ON IGUSA ZETA FUNCTIONS OF MONOMIAL IDEALS

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Abstract. We show that the real parts of the poles of the Igusa zeta function of a monomial ideal can be computed from the torus-invariant divisors on the normalized blow-up of the affine space along the ideal. Moreover, we show that every such number is a root of the Bernstein-Sato polynomial associated to the monomial ideal.

1. Introduction

If \( f \) is a nonconstant polynomial in \( \mathbb{Z}[x_1, \ldots, x_n] \) and \( p \) is a fixed prime, then the Igusa zeta function of \( f \) is defined by

\[
Z_f(s) = \int_{(\mathbb{Z}_p)^n} |f(y)_p^s| |dy|,
\]

for every \( s \in \mathbb{C} \) with \( \text{Re}(s) > 0 \). Here \( \mathbb{Z}_p \) denotes the ring of \( p \)-adic integers, endowed with the discrete valuation \( \text{ord}_p \) and with the \( p \)-adic absolute value defined by

\[
|a|_p = \left( \frac{1}{p} \right)^{\text{ord}_p(a)}.
\]

The measure \( \mu \) on \((\mathbb{Z}_p)^n\) that is used in the above integral is the Haar measure characterized by

\[
\mu \left( \prod_{i=1}^n p^{m_i} \mathbb{Z}_p \right) = \left( \frac{1}{p} \right)^{\sum_i m_i}.
\]

As defined, \( Z_f \) is a holomorphic function on the half plane \( \{ s \mid \text{Re}(s) > 0 \} \), and one can show that it admits a meromorphic extension to \( \mathbb{C} \). In fact, \( Z_f \) is a rational function of \( \left( \frac{1}{p} \right)^s \). A proof of rationality that also gives information on the real parts of the possible poles of \( Z_f \) proceeds as follows. Let \( \pi: Y \to X = \mathbb{A}^n \) be an embedded resolution of singularities of \( f \) defined over \( \mathbb{Q} \). This means that \( \pi \) is proper and birational, \( Y \) is smooth, and the union of \( \pi^*(\text{div}(f)) \) and of the exceptional locus of \( \pi \) is a divisor with simple normal crossings. For every prime divisor \( E \) in this union, denote by \( a_E(f) \) the order of \( E \) in \( \pi^*(\text{div}(f)) \) and by \( k_E \) the order of \( E \) in the relative canonical class \( K_{Y/X} \) (this is the divisor locally defined by \( \text{det}(\text{Jac}(\pi)) \)). Using the Change of Variable formula for \( p \)-adic integrals to express
$Z_f$ as an integral over $Y(\mathbb{Z}_p)$, Igusa obtained the rationality of $Z_f$ as a function of $(\frac{1}{p})^s$ and the fact that if $s$ is a pole of $Z_f$, then $\text{Re}(s) = -\frac{k_{E}+1}{a_{E}(f)}$ for some divisor $E$ as above. Our main reference for Igusa zeta functions is Igusa’s book [Ig] (see also Denef’s Bourbaki report [De]).

While every divisor on a log resolution of $f$ gives a candidate for the real part of a pole of $Z_f$, examples show that most of these numbers do not come actually from poles of $Z_f$. In fact an outstanding open problem in the field is the following conjecture of Igusa, relating the poles to one of the basic invariants of the singularities of $f$, its Bernstein-Sato polynomial.

**Conjecture 1.1.** Let $f$ be a nonconstant polynomial in $\mathbb{Z}[X_1, \ldots, X_n]$. For almost all primes $p$ the following holds: if $s$ is a pole of $Z_f$, then the real part of $s$ is a root of the Bernstein-Sato polynomial $b_f$ of $f$.

We recall that the Bernstein-Sato polynomial of $f$ is a polynomial in one variable introduced independently in [De] and [SS]. It is a subtle but very fundamental invariant of the singularities of $f$. We do not give its definition as we will not need it, but we mention that its roots are related to the eigenvalues of the monodromy of the hypersurface $f^{-1}(0)$. There is, in fact, a weaker version of the above conjecture that is stated in terms of these eigenvalues and that is known as the Monodromy Conjecture (see [De] for more on these conjectures and also [Ve] for some recent work in this direction).

