ABSTRACT. Mehta and van der Kallen put a Frobenius splitting on the type A cotangent bundle $T^\ast \GL_n/B$, thereby defining a stratification by compatibly split subvarieties, and they determined a few of the elements of this stratification. We embed $T^\ast \GL_n/B$ as a stratum in a larger stratified (and Frobenius split) space $GL_n/B \times \Mat_n$ whose stratification we determine, thereby giving a full description of the one of Mehta–van der Kallen. The main technique is to endow $GL_n/B \times \Mat_n$ with a Bruhat atlas, covering it with open sets that are stratified-isomorphic to Bruhat cells (in $GL_{2n}/B_{2n}$). Among the consequences are that each stratum closure is normal and Cohen–Macaulay.

1. Statement of results

The Bruhat decomposition of the flag manifold $Fl(n)$ is induced from the $n^2$ statistics

$$F^* \mapsto \dim(F^i \cap E^j), \quad i, j \in [n]$$

where $(E^i)_{i \in [n]}$ (respectively $(E_i)_{i \in [n]}$) denotes the standard flag on affine space $A^n$, invariant under the upper triangular matrices $B = B_n \leq GL_n$ (respectively the anti-standard flag, invariant under the lower triangular matrices $B^{-}$). If we identify $Fl(n) \cong GL_n/B$ as usual, we can collect these statistics into the single map

$$gB/B \mapsto BgB \in B\setminus GL_n/B \cong S_n$$

which takes the closure relation on strata to the (opposite) Bruhat order on $S_n$.

In this paper we introduce and study a similar story on $Fl(n) \times \Mat_n$, where $\Mat_n$ denotes the space of $n \times n$ matrices:

**Proposition 1.1.** The map

$$v: Fl(n) \times \Mat_n \to B_{2n}^{-}\setminus GL_{2n}/B_{2n} \cong S_{2n}$$

$$(gB/B, X) \mapsto B_{2n}^{-}\left[ \begin{array}{c} X \\ w_0g^{-1} \\ 0 \end{array} \right] B_{2n}$$

Date: October 11, 2021.

AK was partially supported by NSF DMS-1700372.

SS was partially supported by NSF DMS-1812462.
is well-defined (independent of the choice of lift of gB/B to g), and both determines and is determined by the $4n^2$ statistics

\[
\text{rank}(E^i \hookrightarrow A^n \xrightarrow{X} A^n \rightarrow A^n/E_i), \quad \text{rank}(A^n \xrightarrow{X} A^n \rightarrow A^n/(E_{n-i} + F^i)) - \dim(E_{n-i} \cap F^i),
\]

\[
\text{rank}(F^{n-i} \cap E^i \hookrightarrow A^n \xrightarrow{X} A^n) - \dim(E^i \cap F^{n-i}), \quad \text{rank}(F^i \hookrightarrow A^n \xrightarrow{X} A^n \rightarrow A^n/F^i).
\]

Let $\mathcal{Y}$ be the poset of closures of the level sets of $v$. Proposition 1.1 implies that $\mathcal{Y}$ bears a ranked embedding $\mathcal{Y} \hookrightarrow S_{2n}$, again called $v$. Our motivating example of a $Y \in \mathcal{Y}$ is defined by the conditions

\[
\text{rank}(F^i \hookrightarrow A^n \xrightarrow{X} A^n \rightarrow A^n/F^i) = 0, \quad i \in [n],
\]

which are equivalent to $X(F^i) \leq F^{i-1}$. This is the ubiquitous Springer space $Y_{spr}$ of type $A$, isomorphic to $T^*GL_n/B$, whose projection $(F^i, X) \mapsto X$ is the Springer resolution of the nilpotent cone. Another motivating example is the Grothendieck–Springer family $Y_{GS}$ defined by $X(F^i) \leq F^i$. Each of these inherit stratifications, which are described at the end of Proposition 1.3 below.

Example: $n = 2$. Below is drawn the poset $\mathcal{Y}$ of strata inside $\mathbb{P}^1 \times \text{Mat}_2$, indexed by their elements in $S_{2n} = S_4$. We have grouped them according to their projections to $\text{Mat}_2$ (which are automatically compatibly Frobenius split therein), whose equations are written in the grouping; additional equations involving $F^1 \in \mathbb{P}^1$ are indicated by the superscripts, legend at right. We have drawn in blue only the Hasse diagram edges for those projections; the full partial order is just the restriction of $S_4$ Bruhat order.

