Optimal Learning Dynamics of Multi Agents in Restless Multiarmed Bandit Game

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Abstract
Social learning is learning through the observation of or interaction with other individuals; it is critical in the understanding of the collective behaviors of humans in social physics. We study the learning process of agents in a restless multiarmed bandit (rMAB). The binary payoff of each arm changes randomly and agents maximize their payoffs by exploiting an arm with payoff 1, searching the arm at random (individual learning), or copying an arm exploited by other agents (social learning). The system has Pareto and Nash equilibria in the mixed strategy space of social and individual learning. We study several models in which agents maximize their expected payoffs in the strategy space, and demonstrate analytically and numerically that the system converges to the equilibria. We also conducted an experiment and investigated whether human participants adopt the optimal strategy. In this experiment, three participants play the game. If the reward of each group is proportional to the sum of the payoffs, the median of the social learning rate almost coincides with that of the Pareto equilibrium.

Keywords: Nash equilibrium, Pareto optimum, optimal learning dynamics, multi agent system, multiarmed bandit

1. Introduction
A multiagent system is an active research field and it is applicable to many academic disciplines. In physics, it has been employed to study the collective behaviors of humans in econophysics[1, 2] and socio-physics[3–5]. If the system exhibits intriguing macroscopic behaviors, it is challenging to solve the system theoretically. For example, the voter model abstracts the influence process among humans and exhibits rich mathematical structures, as a consensus formation[6–8]. In the studies, the model parameters are typically constant and the agents do not learn from their experiences. If the agents learn from their experience, the problem becomes an reinforcement learning multiagent system. The model parameters change as the agents learn; therefore, it becomes more difficult to solve the system than the nonlearning agent system.

Social learning is learning through the observation of or interaction with other individuals[10–14]. As it causes the tendency to follow others’ behaviors, the system of social learning agents becomes a strongly correlated system. As a toy model, we have proposed a multiagent system[15] in a restless multiarmed bandit (rMAB). A restless multiarmed bandit[12] is a slot machine with multiple arms. The term “restless” implies that the payoffs change randomly. We call an arm with a high payoff a good arm. Each agent maximizes his payoffs by exploiting an arm, searching for a good arm (individual learning), and copying an arm exploited by other agents (social learning). The system exhibits a phase transition at a critical value of the social learning probability[15]. If the social learning probability is below the threshold value, the variance of the number of agents who have found the good arm is proportional to the number of agents. If it exceeds the threshold value, the variance becomes proportional to the power of the number of agents with a critical exponent larger than 1. The system shows an oscillatory behavior between the state where almost all agents know the good arms and the state where none of the agents know them. Further, the optimal value of the social learning probability is studied. In[16], we studied an rMAB with only one good arm and demonstrated that the system yields the unique Nash equilibrium and Pareto optimal states, thus solving the famous Rogers paradox[17–20]. The question that arises is how the agents adopted the optimal strategies. If the agents can change their social learning
probability, the optimization problem can be formulated as a reinforcement learning problem. It is a difficult problem and an efficient approach does not exist. In this study, we investigate the type of strategies that agents adopt when they attempt to maximize their fitness. Hence, we first analytically and numerically study the best response dynamics and its natural variants, that is, whether they reach one of the natural equilibria, the Nash equilibrium, and the Pareto optimal states. Next, we replace the agents with humans and perform an experiment to examine whether humans tend to employ the optimal strategy \cite{21}. In this experiment, three human players play the rMAB game.

This manuscript is organized as follows. In the next two sections, we state the definition of our model and present the necessary results from our previous work \cite{16}. In section 4, we study the best response dynamics. We demonstrate that they converge to the Nash equilibrium or the Pareto optimal state. In section 5, we elaborate the experimental setup and present the results. The last section presents the concluding remarks.

2. Model

The rMAB contains only one good arm and infinite number of bad arms. There are $N$ agents labeled by $n = 1, \cdots, N$ (see fig. 1). The system evolves in time as follows.

In each turn, an agent (e.g., agent $n$) is chosen randomly. He exploits his arm and obtains payoff 1 if he knows a good arm. If he does not know a good arm, he searches for it by a random search (individual learning) with probability $1 - r_n$, or copies the information of other agents’ good arms (social learning) with probability $r_n$. The random search always succeeds with probability $q_I$. Meanwhile, the copy process succeeds with probability $q_O$ if and only if at least one agent knows a good arm.

Subsequently, with probability $q_C/N$, the good arm changes to a bad arm and another good arm appears (environmental change). When an environmental change occurs, the agents who know the good arm are forced to forget it and know a bad one.

This completes the turn. The system will proceed to the next turn.

We call $N$ consecutive turns a step. It is expected that each agent performs one action per step. The probability that the environment does not change during one step is $(1 - q_C/N)^N$ which is close to $e^{-q_C}$ when $N$ is large. Therefore, this probability must not be small for the copy process to be meaningful.

Figure 2 shows the schematics of the time evolution of the system.

![Figure 1: Bandits and agents.](image_url)
Mathematical Formulation. Next, we formulate a stochastic model. We introduce the random variable $\sigma_n$ defined by

$$\sigma_n = \begin{cases} 1, & \text{if agent } n \text{ knows a good arm}, \\ 0, & \text{if agent } n \text{ does not know a good arm}. \end{cases} \quad (1)$$

We regard $\sigma_n$ as agent $n$’s knowledge of a good arm. For each turn $t$, we have a joint probability function $P(\sigma_1, \cdots, \sigma_N|t)$. For simplicity we use vector notation: $P(\vec{\sigma}|t)$. Time evolution is described by the Chapman–Kolmogorov equation:

$$P(\vec{\sigma}'|t+1) = \sum_{\vec{\sigma}''} T(\vec{\sigma}'|\vec{\sigma}'') P(\vec{\sigma}|t), \quad (2)$$

where $T(\vec{\sigma}'|\vec{\sigma})$ is the stochastic matrix (transfer matrix) of the system. The matrix is expressed as the product of the agent action part and the environmental change part:

$$T(\vec{\sigma}'|\vec{\sigma}) = \sum_{\vec{\sigma}''} T^C(\vec{\sigma}'|\vec{\sigma}'') T^A(\vec{\sigma}'|\vec{\sigma}''). \quad (3)$$

