On certain multiplier projections

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Abstract

Let $M(C(\prod_{j=1}^{\infty} S^2, \mathcal{K}))$, denote the multiplier algebra over $C(\prod_{j=1}^{\infty} S^2, \mathcal{K})$, the algebra of continuous functions into the compact operators with spectrum the infinite product of two-spheres. We consider multiplier projections in $M(C(\prod_{j=1}^{\infty} S^2, \mathcal{K}))$ of a certain diagonal form. We show that, while for each multiplier projection $Q$ of the special form, we have that $Q(x) \in B(\mathcal{H}) \setminus \mathcal{K}$ for all $x \in \prod_{j=1}^{\infty} S^2$, the ideal generated by $Q$ in $M(C(\prod_{j=1}^{\infty} S^2, \mathcal{K}))$ might be proper. We further show that the ideal generated by a multiplier projection of the special form is proper if and only if the projection is stably finite.

1 Introduction

The C*-algebra $C(\prod_{j=1}^{\infty} S^2, \mathcal{K})$ of continuous functions into the compact operators with spectrum the infinite product of two-spheres has been of interest in the construction of C*-algebras with non-regular behavior. M. Rørdam used this algebra in [R1] to construct a separable simple C*-algebra with both a finite and a (non-zero) infinite projection. In [R3] Rørdam used it to construct an extension

$$0 \to C(\prod_{j=1}^{\infty} S^2, \mathcal{K}) \to B \to \mathcal{K} \to 0$$

such that $B$ is not stable (despite the fact that both, ideal and quotient, are stable C*-algebras). Also, Rørdam’s construction in [R2] of a non-stable C*-algebra, which becomes stable after tensoring it with large enough (non-zero) matrix algebras, can be altered to using comparability properties of projections in matrix algebras over $C(\prod_{j=1}^{\infty} S^2, \mathcal{K})$.

Most constructions have in common that they take advantage of special multiplier projections of a certain diagonal form. The projections considered are infinite direct sums

$$Q = \bigoplus_{j=1}^{\infty} p_{I_j}, \quad (*)$$

where each direct summand $p_{I_j}$ is a finite tensor product of Bott projections over coordinates specified by a finite subset $I_j$ of the natural numbers. (We remind the reader of the detailed construction in the following section.) Using multiplier projections of this certain form, Rørdam proves in [R1] that there exists a finite full multiplier projection in $M(C(\prod_{j=1}^{\infty} S^2, \mathcal{K}))$ (and thereby showing that the C*-algebra $C(\prod_{j=1}^{\infty} S^2, \mathcal{K})$ does not have the corona factorization property). Recall that a projection in a C*-algebra is called full, if the closed two-sided ideal generated by it is the whole C*-algebra. Fullness of a projection in the multiplier algebra implies that some multiple of it is equivalent to the identity ([RLL], Exercise 4.8). But the multiplier unit of a stable C*-algebra is properly infinite ([R5], Lemma 3.4). Hence, Rørdam’s finite full projection is not stably finite. (A projection is stably finite if any multiple of it is a finite projection).

In this paper we investigate non-full multiplier projections in $M(C(\prod_{j=1}^{\infty} S^2, \mathcal{K}))$ of the special form as in $(*)$. Firstly, it is all but obvious that there exist non-full projections of this diagonal form at all. Identifying $M(C(\prod_{j=1}^{\infty} S^2, \mathcal{K}))$ with the strictly continuous functions from $\prod_{j=1}^{\infty} S^2$ into $B(\mathcal{H})$,
any multiplier projection $Q$ of the certain diagonal form satisfies that $Q(x) \in \mathcal{B(H)} \setminus \mathcal{K}$. In particular, locally, $Q(x)$ is full in $\mathcal{B(H)}$ for all $x \in X$. (It follows from the results of Pimsner, Popa and Voiculescu [PPV] that such a projection cannot be found when the spectrum is finite-dimensional.)

Using the techniques from [R1] we then prove the following result:

**Theorem:** Let

$$Q = \bigoplus_{j=1}^{\infty} p_{I_j} \in \mathcal{M}(\prod_{j=1}^{\infty} S^2, \mathcal{K}).$$

Then $Q$ is non-full if, and only if, $Q$ is stably finite.

The paper is organized as follows. In Section 2 we recall notation and constructions from [R1] and specify the multiplier projections the paper is devoted to. Section 3 contains the technical tool to prove our main results. In Section 4 we characterize non-fullness of multiplier projections in a combinatorial way. Finally, Section 5 contains the proof of the main theorem, i.e. we show that all non-full projections from section 4 are stably finite.

## 2 Preliminaries

Consider the following setting (and notation), which is adapted from [R1]. We will consider the Hausdorff space given by an infinite product of two-spheres, $X = \prod_{j=1}^{\infty} S^2$, equipped with the product topology. Since $S^2$ is compact, it follows from Tychonoff’s Theorem (see for example [M]) that $X$ is compact. Let further

$$p \in C(S^2, M_2(\mathbb{C}))$$

denote the Bott projection, i.e., the projection corresponding to the ‘Hopf bundle’ $\xi$ over $S^2$ with total Chern class $c(\xi) = 1 + x$ (see e.g. [K]).

With $\pi_n : X \to S^2$ denoting the coordinate projection onto the $n$-th coordinate, consider the (orthogonal) projection

$$p_n := p \circ \pi_n \in C(\prod_{j=1}^{\infty} S^2, M_2(\mathbb{C})).$$

If $I \subseteq \mathbb{N}$ is a finite subset, $I = \{n_1, n_2, \ldots, n_k\}$, then let $p_I$ denote the pointwise tensor product

$$p_I := p_{n_1} \otimes p_{n_2} \otimes \ldots \otimes p_{n_k} \in C \left( \prod_{j=1}^{\infty} S^2, M_2(\mathbb{C}) \otimes M_2(\mathbb{C}) \otimes \ldots \otimes M_2(\mathbb{C}) \right).$$

It is shown in [R1] that the projection $p_n$ corresponds to the pull-back of the Hopf bundle via the coordinate projection $\pi_n$, denoted by $\xi_n := \pi_n^*(\xi)$, and that the projection $p_I$ corresponds to the tensor product of vector bundles $\xi_{n_1} \otimes \xi_{n_2} \otimes \ldots \otimes \xi_{n_k}$.

