Abstract. Let \((M,g)\) be a Riemannian manifold, \(L(M)\) its frame bundle. We construct new examples of Riemannian metrics, which are obtained from Riemannian metrics on the tangent bundle \(T M\). We compute the Levi–Civita connection and curvatures of these metrics.

1. Introduction

Let \((M,g)\) be a Riemannian manifold, \(L(M)\) its frame bundle. The first example of a Riemannian metric on \(L(M)\) was considered by Mok [12]. This metric, called the Sasaki–Mok metric or the diagonal lift \(g^d\) of \(g\), was also investigated in [5] and [6]. It is very rigid, for example, \((L(M), g^d)\) is never locally symmetric unless \((M,g)\) is locally Euclidean. Moreover, with respect to the Sasaki–Mok metric vertical and horizontal distributions are orthogonal. A wider and less rigid class of metrics \(\bar{g}\), in which vertical and horizontal distributions are no longer orthogonal, has been recently considered by Kowalski and Sekizawa in the series of papers [9, 10, 11]. These metrics are defined with respect to the decomposition of the vertical distribution \(\mathcal{V}\) into \(n = \dim M\) subdistributions \(\mathcal{V}_1, \ldots, \mathcal{V}_n\).

In this short paper we introduce a new class of Riemannian metrics on the frame bundle. We identify distributions \(\mathcal{V}_i\) with the vertical distribution in the second tangent bundle \(TT M\). Namely, each map \(R_i : L(M) \to TM\), \(R_i(u_1, \ldots, u_n) = u_i\) induces a linear isomorphism \(R_{i*} : \mathcal{H} \oplus \mathcal{V}_i \to TT M\), where \(\mathcal{H}\) is a horizontal distribution defined by the Levi–Civita connection \(\nabla\) on \(M\). By this identification we pull–back the Riemannian metric from \(TM\). We pull–back natural metrics, in the sense of Kowalski and Sekizawa [8], from \(TM\) and study the geometry of such Riemannian manifolds. We compute the Levi–Civita connection, the curvature tensor, sectional and scalar curvature.
2. Riemannian metrics on frame bundles

Let $(M, g)$ be a Riemannian manifold. Its frame bundle $L(M)$ consists of pairs $(x, u)$ where $x = \pi_{L(M)}(u) \in M$ and $u = (u_1, \ldots, u_n)$ is a basis of a tangent space $T_x M$. We will write $u$ instead of $(x, u)$. Let $(x_1, \ldots, x_n)$ be a local coordinate system on $M$. Then, for every $i = 1, \ldots, n$ we have

$$u_i = \sum_j u^j_i \frac{\partial}{\partial x_j}$$

for some smooth functions $u^j_i$ on $L(M)$. Putting $\alpha_i = x_i \circ \pi_{L(M)}$, $(\alpha_i, u^j_i)$ is a local coordinate system on $L(M)$. Let $\omega$ be a connection form of $L(M)$ corresponding to Levi–Civita connection $\nabla$ on $M$. We have a decomposition of the tangent bundle $TL(M)$ into the horizontal and vertical distribution:

$$T_u L(M) = \mathcal{H}_u^{L(M)} \oplus \mathcal{V}_u^{L(M)},$$

where $\mathcal{H}^{L(M)} = \ker \omega$ and $\mathcal{V}^{L(M)} = \ker \pi_{L(M)*}$. Let $X^h$ denotes the horizontal lift of a vector field $X$ on $M$.

Decompose the second tangent bundle $TTM$ into horizontal and vertical part, $T_* TM = \mathcal{H}^{TM}_* \oplus \mathcal{V}^{TM}_*$, with respect to the connection map $K : TTM \to TM$ and the projection in the tangent bundle $\pi_{TM} : TM \to M$, see for example [2]. Let $X^{h,TM}$ and $X^{v,TM}$ denote the horizontal and vertical lifts to $TTM$ of a vector field $X$ on $M$.

