Lyapunov-Net: A Deep Neural Network Architecture for Lyapunov Function Approximation

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Abstract—We develop a versatile deep neural network architecture, called Lyapunov-Net, to approximate Lyapunov functions of dynamical systems in high dimensions. Lyapunov-Net guarantees positive definiteness, and thus can be easily trained to satisfy the negative orbital derivative condition, which only renders a single term in the empirical risk function in practice. This significantly simplifies parameter tuning and results in greatly improved convergence during network training and approximation quality. We also provide comprehensive theoretical justifications on the approximation accuracy and certification guarantees of Lyapunov-Nets. We demonstrate the efficiency of the proposed method on nonlinear dynamical systems in high dimensional state spaces, and show that the proposed approach significantly outperforms the state-of-the-art methods.

I. INTRODUCTION

Lyapunov functions play central roles in dynamical systems and control theory. They have been used to characterize asymptotic stability of equilibria of nonlinear systems, estimate regions of attraction, investigate system robustness against perturbations, and more [6], [11], [12], [30]. In addition, Control Lyapunov functions are used to derive stabilizing feedback control laws [2]. The properties that Lyapunov functions must satisfy can be viewed as constrained partial differential inequations (PDIs). However, like partial differential equations (PDEs), for problems with high dimensional state spaces, it remains very challenging to compute approximated Lyapunov functions using classical numerical methods, such as finite-difference method (FDM) and finite element method (FEM). The main issue with these classical methods is the sizes of variables to solve grows exponentially fast in terms of the state space dimension of the problem. Such issue is known as the curse of dimensionality [4].

Recent years have witnessed a tremendous success of deep learning (DL) methods in solving high-dimensional PDEs [9], [14], [29], [38]. Backed by the provable approximation power [15], [16], [22], [27], [37], a (deep) neural network can approximate a general set of functions at any prescribed accuracy level if the network size (e.g., width and depth) is sufficiently large. Successful training of the parameters (e.g., weights and biases) of the network approximating the solution of a PDE also require a properly designed loss function and an efficient (nonconvex) optimization algorithm.

However, unlike traditional DL-based methods in many classification and regression applications, the methods developed in [9], [14], [29], [38] parameterize the solutions of specific PDEs as deep networks, and the training of the these networks only require function and partial derivative (up to certain order depending on the PDE’s order) evaluations at randomly sampled collocation points.

Recently, deep neural networks (DNNs) also emerged to approximate Lyapunov functions of nonlinear and non-polynomial dynamical systems in high-dimensional spaces [6], [7], [12], [30], [36]. For control Lyapunov functions, DNNs can also be used to approximate the control laws, eliminating the restrictions on control laws to specific function type (e.g., affine functions) in classical control methods such as linear-quadratic regulator (LQR).

In this work, we propose a general framework to approximate Lyapunov functions and control laws using deep neural nets. Our main contributions lie in the following aspects:

• We propose a highly versatile network architecture, called Lyapunov-Net, to approximate Lyapunov functions. Specifically, this network architecture guarantees the desired positive definiteness property. This leads to simple parameter tuning, significantly accelerated convergence during network training, and greatly improved solution quality. This will allow for fast adoption of neural networks in (control) Lyapunov function approximation in a broad range of applications.

• We show that the proposed Lyapunov-Nets are dense in a general class of Lyapunov functions. More importantly, we prove that the Lyapunov-Nets trained by minimizing empirical risk functions using finitely many collocation points are Lyapunov functions, which provides a theoretical certification guarantee of neural network based Lyapunov function approximation.

• We test the proposed Lyapunov-Net to solve several benchmark problems numerically. We show that our method can effectively approximate Lyapunov functions in these problems with very high state dimensionality. Moreover, we demonstrate that our method can be used to find control laws in the control Lyapunov problem setting.

With the accuracy and efficiency of Lyapunov-Nets demonstrated in this work, we anticipate that our method can be applied to a much broader range of dynamical system and control problems in high dimensional spaces.

