A one-dimensional diffusion model for overloaded queues with customer abandonment

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Abstract

We use an Ornstein–Uhlenbeck (OU) process to approximate the queue length process in a GI/GI/n + M queue. This one-dimensional diffusion model is able to produce accurate performance estimates in two overloaded regimes: In the first regime, the number of servers is large and the mean patience time is comparable to or longer than the mean service time; in the second regime, the number of servers can be arbitrary but the mean patience time is much longer than the mean service time. Using the diffusion model, we obtain Gaussian approximations for the steady-state queue length and the steady-state virtual waiting time. Numerical experiments demonstrate that the approximate distributions are satisfactory for queues in these two regimes.

To mathematically justify the diffusion model, we formulate the two overloaded regimes into an asymptotic framework by considering a sequence of queues. The mean patience time goes to infinity in both asymptotic regimes, whereas the number of servers approaches infinity in the first regime but does not change in the second. The OU process is proved to be the diffusion limit for the queue length processes in both regimes. A crucial tool for proving the diffusion limit is a functional central limit theorem for the superposition of time-scaled renewal processes. We prove that the superposition of n independent, identically distributed stationary renewal processes, after being centered and scaled in both space and time, converges in distribution to a Brownian motion as n goes to infinity.

1 Introduction

Consider a GI/GI/n + M queue. The customer arrival process of this system is a renewal process and the service times are independent, identically distributed (iid) nonnegative random variables. Customers are served by n identical servers. Upon arrival, a customer gets into service if an idle server is available; otherwise, he waits in a buffer with infinite room. Waiting customers are served on the first-come, first-served basis, and the servers are not allowed to idle if there are customers waiting. Each customer has a random patience time. When a customer’s waiting time exceeds his patience time, the customer abandons the system without being served. The patience times are iid following an exponential distribution, and the sequences of interarrival, service, and patience times are mutually independent.
We are interested in the performance of this queue when it is overloaded, i.e., the customer arrival rate is greater than the service capacity. In this case, not all customers are able to receive service and a fraction of them must abandon the system. We use a simple one-dimensional diffusion process to approximate the scaled queue length process. This diffusion process is an Ornstein–Uhlenbeck (OU) process. The diffusion model is able to produce accurate performance estimates when the queue is operated in either of the following two overloaded regimes. In the first regime, the number of servers is large and the mean patience time is comparable to or longer than the mean service time. We call it the many-server overloaded regime. In the second regime, the number of servers can be arbitrary, but the mean patience time is much longer (i.e., on a higher order) than the mean service time. This regime is referred to as the long patience overloaded regime. In this paper, most efforts are focused on queues in the many-server overloaded regime.

Queues with customer abandonment are used to model service systems. A call center with many service agents is a typical example; see Gans et al. (2003) for a comprehensive review. Because the rate of incoming calls changes over time, a call center may become overloaded during the peak hours of a day. Waiting on a phone line, a customer may hang up the phone before being connected to an agent. Although customer abandonment is present in most call centers, empirical studies suggest that customers are generally patient when they hold the line. It was reported in Mandelbaum et al. (2001) and Mandelbaum and Zeltyn (2013) that in the call center of an Israeli bank, the mean customer patience time was at least several times longer than the mean service time. The many-server overloaded regime is thus relevant to call center operations. As pointed out by Whitt (2006), in service-oriented call centers, staffing costs usually dominate the expenses of customer delay and abandonment. The rational operational regime for these systems is the efficiency-driven (ED) regime that emphasizes server utilization over the quality of service. In the ED regime, the service capacity is set below the customer arrival rate by a moderate fraction. Because the lost service demands of abandoning customers compensate for the excess in the arrival rate over the service capacity, a many-server queue operated in the ED regime can still achieve reasonable performance. More specifically, the mean waiting time is comparable to the mean service time, a moderate fraction of customers abandon the system, and all servers are almost always busy. The ED regime is closely related to the many-server overloaded regime studied in this paper.

For queues in the ED regime, a fluid model proposed by Whitt (2006) is useful in estimating several performance measures, including the fraction of abandonment, the mean queue length, and the mean virtual waiting time. In the M/M/n + GI setting, the accuracy of the fluid model was studied by Bassamboo and Randhawa (2010). They proved that in the steady state, the accuracy gaps of the fluid approximations for the mean queue length and the rate of customer abandonment do not increase with the arrival rate. This implies that fluid approximations could be particularly accurate when the queue is operated in the ED regime. Such a deterministic model, however, cannot be used to estimate any nontrivial probability or distribution. In other words, we cannot estimate the distribution of queue length or customer waiting time using the fluid model. The performance targets of a service system may require a tail probability to be less than a specified value, e.g., “80% of customers wait less than 2 minutes.” A refined model is thus necessary to obtain such an estimate. The proposed one-dimensional diffusion model offers a simple yet accurate refinement for the fluid model. Although the exponential patience time assumption is somewhat restrictive, by certain modification the diffusion model may extend to GI/GI/n + GI queues, allowing for a
general patience time distribution; see Section 7 for an illustration.

Customers would wait long if the service is of critical importance. When such a system gets overloaded, customer waiting times will be prolonged significantly. The long patience overloaded regime is relevant to this type of systems. One important example is an organ transplant waiting list: The transplant candidates on the list form a queue; when an organ is found, the candidate at the top of the list receives the organ. These candidates may abandon the waiting list either because of death, or because their health has deteriorated so that transplantation is no longer appropriate. As the need for organs usually far exceeds the supply of donors, a transplant candidate may have to wait for years before transplantation. Such a system must be operated in the long patience overloaded regime; see Su and Zenios (2004, 2006). Jennings and Reed (2012) studied fluid and diffusion models for the virtual waiting time process of a single-server queue in this regime.

As an OU process, the diffusion model has a Gaussian stationary distribution. This fact allows us to approximate the steady-state queue length and virtual waiting time distributions by Gaussian distributions. The proposed diffusion model, whether for a many-server queue or for a queue with one or several servers, depends on the interarrival and service time distributions only through their first two moments. This is in sharp contrast to the approximate models for many-server queues in the literature, where the entire service time distribution is built into the fluid or diffusion equations; see, e.g., Whitt (2006), Kang and Ramanan (2010), Mandelbaum and Momčilović (2012), and Zhang (2013). With a general service time distribution, these approximate models are either non-Markovian or deterministic. It is difficult to obtain the steady-state queue length and virtual waiting time distributions using these models. When the service time distribution is phase-type, Dai et al. (2010) proved a multi-dimensional diffusion limit for many-server queues in an overloaded regime. As phase-type distributions can approximate any positive-valued distribution, this model is still relevant to queues with a general service time distribution. Using this multi-dimensional model, Dai and He (2013) proposed a finite element algorithm for computing the steady-state queue length distribution. Although the algorithm is able to produce accurate performance estimates, the computational complexity increases exponentially as the dimension of the diffusion model grows. The curse of dimensionality is a serious issue when the dimension is not small. In contrast, the diffusion model proposed in this paper is a one-dimensional process that has an explicit stationary distribution, thus leading to simple performance formulas.

The one-dimensional diffusion model is rooted in the limit theorems presented in Section 3. The diffusion limits for queues in the many-server overloaded regime and in the long patience overloaded regime can be found in Theorems 1 and 4, respectively. Although the two limit processes are identical, it is more challenging to prove the diffusion limit in the many-server regime. In this regime, we consider a sequence of queues indexed by the number of servers $n$, and assume that the mean patience time goes to infinity as $n$ goes large. The queue length processes within this asymptotic framework are scaled in both space and time, with the number of servers and the mean patience time being the respective scaling factors. This space-time scaling is essential to obtain a one-dimensional diffusion limit when the service time distribution is general. In the asymptotic regimes specified in Dai et al. (2010) and Mandelbaum and Momčilović (2012), by contrast, the queue length processes are scaled only in space and not in time. In this case, only when the service time distribution is exponential, will the scaled queue length processes converge to a one-dimensional Markov process.
The technique of scaling in both space and time has been used in [Whitt 2003, 2004], Gurvich (2004), and Atar (2012) for many-server queues with an exponential service time distribution. It is not surprising that the diffusion limits in these papers are one-dimensional. Theorem 1 in our paper demonstrates that by the means of space-time scaling, many-server queues with a general service time distribution may also have a one-dimensional diffusion limit when they are overloaded. The space-time scaling used in our model is similar to the scaling used in Theorem 4.1 in Whitt (2004), where a sequence of M/M/n/r + M queues is studied in an overloaded regime. This regime allows either the number of servers or the mean patience time or both of them to go to infinity, all of which lead to an OU limit process. The latter two cases of this regime correspond to the long patience overloaded regime and the many-server overloaded regime, respectively. In this sense, Theorems 1 and 4 in our paper have extended the OU limit for overloaded queues to a much more general setting. A critically loaded regime, known as the nondegenerate slowdown regime, is studied by Whitt (2003); Gurvich (2004), and Atar (2012) for many-server queues with an exponential service time distribution. In this regime, the diffusion limit for the queue length processes, which is also scaled in both space and time, is either a reflected OU process when the patience time distribution is exponential, or a reflected Brownian motion when there is no abandonment.

The most important tool for proving the diffusion limit in the many-server regime is a functional central limit theorem (FCLT) for the superposition of time-scaled, stationary renewal processes, which is presented in Theorem 3. The well-known FCLT for renewal processes states that as the scaling factor goes to infinity, a time-scaled renewal process converges in distribution to a Brownian motion. Whitt (1985) proved an FCLT for the superposition of renewal processes, which states that the superposition of n iid stationary renewal processes, after being scaled in space, converges in distribution to a Gaussian process. In general, this Gaussian process is not a Brownian motion. Theorem 3 in our paper is a supplement to these results. We prove that the superposition of n iid stationary renewal processes, after being scaled in both space and time, converges in distribution to a Brownian motion again. This theorem allows us to approximate the scaled service completion process by a Brownian motion, which is the key to approximating the scaled queue length process by a one-dimensional diffusion process. To apply this theorem, we consider a sequence of perturbed systems that are asymptotically equivalent to the original queues but have simpler dynamics. We assume that servers in a perturbed system are always busy so that the service completion process is the superposition of n renewal processes. Using the simplified dynamics of perturbed systems, we prove the many-server diffusion limit by a standard continuous mapping approach.

The remainder of the paper is organized as follows. The diffusion model and the approximate formulas are introduced in Section 2. Their underlying limit theorems are presented in Section 3. We examine the performance formulas by numerical examples in Section 4. Sections 5 and 6 are dedicated to the respective proofs of Theorems 1 and 2. Future research topics are discussed in Section 7. We leave the proof of Theorem 3 to the appendix.

