W_{0}^{1,1} MINIMA OF NON COERCIVE FUNCTIONALS

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Abstract. We study an integral non coercive functional defined on $H_{0}^{1}(\Omega)$, proving the existence of a minimum in $W_{0}^{1,1}(\Omega)$.

In this paper we study a class of integral functionals defined on $H_{0}^{1}(\Omega)$, but non coercive on the same space, so that the standard approach of the Calculus of Variations does not work. However, the functionals are coercive on $W_{0}^{1,1}(\Omega)$ and we will prove the existence of minima, despite the non reflexivity of $W_{0}^{1,1}(\Omega)$, which implies that, in general, the Direct Methods fail due to lack of compactness.

Let $J$ be the functional defined as

$$J(v) = \int_{\Omega} \frac{j(x, \nabla v)}{1 + b(x)|v|^{2}} + \frac{1}{2} \int_{\Omega} |v|^{2} - \int_{\Omega} f v, \quad v \in H_{0}^{1}(\Omega).$$

We assume that $\Omega$ is a bounded open set of $\mathbb{R}^{N}$, $N > 2$, that $j : \Omega \times \mathbb{R}^{N} \to \mathbb{R}$ is such that $j(\cdot, \xi)$ is measurable on $\Omega$ for every $\xi$ in $\mathbb{R}^{N}$, $j(x, \cdot)$ is convex and belongs to $C^{1}(\mathbb{R}^{N})$ for almost every $x$ in $\Omega$, and

$$\alpha|\xi|^{2} \leq j(x, \xi) \leq \beta|\xi|^{2}, \quad (1)$$

$$|j_{\xi}(x, \xi)| \leq \gamma|\xi|, \quad (2)$$

for some positive $\alpha$, $\beta$ and $\gamma$, for almost every $x$ in $\Omega$, and for every $\xi$ in $\mathbb{R}^{N}$. We assume that $b$ is a measurable function on $\Omega$ such that

$$0 \leq b(x) \leq B, \quad (3)$$

where $B > 0$, while $f$ belongs to some Lebesgue space. For $k > 0$ and $s \in \mathbb{R}$, we define the truncature function as $T_{k}(s) = \max(-k, \min(s, k))$.

In [3] the minimization in $H_{0}^{1}(\Omega)$ of the functional

$$I(v) = \int_{\Omega} \frac{j(x, \nabla v)}{1 + |v|^{2}} - \int_{\Omega} f v, \quad 0 < \theta < 1, \quad f \in L^{m}(\Omega),$$

was studied. It was proved that $I(v)$ is coercive on the Sobolev space $W_{0}^{1,q}(\Omega)$, for some $q = q(\theta, m)$ in $(1, 2)$, and that $I(v)$ achieves its minimum on $W_{0}^{1,q}(\Omega)$. This approach does not work for $\theta > 1$ (see Remark 7 below). Here we will able to overcome this difficulty thanks to the presence of the lower order term $\int_{\Omega} |v|^{2}$, which will yield the coercivity of $J$ on $W_{0}^{1,1}(\Omega)$; then we will prove the existence of minima in $W_{0}^{1,1}(\Omega)$, even if it is a non reflexive space.

Integral functionals like $J$ or $I$ are studied in [1], in the context of the Thomas-Fermi-von Weizsäcker theory.
We are going to prove the following result.

**Theorem 1.** Let \( f \in L^2(\Omega) \). Then there exists \( u \) in \( W_0^{1,1}(\Omega) \cap L^2(\Omega) \) minimum of \( J \), that is,

\[
\frac{\partial}{\partial x_j} \left( \frac{a(x) \nabla u}{(1 + b(x)|u|^2)} \right) + u = f \quad \text{in} \ \Omega,
\]

\[
0 = \frac{\partial}{\partial x_j} \left( \frac{a(x) \nabla u}{(1 + b(x)|u|^2)} \right) \quad \text{on} \ \partial \Omega.
\]