The remainder of the paper is organized as follows. In Section II, we review the recent developments in approximating Lyapunov functions and control laws using deep learning methods as well as briefly discuss some results in neural

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network approximation power. In Section III, we present our proposed network architecture and training strategies as well as critical theory providing certification guarantees for Lyapunov-Net in approximating Lyapunov functions. In Section IV, we conduct numerical experiments using the proposed method on several nonlinear dynamical systems, demonstrating promising results on these tests. Section V concludes this paper.

II. RELATED WORK

a) Approximating Lyapunov function using neural networks: Approximating Lyapunov functions using neural networks can be dated back to [23], [33]. In [23] the authors attempted the idea assuming that a shallow neural network can exactly represent the target Lyapunov function. In [33], stabilization problems in control using neural networks with one or two hidden layers are considered. In [28], the authors propose a special shallow network to approximate Lyapunov functions, however, the determination of the network parameters involves a series of constraints and Hessian computations. In [18], the control Lyapunov functions (CLF) using quadratic Lyapunov function candidate is considered. A DNN approach to CLF is considered in [1], [6] which are similar. Approximating stabilizing controllers using neural networks is considered in [20]. Time discretized dynamics and successive parameter updates are considered in [30], [32]. Specifically, [30] considers jointly learning the Lyapunov function and its decreasing region which is expected to match the Region of Attraction (ROA).

Special network architectures to approximate Lyapunov functions are also considered in [30]. In [30], a Lyapunov neural network of form \( \| \phi_0(\cdot) \|^2 \) is proposed to ensure positive semi-definiteness, where \( \phi_0(\cdot) \) is a DNN. To ensure positive definiteness, the authors restrict \( \phi_0 \) to be a feedforward neural network, all weight matrices to have full column rank, all biases to be zero, and all activation functions to have trivial null space (e.g., tanh or leaky ReLU but not sigmoid, swish, softmax, ReLU, or RePU). Compared to the architecture in [30], the network architecture in the present work does not have any of these restrictions. Moreover, the positive definite weight matrix constructions in [30] can make \( \| \phi_0(x) \|^2 \) grow excessively fast as \( \| x \| \) increases, whereas ours does not have this issue.

In [12], the author considers dynamical systems with small-gain property, which yields a compositional Lyapunov function structure that have decoupled components. In this case, it is shown that the size of the DNN used to approximate such Lyapunov functions can be dependent exponentially on the maximal dimension of these components rather than the original state space dimension.

Still other authors propose neural networks for other control certificates such as barrier functions [7], [31], [36] and contraction metrics [34], [35]. While these are a different direction compared to our approach, they present other promising angles for neural networks in control.

b) Universal approximation theory of neural networks: The approximation powers of neural networks have been studied since the early 1990s. The asymptotic analysis justifies that shallow networks with one hidden layer and any non-polynomial differentiable activation function is dense in \( C^1(\Omega) \) for any compact set \( \Omega \subset \mathbb{R}^d \) [15]. However, the number of neurons needed to obtain a prescribed approximation accuracy of \( \varepsilon > 0 \) may grow exponentially fast in \( d \). Specifically, for shallow networks with sigmoid activation function, it is shown that a universal approximator of \( C^1([-1,1]^d) \) needs \( O(\varepsilon^{-d}) \) computing neurons [15], [16]. The results are extended to (deep) ReLU neural networks in [22], [37], and RePU networks in [21].

There are also works developed to investigate the power of DNNs in approximating other classes of functions. In the class of analytic functions, authors in [8], [26] showed the exponential convergence speed based on the size of the approximating network. The study of approximation power of DNN remains an active research field [3], [5], [24], as the theory continues to lag behind the success of neural networks in practice [13].

III. PROPOSED METHOD

In this section, we propose the Lyapunov-Net architecture to approximate Lyapunov functions, and discuss its key properties and associated training strategies. We first recall the definition of Lyapunov function for a given general dynamical system \( x' = f(x) \) with Lipschitz continuous \( f \) on the problem domain \( \Omega \). We assume that \( x = 0 \) is the unique equilibrium of the system dynamics that lies within \( \Omega \). If \( x = 0 \) is asymptotically stable, then by the converse Lyapunov theory, we can find a Lyapunov function \( V \) defined as below.

