Abstract

In this paper we study how to distinguish two embeddings of a finite collection of disjoint circles into the plane up to planar isotopy. We adopt the spirit of the approach by V. Turaev, Operator Invariants of Tangles, Math. USSR-Izv. 35 (1990), 411–444, by considering a category of planar tangles and representing it in an “algebraic” category. From this we can extract a numerical invariant for embeddings of a finite collection of disjoint circles and this invariant is, up to certain choices, complete.

1 Introduction

Let $\mathbf{PT}$ be the category of non-singular planar tangles whose objects are finite sets of points in the real line identified up to 1-dimensional isotopies\(^1\), and whose morphisms between two objects $O_1$ and $O_2$ are piecewise regular.

\(^1\)these objects can be regarded as finite ordinal numbers $\emptyset, \{0\}, \{0,1\}, ...$
1-dimensional manifolds, with boundary $O_1 \times \{1\} \cup O_2 \times \{0\}$, embedded in $\mathbb{R} \times [0, 1]$ identified up to planar isotopies:

The composition of two morphisms $t_1$ and $t_2$ is defined by

$$t_2 \circ t_1 := g(f(t_1) \cup t_2)$$

where $f(x, y) = (x, y + 1)$ and $g(x, y) = (x, y/2)$.

In this paper, the downward direction composition is used, some authors use the opposite direction.

This category has the following presentation:

The generators are morphisms $\hat{t}_{n,k} \in \text{hom}(\{1, \ldots, n - 1\}, \{1, \ldots, n + 1\})$ connecting $(i, 1)$ to $(i, 0)$ if $i \leq k - 2$, $(i, 1)$ to $(i + 2, 0)$ if $i \geq k - 1$ and $(k - 1, 0)$ to $(k, 0)$:
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and $\tilde{t}_{n,k} \in \text{hom}(\{1, \ldots, n+1\}, \{1, \ldots, n-1\})$ connecting $(i, 1)$ to $(i, 0)$ if $i \leq k-2$, $(i+2, 1)$ to $(i, 0)$ if $i \geq k-1$ and $(k-1, 1)$ to $(k, 1)$:

$$\tilde{t}_{n,k} = \begin{array}{cccccccc}
1 & 2 & \cdots & k-2 & k & k+1 & \cdots & n & n+1 \\
1 & 2 & \cdots & k-2 & k-1 & \cdots & n-2 & n-1
\end{array}$$

for any $k, n \in \mathbb{N}$ with $2 \leq k \leq n+1$.

For the rest of this paper it is better to number the intervals instead of the points:

$$\hat{t}_{n,k} = \begin{array}{cccccccc}
1 & 2 & \cdots & k-2 & k-1 & k & \cdots & n+1 & n+2 \\
1 & 2 & \cdots & k-2 & k-1 & k & k+1 & k+2 & \cdots & n+1 & n+2
\end{array}$$

These generators satisfy the following relations:

$$\hat{t}_{n,k+1} \circ \hat{t}_{n,k} = \hat{t}_{n,k-1} \circ \hat{t}_{n,k} = id_n$$

where $id_n := \{1, \ldots, n\} \times [0, 1]$ is the identity morphism on $\{1, \ldots, n\}$;
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\[ \hat{t}_{n+2,l} \circ \hat{t}_{n,k} = \hat{t}_{n+2,k} \circ \hat{t}_{n,l-2} \]

for \( l \geq k + 2; \)

\[ \hat{t}_{n-2,l-2} \circ \hat{t}_{n,k} = \hat{t}_{n-2,k} \circ \hat{t}_{n,l} \]

for \( l \geq k + 2; \)

\[ \hat{t}_{n-2,l-2} \circ \hat{t}_{n-2,k} = \hat{t}_{n,k} \circ \hat{t}_{n,l} \]

for \( l \geq k + 2 \) and

\[ \hat{t}_{n,l} \circ \hat{t}_{n,k} = \hat{t}_{n-2,k} \circ \hat{t}_{n-2,l-2} \]

for \( l \geq k + 2. \)

Now we want to represent non-singular planar tangles by functions. That is, we want to find functions \( \hat{T}_{n,k} \) and \( \hat{T}_{n,k} \) satisfying the following relations:

\[ \hat{T}_{n,k+1} \circ \hat{T}_{n,k} = \hat{T}_{n,k-1} \circ \hat{T}_{n,k} = id \]
\[ \hat{T}_{n+2,l} \circ \hat{T}_{n,k} = \hat{T}_{n+2,k} \circ \hat{T}_{n,l-2} \]

for \( l \geq k + 2; \)

\[ \hat{T}_{n-2,l-2} \circ \hat{T}_{n,k} = \hat{T}_{n-2,k} \circ \hat{T}_{n,l} \]

for \( l \geq k + 2; \)

\[ \hat{T}_{n-2,l-2} \circ \hat{T}_{n-2,k} = \hat{T}_{n,k} \circ \hat{T}_{n,l} \]

for \( l \geq k + 2 \) and

\[ \hat{T}_{n,l} \circ \hat{T}_{n,k} = \hat{T}_{n-2,k} \circ \hat{T}_{n-2,l-2} \]

for \( l \geq k + 2. \)

Our proposal is to represent \( \mathbf{PT} \) by the following category \( \mathbf{PI}_M \) whose objects are \( \mathbf{O}_1, \mathbf{O}_2, \mathbf{O}_3, ... \) where \( \mathbf{O}_n \) is the set of pairs \((R, \mathbf{v})\) such that \( R = [r_{i,j}] \) is a \( n \times n \) Boolean matrix satisfying the following properties:

**E1.** \( R \geq I \) (where \( I \) is the identity matrix);

**E2.** \( R^t = R \) (the matrix is symmetric);

**E3.** \( R^2 = R \) (the matrix is idempotent);

**T1.** If \( r_{i,j} = 1 \) then \( |i - j| \) is even;

**T2.** For any \( \alpha \leq \beta \leq \gamma \leq \delta \), \( r_{\alpha,\gamma}r_{\beta,\delta} \leq r_{\alpha,\beta}r_{\beta,\gamma}r_{\gamma,\delta}; \)

**T3.** For any \( \alpha < \beta \) if \( r_{\alpha,\beta} = 1 \) then either \( r_{\alpha+1,\beta-1} = 1 \) or there exists \( \gamma \) between \( \alpha \) and \( \beta \) such that \( r_{\alpha,\gamma} = 1. \)

and \( \mathbf{v} \) is an array of \( n \) entries with values in a chosen lattice ordered monoid \( \mathbb{M} \) such that it is fixed by the action induced by \( R \) which we will define later:

**EC.** \( R \star \mathbf{v} = \mathbf{v}. \)

Note: The properties E1, E2 and E3 represent an equivalence relation, and the properties T1, T2 and T3 have topological motivations (see the explanation in section 3.1).

A morphism between \( \mathbf{O}_m \) and \( \mathbf{O}_n \) is just a set function between the sets \( \mathbf{O}_m \) and \( \mathbf{O}_n. \)
2 Algebraic interlude

Definition 1 The canonical Boolean algebra $B$ is the set $\{0, 1\}$ with two binary operations: the sum $+$ and the multiplication $\cdot$, and a unary operation the negation $\neg$ such that $(\{0, 1\}, +, \cdot)$ is the (unique) semi-ring with $1+1 = 1$, $\neg 0 = 1$ and $\neg 1 = 0$.

Definition 2 A Boolean matrix is a matrix with values in the canonical Boolean algebra.

We define the operations sum, multiplication and transpose in the same way as on real matrices:

Sum: $[a_{i,j}] + [b_{i,j}] := [c_{i,j}]$ where $c_{i,j} = a_{i,j} + b_{i,j}$

Multiplication: $[a_{i,j}] [b_{i,j}] := [c_{i,j}]$ where $c_{i,j} = \sum_{k=1}^{n} a_{i,k} b_{k,j}$

Transpose: $[a_{i,j}]^t := [a_{j,i}]$

There is a natural partial order relation on these matrices given in the following way:

$[a_{i,j}] \leq [b_{i,j}]$ iff $a_{i,j} \leq b_{i,j} \forall i,j$

These matrices have many of the properties of real matrices.

Proposition 1 Let $A$, $B$ and $C$ be Boolean matrices with appropriate dimensions. We have:

1. (commutativity of the sum) $A + B = B + A$;
2. (associativity) $(A + B) + C = A + (B + C)$ and $(AB)C = A(BC)$;
3. (distributivity) $A(B + C) = AB + AC$ and $(A + B)C = AC + BC$;
4. (existence of the zero matrix) $A + O = A$, $AO = O$ and $OA = O$ where $O$ is the matrix with all entries equal to zero;
5. (existence of the identity matrix) $AI = A$ and $IA = A$ where $I = [\delta_{i,j}]$ with $\delta_{i,j} = 1 \Leftrightarrow i = j$;
6. (idempotency of the sum) $A + A = A$;
8. $A \leq B \iff A + B = B$;

9. $(AB)^t = B^tA^t$ and $(A + B)^t = A^t + B^t$;

10. $A \leq B \Rightarrow A + C \leq B + C$ and $CA \leq CB$ and $AC \leq BC$ and $A^t \leq B^t$.

We can regard a square Boolean matrix $R = [r_{i,j}]$ of dimension $n$ as a binary relation $\sim_R$ on the set $\{1, \ldots, n\}$:

$$i \sim_R j \iff r_{i,j} = 1$$

Then:

**Proposition 2** The binary relation $\sim_R$ represented by the matrix $R$ is:

i. reflexive iff $I \leq R$;

ii. symmetric iff $R^t = R$;

iii. transitive iff $R^2 \leq R$.

Thus we can transpose the notions of reflexivity, symmetry and transitivity from the binary relations to square Boolean matrices. Notice that reflexivity and transitivity imply idempotency of the product for Boolean matrices.

**Proposition 3** (definition) Let $A$ be a square Boolean matrix and let

$$\overline{A} := \sum_{n=1}^{\infty} A^n = A + A^2 + \ldots$$

Then:

i. $\overline{A}$ is transitive;

ii. For any transitive matrix $B$, $A \leq B \Rightarrow A \leq \overline{A} \leq B$;

iii. $A \leq B \Rightarrow \overline{A} \leq \overline{B}$;

iv. $\overline{\overline{A}} = A$;
v. If $A \geq I$ then $\overline{A} = A^n$ for some natural $n$.\(^2\)

$\overline{A}$ is called the transitive closure of $A$.

Next we will define a \textit{lattice ordered additive monoid} to be a commutative monoid $(\mathbb{M}, \oplus, \emptyset)$, where $\oplus$ is the binary operation of the monoid and $\emptyset$ is the zero element, with a partial order relation $\leq$ such that $(\mathbb{M}, \leq)$ is a distributive lattice with minimum $\emptyset$ and where the sum $\oplus$ is distributive over the operations meet $\land$ and join $\lor$. Formally, it is a set $\mathbb{M}$ with three binary operations $\oplus$, $\lor$ and $\land$ and an element $\emptyset$ such that for any $a, b, c \in \mathbb{M}$:

\begin{itemize}
  \item[M.] $(\mathbb{M}, \oplus)$ is a commutative monoid:
    \begin{itemize}
      \item[M1.] (commutativity) $a \oplus b = b \oplus a$;
      \item[M2.] (associativity) $(a \oplus b) \oplus c = a \oplus (b \oplus c)$;
      \item[M3.] (existence of the zero element) $\emptyset \oplus a = a$;
    \end{itemize}
  \item[L.] $(\mathbb{M}, \lor, \land)$ is a distributive lattice:
    \begin{itemize}
      \item[L1.] (idempotency) $a \lor a = a$ and $a \land a = a$;
      \item[L2.] (commutativity) $a \lor b = b \lor a$ and $a \land b = b \land a$;
      \item[L3.] (associativity) $(a \lor b) \lor c = a \lor (b \lor c)$ and $(a \land b) \land c = a \land (b \land c)$;
      \item[L4.] (absorption) $a \land (a \lor b) = a$ and $a \lor (a \land b) = a$;
      \item[L5.] (distributivity) $a \land (b \lor c) = (a \land b) \lor (a \land c)$ and $a \lor (b \land c) = (a \lor b) \land (a \lor c)$;
    \end{itemize}
  \item[C.] The lattice and monoid structures of $\mathbb{M}$ are compatible by the following axioms:
    \begin{itemize}
      \item[C1.] $\emptyset \lor a = a$ and $\emptyset \land a = \emptyset$;
      \item[C2.] $a \oplus (b \lor c) = (a \oplus b) \lor (a \oplus c)$ and $a \oplus (b \land c) = (a \oplus b) \land (a \oplus c)$.
    \end{itemize}
\end{itemize}

Remember that by definition $a \lor b = \sup\{a, b\}$ and $a \land b = \inf\{a, b\}$. Also we have $a \leq b \iff a \lor b = b \iff a \land b = a$. Using the axioms of such monoids\(^3\) we have the following properties:

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\(^2\)This is not true for matrices with infinite dimension.

\(^3\)In this paper we will only consider monoids of this type and will refer to them simply as monoids.
P1. $a \leq a \oplus b$;

P2. $(a \lor b) \oplus (a \land b) = a \oplus b$.

The property P1 is very easy to prove and the proof of P2 follows from the following inequalities:

$$(a \lor b) \oplus (a \land b) = [a \oplus (a \land b)] \lor [b \oplus (a \land b)]$$

$$= [(a \oplus a) \land (a \oplus b)] \lor [(b \oplus a) \land (b \oplus b)]$$

$$\leq (a \oplus b) \lor (b \oplus a) = a \oplus b$$

$$(a \lor b) \oplus (a \land b) = [(a \lor b) \oplus a] \land [(a \lor b) \oplus b]$$

$$= [(a \lor a) \lor (b \lor a)] \land [(a \lor b) \lor (b \lor b)]$$

$$\geq (b \lor a) \land (a \lor b) = a \lor b$$

Examples:

1. $\mathbb{M} := \mathbb{N}_0 = \{0, 1, 2, \ldots\}$, $\emptyset := 0$, $a \oplus b := a + b$, $a \lor b := \max\{a, b\}$ and $a \land b := \min\{a, b\}$;

2. $\mathbb{M} := \mathbb{N}_1 = \{1, 2, \ldots\}$, $\emptyset := 1$, $a \oplus b := ab$, $a \lor b := \text{l.c.m.}\{a, b\}$ and $a \land b := \text{g.c.d.}\{a, b\}$;

3. $\mathbb{M}$ a distributive lattice with minimum and $\oplus := \lor$.

Now we consider the following action of the canonical Boolean algebra $\mathcal{B}$ on a monoid $\mathbb{M}$:

$$\mathcal{B} \times \mathbb{M} \longrightarrow \mathbb{M}$$

$$(v, m) \longmapsto v \ast m$$

where

$$v \ast m := \begin{cases} m & \text{if } v = 1 \\ \emptyset & \text{if } v = 0 \end{cases}$$

Then we have:

i. $(v_1v_2) \ast m = v_1 \ast (v_2 \ast m)$ $\forall v_1, v_2 \in \mathcal{B}; m \in \mathbb{M}$;
ii. \((v_1 + v_2) * m = (v_1 * m) \lor (v_2 * m) \forall v_1, v_2 \in B; m \in M;\)

iii. \(v * (m_1 \lor m_2) = (v * m_1) \lor (v * m_2) \forall v \in B; m_1, m_2 \in M;\)

iv. \(v * (m_1 \oplus m_2) = (v * m_1) \oplus (v * m_2) \forall v \in B; m_1, m_2 \in M.\)

Now we can define an action of Boolean matrices on arrays with values in the monoid \(M.\)

**Definition 3** Let \([v_{i,j}]_{m \times n}\) be a Boolean matrix and \((a_j)_{j=1,...,n}\) be an array in \(M^n.\) We define

\[
[v_{i,j}] * (a_j) := (b_i)_{i=1,...,m} \text{ where } b_i = \bigvee_{j=1}^n v_{i,j} * a_j
\]

**Proposition 4** For any Boolean matrices \(A\) and \(B\) and any arrays \(\vec{x}\) and \(\vec{y}\) with values in \(M,\) we have:

1. \((AB) * \vec{x} = A * (B * \vec{x});\)
2. \((A + B) * \vec{x} = (A * \vec{x}) \lor (B * \vec{x});\)
3. \(I * \vec{x} = \vec{x}\) and \(O * \vec{x} = \vec{0};\)
4. \(A * (\vec{x} \lor \vec{y}) = (A * \vec{x}) \lor (A * \vec{y});\)
5. \(A * (\vec{x} \oplus \vec{y}) \leq (A * \vec{x}) \oplus (A * \vec{y}).\)

where \(I\) is the identity matrix, \(O\) is the zero matrix, the operators \(\lor\) and \(\oplus\) are defined coordinate by coordinate in \(M^n\) and \(\vec{0} = (0, ..., 0).\)

3 **Representation of the category PT on PI_M**

To each object of \(PT\) with cardinality \(n\) we associate the object \(O_{n+1}\) of \(PI_M.\)

The motivation is the following. An object \(O\) of \(PT\) gives a decomposition of the real line into intervals, and each planar tangle that ends on \(O\) decomposes the strip \(\mathbb{R} \times [0,1]\) into regions whose boundaries contain these intervals. Ordering the intervals in the natural way we will store in a Boolean matrix the information about which intervals are in the same region, that
is, the intervals $i$ and $j$ are in the same region if and only if the $(i, j)$ entry of the Boolean matrix is 1. Also to each interval we associate a value (in the given monoid $M$) which is specific for the region to which the interval belongs. Thus in this way, intervals in the same region have the same value and therefore the array of values is fixed by the action of the matrix.

This should make clear the reason for the properties that the matrices in $\mathcal{O}_n$ have to satisfy. Indeed, the author conjectures that any matrix with the properties E1, E2, E3, T1, T2 and T3 has a geometric realization in this form.

To obtain a functor from the category $\mathbf{PT}$ to the category $\mathbf{PI}_M$ we need to associate to each elementary tangle $\hat{t}_{n,k}$ and $\check{t}_{n,k}$ functions $\hat{T}_{n,k} : \mathcal{O}_n \rightarrow \mathcal{O}_{n+2}$ and $\check{T}_{n,k} : \mathcal{O}_{n+2} \rightarrow \mathcal{O}_n$ that satisfy the same relations as $\hat{t}_{n,k}$ and $\check{t}_{n,k}$.

We want these functions to preserve the motivation for the definition of $\mathcal{O}_n$. Specifically if $(R, \nu)$ is an element of $\mathcal{O}_n$, and $R$ is the matrix of connectivity of the intervals for a specific tangle that ends on the object associated with $\mathcal{O}_n$ then the image $(R', \nu')$ of $(R, \nu)$ by $\hat{T}_{n,k}$ (or $\check{T}_{n,k}$) has $R'$ as the matrix of connectivity of the intervals which terminate the composition of the tangle $\hat{t}_{n,k}$ (or $\check{t}_{n-2,k}$) with the specific tangle. Furthermore, if $\nu$ gives the values assigned to the intervals, then $\nu'$ gives the values assigned to the intervals after composition with the tangle $\hat{t}_{n,k}$ (or $\check{t}_{n-2,k}$).

We will define $\hat{T}_{n,k}$ and $\check{T}_{n,k}$ as follows

$$\hat{T}_{n,k}(R, \nu) = (R', \nu')$$

where

$$R' = B_{n,k}RB_{n,k}^t + D_{n+2,k}$$

and

$$\nu' = B_{n,k} \ast \nu$$

$B_{n,k}$ is a Boolean matrix with $n + 2$ rows and $n$ columns defined by

$$B_{n,k} := [b_{i,j}] \text{ with } b_{i,j} = 1 \text{ iff } i = j < k \text{ or } i = j + 2 > k.$$  

$D_{n,k}$ is the diagonal square Boolean matrix of dimension $n$ defined by

$$D_{n,k} := [d_{i,j}] \text{ with } d_{i,j} = 1 \text{ iff } i = j = k.$$  

We can regard the matrix $B_{n,k}$ as the connectivity relation between the upper and lower intervals of the tangle $\hat{t}_{n,k}$, that is, $b_{i,j} = 1$ iff the upper
interval \(j\) and the lower interval \(i\) are in the same region for the tangle \(\hat{t}_{n,k}\) (or equivalently, iff the upper interval \(i\) and the lower interval \(j\) are in the same region for the tangle \(\check{t}_{n,k}\)).

In this sense the formula \(R' = B_{n,k} R B_{n,k}^t + D_{n+2,k}\) means that two distinct intervals \(i\) and \(j\) are in the same region after the composition with \(\hat{t}_{n,k}\) if \(i \neq k\) and \(j \neq k\) and the intervals \(\hat{k}(i)\) and \(\hat{k}(j)\) (\(\hat{k}(i) = i\) if \(i < k\), \(\hat{k}(i) = i + 2\) if \(i > k\)) are in the same region before the composition by \(\hat{t}_{n,k}\). In other words two intervals which not \(k\) are in the same region if the respective intervals above them in the tangle \(\hat{t}_{n,k}\) are in the same region before the composition with \(\hat{t}_{n,k}\).

