Parity considerations for drops in cycles on \{1, 2, \ldots, n\}

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Abstract. In 2019, A. Lazar and M. L. Wachs conjectured that the number of cycles on \([2n]\) with only even-odd drops equals the \(n\)-th Genocchi number. In this paper, we restrict our attention to a subset of cycles on \([n]\) that in all drops in the cycle, the latter entry is odd. We deduce two bivariate generating functions for such a subset of cycles with an extra variable introduced to count the number of odd-odd and even-odd drops, respectively. One of the generating function identities confirms Lazar and Wachs’ conjecture, while the other identity implies that the number of cycles on \([2n - 1]\) with only odd-odd drops equals the \((n - 2)\)-th Genocchi median.

Keywords. Cycle, odd-odd drop, even-odd drop, bivariate generating function, Genocchi number, Genocchi median.

2020MSC. 05A05, 05A15, 35C10.

1. Introduction

Let \(\mathfrak{S}_n\) denote the set of permutations on \{1, 2, \ldots, n\} =: [n]. There is a natural equivalence relation \(\sim_{\text{cyc}}\) on \(\mathfrak{S}_n\) defined as follows. For \(\pi = \pi_1\pi_2 \cdots \pi_n\) and \(\hat{\pi} = \hat{\pi}_1\hat{\pi}_2 \cdots \hat{\pi}_n\), two permutations in \(\mathfrak{S}_n\), we say \(\pi \sim_{\text{cyc}} \hat{\pi}\) if there exists a nonnegative integer \(m\) such that \(\pi_i = \hat{\pi}_{m+i}\) for all \(1 \leq i \leq n\); here we assume that \(\hat{\pi}_K = \hat{\pi}_k\) where \(1 \leq k \leq n\) is the unique index such that \(K \equiv k \pmod{n}\). In other words, we treat the entries in a permutation in a cyclic way. From this perspective, we may define cycles on \([n]\) as equivalence classes in the quotient set \(\mathfrak{S}_n/\sim_{\text{cyc}}\).

Definition 1.1. Let \(\sim_{\text{cyc}}\) be the equivalence relation defined as above.

(i). We say \([\pi]\) is a cycle on \{1, 2, \ldots, n\} if \([\pi]\) is a member of the quotient set \(\mathfrak{S}_n/\sim_{\text{cyc}}\). We also denote the quotient set \(\mathfrak{S}_n/\sim_{\text{cyc}}\) by \(\mathcal{C}_n\), the set of cycles on \([n]\).

(ii). For each \([\pi] \in \mathcal{C}_n\), we assume that the representative \(\pi = \pi_1\pi_2 \cdots \pi_n\) is a permutation in \(\mathfrak{S}_n\) that starts with \(\pi_1 = 1\). Also, for any positive index \(K\), we put \(\pi_K = \pi_k\) where \(1 \leq k \leq n\) is the unique index such that \(K \equiv k \pmod{n}\).

Our starting point is a conjecture of A. Lazar and M. L. Wachs [2, Conjecture 6.4] on cycles with only even-odd drops; this conjecture was recently proved by Z. Lin and S. H. F. Yan [3] through a bijective approach, and independently by Q. Pan and J. Zeng [4] using continued fractions.

