Initial-boundary value problems for fractional diffusion equations with time-dependent coefficients

Adam Kubica, Masahiro Yamamoto

February 6, 2018

Abstract

We discuss an initial-boundary value problem for a fractional diffusion equation with Caputo time-fractional derivative where the coefficients are dependent on spatial and time variables and the zero Dirichlet boundary condition is attached. We prove the unique existence of weak and regular solutions.

1 Introduction

In this paper we study a parabolic type equation with time fractional Caputo derivative and general elliptic operator. This problem were considered in many papers (see [1], [2], [9], [15], [16], [17]), however, in our opinion, it is not completely understand yet. The main issue which should be explored more deeply is the meaning of initial condition $u(t=0)$ and the correctness of weak formulation of the Caputo derivative given for example in [17]. In this paper we solved this problem only partially and we will address to it in another paper. Our results suggest that equations with the Caputo derivative of order $\alpha \in (0, 1)$ requires more regularity of data if $\alpha$ is equal to or less than $\frac{1}{2}$. Under additional assumptions on data, we obtain the continuity of solution, but the continuity holds in some dual space which order depends on $\alpha$.

The second contribution of our work is the study of general elliptic operator for which one can not apply Fourier expansion of solution (see [13], [14]) and it is impossible to reduce the problem to ordinary fractional equation.

---

*Department of Mathematics and Information Sciences, Warsaw University of Technology, ul. Koszykowa 75, 00-662 Warsaw, Poland, E-mail addresses: A.Kubica@mini.pw.edu.pl
†Department of Mathematical Sciences, The University of Tokyo, Komaba, Meguro, Tokyo - 153, Japan
‡Corresponding author. E-mail addresses: myama@ms.u-tokyo.ac.jp (M. Yamamoto)
Finally, our approach follows standard procedure for classical parabolic problems: first we construct approximate solution, next we obtain a priori estimate and further we obtain solution by the weak compactness argument.

Now we recall the definitions of the fractional integration $I^\alpha$ and the fractional Riemann-Liouville derivative

\[ I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)d\tau \quad \text{for } \alpha > 0, \]

\[ \partial^\alpha f(t) = \frac{d}{dt} I^{1-\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-\tau)^{-\alpha} f(\tau)d\tau \quad \text{for } \alpha \in (0, 1). \]

The formula for $I^\alpha f$ is meaningful for $f \in L^1$. However the formula for the Riemann-Liouville derivative requires more regularity of $f$ and is well defined at least for absolutely continuous $f$ (see proposition 3 in the appendix) and then $\partial^\alpha f$ is in $L^1$.

The problem which we shall consider, involves the fractional Caputo derivative

\[ D^\alpha f(t) = \partial^\alpha [f(\cdot) - f(0)](t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-\tau)^{-\alpha} [f(\tau) - f(0)]d\tau, \]

and this formula is again meaningful for absolutely continuous function $f$.

The aim of this paper is to analyze partial differential equations of parabolic type which contain the fractional Caputo derivatives. If we deal with weak solutions, then the Caputo fractional derivative should be understood in a suitable way. To be more precise we have to formulate the problem which we analyze in this paper.

Assume that $T < \infty$ and $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary, where $N \geq 2$. We set

\[ \Omega^T = \Omega \times (0, T). \]

We shall consider the following problem

\[
\begin{aligned}
D^\alpha u &= Lu + f \quad \text{in } \Omega^T, \\
\left. u \right|_{\partial \Omega} &= 0 \quad \text{for } t \in (0, T) \\
\left. u \right|_{t=0} &= u_0 \quad \text{in } \Omega,
\end{aligned}
\]

where

\[ Lu(x, t) = \sum_{i,j=1}^N \partial_i (a_{i,j}(x,t) \partial_j u(x,t)) + \sum_{j=1}^N b_j(x,t) \partial_j u(x,t) + c(x,t) u(x,t), \]

\[ \partial_i = \frac{\partial}{\partial x_i} \text{ for } i = 1, \ldots, N, \text{ and by } D^\alpha \text{ we denote the Caputo fractional time derivative, i.e.} \]

\[ D^\alpha u(x,t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-\tau)^{-\alpha} [u(x,\tau) - u(x,0)]d\tau. \]

In the whole paper, the fractional integration and the fractional differentiation are related only with time variable, and

\[ I^\alpha w(x,t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} w(x,\tau)d\tau, \quad \partial^\alpha u(x,t) = \frac{d}{dt} I^{1-\alpha} [u(x,\cdot)](t). \]
Therefore we have to guarantee the existence of \( u \), we could set
\[
I^{1-\alpha}[u(x, \cdot) - u(x, 0)](t),
\]
and \( u(x, 0) \) is involved. Therefore we have to guarantee the existence of \( u|_{t=0} \) in some sense and initial condition \( h_3 \) should be fulfilled. If these two demands are satisfied, then for problem \( \mathbf{4} \) we could set
\[
D^\alpha u(x, t) = \frac{d}{dt} I^{1-\alpha}[u(x, \cdot) - u_0(x)](t).
\]
The above formula is a starting point in formulating a weak form of the Caputo derivative related with the problem \( \mathbf{4} \) (we follow \([17]\)). We shall show that our construction of the solution of \( \mathbf{4} \) will fulfill these two demands, at least in the case of \( L = \Delta \) (see theorem \( \mathbf{3} \)). This issue for the general elliptic operator will be examined in another paper.

We assume that the operator \( L \) is uniform elliptic, i.e., there exist positive constants \( \lambda, \mu \) such that
\[
\lambda |\xi|^2 \leq \sum_{i,j=1}^{N} a_{i,j}(x, t)\xi_i\xi_j \leq \mu |\xi|^2 \quad \text{for} \quad \xi \in \mathbb{R}^n, \ t \in [0, T],
\]
with measurable coefficients \( a_{i,j} \) and \( a_{i,j} = a_{j,i} \).

We recall the result by Zacher \([17]\) concerning weak solutions of \( \mathbf{4} \). We introduce notation
\[
W^\alpha(u_0, H^1_0(\Omega), L^2(\Omega)) = \{ u \in L^2(0,T; H^1_0(\Omega)) : I^{1-\alpha}(u-u_0) \in 0 H^1(0,T; H^{-1}(\Omega)) \},
\]
where the subscript 0 of \( H^1 \) means vanishing of the trace for \( t = 0 \) and \( AC := AC[0,T] \) denotes the space of absolutely continuous functions defined on \( [0,T] \) (see definition 1.2, chap. 1 \([10]\)). The following theorem is a special case of theorem 3.1 \([17]\) (see also corollary 4.1 in \([17]\)).

**Theorem 1** \((\mathbb{17})\). Assume that \( \Omega \subseteq \mathbb{R}^N \) is a smooth bounded domain, \( u_0 \in L^2(\Omega), f \in L^2(0,T; H^{-1}(\Omega)), b_j, c \in L^\infty(\Omega) \) and \((8)\) holds. Then there exists a unique weak solution \( u \in W^\alpha(u_0, H^1_0(\Omega), L^2(\Omega)) \) of \((\mathbb{4})\), i.e.,
\[
\frac{d}{dt} \int_\Omega I^{1-\alpha}[u(x, t) - u_0(x)]\varphi(x)dx + \sum_{i,j=1}^{N} \int_\Omega a_{i,j}(x, t)\partial_j u(x, t)\partial_i \varphi(x)dx
\]
\[
= \sum_{j=1}^{N} \int_\Omega b_j(x, t)\partial_j u(x, t)\varphi(x)dx + \int_\Omega c(x, t)u(x, t)\varphi(x)dx + \langle f(t), \varphi \rangle_{H^{-1} \times H^1_0(\Omega)}
\]
holds for all \( \varphi \in H^1_0(\Omega) \) and a.a. \( t \in (0, T) \). Furthermore, the following estimate
\[
||I^{1-\alpha}(u - u_0)||_{H^1(0,T; H^{-1}(\Omega))} + ||u||_{L^2(0,T; H^1_0(\Omega))} \leq C[||u_0||_{L^2(\Omega)} + ||f||_{L^2(0,T; H^{-1}(\Omega))}]
\]
holds.
Remark 1. By theorem \[ \text{[1]} \] for given \( u_0 \in L^2(\Omega) \) u satisfying \( \text{(3)} \) exists uniquely. However this result does not guarantee that \( u_{|t=0} = u_0 \). In particular, it is not clear that \( u_{|t=0} = u_0 \). In other words, the first term on left-hand side of \( \text{(3)} \) may not represent the Caputo derivative. However, in the paper \[ \text{[17]} \], it is remarked (see p.8) that if \( \frac{d}{dt} I^{1-\alpha}[u(x,t) - u_0(x)] \) is in \( C([0,T];H^{-1}(\Omega)) \), then \( u \in C([0,T];H^{-1}(\Omega)) \) and \( u(0) = u_0 \). In this paper we develop this idea in order to overcome the difficulties related to the definition of the initial value of solution (see proposition \[ \text{[2]} \] in appendix).

In the present paper, we first obtain a result similar to theorem \[ \text{[1]} \] but its proof is based on special approximating sequence, which further enables us to improve the regularity of the solutions.

Theorem 2. Assume that \( \alpha \in (0,1) \), \( T > 0 \), \( u_0 \in L^2(\Omega) \) and \( f \in L^2(0,T;H^{-1}(\Omega)) \)  
Assume that \( \text{(3)} \) holds and for some \( p_1, p_2 \in [2, \frac{2N}{N-2}] \) we have \( b \in L^\infty(0,T;L^{\frac{2p_1}{p_1-2}}(\Omega)) \), \( c \in L^\infty(0,T;L^{\frac{2p_2}{p_2-2}}(\Omega)) \). Then there exists a unique weak solution \( u \in W^\alpha(u_0, H^1_0(\Omega), L^2(\Omega)) \) of \( \text{(4)} \), i.e., \( \text{(3)} \) holds and \( u \) satisfies the following estimate

\[
\|I^{1-\alpha}(u - u_0)\|_{H^1(0,T;H^{-1}(\Omega))} + \|u\|_{L^2(0,T;H^1_0(\Omega))} + \|u\|_{H^\frac{2}{p}(0,T;L^2(\Omega))} \leq C \left( \|u_0\|_{L^2(\Omega)} + \|f\|_{L^2(0,T;H^{-1}(\Omega))} \right),
\]

(11)

where \( C \) depends only on \( \alpha, \mu, \lambda, T \|b\|_{L^\infty(0,T;L^{\frac{2p_1}{p_1-2}}(\Omega))}, \|c\|_{L^\infty(0,T;L^{\frac{2p_2}{p_2-2}}(\Omega))} \).

Furthermore, if \( \alpha > \frac{1}{2} \), then \( u \in C([0,T];H^{-1}(\Omega)) \) and \( u_{|t=0} = u_0 \).

Here and henceforth we set \( \frac{2p}{p-2} = \infty \) if \( p = 2 \).

In the case of \( L = \Delta \) we are able to define \( u_{|t=0} \) for \( \alpha \leq \frac{1}{2} \). To formulate the result we need the following notation.

\[ \mathcal{H}^k = \left\{ w \in H^k(\Omega) : \Delta^a w|_{\partial \Omega} = 0 \text{ for } a = 0, 1, \ldots, \left[ \frac{k-1}{2} \right] \right\}, \]

(12)

and \( \mathcal{H}^k \) denotes the dual space to \( \mathcal{H}^k \).

Theorem 3. Assume that \( u_0 \in L^2(\Omega), f \in L^2(0,T;H^{-1}(\Omega)) \) and \( u \) is a solution of \( \text{(4)} \) for \( L = \Delta \) given by theorem \[ \text{[2]} \]. Then

- if \( \alpha > \frac{1}{2} \), then \( I^{1-\alpha}[u - u_0] \in \mathcal{H}^1(0,T;H^{-1}(\Omega)) \) and \( u \in C([0,T];H^{-1}(\Omega)), u_{|t=0} = u_0 \).

- if \( \alpha = \frac{1}{2} \) and in addition \( \partial^k f \in L^p(0,T;H^3) \) for some \( p \in (1,2) \), then \( u - u_0 = I^{1-2\alpha}[u - u_0] \in \mathcal{H}^{1,p}(0,T;H^3) \) and \( u \in C([0,T];H^3) \), \( u_{|t=0} = u_0 \).

If \( \alpha \in (0, \frac{1}{2}) \) and \( k \in \mathbb{N} \) is the smallest number such that \( \frac{1}{2} \leq (k+1)\alpha < 1 \), then

- if \( \frac{1}{2} < (k+1)\alpha \) and in addition \( \partial^m f \in L^2(0,T;H^{2m+1}) \) for \( m = 1, \ldots, k \), then \( \frac{d}{dt} I^{-(k+1)\alpha}[u - u_0] \in L^2(0,T;H^{2k+1}) \) and \( u \in C([0,T];H^{2k+1}) \), \( u_{|t=0} = u_0 \).
• if \( \frac{1}{\alpha} = (k + 1) \alpha \) and in addition \( \partial^{m \alpha} f \in L^2(0, T; (H^{2m+1})^*) \) for \( m = 1, \ldots, k \), 
\( \partial^{(k+1) \alpha} f \in L^p(0, T; (H^{2k+3})^*) \) for some \( p \in (\frac{2}{1+2 \alpha}, 2) \), then \( I^{1-(k+1) \alpha}[u - u_0] \in \mathcal{O} \) and \( u \in C([0, T]; (H^{2k+1})^*) \), \( u_{t=0} = u_0 \).

The above assumption concerning \( f \) seems to be essential in any problems with the Caputo fractional derivative. To illustrate this, we focus on the case of \( \alpha \in \left( \frac{1}{2}, \frac{3}{2} \right) \) \((k = 1)\). We shall consider simple equation

\[
D^\alpha w(t) = f(t) \quad \text{on} \quad [0, T].
\]

We shall show that the assumption \( \partial^\alpha f \in L^2(0, T) \) is crucial in the problem (13). For this purpose, we shall find \( f \in L^2(0, T) \) such that \( \partial^\alpha f \not\in L^2(0, T) \), for which the problem (13) can not have a continuous solution. We recall that the Caputo derivative has not a continuous solution with arbitrary \( \alpha \in \left( \frac{1}{2}, \frac{3}{2} \right) \).

Suppose the contrary, i.e., there exists a continuous function \( w \) such that

\[
\frac{d}{dt}I^{1-\alpha}[w(w(0))] = f(t)
\]

holds. Then applying \( I^{1+\alpha} \) to both sides of the equality, we obtain \( I[w(w(0))] = I^{1+\alpha}f(t) \). For \( \beta \in (-\frac{1}{2}, -\alpha) \) we set \( f(t) = t^\beta \). Then \( f \in L^2(0, T) \), but \( \partial^\alpha f \not\in L^2(0, T) \). Thus \( I[w(w(0))] = c_\alpha \beta t^{1+\alpha+\beta} \) and we see that \( w - w(0) = c_\alpha \beta t^{\alpha+\beta} \). The right-hand side is unbounded if \( t \to 0^+ \), and so \( w \) can not be continuous. Therefore, the problem (13) with the Caputo derivative has not a continuous solution with arbitrary \( f \in L^2(0, T) \).

Now we formulate the result concerning more regular solution.