The formula \(\vec{v}' = B_{n,k} \star \vec{v}'\) means that the extended regions (after the composition with the tangle \(\hat{t}_{n,k}\)) preserve the old values and the new region created over the interval \(k\) receives the value \(\emptyset\).

Now we define \(\check{T}_{n,k}\):

\[
\check{T}_{n,k}(R, \vec{v}) = (R', \vec{v}')
\]

where

\[
R' = (B_{n,k}^t R B_{n,k})^2
\]

and

\[
\vec{v}' = R' \star [(B_{n,k}^t \star \vec{v}) \oplus (e_{n,k-1} \star x_k)]
\]

where \(e_{n,k-1}\) is a 1-column Boolean matrix of dimension \(n\) defined by \(e_{n,k-1} := [\epsilon_i]\) with \(\epsilon_i = 1\) iff \(i = k - 1\) and \(x_k\) is a monoid value which depends on \(R\) and \(\vec{v}\) (despite this, we use the symbol \(x_k\) instead of \(x_k(R, \vec{v})\) to simplify the notation), given by the following formula:

\[
x_k = [r_{k-1,k+1} \varphi(v_k)] \oplus [(\lnot r_{k-1,k+1}) \varphi(v_{k-1} \land v_{k+1})] = \begin{cases} 
\varphi(v_k) & \text{if } r_{k-1,k+1} = 1 \\
v_{k-1} \land v_{k+1} & \text{if } r_{k-1,k+1} = 0 
\end{cases}
\]

where \(r_{k-1,k+1} = e_{n,k-1}^t R e_{n,k+1}\) (the \((k-1, k+1)\) entry of \(R\)) and \(\varphi : \mathbb{M} \rightarrow \mathbb{M}\) is a fixed function (without structure) independent\(^4\) of \((R, \vec{v})\).

The idea behind the formula \(R' = (B_{n,k}^t R B_{n,k})^2\) is the same as before. The matrix \(B_{n,k}^t R B_{n,k}\) transfers the relation between two intervals of belonging to the same region from the top of the tangle \(\hat{t}_{n,k}\) to the bottom, and we need to take the square power because the matrix \(B_{n,k}^t R B_{n,k}\) may not be transitive,\(^4\)The representation depends on the choice of the function \(\varphi\) i.e. a different function \(\varphi\) gives a different representation.
since $\tilde{t}_{n,k}$ joins the regions associated to the intervals $k - 1$ and $k + 1$ (which may or may not be the same).

The formula $\vec{v'} = R' \ast [(B'_{n,k} \ast \vec{v}) \oplus (e_{n,k-1} \ast x_k)]$ plays a crucial role in the construction and needs a more careful explanation. What it says is that the interval $k - 1$ receives the values of the old intervals $k - 1$ and $k + 1$ and if these intervals are in distinct regions then we sum them by the operation $\oplus$ (since $(v_{k-1} \vee v_{k+1}) \oplus (v_{k-1} \wedge v_{k+1}) = (v_{k-1} \oplus v_{k+1})$ by P2).

If they are in the same region then we take their common value and sum to it some modification (given by the function $\phi$) of the value of the interval $k$ corresponding to a region which is closed after the composition with $\tilde{t}_{n,k}$.

The other intervals receive their former value if they are not in the region associated to the interval $k - 1$ or receive the new value of the interval $k - 1$ if they are in the same region as that interval. This is why we take the action of the matrix $R'$ on the array $(B'_{n,k} \ast \vec{v}) \oplus (e_{n,k-1} \ast x_k)$ so as to transfer the value of the interval $k - 1$ to others connected with it.

A better way of thinking about this may be that the array of values describes the histories of the regions associated to each interval, which keep track of the histories of any closed region inside them by means of the function $\phi$. The Boolean matrix essentially plays the role of an assistant storing the
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information about which intervals are in the same region. For example, in
the case of closed planar curves, which are morphisms from the empty set to
itself, we get in the end a one-dimensional square matrix (which is unique by
the condition E1) and a one-dimensional array (or simply a monoid value). So
in this case the Boolean matrix doesn’t matter at all and the only significant
content is the monoid value.

So as to simplify the notation we will always substitute ġtn,k, ĝtn,k, ĝTn,k,
ĝTn,k, Bn,k, Dn,k and en,k by ġtk, ĝtk, ĝT, ĝT, Bk, Dk and ek when n is implicit.

3.1 The well-definedness of the functions ĝtn,k and ĝTn,k

ĝTn,k : Oₙ → Oₙ₊₂
(R, v) → (R′, v′)

with R′ = Bk RBt k + Dk and v′ = Bk * v.

We need to check that if R = [rᵢ,j] is an n-dimensional matrix that satisfies
the conditions:

E1. R ≥ I;
E2. Rt = R;
E3. R² = R;
T1. If rᵢ,j = 1 then |i − j| is even;
T2. For any α ≤ β ≤ γ ≤ δ, rᵢ,γrᵢ,δ ≤ rᵢ,βrᵢ,γ;
T3. For any α < β if rᵢ,β = 1 then rᵢ+1,β−1 = 1 or there exists γ between α
and β such that rᵢ,γ = 1.

then the matrix R′ = Bk RBt k + Dk is an n + 2-dimensional matrix that
satisfies the same conditions.

Also we need to prove that if v is fixed by the action of R:

EC. R * v = v

then v′ = Bk * v is likewise fixed by the action of R′.

First we will check that R′ satisfies the conditions E1, E2, E3, T1, T2
and T3.
E1:

\[ R \geq I \Rightarrow R' = B_kRB_k^t + D_k \geq B_kIB_k^t + D_k \geq (I - D_k) + D_k = I \]

The operation \textit{minus} on Boolean matrices is defined in the following way:

\[
[a_{i,j}] - [b_{i,j}] = [c_{i,j}] \text{ where } c_{i,j} = 1 \text{ iff } a_{i,j} > b_{i,j} (\text{i.e. } a_{i,j} = 1 \text{ and } b_{i,j} = 0)
\]

It easy to see that, for any matrices \( A \) and \( B \), \((A - B) + B \geq A\) and \( A \geq B \Rightarrow (A - B) + B = A \). We leave it to the reader to check that \( B_kB_k^t \geq I - D_k \).

E2:

\[
R = R' \Rightarrow R'^t = (B_kRB_k^t + D_k)^t = B_kRB_k^t + D_k = B_kRB_k^t + D_k = R
\]

E3:

\[
R^2 = R \Rightarrow R'^2 = (B_kRB_k^t + D_k)^2 = B_kRB_k^tB_kRB_k^t + D_k = B_kRB_k^t + D_k^2 \\
= B_kRB_k^t + B_kRO + ORB_k + D_k \\
= B_kRB_k^t + D_k \\
= R' 
\]

We leave it to the reader to check that \( B_kB_k^t = I \), \( B_k^tD_k = O \) and \( D_kB_k = O \).

T1: The condition

\[ r_{i,j} = 1 \Rightarrow i - j \in 2\mathbb{Z} \]

is equivalent to the condition

\[ R \leq C_{n \times n} \]

where \( C_{m \times n} \) is the \textit{chess board matrix} of \textit{dimension} \( m \times n \) defined in the following way:

\[
C_{m \times n} = [c_{i,j}]_{l=1,\ldots,m} \text{ with } c_{i,j} = 1 \text{ iff } i - j \in 2\mathbb{Z} 
\]

It is easy to see that \( C_{l \times m}C_{m \times n} \leq C_{l \times n} \) (in fact, this is an equality unless \( m = 1 \)), and also we have \( B_kB_k^t \leq C_{(n+2)\times n} \) and \( D_k \leq C_{(n+2)\times (n+2)} \).

Thus \( R \leq C_{n \times n} \Rightarrow R' = B_kRB_k^t + D_k \leq C_{(n+2)\times n}C_{n \times n}C_{n \times (n+2)} + C_{(n+2)\times (n+2)} \leq C_{(n+2)\times (n+2)} \).
**T2:** Let \([r_{i,j}] = R\) and \([r'_{i,j}] = R' = B_k RB_k^t + D_k\).

Suppose that:

Hypothesis: \(\forall_{\alpha \leq \beta \leq \gamma \leq \delta} r_{\alpha,\gamma} r_{\beta,\delta} \leq r_{\alpha,\beta} r_{\beta,\gamma} r_{\gamma,\delta} r_{\gamma,\delta}^t\).

We want to prove that:

Thesis: \(\forall_{\alpha \leq \beta \leq \gamma \leq \delta} r'_{\alpha,\gamma} r'_{\beta,\delta} \leq r'_{\alpha,\beta} r'_{\beta,\gamma} r'_{\gamma,\delta}\).

In the case \(\alpha = \beta\) or \(\beta = \gamma\) or \(\gamma = \delta\) this assertion is true if \([r'_{i,j}] = R'\) satisfies the conditions of an equivalence relation (E1, E2 and E3):

1. \(r'_{i,i} = 1\) for any \(i\);
2. \(r'_{i,j} = r'_{j,i}\) for any \(i\) and \(j\);
3. \(r'_{i,j} r'_{j,k} \leq r'_{i,k}\) for any \(i, j\) and \(k\).

which we have already seen to be true.

In fact, if \(\alpha = \beta\) we have

\[
r'_{\alpha,\gamma} r'_{\beta,\delta} = r'_{\alpha,\gamma} r'_{\alpha,\delta} = r'_{\alpha,\gamma} r'_{\alpha,\delta} \leq r'_{\gamma,\delta}
\]

and

\[
r'_{\alpha,\gamma} r'_{\beta,\delta} \leq r'_{\alpha,\gamma} = r'_{\beta,\gamma}
\]

and

\[
r'_{\alpha,\gamma} r'_{\beta,\delta} \leq 1 = r'_{\alpha,\alpha} = r'_{\alpha,\beta}
\]

thus

\[
r'_{\alpha,\gamma} r'_{\beta,\delta} \leq r'_{\alpha,\beta} r'_{\beta,\gamma} r'_{\gamma,\delta}
\]

if \(\beta = \gamma\) we have

\[
r'_{\alpha,\gamma} r'_{\beta,\delta} = r'_{\alpha,\beta} r'_{\beta,\delta} = r'_{\alpha,\beta} r'_{\beta,\delta} r'_{\gamma,\delta} = r'_{\alpha,\beta} r'_{\beta,\gamma} r'_{\gamma,\delta}
\]

and if \(\gamma = \delta\) we have

\[
r'_{\alpha,\gamma} r'_{\beta,\delta} = r'_{\alpha,\gamma} r'_{\beta,\gamma} = r'_{\alpha,\gamma} r'_{\gamma,\beta} \leq r'_{\alpha,\beta}
\]

and

\[
r'_{\alpha,\gamma} r'_{\beta,\delta} \leq r'_{\beta,\delta} = r'_{\beta,\gamma}
\]
and
\[ r'_{\alpha,\gamma} r'_{\beta,\delta} \leq 1 = r'_{\gamma,\gamma} = r'_{\gamma,\delta} \]
thus
\[ r'_{\alpha,\gamma} r'_{\beta,\delta} \leq r'_{\alpha,\beta} r'_{\beta,\gamma} r'_{\gamma,\delta} \]
So we are left with the case \( \alpha < \beta < \gamma < \delta \). It is easy to see that
\[
R'_{i,j} = \begin{cases} 
R'_{k(i),k(j)} & \text{if } i, j \neq k \\
\delta_{k,j} & \text{if } i = k \\
\delta_{i,k} & \text{if } j = k 
\end{cases}
\]
where \( \hat{k}(i) = \begin{cases} 
i & \text{if } i < k \\
i - 2 & \text{if } i > k 
\end{cases} \) and \( \delta_{i,k} = \begin{cases} 1 & \text{if } i = k \\
0 & \text{if } i \neq k 
\end{cases} \)
if \( k \in \{\alpha, \beta, \gamma, \delta\} \) (with \( \alpha < \beta < \gamma < \delta \)) then
\[ r'_{\alpha,\gamma} r'_{\beta,\delta} = 0 \leq r'_{\alpha,\beta} r'_{\beta,\gamma} r'_{\gamma,\delta} \]
if \( k \not\in \{\alpha, \beta, \gamma, \delta\} \) we have \( \hat{k}(\alpha) \leq \hat{k}(\beta) \leq \hat{k}(\gamma) \leq \hat{k}(\delta) \) and then
\[ r'_{\alpha,\gamma} r'_{\beta,\delta} = r'_{\hat{k}(\alpha),\hat{k}(\gamma)} r'_{\hat{k}(\beta),\hat{k}(\delta)} \leq r'_{\hat{k}(\alpha),\hat{k}(\beta)} r'_{\hat{k}(\beta),\hat{k}(\gamma)} r'_{\hat{k}(\gamma),\hat{k}(\delta)} = r'_{\alpha,\beta} r'_{\beta,\gamma} r'_{\gamma,\delta} \]

**T3:** Suppose by hypothesis that:

Hypothesis: \( \forall \alpha < \beta \quad r_{\alpha,\beta} = 1 \Rightarrow r_{\alpha + 1,\beta - 1} = 1 \) or \( \exists \alpha < \gamma < \beta : r_{\alpha,\gamma} = 1 \).

We want to prove

Thesis: \( \forall \alpha < \beta \quad r'_{\alpha,\beta} = 1 \Rightarrow r'_{\alpha + 1,\beta - 1} = 1 \) or \( \exists \alpha < \gamma < \beta : r'_{\alpha,\gamma} = 1 \).

where \([r'_{i,j}] = R'\) and \([r'_{i,j}] = R' = B_k R B_k^t + D_k\). If \( \beta = \alpha + 1 \) then the thesis is true by the condition E2 or by the condition T1.

If \( \beta = \alpha + 2 \) then the thesis is true by the condition E1.

Now, we consider \( \beta \geq \alpha + 3 \). If \( k = \alpha \) or \( k = \beta \) we have \( r'_{\alpha,\beta} = 0 \), so the thesis is true. If \( k = \alpha + 1 \) then \( r'_{\alpha,k+1} = r'_{k-1,k+1} = r_{k(k-1),k(k+1)} = r_{k-1,k-1} = 1 \) and we have \( \alpha = k - 1 < k + 1 < \beta \), so the thesis is true. If \( k = \beta - 1 \) then \( r'_{\alpha,k-1} = r'_{\alpha,\beta} r'_{\beta,k-1} = r'_{\alpha,\beta} r'_{k+1,k-1} = r'_{\alpha,\beta} \) and we have \( \alpha < k - 1 < k + 1 = \beta \), so the thesis is true. Now suppose that
$k \notin \{\alpha, \alpha + 1, \beta - 1, \beta\}$. If $r'_{\alpha, \beta} = r_{k(\alpha), k(\beta)} = 1$ then, by hypothesis, $r_{k(\alpha)+1, k(\beta)-1} = 1$ or there exists $\hat{k}(\alpha) < \gamma' < \hat{k}(\beta)$ s.t. $r_{k(\alpha), \gamma'} = 1$.

Since $k \notin \{\alpha, \alpha + 1, \beta - 1, \beta\}$ we have that $\hat{k}(\alpha) + 1 = \hat{k}(\alpha + 1)$ and $\hat{k}(\beta) - 1 = \hat{k}(\beta - 1)$ ($k \notin \{\alpha, \alpha + 1\}$ or $k < \alpha$ or $k < \alpha + 1 = \hat{k}(\alpha)$ or $k + 1 = \hat{k}(\alpha + 1)$, and the same argument for $\hat{k}(\beta) - 1 = \hat{k}(\beta - 1)$). Thus $r_{k(\alpha)+1, k(\beta)-1} = r'_{\alpha+1, \beta-1}$. Since $k : \mathbb{N} \setminus \{k\} \to \mathbb{N}$ is surjective and monotone, for any $\gamma'$ between $\hat{k}(\alpha)$ and $\hat{k}(\beta)$ there exists $\gamma$ between $\alpha$ and $\beta$ such that $\gamma' = \hat{k}(\gamma)$. And then $r_{k(\alpha), \gamma'} = r'_{\alpha, \gamma}$. Thus the thesis is true.

Now we only have to check the extra condition:

**EC.** $R'*\overrightarrow{v}' = \overrightarrow{v}'$

assuming that $\overrightarrow{v}$ is fixed by the action of $R$.

$$R'*\overrightarrow{v}' = (B_kRB_k^t + D_k) * B_k * \overrightarrow{v}$$
$$= [(B_kRB_k^t + D_k)B_k] * \overrightarrow{v}$$
$$= (B_kRB_k^tB_k + D_kB_k) * \overrightarrow{v}$$
$$= (B_kR) * \overrightarrow{v}$$
$$= B_k * (R * \overrightarrow{v})$$
$$= B_k * \overrightarrow{v}$$
$$= \overrightarrow{v}'$$

Next we show that the function $\hat{T}_k$ is well-defined. Recall that:

$$\hat{T}_{n,k} : O_{n+2} \to O_n$$

$$(R, \overrightarrow{v}) \mapsto (R', \overrightarrow{v}')$$

with

$$R' = (B_k^tRB_k)^2$$

and

$$\overrightarrow{v}' = R' * [(B_k^t * \overrightarrow{v}) \oplus (e_{k-1} * x_k)]$$

where

$$x_k = [r_{k-1, k+1} * \varphi(v_k)] \oplus [(\neg r_{k-1, k+1}) * (v_{k-1} \land v_{k+1})] = \begin{cases} 
\varphi(v_k) & \text{if } r_{k-1, k+1} = 1 \\
v_{k-1} \land v_{k+1} & \text{if } r_{k-1, k+1} = 0 
\end{cases}$$

Let us prove that $R'$ satisfies the six conditions in $O_n$ and that $\overrightarrow{v}'$ is fixed by the action of $R'$.
E1: 
\[ R \geq I \Rightarrow R' = (B_k^tRB_k)^2 \geq (B_k^tIB_k)^2 = I^2 = I \]

E2: 
\[ R = R' \Rightarrow R'' = [(B_k^tRB_k)^2]^t = [(B_k^tRB_k)^t]^2 = (B_k^tR^tB_k)^2 = (B_k^tRB_k)^2 = R' \]

E3: To check the transitivity of \( R' \) it is sufficient to prove that \( (B_k^tRB_k)^3 \leq (B_k^tRB_k)^2 \) (assuming that \( R^2 = R \)).

\[
(B_k^tRB_k)^3 = B_k^tRB_kB_k^tRB_kB_k^tRB_k \\
\leq B_k^tR(I + B_kD_{k-1}B_k)RB_kB_k^tRB_k \\
= B_k^tRIB_kB_k^tRB_k + B_k^tRB_kB_kD_{k-1}B_k^tR(I + B_kD_{k-1}B_k)RB_k \\
= B_k^tRB_kB_k^tRB_k + B_k^tRB_kB_kD_{k-1}B_k^tR^2B_k \\
\quad + B_k^tRB_kD_{k-1}B_k^tRB_k
\leq B_k^tRB_kB_k^tRB_k + B_k^tRB_kB_kD_{k-1}B_k^tRB_k + B_k^tRB_kB_kD_{k-1}B_k^tRB_k \\
= B_k^tRB_kB_k^tRB_k \\
= (B_k^tRB_k)^3
\]

We leave it to the reader to check that \( B_kB_k^t \leq I + B_kD_{k-1}B_k \) and \( D_{k-1}B_k^tRB_kD_{k-1} \leq D_{k-1} \).

T1:
\[ R \leq C_{(n+2)\times(n+2)} \Rightarrow R' = (B_k^tRB_k)^2 \leq (C_{n\times(n+2)}C_{(n+2)\times(n+2)}C_{(n+2)\times n})^2 \\
\leq (C_{n\times n})^2 = C_{n\times n} \]

T2: Using the "equality" \( r_{i,j} = e_i^tRe_j \) we need to check that:

Thesis: \( \forall \alpha \leq \beta \leq \gamma \leq \delta \) \[ e_\alpha^tR'e_\gamma e_\beta^tR'e_\delta \leq e_\alpha^tR'e_\beta e_\gamma e_\delta R'e_\gamma R'e_\delta. \]

assuming the hypothesis:

Hypothesis: \( \forall \alpha \leq \beta \leq \gamma \leq \delta \) \[ e_\alpha^tRe_\gamma e_\beta^tRe_\delta \leq e_\alpha^tRe_\beta e_\gamma e_\delta Re_\gamma e_\delta. \]
Since \( R' \) satisfies the equivalence relation conditions (E1, E2 and E3), the thesis is satisfied for \( \alpha = \beta \) or \( \beta = \gamma \) or \( \gamma = \delta \). So we can assume that \( \alpha < \beta < \gamma < \delta \).

We will substitute the hypothesis by a more appropriate hypothesis. But for that we need to introduce a new definition and same properties. Let \( u \) and \( v \) be two one-column non-zero matrices. We define:

\[
u < v \text{ iff } \max\{i : u_i = 1\} < \min\{i : v_i = 1\}.
\]

**Proposition 5.1.** \( \prec \) defines a strict order relation on the set of non-zero one-column matrices;

2. \( u \prec v \), \( e_\alpha \leq u \) and \( e_\beta \leq u \) \( \Rightarrow \alpha < \beta \);

3. \( (\forall \alpha \prec < \gamma \prec < \delta) \ e_\alpha R e_\gamma e_\beta R e_\delta \leq e_\alpha R e_\beta e_\gamma R e_\delta \) \( \Rightarrow \ (\forall u < v < w < x) \ u^t R w v^t R x \leq u^t R w v^t R x \);

4. \( \alpha < \beta \Rightarrow B_k e_\alpha < B_k e_\beta \).

