Universal aspects of critical percolation on random half-planar maps

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Abstract

We study a large class of Bernoulli percolation models on random lattices of the half-plane, obtained as local limits of uniform planar triangulations or quadrangulations. We first compute the exact value of the site percolation threshold in the quadrangulation case using the so-called peeling techniques. Then, we generalize a result of Angel about the scaling limit of crossing probabilities, that are a natural analogue to Cardy’s formula in (non-random) plane lattices. Our main result is that those probabilities are universal, in the sense that they do not depend on the percolation model neither on the degree of the faces of the map.

1 Introduction

In this work, we consider several aspects of Bernoulli percolation models (site, bond and face percolation) on Uniform Infinite Half-Planar Maps, more precisely on the Uniform Infinite Planar Triangulation and Quadrangulation of the half-plane (UIHPT and UIHPQ in short), which are defined as the so-called local limit of random planar maps. Those maps, or rather their infinite equivalents (UIPT and UIPQ), were first introduced by Angel & Schramm ([5]) in the case of triangulations and by Krikun ([16]) in the case of quadrangulations (see also [12], [18] and [13] for an alternative approach). Angel later defined in [2] the half-plane models which have nicer properties. The Bernoulli percolation models on these maps are defined as follows: every site (resp. edge, face) is open with probability $p$ and closed otherwise, independently of every other sites (resp. edges, faces). All the details concerning the map and percolation models are postponed to Section 2. More specifically, we will focus on the site percolation threshold for quadrangulations and the scaling limits of crossing probabilities.

In Section 3 we compute the site percolation threshold on the UIHPQ, denoted by $p_{c,\text{site}}$. This problem was left open in [3], where percolation thresholds are given for any percolation and map model (site, bond and face percolation on the UIHPT and UIHPQ), except for the site percolation on the UIHPQ. Roughly speaking, it is made harder by the fact that peeling techniques are not well defined a priori in this setting, as we will discuss later. This work is also useful in order to study the problem of the last section in the special case of site percolation on the UIHPQ. Namely, we will prove the following.

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Theorem 1.1. For Bernoulli site percolation on the UIHPQ, we have

\[ p_{c,\text{site}} = \frac{5}{9}. \]

Moreover, there is no percolation at critical point almost surely.

Due to the fact that Uniform Infinite Half-Planar Maps have a boundary, it is natural to consider boundary conditions. Here, the result holds for a free boundary condition. We believe that the percolation threshold is independent of these boundary conditions but did not investigate this further. We will discuss this again in Section 3.

The last section focuses on percolation models on Uniform Infinite Half-Planar Maps at their critical point, more precisely on crossing events. We suppose that the boundary of the map is coloured as in Figure 1 (in the case of bond percolation), where black edges are open, \( a, b \) are fixed and positive and \( \lambda \) is a positive scaling parameter.

\[ \lambda a \]
\[ \lambda b \]

Figure 1: The initial colouring of the boundary for crossing probabilities problems.

Starting from now, we use the notation of [3] and let * denote either of the symbols \( \triangle \) or \( \square \), where \( \triangle \) stands for triangulations and \( \square \) for quadrangulations. The so-called crossing event we are interested in is the event that the two black segments are part of the same percolation cluster. We denote by \( C^*(\lambda a, \lambda b) \) this event and study the scaling limit of its probability when \( \lambda \) goes to infinity. The first result was proved by Angel in the case of site percolation on triangulations.

**Theorem.** (Theorem 3.3 in [2]) Let \( a, b \geq 0 \). For site percolation at the critical point on the UIHPT,

\[ \lim_{\lambda \to +\infty} \mathbb{P} \left( C_{\text{site}}^\triangle (\lambda a, \lambda b) \right) = \frac{1}{\pi} \arccos \left( \frac{b - a}{a + b} \right). \]

Our motivation is that this problem is a natural equivalent to the famous Cardy’s formula in regular lattices, which has been proved by Smirnov in the case of the triangular lattice (see [20]). We are interested in the universal aspect of this scaling limit, in the sense that it is preserved for site, bond and face percolation on the UIHPT and the UIHPQ. Our main result is the following.

**Theorem 1.2.** Let \( a, b \geq 0 \). We have for critical site, bond and face percolation models on the UIHPT and the UIHPQ,

\[ \lim_{\lambda \to +\infty} \mathbb{P} \left( C^*(\lambda a, \lambda b) \right) = \frac{1}{\pi} \arccos \left( \frac{b - a}{a + b} \right). \]

In other words, asymptotic crossing probabilities are equal for site, bond and face percolation on the UIHPT and the UIHPQ at their critical percolation threshold.
We recall that the question of universality of Cardy’s formula for periodic plane graphs is an important open problem in probability theory. Here, the randomness of the planar maps we consider makes the percolation models easier to study.

**Remarks.** We believe that for both the computation of the site percolation threshold and the asymptotics of crossing probabilities, our methods apply in more general settings as long as the *spatial Markov property* of Section 2.2 holds. This includes the generalized half-planar maps of [4], see also [19]. This should also cover the models defined in [10] and [21], which we believe to satisfy a version of the spatial Markov property, although this has not been investigated so far.

Moreover, as we were finishing writing this paper, we became aware of the very recent preprint [9] by Björnberg and Stefánsson, that also deals with site percolation on the UIHPQ. This paper provides upper and lower bounds for the percolation threshold $p_c^{\square}$, but not the exact value $5/9$. Our study of the universality of crossing probabilities is also totally independent of [9].

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## 2 Random planar maps and percolation models

We first recall the construction of random planar maps we will focus on in the sequel, and some important properties of these maps.

### 2.1 Definitions and distributions on planar maps

Let us first consider finite planar maps, i.e. proper embeddings of finite connected graphs in the sphere $S^2$ (more precisely their equivalence class up to orientation-preserving homeomorphisms of the sphere). The faces of a planar map are the connected components of the complement of the embedding, and the degree of a face is the number of oriented edges this face is incident to (with the convention that the face incident to an oriented edge is the face on its left). Every planar map we consider is *rooted*: there is a distinguished oriented edge called the *root* of the map, and the tail vertex of this edge is the *origin* of the map.

