We introduce Backström pairs and Backström rings, study their derived categories and construct for them a sort of categorical resolutions. For the latter we define the global dimension, construct a sort of semiorthogonal decomposition of the derived category and deduce that the derived dimension of a Backström ring is at most 2. Using this semiorthogonal decomposition, we define a description of the derived category as the category of elements of a special bimodule. We also construct a partial tilting for a Backström pair to a ring of triangular matrices and define the global dimension of the latter.

Introduction

Backström orders were introduced in [Ringel and Roggenkamp 1979], where it was shown that their representations are in correspondence with those of quivers or species. A special class of Backström orders are nodal orders, which appeared (without this name) in [Drozd 1990] as such pure noetherian algebras that the classification of their finitely generated modules is tame. In [Burban and Drozd 2004] tameness was also proved for the derived categories of nodal orders. Global analogues of nodal algebras, called nodal curves, were considered in [Burban and Drozd 2011; Drozd and Voloshyn 2012; Voloshyn and Drozd 2013]. Namely, in [Burban and Drozd 2011] a sort of tilting theory for such curves was developed, which related them to some quasihereditary finite dimensional algebras. In [Drozd and Voloshyn 2012] a criterion was found for a nodal curve to be tame with respect to the classification of vector bundles, and in [Voloshyn and Drozd 2013] it was proved that the same class of curves has tame derived categories. It was clear that the tilting theory of [Burban and Drozd 2011] can be extended to a general situation, namely, to Backström curves, i.e., noncommutative curves having Backström orders as their localizations. Nodal orders and related gentle algebras appear in studying mirror symmetry, see for instance, [Lekili and Polishchuk 2018]. Finite dimensional
analouges of nodal orders, called *nodal algebras*, were introduced in [Drozd and Zembyk 2013; Zembyk 2014]. In the latter paper their structure was completely described. In [Zembyk 2015] it was shown that certain important classes of algebras, such as gentle and skewed-gentle algebras, are nodal. In [Burban and Drozd 2017] a tilting theory was developed for nodal algebras, which was applied to the study of derived categories of gentle and skewed-gentle algebras.

This paper is devoted to a tilting theory for *Backström rings*, which are a straightforward generalization of Backström orders and algebras.

In Section 1, we propose a variant of partial tilting, which generalizes the technique of minors from [Burban et al. 2017].

In Section 2, we introduce *Backström pairs*, which are pairs of semiperfect rings $H \supseteq A$ with a common radical; (piecewise) *Backström rings* are likewise introduced as those rings $A$ that occur in (piecewise) Backström pairs with (piecewise) hereditary $H$. We construct the *Auslander envelope* $\tilde{A}$ of a Backström pair and calculate its global dimension. It turns out that this global dimension only depends on the global dimension of $H$. In particular, Auslander envelopes for Backström rings are of global dimension at most 2.

In Section 3, we apply the tilting technique to show that the derived category of the algebra $A$ is connected by a recollement with the derived category of its Auslander envelope. This implies that the derived dimension of $A$ in the sense of [Rouquier 2008] is not greater than that of the Auslander envelope.

In Section 4, we consider a recollement between the derived categories of the algebra $H$ and of the Auslander envelope. It is used to calculate the derived dimension of the Auslander envelope, thus obtaining an upper bound for the derived dimension of the algebra $A$. In particular, we prove that the derived dimension of a Backström or piecewise Backström algebra is at most 2. Moreover, if $A$ is a Backström or piecewise Backström algebra of Dynkin type, then either it is piecewise hereditary of Dynkin type, so der.dim $A = 0$, or else der.dim $A = 1$.

In Section 5, we establish an equivalence between the category $\mathcal{D}(\tilde{A})$ and a bimodule category. This gives a useful instrument for calculations in this derived category. (See, for instance, [Bekkert et al. 2003; Bekkert and Merklen 2003; Burban and Drozd 2004; 2006; 2017; Voloshyn and Drozd 2013].)

In Section 6, we consider another partial tilting for the Auslander envelope $\tilde{A}$ of a Backström pair, relating its derived category by a recollement to the derived category of an algebra $B$ of triangular matrices which looks simpler than the Auslander algebra. In this case, we calculate explicitly the global dimension of $B$ and the kernel of the partial tilting functor

\[ F : \mathcal{D}(B) \to \mathcal{D}(A). \]
1. Partial tilting

Let $\mathcal{T}$ be a triangulated category, $\mathcal{R} \subseteq \text{Ob} \mathcal{T}$. We denote by $\text{Tri}(\mathcal{R})$ the smallest strictly full triangulated subcategory containing $\mathcal{R}$ that is closed under coproducts (this means that if a coproduct of objects from $\text{Tri}(\mathcal{R})$ exists in $\mathcal{T}$, it belongs to $\text{Tri}(\mathcal{R})$). For a DG-category $\mathcal{R}$ we denote by $\mathcal{D}(\mathcal{R})$ its derived category [Keller 1994]. The following result is a generalization of [Lunts 2010, Proposition 2.6]:

**Theorem 1.1.** Let $\mathcal{R}$ be a subset of the set of compact objects of $\text{Ob} \mathcal{D}(\mathbb{A})$, where $\mathbb{A}$ is a Grothendieck category. We consider the DG-category $\mathcal{R}$ with the set of objects $\mathcal{R}$ and the sets of morphisms $\mathcal{R}(T, R) = \mathbb{R}\text{Hom}(T, R)$. Define the functor $F: \mathcal{D}(\mathbb{A}) \rightarrow \mathcal{D}(\mathcal{R}^{\text{op}})$ by mapping a complex $C$ to the DG-module $F_C = \mathbb{R}\text{Hom}_{\mathcal{D}(\mathbb{A})}(-, C)$ restricted onto $\mathcal{R}$.

(1) The restriction of $F$ onto $\text{Tri}(\mathcal{R})$ is an equivalence $\text{Tri}(\mathcal{R}) \rightarrow \mathcal{D}(\mathcal{R}^{\text{op}})$.

(2) There is a recollement diagram in the sense of [Beilinson et al. 1982, 1.4.3]

(1-1) $\text{Ker} F \otimes_{\mathcal{C}} \mathcal{D}(\mathbb{A}) \otimes_{\mathcal{C}} \mathcal{D}(\mathcal{R}^{\text{op}})$

where $\otimes$ is the embedding.\(^1\)

Recall that this means that the following conditions hold:

(a) $F$ and $I$ are exact.
(b) $FI = 0$.
(c) $F^*$ and $F^!$ are left and right adjoint functors to $F$, respectively.
(d) Both adjunction morphisms $\eta: \text{Id}_{\mathcal{D}(\mathcal{R}^{\text{op}})} \rightarrow FF^*$ and $\zeta: FF^! \rightarrow \text{Id}_{\mathcal{D}(\mathcal{R}^{\text{op}})}$ are isomorphisms.
(e) The same holds for the triple $(I, 1^*, 1^!)$.

(Note that Condition 1.4.3.4 from [Beilinson et al. 1982] is a consequence of the other ones; see [Neeman 2001, 9.2].)

If $\mathcal{R}$ generates $\mathcal{D}(\mathbb{A})$, we obtain an equivalence $\mathcal{D}(\mathbb{A}) \simeq \mathcal{D}(\mathcal{R}^{\text{op}})$, as in [Lunts 2010]. If $\mathcal{R}$ consists of one object $R$, we obtain an equivalence $\text{Tri}(R) \simeq \mathcal{D}(R^{\text{op}})$, where $R = \mathbb{R}\text{Hom}(R, R)$.

**Proof.** (1) We identify $\mathcal{D}(\mathbb{A})$ with the homotopy category $\mathcal{I}(\mathbb{A})$ of $K$-injective complexes, i.e., complexes $I$ such that $\text{Hom}(C, I)$ is acyclic for every acyclic complex $C$, and suppose that $\mathcal{R} \subseteq \mathcal{I}(\mathbb{A})$. Then, $\mathbb{R}\text{Hom}$ coincides with Hom within the category $\mathcal{I}(\mathbb{A})$; so, for $C \in \mathcal{I}(\mathbb{A})$, $FC = \text{Hom}_{\mathcal{I}(\mathbb{A})}(-, C)$ restricted onto $\mathcal{R}$. The full subcategory of $\mathcal{I}(\mathbb{A})$ consisting of complexes $C$ such that the natural map $\text{Hom}_{\mathcal{I}(\mathbb{A})}(R, C) \rightarrow \text{Hom}_{\mathcal{D}(\mathcal{R}^{\text{op}})}(FR, FC)$ is bijective for all $R \in \mathcal{R}$ contains $\mathcal{R}$, is strictly full, triangulated and closed under coproducts, since all objects from $\mathcal{R}$ are

\(^{1}\)Note that $\mathcal{R}$ is not necessarily *recollement-defining* in the sense of [Nicolás and Saorín 2009].
compact. Therefore, it contains \( \text{Tri}(\mathcal{R}) \). Quite analogously, the full subcategory of complexes \( C \) such that the natural map \( \text{Hom}_{\mathcal{A}}(C, C') \to \text{Hom}_{\mathcal{A}}(FC, FC') \) is bijective for every \( C' \in \text{Tri}(\mathcal{R}) \) also contains \( \text{Tri}(\mathcal{R}) \). Hence, the restriction of \( F \) onto \( \text{Tri}(\mathcal{R}) \) is fully faithful. Moreover, as the functors \( \text{Hom}_{\mathcal{A}}(-, R) \), where \( R \) runs through \( \mathcal{R} \), generate \( \mathcal{D}(\mathcal{R}^{\text{op}}) \), the functor \( F \) is essentially surjective. Therefore, restricted to \( \text{Tri}(\mathcal{R}) \), it gives an equivalence \( \text{Tri}(\mathcal{R}) \to \mathcal{D} \).

