Existence of positive solutions to boundary value problems of Caputo fractional difference equations

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Abstract

In this paper, we studied the sufficient conditions for the existence of positive solutions to the boundary value problems of Caputo fractional difference equations depending on parameters with non local boundary conditions. We construct and analyse the Green’s function to the corresponding boundary value problem and then established the existence results using Krasnosel’skii fixed point theorem.

Keywords:  Fractional difference equation; existence; fixed point theorem; Green’s function

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1 Introduction

The theory of continuous fractional calculus and their applications have seen tremendous growth since long time. However, the theory of discrete fractional calculus have seen very slow progress. In the last few years or so, various authors studied the theory of fractional difference equations in their research work.

Atici and Eloe [1, 2, 3] studied variety of basic concepts of theory of discrete fractional calculus. Initially they started to use Green’s function approach to establish the existence of solution to discrete fractional boundary value problems. Furthermore, Goodrich [6, 7, 9, 10] extended his valuable contribution to discrete fractional calculus
by establishing some results on discrete fractional boundary value problems, where he used Krasnosel’kii fixed point theorem to prove the existence. In [8], he gave the existence of solution to fractional boundary value problems by means of Brouwer theorem and uniqueness by means of contraction mapping principle.

S. Kang et al. [13, 14], J. Wang et al. [18] followed this trend and gave the existence of positive solutions to discrete fractional boundary value problems. D. Pachpatte et al. [15, 16] in their work also discussed about the existence of positive solutions to some fractional boundary value problems. Pan et al. [17], in their work established the existence and uniqueness of two boundary value problems of fractional difference equations for $2 < v \leq 3$. In [5], Chen et al. studied the Caputo fractional difference boundary value problems by means of cone theoretic fixed point theorems. They considered fractional boundary value problems with $2 < v \leq 3$.

In this paper, we consider a discrete fractional boundary value problem of the form,

$$\Delta^v_{C} y(t) = -\lambda f(t + v - 1, y(t + v - 1)), \quad (1)$$

$$y(v - 3) = 0, \quad \Delta y(v + b) = \phi(y), \quad \Delta^2 y(v - 3) = 0, \quad (2)$$

where, $t \in [0, b + 1], f : [v - 3, v - 2, \ldots, v + b]_{N_{v-3}} \times [0, \infty) \to [0, \infty)$ is continuous and is not identically zero, $2 < v \leq 3$, $\lambda$ is a positive parameter and $\Delta^v_{C} y(t)$ is the standard Caputo difference.

The present paper is organized as follows. In section 2, we will demonstrate some useful lemmas and theorems in order to prove our main results along with some basic but important definitions. In section 3, we establish our main result for existence of positive solutions to the fractional boundary value problem (1)–(2), followed by few examples.

## 2 Preliminary results

In this section, let us first recall some useful definitions and basic lemmas that are very much important to us in the sequel.

**Definition 1 ([8, 3])** We define,

$$t^v = \frac{\Gamma(t + 1)}{\Gamma(t + 1 - v)}, \quad (3)$$

for any $t$ and $v$ for which right hand side is defined. We also appeal to the convention that if $t + v - 1$ is a pole of the Gamma function and $t + 1$ is not a pole, then $t^v = 0$.

**Definition 2 ([5, 12])** The $v$-th fractional sum of a function $f$, for $v > 0$ is defined as

$$\Delta^v_{a} f(t) = \Delta^v_{a} f(t, a) := \frac{1}{\Gamma(v)} \sum_{s=a}^{t-v} (t - s - 1)^{v-1} f(s), \quad (4)$$
for $t \in \mathbb{N}_{a+N-v}$. We also define the $v$-th Caputo fractional difference of $f$ for $v > 0$ by

$$
\Delta_{c}^{\nu} f(t) = \Delta^{-(n-\nu)} \Delta^{n} f(t) = \frac{1}{\Gamma(n-\nu)} \sum_{s=a}^{t-n} (t-s-1)^{n-\nu-1} \Delta^{n} f(s),
$$

(5)

where $t \in \mathbb{N}_{a+N-v}$ and $N \in \mathbb{N}$ is chosen so that $0 \leq N - 1 < v \leq N$.

Now we give some important lemmas that can be found in recent articles.

