PATH INTEGRAL FORMULATION OF THE CONFORMAL
WESS-ZUMINO-WITTEN → TODA REDUCTIONS

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Abstract

The phase space path integral Wess-Zumino-Witten → Toda reductions are formulated in a manifestly conformally invariant way. For this purpose, the method of Batalin, Fradkin, and Vilkovisky, adapted to conformal field theories, with chiral constraints, on compact two dimensional Riemannian manifolds, is used. It is shown that the zero modes of the Lagrange multipliers produce the Toda potential and the gradients produce the WZW anomaly. This anomaly is crucial for proving the Fradkin-Vilkovisky theorem concerning gauge invariance.

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I. INTRODUCTION AND THE STATEMENT OF THE PROBLEM

It is well known that classical Wess-Zumino-Witten (WZW) models based on simple finite dimensional groups can be reduced to Toda theories by imposing linear first class constraints on the WZW currents [1]. The quantised version of the reduction process was also considered earlier, but mainly within the framework of canonical quantisation [2]. The elegance of the classical reduction process suggests, however, that the most natural framework for quantisation is through the path integral. In a recent paper [3], we presented the path integral formulation of the simplest of such reductions, namely, the reduction of the $sl(2, \mathbb{R})$ WZW model to the Liouville theory. In that paper we stressed the importance of the zero modes in proving gauge invariance and in producing the WZW anomaly and the Liouville potential. The zero modes occur because we must work on a compact manifold since otherwise it is not possible to choose configurations for which the kinetic term and the exponential potential are both finite. In the present paper we present a generalisation of these results to the reduction of WZW theories to Toda theories.

The generalisation from the Liouville to the Toda case is not trivial for a number of reasons. First, the usual off-diagonal parameters in the Gauss decomposition for the group valued fields are not the natural fields from the path integral point of view because the Lagrangian density is not quadratic in these fields. In addition, with respect to the ‘improved’ Virasoro generators which are necessary to implement the constraints, these fields are not primary. However, they can be converted into primary fields for which the Lagrangian density is bilinear and remains local. The new fields have the further property that the first class constraints can be expressed as linear conditions on their conjugate momenta. Second, the separation of the Lagrange multipliers into their zero modes and their orthogonal complements requires the introduction of auxiliary fields. The auxiliary fields are only needed to define orthogonality in a conformally invariant
manner and can be eliminated once the zero modes are separated. Third, it turns out that the zero modes occur only at grade one. Finally, unlike the case in the \( sl(2, R) \)
WZW \( \rightarrow \) Liouville reduction, there are ghost fields of non-zero conformal weights and these give a non-trivial contribution to the Polyakov term and hence to the reduced Virasoro centre.

Another interesting feature is that the operations of reducing and quantising do not commute in the sense that they lead to Toda theories with different coupling constants depending on the order in which the operations are performed. The coupling constants are \( k \) and \( k - \gamma \) respectively, where \( k \) is the WZW coupling constant and \( \gamma \) is the dual Coxeter number of the group. As might be expected, the shift in the coupling constant originates in the WZW anomaly.

As in the WZW \( \rightarrow \) Liouville reduction, we use the Batalin-Fradkin-Vilkovisky (BFV) generalisation of the Faddeev-Popov formalism [4]. This is because the BFV formalism allows us to use the WZW gauge in which the Lagrange multipliers are zero (the analogue of the temporal gauge in QED). Since the manifold is compact, and the constraints are chiral, we use the modification of the standard BFV formalism which was introduced in [3]. As in the Liouville case, the integration over the zero modes produces the exponential potential and the gauge variant gradient parts produce the WZW anomaly. The anomaly is crucial for proving the Fradkin-Vilkovisky theorem regarding the gauge independence of the reduction.

The paper is organised as follows. In section II, we briefly recall the basics of simple Lie groups and define the principal \( sl(2, R) \) embedding. In section III, we review the classical WZW \( \rightarrow \) Toda reductions, based on simple Lie groups with principal \( sl(2, R) \) embeddings, to Toda theories. In section IV, we summarise the basics of the BFV formalism. Section V contains the main results of this paper and establishes the quantum WZW \( \rightarrow \) Toda reductions in a gauge independent manner, by using a
modification of the BFV path integral that takes into account the chiral nature of the constraints and the compactness of the manifold. In section VI, we examine the results of the previous section in two special gauges which highlight the general results. In section VII we present our conclusions.

II. SIMPLE LIE GROUPS AND THE PRINCIPAL sl(2, R) EMBEDDING

In this section we briefly recall the necessary properties of simple finite dimensional Lie groups \( G \) [5]. The standard Cartan-Weyl basis for the Lie algebra \( \mathcal{G} \) is

\[
[H_i, H_j] = 0, \quad [H_i, E_\alpha] = \alpha_i E_\alpha, \quad 1 \leq i, j \leq l \tag{2.1a}
\]

\[
[E_\alpha, E_\beta] = N_{\alpha\beta} E_{\alpha+\beta}, \quad \text{if} \quad \alpha + \beta \in \Delta
\]

\[
= \frac{2\alpha \cdot H}{\alpha^2} \delta_{\alpha, -\beta} \tag{2.1b}
\]

where the \( \alpha_i \) are components of root vectors, \( \Delta \) is the lattice of root vectors, and the \( N_{\alpha\beta} \) are constants. In the orthonormal basis for the Cartan subalgebra,

\[
Tr(H_i H_j) = \delta_{ij}, \quad Tr(E_\alpha, E_\beta) = \frac{2}{\alpha^2} \delta_{\alpha, -\beta} \tag{2.2}
\]

The second equation in (2.2), which is actually a consequence of the first one, and (2.1), fix the normalisation of \( E_\alpha \). The Lie algebra \( \mathcal{G} \) is said to be simply-laced if all the roots \( \alpha \) have the same squared length \( \alpha^2 = \alpha_i \alpha_i \) (as is the case for the \( A, D \) and \( E \) series) and non-simply laced otherwise. The real span of the Cartan-Weyl basis yields a maximally non-compact real form \( \mathcal{G}_R \) of the Lie algebra \( \mathcal{G} \). These are the Lie algebras in which we are interested. For the classical Lie algebras \( A_n, B_n, C_n \) and \( D_n \), these forms are given by the real Lie algebras \( \mathfrak{sl}(n, R), \mathfrak{so}(p + 1, p, R), \mathfrak{sp}(2n, R), \) and \( \mathfrak{so}(p, p, R) \).

