BURNIAT SURFACES II: SECONDARY BURNIAT SURFACES FORM THREE CONNECTED COMPONENTS OF THE MODULI SPACE.

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This article is dedicated, with gratitude and admiration, to David Mumford on the occasion of his $2^3 \cdot 3^2$-th birthday.

INTRODUCTION

The so-called Burniat surfaces were constructed by Pol Burniat in 1966 ([Bu66]), where the method of singular bidouble covers was introduced in order to solve the geography problem for surfaces of general type.

The special construction of surfaces with geometric genus $p_g(S) = 0$, done in [Bu66], was brought to attention by [Pet77], which explained Burniat’s calculation of invariants in the modern language of algebraic geometry, and nowadays the name of Burniat surfaces is reserved for these surfaces with $p_g(S) = 0$.

Burniat surfaces are especially interesting examples for the non birationality of the bicanonical map (see [Cil97]). For all the Burniat surfaces $S$ with $K_S^2 \geq 3$ the bicanonical map turns out to be a Galois morphism of degree 4.

We refer to our joint paper with Grunewald and Pignatelli ([BCGP09]) for a general introduction on the classification and moduli problem for surfaces with $p_g(S) = 0$ and its applications: as an example we mention our final corollary here on the validity of Bloch’s conjecture for all deformations of secondary Burniat surfaces.

The main achievement of the present series of three articles is to completely solve the moduli problem for Burniat surfaces, determining the connected components of the moduli space of surfaces of general type containing the Burniat surfaces, and describing their geometry.

The minimal models $S$ of Burniat surfaces have as invariant the positive integer $K_S^2$, which can take values $K_S^2 = 6, 5, 4, 3, 2$.

We get a rationally parametrized family of dimension $K_S^2 - 2$ for each value of $K_S^2 = 6, 5, 3, 2$, and two such families for $K_S^2 = 4$, one called of non nodal type, the other of nodal type. We proposed in [BC09b] to call primary Burniat surfaces those with $K_S^2 = 6$, secondary Burniat surfaces those with $K_S^2 = 5, 4$, and tertiary Burniat surfaces those with...
$K_S^2 = 3$. The reason not to consider the Burniat surface with $K_S^2 = 2$ is that it is just one special element of the family of standard Campedelli surfaces (i.e., with torsion group $(\mathbb{Z}/2\mathbb{Z})^3$) (see [Ku04] and [BC09b]), whose geometry is completely understood (see [Miy77] and [Rei]).

An important result was obtained by Mendes Lopes and Pardini in [MLP01] who proved that primary Burniat surfaces form a connected component of the moduli space of surfaces of general type. A stronger result concerning primary Burniat surfaces was proved in part one ([BC09b]), namely that any surface homotopically equivalent to a primary Burniat surface is a primary Burniat surface. Alexeev and Pardini (cf. [AlPar09]) reproved the result of Mendes Lopes and Pardini by studying more generally the component of the moduli space of stable surfaces of general type containing primary Burniat surfaces.

Here, we shall prove in one go that each of the 4 families of Burniat surfaces with $K_S^2 \geq 4$, i.e., of primary and secondary Burniat surfaces, is a connected component of the moduli space of surfaces of general type.

The case of tertiary Burniat surfaces will be treated in the third one of this series of papers, and we limit ourselves here to say that the general deformation of a Burniat surface with $K_S^2 = 3$ is not a Burniat surface, but it is always a bidouble cover (through the bicanonical map) of a cubic surface with three nodes.

At the moment when we started the redactional work for the present paper we became aware of the fact that a weaker result was stated in [Ku04], namely that each family of Burniat surfaces of secondary type yields a dense set in an irreducible component of the moduli space. The result is derived by Kulikov from the assertion that the base of the Kuranishi family of deformations is smooth. This result is definitely false for the Burniat surfaces of nodal type (proposition 4.12 and corollary 4.23 (iii) of [Ku04]), as we shall now see.

Indeed one of the main technical contributions of this paper is the study of the deformations of secondary Burniat surfaces, through diverse techniques.

A very surprising and new phenomenon occurs for nodal surfaces, confirming Vakil’s ‘Murphy’s law’ philosophy ([Va06]). To explain it, recall that indeed there are two different scheme structures for the moduli spaces of surfaces of general type.

One is the moduli space $\mathfrak{M}^{\text{min}}_{\chi, K^2}$ for minimal models $S$ having $\chi(O_S) = \chi$, $K_S^2 = K^2$, the other is the Gieseker moduli space ([Gies77]) $\mathfrak{M}^{\text{can}}_{\chi, K^2}$ for canonical models $X$ having $\chi(O_X) = \chi$, $K_X^2 = K^2$. Both are quasi projective schemes and there is a natural morphism $\mathfrak{M}^{\text{min}}_{\chi, K^2} \to \mathfrak{M}^{\text{can}}_{\chi, K^2}$ which is a bijection. Their local structure as complex analytic spaces is the quotient of the base of the Kuranishi family by the action of the finite group $\text{Aut}(S) = \text{Aut}(X)$. 
In [Cat89] series of examples were exhibited where $\mathcal{M}^{can}_{\chi,K^2}$ was smooth, but $\mathcal{M}^{min}_{\chi,K^2}$ was everywhere non reduced.

For nodal Burniat surfaces with $K^2_S = 4$ both spaces are everywhere non reduced, but the nilpotence order is higher for $\mathcal{M}^{min}_{\chi,K^2}$; this is a further pathology, which adds to the ones presented in [Cat89] and in [Va06].

More precisely, this is one of our two main results:

**Theorem 0.1.** The subset of the Gieseker moduli space $\mathcal{M}^{can}_{1,4}$ of canonical surfaces of general type $X$ corresponding to Burniat surfaces $S$ with $K^2_S = 4$ and of nodal type is an irreducible connected component of dimension 2, rational and everywhere non reduced.

More precisely, the base of the Kuranishi family of $X$ is locally analytically isomorphic to $\mathbb{C}^2 \times \text{Spec}(\mathbb{C}[t]/(t^m))$, where $m$ is a fixed integer, $m \geq 2$.

The corresponding subset of the moduli space $\mathcal{M}^{min}_{1,4}$ of minimal surfaces $S$ of general type is also everywhere non reduced.

More precisely, the base of the Kuranishi family of $S$ is locally analytically isomorphic to $\mathbb{C}^2 \times \text{Spec}(\mathbb{C}[t]/(t^{2m}))$.

Whereas for the non nodal case we get the following second main result:

**Theorem 0.2.** The three respective subsets of the moduli spaces of minimal surfaces of general type $\mathcal{M}^{min}_{1,K^2}$ corresponding to Burniat surfaces with $K^2 = 6$, resp. with $K^2 = 5$, resp. Burniat surfaces with $K^2 = 4$ of non nodal type, are irreducible connected components, normal, rational of respective dimensions 4, 3, 2.

Moreover, the base of the Kuranishi family of such surfaces $S$ is smooth.

Theorem 0.1 poses the challenging deformation theoretic question to calculate the number $m$ giving the order of nilpotence of the local moduli space (and also of the moduli space at the general point).

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Thanks also to Rita Pardini for spotting a mistake in a previous version of proposition 2.1.

1. **The local moduli spaces of Burniat surfaces**

Burniat surfaces are minimal surfaces of general type with $K^2 = 6, 5, 4, 3, 2$ and $p_g = 0$, which were constructed in [Bu66] as singular
Figure 1. Configurations of lines
bidouble covers (Galois covers with group \((\mathbb{Z}/2\mathbb{Z})^2\)) of the projective plane branched on 9 lines.

We briefly recall their construction: this will also be useful to fix our notation. For more details, and for the proof that Burniat surfaces are exactly certain Inoue surfaces we refer to [BC09b].

Let \(P_1, P_2, P_3 \in \mathbb{P}^2\) be three non collinear points (which we assume to be the points \((1 : 0 : 0), (0 : 1 : 0)\) and \((0 : 0 : 1)\)) and let’s denote by \(\epsilon:= \hat{\text{type}} \left(1 \right)\) and \(u\)

We denote by \(\epsilon: \mathbb{P}^2 \rightarrow \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1\)

such that

\[
\epsilon(y_1 : y_2 : y_3) = \left(\left(y_2 : y_3\right)\left(y_3 : y_1\right)\left(y_1 : y_2\right)\right).
\]

One sees immediately that \(Y \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1\) is the hypersurface of type \((1, 1, 1)\):

\[
Y = \{(x'_1: x_1), (x'_2: x_2), (x'_3: x_3) \mid x_1 x_2 x_3 = x'_1 x'_2 x'_3\}.
\]

We denote by \(L\) the total transform of a general line in \(\mathbb{P}^2\), by \(E_i\) the exceptional curve lying over \(P_i\), and by \(D_i,1\) the unique effective divisor in \(|L - E_i - E_{i+1}|\), i.e., the proper transform of the line \(y_{i-1} = 0\), side of the triangle joining the points \(P_i, P_{i+1}\).

Consider on \(Y\), for each \(i \in \mathbb{Z}/3\mathbb{Z} \cong \{1, 2, 3\}\), the following divisors

\[
D_i = D_{i,1} + D_{i,2} + D_{i,3} + E_{i+2} \in |3L - 3E_i - E_{i+1} + E_{i+2}|,
\]

where \(D_{i,j} \in |L - E_j|\), for \(j = 2, 3\), \(D_{i,j} \neq D_{i,1}\), is the proper transform of another line through \(P_i\) and \(D_{i,1} \in |L - E_i - E_{i+1}|\) is as above. Assume also that all the corresponding lines in \(\mathbb{P}^2\) are distinct, so that \(D := \sum_i D_i\) is a reduced divisor.

Note that, if we define the divisor \(\mathcal{L}_i := 3L - 2E_{i-1} - E_{i+1}\), then

\[
D_{i-1} + D_{i+1} = 6L - 4E_{i-1} - 2E_{i+1} = 2\mathcal{L}_i,
\]

and we can consider (cf. [Cat99]) the associated bidouble cover \(X' \rightarrow Y\) branched on \(D := \sum_i D_i\) (but we take a different ordering of the indices of the fibre coordinates \(u_i\), using the same choice as the one made in [BC09b], where however \(X'\) was denoted by \(X\)).