Now we take a new (weaker) hypothesis:

\[
\text{N.H.: } \forall \alpha \prec < \beta \prec < \gamma \prec < \delta \ e_\alpha B^t_k R B_k e_\gamma e_\beta e_\delta B^t_k R B_k e_\delta \leq e_\alpha B^t_k R B_k e_\beta e_\gamma B^t_k R B_k e_\delta.
\]

\[
e_\alpha R^t e_\gamma e_\beta R^t e_\delta = e_\alpha B^t_k R B_k e_\gamma e_\beta e_\delta B^t_k R B_k e_\delta \leq e_\alpha B^t_k R (I + A_k) B_k e_\gamma e_\beta e_\delta R (I + A_k) R B_k e_\delta
\]

\[
= e_\alpha B^t_k R B_k e_\gamma e_\beta e_\delta B^t_k R B_k e_\delta + e_\alpha B^t_k R A_k e_\gamma e_\beta B^t_k R A_k e_\delta + e_\alpha B^t_k R A_k e_\gamma e_\beta B^t_k R B_k e_\delta
\]

\[
\text{where}
\]

\[
A_k = B_k D_{k-1} B^t_k = B_k e_{k-1} e^t_{k-1} B^t_k
\]

Now, we only need to prove that:

(i) \( e_\alpha B^t_k R B_k e_\gamma e_\beta e_\delta B^t_k R B_k e_\delta \leq e_\alpha R^t e_\beta e_\gamma e_\delta R^t e_\gamma e_\beta R e_\delta \);

(ii) \( e_\alpha B^t_k R A_k e_\gamma e_\beta e_\delta B^t_k R B_k e_\delta \leq e_\alpha R^t e_\beta e_\gamma e_\delta R^t e_\gamma e_\beta R e_\delta \);

(iii) \( e_\alpha B^t_k R B_k e_\gamma e_\beta e_\delta B^t_k R A_k e_\delta \leq e_\alpha R^t e_\beta e_\gamma e_\delta R^t e_\gamma e_\beta R e_\delta \);

(iv) \( e_\alpha B^t_k R A_k e_\gamma e_\beta e_\delta B^t_k R A_k e_\delta \leq e_\alpha R^t e_\beta e_\gamma e_\delta R^t e_\gamma e_\beta R e_\delta \).

For that it is useful to observe that, for arbitrary square matrices,

\[
e^t_i X e_j e^t_i Y e_l = e^t_k Y e_l e^t_i X e_j,
\]
Representations of non-singular planar tangles by operators

\[ e_i^t X e_j = e_j^t X^t e_i \]

and

\[ (e_i^t X e_j)^2 = e_i^t X e_j, \]

since \( e_i^t X e_j \) and \( e_i^t Y e_i \) are one-dimensional square matrices.

(i) \[
\begin{align*}
e^t_\alpha B^t_k RB_k e_\gamma e^t_\beta B^t_k RB_k e_\delta & \leq e^t_\alpha B^t_k RB_k e_\beta e^t_\beta B^t_k RB_k e_\gamma e^t_\gamma B^t_k RB_k e_\delta \\
& \leq e^t_\alpha R' e_\beta e^t_\beta R' e_\gamma e^t_\gamma R' e_\delta
\end{align*}
\]

Observe that

\[ B^t_k RB_k \leq (B^t_k RB_k)^2 = R' \Rightarrow e^t_\ell B^t_k RB_k e_j \leq e^t_\ell R' e_j \]

(ii) \[
\begin{align*}
e^t_\alpha B^t_k RA_k RB_k e_\gamma e^t_\beta B^t_k RB_k e_\delta &= e^t_\alpha B^t_k RB_k e_{k-1} e^t_\alpha B^t_k RB_k e_\gamma e^t_\beta B^t_k RB_k e_\delta. \\
& \quad \text{If } k - 1 < \beta \text{ then} \]

\[ e^t_\alpha B^t_k RB_k e_{k-1} e^t_\alpha B^t_k RB_k e_\gamma e^t_\beta B^t_k RB_k e_\delta \leq e^t_\alpha B^t_k RB_k e_\beta e^t_\beta B^t_k RB_k e_\gamma e^t_\gamma B^t_k RB_k e_\delta \]

\[ \leq e^t_\alpha R' e_\beta e^t_\beta R' e_\gamma e^t_\gamma R' e_\delta \]

If \( k - 1 = \beta \) then

\[ e^t_\alpha B^t_k RB_k e_{k-1} e^t_\alpha B^t_k RB_k e_\gamma e^t_\beta B^t_k RB_k e_\delta \]

\[ = e^t_\alpha B^t_k RB_k e_\beta e^t_\beta B^t_k RB_k e_\gamma e^t_\gamma B^t_k RB_k e_\delta \]

\[ \leq e^t_\alpha R' e_\beta e^t_\beta R' e_\gamma e^t_\gamma R' e_\delta \]

If \( \beta < k - 1 < \delta \) then

\[ e^t_\alpha B^t_k RB_k e_{k-1} e^t_\alpha B^t_k RB_k e_\gamma e^t_\beta B^t_k RB_k e_\delta \]

\[ = e^t_\alpha B^t_k RB_k e_\beta e^t_\beta B^t_k RB_k e_\gamma e^t_\gamma B^t_k RB_k e_\delta \]

\[ \leq e^t_\alpha B^t_k RB_k e_\beta e_\beta e^t_\beta B^t_k RB_k e_\gamma e^t_\gamma B^t_k RB_k e_\delta \]

\[ \leq e^t_\alpha R' e_\beta e^t_\beta R' e_\gamma e^t_\gamma e^t_\gamma R' e_\delta \]

\[ \leq e^t_\alpha R' e_\beta e^t_\beta R' e_\gamma e^t_\gamma R' e_\delta \]
If \( k - 1 = \delta \) then
\[
e'_\alpha B^t_k R B_k e_{k-1} e'_{k-1} B^t_k R B_k e_\gamma e'_\beta B^t_k R B_k e_\delta
\]
\[
= e'_\alpha B^t_k R B_k e_\delta e'_\beta B^t_k R B_k e_\gamma e'_\gamma B^t_k R B_k e_\delta
\]
\[
\leq e'_\alpha R' e_{k-1} e'_\beta R' e_\gamma e'_\gamma R' e_\delta e'_\delta R' e_{k-1}
\]
\[
= e'_\alpha R' e_{k-1} e'_\beta R' e_\gamma e'_\gamma R' e_\delta
\]
\[
\leq e'_\alpha R' e_\beta e'_\beta R' e_\gamma e'_\gamma R' e_\delta
\]
\[
(\text{iii}) \quad e'_\alpha B^t_k R B_k e_\gamma e'_\beta B^t_k R A_k R B_k e_\delta = e'_\delta B^t_k R A_k R B_k e_\beta e'_\gamma B^t_k R B_k e_\alpha.
\]
Since all arguments in case (ii) are valid reversing the order of \( \alpha, \beta, \gamma \) and \( \delta \), we have that: \( e'_\alpha B^t_k R B_k e_\gamma e'_\beta B^t_k R A_k R B_k e_\delta \leq e'_\alpha R' e_\beta e'_\beta R' e_\gamma e'_\gamma R' e_\delta. \]

(iv)
\[
e'_\alpha B^t_k R A_k R B_k e_\gamma e'_\beta B^t_k R A_k R B_k e_\delta
\]
\[
= e'_\alpha B^t_k R B_k e_{k-1} e'_{k-1} B^t_k R B_k e_\gamma e'_\beta B^t_k R B_k e_{k-1} e'_{k-1} B^t_k R B_k e_\delta
\]
\[
\leq e'_\alpha R' e_{k-1} e'_\beta R' e_\gamma e'_\gamma R' e_{k-1} R' e_\delta
\]
\[
= e'_\alpha R' e_{k-1} e'_\beta R' e_\gamma e'_\gamma R' e_{k-1} R' e_\delta
\]
\[
\leq e'_\alpha R' e_\beta e'_\beta R' e_\gamma e'_\gamma R' e_\delta
\]

T3: Lemma 6 If \( R = [r_{i,j}] \) represents an equivalence relation, then the following statements are equivalent:

(i) \( \forall_{\alpha < \beta} \quad r_{\alpha, \beta} = 1 \Rightarrow r_{\alpha+1, \beta-1} = 1 \) or \( \exists_{\alpha < \gamma < \beta} : \quad r_{\alpha, \gamma} = 1; \)

(ii) \( \forall_{\alpha < \beta} \quad r_{\alpha, \beta} = 1 \Rightarrow \exists_{\alpha < \gamma \leq \beta} : \quad r_{\alpha, \gamma} = r_{\alpha+1, \gamma-1} = 1; \)

(iii) \( \forall_{\alpha < \beta} \quad r_{\alpha, \beta} = 1 \Rightarrow r_{\alpha+1, \beta-1} = 1 \) or \( \exists_{\alpha < \gamma < \beta} : \quad r_{\gamma, \beta} = 1; \)

(iv) \( \forall_{\alpha < \beta} \quad r_{\alpha, \beta} = 1 \Rightarrow \exists_{\alpha \leq \gamma < \beta} : \quad r_{\gamma, \beta} = r_{\gamma+1, \beta-1} = 1. \)
Proof. It easy to see that \((ii) \Rightarrow (i)\) and \((iv) \Rightarrow (iii)\). To see that \((i) \Rightarrow (ii)\) we take \(\gamma = \inf\{\delta \leq \beta : r_{\alpha,\delta} = 1\}\) and to see that \((iii) \Rightarrow (iv)\) we take \(\gamma = \sup\{\delta \geq \alpha : r_{\delta,\beta} = 1\}\). \((i) \Leftrightarrow (iii)\) results from the transitivity of \([r_{i,j}]\).

Now we want to see that \([r'_{i,j}] = (B_k^i RB_k)^2\) satisfies one of the statements of the lemma (assuming that \(R = [r_{i,j}]\) also satisfies the same statements and the remaining conditions on \(\mathcal{O}_{n+2}\)). Using the properties of \(R = [r_{i,j}]\) and the relation \(R' = [r'_{i,j}] = (B_k^i RB_k)^2\) we can set:

\[
r'_{\alpha,\beta} = \begin{cases} 
  r_{\alpha,\beta} + r_{\alpha,k+1}r_{k-1,\beta} & \text{if } k-1 > \beta \\
  r_{\alpha,k-1} + r_{\alpha,k+1} & \text{if } k-1 = \beta \\
  r_{\alpha,\beta+2} + r_{\alpha,k-1}r_{k+1,\beta+2} & \text{if } \alpha < k-1 < \beta \\
  r_{k-1,\beta+2} + r_{k+1,\beta+2} & \text{if } k-1 = \alpha \\
  r_{\alpha+2,\beta+2} + r_{\alpha+2,k+1}r_{k-1,\beta+2} & \text{if } k-1 < \alpha 
\end{cases}
\]

Case A: \(k-1 > \beta\). \(r'_{\alpha,\beta} = r_{\alpha,\beta} + r_{\alpha,k+1}r_{k-1,\beta} = 1 \Rightarrow r_{\alpha,\beta} = 1\) or \(r_{\alpha,k+1} = r_{k-1,\beta} = 1\).

A.1: \(r_{\alpha,\beta} = 1 \Rightarrow \exists_{\alpha<\gamma_1 \leq \beta} : r_{\alpha+1,\gamma_1-1} = r_{\alpha,\gamma_1} = 1 \Rightarrow \exists_{\alpha<\gamma_1 \leq \beta} : r'_{\alpha+1,\gamma_1-1} = r'_{\alpha,\gamma_1} = 1\).

A.2: \(r_{\alpha,k+1} = r_{k-1,\beta} = 1 \Rightarrow \exists_{\alpha<\gamma_2 \leq k+1} : r_{\alpha+1,\gamma_2-1} = r_{\alpha,\gamma_2} = 1\).

A.2.1: If \(\gamma_2 \leq \beta\) then we have \(r'_{\alpha+1,\gamma_2-1} = r'_{\alpha,\gamma_2} = 1\) with \(\alpha < \gamma_2 \leq \beta\).

A.2.2: If \(\beta < \gamma_2 \leq k-1\) we can use the condition T2 to get \(r_{\alpha,\beta} = 1\) (case A.1) since \(r_{\alpha,\gamma_2} = 1\) and \(r_{\beta,k-1} = 1\).

A.2.3: If \(\gamma_2 = k\) we have \(r_{\alpha,k} = 1\) which together with \(r_{\alpha,k+1} = 1\) contradicts the condition T1.

A.2.4: If \(\gamma_2 = k+1\) then \(r_{\alpha+1,k} = 1 \Rightarrow \exists_{\alpha+1 \leq \gamma_3 < k} : r_{\gamma_3+1,k-1} = r_{\gamma_3,k} = 1\).

A.2.4.1: If \(\beta \leq \gamma_3 \leq k-1\) we can use the condition T2 to get \(r_{k-1,k} = 1\) (which contradicts the condition T1) since \(r_{\beta,k-1} = 1\) and \(r_{\gamma_3,k} = 1\).

A.2.4.2: If \(\alpha+1 \leq \gamma_3 \leq \beta-1\) we have \(r'_{\gamma_3+1,\alpha} = 1\) (since \(r_{\gamma_3+1,k-1} = 1\) and \(r_{k+1,\alpha} = 1\)) and \(r_{\gamma_3,\alpha+1} = 1\) (since \(r_{\gamma_3,k} = 1\) and \(r_{k,\alpha+1} = 1\)). Taking \(\gamma_4 = \gamma_3 + 1\) we have \(r'_{\alpha+1,\gamma_4-1} = r'_{\alpha,\gamma_4} = 1\) with \(\alpha < \gamma_4 \leq \beta\).
Case B: \( k - 1 = \beta \). \( r'_{a,\beta} = r_{a,k-1} + r_{a,k+1} = 1 \Rightarrow r_{a,k-1} = 1 \) or \( r_{a,k+1} = 1 \).

B.1: \( r_{a,k-1} = 1 \Rightarrow \exists \alpha < \gamma_1 \leq k - 1 : r_{a+1,\gamma_1-1} = r_{a,\gamma_1} = 1 \Rightarrow \exists \alpha < \gamma_1 \leq k - 1 : \alpha < \gamma_1 \leq k - 1 \).

B.2: \( r_{a,k+1} = 1 \Rightarrow \exists \alpha < \gamma_2 \leq k + 1 : r_{a+1,\gamma_2-1} = r_{a,\gamma_2} = 1. \)

B.2.1: If \( \gamma_2 \leq k - 1 \) then we have \( r'_{a+1,\gamma_2-1} = r'_{a,\gamma_2} = 1 \) with \( \alpha < \gamma_2 \leq k - 1 = \beta \).

B.2.2: If \( \gamma_2 = k \) we have \( r_{a,k} = 1 \) which together with \( r_{a,k+1} = 1 \) contradicts the condition T1.

B.2.3: If \( \gamma_2 = k+1 \) then \( r_{a+1,k} = 1 \Rightarrow \exists \alpha \leq \gamma_3 < k : r_{\gamma_3+1,k-1} = r_{\gamma_3,k} = 1. \)

B.2.3.1: If \( \gamma_3 = k - 1 \) then \( r_{\gamma_3+1,k-1} = r_{k,k-1} = 1 \) which contradicts the condition T1.

B.2.3.2: If \( \alpha + 1 \leq \gamma_3 \leq k - 2 \) we have \( r'_{\gamma_3+1,k} = 1 \) (since \( r_{\gamma_3+1,k-1} = 1 \) and \( r_{k+1,\alpha} = 1 \)) and \( r_{\gamma_3,\alpha+1} = 1 \) (since \( r_{\gamma_3,k} = 1 \) and \( r_{k,\alpha+1} = 1 \)). Taking \( \gamma_4 = \gamma_3 + 1 \) we have \( r'_{\alpha+1,\gamma_4-1} = r'_{\alpha,\gamma_4} = 1 \) with \( \alpha < \gamma_4 \leq k - 1 \).

Case C: \( \alpha < k - 1 < \beta \). \( r'_{a,\beta} = r_{a,\beta+2} + r_{a,k+1}r_{k-1,\beta+2} = 1 \Rightarrow r_{a,\beta+2} = 1 \) or \( r_{a,k+1} = r_{k-1,\beta+2} = 1 \).

C.1: \( r_{a,\beta+2} = 1 \Rightarrow \exists \alpha < \gamma_1 \leq \beta + 2 : r_{a+1,\gamma_1-1} = r_{a,\gamma_1} = 1 \) and \( \exists \alpha \leq \gamma_2 < \beta + 2 : r_{\gamma_2,\beta+2} = r_{\gamma_2+1,\beta+1} = 1. \)

C.1.1: If \( \gamma_1 \geq \gamma_2 + 1 \) we can use the condition T2 to get \( r_{a,\beta+1} = 1 \) since \( r_{a,\gamma_1} = 1 \) and \( r_{\gamma_2+1,\beta+1} = 1 \). This together with \( r_{a,\beta+2} = 1 \) contradicts the condition T1.

C.1.2: If \( \gamma_1 \leq \gamma_2 \) then \( \gamma_1 < k \) or \( \gamma_2 > k \) or \( \gamma_1 = \gamma_2 = k \).

C.1.2.1: If \( \gamma_1 < k \) then \( r_{a+1,\gamma_1-1} = r_{a,\gamma_1} = 1 \Rightarrow r'_{a+1,\gamma_1-1} = r'_{a,\gamma_1} = 1 \) with \( \alpha < \gamma_1 < k \leq \beta \).

C.1.2.2: If \( \gamma_2 > k \) then \( r_{\gamma_2,\beta+2} = r_{\gamma_2+1,\beta+1} = 1 \Rightarrow r'_{\gamma_2,\beta+2} = r'_{\gamma_2+1,\beta+1} = 1 \) with \( \alpha \leq k - 2 < \gamma_2 - 2 < \beta \).

C.1.2.3: If \( \gamma_1 = \gamma_2 = k \) then \( r_{a+1,\gamma_1-1} = r_{a+1,k-1} = 1 \) and \( r_{\gamma_2+1,\beta+1} = r_{k+1,\beta+1} = 1 \) which implies \( r'_{a+1,\beta+1} = 1 \).

C.2: \( r_{a,k-1} = r_{k+1,\beta+2} = 1 \Rightarrow r'_{a,k-1} = 1 \) with \( \alpha < k - 1 < \beta \).

Case D: \( k - 1 = \alpha \). This case is analogous to case B.
Case E: $k - 1 < \alpha$. This case is analogous to case A.

Now we only have to check the condition:

**EC.** $R' \ast \overrightarrow{v} = \overrightarrow{v}'$

which is obvious from the definition of $\overrightarrow{v}'$ because $R'$ is idempotent.

### 3.2 Checking the relations on $\hat{T}_k$ and $\tilde{T}_k$

Now we are going to prove that $\hat{T}_k$ and $\tilde{T}_k$ satisfy the following relations:

1. $\hat{T}_{k+1} \circ \hat{T}_k = \hat{T}_{k-1} \circ \hat{T}_k = id$;
2. $\tilde{T}_l \circ \hat{T}_k = \hat{T}_k \circ \tilde{T}_{l-2}$ for $l \geq k + 2$;
3. $\tilde{T}_k \circ \tilde{T}_l = \tilde{T}_{l-2} \circ \hat{T}_k$ and $\tilde{T}_l \circ \hat{T}_k = \hat{T}_k \circ \tilde{T}_{l-2}$ for $l \geq k + 2$;
4. $\tilde{T}_k \circ \tilde{T}_l = \tilde{T}_{l-2} \circ \tilde{T}_k$ for $l \geq k + 2$.

We begin by checking the first relation.

1. $\tilde{T}_{k+1} \circ \hat{T}_k = \tilde{T}_{k-1} \circ \hat{T}_k = id$.

   Let $(R_1, \overrightarrow{a}) \in O_n$, $(R_2, \overrightarrow{b}) = \hat{T}_k(R_1, \overrightarrow{a})$ and $(R_3, \overrightarrow{c}) = \tilde{T}_{k+1}(R_2, \overrightarrow{b}) = \tilde{T}_{k+1} \circ \tilde{T}_k(R_1, \overrightarrow{a})$.