In \([2]\) we studied the following elliptic boundary problem:

\[
\begin{aligned}
&-\text{div} \left( \frac{a(x) \nabla u}{(1 + b(x)|u|^2)} \right) + u = f \quad \text{in} \ \Omega, \\
&u = 0 \quad \text{on} \ \partial \Omega,
\end{aligned}
\]

under the same assumptions on \( \Omega, b, f \), with \( 0 < \alpha \leq a(x) \leq \beta \). It is easy to see that the Euler equation of \( J \), with \( j(x, \xi) = \frac{1}{2} a(x)|\xi|^2 \), is not equation \((5)\). Therefore Theorem \(1\) cannot be deduced from \([2]\).

Nevertheless some technical steps of the two papers (for example, the a priori estimates) are similar.

We will prove Theorem \(1\) by approximation. Therefore, we begin with the case of bounded data.

**Lemma 2.** If \( f \) belongs to \( L^\infty(\Omega) \), then there exists a minimum \( w \) belonging to \( H^1_0(\Omega) \cap L^\infty(\Omega) \) of the functional

\[
v \in H^1_0(\Omega) \mapsto \int_\Omega \frac{j(x, \nabla v)}{1 + b(x)|v|^2} + \frac{1}{2} \int_\Omega |v|^2 - \int_\Omega g v.
\]

**Proof.** Since the functional is not coercive on \( H^1_0(\Omega) \), we cannot directly apply the standard techniques of the Calculus of Variations. Therefore, we begin by approximating it. Let \( M > 0 \), and let \( J_M \) be the functional defined as

\[
J_M(v) = \int_\Omega \frac{j(x, \nabla v)}{1 + b(x)|T_M(v)|^2} + \frac{1}{2} \int_\Omega |v|^2 - \int_\Omega g v, \quad v \in H^1_0(\Omega).
\]

Since \( J_M \) is both weakly lower semicontinuous (due to the convexity of \( j \) and to De Giorgi’s theorem, see \([4]\)) and coercive on \( H^1_0(\Omega) \), for every \( M > 0 \) there exists a minimum \( w_M \) of \( J_M \) on \( H^1_0(\Omega) \). Let \( A = \|g\|_{L^\infty(\Omega)} \), let \( M > A \), and consider the inequality \( J_M(w_M) \leq J_M(T_A(w_M)) \), which holds true since \( w_M \) is a minimum of \( J_M \). We have

\[
\begin{align*}
&\int_\Omega \frac{j(x, \nabla w_M)}{1 + b(x)|T_M(w_M)|^2} + \frac{1}{2} \int_\Omega |w_M|^2 - \int_\Omega g w_M \\
&\quad \leq \int_\Omega \frac{j(x, \nabla T_A(w_M))}{1 + b(x)|T_M(T_A(w_M))|^2} + \frac{1}{2} \int_\Omega |T_A(w_M)|^2 - \int_\Omega g T_A(w_M) \\
&\quad = \int_{\{|w_M| \leq A\}} \frac{j(x, \nabla w_M)}{1 + b(x)|T_M(w_M)|^2} + \frac{1}{2} \int_\Omega |T_A(w_M)|^2 - \int_\Omega g T_A(w_M),
\end{align*}
\]
where, in the last passage, we have used that \(T_M(T_A(w_M)) = T_M(w_M)\) on the set \(\{|w_M| \leq A\}\), and that \(j(x, 0) = 0\). Simplifying equal terms, we thus get

\[
\int_{\{|w_M| \geq M\}} \frac{j(x, \nabla w_M)}{1 + b(x)|T_M(w_M)|^2} + \frac{1}{2} \int_{\Omega} [w_M^2 - |T_A(w_M)|^2] \leq \int_{\Omega} g [w_M - T_A(w_M)].
\]

Dropping the first term, which is nonnegative, we obtain

\[
\frac{1}{2} \int_{\Omega} [w_M - T_A(w_M)] [w_M + T_A(w_M)] \leq \int_{\Omega} g [w_M - T_A(w_M)],
\]

which can be rewritten as

\[
\frac{1}{2} \int_{\Omega} [w_M - T_A(w_M)] [w_M + T_A(w_M) - 2g] \leq 0.
\]