We focus more precisely on $p$-*angulations* (the case $p = 3$ corresponds to *triangulations* and $p = 4$ to *quadrangulations*), i.e. finite planar maps whose faces all have the same degree $p$, and *generalized* $p$-*angulations*, i.e. planar maps whose faces all have degree $p$, except possibly for a finite number of distinguished faces which can have arbitrary degrees. These faces are called *external faces* of the $p$-angulation (by contrast with *internal faces*), and are constrained to have a simple boundary (i.e. its embedding is a Jordan curve). In this setting, an inner vertex of the map is a vertex that is not on the boundary. Finally, triangulations are supposed to be 2-connected (or type-2), that is to say, multiple edges are allowed but loops are not.

Recall that $*$ denote either of the symbols $\triangle$ or $\Box$, where $\triangle$-*angulations* stands for type-2 triangulations and $\Box$-*angulations* for quadrangulations. For $n \geq 1$, denote by $\mathcal{M}_n^*$ the set of generalized $*$-*angulations* that have $n$ inner vertices, so that $\mathcal{M}_n^* := \bigcup_{n \geq 1} \mathcal{M}_n^*$ is the set of finite generalized $*$-*angulations*. By convention, we define $\mathcal{M}_0^*$ as the singleton given by the
map with a single edge. We endow the set $\mathcal{M}^*_f$ of finite ∗-angulations with the local topology, more precisely the distance $d_{\text{loc}}$ defined for every $M, M' \in \mathcal{M}^*_f$ by

$$d_{\text{loc}}(M, M') := (1 + \sup\{r | B_r(M) \sim B_r(M')\})^{-1},$$

where $B_r(M)$ is the planar map given by the ball of radius $r$ around the origin for usual graph distance in the following sense: $B_0$ contains only the origin of the map, and $B_r$ is made of all the vertices at graph distance less than $r$ from the origin, with all the edges linking them.

With this topology, $(\mathcal{M}^*_f, d_{\text{loc}})$ is a metric space, and we define the set of (finite and infinite) ∗-angulations to be the completed space $\mathcal{M}^*$ of $(\mathcal{M}^*_f, d_{\text{loc}})$. An infinite ∗-angulation, that is, an element of $\mathcal{M}^*_\infty := \mathcal{M}^* \setminus \mathcal{M}^*_f$, can also be seen as infinite planar maps, in the sense that they can be defined as the proper embedding of an infinite, locally finite graph in a non-compact surface, dissecting the latter into a collection of simply connected regions (see also the Appendix of [13] for greater details). In this representation, a face of an infinite planar map can have infinite degree in the sense that its boundary contains an accumulation point, and as a consequence an infinite number of edges and vertices.

Finally, for $m \geq 2$, a (finite or infinite) generalized ∗-angulation and a unique external face of degree $m$ is called ∗-angulation of the $m$-gon. For $n \geq 0$ and $m \geq 2$, we denote by $\mathcal{M}^*_{n,m}$ the set of ∗-angulations of the $m$-gon with $n$ inner vertices rooted on the boundary, and $\varphi_{n,m}^*$ its cardinality. By convention, $\mathcal{M}^*_{0,2} = \mathcal{M}^*_2$. Note that $\varphi_{n,m}^* = 0$ for $m$ odd, so that we implicitly restrict ourselves to the cases where $m$ is even for quadrangulations. The quantity $\varphi_{n,m}^*$ being finite for $n \geq 0$ and $m \geq 2$ (see [22] for instance), we can define the uniform probability measure on $\mathcal{M}^*_{n,m}$, denoted by $\nu_{n,m}$. Asymptotics for the numbers $(\varphi_{n,m}^*)_{n \geq 0, m \geq 2}$ are known and universal in the sense that

$$\varphi_{n,m}^* \sim C^*_{\varphi}(m) \rho_n^\varphi n^{-5/2} \quad \text{and} \quad C^*_{\varphi}(m) \sim K^*_{\varphi} \alpha_m^\varphi \sqrt{m},$$

where $\rho_\varphi = 27/2$, $\alpha_\varphi = 9$, $\rho_\square = 12$, $\alpha_\square = \sqrt{54}$ and $K^*_{\varphi} > 0$ (see for instance the work of Gao, or more precisely to [17] for 2-connected triangulations, and [11] for quadrangulations).

We can define another measure on finite planar maps (which gives an equal weight to maps with a fixed number of faces or, equivalently, of inner vertices), called the Boltzmann measure (or free measure in [1]). Let $m \geq 2$ and set

$$Z^*_m := \sum_{n \geq 0} \varphi_{n,m}^* \rho_n^{-m}. $$

The ∗-Boltzmann distribution of the $m$-gon, denoted by $\mu^*_m$, is the probability measure on finite ∗-angulations of the $m$-gon defined for every $M \in \mathcal{M}^*_{n,m}$ by

$$\mu^*_m(M) := \frac{\rho_n^{-m}}{Z^*_m}. \quad (1)$$

A random variable with law $\mu^*_m$ is called a Boltzmann ∗-angulation of the $m$-gon. Note that $(Z^*_m)_{m \geq 2}$ is the partition function of the Boltzmann measure. Moreover, the asymptotic behaviour of $(Z^*_m)_{m \geq 2}$ is also known and given by

$$Z^*_m \sim \iota_m m^{-5/2} \alpha^m, \quad (2)$$

for some constants $\iota_*, \alpha_* > 0$.  

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We now recall the construction of the uniform infinite half-planar maps by Angel and Schramm (see [5]). The Uniform Infinite Planar *-angulation of the Half-Plane, or UIHP-*, is a probability measure supported on infinite half-planar *-angulations defined by the following limits.

**Theorem.** (Theorem 1.8 in [5]) Let \( m \geq 2 \). For \( * = \triangle \) and \( * = \square \) we have

\[
\nu^*_{n,m} \xrightarrow{n \to +\infty} \nu^*_{\infty,m},
\]

in the sense of narrow convergence for the local topology. The probability measure \( \nu^*_{\infty,m} \) is supported on infinite *-angulations of the \( m \)-gon, respectively called the law of the UIPT and UIPQ of the \( m \)-gon. Moreover, we have

\[
\nu^*_{\infty,m} \xrightarrow{m \to +\infty} \nu^*_{\infty},
\]

in the sense of narrow convergence for the local topology. The probability measure \( \nu^*_{\infty} \) is respectively called the law of the half-plane UIPT and UIPQ, or also UIHPT and UIHPQ.

We now give an important property of the previous measure, which justifies the naming “half-plane”.

