A short note on the weak Lefschetz property for Chow groups

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Abstract Motivated by the Bloch–Beilinson conjectures, we formulate a certain covariant weak Lefschetz property for Chow groups. We prove this property in some special cases, using Kimura’s nilpotence theorem.

Keywords Algebraic cycles · Chow groups · Finite–dimensional motives · Weak Lefschetz

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1 Introduction

Let $X$ be a smooth projective variety over $\mathbb{C}$ of dimension $n$. The Chow groups $A^i X$ (of codimension $i$ algebraic cycles modulo rational equivalence) are notoriously hard to understand. For instance, the following conjecture dating from 1974 is still completely open for $i > 1$:

Conjecture 1 (Hartshorne [9]) If $Y \subset X$ is a smooth hyperplane section, restriction induces isomorphisms

$$A^i X_Q \xrightarrow{\sim} A^i Y_Q$$

for $2i < n - 1$.

Since this seems a very difficult problem, in this note we try and formulate a covariant weak Lefschetz property for Chow groups and hope this is easier. To emphasize that we consider the Chow groups as a homology theory, we now switch to the notation $A_i X = A^{n-i} X$. Let $A^i_{hom}$ and $A^i_{AJ}$ denote the subgroup of homologically trivial resp. Abel–Jacobi trivial cycles.

To fix ideas, let’s now consider $A_0 X$, the Chow group of 0–cycles. Since $H^{2n-1}(X, \mathbb{Q})$ is one–dimensional, obviously

$$A_0 Y_Q \to A_0 X_Q / A^0_{hom} X_Q$$

is surjective for any point $Y$ of $X$—and in particular, for a 0–dimensional complete intersection $Y \subset X$. The next step is that (by weak Lefschetz applied to $H^{2n-1}(X, \mathbb{Q})$)

$$A_0 Y_Q \to A_0 X_Q / A^0_{AJ} X_Q$$

is surjective, for any smooth complete intersection curve $Y \subset X$. Going beyond the Abel–Jacobi map, it is conjectured there is a filtration $F^*$ on $A_0$, of which the first two steps are $F^1 = A^0_{hom}$ and $F^2 = A^0_{AJ}$ (cf. [12], [19], [20], [26]). One can then ask:
Question 1 Is it true that \( A_0 Y_\mathbb{Q} \to A_0 X_\mathbb{Q}/F^{\ell+1} \) is surjective, for any smooth complete intersection \( Y \subset X \) of dimension \( \ell \)?

This question is motivated (pun intended) by the expectation that the quotient \( A_0 X_\mathbb{Q}/F^{\ell+1} \) is determined by the cohomology groups \( H^{2n} X, H^{2n-1} X, \ldots, H^{2n-\ell} X \). Since the filtration \( F^* \) only exists conjecturally, this question is not falsifiable. However, it is expected that \( F^{\ell+1} \) vanishes exactly when \( H^n X, \ldots, H^{\ell+1} X \) are supported in codimension 1. This gives the following conjecture, in which \( F^* \) does not appear:

Conjecture 2 Let \( X \) be a smooth projective variety, and suppose \( H^i(X, \mathbb{Q}) = N^1 H^i(X, \mathbb{Q}) \) for \( i \in [\ell+1, n] \). Then

\[ A_0 Y_\mathbb{Q} \to A_0 X_\mathbb{Q} \]

is surjective, for any smooth complete intersection \( Y \subset X \) of dimension \( \ell \).

The main result of this note provides a verification of this conjecture in some special cases. As a by–product, we also get the injectivity part of conjecture 1 in these special cases:

Theorem (cf. Theorem 3) Suppose the Voisin standard conjecture (conjecture 4) holds. Let \( X \) be a smooth projective variety of dimension \( n \), and suppose

(i) Either the motive of \( X \) is finite–dimensional, or \( \text{Griff}^n (X \times X)_\mathbb{Q} = 0 \);
(ii) The Lefschetz standard conjecture \( B(X) \) holds;
(iii) There exists \( r \) such that \( H^i(X, \mathbb{Q}) = N^r H^i(X, \mathbb{Q}) \) for all \( i \in [n-r+1, n] \).

