Encoded Recoupling and Decoupling: An Alternative to Quantum Error Correcting Codes, Applied to Trapped Ion Quantum Computation

D.A. Lidar and L.-A. Wu
Chemical Physics Theory Group, Chemistry Department, University of Toronto,
80 St. George St., Toronto, Ontario M5S 3H6, Canada

A recently developed theory for eliminating decoherence and design constraints in quantum computers, “encoded recoupling and decoupling”, is shown to be fully compatible with a promising proposal for an architecture enabling scalable ion-trap quantum computation [D. Kielpinski et al., Nature 417, 709 (2002)]. Logical qubits are encoded into pairs of ions. Logic gates are implemented using the Sørensen-Mølmer (SM) scheme applied to pairs of ions at a time. The encoding offers continuous protection against collective dephasing. Decoupling pulses, that are also implemented using the SM scheme directly to the encoded qubits, are capable of further reducing various other sources of qubit decoherence, such as due to differential dephasing and due to decohered vibrational modes. The feasibility of using the relatively slow SM pulses in a decoupling scheme quenching the latter source of decoherence follows from the observed 1/f spectrum of the vibrational bath.

PACS numbers: 03.67.Lx, 89.70.+c, 42.50.Lc

I. INTRODUCTION

In the quest to construct a scalable quantum computer, a recent proposal [1] advocating an array-based approach to quantum computing (QC) with trapped ions appears particularly promising. Ions are stored for later processing in a memory region, then transported to an interaction region, where pairs are coupled in order to enact quantum logic gates. This proposal overcomes some of the design constraints that plagued the original Cirac-Zoller (CZ) ion-trap-QC proposal [2], which prevented the latter from becoming fully scalable. In the new proposal an encoding of a single logical qubit into the states \(|0_L\rangle = |↓↑\rangle, |1_L\rangle = |↑↓\rangle\) of two trapped-ion (physical) qubits is used. Quantum logic gates are implemented using the Sørensen-Mølmer (SM) scheme [3, 4] (see also related schemes by Milburn et al. [5, 6]), which has the advantage of reduced sensitivity to motional state heating compared to the CZ proposal. The encoding into \(|↓↑\rangle, |↑↓\rangle\) is useful because these states form a decoherence-free subspace (DFS) \(\{|↑↓\rangle, |↓↑\rangle\}\) with respect to collective dephasing, a process whereby the environment introduces identical random phase modulations on groups of physical qubits [7]. In the context of the “quantum charge coupled device” (QCCD) proposed in [8], such a process is one of two dominant sources of decoherence. The DFS encoding reduces the collective dephasing problem by several orders of magnitude [9]. A method to perform universal QC using the SM scheme on these DFS qubits was proposed in [10], and independently in [11].

The DFS encoding \(|↓↑\rangle, |↑↓\rangle\) is well known [12, 13, 14], and its utility against collective dephasing was demonstrated experimentally using photons [17] and trapped ions [18]. The notion of universal QC using a DFS has been explored in general in [19, 20, 21, 22] for Hamiltonians that always preserve the DFS, in [23, 24] using a combination of DFS and active quantum error correction methods [25, 26, 27, 28, 29, 30, 31], and in [32], using an approach wherein transitions out of the code space are suppressed by continuous observation. Still more generally, the notion of universal QC while overcoming decoherence as well as design constraints has been explored by us and coworkers at a theoretical level in a series of recent papers [33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50], and in [51, 52] using an approach wherein transitions out of the code space are suppressed by continuous observation. Still more generally, the notion of universal QC while overcoming decoherence as well as design constraints has been explored by us and coworkers at a theoretical level in a series of recent papers [33, 34, 35, 36, 37, 38, 39, 40, 41], using an approach wherein transitions out of the code space are suppressed by continuous observation. Still more generally, the notion of universal QC while overcoming decoherence as well as design constraints has been explored by us and coworkers at a theoretical level in a series of recent papers [33, 34, 35, 36, 37, 38, 39, 40, 41].

This theory uses a combination of qubit encoding into a DFS with selective recoupling [11] and dynamical decoupling [12, 13, 14, 15, 16, 17, 18, 19, 20, 21] operations, so we refer to it as “encoded recoupling and decoupling” (ERD). The utility of ERD as a general method for quantum simulation, universal QC, and decoherence suppression has also been stressed and explored by Viola [53].

The main focus of ERD so far was on solid state [53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72] and NMR [72, 73] QC proposals. Here we present a unified treatment of the ERD ideas, and show that they apply also in an atomic physics setting, namely the QCCD ion-trap proposal [1]. Specifically, we show here how to perform universal QC on DFS qubits comprised of pairs of trapped ions, by using the SM scheme for quantum logic, in a manner that involves manipulating only pairs of ions at any given time, while always perfectly preserving the DFS encoding (see Brown et al. [53] for an interesting alternative set of ion pairs-only logic gates, which, however, does not preserve the DFS at all times). By applying strong and fast dynamical decoupling (“bang-bang” [12, 13]) SM pulses we show how to further drastically reduce sources of decoherence beyond collective dephasing. While a qubit is being stored an example of such a source of decoherence is deviations from the collective dephasing approximation. While a qubit is being manipulated for the purposes of QC, coupling to vibrational modes is necessary [2, 3, 4, 5, 6], and decoherence of these vibrational modes due to patch-potential noise is the second dominant source of qubit decoherence [18, 54, 55, 56]. A method to suppress vibrational mode decoherence (as well as heating, which is not as serious a problem when
the SM scheme is used), employing a version of the dynamic decoupling method known as “parity kicks”, was proposed and discussed in detail by Vitali & Tombesi (VT) [46, 47]. This method uses a fast+strong modulation of the trapping potential. We present here an alternative decoupling method for suppressing decoherence of ion-trap qubits due to their coupling to decohered vibrational modes, that operates directly on the qubit (spin-) states. The feasibility of this scheme, in spite of the relative slowness of the SM pulses, follows from the observed 1/f spectrum of the vibrational bath [34, 57]. The concentration of most of the bath spectral density in the vicinity of the low, rather than the high-frequency cutoff, implies much relaxed constraints on the decoupling pulses compared to those usually assumed [12, 13, 14, 17]. A full analysis of this result will be presented in a forthcoming publication [58].

More generally, we show here how all sources of decoherence beyond collective dephasing can in principle be suppressed using sufficiently strong+fast SM pulses. This includes so-called bath-induced “leakage errors”, wherein the system-bath coupling induces transitions into or out of the qubit subspace [12, 24, 15]. We provide feasibility estimates for the decoupling pulses and find that they are within current experimental reach. The overall picture emerging from this work is that ERD provides a means for a robust, decoherence-resistant implementation of universal QC with trapped ions. Experimental implementation of the ERD method should be possible with current ion-trap technology and we suggest a few experiments.

The structure of the paper is as follows. In section II we review the DFS encoding into two spins and the associated logic gates. We show how our previous formulation thereof can be reinterpreted in the context of acting on pairs of trapped ions within the SM scheme. We also present a method for coupling pairs of encoded qubits using pulses that involve controlling only pairs of ions at a time, while always preserving the DFS encoding. In the subsequent sections we discuss how to reduce decoherence. In section III we review the decoupling method, emphasizing its application to trapped ion arrays. We then proceed to apply the ERD method: in section IV we show how to eliminate the residual differential dephasing contribution to decoherence using SM pulses; and, in section V we discuss how to reduce all further sources of decoherence, including the component that arises due to coupling to decohered motional states. Then, in section VI we show how to fully implement ERD, i.e., we show how to combine universal QC via recoupling over DFS-encoded qubits with decoherence suppression via encoded decoupling. To make ERD fully effective for trapped ions we suggest to combine it with the VT potential-modulation method. Concluding remarks are presented in section VII.