**Proposition.** The measure \( \nu^*_{\infty} \), is supported on infinite *-angulations with infinite boundary, that satisfy the following properties almost surely:

- The map can be embedded in the sphere \( \mathbb{S}^2 \) with a unique accumulation point.
- With the stereographic projection sending the accumulation point in \( \mathbb{S}^2 \) to infinity, the map is embedded in the plane \( \mathbb{R}^2 \), with a unique infinite face sent on the lower half-plane \( \mathbb{H}_- \), the boundary sent on the \( x \)-axis and the rest of the map sent on the upper half-plane \( \mathbb{H}_+ \) without any accumulation point.

In that sense, we say that the measure \( \nu^*_{\infty} \) is supported on infinite *-angulations of the half-plane. Finally, \( \nu^*_{\infty} \) enjoys a re-rooting invariance property, in the sense that it is preserved under shifts of the root by one edge on the boundary.

We finally notice that the UIHP-* can also be obtained as the limit of Boltzmann measures of the \( m \)-gon when \( m \) becomes large.

**Theorem.** (Theorem 2.1 in [2]) We have \( \mu^*_{m} \xrightarrow{m \to +\infty} \nu^*_{\infty} \) in the sense of narrow convergence for the local topology.

### 2.2 Spatial Markov property and configurations

We now describe the so-called peeling argument (introduced by Angel), whose principle is to suppose the whole map unknown and to reveal it face by face.

Let us consider a random map \( M \) which has the law of the UIHP-*, and a face \( A \) of \( M \) which is incident to the root. We now reveal the face \( A \), in the sense that we suppose the whole map unknown and work conditionally on the configuration of this face. On the one hand, some edges may lie on the boundary of the infinite connected component of \( M \setminus A \) (where \( M \setminus A \) is the map \( M \) without the edge(s) of \( A \) incident to the infinite face). These
edges are called exposed edges, and the (random) number of exposed edges is denoted by $E^*$. On the other hand, some edges lying on the boundary may be enclosed in a finite connected component of the map $M \setminus A$. We call them swallowed edges, and the number of swallowed edges is denoted by $R^*$. We will sometimes use the notation $R^*_l$ (resp. $R^*_r$) for the number of swallowed edges on the left (resp. right) of the root.

We can now state a remarkable property of the UIHPT and the UIHPQ that will be very useful for our purpose.

**Theorem.** (Spatial Markov Property, Theorem 2.2 in [2]) Let $M$ be a random variable with law $\nu^*_\infty$. We denote by $A$ the face incident to the root edge of $M$.

Then, $M \setminus A$ has a unique infinite connected component, say $M'$, and at most one (if $* = \triangle$) or two (if $* = \square$) finite connected component(s). Moreover, $M'$ has the law $\nu^*_{\infty}$ of the UIHP-*, and the finite connected components are Boltzmann -*-angulations of a m-gon (for the appropriate value of $m \geq 2$).

Finally, all those -*-angulations are independent, in particular $M'$ is independent of $M \setminus M'$. This property still holds replacing the root edge by another oriented edge on the boundary, chosen independently of $M$.

We now describe all the possible configurations for the face incident to the root in the UIHP-*, and the corresponding probabilities. All these results can be found in [3].

**Proposition.** (Triangulations case, [3]) There exists two configurations for the triangle incident to the root in the UIHPT (see Figure 2).

- The third vertex of the face is an inner vertex ($E^\triangle = 2, R^\triangle = 0$). This event has probability $q^\triangle_{-1} = 2/3$.
- The third vertex of the face is on the boundary of the map, $k \geq 1$ edges on the left (resp. right) of the root ($E^\triangle = 1, R^\triangle = k$). This event has probability $q^\triangle_k = Z^\triangle_{k+1} 9^{-k}$.

![Figure 2](#) All the possible configurations when a triangle is revealed.

**Remark.** In particular, by direct computation, we have $\sum_{k \geq 1} q^\triangle_k = 1/6$ (because $2 \sum_{k \geq 1} q^\triangle_k + q^\triangle_{-1} = 1$) and $\sum_{k \geq 1} k q^\triangle_k = \mathbb{E}(R^\triangle_l) = 1/3$. 


Proposition. (Quadrangulations case, [3]) There exists three configurations for the quadrangle $A$ incident to the root in a map $M$ which has the law of the UIHPQ (see Figure 3).

- The face has two vertices lying inside the map ($E^\square = 3, R^\square = 0$). This event has probability $q^\square_{1} = 3/8$

- The face has three vertices on the boundary of the map, the third one being $k \geq 1$ edges on the left (resp. right) of the root. This event has probability $q^\square_{k}$, given by:

  - If $k$ is odd, the fourth vertex is incident to the infinite connected component of $M \setminus A$ ($E^\square = 2, R^\square = k$). Then
    
    $$q^\square_{k} = \frac{Z_{k+1}^{\square} \alpha^{1-k}}{\rho^\square}.$$ 

  - If $k$ is even, the fourth vertex is incident to the finite connected component of $M \setminus A$ ($E^\square = 1, R^\square = k$). Then
    
    $$q^\square_{k} = \frac{Z_{k+2}^{\square} \alpha^{-k}}{\rho^\square}.$$ 

- The face has all of its four vertices lying on the boundary of the map, and the quadrangle defines two segments along the boundary of length $k_1$ and $k_2$ (both odd) ($E^\square = 1, R^\square = k_1 + k_2$). This event has probability $q^\square_{k_1, k_2} = Z_{k_1+1}^{\square} Z_{k_2+1}^{\square} \alpha^{-k_1-k_2}$.

  (This situation should be splitted in two subcases, depending on the number of vertices of the face (1 or 2) that are lying on the same side of the root edge).

We finally give (see [3]) the mean of $E^\square$ and $R^\square$, obtained by direct computation using the previous cases and their respective probabilities.

Proposition. We have

- $\mathbb{E}(E^\triangle) = 5/3$ and $\mathbb{E}(R^\triangle) = 2/3$

- $\mathbb{E}(E^\square) = 2$ and $\mathbb{E}(R^\square) = 1$

Moreover, the distribution of $E^\square$ can be explicitly computed.

- $E^\square = \begin{cases} 3 & \text{with probability } 3/8 \\ 2 & \text{with probability } 1/4 \\ 1 & \text{with probability } 3/8 \end{cases}$

Remark. The configurations being completely symmetric, we have $R^*_t \overset{(d)}{=} R^*_r$, so that $\mathbb{E}(R^*_t) = \mathbb{E}(R^*_r) = \mathbb{E}(R^*)/2$. 

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Figure 3: All the possible configurations when a quadrangle is revealed.