The environmental change part, $T^C(\vec{\sigma}'|\vec{\sigma})$, is obtained as follows. When no environmental change occurs, each agent stores his knowledge on the good arm. If an environmental change occurs, all the agents lose their knowledge. Thus, we have

$$T^C(\vec{\sigma}'|\vec{\sigma}) = \left(1 - \frac{q_C}{N}\right) \prod_{n=1}^{N} \delta_{\sigma_n' \sigma_n} + \frac{q_C}{N} \prod_{n=1}^{N} \delta_{\sigma_n' 0}, \quad (4)$$

where $\delta_{\sigma_n' \sigma_n}$ is the Kronecker delta. Next, we examine the agent action part, $T^A(\vec{\sigma}'|\vec{\sigma})$. The probability that agent $n$ who does not know the good arm is chosen and finds a good arm is

$$p_n(\vec{\sigma}) = \frac{\delta_{\sigma_n 0}}{N} \left(r_n (1 - N_{1,0}) q_O + (1 - r_n) q_I\right), \quad (5)$$

$$N_1 = \sum_{n=1}^{N} \sigma_n. \quad (6)$$

The matrix $T^A(\vec{\sigma}'|\vec{\sigma})$ is the sum of the probability that no agent changes his knowledge and the probability that one of the agents changes his knowledge. Thus, we have

$$T^A(\vec{\sigma}'|\vec{\sigma}) = p_{NC}(\vec{\sigma}) \prod_{n=1}^{N} \delta_{\sigma_n' \sigma_n} + \sum_{n=1}^{N} p_n(\vec{\sigma}) \delta_{\sigma_n' \sigma_n+1} \prod_{\ell \neq n} \delta_{\sigma_\ell' \sigma_\ell}, \quad (7)$$

$$p_{NC}(\vec{\sigma}) = 1 - \sum_{n=1}^{N} p_n(\vec{\sigma}). \quad (8)$$

This completes the formulation of our stochastic model.
We make a short comment on the assumption of social learning. In our model, the copy process is assumed to succeed with probability \( q_0(1 - \delta_{N,0}) \). As already mentioned, this means that social learning succeeds with probability \( q_0 \) if and only if at least one agent knows the correct answer. However, what we want to emphasize is that the copy process always fails if no agent knows the answer. Our intention is to rule out copying of incorrect information\[12, 19]. For this purpose, it is possible to replace the factor \( "1 - \delta_{N,0}" \), for example, with \( N_1/N \), the proportion of agents who know the correct answer. The new model is a natural replacement for the original one. Unfortunately, it can not be solved analytically. A numerical simulation indicates that the new model seems to behave similarly to the original one with a smaller \( q_0 \).

**Asymptotic Property of \( P(\vec{\sigma}|t) \).** It is proven that the matrix \( T(\vec{\sigma}^t|\vec{\sigma}) \) is irreducible and primitive\[22] if and only if

\[
q_C > 0, \quad q_I > 0, \quad \forall r_n < 1. \tag{9}
\]

Under this assumption, the unique Perron vector \( P(\vec{\sigma}) \) exists. The Perron vector has the remarkable feature that it is the long-time limit of an arbitrary solution \( P(\vec{\sigma}|t) \) of the Chapman–Kolmogorov equation, that is, \( \lim_{t \to \infty} P(\vec{\sigma}|t) = P(\vec{\sigma}) \). We say that the system is in the steady state when the probability function is \( P(\vec{\sigma}) \).

The irreducibility of the matrix \( T(\vec{\sigma}^t|\vec{\sigma}) \) presents a physical meaning. Let \( \vec{q} \) be a knowledge vector at an instant turn \( t \). Then, for an arbitrary knowledge vector \( \vec{q}^t \), the probability that the change \( \vec{q} \to \vec{q}^t \) occurs after a few turns is positive (this is merely the definition of irreducibility). Indeed, \( q_C = 0 \) implies that the knowledge on a good arm will never be lost and the dynamics of the system may become trivial. In other words, a transition of type \( \vec{q} \neq \vec{0} \to \vec{0} \) does not exist. When \( q_I(1 - r_n) = 0 \), the transition \( \vec{0} \to (0, \ldots, 0, 1, 0, \ldots, 0) \) is impossible. Thus, no agent will obtain a correct information on a good arm when the initial condition is given by \( \vec{q} = \vec{0} \). Conversely, if \( (9) \) is satisfied, an arbitrary transition \( \vec{q} \to \vec{q}^t \) is possible through \( \vec{0} \). This proves the irreducibility of the probability matrix.

When the matrix \( T \) is irreducible, it is also primitive because \( \text{tr} T \geq T(\vec{0}|\vec{0}) > 0 \). Primitivity ensures that the long-time limit of the probability function \( \lim_{t \to \infty} P(\vec{\sigma}|t) \) coincides with the Perron vector \( P(\vec{\sigma}) \) regardless of the initial probability function \( P(\vec{\sigma}|0) \).

In the following, we assume that the condition \( (9) \) holds, in addition to the inequality

\[
q_0 > 0 \tag{10}
\]

for nontriviality.

3. Nash Equilibrium and Pareto Optimality

**Fitness Function.** We summarize the necessary results from our previous work\[16]. The expected payoff for each agent in the steady state is defined by

\[
w_n = E[\sigma_n] = \sum_{\vec{\sigma}} P(\vec{\sigma})\sigma_n, \quad n = 1, \ldots, N. \tag{11}
\]

We call \( w_n \) the fitness of agent \( n \). It is shown that a function \( w(r, \vec{r}) \) exists such that \( w_n \) is expressed as

\[
w_n = w(r_n, \vec{r}_n), \quad \vec{r}_n = \frac{1}{N - 1} \sum_{k \neq n} r_k. \tag{12}
\]

We call \( w(r, \vec{r}) \) the fitness function, which is expressed explicitly by

\[
w(r, \vec{r}) = \frac{1}{a + q_I + (q_0 - q_I)r} \left\{ q_I + (q_0 - q_I)r - \frac{aq_0r}{a + \kappa} \right\}, \quad \kappa = (N - 1)q_I(1 - \tau) + q_I(1 - r), \quad a = \frac{q_C}{1 - q_C/N}. \tag{13, 14}
\]

