\textbf{sl}_3\text{-FOAM HOMOLOGY CALCULATIONS}

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\textbf{Abstract.} We exhibit a certain infinite family of three-stranded quasi-alternating pretzel knots which are counterexamples to Lobb’s conjecture that the \textit{sl}_3–knot concordance invariant \(s_3\) (suitably normalised) should be equal to the Rasmussen invariant \(s_2\). For this family, \(|s_3| < |s_2|\). However, we also find other knots for which \(|s_3| > |s_2|\). The main tool is an implementation of Morrison and Nieh’s algorithm to calculate Khovanov’s \(sl_3\)-foam link homology. Our C++-program is fast enough to calculate the integral homology of e.g. the (6,5)-torus knot in six minutes. Furthermore, we propose a potential improvement of the algorithm by gluing sub-tangles in a more flexible way.

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1. INTRODUCTION

During the last decade, \textit{sl}_n\text{-polynomials were categorified one after the other: beginning, of course, with Khovanov’s categorification of the Jones polynomial [Kho00] using cobordisms, followed by his categorification of the \textit{sl}_3\text{-polynomial [Kho04] using foams; and finally, Khovanov and Rozansky’s categorification of the \textit{sl}_N\text{-polynomials for arbitrary N [KR08a] and of the Homflypt–polynomial itself [KR08b] (see also Khovanov [Kho07] and Webster [Web07]) using matrix factorisations.

All these homologies are completely combinatorial in nature – unlike e.g. the original knot Floer homology – meaning that their definition is in itself a description how to compute them. By hand, this direct way of computation is hardly feasible for any but the smallest knots, and using skein long exact sequences and criteria for thinness is much more efficient (see e.g. a computation of Mackaay and Vaz [MV08]). On a computer, however, even the straight-forward method, as implemented with some tweaks in the program KhHo [Shu03] by Shumakovitch, could already compute the \textit{sl}_2\text{-homology of knots with up to ca. 20 crossings. Matrix factorisations, on the other hand, are more reluctant to efficient treatment.

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Figure 1. On the left, the \((5, -3, 2)\)-pretzel knot (aka \(10_{125}\)), the smallest knot with \(s_3 \neq s_2\); on the right (drawn with Knotscape [HT99]), \(12n_{340}\), the smallest knot with \(|s_3| > |s_2|\).

by a computer. Nevertheless, Carqueville and Murfet [CM11] wrote a program that is able to compute \(\mathfrak{sl}_N\)-homology of links with up to six crossings and \(N \approx 18\). Webster’s program [Web05] computes Homflypt-homology of short braids, but is largely limited to 3-stranded ones.

Yet it is desirable to be able to compute the Khovanov–Rozansky homologies of much larger knots, since some phenomena require a certain complexity of the knot to occur; see e.g. Hedden and Ording’s 15-crossing knot which demonstrates that the Rasmussen and Floer concordance invariants differ [HO05]. Or to mention an extreme example, see Freedman et al. [FGMW10] for a link with 222 crossings whose Rasmussen invariant was worthwhile to compute at the time, because it could have provided a counterexample to the four-dimensional smooth Poincaré conjecture (this approach was later rebuted by Akbulut [Akb10]).

Bar-Natan’s extension of \(\mathfrak{sl}_2\)-homology from link diagrams to tangles [BN05] (see also Khovanov [Kho02]) led subsequently to a divide-and-conquer algorithm to compute \(\mathfrak{sl}_2\)-homology [BN07]. The speed of this algorithm depends primarily on the girth of the link diagram: this is the maximal number of intersection points of a horizontal line with the diagram (see e.g. [Fre09], and cf. section 3.1 for details). An implementation by Green and Morrison called JavaKh [GM05] is able to compute the homology of knots of girth up to 14, e.g. the \((8, 7)\)-torus knot. Mackaay and Vaz [MV07] and Morrison and Nieh [MN08] then extended \(\mathfrak{sl}_3\)-homology to tangles, and the latter describe in detail the ensuing algorithm.

In this text, we present an implementation of this algorithm as a C++-program called FoamHo [Lew12].

The algorithm can be improved by gluing sub-tangles in a more flexible way, along a sub-tangle tree instead of one after the other. This leads to the notion of the recursive girth of a link, which replaces the girth as main factor limiting calculation speed. Even though this improvement is not implemented in FoamHo yet, the program is still fast enough to calculate the integral homology, reduced or unreduced, of links with girth up to 10, such as the \((6, 5)\)-torus knot. The \(\mathfrak{sl}_3\)-concordance invariant \(s_3\), as defined by Lobb [Lob12] (see also Wu [Wu07] and Lobb [Lob09]), may (in most cases) be extracted from the \(\mathfrak{sl}_3\)-homology by means of the spectral sequence converging to the filtered version of homology. This method was used for the \(\mathfrak{sl}_2\)-homology by Freedman et al. [FGMW10]. It does not depend on the conjectured convergence of the spectral sequence on the second page.

The most striking calculatory result obtained with FoamHo concerns this \(s_3\)-invariant, which is the \(\mathfrak{sl}_3\)-analogue of the Rasmussen invariant \(s_2\). These two invariants share many properties, e.g. they agree on homogeneous and quasi-positive knots, and until now it was not known whether they were actually equal or not (see Lobb [Lob12, Conjectures 1.5, 1.6]).
Conjecture. Let $s_3$ be the $\mathfrak{sl}_3$–concordance invariant, normalised to take the same value as the Rasmussen invariant on the trefoil. If $a > b \geq 3$, $c \geq 2$, and $a + 1 \equiv b + 1 \equiv c \equiv 0 \pmod{2}$, then the $(a,-b,c)$–pretzel knot $P(a,-b,c)$ has $s_3$–invariant

$$s_3(P(a,-b,c)) = a - b + \delta_c,$$

where $\delta_2 = -1$ and $\delta_c = -2$ for $c > 2$.

This statement is called a “conjecture” since it is established by FoamHo-calculations only for small values of $a$, $b$ and $c$. In addition, however, we have a prove for the case $c > 2$ which does not rely on computer calculations [Lew13]. See section 3.5 for further details.

