Quenched limit theorems for expanding on average cocycles

Davor Dragičević, Julien Sedro

November 25, 2021

Dedicated to the memory of Vlado Dragičević (1945-2021)

Abstract

We prove quenched versions of a central limit theorem, a large deviations principle as well as a local central limit theorem for expanding on average cocycles. This is achieved by building an appropriate modification of the spectral method for nonautonomous dynamics developed by Dragičević et al. (Comm Math Phys 360: 1121–1187, 2018), to deal with the case of random dynamics that exhibits nonuniform decay of correlations, which are ubiquitous in the context of the multiplicative ergodic theory. Our results provide an affirmative answer to a question posed by Buzzi (Comm Math Phys 208: 25-54, 1999).

1 Introduction

In the recent years, a great deal of effort has been devoted to the investigation of the statistical properties of random dynamical systems. Indeed, those are key tools to model many natural phenomena, including the transport in complex environments such as in the ocean or the atmosphere [3]: it is therefore crucial to understand their long term quantitative behavior. Among many remarkable contributions, we particularly emphasize those dealing with the decay of correlations [2, 6, 7, 10, 13, 16], various (quenched or annealed) limit laws [1, 4, 17, 18, 19, 20, 21, 32, 33, 34, 41, 45, 46, 50], as well as recent
results devoted to the linear response of random dynamical systems [9, 25, 49]. For similar results in the closely related context of sequential dynamical systems, we refer to [11, 12, 35, 37, 38, 39, 31, 44] and references therein.

1.1 Spectral method for limit theorems

The Nagaev-Guivarc’h spectral method is a powerful approach to establish limit theorems for dynamical systems. Originally introduced in the context of Markov chains [42, 43] and for the purpose of establishing the central limit theorem in autonomous dynamics [30], the method was broadly extended to cover many types of limit theorems: large deviations, Berry-Esseen estimates, local central limit theorems or almost-sure invariance principle, to name a few (we refer to the excellent survey [29], and references therein, for more details).

In [18] (see also [34] for a similar approach), this method was further extended to the non-autonomous case of random compositions of expanding maps, allowing to obtain fiberwise (or quenched) limit theorems, and relying on the careful study of the Lyapunov spectrum, and the associated Oseledets spaces, of a so-called twisted transfer operator cocycle. More precisely, given a measurable family \((T_\omega)_{\omega \in \Omega}\) of (say) uniformly expanding maps of the unit interval \(I\), parametrized by a probability space \((\Omega, \mathcal{F}, P)\) endowed with an invertible, measure-preserving and ergodic transformation \(\sigma : \Omega \xrightarrow{\text{erg}} \Omega\), one forms the cocycle above \(\sigma\):

\[
T^n_\omega := T_{\sigma^{n-1}\omega} \circ \cdots \circ T_\omega, \quad \omega \in \Omega, \; n \in \mathbb{N},
\]

which is known to admit a family of equivariant measures \((\mu_\omega)_{\omega \in \Omega}\) (i.e. \(\mu_\omega \circ T_\omega^{-1} = \mu_{\sigma\omega}\) for P-a.e \(\omega \in \Omega\)). Given a suitable observable \(\psi : \Omega \times I \to \mathbb{R}\), one wants to study the asymptotic behavior of the random process \((\psi_{\sigma^n\omega} \circ T^n_\omega)_{n \geq 0}\) for fixed \(\omega\) in a subset of full measure of \(\Omega\), where \(\psi_\omega = \psi(\omega, \cdot)\). In this endeavor, one associates to each transformation \(T_\omega\), the corresponding transfer operator \(\mathcal{L}_\omega := \mathcal{L}_{T_\omega}\).\(^1\) In addition, one considers the twisted transfer operator \(\mathcal{L}_\omega^\theta\), with twist parameter \(\theta \in \mathbb{C}\), defined for \(\phi \in L^1(I)\) by:

\[
\mathcal{L}_\omega^\theta(\phi) := \mathcal{L}_\omega(e^{i\theta\omega} \phi).
\]

The spectral method for random dynamics may then be crudely summarized in three key steps:

1. Relate the characteristic functions of the (random) Birkhoff sums \(S_n \psi_\omega, n \in \mathbb{N}\) (see (61)) to the \(n\)-th iterate of the twisted transfer cocycle.

\(^1\)i.e., for each \(f \in L^1(I)\), \(g \in L^\infty(I)\), \(\int_I f \cdot g \circ T_\omega \, dm = \int_I f \, \mathcal{L}_\omega^\theta \cdot g \, dm\), where \(m\) is the Lebesgue measure
2. Establish the quasi-compactness (on a suitable Banach space) of the twisted transfer operator cocycle \((L_0^\theta)_{\omega \in \Omega}\) for all parameters \(\theta\) sufficiently close to 0.

3. Study the regularity w.r.t. the twist parameter \(\theta\) of the top Lyapunov exponent \(\Lambda(\theta)\) of the cocycle \((L_0^\theta)_{\omega \in \Omega}\).

Once established, these properties allow to prove various (quenched) limit theorems, such as large deviations or (local) central limit theorem. More recently, this spectral approach was used to establish quenched limit theorems for random hyperbolic dynamics [20] and random \(U(1)\)-extension of one-dimensional expanding maps [14].

1.2 Main contributions of the present paper

Despite those developments, several natural and interesting types of random systems remained out of reach of the method presented in [18]; most notably, systems exhibiting non-uniform in \(\omega\) decay of correlations such as the (general case of) random dynamical systems expanding on average studied by Buzzi [13].

In the setting when all maps \(T_\omega\) are chosen in a vicinity of a single (deterministic and transitive) map \(T\) (which is precisely the setup of [14, 20]), the associated random system will exhibit uniform (in \(\omega\)) decay of correlations (with respect to suitable observables). However, if this is not the case, requiring the presence of a uniform decay of correlations becomes a restrictive condition (see [19, Section 2.2]).

In sharp contrast to [18], where contracting behavior in the family \((T_\omega)_{\omega \in \Omega}\) is only allowed on a set of zero measure (see [18, Eq. (20)]), the results in the present paper cover the general case of expanding on average random dynamics, that doesn’t exhibit the uniform in \(\omega\) decay of correlations, providing a positive answer to a problem posed in [13, p. 30]. We emphasize that, compared with the precise class of expanding on average random dynamical systems introduced in [13], we only need the additional mild assumption that the map \(\omega \mapsto T_\omega\) has a countable range: this is to ensure that we can apply a suitable version of the multiplicative ergodic theorem [26].

We note that there are previous works devoted to limit theorems for random dynamical system that allow contracting behavior on large measure sets [1],

---

\(^2\)Allowing to choose among uncountably many maps would require to work with observables living in a separable space, e.g., fractional Sobolev spaces. In this aspect, we are limited by our choice to work with observables of bounded variation, which form a non-separable Banach space; in turns, this choice has other advantages.
but only under the condition that the family \((T_\omega)_{\omega \in \Omega}\) only takes finitely many values (and assuming that \((\Omega, \mathcal{F}, \mathbb{P}, \sigma)\) is a Bernoulli shift), or don’t require the presence of a uniform decay of correlations, such as the work of Kifer [41] (partially inspired by the work of Cogburn [15]), as well as the first author and Hafouta [22]. Roughly speaking, the main idea in those papers is to pass to the associated induced system, where the inducing is done with respect to the region of \(\Omega\) on which one has the uniform decay of correlations.

Provided that we have a good control for the first entry time to this region, one can deduce limit theorems for the original system from that of the induced one. The problematic aspect of this approach is the following: it is hard to ensure good control of the first entry time since a priori we cannot say anything about the region with respect to which we induce. Indeed, it remains an interesting open problem to build concrete examples to which the results of [41, 22] are applicable, that don’t exhibit the uniform decay of correlations.

In the present paper we rather modify the approach introduced in [18]. The main technical novelty is the introduction of suitable “adapted norms” which depend on the random parameter \(\omega\), and allow us to “absorb the non-uniformity” present in the constants appearing in the decay of correlations estimate. The precise construction relies on the Oseledets splitting of the (unperturbed) transfer cocycle, and is reminiscent of the classical construction due to Pesin for non-uniformly hyperbolic dynamics (see also [3, Chapter 4] for random product of matrices).

Naturally, this will require some restrictions for the class of observables to which our limit theorems will apply. Indeed, we demand that the BV-norm of our observable \(\psi_\omega\) is dominated by a tempered random variable \(K\), that reflects the non-uniformity in the decay of correlations and is explicitly constructed from so-called “Oseledets regularity functions”, which are notoriously hard to compute: this can make our condition difficult to check in practice.

Once the construction of adapted norms is carried out, the previously described steps follow by adapting classical perturbation techniques, based on the analytic version of the implicit function theorem, to our setting. In particular, this greatly simplify (when compared to [18]) the study of the regularity w.r.t. the twist parameter of the top Lyapunov exponent and the associated top Oseledets space generator. Apart from this, our proofs follow closely previous works, most notably [18].
1.3 Statement of main theorems

We now turn to the statement of our main results. Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, endowed with an invertible, measure-preserving ergodic transformation $\sigma: \Omega \to \Omega$, we let $(T_\omega)_{\omega \in \Omega}$ be a measurable family of maps, with the associated transfer operator cocycle $(L_\omega)_{\omega \in \Omega}$ satisfying the requirements of Definition 13. The maps $T_\omega$ act on a set $X$ endowed with a probability measure $m$ and a notion of variation $\text{Var}$ as in Section 2.3. Then, there is (see Theorem 20) an equivariant family $(\mu_\omega)_{\omega \in \Omega}$ of absolutely continuous probability measures, $d\mu_\omega = v_\omega^0 dm$, where $v_\omega^0 \in BV$.

For an observable $\psi: \Omega \times X \to \mathbb{R}$, satisfying $\psi_\omega \in BV$, $\text{esssup}_{\omega \in \Omega} \|\psi_\omega\|_{BV} < +\infty$ (3) and a centering condition (50), we introduce the scaled observable $\psi_K: \Omega \times X \to \mathbb{R}$, defined by

$$\psi_K(\omega, \cdot) := \frac{1}{K^2(\omega)} \psi(\omega, \cdot),$$

where the random variable $K$ is given by (37). We will be concerned with the asymptotic behavior of the Birkhoff sum associated to $\psi_K$, $S_n\psi_K(\omega, \cdot)$, defined by

$$S_n\psi_K(\omega, x) := \sum_{i=0}^{n-1} \psi_K(\sigma^i \omega, T^i_{\omega}).$$

Our first main result is a quenched Central Limit Theorem:

**Theorem 1 (Central Limit Theorem).** Under the previous assumptions, and if the variance $\Sigma^2$, given by (66), satisfies $\Sigma^2 > 0$, then for any bounded continuous $\phi: \mathbb{R} \to \mathbb{R}$, for $\mathbb{P}$-a.e $\omega \in \Omega$, one has

$$\lim_{n \to \infty} \int_X \phi \left( \frac{S_n\psi_K(\omega, x)}{\sqrt{n}} \right) d\mu_\omega(x) = \int \phi d\mathcal{N}(0, \Sigma^2),$$

where $\mathcal{N}(0, \Sigma^2)$ is the centered normal law with variance $\Sigma^2$.

We then prove a Large Deviations Theorem:

**Theorem 2 (Large Deviations Theorem).** Under the previous assumptions, there exists $\varepsilon_0 > 0$ and a non-negative, continuous, strictly convex function $c: (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}$, vanishing only at 0 and such that for any $0 < \varepsilon < \varepsilon_0$, and $\mathbb{P}$-a.e $\omega \in \Omega$,

$$\lim_{n \to \infty} \frac{1}{n} \log \mu_\omega (S_n\psi_K(\omega, \cdot) > n\varepsilon) = -c(\varepsilon)$$

(7)
Finally, we establish the quenched Local Central Limit Theorem, under an aperiodicity condition:

**Theorem 3 (Local Central Limit Theorem).** Under the previous assumptions, and if the aperiodicity condition (76) is satisfied, then for $\mathbb{P}$-a.e $\omega \in \Omega$, every bounded interval $J \subset \mathbb{R}$, one has:

$$\lim_{n \to \infty} \sup_{s \in \mathbb{R}} \left| \sum \sqrt{n} \mu_\omega(s + S_n \psi_K(\omega, \cdot) \in J) - \frac{|J|}{\sqrt{2\pi}} e^{-\frac{s^2}{2\sigma^2}} \right| = 0. \quad (8)$$

We emphasize that the above results generalize those in [18], where $K$ is a constant random variable (see Remark 34) and consequently one can take $\psi_K = \psi$. Let us also emphasize that, as the example discussed in [24, Appendix] illustrate, for observables satisfying only condition (3) the asymptotic variance (63) may fail to exist: this shows that our ‘scale restriction’ (4) is necessary, and that renormalized limit theorems such as Theorems 1, 2 and 3 are the best one can hope for in general. We also note that examples of observables for which our limit theorems hold may be constructed without the above scaling procedure (see Example 35 for details).

Finally, we would like to end this introduction by noting that our methods can be used to obtain other type of limit theorems (e.g. Berry-Esseen estimates), but to keep the length of this paper within reasonable bounds, we refrain to give a complete list of those (for the specific instance of Berry-Esseen estimate in our setting, we may argue exactly as in [21, Section 4.4]). In fact, among the ‘classical’ limit theorems, only the almost sure invariance principle required a non-trivial addition to our approach, which was carried out in [24].

## 2 Preliminaries

### 2.1 Multiplicative ergodic theorem

We begin by recalling the notion of a (linear) cocycle.

**Definition 4.** A tuple $\mathcal{R} = (\Omega, \mathcal{F}, \mathbb{P}, \sigma, \mathcal{B}, \mathcal{L})$ is said to be a linear cocycle or simply a cocycle if the following conditions hold:

- $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space and $\sigma: \Omega \to \Omega$ is an invertible ergodic measure-preserving transformation;
- $(\mathcal{B}, \| \cdot \|)$ is a Banach space and $\mathcal{L}: \Omega \to L(\mathcal{B})$ is a family of bounded linear operators.
We say that \( \mathcal{L} \) is the generator of \( \mathcal{R} \).

We will often identify a cocycle \( \mathcal{R} = (\Omega, \mathcal{F}, \mathbb{P}, \sigma, \mathcal{B}, \mathcal{L}) \) with its generator \( \mathcal{L} \). Moreover, we will write \( \mathcal{L}_\omega \) instead of \( \mathcal{L}(\omega) \).

**Definition 5.** Let \( \mathcal{R} = (\Omega, \mathcal{F}, \mathbb{P}, \sigma, \mathcal{B}, \mathcal{L}) \) be a cocycle. We say that \( \mathcal{R} \) is \( \mathbb{P} \)-continuous if the following holds:

\( \bullet \) \( \Omega \) is a Borel subset of a separable, complete metric space and \( \sigma \) is a homeomorphism;

\( \bullet \) \( \mathcal{L} \) is continuous on each of countably many Borel sets whose union is \( \Omega \).

Let \( \mathcal{R} = (\Omega, \mathcal{F}, \mathbb{P}, \sigma, \mathcal{B}, \mathcal{L}) \) be a \( \mathbb{P} \)-continuous cocycle such that

\[
\int_{\Omega} \log^+ \|L_\omega\| \, d\mathbb{P}(\omega) < +\infty.
\]

(9)

For \( \omega \in \Omega \) and \( n \in \mathbb{N} \), set

\[
\mathcal{L}_\omega^n = L_{\sigma^{n-1}\omega} \circ \ldots \circ L_{\sigma\omega} \circ L_\omega.
\]

It follows from Kingman’s subadditive ergodic theorem that, for \( \mathbb{P} \)-a.e. \( \omega \in \Omega \), the following limits exists:

\[
\Lambda(\mathcal{R}) = \lim_{n \to \infty} \frac{1}{n} \log \|\mathcal{L}_\omega^n\|
\]

and

\[
\kappa(\mathcal{R}) = \lim_{n \to \infty} \frac{1}{n} \log \text{ic}(\mathcal{L}_\omega^n),
\]

where

\[
\text{ic}(A) := \inf \left\{ r > 0 : A(B_B) \text{ admits a finite covering by balls of radius } r \right\},
\]

\( B_B \) is the unit ball of \( \mathcal{B} \), and

\[-\infty \leq \kappa(\mathcal{R}) \leq \Lambda(\mathcal{R}) < +\infty\]

**Definition 6.** We say that \( \Lambda(\mathcal{R}) \) is the top Lyapunov exponent of a cocycle \( \mathcal{R} \), and that \( \kappa(\mathcal{R}) \) is the index of the compactness of \( \mathcal{R} \).

**Definition 7.** Let \( \mathcal{R} = (\Omega, \mathcal{F}, \mathbb{P}, \sigma, \mathcal{B}, \mathcal{L}) \) be a \( \mathbb{P} \)-continuous cocycle such that (9) holds. We say that \( \mathcal{R} \) is quasi-compact if \( \kappa(\mathcal{R}) < \Lambda(\mathcal{R}) \).
The following result gives sufficient conditions under which a cocycle is quasi-compact.

