Tikhonov regularization of dynamical systems associated with nonexpansive operators defined in closed and convex sets

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Received: date / Accepted: date

Abstract In this paper, we propose a Tikhonov-like regularization for dynamical systems associated with non-expansive operators defined in closed and convex sets of a Hilbert space. We prove the well-posedness and the strong convergence of the proposed dynamical systems to a fixed point of the non-expansive operator. We apply the obtained result to dynamical system associated with the problem of finding the zeros of the sum of a cocoercive operator with the subdifferential of a convex function.

Keywords Dynamical systems · Nonexpansive operator · Tikhonov regularization

Mathematics Subject Classification (2010) 34G25 · 47A52 · 47H05 · 47J35 · 90C25

1 Introduction

Let $D$ be a closed and convex set of a Hilbert space $H$. In this paper, we are interested in the study of the following dynamical system

$$\begin{cases}
-x(t) = x(t) - T(x(t)) & \text{a.e. } t \geq 0, \\
x(0) = x_0 \in D,
\end{cases}$$

where $T : D \to D$ is a non-expansive operator (see Definition 2.1 below) and $x_0 \in D$. The consideration of this dynamical system is motivated by the study of the set of fixed points of the operator $T$. Indeed, every equilibrium point of the dynamical system (1) is a fixed point of $T$. 

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The framework (1) includes the dynamical system proposed by Antipin in [2]
\[
-\dot{x}(t) = x(t) - \text{proj}_C(x(t) - \mu \nabla \varphi(x(t))) \quad \text{a.e. } t \geq 0,
\]
\[
x(0) = x_0,
\]
where $\varphi : H \to \mathbb{R}$ is a convex $C^1$ function defined on a real Hilbert space $H$, $C$ is a nonempty closed and convex set of $H$, $x_0 \in H$, $\mu > 0$ and $\text{proj}_C$ denotes the projection operator on the set $C$. In this context, it was shown in [6] that the trajectory of (2) converges weakly to a minimizer of the optimization problem
\[
\inf_{x \in C} \varphi(x),
\]
provided that the latter is solvable. Latter, Abbas and Attouch [1] considered the following generalization of the dynamical system (2)
\[
-\dot{x}(t) = x(t) - \text{prox}_{\mu \Phi}(x(t) - \mu B(x(t))) \quad \text{a.e. } t \geq 0,
\]
\[
x(0) = x_0,
\]
where $\Phi : H \to \mathbb{R} \cup \{+\infty\}$ is a proper, convex and lower semicontinuous functions defined on a real Hilbert space $H$, $B : H \to H$ is a cocoercive operator, $x_0 \in H$, $\mu > 0$ and $\text{prox}_{\mu \Phi} : H \to H$
\[
\text{prox}_{\mu \Phi}(x) := \arg\min_{y \in H} \left\{ \Phi(y) + \frac{1}{2\mu} \|y - x\|^2 \right\},
\]
denotes the proximal point operator of $\Phi$. Finally, in [7], the authors prove the weak convergence of the orbits of the dynamical system (1) to a fixed point for the operator $T : H \to H$, extending the results mentioned above.

In this paper, we study following variant of the dynamical system (1)
\[
-\dot{x}(t) = x(t) - T(x(t)) + \epsilon(t)(x(t) - y) \quad \text{a.e. } t \geq 0,
\]
\[
x(0) = x_0 \in D,
\]
where $y \in D$ and $\epsilon : [0, \infty) \to [0, \infty)$ is an appropriate function. The system (4) corresponds to a Tikhonov-like regularization of the dynamical system (1). This kind of regularization has been considered by several authors (see, e.g., [8]). In [8], the authors consider the system
\[
-\dot{x}(t) = Ax(t) + \epsilon(t)x(t),
\]
where $A$ is a maximal monotone operator defined on a Hilbert space and $\epsilon(t)$ tends to 0 as $t \to +\infty$ with $\int_0^{+\infty} \epsilon(s)ds = +\infty$. They prove the strong convergence towards the least-norm point in $A^{-1}(0)$ provided that the function $\epsilon(t)$ has bounded variation. This setting includes the operator $A = I - T$ when $T$ is defined in all the Hilbert space $H$. However, when the operator $T$ is defined only in a closed convex subset $D \subset H$, it is not clear that the dynamical system (5) is well defined. One of the contributions of this paper is to prove that, under mild assumptions, the system (4) is well defined (see
Proposition 4.1) and that the orbits of (4) converge strongly to the point \( \text{proj}_{\text{Fix}T}(y) \) provided the set \( \text{Fix}T \) is nonempty (see Theorem 4.1).