**Nash Equilibrium.** The strategy space of the agents is an \( N \)-dimensional unit cube,

\[
J = \{ \vec{r} = (r_1, \ldots, r_N) \mid 0 \leq r_i \leq 1 \}. \tag{15}
\]

It is noteworthy that the limit \( r_i \to 1 \) is not excluded. In this space, the unique Nash equilibrium point \( \vec{r}_{\text{Nash}} \) exists. It exhibits the following properties: (i) it is symmetric, \( \vec{r}_{\text{Nash}} = (r_{\text{Nash}}, \ldots, r_{\text{Nash}}) \), (ii) \( r_{\text{Nash}} < 1 \), (iii) \( \vec{r}_{\text{Nash}} \) is strict, \( w(r_{\text{Nash}}, \vec{r}_{\text{Nash}}) < w(r_{\text{Nash}}, \vec{r}_{\text{Nash}}) \), \( \forall r_0 \neq r_{\text{Nash}} \), (iv) \( r_{\text{Nash}} \to 0 \) as \( \{ N(q_0 - q_I) - (a + q_0) \} \to +0 \). The specific form of \( r_{\text{Nash}} \) is as follows:

\[
r_{\text{Nash}} = \begin{cases} 1 - \eta, & N(q_0 - q_I) > a + q_0, \\ 0, & N(q_0 - q_I) \leq a + q_0, \end{cases} \tag{17}
\]


where
\[ \eta = \frac{2(a + q_o)^2}{(q_o - q_t N)(a + q_o) + (a N + q_o)(q_o - q_t) + \sqrt{D_1}} \]
\[ D_1 = ((q_o - q_t N)(a + q_o) - (a N + q_o)(q_o - q_t))^2 + 4(N - 1)(q_o - q_t)(a + q_o)q_o(a + N q_t). \]

Because \( \vec{r}_{Nash} \) is strict, it is an evolutionarily stable strategy (ESS)\(^{23}\). Moreover, this is an ESS based on Thomas\(^{24}\). Indeed, the following inequality is true:
\[ w(r_{Nash}, r_0) > w(r_0, r_0), \quad \forall r_0 \neq r_{Nash}. \quad (18) \]

**Pareto Optimality.** The concept of Pareto optimality is defined in the context of resource allocation. We have regarded the fitness \( w_n = w(r_n, \vec{r}_n) \) as the utility of agent \( n \). Let us also consider this entity as the amount of resources acquired by agent \( n \). At this time, the maximum point of the total utility function,
\[ I(\vec{r}) = \sum_{n=1}^{N} w_n(r_n, \vec{r}_n), \quad (19) \]
is suitable as a Pareto optimal point. It is shown in the Appendix that only one maximum point exists, which we call \( r_{Pareto} \). It is strictly maximal and symmetric: \( r_{Pareto} = (r_{Pareto}, \cdots, r_{Pareto}) \).

It is clear that \( r_{Pareto} \) is the only maximum point of \( w(r, \vec{r}) \). Our result is that
\[ r_{Pareto} = \begin{cases} \frac{(a + q_t N)X - (a + q_t)Y}{q_t N X + (q_o - q_t)Y}, & N(q_o - q_t) > a + q_o, \\ 0, & N(q_o - q_t) \leq a + q_o, \end{cases} \quad (20) \]

where
\[ X = \sqrt{(N - 1)(a + q_o)(q_o - q_t)}, \]
\[ Y = \sqrt{N q_o(a + N q_t)}. \]

It is not difficult to verify that (i) \( r_{Pareto} < 1 \), and (ii) \( r_{Pareto} \to 0 \) as \( \{N(q_o - q_t) - (a + q_o)\} \to +0 \).

**Comparison of Fitness.** We consider three types of fitness per agent: the fitness of individual learners, that in the Nash equilibrium state, and that in the Pareto optimal state. They are defined as follows, respectively:
\[ w_I = w(0, \vec{r}), \quad w_{Nash} = w(r_{Nash}, r_{Nash}), \quad w_{Pareto} = w(r_{Pareto}, r_{Pareto}). \quad (21) \]

It is proven that the inequality, \( w_I \leq w_{Nash} \leq w_{Pareto} \), is true. The equality holds if and only if \( N(q_o - q_t) \leq a + q_o \).

### 4. Optimal Strategies of Agents

Nash equilibrium and Pareto optimality are concepts in equilibrium. Thus, it is important to demonstrate that some learning processes converge to the Nash equilibrium and Pareto optimal states. A standard model is the best response dynamics, which is expected to reach the Nash equilibrium state. In this section, we study the best response dynamics and its variants. Best response dynamics seems to have been appeared by simplifying fictitious play\(^{25–27}\). One of the purpose of both learning models is to explain Nash equilibria by each agent’s local search method of strategy\(^{28}\).

#### 4.1. Best Response Dynamics

Several versions of the best response dynamical system exist. Herein, we consider the continuous-time best response dynamical system.

First, we introduce the best response function,
\[ \beta_N(\vec{r}) = \text{argmax}_{0 \leq r \leq 1} w(r, \vec{r}). \quad (22) \]
The best response function of agent \( n \) is expressed as \( \beta_n(\vec{r}_n) \). See eqs. \(^{12}\) and \(^{13}\). The properties of the function \( \beta_N(r) \) are summarized in fig.\(^{*}\).

The continuous-time best response dynamics is a learning process defined by the following dynamical system:
\[ \frac{dr_n(t)}{dt} = \beta_N(\vec{r}_n(t)) - r_n(t), \quad n = 1, \cdots, N. \quad (23) \]
Thus, it is proven that an arbitrary solution to equation (23) converges to the unique Nash equilibrium point $\bar{r}_{\text{Nash}}$. Nonetheless, we shall demonstrate that any solution to equation (23) converges to the unique Nash equilibrium point $\bar{r}_{\text{Nash}}$.