**Lemma 1 ([7],[15])** Let $t$ and $v$ be any numbers for which $t^v$ and $t^{v-1}$ are defined. Then $\Delta^{\nu} = vt^{v-1}$.

**Lemma 2 ([5])** Assume that $v > 0$ and $f$ is defined on domain $\mathbb{N}_{a}$, then

$$
\Delta_{a^{+}(n-\nu)}^{\nu} \Delta_{c}^{\nu} f(t) = f(t) - \sum_{k=0}^{n-1} C_{k}(t-a)^{k},
$$

(6)

where $C_{k} \in \mathbb{R}$, $k = 0, 1, 2, \ldots, n-1$ and $n - 1 < v \leq n$.

**Lemma 3 ([5])** Let $f : \mathbb{N}_{a+v} \times \mathbb{N}_{a} \rightarrow \mathbb{R}$ be given. Then

$$
\Delta \left( \sum_{s=a}^{t-v} f(t,s) \right) = \sum_{s=a}^{t-v} \Delta_{c} f(t,s) + f(t+1, t+1-v), \text{ for } t \in \mathbb{N}_{a+v}.
$$

(7)

In order to get the main results, now we state and prove an important lemma, which will provide us a representation for the solution of (1)–(2), provided that the solution exists.

**Lemma 4** Let $2 < v \leq 3$ and $h : [v-2, v+b]_{\mathbb{N}_{v-2}} \rightarrow \mathbb{R}$ be given. Then the solution of discrete fractional boundary value problem

$$
\begin{align*}
\Delta_{c}^{v} y(t) &= -h(t+v-1) \\
y(v-3) &= 0, \quad \Delta y(v+b) = \phi(y), \quad \Delta^{2} y(v-3) = 0
\end{align*}
$$

(8)

is given by

$$
y(t) = \sum_{s=0}^{b+1} G(t,s) h(t+v-1) + \alpha(t) \cdot \phi(y)
$$

(9)

where the Green’s function $G : [v-2, v+b]_{\mathbb{N}_{v-2}} \times [0, b+1]_{\mathbb{N}_{0}} \rightarrow \mathbb{R}$ is defined by

$$
G(t,s) = \frac{1}{\Gamma(v)} \begin{cases} 
(v-1)(t+v-3)(v+b-s-1)^{v-2} - (t-s-1)^{v-1}, & 0 \leq s < t-v+1 \leq b+1 \\
(v-1)(t+v-3)(v+b-s-1)^{v-2}, & 0 \leq t-v+1 < s \leq b+1
\end{cases},
$$

(10)

and $\alpha(t) = (t-v+3)$. 

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Proof: From the Lemma 2, we get that a general solution for (8) is the function

\[ y(t) = -\frac{1}{\Gamma(v)} \sum_{s=0}^{t-v} (t-s-1)^{\frac{v-1}{2}} \cdot h(s+v-1) + C_1 + C_2 t + C_3 t^2 \]  

(11)

for some \( C_i \in \mathbb{R} \), \( i = 1, 2, 3 \). Using Lemma 2, we obtain

\[ \Delta y(t) = -\frac{1}{\Gamma(v)} \sum_{s=0}^{t-v+1} (v-1) (t-s-1)^{\frac{v-2}{2}} \cdot h(s+v-1) + C_2 + C_3 2t, \]

\[ \Delta^2 y(t) = -\frac{1}{\Gamma(v)} \sum_{s=0}^{t-v+2} (v-1) (v-2) (t-s-1)^{\frac{v-3}{2}} \cdot h(s+v-1) + 2C_3, \]

From the boundary conditions of (8), we get

\[ C_3 = 0, \quad C_2 = \frac{1}{\Gamma(v-1)} \sum_{s=0}^{b+1} (v+b-s-1)^{\frac{v-2}{2}} \cdot h(s+v-1) + \phi(y) \]

and \( C_1 = -\frac{(v-3)}{\Gamma(v-1)} \sum_{s=0}^{b+1} (v+b-s-1)^{\frac{v-2}{2}} \cdot h(s+v-1) - (v-3) \phi(y) \)