We recall that this precisely means the following: let \(D_i = \text{div}(\delta_i)\), and let \(u_i\) be a fibre coordinate of the geometric line bundle \(\mathbb{L}_{i+1}\), whose sheaf of holomorphic sections is \(\mathcal{O}_Y(\mathcal{L}_{i+1})\).

Then \(X \subset \mathbb{L}_1 \oplus \mathbb{L}_2 \oplus \mathbb{L}_3\) is given by the equations:

\[
\begin{align*}
    u_1 u_2 &= \delta_1 u_3, \quad u_1^2 = \delta_3 \delta_1; \\
    u_2 u_3 &= \delta_2 u_1, \quad u_2^2 = \delta_1 \delta_2; \\
    u_3 u_1 &= \delta_3 u_2, \quad u_3^2 = \delta_2 \delta_3.
\end{align*}
\]
From the birational point of view, as done by Burniat, we are simply adjoining to the function field of \( \mathbb{P}^2 \) two square roots, namely \( \sqrt{\Delta_1} \), \( \sqrt{\Delta_2} \), where \( \Delta_i \) is the cubic polynomial in \( \mathbb{C}[x_0, x_1, x_2] \) whose zero set has \( D_i - E_{i+2} \) as strict transform.

This shows clearly that we have a Galois cover \( X' \to Y \) with group \((\mathbb{Z}/2\mathbb{Z})^2\).

The equations above give a biregular model \( X' \) which is nonsingular exactly if the divisor \( D \) does not have points of multiplicity 3 (there cannot be points of higher multiplicities). These points give then quotient singularities of type \( 1/4(1,1) \), i.e., isomorphic to the quotient of \( \mathbb{C}^2 \) by the action of \((\mathbb{Z}/4\mathbb{Z})\) sending \((u,v) \mapsto (iu, iv)\) (or, equivalently, the affine cone over the 4-th Veronese embedding of \( \mathbb{P}^1 \)).

**Definition 1.1.** A primary Burniat surface is a surface constructed as above, and which is moreover smooth. It is then a minimal surface \( S \) with \( K_S \) ample, and with \( K_S^2 = 6 \), \( p_g(S) = q(S) = 0 \).

A secondary Burniat surface is the minimal resolution of a surface \( X' \) constructed as above, and which moreover has \( 1 \leq m \leq 2 \) singular points (necessarily of the type described above). Its minimal resolution is then a minimal surface \( S \) with \( K_S \) nef and big, and with \( K_S^2 = 6 - m \), \( p_g(S) = q(S) = 0 \).

A tertiary (respectively, quaternary) Burniat surface is the minimal resolution of a surface \( X' \) constructed as above, and which moreover has \( m = 3 \) (respectively \( m = 4 \)) singular points (necessarily of the type described above). Its minimal resolution is then a minimal surface \( S \) with \( K_S \) nef and big, but not ample, and with \( K_S^2 = 6 - m \), \( p_g(S) = q(S) = 0 \).

**Remark 1.2.** 1) We remark that for \( K_S^2 = 4 \) there are two possible types of configurations. The one where there are three collinear points of multiplicity at least 3 for the plane curve formed by the 9 lines leads to a Burniat surface \( S \) which we call of nodal type, and with \( K_S \) not ample, since the inverse image of the line joining the 3 collinear points is a \((-2)\)-curve (a smooth rational curve of self intersection \(-2\)).

In the other cases with \( K_S^2 = 4, 5, 6 \), instead, \( K_S \) is ample.

2) In the nodal case, if we blow up the two \((1,1,1)\) points of \( D \), we obtain a weak Del Pezzo surface \( \tilde{Y} \), since it contains a \((-2)\)-curve. Its anticanonical model \( Y' \) has a node (an \( A_1 \)-singularity, corresponding to the contraction of the \((-2)\)-curve). In the non nodal case, we obtain a smooth Del Pezzo surface \( \tilde{Y} = Y' \) of degree 4.

3) We illustrated the possible configurations of the lines in the plane in figure 1.

We will mostly restrict ourselves in the following to secondary Burniat surfaces. Therefore the branch divisor \( D \) on \( Y \) has one \((P_4)\) or two
singular points \((P_4, P_5)\) of type \((1, 1, 1)\) according to whether we are in the case \(K_S^2 = 5\) or in the case \(K_S^2 = 4\).

Since looking at the graphical picture might not be sufficient, we describe our situation also through an appropriate mathematical notation. Let \(\widetilde{Y} \to Y\) be the blow up of \(Y\) in \(P_4\) (if \(K_S^2 = 5\)), respectively in the points \(P_4, P_5\) (if \(K^2 = 4\)). Let \(E_4\) (resp. \(E_5\)) be the exceptional curve lying over \(P_4\) (resp. over \(P_5\)).

We have summarized in the tables (1), (2), (3) the linear equivalence classes of the divisors \(D_{i,j}\), which are the strict transforms of lines \(D'_{i,j}\) in \(\mathbb{P}^2\).

| Table 1. \(K_S^2 = 5\) |
|---|
| \((i, 1)\) | \(L - E_i - E_{i+1}\) |
| \((i, 2)\) | \(L - E_i - E_4\) |
| \((i, 3)\) | \(L - E_i\) |

| Table 2. \(K_S^2 = 4\): non nodal |
|---|
| \((i, 1)\) | \(L - E_i - E_{i+1}\) |
| \((i, 2)\) | \(L - E_i - E_4\) |
| \((i, 3)\) | \(L - E_i - E_5\) |

| Table 3. \(K_S^2 = 4\): nodal |
|---|
| \((1, 1)\) | \(L - E_1 - E_{i+1}\) |
| \((1, 2)\) | \(L - E_1 - E_4 - E_5\) |
| \((1, 3)\) | \(L - E_1\) |
| \((2, 2)\) | \(L - E_2 - E_4\) |
| \((2, 3)\) | \(L - E_2 - E_5\) |
| \((3, 2)\) | \(L - E_3 - E_4\) |
| \((3, 3)\) | \(L - E_3 - E_5\) |

We see easily from tables (1), (2), (3) some formulae which hold uniformly for all Burniat surfaces:

**Remark 1.3.**

i) \(D_i \equiv -K_{\widetilde{Y}} - 2E_i + 2E_{i+2}\)

ii) \(L_i \cong \mathcal{O}_{\widetilde{Y}}(-K_{\widetilde{Y}} + E_i - E_{i-1})\) since \(L_i \cong \mathcal{O}_{\widetilde{Y}}(L_i)\), where \(L_i \equiv \frac{1}{2}(D_{i-1} + D_{i+1}).\)

This yields \(L_i \equiv 3L - E_{i+1} - 2E_{i-1} - E_4\) for \(K^2 = 5\), and \(L_i = 3L - E_{i+1} - 2E_{i-1} - E_4 - E_5\), for \(K^2 = 4\).

iii) \(D_i - L_i \equiv -3E_i + 3E_{i-1}.\)
We have the following

**Lemma 1.4.** Let $L_i$, $i = 1, 2, 3$ be as in the above remark. Then $H^1(\tilde{Y}, \mathcal{O}_Y(-L_i)) = 0$.

**Proof.** If one has a reduced connected curve $C$ on a surface $Y$ with $H^1(\mathcal{O}_Y) = 0$, then necessarily $H^1(\mathcal{O}_Y(-C)) = 0$.

Since $L_i \equiv -K_{\tilde{Y}} + E_i - E_{i-1}$ we easily find this divisor $C$ in the linear subsystem

$$D_{i-1,2} + D_{i-1,3} + |L - E_{i+1}|,$$

taking a general line in $|L - E_{i+1}|$. 

\[ \square \]

**Remark 1.5.** From the long exact cohomology sequence associated to the short exact sequence of sheaves

$$0 \to \mathcal{O}_{\tilde{Y}}(-L_i) \to \mathcal{O}_{\tilde{Y}}(D_i - L_i) \to \mathcal{O}_{D_i}(D_i - L_i) \to 0,$$

and lemma [1.4] it follows that

$$H^0(\tilde{Y}, \mathcal{O}_{\tilde{Y}}(D_i - L_i)) \cong H^0(D_i, \mathcal{O}_{D_i}(D_i - L_i)),$$

for all $i \in \{1, 2, 3\}$.

We refer to [Cat84b], def. 2.8, page 494, and to [Cat99], p. 106 for the definition of the family of natural deformations of a bidouble cover.

**Proposition 1.6.** Let $S$ be the minimal model of a Burniat surface, given as Galois $(\mathbb{Z}/2\mathbb{Z})^2$-cover of the (weak) Del Pezzo surface $\tilde{Y}$. Then all natural deformations of $\pi: S \to \tilde{Y}$ are Galois $(\mathbb{Z}/2\mathbb{Z})^2$-covers of $\tilde{Y}$.

**Proof.** The natural deformations of a bidouble cover are parametrized by the direct sum of the vector spaces $H^0(\tilde{Y}, \mathcal{O}_{\tilde{Y}}(D_i))$ with the vector spaces $H^0(\tilde{Y}, \mathcal{O}_{\tilde{Y}}(D_i - L_i))$. The second summand is zero exactly when all the natural deformations are Galois.

As observed in iii) of remark [1.3] in all the cases we have

$$D_i - L_i \equiv -3E_i + 3E_{i+2}, \ \forall i \in \{1, 2, 3\}$$

Assume that there exists an effective divisor $C \in |D_i - L_i|$. Then $C \cdot E_{i+2} = -3$, whence $C \geq 3E_{i+2}$. Therefore we can write $C = C' + 3E_{i+2}$, with $C' \in |-3E_i|$, a contradiction. This implies that $|D_i - L_i| = \emptyset$.

\[ \square \]

**Remark 1.7.** It is easy to see that the respective dimensions of the families of Burniat surfaces are

- 4 for $K^2 = 6$;
- 3 for $K^2 = 5$;
- 2 for $K^2 = 4$, non nodal;
- 2 for $K^2 = 4$, nodal.
- 1 for $K^2 = 3$.

