Clifford Wavelet Transform and the Uncertainty Principle

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Abstract. The present paper lies in the whole topic of wavelet harmonic analysis on Clifford algebras. In which we derive a Heisenberg-type uncertainty principle for the continuous Clifford wavelet transform. A brief review of Clifford algebra/analysis, wavelet transform on $\mathbb{R}$ and Clifford Fourier transform and their properties is conducted. Next, such concepts are applied to develop an uncertainty principle based on Clifford wavelets.

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1. Introduction

Transformations such as Fourier and wavelet types are powerful methods for signal/image processing. In Fourier analysis, the signals are transformed from the original domain to the spectral or frequency one. In the frequency domain many characteristics of a signal are seen more clearly. In a counterpart, wavelet bases functions are localized in both spatial and frequency domains and thus yield very sparse and well-structured representations of signals which are important facts in the processing. The first work on wavelet analysis has been developed by Morlet in Ref. [62] to study seismic waves. He also, with Grossman, developed a mathematical study of a continuous wavelet transform (see [33]). In Ref. [61] Meyer recognised the link between harmonic analysis and Morlet’s theory and gave a mathematical foundation to the continuous wavelet theory and thus gave rise to wavelet analysis. The continuous wavelet analysis of a square integrable function $f$ begins by a

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convolution with copies of a given “mother wavelet” $\psi$ translated and dilated respectively by $b \in \mathbb{R}$ and $a > 0$. Such a function $\psi$ has to fulfil an admissibility condition such as

$$A_\psi = \int_{\mathbb{R}} \left| \hat{\psi}(\xi) \right|^2 \frac{d\xi}{|\xi|} < +\infty$$

where $\hat{\psi}$ is the classical Fourier transform of $\psi$. More information on real wavelets can be found in Refs. [20] and [32].

Wavelet theory has a great success especially in the applied fields such as image/signal processing, statistics, finance and engineering. This success encourages researchers to develop new and more efficient tools in this theory to be adapted to understand different problems. Recall that nowadays 3D image processing for example is a challenging topic in many fields such as medicine, arts, physics, informatics, biology and thus needs more theoretical developments. Wavelets and Clifford algebras/analysis are ones of the powerful tools in this subject.

The authors in Ref. [15] applied Clifford algebra concepts for the embedding of color information in images. Similarly, in Refs. [25] and [70] Clifford algebra has been used for image compression. In Ref. [69] a quaternionic wavelet method has been developed for face image features and correlation filters combination for properly and non-properly illuminated face images (see also [75]).

Already with the eventual link between wavelets, Clifford framework and signal processing, the concept of monogenic signals is well known. Monogenic signals are generalized forms of analytic ones. In Ref. [73] a sparse representation based on Marr and Mallat-Zhong wavelets has been developed for monogenic signals. An efficient contour shape and color characterization have been obtained. In Ref. [39] quaternionic and Clifford algebra valued Fourier transforms have been developed and applied for nuclear magnetic resonance, electric engineering, color image and signal processing. In Ref. [79] the energy concentration problem for 2D hypercomplex signals has been extended to Clifford case, in particular to quaternionic signals. Fletcher applied in Ref. [30] matrix representations of both quaternions and Clifford algebras for solving matrix equations. The results have been applied next to Daubechies quaternion and Clifford scaling filters.

In Ref. [50] quaternion-valued wavelets have been introduced in the context of the duplex matrix-valued functions by providing both quaternion scaling and wavelet functions as well as quaternion multiresolution analysis. Associated high-pass and low-pass filters have been provided also.

On the other hand, Clifford analysis leads to the generalization of real and harmonic analysis to higher dimensions. Clifford algebra accurately treats geometric entities depending on their dimensions such as scalars, vectors, bivectors and volume elements, etc. The distinction of axial and polar vectors in physics is resolved in the form of vectors and bivectors. For example, the quaternion description of rotations is fully incorporated in the form of rotors.
With respect to the geometric product of vectors, division by non-zero vectors is defined.

Clifford analysis may be also seen as a generalization of the theory of holomorphic functions of complex variables to the theory of monogenic functions in higher dimensions by means of Clifford vector-valued Dirac operator. The rotation-invariant Dirac operator factorizes the \( m \)-dimensional Laplace operator similarly to Riemann–Liouville operator in complex holomorphic case. For more details, backgrounds and review we may refer also to \([11,17,24,26,34,35,72]\).

The present paper lies in the same topic of wavelet harmonic analysis in the framework of Clifford algebra/analysis applications. We aim to develop an uncertainty principle proof in the Clifford framework based on Clifford wavelets. The paper is organized as follows. In Sect. 2 we give a brief review of Clifford analysis, introduce the notion of wavelet transform in \( \mathbb{R} \) and the uncertainty principle. Eventual link with signal processing and monogenic signals is also reviewed. The third section is devoted to show some important results and properties of Clifford Fourier and Clifford wavelet transforms. In Sect. 4, the uncertainty principle for the Clifford wavelet transform is established.

2. Preliminaries

In this section, we aim to recall the basic properties of Clifford algebras (see \([34,72]\) and the references therein). Next, a review of continuous wavelet transform in \( \mathbb{R} \) and the Heisenberg uncertainty principle are developed already with a discussion of eventual links/applications with signal processing and existing results.

2.1. Clifford Algebras

Clifford analysis may be seen as a generalization of Fourier one in signal processing as it employs real, complex and quaternion numbers. It may be also described with the algebra of Pauli and Dirac matrices for physical space and Minkowski space-time. Thus a unifying language for mathematics and physics can be developed (see \([7,9]\)). In Ref. \([10]\) basic concepts on the historical development of quaternion and Clifford Fourier transforms and wavelets have been displayed.

In what follows, a mathematical review of Clifford algebra/analysis is discussed. The Clifford algebra \( \mathbb{R}_n \) associated to \( \mathbb{R}^n \) is an associative algebra generated by an orthonormal (the canonical) basis \( \{e_1, e_2, \ldots, e_n\} \) by means of a non commutative product

\[
e_i e_j + e_j e_i = -2\delta_{ij},
\]

where \( \delta \) is the Kronecker symbol. This yields a finite \( 2^n \)-dimensional algebra known as the Clifford algebra \( \mathbb{R}_n \). It is decomposed as a direct sum

\[
\mathbb{R}_n = \mathbb{R}_n^0 \oplus \mathbb{R}_n^1 \oplus \cdots \oplus \mathbb{R}_n^n,
\]
where for $k \in \mathbb{N}$, $\mathbb{R}^k_n$ are the spaces of $k$-multi vectors defined by

$$R^k_n = \text{span}_{\mathbb{R}} \{ e_A = e_{i_1 i_2 \ldots i_k}, A = (i_1, i_2, \ldots, i_k), 1 \leq i_1 < i_2 < \cdots < i_k \leq n \},$$

with $e_{i_1 i_2 \ldots i_k} = e_{i_1} e_{i_2} \cdots e_{i_k}$ and $e_{\emptyset} = 1$. We may also decompose $\mathbb{R}_n$ as a direct sum of two sub-algebras

$$\mathbb{R}_n = \mathbb{R}_n^+ \oplus \mathbb{R}_n^- = \bigoplus_{k \text{ even}} \mathbb{R}^k_n \oplus \bigoplus_{k \text{ odd}} \mathbb{R}^k_n$$
called respectively the even and odd sub-algebras. Consequently, any Clifford number $a \in \mathbb{R}_n$ has a representation of the form

$$a = \sum_A a_A e_A, \ a_A \in \mathbb{R}.$$

Denoting $|A|$ the length or the cardinality for the multi-index $A$, the element $a$ may be written as

$$a = \sum_{k=0}^n \sum_{|A|=k} a_A e_A.$$

On the algebra $\mathbb{R}_n$ we may introduce some involutive operators such as

- **Main-involution** $\tilde{e}_j = -e_j$, $\forall j$, which yields that $\tilde{e}_A = (-1)^{|A|} e_A$ and consequently, $\tilde{a}b = \tilde{b}\tilde{a}$, $\forall a, b \in \mathbb{R}_n$.

- **Reversion** $e_j^* = e_j$, $\forall j$, which in turn yields that $e_A^* = (-1)^{|A|(|A|-1)/2} e_A$ and thus $(ab)^* = b^*a^*$, $\forall a, b \in \mathbb{R}_n$.

