BIJECTIONS ON TWO VARIATIONS OF NONCROSSING PARTITIONS

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Abstract. We find bijections on 2-distant noncrossing partitions, 12312-avoiding partitions, 3-Motzkin paths, UH-free Schröder paths and Schröder paths without peaks at even height. We also give a direct bijection between 2-distant noncrossing partitions and 12312-avoiding partitions.

1. Introduction

A (set) partition of $[n] = \{1, 2, \ldots, n\}$ is a collection of mutually disjoint nonempty subsets, called blocks, of $[n]$ whose union is $[n]$. We will write a partition as a sequence of blocks $(B_1, B_2, \ldots, B_k)$ such that $\min(B_1) < \min(B_2) < \cdots < \min(B_k)$.

There are two natural representations of a partition. Let $\pi = (B_1, B_2, \ldots, B_k)$ be a partition of $[n]$. The partition diagram of $\pi$ is the simple graph with vertex set $V = [n]$ and edge set $E$, where $(i, j) \in E$ if and only if $i$ and $j$ are in the same block which does not have an integer between them. For example, see Figure 1. The canonical word of $\pi$ is the word $a_1a_2\cdots a_n$, where $a_i = j$ if $i \in B_j$. For instance, the canonical word of the partition in Figure 1 is 123124412. For a word $\tau$, a partition is called $\tau$-avoiding if its canonical word does not contain a subword which is order-isomorphic to $\tau$.

A partition is noncrossing if the edges of its partition diagram do not intersect. It is easy to see that a partition is noncrossing if and only if it is 1212-avoiding.

Let $\pi$ be a partition and let $k$ be a nonnegative integer. A $k$-distant crossing of $\pi$ is a set of two edges $(i_1, j_1)$ and $(i_2, j_2)$ of the partition diagram of $\pi$ satisfying $i_1 < i_2 \leq j_1 < j_2$ and $j_1 - i_2 \geq k$. A partition $\pi$ is called $k$-distant noncrossing if it has no $k$-distant crossings. Note that 1-distant noncrossing partitions are just noncrossing partitions.

Our main objects are 2-distant noncrossing partitions and 12312-avoiding partitions. Let $\text{NC}_2(n)$ denote the set of 2-distant noncrossing partitions of $[n]$. Let $P_{12312}(n)$ denote the set of 12312-avoiding partitions of $[n]$.

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Figure 1. The partition diagram of $\{(1, 4, 8), (2, 5, 9), (3), (6, 7)\}$. 

1 2 3 4 5 6 7 8 9
Drake and Kim \[1\] found the following generating function for the number of 2-distant noncrossing partitions:

\[
\sum_{n \geq 0} \#NC_2(n)x^n = \frac{3 - 3x - \sqrt{1 - 6x + 5x^2}}{2(1 - x)}.
\]

Using the kernel method, Mansour and Severini \[4\] found the generating function for the number of 12312-avoiding partitions of \([n]\). Interestingly, as a special case of their result, the generating function for the number of 12312-avoiding partitions of \([n]\) is the same as \([1]\), which implies \(\#NC_2(n) = \#P_{12312}(n)\). Moreover, this number also counts several kinds of lattice paths.

A lattice path of length \(n\) is a sequence of integer lattice points \((i, j)\) in the xy-plane starting with \((0, 0)\) and ending with \((n, 0)\). A lattice path is nonnegative if it never goes below the x-axis. Since we will only consider nonnegative lattice paths, throughout this paper, we will write simply a lattice path instead of a nonnegative lattice path.

Let \(L = ((x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n))\) be a lattice path. Then each \(S_i = (x_i - x_{i-1}, y_i - y_{i-1})\) is a step of \(L\). The height of the step \(S_i\) is defined to be \(y_i - 1\). Sometimes we will identify a lattice path \(L\) with the word \(S_1S_2\cdots S_k\) of its steps. Note that the number of steps is not necessarily equal to the length of the lattice path.