If one appends $b > c$ to the hypotheses of the conjecture, then the $(a,-b,c)$–pretzel knot is quasi-alternating (see Champanerkar and Kofman [CK09] and Greene [Gre10]), and hence its $s_2$–invariant equals its classical knot signature (see Manolescu and Ozsváth [MO08]): $s_2(P(a,-b,c)) = \sigma(P(a,-b,c)) = a - b$. This value of the signature can be easily computed using Göritz matrices and the formula of Gordon and Litherland [GL78]. So for this infinite family of pretzel knots, the $s_2$– and $s_3$–invariant differ, and the latter gives a weaker bound for the slice genus than the former. However, there are knots for which this is different, e.g. $s_2(12n_{340}) = 1$ and $s_2(12n_{340}) = 0$.

The rest of the paper is organised as follows: section 2 provides a brief, but essentially self-contained definition of the $\mathfrak{sl}_3$–foam homology as defined by Khovanov [Kho04], Mackaay and Vaz [MV07], and Morrison and Nieh [MN08]. We also define reduced homology in the framework of foams. In section 3.1, we give an account of the divide-and-conquer algorithm, including the computation of integral homology and the acceleration of the algorithm using a sub-tangle tree. In section 3.2, we discuss how to extract $s_3$ from the $\mathfrak{sl}_3$-homology. Section 3.3 addresses some particular implementation issues and their resolution in FoamHo, and section 3.4 presents the usage and characteristics of the program itself. Finally, section 3.5 states some results of FoamHo calculations, and compares them against previously known results.

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2. The $\mathfrak{sl}_3$–foam homology of tangle diagrams

This section gives a definition of the $\mathfrak{sl}_3$–polynomial and its categorification, the $\mathfrak{sl}_3$–foam homology. Except for the use of a generalised definition of planar algebras (sec. 2.2) to formalise localness, and the definition of reduced homology using foams (sec. 2.5), this section contains nothing essentially new. We just review the parts of [Kho04, MN08] which are relevant to the purpose of this section – which is to provide a self-contained definition of $\mathfrak{sl}_3$–homology with the objective of calculation in mind. Instead of choking tori we use dots, like Khovanov [Kho04] and Mackaay and Vaz [MV07]. The origin of webs and the $\mathfrak{sl}_N$–polynomials lie in representation theory [RT90, Kup96], an aspect we we will not dwell on.

2.1. The $\mathfrak{sl}_3$–polynomial, naively. The $\mathfrak{sl}_3$–polynomial can be defined by a single skein relation involving only link diagrams. We will instead use the two skein relations $(\text{Sk}^+)$ and $(\text{Sk}^-)$, see below, because this allows a categorification using foams. These skein relations involve webs: a closed web is a plane oriented trivalent graph whose every vertex is either a source or a sink, and that may have vertex-less circles as additional edges.

A tangle diagram is the generic intersection of a link diagram with a disc; generic means that the disc’s border intersects the diagram’s strands transversely, and does not pass through a crossing. Let us define a map $V$ from the set of smooth isotopy classes of tangle diagrams to the free $\mathbb{Z}[q^\pm]$–module on the set of smooth isotopy classes of closed
webs. The map $V$ is uniquely determined by the following two local relations, which are interpreted naively for now (i.e. apply these relations to all crossings of the link at once, then expand):

\begin{align*}
(Sk^-) & \quad V\left(\begin{array}{c} \circlearrowleft \\ \circlearrowright \end{array}\right) = q^2 \cdot V\left(\begin{array}{c} \circlearrowleft \\ \circlearrowright \end{array}\right) - q^3 \cdot V\left(\begin{array}{c} \circlearrowright \\ \circlearrowleft \end{array}\right) \\
(Sk^+) & \quad V\left(\begin{array}{c} \circlearrowleft \\ \circlearrowright \end{array}\right) = q^{-2} \cdot V\left(\begin{array}{c} \circlearrowleft \\ \circlearrowright \end{array}\right) - q^{-3} \cdot V\left(\begin{array}{c} \circlearrowright \\ \circlearrowleft \end{array}\right).
\end{align*}

Next, we define an evaluation $\langle \cdot \rangle$ of closed webs, called the Kuperberg bracket [Kup96, Jae92], which associates to a closed web a Laurent polynomial in $q$. This evaluation is given by the four relations

\begin{align*}
(C) & \quad \langle \begin{array}{c} \circlearrowleft \\ \circlearrowright \end{array} \rangle = \langle \begin{array}{c} \circlearrowright \\ \circlearrowleft \end{array} \rangle = (q^{-2} + 1 + q^2) \cdot \langle \begin{array}{c} \circlearrowleft \\ \circlearrowright \end{array} \rangle, \\
(D) & \quad \langle \begin{array}{c} \circlearrowleft \\ \circlearrowright \end{array} \rangle = (q^{-1} + q) \cdot \langle \begin{array}{c} \circlearrowleft \\ \circlearrowright \end{array} \rangle, \\
(S) & \quad \langle \begin{array}{c} \circlearrowleft \\ \circlearrowright \end{array} \rangle = \langle \begin{array}{c} \circlearrowleft \\ \circlearrowright \end{array} \rangle + \langle \begin{array}{c} \circlearrowleft \\ \circlearrowright \end{array} \rangle \quad \text{and}
\end{align*}

\begin{align*}
(U) & \quad \langle W_1 \sqcup W_2 \rangle = \langle W_1 \rangle \cdot \langle W_2 \rangle.
\end{align*}

The $\mathfrak{sl}_3$-polynomial, which associates a Laurent polynomial in $q$ to a link diagram, can now be obtained by composing $V$ with $\langle \cdot \rangle$, and identifying the empty web with 1. Categorifying this construction is going to yield the $\mathfrak{sl}_3$-homology. However, it is advantageous to formalise the localness of the relations (Sk$\pm$, C, D, S) before proceeding.

2.2. Planar algebras. While 2-categories do give a framework for webs and foams, they make sense only if one aims at interpreting webs as maps between oriented 0-manifolds; this aspect is not essential to the calculation of $\mathfrak{sl}_3$-homology, and so we use planar algebras instead. Planar algebras were introduced by Jones [Jon98] to identify subfactors. They were subsequently used to describe locally defined knot invariants such as the Jones polynomial. Bar-Natan [BN05] introduced a categorified version of planar algebras called canopolis to describe Khovanov homology, a method adaptable to $\mathfrak{sl}_3$-homology [MN08]. We will use a slightly generalised version of planar algebras, working over arbitrary monoidal categories instead of over the category of vector spaces over a fixed field. In this way, a canopolis is a planar algebra as well.