**Theorem 4.** Assume that \( u_0 \in H_0^1(\Omega) \), \( f \in L^2(0, T; L^2(\Omega)) \), \( \text{(3)} \) holds, \( \max_{i,j} \| \nabla a_{i,j} \|_{L^\infty(\Omega^T)} < \infty \) and for some \( p_1 \in [2, \frac{2N}{N-2}], p_2 \in [2, 4] \) \( \cap \frac{2N}{N-2} \) we have \( b \in L^\infty(0, T; L^{\frac{2p_1}{p_1-2}}(\Omega)) \), \( c \in L^\infty(0, T; L^{\frac{2p_2}{p_2-2}}(\Omega)) \). Then problem (3) has exactly one solution \( u \in L^2(0, T; H^2(\Omega)) \cap H_0^1(\Omega) \) such that \( I^{1-\alpha}[u - u_0] \in \mathcal{O} \) and (3) holds almost everywhere in the sense of (2), where the Caputo derivative \( D^\alpha u \) is interpreted as weak time derivative of \( I^{1-\alpha}[u - u_0] \) and the following estimates

\[
\|u\|_{L^2(0, T; H^2(\Omega))} + \|u\|_{H_0^2(0, T; H_0^2(\Omega))} \leq C_0(\|u_0\|_{H_0^1(\Omega)} + \|f\|_{L^2(0, T; L^2(\Omega))}),
\]

\[
\|I^{1-\alpha}[u - u_0]\|_{H_0^1(0, T; L^2(\Omega))} \leq C_0(\|u_0\|_{H_0^1(\Omega)} + \|f\|_{L^2(0, T; L^2(\Omega))}),
\]

hold, where \( C_0 \) depends only on \( \alpha, \lambda, \mu, p_1, p_2\), \( T \), \( \|\nabla a_{i,j}\|_{L^\infty(\Omega^T)} \), the Poincaré constant and the \( C^2 \)-regularity of \( \partial \Omega \) and the norms \( \|b\|_{L^\infty(0, T; L^{\frac{2p_1}{p_1-2}}(\Omega))}, \|c\|_{L^\infty(0, T; L^{\frac{2p_2}{p_2-2}}(\Omega))}. \)

Furthermore, if \( \alpha > \frac{1}{2} \), then \( u \in C([0, T]; L^2(\Omega)) \) and \( u_{t|t=0} = u_0 \).

2. **Notations**

First we introduce the space

\[
Y_\alpha(T) = \{ h \in C^1(0, T) : t^{1-\alpha}h'(t) \in C[0, T] \}
\]

(16)
with the norm \( \| h \|_{Y_\alpha(T)} = \| h \|_{C[0,T]} + \| t^{1-\alpha} h' \|_{C[0,T]} \). Then \( Y_\alpha(T) \) is a Banach space. If \( H = (h_1, \ldots, h_k) \), then we shall write \( H \in Y_\alpha(T) \), if \( |H| \in Y_\alpha(T) \), where \( |\cdot| \) means the maximum norm on \( \mathbb{R}^k \).

By assumption (8) we have \( a_{i,j} \in L^\infty(\Omega^T) \) (proposition 8). We denote by \( \eta_k(t) \) the standard smoothing kernel, i.e. \( \eta_k \in C_0^\infty(-\varepsilon/T, \varepsilon/T) \), \( \eta_k \) is nonnegative, \( \int_{\mathbb{R}} \eta_k(t)dt = 1 \) and in addition \( \eta_k(t) = \eta_k(-t) \). Then we set

\[
a_{i,j}^n(x,t) = \eta_n(\cdot) * a_{i,j}(x,\cdot)(t), \tag{17}
\]

where we extend \( a_{i,j}(x,t) \) by even reflection for \( t \notin (0,T) \). Then

\[
a_{i,j}^n \longrightarrow a_{i,j} \text{ in } L^2(\Omega^T), \tag{18}
\]

and by definition (17) and (8) we obtain

\[
\lambda |\xi|^2 \leq \sum_{i,j=1}^N a_{i,j}^n(x,t)\xi_i\xi_j \leq \mu |\xi|^2 \quad \forall t \in [0,T], \quad \forall \xi \in \mathbb{R}^n. \tag{19}
\]

As a result we have

\[
a_{i,j}^n(g)(t) \equiv \int_\Omega a_{i,j}^n(x,t)g(x)dx \in Y_\alpha(T) \quad \text{for } n \in \mathbb{N}, \quad i,j \in \{1, \ldots, N\}, \quad g \in L^1(\Omega). \tag{20}
\]

If we extend function \( b_j, c \) by zero for \( t \notin (0,T) \), then the functions \( b_j^n(x,t) \) and \( c^n(x,t) \) are defined analogously, i.e.

\[
b_j^n(x,t) = \eta_n(\cdot) * b_j(x,\cdot)(t), \quad c^n(x,t) = \eta_n(\cdot) * c(x,\cdot)(t), \tag{21}
\]

and we have

\[
b_j^n(g)(t) \equiv \int_\Omega b_j^n(x,t)g(x)dx \in Y_\alpha(T), \quad c^n(g)(t) \equiv \int_\Omega c^n(x,t)g(x)dx \in Y_\alpha(T). \tag{22}
\]

### 3 Approximate solutions

In this section we shall define a special approximate solution for which we will be able to obtain appropriate uniform estimates. We shall assume that

\[
f \in L^2(0,T;H^{-1}(\Omega)), \quad u_0 \in L^2(\Omega), \tag{23}
\]

\[
b_j \in L^1(\Omega^T), \quad c \in L^1(\Omega^T), \tag{24}
\]

and \( a_{i,j} \) are measurable and satisfy (8).

Let \( \{\varphi_n(x)\}_{n \in \mathbb{N}} \) be an orthonormal basis of \( L^2(\Omega) \) such that \( -\Delta \varphi_n = \lambda_n \varphi_n \) in \( \Omega \) and \( \varphi_n|_{\partial\Omega} = 0 \). We will find approximate solution in the form

\[
u^n(x,t) = \sum_{k=1}^n c_{n,k}(t)\varphi_k(x). \tag{25}
\]
Therefore we have to determine the coefficients $c_{n,k}$. For this purpose we extend function $f$ by odd reflection to the interval $(-T, T)$ and we set zero elsewhere. Then

$$f^n(x,t) = \sum_{k=1}^{n} \langle f^{1/2}_n(y,t)\varphi_k(y) \rangle_{H^{-1}\times H^1_0(\Omega)} \varphi_k(x).$$

We denote

$$L^n u(x,t) = \sum_{i,j=1}^{N} \partial_i (a_{i,j}^n(x,t) \partial_j u(x,t)) + \sum_{j=1}^{N} b_j^n(x,t) \partial_j u(x,t) + c^n(x,t) u(x,t), \quad (26)$$

where $a_{i,j}^n$ are defined in (17) and $b_j^n$, $c^n$ in (21).

In order to determine the coefficients $c_{n,k}$, we shall consider the following system

$$\begin{cases}
D^n c_{n,m}(t) = - \sum_{k=1}^{n} \sum_{i,j=1}^{N} c_{n,k}(t) \int_{\Omega} a_{i,j}^n(x,t) \partial_j \varphi_k(x) \partial_i \varphi_m(x) dx \\
u_{|t=0} = u_0^n \quad \text{in} \quad \Omega,
\end{cases} \quad (27)$$

where $u_0^n(x) = \sum_{k=1}^{n} \int_{\Omega} u_0(y) \varphi_k(y) dy \varphi_k(x)$. We define the coefficients $c_{n,k}$ by a projection of the problem (27) onto a finite dimensional space span by $\{\varphi_1, \ldots, \varphi_n\}$. More precisely, we multiply (27) by $\varphi_m$ and integrate over $\Omega$. Then after integrating by parts we have

$$D^n c_{n,m}(t) = - \sum_{k=1}^{n} \sum_{i,j=1}^{N} c_{n,k}(t) \int_{\Omega} a_{i,j}^n(x,t) \partial_j \varphi_k(x) \partial_i \varphi_m(x) dx \\
+ \sum_{k=1}^{n} \sum_{j=1}^{N} c_{n,k}(t) \int_{\Omega} b_j^n(x,t) \partial_j \varphi_k(x) \varphi_m(x) dx \\
+ \sum_{k=1}^{n} c_{n,k}(t) \int_{\Omega} c^n(x,t) \varphi_k(x) \varphi_m(x) dx \\
+ \langle f^{1/2}_n(x,t) \varphi_m(x) \rangle_{H^{-1}\times H^1_0(\Omega)}, \quad (28)$$

where $m = 1, \ldots, n$. By (19), (23), (24) and proposition 8 we deduce that the integrals on the right-hand side are finite. We introduce the following notations:

$$c_n(t) = (c_{n,1}(t), \ldots, c_{n,n}(t)),$$

$$A^n(t) = \left\{ A^n_{m,k}(t) \right\}_{k,m=1}^{n},$$

$$B^n(t) = \left\{ B^n_{m,k}(t) \right\}_{k,m=1}^{n},$$

$$C^n(t) = \left\{ C^n_{m,k}(t) \right\}_{k,m=1}^{n},$$

where $m = 1, \ldots, n$. By (19), (23), (24) and proposition 8 we deduce that the integrals on the right-hand side are finite. We introduce the following notations:

$$A^n(t) = \left\{ A^n_{m,k}(t) \right\}_{k,m=1}^{n},$$

$$B^n(t) = \left\{ B^n_{m,k}(t) \right\}_{k,m=1}^{n},$$

$$C^n(t) = \left\{ C^n_{m,k}(t) \right\}_{k,m=1}^{n},$$

where $m = 1, \ldots, n$. By (19), (23), (24) and proposition 8 we deduce that the integrals on the right-hand side are finite.
\[ F_n(t) = \left( \int_\Omega f_\frac{1}{n}(y,t) \varphi_1(y) dy, \ldots, \int_\Omega f_\frac{1}{n}(y,t) \varphi_n(y) dy \right), \]

\[ c_{n,0} = \left( \int_\Omega u_0(y) \varphi_1(y) dy, \ldots, \int_\Omega u_0(y) \varphi_n(y) dy \right). \]

Then system (28) can be written in the following form

\[
\begin{align*}
D^\alpha c_n(t) &= -A^n(t)c_n(t) + B^n(t)c_n(t) + C^n(t)c_n(t) + F_n(t), \\
c_n(0) &= c_{n,0}.
\end{align*}
\] (29)

We shall show that the above system has an absolutely continuous solution and then under the assumption \( c_n \in AC[0,T] \). By proposition 1 the problem (29) is equivalent to the following integral equation

\[ c_n(t) = c_{n,0} - I^\alpha(A^n c_n)(t) + I^\alpha(B^n c_n)(t) + I^\alpha(C^n c_n)(t) + I^\alpha F_n(t), \] (30)

where by assumption (23), the function \( F_n \) is smooth. By (20) and (22) we have \( A^n, B^n, C^n \in Y_\alpha(T) \). Hence for \( \tilde{A}^n = A^n - B^n - C^n \) we also have

\[ \tilde{A}^n \in Y_\alpha(T). \] (31)

Furthermore we define the space

\[ X(T) = \{ c \in C^1((0,T]; \mathbb{R}^n) : c(0) = c_{0,n}, \ t^{1-\alpha}c'(t) \in C([0,T]; \mathbb{R}^n) \}. \] (32)

Then, for \( c_1, c_2 \in X(T) \), defining the distance \( g(c_1, c_2) = \| c_1 - c_2 \|_{Y_\alpha(T)} \), this is a distance yielding a complete metric on \( X(T) \). We note that \( X(T) \subset AC([0,T]; \mathbb{R}^n) \).

**Lemma 1.** For any \( n \in \mathbb{N} \) and \( T > 0 \) the system (30) has a unique solution in \( X(T) \).

**Proof.** We shall use the Banach fixed point theorem in order to prove the solvability of (30) in the space (32). At the first step we shall obtain the solution on some interval \([0,T_1]\) and further we shall extend the solution. Hence at the beginning we define the operator \( P \) on \( X(T_1) \) by formula

\[ Pc(t) = c_{n,0} - I^\alpha(\tilde{A}^n c)(t) + I^\alpha F_n(t). \] (33)

Under some smallness assumption on \( T_1 \), we shall obtain the fixed point of \( P \). Hence we first have to show that \( Pc \in X(T_1) \), provided \( c \in X(T_1) \). Clearly we have \( Pc(0) = c_{n,0} \) and \( \tilde{A}^n c \) is continuous and by proposition 2 we obtain the continuity of \( Pc \) on \([0,T]\). From (31) we have \( \tilde{A}^n c \in Y_\alpha(T_1) \) and by propositions 3 and 4 we obtain \( t^{1-\alpha}(Pc)' \in C^{0,\alpha}[0,T_1] \), that is, \( Pc \in X(T_1) \) for arbitrary \( T_1 \). Now we shall show that \( P \) is a contraction on \( X(T_1) \), provided \( T_1 \) is small enough. Indeed, we first we note that the operator \( I^\alpha \) is bounded on \( Y_\alpha(T_1) \), and more precisely from proposition 4 we have

\[ \| I^\alpha h \|_{Y_\alpha(T_1)} \leq C(\alpha)T_1^{\alpha} \| h \|_{Y_\alpha(T_1)}, \quad h \in Y_\alpha(T_1). \] (34)
Secondly, we see that
\[ ||h_1 h_2||_{Y_\alpha(T_1)} \leq ||h_1||_{Y_\alpha(T_1)} \cdot ||h_2||_{Y_\alpha(T_1)}, \quad h_1, h_2 \in Y_\alpha(T_1). \] (35)
Therefore, if \( c_1, c_2 \in X(T_1) \), then form (34) and (35) we have
\[ ||Pc_1 - Pc_2||_{Y_\alpha(T_1)} \leq ||I_1^\alpha (\tilde{A}^\alpha (c_1 - c_2))||_{Y_\alpha(T_1)} \]
\[ \leq C(\alpha) T_1^\alpha ||\tilde{A}^\alpha (c_1 - c_2)||_{Y_\alpha(T_1)} \leq C(\alpha) T_1^\alpha ||\tilde{A}^\alpha||_{Y_\alpha(T_1)} \cdot ||c_1 - c_2||_{Y_\alpha(T_1)}. \]
Hence \( P \) is a contraction on \( X(T_1) \), provided
\[ C(\alpha) T_1^\alpha ||\tilde{A}^\alpha||_{Y_\alpha(T_1)} < 1, \] (36)
and finally, we obtained a solution of (30) on \( X(T_1) \).

In order to extend the solution, assume that we have already defined a solution \( \hat{c} \) of (30) on \( [0, T_k] \), where \( T_k > 0 \). We shall define the solution for \( t \in [T_k, T_{k+1}] \) with \( T_{k+1} > T_k \). Therefore we define the set
\[ X_k(T_{k+1}) = \{ c \in C^1((0, T_{k+1}); \mathbb{R}^n) : c(t) = \hat{c}(t) \text{ for } t \in [0, T_k] \}. \]
Then \( X_k(T_{k+1}) \) becomes a complete metric space with the metric \( \rho(c_1, c_2) \equiv ||c_1 - c_2||_{X_k(T_{k+1})} = ||c_1 - c_2||_{C([T_k, T_{k+1}])} \). Then we define an operator \( P \) on \( X_k(T_{k+1}) \) again by formula (33). If \( c \in X_k(T_{k+1}) \), then by definition of \( \hat{c} \), we have \( Pc(t) = \hat{c}(t) \) for \( t \in [0, T_k] \) and by the same reasoning as the previous for \( X(T_1) \), we deduce that \( Pc \in X_k(T_{k+1}) \).