Let \(U\), \(D\) and \(H\) denote an up step, a down step and a horizontal step respectively, i.e., \(U = (1, 1)\), \(D = (1, -1)\) and \(H = (1, 0)\).

A Schröder path of length \(2n\) is a lattice path of length \(2n\) consisting of steps \(U\), \(D\) and \(H = HH = (2, 0)\). Let \(L = S_1S_2\cdots S_k\) be a Schröder path. A UH-pair of \(L\) is a pair \((S_i, S_{i+1})\) of consecutive steps such that \(S_i = U\) and \(S_{i+1} = H\). We say that \(L\) is UH-free if it does not have a UH-pair. A peak of \(L\) is a pair \((S_i, S_{i+1})\) of consecutive steps such that \(S_i = U\) and \(S_{i+1} = D\). The height of a peak \((S_i, S_{i+1})\) is the height of \(S_{i+1} = D\).

Let \(\text{SCH}_{\text{UH}}(n)\) denote the set of UH-free Schröder paths of length \(2n\). Let \(\text{SCH}_{\text{even}}(n)\) (resp. \(\text{SCH}_{\text{odd}}(n)\)) denote the set of Schröder paths of length \(2n\) which have no peaks of even (resp. odd) height.

A labeled step is a step together with an integer label. Let \(D_i\) (resp. \(H_i\)) denote a labeled down step (resp. a labeled horizontal step) with label \(i\).

Let \(\text{CH}_2(n)\) denote the set of lattice paths \(L = S_1S_2\cdots S_n\) of length \(n\) consisting of \(U\), \(D_1, D_2, H_0, H_1\) and \(H_2\) such that

- if \(S_i = H_\ell\) or \(S_i = D_\ell\), then \(S_i\) is of height at least \(\ell\),
- if \(S_i = H_2\) or \(S_i = D_2\), then \(i \geq 2\) and \(S_{i-1} \in \{U, H_1, H_2\}\).

Drake and Kim \[1\] showed that the well known bijection \(\varphi\) between partitions and Charlier diagrams, see \[2\] \[3\], yields a bijection \(\varphi : \text{NC}_2(n) \to \text{CH}_2(n)\). Yan \[8\] found a bijection \(\varphi : \text{SCH}_{\text{UH}}(n-1) \to P_{12312}(n)\) and a bijection between \(\text{SCH}_{\text{UH}}(n)\) and \(\text{SCH}_{\text{even}}(n)\).

Thus the cardinalities of \(\text{NC}_2(n)\), \(\text{CH}_2(n)\), \(\text{SCH}_{\text{even}}(n-1)\), \(\text{SCH}_{\text{UH}}(n-1)\) and \(P_{12312}(n)\) are the same whose sequence appears as A007317 in \[6\].

Let us define the following refined sets.

- \(\text{NC}_2'(n) = \{\pi \in \text{NC}_2(n) : n\ is\ not\ a\ singleton\}\)
- \(\text{CH}_2'(n) = \{L \in \text{CH}_2(n) : \text{the\ last\ step\ of}\ L\ is\ D_1\}\)
- \(\text{SCH}^\prime_{\text{even}}(n) = \{L \in \text{SCH}_{\text{even}}(n) : \text{the\ first\ step\ of}\ L\ is\ U\}\)
- \(\text{SCH}^\prime_{\text{UH}}(n) = \{L \in \text{SCH}_{\text{UH}}(n) : \text{the\ first\ step\ of}\ L\ is\ U\}\)
There are obvious bijections proving the following. For $n \geq 2$,

\[
\#NC_2(n) = 1 + \sum_{i=2}^{n} \#NC_2'(i),
\]

\[
\#CH_2(n) = 1 + \sum_{i=2}^{n} \#CH_2'(i),
\]

\[
\#SCH_{even}(n-1) = 1 + \sum_{i=2}^{n} \#SCH_{even}^{'(i-1)},
\]

\[
\#SCH_{UH}(n-1) = 1 + \sum_{i=2}^{n} \#SCH_{UH}^{'}(i-1),
\]

\[
\#P_{12312}(n) = 1 + \sum_{i=2}^{n} \#P_{12312}^{'}(i).
\]