Notation

All random variables and processes are defined on a common probability space \((\Omega, \mathcal{F}, \mathbb{P})\). We reserve \(\mathbb{E}[\cdot]\) for expectation. The symbols \(\mathbb{N}, \mathbb{N}_0, \mathbb{R}, \) and \(\mathbb{R}_+\) are used to denote the sets of positive integers, nonnegative integers, real numbers, and nonnegative real numbers, respectively. The space of functions \(f: \mathbb{R}_+ \to \mathbb{R}\) that are right-continuous on \([0, \infty)\) and have left limits on \((0, \infty)\) is.
denoted by $\mathbb{D}$, which is endowed with the Skorohod $J_1$ topology. Given an arbitrary function $f \in \mathbb{D}$ and a function $g \in \mathbb{D}$ that is nondecreasing and takes values in $\mathbb{R}_+$, $f \circ g$ denotes the composed function in $\mathbb{D}$ with $(f \circ g)(t) = f(g(t))$ for $t \geq 0$. For a sequence of random variables (or processes) $\{\xi_n : n \in \mathbb{N}\}$ taking values in $\mathbb{R}$ (or $\mathbb{D}$), we write $\xi_n \xrightarrow{a.s.} \xi$ for the almost sure convergence of $\xi_n$ to $\xi$ and write $\xi_n \Rightarrow \xi$ for the convergence of $\xi_n$ to $\xi$ in distribution, where $\xi$ is a random variable with values in $\mathbb{R}$ (or a process with values in $\mathbb{D}$). For a random variable $\xi$ with mean $m_\xi > 0$ and variance $\sigma^2_\xi \geq 0$, the squared coefficient of variation of $\xi$ is defined by $c^2_\xi = \sigma^2_\xi / m^2_\xi$. For any $a, b \in \mathbb{R}$, $a^+ = \max\{a, 0\}$, $a^- = \max\{-a, 0\}$, $a \lor b = \max\{a, b\}$, and $a \land b = \min\{a, b\}$. We use $e$ for the identity function on $\mathbb{R}_+$ and $\chi$ for the constant one function on $\mathbb{R}_+$, i.e., $e(t) = t$ and $\chi(t) = 1$ for $t \geq 0$. For a fixed $s \geq 0$, we use $e_s$ to denote the identity function on $\mathbb{R}_+$ that is capped by $s$, i.e., $e_s(t) = s \land t$ for $t \geq 0$.

2 Diffusion model and performance formulas

Let $\lambda$ be the customer arrival rate and $\mu$ be the service rate of each server. Assume that both interarrival times and service times have finite variances, with squared coefficients of variations $c^2_A$ and $c^2_S$, respectively. As the queue is overloaded, the traffic intensity satisfies $\rho = \lambda / (n\mu) > 1$. If all servers are almost always busy, the fraction of abandoning customers can be approximated by

$$\alpha \approx \rho - 1. \tag{2.1}$$

Let $\gamma$ be the mean patience time. Since patient times are exponentially distributed, each waiting customer abandons the system at rate $1/\gamma$. When the queue is in the steady state, the total abandonment rate from the buffer must be around $n\mu(\rho - 1)$ by the conservation of flow. Hence, the mean queue length (i.e., the mean number of customers in the buffer) can be approximated by

$$q \approx n\mu(\rho - 1)\gamma. \tag{2.2}$$

Let $X(t)$ be the number of customers in the system at time $t$, which fluctuates around $n + q$ as the queue comes into the steady state. To describe the evolution of queue length around the mean, we introduce a scaled version of $X$ by

$$\tilde{X}(t) = \frac{1}{\sqrt{n\gamma}}(X(\gamma t) - n - q).$$

We call $\tilde{X}$ the scaled queue length process. Note that after the mean is removed, $X$ is scaled in both space and time. Besides the commonly used scaling in space by the number of servers, we also change the time scale of the process with the mean patience time as the factor. We propose to use a one-dimensional diffusion process $\hat{X}$ to approximate the scaled queue length process. The initial value of $\hat{X}$ may be taken as $\hat{X}(0) = \tilde{X}(0)$. This diffusion process is an OU process that satisfies the following stochastic differential equation

$$\hat{X}(t) = \hat{M}(t) - \int_0^t \hat{X}(u) \, du \quad \text{for } t \geq 0. \tag{2.3}$$
Here, $\hat{M}$ is a driftless Brownian motion with variance $\mu(\rho c_A^2 + c_S^2 + \rho - 1)$ and $\hat{M}(0) = \hat{X}(0)$.

The OU process is a reasonable model because the queue length process is mean-reverting: At any time, the instantaneous customer abandonment rate from the buffer is proportional to the queue length; when the queue length is either too long or too short, the increased or decreased abandonment rate will pull it back to the equilibrium level. For the diffusion model to be accurate, the mean patience time $\gamma$, serving as the scaling factor in time, should be relatively long compared with the mean service time. More specifically, this model is able to produce satisfactory performance approximations for queues in the many-server overloaded regime and queues in the long patience overloaded regime. The diffusion model in these two regimes is formalized by Theorems 1 and 4, where both regimes are built into an asymptotic framework and $\hat{X}$ is proved to be the limit of the scaled queue length processes. Although the mean patience time goes to infinity in the asymptotic framework, the diffusion model may still work well for a many-server queue when the mean patience time is comparable to or just several times longer than the mean service time. If the number of servers is not many, however, the mean patience time is usually required to be much longer than the mean service time. See Section 4 for further discussion.

The one-dimensional diffusion model yields useful performance approximations. It is well known that the stationary distribution of the OU process is Gaussian. In particular, $\hat{X}$ has a Gaussian stationary distribution with mean 0 and variance $\mu(\rho c_A^2 + c_S^2 + \rho - 1)/2$. Let $X(\infty)$ be the stationary number of customers in the system and $\tilde{X}(\infty)$ be the scaled version. Because $\hat{X}$ is an approximation of $\tilde{X}$, their steady-state distributions are expected to be close, i.e.,

$$\mathbb{P}[X(\infty) > a] \approx 1 - \Phi\left(\frac{\sqrt{2}a}{\sqrt{\mu(\rho c_A^2 + c_S^2 + \rho - 1)}}\right)$$

for $a \in \mathbb{R}$, (2.4)

where $\Phi$ is the standard Gaussian distribution function. As a result, the steady-state queue length approximately follows a Gaussian distribution with mean $q$ and variance

$$\sigma_Q^2 \approx \frac{n\gamma\mu}{2} (\rho c_A^2 + c_S^2 + \rho - 1).$$

(2.5)

Suppose that at time $s \geq 0$, a hypothetical customer with infinite patience arrives at the queue. Let $W(s)$ be the amount of time this hypothetical customer has to wait before getting into service. This waiting time is called the virtual waiting time at $s$. In the steady state, the virtual waiting time process fluctuates around its mean $w$, which can be determined as follows. As the patience time distribution is exponential with mean $\gamma$, the fraction of customers whose patience times are longer than $w$ is $\exp(-w/\gamma)$. This fraction should be approximately equal to the fraction of customers who eventually receive service, so that $\exp(-w/\gamma) \approx 1/\rho$, or

$$w \approx \gamma \log \rho.$$  

(2.6)

We are interested in the distribution of the steady-state virtual waiting time. Let $W(\infty)$ be the virtual waiting time in the steady state, which has a scaled version

$$\tilde{W}(\infty) = \sqrt{n\gamma^{-1}}(W(\infty) - w).$$

Theorems 2 and 5 in Section 3 imply that $\tilde{W}(\infty)$ approximately follows a Gaussian distribution
with mean 0 and variance \((c_A^2 + \rho c_S^2 + \rho - 1)/(2\mu\rho)\), i.e.,

\[
P[\tilde{W}(\infty) > a] \approx 1 - \Phi\left(\frac{a\sqrt{2\mu\rho}}{\sqrt{c_A^2 + \rho c_S^2 + \rho - 1}}\right) \quad \text{for } a \in \mathbb{R}. \tag{2.7}
\]

Hence, the virtual waiting time in the steady state approximately follows a Gaussian distribution with mean \(w\) and variance

\[
\sigma_W^2 \approx \frac{\gamma}{2n\mu\rho}(c_A^2 + \rho c_S^2 + \rho - 1). \tag{2.8}
\]

Formulas (2.4) and (2.7) provide approximate distributions for the queue length and virtual waiting time in the steady state. They will be examined in Section 4.

3 Limit theorems

In this section, we state the underlying limit theorems for the diffusion model and the approximate formulas. The theorems for queues in the many-server overloaded regime and in the long patience overloaded regime are presented in Sections 3.1 and 3.2 respectively.

3.1 Limits in the many-server overloaded regime

To formulate the many-server overloaded regime, let us consider a sequence of G/GI/n + M queues indexed by the number of servers \(n\). The arrival processes in these queues are not required to be renewal. In each queue, the number of initial customers, the arrival process, the sequence of service times, and the sequence of patience times are mutually independent. All these queues have the same traffic intensity \(\rho > 1\) and the same service time distribution. Because the service rate \(\mu\) is invariant, the arrival rate of the \(n\)th system is

\[
\lambda_n = n\rho\mu. \tag{3.1}
\]

We assume that the mean patience time goes to infinity as \(n\) goes large, i.e.,

\[
\gamma_n \to \infty \quad \text{as } n \to \infty. \tag{3.2}
\]

Let \(F\) be the distribution function of service times. As in Whitt (1985), a mild regularity condition is imposed on \(F\), i.e.,

\[
\limsup_{t \downarrow 0} t^{-1}(F(t) - F(0)) < \infty. \tag{3.3}
\]

We also assume that \(F\) has a finite third moment, i.e.,

\[
\int_0^\infty t^3 dF(t) < \infty. \tag{3.4}
\]

Then, the equilibrium distribution of \(F\) is given by

\[
F_c(t) = \mu \int_0^t (1 - F(u)) du \quad \text{for } t \geq 0.
\]
We assign service times to customers according to the following procedure. Let \( \{\xi_{j,k} : j, k \in \mathbb{N}\} \) be a double sequence of independent nonnegative random variables. For each \( j \in \mathbb{N} \), we assume that

\[ \xi_{j,1} \text{ follows distribution } F_e \text{ and } \xi_{j,k} \text{ follows distribution } F \text{ for } k \geq 2. \] (3.5)

In the \( n \)th system, assume that all \( n \) servers are busy at time 0. For \( j = 1, \ldots, n \), \( \xi_{j,1} \) is assigned to the initial customer served by the \( j \)th server as the residual service time at time 0. For \( k \geq 2 \), \( \xi_{j,k} \) is the service time of the \( k \)th customer served by the \( j \)th server. By this assignment, for all \( j, k \in \mathbb{N} \), the \( k \)th service time by the \( j \)th server is identical in all systems that have at least \( j \) servers.

Let \( E_n(t) \) be the number of arrivals in the \( n \)th system during time interval \((0, t]\). Define the diffusion-scaled arrival process \( \tilde{E}_n \) by

\[ \tilde{E}_n(t) = \frac{1}{\sqrt{n \gamma_n}} (E_n(\gamma_n t) - \lambda_n \gamma_n t). \]

Let \( N \) be a renewal process whose interrenewal times have mean 1 and variance \( c^2_A \). If \( E_n \) is renewal with \( E_n(t) = N(\lambda_n t) \), it follows from (3.1) and the FCLT for renewal processes that

\[ \tilde{E}_n \Rightarrow \hat{E} \quad \text{as } n \to \infty, \] (3.6)

where \( \hat{E} \) is a driftless Brownian motion with variance \( \rho \mu c^2_A \) and \( \hat{E}(0) = 0 \). To allow for more general arrival processes, we take (3.6) as an assumption rather than require each \( E_n \) to be renewal. Let \( X_n(t) \) be the number of customers in the \( n \)th system at time \( t \), which has a diffusion-scaled version

\[ \tilde{X}_n(t) = \frac{1}{\sqrt{n \gamma_n}} (X_n(\gamma_n t) - n - n \mu (\rho - 1) \gamma_n). \]

We assume that there exists a random variable \( \tilde{X}(0) \) such that

\[ \tilde{X}_n(0) \Rightarrow \tilde{X}(0) \quad \text{as } n \to \infty. \] (3.7)

The first theorem states the diffusion limit for queue length processes in the many-server overloaded regime. It justifies the diffusion model when the queue has many servers.

**Theorem 1.** Let \( \tilde{X} \) be the OU process given by [2.3]. Assume that the sequence of \( G/GI/n + M \) queues, each indexed by the number of servers \( n \), satisfies (3.1)–(3.7) with \( \rho > 1 \). Then,

\[ \tilde{X}_n \Rightarrow \tilde{X} \quad \text{as } n \to \infty. \]

The second theorem concerns virtual waiting times in these queues. Let \( W_n(s) \) be the virtual waiting time at \( s \geq 0 \) in the \( n \)th queue. A scaled version is defined by

\[ \tilde{W}_n(s) = \gamma_n^{-1} W_n(\gamma_n s). \]

By [2.6], we expect \( \tilde{W}_n(s) \) to be close to \( \log \rho \). To obtain a refined approximation, we further define

\[ \hat{W}_n(s) = \sqrt{n \gamma_n} (W_n(s) - \log \rho), \]
which describes the variation of the virtual waiting time around the mean. Theorem 2 states that $\tilde{W}_n(s)$ converges in distribution as $n$ goes large.