(2) Note that \( \mathcal{D}(\mathcal{R}^{\text{op}}) \) is cocomplete and compactly generated, hence satisfies the Brown representability theorem [Neeman 2001, Theorem 8.3.3]. Therefore, it is true for \( \text{Tri}(\mathcal{R}) \) too. Then, [Neeman 2001, Proposition 9.1.19] implies that a Bowsfield localization functor exists for \( \text{Tri}(\mathcal{R}) \subseteq \mathcal{D}(\mathcal{A}) \) and [Neeman 2001, Proposition 9.1.18] implies that the embedding \( \mathcal{E} : \text{Tri}(\mathcal{R}) \to \mathcal{D}(\mathcal{A}) \) has a right adjoint \( \Theta : \mathcal{D}(\mathcal{A}) \to \text{Tri}(\mathcal{R}) \). Let \( F' : \mathcal{D}(\mathcal{R}^{\text{op}}) \to \text{Tri}(\mathcal{R}) \) be a quasi-inverse to the restriction of \( F \) onto \( \text{Tri}(\mathcal{R}) \). In particular, \( F' \) is a left adjoint to this restriction and the adjunction \( FF' \to \text{Id}_{\mathcal{D}(\mathcal{R}^{\text{op}})} \) is an isomorphism. Then,

\[
FC = \text{Hom}_{\mathcal{A}}(-, C)|_{\mathcal{R}} \simeq \text{Hom}_{\mathcal{A}}(-, \Theta C)|_{\mathcal{R}} = F\Theta C.
\]

Set \( F^* = EF' \). Since \( F'M \in \text{Tri}(\mathcal{R}) \) for every \( M \in \mathcal{D}(\mathcal{R}^{\text{op}}) \),

\[
\text{Hom}_{\mathcal{A}}(F^* M, C) \simeq \text{Hom}_{\text{Tri}(\mathcal{R})}(F'M, \Theta C) \\
\simeq \text{Hom}_{\mathcal{D}(\mathcal{R}^{\text{op}})}(M, F\Theta C) \simeq \text{Hom}_{\mathcal{D}(\mathcal{R}^{\text{op}})}(M, FC),
\]

for any \( M \in \mathcal{D}(\mathcal{R}^{\text{op}}) \) and \( C \in l(\mathcal{A}) \). Hence, \( F^* \) is a left adjoint to \( F \). If, moreover, \( C \in \text{Tri}(\mathcal{R}) \), we obtain

\[
\text{Hom}_{\mathcal{D}(\mathcal{R}^{\text{op}})}(F F^* M, FC) \simeq \text{Hom}_{\mathcal{A}}(F^* M, C) \simeq \text{Hom}_{\mathcal{D}(\mathcal{R}^{\text{op}})}(M, FC).
\]

As \( F \) is essentially surjective, this implies that \( \eta : FF^* \to \text{Id}_{\mathcal{D}(\mathcal{R}^{\text{op}})} \) is an isomorphism. As all objects from \( \mathcal{R} \) are compact, \( F \) respects coproducts, hence has a right adjoint \( F^! \) [Neeman 2001, Theorem 8.4.4]. Now it follows from [Burban et al. 2017, Corollary 2.3] that \( \xi : FF^! \to \text{Id}_{\mathcal{D}(\mathcal{R}^{\text{op}})} \) is an isomorphism and there is a recollement diagram (1-1).

Note that \( \text{Im } F^* = \text{Tri}(\mathcal{R}) \) by construction, but usually \( \text{Im } F^! \not= \text{Tri}(\mathcal{R}) \), though it is equivalent to \( \text{Tri}(\mathcal{R}) \).

**Corollary 1.2.** Under the conditions and notations of the preceding theorem, suppose that \( \text{Hom}_{\mathcal{D}(\mathcal{A})}(R, T[m]) = 0 \) for \( R, T \in \mathcal{R} \) and \( m \neq 0 \). Then, the functor \( F \) induces an equivalence \( \text{Tri}(R) \xrightarrow{\sim} \mathcal{D}(\mathcal{R}^{\text{op}}) \), where \( \mathcal{R} \) is the category with the set of objects \( \mathcal{R} \) and the sets of morphisms \( \mathcal{A}(A, B) = \text{Hom}_{\mathcal{A}}(A, B) \).

In this situation, we call the functor \( F \) a partial tilting functor.
2. Backström pairs

Recall from [Bass 1960; Lambek 1976] that a semiperfect ring is a ring $A$ such that $A/\text{rad } A$ is a semisimple artinian ring and idempotents can be lifted modulo $\text{rad } A$. Equivalently, as a left (or as a right) $A$-module, $A$ decomposes into a direct sum of modules with local endomorphism rings.

**Definition 2.1.** (1) A Backström pair is a pair of semiperfect rings $H \supseteq A$ such that $\text{rad } A = \text{rad } H$. We denote by $\text{C}(H, A)$ the conductor of $H$ in $A$:

$\text{C}(H, A) = \{\alpha \in A \mid H\alpha \subseteq A\} = \text{ann}(H/A)_A$  

(the right subscript $A$ means that we consider $H/A$ as a right $A$-module). Obviously, $\text{C}(H, A) \supseteq \text{rad } A$, so both $A/C$ and $H/C$ are semisimple rings.

(2) We call a ring $A$ a (left) Backström ring (resp. piecewise Backström ring) if there is a Backström pair $H \supseteq A$, where the ring $H$ is left hereditary (resp. left piecewise hereditary [Happel 1988], i.e., derived equivalent to a left hereditary ring). If, moreover, both $A$ and $H$ are finite dimensional algebras over a field $k$, we call $A$ a Backström algebra (resp. piecewise Backström algebra).

**Remark 2.2.** If $e$ is an idempotent in $A$, then $\text{rad}(eAe) = e(\text{rad } A)e$, hence, if $H \supseteq A$ is a Backström pair, so is $eHe \supseteq eAe$. This implies that if $P$ is a finitely generated projective $A$-module, $A' = \text{End}_A P$ and $H' = \text{End}_H(H \otimes_A P)$, then $H' \supseteq A'$ is also a Backström pair. Note that if $H$ is left hereditary (or piecewise hereditary), so is $H'$, hence $A'$ is a Backström ring (piecewise Backström ring) whenever $A$ is. In particular, the notion of Backström (or piecewise Backström) ring is Morita invariant. Note also that if $H$ is left hereditary and noetherian, it is also right hereditary, so $A^{\text{op}}$ is also a Backström ring (piecewise Backström ring).

**Examples 2.3.** (1) An important example of Backström algebras are nodal algebras introduced in [Drozd and Zembyk 2013; Zembyk 2014]. By definition, they are finite dimensional algebras such that there is a Backström pair $H \supseteq A$, where $H$ is a hereditary algebra and $\text{length}_A(H \otimes_A U) \leq 2$ for every simple $A$-module $U$. Their structure was completely described in [Zembyk 2014].

(2) Recall that a $k$-algebra $A$ is called gentle [Assem and Skowroński 1987] if $A \simeq k\Gamma/J$, where $\Gamma$ is a finite quiver (oriented graph) and $J$ is an ideal in the path algebra $k\Gamma$ such that $(J_+)^2 \supseteq J \supseteq (J_+)^k$ for some $k$, where $J_+$ is the ideal generated by all arrows, and the following conditions hold:

(a) For every vertex $i \in \text{Ver } \Gamma$ there are at most two arrows starting at $i$ and at most two arrows ending at $i$.

(b) If an arrow $a$ starts at $i$ (resp. ends at $i$) and arrows $b_1, b_2$ end at $i$ (resp. start at $i$), then either $ab_1 = 0$ or $ab_2 = 0$ (resp. either $b_1a = 0$ or $b_2a = 0$), but not both.
(c) The ideal $J$ is generated by products of arrows of the sort $ab$.

It is proved in [Zembyk 2015] that such algebras are nodal, hence Backström algebras. The same is true for skewed-gentle algebras [Geißland de la Peña 1999] obtained from gentle algebras by blowing up some vertices.

(3) **Backström orders** are orders $A$ over a discrete valuation ring such that there is a Backström pair $H \supseteq A$, where $H$ is a hereditary order. They were considered in [Ringel and Roggenkamp 1979].

(4) Let $H = T(n, \kappa)$ be the ring of upper triangular $n \times n$ matrices over a field $\kappa$ and $A = UT(n, \kappa)$ be its subring of unitriangular matrices $M$, i.e., such that all diagonal elements of $M$ are equal. Then, $H$ is hereditary and $\text{rad } H = \text{rad } A$, hence $A$ is a Backström algebra. In this case, $C(H, A) = \text{rad } A$.