Consequently, we deduce that the solution of fractional boundary value problem (8) has the form,

\[ y(t) = -\frac{1}{\Gamma(v)} \sum_{s=0}^{t-v} (t-s-1)^{\frac{v-1}{2}} \cdot h(s+v-1) \]

\[-\frac{(v-3)}{\Gamma(v-1)} \sum_{s=0}^{b+1} (v+b-s-1)^{\frac{v-2}{2}} \cdot h(s+v-1) - (v-3) \phi(y) \]

\[ + \frac{t}{\Gamma(v-1)} \sum_{s=0}^{b+1} (v+b-s-1)^{\frac{v-2}{2}} \cdot h(s+v-1) + t \cdot \phi(y) \]

\[ = \frac{1}{\Gamma(v)} \sum_{s=0}^{t-v} \left\{ (v-1) (t-v) (v+b-s-1)^{\frac{v-2}{2}} - (t-s-1)^{\frac{v-1}{2}} \right\} \cdot h(s+v-1) \]

\[ + \frac{1}{\Gamma(v)} \sum_{s=t-v+1}^{b+1} \left\{ (v-1) (t-v+3) (v+b-s-1)^{\frac{v-2}{2}} - (t-s-1)^{\frac{v-1}{2}} \right\} \cdot h(s+v-1) \]

\[ + (t-v+3) \cdot \phi(y) \]

\[ = \sum_{s=0}^{b+1} G(t,s) h(t+v-1) + \alpha(t) \cdot \phi(y). \]

Where \( G(t,s) \) and \( \alpha(t) \) are defined in (10), which shows that if (8) has a solution, then it can be represented by (9) and that every function of the form (9) is a solution of (8). This completes the proof.
Remark 1 ([5]) Notice that $G(v - 3, s) = 0$, $G(t, b + 2) = 0$. $G$ could be extended to $[v - 3, v + b]_{N_{v - 3}} \times [0, b + 2]_{N_0}$, so we only discuss $(t, s) \in [v - 2, v + b]_{N_{v - 2}} \times [0, b + 1]_{N_0}$.

Lemma 5 ([5]) The Green’s function $G(t, s)$ satisfies the following properties:

1. $G(t, s) \geq 0$ for each $(t, s) \in [v - 2, v + b]_{N_{v - 2}} \times [0, b + 1]_{N_0}$

2. $\max_{t \in [v - 2, v + b]_{N_{v - 2}}} G(t, s) = G(v + b, s)$ for each $s \in [0, b + 1]_{N_0}$ and

3. There exists a number $\gamma \in (0, 1)$ such that

$$\min_{t \in \left[\frac{v + b}{3(v + b)}\right]_{N_{v - 2}}} G(t, s) \geq \gamma \max_{t \in [v - 2, v + b]_{N_{v - 2}}} G(t, s) = \gamma G(v + b, s), \ s \in [0, b + 1]_{N_0}$$

Lemma 6 ([10, 14]) The function $\alpha(t)$ is strictly increasing in $t$ for $t \in [v - 2, v + b]_{N_{v - 2}}$. In addition, $\min_{t \in [v - 2, v + b]_{N_{v - 2}}} \alpha(t) = 0$ and $\max_{t \in [v - 2, v + b]_{N_{v - 2}}} \alpha(t) = b + 3$.

Proof: The proof of this lemma is similar to that of Lemma (2.5) in [5]. Hence it is omitted.

Corollary 1 ([10, 14]) There is a constant $A_\alpha \in (0, 1)$ such that

$$\min_{t \in \left[\frac{v + b}{3(v + b)}\right]_{N_{v - 2}}} \alpha(t) = A_\alpha \cdot \|\alpha\|,$$

where $\|\cdot\|$ is the usual maximum norm.

Theorem 1 ([3, 16]) Let $E$ be a Banach space, and let $\mathcal{K} \subset E$ be a cone in $E$. Assume that $\Omega_1$ and $\Omega_2$ are bounded open sets contained in $E$ such that $0 \in \Omega_1$ and $\overline{\Omega}_1 \subseteq \Omega_2$. Further assume that $S : \mathcal{K} \cap (\overline{\Omega}_2 \setminus \Omega_1) \to \mathcal{K}$ be a completely continuous operator. If either

1. $\|Sy\| \leq \|y\|$ for $y \in \mathcal{K} \cap \partial \Omega_1$ and $\|Sy\| \geq \|y\|$ for $y \in \mathcal{K} \cap \partial \Omega_2$; Or