Let $B_0 \subset \mathbb{R}^2$ be a closed disc, and $B_1, \ldots, B_n \subset B_0^\circ$ be $n$ pairwisely disjoint smaller closed discs. Punching out these discs yields a disc with holes, $H = B_0 \setminus \bigcup_{i=1}^n B_i^\circ$. Let $M$ be a compact oriented one-dimensional smooth submanifold of $H$ with $M \cap \partial H = \partial M$; in other words, $M$ is a collection of circles and of intervals whose endpoints lie on the boundary of the big discs or one of the smaller discs. An input diagram consists of $M$, $H$, and on each boundary component of $H$ a base point which is not in $\partial M$. We consider input diagrams up to smooth isotopy, in the course of which boundary points of $M$ may not cross the base points.

For every $i \in \{0, \ldots, n\}$, the intersection $M \cap B_i$ is a finite set; at each of its points, the corresponding interval of $M$ is either oriented towards the boundary (+ for $i > 0$, − for $i = 0$), or away from it (− for $i > 0$, + for $i = 0$). Moreover, these points have a canonical order, given by starting from the base point and walking once around the circle in the counterclockwise direction. Thus the isotopy type of $M \cap B_i$ may be written as a sign-word $\varepsilon_i$, i.e. a word over the alphabet $\{+, -\}$. The boundary of $H$ is the tuple $(\varepsilon_0, \ldots, \varepsilon_n)$.

Now suppose $(M, H)$ and $(M', H')$ are two input diagrams, such that $\varepsilon'_k = \varepsilon_k$ for a fixed $k \in \{1, \ldots, n\}$. Then $(M', H')$ may be shrunk and glued into $B_k$, base point on base point
and boundary points on boundary points, resulting in a new input diagram with \( n + n' - 1 \) holes.

Let \( I \) be a subset of the set of all sign-words. Let \( C \) be a monoidal category, in the easiest case just the category \( \text{Set} \) of sets, and in the classical case the category of vector spaces over a fixed field. Then a planar algebra \( \mathcal{P} \) over \( I \) and \( C \) consists of the following data:

- For each \( \varepsilon \in I \), an object \( \mathcal{P}_\varepsilon \in C \).
- For each input diagram \( H \) with boundary \((\varepsilon_0, \ldots, \varepsilon_n)\) such that \( \forall i : \varepsilon_i \in I \), a \( C \)-morphism
  \[ \mathcal{P}_H : \bigotimes_{i=1}^n \mathcal{P}_{\varepsilon_i} \to \mathcal{P}_{\varepsilon_0}. \]

This data is required to satisfy the following axioms:

- Suppose \( H \) is an input diagram with \( n = 1 \) and \( \varepsilon_0 = \varepsilon_1 \) that consists only of appropriately oriented radial strands. Then \( \mathcal{P}_H : \mathcal{P}_{\varepsilon_0} \to \mathcal{P}_{\varepsilon_1} \) is the identity morphism.
- Let \( H \) and \( H' \) be two input diagrams with boundary \((\varepsilon_0, \ldots, \varepsilon_n)\) and \((\varepsilon_0', \ldots, \varepsilon_n')\), respectively. Suppose that for a fixed \( k \in \{1, \ldots, n\} \), \( \varepsilon_0' = \varepsilon_k \). Let \( H'' \) be the input diagram obtained from gluing \( H' \) into the \( k \)-th hole of \( H \). Then the morphism \( \mathcal{P}_{H''} \) is equal to the composition of the morphisms \( \mathcal{P}_H \) and \( \mathcal{P}_{H'} \), i.e.
  \[ \mathcal{P}_{H''} = \mathcal{P}_H \circ (\text{id} \otimes \bigotimes_{i=k+1}^n \mathcal{P}_{\varepsilon_i} \otimes \mathcal{P}_{H'} \otimes \text{id} \otimes \bigotimes_{i=k+1}^n \mathcal{P}_{\varepsilon_i}). \]

If \( F : C \to C' \) is a monoidal functor, one may define the planar algebra \( F(\mathcal{P}) \) over \( I \) and \( C' \) by \( F(\mathcal{P})_\varepsilon = F(\mathcal{P}_\varepsilon) \) and

\[ F(\mathcal{P})_H : \bigotimes_{i=1}^n F(\mathcal{P})_{\varepsilon_i} \to F(\mathcal{P})_{\varepsilon_0} \]

to be the composition \( F(\mathcal{P}_H) \circ \gamma \), where \( \gamma \) is the natural transformation

\[ \bigotimes_{i=1}^n F(\mathcal{P}_{\varepsilon_i}) \to F \left( \bigotimes_{i=1}^n \mathcal{P}_{\varepsilon_i} \right) \]

which comes with the functor \( F \) because it is monoidal. Examples of this construction include, for a planar algebra \( \mathcal{P} \) over \( \text{Set} \), replacing for all \( \varepsilon \in I \) the set \( \mathcal{P}_\varepsilon \) by the free \( R \)-modules for some ring \( R \) by means of applying the left-adjoint of the forgetful functor from the category of \( R \)-modules to \( \text{Set} \); or the quotient \( \mathcal{P} \) by an equivalence relation on
The unit circle intersects the edges of the closed web transversely, and does not pass through the corresponding hole of $H$. Planar algebra structure.

Suppose $P$ and $P'$ are planar algebras over $I$, $I'$ and $C$, $C'$, respectively, such that $I \subset I'$. A planar algebra morphism from $P$ to $P'$ consists of a functor $F : C' \rightarrow C$ and an $I$-indexed collection of $C$-morphisms $P_x \rightarrow F(P'_x)$ which respect the planar algebra structure, i.e. commute with the maps $P_H$ and $P'_H$. The functor $F$ will typically be a forgetful functor.

Tangle diagrams with a base point on the boundary, considered as input diagrams, up to smooth isotopy, form a planar algebra $T$ over $\text{Set}$ and the set $I_0$ of sign-words $\varepsilon_1 \cdots \varepsilon_m$ with $\sum_{j=1}^m \varepsilon_j = 0$. Let us elaborate this example: the planar algebra $T$ associates to a sign-word $\varepsilon$ with an equal number of both signs the (countably infinite) set $T_\varepsilon$ of all tangle diagrams with boundary $\varepsilon$, modulo smooth isotopy; and to an input diagram $H$ with $n$ holes (see e.g. fig. 2) a function which maps a tuple of $n$ tangle diagrams with appropriate boundaries to a new, bigger tangle diagram, by gluing each of the $n$ tangle diagrams into the corresponding hole of $H$. One easily verifies that the planar algebra axioms are satisfied.

