A model for neutrino masses and mixing based on the non-abelian discrete symmetry $A_4$

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Abstract. In this talk I discuss a see-saw $A_4$ model [1] for Tri-Bimaximal mixing which is based on a very economical flavour symmetry and field content and still possesses all the good features of $A_4$ models. In particular the charged lepton mass hierarchies are determined by the $A_4 \times Z_4$ flavour symmetry itself without invoking a Froggatt-Nielsen $U(1)$ symmetry. Tri-Bimaximal mixing is exact in leading order while all the mixing angles receive corrections of the same order in next-to-the-leading approximation. As a consequence the predicted value of $\theta_{13}$ is within the sensitivity of the experiments which will take data in the near future. The light neutrino spectrum with its phenomenological implications is discussed in detail.

1. Introduction

It is an experimental fact [2] that within measurement errors the observed neutrino mixing matrix [3] is compatible with the so called Tri-Bimaximal (TB) form [4]. The best measured neutrino mixing angle $\theta_{12}$ is just about $1\sigma$ below the TB value $\tan^2 \theta_{12} = 1/2$, while the other two angles are well inside the $1\sigma$ interval. It has been pointed out [5] that a broken flavour symmetry based on the discrete group $A_4$ appears to be particularly suitable to reproduce this specific mixing pattern in leading order (LO). Other solutions based on alternative discrete or continuous flavour groups have also been considered, but the $A_4$ models have a very economical and attractive structure, e.g. in terms of group representations and of field content. In most of the models $A_4$ is accompanied by additional flavour symmetries, either discrete like $Z_N$ or continuous like $U(1)$, which are necessary to eliminate unwanted couplings, to ensure the needed vacuum alignment and to reproduce the observed charged lepton mass hierarchies. Given the set of flavour symmetries and having specified the field content, the non leading corrections to the TB mixing arising from loop effects and higher dimensional operators can be evaluated in a well defined expansion. In this talk I present an $A_4$ model for TB mixing which is based on a most economical flavour symmetry and field content and still possesses all the features described above. In particular TB mixing is exact in LO while all mixing angles receive corrections at higher orders. The charged lepton mass hierarchies are determined by the $A_4 \times Z_4$ flavour symmetry itself without invoking a Froggatt-Nielsen $U(1)$ symmetry, as a consequence of a particular alignment as proposed in refs. [6]. Our model, which is of the see-saw type, differs from those in ref.[6] because the flavour symmetry is smaller and the pattern of corrections to TB mixing is more general and flexible.
2. The structure of the model

The model is formulated in terms of the $A_4$ realization in the T diagonal basis (see [1] for more details). Since we want to work in the framework of the see-saw mechanism, we assign the 3 generations of left-handed (LH) lepton doublets $l$ and of right-handed (RH) neutrinos $\nu$ to two triplets 3, while the RH charged leptons $e^c$, $\mu^c$ and $\tau^c$ all transform as 1 (while the most usual classification in $A_4$ models is as 1, $1^c$ and $1^c$). The $A_4$ symmetry is then broken by suitable flavons. All the flavon fields are singlets under the Standard Model gauge group. The complete flavour symmetry is $A_4 \times Z_4$. We adopt a supersymmetric context, so that two Higgs doublets $h_{u,d}$, invariant under $A_4$, are present in the model. A $U(1)_R$ symmetry related to R-parity and the presence of driving fields in the flavon superpotential are common features of supersymmetric formulations. The field content and the symmetry assignments are as in Tab.1. For the class of models of ref.[6] the crucial feature is the alignment

$$\langle \varphi_S \rangle = (v_s, v_S, 0)$$

$$\langle \xi \rangle = u$$

$$\langle \xi' \rangle = u' \neq 0 \ , \ \langle \varphi_T \rangle = (0, v_T, 0) \ , \ \langle \varphi_T \rangle = -\frac{h_u u'}{2h_v} .$$