**Definition 1** (Lyapunov function). Let \( \Omega \subset \mathbb{R}^d \) be a bounded open set and \( \delta \in \Omega \), and \( f : \Omega \rightarrow \mathbb{R}^d \) a Lipschitz function. Then \( V : \Omega \rightarrow \mathbb{R} \) is called a Lyapunov function if (i) \( V \) is positive definite, i.e., \( V(x) \geq 0 \) for all \( x \in \Omega \) and \( V(x) = 0 \) if and only if \( x = 0 \); and (ii) \( V \) has negative orbital-derivative, i.e., \( DV(x) \cdot f(x) < 0 \) for all \( x \neq 0 \).

Denote \( B(x; \delta) := \{ y \in \mathbb{R}^d : \| y - x \| < \delta \} \) as the open ball of radius \( \delta > 0 \) centered at \( x \), and \( \Omega_\delta := \Omega \setminus B(0; \delta) \) the problem domain with \( B(0; \delta) \) excluded. Our approach is to approximate a Lyapunov function using a specially designed deep neural network. Due to limited network size and finite collocation points for training in practice, we only guarantee that our approximation function \( V \) is positive definite in \( \Omega \) and satisfies a slightly relaxed condition of Definition 1 (ii) as follows:

\[
DV(x) \cdot f(x) < 0, \quad \text{for all } x \in \Omega_\delta.
\]  

(1)

where \( \delta > 0 \) is arbitrary and prescribed by user. We term such a function \( V \) as a \( \delta \)-accurate Lyapunov function.

A \( \delta \)-accurate function can be used as a Lyapunov function for \( x' = f(x) \) (or control-Lyapunov function for \( f(x, u(x)) \)) where the control \( u \) is to be found jointly with the Lyapunov function, see Section III-D) to prove that the solution \( x(t) \) will be ultimately bounded within a small compact set (i.e., \( B(0; \delta) \)) [17]. As \( \delta \rightarrow 0 \), the size of this compact set will converge to 0, hence asymptotic stability is established. In
practice, the smaller the value of $\delta$, the larger the network size and number of collocation points are needed to train such a $\delta$-accurate Lyapunov function.

A. Lyapunov-Net and its properties

We first construct an arbitrary network $\phi_\theta(\cdot): \mathbb{R}^d \to \mathbb{R}$ with the set of all its $m$ trainable parameters denoted by $\theta \in \mathbb{R}^m$. This network has input dimension $d$ and output dimension 1. Then we build a scalar-valued network $V_\theta: \mathbb{R}^d \to \mathbb{R}$ from $\phi_\theta$ as follows:

$$V_\theta(x) := |\phi_\theta(x) - \phi_\theta(0)| + \alpha \|x\|, \quad (2)$$

where $\alpha > 0$ is a small user-chosen parameter and $\|\cdot\|$ is the standard 2-norm in Euclidean space. Then it is easy to verify that $V_\theta(0) = 0$ and

$$V_\theta(x) \geq \alpha \|x\| > 0, \quad \forall x \neq 0.$$  

In other words, $V_\theta$ is a candidate Lyapunov function that already satisfies condition (i) in Definition 1: for any network structure $\phi_\theta$ with any parameter $\theta$, $V_\theta$ is positive definite and only vanishes at the equilibrium $0$. We call the neural network $V_\theta$ with architecture specified in (2) a Lyapunov-Net.

We make several remarks regarding the Lyapunov-Net architecture (2) below.

First, we use the augment term $\alpha \|x\|$ to lower bound the function $V_\theta(x)$ in order to ensure positive definiteness. Other positive definite function $r: \mathbb{R}^d \to \mathbb{R}_+$ such that $r(x) = 0$ if and only if $x = 0$ can be chosen as such lower bound, such as $\alpha \|x\|^2$ or $\alpha \log(1 + \|x\|^2)$, etc.

