CYLINDRIC MULTIPARTITIONS AND LEVEL-RANK DUALITY

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ABSTRACT. We show that a multipartition is cylindric if and only if its level rank-dual is a source in the corresponding affine type $A$ crystal. This provides an algebraic interpretation of cylindricity, and completes a similar result for FLOTW multipartitions.

INTRODUCTION

Cylindric multipartitions are a particular class of tuples of partitions that were first introduced by Gessel and Krattenthaler [11], in order to study basic hypergeometric series in affine type $A_1$. They turned up in representation theory in the work of Foda, Leclerc, Okado, Thibon and Welsh [4], building on earlier results of Jimbo, Misra, Miwa and Okado [15]. In fact, cylindric multipartitions that verify an additional property, called the FLOTW multipartitions, appear as vertices in the crystal graph of certain irreducible highest weight representations of affine type $A$ quantum groups [4, Theorem 2.10].

At the combinatorial level, the study of cylindric multipartitions gave rise to many different and interesting results. A formula for the generating functions for cylindric multipartitions was first given in [11] in a special case, and later by Borodin [3, Proposition 5.1] in the general case, using a probabilistic tool called the periodic Schur process. Independently, Tingley used the representation-theoretic interpretation of cylindric multipartitions and the Weyl-Kac character formula to give an alternative formula for this generating function [19, Theorem 4.17], and showed that it agrees with Borodin’s formula. More recently, Foda and Welsh [5] rederived the Andrews-Gordon and Bressoud identities (generalizing the Rogers-Ramanujan identities) via a systematic study of cylindric multipartitions. There, they also exploited the relationships with certain characters of the generalized Virasoro algebras.

At the representation-theoretical level, the study of cylindric and FLOTW multipartitions have very important applications. First of all, by Ariki’s theorem [1], FLOTW multipartitions parametrize the irreducible representations of modular cyclotomic Hecke algebras, see [4, Section 3.4] and further investigations [12], [6], [13], [7]. In fact, one can explicitly construct three commuting crystals on multipartitions using a combinatorial level-rank duality: two Kac-Moody crystals (of affine type $A$), dual to each other, and a so-called Heisenberg crystal [9]. By the works of Ariki [2], Shan [17], Shan and Vasserot [18] and Losev [16], these crystals are categorized by certain branching rules for representations of cyclotomic Hecke algebras and rational Cherednik algebras (in the category $O$). In particular, sources in the crystals also have an important meaning. For instance, finite-dimensional irreducible representations of cyclotomic Cherednik algebras are indexed by multipartitions that are sources in the Kac-Moody and Heisenberg crystals simultaneously [18, Proposition 5.18]. In turn, these correspond to multipartitions whose level-rank dual is FLOTW [9, Theorem 7.7].

In this short note, we complete the latter result by showing that cylindric multipartitions are precisely those multipartitions whose level-rank dual is a source in the dual Kac-Moody crystal. We use exclusively combinatorial arguments, relying on the abacus representation of multipartitions, which we recall in Section 1. In Section 2 we give the definition of cylindric and FLOTW multipartitions and explain how to detect cylindricity in an abacus. Section 3 provides the expected result, namely Theorem 3.3.

1 Note that in [11], the terminology used is “cylindric partitions”, and these objects are not directly defined as tuples of partitions. However, it is easy to see them as such, see for instance [5, Appendix C]. In fact, we will not need Gessel and Krattenthaler’s original definition in this paper.
In the rest of the paper, we fix $e, \ell \in \mathbb{Z}_{>1}$ and $s \in \mathbb{Z}$.

1.1. Multipartitions and abaci. Let $x \in \mathbb{Z}_{>1}$. Later, we will take either $x = \ell$ or $x = e$. An $x$-abacus is a subset $\mathcal{A}$ of $\mathbb{Z} \times \{1, \ldots, x\}$ such that there exists $m_-, m_+ \in \mathbb{Z}$ verifying:

- For all $\beta \leq m_-$ and for all $j \in \{1, \ldots, x\}$, $(\beta, j) \in \mathcal{A}$.
- For all $\beta \geq m_+$ and for all $j \in \{1, \ldots, x\}$, $(\beta, j) \notin \mathcal{A}$.