2.3. The $sl_3$-polynomial in the context of planar algebras. Suppose the unit circle intersects a closed web generically; as in the definition of tangle diagrams, this means that the circle intersects the edges of the closed web transversely, and does not pass through a vertex. Then the intersection of the closed web with the unit disc is called a web. As for input diagrams, we fix a base point on the boundary of a web, and encode the isotopy type of the boundary by a sign-word, in which + stands for a strand oriented away from the boundary, – for a strand oriented towards it. Note that a sign-word $\varepsilon_1 \cdots \varepsilon_m$ is the boundary of some web if and only if $\sum_{j=1}^m \varepsilon_j \equiv 0 \pmod{3}$. Denote by $I_3$ the set of all such sign-words. Webs, up to smooth isotopy, form a planar algebra $W$ over $I_3$ and $\text{Set}$.

Let $W_2$ be the free $\mathbb{Z}[q^{\pm 1}]$-module on $W$. Then $W_2$ forms a planar algebra over $I_3$ and the category of $\mathbb{Z}[q^{\pm 1}]$-modules. In $W_2$, we may interpret the relations (C), (D) and (S) as relations on $W_2$, $W_2^+$ or $W_2^-$ and $W_2^{+-}$ or $W_2^{-+-}$, respectively. Denote by $W^{qr}$ the quotient by the generated equivalence relation. In this context, the relation (U) is implied by the compatibility of the equivalence relation with the planar algebra structure.

Let $W, W'$ be two webs and $\varphi : W \rightarrow W'$ a diffeomorphism just of the webs themselves, not taking into account the ambient discs. We call $\varphi$ a web diffeomorphism if it preserves the order of the boundary points, and the cyclic ordering of edges around vertices. Note that in the quotient $W^{qr}$, the equivalence class of a web is already determined by its web diffeomorphism type. In $W$, this distinction is slightly coarser than the isotopy type, since e.g. web diffeomorphisms do not take the orientation and relative position of closed components into account.

The two skein relations $(\text{Sk}^\pm)$ determine a unique morphism $V : T \rightarrow W_2$ of planar algebras, $T$ being the planar algebra of tangles. A link diagram $L$ may be seen as element of $T_\varepsilon$. The equivalence class $[V(L)] \in W^{qr}$ has a unique member that is a $\mathbb{Z}[q^{\pm 1}]$-multiple of the empty web. The coefficient equals the $sl_3$–polynomial of the link diagram. Reidemeister invariance may be shown by proving that the tangle diagrams with two, four and six boundary points corresponding to the Reidemeister moves I, II and III have in each case the same image under $V$.

2.4. The $sl_3$-homology in the context of a special kind of planar algebras: canopolis. To categorify the $sl_3$–polynomial, one needs to understand foams, the cobordisms of webs. Suppose that for all $i \in \{1, 2, 3\}$, $\Sigma^i$ are compact oriented smooth (generally not connected) 2–manifolds with $m$ boundary components $S_1^i, \ldots, S_m^i$ each. Let $\varphi_{j,i} : S_j^i \rightarrow S_j^\prime$ and $\psi_{j,i} : S_j^i \rightarrow S_j^\prime$ be orientation preserving diffeomorphisms. Consider the quotient of $\Sigma^1 \cup \Sigma^2 \cup \Sigma^3$ by the equivalence relation generated by $[x] = [\varphi_{j,i}(x)] = [\psi_{j,i}(x)]$ for all $j$. The images of the $S_j^i$ in the quotient are called singular circles, and the images of connected components of the $\Sigma^i$ are called facets. There are three facets adjacent to each singular
circle. Associate a non-negative integer to each facet. Such an integer $d$ will graphically be represented by drawing $d$ dots on the facet, which may roam the facet freely, but may not cross a singular circle. Such a quotient, together with the dots and with a choice of cyclic ordering of the three facets around each singular circle is called a prefoam.

Now consider a smooth embedding of a prefoam into $\mathbb{R}^3$, i.e. an embedding that is smooth on the facets and on the singular circles. Such an embedding induces cyclic orderings of the facets around each singular circle by the left-hand rule (see figure 3). If these cyclic orderings agree with those given by the prefoam, the image of the embedding is called a closed foam. Under the following conditions, the intersection of a closed foam with the cylinder $B \times [0, 1] = \{(x, y, z) \mid x^2 + y^2 \leq 1 \text{ and } 0 \leq z \leq 1\}$ is called a foam:

- The boundary of the cylinder intersects the facets and singular circles of the closed foam transversely.
- The side $(\partial B) \times [0, 1]$ of the cylinder intersects the closed foam in finitely many vertical lines, and is disjoint from all singular circles.
- The intersections with the top and bottom of the cylinder are webs.
- The base point of the top and the base point of the bottom web have the same $x$– and $y$–coordinates.

We consider foams up to isotopies which, on the side of the cylinder, do not depend on the $z$–coordinate. A connected component of the intersection of a singular circle with the cylinder is called a singular edge. The tangle diagrams on the bottom of the cylinder is called domain of the foam, and the codomain of the foam is defined as the tangle diagram on the top of the cylinder, with the orientation of each edge reversed. As usual, let $\chi$ denote the Euler characteristic. Then the degree of a foam $f$ is defined by

$$\deg f = \chi(\text{domain of } f) + \chi(\text{codomain of } f) + 2(\text{total number of dots on } f) - 2\chi(f).$$

Foams can be glued in two ways: if the domain of one foam agrees with the codomain of another, by stacking them on top of each other. Or, by gluing them into the cylindrical holes of a thickened input diagram. The degree is additive with respect to both of these operations.

Webs with a fixed boundary and the foams between them thus constitute a graded category, i.e. a category whose morphisms have an integral rank which is additive under composition. Let us define a planar algebra $W^\infty$ over $I_3$ and the category $\text{GCat}$ of small graded categories: $^1$ to $\varepsilon \in I_3$, associate the category whose set of objects is $W_\varepsilon$, and whose morphisms $W \to W'$ between two webs $W, W' \in W_\varepsilon$ are the foams with domain $W$ and

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$^1$The $c$ superscript stands for categorification.
codomain $W'$. If $H$ is a planar input diagram, then $W_H$ is the functor that acts as $W_H$ on the objects, and glues foams into a thickened version of $H$.