Now we shall show that \( P \) is a contraction on \( X_k(T_{k+1}) \), provided \( T_{k+1} - T_k \) is small enough. Indeed, if \( c_1, c_2 \in X_k(T_{k+1}) \), then
\[ ||Pc_1 - Pc_2||_{X_k(T_{k+1})} = ||[I_1^\alpha (\tilde{A}^\alpha (c_1 - c_2))']_{C([T_k, T_{k+1}])}, \]
where \( I_1^\alpha \) denotes the fractional integration operator with beginning point \( T_k \). Using the analog of proposition 3 for \( I_1^\alpha \) and the equality \( c_1(T_k) = c_2(T_k) \), we obtain
\[ ||Pc_1 - Pc_2||_{X_k(T_{k+1})} = ||I_1^\alpha [[\tilde{A}^\alpha (c_1 - c_2)]']_{C([T_k, T_{k+1}])} \]
\[ \leq C(\alpha) ||\tilde{A}^\alpha||_{C([T_k, T_{k+1}])} [T_{k+1} - T_k]^{\alpha} \cdot ||c_1 - c_2||_{X_k(T_{k+1})} \]
\[ + C(\alpha) ||(\tilde{A}^\alpha)'||_{C([T_k, T_{k+1}])} [T_{k+1} - T_k]^{\alpha} \cdot ||c_1 - c_2||_{C([T_k, T_{k+1}])}. \]
Using the inequality \( ||c_1 - c_2||_{C([T_k, T_{k+1}])} \leq [T_{k+1} - T_k] ||c_1 - c_2||_{X_k(T_{k+1})} \), we deduce that \( P \) is a contraction on \( X_k(T_{k+1}) \), provided
\[ C(\alpha) \left[ ||\tilde{A}^\alpha||_{C([T_k, T_{k+1}])} + [T_{k+1} - T_k]^{\alpha} ||(\tilde{A}^\alpha)'||_{C([T_k, T_{k+1}])} \right] [T_{k+1} - T_k]^{\alpha} < 1. \] (37)
By (34) the quantities \( ||\tilde{A}^\alpha||_{C([T_k, T_{k+1}])}, ||(\tilde{A}^\alpha)'||_{C([T_k, T_{k+1}])} \) are bounded by \( T_1^{\alpha - 1} ||\tilde{A}^\alpha||_{Y_\alpha(T)} \) and by iteration we obtain the solution of (30) which belongs to the space \( X(T) \).
The uniqueness follows from the uniqueness of the fixed point given by the Banach theorem. \( \square \)
Corollary 1. If \( n \in \mathbb{N} \) and \( T > 0 \), then \( u^n \) given by (25) and (30) satisfies

\[
\int_{\Omega} D^\alpha u^n(x,t) \varphi_m(x) dx + \sum_{i,j=1}^{N} \int_{\Omega} a^n_{i,j}(x,t) \partial_j u^n(x,t) \partial_i \varphi_m(x) dx
\]

\[
= \sum_{j=1}^{N} \int_{\Omega} b^n_j(x,t) \partial_j u^n(x,t) \varphi_m(x) dx + \int_{\Omega} c^n(x,t) u^n(x,t) \varphi_m(x) dx
\]

\[
+ \langle f^n(x,t), \varphi_m(x) \rangle_{H^{-1} \times H^1_0(\Omega)}, \tag{38}
\]

for \( m = 1, \ldots, n \). Furthermore, if \( x \in \Omega \) and \( \beta \in \mathbb{N}^N \), then

\[ \partial^{\beta} x u^n(x, \cdot) \in AC[0, T] \]

\[ t^{1-\alpha} \partial^\beta_x u^n \in C(\bar{\Omega} \times [0, T]) \]

provided \( \partial \Omega \) is sufficiently smooth (e.g. \( \partial \Omega \in C^{|\beta|+1} \)).

4 Weak solutions

We shall apply the standard energy method. Briefly speaking, we multiply the approximate problem (38) by its solution. In order to deal with the Caputo derivative we need the following lemma.

Lemma 2. Assume that \( w \in L^2(\Omega T) \) and

\[ w(x, \cdot) \in AC[0, T] \quad \text{for} \quad x \in \Omega, \tag{39} \]

and

\[ t^{1-\alpha} w_t \in L^\infty(\Omega T). \tag{40} \]

Then the following equality

\[
D^\alpha \|w(\cdot, t)\|_{L^2(\Omega)}^2 + \frac{\alpha}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha-1} \int_{\Omega} |w(x,t) - w(x,\tau)|^2 dx d\tau
\]

\[
+ \frac{1}{\Gamma(1-\alpha)} t^{-\alpha} \int_{\Omega} \left| w(x,t) - w(x,0) \right|^2 dx = 2 \int_{\Omega} D^\alpha w(x,t) \cdot w(x,t) dx \tag{41} \]

holds.

Proof. By the definition, we have

\[
2 \int_{\Omega} D^\alpha w(x,t) \cdot w(x,t) dx - D^\alpha \|w(\cdot, t)\|_{L^2(\Omega)}^2
\]

\[
= \frac{2}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} \int_{\Omega} w_t(x,\tau)[w(x,t) - w(x,\tau)] dx d\tau
\]

\[
= \frac{2}{\Gamma(1-\alpha)} \left( \int_{0}^{t-h} + \int_{t-h}^{t} \right) = I_1 + I_2.
\]

Then

\[
I_1 = -\frac{1}{\Gamma(1-\alpha)} \int_0^{t-h} (t-\tau)^{-\alpha} \int_{\Omega} \left( |w(x,t) - w(x,\tau)|^2 \right)' dx d\tau
\]
We denote the last integral by $I_3$. Then using assumption (40) we obtain

$$I_3 \leq h^{2-\alpha} |\Omega| \|t^{1-\alpha} w_t\|_{L^\infty(\Omega^T)} (t-h)^{2(\alpha-1)} \rightarrow 0,$$

if $h \rightarrow 0$. Again using (40) we have the estimate for $I_2$

$$|I_2| \leq 2\|t^{1-\alpha} w_t\|_{L^\infty(\Omega^T)} \|w\|^2_{L^2(\Omega^T)} \int_{t-h}^t (t-\tau)^{-\alpha-1} d\tau \rightarrow 0,$$

provided $h \rightarrow 0$. Therefore we obtain (41). \hfill \Box

Now we can prove the first energy estimate for approximate solutions.

**Lemma 3.** Assume that $u_0 \in L^2(\Omega)$ and $f \in L^2(0,T; H^{-1}(\Omega))$, and for some $p_1, p_2 \in [2, \frac{2N}{N-2}]$ we have $b \in L^\infty(0,T; L^\frac{2p_1}{p_1-2}(\Omega))$, $c \in L^\infty(0,T; L^\frac{2p_2}{p_2-2}(\Omega))$. Then for each $t \in [0,T]$ and $n \in \mathbb{N}$ the approximate solution $u^n$ satisfies the following estimate

$$I^{1-\alpha} \|u^n(\cdot,t)\|^2_{L^2(\Omega)} + \frac{\alpha}{\Gamma(1-\alpha)} \int_0^t \int_0^\tau (\tau-s)^{-\alpha-1} \|u^n(\cdot,s)\|^2_{L^2(\Omega)} d\tau ds + \frac{1}{\Gamma(1-\alpha)} \int_0^t \tau^{-\alpha} \|u^n(\cdot,\tau) - u^n_0(\cdot,\tau)\|^2_{L^2(\Omega)} d\tau + \lambda \int_0^t \|Du^n(\cdot,\tau)\|^2_{L^2(\Omega)} d\tau \leq C_0 \left( \|u_0\|^2_{L^2(\Omega)} + \int_0^t \|f(\cdot,\tau)\|^2_{H^{-1}(\Omega)} d\tau + \delta_n \right),$$

where $C_0$ depends only on $\|b\|_{L^\infty(0,T; L^\frac{2p_1}{p_1-2}(\Omega))}$, $\|c\|_{L^\infty(0,T; L^\frac{2p_2}{p_2-2}(\Omega))}$, $\lambda$, $\alpha$ and $T$ and $\delta_n \rightarrow 0$ uniformly with respect to $t$, if $n \rightarrow \infty$.

**Proof.** We multiply (38) by $c_{n,m}(t)$ and sum over $m = 1, \ldots, n$. Then we have

$$\int_{\Omega} D^n u^n(x,t) u^n(x,t) dx + \sum_{i,j=1}^N \int_{\Omega} a_{i,j}^n(x,t) \partial_j u^n(x,t) \partial_i u^n(x,t) dx$$

$$= \sum_{j=1}^N \int_{\Omega} b_j^\alpha(x,t) \partial_j u^n(x,t) u^n(x,t) dx + \int_{\Omega} c^n(x,t) |u^n(x,t)|^2 dx$$

$$+ \langle f_{\frac{n}{m}}(x,t), u^n(x,t) \rangle_{H^{-1} \times H_0^1(\Omega)}.$$
\[
+ \frac{1}{2\Gamma(1 - \alpha)} t^{-\alpha} \| u^n(\cdot, t) - u^n_0(\cdot) \|_{L^2(\Omega)}^2 + \lambda \| Du^n(\cdot, t) \|_{L^2(\Omega)}^2 \leq \sum_{j=1}^N \int_{\Omega} b_j^n(x, t) \partial_j u^n(x, t) u^n(x, t) \, dx + \int_{\Omega} c^n(x, t) |u^n(x, t)|^2 \, dx \\
+ \frac{\lambda}{2} \| Du^n(\cdot, t) \|_{L^2(\Omega)}^2 + \frac{1}{2\lambda} \| f_\frac{1}{\alpha}(\cdot, t) \|_{H^{-1}(\Omega)}^2. \quad (43)
\]

First we obtain the estimate for the lower-order terms. In particular, if we denote \( b^n = (b_1^n, \ldots, b_N^n) \), then we have

\[
D^\alpha \| u^n(\cdot, t) \|_{L^2(\Omega)}^2 + \lambda \| Du^n(\cdot, t) \|_{L^2(\Omega)}^2 \leq \int_{\Omega} |b^n(x, t)| \| Du^n(x, t) \|_{L^2(\Omega)} \| u^n(x, t) \|_{L^2(\Omega)} \, dx + 2 \int_{\Omega} |c^n(x, t)| \| u^n(x, t) \|_{L^2(\Omega)}^2 \, dx + \frac{1}{\lambda} \| f_{\frac{1}{\alpha}}(\cdot, t) \|_{H^{-1}(\Omega)}^2 \leq \frac{\lambda}{4} \| Du^n(\cdot, t) \|_{L^2(\Omega)}^2 + \frac{1}{\lambda} \| b^n(\cdot, t) \|_{L^p\Omega(\Omega)}^2 \| u^n(\cdot, t) \|_{L^2(\Omega)}^2 \leq \frac{\lambda}{4} \| Du^n(\cdot, t) \|_{L^2(\Omega)}^2 + \frac{1}{\lambda} \| b^n(\cdot, t) \|_{L^p\Omega(\Omega)}^2 \left[ \varepsilon_1 \| Du^n(\cdot, t) \|_{L^2(\Omega)}^2 + c(\varepsilon_1, p_1) \| u^n(\cdot, t) \|_{L^2(\Omega)}^2 \right] + \| c^n(\cdot, t) \|_{L^p\Omega(\Omega)}^2 \left[ \varepsilon_2 \| Du^n(\cdot, t) \|_{L^2(\Omega)}^2 + c(\varepsilon_2, p_2) \| u^n(\cdot, t) \|_{L^2(\Omega)}^2 \right] + \frac{1}{\lambda} \| f_{\frac{1}{\alpha}}(\cdot, t) \|_{H^{-1}(\Omega)}^2.
\]

If we take \( \varepsilon_1, \varepsilon_2 \) small enough, then

\[
D^\alpha \| u^n(\cdot, t) \|_{L^2(\Omega)}^2 + \lambda \| Du^n(\cdot, t) \|_{L^2(\Omega)}^2 \leq h_n(t) \| u^n(\cdot, t) \|_{L^2(\Omega)}^2 + \frac{\lambda}{2} \| Du^n(\cdot, t) \|_{L^2(\Omega)}^2 + \frac{1}{\lambda} \| f_{\frac{1}{\alpha}}(\cdot, t) \|_{H^{-1}(\Omega)}^2. \quad (44)
\]

where the function \( h_n(t) \) depends continuously on some powers of \( \| b^n(\cdot, t) \|_{L^p\Omega(\Omega)}^2 \), \( \| c^n(\cdot, t) \|_{L^p\Omega(\Omega)}^2 \) and \( \lambda \). If we apply \( I^\alpha \) to the sides of (44), then

\[
\| u^n(\cdot, t) \|_{L^2(\Omega)}^2 \leq \| u^n(\cdot, 0) \|_{L^2(\Omega)}^2 + \frac{1}{\lambda} I^\alpha \| f_{\frac{1}{\alpha}}(\cdot, t) \|_{H^{-1}(\Omega)}^2 + g_n(t) I^\alpha \| u^n(\cdot, t) \|_{L^2(\Omega)}^2, \quad (45)
\]

where the function \( g_n(t) \) depends continuously on some powers of \( \| b^n \|_{L^\infty(0, t; L^p\Omega(\Omega))}^2 \), \( \| c^n \|_{L^\infty(0, t; L^p\Omega(\Omega))}^2 \) and \( \lambda \). We apply a generalized Gronwall lemma (proposition in appendix) to obtain

\[
\| u^n(\cdot, t) \|_{L^2(\Omega)}^2 \leq \| u^n(\cdot, 0) \|_{L^2(\Omega)}^2 \sum_{k=0}^\infty g_n(t) \frac{t^{\alpha k}}{\Gamma(1 + \alpha k)}
\]
If we estimate the last two integrals similarly to the previous, then

\begin{align*}
I^{1-\alpha} \| u^n(\cdot, t) \|_{L^2(\Omega)}^2 & + \frac{\alpha}{\Gamma(1-\alpha)} \int_0^t \int_0^\tau \frac{\| u^n(\cdot, \tau) - u^n(\cdot, s) \|_{L^2(\Omega)}^2}{|\tau - s|^{\alpha+1}} ds d\tau \\
+ & \frac{1}{\Gamma(1-\alpha)} \int_0^t \tau^{-\alpha} \| u^n(\cdot, \tau) - u^n(\cdot, 0) \|_{L^2(\Omega)}^2 d\tau + \lambda \int_0^t \| D u^n(\cdot, \tau) \|_{L^2(\Omega)}^2 d\tau \\
& \leq \frac{2t^{1-\alpha}}{\Gamma(2-\alpha)} \| u^n(\cdot, 0) \|_{L^2(\Omega)}^2 + \frac{2}{\lambda} \int_0^t \| f_n(\cdot, \tau) \|_{H^{-1}(\Omega)}^2 d\tau + g_n(t) \int_0^t \| u^n(\cdot, \tau) \|_{L^2(\Omega)}^2 d\tau.
\end{align*}

(46)

The convergence of the series follows from the d’Alembert criterion and \( \lim_{k \to \infty} \frac{\Gamma(x+\alpha)}{\Gamma(x+\beta)} = 1 \).