Let us consider the following two different graphical representations of $x$-abaci. Firstly, we can represent an $x$-abacus $\mathcal{A}$ by $x$ rows of beads, numbered from 1 at the bottom to $x$ at the top, where we put a black (respectively white) bead in column $\beta$ and row $j$ if $(\beta, j) \in \mathcal{A}$ (respectively $(\beta, j) \notin \mathcal{A}$). We call this the horizontal representation of $\mathcal{A}$. In a dual fashion, we can represent $\mathcal{A}$ by $x$ columns of beads, numbered from 1 at the left to $x$ at the right, where we put a white (respectively black) bead in row $\beta$ and column $j$ if $(\beta, j) \in \mathcal{A}$ (respectively $(\beta, j) \notin \mathcal{A}$). We call this the vertical representation of $\mathcal{A}$, and we will use the notation $\mathcal{A}$ instead. In the rest of this paper, we will use the horizontal representation for $\ell$-abaci and the vertical representation for $e$-abaci.

Let $\mathcal{A}$ be an $x$-abacus. The charge of $\mathcal{A}$ is the element $s = (s_1, \ldots, s_x) \in \mathbb{Z}^x$ such that in the horizontal representation, the $x$-abacus obtained from $\mathcal{A}$ by pushing all black beads as far to the left as possible, the rightmost bead in row $j$, say $(\beta, j)$, verifies $\beta = s_j$, for all $j \in \{1, \ldots, \ell\}$. We denote $|s| = \sum_{i=1}^{\ell} s_i$. Remember that we had fixed $s \in \mathbb{Z}$. We will denote $\mathbb{Z}^x(s) = \{s \in \mathbb{Z}^x \mid |s| = s\}$.

An $x$-partition is an $x$-tuple of partitions. Denote $\Pi^x$ the set of all $x$-partitions. Let $s = (s_1, \ldots, s_x) \in \mathbb{Z}^x$. The set of $x$-abaci with charge $s$ is in bijection with the set of $x$-partitions via the map $\mathcal{A} \mapsto \lambda = ((\lambda_1^{(1)}, \lambda_2^{(1)}, \ldots), \ldots, (\lambda_1^{(\pi)}, \lambda_2^{(\pi)}, \ldots))$ defined by

$$(\beta, j) \mapsto \lambda_k^{(j)} = \beta - s_j + k - 1$$

for all $(\beta, j) \in \mathcal{A}$.

We write $|\lambda, s|$ for the data consisting of an element $s \in \mathbb{Z}^x$ and an $x$-partition $\lambda$, and call it a charged $x$-partition. Further, we write $\mathcal{A} = \mathcal{A}(\lambda, s)$ for the corresponding $x$-abacus, and will often identify $\mathcal{A}$ with $|\lambda, s|$.

**Example 1.1.** Let $\ell = 3$, $\lambda = (10,9,1,9,3,7,6,4,1,6,3^2)$ be an $\ell$-abacus and $s = (-4,0,-5)$ a charge. The horizontal representation of $\mathcal{A}(\lambda, s)$ is given in Figure 1. The dashed line is placed at position $\frac{1}{2}$ in order to keep track of the horizontal grading.

1.2. Level-rank duality. Let $\mathcal{A}$ be an $\ell$-abacus with charge $s$ and consider the following procedure. Represent $\mathcal{A}$ horizontally. Stack copies of $\mathcal{A}$ on top of each other, so that each new copy is shifted $e$ steps to the right. This results in a filling of $\mathbb{Z}^2$ by white and black beads, denoted by $\overline{\mathcal{A}}$. Choose $e$ consecutive columns of beads such that the index of the rightmost column is divisible by $e$. This is the vertical representation of a unique abacus, which we denote $\overline{\mathcal{A}}$. The corresponding charge $\overline{s}$ verifies $|\overline{s}| = -|s|$. We denote $\lambda$ the $e$-partition such that $\mathcal{A}(\lambda, \overline{s}) = \overline{\mathcal{A}}$.