2. $\|Sy\| \geq \|y\|$ for $y \in \mathcal{K} \cap \partial \Omega_1$ and $\|Sy\| \leq \|y\|$ for $y \in \mathcal{K} \cap \partial \Omega_2$

Then the operator $S$ has at least one fixed point in $\mathcal{K} \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Let $\mathcal{B}$ be the Banach space of all functions $y : [v - 2, v + b]_{N_{v - 2}} \to \mathbb{R}$ with respect to the norm $\|y\| = \max \left\{|y(t)| : t \in [v - 2, v + b]_{N_{v - 2}}\right\}$.
Now, we define a cone $\mathcal{K} \subset \mathcal{B}$ by

$$
\mathcal{K} = \left\{ y \in \mathcal{B} : y(t) > 0, \min_{t \in \left[\frac{v+b}{4}, \frac{v+2+b}{4}\right]} y(t) \geq \mathcal{F} \|y\|, \phi(y) \geq 0 \right\},
$$

where $\mathcal{F} = \min \{\gamma, A\alpha\}$.

### 3 Main Result

In the sequel, we now present the next structural assumptions that to be impose on (1)–(2) to get the existence of positive solution.

**F1:** $h(t)$ be a positive function and $g(y)$ be a non negative functional such that

$$f(t, y) = h(t) \cdot g(y).$$

**F2:**

$$\lim_{\|y\| \to 0^+} \frac{g(y)}{\|y\|} = 0, \quad \lim_{\|y\| \to \infty} \frac{g(y)}{\|y\|} = +\infty;$$

**F3:**

$$\lim_{\|y\| \to 0^+} \frac{g(y)}{\|y\|} = +\infty, \quad \lim_{\|y\| \to \infty} \frac{g(y)}{\|y\|} = 0;$$

**G1:** The functional $\phi$ is linear, in particular assume $\phi(y) = \sum_{i=v-3}^{v+b} \frac{c_i}{(b+3)} y(i)$ for $c_i \in \mathbb{R}$.

**G2:** We have $\sum_{i=v-3}^{v+b} \frac{c_i}{(b+3)} G(i, s) \geq 0$ for each $s \in [0, b+1]_{\mathbb{N}_0}$ and

$$\sum_{i=v-3}^{v+b} \frac{c_i}{(b+3)} \leq \frac{1}{2}.$$

**G3:** $\phi(\alpha)$ is non negative.

We also let,

$$\eta := 2 \max_{t \in [v-2, v+b]} \sum_{s=0}^{b+1} G(t, s) h(s+v-1),$$

$$\rho := \min_{t \in [v-2, v+b]} \sum_{s=\left[\frac{b+v+1}{4}\right]}^{\frac{b+v+1}{4}} G(t, s) h(s+v-1),$$

Now, let $F$ be the operator defined by,

$$
(Fy)(t) = \lambda \sum_{s=0}^{b+1} G(t, s) f(s+v-1, y(s+v-1)) + \alpha(t) \cdot \phi(y).
$$

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It can be easily verify that \( y(t) \) is a fixed point of \( F \) if and only if \( y(t) \) is a solution of fractional boundary value problem \( (1)-(2) \).

**Theorem 2** Assume that \( G1 - G3 \) hold and let \( F \) be the operator defined in \( (15) \) and \( \mathcal{K} \) be the cone defined in \( (12) \). Then \( F : \mathcal{K} \rightarrow \mathcal{K} \).

Proof: From \( G1 \), we first show that for every \( y \in \mathcal{K} \),

\[
\phi (Fy) = \sum_{i=0}^{v+b} \frac{c_{i-v+3}}{(b+3)} (Fy) (i)
\]

\[
= \sum_{i=0}^{v+b} \frac{c_{i-v+3}}{(b+3)} \lambda \sum_{s=0}^{b+1} G(i,s) h(s+v-1) g(y(s+v-1))
\]

\[
+ \sum_{i=0}^{v+b} \sum_{j=0}^{v+b} \frac{c_{i-v+3} c_{j-v+3}}{(b+3) (b+3)} y(j) \alpha (i)
\]

\[
= \phi \left\{ \lambda \sum_{s=0}^{b+1} G(t,s) h(s+v-1) g(y(s+v-1)) \right\} + \phi (\alpha) \cdot \phi (y)
\]

From \( G2 \) and \( G3 \) together with the non negativity of \( g(y) \), we get \( \phi (Fy) \geq 0 \).