We then have, since \(w_M = T_A(w_M)\) on the set \(\{|w_M| \leq A\}\),

\[
\frac{1}{2} \int_{\{|w_M| > A\}} [w_M - A][w_M + A - 2g] + \frac{1}{2} \int_{\{|w_M| < -A\}} [w_M + A][w_M - A - 2g] \leq 0.
\]

Since \(|g| \leq A\), we have \(A - 2g \leq -A\), and \(-A - 2g < A\), so that

\[
0 \leq \frac{1}{2} \int_{\{|w_M| > A\}} [w_M - A]^2 + \frac{1}{2} \int_{\{|w_M| < -A\}} [w_M + A]^2 \leq 0,
\]

which then implies that \(\text{meas}(\{|w_M| \geq A\}) = 0\), and so \(|w_M| \leq A\) almost everywhere in \(\Omega\). Recalling the definition of \(A\), we thus have

\[
\|w_M\|_{L^\infty(\Omega)} \leq \|g\|_{L^\infty(\Omega)}.
\]

Since \(M > \|g\|_{L^\infty(\Omega)}\), we thus have \(T_M(w_M) = w_M\). Starting now from \(J_M(w_M) \leq J_M(0) = 0\) we obtain, by \((9)\),

\[
\int_{\Omega} \frac{j(x, \nabla w_M)}{1 + b(x)|w_M|^2} + \frac{1}{2} \int_{\Omega} |w_M|^2 \leq \int_{\Omega} g w_M \leq \text{meas}(\Omega) \|g\|_{L^\infty(\Omega)}^2,
\]

which then implies, by \((11)\) and \((13)\), and dropping the nonnegative second term,

\[
\frac{\alpha}{[1 + B\|g\|_{L^\infty(\Omega)}]} \int_{\Omega} \nabla w_M^2 \leq \text{meas}(\Omega) \|g\|_{L^\infty(\Omega)}^2.
\]

Thus, \(\{w_M\}\) is bounded in \(H^1_0(\Omega) \cap L^\infty(\Omega)\), and so, up to subsequences, it converges to some function \(w\) in \(H^1_0(\Omega) \cap L^\infty(\Omega)\) weakly in \(H^1_0(\Omega)\), strongly in \(L^2(\Omega)\), and almost everywhere in \(\Omega\). We prove now that

\[
\int_{\Omega} \frac{j(x, \nabla w)}{1 + b(x)|w|^2} \leq \liminf_{M \to +\infty} \int_{\Omega} \frac{j(x, \nabla w_M)}{1 + b(x)|w_M|^2}.
\]
Indeed, since \( j \) is convex, we have
\[
\int_{\Omega} \frac{j(x, \nabla w_M)}{[1 + b(x)]|w_M|^2} \geq \int_{\Omega} \frac{j(x, \nabla w)}{[1 + b(x)]|w|^2} - \int_{\Omega} \frac{j(x, \nabla w)}{[1 + b(x)]|w_M|^2} \cdot \nabla[w_M - w].
\]

Using assumption (1), the fact that \( w \) belongs to \( H^1_0(\Omega) \), the almost everywhere convergence of \( w_M \) to \( w \) and Lebesgue’s theorem, we have
\[
\lim_{M \to +\infty} \int_{\Omega} \frac{j(x, \nabla w)}{[1 + b(x)]|w_M|^2} = \int_{\Omega} \frac{j(x, \nabla w)}{[1 + b(x)]|w|^2},
\]
strongly in \( (L^2(\Omega))^N \).