An important feature of each family of Burniat surfaces is that the canonical models do not get worse singularities for special elements of the family.

The minimal model $S$ of a Burniat surface is a smooth bidouble cover of a smooth weak Del Pezzo surface $	ilde{Y}$, branched over a normal crossings divisor. $K_S$ is ample for $K_S^2 \geq 4$ unless we are in the nodal case with $K_S^2 = 4$.

In this nodal case one has a singular Del Pezzo surface $Y'$ with an $A_1$-singularity obtained contracting the $(-2)$ curve $D_{1,2}$.

The canonical model $X$ of $S$ is obtained contracting the $(-2)$ curve $E$ which is the inverse image of $D_{1,2}$. $X$ is a finite bidouble cover of $Y'$.

In this last case we shall preliminarily investigate numerical invariants of the Kuranishi family of $S$ and then use them to describe the Kuranishi family of $X$.

Our first goal will be to determine $\dim H^1(S, \Theta_S)$, using the following special case of theorem 2.16 of [Cat84b]:

**Proposition 1.8.** Let $\pi : S \to \tilde{Y}$ be a Galois $(\mathbb{Z}/2\mathbb{Z})^2$-cover of smooth projective surfaces with branch divisor $D := D_1 + D_2 + D_3$. Then

$$
\pi_*(\Omega^1_S \otimes \Omega^2_S) = (\Omega^1_{\tilde{Y}}(\log D_1, \log D_2, \log D_3) \otimes \Omega^2_{\tilde{Y}}) \oplus \\
\bigoplus_{i=1}^3 \Omega^1_{\tilde{Y}}(\log D_i) \otimes \Omega^2_{\tilde{Y}} \otimes O_{\tilde{Y}}(L_i),
$$

where $\Omega^1_{\tilde{Y}}(\log D_1, \log D_2, \log D_3)$ is the subsheaf of the sheaf of rational 1-forms generated by $\Omega^1_{\tilde{Y}}$ and by $d\log(\delta_1), d\log(\delta_2), d\log(\delta_3)$, and where $D_i = \text{div}(\delta_i)$.

Moreover the first summand is the invariant one, and the other three correspond to the three non trivial characters of $(\mathbb{Z}/2\mathbb{Z})^2$.

We are able to use the above result observing in fact that the sheaf $\Omega^1_S \otimes \Omega^2_S = \Omega^1_S(K_S)$ is the Serre dual of $\Theta_S$, and that for each locally free sheaf $\mathcal{F}$ on $S$ we have (the second formula is duality for a finite map, cf. [Har77], exercise 6.10, page 239):

- $H^i(\mathcal{F}) = H^i(\pi_*(\mathcal{F}))$,
- $\pi_*(\mathcal{F}^\vee(K_S)) \cong (\pi_*\mathcal{F})^\vee(K_{\tilde{Y}})$,
- $K_S = \pi^*(K_{\tilde{Y}} + L_1 + L_2 + L_3)$
- $H^i(\Theta_S)^\vee = H^{2-i}(\pi_*(\Omega^1_S \otimes \Omega^2_S))$.

Moreover, we use the following exact residue sequence

$$
0 \to \Omega^1_{\tilde{Y}} \to \Omega^1_{\tilde{Y}}(\log D_1, \ldots, \log D_k) \to \bigoplus_{i=1}^k O_{D_i} \to 0
$$

holding more generally if the divisors $D_i$ are reduced and $\tilde{Y}$ is a factorial variety (see e.g. lemma 3, page 675 of [CHKS06]).
We are left with the calculation of the cohomology groups of the sheaves:

$$
\Omega^1_Y(\log D_1, \log D_2, \log D_3)(K_Y),
$$

respectively

$$
\Omega^1_Y(\log D_i)(K_Y + L_i).
$$

However, the second cohomology groups vanish since $S$ is of general type hence $H^0(\Theta_S) = 0$. The Riemann Roch theorem tells us what are the alternating sums of the dimensions, thus in the end it suffices to calculate the $H^0$ of these sheaves.

Let us look at the invariant part, using the exact sequence

$$
0 \to \Omega^1_Y(K_Y) \to \Omega^1_Y(\log D_1, \log D_2, \log D_3)(K_Y) \to \bigoplus_{i=1}^3 O_{D_i}(K_Y) \to 0.
$$

The space $H^0(\Omega^1_Y(K_Y))$ vanishes since $H^0(\Omega^1_Y) = 0$ and $-K_Y$ is effective.

Moreover, if $\tilde{Y}$ is a Del Pezzo surface, then $-K_{\tilde{Y}}$ is ample and also $H^0(O_{D_i}(K_{\tilde{Y}})) = 0$.

Thus $H^0(\Omega^1_Y(\log D_1, \log D_2, \log D_3)(K_Y)) = 0$ unless we are in the nodal case. Here there is the (-2) curve $D_{1,2}$ which is a connected component of $D_1$, hence in this case $H^0(O_{D_i}(K_{\tilde{Y}})) \cong \mathbb{C}$.

On the other hand the coboundary in the long exact cohomology sequence is given by cup product with the extension class, which is the direct sum of the Chern classes of the divisors $D_i$, $c_1(D_i) \in H^1(\Omega^1_Y)$.

Note, in the nodal case, that $-K_{\tilde{Y}} = |3L - \sum_{i=1}^5 E_i|$ is base point free. Therefore there is a morphism $O_{\tilde{Y}}(K_{\tilde{Y}}) \to O_{\tilde{Y}}$, which is not identically zero on any component of the $D_i$’s.

We get the commutative diagram with exact rows

$$
\begin{array}{cccc}
0 & \to & \Omega^1_Y(K_Y) & \to & \Omega^1_Y(\log D_1, \log D_2, \log D_3)(K_Y) & \to & \bigoplus_{i=1}^3 O_{D_i}(K_Y) & \to & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \Omega^1_Y & \to & \Omega^1_Y(\log D_1, \log D_2, \log D_3) & \to & \bigoplus_{i=1}^3 O_{D_i} & \to & 0 \\
\end{array}
$$

From this we get the commutative diagram

$$
\begin{array}{c}
\mathbb{C} \cong H^0(\tilde{Y}, \bigoplus_{i=1}^3 O_{D_i}(K_{\tilde{Y}})) \xrightarrow{\delta} H^1(\tilde{Y}, \Omega^1_{\tilde{Y}}(K_{\tilde{Y}})) \\
\xrightarrow{\psi_2} H^0(\tilde{Y}, \bigoplus_{i=1}^3 O_{D_i}) \xrightarrow{\psi_1} H^1(\tilde{Y}, \Omega^1_{\tilde{Y}}). \\
\end{array}
$$

Note that by a straightforward extension of the argument given in [Cat84b], lemma 3.7, the image of the function identically equal to 1 on $D_{1,2}$ maps under $\psi_1$ to the first Chern class of $D_{1,2}$. Hence $\varphi = \psi_1 \circ \psi_2 \neq 0$, hence also $\delta$ is non zero.
Lemma 1.9. For a primary or secondary Burniat surface the $G := (\mathbb{Z}/2\mathbb{Z})^2$-invariant part $H^0(\Omega^1_S \otimes \Omega^2_S))^G$ of $H^0(\Omega^1_S \otimes \Omega^2_S))$ vanishes.

Let us now turn to the other characters. We have then the other exact sequence

$$0 \to \Omega^1_Y(K_\tilde{Y} + L_i) \to \Omega^1_Y(\log D_i)(K_\tilde{Y} + L_i) \to \mathcal{O}_{D_i}(K_\tilde{Y} + L_i) \to 0$$

and we recall that, by remark 1.3

$$\Omega^1_Y(\log D_i)(K_\tilde{Y} + L_i) \cong \Omega^1_Y(\log D_i)(E_i - E_{i+2}).$$

We shall calculate the dimension of the space

$$H^0(\Omega^1_Y(\log D_i)(E_i - E_{i+2}))$$

taking the direct image sheaf on $\mathbb{P}^2$.

We need a lemma which we state for simplicity in the case of dimension 2: it shows what is the effect of blowing down a (-1) curve.

Lemma 1.10. Consider a finite set of distinct linear forms

$$l_\alpha := y - c_\alpha x, \alpha \in A$$

vanishing at the origin in $\mathbb{C}^2$. Let $p: Z \to \mathbb{C}^2$ be the blow up of the origin, let $D_\alpha$ be the strict transform of the line $L_\alpha := \{l_\alpha = 0\}$, and let $E$ be the exceptional divisor.

Let $\Omega^1_C_Z((d \log l_\alpha)_{\alpha \in A})$ be the sheaf of rational 1-forms $\eta$ generated by $\Omega^1_C$ and by the differential forms $d \log l_\alpha$ as an $\mathcal{O}_C_Z$-module and define similarly $\Omega^1_C((d \log D_\alpha)_{\alpha \in A})$. Then:

1. $p_*(\Omega^1_C((d \log l_\alpha)_{\alpha \in A})) = \Omega^1_C$,
2. $p_*(\Omega^1_C((d \log l_\alpha)_{\alpha \in A})) = p_*(\Omega^1_C((d \log l_\alpha)_{\alpha \in A})) = \Omega^1_C((d \log l_\alpha)_{\alpha \in A})$,
3. $p_*(\Omega^1_C((d \log l_\alpha)_{\alpha \in A})) = \{\eta \in \Omega^1_C((d \log l_\alpha)_{\alpha \in A})| \eta = \sum_\alpha g_\alpha d \log l_\alpha + \omega, \omega \in \Omega^1_C, \sum_\alpha g_\alpha(0) = 0\}$.

Proof. The sheaf $\Omega^1_C((d \log l_\alpha)_{\alpha \in A})$ is locally free outside of the origin, and torsion free in view of the residue sequence, since $\bigoplus_{\alpha \in A} O_{L_\alpha}$ has no section with a 0-dimensional support.

Likewise, all other direct image sheaves are torsion free, and those in 2. and 3. are equal to $\Omega^1_C((d \log l_\alpha)_{\alpha \in A})$ outside of the origin.