Then for any codimension \( r \) smooth complete intersection \( Y \subset X \) of class \( [Y] = L^r \in H^{2r}(X, \mathbb{Q}) \) with \( L \) ample, push–forward maps

\[ A_i(Y)_\mathbb{Q} \to A_i(X)_\mathbb{Q} \]

are surjective for \( i < r \). Moreover, restriction maps

\[ A^i_{AJ}(X)_\mathbb{Q} \to A^i_{AJ}(Y)_\mathbb{Q} \]

are injective for \( i \leq r + 1 \).

In certain cases some of the hypotheses are automatically satisfied, and the statement simplifies:

Corollary (cf. Corollary 7) Let \( X \) be a smooth projective 3fold which is dominated by a product of curves. Suppose

\[ H^3(X, \mathbb{Q}) = N^1 H^3(X, \mathbb{Q}) . \]

Then for any smooth ample hypersurface \( Y \subset X \), the push–forward map

\[ A_0(Y)_\mathbb{Q} \to A_0(X)_\mathbb{Q} \]

is surjective, and

\[ A^2_{AJ}(X)_\mathbb{Q} \to A^2_{AJ}(Y)_\mathbb{Q} \]

is injective.

Corollary (cf. Corollary 2) Let \( X \) be a product of smooth projective surfaces

\[ X = S_1 \times \cdots \times S_m , \]

where each \( S_j \) is either a \( K3 \) surface of Picard number 19 or 20, or has \( A^0_{AJ}(S_j)_\mathbb{Q} = 0 \). Suppose at least one \( S_j \) has \( A^0_{AJ}(S_j)_\mathbb{Q} = 0 \). Then for any smooth ample hypersurface \( Y \subset X \), the push–forward map

\[ A_0(Y)_\mathbb{Q} \to A_0(X)_\mathbb{Q} \]

is surjective, and

\[ A^2_{AJ}(X)_\mathbb{Q} \to A^2_{AJ}(Y)_\mathbb{Q} \]

is injective.
It was already known that in situations like these two corollaries, $A_0X_\mathbb{Q}$ is supported on some divisor (this follows for instance from [26 Theorem 3.32]); thus, our only contribution is the precision that any ample hypersurface does the job. The injectivity statement, on the other hand, seems to be genuinely new: as far as we know, these are the first examples of varieties with non–trivial $A_{n-1}^\mathbb{Q}$ for which this injectivity is known to hold. The proof of the theorem is an easy exercise in using the meccano of correspondences; the only “deep” ingredient is Kimura’s nilpotence theorem [16].

We end this introduction with a challenge. As is well–known [5], the hypothesis of conjecture 2 is verified when $A_0X_\mathbb{Q}$ is supported in dimension $\ell$. This gives the following special case of conjecture 4:

**Conjecture 3** Let $X$ be a smooth projective variety, and suppose $A_0X_\mathbb{Q}$ is supported on a closed subvariety of dimension $\ell$. Then any smooth complete intersection $Y \subset X$ of dimension $\ell$ supports $A_0X_\mathbb{Q}$.

This is true for $\ell \leq 1$, but for $\ell > 1$ I have no idea how to prove this...

**Conventions** In this note, the word variety refers to a quasi–projective algebraic variety over $\mathbb{C}$. A subvariety will be a (possibly reducible) reduced subscheme which is equidimensional. The Chow group of $i$–dimensional cycles on $X$ is denoted $A_iX$; for $X$ smooth of dimension $n$ the notations $A_iX$ and $A^{n-i}X$ will be used interchangeably. The Griffiths group $Griff$ is the group of $i$–dimensional cycles that are homologically trivial modulo algebraic equivalence. In diagrams, we will sometimes write $H^jX$ or $H_jX$ to designate singular cohomology $H^j(X, \mathbb{Q})$ resp. Borel–Moore homology $H_j(X, \mathbb{Q})$.