- **Conjugation** $\overline{e}_j = -e_j$, $\forall j$, yielding that $\overline{e}_A = (-1)^{|A|(|A|+1)/2} e_A$ and consequently, $\overline{ab} = \overline{b}\overline{a}$, $\forall a, b \in \mathbb{R}_n$.

The concept of real Clifford algebra can be extended to the complex Clifford algebra $\mathbb{C}_n = \mathbb{R}_n + i\mathbb{R}_n$. An element $\lambda \in \mathbb{C}_n$ may be written on the form $\lambda = \sum_A \lambda_A e_A$, $\lambda_A \in \mathbb{C}$ and thus possesses the decomposition $\lambda = a + ib$, $a, b \in \mathbb{R}_n$. This induces the

- **Hermitian conjugation** $\lambda^\dagger = \overline{a} - i\overline{b}$.

In this context, a vector $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ may be identified to the Clifford element in $\mathbb{R}_n$, $x = \sum_{j=1}^n x_j e_j$. This permits to define the Clifford product of two vectors by

$$xy = x \cdot y + x \wedge y,$$

where the $\cdot$ product is an analogous of the classical inner product on $\mathbb{R}^n$,

$$x \cdot y = - \langle x, y \rangle = -\sum_{j=1}^n x_j y_j,$$

and where the $\wedge$ product is the outer product

$$x \wedge y = \sum_{j<k} e_j e_k (x_j y_k - x_k y_j).$$

This yields that

$$x \cdot y = \frac{1}{2}(xy + yx) \quad \text{and} \quad x \wedge y = \frac{1}{2}(xy - yx).$$
In particular we have
\[ x^2 = -|x|^2 = -\sum_{j=1}^{n} |x_j|^2. \]

Any vector \( x \) is decomposed as \( x = x_{\|\omega} + x_{\perp\omega} \) for a \( \omega \in S^{n-1} \) with \( \omega \cdot x_{\perp\omega} = 0 \) and \( \omega \wedge x_{\|\omega} = 0 \) where \( S^{n-1} \) stands for the unit sphere in \( \mathbb{R}^n \).

This in turn induces that \( x_{\|\omega} = \langle x, \omega \rangle \omega \) and \( x_{\perp\omega} = \omega (x \wedge \omega) \) which permits next to characterize the reflection \( R_{\omega} \) with respect to the hyperplane \( \omega \perp \) as \( R_{\omega}(x) = \omega x \omega \). Cartan–Dieudonné Theorem ([16]) relates the reflection to the so-called spinors. It states that there exists \( \omega_1, \omega_2, \ldots, \omega_{2l} \in S^{n-1} \) with \( \omega_j^2 = -1, 1 \leq j \leq 2l \) \((l \in \mathbb{N})\) and a rotation \( T \in SO(n) \) \((SO(n)\) being the rotation group on \( \mathbb{R}^n \)) such that
\[ T(x) = [R_{\omega_1} \circ R_{\omega_2} \circ \cdots \circ R_{\omega_{2l}}](x) = \omega_1 \omega_2 \cdots \omega_{2l} x. \]

Denoting \( s = \omega_1 \omega_2 \cdots \omega_{2l} \) and \( \bar{s} = \omega_{2l} \omega_{2l-1} \cdots \omega_1 \) we have \( T(x) = sx\bar{s} \).

The element \( s \) is called a spinor. Generally speaking, the spin group of order \( n \) is
\[ \text{Spin}(n) = \left\{ s \in \mathbb{R}^n ; s = \prod_{j=1}^{2l} \omega_j, \omega_j^2 = -1, 1 \leq j \leq 2l \right\}. \]

To finish with this brief overview on Clifford algebra/analysis, it remains to recall the functional framework. Let \( f : \mathbb{R}^n \longrightarrow \mathbb{C}_n \). It may be expressed as
\[ f(x) = \sum_A f_A(x) e_A \]
where \( f_A \) are generally \( \mathbb{C} \)-valued functions and \( A \subset \{1, 2, \ldots, n\} \). The inner product of two functions \( f = \sum_A f_A(x) e_A \) and \( g = \sum_B g_B(x) e_B \) is defined by
\[ \langle f, g \rangle_{L^2(\mathbb{R}^n, dV(x))} = \int_{\mathbb{R}^n} \overline{f(x)} g(x) dV(x) \tag{2.1} \]
and the associated norm by
\[ \| f \|_{L^2(\mathbb{R}^n, dV(x))} = \langle f, f \rangle_{L^2(\mathbb{R}^n, dV(x))}^{1/2}. \]

We denote also
\[ \| f \|_{L^1(\mathbb{R}^n, dV(x))} = \int_{\mathbb{R}^n} |f(x)| dV(x) \]
where \( dV(x) \) stands for the Lebesgue measure. The inner product (2.1) satisfies the Cauchy-Schwartz inequality
\[ |\langle f, g \rangle_{L^2(\mathbb{R}^n, dV(x))}| \leq \| f \|_{L^2(\mathbb{R}^n, dV(x))} \| g \|_{L^2(\mathbb{R}^n, dV(x))}. \tag{2.2} \]
2.2. Wavelets on $\mathbb{R}$

A wavelet is in its simple sense a wave function that decays rapidly and has a zero average value. Wavelet analysis consists of breaking up a signal into approximating functions (shifted and dilated versions of a wavelet) contained usually in bounded domains [19]. Wavelet analysis was introduced in the early 1980s in the context of signal analysis and petroleum exploration to give a representation of signals and detect their characteristics. Several methods previously have been used for this aim, the most known is the Fourier one. In Fourier transform description of signals is limited to the overall behavior and can not provide any information on the details. In digital signal processing, Fourier analysis often requires linear calculation algorithms. In 1940, Denis Gabor introduced the windowed Fourier transform (WFT) to address the problem of time-frequency localization. It consists of calculating the Fourier transform of the signal by multiplying it by a function localized in time (Gaussian window) and then calculating the transform. But the situation has changed with the emergence of new problems especially those related to irregular variations. The major drawback of the WFT is the shape stability and the window’s size. Gaussian type windows can not for example model many non stationary properties. Henceforth the need for analysis using non-linear algorithms, non-stationary signals and/or non-periodical bases became necessary. Specific analysis with well-specified characteristics such as localization in time and frequency, adaptivity to the data, easily implemented advanced algorithms and optimum computation time needs to be developed. This was how wavelet theory was born. It had subsequently renewed interest and has been steadily developed in theory and application.

Wavelets differ from Fourier methods as they allow the localization of a signal in both time and frequency. It is a tool which breaks up data into different frequency components or sub-bands and then studies each component with a resolution that is matched to specific or proper scale. Unlike the Fourier series, it can be used on non-stationary transient signals with more precise results. This section is devoted to present the main ideas on wavelet analysis namely wavelet transforms, multi-resolution analysis, wavelet bases and algorithms of construction and reconstruction.

Wavelet analysis of functions/signals is based primarily on an effective representation for standard functions, robustness to the specification models, a reduction in the computation time, simplicity of the analysis, both easy generalization and efficient depending on the dimension and finally a location in time and frequency. Mathematically and quantitatively speaking wavelet analysis of functions starts by computing a type of transform known as wavelet transform similar to Fourier one and which consists of a convolution product of the function with special copies of one source analyzing function called mother wavelet which plays the role of the Fourier mode $e^{ix}$. Denote $\psi$ as such function. It should satisfy
- A finite energy or space/time localization assumption: \( \psi \in L^2(\mathbb{R}, dx) \).
- An admissibility assumption stating that
  \[
  0 < A_\psi = \int_{\mathbb{R}_+^*} \frac{|\hat{\psi}(\xi)|^2}{|\xi|} d\xi < \infty
  \]
  where \( \hat{\psi}(\xi) = \int_{\mathbb{R}} \psi(x) e^{-ix\xi} dx \) is the classical Fourier transform of \( \psi \) and \( \mathbb{R}_+^* = (0, +\infty) \).
- Vanishing moments:
  \[
  \exists N \in \mathbb{N}, \int_{\mathbb{R}} x^n \psi(x) dx = 0, \forall n \leq N.
  \]

To analyze a signal by wavelets, one passes as for the analysis of Fourier by its wavelet transform. There are two types of processing; The continuous wavelet transform (CWT) and discrete wavelet transform (DWT).

