Optimal $C^{1,\alpha}$ regularity for degenerate fully nonlinear elliptic equations with Neumann boundary condition

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Abstract

In the present paper, we study sharp $C^{1,\alpha}$ regularity results with boundary Neumann condition for viscosity solutions for a class of degenerate fully non-linear elliptic equations with Neumann boundary conditions.

Keywords: Regularity theory, optimal a priori estimates, fully nonlinear elliptic equations.

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1. Introduction

In this paper, we study sharp $C^{1,\alpha}$ regularity estimates of viscosity solutions for

$$|\nabla u|^\gamma F(D^2 u) = f,$$

(1.1)

with Neumann boundary condition $\nabla u \cdot v = g$ when $\Omega \subset \mathbb{R}^n$ is a bounded domain with $\partial \Omega \in C^{1,1}$, $v$ denotes the inner normal of $\Omega$, $f$ is a function defined in $\Omega$, $g$ are functions defined on $\partial \Omega$ and $\gamma > 0$, $F$ is uniformly elliptic, $F(0) = 0$. Notice that solutions $u$ of (1.1) cannot be more regular than $C^{1,\alpha}$. More precisely for $0 < \alpha < 1$, the function

$$u(x) = |x|^{1+\alpha}$$

(as mentioned in [1,17]) satisfies

$$|\nabla u|^\gamma \Delta u = C |x|^{(1+\alpha)(\gamma+1)-(\gamma+2)}$$

where $C = (1 + \alpha)^{1+\gamma}(n + \alpha - 1)$. The RHS is in $L^\infty$, if we choose $\alpha = \frac{1}{1+\gamma}$. This example shows that the best regularity that one can expect for solutions to (1.1) is $C^{1,\frac{1}{1+\gamma}}$. The specificity of these equations is that they are not uniformly elliptic; they are either singular or degenerate (in a way to be made precise). Singular/degenerate fully non-linear elliptic equations of the type (1.1) makes part of a class of non-linear elliptic equations studied in a series of papers by Birindelli and Demengel, stating with [3] (for singular case) and [4,5,6] for the degenerate case. Interior regularity properties of viscosity solutions of (1.1) have been studied since a long time, starting with the seminal paper of C. Imbert and L. Silvestre [17], which contains interior $C^{1,\alpha}$ estimates for (1.1) when $f \in L^\infty(B_1)$. Very recently, optimal regularity was proved in [1], with the optimal Hölder’s coefficient $\alpha = \frac{1}{1+\gamma}$.
For Dirichlet boundary data, the regularity up to the boundary is fairly well understood. To know, Berindelli and Demengel [7] proved $C^{1,\alpha}(\overline{\Omega})$ estimates in the presence of a regular boundary datum. However, for the Neumann problem, there are still not many results. Regularity properties of viscosity solutions of fully nonlinear elliptic equations $F(D^2u) = f$ with Neumann boundary condition have been with the seminal paper L. Silvestre and E. Milakes [22], which contains in particular $C^{1,\alpha}$ estimates to viscosity solutions of
\[
\begin{cases}
F(D^2u) = f & \text{in } \Omega,
\end{cases}
\]
when $f \in L^p(\Omega)$, $p > n$ and $g \in C^\beta(\partial\Omega)$. Very much inspired by that breakthrough, we wanted to complete the work, showing that a similar result holds at the flat boundary (see Theorems 2.1).

Very recently, the $C^{1,\alpha}$ estimate for degenerate fully nonlinear equations of the type
\[
\begin{cases}
|\nabla u|^\gamma F(D^2u) = f & \text{in } \Omega, \\
\nabla u \cdot \nu = g & \text{in } \partial\Omega,
\end{cases}
\] (1.2)
has been solved in [2]. In this paper we will develop the optimal regularity theory for (1.2). Precisely, we will apply the technique presented in [1] to prove that viscosity solution to (1.2) are $C^{1,\alpha}$, where $\alpha = \min\{\alpha_0, \beta, \gamma^\frac{1}{\gamma}\}$ (see (4.2)). The main difference between this article and [2] is bases on the following fundamental steps to achieve this property are: ABP type estimate with boundary Neumann condition “when the gradient is large” (see Section 3.1) and $C^{1,\alpha_0}$ estimate for $F$-harmonic functions combined with a certain geometric oscillation estimate which has its roots in the paper [1]. In our problem, the structure of the operator requires some changes.

The paper is organized as follows. In Section 2 we specify the notation to be used in the paper and main results. In Section 3.1, we will prove $C^{0,\alpha}$ regularity for (1.1). The method of proof is similar to that of [1,22]: Getting $C^{0,\alpha}$ estimates consists in proving an Alexandrov-Bekelman-Pucci estimate for (1.1) which says that if a positive solution is small at some point, then it is small in a set of positive measure. Interior Alexandrov-Bakelman-Pucci (ABP) estimate were obtained for such equations independently in [11] and [16]. The authors prove that such functions satisfy a Harnack inequality and are Hölder continuous. The idea is that the function is already regular where the gradient is small, and where the gradient is large it solves an equation. The difficulty is that we don’t know apriori where the gradient is large. The key step is an ABP-type estimate which says that if a positive solution is small at some point, then it is small in a set of positive measure. It was used to derive Harnack inequality in the singular case in [12] and in both cases in [16]. From Harnack inequality, it is classical to derive Hölder estimate [12] in the singular case, [16] in both cases. In Section 4 we will $C^{1,\alpha}$ regularity. Getting $C^{1,\alpha}$ estimates consists in proving that the graph of the function $u$ can be approximated by planes with an error bounded by $Cr^{1+\alpha}$ in semi-balls of radius $r$. The proof is based on an iterative argument, in which we show that the graph of $u$ gets flatter in smaller semi-balls. When we consider the problem (1.1) the principal difficulty lies in the following fact: if $\ell(x) = a + \vec{p} \cdot x$ is a affine function and $u$ is a viscosity solution for the problem (1.1), we can not conclude that $u + \ell$ is a viscosity solution for the problem (1.1). In [22], this fact is important because it allows us to apply regularity theory for $v = u + \ell$ which is crucial to reach a improvement of flatness for the problem
\[
|\nabla u + \vec{q}|^\gamma F(D^2u) = f & \text{in } B_1,
\]
where $\vec{q} \in \mathbb{R}^n$. In Section 4 is devoted to obtaining an extension of Lemmata 4, 5 and 6 in [17] to Neumann boundary conditions.
2. Background results

In this section, we will present notations and main assumptions which we will work throughout this article. Furthermore, we will also collect some preliminary results for future references. For the reader’s convenience we recall the definition of viscosity solutions of fully nonlinear elliptic equations and provide a brief collection of basic results related to this notion. We always assume that $F$ is $(\lambda,\Lambda)$-elliptic, i.e., there exists constants $0 \leq \lambda \leq \Lambda < \infty$ such that

$$\lambda \|N\| \leq F(M + N, x) - F(M, x) \leq \Lambda \|N\|$$

holds for $M, N \in \text{Sym}(n), N \geq 0$ and $x \in \Omega$, where $\|M\| := \sup_{|x|=1} |Mx|$. Also $\nabla u$ will denote the total gradient of $u$. We will introduce the well-known Pucci’s extremal operators: Let $0 < \lambda \leq \Lambda$ be given constants. For $M \in \mathcal{S}(n)$ we define:

$$\mathcal{P}^+_{\lambda, \Lambda}(M) := \lambda \sum_{e_{i} > 0} e_{i} + \lambda \sum_{e_{i} < 0} e_{i} \quad \text{and} \quad \mathcal{P}^+_{\lambda, \Lambda}(M) := \lambda \sum_{e_{i} > 0} e_{i} + \Lambda \sum_{e_{i} < 0} e_{i},$$

where $e_{i} = e_{i}(M)$ denote the eigenvalues of $M$. We say that $u$ is a viscosity subsolution (supersolution) of $(1.1)$ if for any $\varphi \in C^{2}(\Omega \cup \partial \Omega)$ touching $u$ by above (below) at $x_{0}$ in $\Omega \cup \partial \Omega$, we have that

$$|\nabla \varphi(x_{0})|^{2}F(D^{2}\varphi(x_{0})) \geq (\leq) f(x_{0}) \quad \text{if} \quad x_{0} \in \Omega$$

and

$$\nabla \varphi(x_{0}) \cdot \nu \geq (\leq) g(x_{0}) \quad \text{if} \quad x_{0} \in \partial \Omega,$$