1.: $p_*(\Omega^1_C((d \log l_\alpha)_{\alpha \in A})) \subset \Omega^1_C$ holds since the left hand side is torsion free and coincides with the right hand side outside the origin. But $\Omega^1_C$ is locally free, hence it enjoys the Hartogs property, so the desired inclusion holds. It suffices then to show that $p^*(\Omega^1_C) \subset \Omega^1_C((d \log l_\alpha)_{\alpha \in A})$. This follows since in the affine chart $(x, t) \mapsto (x, y = xt)$ of the blow up, we have $dx = xd \log x, dy = x(dt + td \log x)$ (and similarly on the other chart).
Proof. The previous lemma shows that, since $f$ implies that 

The condition that 

The first inclusion follows, the two sheaves being torsion free and 
equal outside of the origin, from the assertion that $p^*\Omega^1_{\mathbb{C}^2}((d\log l_a)_{a\in A}) \subset \Omega^1_Z((d\log l_a)_{a\in A})$.
This assertion is easily verified in each affine chart, since $d\log l_a = d\log x + d\log(t - c_a)$.

The second inclusion is obvious, while, for the third, 

consists of rational differential 1-forms $\omega$ which, when restricted to 
$\mathbb{C}^2 \setminus \{0\}$, yield sections of $\Omega^1_{\mathbb{C}^2}((d\log l_a)_{a\in A})$.

Therefore in particular $\omega \prod_{a\in A} l_a$ is a regular holomorphic 1-form on 
$\mathbb{C}^2$.

Thus, modulo holomorphic 1-forms, we can write 

$$\omega = \frac{f}{\prod_{a\in A} l_a} dx + \frac{g}{\prod_{a\in A} l_a} dy,$$

where $f, g$ are pseudopolynomials of degree in $y$ less than $r := \text{card}(A)$.

By Hermite interpolation we can write $f = \sum_{a\in A} f_a l_a^{-1} \prod_{\beta\in A} l_\beta$, $g = \sum_{a\in A} g_a l_a^{-1} \prod_{\beta\in A} l_\beta$, so that finally, up to a holomorphic 1-form, 

$$\omega = \sum_{a\in A} \frac{f_a dx + g_a dy}{l_a}.$$

The condition that $\omega$ restricted to $\mathbb{C}^2 \setminus \{0\}$ yields a section of $\Omega^1_{\mathbb{C}^2}((d\log l_a)_{a\in A})$ implies that $f_a = -c_a g_a$.

Whence, finally, modulo holomorphic 1-forms, we can write $\omega = \sum_{a\in A} g_a d\log l_a$.

To prove the last statement, pull back such a 1-form $\omega$: $p^*\omega = \sum_{a\in A} p^*(g_a d\log l_a) = (\sum_{a\in A} g_a) d\log x + \sum_{a\in A} g_a d\log(t - c_a)$.

This form lies in $\Omega^1_Z((\log D_a)_{a\in A})$ if and only if $p^*(\sum_{a\in A} g_a(0)) = 0$.

\[ \square \]

Corollary 1.11. The dimension of the space $H^0(\Omega^1_{\mathbb{C}^2}(\log D_{i})(E_i-E_{i+2}))$ is equal to 

- 2 in the case $K^2_S = 6$,
- 1 in the case $K^2_S = 5$,
- 0 in the non nodal case $K^2_S = 4$,
- 0, 1 in the nodal case $K^2_S = 4$, according to $i \neq 1, i = 1$.

\[ \text{Proof.} \] The previous lemma shows that, since $D_i = D_{i,1} + D_{i,2} + D_{i,3} + E_{i+2}$, which by the way consists of four disjoint curves, then $H^0(\Omega^1_{\mathbb{C}^2}(\log D_i)(E_i-
the tangent sheaf \( \Theta_{E_i} \) maps onto \( H^0(\Omega^1_{\mathcal{E}}(\log D'_{i,1}, \log D'_{i,2}, \log D'_{i,3})) \), where \( D'_{i,j} \) is the line image of the curve \( D_i \).

By the residue exact sequence
\[
H^0(\Omega^1_{\mathcal{E}}(\log D'_{i,1}, \log D'_{i,2}, \log D'_{i,3})) = \{(c_j) \in \mathbb{C}^3 | \sum_j c_j = 0 \} \cong \mathbb{C}^2.
\]

By 3. we get the subspace of \( \{(c_j) \in \mathbb{C}^3 | \sum_j c_j = 0 \} \) such that \( c_j = 0 \) iff \( D'_{i,j} \) contains \( P_4 \) or \( P_5 \). The rest is a trivial verification.

\[ \square \]

**Lemma 1.12.** i) \( \chi(\mathcal{O}_{D_1}(E_i - E_{i+2})) = 8 \),

ii) \( \chi(\Omega^1_{\tilde{Y}}(E_i - E_{i+2})) = -e(\tilde{Y}) = K^2_{\tilde{Y}} - 12. \)

In particular, it follows that \( \chi(\Omega^2_{\tilde{Y}}(\log D_i)(E_i - E_{i+2})) = 8 - e(\tilde{Y}) = K^2_{\tilde{Y}} - 4. \)

**Proof.** The third assertion follows from the first two in view of the exact sequence of locally free sheaves on \( \tilde{Y} \):
\[
0 \to \Omega^1_{\tilde{Y}}(E_i - E_{i+2}) \to \Omega^1_{\tilde{Y}}(\log D_i)(E_i - E_{i+2}) \to \mathcal{O}_{D_i}(E_i - E_{i+2}) \to 0.
\]

i) Observe that for \( 1 \leq i, j \leq 3 \), we have \( (E_i - E_{i+2}) \cdot D_{i,j} = 1 = (E_i - E_{i+2}) \cdot E_{i+2} \), whence \( \chi(\mathcal{O}_{D_i}(E_i - E_{i+2})) = 4 \cdot \chi(\mathcal{O}_{\mathbb{P}^1}(1)) = 8. \)

ii) In order to calculate \( \chi(\Omega^1_{\tilde{Y}}(E_i - E_{i+2})) \) we use the splitting principle and write formally \( \Omega^1_{\tilde{Y}} = \mathcal{O}_{\tilde{Y}}(A_1) \oplus \mathcal{O}_{\tilde{Y}}(A_2) \), where \( A_1, A_2 \) are “divisors” such that \( A_1 + A_2 \equiv K_{\tilde{Y}}, A_1 \cdot A_2 = -e(\tilde{Y}) = 12 - K^2_{\tilde{Y}} \). Using that \( (E_i - E_{i+2})^2 = -2 \), \( K_{\tilde{Y}} \cdot (E_i - E_{i+2}) = 0 \), we obtain
\[
\chi(\Omega^1_{\tilde{Y}}(E_i - E_{i+2})) = \chi(\mathcal{O}_{\tilde{Y}}(A_1 + E_i - E_{i+2})) + \chi(\mathcal{O}_{\tilde{Y}}(A_2 + E_i - E_{i+2})) =
\]
\[
= 2 + \frac{1}{2}((A_1 + E_i - E_{i+2})(E_i - E_{i+2} - A_2) + (A_2 + E_i - E_{i+2})(E_i - E_{i+2} - A_1)) =
\]
\[
= 2 + \frac{1}{2}(-2 - 2 - 2A_1 \cdot A_2) = -e(\tilde{Y}).
\]

\[ \square \]

**Corollary 1.13.** Let \( S \) be the minimal model of a Burniat surface.

Then the dimensions of the eigenspaces of the cohomology groups of the tangent sheaf \( \Theta_S \) (for the natural \((\mathbb{Z}/2\mathbb{Z})^2\)-action) are as follows.

- \( K^2 = 6: h^1(S, \Theta_S)^{inv} = 4, h^2(S, \Theta_S)^{inv} = 0, h^1(S, \Theta_S)^i = 0, h^2(S, \Theta_S)^i = 2 \), for \( i \in \{1, 2, 3\} \);
- \( K^2 = 5: h^1(S, \Theta_S)^{inv} = 3, h^2(S, \Theta_S)^{inv} = 0, h^1(S, \Theta_S)^i = 0, h^2(S, \Theta_S)^i = 1 \), for \( i \in \{1, 2, 3\} \);
- \( K^2 = 4 \) of non nodal type: \( h^1(S, \Theta_S)^{inv} = 2, h^2(S, \Theta_S)^{inv} = 0, h^1(S, \Theta_S)^i = h^2(S, \Theta_S)^i = 0 \), for \( i \in \{1, 2, 3\} \);
- \( K^2 = 4 \) of nodal type: \( h^1(S, \Theta_S)^{inv} = 2, h^2(S, \Theta_S)^{inv} = 0, h^1(S, \Theta_S)^i = 1 = h^2(S, \Theta_S)^i, h^j(S, \Theta_S)^i = 0 \), for \( i \in \{2, 3\} \).
Proof. It is a straightforward consequence of corollary 1.11 of lemma 1.12 and of the Enriques-Kuranishi formula ̂(ΘS) = -10(̂OS) + 2K2S.

We are in the position to state the main results of this section.

Recall that for surfaces of general type we have two moduli spaces: one is the moduli space Mnχ,K2 for minimal models S having ̂(̂OS) = ̂χ, K2S = K2, the other is the moduli space Mcanχ,K2 for canonical models X having ̂(̂OX) = ̂χ, K2X = K2; the latter is called the Gieseker moduli space. Both are quasi projective schemes by Gieseker’s theorem (Gies77) and by the fact that there is a natural morphism Mnχ,K2 → Mcanχ,K2 which is a bijection. Their local structure as complex analytic spaces is the quotient of the base of the Kuranishi family by the action of the finite group Aut(S) = Aut(X). Usually the scheme structure of Mnχ,K2 tends to be more singular than the one of Mcanχ,K2 (see e.g. Cat89).

Recall moreover that in the following theorem K is always ample, thus the minimal and canonical model coincide. Instead, later on, for surfaces with K2 = 4 of nodal type, S contains exactly one -2 curve E, thus the canonical model X has always exactly one singular point, an A1-singularity.