Since the number of root pairs \( \pm \alpha \), in general, exceeds the rank \( l \), it is convenient to choose a set of roots \( \alpha^{(s)}, 1 \leq s \leq l \), which constitute a basis for the \( l \)-dimensional space of roots. This basis can be chosen in such a way that an arbitrary root \( \alpha \) can
always be expressed as
\[ \alpha = \sum_{s=1}^{l} n_s \alpha^{(s)} \]  
where each \( n_s \in \mathbb{Z} \) and either \( n_s \geq 0 \) or \( n_s \leq 0 \). The two cases correspond to \( \alpha \) being a positive or a negative root respectively. The \( \alpha^{(s)} \) are said to constitute a basis of simple roots which we will denote be \( \Delta_s \), and the subspaces of positive and negative roots have the obvious notation \( \Delta^+ \) and \( \Delta^- \). For the simple roots
\[ [E_{\alpha^{(s)}}, E_{-\alpha^{(r)}}] = \frac{2\alpha^{(s)} \cdot H}{|\alpha^{(s)}|^2} \delta_{rs} \]
The principal \( sl(2, R) \) embedding in \( \mathcal{G}_R \) is defined by choosing an element \( M_0 \) in the Cartan subalgebra, with respect to which all the simple roots have grade one. This element is given uniquely by
\[ M_0 = \rho \cdot H, \quad \text{where} \quad \rho = \sum_{s=1}^{l} \mu^{(s)} = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha \]
where \( \mu^{(s)} \) are the fundamental weights. The other components of the principal \( sl(2, R) \) embedding in \( \mathcal{G}_R \) are got by taking suitable linear combinations \( M_\pm \) of \( E_{\pm\alpha^{(s)}} \).
\[ M_+ = \sum_s p_s E_{\alpha^{(s)}} \quad \text{and} \quad M_- = \sum_s q_s E_{-\alpha^{(s)}} \]
where the sum extends only over the simple roots because the \( M_\pm \) are step operators with grade \( \pm 1 \) with respect to \( M_0 \). The requirement that \( \{M_0, M_\pm\} \) constitute an \( sl(2, R) \) subalgebra constrains the combination \( p_s q_s \) to be of the form
\[ p_s q_s = \alpha^{(s)} (K^{-1})_{sr} \rho^{(r)} \quad \text{where} \quad K_{rs} = 2\frac{\alpha^{(r)} \cdot \alpha^{(s)}}{|\alpha^{(s)}|^2} \]
\( K \) being the Cartan matrix. In conjunction with (2.3) we see that \( M_0 \) defines an integer grading for the entire Lie algebra \( \mathcal{G}_R \). We will now review the basics of the WZW models based on the real forms of simple, non-compact, groups \( G_R \) and and use the principal \( sl(2, R) \) embedding defined above to reduce them classically to Toda theories.
III. THE CLASSICAL WZW → TODA REDUCTIONS

The WZW model based on a group $G_R$ is defined on a two dimensional compact manifold $\partial \Sigma$ by the Action [6]

$$S = k \int_{\partial \Sigma} Tr (g^{-1}dg) \cdot (g^{-1}dg) - \frac{2k}{3} \int_\Sigma Tr (g^{-1}dg) \wedge (g^{-1}dg) \wedge (g^{-1}dg) \quad (3.1)$$

Here $g \in G_R$. The two dimensional manifold is parametrised by the light-cone coordinates $z_r$ and $z_l$ defined by

$$z_r = \frac{z_0 + z_1}{2}, \quad z_l = \frac{z_0 - z_1}{2} \quad (3.2)$$

respectively. The Action is invariant under

$$g \to gu(z_r), \quad g \to v(z_l)g \quad (3.3)$$

where $u(z_r), v(z_l) \in G_R$. The conserved Noether currents which generate the above transformations,

$$J_r = -(\partial_r g)g^{-1}, \quad J_l = g^{-1}(\partial_l g) \quad (3.4)$$

take their values in the Lie algebra $G_R$. As is well-known, the components $J^a_r$ of the currents satisfy the classical version of the Kac-Moody algebra, given in terms of the Poisson brackets of the currents by

$$\{J^a_r(z), J^b_r(z')\} = i f^{ab}_c J^c_r \delta(z - z') + k \delta^{ab} \delta(z - z') \quad (3.5)$$

A similar equation also holds for the components of the left currents.

Since we are interested only in the maximally non-compact real form of the Lie algebra, the group element $g$ admits a unique, local, Gauss decomposition of the form

$$g = ABC \quad (3.6)$$
where

\[ A = e^{\sum \hat{a}^\alpha E_\alpha}, \quad B = e^{\phi H}, \quad C = e^{\sum \hat{a}^{-\alpha} E_{-\alpha}} \quad \text{where} \quad \alpha \in \Delta^+ \quad (3.7) \]

In the above equation, \( A \) and \( C \) are nilpotent subgroups each with dimension \((\dim G - l)/2\), and \( B \) is the maximal abelian subgroup. This property makes the WZW models based on real, non-compact, groups, the natural generalisations of the \( SL(2, R) \) model studied in [3]. The fields \( \hat{a}^\alpha, \hat{a}^{-\alpha}, \) and \( \phi^i \) parametrise the group manifold. As is well-known, the Gauss decomposition is not valid globally. This issue has been dealt with in detail in [7]. For simplicity, we restrict our present considerations to the coordinate patch that contains the identity. Similar results hold for the other patches. The above decomposition is very useful for setting up the Hamiltonian formalism to which we now pass.

The Polyakov-Wiegmann factorisation formula [8] for the WZW model states that

\[ S(XY) = S(X) + S(Y) + \int d^2z \text{Tr}[(X^{-1}\partial_l X)(\partial_r Y)Y^{-1}] \quad (3.8) \]

where \( X \in G_1 \) and \( Y \in G_2 \), \( G_1 \) and \( G_2 \) being arbitrary simple Lie groups. Using this formula recursively, and from the nilpotency of \( A \) and \( C \), we find

\[ S(ABC) = S(B) + \int d^2z \text{Tr}[(A^{-1}\partial_l A)B(\partial_r C)C^{-1}B^{-1}] \quad (3.9) \]

Substituting for \( A, B \) and \( C \) from (3.7) in the above equation and evaluating the traces gives

\[ S = \frac{k}{2} \int_{\partial \Sigma} d^2z \left[ (\partial_l \phi^i)(\partial_r \phi^i) + \frac{4}{\alpha^2} V_\alpha U_{-\alpha} e^{-\alpha \cdot \phi}(\partial_l \hat{a}^\alpha)(\partial_r \hat{a}^{-\alpha'}) \right] \quad (3.10) \]

where \( U \) and \( V \) are defined by the left and right currents of the nilpotent subgroups \( A \) and \( C \) through the relations

\[ A^{-1}dA = V_\alpha (\hat{a}^\alpha) d\hat{a}^\alpha = V_\alpha^\beta (\hat{a}^\alpha) E_\beta d\hat{a}^\alpha \quad (3.11a) \]

\[ (dC)C^{-1} = U_{-\alpha} (\hat{a}^{-\alpha}) d\hat{a}^{-\alpha} = U_{-\alpha}^\beta (\hat{a}^{-\alpha}) E_{-\beta} d\hat{a}^{-\alpha} \quad (3.11b) \]
It is clear that because $U$ and $V$ are functions of $\hat{a}^{-\alpha}$ and $\hat{a}^\alpha$ respectively, the Action (3.10) is not quadratic in these fields. Since we would finally like to integrate out these fields, the off-diagonal parameters in the Gauss decomposition of the group valued fields are not the natural ones to use as fields. However, the functions $U$ and $V$ can be used as kernels to define new fields $a^\alpha$ and $a^{-\alpha}$ expressed in terms of the old fields $\hat{a}^\alpha$ and $\hat{a}^{-\alpha}$ through the integral equations

$$a^\alpha = \hat{a}^\alpha + \int \sum_{\beta < \alpha} V^\alpha_\beta (\hat{a}^\alpha) d\hat{a}^\beta, \quad \text{and} \quad a^{-\alpha} = \hat{a}^{-\alpha} + \int \sum_{\beta < \alpha} U^{-\alpha}_\beta (\hat{a}^{-\alpha}) d\hat{a}^{-\beta} \quad (3.12)$$

All the results will be completely independent of these kernels. In terms of the new fields the Action takes the simple form

$$S = \frac{k}{2} \int_{\partial \Sigma} d^2 z \left[ (\partial_l \phi^i) (\partial_r \phi^i) + \frac{4}{\alpha^2} e^{-\alpha \cdot \phi} (\partial_l a^\alpha) (\partial_r a^{-\alpha}) \right] \quad (3.13)$$