Considering the compact operators $\mathcal{K}$ on a separable Hilbert space as an AF algebra, the inductive limit of the sequence

$$\mathbb{C} \to M_2(\mathbb{C}) \to M_3(\mathbb{C}) \to M_4(\mathbb{C}) \to \ldots,$$

with connecting *-homomorphisms mapping each matrix algebra into the upper left corner of any larger matrix algebra at a later stage, we get an embedding of each matrix algebra over $\mathbb{C}$ into the compact operators $\mathcal{K}$. In this way we can consider all the projections $p_n$ and $p_I$, defined as above, as projections in $C(\prod_{j=1}^{\infty} S^2, \mathcal{K})$.

In addition to the setting of [R1], let us denote by $p^-$ the projection corresponding to the complex line bundle $\xi^-$ over $S^2$ with total Chern class $c(\xi^-) = 1 - x$. (Recall that the first Chern class is a
complete invariant for complex line bundles (see Proposition 3.10 of [H]). The tensor product $\xi \otimes \xi^-$ is isomorphic to the one-dimensional trivial bundle, because its Euler class can be computed, using [R1] Equation 3.3, to be
\[ e(\xi \otimes \xi^-) = x - x = 0 \]
and the only line bundle with zero Euler class is the trivial bundle $\theta_1$ ([H] Proposition 3.10). Accordingly, the projection in $C(X, M_4(\mathbb{C}))$ that is given by the pointwise tensor product of $p$ and $p^-$ is equivalent to a 1-dimensional constant projection.

We finally define $p^-_n \in C(\prod_{j=1}^{\infty} S^2, \mathcal{K})$ as $p^-_n := p^- \circ \pi_n$.

The following well known result can be found for example in [L].

**Lemma 2.1:** Let $X$ be a compact Hausdorff space. Let $C(X, \mathcal{K})$ denote the continuous functions from $X$ into the compact operators $\mathcal{K}$ on a separable Hilbert space $\mathcal{H}$. Let further $C_{*\text{-}s}(X, \mathcal{B}(\mathcal{H}))$ denote the $\ast$-strongly continuous (or strictly continuous) functions from $X$ into the bounded operators on the Hilbert space $\mathcal{H}$. Then
\[ \mathcal{M}(C(X, \mathcal{K})) \cong C_{*\text{-}s}(X, \mathcal{B}(\mathcal{H})). \]

We will often take advantage of identifying $C(X, \mathcal{K})$ with $C(X) \otimes \mathcal{K}$. For instance it is then immediate to see stability of $C(X, \mathcal{K})$.

For any stable $C^*$-algebra $A \cong A \otimes \mathcal{K}(\mathcal{H})$ we can embed the algebra of all bounded operators
\[ \mathcal{B}(\mathcal{H}) \cong 1_{\mathcal{M}(A)} \otimes \mathcal{B}(\mathcal{H}) \]
into $\mathcal{M}(A)$ (see e.g. [L] Chapter 4). Hence, we can find a sequence $\{S_j\}_{j=1}^{\infty}$ of isometries with orthogonal range projections in $\mathcal{M}(A)$ such that the range projections sum up to the identity of $\mathcal{M}(A)$ in the strict topology:
\[ S_j^* S_j = 1_{\mathcal{M}(A)} \text{ for all } j, \text{ and } \sum_{j=1}^{\infty} S_j S_j^* = 1_{\mathcal{M}(A)}. \]

Using such a sequence we can define infinite direct sums of projections in $A$:

For a sequence $\{p_j\}_{j=1}^{\infty}$ of projections in $A$ we define
\[ \bigoplus_{j=1}^{\infty} p_j := \sum_{j=1}^{\infty} S_j p_j S_j^* \in \mathcal{M}(A). \]

The sum is strictly convergent and hence defines a projection in the multiplier algebra of $A$, which, up to unitary equivalence, is independent of the chosen isometries ([R1], page 10). Also, its unitary equivalence class in the ordered Murray-von Neumann semigroup is independent of permutations of the direct summands (see Lemma 4.2 of [R1]).

For fixed projections $Q \in \mathcal{M}(A)$ we will denote the direct sum $Q \oplus Q \oplus \ldots \oplus Q$ of $Q$ with itself $m$ times by $m \cdot Q$.

We are now ready to specify the multiplier projections this paper is devoted to and which were considered by Rørdam in [R1] and [R3]: All our results are for multiplier projections given by
\[ Q = \bigoplus_{j=1}^{\infty} p_{t_j}, \quad (*) \]
where each $p_{t_j}$ is a tensor product of Bott projections as above.
3 Technical result

The following result is basically contained in [R1] by a combination of Proposition 3.2 with Proposition 4.5 from that paper. It makes it possible to check minorization of projections as in (\ast) by trivial projections in \(C(\prod_{j=1}^{\infty} S^2, K)\) in purely combinatorial terms. By a trivial projection we mean a projection that is equivalent to a constant one (i.e., any projection that corresponds to a trivial complex vector bundle). We will denote trivial 1-dimensional projections in \(C(\prod_{j=1}^{\infty} S^2, K)\) by \(g\). Recall that for any non-empty finite subset \(I\) of \(N\) we denote by \(p_I\) the tensor product of Bott projections over the coordinates given by \(I\).

**Proposition 3.1:** Let \(I_j, j \in N\), be finite subsets of \(N\), and consider the multiplier projection \(Q\) in \(M(C(\prod_{j=1}^{\infty} S^2, K))\) given by
\[
Q = \bigoplus_{j=1}^{\infty} p_{I_j}.
\]
Then the following statements are equivalent:

(i) \(g \not\leq Q = \bigoplus_{j=1}^{\infty} p_{I_j}\).

(ii) \(|\bigcup_{j \in F} I_j| \geq |F|\) for all finite subsets \(F \subseteq N\).