The CWT is based firstly on the introduction of a translation parameter \( b \in \mathbb{R} \) and a scale parameter \( a > 0 \) to a mother wavelet \( \psi \). The translation parameter determines the position or the time around which we want to assess the behavior of the signal, while the scale factor is used to assess the signal behavior around the position or the frequency.

**Definition 2.1. (Continuous-wavelet transform)** Let \( f \in L^2(\mathbb{R}, dx) \). Its continuous wavelet transform is defined as

\[
T_\psi[f](a, b) = \int_{\mathbb{R}} f(x) \overline{\varphi^{a,b}(x)} dx = (f * \overline{\varphi}_a)(b),
\]

where \( \varphi^{a,b}(x) = \frac{1}{\sqrt{a}} \psi\left(\frac{x-b}{a}\right) \), \( a > 0, b \in \mathbb{R} \) and \( \overline{\varphi}_a(x) = \frac{1}{\sqrt{a}} \overline{\varphi\left(\frac{-x}{a}\right)} \).

By varying the parameters \( a \) and \( b \) we may cover completely all the time-frequency plane. This gives a full and redundant representation of the whole signal to be analyzed (see [49]). This transform is called continuous because of the nature of the parameters \( a \) and \( b \) that may operate at all levels and positions. Using the admissibility assumption above we immediately affirm that

\[
\int_{\mathbb{R}_+^*} \int_{\mathbb{R}} |T_\psi[f](a, b)|^2 db \frac{da}{a^2} < \infty
\]

where \( \mathbb{R}_+^* = (0, +\infty) \). More precisely, an inner product may be defined for the wavelet transform as

\[
[T_\psi[f], T_\psi[g]] = \frac{1}{A_\psi} \int_{\mathbb{R}_+^*} \int_{\mathbb{R}} \overline{T_\psi[f](a, b)} T_\psi[g](a, b) db \frac{da}{a^2}
\]

which in turn may be related to the inner product of the analyzed functions \( f \) and \( g \) by means of a Fourier-Plancherel type formula as

\[
[T_\psi[f], T_\psi[g]] = <f, g >_{L^2(\mathbb{R}, dx)}.
\]
Moreover, a reconstruction formula may also be proved stating that

$$f(x) = \frac{1}{\mathcal{A}_\psi} \int_{\mathbb{R}_+^*} \int_{\mathbb{R}} \psi_{a,b}(x) T_{\psi}[f](a,b) \frac{da}{a^2}. $$

To analyze statistical series or discrete time signals and avoid redundancy problems and integrals calculations appearing in CWT, one makes use of the discrete wavelet transform by restricting to discrete calculations grids for scale and position parameters such as the most used dyadic grid $a = 2^{-j}$ and $b = k2^{-j}$, $j, k \in \mathbb{Z}$. In this case, the wavelet copy $\psi_{a,b}$ is usually denoted by $\psi_{j,k}(x) = 2^{j/2} \psi(2^j x - k)$, $j, k \in \mathbb{Z}$.

The continuous wavelet transform of a signal $S$ will be known as the discrete wavelet transform (called also detail coefficient) expressed by

$$d_{j,k} = \int_{-\infty}^{\infty} \psi_{j,k}(t) S(t) dt \quad (2.3)$$

It holds that the set $(\psi_{j,k})_{j,k\in\mathbb{Z}}$ constitutes an orthonormal basis of $L^2(\mathbb{R})$ called wavelet basis and that a signal $S \in L^2(\mathbb{R})$ is decomposed according to this basis into a series

$$S(t) = \sum_{j=0}^{\infty} \sum_{k} d_{j,k} \psi_{j,k}(t) \quad (2.4)$$

called the wavelet series of $S$ which replaces the reconstruction formula for the CWT.

It holds in wavelet analysis of signals that the last series decomposition may be constructed using the so-called multiresolution analysis (MRA). It is a functional framework for representing signals in different levels called resolutions (see [49]). MRA is a countable set of closed subsets $(V_j)_{j\in\mathbb{Z}}$ (and $(W_j)_{j\in\mathbb{Z}}$) of $L^2(\mathbb{R})$ satisfying some special properties such as

$$V_{j+1} = V_j \oplus W_j \text{ and } L^2(\mathbb{R}) = \bigoplus_{j\in\mathbb{Z}} W_j. \quad (2.5)$$

In wavelet theory, it holds that $\psi \in W_0$ and that $(\psi_{j,k}(t) = 2^{j/2} \psi(2^j t - k))_k$ is an orthogonal basis of $W_j$. Besides there exists a function $\varphi \in V_0$ (called father wavelet or scaling function) such that $(\varphi_{j,k})_k$ is an orthogonal (generally Riesz) basis of $V_j$. The strongest point in MRA and wavelet theory (which is also a crucial point in signal analysis) is that $\varphi$ and $\psi$ leads each one to the other by means of the so-called 2-scale relation

$$\varphi(x) = \sqrt{2} \sum_k h_k \varphi(2x - k) \text{ with } h_k = \int_{\mathbb{R}} \varphi(x) \varphi(2x - k) dx \quad (2.6)$$

which yields that

$$\psi(x) = \sqrt{2} \sum_k g_k \varphi(2x - k) \text{ with } g_k = (-1)^k h_{1-k}.$$ (see [19,36,49]). The sequences $H = (h_k)_k$ and $G = (g_k)_k$ are called respectively the discrete low-pass and high-pass filters. They play a crucial role in
wavelet analysis of signals. Indeed, consider a signal $S$ and its approximation coefficients $a_{j,k}$ (relatively to $\varphi_{j,k}$) and details coefficients $d_{j,k}$. Using Eq. (2.6) we get for $j, k \in \mathbb{Z}$

$$a_{j,k} = \sum_l h_l a_{j+1,2k+l}, \quad d_{j,k} = \sum_l g_l a_{j+1,l+2k}$$

and

$$a_{j+1,n} = \sum_k h_{n-2k} a_{j,m} + \sum_k g_{n-2k} d_{j,m}.$$

This means that the approximation at level $j$ is obtained from the level $j+1$ by the intermediate of the filter $H$. The details at level $j$ are obtained from the level $j+1$ by the intermediate of the filter $G$. Furthermore, the approximation at the level $j+1$ may be obtained by superposing the two observations of approximation and details at the level $j$. Using these relations, a signal $S$ may be decomposed into a (splitted series) as

$$S = \sum_k a_{J,k} \varphi_{J,k} + \sum_{j=J}^{\infty} \sum_k d_{j,k} \psi_{j,k} \quad (2.7)$$

for any $J \in \mathbb{Z}$ known as the wavelet series decomposition of $S$ at the level $J$. The first sum belongs to $V_J$ and it reflects the global behavior of the signal $S$ and the second component relative to the $d_{j,k}$’s represents the details of $S$ and thus reflects the dynamic behavior of the signal. More about signal analysis with wavelets may be found in Ref. [19,36,49].

2.3. Uncertainty Principle Revisited

The uncertainty principle also known as Heisenberg’s uncertainty principle discovered in 1927 by Heisenberg in Ref. [37] and revisited next in Ref. [38] is one of the most famous and important concepts in quantum mechanics. It plays an important role in the development and understanding of quantum physics. The physical origin of uncertainty principle is related to quantum systems and states that: the determination of positions by performing measurement on the system disturbs it sufficiently to make the determination of momentum imprecise and vice-versa.

Using Fourier transform, the authors in Ref. [3] established a Stein–Weiss type inequality for the Riesz type potential generated by a Riemann–Liouville operator. Pitt’s and Beckner logarithmic uncertainty inequalities have been also proved. The same authors investigated in Ref. [4] a Hausdorff–Young inequality for the Fourier transform connected with Riemann–Liouville operator. Such inequality has been applied next to prove an entropy based uncertainty principle and a Heisenberg–Pauli–Weyl inequality (see also [44]). In Refs. [63,64], two types of uncertainty principle such as Heisenberg–Pauli–Weyl and Beurling–Hörmander have been established for the Fourier transform associated with the spherical mean operator in some local framework.

Using real wavelet transform, in Ref. [66] continuous wavelet transform associated with the spherical mean operator has been introduced yielding
a Plancherel formula as well as its inversion. Such findings have been applied next to prove an analogue of Heisenberg’s inequality for the introduced wavelet transform (see also [67,68]).

In Ref. [18] the continuous shearlet transform has been investigated to construct mother shearlet function applied next for an associated general uncertainty principle. Minimizers of such uncertainty have been also developed by means of the new wavelets.