Notice that the Lagrangian density in (3.13) is quadratic in the new fields $a^{\pm \alpha}$. Moreover, the Lagrangian density remains local although the transformation in (3.12) is not. Furthermore, the transformation is idempotent and hence the Jacobian of the transformation is unity. Thus there will be no change in the standard symplectic measure used to define the phase space path integral in section V. The momenta canonically conjugate to $a^\alpha, \phi^i, a^{-\alpha}$ respectively are

$$\pi_\alpha = \frac{2k}{\alpha^2} (\partial_r a^{-\alpha}) e^{-\alpha \cdot \phi}, \quad \pi_{-\alpha} = \frac{2k}{\alpha^2} (\partial_l a^\alpha) e^{-\alpha \cdot \phi}, \quad \pi_i = k \partial_0 \phi^i \quad (3.14)$$

The canonical Hamiltonian density $H$ is

$$H = \frac{1}{2k} \pi_i^2 + \frac{k}{2} (\phi^i)'^2 + \frac{\alpha^2}{2k} \pi_\alpha \pi_{-\alpha} e^\alpha \cdot \phi + \pi_\alpha (a^\alpha)' - \pi_{-\alpha} (a^{-\alpha})' \quad (3.15)$$

The left and right conserved currents are given by

$$J_i = g^{-1} \partial_t g = C^{-1} [e^{-\alpha \cdot \phi} (\partial_t a^\alpha) E_\alpha + (\partial_t \phi^i) H_i + (\partial_t a^{-\alpha}) E_{-\alpha}] C \quad (3.16a)$$
\[ J_r = - (\partial_r g) g^{-1} = - A (\partial_r a^\alpha) E_\alpha + (\partial_r \phi_i) H_i + (\partial_r a^{-\alpha}) BCE_{-\alpha} C^{-1} B^{-1} A^{-1} \quad (3.16b) \]

It is straightforward to check that, in terms of the currents, the Hamiltonian density takes the Sugawara form \textit{viz.}

\[ H = \mathcal{T}_r + \mathcal{T}_l \quad \text{where} \quad \mathcal{T}_r = \frac{1}{2} Tr [J_r^2] \quad \text{and} \quad \mathcal{T}_l = \frac{1}{2} Tr [J_l^2] \quad (3.17) \]

The currents may also be expressed completely in terms of the phase space variables \(a^\alpha, \phi^i, a^{-\alpha}\) and their conjugate momenta using the relations in (3.14). Further, by using canonical Poisson brackets for the phase space variables \textit{viz.}

\[ \{a^\alpha(z), \pi_\beta(z')\} = \{a^{-\alpha}(z), \pi_{-\beta}(z')\} = \delta^\alpha_\beta \delta(z-z'); \quad \{\phi^i(z), \pi_j(z')\} = \delta^i_j \delta(z-z') \quad (3.18) \]

the rest being zero, we can check explicitly that the currents satisfy two independent copies of the standard Kac-Moody algebra (3.5). This is a further proof of the fact that the measure for the phase space path integral in terms of the new fields \(a^{\pm \alpha}\) is the standard symplectic measure.

The constraints we want to impose are

\[ \Phi_\alpha \equiv J^r_\alpha - M_\alpha = 0 \quad \text{where} \quad M_\alpha \neq 0 \quad \text{iff} \quad \alpha \in \Delta_s \quad (3.19a) \]

and

\[ \Phi_{-\alpha} \equiv J^l_{-\alpha} - M_{-\alpha} = 0 \quad \text{where} \quad M_{-\alpha} \neq 0 \quad \text{iff} \quad -\alpha \in \Delta_s \quad (3.19b) \]

where \(M_\alpha\) and \(M_{-\alpha}\) are the components of the step operators of the principal \(sl(2,\mathbb{R})\) embedding,

\[ M_\alpha = Tr (E_{-\alpha} M_+) \quad \text{and} \quad M_{-\alpha} = Tr (E_\alpha M_-) \quad (3.19c) \]

It may be worth mentioning in passing here that, for the purposes of this paper, the above requirement is not very strict and \(M_\alpha\) and \(M_{-\alpha}\) can be allowed to be completely arbitrary, but non-zero, constants. That they are the components of \(M_\pm\) is necessary.
only to ensure that the reduced Kac-Moody algebra is a W-algebra \( i.e. \) has a primary basis.

The virtue of using the new variables \( a^{\pm \alpha} \) is that the constraints (3.19) can be directly expressed in terms of their conjugate momenta.

\[
\Phi_{\alpha} \equiv \pi_{\alpha} - \frac{2k}{\alpha^2} M_{\alpha} = 0 \text{ where } M_{\alpha} \neq 0 \text{ iff } \alpha \in \Delta_s \quad (3.20a)
\]

and

\[
\Phi_{-\alpha} \equiv \pi_{-\alpha} - \frac{2k}{\alpha^2} M_{-\alpha} = 0 \text{ where } M_{-\alpha} \neq 0 \text{ iff } -\alpha \in \Delta_s \quad (3.20b)
\]

This is possible because the constraints (3.19) reduce the relationship between the currents and the momenta \( \pi_{\alpha} \) and \( \pi_{-\alpha} \), which in general is quite complicated, to a simple linear relation. Upon imposing the constraints (3.20) on the classical Hamiltonian density (3.15) of the WZW model, we get, apart from boundary terms,

\[
H = \frac{1}{2k} \pi_i^2 + \frac{k}{2} (\phi^i)'^2 + k \Lambda_{\alpha} e^{\alpha \cdot \phi} \quad (3.21)
\]

where

\[
\Lambda_{\alpha} = \frac{2}{\alpha^2} M_{\alpha} M_{-\alpha} \quad (3.22)
\]

This is easily recognised as the expression for the Hamiltonian density of the classical Toda theory. Although the two sets of constraints (3.19) and (3.20) are completely equivalent physically in the above sense, it is considerably simpler to work in terms of the latter set because, unlike the Poisson bracket of two current components, the Poisson bracket of two momenta is strictly zero. This fact implies that the BRS charge for the reduction, to be defined in the next section, does not have terms which involve higher powers of the ghost fields.

The constraints in (3.20) set the grade one momenta \( \pi_{\pm \alpha(s)} \) equal to non-zero constants. As is well-known, this is not consistent with the conformal invariance, defined
by the two Sugawara Virasoro operators in (3.17), because the momenta, like the corresponding currents $J^r_{-\alpha(s)}$ and $J^l_{\alpha(s)}$ have conformal dimension one. Hence, the Virasoro generators in (3.17) are replaced by the ‘improved’ generators $T_r$ and $T_l$ defined by

$$T_r = T_r - \frac{2}{\alpha^2} \partial_r J_0^r \quad \text{and} \quad T_l = T_l - \frac{2}{\alpha^2} \partial_l J_0^l$$  \hspace{1cm} (3.23)$$

where

$$J_0^r = Tr [M_0 J_r] \quad \text{and} \quad J_0^l = Tr [M_0 J_l]$$  \hspace{1cm} (3.24)$$

As will be seen in section V, this improvement may be implemented by coupling the currents to a fixed background metric in a specific, non-minimal, way. At this stage another advantage of using the new fields $a^{\pm \alpha}$ becomes clear. With respect to the ‘improved’ Virasoros defined above, the new fields $a^{\pm \alpha}$ and hence their conjugate momenta $\pi^{\pm \alpha}$ are primary fields, unlike the old fields which do not have specific conformal transformation properties. With respect to the conformal group generated by the ‘improved’ Virasoros, the currents we wish to constrain, namely, $J^r_{-\alpha}$ and $J^l_{\alpha}$, or equivalently the momenta $\pi^{\pm \alpha}$, have the following conformal weights,

$$\omega(J^r_{-\alpha}) = \omega(\pi_{\alpha}) = (0, 1 - m_\alpha), \quad \omega(J^l_{\alpha}) = \omega(\pi_{-\alpha}) = (1 - m_\alpha, 0)$$  \hspace{1cm} (3.25)$$

where the positive integer $m_\alpha$ is defined by $m_\alpha = \rho \cdot \alpha$. It follows that for simple roots ($\alpha \in \Delta_s$) which have grade $m_\alpha(s)$ equal to one, the above currents and momenta are conformal scalars. The constraints in (3.19, 3.20) are, therefore, compatible with this conformal group.

