A novel finite element analysis of three-dimensional circular crack

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Abstract. A novel singular element containing a part of the circular crack front is established to solve the singular stress fields of circular cracks by using the numerical series eigen-solutions of singular stress fields. The element is derived from the Hellinger-Reissner variational principle and can be directly incorporated into existing 3D brick elements. The singular stress fields are determined as the system unknowns appearing as displacement nodal values. The numerical studies are conducted to demonstrate the simplicity of the proposed technique in handling fracture problems of circular cracks. The usage of the novel singular element can avoid mesh refinement near the crack front domain without loss of calculation accuracy and velocity of convergence. Compared with the conventional finite element methods and existing analytical methods, the present method is more suitable for dealing with complicated structures with a large number of elements.

1. Introduction

Traditionally, fracture mechanics mainly considers two plane states, i.e., plane stress and plane strain states, these simple two-dimensional (2D) cases make the equations for singular stress fields much easy to obtain by analytical methods [1]. Such essentially 2D theories have been very successful in dealing with fracture and fatigue on the macro-scale where the crack front can often be assumed to be a straight line. In fact, the crack front is usually curved, thereby, the three-dimensional (3D) character of the crack problem must be taken into account. A small number of fracture problems in 3D cracks have analytic solutions in close form due to the difficulty of mathematics [2-5]. For curved cracks in finite specimens, different numerical methods were presented to solve the fracture parameters [6-10]. The finite element method (FEM) is one of the most widely used numerical methods for this problem, in which special elements were developed for stress intensity factor (SIF) evaluation. The shape functions with a 1/\(r^{1/2}\) singularity at the crack front were used to establish the singular quarter-point element, which make it mainly be used for the crack problems of homogeneous materials [10-12]. Some scholars constructed interpolation functions for special elements by using the leading-order terms of asymptotic solutions [13]. There exist three shortcomings in the special elements with built-in leading-order singularity and conventional elements: the first one is that the numerical results are still dependent on the special element’s size; the second one is that these enriched elements usually need transition elements in order to improve the accuracy of the numerical results; thirdly, inter-element compatibility between the special and conventional elements can’t be satisfied due to different number of degrees of freedom at nodes. In contrast, special elements with built-in series solutions of
asymptotic displacement and stress fields is insensitive to the element sizes, therefore, highly refined meshes are not necessary near the crack tips, and the computational efficiency in the finite element analysis can be improved [14,15]. However, a special element with built-in series solutions for 3D curved crack problems is still needed considering all of the existing special elements are for 2D cracks or 3D straight cracks. For this purpose, a novel singular element containing a part of a circular crack front is developed herein based on the numerical series eigen-solutions as well as Hellinger-Reissner variational principle. Numerical benchmark examples are presented to demonstrate the accuracy and efficiency of the super crack front elements. The first validation problem involves a circular penny-shaped crack in a 3D solid. The second validation problem is a cracked bar under tension and torsion. The last example is an elliptical crack in an infinite solid. The current investigation will pay close attention to the singular stress field along the circular crack front and all possible fracture modes in 3D circular cracks.

2. Establishment of a super corner front element model

Figure 1 shows a part of a circular crack front and the attached Cartesian (x, y, z) and cylindrical polar (r, θ, z) coordinates originating from the center of the circular crack. The key step is the selection an appropriate local (curvilinear) coordinate system (ρ, φ, θ), which is convenient to describe the local deformation behavior in the vicinity of the crack front (locus of point o shown in figure 1). The local coordinate system is, as mentioned above, comprised of ρ, which denotes the radial direction from a point located on the crack front line, φ which denotes the angular direction measured counterclockwise from the x-y plane, and θ which is positive counterclockwise (looking from top) along the circumferential line of the crack tip.