For those familiar with the interplay of stratifications under the projection $G/B \rightarrow G/P$ (studied in [KnLamS14]), we mention two ways this projection $\text{Fl}(n) \times \text{Mat}_n \rightarrow \text{Mat}_n$ is unlike that one:

Legend
a. $XF^1 \leq F^1$

b. $\text{im}(X) \leq F^1$
c. $XF^1 = 0$
d. $F^1 = E_1$
e. $F^1 = E^1$
(1) Not every compatibly split subvariety of Mat\(_n\) is the image of a stratum in Fl(\(n\)) × Mat\(_n\); for example, \(\{ X : \text{Tr}(X) = 0 \} \) is not such an image.

(2) For some compatibly split strata \(Z\) in Mat\(_n\) that are images of compatibly split strata \(\tilde{Z}\) in Fl(\(n\)) × Mat\(_n\), there don’t exist \(\tilde{Z}\) mapping birationally to (or even of the same dimension as) \(Z\).

Of particular interest are the point strata \(Y\)_\text{min} \(\cong S_n\), of the form \((\pi B/B, 0), \pi \in S_n\). Restricted to those, our map \(\nu\) gives an injection \(S_n \rightarrow S_{2n}\) yet again denoted \(\nu\), with

\[\ell(\nu(\pi)) = \dim(Fl(\pi)) \times Mat_n) = \binom{n}{2} + n^2\]

for all \(\pi \in S_n\).

**Theorem 1.2.** The following properties hold for the poset \(Y\) of strata:

(1) This \(Y\) is a **stratification by closed subvarieties**, i.e., each \(Y \in Y\) is irreducible, and the scheme-theoretic intersection \(Y_1 \cap Y_2\) of two subvarieties \(Y_1, Y_2 \in Y\) is reduced and is a union of other \(Z\) \(\in Y\).

(2) The image of \(\nu:\ Y_{\text{op}} \hookrightarrow S_{2n}\) is an order ideal in the Bruhat order of \(S_{2n}\), with maximal elements \(\{ (\nu(\pi)) \in S_{2n} : \pi \in S_n\}\).

(3) For \(\pi \in S_n\), let \(\Gamma = \{ \gamma \leq \pi \}\) be the maximal biGrassmannian elements below \(\pi\), computable (as in [Ko10]) from the “Fulton essential set” of \(\pi\). Then \(Y_\pi\) is the scheme-theoretic intersection \(\bigcap_{\gamma \in \Gamma} Y_\gamma\), i.e., to find the equations for a general stratum it is enough to understand the biGrassmannian case.

(4) After reducing these schemes (defined over \(\mathbb{Z}\)) modulo any prime \(p\), there is a (unique) Frobenius splitting on Fl(\(n\)) × Mat\(_n\) with respect to which \(Y\) consists of exactly the compatibly split subvarieties.

(5) Identify Fl(\(n\)) \(\cong GL_n/B\), so Fl(\(n\)) bears an open cover \(\bigcup_{\pi \in S_n} \pi B/B \rightarrow B/B\) by translates of the big cell. There is a “Bruhat atlas” on Fl(\(n\)) × Mat\(_n\) identifying

\[\pi B/B \times Mat_n \cong X_\nu(\pi) := B_{2n} \nu(\pi) B_{2n}/B_{2n}\]

as stratified spaces, i.e., taking

\[Y_\rho \cap (\pi B/B \times Mat_n) \hookrightarrow X_\nu(\pi) \cap X_\rho\]  

for all \(\rho \leq \nu(\pi)\), where \(X_\rho := \overline{B_{2n} \rho B_{2n}/B_{2n}}\).

Ergo, every stratum in \(Y\) is normal and Cohen–Macaulay, with rational singularities.

In particular (5) implies (1), (2), (4), and (4) implies (3). So (5) is really the key statement. We recall the definition of Bruhat atlases in §3.

There had not previously been a full determination of the strata in this stratification of \(T^*GL_n/B\); see [MvdK92, VdK08] for examples. Other Frobenius splittings of cotangent bundles of flag varieties were studied in [KuLauT98, Ha13].