Next, $W^{eq}$ may be constructed by applying a functor from $\mathbf{GCat}$ to $\mathbf{ACat}$, the category of small additive categories: replace webs by $\mathbb{Z}[q^{\pm 1}]$–linear combinations of webs, and foams by matrices of $\mathbb{Z}$–linear combination of foams, where morphisms from $q^\alpha \cdot W$ to $q^\beta \cdot W'$ are the foams with degree $\beta - \alpha$. So the categories $W_{\varepsilon}^{eq}$ are not graded, but have instead a shift operator for their objects. In this planar algebra, consider the following morphisms:

\[(C') \quad \begin{array}{c} (\quad \bullet \quad \bullet \quad \bullet \quad )^T \\ (\quad \circ \quad \circ \quad \circ \quad ) \end{array} \quad (q^{-2} + 1 + q^{3}) \cdot (\quad \circ \quad )\]

\[(D') \quad \begin{array}{c} (\quad \bullet \quad \bullet \quad \bullet \quad )^T \\ (\quad \circ \quad \circ \quad \circ \quad ) \end{array} \quad (q^{-1} + q) \cdot (\quad \circ \quad ) \quad \text{and} \quad (S') \quad \begin{array}{c} (\quad \bullet \quad \bullet \quad \bullet \quad )^T \\ (\quad \circ \quad \circ \quad \circ \quad ) \end{array} \quad (q^{-1} + q) \cdot (\quad \circ \quad )

Demanding that these three pairs of morphisms are mutually inverse – for any placement of the base point – generates an equivalence relation on the planar algebra $W^{eq}$. For example, composing the two morphisms in $(C')$ one way yields the so-called surgery relation, and composing them the other way yields the evaluation of the sphere with four or less dots.

Let $W^{eq}$ be the quotient of $W^{eq}$ by this equivalence relation. It is a consequence of [Kho04] that the relation set $(C')$, $(D')$, $(S')$ is equivalent to the original relation set given by Khovanov. Thus the elegant, but non-constructive universal BHMV-construction [BHMV95] may be circumvented.

Let us consider some examples of $W_{\varepsilon}^{eq}$. For $\varepsilon = \emptyset$, the isomorphism classes of objects of this category are in correspondence with the elements of the free $\mathbb{Z}[q^{\pm 1}]$–module generated by the empty web $\emptyset$. Non-zero morphisms $q^\alpha \cdot \emptyset \rightarrow q^\beta \cdot \emptyset$ exist only if $\alpha = \beta$, and in this case the morphism $\mathbb{Z}$–module is just $\mathbb{Z}$. The $\mathbb{Z}[q^{\pm 1}]$–module Khovanov [Kho04] associates to a closed web $W$ can be recovered as

$$\bigoplus_{\alpha \in \mathbb{Z}} \text{Hom}_{W_{\varepsilon}^{eq}}(q^0 \cdot \emptyset, q^\alpha \cdot W).$$

Similarly, the isomorphism classes of objects of $W_{\varepsilon}^{eq}$ are in correspondence with the elements of the free $\mathbb{Z}[q^{\pm 1}]$–module generated by the web $\begin{array}{c} \bullet \quad \bullet \quad \bullet \quad \end{array}$, which is just an interval. The morphisms are more complicated, however, since there are non-zero foams of three different degrees: rectangles with none, one, or two dots. So the morphism $\mathbb{Z}$–module from $q^\alpha \cdot \begin{array}{c} \bullet \quad \bullet \quad \bullet \quad \end{array} \rightarrow q^\beta \cdot \begin{array}{c} \bullet \quad \bullet \quad \bullet \quad \end{array}$ is $\mathbb{Z}$ if $\beta - \alpha \in \{0, 2, 4\}$, and trivial otherwise.
Let a diffeomorphism between two foams be called a foam diffeomorphism if its restriction to the top (bottom) of the cylinder constitutes a web diffeomorphism between the domains (codomains) of the two foams. In $W^{cqr}$, the equivalence class of a foam is already determined by its foam diffeomorphism type; this is not the case in $W^{cq}$, where e.g. a cylindrical foam tied into a knot is not the identity of the circular web.

Finally, let $W^{cqrt}$ be the planar algebra obtained from $W^{cqr}$ by setting $W^{cqrt}_{\epsilon}$ to be the category of bounded chain complexes (up to chain homotopy) over $W^{cqr}_{\epsilon}$. More formally, this means applying a monoidal functor $K: ACat \rightarrow ACat$. The natural transformation $K(C_1) \otimes K(C_2) \rightarrow K(C_1 \otimes C_2)$ is given by

\[
(*) \quad (P_i, g_i) \otimes (Q_j, h_j) \mapsto \left( \bigoplus_{i+j=k} P_i \otimes Q_j, \sum_{i+j=k} g_i \otimes \text{id}Q_j + (-1)^i \text{id}P_i \otimes h_j \right)_k.
\]

In the notation of complexes, the module at $t$–degree 0 is underlined. We may now define the $\mathfrak{sl}_3$–chain complex as the planar algebra morphism $V^c: T \rightarrow W^{cqrt}$ uniquely determined by the skein relations

\[
\text{(Sk}^{-c}\text{)} \quad V^c \left( \begin{array}{c}
\text{x} \\
\text{y}
\end{array} \right) = q^2 \left( \begin{array}{c}
\text{x} \\
\text{y}
\end{array} \right) \quad \text{and}
\]

\[
\text{(Sk}^{+c}\text{)} \quad V^c \left( \begin{array}{c}
\text{x} \\
\text{y}
\end{array} \right) = q^{-3} \left( \begin{array}{c}
\text{x} \\
\text{y}
\end{array} \right) \quad \Rightarrow \quad q^{-2} \left( \begin{array}{c}
\text{x} \\
\text{y}
\end{array} \right).
\]

Note that these relations each just give the complex of one tangle diagram, and no cone or flattening is involved. The minus-sign which must intervene to arrange anti-commutativity of differentials along squares is hidden in (*). Although the placement of this sign is somewhat arbitrary, $V^c$ is still unique, since chain complexes are considered only up to homotopy.

