Renormalization constants and beta functions for the gauge couplings of the Standard Model to three-loop order

Luminita N. Mihaila, Jens Salomon, Matthias Steinhauser

1Institut für Theoretische Teilchenphysik, Karlsruhe Institute of Technology (KIT), D-76128 Karlsruhe, Germany

We compute the beta functions for the three gauge couplings of the Standard Model in the minimal subtraction scheme to three loops. We take into account contributions from all sectors of the Standard Model. The calculation is performed using both Lorenz gauge in the unbroken phase of the Standard Model and background field gauge in the spontaneously broken phase. Furthermore, we describe in detail the treatment of $\gamma_5$ and present the automated setup which we use for the calculation of the Feynman diagrams. It starts with the generation of the Feynman rules and leads to the bare result for the Green’s function of a given process.

PACS numbers: 11.10.Hi 11.15.Bt
I. INTRODUCTION

Renormalization group functions are fundamental quantities of each quantum field theory. They provide insights in the energy dependence of cross sections, hints to phase transitions and can provide evidence to the energy range in which a particular theory is valid. The renormalization group functions of the gauge couplings in the Standard Model (SM) are of particular importance in the context of Grand Unified Theories allowing the extrapolation of low-energy precision data to high energies, not accessible by collider experiments.

Important milestones for the calculation of the gauge coupling beta functions in the Standard Model are the following computations:

- The one-loop beta functions in gauge theories along with the discovery of asymptotic freedom have been presented in Refs. [1, 2].
- The corresponding two-loop corrections
  - in gauge theories without fermions [3, 4],
  - in gauge theories with fermions neglecting Yukawa couplings [5, 7],
  - with corrections involving Yukawa couplings [8],
are also available.
- The two-loop gauge coupling beta functions in an arbitrary quantum field theory have been considered in Ref. [9, 10].
- The contribution of the scalar self-interaction at three-loop order has been computed in [11, 12].
- The gauge coupling beta function in quantum chromodynamics (QCD) to three loops is known from Ref. [13, 14].
- The three-loop corrections to the gauge coupling beta function involving two strong and one top quark Yukawa coupling have been computed in Ref. [15].
- The three-loop corrections for a general quantum field theory based on a single gauge group have been computed in [16].
- The four-loop corrections in QCD are known from Refs. [17, 18].

Two-loop corrections to the renormalization group functions for the Yukawa and Higgs boson self-couplings in the Standard Model are also known [10, 19, 23]. Recently, the dominant three-loop corrections to the renormalization group functions of the top quark Yukawa and the Higgs boson self-coupling have been computed in Ref. [24]. In that calculation the gauge couplings and all the Yukawa couplings except the one of the top quark have been set to zero.

In this paper we present details to the three-loop calculation of the gauge coupling renormalization constants and the corresponding beta functions in the SM taking into account all sectors. The results have already been presented in Ref. [25].

The remainder of the paper is organized as follows: In the next Section we introduce our notation and describe in detail how we proceed to obtain the beta functions of the gauge
II. THE CALCULATION OF THE BETA FUNCTIONS

In this paper we present the beta functions for the three gauge couplings of the SM up to three loops in the modified minimal subtraction (\(\overline{\text{MS}}\)) renormalization scheme. In the corresponding calculation we take into account contributions involving the three gauge couplings of the SM, the top, the bottom, and the tau Yukawa couplings and the Higgs self-coupling. We derive the beta functions for a general SM Yukawa sector from the calculation involving the aforementioned seven couplings. We postpone the discussion of this issue to the next Section. In the following, we give details on the computation at the three-loop order.

We define the beta functions as

\[
\mu^2 \frac{d}{d\mu^2} \frac{\alpha_i}{\pi} = \beta_i(\{\alpha_j\},\epsilon),
\]

where \(\epsilon = (4 - d)/2\) is the regulator of Dimensional Regularization with \(d\) being the space-time dimension used for the evaluation of the momentum integrals. The dependence of the couplings \(\alpha_i\) on the renormalization scale is suppressed in the above equation.

The three gauge couplings \(\alpha_1, \alpha_2\) and \(\alpha_3\) used in this paper are related to the quantities usually used in the SM by the following all-order relations

\[
\begin{align*}
\alpha_1 &= \frac{5}{3} \frac{\alpha_{\text{QED}}}{\cos^2 \theta_W}, \\
\alpha_2 &= \frac{\alpha_{\text{QED}}}{\sin^2 \theta_W}, \\
\alpha_3 &= \alpha_s,
\end{align*}
\]

where all quantities are defined in the \(\overline{\text{MS}}\) scheme. \(\alpha_{\text{QED}}\) is the fine structure constant, \(\theta_W\) the weak mixing angle and \(\alpha_s\) the strong coupling. We adopt the SU(5) normalization which leads to the factor 5/3 in the equation for \(\alpha_1\). Eq. (2) serves as a definition for \(\alpha_{\text{QED}}\) and \(\theta_W\).

To lowest order, the Yukawa couplings are given by

\[
\alpha_x = \frac{\alpha_{\text{QED}} m_x^2}{2 \sin^2 \theta_W M_W^2} \quad \text{with} \quad x = t, b, \tau,
\]
where \( m \) and \( M_W \) are the fermion and W boson mass, respectively. In the numerical analysis below also one-loop corrections to Eq. (3) are taken into account [26].

We denote the Higgs boson self-coupling by \( \hat{\lambda} \), where the Lagrange density contains the following term

\[
L_{SM} = \ldots - (4\pi \hat{\lambda})(H^\dagger H)^2 + \ldots ,
\]

(4)
describing the quartic Higgs boson self-interaction.

The beta functions are obtained by calculating the renormalization constants relating bare and renormalized couplings via

\[
\alpha_i^{\text{bare}} = \mu^2 \epsilon Z_{\alpha_i} \left( \{ \alpha_j \}, \epsilon \right) \alpha_i .
\]

(5)
Taking into account that \( \alpha_i^{\text{bare}} \) does not depend on \( \mu \), Eqs. (1) and (5) lead to

\[
\beta_i = -\left[ \frac{\alpha_i}{\pi} + \frac{\alpha_i}{Z_{\alpha_i}} \sum_{j=1,j\neq i}^7 \frac{\partial Z_{\alpha_i}}{\partial \alpha_j} \beta_j \right] \left( 1 + \frac{\alpha_i}{Z_{\alpha_i}} \frac{\partial Z_{\alpha_i}}{\partial \alpha_i} \right)^{-1} ,
\]

(6)
where \( i = 1, 2 \) or \( 3 \). We furthermore set \( \alpha_4 = \alpha_1, \alpha_5 = \alpha_2, \alpha_6 = \alpha_\tau \) and \( \alpha_7 = \hat{\lambda} \).

The first term in the first factor of Eq. (6) originates from the term \( \mu^2 \epsilon \) in Eq. (5) and vanishes in four space-time dimensions. The second term in the first factor contains the beta functions of the remaining six couplings of the SM. Note that (for the gauge couplings) the one-loop term of \( Z_{\alpha_4} \) only contains \( \alpha_4 \), whereas at two loops all couplings are present, except \( \hat{\lambda} \). The latter appears for the first time at three-loop level. As a consequence, it is necessary to know \( \beta_j \) for \( j = 4, 5, 6 \) to one-loop order and only the \( \epsilon \)-dependent term for \( \beta_7 \), namely \( \beta_7 = -\epsilon \alpha_7 / \pi \). From the second term in the first factor and the second factor of Eq. (6) one can read off that three-loop corrections to \( Z_{\alpha_i} \) are required for the computation of \( \beta_i \) to the same loop order.

We have followed two distinct paths to obtain our results for the three-loop renormalization constants, which we discuss in the following Subsections, where we discuss their features and differences.

**A. Lorenz gauge in the unbroken phase of the SM**

The first method used for the calculation of the renormalization constants is based on Feynman rules derived for the SM in the unbroken phase in a general Lorenz gauge with three independent gauge parameters corresponding to the three simple gauge groups. All building blocks of our calculation are evaluated for general gauge parameters in order to have a strong check of the final results for the beta functions which have to be gauge parameter independent. It is possible to use the unbroken phase of the SM since the gauge beta functions in the \( \overline{\text{MS}} \) scheme are independent of all mass parameters and thus the spontaneous symmetry breaking does not affect the final result. Note that this choice is advantageous for the calculation because in the unbroken phase much less different types of vertices have to be considered as compared to the phase in which the spontaneous symmetry breaking is present.

In principle each vertex containing the coupling \( g_i = \sqrt{4\pi \alpha_i} \) can be used to determine the corresponding renormalization constant via

\[
Z_{\alpha_i} = \frac{(Z_{\text{vrtx}})^2}{\prod_k Z_{k,\text{wf}}} ,
\]

(7)
where $Z_{vrtx}$ stands for the renormalization constant of the vertex and $Z_{k,\text{wf}}$ for the wave function renormalization constant; $k$ runs over all external particles.

We have computed $Z_{\alpha_3}$ using both the ghost-gluon and the three-gluon vertex. $Z_{\alpha_2}$ has been evaluated with the help of the ghost-$W_3$, the $W_1W_2W_3$ and the $\phi^+\phi^-W_3$ vertices where $\phi^\pm$ is the charged component of the Higgs doublet corresponding to the charged Goldstone boson in the broken phase and $W_1$, $W_2$ and $W_3$ are the W boson components. As to $Z_{\alpha_1}$, a Ward identity guarantees that there is a cancellation between the vertex and some of the wave function renormalization constants yielding

$$Z_{\alpha_1} = \frac{1}{Z_B},$$

where $Z_B$ is the wave function renormalization constant for the gauge boson of the $U(1)$ subgroup of the SM in the unbroken phase.

In Fig. 1 we show several one-, two- and three-loop sample diagrams contributing to the considered two- and three-point functions.

We have not used vertices involving fermions as external particles as they may lead to problems in connection with the treatment of $\gamma_5$ in $d \neq 4$ dimensions. The vertices selected by us are safe in this respect. A detailed discussion of our prescription for $\gamma_5$ is given below.

In order to compute the individual renormalization constants entering Eq. (7) we proceed as outlined, e.g., in Ref. [15]. The underlying formula can be written in the form

$$Z_\Gamma = 1 - K_\epsilon (Z_\Gamma \Gamma),$$

where $\Gamma$ represents the two- or three-point function corresponding to the renormalization constant $Z_\Gamma$ and the operator $K_\epsilon$ extracts the pole part of its argument. From the structure of Eq. (9) it is clear that $Z_\Gamma$ is computed order-by-order in perturbation theory in a recursive way. It is understood that the bare parameters entering $\Gamma$ on the right-hand side are replaced by the renormalized ones before applying $K_\epsilon$. The corresponding counterterms are only needed to lower loop orders than the one which is requested for $\Gamma$. In our approach the three-loop calculation of $Z_{\alpha_1}$ requires — besides the result for $Z_{\alpha_1}$ to two loops — the one-loop renormalization constants for the other two gauge and the Yukawa couplings. Furthermore we have to renormalize the gauge parameters; the corresponding renormalization constants are given by the wave function renormalization constants of the corresponding gauge bosons which we anyway have to evaluate in the course of our calculation.

B. Background field gauge in the spontaneously broken phase

The second method that we used in order to get an independent result for the renormalization constants of the gauge couplings is a calculation in the BFG [27, 28]. The basic idea of the BFG is the splitting of all gauge fields in a “quantum” and a “classical” part where in practical calculations the latter only occurs as external particle.

The BFG has the advantage that Ward identities guarantee that renormalization constants for gauge couplings can be obtained from the exclusive knowledge of the corresponding wave function renormalization constants \footnote{In Lorenz gauge, this only works for $U(1)$ gauge groups, cf. Subsection IIA.} Thus we have the following formula
FIG. 1. Sample Feynman diagrams contributing to the Green’s functions which have been used for our calculation of the gauge coupling renormalization constants. Solid, dashed, dotted, curly and wavy lines correspond to fermions, Higgs bosons, ghosts, gluons and electroweak gauge bosons, respectively.
\[
Z_{\alpha_i} = \frac{1}{Z_{A_i,\text{wf}}}, \tag{10}
\]

where \( A \) denotes the gauge boson corresponding to the gauge coupling \( \alpha_i \).

In contrast to the calculation using Lorenz gauge, we performed the calculation in the BFG in the spontaneously broken phase of the SM.\(^2\) As discussed in the last Subsection, such a calculation is more involved than a calculation in the unbroken phase since more vertices are present. On the other hand, it constitutes an additional check of our result, that allows us not only to compare the BFG and the Lorenz gauge but also to switch from the broken to the unbroken phase of the SM.

Since the calculation has been performed in the broken phase we have computed the transverse part of the two-point functions of the (background) photon, \( Z \) boson, photon-\( Z \) mixing, \( W \) boson and gluon which we denote by \( \Pi_{\gamma}, \Pi_{Z}, \Pi_{\gamma Z}, \Pi_{W}, \Pi_{g} \), respectively. Sample Feynman diagrams up to three loops can be found in the first two lines of Fig. 1.

\( \Pi_W \) and \( \Pi_g \) can be used in analogy to Subsection II A in order to obtain the corresponding renormalization constants which leads in combination with Eq. (10) to the renormalization constants for \( \alpha_2 \) and \( \alpha_s \). We found complete agreement with the calculation performed in Lorenz gauge.

As far as the self energies involving photon and \( Z \) boson are concerned, we consider at the bare level appropriate linear combinations in order to obtain the \( B \) and \( W \) boson self-energy contributions. To be precise, we have

\[
\begin{align*}
\Pi_B &= \cos^2 \theta_W \Pi_\gamma + 2 \cos \theta_W \sin \theta_W \Pi_{\gamma Z} + \sin^2 \theta_W \Pi_Z, \\
\Pi_W &= \sin^2 \theta_W \Pi_\gamma - 2 \cos \theta_W \sin \theta_W \Pi_{\gamma Z} + \cos^2 \theta_W \Pi_Z. \tag{11}
\end{align*}
\]

The second linear combination can immediately be compared with the bare result obtained from the charged \( W \) boson self energy and complete agreement up to the three-loop order has been found. This constitutes a strong consistency check on the implementation of the BFG Feynman rules. \( \Pi_B \) is used together with Eq. (10) in order to obtain the renormalization constant for \( \alpha_1 \). Again, complete agreement with the calculation described in the previous Subsection has been found.

In our BFG calculation we want to adopt Landau gauge in order to avoid the renormalization of the gauge parameters \( \xi_i \). However, it is not possible to choose Landau gauge from the very beginning since some Feynman rules for vertices involving a background gauge boson contain terms proportional to \( 1/\xi_i \) where \( \xi_i = 0 \) corresponds to Landau gauge. To circumvent this problem we evaluate the bare integrals for arbitrary gauge parameters. In the final result all inverse powers of \( \xi_i \) cancel and thus the limit \( \xi_i = 0 \) can be taken at the bare level.

\[\boxed{\text{C. Automated Calculation}}\]

Higher order calculations in the SM taking into account all contributions are quite involved. Apart from the complicated loop integrals, there are many different Feynman rules...
and plenty of Feynman diagrams which have to be considered. In our calculation we have used a setup which to a large extend avoids manual interventions in order to keep the error-proneness to a minimum.

As far as the loop integrals are concerned we exploit the fact that the beta function in the $\overline{\text{MS}}$ scheme is independent of the external momenta and the particle masses. Thus, we can choose a convenient kinematical configuration which leads to simple loop integrals as long as the infra-red structure is not modified. In our case we set all particle masses to zero and only keep one external momentum different from zero. We have checked that no infra-red divergences are introduced as we will discuss in detail in Section IV. In this way all loop-integrals are mapped to massless two-point functions that up to three loops can be computed with the help of the package MINCER \cite{29}.

As core of our setup we use a well-tested chain of programs that work hand-in-hand: QGRAF \cite{30} generates all contributing Feynman diagrams. The output is passed via q2e \cite{31, 32}, which transforms Feynman diagrams into Feynman amplitudes, to exp \cite{31, 32} that generates FORM \cite{33} code. The latter is processed by MINCER \cite{29} and/or MATAD \cite{34} that compute the Feynman integrals and output the $\epsilon$ expansion of the result. The parallelization of the latter part is straightforward as the evaluation of each Feynman diagram corresponds to an independent calculation. We have also parallelized the part performed by q2e and exp which is essential for our calculation since it may happen that a few times $10^5$ diagrams contribute at three-loop level to a single Green’s function. The described workflow is illustrated on the top of Fig. 2.

In order to perform the calculation described in this paper we have extended the above setup by the vertical program chain in Fig. 2. The core of the new part is the program FeynArtsToQ2E which translates FeynArts \cite{35} model files into model files processable by QGRAF and q2e. In this way we can exploit the well-tested input files of FeynArts in our
effective and flexible setup based on QGRAF, q2e, exp and MINCER. This avoids the coding of
the Feynman rules by hand which for the SM would require a dedicated debugging process.

For the part of our calculation based on the BFG we have used the FeynArts model files
which come together with version 3.5. However, for Lorenz gauge in the unbroken phase
there is no publicly available FeynArts model. For this reason we have used the package
FeynRules \cite{36} in order to generate such a file which is also indicated in Fig. 2.

Let us mention that FeynArtsToQ2E is not restricted to the SM but can process all model
files available for FeynArts.

D. Treatment of $\gamma_5$

An important issue in multi-loop calculations is the definition of $\gamma_5$ away from $d = 4$
dimensions. A first possibility is the naive regularization that requires that $\gamma_5$ anti-commutes
with all other $\gamma$-matrices. This approach has the advantage that its implementation is very
simple. However, it can lead to wrong results, especially for Feynman diagrams involving
several fermion loops. For example, the naive regularization of $\gamma_5$ leads to the problematic
result (see, e.g., Ref. \cite{37})

$$\text{tr}(\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma_5) = 0 \quad (d \neq 4).$$

(12)

The limit of this expression for $d \to 4$ does not agree with its value in the physical case,
when the regularization is turned off

$$\text{tr}(\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma_5) = -4i\varepsilon^{\mu\nu\rho\sigma} \quad (d = 4).$$

(13)

Here the totally anti-symmetric Levi-Civita tensor is defined by $\varepsilon^{0123} = 1$.

It is therefore reassuring that one can show explicitly that in the computation via the
ghost-ghost-gauge boson vertex\cite{3} all contributions stemming from this kind of traces vanish.
To prove this, we notice that this kind of traces can only lead to non-vanishing contributions
if there are at least two of them in a diagram. Only in this case the $\varepsilon$-tensors originating
from Eq. (13) can be contracted, providing Lorentz structures that may contribute to the
renormalization constants. We observe that the fermion loops can only yield problematic
non-vanishing contributions if at least three lines are attached to them. Otherwise, there are
too few external momenta and too few open Lorentz indices available. A general three-loop
diagram with at least two closed fermion loops has the following form

\[ \text{Diagram} \]

Here the solid lines represent fermions while the double lines can be either scalar bosons or
gauge bosons.

One can easily show that one cannot attach ghosts as external particles in the above dia-
gram, since there are no vertices involving ghosts and fermions. Thus, the ghost self-energy

\[ \text{Diagram} \]

We restrict the discussion in this Subsection to the ghost-ghost-gauge boson vertex. For all other vertices
without external fermions the reasoning is in full analogy.
and the ghost-ghost-gauge boson vertex do not contain diagrams of this type. Therefore, we need to consider only diagrams with two external gauge bosons for the following discussion. So the diagrams which still have to be discussed have the structure

![Diagram](image)

There are three ways to replace the double lines:

![Diagram](image)

The first diagram type cannot yield problematic contributions for the same reason as mentioned above. The second diagram type vanishes as the fermion traces involve exactly five $\gamma$-matrices, not counting $\gamma_5$ matrices. (Remember that we deal with diagrams in which all the propagators are massless.) Finally, the third diagram type involves fermion loops with three external gauge bosons. Such diagrams can indeed contain contributions originating from traces of $\gamma_5$ and an even number of other $\gamma$-matrices. However, the sum over all possible fermion species that can circulate in the loops, including also the diagrams in which the fermions circle in opposite directions, vanishes. This is of course a consequence of the cancellation of the Adler-Bell-Jackiw anomaly within the SM, as required by gauge invariance. Therefore, we are allowed to calculate the Feynman diagrams contributing to $Z_{\alpha_i}$ using a naive regularization prescription for $\gamma_5$, in which the diagrams containing triangle anomalies are set to zero from the very beginning, according to Eq. (12).

As an additional check of the calculation we implemented also a “semi-naive” regularization prescription for $\gamma_5$. Explicitly, we evaluate the expression $\text{tr}(\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma_5)$ by applying the formal replacement

$$\text{tr}(\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma_5) = -4i \tilde{\epsilon}^{\mu\nu\rho\sigma} + \mathcal{O}(\epsilon) \right) .$$

The tensor $\tilde{\epsilon}^{\mu\nu\rho\sigma}$ has some similarities with the four-dimensional Levi-Civita tensor: $(i)$ it is completely antisymmetric in all indices; $(ii)$ when contracted with a second one of its kind one obtains the following result

$$\tilde{\epsilon}^{\mu\nu\rho\sigma} \tilde{\epsilon}_{\mu'\nu'\rho'\sigma'} = g_{[\mu'} g_{\nu'} g_{\rho'] g_{\sigma]} ,$$

where the square brackets denote complete anti-symmetrization. When taking the limit $d \to 4$, $\tilde{\epsilon}^{\mu\nu\rho\sigma}$ converts into the four-dimensional Levi-Civita tensor and Eqs. (14) and (15) ensure that it provides the correct four-dimensional result.

At this point a comment on Eq. (14) is in order. It is straightforward to see that the combination of this equation and the cyclic property of traces leads to an ambiguity of order $\mathcal{O}(\epsilon)$. Therefore, we made sure that the terms that need to be treated in this way generate at most simple poles in $\epsilon$, which are thus unambiguous, and the above procedure can be applied directly without introducing additional finite counterterms.

---

4 The proof of this statement can be based only on considerations about group theoretic invariants. For details see Chapter 20 of Ref. [38].
TABLE I. The number of Feynman diagrams contributing to the Green’s functions evaluated in this work. Left table: two- and three-point functions computed in Lorenz gauge; right table: two-point functions computed in BFG. The superscript “B” denotes background fields. The first column indicates the external legs of the Green’s function, the other columns show the number of diagrams at the individual loop orders. Note that the BBB vertex is computed in order to have a cross check as we will explain in Section IV.

Let us stress again that we find the same result for the renormalization constants $Z_{\alpha_i}$ both from the ghost–ghost–gauge boson vertex and by using other vertices and both by applying the “naive” as well as the “semi-naive” scheme. These findings strongly support the above reasoning.