Note that this differs from the usual $A_4$ alignment in that $\langle \varphi_T \rangle = (0, v_T, 0)$ replaces $\langle \varphi_T \rangle = (v_T, 0, 0)$. The difference is that, while $(1, 0, 0)^n = (1, 0, 0)$ (i.e. all positive powers are aligned in the same direction), for $(0, 1, 0)$ we have $(0, 1, 0)^2 = (0, 0, 1)$ and $(0, 1, 0)^3 = (1, 0, 0)$. These 3 directions are important in order to obtain the observed hierarchy of charged lepton masses: the electron, muon and tauon masses arise at order $(\langle \varphi_T \rangle / \Lambda)^3$, $(\langle \varphi_T \rangle / \Lambda)^2$, and $\langle \varphi_T \rangle / \Lambda$, respectively, where $\Lambda$ is the cutoff. For $\langle \varphi_T \rangle / \Lambda \sim O(\lambda^3)$, with $\lambda_C$ being the Cabibbo angle, the correct hierarchy is reproduced. In the following we first assume that the stated alignment actually occurs and describe the LO structure of the model. Then in subsect. 2.3 we will show that the alignment is indeed naturally realized at LO from the most general superpotential allowed by the symmetry of the model.

2.1. Charged leptons

The leading order structure of the vacua in eqs.(1,2) automatically generates a diagonal charged lepton matrix, through the following superpotential terms:

$$w_l = \frac{y_{e}}{\Lambda} e^c(\ell \varphi_T) h_d +$$

$$\frac{y_{\mu}}{\Lambda^2} \mu^c(\ell \varphi_T \varphi_T) h_d + \frac{y_{\mu}}{\Lambda^2} \mu^c(\ell \varphi_T \varphi_T)^{\prime} \xi' h_d +$$

$$\frac{y_{\tau}}{\Lambda^3} e^c(\ell \varphi_T \varphi_T)^{\prime} \xi' h_d + \frac{y_{\mu}^\prime}{\Lambda^3} e^c(\ell \varphi_T \varphi_T)^{\prime} \xi^2 h_d + \frac{y_{\mu}^\prime}{\Lambda^3} e^c(\ell \varphi_T \varphi_T)^{\prime} \xi^3 h_d +$$

$$\frac{y_{\tau}^\prime}{\Lambda^3} e^c(\ell \varphi_T \varphi_T)^{\prime} \xi h_d + \frac{y_{\tau}^\prime}{\Lambda^3} e^c(\ell \varphi_T \varphi_T)^{\prime} \xi h_d + \ldots .$$

| Field | $\nu^c$ | $\ell^c$ | $\mu^c$ | $\tau^c$ | $h_u$ | $h_d$ | $\varphi_T$ | $\xi'$ | $\varphi_S$ | $\xi$ | $\varphi_{0}^\prime$ | $\varphi_{0}''$ | $\xi_0$ |
|-------|--------|--------|--------|--------|-----|-----|---------|------|---------|-----|--------|--------|-----|
| $A_4$ | 3      | 3      | 1      | 1      | 1   | 1   | 3       | 1    | 3       | 3   | 3      | 1      | 1   |
| $Z_4$ | -i     | i      | i      | -i     | 1   | 1   | i       | i    | 1       | 1   | -1     | 1      | 1   |
| $U(1)_R$ | 1    | 1      | 1      | 1      | 0   | 0   | 0       | 0    | 0       | 0   | 2      | 2      | 2   |

Table 1. Transformation properties of leptons, electroweak Higgs doublets and flavons under $A_4 \times Z_4$ and $U(1)_R$.
In the above expression for the superpotential $w_l$, for each charged lepton flavour, only the lowest order operators in an expansion in powers of $1/\Lambda$ are explicitly shown. Dots stand for higher dimensional operators that will be discussed later on. After symmetry breaking, the mass matrix has the form:

$$m_\ell = \begin{pmatrix} \frac{x_{\ell}}{\Lambda} (2y_{\ell} v_T^u u' + y_{\ell}' u'^2 + y_{\ell}'' v_T^2) & 0 & 0 \\ 0 & \frac{x_{\ell}}{\Lambda} (2y_{\ell} v_T^u + y_{\ell}' u') & 0 \\ 0 & 0 & \frac{y_{\ell} v_T^u v_T^T}{} \end{pmatrix},$$

where $v_d = \langle h_d \rangle$. As a result, the charged lepton mass matrix is diagonal and with hierarchical entries. To estimate the order of magnitude of $v_T$ and $u'$, we can use the experimental information on the ratio of lepton masses. Assuming that all the $y$ coefficients are of $O(1)$, one obtains:

$$\frac{m_\mu}{m_\tau} \sim 2 \varepsilon + \varepsilon_u \approx 0.06 \quad \frac{m_e}{m_\tau} \sim 2 \varepsilon \varepsilon_u + \varepsilon^2 \approx 0.0003$$

where we introduced the small quantities $\varepsilon = v_T/\Lambda$ and $\varepsilon_u = u'/\Lambda$. These relations are satisfied for both sets of values $(\varepsilon, \varepsilon_u) = (0.043, -0.025)$ and $(\varepsilon, \varepsilon_u) = (0.077, -0.094)$. As we see, we can roughly assume that both $\varepsilon$ and $\varepsilon_u$ are of the same order of magnitude, $O(\Lambda_C^2)$.