II. ENCODED UNIVERSAL LOGIC GATES IN ION TRAPS

To fix terminology we first connect the methods developed in [33, 34, 35] to the gates proposed for trapped ions in [1]. Let $X_i, Y_i, Z_i$ denote the standard Pauli matrices $\sigma^x_i, \sigma^y_i, \sigma^z_i$, acting on the $i$th physical qubit (we will use both notations interchangeably). In [33] it was shown that for the code $\{|0_L\rangle = |\uparrow\rangle, |1_L\rangle = |\downarrow\rangle\}$ the encoded logical operations (involving the first two physical qubits) are

$$\begin{align*}
X_{12} &= \frac{1}{2}(X_1X_2 + Y_1Y_2), \\
Y_{12} &= \frac{1}{2}(Y_1X_2 - X_1Y_2), \\
Z_{12} &= \frac{1}{2}(Z_1 - Z_2).
\end{align*}$$

These operations form an $su(2)$ algebra (i.e., we think of them as Hamiltonians rather than unitary operators). We use a bar to denote logical operations on the encoded qubits. In [33, 34, 35] these logical operations were denoted by $T^{\alpha}_{ij}$, $\alpha \in \{x, y, z\}$, and a detailed analysis was given on how to use typical solid-state Hamiltonians (Heisenberg, XXZ, and XY models) to implement quantum logic operations using this DFS encoding. E.g., the term $X_1X_2 + Y_1Y_2$ is the spin-spin interaction in the XY model, and $Z_1 - Z_2$ represents a Zeeman splitting. A static Zeeman splitting and a controllable XY interaction can be used to generate a universal set of logic gates, a result that has very recently been applied in the context of spin-based QC using semiconductor quantum dots and cavity QED [59]. Similar conclusions hold when the XXY interaction is replaced by a Heisenberg [44, 50, 51] or XXZ interaction [44], or even by a Heisenberg interaction that includes an anisotropic spin-orbit term [58]. We remark that, as first shown in [19], the various types of exchange interactions can be made universal also without any single-qubit terms (such as a Zeeman splitting), by encoding into three or more qubits [44, 52, 53, 54, 55], a result that has been termed “encoded universality” [60].

A. Logic gates on two ions encoding a single logical qubit

Sørensen and Mølmer proposed a quantum logic gate that couples two ions via a two photon process that virtually populates the excited motional states of the ions [4]. The SM scheme works well even for ions in thermal motion, while the CZ scheme requires cooling the ions to their motional ground state. The SM scheme involves applying two lasers of opposite detuning $\delta$ to the two ions. Ideally the Lamb-Dicke limit should be satisfied:

$$(n + 1)\eta^2 \ll 1,$$ (2)
where $\eta$ is the Lamb-Dicke parameter and $n$ is the mean vibrational number. Deviations from the Lamb-Dicke limit lead to fidelity reduction that is proportional to $\eta^4$. The time required to prepare a maximally entangled state using the SM scheme is

$$\tau_{SM} = \frac{\pi}{\eta \Omega} \sqrt{K}$$  

(3)

where $\Omega$ is the Rabi frequency and $K$ is an integer determined by the ion trap. For realistic parameters, in the strong field limit ($K = 1$ in Eq. (12) of [4]), $\tau_{SM}$ can be made as short as 100usec.

In [4], it was shown that the SM two-ion gate can be expressed as follows. The unitary gate $U_{2i}(\theta, \phi_1, \phi_2)$ was introduced, which we here rename $U_{ij}(\theta, \phi_i, \phi_j)$:

$$U_{ij}(\theta, \phi_i, \phi_j) = \exp(i\theta X_{\phi_i} X_{\phi_j}) = \cos \theta I_i I_j + i \sin \theta X_{\phi_i} X_{\phi_j},$$

(4)

where $X_{\phi} \equiv X \cos \phi + Y \sin \phi$.

The phase $\phi_i$ is the phase of the driving laser at the $i$th ion, while $\theta$ is a parameter and can be set over a wide range of values [4-67]. Introducing the operators

$$\tilde{X}_{ij} \equiv \frac{1}{2}(X_i X_j - Y_i Y_j), \quad \tilde{Y}_{ij} \equiv \frac{1}{2}(Y_i X_j + X_i Y_j)$$

(5)

(denoted $R^x_{ij}$, $R^y_{ij}$ respectively in [33, 34, 35]) we can express

$$U_{ij}(\theta, \phi_i, \phi_j) = \cos \theta I + i \sin \theta (\cos \Delta \phi_{ij} \tilde{X}_{ij} + \sin \Delta \phi_{ij} \tilde{Y}_{ij} + \cos \Phi_{ij} \tilde{X}_{ij} + \sin \Phi_{ij} \tilde{Y}_{ij}),$$

(6)

where $\Phi_{ij} = \phi_i + \phi_j$. It is simple to check that $\tilde{X}_{ij}$ and $\tilde{Y}_{ij}$ annihilate the code subspace $\{\{0_i\} = \{|\downarrow\rangle, |1_i\rangle = |\uparrow\rangle\}$ and have non-trivial action (as encoded $X$ and $Y$) on the orthogonal subspace $\{|\downarrow\rangle, |\uparrow\rangle\}$. Therefore, as also observed in [4] and [3], upon restriction to the DFS we can write:

$$U_{ij}(\theta, \phi_i, \phi_j) \rightarrow_{\text{DFS}} \tilde{U}_{ij}(\theta, \Delta \phi_{ij}) = \exp(i\theta \tilde{X}_{\Delta \phi_{ij}}) = \cos \theta I + i \sin \theta \tilde{X}_{\Delta \phi_{ij}}.$$  

(7)

The fact that $\tilde{U}_{ij}$ depends only on the relative phase $\Delta \phi_{ij}$ is crucial: this quantity can be controlled by adjusting the angle between the driving laser and the interatomic axis, as well as by small adjustments of the trap voltages (which, in turn, control the trap oscillation frequency, and hence the ion spacing), whereas it is much harder to control the absolute phase $\phi_i$ [4, 52, 58], and hence also $\Phi_{ij}$. This is why the code subspace $\{|\downarrow\rangle, |\uparrow\rangle\}$ enjoys a preferred status over the subspace $\{|\downarrow\rangle, |\uparrow\rangle\}$. A thorough theoretical analysis of the approximations leading to the gate $U_{ij}(\theta, \phi_i, \phi_j)$ is given in [4] (see also [3] for an abbreviated exposition that emphasizes the connection to computation in a DFS).

Let us establish the connection between the seemingly distinct sets of logic operations in Eqs. (1, 4). To do so, we only need to use Eqs. (3), (4) while ignoring the component that annihilates the DFS $\{X, Y\}$. Then:

$$\exp(i\theta \tilde{X}_{12}) = U_{12}(\theta, \phi, \phi) = \tilde{U}_{12}(\theta, 0),$$  

$$\exp(i\theta \tilde{Y}_{12}) = U_{12}(\theta, \phi, \phi + \pi) = \tilde{U}_{12}(\theta, \pi)$$  

$$\exp(i\theta \tilde{Z}_{12}) = \exp(i\tilde{Z}_{12}) \exp(i\tilde{X}_{12}) \exp(-i\tilde{Z}_{12})$$  

$$= \tilde{U}_{12}(\frac{\pi}{4}, \frac{\pi}{2}) \tilde{U}_{12}(\theta, 0) \tilde{U}_{12}(\frac{\pi}{4}, \frac{\pi}{2}).$$

(8)

The third line follows from the elementary operator identity

$$X_{\phi} = X \cos \phi + Y \sin \phi = e^{-i\phi Z/2} X e^{i\phi Z/2}$$

(9)

which holds for any $su(2)$ angular momentum set $\{X, Y, Z\}$, i.e., operators that satisfy the commutation relation $[X, Y] = 2iZ$ (and cyclic permutations thereof), in particular also the encoded operators $\{\tilde{X}, \tilde{Y}, \tilde{Z}\}$.

This proves the equivalence of the two sets of operators. Using these results and Eq. (3), a more direct connection can be written in terms of the Hamiltonians:

$$\tilde{X}_{12} \iff X_{\phi} X_{\phi}$$  

(10)

$$\tilde{Y}_{12} \iff X_{\phi} X_{\phi + \pi/2}.$$  

(11)

where the equivalence is meant in terms of a projection of the RHS Hamiltonians to the DFS. In the context of ion-trap QC the logic gate $U(\theta, \Delta \phi)$ can be performed directly, so it may be more intuitively useful than the $\{X, Y, Z\}$ set. Eq. (8) shows that by properly choosing $\theta$ and $\Delta \phi_{ij}$ all single DFS-encoded qubit gates can be performed.