(5) $\Lambda_n = \kappa[x_1, x_2, \ldots, x_n]/(x_1, x_2, \ldots, x_n)^2$ embeds into $H = \prod_{i=1}^n \kappa \Gamma_i$, where $\Gamma_i = \{a_i \mapsto (x_i \text{ maps to } a_i)\}$. Obviously, under this embedding $\text{rad } \Lambda_n = \text{rad } H$, so $\Lambda_n$ is a Backström algebra.

We consider a fixed Backström pair $H \supseteq A$, set $e = \text{rad } A = \text{rad } H$ and denote by $C$ the conductor $C(H, A)$. Obviously, $C$ is a two-sided $A$-ideal and the biggest left $H$-ideal contained in $A$. Actually, it even turns out to be a two-sided $H$-ideal and its definition is left-right symmetric.

**Lemma 2.4.** Let $R \subseteq S$ be semisimple rings, $I = \{\alpha \in R \mid S\alpha \subseteq R\}$. Then, $I$ is a two-sided $S$-ideal.

**Proof.** Obviously, $I$ is a left $S$-ideal and a two-sided $R$-ideal. As $R$ is semisimple, $I = Re$ for some central idempotent $e \in R$. Then, $Se \subseteq Re$, so $Se = Re = eR$ and $(1 - e)Se = 0$. Hence, $eS(1 - e)$ is a left ideal in $S$ and $(eS(1 - e))^2 = 0$, so $eS(1 - e) = 0$ and $I = Se = eS$ is also a right $S$-ideal. $\square$

**Proposition 2.5.** $C$ is a two-sided $H$-ideal. It is the biggest $H$-ideal contained in $A$. Therefore, it coincides with the set $\{\alpha \in A \mid \alpha H \subseteq A\}$ or with $\text{ann}_A(H/A)$ considered as a left $A$-module.

**Proof.** It follows from the preceding lemma applied to the rings $A/\text{rad } A$ and $H/\text{rad } H$. $\square$
$A^a_b \subseteq \text{rad } A$ if $a \neq b$. Hence, if we set $\tau^a = \text{rad } A^a (a = 0, 1)$ and consider the Pierce decomposition of the ring $A$

$$A = \begin{pmatrix} A^0_0 & A^1_0 \\ A^0_1 & A^1_1 \end{pmatrix},$$

the Pierce decomposition of the ideal $\tau$ becomes

$$\tau = \begin{pmatrix} \tau^0_0 & \tau^1_0 \\ \tau^0_1 & \tau^1_1 \end{pmatrix},$$

where $\tau^a_a = \text{rad } A^a_a$, $a = 0, 1$. This implies that $H^0$ and $H^1$ have no isomorphic direct summands, the Pierce decomposition of $H$ is

$$H = \begin{pmatrix} H^0_0 & A^1_0 \\ A^0_1 & A^1_1 \end{pmatrix}$$

and $\tau^0_0 = \text{rad } H^0_0$. Now, one easily sees that an element $a = (\alpha \beta \gamma \delta)$ belongs to $C$ if and only if $H^0 \alpha \subseteq A^0$. We claim that in that case $H^0 \alpha \subseteq \text{rad } A^0$. Otherwise $H^0 \alpha$ contains an idempotent, hence a direct summand of $A^0$, which is isomorphic to some $A_i$ with $1 \leq i \leq r$. This is impossible, since $HA_i \neq A_i$. Therefore, $a \in \tau^0_0$ and we obtain the following result:

**Proposition 2.6.** The Pierce decomposition of the ideal $C$ is

$$C = \begin{pmatrix} \tau^0_0 & A^1_0 \\ A^0_1 & A^1_1 \end{pmatrix}.$$

**Definition 2.7.** Analogously to [Burban and Drozd 2011], we define the Auslander envelope of the Backström pair $H \supseteq A$ as the ring $\tilde{A}$ of $2 \times 2$ matrices of the form

$$\tilde{A} = \begin{pmatrix} A & H \\ C & H \end{pmatrix}$$

with the usual matrix multiplication.

Using Pierce decompositions of $A$, $H$ and $C$, we also present $\tilde{A}$ as the ring of $4 \times 4$ matrices

$$\tilde{A} = \begin{pmatrix} A^0_0 & A^1_0 & H^0_0 & A^1_0 \\ A^0_1 & A^1_1 & A^0_1 & A^1_1 \\ \tau^0_0 & \tau^1_0 & H^0_0 & A^1_0 \\ \tau^0_1 & \tau^1_1 & A^0_1 & A^1_1 \end{pmatrix}.$$  

(2-1)
We also define $\tilde{H}$ as the ring of $4 \times 4$ matrices of the form

\[
\tilde{H} = \begin{pmatrix} H & H \\ C & H \end{pmatrix} \quad \text{or} \quad \tilde{H} = \begin{pmatrix} H^0 & A^1_0 & H^0_0 & A^1_0 \\ A^0_1 & A^1_1 & A^0_1 & A^1_1 \\ t^0_0 & A^1_0 & H^0_0 & A^1_0 \\ A^0_1 & A^1_1 & A^0_1 & A^1_1 \end{pmatrix}.
\]

Obviously, $\text{rad}\, \tilde{H} = \text{rad}\, \tilde{A}$, so $\tilde{H} \supseteq \tilde{A}$ is also a Backström pair. $\tilde{A}$ is left noetherian if and only if $A$ is left noetherian and $H$ is finitely generated as a left $A$-module.

In the noetherian case one can calculate the global dimensions of $\tilde{A}$ and $\tilde{H}$. It turns out that it only depends on $H$.

**Theorem 2.8.** Suppose that either $A$ (hence also $H$) is left perfect or $A$ is left noetherian and $H$ is finitely generated as a left $A$-module. Then

\[
1.\text{gl.dim} \tilde{A} = 1 + \max(1 + \text{pr.dim}_H t^0, \text{pr.dim}_H t^1)
\]

\[= \begin{cases} 
1 + 1.\text{gl.dim} H & \text{if } \text{pr.dim}_H t^0 \geq \text{pr.dim}_H t^1, \\
1.\text{gl.dim} H & \text{if } \text{pr.dim}_H t^0 < \text{pr.dim}_H t^1
\end{cases}
\]

and

\[
1.\text{gl.dim} \tilde{H} = 1.\text{gl.dim} H,
\]

where we set $\text{pr.dim} 0 = -1$. In particular, if $A$ is a Backström ring, so is $\tilde{A}$, and if $A$ is not left hereditary, then $1.\text{gl.dim} \tilde{A} = 2$.  

For instance, this is the case for nodal (in particular, gentle or skewed-gentle) algebras (Examples 2.3).

**Proof.** Under these conditions $\tilde{A}$ and $\tilde{H}$ are either left perfect or left noetherian. We recall that if a ring $\Lambda$ is left perfect or left noetherian and semiperfect, then $1.\text{gl.dim} \Lambda = \text{pr.dim}_\Lambda (\Lambda/\text{rad} \Lambda) = 1 + \text{pr.dim}_\Lambda \text{rad} \Lambda$. The $4 \times 4$ matrix presentation (2-1) of $\tilde{A}$ implies that the corresponding presentation of $\text{rad}\, \tilde{A}$ is

\[
(2-2) \quad \text{rad}\, \tilde{A} = \begin{pmatrix} t^0_0 & A^1_0 & H^0_0 & A^1_0 \\ A^0_1 & A^1_1 & A^0_1 & A^1_1 \\ t^0_0 & A^1_0 & t^0_0 & A^1_0 \\ A^0_1 & t^1_1 & A^0_1 & t^1_1 \end{pmatrix}.
\]

An $\tilde{A}$-module $M$ is given by a quadruple $(M', M'', \psi, \psi)$, where $M'$ is an $A$-module, $M''$ is an $H$-module, $\psi : M'' \to M'$ is a homomorphism of $A$-modules and $\phi : C \otimes_A M' \to M''$ is a homomorphism of $H$-modules. Namely, $M' = e'M$, $M'' = e''M$, where $e' \equiv (1, 0, 0, 0)$, $e'' \equiv (0, 0, 1, 0)$, $\psi(m'') = (0, 1, 0, 0)m''$ and $\phi(c \otimes m') = (0, 0, 0, 0)m'$. 

---

Note that if $\tilde{A}$ is left hereditary, so is $A = e'\tilde{A}e'$ [Sandomierski 1969].
We frequently write \( M = \left( \frac{M'}{M''} \right) \), not mentioning \( \phi \) and \( \psi \). For an \( H \)-module \( N \) we define the \( A \)-module \( N^+ = \left( \frac{N}{N'} \right) \). Then, \( N \mapsto N^+ \) is an exact functor mapping projective modules to projective ones, since \( H^+ = \left( \frac{H}{H} \right) \) is a projective \( A \)-module.