**Lemma 8.** Let $\mathcal{R} = (\Omega, \mathcal{F}, \mathbb{P}, \sigma, \mathcal{B}, \mathcal{L})$ be a $\mathbb{P}$-continuous cocycle such that (9) holds. Furthermore, let $(\mathcal{B}', |\cdot|)$ be a Banach space such that $\mathcal{B} \subset \mathcal{B}'$ and that the inclusion $(\mathcal{B}, \|\cdot\|) \hookrightarrow (\mathcal{B}', |\cdot|)$ is compact. Finally, assume the following:

- $\mathcal{L}_\omega$ can be extended continuously to $(\mathcal{B}', |\cdot|)$ for $\mathbb{P}$-a.e. $\omega \in \Omega$;
- there exist $N \in \mathbb{N}$ and measurable functions $\alpha_\omega, \beta_\omega, \gamma_\omega : \Omega \to \mathbb{R}$ such that for $f \in \mathcal{B}$ and $\mathbb{P}$-a.e. $\omega \in \Omega$,

$$\|\mathcal{L}_\omega^N f\| \leq \alpha_\omega \|f\| + \beta_\omega |f| \quad (10)$$

and

$$\|\mathcal{L}_\omega\| \leq \gamma_\omega; \quad (11)$$

- we have that

$$\int_\Omega \log \alpha_\omega \, d\mathbb{P}(\omega) < N \Lambda(\mathcal{R}) \text{ and } \int_\Omega \log \gamma_\omega \, d\mathbb{P}(\omega) < \infty. \quad (12)$$

Then,

$$\kappa(\mathcal{R}) \leq \frac{1}{N} \int_\Omega \log \alpha_\omega \, d\mathbb{P}(\omega).$$

In particular, $\mathcal{R}$ is quasi-compact.

**Proof.** If $N = 1$, the desired conclusion follows from [18, Lemma 2.1.]. In the general case, we consider the cocycle $\mathcal{R}^N = (\Omega, \mathcal{F}, \mathbb{P}, \sigma^N, \mathcal{B}, \mathcal{L})$ whose generator $\mathcal{L}$ is given by $\mathcal{L}_\omega := \mathcal{L}_\omega^N$, $\omega \in \Omega$. It is easy to show that

$$\kappa(\mathcal{R}^N) = N \kappa(\mathcal{R}) \quad \text{and} \quad \Lambda(\mathcal{R}^N) = N \Lambda(\mathcal{R}). \quad (13)$$

It follows from our assumptions and [18, Lemma 2.1.] that

$$\kappa(\mathcal{R}^N) \leq \int_\Omega \log \alpha_\omega \, d\mathbb{P}(\omega),$$

which together with (13) implies the desired conclusion.

We are now in a position to recall the version of the multiplicative ergodic theorem (MET) established in [26].

**Theorem 9.** Let $\mathcal{R} = (\Omega, \mathcal{F}, \mathbb{P}, \sigma, \mathcal{B}, \mathcal{L})$ be a quasi-compact $\mathbb{P}$-continuous cocycle. Then, there exist $1 \leq l \leq \infty$ and a sequence of exceptional Lyapunov exponents

$$\Lambda(\mathcal{R}) = \lambda_1 > \lambda_2 > \ldots > \lambda_l > \kappa(\mathcal{R}) \quad (if \ 1 \leq l < \infty)$$

8
or

\[ \Lambda(\mathcal{R}) = \lambda_1 > \lambda_2 > \ldots \quad \text{and} \quad \lim_{n \to \infty} \lambda_n = \kappa(\mathcal{R}) \quad (\text{if } l = \infty). \]

Furthermore, for \( P\)-a.e. \( \omega \in \Omega \) there exists a unique splitting (called the Oseledets splitting) of \( \mathcal{B} \) into closed subspaces

\[ \mathcal{B} = V(\omega) \oplus \bigoplus_{j=1}^{l} Y_j(\omega), \quad (14) \]

depending measurably on \( \omega \) and such that:

1. For each \( 1 \leq j \leq l \), \( Y_j(\omega) \) is finite-dimensional (i.e. \( m_j := \dim Y_j(\omega) < \infty \)), \( Y_j \) is equivariant i.e. \( L_\omega Y_j(\omega) = Y_j(\sigma \omega) \) and for every \( y \in Y_j(\omega) \setminus \{0\} \),

\[ \lim_{n \to \infty} \frac{1}{n} \log \|L_\omega^n y\| = \lambda_j. \]

2. \( V \) is weakly equivariant i.e. \( L_\omega V(\omega) \subseteq V(\sigma \omega) \) and for every \( v \in V(\omega) \),

\[ \lim_{n \to \infty} \frac{1}{n} \log \|L_\omega^n v\| \leq \kappa(\mathcal{R}). \]

### 2.2 Dual cocycles

In the following, we will need to establish ‘parallel’ properties for a cocycle and its dual; hence, we recall in this section some basic facts related to this notion.

Given a cocycle \( \mathcal{R} = (\Omega, \mathcal{F}, \mathbb{P}, \sigma, \mathcal{B}, \mathcal{L}) \), one defines the adjoint cocycle \( \mathcal{R}^* = (\Omega, \mathcal{F}, \mathbb{P}, \sigma^{-1}, \mathcal{B}, \mathcal{L}^*) \), where \( (\mathcal{L}^*)_\omega = \mathcal{L}_{\sigma^{-1}\omega}^* \), \( \omega \in \Omega \). We will write \( \mathcal{L}_\omega^* \) for \( (\mathcal{L}^*)_\omega \), which is thus the adjoint operator of \( \mathcal{L}_{\sigma^{-1}\omega} \). We notice that \( \mathcal{R}^* \) is \( \mathbb{P} \)-continuous if and only if \( \mathcal{R} \) is.

Furthermore, it is easy to see that \( \Lambda(\mathcal{R}^*) = \Lambda(\mathcal{R}) \) and that \( \kappa(\mathcal{R}) = \kappa(\mathcal{R}^*) \). In particular, \( \mathcal{R} \) is quasi-compact if and only if \( \mathcal{R}^* \) is.

One also has a natural relation between the Oseledets splitting and Lyapunov exponents of \( \mathcal{R} \) and those of \( \mathcal{R}^* \). Recall that, given a subspace \( S \subset \mathcal{B} \), the annihilator \( S^o \) of \( S \) is defined by \( \{ \ell \in \mathcal{B}^*, \ell(f) = 0 \ \forall \ f \in S \} \); similarly, given \( S^* \subset \mathcal{B}^* \) we set \( (S^*)^o := \{ f \in \mathcal{B}, \ell(f) = 0 \ \forall \ell \in S^* \} \).

**Theorem 10** ([18], Cor. 2.5 and Lemma 2.6). Under the assumption of Theorem 9, the adjoint cocycle admits a unique measurable equivariant splitting

\[ \mathcal{B}^* = V^*(\omega) \oplus \bigoplus_{i=1}^{l} Y_i^*(\omega), \quad (15) \]
with the same exponents \((\lambda_i)_{i\in\{1,\ldots,l\}}\) and multiplicities \((m_i)_{i\in\{1,\ldots,l\}}\) as \(R\). Furthermore, denoting by \(H(\omega) := V(\omega) \oplus \bigoplus_{i=2}^{l} Y_i(\omega)\) (resp. \(H^*(\omega) := V^*(\omega) \oplus \bigoplus_{i=2}^{l} Y_i^*(\omega)\)), one has
\[ Y_i^*(\omega) = H(\omega)^\circ \quad \text{and} \quad Y_1(\omega) = (H^*(\omega))^\circ. \] (16)

### 2.3 Cocycles expanding on average

We begin by recalling the setup from [13]. Let \((X, \mathcal{G})\) be a measurable space endowed with a probability measure \(m\) and a notion of a variation \(\text{Var}\) on \(L^1(X, m)\) which satisfies the following conditions:

- **(V1)** \(\text{Var}(th) = |t| \text{Var}(h)\);
- **(V2)** \(\text{Var}(g + h) \leq \text{Var}(g) + \text{Var}(h)\);
- **(V3)** \(\|h\|_{L^\infty} \leq C_{\text{Var}}(\|h\|_1 + \text{Var}(h))\) for some constant \(1 \leq C_{\text{Var}} < \infty\);
- **(V4)** for any \(C > 0\), the set \(\{h: X \to \mathbb{R} : \|h\|_1 + \text{Var}(h) \leq C\}\) is \(L^1(m)\)-compact;
- **(V5)** \(\text{Var}(1_X) < \infty\), where \(1_X\) denotes the function equal to 1 on \(X\);
- **(V6)** \(\{h: X \to \mathbb{R}_+ : \|h\|_1 = 1 \text{ and } \text{Var}(h) < \infty\}\) is \(L^1(m)\)-dense in \(\{h: X \to \mathbb{R}_+ : \|h\|_1 = 1\}\).
- **(V7)** for any \(f \in L^1(X, m)\) such that \(\text{essinf } f > 0\), we have
  \[ \text{Var}(1/f) \leq \frac{\text{Var}(f)}{(\text{essinf } f)^2}. \]

- **(V8)** \(\text{Var}(fg) \leq \|f\|_{L^\infty} \cdot \text{Var}(g) + \|g\|_{L^\infty} \cdot \text{Var}(f)\).
- **(V9)** for \(M > 0\), \(f: X \to [-M, M]\) measurable and every \(C^1\) function \(h: [-M, M] \to \mathbb{C}\), we have \(\text{Var}(h \circ f) \leq \|h'\|_{L^\infty} \cdot \text{Var}(f)\).

We define
\[ BV = BV(X, m) = \{g \in L^1(X, m) : \text{Var}(g) < \infty\}. \]

Then, \(BV\) is a Banach space with respect to the norm
\[ \|g\|_{BV} = \|g\|_1 + \text{Var}(g). \]
Remark 11. In the rest of the paper we will use (V1)-(V9) without explicitly referring to those properties. In particular, we will often use that

\[ \|fg\|_{BV} \leq C_{\text{var}} \|f\|_{BV} \cdot \|g\|_{BV} \quad \text{for } f, g \in BV, \]

which follows readily from (V3) and (V8).

Example 12. It is well-known (see [18, Section 2.2.]) that in the case when \( X = [0,1] \) and \( m \) is the Lebesgue measure, \( \text{Var} : L^1(X,m) \to [0,\infty] \) given by

\[ \text{Var}(g) = \inf_{h = g(\text{mod} \ m)} \sup_{0 = s_0 < s_1 < \ldots < s_n = 1} \sum_{k=1}^{n} |h(s_k) - h(s_{k-1})| \quad g \in L^1(X,m), \tag{17} \]

satisfies properties (V1)-(V9).

In higher dimensions, when \( X \subset \mathbb{R}^n, n > 1 \) and \( m \) is again the Lebesgue measure, \( \text{Var} : L^1(X,m) \to [0,\infty] \) defined by

\[ \text{Var}(f) = \sup_{0 < \varepsilon \leq \varepsilon_0} \frac{1}{\varepsilon^n} \int_{\mathbb{R}^d} \text{osc}(f, B_\varepsilon(x)) \, dm(x), \tag{18} \]

where

\[ \text{osc}(f, B_\varepsilon(x)) = \text{esssup}_{x_1, x_2 \in B_\varepsilon(x)} |f(x_1) - f(x_2)|, \]

also fulfills conditions (V1)-(V9). We refer to [18, Section 2.2.] and [48] for details.

Let \((\Omega, \mathcal{F}, \mathbb{P}, \sigma)\) be a probability space and \( \sigma : \Omega \to \Omega \) an invertible ergodic measure-preserving transformation. Let \( T_\omega : X \to X, \omega \in \Omega \) be a collection of non-singular transformations (i.e. \( m \circ T^{-1}_\omega \ll m \) for each \( \omega \)) acting on \( X \). Each transformation \( T_\omega \) induces the corresponding transfer operator \( L_\omega \) acting on \( L^1(X,m) \) and defined by the following duality relation

\[ \int_{X} (L_\omega \phi) \psi \, dm = \int_{X} \phi(\psi \circ T_\omega) \, dm, \quad \phi \in L^1(X,m), \psi \in L^\infty(X,m). \]

Thus, we obtain a cocycle of transfer operators \((\Omega, \mathcal{F}, \mathbb{P}, \sigma, L^1(X,m), L)\) that we denote by \( L = (L_\omega)_{\omega \in \Omega} \).

Definition 13. A cocycle \( L = (L_\omega)_{\omega \in \Omega} \) of transfer operators is said to be **good** if the following conditions hold:

- \( \Omega \) is a Borel subset of a separable, complete metric space and \( \sigma \) is a homeomorphism. Moreover, \( L \) is \( \mathbb{P} \)-continuous;
there exists a random variable $C: \Omega \to (0, +\infty)$ such that $\log C \in L^1(\Omega, P)$ and
\[
\|\mathcal{L}_\omega h\|_{BV} \leq C(\omega)\|h\|_{BV}, \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega \text{ and } h \in BV; \quad (19)
\]

there exist $N \in \mathbb{N}$ and random variables $\alpha^N, K^N: \Omega \to (0, +\infty)$ such that
\[
\int_\Omega \log \alpha^N \, d\mathbb{P} < 0, \quad \log K^N \in L^1(\Omega, \mathbb{P}) \quad (20)
\]
and, for $\mathbb{P}$-a.e. $\omega \in \Omega$ and $h \in BV$,
\[
\text{Var}(\mathcal{L}_\omega^N h) \leq \alpha^N(\omega) \text{Var}(h) + K^N(\omega)\|h\|_1; \quad (21)
\]
for each $a > 0$ and $\mathbb{P}$-a.e. $\omega \in \Omega$, there exist random numbers $n_c(\omega) < +\infty$ and $\alpha_0(\omega), \alpha_1(\omega), \ldots$ such that for every $h \in C_a$,
\[
\text{essinf}_x (\mathcal{L}_\omega^n h)(x) \geq \alpha_n\|h\|_1 \quad \text{for } n \geq n_c,
\]
where
\[
C_a := \{ h \in L^\infty(X, m) : h \geq 0 \text{ and } \text{Var}(h) \leq a\|h\|_1 \}.
\]

**Remark 14.** Let us make a few comments on those assumptions:

- We note that Definition 13 almost coincides with [13, Definition 1.1], the only difference being that the first requirement in Definition 13 is absent in [13, Definition 1.1]. We stress that the first requirement in Definition 13 ensures that we can apply Theorem 9 to any good cocycle of transfer operators.

- We have that $\mathcal{L}$ is $\mathbb{P}$-continuous whenever the map $\omega \mapsto T_\omega$ has a countable range $\{T_1, T_2, \ldots\}$, and for each $j$, $\{\omega \in \Omega : T_\omega = T_j\} \in \mathcal{F}$ (see [26, Section 4.1]).

- Observe that (19) implies that $\mathcal{L}_\omega$ is a bounded operator on $BV$ for $\mathbb{P}$-a.e. $\omega \in \Omega$. Thus, from now on we will view $\mathcal{L}$ as a cocycle acting on $BV$ (and not $L^1(X, m)$).

Let us now describe examples of systems having good cocycles of transfer operators:

**Example 15 (Lasota-Yorke cocycles).** Consider $X = [0, 1]$, endowed with Lebesgue measure and the classical notion of variation $\text{Var}$ given by (17). We say that $T : X \to X$ is a piecewise monotonic non-singular map (p.m.n.s. map for short) if the following conditions hold:
• $T$ is piecewise monotonic, i.e. there exists a subdivision $0 = a_0 < a_1 < \cdots < a_N = 1$ such that for each $i \in \{0, \ldots, N - 1\}$, the restriction $T_i = T_{(a_i, a_{i+1})}$ is monotonic (in particular it is a homeomorphism on its image).

• $T$ is non-singular, i.e. there exists $|T'| : [0, 1] \to \mathbb{R}_+$ such that for any measurable $E \subset (a_i, a_{i+1})$, $m(T(E)) = \int_E |T'| dm$.

The intervals $(a_i, a_{i+1})_{i \in \{0, \ldots, N-1\}}$ are called the intervals of $T$. We also set $N(T) := N$ and $\lambda(T) := \text{essinf}_{[0,1]} |T'|$.

We consider a family $(T_\omega)_{\omega \in \Omega}$ of random p.m.n.s as above, and such that $T : \Omega \times [0, 1] \to [0, 1]$, $(\omega, x) \mapsto T_\omega(x)$ is measurable. Denoting $N_\omega = N(T_\omega)$ and $\lambda_\omega = \lambda(T_\omega)$, we assume that

• The map $\omega \mapsto \left( \text{Var} \left( \frac{1}{|T'|} \right), N_\omega, \lambda_\omega, a_1, \ldots, a_{N-1} \right)$ is measurable.

