A note on the joint measurability of POVMs and its implications for contextuality

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The purpose of this note is to clarify the logical relationship between joint measurability and contextuality for quantum observables in view of recent developments [1–4].

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INTRODUCTION

In a recent work [2], a new proof of contextuality—in the generalized sense of Spekkens [1, 3]—was provided using positive operator-valued measures (POVMs) and the connection between joint measurability of POVMs and contextuality was explicited. It was later shown in [2] that any joint measurability structure can be realized in quantum theory, leaving open the question of whether contextuality can always be demonstrated in these joint measurability structures. Subsequent to these two developments, in Ref. [4] a peculiar feature of POVMs with respect to joint measurability was pointed out: that there exist three measurements which are pairwise jointly measurable and triplewise jointly measurable but for which there exist pairwise joint measurements which do not admit a triplewise joint measurement. In this note, I will briefly put these results in context and point out the logical relationship between joint measurability and the possibility of contextuality. Also, throughout this note, ‘sharp measurement’ will be synonymous with projection-valued measures (PVMs) and ‘unsharp measurement’ will be synonymous with POVMs that are not PVMs.

UNIQUENESS OF JOINT MEASUREMENT FOR PROJECTION-VALUED MEASURES

Since the peculiarity of positive-operator valued measures (POVMs) in cases of interest here arises from the nonuniqueness of joint measurements, I will first prove the uniqueness of joint measurements for projection-valued measures (PVMs). This will help clarify how the distinction between sharp and unsharp measurements comes to play a role in Specker’s scenario [2].

Consider a nonempty set Ωi and a σ-algebra Fi of subsets of Ωi, for i ∈ {1, . . . , N}. The POVM Mi is defined as the map Mi : Fi → B+(H), where \( \sum_{X_i \in F_i} M_i(X_i) = I \) and B+(H) denotes the set of positive semidefinite operators on a Hilbert space H. I is the identity operator on H. Therefore: \( M_i ≡ \{ M_i(X_i) | X_i \in F_i \} \), where Xi labels the elements of POVM Mi. Mi becomes a projection-valued measure (PVM) under the additional constraint \( M_i(X_i)^2 = M_i(X_i) \) for all \( X_i \in F_i \).

Theorem 1. Given a set of POVMs, \{M1, . . . , MN\}, all of which except at most one—say MN—are PVMs, so that for i ∈ {1, . . . , N − 1}

\[ M_i ≡ \{ M_i(X_i) | X_i \in F_i, M_i(X_i)^2 = M_i(X_i) \} \]

and

\[ M_N ≡ \{ M_N(X_N) | X_N \in F_N \}, \]

the set of POVMs, \{M1, . . . , MN\}, is jointly measurable if and only if they commute pairwise, i.e.,

\[ M_j(X_j)M_k(X_k) = M_k(X_k)M_j(X_j), \]

for all j, k ∈ {1, . . . , N} and \( X_j \in F_j, X_k \in F_k \). In this case, there exists a unique joint POVM M, defined as a map

\[ M : F_1 \times F_2 \times \cdots \times F_N \rightarrow B+(H), \]

such that

\[ M(X_1 \times \cdots \times X_N) = M_1(X_1)M_2(X_2) \cdots M_N(X_N), \]

for all \( X_1 \times \cdots \times X_N \in F_1 \times \cdots \times F_N \).

Proof. —This proof is adapted from, and is a generalization of, the proof of Proposition 8 in the Appendix of Ref. [2].

The first part of the proof is for the implication: joint measurability ⇒ pairwise commutativity—A joint POVM for \{M1, . . . , MN\} is defined as a map \( M : F_1 \times F_2 \times \cdots \times F_N \rightarrow B+(H) \), such that

\[ M_i(X_i) = \sum_{\{X_j \in F_j | j \neq i\}} M(X_1 \times \cdots \times X_N) \tag{1} \]

for all \( X_i \in F_i, i \in \{1, \ldots, N\} \). Also, \( M(X_1 \times \cdots \times X_N) ≤ M_i(X_i) \), so the range of \( M(X_1 \times \cdots \times X_N) \) is contained in the range of \( M_i(X_i) \), and therefore:

\[ M_1(X_1)M(X_1 \times \cdots \times X_N) = M(X_1 \times \cdots \times X_N). \tag{2} \]

Using this relation for the complement \( Ω_i \setminus X_i \in F_i \):

\[ M_1(X_1)M(Ω_i \setminus X_1 \times \cdots \times X_N) = (I - M_1(Ω_i \setminus X_1))M(Ω_i \setminus X_1 \times \cdots \times X_N) = 0. \tag{3} \]
Taking the adjoints, it follows that
\[ M(X_1 \times \cdots \times X_N)M_1(X_1) = M(X_1 \times \cdots \times X_N), \] (4)
and
\[ M(\Omega_1 \setminus X_1 \times \cdots \times X_N)M_1(X_1) = 0. \] (5)