Our goal in this note is to prove the analogue of Conjecture 1.1 when we replace $f$ by a monomial ideal. Though less studied, Igusa zeta functions for nonnecessarily principal ideals in $\mathbb{Z}[x_1, \ldots, x_n]$ can be defined in a very similar way with (1). More precisely, if $I$ is a nonzero proper ideal of $\mathbb{Z}[x_1, \ldots, x_n]$ and if we put for $y \in (\mathbb{Z}_p)^n$

$$\text{ord}_p I(y) = \min \{ \text{ord}_p(f(y))| f \in I \},$$

then we have

$$Z_I(s) = \int_{(\mathbb{Z}_p)^n} \left( \frac{1}{p} \right)^{\text{ord}_p I(y)} |dy|.$$

The above-mentioned results in the case of one polynomial extend in a straightforward way to the case of an arbitrary ideal. Note that in order to prove rationality, we need to consider a log resolution of $I$: this is a morphism $\pi: Y \to \mathbb{A}^n$ as before, such that $\pi^{-1}(V(I))$ is a Cartier divisor and its union with the exceptional locus of $\pi$ is a divisor with simple normal crossings. If $E$ is a prime divisor on $Y$ contained in this union, then $a_E(I)$ is by definition the coefficient of $E$ in $\pi^{-1}(V(I))$. As in the case of a principal ideal, one can show that given a log resolution $\pi$, for every pole $s$ of $Z_I$ there is a divisor $E$ on $Y$ such that $\text{Re}(s) = -\frac{k_{E}+1}{a_{E}(I)}$.

On the other hand, the definition of the Bernstein-Sato polynomial has been extended in [BMS3] from the case of one polynomial to that of an arbitrary ideal. This is again a polynomial in one variable, and therefore the analogue of Conjecture 1.1 makes sense in this case. We will prove the monomial case, i.e. when $I$ is generated by monomials.

**Theorem 1.2.** If $I$ is a nonzero proper monomial ideal of $\mathbb{Z}[x_1, \ldots, x_n]$, then for every prime $p$ and every pole $s$ of $Z_I$, the real part of $s$ is a root of the Bernstein-Sato polynomial of $I$. 
The key ingredient in the proof of the above theorem is a result on the poles of Igusa-type zeta functions associated to cones. Suppose that $N \simeq \mathbb{Z}^n$ is a lattice and that $\sigma$ is a pointed, rational, polyhedral cone in $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$. We denote by $H^\sigma$ the relative interior of the cone $\sigma$. If $\sigma^\vee$ is the dual cone in $M_{\mathbb{R}}$, where $M = \text{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$, and if $\ell_1, \ell_2$ are elements in $\sigma^\vee \cap M$ such that $\sigma \cap \{v \mid \ell_2(v) = 0\} = \{0\}$, then we put

$$Z_{\sigma, \ell_1, \ell_2}(s) := \sum_{v \in \sigma^\vee \cap N} \left(\frac{1}{p}\right)^{\ell_1(v)s + \ell_2(v)}.$$

It is easy to see (and it will follow from our computations) that this is well-defined and holomorphic in $\{s \in \mathbb{C} \mid \text{Re}(s) > 0\}$. We prove the following

**Theorem 1.3.** For every $\sigma$, $\ell_1$ and $\ell_2$ as above, $Z_{\sigma, \ell_1, \ell_2}$ is a rational function of $\left(\frac{1}{p}\right)^s$, and therefore can be meromorphically extended to $\mathbb{C}$. Moreover, for every pole $s$ of $Z_{\sigma, \ell_1, \ell_2}$ there is a primitive generator $v$ of a ray of $\sigma$ such that $\text{Re}(s) = -\frac{\ell_2(v)}{\ell_1(v)}$.

Given a monomial ideal $I$, we give in the next section a formula for $Z_I$ in terms of suitable zeta functions for the cones in the normal fan to the Newton polyhedron of $I$ (we refer for the relevant definition and for the precise formula to that section). Let us just mention that this fan defines the toric variety that is the normalized blow-up of $\mathbb{A}^n$ along the ideal $I$. Using this formula and Theorem 1.3 we will show in §3 that the real part of every pole of $Z_I$ corresponds to a torus-invariant divisor in the normalized blow-up of $\mathbb{A}^n$ along $I$ (despite the fact that the normalized blowing-up is not a log resolution of $I$).