**Proof.** That (ii) implies (i) is the content of Proposition 4.5 (i) of [R1].

If, on the other hand, there is a finite subset \(F \subseteq N\) such that \(|\bigcup_{j \in F} I_j| < |F|\), consider the subprojection \(\bigoplus_{j \in F} p_{I_j}\) in \(C(\prod_{j=1}^{\infty} S^2, K)\). Let \(J := \bigcup_{j \in F} I_j\). With \(\pi_J\) denoting the projection onto the coordinates given by \(J\), we have \(\bigoplus_{j \in F} p_{I_j} = \pi_J^* (q)\) for some projection \(q \in C(\prod_{j=1}^{\mid J \mid} S^2, K)\). The projection \(q\) corresponds to a vector bundle of dimension \(|F|\) over \(|J| = |\bigcup_{j \in F} I_j|\)-many copies of \(S^2\). But then by [Hu], Theorem 8.1.2, this vector bundle majorizes a trivial bundle. In terms of projections this implies
\[
g = \pi_J^* (g) \preceq \pi_J^* (q) = \bigoplus_{j \in F} p_{I_j} \preceq Q.
\]

\[\Box\]

It is possible to generalize this result. The following proposition allows to count the precise number of trivial subprojections (while Proposition 3.1 is only good to check existence of some trivial subprojection).

**Proposition 3.2:** Let \(I_j, j \in N\), be finite subsets of \(N\), and consider the multiplier projection \(Q\) in \(M(C(\prod_{j=1}^{\infty} S^2, K))\) given by
\[
Q = \bigoplus_{j=1}^{\infty} p_{I_j}.
\]
Let \(m \in N\).

Then the following statements are equivalent:

(i) \(m \cdot g \not\leq Q \sim \bigoplus_{j=1}^{\infty} p_{I_j}\).

(ii) \(|F| < |\bigcup_{j \in F} I_j| + m\) for all finite subsets \(F \subseteq N\).
Proof. The implication from (i) to (ii) can be seen from standard stability properties of vector bundles, as follows: Assume there is some finite subset \( F \) such that

\[
|F| \geq \left| \bigcup_{j \in F} I_j \right| + m.
\]

Then \( \bigoplus_{j \in F} p_I \) is an \(|F|\)-dimensional subprojection of \( Q \) that can be considered, using the identification of projections with vector bundles and using a pullback by the appropriate coordinate projection (as in the proof of Proposition 3.1), as an \(|F|\)-dimensional vector bundle over a base space consisting of the product of \( \bigcup_{j \in F} I_j \) copies of \( S^2 \). Then Theorem 8.1.2 from [Hu] proves the existence of a trivial \( \left( |F| - \bigcup_{j \in F} I_j \right) \)-dimensional subbundle. This implies (again in terms of projections in \( \mathcal{M}(\Pi_{j=1}^{\infty} S^2, K) \)):

\[
m \cdot g \leq \left( |F| - \left| \bigcup_{j \in F} I_j \right| \right) \cdot g \leq \bigoplus_{j \in F} p_I \leq \bigoplus_{j=1}^{\infty} p_I = Q.
\]

Let us now prove that (ii) implies (i): By hypothesis all finite subsets \( F \subseteq \mathbb{N} \) satisfy

\[
|F| < \left| \bigcup_{j \in F} I_j \right| + m.
\]

Assume \( m \cdot g \leq Q \). Then \( m \cdot g \leq \bigoplus_{j=1}^{N} p_I \) for some \( N \in \mathbb{N} \) by Lemma 4.4 of [R1]. Let \( k_1, k_2, \ldots, k_{m-1} \) be natural numbers in \( \mathbb{N} \setminus \bigcup_{j=1}^{N} I_j \). Then by Lemma 2.3 of [KN] there exists a projection \( q \) such that

\[
q \oplus \left( p_{k_1}^{-1} \otimes p_{k_2}^{-1} \otimes \ldots \otimes p_{k_{m-1}}^{-1} \right) \sim m \cdot g \leq Q.
\]

Tensoring (pointwise) both sides by \( p_K := p_{k_1} \otimes p_{k_2} \otimes \ldots \otimes p_{k_{m-1}} \), it follows that

\[
\left( q \otimes p_{k_1} \otimes p_{k_2} \otimes \ldots \otimes p_{k_{m-1}} \right) \oplus g \leq \bigoplus_{j=1}^{\infty} p_I \otimes p_{k_1} \otimes p_{k_2} \otimes \ldots \otimes p_{k_{m-1}}.
\]

In particular,

\[
g \leq \bigoplus_{j=1}^{\infty} p_I \otimes p_K = \bigoplus_{j=1}^{\infty} p_{I_j \cup K}.
\]

By Proposition 3.1 this entails that there is some finite subset \( F \subseteq \mathbb{N} \) such that

\[
\left| \bigcup_{j \in F} I_j \cup K \right| < |F|.
\]

Hence,

\[
|F| > \left| \bigcup_{j \in F} I_j \cup K \right| = \left| \bigcup_{j \in F} I_j \right| + |K| = \left| \bigcup_{j \in F} I_j \right| + (m - 1).
\]

But the existence of a finite subset \( F \) satisfying

\[
|F| \geq \left| \bigcup_{j \in F} I_j \cup K \right| + 1 = \left| \bigcup_{j \in F} I_j \right| + m
\]

contradicts the hypothesis. \( \Box \)
If we want to consider multiples of the multiplier projection as well, we can apply

**Corollary 3.3:** Let $I_j, j \in \mathbb{N}$, be finite subsets of $\mathbb{N}$, and consider the multiplier projection $Q$ in $\mathcal{M}(\prod_{j=1}^{\infty} S^2, \mathcal{K})$ given by

$$Q = \bigoplus_{j=1}^{\infty} p_{I_j}.$$ 

Let $m, n \in \mathbb{N}$.

Then the following statements are equivalent:

(i) $m \cdot g \preceq n \cdot Q \sim \bigoplus_{j=1}^{\infty} n \cdot p_{I_j}$.

(ii) $n|F| < \left| \bigcup_{j \in F} I_j \right| + m$ for all finite subsets $F \subseteq \mathbb{N}$.