Here we say a consecutive pair \((\pi_i, \pi_{i+1})\) (with \(1 \leq i \leq n\)) is a drop in a cycle \([\pi] \in \mathcal{C}_n\) if \(\pi_i > \pi_{i+1}\). Notice that drops in cycles will not be affected by the choice of the representative of each equivalence class in \(\mathcal{C}_n\). We also say a drop \((\pi_i, \pi_{i+1})\) is odd-odd (resp. even-odd) if \(\pi_i\) is odd (resp. \(\pi_i\) is even) and \(\pi_{i+1}\) is odd.
Let 
\[
\text{DROP}(\pi) := \{(\pi_i, \pi_{i+1}) : \pi_i > \pi_{i+1} \text{ with } 1 \leq i \leq n\},
\]
the set of drops in \([\pi]\). Further, for the unique cycle \([(1)]\) in \(C_1\), we assume that it has a unique drop:
\[
\text{DROP}([(1)]) = \{(\ast, 1)\},
\]
where we tactically assume that the parity of \(\ast\) is neither even nor odd. Now, for each cycle \([\pi] \in C_n\), we define
\[
\begin{align*}
\text{drop}_{oo}(\pi) := \text{card}\{& (\pi_i, \pi_{i+1}) \in \text{DROP}(\pi) : (\pi_i, \pi_{i+1}) \text{ is odd-odd} \} \\
\text{drop}_{eo}(\pi) := \text{card}\{& (\pi_i, \pi_{i+1}) \in \text{DROP}(\pi) : (\pi_i, \pi_{i+1}) \text{ is even-odd} \}
\end{align*}
\]
Since each \([\pi] \in C_n\) must have a drop of the form \((a, 1)\) for some \(a\), our attention is then restricted to a subset of \(C_n\):
\[
C^n_o := \{\pi \in C_n : \pi_{i+1} \text{ is odd for all drops } (\pi_i, \pi_{i+1}) \text{ in } [\pi]\}.
\]
We remark that the set \(C^n_o\) includes cycles with only even-odd drops in the conjecture of Lazar and Wachs as a subset.

The first object of this paper is the following bivariate generating function identity.

**Theorem 1.1.**

\[
\sum_{n \geq 1} \sum_{[\pi] \in C^n_o} x^{\text{drop}_{oo}(\pi)} t^n = \sum_{m \geq 1} \frac{m!(m-1)!t^{2m}}{\prod_{k=1}^{m} (1 + k^2(1 - x)t^2)} + \sum_{m \geq 1} \frac{(m-1)!(2)^{2m-1}}{\prod_{k=1}^{m} (1 + k^2(1 - x)t^2)}. \tag{1.1}
\]

Recall from [1] that the (unsigned) Genocchi numbers
\[
\{g_n\}_{n \geq 1} = \{1, 1, 3, 17, 155, 2073, 38227, 929569, \ldots\},
\]
which enumerate the number of Dumont permutations on \([2n - 2]\), are given by the generating function
\[
\sum_{n \geq 1} g_n t^n = \sum_{m \geq 1} \frac{m!(m-1)!t^m}{\prod_{k=1}^{m} (1 + k^2t)}. \tag{1.1}
\]

Letting \(x = 0\) in Theorem 1.1 immediately yields an alternative proof of Lazar and Wachs’ conjecture [2, Conjecture 6.4].

**Corollary 1.2.** For all \(n \geq 1\), \(g_n\) is equal to the number of cycles on \([2n]\) with only even-odd drops.

Analogously, we may introduce another variable \(y\) to count the number of even-odd drops.

**Theorem 1.3.**

\[
\sum_{n \geq 1} \sum_{[\pi] \in C^n_o} y^{\text{drop}_{eo}(\pi)} t^n
\]
Parity considerations for drops in cycles on \{1, 2, \ldots, n\}

\[
= (y - 1)t^2 + \sum_{m \geq 1} \frac{m!(m-1)t^{2m}}{\prod_{k=1}^{m} (1 + k(k+1)(1-y)t^2)} + \sum_{m \geq 1} \frac{((m-1)!)^2 t^{2m-1}}{\prod_{k=1}^{m} (1 + k(k-1)(1-y)t^2)}.
\]  

(1.2)

Recall also from [1] that the (unsigned) Genocchi medians

\[ \{h_n\}_{n \geq 0} = \{1, 2, 8, 56, 608, 9440, 198272, 5410688, \ldots\}, \]

which enumerate the number of Dumont derangements on \([2n + 2]\), are given by

\[
1 + \sum_{n \geq 1} h_{n-1}t^n = t^{-1}\sum_{m \geq 1} \frac{((m-1)!)^2 t^m}{\prod_{k=1}^{m} (1 + k(k-1)t)}.
\]

Therefore, letting \(y = 0\) in Theorem 1.3, we have the following result.