The strata thus constructed are each invariant under the action of the subgroup B (by standard action on \(F^*\) and conjugation on \(X\)). Our other result characterizes those strata which are GL\(_n\)-invariant. Recall that the **diagram of a permutation** \(\sigma \in S_m\) is the set

\[D(\sigma) := \{(i, j) \in [m] \times [m] | \sigma(i) > j, \sigma^{-1}(j) > i\}\]

and the **Fulton essential set** of \(\sigma\) is the Southeast corners of \(D(\sigma)\), i.e. the set of matrix entries \((i, j) \in D(\sigma)\) such that \((i + 1, j) \notin D(\sigma)\) and \((i, j + 1) \notin D(\sigma)\).

\(^1\)It is more usual to axiomatize stratifications using disjoint locally closed subvarieties, but (except for the reducedness requirement) this approach is equivalent and generally more convenient.
Proposition 1.3. Let \( Y \) be a stratum, let \( \sigma = v(Y) \), and let \( \rho \) be the partial permutation of size \( n \) given by the bottom right \( n \times n \) submatrix of \( \sigma \). The following are equivalent:

1. \( Y \) is \( \text{GL}_n \)-invariant.
2. The Fulton essential set of \( \sigma \) lies in the bottom right \( (n+1) \times (n+1) \) submatrix.
3. Neither \( \sigma \) nor \( \sigma^{-1} \) have descents in \( \{1, \ldots, n\} \).
4. We have an isomorphism \( \text{GL}_n \times B \rightarrow Y \) where the closure of \( B\rho^T w_0 B \) is taken inside of \( \text{Mat}_n \).

Moreover, each \( n \times n \) partial permutation \( \rho \) arises from a unique such (full) permutation \( \sigma \in S_{2n} \).

Two examples are of special note:

\[
v(Y_{\text{GS}}) = 1 \cdot 2 \cdot \ldots \cdot n \cdot 2n \cdot (2n-1) \cdot \ldots \cdot (n+1)
v(Y_{\text{Spr}}) = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot (n-1) \cdot 2n \cdot (2n-1) \cdot \ldots \cdot n
\]

whose induced stratifications match those defined by Mehta and van der Kallen in \([\text{MvdK}92]\) (recalled in \( \S 2 \)).

Remark 1.4. It would be of great interest if the conormal varieties to Schubert varieties could be made compatibly split with respect to some splitting on the cotangent bundle. Their union (inside \( Y_{\text{Spr}} \)) is given by the moment map equations “the Southwest triangle of \( X \) vanishes”. Unfortunately the conditions we impose on \( X \) are Northwest rank conditions, so we have nothing to say about the conormal varieties.

We comment briefly on the approach that suggested this space and this Bruhat atlas. For a very long time we tried to cover \( T^* \text{GL}_n / B \) itself with charts \( X_{\rho}^\circ \) from the flag variety of a Kac–Moody group \( H \). This would require embedding the Mehta–van der Kallen poset as an order ideal inside \( W_H \). As far as we could tell, that would require that \( H \) have a rather fearsome Dynkin diagram, a sort of “broom” with handle of length \( n-2 \) plus \( n+1 \) bristles (nicely indexed by the covers of \( v(Y_{\text{Spr}}) \)). It seems very hard to compute the defining equations of strata inside such general Kac–Moody flag varieties; the closest work seems to be \([\text{ELu19}]\). Much more recently we tried instead to embed the M–vdK poset (for small \( n \)) as an order ideal inside \( \{ \rho \in S_m : \rho \geq \sigma \} \) for some fixed \( \sigma \) with \( X_{\sigma} \) smooth (so that each \( X_{\rho}^\circ \cap X_{\sigma} \) is a cell). This succesful approach led to the discovery of \( v(Y_{\text{Spr}}) \). That guess in turn prompted the question of whether the ambient variety should be chosen larger than \( T^* \text{GL}_n / B \), larger by dimension \( \ell(v(Y_{\text{Spr}})) = \binom{n+1}{2} \), and there was an obvious such choice.

2. Stratifications and Frobenius splittings

We start with three equivalent axiomatizations of “stratification of \( M \) by irreducible subvarieties”. The first and most traditional is a finite disjoint decomposition \( M = \bigsqcup_{Y \in \mathcal{Y}} Y^\circ \) with the property that \( \overline{Y^\circ} = \bigsqcup_{Z \in Y^\circ, Z \subseteq Y^\circ} Z^\circ \). We avoid using this definition because it is more pleasant to work with closed subschemes rather than with locally closed subschemes.