**Proof.** Note, that in $n \cdot Q$ each index set $I_j$ appears $n$ times. Choosing the same set $I_j$ several times does not increase the left-hand side of the inequality (ii) of Proposition 3.2, while it does increase the right-hand side of that inequality. Now the statement follows immediately from Proposition 3.2. \qed

### 4 Non-full multiplier projections

The combinatorial description of subequivalences makes it possible to prove the following useful result.

**Lemma 4.1:** If $N \cdot g \preceq \bigoplus_{j=1}^{\infty} p_{I_j}$ for all $N \in \mathbb{N}$, then

$$1 \preceq Q.$$ 

**Proof.** By Proposition 3.2 the hypothesis is equivalent to:

For all $N \in \mathbb{N}$ there is some finite subset $F \subseteq \mathbb{N}$ such that

$$|F| \geq \left| \bigcup_{j \in F} I_j \right| + N. \quad (***)$$

Let $G \subseteq \mathbb{N}$ be any finite subset of the natural numbers. We claim that there is then some finite subset $H \subseteq (\mathbb{N} \setminus G)$ such that

$$g \preceq \bigoplus_{j \in H} p_{I_j}.$$ 

To show this, apply the hypothesis (**) to the choice $|G| + 1$ for $N$: we obtain a finite subset $F \subseteq \mathbb{N}$ such that

$$|F| \geq \left| \bigcup_{j \in F} I_j \right| + |G| + 1.$$ 

Then

$$\left| \bigcup_{j \in F \setminus G} I_j \right| + 1 \leq \left| \bigcup_{j \in F} I_j \right| + 1 \leq |F| - |G|.$$ 

By Proposition 3.1 this implies that

$$g \preceq \bigoplus_{j \in F \setminus G} p_{I_j},$$ 

and we can take $H := F \setminus G$. 


Using this intermediate result we begin to iterate:

Firstly by assumption we have $g \preceq \bigoplus_{j=1}^{m_1-1} p_{I_j}$, and therefore by Lemma 4.4 of \cite{R1},

$$g \preceq \bigoplus_{j=1}^{m_1-1} p_{I_j}$$

for some $m_1 \in \mathbb{N}$. But then with $G = \{1, 2, \ldots, (m_1 - 1)\}$ we can find, by application of the proven claim, some natural number $m_2 > m_1$ and $H \subseteq \{m_1, (m_1 + 1), \ldots, (m_2 - 1)\}$ such that

$$g \preceq m_2 - 1 \bigoplus_{j=m_1}^{m_2-1} p_{I_j}.$$ 

Iterating, we get a strictly increasing sequence of natural numbers $1 = m_0 < m_1 < m_2 < m_3 < \ldots$ and, for all $i \in \mathbb{N}$, we get a partial isometry $v_i \in C(\prod_{j=1}^\infty S^2, K)$ such that

$$g = v_i^* v_i \sim v_i v_i^* \preceq \bigoplus_{j=m_i-1}^{m_i-1} p_{I_j}.$$ 

The multiplier

$$V = \bigoplus_{i=1}^\infty v_i$$

then implements the subequivalence

$$\mathbb{1} \sim \infty \cdot g \preceq \bigoplus_{j=1}^\infty p_{I_j} = Q.$$ 

We can now prove the main theorem of this section, which is a combinatorial characterization for multiplier projections of the special form to be non-full.

**Theorem 4.2:** Let $Q = \bigoplus_{j=1}^\infty p_{I_j} \in \mathcal{M}(C(\prod_{j=1}^\infty S^2, K))$ be as above. Then the following statements are equivalent:

(i) $Q$ is non-full.

(ii) $\forall m \in \mathbb{N} \exists N(m) \in \mathbb{N}$ such that $N(m) \cdot g \not\preceq m \cdot Q$.

(iii) $\forall m \in \mathbb{N} \exists N(m) \in \mathbb{N}$ such that $m |F| < |\bigcup_{j \in F} I_j| + N(m)$ for all finite subsets $F \subseteq \mathbb{N}$.

**Proof.** The equivalence between (ii) and (iii) follows from Proposition 3.3.

If we are in the situation of the condition (ii), then, in particular, $\mathbb{1} \not\preceq m \cdot Q$ for any natural number $m$ and so $Q$ cannot be full (see \cite{RLL}, Exercise 4.8). This proves that (ii) implies (i).

Finally assume that there exists some $m \in \mathbb{N}$ such that for all $N \in \mathbb{N}$ we have $N \cdot g \preceq m \cdot Q$. Then by Lemma 4.4 also $\mathbb{1} \preceq m \cdot Q$ and $Q$ is full. So (i) implies (ii). \qed

Rephrasing the content of Theorem 4.2 we get the following interesting result.

**Corollary 4.3:** There exists a compact Hausdorff space $X$ and a projection $Q$ in $C_{\ast}(X, \mathcal{B}(\mathcal{H}))$, the multiplier algebra of $C(X, K)$, such that $Q(x) \in \mathcal{B}(\mathcal{H}) \setminus K$ for all $x \in X$, and $Q$ is not full in $C_{\ast}(X, \mathcal{B}(\mathcal{H}))$. 

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In particular, the projection $Q(x)$ is full in the fiber over each $x \in X$, but $Q$ is itself non-full. It follows from the results of Pimsner, Popa and Voiculescu in [PPV] that for obtaining such an example the space $X$ is necessarily of infinite dimension.

**Proof.** Let $X = \prod_{j=1}^{\infty} S^2$. To show existence of the projection $Q$, choose pairwise disjoint subsets $I_j \subseteq \mathbb{N}$ such that $|I_j| = n$ and set

$$Q := \bigoplus_{j=1}^{\infty} p_{I_j} \in \mathcal{M}(C(X, \mathcal{K})).$$

We then have that $Q(x) \in \mathcal{B}(\mathcal{H}) \setminus \mathcal{K}$, since $\|p_j(x)\| = 1$ for all $x \in X$ and all $j \in \mathbb{N}$ (and since a strictly convergent sum of pairwise orthogonal elements in the compact operators $\mathcal{K}$ belongs to $\mathcal{K}$ if, and only if, the elements converge to 0 in norm (cf. [R1] Proof of Proposition 5.2)). So we only need to show that the index sets $I_j$ satisfy the condition (iii) of Theorem 4.2; that is, we need to show that

$$\forall m \in \mathbb{N} \exists N(m) \in \mathbb{N} \text{ such that } m < \left| \bigcup_{j \in F} I_j \right| + N(m) \frac{m(m-1)}{|F|}$$

for all finite subsets $F \subseteq \mathbb{N}$.