For an index $i = 1, \ldots, n$ define a map $R_i : L(M) \to TM$ as follows

$$R_i(u) = u_i, \quad u = (u_1, \ldots, u_n) \in L(M).$$

Proposition 2.1. The operator $R_i$ has the following properties.

1. We have

$$R_i^* X^h = X^{h,TM}.$$  

In particular, $R_\ast^*$ is an isomorphism of $\mathcal{H}^{L(M)}$ and $\mathcal{H}^{TM}$.

2. Let $\mathcal{V}^i$ be a linear subspace of $\mathcal{V}^{L(M)}$ spanned by fundamental vertical vectors $A^\ast$, where the matrix $A \in \mathfrak{gl}(n)$ has only nonzero $i$–th column. Then $R_\ast^*$ is an isomorphism of $\mathcal{V}^i$ and $\mathcal{V}^{TM}$, and is zero on $\mathcal{V}^j$ for $j \neq i$. Moreover, there is a decomposition

$$\mathcal{V}^{L(M)} = \mathcal{V}^1 \oplus \ldots \oplus \mathcal{V}^m.$$  

Proof. Easy computations left to the reader. □

By Proposition 2.1 we have natural identifications

$$\mathcal{H}^{L(M)} \leftrightarrow \mathcal{H}^{TM} \leftrightarrow TM \quad X^h \leftrightarrow X^{h,TM} \leftrightarrow X$$
and
\begin{equation}
(2.2) \quad V^i \leftrightarrow V^{TM} \leftrightarrow TM \leftrightarrow X^v,i \leftrightarrow X^{v,TM} \leftrightarrow X
\end{equation}

Hence, we have defined the vertical lift $X^{v,i} \in V^i$ of the vector $X \in TM$ satisfying the property

$$R_{v*} X^{v,i} = X^{v,TM}.$$ 

Let $c = (c_1, \ldots, c_n) \in \mathbb{R}^n$ and $C = (c_{ij})$ be an $n \times n$ matrix. We assume that the $(n + 1) \times (n + 1)$ matrix $ar{C} = \begin{pmatrix} 1 & c \\ c^T & C \end{pmatrix}$ is symmetric and positive definite. Let $g_{TM}$ be a Riemannian metric on $TM$.

Now, we are able to define a new class of Riemannian metrics $\bar{g} = \bar{g}_{c, C}$ on $L(M)$. Let $F : L(M) \to TM$ be any smooth function. Put

$$\bar{g}(X^h, Y^h)_{u} = g_{TM}(X^{h,TM}, Y^{h,TM})_{F(u)},$$

$$\bar{g}(X^h, Y^{v,i})_{u} = c_i g_{TM}(X^{h,TM}, Y^{v,TM})_{F(u)},$$

$$\bar{g}(X^{v,i}, Y^{v,j})_{u} = c_{ij} g_{TM}(X^{v,TM}, Y^{v,TM})_{F(u)}.$$ 

Fix $u \in L(M)$. Let $e_1, \ldots, e_n$ be a basis in $T_x M$, $\pi(u) = x$, such that $(e_1)^{h,TM}_F, \ldots, (e_n)^{h,TM}_F$ is an orthonormal basis in $\mathcal{H}^{TM}_{F(u)}$. Then

\begin{equation}
(2.3) \quad e_1^h, \ldots, e_n^h, e_1^{v,1}, \ldots, e_n^{v,1}, e_1^{v,n}, \ldots, e_n^{v,n}
\end{equation}

is a basis in $T_u L(M)$. Let $G$ be a matrix of the Riemannian metric $g_{TM}$ with respect to the basis $e_1^{h,TM}, \ldots, e_n^{h,TM}, e_1^{v,TM}, \ldots, e_n^{v,TM}$. The fact that $\bar{g}$ is positive definite follows from the following lemma.

**Lemma 2.2.** Let

$$G = \begin{pmatrix} I & g^{hv} \\ g^{vh} & \hat{g} \end{pmatrix}$$

be a positive definite symmetric $2n \times 2n$ block matrix. Then the matrix

$$\bar{G} = \begin{pmatrix} I & c \otimes g^{vh} \\ c^T \otimes g^{hv} & C \otimes \hat{g} \end{pmatrix}$$

is positive definite.