Since \( \nabla w_M \) tends to \( \nabla w \) weakly in the same space, we thus have that
\[
\lim_{M \to +\infty} \int_{\Omega} \frac{j(x, \nabla w)}{[1 + b(x)]|w_M|^2} \cdot \nabla[w_M - w] = 0.
\]

Using (8) and (9), we have that (7) holds true. On the other hand, using (1) and Lebesgue’s theorem again, it is easy to see that
\[
\lim_{M \to +\infty} \int_{\Omega} \frac{j(x, \nabla v)}{[1 + b(x)]|T_M(v)|^2} = \int_{\Omega} \frac{j(x, \nabla v)}{[1 + b(x)]|v|^2}, \quad \forall v \in H^1_0(\Omega).
\]

Thus, starting from \( J_M(w_M) \leq J_M(v) \), we can pass to the limit as \( M \) tends to infinity (using also the strong convergence of \( w_M \) to \( w \) in \( L^2(\Omega) \)), to have that \( w \) is a minimum. \( \square \)

As stated before, we prove Theorem 1 by approximation. More in detail, if \( f_n = T_n(f) \) then Lemma 2 with \( g = f_n \) implies that there exists a minimum \( u_n \) in \( H^1_0(\Omega) \cap L^\infty(\Omega) \) of the functional
\[
J_n(v) = \int_{\Omega} \frac{j(x, \nabla v)}{[1 + b(x)]|v|^2} + \frac{1}{2} \int_{\Omega} |v|^2 - \int_{\Omega} f_n v, \quad v \in H^1_0(\Omega).
\]

In the following lemma we prove some uniform estimates on \( u_n \).

**Lemma 3.** Let \( u_n \) in \( H^1_0(\Omega) \cap L^\infty(\Omega) \) be a minimum of \( J_n \). Then
\[
\int_{\Omega} \frac{|\nabla u_n|^2}{[1 + b(x)]|u_n|^2} \leq \frac{1}{2\alpha} \int_{\Omega} |f|^2; \tag{10}
\]
\[
\int_{\Omega} |\nabla T_k(u_n)|^2 \leq \frac{(1 + B k)^2}{2\alpha} \int_{\Omega} |f|^2; \tag{11}
\]
\[
\int_{\Omega} |u_n|^2 \leq 4 \int_{\Omega} |f|^2; \tag{12}
\]
We are left with estimate (14). Since (13) \[ \int_\Omega |\nabla u_n| \leq \left[ \frac{1}{2\alpha} \int_\Omega |f|^2 \right]^{\frac{1}{2}} \left( \text{meas}(\Omega)^{\frac{1}{2}} + 2B \left[ \int_\Omega |f|^2 \right]^{\frac{1}{2}} \right); \]

(14) \[ \int_\Omega |G_k(u_n)|^2 \leq 4 \int_{\{u_n \geq k\}} |f|^2, \]

where \( G_k(s) = s - T_k(s) \) for \( k \geq 0 \) and \( s \) in \( \mathbb{R} \).

**Proof.** The minimality of \( u_n \) implies that \( J_n(u_n) \leq J_n(0) \), that is,

(15) \[ \int_\Omega \frac{j(x, \nabla u_n)}{1 + b(x)|u_n|^2} + \frac{1}{2} \int_\Omega u_n^2 \leq \int_\Omega f_n u_n. \]

Using (1) on the left hand side, and Young’s inequality on the right hand side gives

\[ \alpha \int_\Omega \frac{|\nabla u_n|^2}{1 + b(x)|u_n|^2} + \frac{1}{2} \int_\Omega u_n^2 \leq \frac{1}{2} \int_\Omega u_n^2 + \frac{1}{2} \int_\Omega f_n^2, \]

which then implies (10). Let now \( k \geq 0 \). The above estimate, and (3), give

\[ \frac{1}{(1+Br)^2} \int_\Omega |\nabla T_k(u_n)|^2 \leq \int_{\{u_n \leq k\}} \frac{|\nabla u_n|^2}{1 + b(x)|u_n|^2} \leq \frac{1}{2\alpha} \int_\Omega |f|^2, \]

and therefore (11) is proved. On the other hand, dropping the first positive term in (15) and using Hölder’s inequality on the right hand side, we have

\[ \frac{1}{2} \int_\Omega |u_n|^2 \leq \int_\Omega |f_n u_n| \leq \left[ \int_\Omega |f_n|^2 \right]^{\frac{1}{2}} \left[ \int_\Omega |u_n|^2 \right]^{\frac{1}{2}}, \]

that is, (12) holds. Hölder’s inequality, assumption (3), and estimates (10) and (12) give (13):

