Hölder continuity of random processes

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Abstract

For a Young function \( \varphi \) and a Borel probability measure \( m \) on a compact metric space \((T, d)\) the minorizing metric is defined by

\[
\tau_{m, \varphi}(s, t) := \max\{\int_0^{d(s, t)} \varphi^{-1}\left(\frac{1}{m(B(s, \varepsilon))}\right)d\varepsilon, \int_0^{d(s, t)} \varphi^{-1}\left(\frac{1}{m(B(t, \varepsilon))}\right)d\varepsilon\}.
\]

In the paper we extend the result of Kwapien and Rosinski relaxing the conditions on \( \varphi \) under which there exists a constant \( K \) such that

\[
E \sup_{s, t \in T} \varphi\left(\frac{|X(s) - X(t)|}{K \tau_{m, \varphi}(s, t)}\right) \leq 1,
\]

for each separable process \( X(t), t \in T \) which satisfies \( \sup_{s, t \in T} E \varphi\left(\frac{|X(s) - f(t)|}{d(s, t)}\right) \leq 1 \).

In the case of \( \varphi_p(x) \equiv x^p, p \geq 1 \) we obtain the somewhat weaker results.

Key words: majorizing measures, minorizing metric, regularity of samples

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1 Introduction

Let \( X \) be a topological space and \( \mathcal{B}(X) \) its Borel \( \sigma \)-field. We denote by \( \mathcal{B}(X), \mathcal{B}_b(X), C(X), C_b(X) \) the set of all measurable, bounded measurable, continuous and bounded continuous functions respectively. Furthermore \( \mathcal{P}(X) \) denotes the family of all Borel, probability measures on \( X \). For each \( \mu \in \mathcal{P}(X) \), \( f \in \mathcal{B}_b(X) \) and \( A \in \mathcal{B}(X) \) we define

\[
\int_A f(u)\mu(du) := \frac{1}{\mu(A)} \int_A f(u)\mu(du),
\]

where, we have used the convention \( 0/0 = 0 \) (as we do throughout the whole paper). By \( \text{supp}(\mu) \) we denote the support of \( \mu \).

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In the paper we consider finite Young functions; that is increasing convex functions \( \varphi: [0, \infty) \to [0, \infty) \) satisfying \( \varphi(0) = 0, \lim_{x \to \infty} \varphi(x) = \infty. \) For a simplicity we will be assuming also that \( \varphi(1) = 1. \) As in ([3], Def. 5, page 40), we let \( \Delta^2 \) denote the set of all finite Young functions satisfying for some \( c \geq 0, r > 1 \)

\[
\varphi(x)^2 \leq \varphi(rx), \quad \text{for some} \quad x \geq c.
\]  

\((\Delta^2)\)

and let \( \nabla' \) (see [3], Def 7, page 28) denote the set of all finite Young functions \( \varphi \) verifying for some \( c \geq 0, r > 1 \)

\[
\varphi(x) \varphi(y) \leq \varphi(x y), \quad \text{for} \quad x, y \geq c.
\]  

\((\nabla')\)

Note that if \( (\Delta^2) \), resp. \( (\nabla') \) holds for some \( c > 0 \), then \( (\Delta^2) \), resp. \( (\nabla') \), holds for every \( c' > 0 \) with appropriate choice of \( r' \). If \( h \in \mathcal{B}(X) \) we let

\[
|h|_\varphi^p := \inf\{a > 0 : \int_X \varphi\left(\frac{|h(s)|}{a}\right) \mu(ds) \leq 1\}, \quad \|h\|_\varphi^p := \inf a(1 + \int_X \varphi\left(\frac{|h(s)|}{a}\right)) \mu(ds).
\]

denote the two Orlicz norms of \( h \). Then \( |\cdot|_\varphi^p \) and \( \|\cdot\|_\varphi^p \) are semi-norms on \( \mathcal{B}(X) \), satisfying \( |h|_\varphi^p = 0 \iff \|h\|_\varphi^p = 0 \iff h = 0, \mu\text{-a.e.} \) Note that \( |h|_\varphi^\infty < \infty \iff \int_X \varphi\left(\frac{|h|}{a}\right) < \infty \) for some \( 0 < a < \infty \iff \|h\|_\varphi^\infty < \infty \) and recall that the Orlicz space \( L^\varphi(\mu) \) is the set of all measurable functions satisfying one of the three equivalent conditions (see [3]). Then \( (L^\varphi(\mu), |\cdot|_\varphi) \) is a complete semi-normed space. As we prove in Lemma 1 semi-norms \( |\cdot|_\varphi^p \) and \( \|\cdot\|_\varphi^p \) are comparable.

Let \( (T, d) \) be a fixed compact, metric space and \( m \) a fixed probability measure (defined on Borel subsets) on \( T \). For \( x \in T \) and \( \varepsilon \geq 0 \), \( B(x, \varepsilon), B^c(x, \varepsilon) \) denote respectively the closed and the open ball with the center at \( x \) and the radius \( \varepsilon \) i.e. \( B(x, \varepsilon) = \{ y \in T : d(x, y) \leq \varepsilon \} \), \( B^c(x, \varepsilon) = \{ y \in T : d(x, y) < \varepsilon \} \). The diameter of \( T \), i.e. \( \sup\{d(s, t) : s, t \in T\} \) is denoted by \( D(T) \). We define the minorizing metric

\[
\tau_{m,\varphi}(s, t) := \max\left\{ \int_0^{d(s, t)} \varphi^{-1}\left(\frac{1}{m(B(s, \varepsilon))}\right)d\varepsilon, \int_0^{d(s, t)} \varphi^{-1}\left(\frac{1}{m(B(t, \varepsilon))}\right)d\varepsilon \right\} \quad \text{for} \quad s, t \in T.
\]

Kwapień and Rosinski [2] introduced these metrics to prove results on Hölder continuity of random processes with bounded increments. However their method requires that \( \varphi \) verifies \( (\Delta^2) \) which means the exponential growth of \( \varphi \). The goal of this paper is to obtain similar results, yet under relaxed conditions imposed on \( \varphi \).

**Theorem 1** Let \( \varphi \) and \( \psi \) be Young functions (verifying \( \varphi(1) = \psi(1) = 1 \)) and for some \( R > 1, n_0 \geq 1, n_0 \in \mathbb{N} \)

\[
\frac{\varphi(R^k)}{\varphi(R^{k+1})} \leq \frac{\varphi(R^{k-1})}{\varphi(R^k)}, \quad \text{for} \quad k \geq 1, \quad k \in \mathbb{N}.
\]

\((1)\)

\[
\sum_{k=0}^{\infty} \frac{\varphi(R^k)}{\psi(R^{k+n_0})} < \infty.
\]

\((2)\)
Let $\psi_+(x) = (\psi(x) - 1)_+$ for all $x > 0$. Then there exists a Borel probability measure $\nu$ on $T \times T$ and a constant $0 < K < \infty$ only depending on $(\varphi, \psi)$ such that for every continuous function $f : T \to \mathbb{R}$ holds

$$|f(s) - f(t)| \leq K|f|^\nu_{\psi_+}(s, t),$$

for $s, t \in T$, where $f^\nu_{\psi}(u, v) = \frac{|f(u) - f(v)|}{d(u, v)}$. (3)

and if $\psi \in \nabla'$, then we have

$$\sup_{s, t \in T} \psi_+(\frac{|f(s) - f(t)|}{K \tau_{m, \varphi}(s, t)}) \leq \int_{T \times T} \psi_+(\frac{|f(u) - f(v)|}{d(u, v)}) \nu(du, dv),$$

(4)

where $r$ is chosen such that condition (\nabla'') holds with $c = 1$.