**Theorem 1.4.** Let $I$ be a nonzero proper monomial ideal of $\mathbb{Z}[x_1, \ldots, x_n]$. For every pole $s$ of $Z_I$, there is a torus-invariant divisor $E$ on the normalized blowing-up of $\mathbb{A}^n$ along $I$ such that

$$\text{Re}(s) = -\frac{k_E + 1}{a_E(I)}.$$

On the other hand, explicit descriptions of the roots of the Bernstein-Sato polynomial of a monomial ideal have been obtained in [BMS1] and [BMS2]. We use the description in [BMS2] and Theorem 1.3 to prove Theorem 1.2 in the last section.

We mention that a description for the Igusa zeta function of a monomial ideal has also been obtained by Zúñiga-Galindo in [Zu]. Moreover, similar results appear in the work of Denef and Hoornert [DH], in which one describes the poles of the Igusa zeta functions for nondegenerate hypersurfaces with respect to their Newton polyhedron. It is shown in loc. cit. that for such $f$ the real part of essentially any pole corresponds to a facet of the Newton polyhedron of $f$, as above. Moreover, Loeser [Lo] showed that under some mild extra assumptions these numbers are roots of the Bernstein-Sato polynomial of $f$, thus proving Conjecture 1.1 for such nondegenerate hypersurfaces. On the other hand, note that the relations between the respective Igusa zeta functions and between the Bernstein-Sato polynomials of $f$ and of the corresponding monomial ideal are not clear in general.

2. **Igusa zeta functions of monomial ideals**

Let $I$ be a nonzero proper ideal of $\mathbb{Z}[x_1, \ldots, x_n]$ generated by monomials. If $u = (u_1, \ldots, u_n) \in \mathbb{N}^n$, we denote by $x^u$ the corresponding monomial $x_1^{u_1} \cdots x_n^{u_n}$.
The Newton polyhedron $P_I$ of $I$ is the convex hull of those $u$ in $\mathbb{N}^n$ such that $x^u$ is in $I$.

If $N = \mathbb{Z}^n$ and $N_\mathbb{R} = N \otimes_{\mathbb{Z}} \mathbb{R}$, we think of $P_I$ as lying in $M_\mathbb{R}$, where $M = \text{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$. We denote by $\langle \cdot, \cdot \rangle$ the canonical pairing between $M$ and $N$. If we consider in $N_\mathbb{R}$ the cone generated by the elements of the standard basis $e_1, \ldots, e_n$, then the corresponding toric variety is the affine space $\mathbb{A}^n$ and the subscheme $V(I)$ is invariant under the torus action (we refer to [Fu] for the basic notions on toric varieties). Hence the normalized blowing-up of $\mathbb{A}^n$ along $I$ is again a toric variety, and therefore it corresponds to a fan subdividing the above cone. This is the normal fan to the polyhedron $P_I$, that we will denote by $\Delta_I$. It is defined as follows: to each face $Q$ of $P_I$ one associates the cone

$$\sigma_Q := \{v \in N_\mathbb{R} | \langle u, v \rangle \leq \langle u', v \rangle \text{ for every } u \in Q \text{ and } u' \in P_I \}.$$  

The fan $\Delta_I$ consists of the cones $\sigma_Q$, when $Q$ varies over the faces of $P_I$. Note that $\dim(\sigma_Q) = n - \dim(Q)$, so the rays of $\Delta_I$ correspond to the facets of $P_I$, and the maximal cones of $\Delta_I$ correspond to the vertices of $P_I$.

Let $p$ be a fixed prime. We now proceed to the computation of $Z_I$. For every $a = (a_1, \ldots, a_n) \in \mathbb{N}^n$ we consider the set $C_a = \prod_{i=1}^n (p^{a_i} \mathbb{Z}_p \setminus p^{a_i+1} \mathbb{Z}_p)$. Since each $p^{a_i} \mathbb{Z}_p \setminus p^{a_i+1} \mathbb{Z}_p$ is a disjoint union of $(p-1)$ translates of $p^{a_i+1} \mathbb{Z}_p$, we see that

$$\mu(C_a) = (p-1)^n \frac{1}{p^{\sum a_i}}.$$  

We denote by $e$ the vector $(1, \ldots, 1)$, so $\langle e, a \rangle = \sum_i a_i$.