Thus, for $n \geq 2$, the cardinalities of the following sets are also the same, whose sequence appears as A002212 in [6] : $NC_2'(n)$, $CH_2'(n)$, $SCH_{even}(n-1)$, $SCH_{UH}(n-1)$ and $P_{12312}'(n)$. This number is also equal to both $\#SCH_{odd}(n-1)$ and the number of 3-Motzkin paths of length $n - 2$, where a 3-Motzkin path of length $n$ is a lattice path of length $n$ consisting of $U$, $D$, $H_0$, $H_1$ and $H_2$.

Let $MOT_3(n)$ denote the set of 3-Motzkin paths of length $n$. Our main purpose is to find bijections between these objects. For the overview of our bijections see Figure 2 where $\varphi$ is the known bijection between partitions and Charlier diagrams [2, 3], and $\phi$ is Yan’s bijection [8]. We note that our bijection $g$ in Figure 2 is also discovered by Shapiro and Wang [5]. We also provide a direct bijection between $NC_2(n)$ and $P_{12312}(n)$.

The rest of this paper is organized as follows. In Section 2 we find bijections in Figure 2. In Section 3 we review Yan’s bijection and its consequences for self containment. In Section 4 we obtain refined results following from our bijections. In Section 5 we provide a direct bijection between $NC_2(n)$ and $P_{12312}(n)$.

2. Bijections

In this section we find the bijections $f, g, h$ and $\iota$ in Figure 2.

2.1. The bijection $f : CH_2'(n) \rightarrow MOT_3(n - 2)$. Recall that $CH_2'(n)$ is the set of lattice paths $L = S_1S_2\cdots S_n$ of length $n$ consisting of $U$, $D_1$, $D_2$, $H_0$, $H_1$ and $H_2$ such that

- if $S_i = H_1$ or $S_i = D_1$, then $S_i$ is of height at least $\ell$,
- if $S_i = H_2$ or $S_i = D_2$, then $i \geq 2$ and $S_{i-1} \in \{U, H_1, H_2\}$,
Theorem 2.1. The map \( f : \text{CH}_2(n) \rightarrow \text{MOT}_3(n-2) \) is a bijection.
See Figure 5 for an example of \( g \). Let \( S \) correspond to \( H \) Schröder paths. Thus \( g \) is invertible.

**Proof.** Each \( L \in \text{MOT}_3(n-2) \) is uniquely decomposed as

\[
H_2^{k_1} L_1 (H_1 H_2^{k_2+1} L_2) (H_1 H_2^{k_3+1} L_3) \cdots (H_1 H_2^{k_r+1} L_r),
\]

where \( L_i \in B(n_i) \) for some \( k_i, n_i \geq 0 \) and \( r \geq 1 \). Thus we have the inverse \( f^{-1}(L) \) which is decomposed as

\[
H_2^{k_1} (U f_0^{-1}(L_1) D_1) H_2^{k_2} (U f_0^{-1}(L_2) D_1) \cdots H_2^{k_r} (U f_0^{-1}(L_r) D_1).
\]

\( \square \)

### 2.2. The bijection \( g : \text{MOT}_3(n) \to \text{SCH}_{\text{odd}}(n+1) \)

We define \( g : \text{MOT}_3(n) \to \text{SCH}_{\text{odd}}(n+1) \) as follows. Let \( L \in \text{MOT}_3(n) \). Then \( g(L) \) is the lattice path obtained from \( L \) by doing the following.

1. Change \( U \) to \( UU \), \( D \) to \( DD \), \( H_0 \) to \( H^2 \), \( H_1 \) to \( DU \), and \( H_2 \) to \( UD \).
2. Add \( U \) at the beginning and \( D \) at the end.
3. Change all the consecutive steps \( UD \) which form a peak of odd height to \( H^2 \).

See Figure 5 for an example of \( g \).

