A REMARK ON FROBENIUS DESCENT FOR VECTOR BUNDLES

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Abstract. We give a class of examples of vector bundles on a relative smooth projective curve over \( \text{Spec} \, \mathbb{Z} \) such that for infinitely many prime reductions the bundle has a Frobenius descent, but the restriction to the generic fiber in characteristic zero is not semistable. In the third section of the paper we prove for a large class of varieties (including abelian varieties) that any vector bundle with this Frobenius descent property is generically semistable.

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1. Introduction

Let \( X \) be a smooth projective variety defined over an algebraically closed field of characteristic \( p > 0 \) with a fixed very ample line bundle \( \mathcal{O}_X(1) \). Further we denote by \( F \) the absolute Frobenius morphism \( F : X \to X \) which is the identity on the topological space underlying \( X \) and the \( p \)th power map on the structure sheaf \( \mathcal{O}_X \). A vector bundle \( \mathcal{E} \) on \( X \) descends under \( F \) if there exists another vector bundle \( \mathcal{F} \) such that \( \mathcal{E} \cong \mathcal{F}^\ast(F) \). This note is inspired by the recent preprint [6] of K. Joshi. In the relative situation, where a morphism \( X \to \text{Spec} \, R \) with generic fiber \( X := X_0 \) is given and \( R \) is a \( \mathbb{Z} \)-domain of finite type, Joshi asked the following interesting question: “assume \( X \) is a smooth projective variety and suppose \( V \) is a vector bundle which descends under Frobenius modulo an infinite set of primes then is it true that \( V \) is semistable (with respect to any ample line bundle on \( X \))?”. He gives a positive answer to this question for rank two vector bundles under the additional assumption that \( \text{Pic}(X) = \mathbb{Z} \).

In section 2 of this paper we provide a class of examples which give a negative answer to this question in general. We show that on the relative Fermat curve \( C = V_+(X^d + Y^d + Z^d) \to \text{Spec} \, \mathbb{Z} \), with \( d \geq 5 \) odd, there exists a vector bundle \( \mathcal{E} \) of rank two such that for infinitely many prime numbers \( p \) the reduction \( \mathcal{E}_p = \mathcal{E}|_{C_0} \) modulo \( p \) has a Frobenius descent, but \( \mathcal{E}_0 = \mathcal{E}|_{C_0} \) is not semistable on the fiber over the generic point. In section 3 we give an
affirmative answer to this question under the assumption that for every closed point \( m \in \text{Spec } R \) every semistable vector bundle on the fiber \( X_m \) is strongly semistable. We recall that a semistable vector bundle \( E \) is strongly semistable if \( F^e(E) \) is semistable for all integers \( e \geq 0 \). This provides further examples of varieties with \( \text{Pic}(X) \neq \mathbb{Z} \) (for example abelian varieties) for which the question of Joshi still has a positive answer.

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2. A counterexample for vector bundles on curves

In this section we give an example of a rank two vector bundle on a generically smooth projective relative curve over \( \text{Spec } \mathbb{Z} \) such that infinitely many prime reductions have a Frobenius descent but the bundle is not semistable on the generic fiber in characteristic zero.

Our example will use the syzygy bundle \( \text{Syz}(X^2, Y^2, Z^2)(m) \) on Fermat curves \( C = V_+(X^d + Y^d + Z^d) \subset \mathbb{P}^2 \) defined over a field \( K \). This vector bundle is defined by the short exact sequence

\[
0 \rightarrow \text{Syz}(X^2, Y^2, Z^2)(m) \rightarrow \mathcal{O}_C(m - 2)^3 \rightarrow \mathcal{O}_C(m) \rightarrow 0,
\]

where the penultimate mapping is given by \( (s_1, s_2, s_3) \mapsto s_1 X^2 + s_2 Y^2 + s_3 Z^2 \).

The bundle \( \text{Syz}(X^2, Y^2, Z^2)(m) \) is semistable for \( d \geq 5 \) by [2, Proposition 6.2]. In positive characteristic \( p > 0 \), since the presenting sequence only involves locally free sheaves, it is easy to see that the Frobenius pull-back \( F^*(\text{Syz}(X^2, Y^2, Z^2)(m)) \cong \text{Syz}(X^{2p}, Y^{2p}, Z^{2p})(mp) \).