Let us introduce several processes to state this theorem. Fix $s \geq 0$. Let $\hat{G}^s$ be a standard Brownian motion (the superscript emphasizes that the process may change with $s$) and $\hat{B}$ be a driftless Brownian motion with variance $\mu c_S^2$ and $\hat{B}(0) = 0$. Assume that $\hat{X}(0), \hat{E}, \hat{G}^s,$ and $\hat{B}$ are mutually independent. Define a function $y^s : \mathbb{R}_+ \to \mathbb{R}$ by

$$y^s(t) = \begin{cases} 
  (\rho - 1)\mu & \text{for } 0 \leq t < s, \\
  (\rho \exp(s - t) - 1)\mu & \text{for } s \leq t < s + \log \rho, \\
  -\mu(t - s - \log \rho) & \text{for } t \geq s + \log \rho.
\end{cases} \quad (3.8)$$

Theorem 2. Under the conditions of Theorem 1, for any given $s \geq 0$,

$$\tilde{W}_n(s) \Rightarrow \mu^{-1} \hat{Y}^s(s + \log \rho) \quad \text{as } n \to \infty,$$

where $\hat{Y}^s$ satisfies the stochastic differential equation

$$\hat{Y}^s(t) = \hat{X}(0) + \hat{E}(s \wedge t) - \hat{B}(t) - \hat{G}^s\left(\int_0^t y^s(u) \, du\right) - \int_0^t \hat{Y}^s(u) \, du \quad \text{for } 0 \leq t \leq s + \log \rho. \quad (3.9)$$

In particular,

$$\hat{Y}^s(s + \log \rho) = \exp(-s - \log \rho)\left(\hat{X}(0) + \int_0^s \exp(u) \, d\hat{E}(u) - \int_0^{s+\log \rho} \exp(u) \, d\hat{B}(u) - \int_0^{s+\log \rho} y^s(u)^{1/2} \exp(u) \, d\hat{G}^s(u)\right). \quad (3.9)$$

Put $\hat{W}(s) = \hat{Y}^s(s + \log \rho)/\mu$. As $s$ goes large, $\hat{W}(s)$ converges in distribution to a Gaussian random variable with mean 0 and variance $(c_A^2 + \rho c_S^2 + \rho - 1)/(2\mu \rho)$, which leads to formula (2.7).

The third theorem plays an essential role in proving Theorems 1 and 2. It is an FCLT for the superposition of time-scaled, stationary renewal processes. These renewal processes are defined as follows. For $t \geq 0$ and $j \in \mathbb{N}$, let

$$N_j(t) = \max\{k \in \mathbb{N}_0 : \xi_{j,1} + \cdots + \xi_{j,k} \leq t\}. \quad (3.10)$$

As a convention, we take $N_j(t) = 0$ if $\xi_{j,1} > t$. By (3.5), each $N_j$ is a delayed renewal process with delay distribution $F_\varepsilon$ and interrenewal distribution $F$. Because $F_\varepsilon$ is the equilibrium distribution of $F$, $\{N_j : j \in \mathbb{N}\}$ is a sequence of iid stationary renewal processes.

Theorem 3. Let $\{N_j : j \in \mathbb{N}\}$ be a sequence of iid stationary renewal processes, i.e., the delay distribution $F_\varepsilon$ of each renewal process is the equilibrium distribution of the interrenewal distribution $F$. Assume that $F$ has mean $1/\mu$ and satisfies (3.3) and (3.4). Let

$$B_n(t) = \sum_{j=1}^n N_j(t) \quad (3.11)$$
and \( \{ \gamma_n : n \in \mathbb{N} \} \) be a sequence of positive numbers such that \( \gamma_n \to \infty \) as \( n \to \infty \). Then,

\[
\tilde{B}_n \Rightarrow \hat{B} \quad \text{as} \quad n \to \infty,
\]

where

\[
\tilde{B}_n(t) = \frac{1}{\sqrt{n\gamma_n}} (B_n(\gamma_nt) - n\mu \gamma_nt)
\]

and \( \hat{B} \) is a driftless Brownian motion with variance \( \mu c^2_S \) and \( \hat{B}(0) = 0 \).

Let us compare Theorem 3 with two other FCLTs. Consider the sequence of iid stationary renewal processes \( \{N_j : j \in \mathbb{N} \} \). By the FCLT for renewal processes, \( \{N_1(\ell t) - \ell \mu t) / \sqrt{\ell} : t \geq 0 \} \) converges in distribution to a Brownian motion as \( \ell \) goes to infinity; see Theorem 5.11 in Chen and Yao (2001). Clearly, the increments of this time-scaled renewal process become independent of its history as the scaling factor gets large. Whitt (1985) proved an FCLT for the superposition of stationary renewal processes. It states that \( \{ \sum_{j=1}^n (N_j(t) - \mu t) / \sqrt{n} : t \geq 0 \} \) converges in distribution to a zero-mean Gaussian process that has stationary increments and continuous paths. In this FCLT, the superposition process is scaled in space only. The covariance function of each stationary renewal process is retained in the limit Gaussian process, which, in general, is not a Brownian motion; see Theorem 2 in Whitt (1985). In our theorem, each superposition process is scaled in both space and time. Squeezing the time scale erases the dependence of the increments of \( \tilde{B}_n \) to its history. The limit of these space-time scaled superposition processes is thus a Gaussian process with independent, stationary increments and continuous paths, which must be a Brownian motion.

In the many-server overloaded regime, all servers of a queue are nearly always busy. The service completion process is thus almost identical to a superposition of many renewal processes. Theorem 3 implies that it is possible to approximate the scaled service completion process by a Brownian motion. This approximation enables us to explore a simple one-dimensional diffusion model, which is able to capture the dynamics of a many-server queue with a general service time distribution, by zooming out our view in both space and time.

### 3.2 Limits in the long patience overloaded regime

To formulate the long patience overloaded regime, we fix the number of servers \( n \) and consider a sequence of G/GI/n+M queues indexed by \( k \in \mathbb{N} \). All these queues share the same arrival process, the same service distribution, and thus the same traffic intensity \( \rho > 1 \). We assume that the mean patience time in the \( k \)th queue goes to infinity as \( k \) goes large, i.e.,

\[
\gamma_k \to \infty \quad \text{as} \quad k \to \infty.
\]

Let \( E \) be the common arrival process of these queues, which has a diffusion-scaled version

\[
\tilde{E}_k(t) = \frac{1}{\sqrt{n\gamma_k}} (E(\gamma_k t) - \lambda \gamma_k t).
\]

Assume that

\[
\tilde{E}_k \Rightarrow \hat{E} \quad \text{as} \quad k \to \infty,
\]

where \( \hat{E} \) is a drift less Brownian motion with variance \( \rho \mu c^2_A \) and \( \hat{E}(0) = 0 \). Let \( X_k(t) \) be the number
of customers in the $k$th system at time $t$. Put

$$\tilde{X}_k(t) = \frac{1}{\sqrt{n\gamma_k}} (X_k(\gamma_k t) - n - n\mu(\rho - 1)\gamma_k).$$

We assume that there exists a random variable $\tilde{X}(0)$ such that

$$\tilde{X}_k \Rightarrow \tilde{X}(0) \quad \text{as} \quad k \to \infty. \quad (3.15)$$

**Theorem 4.** Let $\tilde{X}$ be the OU process given by (2.3) and $n$ be a fixed positive integer. Assume that the sequence of $G/GI/n + M$ queues, indexed by $k \in \mathbb{N}$, satisfies (3.13)--(3.15) with $\rho > 1$. Then,

$$\tilde{X}_k \Rightarrow \tilde{X} \quad \text{as} \quad k \to \infty.$$  

In the $k$th queue, let $W_k(s)$ be the virtual waiting time at $s \geq 0$, which has a scaled version

$$\tilde{W}_k(s) = \sqrt{n\gamma_k}(\gamma_k^{-1}W_k(\gamma_k s) - \log \rho).$$

**Theorem 5.** Under the conditions of Theorem 4, for any given $s \geq 0$,

$$\tilde{W}_k(s) \Rightarrow \mu^{-1}\hat{Y}^s(s + \log \rho) \quad \text{as} \quad k \to \infty,$$

where $\hat{Y}^s$ is the diffusion process defined in Theorem 2.

Because all queues have the same number of servers, we need the FCLT for renewal processes, instead of Theorem 3, in proving Theorems 4 and 5. With minor modification, one can follow the proofs of Theorems 1 and 2 to finish these proofs. We would not include them in this paper.

## 4 Numerical examples

In this section, we examine the approximate formulas obtained from the diffusion model by simulation. We assume a Poisson arrival process and an exponential patience time distribution. All numerical examples have the same traffic intensity $\rho = 1.2$. Different service time distributions, all with mean $1/\mu = 1.0$, are tested in the many-server and long patience overloaded regimes.

In the simulation examples, the service time distribution may be deterministic, Erlang (with two stages), or log-normal. These three distributions are denoted by D, E$_2$, and LN, respectively. With $c^2_S = 0$ and 0.5, respectively, the deterministic and Erlang distributions are used to represent scenarios where service times have small to moderate variability. It was reported in Brown et al. (2005) that a log-normal distribution provides a good fit for the service time data from the call center of an Israeli bank. We also test such a distribution that yields more variable service times. The log-normal distribution has $c^2_S = 1.52$, which is identical to the value from the data in Brown et al. (2005). All simulation results are obtained by averaging 30 independent runs and in each run, the queue is simulated for $1.0 \times 10^6$ time units.
4.1 Examples in the many-server overloaded regime

Consider an M/GI/100 + M queue. With \( n = 100 \), the customer arrival rate is \( \lambda = np \mu = 120 \). We evaluate the performance of this queue with mean patience time \( \gamma = 1.0, 5.0, \) and 10, respectively.

The estimates of several performance measures, including the abandonment fraction, the mean and variance of the steady-state queue length, and the mean and variance of the steady-state virtual waiting time, are listed in Table 1. We use (2.1), (2.2), (2.5), (2.6), and (2.8) to obtain the approximate results. Formulas (2.1), (2.2), and (2.6) can be obtained from the fluid model proposed by Whitt (2006). This fluid model, however, cannot be used to estimate variances.

In Table 1 the approximate results of the abandonment fraction, the mean queue length, and the mean virtual waiting time agree with the simulation results very well. This is consistent with the conclusion drawn by Whitt (2006): The fluid model is able to produce accurate approximations for mean performance measures in an overloaded queue with many servers. As the scaling
Erlang service time distribution, the approximate variances are satisfactory even if the variability of service times is not large, a moderate scaling factor in time increases. Comparing the variance results in the table, however, we can tell that an adequate diffusion approximation may not require a long mean patience time: With a mean patience time that is comparable to the mean service time, the approximate variances are satisfactory when the service times are deterministic or follow an Erlang distribution. We may explain this observation as follows. Because the service completion process is close to a superposition of renewal processes, a Brownian motion is used implicitly in the diffusion model to approximate its fluctuation (see Section 3.1 for more details). This replacement is supported by Theorem 3. As we discussed in Section 5 for more details). This replacement is supported by Theorem 3. As we discussed in Section 5 for more details). This replacement is supported by Theorem 3. As we discussed in Section 5 for more details). This replacement is supported by Theorem 3. As we discussed in Section 5 for more details). This replacement is supported by Theorem 3. As we discussed in Section 5 for more details).
To get adequate approximations, the mean patience time should be at least several times longer than the mean service time. The approximate variances are satisfactory when $\gamma = 5.0$ and 10.