2.3 Percolation models

We now specify the percolation models we focus on. Recall that we are interested in Bernoulli percolation on the random maps we previously introduced, i.e. every site (resp. edge, face) is open (we will say coloured black, or refer to the value 1 in the sequel) with probability \( p \) and closed (coloured white, or taking value 0) otherwise, independently of every other sites (resp. edges, faces). Note that this colouring convention is the same as in [1], but opposed to that of [3]. The convention for face percolation (i.e. site percolation on the dual graph) is that two faces are adjacent if they share an edge.

Here is a more precise definition of the probability measure \( \mathbb{P}_p \) induced by our model. Denote by \( \mathcal{M} \) the set of planar maps, and for a given map \( m \in \mathcal{M} \), define the following measure on the set \( \{0, 1\}^{e(m)} \) of colourings of this map (where \( e(m) \) is the set of the elements (vertices, edges or faces) of \( m \)):

\[
\mathbb{P}_p^{e(m)} := (p\delta_1 + (1 - p)\delta_0)^{e(m)}.
\]
We then define $P_p$ as the following measure on the set $\{(m, c) | m \in M, c \in \{0, 1\}^{c(m)}\}$ of coloured maps:

$$P_p(dm, dc) := \nu_\infty^e(dm) P_p^e(dc).$$

In other words, $P_p$ is the measure on coloured planar maps such that the map has the law of the UIHP-$*$ and conditionally on this map, the colouring is a Bernoulli percolation with parameter $p$. We slightly abuse notation here, since we denote by $P_p$ the probability measure induced by every map and percolation model considered in this paper, but there is little risk of confusion. In the sequel, we will often work conditionally on the colouring of the boundary of the map.

We finally define the percolation threshold (or critical point) in this model. Denote by $C^*$ the open percolation cluster of the origin of the map (resp. the root edge for bond percolation and the face incident to it for face percolation) and recall that the percolation event is the event that $C^*$ is infinite. The percolation probability is defined for $p \in [0, 1]$ by

$$\Theta^*(p) := P_p(|C^*| = +\infty).$$

A standard coupling argument proves that the function $\Theta^*$ is non-decreasing, so that there exists a critical point $p^*_c$, called the percolation threshold, such that

$$\begin{cases} 
\Theta^*(p) = 0 & \text{if } p < p^*_c \\
\Theta^*(p) > 0 & \text{if } p > p^*_c.
\end{cases}$$

Thus, the percolation threshold $p^*_c$ can also be defined by the identity $p^*_c := \inf\{p \in [0, 1] : \Theta^*(p) > 0\} = \sup\{p \in [0, 1] : \Theta^*(p) = 0\}$. Note that both $C^*$ and $p^*_c$ depend on the law of the infinite planar map and on the percolation model we consider.

### 2.4 Lévy $3/2$-stable process

Let us define the (spectrally negative) $3/2$-stable process and give some important properties that will be used later. All the results can be found in [6].

**Definition 2.1.** The Lévy spectrally negative $3/2$-stable process is a Lévy process $(S_t)_{t \geq 0}$ whose Laplace transform is given by $E(e^{\lambda S_t}) = e^{t \lambda^{3/2}}$. Its Lévy measure is supported on $\mathbb{R}_-$ and given by

$$\Pi(dx) = \frac{3}{4 \sqrt{\pi}} |x|^{-5/2} dx 1_{\{x < 0\}}.$$

In particular, this process has no positive jumps. Finally, the Lévy spectrally negative $3/2$-stable process has a scaling property with parameter $3/2$, i.e. for every $\lambda > 0$ the processes $(S_t)_{t \geq 0}$ and $(\lambda^{-3/2} S_{\lambda t})_{t \geq 0}$ have the same law.

Note that this process is also called Airy-stable process (ASP in short). We will need the so-called positivity parameter of the process $S$, defined by

$$\rho := P(S_1 \geq 0)$$

Applying results of [6] in our setting, we get the following.
Lemma 2.2. (Chapter 8 of [6]) The positivity parameter of the Lévy 3/2-stable process is given by \( \rho = \frac{2}{3} \).

This process will be very useful for our purpose because it is the scaling limit of a large class of random walks.

Proposition 2.3. (Section 4 in [3], Chapter 8 of [8], [7]) Let \( X \) being a centered real-valued random variable such that \( P(X > t) = o(t^{-3/2}) \) and \( P(X < -t) = ct^{-3/2}(1 + o(t^{-3/2})) \) for a positive constant \( c \), and for every \( n \geq 1 \), \( S_n = \sum_{i=1}^{n} X_i \) where the random variables \( (X_i)_{i \geq 1} \) are independent and have the same law as \( X \), we have

\[
\left( \frac{S_{[\lambda t]}}{\lambda^{2/3}} \right)_{t \geq 0} \xrightarrow{d} \kappa(S_t)_{t \geq 0},
\]

in the sense of convergence in law for Skorokhod topology, where \( \kappa \) is an explicit constant (depending on \( c \)).

We end this section with an important property of the 3/2-stable process, which concerns the distribution of the so-called overshoot at the first entrance in \( \mathbb{R}_- \). It is a consequence of the fact that the so-called ladder height process of \(-S_t)_{t \geq 0} \) is a stable subordinator with index 1/2 (see [6] for details). We use the notation \( P_a \) for the law of the process started at \( a \).

Proposition 2.4. (Section 3.3 in [2], Chapter 3 of [6], Example 7 in [15]) Let \( \tau := \inf\{t \geq 0 : S_t \leq 0\} \) denote the first entrance time of the 3/2-stable process \( S \) in \( \mathbb{R}_- \). Then, the distribution of the overshoot \( |S_\tau| \) of \( S \) at the first entrance in \( \mathbb{R}_- \) is given for every \( a, b > 0 \) by

\[
P_a(|S_\tau| > b) = \frac{1}{\pi} \arccos \left( \frac{b - a}{a + b} \right).
\]

Moreover, the joint distribution of the undershoot and the overshoot \((S_{\tau-}, |S_\tau|)\) of \( S \) at the first entrance in \( \mathbb{R}_- \) is absolutely continuous with respect to the Lebesgue measure on \( (\mathbb{R}_+)^2 \).

3 Site percolation threshold on the UIHPQ

Throughout this section, we focus on the computation of the site percolation threshold on the UIHPQ, denoted by \( p_{\square, \text{site}} \). We consider Bernoulli site percolation on a random map which has the law of the UIHPQ.