### 2.2. Neutrinos

In the neutrino sector the superpotential is given by:

$$w_\nu = y_\nu (\nu^c \ell) h_u + (M + a \xi) \nu^c \nu^c + b \nu^c \nu^c \varphi_S$$

where $a$ and $b$ are generic coefficients and $M$ is a constant with dimension of mass. Note that we also included in the LO neutrino superpotential linear terms in $\varphi_S$ and $\xi$.

The Dirac mass matrix is obtained from the first term in eq.(5) and it is given by:

$$m_D = y_\nu v_u \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = v_u Y_\nu.$$

The other terms lead to the Majorana mass matrix:

$$m_M = \begin{pmatrix} M + a u + 2 b v_S & -b v_S & -b v_S \\ -b v_S & 2 b v_S & M + a u - b v_S \\ -b v_S & M + a u - b v_S & 2 b v_S \end{pmatrix}.$$  

The light neutrino mass matrix is then given by the see-saw formula $m_{\text{light}} = -m_D^T m_M^{-1} m_D$. Note that all matrices $m_D$, $m_M$, $m_M^{-1}$ and $m_{\text{light}}$ are of the general form

$$m = \begin{pmatrix} x & y & y \\ y & x + v & y - v \\ y & y - v & x + v \end{pmatrix}$$

and therefore are diagonalized by $U_{\text{TB}}$:

$$U_{\text{TB}} = \begin{pmatrix} \sqrt{2/3} & 1/\sqrt{3} & 0 \\ -1/\sqrt{6} & 1/\sqrt{3} & -1/\sqrt{2} \\ -1/\sqrt{6} & 1/\sqrt{3} & +1/\sqrt{2} \end{pmatrix}. $$
The $m_{\text{light}}$ eigenvalues are:

\[
\begin{align*}
    m_1 &= -\left(\frac{v_u^2 g_y^2}{M + au + 3bv_S}\right), \\
    m_2 &= -\left(\frac{v_u^2 g_y^2}{M + au}\right) \\
    m_3 &= \left(\frac{v_u^2 g_y^2}{M + au - 3bv_S}\right) .
\end{align*}
\] (10)

2.3. Alignment
At LO the most general driving superpotential $w_d$ invariant under $A_4 \times Z_4$ with $R = 2$ is given by

\[
w_d = M(\varphi_0^S \varphi_S) + g_1(\varphi_0^S \varphi_S \varphi_S) + g_2 \xi(\varphi_0^S \varphi_S) + g_3 \xi_0(\varphi_S \varphi_S) + g_4 \xi \xi^2 + M_\xi \xi_0 \xi (11)
\] + $M_0^2 \xi_0 + h_1 \xi'(\varphi_0^T \varphi_T)^n + h_2(\varphi_0^T \varphi_T \varphi_T) .
\]

The equations giving the vacuum structure for the fields $\varphi_T$ and $\xi'$ are:

\[
\begin{align*}
    \frac{\partial w}{\partial \varphi_{01}} &= 2h_2(\varphi_{T1}^2 - \varphi_{T2} \varphi_{T3}) + h_1 \xi' \varphi_{T3} = 0 \\
    \frac{\partial w}{\partial \varphi_{02}} &= 2h_2(\varphi_{T2}^2 - \varphi_{T1} \varphi_{T3}) + h_1 \xi' \varphi_{T2} = 0 (12) \\
    \frac{\partial w}{\partial \varphi_{01}} &= 2h_2(\varphi_{T3}^2 - \varphi_{T1} \varphi_{T2}) + h_1 \xi' \varphi_{T3} = 0
\end{align*}
\]

whose solutions are:

\[
\langle \xi' \rangle = u' \neq 0 , \quad \langle \varphi_T \rangle = (0, v_T, 0) , \quad v_T = -\frac{h_1 u'}{2h_2} , \quad (13)
\]
with $u'$ undetermined.