   We want to show that $(R_3, \overrightarrow{c}) = (R_1, \overrightarrow{a})$.

   \[ \begin{align*}
   R_2 &= B_k R_1 B_k^t + D_k = B_k R_1 B_k^t + I \Rightarrow R_3 = [B_{k+1}^t (B_k R_1 B_k^t + I) B_{k+1}]^2 \\
   R_3 &= (B_{k+1}^t B_k R_1 B_k^t B_{k+1} + B_{k+1}^t B_{k+1})^2 = (R_1 + I)^2 = R_1^2 = R_1
   \end{align*} \]

   We leave it to the reader to check the identities $B_{k+1}^t B_k = I$ and $B_{k+1}^t B_{k+1} = I$.

\[
\begin{align*}
\overrightarrow{c} &= R_3 \ast [(B_{k+1}^t \ast \overrightarrow{b}) \oplus (e_k \ast x_{k+1})] \Rightarrow \overrightarrow{c} = R_3 \ast [(B_{k+1}^t \ast \overrightarrow{a}) \oplus (e_k \ast x_{k+1})] \\
\overrightarrow{b} &= (B_k \ast \overrightarrow{d})
\end{align*}
\]

where $x_{k+1} = [r_{k,k+2} \ast \varphi(b_{k+1})] \oplus [(-r_{k,k+2}) \ast (b_k \land b_{k+2})]$ with $r_{k,k+2} = e_k^t R_2 e_{k+2} = e_k^t (B_k R_1 B_k^t + D_k)e_{k+2} = e_k^t B_k R_1 B_k^t e_{k+2} + e_k^t D_k e_{k+2} = 0$
thus \(x_{k+1} = b_k \land b_{k+2} = \emptyset \land a_k = \emptyset\).

Then we have
\[
\overrightarrow{c} = R_3 \ast (((B'_{k+1}B_k) \ast \overrightarrow{d}) \oplus (e_k \ast \emptyset)) = R_1 \ast (\overrightarrow{d} \oplus \emptyset) = R_1 \ast \overrightarrow{d} = \overrightarrow{d}
\]

To check the identity \(\hat{T}_{k-1} \circ \hat{T}_k = id\) we use the same procedure.

2. \(\hat{T}_l \circ \hat{T}_k = \hat{T}_k \circ \hat{T}_{l-2}\) for \(l \geq k + 2\).
Let \((R_1, \overrightarrow{d'}) \in \mathcal{O}_n, (R_2, \overrightarrow{b'}) = \hat{T}_k(R_1, \overrightarrow{d'})\) and \((R_3, \overrightarrow{c'}) = \hat{T}_l(R_2, \overrightarrow{b'}) = \hat{T}_l \circ \hat{T}_k(R_1, \overrightarrow{d'})\), and let \((R'_2, \overrightarrow{b'''}) = \hat{T}_{l-2}(R_1, \overrightarrow{d'})\) and \((R'_3, \overrightarrow{c''}) = \hat{T}_k(R'_2, \overrightarrow{b'''}) = \hat{T}_k \circ \hat{T}_{l-2}(R_1, \overrightarrow{d'})\). We want to check that \((R'_3, \overrightarrow{c''}) = (R_3, \overrightarrow{c'})\).

\[
\begin{align*}
R_3 &= B_tR_2B'_1 + D_l \\
&= B_l(B_kR_1B'_k + D_k)B'_1 + D_l \\
&= B_lB_kB'_1B'_1B'_1 + B_lD_kB'_1 + D_l \\
&= (B_lB_k)R_1(B_lB_k) + D_k + D_l
\end{align*}
\]

and
\[
\begin{align*}
R'_3 &= B_kR'_2B'_1 + D_k \\
&= B_k(B_{l-2}R_1B'_{l-2} + D_{l-2})B'_1 + D_k \\
&= B_kB_{l-2}B'_1B'_1B'_1 + B_kD_{l-2}B'_1 + D_k \\
&= (B_kB_{l-2})R_1(B_kB_{l-2}) + D_l + D_k
\end{align*}
\]

Thus \(R'_3 = R_3\). We leave it to the reader to check \(B_lD_kB'_1 = D_k, B_kD_{l-2}B'_1 = D_l\) and \(B_lB_k = B_kB_{l-2}\).

\[
\begin{cases}
\overrightarrow{c'} = B_l \ast \overrightarrow{b'} = B_l \ast (B_k \ast \overrightarrow{d'}) = (B_lB_k) \ast \overrightarrow{d'} \\
\overrightarrow{c''} = B_k \ast \overrightarrow{b''} = B_k \ast (B_{l-2} \ast \overrightarrow{d'}) = (B_kB_{l-2}) \ast \overrightarrow{d'} \Rightarrow \overrightarrow{c''} = \overrightarrow{c'}.
\end{cases}
\]

3. \(\hat{T}_k \circ \hat{T}_l = \hat{T}_{l-2} \circ \hat{T}_k\) and \(\hat{T}_l \circ \hat{T}_k = \hat{T}_k \circ \hat{T}_{l-2}\) for \(l \geq k + 2\).
Let \((R_1, \overrightarrow{d'}) \in \mathcal{O}_n, (R_2, \overrightarrow{b'}) = \hat{T}_l(R_1, \overrightarrow{d'})\) and \((R_3, \overrightarrow{c'}) = \hat{T}_k(R_2, \overrightarrow{b'}) = \hat{T}_k \circ \hat{T}_l(R_1, \overrightarrow{d'})\), and let \((R'_2, \overrightarrow{b'''}) = \hat{T}_{l-2}(R_1, \overrightarrow{d'})\) and \((R'_3, \overrightarrow{c''}) = \hat{T}_k(R'_2, \overrightarrow{b'''}) = \hat{T}_{l-2} \circ \hat{T}_k(R_1, \overrightarrow{d'})\).

We want to check that \((R'_3, \overrightarrow{c''}) = (R_3, \overrightarrow{c'})\).

First the case \(l > k + 2\) (where we have the identity \(B'_kB_l = B_{l-2}B'_k\)).
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\[ R_3 = (B^*_i R^*_i B_k)^2 \]
\[ = [B^*_i (B^*_i R^*_i B^*_i + D_i) B_k]^2 \]
\[ = (B^*_i B^*_i R^*_i B^*_i B_k + B^*_i D_i B_k)^2 \]
\[ = (B^*_i B^*_i R^*_i B^*_i B_k + D_i - 2)^2 \]
\[ = B^*_i B^*_i R^*_i B^*_i B_k B^*_i B^*_i B_k + B^*_i B^*_i R^*_i B_k B^*_i D_i - 2 \]
\[ + D_i - 2 B^*_i B^*_i R^*_i B_k B^*_i B_k + D_i - 2 \]
\[ = B^*_i B^*_i R^*_i B_k I B^*_i B^*_i B_k + B^*_i B^*_i R^*_i B_k O + O B^*_i R^*_i B_k B^*_i B_k + D_i - 2 \]
\[ = B^*_i B^*_i R^*_i B_k B^*_i B_k B^*_i D_i - 2 \]
\[ = B^*_i B^*_i R^*_i B^*_i B_k B^*_i B_k + D_i - 2 \]
\[ = R^*_i \]

We leave it to the reader to check \( B^*_i D_i B_k = D_i - 2 \).

\[ \overrightarrow{c} = R_3 * [(B^*_i * \overrightarrow{b} \oplus (e_{k-1} * x_k)] \text{ where } x_k = [r_{k-1,k+1}^{(2)} * \varphi(b_k)] \oplus (\neg r_{k-1,k+1}^{(2)}) \ni (b_{k-1} \wedge b_{k+1}) \text{ with } \]
\[ r_{k-1,k+1}^{(2)} = e_{k-1}^t R^*_2 e_{k+1} \text{ and } \overrightarrow{b} = B^*_i \overrightarrow{a}. \]

In this way:

\[ \overrightarrow{c} = R_3 * [(B^*_i B^*_i \overrightarrow{a} \oplus (e_{k-1} * x_k)] \]
\[ = R^*_i * [(B^*_i B^*_i \overrightarrow{a} \oplus (e_{k-1} * x_k)] \]
\[ = (B^*_i B^*_i R^*_i B_k B_k + D_i - 2) * [(B^*_i B^*_i \overrightarrow{a} \oplus (e_{k-1} * x_k)] \]
\[ = \{(B^*_i B^*_i R^*_i B^*_i B_k \overrightarrow{a} \oplus (e_{k-1} * x_k)] \}
\[ \vee (D_i - 2 * [(B^*_i B^*_i \overrightarrow{a} \oplus (e_{k-1} * x_k)] \}

Lemma 7. Let \( M = [\mu_{i,j}] \) be a Boolean matrix and let \( \overrightarrow{v} \) and \( \overrightarrow{w} \) be two arrays. If, for each index \( i \), we have one of the following situations:

1. \( \mu_{i,j} \neq w_j = \emptyset \) for all \( j \) (or \( \mu_{i,j} \neq v_j = \emptyset \) for all \( j \));
2. \( \mu_{i,j} = 1 \) for, at most, a single index \( j \);

then \( M * (\overrightarrow{v} \oplus \overrightarrow{w}) = (M * \overrightarrow{v}) \oplus (M * \overrightarrow{w}) \).

Proof. Let \( \overrightarrow{x} := M * (\overrightarrow{v} \oplus \overrightarrow{w}) \) and \( \overrightarrow{y} := (M * \overrightarrow{v}) \oplus (M * \overrightarrow{w}) \). We have
\[ x_i = \bigvee_{j=1}^{n} \mu_{i,j} \neq v_j \oplus w_j = \bigvee_{j=1}^{n} [(\mu_{i,j} \neq v_j) \oplus (\mu_{i,j} \neq w_j)] \]
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and

\[ y_i = (\bigvee_{j=1}^{n} \mu_{i,j} \ast v_j) \oplus (\bigvee_{j=1}^{n} \mu_{i,j} \ast w_j) \]

(1) If \( \mu_{i,j} \ast w_j = \emptyset \) for all \( j \) then

\[ x_i = \bigvee_{j} [(\mu_{i,j} \ast v_j) \oplus \emptyset] = \bigvee_{j} \mu_{i,j} \ast v_j \]

and

\[ y_i = (\bigvee_{j} \mu_{i,j} \ast v_j) \oplus (\bigvee_{j} \emptyset) = \bigvee_{j} \mu_{i,j} \ast v_j \]

(2) If there exists \( k \) such that \( \mu_{i,j} = 1 \Rightarrow j = k \) then

\[ x_i = \bigvee_{j} [(\mu_{i,j} \ast v_j) \oplus (\mu_{i,j} \ast w_j)] = (\mu_{i,k} \ast v_k) \oplus (\mu_{i,k} \ast w_k) \]

and

\[ y_i = (\bigvee_{j} \mu_{i,j} \ast v_j) \oplus (\bigvee_{j=1}^{n} \mu_{i,j} \ast w_j) = (\mu_{i,k} \ast v_k) \oplus (\mu_{i,k} \ast w_k) \]

With this lemma we have that:

\[ B_{l-2}^t \left[ (B_{l-2} B_{k}^t \ast \overrightarrow{a}) \oplus (e_{k-1} \ast x_k) \right] = (B_{l-2} B_{l-2} B_{k}^t \ast \overrightarrow{a}) \oplus (B_{l-2} e_{k-1} \ast x_k) \]

by the condition (2) of the lemma for \( i \neq l - 3 \), and by the condition (1) for \( i = l - 3 \); and

\[ D_{l-2} \left[ (B_{l-2} B_{k}^t \ast \overrightarrow{a}) \oplus (e_{k-1} \ast x_k) \right] = (D_{l-2} B_{l-2} B_{k}^t \ast \overrightarrow{a}) \oplus (D_{l-2} e_{k-1} \ast x_k) \]

by the condition (2) of the lemma.

Thus

\[ \overrightarrow{c} = \{ (B_{l-2} R_{2}^t B_{l-2}^t) \ast [(B_{l-2} B_{k}^t \ast \overrightarrow{a}) \oplus (e_{k-1} \ast x_k)] \} \]

\[ \vee \{ D_{l-2} \ast [(B_{l-2} B_{k}^t \ast \overrightarrow{a}) \oplus (e_{k-1} \ast x_k)] \} \]

\[ = \{ (B_{l-2} R_{2}^t) \ast [(B_{l-2} B_{l-2} B_{k}^t \ast \overrightarrow{a}) \oplus (B_{l-2} e_{k-1} \ast x_k)] \} \]

\[ \vee \{ [D_{l-2} B_{l-2} B_{k}^t \ast \overrightarrow{a}) \oplus (D_{l-2} e_{k-1} \ast x_k)] \} \]

\[ = \{ (B_{l-2} R_{2}^t) \ast [(B_{k}^t \ast \overrightarrow{a}) \oplus (B_{l-2} e_{k-1} \ast x_k)] \} \]

\[ \vee \{ [O B_{k}^t \ast \overrightarrow{a}) \oplus (O x_k)] \}

\[ = (B_{l-2} R_{2}^t) \ast [(B_{k}^t \ast \overrightarrow{a}) \oplus (e_{k-1} \ast x_k)] \]
On the other hand
\[ \overrightarrow{c'} = B_{l-2} \ast \overrightarrow{b'} \]
\[ = B_{l-2} \ast \{(B_k^t \ast \overrightarrow{a}) \oplus (e_{k-1} \ast x')\} \]
\[ = (B_{l-2}R_2^t) \ast \{(B_k^t \ast \overrightarrow{a}) \oplus (e_{k-1} \ast x')\} \]

where \( x_k' = [r_{k-1,k+1}^t \ast \varphi(a_k)] \oplus [(-r_{k-1,k+1}^t) \ast (a_{k-1} \land a_{k+1})] \) with \( r_{k-1,k+1}^t = e_{k-1}^t R_1 e_{k+1} \).

To check \( \overrightarrow{c'} = \overrightarrow{c} \) we only need to prove that \( x_k' = x_k \).

\[
r_{k-1,k+1}^{(2)} = e_{k-1}^t R_2 e_{k+1} = e_{k-1}^t (B_l R_1 B_l^t + D_l) e_{k+1} = e_{k-1}^t B_l R_1 B_l^t e_{k+1} + e_{k-1}^t D_l e_{k+1} = e_{k-1}^t R_1 e_{k+1} = r_{k-1,k+1}^{(1)}
\]

\[ \overrightarrow{b'} = B_l \ast \overrightarrow{a} \] and \( l > k + 2 \) implies that \( b_{k-1} = a_{k-1} \), \( b_k = a_k \) and \( b_{k+1} = a_{k+1} \).

Then
\[
x_k = [r_{k-1,k+1}^{(2)} \ast \varphi(b_k)] \oplus [(-r_{k-1,k+1}^{(2)}) \ast (b_{k-1} \land b_{k+1})] = [r_{k-1,k+1}^{(1)} \ast \varphi(a_k)] \oplus [(-r_{k-1,k+1}^{(1)}) \ast (a_{k-1} \land a_{k+1})] = x_k'
\]

Now, let us study the case \( l = k + 2 \).

In this case, in contrast with the case \( l > k + 2 \), we don’t have \( B_l^t B_l = B_{l-2} B_{l-2} \). In fact, \( B_l^t B_{k+2} = (I - D_k) + Q_{k-1,k+1} \) and \( B_k B_l^t = (I - D_k) + Q_{k-1,k+1} + Q_{k+1,k-1} \) where \( Q_{\alpha,\beta} = e_{\alpha} e_{\beta}^t \) (i.e. all entries of \( Q_{\alpha,\beta} \) are zero except the entry \( (\alpha, \beta) \)). Thus \( B_l^t B_{k+2} < B_k B_l^t \).

\[
R_3 = (B_l^t R_2 B_k)^2 = [B_k^t (B_{k+2} R_1 B_{k+2}^t + D_{k+2}) B_k]^2 = (B_l^t B_{k+2} R_1 B_{k+2}^t B_k + B_l^t D_{k+2} B_k)^2 = (B_l^t B_{k+2} R_1 B_{k+2}^t B_k + D_k)^2 = B_k^t B_{k+2} R_1 B_{k+2}^t B_k + D_k
\]
because

\[ B_t^t B_{k+2} R_1 B_{k+2}^t B_k D_k \leq B_t^t B_{k+2} R_1 B_k B_t^t D_k = B_t^t B_{k+2} R_1 B_k O = O \]

and

\[ D_k B_t^t B_{k+2} R_1 B_{k+2}^t B_k = (B_t^t B_{k+2} R_1 B_{k+2}^t B_k D_k)^t = O \]

Thus

\[ R_3' = B_k R_2 B_k' + D_k \]

On the other hand, since \( B_t^t B_{k+2} = B_k B_t^t \), we have

\[ R_3 = B_k B_{k+2} R_1 B_{k+2}^t B_k B_t^t B_{k+2} R_1 B_{k+2}^t B_k + D_k \]

Thus \( R_3 = R_3' \).

Here we make use of the following inequalities:

\[ B_t^t B_k D_k B_t^t B_{k+2} \leq B_k B_t^t D_k B_t^t B_{k+2} = O \]

and

\[ R_3 = (B_t^t R_2 B_k)^2 \]

Since \( B_t^t B_{k+2} \leq B_k B_t^t \), we have

\[ R_3' \leq R_3 + B_k B_t^t B_k' + D_k \]

Therefore

\[ R_3' = B_k R_3 B_t^t B_k' + D_k \]

Thus \( R_3 = R_3' \).
$$
\overline{c} = R_3 * [(B_k^t \overline{b'} + (e_{k-1} * x_k)]
= R'_3 * [(B_k^t B_{k+2}^t \overline{d'}) + (e_{k-1} * x_k)]
= (B_k R'_2 B_k^t + D_k) * [(B_k^t B_{k+2}^t \overline{d'}) + (e_{k-1} * x_k)]
= \{(B_k R'_2 B_k^t) * [(B_k^t B_{k+2}^t \overline{d'}) + (e_{k-1} * x_k)]\}
\cup \{D_k * [(B_k^t B_{k+2}^t \overline{d'}) + (e_{k-1} * x_k)]\}
= (B_k R'_2 B_k^t) * [(B_k^t B_{k+2}^t \overline{d'}) + (e_{k-1} * x_k)]
$$

because, using lemma 7, we have

$$D_k * [(B_k^t B_{k+2}^t \overline{d'}) + (e_{k-1} * x_k)] = (D_k B_k^t B_{k+2}^t \overline{d'}) + (D_k e_{k-1} * x_k) = \emptyset$$

since $D_k B_k^t B_{k+2} \leq D_k B_k B_k^t = O$ and $D_k e_{k-1} = O$.

$$\overline{c}' = B_k * \overline{b'}
= B_k \{R'_2 * [(B_k^t \overline{d'}) + (e_{k-1} * x_k']\}
= (B_k R'_2) * [(B_k^t \overline{d'}) + (e_{k-1} * x_k]$$

**Lemma 8** Let $\overline{v}$ be an array with values in a monoid. If $v_{k+1} \leq v_{k-1}$ then $B_k^t \overline{v} = X_k \overline{v}$ where $X_k = B_k^t (I - D_{k+1})$.

**Proof.** $X_k$ differs from $B_k^t$ only in the entry $(k - 1, k + 1)$ which is 0 in $X_k$ and 1 in $B_k^t$. Thus we only need to check the equality $B_k^t \overline{v} = X_k \overline{v}$ for the $k - 1$ coordinate which is $v_{k-1} \lor v_{k+1}$ in $B_k^t \overline{v}$ and $v_{k-1}$ in $X_k \overline{v}$. Since, by hypothesis, $v_{k+1} \leq v_{k-1}$ we have the equality. $\blacksquare$

The $k + 1$ coordinate of $(B_k^t B_{k+2}^t \overline{d'}) + (e_{k-1} * x_k)$ is

$$e_{k+1}' * [(B_k^t B_{k+2}^t \overline{d'}) + (e_{k-1} * x_k)] = (e_{k+1}' B_k^t B_{k+2}^t \overline{d'}) + (e_{k+1} e_{k-1} * x_k)
= (e_{k+1}' \overline{d'}) + (0 * x_k)
= a_{k+1}$$

and the $k - 1$ coordinate of $(B_k^t B_{k+2}^t \overline{d'}) + (e_{k-1} * x_k)$ is

$$e_{k-1}' * [(B_k^t B_{k+2}^t \overline{d'}) + (e_{k-1} * x_k)] = (e_{k-1}' B_k^t B_{k+2}^t \overline{d'}) + (e_{k-1} e_{k-1} * x_k)
= [(e_{k-1}' + e_{k+1}') \overline{d'} + (1 * x_k)
= (a_{k-1} \lor a_{k+1}) + x_k$$
Thus, we may apply the previous lemma
\[
B_k^t \ast [(B_k^t B_{k+2} \ast \vec{a}') \oplus (e_{k-1} \ast x_k)] = X_k \ast [(B_k^t B_{k+2} \ast \vec{a}') \oplus (e_{k-1} \ast x_k)]
\]
\[
= (X_k B_k^t B_{k+2} \ast \vec{a}') \oplus (X_k e_{k-1} \ast x_k)
\]
\[
= (B_k^t \ast \vec{a}') \oplus (e_{k-1} \ast x_k)
\]

We leave it to the reader to check that \(X_k B_k^t B_{k+2} = B_k^t\). Therefore
\[
\vec{c} = (B_k^t R'_2 B_k) \ast [(B_k^t B_{k+2} \ast \vec{a}') \oplus (e_{k-1} \ast x_k)]
\]
\[
= (B_k^t R'_2) \ast [(B_k^t \ast \vec{a}') \oplus (e_{k-1} \ast x_k)]
\]
and thus \(\vec{c} = \vec{c}''\) if \(x_k = x'_k\) and we can check this using the same proof as was used in the case \(l > k + 2\).