If $u$ is both subsolution and supersolution, we call it a viscosity solution. We will denote by $C^{0,\alpha}(x_{0})$ (or $C^{1,\alpha}(X_{0})$) the class of functions $\nu$ which are Hölder continuous (or which have Hölder continuous first order derivatives with Hölder exponent $\alpha \in (0,1)$). By $\| \cdot \|_{C^{0,\alpha}}$ we mean the maximum of $L^{\infty}$-norm and the $\alpha$-Hölder semi-norm. Also by $\| \cdot \|_{C^{1,\alpha}}$ we mean the maximum of $L^{\infty}$-norm and the $\| \cdot \|_{C^{\alpha}}$-norm of the gradient. Finally, an arbitrary point in $\mathbb{R}^{n}$ will be denoted by $x' = (x_{1}, \ldots, x_{n-1})$, $x = (x', x_{n})$ and $B_{r}(x)$ will denote the Euclidean ball in $\mathbb{R}^{n}$ centered at $x$ of radius $r$. Let us set $B_{r}^{+} = B_{r}(0) \cap \{x_{n} > 0\}$ and $Y = B_{r}(0) \cap \{x_{n} = 0\}$. In the present paper we are going to show that a similar result holds at the flat boundary. Thus, we assume that $u$ solves

$$\begin{cases}
|\nabla u|^{2}F(D^{2}u) = f & \text{in } B_{r}^{+} \\
\nabla u \cdot \nu = g & \text{in } Y
\end{cases}$$

\hspace{1cm} (2.2)

where $\nu = \tilde{e}_{n} = (0,0,\ldots,1)$.

Now, we state our mains results.

**Theorem 2.1** ($C^{1,\alpha}$ regularity). Let $f \in L^{\infty}(B_{1}^{+})$, $g \in C^{\beta}(Y)$ and $u$ be a viscosity solution to $(1.1)$. Then $u \in C^{1,\alpha}(B_{1}^{+})$ where $\alpha = \min \left( \alpha_{0}, \beta, \frac{1}{1-\gamma} \right)$ and $\alpha_{0}$ is the optimal Hölder exponent for solutions to constant coefficient, homogeneous equation. Moreover, we have the estimate

$$\|u\|_{C^{1,\alpha}(B_{1/2}^{+})} \leq C \left( \|u\|_{C(B_{1}^{+})} + \|g\|_{C^{\beta}(Y)} + \|f\|_{L^{\infty}(B_{1}^{+})} \right)$$

for a universal constant $C$. 

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3. Some important results

In the present section we intend to prove $C^{0,\alpha}$ regularity for the solution up to the boundary for \((1.1)\). For this, we obtain an extension to the ABP estimate to Neumann boundary conditions for degenerate elliptic equations when the gradient is large. The idea is that the function is already regular where the gradient is small, and where the gradient is large it solves an equation. The difficulty is that we dont know apriori where the gradient is large. As a consequence, applying the theory developed in [16], we obtain Hölder regularity.

3.1. ABP estimate with boundary Neumann condition \textit{“when the gradient is large”}

In our discussion in this section, we will instead be considering slightly more general degenerate elliptic operators. We generalize to the ABP-estimate to Neumann boundary conditions to the case of fully non-linear elliptic equations in non-divergence form

\[
\begin{cases}
F(x, \nabla u, D^2u) = 0, & \text{in } B^+_1 \\
\nabla u \cdot \nu = g & \text{in } \Gamma.
\end{cases}
\] (3.1)

which can be either degenerate or singular. We study functions which are viscosity solutions to (3.1), but only at points where the gradient is large. More precisely, we consider operators $F$ satisfying:

\[
\begin{cases}
|\vec{p}| \geq M_F \implies \mathcal{D}^+(M) + \sigma(x)|\vec{p}| + f(x) \geq 0, \\
F(x, \vec{p}, M) \geq 0 & (3.2)
\end{cases}
\]

\[
\begin{cases}
|\vec{p}| \geq M_F \implies \mathcal{D}^-(M) - \sigma(x)|\vec{p}| - f(x) \leq 0, \\
F(x, \vec{p}, M) \leq 0 & (3.3)
\end{cases}
\]

for some continuous functions $\sigma, f : B^+_1 \to \mathbb{R}$ and $M_F, \sigma_2$ is a non-negative constant. Throughout the section, we assume the following assumptions:

Assumption (A).

- $F$ is continuous on $\Omega \times \mathbb{R}^n \setminus B_{M_F} \times S(n)$ for some $M_F \geq 0$;
- $F$ is (degenerate) elliptic, i.e. for all $X \in \Omega, \vec{p} \in \mathbb{R}^n$ ($\vec{p} \neq 0$ for singular equation) and $M, N \in \mathcal{S}(n)$,

\[
M \leq N \implies F(x, \vec{p}, N) \leq F(x, \vec{p}, M).
\]

Silvestre and Milakis recently studied ABP-estimates to Neumann boundary conditions for Fully nonlinear equations in [22]. Our first main contribution in this paper is to prove the following result:

Theorem 3.1. Consider a non-linearity $F$ which satisfies (A) and (3.2) and (3.3). Let $u$ be a supersolution of \((3.1)\). Then

\[
\inf_{\partial B_{r}\cap \{u>0\}} u - \inf_{B^+_1} u \leq C r \left(M^n_F + \int_{\{u=\Gamma_u\}} |f^+(x)|^n dx\right)^{\frac{1}{n}} + C r \sup_{\Gamma} g.
\] (3.4)

where $\Gamma_u$ is the convex envelop of $u$ in $\Omega$ and $C = C(n, \lambda, \Lambda) > 0$.

Proof. The proof goes in the lines of [22]. Without loss of generality we can assume $\inf_{\partial B_{r}\cap \{u>0\}} u = 0$. Next we will analyze two cases:
1. $-\inf_{B_{\gamma}^+} u \leq 4r \sup_{\gamma} g$. In this case the proof is immediate.
2. $-\inf_{B_{\gamma}^+} u > 4r \sup_{\gamma} g$. Define the following set
   \[ \Theta = \left\{ Z \in \mathbb{R}^n : \sup_{\gamma} g \leq \langle Z, v \rangle, |Z| \leq \frac{-\inf_{B_{\gamma}^+} u}{2r} \right\}. \]

Let $u(x_0) = \inf_{B_{\gamma}^+} u$. Notice that given $Z \in \Theta$ we have:
   
   i. The linear function $\langle Z, x - x_0 \rangle + u(x_0)$ touches $u$ at $x_0$ and is below $u$ in $\partial B \cap \{ x_n > 0 \}$.
   
   ii. From i. there is $b \in \mathbb{R}$ such that the translation $\langle Z, x \rangle + b$ touches $u$ from below in a point that is not in $\partial B \cap \{ x_n > 0 \}$. Thus, since $\langle Z, v \rangle \geq \sup_{\gamma} g$, the function $\langle Z, x \rangle + b$ can not touch $u$ in a point of $\Upsilon$.

From i. and ii. we conclude that
   \[ \Theta \subset \nabla \Gamma \left( B_{\gamma}^+ \right). \]

Thus, we obtain $\Theta \setminus B_{M_F} \subset \nabla \Gamma \left( B_{\gamma}^+ \right) \setminus B_{M_F}$ and since $\Gamma_u$ is $C_{1,1}$ in $\mathcal{B} = \{ x \in B_{\gamma}^+ : |\nabla \Gamma(x)| \geq M_F \}$ (see [16]) we estimate
   \[ c |R^n - M_F^\gamma| \leq |\Theta \setminus B_{M_F}| \leq |\nabla \Gamma \left( B_{\gamma}^+ \right) \setminus B_{M_F}| \leq \int_{\{ u = \Gamma_u \}} |f^+ + \nu| \alpha dx, \]
   where $R = \frac{-\inf_{B_{\gamma}^+} u}{2r}$ and $c = c(n)$ is a positive constant. The Proposition is concluded.