## B. Entangling gate between pairs of encoded qubits involving four ions

In [4] the following unitary gate was introduced

$$U_4 = \exp(-i\frac{\pi}{4} X_{\phi_1} X_{\phi_2} X_{\phi_3} X_{\phi_4})$$

$$= \frac{1}{\sqrt{2}} (I_1 I_2 I_3 I_4 - iX_{\phi_1} X_{\phi_2} X_{\phi_3} X_{\phi_4})$$

$$\rightarrow_{\text{DFS}} \frac{1}{\sqrt{2}} (I_1 I_2 I_3 I_4 - i\tilde{X}_{\Delta \phi_{12}} X_{\Delta \phi_{34}})$$

$$= \exp(-i\frac{\pi}{4} X_{\phi_1} X_{\phi_2} X_{\phi_3} X_{\phi_4}).$$

(12)

This gate, also considered in slightly less general form in [3], can be used to entangle two DFS-qubits. It involves simultaneous control over two phase differences $\Delta \phi_{12}$, $\Delta \phi_{34}$, and thus control over the motion of two pairs of ions. The case $\Delta \phi_{12} = \Delta \phi_{34} = 0$ was used in [3].
to demonstrate entanglement of four trapped-ion qubits, but this choice is not unique.

We now come to an important point that was not addressed in [1]: in order for the DFS encoding to offer protection against collective dephasing during the execution of the entangling gate, collective dephasing conditions must prevail over all four ions. To see this, note that a differential dephasing term such as \((Z_i - Z_j) \otimes B\) (where \(B\) is a bath operator) does not commute with \(U_4\), so that if such a term exists during gate execution then the DFS will not be preserved, according to a theorem in [14]. On the other hand, collective dephasing over all four ions, expressed by a system-bath coupling of the form \((\sum_{i=1}^4 Z_i) \otimes B\), does commute with \(U_4\), so that in this case the DFS is preserved [14]. While deviations from collective dephasing over pairs of ions have been shown experimentally to be small [18], this may not be the case over the length scales involving four ions [19]. We discuss in section [3] how to create such extended collective dephasing conditions.

Taken together, the results in this section show how universal QC can be implemented using trapped ions by applying the SM scheme to pairs of ions at a time, each encoding a DFS qubit. The DFS encoding takes care of protecting the encoded information against collective dephasing. We now move on to a discussion of how to reduce additional source of decoherence.

III. DYNAMICAL DECOUPLING PULSES AND THEIR APPLICATION TO TRAPPED IONS

Let us briefly review the decoupling technique, as it pertains to our problem (for an overview see, e.g., [20]). Decoupling, as proposed by Viola and Lloyd [22, 23], relies on the ability to apply strong and fast pulses, in a manner which effectively averages the system-bath interaction Hamiltonian \(H_{SB}\) to zero. A quantitative analysis was first performed in [22, 23] for pure dephasing in the linear spin-boson model (which corresponds to the Ohmic case of the Caldeira-Leggett model [24]): \(H_{SB} = \gamma \sigma^2 \otimes B\), where \(B\) is a Hermitian boson operator. The analysis was recently extended to the nonlinear spin-boson model, with similar conclusions about performance [25]. Imperfections in the pulses were considered in [22, 23], and it was shown that an optimal value for the pulse period can be found. Since the decoupling pulses are strong one ignores the evolution under \(H_{SB}\) while the pulses are on, and since the pulses are fast one ignores the evolution of the bath under its free Hamiltonian \(H_B\) during the pulse cycle. The simplest example of eliminating an undesired unitary evolution \(U = \exp[-it(H_{SB} + H_B)]\), is the “parity-kick sequence” [22, 23, 25]. Suppose we have at our disposal a fully controllable interaction generating a gate \(R\) such that “\(R\) conjugates \(U\)”: \(R^\dagger UR = U\). Then the sequence \(UR^\dagger UR = I\) serves to eliminate \(U\). A simple example of a parity kick sequence is the following. Assume we can turn on the single-qubit Hamiltonian \(\Omega X_i\) for a time \(\pi/2\Omega\). This generates the single-qubit gate \(X_j = \exp(-i\pi/2 X_j)\).

Suppose that \(H_{SB} = \sum_{i=1}^N \sum_{\alpha \in \{X,Y,Z\}} \gamma^\alpha \sigma_i^\alpha \otimes B_i^\alpha\). Each term in \(H_{SB}\) either commutes or anti-commutes with \(X_j\). If a term \(A\) in \(H_{SB}\) anticommutes with \(X_j\) then the evolution under it will be conjugated by the gate \(X_j\): \(X_j \exp(-iA\Delta t)X_j = \exp(-iX_jAX_j\Delta t) = \exp(iA\Delta t)\). This allows for selectively removing this term using the parity-kick cycle, which we write as: \([\Delta t, X_j, \Delta t, X_j]\). Reading from right to left, this notation means: apply \(X_j\) pulse, free evolution for time \(\Delta t\), repeat. Suppose that we can also apply the single qubit gate \(Y_j\). Then, since every system factor in the above \(H_{SB}\) contains a single-qubit operator, it follows that we can selectively keep or remove each term in \(H_{SB}\) by using the parity-kick cycle. Note, however, that without additional symmetry assumptions, this procedure, if used to eliminate all errors, requires a number of pulses that is exponential in the number of qubits \(N\) [24, 25]. The reason is that without symmetry assumptions we will need at least two non-commuting single-qubit operators per qubit (e.g., \(X, Y\)), and we will need to concatenate decoupling pulse sequences. Below we show how to dynamically generate such symmetries, in a way that avoids this exponential scaling (for a discussion of this point see the Conclusions section).

Note that in the analysis of the parity kick cycle we ignored \(H_{SB}\) and \(H_B\) while \(R\) was operating; this is justified by the strength assumption. The bath Hamiltonian \(H_B\) commutes with the applied pulses, but its effect is very important since \([H_B, H_{SB}] \neq 0\) in general. Therefore if the bath has spectral components at frequencies higher than the inverse of the interval between decoupling pulses, then the bath density matrix will pick up phases that are essentially random, and this effect will show up as decoherence (for a quantitative analysis see [24, 25]). Hence it is commonly assumed that the pulse interval, \(\Delta t\), should be small compared to the inverse of the high-frequency cutoff \(\omega_c\) of the bath spectral density \(I(\omega)\) [24, 25], which also sets the scale of the bath-induced noise correlation time \(t_c\) (the speed assumption). It can be shown that the overall system-bath coupling strength \(\gamma_{SB}\) is then renormalized by a factor \(\Delta t/\omega_c\) after a cycle of decoupling pulses [15], or that the bath-induced error rate is reduced by a factor proportional to \((\Delta t/\omega_c)^2\) [15]. Using a Magnus expansion [71], it can be shown that there is a hierarchy of decoupling schemes, whereby \(\gamma_{SB}\) is renormalized by a factor \((\Delta t/\omega_c)^k\), where \(k \geq 1\) is the order of the decoupling scheme [14]. The implication for single-qubit dephasing, \(H_{SB} = \sum_{i=1}^4 \gamma_{SB}Z \otimes B\) (\(B\) is a dimensionless bath operator), is that the dephasing time \(T_2\) is increased by a factor \(1/(\Delta t\omega_c)^{2k}\) [14].

It seems crucial to be able to apply pulses at intervals \(\Delta t \ll 1/\omega_c\). However, as shown first by Viola & Lloyd [22], and then by VT in their quantitative analysis of a vibrational mode linearly coupled to a boson bath, a finite-temperature bath sets another, thermal timescale that must be beat in order for the decoupling method to
work [47]. Let the system-bath coupling be

$$H_{SB}^{\text{ vib}} = \gamma \sum_k (ab^*_k + a^*b_k),$$  \hspace{1cm} (13)$$

where $a$ ($b$) is an annihilation operator for the system ($k$th bath) vibrational mode, and $\gamma$ is the (for simplicity uniform) energy damping rate. In the context of trapped ions the bath is provided by fluctuating patch-potentials (due, e.g., to randomly oriented domains at the surface of the electrodes or adsorbed materials on the electrodes) [56]. Then VT showed that the decoupling pulse interval (in fact, the entire cycle time) must be shorter also than the thermal decoherence time

$$t_{\text{ dec}}(T) = \{\gamma[1 + 2n(T)]\}^{-1},$$

where $n(T) = [e^{\hbar \omega_0 / kT} - 1]^{-1}$ is the mean vibrational number of the system oscillator at thermal equilibrium with temperature $T$, and $\omega_0$ is the frequency of the oscillator, i.e., the system is described by the harmonic oscillator Hamiltonian $H_S = \hbar \omega_0 a^*a$. Thus the timescale condition for successful decoupling is

$$\Delta t \ll \min\{1/\omega_c, t_{\text{ dec}}(T)\}.$$  

As shown in the VT analysis, the timescale $t_{\text{ dec}}(T)$ is especially relevant for vibrational mode decoherence in ion traps, which as already mentioned above, is responsible for qubit decoherence during quantum logic gate operations.