Now, let us study the other relation \(\hat{T}_l \circ \hat{T}_k = \hat{T}_k \circ \hat{T}_{l-2}\) for \(l \geq k + 2\).
We know that \(\hat{T}_l' \circ \hat{T}_k' = \hat{T}_k' \circ \hat{T}_l'\) for \(k' \geq l' + 2\) (substituting \(k\) by \(l'\) and \(l\) by \(k'\) in the relation we have already proved).

We will use a mirror symmetry between these two relations. For that purpose, let us introduce the function *mirror symmetry* defined as follow:
\[
M_n : \mathcal{O}_n \rightarrow \mathcal{O}_n \quad (R, \vec{v}) \mapsto (S_n RS_n, S_n \ast \vec{v})
\]
where \(S_n\) is the \(n\)-dimensional square matrix defined by
\[
S_n := [s_{i,j}] \text{ with } s_{i,j} = 1 \text{ iff } i + j = n + 1.
\]
We have that \(M_n \circ M_n\) is the identity function on \(\mathcal{O}_n\) since \(S_n^2 = I\). Also, hearing in mind the relations \(S_{n+2} B_{n,k} S_n = B_{n,n+3-k}\) and \(S_{n+2} D_{n+2,k} S_{n+2} = D_{n+2,n+3-k}\), it is not too hard\(^5\) to check the commutativity of the following squares:
\[
\begin{align*}
\mathcal{O}_n \xrightarrow{\hat{T}_k} \mathcal{O}_{n+2} \xrightarrow{M_n} \mathcal{O}_n \\
\mathcal{O}_n \xrightarrow{\hat{T}_l} \mathcal{O}_{n+2} \xrightarrow{M_n} \mathcal{O}_n \\
\mathcal{O}_{n+2} \xrightarrow{\hat{T}_{l}} \mathcal{O}_n \xrightarrow{M_n} \mathcal{O}_{n+2}
\end{align*}
\]

\(^5\)It is only necessary to check the commutativity of the two first squares since the other two are obtained from these by a change of variables. For the second square it is useful to check first the identity \(x_{n+3-l}(S_{n+2} RS_{n+2}, S_{n+2} \ast \vec{v}) = x_l(R, \vec{v})\).
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Thus the outside square in the following diagram:

commutes (i.e. \( \hat{T}_l \circ \hat{T}_k = \hat{T}_k \circ \hat{T}_{l-2} \)) if and only if the inside square commutes (i.e. \( \hat{T}_{n+3-l} \circ \hat{T}_{n+3-k} = \hat{T}_{n+1-k} \circ \hat{T}_{n+3-l} \)) if and only if the inside square commutes \( \left( \hat{T}_l' \circ \hat{T}_k' = \hat{T}_{k'-2} \circ \hat{T}_{l'} \right) \) by making the change of variables: \( l' = n + 3 - l \) and \( k' = n + 3 - k \) which we know to be true since \( l \geq k + 2 \Rightarrow k' \geq l' + 2 \).

4. \( \hat{T}_k \circ \hat{T}_l = \hat{T}_{l-2} \circ \hat{T}_k \) for \( l \geq k + 2 \).

Let \( (R_1, \overrightarrow{a}) \in \mathcal{O}_n \), \( (R_2, \overrightarrow{b}) = \hat{T}_l(R_1, \overrightarrow{a}) \) and \( (R_3, \overrightarrow{c}) = \hat{T}_k(R_2, \overrightarrow{b}) = \hat{T}_k \circ \hat{T}_l(R_1, \overrightarrow{a}) \), and let \( (R_2', \overrightarrow{b'}) = \hat{T}_l(R_1, \overrightarrow{a}) \) and \( (R_3', \overrightarrow{c'}) = \hat{T}_{l-2}(R_2, \overrightarrow{b'}) = \hat{T}_{l-2} \circ \hat{T}_k(R_1, \overrightarrow{a}) \).

We want to see that \( (R_3', \overrightarrow{c'}) = (R_3, \overrightarrow{c}) \).

\[
R_3 = (B_k^t R_2 B_k)^2 = B_k^t R_2 B_k = B_k^t B_k^t R_1 B_k = B_k^t B_k^t B_1 B_1 B_k \tag{4}
\]

\[
R_3' = B_{l-2}^t R_2 B_{l-2} = B_{l-2}^t B_k^t R_1 B_k B_{l-2} = B_{l-2}^t B_k^t R_1 B_k B_{l-2} \tag{5}
\]

and since \( B_k B_{l-2} = B_k B_{l-2} \) we have \( R_3 = R_3' \).

Note that \( A \) means the transitive closure of \( A \).
(*) Now let us prove that $B_t^k B_t^l R_2 B_t^k B_k = B_t^k B_t^l R_2 B_t B_k$ and $B_{k-2} B_t^l R_2 B_k B_{k-2} = B_t^l R_2 B_k B_{k-2}$.

**Lemma 9** Let $A$ be a square matrix, then

$$B_t^k A B_k = B_t^k (I - D_k) A (I - D_k) B_k$$

**Proof.** For any natural number $n$, we have

$$B_t^k A B_k \geq (B_t^k A B_k)^n$$

$$= B_t^k A B_k B_t^k A B_k \cdots B_t^k A B_k B_t^k A B_k$$

$$\geq B_t^k A (I - D_k) A (I - D_k) \cdots (I - D_k) B_k$$

$$= B_t^k (I - D_k) A (I - D_k) \cdots (I - D_k) B_k$$

$$= B_t^k [(I - D_k) A (I - D_k)]^n B_k$$

therefore $B_t^k A B_k \geq B_t^k (I - D_k) A (I - D_k) B_k$ and thus

$$B_t^k A B_k \geq B_t^k (I - D_k) A (I - D_k) B_k$$

On the other hand,

$$B_t^k A B_k = B_t^k (I - D_k) A (I - D_k) B_k \leq B_t^k (I - D_k) A (I - D_k) B_k$$

and then $B_t^k A B_k \leq B_t^k (I - D_k) A (I - D_k) B_k$. 

**Corollary 10** If $(I - D_k) A (I - D_k) \leq (I - D_k) A (I - D_k)$ then $B_t^k A B_k = B_t^k A B_k$.

**Proof.**

$$B_t^k A B_k = B_t^k (I - D_k) A (I - D_k) B_k$$

$$\leq B_t^k (I - D_k) A (I - D_k) B_k$$

$$= B_t^k A B_k$$

On the other hand,

$$\overline{A} \geq A \Rightarrow B_t^k A B_k \geq B_t^k A B_k \Rightarrow B_t^k A B_k \geq B_t^k A B_k$$

\[ \blacksquare \]
Claim: \((I-D_k)B_l^t R_1 B_l (I-D_k) \leq (I-D_k)B_l^t R_1 B_l (I-D_k)\).

Proof.

\[
(I-D_k)B_l^t R_1 B_l (I-D_k) \geq [(I-D_k)B_l^t R_1 B_l (I-D_k)]^2 \\
= (I-D_k)B_l^t R_1 B_l (I-D_k)
\]

\[
(I-D_k)B_l^t R_1 B_l (I-D_k) = (I-D_k)B_l^t R_1 B_l (I-D_k) \\
= (I-D_k)B_l^t R_1 B_l (I-D_k)
\]

\[
(I-D_k)B_l^t R_1 B_l (I-D_k) \geq (I-D_k)B_l^t R_1 B_l (I-D_k) \\
= (I-D_k)B_l^t R_1 B_l (I-D_k)
\]

Thus

\[
(I-D_k)B_l^t R_1 B_l (I-D_k) \leq (I-D_k)B_l^t R_1 B_l (I-D_k)
\]

And therefore,

\[
(I-D_k)B_l^t R_1 B_l (I-D_k) = (I-D_k)B_l^t R_1 B_l (I-D_k) \\
\leq (I-D_k)B_l^t R_1 B_l (I-D_k)
\]

Using the same argument we can also prove the following claim.

Claim: \((I-D_{t-2})B_l^t R_1 B_k (I-D_{t-2}) \leq (I-D_{t-2})B_l^t R_1 B_k (I-D_{t-2})\).

Therefore we have \(B_l^t B_k^t R_2 B_l B_k = B_l^t B_k^t R_2 B_l B_k\) and \(B_{t-2}^t B_k^t R_2 B_k B_{t-2} = B_{t-2}^t B_k^t R_2 B_k B_{t-2}\).

Now let us see that \(\overline{c'} = \overline{c}\). We will check \(c'_i = c_i\) for each index \(i\).
Lemma 11 Let $R = [r_{i,j}]$ be a matrix with the properties E1, E2 and E3 (i.e. an equivalence relation) and $\overrightarrow{v}$ an array fixed by the action of $R$.

1. If $r_{i,j} = 1$ then $e_i^t \ast \overrightarrow{v} = e_j^t \ast \overrightarrow{v}$ (i.e. $v_i = v_j$).

Now let $\hat{R} = [\hat{r}_{i,j}] = (B_\alpha^t R B_\alpha)^2$.

2. If $\hat{r}_{i,\alpha-1} = 0$ then

$$e_i^t \{ \hat{R} \ast [(B_\alpha^t \ast \overrightarrow{v}) \oplus (e_{\alpha-1} \ast x)] \} = e_i^t B_\alpha^t \overrightarrow{v} = \begin{cases} v_i & \text{if } i < \alpha - 1 \\ v_{i-2} & \text{if } i > \alpha - 1 \end{cases}$$

3. $e_{\alpha-1}^t \{ \hat{R} \ast [(B_\alpha^t \ast \overrightarrow{v}) \oplus (e_{\alpha-1} \ast x)] \} = (v_{\alpha-1} \lor v_{\alpha+1}) \oplus x$.

Proof.

1. 

$$r_{i,j} = 1 \Rightarrow e_i^t \ast \overrightarrow{v} = e_i^t R \ast \overrightarrow{v} \quad (\text{since } \overrightarrow{v} = R \ast \overrightarrow{v})$$

$$\geq e_i^t R e_j e_j^t \ast \overrightarrow{v} \quad (\text{since } e_j e_j^t = D_j \leq I)$$

$$= e_j^t \ast \overrightarrow{v} \quad (\text{since } e_i R e_j^t = r_{i,j} = 1)$$

Since $R$ is symmetric, we have also $e_j^t R \ast \overrightarrow{v} \geq e_i^t R \ast \overrightarrow{v}$.

2. 

$$\hat{r}_{i,\alpha-1} = 0 \Rightarrow e_i^t \hat{R} = e_i^t \hat{R}[(I - D_{\alpha-1}) + D_{\alpha-1}]$$

$$= e_i^t \hat{R}(I - D_{\alpha-1}) + e_i^t \hat{R} D_{\alpha-1}$$

$$= e_i^t \hat{R}(I - D_{\alpha-1}) + e_i^t \hat{R} e_{\alpha-1} e_{\alpha-1}^t$$

Thus

$$e_i^t \hat{R} [(B_\alpha^t \ast \overrightarrow{v}) \oplus (e_{\alpha-1} \ast x)] = e_i^t \hat{R}(I - D_{\alpha-1}) [(B_\alpha^t \ast \overrightarrow{v}) \oplus (e_{\alpha-1} \ast x)]$$

$$= e_i^t \hat{R} \{ [(I - D_{\alpha-1}) B_\alpha^t \ast \overrightarrow{v}] \}$$

$$\oplus [(I - D_{\alpha-1}) e_{\alpha-1} \ast x]$$

$$= e_i^t \hat{R}(I - D_{\alpha-1}) B_\alpha^t \ast \overrightarrow{v}$$

$$= e_i^t B_\alpha^t R B_\alpha^t B_\alpha^t R B_\alpha^t (I - D_{\alpha-1}) B_\alpha^t \ast \overrightarrow{v}$$

$$\leq e_i^t B_\alpha^t R B_\alpha^t B_\alpha^t R \ast \overrightarrow{v}$$

$$= e_i^t B_\alpha^t R B_\alpha^t \ast \overrightarrow{v}$$
Since
\[e_i^t B^t_a R B_a e_{a-1} \leq e_i^t \hat{R} e_{a-1} = 0 \Rightarrow e_i^t B^t_a R B_a = e_i^t B^t_a R B_a (I - D_{a-1})\]
we have that
\[e_i^t \hat{R} \ast [(B^t_a \ast \overrightarrow{v}) \oplus (e_{a-1} \ast x)] \leq e_i^t B^t_a R B_a B^t_a \ast \overrightarrow{v} = e_i^t B^t_a R B_a (I - D_{a-1}) B^t_a \ast \overrightarrow{v} \leq e_i^t B^t_a R \ast \overrightarrow{v} = e_i^t B^t_a \ast \overrightarrow{v}\]

On the other hand, we have
\[e_i^t \hat{R} \ast [(B^t_a \ast \overrightarrow{v}) \oplus (e_{a-1} \ast x)] \geq e_i^t \hat{R} \ast (B^t_a \ast \overrightarrow{v}) \geq e_i^t B^t_a \ast \overrightarrow{v}\]

3.
\[e_{a-1}^t \hat{R} \ast [(B^t_a \ast \overrightarrow{v}) \oplus (e_{a-1} \ast x)] \geq e_{a-1}^t \hat{R} \ast [(B^t_a \ast \overrightarrow{v}) \oplus (e_{a-1} \ast x)] = \left( e_{a-1}^t B^t_a \ast \overrightarrow{v} \right) \oplus (e_{a-1}^t e_{a-1} \ast x) = (v_{a-1} \lor v_{a+1}) \oplus x\]

On the other hand
\[e_{a-1}^t \hat{R} \ast [(B^t_a \ast \overrightarrow{v}) \oplus (e_{a-1} \ast x)] \leq (e_{a-1}^t \hat{R} B^t_a \ast \overrightarrow{v}) \oplus (e_{a-1}^t \hat{R} e_{a-1} \ast x) = \left( e_{a-1}^t \hat{R} B^t_a \ast \overrightarrow{v} \right) \oplus x \overset{(*)}{=} (v_{a-1} \lor v_{a+1}) \oplus x\]

\((*)\) \[e_{a-1}^t B^t_a \ast \overrightarrow{v} \geq e_{a-1}^t B^t_a \ast \overrightarrow{v} = v_{a-1} \lor v_{a+1}, \text{ and on the other hand}\]

\[e_{a-1}^t \hat{R} B^t_a \ast \overrightarrow{v} = \left( e_{a-1}^t B^t_a R B_a B^t_a R B_a B^t_a \ast \overrightarrow{v} \right) \leq e_{a-1}^t B^t_a R B_a B^t_a R (I + B_a D_{a-1} B^t_a) \ast \overrightarrow{v} = \left( e_{a-1}^t B^t_a R B_a B^t_a R \ast \overrightarrow{v} \right) \lor \left( e_{a-1}^t B^t_a R B_a B^t_a e_{a-1} e_{a-1} B^t_a \ast \overrightarrow{v} \right) = (e_{a-1}^t B^t_a R \ast \overrightarrow{v}) \lor \left( e_{a-1}^t B^t_a R B_a B^t_a e_{a-1} e_{a-1} B^t_a \ast \overrightarrow{v} \right) \lor \left( e_{a-1}^t B^t_a \ast \overrightarrow{v} \right) = e_{a-1}^t B^t_a \ast \overrightarrow{v}\]
Lemma 12 Let \( R = [r_{i,j}] \) be a matrix with the properties E1, E2 and E3 (i.e. an equivalence relation) and let \( \hat{R} = [\hat{r}_{i,j}] = (B^t \alpha RB_\alpha)^2 \). Then:

(a) For any \( j \neq \alpha - 1, \hat{r}_{\alpha-1,j} = r_{\alpha-1,\hat{\alpha}(j)} + r_{\alpha+1,\hat{\alpha}(j)} \) where \( \hat{\alpha}(j) = \begin{cases} j & \text{if } j < \alpha - 1, \\ j + 2 & \text{if } j > \alpha - 1; \end{cases} \)

(b) If \( i \neq \alpha - 1 \) and \( j \neq \alpha - 1 \) then \( \hat{r}_{i,j} \geq r_{\hat{\alpha}(i),\hat{\alpha}(j)}; \)

(c) If \( i \neq \alpha - 1, j \neq \alpha - 1 \) and \( \hat{r}_{i,\alpha-1} = 0 \) (or \( \hat{r}_{j,\alpha-1} = 0 \) then \( \hat{r}_{i,j} = r_{\hat{\alpha}(i),\hat{\alpha}(j)} \).

Proof.

(a)
\[
\hat{r}_{\alpha-1,j} = e^t_{\alpha-1} \hat{R} e_j \\
= e^t_{\alpha-1}(B^t \alpha RB_\alpha)^2 e_j \\
\geq e^t_{\alpha-1} B^t \alpha RB_\alpha e_j \\
= (e^t_{\alpha-1} + e^t_{\alpha+1}) RB_\alpha e_j \\
= r_{\alpha-1,\hat{\alpha}(j)} + r_{\alpha+1,\hat{\alpha}(j)}
\]

On the other hand
\[
\hat{r}_{\alpha-1,j} = e^t_{\alpha-1} B^t \alpha RB_\alpha B^t \alpha RB_\alpha e_j \leq e^t_{\alpha-1} B^t \alpha R(I + B_\alpha D_{\alpha-1} B^t \alpha) RB_\alpha e_j = e^t_{\alpha-1} B^t \alpha RB_\alpha e_j + e^t_{\alpha-1} B^t \alpha RB_\alpha e_{\alpha-1} e^t_{\alpha-1} B^t \alpha RB_\alpha e_j = e^t_{\alpha-1} B^t \alpha RB_\alpha e_j = r_{\alpha-1,\hat{\alpha}(j)} + r_{\alpha+1,\hat{\alpha}(j)}
\]

(b)
\[
\hat{r}_{i,j} = e^t_i (B^t \alpha RB_\alpha)^2 e_j \\
\geq e^t_i B^t \alpha RB_\alpha e_j = e^t_i \hat{R} e_{\hat{\alpha}(j)} = r_{\hat{\alpha}(i),\hat{\alpha}(j)}
\]

(c)
\[
\hat{r}_{i,j} = e^t_i (B^t \alpha RB_\alpha)^2 e_j = e^t_i B^t \alpha RB_\alpha e_j \leq e^t_i B^t \alpha R(I + B_\alpha D_{\alpha-1} B^t \alpha) RB_\alpha e_j = e^t_i B^t \alpha RB_\alpha e_j + e^t_i B^t \alpha RB_\alpha e_{\alpha-1} e^t_{\alpha-1} B^t \alpha RB_\alpha e_j = e^t_i B^t \alpha RB_\alpha e_j = r_{\hat{\alpha}(i),\hat{\alpha}(j)}
\]
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\[
(*) \quad e_i^t B_i^t R B_o e_{\alpha-1} \leq e_i^t (B_i^t R B_o)^2 e_{\alpha-1} = \dot{r}_{i,\alpha-1} = 0 \quad \text{(or } e_{\alpha-1}^t B_i^t R B_o e_j \leq \dot{r}_{\alpha-1,j} = 0)\).
\]

And thus, by (b), we have \(\dot{r}_{i,j} = r_{\alpha(i),\alpha(j)}\).

\[\text{Convention: } R_1 = [r_{i,j}], \quad R_2 = [\dot{r}_{i,j}] = (B_i^t R_1 B_i)^2, \quad R_3 = [\ddot{r}_{i,j}] = (B_i^t R_2 B_k)^2, \quad R'_2 = [\dot{r}'_{i,j}] = (B_i^t R_1 B_k)^2 \quad \text{and} \quad R'_3 = [\dot{r}'_{i,j}] = (B_i^t R_2 B_k)^2.\]

By lemma 11, we have:

\[
c_{k-1} = e_k^t R_3 [\{ (B_i^t \ast \overrightarrow{b} ) \oplus (e_{l-1} \ast x_l ) \}]
= (b_{k-1} \vee b_{k+1}) \oplus x_k
\]

where \(x_k = [\dot{r}_{k-1,k+1} \ast \varphi(b_k)] \oplus (\dot{r}_{k-1,k+1} \ast (b_{k-1} \wedge b_{k+1}))\) with \(\dot{r}_{k-1,k+1} = e_{k-1}^t R_2 e_{k+1}\).

By lemma 12, \(\dot{r}_{k-1,l-3} = 0 \Rightarrow \dot{r}_{k-1,l-1} = 0 \quad \text{and} \quad \dot{r}_{k+1,l-1} = 0\). And then, by lemma 11:

\[
b_{k-1} = e_{k-1}^t R_2 [\{ (B_i^t \ast \overrightarrow{a} ) \oplus (e_{l-1} \ast x_l ) \}]
= a_{k-1}
\]

and

\[
b_{k+1} = e_{k+1}^t R_2 [\{ (B_i^t \ast \overrightarrow{a} ) \oplus (e_{l-1} \ast x_l ) \}]
= a_{k+1}
\]

Thus

\[
c_{k-1} = (a_{k-1} \vee a_{k+1}) \oplus x_k
\]

Since \(R'_3 = R_3, \quad \dot{r}'_{k-1,l-3} = \ddot{r}_{k-1,l-3} = 0\). Then, by lemma 11:

\[
c'_{k-1} = e_{k-1}^t R'_3 [\{ (B_i^t \ast \overrightarrow{b} ) \oplus (e_{l-3} \ast x_{l-2} ) \}]
= b'_{k-1}
= e_{k-1}^t R'_2 [\{ (B_i^t \ast \overrightarrow{a} ) \oplus (e_{k-1} \ast x_k' ) \}]
= (a_{k-1} \vee a_{k+1}) \oplus x_k'
\]

where

\[
x_k' = [r_{k-1,k+1} \ast \varphi(a_k)] \oplus (\dot{r}_{k-1,k+1} \ast (a_{k-1} \wedge a_{k+1}))
\]

with \(r_{k-1,k+1} = e_{k-1}^t R_1 e_{k+1}\). 

