Noncommutative localization

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Abstract

The main result of this note says that the Cuntz-Krieger algebras are local rings from a standpoint of noncommutative geometry. The fact is established by localization of a functor between the elliptic curves with complex multiplication and the noncommutative tori with real multiplication.

Key words and phrases: elliptic curve, noncommutative torus

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1 Introduction

A. The noncommutative tori. Let $0 < \theta < 1$ be an irrational number. Recall, that noncommutative torus $\mathcal{A}_\theta$ is a universal $C^*$-algebra, generated by the unitaries $u, v$ satisfying the commutation relation $vu = e^{2\pi i \theta} uv$. It is useful to add to $\mathcal{A}_\theta$ its Effros-Shen algebra $\mathbb{A}_\theta$:

\[
\begin{array}{c}
a_0 & a_1 & \ldots \\
\end{array}
\]

\[\theta = a_0 + \frac{1}{\frac{1}{a_1} + \frac{1}{a_2} + \ldots}\]

Figure 1: The Bratteli diagram of an Effros-Shen algebra.

The $K$-theory cannot (essentially) distinguish between the $\mathcal{A}_\theta$ and $\mathbb{A}_\theta$ [13]: either do we by using the same name – noncommutative torus, and the same

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notation – $\kappa_\theta$ – in the both cases. Recall, that the $\kappa_\theta, \kappa_\theta'$ are said to be *stably isomorphic*, whenever $\kappa_\theta \otimes K \cong \kappa_\theta' \otimes K$, where $K$ is the $C^*$-algebra of compact operators. It is a beautiful and deep fact, that the $\kappa_\theta, \kappa_\theta'$ are stably isomorphic, if and only if, $\theta' \equiv \theta \mod SL(2, \mathbb{Z})$, i.e. $\theta' = (a\theta + b)/(c\theta + d)$, where $a, b, c, d \in \mathbb{Z}$ and $ad - bc = 1$ [4]. The $\kappa_\theta$ is said to have a real *multiplication*, whenever $\theta$ is a quadratic irrationality [7]. Such a property signals a fact that the ring $\text{End}(\kappa_\theta)$ exceeds the ring $\mathbb{Z}$; we shall denote such a case by $\kappa_{RM}$. Any $\kappa_{RM}$ has a periodic Bratteli diagram with the incidence matrix $A$. The latter is connected to $\theta$ by an explicit formula $A = \prod_{i=1}^{n}(a_i, 1, 1, 0)$, where $\theta = [a_1, \ldots, a_n]$ is a purely periodic fraction. (Here a line notation for the matrices and the Lagrange’s Theorem for the continued fractions have been tacitly used.)

B. The Teichmüller functor. The noncommutative tori make up a boundary of the space of complex elliptic curves. To give an idea, let $H = \{x + iy \in \mathbb{C} \mid y > 0\}$ be the upper half-plane and $\mathbb{C}/(\mathbb{Z} + \tau \mathbb{Z})$, $\tau \in H$ a complex torus; we routinely identify the latter with a non-singular elliptic curve via the Weierstrass $\wp$ function. It is classically known, that two complex tori are isomorphic, whenever $\tau' \equiv \tau \mod SL(2, \mathbb{Z})$. The action of the group $SL_2(\mathbb{Z})$ on $H$ extends to the boundary $\partial H = \{y = 0 \mid x + iy \in \mathbb{C}\}$, where it coincides with the stable isomorphisms of the noncommutative tori. Here the irrational points $x$ of the boundary are identified with the noncommutative tori by the formula $x \mapsto \theta = |x|/(1 + |x|)$. It is remarkable, that this correspondence is categorical: there exists a covariant functor $F$ (the Teichmüller functor), which maps isomorphic elliptic curves to the stably isomorphic noncommutative tori. In what follows, we restrict to a special family of elliptic curves of the following nature. Recall, that if $\tau$ is imaginary and quadratic, the elliptic curve is said to have a complex *multiplication*. The latter is equivalent to the condition, that the ring of endomorphisms of $\mathbb{C}/(\mathbb{Z} + \tau \mathbb{Z})$ exceeds the ring $\mathbb{Z}$. The elliptic curves with complex multiplication are fundamental in arithmetic algebraic geometry; we shall denote such curves by $E_{CM}$. It is a beautiful and important fact, that the Teichmüller functor maps the $E_{CM}$’s into the $\kappa_{RM}$’s – the noncommutative tori with real multiplication. In fact, the functor $F$ establishes an equivalence between the two categories; under the equivalence, any arithmetical property of the $E_{CM}$ is reflected (and looks simpler!) in terms of the corresponding $\kappa_{RM}$. A correspondence between the complex and noncommutative tori has been reported by Connes-Douglas-Schwarz [1], Kontsevich [5], Manin [6] and others; it was
anticipated, that the correspondence is a functor between the two categories, yet with no proof. The proof was eventually given in [8] and extended to the higher genus Riemann surfaces in [9]; in both cases a compactification of the Teichmüller space by the projective classes of measured foliations has been used [15]. The fact that the complex multiplication transforms into a real multiplication was (implicitly) conjectured in [7] and proved in [10]. It was admitted, that “... a comprehensive revision from the noncommutative point of view of everything we used to think about in the commutative terms is required...” [6], p.16.