We denote by \( L_i \) and by \( R_i \) the \( i \)-th column of the presentations \((2\cdot1)\) and \((2\cdot2)\), respectively. Then, \( R_1 = (r_0) + \) and \( R_2 = R_4 = (r_1) + \), where \( r_i = te_i \). If 
\[
\cdots \to F_k \to \cdots \to F_1 \to F_0 \to N \to 0
\]
is a minimal projective resolution of an \( H \)-module \( N \),
\[
\cdots \to F_k^+ \to \cdots \to F_1^+ \to F_0^+ \to N^+ \to 0
\]
is a minimal projective resolution of \( N^+ \), so \( \text{pr.dim}_A N^+ = \text{pr.dim}_H N \). In particular, \( \text{pr.dim}_A R_1 = \text{pr.dim}_H r_0 \) and \( \text{pr.dim}_A R_2 = \text{pr.dim}_H r_1 \). For the module \( R_3 \) we have an exact sequence
\[
(2\cdot3) \quad 0 \to (r_0)^+ \to R^3 \to \left( \frac{H^0/r^0}{0} \right) \to 0.
\]
Note that \( H^0/r_0 \) is a semisimple \( A \)-module and \( e_1(H^0/r_0) = 0 \), hence it contains the same simple direct summands as \( A^0/r_0 \). The same is true for
\[
\left( \frac{H^0/r^0}{0} \right) \quad \text{and} \quad \left( \frac{A^0/r^0}{0} \right) = L^1/R^1.
\]
Hence,
\[
\text{pr.dim}_A \left( \frac{H^0/r^0}{0} \right) = 1 + \text{pr.dim}_A R_1 = 1 + \text{pr.dim}_H r_0.
\]
Therefore, the exact sequence \((2\cdot3)\) shows that \( \text{pr.dim}_A R_3 = 1 + \text{pr.dim}_H r_0 \) and
\[
\text{pr.dim}_A \text{rad} \tilde{A} = \text{max}(1 + \text{pr.dim}_H r_0, \text{pr.dim}_H r_1),
\]
which gives the necessary result for \( \tilde{A} \). On the other hand, \( R_3 \) is a projective \( \tilde{H} \)-module, whence \( \text{l.gl.dim} \tilde{H} = 1.\text{gl.dim} H \).  

\[\square\]

3. The structure of derived categories

In what follows we denote by \( \mathcal{D}(A) \) the derived category \( \mathcal{D}(A\text{-Mod}) \). We denote by \( \mathcal{D}_f(A) \) the full subcategory of \( \mathcal{D}(A) \) consisting of complexes quasi-isomorphic to complexes of finitely generated projective modules. If \( A \) is left noetherian, it coincides with the derived category of the category \( A\text{-mod} \) of finitely generated \( A \)-modules. We also use the usual superscripts \( +, -, b \). By \( \text{Perf}(A) \) we denote the full subcategory of perfect complexes from \( \mathcal{D}(A) \), i.e., complexes quasi-isomorphic to finite complexes of finitely generated projective modules. It coincides with the full subcategory of compact objects in \( \mathcal{D}(A) \) [Rouquier 2008]. If \( A \) is left
noetherian, an $A$-module $M$ belongs to $\text{Perf}(A)$ if and only if it is finitely generated and of finite projective dimension.

There are close relations between the categories $\mathcal{D}(A)$, $\mathcal{D}(H)$ and $\mathcal{D}(\tilde{A})$ based on the following construction [Burban et al. 2017].

Let $P = \left( \begin{array}{c} A \\ C \end{array} \right)$. It is a projective $\tilde{A}$-module and $\text{End} P \simeq A^{op}$, so it can be considered as a right $A$-module. Consider the functors

$$F = \text{Hom}_{\tilde{A}}(P, -) \simeq P^{\vee} \otimes_{\tilde{A}} - : \tilde{A}\text{-Mod} \rightarrow A\text{-Mod},$$

$$F^* = P \otimes_A - : A\text{-Mod} \rightarrow \tilde{A}\text{-Mod},$$

$$F' = \text{Hom}_A(P^{\vee}, -) : A\text{-Mod} \rightarrow \tilde{A}\text{-Mod},$$

where $P^{\vee} = \text{Hom}_{\tilde{A}}(P, \tilde{A}) \simeq (A \ H)$ is the dual right projective $\tilde{A}$-module, the functor $F$ is exact, $F^*$ is its left adjoint and $F'$ is its right adjoint. Moreover, the adjunction morphisms $FF^* \rightarrow \text{Id}_{\tilde{A}\text{-Mod}}$ and $\text{Id}_{A\text{-Mod}} \rightarrow FF'$ are isomorphisms [Burban et al. 2017, Theorem 4.3]. The functors $F^*$ and $F'$ are fully faithful and $F$ is essentially surjective, i.e., every $A$-module is isomorphic to $FM$ for some $\tilde{A}$-module $M$. $\text{Ker} F$ is a Serre subcategory of $\tilde{A}\text{-Mod}$ equivalent to $H\text{-Mod}$, where $H = H/C \simeq \tilde{A}/(A \ H)$. The embedding functor $l : \text{Ker} F \rightarrow \tilde{A}\text{-Mod}$ has a left adjoint $l^*$ and a right adjoint $l'$ and we obtain a recollement diagram

$$\begin{array}{ccc}
\text{Ker} F & \xrightarrow{l} & \tilde{A}\text{-Mod} \\
\xrightarrow{l'} & & \xleftarrow{F^*} \text{A-Mod} \\
\xleftarrow{F'} & & \\
& & \end{array}$$

As the functor $F$ is exact, it extends to the functor between the derived categories $DF : \mathcal{D}(\tilde{A}) \rightarrow \mathcal{D}(A)$ acting on complexes componentwise. The derived functors $\text{L}F^*$ and $\text{R}F'$ are its left and right adjoints, respectively, the adjunction morphisms $\text{Id}_{\mathcal{D}(A)} \rightarrow DF \cdot \text{L}F^*$ and $DF \cdot \text{L}F^* \rightarrow \text{Id}_{\mathcal{D}(A)}$ are again isomorphisms and we have a recollement diagram

$$\begin{array}{ccc}
\text{Ker} DF & \xrightarrow{\text{L}F^*} & \mathcal{D}(\tilde{A}) \\
\xleftarrow{\text{R}F'} & & \xrightarrow{DF} \mathcal{D}(A) \\
& \xleftarrow{\text{Id}} & \\
& & \end{array}$$

(It also follows from Corollary 1.2.) Here $\text{Ker} DF = \mathcal{D}_H(\tilde{A})$, the full subcategory of complexes whose cohomologies are $H$-modules, i.e., are annihilated by the ideal $(A \ H)$. Note that, as a rule, it is not equivalent to $\mathcal{D}(H)$. From the definition of $F$ it follows that

$$\text{Ker} DF = P^\perp = \{ C \in \mathcal{D}(\tilde{A}) \mid \text{Hom}_{\mathcal{D}(\tilde{A})}(P, C[k]) = 0 \text{ for all } k \}.$$  

Obviously, $DF$ maps $\mathcal{D}^\sigma(\tilde{A})$ to $D^\sigma(A)$ for $\sigma \in \{ +, -, b \}$, $\text{L}F^*$ maps $\mathcal{D}^-(A)$ to $\mathcal{D}^-(\tilde{A})$ and $\text{R}F'$ maps $\mathcal{D}^+(A)$ to $\mathcal{D}^+(\tilde{A})$. If $\tilde{A}$ is left noetherian, $DF$ maps $\mathcal{D}_f(\tilde{A})$ to $\mathcal{D}_f(A)$ and $\text{L}F^*$ maps $\mathcal{D}_f(A)$ to $\mathcal{D}_f(\tilde{A})$. Finally, both $DF$ and $\text{L}F^*$ have right adjoints, hence map compact objects (i.e., perfect complexes) to compact ones.
On the contrary, usually $\mathcal{F}^*$ does not map $\mathcal{D}^b(A)$ to $\mathcal{D}^b(\tilde{A})$. For instance, it is definitely so if $\text{1.gl.dim } A < \infty$ while $\text{1.gl.dim } A = \infty$ as in Examples 2.3 (4, 5). If $\text{1.gl.dim } H$ is finite, so is $\text{1.gl.dim } \tilde{A}$, thus this recollement can be considered as a sort of categorical resolution of the category $\mathcal{D}(A)$. In any case, it is useful for studying the categories $A\text{-Mod}$ and $\mathcal{D}(A)$ if we know the structure of the categories $\tilde{A}\text{-Mod}$ and $\mathcal{D}(\tilde{A})$. For instance, it is so if we are interested in the derived dimension, i.e., the dimension of the category $\mathcal{D}^b_f(A)$ in the sense of [Rouquier 2008].

**Definition 3.1.** Let $\mathcal{F}$ be a triangular category and $\mathcal{M}$ be a set of objects from $\mathcal{F}$.

1. We denote by $\langle \mathcal{M} \rangle$ the smallest full subcategory of $\mathcal{F}$ containing $\mathcal{M}$ and closed under direct sums, direct summands and shifts (not closed under cones, so not a triangulated subcategory).

2. If $\mathcal{N}$ is another subset of $\mathcal{F}$, we denote by $\mathcal{M} \triangledown \mathcal{N}$ the set of objects $C$ from $\mathcal{F}$ such that there is an exact triangle $A \rightarrow B \rightarrow C \rightarrow$, where $A \in \mathcal{M}$, $B \in \mathcal{N}$.

3. We define $\langle \mathcal{M} \rangle_k$ recursively, setting $\langle \mathcal{M} \rangle_1 = \langle \mathcal{M} \rangle$ and $\langle \mathcal{M} \rangle_{k+1} = \langle \langle \mathcal{M} \rangle \triangledown \langle \mathcal{M} \rangle_k \rangle$.