It is easy to see that the map

$$\Pi^x \times \mathbb{Z}^x(s) \longrightarrow \Pi^x \times \mathbb{Z}^x(-s)$$

$$(\lambda, s) \longmapsto (\lambda, \overline{s})$$

is a bijection. We call it the **level-rank duality**.

\[\text{Figure 1. Drawing the horizontal abacus from a charged multipartition.}\]
Example 1.2. Take \( \ell = 4, e = 3, \lambda = (3.1, -1^3, 4.2.1) \) and \( s = (0, 1, 1, 2) \). Then the level-rank duality is illustrated in Figure 2.

Remark 1.3. A slightly simpler version of the combinatorial level-rank duality was introduced by Uglov [20], and used in the works of Tingley [19] and Foda and Welsh [5]. Here, we have twisted Uglov’s level-rank duality by taking the transpose, in the spirit of [9]. This is essential to prove Theorem 3.3 since we use the commutation of the crystals [9, Theorem 4.8] that requires this convention.

2. CYLINDRIC AND FLOTW MULTIPARTITIONS

2.1. Cylindricity and FLOTW property. Let \( e, \ell \in \mathbb{Z}_{>1} \) and \( s \in \mathbb{Z} \). Set

\[
D(s) = \left\{ (s_1, \ldots, s_\ell) \in \mathbb{Z}_\ell \left| \sum_{j=1}^\ell s_j = s \text{ and } s_1 \leq s_2 \leq \cdots \leq s_\ell \leq s_1 + e \right. \right\}.
\]

Note that this set depends on \( \ell \) and \( e \). Further, for \( \lambda = ((\lambda_1^{(1)}, \lambda_2^{(1)}), \ldots, (\lambda_1^{(2)}, \lambda_2^{(2)}), \ldots, (\lambda_1^{(\ell)}, \lambda_2^{(\ell)})) \in \Pi^\ell, s \in \mathbb{Z}_\ell \) and \( \alpha \in \mathbb{Z}_{>0} \), denote

\[
\mathcal{R}(\lambda, s, \alpha) = \left\{ \lambda_k^{(j)} - k + s_j \mod e \mid \lambda_k^{(j)} = \alpha ; \ 1 \leq j \leq \ell, k \geq 1 \right\}.
\]

Definition 2.1. Let \( s \in D(s) \) and \( \lambda \) be an \( \ell \)-partition.

1. We say that \( |\lambda, s\rangle \) is e-cylindric if \( \forall 1 \leq j \leq \ell - 1, \lambda_k^{(j)} \geq \lambda_k^{(j+1)} \forall k \geq 1 \) and \( \lambda_k^{(\ell)} \geq \lambda_k^{(1)} + e + s_1 - s_\ell \forall k \geq 1 \).

2. We say that \( \lambda \) is e-FLOTW if the two following conditions hold:
   (a) \( |\lambda, s\rangle \) is e-cylindric.
   (b) For all \( \alpha \in \mathbb{Z}_{>0} \), \( \mathcal{R}(\lambda, s, \alpha) \neq \{0, \ldots, e - 1\} \).
Iterate for each remaining black bead of row $\beta$ beads (at black yoke (respectively a white yoke). We claim that $A^{(\ell)}_j$ if

$A^{(\ell)}$ has its horizontal representation. Simply put, $A^{(\ell)}$ is just a collection of $\ell+1$ consecutive rows of $A$.

The following procedure is inspired by [19, Section 3.3] and [5, Section 4]. In $A$, yoke the leftmost white beads in every row to each other, and repeat the procedure recursively on the remaining unyoked white beads.

**Remark 2.2.** FLOTW multipartitions were introduced in [4] in order to give a combinatorial realization of

irreducible highest weight crystals for the quantum group associated to the affine Kac-Moody algebra $\widehat{sl}_e$, see

[4] Theorem 2.10 [4].

