A SUBCLASS OF PSEUDO-TYPE MEROMORPHIC BI-UNIVALENT FUNCTIONS

Adnan Ghazy ALAMOUSH
Faculty of Science, Taibah University, SAUDI ARABIA

ABSTRACT. In this paper, a new subclass of pseudo-type meromorphic bi-univalent functions is defined on \( \Delta = \{ z : z \in \mathbb{C} \text{ and } 1 < |z| < \infty \} \), we derive estimates on the initial coefficient \( |b_0|, |b_1| \) and \( |b_2| \). Relevant connections of the new results with various well-known results are indicated.

1. Introduction

Let \( A \) denote the class of functions \( f(z) \) of the form:

\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n
\]

which are analytic in the open unit open disk \( U = \{ z : z \in \mathbb{C}, |z| < 1 \} \). Also, let the class of univalent and normalized analytic function in the unit disc \( U \) be denoted by \( S \) with the normalization conditions

\[
f(0) = 0 = f'(0) - 1.
\]

Furthermore, bi-univalency concept is extended to the class of meromorphic functions defined on \( \Delta = \{ z : z \in \mathbb{C}, 1 < |z| < \infty \} \). For this aim, let \( \Sigma \) denote the class of meromorphic univalent functions \( g \) of the form

\[
g(z) = z + \sum_{n=0}^{\infty} \frac{b_n}{z^n}
\]

defined on the domain \( \Delta \). It is well known that every function \( g \in \Sigma \) has an inverse \( g^{-1} = h \), defined by

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\[ g^{-1}(g(z)) = z \quad (z \in \triangle), \]

and

\[ g^{-1}(g(w)) = w \quad (M < |w| < \infty, \ M > 0), \]

where

\[ g^{-1}(w) = h(w) = w + \sum_{n=0}^{\infty} B_n \frac{w^n}{w^n} = w - b_0 - \frac{b_1}{w} b_0 + b_2 \frac{1}{w^2} - \frac{b_1 b_2^2 + 2 b_0 b_2 + b_3}{w^3} + \ldots. \]  

A simple computation shows that

\[ w = g(h(w)) = (b_0 + B_0) + w + \frac{b_1 + B_1}{w} + \frac{B_2 - b_1 B_0 + b_2}{w^2} + \frac{B_3 - b_1 B_1 + b_1 B_0^2 - 2 b_2 B_0 + b_3}{w^3} + \ldots. \]  

Comparing the initial coefficients in (4), we find that

\[ b_0 + B_0 = 0 \quad \Rightarrow \quad B_0 = -b_0 \]

\[ b_1 + B_1 = 0 \quad \Rightarrow \quad B_1 = -b_1 \]

\[ B_2 - b_1 B_0 + b_2 = 0 \quad \Rightarrow \quad B_2 = -(b_2 + b_1 b_0) \]

\[ B_3 - b_1 B_1 + b_1 B_0^2 - 2 b_2 B_0 + b_3 = 0 \quad \Rightarrow \quad B_3 = -(b_3 + 2 b_0 b_1 + b_1 b_0^2 + b_1^2). \]

A function \( f \in \Sigma \) is said to be meromorphic bi-univalent if \( f^{-1} \in \Sigma' \). The family of all meromorphic bi-univalent functions is denoted by \( \Sigma' \). Estimates on the coefficient of meromorphic univalent functions were investigated by some researchers recently; for example, Schiffer \[11\] obtained the estimate \( |b_2| < \frac{3}{2} \) for meromorphic univalent functions \( f \in S \) with \( b_0 = 0 \). Also, Duren \[12\] obtained the inequality \( |b_2| < \frac{2}{n+1} \) for \( f \in S \) with \( b_k = 0, 1 \leq k \leq \frac{n}{2} \). Springer \[8\] used variational methods to prove that proved that

\[ |B_3| < 1 \quad \text{and} \quad |B_3 + \frac{1}{2} B_1^2| < \frac{1}{2}, \]

and conjectured that

\[ |B_{2n-1}| \leq \frac{(2n-2)!}{n!(n-1)!} \quad (n = 1, 2, \ldots). \]

Later on, Kubota \[16\] has proved that the Springer conjecture is true for \( n = 3, 4, 5 \). Furthermore Schober \[7\] obtained sharp bounds for \( |B_{2n-1}| \) if \( 1 \leq n \leq 7 \). Recently, Kapoor and Mishra \[5\] introduced new subclasses of meromorphically bi-univalent functions and obtained estimates on the initial coefficients for functions belonging to these subclasses.