## 2 Preliminary

**Definition 1 (Coniveau filtration [4])** Let $X$ be a quasi–projective variety. The coniveau filtration on cohomology and on homology is defined as

$$N^cH^i(X, \mathbb{Q}) = \sum \text{Im}(H^1_Y(X, \mathbb{Q}) \to H^i(X, \mathbb{Q}));$$

$$N_cH_i(X, \mathbb{Q}) = \sum \text{Im}(H_i(Z, \mathbb{Q}) \to H_i(X, \mathbb{Q})), $$

where $Y$ runs over codimension $\geq c$ subvarieties of $X$, and $Z$ over dimension $\leq c$ subvarieties.

We recall the statement of the “Voisin standard conjecture”:

**Conjecture 4 (Voisin standard conjecture [25])** Let $X$ be a smooth projective variety, and $Y \subset X$ closed with complement $U$. Then the natural sequence

$$N_iH_{2i}(Y, \mathbb{Q}) \to N_iH_{2i}(X, \mathbb{Q}) \to N_iH_{2i}(U, \mathbb{Q}) \to 0$$

is exact for any $i$.

**Remark 1** Hodge theory gives an exact sequence

$$\text{Gr}^W_{2i}H_{2i}Y \cap F^{-i} \to H_{2i}X \cap F^{-i} \to \text{Gr}^W_{2i}H_{2i}U \cap F^{-i} \to 0,$$

where $W$ denotes Deligne’s weight filtration, and $F$ the Hodge filtration on $H_*(\cdot, \mathbb{Q})$. Hence if the Hodge conjecture (that is, its homology version for singular varieties [11]) is true, then conjecture 4 is true.

What’s more, this conjecture fits in very neatly with the classical standard conjectures: Voisin shows that conjecture 4 plus the algebraicity of the Künneth components of the diagonal is equivalent to the Lefschetz standard conjecture [25 Proposition 1.6].

**Remark 2** Conjecture 4 is obviously true for $i \leq 1$ (this follows from the truth of Hodge conjecture for curve classes), and for $i \geq \dim Y - 1$ (where it follows from the Hodge conjecture for divisors).

The main ingredient used in this note is Kimura’s nilpotence theorem:

1. This is not strictly true: indeed, [1] Corollary 5] gives non–trivial examples of varieties where the injectivity part of conjecture 4 is verified.
Theorem 1 (Kimura [14]) Let $X$ be a smooth projective variety of dimension $n$ with finite-dimensional motive. Let $\Gamma \in A^n(X \times X)_Q$ be a correspondence which is homologically trivial. Then there is $N \in \mathbb{N}$ such that

$$\Gamma^\circ N = 0 \quad \in A^n(X \times X)_Q.$$

Remark 3 We refer to [14], [1], [20] for the definition of finite-dimensional motive. Conjecturally, any variety has finite-dimensional motive [14]. What mainly concerns us in the scope of this note, is that there are quite a few examples which are known to have finite-dimensional motive: varieties dominated by products of curves [14], $K3$ surfaces with Picard number 19 or 20 [21], any surface with vanishing geometric genus for which Bloch’s conjecture has been verified [8, Theorem 2.11], 3folds with nef tangent bundle [10], certain 3folds of general type [22, Section 8].

There is also the following nilpotence result, which predates Kimura’s theorem:

Theorem 2 (Voisin [24], Voevodsky [23]) Let $X$ be a smooth projective algebraic variety of dimension $n$, and $\Gamma \in A^n(X \times X)_Q$ a correspondence which is algebraically trivial. Then there is $N \in \mathbb{N}$ such that

$$\Gamma^\circ N = 0 \quad \in A^n(X \times X)_Q.$$

3 Main

We now proceed with the proof of the main result of this note:

Theorem 3 Suppose the Voisin standard conjecture holds. Let $X$ be a smooth projective variety of dimension $n$, and suppose

(i) Either the motive of $X$ is finite-dimensional, or $\text{Griff}^n(X \times X)_Q = 0$;

(ii) The Lefschetz standard conjecture $B(X)$ holds;

(iii) $H^i(X, \mathbb{Q}) = N^{i-r}H^i(X, \mathbb{Q})$ for all $i \in [n-r+1, n]$.