• We have the following expanding-on-average property:

$$\int_{\Omega} \log \lambda_\omega \, d\mathbb{P}(\omega) > 0. \quad (22)$$

• The map $\log^+ \left( \frac{N_\omega}{\lambda_\omega} \right)$ is integrable.

• The map $\log^+ \left( \text{Var} \left( \frac{1}{|T'|} \right) \right)$ is integrable.

• $T_\omega$ is covering, i.e. for any interval $I \subset [0, 1]$, there exists a random number $n_c(\omega) > 0$ such that for any $n \geq n_c$, one has

$$\text{essinf}_{[0,1]} \mathcal{L}_\omega^{(n)}(I_I) > 0.$$ 

We will call a cocycle satisfying the previous assumptions an expanding on average Lasota-Yorke cocycle. For a countably-valued measurable family $(T_\omega)_{\omega \in \Omega}$ of expanding on average Lasota-Yorke cocycle, Remark 14 and [13, Section 1.2] tells us that the associated family of transfer operators is good in the sense of Definition 13.

**Example 16 (Smooth expanding on average cocycles).** We describe here an example, inspired by Kifer [40], of cocycles to which the previous definition apply. Consider $X = S^1$ be the unit circle, endowed with the Lebesgue measure $m$ and the notion of variation given by $\text{Var}(\phi) := \int_X |\phi'| \, dm = \|\phi\|_{L^1}$. We consider a family of maps $(T_\omega)_{\omega \in \Omega}$, such that $\Omega$ is a Borel subset of a separable, complete metric space, endowed with a homeomorphism $\sigma$,
and $T : \Omega \times X \to X$ is measurable and $T_\omega := T(\omega, \cdot)$ is $C^r$, $r \geq 2$. Using the same notations as the previous example (note that here, $N_\omega = 1$ a.s), we make the following assumptions on this family:

1. The map $\omega \in \Omega \mapsto \left( \int_X \frac{|T''(\omega)|}{(T')^2} dm, \lambda_\omega \right)$ is measurable, where $\lambda_\omega = \operatorname{essinf}_{[0,1]} |T''|$. 

2. The expanding on average property \((22)\) holds.

3. The map $\log \left( \int_X \frac{|T''(\omega)|}{(T')^2} dm \right)$ is $P$-integrable.

We call a family $(T_\omega)_{\omega \in \Omega}$ satisfying the previous assumption a smooth expanding on average cocycle.

Let us show that the transfer operator cocycle $(L_\omega)_{\omega \in \Omega}$ associated to a countably valued, smooth expanding on average cocycle is a good family of transfer operators in the sense of Definition 13. We first observe that the $P$-continuity property of $L$ follows, as in the previous example, from Remark 14. Second, we show that for the norm\(^3\) $\| \cdot \|_{W^{1,1}}$ given by $\| \phi \|_{W^{1,1}} := \| \phi \|_{L^1} + \| \phi' \|_{L^1}$, the Lasota-Yorke inequalities \((19)\) and \((21)\) hold, for $N = 1$. We start by recalling deterministic Lasota-Yorke inequalities: for a $C^2$ map $T : X \to X$, such that $D(T) := \sup \frac{|T''|}{(T')^2} < \infty$ (note that $D(T)$ is measurable in $T$), the well-known formula

$$
(L_T \phi)' = L_T \left( \frac{\phi'}{T'} \right) + L_T \left( \frac{T''}{(T')^2} \phi \right)
$$

\((23)\)

easily implies that (see also [27, Lemma 29])

$$
\| (L_T \phi)' \|_{L^1} \leq \frac{1}{\lambda(T)} \| \phi' \|_{L^1} + D(T) \| \phi \|_{L^1},
$$

\((24)\)

where $\lambda(T) = \operatorname{essinf}_{[0,1]} |T'|$. In the spirit of Buzzi [13, Lemma 1.3], one has:

**Lemma 17. For a $C^2$ map $T : X \subseteq$**

$$
\int_X |(L_T \phi)'| \, dm \leq C_0(T) \| \phi \|_{W^{1,1}}.
$$

\((25)\)

with

$$
C_0(T) := 2 \max \left( \frac{1}{\lambda(T)}, \int_X \frac{|T''|}{(T')^2} \, dm \right)
$$

\(^3\)This is of course the classical $(1,1)$ Sobolev norm.
Proof. Starting from (23), one gets
\[
\int_X |(\mathcal{L}_T \phi)'| \, dm \\
\leq \frac{1}{\lambda(T)} \int_X |\phi'| \, dm + \left( \int_X \frac{|T''|}{(T')^2} \, dm \right) \max |\phi| \\
\leq \left( \frac{1}{\lambda(T)} + \int_X \frac{|T''|}{(T')^2} \, dm \right) \|\phi\|_{L^1} + \int_X \frac{|T''|}{(T')^2} \, dm \|\phi\|_{L^1},
\]
which immediately gives the result. \(\square\)

Now, (19) follows easily from the previous lemma and the definition of a smooth expanding on average cocycle. To obtain (21), one needs a little more work and once again, we follow closely [13, p.33]. We denote \(\lambda_* = \int_\Omega \log \lambda_0 \, dP\) and \(C_0(\omega) = C_0(T_\omega)\). By integrability of \(\log C_0(\omega)\) and \(\log \lambda_\omega\), we may choose \(\varepsilon_0 > 0\) sufficiently small so that, for any measurable \(E \subset \Omega\) with \(P(E) < \varepsilon_0\),

\[
\int_E \log C_0(\omega) \, dP \leq \frac{\lambda_*}{4} \quad \text{and} \quad \int_E \log \lambda_\omega \, dP \leq \frac{\lambda_*}{4}.
\]

We now let \(\Delta\) be a constant so large that the set \(\Omega_b := \{\omega \in \Omega, \, D(\omega) > \Delta\}\) has measure smaller than \(\varepsilon_0\), where \(D(\omega) = D(T_\omega)\). We then set:
\[
\alpha(\omega) := \begin{cases} 
1/\lambda_\omega & \text{if } \omega \notin \Omega_b \\
C_0(\omega) & \text{otherwise}
\end{cases}
\]
and \(K(\omega) := \max(C_0(\omega), \alpha(\omega), \Delta, 6)\). One then has:

- if \(\omega \notin \Omega_b\), then \(\alpha(\omega) = \frac{1}{\lambda_\omega}\) and \(K(\omega) \geq \Delta \geq D(\omega)\). Hence, by (24), (21) holds in this case.

- If \(\omega \in \Omega_b\), then \(\alpha(\omega) = C_0(\omega)\) and \(K(\omega) \geq C_0(\omega)\), hence by Lemma 17, (21) also holds in this case.

Finally, we observe that by our assumptions, \(\alpha\) is log-integrable (from which \(K\) is also log-integrable), with
\[
\int_\Omega \log \alpha(\omega) \, dP(\omega) = -\int_\Omega \log \lambda_\omega \, dP(\omega) + \int_{\Omega_b} (\log \lambda_\omega + \log C_0(\omega)) \, dP(\omega) \\
\leq -\lambda_* + \frac{\lambda_*}{4} + \frac{\lambda_*}{4} \\
< 0.
\]
This completes the proof of the Lasota-Yorke inequalities for smooth expanding on average cocycles. As for the last assertion in Definition 13, we note that is shown in [24, Example 6] that for each non-trivial interval $I \subset X$, for $\mathbb{P}$-a.e $\omega \in \Omega$, there is a $n_c := n_c(\omega, I) < \infty$ such that for all $n \geq n_c$,

$$T^n_\omega(I) = X.$$ 

Now it remains to argue as in [13, Claim p.32], taking into account [13, Remark 0.1, 2.].

**Example 18 (Multidimensional piecewise affine maps.)** Our abstract setup also covers multidimensional examples. The one we describe now is due to Buzzi [13, Appendix B].

Recall that a polytope in $\mathbb{R}^d$ is defined as the intersection of half-spaces. If $X \subset \mathbb{R}^d$, let $P$ be a finite collection of pairwise disjoint, open polytopes $A$ of $\mathbb{R}^d$, such that $Y = \bigcup_{A \in P} A$ is dense in $X$. We now let $f : Y \to X$ be such that for any $A \in P$, $f : A \to f(A) \subset X$ is the restriction of an affine map $f_A$ of $\mathbb{R}^d$: we say that $(X, P, f)$ is a piecewise affine map. We will also assume that each $f_A$ is invertible.

We define the expansion rate of $f$,

$$\lambda(f) := \inf_{x \in Y} \inf_{\|v\| = 1} \|Df(x) \cdot v\|.$$ 

We also recall that, given a polytope $A \subset \mathbb{R}^d$, we can define the $\varepsilon$-multiplicity of its boundary $\partial A$ at $x \in X$, $\text{mult}(\partial A, \varepsilon, x)$, as the number of hyperplanes in $\partial A$ having non-empty intersection with $B(x, \varepsilon)$ the ball of radius $\varepsilon$ centered at $x$. We then set

$$\text{mult}(\partial P, \varepsilon) := \sup_{x \in X} \sum_{x \in A \in P} \text{mult}(\partial A, \varepsilon, x)$$

$$\text{mult}(\partial P) := \lim_{\varepsilon \to 0} \text{mult}(\partial A, \varepsilon).$$

Finally we notice that there are some $\varepsilon > 0$ for which $\text{mult}(\partial P, \varepsilon) = \text{mult}(\partial P)$. We denote by $\varepsilon(\partial P)$ the supremum of such $\varepsilon$.

Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, endowed as usual with an invertible, measure-preserving and ergodic self map $\sigma$, we consider countably-valued, measurable families of polytopes $(A_\omega)_{\omega \in \Omega}$ of $X \subset \mathbb{R}^d$ and affine maps $(f_{A_\omega})_{\omega \in \Omega}$: this data defines a cocycle of random piecewise affine map $(X, P_\omega, f_\omega)$, for which we assume:

1. For any $n \in \mathbb{N}$, the map $\omega \mapsto (\lambda(f^n_\omega), |P^n_\omega|, \text{mult}(\partial P^n_\omega), \varepsilon(\partial P^n_\omega))$ is measurable.
2. The map $\frac{|P|}{\lambda}$ is log$^+\ P$-integrable.

3. The following expansion on average condition holds:

$$\Lambda := \lim_{n \to \infty} \lim_{K \to \infty} \int_{\Omega} \frac{1}{n} \log \min \left( \frac{\lambda^n}{\mu \text{mult}(P^n)}, K \right) d\mathbb{P} > 0.$$ 

4. The following random covering condition holds:

For any ball $B \subset X$, $\mathbb{P}$-a.e $\omega \in \Omega$ there is a $n_c := n_c(\omega, B)$ such that $f^n_\omega(B) = X$ (modulo a null set for Lebesgue measure) for $n \geq n_c$.

Under the previous assumptions, and for the notion of variation on $X$ given by (18), it is established in [13, Prop B.1] that a random piecewise affine map has a good random transfer operator, in the sense of [13, Def.1.1]. Together with the assumption that this map is countably valued, this shows that the associated transfer operator cocycle is good in the sense of Definition 13.

Remark 19 (Multidimensional piecewise expanding maps, beyond the affine case). The recent paper [8], which studies statistical properties of so-called ‘random Saussol maps’, a class of piecewise expanding on average multidimensional systems, opens up the possibility to apply our approach to this much larger family of multidimensional examples, by establishing (19) and (21) for the transfer operator cocycle acting on spaces on bounded oscillation. Despite this progress, the question of mixing remains open in this setting, so that we cannot directly apply our approach to this type of system.

For any cocycle having a good family of transfer operators in the sense of Definition 13 (in particular, for the three previous examples), the following result holds:

**Theorem 20** ([13], Main Theorem). Let $\mathcal{L} = (\mathcal{L}_\omega)_{\omega \in \Omega}$ be a good cocycle of transfer operators. Then, we have the following:

1. there exists an essentially unique measurable family $(v_0^\omega)_{\omega \in \Omega} \subset BV$ such that $v_\omega \geq 0$, $\int_X v_0^\omega \, dm = 1$ and

$$\mathcal{L}_\omega v_0^\omega = v_0^\sigma_\omega, \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega.$$

2. There exists a random variable $Z: \Omega \to (0, +\infty)$ and $\rho \in (0,1)$ such that for $\mathbb{P}$-a.e $\omega \in \Omega$, any $\phi \in BV$ and any $n \geq 0$,

$$\left\| \mathcal{L}_\omega^n \phi - \left( \int_X \phi \, dm \right) v_0^\sigma_\omega \right\|_{L^\infty} \leq Z(\omega) \rho^n \|\phi\|_{L^1}.$$

(26)
3. Furthermore,
\[ \left| \int_X L^n_{\omega}(\varphi v^0_{\omega}) \psi \, dm \right| \leq Z(\omega) \rho^n \| \psi \|_{\infty} \cdot \| \varphi \|_{BV}, \] (27)

for P-a.e. \( \omega \in \Omega \), \( \psi \) bounded and \( \varphi \in BV \) such that \( \int_X \varphi \, d\mu_\omega = 0 \), where \( d\mu_\omega = v^0_{\omega} \, dm \).

Remark 21. The above stated Theorem 20, particularly its second assertion, is slightly different from the one stated in [13]: let us explain how to get from the former to the latter:

- [13, Main Theorem, 2.] is stated at \( \sigma^{-n}\omega \): to get it at \( \omega \) instead, we simply remark that we may take \( p = 0 \) in [13, Prop. 4.1] (instead of \( p = -n \)), which is possible since the proposition is valid for any \( |p| \leq n \).

- [13, Main Theorem, 2.] is stated for non-negative \( \phi \) such that \( \| \phi \|_1 = 1 \). Hence, to get our version, starting from \( \phi \geq 0 \), simply replace \( \phi \) by \( \frac{\phi}{\| \phi \|_1} \), and multiply both sides by \( \| \phi \|_1 \). For arbitrary real-valued \( \phi \in BV \), decompose it as the sum of its positive and negative parts:
\[ \phi = \phi^+ - \phi^- \]
and using the inequality \( \| u-v \|_{\infty} \leq \| u+v \|_{\infty} \) for non-negative, bounded functions, we get:
\[ \left\| L^n_{\omega}\phi - \left( \int_X \phi \, dm \right) v^0_{\omega} \right\|_{\infty} \leq \left\| L^n_{\omega}|\phi| - \left( \int_X |\phi| \, dm \right) v^0_{\omega} \right\|_{\infty} \]
which immediately gives the announced result.

From Theorem 20, one easily derives the following:

Lemma 22. Let \( L = (L_\omega)_{\omega \in \Omega} \) be a good cocycle of transfer operators. Then, \( \Lambda(L) \geq 0 \) and \( \kappa(L) < 0 \). In particular, \( L \) is quasi-compact.

Proof. Using the same notation as in the statement of Theorem 20, we have that
\[ \limsup_{n \to \infty} \frac{1}{n} \log \| L^n_{\omega}\|_{BV} \geq \limsup_{n \to \infty} \frac{1}{n} \log \| L^n_{\omega} v^0_{\omega}\|_{BV} \]
\[ = \limsup_{n \to \infty} \frac{1}{n} \log \| v^0_{\sigma^n\omega}\|_{BV} \]
\[ \geq \limsup_{n \to \infty} \frac{1}{n} \log \| v^0_{\sigma^n\omega}\|_1 \]
\[ = 0, \]
\footnote{We note that, by decomposing it as the sum of its real and imaginary part, this also holds for general complex-valued \( \phi \in BV \).}
for \( P\text{-a.e. } \omega \in \Omega \). This implies that \( \Lambda(\mathcal{L}) \geq 0 \).

We now apply Lemma 8 with \( \| \cdot \| = \| \cdot \|_{BV} \) and \( | \cdot | = \| \cdot \|_1 \). Indeed, observe that (19) and (21) imply that (10) and (11) hold with

\[ \alpha_\omega = \alpha^N(\omega), \quad \beta_\omega = K^N(\omega) + 1 \quad \text{and} \quad \gamma_\omega = C(\omega). \]

Since \( \log C \in L^1(\Omega, \mathbb{P}) \), \( \int_\Omega \log \alpha^N \, d\mathbb{P} < 0 \) and \( \Lambda(\mathcal{L}) \geq 0 \), we have that (12) holds. By Lemma 8, we obtain the desired conclusion.

### Proposition 23.

Let \( \mathcal{L} = (\mathcal{L}_\omega)_{\omega \in \Omega} \) be a good cocycle of transfer operators. Then, \( \Lambda(\mathcal{L}) = 0 \).

**Proof.** By Lemma 22 we have that \( \Lambda(\mathcal{L}) \geq 0 \). By using that \( \| \mathcal{L}_\omega \|_1 \leq 1 \) and applying [28, Lemma C.5] (which we can due to Lemma 22) for the cocycle \( (\mathcal{L}_\omega^N)_{\omega \in \Omega} \) over \( \sigma^N \), we conclude that \( \Lambda(\mathcal{L}) \leq 0 \).