Theorem 1 has an application to the stochastic analysis. We say that process $X(t)$, $t \in T$ has $\varphi$-bounded increments if it verifies

$$\mathbb{E}\sup_{s, t \in T} \frac{|X(s) - X(t)|}{d(s, t)} \leq 1.$$  (5)

**Corollary 1** Suppose $(\varphi, \psi)$ verify conditions (3) and (2). For each separable stochastic process $X(t)$, $t \in T$ which has $\psi$-bounded increments there holds

$$\mathbb{E}\sup_{s, t \in T} \frac{|X(s) - X(t)|}{2K \tau_{m, \varphi}(s, t)} \leq 1$$

and if $\psi \in \nabla'$ then also

$$\mathbb{E}\sup_{s, t \in T} \psi\left(\frac{|X(s) - X(t)|}{2K \tau_{m, \varphi}(s, t)}\right) \leq 1$$

where $K$ is the same constant as in Theorem 1.

**Proof.** Following arguments from the proof of Theorem 2.3 in Talagrand [5] it is enough to prove the result assuming that $X(t)$, $t \in T$ has a.s. continuous samples. Theorem 1, namely (3) the Fubini theorem and the definition of $|\cdot|_{\psi_+}$ give

$$\mathbb{E}\sup_{s, t \in T} \frac{|X(s) - X(t)|}{K \tau_{m, \varphi}(s, t)} \leq 1 + \int_{T \times T} \psi_+(\frac{|X(u) - X(v)|}{d(u, v)}) \nu(du, dv) \leq 2.$$

It proves the first thesis. If $\psi \in \nabla'$, then we can apply (3) instead of (3) obtaining

$$\mathbb{E}\sup_{s, t \in T} \psi\left(\frac{|X(s) - X(t)|}{K \tau_{m, \varphi}(s, t)}\right) \leq 1 + \mathbb{E}\sup_{s, t \in T} \psi_+(\frac{|X(s) - X(t)|}{K \tau_{m, \varphi}(s, t)}) \leq$$

$$\leq 1 + \int_{T \times T} \psi_+(\frac{|X(u) - X(v)|}{d(u, v)}) \nu(du, dv) \leq 2.$$  

By the convexity of $\varphi$, we derive the second claim.
Remark 1 Note that if \( \sum_{k=0}^{\infty} \frac{\varphi(R^k)}{\varphi(R^{k+n_0})} < \infty \), for some \( R > 1, n_0 \geq 1 \) then we can take \( \psi \equiv \varphi \) in Theorem 1. Thus all processes which verify (5) (for \( \varphi \)) are Hölder continuous with respect to \( \tau_{m,\varphi}(s,t) \). If \( \varphi(x) \equiv x^p \) we can take \( \psi(x) \equiv x^{p+\epsilon} \), where \( \epsilon > 0 \) and consequently obtain a generalization of basic Kolmogorov result [4].

We then prove the converse statement that minorizing metrics are optimal when considering Hölder continuity of processes with bounded increments.

**Theorem 2** Assume \((\varphi, \psi)\) verify for some \( R, n_0 \geq 1 \)

\[
\sum_{k=0}^{\infty} \frac{\psi(R^k)}{\varphi(R^{k+n_0})} < \infty. \tag{6}
\]

Suppose \( \rho \) is a metric on \( T \) such that for each separable process \( X(t), t \in T \) which has \( \psi \)-bounded increments (verifies condition (5) for \( \psi \)), we have

\[ P(\sup_{s,t \in T} |X(s) - X(t)|/\rho(s,t) < \infty) = 1, \]

then there exist a constant \( K \) and a Borel probability measure \( m \) (which depends on \((\varphi, \psi)\) only) such that \( \tau_{m,\varphi}(s,t) \leq K\rho(s,t) \).

**Remark 2** If \( \sum_{k=0}^{\infty} \frac{\varphi(R^k)}{\varphi(R^{k+n_0})} < \infty \) then we can take \( \psi = \varphi \) in Theorem 2. That means there exists \( m \in \mathcal{P}(T) \) such \( \tau_{m,\varphi}(s,t) \leq K\rho(s,t) \) for each \( \rho \) with respect to which all process with \( \varphi \)-bounded increments are Hölder continuous.

We also prove some generalization of Talagrand’s Theorem 4.2 [5] and the author’s Theorem 1 in [1].

**Theorem 3** Assume that \( \varphi \) verifies (7) for some \( R > 1 \). There exist constants \( C, K \) (depending on \( \varphi \) only) and a Borel probability measure \( \nu \) on \( T \times T \) such that for each continuous function \( f \) on \( T \) the inequality holds

\[
\sup_{s,t \in T} \varphi_+(\frac{|f(s) - f(t)|}{C\tau_{m,\varphi}(s,t)\varphi^{-1}_+(M(m,\varphi))}) \leq \int_{T \times T} \varphi_+(\frac{|f(u) - f(v)|}{d(u,v)})\nu(du, dv),
\]

where \( M(m,\varphi) := \int_T \int_T \varphi^{-1}((\frac{1}{m(B(t,\epsilon))}))d\xi\nu(dt, \xi) < \infty. \)

**Corollary 2** For each separable process \( X(t), t \in T \) which satisfies (6) (for \( \varphi \)) there holds

\[
\mathbb{E} \sup_{s,t \in T} \varphi(\frac{|X(s) - X(t)|}{C\tau_{m,\varphi}(s,t)\varphi^{-1}_+(M(m,\varphi))}) \leq 1.
\]
Proof. As in the proof of Corollary 1 it is enough to show the result for $X(t)$, $t \in T$ with a.s. continuous samples. Note that $\varphi(x) \leq 1 + \varphi_+(x)$, thus due to Theorem 3 the Fubini theorem we obtain

$$
E \sup_{s,t \in T} \varphi\left(\frac{|X(s) - X(t)|}{C_{\tau_m,\varphi}(s,t)\varphi^{-1}(\frac{M(m,\varphi)}{K_{\tau_m,\varphi}(s,t)})}\right) \leq 1 + \int_{T \times T} E\varphi\left(\frac{|X(u) - X(v)|}{d(u,v)}\right)\nu(du, dv) \leq 2.
$$

Now by the convexity we establish the result. ■

In the paper we follow methods from [1]. For a completeness we repeat from there some of the arguments.