4. The *dimension* $\dim \mathcal{F}$ of $\mathcal{F}$ is the smallest $k$ such that there is a finite set of objects $\mathcal{M}$ such that $\langle \mathcal{M} \rangle_{k+1} = \mathcal{F}$ (if it exists). We call the dimension $\dim \mathcal{D}^b_f(A)$ the *derived dimension* of the ring $A$ and denote it by $\text{der.dim } A$.

As the functor $F$ is exact and essentially surjective, the next result is evident:

**Proposition 3.2.** We have $\text{der.dim } A \leq \text{der.dim } \tilde{A}$. Namely, if $\mathcal{D}^b_f(\tilde{A}) = \langle \mathcal{M} \rangle_{k+1}$, then $\mathcal{D}^b_f(A) = \langle DF(\mathcal{M}) \rangle_{k+1}$.

4. **Semiorthogonal decomposition**

There is another recollement diagram for $\mathcal{D}(\tilde{A})$ related to the projective module $Q = (H^h)$ with $\text{End } Q \simeq H^{\text{op}}$. Namely, we set

\[
\begin{align*}
G &= \text{Hom}_\tilde{A}(Q, -) \simeq Q^\vee \otimes \tilde{A} : \tilde{A}\text{-Mod} \rightarrow H\text{-Mod}, \\
G^* &= Q \otimes H - : H\text{-Mod} \rightarrow \tilde{A}\text{-Mod}, \\
G^! &= \text{Hom}_H(Q^\vee, -) : H\text{-Mod} \rightarrow \tilde{A}\text{-Mod},
\end{align*}
\]

where $Q^\vee = \text{Hom}_\tilde{A}(Q, \tilde{A}) \simeq (C H)$,

\[
\begin{align*}
DG : \mathcal{D}(\tilde{A}) &\rightarrow \mathcal{D}(H) \text{ is } G \text{ applied componentwise,} \\
LG^* : \mathcal{D}(A) &\rightarrow \mathcal{D}(\tilde{A}) \text{ is the left adjoint of } DG, \\
RG^! : \mathcal{D}(A) &\rightarrow \mathcal{D}(\tilde{A}) \text{ is the right adjoint of } DG.
\end{align*}
\]
We also set $\tilde{A} = A/C \simeq \hat{A}/(C \hat{H})$. Then, we have recollement diagrams

$$\xymatrix{\text{Ker } G \ar[r]^-j & \tilde{A}\text{-Mod} \ar[r]^-G & H\text{-Mod}}$$

and

$$\xymatrix{\text{Ker } \text{DG} \ar[r]^-{Lj} & \mathcal{D}(\tilde{A}) \ar[r]^-{DG} & \mathcal{D}(H),}$$

where $\text{Ker } G \simeq \tilde{A}\text{-Mod}$. Since the $\tilde{A}$-ideal $(C \ hat{H})$ is projective as a right $\tilde{A}$-module, [Burban et al. 2017, Theorem 4.6] implies that $\text{Ker } \text{DG} \simeq \mathcal{D}(\tilde{A})$.

As usual, this recollement diagram gives semiorthogonal decompositions [Burban et al. 2017, Corollary 2.6]

$$\mathcal{D}(\tilde{A}) = (\text{Ker } \text{DG}, \text{Im } \text{LG}^*) = (\text{Im } \text{RG}^1, \text{Ker } \text{DG})$$

with $\text{Ker } \text{DG} \simeq \mathcal{D}(\tilde{A})$ and $\text{Im } \text{LG}^* \simeq \text{Im } \text{RG}^1 \simeq \mathcal{D}(H)$ (though usually $\text{Im } \text{LG}^* \neq \text{Im } \text{RG}^i$).

Recall from [Kuznetsov and Lunts 2015] that a *semiorthogonal decomposition* $\mathcal{I} = (\mathcal{I}_1, \mathcal{I}_2)$, where $\mathcal{I}_1, \mathcal{I}_2$ are full triangulated subcategories of $\mathcal{I}$, means that

$$\text{Hom}_{\mathcal{I}}(T_2, T_1) = 0 \quad \text{for all } T_1 \in \mathcal{I}_1 \text{ and } T_2 \in \mathcal{I}_2,$$

and for every object $T \in \mathcal{I}$ there is an exact triangle $T_1 \to T_2 \to T \to$, where $T_i \in \mathcal{I}_i$.

**Lemma 4.1.** If $\mathcal{I} = (\mathcal{I}_1, \mathcal{I}_2)$ is a semiorthogonal decomposition of a triangulated category $\mathcal{I}$, then

$$\dim \mathcal{I} \leq \dim \mathcal{I}_1 + \dim \mathcal{I}_2 + 1.$$

**Proof.** First we show that for any subsets $\mathcal{M}, \mathcal{N}$ of objects of the category $\mathcal{I}$

$$\langle \mathcal{M} \rangle_{k+1} \uparrow \mathcal{N} \subseteq \langle \mathcal{M} \rangle \uparrow \langle \langle \mathcal{M} \rangle \uparrow N \rangle \subseteq \langle \mathcal{M} \rangle \uparrow \langle \langle \mathcal{M} \rangle \uparrow \langle \langle \mathcal{M} \rangle \uparrow \cdots \rangle \cdots \rangle.$$  \hspace{1cm} (4-2)

Indeed, let $C \in \langle \mathcal{M} \rangle_{k+1} \uparrow \mathcal{N}$, i.e., there is an exact triangle $A \to B \to C \to$, where $A \in \langle \mathcal{M} \rangle_{k+1}, B \in \mathcal{N}$. There is also an exact triangle $A_1 \to A \to A_2 \to$, where $A_1 \in \langle \mathcal{M} \rangle_k, A_2 \in \langle \mathcal{M} \rangle$. The octahedron axiom implies that there are exact triangles $A_1 \to B \to B' \to$ and $A_2 \to B' \to C \to$. Therefore, $B' \in \langle \mathcal{M} \rangle_k \uparrow \mathcal{N}$ and $C \in \langle \mathcal{M} \rangle \uparrow \langle \langle \mathcal{M} \rangle \uparrow \mathcal{N} \rangle$.

Now, let $\langle \mathcal{M} \rangle_{k+1} \uparrow \mathcal{I}_1$ and $\langle \mathcal{M} \rangle_{l+1} \uparrow \mathcal{I}_2$. Then, for every $T \in \mathcal{I}$ there is an exact triangle $T_1 \to T_2 \to T \to$, where $T_1 \in \langle \mathcal{M} \rangle_{k+1}, T_2 \in \langle \mathcal{M} \rangle_{l+1}$. But, according to (4-2), $\langle \mathcal{M} \rangle_{k+1} \uparrow \langle \mathcal{M} \rangle_{l+1} \subseteq \langle \mathcal{M} \cup \mathcal{N} \rangle_{k+l+2}$, so $\mathcal{I} = \langle \mathcal{M} \cup \mathcal{N} \rangle_{k+l+2}$ and $\dim \mathcal{I} \leq k+l+1$. \hspace{1cm} \Box

---

\footnote{In [Psaroudakis 2014, Theorem 7.4] this result is proved in the case when this decomposition arises from a recollement.}
Since $\tilde{A}$ is semisimple, any indecomposable object from $\mathcal{D}(\tilde{A})$ is just a shifted simple module, so $\mathcal{D}_f^b(\tilde{A}) = \langle \tilde{A} \rangle$ and $\text{der.dim} \tilde{A} = 0$. If $H$ is hereditary, every indecomposable object from $\mathcal{D}_f^b(H)$ is a shift of a module. For every module $M$ there is an exact sequence $0 \to P' \to P \to M \to 0$ with projective modules $P$, $P'$ and, since $H$ is semiperfect, every indecomposable projective $H$-module is a direct summand of $H$. Hence, $\mathcal{D}_f^b(H) = \langle H \rangle_2$ and $\text{der.dim} H \leq 1$.

**Corollary 4.2.** We have $\text{der.dim} A \leq \text{der.dim} H + 1$. In particular, if $A$ is a Backström (or piecewise Backström) ring, $\text{der.dim} A \leq 2$.

A finite dimensional hereditary algebra is said to be of *Dynkin type* if it has finitely many isomorphism classes of indecomposable modules. Such algebras, up to Morita equivalence, correspond to Dynkin diagrams [Dlab and Ringel 1976; Gabriel 1972]. If the derived category of an algebra $H$ is equivalent to the derived category of a hereditary algebra of Dynkin type, we say that $H$ is *piecewise hereditary of Dynkin type*. We say that a Backström (or piecewise Backström) algebra $A$ is of Dynkin type if there is a Backström pair $H \supseteq A$, where $H$ is a hereditary (piecewise hereditary) algebra of Dynkin type. For instance, it is so if $A$ is a gentle or skewed-gentle algebra [Zembyk 2015], or the algebra $\text{UT}(n|k)$ of unitriangular matrices over a field (Examples 2.3 (4)), or the algebra $\Lambda_n$ from Examples 2.3 (5). In this case, $\mathcal{D}_f^b(H) = \langle M_1, M_2, \ldots, M_m \rangle_1$, where $M_1, M_2, \ldots, M_m$ are all pairwise nonisomorphic indecomposable $H$-modules, so $\text{der.dim} H = 0$.