C. The noncommutative reciprocity. Let $L(E_{CM}, s)$ be the Hasse-Weil $L$-function ([14]) of an elliptic curve $E_{CM}$ and $\mathbb{A}_{RM} = F(E_{CM})$ an image of the $E_{CM}$ under the Teichmüller functor. Denote by

$$\zeta_p(\mathbb{A}_{RM}, z) := \exp \left( \sum_{n=1}^{\infty} \frac{|K_0(\mathcal{O}_{\varepsilon_n})|}{n} z^n \right), \quad \varepsilon_n = \begin{cases} L_p^n, & \text{if } p \nmid tr^2(A) - 4 \\ 1 - \alpha^n, & \text{if } p \mid tr^2(A) - 4 \end{cases},$$

a local zeta function of the noncommutative torus $\mathbb{A}_{RM}$; here $L_p = (tr(A^p), p, -1, 0)$ is an integer matrix, $\alpha \in \{-1, 0, 1\}$ and $\mathcal{O}_{\varepsilon_n}$ a Cuntz-Krieger algebra given by the matrix $\varepsilon_n$ [2]. An infinite product $L(\mathbb{A}_{RM}, s) = \prod_p \zeta_p(\mathbb{A}_{RM}, p^{-s})$ will be called an $L$-function of the torus $\mathbb{A}_{RM}$. Our starting point is the following

**Lemma 1 ([11])** $L(E_{CM}, s) \equiv L(\mathbb{A}_{RM}, s)$.

D. An objective. Let $p$ be a prime number. Let $E_{CM}(\mathbb{F}_p)$ be a localization of the $E_{CM}$ at the prime ideal $\mathfrak{p}$ over $p$. Where does the $E_{CM}(\mathbb{F}_{p^n})$ go under the Teichmüller functor? In other words, what is a proper notion of localization for the (noncommutative) ring $\mathbb{A}_{RM}$? To answer this question, let us compare the local zetas for the $L(E_{CM}, s)$ and $L(\mathbb{A}_{RM}, s)$; it follows from lemma [1] that for every $n \geq 1$ it holds $|E_{CM}(\mathbb{F}_{p^n})| = |K_0(\mathcal{O}_{\varepsilon_{n}})|$ [3]. This simple observation tells us, that the $\mathcal{O}_{\varepsilon_n}$ is a localization (at a prime $p$) of the ring $\mathbb{A}_{RM}$; in other words, the Cuntz-Krieger algebra $\mathcal{O}_{\varepsilon_n}$ is a local ring from a standpoint of noncommutative geometry. Our modest goal is to address the following elementary, but important

**Correctness question.** To show that the assignment $F(p) : E_{CM}(\mathbb{F}_{p^n}) \mapsto \mathcal{O}_{\varepsilon_n}$ does not depend on the choice of a representative in the isomorphism class of the elliptic curve $E_{CM}(\mathbb{F}_{p^n})$; see Fig.2.

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1In fact, a stronger result is true: the $E_{CM}(\mathbb{F}_{p^n}) \cong K_0(\mathcal{O}_{\varepsilon_n})$ are isomorphic as abelian groups [11].
E. The result. We shall denote by $E(p)$ a category of all elliptic curves with complex multiplication, reduced modulo $p$; the arrows (morphisms) of $E(p)$ are isomorphisms of the elliptic curves over the algebraic closure $\mathbb{F}_{p^\infty}$ of the prime field $\mathbb{F}_p$. Likewise, let $\mathcal{O}(p)$ be an image of the set $E(p)$ under the map $F(p) : E_{CM}(\mathbb{F}_{p^n}) \rightarrow \mathcal{O}_{\varepsilon_n}$; the arrows of $\mathcal{O}(p)$ are trivial isomorphisms (the identities) of the Cuntz-Krieger algebras $\mathcal{O}_{\varepsilon_n}$. Our main result can be formulated as follows.