**Corollary 1.4.** For all \(n \geq 2\), \(h_{n-2}\) is equal to the number of cycles on \([2n - 1]\) with only odd-odd drops.

### 2. A generating tree

For \(n \geq 2\), we consider an arbitrary cycle \([\pi]\) in \(C_{n}^{o}\). First, in \(\pi\), \(n\) must be placed right before an odd entry. Removing \(n\) from \(\pi\), we are led to a cycle on \([n - 1]\). It is also obvious that this cycle is in \(C_{n-1}^{o}\). This means that all cycles in \(C_{n}^{o}\) can be generated by cycles in \(C_{n-1}^{o}\).

Now, given any cycle in \(C_{n-1}^{o}\), it generates \(\lfloor \frac{n+1}{2} \rfloor\) distinct cycles in \(C_{n}^{o}\) by inserting \(n\) right before an odd entry.

Assume that \(n\) is even. If \(n\) is inserted to the middle of an even-odd drop, then the numbers of even-odd and odd-odd drops remain the same; if \(n\) is inserted to the middle of an odd-odd drop, then the number of even-odd drops increases by 1 and the number of odd-odd drops decreases by 1; otherwise, the number of even-odd drops increases by 1 and the number of odd-odd drops remains the same.

Next, assume that \(n\) is odd. If \(n\) is inserted to the middle of an odd-odd drop, then the number of odd-odd drops increases by 1 and the number of even-odd drops decreases by 1; otherwise, the number of odd-odd drops increases by 1 and the number of even-odd drops remains the same.

The above arguments lead to a generating tree:

(i). \(C_{2n-1}^{o} \rightarrow C_{2n}^{o}\). Let \([\pi]\) be in \(C_{2n-1}^{o}\) with \(\text{drop}_{oo}(\pi) = i\) and \(\text{drop}_{eo}(\pi) = j\).

\[
[\pi] \mapsto \begin{cases} 
  i \text{ cycles with } (i - 1) \text{ odd-odd drops and } (j + 1) \text{ even-odd drops}, \\
  j \text{ cycles with } i \text{ odd-odd drops and } j \text{ even-odd drops}, \\
  (n - i - j) \text{ cycles with } i \text{ odd-odd drops and } (j + 1) \text{ even-odd drops}.
\end{cases}
\]
(ii). $\mathbf{C}^n_{2n} \to \mathbf{C}^n_{2n+1}$. Let $[\pi]$ be in $\mathbf{C}^n_{2n}$ with $\text{drop}_{oo}([\pi]) = i$ and $\text{drop}_{eo}([\pi]) = j$.

\[
[\pi] \to \begin{cases} 
  i \text{ cycles with } i \text{ odd-odd drops and } j \text{ even-odd drops,} \\
  j \text{ cycles with } (i + 1) \text{ odd-odd drops and } (j - 1) \text{ even-odd drops,} \\
  (n - i - j) \text{ cycles with } (i + 1) \text{ odd-odd drops and } j \text{ even-odd drops.}
\end{cases}
\]

In other words, the following result holds true.

**Lemma 2.1.** For $n \geq 1$,

(i). \[
\sum_{[\pi] \in \mathbf{C}^n_{2n}} x^{\text{drop}_{oo}([\pi])} y^{\text{drop}_{eo}([\pi])} = \sum_{[\pi'] \in \mathbf{C}^n_{2n-1}} x^{\text{drop}_{oo}([\pi'])} y^{\text{drop}_{eo}([\pi'])} \times \left[ \text{drop}_{oo}([\pi']) x^{-1} y + \text{drop}_{eo}([\pi']) + (n - \text{drop}_{oo}([\pi']) - \text{drop}_{eo}([\pi'])) y \right].
\]