**Proof.** It suffices to show that each principal minor $\bar{G}_k$, $k = 1, \ldots, n+n^2$, of $\bar{G}$ is positive. Obviously $\det\bar{G}_k = 1 > 0$ for $k = 1, \ldots, n$. Hence we assume $k > n$. Then each minor $\bar{G}_k$ is of the same form as the whole matrix $\bar{G}$, thus
we will make calculations using matrix $\tilde{G}$. Computing the determinant of the block matrix we get

$$
\det \tilde{G} = \det(C \otimes \hat{g} - (c^T \otimes g^{vh})(c \otimes g^{hv}))
$$

$$
= \det(C \otimes \hat{g} - (c^T c) \otimes (g^{vh} g^{hv}))
$$

$$
= \det((C - c^T c) \otimes \hat{g} + (c^T c) \otimes (\hat{g} - g^{vh} g^{hv})).
$$

Since

$$
\det(C - c^T c) = \det \bar{C} > 0,
$$

$$
\det \hat{g} > 0,
$$

$$
\det(c^T c) \geq 0,
$$

$$
\det(\hat{g} - g^{vh} g^{hv}) = \det G > 0,
$$

it follows that matrices $(C - c^T c) \otimes \hat{g}$ and $(c^T c) \otimes (\hat{g} - g^{vh} g^{hv})$ are positive definite. Hence theirs sum is positive definite. □

If $\bar{C} = I$ and $g_{TM}$ is the Sasaki metric, then we get Sasaki–Mok metric $\bar{g}^d$.

Assume now $\bar{C} = I$ and $g_{TM}$ is a natural Riemannian metric on $TM$ such that $g_{TM}(X^h, Y^h) = g(X, Y)$ and distributions $\mathcal{H}^{TM}, \mathcal{V}^{TM}$ are orthogonal. Hence, there are two smooth functions $\alpha, \beta : [0, \infty) \to \mathbb{R}$ such that

$$
\bar{g}(X^h, Y^h)_u = g(X, Y),
$$

$$
\bar{g}(X^h, Y^{v,i})_u = 0,
$$

$$
\bar{g}(X^{v,i}, Y^{v,j})_u = 0, \quad i \neq j,
$$

$$
\bar{g}(X^{v,i}, Y^{v,i})_u = \alpha(|u_i|^2)g(X, Y)
$$

$$
+ \beta(|u_i|^2)g(X, u_i)g(Y, u_i).
$$

(2.4)

The above Riemannian metric does not ”see” the index $i$ of the distribution $\mathcal{V}^i$. Since all distributions $\mathcal{H}^{L(M)}, \mathcal{V}^1, \ldots, \mathcal{V}^n$ are orthogonal, it follows that we may put $F_i(u) = u_i$, that is consider a family of maps $F_1, \ldots, F_n$ rather than one map $F$, in the last condition, to obtain the positive definite bilinear form, hence the Riemannian metric,

$$
\bar{g}(X^h, Y^h)_u = g(X, Y),
$$

$$
\bar{g}(X^h, Y^{v,i})_u = 0,
$$

$$
\bar{g}(X^{v,i}, Y^{v,j})_u = 0, \quad i \neq j,
$$

$$
\bar{g}(X^{v,i}, Y^{v,i})_u = \alpha(|u_i|^2)g(X, Y) + \beta(|u_i|^2)g(X, u_i)g(Y, u_i).
$$

(2.5)

We will write $\alpha_i$ and $\beta_i$ instead of $\alpha(|u_i|^2)$ and $\beta(|u_i|^2)$, respectively.
3. Geometry of $\bar{g}$

Let $(M, g)$ be a Riemannian manifold, $(L(M), \bar{g})$ its frame bundle equipped with the metric $\bar{g}$ of the form (2.5). Let $\nabla$ and $R$ denote the Levi-Civita connection and the curvature tensor of $\bar{g}$, respectively.