Thus $c_{k-1} = c'_{k-1}$ if $x_k = x'_k$.

\[
x_k = \left[ \dot{r}_{k-1,k+1} * \varphi(b_k) \right] \oplus \left[ (\neg \dot{r}_{k-1,k+1}) * (b_{k-1} \land b_{k+1}) \right]
\]

\[
= \left[ \dot{r}_{k-1,k+1} * \varphi(b_k) \right] \oplus \left[ (\neg \dot{r}_{k-1,k+1}) * (a_{k-1} \land a_{k+1}) \right]
\]

\[
\Rightarrow \left[ \dot{r}_{k-1,k+1} * \varphi(b_k) \right] \oplus \left[ (\neg \dot{r}_{k-1,k+1}) * (a_{k-1} \land a_{k+1}) \right]
\]

\[
\Rightarrow \left[ r_{k-1,k+1} * \varphi(a_k) \right] \oplus \left[ (\neg r_{k-1,k+1}) * (a_{k-1} \land a_{k+1}) \right]
\]

\[
= x'_k
\]

(*) If $\dot{r}_{k-1,k+1} = 0$ then $\dot{r}_{k-1,k+1} * \varphi(b_k) = \emptyset = \dot{r}_{k-1,k+1} * \varphi(a_k)$.

If $\dot{r}_{k-1,k+1} = 1$ then $\dot{r}_{k-1,k+1} = 0$ by the properties T1 and T2 of $R_2$.

Thus $b_k = c'_{k}R_2 * [(B_1^* \land \alpha^*) \oplus (e_{t-1} \land \alpha_{t})] = a_k$.

(†) $\dot{r}_{k-1,l-3} = 0 \Rightarrow \dot{r}_{k-1,l-1} = 0$ and $\dot{r}_{k+1,l-1} = 0$ by lemma 11.

$\dot{r}_{k-1,l-1} = 0 \Rightarrow \dot{r}_{k-1,k+1} = r_{k-1,k+1}$ by lemma 11.

We now prove $c_{k-1} = c'_{k-1}$ for the case $\dot{r}_{k-1,l-3} = 1$.

By lemma 12,

\[
\dot{r}_{k-1,l-3} = 1 \Rightarrow \dot{r}_{k-1,l-1} = 1 \text{ or } r_{k-1,l-1} = 1 \text{ or } r_{k-1,l+1} = 1 \text{ or } r_{k+1,l-1} = 1 \text{ or } r_{k+1,l+1} = 1
\]

Now we have, in theory, 15 cases to study:

\[
(r_{k-1,l-1}, r_{k-1,l+1}, r_{k+1,l-1}, r_{k+1,l+1}) \in \{0, 1\}^4 \setminus \{(0, 0, 0, 0)\}
\]

but we can exclude the cases $(r_{k-1,l-1}, r_{k-1,l+1}, r_{k+1,l-1}, r_{k+1,l+1}) = (1, 1, 1, 0)$, $(1, 1, 0, 1)$, $(1, 0, 1, 1)$ and $(0, 1, 1, 1)$, because $R_1 = [r_{i,j}]$ satisfies the properties E1, E2 and E3, and the case $(r_{k-1,l-1}, r_{k-1,l+1}, r_{k+1,l-1}, r_{k+1,l+1}) = (1, 0, 0, 1)$, because the inequalities $k - 1 < k + 1 \leq l - 1 < l + 1$ and property T2 of $R_1$ imply that if $r_{k-1,l-1} = r_{k+1,l+1} = 1$ then $r_{k+1,l-1} = 1$ and $r_{k-1,l+1} = 1$.

Thus, we have the following ten cases to study:
Using the properties of $R_1 = [r_{i,j}]$ and $R_2 = [\dot{r}_{i,j}]$, and the relations between them ($R_2 = (B_l^2 R_1 B_l)^2$), it is easy to prove the following statements:

1. If $r_{k-1,l-1} = 1$ or $r_{k-1,l+1} = 1$ or $r_{k+1,l-1} = 1$ or $r_{k+1,l+1} = 1$ then:
   \[ r_{k-1,k+1} = 1 \Leftrightarrow (r_{k-1,l-1} = r_{k+1,l-1} \text{ and } r_{k-1,l+1} = r_{k+1,l+1}); \]

2. If $r_{k-1,l-1} = 1$ or $r_{k-1,l+1} = 1$ or $r_{k+1,l-1} = 1$ or $r_{k+1,l+1} = 1$ then:
   \[ r_{l-1,l+1} = 1 \Leftrightarrow (r_{k-1,l-1} = r_{k+1,l-1} \text{ and } r_{k+1,l-1} = r_{k+1,l+1}); \]

3. $\dot{r}_{k-1,k+1} = r_{k-1,k+1} + r_{k-1,l+1}r_{k+1,l-1}$;
4. $\dot{r}_{k-1,l-1} = r_{k-1,l-1} + r_{k-1,l+1}$;
5. $\dot{r}_{k+1,l-1} = r_{k+1,l-1} + r_{k+1,l+1}$.

Also, using lemma 11, we can easily check:
$c_{k-1} = (b_{k-1} \lor b_{k+1}) \oplus x_k = \begin{cases} b_{k-1} \oplus b_{k+1} & \text{if } \hat{r}_{k-1,k+1} = 0 \\ b_{k-1} \oplus \varphi(b_k) & \text{if } \hat{r}_{k-1,k+1} = 1 \end{cases}$

\[ b_{k-1} = \begin{cases} a_{k-1} & \text{if } \hat{r}_{k-1,l-1} = 0 \\ b_{l-1} & \text{if } \hat{r}_{k-1,l-1} = 1 \end{cases} \]

\[ b_{k+1} = \begin{cases} a_{k+1} & \text{if } \hat{r}_{k+1,l-1} = 0 \\ b_{l-1} & \text{if } \hat{r}_{k+1,l-1} = 1 \end{cases} \]

$b_{l-1} = (a_{l-1} \lor a_{l+1}) \oplus x_l = \begin{cases} a_{l-1} \oplus a_{l+1} & \text{if } r_{l-1,l+1} = 0 \\ a_{l-1} \oplus \varphi(a_l) & \text{if } r_{l-1,l+1} = 1 \end{cases}$

Since $\hat{r}_{k-1,l-3} = 1 \Rightarrow k - 1 - (l - 3) \in 2\mathbb{Z}$ we have $\hat{r}_{k,l-1} = 0$ and then $b_k = a_k$.

With this, we can construct the following table:

| $r_{k-1,l-1}$ | $r_{k-1,l+1}$ | $r_{k+1,l-1}$ | $r_{k+1,l+1}$ | $r_{k-1,k+1}$ | $r_{k-1,k+1}$ |
|---------------|---------------|---------------|---------------|---------------|---------------|
| 1             | 0             | 0             | 0             | 1             | 1             |
| 0             | 1             | 0             | 1             | 0             | 1             |
| 1             | 1             | 0             | 0             | 0             | 0             |
| 0             | 1             | 1             | 1             | 0             | 0             |
| 0             | 0             | 0             | 0             | 0             | 0             |
| 0             | 0             | 1             | 0             | 0             | 0             |
| 0             | 0             | 0             | 1             | 0             | 0             |
| 0             | 0             | 0             | 0             | 0             | 0             |
| 0             | 0             | 1             | 0             | 0             | 0             |

We can construct an analogous table for the value $c'_{k-1}$. All we need to know is that:

1. If $r_{k-1,l-1} = 1$ or $r_{k-1,l+1} = 1$ or $r_{k+1,l-1} = 1$ or $r_{k+1,l+1} = 1$ then:

\[ r_{k-1,k+1} = 1 \Leftrightarrow (r_{k-1,l-1} = r_{k+1,l-1} \text{ and } r_{k-1,l+1} = r_{k+1,l+1}) \]
2. If \( r_{k-1,l-1} = 1 \) or \( r_{k-1,l+1} = 1 \) or \( r_{k+1,l-1} = 1 \) or \( r_{k+1,l+1} = 1 \) then:

\[
r_{l-1,l+1} = 1 \iff (r_{k-1,l-1} = r_{k-1,l+1} \text{ and } r_{k+1,l-1} = r_{k+1,l+1});
\]

3. \( \hat{r}_l^{t-3,l-1} = r_{l-1,l+1} + r_{k-1,l+1} r_{k+1,l-1} \);

4. \( \hat{r}_k^{t-1,l-3} = r_{k-1,l-1} + r_{k+1,l-1} \);

5. \( \hat{r}_k^{t-1,l-1} = r_{k-1,l+1} + r_{k+1,l+1} \).

\[
\hat{r}_k^{t-1,l-3} = \hat{r}_l^{t-3,l-1} = 1 \Rightarrow \\
c_k^{t-1} = c_{l-3}^{t-3} = (b_{l-3}^{t-3} \lor b_{l-1}^{t-1}) \oplus x_{l-2}^{t-2} = \\
\begin{cases}
  b_{l-3}^{t-3} \oplus b_{l-1}^{t-1} & \text{if } \hat{r}_l^{t-3,l-1} = 0 \\
  b_{l-3}^{t-3} \oplus \varphi(b_{l-1}^{t-1}) & \text{if } \hat{r}_l^{t-3,l-1} = 1
\end{cases}
\]

\[
b_{l-3}^{t-3} = \\
\begin{cases}
  a_{l-1} & \text{if } \hat{r}_l^{t-1,l-3} = 0 \\
  b_{l-1}^{t-1} & \text{if } \hat{r}_l^{t-1,l-3} = 1
\end{cases}
\]

\[
b_{l-1}^{t-1} = \\
\begin{cases}
  a_{l+1} & \text{if } \hat{r}_l^{t-1,l-1} = 0 \\
  b_{l-1}^{t-1} & \text{if } \hat{r}_l^{t-1,l-1} = 1
\end{cases}
\]

\[
b_{l-1}^{t-1} = (a_{k-1} \lor a_{k+1}) \oplus x_k = \\
\begin{cases}
  a_{k-1} \oplus a_{k+1} & \text{if } r_{k-1,k+1} = 0 \\
  a_{k-1} \lor \varphi(a_k) & \text{if } r_{k-1,k+1} = 1
\end{cases}
\]

and \( b_{l-2}^{t-2} = a_l \) since \( \hat{r}_k^{t-1,l-3} = 1 \Rightarrow \hat{r}_k^{t-1,l-2} = 0. \)
In order to facilitate the comparison of \( c_{k-1} \) and \( c'_{k-1} \) we will substitute each \( a_i \) (with \( i \in \{k+1, l-1, l+1\} \)) appearing in the expressions of \( c_{k-1} \) and \( c'_{k-1} \) by \( a_j \) where \( j \in \{k-1, k+1, l-1, l+1\} \) is the smallest index such that \( r_{i,j} = 1 \).

The results can be seen in the following table:

| \( r_{k-1,l-1} \) | \( r_{k-1,l+1} \) | \( r_{k+1,l-1} \) | \( r_{k+1,l+1} \) | \( r_{k-1,k+1} \) | \( r_{l-1,l+1} \) |
|-------------------|-------------------|-------------------|-------------------|-------------------|-------------------|
| 1                 | 1                 | 1                 | 1                 | 1                 | 1                 |
| 1                 | 0                 | 1                 | 0                 | 1                 | 0                 |
| 0                 | 1                 | 0                 | 1                 | 1                 | 0                 |
| 1                 | 0                 | 1                 | 0                 | 0                 | 1                 |
| 0                 | 0                 | 1                 | 0                 | 0                 | 0                 |
| 1                 | 0                 | 0                 | 0                 | 0                 | 0                 |
| 0                 | 0                 | 1                 | 0                 | 0                 | 0                 |
| 0                 | 0                 | 0                 | 1                 | 0                 | 0                 |

| \( a_{k+1} \) | \( a_{l-1} \) | \( a_{l+1} \) | \( c_{k-1} \) | \( c'_{k-1} \) |
|---------------|---------------|---------------|--------------|--------------|
| \( a_{k-1} \) | \( a_{k-1} \) | \( a_{k-1} \) | \( a_{k-1} \oplus \varphi(a_l) \oplus \varphi(a_k) \) | \( a_{k-1} \oplus \varphi(a_k) \oplus \varphi(a_l) \) |
| \( a_{k-1} \) | \( a_{k-1} \) | \( a_{l+1} \) | \( a_{k-1} \oplus a_{l+1} \oplus \varphi(a_k) \) | \( a_{k-1} \oplus \varphi(a_k) \oplus a_{l+1} \) |
| \( a_{k-1} \) | \( a_{l-1} \) | \( a_{k-1} \) | \( a_{l-1} \oplus a_{k-1} \oplus \varphi(a_k) \) | \( a_{l-1} \oplus a_{k-1} \oplus \varphi(a_k) \) |
| \( a_{k+1} \) | \( a_{k-1} \) | \( a_{k-1} \) | \( a_{k-1} \oplus \varphi(a_l) \oplus a_{k+1} \) | \( a_{k-1} \oplus a_{k+1} \oplus \varphi(a_l) \) |
| \( a_{k+1} \) | \( a_{k+1} \) | \( a_{k+1} \) | \( a_{k+1} \oplus a_{k+1} \oplus \varphi(a_l) \) | \( a_{k+1} \oplus a_{k+1} \oplus \varphi(a_l) \) |
| \( a_{k+1} \) | \( a_{k-1} \) | \( a_{l+1} \) | \( a_{k-1} \oplus a_{l+1} \oplus a_{k+1} \) | \( a_{k-1} \oplus a_{k+1} \oplus a_{l+1} \) |
| \( a_{k+1} \) | \( a_{l-1} \) | \( a_{k-1} \) | \( a_{l-1} \oplus a_{k-1} \oplus a_{k+1} \) | \( a_{l-1} \oplus a_{k-1} \oplus a_{k+1} \) |

We can easily see that \( c_{k-1} = c'_{k-1} \) for all rows except the sixth row where \( c_{k-1} = c'_{k-1} \) if \( a_k = a_l \). But, since \( r_{k-1,l+1} = r_{k+1,l-1} = 1 \) and \( r_{k-1,k+1} = 0 \) in this case, we have, by the topological properties T2 and T3, that \( r_{k,l} = 1 \) and then \( a_k = a_l \).

Now, to see that \( c_{l-3} = c'_{l-3} \) we proceed in the same way as we did to show \( c_{k-1} = c'_{k-1} \) for the case \( r_{k-1,l-3} = 0 \). For the case \( r_{k-1,l-3} = 1 \), we have \( c_{l-3} = c_{k-1} = c'_{k-1} = c'_{l-3} \) (by lemma 11).
Representations of non-singular planar tangles by operators

For another generic index $i$, we have:

If $\dot{r}_{i,k-1} = 1$ then $c_i = c_{k-1} = c'_{k-1} = c'_i$.

If $\dot{r}_{i,l-3} = 1$ then $c_i = c_{l-3} = c'_{l-3} = c'_i$.

If $\dot{r}_{i,k-1} = \dot{r}_{i,l-3} = 0$ then we have:

$$c_i = e_i^t R_3 \ast (B_k^t \ast \vec{b}) \oplus (e_{k-1} \ast x_k) = e_i^t B_k^t \ast \vec{b} = b_{k(i)}$$

where $k(i) = \begin{cases} i & \text{if } i < k - 1 \\ i - 2 & \text{if } i > k - 1 \end{cases}$

$\dot{r}_{i,l-3} = 0 \Rightarrow \dot{r}_{k(i),l-1} = 0$, by lemma 11. Thus

$$c_i = b_{k(i)} = e_{k(i)}^t R_2 \ast (B_i^t \ast \vec{a} \oplus (e_{l-1} \ast x_l)) = e_{k(i)}^t B_i^t \ast \vec{a} = e_i^t B_k^t B_i^t \ast \vec{a}$$

$$c'_i = e_i^t R_3' \ast (B_{l-2}^t \ast \vec{b}') \oplus (e_{l-3} \ast x_{l-2}) = e_i^t B_{l-2}^t \ast \vec{b}' = b'_{l-2(i)}$$

$\dot{r}'_{i,k-1} = \dot{r}_{i,k-1} = 0 \Rightarrow \dot{r}'_{l-2(i),k-1} = 0$, by lemma [1]. Thus

$$c'_i = b'_{l-2(i)} = e_{l-2(i)}^t R_2 \ast (B_k^t \ast \vec{a} \oplus (e_{k-1} \ast x_k)) = e_{l-2(i)}^t B_k^t \ast \vec{a} = e_i^t B_{l-2} B_k^t \ast \vec{a}$$

Since $B_{l-2}^t B_k^t = B_k^t B_l^t$ we have $c_i = c'_i$.

4 Systems of non-singular planar curves

Now, we are going to study this representation in the particular case of systems of non-singular planar curves. We mean by a system of non-singular planar curves an immersion in the plane of a finite number of disjoint circles which may be regarded as a morphism from the empty partition of the line to itself (i.e. an element of $\text{hom} (\emptyset, \emptyset)$).

Any morphism $t \in \text{hom} (\emptyset, \emptyset)$ is a word made of generators $\hat{t}_{n,k}$ and $\check{t}_{n,k}$. Let us make the substitution: $\hat{t}_{n,k} \mapsto (2, 2k-n-3)$ and $\check{t}_{n,k} \mapsto (-2, 2k-n-3)$ for each word. The number $2k - n - 3$ counts the number of strings on the
left of the local maximum (minimum) minus the number of strings on the right.

The following lemma shows that no information is lost after this substitution. But first, let us introduce a notation for the following sets:

\[ S_+ := \{(2, n) : n \in 2\mathbb{Z}\} \]
and
\[ S_- := \{(-2, n) : n \in 2\mathbb{Z}\} \]

**Lemma 13** After the substitution: \( \hat{t}_{n,k} \mapsto (2, 2k-n-3) \) and \( \hat{t}_{n,k} \mapsto (-2, 2k-n-3) \) of a morphism \( t \in \text{hom}(\emptyset, \emptyset) \) we get a word \((c_1, d_1)(c_2, d_2)...(c_n, d_n)\) in \( S_+ \cup S_- \) satisfying the following condition:

**C.** For any index \( i \):

- if \( c_i = 2 \) then \( |d_i| \leq -(\sum_{j<i} c_j) - 2 = \sum_{j>i} c_j \),
- if \( c_i = -2 \) then \( |d_i| \leq -\sum_{j<i} c_j = (\sum_{j>i} c_j) - 2 \).

Also, if a word in \( S_+ \cup S_- \) satisfies this condition then there exists a unique morphism \( t \in \text{hom}(\emptyset, \emptyset) \) that becomes this word after the substitution: \( \hat{t}_{n,k} \mapsto (2, 2k-n-3) \) and \( \hat{t}_{n,k} \mapsto (-2, 2k-n-3) \).

**Proof.** Observing that each \((c_i, d_i)\) substitutes one generator \( \hat{t}_{n_i,k_i} \) or \( \hat{t}_{n_i,k_i} \), it is easy to see that \(-\sum_{j<i} c_j\) is the number of points at the bottom of the generator \( (\hat{t}_{n_i,k_i} \text{ or } \hat{t}_{n_i,k_i}) \) and that \( \sum_{j>i} c_j \) is number of points at the top of the generator. The condition follows naturally from this fact.

For the second part of the lemma, we make the inverse substitution \((c_i, d_i) = (2, k) \mapsto \hat{t}_{n',k'}\) with \( n' = (\sum_{j>i} c_j) + 1 \) and \( k' = \frac{k+n'+3}{2} \), and \((c_i, d_i) = (-2, k) \mapsto \hat{t}_{n',k'}\) with \( n' = (\sum_{j>i} c_j) - 1 \) and \( k' = \frac{k+n'+3}{2} \).

So we have a new language for morphisms in \( \text{hom}(\emptyset, \emptyset) \) and in this language the local relations become:

1. \( ...(c_i, d_i)(-2, k)(2, k+2)(c_{i+3}, d_{i+3})... = ...(c_i, d_i)(c_{i+3}, d_{i+3})... = (c_i, d_i)(-2, k)(2, k-2)(c_{i+3}, d_{i+3})... \) (i.e. \( \hat{t}_{k'+1} \circ \hat{t}_{k'} = \text{id} = \hat{t}_{k'-1} \circ \hat{t}_{k'} \));

2. \( ...(2, k)(2, l)... = ...(2, l+2)(2, k+2)... \) for \( k \leq l - 2 \) (i.e. \( \hat{t}_{k'} \circ \hat{t}_{l'} = \hat{t}_{l'+2} \circ \hat{t}_{k'} \) for \( l' \geq k' + 2 \));

3. \( ...(2, k)(2, l)... = ...(2, l-2)(-2, k+2)... \) for \( k \leq l - 4 \) (i.e. \( \hat{t}_{k'} \circ \hat{t}_{l'} = \hat{t}_{l'-2} \circ \hat{t}_{k'} \) for \( l' \geq k' + 2 \)).
3.2 \((2, k)(-2, l)\) = \((2, k-2)(-2, l+2)\) for \(k \leq l\) (i.e. \(\hat{t}_{k'} \circ \tilde{t}_{l'} = \tilde{t}_{l'+2} \circ \hat{t}_{k'}\) for \(l' \geq k' + 2\))

4 \((-2, k)(-2, l)\) = \((-2, l-2)(-2, k-2)\) for \(k \leq l-2\) (i.e. \(\hat{t}_{k-2} \circ \tilde{t}_{k'} = \tilde{T}_{k'} \circ \hat{T}_{l'}\) for \(l' \geq k' + 2\)).