We once again use the inequality \([13]\). We apply the operator \( I \) to both sides of \([13]\). Then using the identity \( I = I^{1-\alpha} I^\alpha \) (see theorem 2.5 in \([10]\)) and \( u^n(x, \cdot) \in AC[0, T] \) for each \( x \in \Omega \) and applying proposition \([14]\) we obtain

\begin{align*}
I^{1-\alpha} \| u^n(\cdot, t) \|_{L^2(\Omega)}^2 & + \frac{\alpha}{\Gamma(1-\alpha)} \int_0^t \int_0^\tau \frac{\| u^n(\cdot, \tau) - u^n(\cdot, s) \|_{L^2(\Omega)}^2}{|\tau - s|^{\alpha+1}} ds d\tau \\
+ & \frac{1}{\Gamma(1-\alpha)} \int_0^t \tau^{-\alpha} \| u^n(\cdot, \tau) - u^n(\cdot, 0) \|_{L^2(\Omega)}^2 d\tau + \lambda \int_0^t \| D u^n(\cdot, \tau) \|_{L^2(\Omega)}^2 d\tau \\
& \leq \frac{2t^{1-\alpha}}{\Gamma(2-\alpha)} \| u^n(\cdot, 0) \|_{L^2(\Omega)}^2 + \frac{2}{\lambda} \int_0^t \| f_n(\cdot, \tau) \|_{H^{-1}(\Omega)}^2 d\tau + g_n(t) \int_0^t \| u^n(\cdot, \tau) \|_{L^2(\Omega)}^2 d\tau.
\end{align*}

(47)

Using \([46]\) we have

\begin{align*}
g_n(t) \int_0^t \| u^n(\cdot, \tau) \|_{L^2(\Omega)}^2 d\tau & \leq \| u^n(\cdot, 0) \|_{L^2(\Omega)}^2 \sum_{k=0}^\infty g_n^{k+1}(t) \frac{\Gamma(2+\alpha)}{\Gamma(2+\alpha k)} \\
& \quad + \frac{1}{\lambda} \sum_{k=0}^\infty g_n^{k+1}(t) \left( I^{\alpha(k+1)} \| f_n(\cdot, t) \|_{H^{-1}(\Omega)}^2 \right)(t).
\end{align*}

Using the Mittag-Leffler function \( E_{\alpha, \beta}(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(\alpha k + \beta)} \), we can write

\begin{align*}
\sup_n \sup_{t \in (0, T)} \sum_{k=0}^\infty g_n^{k+1}(t) \frac{t^{\alpha k+1}}{\Gamma(2+\alpha k)} & = \sup_n \sup_{t \in (0, T)} g_n(t) t E_{\alpha, 2}(t^\alpha g_n(t)) \\
& = \sup_n g_n(T) T E_{\alpha, 2}(T^\alpha g_n(T)) \equiv d_0 < \infty,
\end{align*}

(48)
where \( d_0 \) depends only on \( \|b\|_{L^\infty(0,T;L^\frac{2p_1}{p_1-2}(\Omega))} , \|c\|_{L^\infty(0,T;L^\frac{p_2}{p_2-2}(\Omega))} \), \( \lambda, \alpha, T \), and the convergence of the series follows by \( \lim_{x \to \infty} \frac{G(x+\alpha)}{T(x)^\alpha} = 1 \). The second sum is estimated as follows

\[
\sum_{k=0}^\infty g_n^{k+1}(t) \left( \int_0^t (t - \tau)^{\alpha(k+1)} g_n^{k+1}(t) \right) (t) = \sum_{k=0}^\infty \int_0^t \frac{(t - \tau)^{\alpha(k+1)} g_n^{k+1}(t)}{\Gamma(\alpha(k+1) + 1)} ||f_n^k(\cdot, \tau)||^2_{H^{-1}(\Omega)} d\tau \leq \sum_{k=0}^\infty \frac{t^{\alpha(k+1)} g_n^{k+1}(t)}{\Gamma(\alpha(k+1) + 1)} \int_0^t ||f_n^k(\cdot, \tau)||^2_{H^{-1}(\Omega)} d\tau \leq E_{\alpha,1}(t^n g_n(t)) \int_0^t ||f_n^k(\cdot, \tau)||^2_{H^{-1}(\Omega)} d\tau.
\]

Next we denote

\[
\sup_n \sup_{t \in (0,T)} E_{\alpha,1}(t^n g_n(t)) = \sup_n E_{\alpha,1}(T^n g_n(T)) \equiv d_1 < \infty, \tag{49}
\]

where \( d_1 \) depends only on \( \|b\|_{L^\infty(0,T;L^\frac{2p_1}{p_1-2}(\Omega))} , \|c\|_{L^\infty(0,T;L^\frac{p_2}{p_2-2}(\Omega))} \), \( \lambda, \alpha \) and \( T \). We note that

\[
\int_0^t ||f_n^k(\cdot, \tau)||^2_{H^{-1}(\Omega)} d\tau \leq \int_0^t ||f(\cdot, \tau)||^2_{H^{-1}(\Omega)} d\tau + \int_t^{t+\frac{1}{n}} ||f(\cdot, \tau)||^2_{H^{-1}(\Omega)} d\tau.
\]

Thus, setting

\[
\delta_n = \sup_{t \in (0,T-\frac{1}{n})} \int_t^{t+\frac{1}{n}} ||f(\cdot, \tau)||^2_{H^{-1}(\Omega)} d\tau,
\]

and using the assumption concerning \( f \), we see that \( \delta_n \to 0 \) uniformly with respect to \( t \) as \( n \to \infty \). Therefore

\[
g_n(t) \int_0^t ||u^n(\cdot, \tau)||^2_{L^2(\Omega)} d\tau \leq d_0 ||u_0||^2_{L^2(\Omega)} + \frac{d_1}{\lambda} ||f||^2_{L^2(0,t;H^{-1}(\Omega))} + \frac{d_1}{\lambda} \delta_n, \tag{50}
\]

for each \( t \in [0,T] \). This estimate together with (47) give (12).

**Proof of theorem 3** Denote by \( \bar{c}_0 \) the right-hand side of (11). Lemma 3 yields a bound for \( u^n \)

\[
||\nabla u^n||_{L^2(0,T;L^2(\Omega))} + ||u^n||_{H^2(0,T;L^2(\Omega))} \leq \bar{c}_0, \tag{51}
\]

where

\[
||w||_{H^\frac{1}{2}(0,T;L^2(\Omega))} = \left( \int_0^T \int_0^T \frac{||w(\cdot, \tau) - w(\cdot, s)||^2_{L^2(\Omega)}}{||\tau - s||^{1+\alpha}} ds d\tau \right)^{\frac{1}{2}}. \tag{52}
\]

Now we estimate the fractional derivative \( D^\alpha u^n \). If \( w \in H^1_0(\Omega) \), then \( w(x) = \sum_{m=1}^\infty d_m \varphi_m(x) \), where \( d_m \) are some numbers and the series converge in \( H^1_0(\Omega) \). We
denote \( w^n(x) = \sum_{m=1}^n d_m \varphi_m(x) \). Multiplying \( d_m \) by \( d_m \) and summing from \( m = 1 \) to \( n \), we obtain

\[
\int_\Omega D^\alpha u^n(x,t) w(x) dx + \sum_{i,j=1}^N \int_\Omega a^n_{ij}(x,t) \partial_i u^n(x,t) \partial_j w^n(x) dx
\]

\[= \sum_{j=1}^N \int_\Omega b^n_j(x,t) \partial_j u^n(x,t) w^n(x) dx + \int_\Omega e^n(x,t) u^n(x,t) w^n(x) dx
\]

\[\quad + \langle f_n^\alpha(x,t), w(x) \rangle_{H^{-1} \times H^1_0(\Omega)}.\]

Hence, using proposition 8 and the Hölder inequality, we have

\[
\left| \int_\Omega D^\alpha u^n(x,t) w(x) dx \right| \leq \mu \| Du^n(\cdot,t) \|_{L^2(\Omega)} \| Dw^n \|_{L^2(\Omega)} + \| b \|_{L^{p_1}L^{q_1}(\Omega)} \| \nabla u^n(\cdot,t) \|_{L^2(\Omega)} \| w^n \|_{L^p(\Omega)}
\]

\[+ \| c \|_{L^{p_2}L^{q_2}(\Omega)} \| u^n \|_{L^p(\Omega)} \| w^n \|_{L^q(\Omega)} + \| f_n^\alpha(\cdot,t) \|_{H^{-1}(\Omega)} \| w \|_{H^1_0(\Omega)}.\]

The function \( u^n \) is absolutely continuous, and so we have \( D^\alpha u^n(x,t) = \frac{d}{dt} I^{1-\alpha} [u^n(x,t) - u^n_0(x)] \) and

\[
\left\| \frac{d}{dt} I^{1-\alpha} [u^n(x,t) - u^n_0(x)] \right\|_{H^{-1}(\Omega)} = \sup_{\| w \|_{H^1_0(\Omega)} = 1} \left| \int_\Omega D^\alpha u^n(x,t) w(x) dx \right|.
\]

Thus, from the above inequality together with (51), the Sobolev embedding and the Poincaré inequality yield

\[
\sup_n \left| \frac{d}{dt} I^{1-\alpha} [u^n - u_0^n] \right|_{L^2(0,T;H^{-1}(\Omega))} < \infty. \quad (52)
\]

Therefore, the sequence \( I^{1-\alpha} [u^n - u_0^n] \) is uniformly bounded in \( \theta H^1(0,T;H^{-1}(\Omega)) \).

By estimates (51), (52) and the weak compactness argument we obtain \( u \in L^2(0,T;H^1_0(\Omega)) \cap H^{\frac{\alpha}{2}}(0,T;L^2(\Omega)) \) and \( v \in \theta H^1(0,T;H^{-1}(\Omega)) \) such that there exists a subsequence \( u^{n_k} \) such that

\[
u^{n_k} \rightharpoonup u, \quad \nabla u^{n_k} \rightharpoonup \nabla u \quad \text{in} \quad L^2(0,T;L^2(\Omega)), \quad (53)
\]

\[
I^{1-\alpha} [u^{n_k} - u_0^n] \rightharpoonup v \quad \text{in} \quad H^1(0,T;H^{-1}(\Omega)), \quad (54)
\]

\[
\text{if} \quad p_2 > 4, \quad \text{then} \quad u^{n_k} \rightharpoonup u, \quad \text{in} \quad L^2(0,T;L^{\frac{2p_2}{p_2-2}}(\Omega)), \quad (55)
\]

where the last weak limit is a consequence of the interpolation inequality

\[
\| u^{n_k} \|_{L^2(0,T;L^{\frac{2p_2}{p_2-2}}(\Omega))} \leq C_0 \| \nabla u^{n_k} \|_{L^2(0,T;L^2(\Omega))} \| u^{n_k} \|_{L^2(0,T;L^2(\Omega))}^{1-\theta} \| u^{n_k} \|_{L^2(0,T;L^2(\Omega))}^\theta,
\]

which holds by \( \frac{p_2}{2} \in (2, \frac{2N}{N-2}) \).
First we would like to show that \( \frac{d}{dt}I^{1-\alpha}[u-u_0] \) exists in the weak sense in \( L^2(0, T; H^{-1}(\Omega)) \) and \( \frac{d}{dt}I^{1-\alpha}[u-u_0] = \frac{d}{dt}v \). Indeed, we take \( \phi \in C_0^\infty(0, T) \) and \( \varphi \in H^1_0(\Omega) \) and by the weak convergence we have

\[
\int_0^T \phi(t) \left< \frac{d}{dt}v(\cdot, t), \varphi \right>_{H^{-1} \times H^1_0(\Omega)} dt = \lim_{k \to \infty} \int_0^T \phi(t) \left< \frac{d}{dt}I^{1-\alpha}[u^{n_k}(\cdot, t) - u_0^{n_k}], \varphi \right>_{H^{-1} \times H^1_0(\Omega)} dt
\]

\[
= \lim_{k \to \infty} \int_0^T \phi(t) \int_\Omega \frac{d}{dt}I^{1-\alpha}[u^{n_k}(x, t) - u_0^{n_k}(x)] \varphi(x) dx dt
\]

\[
= \lim_{k \to \infty} \int_\Omega \int_0^T \phi(t) \frac{d}{dt}I^{1-\alpha}[u^{n_k}(x, t) - u_0^{n_k}] dt \varphi(x) dx
\]

\[
= - \lim_{k \to \infty} \int_\Omega \int_0^T \phi'(t)I^{1-\alpha}[u^{n_k}(x, t) - u_0^{n_k}] dt \varphi(x) dx
\]

\[
= - \int_\Omega \int_0^T \phi'(t)I^{1-\alpha}[u(x, t) - u_0(x)] dt \varphi(x) dx,
\]

where the last equality is a consequence of the weak continuity of \( I^{1-\alpha} \) on the \( L^2 \)-spaces (theorem 2.6, [10]) and in the previous one we were allowed to integrate by parts, because \( u^{n_k}(x, \cdot) \in AC[0, T] \) and so \( I^{1-\alpha}u^{n_k}(x, \cdot) \in AC[0, T] \) by proposition 3 in appendix. Thus we obtain

\[
\int_0^T \phi(t) \left< v(\cdot, t), \varphi \right>_{H^{-1} \times H^1_0(\Omega)} dt = - \int_0^T \phi'(t) \left< I^{1-\alpha}[u(\cdot, t) - u_0(\cdot)], \varphi \right>_{H^{-1} \times H^1_0(\Omega)} dt,
\]

and so \( \frac{d}{dt}I^{1-\alpha}[u-u_0] = \frac{d}{dt}v \in L^2(0, T; H^{-1}(\Omega)) \) in the weak sense and estimate (11) holds.

Now we shall show the identity (3). By the density argument it is enough to prove it for \( w(x) = \sum_{m=1}^K d_m \varphi_m(x) \), where \( d_m \) are arbitrary numbers. We multiply (33) by \( d_m \) and sum from \( m = 1 \) to \( K \). Then, for fixed \( t_0 \in (0, T) \), we multiply the sides by \( \eta_\varepsilon(t + t_0) \), where \( \eta_\varepsilon \) is a standard mollifier function and finally we integrate with respect to \( t \in (0, T) \). Hence

\[
\int_0^T \eta_\varepsilon(t + t_0) \int_\Omega \frac{d}{dt}I^{1-\alpha}[u^{n_k}(x, t) - u_0^{n_k}(x)] w(x) dx dt
\]

\[
+ \sum_{i,j=1}^N \int_0^T \int_\Omega a_{i,j}^{n_k}(x, t) \partial_j u^{n_k}(x, t) \partial_i w(x) \eta_\varepsilon(t + t_0) dx dt
\]

\[
= \sum_{j=1}^N \int_0^T \int_\Omega b_j^{n_k}(x, t) \partial_j u^{n_k}(x, t) w(x) \eta_\varepsilon(t + t_0) dx dt
\]

\[
+ \int_0^T \int_\Omega c^{n_k}(x, t) u^{n_k}(x, t) w(x) \eta_\varepsilon(t + t_0) dx dt
\]

\[
+ \int_0^T \int_\Omega f_{1/n_k}(x, t) w(x) \eta_\varepsilon(t + t_0) dx dt.
\]
We first take the limits with as $k \to \infty$ and next as $\varepsilon \to 0$. For $\varepsilon < T - t_0$, integrating by parts and using (53) and continuity of $I^{1-\alpha}$ on $L^2$ (Theorem 2.6 in [10]), we obtain
\[
\int_0^T \eta_\varepsilon(t+t_0) \int_\Omega \frac{d}{dt} I^{1-\alpha}[u^{n_k}(x,t) - u_0^{n_k}(x)]w(x)dxdt
\]
\[
= - \int_0^T \eta_\varepsilon'(t+t_0) \int_\Omega I^{1-\alpha}[u^{n_k}(x,t) - u_0^{n_k}(x)]w(x)dxdt
\]
\[
\to - \int_0^T \eta_\varepsilon'(t+t_0) \int_\Omega I^{1-\alpha}[u(x,t) - u_0(x)]w(x)dxdt
\]
\[
= \int_0^T \eta_\varepsilon(t+t_0) \frac{d}{dt} \int_\Omega I^{1-\alpha}[u(x,t) - u_0(x)]w(x)dxdt
\]
\[
\to \frac{d}{dt} \int_\Omega I^{1-\alpha}[u(x,t_0) - u_0(x)]w(x)dx \quad \text{for a.a. } t_0 \in (0,T).
\]
For the next term we proceed similarly. The function $\partial_i w(x)\eta_\varepsilon(t+t_0)$ is smooth in $\Omega^T$, and (18) and (53) yield
\[
\int_0^T \int_\Omega a_{i,j}^{n_k}(x,t) \partial_j u^{n_k}(x,t) \partial_i w(x)\eta_\varepsilon(t+t_0)dxdt
\]
\[
\to \int_0^T \int_\Omega a_{i,j}(x,t) \partial_j u(x,t) \partial_i w(x)\eta_\varepsilon(t+t_0)dxdt
\]
\[
\to \int_\Omega a_{i,j}(x,t_0) \partial_j u(x,t_0) \partial_i w(x)dx \quad \text{for a.a. } t_0 \in (0,T).
\]
The first term on the right-hand side also converges, because from the assumption we have $b_j^{n_k} \to b_j$ in $L^2(\Omega^T)$.