We recall the identities concerning Lie bracket of horizontal and vertical vector fields [9]

\[ [X^h, Y^h]_u = [X, Y]_u - \sum_i (R(X, Y)u_i)^{v,i}, \]

(3.1) \[ [X^h, Y^{v,i}]_u = (\nabla_X Y)^{v,i}_u, \]

\[ [X^{v,i}, Y^{v,j}]_u = 0. \]

Moreover, in the local coordinates, for $X = \sum_i \xi_i \partial_{x_i}$ we have

\[ X^h(u_i) = -\sum_{a,b} \Gamma^j_{ab} u^a_i \xi_b \]

(3.2) \[ X^{v,k}(u_i) = \xi_j \delta_{ik} \]

(3.3) where $\Gamma^j_{ab}$ are Christoffel’s symbols [9].

**Proposition 3.1.** Connection $\nabla$ satisfies the following relations

\[
\left( \nabla_{X^h} Y^h \right)_u = \left( \nabla_X Y \right)_u^h - \frac{1}{2} \sum_i (R(X, Y)u_i)^{v,i},
\]

\[
\left( \nabla_{X^h} Y^{v,i} \right)_u = \frac{\alpha_i}{2} (R(u_i, Y)X)_u^h + \left( \nabla_X Y \right)^{v,i}_u,
\]

\[
\left( \nabla_{X^{v,i}} Y^h \right)_u = \frac{\alpha_i}{2} (R(u_i, X)Y)_u^h,
\]

\[
\left( \nabla_{X^{v,i}} Y^{v,j} \right)_u = 0 \text{ for } i \neq j,
\]

\[
\left( \nabla_{X^{v,i}} Y^{v,j} \right)_u = \frac{\alpha_j^l}{\alpha_i} \left( g(X, u_i)Y^{v,j} + g(Y, u_i)X^{v,j} \right) + \frac{\beta_i^j - \alpha_i^l}{\alpha_i + \alpha_i^l} g(X, u_i)g(Y, u_i) \frac{\beta_i}{\alpha_i + \alpha_i^l} g(X, Y) U^i,
\]

where $U^i_u = u_i^{v,i}.

**Proof.** Follows from the formula for the Levi-Civita connection

\[ 2\bar{g}(\nabla_A B, C) = A\bar{g}(B, C) + B\bar{g}(A, C) - C\bar{g}(A, B) \]

\[ + \bar{g}([A, C], B) + \bar{g}([B, C], A) + \bar{g}([A, B], C) \]

relations (3.1) and the following equalities

\[ X_u^{v,i}(g(u_i, Y)) = g(X, Y), \]

\[ X_u^{v,i}(|u_i|^2) = 2(X, u_i), \]

\[ X_u^h(g(u_i, Y)) = g(u_i, \nabla_X Y). \]
Before we compute the curvature tensor, we will need some formulas concerning the Levi–Civita connection $\nabla$ of certain vector fields.

**Lemma 3.2.** The following equalities hold

\[
\nabla_{X^i} U^j = 0, \\
\nabla_{X^j} U^i = 0, \\
\nabla_{X^j} U^i = \frac{\alpha_i}{\alpha_i} |u_i|^2 \alpha'_i X^{v,i} + \frac{|u_i|^2 (\alpha_i \beta'_i - \alpha'_i \beta_i) + \alpha_i \beta_i}{\alpha_i (\alpha_i + |u_i|^2 \beta_i)} g(X, u_i) U^j.
\]

and

\[
\nabla_W (R(u_i, X)Y)^Q = \sum_j W(u_j^i) (R(u_i, X)Y)^Q + \sum_j u_j^i \nabla_W (R(\frac{\partial}{\partial x_j}, X)Y)^Q
\]

for any $W \in TL(M)$ and $Q$ denoting the horizontal or vertical lift.