To examine the steady-state queue length and virtual waiting time distributions, we list some tail probabilities in Table 2. The distributions of the scaled queue length and virtual waiting time are compared with the Gaussian distributions in (2.4) and (2.7). The results in this table are consistent with what we found in Table 1. With the deterministic or Erlang service time distribution, the approximate distributions are satisfactory when the mean patience time is comparable to or longer than the mean service time; when service times follow the log-normal distribution that has a larger variance, the mean patience time is required to be at least several times longer than the mean service time for the Gaussian distributions to be accurate.

To illustrate how the scaled queue length converges to a Gaussian random variable, let us examine an M/H_2/100 + M queue that has a hyperexponential service time distribution with $1/\mu = 1.0$ and $c_2^S = 4.0$. There are two types of customers in this system. The service times of either type are iid following an exponential distribution. The fraction of the first type is 67.41% and its mean service time is 0.1484, and the fraction of the second type is 32.59% and its mean service time is 2.761. The distribution of the stationary number of customers in this system can be computed by the matrix-analytic method (see Latouche and Ramaswami (1999)). By (2.4), we can approximate this distribution by

$$
P[X(\infty) = i] \approx \frac{1}{\sqrt{n\gamma\mu(\rho c_A^2 + c_S^2 + \rho - 1)/2}} \phi\left(\frac{i - n - n\mu(\rho - 1)\gamma}{\sqrt{n\gamma\mu(\rho c_A^2 + c_S^2 + \rho - 1)/2}}\right) \quad \text{for } i \in \mathbb{N}_0,$$

where $\phi$ is the standard Gaussian density function. We compare the distribution produced by the matrix-analytic method with the approximate distribution in Figure 1. Although the Gaussian approximation does not capture the exact distribution with $\gamma = 1.0$, it is a good fit with $\gamma = 10$. 

Figure 1: The steady-state distribution of the number of customers in an M/H_2/100+M queue with $\mu = 1.0$, $\rho = 1.2$, and $c_2^S = 4.0$: the exact distribution by the matrix-analytic method is compared with the Gaussian approximation from the diffusion model.

(a) $\gamma = 1.0$

(b) $\gamma = 10$

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Table 3: Performance estimates for an M/GI/5 + M queue with \( \mu = 1.0 \) and \( \rho = 1.2 \); simulation results (with 95% confidence intervals) are compared with approximate results (in italics).

| Patience | Abd. fraction | Queue length | Virtual waiting time |
|----------|---------------|--------------|----------------------|
|          | Mean          | Variance     | Mean                 | Variance     |
| M/D/5 + M |               |              |                      |              |
| \( \gamma = 5.0 \) | 0.1814 | 5.441 | 14.93 | 1.041 | 0.4121 |
|          | \( \pm 0.000094 \) | \( \pm 0.0037 \) | \( \pm 0.020 \) | \( \pm 0.00066 \) | \( \pm 0.00050 \) |
|          | 0.1667 | 5.000 | 17.50 | 0.9116 | 0.5000 |
| \( \gamma = 20 \) | 0.1672 | 20.06 | 68.95 | 3.708 | 1.9649 |
|          | \( \pm 0.00011 \) | \( \pm 0.017 \) | \( \pm 0.18 \) | \( \pm 0.0029 \) | \( \pm 0.0049 \) |
|          | 0.1667 | 20.00 | 70.00 | 3.646 | 2.000 |
| \( \gamma = 50 \) | 0.1666 | 49.99 | 175.5 | 9.164 | 5.017 |
|          | \( \pm 0.00011 \) | \( \pm 0.043 \) | \( \pm 0.66 \) | \( \pm 0.0074 \) | \( \pm 0.018 \) |
|          | 0.1667 | 50.00 | 175.0 | 9.116 | 5.000 |
| M/E_2/5 + M |               |              |                      |              |
| \( \gamma = 5.0 \) | 0.1896 | 5.689 | 18.66 | 1.109 | 0.5939 |
|          | \( \pm 0.00012 \) | \( \pm 0.0044 \) | \( \pm 0.020 \) | \( \pm 0.00080 \) | \( \pm 0.00065 \) |
|          | 0.1667 | 5.000 | 23.75 | 0.9116 | 0.7500 |
| \( \gamma = 20 \) | 0.1687 | 20.25 | 90.30 | 3.766 | 2.852 |
|          | \( \pm 0.00014 \) | \( \pm 0.021 \) | \( \pm 0.16 \) | \( \pm 0.0036 \) | \( \pm 0.0050 \) |
|          | 0.1667 | 20.00 | 95.00 | 3.646 | 3.000 |
| \( \gamma = 50 \) | 0.1668 | 50.04 | 237.0 | 9.198 | 7.491 |
|          | \( \pm 0.00013 \) | \( \pm 0.053 \) | \( \pm 0.77 \) | \( \pm 0.0093 \) | \( \pm 0.025 \) |
|          | 0.1667 | 50.00 | 237.5 | 9.116 | 7.500 |
| M/LN/5 + M |               |              |                      |              |
| \( \gamma = 5.0 \) | 0.1985 | 5.953 | 23.90 | 1.191 | 0.8979 |
|          | \( \pm 0.00013 \) | \( \pm 0.0044 \) | \( \pm 0.028 \) | \( \pm 0.00087 \) | \( \pm 0.0014 \) |
|          | 0.1667 | 5.000 | 36.50 | 0.9116 | 1.260 |
| \( \gamma = 20 \) | 0.1716 | 20.59 | 126.4 | 3.875 | 4.458 |
|          | \( \pm 0.00015 \) | \( \pm 0.019 \) | \( \pm 0.27 \) | \( \pm 0.0035 \) | \( \pm 0.010 \) |
|          | 0.1667 | 20.00 | 146.0 | 3.646 | 5.040 |
| \( \gamma = 50 \) | 0.1670 | 50.09 | 354.5 | 9.258 | 12.32 |
|          | \( \pm 0.00016 \) | \( \pm 0.052 \) | \( \pm 1.1 \) | \( \pm 0.0095 \) | \( \pm 0.042 \) |
|          | 0.1667 | 50.00 | 365.0 | 9.116 | 12.60 |

4.2 Examples in the long patience overloaded regime

Let us examine an M/GI/5 + M queue with \( \lambda = 6.0 \) and \( \gamma = 5.0, 20, \) and \( 50 \), respectively. Since the mean patience time is much longer than the mean service time, this queue is in the long patience overloaded regime. The corresponding performance estimates are listed in Tables 3 and 4. As in Section 4.1, we obtain the approximate results in Table 3 by (2.1), (2.2), (2.5), (2.6), and (2.8), and obtain the approximate tail probabilities in Table 4 by (2.4) and (2.7).

The diffusion model approximates a queue whose servers are almost always busy. This condition may not hold if the traffic intensity is not significantly greater than 1, the queue has only one or several servers, and the mean patience time is not very long. With \( \rho = 1.2, n = 5, \) and \( \gamma = 5.0 \), the abandonment fraction of the queue is notably greater than the approximate fraction for all three service time distributions. This implies that the idling time of servers is no longer negligible. In this case, the diffusion model may not produce adequate results. The idling time of servers can
be reduced by increasing the mean patience time: As customers become more patient, the queue length grows longer in the overloaded system, which in turn prevents the servers from idling. In Tables 3 and 4, the approximate results become much more accurate with $\gamma$ length grows longer in the overloaded system, which in turn prevents the servers from idling. In

| Patience | $P[X(\infty) > a]$ | $P[W(\infty) > a]$ |
|----------|--------------------|--------------------|
| $\gamma = 5.0$ | $0.2794$ | $0.2774$ |
| $\gamma = 20$ | $0.2895$ | $0.2513$ |
| $\gamma = 50$ | $0.2811$ | $0.2464$ |
| $\gamma = 50$ | $0.2750$ | $0.2398$ |
| $\gamma = 20$ | $0.3147$ | $0.2979$ |
| $\gamma = 50$ | $0.3106$ | $0.2906$ |
| $\gamma = 20$ | $0.3432$ | $0.3645$ |
| $\gamma = 50$ | $0.3419$ | $0.3328$ |
| $\gamma = 20$ | $0.3522$ | $0.3411$ |
| $\gamma = 50$ | $0.3495$ | $0.3280$ |
| $\gamma = 20$ | $0.3500$ | $0.3274$ |
| $\gamma = 50$ | $0.3495$ | $0.3274$ |

When an overloaded queue has one or several servers, the Brownian approximation used in the diffusion model also requires a large mean patience time. Note that the service completion process is close to the superposition of $n$ renewal processes. When $n$ is a small integer, by the FCLT for renewal processes, a large scaling factor in time is a prerequisite for the scaled superposition process to behave like a Brownian motion. In contrast, when $n$ is a large integer, the scaling in space renders the superposition process close to Gaussian (see Theorem 2 in Whitt (1985)). Then, as long as the scaling in time can sufficiently reduce the dependence of the increments to the history, the space-time scaled superposition process will be close to a Brownian motion. A moderate scaling factor in time is usually sufficient if the variability of service times is not large. This contrast can be confirmed by comparing Tables 1 and 2 with Tables 3 and 4. A mean patience time that is several times longer than the mean service time leads to satisfactory approximate results for a queue with one hundred servers; in a queue with merely five servers, however, the mean patience time has to be tens of times longer than the mean patience time for the diffusion model to work well.
5 Proof of Theorem 1

A sequence of perturbed systems is introduced in Section 5.1. In Section 5.2, we first show that the perturbed systems are asymptotically equivalent to the original queues, and then prove the diffusion limit for the perturbed systems.

5.1 A perturbed system

In the \( n \)th system, the number of customers at time \( t \) follows the dynamical equation

\[
X_n(t) = X_n(0) + E_n(t) - A_n(t) - D_n(t) \quad \text{for } t \geq 0,
\]

(5.1)

where \( A_n(t) \) is the number of customers who have abandoned the system during \((0, t]\) and \( D_n(t) \) is the number of service completions during \((0, t]\). The abandonment process \( A_n \) can be generated via the following standard procedure. Let \( G \) be a unit-rate Poisson process that is independent of \( X_n(0), E_n, \) and \( N_1, \ldots, N_n \) in \([3.10]\). Let \( Q_n(t) \) be the queue length at time \( t \), i.e.,

\[
Q_n(t) = (X_n(t) - n)^+.
\]

(5.2)

Because the patience time distribution is exponential with mean \( \gamma_n \), the instantaneous abandonment rate at \( t \) is \( \gamma_n^{-1}Q_n(t) \). We may generate the abandonment process \( A_n \) by

\[
A_n(t) = G \left( \gamma_n^{-1} \int_0^t Q_n(u) \, du \right).
\]

(5.3)

For the departure process \( D_n \), because \( \{\xi_{j,k} : k \in \mathbb{N}\} \) is the sequence of service times to be finished by the \( j \)th server, the service completion process from this server is identical to \( N_j \) until the \( j \)th server begins to idle. Therefore, \( D_n \) is identical to the superposition of \( N_1, \ldots, N_n \) until the first idle server appears. Let

\[
\tau_n = \inf \{t \geq 0 : X_n(t) < n\},
\]

which is the time that the first idle server appears. Because all servers are busy at time \( 0 \), we have \( \tau_n > 0 \). The departure process satisfies

\[
D_n(t) = B_n(t) \quad \text{for } 0 \leq t \leq \tau_n,
\]

(5.4)

with \( B_n \) given by \([3.11]\). As the superposition of \( n \) iid stationary renewal processes, \( B_n \) is more analytically tractable than \( D_n \). The equivalence between these two processes up to \( \tau_n \) allows us to introduce a perturbed system that has simplified dynamics. This perturbed system is asymptotically equivalent to the original queue as \( n \) goes large.