Then for any codimension $r$ smooth complete intersection $Y \subset X$ of class $[Y] = L^r \in H^{2r}(X, \mathbb{Q})$ with $L$ ample, push–forward maps

$$A_i(Y)_Q \rightarrow A_i(X)_Q$$

are surjective for $i < r$. Moreover, restriction maps

$$A^i_{A,J}(X)_Q \rightarrow A^i_{A,J}(Y)_Q$$

are injective for $i \leq r+1$.

In certain cases, some of the hypotheses can be removed:

Corollary 1 Let $X$ be a smooth projective variety of dimension $n \leq 3$, and suppose

(i) Either the motive of $X$ is finite–dimensional, or $\text{Griff}^n(X \times X)_Q = 0$;

(ii) $H^n(X, \mathbb{Q}) = N^1H^n(X, \mathbb{Q})$.

Then for any smooth ample hypersurface $Y \subset X$, push–forward maps

$$A_0(Y)_Q \rightarrow A_0(X)_Q$$

are surjective, and restriction

$$A^2_{A,J}(X)_Q \rightarrow A^2_{A,J}(Y)_Q$$

is injective.

Corollary 2 Let $X$ be a product of smooth projective surfaces

$$X = S_1 \times \cdots \times S_m,$$

where each $S_j$ is either a $K3$ surface of Picard number 19 or 20, or has $A^2_{A,J}(S_j)_Q = 0$. 
Lemma 1

For each $B$ that is inverse to denote the result of cupping with a power of theorem. Let isomorphism Manin’s identity principle, this action. Since injective for $i \leq 1$, and

$$A_{i}(Y)_{\mathbb{Q}} \rightarrow A_{i}(X)_{\mathbb{Q}}$$

is injective. Suppose there are at least 4 surfaces $S_j$ with $A^{0}_{0,j}(S_j)_{\mathbb{Q}} = 0$. Let $Y \subset X$ be a codimension 2 complete intersection of class $[Y] = L^{2} \in H^{4}(X, \mathbb{Q})$ with $L$ ample. Then

$$A_{i}(Y)_{\mathbb{Q}} \rightarrow A_{i}(X)_{\mathbb{Q}}$$

is surjective for $i \leq 1$, and

$$A_{i}^{j}_{X}(Y)_{\mathbb{Q}} \rightarrow A_{i}^{j}_{X}(Y)_{\mathbb{Q}}$$

is injective for $i \leq 3$.

Proof (of theorem 3) Let $\tau : Y \hookrightarrow X$ be a smooth complete intersection of class $L'$ as in the statement of the theorem. Let

$$L^{j} : H^{i}X(\mathbb{Q}) \rightarrow H^{i+2j}(X, \mathbb{Q})$$

denote the result of cupping with a power of $L$; we use the same notation $L^{j}$ for the correspondence inducing this action. Since $B(X)$ is true, for any $i < n$ there exists a correspondence $C_{i} \in A^{i}(X \times X)_{\mathbb{Q}}$ inducing an isomorphism

$$(C_{i})_{*} : H^{2n-i}(X, \mathbb{Q}) \cong H^{i}(X, \mathbb{Q})$$

that is inverse to $L^{n-i}$.