Denoting
\[ M^{(i)}(X_{i+1} \times \cdots \times X_N) \equiv \sum_{\{X_j \in \mathcal{F}_i \mid j \leq i\}} M(X_1 \times \cdots \times X_N), \]
this implies:
\[ M_1(X_1)M^{(i)}(X_2 \times \cdots \times X_N) \]
\[ = M_1(X_1)M(X_1 \times \cdots \times X_N) \]
\[ + M_1(X_1)M(\Omega_1 \setminus X_1 \times \cdots \times X_N) \]
\[ = M_1(X_1)M(X_1 \times \cdots \times X_N) \]
\[ = M(X_1 \times \cdots \times X_N). \] (6)

Taking the adjoint,
\[ M^{(i)}(X_2 \times \cdots \times X_N)M_1(X_1) = M(X_1 \times \cdots \times X_N). \] (7)

Therefore:
\[ M_1(X_1)M^{(i)}(X_2 \times \cdots \times X_N) \]
\[ = M^{(i)}(X_2 \times \cdots \times X_N)M_1(X_1) \]
\[ = M(X_1 \times \cdots \times X_N). \] (8)

Noting that \( M^{(i-1)}(X_1 \times \cdots \times X_N) \leq M_i(X_i) \), one can repeat the above procedure for \( M_i, i \in \{2, \ldots, N-1\} \), to obtain:
\[ M^{(i-1)}(X_1 \times \cdots \times X_N) \]
\[ = M_i(X_i)M^{(i)}(X_{i+1} \times \cdots \times X_N) \]
\[ = M^{(i)}(X_{i+1} \times \cdots \times X_N)M_i(X_i). \] (9)

Doing this recursively until \( i = N - 1 \) and noting that \( M^{(N-1)}(X_N) = M_N(X_N) \), it follows:
\[ M(X_1 \times \cdots \times X_N) \]
\[ = M_1(X_1)M^{(1)}(X_2 \times \cdots \times X_N) \]
\[ = M^{(1)}(X_2 \times \cdots \times X_N)M_1(X_1) \]
\[ \vdots \]
\[ = M_1(X_1)M_2(X_2) \cdots M_{N-1}(X_{N-1})M_N(X_N) \]
\[ = M_N(X_N)M_{N-1}(X_{N-1}) \cdots M_2(X_2)M_1(X_1). \] (10)

For the last equality to hold, the POVM elements must commute pairwise, so that
\[ M(X_1 \times \cdots \times X_N) = \prod_{i=1}^{N} M_i(X_i). \] (11)

This concludes the proof of the implication, joint measurability \( \Rightarrow \) pairwise commutativity. The converse is easy to see since the joint POVM defined by taking the product of commuting POVM elements,
\[ \{M(X_1 \times \cdots \times X_N) = \prod_{i=1}^{N} M_i(X_i) | X_i \in \mathcal{F}_i\}, \]
is indeed a valid POVM which coarse-grains to the given POVMs, \( \{M_1, \ldots, M_N\} \).

Indeed, pairwise commutativity \( \Rightarrow \) joint measurability for any arbitrary set of POVMs, \( \{M_1, \ldots, M_N\} \), and it is only when all but (at most) one of these POVMs are PVMs that the converse—and the uniqueness of the joint POVM—holds.

**SPECKER’S SCENARIO**

Specker’s scenario requires a set of three POVMs, \( \{M_1, M_2, M_3\} \), that are pairwise jointly measurable, i.e., \( \exists \) POVMs \( M_{12}, M_{23}, \) and \( M_{31} \) which measure the respective pairs jointly. An immediate consequence of the requirement of pairwise joint measurability of \( \{M_1, M_2, M_3\} \) is that in quantum theory these three measurements cannot be realized as projective measurements (PVMs) and still be expected to show any contextuality. This is because for projective measurements or projection-valued measures (PVMs), a set of three measurements that are pairwise jointly measurable—and therefore admit unique pairwise joint measurements—are also triplewise jointly measurable in the sense that there exists a unique triplewise joint measurement which coarse-grains to each pairwise implementation of the three measurements and therefore also to the single measurements.