(ii). \[
\sum_{[\pi] \in \mathbf{C}^n_{2n+1}} x^{\text{drop}_{oo}([\pi])} y^{\text{drop}_{eo}([\pi])} = \sum_{[\pi'] \in \mathbf{C}^n_{2n}} x^{\text{drop}_{oo}([\pi'])} y^{\text{drop}_{eo}([\pi'])} \times \left[ \text{drop}_{oo}([\pi']) + \text{drop}_{eo}([\pi']) xy^{-1} + (n - \text{drop}_{oo}([\pi']) - \text{drop}_{eo}([\pi'])) x \right].
\]

3. Odd-odd drops

Letting $y = 1$ in Lemma 2.1 yields the following relations.

**Lemma 3.1.** Let \[ f_n = f_n(x) := \sum_{[\pi] \in \mathbf{C}^n} x^{\text{drop}_{oo}([\pi])}. \] Then $f_1 = 1$ and for $n \geq 1$,

(i). \[ f_{2n} = nf_{2n-1} - x \frac{d}{dx} f_{2n-1} + \frac{d}{dx} f_{2n-1}. \]

(ii). \[ f_{2n+1} = nf_{2n} - x^2 \frac{d}{dx} f_{2n} + x \frac{d}{dx} f_{2n}. \]

3.1. Cycles in $\mathbf{C}^n_{2n}$. By Lemma 3.1, we have $f_2 = 1$ and for $n \geq 1$,

\[ f_{2n+2} = n^2 \cdot xf_{2n} + n \left( f_{2n} + 2x \frac{d}{dx} f_{2n} - 2x^2 \frac{d}{dx} f_{2n} \right) + \left( \frac{d}{dx} f_{2n} - 2x \frac{d}{dx} f_{2n} + x^2 \frac{d}{dx} f_{2n} \right). \]
Parity considerations for drops in cycles on \( \{1, 2, \ldots, n\} \)

\[ + x \frac{d^2}{dx^2} f_{2n} - 2x^2 \frac{d^2}{dx^2} f_{2n} + x^3 \frac{d}{dx} f_{2n} \).

Defining

\[ F = F(x, t) := \sum_{n \geq 1} f_{2n} t^n, \]

the above recurrence relation then yields a PDE:

\[ t^{-1}(F - t) = x(1 - x)^2 \frac{\partial^2}{\partial x^2} F + 2x(1 - x)t \frac{\partial^2}{\partial x \partial t} F + xt^2 \frac{\partial^2}{\partial t^2} F \]

\[ + (1 - x)^2 \frac{\partial}{\partial x} F + (1 + x)t \frac{\partial}{\partial t} F. \]

(3.1)

Notice that \( F(x, t) \) is also the unique power series solution of the above PDE at \( t = 0 \).

Making the following change of variables

\[
\begin{cases}
\xi = \frac{1}{1-x} \\
\eta = \sqrt{(1-x)t}
\end{cases}
\implies
\begin{cases}
x = 1 - \frac{1}{\xi} \\
t = \xi \eta^2
\end{cases}
\]

and using the chain rule (see, for instance, [5, p. 65]), we are led to

\[ \eta^{-2} \Phi - \xi = \xi^2 (\xi - 1) \frac{\partial^2}{\partial \xi^2} \Phi + \xi (2\xi - 1) \frac{\partial}{\partial \xi} \Phi, \]

where

\[ \Phi = \Phi(\xi, \eta) := F(1 - \frac{1}{\xi}, \xi \eta^2) = F(x, t). \]

Recall that \( F(x, t) \) is the unique power series solution of (3.1) at \( t = 0 \), then \( \Phi(\xi, \eta) \) is also the unique power series solution of (3.2) at \( \eta = 0 \). Writing

\[ \Phi(\xi, \eta) := \sum_{n \geq 0} u_n \eta^n \]

where \( u_n := u_n(\xi) \), then by (3.2), \( u_0 = 0, u_1 = 0, u_2 = \xi, \) and for \( n \geq 1, \)

\[ u_{n+2} = \xi^2 (\xi - 1) \frac{d^2}{d \xi^2} u_n + \xi (2\xi - 1) \frac{d}{d \xi} u_n. \]

Thus, \( u_n \in \mathbb{Z}[\xi] \) for all \( n \geq 0 \). This implies that \( \Phi(\xi, \eta) \) is in \( \mathbb{Z}[\eta][[\xi]] \), and therefore in \( \mathbb{Z}[[\eta]][[\xi]] \).