E. Comparison of the methods

This Subsection is devoted to a brief comparison of the calculation via the Lorenz gauge and the one involving the BFG. As has been mentioned before, in the BFG it is sufficient to consider only the gauge boson propagators. This is advantageous as the use of Lorenz gauge also requires the evaluation of three- (or four-) point functions and in most cases it also demands the consideration of additional two-point functions apart from the gauge boson ones. Disadvantages of the BFG are the increased number of vertices and the more involved structure of the vertices containing a background field.

In Tab. I we list the number of diagrams for each Green’s function contributing to the one-, two- and three-loop order. The number of diagrams computed in this work is obtained from the sum of the numbers in these columns. For comparison we also provide the corresponding number of diagrams which contribute to the four-loop order.

It is tempting to compare the number of contributing Feynman diagrams in Lorenz gauge and in BFG which is, however, not straightforward since we use the former in the broken and the latter in the unbroken phase. Nevertheless, one observes that in the case of $\beta_3$ the number of diagrams entering the BFG calculation is roughly the same as in case the gluon-
ghost vertex is used in Lorenz gauge, even up to four-loop order. All other vertices lead to significantly more diagrams. In the case of $\beta_2$ there are about three to four times more diagrams. In the case of $\beta_2$ there are about three to four times more diagrams to be considered in the BFG as compared to Lorenz gauge. Whereas at three-loop order the difference between approximately 70 000 and 250 000 diagrams is probably not substantial it is striking at four-loop order where the number of Feynman diagrams goes from about 10 000 000 ($W_3 W_3$, $c W_2 \bar{c} W_3$ and $c W_3 \bar{c} W_2 W_3$ Green’s function) to 42 000 000 ($W + B W^- B$ Green’s function) when switching from Lorenz gauge to BFG. Thus, starting from four loops it is probably less attractive to use the BFG.

Let us add that the precise number of Feynman diagrams depends on the detailed setup as, e.g., on the implementation of the four-particle vertices. Because of the colour structure we split in our calculation the four-gluon vertex into two cubic vertices by introducing non-propagating auxiliary particles, whereas all other four-particle vertices are left untouched.

Let us finally mention that the CPU time for the evaluation of an individual diagram ranges from less than a second to a few minutes. For general gauge parameters it may take up to the order of an hour. Thus the use of about 100 cores leads to a wall-clock time which ranges from a few hours for a calculation in Feynman gauge up to about one day for general gauge parameters. For the preparation of the FORM files using QGRAF, q2e and exp also a few hours of CPU time are needed which is because of the large amount of Feynman diagrams. The use of about 100 cores leads to a wall-clock time of a few minutes.

**III. ANALYTICAL RESULTS**

In this Section we present our analytical results for the beta functions. As mentioned before, we are able to present the results involving all contributions of the SM Yukawa sector.

The SM Yukawa interactions are described by (see, e.g., Chapter 11 of Ref. [41])

$$L_{\text{Yukawa}} = -\bar{Q}_i U_{ij} \epsilon H^* u_j^R - \bar{Q}_i D_{ij} H d_j^R - \bar{L}_i L_{ij} H l_j^R + \text{h.c.},$$

where $Y^{U,D,L}$ are complex $3 \times 3$ matrices, $i,j$ are generation labels, $H$ denotes the Higgs field and $\epsilon$ is the $2 \times 2$ antisymmetric tensor. $Q^L, L^L$ are the left-handed quark and lepton doublets, and $u^R, d^R, l^R$ are the right-handed up- and down-type quark and lepton singlets, respectively. The physical mass-eigenstates are obtained by diagonalizing $Y^{U,D,L}$ by six unitary matrices $V^{U,D,L}_{L,R}$ as follows

$$\hat{Y}_f^{IJ} = V^I_L Y^{IJ} V^{f\dagger}_L, \quad f = U, D, L.$$  

As a result the charged-current $W^\pm$ couples to the physical quark states with couplings parametrized by the Cabibbo-Kobayashi-Maskawa (CKM) matrix $V_{CKM} \equiv V^U_L V^{D\dagger}_L$. We furthermore introduce the notation

$$\hat{T} = \frac{1}{4\pi} Y^{U\dagger} Y^U, \quad \hat{B} = \frac{1}{4\pi} Y^{D\dagger} Y^D, \quad \hat{L} = \frac{1}{4\pi} Y^{L\dagger} Y^L.$$  

In order to reconstruct the results for a general Yukawa sector, we have multiplied each Feynman diagram by a factor $(n_h)^m$, where $m$ denotes the number of fermion loops involving Yukawa couplings. After analyzing the structure of the diagrams that can arise, we could establish the following set of replacements that have to be performed in order to take into account a generalized Yukawa sector

$$n_h \alpha_t \rightarrow \text{tr} \hat{T}, \quad n_h \alpha_b \rightarrow \text{tr} \hat{B},$$
The addition of a fourth generation of fermions to the SM particle content can be also easily accounted for by this general notation. In this case, the Yukawa matrices become $4 \times 4$ dimensional. If we assume that the fourth generation is just a repetition of the existing generation pattern but much heavier and if we neglect all SM Yukawa interactions, then the explicit form of Yukawa matrices reads

$$
\hat{F}_4 = \begin{pmatrix} 0_{3 \times 3} & 0 \\ 0 & 0_{1 \times 1} \end{pmatrix}, \quad \text{with} \quad F = T, B, L.
$$

Of course, only traces over products of Yukawa matrices can occur because they arise from closed fermion loops. Using Eqs. (17) and (18), it is straightforward to see that in Eq. (19) only traces of diagonal matrices have to be taken except for $\text{tr} \hat{T} \hat{B}$ which is given by

$$
\text{tr} \hat{T} \hat{B} = \text{tr} \left[ \begin{pmatrix} \alpha_t & 0 & 0 \\ 0 & \alpha_c & 0 \\ 0 & 0 & \alpha_t \end{pmatrix} V_{CKM}^{\dagger} \begin{pmatrix} \alpha_d & 0 & 0 \\ 0 & \alpha_s & 0 \\ 0 & 0 & \alpha_b \end{pmatrix} V_{CKM} \right].
$$

We are now in the position to present our results for the beta functions of the gauge couplings. They are given by

$$
\beta_1 = \frac{\alpha_t^2}{(4\pi)^2} \left\{ \frac{2}{5} + \frac{16n_G}{3} \right\} + \frac{\alpha_t^2}{(4\pi)^3} \left\{ \frac{18\alpha_1}{25} + \frac{18\alpha_2}{5} - \frac{34\alpha_t}{5} - 2\text{tr} \hat{B} - 6\text{tr} \hat{L} + n_G \left[ \frac{76\alpha_1}{15} + \frac{12\alpha_2}{5} + \frac{176\alpha_3}{15} \right] \right\} \\
+ \frac{\alpha_t^2}{(4\pi)^4} \left\{ \frac{489\alpha_1^2}{2000} + \frac{783\alpha_1\alpha_2}{200} + \frac{3401\alpha_2^2}{80} + \frac{54\alpha_1\hat{\lambda}}{25} + \frac{18\alpha_2\hat{\lambda}}{5} - \frac{36\hat{\lambda}^2}{5} - \frac{2827\alpha_1\alpha_t}{200} \\
- \frac{471\alpha_2\text{tr} \hat{T}}{8} - \frac{116\alpha_3\text{tr} \hat{T}}{5} - \frac{1267\alpha_1\text{tr} \hat{B}}{200} - \frac{1311\alpha_2\text{tr} \hat{B}}{40} - \frac{68\alpha_3\text{tr} \hat{B}}{5} - \frac{2529\alpha_1\text{tr} \hat{L}}{200} \\
- \frac{1629\alpha_2\text{tr} \hat{L}}{40} + \frac{183\text{tr} \hat{B}^2}{20} + \frac{51\text{tr} \hat{B} \text{tr} \hat{L}}{10} + \frac{157\text{tr} \hat{B} \text{tr} \hat{L}}{5} + \frac{261\text{tr} \hat{L}^2}{20} + \frac{99\text{tr} \hat{L}^2}{10} \\
+ \frac{3\text{tr} \hat{T} \hat{B}}{2} + \frac{339\text{tr} \hat{T}^2}{20} + \frac{177\text{tr} \hat{T} \text{tr} \hat{L}}{5} + \frac{199\text{tr} \hat{T} \text{tr} \hat{L}}{5} + \frac{303\text{tr} \hat{T}^2}{10}
\right\}
$$
\[\begin{align*}
+ n_G & \left[ - \frac{232\alpha_1^2}{75} - \frac{7\alpha_1\alpha_2}{25} + \frac{166\alpha_2^2}{15} - \frac{548\alpha_1\alpha_3}{225} - \frac{4\alpha_2\alpha_3}{5} + \frac{1100\alpha_3^2}{9} \right] \\
+ n_G^2 & \left[ - \frac{836\alpha_1^2}{135} - \frac{44\alpha_2^2}{15} - \frac{1936\alpha_3^2}{135} \right].
\end{align*}\]

(22)

\[\begin{align*}
\beta_2 &= \frac{\alpha_2^2}{(4\pi)^2} \left\{ - \frac{86}{3} + \frac{16n_G}{3} \right\} \\
&+ \frac{\alpha_2^2}{(4\pi)^3} \left\{ \frac{6\alpha_1}{5} - \frac{518\alpha_2}{3} - 6\tr^T - 6\tr^B - 2\tr L + n_G \left[ \frac{4\alpha_1}{5} + \frac{196\alpha_2}{3} + 16\alpha_3 \right] \right\} \\
&+ \frac{\alpha_2^2}{(4\pi)^4} \left\{ \frac{163\alpha_1^2}{400} + \frac{561\alpha_1\alpha_2}{40} - \frac{66711\alpha_2^2}{432} + \frac{6\alpha_1\hat{\lambda}}{5} + 6\alpha_2\hat{\lambda} - 12\hat{\lambda}^2 - \frac{593\alpha_1\tr^T}{40} \\
&- \frac{729\alpha_2\tr^T}{8} - 28\alpha_3\tr T - \frac{533\alpha_1\tr B}{40} - \frac{729\alpha_2\tr B}{8} - 28\alpha_3\tr B - \frac{51\alpha_1\tr L}{8} \\
&- \frac{243\alpha_2\tr L}{8} + \frac{57\tr^B}{4} + \frac{45(\tr^B)^2}{2} + 15\tr B\tr L + \frac{19\tr L^2}{4} + \frac{5(\tr L)^2}{2} + \frac{27\tr^T\tr B}{2} \\
&+ \frac{57\tr^T}{4} + 45\tr^T\tr B + 15\tr^T\tr L + \frac{45(\tr^T)^2}{2} \\
&+ n_G \left[ - \frac{28\alpha_1^2}{15} + \frac{13\alpha_1\alpha_2}{5} + \frac{2564\alpha_2^2}{27} - \frac{4\alpha_1\alpha_3}{15} + 52\alpha_2\alpha_3 + \frac{50\alpha_3^2}{3} \right] \\
&+ n_G^2 \left[ - \frac{44\alpha_1^2}{45} - \frac{1660\alpha_2^2}{27} - \frac{176\alpha_3^2}{9} \right].
\end{align*}\]

(23)

And

\[\begin{align*}
\beta_3 &= \frac{\alpha_3^2}{(4\pi)^2} \left\{ - 44 + \frac{16n_G}{3} \right\} \\
&+ \frac{\alpha_3^2}{(4\pi)^3} \left\{ - 408\alpha_3 - 8\tr^T - 8\tr^B + n_G \left[ \frac{22\alpha_1}{15} + 6\alpha_2 + \frac{304\alpha_3}{3} \right] \right\} \\
&+ \frac{\alpha_3^2}{(4\pi)^4} \left\{ - 5714\alpha_3^2 - \frac{101\alpha_1\tr^T}{10} - \frac{93\alpha_2\tr^T}{2} - \frac{93\alpha_2\tr B}{10} - \frac{93\alpha_2\tr L}{2} \\
&- 160\alpha_3\tr B + 18\tr^B + 42(\tr^B)^2 + 14\tr B\tr L - 12\tr^T\tr B + 18\tr^T\tr B + 84\tr^T\tr B \\
&+ 14\tr^T\tr L + 42(\tr^T)^2 \\
&+ n_G \left[ - \frac{13\alpha_1^2}{30} - \frac{\alpha_1\alpha_2}{6} + \frac{241\alpha_2^2}{45} + \frac{308\alpha_1\alpha_3}{28\alpha_2\alpha_3} + \frac{20132\alpha_3^2}{9} \right] \\
&+ n_G^2 \left[ - \frac{242\alpha_1^2}{135} - \frac{22\alpha_2^2}{3} - \frac{2600\alpha_3^2}{27} \right].
\end{align*}\]

(24)

In the above formulas \(n_G\) denotes the number of fermion generations. It is obtained by labeling the closed quark and lepton loops present in the diagrams.

To obtain the results for the three-loop gauge beta functions, one also needs the one-loop beta functions of the Yukawa couplings, cf. Eq. (6). They can be found in the literature, of course. Nevertheless we decided to re-calculate them as an additional check of our setup.
For completeness, we present the analytical two-loop expressions which read

\[
\beta_{\alpha_t} = - \frac{\epsilon \alpha_t}{\pi} + \frac{\alpha_t}{(4\pi)^2} \left\{ -6\alpha_t + 6\alpha_t + 4\text{tr} \hat{L} + 12\text{tr} \hat{B} + 12\text{tr} \hat{T} - \frac{17}{5} \alpha_1 - 9\alpha_2 - 32\alpha_3 \right\}
\]

\[
+ \frac{\alpha_t}{(4\pi)^3} \left\{ -29\alpha_1^2 \right. 
\begin{align}
&+ \frac{27\alpha_1 \alpha_2}{5} - 35\alpha_2^2 + \frac{124\alpha_1 \alpha_3}{15} + 36\alpha_2 \alpha_3 - \frac{1616\alpha_3^2}{3} \\
&+ \left. n_C \left[ -\frac{4\alpha_1^2}{45} + 4\alpha_2^2 + \frac{320\alpha_3^2}{9} \right] + 24\hat{\alpha}_t^2 + \frac{91\alpha_1 \alpha_t}{20} + \frac{99\alpha_2 \alpha_t}{4} + 16\alpha_3 \alpha_t - \alpha_t^2 + \frac{237\alpha_1 \alpha_b}{20} \\
&+ \frac{225\alpha_2 \alpha_b}{4} + 144\alpha_3 \alpha_b - 48\hat{\lambda}_\alpha - 11\alpha_t \alpha_b - 48\hat{\alpha}_b + \frac{15\alpha_1 \alpha_x}{2} + \frac{15\alpha_2 \alpha_x}{2} + 5\alpha_t \alpha_x \\
&- 9\alpha_b \alpha_x - 9\hat{\alpha}_x \right\},
\]

(25)

and

\[
\beta_{\alpha_b} = - \frac{\epsilon \alpha_b}{\pi} + \frac{\alpha_b}{(4\pi)^2} \left\{ 6\alpha_b - 6\alpha_t + 4\text{tr} \hat{L} + 12\text{tr} \hat{B} + 12\text{tr} \hat{T} - \alpha_1 - 9\alpha_2 - 32\alpha_3 \right\}
\]

\[
+ \frac{\alpha_b}{(4\pi)^3} \left\{ -29\alpha_1^2 \right. 
\begin{align}
&+ \frac{27\alpha_1 \alpha_2}{5} - 35\alpha_2^2 + \frac{124\alpha_1 \alpha_3}{15} + 36\alpha_2 \alpha_3 - \frac{1616\alpha_3^2}{3} \\
&+ \left. n_C \left[ -\frac{4\alpha_1^2}{45} + 4\alpha_2^2 + \frac{320\alpha_3^2}{9} \right] + 24\hat{\alpha}_t^2 + \frac{91\alpha_1 \alpha_t}{20} + \frac{99\alpha_2 \alpha_t}{4} + 16\alpha_3 \alpha_t - \alpha_t^2 + \frac{237\alpha_1 \alpha_b}{20} \\
&+ \frac{225\alpha_2 \alpha_b}{4} + 144\alpha_3 \alpha_b - 48\hat{\lambda}_\alpha - 11\alpha_t \alpha_b - 48\hat{\alpha}_b + \frac{15\alpha_1 \alpha_x}{2} + \frac{15\alpha_2 \alpha_x}{2} + 5\alpha_t \alpha_x \\
&- 9\alpha_b \alpha_x - 9\hat{\alpha}_x \right\},
\]

(26)

The one-loop results have been expressed in terms of Yukawa matrices since these expressions enter the three-loop beta functions. At two-loop order, however, we refrain from reconstructing the general expression which would require an extension of the rules given in Eq. (19).

Our independent calculation of the two-loop Yukawa beta functions is also interesting as there has been a discrepancy between \[20\] and \[23\] concerning the absence of terms proportional to \(\alpha_b \hat{\alpha}_1\) in Eqs. (25) and (26). We were able to confirm the results of Ref. \[23\].

In Appendices \[A\] and \[B\] we provide the results for the renormalization constants which lead to the beta functions discussed in this Section.
IV. CHECKS

We have successfully performed a number of consistency checks and compared our results with those already available in the literature. We describe these checks in detail in this Section.

The consistency checks show that all computed renormalization constants are local (i.e. there are no $\ln \mu$ terms in the final expression), that the renormalization constants of the gauge couplings are gauge parameter independent and that the beta functions are finite. We also find that the beta functions calculated by considering different vertices in Lorenz gauge agree among themselves and with the results of the computation in BFG.

In order to test that the program *FeynArtsToQ2E* correctly translates *FeynArts* model files into model files for *QGRAF/q2e*, we reproduced the beta function for the Higgs self-coupling to one-loop order and the beta functions for the top and bottom quark, and the tau lepton Yukawa couplings to two-loop order (cf. previous Section). We have even considered quantities within the Minimal Supersymmetric Standard Model, like the relation between the squark masses within one generation, which is quite involved in case electroweak interactions are kept non-zero. Furthermore, we find that the divergent loop corrections to the $BBB$ vertex vanish in the Lorenz gauge, as expected since for this vertex no renormalization is required. We performed the latter check up to three-loop order.

Another check consists in verifying that in the vertex diagrams no infra-red divergences are introduced although one external momentum is set to zero. We do not have to consider two-point functions since they are infra-red safe. One can avoid infra-red divergences by introducing a common mass for the internal particles. Afterwards, the resulting integrals are evaluated in the limit $q^2 \gg m^2$ where $q$ is the non-vanishing external momentum of the vertex diagrams. This is conveniently done by applying the rules of asymptotic expansion [42] which are encoded in the program *exp*. The setup described in Section II.C is particularly useful for this test since *exp* takes over the task of generating *FORM* code for all relevant sub-diagrams which can be up to 35 for some of the diagrams; this makes the calculation significantly more complex. In our case the asymptotic expansion either leads to massless two-point functions or massive vacuum integrals or a combination of both. The former are computed with the help of the package *MINCER*, for the latter the package *MATAD* is used. As a result one obtains a series in $m^2/q^2$ where the coefficients contain numbers and $\ln(m^2/q^2)$ terms. For our purpose it is sufficient to restrict ourselves to the term $(m^2/q^2)^0$ and check that no logarithms appear in the final result. With this method we have explicitly checked that the $W_1W_2W_3$ and three-gluon vertices are free from infra-red divergences. Since the results for the gauge coupling renormalization constants agree with the ones obtained from the other vertices also the latter are infra-red safe.

Let us finally comment on the comparison of our findings with the literature. We have successfully compared our results for the two-loop gauge beta functions with [9] and the two-loop Yukawa beta functions with [23]. Even partial results for the three-loop gauge beta functions were available in the literature [16]. That paper comprises all hitherto known three-loop corrections in a general quantum field theory based on a single gauge group, however, the presentation of the results relies on a quite intricate notation. Its specification to the SM is straightforward, however, a bit tedious. For convenience, the translation rules needed to convert the notation of [16] to ours is given in Appendix D.

After converting the notation of Ref. [16] to ours we find complete agreement, taking into
account the following modifications in Eq. (33) of [16]:

The terms

\[-7g^2 \text{tr} (Y^a \bar{Y}^b) \text{tr} (\bar{Y}^b Y^a C(R)) \frac{12r}{12r} + g^2 \text{tr} (Y^a \bar{Y}^b) \text{tr} (Y^b \bar{Y}^c) C(S)_{ca} \frac{12r}{12r} \]

have to be “symmetrized”, so that they read

\[
\frac{1}{2} \left( -\frac{7g^2 \text{tr} (Y^a \bar{Y}^b) \text{tr} (\bar{Y}^b Y^a C(R))}{12r} - \frac{7g^2 \text{tr} (Y^a \bar{Y}^b) \text{tr} (\bar{Y}^b Y^a C(R))}{12r} \right) + \frac{1}{2} \left( g^2 \text{tr} (Y^a \bar{Y}^b) \text{tr} (Y^b \bar{Y}^c) C(S)_{ca} \frac{12r}{12r} + g^2 \text{tr} (Y^a \bar{Y}^b) \text{tr} (\bar{Y}^b Y^c) C(S)_{ca} \frac{12r}{12r} \right).
\]

Furthermore, one has to correct the obvious misprint

\[
+ \frac{5g^4 \text{tr} (C(R)^2 \bar{Y}^a Y^b)}{12r} \rightarrow + \frac{5g^4 \text{tr} (C(R)^2 \bar{Y}^a Y^a)}{12r}.
\]

V. NUMERICAL ANALYSIS

In this Section we discuss the numerical effect of the new contributions to the gauge beta functions. We solve the corresponding renormalization group equations of the gauge couplings numerically and take into account the contributions from the Yukawa couplings and the Higgs self-coupling to two-loops. As boundary conditions we choose

\[
\begin{align*}
\alpha_{1\text{MS}}(M_Z) &= 0.0169225 \pm 0.0000039, \\
\alpha_{2\text{MS}}(M_Z) &= 0.033735 \pm 0.000020, \\
\alpha_{3\text{MS}}(M_Z) &= 0.1173 \pm 0.00069, \\
\alpha_t\text{MS}(M_Z) &= 0.07514, \\
\alpha_b\text{MS}(M_Z) &= 0.00002064, \\
\alpha_{\tau}\text{MS}(M_Z) &= 8.077 \cdot 10^{-6}, \\
4\pi \hat{\lambda} &= 0.13,
\end{align*}
\]

where the first six entries correspond to experimentally determined values while the value for the Higgs coupling is determined assuming a Higgs boson with mass 125 GeV:

\[
4\pi \hat{\lambda} = \frac{m_H^2}{2v^2} \approx \frac{125^2 \text{GeV}^2}{2 \cdot (\sqrt{2} \cdot 174)^2 \text{GeV}^2} \approx 0.13.
\]

Since \( \hat{\lambda} \) only occurs at three loops the tree-level relation (32) is sufficient for our purpose. Note that the values in Eq. (31) are given in the full SM, the top quark being not integrated

\footnotetext{We want to thank the authors of [16] for pointing out the issue of symmetrization and for assistance in deriving the translation rules.}
FIG. 3. The running of the gauge couplings at three loops. The curve with the smallest initial value corresponds to $\alpha_1$, the middle curve to $\alpha_2$, and the curve with the highest initial value to $\alpha_3$.

out. For a description how these values are determined from the knowledge of their directly measured counterparts [41], we refer to [41, 43].