Now

$$\frac{\left| \bigcup_{j \in F} I_j \right| + \frac{m(m-1)}{2}}{|F|} \geq \frac{\sum_{j=1}^{|F|} j + \frac{m(m-1)}{2}}{|F|} = \frac{|F||F|+1}{2} + \frac{m(m-1)}{2} = \frac{1}{2} \left( 1 + |F| + \frac{m(m-1)}{|F|} \right)$$

and the last expression is minimized when $|F| \in \{(m-1), m\}$.

Hence,

$$\frac{\left| \bigcup_{j \in F} I_j \right| + \frac{m(m-1)}{2} + 1}{|F|} > \frac{\left| \bigcup_{j=1}^{m-1} I_j \right| + \frac{m(m-1)}{2}}{m-1} = \frac{m(m-1)}{2} + \frac{m(m-1)}{2} = m.$$ 

So we can choose $N(m) = \frac{m(m-1)}{2} + 1$.\hfill\Box

5 **Stably finite multiplier projections**

In this section we will show that every multiple of a non-full projection

$$Q = \bigoplus_{j=1}^{\infty} p_{I_j}$$

constructed as in Theorem 4.2 above (and, in particular, every multiple of the explicit projection of Corollary 4.3), is a finite projection. In fact, our results show that a multiplier projection $Q$ of the special form is non-full if, and only if, it is stably finite (Corollary 5.4).

It is fairly easy to see that the projections $m \cdot Q$, where $Q$ is one of the non-full projections from Theorem 4.2, cannot be properly infinite. This follows from the following lemma, together with the existence (Theorem 4.2) of a number $N(m) \in \mathbb{N}$ such that

$$N(m) \cdot g \notin m \cdot Q,$$ 

but $N(m) \cdot g \preceq l \cdot Q$ for sufficiently large $l$. 

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**Lemma 5.1:** Let $A$ be a $C^*$-algebra and $p$ and $q$ two projections in $A \otimes K$ such that $p \preceq k \cdot q$, but $p \not\preceq m \cdot q$ for some $m < k$. Then $q$ is not properly infinite.

*Proof.* Assume that $q$ is properly infinite. Then

$$p \preceq k \cdot q = (q \oplus q) \oplus \ldots \oplus q$$

$$\preceq (q) \oplus q \oplus \ldots \oplus q = (k - 1) \cdot q$$

$$\preceq (k - 2) \cdot q \preceq \ldots \preceq m \cdot q,$$

in contradiction with the assumption.

It does not seem possible to see finiteness of these projections in a similarly easy way. To show finiteness we will need to give a somewhat complicated proof. The idea is the content of the following lemma and is essentially contained in the proof of Theorem 5.6 of [R1].

**Lemma 5.2:** Let $B$ be a simple inductive limit $C^*$-algebra,

$$B_1 \xrightarrow{\varphi_1} B_2 \xrightarrow{\varphi_2} \ldots \xrightarrow{\varphi_i} B_i \xrightarrow{\varphi} \ldots \xrightarrow{\varphi_i} B$$

with injective connecting $*$-homomorphisms $\varphi_j$. Let $q$ be a projection in $B_1$. If the image $\varphi_{i,1}(q)$ of the projection $q$ is not properly infinite in any building block algebra $B_i$, then $q$ must be finite.

*Proof.* The hypothesis that $\varphi_{i,1}(q) \in B_i$ is not properly infinite for any $i \in \mathbb{N}$ together with Proposition 2.3 of [R1] applied to the inductive sequence

$$qB_1q \xrightarrow{\varphi_{2,1}(q)B_2\varphi_{2,1}(q)} \ldots \xrightarrow{\varphi_{i,1}(q)B_i\varphi_{i,1}(q)} \ldots$$

implies that the image of $q$ in the inductive limit algebra $B$ cannot be properly infinite either. Now $B$ is a simple $C^*$-algebra, in which by a result of Cuntz in [C], every infinite projection is properly infinite. Hence, the image of $q$ in $B$ is finite.

Now, injectivity of the connecting maps $\varphi_j$ implies that $q$ must be finite, too.

We can now prove the main result.

**Theorem 5.3:** Let

$$Q = \bigoplus_{j=1}^{\infty} p_{I_j} \in \mathcal{M}(C(\prod_{j=1}^{\infty} S^2, K))$$

be a multiplier projection as before. Suppose there is some $k \in \mathbb{N}$ such that $k \cdot g \not\preceq Q$.

Then $Q$ is finite.

*Proof.* First we reduce to the case that $\mathbb{N} \setminus \bigcup_{j=1}^{\infty} I_j$ is infinite:

Consider the projection map $\rho : \prod_{j=1}^{\infty} S^2 \to \prod_{j=1}^{\infty} S^2$ onto the odd coordinates:

$$\rho(x_1, x_2, x_3, x_4, x_5, \ldots) = (x_1, x_3, x_5, \ldots).$$

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Then the induced mapping $\rho^* : C(\prod_{j=1}^{\infty} S^2, K) \to C(\prod_{j=1}^{\infty} S^2, K)$ given by

$$\rho^*(f) = f \circ \rho$$

is injective and extends to an injective mapping between the multiplier algebras

$$\rho^* : \mathcal{M} \left( C(\prod_{j=1}^{\infty} S^2, K) \right) \to \mathcal{M} \left( C(\prod_{j=1}^{\infty} S^2, K) \right)$$

(to see this consult [L] Proposition 2.5 and use that $\rho^*(n \cdot g) \xrightarrow{n \to \infty} 1$, where $g$ denotes a constant one-dimensional projection as before).