### Proposition 24.

Let \( \mathcal{L} = (\mathcal{L}_\omega)_{\omega \in \Omega} \) be a good cocycle of transfer operators. Furthermore, set

\[ BV^0 = \left\{ h \in BV : \int_X h \, dm = 0 \right\}. \]

Then, using the same notation as in the statement of Theorem 9, we have the following:

- **For \( P\text{-a.e. } \omega \in \Omega \),**
  
  \[ BV = Y_1(\omega) \oplus BV^0, \quad (28) \]
  
  where \( Y_1(\omega) \) is a one-dimensional subspace of \( BV \) spanned by \( v_0^\omega \).

- **For \( P\text{-a.e. } \omega \in \Omega \),**
  
  \[ BV^0 = V(\omega) \oplus \bigoplus_{j=2}^l Y_j(\omega). \quad (29) \]

**Proof.** We claim that

\[ \lim_{n \to \infty} \frac{1}{n} \log \| \mathcal{L}_\omega^n h \|_{BV} < 0, \quad \text{for } P\text{-a.e. } \omega \in \Omega \text{ and } h \in BV^0. \quad (30) \]

Once we establish (30), both assertions of the proposition will follow from the uniqueness of the Oseledets splitting (14) and the simple observation that for any \( h \in BV \) and \( P\text{-a.e. } \omega \in \Omega \),

\[ h = \left( \int_X h \, dm \right) v_0^\omega + \left( h - \left( \int_X h \, dm \right) v_0^\omega \right) \in \text{span}(v_0^\omega) + BV^0. \]
We first observe that the existence of the limit in (30) follows from Theorem 9. For $h \in BV^0$, one has from (26) that

$$\limsup_{n \to \infty} \frac{1}{nN} \log \|\mathcal{L}_{\omega}^{nN} h\|_{\infty} \leq \log \rho < 0.$$  

Choose $a > 0$ sufficiently small such that $a + N \log \rho < 0$ and

$$a + \int_{\Omega} \log \sigma^N d\mathbb{P} < 0.$$

By applying [28, Lemma C.5] for the cocycle $(\tilde{\mathcal{L}}_{\omega})_{\omega \in \Omega}$ over $\sigma^N$ where $\tilde{\mathcal{L}}_{\omega} = e^{a \mathcal{L}_{\omega}^N}$, we conclude that from (31) that

$$\limsup_{n \to \infty} \frac{1}{nN} \log \|\mathcal{L}_{\omega}^{nN} h\|_{BV} < 0,$$

which implies (30).

Notice that Theorem 10, Lemma 22, Propositions 23 and 24 imply the following result for the dual cocycle $\mathcal{L}^*$:

**Corollary 25.** If $\mathcal{L}$ is a cocycle of good transfer operators, then $\mathcal{L}^*$ satisfies:

1. $0 = \Lambda(\mathcal{L}^*) > \kappa(\mathcal{L}^*)$. In particular, $\mathcal{L}^*$ is quasi-compact.

2. One has the following splitting for $BV^*$:

$$BV^* = \text{span}(m) \oplus Y_1(\omega)^\circ,$$

where $\text{span}(m)$ is the one-dimensional subspace of $BV^*$ spanned by Lebesgue measure$^5$ and $Y_1(\omega)^\circ$ is the annihilator of $v_0^\omega$.

We now recall the notion of a tempered random variable.

**Definition 26.** Let $K: \Omega \to (0, +\infty)$ be a measurable map. We say that $K$ is tempered if

$$\lim_{n \to \pm \infty} \frac{1}{n} \log K(\sigma^n \omega) = 0, \text{ for } \mathbb{P}-a.e. \ \omega \in \Omega.$$

We will need the following classical result (see [3, Proposition 4.3.3]).

$^5$we identify the Lebesgue measure $m$ with the functional $\phi \to \int_X \phi dm$ on $BV$
Proposition 27. Let $K : \Omega \rightarrow (0, +\infty)$ be a tempered random variable. For each $\varepsilon > 0$, there exists a tempered random variable $K_\varepsilon : \Omega \rightarrow (1, +\infty)$ such that
\[
\frac{1}{K_\varepsilon(\omega)} \leq K(\omega) \leq K_\varepsilon(\omega) \quad \text{and} \quad K_\varepsilon(\omega)e^{-\varepsilon|n|} \leq K_\varepsilon(\sigma^n \omega) \leq K_\varepsilon(\omega)e^{\varepsilon|n|},
\]
for $\mathbb{P}$-a.e. $\omega \in \Omega$ and $n \in \mathbb{Z}$.

Proposition 28. Let $L = (L_\omega)_{\omega \in \Omega}$ be a good cocycle of transfer operators. Furthermore, let $\Pi(\omega) : BV \rightarrow BV^0$, $\omega \in \Omega$ be a family of projections corresponding to the splitting (28), and let $\lambda_2 < 0$ be the second Lyapunov exponent of $L$. For any $\varepsilon \in (0, -\lambda_2)$, there exist tempered random variables $D_i : \Omega \rightarrow [1, +\infty)$, $i \in \{1, 2\}$ such that:

- for $\mathbb{P}$-a.e. $\omega \in \Omega$, $n \geq 0$ and $\phi \in BV$,
  \[
  \|L^n_\omega \Pi (\omega) \phi\|_{BV} \leq D_1(\omega)e^{-\lambda n}\|\phi\|_{BV}, \quad (32)
  \]
  where we set $\lambda = -\lambda_2 - \varepsilon > 0$;
- for $\mathbb{P}$-a.e. $\omega \in \Omega$, $n \geq 0$ and $\phi \in BV$,
  \[
  \|L^n_\omega (I - \Pi(\omega)) \phi\|_{BV} \leq D_2(\omega)e^{\varepsilon n}\|\phi\|_{BV}; \quad (33)
  \]

Proof. This is a direct consequence of Propositions 23, 24 and [5, Proposition 3.2]. Indeed, the proof of [5, Proposition 3.2] implies that we can take
\[
D_1(\omega) = (1 + \|v_0^0\|_{BV}) \sup_{n \geq 0}(\|L^n_\omega \|_{BV^0} \|BV e^{\lambda n}\), \quad (34)
\]
and
\[
D_2(\omega) = \sup_{n \geq 0}(\|v_{\sigma^n \omega}^0\|_{BV} e^{-\varepsilon n}). \quad (35)
\]

Remark 29. Observe that:

- One can replace $D_1$ in (32) and $D_2$ in (33) by $D = \max\{D_1, D_2\}$. However, in order to construct adapted norms, it is convenient to work with $D_1$ and $D_2$ separately.
- For $\mathbb{P}$-a.e. $\omega \in \Omega$,
  \[
  D_2(\omega)e^\varepsilon \leq D_2(\sigma \omega). \quad (36)
  \]
3 Twisted transfer operator cocycles

3.1 Adapted norms

Throughout this section, we take a good cocycle of transfer operators \( \mathcal{L} = (\mathcal{L}_\omega)_{\omega \in \Omega} \). Let \( \varepsilon, \lambda > 0, D_i : \Omega \to [1, +\infty), i = 1,2 \) and \( \Pi(\omega), \omega \in \Omega \) be as in statement of Proposition 28. For \( \mathbb{P} \)-a.e. \( \omega \in \Omega \) and \( \phi \in BV \), let

\[
\|\phi\|_\omega = \sup_{n \geq 0} (e^{\lambda n} \|L_n^\omega \Pi(\omega) \phi\|_{BV}) + \frac{1}{D_1(\omega)} \sup_{n \geq 0} (e^{-\varepsilon n} \|L_n^\omega (\phi - \Pi(\omega) \phi)\|_{BV}) + \left| \int_X \phi dm \right|.
\]

Observe that \( \omega \mapsto \|\phi\|_\omega \) is measurable for each \( \phi \in BV \). The main properties of adapted norms are collected in the following result.

**Proposition 30.** The following holds:

1. There exists a tempered random variable \( K : \Omega \to [1, +\infty) \) such that, for \( \mathbb{P} \)-a.e. \( \omega \in \Omega \) and \( \phi \in BV \),

\[
\frac{1}{K(\omega)} \|\phi\|_{BV} \leq \|\phi\|_\omega \leq K(\omega) \|\phi\|_{BV}.
\]

2. For \( \mathbb{P} \)-a.e. \( \omega \in \Omega \) and \( \phi \in BV \),

\[
\|\phi\|_1 \leq \|\phi\|_\omega.
\]

3. For \( \mathbb{P} \)-a.e. \( \omega \in \Omega \) and \( \phi \in BV \),

\[
\|L_n^\omega \Pi(\omega) \phi\|_{\sigma_n^\omega} \leq e^{-\lambda n} \|\phi\|_\omega.
\]

4. There exists a constant \( C' > 0 \) such that for \( \mathbb{P} \)-a.e. \( \omega \in \Omega \) and \( \phi \in BV \):

\[
\|\mathcal{L}_\omega \phi\|_{\sigma_\omega} \leq C' \|\phi\|_\omega.
\]

5. We have that

\[
\text{esssup} \|v_\omega^0\|_\omega < +\infty.
\]

**Proof.** It follows from (32), (33) and \( \| \cdot \|_1 \leq \| \cdot \|_{BV} \) that

\[
\|\phi\|_{\omega} \leq (D_1(\omega) + 2) \|\phi\|_{BV},
\]

...
for \(P\)-a.e. \(\omega \in \Omega\) and \(\phi \in BV\). Moreover,

\[
\|\phi\|_\omega \geq \|\Pi(\omega)\phi\|_{BV} + \frac{1}{D_2(\omega)} \|\phi - \Pi(\omega)\phi\|_{BV} \geq \frac{1}{D_2(\omega)} \|\phi\|_{BV},
\]

for \(P\)-a.e. \(\omega \in \Omega\) and \(\phi \in BV\). From these two estimates we conclude that \((37)\) holds with

\[
K(\omega) = \max\{D_1(\omega) + 2, D_2(\omega)\},
\]

which is clearly tempered (since \(D_1\) and \(D_2\) are tempered).

Furthermore, noting that \(\|\phi - \Pi(\omega)\phi\|_1 = \left| \int_X \phi \, dm \right|\), for \(P\)-a.e. \(\omega \in \Omega\) and \(\phi \in BV\) we have that

\[
\|\phi\|_\omega \geq \sup_{n \geq 0} (\|L_n^\omega \Pi(\omega)\phi\|_{BV} \cdot e^{\lambda n}) + \|\phi - \Pi(\omega)\phi\|_1 \\
\geq \|\Pi(\omega)\phi\|_{BV} + \|\phi - \Pi(\omega)\phi\|_1 \\
\geq \|\Pi(\omega)\phi\|_1 + \|\phi - \Pi(\omega)\phi\|_1 \\
\geq \|\phi\|_1,
\]

which yields \((38)\).

Next, observe that

\[
\|L^\omega_n \Pi(\omega)\phi\|_{\sigma^n_\omega} = \sup_{m \geq 0} (\|L^\omega_m L^\omega_n \Pi(\omega)\phi\|_{BV} e^{\lambda m}) \\
= \sup_{m \geq 0} (\|L^\omega_{n+m} \Pi(\omega)\phi\|_{BV} e^{\lambda m}) \\
= e^{-\lambda n} \sup_{m \geq 0} (\|L^\omega_{n+m} \Pi(\omega)\phi\|_{BV} e^{\lambda (m+n)}) \\
\leq e^{-\lambda n} \|\phi\|_\omega,
\]

for \(P\)-a.e. \(\omega \in \Omega\) and \(\phi \in BV\). Consequently, \((39)\) holds.
In addition, for \( \mathbb{P}\text{-a.e. } \omega \in \Omega \) and \( \phi \in BV \) one has (using (36)) that

\[
\|L_\omega \phi\|_{\sigma_\omega} = \sup_{n \geq 0} (\|L^{n+1}_\omega \Pi(\omega) \phi\|_{BV} \cdot e^{\lambda n}) \\
+ \frac{1}{D_2(\sigma_\omega)} \cdot \sup_{n \geq 0} (\|L^{n+1}_\omega (\phi - \Pi(\omega) \phi)\|_{BV} \cdot e^{-\varepsilon n}) \\
+ \left| \int_X L_\omega \phi \, dm \right| \\
\leq e^{-\lambda} \sup_{n \geq 0} (\|L^{n+1}_\omega \Pi(\omega) \phi\|_{BV} \cdot e^{\lambda(n+1)}) \\
+ e^{-\varepsilon} \frac{1}{D_2(\omega)} \cdot \sup_{n \geq 0} (\|L^{n+1}_\omega (\phi - \Pi(\omega) \phi)\|_{BV} \cdot e^{-\varepsilon(n+1)}) \\
+ \left| \int_X \phi \, dm \right| \\
\leq \|\phi\|_{\omega},
\]

and therefore (40) holds.

Finally, (35) implies that

\[
\|v_\omega^0\|_{\omega} = \frac{1}{D_2(\omega)} \cdot \sup_{n \geq 0} (\|L^0_n \phi\|_{BV} \cdot e^{-\varepsilon n}) + \left| \int_X v_\omega^0 \, dm \right| \\
= \frac{1}{D_2(\omega)} \cdot \sup_{n \geq 0} (\|v_\omega^0\|_{BV} \cdot e^{-\varepsilon n}) + 1 \\
= 2,
\]

for \( \mathbb{P}\text{-a.e. } \omega \in \Omega \), which yields (41). The proof of the proposition is completed. \( \square \)

**Remark 31.** Our construction of adapted norms is somewhat similar to the classical construction due to Pesin (see for example [5, Section 3]). However, our construction is different since besides transforming (32) and (33) into uniform type conditions (see (39) and (40)), an important feature of our adapted norms is given by properties (38) and (41). As we will see later on, these properties will be of central importance for our arguments.

**Remark 32.** By putting (34) and (35) in (42), we obtain an explicit expression for the random variable \( K \).

Let us now introduce “dual adapted norms”: we set, for \( \ell \in BV^* \) and \( \mathbb{P}\text{-a.e. } \omega \in \Omega \),

\[
\|\ell\|_{\omega} := \inf \{ C > 0 : |\ell(\phi)| \leq C \|\phi\|_{\omega} \text{ for } \phi \in BV \}.
\]
Observe that \( \omega \mapsto \| \ell \|_{\omega}^* \) is measurable for each \( \ell \in BV^* \). The dual adapted norms \( \| \cdot \|_{\omega}^* \) satisfy properties similar to the adapted norms \( \| \cdot \|_{\omega} \). We summarize them in the following proposition.

**Proposition 33.** The following holds:

1. For \( \mathbb{P}\text{-}a.e \ \omega \in \Omega \) and \( \ell \in BV^* \),
\[
\frac{1}{K(\omega)}\| \ell \|_{BV^*} \leq \| \ell \|_{\omega}^* \leq K(\omega)\| \ell \|_{BV^*},
\]
with \( K(\omega) \) as in (37).

2. For \( \mathbb{P}\text{-}a.e \ \omega \in \Omega \) and \( \ell \in L^\infty(X) \),
\[
\| \ell \|_{\omega}^* \leq \| \ell \|_{\infty}.
\]

3. For \( \mathbb{P}\text{-}a.e \ \omega \in \Omega \) and \( \ell \in BV^* \),
\[
\| \mathcal{L}_\omega^* \ell \|_{\omega}^* \leq C'(\| \ell \|_{\sigma^\omega}),
\]
where \( C' \) as in (40).

4. For \( \mathbb{P}\text{-}a.e \ \omega \in \Omega \), \( \ell \in BV^* \) and \( n \geq 0 \),
\[
\|(\mathcal{L}_\omega^n)^*\Pi^*(\sigma^n\omega)\ell\|_{\omega}^* \leq e^{-\lambda n}\| \ell \|_{\sigma^n\omega}^*,
\]
where \( \lambda \) is given by Proposition 28 and \( \Pi^*(\omega)\ell := \ell - \ell(v^0_\omega)m \).

5. The Lebesgue measure \( m \) satisfies, for \( \mathbb{P}\text{-}a.e \ \omega \in \Omega \)
\[
\| m \|_{\omega}^* \leq 1.
\]
In particular, \( \text{esssup}_{\omega \in \Omega} \| m \|_{\omega}^* < +\infty \).

**Proof.** Using (37) we have that for \( \mathbb{P}\text{-}a.e \ \omega \in \Omega \), \( \ell \in BV^* \) and \( \phi \in BV \),
\[
|\ell(\phi)| \leq \| \ell \|_{BV^*} \cdot \| \phi \|_{BV} \leq K(\omega)\| \ell \|_{BV^*} \cdot \| \phi \|_{\omega},
\]
which implies the second inequality in (44). The first inequality can be established similarly.