With the same procedure, from Eq. (11) we can obtain the equations from which to extract the vacuum expectation values for $\varphi_S$ and $\xi$; an extremum solution is of the form:

\[
\begin{align*}
    \langle \varphi_S \rangle &= (v_S, v_S, v_S) \\
    \langle \xi \rangle &= u . \quad (14)
\end{align*}
\]

Note that we expect a common order of magnitude for the VEV’s (scaled by the cutoff $\Lambda$):

\[
\frac{v_T}{\Lambda} \sim \frac{u'}{\Lambda} \sim \varepsilon, \quad \frac{v_S}{\Lambda} \sim \frac{u}{\Lambda} \sim \varepsilon' . \quad (15)
\]

However, as the minimization equations for the two sets are separate, we can tolerate a moderate hierarchy between $\varepsilon$ and $\varepsilon'$.

3. Beyond the leading order
At the next level of approximation each term $w_l$, $w_\nu$, and $w_d$ of the superpotential is corrected by operators of higher dimension whose contributions are suppressed by at least one power of VEV's$/\Lambda$. The corrections to $w_d$ determine small deviations from the LO VEV alignment configuration. The next to the leading order (NLO) corrections to mass and mixing matrices are
obtained by inserting the corrected VEV alignment in the LO operators plus the contribution of the new operators evaluated with the unperturbed VEV’s.

The VEV configuration obtained from \( w_d + \Delta w_d \), where \( \Delta w_d \) is the most general set of terms suppressed by one power of the cutoff is given by:

\[
\langle \varphi_S \rangle = (v_S + \delta v_S, v_S + \delta v_S, v_S + \delta v_S)
\]

\[
\langle \varphi_T \rangle = (\delta v_{T_1}, v_T + \delta v_{T_2}, \delta v_{T_3})
\]

\[
\langle \xi \rangle = u + \delta u
\]

(16)

and \( u' \) is still undetermined. Thus \( \langle \varphi_S \rangle \) acquires \( \mathcal{O}(1/\Lambda) \) corrections in the same direction, whereas all components of \( \langle \varphi_T \rangle \) acquire different corrections so that its alignment is tilted.

In the charged lepton sector, the correction \( \Delta w_l \) is obtained by adding to each term of \( w_l \) one factor of \( \varphi_S/\Lambda \) or \( \xi/\Lambda \) in all possible ways with arbitrary coefficients. As a result, each diagonal entry gets a small correction, while all non-diagonal entries become non-vanishing and of the order of the diagonal term in each row multiplied by \( \varepsilon' \):

\[
m_\ell = \varepsilon v_d \begin{pmatrix}
a_1 \varepsilon^2 & a_2 \varepsilon^2 \varepsilon' & a_3 \varepsilon^2 \varepsilon' \\
b_1 \varepsilon \varepsilon' & b_2 \varepsilon & b_3 \varepsilon \varepsilon' \\
c_1 \varepsilon' & c_2 \varepsilon' & c_3 \varepsilon'
\end{pmatrix}
\]

where the coefficients \( a_i, b_i \) and \( c_i \) are \( \mathcal{O}(1) \) unspecified constants. This pattern is not altered when one adds the corrections from inserting the shifted VEV’s in the LO expression of \( w_l \). The matrix \( m_\ell \) can be diagonalized to \( \text{Diag}[|a_1^2 \varepsilon^4|, |b_2^2 \varepsilon^2|, |c_3^2 \varepsilon'|] \) by the unitary transformation

\[
U_\ell = \begin{pmatrix}
1 & (\frac{b_2}{c_3}) \varepsilon' & (\frac{a_2}{c_3}) \varepsilon' \\
-\frac{b_2}{c_3} \varepsilon' & 1 & (\frac{a_2}{c_3}) \varepsilon' \\
-\frac{a_1}{c_3} \varepsilon' & \frac{a_1}{c_3} \varepsilon' & 1
\end{pmatrix}
\]

Note that, at this order, the coefficients of the electron row in eq. (17) do not enter in \( U_\ell \). From the matrix in eq. (18), we can compute the corresponding corrections to the TB mixing matrix according to \( U_{PMNS} = U_\ell^T U_{TB} \) and all entries of \( U_{TB} \) get corrected to \( \mathcal{O}(1/\Lambda) \).