The currents $J^r_{\alpha}$ and $J^l_{-\alpha}$ now have conformal weights $(0, 1 + m_\alpha)$ and $(1 + m_\alpha, 0)$ respectively. The phase space variables $a^{\alpha}$ and $a^{-\alpha}$ become primary fields of conformal weights $(0, m_\alpha)$ and $(m_\alpha, 0)$ respectively, the field $\alpha \cdot \phi$ becomes a conformal connection, while $e^{\alpha \cdot \phi}$ becomes a primary field of weight $(m_\alpha, m_\alpha)$. It follows that for $\alpha \in \Delta_s$, it has a weight $(1, 1)$ i.e. it has the opposite conformal weight to the volume element $d^2z$. 

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in the two dimensional space. This completes our discussion of the classical aspects of the WZW → Toda reduction. The classical Toda theory (3.21) can be quantised by any standard procedure [2] and its Virasoro centre is given by

\[ c(k) = \frac{l}{6} + \frac{2p^2}{k} \left[ 1 + k \right]^2 \]  

(3.26)

where \( l = \dim H \). This expression for the Virasoro centre, obtained by first reducing and then quantising, may be contrasted with the one obtained by first quantising using the path integral formulation and then reducing. The rest of the paper deals with this issue.

**IV. THE GENERAL PATH INTEGRAL REDUCTION PROCEDURE**

In this section we first give a brief sketch of the BFV formalism. Let \( p \) and \( q \) be any set of canonically conjugate variables, \( H \) the canonical Hamiltonian, and

\[ Z = \int d(pq) \ e^{-\int dx dt \left[ p\dot{q} - H(p, q) \right]} , \]  

(4.1)

the phase space path integral which is to be reduced by a set of first class constraints \( \Phi(q, p) \). Let \( A \) be a set of Lagrange multipliers, \( B \) their canonically conjugate momenta, and \( b, \bar{c} \) and \( c, \bar{b} \) be conjugate ghost pairs. Then define the nilpotent BRST charge \( \Omega \) by

\[ \Omega = \int dx \left[ c\Phi + bB \right] + \cdots \]  

where \( \{ \Omega, \Omega \} = 0 \)  

(4.2)

Here the dots refer to terms which involve higher powers of ghost fields (which actually do not occur in the present case). A minimal choice for the gauge fixing fermion \( \bar{\Psi} \) is given by

\[ \bar{\Psi} = \bar{c}\chi + \bar{b}A \]  

(4.3)

where \( \chi(p, q, A, B) \) is a set of gauge-fixing conditions. The BFV procedure then consists of inserting the reduction factor

\[ F = \int d(AB\bar{b}\bar{c}\bar{c}) e^{-\int dx dt \left[ \bar{b}\dot{\bar{c}} - \{ \Omega, \bar{\Psi} \} \right]} \]  

(4.4)
into the path integral in (4.1). The Fradkin-Vilkovisky theorem states that the reduced path integral is independent of the choice of the gauge fixing fermion $\bar{\Psi}$. There are some exceptions to this theorem, mainly because of the Gribov problem [9]. However, for the example we are considering, the gauge group is nilpotent, and the path integral is shown to be independent of the gauge fixing conditions by explicit calculation. In the definition of the reduction factor above, it is not necessary to include the term $B\dot{A} + \dot{c}\bar{b}$ in the Action because such a term can always be generated by letting $\chi \rightarrow \chi + \bar{c}\dot{A}$. The standard non-zero Poisson brackets for the variables

$$\{q(x), p(x')\} = \{A(x), B(x')\} = \{b(x), \bar{c}(x')\} = \{c(x), \bar{b}(x')\} = \delta(x - x')$$  (4.5)

imply that

$$\{\Omega, \bar{\Psi}\} = (A\Phi + B\chi) - (\bar{b}\bar{b} - \bar{c}[FP]c - \bar{c}[BFV]b)$$  (4.6)

where the FP and BFV terms are defined by

$$\{\Phi(x), \chi(x')\} = [FP]\delta(x - x'), \quad \{B(x), \chi(x')\} = [BFV]\delta(x - x') = -\frac{\partial\chi}{\partial A}\delta(x - x')$$  (4.7)

Substituting for $\{\Omega, \bar{\Psi}\}$ in $F$ and doing the $\bar{b}\bar{b}$ integrations yields

$$F = \int d(AB\bar{c}\bar{c})e^{\int dxdt \left[A\Phi + B\chi + \bar{c}([FP] + [BFV]\partial_t)\right]c}$$  (4.8)

Assuming that $\chi$ is independent of the $B$-fields, as is usually the case, we may also integrate over them to get

$$F = \int d(A\bar{c}\bar{c})\delta(\chi) e^{\int dxdt \left[A\Phi + \bar{c}([FP] + [BFV]\partial_t)\right]c}$$

$$= \int dA\delta(\chi)det([FP] + [BFV]\partial_t) e^{\int dxdt \left[A\Phi\right]}$$  (4.9)

Note that if $[BFV]$ is equal to zero, we recover the standard Faddeev-Popov insertion [10]. On the other hand, as is clear from (4.7), it is the presence of the $[BFV]$ term that allows the gauge fixing function $\chi$ to depend on the Lagrange multipliers. Thus
the BFV path integral allows us to consider gauge fixing conditions that depend on the
Lagrange multipliers on an equal footing with those which depend only on the phase
space variables. In the next section we present a slightly modified BFV path integral
for establishing the gauge independent quantum WZW → Toda reductions.

V. THE QUANTUM WZW → TODA REDUCTIONS

In order to set up the quantum WZW → Toda reductions through the path integral
method, we first note that since the constraints are linear in the momenta, it is natural
to start with the unconstrained WZW phase space path integral, namely,

\[ I(j) = \int d(\phi^i \pi_i a^\pm \alpha \pi_{\pm\alpha}) \ e^{-\int d^2 z \left[ \pi_\alpha \dot{a}^\alpha + \pi_i \dot{\phi}^i + \pi_{-\alpha} \dot{a}^{-\alpha} - H + j \cdot \phi \right]} \]  \hspace{1cm} (5.1)

and to use the BFV formalism for the reduction. The standard symplectic measure
d(\phi^i \pi_i a^\pm \alpha \pi_{\pm\alpha}) used above is the correct phase space measure because an integration
over the momenta with this measure produces the configuration space path integral with
the correct (group invariant) measure \( viz. \ d(e^{-\alpha \cdot \phi a^\pm \alpha \phi^j}) \). Here the external source, \( j \),
is attached only to \( \phi \) since the other variables will be eliminated by the reduction.

We now apply the BFV formalism to WZW → Toda reductions. The application
will differ from the standard BFV formalism in two respects. First, since we are dealing
with independent left handed and right handed constraints, it is convenient to replace
the standard BFV formalism with a light-cone version. The light-cone version of the
BFV formalism is introduced by replacing the space and time directions by the two
branches of the light-cone parametrised by the light-cone coordinates \( z_r \) and \( z_l \), using
a different branch as the time for each of the two constraints. However, since we will
use the Euclidean version of the theory in the path integral, these coordinates actually
get converted into holomorphic and anti-holomorphic coordinates. Thus all the fields
will be functions of the holomorphic and anti-holomorphic variables and functions which
depend only on one variable will be holomorphic or anti-holomorphic functions. Second, for reasons already explained, we must work on a compact manifold and thus we need a curved space generalisation of the BFV formalism. The compactness of the manifold also entails the presence of zero modes for the Lagrange multiplier fields. As will be seen later, the need to define the orthogonality condition between the zero modes and the gauge variant modes in a conformally invariant manner requires us to introduce auxiliary fields. These auxiliary fields are important in picking out the correct zero modes \( i.e., \) those which have the proper gauge and conformal properties, and can be eliminated as soon as this is done.