Our aim is to prove Theorem [14]. In this statement, there is no condition on the initial colouring of the boundary, which is completely free (a free vertex is by definition open with probability \( p \), closed otherwise, independently of all other vertices in the map). In order to simplify the proof, but also for the purpose of Section 4 we first work conditionally on the “Free-Black-White” colouring of the boundary presented in Figure 4.

Figure 4: The initial colouring of the boundary for the site percolation threshold problem.

The key here is to keep as much randomness as we can on the colour of the vertices and to use an appropriate peeling process, following the ideas of [3].
3.1 Peeling process

We now want to reveal the map face by face in a proper way, which we call a peeling process or an exploration process. The strategy here is to reveal the colour of the free vertices of the boundary, and to "discard" or "peel" the white vertices that are discovered in a sense we now make precise. To do so, we need an alternative peeling process defined as follows.

**ALGORITHM 1.** (Vertex-peeling process) Suppose that we want to peel a marked (say, white) vertex on the boundary of the UIHPQ coloured as in Figure 5 (the colouring on the right of the vertex we peel is fixed but arbitrary and we call it the “right boundary”).

Figure 5: The boundary of the UIHPQ we consider and the marked vertex we want to peel.

Reveal the face incident to the edge of the boundary that links the marked vertex to a free vertex, and denote by $R_r$ the number of swallowed edges on the right of this edge (do not reveal the colour of the vertices that are discovered).

- If $R_r > 0$, the algorithm ends.
- If $R_r = 0$, repeat the algorithm on the UIHPQ given by the unique infinite connected component of the map deprived of the revealed face.

Let us now give the main properties of this algorithm. Since we work on the UIHPQ and in order to make the notation less cluttered, we denote by $R_r$ a random variable which has the same law as the random variable $\mathcal{R}_r^\square$ of Section 2.

**Proposition 3.1.** The vertex-peeling process is well defined, in the sense that the pattern of the boundary ("Free-Marked vertex-Right boundary") is preserved as long as the algorithm does not end, which occurs in finite time almost surely. Moreover, when the algorithm ends:

- The number of swallowed edges on the right of the "Free-White" edge in the initial map has the law $\mathbb{P}_p(\mathcal{R}_r \in \cdot | \mathcal{R}_r > 0)$ of the random variable $\mathcal{R}_r$ conditioned to be positive.
- The unique infinite connected component of the map deprived of the revealed faces has the law of the UIHPQ and the marked white vertex is not in this map. In that sense, the vertex has been peeled by the process.

**Proof.** The invariance of the pattern on the boundary is a consequence of the fact that $R_r = 0$ as long as the process is not over and that we do not reveal the colour of the vertices that are discovered. Moreover, the spatial Markov property of Section 2.2 implies that the sequence of swallowed edges to the right of the Free-White edge is an i.i.d. sequence of random variables which have the same law as $\mathcal{R}_r$. The algorithm ends when we reach the first positive variable in that sequence (which happens in finite time almost surely). Thus, the law of this variable is clearly $\mathbb{P}_p(\mathcal{R}_r \in \cdot | \mathcal{R}_r > 0)$, and it equals the number of swallowed edges to the right in the initial map by construction. Finally, since $\mathcal{R}_r > 0$ at the last step, the white vertex is not on the boundary of the unique infinite connected component at the end of the process. □
We are now able to describe the complete peeling process we focus on, recalling that we consider a map which has the law of the UIHPQ conditionally on the colouring of the boundary presented in Figure 4.

**ALGORITHM 2.** (Peeling process) Reveal the colour of the rightmost free vertex on the boundary.

- If it is black, repeat the algorithm.
- If it is white, mark this vertex and execute the vertex-peeling process. Then, repeat the algorithm on the unique infinite connected component of the map deprived of the faces revealed by the vertex-peeling process.

The algorithm ends when the initial finite black segment has been completely swallowed.

**Remark.** In the previous algorithms, the map we consider at each step of the peeling process is implicitly rooted at the next edge we have to peel, which is determined by the colouring of the boundary.

As a consequence of the properties of the vertex-peeling process, we get that the peeling process is well defined, in the sense that the pattern of the boundary (Free-Black-White) is preserved as long as the algorithm does not end. Moreover, at each step, the planar map we consider has the law of the UIHPQ and does not depend on the revealed part of the map: the peeling process transitions are independent and have the same law. In particular, if we denote by $\mathcal{H}_n, c_n$ the number of swallowed edges at the right of the root edge and the colour of the revealed vertex at step $n$ of the exploration, then $(\mathcal{H}_n, c_n)_{n \geq 0}$ are independent and identically distributed.

The quantity we are interested in is the **length of the finite black segment** at step $n$ of the process, denoted by $B_n$. The process $(B_n)_{n \geq 0}$ is related to the percolation event by the following lemma, whose proof is omitted, since $B_n = 0$ implies that the black cluster is enclosed in a finite region of the map.
Lemma 3.2. Denote by $C_{\text{site}}$ the open cluster of the origin (black) vertex in our map. Then

- $\{B_n \xrightarrow{n\to\infty} +\infty\} \subset \{|C_{\text{site}}| = +\infty\}$
- $\{\exists n \geq 0 | B_n = 0\} \subset \{|C_{\text{site}}| < +\infty\}$

It is then sufficient to determine the behaviour of the process $(B_n)_{n \geq 0}$, which is remarkably simple as a consequence of the very definition of the peeling process.

Proposition 3.3. The process $(B_n)_{n \geq 0}$ is a Markov chain with initial law $\delta_1$, whose transitions are given for $n \geq 0$ by:

$$B_{n+1} = (B_n + 1_{c_n=1} - 1_{c_n=0}(\mathcal{H}_n - 1))1_{\{B_n>0, B_n+1_{c_n=1} - 1_{c_n=0}(\mathcal{H}_n-1)\geq0\}}.$$

Moreover, $(\mathcal{H}_n)_{n \geq 0}$ is sequence of i.i.d. random variables and for $n \geq 0$, conditionally on the event $\{c_n = 0\}$, $\mathcal{H}_n$ has the law of $\mathcal{R}_r$ conditioned to be positive.

In particular, the process $(B_n)_{n \geq 0}$ has the same law as a random walk started at $1$ and killed at its first entrance in $\mathbb{Z}_-$, with steps distributed as $1_{c_0=1} - 1_{c_0=0}(\mathcal{H}_0 - 1)$.

3.2 Computation of the percolation threshold

We now compute the percolation threshold for the “Free-Black-White” initial colouring of the boundary we are interested in.

