \leq n\), this defines categories \(\Omega_{c}^{n,s,\lambda}(A_{j})\) via pullbacks (note that the \(s \leq n\) restriction guarantees that the target of the lower \(V^{n}\) is indeed \(\Sigma : \Sigma c\))
\[
\Omega_{c}^{n,s,\lambda}(A_{j}) \xrightarrow{V^{n}} \Sigma : \cup j A_{j}
\]
along with analogues of \(d_{i}\) (for \(i < n\)), \(s_{j}\), \(V^{k}\), \(\pi_{i,k}\) and of the forgetful functors in (A.38) (cf. the discussion following (A.13)).
Proposition A.42. A tuple of commutative squares
\[
\begin{array}{c}
A_j \xrightarrow{\varphi} B_j \\
\downarrow \Sigma e \xrightarrow{\gamma} \Sigma D
\end{array}
\]
induces natural maps \( \varphi : \Omega^{n,\lambda}_e (A_j) \to \Omega^{n,\lambda}_D (B_j) \).

Similarly, a map of tuples \( A_j \to B_{g(i)} \) for \( \gamma : \{1, \ldots, l\} \to \{1, \ldots, l'\} \) induces natural maps
\( \gamma : \Omega^{n,\lambda}_e (A_j) \to \Omega^{n,\lambda'}_e (B_{j'}) \).

Moreover, both \( \varphi \) and \( \gamma \) satisfy the analogues of the commutativity properties in Proposition A.15. In particular, the diagram below commutes.
\[
\begin{array}{c}
\Omega^{n,\lambda}_e (A_j) \xrightarrow{V^k} \Sigma : \Omega^{n-k,\lambda}_e (A_j) \\
\downarrow \varphi \quad \quad \quad \downarrow \Omega^{n,\lambda}_e (B_{j'}) \xrightarrow{\gamma} \Omega^{n-k,\lambda'}_e (B_{j'}) \\
\end{array}
\]

Proof. This follows by repeating the argument in the proof of Proposition A.15.

Corollary A.43 (cf. [BP21, Cor. 5.35]). Let \(-1 \leq k, -1 \leq s \leq n\). There are natural identifications
\[
\Omega^k : \Omega^{n,\lambda}_e (A_j) \simeq \Omega^{n+k,\lambda+1}_e (A_j), \quad \Omega^{n+k,s,\lambda}_e (A_j) \simeq \Omega^{n+k+1,s,\lambda}_e (A_j)
\]
which are unital and associative in the natural ways. Moreover, these induce identifications
\[
d_i : \Omega^{n,\lambda}_e (A_j) \simeq \Omega^{n+k,s,\lambda}_e (A_j) \quad \pi_{i,k} : \Omega^{n,\lambda}_e (A_j) \simeq \pi_{i,k} (A_j) \quad s_j : \Omega^{n,\lambda}_e (A_j) \simeq s_j (A_j)
\]
\[
\Omega^k : (d_i) \simeq (\pi_{i,k}) \quad \Omega^k : (s_j) \simeq (s_{k+j+1})
\]
Proof. Much as in Proposition A.17, this follows by noting that all squares in the following diagrams are pullback squares.
\[
\begin{array}{c}
\Omega^{n+k,\lambda+1}_e (A_j) \xrightarrow{V^k} \Sigma : \Omega^{n,\lambda}_e (A_j) \xrightarrow{V^n} \Sigma : \Omega^{n+k,\lambda+1}_e (A_j) \\
\downarrow \quad \quad \quad \downarrow \quad \quad \quad \downarrow \\
\Omega^k : \Omega^{n+k,\lambda+1}_e (A_j) \xrightarrow{V^k} \Sigma : \Omega^{n,\lambda}_e (A_j) \xrightarrow{V^n} \Sigma : \Omega^{n+k,\lambda+1}_e (A_j) \\
\downarrow \quad \quad \quad \downarrow \quad \quad \quad \downarrow \\
\Omega^{n+k,\lambda+1}_e (A_j) \xrightarrow{V^k} \Sigma : \Omega^{n,\lambda}_e (A_j) \xrightarrow{V^n} \Sigma : \Omega^{n+k,\lambda+1}_e (A_j) \\
\downarrow \quad \quad \quad \downarrow \quad \quad \quad \downarrow \\
\Omega^{n+k,\lambda+1}_e (A_j) \xrightarrow{V^k} \Sigma : \Omega^{n,\lambda}_e (A_j) \xrightarrow{V^n} \Sigma : \Omega^{n+k,\lambda+1}_e (A_j) \\
\downarrow \quad \quad \quad \downarrow \quad \quad \quad \downarrow \\
\Omega^{n+k,\lambda+1}_e (A_j) \xrightarrow{V^k} \Sigma : \Omega^{n,\lambda}_e (A_j) \xrightarrow{V^n} \Sigma : \Omega^{n+k,\lambda+1}_e (A_j) \\
\downarrow \quad \quad \quad \downarrow \quad \quad \quad \downarrow \\
\Omega^{n+k,\lambda+1}_e (A_j) \xrightarrow{V^k} \Sigma : \Omega^{n,\lambda}_e (A_j) \xrightarrow{V^n} \Sigma : \Omega^{n+k,\lambda+1}_e (A_j) \\
\end{array}
\]