However, for trapped ions experimental evidence so far points to a $1/f^\alpha$ spectrum for the vibrational bath over a range $1 - 100$ MHz [56, p.5], implying that there is no clear high-frequency cutoff $\omega_c$. Encouragingly, in a recent experiment involving a charge qubit in a small superconducting electrode (Cooper-pair box), a version of parity-kick decoupling was successfully used to suppress low-frequency energy-level fluctuations (causing dephasing) due to $1/f$ charge noise [72]. This suggests that decoupling can help even in the absence of a clear cutoff frequency. Recent theoretical results suggest this observation [28]: for $1/f$ noise most of the bath spectral density $I(\omega)$ is concentrated in the low, rather than the high-end of the frequency range.

In spite of the apparent $1/f^\alpha$ spectrum in trapped ions, VT used a cutoff estimate of $\omega_c \lesssim 100$ MHz [47], and showed that suppression of vibrational decoherence can be accomplished by pulsing the oscillation frequency $\omega_0$ of the ion chain (i.e., by pulsing the trapping potential), provided $\Delta t < 1/\omega_c \sim 1$ nsec, and $T < 10$ mK.

Given the estimate in Section [11A] of $\tau_{SM} \gtrsim 1$ $\mu$s for the SM gate, it is clear that we cannot hope to satisfy the strict $\Delta t < 1$ nsec timescale requirement which would be needed in order to use decoupling directly on the qubit, rather than the vibrational modes, assuming the VT estimate of $\omega_c$. However, the theoretical analysis [38] and the success of parity-kick decoupling in the presence of $1/f$ noise in the charge qubit case [72] suggests that it may well be worthwhile to apply decoupling pulses on the qubit in addition to, or perhaps instead of, the VT trapping-potential-modulation scheme.

Now let us comment on the strength assumption. Here we must make sure that the amplitude of the decoupling pulses, $\Omega$, can be made much stronger than the system-bath interaction $\gamma$ in Eq. (13), i.e., the heating rate from the vibrational ground state of the ion chain. Experimental measurements of $\gamma$ are very sensitive to trap geometry, secular frequency, and size [56], and range from a few Hz to a few tens of KHz [56, 73]. On the other hand, one can have $\Omega$ as high as 1 MHz [47], so the strength assumption can be comfortably satisfied. This does come at a price, however, since in the strong field limit the SM gate is perturbed by a term that yields direct, off-resonant coupling of the qubit $|\uparrow\rangle$ and $|\downarrow\rangle$ states without changes in the vibrational motion [56]. This is a unitary gate error that decreases the gate fidelity by $(N/2)(\Omega/\delta)^2$, where $N$ is the number of ions participating in the gate [Table II]. This forces us to be in a parameter regime where $\Omega \ll \delta$. In principle it is possible to exactly cancel this effect if the duration of the laser pulses is chosen so that both Eq. (3) and the condition $\tau_{SM} = K^\prime \pi/\delta$ are satisfied, where $K^\prime$ is an integer and $\delta$ is the detuning. However, in the context of decoupling we will also need to satisfy conditions such as $\Omega \tau_{SM} = \pi/m$ where $m$ is an integer. Putting these conditions together yields

$$\Omega K^\prime \pi = \pi/m \Rightarrow \delta = mK^\prime \Omega,$$

$$\Omega \pi \sqrt{K} \eta = \pi/m \Rightarrow \eta = m\sqrt{K}.$$  

While there is no problem with the first of these, the second condition implies that we cannot be in the Lamb-Dicke limit, Eq. (3). Therefore exact cancellation is not possible in our case, and we must resort to $\Omega \ll \delta$ in order to keep the fidelity reduction to a minimum. On the other hand, the kind of unitary error that is caused by off-resonant coupling can be corrected by optimal control pulse shaping methods [66].

Finally, we note that fluctuations of various experimental parameters, such as intensity and phase fluctuations of the exciting lasers, can cause pure dephasing of the vibrational modes, in addition to the dissipative coupling described above [70]. Clearly, the success of decoupling strategies hinges on strong suppression of such fluctuations, as in the threshold theorem of fault tolerant quantum error correction [52, 26, 44, 41].

To conclude, the discussion in this section indicates that the experimental viability of decoupling schemes in ion traps is rather promising, although it is hard to estimate their success at this point. In the following sections the analysis will be carried out at a more abstract level, emphasizing the algebraic conditions for a successful implementation of ERD. In the end it will be up to an experiment to decide the usefulness of the proposed schemes.
IV. Creating Collective Dephasing Conditions Using Decoupling Pulses: Reducing decoherence during Storage

One of the important advantages of the DFS encoding (\{|\uparrow\rangle,|\downarrow\rangle\}) is that it is immune to collective dephasing. However, other sources of decoherence inevitably remain. In this and the following section, we algebraically classify all additional decoherence effects and show how they can be eliminated, in particular in the context of trapped ions.

A. Creating collective dephasing on a pair of ions

First, let us analyze the effect of breaking the collective dephasing symmetry, by considering a system-bath interaction of the form

\[ H_{SB}^{\text{dep}(2)} = Z_1 \otimes B_1^z + Z_2 \otimes B_2^z \]

where \(B_1^z, B_2^z\) are arbitrary bath operators. This describes a general dephasing interaction on two qubits, and we can expect this to be the case during storage of trapped ion qubits in the QCCD proposal. The source of such dephasing during storage is long wavelength, randomly fluctuating ambient magnetic fields [18], that randomly shift the relative phase between the qubit \(|\uparrow\rangle\) and \(|\downarrow\rangle\) states through the Zeeman effect. The interaction can be rewritten as a sum over a collective dephasing term \(Z_1 + Z_2\) and another, differential dephasing term \(Z_1 - Z_2\), that is responsible for errors on the DFS:

\[ H_{SB}^{\text{dep}(2)} = (Z_1 + Z_2) \otimes B_{\text{col}}^z + (Z_1 - Z_2) \otimes B_{\text{dif}}^z \]

Here \(B_{\text{col}}^z = (B_1^z + B_2^z)/2\) and \(B_{\text{dif}}^z = (B_1^z - B_2^z)/2\). If \(B_{\text{dif}}^z\) were zero then there would only be collective dephasing and the DFS encoding would offer perfect protection [54]. However, in general \(B_{\text{dif}}^z \neq 0\), and the DFS encoding will not suffice to offer complete protection.

The crucial observation is that, since \(Z_1 - Z_2 \propto \mathbf{x}_{12}\) [recall Eq. (6)], the offending term causes logical errors on the DFS [33]. Then the problem of \(B_{\text{dif}}^z \neq 0\) can be solved using a series of pulses that symmetrize \(H_{SB}^{\text{dep}(2)}\) such that only the collective term remains, as shown in [33, 35, 36]. To do so note that since the offending term \(\propto X_{12}\), it anticommutes with \(X_{12} = \frac{1}{2}(X_1X_2 + Y_1Y_2)\).