Proposition 3.4. For Bernoulli site percolation on the UIHPQ with a “Free-Black-White” boundary condition, we have

$$p_{\text{c,site}}^\square = \frac{5}{9}.$$

Moreover, there is no percolation at critical point almost surely: $\Theta_{\text{site}}(p_{\text{c,site}}^\square) = 0$.

Proof. The quantity which rules the behaviour of $(B_n)_{n \geq 0}$ is $\mathbb{E}_p(1_{c_0=1} - 1_{c_0=0}(\mathcal{H}_0 - 1))$. It is obvious that $\mathbb{E}_p(c_0) = p$. Now the law of $\mathcal{H}_0$ conditionally on $\{c_0 = 0\}$ do not depend on $p$ by construction and we have:

$$\mathbb{E}_p(\mathcal{H}_0 | c_0 = 0) = \mathbb{E}(\mathcal{R}_r | \mathcal{R}_r > 0) = \frac{\mathbb{E}(\mathcal{R}_r 1_{\{\mathcal{R}_r>0\}})}{\mathbb{P}(\mathcal{R}_r > 0)} = \frac{\mathbb{E}(\mathcal{R}_r)}{\mathbb{P}(\mathcal{R}_r > 0)} = \frac{1}{2} = \frac{9}{4}.$$

Thus,

$$\mathbb{E}_p(1_{c_0=1} - 1_{c_0=0}(\mathcal{H}_0 - 1)) = p - (1 - p)(\mathbb{E}_p(\mathcal{H}_0 | c_0 = 0) - 1) = p - \frac{5}{4}(1 - p).$$

We get that $\mathbb{E}_p(1_{c_0=1} - 1_{c_0=0}(\mathcal{H}_0 - 1)) = 0$ if and only if $p = 5/9$. In the case where $p \neq 5/9$, standard arguments on the behaviour of simple random walks and Lemma 3.2 yield the first statement. Finally, when $p = 5/9$, the random walk with steps distributed as $1_{c_0=1} - 1_{c_0=0}(\mathcal{H}_0 - 1)$ is null recurrent, so that almost surely, there exists $n \geq 0$ such that $B_n = 0$. This concludes the proof of the second assertion. \qed
3.3 Percolation threshold for a free boundary condition

The last question we want to discuss in this section is the universality of the percolation threshold with respect to the initial colouring of the boundary. Namely, we now consider Bernoulli site percolation on the UIHPQ in the most natural setting, i.e. with a free boundary condition, and prove Theorem 1.1.

Proof of Theorem 1.1. First of all, we can work without loss of generality conditionally on the fact that the origin vertex of the map is open. We then use a peeling process which is roughly the same as before. We reveal the colour of the rightmost free vertex on the left of the origin (on the boundary).

- If it is black, repeat the algorithm.
- If it is white, mark this vertex and execute the vertex-peeling process. Then, repeat the algorithm on the unique infinite connected component of the map deprived of the faces revealed by the vertex-peeling process.

The algorithm ends when the finite black segment on the boundary has been completely swallowed. At each step, the map is implicitly rooted at the next edge we have to peel.

When the algorithm ends, two situations may occur, depending on the colour of the rightmost vertex of the boundary that is part of the last revealed face (see Figure 7). By properties of the vertex-peeling process, such a vertex exists and lies on the right of the root edge.

1. If it is white, then the open cluster of the origin is enclosed in a finite region of the map and the percolation event do not occur.
2. If it is black, then the percolation event occurs only if this vertex is part of an infinite open cluster. Thus, we end up in the initial situation and we can repeat the previous algorithm with this vertex being the origin of the new map.

Figure 7: The situation when the finite black segment on the boundary is swallowed.

Now, if \( p \leq \frac{5}{9} \), the same arguments as in the proof of Proposition 3.4 ensure that the first algorithm ends almost surely. Since the probability that we then end up in the first situation is \( 1 - p > 0 \), we immediately get that \( \Theta_{\text{site}}(p) = 0 \). On the other hand, if \( p > \frac{5}{9} \), \( \Theta_{\text{site}}(p) > 0 \) using directly the result of Proposition 3.4 and a standard monotone coupling argument. This yields the expected result. \( \Box \)
Scaling limits of crossing probabilities in half-plane random maps

Throughout this section, we focus on the problem of scaling limits of crossing probabilities, and aim at generalizing the results of [2]. Despite the fact that we also use a peeling process, the problem is much harder because the models we consider are less well-behaved than site percolation on the UIHPT.

More precisely, we consider site, bond and face percolation on the UIHPT and the UIHPQ, and suppose that the boundary of the map has the colouring of Figure 8 (for the bond percolation case).

![Figure 8: The initial colouring of the boundary for the crossing probabilities problem.](image)

In other words, the boundary is “White-Black-White-Black”, with two infinite segments and two finite ones, of lengths $|\lambda a|$ and $|\lambda b|$ respectively. The crossing event we focus on is the following: “there exists a black path linking the two black segments of the boundary”, or in an equivalent way, “the two black segments are part of the same percolation cluster”. We denote by $C_{\text{bond}}(\lambda a, \lambda b)$ (resp. $C_{\text{face}}(\lambda a, \lambda b)$ and $C_{\text{site}}(\lambda a, \lambda b)$) this event where * denote either of the symbols $\triangle$ or $\square$. The quantity we are interested in is the scaling limit of the crossing probability $\lim_{\lambda \to +\infty} P_p(C^*(\lambda a, \lambda b))$.

This quantity has no reason to be universal, unless we consider the percolation models at their critical point, i.e when $p$ is exactly the percolation threshold. We recall those values that can be found in [3] and the previous section.