The paper is organized as follows. In Section 2, we set the notation of the paper and prove preliminary results on non-expansive operators. In Section 3, we present the main properties of the dynamical system (1). In Section 4, we present the main result of the paper (see Theorem 4.1), namely, the strong convergence of the trajectories of the dynamical system (4) to a point in the set \( \text{Fix}T \). Then, we give some applications of the main result to the dynamical system (3). The paper ends with conclusions and final remarks.

2 Notation and preliminaries

Let \( H \) be a Hilbert space endowed with a scalar product \( \langle \cdot, \cdot \rangle \) and unit ball \( B \). Given a closed and convex set \( S \subset H \) we define the distance function \( d_S \) and the projection over \( S \) as the maps

\[
d_S(x) := \inf_{y \in S} \|y - x\| \quad \text{and} \quad \text{proj}_S(x) := \{ y \in H : d_S(x) = \|x - y\| \}.
\]

It is not difficult to prove that the map \( x \mapsto d_S^2(x) \) is differentiable with

\[
\nabla d_S^2(x) = 2(x - \text{proj}_S(x)) \quad \text{for all } x \in H.
\]

Moreover, the following inequality holds

\[
\langle x - \text{proj}_S(x), y - \text{proj}_S(x) \rangle \leq 0 \quad \text{for all } y \in S.
\] (6)

We refer to [4] for more details.

Let \( \Phi : H \to \mathbb{R} \cup \{+\infty\} \) be a proper, convex and lower semicontinuous function and \( \mu > 0 \). The proximal point operator of \( \Phi \) is defined as

\[
\text{prox}_{\mu \Phi}(x) := \arg\min_{y \in H} \left\{ \Phi(y) + \frac{1}{2\mu} \|y - x\|^2 \right\}.
\] (7)

The proximal point operator is everywhere well defined and the map \( x \mapsto \text{prox}_{\mu \Phi}(x) \) is Lipschitz of constant 1 (see [5, Section 12.4]). Moreover, when \( \Phi \) is the indicator function of a closed and convex set \( C \), then the proximal point operator coincides with the projection operator over \( C \), that is, \( \text{prox}_{\mu \Phi}(x) = \text{proj}_C(x) \).

The proximal point operator plays a fundamental role in optimization theory. Indeed, the proximal point operator is the basis of several optimization algorithms (see, e.g., [5]). Moreover, it is well known (see [5, Proposition 12.29]) that the set of fixed point of this operator coincides with the set of solution of the problem

\[
\inf_{x \in H} \Phi(x).
\]

The following definitions will be used throughout the paper.

**Definition 2.1** An operator \( T : D \subset H \to H \) is called
1. \( \beta \)-cocoercive on \( D \), if
\[
\langle T(x) - T(y), x - y \rangle \geq \beta \|T(x) - T(y)\|^2 \quad \text{for all } x, y \in D.
\]

2. Non-expansive on \( D \) if
\[
\|T(x) - T(y)\| \leq \|x - y\| \quad \text{for all } x, y \in D.
\]

3. Firmly non-expansive on \( D \) if
\[
\|T(x) - T(y)\|^2 + \|(\text{Id} - T)(x) - (\text{Id} - T)(y)\|^2 \leq \|x - y\|^2 \quad \text{for all } x, y \in D.
\]

It is important to mention that if \( D \) is closed and convex and \( T: D \to D \) is non-expansive, then the set \( \text{Fix} T \) is closed and convex (see, e.g., [5, Corollary 4.24]). Moreover, if, in addition, \( D \) is bounded, the Browder-Göhde-Kirk’s Theorem (see, e.g., [5, Theorem 4.29]) asserts that the set \( \text{Fix} T \) is nonempty. On the other hand, if \( T \) is \( \alpha \)-Lipschitz with \( \alpha \in [0, 1) \), then \( \text{Fix} T \) is a singleton.

Let us consider a non-expansive operator \( T: D \to D \) and define the operator \( G: D \to H \) given by
\[
G(x) = x - T(x).
\]

It is clear that the set of fixed point of \( T \) coincides with the set of zeros of \( G \). Moreover, according to [5, Proposition 4.4], if the operator \( T \) is non-expansive, then \( T \) is monotone. The following lemma gives the existence of approximate zeros of \( G \).