El-Haoui et al. [27] introduced the quaternionic offset linear canonical transform and derived a relationship with the quaternion Fourier transform to establish next Plancherel like rules. These findings have been applied next to prove different uncertainty principles including Heisenberg-Weyls, Hardys, Beurlings and logarithmic ones in the case of the new framework. Recently El-Houi and Fahlaoui established in [28] several uncertainty inequalities in the real Clifford algebra $Cl(p, q)$ such as Hausdorf–Young inequality and qualitative uncertainty principles of Donoho-Stark.

In Ref. [29] expansions of signals with respect to Gabor wavelets and short time Fourier transform have been investigated. Using Heisenberg group techniques, stable iterative algorithms for signal analysis and synthesis have been developed. These algorithms have been shown to be convergent for a variety of norms. Besides, compatibility with the time-frequency localization of signals has been proved.

In Ref. [40], based on some similitude group, Hitzer introduced the local Clifford geometric algebra wavelet transform such as Clifford Gabor wavelets and proposed a generalized Clifford wavelet uncertainty principle. In Ref. [41] the same author derived a new directional uncertainty principle for quaternion valued functions by means of quaternion Fourier transformation generalized to the case of Clifford geometric algebras.

In Ref. [43] basic concepts of multivector functions and their vector derivative in geometric algebra have been introduced. Concepts of Fourier and Clifford transforms and some useful properties have been also investigated in the same framework of geometric algebras. An uncertainty principle has been next developed for many cases of Clifford wavelets and shown to be useful for signal processing.

In Ref. [42] a generalization of the Fourier transform in some Clifford geometric algebras has been extended and adopted for real Clifford geometric algebra Fourier transform. This has been applied next to define and prove the uncertainty principle for multivector functions in the new Clifford geometric algebras.

In Ref. [48] the quaternion ridgelet transform and curvelet transform associated to the quaternion Fourier transform have been investigated and applied to derive associated reconstruction formulae, reproducing kernels and uncertainty principles.

In Ref. [51] an uncertainty principle associated with the quaternion linear canonical transform has been proved by considering the fundamental relationship between such transform and the quaternion Fourier one. The new
principle has been applied to derive an inverse transform and both Parseval and Plancherel formulas associated with the quaternion linear canonical transform (see also [52,53] for the same authors and similar subject).

Mawardi and Hitzer proposed in Refs. [54–56] a construction of some Clifford-algebra-valued wavelets using the similitude groups in a special case. The new framework includes complex Gabor wavelets and extends them to multivectors Clifford Gabor wavelets. A new uncertainty principle for the Clifford Gabor wavelet transform has been proved in the new framework. Generalizations of these results have been conducted by the same authors in Ref. [57].

In Ref. [58] a right-sided quaternionic Fourier transform has been applied to establish an uncertainty principle. Such uncertainty principle has been shown to prescribe a lower bound on the product of the effective widths of quaternion-valued signals in the spatial and frequency domains. Furthermore, Gaussian quaternion signals have been shown to be the only ones minimizing the uncertainty. In Ref. [59] the continuous quaternion wavelet transform has been introduced with admissibility condition expressed by means of the right-sided quaternion Fourier transform. An application has been derived to establish a Heisenberg type uncertainty principle for the new extended wavelets.

In the same direction in Ref. [47] the linear canonical transform has been revisited and next generalized to quaternion-valued signals. Such generalization has been applied to establish an uncertainty principle in the new framework prescribing a lower bound on the product of the effective widths of quaternion-valued signals in the spatial and frequency domains. Gaussian quaternion signals have been shown here also to be the only ones minimizing the new uncertainty.

Recently Mejjaoli et al. [60] a continuous wavelet transform associated with the spherical mean operator relatively to some real parameters. Donoho-Stark and Benedicks-type uncertainty principles have been developed.

Yang et al. [77] investigated a stronger uncertainty principles in terms of covariance and absolute covariance based on Fourier transform in both directional and spatial cases for real para-vector-valued signals. Conditions of equality of the studied uncertainty principles have been discussed.

Finally, Yang and Kou [78] applied the so-called linear canonical transforms to extend the uncertainty principle for hypercomplex signals in the linear canonical transform domains. Minimizers have been shown to be Gaussian signals, which joins several works mentioned above.

Mathematically, the uncertainty principle states that a non-zero function and its Fourier transform cannot both be sharply localized. In quantum mechanics, for example, considering the position and the momentum of a particle, the uncertainty principle asserts that there is a fundamental limit to the precision with which it is possible to simultaneously know the position and the momentum of the particle. We say that these two variables are complementary.

The next theorem formally summarizes the Heisenberg’s uncertainty principle.
Theorem 2.2. (Uncertainty Principle [76]) Let $A$ and $B$ be two self-adjoint operators on a Hilbert space $X$ with domains $D(A)$ and $D(B)$ respectively and denote $[A, B] = AB - BA$ their commutator. Then

$$\|Af\|_2 \|Bf\|_2 \geq \frac{1}{2} |< [A, B] f, f >|, \forall f \in D([A, B]). \quad (2.8)$$

For more backgrounds on the uncertainty principle, its variants, Fourier and wavelet transforms on the Euclidian space $\mathbb{R}^n$ the readers may be refered also to [46,65,71,74].

3. Clifford Fourier and Clifford Wavelet Transforms

Clifford analysis constitutes an extension of harmonic analysis to higher dimensional Euclidean spaces. Signals and/or time series in the real case are extended to analytic ones in complex analysis. Furthermore these are extended to mongenic signals in Clifford case. These extensions are provided with corresponding tranforms such as Fourier one. See [1,2,5–9,12–14,21–23].

In this section we propose to review some basic concepts of the Clifford Fourier transform. Fore more details we may refer to [31] and [45]. The Clifford Fourier transform of a Clifford-valued function $f \in L^1 \cap L^2(\mathbb{R}^n, dV(x))$ is

$$\mathcal{F} [f](\xi) = \hat{f}(\xi) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-i<x, \xi>} f(x) dV(x).$$

It is an invertible transform and its inverse is

$$\mathcal{F}^{-1} [f](x) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{i<x, \xi>} \hat{f}(\xi) dV(\xi).$$

In the sequel, we shall use the two operators 

$$A_k f(x) = x_k f(x) \quad \text{and} \quad B_k f(x) = \partial_{x_k} f(x), \quad k = 1, 2, \ldots, n.$$ 

Using Theorem 2.2, we obtain

$$\|A_k f\|_{L^2(\mathbb{R}^n, dV(x))} \|B_k f\|_{L^2(\mathbb{R}^n, dV(x))} \geq \frac{1}{2} |< [A_k, B_k] f, f >|,$$

which reads otherwise as

$$\|x_k f\|_{L^2(\mathbb{R}^n, dV(x))} \|\xi_k \hat{f}\|_{L^2(\mathbb{R}^n, dV(x))} \geq \frac{1}{2} \|f\|^2_{L^2(\mathbb{R}^n, dV(x))}. \quad (3.1)$$

In the sequel we introduce the concept of the Clifford wavelet transform and some of its important properties to be used later. In this context, a function $\psi \in L^1 \cap L^2(\mathbb{R}^n, dV(x))$ will be considered as a Clifford mother wavelet. To join the admissibility assumptions in the case of wavelets on $\mathbb{R}$, here-also we assume that

- $\psi$ is a Clifford-algebra-valued function.
- $\hat{\psi}(\xi) \left[\hat{\psi}(\xi)\right]^{\dagger}$ is scalar.
- $A_\psi = (2\pi)^n \int_{\mathbb{R}^n} \frac{\hat{\psi}(\xi) \left[\hat{\psi}(\xi)\right]^{\dagger}}{|\xi|^n} dV(\xi) < \infty$. 
For \((a, b, s) \in \mathbb{R}_+ \times \mathbb{R}^n \times Spin(n),\) we denote
\[
\psi^{a,b,s}(x) = \frac{1}{a^{n/2}} s \psi \left( \frac{b \cdot s - x}{a} \right) s.
\]
It holds in fact that these copies are also admissible and that
\[
A_{\psi^{a,b,s}} = \frac{a^{n/2}}{(2\pi)^n} A_{\psi} < \infty.
\]

**Proposition 3.1.** The set \(\{\psi^{a,b,s}, a > 0, b \in \mathbb{R}^n, s \in Spin(n)\}\) is dense in the space \(L^2(\mathbb{R}^n, dV(x))\).