(16) \[ \int_\Omega |\nabla u_n| \leq \left[ \int_\Omega \frac{|\nabla u_n|^2}{1 + b(x)|u_n|^2} \right]^{\frac{1}{2}} \left[ \int_\Omega [1 + b(x)|u_n|^2] \right]^{\frac{1}{2}} \leq \left[ \frac{2\alpha}{\alpha} \int_\Omega |f|^2 \right]^{\frac{1}{2}} \left( \text{meas}(\Omega)^{\frac{1}{2}} + 2B \left[ \int_\Omega |f|^2 \right]^{\frac{1}{2}} \right). \]

We are left with estimate (14). Since \( J_n(u_n) \leq J_n(T_k(u_n)) \) we have

\[ \frac{1}{2} \int_\Omega \frac{j(x, \nabla u_n)}{1 + b(x)|u_n|^2} + \frac{1}{2} \int_\Omega |u_n|^2 - \int_\Omega f_n u_n \]

\[ \leq \frac{1}{2} \int_\Omega \frac{j(x, \nabla T_k(u_n))}{1 + b(x)|T_k(u_n)|^2} + \frac{1}{2} \int_\Omega |T_k(u_n)|^2 - \int_\Omega f_n T_k(u_n). \]

Recalling the definition of \( G_k(s) \), and using that \( |s|^2 - |T_k(s)|^2 \geq |G_k(s)|^2 \), the last inequality implies

\[ \frac{1}{2} \int_\Omega \frac{j(x, \nabla G_k(u_n))}{1 + b(x)|u_n|^2} + \frac{1}{2} \int_\Omega |G_k(u_n)|^2 \leq \int_\Omega f_n G_k(u_n). \]
Dropping the first term of the left hand side and using Hölder’s inequality on the right one, we obtain
\[
\frac{1}{2} \int_{\Omega} |G_k(u_n)|^2 \leq \left[ \int_{\{\{|u_n|\geq k\}} |f|^2 \right]^{\frac{1}{2}} \left[ \int_{\Omega} |G_k(u_n)|^2 \right]^{\frac{1}{2}},
\]
that is, (14) holds.

**Lemma 4.** Let \( u_n \) in \( H^1_0(\Omega) \cap L^\infty(\Omega) \) be a minimum of \( J_n \). Then there exists a subsequence, still denoted by \( \{u_n\} \), and a function \( u \) in \( W^{1,1}_0(\Omega) \cap L^2(\Omega) \), with \( T_k(u) \) in \( H^1_0(\Omega) \) for every \( k > 0 \), such that \( u_n \) converges to \( u \) almost everywhere in \( \Omega \), strongly in \( L^2(\Omega) \) and weakly in \( W^{1,1}_0(\Omega) \), and \( T_k(u_n) \) converges to \( T_k(u) \) weakly in \( H^1_0(\Omega) \). Moreover, \( u \) belongs to \( L^2(\Omega) \).

**Proof.** By (13), the sequence \( u_n \) is bounded in \( W^{1,1}_0(\Omega) \). Therefore, it is relatively compact in \( L^1(\Omega) \). Hence, up to subsequences still denoted by \( u_n \), there exists \( u \) in \( L^1(\Omega) \) such that \( u_n \) almost everywhere converges to \( u \). From Fatou’s lemma applied to (12) we then deduce that \( u \) belongs to \( L^2(\Omega) \).