Reidemeister invariance may, just as for the $\mathfrak{sl}_3$–polynomial, be shown by inspecting the small tangle diagrams corresponding to Reidemeister moves [MN08, section 4.2]. So the $\mathfrak{sl}_3$–chain complex is, up to chain homotopy, an invariant of tangles.

A link diagram lives in $T_\emptyset$, and is mapped to a chain complex in the category of $\mathbb{Z}[q^{\pm 1}]$–linear combination of closed webs and matrices of $\mathbb{Z}$–linear combination of foams between them. Due to the relations, there is a homotopy to a chain complex which contains only empty webs and closed foams. The latter are in turn just integral multiples of the empty foam. Setting the empty foam to 1, a chain complex in the category of free graded abelian groups is obtained. Its homology is the $\mathfrak{sl}_3$–homology of the link.

2.5. Reduced $\mathfrak{sl}_3$–homology. A reduced version of $\mathfrak{sl}_N$–homology has been introduced by Khovanov and Rozansky [KR08a] in the context of matrix factorisations. This section contains a definition in the context of foams.

Let $D$ be a diagram of a link $L$ with a marked component. Cutting $D$ open at an arbitrary point on the marked component, one obtains a tangle diagram $D'$ with boundary $+\ldots$. It is well known that two such tangle diagrams represent the same link with a marked component if and only if they are connected by a finite sequence of Reidemeister moves. So, the homotopy type of the $\mathfrak{sl}_3$–chain complex $C$ of $D'$ is an invariant of links with a marked component; in particular, it is a knot invariant. As noted in section 2.4, if $C$ is fully simplified, it contains only one kind of web – an interval –, and three kinds of foams: the identity foam of the interval with none, one or two dots. Now map the identity foam to one, and foams of higher degree to zero. Thus one obtains a chain complex of free graded abelian groups. Its homology is the reduced $\mathfrak{sl}_3$–homology of $L$, an invariant of links with a marked component.
As simple examples show, its value may indeed depend on the choice of marked component, and it does neither determine the unreduced version, nor is it determined by it (see section 3.5).

3. Calculations

3.1. The algorithm. The above definition of $\mathfrak{sl}_3$-homology gives a straightforward way of practical calculation: for an $n$–crossing diagram, take $n$ copies of the chain complex of $\text{Sk}^{\pm c}$, and take the tensor product (*) of these complexes as determined by the diagram. This is the same as forming the cube of resolutions. Transform this chain complex of closed webs and foams between them into a homotopic chain complex of empty webs and closed foams using the three relations ($C^c$), ($D^c$), ($S^c$); finally, evaluate the closed foams, and calculate the homology of the emerging integral chain complex.

In the algorithm described by Morrison and Nieh [MN08], it is just the order in which these steps are taken which is changed: we do not apply (Sk$^{\pm c}$) to all crossings at once, but to one after the other, and try to simplify the chain complex at each step as much as possible. Examples for the manual application of this algorithm can be found in [MN08].

At each step, one manipulates a chain complex of tangles; if these tangles have less boundary points, then there are fewer different tangles and cobordisms between them and thus the calculations demand less memory and will, heuristically, go faster. Therefore, one glues the crossings in such an order that the cardinal of the boundary of the intermediate tangles is minimised (or at least, as low as one sees possible). This minimum is precisely the girth of the link diagram, which is thus the main factor limiting the speed of the algorithm.

In this text, we propose a variation of this algorithm based on sub-tangle trees, which is potentially faster, because intermediate tangles will have smaller boundary.

**Definition 3.1.** Let $D$ be a non-split link diagram with crossings enumerated from 1 to $n$. Decompose $D$ into small tangle diagrams $D_i$ that contain the $i$–th crossing, respectively. A sub-tangle tree of $D$ is a full binary tree with a tangle diagram at each node, such that

- The leaves are decorated by the $D_i$.
- The root is decorated by $D$.
- Every node which is not a leaf has two children decorated with adjacent tangle diagrams, and is decorated itself with the union of those two tangle diagrams.

Let the girth of a sub-tangle tree be the maximum number of boundary points of the tangle diagrams at all nodes. The recursive girth of $D$ is the minimum of the girths of all its sub-tangle trees.
In general, the recursive girth of a diagram is smaller than its girth. For example, pretzel links have recursive girth 4, and girth 6; and the 222–crossing link considered in [FGMW10] has girth $\approx 24$, and recursive girth at most 16.

Let us give a description of the algorithm. Given a link diagram $D$, find a sub-tangle tree of $D$ with small girth. The algorithm consists in simplifying the $\mathfrak{s}_3$–chain complex of the tangle diagram at each node, one after the other. In the final step, the chain complex of the diagram at the root is handled, which is $D$ itself. After simplifications, this will be a chain complex of vectors of $q$–shifted empty webs and matrices whose entries are integer multiples of the empty foam; identifying the empty foam with 1, this becomes a complex of free graded abelian groups, and its homology may be calculated – separately for each $q$–degree – using the Smith normal form.

In the beginning, only the complexes at the leaves are known, which are given by the relations $(\text{Sk}^{k\epsilon})$. During each step, fix a node whose two children have complexes $C_1$ and $C_2$ that are already known and simplified. Now, decorate this node with the tensor product $C_1 \otimes C_2$ (see (*)) and simplify this chain complex as described below. After as many steps as $D$ has crossings, the process reaches the root and is finished.

There are two ways to simplify a chain complex: firstly, apply the circle, digon or square relation wherever possible. Secondly, apply Gauß’ homotopy to all matrix entries which are plus or minus an invertible foam:

**Lemma 3.2** (Gauß’ homotopy, [BN07]). Over an additive category with an isomorphism $h$, the chain complex

$$
\cdots \longrightarrow P \xrightarrow{g} Q \oplus R \xrightarrow{h} S \oplus T \xrightarrow{i} U \xrightarrow{j} \cdots
$$

is homotopic to

$$
\cdots \longrightarrow P \xrightarrow{g} R \xrightarrow{k} T \xrightarrow{l} U \xrightarrow{m} \cdots
$$

The additive category in question is $W_{cqr}^{\epsilon}$, whose objects are $R[q^{\pm 1}]$–linear combinations of webs with boundary $\epsilon$, and whose morphisms are matrices of $R$–linear combinations of foams. The base ring $R$ is usually $\mathbb{Z}$; but one may also just calculate homology over some field. Gauß’ homotopy may then be applied more often, namely to all non-zero multiples of invertible foams instead of just plus or minus an invertible foam. This may increase the algorithm’s speed.