Consider the system equation (5.1). By (5.2), (5.3), and (5.4),

\[
X_n(t) = X_n(0) + E_n(t) - G \left( \gamma_n^{-1} \int_0^t (X_n(u) - n)^+ \, du \right) - B_n(t) \quad \text{for } 0 \leq t \leq \tau_n.
\]
From this equation, we introduce a new process $Y_n$ by

$$Y_n(t) = Y_n(0) + E_n(t) - G_n \left( \gamma_n^{-1} \int_0^t (Y_n(u) - n)^+ \, du \right) - B_n(t) \quad \text{for } t \geq 0,$$

(5.5)

where we set $Y_n(0) = X_n(0)$. We refer to (5.5) as the perturbed system equation. Clearly,

$$Y_n(t) = X_n(t) \quad \text{for } 0 \leq t \leq \tau_n$$

(5.6)
on each sample path. Thus, $\tau_n$ can be defined alternatively by

$$\tau_n = \inf \{ t \geq 0 : Y_n(t) < n \}.$$ 

(5.7)

The perturbed system can be envisioned as a queue where no server is allowed to idle. If a server finds the buffer empty upon a service completion, she begins to serve a customer who has not arrived yet. In the perturbed system, all servers are always busy and the departure process from each server is a stationary renewal process.

### 5.2 Limit processes for perturbed systems and asymptotic equivalence

We will prove Theorem 1 by a continuous mapping approach where two continuous maps are involved. The first map is used to prove a fluid limit, and the second is for a diffusion limit. The fluid limit enables us to establish the asymptotic equivalence between the original queues and the perturbed systems, which implies that these two sequences of systems have the same diffusion limit.

For any $f \in \mathbb{D}$, let $x$ and $z$ be two functions in $\mathbb{D}$ such that

$$x(t) = f(t) - \int_0^t x(u) \, du \quad \text{and} \quad z(t) = f(t) - \int_0^t z(u) \, du.$$ 

(5.8)

By Theorem 4.1 in Pang et al. (2007), each integral equation defines a continuous map.

**Lemma 1.** For each $f \in \mathbb{D}$, there is a unique $(x, z) \in \mathbb{D} \times \mathbb{D}$ such that (5.8) holds. Let $\varphi : \mathbb{D} \to \mathbb{D}$ be the function that maps $f$ to $x$ and $\psi : \mathbb{D} \to \mathbb{D}$ be the function that maps $f$ to $z$. Then, $\varphi$ and $\psi$ are continuous maps when $\mathbb{D}$ (as both the domain and the range) is endowed with the $J_1$ topology.

In the fluid scaling, the perturbed system equation (5.5) can be written as

$$\bar{Y}_n(t) = \bar{Y}_n(0) + \bar{E}_n(t) - \bar{G}_n \left( \int_0^t \bar{Y}_n(u) \, du \right) - \bar{B}_n(t) - \int_0^t \bar{Y}_n(u) \, du,$$

where

$$\bar{E}_n(t) = \frac{1}{n \gamma_n} E_n(\gamma_n t), \quad \bar{G}_n(t) = \frac{1}{n \gamma_n} (G(n \gamma_n t) - n \gamma_n t), \quad \bar{B}_n(t) = \frac{1}{n \gamma_n} B_n(\gamma_n t),$$

(5.9)

and

$$\bar{Y}_n(t) = \frac{1}{n \gamma_n} (Y_n(\gamma_n t) - n).$$

(5.10)
Lemma 2. Under the conditions of Theorem 1,
\[ \bar{Y}_n \Rightarrow \mu(\rho - 1)\chi \quad \text{as } n \to \infty. \]

Proof. By (3.1) and (3.6), \( \bar{Y}_n \Rightarrow \mu e \) as \( n \to \infty \). Since \( Y_n(0) = X_n(0) \), we have \( \bar{Y}_n(0) \Rightarrow (\rho - 1)\mu \) as \( n \to \infty \) by (3.7). Because \( \bar{Y}_n(t) \leq \bar{Y}_n(0) + \bar{E}_n(t) \),
\[
\lim_{a \to \infty} \lim_{n \to \infty} \sup_{0 \leq t \leq T} \Pr\left[ \sup_{0 \leq t \leq T} \bar{Y}_n(t) > a \right] = 0 \quad \text{for all } T > 0. \tag{5.11}
\]

The functional law of large numbers (see Theorem 5.10 in Chen and Yao (2001)) implies that \( \bar{G}_n \Rightarrow 0 \) as \( n \to \infty \), which, along with (5.11), implies that \( \{ \bar{G}_n \left( \int_0^t \bar{Y}_n(u)^+ \, du \right) : t \geq 0 \} \Rightarrow 0 \) as \( n \to \infty \).

Proposition 1 in the appendix states that \( \bar{B}_n \Rightarrow \mu e \) as \( n \to \infty \). Put
\[
\bar{M}_n(t) = \bar{Y}_n(0) + \bar{E}_n(t) - \bar{G}_n \left( \int_0^t \bar{Y}_n(u)^+ \, du \right) - \bar{B}_n(t).
\]

We deduce from the previous convergence results that \( \bar{M}_n \Rightarrow \mu(\rho - 1)(\chi + e) \) as \( n \to \infty \). Note that \( \varphi(\mu(\rho - 1)(\chi + e)) = \mu(\rho - 1)\chi \). Because \( \bar{Y}_n = \varphi(\bar{M}_n) \), the fluid limit follows from Lemma 1 and the continuous mapping theorem (see Theorem 5.2 in Chen and Yao (2001)). \qed

Let
\[
\bar{\tau}_n = \gamma_n^{-1}\tau_n. \tag{5.12}
\]

Then, \( \bar{\tau}_n \) is the instant when the first idle server appears in the time-scaled system. By (5.6),
\[
\bar{X}_n(t) = \bar{Y}_n(t) \quad \text{for } 0 \leq t \leq \bar{\tau}_n, \tag{5.13}
\]

where
\[
\bar{Y}_n(t) = \frac{1}{\sqrt{n\gamma_n}}(Y_n(\gamma_n t) - n - n\mu(\rho - 1)\gamma_n).
\]

The next lemma states that \( \bar{\tau}_n \to \infty \) in probability as \( n \to \infty \), which implies that \( \bar{X}_n \) and \( \bar{Y}_n \) are asymptotically equal over any finite time interval.

Lemma 3. Under the conditions of Theorem 1,
\[
\lim_{n \to \infty} \Pr[\bar{\tau}_n \leq T] = 0 \quad \text{for all } T > 0.
\]

Proof. By (5.7), (5.10), and (5.12), \( \bar{\tau}_n = \inf\{t \geq 0 : \bar{Y}_n(t) < 0\} \), which yields
\[
\Pr[\bar{\tau}_n \leq T] = \Pr\left[ \inf_{0 \leq t \leq T} \bar{Y}_n(t) < 0 \right].
\]

Then, the assertion follows from Lemma 2. \qed
Put
\[
\tilde{A}_n(t) = \frac{1}{\sqrt{n\gamma_n}} \left( G\left( \gamma_n^{-1} \int_0^{\gamma_n t} (Y_n(u) - n)^+ \, du \right) - \gamma_n^{-1} \int_0^{\gamma_n t} (Y_n(u) - n)^+ \, du \right),
\]
\[
\tilde{\Delta}_n(t) = \frac{1}{\sqrt{n\gamma_n}} \gamma_n^{-1} \int_0^{\gamma_n t} (Y_n(u) - n)^- \, du.
\]

With these processes, we can derive a diffusion-scaled version of the dynamical equation (5.5),
\[
\tilde{Y}_n(t) = \tilde{Y}_n(0) + \tilde{E}_n(t) - \tilde{A}_n(t) - \tilde{\Delta}_n(t) - \tilde{B}_n(t) - \int_0^t \tilde{Y}_n(u) \, du.
\]

**Lemma 4.** Under the conditions of Theorem 1
\[
\tilde{Y}_n \Rightarrow \hat{X} \quad \text{as } n \to \infty.
\]

**Proof.** Let
\[
\tilde{M}_n(t) = \tilde{Y}_n(0) + \tilde{E}_n(t) - \tilde{A}_n(t) - \tilde{\Delta}_n(t) - \tilde{B}_n(t).
\]
Because \(\tilde{Y}_n = \psi(\tilde{M}_n)\) and \(\hat{X} = \psi(M)\), Lemma 1 and the continuous mapping theorem will lead to the assertion once we prove \(\tilde{M}_n \Rightarrow \hat{M}\) as \(n \to \infty\).

Put
\[
\tilde{G}_n(t) = \frac{1}{\sqrt{n\gamma_n}} \left( G(n\gamma_n (\rho - 1)\mu t) - n\gamma_n (\rho - 1)\mu t \right).
\]

By the FCLT for renewal processes, \(\tilde{G}_n \Rightarrow \hat{A}\) as \(n \to \infty\) where \(\hat{A}\) is a driftless Brownian motion with variance \((\rho - 1)\mu\) and \(\hat{A}(0) = 0\). Recall that \(Y_n(0) = X_n(0)\) and \(\tilde{Y}_n(0), \tilde{E}_n, \tilde{G}_n, \tilde{B}_n\) are mutually independent. By (3.6), (3.7), and Theorem 3
\[
\tilde{Y}_n(0) + \tilde{E}_n - \tilde{G}_n - \tilde{B}_n \Rightarrow \hat{M} \quad \text{as } n \to \infty. \tag{5.14}
\]

Put
\[
\tilde{\zeta}_n(t) = \frac{1}{(\rho - 1)\mu} \int_0^t Y_n(u)^+ \, du.
\]

Then, \(\tilde{A}_n = \tilde{G}_n \circ \tilde{\zeta}_n\). By Lemma 2 \(\tilde{\zeta}_n \Rightarrow e\) as \(n \to \infty\). Because \(\tilde{G}_n \Rightarrow \hat{A}\) and \(\hat{A}\) has continuous paths almost surely, it follows that
\[
\tilde{A}_n - \tilde{G}_n \Rightarrow 0 \quad \text{as } n \to \infty. \tag{5.15}
\]

Moreover,
\[
P \left[ \sup_{0 \leq t \leq T} \tilde{\Delta}_n(t) > 0 \right] \leq P \left[ \inf_{0 \leq t \leq T} Y_n(\gamma_n t) < n \right] = P[\tilde{\tau}_n \leq T] \quad \text{for all } T > 0.
\]

Then, Lemma 3 implies that
\[
\tilde{\Delta}_n \Rightarrow 0 \quad \text{as } n \to \infty. \tag{5.16}
\]

It follows from (5.14)–(5.16) and the convergence-together theorem (see Theorem 5.4 in Chen and Yao (2001)) that \(M_n \Rightarrow M\) as \(n \to \infty\).
Proof of Theorem 1. By (5.13),
\[
\mathbb{P} \left[ \sup_{0 \leq t \leq T} |\tilde{X}_n(t) - \tilde{Y}_n(t)| > 0 \right] \leq \mathbb{P}[\tilde{\tau}_n \leq T] \quad \text{for all } T > 0.
\]

Lemma 3 implies that \( \tilde{X}_n - \tilde{Y}_n \Rightarrow 0 \) as \( n \to \infty \). Then, the theorem follows from Lemma 4 and the convergence-together theorem.

6 Proof of Theorem 2

The proof of Theorem 2 also relies on the analysis of perturbed systems. In Section 6.1, using a perturbed system that has a stopped arrival process, we introduce an asymptotically equivalent representation for a virtual waiting time in the original queue. In Section 6.2, we establish an asymptotic relationship between the virtual waiting time and the queue length at a certain time in the perturbed system with arrival stopping. We prove Theorem 2 by using a diffusion limit for the queue length processes in the perturbed systems.