Now, we may write

\[ \Phi(\xi, \eta) := \sum_{m \geq 0} \phi_m \xi^m, \]

where \( \phi_m := \phi_m(\eta) \). From (3.2), we find that

\[ \phi_0 = 0, \]

\[ \phi_1 = \frac{\eta^2}{1 + \eta^2}, \]

and for \( m \geq 2, \)

\[ \phi_m = \frac{m(m - 1)\eta^2}{1 + m^2\eta^2} \phi_{m-1}. \]
It follows that, for \( m \geq 1 \),
\[
\phi_m = \frac{m!(m-1)!\eta^2}{\prod_{k=1}^{m}(1 + k^2\eta^2)}.
\]

We conclude that
\[
\Phi(\xi, \eta) = \sum_{m \geq 1} \frac{m!(m-1)!\eta^2}{\prod_{k=1}^{m}(1 + k^2\eta^2)} \xi^m,
\]
and therefore,
\[
F(x, t) = \sum_{m \geq 1} \frac{m!(m-1)!t^m}{\prod_{k=1}^{m}(1 + k^2(1-x)t)}.
\]

3.2. Cycles in \( C_{2n-1} \). We deduce from Lemma 3.1 that \( f_1 = 1 \) and for \( n \geq 1 \),
\[
f_{2n+1} = n^2 \cdot x f_{2n-1} + n \left( 2x \frac{d}{dx} f_{2n-1} - 2x^2 \frac{d}{dx} f_{2n-1} \right) + \left( -x \frac{d}{dx} f_{2n-1} + x^2 \frac{d}{dx} f_{2n-1} \right)
+ x \frac{d^2}{dx^2} f_{2n-1} - 2x^2 \frac{d^2}{dx^2} f_{2n-1} + x^3 \frac{d^2}{dx^2} f_{2n-1} \right).
\]

Defining
\[
\tilde{F} = \tilde{F}(x, t) := \sum_{n \geq 1} f_{2n-1} t^n,
\]
the above recurrence relation then yields a PDE:
\[
t^{-1}(\tilde{F} - t) = x(1-x)^2 \frac{\partial^2}{\partial x^2} \tilde{F} + 2x(1-x)t \frac{\partial^2}{\partial x \partial t} \tilde{F} + xt^2 \frac{\partial^2}{\partial t^2} \tilde{F} - x(1-x) \frac{\partial}{\partial x} \tilde{F} + xt \frac{\partial}{\partial t} \tilde{F}.
\]

Notice that \( \tilde{F}(x, t) \) is also the unique power series solution of the above PDE at \( t = 0 \).

Making the following change of variables
\[
\begin{cases}
\xi = \frac{1}{1-x} \\
\eta = \sqrt{(1-x)t}
\end{cases}
\quad \iff 
\begin{cases}
x = 1 - \frac{1}{\xi} \\
t = \xi \eta^2
\end{cases}
\]
and using the chain rule, we have
\[
\eta^{-2} \tilde{\Phi} - \xi = \xi^2 (\xi - 1) \frac{\partial^2}{\partial \xi^2} \tilde{\Phi} + \xi (\xi - 1) \frac{\partial}{\partial \xi} \tilde{\Phi},
\]
where
\[
\tilde{\Phi} = \tilde{\Phi}(\xi, \eta) := \tilde{F}(1 - \frac{1}{\xi}, \xi \eta^2) = \tilde{F}(x, t).
\]

Further, we observe that \( \tilde{\Phi}(\xi, \eta) \) is in \( \mathbb{Z}[\xi][[\eta]] \), and therefore in \( \mathbb{Z}[[\eta]][[\xi]] \).