It is noteworthy that for all three gauge couplings the sum of all three-loop terms involving at least one of the couplings $\alpha_b$, $\alpha_\tau$ or $\lambda$ leads to corrections which are less than 0.1% of the difference between the two- and three-loop prediction.

In Fig. 3 the running of the couplings $\alpha_1$, $\alpha_2$ and $\alpha_3$, is shown from $\mu = M_Z$ up to high energies. At this scale no difference between one, two and three loops is visible, all curves lie on top of each other.

The differences between the loop orders can be seen in Fig. 4 which magnifies the intersection point between $\alpha_1$ and $\alpha_2$. There is a clear jump between the one- (dotted) and two-loop (dashed) prediction. The difference between two and three loops (solid curves) is significantly smaller which implies that perturbation theory converges very well.

The experimental uncertainties for $\alpha_1(M_Z)$ and $\alpha_2(M_Z)$ as given in Eq. (31) are reflected by the bands around the three-loop results. Defining the difference between the two- and three-loop result as theoretical uncertainty one observes that it is smaller than the experimental one, however, of the same order of magnitude. Without the new three-loop calculation performed in this paper the theory uncertainty is much larger than the experimental one.

Also in the case of $\alpha_3$ perturbation theory seems to converge well. However, in contrast to $\alpha_1$ and $\alpha_2$ the experimental uncertainty turns out to be much larger than the theoretical uncertainty. This is not surprising as the relative experimental uncertainty of $\alpha_3$ at the electroweak scale is quite large compared to its electroweak counterparts. The relative experimental and theoretical uncertainty of $\alpha_3$ is plotted as a function of the renormalization
FIG. 4. The running of the electroweak gauge couplings in the SM. The lines with positive slope correspond to $\alpha_1$, the lines with negative slope to $\alpha_2$. The dotted, dashed and solid lines correspond to one-, two- and three-loop precision, respectively. The bands around the three-loop curves visualize the experimental uncertainty.

scale in Fig. 5. Note that by construction we have that $\Delta\alpha_3/\alpha_3|_{\text{theory}}$ approaches zero for $\mu \to M_Z$.

Let us finally identify the numerically most important contributions. This is done by running from $\mu = M_Z$ to $\mu = 10^{16}$ GeV and by comparing the contribution of each individual term to the total difference between the two- and three-loop prediction. Similar results are also obtained for lower scales.

About 90% of the three-loop corrections to the running of the gauge couplings arises from only a few terms. In the case of $\alpha_1$ there is only one term which dominates, namely the one of order $\alpha_1^2\alpha_3^2$. For $\alpha_2$ one has a contribution of +56% from the $\mathcal{O}(\alpha_2^2\alpha_3^3)$ term, +13% from order $\alpha_3^3\alpha_3$ and +37% from $\mathcal{O}(\alpha_4^3)$. All other terms contribute at most 5% and partly also cancel each other. Except for the term of $\mathcal{O}(\alpha_4^2)$ all these terms are presented in this paper for the first time.

The beta function $\beta_3$ is dominated by the strong corrections, however, large cancellations between the $\alpha_3^4$ (+137%), $\alpha_3^3\alpha_2$ (+45%), $\alpha_3^2\alpha_1^2$ (+28%) and $\alpha_3^2\alpha_2^2$ (+17%) terms on the one hand and the $\alpha_3^4\alpha_4$ ($-112\%$) and $\alpha_3^2\alpha_2\alpha_4$ ($-16\%$) on the other hand are observed. It is worth noting that the four-loop term of order $\alpha_5^4$ amounts to $58\%$ of the difference between the two- and three-loop predictions. This number has been obtained by adding the four-loop QCD term [17, 18] to $\beta_3$. 
VI. CONCLUSIONS

In this paper we present three-loop results for renormalization constants that are used to compute the three SM gauge coupling beta functions, taking into account all contributions. We have checked that our expressions agree with all partial results present in the literature. Furthermore the two-loop corrections to the Yukawa couplings have been computed. We have performed the calculation using both Lorenz gauge within the unbroken phase of the SM and BFG in the broken phase. Our final result is valid for a generic flavour structure with an arbitrary CKM matrix. It is furthermore sufficiently general to consider a fourth generation of quarks and leptons.

In order to perform the calculation in an automated way we have written an interface, FeynArtsToQ2E, between the package FeynArts and our chain of programs (QGRAF, q2e, exp, MATAD, MINCER) allowing to handle the $O(10^6)$ Feynman diagrams, which have to be considered in the course of the calculation, in an effective way. Thus, we could perform several checks involving various different Green’s functions. FeynArtsToQ2E is not limited to the SM but can easily be used for extensions like supersymmetric models.

ACKNOWLEDGMENTS

We thank David Kunz for tests concerning FeynArtsToQ2E and Mikolaj Misiak for suggesting to generalize the Yukawa structure of the final result. We thank Esben Mølgaard for communications concerning Eq. (29). This work is supported by DFG through SFB/TR 9.
“Computational Particle Physics”.

Appendix A: Renormalization constants

In this Appendix we present analytical results for the renormalization constants $Z_{\alpha_1}, Z_{\alpha_2}, Z_{\alpha_3}, Z_B, Z_W, Z_G, Z_H, Z_{cW}, Z_{cG}, Z_{WWW}, Z_{GGG}, Z_{HHH}$, where the definition of $Z_{\alpha_i}$ is given in Eq. (5) and the field renormalization is defined through

$$
B_{\text{bare}} = Z_B B, \quad W_{\text{bare}} = Z_W W, \quad G_{\text{bare}} = Z_G G, \quad c_{W,\text{bare}} = Z_{cW} c_W, \quad c_{G,\text{bare}} = Z_{cG} c_G, \quad H_{\text{bare}} = Z_H H. \quad (A1)
$$

$B, W, G, c_W$ and $c_G$ denote the gauge boson and ghost fields. The scalar field $H$ is defined in Eq. (15). The renormalization constants for the three-particle vertices are also defined in a multiplicative way.

Some of the results listed below contain the gauge parameters $\xi_B, \xi_W$ and $\xi_G$. They are conveniently defined via the corresponding gauge boson propagator which is given by

$$
D_{X}^{\mu\nu}(q) = \frac{i}{q^2} \left( g^{\mu\nu} + (1 - \xi_X) \frac{g^{\mu\nu}q^2}{q^2} \right), \quad (A2)
$$

with $X = B, W, G$. Note that $\xi_X = 1$ corresponds to Feynman and $\xi_X = 0$ to Landau gauge. Our analytical results read

$$
Z_{\alpha_1} = 1 + \frac{\alpha_1}{4\pi} \left\{ \frac{1}{10} + \frac{4n_G \alpha_1}{3} + \frac{\alpha_1}{(4\pi)^3} \left\{ \frac{1}{100} + \frac{4n_G \alpha_1}{15} + \frac{16n_G^2 \alpha_1}{9} \right\} \right. \\
+ \frac{\alpha_1}{(4\pi)^3} \left\{ \frac{1}{100} + \frac{8n_G^2 \alpha_1^2}{15} + \frac{64n_G^3 \alpha_1^3}{27} \right\} \\
+ \frac{1}{\epsilon^2} \left[ \frac{21\alpha^2_1}{1000} + \frac{9\alpha_1 \alpha_2}{20} - \frac{43\alpha_2^2}{40} + \frac{17\alpha_1 \alpha_3}{20} + \frac{17\alpha_1 \alpha_3}{20} + \frac{51\alpha_2 \alpha_3}{80} + \frac{34\alpha_3 \alpha_3}{15} - \frac{7\alpha_1 \alpha_3}{240} \right. \\
+ \frac{3\alpha_2 \alpha_3 \dot{B}}{16} + \frac{2\alpha_3 \alpha_3 \dot{B}}{3} + \frac{33\alpha_1 \alpha_3 \dot{L}}{80} - \frac{9\alpha_2 \alpha_3 \dot{L}}{16} - \frac{\left( \alpha_3 \alpha_3 \dot{B} \right)^2}{4} - \frac{5\alpha_3 \alpha_3 \dot{B} \dot{L}}{6} \\
- \frac{3\alpha_2 \alpha_3 \dot{B}}{8} - \frac{\left( \dot{L} \right)^2}{4} + \frac{11\dot{B} \dot{B}}{20} - \frac{17\dot{L} \dot{B}}{40} - \frac{11\dot{B} \dot{B}}{10} - \frac{31\dot{L} \dot{B}}{30} - \frac{17(\dot{L} \dot{B})^2}{20} \\
+ \frac{n_G \left( \frac{77\alpha^2_1}{180} + \frac{63\alpha_1 \alpha_2}{50} - \frac{31\alpha^2_2}{60} - \frac{22\alpha_1 \alpha_3}{75} - \frac{242\alpha^2_3}{45} - \frac{34\alpha_1 \alpha_3}{15} - \frac{2\alpha_3 \alpha_3 \dot{B}}{3} - \frac{2\alpha_1 \alpha_3 \dot{L}}{2} \right)}{15} \\
+ \frac{n_G^2 \left( \frac{266\alpha^2_1}{15} + \frac{4\alpha_1 \alpha_2}{5} + \frac{2\alpha^2_2}{15} + \frac{176\alpha_1 \alpha_3}{45} + \frac{88\alpha^2_3}{135} \right)}{135} \left. \right\} \right\}
$$

$$
+ \frac{1}{\epsilon} \left[ \frac{163\alpha^2_1}{800} + \frac{261\alpha_1 \alpha_2}{800} + \frac{3401\alpha^2_2}{960} + \frac{9\alpha_1 \lambda}{50} + \frac{3\alpha_2 \lambda}{10} - \frac{3\lambda^2}{5} - \frac{2827\alpha_1 \dot{L}}{2400} - \frac{157\alpha_2 \dot{L}}{32} \right. \\
- \frac{29\dot{B} \dot{B}}{15} - \frac{1267\dot{B} \dot{B}}{2400} - \frac{437\dot{B} \dot{B}}{160} - \frac{17\dot{B} \dot{B}}{15} - \frac{843\alpha_1 \dot{L}}{800} - \frac{543\alpha_2 \dot{L}}{160} \right. \left. \right\}
$$

where the definition of $\xi_X = 1$ corresponds to Feynman and $\xi_X = 0$ to Landau gauge. Our analytical results read
Z_{\alpha_2} = 1 + \frac{\alpha_2}{4\pi} \left\{ -\frac{43}{6} + \frac{4n_G}{3} \right\} + \frac{\alpha_2}{20} \left\{ \frac{1}{2}\left[\frac{1849\alpha_2}{36} - \frac{172n_G\alpha_2}{9} + \frac{16n_G^2\alpha_2}{9} \right] \right\} \\
+ \frac{1}{\epsilon} \left\{ \frac{3\alpha_1}{20} - \frac{259\alpha_2}{12} - \frac{3r\hat{T}}{4} - \frac{3\hat{B}}{4} - \frac{\hat{L}}{4} + n_G \left( \frac{\alpha_1}{10} + \frac{49\alpha_2}{6} + 2\alpha_3 \right) \right\} \\
+ \frac{\alpha_2}{(4\pi)^2} \left\{ \frac{1}{\epsilon^2} \left[ -\frac{79507\alpha_2}{216} + \frac{1849n_G\alpha_2}{9} - \frac{344n_G^2\alpha_2}{9} + \frac{64n_G^3\alpha_2}{27} \right] \right\} \\
+ \frac{1}{\epsilon^2} \left\{ \frac{\alpha_1}{200} - \frac{43\alpha_1\alpha_2}{20} + \frac{7795\alpha_2^2}{216} + \frac{17\alpha_1\hat{T}}{80} + \frac{181\alpha_2\hat{T}}{16} + \frac{2\alpha_3\hat{T}}{16} + \frac{\alpha_1\hat{B}}{16} \\
+ \frac{181\alpha_2\hat{B}}{16} + \frac{2\alpha_3\hat{B}}{16} + \frac{3\alpha_1\hat{L}}{8} + \frac{181\alpha_2\hat{L}}{48} - \frac{3\hat{B}^2}{8} - \frac{3(\hat{T})^2}{4} - \frac{\hat{B}\hat{L}}{2} \right\} \\
+ \frac{(\hat{L})^2}{12} - \frac{3\hat{L}\hat{T}}{8} - \frac{3\hat{T}\hat{B}}{2} - \frac{\hat{T}\hat{L}}{2} - \frac{3(\hat{T})^2}{4} - \frac{\hat{L}^2}{8} \\
+ n_G \left( \frac{2\alpha_1^2}{100} - \frac{31\alpha_1\alpha_2}{30} - \frac{22001\alpha_2^2}{108} - \frac{86\alpha_2\alpha_3}{3} - \frac{22\alpha_3^2}{3} - \frac{2\alpha_2\hat{T}}{3} - \frac{2\alpha_2\hat{B}}{3} - \frac{2\alpha_2\hat{L}}{3} \right) \\
+ n_G^2 \left( \frac{2\alpha_1^2}{45} + \frac{4\alpha_1\alpha_2}{15} + \frac{68\alpha_2^2}{27} + \frac{16\alpha_2\alpha_3}{3} + \frac{8\alpha_3^2}{9} \right) \right\} \\
+ \frac{1}{\epsilon} \left\{ \frac{163\alpha_1^2}{4800} + \frac{187\alpha_2^2}{160} - \frac{6671\alpha_2^2}{5184} + \frac{\alpha_1\hat{\lambda}}{10} + \frac{\alpha_2\hat{\lambda}}{2} - \hat{\lambda}^2 - \frac{593\alpha_1\hat{\hat{T}}}{480} - \frac{243\alpha_2\hat{\hat{T}}}{32} \right\} \\
+ \frac{7\alpha_3\hat{T}}{3} - \frac{533\alpha_3\hat{B}}{480} - \frac{243\alpha_2\hat{B}}{32} - \frac{7\alpha_3\hat{\hat{L}}}{3} - \frac{17\alpha_1\hat{\hat{L}}}{17} - \frac{8\alpha_2\hat{\hat{L}}}{8} - \frac{19\hat{\hat{B}}^2}{16} \\
+ \frac{15(\hat{B})^2}{8} + \frac{5\hat{B}\hat{L}}{4} + \frac{19\hat{\hat{L}}^2}{4} + \frac{5(\hat{\hat{L}})^2}{24} + \frac{9\hat{\hat{T}}\hat{\hat{B}}}{8} + \frac{19\hat{\hat{T}}^2}{16} + \frac{15\hat{\hat{T}}\hat{\hat{B}}}{4} \\
+ \frac{5\hat{\hat{T}}\hat{\hat{L}}}{4} + \frac{15(\hat{\hat{T}})^2}{8} + n_G \left( -\frac{7\alpha_1^2}{135} + \frac{13\alpha_1\alpha_2}{81} + \frac{6412\alpha_2^2}{81} - \frac{\alpha_1\alpha_3}{45} + \frac{13\alpha_2\alpha_3}{3} + \frac{125\alpha_3^2}{9} \right) \\
+ n_G^2 \left( -\frac{11\alpha_1^2}{135} - \frac{415\alpha_2^2}{81} - \frac{44\alpha_3^2}{27} \right) \right\} , \tag{A4}
\]

Z_{\alpha_3} = 1 + \frac{\alpha_3}{4\pi} \left\{ -11 + \frac{4n_G}{3} \right\} + \frac{\alpha_3}{(4\pi)^2} \left\{ \frac{1}{\epsilon^2} \left[ \frac{121\alpha_3}{3} - \frac{88n_G\alpha_3}{3} + \frac{16n_G^2\alpha_3}{9} \right] \right\} \\
+ \frac{1}{\epsilon} \left\{ -5\alpha_3 - \hat{T} - \hat{\hat{B}} + n_G \left( \frac{11\alpha_1}{60} + \frac{3\alpha_2}{4} + \frac{38\alpha_3}{3} \right) \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\}.
\[
Z_B = 1 + \frac{\alpha_3}{(4\pi)^3} \left\{ \frac{1}{e^3} \left[ -1331\alpha_3^2 + 484n_G\alpha_3^2 - \frac{176n_G^2\alpha_3^2}{3} + \frac{64n_G^3\alpha_3^2}{27} \right] \right.
\]
\[
+ \frac{1}{e^2} \left[ 1309\alpha_3^2 + \frac{17\alpha_3\hat{T}}{60} + \frac{3\alpha_2\hat{T}}{4} + \frac{74\alpha_3\hat{T}}{3} \right. + \frac{\alpha_1\hat{B}}{12} + \frac{3\alpha_2\hat{B}}{4} + \frac{74\alpha_3\hat{B}}{3}
\]
\[
- \frac{\hat{B}^2}{2} - (\hat{B})^2 - \frac{\hat{B}\hat{T}\hat{L}}{3} + \hat{T}\hat{B} - \frac{\hat{T}^2}{2} - 2\hat{T}\hat{B} - \frac{\hat{T}\hat{L}}{3} - (\hat{T})^2
\]
\[
+ n_G \left[ \frac{11\alpha_3^2}{1800} - \frac{43\alpha_3^2}{24} - \frac{33\alpha_3\alpha_2}{30} - \frac{4354\alpha_3^2}{9} \right.
\]
\[
- \frac{8\alpha_3\hat{T}}{3} - \frac{8\alpha_3\hat{B}}{3}
\]
\[
+ n_G \left[ \frac{11\alpha_3^2}{135} + \frac{22\alpha_3}{45} + \frac{1064\alpha_3^2}{27} \right]
\]
\[
+ \frac{1}{\epsilon} \left[ -\frac{2857\alpha_3^3}{6} + \frac{101\alpha_1\hat{T}}{120} - \frac{31\alpha_2\hat{T}}{8} - \frac{40\alpha_3\hat{T}}{3} - \frac{89\alpha_1\hat{B}}{120} - \frac{31\alpha_2\hat{B}}{8} - \frac{40\alpha_3\hat{B}}{3} \right]
\]
\[
+ \frac{3\hat{B}^2}{2} + \frac{7(\hat{B})^2}{2} + \frac{7\hat{B}\hat{L}}{6} - \hat{T}\hat{B} + \frac{3\hat{T}^2}{2} + \frac{7\hat{T}\hat{B}}{6} + \frac{7(\hat{T})^2}{2}
\]
\[
+ n_G \left[ -\frac{13\alpha_1^2}{360} - \frac{241\alpha_2^2}{72} + \frac{77\alpha_1\alpha_3}{135} + \frac{7\alpha_2\alpha_3}{3} + \frac{5033\alpha_3^2}{27} \right]
\]
\[
+ n_G \left[ -\frac{121\alpha_3^2}{810} - \frac{650\alpha_3^2}{81} \right] \right\}, \quad (A5)
\]
\[ + n_G^2 \left( \frac{209\alpha_1^2}{405} + \frac{11\alpha_2^2}{45} + \frac{484\alpha_3^2}{405} \right) \right\}, \]

\[ Z_W = 1 + \frac{\alpha_2}{4\pi} \left\{ \frac{25}{6} - \frac{4n_G}{3} - \xi_W \right\} + \frac{\alpha_2}{(4\pi)^2} \left\{ \frac{1}{\epsilon^2} \left[ - \frac{25\alpha_2}{4} - \frac{8\xi_W\alpha_2}{3} + \xi_W^2 \alpha_2 + n_G \left( 2\alpha_2 + \frac{4\xi_W \alpha_2}{3} \right) \right] \right\}

\[ + \frac{1}{\epsilon} \left[ - \frac{3\alpha_1}{20} + \frac{113\alpha_2}{8} + \frac{3\text{tr} \hat{T}}{4} + \frac{3\text{tr} \hat{B}}{4} + \frac{\text{tr} \hat{L}}{4} - \frac{11\xi_W \alpha_2}{4} - \frac{\xi_W^2 \alpha_2}{2} \right]

\[ + n_G \left( - \frac{\alpha_1}{10} - \frac{13\alpha_2}{2} - 2\alpha_3 \right) \right\} + \frac{\alpha_2}{(4\pi)^4} \left\{ \frac{1}{\epsilon^3} \left[ \frac{1525\alpha_2^2}{72} + \frac{8\xi_W \alpha_2^2}{9} + \frac{89\xi_W \alpha_2^2}{12} + \frac{7\xi_W \alpha_2^2}{6} - \xi_W \alpha_2 \right] \right\}

\[ + n_G \left( - \frac{86\alpha_2^2}{9} - \frac{10\xi_W \alpha_2^2}{9} - \frac{4\xi_W \alpha_2^2}{9} \right) \right\}

\[ + \frac{1}{\epsilon^2} \left[ - \frac{\alpha_1^2}{200} + \frac{3\alpha_1 \alpha_2}{20} - \frac{29629\alpha_2^2}{432} - \frac{17\alpha_1 \text{tr} \hat{T}}{80} - \frac{21\alpha_2 \text{tr} \hat{T}}{16} - 2\alpha_3 \text{tr} \hat{T} \right]

\[ - \frac{\alpha_1 \text{tr} \hat{B}}{16} - \frac{2\alpha_2 \text{tr} \hat{B}}{16} - 2\alpha_3 \text{tr} \hat{B} - \frac{3\alpha_1 \text{tr} \hat{L}}{16} - \frac{7\alpha_2 \text{tr} \hat{L}}{16} - \frac{3\text{tr} \hat{B}^2}{8} + \frac{3(\text{tr} \hat{B})^2}{4} \right]

\[ + \frac{\text{tr} \hat{B} \text{tr} \hat{L}}{2} + \frac{\text{tr} \hat{B}^2}{8} + \frac{(\text{tr} \hat{L})^2}{12} - \frac{3\text{tr} \hat{B} \text{tr} \hat{L}}{4} - \frac{3\text{tr} \hat{B}^2}{8} + \frac{3\text{tr} \hat{L}^2}{2} + \frac{3(\text{tr} \hat{L})^2}{4} \right]

\[ + \xi_W \left( \frac{3\alpha_1 \alpha_2}{20} - \frac{271\alpha_2^2}{24} - \frac{3\alpha_2 \text{tr} \hat{T}}{4} - \frac{3\alpha_2 \text{tr} \hat{B}}{4} - \frac{\alpha_2 \text{tr} \hat{L}}{4} \right) + \frac{53\xi_W \alpha_2^2}{12} + \frac{7\xi_W \alpha_2^2}{6} \right]