At the same time \(X_{12}\) commutes with \(Z_1 + Z_2\). This allows us to flip the sign of the offending term by using a pair of \(\pm \pi/2\) pulses in \(X_{12}\), while leaving only the collective term. Evolution with the flipped sign followed by unaltered evolution leads to cancellation of the offending term. Specifically [35]:

\[ e^{-iH_{SB}^{\tau}}e^{-i\frac{\pi}{2}X_{12}}e^{-iH_{SB}^{\tau}}e^{i\frac{\pi}{2}X_{12}} = e^{-i(Z_1+Z_2)\otimes B_{\text{col}}^z2\tau} \]

or, in ion-trap terms:

\[ e^{-iH_{SB}^{\tau}}U_{12}(\frac{\pi}{2},0)e^{-iH_{SB}^{\tau}}U_{12}(\frac{\pi}{2},0) = e^{-i(Z_1+Z_2)\otimes B_{\text{col}}^z2\tau}, \]

where \(U_{ij}(\theta,\Delta\phi_{ij})\) was defined in Eq. (3), and we used the identification found in Eq. (8). This equation means that the system-bath coupling effectively looks like collective dephasing at the end of the pulse sequence. Thus, the system is periodically \(2\pi\) projected into the DFS.

In order for the procedure described in Eq. (14) to work, the SM gate \(U_{12}(\pm \frac{\pi}{2},0)\) must be executed at a timescale faster than the cutoff frequency associated with the fluctuating magnetic fields causing the differential dephasing term in \(H_{SB}^{\text{dep}(2)}\). This cutoff has not yet been characterized experimentally, but the decay rate of the DFS-encoded state of two ions has been measured to be 2.2 KHz [18]. Using this as a rough estimate for the cutoff frequency, we see that the procedure of Eq. (14) is likely to be attainable with fast \((\gamma_{SM} \approx 1\mu s)\) SM pulses.

B. Creating collective dephasing on a block of four ions

So far we have discussed creation of collective dephasing conditions on a single DFS qubit. However, as mentioned in Section II, it is essential for the reliable execution of an entangling logic gate to have collective dephasing over all four ions participating in the gate, even if only two are coupled at a time. A procedure for creating collective decoherence conditions over blocks of 3, 4, 6 and 8 qubits was given in [36]. Here we show how to do the same for a block of 4 qubits with collective dephasing.

Let us start with a general dephasing Hamiltonian on \(N\) ions, and rewrite it in terms of nearest-neighbor sums and differences:

\[ H_{SB}^{\text{dep}} = \sum_{i=1}^{N} Z_i \otimes B_i = \sum_{j=1}^{N/2} (Z_{2j} + Z_{2j-1}) \otimes B_{2j}^+ + (Z_{2j} - Z_{2j-1}) \otimes B_{2j}^- , \]

where \(B_{2j}^\pm = (B_{2j} \pm B_{2j-1})/2\). As noted above, \(Z_{2j} \propto \mathbf{x}_{2j-1,2j}\), so that to eliminate all nearest-neighbor differences of the form \(Z_{2j} - Z_{2j-1}\) we can use the collective decoupling pulse \(X_{nn} = \bigotimes_{j=1}^{N/2} e^{i\frac{\pi}{2}X_{2j-1,2j}}\):

\[ e^{-iH_{SB}^{\tau}}X_{nn}e^{-iH_{SB}^{\tau}}X_{nn}^\dagger = e^{-i2\tau \sum_{j=1}^{N/2}(Z_{2j}+Z_{2j-1})\otimes B_{2j}^+} \]

or, in ion-trap terms:
where \( B_{2j}^{+} \equiv (B_{2j+2}^{+} \pm B_{2j}^{+})/2 \). The term in the first line contains only the desired block-collective dephasing over 4 ions. The term in the second line contains undesired differential dephasing terms that we wish to eliminate. But these terms once again have the appearance of encoded \( Z \) operators, between next-nearest neighbor ion pairs. Therefore we need to apply a second collective pulse \( X_{nnn} = \bigotimes_{j=1}^{N/2} e^{i\pi/2 \bar{X}_{j-1,2j+1}} e^{i\pi/2 \bar{X}_{2j,2j+2}} \), that applies encoded \( X \) operators on these ion pairs. At this point we are left just with collective dephasing terms on blocks of 4 ions, as required:

\[
e^{-i\tau \sum_{j=1}^{N/2} (Z_{2j} + Z_{2j-1}) \otimes B_{2j}^{+}} e^{-i\tau \sum_{j=1}^{N/2} (Z_{2j+2} + Z_{2j+1} + Z_{2j} + Z_{2j-1}) \otimes B_{2j}^{+}} + e^{-i\tau \sum_{j=1}^{N/2} [(Z_{2j+2} - Z_{2j}) + (Z_{2j+1} - Z_{2j-1})] \otimes B_{2j}^{+}}.
\]

(15)

This pulse sequence is important to ensure that collective dephasing conditions will prevail during the execution of logic gates between DFS qubits, as emphasized in Section IIIB.

To conclude, the procedures discussed in this section provide a means for engineering collective dephasing conditions in an ion trap experiment. We propose here to implement these symmetrization schemes experimentally. Moreover, we propose to combine them with the logic gates described in Sec. [III]. How to do this efficiently is discussed in Sec. [V] below.

V. REDUCTION OF ALL REMAINING DECOHERENCE ON A SINGLE DFS QUBIT DURING LOGIC GATE EXECUTION

The reduction of differential dephasing errors, as in the previous subsection, is particularly relevant for storage errors. However, this is only the first step. Additional sources of decoherence may take place during storage, and in particular during the execution of logic gates, the most dominant of which is qubit decoherence due to coupling to decohered vibrational modes, as discussed above. It is useful to provide a complete algebraic classification of the possible decoherence processes. This will allow us to see what can be done using SM-decoupling pulses. To this end let us now write the system-bath Hamiltonian on two physical qubits in the general form

\[ H_{SB} = H_{\text{Leak}} + H_{\text{Logi}} + H_{\text{DFS}} \]

where

\[ H_{\text{DFS}} = \text{Span}\{ZI + IZ, XY + YX, XX - YY, ZZ, II\} \]

\[ H_{\text{Leak}} = \text{Span}\{XI, IX, YI, IY, ZX, ZX, YZ, ZY\} \]

\[ H_{\text{Logi}} = \text{Span}\{\bar{X} = XX + YY, \bar{Y} = YX - XY\} \]

\[ \bar{Z} = \frac{ZI - IZ}{2} \]

(16)

where \( I \) is the identity operator, \( XZ \equiv X_1 Z_2 \) (etc.), and where Span means a linear combination of these operators tensors with bath operators. The 16 operators in Eq. (16) form a complete basis for all 2-qubit operators. This classification, first introduced in [27], has the following significance. The operators in \( H_{\text{DFS}} \) either vanish on the DFS, or are proportional to identity on it. In either case their effect is to generate an overall phase on the DFS, so they can be safely ignored from now on. The operators in \( H_{\text{Leak}} \) are the leakage errors: terms that cause
transitions between states inside and outside of the DFS. A universal and efficient decoupling method for eliminating such errors, for arbitrary numbers of (encoded) qubits was given in [39]. Finally, the operators in $H_{\text{Logi}}$ have the form of logic gates on the DFS. However, these are undesired logic operations, since they are coupled to the bath, and thus cause decoherence.

It is worthwhile to already emphasize that the operator $YI + IY \in H_{\text{Logi}}$ is of particular importance: As shown in [4, Eq.43], this is the operator that describes qubit decoherence due to motional decoherence during the application of the SM gate.

In the previous subsection we showed how to eliminate the logical error $\bar{Z}$, but we see now that this was only one error in a much larger set. To deal with this error we need to introduce one more pulse pair that anticommutes with $\bar{X}_{12}$, e.g., $\exp(\pm i\frac{\pi}{2}\bar{X}_{12}) = \bar{U}_{12}(\pm \frac{\pi}{2})$.

Let us now see how to combine all the decoherence elimination pulses into one efficient sequence. First we introduce the abbreviations

$$\Pi \equiv \bar{U}_{12}(\pm \pi, 0) = \exp(\pm i\pi \bar{X}_{12}) = \Pi^\dagger = PP$$
$$Q \equiv \bar{U}_{12}(\pm \frac{\pi}{2}, \frac{\pi}{2}) = \exp(\pm i\frac{\pi}{2}\bar{Y}_{12})$$
$$\Lambda \equiv \bar{U}_{12}(\pm \pi, \frac{\pi}{2}) = \exp(\pm i\pi \bar{Y}_{12}) = \Lambda^\dagger = QQ$$

As argued above, the $\pi$ pulse $\Pi$ eliminates $H_{\text{Leak}}$: \[ \exp[-i(H_{\text{Logi}} + H_{\text{DFS}})2\tau] = [\tau, \Pi, \tau, \Pi]. \]

This may be sufficient in practice, since as argued above this pulse sequence eliminates the $YI + IY$ term, and the DFS encoding takes care of collective dephasing. Thus we expect that using cycles of two pulses we can almost entirely eliminate the two most important sources of decoherence. This expectation of course depends on the time scale requirement for decoupling being satisfied, as discussed in detail in Section III above. In practice it may well be advantageous to combine the DFS encoding and $[\tau, \Pi, \tau, \Pi]$ pulse sequence with the VT method of pulsing the trapping potential [43, 44].