Recently, some several researchers such as \( \text{see} \ [1, 2, 3, 4, 6, 9, 13, 14] \) introduced new subclasses of meromorphically bi-univalent functions and obtained estimates on the initial coefficients for functions belonging to these subclasses.
In 2013, Babalola [10] defined a new subclass $\lambda$-pseudo starlike function of order $0 \leq \beta < 1$ satisfying the analytic condition
\[
\Re \left\{ \frac{zf(z)^\lambda}{g(z)} \right\} > \beta \quad (\lambda \geq 1, \quad z \in U).
\] (5)

In particular, Babalola [10] proved that all $\lambda$-pseudo-starlike functions are Bazilevic of type $\frac{1}{2}$ and order $\beta$ and are univalent in open unit disk $U$.

Motivated by the earlier work of (9), (15), in the present paper, we introduce a new subclasses of the class $\Sigma' \subset \Sigma$ and the estimates for the coefficients $|b_0|, |b_1|$ and $|b_2|$ are investigated. Some new consequences of the new results are also pointed out.

2. Coefficient Bounds for the Function Class $\Sigma_{h,p}(\lambda, \mu)$

We begin by introducing the function class $\Sigma_{h,p}(\lambda, \mu)$ by means of the following definition.

**Definition 2.1.** Let the functions $h, p : \Delta \to C$ be analytic functions and
\[
h(z) = 1 + \frac{h_1}{z} + \frac{h_2}{z^2} + \frac{h_3}{z^3} + \cdots, \quad p(z) = 1 + \frac{p_1}{z} + \frac{p_2}{z^2} + \frac{p_3}{z^3} + \cdots,
\]
such that
\[
\min\{\Re(h(z)), \Re(p(z))\} > 0, \quad z \in \Delta.
\]

A function $g(z) \in \Sigma'$ given by (2) is said to be in the class $\Sigma_{h,p}(\lambda, \mu)$ if the following conditions are satisfied:
\[
g \in \Sigma' \text{ and } 1 + \frac{1}{\gamma} \left[ (1 - \lambda) \left( \frac{g(z)}{z} \right)^\mu + \lambda \left( \frac{z(g(z)^\mu}{g(z)} - 1 \right) \right] \in h(\Delta),
\]
\[
(0 < \lambda \leq 1, \mu \geq 1, \quad z \in \Delta), \quad (6)
\]
and
\[
1 + \frac{1}{\gamma} \left[ (1 - \lambda) \left( \frac{h(w)}{w} \right)^\mu + \lambda \left( \frac{w(h(w)^\mu}{h(w)} - 1 \right) \right] \in p(\Delta)
\]
\[
(0 < \lambda \leq 1, \mu \geq 1, \quad w \in \Delta), \quad (7)
\]

where $g \in \Sigma'$ and $\gamma \in C \setminus \{0\}$ and the function $h$ is given by (3).

**Remark 2.1.** There are many choices of $h$ and $p$ which would provide interesting subclasses of class $\Sigma_{h,p}(\lambda, \mu)$.

1. If we take
\[
h(z) = p(z) = \left( \frac{1 + \frac{1}{z}}{1 - \frac{1}{z}} \right)^\alpha = 1 + \frac{2\alpha}{z} + \frac{2\alpha^2}{z^2} + \cdots, \quad (0 < \alpha \leq 1, \quad z \in \Delta).
\]
So it is easy to verify that the functions \( h(z) \) and \( p(z) \) satisfy the hypotheses of Definition 2.1. If \( f \in \Sigma'_\alpha(\lambda, \mu) \). Then

\[
\left| \arg \left( 1 + \frac{1}{\gamma} \left[ (1 - \lambda) \left( \frac{g(z)}{z} \right)^\mu + \lambda \left( \frac{z(g(z)^m}{g(z)} - 1 \right) \right] \right) \right| < \frac{\alpha \pi}{2}
\]

\((0 < \lambda \leq 1, 0 < \alpha \leq 1, \mu \geq 1, z \in \Delta),\)

and

\[
\left| \arg \left( 1 + \frac{1}{\gamma} \left[ (1 - \lambda) \left( \frac{h(w)}{w} \right)^\mu + \beta \left( \frac{w(h(w)^m}{h(w)} - 1 \right) \right] \right) \right| < \frac{\alpha \pi}{2}
\]

\((0 < \lambda \leq 1, 0 < \alpha \leq 1, \mu \geq 1, w \in \Delta),\)

where \( g(z) \in \Sigma' \) and \( \gamma \in C \setminus \{0\} \) and the function \( h \) is given by (3).