In the present section we intend to prove $C^{0,\alpha}$ regularity for the solution up to the boundary. The classical ABP estimate (Theorem 3.1) says the following

**Lemma 3.2.** Let $\bar{q} \in \mathbb{R}^n$, $f \in C(B_{\gamma}^+)$, $g \in L^\infty(\Upsilon)$ and $u$ be a viscosity solution of

\[
\begin{align*}
|\nabla u + \bar{q}|^2 F(D^2 u) &= f & \text{in } B_{\gamma}^+, \\
\nabla u \cdot v &= g & \text{in } \Upsilon,
\end{align*}
\]

(3.5)

Suppose that $|\bar{q}| \leq a_0$ with $a_0 = a_0(\lambda, \Lambda, n, \gamma) > 0$. Then $u$ is $C^{\beta_1}(B_{1/2}^+) \text{ up to boundary}$, for a universal constant $\beta_1 \in (0, 1)$. Moreover,

\[
|u|_{C^{\alpha \beta_1}(B_{1/2}^+)} \leq C \left( |u|_{L^\infty(B_{\gamma}^+)} + |g|_{L^\infty(\Upsilon)} + \max(M_F, |f|_{L^\infty(B_{\gamma}^+)}) \right)
\]

(3.6)

for a universal constant $C$ and $M_F = M_F(n, \lambda, \Lambda, \gamma, r) > 0$.

**Proof.** If we can show that any function $u \in C(B_{\gamma}^+)$ such that $|\nabla u|^2 F(D^2 u) = f$ in $B_{\gamma}^+$ and $\nabla u \cdot v = g$ in $\Upsilon \cap B_{r}$ satisfies

\[
\text{osc}_{B_{1/2}^+} u \leq (1 - \mu) \text{osc}_{B_{1/2}^+} v + r \left( M_F + |f|_{L^\infty\Upsilon} + \sup_{\Upsilon} g \right) + r^2 |F(0)|,
\]

(3.7)

for universal constant $\mu > 0$, $M_F > 0$ and $C$. Then applying (3.7) to translations of $u$ we obtain a $C^\alpha$ modulus of continuity for $u$ at the bottom $\Upsilon$ by a standard iterative argument. Then, (3.6) follows by interior regularity. So we are going to show (3.7). Let $v$ be the solution of the problem

\[
\begin{align*}
F(D^2 v) &= 0 & \text{in } B_{\gamma}^+, \\
v &= u & \text{in } \partial B_{\gamma} \cap \{ x_n > 0 \}
\end{align*}
\]

\[
\nabla v \cdot v = 0 & \text{ in } \Upsilon \cap B_{r}
\]

(3.8)
As in [22] we obtain
\[
\text{osc}_{B_r^{1/2}} v \leq (1 - \mu) \text{osc}_{B_r} u + C|F(0)| r^2, \tag{3.9}
\]
where \(0 < \mu < 1\). Since \(v \in C^{1,\alpha}(B_r^+)\) (see, [22]), \(w = u - v + \langle \nabla v(0), x \rangle\) is a solution to
\[
\begin{align*}
\{ \nabla w + (\nabla v - \nabla v(0) + \tilde{q})|t|^7 F(D^2w) &= f \quad \text{in } B_r^+ \\
w &= \langle \nabla v(0), x \rangle \quad \text{in } \partial B_r \cap \{x_n > 0\} \\
\nabla w \cdot v &= g + \nabla v(0) \cdot v \quad \text{in } Y \cap B_r
\end{align*}
\tag{3.10}
\]
and
\[
|\nabla v - \nabla v(0) + \tilde{q}|_\infty \leq c_0(n, \lambda, \Lambda) + a_0 : = M_F^{-1},
\]
where \(c_0 = [v|_{C^{1,\alpha}(B_r^+)}^+}\). The equation (3.10) can be written as
\[
\begin{align*}
G(Dw, D^2w) &= f \quad \text{in } B_r^+ \\
w &= \langle \nabla v(0), x \rangle \quad \text{in } \partial B_r \cap \{x_n > 0\} \\
\nabla w \cdot v &= g + D\nabla v(0) \cdot v \quad \text{in } Y \cap B_r
\end{align*}
\]
where \(G(\tilde{p}, X) = |\tilde{p} + (\nabla v - \nabla v(0) + \tilde{q})|F(X)\). In particular, if \(|\tilde{p}| \geq 2M_F^{-1}\) then
\[
|\tilde{p} + (\nabla v - \nabla v(0) + \tilde{q})|^7 \geq M_F^{-7}.
\]
In particular,
\[
\begin{align*}
G(\tilde{q}, M) &= f(x) \\
|\tilde{p}| \geq 2M_F^{-1} &\implies \begin{cases} 
\partial^+(D^2w) + M_F^{-1} \frac{|f|}{M_F} \geq 0 \\
\partial^-(D^2w) - M_F^{-1} \frac{|f|}{M_F} \leq 0
\end{cases}
\end{align*}
\]
Thus, applying the Lemma[5.1] to \(w\) and \(-w\) we obtain
\[
\sup_{B_r^+} |w| \leq Cr \left( M_F + \|f\|_{L^p} \right) + Cr \sup_{\Gamma} g + \sup_{B_r^+} |\langle \nabla v(0), x \rangle|.
\tag{3.11}
\]
Re-scaling of the \(C^{1,\alpha}\) estimates we find
\[
|\nabla v(0)| \leq C \left( \frac{\text{osc}_{B_r^{-1/2}} v}{r} + Cr |F(0)| \right).
\tag{3.12}
\]
From (3.11) and (3.12) we estimate
\[
\sup_{B_r^+} |w| \leq Cr \left( M_F + \|f\|_{L^p} \right) + Cr \sup_{\Gamma} g + C\text{osc}_{B_r^{1/2}} v + Cr^2 |F(0)|.
\tag{3.13}
\]
Then, we get
\[
\text{osc}_{B_r^{1/2}} u \leq (1 + C) \text{osc}_{B_r^{1/2}} v + Cr \left( M_F + \|f\|_{L^p} \right) + Cr \sup_{\Gamma} g + Cr^2 |F(0)| \leq (1 + C)(1 - \mu) \text{osc}_{B_r} u + Cr \left( M_F + \|f\|_{L^p} \right) + Cr \sup_{\Gamma} g + Cr^2 |F(0)|.
\tag{3.14}
\]
Finally, we find
\[ \text{osc}_{B_{r/2}} u \leq (1 - \mu) \text{osc}_{B_r} u + r \left( M_F + \| f \|_{L^1} + \sup_{\Omega} g \right) + r^2 |F(0)|, \]  
where \( \mu = \frac{u}{1 + C} \). By a standard iterative argument the corollary is concluded.

In [2], Eq. (3.5) is considered when \( |q| \geq 1 \). This is achieved in [17] via the Ishii-Lion’s type doubling variable argument using which the authors were able to obtain uniform Lipchitz estimates for solutions to (3.5) when \( |q| \) is large enough. Precisely, we use the following result:

**Lemma 3.3** (See [2], Lemma 5.1). Let \( \gamma \in (0, +\infty) \). Assume that \( |q| \geq 1 \) and let \( w \) be a viscosity solution to equation (3.5) with Neumann boundary condition \( \nabla w \cdot \nu = g \) in \( \Omega \). For all \( r \in (0, 1) \), there exists a constant \( C = C(n, \gamma, \beta_1) > 0 \) such that for all \( x, y \in B_r \),
\[ |w(x) - w(y)| \leq C \left( \| w \|_{L^\gamma(B_r)} + \| g \|_{L^\gamma(B_r)} + \| f \|_{L^\gamma(B_r)} \right) \cdot |x - y|. \]

Since the Hölder estimates depends only on the bounds of the solution, of \( f, g \) and on the structural constants, an immediate consequence of Lemmas 3.2 and 3.3 is the following compactness criterion that will be useful in the last section.