The function $\text{ord}_p I$ is constant on $C_a$ with value

$$\nu(a) := \min \{\langle u, a \rangle | x^u \in I \} = \min \{\langle u, a \rangle | u \in P_I \}.$$  

Since the sets $C_a$ are disjoint and the complement of their union has measure zero, we deduce

$$(4) \quad Z_I(s) = \sum_{a \in \mathbb{N}^n} \left(1 - \frac{1}{p}\right)^n \cdot \left(\frac{1}{p}\right)^{\langle e, a \rangle + \nu(a)}.$$  

Note that $\nu$ is a linear function on each of the cones in $\Delta_I$. Indeed, if $w$ is a vertex of $P_I$, then $\nu(a) = \langle w, a \rangle$ whenever $a$ is in $\sigma_w$.

If $\sigma$ is a cone in $\Delta_I$, choose a vertex $w$ of $P_I$ such that $\sigma$ is contained in $\sigma_w$ and put $\ell_\sigma := w$. By letting the $a$ in (4) vary inside the relative interior of each cone in $\Delta_I$, and using the definition in the Introduction, we get the following

**Proposition 2.1.** With the above notation, we have

$$(5) \quad Z_I(s) = \left(1 - \frac{1}{p}\right)^n \cdot \sum_{\sigma \in \Delta_I} Z_{\sigma, \ell_\sigma, e}(s).$$  

3. **Igusa zeta functions for cones**

Our goal in this section is to discuss Igusa-type zeta functions associated to cones and prove Theorem 1.3. Let $N$ be a lattice, $M$ its dual, and $\sigma$ a pointed, rational polyhedral cone in $N_\mathbb{R}$. We consider $\ell_1$ and $\ell_2$ in $\sigma^\vee \cap M$, where $\sigma^\vee$ is the dual cone of $\sigma$, such that $\sigma \cap \{v | \ell_2(v) = 0\} = \{0\}$. We want to study the function $Z_{\sigma, \ell_1, \ell_2}$ and its poles.
The definition of $Z_{\sigma,\ell_1,\ell_2}$ was motivated by the formula in Proposition 2.1 but sometimes it is more natural to consider a version of this function in which we sum over all the integer points in $\sigma$:

$$Z_{\sigma,\ell_1,\ell_2}(s) := \sum_{v \in \sigma \cap \mathbb{N}} \left(\frac{1}{p}\right)^{\ell_1(v)s + \ell_2(v)}.$$  

(6)

Again, this is well-defined if $\text{Re}(s) > 0$ and we have $Z_{\sigma,\ell_1,\ell_2} = \sum_{\tau} Z_{\tau,\ell_1,\ell_2}$, where the sum is over all faces $\tau$ of $\sigma$. Moreover, it follows from this formula that $Z_{\sigma,\ell_1,\ell_2}$ can be computed in terms of the functions $Z_{\tau,\ell_1,\ell_2}$, where $\tau$ varies over the faces of $\sigma$.

We start with the following

**Lemma 3.1.** Let $v_1, \ldots, v_r$ in $\mathbb{N}$ be linearly independent over $\mathbb{Q}$. If $w$ is in $\mathbb{N}$ and $\ell_1, \ell_2$ are elements in $M$, with $\ell_1$ nonnegative and $\ell_2$ positive on all the $v_i$, then we put

$$A(s) := \sum_{v \in S} \left(\frac{1}{p}\right)^{\ell_1(v)s + \ell_2(v)},$$

where $S = \{w + a_1 v_1 + \ldots + a_r v_r \mid a = (a_i) \in \mathbb{N}^r\}$. The function $A$ is well-defined and holomorphic for $\text{Re}(s) > 0$ and it is a rational function in $\left(\frac{1}{p}\right)^s$, so it has a meromorphic continuation to $\mathbb{C}$. Moreover, if $s$ is a pole of $A$, then there is $i$ such that $\text{Re}(s) = -\frac{\ell_2(v_i)}{\ell_1(v_i)}$.