## 2 Notation and Preliminaries

### Young functions

**Lemma 1** There holds $|h|_\varphi^\alpha \leq \|h\|_\varphi^\mu \leq 2|h|_\varphi^\mu$ for every $h \in \mathfrak{B}(X)$.

**Proof.** First note either $\int_X \varphi(\frac{|h|}{\alpha})d\mu \leq 1$ or $\int_X \varphi(\frac{|h|}{\alpha})d\mu > 1$ and in this case using that $\alpha \to \alpha \varphi(\frac{\alpha}{x})$ is decreasing we derive

$$
\int_X \varphi\left(\frac{|h|}{\alpha} \right) d\mu \leq \frac{\int_X \varphi\left(\frac{|h|}{\alpha} \right) d\mu}{\int_X \varphi\left(\frac{|h|}{\alpha} \right) d\mu} = 1.
$$

Consequently $|h|_\varphi^\mu \leq a + a \int_X \varphi\left(\frac{|h|}{\alpha} \right) d\mu$ for all $a > 0$. That means $|h|_\varphi^\mu \leq \|h\|_\varphi^\mu$. The last inequality follows by taking $a = |h|_\varphi^\mu$ in the definition of $\|h\|_\varphi^\mu$. ■

**Lemma 2** Let $\varphi$ be a Young function satisfying condition (V') with $c = 0$ and $r > 0$. Then we have $\varphi\left(\frac{1}{r} |h|_\varphi^\mu\right) \leq \int_S \varphi(|h|)d\mu$ for every $h \in \mathfrak{B}(X)$.

**Proof.** If $\int_S \varphi(|h|)d\mu$ is either 0 or $\infty$, then the inequality holds trivially. Suppose that $0 < \int_X \varphi(|h|)d\mu < \infty$ and let us take $C > 0$ so that $\varphi(C) = \int_X \varphi(|h|)d\mu$. By (V') property we have $\varphi(C)\varphi(\frac{|h|}{rC}) \leq \varphi(x)$ for all $x \geq 0$ and consequently

$$
\int_X \varphi\left(\frac{|h|}{rC}\right) d\mu \leq \frac{1}{\varphi(C)} \int_X \varphi(|h|)d\mu = 1.
$$

Hence, we see that $\|h\|_\varphi^\mu \leq rC$ which proves the lemma. ■
Observe that for each Young function \( \varphi \) there holds

\[
\frac{x}{y} \leq \frac{\varphi(x)}{\varphi(y)} \quad \text{for } \frac{x}{y} \geq 1.
\] (7)

**Lemma 3** If \( \varphi \) satisfies (\ref{eq:condition}) then \( \varphi \in \nabla' \) with \( r = R^2 \) and \( c = 1 \).

**Proof.** By (\ref{eq:condition}) we have

\[
\frac{\varphi(R_i^k)}{\varphi(R_i^{k+1})} \leq \frac{\varphi(R_i^{k-1})}{\varphi(R_i^{k+1})}, \quad \text{for } k \geq 1,
\]

Let \( i, j \geq 0 \) be such that \( R^i \leq x < R^{i+1} \) and \( R^j \leq y < R^{j+1} \). Clearly

\[
\frac{\varphi(R_i^{i+1})}{\varphi(R_i^{i+1}R_j^{i+1})} = \frac{\varphi(R_i^{i+1}R_j^i)}{\varphi(R_i^{i+1}R_j^{i+1})} \leq \frac{\varphi(R_i^i)}{\varphi(R_i^{i+1})} \leq \frac{\varphi(R_j^j)}{\varphi(R_j^{i+1})} = \frac{1}{\varphi(R_i^{i+1})}
\]

and hence \( \varphi(x)\varphi(y) \leq \varphi(R_i^{i+1})\varphi(R_j^{i+1}) \leq \varphi(R_i^{i+1}R_j^{i+1}) \leq \varphi(R_i^{i+1}R_j^{i+1}) \).

\( \blacksquare \)

**The main construction**

Fix any \( R > 2 \). For \( k \geq 0 \) and \( x \in T \) we define \( r_0(x) = D(T) \) and

\[
r_k(x) := \min\{\varepsilon \geq 0 : \frac{1}{m(B(x, \varepsilon))} \leq \varphi(R_k)\}.
\] (8)

Let us notice that \( r_k \leq D(T) \), for \( k \geq 0 \).

**Lemma 4** The functions \( r_k \) verify the Lipschitz condition with constant 1.

**Proof.** Clearly \( r_0 \) is a constant function so it is 1-Lipschitz. For \( k > 0 \) and \( s, t \in T \) it is

\[
\frac{1}{m(B(s, r_k(t) + d(s, t)) \leq \varphi(R_k), \quad \text{and} \quad \frac{1}{m(B(t, r_k(s) + d(s, t)) \leq \varphi(R_k)}.
\]

Hence \( r_k(s) \leq r_k(t) + d(s, t) \), \( r_k(t) \leq r_k(s) + d(s, t) \), thus \( r_k \) is 1-Lipschitz.

\( \blacksquare \)

**Remark 3** Note that if \( r(x) := \lim_{k \to \infty} r_k(x) \), we have \( r(x) = \inf\{\varepsilon \geq 0 : m(B(x, \varepsilon)) > 0\} = \essinf d(x, \cdot) \) where the essential infimum is taken with respect to the probability measure \( m \). In particular \( r(x) = 0 \) if and only if \( x \in \text{supp}(m) \).
For each positive integer $c$ we have

$$\frac{R-1}{R} \sum_{k \geq c} r_k(x) R^k \leq \sum_{k \geq c} r_k(x)(R^k - R^{k-1}) \leq \sum_{k \geq c} (r_k(x) - r_{k+1}(x)) R^k +$$

$$+ \limsup_{k \to \infty} r_{k+1}(x) R^{k+1} \leq \sum_{k \geq c} \int_{r_{k+1}(x)} r_k(x) \varphi^{-1}\left(\frac{1}{m(B(x, \varepsilon))}\right) d\varepsilon +$$

$$+ \limsup_{k \to \infty} \int_0^{r_{k+1}(x)} \varphi^{-1}\left(\frac{1}{m(B(x, \varepsilon))}\right) d\varepsilon = \int_0^{r_c(x)} \varphi^{-1}\left(\frac{1}{m(B(x, \varepsilon))}\right) d\varepsilon.$$

Thus

$$\sum_{k \geq c} r_k(x) R^k \leq \frac{R}{R-1} \int_0^{r_c(x)} \varphi^{-1}\left(\frac{1}{m(B(x, \varepsilon))}\right) d\varepsilon. \quad (9)$$

Let us abbreviate $B(x, r_k(x))$ by $B_k(x)$ and $B^c(x, r_k(x))$ by $B^c_k(x)$ for $k > 0$. For $k = 0$ we put $B^c_0(x) = B_0(x) = T$. Due to (8) it is clear that

$$\frac{1}{m(B_k(x))} \leq \varphi(R^k) \leq \frac{1}{m(B^c_k(x))}, \quad \text{for } k \geq 0. \quad (10)$$

For each $k \geq 0$ we define the linear operator $S_k : \mathfrak{B}_b(T) \to \mathfrak{B}_b(T)$ by the formula

$$S_k f(x) := \int_{B_k(x)} f(u) m(du) = \frac{1}{m(B_k(x))} \int_{B_k(x)} f(u) m(du).$$

If $f, g \in \mathfrak{B}_b(T), k \geq 0$, then we easily check that:

(i) $S_k 1 = 1$;

(ii) if $f \leq g$ then $S_k f \leq S_k g$ and hence $|S_k f| \leq S_k |f|$;

(iii) if $f \in C(T)$ and $\lim_{k \to \infty} r_k(x) = 0$, then $\lim_{k \to \infty} S_k f(x) = f(x)$.