The following relation derived from fig. 3 is noteworthy:

$$\beta_N(r) - \beta_N(r_{\text{Nash}}) = -(N - 1)\gamma(r)(r - r_{\text{Nash}}), \quad 0 \leq \gamma(r) \leq \frac{1}{2},$$

(24)

We write $\gamma(\bar{r}_n(t))$ as $\gamma_n(t)$ for brevity. Let $x_n(t) = r_n(t) - r_{\text{Nash}}$. Because $r_{\text{Nash}}$ is the fixed point of $\beta_N(r)$, the next equation is derived,

$$\beta_N(\bar{r}_n(t)) - r_n(t) = -\gamma_n(t) \sum_{k \neq n} x_k(t) - x_n(t).$$

(25)

We introduce the positive definite quadratic form:

$$V(\vec{x}) = \frac{1}{2} \sum_{n=1}^{N} x_n^2 + \frac{1}{2} \left( \sum_{n=1}^{N} x_n \right)^2.$$  

(26)

The function $V(\vec{x})$ contains a unique minimum at $\vec{x} = \vec{0}$. We differentiate $V(\vec{x})$ along a solution of (23),

$$\frac{d}{dt} V(\vec{x}(t)) = \sum_{n=1}^{N} \frac{dx_n}{dt}(t) \left( x_n(t) + \sum_{k=1}^{n} x_k(t) \right)$$

$$= - \sum_{n=1}^{N} (1 - \gamma_n(t)) x_n(t)^2 + \left( 1 + \sum_{n=1}^{N} \gamma_n(t) \right) \left( \sum_{k=1}^{n} x_k(t) \right)^2$$

$$\leq \frac{1}{2} \sum_{n=1}^{N} x_n(t)^2 - \frac{1}{2} \left( \sum_{n=1}^{N} x_n(t) \right)^2$$

$$= -V(\vec{x}(t)).$$

This yields

$$0 \leq V(\vec{x}(t)) \leq V(\vec{x}(0)) e^{-t} \to 0 \quad (t \to \infty).$$

Thus, it is proven that an arbitrary solution $\vec{r}(t)$ of (23) converges to $\vec{r}_{\text{Nash}}$.

It is noteworthy that the discrete best response dynamics,

$$r_n(t + 1) = r_n(t) + \alpha \{ \beta_N(\bar{r}_n(t)) - r_n(t) \}, \quad n = 1, \cdots, N,$$

(27)
also converges to the Nash equilibrium irrespective of the initial condition \( \vec{r}(0) \in J \) if the learning rate \( \alpha \) is sufficiently small. This is evident because (23) is a continuous limit \( \alpha \rightarrow +0 \) with substitution \( r_n(t + 1) \leftarrow r_n(t + \alpha) \) in the left-hand side of (27).

Further, in eq. (27), the strategies of all agents are updated simultaneously. Meanwhile, a model exists in which only the randomly selected agent’s strategy is updated. The latter is represented by replacing the learning rate \( \alpha \) with \( \alpha/N \) in eq. (27).

### 4.2 Derivative Best Response Dynamics

The best response dynamics is a natural and straightforward learning procedure that can lead the system to the Nash equilibrium. However, when no agent can obtain other agents’ strategies, it is impossible for agent \( n \) to know the value, \( \beta_N(\vec{r}_n(t)) \), of the best response function.

Therefore, as a “realistic” model, we introduce the following system:

\[
\frac{dr_n}{dt}(t) = \frac{\partial w}{\partial r}(r_n(t), \vec{r}_n(t)), \quad n = 1, \ldots, N,
\]

where \([r] \equiv \max(r_*, \min(r^*, r))\) is the truncation of \( r \) to the closed interval \([r_*, r^*]\) (see fig. 4). Because \( w(r_n, \vec{r}_n) \) is the expected income of agent \( n \), he could estimate the differential coefficient, \( \frac{\partial w}{\partial r}(r_n(t), [\vec{r}_n(t)]) \).

In reinforcement learning, it is natural to replace the estimate by a numerical derivative. Agent \( n \) estimates \( w(r_n, \vec{r}_n) \) by the time average of his income \( \sigma_n \) for \( r_n, \vec{r}_n \). He changes \( r_n \) to \( r_n + \Delta r_n \) and estimates the time average. We divide the difference in the time averages by \( \Delta r_n \) and obtain the derivative.

The fitness function \( w(r, \vec{r}) \) contains a unique maximal point as a function of the first variable \( r \). This point is a strictly decreasing function of the second variable \( r \). Let \( \beta_N(\vec{r}) \) be the function thereof. Subsequently, the following relation is true:

\[
\beta_N(\vec{r}) = \beta_N([\vec{r}]) = \max(0, \min(1, \beta_N(\vec{r}))).
\]

As a consequence, the crucial condition

\[
0 \leq \frac{\partial w}{\partial r}(0, [\vec{r}_n(t)]), \quad \text{and} \quad \frac{\partial w}{\partial r}(1, [\vec{r}_n(t)]) \leq 0, \quad n = 1, \ldots, N,
\]

holds. This ensures that any solution \( \vec{r}(t) \) of the system (28) remains in the domain \( J \). Therefore, the introduction of the function \([r]\) renders (28) well-defined.

We wish to demonstrate that \( \vec{r}(t) \) converges to the Nash equilibrium point. When \( N(q_O - q_I) \leq a + q_O, \beta(\vec{r}_n(t)) = 0 \) such that the right-hand side of (28) is negative whenever \( r_n(t) > 0 \). Therefore, \( r_n(t) \rightarrow 0 \) as \( t \rightarrow \infty \). In the \( N(q_O - q_I) > a + q_O \) case, we first describe the analytical property of the Nash equilibrium point. It is asymptotically

\[\text{A sufficient condition is that } \alpha \leq \sqrt{\frac{2}{4 + N(N + 2 + (N + 3)^2)}}.\]
Figure 5: Numerical integration of the derivative best response dynamical system for various initial points. Parameters: $q_C = 0.1$, $q_I = 0.2$, $q_O = 0.8$, $N = 2$. Our analytical result is that $r_{\text{Nash}} = 0.1537\ldots$.