We have to consider the two cases to deal with the next term on the right-hand side. If $p_2 \in [2,4]$, then $c \in L^2(\Omega^T)$ and $c_{n,k} \to c$ in $L^2(\Omega^T)$. Thus $c_{n,k} u^{n_k} \to cu$ in $L^2(\Omega^T)$. In the case of $p_2 > 4$ we can write
\[
\left| \int_0^T \int_\Omega (c^{n_k}(x,t)u^{n_k}(x,t) - c(x,t)u(x,t)) w(x)\eta_\varepsilon(t+t_0)dxdt \right|
\]
\[
\leq \int_0^T \int_\Omega |c^{n_k}(x,t) - c(x,t)||u^{n_k}(x,t)||w(x)||\eta_\varepsilon(t+t_0)dxdt
\]
\[
+ \left| \int_0^T \int_\Omega c(x,t)(u^{n_k}(x,t) - u(x,t))w(x)\eta_\varepsilon(t+t_0)dxdt \right|.
\]
The first term converges to zero, because $c_{n,k} \to c$ in $L^2(0,T;L^{p_2}(\Omega))$ and $u^{n_k}$ is bounded in $\left(L^2(0,T;L^{p_2}(\Omega))\right)^* = L^2(0,T;L^{p_2^*}(\Omega))$ by (55). In terms of (55), we can deal with the second term.
Finally we obtain

\[
\int_0^T \int_\Omega f_{1/n_k}(x,t)w(x)\eta_\epsilon(t + t_0)dxdt = \int_0^T \langle f_{1/n_k}(t), w \rangle_{H^{-1} \times \dot{H}^1_0(\Omega)} \eta_\epsilon(t + t_0)dt
\]

and \([9]\) is proved for \(\varphi = \sum_{m=1}^K d_m \varphi_m\) and a.a. \(t \in (0, T)\). By density argument we deduce \([9]\) for all \(\varphi \in \dot{H}^1_0(\Omega)\) and a.a. \(t \in (0, T)\).

In the case of \(\alpha \in (\frac{1}{2}, 1)\), the continuity of \(u\) and the equality \(u_{|t=0} = u_0\) immediately follow from proposition 7.

It remains to prove the uniqueness of solutions in theorem 2. Assume that \(u \in W^\alpha(0, \dot{H}^1_0(\Omega), H^{-1}(\Omega))\) satisfies \([9]\) with \(f \equiv 0\) and \(u_0 \equiv 0\). Then we set

\[u_n(x, \tau) = \sum_{k=1}^n d_k(\tau) \varphi_k(x),\]

where \(d_k(\tau) = \int_\Omega u(x, \tau) \varphi_k(x) dx\). Setting \(\varphi = \varphi_k\) in \([9]\) and multiplying by \(d_k(\tau)\) and summing from \(k = 1\) to \(n\), we have

\[
\int_0^t \int_\Omega b_j(x, \tau)\partial_j u(x, \tau)u_n(x, \tau)dxdt + \int_0^t \int_\Omega c(x, \tau)u(x, \tau)u_n(x, \tau)dxdt.
\]

The convergence \(u_n \longrightarrow u\) in \(L^2(0, t; \dot{H}^1_0(\Omega))\) yields

\[
\lim_{n \to \infty} \int_0^t \int_\Omega b_j(x, \tau)\partial_j u(x, \tau)u_n(x, \tau)dxdt + \int_0^t \int_\Omega c(x, \tau)u(x, \tau)u_n(x, \tau)dxdt.
\]

We have

\[
\int_0^t \int_\Omega b_j(x, \tau)\partial_j u(x, \tau)u_n(x, \tau)dxdt = \int_0^t \int_\Omega b_j(x, \tau)\partial_j I^{1-\alpha}u(x, \tau)dxdt.
\]

Indeed,

\[
\int_0^t \langle d - I^{1-\alpha}u, \varphi_k \rangle dt = \int_0^t \sum_{k=1}^n d_k(\tau) \int_\Omega b_j(x, \tau)\partial_j u(x, \tau)\varphi_k(x)dxdt
\]

Finally, we obtain

\[
\int_0^T \int_\Omega f_{1/n_k}(x,t)w(x)\eta_\epsilon(t + t_0)dxdt = \int_0^T \langle f_{1/n_k}(t), w \rangle_{H^{-1} \times \dot{H}^1_0(\Omega)} \eta_\epsilon(t + t_0)dt
\]
\[
\begin{align*}
&= \int_0^t \sum_{k=1}^n d_k(\tau) \frac{d}{d\tau} I^{1-\alpha} d_k(\tau) d\tau = \int_0^t \sum_{k,m=1}^n d_m(\tau) \frac{d}{d\tau} I^{1-\alpha} d_k(\tau) \int_\Omega \varphi_k(x) \varphi_m(x) dx d\tau \\
&= \int_0^t \sum_{k,m=1}^n d_m(\tau) \frac{d}{d\tau} I^{1-\alpha} d_k(\tau) \langle \varphi_k(\cdot), \varphi_m(\cdot) \rangle d\tau = \int_0^t \left\langle \frac{d}{d\tau} I^{1-\alpha} u_n(\cdot, \tau), u_n(\cdot, \tau) \right\rangle d\tau.
\end{align*}
\]

Then we can write
\[
\lim_{n \to \infty} \int_0^t \left\langle \frac{d}{d\tau} I^{1-\alpha} u_n(\cdot, \tau), u_n(\cdot, \tau) \right\rangle d\tau = \lim_{n \to \infty} \int_0^t \left\langle \frac{d}{d\tau} I^{1-\alpha} u_n(\cdot, \tau), u_n(\cdot, \tau) \right\rangle d\tau
\]
\[
= \lim_{n \to \infty} \int_\Omega \int_0^t \frac{d}{dt} I^{1-\alpha} u_n(x, \tau) \cdot u_n(x, \tau) dx d\tau \geq \frac{t^{-\alpha}}{2 \Gamma(1 - \alpha)} \lim_{n \to \infty} \int_0^t |u_n(x, \tau)|^2 d\tau dx
\]
\[
\geq \frac{t^{-\alpha}}{2 \Gamma(1 - \alpha)} \int_0^t \int_\Omega |u(x, \tau)|^2 dx d\tau
\]
for a.a. \( t \in [0, T] \), where we applied corollary 2. Using ellipticity condition (53), we obtain
\[
\frac{t^{-\alpha}}{2 \Gamma(1 - \alpha)} \int_0^t \|u(\cdot, \tau)\|^2_{L^2(\Omega)} d\tau + \frac{\lambda}{2} \int_0^t \|Du(\cdot, \tau)\|^2_{L^2(\Omega)} d\tau \leq C_0 \int_0^t \|u(\cdot, \tau)\|^2_{L^2(\Omega)} d\tau,
\]
where \( C_0 \) is a constant which depends only on the norms of \( b \) in \( L^\infty(0, T; L^\frac{2m}{m-2}(\Omega)) \) and \( c \) in \( L^\infty(0, T; L^\frac{2p}{p-2}(\Omega)) \). Therefore we deduce that \( u \equiv 0 \) on \( \Omega \times (0, t) \), provided \( t \) is small enough. Repeating this argument, we deduce that \( u \equiv 0 \) on \( \Omega \times (0, T) \). \( \square \)

**Proof of theorem 3.** In the case of \( L = \Delta \), the equality (58) has the following form
\[
\int_\Omega D^\alpha u^n(x, t) \varphi_m(x) dx = \int_\Omega u^n(x, t) \Delta \varphi_m(x) dx + \int_\Omega f_\frac{1}{m}(x, t) \varphi_m(x) dx.
\] (57)
For \( \alpha > \frac{1}{2} \) the result is contained in theorem 2. Now assume that \( \alpha \in (\frac{1}{2}, \frac{3}{4}) \). Then we can apply the Riemann-Liouville derivative \( \partial^\alpha \) to both sides of (57) and by proposition 5 we obtain
\[
\int_\Omega D^{2\alpha} u^n(x, t) \varphi_m(x) dx = \int_\Omega \partial^\alpha u^n(x, t) \Delta \varphi_m(x) dx + \int_\Omega \partial^\alpha f_\frac{1}{m}(x, t) \varphi_m(x) dx
\]
\[
= \int_\Omega D^\alpha u^n(x, t) \Delta \varphi_m(x) dx + \frac{t^{-\alpha}}{\Gamma(1 - \alpha)} \int_\Omega u^n(x, 0) \Delta \varphi_m(x) dx + \int_\Omega \partial^\alpha f_\frac{1}{m}(x, t) \varphi_m(x) dx.
\] (58)
If \( w \in \tilde{H}^3 \), then there exist constants \( d_m \) such that \( w(x) = \sum_{m=1}^\infty d_m \varphi_m(x) \) in \( \tilde{H}^3 \). Multiplying (58) by \( d_m \) and summing over \( m \), we have
\[
\int_\Omega D^{2\alpha} u^n(x, t) w(x) dx = \int_\Omega D^\alpha u^n(x, t) \Delta w(x) dx + \frac{t^{-\alpha}}{\Gamma(1 - \alpha)} \int_\Omega u^n(x, 0) \Delta w(x) dx
\]
\[
+ \int_\Omega \partial^\alpha f_\frac{1}{m}(x, t) w(x) dx.
\] (59)
Then, because \( \Delta w|_{\partial \Omega} = 0 \), we can write
\[
\|D^{2\alpha}u^n(\cdot, t)\|_{(H^3)^*} = \sup_{\|w\|_{H^3} = 1} \left| \int_\Omega D^{2\alpha}u^n(x, t)w(x)dx \right| \leq \|\partial^\alpha f^n_1(\cdot, t)\|_{(H^3)^*},
\]
\[
+ \sup_{\|w\|_{H^3} = 1} \left( \|D^\alpha u^n(\cdot, t)\|_{H^{-1}(\Omega)}\|\Delta w\|_{H^0(\Omega)} + \frac{t^{-\alpha}}{\Gamma(1 - \alpha)}\|u^n(\cdot, 0)\|_{L^2(\Omega)}\|\Delta w\|_{L^2(\Omega)} \right)
\]
\[
\leq \|D^\alpha u^n(\cdot, t)\|_{H^{-1}(\Omega)} + \frac{t^{-\alpha}}{\Gamma(1 - \alpha)}\|u_0\|_{L^2(\Omega)} + \|\partial^\alpha f^n_1(\cdot, t)\|_{(H^3)^*}. \tag{60}
\]
We have to consider two cases. If \( \alpha \in \left( \frac{1}{4}, \frac{1}{2} \right) \), then squaring and integrating both sides of (60), we obtain
\[
\|D^{2\alpha}u^n\|_{L^2(0, T; (H^3)^*)} \leq \sqrt{3}\|D^\alpha u^n\|_{L^2(0, T; H^{-1}(\Omega))} + \frac{\sqrt{3}}{\Gamma(1 - \alpha)} \left( \frac{T^{1-2\alpha}}{1 - 2\alpha} \right)^{\frac{1}{2}}\|u_0\|_{L^2(\Omega)} + c(\alpha)\|\partial^\alpha f\|_{L^2(0, T; (H^3)^*)}, \tag{61}
\]
where we applied the inequality
\[
\|\partial^\alpha f^n_1\|_{L^2(0, T; (H^3)^*)} \leq c(\alpha)\|\partial^\alpha f\|_{L^2(0, T; (H^3)^*)}, \tag{62}
\]
given in proposition 13 in the appendix. By the assumption concerning \( f \) and (52), we have a uniform bound for the right-hand side and proceeding as in the proof of theorem 2, we obtain that a weak solution \( u \) of (41) satisfies \( \mathcal{I}^{1-2\alpha}[u - u_0] \in \overset{\circ}{H}^1(0, T; (H^3)^*) \). Hence, by proposition 7 we see that \( u \in C([0, T]; (H^3)^*) \) and \( u|_{t=0} = u_0 \).

If \( \alpha = \frac{1}{2} \), then we take the \( p \)-th power of both sides of (60), where \( p \in (1, 2) \). Then we have \( u - u_0 \in \overset{\circ}{W}^{1, p}(0, T; (H^3)^*) \), so that \( u \in C([0, T]; (H^3)^*) \) and \( u|_{t=0} = u_0 \).

For \( \alpha \in \left( \frac{1}{4}, \frac{1}{2} \right) \), we proceed similarly. We apply the Riemann-Liouville derivative \( \partial^{2\alpha} \) to both sides of (41) and taking the \( \overset{\circ}{H}^3 \)-norm by duality we obtain
\[
\|D^{2\alpha}u^n(\cdot, t)\|_{(H^3)^*} \leq \|D^{2\alpha}u^n(\cdot, t)\|_{(H^3)^*} + \frac{t^{-2\alpha}}{\Gamma(1 - 2\alpha)}\|u_0\|_{L^2(\Omega)} + \|\partial^{2\alpha} f^n_1(\cdot, t)\|_{(H^3)^*}, \tag{63}
\]
where we used the equality \( \Delta^k w|_{\partial \Omega} = 0 \) for \( k = 1, 2 \). Next, applying (61) and proceeding as in the previous case, we prove the claim.

In general, if \( \alpha \in (0, \frac{1}{2}) \) and \( k \) is the smallest number such that \( \frac{1}{2} < \alpha(k + 1) < 1 \), then applying the Riemann-Liouville derivative \( \partial^m \) to both sides of (54), we obtain
\[
\int_\Omega D^{(m+1)\alpha}u^n(x, t)w(x)dx = \int_\Omega D^{m\alpha}u^n(x, t)\Delta w(x)dx + \frac{t^{-m\alpha}}{\Gamma(1 - m\alpha)}\int_\Omega u^n(x, 0)\Delta w(x)dx
\]
\[+ \int_\Omega \partial^{m\alpha} f^n_1(x, t)w(x)dx,
\]
where \( w \in \overset{\circ}{H}^{2m+1} \) and \( m = 1, \ldots, k \). Then we have
\[
\|D^{(m+1)\alpha}u^n(\cdot, t)\|_{(H^{2m+1})},
\]

20
Using these inequalities for \( m = 1, \ldots, k \); we have
\[
\| D^{(k+1)\alpha} u^n(\cdot, t) \|_{(H^{2k+1})^*} \leq \| D^\alpha u^n(\cdot, t) \|_{(H^{1})^*} + t^{-k\alpha} \| u_0 \|_{L^2(\Omega)} \sum_{m=1}^k \frac{t^{(k-m)\alpha}}{\Gamma(1-m\alpha)} + \sum_{m=1}^k \| \partial^{m\alpha} f_n(\cdot, t) \|_{(H^{2m+1})^*}. \tag{64}
\]
We recall that \((H^1)^* = (H^1_0(\Omega))^* = H^{-1}(\Omega)\). Hence using the assumption, the estimate \(52\), the condition \( k\alpha < \frac{1}{2} \) and proposition \(13\) we obtain a uniform bound for \( \frac{d}{dt} I^{1-(k+1)\alpha} [u^n - u_0^n] \) in \( L^2(0, T; (H^{2k+1})^*) \). Hence \( \frac{d}{dt} I^{1-(k+1)\alpha} [u - u_0] \in L^2(0, T; (H^{2k+1})^*) \), and applying proposition \(7\) we finish the proof in this case.