We are going to prove that \( u_n \) strongly converges to \( u \) in \( L^2(\Omega) \). Let \( E \) be a measurable subset of \( \Omega \); then by (14) we have
\[
\int_E |u_n|^2 \leq 2 \int_E |T_k(u_n)|^2 + 2 \int_E |G_k(u_n)|^2
\]
\[
\leq 2k^2 \text{meas}(E) + 2 \int_{\{\{|u_n|\geq k\}} |f|^2 \right]^{\frac{1}{2}} \left[ \int_{\Omega} |G_k(u_n)|^2 \right]^{\frac{1}{2}}
\]
\[
\leq 2k^2 \text{meas}(E) + 8 \int_{\{\{|u_n|\geq k\}} |f|^2.
\]

Since \( u_n \) is bounded in \( L^2(\Omega) \) by (12), we can choose \( k \) large enough so that the second integral is small, uniformly with respect to \( n \); once \( k \) is chosen, we can choose the measure of \( E \) small enough such that the first term is small. Thus, the sequence \( \{u_n^2\} \) is equiintegrable and so, by Vitali’s theorem, \( u_n \) strongly converges to \( u \) in \( L^2(\Omega) \).

Now we prove that \( u_n \) weakly converges to \( u \) in \( W^{1,1}_0(\Omega) \). Let \( E \) be a measurable subset of \( \Omega \). By Hölder’s inequality, assumption (3), and (10), one has, for \( i \in \{1, \ldots, N\} \),
\[
\int_E \left| \frac{\partial u_n}{\partial x_i} \right| \leq \int_E \left| \nabla u_n \right| \leq \left[ \int_E \frac{|\nabla u_n|^2}{1 + b(x)|u_n|^2} \right]^{\frac{1}{2}} \left[ \int_{\Omega} \frac{1 + b(x)|u_n|^2}{[1 + B|u_n|^2]} \right]^{\frac{1}{2}}
\]
\[
\leq \left[ \frac{1}{2\alpha} \int_{\Omega} |f|^2 \right]^{\frac{1}{2}} \left[ \int_{\Omega} \frac{1 + B|u_n|^2}{[1 + B|u_n|^2]} \right]^{\frac{1}{2}}.
\]

Since the sequence \( \{u_n\} \) is compact in \( L^2(\Omega) \), this estimate implies that the sequence \( \left\{ \frac{\partial u_n}{\partial x_i} \right\} \) is equiintegrable. Thus, by Dunford-Pettis
and then use (18) to choose \( n \) arbitrarily small. Hence, we have proved that
\[
\begin{align*}
\lim_{n \to +\infty} u_n & \text{ belongs to } H^1(\Omega), \\
\text{and since } u_n \text{ tends to } u \text{ almost everywhere in } \Omega, \\
\text{then } T_k(u_n) \text{ weakly converges to } T_k(u) \text{ in } H^1_0(\Omega), \\
\text{and } T_k(u) \text{ belongs to } H^1_0(\Omega) \text{ for every } k \geq 0.
\end{align*}
\]

Finally, we prove (17). Let \( \Phi \) be a fixed function in \((L^\infty(\Omega))^N\). Since \( u_n \) almost everywhere converges to \( u \) in \( \Omega \), we have
\[
\lim_{n \to +\infty} \frac{\Phi}{1 + b(x)|u_n|} = \frac{\Phi}{1 + b(x)|u|} \quad \text{almost everywhere in } \Omega.
\]

By Egorov’s theorem, the convergence is therefore quasi uniform; i.e., for every \( \delta > 0 \) there exists a subset \( E_\delta \) of \( \Omega \), with \( \text{meas}(E_\delta) < \delta \), such that
\[
(18) \quad \lim_{n \to +\infty} \frac{\Phi}{1 + b(x)|u_n|} = \frac{\Phi}{1 + b(x)|u|} \quad \text{uniformly in } \Omega \setminus E_\delta.
\]