**Corollary 3.4.** Assume the hypothesis of Lemmas 3.2 and 3.3. Let \((\bar{q}_k)\) is a sequence in \( \mathbb{R}^n \) and \((f_k)_k, (g_k)_k\) are sequences of continuous and uniformly bounded functions and \((u_k)_k\) is a sequence of uniformly bounded viscosity solutions of,
\[
\begin{cases}
|\nabla u_k + \bar{q}_k| F(D^2 u_k) = f_k(x), & \text{in } B^+_1, \\
\nabla u_k \cdot \nu = g_k & \text{in } \Omega
\end{cases}
\]

Then the sequence \((u_k)_k\) is relatively compact in \( C(B^+_1) \). In particular, if \( u_k \to u_\infty, \bar{q}_k \to \bar{q}_\infty \) and \( f_k \to f_\infty \) and \( g_k \to g_\infty \), then \( u_\infty \) is a viscosity solution to
\[
\begin{cases}
|\nabla u_\infty + \bar{q}_\infty| F(D^2 u_\infty) = f_\infty(x), & \text{in } B^+_1, \\
\nabla u_\infty \cdot \nu = g_\infty & \text{in } \Omega
\end{cases}
\]

To end the section, as in [17], we used the following lemma.

**Lemma 3.5.** Let \( \bar{q} \in \mathbb{R}^n, \varphi \in C(\partial B_1 \cap \{ x_n > 0 \}), g \in L^\infty(\Omega) \) and \( u \) be a viscosity solution of
\[
\begin{cases}
|\nabla u + \bar{q}| F(D^2 u) = 0 & \text{in } B^+_1 \\
u = \varphi & \text{in } \partial B_1 \cap \{ x_n > 0 \} \\
\nabla u \cdot \nu = g & \text{in } \Omega
\end{cases}
\]

Then \( u \) is a viscosity solution of
\[
\begin{cases}
F(D^2 u) = 0 & \text{in } B^+_1 \\
u = \varphi & \text{in } \partial B_1 \cap \{ x_n > 0 \} \\
\nabla u \cdot \nu = g & \text{in } \Omega
\end{cases}
\]

**Proof.** We can reduce the problem to the case \( \bar{p} = 0 \) since \( v = u^\gamma \bar{p} \cdot x \) solves \( |\nabla v| F(D^2 v) = 0 \). It is sufficient to prove the super-solution property since the sub-solution property is similar. Suppose that \( P(x) \) touches \( u \) from below (resp. above) at a point \( x_0 \in B^+_1 \). If \( x_0 \in B^+_1 \), then by Lemma 6 in [17], \( F(D^2 P(x_0)) \geq 0 \) (resp. \( \leq 0 \)). If \( x_0 \in \Omega \), we know that \( P \) satisfy also the same inequality for the normal derivatives in the point \( x_0 \in \Omega \).
4. Proof of Main Result

In this section we give a proof for Theorem 2.1. We assume that \( \gamma > 0 \) and \( f \in L^\infty(B_1^+) \), \( g \in C^\beta(\Upsilon) \), \( \varphi \in C(\partial B_1 \cap \{ x_n > 0 \}) \), and we want to show that any viscosity solution \( u \) of

\[
\begin{cases}
|\nabla u|^\gamma F(D^2 u) = f & \text{in } B_1^+ \\
u = \varphi & \text{in } \partial B_1 \cap \{ x_n > 0 \} \\
\nabla u \cdot \nu = g & \text{in } \Upsilon.
\end{cases}
\tag{4.1}
\]

is in \( C^{1,\alpha}(\overline{B_{1/2}}) \), where

\[
\alpha := \min \left\{ \alpha_0, \beta, \frac{1}{1+\gamma} \right\}
\tag{4.2}
\]

where \( \alpha_0 \) is the optimal exponent of regularity theory for homogeneous equations \( F(D^2 u) = 0 \) in \( B_1^+ \) with Neumann boundary condition \( \nabla u \cdot \nu = 0 \) in \( \Upsilon \) (see Theorem 6.1 in [17]), where the estimate indicated in (4.2) should be read as

\[
\begin{cases}
\text{If } \min \left\{ \beta, \frac{1}{1+\gamma} \right\} < \alpha_0, \text{ then } u \in C^{1,\min\{\beta, \frac{1}{1+\gamma}\}} \\
\text{If } \min \left\{ \beta, \frac{1}{1+\gamma} \right\} \geq \alpha_0, \text{ then } u \in C^{1,\sigma}, \text{ for any } 0 < \sigma < \alpha_0
\end{cases}
\]

4.1. Reduction of the problem

In this section, we first show that a simple rescaling reduces the proof of the problem to the case that \( \|u\|_{L^\infty(B_{1/2}^+)} \leq 1 \), \( \|g\|_{C^\beta(\Upsilon)} \leq \varepsilon_0 \) and \( \|f\|_{L^\infty(B_{1/2}^+)} \leq \varepsilon_0 \) for some small constant \( \varepsilon_0 \) which will be chosen later. We then further reduce the proof to an improvement of flatness lemma. We work with the arbitrary normalization \( \|u\|_{L^\infty(B_{1/2}^+)} \leq 1 \) because that implies that \( \text{osc } u \leq 1 \) and that will be a good starting point for our iterative proof of \( C^{1,\alpha} \) regularity.

**Proposition 4.1.** In order to prove Theorem 2.1 it is enough to prove that

\[
\|u\|_{C^{1,\alpha}(\overline{B_{1/2}})} \leq C
\]

assuming \( \|u\|_{L^\infty(B_{1/2}^+)} \leq 1 \), \( \|g\|_{C^\beta(\Upsilon)} \leq \varepsilon_0 \) and \( \|f\|_{L^\infty(B_{1/2}^+)} \leq \varepsilon_0 \) for some \( \varepsilon_0 > 0 \) depends on the ellipticity constants, dimension and \( \gamma \).

**Proof.** Given any function \( u \) under the assumptions of Theorem 2.1 we can take

\[
\eta = \frac{1}{\|u\|_{L^\infty(B_{1/2}^+)} + \varepsilon_0^{-1} (\|g\|_{C^\beta(\Upsilon)} + \|f\|_{L^\infty(B_{1/2}^+)})}
\]

and consider the scaled function \( \tilde{u}(x) = \eta u(x) \) solving the equation

\[
\begin{cases}
|\nabla \tilde{u}|^\gamma F_\eta(D^2 \tilde{u}) = \tilde{f}, & \text{in } \Omega \\
\nabla \tilde{u} \cdot \nu = \tilde{g} & \text{in } \Upsilon
\end{cases}
\]
where \( F_\eta(M) = \eta F(\eta^{-1}M), \) \( \tilde{f}(x) = \eta^1 k f(x) \) and \( \tilde{g}(x) = \eta g(x) \). Note that, the function \( F_\eta(\cdot) \) has the same ellipticity constant as \( F(M) \). But now \( \| \tilde{u} \|_{L^2(B_1^+)} \leq 1, \| \tilde{g} \|_{C^\beta(B_1^+)} \leq \varepsilon_0 \) and \( \| \tilde{f} \|_{L^\infty(B_1^+)} \leq \varepsilon_0 \). Therefore, if

\[
\| \tilde{u} \|_{C^{1,\alpha}(\overline{B_{1/2}^+})} \leq C,
\]

by scaling back to \( u \), we get

\[
\| u \|_{C^{1,\alpha}(\overline{B_{1/2}^+})} \leq C \left( \| u \|_{C(B_1^+)} + \| g \|_{C^\beta(\overline{\Upsilon})} + \| f \|_{L^\infty(B_1^+)} \right)
\]

\[\Box\]

4.2. A geometric tangential approach

We proceed in a way similar to [8] and prove an approximation lemma with Neumann boundary condition first. Using this result we iterate the estimates of the theory to obtain \( C^{1,\alpha} \)-estimates adapting Lemma 5.1 in [1] to our case.