2.2. **Yoking beads.** Cylindricity can be easily detected on an abacus whose charge $s$ is in $D(s)$. Let $A'$ be the $(\ell+1)$-abacus defined by $(\beta, j) \in A' \iff (\beta, j) \in A$ if $j \leq \ell$ and $(\beta, \ell + 1) \in A' \iff (\beta - e, 1) \in A$, and consider its horizontal representation. Simply put, $A'$ is just a collection of $\ell+1$ consecutive rows of $A$.

The following procedure is inspired by [19, Section 3.3] and [5, Section 4]. In $A'$, yoke the leftmost white beads in every row to each other, and repeat the procedure recursively on the remaining unyoked white beads.

**Example 2.3.** Take $\ell = 4$, $e = 3$, $\lambda = (3.1, \ldots, 1^3, 4.2.1)$ and $s = (0, 1, 1, 2)$. The yoking procedure is illustrated in Figure 5

**Lemma 2.4.** The abacus $A$ is cylindric if and only every pair of yoked white beads $((\beta_1, j + 1), (\beta_2, j))$ in $A'$

verifies $\beta_1 \geq \beta_2$.

In other words, an $\ell$-abacus is cylindric if and only if its charge is in $D(s)$ and all yokes have a “north-east/south-west” direction.

**Proof.** Consider the following dual yoking procedure on $A'$. Let $\beta_0$ be the index of the rightmost column of $A'$ such that $(\beta, j)$ is black for all $\beta \leq \beta_0$ and for all $j = 1, \ldots, \ell + 1$. For all $\beta \leq \beta_0$, yoke the black beads in consecutive rows of column $\beta$ to each other. By construction, there is a black bead in row $\ell + 1$ which is not yoked yet. Yoke the leftmost such bead to the leftmost black bead in row $\ell$ which is not yoked yet (if it exists). Iterate for each remaining black bead of row $\ell + 1$ of $A'$. Since $s \in D(s)$, the only black beads that remain unyoked appear on row $\ell + 1$. Figure 4 illustrates the dual yoking procedure for the abacus of Example 2.3.

To differentiate the two yoking procedures, let us call a yoke between black beads (respectively white beads) a black yoke (respectively a white yoke). We claim that $A$ is cylindric if and only if every pair of yoked black beads $((\beta_1, j + 1), (\beta_2, j))$ in $A'$ verifies $\beta_1 \leq \beta_2$. Indeed, if $1 \leq j \leq \ell - 1$, we can use the bijection between $\ell$-abaci and $\ell$-partitions given by Formula (1.1). In this case, for yoked beads $((\beta_1, j + 1), (\beta_2, j))$ in $A'$, we have

$$\beta_1 \leq \beta_2 \iff \beta_1 - s_{j+1} + (k + s_{j+1} - s_j) - 1 \leq \beta_2 - s_j + k - 1$$

$$\iff \lambda^{(j)}_{k+s_{j+1}-s_j} \leq \lambda^{(j)}_k.$$

If $j = \ell$, we have to substract $e$ to $\beta_1$ by definition of $A'$. So in this case, we have

$$\beta_1 \leq \beta_2 \iff \beta_1 - e + s_1 + (k + e + s_1 - s_\ell) - 1 \leq \beta_2 - s_\ell + k - 1$$