\[ + n_G \left( - \frac{7\alpha_1}{100} + \frac{\alpha_1 \alpha_2}{10} + \frac{4273\alpha_2^2}{108} + 2\alpha_2 \alpha_3 + \frac{22\alpha_2^3}{3} \right) + \xi_W \left( \frac{\alpha_1 \alpha_2}{10} + \frac{47\alpha_2^2}{6} + 2\alpha_2 \alpha_3 \right) + \frac{2\xi_W \alpha_2^2}{3} \right] + n_G^2 \left( - \frac{2\alpha_1^3}{45} - \frac{118\alpha_2^3}{27} - \frac{8\alpha_3^3}{9} \right) \right]\n
\[ + \frac{1}{\epsilon} \left[ - \frac{163\alpha_1^2}{4800} - \frac{11\alpha_1 \alpha_2}{32} - \frac{3\zeta_3 \alpha_1 \alpha_2}{10} + \frac{14353\alpha_2^2}{1728} + \frac{\zeta_3 \alpha_2^2}{2} - \frac{\alpha_1 \lambda}{10} - \frac{\alpha_2 \lambda}{2} + \hat{\lambda}^2 \right]

\[ + \frac{593\alpha_1 \text{tr} \hat{T}}{480} + \frac{79\alpha_2 \text{tr} \hat{T}}{32} + \frac{7\alpha_3 \text{tr} \hat{T}}{3} + \frac{533\alpha_1 \text{tr} \hat{B}}{480} + \frac{79\alpha_2 \text{tr} \hat{B}}{32} + \frac{7\alpha_3 \text{tr} \hat{B}}{3} + \frac{17\alpha_1 \text{tr} \hat{L}}{32} \right]

\[ + \frac{79\alpha_2 \text{tr} \hat{L}}{96} + \frac{19\text{tr} \hat{B}^2}{16} + \frac{15(\text{tr} \hat{B})^2}{4} + \frac{5\text{tr} \hat{B} \text{tr} \hat{L}}{4} - \frac{19\text{tr} \hat{L}^2}{48} - \frac{5(\text{tr} \hat{L})^2}{24} + \frac{9\text{tr} \hat{B} \text{tr} \hat{L}}{8} \right]

\[ + \frac{19\text{tr} \hat{T}^2}{8} - \frac{15\text{tr} \hat{T} \text{tr} \hat{B}}{4} - \frac{5\text{tr} \hat{T} \text{tr} \hat{L}}{4} - \frac{15(\text{tr} \hat{T})^2}{8} \right]

\[ + \xi_W \left( - \frac{105\alpha_2^2}{8} - 2\zeta_3 \alpha_2^2 \right) + \xi_W \left( - \frac{11\alpha_2^2}{4} - \frac{\zeta_3 \alpha_2^2}{2} \right) - \frac{7\xi_W \alpha_2^2}{12} \right]

\[ + n_G \left( \frac{7\alpha_1^2}{45} + \frac{8\alpha_1 \alpha_2}{15} - \frac{4\zeta_3 \alpha_1 \alpha_2}{5} - \frac{7025\alpha_2^2}{108} + 12\zeta_3 \alpha_2^2 + \frac{\alpha_1 \alpha_3}{45} + \frac{32\alpha_2 \alpha_3}{3} \right) \right]

\[ - 16\zeta_3 \alpha_2 \alpha_3 - \frac{125\alpha_3^2}{9} + \frac{8\xi_W \alpha_2^2}{3} \right] + n_G \left( \frac{11\alpha_1^2}{135} + \frac{185\alpha_2^2}{27} + \frac{44\alpha_3^2}{27} \right) \right\}, \]
\[
Z_G = 1 + \frac{\alpha_3}{4\pi\epsilon} \left\{ \frac{13}{2} - \frac{4n_G}{3} - \frac{3\xi_G}{2} \right\} \\
+ \frac{\alpha_3}{(4\pi)^2} \left\{ \frac{1}{\epsilon^2} \left[ -\frac{117\alpha_3}{8} - \frac{51\xi_G\alpha_3}{8} + \frac{9\xi_G^2\alpha_3}{4} + n_G \left( 3\alpha_3 + 2\xi_G\alpha_3 \right) \right] \\
+ \frac{1}{\epsilon} \left[ + \frac{531\alpha_3}{16} + \text{tr}\hat{T} + \text{tr}\hat{B} - \frac{99\xi_G\alpha_3}{16} - \frac{9\xi_G^2\alpha_3}{8} + n_G \left( -\frac{11\alpha_3}{60} - \frac{3\alpha_2}{4} - \frac{61\alpha_3}{6} \right) \right] \\
+ \frac{\alpha_3}{(4\pi)^2} \left\{ \frac{1}{\epsilon^3} \left[ -\frac{1209\alpha_3^2}{16} + \frac{4n_G\alpha_3^2}{3} + \frac{423\xi_G\alpha_3^2}{16} + \frac{9\xi_G^2\alpha_3^2}{2} - \frac{27\xi_G^3\alpha_3^2}{8} \\
+ n_G \left( -\frac{22\alpha_3^2}{2} - \frac{15\xi_G\alpha_3^2}{2} - \frac{3\xi_G^2\alpha_3^2}{2} \right) \right] \\
+ \frac{1}{\epsilon^2} \left[ -\frac{7957\alpha_3^2}{32} - \frac{17\alpha_1\text{tr}\hat{T}}{60} - \frac{3\alpha_2\text{tr}\hat{T}}{4} - \frac{25\alpha_3\text{tr}\hat{T}}{6} - \frac{\alpha_1\text{tr}\hat{B}}{12} - \frac{3\alpha_2\text{tr}\hat{B}}{4} - \frac{25\alpha_3\text{tr}\hat{B}}{6} \\
+ \frac{\text{tr}\hat{B}^2}{2} + (\text{tr}\hat{B})^2 + \frac{\text{tr}\hat{B}\text{tr}\hat{L}}{3} - \text{tr}\hat{B}\hat{B} + \frac{\text{tr}\hat{T}^2}{2} + 2\text{tr}\hat{T}\text{tr}\hat{B} + \frac{\text{tr}\hat{T}\text{tr}\hat{L}}{3} + (\text{tr}\hat{T})^2 \\
+ \xi_G \left( -\frac{1287\alpha_3^2}{32} - \frac{3\alpha_3\text{tr}\hat{T}}{2} - \frac{3\alpha_3\text{tr}\hat{B}}{2} \right) + \frac{117\xi_G\alpha_3^2}{8} + \frac{63\xi_G^2\alpha_3^2}{16} \\
+ n_G \left( -\frac{11\alpha_1^2}{1800} + \frac{43\alpha_2^2}{24} + \frac{11\alpha_1\alpha_3}{40} + \frac{9\alpha_2\alpha_3}{8} + \frac{169\alpha_3^2}{18} \\
+ \xi_G \left( \frac{11\alpha_1\alpha_3}{40} + \frac{9\alpha_2\alpha_3}{8} + \frac{73\alpha_3^2}{4} \right) + \frac{3\xi_G^2\alpha_3^2}{2} \right) + n_G^2 \left( -\frac{11\alpha_1^2}{135} - \frac{\alpha_2^2}{3} - \frac{182\alpha_3^2}{27} \right) \\
+ \frac{1}{\epsilon} \left[ \frac{9965\alpha_3^2}{32} - \frac{81\xi_G\alpha_3^2}{16} + \frac{101\alpha_1\text{tr}\hat{T}}{120} + \frac{31\alpha_2\text{tr}\hat{T}}{8} + \frac{91\alpha_3\text{tr}\hat{T}}{12} + \frac{89\alpha_1\text{tr}\hat{B}}{120} + \frac{31\alpha_2\text{tr}\hat{B}}{8} \\
+ \frac{91\alpha_3\text{tr}\hat{B}}{12} - \frac{3\alpha_3\text{tr}\hat{B}^2}{2} - 7(\text{tr}\hat{B})^2 - 7\text{tr}\hat{B}\text{tr}\hat{L}}{6} + \text{tr}\hat{B}\hat{B} - \frac{3\text{tr}\hat{T}^2}{2} - 7\text{tr}\hat{T}\text{tr}\hat{B} - \frac{7\text{tr}\hat{T}\text{tr}\hat{L}}{6} \\
- \frac{7(\text{tr}\hat{T})^2}{2} + \xi_G \left( -\frac{1503\alpha_3^2}{32} + \frac{87\alpha_3^2}{4} \right) + \xi_G^2 \left( -\frac{297\alpha_3^2}{32} - \frac{27\xi_G\alpha_3^2}{16} \right) - \frac{63\xi_G^2\alpha_3^2}{32} \\
+ n_G \left( \frac{13\alpha_1^2}{360} + \frac{\alpha_1\alpha_2}{120} - \frac{41\alpha_2^2}{72} + \frac{3223\alpha_1\alpha_3}{2160} - \frac{11\xi_G\alpha_1\alpha_3}{5} + \frac{293\alpha_2\alpha_3}{48} - 9\xi_G\alpha_2\alpha_3 \\
- \frac{8155\alpha_3^2}{54} + 22\xi_G\alpha_3^2 + 6\xi_G^2\alpha_3^2 \right) + n_G^2 \left( \frac{121\alpha_1^2}{810} + \frac{11\alpha_2^2}{18} + \frac{860\alpha_3^2}{81} \right) \right\}, \tag{A8}
\]

\[
Z_{cw} = 1 + \frac{\alpha_2}{4\pi\epsilon} \left\{ \frac{3}{2} - \frac{\xi_{W}}{2} \right\} \\
+ \frac{\alpha_2}{(4\pi)^2} \left\{ \frac{1}{\epsilon^2} \left[ -\frac{17\alpha_2}{4} + n_G\alpha_2 + \frac{3\xi_{W}\alpha_2}{8} \right] + \frac{1}{\epsilon} \left[ \frac{179\alpha_2}{48} - \frac{5n_G\alpha_2}{6} + \xi_{W}\alpha_2 \right] \right\} \\
+ \frac{\alpha_2}{(4\pi)^2} \left\{ \frac{1}{\epsilon^3} \left[ \frac{1309\alpha_2^2}{72} + \frac{8n_G^2\alpha_2^2}{9} + \frac{17\xi_{W}\alpha_2^2}{24} - \frac{9\xi_{W}^2\alpha_2^2}{16} - \frac{5\xi_{W}\alpha_2^2}{16} \right] \\
+ n_G \left( -\frac{145\alpha_2^2}{18} - \frac{\xi_{W}\alpha_2^2}{6} \right) \right\} + \frac{1}{\epsilon^2} \left[ \frac{3\alpha_1\alpha_2}{20} - \frac{29209\alpha_2^2}{864} - \frac{20n_G^2\alpha_2^2}{27} - \frac{3\alpha_2\text{tr}\hat{T}}{4} - \frac{3\alpha_2\text{tr}\hat{B}}{4} \right].
\]
\[ Z_{cc} = 1 + \frac{\alpha_3}{4\pi} \{ \frac{9}{4} - \frac{3\xi G}{4} \} \]
\[ + \frac{\alpha_3}{(4\pi)^2} \{ \frac{1}{\epsilon^2} \left[ \frac{315}{32} + \frac{3n_G a_3}{2} + \frac{27\xi_G a_3}{32} \right] + \frac{1}{\epsilon} \left[ \frac{285}{32} - \frac{5n_G a_3}{4} + \frac{9\xi_G a_3}{32} \right] \} \]
\[ + \frac{\alpha_3}{(4\pi)^3} \{ \frac{1}{\epsilon^3} \left[ \frac{8295}{128} + \frac{4n_G^2 a_3^2}{3} + \frac{315\xi_G a_3^2}{128} - \frac{243\xi_G^2 a_3^2}{128} - \frac{135\xi_G^3 a_3^2}{128} \right] \}
\[ + n_G \left( -\frac{149a_3^2}{8} + \frac{3\xi_G a_3^2}{8} \right) \]
\[ + \frac{1}{\epsilon} \left[ - \frac{93\alpha_1^2}{1600} - \frac{27\alpha_1\alpha_2}{160} + \frac{51\alpha_2^2}{64} - 3\lambda^2 - \frac{17\alpha_1 \hat{T} \hat{T}}{16} - \frac{45\alpha_2 \hat{T} \hat{T}}{16} - \frac{10\alpha_3 \hat{B} \hat{B}}{16} \right] \\
\[ - \frac{45\alpha_2 \hat{B} \hat{B}}{16} - \frac{10\alpha_3 \hat{B} \hat{B}}{16} - \frac{15\alpha_1 \hat{L} \hat{L}}{16} - \frac{15\alpha_2 \hat{L} \hat{L}}{16} + \frac{27\hat{B}^2}{8} + \frac{9\hat{L}^2}{8} - \frac{3\hat{T} \hat{T} \hat{B}}{4} + \frac{27\hat{T} \hat{T} \hat{B}^2}{8} \]
\[ - \frac{3\xi_{W}\alpha_2^2}{2} - \frac{3\xi_{W}\alpha_2^2}{2} + n_G \left( - \frac{\alpha_1^2}{4} - \frac{5\alpha_2^2}{4} \right) \right] \]
\[ + \frac{1}{(4\pi)^3} \left[ \frac{1}{c^3} \left[ \frac{429\alpha_1^3}{16000} + \frac{891\alpha_1^2\alpha_2}{3200} - \frac{159\alpha_1\alpha_2^2}{640} + \frac{855\alpha_2^3}{384} - \frac{93\alpha_1^2 \hat{T} \hat{T}}{800} - \frac{9\alpha_1\alpha_2 \hat{T} \hat{T}}{16} \right] \right] \\
\[ + \frac{435\alpha_2^3 \hat{T} \hat{T}}{32} - \frac{7\alpha_1 \alpha_2 \hat{T} \hat{T}}{5} + \frac{9\alpha_2 \alpha_3 \hat{B} \hat{B}}{800} - \frac{177\alpha_2^2 \hat{T} \hat{T}}{80} - \frac{9\alpha_1 \alpha_2 \hat{T} \hat{T}}{80} \\
\[ + \frac{435\alpha_2^3 \hat{B} \hat{B}}{32} + \frac{17\alpha_1 \alpha_2 \hat{B} \hat{B}}{5} - \frac{9\alpha_2 \alpha_3 \hat{B} \hat{B}}{800} + \frac{27\alpha_1 \alpha_2 \hat{B} \hat{B}}{80} \\
\[ + \frac{145\alpha_2^3 \hat{L} \hat{L}}{32} + \frac{9\alpha_1 \alpha_2 \hat{L} \hat{L}}{5} - \frac{9\alpha_2 \alpha_3 \hat{L} \hat{L}}{800} + \frac{27\alpha_1 \alpha_2 \hat{L} \hat{L}}{80} \\
\[ - \frac{3\alpha_1 \alpha_2 \hat{T} \hat{T}}{4} - \frac{20\alpha_1 \alpha_2 \hat{B} \hat{B}}{4} - \frac{20\alpha_2 \alpha_3 \hat{B} \hat{B}}{4} + \frac{27\alpha_1 \alpha_2 \hat{B} \hat{B}}{4} \\
\[ - \frac{9\alpha_1 \alpha_3 \hat{T} \hat{T}}{20} + \frac{36\alpha_2 \alpha_3 \hat{T} \hat{T}}{20} - \frac{9\alpha_1 \alpha_3 \hat{T} \hat{T}}{2} + \frac{9\alpha_2 \alpha_3 \hat{T} \hat{T}}{2} + \frac{9\alpha_3 \alpha_2 \hat{T} \hat{B} \hat{T}}{4} \\
\[ + \frac{9\alpha_1 \alpha_3 \hat{T} \hat{T}}{10} + \frac{18\alpha_2 \alpha_3 \hat{T} \hat{T}}{4} - \frac{9\alpha_1 \alpha_3 \hat{T} \hat{T}}{2} - \frac{9\alpha_2 \alpha_3 \hat{T} \hat{T}}{2} + \frac{9\alpha_3 \alpha_2 \hat{T} \hat{B} \hat{T}}{4} \\
\[ + \frac{\xi_B \left( \frac{297\alpha_1^3}{16000} - \frac{243\alpha_1^2\alpha_2}{1600} - \frac{531\alpha_1\alpha_2^2}{640} - \frac{63\xi_{W}\alpha_1\alpha_2^2}{640} - \frac{81\alpha_1\alpha_2 \hat{T} \hat{T}}{800} + \frac{81\alpha_2 \alpha_3 \hat{T} \hat{T}}{160} \right) \right] \]
\[ + \frac{9\alpha_1 \alpha_3 \hat{T} \hat{T}}{5} + \frac{117\alpha_2^3 \hat{T} \hat{T}}{800} - \frac{81\alpha_1 \alpha_2 \hat{B} \hat{B}}{160} - \frac{9\alpha_2 \alpha_3 \hat{B} \hat{B}}{800} + \frac{27\alpha_1 \alpha_2 \hat{B} \hat{B}}{160} \\
\[ + \frac{27\alpha_1 \alpha_2 \hat{B}^2}{80} + \frac{9\alpha_1 \alpha_2 \hat{L} \hat{L}}{80} - \frac{27\alpha_1 \alpha_2 \hat{B} \hat{B}}{80} - \frac{9\alpha_1 \alpha_2 \hat{L} \hat{L}}{80} - \frac{9\alpha_1 \alpha_2 \hat{B} \hat{T} \hat{T}}{80} \\
\[ + \frac{\xi_W \left( \frac{81\alpha_1^2\alpha_2}{1600} + \frac{27\alpha_1^2\alpha_2}{320} - \frac{27\alpha_1 \alpha_2 \hat{T} \hat{T}}{80} - \frac{27\alpha_2 \alpha_2 \hat{B} \hat{B}}{80} - \frac{9\alpha_1 \alpha_2 \hat{L} \hat{L}}{80} \right) \right] \]
\[ + \frac{\xi_W \left( \frac{297\alpha_1^3}{3200} - \frac{81\alpha_1^2\alpha_2}{320} - \frac{363\alpha_1^2\alpha_2}{128} + \frac{9\alpha_1 \alpha_2 \hat{T} \hat{T}}{160} - \frac{27\alpha_2 \alpha_2 \hat{T} \hat{T}}{32} + \frac{9\alpha_2 \alpha_3 \hat{T} \hat{T}}{32} + \frac{27\alpha_2 \alpha_2 \hat{B} \hat{B}}{16} \right) \right] \\
\[ + \frac{\xi_B \left( \frac{189\alpha_1 \alpha_2^2}{640} - \frac{63\alpha_1 \alpha_2 \hat{T} \hat{T}}{128} - \frac{63\alpha_2 \alpha_2 \hat{B} \hat{B}}{32} - \frac{27\alpha_2 \alpha_2 \hat{T} \hat{B} \hat{B}}{32} - \frac{9\alpha_3 \alpha_2 \hat{T} \hat{T}}{16000} - \frac{77\xi_{W}\alpha_2^3}{128} \right) \right] \\
\[ + n_G \left( \frac{7\alpha_1^3}{40} + \frac{27\alpha_1 \alpha_2^2}{40} - \frac{263\alpha_2^3}{24} - \frac{\alpha_2 \hat{T} \hat{T}}{3} - \frac{3\alpha_2 \hat{T} \hat{T} \hat{B}}{3} + \frac{16\alpha_2 \hat{T} \hat{T} \hat{B} \hat{B}}{3} - \frac{11\alpha_2 \hat{T} \hat{T}}{15} \right) \]
\[-3\alpha_2^2 \text{tr} \hat{B} + \frac{16\alpha_2^3 \text{tr} \hat{B}}{3} + \frac{\alpha_2^2 \text{tr} \hat{L}}{5} - \alpha_2^2 \text{tr} \hat{L} + \xi_B \left( -\frac{9\alpha_1^3}{200} - \frac{9\alpha_1 \alpha_2^2}{40} \right) \]

\[+ \xi_W \left( -\frac{9\alpha_1^2 \alpha_2}{40} - \frac{5\alpha_2^3}{8} \right) + n_G \left( \frac{4\alpha_1^3}{15} + \frac{4\alpha_2^3}{3} \right) \]

\[+ \frac{1}{\varepsilon^2} \left[ -\frac{97\alpha_1^3}{3200} - \frac{99\alpha_2^3}{1280} + \frac{4917\alpha_1 \alpha_2^2}{1280} - \frac{121093\alpha_2^3}{2304} - \frac{27\alpha_1^2 \lambda}{200} - \frac{9\alpha_1 \alpha_2 \lambda}{20} - \frac{9\alpha_2^2 \lambda}{8} \right] \]

\[+ \frac{9\alpha_1 \lambda^2}{20} + \frac{9\alpha_2 \lambda^2}{4} - 24\lambda^3 - \frac{541\alpha_2^3 \text{tr} \hat{T}}{1600} - \frac{177\alpha_1 \alpha_2 \text{tr} \hat{T}}{160} - \frac{885\alpha_2^3 \text{tr} \hat{T}}{64} - \frac{\alpha_1 \alpha_3 \text{tr} \hat{T}}{10} \]

\[- \frac{33\alpha_2^3 \alpha_3 \text{tr} \hat{T}}{2} + \frac{198\alpha_2 \alpha_3 \text{tr} \hat{T}}{10} - \frac{9\lambda^2 \text{tr} \hat{T}}{8} + \frac{191\alpha_1 \alpha_2 \text{tr} \hat{B}}{1600} + \frac{57\alpha_1 \alpha_2 \text{tr} \hat{B}}{160} - \frac{885\alpha_2 \text{tr} \hat{B}}{64} \]

\[- \frac{295\alpha_2^3 \text{tr} \hat{L}}{64} - \frac{3\lambda^2 \text{tr} \hat{L}}{8} + \frac{15\alpha_2 \alpha_3 \text{tr} \hat{T}}{8} - \frac{123\alpha_1 \alpha_2 \text{tr} \hat{B}^2}{80} - \frac{117\alpha_2 \alpha_3 \text{tr} \hat{B}}{16} - \frac{39\alpha_2 \alpha_3 \text{tr} \hat{B}}{16} + \frac{18\lambda \text{tr} \hat{B}^2}{32} \]

\[+ \frac{81\alpha_2 \text{tr} \hat{B} \text{tr} \hat{L}^2}{8} + \frac{27\alpha_2 \text{tr} \hat{L} \text{tr} \hat{L}^2}{8} + \frac{5\lambda \text{tr} \hat{L}^3}{8} - \frac{261\alpha_1 \text{tr} \hat{L}^2}{80} - \frac{39\alpha_2 \text{tr} \hat{L}^2}{16} + \frac{6\lambda \text{tr} \hat{L}^2}{20} \]

\[+ \frac{27\alpha_2 \text{tr} \hat{L} \text{tr} \hat{L}^2}{8} + \frac{27\alpha_2 \text{tr} \hat{L} \text{tr} \hat{L}^2}{8} + \frac{9\alpha_1 \text{tr} \hat{L}^2}{8} + \frac{15\lambda \text{tr} \hat{L}^3}{8} + \frac{43\alpha_1 \text{tr} \hat{L}}{20} \]