Now $Q$ must be finite in $\mathcal{M}(C(\prod_{j=1}^{\infty} S^2, K))$, if $\rho^*(Q)$ is. Indeed, on supposing $Q$ to be infinite, i.e. $Q \sim Q_0 < Q$ for some projection $Q_0$, injectivity of $\rho^*$ implies $\rho^*(Q - Q_0) > 0$ and hence infiniteness of $\rho^*(Q)$.

But now $\rho^*(Q)$ is of the same form as $Q$, i.e.,

$$\rho^*(Q) = \bigoplus_{j=1}^{\infty} p_{I_j},$$

and the sets $I_j$ of indices being used satisfy $\mathbb{N} \setminus \bigcup_{j=1}^{\infty} I_j \supseteq 2\mathbb{N}$, and in particular $\mathbb{N} \setminus \bigcup_{j=1}^{\infty} I_j$ is infinite, as desired.

After this reduction step we start the main part of the proof. By assumption we can find $k \in \mathbb{N} \cup \{0\}$ such that $k \cdot g \preceq Q$, but $(k+1) \cdot g \not\preceq Q$. Choose a partition $\{A_i\}_{i=-\infty}^{\infty}$ of $\mathbb{N}$ such that each $A_i$ is infinite and such that $A_0 = \bigcup_{j=1}^{\infty} I_j$, i.e., $A_0$ contains exactly all the indices used in our multiplier projection $Q$. Also, choose a partition $\{B_i\}_{i=-\infty}^{\infty}$ of $A_{-1}$ with each $B_i$ of cardinality $k$, except in the case $k = 0$ where we do not need the sets $B_i$ at all.

For each $r \geq 0$, choose an injective map

$$\gamma_r : \mathbb{Z} \times A_r \to A_{r+1}.$$  

We can now define an injective map $\nu : \mathbb{Z} \times \mathbb{N} \to \mathbb{N}$, by

$$\nu(j, l) = \gamma_r(j, l), \text{ for every } l \in A_r.$$  

Injectivity of $\nu$ follows from injectivity of each $\gamma_r$ and disjointness of the sets $A_j$.

Using the injective map $\nu$, let us now define a *-homomorphism

$$\varphi : \mathcal{M}(C(\prod_{j=1}^{\infty} S^2, K)) \to \mathcal{M}(C(\prod_{j=1}^{\infty} S^2, K)).$$

The construction of this *-homomorphism is only a small variation of a mapping that M. Rørdam defined in his paper [R1] to construct “A simple C*-algebra with a finite and an infinite projection”. $\varphi$ will depend on the natural number $k$ from the hypothesis of the theorem. But the change of $\varphi$ for varying $k$ is minor, so we can take care of all cases at once. (Only the case $k = 0$ has to be treated separately, but this is actually exactly Rørdam’s map from [R1].)

For $j \leq 0$ and in the case $k \geq 1$ we define $\varphi_j : C(\prod_{j=1}^{\infty} S^2, K) \to C(\prod_{j=1}^{\infty} S^2, K)$ by

$$\varphi_j(f)(x_1, x_2, x_3, \ldots) = \tau(f(x_{\nu(j,1)}), x_{\nu(j,2)}, x_{\nu(j,3)}, x_{\nu(j,4)}, \ldots) \otimes p_{B_i})$$
with the finite sets $B_j \subseteq \mathbb{N}$ chosen above, and a chosen isomorphism $\tau : \mathcal{K} \otimes \mathcal{K} \to \mathcal{K}$. In the case $k = 0$ we simply define $\varphi_j$ by

$$\varphi_j(f)(x_1, x_2, x_3, \ldots) = f(x_{\nu(j,1)}, x_{\nu(j,2)}, x_{\nu(j,3)}, x_{\nu(j,4)}, \ldots).$$

For $j \geq 1$ we define $\varphi_j : C(\prod_{j=1}^{\infty} S^2, \mathcal{K}) \to C(\prod_{j=1}^{\infty} S^2, \mathcal{K})$ by

$$\varphi_j(f)(x_1, x_2, x_3, \ldots) = \tau(f(c_{j,1}, c_{j,2}, \ldots, c_{j,j}, x_{\nu(j,j+1)}, x_{\nu(j,j+2)}, \ldots) \otimes p_{B_j \cup \{\nu(j,1), \nu(j,2), \ldots, \nu(j,j)\}})$$

with points

$$c_{1,1}, c_{2,1}, c_{2,2}, c_{3,1}, c_{3,2}, c_{3,3}, c_{4,1}, c_{4,2}, c_{4,3}, c_{4,4}, \ldots.$$ 

in $S^2$ chosen in such a way that for all $j \in \mathbb{N}$,

$$\{(c_{k,1}, c_{k,2}, \ldots, c_{k,j}) \mid k \geq j\} \text{ is dense in } \prod_{i=1}^{j} S^2.$$ 

(Here the case $k = 0$ just means that every set $B_j$ is taken to be the empty set.)

After choosing a sequence of isometries $\{S_j\}_{j=-\infty}^{\infty}$ in $\mathcal{M}(C(\prod_{j=1}^{\infty} S^2, \mathcal{K}))$ such that

$$S_j^* S_j = 1 \text{ for all } j \in \mathbb{Z} \text{ and } \sum_{j=-\infty}^{\infty} S_j S_j^* = 1,$$

define $\tilde{\varphi} : C(\prod_{j=1}^{\infty} S^2, \mathcal{K}) \to \mathcal{M}(C(\prod_{j=1}^{\infty} S^2, \mathcal{K}))$ by

$$\tilde{\varphi} := \sum_{j=-\infty}^{\infty} S_j \varphi_j S_j^*.$$

Then by Proposition 8.3 recalling that the cardinality of each set $B_j$ was chosen to be equal to $k$, and by the fact that $\varphi_j(g) \sim p_{B_j}$ for all $j \leq 0$, we get

$$\tilde{\varphi} ((k+1) \cdot g) \geq \bigoplus_{j=-\infty}^{0} (k+1) \cdot p_{B_j} \geq \bigoplus_{j=-\infty}^{0} g \sim 1.$$ 