On the other hand, from the Lemma 5 and Corollary 1 it follows that

\[
\min_{t \in \left[ \frac{v+b}{4}, \frac{3(v+b)}{4} \right]_{N_{v-2}}} (Fy) (t) \geq \min_{t \in \left[ \frac{v+b}{4}, \frac{3(v+b)}{4} \right]_{N_{v-2}}} \lambda \sum_{s=0}^{b+1} G(t,s) f(s+v-1,y(s+v-1))
\]

\[
+ \min_{t \in \left[ \frac{v+b}{4}, \frac{3(v+b)}{4} \right]_{N_{v-2}}} \alpha (t) \phi (y)
\]

\[
\geq \mathcal{F} \max_{t \in \left[ v-2, v+b \right]_{N_{v-2}}} \lambda \sum_{s=0}^{b+1} G(t,s) h(s+v-1) g(y(s+v-1))
\]

\[
+ A \alpha \| \phi (y)
\]

\[
\geq \mathcal{F} \max_{t \in \left[ v-2, v+b \right]_{N_{v-2}}} \lambda \sum_{s=0}^{b+1} G(t,s) h(s+v-1) g(y(s+v-1))
\]

\[
+ \mathcal{F} \| \phi (y)
\]

\[
\geq \mathcal{F} \| Fy \|.
\]

Thus,

\[
\min_{t \in \left[ \frac{v+b}{4}, \frac{3(v+b)}{4} \right]_{N_{v-2}}} (Fy) (t) \geq \mathcal{F} \| Fy \|.
\]

Also, for every \( y \in \mathcal{K} \), \((Fy) (t) \geq 0\). Hence, we conclude that \( F : \mathcal{K} \rightarrow \mathcal{K} \).

This completes the proof.
Theorem 3 Suppose that the condition $F1, F2$ and $G1 - G3$ hold. Then the fractional boundary value problem \([1]-[2]\) has at least one positive solution.

Proof: From the condition $F2$, there exists $r_1 > 0$ such that $g(y) \leq \frac{\lambda}{\eta}$, for $0 < \|y\| \leq r_1$. Let $\Omega_1 = \{y \in \mathcal{X} : \|y\| < r_1\}$ then for any $y \in \mathcal{X} \cap \partial \Omega_1$, we have

\[
(Fy)(t) = \lambda \sum_{s=0}^{b+1} G(t,s) f(s + v - 1, y(s + v - 1)) + \alpha(t) \cdot \phi(y)
\]

\[
\leq \lambda \max_{t \in [\nu - 2, v + b][\eta - 2]} \sum_{s=0}^{b+1} G(t,s) h(s + v - 1) g(y(s + v - 1)) + (b + 3) \phi(y)
\]

\[
\leq \lambda \cdot \frac{\eta}{\lambda} \cdot \max_{t \in [\nu - 2, v + b][\eta - 2]} \sum_{s=0}^{b+1} G(t,s) h(s + v - 1) \cdot \frac{r_1}{\lambda} + (b + 3) \sum_{i=\nu - 3}^{v + b} \frac{c_i - v + 3}{(b + 3)} y(i)
\]

\[
\leq \frac{r_1}{2} + \frac{r_1}{2} = \|y\|.
\]

Thus, we have $\|Fy\| \leq \|y\|$ for $y \in \mathcal{X} \cap \partial \Omega_1$.

On the other hand, from the condition $F2$, there exists $r_2 > 0$ such that $0 < r_1 < r_2$ and $g(y) \geq \frac{\lambda}{\rho}$.

We consider $\Omega_2 = \{y \in \mathcal{X} : \|y\| < r_2\}$, then for any $y \in \mathcal{X} \cap \partial \Omega_2$, we have

\[
(Fy)(t) = \lambda \sum_{s=0}^{b+1} G(t,s) f(s + v - 1, y(s + v - 1)) + \alpha(t) \cdot \phi(y)
\]

\[
\geq \lambda \sum_{s=0}^{b+1} \min_{s=\nu - 2, v + b} \frac{3(b + 1)}{b(v + 1)} G(t,s) h(s + v - 1) g(y(s + v - 1))
\]+ \min_{t \in [\nu - 2, v + b][\eta - 2]} \alpha(t) \phi(y)

\[
\geq \lambda \cdot \min_{t \in [\nu - 2, v + b][\eta - 2]} \sum_{s=\nu - 3}^{v + b} \frac{3(b + 1)}{b(v + 1)} G(t,s) h(s + v - 1) \cdot \frac{r_2}{\lambda \rho}
\]

\[
= r_2 = \|y\|.
\]

Thus, we have $\|Fy\| \geq \|y\|$ for $y \in \mathcal{X} \cap \partial \Omega_2$.