**Lemma 2.1** Assume that \( T: D \to D \) is non-expansive and fix \( y \in D \). Then, for every \( \varepsilon > 0 \) there exists a unique \( x_\varepsilon y \in D \) such that
\[
\varepsilon x_\varepsilon y + G(x_\varepsilon y) = \varepsilon y.
\]

**Proof** Since the operator \( T \) is non-expansive on \( D \), we can apply [5, Proposition 4.30], to obtain that for all \( \eta \in (0, 1) \) there exists a unique \( x_\eta y \in D \) such that
\[
x_\eta y = \eta y + (1 - \eta)T(x_\eta y).
\]

In particular, taking \( \eta = \frac{\varepsilon}{1+\varepsilon} \), we obtain the existence of a unique \( x_\varepsilon y \in D \) such that
\[
x_\varepsilon y = \frac{\varepsilon}{1+\varepsilon} y + \left(1 - \frac{\varepsilon}{1+\varepsilon}\right) T(x_\varepsilon y),
\]

which implies the result.

Now, let us define \( \mathcal{F}: (0, +\infty) \times D \to D \), given by
\[
\mathcal{F}(\varepsilon, y) = x_\varepsilon y,
\]

where \( x_\varepsilon y \) is the unique solution of (9) given by Lemma 2.1.

The next results give us some properties of the trajectory \( x_\varepsilon y \).

**Lemma 2.2** Consider the function \( \mathcal{F} \) defined in (10). Then,

i) For all \( \varepsilon > 0 \) the function \( \mathcal{F}(\varepsilon, \cdot) \) is firmly nonexpansive on \( D \).

ii) If \( \text{Fix} T = \emptyset \), then for all \( y \in D \), \( \lim_{\varepsilon \to 0} \|\mathcal{F}(\varepsilon, y)\| = \infty \).
iii) For all \( \varepsilon > 0 \), and all \( x^* \in \text{Fix} T \)
\[
\|y - \mathcal{F}(\varepsilon,y)\|^2 + \|\mathcal{F}(\varepsilon,y) - x^*\|^2 \leq \|y - x^*\|^2.
\]

iv) If \( \text{Fix} T \neq \emptyset \), then
\[
\lim_{\varepsilon \to 0^+} \mathcal{F}(\varepsilon,y) = \text{proj}_{\text{Fix} T}(y).
\]

v) For all \( y \in D \), the function \( \varepsilon \to \|y - \mathcal{F}(\varepsilon,y)\| \) is decreasing.

vi) For all \( y \in D \), the function \( \mathcal{F}(\cdot,y) \) is continuous.

Proof See [5, Proposition 4.30].

The following result is fundamental to establish continuity properties for the map \( \varepsilon \mapsto \mathcal{F}(\varepsilon,y) \) for \( y \in D \) fixed.

Lemma 2.3 Consider \( \mu > \lambda > 0 \). Then for every \( y \in D \)
\[
\mathcal{F}(\lambda,y) = \mathcal{F} \left( \mu, \frac{\lambda}{\mu} y + \left( 1 - \frac{\lambda}{\mu} \right) \mathcal{F}(\lambda,y) \right). \tag{11}
\]

Proof We observe that
\[
\mathcal{F}(\lambda,y) \in D, \quad \text{and} \quad \frac{\lambda}{\mu} y + \left( 1 - \frac{\lambda}{\mu} \right) \mathcal{F}(\lambda,y) \in D,
\]
thus (11) is well-defined.

To end the proof, it is enough to verify that \( \mathcal{F}(\lambda,y) \) satisfies (9) with
\[
\varepsilon = \mu \quad \text{and} \quad z = \frac{\lambda}{\mu} y + \left( 1 - \frac{\lambda}{\mu} \right) \mathcal{F}(\lambda,y).
\]

Indeed,
\[
\mu \mathcal{F}(\lambda,y) + G(\mathcal{F}(\lambda,y)) = \mu \mathcal{F}(\lambda,y) + \lambda \mathcal{F}(\lambda,y) + G(\mathcal{F}(\lambda,y)) - \lambda \mathcal{F}(\lambda,y)
\]
\[
= \mu \mathcal{F}(\lambda,y) + \lambda y - \lambda \mathcal{F}(\lambda,y)
\]
\[
= \mu \left( \frac{\lambda}{\mu} y + \left( 1 - \frac{\lambda}{\mu} \right) \mathcal{F}(\lambda,y) \right),
\]

which ends the proof.