Now let us discuss adding the extra pulses needed to achieve full decoherence elimination. The $\pi/2$ pulse $P$ eliminates $\bar{Y}$ and $\bar{Z}$ in $H_{\text{Logi}}$. Combining this with the sequence for leakage elimination we have the sequence of 4 pulses:

$$e^{-iH_{\text{DFS}}8\tau} = [U(\tau)\Pi U(\tau)]^{P_1}[U(\tau)\Pi U(\tau)]^{P_1}[U(\tau)\Pi U(\tau)]^{P_1}[U(\tau)\Pi U(\tau)]^{P_1}Q$$

(18)

(where we have used $\Pi P_1 = P, \Pi P = P_1$).

If we wish to entirely eliminate decoherence then we are left just with getting rid of the logical error due to $\bar{X}$. To eliminate it we now combine with the $\bar{Y}$-direction, $\pi/2$ pulse, $Q$:

At this point we are left with just a single error: $\bar{X}_{12} \otimes B$ itself, in $H_{\text{Logi}}$. Clearly, we cannot use a pulse generated by $\bar{X}_{12}$ to eliminate this error. Instead, to deal with this error we need to introduce one more pulse pair that anticommutes with $\bar{X}_{12}$, e.g., $\exp(\pm i\frac{\pi}{2}\bar{Y}) = \bar{U}_{12}(\pm \frac{\pi}{2})$.

$$\bar{U}_{ij}(\theta, \Delta \phi_{ij}) = \cos \theta \bar{I} + i \sin \theta \bar{X}_{\Delta \phi_{ij}}.$$ Finally, note that in principle the last pulse sequence is applicable also to other QC proposals, such as NMR and quantum dots.

One important caveat (mentioned in Section III above)
is that, because we need very strong and fast pulses, our gate operation may become imperfect. Specifically, off-resonant coupling and deviations from the Lamb-Dicke approximation may become important. The former introduces a term $XI + IX$ into the Hamiltonian generating the $U_{ij}(\theta, \phi_i, \phi_j)$ gate \[\text{(Sec. IIIA)},\] which can cause unitary leakage errors from the DFS. These can in turn be reduced using the methods in \[\text{(19), (20)}\]. Whether the decoupling method we have proposed offers an improvement will have to be put to an experimental test.

VI. COMBINING LOGIC GATES WITH DECOUPLING PULSES

So far we have discussed computation using the encoded recoupling method (Section \[\text{IV}\]), and encoded decoupling (Sections \[\text{VII}\]). We now put the two together in order to obtain the full ERD picture. At least two methods are available for combining quantum computing operations with the sequences of decoupling pulses we have presented above. For a general analysis of this issue see \[\text{(21)}\].

A. Fast + Strong Gates Method

The decoupling pulse sequences given in Sec. \[\text{VII}\] “stroboscopically” create collective dephasing conditions at the conclusion of each cycle. As noted above, this is equivalent to a periodic projection into the DFS. This property allows for “stroboscopic” quantum computation at the corresponding projection times \[\text{(21)}\]. Here the computation pulses need to be synchronized with the decoupling pulses, and inserted at the end of each cycle. The amount of time available for implementation of a logic gate is no more than the bath correlation time \[\tau_c = 2\pi/\omega_c\]. Assuming the dominant decoherence contributions not accounted for by the DFS encoding to come from differential dephasing (setting the \[\tau_c\] time-scale) and \[1/f\] noise, and that we already assumed that we can use pulses with interval \[\Delta t \ll \tau_c\], it is consistent to assume that we can then also perform logic gates on the same time scale.

B. Fast + Weak Gates Method

There may be an advantage to using fast but weak pulses for the logic gates, while preserving the fast + strong property of the decoupling pulses. To see how to combine logic gates with decoupling in this case, let us denote by \[H_S = X_{\phi_i}X_{\phi_j}\] the controllable system Hamiltonian that generates the entangling gate $U_{ij}(\theta, \phi_i, \phi_j)$ [recall Eq. \[\text{(4)}\]]. Suppose first that we turn on this logic-gate generating Hamiltonian in a manner that is neither very strong nor very fast, so that the system-bath interaction is not negligible while \[H_S\] is on (this obviously puts less severe demands on experimental implementation). Then the corresponding unitary operator describing the dynamics of system plus bath is:

\[
\hat{U}(t) = \exp[-i t(H_S + H_{SB} + H_B)].
\]

Now, if we choose \[H_S\] so that it commutes with the decoupling pulses, then we can show that after decoupling

\[
\hat{U}(t) \mapsto \exp[-i 2t(H_S + H_B)],
\]

provided \[t\] is sufficiently small. Tracing out the bath then leaves a purely unitary, decoherence-free evolution on the system. To prove this, assume we have chosen \[t\] and the decoupling Hamiltonian \[H_S\] so that (i) \[\exp(-itH_S)H_SB\exp(itH_S) = -H_SB\] (the parity kick transformation), and (ii) \[\{H_S, H_B\} = 0\]. Then

\[
\hat{U}(t) e^{-it' H_S} \hat{U}(t) e^{-it' H_S'} = \hat{U}(t) e^{-it[H_S + e^{-it' H_S} H_{SB} e^{it' H_S'} + H_B]}
\]

\[
= e^{-it[H_S + H_{SB} + H_B]} e^{-it[H_S - H_{SB} + H_B]}
\]

\[
= e^{-i [2t(H_S + H_B) + t^2(|H_SB, H_S| + |H_B, H_S|)] + O(t^3)},
\]

where we have used the Baker-Campbell-Hausdorff formula, \[\exp(\alpha A) \exp(\alpha B) = \exp(\alpha (A + B) + \frac{\alpha^2}{2}[A, B] + O(\alpha^3))\].

Setting \[H_S = \Omega S\] and \[H_{SB} = \gamma_{SB} S' \otimes B\] we have the condition \[t \ll 1/\sqrt{\gamma_{SB}}\], in order to be able to neglect the \[O(t^2)\] term \[|H_{SB}, H_S|\]. Using \[\Omega \sim 1\text{MHz}, \gamma_{SB} \sim 10\text{KHz}\] as in Section \[\text{III}\], we find \[t \ll 10\text{msec}\]. However, the more stringent constraint comes from the \[|H_{SB}, H_B|\] term, since \[H_B\] is not bounded for a harmonic oscillator. A more careful analysis then shows the familiar conclusion, that the bath should not be allowed to evolve for longer than its correlation time \[\tau_c\]. Hence the actual requirement may still be the far more stringent condition \[t \ll 1/\omega_c \leq 1\text{sec}\] for the decoupling pulse interval; see Sec. \[\text{III}\]. This cannot be satisfied with SM pulses, but in this case we can resort to the VT potential modulation method. When we do this in conjunction with SM decoupling pulses we must be sure that Eq. \[\text{(21)}\]
is an excellent approximation. On the other hand, the requirements for a $1/f$ bath spectral density are far less stringent and may be satisfied even with SM pulses alone [58]. Furthermore, for the rotation angle $\theta = \Omega$ describing the computation we have $\theta \ll \sqrt{\Omega/\gamma_{SB}} \leq 10$, which means that there is no restriction on applying large rotations.