**Lemma 2.1.** Let \( d = 2\ell + 1 \) with \( \ell \geq 2 \) and let \( C := \text{Proj } K[X, Y, Z]/(X^d + Y^d + Z^d) \) be the Fermat curve of degree \( d \) defined over a field \( K \) of characteristic \( p \equiv \ell \mod d \). Then the Frobenius pull-back of \( \text{Syz}(X^2, Y^2, Z^2)(3) \) sits inside the short exact sequence

\[
0 \rightarrow \mathcal{O}_C(\ell - 1) \rightarrow \text{Syz}(X^{2p}, Y^{2p}, Z^{2p})(3p) \rightarrow \mathcal{O}_C(-\ell + 1) \rightarrow 0.
\]

In particular, the Frobenius pull-back is not semistable and this sequence constitutes its Harder-Narasimhan filtration.

**Proof.** We write \( 2p = dk + 2\ell \) with \( k \) even. The pull-back \( \text{Syz}(X^{2p}, Y^{2p}, Z^{2p}) \) of \( \text{Syz}(X^2, Y^2, Z^2) \) has a non-trivial global section in total degree \( d(k + 1 + k/2) \) by [3, Proof of Proposition 1.2]. From the presenting sequence of the pull-back one reads off the degree as follows:

\[
\deg(\text{Syz}(X^{2p}, Y^{2p}, Z^{2p})(d(k + 1 + k/2))) = d(2d(k + 1 + k/2) - 6p) = d(2d(k + 1 + k/2) - 3(dk + 2\ell)) = d(2d - 6\ell) = d(-2\ell + 2) < 0.
\]
Since a semistable vector bundle of negative degree cannot have non-trivial global sections, the Frobenius pull-back \( \text{Syz}(X^{2p}, Y^{2p}, Z^{2p}) \) is not semistable. We obtain a non-trivial mapping \( \mathcal{O}_C(\ell - 1) \rightarrow \text{Syz}(X^{2p}, Y^{2p}, Z^{2p})(3p) \). We want to show that this mapping constitutes the Harder-Narasimhan filtration of the pull-back, meaning that this mapping has no zeros. Hence, assume that we have a factorization

\[
\mathcal{O}_C(\ell - 1) \rightarrow \mathcal{L} \rightarrow \text{Syz}(X^{2p}, Y^{2p}, Z^{2p})(3p),
\]

where \( \mathcal{L} \) is a subbundle of the syzygy bundle and has degree \( \deg(\mathcal{L}) := \alpha \geq (\ell - 1)d \). We have the short exact sequence

\[
0 \rightarrow \mathcal{L} \rightarrow \text{Syz}(X^{2p}, Y^{2p}, Z^{2p})(3p) \rightarrow \mathcal{L}' \rightarrow 0,
\]

where \( \mathcal{L}' \) is a line bundle of degree \(-\alpha\). By [15, Corollary 2] (or [16, Theorem 3.1]) the inequality

\[
\mu_{\text{max}}(\mathcal{S}) - \mu_{\text{min}}(\mathcal{S}) = \alpha - (-\alpha) = 2\alpha \leq 2g - 2
\]

holds, where \( \mathcal{S} := \text{Syz}(X^{2p}, Y^{2p}, Z^{2p})(3p) \) and \( g \) denotes the genus of \( C \). The genus formula for plane curves yields

\[
2g - 2 = (d - 1)(d - 2) - 2 = d(d - 3) = 2d(\ell - 1).
\]

Therefore, we obtain \( \alpha = d(\ell - 1) \). Hence, \( \mathcal{O}_C(\ell - 1) \cong \mathcal{L} \) and the Harder-Narasimhan filtration is indeed \( 0 \subset \mathcal{O}_C(\ell - 1) \subset \text{Syz}(X^{2p}, Y^{2p}, Z^{2p})(3p) \). □