In the neutrino sector the corrections due to inserting the VEV shifts in the LO operators do not affect the Dirac mass at all while the changes in the Majorana mass are still of the form that is diagonalized by the TB mixing matrix. However the corrections from operators of higher dimension obtained by inserting one extra power of \( \varphi_S/\Lambda \) or \( \xi/\Lambda \), of order \( \varepsilon' \), alter both the Dirac and the Majorana mass in such a way that the TB mixing pattern is completely violated by small corrective terms. As a result, the overall correction to TB mixing arises from the most general symmetric matrix of order \( v_S \varepsilon' \):

\[
m_\nu = -m_D^T m_M^{-1} m_D = (m_\nu)_{TB} + v_S \varepsilon' \begin{pmatrix}
A & B & C \\
B & D & F \\
C & F & E
\end{pmatrix}
\]

where by \( (m_\nu)_{TB} \) we denote the matrix, diagonalizable by \( U_{TB} \), which is obtained from the LO term plus the corrective terms that can be reabsorbed in the LO coefficients as they preserve TB mixing. In conclusion when the NLO corrections are included TB mixing is violated by small terms and one expects:

\[
\sin^2 \theta_{12} = \frac{1}{3} + \mathcal{O}(\varepsilon') \\
\sin^2 \theta_{23} = \frac{1}{2} + \mathcal{O}(\varepsilon') \\
\sin \theta_{13} = \mathcal{O}(\varepsilon')
\]

(20)

We have already noted that the data require that \( \varepsilon' \leq \mathcal{O}(\lambda_3^2) \).
4. Light neutrino spectrum and constraints from $r$

It is useful to study the constraints on the model imposed by the observed values of $\Delta m^2_{\text{atm}}$ and of the ratio $r = \Delta m^2_{\text{sol}}/|\Delta m^2_{\text{atm}}|$. Here $\Delta m^2_{\text{sol}} = |m_2|^2 - |m_1|^2 > 0$, $\Delta m^2_{\text{atm}} = |m_3|^2 - |m_1|^2$. We do this in the LO approximation. In fact the results of the previous section indicate that the corrections to the spectrum are sufficiently small to be neglected for a first orientation. For this discussion we adopt the following parameterization:

$$
A = M + au = |A| e^{i\phi_A},
B = 3bu_S = |B| e^{i\phi_B},
\alpha = \frac{|B|}{|A|},
\phi = \phi_B - \phi_A.
$$

In LO the masses can then be written as:

$$
m_1 = -\frac{v_u^2 y_u^2}{|A|} e^{i\phi_A} \left( \frac{1}{1 + \alpha e^{i\phi}} \right),
m_2 = -\frac{v_u^2 y_u^2}{|A|} e^{i\phi_A},
m_3 = \frac{v_u^2 y_u^2}{|A|} e^{i\phi_A} \left( \frac{1}{1 + \alpha e^{i\phi}} \right).
$$

In terms of the parameters $\alpha$ (which is real and positive) and $\phi$, the ratio $r$ is:

$$
r = \frac{\Delta m^2_{\text{sol}}}{|\Delta m^2_{\text{atm}}|} = \frac{(1 + \alpha^2 - 2\alpha \cos \phi)(\alpha + 2\cos \phi)}{4|\cos \phi|}.
$$

The limit $\alpha \to 0$ gives $r = 1/2$, which is too large compared to the experimental value. Thus in order to accommodate $r \sim 1/30$ one needs a value of $\alpha$ of $O(1)$, which implies that (see eq. (21)) $|A| = |M + au| \sim |B| = 3|bu_S|$. With $|a|, |b| \sim O(1)$ this is obtained if $|M| \sim u, v_S$, or, in other words, $|M|$ must be sizeably smaller than the cutoff $\Lambda$. We interpret this result as related to the fact that the RH neutrino Majorana mass $M$ must empirically be close to $M_{\text{GUT}}$. This means that in the context of a grand unified theory $M$ must be of $O(M_{\text{GUT}})$ rather than of $O(M_{\text{Planck}})$. More precisely, from Eq. (23) one recognizes that a small $r$ can be reproduced if

$$
a) \quad \cos \phi \sim \alpha \sim 1
$$

$$
b) \quad \cos \phi = -\frac{\alpha}{2} + \delta\alpha \quad \delta\alpha \sim O(r).
$$

The first condition $a)$ corresponds to a normal hierarchy spectrum whereas the second condition $b)$ leads to an inverted hierarchy scheme.