Note that, by the previous lemma, these relations preserve the condition C.

**Proposition 14** Any word in \(S_+ \cup S_-\) satisfying the condition C is equivalent, by the previous relations, to a word of symbols \((2, 0)\) and \((-2, 0)\).

**Proof.** We use the following algorithm to transform any word in \(S_+ \cup S_-\) satisfying the condition C into a word of symbols \((2, 0)\) and \((-2, 0)\).

**ALGORITHM:**

**input:** Take a word in \(S_+ \cup S_-\) satisfying the condition C;

**step 1.** Apply the relations 3.1 \((2, l-2)(-2, k+2)\) = \((2, k)(2, l)\) for \(k \leq l-4\) and 3.2 \((2, k)(-2, l)\) = \((2, l+2)(2, k-2)\) for \(k \leq l\) so as to put all the symbols of the form \((-2, k)\) on the left side of the word and all the symbols of the form \((2, k)\) on the right;

**step 2.** Use the relations 2 \((2, k)(2, l)\) = \((2, l+2)(2, k+2)\) for \(k \leq l-2\) and 4 \((-2, k)(-2, l)\) = \((-2, l-2)(-2, k-2)\) for \(k \leq l-2\) to order in semi-decreasing order (according to \(d_i\)) each block consisting of \((c_i, d_i)\) of the form \((2, k)\) or \((-2, k)\);

**step 3.** If there exists a sequence of the type \((-2, k)(2, k+2)\) in the word we apply the relation 1 \((-2, k)(2, k+2)(c_{i+3}, d_{i+3})\) = \((c_i, d_i)(c_{i+3}, d_{i+3})\) and go back to step 1;

**step 4.** If there exists a sequence of the type \((-2, k)(2, l)\) with \(k \leq l-4\) then we apply the relation 3.1 \((-2, k)(2, l)\) = \((2, l-2)(-2, k+2)\) and go back to the step 3;

**step 5.** If there doesn’t exist a sequence of the type \((-2, k)(2, l)\) with \(k \leq l-2\) in the word we output this.

**Claim 1.** The algorithm always terminates in a finite number of steps.
We will prove this by induction on the number of symbols since the algorithm doesn’t increase this.

It easy to see that any word satisfying the condition C has the same number of symbols of the form \((2, k)\) as of the form \((-2, k)\), and always begins with \((-2, 0)\) and ends with \((2, 0)\). So \((-2, 0)(2, 0)\) is the unique word of two symbols and this passes through the algorithm without any changes.

Now, assuming that the algorithm terminates for any word with \(2^n\) symbols, we take a word with \(2^{n+2}\) symbols. We consider for any word \(w = (c_1, d_1)(c_2, d_2)...(c_n, d_n)\) the "potential"

\[
E(w) = \left( \sum_{c_i=2} i, \sum_{c_i=-2} d_i - \sum_{c_i=2} d_i \right)
\]

which takes values in \(\mathbb{Z}^2\) with lexicographic order (i.e. \((a, b) \leq (c, d)\) iff \(a \leq c\) or \(a = c\) and \(b \leq d\)). We can see that, after being increased in step 1, the potential of the word is always decreased until (if it occurs) the algorithm returns to step 1 (after step 3), but in this case the word has \(2^n\) symbols and then by the induction hypothesis the algorithm terminates. The condition C implies that there are a finite number of potentials for a word with \(2^n + 2\) symbols (or less), and thus the algorithm terminates in a finite number of steps.

**Claim 2.** The output word has only \((2, 0)\) and \((-2, 0)\) as symbols.

For a word \(w = (c_1, d_1)...(c_{2n}, d_{2n})\) let \(\alpha_1 < ... < \alpha_n\) be the indices such that \(c_{\alpha_i} = 2\) and \(\beta_1 < ... < \beta_n\) be the indices such that \(c_{\beta_i} = -2\). We consider a new ”potential” \(E_2\) defined by the formula:

\[
E_2(w) = \max \left( \{d_{\alpha_{i+1}} - d_{\alpha_i} - 2(\alpha_{i+1} - \alpha_i - 1) : 1 \leq i \leq n - 1\} \cup \{d_{\beta_{i+1}} - d_{\beta_i} - 2(\beta_{i+1} - \beta_i - 1) : 1 \leq i \leq n - 1\} \right)
\]

After step 2 the potential of the word becomes non-positive and step 4 doesn’t change this. This means that the output word contains no sequence of the type \((2, k)(2, l)\) or \((-2, k)(-2, l)\) with \(k < l\). It is a condition of the algorithm that the output word contains no sequence of the type \((-2, k)(2, l)\) with \(k < l\).

Thus, if it does not contain a sequence of the type \((2, k)(-2, l)\) with \(k < l\), then the word is ordered by semi-decreasing order and, since it has to begin
with \((-2, 0)\) and to end with \((2, 0)\), the word has only \((2, 0)\) and \((-2, 0)\) as symbols.

To show that the output word contains no sequence of the type \((2, k)(-2, l)\) with \(k < l\), it is enough to observe that after step 1 there is no sequence of the type \((2, k)(-2, l)\) and that steps 2 and 4 do not produce any new sequences of the type \((2, k)(-2, l)\) with \(k < l\).}

**NOTE:** This algorithm was not conceived to be the most efficient but to guarantee an easy proof that it terminates. The author conjectures that there exist more efficient algorithms.

Next, we observe that the family of non-singular planar curves (i.e. \(\text{hom}(\emptyset, \emptyset)\)) together with the composition has a structure of a commutative monoid. We will see that any irreducible element of this monoid (i.e. a system of curves that is not a composition of other systems of curves) is a system of curves encircled by another curve. Note that to encircle a system of curves by another curve is, in the \((\pm2, k)\) words language, the same as adding a \((-2, 0)\) at the beginning and a \((2, 0)\) at the end of the corresponding word. Thus if a word \((c_1,0)(c_2,0)...(c_n,0)\) of symbols \((-2,0)\) and \((2,0)\)\(^6\) satisfying the condition C (and thus \((c_1,0) = (-2,0)\) and \((c_n,0) = (2,0)\)) doesn’t come from a system of curves encircled by another curve (this means that the word \((c_2,0)...(c_{n-1},0)\) doesn’t satisfy the condition C) then there exists \(2 < k < n\) such that \(\sum_{j<k} c_j = 0\), which implies that \((c_{k-1},0) = (2,0)\), \((c_k,0) = (-2,0)\) and the words \((c_1,0)...(c_{k-1},0)\) and \((c_k,0)...(c_n,0)\) satisfy the condition C. Therefore the morphism in \(\text{hom}(\emptyset, \emptyset)\) (corresponding to the word \((c_1,0)...(c_n,0)\)) is the composition of two morphisms in \(\text{hom}(\emptyset, \emptyset)\) (corresponding to the words \((c_1,0)...(c_{k-1},0)\) and \((c_k,0)...(c_n,0)\)).

Note that we have just proved that an irreducible morphism is another morphism encircled by an exterior curve, but we haven’t proved yet that a morphism encircled by an exterior curve is an irreducible morphism. This is because the last proposition says nothing about when two words of symbols \((-2,0)\) and \((2,0)\) represent equivalent words (by the relations 1, 2, 3 and 4).

However we do know that, using the composition of morphisms and the operation of encircling, we can generate all morphisms in \(\text{hom}(\emptyset, \emptyset)\) from the identity.

\(^6\)By the previous proposition, any system of non-singular planar curves can be represented in this way.
With this in mind, we are going to see what happens in the representation when we compose two morphisms (see corollary 17) or encircle one by a circle (see corollary 21).

Consider, for a fixed value \( m \in \mathbb{M} \) and for each natural number \( n \), the following morphism:

\[
\Psi_{n, m} : \mathcal{O}_n \longrightarrow \mathcal{O}_n
\]

\[
(R, \overrightarrow{v}) \longmapsto (R, R \ast (e_1 \ast m \oplus \overrightarrow{v}))
\]

The purpose of this morphism is to add the value \( m \) to the region associated to the first interval.

**Lemma 15** For any \( 2 \leq k \leq n + 1 \), \( \hat{T}_{n,k} \circ \Psi_{n,m} = \Psi_{n+2,m} \circ \hat{T}_{n,k} \) and \( \hat{T}_{n,k} \circ \Psi_{n+2,m} = \Psi_{n,m} \circ \hat{T}_{n,k} \).

**Proof.**

First equation: \( \hat{T}_{n,k} \circ \Psi_{n,m} = \Psi_{n+2,m} \circ \hat{T}_{n,k} \).

For an arbitrary \((R, \overrightarrow{v}) \in \mathcal{O}_n\), let \((R_1, \overrightarrow{v}_1) = \Psi_{n,m}(R, \overrightarrow{v}), (R_2, \overrightarrow{v}_2) = \hat{T}_{n,k}(R_1, \overrightarrow{v}_1), (R'_1, \overrightarrow{v}'_1) = \hat{T}_{n,k}(R, \overrightarrow{v})\) and \((R'_2, \overrightarrow{v}'_2) = \Psi_{n+2,m}(R'_1, \overrightarrow{v}'_1)\).

We want to check \((R_2, \overrightarrow{v}_2) = (R'_2, \overrightarrow{v}'_2)\).

Since \( \Psi \) does not change the matrices and the changes of the matrices under the morphisms \( \hat{T} \) do not depend on the choice of the array of values, it is obvious that \( R_2 = R'_2 \).

\[
\overrightarrow{v}_2 = B_{n,k} \ast \overrightarrow{v}_1
\]

\[
= B_{n,k} \ast [R \ast (e_1 \ast m \oplus \overrightarrow{v}_1)]
\]

\[
= B_{n,k} R \ast (e_1 \ast m \oplus \overrightarrow{v}_1)
\]

\[
= B_{n,k} R B_{n,k}' B_{n,k} \ast (e_1 \ast m \oplus \overrightarrow{v}_1)
\]

\[
= [B_{n,k} R B_{n,k}' B_{n,k} \ast (e_1 \ast m \oplus \overrightarrow{v}_1)] \lor [D_{n+2,k} B_{n,k} \ast (e_1 \ast m \oplus \overrightarrow{v}_1)]
\]

\[
= (B_{n,k} R B_{n,k}' + D_{n+2,k}) \ast [B_{n,k} \ast (e_1 \ast m \oplus \overrightarrow{v}_1)]
\]

\[
= R'_1 \ast [(B_{n,k} \ast e_1 \ast m) \oplus (B_{n,k} \ast \overrightarrow{v}_1)]
\]

\[
= R'_1 \ast [(B_{n,k} \ast e_1 \ast m) \oplus \overrightarrow{v}'_1]
\]

On the other hand,

\[
\overrightarrow{v}'_2 = R'_1 \ast (e_1 \ast m \oplus \overrightarrow{v}'_1)
\]
If \( k > 2 \) then \( B_{n,k} e_1 = e_1 \) and therefore \( \vec{v}_2 = \vec{v}'_2 \).

If \( k = 2 \) then

\[
\vec{v}_2 = R'_1 \cdot [(B_{n,k} e_1 \cdot m) \oplus \vec{v}'_1] = R'_1 \cdot [(e_1 \cdot m \lor e_3 \cdot m) \oplus \vec{v}'_1] = R'_1 \cdot (e_1 \cdot m \oplus \vec{v}'_1) \lor R'_1 \cdot (e_3 \cdot m \oplus \vec{v}'_1) = R'_1 \cdot (e_1 \cdot m \oplus \vec{v}'_1) = \vec{v}'_2
\]

because \( R'_1 \cdot (e_3 \cdot m \oplus \vec{v}'_1) = R'_1 \cdot (e_1 \cdot m \oplus \vec{v}'_1) \) as we will prove next:

\[
R'_1 e_1 = (B_{n,2} R B_{n,2}' + D_{n+2,2}) e_1 = B_{n,2} R B_{n,2}' e_1 \lor D_{n+2,2} e_1 = B_{n,2} R e_1
\]

\[
R'_1 e_3 = (B_{n,2} R B_{n,2}' + D_{n+2,2}) e_3 = B_{n,2} R B_{n,2}' e_3 \lor D_{n+2,2} e_3 = B_{n,2} R e_3
\]

thus \( R'_1 e_1 \cdot m = R'_1 e_3 \cdot m \), and using the next lemma we conclude

\[ R'_1 \cdot (e_3 \cdot m \oplus \vec{v}'_1) = R'_1 \cdot (e_1 \cdot m \oplus \vec{v}'_1) \]

Lemma 16 If \( R = R' \) and \( R \cdot \vec{v} = \vec{v}' \) then \( R \cdot (\vec{v} \oplus \vec{v}') = R \cdot \vec{v} \oplus \vec{v}' \).

**Proof.** Let \( \vec{x} := R \cdot (\vec{v} \oplus \vec{v}') \) and \( \vec{y} := R \cdot \vec{u} \oplus \vec{v} \), then

\[
x_i = \bigvee_j r_{i,j} \cdot (u_j \oplus v_j) \quad \text{and} \quad y_i = \bigvee_j r_{i,j} \cdot u_j \oplus v_i
\]

\[
x_i = \bigvee_j r_{i,j} \cdot (u_j \oplus v_j)
\]

\[
= \bigvee_{j: r_{i,j}=1} (u_j \oplus v_j)
\]

\[
= \bigvee_{j: r_{i,j}=1} (u_j \oplus v_i)
\]

\[
= (\bigvee_{j: r_{i,j}=1} u_j) \oplus v_i
\]

\[
= (\bigvee_{j: r_{i,j}=1} u_j) \oplus v_i
\]

\[
y_i
\]

Note that, since \( R \cdot \vec{v}' = \vec{v}' \), we have that \( v_j = v_i \) whenever \( r_{i,j} = 1 \).

Second equation \( T_{n,k} \circ \Psi_{n+2,m} = \Psi_{n,m} \circ T_{n,k} \).

For an arbitrary \((R, \vec{v}) \in \mathcal{O}_n\), let \((R_1, \vec{a}') = \Psi_{n+2,m}(R, \vec{v}), (R_2, \vec{b}') = T_{n,k}(R_1, \vec{a}'), (R'_1, \vec{a}'') = T_{n,k}(R, \vec{v})\) and \((R'_2, \vec{b}'') = \Psi_{n,m}(R'_1, \vec{a}'')\). We want to check \((R_2, \vec{b}') = (R'_2, \vec{b}'')\).
Since $\Psi$ does not change the matrices and the changes of the matrices under the morphisms $\hat{T}$ do not depend on the choice of the array of values, it is obvious that $R_2 = R'_2$.

$$\overrightarrow{b} = R_2 \ast [(B_k \ast \overrightarrow{a'}) \oplus e_{k-1} \ast \tilde{x}_k]$$

with

$$\overrightarrow{a'} = R \ast (e_1 \ast m \oplus \overrightarrow{v'}) = Re_1 \ast m \oplus \overrightarrow{v}$$

and

$$\tilde{x}_k = \begin{cases} a_{k-1} \land a_{k+1} & \text{if } r_{k-1,k+1} = 0 \\ \varphi(a_k) & \text{if } r_{k-1,k+1} = 1 \end{cases}$$

On the other hand

$$\overrightarrow{b'} = R'_2 \ast (e_1 \ast m \oplus \overrightarrow{a''}) = R'_2 e_1 \ast m \oplus \overrightarrow{a''}$$

We want to check $\overrightarrow{b} = \overrightarrow{b'}$.

**A) $b_{k-1} = b'_{k-1}$**

By lemma 1, we have

$$b_{k-1} = (a_{k-1} \lor a_{k+1}) \oplus \tilde{x}_k = \begin{cases} a_{k-1} \lor a_{k+1} & \text{if } r_{k-1,k+1} = 0 \\ a_{k-1} \lor \varphi(a_k) & \text{if } r_{k-1,k+1} = 1 \end{cases}$$

Bearing in mind that

$$a_{k-1} = e_{k-1} \ast (Re_1 \ast m \oplus \overrightarrow{v'}) = r_{k-1,1} \ast m \oplus v_{k-1}$$
$$a_{k+1} = e_{k+1} \ast (Re_1 \ast m \oplus \overrightarrow{v'}) = r_{k+1,1} \ast m \oplus v_{k+1}$$
$$a_k = e_k \ast (Re_1 \ast m \oplus \overrightarrow{v'}) = r_{k,1} \ast m \oplus v$$

we have that

$$b_{k-1} = \begin{cases} (r_{k-1,1} \ast m) \oplus v_{k-1} \oplus (r_{k+1,1} \ast m) \oplus v_{k+1} & \text{if } r_{k-1,k+1} = 0 \\ (r_{k-1,1} \ast m) \oplus v_{k-1} \oplus \varphi(r_{k,1} \ast m \oplus v_k) & \text{if } r_{k-1,k+1} = 1 \end{cases}$$

Now we note that $r_{k-1,k+1} = 0$ implies that $r_{k-1,1} = 0$ or $r_{k+1,1} = 0$ and therefore $(r_{k-1,1} \ast m) \oplus (r_{k+1,1} \ast m) = (r_{k-1,1} + r_{k+1,1}) \ast m$. 
On the other hand, $r_{k-1,k+1} = 1$ implies that $r_{k,1} = 0$ and $r_{k-1,1} = r_{k+1,1} = r_{k-1,1} + r_{k+1,1}$.

Thus

\[
b_{k-1} = \begin{cases} 
(r_{k-1,1} + r_{k+1,1}) \ast m \oplus v_{k-1} \oplus v_{k+1} & \text{if } r_{k-1,k+1} = 0 \\
(r_{k-1,1} + r_{k+1,1}) \ast m \oplus v_{k-1} \oplus \varphi(v_k) & \text{if } r_{k-1,k+1} = 1 
\end{cases}
\]

That means

\[
b_{k-1} = (r_{k-1,1} + r_{k+1,1}) \ast m \oplus a'_{k-1}
\]

by lemma 11. Now

\[
\vec{b'} = R'_1 \ast (e_1 \ast \overline{a'}) = R'_1 e_1 \ast m \oplus \overline{a'}
\]

i.e.

\[
b'_{k-1} = e'_{k-1} \ast (R'_1 e_1 \ast m \oplus \overline{a'}) = e'_{k-1} R'_1 e_1 \ast m \oplus a'_{k-1} = r'_{k-1,1} \ast m \oplus a'_{k-1}
\]

and using lemma 12 we have

\[
b'_{k-1} = (r_{k-1,1} + r_{k+1,1}) \ast m \oplus a'_{k-1} = b_{k-1}
\]

B) $b_i = b'_i$ with $r'_{i,k-1} = 0$.

By lemma 11, $b_i = e'_i B^*_k \ast \overline{a'}$.

\[
b_i = e'_i B^*_k \ast (Re_1 \ast m \oplus \overline{v}) = e'_i B^*_k R e_1 \ast m \oplus e'_i B^*_k \overline{v} = r_{k(i),1} \ast m \oplus e'_i B^*_k \overline{v}
\]

On the other hand,

\[
b'_i = e'_i (R'_1 e_1 \ast m \oplus \overline{a'}) = e'_i R'_1 e_1 \ast m \oplus e'_i \overline{a'} = r'_{i,1} \ast m \oplus e'_i B^*_k \overline{v}
\]

Thus we have $b_i = b'_i$ if $r'_{i,1} = r_{k(i),1}$. 


If $k - 1 \neq 1$ then, by lemma 12, $r'_{i,1} = r_{k(i),k(1)} = r_{k(i),1}$.

If $k - 1 = 1$ then, by lemma 12, $r'_{i,1} = r'_{i,k-1} \geq r_{k(i),k-1} = r_{k(i),1}$

and since $r'_{i,1} = r'_{i,k-1} = 0$ we have $r'_{i,1} = r_{k(i),1}$.

C) $b_i = b'_i$ with $r'_{i,k-1} = 1$.

Since, by lemma 11, $b_i = b_{k-1}$ and $b'_i = b'_{k-1}$ this shows that $b_i = b'_i$.

\[\text{Corollary 17} 1. \text{ For any } T \in \hom(O_m, O_n), \; T \circ \Psi_{m,m} = \Psi_{n,m} \circ T;\]

2. For any $T \in \hom(O_1, O_1)$ (corresponding to a system of non-singular planar curves) $T([1], \emptyset) = ([1], x) \Rightarrow T([1], m) = ([1], x \oplus m)$;

3. For any $T_1, T_2 \in \hom(O_1, O_1)$ $T_2 \circ T_1([1], \emptyset) = ([1], m_2 \oplus m_1$ where $([1], m_i) = T_i([1], \emptyset), \; i = 1, 2$.

\[\text{Proof.}\]

1. It is obvious that if $\{\Psi_{n,m}\}_{n \in \mathbb{N}}$ commute with the generators $\mathcal{T}_{n,k}$ and $\mathcal{T}_{n,k}$ then they commute with any morphism $T$.