Finally, if \( \frac{1}{2} = (k + 1)\alpha \) and \( w \in H^{2k+3} \), then applying the Riemann-Liouville derivative \( \partial^{(k+1)\alpha} \) to both sides of \(57\), we obtain
\[
\int_\Omega D^{\alpha + \frac{1}{2}} u^n(x, t) w(x) dx = \int_\Omega D^{\frac{3}{2}} u^n(x, t) \Delta w(x) dx + \frac{t^{-\frac{1}{2}}}{\Gamma(\frac{1}{2})} \int_\Omega u^n(x, 0) \Delta w(x) dx + \int_\Omega \partial^{\frac{3}{2}} f_n(x, t) w(x) dx.
\]
Then using \(64\) and taking the \( p \)-th power of both sides, we obtain a uniform bound for \( D^{\alpha + \frac{1}{2}} u^n \) in \( L^p(0, T; (H^{2k+3})^*) \), provided \( p \in [1, 2) \). In order to apply proposition \(7\) we choose \( p \) such that \( \frac{1}{p} < \frac{1}{2} + \alpha \) and the proof is completed. \(\square\)

## 5 Regular solutions

Now we shall prove the existence of regular solution of problem \(41\). We start with the proof of the second energy estimate for approximating solutions.

**Lemma 4.** Assume that \( u_0 \in H^1_0(\Omega), f \in L^2(0, T; L^2(\Omega)) \) and \( \max_{i,j} \| \nabla a_{i,j} \|_{L^\infty(\Omega)^2} < \infty \) and for some \( p_1, p_2 \in [2, \frac{2N}{N-2}] \) we have \( b \in L^\infty(0, T; L^\frac{2p_1}{p_1-2}(\Omega)), \ c \in L^\infty(0, T; L^\frac{2p_2}{p_2-2}(\Omega)) \). Then for each \( t \in [0, T] \) and \( n \in \mathbb{N} \) the approximate solution \( u^n \) satisfies the following estimate
\[
I^{1-\alpha} \| \nabla u^n(\cdot, t) \|_{L^2(\Omega)}^2 + \frac{\alpha}{\Gamma(1-\alpha)} \int_0^t \int_0^\tau (\tau - s)^{-\alpha - 1} \| \nabla u^n(\cdot, \tau) - \nabla u^n(\cdot, s) \|_{L^2(\Omega)}^2 ds d\tau
+ \frac{1}{\Gamma(1-\alpha)} \int_0^t \tau^{-\alpha} \| \nabla u^n(\cdot, \tau) - \nabla u_0^n(\cdot) \|_{L^2(\Omega)}^2 d\tau + \frac{\lambda}{32} \int_0^t \| D^2 u^n(\cdot, \tau) \|_{L^2(\Omega)}^2 d\tau
\leq \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} \| \nabla u_0^n \|_{L^2(\Omega)}^2 + 4\lambda^{-1} \int_0^t \| f(\cdot, \tau) \|_{L^2(\Omega)}^2 d\tau + 4\lambda^{-1} \tilde{\delta}_n + \tilde{C}_0 \int_0^t \| D u^n(\cdot, \tau) \|_{L^2(\Omega)}^2 d\tau,
\tag{65}
\]
where \( \tilde{\delta}_n \rightarrow 0 \) uniformly with respect to \( t \) as \( n \rightarrow \infty \) and \( \tilde{C}_0 \) depends only on \( \max_{i,j} \| \nabla a_{i,j} \|_{L^\infty(\Omega)^2}, \ the \ regularity \ of \ \partial_\Omega, \ p_1, p_2, \lambda \) and norms \( \| b \|_{L^\infty(0, T; L^{\frac{2p_1}{p_1-2}}(\Omega))}, \ \| c \|_{L^\infty(0, T; L^{\frac{2p_2}{p_2-2}}(\Omega))} \).
Proof. We multiply \( 38 \) by \( c_{n,m}(t) \lambda_m \) and sum over \( m = 1, \ldots, n \). Then

\[
- \int_{\Omega} D^n u^n(x,t) \Delta u^n(x,t) dx - \sum_{i,j=1}^{N} \int_{\Omega} a_{i,j}^n(x,t) \partial_j u^n(x,t) \partial_i \Delta u^n(x,t) dx
\]

\[
= - \sum_{j=1}^{N} \int_{\Omega} b_j^n(x,t) \partial_j u^n(x,t) \Delta u^n(x,t) dx - \int_{\Omega} c^n(x,t) u^n(x,t) \Delta u^n(x,t) dx
\]

\[- \int_{\Omega} f^n(x,t) \Delta u^n(x,t) dx.
\]

Using the boundary condition, we have \( D^n u^n_{|\partial \Omega} = 0 \), \( \Delta u^n_{|\partial \Omega} = 0 \) and integrating by parts, we obtain

\[
\int_{\Omega} D^n \nabla u^n(x,t) \nabla u^n(x,t) dx + \sum_{i,j=1}^{N} \int_{\Omega} \partial_i \left( a_{i,j}^n(x,t) \partial_j u^n(x,t) \right) \Delta u^n(x,t) dx
\]

\[
= - \sum_{j=1}^{N} \int_{\Omega} b_j^n(x,t) \partial_j u^n(x,t) \Delta u^n(x,t) dx - \int_{\Omega} c^n(x,t) u^n(x,t) \Delta u^n(x,t) dx
\]

\[- \int_{\Omega} f^n(x,t) \Delta u^n(x,t) dx.
\]

Applying proposition 3 from the appendix and the Young inequality, we obtain

\[
\int_{\Omega} D^n \nabla u^n(x,t) \nabla u^n(x,t) dx + \frac{\lambda}{16} \| \nabla^2 u^n(\cdot,t) \|^2_{L^2(\Omega)}
\]

\[
\leq C_{0,n} \| \nabla u^n(\cdot,t) \|^2_{L^2(\Omega)} + \frac{4}{\lambda} \| f^n(\cdot,t) \|^2_{L^2(\Omega)} + \frac{4}{\lambda} \| b^n \nabla u^n \|^2_{L^2(\Omega)} + \frac{4}{\lambda} \| c^n u^n \|^2_{L^2(\Omega)},
\]

where \( C_{0,n} \) depends only on the regularity of \( \partial \Omega \) and \( \kappa^n(t) \equiv \max_{i,j} \| \nabla a_{i,j}^n(\cdot,t) \|_{L^\infty(\Omega^r)} \) and \( C_{0,n} \) are uniformly estimated by some \( C_0 \), which depends only on \( \max_{i,j} \| \nabla a_{i,j} \|_{L^\infty(\Omega^r)} \) and the regularity of \( \partial \Omega \).

Similarly as in the proof of lemma 3 using the Sobolev embedding, we obtain

\[
\| b^n \nabla u^n \|^2_{L^2(\Omega)} + \| c^n u^n \|^2_{L^2(\Omega)}
\]

\[
\leq \left( \| b^n(\cdot,t) \|^2_{\frac{2p_1}{L^\frac{2p_1}{2}}} + \| c^n(\cdot,t) \|^2_{L^\frac{2p_2}{2}(\Omega)} \right) \left[ \| \nabla^2 u^n(\cdot,t) \|^2_{L^2(\Omega)} + C \| u^n(\cdot,t) \|^2_{L^2(\Omega)} \right],
\]

where \( C \) depends only on \( \varepsilon, p_1 \) and \( p_2 \). Taking \( \varepsilon \) small enough, we have

\[
\int_{\Omega} D^n \nabla u^n(x,t) \nabla u^n(x,t) dx + \frac{\lambda}{32} \| \nabla^2 u^n(\cdot,t) \|^2_{L^2(\Omega)}
\]

\[
\leq C_0 \| \nabla u^n(\cdot,t) \|^2_{L^2(\Omega)} + \frac{4}{\lambda} \| f^n(\cdot,t) \|^2_{L^2(\Omega)},
\]

22
where $C_0$ depends only on $\max_{i,j} \|\nabla a_{i,j}\|_{L^\infty(\Omega)}$, the regularity of $\partial \Omega$, $p_1$, $p_2$, $\lambda$ and norms $\|b\|_{L^\infty(\partial \Omega)}$, $\|c\|_{L^\infty(\partial \Omega)}$. 

By corollary [41] the function $\nabla u^n$ satisfies the assumption of lemma 2 and from [41] we obtain

$$
\frac{1}{2} D^\alpha \|\nabla u^n(\cdot, t)\|_{L^2(\Omega)}^2 + \frac{\alpha}{2\Gamma(1 - \alpha)} \int_0^t (t - \tau)^{-\alpha - 1} \|\nabla u^n(\cdot, t) - \nabla u^n(\cdot, \tau)\|_{L^2(\Omega)}^2 d\tau
$$

$$
+ \frac{1}{2\Gamma(1 - \alpha)} \int_0^t (t - \tau)^{-\alpha} \|\nabla u^n(\cdot, t) - \nabla u^n(\cdot, \tau)\|_{L^2(\Omega)}^2 d\tau + \frac{\lambda}{32} \|\nabla^2 u^n(\cdot, t)\|_{L^2(\Omega)}^2
$$

$$
\leq C_0 \|\nabla u^n(\cdot, t)\|_{L^2(\Omega)}^2 + \frac{4}{\lambda} \|f_\alpha(\cdot, t)\|_{L^2(\Omega)}^2. 
$$

We integrate both sides of (66) with respect to $t \in (0, T)$ and use the identity $I = I^{1 - \alpha} I^\alpha$ and $\nabla u^n(x, \cdot) \in AC[0, T]$ for each $x \in \Omega$. We estimate the second term on the right-hand side as in the proof of lemma 3 and after applying propositions 1 and [41] we have [65].

**Proof of theorem 4.** Under the assumptions of theorem the existence of a weak solution $u$ is guaranteed by theorem 2. Therefore we have to obtain the additional estimates. By [42], [65] and the weak compactness argument, we obtain the bound [44]. Reasoning similarly as in the proof of [52], we obtain

$$
\left\| \frac{d}{dt} I^{1 - \alpha}[u^n - u^n_0] \right\|_{L^2(0, T; L^2(\Omega))} \leq C(\|u_0\|_{H^1_0(\Omega)} + \|f\|_{L^2(0, T; L^2(\Omega))}).
$$

Hence there exist $w \in L^2(\Omega^T)$ and a subsequence, denoted again by $u^n$, such that $\frac{d}{dt} I^{1 - \alpha}[u^n - u^n_0] \rightharpoonup w$ in $L^2(\Omega^T)$. As in the proof of theorem 2, we see that $\frac{d}{dt} I^{1 - \alpha}[u^n - u^n_0] = w$, where the time derivative is understood in the weak sense.

Finally, from proposition 7 we obtain the the continuity of $u$ with the values in $L^2(\Omega)$, provided $\alpha > \frac{1}{2}$.  

**6 Appendix**

In this section we collect useful propositions for the proofs. The basic equality for the fractional integral is $I^\alpha I^\beta f = I^{\alpha + \beta} f$ and holds for $f \in L^1(0, T)$, where $a, b$ are positive numbers (see theorem 2.5 in [10]). We also have (see equalities (2.4.33) and (2.4.44) in [10])

**Proposition 1.** If $f \in AC[0, T]$ and $\alpha \in (0, 1]$, then $I^\alpha D^\alpha f(t) = f(t) - f(0)$ and $D^\alpha I^\alpha f(t) = f(t)$.

By direct calculation we have

**Proposition 2.** If $f \in C[0, T]$ and $\alpha \in (0, 1)$, then $I^\alpha f \in C[0, T]$.

**Proposition 3** (lemma A.1 in [3]). If $f \in AC[0, T]$ and $\alpha \in (0, 1)$, then $I^\alpha f \in AC[0, T]$ and $(I^\alpha f)'(t) = I^\alpha f(t) + \frac{\alpha - 1}{\Gamma(\alpha)} f(0)$. 

23
Proposition 4 (lemma A.4 in [5]). Assume that \( \alpha \in (0, 1) \), \( f \in AC[0, T] \) and \( t^{1-\alpha}f' \in L^\infty(0, T) \). Then
\[
|t_2^{1-\alpha}(I^\alpha f')(t_2) - t_1^{1-\alpha}(I^\alpha f')(t_1)| \leq C_0 \|t^{1-\alpha}f'\|_{L^\infty(0, T)}|t_2 - t_1|^{\alpha},
\] (68)
where \( C_0 \) depends only on \( \alpha \). In particular, \( t \mapsto t^{1-\alpha}(I^\alpha f')(t) \in C^{0,\alpha}[0, T] \) and \( D^{1-\alpha}f \in C^{0,\alpha}(0, T) \).

In the formulation of lemma A.4 in [5] it should be \( D^{1-\alpha}f \in C^{0,\alpha}(0, T) \) instead of \( D^{\alpha}f \in C^{0,1-\alpha}(0, T) \).

Proposition 5. Assume that \( \alpha, \beta \in (0, 1) \), \( \alpha + \beta \leq 1 \), \( f \in AC[0, T] \). Then, for the Caputo derivative \( D^\beta \) and the Riemann-Liouville derivative \( \partial^\beta \), defined by (3) and (2) respectively, the equality \( \partial^\beta D^\alpha f(t) = D^{\alpha+\beta}f(t) \) holds.

Proof. We can write
\[
\partial^\beta D^\alpha f(t) = \frac{d}{dt}I^{1-\beta}D^\alpha f(t) = \frac{d}{dt}I^{1-\beta} \frac{d}{dt}I^{1-\alpha}[f - f(0)].
\]
Applying proposition 3 we have
\[
\frac{d^2}{dt^2}I^{1-\beta}I^{1-\alpha}[f - f(0)] = \frac{d}{dt}I^{1-\alpha-\beta}[f - f(0)] = D^{\alpha+\beta}f(t).
\]

Proposition 6 (theorem 1 in [12]). Assume that \( \alpha > 0 \), \( T \in (0, \infty] \), \( a, w \in L^1_{\text{loc}}[0, T] \), \( a, g, w \) are nonnegative and \( g \) is nondecreasing and bounded. If \( w \) satisfies inequality
\[
w(t) \leq a(t) + g(t)(I^\alpha w)(t) \quad \text{for } t \in [0, T),
\]
then
\[
w(t) \leq \sum_{k=0}^{\infty} g^k(t)(I^{\alpha k}a)(t) \quad \text{for } t \in [0, T).
\]

For convenience of readers, we recall a simple proof from [12].

Proof. If we apply the operator \( (g(t)I^\alpha)^n \) to both sides of (69) and using the properties of \( g \) we deduce that
\[
w(t) \leq \sum_{k=0}^{n-1} g^k(t)I^{\alpha k}a(t) + g^n(t)I^{\alpha n}w(t).
\]
The last term uniformly tends to 0, when \( n \to \infty \).
Proposition 7. Assume that \( X \) is a normed vector space, \( u \in L^1(0,T; X) \), \( p \in (1,\infty) \) and \( \frac{d}{dt}(I^{1-\alpha}[u-u_0]) \in L^p(0,T; X) \) and \( I^{1-\alpha}[u-u_0](0) = 0 \). If \( \alpha \in (\frac{1}{p},1] \), then \( u \in C([0,T]; X) \) and \( u(0) = u_0 \).