Figure 6: Histogram of $\{W(r_1), \cdots, W(r_{1000})\}$. The number of classes is 30. Parameters: $q_C = 0.2$, $q_I = 0.3$, $q_O = 0.8$, $N = 10$.

stable. Indeed, substituting $r_n = r_{\text{Nash}} + \delta r_n$ into (28) yields

$$\frac{d}{dt} \delta \vec{r} = A \delta \vec{r}, \quad A = \begin{pmatrix} \mu & \nu & \cdots & \nu \\ \nu & \mu & \cdots & \vdots \\ \vdots & \cdots & \ddots & \nu \\ \nu & \cdots & \nu & \mu \end{pmatrix},$$

where

$$\mu = \frac{\partial^2 w}{\partial r^2}(r_{\text{Nash}}, r_{\text{Nash}}) < \nu = \frac{\partial^2 w}{\partial r \partial \bar{r}}(r_{\text{Nash}}, r_{\text{Nash}}) < 0.$$
Next, we differentiate the function \( V(\vec{x}) \), defined in (20), along a solution of the equation (28),

\[
\frac{d}{dt}V(\vec{x}(t)) = W(\vec{r}(t)), \quad W(\vec{r}) = \sum_{n=1}^{N} \frac{\partial w}{\partial r_n}(r_n, [\vec{r}]_n) \left\{ x_n + \sum_{k=1}^{N} x_k \right\}.
\]

We randomly choose 1000 points, \( \vec{r}_k \in J, k = 1, \ldots, 1000 \) and calculate \( W(\vec{r}_k) \). Figure 6 shows the histogram of data \( \{W(\vec{r}_k)\}_{k=1, \ldots, 1000} \). This suggests that \( W(\vec{r}) \) is negative on \( J \).

The results above strongly indicate that \( \vec{r}(t) \) converges to the Nash equilibrium point.

### 4.3. Cooperative Dynamics

When agents are cooperative, the system may converge to the Pareto optimal state. A possible dynamical system that approaches the Pareto optimal point can be expressed as follows:

\[
\frac{dr_n}{dt} = \frac{I(r)}{\partial r_n}, \quad n = 1, \ldots, N,
\]

where \( I(\vec{r}) \) is the total fitness function defined in (19). When the initial point is close to the Pareto optimal point \( \vec{r}_{\text{Pareto}} \), this system will converge to it because \( \vec{r}_{\text{Pareto}} \) is the asymptotically stable unique fixed point of the system. Indeed, the Hessian of \( I \) at \( \vec{r}_{\text{Pareto}} \) is positive definite. See Appendix.

However, this intuitive Pareto dynamics presents some deficiencies. First, the solution is not guaranteed to remain in \( J \). Next, it is difficult to know whether the Pareto optimal point is a global attraction point.

Therefore, we will introduce another dynamical system with better properties, that is, a Pareto version of the best response dynamics. A natural definition of the best response function of the Pareto type for agent \( n \) may be \( \text{argmax} I(\vec{r}) \). This is the larger zero \( r_n \) of the function,

\[
\begin{align*}
\frac{\partial I}{\partial r_n}(\vec{r}) &= \frac{\partial w}{\partial r_n}(r_n, [\vec{r}]_n) + \frac{1}{N-1} \sum_{k \neq n} \frac{\partial w}{\partial r_k}(r_k, [\vec{r}]_k), \\
&\text{truncated to the closed interval } [0, 1].
\end{align*}
\]

Unfortunately, the resulting function is not desirable because of its complexity. Instead, we adopt the following simple definition of the best response function \( \beta_P(\vec{r}) \) of the Pareto type,

\[
\beta_P(\vec{r}) = \max(0, \min(1, |\vec{r} - \vec{r}_{Pareto}|)), \quad \text{where} \quad |\vec{r} - \vec{r}_{Pareto}| = \text{larger zero } r \text{ of } \frac{\partial w}{\partial r_n}(r_n, \vec{r}) + \frac{\partial w}{\partial r_k}(r_k, \vec{r}).
\]

This is an extension of \( \beta_N(\vec{r}) \), the best response function of the Nash type. The features of the function \( \beta_P(r) \) are summarized in fig. 7. Additionally, it exhibits the following properties: (i) the inequality \( \beta_P(r) \leq \beta_N(r) \) is true, and (ii) \( r_{\text{Pareto}} \) defined in (20) is the unique fixed point.

(a) \( q_O \geq Nq_I \) Case

\[
\frac{(N-1)(1-r) - \frac{a+q_I}{q_O-q_I}}{\sqrt{\frac{Nq_O}{q_I} + 1}} - \frac{\frac{Nq_O}{q_I} + 1}{N-1} \left( r - r^* \right)
\]

(b) \( q_O < Nq_I \) Case

\[
\frac{(N-1)(1-r) - \frac{a+q_I}{q_O-q_I}}{\sqrt{\frac{Nq_O}{q_I} + 1}} - \frac{\frac{Nq_O}{q_I} + 1}{N-1} \left( r - r^* \right)
\]

Figure 7: Plots of the function \( \beta_P(r) \). The entity \( r_{**} \) is given by \( r_{**} = 1 - \frac{2(a+q_I)-a(q_O-q_I) + \sqrt{D_A}}{2(N-1)q_I(q_O-q_I)} \), where \( D_A = \{q_I(a + q_O) - a(q_O - q_I) \}^2 + 4q_I(q_O - q_I)(a + q_O)(a + Nq_O) \). The entity \( r^* \) is defined in fig. 3.

Now, we introduce the cooperative dynamical system, that is, the best response dynamical system of the Pareto type:

\[
\frac{dr_n}{dt} = \beta_P(\vec{r}_n) - r_n, \quad n = 1, \ldots, N.
\]

\( ^2 \)Saddle points exist outside the domain \( J \).
This is well defined because $0 \leq \beta_P \leq 1$ such that $r(t) \in J$ is guaranteed. The inequality $\beta_P(r) \leq \beta_N(r)$ suggests that agents obeying (30) can be described as more cooperative than in the case of (23).