**Lemma 4.2 (Approximation Lemma).** Let \( \bar{q} \in \mathbb{R}^n \) be an arbitrary vector. Suppose that \( \varphi \in C(\partial B_1 \cap \{x_n > 0\}) \), has \( \rho = \rho(x) \) as modulus of continuity on \( \partial B_1 \cap \{x_n > 0\} \) and satisfy \( \| \varphi \|_{L^\infty(\partial B_1 \cap \{x_n > 0\})} \leq K \), for some positive constant \( K \). Then, given \( \delta > 0 \), there exists \( \varepsilon_0 > 0 \) depending only on \( \delta, n, \lambda, \Lambda, \rho, K \) such that if \( f \) is Hölder continuous in \( B_1^+ \), \( g \) is Hölder continuous on \( \overline{\Upsilon} \),

\[
\| f \|_{L^\infty(B_1^+)} \leq \varepsilon_0 \quad \text{and} \quad \| g \|_{L^\infty(\overline{\Upsilon})} \leq \varepsilon_0
\]

then any two viscosity solutions \( v \) and \( w \) of, respectively

\[
\begin{align*}
|\nabla w + \bar{q}|^\gamma F(D^2w) &= f \quad \text{in} \ B_1^+ \\
w &= \varphi \quad \text{on} \ \partial B_1 \cap \{x_n > 0\} \\
\nabla w \cdot v &= g \quad \text{in} \ \overline{\Upsilon}
\end{align*}
\]

and

\[
\begin{align*}
F(D^2h) &= 0 \quad \text{in} \ B_1^+ \\
h &= \varphi \quad \text{on} \ \partial B_1 \cap \{x_n > 0\} \\
\nabla h \cdot v &= 0 \quad \text{in} \ \overline{\Upsilon}
\end{align*}
\]

satisfy

\[
\| w - h \|_{L^\infty(B_1^+)} \leq \delta.
\]

**Proof.** We argue by contradiction. Suppose that the lemma is not true. Then there exist \( \delta_0 > 0 \) and a sequence of operators \( F_k \) and functions \( v_k, w_k \in C(\partial B_1 \cap \{x_n > 0\}) \), \( g_k \in C^\beta(\overline{\Upsilon}) \), \( \tilde{q}_k \) and \( \tilde{f}_k \) (which satisfy the hypothesis for \( F, \varphi, g \) and \( f \), respectively, in the lemma) for which there are viscosity solutions \( w_k \) and \( h_k \) of

\[
\begin{align*}
\begin{cases}
|\nabla w_k + \tilde{q}_k|^\gamma F(D^2w_k) &= f_k \quad \text{in} \ B_1^+ \\
w_k &= \varphi_k \quad \text{on} \ \partial B_1 \cap \{x_n > 0\} \\
\nabla w_k \cdot v &= g_k \quad \text{in} \ \overline{\Upsilon}
\end{cases}
\end{align*}
\]

and

\[
\begin{align*}
\begin{cases}
F(D^2h_k) &= 0 \quad \text{in} \ B_1^+ \\
h_k &= \varphi_k \quad \text{on} \ \partial B_1 \cap \{x_n > 0\} \\
\nabla h_k \cdot v &= 0 \quad \text{in} \ \overline{\Upsilon}
\end{cases}
\end{align*}
\]

\[\Box\]
such that
\[ \| f_k \|_{L^r(B^+_1)} \to 0, \| g_k \|_{L^r(\Gamma)} \to 0 \quad \text{as} \quad k \to \infty \] (4.3)
and
\[ \| w_k - h_k \|_{L^r(B^+_1)} > \delta_k \quad \forall k. \] (4.4)

Since all \( \varphi_k \)'s have the same modulus of continuity \( \rho \) on \( \partial B_1 \cap \{ x_n > 0 \} \), \( \| \varphi_k \|_{L^r(\partial B_1 \setminus \{ x_n > 0 \})} \leq K \) for any \( k \), we may suppose that \( \varphi_k \to \varphi_\infty \) uniformly in \( \partial B_1 \cap \{ x_n > 0 \} \) and apply Lemma 9.1 in [22], conclude that \( \{ h_k \} \) is equicontinuous (and uniformly bounded) sequence of functions in \( B^+_1 \). Therefore, taking subsequence, we may assume that \( h_k \to h_\infty \) uniformly in \( B^+_1 \) and \( h_\infty = \varphi_\infty \) in \( \partial B_1 \cap \{ x_n > 0 \} \). We may also assume (again by the Arzela-Ascoli theorem) that \( F_k(\cdot) \to \tilde{\varphi}_\infty(\cdot) \) as \( k \to \infty \) uniformly in compact of \( \text{Sym}(n) \) (the space of symmetric matrices), for some uniformly elliptic operator \( \tilde{\varphi}_\infty \). Thus, \( h_\infty \in C(B^+_1) \) is viscosity solution of
\[
\begin{cases}
\tilde{\varphi}_\infty(D^2 h_\infty) = 0 & \text{in } B^+_1 \\
h_\infty = \varphi_\infty & \text{in } \partial B_1 \cap \{ x_n > 0 \} \\
\nabla h_\infty \cdot v = 0 & \text{in } \Gamma
\end{cases}
\]

Our goal is to show that the sequence \( w_k \) is pre-compact in the \( C^0(B^+_1) \)-topology. We can obtain a universally large constant \( A_0 > 0 \), such that, given a subsequence \( \{ \tilde{q}_k \} \), where
\[ |\tilde{q}_k| \geq A_0, \quad \forall k \in \mathbb{N}, \]
then, by Lemma [4.4] \( \{ w_k \}_{j \in \mathbb{N}} \) is bounded in \( C^{0,1}(\overline{B^+_1}) \). For the case where,
\[ |\tilde{q}_k| < A_0, \quad \forall k \in \mathbb{N}, \]
easily follows (see Lemma [4.3]) that the sequence \( \{ w_k \}_{k \in \mathbb{N}} \) is bounded in \( C^{0,\beta_1}(\overline{B^+_1}) \) for some \( 0 < \beta_1 < 1 \). Therefore in general, the family \( \{ w_k \}_{j \in \mathbb{N}} \) is bounded in \( C^{0,\beta_1}(\overline{B^+_1}) \), which gives us the desired compactness. From the compactness previously established, up to a subsequence, \( w_k \to w_\infty \) locally uniformly in \( B^+_1 \). Therefore, taking subsequences, we may assume that
\[ w_k \to w_\infty \quad \text{and} \quad h_k \to h_\infty \quad \text{uniformly in } \overline{B^+_1} \quad \text{as} \quad k \to \infty, \]
for some \( w_\infty, h_\infty \in C(B^+_1) \) such that
\[ w_\infty = h_\infty = \varphi_\infty \quad \text{in } \partial B^+_1 \cap \{ x_n > 0 \}. \]

Our ultimate goal is to prove that the limiting function \( w_\infty \) is a solution to a constant coefficient, homogeneous, \( (\lambda, \Lambda) \)-uniform elliptic equation. For that we also divide our analysis in two cases.

- If \( |\tilde{q}_j| \) bounded, we can extract a subsequence of \( \{ \tilde{q}_j \} \), that converges to some \( \tilde{w}_\infty \in \mathbb{R}^n \). Finally, by Corollary [4.3] we conclude
\[
\begin{cases}
|\nabla w_\infty + \tilde{w}_\infty| \cdot \tilde{\varphi}_\infty(D^2 w_\infty) = 0 & \text{in } B^+_1, \\
w_\infty = \varphi_\infty & \text{in } \partial B_1 \cap \{ x_n > 0 \} \\
\nabla w_\infty \cdot v = 0 & \text{in } \Gamma
\end{cases}
\]
For some $\tilde{\gamma}_\infty (\lambda, \Lambda)$-elliptic. So, by Lemma 3.5, we conclude that $w_\infty$ also satisfies

\[
\begin{cases}
\tilde{\gamma}_\infty(D^2w_\infty) = 0, & \text{in } B^+_1, \\
w_\infty = \varphi_\infty & \text{in } \partial B_1 \cap \{x_n > 0\} \\
\nabla w_\infty \cdot \nu = 0 & \text{in } \varGamma
\end{cases}
\]

in the viscosity sense.