The following proposition establishes the continuity and differentiability almost everywhere of the map \( \varepsilon \mapsto \mathcal{F}(\varepsilon,x) \) for \( x \in D \) fixed.

Proposition 2.1 For every \( \varepsilon_1, \varepsilon_2 > 0 \) and \( x \in D \)
\[
\|\mathcal{F}(\varepsilon_2,x) - \mathcal{F}(\varepsilon_1,x)\| \leq \frac{|\varepsilon_2 - \varepsilon_1|}{\min\{\varepsilon_1, \varepsilon_2\}} \|x - \mathcal{F}(\min\{\varepsilon_1, \varepsilon_2\},x)\|. \tag{12}
\]

Consequently, for every \( x \in D \) the function \( \mathcal{F}(\cdot,x) \) is locally Lipschitz and for all \( x \in D \) and a.e. \( t > 0 \)
\[
\left\| \frac{d}{dt} \mathcal{F}(\cdot,x)(t) \right\| \leq \frac{1}{t} \|x - \mathcal{F}(t,x)\|.
\]
Proof Fix \( x \in D \) and assume that \( \varepsilon_2 > \varepsilon_1 \). Then, according to Lemma 2.3,

\[
\| \mathcal{F}(\varepsilon_2, x) - \mathcal{F}(\varepsilon_1, x) \| = \left\| \frac{\varepsilon_1}{\varepsilon_2} x + (1 - \frac{\varepsilon_1}{\varepsilon_2}) \mathcal{F}(\varepsilon_2, x) \right\|.
\]

Next, due to Lemma 2.2 i), we know that \( \mathcal{F}(\varepsilon_2, \cdot) \) is nonexpansive. Thus,

\[
\| \mathcal{F}(\varepsilon_2, x) - \mathcal{F}(\varepsilon_1, x) \| \leq \left( 1 - \frac{\varepsilon_1}{\varepsilon_2} \right) \| x - \mathcal{F}(\varepsilon_2, x) \| = \frac{\varepsilon_2 - \varepsilon_1}{\varepsilon_2} \| x - \mathcal{F}(\varepsilon_2, x) \|.
\]

Repeating the argument for \( \varepsilon_1 > \varepsilon_2 \), we obtain the result.

3 Dynamical systems associated with nonexpansive operators

In this section, we study the following dynamical system

\[
\begin{cases}
-\dot{x}(t) = x(t) - T(x(t)) & \text{a.e. } t \geq 0, \\
x(0) = x_0 \in D,
\end{cases}
\]

where \( T : D \to D \) is a nonexpansive operator defined over a closed and convex set \( D \subset H \). The dynamical system (13) was considered in [7] for a non-expansive operator \( T : H \to H \). The following proposition establishes the well-posedness of (13).

**Proposition 3.1** If \( T : D \to D \) is a nonexpansive operator, then the dynamical system (13) admits a unique solution \( x \in AC_{\text{loc}}([0, +\infty); H) \). Moreover, this solution satisfies \( x(t) \in D \) for all \( t \geq 0 \)

**Proof** See the proof of Proposition 4.1.

The following proposition establishes convergence properties of the dynamical system (13). Its proof follows in the same way as [7] Theorem 6.

**Proposition 3.2** Let \( T : D \to D \) be a nonexpansive operator such that \( \text{Fix} T \neq \emptyset \). Let \( x(\cdot) \) be the unique solution of (13). Then the following assertions hold:

(i) the trajectory \( x \) is bounded and \( \int_0^{+\infty} \| x(t) \|^2 dt < +\infty \);
(ii) \( \lim_{t \to +\infty} (T(x(t)) - x(t)) = 0 \) and for all \( t > 0 \)

\[
\| x(t) - Tx(t) \| \leq \frac{1}{\sqrt{t}} \text{dist}(x_0, \text{Fix} T);
\]

(iii) \( \lim_{t \to +\infty} k(t) = 0 \);
(iv) \( x(t) \) converges weakly to a point in \( \text{Fix} T \), as \( t \to +\infty \).