Since the left and right-handed constraints are independent, it is easy to see that, in the light-cone version, the BFV reduction factor \( F \) is just the product of two factors \( F_\alpha \) and \( F_{-\alpha} \) where

\[
F_\alpha = \int dA^\alpha \delta(\chi^\alpha) \det \left( [FP]_{a\alpha} + [BFV]_{a\alpha} \partial \right) e^{\int dx dt \left[ A^\alpha \Phi_\alpha \right]} \quad (5.2)
\]

and similarly for \( F_{-\alpha} \). Furthermore, because \( \Phi_\alpha = \pi_\alpha - \frac{2k}{\alpha^2} M_\alpha \), we see that the argument in the determinant in (5.2) is

\[
[FP]_{a\alpha} + [BFV]_{a\alpha} \partial t = - \left( \frac{\partial \chi^\alpha}{\partial a^\alpha} + \frac{\partial \chi^\alpha}{\partial A^\alpha} \partial t \right) \quad (5.3)
\]

According to the BFV prescription, the reduction factor (5.2) is to be inserted into the unconstrained WZW path integral (5.1). Thus, integrating over \( \pi_i \) and regarding \( \phi^i \) as a background field for the time being, the reduced path integral is

\[
I = \int d(a^{\pm \alpha} \pi_{\pm \alpha} A^{\pm \alpha}) \delta(\chi^\alpha) \delta(\chi^{-\alpha}) \det \left[ \left( \frac{\partial \chi^\alpha}{\partial a^\alpha} + \frac{\partial \chi^\alpha}{\partial A^\alpha} \partial t \right) \left( \frac{\partial \chi^{-\alpha}}{\partial a^{-\alpha}} + \frac{\partial \chi^{-\alpha}}{\partial A^{-\alpha}} \partial r \right) \right] e^{-S_A} \quad (5.4a)
\]

where

\[
S_A = \int d^2 z \left[ \frac{k}{2} \left( \partial_r \phi^i \right) \left( \partial_t \phi^i \right) + \pi_\alpha \partial_t a^\alpha + \pi_{-\alpha} \partial_r a^{-\alpha} - \frac{\alpha^2}{2k} \pi_\alpha \pi_{-\alpha} e^{a^\alpha \cdot \phi} - A^\alpha (\pi_\alpha - \frac{2k}{\alpha^2} M_\alpha) - A^{-\alpha} (\pi_{-\alpha} - \frac{2k}{\alpha^2} M_{-\alpha}) \right] \quad (5.4b)
\]
Integrating over the momenta $\pi_\alpha$ and $\pi_{-\alpha}$ gives the configuration space version of the BFV path integral for the gauged WZW model

$$ I = \int d(e^{-\alpha \cdot \phi} a^{\pm \alpha} A^{\pm \alpha}) \delta(\chi^\alpha) \delta(\chi^{-\alpha}) \det \left[ \left( \frac{\partial \chi^\alpha}{\partial a^\alpha} + \frac{\partial \chi^{-\alpha}}{\partial A^{\alpha}} \right) \left( \frac{\partial \chi^{-\alpha}}{\partial a^{-\alpha}} + \frac{\partial \chi^{-\alpha}}{\partial A^{-\alpha}} \right) \right] e^{-S_G} $$

where $S_G$ stands for the Action of the gauged WZW model and is given by

$$ S_G = \int d^2z \frac{k}{2} (\partial_r \phi^i)(\partial_t \phi^i) + \frac{2k}{\alpha^2} \left[ e^{-\alpha \cdot \phi}(\partial_r a^{-\alpha} - A^{-\alpha})(\partial_t a^{\alpha} - A^{\alpha}) + A^{\alpha} M_\alpha + A^{-\alpha} M_{-\alpha} \right] $$

Equations (5.5a,b) are the standard BFV results for the reduced path integral in Euclidean coordinates. It is obvious that the Action (5.5b) is invariant under the gauge transformations

$$ a^{\alpha} \rightarrow a^{\alpha} + \lambda^{\alpha}, \quad A^{\alpha} \rightarrow A^{\alpha} + \partial_\lambda^{\alpha} $$

and similarly for $a^{-\alpha}$ and $A^{-\alpha}$.

We can now discuss the zero modes of the $A$’s. This we can do by taking into account the conformal spins $\omega(s_l, s_r)$ of the fields tabulated below

| $\omega(e^{a \cdot \phi})$ | $\omega(a^{\alpha})$ | $\omega(a^{-\alpha})$ | $\omega(A^{\alpha})$ | $\omega(A^{-\alpha})$ |
|--------------------------|-------------------|-------------------|-------------------|-------------------|
| $(m_\alpha, m_\alpha)$   | $(0, m_\alpha)$   | $(m_\alpha, 0)$   | $(1, m_\alpha)$   | $(m_\alpha, 1)$   |

The weights of $a^{\alpha}$, $a^{-\alpha}$ and $e^{a \cdot \phi}$ were determined following (3.25) and the natural choice of weights for the $A$ fields above follows from the gauge transformations (5.6). Consider, for definiteness, $A^{\alpha}$, and decompose it into a maximally gauge invariant part $A^{\alpha}_0$ and its orthogonal complement $\hat{A}^{\alpha}$ which is gauge variant and can be gauged away i.e. let

$$ A^{\alpha} = A^{\alpha}_0 + \hat{A}^{\alpha} \quad \text{where} \quad \hat{A}^{\alpha} = \partial_\lambda^{\alpha} $$

In the above equation the gauge transformation parameter $\lambda^{\alpha}$ has a conformal weight $\omega(\lambda^{\alpha}) = (0, m_\alpha)$. The requirement that the $A^{\alpha}_0$ and $\hat{A}^{\alpha}$ be orthogonal to each other can then be written as

$$ \int d^2z \, h_\alpha A^{\alpha}_0 \hat{A}^{\alpha} = 0 $$

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where $h_\alpha$ is a set of auxiliary fields with conformal weights

$$\omega(h_\alpha) = (-1, 1 - 2m_\alpha)$$

(5.10)

The factor $h_\alpha$ in (5.9) comes from the requirement that the orthogonality be defined in a conformally invariant manner. As is obvious from the weights of the $A$ fields defined in (5.7), it is not possible to define this condition in a conformally invariant manner without introducing auxiliary fields with appropriate conformal properties. It will become clear presently, however, that these fields are necessary only to pick the true zero modes of the gauge fields. Once this is done, they can be easily eliminated.

Substituting for $\hat{A}_\alpha$ from (5.8) in (5.9), and using the fact that the orthogonality must be valid for arbitrary $\lambda_\alpha$, it follows from a simple partial integration that

$$\partial_l(h_\alpha A_0^\alpha) = 0 \quad \text{or} \quad A_0^\alpha = h_\alpha^{-1} f_\alpha(z_r)$$

(5.11)

where $f_\alpha(z_r)$ are arbitrary holomorphic functions with a conformal weights given by

$$\omega(f_\alpha(z_r)) = (0, 1 - m_\alpha)$$

(5.12)

However, since there are no holomorphic (or anti-holomorphic) functions on a compact Riemann surface except the constant functions [11], we see that $f_\alpha(z_r)$ must be constant. Furthermore, for $m_\alpha \neq 1$, $f_\alpha(z_r)$ are not conformal scalars and hence the only constant they can be set equal to, without breaking conformal invariance, is zero. Thus there are no zero modes which have grade greater than one and only one zero mode of grade one for each positive root, namely,

$$A_0^\alpha^{(s)} \sim h_\alpha^{-1}$$

(5.13)

We may now eliminate the auxiliary field by setting

$$h_\alpha^{(s)} = e^{\alpha^{(s)} \phi}$$

(5.14)
This is the most natural choice for $h_\alpha$ since $c^{(s)}_{\alpha} \cdot \phi$ are the only local fields, apart from the background metric, which has the correct conformal weight.