- If $|\vec{q}_j|$ unbounded, then without loss of generality $|\vec{q}_j| \to \infty$. In this case, define $\vec{e}_k = \vec{q}_k/|\vec{q}_k|$ and so $w_k$ satisfies

\[
\begin{cases}
|\vec{e}_k + 0 \cdot \nabla w_k| \tilde{\gamma}_\infty(D^2w_k) = f_k/|\vec{q}_k|^j & \text{in } B^+_1, \\
w_k = \phi_k & \text{in } \partial B_1 \cap \{x_n > 0\} \\
\nabla w_k \cdot \nu = g_k & \text{in } \varGamma
\end{cases}
\]

We get at the limit

\[
\begin{cases}
|\vec{e}_\infty + 0 \cdot \nabla w_\infty| \tilde{\gamma}_\infty(D^2w_\infty) = 0 & \text{in } B^+_1, \\
w_\infty = \phi_\infty & \text{in } \partial B_1 \cap \{x_n > 0\} \\
\nabla w_\infty \cdot \nu = 0 & \text{in } \varGamma
\end{cases}
\]

for $|\vec{e}_\infty| = 1$. So, $w_\infty$ satisfies

\[
\begin{cases}
\tilde{\gamma}_\infty(D^2w_\infty) = 0 & \text{in } B^+_1, \\
w_\infty = \phi_\infty & \text{in } \partial B_1 \cap \{x_n > 0\} \\
\nabla w_\infty \cdot \nu = 0 & \text{in } \varGamma
\end{cases}
\]

(4.5)

in the viscosity sense. Since (4.5) is uniquely solvable we get $v_\infty = w_\infty$ in $B^+_1$, which also yields a contradiction to (4.4).

\[ \square \]

4.3. Universal flatness improvement

In this Section, we deliver the core sharp oscillation decay that will ultimately imply the optimal $C^{1, \alpha}$ regularity estimate for solutions to Eq. (1.1). The first task is a step-one discrete version of the aimed optimal regularity estimate. A quick inference on the structure of equation (1.1) reveals that no universal regularity theory for such equation could go beyond $C^{1, \alpha_0}$, where $\alpha_0$ denote the optimal Hölder continuity exponent for solutions constant coefficients, homogeneous, elliptic equation

\[ F(D^2 h) = 0 \quad \text{in } B^+_1 \]

with $\nabla h \cdot \nu = 0$ on $\varGamma$. In fact the degeneracy term forces solutions to be less regular than solutions to the uniformly elliptic problem near its set of critical points. In this present section we show that a viscosity solution, $u$, to (1.1) is pointwise differentiable and its gradient, $\nabla u$, is locally of class $C^{0, \min\{\alpha_0^-, \beta, \frac{1}{1+\gamma}\}}$, which is precisely the optimal regularity for degenerate equations of the type (1.1).

This is the contents of next Lemma.
Lemma 4.3. Let \( \vec{q} \in \mathbb{R}^n \) be an arbitrary vector and suppose that \( \varphi \in C(\partial B_1 \cap \{ x_n > 0 \}) \). Let \( w \) a normalized, i.e., \( \| w \|_{L^\infty(B_1^+)} \leq 1 \), viscosity solution to
\[
\left\{ \begin{array}{l}
|\nabla w + \vec{q}| F(D^2w) = f \quad \text{in } B_1^+ \\
\nabla w \cdot \nu = g \quad \text{in } \Upsilon,
\end{array} \right.
\]
Given \( \alpha \in (0, \alpha_0) \cap (0, \frac{1}{1+\gamma}] \), there exists \( 0 < \rho_0 < 1/2 \) and \( \varepsilon_0 > 0 \), depending only upon \( n, \lambda, \Lambda, \gamma \) and \( \alpha \), such that if
\[
\| f \|_{L^\infty(B_1^+)} + \| g \|_{C^\beta(\Upsilon)} \leq \varepsilon_0,
\]
then there exists an affine function \( \ell(X) = a + \vec{b} \cdot X \) such that
\[
\| w - \ell \|_{L^\infty(B_{\rho_0}^+)} \leq \rho_0^{1+\alpha}
\]
\[
\vec{b} \cdot \nu = 0.
\]
Furthermore,
\[
|a| + |\vec{b}| \leq C(n, \lambda, \Lambda)
\]
Proof. For a \( \delta > 0 \) to be chosen a posteriori, let \( h \) be a solution to a constant coefficient, homogeneous, \((\lambda, \Lambda)\)-uniform elliptic equation that is \( \delta \)-close to \( v \) in \( L^\infty(B_1^+) \). The existence of such a function is the thesis of Lemma 4.2, provided \( \varepsilon_0 \) is chosen small enough, depending only on \( \delta \) and universal parameters. Since our choice for \( \delta \) later in the proof will depend only upon universal parameters, we will conclude that the choice of \( \varepsilon_0 \) is too universal. From normalization of \( v \), it follows that \( \| h \|_{L^\infty(B_1^+)} \leq 2 \). Therefore, from the regularity theory available for \( w \), see for instance [22], Theorem 9.3, we can estimate
\[
\sup_{B_{1/2}} |h(X) - (\nabla h(0) \cdot X + h(0))| \leq C(n, \lambda, \Lambda, \alpha) \cdot r^{1+\alpha_0}
\]
\[
|\nabla h(0)| + |h(0)| \leq C(n, \lambda, \Lambda, \alpha_0)
\]
and by boundary condition \( \nabla h(0) \cdot \nu = 0 \). Remark that, by Theorem 6.1 in [22], \( \nabla h \) is well defined up to the boundary \( \Upsilon \). Let us label
\[
\ell(X) = \nabla h(0) \cdot X + h(0).
\]
It readily follows from triangular inequality that
\[
\sup_{B_{\rho_0}} |w(x) - \ell(x)| \leq \delta + C(n, \lambda, \Lambda) \cdot \rho_0^{1+\alpha_0}
\]
Now, fixed an exponent \( \alpha < \alpha_0 \), we select \( \rho_0 \) and \( \delta \) as
\[
\rho_0 = \sqrt[\alpha_0-\alpha]{\frac{1}{2C(n, \lambda, \Lambda)}}
\]
\[
\delta = \frac{1}{2} \left( \frac{1}{2C(n, \lambda, \Lambda)} \right)^{\frac{1+\alpha}{\alpha_0-\alpha}}
\]
where $C$ is the universal constant appearing in (4.6). We highlight that the above choices depend only upon $n, \lambda, \Lambda$ and the fixed exponent $0 < \alpha < \alpha_0$. Finally, combining (4.6), (4.7), (4.8) and (4.9), we obtain

\[
\sup_{X \in B_{\rho_0}} |w(X) - \ell(X)| \leq \frac{1}{2} \left( \frac{1}{2C(n, \lambda, \Lambda)} \right)^{1+\alpha_0} + C(n, \lambda, \Lambda) \cdot \rho_0^{1+\alpha} \cdot \rho_0^{1+\alpha} = \frac{1}{4} \rho_0^{1+\alpha} + \frac{1}{2} \rho_0^{1+\alpha} = \rho_0^{1+\alpha},
\]

and the Lemma is proven.

Now, we can derive the pointwise $C^{1,\alpha}$ regularity for (2.2) in the general form.