$B(X)$ being true, the Künneth components $\pi_{i}$ of the diagonal of $X$ are algebraic [16]. Since $B(X)$ implies $B(Y)$ [16], the same holds for the Künneth components $\pi_{i}^{Y}$ of $Y$. We now proceed to relate them:

**Lemma 1** For each $i \leq n - r$, define

$$\Pi_{i} := (C_{i}) \circ (L^{n-i-r}) \circ ((\tau \times \tau)_{*}(\pi_{i}^{Y})) \in A^{n}(X \times X, \mathbb{Q}).$$

Then for each $i \leq n - r$, we have equality

$$\Pi_{i} = \pi_{i} \in H^{2n}(X \times X, \mathbb{Q}).$$

Proof We consider the action on $H^{i}(X, \mathbb{Q})$. There is a factorization

$$H^{j}X \xrightarrow[\downarrow]{(\tau \times \tau)_{*}(\pi_{i}^{Y})} H^{j+2r}X \xrightarrow{L^{n-i-r}} H^{2n-2i+j}X \xrightarrow{(C_{i})_{*}} H^{i}X$$

Hence, if $j \neq i$ then

$$(\Pi_{i})_{*}H^{j}X = 0,$$

and for $j = i$ we have

$$\Pi_{i} = (C_{i}) \circ (L^{n-i-r}) \circ (L^{i}) = \text{id} : H^{i}X \rightarrow H^{i}X.$$

It follows that for any variety $Z$, the action of $\Pi_{i}$ on $H^{i}(X \times Z)$ is projection on $H^{i}X \otimes H^{j-i}Z$; thus by Manin’s identity principle, $\Pi_{i}$ and $\pi_{i}$ coincide as homological correspondences.

**Lemma 2** For each $i \leq n - r$, and each $j < r$, we have

$$(\Pi_{i})_{*}A_{j}X_{\mathbb{Q}} = 0.$$
Proof For any correspondence \( C \in A^{n-r}(Y \times Y)_Q \), there is a factorization

\[
\begin{array}{ccc}
A_jX_Q & \xrightarrow{((\tau \times \tau)_*)C} & A_{j-r}X_Q \\
\downarrow & & \uparrow \\
A_{j-r}Y_Q & \xrightarrow{C_*} & A_{j-r}Y_Q
\end{array}
\]

In particular, taking \( C = \pi_i^Y \), we see that the action of \((\tau \times \tau)_*(\pi_i^Y)\) on \( A_jX_Q \) factors over \( A_{j-r}Y_Q \), hence is 0 for \( j < r \).

\[\square\]

Lemma 3 Let \( t\Pi_i \) denote the transpose of \( \Pi_i \). For each \( i \leq n - r \), and each \( j \), we have

\[
(t\Pi_i)_*A_jX_Q \subset \text{Im}(A_jY_Q \rightarrow A_jX_Q).
\]

Moreover, for each \( j \leq r + 1 \), we have

\[
(t\Pi_i)_*A_{A,j}X_Q = 0.
\]

Proof It is immediate from the definition that

\[
t\Pi_i = ((\tau \times \tau)_*(\pi_i^Y)) \circ (L^{n-i-r}) \circ tC_i \in A^n(X \times X)_Q.
\]

Using the same diagram as in the proof of lemma\[2\] one can find a factorization

\[
\begin{array}{ccc}
A_jX_Q & \xrightarrow{(t(L^{n-i-r})\circ tC_i)_*} & A_{j+r}X_Q \\
\downarrow & & \uparrow \\
A_{j+r}Y_Q & \xrightarrow{t(\pi_i^Y)_*} & A_{j+r}Y_Q
\end{array}
\]

and the lemma is proven.

\[\square\]

By hypothesis (iii), we have

\[
H^i(X, Q) = N^rH^i(X, Q) \quad \forall n - r < i \leq n.
\]

Applying hard Lefschetz, one finds

\[
H^i(X, Q) = N^rH^i(X, Q) \quad \forall n - r < i < n + r.
\]

This means that in the range \( n - r < i < n + r \), the Künneth component \( \pi_i \) is supported in codimension \( r \). That is, there exists a subvariety \( Z \subset X \) of codimension \( r \), such that for each \( n - r < i < n + r \), \( \pi_i \) goes to 0 under the restriction

\[
H^{2n}(X \times X, Q) \rightarrow H^{2n}((X \times X) \setminus (Z \times Z), Q).
\]