A more intuitive argument would be to note that any gauge-invariant part $A^0_\alpha$ of $A^\alpha$ would have conformal spin (the difference of the left and right conformal weights) equal to $1 - m_\alpha$ and be both primary and local. But the only residual fields out of which they could be constructed are the fields $e^{\alpha} \cdot \phi$ and since these fields have spin zero, the only permissible zero modes are those that have spin zero i.e grade equal to one. The Lagrange multiplier fields can therefore be written as

$$A^\alpha = \mu^\alpha e^{\alpha} \cdot \phi + \partial_\mu \lambda^\alpha, \quad A^{-\alpha} = \mu^{-\alpha} e^{\alpha} \cdot \phi + \partial_\nu \lambda^{-\alpha}$$

where $\mu^\alpha, \mu^{-\alpha} \neq 0$ iff $\alpha \in \Delta_s$ (5.15) $\mu^\alpha$ and $\mu^{-\alpha}$ being constants.

As has already been mentioned, it is desirable to have a curved space generalisation of the path integral in (5.5) because the manifold is compact. We choose a fixed background metric $g^{\mu\nu}$ for this purpose. An interesting property of the Action (5.5b) is that, if we use conformal coordinates $g_{\mu\nu} = e^{\sigma(x)} \eta_{\mu\nu}$, the metric does not appear explicitly. In particular, since the partial derivatives act on the sides of the fields that have conformal weight zero, they remain ordinary derivatives i.e. there is no need to modify them with the spin connection $\partial \sigma$. Furthermore, this continues to be the case when we change from the Sugawara Virasoro to the improved one. The reason for the invariance under the change of Virasoro is that the change of $a^\alpha$ and $a^{-\alpha}$ from scalars to fields of weights $(0, m_\alpha)$ and $(m_\alpha, 0)$ respectively is exactly compensated by the change in $e^{\alpha} \cdot \phi$ from a conformal scalar to a primary field of weight $(m_\alpha, m_\alpha)$.

As mentioned earlier, the improvement terms in the Virasoro can be incorporated explicitly in the presence of a background metric. This is done by adding to the Lagrangian density, a term of the form $\sqrt{g} g^{\mu\nu} \nabla_\mu J_\nu^0$, where $\nabla_\mu = \partial_\mu + (\partial_\mu \sigma)$ is the covariant derivative. In conformal coordinates this reduces to $(\partial_\mu \sigma) J^0_\mu$ apart from a total derivative term. However, since the field $\alpha \cdot \phi$ is no longer a scalar but a spin-zero connection,
the current \( J^\mu_0 \) is no longer a vector but a spin-one connection. To restore the vectorial properties of \( J^\mu_0 \), it is necessary to let \( J^\mu_0 \rightarrow J^\mu_0 - \rho \partial_\mu \sigma \). In that case, the cross-terms in \( Tr (J^i)^2 + (\partial_\mu \sigma) J^\mu_0 \) exactly cancel leaving a net addition to the Lagrangian density of a Polyakov term \(-k\rho^2(\partial \sigma)^2/2\). The Polyakov term cannot be ignored because it is this term that produces the known classical centre \( c = -k\rho^2 \) for the improved Virasoro algebra according to the standard formula \( \partial S/\partial \sigma(x) = cR(x) \), where \( R(x) \) is the Ricci scalar. Thus the net effect of introducing curvilinear coordinates is simply to add a Polyakov term \( k\rho^2R \sigma/2 \) to the Action.

We also have to consider the effect of the change of Virasoro on the measure in (5.5). The factor \( (e^{-\alpha \cdot \phi} a^\alpha a^{-\alpha}) \) in the measure remains a scalar under the change of Virasoro. Hence the curved space generalisation of the \( a^\alpha a^{-\alpha} \) integral requires only the usual factor \( \sqrt{g} \). On the other hand, since the \( A^\alpha \) and \( A^{-\alpha} \) fields have weights \((1, m_\alpha)\) and \((m_\alpha, 1)\) respectively, their measure requires a factor \((\sqrt{g})^{m_\alpha}\).

Substituting (5.15) in the gauged WZW path integral (5.5a,b), and incorporating the above mentioned modifications because of the curved space generalisation, we get

\[
I = \int d(\sqrt{g}e^{-\alpha \cdot \phi} a^\alpha a^{-\alpha}) d\left[ (\frac{1}{\sqrt{g}})^m \partial_{\lambda^\alpha} \partial_{\sigma} \lambda^{-\alpha}\right] \\
\delta(\chi^\alpha) \delta(\chi^{-\alpha}) det \left[ \left( \frac{\partial \chi^\alpha}{\partial a^\alpha} + \frac{\partial \chi^\alpha}{\partial \lambda^\alpha} \right) \left( \frac{\partial \chi^{-\alpha}}{\partial a^{-\alpha}} + \frac{\partial \chi^{-\alpha}}{\partial \lambda^{-\alpha}} \right) \right] e^{-\hat{S}_G} \times I_0
\]

where \( \hat{S}_G \) is the Action for the fluctuations and \( I_0 \) is the path integral for the zero modes. Since the cross-terms between the \( A^\alpha \) and \( \hat{A}^\alpha \) terms, as well as the \( M_\alpha \hat{A}^\alpha \) and \( M^{-\alpha} \hat{A}^{-\alpha} \) terms are pure divergences, these terms drop out and \( \hat{S}_G \) and \( I_0 \) may be written as

\[
\hat{S}_G = \int d^2 z \left[ \frac{k}{2} (\partial_t \phi^i) (\partial_t \phi^i) + \frac{2k}{\alpha^2} e^{-\alpha \cdot \phi} \left( \partial_t (a^\alpha - \lambda^\alpha) \right) \left( \partial_t (a^{-\alpha} - \lambda^{-\alpha}) \right) - \frac{k\rho^2}{2} (\partial \sigma)^2 \right] 
\]

(5.16b)
\[ I_0 = \int d(\mu^\alpha \mu^{-\alpha}) e^{-\int d^2 z \left( \frac{\partial}{\partial a} e^{a\phi} \mu^\alpha \mu^{-\alpha} - \frac{\partial}{\partial \chi} e^{a\phi} (\mu^\alpha M_\alpha + \mu^{-\alpha} M_{-\alpha}) \right)} = e^{k\Lambda_\alpha} \int d^2 z \ e^{a\phi} \]  

(5.16c)

respectively. Note that the integral over the zero modes \( \mu^{\pm \alpha} \) has produced the Toda potential term \( k\Lambda_\alpha e^{a\phi} \), \( \Lambda_\alpha \neq 0, \) iff \( \alpha \in \Delta_s \). The determinant in (5.16a) may be simplified by using

\[
\frac{\partial \chi^\alpha}{\partial a^\alpha} + \frac{\partial \chi^\alpha}{\partial \lambda^\alpha} = 2 \frac{\partial \chi^\alpha}{\partial (a^\alpha + \lambda^\alpha)}
\]  