Moreover, using (38) we have that
\[
|\ell(\phi)| \leq \| \ell \|_{\infty} \cdot \| \phi \|_{1} \leq \| \ell \|_{\infty} \cdot \| \phi \|_{\omega},
\]

\( ^6 \) we identify \( \ell \) with the functional \( \phi \to \int_X \phi \ell \ dm \) on \( BV \)
for $\mathbb{P}$-a.e $\omega \in \Omega$, $\ell \in L^\infty(X)$ and $\phi \in BV$. Hence, (45) holds.

Furthermore, it follows from (40) that

$$|L^*_\omega \ell(\phi)| = |\ell(L_\omega \phi)| \leq \|\ell\|^*_\omega \cdot \|L_\omega \phi\|_{\sigma\omega} \leq C'\|\ell\|^*_\omega \cdot \|\phi\|_{\omega},$$

for $\mathbb{P}$-a.e $\omega \in \Omega$, $\ell \in BV^*$ and $\phi \in BV$. We conclude that (46) holds.

On the other hand, by (39) we have that

$$|(L^n_\omega)^* \Pi^*(\sigma^n \omega) \ell(\phi)| = |\ell(L^n_\omega \Pi(\omega) \phi)| \leq \|\ell\|^*_\omega \cdot \|L^n_\omega \Pi(\omega) \phi\|_{\sigma^n \omega} \leq e^{-\lambda n}\|\ell\|^*_\omega \cdot \|\phi\|_{\omega},$$

for a.e $\omega \in \Omega$, $\ell \in BV^*$ and $\phi \in BV$. Thus, we proved (47).

Finally, (38) implies that

$$|m(\phi)| = \left| \int_X \phi \, dm \right| \leq \|\phi\|_1 \leq \|\phi\|_{\omega},$$

for $\mathbb{P}$-a.e. $\omega \in \Omega$ and $\phi \in BV$ which implies (48). \qed

3.2 Twisted transfer operators

We start by fixing an observable. Let $\psi: \Omega \times X \to \mathbb{R}$ be a measurable map such that the following conditions hold:

- $\psi_\omega := \psi(\omega, \cdot) \in BV$ for $\omega \in \Omega$;
- we have that

$$\text{esssup}_{\omega \in \Omega} \left( K(\omega)^2 \|\psi_\omega\|_{BV} \right) < +\infty, \quad (49)$$

where $K$ is given by Proposition 30;
- for $\mathbb{P}$-a.e. $\omega \in \Omega$,

$$\int_X \psi_\omega v_0^\omega \, dm = 0. \quad (50)$$

Remark 34. Let us make a few comments on those assumptions:

- Since $K(\omega) \geq 1$, it follows from (49) that

$$\text{esssup}_{\omega \in \Omega} \|\psi_\omega\|_{BV} < +\infty. \quad (51)$$
• We observe that (49) is closed under centering. More precisely, if \( \psi \) satisfies (49) than so does \( \psi' \) given by

\[
\psi'_\omega = \psi_\omega - \int_X \psi \, d\mu_\omega \quad \omega \in \Omega.
\]

• In the setting of [18], (32) and (33) hold with \( D \) a constant (see [18, Definition 2.8] and [18, Lemma 2.11]). Moreover, \( \operatorname{esssup}_{\omega \in \Omega} \| v^0_\omega \|_{BV} < +\infty \). Consequently, it follows from the proof of Proposition 30 that (37) holds with \( K \) being a constant. Therefore, in this setting (49) and (51) are equivalent.

Next, we describe examples of observables satisfying (49).

**Example 35.**

• Let \( \psi : \Omega \times X \to \mathbb{R} \) be such that (50) and (51) hold. Then the 'rescaled' observable \( \psi_K := K^{-2} \psi \) satisfies (49) and (50).

• Due to the somewhat complicated expression for \( K \) (see Remark 32), it is of interest to describe alternative ways of constructing observables satisfying (49). In order to do so, take an arbitrary \( \delta > 0 \). By Proposition 27, there exists a random variable \( K_\delta : \Omega \to [1, +\infty) \) such that

\[
K(\omega) \leq K_\delta(\omega) \quad \text{and} \quad K_\delta(\sigma^{-n}\omega) \leq e^{\delta n} K_\delta(\omega),
\]

for \( \mathbb{P} \)-a.e. \( \omega \in \Omega \) and \( n \in \mathbb{N} \). Take an arbitrary \( C > 0 \). Furthermore, choose \( T > 0 \) such that \( \mathbb{P}(\Omega_T^0) > 0 \), where

\[
\Omega_T^\prime := \{ \omega \in \Omega : K_\delta(\omega) \leq T \}.
\]

We note that this holds for all \( T \) sufficiently large. Set \( \Omega_T^0 = \Omega_T^\prime \) and for \( n \in \mathbb{N} \), let \( \Omega_T^n = \sigma^{-n}(\Omega_T^\prime) \setminus \bigcup_{k=0}^{n-1} \sigma^{-k}(\Omega_T^\prime) \). Clearly, \( \Omega_T^n \cap \Omega_T^m = \emptyset \) for \( n \neq m \). Moreover, the ergodicity of \( \sigma \) implies that \( \mathbb{P}(\bigcup_{n=0}^{\infty} \Omega_T^n) = 1 \). If we choose \( \psi \) so that

\[
\| \psi_\omega \|_{BV} \leq \frac{C}{T^2} e^{-2\delta n} \quad \text{for} \quad \omega \in \Omega_T^n \quad \text{and} \quad n \geq 0,
\]

we have that

\[
\operatorname{esssup}_{\omega \in \Omega} \left( K(\omega)^2 \| \psi_\omega \|_{BV} \right) \leq C.
\]

In particular, (49) holds. By centering our observable \( \psi \), we can ensure that (50) also holds (see Remark 34).

For \( \theta \in \mathbb{C} \) and \( \omega \in \Omega \), set

\[
\mathcal{L}_\omega^\theta \phi = \mathcal{L}_\omega(e^{\theta \psi_\omega} \phi), \quad \phi \in BV.
\]
Lemma 36. There exists a constant $C'' > 0$ such that for $|\theta| \leq 1$, $\mathbb{P}$-a.e. $\omega \in \Omega$ and $\phi \in BV$, 

$$\|L^\theta_\omega \phi\|_{\sigma_\omega} \leq C'' \|\phi\|_{\omega}.$$ 

Proof. By (37) and (40), we have that 

$$\|L^\theta_\omega \phi - L_\omega \phi\|_{\sigma_\omega} = \|L_\omega ((e^{\theta \psi_\omega} - 1)\phi)\|_{\sigma_\omega}$$

$$\leq C'' \|(e^{\theta \psi_\omega} - 1)\phi\|_{BV}$$

$$\leq C'K(\omega)\|(e^{\theta \psi_\omega} - 1)\phi\|_{BV}$$

$$\leq C'\text{Var}(\psi_\omega)\|e^{\theta \psi_\omega} - 1\|_{BV} \cdot \|\phi\|_{BV}.$$

On the other hand, the arguments in the proof of [18, Lemma 3.13] give that 

$$\|(e^{\theta \psi_\omega} - 1)\|_{BV} \leq e\|\psi_\omega\|_{\infty}(\|\psi_\omega\|_{\infty} + \text{Var}(\psi_\omega))$$

$$\leq (C_{\text{var}} + 1)e\|\psi_\omega\|_{\infty} \cdot \|\psi_\omega\|_{BV},$$

and therefore 

$$\|L^\theta_\omega \phi - L_\omega \phi\|_{\sigma_\omega} \leq C'C_{\text{var}}(C_{\text{var}} + 1)K(\omega)\|\psi_\omega\|_{BV} \cdot \|\phi\|_{\omega}.$$ 

From (49) (and (51)), we conclude that there exists a constant $N > 0$ such that 

$$\|L^\theta_\omega \phi - L_\omega \phi\|_{\sigma_\omega} \leq N \|\phi\|_{\omega},$$

for $|\theta| \leq 1$, $\mathbb{P}$-a.e. $\omega \in \Omega$ and $\phi \in BV$. The conclusion of the lemma now follows readily from (40) and the triangle inequality. \square 

Remark 37. From Lemma 36, we conclude immediately that the dual twisted transfer operator acts boundedly, i.e. that 

$$\|(L^\theta_\omega)^* \ell\|_{\omega} \leq C'' \|\ell\|_{\sigma_\omega},$$

for $\mathbb{P}$-a.e $\omega \in \Omega$ and every $|\theta| \leq 1$, $\ell \in BV^*$, and with $C''$ as in Lemma 36.

3.3 An auxiliary regularity result 

Let $S$ consists of all measurable maps $V: \Omega \times X \to \mathbb{C}$ such that $V(\omega, \cdot) \in BV$ for $\omega \in \Omega$ and $\|V\|_S := \text{esssup}_{\omega \in \Omega} \|V(\omega, \cdot)\|_{\omega} < +\infty$. For $V \in S$, we will often write $V_\omega$ instead of $V(\omega, \cdot)$.

The proof of the following result is a straightforward consequence of (37) (and completeness of $BV$).
Lemma 38. We have that $(\mathcal{S}, \| \cdot \|_S)$ is a Banach space.

Remark 39. It follows from (41) that the map $(\omega, x) \mapsto v_0^0(x)$ belongs to $\mathcal{S}$. From now on, we will denote this map by $v_0$.

Let $\mathcal{S}_0$ denote the closed subspace of $\mathcal{S}$ which consists of all $\mathcal{V} \in \mathcal{S}$ such that

$$\int_X \mathcal{V}(\omega, \cdot) \, dm = 0, \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega.$$  

Let $B_{\mathbb{C}}(0,1)$ denote the unit ball in $\mathbb{C}$. We define $G: B_{\mathbb{C}}(0,1) \times \mathcal{S}_0 \to \mathcal{S}$ by

$$G(\theta, \mathcal{W})(\omega, \cdot) = L_{\sigma^{-1}\omega}^\theta (\mathcal{W}(\sigma^{-1}\omega, \cdot) + v_0^0(\sigma^{-1}\omega)).$$

We claim that $G$ is well-defined. Indeed, Lemma 36 implies that

$$\|G(\theta, \mathcal{W})(\omega, \cdot)\|_\omega \leq C''(\|\mathcal{W}(\sigma^{-1}\omega, \cdot)\|_{\sigma^{-1}\omega} + \|v_0^0\|_{\sigma^{-1}\omega}),$$

for $\mathbb{P}$-a.e. $\omega \in \Omega$. Therefore, using (41) we conclude that

$$\text{esssup}_{\omega \in \Omega} \|G(\theta, \mathcal{W})(\omega, \cdot)\|_\omega < +\infty.$$ 

Hence, $G$ is well-defined. We also define $H: B_{\mathbb{C}}(0,1) \times \mathcal{S}_0 \to L^\infty(\Omega)$ by

$$H(\theta, \mathcal{W})(\omega) = \int_X L_{\sigma^{-1}\omega}^\theta (\mathcal{W}(\sigma^{-1}\omega, \cdot) + v_0^0(\sigma^{-1}\omega)) \, dm.$$ 

We note that $H$ is well-defined since $G$ is well-defined and due to (38). Some finer properties of $G$ and $H$ are given in the following result.

Lemma 40. There exists a neighborhood $\mathcal{U} \subset B_{\mathbb{C}}(0,1) \times \mathcal{S}_0$ of $(0,0)$ such that $G$ and $H$ are analytic on $\mathcal{U}$ and that $\text{essinf}_{\mathcal{U}} \|H\|_{L^\infty} \geq \frac{1}{2}$.

Proof. For any $\theta \in \mathbb{C}$, $|\lambda| < 1$, $\mathbb{P}$-a.e. $\omega \in \Omega$ and $\mathcal{W} \in \mathcal{S}$, we have that

$$G(\theta, \mathcal{W})(\omega) = L_{\sigma^{-1}\omega}^\theta (\mathcal{W}(\sigma^{-1}\omega) + v_0^0(\sigma^{-1}\omega))$$

$$= L_{\sigma^{-1}\omega}(e^{\theta\psi_{\sigma^{-1}\omega}} (\mathcal{W}(\sigma^{-1}\omega) + v_0^0))$$

$$= \sum_{k=0}^\infty \frac{\theta^k}{k!} L_{\sigma^{-1}\omega} (\psi_{\sigma^{-1}\omega}^k (\mathcal{W}(\sigma^{-1}\omega) + v_0^0)),$$

where $\psi_{\sigma^{-1}\omega}^k$ denotes the $k$-th power of $\psi_{\sigma^{-1}\omega}$. We now remark that the maps $\phi_k: B_{\mathbb{C}}(0,1) \times \mathcal{S}_0 \to \mathcal{S}$ given by

$$\phi_k(\theta, \mathcal{W}) := \frac{\theta^k}{k!} L_{\sigma^{-1}\omega} (\psi_{\sigma^{-1}\omega}^k (\mathcal{W}(\sigma^{-1}\omega) + v_0^0))$$

29
are analytic. Moreover, (37) and (40) imply that

\[
\|L_{\sigma^{-1}\omega}(\psi_{\sigma^{-1}\omega}^k(W^{\sigma^{-1}\omega}) + v_0^{\sigma^{-1}\omega})\|_\omega \\
\leq C\|\psi_{\sigma^{-1}\omega}(W^{\sigma^{-1}\omega}) + v_0^{\sigma^{-1}\omega}\|_{\sigma^{-1}\omega} \\
\leq C'K(\sigma^{-1}\omega)C_{var}\|\psi_{\sigma^{-1}\omega}^k\|_{BV} \cdot \|W^{\sigma^{-1}\omega}) + v_0^{\sigma^{-1}\omega}\|_{BV} \\
\leq C'K(\sigma^{-1}\omega)^2C_{var}\|\psi_{\sigma^{-1}\omega}^k\|_{BV} \cdot (||W||_s + \|v_0\|_s) \\
\leq C'[\text{esssup}_{\omega\in\Omega}(C_{var}K(\sigma^{-1}\omega)^2\|\psi_{\sigma^{-1}\omega}^k\|_{BV})]^k \left(||W||_s + \|v_0\|_s\right),
\]

where we used that \(K(\omega)^{2/k} \leq K(\omega)^2\) (which holds since \(K(\omega) \geq 1\)). We also note that (49) implies that \(\text{esssup}_{\omega\in\Omega}(C_{var}K(\sigma^{-1}\omega)^2\|\psi_{\sigma^{-1}\omega}^k\|_{BV}) < +\infty\). In particular, we obtain that, for \(\rho \in (0, 1)\) sufficiently small, \(k \mapsto \|\rho^k \psi_k(\theta, W)\|_s\) is summable, and thus \(G\) is a uniform limit of analytic maps on some neighborhood \(U \subset B_C(0, 1) \times S_0\) of \((0, 0)\).

A similar argument can be applied to \(H\). In particular, \(H\) is continuous and thus by shrinking \(U\) if necessary, we can assume that for \((\theta, W) \in U\) we have that

\[
\|H(\theta, W) − H(0, 0)\|_{L^\infty} \leq \frac{1}{4}.
\]

Since \(H(0, 0)(\omega) = \int_X L_{\sigma^{-1}\omega}v_0^{\sigma^{-1}\omega}dm = 1\) for \(\mathbb{P}\) a.e. \(\omega \in \Omega\), we easily obtain that

\[
\text{essinf}_U \|H\|_{L^\infty} \geq \frac{1}{2},
\]

which completes the proof of the lemma. 

\[\square\]

**Lemma 41.** The map \(F : U \subset B_C(0, 1) \times S_0 \to S_0\) given by

\[
F(\theta, W) := \frac{G(\theta, W)}{H(\theta, W)} - W - v_0,
\]

is well-defined, analytic and satisfies \(F(0, 0) = 0\). Furthermore, its differential w.r.t \(W\) at \((0, 0)\), \(D_2F(0, 0) : S_0 \to S_0\) is invertible.

**Proof.** The first part of the lemma follows readily from Lemma 40. Furthermore, note that

\[
(D_2F(0, 0)W)_{\omega} = L_{\sigma^{-1}\omega}W_{\sigma^{-1}\omega} - W_{\omega},
\]

for \(\omega \in \Omega\) and \(W \in S_0\).

We now prove that \(D_2F(0, 0)\) is invertible. Assume that \(W \in S_0\) satisfies that \(D_2F(0, 0)W = 0\). Hence, \(W_{\omega} = L_{\sigma^{-1}\omega}W_{\sigma^{-1}\omega}\) for \(\mathbb{P}\) a.e. \(\omega \in \Omega\). Consequently, it follows from (39) that for each \(n \in \mathbb{N}\) and \(\mathbb{P}\) a.e. \(\omega \in \Omega\),

\[
\|W_{\omega}\|_\omega = \|L_{\sigma^{-n}\omega}W_{\sigma^{-n}\omega}\|_\omega \\
\leq e^{-\lambda n}\|W_{\sigma^{-n}\omega}\|_{\sigma^{-n}\omega} \\
\leq e^{-\lambda n}\|W\|_s.
\]

30
Letting $n \to \infty$, we conclude that $\mathcal{W} = 0$. Thus, $D_2 F(0, 0)$ is injective.

In order to establish the surjectivity of $D_2 F(0, 0)$, let us take an arbitrary $\mathcal{W} \in S_0$. We define $\mathcal{V}$ by

$$\mathcal{V}(\omega, \cdot) = -\sum_{n=0}^{\infty} L_{\sigma^{-n}\omega}^n \mathcal{W}(\sigma^{-n}\omega, \cdot), \quad \omega \in \Omega.$$  

Using (39), it is easy to show that $\mathcal{V} \in S_0$. Moreover, a simple computation yields that $D_2 F(0, 0) \mathcal{V} = \mathcal{W}$. Together with the open mapping theorem, this completes the proof of the lemma.