We shall demonstrate that all the solutions of (30) converge to the Pareto optimal point $\tilde{r}_{\text{Pareto}}$. Hence, we present the following estimation derived from fig. 7:

$$\beta_P(r) - \beta_P(\tilde{r}_{\text{Pareto}}) = -(N-1)\gamma_P(r - r_{\text{Pareto}}),$$

$$0 \leq \gamma_P(r) \leq \max \left( \frac{1}{N+1}, \frac{1}{\sqrt{\frac{\gamma_P}{\gamma_N} + 1}} \right) \equiv c < 1.$$  

This is an analogy of (24). Let $x_n = r_n - r_{\text{Pareto}}$ and let $\gamma_n(t) = \gamma_P(\tilde{r}_n(t))$. Subsequently, we have a relation analogous to (25):

$$\beta_P(\tilde{r}_n(t)) - r_n(t) = -\gamma_n(t) \sum_{k \neq n} x_k(t) - x_n(t).$$

Further, the time derivative of the function $V(\tilde{x})$ defined in (20) along the solution of (30) reads

$$\frac{d}{dt}V(\tilde{x}(t)) \leq -2(1-c)V(\tilde{x}(t)), \quad 0 \leq V(\tilde{x}(t)) \leq V(\tilde{x}(0))e^{-2(1-c)t} \to 0 \quad (t \to \infty).$$

Therefore, we conclude that all solutions of (30) converge to the Pareto optimal point $\tilde{r}_{\text{Pareto}}$.

It is noteworthy that a discrete version of the system (30),

$$r_n(t+1) = r_n(t) + \alpha \{\beta_P(\tilde{r}_n(t)) - r_n(t)\}, \quad n = 1, \cdots, N,$$

converges to the Pareto optimal point as well if the learning rate $\alpha$ is sufficiently small.

Equation (33) is a model equation of the simultaneous-update type (see also eq. (27)). An individual update-type model is realized by replacing the learning rate $\alpha$ with $\alpha/N$ in eq. (33).

5. Experimental Studies

We have performed an experiment to study whether human adopts the Pareto equilibrium. It is a laboratory experiment performed in Kitasato and Hirosaki University. In the experiment, multiple human players participate in the game and compete for the number of coins to be acquired. A total of thirty three subjects (1 female and 32 males; mean age(1 s.d.) =29.2(1.6)) participated and we label them as $i \in \{1, \cdots, 33\}$. There are 11 groups of three subjects ($N = 3$) and we label them as $G \in \{1, \cdots, 11\}$. The subjects in each group know each other and the total reward given to the participants is in proportion to the total number of coins acquired by them (3 yen/coin). It is an incentive to cooperate with each other in the same group to maximize the number of coins. The optimal strategy of the players is $r_{\text{Pareto}}$. The reward is in the range of 2000 yen and 3000 yen per subject.

There are three rounds of thirty minutes and we label them as $R \in \{1, 2, 3\}$. Between the rounds, there are two intervals of 10 minutes and the players can discuss how to maximize the number of coins. The optimal strategy is $r_{\text{Pareto}}$. We adopt the parameter setting $q_1 = 0.1, q_0 = 0.8, q_C = 0.2$ so that $r_{\text{Pareto}} = 0.23$ and $r_{\text{Nash}} = 0.34$. Table 1 summarizes the settings of the experiment.

| $N$ | $S$ | $T$ | $R$ | Reward | Date      | Subject | Pool       |
|-----|-----|-----|-----|--------|-----------|---------|------------|
| 3   | 33  | 263 | 1   | 3 yen/coin | 2017/6 and 2017/11 | Both Univ. |
| 3   | 33  | 319 | 2   | 3 yen/coin | 2017/6 and 2017/11 | Both Univ. |
| 3   | 33  | 322 | 3   | 3 yen/coin | 2017/6 and 2017/11 | Both Univ. |

Table 1: Experimental design. $N$: the number of subjects in each group, $S$: the total number of subjects, $T$: average number of turns (actions) of the subjects, $R$: the round number.

5.1. Method

We have developed a browse rMAB game. The game interface is shown in Figure 8. The experiments were performed in the laboratory. Experimenter explained the experimental procedures and the rewards, the subjects were asked to sit on chairs that are located far from each others. While playing the game, it was forbidden to talk to other subjects so that it was impossible to share the information whether they know the bandit with $\sigma = 1$ or not. During the intervals, the subjects could freely talk with others and discussed how to get more coins.

Figure 8 shows the screenshot of the game. When the player does not know the bandit with $\sigma = 1$, he sees the left figure. There are two options, individual learning and social learning, which are chosen by pushing the buttons with labels, ”search!” and ”cheat!”, respectively. When the player knows the bandit with $\sigma = 1$, he sees the right
Individual learning and social learning are chosen by pushing the button, "Search!" and "Cheat!", respectively. If he knows the bandit with $\sigma = 1$, he sees the screen in the right figure. Usually, he exploits the bandit by pushing the button "Get Coin!" and get some coins. The number of coins is in the range $\{1, 2, 3\}$ and it is fixed at random with probability $1/3$ when one finds the bandit. This mechanism was introduced to make the game interesting. The other two buttons provide the player to changes the bandit. If the number of coins in the exploit is one, one should want to changes the bandit. If one push "Search" or "Cheat!", one searches the bandit with $\sigma = 1$ again. If he succeeds, he can change the number of the coins. "Cheat" button is better than "Search" button when $q_O > q_I$. The subjects were taught about the function before the start of the experiment. Hereafter, we explain the results of the experiment. Summary of the statistics are given in Table 2.

5.2. Results

The left figure of Figure 9 shows the scatter plot of $(r_i, \pi_i)$. There are three rounds in the experiment, $R \in \{1, 2, 3\}$, we show the plots with different symbols for the rounds. The black triangle shows $(r_{\text{Pareto}}, w_{\text{Pareto}})$. The right figure shows the boxplots of $r_i$. The broken and dotted lines shows $r_{\text{Nash}}$ and $r_{\text{Pareto}}$. The collective average of $r_i$ in round 2 and round 3 are 0.31 and 0.30 and they are slightly larger than $r_{\text{Pareto}} = 0.23$. The median (50 percentile) of $r_i$ in round 2 and round 3 are 0.23 and 0.25, which almost coincide with $r_{\text{Pareto}}$. As the experiment proceeds, we observe more collaborative behavior.

| $R$ | $r_{\text{opt}}$ | Mean of $r_i$ (SE) | SD of $r_i$ | Median of $r_i$ | $w(r_{\text{opt}}, r_{\text{opt}})$ | Mean of $\pi_i$ (SE) |
|-----|-----------------|-------------------|-------------|-----------------|--------------------------|-------------------|
| 1   | 0.23            | 0.35(2)           | 0.07        | 0.33            | 0.36                     | 0.40(1)           |
| 2   | 0.23            | 0.31(5)           | 0.08        | 0.23            | 0.36                     | 0.42(1)           |
| 3   | 0.23            | 0.30(4)           | 0.06        | 0.25            | 0.36                     | 0.37(1)           |

Table 2: Summary of experimental results. $r_{\text{opt}}$ is the optimal strategy.