Fix $l \geq 0$. There exists unique $m^l_{x,k} \in \mathcal{P}(T)$ such that for each $f \in \mathfrak{B}_b(T)$ we have

$$S_l S_{l-1} \ldots S_k f(x) = \int_T f(u) m^l_{x,k}(du), \quad \text{for } 0 \leq k \leq l. \quad (11)$$

Let us define

$$r^l_k := \sum_{i=k}^l 2^{i-k} r_i, \quad B^l_k(x) := B(x, r^l_k(x)), \quad \text{for } k \leq l.$$

**Lemma 5** For each $u \in B^l_{k+1}(x) 0 \leq k < l$ we have $B_k(u) \subset B^l_k(x)$ and

$$r_k(u) \leq r_k(x) + r^l_{k+1}(x) \leq r^l_k(x).$$
**Proof.** Fix \( u \in B_{k+1}^l(x) \). Since \( r_k \) are 1-Lipschitz, we get
\[
    r_k(u) \leq r_k(x) + d(x, u) \leq r_k(x) + r_{k+1}^l(x) \leq r_k^l(x).
\]
Clearly \( r_k(u) \leq r_k(x) + r_{k+1}^l(x) \). Furthermore \( d(x, u) \leq r_{k+1}^l(x) \), thus
\[
    B(u, r_k(u)) \subset B(u, r_k(x) + r_{k+1}^l(x)) \subset B(x, r_k(x) + 2r_{k+1}^l(x)) = B(x, r_k^l(x))
\]
and by the definition \( B_k(u) = B(u, r_k(u)) \), \( B_k^l(x) = B(x, r_k^l(x)) \).

\[\square\]

**Lemma 6** For all \( 0 \leq k \leq l \) we have \( m_{x,k}^l(B_k^l(x)) = 1 \) i.e. \( \text{supp}(m_{x,k}^l) \subset B_k^l(x) \).

**Proof.** We prove Lemma 6 by the reverse induction on \( k \). Clearly \( \text{supp}(m_{x,l}^1) = B(x, r_1(x)) = B_1^l(x) \). Suppose that for some \( k < l \) we have \( \text{supp}(m_{x,k+1}) \subset B_{k+1}^l(x) \), then the definition gives
\[
\int_T f(u)m_{x,k}(du) = \int_T \int_{B_k(u)} f(v)m(dv)m_{x,k+1}(du), \quad \text{for } f \in \mathcal{B}_b(T).
\]
Due to Lemma 5 we have \( B_k(u) \subset B_k^l(x) \), for \( u \in B_{k+1}^l(x) \). It ends the proof.

\[\square\]

**Corollary 3** For each \( f \in \mathcal{B}_b(T) \), and \( k \leq l \) the inequality holds
\[
S_1S_{l-1}...S_k|f|(x) = \int_T |f(u)|m_{x,k}(du) \leq \varphi(R^k) \int_{B_k^l(x)} |f(u)|m(du).
\]

**Proof.** If \( k = l \) the inequality is obvious. If \( k < l \), using Lemma 6 and (11) we obtain
\[
S_1S_{l-1}...S_k|f|(x) = \int_T \int_{B_k(u)} |f(v)|m(dv)m_{x,k+1}(du) \leq \varphi(R^k) \int_{B_k^l(x)} |f(v)|m(dv).
\]
Let us notice that for a positive integer \( c \) with \( 0 \leq c < l \) we have
\[
\sum_{k=c}^{l-1} r_k^l R^k = \sum_{k=c}^{l-1} \sum_{i=k}^l \left( \frac{2}{R} \right)^{i-k} r_i R^i \leq \sum_{j=0}^{\infty} \left( \frac{2}{R} \right)^j \sum_{i=c}^l r_i R^i = \frac{R}{R-2} \sum_{i=c}^{\infty} r_i R^i.
\]
Together with (9) it gives
\[
\sum_{k=c}^{l-1} r_k^l(x) R^k \leq \frac{R^2}{(R-1)(R-2)} \int_0^{r_c(x)} \varphi^{-1} \left( \frac{1}{m(B(x, \varepsilon))} \right) d\varepsilon. \quad (12)
\]
3 Proof of Theorem 1

Proof. We may assume that (1) and (2) hold with $R > 5$ (note that if (1) and (2) hold for some $R$ then they hold also for $R^l$, where $l \in \mathbb{N}$). Fix $s, t \in T$, without losing the generality we may assume also $\tau_m, \varphi(s, t) < \infty$, which implies that $\lim_{k \to \infty} r_k(x) = 0$, for $x = s, t$. If $d(s, t) < D(T)$ then there exist positive integers $a, b$ such that

$$r_a(s) \leq d(s, t) < r_{a-1}(s), \quad r_b(t) \leq d(s, t) < r_{b-1}(t),$$

and we can define $c := \max\{a, b\}$. If $d(s, t) = D(T) = r_0$, we put $c := 0$. For a fixed $l > c$ let us denote

$$\tau_x := \max\{k \geq 1 : B^l_k(s) \cup B^l_k(t) \subset B^l_{k-1}(u), \text{ for all } u \in B^l_k(x), \quad x = s, t.\}

and $\tau := \min\{\tau_x, \tau_l\}$. Observe that $B^0_{\tau}(u) = T$, for all $u \in T$ so $\tau_x$ is well defined and clearly $1 \leq \tau \leq c$. For simplicity we put also $r^l_k(s, t) := r^l_k(s) + r^l_k(t)$ and $d_k(s, t) := \min\{r^l_k(s, t) + d(s, t), D(T)\}$. Note that

$$d_r(s, t) \leq r_{r-1}(u), \quad \text{for all } u \in B^l_r(x) \text{ if } \tau = \tau_x. \quad (13)$$

Lemma 7 The inequality holds

$$d_r(s, t)R^r + \sum_{k=r}^{c} R^k r^l_k(s, t) \leq \frac{R}{R - 5} R^c \left(\frac{3}{2} d(s, t) + 2 r^l_c(s, t)\right).$$

Proof. Let $\tau \leq k < c$ be given and let $x$ be either $s$ or $t$. There exist $u_x \in B^l_{k+1}(x), \quad x = s, t$ such that $r_k(u_x) \leq d_k(s, t)$. Indeed, otherwise

$$B^l_{k+1}(s) \cup B^l_{k+1}(t) \subset B(u, d_{k+1}(s, t)) \subset B^l_k(u) \quad \text{for all } u \in B^l_{k+1}(t) \cup B^l_k(s)$$

which is impossible due to the definition of $\tau$.