From Theorem 1 it follows that if \( M_i, i \in \{1, 2, 3\} \), are PVMs then they admit unique pairwise and triplewise joint POVMs:
\[ M_{ij}(X_i \times X_j) = M_i(X_i)M_j(X_j), \] (12)
\[ M(X_1 \times X_2 \times X_3) = M_1(X_1)M_2(X_2)M_3(X_3), \] (13)
corresponding to the maps \( M_{ij} : \mathcal{F}_i \times \mathcal{F}_j \rightarrow \mathcal{B}_+ (\mathcal{H}) \) and \( M : \mathcal{F}_1 \times \mathcal{F}_2 \times \mathcal{F}_3 \rightarrow \mathcal{B}_+ (\mathcal{H}) \), respectively. Intuitively, this is easy to see since joint measurability is equivalent to pairwise commutativity for a set of projective measurements and the joint measurement for each pair is unique. The existence of a unique joint measurement implies that there exists a joint probability distribution realizable via this joint measurement, thus explaining the pairwise statistics of the triple of measurements noncontextually in the traditional Kochen-Specker sense.

\[ ^{1} \text{KS-noncontextuality just means that there exists a joint proba-} \]
Clearly, then, the three measurements \{M_1, M_2, M_3\} must necessarily be unsharp for Specker’s scenario to exhibit KS-contextuality. The uniqueness of joint measurements (pairwise or triplewise) need not hold in this case. I will refer to pairwise joint measurements as “2-joints” and triplewise joint measurements as “3-joints”. Also, I will use the phrases ‘joint measurability’ and ‘compatibility’ interchangeably since they will refer to the same notion. Consider the four propositions regarding the three measurements:

- \(\exists 2\text{-joint}: \{M_1, M_2, M_3\}\) admit 2-joints,
- \(\nexists 2\text{-joint}: \{M_1, M_2, M_3\}\) do not admit 2-joints,
- \(\exists 3\text{-joint}: \{M_1, M_2, M_3\}\) admit a 3-joint,
- \(\nexists 3\text{-joint}: \{M_1, M_2, M_3\}\) do not admit a 3-joint.

The possible pairwise-triplewise propositions for the three measurements are:

- \((\exists 2\text{-joint}, \exists 3\text{-joint})\)
- \((\exists 2\text{-joint}, \nexists 3\text{-joint})\)
- \((\nexists 2\text{-joint}, \exists 3\text{-joint})\)
- \((\nexists 2\text{-joint}, \nexists 3\text{-joint})\)

Note that the proposition \((\nexists 2\text{-joint}, \exists 3\text{-joint})\) is trivially excluded because triplewise compatibility implies pairwise compatibility. Of the three remaining propositions, the ones of interest for contextuality are \((\exists 2\text{-joint}, \exists 3\text{-joint})\) and \((\exists 2\text{-joint}, \nexists 3\text{-joint})\), since the remaining one is simply about observables that do not admit any joint measurement at all and hence no non-trivial measurement contexts exist for this proposition.

It may seem that for purposes of contextuality even the proposition \((\exists 2\text{-joint}, \exists 3\text{-joint})\) is of no interest, but there is a subtlety involved here: one is only considering whether 2-joints or a 3-joint exist for the set \{M_1, M_2, M_3\}. Since the statistics that is of relevance for Specker’s scenario is the pairwise statistics \[1, 2\], one also needs to consider whether a given choice of 2-joints, \{M_{12}, M_{23}, M_{31}\}, admits a 3-joint, i.e., the proposition \((\exists 3\text{-joint} | \text{a choice of 2-joints})\) or its negation \((\nexists 3\text{-joint} | \text{a choice of 2-joints})\). The four possible conjunctions are:

\[\begin{align*}
(\exists 2\text{-joint}, \exists 3\text{-joint}) & \wedge (\exists 3\text{-joint} | \text{a choice of 2-joints}) \\
(\exists 2\text{-joint}, \exists 3\text{-joint}) & \wedge (\nexists 3\text{-joint} | \text{a choice of 2-joints}) \\
(\exists 2\text{-joint}, \nexists 3\text{-joint}) & \wedge (\exists 3\text{-joint} | \text{a choice of 2-joints}) \\
(\exists 2\text{-joint}, \nexists 3\text{-joint}) & \wedge (\nexists 3\text{-joint} | \text{a choice of 2-joints})
\end{align*}\]

Of these, the first conjunction rules out the possibility of KS-contextuality, so it is not of interest for the present purpose. The third conjunction is false since the existence of a 3-joint for a given choice of 2-joints would also imply the existence of a 3-joint for the three measurements, hence contradicting the fact that these admit no 3-joints. Thus the two remaining conjunctions of interest are:

- **Proposition 1**: \((\exists 2\text{-joint}, \exists 3\text{-joint}) \wedge (\nexists 3\text{-joint} | \text{a choice of 2-joints})\)
- **Proposition 2**: \((\exists 2\text{-joint}, \nexists 3\text{-joint}) \wedge (\exists 3\text{-joint} | \text{a choice of 2-joints})\)

These two possibilities lead to the following propositions:

- **Weak**: \((\exists 2\text{-joint}) \wedge (\nexists 3\text{-joint} | \text{a choice of 2-joints})\)
- **Strong**: \((\exists 2\text{-joint}) \wedge (\exists 3\text{-joint} | \text{a choice of 2-joints})\)

where **Weak** \(\iff\) **Proposition 1 \lor\) **Proposition 2**, and **Strong** \(\iff\) **Proposition 2**. The proposition **Weak** relaxes the requirement of proposition **Strong** that the three measurements should themselves be incompatible to only the requirement that there exists a choice of 2-joints that do not admit a 3-joint. Obviously, under **Strong**, there exists no 3-joint for all possible choices of 2-joints: **Strong** \(\Rightarrow\) **Weak**.

**Comment on Ref. [4] vis-à-vis Ref. [2]**

In Ref. [2], contextuality—in the generalized sense of Spekkens [2] and by implication in the Kochen-Specker sense—was shown keeping in mind the proposition **Strong**, i.e., requiring that the three measurements \{M_1, M_2, M_3\} are pairwise jointly measurable but not

\[\text{Note that for the case of PVMS, only the conjunction (3-joint) \land (3-joint) | \text{a choice of 2-joints}) makes sense and that it is, in fact, equivalent to the proposition (3-joint, 3-joint) since there is no "choice of 2-joints" available: the 2-joints, if they exist, are unique and admit a unique 3-joint (cf. Theorem [4]). Consequently, the propositions **Weak** and **Strong** are not admissible for PVMS.}\]
triplewise jointly measurable. This was in keeping with the approach adopted in Ref. [1], where the construction used did not violate the LSW inequality [1, 2]. Indeed, as shown in Theorem 1 of Ref. [2], the construction used in Ref. [1] could not have produced a violation because it sought a state-independent violation.

In Ref. [4], the authors—under Proposition 1—use the construction first obtained in [2] to show a higher violation of the LSW inequality than reported in [2]. It was seen to approach a maximum of 0.0896 for $R_3^Q = 0.9374$ and $\eta \approx 0.4566$.

Another comment in Ref. [4] is the following:

“Interestingly, there are three observables that are not triplewise jointly measurable but cannot violate LSW’s inequality no matter how each two observables are jointly measured.”

That is, Strong $\Rightarrow$ Violation of LSW inequality. Equally, it is also the case that Weak $\Rightarrow$ Violation of LSW inequality. Neither of these is surprising given the discussion in this note. In particular, note the following implications ($0 \leq \eta \leq 1$):

1. Violation of LSW inequality, i.e., $R_3^Q > 1 - \frac{2}{3}$ $\Rightarrow$ Violation of KS inequality, i.e., $R_3^Q > \frac{2}{3}$,

2. Violation of KS inequality, i.e., $R_3^Q > \frac{2}{3}$ $\Rightarrow$ Weak: 

(∃ 2-joint) $\wedge$ (∃ 3-joint | a choice of 2-joints),

3. Strong $\Rightarrow$ Weak.

Therefore, Weak is a necessary condition for a violation of the LSW inequality. It can be satisfied either under Proposition 1 (as done in [4]) or under Proposition 2 (or Strong, as done in [2]).

**JOINT MEASURABILITY STRUCTURES**

I end this note with a comment on the result proven in Ref. [3], where it was shown constructively that any conceivable joint measurability structure for a set of $N$ observables is realizable via binary POVMs. With regard to contextuality, this result proves the admissibility in quantum theory of contextuality scenarios that are not realizable with POVMs alone. This should be easy to see, specifically, from the example of Specker’s scenario, where POVMs do not suffice to demonstrate contextuality, primarily because they possess a very rigid joint measurability structure dictated by pairwise commutativity and their joint measurements are unique (Theorem 1). If one can demonstrate contextuality given the scenarios obtained from more general joint measurability structures then a relaxation of a sort similar to the case of Specker’s scenario (from Strong to Weak) will also lead to contextuality. In this sense, an implication of the result of Ref. [3] is that it allows one to consider the question of contextuality for joint measurability structures which admit no PVM realization in quantum theory on account of Theorem 1.

In particular, for POVMs, pairwise compatibility $\Leftrightarrow$ global compatibility because commutativity is a necessary and sufficient criterion for compatibility. On the other hand, POVMs allow for a failure of the implication pairwise compatibility $\Rightarrow$ global compatibility because pairwise compatibility is not equivalent to pairwise commutativity for POVMs: pairwise commutativity $\Rightarrow$ pairwise compatibility, but not conversely.
CONCLUSION

I hope this note clarifies issues that may have escaped analysis in Refs. [1–4]. In particular, the logical relationship between admissible joint measurability structures and the possibility of contextuality should be clear from the discussion here.

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