The following is the main result of this subsection.

**Theorem 42.** There exists a neighborhood $U \subset B_C(0,1)$ of $0 \in \mathbb{C}$, such that for any $\theta \in U$, there exist $v^\theta \in S$, $\lambda^\theta \in L^\infty(\Omega)$, satisfying:

1. The maps $U \ni \theta \mapsto v^\theta \in S$ and $U \ni \theta \mapsto \lambda^\theta \in L^\infty(\Omega)$ are analytic.

2. For any $\theta \in U$ and $\mathbb{P}$-a.e. $\omega \in \Omega$, $v^\theta_\omega$, $\lambda^\theta_\omega$ satisfy

$$L^\theta_\omega v^\theta_\omega = \lambda^\theta_\omega v^\theta_\omega \quad (52)$$

$$\lambda^\theta_\omega = \int_X L^\theta_\omega v^\theta_\omega dm. \quad (53)$$

**Proof.** By the previous lemma, we may apply the analytic implicit function theorem between Banach spaces to $F$, to construct a neighborhood $U \subset B_C(0,1)$ of $0$, a $w^\theta \in S_0$, analytic in $\theta$, and such that for any $\theta \in U$,

$$0 = F(\theta, w^\theta) = \frac{G(\theta, w^\theta)}{H(\theta, w^\theta)} - w^\theta - v^0.$$

Setting

$$v^\theta := v^0 + w^\theta, \quad (54)$$

one obtains the desired result.

For future use we also note the following:

**Lemma 43.** The first and second derivatives at 0 of the analytic map $\theta \ni \lambda^\theta \in L^\infty(\Omega)$ are given by:

$$\frac{d\lambda^\theta}{d\theta}(0) = \int_X \psi_\omega v^0_\omega dm = 0$$

$$\frac{d^2\lambda^\theta}{d\theta^2}(0) = \int_X \psi^2_\omega v^0_\omega dm + 2 \sum_{j=1}^{\infty} \int_X \psi_\omega \cdot [L^j_{\sigma^{-j}\omega}(\psi_{\sigma^{-j}\omega} v^0_{\sigma^{-j}\omega})] dm.$$  

31
Proof. Classical computation, see e.g. [18, Lemma 4.5]. \(\square\)

Let us now explain how to get the dual of the previous construction: we define
\[
\mathcal{N} := \{ \Phi : \Omega \to BV^* \text{ measurable, } \text{esssup}_{\omega \in \Omega} \| \Phi_{\omega} \|_*^* < \infty \}. \tag{55}
\]
Similarly to \(S\), we may set, for \(\Phi \in \mathcal{N}\),
\[
\| \Phi \|_{\mathcal{N}} := \text{esssup}_{\omega \in \Omega} \| \Phi_{\omega} \|_*^* \omega < \infty \]
which defines a norm that turns \(\mathcal{N}\) into a Banach space. We notice that by (48), \(m \in \mathcal{N}\).

Let us set
\[
\mathcal{N}_0 := \{ \Phi \in \mathcal{N}, \Phi_{\omega}(v_0^\omega) = 0 \text{ for } \mathbb{P}\text{-a.e. } \omega \in \Omega \}. \tag{56}
\]
Then, \(\mathcal{N}_0\) is a closed subspace of \(\mathcal{N}\).

We may now introduce \(G^* : B_{C}(0, 1) \times \mathcal{N}_0 \to \mathcal{N}\) and \(H^* : B_{C}(0, 1) \times \mathcal{N}_0 \to L^\infty(\Omega)\) by
\[
G^*(\theta, \Phi)(\omega) := (\mathcal{L}_\omega^{\theta})^*(\Phi_{\sigma^\omega} + m),
\]
and
\[
H^*(\theta, \Phi)(\omega) := G^*(\theta, \Phi)(\omega)(v_0^\omega),
\]
for \(\omega \in \Omega, (\theta, \Phi) \in B_{C}(0, 1) \times \mathcal{N}_0\).

The proof of the following result can be obtained by repeating the arguments yielding Theorem 42.

**Proposition 44.** There exists a neighborhood \(U'\) of \((0, 0) \in B_{C}(0, 1) \times \mathcal{N}_0\), such that:

- The maps \(G^*\) and \(H^*\) are analytic on \(U'\) and \(\text{essinf}_{U'} \| H \|_{L^\infty} \geq \frac{1}{2}\).
- The map \(F^* : U' \subset B_{C}(0, 1) \times \mathcal{N}_0 \to \mathcal{N}_0\), defined by
  \[
  F^*(\theta, \Phi) := \frac{G^*(\theta, \Phi)}{H^*(\theta, \Phi)} - \Phi - m,
  \]
satisfies \(F^*(0, 0) = 0\), is analytic, and its partial differential w.r.t \(\Phi\) at \((0, 0)\), \(D_2 F^*(0, 0) : \mathcal{N}_0 \to \mathcal{N}_0\) is invertible.
- For any \(\theta\) in some neighborhood \(U'\) of \(0 \in \mathbb{C}\), there exists \(\phi^\theta \in \mathcal{N}\) and \(\tilde{\lambda}^\theta \in L^\infty(\Omega, \mathbb{C})\), both analytic in \(\theta\) and such that
  \[
  (\mathcal{L}_\omega^{\theta})^* \phi^\theta_{\sigma^\omega} = \tilde{\lambda}^\theta_{\omega} \phi^\theta_{\omega} \tag{57}
  \]
  \[
  \tilde{\lambda}^\theta_{\omega} = \phi^\theta_{\sigma^\omega}(\mathcal{L}_\omega^{\theta} v_0^\omega). \tag{58}
  \]

We end this section by a remark on the generalized eigenvalues \(\tilde{\lambda}^\theta\) and \(\lambda^\theta\).
Remark 45. A priori, the quantities $\lambda^\theta$, given by (52) and $\tilde{\lambda}^\theta$, given by (57) have no reason to coincide. Nevertheless, they do. Indeed, by linearity, we may choose a solution $\phi^\theta$ of (57) normalized by $\phi^\theta(\omega) = 1$, $\mathbb{P}$-a.e. Evaluating both sides of (57) at $v^\theta$, one gets, for a.e $\omega \in \Omega$

$$\lambda^\theta(\sigma_\omega)(v^\theta) = \tilde{\lambda}^\theta(\sigma_\omega)(v^\theta),$$

from which $\lambda^\theta = \tilde{\lambda}^\theta$ $\mathbb{P}$-a.e.

### 3.4 Quasi-compactness and regularity of the top Lyapunov exponent of the twisted cocycle

In this subsection, we prove the following theorem:

**Theorem 46.** For $\theta$ sufficiently close to 0, $\mathcal{L}^\theta = (\mathcal{L}^\theta_\omega)_{\omega \in \Omega}$ is a quasi-compact cocycle. Furthermore, the top Oseledets space $Y^1(\omega)$ is one-dimensional, spanned by $v^\theta_\omega$ (given by (52)). We write

$$BV = Y^1(\omega) \oplus H^\theta(\omega),$$

for the associated Oseledets splitting.

We begin with the following lemma.

**Lemma 47.** There exists a random variable $\tilde{C}: \Omega \to \Omega$ such that $\log \tilde{C} \in L^1(\Omega)$ and

$$\|\mathcal{L}^\theta_\omega \phi\|_{BV} \leq \tilde{C}(\omega)\|\phi\|_{BV}, \quad \text{for } \mathbb{P} \text{-a.e. } \omega \in \Omega, \ |\theta| \leq 1 \text{ and } \phi \in BV.$$

**Proof.** By (19), we have that

$$\|\mathcal{L}^\theta_\omega \phi\|_{BV} = \|\mathcal{L}_\omega(e^{\theta \psi_\omega} \phi)\|_{BV} \leq C(\omega)\|e^{\theta \psi_\omega} \phi\|_{BV}.$$

On the other hand, it is shown in the proof of [18, Lemma 3.2] that

$$\|e^{\theta \psi_\omega} \phi\|_{BV} \leq e^{\|\psi_\omega\|_{\infty}} (1 + |\theta|C_{var}(\psi_\omega)) \|\phi\|_{BV} \leq e^{\|\psi_\omega\|_{\infty}} (1 + C_{var}(\psi_\omega)) \|\phi\|_{BV}.$$

Hence,

$$\|\mathcal{L}^\theta_\omega \phi\|_{BV} \leq C(\omega) \esssup_{\omega \in \Omega} \left( e^{\|\psi_\omega\|_{\infty}} (1 + C_{var}(\psi_\omega)) \right) \|\phi\|_{BV},$$

33
for $\mathbb{P}$-a.e. $\omega \in \Omega$, $|\theta| \leq 1$ and $\phi \in BV$. In order to complete the proof, it remains to recall that $\log C \in L^1(\Omega)$ and to note that

$$\text{esssup}_{\omega \in \Omega} \left( e^{\|\psi\|_{\infty}} \left( 1 + C_{\text{var}} \text{Var}(\psi) \right) \right) < +\infty,$$

which follows from (49) (see Remark 34).

**Remark 48.** Observe that in the proof of Lemma 47 we actually showed that $\|L^{\theta}\|_{BV}$ was bounded a positive, log-integrable random variable $C_\theta$, going to $C$ a.s as $\theta \to 0$, and dominated by a (still log-integrable) $\tilde{C}$ of the form $cC$ for some constant $c > 0$.

It follows from Lemma 47 that the top Lyapunov exponent of the twisted cocycle $L^\theta = (L^\theta_\omega)_{\omega \in \Omega}$ exists for each $|\theta| \leq 1$. We will denote it by $\Lambda(\theta)$. Following [18], we introduce an auxiliary quantity. More precisely, for $\theta \in U$ let

$$\hat{\Lambda}(\theta) = \int_{\Omega} \log |\lambda^\theta_\omega| \, d\mathbb{P}(\omega),$$

where $\lambda^\theta_\omega$ is given by (52). For $\theta \in U$ and $\omega \in \Omega$, set

$$L^{\theta,n}_\omega := L^{\theta}_{\sigma^{n-1}\omega} \circ \ldots \circ L^\theta_\omega.$$

We recall (see [18, Lemma 3.3]) that

$$L^{\theta,n}_\omega(\phi) = L^n_\omega(e^{\theta S_n(\omega, \cdot)} \phi),$$

where

$$S_n(\omega, \cdot) = \sum_{i=0}^{n-1} \psi(\sigma^i \omega, T^i_\omega(\cdot)),$$

and

$$T^i_\omega = T^{\sigma^i-1}_\omega \circ \ldots \circ T^1_\omega.$$

The following is a version of [18, Lemma 3.8.] in our setting.

**Lemma 49.** For $\theta \in U$, we have that $\Lambda(\theta) \geq \hat{\Lambda}(\theta)$. 

34
Proof. By (37), (38), (52) and recalling that $K$ is tempered, we have that

$$
\Lambda(\theta) \geq \limsup_{n \to \infty} \frac{1}{n} \log \|L_\omega^{\theta,n}v_\theta\|_{BV}
\geq \limsup_{n \to \infty} \frac{1}{n} \log \left(K(\sigma^n) - 1 \|L_\omega^{\theta,n}v_\theta\|_{\sigma^n}\right)
= \limsup_{n \to \infty} \frac{1}{n} \log \|L_\omega^{\theta,n}v_\theta\|_{\sigma^n}
\geq \limsup_{n \to \infty} \frac{1}{n} \log \|L_\omega^{\theta,n}v_\theta\|_1
= \limsup_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \log |\lambda^{\theta}_{\sigma^k}\| + \limsup_{n \to \infty} \frac{1}{n} \log \|v_{\sigma^n}\|_1
\geq \limsup_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \log |\lambda^{\theta}_{\sigma^k}|,
$$

for $\mathbb{P}$-a.e. $\omega \in \Omega$, where in the last step we have used that

$$
1 = \left| \int_X v_\omega^0 \, dm \right| \leq \|v_\omega^0\|_1.
$$

It remains to observe that Birkhoff ergodic theorem implies that

$$
\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \log |\lambda^{\theta}_{\sigma^k}| = \hat{\Lambda}(\theta),
$$

for $\mathbb{P}$-a.e. $\omega \in \Omega$. \hfill $\square$

Lemma 50. We have that $\hat{\Lambda}$ is real-analytic and harmonic on a neighborhood of $0$. Moreover, $\hat{\Lambda}'(0) = 0$.

Proof. The first part follows by noting that, by Lemma 43 and the dominated convergence theorem, $\hat{\Lambda}$ is the real-part of complex-analytic function. The second part is an easy consequence of (50) (see [18, Lemma 3.11]). \hfill $\square$

We are now able to prove the main theorem of this section:

Proof of Theorem 46. Throughout this proof $D > 0$ will denote a generic constant (independent on $\omega$, $\theta$ and $\phi$) that can change from line to line. We begin by noting that (21) implies that

$$
\|L_\omega^{\theta,N}\|_{BV} \leq \|L_\omega^N\|_{BV} + \|(L_\omega^{\theta,N} - L_\omega^N)\|_{BV}
\leq \alpha^N(\omega)\|\phi\|_{BV} + (K^N(\omega) + 1)\|\phi\|_1 + \|(L_\omega^{\theta,N} - L_\omega^N)\|_{BV}.
$$
By using the same arguments as in the proof of Lemma 36 we have that
\[
\| (L^\theta_{\omega} - L_{\omega}) \phi \|_{BV} = \| L_{\omega} (e^{\theta \psi_{\omega}} - 1) \phi \|_{BV} \\
\leq C(\omega) \| (e^{\theta \psi_{\omega}} - 1) \phi \|_{BV} \\
\leq C(\omega) C_{\text{var}} |\theta| c^{\psi_{\omega} \infty} (\| \psi_{\omega} \|_{\infty} + \text{Var}(\psi_{\omega})) \| \phi \|_{BV}.
\]
Moreover, we have
\[
(L^\theta_{\omega,N} - L^\theta_{\omega}) \phi = \sum_{j=0}^{N-1} L^\theta_{\sigma^{N-j} \omega} (L^\theta_{\sigma^{N-1-j} \omega} - L^\theta_{\sigma^{N-1-j} \omega}) L_{\omega}^{N-1-j} \phi.
\]
By combining the above facts together with (19), Lemma 47 (see also Remark 48) and (51), we obtain that
\[
\| L^\theta_{\sigma^{N-j} \omega} (L^\theta_{\sigma^{N-1-j} \omega} - L^\theta_{\sigma^{N-1-j} \omega}) L_{\omega}^{N-1-j} \phi \|_{BV} \\
\leq D |\theta| \prod_{k=0}^{N-1} C(\sigma^k \omega) \| \phi \|_{BV} e^{c^{\psi_{\sigma^{N-1-j} \omega} \infty} (\| \psi_{\sigma^{N-1-j} \omega} \|_{\infty} + \text{Var}(\psi_{\sigma^{N-1-j} \omega}))} \\
\leq D |\theta| \prod_{k=0}^{N-1} C(\sigma^k \omega) \| \phi \|_{BV},
\]
for \( \mathbb{P}\)-a.e. \( \omega \in \Omega, |\theta| \leq 1, \phi \in BV \) and \( j \in \{0, \ldots, N-1\} \). Hence,
\[
\| (L^\theta_{\omega} - L^\theta_{\omega}) \phi \|_{BV} \leq D |\theta| \prod_{k=0}^{N-1} C(\sigma^k \omega) \| \phi \|_{BV},
\]
and thus
\[
\| L^\theta_{\omega,N} \phi \|_{BV} \leq \left( \alpha^N(\omega) + D |\theta| \prod_{k=0}^{N-1} C(\sigma^k \omega) \right) \| \phi \|_{BV} + (K^N(\omega) + 1) \| \phi \|_{1}, \tag{62}
\]
for \( \mathbb{P}\)-a.e. \( \omega \in \Omega, |\theta| \leq 1 \) and \( \phi \in BV \). From Lemma 8, Lemma 47 and (62), by arguing as in the proof of [18, Theorem 3.12], we get that the twisted transfer operator cocycle is quasi-compact.

The fact that the top Oseledets space is one-dimensional now follows from [18, Lemma A.3], arguing as in the second part of the proof of [18, Theorem 3.12].