\[\square\]

51
Filtration of free extensions

We now return to proving Lemma 5.8, describing the free extensions (A.33). As discussed after (A.36), let $\lambda_l$ denote the partition on

$$
\{(l)\} = \{-\infty, -l, \ldots, 0, 1, \ldots, +\infty\}
$$

such that $(\lambda_l)_a = \{-\infty\}$, and define $N_{n,l}^{(O,X,Y)}$ to be the opposite of the composite

$$
\Omega_{\epsilon}^{n,0,\lambda_l} \xrightarrow{(V^\epsilon)_n^{n+1}} \sum_{l} \sum_{\epsilon} \sum_{\epsilon}^{(O,X,\ldots,Y)} \sum_{l} v_{O,\epsilon}^{op} \otimes v_{Y,op}^{op}.
$$

The upshot of §A.4, in particular Corollary A.43, is that $N \left( N_v X^{\text{Set}} \right)$ is the span $\sum_{l} \sum_{\epsilon}^{op} \sum_{\epsilon} \sum_{\epsilon} \sum_{\epsilon}^{(O,X,Y)} \sum_{l} v_{O,\epsilon}^{op} \otimes v_{Y,op}^{op}$ and hence, following (A.35), we conclude that

$$
\mathcal{O}[u] \simeq \text{colim}_{(\Delta\times\Delta)^{op}} \left( \text{Lan}_{(\Omega_{\epsilon}^{n,0,\lambda_l} \rightarrow \Sigma_{\epsilon})^{op}} N_{n,l}^{(O,X,Y)} \right).
$$

Moreover, the simplicial operators in the $l$ direction are described by antisymmetric functions $\{(l)\} \rightarrow \{(l')\}$ which are given by (A.39) on non-negative values.

In what follows, the realization $|\mathcal{C}|$ of a simplicial object $\mathcal{C} \in \text{Cat}^{\Delta^op}$ is as in [BP21, (A.1)].

**Proposition A.45.** The double simplicial realization $[\Omega_{\epsilon}^{n,\lambda_l}]$, denoted $\Omega_{\epsilon}$ and called the extension tree category, has objects the $\{O,X,Y\}$-labeled trees, and arrows the tall maps $f : \mathcal{T} \rightarrow \mathcal{S}$ such that

(i) if $T_a$ has a $X$-label, then $S_a \in \Sigma_e$ and $S_a$ has a $X$-label;

(ii) if $T_a$ has a $Y$-label, then $S_a \in \Sigma_e$ and $S_a$ has either a $X$-label or a $Y$-label;

(iii) if $T_a$ has a $O$-label, then $S_a$ has only $X$ and $O$ labels.