**Theorem 1** The map $F(p) : E(p) \rightarrow \mathcal{O}(p)$ is a functor, which maps every pair of isomorphic elliptic curves into a pair of trivially isomorphic Cuntz-Krieger algebras; see Fig.2.

The functor $F(p)$ can be viewed as a localization at a prime $p$ of the Teichmüller functor $F$, restricted to the elliptic curves with complex multiplication; hence the title of our note. The approach is in line with a philosophy of [12].

2 Proof of theorem 1

Case 1. $p \nmid tr^2(A) - 4$ (a good prime). It was shown ([11], lemma 9), that in this case the elliptic curve $E_{CM}(K)$ has a good reduction at $p$.

**Lemma 2** ([3], Prop. 8.4.2) Let $E(K)$ and $E'(K)$ be a pair of isomorphic elliptic curves over the algebraic field $K$, where $\text{char } (K) = 0$; let $p$ be a prime number, such that $E(K)$ and $E'(K)$ have a good reduction at $p$. Then the curves $E(\mathbb{F}_{p^n})$ and $E'(\mathbb{F}_{p^n})$ are isomorphic over the closure $\mathbb{F}_{p^\infty}$ of the field $\mathbb{F}_p$. 

![Figure 2: The functor $F(p)$.](image)
Proof. Let $k$ be an arbitrary field. Recall that any elliptic curve $E(k)$ can be written in the form

$$E : y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6,$$  

(1)

where $a_i \in k$. A general admissible change of variable is

$$x = u^2x' + r, \quad y = u^3y' + su^2x' + t, \quad u \neq 0,$$  

(2)

and $u, r, s, t \in k$. The general admissible change of variable establishes an isomorphism between the elliptic curves $E(k)$ and $E'(k)$. In particular, the discriminant $\Delta$ of $E(k)$ transforms to $\Delta' = u^{-12}\Delta$.

Let $k = \mathbb{F}_p\infty$. Since $E(K)$ and $E'(K)$ have a good reduction at $p$, the discriminants $\Delta$ and $\Delta'$ do not vanish after the reduction; therefore, $u \neq 0$ and $u \in \mathbb{F}_p\infty$.

Similarly, the coefficients $r, s, t$ from the admissible change of variable, which takes $E(K)$ to $E'(K)$, can be reduced modulo $p$, so that $r, s, t \in \mathbb{F}_p\infty$.

Thus, one obtains an admissible change of variable defined over the field $\mathbb{F}_p\infty$, which takes the elliptic curve $E(\mathbb{F}_p\infty)$ to an isomorphic elliptic curve $E'(\mathbb{F}_p\infty)$. □

Let $E_{CM}(K)$ be an elliptic curve with complex multiplication; here $K$ is the minimal field of definition of the $E_{CM}$. The $\text{char}(K) = 0$ and it is known that $K$ is the maximal abelian extension of an imaginary quadratic field. Let $E_{CM}(\mathbb{F}_p\infty)$ be a reduction of the $E_{CM}(K)$ at a prime $p$. If $E_{CM}(\mathbb{F}_p\infty) \cong E'_{CM}(\mathbb{F}_p\infty)$ is a pair of isomorphic elliptic curves, then by lemma 2, there exists an elliptic curve $E'_{CM}(K)$, isomorphic to the $E_{CM}(K)$, for which the curve $E'_{CM}(\mathbb{F}_p\infty)$ is a good reduction modulo $p$.

On the other hand, if $E_{CM}(K) \cong E'_{CM}(K)$ are isomorphic (over $\mathbb{C}$), then the Teichmüller functor $F$ maps the $E_{CM}(K)$ and $E'_{CM}(K)$ into a pair of the stably isomorphic noncommutative tori $\mathbb{A}_{RM}$ and $\mathbb{A}'_{RM}$, respectively. The situation is described by a commutative diagram in Fig.3.

**Lemma 3** The noncommutative tori $\mathbb{A}_{RM}$ and $\mathbb{A}'_{RM}$ are stably isomorphic if and only if $A$ and $A'$ are similar matrices.