Theorem 1.14. The three respective subsets of the moduli spaces of minimal surfaces of general type Mnχ,K2 corresponding to Burniat surfaces with K2 = 6, resp. with K2 = 5, resp. Burniat surfaces with K2 = 4 of non nodal type, are irreducible open sets, normal, unirational of respective dimensions 4,3,2.

More precisely, the base of the Kuranishi family of S is smooth.

Proof. By corollary 1.13 the tangent space to the Kuranishi family of S, H1(ΘS), consists of invariants for the action of the group G := (Z/2Z)2.

It follows then (cf. Cat88 lecture three, page 23) that all the local deformations admit the G-action, hence they are bidouble covers of a deformation of the Del Pezzo surface Y.

Moreover, the dimension of H1(ΘS) coincides with the dimension of the image of the Burniat family containing S in the moduli space Mnχ,K2, hence the Kuranishi family of S is smooth, and coincides with the Burniat family by the above argument.

Alternatively, one could show directly that the Kodaira Spencer map is bijective, or simply observe that a finite morphism between smooth manifolds of the same dimension is open.

Observe then that the quotient of a smooth variety by the action of a finite group is normal.

Finally, the Burniat family is parametrized by a (smooth) rational variety. 

□
We shall prove in the final section that these irreducible components are not only unirational, but indeed rational.

**Theorem 1.15.** The subset of the Gieseker moduli space $\mathcal{M}_{\text{can}}$ of canonical surfaces of general type $X$ corresponding to Burniat surfaces $S$ with $K_S^2 = 4$ and of nodal type is an irreducible open set of dimension 2, unirational and everywhere non reduced.

More precisely, the base of the Kuranishi family of $X$ is locally analytically isomorphic to $\mathbb{C}^2 \times \text{Spec}(\mathbb{C}[t]/(t^m))$, where $m$ is a fixed integer, $m \geq 2$.

The corresponding subset of the moduli space $\mathcal{M}_{\text{min}}$ of minimal surfaces $S$ of general type is also everywhere non reduced.

More precisely, the base of the Kuranishi family of $S$ is locally analytically isomorphic to $\mathbb{C}^2 \times \text{Spec}(\mathbb{C}[t]/(t^{2m}))$.

The rest of this section is devoted to the proof of theorem 1.15, the first argument, as in theorem 1.14, being that all the local deformations admit the $G$-action. This is more difficult to show since $H^1(\Theta_S)$ consists of the direct sum of the space of $G$-invariants, which has dimension 2, and of a 1-dimensional character space.

Let $S$ be the minimal model of a Burniat surface with $K^2 = 4$ of nodal type, let $X$ be its canonical model, and denote by $\pi: S \rightarrow X$ the blow down of the unique $(-2)$-curve $E$ of $S$ (lying over $D_{1,2}$).

By [BW74] we know that $\pi_* \Theta_S = \Theta_X$ and that $H^1_{\mathcal{E}}(\Theta_S)$ has dimension 1.

It follows then from the Leray spectral sequence for $\pi_*:

- $H^2(S, \Theta_S) = H^2(X, \Theta_X)$,
- $H^1(S, \Theta_S) = H^1(X, \Theta_X) \oplus R^1 \pi_* \Theta_S = H^1(\Theta_X) \oplus H^1_{\mathcal{E}}(\Theta_S)$.

In particular $h^1(\Theta_X) = 2$.

On the other hand, by the local to global Ext-spectral sequence, we have the “five term exact sequence”:

$$0 \rightarrow H^1(X, \Theta_X) \rightarrow \text{Ext}^1_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X) \rightarrow H^0(X, \mathcal{E}xt^1_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X)) \rightarrow H^2(X, \Theta_X) \rightarrow \text{Ext}^2_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X) \rightarrow 0.$$

Note that the above exact sequence is a $G = (\mathbb{Z}/2\mathbb{Z})^2$-equivariant sequence of $\mathbb{C}$-vector spaces, since all sheaves have a natural $G$-linearization.

We proceed now to calculate the decomposition in character spaces of the single terms of the exact sequence.

**Lemma 1.16.** The 1-dimensional space $H^0(X, \mathcal{E}xt^1_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X))$ is a space of invariants for the $G$-action.

**Proof.** Recall that $D_{1,2}$ is a $(-2)$-curve on $\tilde{Y}$ and $X$ is a bidouble cover of the nodal Del Pezzo surface $Y'$ of degree 4 obtained contracting $D_{1,2}$.

Moreover, the curve $D_{1,2}$ intersects exactly two other irreducible components of the branch locus, namely, $D_{2,1}$ and $E_1$, which is also a component of $D_2$. 
We want to describe the structure of the morphism \( f: X \to Y' \) locally around the \( A_1 \)-singular point \( P' \).

Locally around \( P' \) we can assume that \( Y' = \{ z^2 = xy \} \).

Then, locally around the node \( P \) of \( X \), \( X = \{ w^2 = uv \} \), and the bidouble covering \( f: X \to Y' \) is given by the equations: \( w^2 = z \), \( u^2 = x \), \( v^2 = y \).

In fact, the intermediate double cover branched only on \( D_1,2 \) corresponds to the double cover branched only on \( P' \), and given by \( \Phi: \mathbb{C}^2 \to Y' \), such that \( \Phi(u, v) = (u^2, v^2, uv) := (x, y, z) \), while \( X \) is the double cover \( w^2 = uv \) branched on the inverse images of the lines \( x = z = 0 \) and \( y = z = 0 \) (observe that for \( A_1 \) the two \( G \) actions listed in Table 3 of \cite{Cat87}, page 93 are conjugate to each other).

The local deformation of the \( A_1 \)-singularity on \( X \) is given by \( X_t = \{ w^2 = uv + t \} \).

Then \( t \in \mathbb{C} \) is a trivial representation of \( G \) and therefore \( X_t \) yields a family of \( G \)-coverings of \( Y' \) described by the equations \( w^2 = z + t \), \( u^2 = x \), \( v^2 = y \). This proves the claim.

\( \square \)

Corollary 1.17. The obstruction map
\[ \text{ob}: H^0(X, \mathcal{E}xt^1_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X)) \to H^2(X, \Theta_X) \]

is identically zero.

Proof. Recall that “ob” is a \( G \)-equivariant homomorphism. Since \( H^0(X, \mathcal{E}xt^1_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X)) \) is a trivial \( G \)-representation, while \( H^2(X, \Theta_X) = H^2(S, \Theta_S) \) is a nontrivial \( G \)-representation (cf. corollary 1.13), it follows that \( \text{ob} = 0 \).

\( \square \)

Corollary 1.18. \( \text{Ext}^2_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X) = \text{Ext}^2_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X)^1 \).

Proof. Follows immediately from the exact sequence (2) and the above corollary.

\( \square \)

Lemma 1.19. \( H^0(X, \mathcal{R}^1\pi_* \Theta_S) = H^1_E(\Theta_S) \) is a non trivial character of \( G \).

Proof. By the theorem of Brieskorn-Tjurina \( \cite{Brie68, Tju70} \), the simultaneous resolution of the node on \( X \) is given by \( \frac{x - \tau}{w} = \frac{t}{w + \tau} \), where one has made the base change \( \tau^2 = t \), using the notation of the proof of lemma 1.16. The action of \( G \) lifts in a unique way to the simultaneous resolution of the family since \( \tau \) must be an eigenvector with character equal to the same character of \( w \) (observe that both \( w - \tau, w + \tau \) are eigenvectors).

Since \( \mathbb{C} \tau \cong H^1_E(\Theta_S) \) as \( G \)-representation, we have proven that \( H^1_E(\Theta_S) \) is an eigenspace corresponding to a non trivial character of \( G \).

\( \square \)
Since $H^1(S, \Theta_S) = H^1(\Theta_X) \oplus H^1_L(\Theta_S)$, the above lemma and corollary immediately imply the following

**Corollary 1.20.** $H^1(X, \Theta_X)$ is a trivial $G$-representation, hence also $\Ext^2_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X)$.

Now we are ready to prove the following

**Proposition 1.21.** Let $X$ be the canonical model of a Burniat surface with $K_S^2 = 4$ of nodal type. Then all deformations of $X$ are deformations of the pair $(X, G)$.

*Proof.* Since, by the above considerations, $G$ acts trivially on the base of the Kuranishi family of $X$, it follows that $\text{Def}(X) = \text{Def}(X)^G$.

□

The consequence is then that also all deformations of $S$ are deformations of the pair $(S, G)$.

The main theorem of this section (thm. 1.15) will now follow once we have proven

**Proposition 1.22.** Burniat surfaces with $K_S^2 = 4$ of nodal type yield an open set in the moduli space.

*Proof.* Let $S$ be the minimal model of a Burniat surface with $K_S^2 = 4$ of nodal type. Then $S/G = \hat{Y}$, where $\hat{Y}$ is a weak Del Pezzo surface.

Now, by the above, any small deformation $S_t$ of $S$ is in fact a deformation of $(S, G)$. It suffices to show that $S_t/G$ is again a weak Del Pezzo surface, i.e., the $(-2)$-curve remains under a small deformation.

We remark that the $(-2)$-curve on $\hat{Y}$ is $D_{1,2}$, which is a connected component of $D_2$, hence $E$ is a connected component of the fixed point set $\text{Fix}(\sigma_2)$ of an element $\sigma_2 \in G$.

Let now $S \to T$ be a one parameter family of minimal models, such that $G$ acts on $S \to T$, with trivial action on $T$ and the given action on the central fibre. Then the component of $\text{Fix}(\sigma_2)$ in $S$ has dimension 2, whence all the deformations $S_t$ of $S$ carry a $-2$ curve $E_t$ deformation of $E$. It follows that the quotient of $E_t$ yields a $-2$ curve on $\hat{Y}$.

In other words, we have shown that all deformations of $X$ are equisingular, therefore $\text{Def}(X) \subset H^1(\Theta_X)$. The Burniat family shows that $\text{dim}(\text{Def}(X)) \geq 2$, whence set theoretically $\text{Def}(X) = H^1(\Theta_X)$.