**Proof.** This is a direct analogue of [BP21, Prop. 5.47], and the proof therein carries through without notable changes, so we only sketch the key arguments. Firstly, it is straightforward [BP21, Rem. 5.41] that, for each fixed $l$, the realization $[\Omega_{\epsilon}^{n,\lambda_l}]$ in the $n$ direction is the category $\Omega_{\epsilon}^{l,\lambda_l}$ with objects the $(\{(l)\})$-labeled trees and maps the tall maps which are inert on colors other than $-\infty/O$. Moreover, maps $\mathcal{T} \rightarrow \mathcal{S}$ in $\Omega_{\epsilon}$ canonically factor as $\mathcal{T} \rightarrow \mathcal{T}' \rightarrow \mathcal{S}$, where the first map is a relabel map (i.e. an underlying isomorphism of trees that just changes labels) and the second map is a label map. Hence, the result follows from the observation that relabel maps in $\Omega_{\epsilon}$ correspond to objects of $\Omega_{\epsilon}^{l,\lambda_l}$ while label maps correspond to maps of $\Omega_{\epsilon}^{l,\lambda_o}$.

We note that the proof of [BP21, Prop. 5.42] holds when replacing $\Omega_{\epsilon}$ with $\Omega_{\epsilon}$, and we thus apply it to (A.44) that the “natural transformation component of differential operators are isomorphisms” condition for the $n$ direction follows from (A.24) and (A.25), while in the $l$ direction it follows since the associated maps of tuples (cf. Proposition A.42) are the identity in each coordinate) to yield

$$
\mathcal{O}[u] \simeq \text{Lan}_{(\Omega_{\epsilon}^{n,0,\lambda_l} \rightarrow \Sigma_{\epsilon})^{op}} N_{n,l}^{(O,X,Y)}.
$$

The desired filtration (A.34) will now be obtained by first replacing $\Omega_{\epsilon}$ in (A.46) with a suitable subcategory $\bar{\Omega}_{\epsilon}$, and then producing a filtration $\bar{\Omega}_{\epsilon}[\leq k]$ of $\bar{\Omega}_{\epsilon}$ itself.
Definition A.47. Let \( \tilde{\Omega}_e \rightarrow \Omega^e \) denote the full subcategory of those labeled trees whose underlying tree is alternating (cf. Example 5.7 and the preceding discussion), active nodes are labeled by \( \mathcal{O} \), and inert nodes are labeled by \( X \) or \( Y \).

We write \( V^{in}(T) \) (resp. \( V^Y(T) \)) for the subset of inert (resp. \( Y \)-labeled) vertices.

Letting \( |\tilde{T}| = |V^{in}(T)| \), \( |\tilde{T}|_Y = |V^Y(T)| \), we define subcategories of \( \tilde{\Omega}_e \) by:

(i) \( \tilde{\Omega}_e[k] \) (resp. \( \tilde{\Omega}_e[k \cdot Y] \)) denotes the full subcategory of those \( \tilde{T} \) with \( |\tilde{T}| \leq k \) (\( |\tilde{T}|_Y = k \));

(ii) \( \tilde{\Omega}_e[k \cdot Y] \) (resp. \( \tilde{\Omega}_e[k \cdot Y] \)) denotes the further full subcategory of those \( \tilde{T} \) with \( |\tilde{T}|_Y \neq k \).

Subcategories \( \tilde{\Omega}_e[k] \), \( \tilde{\Omega}_e[k \cdot Y] \) of the category \( \tilde{\Omega}_e \) of alternating trees are defined similarly.

The following results follow exactly as in the cited results from [BP21] that they adapt.

Lemma A.48 (cf. [BP21, Cor. 5.62, Lemma 5.68]). \( \tilde{\Omega}_e \rightarrow \Omega^e \) is Ran-initial over \( \Sigma_e \). More explicitly, for a functor \( N: \Omega^e \rightarrow \mathcal{V} \) with \( \mathcal{V} \) complete, it is Ran\( \Sigma^e \rightarrow \Sigma \) \( N \sim \) Ran\( \tilde{\Sigma}^e \rightarrow \Sigma \) \( N \).

Similarly, \( \tilde{\Omega}_e[k \cdot Y] \rightarrow \tilde{\Omega}_e[k \cdot Y] \) is Ran-initial over \( \Sigma_e \).

Remark A.49 (cf. [BP21, Remark 5.66]). The following diagram

\[
\begin{array}{ccc}
\tilde{\Omega}_e[k \cdot Y] & \rightarrow & \tilde{\Omega}_e[k] \\
\downarrow & & \downarrow \\
\Omega^e[k] & & \\
\end{array}
\]

is a map of Grothendieck fibrations over \( \Omega^e[k] \) such that fibers over \( \tilde{T} \in \Omega^e[k] \) are the punctured cube and cube categories

\[(Y \rightarrow X)^{\ast V^{in}(T)} - Y^{\ast V^{in}(T)}, \quad (Y \rightarrow X)^{\ast V^Y(T)}, \]

for \( V^{in}(T) \) the set of inert vertices.