Numerical eigen-solutions of 3D displacement and stress fields solved from a one-dimensional finite element formulation for generalized plane strain problems [16] are resorted to herein. The assumed 3D displacement and stress fields in the crack tip domain have satisfied the governing differential equations and compatibility equations in advance. As a result, the displacement and stress vectors in the ρ-θ coordinate plane of the local coordinate system (ρ, φ, θ) (see figure 1) can be written as

\[ u(\rho, \theta) = U\beta \]  \hspace{1cm} (1)

\[ \sigma(\rho, \theta) = \Sigma\beta \]  \hspace{1cm} (2)

in which \( u(\rho, \theta) \) and \( \sigma(\rho, \theta) \) are total asymptotic displacements and stresses near the curved crack front, respectively, \( U \) and \( \Sigma \) include displacement and stress variations, respectively, due to the existence of n-order stress singularities, and \( \beta \) contains unknown multiplicative coefficients to be determined. In the one-dimensional finite element formulation, the angular coordinate \( \theta \) is related to a natural coordinate \( \xi \).

![Figure 1. Coordinate system at the crack front.](image1.png)

![Figure 2. A novel singular element.](image2.png)

In general, the multiplicative coefficients \( \beta_i \) in equations (1) and (2) vary along the crack front for
the specific case of a 3D crack, unlike the 2D case where they are assumed to be a constant value for a given geometry and loading. A novel singular element containing a part of curved crack front line is established to address the influence of 3D effects. As shown in figure 2, the novel singular element is composed of some hybrid four-node quadrilateral element. In order to establish the element stiffness matrix of a four-node quadrilateral isoparametric element, a new natural coordinate \( \eta \) is introduced to represent the position of a normal plane along the crack front, and the coefficients \( \beta_i(\eta) \) are assumed to be a linear function of \( \eta \) in each element, as a result, the assumed displacement and stress components in local coordinate system \((\rho, \phi, \theta)\) can be written as

\[
\begin{align*}
\mathbf{u}(\rho, \phi, \theta) &= (1-\eta)/2 \cdot \mathbf{u}_1(\rho, \theta) + (1+\eta)/2 \cdot \mathbf{u}_2(\rho, \theta)
\end{align*}
\]

\[
\mathbf{\sigma}(\rho, \phi, \theta) = (1-\eta)/2 \cdot \mathbf{\sigma}_1(\rho, \phi) + (1+\eta)/2 \cdot \mathbf{\sigma}_2(\rho, \phi)
\]

in which \( \mathbf{u}_1(\rho, \theta) \) and \( \mathbf{\sigma}_1(\rho, \phi) \) are components corresponding to \( \eta = -1 \), and \( \beta_1, \mathbf{u}_2(\rho, \theta) \) and \( \mathbf{\sigma}_2(\rho, \phi) \) are corresponding to \( \eta = 1 \).

To formulate the element stiffness matrix of the novel singular element by the Hellinger-Reissner principle, the hybrid functional for the problem in crack front domain \( \Omega \) is expressed as the following form [15,17]:

\[
\pi = \int_{\Omega} \left(-\mathbf{\sigma}(x, y, z)^T \mathbf{S} \mathbf{\sigma}(x, y, z)/2 + \mathbf{\sigma}(x, y, z)^T \mathbf{A} \mathbf{u}(x, y, z)\right) \mathrm{d}x \mathrm{d}y \mathrm{d}z - \int_{\Gamma} \mathbf{t}(x, y, z)^T (\mathbf{u}(x, y, z) - \mathbf{\tilde{u}}(x, y, z)) \mathrm{d}A
\]

In which \( \Gamma \) denotes the circumferential boundary surfaces of the crack front element, which is composed of outward surfaces of all four-node quadrilateral elements. \( \mathbf{S} \) is the elastic compliance matrix in the constitutive relation between strains and stresses. \( \mathbf{u}(x, y, z), \mathbf{\sigma}(x, y, z) \) and \( \mathbf{t}(x, y, z) \) are, respectively, the stress, displacement and boundary traction vectors in Cartesian coordinate system \((x, y, z)\).