**Lemma 4.4.** Consider $\varphi \in C(\partial B_1 \cap \{x_n > 0\})$, $g \in C^\beta(\Upsilon)$ for some $\beta \in (0,1)$, $f \in L^\infty(B_1^+)$. Let $u$ be a viscosity solution to

\[
\begin{cases}
|\nabla u|^\gamma F(D^2 u) = f & \text{in } B_1^+ \\
u \cdot D u = \varphi & \text{on } \partial B_1 \cap \{x_n > 0\} \\
abla u \cdot \nu = g & \text{in } \Upsilon,
\end{cases}
\]

with $\|u\|_{L^\infty(B_1^+)} \leq 1$ and $\alpha = \min \left\{ \alpha_0, \beta, \frac{1}{1+\gamma} \right\}$. Then, there exists $0 < \rho_0 < 1/2$ and $\varepsilon_0 \in [0,1]$ only depending on $\lambda, \Lambda, n$ and $\gamma$ such that, if

\[
\|f\|_{L^\gamma(B_1^+)} + \|g\|_{C^\beta(\Upsilon)} \leq \varepsilon_0
\]

then for all $j \in \mathbb{N}$, there exists a sequence of affine functions $\ell_j(x) = a_j + \bar{b}_j \cdot x$ satisfying

\[
|a_{j+1} - a_k| + \rho_0^j \|\bar{b}_j - \bar{b}_k\| \leq C_0 \rho_0^{(1+\alpha)j}
\]

(4.10)

\[
\bar{b}_j \cdot \nu = g(0),
\]

(4.11)

such that

\[
\sup_{x \in B_{\rho_0}} |u(x) - \ell_j(x)| \leq \rho_0^{j(1+\alpha)}.
\]

(4.12)

**Proof.** We argue by finite induction. The case $k = 1$ is precisely the statement of Lemma 4.3. Suppose we have verified (4.12) for $k = 1, 2, \ldots, j$. Define the rescaled function $w : B_1^+ \to \mathbb{R}$

\[
w(x) := \frac{(u - \ell_j)(\rho_0^j x)}{\rho_0^{j(1+\alpha)}}
\]

It readily follows from the induction assumption that $\|w\|_{L^\infty(B_1^+)} \leq 1$. Furthermore, $w$ satisfies

\[
\begin{cases}
|\nabla w + \rho_0^{-\alpha} \bar{b}_j|^\gamma F_j(D^2 w) = \tilde{f} & \text{in } B_1^+ \\
w = \tilde{\varphi} & \text{on } \partial B_1 \cap \{x_n > 0\} \\
abla w \cdot \nu = \tilde{g} & \text{in } \Upsilon.
\end{cases}
\]
where

\[
\tilde{f}(x) = \rho^{j[1-\alpha(1+\gamma)]}f(\rho_0^j x)
\]
\[
\tilde{\phi} = \rho_0^{-j(1+\alpha)}\left(\phi(\rho_0^j x) - \ell_j(\rho_0^j x)\right)
\]
\[
\tilde{g}(x) = \rho_0^{-\alpha_j}g(\rho_0^j x)
\]

and

\[
F_j(M) := \rho_0^{j(1-\alpha)}F\left(\frac{1}{\rho_0^{j(1-\alpha)}}M\right)
\]

It is standard to verify that the operator \(F_j\) is \((\lambda, \Lambda)\)-elliptic. Note that, by induction hypothesis

\[
\|\tilde{\phi}\|_{L^\infty(\partial B_1 \cap \{x_n > 0\})} \leq 1.
\]

Also, one easily estimate

\[
\|f_j\|_{L^\infty(B_1)} \leq \rho_0^{j[1-\alpha(1+\gamma)]}\|f\|_{L^\infty(\partial B_{\rho_0^j})} \tag{4.13}
\]
\[
\|\tilde{g}\|_{C^\beta(Y)} \leq \rho_0^{(\beta-\alpha)j}\|g\|_{C^\beta(Y \cap B_{\rho_0^j})}. \tag{4.14}
\]

Due to the sharpness of the exponent selection \(\alpha = \min\{\alpha_0, \beta, \frac{1}{1+\gamma}\}\), namely \(\alpha \leq \frac{1}{1+\gamma}\) and \(\alpha \leq \beta\), we conclude \((F_j, f_j, g_j)\) satisfies the smallness assumption Lemma 4.3. Thus, there exists a affine function \(\tilde{\ell}(X) := a + \tilde{b} \cdot X\) with

\[
|a| + |\tilde{b}| \leq C(n, \lambda, \Lambda) \tag{4.15}
\]
\[
\tilde{b} \cdot \nu = 0, \tag{4.16}
\]

such that

\[
\sup_{\overline{B_{\rho_0^j}}} |w(X) - \tilde{\ell}(X)| \leq \rho_0^{1+\alpha}. \tag{4.17}
\]

In the sequel, we define the \((j+1)\)th approximating affine function,

\[
\ell_{j+1}(X) := a_{j+1} + \tilde{b}_{j+1} \cdot X,
\]

where the coefficients are given by

\[
a_{j+1} := a_k + \rho_0^{j(1+\alpha)}a \quad \text{and} \quad \tilde{b}_{j+1} := \tilde{b}_j + \rho_0^{\alpha_j/\beta}.
\]

By induction assumption and (4.16), \(b_{j+1} \cdot \nu = g(0)\). Re-scaling estimate (4.17) back, we obtain

\[
\sup_{x \in \overline{B}_{\rho_0^{j+1}}} |u(x) - \ell_{j+1}(x)| \leq \rho_0^{(j+1)(1+\alpha)}
\]

and the proof of Lemma is complete. \(\square\)
4.4. Proof of Theorem 2.1

We now conclude the proof of Theorem 2.1. By interior estimates (see Theorem 1 in [17]) it is enough to find a $C^{1,\alpha}$ estimate for the points in $\Upsilon$. Moreover, it suffices to show the aimed $C^{1,\alpha}$ estimate at the origin for a solution $u$ under the hypotheses of Lemmas 4.3 and 4.4. For a fixed exponent $\alpha$ satisfying the sharp condition, we will establish the existence of an affine function

$$\ell_*(X) := a_* + \vec{b}_* \cdot x,$$

such that

$$|a_*| + |\vec{b}_*| \leq C,$$

and

$$\sup_{x \in B_r^j} |u(x) - \ell_*(x)| \leq C \cdot r^{1+\alpha}, \quad \forall r \ll 1,$$

for a constant $C$ that depends only on $n, \lambda, \Lambda, \kappa$ and $\alpha$. Initially, we notice that it follows from (4.10) that the coefficients of the sequence of affine functions $\ell_k$ generated in Lemma 4.4, namely $\vec{b}_j$ and $a_j$, are Cauchy sequences in $\mathbb{R}^n$ and in $\mathbb{R}$, respectively. Let

$$\vec{b}_* := \lim_{j \to \infty} \vec{b}_j \quad (4.18)$$
$$a_* := \lim_{j \to \infty} a_j \quad (4.19)$$

It also follows from the estimate obtained in (4.10) that

$$|a_* - a_j| \leq \frac{C_0}{1 - \rho_0} \rho_0^{j+1},$$
$$|\vec{b}_* - \vec{b}_j| \leq \frac{C_0}{1 - \rho_0} \rho_0^j \rho_0^j.$$

Now, fixed a $0 < r < \rho_0$, we choose $j \in \mathbb{N}$ such that

$$\rho_0^{j+1} < r \leq \rho_0^j.$$

We estimate

$$\sup_{x \in B_r^j} |u(x) - \ell_*(x)| \leq \sup_{x \in B_r^j} |u(x) - \ell_*(x)|$$
$$\leq \sup_{x \in B_r^j} |u(x) - \ell_j(x)| + \sup_{x \in B_r^j} |\ell_k(x) - \ell_*(x)|$$
$$\leq \rho_0^{j+1} + \frac{C_0}{1 - \rho_0} \rho_0^{j+1}$$
$$\leq \frac{1}{\rho_0^{1+\alpha}} \left[ 1 + \frac{C_0}{1 - \rho_0} \right] \cdot r^{1+\alpha},$$

and the proof of Theorem is finally complete.
5. Some consequences of the main result

In this Section, we will present the some consequences of the main result. An important consequence is the following:

Corollary 5.1. Let \( u \) be a viscosity solutions to

\[
|\nabla u|^\gamma F(D^2 u) = f + B_1^+ \]

with Neumann boundary condition \( \nabla u \cdot \nu = g \) in \( \Omega \). Assume \( f \in L^\infty(B_1^+) \), \( g \in C^\beta(\Omega) \) and \( F \) is uniformly elliptic and concave. Then \( u \in C^{1,\min\{\beta, \frac{1}{1+\gamma}\}}(B_{1/2}^+) \) and this regularity is optimal.