**Proof.** If $\text{Re}(s) > -\frac{\ell_2(v_i)}{\ell_1(v_i)}$ for all $i$ such that $\ell_1(v_i)$ is nonzero, then we have

$$A(s) = \left(\frac{1}{p}\right)^{\ell_1(w)s + \ell_2(w)} \prod_{i=1}^{n} \sum_{a_i \in \mathbb{N}} \left(\frac{1}{p}\right)^{a_i(\ell_1(v_i)s + \ell_2(v_i))}$$

$$= \left(\frac{1}{p}\right)^{\ell_1(w)s + \ell_2(w)} \prod_{i=1}^{n} \frac{1}{1 - \left(\frac{1}{p}\right)^{\ell_1(v_i)s + \ell_2(v_i)}}.$$

The assertions in the lemma are direct consequences of this formula. \hfill \square

We can now give the proof of our result on Igusa-type zeta functions associated to cones.

**Proof of Theorem 3.3.** Arguing by induction on the dimension of $\sigma$, we may assume that the theorem holds for all cones of smaller dimension than $\dim(\sigma)$ (the case when $\dim(\sigma)$ is zero being trivial). In this case, we see that proving the assertions in the theorem for $Z_{\sigma,\ell_1,\ell_2}$ is equivalent with proving them for $Z_{\tau,\ell_1,\ell_2}$.

We first show that it is enough to prove the theorem when $\sigma$ is a simplicial cone. Indeed, it is well-known that one can always find a fan $\Gamma$ refining the cone $\sigma$ such that every cone in $\Gamma$ is simplicial and the one-dimensional cones in $\Gamma$ are precisely the rays of $\sigma$. Since

$$Z_{\sigma,\ell_1,\ell_2} = \sum_{\tau \in \Gamma} Z_{\tau,\ell_1,\ell_2},$$

and since each ray of a cone in $\Gamma$ is a ray of $\sigma$, we see that it is enough to prove the theorem for each (maximal) cone in $\Gamma$. 

Therefore we may assume that \( \sigma \) is simplicial and our goal is to show that \( \mathbb{Z}_{\sigma, \ell_1, \ell_2} \) satisfies the assertions in the theorem. Let \( v_1, \ldots, v_r \) be the primitive generators of the rays of \( \sigma \). Since \( \sigma \) is simplicial, the \( v_i \) are linearly independent over \( \mathbb{Q} \). The semigroup \( \sigma \cap \mathbb{N} \) is finitely generated, so we may choose generators \( w_1, \ldots, w_s \). The \( v_i \) span \( \sigma \) over \( \mathbb{Q} \), hence we can find a positive integer \( m \) such that every \( mw_j \) is in the semigroup generated by the \( v_i \). It follows that after replacing \( \{w_1, \ldots, w_s\} \) by \( \{q_1 w_1 + \ldots + q_s w_s \mid 0 \leq q_j \leq m - 1\} \), we may assume that

\[
\sigma \cap \mathbb{N} = \bigcup_{j=1}^{s} (w_j + \mathbb{N}),
\]

where \( \mathbb{N} \) is the semigroup generated by the \( v_i \).

If \( I \subseteq \{1, \ldots, s\} \), let us put

\[
A_I(s) := \sum_v \left( \frac{1}{p} \right)^{\ell_1(v)s + \ell_2(v)},
\]

where the sum is over \( v \) in \( \bigcap_{j \in I} (w_j + \mathbb{N}) \). We claim that \( \bigcap_{j \in I} (w_j + \mathbb{N}) \) is either empty or it is equal to \( w + \mathbb{N} \) for a suitable \( w \) in \( \mathbb{N} \). Indeed, by an obvious induction on \( |I| \) it is enough to show this when \( I \) has two elements, say \( j \) and \( k \). The intersection of \( w_j + \mathbb{N} \) and \( w_k + \mathbb{N} \) is nonempty if and only if \( w_j - w_k \) lies in the lattice generated by the \( v_i \). If this is the case, let us write \( w_j - w_k = \sum_{i=1}^{r} a_i v_i \) for suitable integers \( a_1, \ldots, a_r \). If we put \( w = w_j + \sum_{i=1}^{r} \max\{0, -a_i\} v_i \), then it is easy to check that \( (w_j + \mathbb{N}) \cap (w_k + \mathbb{N}) = w + \mathbb{N} \), which proves our claim.