We can finally describe the filtration (A.34).

Definition A.50. Let \( \mathcal{O}_k \) denote the left Kan extension, where we abbreviate the restriction of \( N(\mathcal{O}, \mathcal{X}, \mathcal{V}) \) as \( \tilde{N} \)

\[
\begin{array}{ccc}
\tilde{\Omega}^e[k \cdot Y] & \rightarrow & \tilde{\Omega}^e[k] \\
\downarrow & & \downarrow \\
\tilde{\Sigma}^{op} & \rightarrow & \mathcal{V} \\
\Sigma^{op} & \rightarrow & \\
\end{array}
\]

Since \( \tilde{\Omega}^e[0] \sim \Sigma_e \) and the nerve of \( \tilde{\Omega}^e \) is the union of the nerves of the \( \tilde{\Omega}^e[k] \) the desired filtration (A.34) follows. We now finally prove Lemma 5.8.

Proof of Lemma 5.8. Assume first that \( G = * \) is the trivial group.

The desired filtration (5.10) is as described by Definition A.50, so it remains only to check that the filtration stages fit into the pushout diagrams as in (5.11).

To obtain these, we consider the following diagram, where the left square is a pushout at the level of nerves (cf. [BP21, (5.75)])], so that after taking Lan one obtains the right pushout (that
the top right corner is $\mathcal{O}_{k-1}$ follows from Lemma A.48; note that the “ops” convert $\text{Ran}$ to $\text{Lan}$

\[
\begin{align*}
\Omega^c_e[k \times Y] & \longrightarrow \Omega^c_e[k \leq Y] & \text{Lan}_{\Omega^c_e[k \times Y]} \longrightarrow \mathcal{O}_{k-1} \\
\downarrow & & \downarrow \\
\Omega^c_e[k] & \longrightarrow \Omega^c_e[k \leq Y] & \text{Lan}_{\Omega^c_e[k \times Y]} \longrightarrow \mathcal{O}_k
\end{align*}
\]  

(A.51)

But now note that Remark A.49 allows us to iteratively compute the $\text{Lan}$ appearing in (A.51) by first left Kan extending to $\Omega^c_e[k]$, showing that the map between the $\text{Lan}$ terms in (A.51) is

\[
\text{Lan}_{(\Omega^c_e[k] \rightarrow \Omega^c_e)} \left( \bigotimes_{v \in V^{\text{in}}(T)} \mathcal{O}(T_v) \otimes \bigoplus_{v \in V^{\text{in}}(T)} u(T_v) \right)
\]  

(A.52)

which matches (5.12), finishing the proof of the $G = \ast$ case (compare with [BP21, Prop. 5.77]).

For the case of a general group $G$, note first that, since all our constructions are compatible with changes of color $\varphi: \mathcal{C} \rightarrow \mathcal{D}$, the left Kan extension in (A.46) is compatible with the $G$-action on $\mathcal{C}$ (see the discussion following Remark 3.46 concerning cocartesian arrows in $\text{Cat} (\downarrow V)$. Thus, we have the alternative formula

\[
\mathcal{O}[u] \cong \text{Lan}_{(G^{op} \times \Omega^c_e \rightarrow G^{op} \times \Sigma_e)^{op}} N^{(O,X,Y)}.
\]

Moreover, since the subcategories $\overline{\Omega}^c_e$, $\overline{\Omega}^c_e[k \leq Y]$, $\overline{\Omega}^c_e[k]$ are all compatible with the $G$-action, we can replace these categories with $G^{op} \times \overline{\Omega}^c_e$, $G^{op} \times \overline{\Omega}^c_e[k \leq Y]$, $G^{op} \times \overline{\Omega}^c_e[k]$ in Definition A.50 as well as in (A.51) so that the right square is now in $\text{Sym}_V^e(V)^G = \mathfrak{V}_{G^e}^e$. And, since the diagram in Remark A.49 remains a map of Grothendieck fibrations upon applying $G^{op} \times (-)$, one likewise has the $G^{op} \times (-)$ analogue of (A.52), showing that the description in (5.12) holds for a general $G$.