Proof. Corollary follows from Theorem 2.1 since solutions to concave equations with Neumann boundary conditions are of class \( C^{1,1} \) by [17].

Another interesting example to visit is the \( \infty \)-Laplacian operator

\[
\Delta_{\infty}v := (\nabla v)^t \cdot D^2 v \cdot \nabla v.
\]

The theory of infinity-harmonic functions, i.e., \( \Delta_{\infty}v = 0 \), has received great deal of attention. One of the main open problems in the modern theory of PDEs is whether infinity-harmonic functions are of class \( C^{1,1} \). This conjecture has been answered positively by Savin [21] in the plane. Over here, we would like to mention that although the conjecture is open, nevertheless it is well known that that solutions to \( \Delta_{\infty}v = 0 \) are locally of class \( C^{1,\alpha} \) in the plane, for some exponent \( \alpha \) depending only \( n \) for instance, [14] and quite recently, Evans and Smart [15] proved that infinity-harmonic functions are everywhere differentiable regardless the dimension. We remember the famous example of the infinity-harmonic function \( u(x,y) = x^{4/3} - y^{4/3} \) due to Aronsson from the late 1960s sets the ideal optimal regularity theory for such problem. In high dimensions, the situation is quite different. Very recently, the \( C^{1,1/3} \) conjecture has been solved in the context for obstacle problems by Rossi, J., Teixeira, E. and Urbano M. in [20].

Let \( h \in C(B_1) \) be an infinity-harmonic function. For each \( p \gg 1 \), let \( h_p \) be the solution to the boundary value problem

\[
\begin{cases}
\Delta_p h_p &= 0 \text{ in } B_{3/4}^+ \\
h_p &= h \text{ on } \partial B_{3/4}^+
\end{cases}
\]

where

\[
\Delta_p v := |\nabla v|^{p-2} \Delta v + (p-2)|\nabla v|^{p-4} \Delta_{\infty}v
\]

is the \( p \)-Laplacian operator. It is known that \( h_p \to h \) uniformly to \( h \). In particular

\[
\Delta_{\infty} h_p = o(1), \quad \text{as } p \to \infty.
\]

Hereafter, let us call \( h_p \) the \( p \)-harmonic approximation of the infinity-harmonic function \( h \). Our contribution in the context of the conjecture is the following

Theorem 5.2. Let \( f \in L^\infty(B_1^+) \), \( g \in C^\beta(\Omega) \) and \( h \in C(B_1^+) \) satisfy

\[
\begin{cases}
|\Delta_{\infty} h| &= O(p^{-1}) \text{ as } p \to \infty \text{ in } B_1^+ \\
u &= \varphi \text{ on } \partial B_1 \cap \{x_n > 0\} \\
\nabla u \cdot \nu &= g \text{ in } \Omega,
\end{cases}
\]

in the viscosity sense. Assume \( u \) is smooth up to a possible radial singularity. Then \( u \in C^{1,\min\{\beta, \frac{1}{1+\gamma}\}}(B_{1/2}^+) \).
Proof. In fact, since $h_p$ is $p$-harmonic, it satisfies
\[ |\nabla h_p|^2 \Delta h_p = (2 - p) \Delta_{\infty} h_p. \]
From the Corollary 5.1 we deduce $\| h_p \|_{C^{1,\min\{\beta, 1/3\}}(\Omega)} \leq C$. □

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Bibliography

References

[1] Araújo, D., Ricarte, G. and Teixeira, E. Geometric gradient estimates for solutions to degenerate elliptic equations. Calculus of Variations (2015) 53:605-625.

[2] Benerjee, A. and Verma R. B. $C^{1,\alpha}$ regularity for degenerate fully nonlinear elliptic equations with Neumann Boundary Conditions. https://arxiv.org/pdf/1903.11898.pdf (Preprint).

[3] Birindelli, I. and Demengel, F. Comparison principle and Liouville type results for singular fully nonlinear operators, Ann. Fac. Sci. Toulouse Math. (6) 13(2) (2004) 261-287.

[4] Birindelli, I. and Demengel, F. Regularity and uniqueness of the first eigenfunction for singular fully nonlinear operators, J. Differential Equations 249 (5) (2010) 1089-1110.

[5] Birindelli, I. and Demengel, F., Eigenvalue, maximum principle and regularity for fully nonlinear homogeneous operators, Commun. Pure Appl. Anal. 6(2) (2007) 335-366.

[6] Birindelli, I. and Demengel, F. Eigenvalue and Dirichlet problem for fully-nonlinear operators in non-smooth domains, J. Math. Anal. Appl. 352 (2) (2009) 822-835.

[7] Birindelli, I. and Demengel, F. $C^{1,\beta}$ regularity for Dirichlet problems associated to fully nonlinear degenerate elliptic equations, Esaim Cocv, (2014), 20 (40), 1009-1024.

[8] Caffarelli, L.A., Cabre, X., Fully Nonlinear Elliptic Equations. Colloquium Publications 43, American Mathematical Society, Providence, RI, 1995.

[9] Caffarelli, L. A. Interior a priori estimates for solutions of fully nonlinear equations. Ann. of Math. (2) 130 (1989), no. 1, 189–213.

[10] Caffarelli, L.A., Crandall, M.G., Kocan, M. and Swiech, A. On viscosity solutions of fully nonlinear equations with measurable ingredients. Comm. Pure Appl. Math. 49(1996)(4), 365-397.
[11] Dávila, G., Felmer P. and A. Quaas A., Alexandroff-Bakelman-Pucci estimate for singular or degenerate fully nonlinear elliptic equations, C. R. Math. Acad. Sci. Paris, 347 (2009), pp. 1165?1168.

[12] Dávila, G., Felmer P. and A. Quaas A. Harnack inequality for singular fully nonlinear operators and some existence results, Calc. Var. Partial Differential Equations, 39 (2010), pp. 557?578.

[13] Dongsheng, Li, Zhang, Kai Regularity for fully nonlinear elliptic equations with oblique boundary conditions. Archive for Rational Mechanics and Analysis manuscript.

[14] Evans, L.C., Savin, O. $C^{1,\alpha}$ regularity for infinity harmonic functions in two dimensions. Calc. Var. Partial Differ. Equ. 32, 325?347 (2008)

[15] Evans, L.C., Smart, C.K. Everywhere differentiability of infinity harmonic functions. Calc. Var. Partial Differ. Equ. 42, 289?299 (2011)

[16] Imbert, C. Alexandroff-Bakelman-Pucci estimate and Harnack inequality for degenerate/singular fully non-linear elliptic equations. Journal of Differential Equations, Elsevier, 2011, 250 (3), pp.1553-1574.

[17] Imbert, C. and Silvestre, L. $C^{1,\alpha}$ regularity of solutions of degenerate fully non-linear elliptic equations. Advances in Mathematics. 233 (2013), 196-206.

[18] Imbert, C.; Silvestre, L. Estimates on elliptic equations that hold only where the gradient is large. J Eur. Math. Soc., to appear.

[19] Ishii, H and Lions, P. Viscosity solutions of fully nonlinear second-order elliptic partial differential equations, J. Differential Equations, 83 (1990), pp. 26?78.

[20] Rossi, J.D., Teixeira, E. and Urbano M. Optimal regularity at the free boundary for the infinity obstacle problem. Interfaces and Free Boundaries 2015

[21] Savin, O. $C^1$ regularity for infinity harmonic functions in two dimensions. Arch. Rational Mech. Anal. 176, 351?361 (2005)

[22] Silvestre L. and Milakis, E. Regularity for Fully Nonlinear Elliptic Equations with Neumann Boundary Data. Comm. in Partial Differential Equations.

[23] Silvestre, L. and Imbert, C. $C^{1,\alpha}$ regularity of solutions of degenerate fully nonlinear elliptic equations. Advances in Mathematics. 233 (2013), 196-206.

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