**Proposition 51.** For \( \theta \) sufficiently close to 0, \( \Lambda(\theta) = \hat{\Lambda}(\theta) \). In particular, \( \Lambda \) is real-analytic and harmonic on a neighborhood of 0.
Proof. This follows from Lemma 50 and Theorem 46 by arguing exactly as in the proof of [18, Corollary 3.14]. \qed

As a consequence of Theorems 46 and 10, one has:

**Corollary 52.** The dual twisted transfer operator cocycle \((L^0)^*\) is quasi-compact, and its top Oseledets space, \((Y^0_1(\omega))^*\) is one-dimensional, spanned by \(\phi^0_\omega\) given by (57).

### 4 Limit theorems: new results with “old” proofs

#### 4.1 Variance

By \(E_\omega(\phi)\) we will denote \(\int_X \phi \, d\mu_\omega\), where \(\mu_\omega, \omega \in \Omega\) are as in the statement of Theorem 20. Moreover, \(\mu\) will denote the measure on \(\Omega \times X\) given by

\[
\mu(A \times B) = \int_A \mu_\omega(B) \, dP(\omega), \quad \text{for } A \in \mathcal{F} \text{ and } B \in \mathcal{G}.
\]

We note that \(\mu\) is invariant and ergodic for the skew-product transformation \(\tau: \Omega \times X \to \Omega \times X\) given by

\[
\tau(\omega, x) = (\sigma \omega, T_\omega(x)), \quad (\omega, x) \in \Omega \times X.
\]

**Lemma 53.** There exists \(\Sigma^2 \geq 0\) such that

\[
\lim_{n \to \infty} \frac{1}{n} \mathbb{E}_\omega \left( \sum_{k=0}^{n-1} \psi_{\sigma^k \omega} \circ T^k_\omega \right)^2 = \Sigma^2, \quad \text{for a.e. } \omega \in \Omega. \tag{63}
\]

**Proof.** By using (51) (which is a consequence of (49)) and arguing as in the proof of [19, Lemma 12], we find that

\[
\mathbb{E}_\omega \left( \sum_{k=0}^{n-1} \psi_{\sigma^k \omega} \circ T^k_\omega \right)^2 = \sum_{k=0}^{n-1} \mathbb{E}_\omega(\psi^2_{\sigma^k \omega} \circ T^k_\omega) + 2 \sum_{i=0}^{n-1} \sum_{j=i+1}^{n-1} \mathbb{E}_{\sigma^i \omega}(\psi_{\sigma^i \omega}(\psi_{\sigma^j \omega} \circ T^j_{\sigma^i \omega}))
\]

and

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \mathbb{E}_\omega(\psi^2_{\sigma^k \omega} \circ T^k_\omega) = \int_{\Omega \times X} \psi(\omega, x)^2 \, d\mu(\omega, x),
\]

for \(P\)-a.e. \(\omega \in \Omega\). Set

\[
\Psi(\omega) = \sum_{n=1}^{\infty} \int_X \psi(\omega, x) \psi(\tau^n(\omega, x)) \, d\mu_\omega(x) = \sum_{n=1}^{\infty} \int_X L^n_\omega(\psi_\omega \psi_{\sigma^n \omega}^0) \psi_{\sigma^n \omega} \, dm.
\]
By (32) and noting that \( \max \{ D(\omega), \|v_0^\omega\|_{BV} \} \leq K(\omega) \) (see the proof of Proposition 30), we have that

\[
|\Psi(\omega)| \leq \sum_{n=1}^{\infty} \|L_n(\psi_0^\omega)\|_{BV} \cdot \|\psi_0\|_{BV} \\
\leq K(\omega) \text{esssup}_{\omega \in \Omega} \|\psi_0\|_{BV} \sum_{n=1}^{\infty} e^{-\lambda n} \|\psi_0^\omega\|_{BV} \\
\leq C_{\text{var}} K(\omega)^2 \|\psi_0\|_{BV} \cdot \text{esssup}_{\omega \in \Omega} \|\psi_0\|_{BV} \frac{1}{1 - e^{-\lambda}}.
\]

for \( \mathbb{P}\)-a.e. \( \omega \in \Omega \). Using (49), we conclude that \( \Psi \in L^\infty(\Omega) \) and thus it follows again from Birkhoff ergodic theorem that, for \( \mathbb{P} \) a.e. \( \omega \in \Omega \),

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \Psi(\sigma^i \omega) = \int \Omega \Psi(\omega) d\mathbb{P}(\omega) = \sum_{n=1}^{\infty} \int_{\Omega \times X} \psi(\omega, x) \psi(\tau^n(\omega, x)) d\mu(\omega, x). \tag{64}
\]

Moreover, by arguing as in the proof of [19, Lemma 12], we have that

\[
\left| \sum_{i=0}^{n-1} \sum_{j=i+1}^{n-1} \mathbb{E}_{\sigma^i \omega}(\psi_{\sigma^i \omega}(\psi_{\sigma^i \omega} \circ T_{\sigma^i \omega}^{j-i})) - \sum_{i=0}^{n-1} \Psi(\sigma^i \omega) \right| \\
\leq \sum_{i=0}^{n-1} \sum_{j=i+1}^{n-1} \int_X L_{\sigma^i \omega}^k(\psi_{\sigma^i \omega} v_{\sigma^i \omega}) \psi_{\sigma^i+i \omega} d\mu \\
\leq \text{esssup}_{\omega \in \Omega} \|\psi_0\|_{BV} \cdot \text{esssup}_{\omega \in \Omega} (K(\omega)^2 \|\psi_0\|_{BV}) \sum_{i=0}^{n-1} \sum_{k=n-i}^{\infty} e^{-\lambda k},
\]

which implies that

\[
\lim_{n \to \infty} \frac{1}{n} \left( \sum_{i=0}^{n-1} \sum_{j=i+1}^{n-1} \mathbb{E}_{\sigma^i \omega}(\psi_{\sigma^i \omega}(\psi_{\sigma^i \omega} \circ T_{\sigma^i \omega}^{j-i})) - \sum_{i=0}^{n-1} \Psi(\sigma^i \omega) \right) = 0, \tag{65}
\]

It follows from (64) and (65) that

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \sum_{j=i+1}^{n-1} \mathbb{E}_{\sigma^i \omega}(\psi_{\sigma^i \omega}(\psi_{\sigma^i \omega} \circ T_{\sigma^i \omega}^{j-i})) = \sum_{n=1}^{\infty} \int_{\Omega \times X} \psi(\omega, x) \psi(\tau^n(\omega, x)) d\mu(\omega, x)
\]

38
for a.e. $\omega \in \Omega$ and therefore (63) holds with

$$\Sigma^2 = \int_{\Omega \times X} \psi(\omega, x)^2 d\mu(\omega, x) + 2 \sum_{n=1}^{\infty} \int_{\Omega \times X} \psi(\omega, x) \psi(\tau^n(\omega, x)) d\mu(\omega, x).$$

(66)

Finally, we note that it follows readily from (63) that $\Sigma^2 \geq 0$ and the proof of the lemma is completed. \qed

The proof of the following result can be done by arguing exactly as in the proof of [19, Proposition 3].

**Proposition 54.** We have that $\Sigma^2 = 0$ if and only if there exist $\phi \in L^2(\Omega \times X, \mu)$ such that $\psi = \phi - \phi \circ \tau$.

**Corollary 55.** One has $\Lambda''(0) = \Sigma^2$. In particular, if $\Sigma^2 > 0$, $\Lambda$ is strictly convex on some neighborhood of 0.

**Proof.** The value of $\Lambda''(0)$ is easily obtained by Lemma 43 and (66) (see also [18, Section 3.6] for a similar argument), and the rest is an elementary fact of real analysis. \qed

### 4.2 Proof of Theorem 1

In this section, we prove that if $\Sigma^2 > 0$ then for $\mathbb{P}$-a.e. $\omega \in \Omega$, the process $(\psi_{\sigma^n \omega} \circ T^n_{\omega})_{n \geq 0}$ satisfies the central limit theorem. More precisely, we show that for $\mathbb{P}$-a.e. $\omega \in \Omega$ and every bounded and continuous $\phi: \mathbb{R} \to \mathbb{R}$ we have that

$$\lim_{n \to \infty} \int_X \phi \left( \frac{S_n \psi(\omega, x)}{\sqrt{n}} \right) d\mu_\omega(x) = \int_{\mathbb{R}} \phi \ d\mathcal{N}(0, \Sigma^2),$$

where $\mathcal{N}(0, \Sigma^2)$ denotes the normal distribution (with parameters 0 and $\Sigma^2$) and $S_n \psi$ is given by (61).

In order to establish the above claim, we essentially follow [18, Proof of Thm B.] and the classical method given by Levy’s continuity theorem:
Writing \( t_n = t/\sqrt{n} \), we have that

\[
\int_X e^{it_n S_n \psi(\omega, \cdot)} v_\omega^0 \, dm = \int_X L^{it_n,n}_\omega v_\omega^0 \, dm
\]

\[
= \int_X L^{it_n,n}_\omega (v_\omega^0 - v^{it_n}_\omega) \, dm + \int_X L^{it_n,n}_\omega v^{it_n}_\omega \, dm
\]

\[
= \int_X L^{it_n,n}_\omega (v_\omega^0 - v^{it_n}_\omega) \, dm + \prod_{j=0}^{n-1} \lambda^{it_n,\sigma_j}_\omega \int_X v^{it_n}_\omega \, dm
\]

\[
= \int_X L^{it_n,n}_\omega (v_\omega^0 - v^{it_n}_\omega) \, dm + \prod_{j=0}^{n-1} \lambda^{it_n,\sigma_j}_\omega
\]

Observing that (38) and (60) imply that

\[
\left| \int_X L^{it_n,n}_\omega (v_\omega^0 - v^{it_n}_\omega) \, dm \right| = \left| \int_X L^n_\omega (e^{it_n S_n \psi(\omega, \cdot)} (v_\omega^0 - v^{it_n}_\omega)) \, dm \right|
\]

\[
= \left| \int_X e^{it_n S_n \psi(\omega, \cdot)} (v_\omega^0 - v^{it_n}_\omega) \, dm \right|
\]

\[
\leq \|v_\omega^0 - v^{it_n}_\omega\|_1
\]

\[
\leq \|v_\omega^0 - v^{it_n}_\omega\|_\omega
\]

\[
\leq \|v^0 - v^{it_n}\|_S,
\]

for \( \mathbb{P} \)-a.e. \( \omega \in \Omega \) and \( n \in \mathbb{N} \). Thus, from Theorem 42 we conclude that

\[
\lim_{n \to \infty} \int_X L^{it_n,n}_\omega (v_\omega^0 - v^{it_n}_\omega) \, dm = 0, \quad \text{for } \mathbb{P} \text{-a.e. } \omega \in \Omega.
\]

Hence, we have to show that

\[
\lim_{n \to \infty} \prod_{j=0}^{n-1} \lambda^{it_n,\sigma_j}_\omega = e^{-t^2 \Sigma^2/2} \quad \text{for } \mathbb{P} \text{-a.e. } \omega \in \Omega,
\]

which is equivalent to

\[
\lim_{n \to \infty} \sum_{j=0}^{n-1} \log \lambda^{it_n,\sigma_j}_\omega = -\frac{t^2 \Sigma^2}{2} \quad \text{for } \mathbb{P} \text{-a.e. } \omega \in \Omega.
\]

Let us consider the map \( H \), defined from a neighborhood of 0 in \( \mathbb{C} \) to \( L^\infty(\Omega) \), by \( H(\theta)(\omega) = \log \lambda^0_\omega \), \( \omega \in \Omega \). Up to shrinking its domain, this map is
analytic, as the composition of two analytic maps, and satisfies by Lemma 43 $H(0)(\omega) = 0$, $H'(0)(\omega) = 0$ and
\[
H''(0)(\omega) = \int_X \psi_v^0 \rho dm + 2 \sum_{j=1}^\infty \int_X \psi_{\omega} \left[ \mathcal{L}_{\omega}^{(j)}(\psi_{\omega} \psi_{v_{\omega}}^0) \right] dm.
\]
By Taylor’s formula at order two, one has
\[
H(it_n)(\sigma^j \omega) = -\frac{t^2}{2n} H''(0)(\omega) + R(it_n)(\omega),
\]
with $R$ the remainder of the series. One then has
\[
\sum_{j=0}^{n-1} H(it_n)(\sigma^j \omega) = -\frac{t^2}{2n} \sum_{j=0}^{n-1} H''(0)(\sigma^j \omega) + \sum_{j=0}^{n-1} R(it_n)(\sigma^j \omega).
\]
By Birkhoff ergodic theorem, as $n \to \infty$,
\[
\frac{1}{n} \sum_{j=0}^{n-1} H''(0)(\sigma^j \omega) \to \int_\Omega H''(0)(\omega) d\mathbb{P}(\omega) = \Sigma^2, \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega.
\]
For the remainder term, we notice that one may write $R(\theta) = \theta^2 \tilde{R}(\theta)$, with $\|\tilde{R}(\theta)\|_{L^\infty} \to 0$ when $\theta \to 0$. In particular, for any $\varepsilon > 0$ and $t \in \mathbb{R} \setminus \{0\}$, one may find a $\delta > 0$, such that $\|\tilde{R}(\theta)\|_{L^\infty} \leq \frac{\varepsilon}{t^2}$ if $|\theta| \leq \delta$. There is also a $n_1$, such that for $n \geq n_1$, $|it_n| \leq \delta$. Putting this together yields, for $n \geq n_1$,
\[
\left| \sum_{j=0}^{n-1} R(it_n)(\sigma^j \omega) \right| \leq \frac{t^2}{n} \sum_{j=0}^{n-1} \varepsilon \leq \varepsilon,
\]
i.e. $\lim_{n \to \infty} \sum_{j=0}^{n-1} R_{\sigma^j \omega}(it_n) = 0$, for $\mathbb{P}$-a.e. $\omega \in \Omega$. The announced result follows.

**Remark 56.** As mentioned, our proof of the quenched C.L.T. is similar to the proof of [18, Theorem B]. However, we note that our initial step (67) is different from that in [18]. In particular, we don’t need the version of [18, Lemma 4.4] in our setting.

### 4.3 Proof of Theorem 2

In this section, we prove a (quenched) large deviations estimate for the process $(S_n \psi(\omega, \cdot))_{n \geq 0}$. We need the following classical result.

41
Theorem 57. (Gärtner-Ellis [36]) For $n \in \mathbb{N}$, let $\mathbb{P}_n$ be a probability measure on a measurable space $(Y, \mathcal{T})$ and let $\mathbb{E}_n$ denote the corresponding expectation operator. Furthermore, let $S_n$ be a real random variable on $(\Omega, \mathcal{T})$ and assume that on some interval $[-\theta_+, \theta_+]$, $\theta_+ > 0$, we have

$$\lim_{n \to \infty} \frac{1}{n} \log \mathbb{E}_n(e^{\theta S_n}) = \psi(\theta),$$

where $\psi$ is a strictly convex continuously differentiable function satisfying $\psi'(0) = 0$. Then, there exists $\varepsilon_+ > 0$ such that the function $c$ defined by

$$c(\varepsilon) = \sup_{|\theta| \leq \theta_+} \{\theta \varepsilon - \psi(\theta)\}$$

is nonnegative, continuous, strictly convex on $[-\varepsilon_+, \varepsilon_+]$, vanishing only at 0 and such that

$$\lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}_n(S_n > n\varepsilon) = -c(\varepsilon), \quad \text{for every } \varepsilon \in (0, \varepsilon_+).$$

We need the following lemma, whose statement and proof are similar to [18, Lemma 4.2]. As such, we will omit the proof.

Lemma 58. Let $\theta \in \mathbb{C}$ be close enough to 0. For any $f \in \text{BV}$ such that $\phi_\omega^\theta(f) \neq 0$, one has

$$\lim_{n \to \infty} \frac{1}{n} \log \left| \int_X e^{\theta S_n\psi(\omega, \cdot)} f d\mu \right| = \Lambda(\theta).$$

Proof of quenched large deviations. We start by remarking that Lemma 58 implies that

$$\lim_{n \to \infty} \frac{1}{n} \log \left| \int_X e^{\theta S_n\psi(\omega, \cdot)} d\mu_\omega \right| = \Lambda(\theta).$$

Indeed, since we may write $\int_X e^{\theta S_n\psi(\omega, \cdot)} d\mu_\omega = \int_X e^{\theta S_n\psi(\omega, \cdot)} v_\omega^0 dm$, it suffices to show that $\phi_\omega^\theta(v_\omega^0) \neq 0$. But this easily follows from $1 = \phi_\omega^0(v_\omega^0) = \int_X v_\omega^0 dm$ and analyticity of $\theta \mapsto \phi_\omega^\theta \in \mathcal{N}$ in some neighborhood of 0.