$$\iff \lambda^{(1)}_{k+e+s_1-s_\ell} \leq \lambda^{(0)}_k.$$
which proves the claim. In other words, $\mathcal{A}$ is cylindric if and only if all black yokes have a “north-west/south-east” direction. Finally, we claim that there is a black yoke between $(\beta_1, j + 1)$ and $(\beta_2, j)$ with $\beta_1 > \beta_2$ if and only if there is a white yoke between $(\beta'_1, j + 1)$ and $(\beta'_2, j)$ for some $\beta'_1, \beta'_2$ such that $\beta'_1 < \beta'_2$. To see this, assume first that such a black yoke exists and consider the leftmost black yoke verifying this property, with minimal $j$. Denote $\delta = \beta_1 - \beta_2 > 0$. Then by minimality, there is a white bead in position $(\beta_1 - 1, j + 1)$. Set $\beta'_1 = \beta_1 - 1$. This bead is yoked to another white bead (since every white bead belongs to a white yoke) in row $j$. By definition of the black yokes, there is the same number of black beads to the left of $(\beta_1, j + 1)$ and to the left of $(\beta_2, j)$. Therefore, by definition of the white yokes, the white bead in row $j$ which is yoked to $(\beta'_1, j + 1)$ has position $(\beta'_2, j)$ with $\beta'_2 \geq \beta_2 + \delta$. Thus, $\beta'_2 = \beta_2 + (\beta_1 - \beta_2) = \beta_1 = \beta'_1 + 1$, i.e. $\beta'_2 > \beta'_1$. Conversely, assume there is a white yoke between $(\beta'_1, j + 1)$ and $(\beta'_2, j)$ with $\beta'_1 < \beta'_2$, and consider the rightmost such yoke, with maximal $j$. By maximality, there is a black bead in position $(\beta'_1 + 1, j + 1)$. Set $\beta_1 = \beta'_1 + 1$. If this bead is yoked to another bead, then the same argument as above applies, and the black bead in row $j$ to which it is yoked is $(\beta'_2, j)$ with $\beta'_2 < \beta_1$. If this bead is not yoked, then it belongs to row $\ell + 1$, but then the white bead $(\beta'_1, \ell + 1)$ is obviously yoked to $(\beta'_1 - e, 1)$. In turn, all the yokes $(\beta', j)$ that connect $(\beta'_1, \ell + 1)$ and $(\beta'_1 - e, 1)$ verify $\beta' < \beta'_1$, which contradicts the hypothesis. 

**Remark 2.5.**

1. The argument of the above proof show that cylindricity for multipartitions behaves nicely with respect to taking the transpose.

2. In [5], Foda and Welsh use a different convention for representing multipartitions by abaci. One recovers their convention by taking the transpose. Also, they only consider abaci corresponding to $e$-cylindric multipartitions, and thus do not have an equivalent statement to Lemma 2.4.

3. In [19], Tingley also uses a different convention, but considers those abaci that essentially verify the combinatorial property of Lemma 2.4. He calls them “descending abaci”. By Lemma 2.4, descending abaci and cylindric abaci are the same (up to the twist in conventions).

**Example 2.6.** Take $\ell = 2$, $e = 4$, $\lambda = (3^2.1, 4, 3.2)$ and $s = (1, 2)$. Then $|\lambda,s|$ is cylindric, as the yoking procedure in Figure 5 shows.

### 3. Crystal characterizations

Following [9], there are three commuting crystal structures on the set of $\ell$-partitions:

- an $\widetilde{\mathfrak{sl}}_\ell$-crystal arising from the integrable action of the quantum group $\mathcal{U}_q(\mathfrak{sl}_\ell)$ on the $\nu$-deformed level $\ell$ Fock space of [15].

- an $\mathcal{H}$-crystal arising from the action of the quantum Heisenberg algebra on the same space [20], [8].

- an $\tilde{\mathfrak{sl}}_\ell$-crystal arising from the integrable action of $\mathcal{U}_{1/\hbar}(\mathfrak{sl}_\ell)$ on an appropriate direct sum of $\nu$-deformed level $\ell$ Fock spaces via level-rank duality [12], [20], [9].

To define these different module structures, one has to fix a charge parameter $s \in \mathbb{Z}'$. Without restriction, we choose $s \in \mathbb{Z}'(s)$. We do not recall here the explicit formulas for computing the different crystals. For the $\widetilde{\mathfrak{sl}}_\ell$-crystal, we refer to [7, Chapter 6]. The case of $\tilde{\mathfrak{sl}}_\ell$-crystal is given by the same formula, after switching the role of $\ell$ and $e$ and using Correspondence 1.2. For the $\mathcal{H}$-crystal, the complete explicit formulas have been given recently in [10].