By Lemma 3 functions $r_k$ are 1-Lipschitz, therefore

$$r_k(x) \leq r_k(u_x) + r^l_{k+1}(x) \leq d_{k+1}(s, t) + r^l_{k+1}(x), \quad x = s, t.$$

Since $r^l_k = r_k + 2r^l_{k+1}$, we obtain $r^l_k(x) \leq d_{k+1}(s, t) + 3r^l_{k+1}(x)$. Consequently

$$r^l_k(s, t) \leq 2d_{k+1}(s, t) + 3r^l_{k+1}(s, t) = 2d(s, t) + 5r^l_{k+1}(s, t).$$

Iterating this inequality, we obtain the following result

$$r^l_k(s, t) \leq 2d(s, t) \leq \sum_{i=0}^{c-k-1} 5^i 5^{c-k} r^l_c(s, t) = \frac{d(s, t)}{2} (5^{c-k} - 1) + 5^{c-k} r^l_c(s, t) \quad (14)$$
for all \( \tau \leq k \leq c \) (observe that inequality holds trivially for \( k = c \)). Hence, we have

\[
\sum_{k=\tau}^{c} r^I_k(s, t) \leq \left( \frac{d(s, t)}{2} + r^I_c(s, t) \right) \sum_{k=\tau}^{c} R^k 5^{c-k} \leq \frac{R}{R-5} R^c \left( \frac{d(s, t)}{2} + r^I_c(s, t) \right)
\]

and by (14) we have (recall that \( R > 5 \))

\[
d_{\tau}(s, t) R^\tau \leq R^\tau (d(s, t) + r^I_c(s, t)) \leq d(s, t) (1 + \frac{1}{2} (5^{c-\tau} - 1)) R^\tau + 5^{c-\tau} R^\tau r^I_c(s, t) \leq 5^{c-\tau} R^\tau (d(s, t) + r^I_c(s, t)) \leq R^c (d(s, t) + r^I_c(s, t)).
\]

Since \( \frac{R}{R-5} > 1 \), we obtain the inequality.

We remind that \( f^d(u, v) = \frac{|f(u) - f(v)|}{d(u, v)} \). For simplicity we denote

\[ F_k := \{(u, v) \in T \times T : f^d(u, v) \geq R^k \}, \quad k \geq 0. \]

**Lemma 8** If \( \varphi \) satisfies (11), then for each positive integer \( n \) and \( f \in C(T) \) there holds

\[
|S_t f(s) - S_t f(t)| \leq d_{\tau}(s, t) R^{\tau+n} + \sum_{x \in \{s, t\}} \sum_{k=\tau}^{l-1} r^I_k(x) R^{k+n} + \sum_{x \in \{s, t\}} \varphi(R^{k+1}) \int_{B_{k+1}(x)} r_k(u) \int_{B_k(u)} f^d(u, v) 1_{F_{k+n}}(dv)m(du) + d_{\tau}(s, t) \varphi(R^{\tau+1}) \int_{B_{\tau}^c(y)} \int_{B_{\tau-1}(u)} f^d(u, v) 1_{F_{\tau+n}}(dv)m(du),
\]

where \( y = t \) if \( \tau = \tau_t \) and \( y = s \) if \( \tau \neq \tau_t \).

**Proof.** Fix \( f \in C(T) \). Without losing the generality generality we can assume that \( \tau = \tau_t \). Clearly

\[
S_t f(s) - S_t f(t) = \sum_{k=\tau}^{l-1} S_i \ldots S_{k+1} (\text{Id} - S_k) f(s) - \sum_{k=\tau}^{l-1} S_i \ldots S_{k+1} (\text{Id} - S_k) f(t) + (S_i \ldots S_{\tau} f(s) - S_i \ldots S_{\tau} f(t)).
\]

We have also

\[
|S_i \ldots S_{k+1} (\text{Id} - S_k) f(x)| \leq \int_T |(\text{Id} - S_k) f(u)| m^I_{x,k+1}(du),
\]
Since $f^d(u,v) \leq R^{k+n} + f^d(u,v)1_{F_{k+n}}$, we obtain

$$|\text{Id} - S_k f(u)| \leq \int_{B_k(u)} |f(u) - f(v)| m(dv) \leq r_k(u) \int_{B_k(u)} f^d(u,v)m(dv) \leq r_k(u) R^{k+n} + r_k(u) \int_{B_k(u)} f^d(u,v)1_{F_{k+n}} m(dv), \text{ for all } u \in T.$$ 

By Lemma 5, $r_k(u) \leq r_k^t(x)$, whenever $u \in B^t_{k+1}(x)$. This, (17) and Corollary 3 imply that

$$|S_l \ldots S_{k+1}(\text{Id} - S_k) f(x)| \leq \int_T |\text{Id} - S_k f(u)| m_{x,k+1}(du) \leq r_k(x) R^{k+n} +$$

$$+ \int_T r_k(u) \int_{B_k(u)} f^d(u,v)1_{F_{k+n}} m(dv)m_{x,k+1}(du) \leq r_k(x) R^{k+n} +$$

$$+ \varphi(R^{k+1}) \int_{B^t_{k+1}(x)} r_k(u) \int_{B_k(u)} f^d(u,v)1_{F_{k+n}} m(dv)m_{x,k+1}(du). \quad (18)$$

To bound the last part in (16) let us observe that

$$|S_l \ldots S_{\tau} f(s) - S_l \ldots S_{\tau} f(t)| \leq \int_T |f(u) - S_{\tau} f(w)| m_{s,\tau+1}(dw) m_{\tau,t}(du). \quad (19)$$

By Lemma 6, $\text{supp}(m_{x,k}^l) \subset B^t_k(x)$, $x \in T$. If $w \in B^t_{\tau+1}(s)$ and $u \in B^t_{\tau}(t)$, then

$$|f(u) - S_{\tau} f(w)| \leq \int_{B^t_{\tau}(w)} |f(u) - f(v)| m(dv).$$

Lemma 5 implies that $B_{\tau}(w) \subset B^t_k(s)$. Hence for each $u \in B^t_{\tau}(t)$, $v \in B_{\tau}(w)$

$$d(u,v) \leq \min\{d(u,t) + d(t,s) + d(s,v), D(T)\} \leq d_{\tau}(s,t). \quad (20)$$

Applying (20) and $f^d(u,v) \leq R^{\tau+n} + f^d(u,v)1_{F_{\tau+n}}$, we obtain

$$|f(u) - S_{\tau} f(w)| \leq d_{\tau}(s,t) \int_{B_{\tau}(w)} f^d(u,v)m(dv) \leq$$

$$\leq d_{\tau}(s,t)(R^{\tau+n} + \int_{B_{\tau}(w)} f^d(u,v)1_{F_{\tau+n}} m(dv)). \quad (21)$$