\[+ \frac{9\alpha_1 \text{tr} \hat{L} \text{tr} \hat{L}^2}{8} + \frac{46\alpha_3 \text{tr} \hat{L} \text{tr} \hat{L}^2}{8} - \frac{11\alpha_1 \text{tr} \hat{L} \text{tr} \hat{L}^2}{112} - \frac{297\alpha_1 \text{tr} \hat{L}^2}{128} \]

\[- \frac{117\alpha_2 \text{tr} \hat{L}^2}{16} - \frac{39\alpha_2 \text{tr} \hat{L}^2}{16} + \frac{18\lambda \text{tr} \hat{L}^2}{8} + \frac{81\alpha_2 \text{tr} \hat{L}^2}{8} + \frac{27\alpha_2 \text{tr} \hat{L}^2}{8} \]

\[+ \xi_B \left( \frac{279\alpha_1^3}{3200} + \frac{81\alpha_2^3}{3200} - \frac{1533\alpha_1 \alpha_2^2}{1280} + \frac{9\xi_W \alpha_1 \alpha_2^2}{40} + \frac{9\xi_W^2 \alpha_1 \alpha_2^2}{320} - \frac{9\alpha_1 \alpha_2 \lambda^2}{20} + \frac{51\alpha_2^3 \text{tr} \hat{T}}{320} \right) \]

\[+ \frac{27\alpha_1 \alpha_2 \text{tr} \hat{T}}{64} + \frac{3\alpha_1 \alpha_2 \text{tr} \hat{T}}{2} + \frac{3\alpha_1 \text{tr} \hat{B}}{64} + \frac{27\alpha_1 \alpha_2 \text{tr} \hat{B}}{64} + \frac{3\alpha_1 \alpha_3 \text{tr} \hat{B}}{2} + \frac{9\alpha_2 \alpha_3 \text{tr} \hat{L}}{64} \]

\[+ \frac{9\alpha_1 \alpha_2 \text{tr} \hat{L}}{64} - \frac{81\alpha_1 \text{tr} \hat{L}^2}{160} - \frac{27\alpha_1 \alpha_2 \text{tr} \hat{L}}{160} - \frac{9\alpha_1 \alpha_3 \text{tr} \hat{L}}{80} - \frac{81\alpha_1 \text{tr} \hat{L}^2}{160} \]

\[+ \xi_W \left( \frac{279\alpha_1^3}{6400} - \frac{351\alpha_1 \alpha_2^2}{640} - \frac{141\alpha_2^3}{256} + \frac{9\alpha_2 \lambda^2}{4} + \frac{51\alpha_1 \alpha_2 \text{tr} \hat{T}}{64} + \frac{423\alpha_2^3 \text{tr} \hat{T}}{64} \right) \]

\[+ \frac{15\alpha_2 \alpha_3 \text{tr} \hat{T}}{32} + \frac{15\alpha_1 \alpha_2 \text{tr} \hat{B}}{64} + \frac{423\alpha_2^3 \text{tr} \hat{B}}{64} + \frac{15\alpha_2 \alpha_3 \text{tr} \hat{B}}{64} + \frac{45\alpha_1 \alpha_2 \text{tr} \hat{L}}{64} + \frac{141\alpha_2^3 \text{tr} \hat{L}}{64} \]

\[+ \frac{81\alpha_2 \text{tr} \hat{B}^2}{32} + \frac{27\alpha_2 \text{tr} \hat{L} \text{tr} \hat{L}^2}{32} + \frac{9\alpha_2 \text{tr} \hat{L} \text{tr} \hat{L}^2}{32} + \frac{81\alpha_2 \text{tr} \hat{L}^2}{32} \]

\[+ \xi_w^2 \left( -\frac{27\alpha_1 \alpha_2^2}{320} + \frac{189\alpha_2^3}{64} + \frac{9\alpha_2 \text{tr} \hat{T}}{16} + \frac{9\alpha_2 \text{tr} \hat{T}}{16} + \frac{3\alpha_2 \text{tr} \hat{L}}{16} + \frac{33\alpha_2^3 \text{tr} \hat{L}}{64} \right) \]

\[+ n_G \left( \frac{11\alpha_1^3}{1200} - \frac{219\alpha_1 \alpha_2}{400} - \frac{39\alpha_1 \alpha_2^2}{80} + \frac{3241\alpha_2^3}{144} + \frac{11\alpha_1^2 \alpha_2}{25} + \frac{3\alpha_2 \alpha_3}{3} - \frac{11\alpha_2 \text{tr} \hat{T}}{30} + \frac{3\alpha_2 \text{tr} \hat{T}}{2} \right) \]

\[+ \frac{40\alpha_2 \text{tr} \hat{T}}{10} + \frac{19\alpha_1 \alpha_2 \text{tr} \hat{B}}{2} + \frac{3\alpha_2 \text{tr} \hat{B}}{3} - \frac{40\alpha_3 \text{tr} \hat{B}}{10} - \frac{\alpha_2 \text{tr} \hat{L}}{2} + \frac{\alpha_2 \text{tr} \hat{L}}{2} + \xi_B \left( \frac{3\alpha_1^3}{80} + \frac{3\alpha_1 \alpha_2^2}{16} \right) \]
\[
+ \xi_w \left( \frac{3\alpha_1^2\alpha_2 - 9\alpha_2^3}{16} \right) + n_G \left( -\frac{2\alpha_1^3}{9} - \frac{10\alpha_2^3}{9} \right) - 3\alpha_6\alpha_t^2 - 3\alpha_5^2\alpha_t
\]

+ \frac{1}{\epsilon} \left[ -\frac{413\alpha_3^4}{6000} + \frac{27\zeta_3\alpha_1^3}{2000} - \frac{279\alpha_1^3\alpha_2}{800} - \frac{27\zeta_3\alpha_2^3\alpha_2}{400} - \frac{51\alpha_1\alpha_2^5}{64} + \frac{9\zeta_3\alpha_1\alpha_2^5}{80} + \frac{70519\alpha_2^3}{1728} \right]

+ \frac{69\zeta_3\alpha_2^3}{16} - \frac{117\zeta_3\lambda^3}{16} + \frac{27\zeta_3\alpha_2^3\lambda}{50} - \frac{39\alpha_1\alpha_2\lambda}{40} + \frac{9\zeta_3\alpha_1\alpha_2\lambda}{16} - \frac{39\alpha_2^3\lambda}{16} + \frac{9\zeta_3\alpha_2^3\lambda}{2}

- 3\alpha_1\lambda^2 - 15\alpha_2\lambda^2 + 12\lambda^3 + \frac{52831\alpha_3^2tr\hat{T}}{28800} - \frac{\zeta_3\alpha_1^2tr\hat{T}}{100} - \frac{371\alpha_1\alpha_2tr\hat{T}}{320}

- \frac{27\zeta_3\alpha_1\alpha_2tr\hat{T}}{10} - \frac{2433\alpha_3^2tr\hat{T}}{128} + \frac{63\zeta_3\alpha_2^3tr\hat{T}}{180} - \frac{2419\alpha_1\alpha_3tr\hat{T}}{180} - \frac{68\zeta_3\alpha_1\alpha_3tr\hat{T}}{5}

+ \frac{163\alpha_2\alpha_3tr\hat{T}}{4} - \frac{36\zeta_3\alpha_2\alpha_3tr\hat{T}}{9} - \frac{910\alpha_2^3tr\hat{T}}{9} + \frac{8\zeta_3\alpha_3^2tr\hat{T}}{2} + \frac{45\lambda^2tr\hat{T}}{2} + \frac{27\alpha_1(tr\hat{T})^2}{20}

+ \frac{27\alpha_2(tr\hat{T})^2}{4} + \frac{5479\alpha_3^2tr\hat{B}}{28800} + \frac{29\zeta_3\alpha_3^2tr\hat{B}}{100} - \frac{671\alpha_1\alpha_2tr\hat{B}}{320} + \frac{9\zeta_3\alpha_1\alpha_2tr\hat{B}}{5}

- \frac{2433\alpha_2^3tr\hat{B}}{128} + \frac{63\zeta_3\alpha_2^3tr\hat{B}}{180} - \frac{99\alpha_1\alpha_3tr\hat{B}}{180} - \frac{4\zeta_3\alpha_1\alpha_3tr\hat{B}}{4} + \frac{163\alpha_2\alpha_3tr\hat{B}}{4}

- \frac{36\zeta_3\alpha_2\alpha_3tr\hat{B}}{9} - \frac{910\alpha_2^3tr\hat{B}}{9} + \frac{8\zeta_3\alpha_3^2tr\hat{B}}{2} + \frac{45\lambda^2tr\hat{B}}{2} + \frac{27\alpha_1tr\hat{T}tr\hat{B}}{10} + \frac{27\alpha_2tr\hat{T}tr\hat{B}}{20}

- \frac{27\alpha_1(tr\hat{B})^2}{4} + \frac{27\alpha_2(tr\hat{B})^2}{4} + \frac{8517\alpha_2^3tr\hat{L}}{3200} - \frac{117\zeta_3\alpha_1^2tr\hat{L}}{100} + \frac{411\alpha_1\alpha_2tr\hat{L}}{320} - \frac{18\zeta_3\alpha_1\alpha_2tr\hat{L}}{5}

- \frac{81\alpha_2^3tr\hat{L}}{128} + \frac{21\alpha_3^2tr\hat{L}}{4} + \frac{15\lambda^2tr\hat{L}}{2} + \frac{9\alpha_1tr\hat{T}tr\hat{L}}{10} + \frac{9\alpha_2tr\hat{T}tr\hat{L}}{2} + \frac{9\alpha_1tr\hat{B}tr\hat{L}}{10}

+ \frac{9\alpha_2tr\hat{B}tr\hat{L}}{2} + \frac{9\alpha_2^3tr\hat{B}^2}{20} + \frac{3\alpha_2^2(tr\hat{L})^2}{4} + \frac{3\alpha_2^2(tr\hat{L})^2}{4} + \frac{25\alpha_2^3}{16} - \frac{3\zeta_3tr\hat{B}^3}{4} + \frac{303\alpha_1tr\hat{B}^2}{80} - \frac{9\zeta_3tr\hat{B}^2}{5}

+ \frac{279\alpha_2^3tr\hat{B}^2}{16} - \frac{9\zeta_3\alpha_2^3tr\hat{B}^2}{2} + \frac{5\alpha_3tr\hat{B}^2}{2} + 24\zeta_3\alpha_3tr\hat{B}^2 - 15\lambda tr\hat{B}^2

- \frac{18\lambda tr\hat{B}^2}{6} + \frac{6tr\hat{L}tr\hat{B}^2}{48} - \frac{25\alpha_2^3}{48} - \frac{\zeta_3tr\hat{L}^3}{5} + \frac{33\alpha_2^3tr\hat{L}^2}{80} + \frac{9\zeta_3\alpha_1tr\hat{L}^2}{9} + \frac{9\zeta_3tr\hat{L}^2}{16}

- \frac{3\zeta_3\alpha_2tr\hat{L}^2}{5} - \frac{5\lambda tr\hat{L}^2}{6} - \frac{6tr\hat{T}tr\hat{L}^2}{6} - \frac{6tr\hat{B}tr\hat{L}^2}{6} - \frac{2tr\hat{L}tr\hat{L}^2}{2} + \frac{25\alpha_2^3}{16}

- \frac{3\zeta_3tr\hat{L}^3}{40} + \frac{3\alpha_1tr\hat{T}tr\hat{B}}{40} - \frac{8\zeta_3tr\hat{T}tr\hat{B}}{8} + \frac{21\alpha_2tr\hat{T}tr\hat{B}}{8} - \frac{19\alpha_3tr\hat{T}tr\hat{B}}{16} + \frac{16\zeta_3\alpha_3tr\hat{T}tr\hat{B}}{8}

+ \frac{tr\hat{L}tr\hat{T}tr\hat{B}}{2} + \frac{21\alpha_1tr\hat{T}^2}{80} + \frac{3\zeta_3\alpha_1tr\hat{T}^2}{5}

+ \frac{279\alpha_2tr\hat{T}^2}{16} - \frac{9\zeta_3\alpha_2tr\hat{T}^2}{2} - \frac{5\alpha_3tr\hat{T}^2}{2} + 24\zeta_3\alpha_3tr\hat{T}^2 - 15\lambda tr\hat{T}^2 - 18\lambda tr\hat{T}^2

- \frac{6tr\hat{T}tr\hat{T}^2}{2} - \frac{507\zeta_2\alpha_3^2}{64} + \frac{\xi_w ( - \frac{39\alpha_3^2}{32} - \frac{3\zeta_3\alpha_2^3}{8} ) - \frac{5\zeta_3\alpha_2^3}{16}}{16}

+ n_G \left( -\frac{158\lambda_1}{225} + \frac{19\zeta_3\alpha_1^3}{25} - \frac{3\alpha_1^2\alpha_2}{40} + \frac{9\zeta_3\alpha_2^3\alpha_2}{25} - \frac{3\alpha_1^2\alpha_2^2}{10} - \frac{3\zeta_3\alpha_1\alpha_2^2}{5} + \frac{238\lambda_3^2}{216} \right)

- \frac{15\zeta_3\alpha_2^3}{20} + \frac{33\alpha_2^3\alpha_3}{25} + \frac{44\zeta_3\alpha_2^3\alpha_3}{4} + \frac{45\alpha_3^2\alpha_3}{120} + \frac{12\zeta_3\alpha_2^3\alpha_3}{8} + \frac{127\alpha_1\alpha_2^2\alpha_3}{120} + \frac{21\alpha_2^3tr\hat{T}}{8}
\[
\frac{32\alpha_3^2 \text{tr} \hat{T}}{3} + \frac{31\alpha_3^2 \text{tr} \hat{B}}{120} + \frac{21\alpha_2^2 \text{tr} \hat{B}}{8} + \frac{32\alpha_3^2 \text{tr} \hat{B}}{3} + \frac{39\alpha_3^2 \text{tr} \tilde{L}}{40} + \frac{7\alpha_3^2 \text{tr} \tilde{L}}{8} + \frac{17\xi_W \alpha_3^2}{8}
\]
\[+ n_G^2 \left( -\frac{7\alpha_3^4}{27} - \frac{35\alpha_3^2}{27} - \frac{277\alpha_2 \alpha_3^2}{16} - \frac{277\alpha_2 \alpha_3^2}{16} \right) \bigg) , \quad (A11)\]

\[Z_{CW} = 1 - \frac{\xi_W \alpha_2}{4\pi} \frac{1}{\epsilon} + \frac{\xi_W \alpha_2}{(4\pi)^3} \left\{ \frac{1}{\epsilon^3} \left[ -\frac{61\alpha_2^2}{12} + \frac{2n_G \alpha_2}{3} - 3\xi_W \alpha_2^2 - \frac{\xi_W \alpha_2^2}{3} \right] \right. \]
\[+ \frac{1}{\epsilon^2} \left[ \frac{221\alpha_2^2}{24} - \frac{5n_G \alpha_2^2}{3} + 4\xi_W \alpha_2^2 + \frac{3\xi_W \alpha_2^2}{4} \right] \]
\[+ \frac{1}{\epsilon} \left[ -\frac{373\alpha_2^2}{48} + \frac{5n_G \alpha_2^2}{2} - \frac{13\xi_W \alpha_2^2}{8} - \frac{5\xi_W^2 \alpha_2^2}{12} \right] \bigg) , \quad (A12)\]

\[Z_{CG} = 1 - \frac{3 \xi_G \alpha_3}{2} \frac{1}{4\pi} \frac{1}{\epsilon} + \frac{\xi_G \alpha_3}{(4\pi)^2} \left\{ \frac{1}{\epsilon^2} \left[ \frac{27\alpha_3}{8} + \frac{9\xi_G \alpha_3}{4} \right] \right. \]
\[+ \frac{1}{\epsilon} \left[ -\frac{45\alpha_3}{16} - \frac{9\xi_G \alpha_3}{16} \right] \bigg) , \quad (A13)\]

\[Z_{WW} = 1 + \frac{\alpha_2}{4\pi} \left\{ \frac{8}{3} - \frac{4n_G}{3} - \frac{3\xi_W}{2} \right\} \]
\[+ \frac{\alpha_2}{(4\pi)^2} \left\{ \frac{1}{\epsilon^2} \left[ -\frac{3\alpha_1}{20} + \frac{499\alpha_2}{48} + \frac{3\text{tr} \hat{T}}{4} + \frac{3\text{tr} \hat{B}}{4} + \text{tr} \tilde{L} \right] \right. \]
\[+ \frac{1}{\epsilon} \left[ -\frac{3\alpha_1}{20} + \frac{499\alpha_2}{48} + \frac{3\text{tr} \hat{T}}{4} + \frac{3\text{tr} \hat{B}}{4} + \text{tr} \tilde{L} \right] \bigg) \]
\[+ n_G \left( -\frac{\alpha_1}{10} - \frac{17\alpha_2}{3} - \alpha_3 \right) \bigg) \]
\[+ \frac{\alpha_2}{(4\pi)^3} \left\{ \frac{1}{\epsilon^3} \left[ \frac{70\alpha_3^2}{3} + \frac{4n_G \alpha_2}{3} + \frac{59\xi_W \alpha_2^2}{8} - \frac{5\xi_W^2 \alpha_2^2}{8} - \frac{35\xi_W \alpha_2^2}{16} \right] \right. \]
\[+ \frac{1}{\epsilon^2} \left[ -\frac{\alpha_1^2}{200} + \frac{9\alpha_1 \alpha_2}{40} - \frac{26057\alpha_2^2}{432} - \frac{17\alpha_1 \text{tr} \hat{T}}{80} - \frac{27\alpha_2 \text{tr} \hat{T}}{16} - \frac{2\alpha_3 \text{tr} \hat{T}}{16} - \frac{\alpha_1 \text{tr} \hat{B}}{16} \right] \]
\[+ \frac{1}{\epsilon} \left[ -\frac{\alpha_1}{200} + \frac{9\alpha_1 \alpha_2}{40} - \frac{26057\alpha_2^2}{432} - \frac{17\alpha_1 \text{tr} \hat{T}}{80} - \frac{27\alpha_2 \text{tr} \hat{T}}{16} - \frac{2\alpha_3 \text{tr} \hat{T}}{16} - \frac{\alpha_1 \text{tr} \hat{B}}{16} \right] \]
\[
\begin{align*}
&\frac{(\text{tr} \hat{L})^2}{12} - \frac{3\text{tr} \hat{T} \hat{B}}{4} + \frac{3\text{tr} \hat{T}^2}{8} + \frac{3\text{tr} \hat{T} \text{tr} \hat{B}}{2} + \frac{\text{tr} \hat{T} \text{tr} \hat{L}}{2} + \frac{3(\text{tr} \hat{T})^2}{4}
\end{align*}
\]
\[
+ \xi_w \left( \frac{9 \alpha_1 \alpha_2}{40} - \frac{165 \alpha_2^2}{32} - \frac{9 \alpha_2 \text{tr} \hat{T}}{8} - \frac{9 \alpha_2 \text{tr} \hat{B}}{8} + \frac{9 \alpha_2 \text{tr} \hat{L}}{8} \right) + \frac{157 \xi_w^2 \alpha_2^2}{16} + \frac{17 \xi_w^3 \alpha_2^2}{8}
\]
\[
+ n_G \left( -\frac{7 \alpha_2^3}{100} + \frac{3 \alpha_1 \alpha_2}{20} + \frac{4433 \alpha_2^2}{108} + 3 \alpha_2 \alpha_3 + \frac{22 \alpha_3^2}{3} \right)
\]
\[
+ \xi_w \left( \frac{3 \alpha_1 \alpha_2}{20} + \frac{21 \alpha_2^2}{2} + 3 \alpha_2 \alpha_3 \right) + \xi_w^2 \alpha_2^2 + n_G \left( -\frac{2 \alpha_1^2}{45} - \frac{128 \alpha_2^2}{27} - \frac{8 \alpha_3^2}{9} \right)
\]
\[
+ \frac{1}{\epsilon} \left[ -\frac{163 \alpha_1^2}{4800} + \frac{11 \alpha_1 \alpha_2}{160} - \frac{9 \zeta_3 \alpha_1 \alpha_2}{20} + \frac{312361 \alpha_2^2}{5184} + \frac{3 \zeta_3 \alpha_2^2}{4} - \frac{\alpha_1 \lambda}{10} - \frac{\alpha_2 \lambda}{2} + \lambda^2 \right]
\]
\[
+ \frac{593 \alpha_1 \text{tr} \hat{T}}{480} - \frac{3 \alpha_2 \text{tr} \hat{T}}{32} + \frac{7 \alpha_3 \text{tr} \hat{T}}{480} - \frac{3 \alpha_2 \text{tr} \hat{B}}{32} + \frac{3 \alpha_2 \text{tr} \hat{B}}{3} + \frac{17 \alpha_1 \text{tr} \hat{L}}{32} - \frac{19 \alpha_2 \text{tr} \hat{L}}{16} - \frac{15(\text{tr} \hat{B})^2}{4} - \frac{5 \text{tr} \hat{B} \text{tr} \hat{L}}{48} - \frac{9 \text{tr} \hat{L}^2}{24} - \frac{\text{tr} \hat{L} \text{tr} \hat{B}}{8}
\]
\[
+ \frac{33 \alpha_2^2}{8} - \frac{3 \zeta_3 \alpha_2^2}{4}
\]
\[
+ \xi_w \left( \frac{7 \alpha_1^2 \alpha_2}{8} + \frac{109 \alpha_1 \alpha_2}{45} + \frac{125 \alpha_3^2}{9} + 4 \xi_w \alpha_2^2 \right)
\]
\[
+ n_G \left( \frac{11 \alpha_1^2}{135} + \frac{625 \alpha_2^2}{81} + \frac{44 \alpha_3^2}{27} \right) \right),
\]
\[
(A14)
\]