(2) If we take

\[
h(z) = p(z) = \frac{1 + \frac{1 - 2\beta}{1 - z}}{1 - \frac{1}{z}} = 1 + \frac{2(1 - \beta)}{z} + \frac{2(1 - \beta)}{z^2}, \quad (0 \leq \beta < 1, z \in \Delta).
\]

So it is easy to verify that the functions \( h(z) \) and \( p(z) \) satisfy the hypotheses of Definition 2.1. If \( f \in \Sigma'_\beta(\lambda, \mu) \). Then

\[
\Re \left( 1 + \frac{1}{\gamma} \left[ (1 - \lambda) \left( \frac{g(z)}{z} \right)^\mu + \lambda \left( \frac{z(g(z)^m}{g(z)} - 1 \right) \right] \right) > \beta
\]

\((0 < \lambda \leq 1, 0 \leq \beta < 1, \mu \geq 1, z \in \Delta),\)

and

\[
\Re \left( 1 + \frac{1}{\gamma} \left[ (1 - \lambda) \left( \frac{h(w)}{w} \right)^\mu + \beta \left( \frac{w(h(w)^m}{h(w)} - 1 \right) \right] \right) > \beta
\]

\((0 < \lambda \leq 1, 0 \leq \beta < 1, \mu \geq 1, w \in \Delta),\)

where \( g \in \Sigma' \) and \( \gamma \in C \setminus \{0\} \) and the function \( h \) is given by (3).

**Theorem 2.1.** Let \( g(z) \) be given by (2) be in the class \( \Sigma'_\alpha(\lambda, \mu) \). Then

\[
|b_0| \leq \min \left\{ \sqrt{\frac{\gamma^2(|h_1|^2 + |p_1|^2)}{2(\mu - \lambda \mu - \lambda)^2}}, \sqrt{\frac{\gamma^2(|h_2|^2 + |p_2|^2)}{\mu(\mu - 1)(1 - \lambda) + 2\lambda}} \right\}
\]

(8)

and

\[
|b_1| \leq \min \left\{ \frac{\gamma(|h_2| + |p_2|)}{2(\mu(\mu - 1)(1 - \lambda) + 2\lambda)}, \frac{|\gamma|}{(|\mu - \lambda - 2\lambda\mu|)} \left( \sqrt{\frac{|h_2|^2 + |p_2|^2}{2}} + \frac{\mu(\mu - 1)(1 - \lambda) + 2\lambda |h_1|^2 + |p_1|^2}{16(\mu - \lambda \mu - \lambda)^2} \right) \right\},
\]

(9)

and

\[
|b_2| \leq \frac{|\gamma|}{2(|\mu - \lambda - 3\lambda\mu|)} \left[ \frac{(\mu(\mu - 1)(\mu - 2)(1 - \lambda) - 6\lambda)\gamma^2 |p_1|^3}{3(|\mu - \lambda \mu - \lambda|^3)} \right]
\]
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Let \( g \in \Sigma^e_0(\lambda, \mu) \). Then, by Definition 2.1 of meromorphically bi-univalent function class \( \Sigma^e_0(\lambda, \mu) \), the conditions (6) and (7) can be rewritten as follows:

\[
1 + \frac{1}{\gamma} (1 - \lambda) \left( g(z) \right)^{\mu} + \lambda \left( \frac{z (g(z))^\mu}{g(z)} - 1 \right) = h(z) \quad (z \in \Delta) \quad (11)
\]

and

\[
1 + \frac{1}{\gamma} \left( 1 - \lambda \right) \left( \frac{h(w)}{w} \right)^{\mu} + \beta \left( \frac{w (h(w))^\mu}{h(w)} - 1 \right) = p(w), \quad (w \in \Delta) \quad (12)
\]

respectively. Here, and in what follows, the functions \( h(z) \in P \) and \( p(w) \in P \) have the following forms:

\[
h(z) = 1 + \frac{p_1}{z} + \frac{p_2}{z^2} + \frac{p_3}{z^3} + \cdots \quad (z \in \Delta) \quad (13)
\]