Hence $\tilde{\varphi}(n \cdot g)$ converges strictly for $n \to \infty$ to a projection

$$F \sim \bigoplus_{j=-\infty}^{\infty} F_j \geq 1,$$

where

$$F_j = \begin{cases} \tau(1 \otimes p_{B_j}) & \text{for } j \leq 0 \text{ and } k \geq 1 \\ 1 & \text{for } j \leq 0 \text{ and } k = 0 \\ \tau \left(1 \otimes p_{B_j \cup \{\nu(j,1), \nu(j,2), \ldots, \nu(j,j)\}}\right) & \text{for } j \geq 1 \text{ and } k \geq 1 \\ \tau \left(1 \otimes p_{\{\nu(j,1), \nu(j,2), \ldots, \nu(j,j)\}}\right) & \text{for } j \geq 1 \text{ and } k = 0. \end{cases}$$

Here the map $\tilde{\tau} : \mathcal{B}(\mathcal{H}) \otimes \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is the extension of $\tau$ to $\mathcal{B}(\mathcal{H}) \otimes \mathcal{B}(\mathcal{H})$, which exists because $\tau(e_n \otimes e_n) \to 1$ strictly (L)).
Since $F \succeq 1$, by Lemma 4.3 of [R1] $F \sim 1$ and hence there is an isometry $V \in \mathcal{M}(C(\prod_{j=1}^{\infty} S^2, \mathcal{K}))$ such that the map
\[ \varphi := V^* \phi V \]
extends to a unital mapping $\varphi : \mathcal{M}(C(\prod_{j=1}^{\infty} S^2, \mathcal{K})) \to \mathcal{M}(C(\prod_{j=1}^{\infty} S^2, \mathcal{K}))$. (Here we are using [L] again.)

For every $0 \neq f$ there is some $\delta > 0$ and some open set
\[ U = U_1 \times U_2 \times U_3 \times \ldots \times U_r \times S^2 \times S^2 \times \ldots \subseteq S^2 \times S^2 \times S^2 \times \ldots \times S^2 \times S^2 \times \ldots \]
such that $\|f_U\| \geq \delta$. By the density condition on the $c_{ij}$ there are infinitely many $j \geq 0$ such that for any $x \in \prod_{j=1}^{\infty} S^2$,
\[ \|\varphi_j(f)(x)\| \geq \delta > 0. \]
Hence $\varphi(f)(x) \in \mathcal{B}(\mathcal{H}) \setminus \mathcal{K}$ for all $x$ and $\varphi(f) \in \mathcal{M}(C(\prod_{j=1}^{\infty} S^2, \mathcal{K})) \setminus C(\prod_{j=1}^{\infty} S^2, \mathcal{K})$.

In particular, $\varphi$ is injective, and $C(\prod_{j=1}^{\infty} S^2, \mathcal{K})\varphi(f)C(\prod_{j=1}^{\infty} S^2, \mathcal{K})$ is norm dense in $C(\prod_{j=1}^{\infty} S^2, \mathcal{K})$. (The latter holds since $\varphi(f)(x) \neq 0$ for all $x \in \prod_{j=1}^{\infty} S^2$.)

We get that $(k + 1) \cdot g$ is an element in $C(\prod_{j=1}^{\infty} S^2, \mathcal{K})\varphi(f)C(\prod_{j=1}^{\infty} S^2, \mathcal{K})$. Further, $\varphi((k + 1) \cdot g) \succeq 1$, and so $\varphi^2(f)$ is full in $\mathcal{M}(C(\prod_{j=1}^{\infty} S^2, \mathcal{K}))$.

This implies the simplicity of the inductive limit
\[ B := \lim_{\to} \mathcal{M} \left( C(\prod_{j=1}^{\infty} S^2, \mathcal{K}) \right), \varphi \right). \]

We have now arrived in the setting of Lemma 5.2 and it suffices to show that $\varphi^m(Q)$ is not properly infinite for all $m \in \mathbb{N}$. For this we define maps
\[ \alpha_j : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N}), \quad j \in \mathbb{Z}, \]
\[ \alpha_j(J) = \nu(j, J) \cup B_j \cup \{\nu(j, 1), \nu(j, 2), \ldots, \nu(j, j)\}, \]
with the convention that $\{\nu(j, 1), \nu(j, 2), \ldots, \nu(j, j)\} = \emptyset$ for $j \leq 0$. To simplify our computations let us introduce new notation and denote from now on $B_j \cup \{\nu(j, 1), \nu(j, 2), \ldots, \nu(j, j)\}$ simply by $\bar{B}_j$.

With these definitions, one has
\[ \varphi(p_I) \sim \bigoplus_{j=-\infty}^{\infty} p_{\alpha_j(I)} = \bigoplus_{j=-\infty}^{\infty} p_{\nu(j,1),\ldots,\nu(j,j)} \bar{B}_j. \]

Set $\Gamma_0 := \{I_s \mid s \in \mathbb{N}\}$ and define inductively
\[ \Gamma_{n+1} := \{\alpha_j(I) \mid j \in \mathbb{Z}, \ I \in \Gamma_n\}. \]

Then
\[ \varphi^m(Q) \sim \bigoplus_{I \in \Gamma_m} p_I. \]

We will prove that $\varphi^m(Q)$ is not properly infinite by applying Rørdam’s criterion (Proposition 3.1), showing that for each $m \geq 1$ there is an injective map
\[ t_m : \Gamma_m \to \mathbb{N} \]
such that $t_m(I) \in I$ for all $I \in \Gamma_m$. Once we have this map, it follows that
\[ \varphi^m(Q) \sim \bigoplus_{I \in \Gamma_m} p_I \not\in g. \]
for any \( m \geq 1 \). But for each \( m \) the projection \( g \) is in the ideal of \( C(\prod_{j=1}^{\infty} S^2, \mathcal{K}) \) given by

\[
\left( C(\prod_{j=1}^{\infty} S^2, \mathcal{K}) \right) \varphi^m(Q) \left( C(\prod_{j=1}^{\infty} S^2, \mathcal{K}) \right).
\]

Then \( g \preceq l \cdot \varphi^m(Q) \) for some \( l \in \mathbb{N} \) ([RLL] Exercise 4.8) and an application of Lemma 5.1 shows that none of the projections \( \varphi^m(Q), m \in \mathbb{N} \), is properly infinite. By Lemma 5.2 this implies that the projection \( Q \) is finite.