These crystals are certainly oriented colored graphs, whose vertices are the $\ell$-partitions, and each of whose connected components have a unique source vertex. Sources in the $\widetilde{\mathfrak{sl}}_\ell$-crystal have a simple characterization. In order to state it, we recall the notion of periods in an abacus. The first $e$-period in $\mathcal{A}$ is, if it exists, the sequence $P = ((\beta_1, j_1), \ldots, (\beta_e, j_e))$ of $e$ elements in $\mathcal{A}$ such that
The first period of $\mathcal{A}\setminus P$, if it exists, is called the second period of $\mathcal{A}$. We define similarly the $k$-th period of $\mathcal{A}$ by induction. An $\ell$-abacus is called totally $e$-periodic if it has infinitely many $e$-periods. The following result was proved by Jacon and Lecouvey [14, Theorem 5.9].

**Theorem 3.1.** An $\ell$-abacus is a source in the $\hat{\mathcal{sl}}_{\ell}$-crystal if and only if it is totally $e$-periodic.

An analogous result for the $\mathcal{H}$-crystal is given in [10, Theorem 4.15 and Example 4.18]. We will not need it here.

**Example 3.2.** Let $e = \ell = 3$, $\lambda = (1, 2^2, 4^2)$ and $s = (1, 2, 3)$. The abacus $\mathcal{A}(\lambda, s)$ is represented horizontally in Figure 6. We have yoked black beads belonging to the same $e$-period. One sees that $\mathcal{A}(\lambda, s)$ is totally $e$-periodic.

We are ready to prove the following result relating cylindricity (respectively FLOTW property) and sources in crystals.

**Theorem 3.3.** Let $\mathcal{A}$ be an $\ell$-abacus.

1. $\mathcal{A}$ is $e$-cylindric if and only if $\hat{\mathcal{A}}$ is a source in the $\hat{\mathcal{sl}}_{\ell}$-crystal.
2. $\mathcal{A}$ is $e$-FLOTW if and only if $\hat{\mathcal{A}}$ is a source in the $\hat{\mathcal{sl}}_{\ell}$-crystal and in the $\mathcal{H}$-crystal.

**Proof.**

(1) Assume $\mathcal{A}$ is $e$-cylindric and extend the yoking procedure of Section 2.2 to $\hat{\mathcal{A}}$. By Lemma 2.4, all white yokes have a “north-east/south-west” direction, so that every yoke in $\mathcal{A}$ spreads on at most $e$ columns. In fact, slicing $\hat{\mathcal{A}}$ horizontally and then vertically shows that every white yoke in $\hat{\mathcal{A}}$ corresponds to a period in $\hat{\mathcal{A}}$. Therefore, $\hat{\mathcal{A}}$ is totally periodic, and the result follows by Theorem 3.1. Conversely, if one represents a totally periodic abacus vertically, periods corresponds to yokes in the horizontal level-rank dual which all have a “north-east/south-west” direction.

(2) This is a direct consequence of [9, Theorem 6.19] and has been shown in [9, Proof of Theorem 7.7] already. Alternatively, one can use the combinatorial characterizations of (1) and [10, Theorem 4.15] to recover this result directly.

**Remark 3.5.** In [19, Definition 3.8], Tingley defines the notion of a “tight” descending abacus. It is proved in [8, Proposition 5.15] that Tingley’s tightening operators correspond to the raising Heisenberg crystal operators. Moreover, we already observed in Remark 2.5(3) that descending abaci correspond to cylindric multipartitions. Therefore by Theorem 3.3(2), tight descending abaci correspond to FLOTW multipartitions. This shall come as no surprise because Tingley’s tight descending abaci are constructed to realize irreducible highest weight crystals [19, Theorem 3.14], and so are FLOTW multipartitions (as mentioned in Remark 2.2).
Figure 7. The level-rank dual of an $e$-cylindric $\ell$-abacus is totally $\ell$-periodic.

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