Since $\tau = \tau_1$ we have $B_{\tau}(w) \subset B^t_{\tau}(s) \subset B^n_{\tau-1}(u)$ for all $w \in B^t_{\tau+1}(t)$. Together with (10) it implies

$$\int_{B_{\tau}(w)} f^d(u,v)1_{F_{\tau+n}} m(dv) \leq \varphi(R^\tau) \int_{B_{\tau}(w)} f^d(u,v)1_{F_{\tau+n}} m(dv) \leq$$

$$\leq \frac{\varphi(R^\tau)}{\varphi(R^{\tau-1})} \int_{B^n_{\tau-1}(u)} f^d(u,v)1_{F_{\tau+n}} m(dv). \quad (22)$$
The condition (1) gives $\frac{\varphi(R^n)}{\varphi(R^{n-1})} \leq \frac{\varphi(R^{n+1})}{\varphi(R^n)}$. Hence, due to (21) and (22) we obtain

$$\left| f(u) - S_\tau f(w) \right| \leq d_\tau(s, t)(R^{n+1} + \frac{\varphi(R^{n+1})}{\varphi(R^n)} \int_{B_{r_\tau}(u)} f^d(u, v)1_{F_{r+n}}(dv)).$$

(23)

Inequalities (19), (23) and Corollary 3 imply

$$|S_1...S_\tau f(s) - S_1...S_\tau f(t)| \leq d_\tau(s, t)(R^{n+1} + \frac{\varphi(R^{n+1})}{\varphi(R^n)} \int_{B_{r_\tau}(u)} f^d(u, v)1_{F_{r+n}}(dv)m_{l, r}(du)) \leq d_\tau(s, t)(R^{n+1} + \varphi(R^{n+1}) \int_{B_{r}(t)} \int_{B_{r_\tau}(u)} f^d(u, v)1_{F_{r+n}}(dv)m(du)).$$

(24)

Note that (18) and (24) give the result.

Lemma 9

If $A = \frac{4R^3}{(R-1)(R-2)(R-3)} + \frac{3R^2}{2(R-3)}$, then we have

$$d_\tau(s, t)(R^{n} + \sum_{x \in \{s, t\}} \sum_{k=\tau}^{l-1} r_k(x)R^k) \leq A\varphi_{m, \varphi}(s, t).$$

Proof. Lemma 7 gives

$$d_\tau(s, t)(R^\tau + \sum_{x \in \{s, t\}} \sum_{k=\tau}^{l-1} r_k(x)R^k) = \sum_{k=\tau}^{c} r_k(s, t)R^k + \sum_{k=c+1}^{l-1} r_k(s, t)R^k \leq R \frac{3}{R-5}(d(s, t) + 2 \sum_{k=c}^{l-1} r_k(s, t)R^k).$$

Clearly $r_c(x) \leq d(s, t)$, $x \in \{s, t\}$, thus by (12) we obtain

$$2 \sum_{k=c}^{l-1} (r_k(s) + r_k(t))R^k \leq \frac{4R^2}{(R-1)(R-2)} \max_{x \in \{s, t\}} \int_0^{d(s, t)} \varphi^{-1}\left(\frac{1}{m(B(x, \varepsilon))}\right) d\varepsilon.$$

Since $d(s, t) < \max\{r_{c-1}(s), r_{c-1}(t)\}$ if $c > 0$ and $d(s, t) = D(T)$ if $c = 0$, we have

$$R^{c-1} \leq \max_{x \in \{s, t\}} \varphi^{-1}\left(\frac{1}{m(B(x, d(s, t)))}\right).$$

It follows that

$$d(s, t)R^c \leq R \max_{x \in \{s, t\}} \int_0^{d(s, t)} \varphi^{-1}\left(\frac{1}{m(B(x, \varepsilon))}\right) d\varepsilon.$$
Hence, due to the definition of $\tau_{m,\varphi}(s, t)$ we deduce

$$d_\tau(s, t)R^\tau + \sum_{x \in \{s, t\}} \sum_{k=\tau}^{l-1} r_k^i(x)R^k \leq A\tau_{m,\varphi}(s, t).$$

Lemma 5 implies $r_k(u) \leq r_k^i(x)$, for $u \in B_k^i(x)$. This observation together with Lemma 8 (with $n = n_0 + 1$) yields

$$|S_t f(s) - S_t f(t)| \leq d_\tau(s, t)R^{\tau+n_0+1} + \sum_{x \in \{s, t\}} \sum_{k=\tau}^{l-1} r_k^i(x)R^{\tau+n_0+1} +$$

$$+ \sum_{x \in \{s, t\}} \sum_{k=\tau}^{l-1} r_k^i(x)R^{\tau+n_0+1}\varphi(R^{k+1}) \int_{B_k^i(x)} \int_{B_k^i(u)} \frac{f^d(u, v)}{R^{k+n}} 1_{F_{k+n}}(dv)m(du) +$$

$$+ d_\tau(s, t)R^{\tau+n_0+1}\varphi(R^{\tau+1}) \int_{B_k^i(y)} \int_{B_k^i(u)} \frac{f^d(u, v)}{R^{\tau+n_0+1}} 1_{F_{k+n_0+1}}(dv)m(du).$$

By Lemma 9 we obtain

$$|S_t f(s) - S_t f(t)| \leq AR^{n_0+1}\tau_{m,\varphi}(s, t)(1 +$$

$$+ \sum_{x \in \{s, t\}} \sum_{k=1}^{\infty} \varphi(R^{k+1}) \int_T \int_{B_k(u)} f^d(u, v) \frac{1}{R^{k+n_0+1}} 1_{F_{k+n_0+1}}(dv)m(du) +$$

$$+ \sum_{k=1}^{\infty} \varphi(R^{k+1}) \int_T \int_{B_k(u)} f^d(u, v) \frac{1}{R^{k+n_0+1}} 1_{F_{k+n_0+1}}(dv)m(du)). \quad (25)$$

For each $k \geq 0$ applying (7) (for $\psi$) we have

$$\frac{f^d(u, v)}{R^k} 1_{F_k} \leq \frac{1}{\psi_+(R^k)} \psi_+(f^d(u, v)) \leq \frac{1}{\psi_+(R^k)} \psi_+(f^d(u, v)). \quad (26)$$

The right hand side of (25) does not depend on $l$, furthermore the property (iii) of $S_t$ gives that $\lim_{t \to \infty} S_t f(x) = f(x)$, for $x \in \{s, t\}$. Hence combining (26) and (25) we obtain

$$\frac{|f(s) - f(t)|}{AR^{n_0+1}\tau_{m,\varphi}(s, t)} \leq 1 + 2 \sum_{k=1}^{\infty} \frac{\varphi(R^{k+1})}{\psi_+(R^{k+n_0+1})} \int_T \int_{B_k(u)} \psi_+(f^d(u, v))m(du)m(du) +$$

$$+ \sum_{k=1}^{\infty} \frac{\varphi(R^{k+1})}{\psi_+(R^{k+n_0+1})} \int_T \int_{B_k(u)} \psi(f^d(u, v))1_{F_k}(dv)m(du). \quad (27)$$