Hence, from \([16], [17]\) and by Theorem \([1]\) we have $F$ has at least one fixed point, $y \in \mathcal{X} \cap (\Omega_2 \setminus \Omega_1)$. This function $y(t)$ is a positive solution of \([1]-[2]\) and satisfies $r_1 \leq \|y\| \leq r_2$. This completes the proof.
Let $\mathbf{Ω} = \{ y \in \mathcal{X} : \| y \| < r_3 \}$ then for any $y \in \mathcal{X} \cap \partial \mathbf{Ω}_{r_3}$, we have

$$
(F y)(t) = \lambda \sum_{s=0}^{b+1} G(t,s) f(s + v - 1, y(s + v - 1)) + \alpha(t) \cdot \phi(y)
$$

$$
\geq \lambda \sum_{s=0}^{b+1} \left[ \frac{3(b+v)}{2} - v + 1 \right] G(t,s) h(s + v - 1) g(y(s + v - 1))
+ \min_{t \in [v-2,v+b]\mathcal{N}_{v-2}} \alpha(t) \cdot \phi(y)
$$

$$
\geq \lambda \sum_{s=0}^{b+1} \left[ \frac{3(b+v)}{2} - v + 1 \right] G(t,s) h(s + v - 1) \cdot \frac{r_3}{\lambda \rho}
$$

$$
= r_3 = \| y \|. \tag{18}
$$

Thus, we have $\| F y \| \geq \| y \|$ for $y \in \mathcal{X} \cap \partial \mathbf{Ω}_{r_3}$.

Now, we consider two cases for the construction of $\Omega_{r_4}$.

Case I: Assume that $g(y)$ is bounded. Then by condition $F3$, there exist $R_1 > r_4$ such that $g(y) \leq \frac{R_1}{\lambda \eta}$ for $y \in \mathcal{X}$.

Let $\mathbf{Ω}_{R_1} = \{ y \in \mathcal{X} : \| y \| < R_1 \}$, then we have

$$
(F y)(t) = \lambda \sum_{s=0}^{b+1} G(t,s) f(s + v - 1, y(s + v - 1)) + \alpha(t) \cdot \phi(y)
$$

$$
\leq \lambda \max_{t \in [v-2,v+b]\mathcal{N}_{v-2}} \sum_{s=0}^{b+1} G(t,s) h(s + v - 1) g(y(s + v - 1)) + (b + 3) \phi(y)
$$

$$
\leq \lambda \max_{t \in [v-2,v+b]\mathcal{N}_{v-2}} \sum_{s=0}^{b+1} G(t,s) h(s + v - 1) \cdot \frac{R_1}{\lambda \eta} + (b + 3) \sum_{i=v-3}^{v+b} \frac{c_{l-v+3}}{(b + 3)} y(i)
$$

$$
\leq \lambda \frac{\eta}{2} \cdot \frac{R_1}{\lambda \eta} + \frac{R_1}{2}
\leq \frac{R_1}{2} + \frac{R_1}{2}
\leq R_1 = \| y \|. \tag{19}
$$

Case II: Assume that $g(y)$ is unbounded. Then, by condition $F3$, there exist some $R_2$ such that $g(y) \leq \frac{R_2}{\lambda \eta}$ for $0 < \| y \| \leq R_2$. 