\[
Z_{GGG} = 1 + \frac{\alpha_3}{\epsilon} \left\{ \frac{17}{4} - \frac{4 n_G}{3} - \frac{9 \xi_G}{4} \right\}
\]
\[
+ \frac{\alpha_3}{(4\pi)^2} \left\{ \frac{1}{\epsilon^2} \left[ -\frac{45 \alpha_3}{32} - \frac{9 \xi_G \alpha_3}{2} + \frac{135 \xi_3 \alpha_3}{32} \right] + n_G \left( \frac{9 \alpha_3}{2} + 3 \xi_G \alpha_3 \right) \right\}
\]
\[
+ \frac{1}{\epsilon} \left[ \frac{777 \alpha_3}{32} + \text{tr} \hat{T} + \text{tr} \hat{B} - \frac{297 \xi \alpha_3}{32} - \frac{27 \xi_3 \alpha_3}{16} + n_G \left( -\frac{11 \alpha_3}{60} - \frac{3 \alpha_2}{4} - \frac{107 \alpha_3}{12} \right) \right] \right\}
\]
\[
+ \frac{\alpha_3}{(4\pi)^2} \left\{ \frac{1}{\epsilon^3} \left[ \frac{10863 \alpha_3^2}{128} + 2 n_G \alpha_3^2 + \frac{3537 \xi \alpha_3^2}{128} - \frac{135 \xi_3 \alpha_3^2}{128} - \frac{945 \xi_3 \alpha_3^2}{128} \right] \right\}
\]
\[
+ n_G \left( -\frac{33 \alpha_3^2}{8} - \frac{117 \xi \alpha_3^2}{8} - \frac{45 \xi_3 \alpha_3^2}{8} \right) \right\}
\]
\[
+ \frac{1}{\epsilon^2} \left[ -\frac{28079 \alpha_3^2}{128} - \frac{17 \alpha_1 \text{tr} \hat{T}}{60} - \frac{3 \alpha_2 \text{tr} \hat{T}}{4} - \frac{59 \alpha_3 \text{tr} \hat{T}}{12} - \frac{\alpha_1 \text{tr} \hat{B}}{12} - \frac{3 \alpha_2 \text{tr} \hat{B}}{4} - \frac{59 \alpha_3 \text{tr} \hat{B}}{12} \right] \right\}
\]
\[
+ \frac{\text{tr} \hat{B}^2}{2} + \frac{(\text{tr} \hat{B})^2}{2} + \frac{\text{tr} \hat{B} \text{tr} \hat{L}}{3} - \frac{\text{tr} \hat{T} \text{tr} \hat{B}}{2} + \frac{\text{tr} \hat{T} \text{tr} \hat{L}}{3} + (\text{tr} \hat{T})^2
\]
\[
+ \xi_G \left( -\frac{621 \alpha_3^2}{32} - \frac{9 \alpha_3 \text{tr} \hat{T}}{4} - \frac{9 \alpha_3 \text{tr} \hat{B}}{4} \right) + \frac{4185 \xi_3 \alpha_3^2}{128} + \frac{45 \xi_3 \alpha_3^2}{64} \right),
\]
\[
(A15)
\]
\begin{align*}
&+ n_G \left( - \frac{11 \alpha_1^2}{1800} + \frac{43 \alpha_2^2}{24} + \frac{33 \alpha_1 \alpha_3}{80} + \frac{27 \alpha_2 \alpha_3}{16} + \frac{1410 \alpha_3^2}{144} + \frac{9 \xi_c^2 \alpha_3^2}{4} \right) \\
&+ \frac{1}{\varepsilon} \left[ \frac{4397 \alpha_3^2}{192} - \frac{243 \zeta \alpha_3^2}{64} + \frac{101 \alpha_1 \text{tr} \hat{T}}{8} + \frac{113 \alpha_3 \text{tr} \hat{T}}{24} + \frac{89 \alpha_1 \text{tr} \hat{B}}{120} + \frac{3 \alpha_3 \text{tr} \hat{B}}{8} \\
&+ \frac{113 \alpha_3 \text{tr} \hat{B}}{24} - \frac{3 \alpha_3 \text{tr} \hat{B}}{2} - \frac{7 (\text{tr} \hat{B})^2}{6} + \frac{3 \alpha_2 \text{tr} \hat{L}}{2} - \frac{7 \text{tr} \hat{B} \text{tr} \hat{L}}{6} - \frac{7 (\text{tr} \hat{T})^2}{2} + \frac{\xi_G}{8} - \frac{89 \alpha_3^2}{64} - \frac{81 \zeta \alpha_3^2}{32} - \frac{189 \xi_c^2 \alpha_3^2}{64} \\
&+ n_G \left( \frac{13 \alpha_1^2}{360} + \frac{121 \alpha_2^2}{120} - \frac{241 \alpha_2^2}{72} + \frac{10901 \alpha_1 \alpha_3}{4320} - \frac{33 \zeta \alpha_3 \alpha_1}{10} + \frac{99 \alpha_2 \alpha_3}{96} - \frac{27 \zeta \alpha_2 \alpha_3}{2} \\
&- \frac{14399 \alpha_3^2}{108} + 9 \xi_c \alpha_3^2 + 33 \zeta \alpha_3^2 \right) + n_G \left( \frac{121 \alpha_2^2}{810} + \frac{11 \alpha_2^2}{18} + \frac{965 \alpha_3^2}{81} \right) \right], \\
&\text{(A15)}
\end{align*}

\begin{align*}
Z_{\text{HHW}} = &+ \frac{1}{4} \frac{1}{\varepsilon} \left\{ \frac{9 \alpha_1}{20} + \frac{3 \alpha_2}{4} - 3 \text{tr} \hat{T} - 3 \text{tr} \hat{B} - \text{tr} \hat{L} - \frac{3 \xi_B \alpha_1}{20} - \frac{5 \xi \alpha_2}{4} \right\} \\
&+ \frac{1}{(4\pi)^2} \left\{ \frac{1}{\varepsilon^2} \left\{ \frac{9 \alpha_1}{800} + \frac{27 \alpha_1 \alpha_2}{80} - \frac{77 \alpha_2}{32} - \frac{3 \alpha_1 \text{tr} \hat{T}}{40} + \frac{9 \alpha_2 \text{tr} \hat{T}}{8} + 12 \alpha_3 \text{tr} \hat{T} - \frac{39 \alpha_1 \text{tr} \hat{B}}{40} \\
&+ \frac{9 \alpha_2 \text{tr} \hat{B}}{8} + 12 \alpha_3 \text{tr} \hat{B} + \frac{27 \alpha_1 \text{tr} \hat{L}}{40} + \frac{3 \alpha_2 \text{tr} \hat{L}}{8} - \frac{9 \text{tr} \hat{B}^2}{4} - \frac{3 \text{tr} \hat{L}^2}{4} + \frac{9 \text{tr} \hat{B} \text{tr} \hat{L}}{2} - \frac{9 \text{tr} \hat{T}^2}{4} \\
&+ \xi_B \left( - \frac{27 \alpha_2^2}{400} - \frac{9 \alpha_1 \alpha_2}{80} + \frac{9 \alpha_1 \text{tr} \hat{T}}{20} + \frac{9 \alpha_2 \text{tr} \hat{L}}{20} + \frac{3 \alpha_1 \text{tr} \hat{L}}{16} + \frac{3 \xi \alpha_1 \alpha_2}{16} \right) \\
&+ \xi_W \left( - \frac{9 \alpha_1 \alpha_2}{16} + \frac{15 \alpha_2^2}{16} + \frac{5 \alpha_2 \text{tr} \hat{L}}{4} + 4 \alpha_3 \text{tr} \hat{L} \right) + \frac{9 \xi \alpha_2 \alpha_1^2}{800} + \frac{45 \xi \alpha_2 \alpha_1^2}{32} \\
&+ n_G \left( \frac{3 \alpha_1^2}{10} + \frac{\alpha_2^2}{2} \right) \right\} \\
&+ \frac{1}{\varepsilon} \left\{ \frac{- 93 \alpha_1^2}{1600} - \frac{27 \alpha_1 \alpha_2}{160} + \frac{817 \alpha_2^2}{192} - \frac{3 \lambda^2}{16} - \frac{17 \alpha_1 \text{tr} \hat{T}}{16} - \frac{45 \alpha_2 \text{tr} \hat{T}}{16} - 10 \alpha_3 \text{tr} \hat{T} \\
&- \frac{5 \alpha_1 \text{tr} \hat{B}}{16} - \frac{45 \alpha_2 \text{tr} \hat{B}}{16} - 10 \alpha_3 \text{tr} \hat{B} - \frac{15 \alpha_3 \text{tr} \hat{L}}{16} + \frac{15 \alpha_3 \text{tr} \hat{L}}{16} + \frac{27 \text{tr} \hat{B}^2}{8} + \frac{9 \text{tr} \hat{L}^2}{8} \\
&- \frac{3 \text{tr} \hat{B} \text{tr} \hat{L}}{4} + \frac{23 \xi \alpha_1 \alpha_2^2}{16} + \frac{7 \xi_W \alpha_2^2}{16} + n_G \left( - \frac{\alpha_1^2}{4} - \frac{5 \alpha_2^2}{12} \right) \right\} \\
&+ \frac{1}{(4\pi)^3} \left\{ \frac{1}{\varepsilon^3} \left\{ \frac{429 \alpha_1^3}{16000} + \frac{297 \alpha_2^3 \alpha_1}{3200} - \frac{693 \alpha_2 \alpha_1^2}{640} + \frac{12551 \alpha_2^3}{1152} - \frac{93 \alpha_1 \alpha_2 \text{tr} \hat{T}}{800} - \frac{9 \alpha_1 \alpha_2 \text{tr} \hat{T}}{20} \\
&- \frac{27 \alpha_2 \text{tr} \hat{T}}{32} - \frac{7 \alpha_1 \alpha_3 \text{tr} \hat{T}}{5} - \frac{9 \alpha_2 \alpha_3 \text{tr} \hat{T}}{76 \alpha_2 \text{tr} \hat{L}} - \frac{177 \alpha_2 \text{tr} \hat{B}}{800} + \frac{9 \alpha_1 \alpha_2 \text{tr} \hat{B}}{40} \\
&- \frac{27 \alpha_2 \text{tr} \hat{B}}{32} + \frac{17 \alpha_1 \alpha_3 \text{tr} \hat{B}}{5} - \frac{9 \alpha_2 \alpha_3 \text{tr} \hat{B}}{76 \alpha_2 \text{tr} \hat{L}} - \frac{339 \alpha_1 \alpha_2 \text{tr} \hat{L}}{800} - \frac{27 \alpha_1 \alpha_2 \text{tr} \hat{L}}{40} \right\}.
\end{align*}
\[- \frac{9\alpha^3_3 tr L}{32} - \frac{9 tr B^3}{4} - \frac{9\alpha_1 tr B^2}{20} + \frac{27\alpha_2 tr B^2}{8} + 18\alpha_3 tr B^2 - \frac{9 tr \hat{L} tr B^2}{4} \]
\[- \frac{9 tr B tr \hat{B}^2}{4} - \frac{3 tr \hat{L} tr B^2}{4} - \frac{3 tr \hat{L}^3}{4} + \frac{27\alpha_1 tr \hat{L}^2}{20} + \frac{9\alpha_2 tr \hat{L}^2}{8} - \frac{9 tr \hat{L} tr \hat{L}^2}{4} \]
\[- \frac{3 tr B tr \hat{L}^2}{4} + \frac{9 tr \hat{L} tr \hat{L}^2}{4} - \frac{9 tr \hat{L} tr \hat{L}^2}{4} - \frac{20}{4} - \frac{9\alpha_3 tr \hat{L}^2}{8} \]
\[- \frac{36\alpha_3 tr \hat{L} tr B}{4} + \frac{27\alpha_2 tr \hat{L}^2}{8} + \frac{18\alpha_3 tr \hat{L}^2}{4} - \frac{9\alpha_1 tr \hat{L}^2}{4} + \frac{9\alpha_1 tr \hat{L}^2}{10} \]
\[+ \frac{27\alpha_2 tr \hat{L}^2}{8} + \frac{18\alpha_3 tr \hat{L}^2}{4} - \frac{9\alpha_1 tr \hat{L}^2}{10} \]
\[+ \xi_B \left( - \frac{29\alpha_1^3}{1600} - \frac{81\alpha_1^2\alpha_2}{1600} + \frac{231\alpha_1\alpha_2^2}{640} + \frac{9\alpha_1^2 tr \hat{L}^2}{800} - \frac{27\alpha_2 \alpha_3 tr \hat{L}^2}{160} \right) - \frac{9\alpha_1 \alpha_3 tr \hat{L}^2}{5} + \frac{117\alpha_3^2 tr \hat{B}^2}{800} - \frac{9\alpha_2 \alpha_3 tr \hat{B}^2}{800} - \frac{9\alpha_1 \alpha_3 tr \hat{B}^2}{800} + \frac{9\alpha_2 \alpha_3 tr \hat{B}^2}{800} \]
\[+ \xi_B \left( - \frac{27\alpha_2^2 tr \hat{L}^2}{16} - \frac{9\alpha_1 \alpha_2^2}{16} + \frac{9\alpha_2 \alpha_3 tr \hat{B}^2}{16} - \frac{9\alpha_1 \alpha_3 tr \hat{B}^2}{16} + \frac{3\alpha_1 \alpha_2 tr \hat{L}^2}{16} - \frac{3\alpha_1 \alpha_3 tr \hat{L}^2}{16} \right) - \frac{27\xi_2 \alpha_1 \alpha_2^2}{128} \]
\[+ \xi_w \left( - \frac{27\alpha_1^3}{128} + \frac{27\alpha_2^3}{64} + \frac{745\alpha_1^3}{384} + \frac{3\alpha_1 \alpha_3 tr \hat{L}^2}{32} + \frac{225\alpha_2 \alpha_3 tr \hat{L}^2}{32} - 15\alpha_2 \alpha_3 tr \hat{L}^2 \right) + \frac{3\alpha_1 \alpha_2 tr \hat{B}^2}{32} \]
\[+ \frac{225\alpha_2 \alpha_3 tr \hat{B}^2}{32} + \frac{15\alpha_2 \alpha_3 tr \hat{B}^2}{32} - \frac{75\alpha_2 \alpha_3 tr \hat{B}^2}{32} + \frac{45\alpha_2 \alpha_3 tr \hat{B}^2}{32} \]
\[- \frac{27\alpha_1^3 tr \hat{L}^2}{16} - \frac{27\alpha_2^3 tr \hat{L}^2}{16} - \frac{9\alpha_2 \alpha_3 tr \hat{B}^2}{16} - \frac{9\alpha_2 \alpha_3 tr \hat{B}^2}{16} - \frac{9\alpha_2 \alpha_3 tr \hat{B}^2}{16} \]
\[- \frac{27\xi_2 \alpha_1 \alpha_2^2}{128} - \frac{40\alpha_1 \alpha_2^2}{128} + \frac{135\alpha_2 \alpha_3 tr \hat{L}^2}{32} - \frac{135\alpha_2 \alpha_3 tr \hat{L}^2}{32} - \frac{45\alpha_2 \alpha_3 tr \hat{L}^2}{32} \]
\[- \frac{27\xi_2 \alpha_1 \alpha_2^2}{128} - \frac{195\xi_2 \alpha_1 \alpha_2^2}{128} \]
\[- \frac{9\xi_2 \alpha_1 \alpha_2^2}{1600} - \frac{128}{128} + n_G \left( \frac{7\alpha_1^3}{40} + \frac{9\alpha_2 \alpha_2}{40} + \frac{9\alpha_1 \alpha_2}{40} - \frac{317\alpha_2^3}{72} - \frac{\alpha_1^2 tr \hat{L}^2}{3} + \frac{16\alpha_2^3 tr \hat{L}^2}{3} - \frac{11\alpha_1^2 tr \hat{B}^2}{15} \right) \]
\[+ \frac{16\alpha_2^3 tr \hat{B}^2}{3} - \frac{\alpha_2 \alpha_3 tr \hat{L}^2}{5} + \xi_B \left( - \frac{9\alpha_1^3}{200} - \frac{3\alpha_1 \alpha_2^2}{40} \right) + \xi_w \left( - \frac{3\alpha_1 \alpha_2^2}{80} - \frac{5\alpha_2^3}{9} \right) \]
\[+ \frac{1}{\epsilon^2} \left[ - \frac{9\alpha_1^3}{3200} + \frac{63\alpha_2 \alpha_2}{640} + \frac{2901\alpha_1 \alpha_2}{1280} - \frac{193057\alpha_2}{6912} - \frac{27\alpha_2 \lambda}{200} - \frac{9\alpha_1 \alpha_2 \lambda}{20} \right] \]
\[- \frac{9\alpha_2 \lambda}{8} + \frac{9\alpha_1 \lambda^2}{20} + \frac{27\alpha_2 \lambda^2}{4} - \frac{24\lambda^3}{1600} + \frac{39\alpha_1 \alpha_2 tr \hat{L}^2}{80} + \frac{149\alpha_2 \alpha_2 tr \hat{L}^2}{64} \]
\[\begin{align*}
&- \frac{\alpha_1 \alpha_3 \text{tr} \hat{T}}{10} - \frac{3\alpha_2 \alpha_3 \text{tr} \hat{T}}{2} + 198\alpha_3^2 \text{tr} \hat{T} - 9\lambda^2 \text{tr} \hat{T} + \frac{191\alpha_3^2 \text{tr} \hat{B}}{1600} + \frac{33\alpha_1 \alpha_3 \text{tr} \hat{B}}{40} \\
&+ \frac{149\alpha_3^2 \text{tr} \hat{B}}{64} - \frac{49\alpha_1 \alpha_3 \text{tr} \hat{B}}{10} - \frac{3\alpha_2 \alpha_3 \text{tr} \hat{B}}{2} + 198\alpha_3^2 \text{tr} \hat{B} - 9\lambda^2 \text{tr} \hat{B} - \frac{3\alpha_3^2 \text{tr} \hat{L}}{1600} \\
&+ \frac{149\alpha_3^2 \text{tr} \hat{L}}{192} - 3\lambda^2 \text{tr} \hat{L} + \frac{15\text{tr} \hat{B}^3}{8} - \frac{123\alpha_1 \text{tr} \hat{B}^2}{80} - \frac{99\alpha_2 \text{tr} \hat{B}^2}{8} - 39\alpha_3 \text{tr} \hat{B}^2 \\
&+ 18\lambda \text{tr} \hat{B}^2 + \frac{81\text{tr} \hat{B} \text{tr} \hat{B}^2}{8} + \frac{27\text{tr} \hat{L} \text{tr} \hat{B}^2}{8} + \frac{5\text{tr} \hat{L}^3}{8} - \frac{261\alpha_1 \text{tr} \hat{L}^2}{80} \\
&- \frac{33\alpha_2 \text{tr} \hat{L}^2}{8} + 6\lambda \text{tr} \hat{L}^2 + \frac{27\text{tr} \hat{L} \text{tr} \hat{L}^2}{8} + \frac{27\text{tr} \hat{B} \text{tr} \hat{L}^2}{8} + 9\lambda \text{tr} \hat{L} \text{tr} \hat{L}^2 \\
&+ \frac{15\text{tr} \hat{L}^3}{8} + \frac{43\alpha_1 \text{tr} \hat{L} \text{tr} \hat{B}}{4} + \frac{9\alpha_2 \text{tr} \hat{L} \text{tr} \hat{B}}{4} + 46\alpha_3 \text{tr} \hat{L} \text{tr} \hat{B} + \frac{81\text{tr} \hat{L} \text{tr} \hat{B}^2}{8} + 27\text{tr} \hat{L} \text{tr} \hat{L}^2 \\
&- \frac{11\text{tr} \hat{L} \text{tr} \hat{L} \text{tr} \hat{B}}{4} - \frac{297\alpha_1 \text{tr} \hat{L}^2}{80} - \frac{99\alpha_2 \text{tr} \hat{L}^2}{8} - 39\alpha_3 \text{tr} \hat{L}^2 + 18\lambda \text{tr} \hat{L}^2 \\
&+ \xi_B \left( \frac{279\alpha_3^2}{32000} + \frac{81\alpha_1^2 \alpha_2}{3200} - \frac{817\alpha_1 \alpha_2^2}{1280} + \frac{9\alpha_1 \lambda^2}{20} + \frac{51\alpha_1^2 \text{tr} \hat{T}}{320} + \frac{27\alpha_1 \alpha_2 \text{tr} \hat{T}}{64} \right) \\
&+ \frac{3\alpha_1 \alpha_3 \text{tr} \hat{T}}{2} + \frac{3\alpha_2 \text{tr} \hat{B}}{64} + \frac{27\alpha_1 \alpha_2 \text{tr} \hat{B}}{64} + \frac{3\alpha_1 \alpha_3 \text{tr} \hat{B}}{64} + \frac{9\alpha_2 \text{tr} \hat{L}}{64} + \frac{9\alpha_2 \text{tr} \hat{L}}{64} \\
&+ \frac{81\alpha_1 \text{tr} \hat{B}^2}{160} - \frac{27\alpha_1 \text{tr} \hat{L}^2}{160} + \frac{9\alpha_1 \text{tr} \hat{B} \text{tr} \hat{B}^2}{160} - \frac{81\alpha_1 \text{tr} \hat{L} \text{tr} \hat{L}^2}{160} + \frac{69\xi_w \alpha_1 \alpha_2 \text{tr} \hat{T}}{160} + \frac{777\alpha_3^2 \text{tr} \hat{T}}{64} \\
&+ \frac{25\alpha_2 \alpha_3 \text{tr} \hat{T}}{2} + \frac{25\alpha_1 \alpha_3 \text{tr} \hat{B}}{64} + \frac{77\alpha_3^2 \text{tr} \hat{B}}{64} + \frac{25\alpha_2 \alpha_3 \text{tr} \hat{B}}{64} + \frac{75\alpha_1 \alpha_3 \text{tr} \hat{L}}{64} \\
&+ \frac{259\alpha_3^2 \text{tr} \hat{L}}{64} - \frac{135\alpha_3 \text{tr} \hat{B}^2}{64} + \frac{45\alpha_2 \text{tr} \hat{L}^2}{32} + \frac{15\alpha_2 \text{tr} \hat{B} \text{tr} \hat{L}^2}{16} - \frac{135\alpha_2 \text{tr} \hat{L}^2}{32} \\
&+ \xi_w \left( -\frac{63\alpha_1 \alpha_2^2}{320} + \frac{473\alpha_3^2}{64} + \frac{21\alpha_2^3 \text{tr} \hat{T}}{16} + \frac{21\alpha_2 \text{tr} \hat{B}^2}{16} + \frac{7\alpha_2 \text{tr} \hat{L}}{16} \right) + \frac{241\xi^3 \text{tr} \hat{T}^3}{192} \\
&+ n_G \left( -\frac{11\alpha_1^3}{1200} - \frac{69\alpha_1 \alpha_2}{64} + \frac{17\alpha_1 \alpha_2^2}{16} + \frac{4123\alpha_3^2}{432} + \frac{11\alpha_1 \alpha_3}{25} + \frac{\alpha_2 \alpha_3}{30} - \frac{11\alpha_2 \text{tr} \hat{T}}{30} \\
- \frac{\alpha_2 \text{tr} \hat{T}}{30} + \frac{19\alpha_2 \text{tr} \hat{B}}{30} - \frac{\alpha_2 \text{tr} \hat{B}}{30} - \frac{40\alpha_2 \text{tr} \hat{B}}{30} + \frac{9\alpha_1 \text{tr} \hat{L}}{30} - \frac{\alpha_2 \text{tr} \hat{L}}{30} \right) \\
&+ \xi_B \left( \frac{3\alpha_1^2}{80} + \frac{\alpha_2 \alpha_3}{16} \right) + \xi_w \left( \frac{5\alpha_1 \alpha_2 \alpha_3}{16} - \frac{103\alpha_3^2}{48} \right) + n_G \left( -\frac{2\alpha_1^3}{9} + \frac{10\alpha_2^3}{27} \right) \\
- 3\alpha_1 \alpha_2 \alpha_3 + 3\alpha_2 \alpha_3 \alpha_2 \\
+ \frac{1}{\epsilon} \left[ -\frac{413\alpha_1^3}{6000} + \frac{27\zeta_3 \alpha_1^3}{2000} - \frac{279\alpha_1 \alpha_2}{800} - \frac{27\zeta_3 \alpha_1 \alpha_2}{400} - \frac{123\alpha_1 \alpha_2^2}{320} - \frac{3\zeta_3 \alpha_1 \alpha_2}{80} \\
+ \frac{93307\alpha_2^3}{5184} + \frac{73\zeta_3 \alpha_2^3}{16} - \frac{117\alpha_2 \lambda^2}{400} + \frac{27\zeta_3 \alpha_2 \lambda^2}{50} - \frac{39\alpha_1 \alpha_2 \lambda}{40} + \frac{9\zeta_3 \alpha_1 \alpha_2 \lambda}{5} - \frac{39\alpha_3 \lambda^2}{16} \right]
\end{align*}\]
\[
\begin{align*}
& + \frac{9\zeta_3\alpha^2}{2} - 3\alpha_1\lambda^2 - 15\alpha_2\lambda^2 + 12\lambda^3 + \frac{52831\alpha^2}{2880}r^T + \frac{\zeta_3\alpha_3^2}{100}r^T - \frac{371\alpha_1\alpha_2}{320}r^T \\
& - \frac{27\zeta_3\alpha_1\alpha_2}{10}r^T + \frac{2761\alpha_2^2}{128}r^T + \frac{63\zeta_3\alpha_3^2}{4}r^T + \frac{2419\alpha_1\alpha_3r}{180}r^T - \frac{68\zeta_3\alpha_3}{5}r^T \\
& + \frac{163\alpha_3\alpha_3}{4}r^T - \frac{36\zeta_3\alpha_2\alpha_3}{9}r^T - \frac{910\zeta_3}{4}r^T + \frac{8\zeta_3\alpha_3^2}{2}r^T + \frac{45\lambda^2}{2}r^T \\
& + \frac{27\alpha_1}{20}(tr^T)^2 + \frac{27\alpha_2}{4}(tr^2)^2 + \frac{5479\alpha_2}{2880}r^B + \frac{29\zeta_3\alpha_3^2}{100}r^B - \frac{671\alpha_1\alpha_2}{320}r^B \\
& + \frac{9\zeta_3\alpha_1\alpha_2}{5}r^B - \frac{2761\alpha_2^2}{128}r^B + \frac{63\zeta_3\alpha_3^2}{4}r^B + \frac{991\alpha_1\alpha_3}{180}r^B - \frac{4\zeta_3\alpha_1\alpha_3}{4}r^B \\
& + \frac{163\alpha_3\alpha_3}{4}r^B - \frac{36\zeta_3\alpha_2\alpha_3}{9}r^B - \frac{910\zeta_3}{4}r^B + \frac{8\zeta_3\alpha_3^2}{2}r^B + \frac{45\lambda^2}{2}r^B \\
& + \frac{27\alpha_1}{10}(tr^T)^2r^B + \frac{27\alpha_2}{2}(tr^2)^2r^B + \frac{27\alpha_1}{20}(tr^B)^2 + \frac{27\alpha_2}{4}(tr^B)^2 + \frac{8517\alpha^2}{320}L \\
& - \frac{117\zeta_3\alpha_2^2}{100}r^L + \frac{411\alpha_1\alpha_2^2}{320}r^L - \frac{18\zeta_3\alpha_1\alpha_2}{5}r^L - \frac{2761\alpha_2^2}{384}r^L + \frac{21\zeta_3\alpha_2^2}{4}r^L \\
& + \frac{15\lambda^2}{2}r^L + \frac{9\alpha_1}{10}r^Ttr^L + \frac{9\alpha_2}{10}r^Ttr^L + \frac{9\alpha_1}{2}r^Btr^L + \frac{9\alpha_2}{2}r^Btr^L \\
& + \frac{3\alpha_1}{20}(tr^L)^2 + \frac{3\alpha_2}{4}(tr^L)^2 + \frac{25tr^L}{16} - \frac{3\zeta_3tr^L}{80} + \frac{303\alpha_1^2}{8} - \frac{9\zeta_3\alpha_1^2}{5} \\
& + \frac{279\alpha_3^2}{16} - \frac{9\zeta_3\alpha_2}{2}B^2 - \frac{5\zeta_3tr^L}{2} + \frac{24\zeta_3\zeta_3}{2}tr^L - \frac{15\lambda}{2}tr^L \\
& - 18tr^Btr^B - 6tr^Ltr^B + \frac{25tr^L}{48} - \frac{3\zeta_3tr^L}{80} + \frac{33tr^L}{8} \\
& + \frac{9\zeta_3\alpha_1^2}{5} - \frac{93\alpha_2^2}{16} - \frac{3\zeta_3tr^L}{80} - 5\lambda tr^L - 6tr^Ttr^L - 6tr^Btr^L \\
& + \frac{25tr^L}{16} - \frac{3\zeta_3tr^L}{80} + \frac{31\alpha_1tr^L}{40} - \frac{8\zeta_3\alpha_1tr^L}{5} \\
& + \frac{21\alpha_2tr^L}{8} - 19\alpha_3tr^L + 16\zeta_3\alpha_3tr^L \\
& + \frac{tr^Ltr^T}{2} + \frac{211\alpha_1tr^L}{80} - \frac{2\zeta_3\alpha_1^2tr^L}{5} + \frac{279\alpha_2tr^L}{16} - \frac{9\zeta_3\alpha_3tr^L}{2} \\
& - \frac{5\zeta_3tr^L}{2} + 24\zeta_3\alpha_3tr^L - 15\lambda tr^L - 18tr^Ttr^L - 6tr^Ltr^T \\
& + \xi_w \left( - \frac{927\alpha_3^2}{64} - \frac{\zeta_3\alpha_3^2}{2} \right) + \xi_w \left( - \frac{83\alpha_3^2}{32} - \frac{5\zeta_3\alpha_3^2}{8} \right) - \frac{29\zeta_3^2\alpha_3^2}{48} \\
& + n_G \left( - \frac{158\alpha_3^2}{225} + \frac{19\zeta_3\alpha_3^2}{25} - \frac{3\alpha_1^2}{40} + \frac{9\zeta_3\alpha_3^2}{25} - \frac{3\alpha_1^2}{40} + \frac{\zeta_3\alpha_1^2}{5} - \frac{1285\alpha_3^2}{324} \\
& - \frac{9\zeta_3\alpha_3^2}{20} - \frac{44\alpha_2^2\alpha_3}{25} - \frac{15\alpha_3^2}{4} + 4\zeta_3\alpha_2^2\alpha_3 + \frac{127\alpha_2^2}{120} + \frac{21\alpha_2^2}{8} \\
& + \frac{32\zeta_3\alpha_2^2}{3} + \frac{31\alpha_2^2tr^L}{120} + \frac{21\alpha_2^2tr^L}{8} + \frac{32\alpha_2^2tr^L}{3} + \frac{39\alpha_2^2tr^L}{40} + \frac{7\alpha_2^2tr^L}{8} + \frac{83\xi_w\alpha_3^2}{24} \right)
\end{align*}
\]
\[ + n_G^2 \left( -\frac{7\alpha_t^3}{27} - \frac{35\alpha_b^3}{81} \right) - \frac{27\alpha_t\alpha_b^2}{16} - \frac{27\alpha_\tau\alpha_t^2}{16} \right) \right]. \quad (A16) \]