Let us now show how to efficiently combine logic operations and decoupling pulses. For simplicity consider only the case where we can neglect the $X$ error, i.e., our decoupling sequence is the 4-pulse one given in Eq. (18). Suppose we wish to implement a logical $Y$ operation, i.e., $\exp(-i\theta X_{12})$. Recall [Eq. (10)] that this involves turning on the Hamiltonian $H_S^Y = \Omega_X X_\phi X_\phi \xrightarrow{\text{DFS}} \Omega_X Y_{12}$ between two physical qubits. Because the decoupling pulses $P = \exp(-i\phi Y_{12})$ and $\Pi = \exp(\pm i\theta X_{12})$ are generated in terms of the same Hamiltonian, they commute with $H_S^Y$ while eliminating $H_{SB}$ (except for the terms in $H_{SB}$ that have trivial action on the DFS). Thus the conditions under which Eq. (21) were shown to hold are satisfied. This allows us to insert the logic gates into the four free evolution periods involved in the pulse sequence of Eq. (18). Thus, the full pulse sequence that combines creation of collective dephasing conditions with execution of the logic gate is:

$$e^{-i(t/4)\Omega Y_{12} + H_{DFS}} = \tilde{U}(t/4)\Pi\tilde{U}(t/4)P\tilde{U}(t/4)\Pi\tilde{U}(t/4)P^\dagger,$$

(21)

with $\tilde{U}(t) = \exp[-it(H_S^Y + H_{SB} + H_B)]$, and which, using the DFS encoding, is equivalent to the desired $\exp(-i\phi Y_{12})$. This involves 8 control pulses, 4 of which are of the fast+strong type (those involving $P$ and $\Pi$), and 4 of which must be fast, but need not be so strong that we can neglect $H_{SB}$.

If we wish to implement logical $Y$ operation, i.e., $\exp(-i\phi Y_{12})$, then we cannot now use $P$ and $\Pi$, since they anticommute with $Y_{12}$ and will eliminate it. Instead we should use decoupling pulses generated in terms of $Y_{12}$, which will also have the desired effect of eliminating $H_{\text{leak}}$, as well as $X$ and $Z$ logical errors, while commuting with the logical gates that create the collective dephasing interaction (and for this reason can of course not eliminate $Y$ errors). These are just the $Q$ and $\Lambda$ pulses defined in Eq. (17). In ion trap terms this implies [recall Eq. (11)] turning on the Hamiltonian $H_S^{Y \phi} = \Omega_Y X_\phi X_{\phi + \pi/2} \xrightarrow{\text{DFS}} \Omega_Y Y_{12}$ between two physical qubits. Thus:

$$e^{-i(t/4)\Omega Y_{12} + H_{DFS}} = \tilde{U}(t/4)\Lambda\tilde{U}(t/4)Q\tilde{U}(t/4)\Lambda\tilde{U}(t/4)Q^\dagger,$$

(22)

with $\tilde{U}(t) = \exp[-it(H_S^{Y \phi} + H_{SB} + H_B)]$, and which, using the DFS encoding, is equivalent to the desired $\exp(-i\phi Y_{12})$.

Finally, to generate single DFS-qubit rotations about an arbitrary axis we can combine Eqs. (21), (22) according to the Euler angles construction. Given that Eqs. (21), (22) each take 8 pulses, the Euler angle method will generate an arbitrary DFS-qubit rotation in at most 24 pulses.

Concerning gates that entangle two DFS-qubits, the situation is more involved, since now the next-nearest neighbor pulses in Eq. (19) that create the collective dephasing conditions on four ions, do not all commute with the $U_4$ gate of Eq. (12). Therefore here we must resort to the strong + fast method of the previous subsection, i.e., we need to synchronize the $U_4$ pulses with the end of the decoupling pulse sequence.

Taken together, the methods described in this section provide an explicit way to implement universal QC using trapped ions in a manner that offers protection against all sources of qubit decoherence, using a fast + strong (or fast + weak) version of the SM scheme, possibly in combination with the VT potential modulation method.

VII. DISCUSSION AND CONCLUSIONS

We have proposed a method of encoded recoupling and decoupling (ERD) for performing decoherence-protected quantum computation in ion traps. Our method combines the Sørensen-Mølmer (SM) scheme for quantum logic gates with an encoding into ion-pair decoherence-free subspaces (each pair yielding one encoded qubit), and sequences of recoupling and decoupling pulses. The qubit encoding protects against collective dephasing processes, while the decoupling pulses symmetrize all other sources of decoherence into a collective dephasing interaction. The recoupling pulses are used to implement encoded quantum logic gates, either during or in between the decoupling pulses. All pulses are generated directly using the SM scheme. We have provided numerical estimates of the feasibility of our scheme, which seem quite favorable. In order to achieve full protection against all decoherence it may be necessary to supplement ERD with the potential modulation method due to Vitali & Tombesi, in order to reduce vibrational mode decoherence. However, it may be worthwhile to test ERD without potential modulation first, as a significant reduction in decoherence can already be expected according to the results presented here. This is so because the vibrational bath has been found experimentally to have a $1/f^n$ spectral density [54], and there exists evidence that in such a case decohering may be possible under moderate timing constraints [58].

As mentioned in Section 11, the dynamical decoupling method requires an exponential number of pulses if the most general form of decoherence is to be suppressed, that can couple arbitrary numbers of qubits to the environment (total decoherence [10]). This exponential scaling is avoided here by focusing on decoherence elimination inside blocks of finite size (e.g., at most four ions) where arbitrary decoherence is allowed. However, we have implicitly assumed that there are no decoherence processes coupling different blocks. This is a reasonable assumption for trapped ions, where the different blocks can be kept sufficiently far apart until they need to be brought together in order to execute inter-block logic
gates. When this happens, ERD can still be efficiently applied on the temporarily larger block.

It may be questioned whether there is any advantage in using ERD compared to methods of active quantum error correcting codes (QECC). Both ERD and QECC are capable of dealing with arbitrary decoherence processes, and are fully compatible with universal quantum computation. There are two main advantages to ERD: First, we need only two ions per qubit, compared to a redundancy of five ions per qubit to handle all single-qubit errors in QECC \[^27\]. So far experiments involving trapped ions have used up to four ions \[^{[6]}\], so that this encoding economy is a distinct advantage for near-term experiments. Second, our method is directly compatible with the SM scheme for logic gates in ion traps. On the other hand it is not clear how to directly use SM gates for QECC. These are general features of ERD: economy of encoding redundancy and use of only the most easily controllable interactions. On the other, the disadvantage of ERD compared to QECC is that there does not exist, at this point, a result analogous to the threshold theorem of fault tolerant quantum error correction. This means that we cannot yet guarantee full scalability of ERD as a stand-alone method. However, in principle it is always possible to concatenate ERD with QECC, as done, e.g., for DFS with QECC in \[^{[5, 2, 23, 24]}\], and then the standard fault tolerance results apply.

Finally, we note that ERD is a general method, that is not limited to trapped ions. We hope that the methods proposed here will inspire experimentalists to implement encoded recoupling and decoupling in the lab, thus demonstrating the possibility of fully decoherence-protected quantum computation, in particular using trapped ions.

Acknowledgments

This material is based on research sponsored by the Defense Advanced Research Projects Agency under the QuIST program and managed by the Air Force Research Laboratory (AFOSR), under agreement F49620-01-1-0468. The U.S. Government is authorized to reproduce and distribute reprints for Governmental purposes notwithstanding any copyright notation thereon. The views and conclusions contained herein are those of the authors and should not be interpreted as necessarily representing the official policies or endorsements, either expressed or implied, of the Air Force Research Laboratory or the U.S. Government. D.A.L. further gratefully acknowledges financial support from PRO, NSERC, and the Connaught Fund. We thank Prof. C. Monroe and Dr. S. Schneider for very useful discussions.