By using the divergence theorem over \( \Omega \) the functional \( \pi \) are simplified as

\[
\pi = \frac{1}{2} \int_{\Omega} \mathbf{\sigma}(x, y, z)^T \mathbf{n} (x, y, z) \mathrm{d}A - \int_{\Gamma} \mathbf{t}(x, y, z)^T \mathbf{\tilde{u}}(x, y, z) \mathrm{d}A
\]

In which \( \mathbf{n} \) is a matrix containing the uniform normal vectors \( n_x, n_y \) and \( n_z \). The displacement \( \mathbf{u}(x, y, z), \) stress \( \mathbf{\sigma}(x, y, z) \) and traction \( \mathbf{t}(x, y, z) \) in equation (6) can be transformed from the corresponding components in equations (3) and (4) in the corresponding local coordinate system \((\rho, \phi, \theta)\), i.e.

\[
\begin{align*}
\mathbf{u}(x, y, z) &= \mathbf{Z}_u \mathbf{u}(\rho, \phi, \theta) = \mathbf{Z}_u \mathbf{U} \beta_x \\
\mathbf{\sigma}(x, y, z) &= \mathbf{Z}_\sigma \mathbf{\sigma}(\rho, \phi, \theta) = \mathbf{Z}_\sigma \mathbf{\Sigma} \beta_x \\
\mathbf{t}(x, y, z) &= \mathbf{Z}_t \mathbf{t}(\rho, \phi, \theta) = \mathbf{Z}_t \mathbf{n} \Sigma \beta_x
\end{align*}
\]

The coordinate system transformation matrices \( \mathbf{Z}_u \) and \( \mathbf{Z}_\sigma \) are from the local coordinate system \((\rho, \phi, \theta)\) to the Cartesian coordinate system \((x, y, z)\).

Based on the hybrid-stress finite element method, the boundary displacements \( \mathbf{\tilde{u}}(x, y, z) \) are assumed separately from \( \mathbf{u}(x, y, z) \), and are expressed in terms of the nodal displacements in the displacement-base finite element methods as

\[
\mathbf{\tilde{u}}(x, y, z) = \mathbf{N} \mathbf{\delta}
\]

where the matrix \( \mathbf{N} \) represents the shape function matrix for a four-node quadrilateral element allowing for 3D displacements at a node, and the vector \( \mathbf{\delta} = [q_x, q_y, q_z] \) includes the nodal
displacements. Substituting components of equations (7) - (10) into equation (6), leads to
\[
\pi = \frac{1}{2} \beta_c^T H \beta_c - \beta_c^T G \delta \tag{11}
\]
where \(G\) and \(H\) are two matrix for areal integrals in the outward boundary faces \(\Gamma\), namely:
\[
H = \frac{1}{2} \int_{\Gamma} \left[ (n Z_\Sigma) U_c + U_c^T (n Z_\Sigma) \right] dA \tag{12}
\]
\[
G = \int_{\Gamma} (n Z_\Sigma N_c)^T N dA \tag{13}
\]
Taking the first variation of equation (11) with respect to \(\beta_c\) and noting that \(\delta \pi = 0\), we determine \(\beta_c\) as follows:
\[
\beta_c = H^{-1} G \delta \tag{14}
\]
Substituting \(\beta_c\) into equation (11) leads to:
\[
\pi = \frac{1}{2} \delta^T G^T H^{-1} G \delta \tag{15}
\]
Invoking the first variation of equation (15) with respect to \(\delta\), we obtain the following element stiffness matrix for the super crack front element:
\[
K = G^T H^{-1} G \tag{16}
\]
3. Numerical examples
Three benchmark problems are studied in this work to verify the proposed method. The first one is a penny-shaped crack in a cylinder, the stress intensity factors corresponding to mode I and III are considered. The second one is a cracked round bar subjected to tensile and torsional loads. The third one considered is an elliptical crack in an infinite cuboid body under tension, the variation of SIFs along the crack front line will be paid close attention to. The convergence of the present method is discussed in the first and second examples to show a dependence of singular stress fields on the number of eigenvalues truncated, and the number of nodes of the super singular hybrid element. In all examples, singular displacement and stress fields are finally determined by equations (3) and (4).