In order to test the above hypothesis, we adopt a hierarchical Bayesian method. We assume $r_i$ for subject $i$ who participate in the experiment of group $G[i] \in \{1, \cdots, 11\}$ in round $R[i] \in \{1, 2, 3\}$ obeys,

$$r_i \sim t(\nu = 3, \mu = r(R[i], G[i]), \text{scale} = \sigma_A)$$

$$r[R, G] = r_A + \Delta r_R[R] + \Delta r_G[G]$$

$$\Delta r_R[R] \sim \mathcal{N}(0, \sigma_R^2)$$

$$\Delta r_G[G] \sim \mathcal{N}(0, \sigma_G^2)$$
Figure 9: Left: Scatter plot of \((r_i, \sigma_i)\). There are three rounds in the experiment, \(R \in \{1, 2, 3\}\). Empty circles, empty diamonds and gray circles show the plot for \(R = 1, 2, 3\), respectively. The black triangle shows \((r_{\text{Pareto}}, \sigma_{\text{Pareto}})\). Right: Boxplot of \(r_i\). The broken (dotted) horizontal line shows \(r_{\text{Nash}} (r_{\text{Pareto}})\).

\(\Delta r_R\) and \(\Delta r_G\) describe the dependence of \(r(R, G)\) on \(R\) and \(G\), respectively. As for \(\sigma_A, \sigma_R, \sigma_G\), we assume the Half-Cauchy prior,

\[
\sigma_A, \sigma_R, \sigma_G \sim \text{Cauchy}_+ (25).
\]

We studied the posterior distribution of \(r_A, \Delta r_R[R], \Delta r_G[G]\) and estimate the 95% Bayesian credible intervals using Stan 2.19.2 in R 3.6.2 software. We have checked the convergence of the sampling by the Gelman-Rubin statistics. Table 3 shows the results for \(r_A + \Delta r_R[R]\).

| \(R\) | \(R = 1\) | \(R = 2\) | \(R = 3\) |
|------|---------|---------|---------|
| mean | 0.34    | 0.24    | 0.26    |
| 95% C.I. | [0.28,0.41] | [0.17,0.30] | [0.20,0.31] |

Table 3: The mean and 95% Bayesian credible intervals of \(r_A + \Delta r_R[R]\).

We see that the credible intervals of \(r_A + \Delta r_R[R]\) for \(R = 2, 3\) does not include \(r_{\text{Nash}} = 0.34\). We also reject the hypothesis that \(\Delta r_R[R = 1] = \Delta r_R[R = 2] = \Delta r_R[R = 3]\) with significance of 1% by estimating \(P(\Delta r_1 < \Delta r_2)\) and \(P(\Delta r_1 < \Delta r_3)\). We observe collaborative behaviors in \(R = 2\) and \(R = 3\).

6. Concluding Remarks

We herein introduced a multiagent system in a restless multiarmed bandit game and studied the optimal learning dynamics of agents theoretically and experimentally.

As is well known, the Nash equilibrium is a typical solution in noncooperative games. We demonstrated that best response dynamics drove the system to equilibrium. In the case of cooperative game, we introduced a new dynamical system, that is, a best response dynamical system of the Pareto type and proved that it converged to the Pareto optimal point. We also conducted a cooperative game type experiment. As shown in fig.9 the distribution of strategies of the participants appeared to be centered around the Pareto optimal point as the round proceeded. This observation was supported by a hierarchical Bayesian analysis.

The following issues can be explored in the future. First, we have introduced a new concept of the best response function in section 4, that is, the best response function of the Pareto type. It is a natural extension of the function of Nash type. It may be possible to define the same type of function in generic systems, and we believe that it is a useful concept. Second, it would be interesting to perform noncooperative game type experiments to examine whether humans adopt the Nash strategy. Third, we investigated the agents’ behavior based on the optimal strategies in equilibrium. This procedure is reasonable because, as shown in section 4 the Nash equilibrium and Pareto optimal states are the natural destinations in the long run. The experiments on Nash equilibrium may be explained better if a time-dependent theory is considered.
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Competing Interests

The authors declare that they have no competing interests.

Ethics Declarations

Informed consent was obtained from all individual participants involved in the study.

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Appendix  Maximum Point of the Total Fitness Function

Derivative of the total fitness function, \( I(\vec{r}) \), is given by

\[
\frac{1}{a} \frac{\partial I}{\partial r_n} (\vec{r}) = \frac{1}{a + \kappa} \left\{ \frac{k(q_0 - q_I) - q_I(a + q_O)}{(a + q_n)^2} - K \right\},
\]

\( K = \sum_{k=1}^{N} \frac{q_I q_0^r k}{(a + q_k)(a + \kappa)}, \quad q_n \equiv q_I + (q_0 - q_I)r_n. \)

Because \( J \) is compact, at least one maximum point of \( I(\vec{r}) \) in \( J \) exists.

Maximum Point in \( J^o \). Let \( \vec{r} \) be one of the maximum points of \( I(\vec{r}) \). First, we assume that \( \vec{r} \) is an inner point of \( J \): \( \vec{r} \in J^o \). This is a zero of \( \vec{r} \). Therefore, we have

\[
(q_0 - q_I)\kappa - q_I(a + q_O) > 0,
\]

\[
a + q_n = \sqrt{(q_0 - q_I)\kappa - q_I(a + q_O)}.
\]

Thus, this point is on the diagonal: \( r_1 = \cdots = r_N = r > 0 \). Therefore, this point is also a maximum point of \( w(r, r) \). This is merely the unique maximal point, \( r_{\text{Pareto}} \), of \( w(r, r) \). Therefore, \( \vec{r} \) is the unique maximum point of \( I(\vec{r}) \) in \( J^o \).