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Choose $R_3$ such that $R_3 > r_4$ and for $0 < ||y|| \leq R_3$, $g(y) \leq g(R_3)$.
Define $R = \max\{R_2, R_3\}$. Now, we set $\Omega_R = \{y \in \mathcal{K} : ||y|| < R\}$, then $g(y) \leq \frac{R}{\lambda \eta}$.
Thus for $y \in \partial \Omega_R$, we have

\[
(Fy)(t) = \lambda \sum_{s=0}^{b+1} G(t, s) f(s + v - 1, y(s + v - 1)) + \alpha(t) \cdot \phi(y) \\
\leq \lambda \max_{t \in [v-2, v+b] \cap \mathbb{N}_{v-2}} \sum_{s=0}^{b+1} G(t, s) h(s + v - 1) g(y(s + v - 1)) + (b + 3) \phi(y) \\
\leq \lambda \max_{t \in [v-2, v+b] \cap \mathbb{N}_{v-2}} \sum_{s=0}^{b+1} G(t, s) h(s + v - 1) \cdot \frac{R}{\lambda \eta} + (b + 3) \sum_{i=v-3}^{v+b} \frac{c_{i-v+3}}{(b + 3)} y(i) \\
\leq \lambda \eta \cdot \frac{R}{\lambda \eta} + \frac{R}{2} \\
\leq \frac{R}{2} + \frac{R}{2} \\
= ||y||.
\]

(20)

Thus, in both Case I and Case II, we have $||Fy|| \leq ||y||$ for $y \in \mathcal{K} \cap \partial \Omega_R$.

Hence, from (18), (19), (20) and by Theorem 1 we have $F$ has at least one fixed point, $y \in \mathcal{K} \cap (\Omega_{r_3} \setminus \Omega_{r_4})$ with $r_3 \leq ||y|| \leq r_4$. This Completes the proof.

**Example 1** Consider the following fractional boundary value problem of Caputo fractional difference equation,

\[
\Delta_{c}^{\frac{8}{3}} y(t) = -\lambda \left( t + \frac{5}{3} \right)^2 y \left( t + \frac{5}{3} \right) \left\{ e^{y(t+\frac{4}{3})} - 1 \right\} \\
y \left( -\frac{1}{3} \right) = 0, \quad \Delta y \left( \frac{35}{3} \right) = \frac{3}{12} y \left( \frac{5}{3} \right) + \frac{5}{24} y \left( \frac{14}{3} \right), \quad \Delta^2 y \left( -\frac{1}{3} \right) = 0,
\]

(21) (22)

where, $v = \frac{8}{3}$, $b = 9$, $h(t) = t^2$, $g(y) = ||y|| \left( e^{||y||} - 1 \right)$, $t \in [0, 9] \cap \mathbb{N}_0$, we have

\[
\lim_{||y|| \to 0^+} \frac{g(y)}{||y||} = \lim_{||y|| \to 0^+} \frac{||y|| \left( e^{||y||} - 1 \right)}{||y||} = 0, \quad \lim_{||y|| \to \infty} \frac{g(y)}{||y||} = \lim_{||y|| \to \infty} \frac{||y|| \left( e^{||y||} - 1 \right)}{||y||} = \infty.
\]

Thus, $g(y)$ and $\phi(y)$ satisfy the conditions of Theorem 3 hence the boundary value problem (21)-(22) has at least one positive solution.
Example 2 Consider the following fractional boundary value problem of Caputo fractional difference equation,

\[
\Delta^\frac{8}{3} y(t) = -\lambda e^{(t+\frac{5}{3})} \sec^2 \left\{ y \left( t + \frac{5}{3} \right) \right\},
\]

(23)

\[
y \left( -\frac{1}{3} \right) = 0, \quad \Delta y \left( \frac{35}{3} \right) = \frac{7}{12} y \left( \frac{2}{3} \right) - \frac{1}{6} y \left( \frac{17}{3} \right), \quad \Delta^2 y \left( -\frac{1}{3} \right) = 0,
\]

(24)

where, \( v = \frac{8}{3}, \ b = 9, \ h(t) = e^t, \ g(y) = \sec^2 \left\{ \|y\| \right\}, \ t \in [0,9]_{\mathbb{N}_0}, \) we have

\[
\lim_{\|y\| \to 0^+} \frac{g(y)}{\|y\|} = \lim_{\|y\| \to 0^+} \frac{\sec^2 \left\{ \|y\| \right\}}{\|y\|} = +\infty, \quad \lim_{\|y\| \to +\infty} \frac{g(y)}{\|y\|} = \lim_{\|y\| \to +\infty} \frac{\sec^2 \left\{ \|y\| \right\}}{\|y\|} = 0
\]

Thus, \( g(y) \) and \( \phi(y) \) satisfy the conditions of Theorem 4, hence the boundary value problem (23)–(24) has at least one positive solution.

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