**Theorem 2.2.** The map \( g : \text{MOT}_3(n) \to \text{SCH}_{\text{odd}}(n+1) \) is a bijection.

**Proof.** Clearly the first and the second steps in the construction of \( g \) are invertible. The third step is also invertible because every step \( H^2 \) of even height always comes from a peak of odd height. Thus \( g \) is invertible. \( \square \)

### 2.3. The bijection \( h : \text{SCH}_{\text{odd}}(n) \to \text{SCH}'_{\text{BH}}(n) \)

Let \( L = S_1 S_2 \cdots S_k \) be a Schröder path. Let \( S_i = U \) be an up step. Then there is a unique down step \( S_j = D \) such that \( i < j \) and \( S_i+1 S_{i+2} \cdots S_{j-1} \) is a (possibly empty) lattice path. We call such \( S_j \) the down step corresponding to \( S_i \). We also call \( S_i \) the up step corresponding to \( S_j \).
In the procedure of Proof, we always get $\xi(S_i, S_{i+1})$ such that $\xi(S_i, S_{i+1}) > \xi(S_{i'}, S_{i'+1})$. Let $(S_i, S_{i+1})$ be a UH-pair and let $S_j$ be the down step corresponding to $S_i$. Then we define the function $\xi$ as follows.

$$\xi(S_i, S_{i+1}) = \begin{cases} i, & \text{if } S_{i+1} \text{ is of even height;} \\ j, & \text{if } S_{i+1} \text{ is of odd height.} \end{cases}$$

If $L$ is not UH-free, we define the $\xi$-maximal UH-pair of $L$ to be the UH-pair $(S_i, S_{i+1})$ satisfying the following: for any UH-pair $(S_{i'}, S_{i'+1})$ with $i' \neq i$, we have $\xi(S_i, S_{i+1}) > \xi(S_{i'}, S_{i'+1})$.

Let $L \in \text{SCH}_{\text{odd}}(n)$. Assume that $L$ is not UH-free. Let $(S_i, S_{i+1})$ be the $\xi$-maximal UH-pair of $L$. Let $S_j$ be the down step corresponding to $S_i$. Then we define $h_0(L)$ as follows.

1. If $S_{i+1}$ is of even height, then $h_0(L)$ is the lattice path obtained from $L$ by replacing $S_iS_{i+1}$ with $U\varepsilon H$.
2. If $S_{i+1}$ is of odd height, then let $L' = S_{i+2}S_{i+3} \cdots S_{j-1}$.
   a. If $L'$ is empty, i.e., $j = i + 2$, then $h_0(L)$ is the lattice path obtained from $L$ by replacing $S_iS_{i+1}S_{i+2}$ with $H^2UD$.
   b. If $L'$ is not empty, then $h_0(L)$ is the lattice path obtained from $L$ by replacing $S_iS_{i+1} \cdots S_j$ with $U\varepsilon DUD$.

See Figure 6.

Now we define $h : \text{SCH}_{\text{odd}}(n) \to \text{SCH}_{\text{UH}}(n)$ as follows. Let $L \in \text{SCH}_{\text{odd}}(n)$ and $L_0 = L$. Then we define $L_i = h_0(L_{i-1})$ for $i \geq 1$ if $L_{i-1}$ is not UH-free. Since the number of UH-free pairs of $L_i$ is one less than that of $L_{i-1}$, or they are the same and

$$\xi(\text{the maximal UH-pair of } L_i) < \xi(\text{the maximal UH-pair of } L_{i-1}),$$

we always get $L_r$ which is UH-free for some $r$. We define $h(L)$ to be $L_r$ if $L_r$ does not start with $H^2$; and the lattice path obtained from $L_r$ by replacing $H^2$ with $UD$ otherwise. For example, see Figure 7.