We then write
\[
\tilde{\Phi}(\xi, \eta) := \sum_{m \geq 0} \tilde{\phi}_m \xi^m,
\]
Parity considerations for drops in cycles on \( \{1, 2, \ldots, n\} \)

where \( \hat{\phi}_m := \hat{\phi}_m(\eta) \). From (3.5), we find that

\[
\hat{\phi}_0 = 0, \\
\hat{\phi}_1 = \frac{\eta^2}{1 + \eta^2},
\]

and for \( m \geq 2 \),

\[
\hat{\phi}_m = \frac{(m - 1)^2 \eta^2}{1 + m^2 \eta^2} \hat{\phi}_{m-1}.
\]

It follows that, for \( m \geq 1 \),

\[
\hat{\phi}_m = \frac{(m-1)!}{\prod_{k=1}^{m}(1 + k^2 \eta^2)} \eta^{2m}.
\]

We conclude that

\[
\hat{\Phi}(\xi, \eta) = \sum_{m \geq 1} \frac{(m-1)!}{\prod_{k=1}^{m}(1 + k^2 \eta^2)} \xi^m,
\]

and therefore,

\[
\hat{F}(x, t) = \sum_{m \geq 1} \frac{(m-1)!}{\prod_{k=1}^{m}(1 + k^2(1-x)t)} t^m.
\] (3.6)

3.3. Proof of Theorem 1.1. We observe that

\[
\sum_{n \geq 1} \sum_{|\pi| \in \mathcal{C}_n} x^{\text{drop}(|\pi|)} t^n = \sum_{n \geq 1} f_{2n} t^{2n} + \sum_{n \geq 1} f_{2n-1} t^{2n-1} = F(x, t^2) + t^{-1} \hat{F}(x, t^2).
\]

Substituting (3.3) and (3.6) into the above gives Theorem 1.1.

4. Even-odd drops

Letting \( x = 1 \) in Lemma 2.1, we arrive at relations as follows.

Lemma 4.1. Let

\[
g_n = g_n(y) := \sum_{|\pi| \in \mathcal{C}_n} y^{\text{drop}(|\pi|)}.
\]

Then \( g_1 = 1 \) and for \( n \geq 1 \),

(i).

\[
g_{2n} = ng_{2n-1} - y \frac{d}{dy} g_{2n-1} + y \frac{d}{dy} g_{2n-1}.
\]

(ii).

\[
g_{2n+1} = ng_{2n} - y \frac{d}{dy} g_{2n} + \frac{d}{dy} g_{2n}.
\]
4.1. Cycles in $\xi_{2n}$. By Lemma 4.1, we have $g_2 = y$ and for $n \geq 1$,
\[
g_{2n+2} = n^2 \cdot y g_{2n} \\
+ n \left( y g_{2n} + 2y \frac{d}{dy} g_{2n} - 2y^2 \frac{d}{dy} g_{2n} \right) \\
+ \left( y \frac{d^2}{dy^2} g_{2n} - 2y^2 \frac{d^2}{dy^2} g_{2n} + y^3 \frac{d^2}{dy^2} g_{2n} \right).
\]
Defining
\[
G = G(y,t) := \sum_{n \geq 1} g_{2n} t^n,
\]
the above recurrence relation then yields a PDE:
\[
t^{-1}(G - yt) = y(1 - y^2) \frac{\partial^2}{\partial y^2} G + 2y(1 - y) t \frac{\partial^2}{\partial y \partial t} G + y t^2 \frac{\partial^2}{\partial t^2} G + 2y t \frac{\partial}{\partial t} G.
\]
(4.1)
Notice that $G(y,t)$ is also the unique power series solution of the above PDE at $t = 0$.