Second, the term $|\phi_\theta(x) - \phi_\theta(0)|$ in (2) can be replaced with $\psi(\phi_\theta(x) - \phi_\theta(0))$ for any non-negative function $\psi: \mathbb{R}^n \to \mathbb{R}_+$ with $\psi(0) = 0$. We chose $\psi(\cdot) = |\cdot|$ for its application to our theory and simplicity. So long as $\phi_\theta$ is Lipschitz continuous then $V_\theta$ is Lipschitz continuous and hence weakly differentiable. In practice, we can also use $|\cdot|^2$ or Huber norm to smooth out $V_\theta$. Note that we can also use vector-valued DNN $\phi_\theta: \mathbb{R}^d \to \mathbb{R}^d$ where $d'$ is arbitrary which can further improve network capacity. In this case, $\psi(\cdot)$ can be set to $|\cdot|$.

Third, if the equilibrium is at $x^*$ instead of 0, then we can simply replace $\phi_\theta(0)$ and $\|x\|$ in (2) with $\phi_\theta(x^*)$ and $\|x - x^*\|$, respectively. Without loss of generality, we assume the equilibrium is at 0 hereafter in this paper.

B. Training of Lyapunov-Net

The training of Lyapunov-Net $V_\theta$ in (2) refers to finding a specific network parameter $\theta$ such that the negative-orbital-derivative condition $DV_\theta(x) \cdot f(x) < 0$ is satisfied at every $x \in \Omega \setminus \{0\}$. This can be achieve by minimizing a risk function that penalize $V_\theta$ if the negative-orbital-derivative condition fails to hold at some $x$. We can choose the following as such risk function:

$$\ell(\theta) := \frac{1}{|\Omega|} \int_\Omega (DV_\theta(x) \cdot f(x) + \gamma \|x\|)^2_+ dx, \quad (3)$$

where $(z)_+ := \max(z, 0)$ for any $z \in \mathbb{R}$. Here $\gamma$ is a user chosen parameter. It is clear that the risk function $\ell(\theta)$ reaches the minimal function value 0 if and only if $DV_\theta(x) \cdot f(x) \leq -\gamma \|x\|$ for all $x \in \Omega$, which, in conjunction with $V_\theta$ already being positive definite, ensures that $V_\theta$ is a Lyapunov function.

In practice, the integral in (3) does not have analytic form, and we have to resort to Monte-Carlo integration which is suitable for high-dimensional problems. To this end we approximate $\ell(\theta)$ in (3) using the empirical expectation

$$\hat{\ell}(\theta) := \frac{1}{N} \sum_{i=1}^N (DV_\theta(x_i) \cdot f(x_i) + \gamma \|x_i\|)^2_+, \quad (4)$$

where $\{x_i: i \in [N]\}$ are drawn from $\Omega$ using a user chosen sampling method. Then we train the Lyapunov-Net $V_\theta$ by minimizing $\hat{\ell}(\theta)$ in (4) with respect to $\theta$. In this case, standard network training algorithms, such as ADAM [19], can be employed. For simplicity, we use uniform sampling in the experiments, and leave improved sampling strategies for future investigation.

The empirical risk function defined using finitely many sampling points introduces inaccuracies near 0, which is common in the literature. Several existing works [6], [12] observed that the deep-learning-based Lyapunov function approximation may violate the condition $DV_\theta(x) \cdot f(x) < 0$ within a small neighborhood of the equilibrium. As such, in this work we will concern ourselves with finding only a $\delta$-accurate Lyapunov function as discussed following (1).

It is worth stressing that the main advantage of the Lyapunov-Net architecture (2) is that the risk function (3) (or the empirical risk function (4)) consists of a single term only with a single parameter. This is in contrast to existing works [6], [12] where the risk functions have multiple terms to penalize the violations of the negative-orbital-derivative condition, positive definiteness condition, bound requirements, etc. Thus, network training in these works requires experienced users to carefully tune the hyperparameters to properly weigh these penalty terms in order to obtain a reasonable solution. On the other hand, the proposed empirical risk function for Lyapunov-Net $V_\theta$ requires little effort in parameter-tuning, and the convergence is much faster in network training, as will be demonstrated in our numerical experiments below.

C. Lyapunov-Net approximation and certification theory

We now present a result that shows the theoretical guarantees of the approximation ability of Lyapunov-Net. This result is stated for the activation function RePU which in this section shall refer to the function $RePU(x) = \max\{0, x\}^2$. We note that the forthcoming result can be extended to other even smoother activation functions (such as tanh or sigmoid).