We now have
\[
\begin{align*}
\left| \int_\Omega \frac{\nabla u_n}{1 + b(x)|u_n|} \cdot \Phi - \int_\Omega \frac{\nabla u}{1 + b(x)|u|} \cdot \Phi \right| \\
\leq \int_{\Omega \setminus E_\delta} \nabla u_n \cdot \frac{\Phi}{1 + b(x)|u_n|} - \int_{\Omega \setminus E_\delta} \nabla u \cdot \frac{\Phi}{1 + b(x)|u|} \\
+ \|\Phi\|_{L^\infty(\Omega)} \int_{E_\delta} [||\nabla u_n|| + ||\nabla u||].
\end{align*}
\]

Using the equiintegrability of \( ||\nabla u_n|| \) proved above, and the fact that \( ||\nabla u|| \) belongs to \( L^1(\Omega) \), we can choose \( \delta \) such that the second term of the right hand side is arbitrarily small, uniformly with respect to \( n \), and then use (18) to choose \( n \) large enough so that the first term is arbitrarily small. Hence, we have proved that
\[
(19) \quad \lim_{n \to +\infty} \frac{\nabla u_n}{1 + b(x)|u_n|} = \frac{\nabla u}{1 + b(x)|u|} \quad \text{weakly in } (L^1(\Omega))^N.
\]

On the other hand, from (10) it follows that the sequence \( \frac{\nabla u_n}{1 + b(x)|u_n|} \) is bounded in \((L^2(\Omega))^N\), so that it weakly converges to some function \( \sigma \).
in the same space. Since (19) holds, we have that $\sigma = \frac{\nabla u}{1+b(x)|u|}$, and (17) is proved. \hfill \Box

Remark 5. The fact that we need to prove (17) is one of the main differences with the paper [2].

Proof of Theorem 4. Let $u_n$ be as in Lemma 3. The minimality of $u_n$ implies that

\begin{equation}
\int \frac{j(x, \nabla u_n)}{1+b(x)|u_n|^2} + \frac{1}{2} \int |u_n|^2 - \int f_n u_n \\
\leq \int \frac{j(x, \nabla v)}{1+b(x)|v|^2} + \frac{1}{2} \int |v|^2 - \int f_n v
\end{equation}

for every $v$ in $H^1_0(\Omega)$. The result will then follow by passing to the limit in the previous inequality. The right hand side of (20) is easy to handle since $f_n$ converges to $f$ in $L^2(\Omega)$. Let us study the limit of the left hand side of (20). The convexity of $j$ implies that

\begin{align*}
\int \frac{j(x, \nabla u_n)}{1+b(x)|u_n|^2} &\geq \int \frac{j(x, \nabla T_k(u))}{1+b(x)|u_n|^2} \\
- \int \frac{j_\varepsilon(x, \nabla T_k(u))}{1+b(x)|u_n|^2} \cdot \left( \frac{\nabla u_n}{1+b(x)|u_n|^2} - \frac{\nabla T_k(u)}{1+b(x)|u_n|^2} \right).
\end{align*}

By (17), assumptions (11) and (2), and Lebesgue’s theorem, we have

\begin{align*}
\liminf_{n \to \infty} \int \frac{j(x, \nabla u_n)}{1+b(x)|u_n|^2} &\geq \int \frac{j(x, \nabla T_k(u))}{1+b(x)|u|^2} \\
- \int \frac{j_\varepsilon(x, \nabla T_k(u))}{1+b(x)|u|^2} \cdot \frac{\nabla(u - T_k(u))}{1+b(x)|u|^2},
\end{align*}

that is, since $j_\varepsilon(x, \nabla T_k(u)) \cdot \nabla(u - T_k(u)) = 0$,

\begin{equation}
\int \frac{j(x, \nabla T_k(u))}{1+b(x)|u|^2} \leq \liminf_{n \to \infty} \int \frac{j(x, \nabla u_n)}{1+b(x)|u_n|^2}.
\end{equation}

Letting $k$ tend to infinity, and using Levi’s theorem, we obtain

\begin{equation}
\int \frac{j(x, \nabla u)}{1+b(x)|u|^2} \leq \liminf_{n \to \infty} \int \frac{j(x, \nabla u_n)}{1+b(x)|u_n|^2}.
\end{equation}