and

\[
p(w) = 1 + \frac{q_1}{w} + \frac{q_2}{w^2} + \frac{q_3}{w^3} + \cdots \quad (w \in \Delta) \quad (14)
\]

upon substituting from (13) and (14) into (11) and (12), respectively, and equating the coefficients, we get

\[
\frac{(\mu - \lambda \mu - \lambda)}{\gamma} b_0 = h_1 \quad (15)
\]

\[
\frac{1}{\gamma} \left( [\mu(\mu - 1)(1 - \lambda) + 2\lambda]b_0^2 + 2(\mu - \lambda - 2\lambda \mu)b_1 \right) = h_2 \quad (16)
\]

\[
\frac{1}{\gamma} [\mu(\mu - 1)(1 - \lambda) - \gamma]b_0^3 + \frac{1}{\gamma} [\mu(\mu - 1)(1 - \lambda) + 2\lambda + \lambda \mu]b_0b_1
\]

\[
+ \frac{1}{\gamma} [\mu - \lambda - 3\mu \lambda]b_2 = h_3 \quad (17)
\]

\[- \frac{(\mu - \lambda \mu - \lambda)}{\gamma} b_0 = p_1 \quad (18)
\]

\[
\frac{1}{\gamma} \left( [\mu(\mu - 1)(1 - \lambda) + 2\lambda]b_0^2 + 2(\lambda - \mu + 2\lambda \mu)b_1 \right) = p_2 \quad (19)
\]

and

\[
\frac{1}{\gamma} \left( 6\lambda - [\mu(\mu - 1)(1 - \lambda)]b_0^3 + 6(\mu(\mu - 1)(1 - \lambda)
\]

\[
- \mu(1 - \lambda) + 3\lambda + 3\lambda \mu]b_0b_1 + 6(\lambda - \mu + 3\mu \lambda)b_2 \right] = p_3. \quad (20)
\]

From (15) and (18), we find that

\[
h_1 = -q_1 \quad (21)
\]

and

\[
2(\mu - \lambda \mu - \lambda)^2 b_0^2 = \gamma^2 (h_1^2 + p_1^2) \quad (22)
\]
that is,
\[ |b_0|^2 \leq \frac{|\gamma|^2(|h_1|^2 + |p_1|^2)}{2(\mu - \lambda \mu - \lambda)^2}. \]  
(23)

Adding (16) and (19), we get
\[ [(\mu(\mu - 1)(1 - \lambda) + 2\lambda)] b_0^2 = \gamma(h_2 + p_2) \]
that is,
\[ |b_0|^2 \leq \frac{|\gamma|(|h_2| + |p_2|)}{|\mu(\mu - 1)(1 - \lambda) + 2\lambda|}. \]  
(25)

From (23) and (25) we get the desired estimate on the coefficient $|b_0|$ as asserted in (8).

Next, in order to find the bound on $|b_0|$, by subtracting the equation (16) from the equation (19), we get
\[ 2(\mu(\mu - 1)(1 - \lambda) + 2\lambda)b_1 = \gamma(h_2 - p_2), \]
that is,
\[ |b_1| \leq \frac{|\gamma|(|h_2| + |p_2|)}{2(\mu(\mu - 1)(1 - \lambda) + 2\lambda)}. \]  
(27)

By squaring and adding (16) and (19), using (22) in the computation leads to
\[ b_1^2 = \frac{\gamma^2}{(\mu - \lambda - 2\lambda \mu)^2} \left( \frac{h_2^2 + p_2^2}{2} - \frac{[\mu(\mu - 1)(1 - \lambda) + 2\lambda h_2^2 + p_2^2]^2}{16(\mu - \lambda \mu - \lambda)^2} \right). \]  
(28)

that is,
\[ |b_1| \leq \frac{|\gamma|^2}{(\mu - \lambda - 2\lambda \mu)} \left( \sqrt{\frac{|h_2|^2 + |p_2|^2}{2} + \frac{[\mu(\mu - 1)(1 - \lambda) + 2\lambda |h_2|^2 + |p_2|^2]^2}{16(\mu - \lambda \mu - \lambda)^2}} \right). \]  
(29)

From (26) and (28) we get the desired estimate on the coefficient $|b_1|$ as asserted in (9).