In the normal hierarchy case, $a)$, by expanding around $\cos \phi \sim \alpha \sim 1$, we obtain:

$$
|m_1|^2 = \frac{1}{3} \Delta m^2_{\text{atm}} r + ...
$$

$$
|m_2|^2 = \frac{4}{3} \Delta m^2_{\text{atm}} r + ...
$$

$$
|m_3|^2 = \left(1 + \frac{r}{3}\right) \Delta m^2_{\text{atm}} + ...
$$

$$
|m_{ee}|^2 = \frac{16}{27} \Delta m^2_{\text{atm}} r + \ldots \sim (0.007 \text{eV})^2,
$$
where we have expressed the parameters in terms of $\Delta m^2_{atm}$ and $r$. Dots denote terms of order $r^2$ as well as corrections beyond the LO. Note that in this model $|m_1|$ cannot vanish. In the last line $|m_{ee}|$ is the effective mass combination controlling the violation of the total lepton number in neutrinoless double beta decay.

In the inverse hierarchy case $b)$, we set $2 \cos \phi = -\alpha + \delta$ with $\delta$ positive and small and vary $\alpha$ in the range between 0.07 and 2 (the lower limit comes from the absolute bound on the squared masses at fixed $\Delta m^2_{atm}$ and $r$, taken indicatively at $|m_i| \leq 0.5$ eV). The quantity $\delta$ is determined in terms of $r$:

$$\delta = \frac{2\alpha r}{1 + 2\alpha^2}$$  \hspace{1cm} (26)

while the scale of the squared masses is fixed by $|\Delta m^2_{atm}|$. In terms of $\alpha$ one obtains:

$$|m_2|^2 = |m_1|^2 + |\Delta m^2_{atm}| r = |\Delta m^2_{atm}| \left( \frac{1 + 2\alpha^2}{2\alpha^2} \right) \left[ 1 + r \left( 1 + \frac{1}{(1 + 2\alpha^2)^2} \right) \right] + ...$$

$$|m_3|^2 = |m_1|^2 - |\Delta m^2_{atm}|.$$  \hspace{1cm} (27)

Note that the $r$ terms are sufficiently small that the corresponding contributions could be overshadowed by the neglected NLO corrections. Omitting these additional linear terms in $r$, $m_{ee}$ is given by:

$$|m_{ee}|^2 = |\Delta m^2_{atm}| \left( \frac{1 + 2\alpha^2}{2\alpha^2} \right) \left( 1 - \frac{2}{9} \alpha^2 \right) + ...$$  \hspace{1cm} (28)

$|m_{ee}|^2$ is a decreasing function of $\alpha$ in the physical range and is close to $|\Delta m^2_{atm}| \sim (0.05 \text{ eV})^2$ for $\alpha = 1$ and to $|\Delta m^2_{atm}|/8$ for $\alpha = 2$. The behaviour of $|m_{ee}|$ as a function of $\alpha$, including the neglected terms in $r$, is shown in the left panel of Fig.(1). The most pronounced inverse hierarchy is realized for $\alpha = 2$ where $|m_3| \sim |m_1|/3$, as it is seen from the right panel of Fig.(1).

![Figure 1. Behaviour of neutrino masses in the inverted hierarchy case (at fixed $\Delta m^2_{atm}$ and $r$) as a function of $\alpha$ in the range between 0.07 and 2 (the lower bound on $\alpha$ corresponds to an upper bound on $|m_i|$). Left panel: $|m_{ee}|$. Right panel: $|m_2|$, $|m_3|$ and the ratio $|m_3|/|m_2|$.]
5. Conclusion
We have presented and discussed an $A_4$ model for TB mixing of the see-saw type which, in spite of being based on a most economical flavour symmetry and field content, still it is phenomenologically viable. In particular TB mixing is exact in LO while all mixing angles receive corrections at higher orders. The charged lepton mass hierarchy is determined by the $A_4 \times Z_4$ flavour symmetry itself without invoking a Froggatt-Nielsen $U(1)$ symmetry. A value of $\theta_{13} \sim \mathcal{O}(\lambda^2)$ is indicated which is within the sensitivity of the experiments which are now in preparation and will take data in the near future. This example shows once more that the results derived from $A_4$ are robust and, in particular, do not depend on the detailed mechanism that produces the hierarchy of charged lepton masses. The model is compatible with either a normal hierarchy or an inverse hierarchy spectrum. We have studied the spectrum in detail in these different cases and discussed the predictions for the mass eigenvalues and the angle $\theta_{13}$.

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