Note that in the results for \( Z_H \) and \( Z_{HHW} \) there are terms which contain explicitly \( \alpha_b \) and \( \alpha_t \) since we have not been able to reconstruct the corresponding expressions in terms of \( \bar{B} \) and \( \bar{T} \). They drop out in the final result for \( Z_{\alpha_2} \).

### Appendix B: Two-loop Yukawa coupling renormalization constants

This Subsection contains the two-loop results for the Yukawa coupling renormalization constants defined through

\[ \alpha_i^{\text{bare}} = Z_{\alpha_i} \alpha_i, \quad (B1) \]

with \( i = t, b, \tau \). They read

\[
Z_{\alpha_t} = 1 + \frac{1}{4\pi \epsilon} \left\{ -\frac{17\alpha_t}{20} - \frac{9\alpha_2}{4} - 8\alpha_3 + \frac{3\alpha_t}{2} - \frac{3\alpha_b}{2} + 3\epsilon\hat{T} + 3\epsilon\hat{B} + \epsilon\hat{\bar{L}} \right\}
\[
+ \frac{1}{(4\pi)^2} \left\{ \frac{1}{\epsilon^2} \left[ \frac{51\alpha_t^2}{80} + \frac{153\alpha_1\alpha_2}{8} + \frac{339\alpha_2^2}{32} + \frac{34\alpha_1\alpha_3}{5} + 18\alpha_2\alpha_3 + 76\alpha_3 - \frac{459\alpha_1\alpha_t}{80} \right]
\right.
\]
\[
- \frac{243\alpha_2\alpha_t}{16} - \frac{54\alpha_3\alpha_t}{4} - \frac{81\alpha_t^2}{8} - \frac{117\alpha_1\alpha_b}{8} - \frac{81\alpha_2\alpha_b}{16} - 18\alpha_3\alpha_b + \frac{45\alpha_t\alpha_b}{4} + \frac{9\alpha_b^2}{2}
\]
\[
- \frac{79\alpha_1\alpha_\tau}{16} - \frac{27\alpha_2\alpha_\tau}{8} - \frac{8\alpha_3\alpha_\tau}{4} + \frac{33\alpha_1\alpha_\tau}{4} + \frac{15\alpha_b\alpha_\tau}{4} + \frac{7\alpha_\tau^2}{4}
\]
\[
+ n_G\left( -\frac{17\alpha_t^2}{30} - \frac{3\alpha_3^2}{2} - \frac{16\alpha_\tau^2}{3} \right) \right]\]
\[
+ \frac{1}{\epsilon} \left[ \frac{9\alpha_t^2}{400} - \frac{9\alpha_1\alpha_2}{40} - \frac{35\alpha_2^2}{8} + \frac{19\alpha_1\alpha_3}{30} + \frac{9\alpha_2\alpha_3}{2} - \frac{202\alpha_2^2}{3} + 3\lambda^2 + \frac{393\alpha_1\alpha_t}{160} \right]
\]
\[
+ \frac{225\alpha_2\alpha_t}{32} + 18\alpha_3\alpha_t - 6\hat{\lambda}\alpha_t - 6\alpha_2^2 + \frac{7\alpha_1\alpha_b}{160} + \frac{99\alpha_2\alpha_b}{32} + 2\alpha_3\alpha_b - \frac{11\alpha_1\alpha_b}{8} - \frac{\alpha_2^2}{8}
\]
\[
+ \frac{15\alpha_1\alpha_\tau}{16} - \frac{15\alpha_2\alpha_\tau}{16} - \frac{9\alpha_1\alpha_\tau}{8} + \frac{5\alpha_2\alpha_\tau}{8} - \frac{9\alpha_2^2}{8} + n_G\left( \frac{29\alpha_1^2}{90} + \frac{\alpha_2^2}{2} + \frac{40\alpha_3^2}{9} \right) \right\}, \quad (B2) \]

\[
Z_{\alpha_b} = 1 + \frac{1}{4\pi \epsilon} \left\{ -\frac{\alpha_1}{4} - \frac{9\alpha_2}{4} - 8\alpha_3 - \frac{3\alpha_t}{2} + \frac{3\alpha_b}{2} + 3\epsilon\hat{T} + 3\epsilon\hat{B} + \epsilon\hat{\bar{L}} \right\}
\]
\[
+ \frac{1}{(4\pi)^2} \left\{ \frac{1}{\epsilon^2} \left[ \frac{3\alpha_1^2}{160} + \frac{9\alpha_1\alpha_2}{16} + \frac{339\alpha_2^2}{32} + 2\alpha_1\alpha_3 + 18\alpha_2\alpha_3 + 76\alpha_3 - \frac{81\alpha_1\alpha_t}{80} - \frac{81\alpha_2\alpha_t}{16} \right]
\right.
\]
\[
- 18\alpha_3\alpha_t + \frac{9\alpha_t^2}{2} - \frac{27\alpha_1\alpha_b}{16} - \frac{243\alpha_2\alpha_b}{16} - 54\alpha_3\alpha_b + \frac{45\alpha_1\alpha_b}{4} + \frac{81\alpha_t^2}{8} - \frac{11\alpha_1\alpha_\tau}{8}
\]
\[
- \frac{27\alpha_2\alpha_\tau}{8} - \frac{8\alpha_3\alpha_\tau}{4} + \frac{15\alpha_t\alpha_\tau}{4} + \frac{7\alpha_\tau^2}{4} + n_G\left( -\frac{\alpha_1}{6} - \frac{3\alpha_2^2}{2} - \frac{16\alpha_3^2}{3} \right) \right]\]
\[
+ \frac{1}{\epsilon} \left[ -\frac{29\alpha_1^2}{400} - \frac{27\alpha_1\alpha_2}{40} - \frac{35\alpha_2^2}{8} + \frac{31\alpha_1\alpha_3}{30} + \frac{9\alpha_2\alpha_3}{2} - \frac{202\alpha_2^2}{3} + 3\lambda^2 + \frac{91\alpha_1\alpha_t}{160} \right]
\]
\[
+ \frac{99\alpha_2\alpha_t}{32} + 2\alpha_3\alpha_t - \frac{\alpha_t^2}{8} + \frac{237\alpha_1\alpha_b}{160} + \frac{225\alpha_2\alpha_b}{32} + 18\alpha_3\alpha_b - 6\hat{\lambda}\alpha_b - \frac{11\alpha_1\alpha_b}{8} - \frac{6\alpha_b^2}{8}
\]
\[ Z_{\alpha} = 1 + \frac{1}{4\pi \epsilon} \left\{ \begin{array}{l} -9a_1^2 - \frac{9a_2}{8} + 3\alpha_T + 3\text{tr} \hat{T} + 3\text{tr} \hat{B} + \text{tr} \hat{L} \\ \frac{1}{(4\pi)^2} \left( \begin{array}{c} 387\alpha_1^2 + \frac{81\alpha_1\alpha_2}{16} + \frac{339\alpha_2^2}{32} - \frac{321\alpha_1\alpha_2}{40} - \frac{81\alpha_2\alpha_t}{8} - 12\alpha_3\alpha_t + \frac{45\alpha_2^2}{4} \\ - \frac{57\alpha_1\alpha_b}{8} - \frac{81\alpha_2\alpha_b}{8} - 12\alpha_3\alpha_b + \frac{27\alpha_t\alpha_b}{2} + \frac{45\alpha_b^2}{4} - \frac{135\alpha_1\alpha_r}{16} - \frac{135\alpha_2\alpha_r}{16} \\ + \frac{51\alpha_1\alpha_r}{4} + \frac{51\alpha_b\alpha_r}{4} + \frac{25\alpha_r^2}{4} + n_G \left( -\frac{3\alpha_1^2}{2} - \frac{3\alpha_2^2}{2} \right) \end{array} \right) \end{array} \right. \]

\[ + \frac{1}{\epsilon} \left[ \begin{array}{l} -\frac{51\alpha_1^2}{400} + \frac{27\alpha_1\alpha_b}{40} - \frac{35\alpha_b^2}{8} + 3\lambda^2 + \frac{17\alpha_1\alpha_t}{16} + \frac{45\alpha_2\alpha_t}{16} + 10\alpha_3\alpha_t - \frac{27\alpha_t^2}{8} \\ + \frac{5\alpha_1\alpha_b}{16} + \frac{45\alpha_2\alpha_b}{16} + \frac{10\alpha_3\alpha_b}{4} - \frac{3\alpha_2}{8} - \frac{27\alpha_t^2}{8} + \frac{537\alpha_1\alpha_r}{160} + \frac{165\alpha_2\alpha_r}{32} - 6\lambda \alpha_r \\ - \frac{27\alpha_1\alpha_r}{8} - \frac{27\alpha_b\alpha_r}{8} - \frac{3\alpha_r^2}{2} + n_G \left( \frac{11\alpha_1^2}{10} + \frac{\alpha_2^2}{2} \right) \end{array} \right] \} \right. \]

**Appendix C: Beta functions for $\alpha_{\text{QED}}$ and $\sin^2 \theta_W$**

This Appendix contains explicit results up to three-loop order for the QED coupling $\alpha_{\text{QED}}$ and the weak mixing angle defined in the $\overline{\text{MS}}$ scheme. We refrain from providing expressions for the renormalization constants but directly list the beta functions. They are obtained in a straightforward way from Eq. (2) and are given by

\[ \beta_{\alpha_{\text{QED}}} = \frac{\alpha_{\text{QED}}}{(4\pi)^2} \left\{ -28 + \frac{128n_G}{9} \right\} + \frac{\alpha_{\text{QED}}^2}{(4\pi)^3} \left\{ -\frac{500\alpha_{\text{QED}}}{3\sin^2 \theta_W} + \frac{4\alpha_{\text{QED}}}{\cos^2 \theta_W} \right\} \]

\[ + \frac{52\text{tr} \hat{T}}{3} - \frac{28\text{tr} \hat{B}}{3} - 12\text{tr} \hat{L} + n_G \left[ \frac{208\alpha_{\text{QED}}}{3\sin^2 \theta_W} + \frac{416\alpha_{\text{QED}}}{27\cos^2 \theta_W} + \frac{320\alpha_3}{9} \right] \}

\[ + \frac{\alpha_{\text{QED}}^2}{(4\pi)^4} \left\{ \frac{137\alpha_{\text{QED}}^2}{4\sin^2 \theta_W \cos^2 \theta_W} - \frac{318251\alpha_{\text{QED}}^2}{216 \sin^4 \theta_W} + \frac{163\alpha_{\text{QED}}^2}{72 \cos^4 \theta_W} + \frac{12\alpha_{\text{QED}} \lambda}{\sin^2 \theta_W} + \frac{8\alpha_{\text{QED}} \hat{\lambda}}{\cos^2 \theta_W} \right\} \]

\[ - 24\lambda^2 - \frac{757\alpha_{\text{QED}} \text{tr} \hat{T}}{36 \sin^2 \theta_W} - \frac{2303\alpha_{\text{QED}} \hat{\alpha}_T}{36 \cos^2 \theta_W} - \frac{200\alpha_{\text{QED}} \hat{T}}{3} - \frac{583\alpha_{\text{QED}} \hat{B}}{36 \cos^2 \theta_W} - \frac{1433\alpha_{\text{QED}} \hat{B}}{36 \cos^2 \theta_W} \]

\[ - \frac{152\alpha_{\text{QED}} \hat{B}}{3} - \frac{393\alpha_{\text{QED}} \hat{L}}{4 \sin^2 \theta_W} - \frac{183\alpha_{\text{QED}} \hat{L}}{4 \cos^2 \theta_W} + \frac{59\hat{B}^2}{2} + 31(\hat{B})^2 + \frac{202\hat{B} \hat{L}}{3} \]

\[ + \frac{53\hat{L}^2}{2} + 19(\hat{L})^2 + 16\hat{B} \hat{L} + \frac{85\hat{T}^2}{2} + 104\hat{T} \hat{B} + \frac{244\hat{T} \hat{L}}{3} + 73(\hat{T} \hat{L})^2 \]

\[ + n_G \left[ \frac{32\alpha_{\text{QED}}^2}{9 \sin^2 \theta_W \cos^2 \theta_W} + \frac{2614\alpha_{\text{QED}}^2}{27 \sin^4 \theta_W} - \frac{1580\alpha_{\text{QED}}^2}{81 \cos^4 \theta_W} + \frac{152\alpha_{\text{QED}} \alpha_3}{3 \sin^2 \theta_W} \right] \]

\[ - \frac{584\alpha_{\text{QED}} \alpha_3}{81 \cos^2 \theta_W} + \frac{10000\alpha_{\text{QED}}^2}{27} \right\} \]