Moreover, if \( T \) is \( \alpha \)-Lipschitz with \( \alpha \in [0, 1) \), then the unique fixed point \( x^* \) of \( T \) is globally exponentially stable, that is,

\[
\| x(t) - x^* \| \leq e^{-(1 - \alpha \varepsilon)} \| x_0 - x^* \| \quad \text{for all } t \geq 0
\]
4 Tikhonov regularization

In this section, we study the Tikhonov regularization of the projected dynamical system (1). Let us consider the following assumptions:

**Assumption 1** Let $\varepsilon : \mathbb{R}_+ \to \mathbb{R}_+$ be a positive function satisfying

(a) $\varepsilon$ is absolutely continuous, nonincreasing and $\lim_{t \to +\infty} \varepsilon(t) = 0$;
(b) $\int_0^{+\infty} \varepsilon(s) ds = +\infty$;
(c) $\lim_{t \to +\infty} \varepsilon(t) = 0$.

We observe that, for example, the function $\varepsilon(t) = \frac{1}{1+\beta t}$ with $\beta \in (0,1)$ satisfy Assumption 1.

Now, we consider the following dynamical system:

\[
\begin{cases}
-\dot{x}(t) = x(t) - T(x(t)) + \varepsilon(t)(x(t) - y) & \text{a.e. } t \geq 0, \\
x(0) = x_0 \in D,
\end{cases}
\]  

(14)

where $T : D \to D$ is a nonexpansive operator, $y \in D$ and $\varepsilon$ is a function satisfying Assumption 1. The following proposition establishes the well-posedness of the dynamical system (14).

**Proposition 4.1** Fix $y \in D$ and assume that $T : D \to D$ is a nonexpansive operator. If $\varepsilon \in L^1_{\text{loc}}(\mathbb{R}_+)$, then the dynamical system (14) admits a unique solution $x \in AC_{\text{loc}}(\mathbb{R}_+;H)$. Moreover, this solution satisfies $x(t) \in D$ for all $t \geq 0$.

**Proof** Let us consider the dynamical system

\[
\begin{cases}
-\dot{x}(t) = \text{proj}_D(x(t)) - T(\text{proj}_D(x(t))) + \varepsilon(t)(\text{proj}_D(x(t)) - y) & \text{a.e. } t \geq 0, \\
x(0) = x_0 \in D,
\end{cases}
\]  

(15)

According to the classical Cauchy-Lipschitz theorem (see, e.g., [9, Proposition 6.2.1]), if $\varepsilon \in L^1_{\text{loc}}(\mathbb{R}_+)$, then the dynamical system (15) has a unique solution $x \in AC_{\text{loc}}(\mathbb{R}_+;H)$. We aim to prove that $x(t) \in D$ for all $t \geq 0$. To do that, we define the function $\psi(t) := \frac{1}{2}d^2_D(x(t))$. This function is absolutely continuous and for a.e. $t \geq 0$

\[
\psi(t) = \langle x(t) - \text{proj}_D(x(t)), \dot{x}(t) \rangle = \langle x(t) - \text{proj}_D(x(t)), T(\text{proj}_D(x(t))) - \text{proj}_D(x(t)) \rangle + \varepsilon(t)(x(t) - \text{proj}_D(x(t)), y - \text{proj}_D(x(t))) \leq 0,
\]

where we have used the inequality (6). Thus, since $\psi(0) = 0$, it follows that $\psi(t) = 0$ for all $t \geq 0$. Therefore, $\text{proj}_D(x(t)) = x(t)$ for all $t \geq 0$, which proves that $x$ is the unique solution of (14).
The next theorem establishes, under mild assumptions, that the point \( \text{proj}_{\text{Fix}T}(y) \) is globally asymptotically stable for the dynamical system (14), that is,

\[
x(t) \to \text{proj}_{\text{Fix}T}(y) \text{ as } t \to +\infty.
\]

The proof is strongly based on [8], where the authors use the Tikhonov regularization technique to deal with the set of zeros of maximal monotone operators. It is worth noting that the results from [8] does not apply in our setting because the operator \( I - T \) is not necessarily maximal monotone.

**Theorem 4.1** Fix \( y \in D \) and suppose that \( T : D \to D \) is a non-expansive operator with \( \text{Fix}T \neq \emptyset \). If Assumption [7] holds, then the unique solution of (14) \( x(t) \) converges strongly to \( \text{proj}_{\text{Fix}T}(y) \), as \( t \to +\infty \).