It is straightforward to verify that

\[
(q_0 - q_I)\kappa - q_I(a + q_O) > 0 \iff N(q_0 - q_I) - (a + q_O) > 0.
\]

Maximum Point on \( J^b \). Next, we assume that the maximum point, \( \vec{r} \), is a boundary point of \( J \): \( \vec{r} \in J^b \).

First, we demonstrate that \( \forall r_n = 0 \) or \( 1 \). Let \( \exists r_j \in (0, 1) \). Subsequently

\[
\frac{1}{a} \frac{\partial I}{\partial r_j} = \frac{1}{a + \kappa} \left\{ \frac{(q_0 - q_I)\kappa - q_I(a + q_O)}{(a + q_j)^2} - K \right\} = 0.
\]

Because \( \vec{r} \) is a boundary point, \( r_n = 0 \) or \( 1 \). However, when \( r_n = 0 \), we have

\[
\frac{1}{a} \frac{\partial I}{\partial r_n} = \frac{1}{a + \kappa} \left\{ \frac{(q_0 - q_I)\kappa - q_I(a + q_O)}{(a + q_n)^2} - K \right\} > 0.
\]

When \( r_n = 1 \), we have

\[
\frac{1}{a} \frac{\partial I}{\partial r_n} = \frac{1}{a + \kappa} \left\{ \frac{(q_0 - q_I)\kappa - q_I(a + q_O)}{(a + q_n)^2} - K \right\} < 0.
\]

This contradicts the maximality of \( \vec{r} \).

Next, we demonstrate that \( \forall r_n = 0 \). Let \( r_1 = \cdots = r_k = 0, r_{k+1} = \cdots = r_N = 1 \). Subsequently, \( \kappa = kq_I \) and

\[
\frac{1}{a} \frac{\partial I}{\partial r_n} = \frac{1}{a + kq_I} \left\{ \frac{(q_0 - q_I)kq_I - q_I(a + q_O)}{(a + q_n)^2} - K \right\}, \quad K = \frac{(N-k)q_Iq_O}{(a + q_O)(a + kq_I)}.
\]

Because \( \vec{r} \) is a maximal point of \( I(\vec{r}) \), we have

\[
\frac{1}{a} \frac{\partial I}{\partial r_1} = \frac{1}{a + kq_I} \left\{ \frac{(q_0 - q_I)kq_I - q_I(a + q_O)}{(a + q_1)^2} - K \right\} \leq 0, \quad (2)
\]

\[
\frac{1}{a} \frac{\partial I}{\partial r_N} = \frac{1}{a + kq_I} \left\{ \frac{(q_0 - q_I)kq_I - q_I(a + q_O)}{(a + q_O)^2} - K \right\} \geq 0. \quad (3)
\]

In the case of \( k = 0 \), \( q_I \) reads

\[
q_I - \frac{(a + q_O)^2}{(a + q_1)^2} \cdot \frac{Nq_Iq_O}{(a + q_O)(a + kq_I)} \geq 0.
\]

This cannot occur. Next, we consider the \( 1 \leq k \leq N - 1 \) case. From \( 2 \) and \( 3 \), we have

\[
q_I \frac{(q_0 - q_I)k - (a + q_O)}{(a + q_I)^2(a + kq_I)} \leq \frac{(N-k)q_Iq_O}{(a + q_O)(a + kq_I)} \leq q_I \frac{(q_0 - q_I)k - (a + q_O)}{(a + q_O)^2(a + kq_I)}.
\]

This is also impossible because \( 0 < q_I < q_O \). Finally, we investigate the \( k = N \) case. The inequality \( 2 \) reduces to

\[
(q_0 - q_I)N - (a + q_O) \leq 0.
\]

This is the only possible case. That is, if the maximum point of \( I(\vec{r}) \) is on the boundary \( J^b \), it is merely the origin \( \vec{r} = \vec{0} \). In this case, the origin is indeed the maximum point of \( I(\vec{r}) \): because

\[
(q_0 - q_I)\kappa - q_I(a + q_O) = q_I((q_0 - q_I)N - (a + q_O)) \leq 0,
\]

\( 1 \) is negative on \( J \) except for the origin.
Strict Maximality. We explicitly demonstrate that $r_{\text{Pareto}}$ is strictly maximal when $r_{\text{Pareto}} \in J^o$. We consider the Hessian of the total fitness function at the Pareto optimal point,

$$H(r_{\text{Pareto}}) = \frac{1}{2} \left( \frac{\partial^2 I}{\partial r_m \partial r_n}(r_{\text{Pareto}}) \right) = -\begin{pmatrix} \alpha + \beta & \beta & \cdots & \beta \\ \beta & \alpha + \beta & \cdots & \cdots \\ \vdots & \vdots & \ddots & \beta \\ \beta & \cdots & \beta & \alpha + \beta \end{pmatrix},$$

where

$$\alpha = a \frac{(q_O - q_I)((q_O - q_I)\kappa_{\text{Pareto}} - q_I(a + q_O))}{(a + q_{\text{Pareto}})^2(a + \kappa_{\text{Pareto}})}$$

$$\beta = \frac{aq_Iq_O(a + q_I)}{(a + q_{\text{Pareto}})^2(a + \kappa_{\text{Pareto}})^2} + \frac{Naq_Iq_O\kappa_{\text{Pareto}}}{(a + q_{\text{Pareto}})(a + \kappa_{\text{Pareto}})^3},$$

and

$$\kappa_{\text{Pareto}} = Nq_I(1 - r_{\text{Pareto}}), \quad q_{\text{Pareto}} = q_I + (q_O - q_I)r_{\text{Pareto}}.$$ 

The Hessian contains two eigenvalues, $-\alpha$ with algebraic multiplicity $(N - 1)$, and $-(\alpha + N\beta)$ with algebraic multiplicity 1. Because

$$(q_O - q_I)N - (a + q_O) > 0 \iff (q_O - q_I)\kappa_{\text{Pareto}} - q_I(a + q_O) > 0,$$

we obtain $\alpha, \beta > 0$. This proves the strict maximality.