The second, which is the one that we will use, is a finite collection \( \mathcal{Y} \) of subvarieties of \( M \) with the property that

\[
A, B \in \mathcal{Y} \implies A \cap B = \bigcup_{Z \in \mathcal{Y}, Z \subseteq A \cap B} Z.
\]

Already there is a subtlety compared to the first definition: should these intersections be required to be reduced?
We mention a third axiomatization of stratification: this is a finite collection of sub-
schemes \(Y^i\) which is closed under union, intersection, and taking geometric components. 
This definition has more of an “algebra” flavor, with two binary operations and a (multi-
valued) unary operation. One benefit of this point of view is to allow for the definition of 
the stratification generated by a hypersurface \(H\), which amounts to decomposing \(H\) into 
its components, intersecting them with one another, and repeating those two operations 
until one stops finding new varieties. There is a new subtlety about reducedness, visible 
in the stratification of the plane generated by the hypersurface \(xy(x + y) = 0\): should the 
intersections of unions of strata be required to be reduced?

For the purposes of this paper, we will consider the most restrictive form of these def-
initions for \(Y\) and require that any intersection of closed subschemes in this stratification 
is reduced. This will arise naturally if \(M\) is endowed with a Frobenius splitting, whose defi-
definition we now recall from [BrKu05, §1.3.1].

If \(R\) is a commutative algebra over \(\mathbb{F}_p\), then a function \(\phi: R \rightarrow R\) is a Frobenius splitting 
if, for all \(a, b \in R\),

1. \(\phi(a + b) = \phi(a) + \phi(b)\),
2. \(\phi(a^p b) = a^p \phi(b)\), and
3. \(\phi(1) = 1\).

If \(R\) admits a Frobenius splitting, then \(R\) is easily seen to be reduced: if \(x \in R\) is nilpotent 
and non-zero, then let \(m\) be the largest power such that \(x^m \neq 0\). Then \(m \geq 1\) and \((x^m)^p = 0\) 
and so \(0 = \phi(0) = \phi((x^m)^p) = x^m \phi(1) = x^m\), a contradiction. One may extend \(\phi\) 
uniquely to any localization \(R[1/b]\) by \(\phi(a/b) = \phi(ab^{p-1}/b^p) = \phi(ab^{p-1})/b\), and this 
locality allows us to define splittings of schemes, not just rings (\(\Leftrightarrow\) affine schemes).

An ideal \(I \subset R\) is compatibly split if \(\phi(I) \subset I\). In that case, \(\phi\) descends to a Frobenius 
splitting on \(R/I\) (hence \(R/I\) is reduced, so \(I\) is radical). A nontrivial theorem (in what is 
still only a 3-page paper [KuM09], see also [Sc09]) is that in a Frobenius split scheme of 
finite type, the number of compatibly split subvarieties is finite. It is easy to show they 
form a stratification in the most restrictive of the three senses.

If \(M\) and \(H\) are both defined over \(\mathbb{Z}_p\), then one can transfer the corresponding reduced-
ness results to characteristic zero by standard spreading-out techniques.

In fact, we won’t need to work carefully with this definition of \(\phi\): as we will see, the 
existence of a Bruhat atlas (defined in the next section) guarantees that our space \(M\) has 
a chart of affine spaces such that the induced stratification on each is generated by a 
 hypersurface in the sense just explained.

In our situation, we can describe the irreducible components of the divisor of the strati-
fication using the map \(v\). More specifically, for \(i = 1, \ldots, n\), the divisor labeled by \(s_i \in S_{2n}\) is 

\[Y_{s_i} = \{(F^*, X) \mid \det(NW_i \times i \text{ submatrix of } X) = 0\}.\]

For \(i = 1, \ldots, n - 1\), the divisor labeled by \(s_{n+i} \in S_{2n}\) is 

\[Y_{s_{n+i}} = \{(F^*, X) \mid \rank(F^{n-i} \to X_i \to \mathbb{A}^n/F^i) < n - i\}.\]

In particular, every stratum can be obtained by the process of intersecting these divisors, 
taking irreducible components, intersecting with more divisors, and repeating.