**Proof.** Let \(f\) be an analyzed function such that
\[
<\psi^{a,b,s}, f >_{L^2(\mathbb{R}^n, dV)} = 0, \forall a > 0, b \in \mathbb{R}^n \text{ and } s \in Spin(n).
\]
We shall prove that \(f = 0\). Due to the Parseval identity of the Clifford Fourier transform, we obtain
\[
<\psi^{a,b,s}, f >_{L^2(\mathbb{R}^n, dV)} = <\hat{\psi}^{a,b,s}, \hat{f} >_{L^2(\mathbb{R}^n, dV)} = 0.
\]
Since,
\[
<\psi^{a,b,s}, f >_{L^2(\mathbb{R}^n, dV)} = a^{n/2} \int_{\mathbb{R}^n} e^{i \cdot b \cdot \xi} \left[ \hat{\psi}(a \xi s) \right]^\dagger \bar{s} \hat{f}(\xi) dV(\xi),
\]
then,
\[
s \left[ \hat{\psi}(a \xi s) \right]^\dagger \bar{s} \hat{f}(\xi) = 0.
\]
Recall now that for a fixed \(\xi \neq 0\) in \(\mathbb{R}^n\),
\[
\{a \xi s, a > 0 \text{ and } s \in Spin(n)\} = \mathbb{R}^n.
\]
It results that
\[
\hat{f} = 0 \quad \text{and} \quad f \equiv 0.
\]

**Definition 3.2.** (Clifford Wavelet Transform) The Clifford wavelet transform of an analyzed function \(f \in L^2(\mathbb{R}^n, dV(x))\) with respect to the mother wavelet \(\psi\) is
\[
T_\psi[f](a, b, s) = \int_{\mathbb{R}^n} \left[ \psi^{a,b,s}(x) \right]^\dagger f(x) dV(x).
\]

**Definition 3.3.** (Inner product relation) Let \(\mathcal{H}_\psi = \{T_\psi[f], f \in L^2(\mathbb{R}^n, dV(x))\}\) be the image of \(L^2(\mathbb{R}^n, dV(x))\) relatively to the operator \(T_\psi\). We define the inner product by
\[
[T_\psi[f], T_\psi[g]] = \frac{1}{A_\psi} \int_{Spin(n)} \int_{\mathbb{R}^n} \int_{\mathbb{R}_+} (T_\psi[f](a, b, s))^\dagger T_\psi[g](a, b, s) \frac{da}{a^{n+1}} dV(b) ds
\]
where \(ds\) stands for the Haar measure on \(Spin(n)\).

**Proposition 3.4.** \(T_\psi : L^2(\mathbb{R}^n, dV(x)) \rightarrow \mathcal{H}_\psi\) is an isometry.
Proof. We have to show that

$$[T_\psi [f], T_\psi [g]] = \langle f, g \rangle_{L^2(\mathbb{R}^n, dV(x))}.$$  \hfill (3.2)

Put

$$\Phi_\psi(a, s, \xi) [f] (-b) = \left[ \hat{\psi}(a \xi s) \right]^\dagger \pi f(\xi) (-b)$$

and similarly

$$\Phi_\psi(a, s, \xi) [g] (-b) = \left[ \hat{\psi}(a \xi s) \right]^\dagger \pi g(\xi) (-b).$$

Hence we obtain

$$T_\psi [f] (a, b, s) = a^{\frac{n}{2}} s (2\pi)^{\frac{n}{2}} \Phi_\psi(a, \xi, s) [f] (-b)$$

and

$$T_\psi [g] (a, b, s) = a^{\frac{n}{2}} s (2\pi)^{\frac{n}{2}} \Phi_\psi(a, \xi, s) [g] (-b).$$

Applying Parseval formula we get

$$\langle \Phi_\psi(a, \cdot, s) [f], \Phi_\psi(a, \cdot, s) [g] \rangle = \langle \Phi_\psi(a, \cdot, s) [f], \Phi_\psi(a, \cdot, s) [g] \rangle.$$

Next,

$$[T_\psi [f], T_\psi [g]]$$

$$= \frac{1}{(2\pi)^n A_\psi} \int_{\text{Spin}(n)} \int_{\mathbb{R}^n} \left\{ \int_{\mathbb{R}^n} \left[ \hat{\psi}(a \xi s) \right]^\dagger \pi f(\xi) \left[ \hat{\psi}(a \xi s) \right]^\dagger \pi g(\xi) dV(\xi) \right\} \frac{da}{a} ds$$

$$= \frac{1}{(2\pi)^n A_\psi} \int_{\text{Spin}(n)} \int_{\mathbb{R}^n} \left\{ \int_{\mathbb{R}^n} \left[ \hat{\psi}(a \xi s) \right]^\dagger \pi f(\xi) \left[ \hat{\psi}(a \xi s) \right]^\dagger \pi g(\xi) dV(\xi) \right\} \frac{da}{a} ds$$

$$= \frac{1}{(2\pi)^n A_\psi} \int_{\mathbb{R}^n} \left[ \hat{f}(\xi) \right]^\dagger \left\{ \int_{\text{Spin}(n)} \int_{\mathbb{R}^n} \left[ \hat{\psi}(a \xi s) \right]^\dagger \pi \frac{da}{a} ds \right\} \pi g(\xi) dV(\xi).$$

Observing now that

$$\int_{\text{Spin}(n)} \int_{\mathbb{R}^n} \left[ \hat{\psi}(a \xi s) \right]^\dagger \pi \frac{da}{a} ds = \frac{A_\psi}{(2\pi)^n},$$ \hfill (3.3)

we get immediately

$$[T_\psi [f], T_\psi [g]] = \frac{1}{(2\pi)^n A_\psi} \int_{\mathbb{R}^n} \left[ \hat{f}(\xi) \right]^\dagger \left\{ \int_{\text{Spin}(n)} \int_{\mathbb{R}^n} \Gamma(t, \nu) dS(\nu) \frac{da}{t} \right\} \pi g(\xi) dV(\xi).$$
where we denoted Ω(t, ν) = \( \hat{\psi}(t\nu) [\hat{\psi}(t\nu)]^\dagger \). Otherwise, by taking \( u = t\nu \), we obtain

\[
[T_\psi [f], T_\psi [g]] = \frac{1}{(2\pi)^n A_\psi} \int_{\mathbb{R}^n} \left[ \hat{f}(\xi) \right]^\dagger \int_{\mathbb{R}^n} \frac{\hat{\psi}(u) [\hat{\psi}(u)]^\dagger}{|u|^n} dV(u) \hat{g}(\xi) dV(\xi)
\]

\[
= \int_{\mathbb{R}^n} \left[ \hat{f}(\xi) \right]^\dagger \hat{g}(\xi) dV(\xi)
\]

\[
= <\hat{f}, \hat{g}>
\]

(3.4)

As a result we get

\[
\int_{Spin(n) \mathbb{R}^n} \int_{\mathbb{R}_+} \|T_\psi [f] (a, b, s)\|^2 \frac{da}{a^{n+1}} dV(b) ds = A_\psi \|f\|^2_2.
\]

(3.5)

□

As a result of the last proposition and as in the real case, we have here a Clifford wavelet reconstruction formula.

**Proposition 3.5.** For all \( f \in L^2(\mathbb{R}^n, dV(x)) \) we have

\[
f(x) = \frac{1}{A_\psi} \int_{Spin(n) \mathbb{R}^n} \int_{\mathbb{R}_+} \int_{\mathbb{R}^n} \psi^{a,b,s}(x) T_\psi [f] (a, b, s) \frac{da}{a^{n+1}} dV(b) ds.
\]

4. Clifford Wavelet Uncertainty Principle

In this section, we establish and prove the Heisenberg uncertainty principle for the Clifford wavelet transform. Backgrounds may be found in Ref. [59].

**Theorem 4.1.** Let \( \psi \in L^2(\mathbb{R}^n, dV(x)) \) be an admissible Clifford mother wavelet. Then for \( f \in L^2(\mathbb{R}^n, dV(x)) \) the following inequality holds

\[
\left( \int_{Spin(n) \mathbb{R}^n} \int_{\mathbb{R}_+} \|b_k T [f] (a, \cdot, s)\|^2 \frac{da}{a^{n+1}} ds \right)^{\frac{1}{2}} \|\xi_k \hat{f}\|_2 \geq \frac{(2\pi)^{\frac{n}{2}}}{2} \sqrt{A_\psi} \|f\|^2_2,
\]

(4.1)

where \( k = 1, 2, \ldots, n \).