Choosing coordinates $(t_1, t_2, t_3)$ for $\Ext^1_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X)$ such that $\{t_3 = 0\}$ is the hyperplane $H^1(\Theta_X)$, we see that the Kuranishi equation is a power of $t_3$, say $t_3^m$. Since the Kuranishi equation has differential vanishing at the origin, it follows that $m \geq 2$.

Now, the local map $H^1(\Theta_S) \to \Ext^1_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X)$ (cf. theorem 2.6 of [BW74], see also [Cat89]) is given by $(s_1, s_2, s_3) \mapsto (s_1, s_2, s_3^2)$, and $\text{Def}(S)$ is the pull back of $\text{Def}(X)$. Hence $\text{Def}(S)$ is the subscheme $s_3^{2m} = 0$. □
2. One parameter limits of secondary Burniat surfaces

In this section we shall show that Burniat surfaces with $K^2 \geq 4$ form a closed set of the moduli space.

This will be accomplished through the study of limits of one parameter families of such Burniat surfaces.

We get a new result only in the case of secondary Burniat surfaces with $4 \leq K^2 \leq 5$. The argument is exactly the same for $K^2 = 6$, but in this case we are just giving a fourth proof after the ones given in [MLP01], in [AlPa09] and in part I ([BC09b]).

Let $Y'$ be a normal $\mathbb{Q}$-Gorenstein surface and denote the dualizing sheaf of $Y'$ by $\omega_{Y'}$.

Then there is a minimal natural number $m$ such that $\omega_{Y'}^\otimes m$ is an invertible sheaf and it makes sense to define $\omega_{Y'}$ to be ample, respectively anti-ample; $Y'$ is Gorenstein iff $m = 1$.

We shall need the following

Proposition 2.1. Let $Y'$ be a normal $\mathbb{Q}$-Gorenstein Del Pezzo surface (i.e., $\omega_{Y'}$ is anti-ample) with $K_{Y'}^2 \geq 4$. Then $Y'$ is in fact Gorenstein.

Proof. Assume that $m \geq 2$. Then (cf. [Rei85], Proposition on page 362), there is a $\mathbb{Z}/m\mathbb{Z}$-Galois covering $p: W \to Y'$ such that $W$ is Gorenstein and such that $K_W = p^* K_{Y'}$, where $\omega_{Y'}$ is the sheaf associated to the Weil divisor $K_{Y'}$. $p$ is only branched on the singular points of $Y'$ which are not Gorenstein.

Since $\omega_{Y'}$ is anti ample, it follows that $K_W$ is anti ample, hence $W$ is a normal Gorenstein Del Pezzo surface. As it is well known (cf. e.g. [CMa08], Theorem 4.3) $W$ is smoothable and in particular $K_W^2 \leq 9$, indeed $K_W^2 \leq 8$ if $W$ is singular.

On the other hand: $K_W^2 = mK_{Y'}^2 \geq 4m$ and this implies that $m = 2$, $K_{Y'}^2 = 4$.

Therefore $K_W^2 = 8$, whence either $W$ is the blow up of the plane in one point, or $W = \mathbb{Q}$ a quadric in $\mathbb{P}^3$.

If $W$ is smooth then $Y' = W/(\mathbb{Z}/2\mathbb{Z})$ has only $A_1$-singularities and is Gorenstein.

It remains therefore to exclude the case that $W$ is the quadric cone.

In this case $Y' = Q/i$, where $i$ is an involution on $Q$: since the quotient is not Gorenstein (see [Cat87], Table 2 and Theorem 2.2, page 90) it acts on the tangent space at the node of $Q$ as $-\text{Id}$.

The involution $i$ on $Q$ acts linearly on the anticanonical model of $Q$, thus $i$ extends to a linear involution $I$ on $\mathbb{P}^3$.

The vertex $v \in Q$ is an isolated fixed point of $I$, and $I$ acts as $-\text{Id}$ on the tangent space of $v$. Therefore $H^0(Q, \mathcal{O}_Q(1))$ splits into two eigenspaces of respective dimensions 3, 1.

In particular there is a pointwise fixed hyperplane $H \subset \mathbb{P}^3$ for $I$. Since then $C := Q \cap H$ is pointwise fixed by $I$, we contradict the fact that $I$ has only isolated fixed points on $Q$. 

Proposition 2.2. Let \( T \) be a smooth affine curve, \( t_0 \in T \), and let \( f : \mathcal{X} \to T \) be a flat family of canonical surfaces. Suppose that \( \mathcal{X}_t \) is the canonical model of a Burniat surface with \( 4 \leq K_{\mathcal{X}_t}^2 \) for \( t \neq t_0 \in T \). Then there is an action of \( G := (\mathbb{Z}/2\mathbb{Z})^2 \) on \( \mathcal{X} \) yielding a one parameter family of finite \((\mathbb{Z}/2\mathbb{Z})^2\)-covers

\[
\begin{array}{ccc}
\mathcal{X} & \to & \mathcal{Y} \\
\downarrow f & & \downarrow \Phi \\
T & \to & \\
\end{array}
\]

(i.e., \( \mathcal{X}_t \to \mathcal{Y}_t \) is a finite \((\mathbb{Z}/2\mathbb{Z})^2\)-cover), such that \( \mathcal{Y}_t \) is a Gorenstein Del Pezzo surface for each \( t \in T \).

Proof. Note that \( \mathcal{X} \) is Gorenstein, since \( T \) is smooth and the fibres have hypersurface singularities.

Since \( \mathcal{X} \setminus f^{-1}(t_0) \to T \setminus \{t_0\} \) is a family of canonical models of Burniat surfaces, we have a \((\mathbb{Z}/2\mathbb{Z})^2\)-action on \( \mathcal{X} \setminus f^{-1}(t_0) \) (this is the Galois group action for the bicanonical map).

By \cite{Cat83}, thm. 1.8, the \((\mathbb{Z}/2\mathbb{Z})^2\)-action extends to \( \mathcal{X} \).

We set \( \mathcal{Y} := \mathcal{X} / (\mathbb{Z}/2\mathbb{Z})^2 \) and we denote by \( \Phi \) the finite morphism \( \mathcal{X} \to \mathcal{Y} \).

We have for all \( t \in T \):

- \( K_{\mathcal{Y}_t} = K_{\mathcal{Y} \mid \mathcal{Y}_t} \);
- \( K_{\mathcal{X}_t} = K_{\mathcal{X} \mid \mathcal{X}_t} \).

Moreover,

\[
2K_{\mathcal{X}} = 2\Phi^*(K_{\mathcal{Y}}) + \mathcal{B},
\]

where \( \mathcal{B} \) is the branch divisor of \( \Phi : \mathcal{X} \to \mathcal{Y} \).

Since for \( t \neq t_0 \) we have \( 2K_{\mathcal{X}_t} = -\Phi^*(K_{\mathcal{Y}_t}) \), it follows that

\[
2K_{\mathcal{X}} + \Phi^*(K_{\mathcal{Y}}) \equiv 0 \quad \text{on} \quad \mathcal{X} \setminus \mathcal{X}_{t_0}.
\]

Since however \( \mathcal{X}_{t_0} = f^{-1}(t_0) \) is irreducible, we obtain (after possibly restricting \( T \)) that \( 2K_{\mathcal{X}} + \Phi^*(K_{\mathcal{Y}}) \equiv 0 \) on \( \mathcal{X} \).

In particular, \( 2K_{\mathcal{X}_t} = -\Phi^*(K_{\mathcal{Y}_t}) \) for all \( t \in T \), which implies that \( -K_{\mathcal{Y}_t} \) is ample for all \( t \in T \).

Moreover, \( K_{\mathcal{X}_t}^2 = K_{\mathcal{Y}_t}^2 \) for all \( t \in T \).

By construction, \( \mathcal{Y}_t \) is a Gorenstein Del Pezzo surface for \( t \neq t_0 \), and \( \mathcal{Y}_{t_0} \) is a normal \( \mathbb{Q} \)-Gorenstein Del Pezzo surface, whence it is Gorenstein by prop. \( \ref{prop:QGorensteinDelPezzo} \).

This implies immediately the following:

Corollary 2.3. Consider a one parameter family of bidouble covers \( \mathcal{X} \to \mathcal{Y} \) as in prop. \( \ref{prop:oneParameterFamily} \). Then the branch locus of \( \mathcal{X}_{t_0} \to \mathcal{Y}_{t_0} \) is the limit of the branch locus of \( \mathcal{X}_t \to \mathcal{Y}_t \), and it is reduced.
Note that the limit of a line on the del Pezzo surfaces $Y_t$ is a line on the del Pezzo surface $Y_{t_0}$, and, as a consequence of the above assertion, two lines in the branch locus in $Y_t$ cannot tend to the same line in $Y_{t_0}$.

**Remark 2.4.** Let $X$ be the canonical model of a Burniat surface with $4 \leq K_X^2 \leq 6$. Recall once more that $X$ is smooth for $K_X^2 = 6, 5,$ and for $K_X^2 = 4$ in the non nodal case. For $K_X^2 = 4$ and the nodal case, $X$ has one ordinary node.

In all three cases the branch locus consists of the union of 3 hyperplane sections, containing $\nu$ lines and $\frac{1}{2}(3K_X^2 - \nu)$ conics, where

- a) $\nu = 6$ for $K_X^2 = 6$,
- b) $\nu = 9$ for $K_X^2 = 5$,
- c) $\nu = 12$ for $K_X^2 = 4$ non nodal,
- d) $\nu = 10$ for $K_X^2 = 4$ nodal.

In fact, in case a) the 6 lines contained in the branch locus are: $D_{i,1}, 1 \leq i \leq 3, E_1, E_2, E_3$. In case b) the 9 lines contained in the branch locus are: $D_{i,j}, 1 \leq i \leq 3, 1 \leq j \leq 2, E_1, E_2, E_3$.

In case c) the 12 lines in the branch locus of the bidouble cover are: $D_{i,j}, 1 \leq i,j \leq 3, E_1, E_2, E_3,$ and finally in case d) the 10 lines are: $D_{1,1}, 1 \leq i \leq 3, D_{2,2}, D_{2,3}, D_{3,2}, D_{3,3}, E_1, E_2, E_3$.