Hence, since $\Lambda$ is convex in some (small enough) real neighborhood of 0 (see Corollary 55), we may apply Theorem 57 for each fixed $\omega \in \Omega$ in some full-measure subset, with $\mathbb{P}_n = \mu_\omega$, $S_n = S_n\psi_\omega$ and $\psi(\theta) = \Lambda(\theta)$.

4.4 Proof of Theorem 3

We first establish the following version of [21, Lemma 4.7] in our setting.
Lemma 59. There exist $C > 0$ and $0 < r < 1$ such that for every $\theta \in \mathbb{C}$ sufficiently close to 0, every $n \in \mathbb{N}$ and $\mathbb{P}$-a.e. $\omega \in \Omega$, we have

$$\left| \int_X \mathcal{L}_{\omega}^{\theta,n}(v_0^\omega - \phi_\omega^\theta(v_0^\omega))\, dm \right| \leq C r^n |\theta|.$$  

Proof. For $\theta$ near 0 and $\omega \in \Omega$, let

$$Q_{\omega}^{\theta,n} h := \mathcal{L}_{\omega}^{\theta,n}(h - \phi_\omega^\theta(h)v_0^\omega), \quad h \in BV.$$  

Then,

$$Q_{\omega}^{\theta,n} h = \mathcal{L}_{\omega}^{\theta,n}(h - \phi_\omega^\theta(h)v_0^\omega), \quad \text{for } \omega \in \Omega, n \in \mathbb{N} \text{ and } h \in BV. \quad (71)$$

Observe that (39) gives that

$$\|Q_{\omega}^{\theta} h\|_{\sigma_\omega} \leq e^{-\lambda} \|h\|_{\sigma_\omega}, \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega \text{ and } h \in BV. \quad (72)$$

Take $r > 0$ such that $e^{-\lambda} < r < 1$. It follows from Theorem 42, Proposition 44 and (72) that for $\theta$ sufficiently close to 0,

$$\|Q_{\omega}^{\theta} h\|_{\sigma_\omega} \leq r \|h\|_{\sigma_\omega}, \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega \text{ and } h \in BV. \quad (73)$$

By iterating (73) and using (71), we obtain that

$$\|\mathcal{L}_\omega^{\theta,n}(h - \phi_\omega^\theta(h)v_0^\omega)\|_{\sigma^n_\omega} \leq r^n \|h\|_{\sigma^n_\omega} \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega, n \in \mathbb{N} \text{ and } h \in BV, \quad (74)$$

whenever $\theta$ is sufficiently close to 0. Now it follows readily from (38), (41) and (74) that there exists $C > 0$ such that for $\theta$ sufficiently close to 0, $\mathbb{P}$-a.e. $\omega \in \Omega$ and $n \in \mathbb{N},$

$$\left| \int_X \mathcal{L}_{\omega}^{\theta,n}(v_0^\omega - \phi_\omega^\theta(v_0^\omega))\, dm \right| \leq C r^n. \quad (75)$$

The conclusion of the lemma now follows from the Cauchy integral formula and a simple observation that the term $\int_X \mathcal{L}_\omega^{\theta,n}(v_0^\omega - \phi_\omega^\theta(v_0^\omega))\, dm$ vanishes at $\theta = 0$. \qed

Remark 60. We note that for the purpose of establishing the quenched local C.L.T (75) is sufficient. However, the finer conclusion given by Lemma 59 is needed for the Berry-Essen estimates (see [21, Section 4.4]).

We now establish the quenched local C.L.T under the following aperiodicity assumption. Namely, we require that for $\mathbb{P}$-a.e. $\omega \in \Omega$ and every compact interval $J \subset \mathbb{R} \setminus \{0\}$, there exist $C = C(\omega) > 0$ and $\rho \in (0, 1)$ such that

$$\|\mathcal{L}_\omega^{t,n}\|_{BV} \leq C \rho^n, \quad \text{for } t \in J \text{ and } n \geq 0. \quad (76)$$
Following [18, 47], it is sufficient to show that for $h \in L^1(\mathbb{R})$ whose Fourier transform $\hat{h}$ has a compact support,

$$
\sup_{s \in \mathbb{R}} \left| \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\sqrt{n} \hat{h} \left( \frac{t}{\sqrt{n}} \right)} \int_{\mathcal{X}} \mathcal{L}_\omega^{\frac{t}{\sqrt{n}}} v_\omega^0 \, dm \, dt - \frac{\hat{h}(0)}{2\pi} \int_{\mathbb{R}} e^{i\sqrt{n} \cdot e^{-\frac{y_2^2}{2}}} \, dt \right| \to 0,
$$

when $n \to \infty$, for $\mathbb{P}$-a.e. $\omega \in \Omega$.

Choose $\delta > 0$ such that the support of $\hat{h}$ is contained in $[-\delta, \delta]$. For any $\delta \in (0, \delta)$, we have that

$$
\begin{align*}
\frac{1}{2\pi} \int_{\mathbb{R}} e^{i\sqrt{n} \hat{h} \left( \frac{t}{\sqrt{n}} \right)} \int_{\mathcal{X}} \mathcal{L}_\omega^{\frac{t}{\sqrt{n}}} v_\omega^0 \, dm \, dt
&= \frac{1}{2\pi} \int_{|t|<\delta\sqrt{n}} e^{i\sqrt{n} \hat{h} \left( \frac{t}{\sqrt{n}} \right)} \int_{\mathcal{X}} \prod_{j=0}^{n-1} \hat{\lambda}_{\sigma_j}^{\frac{t}{\sqrt{n}}} v_\omega^0 - \hat{h}(0) \hat{v}_\omega \, dt \\
&\quad + \frac{\sqrt{n}}{2\pi} \int_{|t|<\delta} e^{i\sqrt{n} \hat{h}(t)} \int_{\mathcal{X}} \prod_{j=0}^{n-1} \hat{\lambda}_{\sigma_j}^{\frac{t}{\sqrt{n}}} (v_\omega^0)_{\sigma_j} \, dm \, dt \\
&\quad + \frac{\sqrt{n}}{2\pi} \int_{\delta \leq |t| < \delta} e^{i\sqrt{n} \hat{h}(t)} \int_{\mathcal{X}} \mathcal{L}_\omega^{\frac{t}{\sqrt{n}}} v_\omega^0 \, dm \, dt \\
&\quad - \frac{\hat{h}(0)}{2\pi} \int_{|t|\geq\delta\sqrt{n}} e^{i\sqrt{n} \cdot e^{-\frac{y_2^2}{2}}} \, dt
= (I) + (II) + (III) + (IV) + (V)
\end{align*}
$$

By repeating the arguments in the proof of [18, Theorem C], we show that the terms $(I) - (V)$ converge to 0 uniformly in $s$, when $n \to \infty$. In particular, the aperiodicity condition (76) will take care of $(IV)$, Lemma 59 enables us to handle $(III)$, while the fact that $(V)$ converges to 0 uniformly in $s$ follows easily from the dominated convergence theorem. In order to handle the terms $(I)$ and $(II)$, one needs the following result whose statement and the proof is the same as [18, Lemma 4.6].

**Lemma 61.** For $\tilde{\delta} > 0$ sufficiently small, there exists $n_0 \in \mathbb{N}$ such that for $\mathbb{P}$-a.e. $\omega \in \Omega$, $n \geq n_0$ and $t$ such that $|t| < \tilde{\delta}\sqrt{n}$,

$$
\left| \prod_{j=0}^{n-1} \hat{\lambda}_{\sigma_j}^{\frac{t}{\sqrt{n}}} \right| \leq e^{-\frac{\beta \gamma t^2}{n}}.
$$

**Remark 62.** Let us comment a bit on the aperiodicity assumption:
• We refer to [18, Section 4.3.2] for certain equivalent formulations of the condition (76) (that readily apply to our setting).

• One can also formulate and proof the periodic version of the local central limit theorem just as in [18, Section 4.4].

5 Acknowledgments

We would like to thank Yeor Hafouta for several useful comments.

DD was supported in part by Croatian Science Foundation under the project IP-2019-04-1239 and by the University of Rijeka under the projects uniri-priod-18-9 and uniri-pr-priod-19-16.

JS was supported by the European Research Council (ERC) under the European Union’s Horizon 2020 research and innovation programme (grant agreement No 787304).

References

[1] Aimino, R., Nicol, M. and Vaienti, S. Annealed and quenched limit theorems for random expanding dynamical systems. Probab. Theory Relat. Fields 162, 233–274 (2015).

[2] J. F. Alves, W. Bahsoun and R. Ruziboev, Almost sure rates of mixing for partially hyperbolic attractors, preprint, https://arxiv.org/abs/1904.12844

[3] L. Arnold, Random dynamical systems, Springer Monogr. Math., Springer, Berlin, 1998.

[4] A. Ayyer, C. Liverani, and M. Stenlund, Quenched CLT for random toral automorphism, Discrete Contin. Dyn. Syst. 24 (2009), 331–348.

[5] L. Backes and D. Dragičević, Periodic approximation of exceptional Lyapunov exponents for semi-invertible operator cocycles, Ann. Acad. Sci. Fenn. Math. 44 (2019), 183–209.

[6] W. Bahsoun and C. Bose, Mixing rates and limit theorems for random intermittent maps, Nonlinearity 29 (2016) 1417–1433.

[7] W. Bahsoun, C. Bose and M. Ruziboev, Quenched decay of correlations for slowly mixing systems, Trans. Amer. Math. Soc. 372 (2019), 6547–6587.

[8] F. Batayneh and C. González-Tokman Invariant measures for random expanding on average Saussol maps, Stoch. Dyn., to appear.

[9] W. Bahsoun, M. Ruziboev and B. Saussol, Linear response for random dynamical systems, Adv. Math. 364 (2020), 107011, 44pp.

[10] V. Baladi, M. Benedicks and V. Maume-Deschamps, Almost sure rates of mixing for i.i.d. unimodal maps, Ann. Sci. Ecole Norm. Sup. Vol. 35 (2002), 77–126.
[11] V.I. Bakhtin, *Random processes generated by a hyperbolic sequence of mappings. I*, Izv. Ross. Akad. Nauk Ser. Mat. 58 (1994) 40–72.

[12] V.I. Bakhtin, *Random processes generated by a hyperbolic sequence of mappings. II*, Izv. Ross. Akad. Nauk Ser. Mat. 58 (1994) 184–195.

[13] J. Buzzi, *Exponential decay of correlations for random Lasota-Yorke maps*, Commun. Math. Phys. 208, 25 – 54 (1999).

[14] Y. Moo Chung, Y. Nakano and J. Wittsten, *Quenched limit theorems for random U(1) extensions of expanding maps*, https://arxiv.org/abs/2104.01606.

[15] R. Cogburn, *On the Central Limit Theorem for Markov Chains in Random Environments*, Ann. Probab. 19 (1991), 587–604.

[16] M. Demers and C. Liverani, *Projective cones for generalized dispersing billiards*, preprint, https://arxiv.org/abs/2104.06947

[17] M. Demers, F. Pène and H-K. Zhang, *Local limit theorem for randomly deforming billiards*, Comm. Math. Phys. 375 (2020), 2281–2334.

[18] D. Dragičević, G. Froyland, C. González-Tokman and S. Vaienti, *A spectral approach for quenched limit theorems for random expanding dynamical systems*, Comm. Math. Phys. 360 (2018), 1121–1187.

[19] D. Dragičević, G. Froyland, C. González-Tokman and S. Vaienti, *Almost sure invariance principle for random piecewise expanding maps*, Nonlinearity 31 (2018), 2252–2280.

[20] D. Dragičević, G. Froyland, C. González-Tokman and S. Vaienti, *A spectral approach for quenched limit theorems for random hyperbolic dynamical systems*, Trans. Amer. Math. Soc. 373 (2020), 629–664.

[21] D. Dragičević and Y. Hafouta, *Limit theorems for random expanding or Anosov dynamical systems and vector-valued observables*, Ann. Henri Poincare 21 (2020), 3869—3917.

[22] D. Dragičević and Y. Hafouta, *Almost sure invariance principle for random distance expanding maps with a nonuniform decay of correlations*, Thermodynamic Formalism, Lecture Notes in Mathematics Vol. 2290, CIRM Jean-Morlet Chair Subseries, Springer-Verlag, 177–204.

[23] D. Dragičević and Y. Hafouta, *Almost sure invariance principle for random dynamical systems via Gouëzel’s approach*, Nonlinearity 34 (2021), 6773–6798.

[24] D. Dragičević, Y. Hafouta and J. Sedro, *A vector-valued almost sure invariance principle for random expanding on average cocycles*, preprint, https://arxiv.org/abs/2108.08714.

[25] D. Dragičević and J. Sedro, *Statistical stability and linear response for random hyperbolic dynamics*, Ergodic Theory Dynam. Systems, to appear.

[26] G. Froyland, S. Lloyd, and A. Quas, *A semi-invertible Oseledets theorem with applications to transfer operator cocycles*, Discrete Contin. Dyn. Syst. 33 (2013), 3835–3860.

[27] S. Galatolo and J. Sedro, *Quadratic response of random and deterministic dynamical systems*, Chaos 30, 023113 (2020).
[28] C. González-Tokman and A. Quas, A semi-invertible operator Oseledets theorem, Ergodic Theory Dynam. Systems 34 (2014), 1230–1272.

[29] S. Gouëzel, Limit theorem in dynamical systems using the spectral method, Proceedings of Symposia in Pure Mathematics Volume 89, (2015) pp. 161–193.

[30] Y. Guivarc’h and J. Hardy, Théorèmes limites pour une classe de chaînes de Markov et applications aux difféomorphismes d’Anosov, Annales de l’IHP Probabilités et statistiques 24 (1988), 73–98.

[31] Y. Hafouta, Limit theorems for some time dependent dynamical systems, Nonlinearity 33 (2020), 6421.

[32] Y. Hafouta, Limit theorems for some skew products with mixing base maps, Ergodic Theory Dynam. Systems 41 (2021), 241–271.

[33] Y. Hafouta, Limit theorems for random non-uniformly expanding or hyperbolic maps with exponential tails, Ann. Henri Poincaré, to appear.

[34] Y. Hafouta and Yu. Kifer, Nonconventional limit theorems and random dynamics, World Scientific, Singapore, 2018.

[35] N. Haydn, M. Nicol, A. Török and S. Vaient, Almost sure invariance principle for sequential and nonstationary dynamical systems, Trans. Amer. Math. Soc. 369 (2017), 5293–5316.

[36] H. Hennion and L. Hervé, Limit theorems for Markov chains and stochastic properties of dynamical systems by quasi-compactness, Lecture Notes in Mathematics vol. 1766, Springer-Verlag, Berlin, 2001.

[37] A. Korepanov and J. Leppänen, Loss of memory and moment bounds for nonstationary intermittent dynamical systems, Comm. Math. Phys., to appear, DOI: 10.1007/s00220-021-04071-5

[38] J. Leppänen and M. Stenlund, Quasistatic dynamics with intermittency, Math. Phys. Anal. Geom. 19 (2016), Art. 8, 23 pp.

[39] J. Leppänen and M. Stenlund, Sunklodas’ approach to normal approximation for time-dependent dynamical systems, J. Stat. Phys. 181 (2020), 1523–1564.

[40] Y. Kifer, Thermodynamic formalism for random transformations revisited, Stochastics and Dynamics, Vol. 8, No. 1 (2008) 77–102.

[41] Y. Kifer, Limit theorems for random transformations and processes in random environments, Trans. Am. Math. Soc. 350(4) (1998), 1481–1518.

[42] S. V. Nagaev, Some limit theorems for stationary Markov chains, Theory of Probability & Its Applications 2 (1957), 378–406.

[43] S. V. Nagaev, More exact statement of limit theorems for homogeneous Markov chains, Theory of Probability & Its Applications 6 (1961), 62–81.

[44] P. Nándori, D. Szász, and T. Varjú, A central limit theorem for time-dependent dynamical systems, J. Stat. Phys. 146 (2012), 1213–1220.

[45] M. Nicol, A. Török and S. Vaient, Central limit theorems for sequential and random intermittent dynamical systems, Ergodic Theory Dynam. Systems 3 (2018), 1127–1153.

47
[46] O. Hella and M. Stenlund, *Quenched normal approximation for random sequences of transformations*, J. Stat. Phys. **178** (2020), 1–37.

[47] J. Rousseau-Egele, *Un théorème de la limite locale pour une classe de transformations dilatantes et monotones par morceaux*, Ann. Probab. **11** (1983) 772–788.

[48] B. Saussol, *Absolutely continuous invariant measures for multidimensional expanding maps*, Israel J. Math. **116** (2000), 223–248.

[49] J. Sedro, and H.H Rugh, *Regularity of characteristic exponents and linear response for transfer operator cocycles*, Comm. Math. Phys. **383** (2021), 1243–1289.

[50] Y. Su, *Random Young Towers and Quenched Limit Laws*, preprint, https://arxiv.org/pdf/1907.12199.