**Theorem 2.3.** The map $h : \text{SCH}_{\text{odd}}(n) \to \text{SCH}_{\text{UH}}(n)$ is a bijection.

**Proof.** In the procedure of $h$, the odd peaks are constructed from right to left. Since $h_0$ is invertible, so is $h$. \hfill $\square$
2.4. The bijection $\iota : \text{SCH}_{\text{odd}}(n) \rightarrow \text{SCH}_{\text{even}}'(n)$. Let $L = S_1 S_2 \cdots S_k \in \text{SCH}_{\text{odd}}(n)$. Then $\iota(L)$ is defined as follows.

1. If $S_k = H^2$, then $\iota(L) = US_1 \cdots S_{k-1}D$.
2. If $S_k = D$, then let $S_i$ be the up step corresponding to $S_k$ and we define $\iota(L) = US_1 \cdots S_{i-1}DS_{i+1} \cdots S_{k-1}$.

Figure 7. An example of $h$. Green (resp. Red) color is for UH-pairs with $H^2$ of even (resp. odd) height. Odd peaks are circled. Dashed arrows indicate the down steps corresponding to the up steps.
Figure 8. The map $\iota$.

See Figure 8.

Then $\iota(L) \in \text{SCH}_{\text{even}}'(n)$. Clearly, $\iota: \text{SCH}_{\text{odd}}(n) \to \text{SCH}_{\text{even}}'(n)$ is a bijection.

3. Yan’s bijection

In this section we describe Yan’s bijection $\phi: \text{SCH}_{\text{UH}}(n-1) \to P_{12312}'(n)$.

Let $L \in \text{SCH}_{\text{UH}}(n-1)$. Then we label each step of $L$ as follows.

1. Find all the peaks of $L$, and label the up step of the $i$-th leftmost peak with $i + 1$ for $i = 1, 2, \ldots$.
2. We label all the unlabeled up steps and the horizontal steps with the largest integers which are left to the steps. If there is no integer which is left to such a step, then we label it with 1.
3. For each down step $D$, we label it with the same integer which is the label of the up step corresponding to $D$.

Then $\phi$ is defined to be the word obtained by reading the labels of downs steps and horizontal steps of $L$ from left to right and by adding 1 at the beginning. See Figure 9 for an example of this map.

Let $\pi = (B_1, B_2, \ldots, B_r)$ be a partition. A pair $(B_i, a)$ of a block and an integer is called inversion if $a > \min(B_i)$ and $a \in B_j$ and for some $j < i$. Let $\text{inv}(B_i)$ denote the number of inversions of form $(B_i, a)$.

From the definition of $\phi$, we get the following immediately for $L$ and $\pi$ with $\phi(L) = \pi$.

- The first step of $L$ is a horizontal step if and only if 1 is connected to 2 in $\pi$.
- If $\pi = (B_1, \ldots, B_k)$, then $L$ has $k - 1$ peaks and their heights are $\text{inv}(B_2) + 1, \text{inv}(B_3) + 1, \ldots, \text{inv}(B_k) + 1$.

Thus $\phi$ is indeed a bijection between $\text{SCH}_{\text{UH}}'(n-1)$ and $P_{12312}'(n)$.

Using this bijection Yan proved the following theorem.

**Theorem 3.1.** Let $n, k \geq 1$. Then the number of 12312-avoiding partitions of $[n+1]$ with $k + 1$ blocks is equal to

$$
\sum_{i=k}^{n} \frac{1}{i} \binom{i}{k-1} \binom{n}{i}.
$$
Label the up steps of peaks

Label the unlabeled up steps and horizontal steps

Label all the down steps

Read the labels of down steps and horizontal steps and add 1 at the beginning

1 1 2 3 3 4 2 1 5 4 4 5

Figure 9. Yan's bijection

Figure 10. All the five crossings are high crossings.

4. Refined results

Let $\pi \in \text{NC}_2'(n)$. Let $\{(a, i + 1), (i, b)\}$ be a crossing of $\pi$. Let $A$ and $B$ be the blocks containing $a$ and $b$ respectively. Let $r = \min(\max(A), \max(B))$. This crossing is called low crossing if it satisfies the following:

- $(a, i + 1)$ is not nested by another edge,
- $i, i+1, \ldots, r$ are contained in $B$ and $A$ alternatively, that is, $i \in B, i+1 \in A, i+2 \in B$, and so on.