We shall use the following:

**Proposition 2.5.** ([Cat84a], prop. 1.7] A weak Del Pezzo surface $W$ is either

- $\mathbb{P}^1 \times \mathbb{P}^1$, or
- $\mathbb{F}_2$, or
- the blow up $\mathbb{P}^2(P_1, \ldots, P_r)$, $r \leq 8$,

at $r$ distinct points $P_1, \ldots, P_r$ satisfying the following three conditions:

i) no more than 3 $P_i$'s are collinear;
ii) no more than 6 $P_i$'s lie on a conic;
iii) the set $\{P_1, \ldots, P_r\}$ can be partitioned into subsets $\{P_{i_1}, \ldots, P_{i_k}\}$ with $P_{i_1} \in \mathbb{P}^2$, $P_{i_{(j+1)}}$ infinitely near to $P_{i_j}$, but not lying on the proper transform of $P_{i_{(j-1)}}$.

Since weak Del Pezzo surfaces $W$ are exactly the minimal resolutions of singularities of normal Gorenstein Del Pezzo surfaces $Z$, we use the above result to show the following technical, possibly well known result:

**Proposition 2.6.** Let $Z$ be a normal Gorenstein Del Pezzo surface of degree $d$.

Then $Z$ contains no line for $d = 9, 8$ unless $Z = \mathbb{F}_1$, which contains one line.

For $d = 7$ $Z$ contains 2 or 3 lines, and is smooth in the latter case.

If $d = 6, 5, 4$ $Z$ contains at most 6, respectively 10, respectively 16 lines. If $Z$ contains at least 6, respectively 9, respectively 13 lines (i.e., irreducible curves $C$ with $C \cdot K_Z = -1$), then $Z$ is smooth.
Assume that $d = 4$ and that $Z$ contains at least 10 lines. Then we have the following possibilities:

i) $Z$ is smooth and has 16 lines;

ii) $Z$ has exactly one singular point, of type $A_{1}$, $Z$ has 12 lines and $Z$ is the anticanonical model of the weak Del Pezzo surface obtained blowing up the plane in 5 distinct points such that three of them are collinear.

Proof. If $W$ is $F_{0} = \mathbb{P}^{1} \times \mathbb{P}^{1}$, $\mathbb{F}_{2}$ or $\mathbb{P}^{2}$, then obviously $W$ contains no line.

Thus we may assume that $W$ is the blow up of the plane at $P_{1}, \ldots P_{r}$, with $r = 9 - d$. For $r = 1$ there is only the line $E_{1}$, where we denote as customary by $E_{i}$ the full transform of the point $P_{i}$.

Any line $C$ is in particular an effective divisor such that $C^{2} = CK_{W} = -1$, and in particular it is contained in some anticanonical divisor $H = 3L - \sum E_{j}$, where $L$ is the nef and big divisor pull back of a line of $\mathbb{P}^{2}$.

Thus $C \equiv aL - \sum b_{j}E_{j}$ and since $LC \geq 0$, $L(H - C) \geq 0$, one gets $0 \leq a \leq 3$.

As usual $C^{2} = CK_{W} = -1$ implies

$$a^2 + 1 = \sum b_{j}^2, \quad \sum j b_{j} = 3a - 1 \Rightarrow \sum j b_{j}(b_{j} - 1) = (a - 1)(a - 2).$$

The right hand side vanishes for $a = 1, 2$ and equals 2 for $a = 0, 3$ while each summand on the left side of the last equality is at least 2 unless $b_{j} = 0$ or $b_{j} = 1$.

Not considering the $b_{j}$’s equal to zero, for $a = 0$ one has one $b_{j} = -1$, for $a = 1$ one has two $b_{j} = 1$, for $a = 2$ one has five $b_{j} = 1$.

While $a = 3$ can only occur for $r \geq 7$, with one $b_{j}$ equal to 2, and six equal to 1.

This gives the a priori bound that the number of lines is at most

$$N(r) := r + \left(\begin{array}{c} r \\ 2 \end{array}\right) + \left(\begin{array}{c} r \\ 5 \end{array}\right).$$

This gives the number of lines in the case where $-K_{W}$ is ample, namely, for $d = 7, 6, 5, 4$ we get $r = 2, 3, 4, 5$ and $N = 3, 6, 10, 16$.

Since, if $-K_{W}$ is ample, each such divisor is linearly equivalent to an unique effective one which is irreducible.

If $-K_{W}$ is not ample but nef, then there are -2 curves $D$, i.e., irreducible divisors $D$ with $D \equiv aL - \sum b_{j}E_{j}$, $0 \leq a \leq 3$, and $D^2 = -2, DK_{W} = 0$. These conditions are equivalent to

$$a^2 + 2 = \sum b_{j}^2, \quad \sum j b_{j} = 3a \Rightarrow \sum j b_{j}(b_{j} - 1) = (a - 1)(a - 2).$$
By the same token $a = 1, 2$ implies $b_j = 1, 0$ and we get for $a = 1$
three $b_j = 1$, for $a = 2$ six $b_j = 1$. For $a = 0$ we get a divisor of the
form $E_i - E_j$, for $a = 3$ must be $r \geq 8$ and one $b_j = 2$, seven $b_j = 1$.
What is left is to show that each -2 curve $D$ makes the number of
lines diminish sufficiently.
For $a = 2$, we must have $r \geq 6$ (and then we lose 6 lines); for $a = 1,$
$D = L - E_i - E_j - E_k$, we lose 3 lines, since $L - E_i - E_j = D + E_k$.
We also lose, if $r \geq 5$, $C(r - 3, 2)$ lines of the form $D + (L - E_h - E_b)$.
Since we assume $r \leq 7$, let us see what happens if $D = E_i - E_j$
is effective. This means that $P_j$ is infinitely near to $P_i$, so we have a
string of infinitely near points as in iii) of proposition 2.5.
Assume that this string is $P_{i_1}, \ldots, P_{i_b}$. Then each $E_{i_k}$ is not irre-
ducible for $h = 1, \ldots, k - 1$. Also the effective divisor $L - E_j - E_{i_k}$ is
not irreducible for $h = 2, \ldots, k$, and for $P_j$ not infinitely near to $P_{i_1}$.
Moreover $L - E_{i_1} - E_{i_k}$ is effective, and contained in $L - E_{i_k} - E_{i_k}$,
whenever $h \leq l$ are not equal to 1, 2.
The loss is therefore at least 
\[(k - 1) + (k - 1)(r - k) + \frac{1}{2}(k + 1)(k - 2) = \]
\[= (k - 1)[r - (k - 1)] + \frac{1}{2}(k + 1)(k - 2). \]

For $k = 2$ we get a loss of $r - 1$ lines, otherwise a bigger loss.
We want to finally show that the case $r = 5$ and $k = 2$ yields the
same surface which is encountered for $r = 5$, no infinitely near points,
but 3 collinear points.
Consider then, as in the nodal case, 5 points such that $P_1, P_4, P_5$ are
collinear, and let $\Psi : \mathbb{P}^2 \rightarrow \mathbb{P}^2$ be the birational standard Cremona
transformation based on the points $P_1, P_2, P_3$. On the Del Pezzo $\tilde{Y}$
and $D$ we correspond to the linear system $2L - E_1 - E_2 - E_5$. This system contracts the -2 curve to a point, as
well as the lines $E_4, E_3, L - E_1 - E_2, L - E_2 - E_5$.
Since the -2 curve intersects $E_4$, we get also a representation of $\tilde{Y}$ as
the blow up of the plane in five points, of which one infinitely near to the
other.

\[\square\]

**Theorem 2.7.** Each family of Burniat surfaces with $K^2 = 4, 5, 6$ yields
a closed subset of the moduli space.

**Proof.** Consider a one parameter family of bidouble covers $X \rightarrow Y$ as in
prop. 2.2 such that $X_t \rightarrow Y_t$ is the bicanonical map of a Burniat surface
for $t \neq t_0$. Then $X_{t_0} \rightarrow Y_{t_0}$ is a bidouble cover of a normal Gorenstein
Del Pezzo surface of degree $K^2_{X_{t_0}}$ and $X_{t_0}$ has canonical singularities.
Moreover, the branch locus of $X_{t_0} \rightarrow Y_{t_0}$ is the limit of the branch
locus of $X_t \rightarrow Y_t$, hence it contains at least $3(8 - K^2_{X_t})$ lines in the non
nodal case, and 10 in the nodal case.
Then by proposition 2.6, \( Y_{t_0} \) is smooth for \( K_{X_{t_0}}^2 \geq 5 \), while for \( K_{X_{t_i}}^2 = 4 \) it has at most one node.

Thus, for \( K_{X_{t_0}}^2 \geq 5 \), \( X_{t_0} \) is again a Burniat surface.

Assume that \( K_{X_{t_i}}^2 = 4 \) and that we are in the non nodal case. We are done unless \( Y_{t_0} \) has a node.

In this case every line of \( Y_{t_0} \) is a component of the branch locus.

Note that through the node of \( Y_{t_0} \) pass 4 lines. By [Cat87], table 3, page 93, a bidouble cover of a node branched in 4 lines is no longer a rational double point, and we have reached a contradiction.

Finally, in the nodal case, we have seen that the family \( Y_t \) is equisingular. By proposition 2.6, the minimal resolution of \( Y_{t_0} \) is the blow up of \( \mathbb{P}^2 \) in 5 distinct points, none infinitely near, with \( P_1, P_4, P_5 \) collinear.

A similar representation holds for the minimal resolution \( W_t \) of \( Y_t \); by the above argument two of the lines passing through the node cannot be part of the branch locus. Thus the branch locus for each \( W_t \) consists of the \(-2\) curve, of 10 (Del Pezzo) lines and a (Del Pezzo) conic. Thus the configuration of the branch locus remains of the same type and the central fibre \( X_{t_0} \) is again a nodal Burniat surface.

\[ \square \]

3. **Proof of the main theorems and corollaries**

All the statements (except the one concerning rationality) of the two main theorems follow combining the two theorems 1.14 and 1.15, showing that the Burniat families for \( K_{S}^2 \geq 4 \) form open sets, with theorem 2.7, showing that they form closed sets.