*Proof.* It follows from the definition of $\mathbb{A}_{RM}$, that $Av_\theta = \lambda_Av_\theta$, where $v_\theta = (1, \theta)$ and $\lambda_A$ the eigenvalue of the matrix $A$. Similarly, for the $\mathbb{A}'_{RM}$, one gets $A'v_{\theta'} = \lambda'_{A}v_{\theta'}$. Note, that whenever $\mathbb{A}_{RM}$ and $\mathbb{A}'_{RM}$ are stably isomorphic, we
have \( v_{\theta'} = Bv_{\theta} \) for a matrix \( B \in GL_2(\mathbb{Z}) \). Using the formula \( v_{\theta} = B^{-1}v_{\theta'} \), one can exclude the \( v_{\theta} \) in the LHS of the equation \( Av_{\theta} = \lambda_A v_{\theta} \); one gets an equation \( AB^{-1}v_{\theta'} = \lambda_A v_{\theta} \). The latter can be multiplied from the left by the matrix \( B \); thus \( BAB^{-1}v_{\theta'} = \lambda_A Bv_{\theta} = \lambda_A v_{\theta'} \). We have the following system of equations:

\[
\begin{align*}
BAB^{-1}v_{\theta'} &= \lambda_A v_{\theta'} \\
A'v_{\theta'} &= \lambda_A' v_{\theta'}.
\end{align*}
\]

(i) Suppose, that \( A \) and \( A' \) are similar, so that \( A' = BAB^{-1} \) for a matrix \( B \in GL_2(\mathbb{Z}) \); then their eigenvalues are equal \( \lambda_A = \lambda_A' \) and the system of equations \((3)\) is satisfied. Therefore, the \( \mathbb{A}_{RM} \) is stably isomorphic to the \( \mathbb{A}'_{RM} \).

(ii) Suppose, that the \( \mathbb{A}_{RM} \) and \( \mathbb{A}'_{RM} \) are stably isomorphic. It is known that \( \lambda_A \) is an invariant of the stable isomorphism; thus, \( \lambda_A = \lambda_A' \). Comparing the first and the second lines of \((3)\), one concludes that \( A' = BAB^{-1} \); therefore, \( A \) and \( A' \) are similar. Lemma 3 is proved. \( \Box \)

Note that if \( A \) and \( A' \) are similar matrices, then \( A^p \) is similar to \( (A')^p \); indeed, \( (A')^p = (BAB^{-1})^p = BAB^{-1}B^p \). On the other hand, the trace of a matrix is an invariant of the similar matrices. Thus, \( tr(A^p) = tr((A')^p) \) and \( L_p = (tr(A^p), p, -1, 0) = (tr((A')^p), p, -1, 0) = L_p' \). In view of the diagram of Fig.3, we conclude that the map \( F(p) \) sends any pair of isomorphic elliptic
curves $E_{CM}(\mathbb{F}_p^n)$ and $E'_{CM}(\mathbb{F}_p^n)$ into the same (i.e. trivially isomorphic) Cuntz-Krieger algebras $\mathcal{O}_{L_p^n} \equiv \mathcal{O}_{(L'_p)^n}$.

**Case 2.** $p \mid tr^2(A) - 4$ (a bad prime). Recall ([11], lemma 8), that the $E_{CM}(K)$ has at $p$:

$$\alpha = \begin{cases} 
1 & \text{a split multiplicative reduction (a node)}; \\
-1 & \text{a non-split multiplicative reduction (a node)}; \\
0 & \text{an additive reduction (a cusp)}. 
\end{cases} \quad (4)$$

First let us show, that the multiplicative and additive reductions are preserved by an admissible change of variable in equation (1). Indeed, let

$$c_4 = (a_1^2 + 4a_2)^2 - 24(a_1a_3 + 2a_4) \quad \text{and} \quad u^4c'_4 = c_4. \quad (5)$$

Then $\Delta = 0, c_4 \neq 0$ for a multiplicative reduction and $\Delta = c_4 = 0$ for an additive reduction [14], p.182; since $u \neq 0$, one concludes that no admissible change of variable can bring the multiplicative to an additive reduction.

The cases $\alpha = 1$ and $\alpha = -1$ can be treated similarly. Indeed, the singular point is a node, whose tangent lines have slope $\beta_1$ and $\beta_2$, respectively. Then $\beta_1$ and $\beta_2$ are rational over $\mathbb{F}_p$ whenever $\alpha = 1$ and $\beta_1$ and $\beta_2$ are quadratic over $\mathbb{F}_p$ whenever $\alpha = -1$; [14], *ibid.* Thus, no admissible change of variable can bring the split multiplicative reduction to a non-split multiplicative reduction. Since all possible cases are exhausted, theorem [1] is proved. \(\square\)

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