Proof. We have \( I^{1-\alpha}[u-u_0](t) = \int_0^t \frac{d}{ds}(I^{1-\alpha}[u-u_0])(s)ds \) and
\[
\| I^{1-\alpha}[u-u_0](t) \|_X \leq \int_0^t \| \frac{d}{ds}(I^{1-\alpha}[u-u_0])(s) \|_X ds \leq t^{1-\frac{1}{p}} \| \frac{d}{dt}I^{1-\alpha}[u-u_0] \|_{L^p(0,T; X)}
\]
for all \( t \), where we used the fact that \( I^{1-\alpha}[u-u_0] \) is absolutely continuous. Thus
\[
t^{\alpha-1} \| I^{1-\alpha}[u-u_0](t) \|_X \leq t^{\alpha-\frac{1}{p}} \| \frac{d}{dt}(I^{1-\alpha}[u-u_0]) \|_{L^p(0,T; X)}.
\]
(70)

On the other hand we have
\[
I^{\alpha} \frac{d}{dt}(I^{1-\alpha}[u-u_0])(t) = u(t) - u_0,
\]
(71)
because \( I^{1-\alpha}[u-u_0] \in AC \) and applying proposition 8 we have
\[
I^{\alpha} \frac{d}{dt}(I^{1-\alpha}[u-u_0])(t) = \frac{d}{dt}I^{\alpha}(I^{1-\alpha}[u-u_0])(t) = \frac{d}{dt}I[u-u_0](t) = u(t) - u_0.
\]

From theorem 3.6 in [10], we see that \( I^{\alpha} \) is continuous from \( L^p(0,T) \) to \( C^{0,\frac{1}{p}-\frac{1}{p}}([0,T]; X) \) and the left-hand side of (71) is Hölder continuous, and so \( u \in C^{0,\frac{1}{p}-\frac{1}{p}}([0,T]; X) \). Therefore \( u(0) \) is well-defined. Using (70) and setting \( C_\alpha = \Gamma(2-\alpha) \), we have
\[
\| u(0) - u_0 \|_X = C_\alpha t^{\alpha-1} \| I^{1-\alpha}[u(0) - u_0](t) \|_X
\]
\[
\leq C_\alpha t^{\alpha-1} \| I^{1-\alpha}[u-u_0](t) \|_X + C_\alpha t^{\alpha-1} \| I^{1-\alpha}[u-u_0](0) \|_X
\]
\[
\leq C_\alpha t^{\alpha-\frac{1}{p}} \| \frac{d}{dt}(I^{1-\alpha}[u-u_0]) \|_{L^p(0,T; X)} + \sup_{\tau \in (0,t)} \| u(\tau) - u(0) \|_X
\]
\[
\leq C_\alpha t^{\alpha-\frac{1}{p}} \| \frac{d}{dt}(I^{1-\alpha}[u-u_0]) \|_{L^p(0,T; X)} + C_{\alpha,p} t^{\alpha-\frac{1}{p}},
\]
where we used the Hölder continuity of \( u \). As \( t \to 0 \), we have \( u(0) = u_0 \). 

Proposition 8. Assume that \( a_{i,j} = a_{j,i} \) and \( \{a_{i,j}(x,t)\}_{i,j=1}^N \) define a uniformly elliptic operator of second order, i.e., there exist \( \lambda, \mu > 0 \) such that
\[
\lambda |\xi|^2 \leq \sum_{i,j=1}^N a_{i,j}(x,t)\xi_i\xi_j \leq \mu |\xi|^2 \quad \text{for} \quad \xi \in \mathbb{R}^N, \quad x \in \Omega, \quad t \in [0,T],
\]
(72)
holds. Then for all \( i,j,x,t \) we have \( |a_{i,j}(x,t)| \leq \mu \).

Proof. If we take \( k \in \{1,\ldots,N\} \) and set \( \xi_k = 1 \) and \( \xi_l = 0 \) for \( l \neq k \), then from (72) we see \( 0 < \lambda \leq a_{k,k} \leq \mu \). If we take \( k,l \in \{1,\ldots,N\} \) and set \( \xi_k = 1, \xi_l = 1 \) and \( \xi_m = 0 \) for \( m \neq k,l \), then we have \( 2\lambda \leq 2a_{k,l} + a_{k,k} + a_{l,l} \leq 2\mu \). Thus \( 2a_{k,l} \leq 2\mu - a_{k,k} - a_{l,l} \leq 2\mu \) and \( 2a_{k,l} \geq 2\lambda - a_{k,k} - a_{l,l} \geq 2\lambda - 2\mu \geq -2\mu \). 

25
Proposition 9. Assume that $\Omega \subset \mathbb{R}^N$ is a bounded domain with the boundary of $C^2$ class and there exist $\lambda, \mu > 0$ such that

$$\lambda |\xi|^2 \leq \sum_{i,j=1}^{N} a_{i,j} (x,t) \xi_i \xi_j \leq \mu |\xi|^2 \quad \text{for} \quad \xi \in \mathbb{R}^N, \ x \in \Omega, \ t \in [0, T], \quad (73)$$

holds, where $a_{i,j} = a_{j,i}$ and $\kappa(t) = \max_{i,j} \|\nabla a_{i,j}(\cdot,t)\|_{L^\infty(\Omega)}$. If $u \in H^3(\Omega)$ and $u$ and $\Delta u$ vanish on $\partial \Omega$, then

$$\frac{\lambda}{4}\|\nabla^2 u\|_{L^2(\Omega)}^2 - C\|\nabla u\|_{L^2(\Omega)}^2 \leq \sum_{i,j=1}^{N} \int_\Omega \partial_i (a_{i,j} (x,t) \partial_j u) \Delta u dx$$

where $C$ depends continuously on $\kappa(t)$ and the $C^2$-norm of $\partial \Omega$ and $\nabla^2 u = \{u_{x_j x_i}\}_{i,j=1}^{N}$.

Proof. We shall follow [6]. Integrating twice by parts, we have

$$\frac{\lambda}{4}\|\nabla^2 u\|_{L^2(\Omega)}^2 - C\|\nabla u\|_{L^2(\Omega)}^2 \leq \sum_{i,j=1}^{N} \int_\Omega \partial_i (a_{i,j} (x,t) \partial_j u) \Delta u dx$$

More precisely, for fixed $i,j$ and $t$, we obtain

$$\frac{\lambda}{4}\|\nabla^2 u\|_{L^2(\Omega)}^2 - C\|\nabla u\|_{L^2(\Omega)}^2 \leq \sum_{i,j=1}^{N} \int_\Omega \partial_i (a_{i,j} (x,t) \partial_j u) \Delta u dx$$

where $C$ depends only on $\kappa(t)$ and the $C^2$-norm of $\partial \Omega$.

For this purpose we first write the function under the integral on the left-hand side of (74) in coordinates related with boundary point $x^0 \in \partial \Omega$. More precisely, for fixed
$t$ and $x^0 \in \partial \Omega$, we define an orthogonal transformation $P = \{p_{m,i}\}_{m,i=1}^N$ such that for $y = P(x - x^0)$ we have $(0, \ldots, 1) = P(n(x^0) - x^0)$, where $n(x^0)$ is the outer normal vector at $x^0$. By the assumption concerning the boundary we have $\Omega \ni x^0 \mapsto P(x^0)$ is $C^1$. Then $y_m = \sum_{i=1}^N p_{m,i} (x_i - x_i^0)$ and since $P^T = P^{-1}$, we have $x_i - x_i^0 = \sum_{m=1}^N p_{m,i} y_m$ and $\frac{\partial}{\partial x_i} = \sum_{m=1}^N p_{m,i} \frac{\partial}{\partial y_m}$. Let $(y_1, \ldots, y_{N-1}, \omega(y_1, \ldots, y_{N-1}))$ be a parametrization of some neighborhood of $x^0 \in \partial \Omega$. Then

$$\omega(0) = 0, \quad \frac{\partial \omega}{\partial y_i}(0) = 0, \quad \text{for } i = 1, \ldots, N - 1. \quad (75)$$

If we denote $\tilde{u}(y, t) = u(x^0 + P^T y, t)$, then using $u_{\partial \Omega} = 0$ we obtain

$$\tilde{u}(y_1, \ldots, y_{N-1}, \omega(y_1, \ldots, y_{N-1}), t) = 0,$$

in some neighborhood of $0 \in \mathbb{R}^{N-1}$. If we take $i, j \in \{1, \ldots, N - 1\}$ and differentiate the above equality with respect to $y_i$ and next $y_j$, then we have

$$\frac{\partial \tilde{u}}{\partial y_i} + \frac{\partial \tilde{u}}{\partial y_j} \frac{\partial \omega}{\partial y_i} = 0, \quad \frac{\partial^2 \tilde{u}}{\partial y_i \partial y_j} + \frac{\partial^2 \tilde{u}}{\partial y_i \partial y_N} \frac{\partial \omega}{\partial y_j} + \frac{\partial^2 \tilde{u}}{\partial y_j \partial y_N} \frac{\partial \omega}{\partial y_i} = 0,$$

in some neighborhood of $0 \in \mathbb{R}^{N-1}$. Hence (75) yields

$$\frac{\partial \tilde{u}}{\partial y_j}(0, t) = 0, \quad \frac{\partial^2 \tilde{u}}{\partial y_i \partial y_j}(0, t) = -\frac{\partial \tilde{u}}{\partial y_N}(0, t) \frac{\partial \omega}{\partial y_i \partial y_j}(0) \quad \text{for } i, j \in \{1, \ldots, N - 1\}. \quad (77)$$

On the other hand, using the equality $\frac{\partial u}{\partial n}(x^0, t) = \frac{\partial \tilde{u}}{\partial y_N}(0, t)$ and (77), we see

$$\partial_i \frac{\partial u}{\partial n}(x^0, t) = \sum_{k=1}^N p_{k,i} \frac{\partial^2 \tilde{u}}{\partial y_N \partial y_k}(0, t), \quad \partial_j u(x^0, t) = \sum_{m=1}^N p_{m,j} \frac{\partial \tilde{u}}{\partial y_m}(0, t) = p_{N,j} \frac{\partial \tilde{u}}{\partial y_N}(0, t).$$

Thus

$$\sum_{i,j=1}^N a_{i,j}(x^0, t) \partial_j u(x^0, t) \partial_i \frac{\partial u}{\partial n}(x^0, t) = \sum_{i,j=1}^N a_{i,j}(x^0, t) p_{N,i} p_{N,j} \frac{\partial \tilde{u}}{\partial y_N}(0, t) \frac{\partial^2 \tilde{u}}{\partial y_N \partial y_k}(0, t)$$

$$= \sum_{i,j=1}^N a_{i,j}(x^0, t) p_{N,i} p_{N,j} \frac{\partial \tilde{u}}{\partial y_N}(0, t) \frac{\partial^2 \tilde{u}}{\partial y_N}(0, t) + \sum_{k=1}^{N-1} \sum_{i,j=1}^N a_{i,j}(x^0, t) p_{N,i} p_{N,j} \frac{\partial \tilde{u}}{\partial y_N}(0, t) \frac{\partial^2 \tilde{u}}{\partial y_N \partial y_k}(0, t).$$

We shall show that the first sum vanishes. Indeed, the Laplace operator is invariant under orthogonal change of variables, so that $\Delta_y \tilde{u}(0, t) = \Delta_x u(x^0, t)$. On the other side, by the boundary condition we have $\Delta_x u(x^0, t) = 0$ and then by (77) we obtain $\frac{\partial^2 \tilde{u}}{\partial y_N}(0, t) = 0$. Thus

$$\sum_{i,j=1}^N \int_{\partial \Omega} a_{i,j}(x^0, t) \partial_j u(x^0, t) \partial_i \frac{\partial u}{\partial n}(x^0, t) dS(x^0)$$

27
Proposition 10. If \( I_k \) holds.

The key observation is that the differentiation with respect to \( C \) where
\[
\frac{\partial}{\partial y_k} \left( \frac{\partial u}{\partial n}(x^0, t) \right)^2 dS(x^0).
\]
and the inequality
\[
\sum_{i,j=1}^N \int_{\partial \Omega} a_{i,j}(x^0, t) \partial_j u(x^0, t) \partial_i \frac{\partial u}{\partial n}(x^0, t) dS(x^0) = \int_{\partial \Omega} K(x, t) \left| \frac{\partial u}{\partial n}(x, t) \right|^2 dS(x),
\]
where \( \Omega \ni x \rightarrow K(x, t) \) is a continuous function and \( \|K(\cdot, t)\|_{C(\partial \Omega)} \) depends only on \( \kappa(t) \) and the \( C^2 \)-regularity of \( \partial \Omega \). Hence using inequality (21) in [7], we have
\[
\sum_{i,j=1}^N \int_{\partial \Omega} a_{i,j}(x, t) \partial_j u(x, t) \partial_i \frac{\partial u}{\partial n}(x, t) dS(x) \leq C \int_{\partial \Omega} \left| \frac{\partial u}{\partial n}(x, t) \right|^2 dS(x)
\]
where \( C(\varepsilon) \) depends only on \( \varepsilon, \kappa(t) \) and the \( C^2 \) regularity of \( \partial \Omega \). If we take \( \varepsilon = \frac{1}{T} \), then we get (74) and the proof is finished.

The following proposition can be obtained formally by integration of the equality (17) in lemma 2.1 [17] with \( k(t) = \frac{1}{\Gamma(1-\alpha)} t^{-\alpha} \). However, the function \( k(t) \) does not belong to \( W^{1,1}(0, T) \) and we can not apply this lemma directly.

**Proposition 10.** If \( w \in AC[0, T] \) then for \( \alpha \in (0, 1) \) the following equality
\[
\int_0^T \partial^\alpha w(t) \cdot w(t) dt = \frac{\alpha}{4} \int_0^T \int_0^T \frac{|w(t) - w(\tau)|^2}{|t - \tau|^{1+\alpha}} d\tau dt + \frac{1}{2 \Gamma(1-\alpha)} \int_0^T [(T-t)^{-\alpha} + t^{-\alpha}] |w(t)|^2 dt
\]  
holds, where \( \partial^\alpha \) denotes the Riemann-Liouville derivative. In particular, for \( t \in (0, T) \) the inequality
\[
\int_0^t \partial^\alpha w(\tau) \cdot w(\tau) d\tau \geq \frac{t^{-\alpha}}{2 \Gamma(1-\alpha)} \int_0^t |w(\tau)|^2 d\tau
\]  
holds.