In [Chen et al. 2008] it was proved that $\text{der.dim} A = 0$ for a finite dimensional algebra $A$ if and only if $A$ is a piecewise hereditary algebra of Dynkin type.

**Corollary 4.3.** If $A$ is a Backström (or piecewise Backström) algebra of Dynkin type (for instance, gentle or skewed-gentle), but is not piecewise hereditary of Dynkin type, then $\text{der.dim} A = 1$.

**Example 4.4.** The path algebra of the commutative quiver

\[
\begin{array}{c}
1 \\
\alpha_0 \\
\alpha_1 \\
\beta_0 \\
\beta_1 \\
\gamma \\
\gamma' \\
\end{array}
\quad
\begin{array}{c}
2 \\
\beta_0' \\
\beta_1' \\
3 \\
4 \\
4' \\
\end{array}
\quad
\alpha_1 \alpha_0 = \beta_1 \beta_0
\]

is a tilted (hence piecewise hereditary) algebra of type $\tilde{D}_5$. At the same time it is a Backström algebra of type $A_4$. Namely, it is a skewed-gentle algebra obtained from the path algebra of the quiver $1 \to 2 \to 3 \to 4$ by blowing up vertices 2 and 4.\(^5\)

\(\text{der.dim} H \leq 1.\)

\[^4\]It is proved in [Happel 1988] that piecewise hereditary algebras of Dynkin type are just iterated tilted algebras of Dynkin type.

\(^5\)See [Zembyk 2014] for the construction of blowing up and its relation to nodal algebras.
5. Relation to bimodule categories

In this section, we explain how a semiorthogonal decomposition allows us to apply to calculations in a triangulated category the technique of matrix problems, namely, of bimodule categories, as in [Drozd 2010].

Let \( \mathcal{A} \) and \( \mathcal{B} \) be additive categories, \( \mathcal{U} \) be an \( \mathcal{A} \mathcal{B} \)-bimodule, i.e., a biadditive functor \( \mathcal{A}^{\text{op}} \times \mathcal{B} \to \text{Ab} \). Recall from [Drozd 2010] that the bimodule category or the category of elements of the bimodule \( \mathcal{U} \) is the category \( \text{El}(\mathcal{U}) \) whose set of objects is \( \bigcup_{A \in \mathcal{A}, B \in \mathcal{B}} \mathcal{U}(A, B) \) and whose morphisms from \( u \in \mathcal{U}(A, B) \) to \( v \in \mathcal{U}(A', B') \) are the pairs \( (\alpha, \beta) \) such that \( u\alpha = \beta v \), where \( \alpha : A' \to A, \beta : B \to B' \). Here, as usual, we wrote \( u\alpha \) and \( \beta v \) instead of \( \mathcal{U}(\alpha, 1_B)u \) and \( \mathcal{U}(1_{A'}, \beta)v \). Bimodule categories appear when there is a semiorthogonal decomposition of a triangulated category.

**Theorem 5.1.** Let \( (\mathcal{A}, \mathcal{B}) \) be a semiorthogonal decomposition of a triangulated category \( \mathcal{C} \). Consider the \( \mathcal{A} \mathcal{B} \)-bimodule \( \mathcal{U} \) such that \( \mathcal{U}(A, B) = \text{Hom}_\mathcal{C}(A, B), A \in \mathcal{A}, B \in \mathcal{B} \). For every \( f : A \to B \) fix a cone \( C_f \), that is, an exact triangle \( A \xrightarrow{\eta} B \xrightarrow{\beta} C_f \xrightarrow{\alpha} A[1] \). The map \( f \mapsto C_f \) induces an equivalence of categories \( \mathcal{C} : \text{El}(\mathcal{U}) \xrightarrow{\sim} \mathcal{C}/\mathcal{J} \), where \( \mathcal{J} \) is the ideal of \( \mathcal{C} \) consisting of morphisms \( \eta \) such that there are factorizations \( \eta = \eta'\xi = \xi\eta'' \), where the source of \( \eta' \) is in \( \mathcal{A} \) and the target of \( \eta'' \) is in \( \mathcal{B} \). Moreover, \( \mathcal{J}^2 = 0 \), so \( \mathcal{C} \) induces a bijection between isomorphism classes of objects from \( \text{El}(\mathcal{U}) \) and from \( \mathcal{C} \).

**Proof.** As \( (\mathcal{A}, \mathcal{B}) \) is a semiorthogonal decomposition of \( \mathcal{C} \), every object from \( \mathcal{C} \) occurs in an exact triangle \( A \xrightarrow{\eta} B \xrightarrow{\beta} C \xrightarrow{\alpha} A[1] \), where \( A \in \mathcal{A}, B \in \mathcal{B}, \) so \( f \) is an object from \( \text{El}(\mathcal{U}) \) and \( C \simeq C_f \). Let \( f' : A' \to B' \) be another object of \( \text{El}(\mathcal{U}) \) and \( (\alpha, \beta) : f \to f' \) be a morphism from \( \text{El}(\mathcal{U}) \). Fix a commutative diagram

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow{\alpha} & & \downarrow{\beta} \\
A' & \xrightarrow{f'} & B'
\end{array}
\quad
\begin{array}{ccc}
C_f & \xrightarrow{f_1} & C_f \\
\downarrow{\gamma} & & \downarrow{\alpha[1]} \\
C_f' & \xrightarrow{f'_2} & A'[1]
\end{array}
\]

It exists, though is not unique. Let \( \gamma' \) be another morphism making the diagram (5-1) commutative and set \( \eta = \gamma - \gamma' \). Then, \( \eta f_1 = 0 \), hence \( \eta \) factors through \( f_2 \), and \( f_2'\eta = 0 \), hence \( \eta \) factors through \( f_1' \). Thus, \( \eta \in \mathcal{J} \). On the other hand, if \( \eta : C_f \to C_f' \) is in \( \mathcal{J} \), the decomposition \( \eta = \eta'\xi \) implies that \( \eta f_1 = \eta'\xi f_1 = 0 \) and the decomposition \( \eta = \xi\eta'' \) implies that \( f_2'\eta = f_2\xi\eta'' = 0 \), hence the morphism \( \gamma' = \gamma + \eta \) makes the diagram (5-1) commutative. Therefore, the class \( \mathcal{C}(\alpha, \beta) \) of \( \gamma \) modulo \( \mathcal{J} \) is uniquely defined, so the maps \( f \mapsto C_f \) and \( (\alpha, \beta) \mapsto \mathcal{C}(\alpha, \beta) \) define a functor \( \mathcal{C} : \text{El}(\mathcal{U}) \to \mathcal{C}/\mathcal{J} \).

---

6This theorem is a partial case of [Drozd 2010, Theorem 1.1].
Proof. (1) Hom\(_A(\tilde{A}, M^+)\) is identified with the set of homomorphisms \(\phi : P \to M^+\) such that \(\phi e = 0\). A homomorphism \(\phi : P \to M^+\) is uniquely defined by an element \(u \in M\) such that \(\phi(\begin{pmatrix} u & \xi \end{pmatrix}) = \begin{pmatrix} au \xi \end{pmatrix}\). Namely, \(\phi(\begin{pmatrix} u & \xi \end{pmatrix}) = \begin{pmatrix} au & \xi \end{pmatrix}\). Obviously, \(\phi e = 0\) if and only if \(Cu = 0\), i.e., \(u \in \text{ann}_M C\).

(2) Ext\(_A^1(\tilde{A}, M^+)\) is identified with the set of homomorphisms \(\phi : P \to M^+\) such that \(\phi\gamma = 0\). A homomorphism \(\phi : P \to M^+\) is uniquely defined by an element \(u \in M\) such that \(\phi(\begin{pmatrix} u & \xi \end{pmatrix}) = \begin{pmatrix} au & \xi \end{pmatrix}\). Namely, \(\phi(\begin{pmatrix} u & \xi \end{pmatrix}) = \begin{pmatrix} au & \xi \end{pmatrix}\). Obviously, \(\phi e = 0\) if and only if \(Cu = 0\), i.e., \(u \in \text{ann}_M C\).

Theorem 5.2. (1) Hom\(_A(\tilde{A}, M^+)\) isomorphic to \(\{u \in M \mid Cu = 0\}\).