As a rule, use the circle/digon/square relations only when it is not possible to apply Gauß’ homotopy. This is to prevent those relations from altering a foam which is plus or minus the identity and could thus be removed.

### 3.2 Extracting the $s_3$–concordance invariant from homology

There is a spectral sequence $E_\bullet$ from the graded $\mathfrak{s}_3$–homology converging to a the filtered version. Given the $\mathfrak{s}_3$–homology of a knot, this allows in many cases to determine its $s_3$–invariant. The same method was used by Freedman et al. for $\mathfrak{s}_2$–homology [FGMW10, section 5.2].

The filtered $\mathfrak{s}_3$–homology has Poincaré polynomial $q^{-2s_3}(q^{-2} + 1 + q^2)$. The differential on the $k$–th page of the spectral sequence has $(t,q)$–degree $(1, -3k)$. Let $KR_3$ be the Poincaré polynomial of the graded $\mathfrak{s}_3$–homology of a fixed knot. Then $E_\bullet$ gives rise to a decomposition of the form

$$
KR_3(t,q) = q^{-2s_3}(q^{-2} + 1 + q^2) + \sum_{k=1}^{\infty} \zeta_k(t,q) \cdot (1 + tq^{3k}),
$$

for some Laurent polynomials $\zeta_k(t,q)$ with non-negative coefficients. Reversely, given the value of $KR_3$, one may easily determine all possible such decompositions. The conjecture
that \( E \) converges on the second page translates as \( \forall k \geq 2 : \zeta_k(t,q) = 0 \). Assuming the conjecture is true, the above decomposition is unique; otherwise, however, it is generally not. But even if there are are several possible decompositions, it may happen they all share the same value for \( s_3 \). This need not be the case, either, as the example \( K \mathcal{R}_3(t,q) = 1 + q^2 + q^4 + q^6 + q^{12} \) demonstrates: either \( s_3 = -1, \zeta_1(t,q) = q^6, \zeta_3(t,q) = 0 \), or \( s_3 = -2, \zeta_2(t,q) = 1, \zeta_3(t,q) = 0 \). The first knots for which \( s_3 \) cannot be uniquely determined from \( sl_3 \)-homology are 12n118, 12n210, 12n214 and 12n318 (see section 3.5 for further examples). At any rate, one obtains a list of possible values for \( s_3 \), and for most small knots, only one value is possible.

3.3. **Implementation issues.** An implementation of the algorithm of section 3.1 is possible because webs and foams need only be considered up to web and foam diffeomorphisms (see sections 2.3 and 2.4), and their diffeomorphism type contains only finitely much information. A web is determined by the following data:
- Its boundary, given as a sign-word.
- The number of edges that are circles.
- For each vertex, the triple of the vertices or boundary points it is connected to, listed in counterclockwise order.

Likewise, a foam is completely encoded by the following information:
- Its domain and codomain web.
- The start and end point of every singular edge that is an interval.
- For each singular edge, the three facets adjacent to it, in the order specified by the left-hand rule (see fig. 3).
- The genus of and number of dots on each facet.
- Each boundary component of each facet, given as an ordered tuple of edges; in this tuple, edges of the domain and codomain web, and singular edges alternate.

In order to reduce the complexity of foams, FoamHo applies the well-known relations shown in fig. 5 whenever possible. In particular, these relations are sufficient to evaluate closed foams. Gauß' homotopy is in fact only applied to plus or minus an identity foam; there are other isomorphisms than those, but they are difficult to detect for a computer program, and so rare that looking for them does not seem worthwhile.

3.4. **FoamHo, a \( sl_3 \)-calculator.** The algorithm described in the previous sections was implemented by the author as a C++-program\(^2\). It computes the integral or rational (homology over finite fields is not yet implemented), reduced or unreduced homology of knot or link diagrams given in braid, planar diagram or Dowker-Thistlethwaite notation; it also attempts to find the value of the \( s_3 \)-concordance invariant, using the spectral sequence from \( sl_3 \)-homology to filtered \( sl_3 \)-homology. If it is impossible to extract the \( s_3 \)-invariant, a list of possible values is printed, and the value which corresponds to the convergence of the spectral sequence on the second page is highlighted.

The current version does not make use of the sub-tangle trees yet, and glues instead one crossing after the other. Apart from that, there is surely some room for rendering the program faster and less memory hungry by optimising the code, without changing the algorithm; e.g., webs and foams are encoded in a redundant way, and tightening this would reduce the memory consumption.

The program was baptised FoamHo in recognition of Shumakovitch’s KhoHo [Shu03]. It has been released under the GPL\(^3\), and its source code as well as compiled versions for Linux and Windows may be downloaded from [Lew12]. There are also tables of the homology and the \( s_3 \)-concordance invariant of small knots and links available.

\(^{2}\)Using the PARI/GP-library [PAR12] to calculate the Smith normal form, and the MPIR library [MPI12] for arbitrary precision integers and rationals.

\(^{3}\)See http://www.gnu.org/licenses/gpl.html.
Let us give a rough idea of the capabilities of the program for large knots. On an AMD Opteron (2.2 GHz), the (6,5)–torus knot’s homology can be calculated in six minutes, using 80 MB of RAM; the (8,5)–torus knot takes 50 minutes and 275 MB of RAM; but the (7,6)–torus knot is out of the reach of 5 GB of RAM. In other words, links of girth 10 can be calculated until ca. 40 crossings, while links of girth 12 would demand a well-equipped computer.

Detailed usage instructions for FoamHo can be found in the README-file which is distributed along with the program’s source code and binaries. The following appetising example session demonstrates the computation of the (4,3)–torus knot’s homology. User input begin with a dollar sign and is printed bold:

$ ./foamho -h
foamho, a sl3-homology calculator, version 1.1.
Usage: foamho [OPTIONS] braid | pd | dt NOTATION

For example, the following three commands all compute the figure-8-knot’s integral homology:
foamho braid aBaB
foamho pd "[[4,2,5,1],[8,6,1,5],[6,3,7,4],[2,7,3,8]]
foamho dt "[4,6,8,2]"

Options:
-q  Compute rational homology instead of integral.
-r  Compute reduced homology instead of unreduced. You may
give a number right after -r to indicate the marked
strand (useful for links).
-g  Do not attempt to optimise the girth.
-v  Display some progress information.
-vv Display more detailed progress information.
-t  Display time and memory consumption.
-h  Display this help message and exit.