Since \(Y_{GS}\) is contained in each of the divisors of the second type, the induced stratifica-
tion on \(Y_{GS}\) that we get is generated by the intersections \(Y_{GS} \cap Y_{s_i}\) for \(i = 1, \ldots, n\). This 
matches with the stratification defined by [MvdK92].
3. The Bruhat Atlas

We recall the definition of Bruhat atlas due to X. He, J.-H. Lu, and the first author [E16, GKL19, Hu19a, Hu19b, LuYu20, BaHe21]. A Bruhat chart on a space $M$ with stratification $\mathcal{Y}(M)$ is a triple $(H, v_0, c)$ where

1. $H$ is a Kac–Moody group with standard Borel subgroups $B_H^+$ and Weyl group $W_H$,
2. $v_0$ is an element of $W_H$; it has an associated finite-dimensional Bruhat cell

$$X_{v_0}^0 := B_H v_0 B_H / B_H$$

of dimension $\ell(v_0)$, which is stratified by its intersection with $B_H^-$-orbit closures

$$X_w := B_H w B_H / B_H,$$

3. $c : X_{v_0}^0 \to M$ is an open embedding, and an isomorphism of stratified spaces with its image, i.e., the strata inside $c(X_{v_0}^0)$ are exactly the subsets $c(X_{v_0}^0 \cap X_w)$ for $w \leq v$.

In particular, each stratum in $M$ meeting the open image of $c$ receives a label $w \leq v_0$. If we can cover $M$ using charts, with the labelings compatible across charts, we call this a Bruhat atlas. Unwrapping this discussion, a Bruhat atlas on $(M, \mathcal{Y}(M))$ is a triple

$$(H, v, \{ c_p \}_{p \in \mathcal{Y}(M)_{\min}})$$

where

1. $H$ is a Kac–Moody group with standard Borel subgroups $B_H^+$ and Weyl group $W_H$,
2. $\mathcal{Y}(M)_{\min}$ is the set of minimal strata, each a point,
3. $\nu : \mathcal{Y}(M)_{\text{op}} \to W_H$ is a ranked embedding of posets, with image $\bigcup_{p \in \mathcal{Y}(M)_{\min}} \{ 1, \nu(p) \}$,
4. for each $p \in \mathcal{Y}(M)_{\min}$ we have an open embedding $c_p : X_{v(p)}^0 \to M$ which is an isomorphism (to its image) of $W_H$-stratified spaces. That is, for each stratum $Y \in \mathcal{Y}$, $c_p$ restricts to an isomorphism

$$X_{v_0}^0 \cap X_{v(Y)} \cong c_p(X_{v(p)}^0) \cap Y,$$

5. these open images $\{ c_p(X_{v(p)}^0) \}$ cover $M$.

This concept (foreshadowed in [Sn11, KnWYo13]) serves as a very efficient organizing principle for describing a stratified space and properties of its strata. Here is a sample:

**Proposition 3.1.** Suppose that $(M, \mathcal{Y}(M))$ has a Bruhat atlas.

1. Each $Y \in \mathcal{Y}(M)$ has rational singularities (and, in particular, is normal and Cohen–Macaulay).
2. The stratification $\mathcal{Y}(M)$ is generated by its hypersurface, i.e. it is the coarsest stratification by closed subvarieties (as defined in Theorem 1.2(1)) that includes the components of the complement of the open stratum.

Indeed, the induced stratification on any stratum $Y \in \mathcal{Y}(M)$ likewise is generated by its hypersurface $\bigcup \{ Z \in \mathcal{Y}(M) : Z \subsetneq Y \}$.

**Proof.** (1) This is true of each $X_{v_0}^0 \cap X_w$, hence it is true of the open cover $\{ c_p(X_{v(p)}^0 \cap X_{v(Y)}) \}$ of $Y$, and these conditions are local so checkable on an open cover.

(2) Since the charts cover, each $Y \in \mathcal{Y}(M)$ meets some chart $(\pi N \cdot B/B) \times \text{Mat}_n$ in some open set $Y'$. As seen in the proof of [BrKu05, Theorem 2.3.1], the stratification on any chart $X_{v(p)}^0$ has this generated-by-hypersurface property. The same proof, based on a combinatorial property of Bruhat order, extends to strata $X_w \cap X_{v(p)}^0$. □
Concretely, this generation consists of taking the known compatibly split subvarieties, intersecting them with one another, picking out the components, and repeating until done. If we have a sequence of such operations on strata in \((\pi N/B) \times \text{Mat}_n\) culminating in the stratum \(Y\) of \((\pi N/B) \times \text{Mat}_n\), we can do the same with their closures in \(\text{Fl}(\pi) \times \text{Mat}_n\), culminating in a union of \(Y\) possibly with some other components (outside the chart). From there we pick out, and have “generated”, \(Y\).