\[\square\]

**Remark A.53** (cf. [BP21, Prop. 5.77]). Similarly to the formulas (3.42) and (3.50) for $\mathcal{F}_X$ and $\mathcal{F}^c_X$, one can use the description of left Kan extensions over maps of groupoids (cf. Remark 3.46) to give an explicit pointwise description of the right pushouts in (A.51). Namely, for each $\mathcal{C}$-profile $\overline{\mathcal{C}} \in \Sigma_e$, one has the following alternative pushout diagrams

\[
\begin{align*}
\bigcup_{[T] \in \text{Iso} (\overline{\mathcal{C}}, \Omega^c_e[k])} \left( \bigotimes_{v \in V^{\text{in}}(T)} \mathcal{O}(T_v) \otimes Q_{\text{in}}(u) \right) \cdot_{\text{Aut}_{\Sigma_e}(\overline{\mathcal{C}})} \text{Aut}_{\Sigma_e}(\overline{\mathcal{C}}) & \longrightarrow \mathcal{O}_{k-1}(\overline{\mathcal{C}}) \\
\downarrow & & \downarrow \\
\bigcup_{[T] \in \text{Iso} (\overline{\mathcal{C}}, \Omega^c_e[k])} \left( \bigotimes_{v \in V^{\text{in}}(T)} \mathcal{O}(T_v) \otimes \bigoplus_{v \in V^{\text{in}}(T)} \mathcal{O}(T_v) \right) \cdot_{\text{Aut}_{\Sigma_e}(\overline{\mathcal{C}})} \text{Aut}_{\Sigma_e}(\overline{\mathcal{C}}) & \longrightarrow \mathcal{O}_k(\overline{\mathcal{C}})
\end{align*}
\]  

(A.54)

\[
\begin{align*}
\bigcup_{[T] \in \text{Iso} (\overline{\mathcal{C}}, G^{op} \times \Omega^c_e[k])} \left( \bigotimes_{v \in V^{\text{in}}(T)} \mathcal{O}(T_v) \otimes Q_{\text{in}}(u) \right) \cdot_{\text{Aut}_{G^{op} \times \Sigma_e}(\overline{\mathcal{C}})} \text{Aut}_{G^{op} \times \Sigma_e}(\overline{\mathcal{C}}) & \longrightarrow \mathcal{O}_{k-1}(\overline{\mathcal{C}}) \\
\downarrow & & \downarrow \\
\bigcup_{[T] \in \text{Iso} (\overline{\mathcal{C}}, G^{op} \times \Omega^c_e[k])} \left( \bigotimes_{v \in V^{\text{in}}(T)} \mathcal{O}(T_v) \otimes \bigoplus_{v \in V^{\text{in}}(T)} \mathcal{O}(T_v) \right) \cdot_{\text{Aut}_{G^{op} \times \Sigma_e}(\overline{\mathcal{C}})} \text{Aut}_{G^{op} \times \Sigma_e}(\overline{\mathcal{C}}) & \longrightarrow \mathcal{O}_k(\overline{\mathcal{C}})
\end{align*}
\]  

(A.55)
where $Q^m_T[u]$ denotes the source of the pushout-product map

$$
\bigtriangleup_{v \in V^m(T)} u(\bar{T}_v) \colon Q^m_T[u] \to \bigotimes_{v \in V^m(T)} Y(\bar{T}_v).
$$

Moreover, just as in Remark 3.49, the contrast between (A.54) and (A.55) is that (A.54) has more coproduct summands while (A.55) has larger inductions.

### A.5 Injective change of color and pushouts of operads

This subsection is dedicated to proving Corollary A.61 below, which will be needed to prove [BPb, Prop. 3.32] in the sequel, and deals with pushouts in $\mathcal{O}_p(V)$ whose maps are injective on colors.