Written in 2012/2013 by Lukas Lewark, lewark@math.jussieu.fr.
All feedback is welcome.

$ ./foamho -t braid abababab

Girth-optimised link diagram (modified pd notation) [[2,4,3,1], [5,7,6,4],
[6,9,8,3], [7,11,10,9], [10,13,12,8], [11,15,14,13], [14,16,1,12],
[15,5,2,16]].
Girth: 6.
Calculating...
Done. Result:
Rational homology: \((q^{-14} + q^{-12} + q^{-10}) + t^{-2}(q^{-16} + q^{-14}) +
t^{-3}(q^{-22} + q^{-20}) + t^{-4}(q^{-20} + 2q^{-18} + q^{-16}) +
t^{-5}(q^{-26} + 2q^{-24} + q^{-22}) + t^{-6}(q^{-22}) + t^{-7}(q^{-28} + q^{-26}) +
t^{-8}(q^{-32})\)
Total rank: 19
Rational homology is not self-dual => the link is chiral.
3-torsion: \(t^{-3}(q^{-18}) + t^{-5}(q^{-22} + q^{-20}) + t^{-7}(q^{-26} + q^{-24}) +
t^{-8}(q^{-30} + q^{-28})\)
The 3-torsion part of homology is not self-dual => the link is chiral.
s_3-concordance invariant: -12
Run time in seconds: 2
Memory consumption in megabytes: 9.5

3.5. Calculatory results.

Pretzel knots: Since pretzel knots have girth only 6, their homology can be computed
particularly quickly. In addition to FoamHo calculations, we use an inequality which is
stated by Lobb [Lob11] (cf. also Kawamura [Kaw07]) for the Rasmussen invariant, but
holds for the \(s_3\)-invariant as well. With these two tools, we can confirm the conjecture (see
introduction) for the following values:
- \(b \leq 49\), any \(a > b\) and any \(c \geq 4\) (an infinite family).
- \(b < a \leq 49\) and \(c = 2\).

Furthermore, we are able to prove the conjecture for \(c > 2\), and prove \(\delta_2 \in \{-1, -2\}\). The
proof (paper in preparation [Lew13]) makes use of the tools developed by Rasmussen in
[Ras06].
s₃ and s₂ for small knots: Of the 59,937 prime knots with 14 crossings or less, there are 361 for which the s₃ and s₂ concordance invariants differ, and for 63 the absolute value of s₃ is greater, i.e. the lower bound to the slice genus coming from s₃ is better than the one coming from s₂. For all these 361 knots, the difference between s₂ and s₃ is ±1. A 16–crossing prime knot example for a greater difference is given by the conjecture (see introduction): the (7, −5, 4)–pretzel knot, with s₃ = 0 and s₂ = 2.

The s₃–concordance invariant cannot be determined for certain knots with 14 crossings or less, so the statistics in the preceding paragraph are based on the assumption that the spectral sequence to the filtered version of homology converges on the second page.

s₃ and the Floer–concordance invariant τ: Let τ be the knot concordance invariant coming from Floer homology, as defined by Ozsváth and Szabó [OS03]. Hedden and Ording [HO05] found examples of knots K for which 2 = s₂(K) ≠ 2τ(K) = 0. FoamHo calculations yield the following results for these knots:

\[ s₃(D⁺(T_{2,3}, 2)) = 2 \]
\[ s₃(D⁺(T_{2,7}, 8)) \in \{2, 3, 4\} \]
\[ s₃(D⁺(T_{2,5}, 5)) \in \{2, 3\} \]
\[ s₃(D⁺(T_{2,7}, 7)) \in \{2, 3, 4, 5\} \]
\[ s₃(D⁺(T_{2,5}, 4)) \in \{2, 3, 4\} \]
\[ s₃(D⁺(T_{2,7}, 6)) \in \{2, 3, 4, 5, 6\} \]

All but the first knot are examples of how the method of section 3.2 may fail to completely determine the value of s₃. Under the assumption that the spectral sequence to the filtered version of homology converges on the second page, however, each of the above knots K satisfies s₃(K) = 2. So these knots cannot be expected to give examples for s₂ ≠ s₃, but they demonstrate that s₃ ≠ 2τ.

Thinness: Khovanov [Kho03] called a knot H-thin if its unreduced Khovanov homology is supported in only one diagonal, or, equivalently, if its reduced Khovanov homology is supported in only one. Thinness was generalised to slₙ–homologies by Rasmussen [Ras07]. Alternating knots are H-thin, but not necessarily N-thin for N ≥ 2. We find the knots 11a263, 12a36, 12a604, 12a604, 12a604, 12a617, 12a829, 12a832 to be the smallest examples of alternating knots that are not 3-thin. None of these knots is two-bridge, which agrees with Rasmussen’s theorem [Ras06] that two-bridge knots are N-thin for all N.

Mutation: Integral reduced slₙ–homology has been proven to be invariant under mutation of knots by Jaeger [Jae11]; for unreduced homology, invariance appears to be an open question. Our calculations confirm that all mutant families up to 13 crossings have the same reduced and unreduced integral slₙ–homology. We use the lists provided by Stoimenow [Sto].

Torsion: All prime knots and links for which the homology was computed, including knots with up to 12 and links with up to 11 crossings, have 3–torsion, with the exception of the Hopf links, whose homology is torsion-free. This is reminiscent of the omnipresence of 2–torsion in Khovanov homology remarked by Shumakovitch [Shu04]. Exemplary calculations also show the existence of 2–, 4–, 5– and 8–torsion, which are rather scarce. Small knots have torsion-free reduced homology, but large enough knots like the (8, 5)–torus knot have not.

Reduced and unreduced homology: Reduced sl₂–homology was conjectured by Khovanov [Kho03] to be determined by its unreduced counterpart; more precisely, that the rank of reduced homology were one less than the rank of unreduced homology. This is true for small knots, but Shumakovitch produced 15–crossing counterexamples [Shu12]. For sl₃–homology even considering only knots with crossing number six is enough to see that there is no linear relationship between the ranks of reduced and unreduced homology.
Previous computations: FoamHo calculations are in agreement with the results of Carqueville and Murfet [CM11], who compute reduced and unreduced rational $\mathfrak{s}_3$–homology of all prime knots and links with up to six crossings.

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