3.1. A penny-shaped crack in a cylinder
Figure 3 presents a cylinder containing penny-shaped crack. Tensile loading \(F\) and torsional loading \(T\) are applied to two end faces of the cylinder. As shown in the figure, the diameter and height of the cylinder are represented by \(D\) and \(H\), respectively. And \(a\) is the radius of the circular crack. As a consequence of the Saint venant principle, the tensile stress and torsional stress in the horizontal cross-section can be calculated as \(\sigma = F/(\pi D^2)\) and \(\tau = 16T/(\pi D^3)\).
Figure 3. A cylinder containing penny-shaped crack.

Figure 4. One quarter of the grid model for the penny-shaped crack problem.

Based on the finite element analysis, a finite element model with different mesh densities is established to simulate the crack in an infinite cylinder. $D$ and $H$ is equal to $300a$ in this model. Figure 4 shows one quarter of the cylinder that can clearly illustrate the meshes near the crack fronts. In this case, a certain number of novel 18 node crack front elements are distributed at the crack fronts, around which are traditional 3D eight-node brick elements. The material Poisson’s ratio is equal to 0.3.

The von Mises stress contours of the cylinder with penny-shaped crack is shown figure 5. Figures 5(a) and 5(b) are the results of the tensile loading and the torsional loading, respectively. No function for post-processing of contours is an existing limitation of user-defined elements in Abaqus. As shown in figure 5, contours do not represent the stresses in the super crack front element. As a result, the program of the user-defined element adds a callable Fortran code in order that the results of singular stresses can be exported to a log file generated by Abaqus. According to the definition of modes I-III fracture modes, the SIFs can be numerically estimated by equations (17) - (19):

$$K_I = \lim_{\rho \to 0} \sqrt{2} \rho \cdot \sigma_\phi (\rho, 0, \theta)$$  \hspace{1cm} (17)

$$K_{II} = \lim_{\rho \to a} \sqrt{2} \rho \cdot \sigma_{\rho \phi} (\rho, 0, \theta)$$  \hspace{1cm} (18)

$$K_{III} = \lim_{\rho \to a} \sqrt{2} \rho \cdot \sigma_{\theta \phi} (\rho, 0, \theta)$$  \hspace{1cm} (19)

Convergence property is researched to validate the effectiveness of the present finite element model. As shown in figure 6, The SIF values calculated from the super crack front element model are
exhibited. For the penny-shaped crack in an infinite cylinder, the normalized SIFs are $K_I/\sqrt{\sigma a}=0.6366$ for tension and $K_{III}/\sqrt{\sigma a}=0$ for torsion. This figure demonstrates that when the element size of the novel crack front elements in a plane perpendicular to the crack front is less than $0.05a \times 0.05a$ and the number of elements along the crack front is larger than 30, the present solutions converge to the reference solutions.

On the other hand, the influence of the number of terms in series solutions $(2N+M)$ on SIFs are listed in Table 1. It shows that when the dimension of vector $\beta$ increases to 26, the present solutions converge to reference values. In terms of an 18-node super crack front element, at least 24 terms from the series representation are needed [15], and the present model still obeys this convergence condition.

![Figure 6. Convergence of the SIF values calculated from the super crack front element model for the penny-shaped crack.](image)

**Table 1.** The effects of the number of terms in series solutions on SIFs for cracks.

| Dim($\beta$) | 8    | 14   | 20   | 26   | 32   | 38   | 44   |
|--------------|------|------|------|------|------|------|------|
| Results for the penny-shaped crack problems | $K_I/\sqrt{\sigma a}=0.6634$ | 0.6566 | 0.6410 | 0.6372 | 0.6373 | 0.6371 | 0.6378 |
| Results for the circumferential crack problems | $K_I/\sqrt{\sigma a}=0.9517$ | 1.0004 | 1.0987 | 1.1227 | 1.1231 | 1.1225 | 1.1226 |
| $K_{III}/\sqrt{\sigma a}=-1.0048$ | -1.0045 | -1.0045 | -1.0057 | -1.006 | -1.0008 | -1.0009 |

3.2. A circumferential crack in a cylinder

Figure 7 presents a cylinder containing circumferential crack. The cylinder is subjected to tensile loading $F$ and torsional loading $T$. $D$ and $H$ denote the diameter and height of the cylinder. $a$ represents the depth of the crack for the circumferential crack. Just like the penny-shaped crack problem, the tensile stress and torsional stress in a horizontal cross-section can be calculated as $\sigma=F/\pi D^2$ and $\tau=16T/(\pi D^3)$.