(2) Ext\(_A^1(\tilde{A}, M^+)\) isomorphic to \(\ker DG \cong \ker AG\), where the quotient \(M/\text{ann}_M C\) embeds into \(\text{Hom}_H(C, M)\) if we map an element \(u \in M\) to the homomorphism \(\mu_u : c \mapsto cu\).
\[ \psi^+: C^+ \to M^+ \] mapping \((a, b)\) to \(\left(\frac{\psi(a)}{\psi(b)}\right)\). Let \(\phi: P \to M^+\) correspond, as above, to an element \(u \in M\). Then,

\[ \phi \varepsilon \left(\begin{array}{c}
a \\
c
\end{array}\right) = \left(\begin{array}{c}
a u \\
c u
\end{array}\right), \]

so it equals \(\mu_u\), and \(\text{Hom}_{\tilde{A}}(P, M^+)\varepsilon\) is identified with \(M / \text{ann}_M C\) embedded into \(\text{Hom}_H(C, M)\) as above. \(\square\)

Actually, in our case an object \(E\) from the category \(\text{El}(\underline{U})\) (therefore, also an object from \(\mathcal{D}^b(\tilde{A})\)) is given by the vertices and solid arrows of a diagram

\[
\begin{array}{ccccccc}
A_n & \xrightarrow{\alpha_n} & A_{n+1} & \xrightarrow{\alpha_{n+1}} & A_{n+2} & \xrightarrow{\alpha_{n+2}} & A_{n+3} \\
M_n & \xrightarrow{\mu_n} & M_{n+1} & \xrightarrow{\mu_{n+1}} & M_{n+2} & \xrightarrow{\mu_{n+2}} & M_{n+3} \\
\gamma_n & \xrightarrow{\gamma_{n+1}} & \gamma_{n+2} & \xrightarrow{\gamma_{n+2}} & \gamma_{n+3} & & & \\
\end{array}
\]

(of arbitrary length), where \(A_i\) are \(\tilde{A}\)-modules, \(M_i\) are \(H\)-modules, \(\mu_i\) belongs to \(\text{Hom}_{\tilde{A}}(A_i, M'^+_i)\) and \(\eta_i\) belongs to \(\text{Ext}^1_{\tilde{A}}(A_i, M'^{+}_{i-1})\). A morphism between \(E\) and \(E'\) is given by the dotted arrows, where

\[
\begin{align*}
\alpha_i & \in \text{Hom}_{\tilde{A}}(A_i, A'_i) \simeq \text{Hom}_{\tilde{A}}(A_i, A'_i), \\
\gamma_i & \in \text{Hom}_H(M_i, M'_i) \simeq \text{Hom}_{\tilde{A}}(M^{+}_i, (M'_i)^+), \\
\beta_i & \in \text{Ext}^1_H(M_i, M'_{i+1}) \simeq \text{Ext}^1_{\tilde{A}}(M^{+}_i, (M'_{i+1})^+).
\end{align*}
\]

These morphisms must satisfy the relations

\[
\mu'_i \alpha_i = \gamma_i \mu_i, \quad \eta'_i \alpha_i = \gamma_{i+1} \eta_i + \beta_i \mu_i.
\]

6. Partial tilting for Backström pairs

Let \(H \subseteq A\) be a Backström pair. Consider the ring \(B\) of triangular matrices of the form

\[
B = \begin{pmatrix}
\tilde{A} & H \\
0 & H
\end{pmatrix}.
\]

Let \(e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\) and \(e_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\), and let \(B_1 = Be_1\) and \(B_2 = Be_2\) be projective \(B\)-modules given by the first and the second column of \(B\), i.e.,

\[
B_1 = \begin{pmatrix} \tilde{A} \\ 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} \tilde{H} \\ H \end{pmatrix}.
\]
A $B$-module $M$ is defined by a triple $\left( \frac{M_1}{M_2}, \chi_M \right)$, where $M_1 = e_1M$ is an $A$-module, $M_2 = e_2M$ is an $H$-module and $\chi_M : M_2 \to M_1$ is an $A$-homomorphism such that $\text{Ker} \, \chi_M \supseteq CM_2$ (it is necessary since $CM_1 = 0$). Namely, $\chi_M$ is multiplication by $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. We write an element $u \in M$ as a column $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$, where $u_1 = e_1u$, $u_2 = e_2u$.

Then,
$$
\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} au_1 + \chi_M(bu_2) \\ cu_2 \end{pmatrix}.
$$

A homomorphism $\alpha : M \to N$ is defined by two homomorphisms $\alpha_1 : M_1 \to N_1$ and $\alpha_2 : M_2 \to N_2$ such that $\alpha_1 \chi_M = \chi_N \alpha_2$. We write $\alpha = (\alpha_1 \alpha_2)$.

**Proposition 6.1.** We have $\text{l.gl.dim } B = \max(\text{l.gl.dim } H, \text{w.dim } \overline{H} + 1)$.

In particular, if $H$ is left hereditary and $\overline{H}$ is not flat as a right $H$-module, then $\text{l.gl.dim } B = 2$.

**Proof.** [Palmér and Roos 1973, Theorem 5] shows that $\text{l.gl.dim } B \leq n$ if and only if $\text{l.gl.dim } H \leq n$ and $\mathbb{R}^n \text{Hom}_\overline{A}(\overline{H} \otimes H, -) = 0$.

As the ring $\overline{A}$ is semisimple,
$$
\mathbb{R}^n \text{Hom}_\overline{A}(\overline{H} \otimes H, -) = \text{Hom}_\overline{A}(\text{Tor}_n^H(\overline{H}, -), -).
$$

This implies the first assertion. The second is obvious, since $\text{Tor}_1^H(\overline{H}, -) = 0$ if and only if $\overline{H}$ is flat as a right $H$-module. \hfill $\square$

We denote by $R$ the $B$-module given by the triple $\left( \frac{H/A}{\pi} \right)$, where $\pi : H \to H/A$ is the natural surjection.

**Proposition 6.2.** (1) $\text{End}_B R \simeq A^{\text{op}}$.

(2) $\text{pr.dim}_B R = 1$.

(3) $\text{Ext}_B^1(R, R) = 0$.

Recall that conditions (2) and (3) mean that $R$ is a partial tilting $B$-module.

**Proof.** The minimal projective resolution of $R$ is
$$
0 \to B_1 \xrightarrow{\varepsilon} B_2 \to R \to 0,
$$
where $\varepsilon$ is the embedding, which gives (2). Any endomorphism $\gamma$ of $R$ induces a commutative diagram:

$$
\begin{array}{ccc}
B_1 & \xrightarrow{\varepsilon} & B_2 \\
\downarrow{\gamma_1} & & \downarrow{\gamma_2} \\
B_1 & \xrightarrow{\varepsilon} & B_2
\end{array}
$$

As $\text{End}_B B_2 \simeq H^{\text{op}}$, $\gamma_2$ is given by multiplication with an element $h \in H$ on the right. If there is a commutative diagram as above, necessarily $h \in A$, which proves (1).
Finally, a homomorphism $\alpha : B_1 \to R$ maps the generator $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ of $B_1$ to an element $\begin{pmatrix} h \\ 0 \end{pmatrix} \in R$. If $h$ is a preimage of $\tilde{h}$ in $H$, then $\alpha$ extends to the homomorphism $B_2 \to R$ that maps the generator $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ of $B_2$ to $\begin{pmatrix} 0 \\ h \end{pmatrix} \in R$. This implies (3). □

Now Theorem 1.1 applied to the module $R$ gives the following result:

**Theorem 6.3.** (1) The functor $F = \mathbb{R}
\text{Hom}(R, -)$ induces an equivalence $\text{Tri}(R) \xrightarrow{\sim} \mathcal{D}(A)$.

(2) $\text{Ker} \ F$ consists of complexes $C$ such that the map $\chi_{\mathcal{H}^i(C)}$ is bijective for all $k$.

(3) There is a recollement diagram

\[
\begin{array}{ccc}
\text{Ker} \ F & \xrightarrow{\iota^*} & \mathcal{D}(B) & \xleftarrow{\iota^!} & \mathcal{D}(A) \\
& & & & \\
\end{array}
\]

Actually, claim (2) means that a complex $C$ is in $\text{Ker} \ F$ if and only if its cohomologies are direct sums of $B$-modules of the form $\left( \begin{pmatrix} U \\ 1 \\ U \end{pmatrix} \right)$, where $U$ is a simple $\mathcal{H}$-module.

$F$ is a partial tilting functor in the sense of Corollary 1.2.

**Proof.** (1) and (3) follow from Proposition 6.2 and Theorem 1.1, since the complex $P : 0 \to B_1 \to B_2 \to 0$ is perfect, hence compact, and isomorphic to $R$ in $\mathcal{D}(B)$. To find $\text{Ker} \ F$, consider a complex

\[ C : \cdots \to C^{k-1} \xrightarrow{d^{k-1}} C^k \xrightarrow{d^k} C^{k+1} \to \cdots, \]

where $C^k$ is defined by a triple $(C_i, d_i^k, \chi_k)$ and $d^k = (d_i^k)$, where $d_i^k \chi_k = \chi_{k+1} d_i^k$ for all $k$. Note that $C_i = (C_i^k, d_i^k)$ ($i = 1, 2$) are complexes, $(\chi_k)$ is a homomorphism of complexes and $H^k(C) = \left( H^k(C_i), \tilde{\chi}_k \right)$, where $\tilde{\chi}_k = \chi_{\mathcal{H}^i(C)}$ is induced by $\chi_k$. A homomorphism $P \to C[k]$ is a pair of homomorphisms $\alpha : B_2 \to C^k$, $\beta : B_1 \to C^{k-1}$ such that $\alpha_1 \pi = \chi_k \sigma_2$, $\beta_2 = 0$, $d^k_1 \alpha_i = 0$ ($i = 1, 2$) and $d^{k-1}_1 \beta_1 = |\sigma_1|$. Let $\alpha_2(1) = x \in C_2^k$ and $\beta_1(1) = y \in C_{k-1}^2$. These values completely define $\alpha$ and $\beta$. The conditions for $\alpha$ and $\beta$ mean that $d^k_2 x = 0$ and $d^k_1 y = \chi_k x$.