To prove this result we need the following lemma.

**Lemma 4.2.**

\[
\int_{Spin(n) \mathbb{R}^n} \int_{\mathbb{R}_+} \int_{\mathbb{R}^n} |\xi_k \hat{T}[f](a, \xi, s)|^2 dV(\xi) \frac{da}{a^{n+1}} ds = \frac{A_\psi}{(2\pi)^n} \|\xi_k \hat{f}\|^2_2.
\]
Proof. As \( \hat{\psi}^\alpha \hat{b} s(\xi) = a^2 \dot{e}^{-i\hat{b} \hat{\xi} s} \hat{\psi}(a \bar{s} \xi s) \bar{s} \), we get

\[
T[f](a, \bar{b}, s) = a^2 \mathcal{F}^{-1} \left[ \bar{s} \left( \hat{\psi}(a \bar{s} \xi s) \right)^\dagger \right] (b)
\]

and

\[
\hat{T}[f](a, \xi, s) = a^2 \bar{s} \left[ \hat{\psi}(a \bar{s} \xi s) \right]^\dagger s \hat{f}(\xi). \tag{4.2}
\]

Therefore,

\[
\int_{\mathbb{R}^n} \left| \xi_k \hat{T}[f](a, \xi, s) \right|^2 dV(\xi) \tag{4.3}
\]

\[
= \int_{\mathbb{R}^n} \left| \xi_k a^2 \bar{s} \left[ \hat{\psi}(a \bar{s} \xi s) \right]^\dagger s \hat{f}(\xi) \right|^2 dV(\xi)
\]

\[
= \int_{\mathbb{R}^n} \xi_k a^2 \bar{s} \left[ \hat{\psi}(a \bar{s} \xi s) \right]^\dagger s \hat{f}(\xi) \left\{ \xi_k a^2 \bar{s} \left[ \hat{\psi}(a \bar{s} \xi s) \right]^\dagger s \hat{f}(\xi) \right\}^\dagger dV(\xi)
\]

\[
= \int_{\mathbb{R}^n} \xi_k^2 a^n \bar{s} \left[ \hat{\psi}(a \bar{s} \xi s) \right]^\dagger s \hat{f}(\xi) \left\{ \bar{s} \left[ \hat{\psi}(a \bar{s} \xi s) \right]^\dagger s \hat{f}(\xi) \right\}^\dagger dV(\xi)
\]

\[
= \int_{\mathbb{R}^n} \xi_k^2 a^n \left\{ \left[ \hat{\psi}(a \bar{s} \xi s) \right]^\dagger \hat{\psi}(a \bar{s} \xi s) \right\} \left\{ \hat{f}(\xi) \left[ \hat{f}(\xi) \right]^\dagger \right\} dV(\xi). \tag{4.4}
\]

Recall next that whenever \( s \in \text{Spin}(n) \), \( s = \omega_1 \omega_2 \cdots \omega_{2l} \) for some \( l \in \mathbb{N} \) with \( \omega_j^2 = -1 \) we get

\[
s \bar{s} = \omega_2 \omega_{2l-1} \cdots \omega_1 \omega_2 \cdots \omega_{2l} = (-1)^{2l} = 1.
\]

Consequently, using (4.4) we obtain

\[
\int_{\text{Spin}(n)} \int_{\mathbb{R}^n} \int \left| \xi_k \hat{T}[f](a, \xi, s) \right|^2 dV(\xi) \frac{da}{a^{n+1}} ds
\]

\[
= \int_{\text{Spin}(n)} \int_{\mathbb{R}^n} \int \xi_k^2 a^n \left\{ \left[ \hat{\psi}(a \bar{s} \xi s) \right]^\dagger \hat{\psi}(a \bar{s} \xi s) \right\} \left\{ \hat{f}(\xi) \left[ \hat{f}(\xi) \right]^\dagger \right\} dV(\xi) \frac{da}{a^{n+1}} ds
\]

\[
= \int_{\text{Spin}(n)} \int_{\mathbb{R}^n} \int \xi_k^2 \left\{ \left[ \hat{\psi}(a \bar{s} \xi s) \right]^\dagger \hat{\psi}(a \bar{s} \xi s) \right\} \left\{ \hat{f}(\xi) \left[ \hat{f}(\xi) \right]^\dagger \right\} dV(\xi) \frac{da}{a} ds
\]

\[
= \int_{\mathbb{R}^n} \left\{ \int_{\text{Spin}(n)} \int \frac{\left[ \hat{\psi}(a \bar{s} \xi s) \right]^\dagger \hat{\psi}(a \bar{s} \xi s)}{a} dV(\xi) \right\} \xi_k \left\{ \hat{f}(\xi) \left[ \hat{f}(\xi) \right]^\dagger \right\} dV(\xi).
\]
Using (3.3), we obtain
\[\int_{\text{Spin}(n) \times \mathbb{R}_+} \left\| \xi_k \hat{T}[f](a, \xi, s) \right\|^2 dV(\xi) \frac{da}{a^{n+1}} ds = \frac{A_\psi}{(2\pi)^n} \left\| \xi_k \hat{f} \right\|^2_2.\]

\[\square\]

**Proof of Theorem 4.1.** Using (3.1) and setting \( \mathbf{x} = \mathbf{b} \in \mathbb{R}^n \), we deduce that
\[\| b_k T[f](a, \cdot, s) \|_2 \left\| \xi_k \hat{T}[f](a, \cdot, s) \right\|_2 \geq \frac{1}{2} \| T[f](a, \cdot, s) \|_2^2.\]

Therefore
\[\int_{\text{Spin}(n) \times \mathbb{R}_+} \| b_k T[f](a, \cdot, s) \|_2 \left\| \xi_k \hat{T}[f](a, \cdot, s) \right\|_2 \frac{da}{a^{n+1}} ds \geq \frac{1}{2} \int_{\text{Spin}(n) \times \mathbb{R}_+} \| T[f](a, \cdot, s) \|_2^2 \frac{da}{a^{n+1}} ds.\]

According to the Cauchy–Schwartz inequality (2.2), it follows that
\[\int_{\text{Spin}(n) \times \mathbb{R}_+} \| b_k T[f](a, \cdot, s) \|_2^2 \frac{da}{a^{n+1}} ds \times \int_{\text{Spin}(n) \times \mathbb{R}_+} \left\| \xi_k \hat{T}[f](a, \cdot, s) \right\|^2_2 \frac{da}{a^{n+1}} ds \geq \left( \frac{1}{2} \int_{\text{Spin}(n) \times \mathbb{R}_+ \times \mathbb{R}^n} \| T[f](a, \mathbf{b}, s) \|^2 dV(\mathbf{b}) \frac{da}{a^{n+1}} ds \right)^2. \quad (4.5)\]

Now, using Lemma 4.2 and the fact that the wavelet-transform is an isometry, we get from (3.5)
\[\int_{\text{Spin}(n) \times \mathbb{R}_+ \times \mathbb{R}^n} (T_\psi[f](a, \mathbf{b}, s))^2 \frac{da}{a^{n+1}} dV(\mathbf{b}) ds = A_\psi \| f \|_2^2.\]

The inequality (4.5) becomes
\[\left( \int_{\text{Spin}(n) \times \mathbb{R}_+} \| b_k T[f](a, \cdot, s) \|_2^2 \frac{da}{a^{n+1}} ds \right)^{\frac{1}{2}} \times \left( \frac{A_\psi}{(2\pi)^n} \left\| \xi_k \hat{f} \right\|^2_2 \right)^{\frac{1}{2}} \geq \frac{1}{2} A_\psi \| f \|_2^2. \quad (4.6)\]

Hence, we obtain
\[\left( \int_{\text{Spin}(n) \times \mathbb{R}_+} \| b_k T[f](a, \cdot, s) \|_2^2 \frac{da}{a^{n+1}} ds \right)^{\frac{1}{2}} \left\| \xi_k \hat{f} \right\|_2 \geq \frac{(2\pi)^{\frac{n}{2}}}{2} \sqrt{A_\psi} \| f \|_2. \quad (4.7)\]

\[\square\]
5. Conclusion

In this paper, an uncertainty principle associated with the continuous wavelet transform in the Clifford algebra’s settings has been formulated and proved. Starting from the definition of real Clifford algebra and the real continuous wavelet transform, we have presented a continuous Clifford wavelet transform, displayed its properties and formulated an associated uncertainty principle.