It follows from our claim and Lemma 3.1 that each \( A_I \) is a rational function of \( \left( \frac{1}{p} \right)^s \). Moreover, if \( s \) is a pole of \( A_I \), then there is \( i \) such that \( \text{Re}(s) = -\frac{\ell_2(v_i)}{\ell_1(v_i)} \). On the other hand, it follows from (7) that

\[
\mathbb{Z}_{\sigma, \ell_1, \ell_2} = \sum_I (-1)^{|I|-1} A_I(s),
\]

where the sum is over all nonempty subsets \( I \) of \( \{1, \ldots, s\} \). Therefore \( \mathbb{Z}_{\sigma, \ell_1, \ell_2} \) satisfies the assertions of the theorem, which completes the proof. \( \square \)

Putting together Theorem 1.3 and the description of the Igusa zeta function of a monomial ideal from the previous section, we can relate the poles of this zeta function with the torus-invariant divisors in the blowing-up along the ideal.

**Proof of Theorem 1.4.** It follows from Proposition 2.1 and Theorem 1.3 that if \( s \) is a pole of \( Z_f \), then there is a primitive generator \( v \) of a ray of the normal fan \( \Delta_f \) to the Newton polyhedron \( P_f \) such that

\[
\text{Re}(s) = \frac{\langle e, v \rangle}{\langle w, v \rangle}.
\]

Here \( w \) is a vertex of \( P_f \) such that \( v \) is contained in the maximal cone \( \sigma_w \) of \( \Delta_f \) corresponding to \( w \).

On the other hand, recall that the torus-invariant divisors on the toric variety defined by \( \Delta_f \) correspond precisely to the rays of \( \Delta_f \). Moreover, if \( E \) is the divisor corresponding to the ray through \( v \), then \( k_E = \langle e, v \rangle - 1 \). Since we also have

\[
a_E(I) = \min\{\langle u, v \rangle \mid x^u \in I\} = \langle w, v \rangle,
\]

as \( v \) lies in \( \sigma_w \), we deduce the assertion in the theorem. \( \square \)
4. Poles and roots of the Bernstein-Sato polynomial

We now show that the real part of any pole of $Z_I$ is a root of the Bernstein-Sato polynomial $b_I$ associated to $I$. In fact, we prove the following stronger statement that together with Theorem 1.4 implies Theorem 1.2.

Proposition 4.1. If $I$ is a nonzero proper monomial ideal and if $E$ is a prime divisor in the normalized blowing-up of the affine space along $I$ such that $a_E(I)$ is nonzero, then $-\frac{k_E+1}{a_E(I)}$ is a root of the Bernstein-Sato polynomial $b_I$.

Proof. The divisor $E$ corresponds to a ray in the normal fan $\Delta_I$ to $P_I$. Let $v$ be a primitive generator of this ray. If $w$ is a vertex of $P_I$ such that the corresponding maximal cone $\sigma_w$ of $\Delta_I$ contains $v$, then we have seen in the proof of Theorem 1.3 that $k_E + 1 = \langle e, v \rangle$ and $a_E(I) = \langle w, v \rangle$. Note that since $\langle w, v \rangle \neq 0$, the facet $Q$ of $P_I$ corresponding to $v$ is not contained in a coordinate hyperplane: if, for example, $Q$ is contained in the hyperplane $(x_i = 0)$, then $v = e_i$, and since $w$ lies in $Q$ we get $\langle w, v \rangle = 0$, a contradiction.

In order to show that $(k_E + 1)/a_E(I)$ is a root of the Bernstein-Sato polynomial $b_I$ associated to $I$, we use the description of the roots of $b_I$ from [BMS2] (in fact, the ones that we need for the theorem are “the most straightforward” of the roots of $b_I$). Since $Q$ is a facet of $P_I$ that is not contained in a coordinate hyperplane, there is a unique linear function $L_Q$ on $M_\mathbb{R}$ such that $Q = P_I \cap L_Q^{-1}(1)$. With this notation, it is shown in [BMS2] (see Remark 4.6) that $-L_Q(e)$ is a root of $b_I$.