Let $\Omega = [-1, 1]^d$ and $\Omega_\delta = \Omega \setminus B(0; \delta)$. Let $x^f = f(x)$ be the dynamical system defined by $f$.

**Assumption 1.** $f$ is $L_f$-Lipschitz continuous on $\Omega$ for some $L_f > 0$ and $f(0) = 0$.

**Assumption 2.** There exist $V^* \in C^1(\Omega; \mathbb{R})$ and constants
\[ \alpha, \beta, \gamma > 0, \text{ such that for all } x \in \Omega \text{ there are} \]
\[ \alpha \|x\| \leq V^\ast(x) \leq \beta \|x\|, \quad (5a) \]
\[ DV^\ast(x) \cdot f(x) \leq -\gamma \|x\|. \quad (5b) \]

With these assumptions we can show the following result:

**Theorem 1.** Let \( \delta > 0 \) and the empirical risk function be given by \( \ell(\theta) = \frac{1}{N} \sum_{i=1}^{N} (DV_\theta(x^{(i)}) \cdot f(x^{(i)}) + \gamma \|x^{(i)}\|) \) for properly chosen sample points \( X := \{x_i \in \Omega_\delta : i = 1, \ldots, N\} \) with \( N \) large enough and neural network \( V_\theta \) given by (2) with RePU activation. Then the minimizer of \( \ell \) must exist and achieve minimum function value 0. Moreover, for any such minimizer \( \hat{\theta} \), the corresponding \( V_{\hat{\theta}} \) is a \( \delta \)-accurate Lyapunov function of \( f \) on \( \Omega \).

The proofs for this theorem and its prerequisite lemmas can be found at [10].

**D. Application to control and others**

In light of the power of Lyapunov functions, we can employ the proposed Lyapunov-Net to many control problems of nonlinear dynamical systems in high-dimension. In this subsection, we instantiate one of such applications of Lyapunov-Net to approximate control Lyapunov function.

Consider a nonlinear control problem \( x' = f(x,u) \) where \( u : \mathbb{R}^d \to \mathbb{R}^n \) (\( d \) is the dimension of the control variable at each \( x \)) is an unknown state-dependent control in order to steer the state \( x \) from any initial to the equilibrium state 0. To this end, we parameterize the control as a deep neural network \( u_\eta : \mathbb{R}^d \to \mathbb{R}^n \) where \( \eta \) represents the network parameters of \( u_\eta \).

Once the network structure \( u_\eta \) is determined, we can define the risk of the control-Lyapunov function (CLF):

\[ \ell_{\text{CLF}}(\theta, \eta) := \frac{1}{|\Omega|} \int_{\Omega} (DV_\theta(x) \cdot f(x,u_\eta(x)) + \gamma \|x\|)^2 dx. \quad (6) \]

Minimizing (6) yields the optimal parameters \( \theta \) and \( \eta \). In practice, we again approximate \( \ell_{\text{CLF}}(\theta, \eta) \) by its empirical expectation \( \hat{\ell}_{\text{CLF}}(\theta, \eta) \) at sampled points in \( \Omega \), as an analogue to \( \ell(\theta) \) versus \( \hat{\ell}(\theta) \) above. Then the minimization can be implemented by alternately updating \( \theta \) and \( \eta \) using (stochastic) gradient descent on \( \hat{\ell}_{\text{CLF}} \). Similar as (3), we have a single term in the loss function in (6), which does not have hyper-parameters to tune and the training can be done efficiently.

**IV. NUMERICAL EXPERIMENTS**

**A. Experiment setting**

We demonstrate the effectiveness of the proposed method through a number of numerical experiments in this section. In our experiments, the value of \( \bar{\alpha} \) used in (2) and the depth and size of \( \theta_0 \) used in \( V_\theta \) for the three test problems are summarized in Table I. We minimize the empirical risk function \( \hat{\ell} \) using the Adam Optimizer with learning rate 0.005 and \( \beta_1 = 0.9, \beta_2 = 0.999 \) and Xavier initializer. In all tests, we iterate until the associated risk of (4) is below a prescribed tolerance of \( 10^{-4} \). We use a sample size \( N \) (values shown in Table I), i.e., the number of uniformly sampled points in \( \Omega \) in (4), such that the associated risk reduces reasonably fast while maintaining good uniform results over the domain.