**Theorem.** (Theorem 1 in [3], [2], Theorem 1.1) The percolation thresholds on the UIHPT are given by

$$p_{c,\text{site}}^\triangle = \frac{1}{2}, \quad p_{c,\text{bond}}^\triangle = \frac{1}{4} \quad \text{and} \quad p_{c,\text{face}}^\triangle = \frac{4}{5}.$$  

The percolation thresholds on the UIHPQ are given by

$$p_{c,\text{site}}^\square = \frac{5}{9}, \quad p_{c,\text{bond}}^\square = \frac{1}{3} \quad \text{and} \quad p_{c,\text{face}}^\square = \frac{3}{4}.$$  

Starting from now, the probability $p$ that an edge (resp. face, vertex) is open is always set at $p_c^\ast$ (resp. $p_c^\ast_{\text{face}}, p_c^\ast_{\text{site}}$), in every model we consider. Most of the arguments being valid for both the UIHPT and the UIHPQ, we will treat those cases simultaneously. Thus, we will often omit the notation * for sake of clarity and underline the differences where required. For technical reasons, we have to treat the cases of bond, face and site percolation separately, even though the proof always uses roughly speaking the same strategy. Our aim is now to prove Theorem 2.
4.1 Crossing probabilities for bond percolation

4.1.1 Peeling process and scaling limit

We start with the case of bond percolation. In order to compute the crossing probability we are interested in, we again use a peeling process that we now describe. The aim of this process is to follow a white path in the dual map, which can prevent a black crossing, since a black crossing and a dual white crossing are complementary events. The algorithm starts with revealing the face incident to the rightmost white edge of the infinite white segment (see Figure 9), without discovering the colour of its incident edges, which are thus free edges. We also say that we peel the rightmost white edge, in the sense that this edge is no longer in the infinite connected component of the map deprived of the revealed face.

![Figure 9: The initial boundary and the first edge to peel.](image)

In general, the algorithm is well defined as soon as the boundary has a “White-(Free)” colouring, meaning that there is an infinite white segment and (eventually) a finite free segment on the left of the boundary.

**ALGORITHM 3.** Consider a half-plane map which has the law of the UIHP-* with a “White-(Free)” colouring on the boundary.

- We reveal the colour of the rightmost free edge, if any:
  - If it is black, repeat the algorithm.
  - If it is white, peel it and reveal the face incident to this edge (without revealing the colour of the other edges of the face).

- If there is no free edge on the boundary of the map, as in the first step, peel the rightmost white edge of the infinite white segment, and reveal the face incident to this edge (without revealing the colour of the other edges of the face).

After each step, repeat the algorithm on the UIHP-* given by the unique infinite connected component of the current map, deprived of the revealed face. We will call revealed part of the map all faces that are not in this infinite connected component (even if all of them may not have been revealed, but rather swallowed in a finite connected component). The map we obtain is implicitly rooted at the next edge we have to peel.

Let us now give some properties of the peeling algorithm, which are essentially the same as in Section 3, apart from the fact that the algorithm never ends. We omit the proof of these properties, that are direct consequences of the spatial Markov property.

**Proposition 4.1.** The peeling process is well defined, in the sense that the pattern of the boundary (White-(Free)) is preserved. Moreover, at each step, the planar map we consider has the law of the UIHP-* and does not depend on the revealed part of the map.
Starting from now, we denote by $\mathcal{E}_n$, resp. $\mathcal{R}_n$, $c_n$, the number of exposed edges, resp. the number of swallowed edges and the color of the revealed edge at step $n$ of the exploration, for every $n \geq 0$. By convention, we set $c_n = 0$ when there is no free edge on the boundary and $\mathcal{E}_n = \mathcal{R}_n = 0$ when no edge is peeled ($c_n = 1$). Recall that $\mathcal{R}_{t,n}$ (resp. $\mathcal{R}_{r,n}$) denotes the number swallowed edges at the left (resp. right) of the root edge at step $n$ of the exploration. The quantity we are interested in is the length of the finite black segment at step $n$. In order to study this quantity, we now introduce two processes.

First, for every $n \geq 0$, we let $F_n$ be the \textbf{length of the free segment} on the boundary at step $n$ of the peeling process. Then, we have $F_0 = 0$ and for every $n \geq 0$,

$$F_{n+1} = F_n - 1_{\{c_n=1\}} + 1_{\{c_n=0\}}(\mathcal{E}_n - \mathcal{R}_{t,n} - 1) \quad \text{if} \quad F_n - 1_{\{c_n=0\}}(\mathcal{R}_{t,n} + 1) \geq 0,$$

and $F_{n+1} = \mathcal{E}_n$ otherwise. The process $(F_n)_{n \geq 0}$ is a Markov chain with respect to the canonical filtration of the exploration process, and we have $c_n = 0$ when $F_n = 0$.

\textbf{Remark.} The Markov chain $(F_n)_{n \geq 0}$ takes value zero at step $n$ only if we have $F_{n-1} = 1$ and $c_n = 1$. If the whole free segment is swallowed by a jump of the peeling process at step $n$, then $F_n = \mathcal{E}_n$ (see Proposition 2 and 3).

Then, we let $B_0 = |\lambda a|$ and for every $n \geq 0$,

$$B_{n+1} = B_n + 1_{\{c_n=1\}} - 1_{\{c_n=0\}}\mathcal{R}_{r,n}.$$

Like $(F_n)_{n \geq 0}$, the process $(B_n)_{n \geq 0}$ is a Markov chain with respect to the canonical filtration of the exploration process. Moreover, if we denote by $T := \inf\{n \geq 0 : B_n \leq 0\}$ the first entrance time of $(B_n)_{n \geq 0}$ in $\mathbb{Z}_-$, then for every $n \in \{0, 1, \ldots, T-1\}$, $B_n$ is the \textbf{length of the finite black segment} at step $n$ of the exploration process. Note that for $n \geq T$, the initial finite black segment has been swallowed by the peeling process. However, the algorithm can continue because the infinite white segment and the free segment are still well defined.

As we will see, and as was stressed in [2], the so-called \textit{overshoot} $B_T$ of the process $(B_n)_{n \geq 0}$ at the first entrance in $\mathbb{Z}_-$ is the central quantity, which governs the behaviour of crossing events.

We now need to describe the law of the random variables involved in the previous definitions of the processes $(F_n)_{n \geq 0}$ and $(B_n)_{n \geq 0}$. This can be done using the very definition of the peeling algorithm and Section 2.2. Note that one has to be careful with the fact that $\mathcal{E}_n = \mathcal{R}_n = 0$ when $c_n = 1$. Thus, we let $(t_k)_{k \geq 1}$ be the sequence of consecutive stopping times such that $c_{t_k} = 0$, and $(s_k)_{k \geq 1}$ the sequence of consecutive stopping times such that $F_{s_k} > 0$. In the sequel, we let $(\mathcal{E}, \mathcal{R}_t, \mathcal{R}_r)$ be the random variables $(\mathcal{E}^*, \mathcal{R}_t^*, \mathcal{R}_r^*)$ defined in Section 2 omitting the notation $\ast$ for sake of clarity. A simple application of the spatial Markov property yields the following result.