Inequality (21) and Lemma 3 imply that

\begin{align*}
\liminf_{n \to \infty} \int \frac{j(x, \nabla u_n)}{1+b(x)|u_n|^2} + \frac{1}{2} \int |u_n|^2 - \int f_n u_n \\
\geq \int \frac{j(x, \nabla u)}{1+b(x)|u|^2} + \frac{1}{2} \int |u|^2 - \int fu.
\end{align*}

Thus, for every $v$ in $H^1_0(\Omega)$,

\begin{equation}
\int \frac{j(x, \nabla u)}{1+b(x)|u|^2} + \frac{1}{2} \int |u|^2 - \int fu \leq \int \frac{j(x, \nabla v)}{1+b(x)|v|^2} + \frac{1}{2} \int |v|^2 - \int fv,
\end{equation}

so that $u$ is a minimum of $J$; its regularity has been proved in Lemma 4.

**Remark 6.** If we suppose that the coefficient $b(x)$ satisfies the stronger assumption

$$0 < A \leq b(x) \leq B,$$

it is possible to prove that $J(u) \leq J(w)$ not only for every $w$ in $H_0^1(\Omega)$, but also for the test functions $w$ such that

$$\begin{cases}
T_k(w) \text{ belongs to } H_0^1(\Omega) \text{ for every } k > 0, \\
\log(1 + A |w|) \text{ belongs to } H_0^1(\Omega), \\
w \text{ belongs to } L^2(\Omega).
\end{cases}$$

Indeed, if $w$ is as in (22), we can use $T_k(w)$ as test function in (4) and we have

$$J(u) \leq J(T_k(w)) = \int_\Omega j(x, \nabla T_k(w)) \, dx + \frac{1}{2} \int_\Omega |T_k(w)|^2 - \int_\Omega f T_k(w).$$

In the right hand side is possible to pass to the limit, as $k$ tends to infinity, so that we have $J(u) \leq J(w)$, for every test function $w$ as in (22).

**Remark 7.** We explicitly point out the differences, concerning the coercivity, between the functionals studied in [3] and the functionals studied in this paper. Indeed, let $0 < \rho < \frac{N - 2}{2}$, and consider the sequence of functions

$$v_n = \exp \left( T_n \left( \frac{1}{|x|^\rho} - 1 \right) \right) - 1,$$

defined in $\Omega = B_1(0)$. Then

$$\log(1 + |v_n|) = T_n \left( \frac{1}{|x|^\rho} - 1 \right),$$

is bounded in $H_0^1(\Omega)$ (since the function $v(x) = \frac{1}{|x|^\rho} - 1$ belongs to $H_0^1(\Omega)$ by the assumptions on $\rho$), but, by Levi’s theorem,

$$\lim_{n \to +\infty} \int_\Omega |\nabla v_n| = \rho \int_\Omega \frac{\exp \left( \frac{1}{|x|^\rho} - 1 \right)}{|x|^{\rho+1}} = +\infty.$$ 

Hence, the functional

$$v \in H_0^1(\Omega) \mapsto \int_\Omega \frac{|\nabla v|^2}{(1 + |v|)^2} = \int_\Omega |\nabla \log(1 + |v|)|^2,$$

which is of the type studied in [3], is non coercive on $W_0^{1,1}(\Omega)$. On the other hand, recalling (16), we have

$$\int_\Omega |\nabla v| = \int_\Omega \frac{|\nabla v|}{1 + |v|} \, (1 + |v|) \leq \frac{1}{2} \int_\Omega \frac{|\nabla v|^2}{(1 + |v|)^2} + \frac{1}{2} \int_\Omega (1 + |v|)^2.$$
Thus, the functional

\[ v \in H_0^1(\Omega) \mapsto \int_\Omega \frac{|\nabla v|^2}{(1 + |v|)^2} + \int_\Omega |v|^2, \]

which is of the type studied here, is coercive on \( W_0^{1,1}(\Omega) \).

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