6.1 A perturbed system with a stopped arrival process

Let \( s \geq 0 \) be a fixed number. Consider the virtual waiting time at \( s \) in the original queue. Because the queue and its perturbed system follow the same dynamics over \([0, \tau_n]\), the asymptotic equivalence proved in Lemma 3 implies an identical limit for the virtual waiting times in both systems. We can thus explore a sequence of perturbed systems to obtain this limit. We follow the approach adopted by Talreja and Whitt (2009), exploiting a sequence of systems with stopped arrival processes.

Suppose that in the queue, the arrival process is “turned off” at time \( s \), i.e., all customers who arrive after \( s \) are rejected. For each \( t \geq 0 \), let \( X^s_n(t) \) be the number of customers at \( t \). Then, \( W_n(s) \) is the amount of time from \( s \) until an idle server appears, i.e.,
\[
W_n(s) = \inf\{u \geq 0 : X^s_n(s + u) < n\}. \tag{6.1}
\]

In such a system with the arrival process stopping at \( s \), the number of customers at \( t \) is given by
\[
X^s_n(t) = X_n(0) + E^s_n(t) - A^s_n(t) - D^s_n(t), \tag{6.2}
\]

where \( E^s_n(t) = E_n(s \wedge t) \), \( A^s_n(t) \) is the number of abandonments by \( t \), and \( D^s_n(t) \) is the number of service completions by \( t \). Let \( G^s \) be a unit-rate Poisson process that is independent of \( X_n(0) \), \( E_n \), and \( N_1, \ldots, N_n \). We may generate the abandonment process \( A^s_n \) by
\[
A^s_n(t) = G^s \left( \frac{1}{\gamma_n} \int_0^t (X^s_n(u) - n)^+ \, du \right) \quad \text{for } t \geq 0.
\]

(We only consider the case that \( s \) is fixed, so that \( G^s \) is allowed to change with \( s \).) Because \( D^s_n(t) = B_n(t) \) for \( 0 \leq s \leq \tau_n \) and \( 0 \leq t \leq s + W_n(s) \), the dynamical equation (6.2) can be written...
as
\[ X^s_n(t) = X_n(0) + E^s_n(t) - G^s_n \left( \gamma_n^{-1} \int_0^t (X^s_n(u) - n)^+ \, du \right) - B_n(t) \]
for \(0 \leq s \leq \tau_n\) and \(0 \leq t \leq s + W_n(s)\). By this equation, we can define a process \(Y^s_n\) by
\[ Y^s_n(t) = X_n(0) + E^s_n(t) - G^s_n \left( \gamma_n^{-1} \int_0^t (Y^s_n(u) - n)^+ \, du \right) - B_n(t) \quad \text{for } t \geq 0. \tag{6.3} \]
Equation (6.3) is the dynamical equation for the \(n\)th perturbed system with the arrival process stopping at \(s\). Clearly,
\[ Y^s_n(t) = X^s_n(t) \quad \text{for } 0 \leq s \leq \tau_n \text{ and } 0 \leq t \leq s + W_n(s). \tag{6.4} \]
Let
\[ V_n(s) = \inf \{ u \geq 0 : Y^s_n(s + u) < n \} \quad \text{for } s \geq 0. \tag{6.5} \]
Then, by (6.1) and (6.4),
\[ V_n(s) = W_n(s) \quad \text{for } 0 \leq s \leq \tau_n. \tag{6.6} \]

### 6.2 Limit processes for perturbed systems with arrival stopping

Following a continuous mapping approach, we first prove a fluid limit for the perturbed systems with arrival stopping. Using (5.9), we can derive a fluid-scaled version of (6.3), given by
\[ \tilde{Y}^s_n(t) = \tilde{Y}_n(0) + \tilde{E}^s_n(t) - \tilde{G}^s_n \left( \int_0^t \tilde{Y}^s_n(u)^+ \, du \right) - \tilde{B}_n(t) - \int_0^t \tilde{Y}^s_n(u)^+ \, du, \tag{6.7} \]
where
\[ \tilde{Y}^s_n(t) = \frac{1}{n\gamma_n} (Y^s_n(\gamma_n t) - n), \quad \tilde{E}^s_n(t) = \frac{1}{n\gamma_n} E^s_n(\gamma_n t), \quad \tilde{G}^s_n(t) = \frac{1}{n\gamma_n} \left( G^s_n(n\gamma_n t) - n\gamma_n t \right). \tag{6.8} \]

**Lemma 5.** Under the conditions of Theorem 2, for all \(s \geq 0\),
\[ \tilde{Y}^s_n \Rightarrow y^s \quad \text{as } n \to \infty, \]
where \(y^s\) is the function given by (3.8).

**Proof.** Write
\[ \tilde{M}^s_n(t) = \tilde{Y}_n(0) + \tilde{E}^s_n(t) - \tilde{G}^s_n \left( \int_0^t \tilde{Y}^s_n(u)^+ \, du \right) - \tilde{B}_n(t). \]
Following the proof of Lemma 2 we obtain \(\tilde{Y}_n(0) \Rightarrow (\rho - 1)\mu, \tilde{E}^s_n \Rightarrow \rho \mu e^s, \tilde{B}_n \Rightarrow \mu e\), and
\[ \left\{ \tilde{G}^s_n \left( \int_0^t \tilde{Y}^s_n(u)^+ \, du \right) : t \geq 0 \right\} \Rightarrow \mu e \quad \text{as } n \to \infty. \]
Then, \(\tilde{M}^s_n \Rightarrow \mu(\rho e^s - e + (\rho - 1)\chi)\) as \(n \to \infty\). Because \(y^s = \varphi(\mu(\rho e^s - e + (\rho - 1)\chi))\) and \(\tilde{Y}^s_n = \varphi(\tilde{M}^s_n)\), the fluid limit follows from Lemma 1 and the continuous mapping theorem.

\[\Box\]
Let
\[ \bar{V}_n(s) = \gamma_n^{-1} V_n(\gamma_n s), \]
which is the virtual waiting time in the time-scaled perturbed system. By (6.5) and (6.8),
\[ \bar{V}_n(s) = \inf\{u \geq 0 : \bar{Y}_n^s(s + u) < 0\}. \] \hfill (6.9)

**Lemma 6.** Under the conditions of Theorem 2, for all \( s \geq 0 \),
\[ \bar{V}_n(s) \Rightarrow \log \rho \text{ as } n \to \infty. \]

**Proof.** Because \( y^s(s + \log \rho - \delta) > 0 \) and \( y^s(s + \log \rho + \delta) < 0 \) for \( \delta > 0 \), Lemma 5 implies that
\[ \lim_{n \to \infty} P[\bar{Y}_n^s(s + \log \rho - \delta) > 0] = 1 \text{ and } \lim_{n \to \infty} P[\bar{Y}_n^s(s + \log \rho + \delta) < 0] = 1. \]
Using (6.9) and the fact that \( \bar{Y}_n^s(t) \) is nonincreasing for \( t \geq s \), we obtain
\[ \lim_{n \to \infty} P[\log \rho - \delta \leq \bar{V}_n(s) \leq \log \rho + \delta] = 1, \]
which completes the proof. \( \square \)

Having established the convergence results in the fluid scaling, let us turn to diffusion-scaled processes. For \( t \geq 0 \), put
\[ \tilde{M}^s_n(t) = \tilde{Y}_n(0) + \tilde{E}^s_n(t) - \tilde{G}^s_n\left( \int_0^t \bar{Y}_n^s(u)^+ \, du \right) - \tilde{B}_n(t), \]
where
\[ \tilde{E}^s_n(t) = \tilde{E}_n(s \land t) \text{ and } \tilde{G}^s_n(t) = \frac{1}{\sqrt{n \gamma_n^s}}(G^s_{\gamma_n^s}(n \gamma_n^s t) - n \gamma_n^s t). \]

**Lemma 7.** Let \( \tilde{E}^s(t) = \tilde{E}(s \land t) \) and \( \tilde{G}^s(t) \) be a standard Brownian motion independent of \( \tilde{X}(0), \tilde{E}^s, \) and \( \tilde{B} \). Under the conditions of Theorem 2, for all \( s \geq 0 \),
\[ \tilde{M}^s_n \Rightarrow \tilde{M}^s \text{ as } n \to \infty, \]
where
\[ \tilde{M}^s(t) = \tilde{X}(0) + \tilde{E}^s(t) - \tilde{G}^s\left( \int_0^t y^s(u)^+ \, du \right) - \tilde{B}(t). \]

**Proof.** By (3.6), \( \tilde{E}^s_n \Rightarrow \tilde{E}^s \text{ as } n \to \infty. \) By the FCLT for renewal processes, Lemma 5, and the random-time-change theorem (see Theorem 5.3 in Chen and Yao (2001)),
\[ \left\{ \tilde{G}^s_n\left( \int_0^t \bar{Y}_n^s(u)^+ \, du \right) : t \geq 0 \right\} \Rightarrow \left\{ \tilde{G}^s\left( \int_0^t y^s(u) \, du \right) : t \geq 0 \right\} \text{ as } n \to \infty. \]
Then, the lemma follows from (3.7) and Theorem 3. \( \square \)
Now consider the diffusion-scaled queue length process, which is defined by
\[
\tilde{Y}_{n}\gamma_{n}\infty(t) = \frac{1}{\sqrt{n\gamma_{n}}}(Y_{n}\gamma_{n}\infty(\gamma_{n}t) - n - n\gamma_{n}y^{s}(t)) \quad \text{for } 0 \leq t \leq s + \log \rho. \tag{6.10}
\]
In the subsequent proofs, \(\tilde{Y}_{n}\gamma_{n}\infty\) is considered only up to time \(s + \log \rho\). For our convenience, we set
\[
\tilde{Y}_{n}\gamma_{n}\infty(t) = \tilde{Y}_{n}\gamma_{n}\infty(s + \log \rho) \quad \text{for } t > s + \log \rho.
\]
Using these processes, we can derive the diffusion-scaled dynamical equation from (6.3),
\[
\tilde{Y}_{n}\gamma_{n}\infty(t) = \tilde{M}_{n}\gamma_{n}\infty(t) - \sqrt{n\gamma_{n}}\int_{0}^{t}(\tilde{Y}_{n}\gamma_{n}\infty(u) + y^{s}(u))\,du \quad \text{for } 0 \leq t \leq s + \log \rho. \tag{6.11}
\]
We will see that the diffusion-scaled virtual waiting time at \(s\) is closely related to the diffusion-scaled queue length at \(s + \log \rho\). The next lemma is a technical result, which states the stochastic boundedness of \(\{\tilde{Y}_{n}\gamma_{n}\infty(s + \log \rho) : n \in \mathbb{N}\}\).

**Lemma 8.** Under the conditions of Theorem 2,
\[
\lim_{a \to \infty} \limsup_{n \to \infty} P[|\tilde{Y}_{n}\gamma_{n}\infty(s + \log \rho)| > a] = 0.
\]

**Proof.** Because \(y^{s}(u) \geq 0\) for \(0 \leq u \leq s + \log \rho\),
\[
\sqrt{n\gamma_{n}}|\tilde{Y}_{n}\gamma_{n}\infty(u) + y^{s}(u)| \leq \sqrt{n\gamma_{n}}|\tilde{Y}_{n}\gamma_{n}\infty(u) - y^{s}(u)| = |\tilde{Y}_{n}\gamma_{n}\infty(u)|.
\]
Then by (6.11),
\[
|\tilde{Y}_{n}\gamma_{n}\infty(t)| \leq |\tilde{M}_{n}\gamma_{n}\infty(t)| + \int_{0}^{t}|\tilde{Y}_{n}\gamma_{n}\infty(u)|\,du \quad \text{for } 0 \leq t \leq s + \log \rho.
\]
It follows from Gronwall’s inequality (see Lemma 21.4 in Kallenberg (2002)) that
\[
|\tilde{Y}_{n}\gamma_{n}\infty(s + \log \rho)| \leq \sup_{0 \leq t \leq s + \log \rho} |\tilde{M}_{n}\gamma_{n}\infty(t)|\rho \exp(s).
\]
Lemma 7 implies that \(\{\tilde{M}_{n} : n \in \mathbb{N}\}\) is stochastically bounded. So is \(\{\tilde{Y}_{n}\gamma_{n}\infty(s + \log \rho) : n \in \mathbb{N}\}\).