Using the Voisin standard conjecture (conjecture\[3\]), this implies the existence of an algebraic cycle \( P'_i \in A_n(Z \times Z)_Q \) such that (denoting by \( P_i \) the push-forward of \( P'_i \) to \( X \times X \)) we have

\[
P_i = \pi_i \in H^{2n}(X \times X, Q) \quad \forall n - r < i < n + r.
\]

Lemma 4 For any \( i \in [n - r + 1, n + r - 1] \), and any \( j < r \), we have

\[
(P_i)_*A_jX_Q = 0.
\]

Moreover, for any \( j \leq r + 1 \), we have

\[
(P_i)_*A_{A,j}X_Q = 0.
\]
Proof Let \( \psi \colon Z \to X \) denote the inclusion, so \( P_t = (\psi \times \psi)_*(P'_t) \). Similar to lemma\(^2\) there is a factorization

\[
\begin{align*}
A_jX_Q \xrightarrow{(P_t)_*} &\ A_jX_Q \\
\downarrow &\ \\
A_{j-r}Z_Q \xrightarrow{(P'_t)_*} &\ A_jZ_Q
\end{align*}
\]

That is, the action of \( P_t \) in the indicated range factors over groups that vanish for dimension reasons and the lemma follows.

Putting together the various parts, we find a decomposition of the diagonal

\[
\Delta = \sum_{i=0}^{n-r} \Pi_i + \sum_{i=n-r+1}^{n-r-1} P_i + \sum_{i=0}^{n-r} i \Pi_i \in H^{2n}(X \times X, \mathbb{Q}) .
\]

This is an equality of cycles modulo homological equivalence. Now, applying Kimura’s nilpotence theorem (theorem\(^1\)), we get that there exists \( N \) such that

\[
\left( \Delta - \sum_{i=0}^{n-r} \Pi_i - \sum_{i=n-r+1}^{n-r-1} P_i + \sum_{i=0}^{n-r} i \Pi_i \right) \circ N = 0 \in \mathcal{A}^n(X \times X)_\mathbb{Q} .
\]

Developing this expression (and noting that \( \Delta \circ N = \Delta \)), we find

\[
\Delta = \sum_j Q_j \in \mathcal{A}^n(X \times X)_\mathbb{Q} ,
\]

where each \( Q_j \) is a composition of elements \( P_{\ell} \) and \( \Pi_{\ell} \), and \( \Pi_{\ell} \). Let \( Q^0_j \) denote the “tail element” of \( Q_j \), i.e. we write

\[
Q_j = Q^0_j \circ Q^1_j \circ \cdots \circ Q^{N_j}_j \in \mathcal{A}^n(X \times X)_\mathbb{Q} ,
\]

with \( Q^0_j \not= \Delta \) (so that \( N' \leq N \)).

Let’s consider the action of \( Q_j \) on \( A_iX_Q \), for \( i < r \):

If \( Q^0_j \) is a \( \Pi_{\ell} \) (for some \( \ell \in [0, n-r] \)), it follows from lemma\(^2\) that

\[
(Q_j)_*(A_iX_Q) = 0 .
\]

Likewise, if \( Q^0_j \) is of the form \( P_{\ell} \) (for some \( n-r < \ell < n+r \)), then applying lemma\(^3\) we find again

\[
(Q_j)_*(A_iX_Q) = 0 .
\]

Finally, if \( Q^0_j \) is of the form \( \Pi_{\ell} \) (for some \( \ell \in [0, n-r] \)), it follows from lemma\(^3\) that

\[
(Q_j)_*(A_iX_Q) \subseteq \text{Im}(A_iY_Q \to A_iX_Q) .
\]

Since \( \Delta \) acts as the identity, we conclude that for \( i < r \), push–forward

\[
A_iY_Q \to A_iX_Q
\]

is surjective.