**Proof** Define the operator

\[
G(x) := x - T(x).
\]

Since \( T \) is nonexpansive on \( D \), it is clear that \( G \) is a monotone operator on \( D \) (see, e.g., [5] Chapter 4).

Denote \( x_\varepsilon := \mathcal{F}(\varepsilon, y) \) (see Lemma 2.2) and consider the function

\[
\theta(t) := \frac{1}{2}||x(t) - x_\varepsilon(t)||^2.
\]

According to Proposition 2.1, \( \theta \) is absolutely continuous and for a.e. \( t > 0 \)

\[
\dot{\theta}(t) = \left\langle x(t) - x_\varepsilon(t), \dot{x}(t) - \dot{\varepsilon}(t) \frac{d}{d\varepsilon} x_\varepsilon(t) \right\rangle
\]

\[
= \left\langle x(t) - x_\varepsilon(t), -G(x(t)) + \varepsilon(t)y - \varepsilon(t)x(t) - \dot{\varepsilon}(t) \frac{d}{d\varepsilon} x_\varepsilon(t) \right\rangle
\]

\[
= \left\langle x(t) - x_\varepsilon(t), -G(x(t)) + G(x_\varepsilon(t)) \right\rangle + \varepsilon(t) \left\langle x(t) - x_\varepsilon(t), x_\varepsilon(t) - x(t) \right\rangle
\]

\[
- \left\langle x(t) - x_\varepsilon(t), \dot{x}(t) \frac{d}{d\varepsilon} x_\varepsilon(t) \right\rangle
\]

where we have used the definition of \( x_\varepsilon(t) \) (see equation (9)). Furthermore, since \( G \) is monotone (see, e.g., [5] Proposition 4.4) and \( x(t) \in D \) for all \( t \geq 0 \), the following inequality holds

\[
\left\langle x(t) - x_\varepsilon(t), -G(x(t)) + G(x_\varepsilon(t)) \right\rangle \leq 0.
\]

Thus, for a.e. \( t \geq 0 \)

\[
\dot{\theta}(t) \leq -2\varepsilon(t) \theta(t) - \varepsilon(t) \left\| \frac{d}{d\varepsilon} x_\varepsilon(t) \right\| \sqrt{2\theta}
\]

\[
\leq -2\varepsilon(t) \theta(t) - \frac{\dot{\varepsilon}(t)}{\varepsilon(t)} ||y - x_\varepsilon(t)|| \sqrt{2\theta}
\]

\[
\leq -2\varepsilon(t) \theta(t) - \frac{\dot{\varepsilon}(t)}{\varepsilon(t)} ||y - \text{proj}_{\text{Fix}T}(y)|| \sqrt{2\theta},
\]
where we have used Proposition 2.1 and assertion v) of Lemma 2.2.

Thus, for a.e. \( t \geq 0 \)

\[
\dot{\theta}(t) + 2\varepsilon(t)\theta(t) \leq -\frac{\dot{\varepsilon}(t)}{\varepsilon(t)}\|y - \text{proj}_{\text{Fix}T}(y)\|\sqrt{2\theta}.
\]

Hence, the function \( \varphi(t) = \sqrt{2\theta} \) satisfies for a.e. \( t \geq 0 \)

\[
\dot{\varphi}(t) + \varepsilon(t)\varphi(t) \leq -\frac{\dot{\varepsilon}(t)}{\varepsilon(t)}\|y - \text{proj}_{\text{Fix}T}(y)\|.
\]

Therefore, for all \( t \geq 0 \)

\[
\varphi(t) \leq \exp\left(-\int_0^t \varepsilon(\tau)d\tau\right) \left(\varphi(0) + \|y - \text{proj}_{\text{Fix}T}(y)\| \int_0^t \frac{\dot{\varepsilon}(s)}{\varepsilon(s)} \exp\left(\int_0^s \varepsilon(\tau)d\tau\right) ds\right)
\]

Moreover, since Assumption 1 holds, the right-hand side in the last inequality goes to zero. Indeed,

\[
\lim_{t \to +\infty} \exp\left(-\int_0^t \varepsilon(\tau)d\tau\right) \int_0^t \frac{\dot{\varepsilon}(s)}{\varepsilon(s)} \exp\left(\int_0^s \varepsilon(\tau)d\tau\right) ds = \lim_{t \to +\infty} \frac{\dot{\varepsilon}(t)}{\varepsilon^2(t)} = 0.
\]

Finally, according to Lemma 2.2 iv,

\[
\limsup_{t \to +\infty} \|x(t) - \text{proj}_{\text{Fix}T}(y)\| \leq \limsup_{t \to +\infty} \left(\|x(t) - x_{\varepsilon(t)}\| + \|x_{\varepsilon(t)} - \text{proj}_{\text{Fix}T}(y)\|\right),
\]

which ends the proof.