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References

[1] Abreu-Blaya, R., Bory-Reyes, J., Bosch, P.: Extension theorem for complex Clifford algebras-valued functions on fractal domains. Bound. Value Probl. (2010). https://doi.org/10.1155/2010/513186
[2] Almeida, J.B.: Can physics laws be derived from monogenic functions? arXiv: physics/0601194 v1 (2006)
[3] Amri, B., Rachdi, L.T.: Beckner logarithmic uncertainty principle for the Riemann–Liouville operator. Int. J. Math. 24(09), 1350070 (2013). https://doi.org/10.1142/s0129167x13500705
[4] Amri, B., Rachdi, L.T.: Uncertainty principle in terms of entropy for the Riemann–Liouville operator. Bull. Malays. Math. Sci. Soc. 39(1), 457–481 (2015). https://doi.org/10.1007/s40840-015-0121-5
[5] Brackx, F., Chisholm, J.S.R., Soucek, V.: Clifford Analysis and Its Applications. NATO Science Series, Series II: Mathematics, Physics and Chemistry, vol. 25. Springer, New York (2000)
[6] Brackx, F., Delanghe, R., Sommen, F.: Clifford Analysis. Pitman Publication, Trowbridge (1982)
[7] Brackx, F., De Schepper, N., Sommen, F.: The two-dimensional Clifford Fourier transform. J. Math. Imaging 26, 5–18 (2006)
[8] Brackx, F., De Schepper, N., Sommen, F.: The Fourier transform in Clifford analysis. Adv. Imaging Electron Phys. 156, 55–201 (2009)
[9] Brackx, F., De Schepper, N., Sommen, F.: Clifford–Hermite and two-dimensional Clifford–Gabor filters for early vision. In: Gürlebeck, K., Könke, C. (eds.) (digital) Proceedings 17th International Conference on the Application of Computer Science and Mathematics in Architecture and Civil Engineering. Bauhaus-Universität Weimar, Weimar (2006). July 12–14
[10] Brackx, F., Hitzer, E., Sangwine, S.J.: History of quaternion and Clifford-fourier transforms and wavelets, in Quaternion and Clifford fourier transforms and wavelets. In: Trends in Mathematics, vol. 27, pp. XI–XXVII (2013)
[11] Brackx, F., Sommen, F.: The continuous wavelet transform in Clifford analysis. In: Clifford Analysis and Its Applications, pp. 9–26. Springer Netherlands, Dordrecht (2001)
[12] Bujack, R., De Bie, H., De Schepper, N., Scheuermann, G.: Convolution products for hypercomplex Fourier transforms. J. Math. Imaging Vis. (2013). https://doi.org/10.1007/s10851-013-0430-y
[13] Bujack, R., Scheuermann, G., Hiter, E.: A general geometric Fourier transform. In: Hitzer, E., Sangwine, S.J. (eds.) Quaternion and Clifford Fourier Transforms and Wavelets, Trends in Mathematics, vol. 27, pp. 155-176. Birkhäuser, Basel. (2013). https://doi.org/10.1007/978-3-0348-0603-9_8

[14] Bujack, R., Scheuermann, G., Hiter, E.: A general geometric fourier transform convolution theorem. Adv. Appl. Clifford Algebra 23(1), 15–38 (2013). https://doi.org/10.1007/s00006-012-0338-4

[15] Carré, P., Berthier, M.: Chapter 6, Color representation and processes with Clifford algebra. In: Fernandez-Maloigne, C. (ed.) Advanced Color Image Processing and Analysis, pp. 147–179. Springer, New York (2013)

[16] Cartan, E.: The theory of spinors. Courier Corporation (1966)

[17] Clifford, W.K.: On the classification of geometric algebras. Mathematical Papers, pp. 397–401 (1882)

[18] Dahkle, S., Kutyniok, G., Maass, P., Sagiv, C., Stark, H.-G., Teschke, G.: The uncertainty principle associated with the continuous sheartlet transform. Int. J. Wavelets Multiresolut. Inf. Process. 6(2), 157–181 (2008)

[19] Daubechies, I.: Ten Lectures on Wavelets, Society for Industrial and Applied mathematics, Philadelphia, PA, USA (1992)

[20] Debnath, L., Shah, F.A.: Wavelet Transforms and Their Applications. Birkhäuser, Boston (2015)

[21] Delanghe, R.: Clifford analysis: history and perspective. Comput. Methods Funct. Theory 1(1), 107–153 (2001)

[22] De Bie, H.: Clifford algebras, Fourier transforms and quantum mechanics. arXiv: 1209.6434v1 (2012)

[23] De Bie, H., Xu, Y.: On the Clifford Fourier transform. ArXiv: 1003.0689 (2010)

[24] De Schepper, N.: Multi-dimensional continuous wavelet transforms and generalized Fourier transforms in Clifford analysis. PhD thesis, Ghent University (2006)

[25] Dian Tunjung, N., Zainal Arifin, A., Soelaiman, R.: Medical image segmentation using generalized gradient vector flow and Clifford geometric algebra. In: International Conference on Biomedical Engineering, Surabaya, Indonesia, November 11 (2008)

[26] Dirac, P.A.M.: The quantum theory of the electron. In: Proceedings of the Royal Society of London. Series A, Containing Papers of a Mathematical and Physical Character, vol. 117(778), pp. 610–624 (1928)

[27] El Haoui, Y., Fahlaoui, S., Hiter, E.: Generalized uncertainty principles associated with the quaternionic offset linear canonical transform. arXiv:1807.04068v2 [math.CA] (2019)

[28] El Haoui, Y., Fahlaoui, S.: Donoho–Stark’s uncertainty principles in real Clifford algebras. arXiv:1902.08465v1 [math.CA] (2019)

[29] Feichtinger, H.G., Gröchenig, K.: Gabor wavelets and the heisenberg group: gabor expansions and short time fourier transform from the group theoretical point of view. Wavelets (1992). https://doi.org/10.1016/b978-0-12-174590-5.50018-6

[30] Fletcher, P.: Discrete wavelets with quaternion and Clifford coefficients. Adv. Appl. Clifford Algebras 28, 59 (2018). https://doi.org/10.1007/s00006-018-0876-5
[31] Fu, Y., Li, L.: Uncertainty principle for multivector-valued functions. Int. J. Wavelets Multiresolut. Inf. Process. 13(01), 1–8 (2015)

[32] Grossmann, A., Morlet, J., Paul, T.: Transforms associated to square integrable group representations. II : examples. Annales de l’IHP Physique théorique 45(3), 293–309 (1986)

[33] Grossmann, A., Morlet, J.: Decomposition of hardy functions into square integrable wavelets of constant shape. SIAM J. Math. Anal. 15, 723–736 (1984)

[34] Hamilton, W.R.: On a new species of imaginary quantities connected with a theory of quaternions. Proc. R. Ir. Acad. 2, 424–434 (1844)

[35] Hamilton, W.R.: Elements of Quaternions. Longmans, Green, & Company, Harlow (1866)

[36] Hardle, W., Kerkyacharian, G., Picard, D., Tsybakov, A.: Wavelets, approximation and statistical applications. Seminar Berlin-Paris (1997)

[37] Heisenberg, W.: Über den anschaulichen inhalt der quantentheoretischen kinematik und mechanik. Zeitschrift für Physik 43, 172–198 (1927)

[38] Heisenberg, W.: Über den anschaulichen inhalt der quantentheoretischen kinematik und mechanik. In: Original Scientific Papers Wissenschaftliche Origin- nalarbeiten, pp. 478–504, Springer, Berlin (1985)

[39] Hitzer, E.: New Developments in Clifford Fourier Transforms. In: Mastorakis, N.E., Pardalos, P.M., Agarwal, R.P., Kocinac, L. (eds.), Advances in Applied and Pure Mathematics, Proceedings of the 2014 International Conference on Pure Mathematics, Applied Mathematics, Computational Methods (PMAMCM 2014), Santorini Island, Greece, July 17-21, 2014, Mathematics and Computers in Science and Engineering Series, vol. 29, pp. 19–25 (2014)

[40] Hitzer, E.: Clifford (geometric) algebra wavelet transform. In: Skala, V., Hildenbrand, D. (eds.) Proc. of GraVisMa 2009, 02-04 Sep. 2009, Plzen, Czech Republic, pp. 94–101 (2009)