**Proof.** Follows by standard computations in local coordinates. \hfill \Box

**Proposition 3.3.** The curvature tensor $\bar{R}$ satisfies the following relations

\[
\bar{R}(X^i, Y^j)Z^h = (R(X, Y)Z)^h + \frac{1}{2} \sum_i (\nabla_Z R)(X, Y)u_i)_{v,i} \\
- \frac{1}{4} \sum_i \alpha_i (R(u_i, R(Y, Z)u_i)X - R(u_i, R(X, Z)u_i)Y \\
- 2R(u_i, R(X, Y)u_i)Z^h, \\
\bar{R}(X^i, Y^j)Z^{v,i} = (R(X, Y)Z)^{v,i} + \frac{\alpha_i}{2} ((\nabla_X R)(u_i, Z)Y - (\nabla_Y R)(u_i, Z)X)^h \\
- \frac{\alpha_i}{4} \sum_j (R(X, R(u_i, Z)Y)u_j - R(Y, R(u_i, Z)X)u_j)^{v,j} \\
+ \frac{\alpha'_i g(Z, u_i)(R(X, Y)u_i)^{v,i} - \beta_i - \alpha'_i}{\alpha_i + |u_i|^2 \beta_i} g(R(X, Y)Z, u_i) U^i, \\
\bar{R}(X^i, Y^{v,i})Z^h = \frac{\alpha_i}{2} ((\nabla_X R)(u_i, Y)Z)^h - \frac{1}{2} (R(Z, X)Y)^{v,i} \\
+ \frac{\alpha'_i}{2 \alpha_i} g(Y, u_i)(R(X, Z)u_i)^{v,i} - \frac{\alpha_i}{4} \sum_j (R(X, R(u_i, Y)Z)u_j)^{v,j} \\
- \frac{\beta_i - \alpha'_i}{2 (\alpha_i + |u_i|^2 \beta_i)} g(R(X, Z)Y, u_i) U^i, \\
\bar{R}(X^i, Y^{v,j})Z^{v,j} = - \frac{\alpha_i \alpha_j}{4} (R(u_i, Y)R(u_j, Z)X)^h \\
\bar{R}(X^i, Y^{v,i})Z^{v,i} = \frac{\alpha'_i}{2} (g(Z, u_i) R(u_i, Y)X - g(Y, u_i) R(u_i, Z)X)^h \\
- \frac{\alpha_i^2}{4} (R(u_i, Y)R(u_i, Z)X)^h - \frac{\alpha_i}{2} (R(Y, Z)X)^h
\]


\[ \mathcal{R}(X^{v,i}, Y^{v,j}) Z^h = \alpha_i (R(X, Y) Z)^h \]
\[ + \frac{\alpha_i^2}{4} (R(u_i, X) R(u_i, Y) Z - R(u_i, Y) R(u_i, X) Z)^h \]
\[ + \alpha_i' g(X, u_i)(R(u_i, Y) Z)^h - g(Y, u_i)(R(u_i, X) Z)^h \]
\[ \mathcal{R}(X^{v,i}, Y^{v,j}) Z^h = \frac{\alpha_i \alpha_j}{4} (R(u_i, X) R(u_j, Y) Z - R(u_j, Y) R(u_i, X) Z)^h \]
\[ \mathcal{R}(X^{v,i}, Y^{v,j}) Z^{v,i} = C_i (g(X, u_i) g(Y, Z) - g(Y, u_i) g(X, Z)) U^i \]
\[ + (A_i g(Y, u_i) g(Z, u_i) + B_i g(Y, Z)) X^{v,i} \]
\[ - (A_i g(X, u_i) g(Z, u_i) + B_i g(X, Z)) Y^{v,i} \]
\[ \mathcal{R}(X^{v,i}, Y^{v,j}) Z^{v,k} = 0 \quad \text{if } i \neq j, j \]