There remains to prove the rationality of the four connected components \( C \) of the moduli space constituted by Burniat surfaces with \( K_{S}^2 \geq 4 \). This is automatical for \( K_{S}^2 = 4 \) since \( C \) has dimension 2, and by Castelnuovo’s criterion every unirational surface (over \( \mathbb{C} \)) is rational.

We deal next with the case \( K_{S}^2 = 5 \).

**Theorem 3.1.** Let \( C \) be the connected component of the moduli space constituted by Burniat surfaces with \( K_{S}^2 = 5 \).

Then \( C \) is a rational 3-fold.

**Proof.** The bicanonical map of \( S \) yields a bidouble cover \( \Phi_2: S \rightarrow \tilde{Y} \), where \( \tilde{Y} \) is the Del Pezzo of degree 5 obtained blowing up the plane in the 4 reference points.

As we saw, the branch locus consists of nine Del Pezzo lines and of 3 Del Pezzo conics. Thus there is exactly one line which is not contained in the branch locus, and we can contract it, obtaining a Del Pezzo surface \( Y \) of degree 6. The branch locus contains now the six lines of \( Y \).

Let us fix an identification of the Galois group of \( \Phi_2 \) with \( G = (\mathbb{Z}/2\mathbb{Z})^3 \). Then these 6 lines, which form an hexagon, are such that each pair of opposite sides is labelled by an element in \( G \setminus \{0\} \).
There are two ways to contract three such lines (one for each pair) and obtain the projective plane $\mathbb{P}^2$, and they are related by the standard Cremona transformation $(x_1 : x_2 : x_3) \mapsto (x_1^{-1} : x_2^{-1} : x_3^{-1})$ associated to the linear system $|2L - E_1 - E_2 - E_3|$.

We chose the points $P_1, P_2, P_3, P_4$ as the reference points ($P_4 = (1 : 1 : 1)$), and we consider now the triples of lines corresponding to $D_{i,3}$, which have necessarily an equation of type $x_{i+2} = a_i x_{i+1}$.

The Cremona transformation acts by $a_i \mapsto a_i^{-1}$, the cyclical permutation of coordinates cyclically permutes the three numbers $a_1, a_2, a_3$, while the transposition exchanging 1 with 2 sends

$$(a_1, a_2, a_3) \mapsto (a_2^{-1}, a_3^{-1}, a_3^{-1}).$$

Composing the action of such a transposition with the action of the Cremona transformation we get the transposition of $a_1$ and $a_2$.

We conclude that there is a subgroup of index two, isomorphic to $S_3$, acting on the three numbers $a_1, a_2, a_3$ via the standard permutation action of the symmetric group $S_3$.

The full group by which we want to divide is generated by this subgroup and by the Cremona transformation. The invariants for the permutation representation of $S_3$ are the three elementary symmetric functions $\sigma_1, \sigma_2, \sigma_3$. The Cremona transformation acts on the field $K$ of $S_3$- invariants by

$$\sigma_3 \mapsto \sigma_3^{-1}, \quad \sigma_1 \mapsto \sigma_2 \sigma_3^{-1}, \quad \sigma_2 \mapsto \sigma_1 \sigma_3^{-1}.$$

Obvious invariants are

$$\sigma_1 + \sigma_2 \sigma_3^{-1} := y_1, \quad \sigma_2 + \sigma_1 \sigma_3^{-1} := y_2, \quad \sigma_3 + \sigma_3^{-1} := y_3.$$

Let $F$ be the field $\mathbb{C}(y_1, y_2, y_3)$: to show that $F$ is the whole field of invariants it will suffice to show that $[K : F] = 2$.

Now, $F(\sigma_3)$ is a quadratic extension of $F$, and the two linear equations in $\sigma_2, \sigma_1$

$$\sigma_1 + \sigma_2 \sigma_3^{-1} = y_1, \quad \sigma_2 + \sigma_1 \sigma_3^{-1} = y_2$$

have determinant $1 - \sigma_3^{-2}$, thus $\sigma_2, \sigma_1 \in F(\sigma_3)$ hence $F(\sigma_3) = K$.

\[\square\]

**Theorem 3.2.** Let $\mathcal{C}$ be the connected component of the moduli space constituted by the primary Burniat surfaces ($K^2_S = 6$).

Then $\mathcal{C}$ is a rational 4-fold.

**Proof.** As in the proof of the previous theorem, we have to divide a parameter space $\cong \mathbb{C}^6$, parametrizing three pairs of lines of equations $x_{i+2} = a_i x_{i+1}, x_{i+2} = b_i x_{i+1}$ by the action of $\mathbb{C}^* \times \cong \mathbb{C}^* \times \cong \mathbb{C}^*$ of $S_3$, of the $\mathbb{Z}/2\mathbb{Z})^3$ generated by the transformations $\tau_i$ such that $\tau_i$ exchanges $a_i$ with $b_i$, and finally of the Cremona transformation (mapping $a_i$ to $a_i^{-1}$, $b_i$ to $b_i^{-1}$).
As before, we can replace the action of $\mathfrak{S}_3$ by the direct sum of two copies of the standard permutation representation (of the $a_i$’s and of the $b_i$’s).

Moreover, we have the action of the subgroup $(\mathbb{C}^*)^2 \subset \text{PGL}(3, \mathbb{C})$ of diagonal matrices $(\mathbb{C}^*)^2 := \{\text{diag}(t_1, t_2, t_3)|t_i \in \mathbb{C}^*\}$:

$$a_i \mapsto a_i \frac{t_i + 1}{t_i + 2}, \quad b_i \mapsto b_i \frac{t_i + 1}{t_i + 2}, \quad i \in \{1, 2, 3\}.$$ 

We set: $\lambda_i := \frac{t_i + 1}{t_i + 2}$, thus $\prod_i \lambda_i = 1$ and our $(\mathbb{C}^*)^2$ is the subgroup of $(\mathbb{C}^*)^3$,

$$(\mathbb{C}^*)^2 = \{(\lambda_1, \lambda_2, \lambda_3)|\prod_i \lambda_i = 1\}.$$ 

The invariants for the $(\mathbb{Z}/2\mathbb{Z})^3$-action are: $u_i := a_i b_i, \quad v_i := a_i + b_i$ and $(\mathbb{C}^*)^3$ acts by

$$u_i \mapsto \lambda_i^2 u_i, \quad v_i \mapsto \lambda_i v_i.$$ 

**Claim 3.3.** The invariants for the $(\mathbb{C}^*)^2$-action are

$$w_i := \frac{u_i}{v_i^2}, \quad i = 1, 2, 3; \quad v := \prod_{i=1}^3 v_i.$$ 

**Proof of the claim.** Clearly the field of $(\mathbb{C}^*)^3$-invariants is generated by the $w_i$’s, and we can replace the generators $u_i, v_i (i = 1, 2, 3)$ by the generators $w_i, v_i (i = 1, 2, 3)$.

Now the exact sequence of algebraic groups

$$1 \rightarrow (\mathbb{C}^*)^2 \rightarrow (\mathbb{C}^*)^3 \rightarrow \mathbb{C}^* \rightarrow 1$$

where $(\lambda_1, \lambda_2, \lambda_3) \mapsto \lambda := \prod_i \lambda_i$, shows that the projection $(\mathbb{C}^*)^3 \rightarrow \mathbb{C}^*$ is the quotient map by the $(\mathbb{C}^*)^2$ action.

Since $\mathbb{C}(v_1, v_2, v_3)$ is the function field of $(\mathbb{C}^*)^3$, the field of invariants for $(\mathbb{C}^*)^3$ acting on $\mathbb{C}(v_1, v_2, v_3)$ is $\mathbb{C}(v)$.

□

Note that $\mathfrak{S}_3$ acts on $\{w_1, w_2, w_3\}$ by the permutation representation, whereas the Cremona transformation acts by $w_i \mapsto u_i, \quad v \mapsto \prod_{i=1}^3 w_i =: \frac{u}{v}$.

In fact, the Cremona transformation sends $u_i$ to $u_i^{-1}$ and $v_i$ to $\frac{1}{a_i} + \frac{1}{b_i} = \frac{a_i + b_i}{a_i b_i} = \frac{u_i}{v_i^2}$.

Since $\frac{u}{v^2} = \prod w_i$ it follows that $u = \prod w_i v^2$, thus $v \mapsto (\prod w_i)^{-1} v^{-1}$. The invariants for the Cremona transformations are therefore

$$w_1, w_2, w_3, v + \frac{1}{v \prod w_i}$$

where the last element is $\mathfrak{S}_3$-invariant.

Finally, the invariants for the action of $\mathfrak{S}_3$ are: the elementary symmetric functions $\sigma_1(w_1, w_2, w_3), \sigma_2(w_1, w_2, w_3), \sigma_3(w_1, w_2, w_3)$, and $v + \frac{1}{v \prod w_i}$.

Thus the field of invariants is rational.
We derive now some easy consequences of the main theorems:

**Corollary 3.4.** All surfaces $S$ which are deformations of Burniat surfaces with $K_S^2 \geq 4$ are again Burniat surfaces, and the bicanonical map of their canonical model is a finite morphism $\Phi_2: X \to Y'$, Galois with group $G = (\mathbb{Z}/2\mathbb{Z})^2$, and with image a Del Pezzo surface $Y'$ of degree $K_S^2$. $Y'$ is singular exactly for the nodal families with $K_S^2 = 4$ (it has precisely one $A_1$ singularity then). In particular, Bloch’s conjecture $A_0(S) = \mathbb{Z}$ holds for for all the surfaces in these 4 connected components of the moduli space.

**Proof.** The last statement follows from our main theorems and the work of Inose and Mizukami [InMi79].

In fact Inose and Mizukami show that Bloch’s conjecture holds for certain classes of Inoue surfaces, which we have shown in part one [BC09b] to coincide with the classes of Burniat surfaces.

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