**Remark 4.1** Theorem 4.1 has several advantages compared with Proposition 3.2.

1. The trajectories are always defined in the set \( D \). Thus, \( T \) needs only to be defined in this set.
2. The cost of discretizing (14) is, relatively, the same as discretizing the dynamical system (13).
3. The convergence obtained in Theorem 4.1 is strong, while the convergence in Proposition 3.2 is weak.
4. All the trajectories of (14) (regardless of the starting point) converge strongly to the point \( \text{proj}_{\text{Fix}T}(y) \).

## 5 Applications

In this section, we present some applications of Theorem 4.1.

Consider a proper, convex and lower semicontinuous function \( \Phi : H \to \mathbb{R} \cup \{+\infty\} \) and its proximal point operator defined in (7). Assume that \( B : \text{dom} \Phi \to H \) is a \( \beta \)-cocoercive operator, that is,

\[
\langle Bx - By, x - y \rangle \geq \beta \|Bx - By\|^2 \quad \text{for all } x, y \in \text{dom} \Phi.
\]

The following result is a well known property
Proposition 5.1 If $\mu \in (0, 2\beta)$, then the operator $T := x \mapsto \text{prox}_{\mu \Phi} (x - \mu Bx)$ is non-expansive. Moreover, for all $x, y \in \text{dom} \Phi$

$$\|Tx - Ty\|^2 + \mu (2\beta - \mu)\|Bx - By\|^2 \leq \|x - y\|^2.$$ 

Thus, as a direct consequence of Theorem 4.1, we have the following result.

Theorem 5.1 Assume that $B : \text{dom} \Phi \to H$ is a $\beta$-cocoercive operator with $\mu \in (0, 2\beta)$. Let $x_0, y \in \text{dom} \Phi$ and $\varepsilon$ be a function satisfying Assumption 1 and suppose that $\text{zer}(\partial \Phi + B) \neq \emptyset$. Let $x : [0, +\infty) \to H$ be the unique solution of

$$\begin{cases}
-\dot{x}(t) = x(t) - \text{prox}_{\mu \Phi} (x(t) - \mu Bx(t)) + \varepsilon(t)(x(t) - y) \\
x(0) = x_0.
\end{cases}$$

Then $x(t)$ converges strongly to $\text{proj}_{\text{zer}(\partial \Phi + B)}(y)$, as $t \to +\infty$.

Remark 5.1 It is worth to emphasize that in the last theorem the operator $B$ must to be defined only in $\text{dom} \Phi$ and not in all the space as in [7].

To end this section, we mention the following result from [10], which is a Baillon-Haddad theorem for convex functions defined in open and convex sets (see also [4, Theorem 3.3] for the twice continuously differentiable case).

Proposition 5.2 Let $C$ be a nonempty open convex subset of $H$, let $f : C \to \mathbb{R}$ be convex and Fréchet differentiable on $C$, and let $\beta > 0$. Then $\nabla f$ is $\beta$-Lipschitz continuous if and only if it is $1/\beta$-cocoercive.

The importance of the following Proposition 5.2 is that it shows a class of cocoercive operators which are not necessarily defined in the whole space.

6 Conclusions and final remarks

In this paper, we propose a Tikhonov-like regularization for dynamical systems associated with non-expansive operators defined in closed and convex sets (possibly not defined in the whole space $H$). Our main contribution is to deal with non-expansive operators defined in a closed and convex set. Moreover, we prove that the Tikhonov regularization converges strongly to a specific point in the set $\text{Fix} T$, provided the latter is not empty. This result extends known results in the literature and, in particular, proposes a dynamical system whose solution is defined in the domain of the non-expansive operator $T$. We expect that the discretization of the dynamical systems in this paper will be the basis for the design of algorithms of Forward-Backward type to find fixed points of non-expansive operators.
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