First note that, by a variation of the arguments in (A.35), (A.44), (A.46), any pushout

$$
\begin{array}{ccc}
A & \longrightarrow & \mathcal{O} \\
\downarrow & & \downarrow \\
B & \longrightarrow & \mathcal{P}
\end{array}
$$

in $\mathcal{O}_p(V)$ has a description

$$
\mathcal{P} \simeq \operatorname{colim}_{[l] \in \Delta^{op}} B_l \left( \mathcal{O}, \mathcal{A}, \mathcal{A}, \mathcal{B} \right)
\simeq \operatorname{colim}_{[l] \in \Delta^{op} \times [n] \in \Delta^{op}} B_l \left( \mathcal{P}^{op+1}, \mathcal{P}^{op+1} \mathcal{A}, \mathcal{P}^{op+1}, \mathcal{P}^{op+1} \mathcal{B} \right)
\simeq \operatorname{colim}_{[l] \in \Delta^{op} \times [n] \in \Delta^{op}} \operatorname{lan} \circ \left( \mathcal{N}^{op+1} \mathcal{V} \cup \mathcal{N}^{op+1} \mathcal{A} \cup \mathcal{N}^{op+1} \mathcal{B} \right)
\simeq \operatorname{colim}_{(\Delta \times \Delta)^{op}} \left( \operatorname{lan}(\mathcal{N}^{op+1} \mathcal{V} \cup \mathcal{N}^{op+1} \mathcal{A} \cup \mathcal{N}^{op+1} \mathcal{B}) \right)
\simeq \operatorname{lan}(\mathcal{N}^{op+1} \mathcal{V} \cup \mathcal{N}^{op+1} \mathcal{A} \cup \mathcal{N}^{op+1} \mathcal{B})
\tag{A.56}
$$

where the partition $\lambda^a$ in the fourth line is the fully-active partition with $(\lambda^a)^a = \langle || \rangle$ and, in analogy to Proposition A.45, the double realization $\Omega^a = \mathcal{N}^{op+1} \mathcal{V} \cup \mathcal{N}^{op+1} \mathcal{A} \cup \mathcal{N}^{op+1} \mathcal{B}$ is the category whose objects are the $\mathcal{O}, \mathcal{A}, \mathcal{B}$-labeled trees, and whose arrows are tall maps $\bar{T} \to \bar{S}$ such that

1. if $v \in V(T)$ is $\mathcal{A}$-labeled, then all vertices in $\bar{S}_v$ are $\mathcal{A}$-labeled;
2. if $v \in V(T)$ is $\mathcal{B}$-labeled, then all vertices in $\bar{S}_v$ are either $\mathcal{A}$-labeled or $\mathcal{B}$-labeled;
3. if $v \in V(T)$ is $\mathcal{O}$-labeled, then all vertices in $\bar{S}_v$ are either $\mathcal{A}$-labeled or $\mathcal{B}$-labeled.

Moreover, one has the formula (where $V^\mathcal{O}$, $V^\mathcal{A}$, $V^\mathcal{B}$ denote $\mathcal{O}, \mathcal{A}, \mathcal{B}$-labeled vertices)

$$
\mathcal{N}^{(\mathcal{O}, \mathcal{A}, \mathcal{B})}(\bar{T}_v) = \left( \bigotimes_{v \in V^\mathcal{O}(T)} \mathcal{O}(\bar{T}_v) \right) \otimes \left( \bigotimes_{v \in V^\mathcal{A}(T)} \mathcal{A}(\bar{T}_v) \right) \otimes \left( \bigotimes_{v \in V^\mathcal{B}(T)} \mathcal{B}(\bar{T}_v) \right).
\tag{A.57}
$$

In the following, recall that if $\varphi \in \mathcal{C} \to \mathcal{D}$ is injective, then $\varphi_1 = \varphi_1$, cf. Remark 3.51.
Lemma A.58. Let \( \varphi : C \to D \) be an injective map of colors, \( A \to B \) a map in \( \mathcal{O}_C(V) \), and \( O \in \mathcal{O}_D(V) \). Then, if the leftmost diagram below in \( \mathcal{O}_D(V) \) is a pushout, so is the adjoint rightmost diagram in \( \mathcal{O}_C(V) \).

\[
\begin{array}{ccc}
\varphi_*A & \to & O \\
\downarrow & & \downarrow \\
\varphi_*B & \to & \varphi^*O
\end{array}
\]

\[
\begin{array}{ccc}
A & \to & \varphi^*O \\
\downarrow & & \downarrow \\
B & \to & \varphi^*P
\end{array}
\]