Fix \(\pi \in S_n\) for the remainder of the section. We recall [KnWYo13, Lemma 2.1]:

**Lemma 3.2.** Let \(E^+ := (\pi N^\pi^{-1}) \cap N^+\). Then each \(g \in \pi N^\pi^{-1}\) can be uniquely factored as \(g = b_- b_+ = c_+c_-\) with \(b_\pm, c_\pm \in E^\pm\), and the map

\[
\Phi: \pi N^\pi^{-1} \to E^+ \times E^-
\]

\[
g \mapsto (b_+, c_-)
\]

is an isomorphism of varieties.

**Proof of Theorem 1.2(5).** We put coordinates on \(X^\nu(\pi)\), first as the free orbit of the group

\[
N_{2n} \cap (\nu(\pi)N_{2n}\nu(\pi)^{-1}) = \left\{ \begin{bmatrix} a & Y \\ 0 & d \end{bmatrix} : a \in N \cap \pi N^\pi^{-1}, d \in N \cap w_0^\pi N^{-\pi w_0} \right\}
\]

through \(\nu(\pi)B_{2n}/B_{2n}\), taking

\[
\begin{bmatrix} a & Y \\ 0 & d \end{bmatrix} \mapsto \begin{bmatrix} a & Y \\ 0 & d \end{bmatrix} \begin{bmatrix} 0 & \pi \end{bmatrix} B_{2n}/B_{2n} = \begin{bmatrix} Yw_0^\pi \alpha \pi \\ dw_0^\pi \pi \end{bmatrix} B_{2n}/B_{2n}
\]

where \(a' \in N^{-\pi N^\pi}, \ y' \in N^\pi \cap \pi N^{-\pi}I\) are the coordinates we will actually use.

Meanwhile we can factor

\[
\begin{bmatrix} X \\ w_0^\gamma \gamma \pi g \end{bmatrix} = \begin{bmatrix} \pi b_- \pi^{-1} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} X' \\ w_0^\gamma \gamma \pi g^{-1} \pi \end{bmatrix}
\]

\[
= \begin{bmatrix} \pi b_- \pi^{-1} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} X'' \\ w_0^\gamma \gamma \pi \end{bmatrix} \begin{bmatrix} 0 \pi c_+^{-1} \pi^{-1} \end{bmatrix}
\]

where \(b_- b_+ = c_+c_- = g\) are the unique factorizations of \(g \in N\) from Lemma 3.2, and \(X'' = \pi b_- \pi \pi X c_+ \pi^{-1}\). We then have maps

\[
c^{-1}_\pi: (\pi g, X) \mapsto \begin{bmatrix} \pi b_- \pi^{-1} \pi X c_+ \pi^{-1} \\ w_0^\gamma \gamma \pi c_+^{-1} \pi g \end{bmatrix} B_{2n}/B_{2n}
\]

\[
(\pi \Phi^{-1}(a', d') \pi b_- \pi^{-1} \pi c_+ \pi^{-1}) \mapsto \begin{bmatrix} Z \\ w_0^\gamma \gamma d' \pi \pi c_+^{-1} \pi^{-1} \end{bmatrix} B_{2n}/B_{2n} : c\pi
\]

giving inverse isomorphisms of \((\pi N B/B) \times \text{Mat}_n\) with \(X^\nu(\pi)\), where in the second line \(b_- := \Phi^{-1}(a', d') \alpha \pi^{-1}, c_+ := \Phi^{-1}(a', d') \pi c_+^{-1}\).