All the implementations and experiments are performed using PyTorch in Python 3.9 in Windows 10 OS on a desktop computer with an AMD Ryzen 7 3800X 8-Core Processor at 3.90 GHz, 16 GB of system memory, and an Nvidia GeForce RTX 2080 Super GPU with 8 GB of graphics memory. The number of iterations needed to reach our stopping criteria in (4) and training time (in seconds) for the three tests are also given in Table I. As discussed in Section II, how the width and depth of a neural network is related to approximation power is an area of active research. In our experiments, we manually selected the width and depth as shown in Table I which yield good results. We leave investigation of how these affect the approximation power of Lyapunov-Net to further study.

**B. Experimental results**

To demonstrate the effectiveness of the proposed method, we apply Lyapunov-Net (2) to three test problems: a two-dimensional (2d) nonlinear system from the curve-tracking application [25], a 10d and 30d synthetic dynamical system (DS) from [12].

\begin{itemize}
  \item \textit{a) 2d DS in curve tracking:} We apply our method to find the Lyapunov function for a 2d nonlinear DS in a curve-tracking problem [25]. The DS of \( x = (\rho, \phi) \) is given by
    \[ \dot{\rho} = -\sin(\phi), \quad (7a) \]
    \[ \phi = (\rho - \rho_0) \cos(\phi) - \mu \sin(\phi) + e. \quad (7b) \]
    We use the following constants in our experiments: \( e = 0.15, \rho_0 = 1, \) and \( \mu = 6.42 \) from [25] as well as RePU activation.

    The problem domain was mapped to Cartesian coordinates around critical point \( x' = (1, 0) \), and converted back when the training is done. The result shows that after only 2 iterations the positive definiteness and negative-orbital-derivative conditions are met. We used RePU activation which is consistent with the theoretical results provided in Section III-C. However, these results still hold by readily modifying the proofs if the common tanh activation is used. We have also tested tanh activation in Lyapunov-Net and obtained similar performance.

  \item \textit{b) 10d Synthetic DS:} We consider a 10d synthetic DS from [12], which is a vector field defined on \([-1, 1]^{10}\) and has an equilibrium at 0. The iteration number and computation time needed for training are shown in Table I.

  \item \textit{c) 30d Synthetic DS:} We consider the same system as the 10d Synthetic DS but concatenate the function \( f \) three times to get a 30d problem. Figure 1 graphed the approximated Lyapunov function \( V_\theta \) (top solid) and \( DV_\theta \cdot f \) (bottom wire) in the \((x_2, x_8)\) and \((x_{10}, x_{13})\) planes, using
\end{itemize}
RePU activation. These plots show that Lyapunov-Net can effectively approximate Lyapunov functions in such high-dimensional problem. We shall also use this example to compare the convergence speeds to the Neural Network structures of [6], [12] in the following section.

C. Comparison with existing DL methods

To demonstrate the significant improvement of Lyapunov-Net over existing DL methods in approximation efficiency, we compare the proposed Lyapunov-Net to two recent approaches [6], [12] that also use deep neural networks to approximate Lyapunov functions in continuous DS setting. Specifically, we use the 30d synthetic DS described above as the test problem in this comparison. Both [6], [12] employ generic deep network structure of $V_{\Theta}$, and thus require additional terms in their risk functions to enforce positive definiteness of the networks. Specifically, the following risk function is used from [12]:

$$\hat{\ell}_1(\Theta) = \frac{1}{N} \sum_{i=1}^{N} \left( (DV_{\Theta}^{DL}(x_i) \cdot f(x_i)) + (||x_i||^2)^2 \right)_+ + (20||x_i||^2 - V_{\Theta}^{DL}(x_i))_+ + \left( V_{\Theta}^{DL}(x_i) - 0.2||x_i||^2 \right)_+^2,$$

which aims at an approximate Lyapunov function $V_{\Theta}^{DL}$ satisfying $0.2||x||^2 \leq V_{\Theta}^{DL}(x) \leq 20||x||^2$ and $DV_{\Theta}^{DL}(x) \cdot f(x) \leq -||x||^2$ for all $x$. In [6], the following risk function is used:

$$\hat{\ell}_2(\Theta) = V_{\Theta}^{NL}(0)^2 + \frac{1}{N} \sum_{i=1}^{N} \left[ (DV_{\Theta}^{NL}(x_i) \cdot f(x_i))_+ + (-V_{\Theta}^{NL}(x_i))_+ \right],$$

which aims at an approximate Lyapunov function $V_{\Theta}^{NL}$ such that $V_{\Theta}^{NL}(0) = 0$, $V_{\Theta}^{NL}(x) \geq 0$ and $DV_{\Theta}^{NL}(x) \cdot f(x) \leq 0$ for all $x$. The activation functions are set to softmax in (8) as suggested in [12] and tanh in (9) as suggested in [6]. We shall label these models as Deep Lyapunov (DL) and Neural Lyapunov (NL) respectively. For Lyapunov-Net we use (4) as it already satisfies the positive definiteness condition. We again do not impose any structural information of the problem into our training, and thus all test methods recognize the DS as a generic 30d system for sake of a fair comparison.

We use the value of the less stringent empirical risk function $\hat{\ell}_2$ with $N = 400,000$ defined in (9) as a metric to evaluate all three methods. Specifically, we plot the values of $\hat{\ell}_2$ (in log scale) versus iteration number and wall-clock training time (in seconds) in Figure 2 using the same learning rate for all methods.

We see in Figure 2 that Lyapunov-Net (LN) has risk value decaying much faster than the other methods. Further, Lyapunov-Net training does not need hyper-parameter tuning to achieve this speed, whereas DL and NL require careful and tedious tuning to balance the different terms in the risk function in order to achieve satisfactory results as shown in Figure 2. This highlights the efficiency and simplicity of Lyapunov-Net to find the desired Lyapunov functions.

We note that the performance of all methods can be further improved using additional structural information as discussed in [12] and falsification techniques in training as in [6]. We leave these improvements to future investigations.

D. Application in Control

In this test we compare Lyapunov-Net and SOS/SDP methods in the problem of estimating the Region of Attraction (ROA) for the classical inverted pendulum control problem. We shall show how the Lyapunov-Net framework allows simultaneous training of the control policy and outperforms SOS/SDP methods by producing a larger ROA.

The inverted pendulum is an often considered problem in control theory see [6], [30]. For this problem we have dynamics $x = (\theta, \dot{\theta})$ governed by $ml^2 \ddot{\theta} = mgl \sin \theta - \beta \dot{\theta} + u$, where $u = u(x)$ is the control. We use $g = 9.82$, $l = 0.5$, $m = 0.15$, and $\beta = 0.1$ for our experiments. We shall use this model to compare SOS/SDP type methods to Lyapunov-Net when it comes to estimating the ROA of this problem using the Lyapunov candidate function they both find within respective valid regions.
In the absence of such a comparison we could make $\bar{u}$ a much more complex and further improved control policy. The training method is the same for both networks.

In Figure 3, we use tanh as activation in $V_0$ because while the error bounds are similar with RePU, tanh has slightly better performance in this case. Once trained the $\bar{u}$-policy found is used for the SOS type method. As SOS methods are only applicable to polynomial systems, we apply a Taylor approximation to the dynamics of the system and then compute a sixth order polynomial candidate using the standard SOS/SDP approach. We finally compute the ROA determined by both algorithms using the level-sets of the functions over the valid regions they found. We find that the area of the region found by Lyapunov-Net is larger than that found by SOS. This result is plotted in Figure 3 where the orange paths are a few sample trajectories. We note the similarity of our results to [6], who in addition to the above, found examples where SOS methods produced incorrect ROA regions on this same problem.

V. CONCLUSIONS

We constructed a versatile deep neural network architecture called Lyapunov-Net to approximate Lyapunov functions for general high-dimensional dynamical systems. We provided theoretical justifications on approximation power and certificate guarantees of Lyapunov-Nets. Applications to control Lyapunov functions are also considered. We demonstrated the effectiveness of our method on several test problems. The Lyapunov-Net framework developed in the present work is expected to be applicable to a much broader range of control and stability problems.

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