**Remark 2.2.** Using Hilbert-Kunz theory and its geometric interpretation developed in [4] and [17] one can give an alternative (but more complicated) proof that the line bundle \( \mathcal{O}_C(\ell - 1) \) is the maximal destabilizing subbundle of the syzygy bundle \( \text{Syz}(X^{2p}, Y^{2p}, Z^{2p})(3p) \). We recall that for a rank two vector bundle the Harder-Narasimhan filtration is already strong in the sense of [9, Paragraph 2.6]. By the formula given in [4, Theorem 3.6] we can compute from the short exact sequence

\[
0 \rightarrow \mathcal{L} \rightarrow \text{Syz}(X^{2p}, Y^{2p}, Z^{2p})(3p) \rightarrow \mathcal{L}' \rightarrow 0,
\]

the Hilbert-Kunz multiplicity \( e_{\text{HK}}(I) \) (see [12]) of the ideal \( I = (X, Y, Z) \) in the homogeneous coordinate ring \( R := K[X, Y, Z]/(X^d + Y^d + Z^d) \) of the curve \( C \) and obtain \( e_{\text{HK}}(I) = 3d + \frac{d^2}{dp^2} \). But, by [13, Theorem 2.3] the Hilbert-Kunz multiplicity of \( I \) equals \( e_{\text{HK}}(I) = 3d + \frac{d(d-3)^2}{4p} \) which implies \( \alpha = d(\ell - 1) \).

**Remark 2.3.** We briefly comment on the situation for \( \ell = 0, 1 \). For \( \ell = 0 \) (and \( p \neq 2 \)) we have \( \text{Syz}(X^2, Y^2, Z^2)(3) \cong \mathcal{O}_p^2 \) and this is also true for its Frobenius pull-back. For \( \ell = 1 \), we get the Fermat cubic which is an elliptic curve. In this case we have an exact sequence

\[
0 \rightarrow \mathcal{O}_C \rightarrow \text{Syz}(X^2, Y^2, Z^2)(3) \rightarrow \mathcal{O}_C \rightarrow 0,
\]
where the (only) global non-trivial section is given by the curve equation. So the syzygy bundle is \( F_2 \) in Atiyah’s classification \([1]\) and is semistable, but not stable. Its Frobenius pull-back is either \( F_2 \) (for \( p \equiv 1 \mod 3 \), i.e. Hasse invariant one) or \( O_C^2 \) (for \( p \equiv 2 \mod 3 \), i.e. Hasse invariant zero).

In the relative situation
\[
C := \text{Proj}(\mathbb{Z}[X, Y, Z]/(X^d + Y^d + Z^d)) \longrightarrow \text{Spec} \mathbb{Z}_d
\]
every fiber \( C_p := C \times_{\text{Spec} \mathbb{Z}_d} \text{Spec} \mathbb{F}_p \) is a smooth projective curve, namely the Fermat curve, defined over the prime field \( \mathbb{F}_p \) (and \( C_p := C \times_{\text{Spec} \mathbb{Z}_d} \overline{\mathbb{F}}_p \) is a smooth projective curve over the algebraic closure of \( \mathbb{F}_p \)) for every prime number \( p \) such that \( p \nmid d \). We remind that by the Theorem of Dirichlet \([14, \text{Chapitre VI, §4, Théorème et Corollaire}] \) there exist infinitely many prime numbers \( p \equiv \ell \mod d \).

**Lemma 2.4.** Let \( d = 2\ell + 1, \ell \geq 2 \), and consider the smooth projective relative curve \( C := \text{Proj}(\mathbb{Z}[X, Y, Z]/(X^d + Y^d + Z^d)) \longrightarrow \text{Spec} \mathbb{Z}_d \). Then the sequence (from Lemma 2.1)
\[
0 \longrightarrow \mathcal{O}_{C_p}(\ell - 1) \longrightarrow \text{Syz}(X^{2p}, Y^{2p}, Z^{2p})(3p) \longrightarrow \mathcal{O}_{C_p}(-\ell + 1) \longrightarrow 0
\]
does not split for almost all primes \( p \equiv \ell \mod d \).