2. $T([1], m) = T \circ \Psi_{1,m}([1], \emptyset) = \Psi_{1,m} \circ T([1], \emptyset) = \Psi_{1,m}([1], x) = ([1], x \oplus m)$.

3. $T_2 \circ T_1([1], \emptyset) = T_2([1], m_1) = ([1], m_2 \oplus m_1)$.

\[\text{This result shows that all information about a morphism } T \in \hom(O_1, O_1) \text{ is contained in a single value } v \in \mathbb{M} \text{ (which is obtained evaluating the morphism } T \text{ on } ([1], \emptyset)). \text{ Thus, we can associate a value in } \mathbb{M} \text{ to each system of non-singular planar curves. Moreover this association is a monoid homomorphism.}\]

Next we will look at what value is obtained when a collection of planar curves is encircled by another curve.

Consider, for each $n$, the following subset of $O_n$

$$\mathcal{U}_n = \{(R, v') \in O_n : e'_1 Re_n = 1\}$$

By the property T1 we have that $\mathcal{U}_n = \emptyset$ for any even number $n$. It is clear that $\mathcal{U}_1 = O_1$ and therefore it easy to see that $\mathcal{U}_n \neq \emptyset$ for any odd number $n$, using the following result.
Proposition 18 For any $n$ and $k$, if $(R, \overrightarrow{v}) \in U_n$ then $\hat{T}_k(R, \overrightarrow{v}) \in U_{n+2}$ and $\tilde{T}_k(R, \overrightarrow{v}) \in U_{n-2}$.

Proof. Let $(R', \overrightarrow{v}') = \hat{T}_k(R, \overrightarrow{v})$. We want to see that if $e_1^t R e_n = 1$ then $e_1^t R' e_{n+2} = 1$.

\[
e_1^t R' e_{n+2} = e_1^t (B_k RB_k^t + D_k) e_{n+2} = e_1^t B_k RB_k^t e_{n+2} = e_1^t R e_n = 1
\]

Let $(R', \overrightarrow{v}') = \tilde{T}_k(R, \overrightarrow{v})$. We want to see that if $e_1^t R e_n = 1$ then $e_1^t R' e_{n-2} = 1$.

\[
e_1^t R' e_{n-2} = e_1^t (B_k^t RB_k)^2 e_{n-2} \geq e_1^t B_k^t RB_k e_{n-2} \geq e_1^t R e_n = 1
\]

Now we consider, for each odd natural number $n$, the following morphism:

$$
\varepsilon_n : U_n \rightarrow U_{n+2} \\
(R, \overrightarrow{v}) \mapsto (\tilde{R}, \overrightarrow{v})
$$

with

$$
\tilde{R} = E_n R E_n^t + F_{n+2}
$$

and

$$
\overrightarrow{v} = E_n \ast \overrightarrow{v}
$$

where $E_n = [e_{i,j}]$ is an $(n+2) \times n$ matrix defined by $e_{i,j} = 1 \iff i - 1 = j$ and $F_{n+2} = [f_{i,j}]$ is an $(n+2) \times (n+2)$ matrix defined by $f_{i,j} = 1 \iff i, j \in \{1, n+2\}$.

Proposition 19 For any odd natural number $n$, $\varepsilon_n$ is well defined.

Proof. We need to see that if $(R, \overrightarrow{v}) \in U_n$ then $\varepsilon_n(R, \overrightarrow{v}) \in U_{n+2}$.

Let $(\tilde{R}, \overrightarrow{v}) = \varepsilon_n(R, \overrightarrow{v})$. We are going to check that $R \ast \overrightarrow{v} = \overrightarrow{v}$ implies $\tilde{R} \ast \overrightarrow{v} = \overrightarrow{v}$, $\tilde{R}$ satisfies the properties E1, E2, E3, T1, T2 and T3 and the $(1, n+2)$ entry of $\tilde{R}$ is 1.

For this purpose, we are going to use the identities: $E_n^t E_n = I$, $E_n^t F_{n+2} = O$ and $F_{n+2} E_n = O$, which we will leave to the reader to check.
Representations of non-singular planar tangles by operators

\[ R \ast \overrightarrow{v} = \overrightarrow{v} \Rightarrow \tilde{R} \ast \overrightarrow{v} = (E_n \ast RE_n^t + F_{n+2}) \ast (E_n \ast \overrightarrow{v}) \]
\[ = (E_n \ast RE_n^t \ast E_n) \ast \overrightarrow{v} \lor (F_{n+2} \ast E_n) \ast \overrightarrow{v} \]
\[ = E_n \ast \overrightarrow{v} \]
\[ = \frac{\overrightarrow{v}}{v} \]

The properties \( E_1, E_2 \) and \( E_3 \) of \( \tilde{R} \) are very easy to verify, so we leave them as an exercise.

For the properties \( T_1, T_2 \) and \( T_3 \), we observe that the \((i, j)\) entry of \( \tilde{R} \) is the \((i - 1, j - 1)\) entry of \( R \) if \( 2 \leq i, j \leq n + 1 \), 1 if \( i, j \in \{1, n + 2\} \) and 0 in all other cases, i.e.:

\[ \tilde{r}_{i,j} = c_i^t (E_n \ast RE_n^t + F_{n+2})e_j = \begin{cases} 
 r_{i-1,j-1} & \text{if} \quad 2 \leq i, j \leq n + 1 \\
 1 & \text{if} \quad i, j \in \{1, n + 2\} \\
 0 & \text{otherwise}
\end{cases} \]

In particular, we have \( \tilde{r}_{1,n+2} = 1 \).

Property \( T_1 \): \( \tilde{r}_{i,j} = 1 \Rightarrow j - i \in 2\mathbb{Z} \).

We have \( \tilde{r}_{i,j} = 1 \Rightarrow i, j \in \{1, n + 2\} \) or \( r_{i-1,j-1} = 1 \). Since we have taken \( n \) to be odd and \( R \) satisfies \( T_1 \) we have \( j - i \in 2\mathbb{Z} \).

Property \( T_2 \): \( \forall \alpha \leq \beta \leq \gamma \leq \delta \tilde{r}_{\alpha,\gamma} = \tilde{r}_{\beta,\delta} = 1 \Rightarrow \tilde{r}_{\alpha,\beta} = \tilde{r}_{\beta,\gamma} = \tilde{r}_{\gamma,\delta} = 1 \).

We only need consider the case \( \alpha < \beta < \gamma < \delta \), and then \( \tilde{r}_{\alpha,\gamma} = \tilde{r}_{\beta,\delta} = 1 \Rightarrow 2 \leq \alpha < \beta < \gamma < \delta \leq n + 1 \). Thus \( \tilde{r}_{\alpha,\gamma} = r_{\alpha-1,\gamma-1} \) and \( \tilde{r}_{\beta,\delta} = r_{\beta-1,\delta-1} \), hence, since \( R \) satisfies \( T_2 \), we have \( r_{\alpha-1,\beta-1} = r_{\beta-1,\gamma-1} = r_{\gamma-1,\delta-1} = 1 \), that is \( \tilde{r}_{\alpha,\beta} = \tilde{r}_{\beta,\gamma} = \tilde{r}_{\gamma,\delta} = 1 \).

Property \( T_3 \): \( \forall \alpha < \beta \tilde{r}_{\alpha,\alpha} = 1 \Rightarrow \tilde{r}_{\alpha+1,\beta-1} = 1 \) or \( \exists \alpha < \gamma < \beta : \tilde{r}_{\alpha,\gamma} = 1 \).

If \( \alpha = 1 \) then \( \tilde{r}_{\alpha,\beta} = 1 \Rightarrow \beta = n + 2 \). Thus \( \tilde{r}_{\alpha+1,\beta-1} = \tilde{r}_{2,n+1} = r_{1,n} = 1 \) since \( R \in U_n \).

If \( \alpha > 1 \) then \( \tilde{r}_{\alpha,\beta} = 1 \Rightarrow \beta < n + 2 \). Thus

\[ \tilde{r}_{\alpha,\beta} = r_{\alpha-1,\beta-1} \Rightarrow r_{\alpha,\beta-2} = 1 \quad \text{or} \quad \exists_{\alpha-1 < \gamma-1 < \beta-1} : r_{\alpha-1,\gamma-1} = 1 \]
\[ \Rightarrow \tilde{r}_{\alpha+1,\beta-1} = 1 \quad \text{or} \quad \exists_{\alpha < \gamma < \beta} : \tilde{r}_{\alpha,\gamma} = 1 \]

\(^7\)This is an abuse of notation, since we should write \((R, \overrightarrow{v}) \in U_n \).
Lemma 20 For any $2 \leq k \leq n + 1$, we have the identities $\hat{T}_{n+2,k+1} \circ \varepsilon_n = \varepsilon_{n+2} \circ \hat{T}_{n,k}$ and $\hat{T}_{n,k+1} \circ \varepsilon_n = \varepsilon_{n-2} \circ \hat{T}_{n-2,k}$.

Proof.

First identity: $\hat{T}_{n+2,k+1} \circ \varepsilon_n = \varepsilon_{n+2} \circ \hat{T}_{n,k}$.

For an arbitrary $(R, \vec{v}) \in \mathcal{U}_n$, let $(R_1, \vec{v}_1) = \varepsilon_n(R, \vec{v})$, $(R_2, \vec{v}_2) = \hat{T}_{n+2,k+1}(R_1, \vec{v}_1)$, $(R'_1, \vec{v}'_1) = \hat{T}_{n,k}(R, \vec{v})$ and $(R'_2, \vec{v}'_2) = \varepsilon_{n+2}(R'_1, \vec{v}'_1)$.

We want to check $(R_2, \vec{v}_2) = (R'_2, \vec{v}'_2)$.

\[
R_2 = B_{n+2,k+1}R_1B^t_{n+2,k+1} + D_{n+4,k+1}
= B_{n+2,k+1}(E_nR E^t_n + F_{n+2})B^t_{n+2,k+1} + D_{n+4,k+1}
= B_{n+2,k+1}E_nR E^t_n B^t_{n+2,k+1} + B_{n+2,k+1}F_{n+2}B^t_{n+2,k+1} + D_{n+4,k+1}
= E_{n+2}B_{n,k}R B^t_{n,k}E^t_{n+2} + F_{n+4} + E_{n+2}D_{n+2,k}E^t_{n+2}
= E_{n+2}(B_{n,k}R B^t_{n,k} + D_{n+2,k})E^t_{n+2} + F_{n+4}
= E_{n+2}R'_1E^t_{n+2} + F_{n+4}
= R'_2
\]

\[
\vec{v}_2 = B_{n+2,k+1} \ast \vec{v}_1
= B_{n+2,k+1} \ast (E_n \ast \vec{v})
= (B_{n+2,k+1}E_n) \ast \vec{v}
= (E_{n+2}B_{n,k}) \ast \vec{v}
= E_{n+2} \ast (B_{n,k} \ast \vec{v})
= E_{n+2} \ast \vec{v}'_1
= \vec{v}'_2
\]

Second identity $\hat{T}_{n,k+1} \circ \varepsilon_n = \varepsilon_{n-2} \circ \hat{T}_{n-2,k}$.

For an arbitrary $(R, \vec{v}) \in \mathcal{U}_n$, let $(R_1, \vec{a}) = \varepsilon_n(R, \vec{v})$, $(R_2, \vec{b}) = \hat{T}_{n,k+1}(R_1, \vec{a})$, $(R'_1, \vec{a}'') = \hat{T}_{n-2,k}(R, \vec{v})$ and $(R'_2, \vec{b}'') = \varepsilon_{n-2}(R'_1, \vec{a}'')$.

We want to check $(R_2, \vec{b}) = (R'_2, \vec{b}'')$. 

\[ R_2 = \left[ B_{n,k+1}^t R_1 B_{n,k+1} \right]^2 \]
\[ = \left[ B_{n,k+1}^t (E_n R E_n^t + F_{n+2}) B_{n,k+1} \right]^2 \]
\[ = \left[ B_{n,k+1}^t E_n R E_n^t B_{n,k+1} + B_{n,k+1}^t F_{n+2} B_{n,k+1} \right]^2 \]
\[ = (E_{n-2} B_{n-2,k}^t R B_{n-2,k} E_{n-2}^t + F_n)^2 \]
\[ = E_{n-2} B_{n-2,k}^t R B_{n-2,k} E_{n-2}^t B_{n-2,k} E_{n-2}^t + E_{n-2} B_{n-2,k}^t R B_{n-2,k} E_{n-2}^t F_n + F_n E_{n-2} B_{n-2,k} R B_{n-2,k} E_{n-2}^t + F_n^2 \]
\[ = E_{n-2} R_1^t E_{n-2}^t + F_n \]
\[ = R_2 \]

\[ \overrightarrow{b} = R_2 \left[ (B_{n,k+1}^t \overrightarrow{d}) \oplus (e_k \overrightarrow{x}_{k+1}) \right] \]
\[ = (E_{n-2} R_1^t E_{n-2}^t + F_n) \left[ (B_{n,k+1}^t \overrightarrow{d}) \oplus (e_k \overrightarrow{x}_{k+1}) \right] \]
\[ = E_{n-2} R_1^t \left[ (E_{n-2} B_{n-2,k}^t \overrightarrow{d}) \oplus (E_{n-2} e_k \overrightarrow{x}_{k+1}) \right] \]
\[ \lor (F_n E_{n-2} B_{n-2,k}^t \overrightarrow{d} \oplus (F_n e_k \overrightarrow{x}_{k+1}) \]
\[ = E_{n-2} R_1^t \left[ (E_{n-2} B_{n-2,k}^t \overrightarrow{d}) \oplus (e_k \overrightarrow{x}_{k+1}) \right] \]
\[ = E_{n-2} R_1^t \left[ (B_{n-2,k}^t \overrightarrow{d}) \oplus (e_k \overrightarrow{x}_{k+1}) \right] \]
\[ = \overrightarrow{b} \]

Here \( \overrightarrow{x}_{k+1} = \overrightarrow{\varphi}_{k,k+2} \left( a_k \land a_{k+2} \right) \oplus \overrightarrow{\varphi}_{k,k+2} \varphi(a_k+1) \) where \( \overrightarrow{\varphi}_{k,k+2} = e_k^t R_1 e_{k+2} \).

Since \( 1 < k, k+2 < n+2 \) we have that \( \overrightarrow{\varphi}_{k,k+2} = r_{k-1,k+1} \) and \( a_k = v_{k-1} \)
\( a_{k+1} = v_k \) and \( a_{k+2} = v_{k+1} \). Thus \( \overrightarrow{x}_{k+1} = x_k \).

\[

\text{Corollary 21} \quad \text{Let} \; F : \mathcal{P}\mathcal{L}_m \rightarrow \mathcal{P}\mathcal{L}_m \text{ be the functor which sends} \; \hat{T}_{n,k} \text{ and} \; \hat{T}_{n,k} \text{ to} \; \hat{T}_{n+2,k+1} \text{ and} \; \hat{T}_{n+2,k+1} \text{ (respectively). Then} \]

1. \( \text{For any} \; T \in \text{hom}(U_n, U_m), \; \varepsilon_m \circ T = F(T) \circ \varepsilon_n. \)

2. \( \text{For any} \; T \in \text{hom}(O_n, O_m) \; (\text{note that} \; U_1 = O_1), \; T([1], \emptyset) = ([1], \varphi) \Rightarrow \overrightarrow{T_{1,2}} \circ F(T) \circ \overrightarrow{T_{1,2}}([1], \emptyset) = ([1], \varphi(\overrightarrow{x})). \)
Proof.

1. This follows immediately from the previous lemma and the definition of the functor $F$.

2.

$$
\hat{T}_{1,2} \circ F(T) \circ \hat{T}_{1,2}([1], \emptyset) = \hat{T}_{1,2} \circ F(T) \left( \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \begin{pmatrix} \emptyset \\ \emptyset \end{pmatrix} \right)
$$

$$
= \hat{T}_{1,2} \circ F(T) \circ \varepsilon_1([1], \emptyset)
$$

$$
= \hat{T}_{1,2} \circ \varepsilon_1 \circ T([1], \emptyset)
$$

$$
= \hat{T}_{1,2} \circ \varepsilon_1([1], x)
$$

$$
= \hat{T}_{1,2} \left( \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \begin{pmatrix} \emptyset \\ \emptyset \end{pmatrix} \right)
$$

$$
= ([1], \varphi(x))
$$

Note that $\hat{T}_{1,2} \circ F(T) \circ \hat{T}_{1,2}$ is the representation of $\hat{t}_{1,2} \circ F(t) \circ \hat{t}_{1,2}$ which corresponds to the encirclement of the morphism $t$.

In summary we see that the representation of a system of non-singular planar curves is a morphism in $\text{hom}(O_1, O_1)$, but this is determined by a single value. This means that the representation gives an application of the monoid of systems of non-singular planar curves $\text{hom}(\emptyset, \emptyset)$ to the monoid of the representation $\mathbb{M}$. To a morphism $t \in \text{hom}(\emptyset, \emptyset)$ we associate the value $v(t) \in \mathbb{M}$ such that $([1], v(t)) = T([1], \emptyset)$ where $T \in \text{hom}(O_1, O_1)$ is the representation of $t$. This application is a monoid morphism $(v(t_1 \circ t_2) = v(t_1) \oplus v(t_2))$ as we have seen (see corollary 17), and furthermore, the function $\varphi$ is what corresponds in $\mathbb{M}$ to the operation of encircling a system of curves by another curve (see corollary 21), i.e. $v([t]) = \varphi(v(t))$ where $\langle t \rangle$ denotes the encirclement of the morphism $t$.

We are going to study this application in the following two particular cases.

**First case:** the monoid is the natural numbers with the usual multiplication as the operation of the monoid and with the division order giving the lattice structure; the function $\varphi$ is the function that sends a number $n$ to the $n^{th}$ prime number.

---

8Here the functor $F$ is defined in the same way for $\text{PT}$ as it was for $\text{PI}_M$. 
We are going to see that, for systems of non-singular planar curves, the application is a monoid isomorphism (and thus is a complete invariant for such systems). This means that two systems of curves with the same value are equivalent (isotopic). Let us prove this by induction on the value \( v(s) = n \) of the system \( s \). Note that it only remains to prove that the application is bijective.

First we observe that if a morphism \( s \in \text{hom}(\emptyset, \emptyset) \) is irreducible (and thus is encircled) then \( v(s) \) is a prime number.

If \( v(s) = 1 \) then \( s \) is the empty system of curves (the identity in \( \text{hom}(\emptyset, \emptyset) \)) because a non-empty system of curves is a non-empty composition of irreducible morphisms therefore it has a non-empty product of prime numbers as value \( v \) (i.e. \( v(s) > 1 \)). This implies that if a morphism has a prime number as value then it is irreducible. Thus an encircled morphism is irreducible.

Now let \( v(s_1) = v(s_2) = n \), and suppose by the induction hypothesis that, for \( k < n \), \( v(s_1) = v(s_2) = k \) implies \( s_1 = s_2 \).

If \( n \) is a prime number (say the \( k^{th} \) prime) then \( s_1 \) and \( s_2 \) are irreducible (encircled), that is \( s_1 = \langle s_3 \rangle \) and \( s_2 = \langle s_4 \rangle \) and \( v(s_3) = v(s_4) = k \). Thus, by the induction hypothesis, \( s_3 = s_4 \) and therefore \( s_1 = s_2 \).

If \( n \) is a composite number then \( s_1 \) and \( s_2 \) factorize into the same irreducible morphisms because if an irreducible morphism \( s_3 \) is a factor of \( s_1 \) then \( v(s_3) \) is a prime number that divides \( n \) and, by the induction hypothesis, \( s_3 \) is the unique morphism with value \( v(s_3) \), therefore \( s_3 \) is also a factor of \( s_2 \). The same argument can be used to prove that each factor appears in \( s_1 \) and \( s_2 \) the same number of times. We only need to do the following exercise to conclude that \( s_1 = s_2 \).

\[ \text{Exercise 1} \quad \text{Show that the monoid } \text{hom}(\emptyset, \emptyset) \text{ is commutative.} \]

\[ \text{Second case:} \quad \text{the monoid is the non-negative integer numbers with the usual sum as the operation of the monoid and with the usual order giving the lattice structure; the function } \varphi \text{ is the function that sends a number } n \text{ to its successor } n + 1. \]

We are going to see that, for a system of non-singular planar curves \( s \), \( v(s) \) is simply the number of curves of \( s \).

This is very easy because, since the application \( v \) is uniquely determined by the relations \( v(s_1 \circ s_2) = v(s_1) + v(s_2) \) and \( v(\langle s \rangle) = \varphi(v(s)) = v(s) + 1 \), we only need to observe that the number of curves \( v(s) \) of a system of curves \( s \) satisfies the relation \( v(s_1 \circ s_2) = v(s_1) + v(s_2) \) and \( v(\langle s \rangle) = v(s) + 1 \).
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