To see that these are *stratified* isomorphisms, we have to compare the matrix \(c^{-1}_\pi(\pi g, X)\) to the matrix \(\begin{bmatrix} X \\ w_0^\gamma \gamma \pi g \end{bmatrix}\) that we used in §1 to define the stratification. This is exactly accomplished in (e), showing that the two lie in the same double coset \(B_{2n}^- \backslash \text{GL}_{2n}/B_{2n}\). \(\square\)
4. The remaining proofs

Proof of Proposition 4.1. Each \((i,j) \in [2n] \times [2n]\) determines a flush-Northwest submatrix with Southeast corner \((i,j)\), indicated in red in the matrices below, and a complete set of \(B_{2n} \times B_{2n}\)-invariants on \(GL_{2n}\) is given by the ranks of these submatrices. The four types of statistics correspond to the four quadrants in the \(2n \times 2n\) matrix \(\nu(B,B,X)\). We write \(V\) for \(A^n\) in this proof, to better distinguish it from \(V^*\). Suppose that \(g\) represents the flag \(F^* \subset V\), i.e., \(F^i\) is the span of the first \(i\) columns of \(g\). The first \(i\) rows of \(w_0g^{-1}\) (thought of as functionals on \(V\) in the natural way) span \(\text{ann}(F^{n-i}) \leq V^*\).

\[
\begin{bmatrix}
X & g \\
\w_0g^{-1} & 0
\end{bmatrix}
\]

The \((i,j)\) in the Northwest quadrant are simplest, giving Northwest rank conditions on \(X\) itself. These compute the first set of numbers,

\[
\text{rank}(E^i \hookrightarrow A^n \xrightarrow{X} A^n \rightarrow A^n/E^i).
\]

Pick \(i \leq n\) and consider the NW \(i \times (n+j)\) submatrix. Then the rank of this matrix is \(\text{rank}(X: V \rightarrow V/(E_{n-i} + F^i))\) plus the rank of the last \(j\) columns. This last quantity is the rank of the map \(F^i \rightarrow V/E_{n-i}\), which is \(j - \dim(F^i \cap E_{n-i})\).

Note that \((V^*/\text{ann}(F^i))^\ast = F^i\). So by taking transpose, the rank of the NW \((n+i) \times n\) matrix is

\[
\begin{bmatrix}
X & g \\
\w_0g^{-1} & 0
\end{bmatrix}
\]

\[i + \text{rank}(X^* : V^* \rightarrow V^*/\text{ann}(F^{n-i})) = i + \text{rank}(X : F^{n-i} \rightarrow V).
\]

Similar remarks as above allow us to recover the rank of the NW \((n+i) \times j\) matrix from knowledge of \(\text{rank}(X : F^{n-i} \cap E^i \rightarrow V)\) and \(\text{rank}(\text{id} : E^i \rightarrow V/F^{n-i})\).

Finally, we want to understand the rank of the NW \((n+i) \times (n+j)\) submatrix. For that, we can use column operations and row operations to say that its rank is \(i+j + \text{rank}(X : F^{n-i} \rightarrow V/F^i)\) (the \(i\) is the contribution from the last \(i\) rows, the \(j\) is the contribution from the last \(j\) columns, and the remainder is what happens when we reduce the NW \(n \times n\) matrix). \(\square\)

Proof of Proposition 4.3. (1) \(\Rightarrow\) (2): The rank conditions given by the pairs in the essential set of \(\sigma\) are non-redundant [Fu91, Lemma 3.14], and hence none of them can be omitted. Now \(Y\) is \(G\)-invariant if its defining conditions do not involve the standard flag \(E\), and from the proof of Proposition 1.1, this is equivalent to all essential boxes being in the bottom right \((n+1) \times (n+1)\) submatrix. This latter sentence also shows (2) \(\Rightarrow\) (1).

(2) \(\Rightarrow\) (3): Suppose that \(i\) is a descent of \(\sigma\) so that \(\sigma(i) > \sigma(i+1)\). Then for some \(j\) such that \(\sigma(i+1) \leq j < \sigma(i)\), the pair \((j,i)\) is in the essential set of \(\sigma\). In particular, if the essential set is contained in the bottom right \((n+1) \times (n+1)\) submatrix, then \(i > n\). Similarly, we see that \(\sigma^{-1}\) has no descents amongst \(1, \ldots, n\).

(3) \(\Rightarrow\) (2): We address here the existence and uniqueness of \(\sigma\) given \(\rho\). Let \(C,R \subseteq [n]\) be the set of columns and rows in \(\rho\) that contain 1s, so \(|C| = |R| = \text{rank}(\rho)\). Then it is easy to determine the rest of \(\sigma:\)

\[
\begin{bmatrix}
|C| & n-|C| \\
|R| & n-|R| \\
\hline
n & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & K & \rho
\end{bmatrix}
\]
where $J$ is a partial permutation matrix with 1s in columns $[n] \setminus C$ running NW/SE, and $K$ is similar, except with 1s in rows $[n] \setminus R$ running NW/SE. From here we see that the diagram in the NW quadrant consists of the $(n - |C|)^2$ square in its SE corner, the diagram in the NE quadrant consists of columns touching the bottom row, and the diagram in the SW quadrant consists of rows touching the right column.