Otherwise, it is called high crossing. For example, see Figure 10 and Figure 11.

Let $L \in \text{CH}_2'(n)$. A step $D_2$ of $L$ is low if its height is 2. Otherwise it is high. A step $H_2$ of $L$ is low if the first step right to it, which is not $H_2$, is a low $D_2$. Otherwise it is high. A high step is either a high $D_2$ or a high $H_2$.
Let us define the following refined sets.

- $\text{NC}_2'(n, k) = \{ \pi \in \text{NC}_2'(n) : \pi \text{ has } k \text{ high crossings} \}$
- $\text{CH}_2'(n, k) = \{ L \in \text{CH}_2'(n) : L \text{ has } k \text{ high steps} \}$
- $\text{MOT}_3(n - 2, k) = \{ L \in \text{MOT}_3(n - 2) : L \text{ has } k \text{ } H_2 \text{'s of height } > 0 \}$
- $\text{SCH}_{\text{odd}}(n - 1, k) = \{ L \in \text{SCH}_{\text{odd}}(n - 1) : L \text{ has } k \text{ peaks of even height } > 2 \}$
- $\text{SCH}_{\text{UH}}(n - 1, k) = \{ L \in \text{SCH}_{\text{UH}}(n - 1) : L \text{ has } k \text{ peaks of even height } > 2 \}$
- $P'_{12312}(n, k) = \{ \pi \in P'_{12312}(n) : \pi \text{ has } k \text{ blocks } B_i \text{ with odd } \text{inv}(B_i) > 1 \}$

It is easy to check that our maps in Figure 2 induce bijections on the sets described above.

Let $p(n, k)$ be the number of one of the objects described above. Let $a(n, k)$ be the number of 3-Motzkin paths of length $n$ with $k$ $H_2$ steps. Let $b(n, k)$ be the number of 3-Motzkin paths of length $n$ with $k$ $H_2$ steps of height at least 1. Let

$$A(x, y) = \sum_{n \geq 0} \sum_{m \geq 0} a(n, m)x^ny^m,$$
$$B(x, y) = \sum_{n \geq 0} \sum_{m \geq 0} b(n, m)x^ny^m,$$
$$F(x, y) = \sum_{n \geq 2} \sum_{m \geq 0} p(n, m)x^ny^m.$$ 

Then it is easy to see that

$$A(x, y) = 1 + (2 + y)xA(x, y) + x^2A(x, y)^2,$$
$$B(x, y) = 1 + 3xB(x, y) + x^2B(x, y)A(x, y),$$
$$F(x, y) = x^2 \cdot B(x, y).$$

Solving these equations, we get

$$F(x, y) = \frac{2x^2}{1 - (4 - y)x + \sqrt{1 - (4 + 2y)x + (4y + y^2)x^2}}.$$ 

If $y = 0$, then

$$F(x, 0) = \frac{2x^2}{1 - 4x + \sqrt{1 - 4x}} = \frac{2x^2}{\sqrt{1 - 4x}} \left( \frac{1}{1 + \sqrt{1 - 4x}} \right) = \frac{2x^2}{\sqrt{1 - 4x}} \left( \frac{1 - \sqrt{1 - 4x}}{4x} \right) = \frac{x}{2} \left( (1 - 4x)^{-\frac{1}{2}} - 1 \right).$$

Note that

$$(1 - 4x)^{-\frac{1}{2}} = \sum_{n \geq 0} \left( -\frac{1}{2} \right)(-4x)^n = \sum_{n \geq 0} \left( \binom{2n}{n} \right)x^n.$$

Thus we have the following.
Proposition 4.1. If \( n \geq 2 \), then
\[
p(n, 0) = \frac{1}{2} \binom{2n - 2}{n - 1}.
\]