(5.17)

and a similar expression for \( \chi^{-\alpha} \) and \( a^{-\alpha} \). Upon using this result, the measure in (5.16a) reduces to

\[
4d(\sqrt{g} e^{-a\phi} a^\alpha a^{-\alpha})d\left[ \frac{1}{\sqrt{g}} \right]^{m_\alpha} \partial_t \lambda^\alpha \partial_r \lambda^\alpha \delta(a^\alpha + \lambda^\alpha) \delta(a^{-\alpha} + \lambda^{-\alpha})
\]  

(5.18)

Eliminating the \( \lambda \) fields by means of the delta functions and rescaling \( a^\alpha \) and \( a^{-\alpha} \) by a factor 2, the path integral becomes

\[
I = I_\alpha \times \det \left[ \frac{1}{\sqrt{g}} \right]^{m_\alpha} \partial_t, \partial_r \right] e^{-\int d^2 z \left[ \frac{1}{2}(\partial_t \phi^i)(\partial_r \phi^i) - k \sum_{\alpha \in \Delta_s} \Lambda_\alpha e^{a\phi} - \frac{k}{2}(\partial \sigma)^2 \right]}
\]  

(5.19)

The \( I_\alpha \) in (5.19) stands for the \( a^\alpha a^{-\alpha} \) part of the integral and is just the well-known one encountered in the computation of the WZW partition function, namely,

\[
I_\alpha = \int d(\sqrt{g} e^{-a\phi} a^{\pm \alpha}) e^{-k \int d^2 z \ e^{-a\phi} (\partial_t a^\alpha)(\partial_r a^{-\alpha})}
\]

\[
= \int d(ad) e^{-\int d^2 z \ a g^{-{\xi \over 2}}(D^\alpha \phi)(D^\alpha \phi T) g^{-{\xi \over 2}}}
\]

\[
= \det \left( \frac{1}{\sqrt{g}} (D^\alpha \phi)^T (D^\alpha \phi) \right)^{-1}
\]  

(5.20)

where \( a = \sqrt{k} g^{\frac{1}{2}} e^{-\frac{\phi}{\sqrt{g}}} a^\alpha \), \( d = \sqrt{k} g^{\frac{1}{2}} e^{-\frac{\phi}{\sqrt{g}}} a^{-\alpha} \), \( D^\alpha \phi = \partial + (\partial \alpha \cdot \phi) \) and \( (D^\alpha \phi)^T = \partial - (\partial \alpha \cdot \phi) \). Thus the path integral (5.19) may be expressed as

\[
I = e^{-k \int d^2 z \left[ \frac{1}{2}(\partial_t \phi^i)(\partial_r \phi^i) - \sum_{\alpha \in \Delta_s} \Lambda_\alpha e^{a\phi} - {\xi \over 2}(\partial \sigma)^2 \right]} \frac{\det \left[ \frac{1}{\sqrt{g}} \right]^{m_\alpha} (\partial_t \partial_r) \right]}{\det \left[ \frac{1}{\sqrt{g}} (D^\alpha \phi)^T (D^\alpha \phi) \right]}
\]  

(5.21)
However, from the identity [12],
\[
det \left[ \frac{1}{\sqrt{g}} \right]^{m_\alpha} (\partial_i \partial_r) = \det \left[ \frac{1}{\sqrt{g}} \right] (\partial_i \partial_r) \times e^{\frac{Q_\alpha}{2}} \int R \Phi R
\]
where \( Q_\alpha = m_\alpha (m_\alpha - 1) \) (5.22)
and the well-known WZW anomaly equation [13]
\[
det \frac{1}{\sqrt{g}} (\partial_i \partial_r) \frac{d}{dg} \int d^2z \sum \alpha \left( (\partial \alpha \cdot \Phi)^2 - \sqrt{f} R \alpha \cdot \Phi \right) = e \int d^2z \left[ \frac{1}{2} (\partial_i (\alpha \cdot \Phi))(\partial_i (\alpha \cdot \Phi)) - \sqrt{f} R \Phi \right]
\]
we obtain
\[
I = e^{- \int d^2z \left[ \frac{(k - \gamma)}{2} (\partial_i \Phi)(\partial_i \Phi) - k \sum_{\alpha \in \Delta_+} \Lambda_\alpha e^{\alpha \cdot \Phi + \sqrt{f}(j + R \rho) \cdot \Phi - \frac{k - (\gamma - 2)}{2} \rho^2(\partial \sigma)^2} \right]}
\]
(5.23)
where we have used \( m_\alpha = \rho \cdot \alpha \) and \( \sum \alpha_i \alpha_i = \gamma \delta_{ij} \), \( \gamma \) being the dual Coxeter number, to get \( \sum Q_\alpha = (\gamma - 2) \rho^2 \). Reintroducing the \( \Phi \)-integration we have finally the reduced configuration space path integral for \( \Phi \)
\[
I = \int d(g^i \Phi^i) e^{- \int d^2z \left[ \frac{(k - \gamma)}{2} (\partial_i \Phi)(\partial_i \Phi) - k \sum_{\alpha \in \Delta_+} \Lambda_\alpha e^{\alpha \cdot \Phi + \sqrt{f}(j + R \rho) \cdot \Phi - \frac{k - (\gamma - 2)}{2} \rho^2(\partial \sigma)^2} \right]}
\]
(5.24)
In the flat space limit this reduces to
\[
I = \int d\Phi^i e^{- \int d^2z \left[ \frac{(k - \gamma)}{2} (\partial_i \Phi)(\partial_i \Phi) - k \sum_{\alpha \in \Delta_+} \Lambda_\alpha e^{\alpha \cdot \Phi + j \cdot \Phi} \right]}
\]
(5.25)
which is easily recognised as the path integral for the Toda theory. Thus (5.25) is just the Toda path integral in a fixed curved background. Writing \( \Phi^i = s^i - \rho^i \sigma \), so that \( s^i \) is a scalar field, (5.24) becomes
\[
I = \int d(g^i \Phi) e^{- \int d^2z \left[ \frac{(k - \gamma)}{2} (\partial_i s)(\partial_i s) - k \sum_{\alpha \in \Delta_+} \Lambda_\alpha \sqrt{f} e^{\alpha \cdot s + \sqrt{f}(1 + (k - \gamma)) R \rho \cdot s + \sqrt{f} j \cdot s} \right]}
\]
(5.26)
where the terms that depend purely on \( \sigma \), including the Polyakov term, have cancelled (except for the \( j \sigma \) term which we have dropped). It is well-known that the Virasoro centre for this theory has the form
\[
c(k - \gamma) = h \left[ \frac{l}{6} + \frac{2 \rho^2}{(k - \gamma)} \right] \left[ 1 + (k - \gamma) \right]^2 = \frac{l}{6} + \frac{2 \rho^2}{(k - \gamma) h} \left[ h + (k - \gamma) h \right]^2
\]
(5.27)
where \( l = \text{dim} H \). The \( h/6 \) in (5.28) comes from the Weyl anomaly for a single scalar field and, to separate the quantum effects, we have recalled from (3.1) that \( k = \kappa/h \) where \( \kappa \sim 1 \). The results are independent of the choice of the gauge fixing conditions – as predicted by the Fradkin-Vilkovisky theorem.