It remains to construct a suitable $\nu \in \mathcal{P}(T \times T)$. For each $g \in C(T \times T)$ we put

$$\nu(g) := \frac{1}{B} \sum_{k=1}^{\infty} \frac{\varphi(R^{k+1})}{\psi_+(R^{k+n_0+1})} (2 \int_T \int_{B_k(u)} g(u, v)m(du)m(du) +$$

$$+ \int_T \int_{B_k(u)} g(u, v)m(du)m(du),$$

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where $B$ is such that $\nu(1) = 1$. This constant exists due to (2), indeed

$$B = 3 \sum_{k=1}^{\infty} \frac{\varphi(R^{k+1})}{\psi_+(R^{k+n_0+1})} = 3 \sum_{k=1}^{\infty} \frac{\varphi(R^k)}{\psi(R^{k+n_0+1})} - 1 \leq \frac{3}{1-R^{-n_0-1}} \sum_{k=1}^{\infty} \frac{\varphi(R^k)}{\psi(R^{k+n_0+1})} < \infty,$$

where we have used that $\psi(x) \leq \psi_+(x) + 1$ and $\psi(R^{k+n_0+1}) - 1 \geq (1-R^{-n_0-1})\psi(R^{k+n_0+1})$ (by convexity). Plugging $\nu$ in (27) and then using homogeneity, we see

$$|f(s) - f(t)| \leq 1 + 2 \int_{T \times T} \psi_+ \left( \frac{f^d(u,v)}{|f^d(u,v)|} \right) \nu(du, dv) \leq 3.$$  \hspace{1cm} (28)

Thus we obtain (3) with $K = 3ABR^{n_0+1}$. Suppose now that $\psi(x_0) \psi(y) \leq \psi(r_{xy})$ for all $x, y \geq 1$. Since $\psi(x) \geq \psi(1) = 1$ for all $x \geq 1$, we have $\psi_+(x) \psi_+(y) \leq \psi_+(r_{xy})$ for all $x, y \geq 0$ and so we see that (4) follows from (3) and Lemma 2.

\[\blacksquare\]

### 4 Proof of Theorem 2

**Proof.** We give a proof which modifies the idea from the paper [2]. In the same way as Theorem 2.3 in [5] it can be proved that the existence of metric $\rho$ on $T \times T$ such that for each separable process $X(t), t \in T$ which satisfies (5) (for $\psi$) there holds

$$\mathbb{P}(\sup_{s,t \in T} \frac{|X(s) - X(t)|}{\rho(s,t)} < \infty) = 1,$$

implies the existence of a constant $K_0$ and a continuous positive functional $\Lambda$ on $C_b(T \times T \setminus \Delta)$ (where $\Delta := \{(t,t) : t \in T\}$) with $\Lambda(1) = 1$ such that for each $f \in C(T)$

$$\sup_{s,t \in T} \frac{|f(s) - f(t)|}{K_0 \rho(s,t)} \leq 1 + \Lambda(f^d), \hspace{1cm} (29)$$

where $f^d(u,v) = \frac{|f(u) - f(v)|}{d_{(u,v)}}$. We define measure $m \in \mathcal{P}(T)$ by the requirement

$$\int_T g(t)m(dt) = \Lambda\left(\frac{g(u) + g(v)}{2}\right), \hspace{1cm} \text{for } g \in C(T). \hspace{1cm} (30)$$

Fix $s, t \in T$ and $l \in \mathbb{N}$. Let us denote

$$h_l(\varepsilon) := \begin{cases} 
R^{-n_0} r_1(t) & \varepsilon \leq r_0(t) \\
R^{k-n_0} r_{k+1}(t) & \varepsilon \leq r_k(t), \hspace{0.5cm} 0 < k \leq l \\
0 & 0 \leq \varepsilon \leq r_{l+1}(t),
\end{cases}$$

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where \( r_k(x) = \min \{ \varepsilon : \frac{1}{m(x, \varepsilon)} \leq \varphi(R^k) \} \), for \( k \geq 0 \) as in our main construction. Observe that \( h_{l}, \ l \geq 1 \) is an increasing family of functions, so \( h := \lim_{l \to \infty} h_{l} \) is well defined. We denote \( f_{l}(x) := \int_{0}^{d(x, x)} h_{l}(\varepsilon) d\varepsilon \) and observe that

\[
\left| \frac{f_{l}(u) - f_{l}(v)}{d(u, v)} \right| \leq \frac{1}{|d(t, u) - d(t, v)|} \int_{d(t, v)}^{d(t, u)} h_{l}(\varepsilon) d\varepsilon = \int_{d(t, v)}^{d(t, u)} h_{l}(\varepsilon) d\varepsilon.
\]

The Jensen’s inequality gives

\[
\psi\left(\frac{f_{l}(u) - f_{l}(v)}{d(u, v)}\right) \leq \int_{d(t, v)}^{d(t, u)} \psi(h_{l}(\varepsilon)) d\varepsilon \leq \psi(h_{l}(d(t, u))) + \psi(h_{l}(d(t, v))),
\]

thus by (30) we have

\[
\Lambda(\psi(f_{l}^{d})) \leq 2 \int_{T} \psi(h_{l}(d(t, u))) m(du).
\] (31)

Using the definition of \( h_{l} \) and (10) we obtain

\[
\int_{T} \psi(h_{l}(d(t, u))) m(du) = \sum_{k=0}^{l} \psi(R^{k-n_{0}}) m(B_{k}(t) \setminus B_{k+1}(t)) \leq \sum_{k=0}^{l} \frac{\psi(R^{k-n_{0}})}{\varphi(R^{k})}.
\] (32)

Applying (10) we derive \( D := \sum_{k=0}^{\infty} \frac{\psi(R^{k-n_{0}})}{\varphi(R^{k})} < \infty \). Consequently (29), (31), (32) yield

\[
\int_{0}^{d(s, t)} h_{l}(\varepsilon) d\varepsilon \leq 1 + \Lambda(\psi(f_{l}^{d})) \leq 1 + 2D.
\]

The right hand side does not depend on \( l \), so

\[
\int_{0}^{d(s, t)} h_{l}(\varepsilon) d\varepsilon \leq 1 + 2D.
\] (33)

The definition of \( h \) gives

\[
\varphi^{-1}\left(\frac{1}{m(B(t, \varepsilon))}\right) \leq R^{k+1} = R^{n_{0}+1} h(\varepsilon), \text{ for } r_{k+1}(t) \leq \varepsilon < r_{k}(t),
\]

thus for \( \delta \in [r_{k+1}(t), r_{k}(t)], k \in \mathbb{N} \)

\[
R^{-n_{0}-1} \int_{r_{k+1}(t)}^{\delta} \varphi^{-1}\left(\frac{1}{m(B(t, \varepsilon))}\right) d\varepsilon \leq \int_{r_{k+1}(t)}^{\delta} h(\varepsilon) d\varepsilon
\]

and hence due to (33) we obtain

\[
\int_{0}^{d(s, t)} \varphi^{-1}\left(\frac{1}{m(B(t, \varepsilon))}\right) d\varepsilon \leq K \rho(s, t),
\]

where \( K = (1 + 2D) R^{n_{0}+1} K_{0} \). Similarly

\[
\int_{0}^{d(s, t)} \varphi^{-1}\left(\frac{1}{m(B(s, \varepsilon))}\right) d\varepsilon \leq K \rho(s, t),
\]

which means \( \tau_{m, \varphi}(s, t) \leq K \rho(s, t) \).
5 Proof of Theorem 3