where
\[ A_i = \frac{3(\alpha'_i)^2 - 2 \alpha_i \alpha''_i}{\alpha_i^2} + \frac{\alpha_i \beta'_i - 2 \alpha'_i \beta_i (\alpha_i + |u_i|^2 \alpha_i')}{\alpha_i^2 (\alpha_i + |u_i|^2 \beta_i)} \]
\[ B_i = \frac{\alpha_i \beta_i - 2 \alpha_i \alpha'_i - (\alpha'_i)^2 |u_i|^2}{\alpha_i (\alpha_i + |u_i|^2 \beta_i)} \]
\[ C_i = -\frac{2 \alpha''_i}{\alpha_i + |u_i|^2 \beta_i} + 3 \alpha_i (\alpha'_i)^2 + 2 (\alpha'_i)^2 \beta_i |u_i|^2 + \alpha_i^2 \beta'_i - \alpha_i \beta_i^2 + \alpha_i \alpha'_i \beta'_i |u_i|^2}{\alpha_i (\alpha_i + |u_i|^2 \beta_i)^2} \]

Proof. Follows from the characterization of the Levi–Civita connection \nand Lemma [3,2] \hfill \Box

Remark 3.4. Notice that
\[ A_i \alpha_i - B \beta_i = C_i (\alpha_i + |u_i|^2 \beta_i), \]
which is equivalent to the condition
\[ \bar{g}(\mathcal{R}(X^{v,i}, Y^{v,i}) Z^{v,i}, W^{v,i}) = \bar{g}(\mathcal{R}(Z^{v,i}, W^{v,i}) X^{v,i}, Y^{v,i}). \]

Corollary 3.5. Let \( X, Y \) be two orthonormal vectors in the tangent space \( T_x M \). Then the scalar curvature \( \mathcal{K} \) of \( (L(M), \bar{g}) \) and \( K \) of \( (M, g) \) are related as follows
\[ \mathcal{K}(X^h, Y^h) = K(X, Y) - \frac{3}{4} \sum_i \alpha_i |R(X, Y) u_i|^2, \]
\[ \mathcal{K}(X^h, Y^{v,i}) = \frac{\alpha_i^2}{4(\alpha_i + \beta_i g(Y, u_i)^2)} |R(u_i, Y) X|^2, \]
\[ \mathcal{K}(X^{v,i}, Y^{v,i}) = \frac{A_i (g(X, u_i)^2 + g(Y, u_i)^2) + B_i}{\alpha_i + \beta_i (g(X, u_i)^2 + g(Y, u_i)^2)} \]
\[ \mathcal{K}(X^{v,i}, Y^{v,j}) = 0 \quad \text{for } i \neq j. \]
In particular, if \((M, g)\) is of constant sectional curvature \(\kappa\), then
\[
\overline{K}(X^h, Y^h) = \kappa - \frac{3}{4} \kappa^2 \sum_i \alpha_i (g(X, u_i)^2 + g(Y, u_i)^2),
\]
\[
\overline{K}(X^h, Y^{v,i}) = \frac{\kappa^2 \alpha_i^2 g(X, u_i)^2}{4(\alpha_i + \beta_i g(Y, u_i))} \geq 0.
\]
If, moreover, \(\sum_i \alpha_i(t_i) t_i < \frac{4}{3\kappa}\) for all \(t_i > 0\), then \(\overline{K}(X^h, Y^h) > 0\).

**Proof.** The formula for \(\overline{K}\) follows by Proposition 3.3. Since \(g(X, u_i)^2 + g(Y, u_i)^2 \leq |u_i|^2\), hence, if \((M, g)\) is of constant sectional curvature and \(\sum_i \alpha_i(t_i) t_i < \frac{4}{3\kappa}\), then
\[
\overline{K}(X^h, Y^h) \geq \kappa - \frac{3}{4} \kappa^2 \sum_i \alpha_i |u_i|^2 > 0.
\]