**Proof.** Since \( \text{Syz}(X^{2p}, Y^{2p}, Z^{2p})(3p) \cong F^*(\text{Syz}(X^2, Y^2, Z^2)(3)) \) holds on every fiber \( C_p \), the bundle \( \text{Syz}(X^{2p}, Y^{2p}, Z^{2p})(3p) \) carries an integrable connection \( \nabla_p \) with \( p \)-curvature zero by the Cartier-correspondence \([7, \text{Theorem 5.1}] \). Assume that the sequence does split for some \( p \equiv \ell \mod d \). Then \( \mathcal{O}_{C_p}(\ell - 1) \) is a direct summand of \( \text{Syz}(X^{2p}, Y^{2p}, Z^{2p})(3p) \). The summand \( \mathcal{O}_{C_p}(\ell - 1) \) carries also a connection with the same properties. Hence, again by the Cartier-correspondence it has a Frobenius descent and so its degree \( d(\ell - 1) \) is divisible by \( p \). But this can only hold for finitely many \( p \). \( \square \)

**Example 2.5.** As above we consider the smooth relative curve
\[
C := \text{Proj}(\mathbb{Z}[X, Y, Z]/(X^d + Y^d + Z^d)) \longrightarrow \text{Spec} \mathbb{Z}_d,
\]
with \( d = 2\ell + 1, \ell \geq 2 \). The Čech-cohomology class \( c = Z^{d-1}/XY \in H^1(C, \mathcal{O}_C(d - 3)) \cong \text{Ext}^1(\mathcal{O}_C(-\ell + 1), \mathcal{O}_C(\ell - 1)) \) defines an extension
\[
0 \longrightarrow \mathcal{O}_C(\ell - 1) \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_C(-\ell + 1) \longrightarrow 0
\]
with the corresponding restrictions to each fiber \( C_p \), where \( p \equiv 0 \) or \( p \equiv \ell \), \( p \mid d \). Note that this extension is non-trivial on every fiber. This vector bundle \( \mathcal{E} \) is our example. As \( \ell \geq 2 \) the bundle \( \mathcal{E}_0 = \mathcal{E}|_{C_0} \) is not semistable on \( C_0 \). By Lemma 2.1 we have for \( p \equiv \ell \mod d \) an extension
\[
0 \longrightarrow \mathcal{O}_{C_p}(\ell - 1) \longrightarrow \text{Syz}(X^{2p}, Y^{2p}, Z^{2p})(3p) \longrightarrow \mathcal{O}_{C_p}(-\ell + 1) \longrightarrow 0
\]
corresponding to \( c' \in H^1(C_p, \mathcal{O}_{C_p}(2\ell - 2)) = H^1(C_p, \mathcal{O}_{C_p}(d - 3)) \), and by Lemma 2.4 we have \( c' \neq 0 \) for almost all \( p \equiv \ell \mod d \). We claim that
in positive characteristic and a further hypothesis on $Y$ that $E$ is semistable rank two vector bundle on $Y$ of arbitrarily large residue characteristic the reduction $E|_X$ of $X$ has Frobenius descent on the fiber $X$. Theorem 3.1. Let $E$ be a vector bundle over $X$. Further

Let $X \to \text{Spec } R$ be a smooth projective morphism of relative dimension $d \geq 1$, where $R$ is a domain of finite type over $\mathbb{Z}$. Typical examples for the base are Spec $\mathbb{Z}$ or arithmetic schemes Spec $D$, where $D$ is the ring of integers in a number field. Let $E$ be a vector bundle over $X$. In [6, Theorem 4.2] K. Joshi proved under the assumptions $\text{Pic}(X) = \mathbb{Z}$ ($X = \mathcal{X}_0$) and $\text{rk}(E) = 2$ that $E_0 = E|_X$ is semistable if for infinitely many closed points $m \in \text{Spec } R$ of arbitrarily large residue characteristic the reduction $E_m$ admits a Frobenius descent on the fiber $X_m = \mathcal{X}_m$. The aim of this section is to prove (using essentially the same methods) this result for vector bundles of arbitrary rank under the assumption that for every closed point $m$ every semistable vector bundle $F$ on $X_m$ is strongly semistable, i.e. $F^{\ast \ast}(F)$ is semistable for all $e \geq 0$ (it is enough to assume this for infinitely many closed points $m$ of arbitrary large residue characteristic). It is interesting to note that Joshi used in [6, Theorem 2.1] the condition $\text{Pic}(Y) = \mathbb{Z}$ on a smooth projective variety $Y$ in positive characteristic and a further hypothesis on $Y$ to prove that every semistable rank two vector bundle on $Y$ is strongly semistable.