Let
\[
\tilde{V}(n)(s) = \sqrt{n\gamma_{n}}(\tilde{V}(n)(s) - \log \rho), \tag{6.12}
\]
which is the diffusion-scaled virtual waiting time in the perturbed system at \(s\). The following lemma states that \(\tilde{V}(n)(s)\) and \(\tilde{Y}_{n}\gamma_{n}\infty(s + \log \rho)/\mu\) are asymptotically close.

**Lemma 9.** Under the conditions of Theorem 2, for all \(s \geq 0\)
\[
\tilde{V}(n)(s) - \mu^{-1}\tilde{Y}_{n}\gamma_{n}\infty(s + \log \rho) \Rightarrow 0 \quad \text{as } n \to \infty.
\]

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Proof. Because $E_n^{s}(s+\bar{V}_n(s)) = E_n^{s}(s+\log \rho) = \bar{E}_n(s)$, it follows from (6.7) that

$$\bar{Y}_n^{s}(s+\log \rho) = \bar{Y}_n^{s}(s+\bar{V}_n(s)) + \bar{B}_n(s+\bar{V}_n(s)) - \bar{B}_n(s+\log \rho)$$

$$+ \bar{G}_n^s\left(\int_0^{s+\bar{V}_n(s)} \bar{Y}_n^{s}(u)^+ \, du\right) - \bar{G}_n^s\left(\int_0^{s+\log \rho} \bar{Y}_n^{s}(u)^+ \, du\right)$$

$$+ \int_0^{s+\bar{V}_n(s)} \bar{Y}_n^{s}(u)^+ \, du - \int_0^{s+\log \rho} \bar{Y}_n^{s}(u)^+ \, du.$$  (6.13)

Multiply both sides of (6.13) by $\sqrt{n\gamma_n}$ and let us consider each term.

By (6.10) and the fact that $y_s(s+\log \rho) = 0$, the left side turns out to be

$$\sqrt{n\gamma_n}\bar{Y}_n^{s}(s+\log \rho) = \bar{Y}_n^{s}(s+\log \rho).$$  (6.14)

Consider the right side. If the arrival process stops at time $\gamma_n s$ for $0 \leq s < \bar{\tau}_n$, the first idle server will appear at $\gamma_n(s+\bar{V}_n(s))$. This must be triggered by a service completion. Because $B_n$ is the superposition of $n$ iid stationary renewal processes, the probability that $B_n$ has a jump of size larger than 1 is 0, which implies that

$$\mathbb{P}\left[\bar{Y}_n^{s}(s+\bar{V}_n(s)) < -\frac{1}{n\gamma_n}\right] \leq \mathbb{P}[\bar{\tau}_n \leq s].$$

Then, by Lemma 3

$$\sqrt{n\gamma_n}\bar{Y}_n^{s}(s+\bar{V}_n(s)) \Rightarrow 0 \text{ as } n \to \infty.$$  (6.15)

By (3.12), (5.9), and (6.12),

$$\sqrt{n\gamma_n}(\bar{B}_n(s+\bar{V}_n(s)) - \bar{B}_n(s+\log \rho)) = \bar{B}_n(s+\bar{V}_n(s)) - \bar{B}_n(s+\log \rho) + \mu\bar{V}_n(s),$$

in which we have

$$\bar{B}_n(s+\bar{V}_n(s)) - \bar{B}_n(s+\log \rho) \Rightarrow 0 \text{ as } n \to \infty$$  (6.16)

by Theorem 3 and Lemma 6. Because $\sqrt{n\gamma_n}\bar{G}_n^s = \bar{G}_n^s$ and $\bar{G}_n^s \Rightarrow G^s$ as $n \to \infty$, it follows from Lemmas 5 and 6 that

$$\sqrt{n\gamma_n}\left(\bar{G}_n^s\left(\int_0^{s+\bar{V}_n(s)} \bar{Y}_n^{s}(u)^+ \, du\right) - \bar{G}_n^s\left(\int_0^{s+\log \rho} \bar{Y}_n^{s}(u)^+ \, du\right)\right) \Rightarrow 0 \text{ as } n \to \infty.$$  (6.17)

Because $\bar{Y}_n^{s}(t)$ is nonincreasing for $t \geq s$,

$$\sqrt{n\gamma_n}\left|\int_0^{s+\bar{V}_n(s)} \bar{Y}_n^{s}(u)^+ \, du - \int_0^{s+\log \rho} \bar{Y}_n^{s}(u)^+ \, du\right|$$

$$\leq |\bar{V}_n(s) - \log \rho||\bar{Y}_n^{s}(s+\log \rho)| + |\bar{V}_n(s) - \log \rho|\sqrt{n\gamma_n}|\bar{Y}_n^{s}(s+\bar{V}_n(s))|.$$

Then, by (6.15) and Lemmas 6 and 8

$$\sqrt{n\gamma_n}\left(\int_0^{s+\bar{V}_n(s)} \bar{Y}_n^{s}(u)^+ \, du - \int_0^{s+\log \rho} \bar{Y}_n^{s}(u)^+ \, du\right) \Rightarrow 0 \text{ as } n \to \infty.$$  (6.18)
We deduce from (6.13)–(6.18) that \( \tilde{V}_n(s) - \tilde{Y}_n(s + \log \rho)/\mu \Rightarrow 0 \) as \( n \to \infty \). □

**Lemma 10.** Under the conditions of Theorem 2, for all \( s \geq 0 \),

\[
\tilde{Y}_n^s \Rightarrow \hat{Y}^s \quad \text{as} \quad n \to \infty,
\]

where

\[
\hat{Y}^s(t) = \hat{M}^s(t) - \int_0^t \hat{Y}^s(u) \, du \quad \text{for} \quad 0 \leq t \leq s + \log \rho
\]

and \( \hat{Y}^s(t) = \hat{Y}^s(s + \log \rho) \) for \( t > s + \log \rho \).

**Proof.** Write

\[
\hat{M}_n^s(t) = \hat{M}_n^s(t) - \sqrt{n\gamma_n} \int_0^t \hat{Y}_n^s(u) \, du.
\]

By (6.11), \( \hat{Y}_n^s(t) = \psi(\hat{M}_n^s)(t) \) for \( 0 \leq t \leq s + \log \rho \). If we can prove that

\[
\sqrt{n\gamma_n} \int_0^{s + \log \rho} \hat{Y}_n^s(u) \, du \Rightarrow 0 \quad \text{as} \quad n \to \infty, \tag{6.19}
\]

then \( \hat{M}_n^s \Rightarrow \hat{M}^s \) as \( n \to \infty \) by Lemma 7 and the convergence-together theorem. The current lemma will follow from Lemma 1 and the continuous mapping theorem.

Because \( \bar{Y}_n^s(t) = \bar{Y}_n(t) \) for \( 0 \leq t \leq s \), Lemma 3 implies that

\[
\lim_{n \to \infty} P \left[ \inf_{0 \leq t \leq s} \bar{Y}_n^s(t) < 0 \right] = 0.
\]

Hence,

\[
\sqrt{n\gamma_n} \int_0^s \bar{Y}_n^s(u) \, du \Rightarrow 0 \quad \text{as} \quad n \to \infty. \tag{6.20}
\]

Note that \( \bar{Y}_n^s(t) \) is nonincreasing for \( s \leq t \leq s + \log \rho \). By (6.9) and (6.10),

\[
\sqrt{n\gamma_n} \int_s^{s + \log \rho} \bar{Y}_n^s(u) \, du = \sqrt{n\gamma_n} \int_{s + \bar{V}_n(s)}^{s + (\bar{V}_n(s) + \log \rho)} \bar{Y}_n^s(u) \, du \leq |(\bar{V}_n(s) - \log \rho)\bar{Y}_n^s(s + \log \rho)|.
\]

Then, using Lemmas 6 and 8 we have

\[
\sqrt{n\gamma_n} \int_s^{s + \log \rho} \bar{Y}_n^s(u) \, du \Rightarrow 0 \quad \text{as} \quad n \to \infty. \tag{6.21}
\]

We obtain (6.19) by combining (6.20) and (6.21). □

**Proof of Theorem 2.** Lemmas 9 and 10 along with the convergence-together theorem, imply that \( \hat{V}_n(s) \Rightarrow \hat{Y}^s(s + \log \rho)/\mu \) as \( n \to \infty \). The convergence of \( W_n \) follows from the asymptotic equivalence between \( \hat{W}_n(s) \) and \( \hat{V}_n(s) \), which can be deduced by (6.6) and Lemma 3. Solution (3.9) can be obtained by Proposition 21.2 in [Kallenberg (2002)]. □
7 Future work

We have demonstrated that in two overloaded regimes, the queue length process of a GI/GI/n + M queue can be approximated by an OU process. One may raise the following questions about this diffusion model: Is the exponential patience time distribution essential for an overloaded queue to have a simple approximate model? With more practical patience time assumptions, can we still approximate the steady-state queue length and the steady-state virtual waiting time by Gaussian random variables?

We will answer these question in our subsequent work. To illustrate this, let us consider a GI/GI/n + GI queue. For call center operations, it is reasonable to assume patience times to be iid since the waiting line is usually invisible to customers. The GI/GI/n + GI queue is thus an important building block for modeling call centers. [Whitt (2006)] obtained the mean queue length and the mean virtual waiting time of this queue by the fluid model. Let $H$ be the distribution function of patience times. Assume that $H$ is absolutely continuous with density $f_H$. The hazard rate function of $H$ is given by

$$h(t) = \frac{f_H(t)}{1 - H(t)} \quad \text{for } t \geq 0.$$  

Let $w$ be the mean virtual waiting time. Then, $H(w)$ is the fraction of patience times that are less than $w$. This fraction should be approximately equal to the abandonment fraction $(\rho - 1)/\rho$. Hence, the mean virtual waiting time can be obtained by solving

$$H(w) = \frac{\rho - 1}{\rho}.$$  

For $0 < s < w$, the probability that a customer who arrived $s$ time units ago is still in the buffer is around $1 - H(s)$. This implies that the mean queue length can be approximated by

$$q = \int_0^w \lambda(1 - H(s)) \, ds.$$  

See [Whitt (2006)] for more details. In the steady state, the virtual waiting time process and the queue length process fluctuate around $w$ and $q$, respectively.

Some observations on queues with an exponential patience time distribution may help us in generalizing the diffusion model. When either $n$ or $\gamma$ is large, it follows from (2.6) and (2.8) that the standard deviation of the virtual waiting time is much smaller than the mean. If this condition holds with a general patience time assumption, the abandonment process will depend on the patience time distribution mostly through a small neighborhood of $w$. As a consequence, the scaled queue length process will be dictated by the patience time hazard rate at $w$. In this case, we use $\gamma = 1/h(w)$ as the scaling factor in time. If the patience time hazard rate changes slowly around $w$, with $\gamma = 1/h(w)$, we may still use (2.4), (2.5), (2.7), and (2.8) to approximate the steady-state distributions and variances. In particular, it was reported in [Mandelbaum and Zeltyn (2013)] that the patience time hazard rate in a large call center was nearly constant after the first several seconds of waiting (see Figure 2 in their paper). We expect that with the above modification, the approximate formulas are useful in performance analysis for such a call center. If the hazard rate changes rapidly around $w$, we may exploit the approximation scheme in [Reed and Ward (2008) and Reed and Tezcan (2012)] to include the hazard rate function on a neighborhood
of w in the diffusion model. The resulting performance approximations would be more complex, but may still have closed-form formulas. To justify the diffusion model for queues with a general patience time distribution, we will modify the current asymptotic regimes to incorporate the hazard rate function. More specifically, we will combine the space-time scaling with the hazard rate scaling proposed by Reed and Ward (2008) in the new asymptotic framework.