In fact, we can prove Proposition 4.1 bijectively as follows. Recall that \( p(n, k) \)

is equal to the number of elements in
\[
\text{SCH}_{\text{odd}}(n - 1, k) = \{ L \in \text{SCH}_{\text{odd}}(n - 1) : L \text{ has } k \text{ peaks of even height} > 2 \}.
\]

Thus it is sufficient to show the following:
\[
\frac{1}{2} \binom{2n}{n} = \# \text{SCH}_{\text{odd}}(n, 0).
\]

If \( L \in \text{SCH}_{\text{odd}}(n, 0) \), then the height of each peak of \( L \) is 2. Let \( L' \) be the Schröder path obtained from \( L \) by changing each horizontal step \( H^2 \) whose height is not 1 to a peak \( \text{UD} \). Then \( L' \) is a Schröder path such that each horizontal step is of height 1. This gives us a bijection between \( \text{SCH}_{\text{odd}}(n, 0) \) and \( \text{SCH}_1(n) \), the set of all Schröder paths of length \( 2n \) such that the height of each horizontal step is 1.

The following proposition finishes our bijective proof of Proposition 4.1.

Proposition 4.2. There is a bijection between \( \text{SCH}_1(n) \) and the set of paths from \((0, 0)\) to \((n, n)\) with steps \((1, 0)\) and \((0, 1)\) starting with a step \((1, 0)\).

Proof. We can consider \( L \in \text{SCH}_1(n) \) as a path from \((0, 0)\) to \((n, n)\) with steps \((1, 0)\), \((0, 1)\) and \((1, 1)\) that never goes below the line \( y = x \). Then \( L \) has steps \((1, 1)\) only on the line \( y = x + 1 \). Let \( L = (S_1, S_2, \ldots, S_m) \) and let \( S_{i_1}, S_{i_2}, \ldots, S_{i_\ell} \) be the steps \((1, 1)\). Then we change \( S_{i_\ell} \) to two consecutive steps \( S = (1, 0), S' = (0, 1), \) and then reflect the path starting from \( S' \) to the last step along the line \( y = x \). We do the same thing with \( S_{i_{\ell-1}} \) on the resulting path, and so on. Then at the end, we get a path with steps \((1, 0)\) and \((0, 1)\) starting with \((1, 0)\). It is easy to see that this procedure is reversible. \( \Box \)

By finding the generating function, Yan [7] showed that the number of UH-free Schröder paths of length \( n \) without horizontal steps of height greater than 1 is equal to
\[
\frac{1}{2} \left( 1 + \sum_{i=0}^{n} \binom{2i}{i} \right).
\]

The following proposition gives us a bijective proof of (3).

Proposition 4.3. There is a bijection between \( \text{SCH}_1(n) \) and the set of UH-free Schröder paths of length \( n \), starting with an up step, and with no horizontal steps of height greater than 1.

Proof. Let \( L \) be a UH-free Schröder path with the conditions. Then we can decompose \( L \) as follows:
\[
L = (UL_1DH^{k_1})(UL_2DH^{k_2})\cdots(UL_\ell DH^{k_\ell}),
\]

where \( k_i \geq 0 \) and \( L_i \) is a (possibly empty) Schröder path with no horizontal steps of height greater than 0 which does not start with a horizontal step. Let
\[
L' = (UH^{k_\ell}L_1D)(UH^{k_2}L_2D)\cdots(UH^{k_1}L_\ell D).
\]

Then \( L' \in \text{SCH}_1(n) \). Clearly, this is invertible. \( \Box \)
5. A direct bijection between $\text{NC}_2(n)$ and $P_{12312}(n)$

Now we have a bijection $\phi \circ h \circ g \circ f : \text{NC}_2(n) \rightarrow P_{12312}(n)$, see Figure 2. Since both $\text{NC}_2(n)$ and $P_{12312}(n)$ are partitions with some conditions, it is natural to ask a direct bijection between them. In this section we find such a direct bijection.

From now on, we will identify a partition in $P_{12312}(n)$ with its canonical word.