On the other hand, since the ray through $v$ corresponds to the facet $Q$ and since $w$ is in $Q$, we have

$$Q = \{ u \in P_I \mid \langle u, v \rangle = \langle w, v \rangle \}.$$  

Therefore $L_Q$ is given by $\frac{1}{(w, v)} \cdot v$, and since $-L_Q(e)$ is a root of $b_I$, we see that $(k_E + 1)/a_E(I)$ is, indeed, a root of $b_I$. \hfill \Box

Remark 4.2. We do not know whether the analogue of Proposition 4.1 holds for a nonnecessarily monomial ideal $I$. Note that if $I = (f)$ is principal, then the assertion is trivial: the divisor $E$ is one of the irreducible components of the divisor $H$ defined by $f$, $k_E = 0$, and $a_E(I)$ is the multiplicity of $E$ in $H$. The fact that $-\frac{1}{a_E(f)}$ is a root of $b_f$ then follows by restricting to an open subset where $E$ is smooth and $H = a_E(f) \cdot E$.

Remark 4.3. The arguments in the previous two sections can also be used to analyze the orders of the possible poles of the Igusa zeta function $Z_I$. Indeed, it follows from Proposition 2.1 and from the proof of Theorem 1.3 that if $s$ is a pole of order $r$ of $Z_I$, then $r \leq n$ and there are $r$ invariant divisors $E_1, \ldots, E_r$ on the normalized blowing-up along $I$ such that $E_1 \cap \ldots \cap E_r \neq \emptyset$ and $\Re(s) = -(k_{E_i} + 1)/a_{E_i}(I)$ for every $i$. We would like to deduce that in this case $\Re(s)$ is a root of order $r$ of $b_I$, but unfortunately, we do not understand well enough the multiplicities of the roots of $b_I$.

Remark 4.4. While Proposition 2.1 gives in principle a formula for the Igusa zeta function of a monomial ideal, and Theorem 1.4 gives an estimate on the denominator of this function (written as a rational function of $1/p^s$), getting a general explicit formula for the denominator is rather difficult. A Maple code for computing $p$-adic and motivic zeta functions of monomial ideals via resolution of singularities is available, upon request, from the first author.
Remark 4.5. Using motivic integration, Denef and Loeser defined in [DL] a motivic analogue of the Igusa zeta function. For concreteness, we preferred to work with $p$-adic integrals. However, as the reader familiar with this topic will certainly notice, all the above results have analogues in the motivic setting, “replacing $p$ by $\mathbb{L}$”. For example, if $\sigma$, $\ell_1$ and $\ell_2$ are as in Theorem 1.3, then the series

$$\sum_{v \in \sigma \cap \mathbb{N}} \mathbb{L}^{-\left(\ell_1(v)s + \ell_2(v)\right)}$$

(9)

can be written as a sum of fractions with numerator in $K[\mathbb{L}^{-s}]$ and denominator of the form

$$\prod_{i=1}^{r} \left(1 - \mathbb{L}^{-\left(\ell_1(v_i)s + \ell_2(v_i)\right)}\right),$$

(10)

where $r \leq \dim(\sigma)$ and $v_1, \ldots, v_r$ are primitive generators of the rays of $\sigma$. Here $K$ is the ring obtained from the Grothendieck ring of varieties over a base field $k$ by inverting $\mathbb{L} = [\mathbb{A}_1^k]$. Similarly, if $I$ is a monomial ideal, then the motivic zeta function of $I$

$$\int_{(\mathbb{A}^n)^{\infty}} \mathbb{L}^{-s \cdot \operatorname{ord}_I}$$

(11)

can be written as a sum of fractions with numerator in $K[\mathbb{L}^{-s}]$ and denominator of the form

$$\prod_{i=1}^{r} \left(1 - \mathbb{L}^{-\left(a_{E_i}(I)s + k_{E_i} + 1\right)}\right),$$

where $r \leq n$ and $E_1, \ldots, E_r$ are divisors on the normalized blowing-up of $\mathbb{A}^n$ along $I$ such that $E_1 \cap \ldots \cap E_r$ is nonempty.

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