Making the following change of variables
\[
\begin{align*}
\xi &= \frac{1}{1 - y} \\
\eta &= \sqrt{(1 - y)t}
\end{align*}
\]
and using the chain rule, we have
\[
\eta^{-2} \Psi - \xi + 1 = \xi^2 (\xi - 1) \frac{\partial^2}{\partial \xi^2} \Psi + 2\xi (\xi - 1) \frac{\partial}{\partial \xi} \Psi,
\]
(4.2)
where
\[
\Psi = \Psi(\xi, \eta) := G\left(1 - \frac{1}{\xi}, \xi \eta^2\right) = G(y,t).
\]
Further, we observe that $\Psi(\xi, \eta)$ is in $\mathbb{Z}[\xi][[\eta]]$, and therefore in $\mathbb{Z}[[\eta]][[\xi]]$.

Now, we write
\[
\Psi(\xi, \eta) := \sum_{m \geq 0} \psi_m \xi^m,
\]
where $\psi_m := \psi_m(\eta)$. From (4.2), we find that
\[
\psi_0 = -\eta^2,
\]
\[
\psi_1 = \frac{\eta^2}{1 + 2\eta^2},
\]
and for $m \geq 2$,
\[
\psi_m = \frac{m(m - 1)\eta^2}{1 + m(m + 1)\eta^2} \psi_{m-1}.
\]
It follows that, for $m \geq 1$,
\[
\psi_m = \frac{m!((m - 1)\eta^2)^m}{\prod_{k=1}^{m} (1 + k(k + 1)\eta^2)}.
\]
We conclude that
\[
\Psi(\xi, \eta) = -\eta^2 + \sum_{m \geq 1} \frac{m!((m - 1)\eta^2)^m}{\prod_{k=1}^{m} (1 + k(k + 1)\eta^2)} \xi^m,
\]
and therefore,
\[ G(y, t) = (y - 1)t + \sum_{m \geq 1} \frac{m!(m - 1)t^m}{\prod_{k=1}^{m} (1 + k(k + 1)(1 - y)t)}. \] (4.3)

4.2. Cycles in \( C_{2n-1}^2 \). It follows from Lemma 4.1 that \( g_1 = 1 \) and for \( n \geq 1 \),
\[ g_{2n+1} = n^2 \cdot yg_{2n-1} + n \left(g_{2n-1} - yg_{2n-1} + 2y \frac{d}{dy} g_{2n-1} - 2y^2 \frac{d}{dy} g_{2n-1} \right) + \left( \frac{d}{dy} g_{2n-1} - 3y \frac{d}{dy} g_{2n-1} + 2y^2 \frac{d}{dy} g_{2n-1} \right) + y \frac{d^2}{dy^2} g_{2n-1} - 2y^2 \frac{d^2}{dy^2} g_{2n-1} + y^3 \frac{d^2}{dy^2} g_{2n-1} \right). \]

Defining
\[ \tilde{G} = \tilde{G}(y, t) := \sum_{n \geq 1} g_{2n-1} t^n, \]
the above recurrence relation then yields a PDE:
\[ t^{-1} (\tilde{G} - t) = y(1 - y)^2 \frac{\partial^2}{\partial y^2} \tilde{G} + 2y(1 - y)t \frac{\partial^2}{\partial y \partial t} \tilde{G} + yt^2 \frac{\partial^2}{\partial t^2} \tilde{G} + (1 - y)(1 - 2y) \frac{\partial}{\partial y} \tilde{G} + t \frac{\partial}{\partial t} \tilde{G}. \] (4.4)

Notice that \( \tilde{G}(y, t) \) is also the unique power series solution of the above PDE at \( t = 0 \).