Appendix: Proof of Theorem 3

Let

\[ S_{j,k} = \sum_{\ell=1}^{k} \xi_{j,\ell} \]  

be the kth partial sum of \( \{\xi_{j,\ell} : \ell \in \mathbb{N}\} \). Take \( S_{j,0} = 0 \) by convention. We first prove a functional strong law of large numbers for the superposition of time-scaled renewal processes.

**Proposition 1.** Let

\[ \bar{B}_n(t) = \frac{1}{n\gamma_n} \sum_{j=1}^{n} N_j(\gamma_n t) \quad \text{for } t \geq 0. \]  

Under the conditions of Theorem 3,

\[ \bar{B}_n \xrightarrow{a.s.} \mu e \quad \text{as } n \to \infty. \]  

**Proof.** Since \( S_{j,N_j(t)} \leq t \leq S_{j,N_j(t)+1} \) for \( t > 0 \), then

\[ \sum_{j=1}^{n} S_{j,N_j(\gamma_n t)} \leq \sum_{j=1}^{n} \frac{n\gamma_n t}{N_j(\gamma_n t)} \leq \sum_{j=1}^{n} \frac{n\gamma_n t}{N_j(\gamma_n t)} \sum_{j=1}^{n} N_j(\gamma_n t) \]  

provided that \( \sum_{j=1}^{n} N_j(\gamma_n t) > 0 \). Note that

\[ \sum_{j=1}^{n} S_{j,N_j(\gamma_n t)+1} = \sum_{j=1}^{n} \xi_{j,1} + \sum_{j=1}^{n} \sum_{k=2}^{N_j(\gamma_n t)+1} \xi_{j,k}. \]  

Because \( N_j(\gamma_n t) \xrightarrow{a.s.} \infty \) as \( n \to \infty \) for \( t > 0 \), then

\[ \frac{\sum_{j=1}^{n} \xi_{j,k}}{\sum_{j=1}^{n} N_j(\gamma_n t)} \xrightarrow{a.s.} \mu^{-1} \quad \text{as } n \to \infty \]  

by the strong law of large numbers. In addition, \( n^{-1} \sum_{j=1}^{n} N_j(\gamma_n t) \xrightarrow{a.s.} \infty \) as \( n \to \infty \) for \( t > 0 \), which implies that

\[ \frac{\sum_{j=1}^{n} \xi_{j,1}}{\sum_{j=1}^{n} N_j(\gamma_n t)} \xrightarrow{a.s.} 0 \quad \text{as } n \to \infty. \]  

Therefore,

\[ \frac{\sum_{j=1}^{n} S_{j,N_j(\gamma_n t)+1}}{\sum_{j=1}^{n} N_j(\gamma_n t)} \xrightarrow{a.s.} \mu^{-1} \quad \text{as } n \to \infty. \]
Also,
\[
\sum_{j=1}^{n} S_{j,N_j(\gamma_n t)} = \frac{\sum_{j=1}^{n} S_{j,N_j(\gamma_n t)}}{\sum_{j=1}^{n} N_j(\gamma_n t)} \cdot \frac{\sum_{j=1}^{n} (N_j(\gamma_n t) - 1)}{\sum_{j=1}^{n} N_j(\gamma_n t)} \xrightarrow{a.s.} \mu^{-1} \quad \text{as } n \to \infty.
\]

Then, \( \bar{B}_n(t) \xrightarrow{a.s.} \mu t \) as \( n \to \infty \) for all \( t \geq 0 \). Because \( \bar{B}_n(t) \) is nondecreasing in \( t \) and \( e \) is a continuous function, the proposition follows from Theorem VI.2.15 in [Jacod and Shiryaev, 2002]. \( \square \)

**Lemma 11.** Let
\[
\bar{L}_n(t) = \frac{1}{\sqrt{n\gamma_n}} \sum_{j=1}^{n} \sum_{k=2}^{N_j(\gamma_n t) + 1} (1 - \mu \xi_{j,k}) \quad \text{for } t \geq 0.
\]  
Under the conditions of Theorem 3,
\[ \bar{L}_n \Rightarrow \hat{B} \quad \text{as } n \to \infty. \]

**Proof.** Let \( \{\eta_k : k \in \mathbb{N}\} \) be a sequence of iid random variables following distribution \( F \). Then, \( \mu \eta_k \) has mean 1 and variance \( c_2^2 S \). Put
\[
\tilde{H}_n(t) = \frac{1}{\sqrt{n\gamma_n}} \sum_{k=1}^{n \gamma_n B_n(t)} (1 - \mu \eta_k) \quad \text{for } t \geq 0.
\]
By Donsker’s theorem, \( \tilde{H}_n \Rightarrow \tilde{H} \) as \( n \to \infty \), where \( \tilde{H} \) is a driftless Brownian motion with variance \( c_2^2 S \) and \( \tilde{H}(0) = 0 \). By (A.2),
\[
\tilde{H}_n(B_n(t)) = \frac{1}{\sqrt{n\gamma_n}} \sum_{k=1}^{n \gamma_n B_n(t)} (1 - \mu \eta_k) = \frac{1}{\sqrt{n\gamma_n}} \sum_{k=1}^{N_1(\gamma_n t) + \cdots + N_n(\gamma_n t)} (1 - \mu \eta_k).
\]
It follows from Proposition 1 and the random-time-change theorem that \( \tilde{H}_n \circ B_n \Rightarrow \mu^{1/2} \tilde{H} \) as \( n \to \infty \). Because \( \hat{L}_n \) has the same distribution as \( \tilde{H}_n \circ B_n \) and \( \mu^{1/2} \tilde{H} \) has the same distribution as \( \hat{B} \), the lemma follows. \( \square \)

**Lemma 12.** Under the conditions of Theorem 3, for all \( 0 \leq r \leq s \leq t \) and \( n \in \mathbb{N} \), there exists \( 0 < c < \infty \) such that
\[
\mathbb{E}[(\tilde{B}_n(s) - \tilde{B}_n(r))(\tilde{B}_n(t) - \tilde{B}_n(s))^2] \leq c(t - r)^2.
\]

**Proof.** Let \( \tilde{N}_j(u) = N_j(u) - \mu u \) for \( u \geq 0 \) and \( j = 1, \ldots, n \). Because \( N_j \) is a stationary renewal process, by inequalities (7) and (8) in [Whitt, 1985], there exists \( c_1 < \infty \) such that
\[
\mathbb{E}[(\tilde{N}_j(s) - \tilde{N}_j(r))^2] \leq c_1(s - r) \tag{A.4}
\]
and
\[
\mathbb{E}[(\tilde{N}_j(s) - \tilde{N}_j(r))^2(\tilde{N}_j(t) - \tilde{N}_j(s))^2] \leq c_1(t - r)^2 \tag{A.5}
\]
for all \( r \leq s \leq t \). (The regularity condition (3.3) is required for inequality (A.5).) In addition, it follows from (A.4) and Hölder’s inequality that

\[
E\left[ |\tilde{N}_j(s) - \tilde{N}_j(r) - N_j(t) - N_j(s)| \right] \leq c_1 (s-r)^{1/2} (t-s)^{1/2} \leq c_1 (t-r).
\] (A.6)

Because \( N_1, \ldots, N_n \) are iid processes,

\[
E[(\tilde{B}_n(s) - \tilde{B}_n(r))^2(\tilde{B}_n(t) - \tilde{B}_n(s))^2] = \frac{1}{n\gamma_n^2} E[(\tilde{N}_1(\gamma_n s) - \tilde{N}_1(\gamma_n r))^2(\tilde{N}_1(\gamma_n t) - \tilde{N}_1(\gamma_n s))^2] \\
+ \frac{n-1}{n\gamma_n^2} E[(\tilde{N}_1(\gamma_n s) - \tilde{N}_1(\gamma_n r))^2] E[(\tilde{N}_1(\gamma_n t) - \tilde{N}_1(\gamma_n s))^2] \\
+ \frac{2(n-1)}{n\gamma_n^2} E[(\tilde{N}_1(\gamma_n s) - \tilde{N}_1(\gamma_n r))(\tilde{N}_1(\gamma_n t) - \tilde{N}_1(\gamma_n s))]^2
\leq c_1 (t-r)^2 + c_1^2 (s-r)(t-s) + 2c_1^2 (t-r)^2,
\]

in which the inequality is obtained by (A.4)–(A.6). The lemma follows with \( c = 3c_1^2 + c_1 \).

\[\square\]

**Proof of Theorem 3.** For \( j \in \mathbb{N} \), let

\[
R_j(t) = S_{j,N_j(t)+1} - t
\] (A.7)

be the recess of \( N_j \) at \( t \geq 0 \). In particular,

\[
R_j(0) = \xi_{j,1}.
\] (A.8)

Because \( N_1, \ldots, N_n \) are iid stationary renewal processes, \( R_1(t), \ldots, R_n(t) \) are iid random variables following distribution \( F_e \) for all \( t \geq 0 \), each having mean

\[
m_e = \int_0^\infty t \, dF_e(t) = \frac{1 + \mu \gamma_n^2}{2\mu}
\]

and variance

\[
\sigma_e^2 = \int_0^\infty t^2 \, dF_e(t) - m_e^2 = \frac{\mu}{3} \int_0^\infty t^3 \, dF(t) - m_e^2.
\]

Note that \( \sigma_e^2 < \infty \) by (3.4). Let

\[
\tilde{R}_n(t) = \frac{1}{\sqrt{n\gamma_n}} \sum_{j=1}^n (R_j(\gamma_n t) - m_e).
\]

Then,

\[
E[\tilde{R}_n(t)^2] = \frac{\sigma_e^2}{\gamma_n} \to 0 \quad \text{as } n \to \infty,
\]

which implies that \( \tilde{R}_n(t) \to 0 \) as \( n \to \infty \) for \( t \geq 0 \). By Theorem 3.9 in [Billingsley (1999)],

\[
(\tilde{R}_n(t_1), \ldots, \tilde{R}_n(t_\ell)) \Rightarrow 0 \quad \text{as } n \to \infty
\] (A.9)
for any $\ell \in \mathbb{N}$ and $0 \leq t_1 < \cdots < t_\ell$. By (A.1), (A.7), and (A.8),

$$R_j(t) = \xi_{j,1} + \sum_{k=2}^{N_j(t)+1} \xi_{j,k} - t = R_j(0) + \sum_{k=2}^{N_j(t)+1} (\xi_{j,k} - \mu^{-1}) + \mu^{-1}N_j(t) - t.$$  

Then, by (3.12) and (A.3), we obtain

$$\tilde{B}_n(t) = -\mu \tilde{R}_n(0) + \mu \tilde{R}_n(t) + \tilde{L}_n(t). \quad \text{(A.10)}$$

We deduced from (A.9), (A.10), and Lemma 11 that

$$(\tilde{B}_n(t_1), \ldots, \tilde{B}_n(t_\ell)) \Rightarrow (\hat{B}(t_1), \ldots, \hat{B}(t_\ell)) \quad \text{as} \ n \to \infty.$$  

Finally, it follows from Lemma 12 and Theorem 13.5 in Billingsley (1999) (with condition (13.13) replaced by (13.14)) that $\tilde{B}_n \Rightarrow B$ as $n \to \infty$.

\[
\square
\]

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