**Proof.** We first note that the left-hand side of (78) is finite, because by proposition 3 \( I^{1-\alpha} w \) is absolutely continuous and \( \partial^\alpha w = \frac{d}{dt} I^{1-\alpha} w \) is in \( L^1(0, T) \). Then we calculate
\[
\int_0^T \partial^\alpha w(t) \cdot w(t) dt = \int_0^T \frac{d}{dt} I^{1-\alpha} [w(t) - w(0)] \cdot w(t) dt + \frac{w(0)}{\Gamma(1-\alpha)} \int_0^T t^{-\alpha} w(t) dt
\]
\[
= \int_0^T I^{1-\alpha} w'(t) \cdot w(t) dt + \frac{w(0)}{\Gamma(1-\alpha)} \int_0^T t^{-\alpha} w(t) dt,
\]
28
Hence using the Lebesgue differential theorem, we have

\[
\frac{1}{\Gamma(1-\alpha)} \int_0^t \int_0^t (t-\tau)^{-\alpha} w'(\tau) \, d\tau \cdot w(t) \, dt + \frac{w(0)}{\Gamma(1-\alpha)} \int_0^t t^{-\alpha} w(t) \, dt
\]

\[
= \frac{1}{\Gamma(1-\alpha)} \int_0^t \int_0^t (t-\tau)^{-\alpha} w'(\tau) \cdot [w(t) - w(\tau)] \, d\tau \, dt + \frac{w(0)}{\Gamma(1-\alpha)} \int_0^t t^{-\alpha} w(t) \, dt
\]

\[
+ \frac{1}{\Gamma(1-\alpha)} \int_0^t \int_0^t (t-\tau)^{-\alpha} w'(\tau) w(\tau) \, d\tau \, dt
\]

\[
= \frac{1}{2} \frac{1}{\Gamma(1-\alpha)} \int_0^t \int_0^t (t-\tau)^{1-\alpha} \left( |w(t) - w(\tau)|^2 \right)_\tau \, d\tau \, dt + \frac{w(0)}{\Gamma(1-\alpha)} \int_0^t t^{-\alpha} w(t) \, dt
\]

\[
+ \frac{1}{2} \frac{1}{\Gamma(1-\alpha)} \int_0^t \int_0^t (t-\tau)^{-\alpha} \left( |w(\tau)|^2 \right)_\tau \, d\tau \, dt
\]

\[
= \frac{\alpha}{2} \frac{1}{\Gamma(1-\alpha)} \int_0^t \int_0^t (t-\tau)^{1-\alpha} \left( |w(t) - w(\tau)|^2 \right)_\tau \, d\tau \, dt + \frac{1}{2} \frac{1}{\Gamma(1-\alpha)} \int_0^t t^{-\alpha} w(t) \, dt + \frac{1}{2} \frac{1}{\Gamma(1-\alpha)} \int_0^t (T-\tau)^{1-\alpha} \left( |w(\tau)|^2 \right)_\tau \, d\tau
\]

\[
= \frac{\alpha}{2} \frac{1}{\Gamma(1-\alpha)} \int_0^t \int_0^t (t-\tau)^{1-\alpha} \left( |w(t) - w(\tau)|^2 \right)_\tau \, d\tau \, dt + \frac{1}{2} \frac{1}{\Gamma(1-\alpha)} \int_0^t t^{-\alpha} w(t) \, dt + \frac{1}{2} \frac{1}{\Gamma(1-\alpha)} \int_0^t (T-\tau)^{1-\alpha} |w(\tau)|^2 \, d\tau
\]

\[
+ \frac{\alpha}{2} \frac{1}{2} \frac{1}{\Gamma(1-\alpha)} \int_0^t \tau^{-\alpha} w(\tau) \, d\tau
\]

Using the Lebesgue differential theorem, we have \(|t-\tau|^{-1} \int_\tau^t w'(s) \, ds \rightarrow w'(t)\) for a.a. \(t\) and thus \(\lim_{\tau \to t^-} |t-\tau|^{-\alpha} |w(t) - w(\tau)|^2 = \lim_{\tau \to t^-} |t-\tau|^{2-\alpha} \left| \int_\tau^t w'(s) \, ds \right|^2 = 0.\)

Hence

\[
\frac{\alpha}{2} \frac{1}{\Gamma(1-\alpha)} \int_0^t \int_0^t (t-\tau)^{1-\alpha} \left( |w(t) - w(\tau)|^2 \right)_\tau \, d\tau \, dt + \frac{1}{2} \frac{1}{\Gamma(1-\alpha)} \int_0^t t^{-\alpha} w(t) \, dt + \frac{1}{2} \frac{1}{\Gamma(1-\alpha)} \int_0^t (T-\tau)^{1-\alpha} \left( |w(\tau)|^2 \right)_\tau \, d\tau
\]

and the proof is finished.

\(\square\)

**Corollary 2.** Assume that \(t \in (0, T)\) and \(\alpha \in (0, 1)\). If \(w \in L^2(0, T)\) and \(I^{1-\alpha} w \in H^1(0, T)\), then (12) holds for \(w\).
Proof. According to [4], there exists a sequence \( \{w_n\} \subset C^1[0,T] \) such that \( w_n(0) = 0 \) and \( w_n \to w \) in \( L^2(0,T) \). Then applying (79) with \( w_n \) and next taking the limit, we obtain (79).

We recall some results from [3] and [4]. We denote

\[
0H^1 = \{ u \in H^1(0,T) : u(0) = 0 \}, \quad X_\alpha = \text{span}\{h_n\}_{n=1}^\infty,
\]

where \( h_n(t) = \sqrt{\frac{2}{n}} \sin\left(\frac{t}{\sqrt{n}}\right) \), \( \lambda_n = \frac{T}{\pi(n+\frac{1}{2})} \), \( n = 0, 1, \ldots \) and \( X_\alpha \) is Hilbert space with the following inner product

\[
(u,v)_{X_\alpha} = \sum_{n=0}^{\infty} \lambda_n^{-2\alpha} (u,h_n)_{L^2(0,T)} (v,h_n)_{L^2(0,T)}.
\]

By lemma 8 in [3] we have \( X_\alpha = 0H^\alpha(0,T) \), where

\[
0H^\alpha(0,T) = \begin{cases} \quad H^\alpha(0,T) & \text{for } \alpha \in \left(0,\frac{1}{2}\right), \\ \{ u \in H^{\frac{1}{2}}(0,T) : \int_0^T \frac{|u(t)|^2}{t} \, dt < \infty \} & \text{for } \alpha = \frac{1}{2}, \\ \{ u \in H^\alpha(0,T) : u(0) = 0 \} & \text{for } \alpha \in \left(\frac{1}{2}, 1\right), \end{cases}
\]

and for \( \alpha \neq \frac{1}{2} \) we have \( \|u\|_{0H^\alpha(0,T)} = \|u\|_{H^\alpha(0,T)} \), but

\[
\|u\|_{0H^{\frac{1}{2}}(0,T)} = \left(\|u\|_{H^{\frac{1}{2}}(0,T)}^2 + \int_0^T \frac{|u(t)|^2}{t} \, dt \right)^{\frac{1}{2}}.
\]

From [3] and [4] we deduce that for \( \alpha \in [0,1] \) the operator \( I^\alpha : L^2(0,T) \to 0H^\alpha(0,T) \) is isomorphism and the following inequalities

\[
e^{-\pi \sqrt{\alpha(1-\alpha)}} \|u\|_{0H^\alpha(0,T)} \leq \|\partial^\alpha u\|_{L^2(0,T)} \leq e^{\pi \sqrt{\alpha(1-\alpha)}} \|u\|_{0H^\alpha(0,T)} \quad \text{for } u \in 0H^\alpha(0,T),
\]

\[
e^{-\pi \sqrt{\alpha(1-\alpha)}} \|I^\alpha f\|_{0H^\alpha(0,T)} \leq \|u\|_{L^2(0,T)} \leq e^{\pi \sqrt{\alpha(1-\alpha)}} \|I^\alpha f\|_{0H^\alpha(0,T)} \quad \text{for } f \in L^2(0,T),
\]

holds. The above estimates are a consequence of Heinz-Kato theorem (see theorem 2.3.4 in [11]).

For measurable \( f \) defined on \( (0,T) \) we set

\[
\tilde{f}(t) = \begin{cases} f(t) & \text{for } t \in (0,T) \\ -f(-t) & \text{for } t \in (-T,0) \\ 0 & \text{elsewhere}. \end{cases}
\]

We define

\[
\Pi_n f(t) = \eta_{\frac{t}{n}} * \tilde{f}(t), \quad n = 1, 2, \ldots,
\]

where \( \eta_\varepsilon \) is mollifier, i.e. \( \eta_\varepsilon \geq 0 \), \( \int_\mathbb{R} \eta_\varepsilon = 1 \), \( \eta_\varepsilon \in C^\infty_0(-\frac{\varepsilon}{T},\frac{\varepsilon}{T}) \) and we assume that in addition \( \eta_\varepsilon(t) = \eta_\varepsilon(-t) \).

30
Proposition 11. For each $\beta \in (0, 1)$ and $n \in \mathbb{N}$ the following inequality holds

$$\|\partial^\beta \Pi_n f\|_{L^2(0,T)} \leq 2e^{2\pi \sqrt{\beta(1-\beta)}}\|\partial^\beta f\|_{L^2(0,T)}, \quad \text{if } I^{1-\beta} f \in \partial H^1(0,T). \quad (82)$$

Proof. By direct calculations we have

$$\|\Pi_n f\|_{\partial H^1(0,T)} \leq 2\|f\|_{\partial H^1(0,T)} \quad \text{for } f \in \partial H^1(0,T),$$

and

$$\|\Pi_n f\|_{L^2(0,T)} \leq 2\|f\|_{L^2(0,T)} \quad \text{for } f \in L^2(0,T).$$

Thus by interpolation argument (see theorem 5.1 and remark 11.5 in [8]) we have

$$\|\Pi_n f\|_{\partial H^\beta(0,T)} \leq 2\|f\|_{\partial H^\beta(0,T)} \quad \text{for } f \in \partial H^\beta(0,T).$$

Applying (80) we obtain (82). □

Proposition 12. Assume that $\beta \in (0, 1)$ and $f \in L^2(0,T)$ satisfies $I^{1-\beta} f \in \partial H^1(0,T)$. Then

$$\partial^\beta \Pi_n f \to \partial^\beta f \quad \text{in } L^2(0,T). \quad (83)$$

Furthermore, this convergence is uniform with respect $\beta \in [0, \delta]$ for any $\delta \in (0, 1)$.

Proof. According to [8], the set $\partial C^1([0,T]) = \{u \in C^1([0,T]) : u(0) = 0\}$ is dense in $\partial H^\beta(0,T)$. We fix $\varepsilon > 0$ and then from (80) we deduce that there exists $\hat{f} \in \partial C^1([0,T])$ such that

$$\|\partial^\beta f - \partial^\beta \hat{f}\|_{L^2(0,T)} \leq \frac{\varepsilon}{13}.$$ 

Then, using (82) we have

$$\|\partial^\beta \Pi_n f - \partial^\beta \hat{f}\|_{L^2(0,T)} 
\leq \|\partial^\beta \Pi_n f - \partial^\beta \Pi_n \hat{f}\|_{L^2(0,T)} + \|\partial^\beta \Pi_n \hat{f} - \partial^\beta \hat{f}\|_{L^2(0,T)} + \|\partial^\beta \hat{f} - \partial^\beta f\|_{L^2(0,T)} 
\leq (2e^{2\pi \sqrt{\beta(1-\beta)}} + 1)\frac{\varepsilon}{13} + \|\partial^\beta \Pi_n \hat{f} - \partial^\beta \hat{f}\|_{L^2(0,T)}.$$ 

To estimate the last term we write

$$\|\partial^\beta \Pi_n \hat{f} - \partial^\beta \hat{f}\|_{L^2(0,T)} \leq e^{\pi \sqrt{\beta(1-\beta)}}\|\Pi_n \hat{f} - \hat{f}\|_{\partial H^\beta(0,T)}$$

$$\leq e^{\pi \sqrt{\beta(1-\beta)}}[1 + (1 - \beta)^{-1}c(T)]^{1/2}\|\Pi_n \hat{f} - \hat{f}\|_{\partial H^1(0,T)},$$

where in the last inequality we applied the continuity of Hardy-Littlewood maximal operator in $L^2$. The last expression is estimated by $\varepsilon/13$, provided $n$ is large enough and the estimate is uniform with respect to $\beta \in [0, \delta]$ for any $\delta \in (0, 1)$.

□

The above results can be extended to the case of vector value functions (see remark 11.5 in [8]) and we have
Proposition 13. If $H$ is a Hilbert space, then for each $\beta \in (0,1)$ and $n \in \mathbb{N}$ the following inequality holds

$$
\|\partial^\beta \Pi_n f\|_{L^2(0,T;H)} \leq 2e^{2\sqrt{\beta(1-\beta)}}\|\partial^\beta f\|_{L^2(0,T;H)}, \quad \text{if} \quad I^{1-\beta} f \in \mathcal{O}H^1(0,T;H). \quad (84)
$$

Furthermore,

$$
\partial^\beta \Pi_n f \rightarrow \partial^\beta f \quad \text{in} \quad L^2(0,T;H) \quad (85)
$$

and this convergence is uniform with respect $\beta \in [0,\delta]$ for any $\delta \in (0,1)$.

Acknowledgment

The research leading to these results has been supported by the People Programme (Marie Curie Actions) of the European Union’s Seventh Framework Programme FP7/2007-2013/ under REA grant agreement no 319012 and the Funds for International Co-operation under Polish Ministry of Science and Higher Education grant agreement no 2853/7.PR/2013/2. Both authors are partially supported by Grants-in-Aid for Scientific Research (S) 15H05740 and (S) 26220702, Japan Society for the Promotion of Science.

References

[1] M. Allen, L. Caffarelli, A. Vasseur, A parabolic problem with a fractional time derivative Arch. Ration. Mech. Anal. 221 (2016), 603-630.

[2] P. Clément, S.O. Londen, G Simonett, Quasilinear evolutionary equations and continuous interpolation spaces, J. Differential Equations 196 (2004), 418-447.

[3] R. Gorenflo, M. Yamamoto, Operator-theoretic treatment of linear Abel integral equations of first kind, Japan J. Indust. Appl. Math. 16 (1999), no. 1, 137-161.

[4] R. Gorenflo, Y. Luchko, M. Yamamoto, Time-fractional diffusion equation in the fractional Sobolev spaces, Fract. Calc. Appl. Anal. 18 (2015), 799-820.

[5] A. Kubica, P. Rybka, K. Ryszewska, Weak solutions of fractional differential equations in non cylindrical domain, Nonlinear Anal. 36 (2017), 154-182.

[6] O.A. Ladyzhenskaya, On integral estimates, convergence, approximate methods, and solution in functionals for elliptic operators, (Russian) Vestnik Leningrad. Univ. 13 (1958), 60-69.

[7] O.A. Ladyzhenskaya, The mathematical theory of viscous incompressible flow, Gordon and Breach Science Publishers, New York 1969.

[8] J.L. Lions, E. Magenes, Non-homogeneous boundary value problems and applications, vol. I, Springer-Verlag, New York-Heidelberg, 1972.

[9] J. Prüss, Evolutionary integral equations and applications, Birkhäuser/Springer Basel AG, Basel, 1993.
[10] S.G. Samko, A.A. Kilbas, O.I. Marichev, *Fractional integrals and derivatives. Theory and applications*, Gordon and Breach Science Publishers, Yverdon, 1993.

[11] H. Tanabe, *Equations of evolution*, Monographs and Studies in Mathematics, 6, Pitman, Boston, Mass.-London, 1979.

[12] H. Ye, J. Gao, Y. Ding, *A generalized Gronwall inequality and its application to a fractional differential equation*, J. Math. Anal. Appl. **328** (2007), 1075-1081.

[13] Z. Li, Y. Liu, M. Yamamoto, *Initial-boundary value problems for multi-term time-fractional diffusion equations with positive constant coefficients*, Appl. Math. Comput. **257** (2015), 381-397.

[14] Y. Liu, W. Rundell, M. Yamamoto, *Strong maximum principle for fractional diffusion equations and an application to an inverse source problem*, Fract. Calc. Appl. Anal. **19** (2016), 888-906.

[15] R. Zacher, *Quasilinear parabolic problems with nonlinear boundary conditions*, Ph.D thesis, Martin-Luther-Universität Halle-Wittenberg, 2003. Available from: https://www.yumpu.com/en/document/view/4926858/quasilinear-parabolicproblems-with-nonlinear-boundary-conditions

[16] R. Zacher, *Maximal regularity of type Lp for abstract parabolic Volterra equations*, J. Evol. Equ. **5** (2005), 79-103.

[17] R. Zacher, *Weak solutions of abstract evolutionary integro-differential equations in Hilbert spaces*, Funkcial. Ekvac. **52** (2009), 1-18.