**Theorem 3.1.** Let $R$ be a $\mathbb{Z}$-domain of finite type and let $f : X \to \text{Spec } R$ be a smooth projective morphism of relative dimension $d \geq 1$ together with a fixed $f$-very ample line bundle $\mathcal{O}_X(1)$ and let $E$ be a vector bundle on $X$. Further
assume that every semistable vector bundle is strongly semistable (with respect to \( O_{X_m}(1) \)) for every fiber \( X_m, m \) a closed point in \( \text{Spec} \, R \). Then the following holds: If \( E_m = E|_{X_m} \) has a Frobenius descent for infinitely many closed points \( m \in \text{Spec} \, R \) of arbitrarily large residue characteristic, then \( E_0 \) is semistable on the generic fiber \( X = X_0 = X_0 \).

**Proof.** One can show by induction over \( \dim R \) that there exists a bound \( b \) such that \( \mu_{\max}(E_m) \leq b \) for all closed points \( m \in \text{Spec} \, R \) (see [5, Lemma 3.1] for an explicit proof). For a closed point \( m \in \text{Spec} \, R \) with descent data \( E_m \cong F^*(F_m) \), \( F_m \) locally free on the fiber \( X_m \), we have

\[
\mu_{\max}(E_m) = \text{char}(\kappa(m)) \mu_{\max}(F_m)
\]

because semistable vector bundles are strongly semistable on every fiber \( X_m \) by assumption. Since we have \( E_m \cong F^*(F_m) \) for infinitely many closed points \( m \) of arbitrarily large residue characteristics, this forces the similar equalities \( \deg(E_0) = \deg(E_m) = \text{char}(\kappa(m)) \deg(F_m) \) (we take the degree always with respect to \( O_{X_m}(1) \)) which implies \( \deg(E_m) = \deg(F_m) = 0 \). Assume the restriction \( E_0 \) to the generic fiber \( X \) is not semistable. Then by the openness of semistability [11, Section 5] every restriction \( E_m \) on \( X_m \) is not semistable. Again by our assumption, \( F_m \) is not semistable either and so \( \mu_{\max}(F_m) \geq 1/r \), \( r = \text{rk}(\mathcal{E}) \). This gives

\[
b \geq \mu_{\max}(E_m) = \text{char}(\kappa(m)) \mu_{\max}(F_m) \geq \frac{\text{char}(\kappa(m))}{r}
\]

which contradicts the assumption that we have Frobenius descent at closed points \( m \in \text{Spec} \, R \) of arbitrarily large residue characteristic. □

**Corollary 3.2.** Let \( R \) be a \( \mathbb{Z} \)-domain of finite type and let \( f : X \to \text{Spec} \, R \) be a smooth projective morphism of relative dimension \( d \geq 1 \) together with a fixed \( f \)-very ample line bundle \( O_X(1) \) and let \( E \) be a vector bundle on \( X \). Suppose that the fibers \( X_m, m \in \text{Spec} \, R \) closed, fulfill at least one of the following (not necessarily independent) properties:

1. \( X_m \) is an abelian variety,
2. \( X_m \) is a homogenous space of the form \( G/P \) where \( P \) is a reduced parabolic subgroup,
3. the cotangent bundle \( \Omega_{X_m} \) fulfills \( \mu_{\max}(\Omega_{X_m}) \leq 0 \).

Then the following holds: If \( E_m \) has a Frobenius descent for infinitely many closed points \( m \in \text{Spec} \, R \) of arbitrarily large residue characteristics, then \( E_0 \) is semistable on \( X = X_0 \).

**Proof.** That every semistable vector bundle is strongly semistable in the case (3) is due to [10, Theorem 2.1]. Condition (3) holds in particular for the varieties occurring in (1) and (2). Other proofs for this property in cases (1) and (2) are given in [15, Corollary 3p] and in case (3) in [9, Corollary 6.3]. Hence, the assertion follows from Theorem 3.1. □
Remark 3.3. On the one hand it is well-known that every semistable vector bundle on an elliptic curve is strongly semistable (cf. [18, Appendix]). So elliptic curves provide an important class of smooth projective varieties with $\text{Pic}(X) \neq \mathbb{Z}$ for which Theorem 3.1 holds. On the other hand it is also known that for every smooth projective curve of genus $g \geq 2$ there exists a semistable vector bundle $F$ such that $F^*(F)$ is not semistable (see [8, Theorem 1]). So we see that Theorem 3.1 is applicable in relative dimension one only for elliptic curves and the projective line $\mathbb{P}^1$.

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