[1] D. Kielpinski, C. Monroe and D.J. Wineland, Nature 417, 709 (2002).
[2] J.I. Cirac and P. Zoller, Phys. Rev. Lett. 74, 4091 (1995).
[3] A. Sørensen and K. Mølmer, Phys. Rev. Lett. 82, 1971 (1999).
[4] A. Sørensen and K. Mølmer, Phys. Rev. A 62, 022311 (2000).
[5] G.J. Milburn, quant-ph/9908037.
[6] G.J. Milburn, S. Schneider and D.F.V. James, Fortschr. Phys. 48, 801 (2000).
[7] P. Zanardi and M. Rasetti, Phys. Rev. Lett. 79, 3306 (1997).
[8] P. Zanardi and M. Rasetti, Mod. Phys. Lett. B 11, 1085 (1997).
[9] L.-M Duan and G.-C. Guo, Phys. Rev. A 57, 737 (1998).
[10] D.A. Lidar, I.L. Chuang and K.B. Whaley, Phys. Rev. Lett. 81, 2594 (1998).
[11] Formally, collective dephasing on \( n \) qubits is defined by writing down the system-bath interaction Hamiltonian as \( H_{SB} = \sum_{i=1}^{n} \sigma_i^z \otimes B \), where \( \sigma_i^z \) is the Pauli-\( z \) matrix acting on the \( i \)th qubit, and \( B \) is an arbitrary bath operator.
[12] The DFS encoding reduces the dephasing rate by about \( 10^3 \) for a string of four ions and about \( 10^4 \) for two ions (D. Kielpinski, private communication).
[13] D.M. Bacon, Ph.D. thesis, Univ. of California, Berkeley (2001). Ch.10. Available at \url{http://www.cs.caltech.edu/~dabacon/research}.
[14] G.M. Palma, K.-A. Suominen and A.K. Ekert, Proc. Roy. Soc. London Ser. A 452, 567 (1996).
[15] D.A. Lidar, D. Bacon and K.B. Whaley, Phys. Rev. Lett. 82, 4556 (1999).
[16] J. Kempe, D. Bacon, D.A. Lidar, and K.B. Whaley, Phys. Rev. A 63, 042307 (2001).
[17] P.G. Kwiat, A.J. Berglund, J.B. Altepeter, and A.G. White, Science 290, 498 (2000).
[18] D. Kielpinski, V. Meyer, M.A. Rowe, C.A. Sackett, W.M. Itano, C. Monroe, and D.J. Wineland, Science 291, 1013 (2001).
[19] D. Bacon, J. Kempe, D.A. Lidar and K.B. Whaley, Phys. Rev. Lett. 85, 1758 (2000).
[20] P. Zanardi, Phys. Rev. A 60, R729 (1999).
[21] L. Viola, E. Knill, and S. Lloyd, Phys. Rev. Lett. 83, 4888 (1999).
[22] K. Khodjasteh and D.A. Lidar, Phys. Rev. Lett., 89, 197904 (2002).
[23] D.A. Lidar, D. Bacon, J. Kempe, and K.B. Whaley, Phys. Rev. A 63, 022307 (2001).
[24] G. Alber, M. Mussinger, A. Delgado, quant-ph/0208177.
[25] D. Aharonov and M. Ben-Or, in Proceedings of 29th Annual ACM Symposium on Theory of Computing (STOC) (ACM, New York, NY, 1997), p. 46, quant-ph/9611025.
[26] J. Preskill, Proc. Roy. Soc. London Ser. A 454, 385 (1998).
[27] R. Laflamme, C. Miquel, J.P. Paz and W.H. Zurek, Phys. Rev. Lett. 77, 198 (1996).
[28] P.W. Shor, Phys. Rev. A 52, 2493 (1995).
[29] A.M. Steane, Phys. Rev. Lett. 77, 793 (1996).
[30] E. Knill, R. Laflamme and W. Zurek, Science 279, 342 (1998).
It is not clear that $1/f$ noise will hold generally: in the event that further experimental progress is made it is possible that the only noise limitation remaining will be thermal electronic (Johnson) noise, with a flat spectrum, as discussed in [41] (D. Wineland, private communication). We expect that the methods proposed here will still be applicable in that case.

[31] A.M. Steane, quant-ph/0207119.
[32] A. Beige, D. Braun, B. Treengena, and P.L. Knight, Phys. Rev. Lett. 85, 1762 (2000).
[33] L.-A. Wu and D.A. Lidar, Phys. Rev. A 65, 042318 (2002).
[34] D.A. Lidar and L.-A. Wu, Phys. Rev. Lett. 88, 017905 (2002).
[35] L.-A. Wu and D.A. Lidar, J. Math. Phys. 43, 4506 (2002).
[36] M.S. Byrd and D.A. Lidar, Quant. Inf. Proc. 1, 155 (2002).
[37] L.-A. Wu and D.A. Lidar, quant-ph/0208118.
[38] L. Viola and S. Lloyd, Phys. Rev. A 58, 2733 (1998).
[39] L. Viola and S. Lloyd, quant-ph/9809058.
[40] L. Viola, E. Knill and S. Lloyd, Phys. Rev. Lett. 82, 2417 (1999).
[41] P. Zanardi, Phys. Lett. A 258, 77 (1999).
[42] D. Vitali and P. Tombesi, Phys. Rev. A 59, 4178 (1999).
[43] D. Vitali and P. Tombesi, Phys. Rev. A 65, 012305 (2002).
[44] L.-M. Duan and G. Guo, Phys. Lett. A 261, 139 (1999).
[45] C. Uchiyama and M. Aihara, Phys. Rev. A 66, 032313 (2002).
[46] M.S. Byrd and D.A. Lidar, Phys. Rev. A 66, 012307 (2001).
[47] D. Bacon, J. Kempe, D.P. DiVincenzo, and K.B. Whaley, Nature 408, 339 (2000).
[48] J. Kempe, D. Bacon, D.P. DiVincenzo and K.B. Whaley, Quant. Inf. Comp. 1, 33 (2001).
[49] J. Kempe and K.B. Whaley, Phys. Rev. A 65, 052330 (2001).
[50] J. Vala and K.B. Whaley, quant-ph/0204016.
[51] D. Bacon, J. Kempe, D.P. DiVincenzo, D.A. Lidar, and K.B. Whaley, in Proceedings of the 1st International Conference on Experimental Implementations of Quantum Computation, Sydney, Australia, edited by R. Clark (Rinton, Princeton, NJ, 2001), p. 257, quant-ph/0102140.
[52] L. Viola, E. Knill, and S. Lloyd, Phys. Rev. Lett. 82, 2417 (1999).
[53] L. Viola, S. Lloyd, and D.A. Lidar, Phys. Rev. A 65, 012305 (2002).
[54] L. Viola and S. Lloyd, Phys. Rev. A 66, 012307 (2002).
[55] E.M. Fortunato, L. Viola, J. Hodges, G. Teklemariam, and D.G. Cory, New J. Phys. 2, 012307 (2002).
[56] A.J. Leggett, S. Chakravarty, M.A.P. Fisher, A. Garg and W. Zwerger, Rev. Mod. Phys. 59, 1 (1987).
[57] G. B. R.R. Ernst and A. Wokaun, Principles of Nuclear Magnetic Resonance in One and Two Dimensions, no. 14 in International Series of Monographs on Chemistry (Oxford University Press, Oxford, 1997).
[58] The reason for the square is that the power spectrum of the system-bath interaction amplitude is $J(\omega) = \gamma_3 \int_{-\infty}^{\infty} e^{-i\omega t} \langle B(t)B(0) \rangle$, where $B(t) = e^{iH_B t} Re^{-iH_B t}$ and angular brackets denote an average with respect to the bath state, and $T_2 = 2/J(0)$: M. Aihara, Phys. Rev. B 25, 53 (1982).
[59] Y. Nakamura, Yu. A. Pashkin, T. Yamamoto, and J.S. Tsai, Phys. Rev. Lett. 88, 047901 (2002).
[60] A. Steane, C.F. Roos, D. Stevens, A. Mundt, D. Leibfried, F. Schmidt-Kaler, and R. Blatt, Phys. Rev. Lett. 83, 4713 (1999).
[61] A. Steane, C.F. Roos, D. Stevens, A. Mundt, D. Leibfried, F. Schmidt-Kaler, and R. Blatt, Phys. Rev. A 62, 042305 (2000).
[62] J.P. Palao and R. Kosloff, Phys. Rev. Lett. 89, 188301 (2002).
[63] S. Schneider and G.J. Milburn, Phys. Rev. A 57, 3748 (1998).
[64] This corresponds to the case $\Delta \alpha = 0$ in Eq. (1) of [1].
[65] The case $B_{\text{SM}}^{(n)} \neq 0$ can also be the result of small departures from equal illumination during the SM logic gate [18], as in the intensity and phase fluctuations of the exciting lasers analyzed in [7]. Hence, in order for the decoupling scheme to work the lasers used to implement the corresponding SM pulses must be more stable than the differential dephasing due to magnetic field fluctuations.
[66] L. Tian and S. Lloyd, Phys. Rev. A 52, 050301 (2000).