In order to find the estimate $|b_2|$, consider the sum of (17) and (20), we have
\[ b_0 b_1 = \frac{\gamma(h_3 + p_3)}{2\mu(\mu - 1)(1 - \lambda) - (1 - \lambda)\mu + 5\lambda + 4\lambda \mu}. \]  
(30)

Subtracting (20) from (17) with $h_1 = -p_1$, we obtain
\[ \frac{2(\mu - \lambda - 3\lambda \mu)}{\gamma} b_2 = h_3 - p_3 - \frac{(\mu - \lambda - 3\lambda \mu) b_0 b_1 - [\mu(\mu - 1)(\mu - 2)(1 - \lambda) - 6\lambda] b_0^3}{3\gamma}. \]  
(31)

Using (21) and (30) in (31) give to
\[ b_2 = \frac{\gamma}{2(\mu - \lambda - 3\lambda \mu)} \left[ \frac{(\mu(\mu - 1)(\mu - 2)(1 - \lambda) - 6\lambda)\gamma^2 p_3^3}{3(\mu - \lambda \mu - \lambda)^3} + \frac{2\mu(\mu - 1)(1 - \lambda) + 5\lambda - 2\mu + 6\lambda}{2\mu(\mu - 1)(1 - \lambda) - (1 - \lambda)\mu + 5\lambda + 4\lambda \mu} h_3 \right]. \]
This evidently completes the proof of Theorem 2.1. \hfill \Box

If we take \( \lambda = 1 \) in Theorem 2.1, we get the following Corollary.

**Corollary 2.2.** Let \( g(z) \) be given by (1.2) be in the class \( \Sigma_{\lambda, \beta}(\alpha) \). Then

\[
|b_0| \leq \min \left\{ \sqrt[\gamma]{2(|h_1|^2 + |p_1|^2)} - \sqrt[\gamma]{2(|h_2|^2 + |p_2|^2)} \right\},
\]

(32)

\[
|b_1| \leq \min \left\{ \frac{|\gamma|(|h_2| + |p_2|)}{2}, \frac{|\gamma|}{|\mu| + 1} \left( \sqrt[\gamma]{\frac{|h_2|^2 + |p_2|^2}{2}} + \sqrt[\gamma]{\frac{|h_1|^2 + |p_1|^2}{2}} \right) \right\},
\]

(33)

and

\[
|b_2| \leq \frac{|\gamma|}{2(|2\mu + 1|)} \times \left[ 2\gamma^2 |p_1|^3 + \frac{6(\mu + 1)}{5 + 4\mu} |h_3| + \frac{2(\mu + 2)}{5 + 4\mu} |p_3| \right].
\]

(34)

If we take

\[
h(z) = p(z) = \left( \frac{1 + \frac{1}{z}}{1 - \frac{1}{z}} \right)^{\alpha} = 1 + \frac{2\alpha}{z} + \frac{2\alpha^2}{z^2} + \cdots, \quad (0 < \alpha \leq 1, \ z \in \Delta),
\]

and

\[
h(z) = p(z) = \frac{1 + \frac{1 - 2\beta}{1 - \frac{1}{z}}}{1 - \frac{1}{z}} = 1 + \frac{2(1 - \mu)}{z} + \frac{2(1 - \mu)}{z^2}, \quad (0 < \mu \leq 1, \ z \in \Delta),
\]

respectively, in the Theorem 2.1, we obtain the following results which is an improvement of estimates obtained by Srivastava et. at [9].

**Corollary 2.3.** Let \( g(z) \) be given by (2) be in the class \( \Sigma_{\lambda, \beta}(\alpha) \). Then

\[
|b_0| \leq 2\alpha
\]

(35)

and

\[
|b_1| \leq \frac{2\sqrt{5\alpha}}{\lambda + 1}.
\]

(36)

**Corollary 2.4.** Let \( g(z) \) be given by (2) be in the class \( \Sigma_{\lambda, \beta}(\mu) \). Then

\[
|b_0| \leq 2(1 - \mu)
\]

(37)

and

\[
|b_1| \leq \frac{2(1 - \mu)\sqrt{4\mu^2 - 8\mu + 5}}{\lambda + 1}.
\]

(38)

**Remark 2.2.** For function \( g \in \Sigma_{h, p}(\lambda, \mu) \) given by (2) by taking \( p(z) = h(z) = \frac{1 + A z}{1 + B z} - 1 \leq B < A \leq 1 \), we obtain the initial coefficient estimates \( |b_0|, |b_1|, \) and \( |b_2| \) which leads to the results discussed in Theorem 2.2 of [15].
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