Proof of Theorem 3. Fix $R > 5$, $s, t \in T$ and $f \in C(T)$. We can assume that $	au_{m, \varphi}(s, t) < \infty$ which implies $\lim_{k \to \infty} r_k(x) = 0$ for $x = s, t$. By Lemma 8 (with $n = 1$) and (13) we have

$$|S_tf(s) - S_tf(t)| \leq d_\tau(s, t)R^{k+1} + \sum_{x \in \{s, t\}} \sum_{k=\tau}^{l-1} r_k(x)R^{k+1} +$$

$$+ \sum_{x \in \{s, t\}} \sum_{k=\tau}^{l-1} \varphi(R^{k+1}) \int_{B^c_{k+1}(x)} r_k(u) \int_{B_k(u)} f^d(u, v)1_{F_{k+1}}m(du)m(du) +$$

$$+ \varphi(R^{k+1}) \int_{B^c_1(y)} r_{\tau-1}(u) \int_{B^c_{\tau-1}(u)} f^d(u, v)1_{F_{\tau+1}}m(du)m(du),$$

where $g = t$ if $\tau = \tau_t$ and $g = s$ if $\tau \neq \tau_t$. By Lemma 9 we obtain

$$|S_tf(s) - S_tf(t)| \leq AR\tau_{m, \varphi}(s, t) +$$

$$+ \sum_{x \in \{s, t\}} \sum_{k=1}^{\infty} \varphi(R^{k+1}) \int_T r_k(u)R^{k+1} \int_{B_k(u)} f^d(u, v)R^{k+1}1_{F_{k+1}}m(du)m(du) +$$

$$+ \sum_{k=1}^{\infty} \varphi(R^{k+1}) \int_T r_{k-1}(u)R^{k+1} \int_{B^c_{k-1}(u)} f^d(u, v)R^{k+1}1_{F_{k+1}}m(du)m(du). \quad (34)$$

The condition (7) gives that for each $k \geq 0$

$$\frac{f^d(u, v)}{R^k} 1_{F_k} \leq \frac{1}{\varphi_+(R^k)} \varphi(f^d(u, v)) 1_{F_k} \leq \frac{1}{\varphi_+(R^k)} \varphi_+(f^d(u, v)). \quad (35)$$

The right hand side of (34) does not depend on $l$ thus we can take the limit on left-hand side which is $\lim_{l \to \infty} S_tf(x) = f(x)$, for all $x \in T$ (by property (iii) of $S_t$). Observe also that by the convexity $\varphi_+(R^{k+1}) - 1 \geq (1 - R^{-1})\varphi(R^{k+1})$. Consequently due to (34) and (35) we obtain

$$\frac{|f(s) - f(t)|}{AR\tau_{m, \varphi}(s, t)} \leq 1 + \frac{1}{1 - R^{-1}}(2 \sum_{k=1}^{\infty} \int_T r_k(u)R^{k+1} \int_{B_k(u)} \varphi_+(f^d(u, v))m(du)m(du) +$$

$$+ \sum_{k=1}^{\infty} \int_T r_{k-1}(u)R^{k+1} \int_{B^c_{k-1}(u)} \varphi_+(f^d(u, v))m(du)m(du)). \quad (36)$$

To construct a probability measure $\nu \in \mathcal{P}(T \times T)$ we put for each $g \in C(T \times T)$

$$\nu(g) := \frac{1}{M(1 - R^{-1})} \sum_{k=1}^{\infty} (2 \int_T r_k(u)R^{k+1} \int_{B_k(u)} g(u, v)m(du)m(du) +$$

$$+ \int_T r_{k-1}(u)R^{k+1} \int_{B^c_{k-1}(u)} g(u, v)m(du)m(du),$$
where $M$ is such that $\nu(1) = 1$. Applying (9) and the definition $M(m, \varphi)$ we get

$$1 = \frac{1}{M(1 - R^{-1})} \sum_{k=1}^{\infty} \left( 2 \int r_k(u)R^{k+1}m(du) + \int r_{k-1}(u)R^{k+1}m(du) \right) \leq \frac{3}{M(1 - R^{-1})} \sum_{k=0}^{\infty} \int r_k(u)R^{k+2}m(du) \leq \frac{3R^4}{M(R - 1)^2} M(m, \varphi).$$

Hence $M \leq BM(m, \varphi)$, where $B = \frac{3R^4}{(R - 1)^2}$. Plugging $\nu$ into (36) we obtain

$$|f(s) - f(t)| \leq AR\tau_{m, \varphi}(s, t) + BM(m, \varphi) \int_{T \times T} \varphi_+(f^d(u, v))\nu(du, dv).$$

By homogeneity we obtain for all $a > 0$

$$\frac{|f(s) - f(t)|}{aR^2|f^d|_{\varphi_+}} \leq AR\tau_{m, \varphi}(s, t) + BM(m, \varphi) \int_{T \times T} \varphi_+(f^d(u, v))\nu(du, dv). \quad (37)$$

Due to Lemma 3 we know that $\varphi \in \nabla'$ with $r = R^2$ and $c = 1$, thus $\varphi_+ \in \nabla'$ with $c = 0$ and $r = R^2$. Consequently by (37) we get

$$\varphi_+(a) \int_{T \times T} \varphi_+(\frac{f^d(u, v)}{aR^2|f^d|_{\varphi_+}})\nu(du, dv) \leq \int_{T \times T} \varphi_+(\frac{f^d(u, v)}{|f^d|_{\varphi_+}})\nu(du, dv) = 1.$$

Using the above inequality in (37) we obtain

$$\frac{|f(s) - f(t)|}{aR^2|f^d|_{\varphi_+}} \leq AR\tau_{m, \varphi}(s, t) + \frac{BM(m, \varphi)}{\varphi_+(a)}, \quad \text{for } a > 0.$$

We can obviously take $a$ such that

$$\frac{BM(m, \varphi)}{\varphi_+(a)} = AR\tau_{m, \varphi}(s, t), \quad \text{i.e. } a = \varphi_+^{-1}\left(\frac{BM(m, \varphi)}{AR\tau_{m, \varphi}(s, t)}\right),$$

thus denoting $K = ARB^{-1}$ we derive

$$\frac{|f(s) - f(t)|}{2AR^3\tau_{m, \varphi}(s, t)^{\varphi_+^{-1}\left(\frac{BM(m, \varphi)}{K\tau_{m, \varphi}(s, t)}\right)}} \leq |f^d|_{\varphi_+}.$$

Lemma 2 gives the result with $C = 2AR^5$. \[\square\]

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