$(1,2,3) \implies (4)$: By $(1)$, the projection $Y \to \text{Fl}(n)$ is a $\text{GL}_n$-equivariant map to a homogeneous $\text{GL}_n$-space, hence determined by its fiber over $B/B$; specifically, the action map

$$\text{GL}_n \times^B (Y \cap ([B/B] \times \text{Mat}_n)) \to Y$$

is an isomorphism. We analyze $Y \cap ([B/B] \times \text{Mat}_n)$ using $\nu : (B/B, X) \mapsto B_{2n}^{-} \begin{bmatrix} X & I \\ w_0 & 0 \end{bmatrix} B_{2n}$.

Asking this to be in $B_{2n}^{-}\sigma B_{2n}$ is (by $(2)$ and [Fu91]) equivalent to asking that, for every $i, j \in [n]$, the rank of the NW $(n + i) \times (n + j)$ submatrix of $\begin{bmatrix} X & I \\ w_0 & 0 \end{bmatrix}$ has rank bounded by that of the corresponding submatrix of $\sigma$. Continuing (**), we break $\sigma$ into blocks as

$$\begin{bmatrix} |C| & n-|C| \\ n-|C| \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} j & n-j \\ n-j & \end{bmatrix}$$

$$\begin{bmatrix} |C| & n-|C| \\ n-|C| \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} j & n-j \\ n-j & \end{bmatrix}$$

with $\begin{bmatrix} 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \end{bmatrix}$.

```latex
\begin{bmatrix}
1 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
\end{bmatrix}
```

where $|C| = a + b + c + d$. Hence the NW $(n + i) \times (n + j)$ submatrix of $\sigma$ has rank

$$|C| + (j - a - c) + (i - a - b) + a = (a + b + c + d) + (j - a - c) + (i - b) = d + j + i.$$ 

Meanwhile, the corresponding submatrix $\begin{bmatrix} X & I \\ w_0 & 0 \end{bmatrix}$ has rank $i + j$ plus the rank of the SW $(n - j) \times (n - i)$ submatrix of $X$.

Together, $(B/B, X)$ is in $Y$ if and only if for each $i, j$,

$$\text{rank}(\text{SW } (n - j) \times (n - i) \text{ rectangle of } X) \leq \text{rank}(\text{SE } (n - i) \times (n - j) \text{ of } \rho) = d = \text{rank}(\text{SE } (n - j) \times (n - i) \text{ of } \rho^T) = \text{rank}(\text{SW } (n - j) \times (n - i) \text{ of } \rho^T w_0)$$

We have reached Fulton’s equations for $B \rho^T w_0 B$.

$(4) \implies (1)$: For any $B$-variety $Z$, $G \times^B Z$ is always $G$-invariant.

It remains to check the two examples, whose permutations we call $\sigma_{GS}$, $\sigma_{Spr}$. In each the southeast quadrant $\rho$ of $\nu(Y)$ is easy to calculate, giving us the spaces $\text{GL}_n \times^B b$ and $\text{GL}_n \times^B n$, respectively. Since (both here and in [MvdK92]) the stratification of $Y_{Spr}$ is restricted from the one on $Y_{GS}$, we need only check that we have the right stratification of $Y_{GS}$, which (thanks to Proposition 3.1 (2)) is determined by its hypersurface.

The codimension 1 strata inside $Y_{GS}$ correspond to the Bruhat covers $w \triangleright \sigma_{GS}$. These come in two types: $\sigma_{s_i} = s_i \sigma$ for $i < n$, and $\sigma \circ (n \leftrightarrow j)$ for $j \in [n + 1, 2n]$. The latter type are exactly the least upper bounds of $\sigma_{GS}$ and $s_n$. Consequently, the codimension 1 strata come from the intersection of $Y_{GS}$ with $Y_{s_i}$ for $i \in [1, n]$. Since $Y_{s_i} = \{(F^*, X) : \text{det}(\text{NW } i \times i \text{ minor of } X) = 0\}$, we have recovered the defining divisor from [MvdK92, §3.4].
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