[41] Hitzer, E.: Directional uncertainty principle for quaternion Fourier transform. Adv. Appl. Clifford Algebra 20, 271–284 (2010). https://doi.org/10.1007/s00006-009-0175-2

[42] Hitzer, E., Mawardi, B.: Uncertainty principle for the Clifford-geometric algebra $C_{3,0}$ based on Clifford Fourier transform. arXiv:1306.2089v1 [math.RA] (2013)

[43] Hitzer, E.: Tutorial on Fourier transformations and wavelet transformations in Clifford geometric algebra. In: Tachibana, K. (ed.) Lecture notes of the International Workshop for Computational Science with Geometric Algebra (FCSGA2007), Nagoya University, Japan, pp. 65–87 (2007)

[44] Hleili, K., Omri, Š., Rachdi, L.T.: Uncertainty principle for the Riemann–Liouville operator. CUBO Math. J. 13(03), 91–115 (2011)

[45] Jday, R.: Heisenberg’s and Hardy’s uncertainty principles in real Clifford algebras. Integral Transforms Spec. Funct. 29(8), 663–677 (2018)

[46] Jorgensen, P., Tian, F.: Non-commutative Analysis. World Scientific Publishing Company, Singapore (2017)

[47] Kou, K.I., Ou, J.-Y., Morais, J.: On uncertainty principle for quaternionic linear canonical transform. Abstract and Applied Analysis. (2013). https://doi.org/10.1155/2013/725952 (Article ID 725952)
[48] Ma, G., Zhao, J.: Quaternion ridgelet transform and curvelet transform. Adv. Appl. Clifford Algebras 28, 80 (2018). https://doi.org/10.1007/s00006-018-0897-0

[49] Mallat, S.: Une exploration des signaux en ondelettes. ISBN 2-7302-0733-3. Les Editions de l’Ecole Polytechnique (2000)

[50] Mawardi, B.: Construction of quaternion-valued wavelets. Matematika 26(1), 107–114 (2010)

[51] Mawardi, B., Ryuichi, A.: A simplified proof of uncertainty principle for quaternion linear canonical transform. Abstract and Applied Analysis. (2016). https://doi.org/10.1155/2016/5874930 (Article ID 5874930)

[52] Mawardi, B., Ashino, R.: Logarithmic uncertainty principle for quaternion linear canonical transform. In: Proceedings of the 2016 International Conference on Wavelet Analysis and Pattern Recognition, Jeju, South Korea (2016)

[53] Mawardi, B., Ashino, R.: A variation on uncertainty principle and logarithmic uncertainty principle for continuous quaternion wavelet transforms. Abstract and Applied Analysis. (2017). https://doi.org/10.1155/2017/3795120 (Article ID 3795120)

[54] Mawardi, B., Hitzer, E.: Clifford algebra $\mathbb{C}l(3,0)$-valued wavelets and uncertainty inequality for clifford gabor wavelet transformation. In: Preprints of Meeting of the Japan Society for Industrial and Applied Mathematics, ISSN: 1345-3378, Tsukuba University. Tsukuba, Japan, pp. 64–65 (2006)

[55] Mawardi, B., Hitzer, E.: Clifford algebra $\mathbb{C}l(3,0)$-valued wavelet transformation, Clifford wavelet uncertainty inequality and Clifford Gabor wavelets. Int. J. Wavelets Multiresolut. Inf. Process. 5(6), 997–1019 (2007)

[56] Mawardi, B., Hitzer, E.: Clifford Fourier transformation and uncertainty principle for the Clifford geometric algebra $\mathbb{C}l_{3,0}$. Adv. Appl. Clifford Algebra 16:41–61. (2006). https://doi.org/10.1007/s00006-006-0003-x

[57] Mawardi, B., Hitzer, E.: Clifford Fourier transform on multivector fields and uncertainty principles for dimensions $n = 2(\text{mod}4)$ and $n = 3(\text{mod}4)$. Adv. Appl. Clifford Algebra 18, 715–736 (2008). https://doi.org/10.1007/s00006-008-0098-3

[58] Mawardi, B., Hitzer, E., Hayashi, A., Ashino, R.: An uncertainty principle for quaternion Fourier transform. Comput. Math. Appl. 56, 2398–2410 (2008)

[59] Mawardi, B., Ashino, R., Vaillancourt, R.: Two-dimensional quaternion wavelet transform. Appl. Math. Comput. 218, 10–21 (2011)

[60] Mejjaoli, H., Ben Hamadi, N., Omri, S.: Localization operators, time frequency concentration and quantitative-type uncertainty for the continuous wavelet transform associated with spherical mean operator. Int. J. Wavelets Multiresolut. Inf. Process. 17(04):1950022. (2019). https://doi.org/10.1142/S02191691950022X

[61] Meyer, Y.: Ondelettes et fonctions splines et analyses graduées. Lectures given at the University of Torino, Italy, 9 (1986)

[62] Morlet, J., Arens, G., Fourgeau, E., Giard, D.: Wave propagation and sampling theory? Part ii: sampling theory and complex waves. Geophysics 47(2), 222–236 (1982)

[63] Msehli, N., Rachdi, L.T.: Heisenberg–Pauli–Weyl uncertainty principle for the spherical mean operator. Mediterr. J. Math. 7(2), 169–194 (2010). https://doi.org/10.1007/s00009-010-0044-1
[64] Msehli, N., Rachdi, L.T.: Beurling-Hörmander uncertainty principle for the spherical mean operator. J. Inequal. Pure Appl. Math. 10(2), 38 (2009)

[65] Nagata, K., Nakamura, T.: Violation of Heisenberg’s uncertainty principle. Open Access Libr. J. 2, e1797 (2015). https://doi.org/10.4236/oalib.1101797

[66] Rachdi, L.T., Meherzi, F.: Continuous wavelet transform and uncertainty principle related to the spherical mean operator. Mediterr. J. Math. (2016). https://doi.org/10.1007/s00009-016-0834-1

[67] Rachdi, L.T., Amri, B., Hammami, A.: Uncertainty principles and time frequency analysis related to the Riemann–Liouville operator. Annali Dell’Università Di Ferrara. (2018). https://doi.org/10.1007/s11565-018-0311-9

[68] Rachdi, L.T., Herch, H.: Uncertainty principles for continuous wavelet transforms related to the Riemann–Liouville operator. Ricerche Di Matematica 66(2), 553–578 (2017). https://doi.org/10.1007/s11587-017-0320-5

[69] Rizo-Rodríguez, D., Mendez-Vazquez, H., Garcia-Reyes, E.: Illumination invariant face recognition using quaternion-based correlation filters. J. Math. Imaging Vis. 45, 164–175 (2013)

[70] Sau, K., Basaka, R.K., Chanda, A.: Image compression based on block truncation coding using Clifford algebra. In: International Conference on Computational Intelligence: Modeling Techniques and Applications (CIMTA), Procedia Technology, vol. 10, pp. 699–706 (2013)

[71] Sen, D.: The uncertainty relations in quantum mechanics. Curr. Sci. 107(2), 203–218 (2018)

[72] Sommen, F, De Schepper, H.: Introductory Clifford analysis, pp. 1–27. Basel (2015)

[73] Soulard, R., Carré, P.: Characterization of color images with multiscale monogenic maxima. IEEE Trans. Pattern Anal. Mach. Intell. 40(10), 2289–2302 (2018)

[74] Stabnikov, P.A.: Geometric interpretation of the uncertainty principle. Nat. Sci. 11(5), 146–148 (2019)

[75] Wietzke, L., Sommer, G.: The signal multi-vector. J. Math. Imaging Vis. 37, 132–150 (2010)

[76] Weyl, H.: The Theory of Groups and Quantum Mechanics, second edn. Dover, New York (1950)

[77] Yang, Y., Dang, P., Qian, T.: Stronger uncertainty principles for hypercomplex signals. Complex Var. Elliptic Equ. 60(12), 1696–1711 (2015). https://doi.org/10.1080/17476933.2015.1041938

[78] Yang, Y., Kou, K.I.: Uncertainty principles for hypercomplex signals in the linear canonical transform domains. Signal Process. 95, 67–75 (2014)

[79] Zou, C., Kou, K. I.: Hypercomplex signal energy concentration in the spatial and quaternionic linear canonical frequency domains, arXiv preprint arXiv:1609.00890 (2016)
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