Both $D$ and $H$ are equal to 300$a$ in the finite element model. Figure 8 shows one quarter of the finite element model that can clearly illustrate the meshes near the crack fronts. A certain number of super 18 node crack front elements are distributed at the crack fronts as before. The material Poisson's ratio is equal to 0.3.
Figure 7. A cylinder containing circumferential crack.

Figure 8. One quarter of the grid model for the circumferential crack problem.

Figure 9 shows the von Mises stress contours of the cylinder with circumferential cracks, and figures 9(a) and 9(b) are results corresponding to the tensile loading and the torsional loading, respectively. According to equations (17)-(19), the SIFs can be numerically estimated. Figure 10 presents the convergence of the present solutions for the circumferential crack problem. The normalized SIFs are $K_I/(\sigma_x a^{1/2})=1.1225$ for tension and $K_{III}/(\sigma_x a^{1/2})=-1.007$ for torsion. This figure indicates that an element size of $0.1 \times 0.1 a$ in the face perpendicular to the crack front are enough to acquire converged solutions provided that 30 or more elements are used. Compared with the penny-shaped crack problem in section 3.1, the present problem is easier to get a convergent solution. This suggests that larger the curvature radius is, the less the numerical solutions are affected by the element size. As shown in table 1, the present solutions also converge to reference results when the dimension of vector $\beta$ increases to 26.
3.3. An elliptical crack in an infinite four prism

Figure 11 shows a four prism containing an embedded elliptical crack, and far-field tension and shear loading are applied to the prism. In the finite element analysis, $2H$, $2W$ and $2B$ are equal to 300a, and the ratio of the elliptical crack’s minor and major axes, $b/a$, is a variable. In the case of uniform tension stress $\sigma^\infty$ and a shear loading $\tau^\infty$, Kassir and Sih [18] gave the accurate analytical expressions of SIFs for an elliptical crack with arbitrary ratio of $a/b$. In the same manner, the novel crack front element is used to solve the singular stress fields. The element size of the super crack front element is set to $0.02a \times 0.02a$, and the number of elements along the crack front is 160. The dimension of $\beta$ is 26.

Figure 12 presents the von Mises stress contours of the four prism with an embedded elliptical crack.

Figure 10. Convergence of the SIF values calculated from the super crack front element model for the circumferential crack problem.

Figure 11. An four prism containing an embedded elliptical crack.
According to equations (18) and (19), the normalized SIFs, $K_{II}/(\tau^*b^{0.5})$ and $K_{III}/(\tau^*b^{0.5})$, and the angle $\phi$ for different ratios of $a/b$ are plotted in figures 13(b) and 13(c). The present solutions are also in very good agreement with the exact solutions [18]. We can’t obtain converged solutions for mixed mode fracture with conventional elements although we have tried many ways to solve this problem.

4. Conclusion
For the circular crack problems, numerical series solutions of singular displacement and stress fields from an ad hoc finite element eigen-analysis method are used to generate a novel 3D crack front element. The built-in series solutions include not only the leading-order terms but also a certain number of high-order terms.

When the curvature radius of a crack front is sufficiently large compared with the element size, the present method is insensitive to the sizes of novel crack front elements in a normal plane of the crack front, so coarse meshes near the crack fronts can be used to acquire converged results. For the engineering analysis, the present novel crack front element can improve the computational efficiency by decreasing the number of elements. Therefore, the novel crack front element is much easier to use.
compared with conventional FEMs.

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