A marked partition is a partition with marking some integers. Each marked integer is called active. Similarly a marked word is a word with marking some letters. Each marked letter is called active.

Let $\pi \in \text{NC}_2(n)$. For $i \in [n]$, let $T_i$ be the marked partition of $[i]$ obtained from $\pi$ by removing all the integers greater than $i$ and by marking integers which are connected to an integer greater than $i$.

Using the sequence $\emptyset = T_0, T_1, T_2, \ldots, T_n = \pi$ of marked partitions, we define a sequence of marked words $w_0, w_1, w_2, \ldots, w_n$ as follows.

Let $w_0$ be the empty word.

For $1 \leq i \leq n$, $w_i$ is defined as follows.

(1) If $i$ is not connected to any integer in $T_i$, then $w_i = w_{i-1}a_m$, where $m = \max(w_{i-1}) + 1$. Otherwise, $i$ is connected to either the largest active integer or the second largest active integer of $T_{i-1}$.
   - If $i$ is connected to the largest active integer of $T_{i-1}$, then let $w_i = w_{i-1}a_1$, where $a_1$ is the rightmost active letter of $w_{i-1}$. And then we make the active letter $a_1$ inactive.
   - If $i$ is connected to the second largest active integer of $T_{i-1}$, then let $w_i = w_{i-1}a_2$, where $a_2$ is the rightmost active letter of $w_{i-1}$. The second rightmost active letter of $w_{i-1}$ remains active, however, we make the rightmost active letter of $w_{i-1}$ inactive in $w_i$.

(2) If $i$ is active in $T_i$, then we find the largest letters in $w_i$ and make the leftmost letter among them active.

For example, see Figure 12. It is easy to check that $w_n$ is 12312-avoiding.

When we know $w_n$, we can reverse this procedure. First, we find $w_i$'s as follows.

For $1 \leq i \leq n$, $w_{i-1}$ is obtained from $w_i$ as follows.

(1) Let $m = \max(w_i)$. Let $t$ be the last letter of $w_i$.
(2) If the leftmost $m$ is active in $w_i$, then make it inactive.
(3) If $t$ appears only once in $w_i$ (equivalently $t$ is greater than any other letters in $w_i$), then we simply remove $t$. Otherwise, find the leftmost $t$ in $w_i$.
   - If the leftmost $t$ is inactive, then we remove the last letter $t$ and make the leftmost $t$ active.
   - If the leftmost $t$ is active, then we must have $t < m$ since we have made the leftmost $m$ inactive. In this case we remove the last $t$, and make the leftmost $t$ still active and the leftmost $m$ active.

Now we construct $T_0, T_1, \ldots, T_n$ as follows.

Let $T_0 = \emptyset$. For $1 \leq i \leq n$, $T_i$ is obtained as follows.

(1) First, let $T_i$ be the marked partition obtained from $T_{i-1}$ by adding $i$.
(2) Let $t$ be the last letter of $w_i$.
(3) If $t$ is equal to the rightmost (resp. the second rightmost) active letter of $w_{i-1}$, then connect $i$ to the largest (resp. the second largest) active integer, say $j$, of $T_{i-1}$, and make $j$ inactive.
Let \( m = \max(w_i) \). If the leftmost \( m \) is active in \( w_i \), then make \( i \) active in \( T_i \).

It is easy to check that this is the inverse map. Thus we get the following theorem.

**Theorem 5.1.** For \( \pi \in NC_2(n) \), the map \( \pi \mapsto w_n \) is a bijection from \( NC_2(n) \) to \( P_{12312}(n) \).

Since this bijection preserves the number of blocks, by Theorem 3.1 we get the following.

**Corollary 5.2.** Let \( n, k \geq 1 \). Then the number of \( 2 \)-distant noncrossing partitions of \([n+1]\) with \( k+1 \) blocks is equal to

\[
\sum_{i=k}^{n} \frac{1}{i!} \binom{i}{k-1} \binom{i}{k} \binom{n}{i}.
\]

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