Note that the WZW anomaly (ratio of determinants) was produced by the fact that the integration over the \( \hat{A} \) fields is restricted by the condition that \( \hat{A} \) be a gradient. Had \( A \) been free, we could have integrated directly over \( A \) in (5.5) and there would have been no WZW anomaly (although there would still be a Weyl anomaly). Thus the WZW anomaly, originates in the fact that the gauge variant parts of the Lagrange multipliers are gradients. The presence of the WZW anomaly means that although the classical reduction converts the WZW theory into a Toda theory with coupling constant \( k \), the quantum reduction converts it into a Toda theory with coupling constant \( k - \gamma \). As a result, although the expressions for the Virasoro centre in (3.26) and (5.28) have the same functional form, their arguments are \( k \) and \( k - \gamma \) respectively. Thus the operations of reducing and quantising do not commute.

\[ \text{VI. THE TODA AND WZW GAUGES} \]

We would now like to investigate what happens in two particular gauges namely, the Toda (physical) gauge and the WZW gauge. The Toda gauge highlights the origin of the WZW anomaly. The WZW gauge provides an interesting interpretation of the formula (5.28) for the Virasoro centre. The key equation for comparison is (5.5), just prior to the separation of the \( A \) fields into their zero mode and gauge variant parts.

The Toda gauge is defined by \( \chi^\alpha \equiv a^\alpha \). In this gauge, one sees from (4.7) that \( [FP] = -1 \) and \( [BFV] = 0 \). Hence the \( a^{\pm\alpha} \) fields are eliminated by the delta functions and we obtain

\[ I = \int \det(e^{-\alpha \cdot \phi}) d(A^{\pm\alpha}) \ e^{-\int d^2z \ \frac{1}{2}(\partial_r \phi)(\partial_t \phi) + \frac{2\kappa}{\alpha} [e^{-\alpha \cdot \phi} A^{\alpha} A^\alpha + A^\alpha M_\alpha + A^{-\alpha} M_{-\alpha}]} \quad (6.1) \]
which on separating the zero modes and integrating over them as in (5.16c) becomes

\[ I = \int \det(e^{-\alpha \cdot \phi}) d(\hat{A}^{\pm \alpha}) e^{-\int d^2 z \sum_{\alpha} \frac{1}{2}(\partial_r \phi^i)(\partial_l \phi^i) + k \sum_{\alpha} e^{\alpha \cdot \phi} + \frac{2k}{\alpha^2} e^{-\alpha \cdot \phi} \hat{A}^{-\alpha} \hat{A}^{\alpha}} \]

(6.2)

Note that if there were no zero modes, the \(M\)-dependent terms would drop out and there would be no Toda potential.

We now use the fact that the \(\hat{A}\) fields are gradients, to obtain

\[ \det(e^{-\alpha \cdot \phi}) d(\hat{A}^{\pm \alpha}) = \det(e^{-\alpha \cdot \phi}) d(\partial_l \lambda^{\alpha} \partial_r \lambda^{-\alpha}) \]

(6.3)

and the rest of the integration proceeds as before. The important point to note is that had the \(\hat{A}\) fields not been gradients, we could have replaced \(\det(e^{-\alpha \cdot \phi}) d(\hat{A}^{\pm \alpha})\) by \(d(e^{-\alpha \cdot \phi} \hat{A}^{\pm \alpha})\) and then there would have been no anomaly. But because \(\hat{A}\) is a gradient, we have

\[ d(e^{-\alpha \cdot \phi} \hat{A}^{\pm \alpha}) = d(e^{-\alpha \cdot \phi} \partial_l \lambda^{\alpha} \partial_r \lambda^{-\alpha}) \]

\[ = d[(D^\alpha)\hat{T}(D^{\alpha} \phi)e^{-\alpha \cdot \phi} \lambda^{\pm \alpha}] = \det[(D^\alpha)\hat{T}(D^{\alpha} \phi)] d(e^{-\alpha \cdot \phi} \lambda^{\pm \alpha}) \]

(6.4)

which is not the same as the correct measure (6.3). Thus the Toda gauge shows explicitly how the zero modes produce the Toda potential and the gauge variant modes produce the WZW anomaly.

The WZW gauge is defined by \(\chi^\alpha \equiv \hat{A}^\alpha\). In this gauge, \(\hat{A}^\alpha = 0\), and one sees from (4.7) that \([FP] = 0\) and \([BFV] = -1\), which, incidentally, shows the necessity of using the BFV formalism for considering this gauge. In this case, the \(\hat{A}\) fields are eliminated by the gauge fixing delta functions and on integrating over the zero modes of the \(A\) fields, the reduced path integral (5.5) becomes

\[ I = \int d(e^{-\alpha \cdot \phi} a^{\pm \alpha}) \times \]

\[ e^{-\int d^2 z \sum_{\alpha} \frac{1}{2}(\partial_r \phi^i)(\partial_l \phi^i) + k \sum_{\alpha} e^{\alpha \cdot \phi} + \frac{2k}{\alpha^2} e^{-\alpha \cdot \phi} \hat{A}^{-\alpha} \hat{A}^{\alpha}} \]

(6.6)
Note that the Action in (6.6) is just the original WZW Action together with the exponential term and the Polyakov term. This form of the Action allows us to read off the Virasoro centre by inspection namely,

$$\frac{6c}{\hbar} = \frac{kr}{k - \gamma} + 12\rho^2k - (r - l + 12\sum Q_\alpha) = l + \frac{\gamma r}{k - \gamma} + 12\rho^2(k - \gamma + 2)$$

$$= l + 24\rho^2 + 12\rho^2\left[\frac{1}{k - \gamma} + k - \gamma\right]$$

(6.7)

where $r = \dim G$ and $l = \dim H$. For the last equality above we have used the ‘strange’ formula $12\rho^2 = \gamma r$ of Freudenthal and deVries [14]. The first term in (6.7) comes from the WZW piece and the second term from the Polyakov term which has contributions from the classical improvement to the Virasoro and from the ghosts. The combination of these terms simplifies to (5.28) and thus gives a simple interpretation of the Toda centre as the sum of the WZW centre and the classical improvement centre $k$, minus the ghost centre $r - l + \sum Q_\alpha$. The expression (6.7) for the centre was obtained earlier in [1] without using the curved background.

VII. CONCLUSIONS

We have shown that the quantum mechanical WZW \rightarrow Toda reductions can be formulated by means of the path integral in a gauge independent manner. For this purpose we have used a modification of the conventional Batalin, Fradkin, Vilkovisky formalism for first class constraints which takes into account the chirality of the constraints and the compactness of the manifold. An interesting feature of the reduction is the role played by the decomposition of the Lagrange multipliers into zero modes which are gauge invariant and gradient parts which are not. The zero modes produce the Toda potential and the gradients produce the WZW anomaly ($k \rightarrow k - \gamma$). This anomaly plays a crucial role in proving the Fradkin-Vilkovisky theorem regarding gauge invariance of the reduction. This is shown explicitly by the fact that if the anomaly is
neglected, the centre of the Virasoro algebra is $c(k)$ and $c(k - \gamma)$ in the Toda and WZW gauges respectively.

Another interesting feature is that the operations of reducing and quantising do not commute in the sense that they lead to Toda theories with different coupling constants. Reduction of the classical WZW theory leads to a Toda coupling constant $k$ – which is not changed by subsequent quantisation – whereas reduction of the quantised WZW theory leads, as we have seen, to a Toda coupling constant $k - \gamma$.

The reduction was simplified by the fact that, in conformal coordinates, the change from the Sugawara to the ‘improved’ energy momentum tensor (which is necessary for conformal invariance) does not show up explicitly in the path integral except for Polyakov terms which appear when the fields are coupled to a fixed background metric. This means that for flat (toroidal) spaces, the path integral remains form invariant under the change of Virasoro. The basic reason for this is that although the individual fields change their conformal properties with respect to the improvement, the combination $e^{-\alpha \cdot \phi} a^\alpha a^{-\alpha}$ which appears in both the measure and the Lagrangian, is invariant under the change.

The modification of the BFV formalism that we have used should be useful for other conformal reductions such as WZW $\rightarrow$ non-abelian Toda reductions and the coset constructions of Goddard and Olive [15]. We hope to address these questions in the future.
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