The argument for the injectivity statement is similar: we consider the action of \( \Delta = \sum_j Q_j \) on \( A^i_{AJ}X_Q \) for \( i \leq r+1 \). If \( Q_j \) is such that its “head” \( Q^{N_j}_j \) is of type \( \Pi_{\ell} \) or \( P_{\ell} \), then \( Q_j \) does not act (by lemma\(^3\) resp. lemma\(^4\)). It follows that we can write

\[
A^i_{AJ}X_Q = \Delta_\ast A^i_{AJ}X_Q = (\sum \text{something} \circ (\tau \times (\tau)_\ast \text{something}))_\ast A^i_{AJ}X_Q ;
\]

the injectivity is then obvious.
Finally, if the hypothesis in (i) of the theorem is that
\[ \text{Griff}^n(X \times X)_\mathbb{Q} = 0, \]
the proof goes as follows: the decomposition of \( \Delta \) is now an equality modulo algebraic equivalence (since by hypothesis, algebraic and homological equivalence coincide on \( X \times X \)). Then, instead of applying Kimura’s theorem, we apply the Voisin/Voevodsky nilpotence theorem (theorem 2). The rest of the proof is verbatim the same.

\[ \square \]

**Proof** (of corollary 1) In case \( n = 2 \), we know \( B(X) \) holds since it holds for any surface [16]. The Voisin standard conjecture is used to get that some Hodge classes in \( H_4(Z \times Z, \mathbb{Q}) \) are algebraic, where \( \dim Z = 1 \); this is trivially true.

Next, the case \( n = 3 \). Under the hypothesis \( H^3X = N^1H^3X \), \( X \) is “motivated by a surface” in the sense of [2], so \( B(X) \) is known to hold [2]. The Voisin standard conjecture is only used to get that some Hodge classes in \( H_6(Z \times Z, \mathbb{Q}) \) are algebraic, where \( \dim Z = 2 \); this is OK by the Hodge conjecture for divisors (remark 2).

\[ \square \]

**Proof** (of corollary 2) As we noted in remark 3 it follows from work of Pedrini [21] and Guletskii–Pedrini [8] that the \( S_j \) have finite–dimensional motive. Hence \( X \) has finite–dimensional motive. We also know \( B(X) \) is true since the Lefschetz standard conjecture is true for all surfaces [16].

In case (i), since there is at least one surface with \( H^2(S_j) = N^1 \), we obviously have
\[ H^{2m}(X, \mathbb{Q}) = N^1H^{2m}(X, \mathbb{Q}). \]

The corollary now follows from theorem 3 note that we don’t need to assume the Voisin standard conjecture, since we can find cycles \( P'_i \) by using the Hodge conjecture on the surfaces with vanishing geometric genus.

In case (ii), the assumptions imply
\[ H^{2m}(X, \mathbb{Q}) = N^2H^{2m}(X, \mathbb{Q}); \]
\[ H^{2m-1}(X, \mathbb{Q}) = N^2H^{2m-1}(X, \mathbb{Q}), \]
and we again apply theorem 3.

**Remark 4** The hypothesis on \( \text{Griff}^n(X \times X) \) in theorem 3 is mainly of theoretical interest, and not practically useful. Indeed, there are precise conjectures predicting when Griffiths groups should vanish [13]; for instance, if \( X \) is a 4fold with \( h^{2,0} = h^{4,0} = h^{3,0} = h^{2,1} = 0 \), Corollary 6.8 implies that if the Bloch–Beilinson conjectures are true then
\[ \text{Griff}^4(X \times X)_\mathbb{Q} = 0. \]

Unfortunately, no non–trivial examples seem to be known. Specifically, I am not aware of any example of a variety \( X \) of dimension \( n \) that satisfies \( \text{Griff}^n(X \times X)_\mathbb{Q} = 0 \), but not \( A_{AJ}X_\mathbb{Q} = 0 \forall i \).

**Remark 5** In [17], I study a certain hard Lefschetz property for Chow groups. Using arguments similar to the present note, this hard Lefschetz property can be proven in some special cases [17].

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