Proof. We start by noting that the top composite in the diagram

\[
\begin{array}{ccc}
\mathcal{O}^p_C & \xrightarrow{\varphi} & \mathcal{O}^p_D \\
\downarrow & & \downarrow \\
\Sigma^{op} & \xrightarrow{\varphi} & \Sigma^{op}_D
\end{array}
\]

is \( N(\varphi^*O,A,B) \) by the last part of Remark 3.51, so that our intended result is equivalent to showing that following map is an isomorphism.

\[
\text{Lan}_{\mathcal{O}^p_C} \to \Sigma^{op}_C N(\varphi^*O,A,B) \cong \text{Lan}_{\mathcal{O}^p_D} \to \Sigma^{op}_D N(\varphi^*O,A,B)
\]

To establish \((A.59)\), first let \( \tilde{\Omega}^p_D \) be the full subcategory of \( \Omega^p_D \) such that, if \( v \in V(T) \) is \( A \) or \( B \)-labeled, then \( \bar{T}_v \in \Sigma_E \) (i.e., all edges of \( \bar{T}_v \) are colored by \( E \)).

It follows from \((A.57)\) that \( N(\varphi^*O,A,B)(\bar{T}) = \emptyset \) whenever \( \bar{T} \notin \tilde{\Omega}^p_D \), and it is straightforward to check that \( \tilde{\Omega}^p_D \) is a sieve of \( \Omega^p_D \), i.e., that for any map \( \bar{T} \to \bar{S} \) in \( \Omega^p_D \) such that \( \bar{S} \in \tilde{\Omega}^p_D \), it is \( \bar{T} \in \tilde{\Omega}^p_D \). It then follows that \( N(\varphi^*O,A,B) \) is the left Kan extension of its restriction to \( \tilde{\Omega}^p_D \), so we are free to replace \( \Omega^p_D \) with \( \tilde{\Omega}^p_D \) in the rightmost Kan extension in \((A.59)\).

Note next that, for each \( \bar{C} \in \Sigma_E \), the inclusion of undercategories \((\bar{C} \downarrow \Omega^p_C) \to (\bar{C} \downarrow \tilde{\Omega}^p_D) \) has a left adjoint given by \( \bar{T} \mapsto \bar{T} - E_{D,C}(\bar{T}) \), where \( E_{D,C}(\bar{T}) \) is the set of edges of \( \bar{T} \) whose colors are not in \( C \) (that this has a natural vertex labeling follows since all the edges being collapsed must connect \( O \)-vertices, so there is no ambiguity as to how to label the vertices of \( \bar{T} - E_{D,C}(\bar{T}) \)). But this shows that the opposite maps \((\bar{C} \downarrow \Omega^p_C)^{op} \to (\bar{C} \downarrow \tilde{\Omega}^p_D)^{op} \) are final and, since \((A.59)\) is computed via colimits over these (opposite) undercategories, the result follows.

Corollary A.60. Suppose that \( F,G \) on the left pushout diagram below are both injective on colors.

\[
\begin{array}{ccc}
A & \xrightarrow{F} & O \\
\downarrow & & \downarrow \\
B & \xrightarrow{F} & \varphi^*O
\end{array}
\]

Then the rightmost diagram is also a pushout diagram.

Proof. This follows by adding to both \( A \) and \( O \) a disjoint trivial operad on the object set \( E_B \setminus E_A \). Doing so does not alter the left pushout, but the map \( G \) now becomes a fixed color map, so that Lemma A.58 can be applied.
Corollary A.61. Suppose that we have a pushout in \( \mathcal{Op}_c(V) \) such that \( F, G \) are both injective on colors.

\[
\begin{array}{ccccc}
A & \xrightarrow{F} & O \\
\downarrow & & \downarrow & & \\
B & \xrightarrow{G} & P \\
\end{array}
\]

If \( F: A \to O \) is a local isomorphism, then so is \( \bar{F}: B \to P \).

Proof. The desired claim can be restated as saying that, if \( A \to F^*O \) is an isomorphism, then so is \( B \to \bar{F}^*O \). But this is immediate from Corollary A.60.

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