\[ + n_G \left[ -\frac{1792\alpha_{\text{QED}}^2}{27 \sin^4 \theta_W} - \frac{22880\alpha_{\text{QED}}^2}{729 \cos^4 \theta_W} - \frac{3520\alpha_{\text{QED}}^2}{81} \right] \}. \]
\[
\beta_{\sin^2 \theta_W} = \mu^2 \frac{d \sin^2 \theta_W}{d \mu^2} \\
= \frac{\alpha_{\text{QED}}}{4\pi} \left\{ \frac{\sin^2 \theta_W}{6} + \frac{43 \cos^2 \theta_W}{6} + n_G \left[ \frac{20 \sin^2 \theta_W}{9} - \frac{4 \cos^2 \theta_W}{3} \right] \right\} \\
+ \frac{\alpha_{\text{QED}}}{(4\pi)^2} \left\{ \alpha_{\text{QED}} + \frac{\alpha_{\text{QED}} \tan^2 \theta_W}{2} + \frac{259 \alpha_{\text{QED}} \cot^2 \theta_W}{6} - \frac{17 \sin^2 \theta_W \text{tr} \hat{T}}{6} \right\} \\
+ 3 \frac{\cos^2 \theta_W \text{tr} \hat{T}}{2} - \frac{5 \sin^2 \theta_W \text{tr} \hat{B}}{6} + \frac{3 \cos^2 \theta_W \text{tr} \hat{B}}{2} - \frac{5 \sin^2 \theta_W \text{tr} \hat{L}}{2} + \frac{\cos^2 \theta_W \text{tr} \hat{L}}{2} \\
+ n_G \left[ \frac{2 \alpha_{\text{QED}}}{3} + \frac{95 \alpha_{\text{QED}} \tan^2 \theta_W}{27} - \frac{49 \alpha_{\text{QED}} \cot^2 \theta_W}{3} + \frac{44 \sin^2 \theta_W \alpha_3}{9} - \frac{4 \cos^2 \theta_W \alpha_3}{3} \right] \\
+ \frac{\alpha_{\text{QED}}}{(4\pi)^3} \left\{ \frac{2279 \alpha_{\text{QED}}^2}{192 \sin^2 \theta_W} + \frac{1403 \alpha_{\text{QED}}^2}{576 \cos^2 \theta_W} + \frac{163 \alpha_{\text{QED}}^2 \tan^2 \theta_W}{576 \cos^2 \theta_W} + \frac{667111 \alpha_{\text{QED}}^2 \cot^2 \theta_W}{1728 \sin^2 \theta_W} \right\} \\
+ \alpha_{\text{QED}} \lambda + \frac{3 \alpha_{\text{QED}} \tan^2 \theta_W \lambda}{2} - \frac{3 \alpha_{\text{QED}} \cot^2 \theta_W \lambda}{2} - 3 \sin^2 \theta_W \lambda^2 + 3 \cos^2 \theta_W \lambda^2 \\
- \frac{881 \alpha_{\text{QED}} \text{tr} \hat{T}}{48} - \frac{2827 \alpha_{\text{QED}} \tan^2 \theta_W \text{tr} \hat{T}}{288} + \frac{729 \alpha_{\text{QED}} \cot^2 \theta_W \text{tr} \hat{T}}{32} - \frac{29 \sin^2 \theta_W \alpha_3 \text{tr} \hat{T}}{3} \\
+ 7 \cos^2 \theta_W \text{tr} \hat{B}^2 - \frac{389 \alpha_{\text{QED}} \text{tr} \hat{B}}{48} - \frac{1267 \alpha_{\text{QED}} \tan^2 \theta_W \text{tr} \hat{B}}{288} + \frac{729 \alpha_{\text{QED}} \cot^2 \theta_W \text{tr} \hat{B}}{32} \\
- \frac{17 \sin^2 \theta_W \alpha_3 \text{tr} \hat{B}}{3} + 7 \cos^2 \theta_W \alpha_3 \text{tr} \hat{B} - \frac{229 \alpha_{\text{QED}} \text{tr} \hat{L}}{16} - \frac{281 \alpha_{\text{QED}} \tan^2 \theta_W \text{tr} \hat{L}}{32} \\
+ \frac{243 \alpha_{\text{QED}} \cot^2 \theta_W \text{tr} \hat{L}}{32} + \frac{61 \sin^2 \theta_W \text{tr} \hat{B}^2}{16} - \frac{57 \cos^2 \theta_W \text{tr} \hat{B}^2}{16} + \frac{17 \sin^2 \theta_W (\text{tr} \hat{B})^2}{8} \\
- \frac{45 \cos^2 \theta_W (\text{tr} \hat{B})^2}{8} + \frac{157 \sin^2 \theta_W (\text{tr} \hat{B} \text{tr} \hat{L})}{16} - \frac{15 \cos^2 \theta_W (\text{tr} \hat{B} \text{tr} \hat{L})}{16} + \frac{87 \sin^2 \theta_W \text{tr} \hat{L}^2}{16} \\
- \frac{19 \cos^2 \theta_W \text{tr} \hat{L}^2}{16} + \frac{33 \sin^2 \theta_W (\text{tr} \hat{L})^2}{8} - \frac{5 \cos^2 \theta_W (\text{tr} \hat{L})^2}{8} + \frac{5 \sin^2 \theta_W \text{tr} \hat{B}}{8} \\
- \frac{27 \cos^2 \theta_W \text{tr} \hat{B} \text{tr} \hat{L}}{8} + \frac{113 \sin^2 \theta_W (\text{tr} \hat{B} \text{tr} \hat{L})}{16} - \frac{57 \cos^2 \theta_W (\text{tr} \hat{B} \text{tr} \hat{L})}{16} + \frac{59 \sin^2 \theta_W \text{tr} \hat{B}}{4} \\
- \frac{45 \cos^2 \theta_W \text{tr} \hat{B} \text{tr} \hat{L}}{4} + \frac{199 \sin^2 \theta_W \text{tr} \hat{B} \text{tr} \hat{L}}{12} - \frac{15 \cos^2 \theta_W \text{tr} \hat{B} \text{tr} \hat{L}}{4} \\
+ \frac{101 \sin^2 \theta_W (\text{tr} \hat{B})^2}{8} - \frac{45 \cos^2 \theta_W (\text{tr} \hat{B})^2}{8} + \frac{127 \alpha_{\text{QED}}^2}{36 \sin^2 \theta_W} + \frac{119 \alpha_{\text{QED}}^2}{108 \cos^2 \theta_W} \\
- \frac{290 \alpha_{\text{QED}} \tan^2 \theta_W}{81 \cos^2 \theta_W} - \frac{6412 \alpha_{\text{QED}} \cot^2 \theta_W}{27 \sin^2 \theta_W} + \frac{2 \alpha_{\text{QED}} \alpha_3}{9} - \frac{137 \alpha_{\text{QED}} \tan^2 \theta_W \alpha_3}{81} \\
- 13 \alpha_{\text{QED}} \cot^2 \theta_W \alpha_3 + \frac{1375 \sin^2 \theta_W \alpha_3^2}{27} - \frac{125 \cos^2 \theta_W \alpha_3^2}{3} \right\} \\
+ n_G \left[ \frac{-11 \alpha_{\text{QED}}}{9 \sin^2 \theta_W} + \frac{55 \alpha_{\text{QED}}}{81 \cos^2 \theta_W} - \frac{5225 \alpha_{\text{QED}} \tan^2 \theta_W}{729 \cos^2 \theta_W} + \frac{415 \alpha_{\text{QED}} \cot^2 \theta_W}{27 \sin^2 \theta_W} \\
- \frac{484 \sin^2 \theta_W \alpha_3^2}{81} + \frac{44 \cos^2 \theta_W \alpha_3^2}{9} \right] \right\}. \quad (C2)
\]


Appendix D: Comparison with Ref. [16]

In this Appendix we provide the explicit form of the expressions needed for the comparison with Ref. [16]. In that paper two-component Weyl spinors were used. To make contact with our convention based on four-component Dirac spinors we define

\[
\Psi_D = \begin{pmatrix} \chi \\ \xi^\dagger \end{pmatrix},
\]

where \( \xi \) and \( \chi \) are left-handed Weyl spinors and \( \Psi_D \) denotes a Dirac spinor. Thus, the Lagrange density of the SM can be expressed in terms of 45 Weyl spinors:

\[
\chi_t, \xi_t, \chi_b, \xi_b, \chi_{c}, \xi_{c}, \chi_{\nu_e}, \chi_{\nu_e}, \chi_{\nu}, \xi_{\nu}, \chi_{u}, \xi_{u}, \chi_{d}, \xi_{d}, \chi_{e}, \xi_{e}, \chi_{\nu_\tau}.
\]

For simplicity we have suppressed the \( SU(3) \) color indices for all quark spinors. Of course, each quark spinor has to be understood as a triplet in color space. In this basis the Yukawa matrices become 45 \( \times \) 45 dimensional.

In the notation of [16] the part of the Lagrange density describing the Yukawa couplings is given by

\[
- \frac{1}{2} \left( Y^{aij} \phi^a \psi_i \bar{\psi}_j + \bar{Y}^{aij} \phi^a \bar{\psi}_i \psi_j \right),
\]

where \( Y^a \) and \( \bar{Y}^a \) are the (complex conjugated) Yukawa matrices, \( \phi^a \) are real scalar fields, \( \bar{\psi}_i \) and \( \psi^i \) are (Hermitian conjugated) spinor fields. There are four real scalar fields in the SM, which means that we have four Yukawa matrices. They are given by

\[
Y^1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0_{3\times 3} & y_t & 0_{3\times 3} & 0_{3\times 1} & 0_{3\times 1} & \cdots \\
y_t & 0_{3\times 3} & 0_{3\times 3} & 0_{3\times 1} & 0_{3\times 1} & \cdots \\
0_{3\times 3} & 0_{3\times 3} & 0_{3\times 3} & 0_{3\times 1} & 0_{3\times 1} & \cdots \\
0_{3\times 3} & 0_{3\times 3} & y_b & 0_{3\times 3} & 0_{3\times 1} & \cdots \\
o_{1\times 3} & o_{1\times 3} & 0_{1\times 3} & 0_{1\times 3} & 0_{1\times 3} & \cdots \\
o_{1\times 3} & 0_{1\times 3} & 0_{1\times 3} & 0_{1\times 3} & 0_{1\times 3} & \cdots \\
o_{1\times 3} & 0_{1\times 3} & 0_{1\times 3} & 0_{1\times 3} & 0_{1\times 3} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},
\]

\[
Y^2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0_{3\times 3} & 0_{3\times 3} & 0_{3\times 3} & y_t & 0_{3\times 3} & 0_{3\times 1} & 0_{3\times 1} & \cdots \\
0_{3\times 3} & 0_{3\times 3} & -y_t & 0_{3\times 3} & 0_{3\times 1} & 0_{3\times 1} & \cdots \\
0_{3\times 3} & -y_t & 0_{3\times 3} & 0_{3\times 3} & 0_{3\times 1} & 0_{3\times 1} & \cdots \\
y_t & 0_{3\times 3} & 0_{3\times 3} & 0_{3\times 3} & 0_{3\times 1} & 0_{3\times 1} & \cdots \\
o_{1\times 3} & o_{1\times 3} & 0_{1\times 3} & 0_{1\times 3} & 0_{1\times 3} & 0 & \cdots \\
o_{1\times 3} & 0_{1\times 3} & 0_{1\times 3} & 0_{1\times 3} & 0_{1\times 3} & 0 & \cdots \\
o_{1\times 3} & 0_{1\times 3} & 0_{1\times 3} & 0_{1\times 3} & 0_{1\times 3} & \cdots & \cdots \cdots \end{pmatrix},
\]
In the SM we have values in the range 1
In the above formulas, all matrix elements not explicitly given are zero. $Y^1/Y^3$ is the Yukawa matrix of the real/the imaginary part of the isospin down component of the SM Higgs doublet. $Y^2/Y^4$ is the Yukawa matrix of the real/the imaginary part of the isospin up component of the SM Higgs doublet. So we have

$$H = \frac{1}{\sqrt{2}} \left( \phi^2 + i\phi^4 \right).$$

The part of the Lagrange density describing the Higgs self-interaction reads in the notation of Ref. [16]

$$-\frac{1}{4!} \lambda_{abcd} \phi^a \phi^b \phi^c \phi^d.$$ 

In the SM we have

$$\lambda_{aaaa} = 6 \times (4\pi \lambda),$$
$$\lambda_{aabb} = \lambda_{abab} = 2 \times (4\pi \lambda) \ (a \neq b),$$
$$\lambda_{abcd} = 0 \ (\text{otherwise}).$$

All relations discussed so far are generic for all three gauge groups. However, there are some expressions which depend on the specific gauge group one wants to consider. The remainder of this Section lists these expressions. For each of the three gauge groups, we give the expressions for the generators in the representations of the scalar fields, $S^A$, and of the Weyl spinors, $R^A$. We also give expressions for the invariants $T(S), C(S), T(R), C(R), C(G), r$, all of which are symbols used in Ref. [16]. They are defined as

$$\text{Tr} \left( S^A S^B \right) = \delta^{AB} T(S),$$
$$\text{Tr} \left( R^A R^B \right) = \delta^{AB} T(R),$$
$$f^{ACD} f^{BCD} = \delta^{AB} C(G),$$

where $f^{ABC}$ are the structure constants of the respective gauge group. The indices $A, B, C$ take values in the range $1, \ldots, r$, where $r$ gives the dimension of the group.
1. $U(1)$

For the case of $U(1)$ gauge group, we have

$$S^1 = \frac{i}{2} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},$$

(D9)

$$R^1 = \text{Diag} \left( \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, -\frac{2}{3}, -\frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, -\frac{1}{2}, 1, -\frac{1}{2}, \ldots \right),$$

(D10)

where the ellipsis is to be replaced twice by the first 15 entries. The derivation of $S^1$ is given below. The entries in $R^1$ are the hypercharges $Y$ of the respective spinors. They can be derived by using the relation $Y = Q - I_3$, where $Q$ corresponds to the electric charge and $I_3$ is the weak isospin. Furthermore, we have

$$T(S) = 1, \quad C(S) = \frac{1}{4} \mathbb{1}_{4 \times 4},$$

$$T(R) = 10, \quad C(R) = (R^1)^2,$$

$$C(G) = 0, \quad r = 1.$$  

(D11)

Let us now show how to derive Eq. (D9). We want to find the $U(1)$ representation transforming the four real scalar fields of the Higgs doublet of the SM (see Eq. (D5)). The matrix $S^1$ is the generator of this transformation,

$$\begin{pmatrix} \phi'_1 \\ \phi'_2 \\ \phi'_3 \\ \phi'_4 \end{pmatrix} = e^{i \omega S^1} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{pmatrix} = \left( \mathbb{1} + i \omega S^1 + \mathcal{O}(\omega^2) \right) \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{pmatrix}.$$

(D12)

with $\omega$ being the transformation parameter. In a next step we take advantage of the fact that it is known how the SM Higgs doublet transforms in order to determine $S^1$. As the SM Higgs doublet has hypercharge 1/2, we have

$$\frac{1}{\sqrt{2}} \begin{pmatrix} \phi^2 + i\phi^4 \\ \phi^1 + i\phi^3 \end{pmatrix} = H' = e^{i \omega (\frac{1}{2} \mathbb{1})} H = e^{i \omega (\frac{1}{2} \mathbb{1})} \frac{1}{\sqrt{2}} \begin{pmatrix} \phi^2 + i\phi^4 \\ \phi^1 + i\phi^3 \end{pmatrix}$$

$$= \left[ \mathbb{1} + i \omega \left( \frac{1}{2} \mathbb{1} \right) + \mathcal{O}(\omega^2) \right] \frac{1}{\sqrt{2}} \begin{pmatrix} \phi^2 + i\phi^4 \\ \phi^1 + i\phi^3 \end{pmatrix}$$

$$= H + \omega \frac{1}{2} \frac{1}{\sqrt{2}} \begin{pmatrix} -\phi^4 + i\phi^2 \\ -\phi^3 + i\phi^1 \end{pmatrix} + \mathcal{O}(\omega^2).$$

(D13)

With the help of the last equation one can determine the transformation of the SM Higgs doublet, like, e.g., $\phi'_1 = \phi_1 - \frac{\omega}{2} \phi_3 + \mathcal{O}(\omega^2)$. It is then straightforward to determine $S^1$ by inserting the equations found in this way in Eq. (D12).
2. \textit{SU}(2)

For the \textit{SU}(2) group, we have

\begin{align*}
S^1 &= -\frac{i}{2} \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, & S^2 &= -\frac{i}{2} \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, & S^3 &= -\frac{i}{2} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad (D14)
\end{align*}

and

\begin{align*}
R^A &= 1/2 \begin{pmatrix} 
\sigma^A_{1,1} \mathbb{I}_{3 \times 3} & 0_{3 \times 3} & \sigma^A_{1,2} \mathbb{I}_{3 \times 3} & 0_{3 \times 3} & 0_{3 \times 1} & 0_{3 \times 1} & \ldots \\
0_{3 \times 3} & 0_{3 \times 3} & 0_{3 \times 3} & 0_{3 \times 3} & 0_{3 \times 1} & 0_{3 \times 1} & \ldots \\
\sigma^A_{2,1} \mathbb{I}_{3 \times 3} & 0_{3 \times 3} & \sigma^A_{2,2} \mathbb{I}_{3 \times 3} & 0_{3 \times 3} & 0_{3 \times 1} & 0_{3 \times 1} & \ldots \\
0_{3 \times 3} & 0_{3 \times 3} & 0_{3 \times 3} & 0_{3 \times 3} & 0_{3 \times 1} & 0_{3 \times 1} & \ldots \\
0_{1 \times 3} & 0_{1 \times 3} & 0_{1 \times 3} & 0_{1 \times 3} & 0_{1 \times 3} & 0_{1 \times 3} & 0_{1 \times 3} \\
0_{1 \times 3} & 0_{1 \times 3} & 0_{1 \times 3} & 0_{1 \times 3} & 0_{1 \times 3} & 0_{1 \times 3} & 0_{1 \times 3} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\end{pmatrix} , \quad (D15)
\end{align*}

In the last equation \( \sigma^A \) are the Pauli matrices. Not all of the matrix elements left out in \( R^A \) vanish, but these elements play no role for the comparison of our results to \([16]\). The derivation of the generators \( S^A \) proceeds in analogy to the derivation of \( S^1 \) explained in the former Subsection. One merely has to substitute the generator of \( U(1) \), \( \frac{1}{2} \mathbb{I} \), by the generators of \( SU(2) \), \( \frac{1}{2} \sigma^A \), in Eq. \((D13)\).

We furthermore have

\begin{align*}
T(S) &= 1, & C(S) &= \frac{3}{4} \mathbb{I}_{4 \times 4}, \\
T(R) &= 6, & C(R) &= \frac{3}{4} \text{Diag} \left( 1, 1, 1, 0, 0, 0, 1, 1, 1, 0, 0, 0, 1, 0, 1, \ldots \right) , \\
C(G) &= 2, & r &= 3. \quad (D16)
\end{align*}

The ellipsis has to be replaced by the first entries twice.

3. \textit{SU}(3)

In the SM the matrices \( S^A \) vanish for the group \( SU(3) \). The matrices \( R^A \) are block-diagonal and read

\begin{equation*}
R^A = \frac{1}{2} \text{BlockDiag} \left( \Lambda^A, -\Lambda^A^T, \Lambda^A, -\Lambda^A^T, 0, 0, 0, \ldots \right). \quad (D17)
\end{equation*}

The ellipsis has to be replaced twice by the former entries, which contain the Gell-Mann matrices \( \Lambda^A \).

Finally, we have

\begin{align*}
T(S) &= 0, & C(S) &= 0_{4 \times 4}.
\end{align*}
\[ T(R) = 6, \quad C(R) = \frac{4}{3} \text{BlockDiag}(\mathbb{1}_{12 \times 12}, 0_{3 \times 3}, \mathbb{1}_{12 \times 12}, 0_{3 \times 3}, \mathbb{1}_{12 \times 12}, 0_{3 \times 3}), \]

\[ C(G) = 3, \quad r = 8. \]  

\[ \text{(D18)} \]

[1] D. J. Gross and F. Wilczek, Phys. Rev. Lett. 30 (1973) 1343.
[2] H. D. Politzer, Phys. Rev. Lett. 30 (1973) 1346.
[3] D. R. T. Jones, Nucl. Phys. B 75, 531 (1974).
[4] O. V. Tarasov and A. A. Vladimirov, Sov. J. Nucl. Phys. 25 (1977) 585 [Yad. Fiz. 25 (1977) 1104].
[5] W. E. Caswell, Phys. Rev. Lett. 33 (1974) 244.
[6] E. Egorian and O. V. Tarasov, Teor. Mat. Fiz. 41 (1979) 26 [Theor. Math. Phys. 41 (1979) 863].
[7] D. R. T. Jones, Phys. Rev. D 25 (1982) 581.
[8] M. S. Fischler and C. T. Hill, Nucl. Phys. B 193 (1981) 53.
[9] M. E. Machacek and M. T. Vaughn, Nucl. Phys. B 222 (1983) 83.
[10] I. Jack and H. Osborn, Nucl. Phys. B 249 (1985) 472.
[11] T. Curtright, Phys. Rev. D 21 (1980) 1543.
[12] D. R. T. Jones, Phys. Rev. D 22 (1980) 3140.
[13] O. V. Tarasov, A. A. Vladimirov and A. Y. Zharkov, Phys. Lett. B 93 (1980) 429.
[14] S. A. Larin and J. A. M. Vermaseren, Phys. Lett. B 303 (1993) 334 [hep-ph/9302208].
[15] M. Steinhauser, Phys. Rev. D 59 (1999) 054005 [arXiv:hep-ph/9809507].
[16] A. G. M. Pickering, J. A. Gracey and D. R. T. Jones, Phys. Lett. B 510 (2001) 347 [Phys. Lett. B 512 (2001 ERRAT,B535,377.2002) 230] [arXiv:hep-ph/0104247].
[17] T. van Ritbergen, J. A. M. Vermaseren and S. A. Larin, Phys. Lett. B 400 (1997) 379 [hep-ph/9701390].
[18] M. Czakon, Nucl. Phys. B 710 (2005) 485 [arXiv:hep-ph/0411261].
[19] M. Fischler and J. Oliensis, Phys. Lett. B 119 (1982) 385.
[20] M. E. Machacek and M. T. Vaughn, Nucl. Phys. B 236 (1984) 221.
[21] M. E. Machacek and M. T. Vaughn, Nucl. Phys. B 249 (1985) 70.
[22] C. Ford, I. Jack and D. R. T. Jones, Nucl. Phys. B 387 (1992) 373 [Erratum-ibid. B 504 (1997) 551] [arXiv:hep-ph/0111190].
[23] M. x. Luo and Y. Xiao, Phys. Rev. Lett. 90 (2003) 011601 [arXiv:hep-ph/0207271].
[24] K. G. Chetyrkin and M. F. Zoller, JHEP 1206 (2012) 033 [arXiv:1205.2892 [hep-ph]].
[25] L. N. Mihaila, J. Salomon and M. Steinhauser, Phys. Rev. Lett. 108 (2012) 151602 [arXiv:1201.5868 [hep-ph]].
[26] D. M. Pierce, J. A. Bagger, K. T. Matchev and R. -j. Zhang, Nucl. Phys. B 491 (1997) 3 [hep-ph/9606211].
[27] L. F. Abbott, Nucl. Phys. B 185 (1981) 189.
[28] A. Denner, G. Weiglein and S. Dittmaier, Nucl. Phys. B 440 (1995) 95 [hep-ph/9410338].
[29] S. A. Larin, F. V. Tkachov and J. A. M. Vermaseren, preprint NIKHEF-H-91-18 (1991).
[30] P. Nogueira, J. Comput. Phys. 105 (1993) 279.
[31] R. Harlander, T. Seidensticker and M. Steinhauser, Phys. Lett. B 426 (1998) 125 [hep-ph/9712228].
[32] T. Seidensticker, hep-ph/9905298.
[33] J. A. M. Vermaseren, arXiv:math-ph/0010025.
[34] M. Steinhauser, Comput. Phys. Commun. 134 (2001) 335 [arXiv:hep-ph/0009029].
[35] T. Hahn, Comput. Phys. Commun. 140 (2001) 418 [hep-ph/0012260].
[36] N. D. Christensen and C. Duhr, Comput. Phys. Commun. 180 (2009) 1614 [arXiv:0806.4194 [hep-ph]].
[37] J. C. Collins, “Renormalization,” Cambridge, Uk: Univ. Pr. (1984) 380p.
[38] M. E. Peskin and D. V. Schroeder, Reading, USA: Addison-Wesley (1995) 842 p.
[39] S. L. Adler, Phys. Rev. 177 (1969) 2426.
[40] J. S. Bell und R. Jackiw, Nuovo Cim. A 60 (1969) 47.
[41] K. Nakamura et al. [Particle Data Group Collaboration], J. Phys. G G 37 (2010) 075021.
[42] V. A. Smirnov, “Applied asymptotic expansions in momenta and masses,” Springer Tracts Mod. Phys. 177 (2002) 1.
[43] W. Martens, L. Mihaila, J. Salomon und M. Steinhauser, Phys. Rev. D 82 (2010) 095013 [arXiv:1008.3070 [hep-ph]].