This morphism is homotopic to zero if and only if there are maps $\sigma : B_2 \to C^{k-1}$ and $\tau : B_1 \to C^{k-2}$ such that $\alpha = d^{k-1}_1 \sigma$ and $\beta = \sigma \varepsilon + d^{k-2}_1 \tau$. Again, $\sigma$ is defined by the element $\tau_1 = C_2^2$ and $\tau$ is defined by the element $\tau_1 = C_2^{k-1}$. Then, the conditions for $\alpha$ and $\beta$ mean that $x = d^{k-1}_1 z$ and $y = \chi_{k-1} z + d^{k-2}_1 \tau$.

Suppose that any homomorphism $P \to C[k]$ is homotopic to zero. Let $\tilde{x}$ in $H^k(C_2)$ be such that $\tilde{x}_k(\tilde{x}) = 0$ and $x \in \ker d^k_2$ be a representative of $\tilde{x}$. Then, $\chi_k(x) = d^k_1 y$ for some $y \in C^{k-1}$, so the pair $(x, y)$ defines a homomorphism $P \to C[k]$. Therefore, there must be $z \in C^{k-1}_2$ such that $x = d^{k-1}_1 z$; thus $\tilde{x} = 0$ and $\tilde{x}_k$ is injective. Let now $\tilde{y} \in H^{k-1}(C_2)$ and $y \in C^{k-1}_2$ be its representative. Then, the pair $(0, y)$ defines a homomorphism $P \to C[k]$, so there must be elements $z \in C^{k-1}_2$.
and \( t \in C_1^{k-2} \) such that \( d_1^{k-1}z = 0 \) and \( y = \chi_{k-1}z + d_1^{k-2}t \). Hence, \( \bar{y} = \tilde{\chi}_{k-1}(\bar{z}) \), so \( \tilde{\chi}_{k-1} \) is surjective. As this holds for all \( k \), we have that all maps \( \tilde{\chi}_k \) are bijective.

On the contrary, suppose that all \( \tilde{\chi}_k \) are bijective. If a pair \((x, y)\) defines a homomorphism \( P \to C[k] \), then \( \chi_k(x) = d_1^{k-1}y \), so \( \tilde{\chi}_k(x) = 0 \). Therefore, \( \bar{x} = 0 \), i.e., \( x = d_2^{k-1}z \) for some \( z \in C_2^{k-1} \) and \( \chi_kx = d_1^{k-1}\chi_{k-1}z \). Then, \( d_1^{k-1}(y - \chi_{k-1}z) = 0 \), hence there is an element \( z' \in C_2^{k-1} \) such that \( d_2^{k-1}z' = 0 \) and the cohomology class of \( y - \chi_{k-1}z \) equals \( \tilde{\chi}_{k-1}\bar{z}' \), i.e., \( y - \chi_{k-1}z = \chi_{k-1}z' + d_1^{k-2}t \) for some \( t \). Then, \( x = d_2^{k-1}(z + z') \) and \( y = \chi_{k-1}(z + z') + d_1^{k-2}t \), so this homomorphism is homotopic to zero. □

As usual, we identify the category \( A\text{-Mod} \) with the full subcategory of \( \mathcal{D}(A) \) consisting of the complexes \( C \) concentrated in degree 0. The following result shows how the partial titling functor \( F \) behaves with respect to modules:

**Corollary 6.4.** Let a \( B \)-module \( M \) be given by the triple \((M_1 \to M_2, \chi_M)\).

1. \( FM \in A\text{-Mod} \) if and only if \( \chi_M \) is surjective. Namely, then \( FM \cong \ker \chi_M \).
2. \( FM \in A\text{-Mod}[1] \) if and only if \( \chi_M \) is injective. Namely, then \( FM \cong \text{cok} \chi_M[1] \).

*Proof.* Note that \( \text{Hom}_B(B_1, M) \cong M_1, \text{Hom}_B(B_2, M) \cong M_2 \) and if \( \phi : B_2 \to M \) maps \((0,1)\) to \((0,1)\), then \( \phi \epsilon \) maps \((1,0)\) to \((\chi_M(x))\). Therefore, \( \mathbb{R}\text{Hom}_B(R, M) \) is the complex

\[
\begin{array}{c}
0 \to M_2 \xrightarrow{\chi_M} M_1 \to 0,
\end{array}
\]

which proves the claim. □

**Remark 6.5.** There are several derived equivalences related to \( \tilde{A} \).

1. If \( A \) is a Backström order, it is known (see [Burban et al. 2017]) that the complex \( T = B_1[1] \oplus H^+ \), where \( B_1 = (\tilde{A}, \tilde{B}) \), is a tilting complex for \( \tilde{A} \) and \( (\text{End}_{\mathcal{D}(\tilde{A})})^{\text{op}}T \cong B \), hence \( \tilde{A} \) is derived equivalent to \( B \). Nevertheless, in the general situation of Backström rings (even of Backström algebras) this is not so. First of all, \( \text{Hom}_{\tilde{A}}(B_1, H^+) \cong \text{ann}_H C \), so it can happen that \( \text{Hom}_{\mathcal{D}(\tilde{A})}(T, T[1]) \neq 0 \). This is so, for instance, for the pair \((T(n, \mathbb{B}), UT(n, \mathbb{B}))\) from Equation (2-3) (4), since in this case the matrix unit \( e_{nn} \) belongs to \( \text{ann}_H C \). This is also so for Equation (2-3) (5). Moreover, even if \( \text{ann}_H C = 0 \), one can see that \( \tilde{H} = \text{Ext}^1_{\tilde{A}}(B_1, H^+) \cong C^{-1}/cH \), where \( C^{-1} = \text{Hom}_H(C, H) \) and \( cH = H/\text{ann}_H C \) is naturally embedded into \( C^{-1} \). Therefore, in this case,

\[
(\text{End}_{\mathcal{D}(\tilde{A})}T)^{\text{op}} \cong B' = \begin{pmatrix} \tilde{A} & \overline{H} \\ 0 & H \end{pmatrix},
\]

which need not coincide with \( B \) (see Example 6.6 below). If \( H \) is a hereditary order, then \( \text{ann}_H C = 0 \) and \( \overline{H} \cong \overline{H}, \) hence \( B' \cong B \), in accordance with [Burban et al. 2017].
(2) On the other hand, set \( T' = (A_{H/A}) \) considered as a left \( \tilde{A} \)-module. One can check it is a tilting module for \( \tilde{A} \) and

\[
\text{End}_{\mathcal{D}(\tilde{A})} T' \cong \mathcal{B} = \begin{pmatrix} A & H/A \\ 0 & \tilde{A}^\circ \end{pmatrix},
\]

hence \( \tilde{A} \) is derived equivalent to \( \tilde{B} \). Unfortunately, this ring can be not so good from the homological point of view. At least, it is not better than \( A \) itself. Namely, as one can easily check,

\[
\text{l.gl. dim } \tilde{B} = \max(\text{l.gl. dim } A, 1 + \text{pr. dim } A(H/A)),
\]

which is either \( \text{l.gl. dim } A \) or (more often) \( \text{l.gl. dim } A + 1 \).

(3) One more observation: Consider the right \( \tilde{A} \)-modules \((\tilde{A} 0)\) and \((C H)\). One can check that \( T'' = (\tilde{A} 0)[1] \oplus (C H) \) is a tilting complex for \( \mathcal{D}(\tilde{A}^\circ) \) and

\[
\text{End}_{\mathcal{D}(\tilde{A}^\circ)} T'' \cong \mathcal{B}'' = \begin{pmatrix} \tilde{A} & 0 \\ \tilde{H} & H \end{pmatrix},
\]

hence \( \tilde{A}^\circ \) is derived equivalent to \( (B'')^\circ \).

Note that the functor \( P \mapsto \text{Hom}_R(P, R) \) induces an exact duality

\[
\text{Perf}(R) \rightarrow \text{Perf}(R^\circ)
\]

for any ring \( R \). Hence, \( \text{Perf}(\tilde{A}) \cong \text{Perf}(B'') \).

**Example 6.6.** Let \( H = \mathbb{T}(3, \mathbb{K}) \) and \( A = \{(a_{ij}) \in H \mid a_{11} = a_{22}\} \). Set \( H_i = H e_{ii} \) and \( U_i = H_i / \text{rad } H_i \). Then, \( C = \{(a_{ij}) \in H \mid a_{11} = a_{22} = 0\} \), hence \( \tilde{H} = U_1 \oplus U_2 \).

On the other hand, \( C = \text{rad } H_2 \oplus H_3 \cong H_1 \oplus H_3 \), so \( C^{-1} = \text{Hom}_H(C, H) \) can be identified with the set of \( 3 \times 2 \) matrices \( (b_{ij}) \) such that \( b_{12} = b_{22} = 0 \). One can check that \( C H \) is identified with the subset \( \{(b_{ij}) \mid b_{11} = 0\} \subset C^{-1} \) and \( \tilde{H} \cong U_2 \not\cong \tilde{H} \) (even \( \text{dim}_K \tilde{H} \neq \text{dim}_K \tilde{H} \)).

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