Making the following change of variables
\[ \begin{cases} \xi = 1 - y \\ \eta = \sqrt{(1 - y)t} \end{cases} \quad \iff \quad \begin{cases} y = 1 - \frac{1}{\xi} \\ t = \xi \eta^2 \end{cases} \]
and using the chain rule, we have
\[ \eta^{-2} \tilde{\Psi} - \xi = \xi^2 (\xi - 1) \frac{\partial^2}{\partial \xi^2} \tilde{\Psi} + \xi^2 \frac{\partial}{\partial \xi} \tilde{\Psi}, \] (4.5)
where
\[ \tilde{\Psi} = \tilde{\Psi}(\xi, \eta) := \tilde{G}(1 - \frac{1}{\xi}, \xi \eta^2) = \tilde{G}(y, t). \]

Further, we observe that \( \tilde{\Psi}(\xi, \eta) \) is in \( \mathbb{Z}[\xi][[\eta]] \), and therefore in \( \mathbb{Z}[[\eta]][[\xi]] \).

Let us write
\[ \tilde{\Psi}(\xi, \eta) := \sum_{m \geq 0} \tilde{\psi}_m \xi^m, \]
where \( \tilde{\psi}_m := \tilde{\psi}_m(\eta) \). From (4.5), we find that
\[ \begin{aligned} \tilde{\psi}_0 &= 0, \\ \tilde{\psi}_1 &= \eta^2, \end{aligned} \]
and for \( m \geq 2 \),
\[ \tilde{\psi}_m = \frac{(m - 1)^2 \eta^2}{1 + m(m - 1)\eta^2} \tilde{\psi}_{m-1}. \]
Therefore, for $m \geq 1$,

$$\tilde{\psi}_m = \frac{((m-1)!)^2 \eta^{2m}}{\prod_{k=1}^{m} (1+k(k-1)\eta^2)}.$$ 

We conclude that

$$\tilde{\Psi}(\xi, \eta) = \sum_{m \geq 1} \frac{((m-1)!)^2 \eta^{2m}}{\prod_{k=1}^{m} (1+k(k-1)\eta^2)} \xi^m,$$

and therefore,

$$\tilde{G}(y, t) = \sum_{m \geq 1} \frac{((m-1)!)^2 t^m}{\prod_{k=1}^{m} (1+k(k-1)(1-y)\eta^2)}.$$ (4.6)

**4.3. Proof of Theorem 1.3.** We observe that

$$\sum_{n \geq 1} \sum_{[\pi] \in C_{2n-1}} y^{\text{drop}_{odd}([\pi])} t^n = \sum_{n \geq 1} g_{2n} t^{2n} + \sum_{n \geq 1} g_{2n-1} t^{2n-1}$$

$$= G(y, t^2) + t^{-1} \tilde{G}(y, t^2).$$

Substituting (4.3) and (4.6) into the above gives Theorem 1.3.

**5. Two identities**

We close this paper with two interesting identities.

**Corollary 5.1.**

$$\sum_{m \geq 1} \frac{((m-1)!)^2 t^m}{\prod_{k=1}^{m} (1+k^2\eta^2)} = t$$ (5.1)

and

$$\sum_{m \geq 1} \frac{m!((m-1)!t^m)}{\prod_{k=1}^{m} (1+k(k+1)t)} = t.$$ (5.2)

**Proof.** For $n \geq 2$, we observe that each cycle in $C_{2n-1}$ has at least one odd-odd drop, in which the former entry is $2n-1$. Recall that

$$\tilde{F}(x, t) = \sum_{n \geq 1} \sum_{[\pi] \in C_{2n-1}} x^{\text{drop}_{even}([\pi])} t^n.$$  

Therefore, $\tilde{F}(0, t) = t$. In light of (3.6), we arrive at (5.1).

Also, for $n \geq 1$, each cycle in $C_{2n}$ has at least one even-odd drop, in which the former entry is $2n$. Since

$$G(y, t) = \sum_{n \geq 1} \sum_{[\pi] \in C_{2n}} y^{\text{drop}_{even}([\pi])} t^n,$$

we have $G(0, t) = 0$. Recalling (4.3) yields (5.2). 

**Acknowledgements.** The author was supported by a Killam Postdoctoral Fellowship from the Killam Trusts.
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