The maps \( t_m \) are defined inductively as follows: For \( m = 1 \), note that

\[
\Gamma_1 = \{ \nu(j, I_s) \cup \tilde{B}_j \mid j \in \mathbb{Z}, s \in \mathbb{N} \}.
\]

For each \( j \in \mathbb{Z} \), set

\[
\Gamma_j^j := \{ \nu(j, I_s) \cup \tilde{B}_j \mid s \in \mathbb{N} \} =: \{ J^j_s \mid s \in \mathbb{N} \}.
\]

Then

\[
\Gamma_1 = \bigcup_{j=-\infty}^{\infty} \Gamma_j^j = \{ J^j_s \mid s \in \mathbb{N}, j \in \mathbb{Z} \}, \quad \text{and} \quad \Gamma_j^j \cap \Gamma_j^{j_2} = \emptyset \quad \text{for} \quad j_1 \neq j_2.
\]

(The latter property holds, because \( \nu \) was chosen to be injective.)

Since \( k \cdot g \preceq Q \), but \( (k + 1) \cdot g \not\preceq Q \), we know by Proposition 3.2 that for any finite subset \( F \subseteq \mathbb{N} \)

\[
\left| \bigcup_{s \in F} I_s \right| + k \geq |F|, \tag{***}
\]

and in the case \( k \geq 1 \) that there is some finite subset \( F_0 \) such that

\[
\left| \bigcup_{s \in F_0} I_s \right| + k = |F_0|.
\]

If \( k = 0 \), we set \( F_0 \) to be the empty set.

After choosing such a finite subset \( F_0 \), for any finite subset \( F \supseteq F_0 \) we must have

\[
\left| \left( \bigcup_{s \in F} I_s \right) \setminus \left( \bigcup_{s \in F_0} I_s \right) \right| \geq |F \setminus F_0|,
\]

since, otherwise, the finite subset \( F \) would violate the inequality (***). By injectivity of \( \nu \) we get for each \( j \in \mathbb{Z} \) that

\[
\left| \bigcup_{s \in F_0} \nu(j, I_s) \right| + k = |F_0|,
\]

and

\[
\left| \left( \bigcup_{s \in F} \nu(j, I_s) \right) \setminus \left( \bigcup_{s \in F_0} \nu(j, I_s) \right) \right| \geq |F \setminus F_0|.
\]

Then by Hall’s marriage theorem one can find for each \( j \) an injective mapping

\[
t_j^j : \Gamma_j^j \to \mathbb{N}
\]
such that for all $J^j_s = (\nu(j, I_s) \cup \tilde{B}_j) \in \Gamma^j_1$,

$$t_1^j(J^j_s) \in J^j_s,$$

and $t_1^j(J^j_s) \notin B_j$ whenever $s \notin F_0$.

(Using the cardinality of each $B_j \subseteq \tilde{B}_j$, $|B_j| = k$, we are able to construct the injective map

$$t_1^j : \{ J^j_s \mid s \in F_0 \} \to \{ J^j_s \mid s \in F_0 \}$$

successively in $s$, choosing different elements of $B_j$ for different values of $s$.)

By injectivity of $\nu$ and pairwise disjointness of the sets $B_j$, $j \in \mathbb{Z}$, there is then an injective map

$$t_1 : \{ J^j_s \mid s \in \mathbb{N}, j \in \mathbb{Z} \} = \Gamma_1 \to \mathbb{N}.$$

We have finished defining an injective map $t_1 : \Gamma_1 \to \mathbb{N}$.

Inductively we define $t_{m+1} : \Gamma_{m+1} \to \mathbb{N}$ after definition of $t_m : \Gamma_m \to \mathbb{N}$ by

$$t_{m+1}(\alpha_j(I)) := \nu(j, t_m(I))$$

for $\alpha_j(I) \in \Gamma_{m+1}$ (and $I \in \Gamma_m$).

With this choice the map $t_{m+1}$ is injective. Indeed, the equations

$$t_{m+1}(\alpha_j(I)) \parallel t_{m+1}(\alpha_j(\tilde{I}))$$

$$\nu(j, t_m(I)) \parallel \nu(\tilde{j}, t_m(\tilde{I}))$$

imply by injectivity of $\nu$ that

$j = \tilde{j}$, and $t_m(I) = t_m(\tilde{I})$.

By the induction hypothesis, $t_m$ was chosen to be injective, and hence

$I = \tilde{I}$.

For each $m \in \mathbb{N}$ we ended up with an injective map

$$t_m : \Gamma_m \to \mathbb{N}$$

such that $t_m(I) \in I$ for all $I \in \Gamma_m$, which is all that was left to construct.

**Corollary 5.4:** Let

$$Q = \bigoplus_{j=1}^{\infty} p_{I_j} \in \mathcal{M}(C(\prod_{j=1}^{\infty} S^2, \mathcal{K})).$$

Then $Q$ is non-full if, and only if, $Q$ is stably finite.

**Proof.** If all multiples $n \cdot Q$ of $Q$ are finite, then $n \cdot Q \not\preceq 1$ for any $n \in \mathbb{N}$ and $Q$ can’t be full. The converse direction follows from combining Theorem 5.3 with Theorem 4.2.

If a multiplier projection of the form

$$Q = \bigoplus_{j=1}^{\infty} p_{I_j}$$

is full, then $1 \preceq m \cdot Q$ for some $m \in \mathbb{N}$. Hence some multiple of $Q$ is properly infinite. The projection $Q$ itself might be finite though (see [R1]).

On the other hand if $Q$ is non-full, then $Q$ is stably finite by Corollary 5.4.

Summarized, the results state that every multiplier projection in $\mathcal{M}(C(\prod_{j=1}^{\infty} S^2, \mathcal{K}))$ of the special form (*) considered above is either non-full and stably finite, or full and stably properly infinite.
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