□

**Corollary 3.6.** The scalar curvature \(\overline{s}\) of \((L(M), \bar{g})\) at \(u \in L(M)\) is of the form
\[
\overline{s} = s - \frac{1}{4} \sum_{i,j,k} \alpha_k |R(e_i, e_j) u_k|^2
+ \sum_k \left( \frac{n(n-1)B_k}{\alpha_k} + \frac{2(nA_k \alpha_k - B_k \beta_k)}{\alpha_k^2} |u_k|^2 + \frac{(n+3)C_k \beta_k}{\alpha_k^2} |u_k|^4
+ \frac{(n-1)\beta_k (B_k(2\alpha_k + \beta_k) + A_k \alpha_k)}{\alpha_k (\alpha_k + |u_k|^2 \beta_k)} + \frac{2C_k \beta_k^2}{\alpha_k^2 (\alpha_k + |u_k|^2 \beta_k)} |u_k|^6 \right),
\]
where \(s\) is the scalar curvature of \((M, g)\) and \(e_1, \ldots, e_n\) is an orthonormal basis in \(T_xM\), \(\pi_{L(M)}(u) = x\).

**Proof.** Fix \(u \in L(M)\) and let \(e_1, \ldots, e_n\) be an orthonormal basis in \(T_xM\), \(\pi_{L(M)}(u) = x\). Then (2.3) forms a basis of \(T_uL(M)\). Put
\[
\bar{g}_i^j = \bar{g}(e_i^{v,k}, e_j^{v,k}) = \alpha_k \delta_{ij} + \beta_k g(e_i, u_k) g(e_j, u_k).
\]
The inverse matrix \((\bar{g}_i^j)\) to \((\bar{g}_i^j)\) is of the form
\[
\bar{g}_i^j \approx \frac{1}{\alpha_k} \delta_{ij} - \frac{\beta_k}{\alpha_k (\alpha_k + |u_k|^2 \beta_k)} g(e_i, u_k) g(e_j, u_k).
\]
Hence
\[
\overline{s} = \sum_{i,j} \bar{g}(\bar{R}(e_i^h, e_j^h) e_j^h, e_i^h) + 2 \sum_{i,j,k} \bar{g}_i^j \bar{g}(\bar{R}(e_i^h, e_j^{v,k}) e_i^{v,k}, e_j^h)
+ \sum_{i,j,k,l,p} \bar{g}_i^j \bar{g}_k^l \bar{g}(\bar{R}(e_i^{v,k}, e_j^{v,k}) e_i^{v,k}, e_j^{v,k})
\]
Follows now from Proposition 3.3 and the equality
\[
\sum_{i,j} |R(e_i, e_j)u_k|^2 = \sum_{i,j} |R(u_k, e_j)e_i)|^2.
\]
□

In the case of a Cheeger–Gromoll type metric we have:

**Corollary 3.7.** Assume

\[\alpha_i(t) = \beta_i(t) = \frac{1}{1+t}, \quad t > 0.\]

Then

\[
K(X^{v,i}, Y^{v,i}) = \frac{-t_i(g(X, u_i)^2 + g(Y, u_i)^2) + t_i^2 + 3t_i + 3}{(1+t_i)^2(1 + g(X, u_i)^2 + g(Y, u_i)^2)},
\]

where \(t_i = |u_i|^2\). In particular, if \((M, g)\) is of constant sectional curvature \(0 < \kappa < \frac{4}{3n}\), then sectional curvature \(K\) is nonnegative.

**Proof.** We have

\[
\sum_i \alpha_i(t_i)t_i = \sum_i \frac{t_i}{1 + t_i} < \frac{4}{3\kappa}
\]

for all \(t_i > 0\) if and only if \(0 < \kappa < \frac{4}{3n}\). Hence, by Corollary 3.5, \(\overline{K}(X^h, Y^h) \geq 0\) for \(X, Y \in T^*_xM\) unit and orthogonal. Moreover, \(g(X, u_i)^2 + (Y, u_i)^2 \leq |u_i|^2 = t_i\).

Thus

\[
\overline{K}(X^{v,i}, Y^{v,i}) \geq \frac{-t_i^2 + t_i^2 + 3t_i + 3}{t_i(1 + t_i)^2} = \frac{3}{t_i(t_i + 1)} > 0.
\]

□

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