\textbf{Lemma 4.2.} The random variables $(\mathcal{E}_{t_k}, \mathcal{R}_{t,k}, \mathcal{R}_{r,k})_{k \geq 1}$ are i.i.d. and have the same law as $(\mathcal{E}, \mathcal{R}_t, \mathcal{R}_r)$. Moreover, the random variables $(c_{s_k})_{k \geq 1}$ are i.i.d. and have the Bernoulli law of parameter $p_{c,\text{bond}}$.

In the sequel, we denote by $c$ a random variable which has the Bernoulli law of parameter $p_{c,\text{bond}}^*$, independent of $(\mathcal{E}, \mathcal{R}_t, \mathcal{R}_r)$. We are now able to describe more precisely the behaviour of the processes $(F_n)_{n \geq 0}$ and $(B_n)_{n \geq 0}$. 

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Behaviour of \((F_n)_{n \geq 0}\). The process \((F_n)_{n \geq 0}\) is not exactly a random walk, but it has the same behaviour when it is far away from zero. More precisely, note that \(F_1 > 0\) almost surely and let \(\hat{\sigma} := \inf\{n \geq 1 : F_n - 1_{\{c_n=1\}} - 1_{\{c_n=0\}}(R_{t,n} + 1) < 0\}\). By construction, as long as \(n \leq \hat{\sigma}\), we have

\[
F_n = F_1 + \sum_{k=1}^{n-1} (-1_{\{c_k=1\}} + 1_{\{c_k=0\}}(E_k - R_{t,k} - 1)).
\]

Thus, \((F_n - F_1)_{1 \leq n \leq \hat{\sigma}}\) is a random walk killed at the random time \(\hat{\sigma}\), whose steps are distributed as \(\hat{X}\) defined by

\[
\hat{X} := -1_{\{c=1\}} + 1_{\{c=0\}}(E - R_{t} - 1),
\]

where we use the definition of \(\hat{\sigma}\) and Lemma 4.2. In particular,

\[
\mathbb{E}(\hat{X}) = -p_{c,bond}^* + (1 - p_{c,bond}^*)(\mathbb{E}(E - R_{t}) - 1) = \begin{cases}
\frac{1}{3} - \frac{4}{3}p_{c,bond}^* & \text{when } * = \triangle \\
\frac{1}{2} - \frac{3}{2}p_{c,bond}^* & \text{when } * = \square
\end{cases}.
\]

In other words, using the percolation thresholds given at the beginning of Section 4, we have \(\mathbb{E}(\hat{X}) = 0\). It is finally easy to check that the random variable \(\hat{X}\) satisfies the conditions of Proposition 2.3. In particular, if we denote by \((S_n)_{n \geq 0}\) a random walk with steps distributed as \(\hat{X}\) we get

\[
\left(\frac{\hat{S}_{\lfloor \lambda t \rfloor}}{\lambda^{2/3}}\right)_{t \geq 0} \xrightarrow{d} \kappa(S_t)_{t \geq 0},
\]

(3)

in the sense of convergence in law for Skorokhod’s topology, where \(S\) is the Lévy 3/2-stable process of Section 2.4.

Behaviour of \((B_n)_{n \geq 0}\). Exactly as before, \((B_n)_{n \geq 0}\) is not a random walk. However, \((B_n)_{n \geq 0}\) behaves as a random walk when \(F_n\) is not zero. More precisely, recall that \(F_1 > 0\) and let us introduce the stopping time \(\sigma := \inf\{k \geq 1 : F_k = 0\}\). As long as \(n \leq \sigma\), we have

\[
B_n = B_1 + \sum_{k=1}^{n-1} (1_{\{c_k=1\}} - 1_{\{c_k=0\}}R_{t,k}).
\]

Thus, \((B_n - B_1)_{1 \leq n \leq \sigma}\) is a random walk killed at the random time \(\sigma\), whose steps are distributed as \(X\) defined by

\[
X := 1_{\{c=1\}} - 1_{\{c=0\}}R_{t}.
\]

In particular, recalling that the law of \(R_{t}\) depends on the model and that its expectation has been given in Section 2.2, we get that

\[
\mathbb{E}(X) = p_{c,bond}^* - (1 - p_{c,bond}^*)\mathbb{E}(R_{t}) = \begin{cases}
\frac{4}{3}p_{c,bond}^* - \frac{1}{3} & \text{when } * = \triangle \\
\frac{3}{2}p_{c,bond}^* - \frac{1}{2} & \text{when } * = \square
\end{cases}.
\]
Since we work at critical point, \( \mathbb{E}(X) = 0 \) in both the UIHPT and the UIHPQ, and properties of the random variable \( X \) yields that if \((S_n)_{n \geq 0}\) is a random walk with steps distributed as \( X \),

\[
\left( \frac{S_{[\lambda t]}}{\lambda^{2/3}} \right)_{t \geq 0} \xrightarrow{\text{(d)}} \kappa(S_t)_{t \geq 0},
\]

in the sense of convergence in law for Skorokhod’s topology, where \( S \) is the Lévy \( 3/2 \)-stable process of Section 2.4.

We are now interested in the scaling limit of the process \((B_n)_{n \geq 0}\). We want to prove the following result:

**Proposition 4.3.** We have, in the sense of convergence in law for Skorokhod’s topology:

\[
\left( \frac{B_{[\lambda^{3/2} t]}}{\lambda} \right)_{t \geq 0} \xrightarrow{\text{(d)}} \kappa(S_t)_{t \geq 0},
\]

where \( S \) is the Lévy spectrally negative \( 3/2 \)-stable process.

The idea is to couple a random walk \((S_n)_{n \geq 0}\) with steps distributed as the random variable \( X \) and the process \((B_n)_{n \geq 0}\) in such a way that \( S_n \) and \( B_n \) are close. To do so, let us set \( S_0 = B_0 \), and introduce a sequence \((\beta_n)_{n \geq 0}\) of i.i.d. random variables with Bernoulli law of parameter \( p_{c,\text{bond}}^{*} \). We then define \((S_n)_{n \geq 0}\) recursively as follows. For every \( n \geq 0 \):

\[
S_{n+1} = S_n + \begin{cases} 
B_{n+1} - B_n = 1_{\{c_n = 1\}} - 1_{\{c_n = 0\}} R_{r,n} & \text{ when } F_n > 0 \\
\beta_n + (1 - \beta_n)(B_{n+1} - B_n) = 1_{\{\beta_n = 1\}} - 1_{\{\beta_n = 0\}} R_{r,n} & \text{ when } F_n = 0 
\end{cases}
\]

Otherwise said, as long as \( F_n > 0 \), \((S_n)_{n \geq 0}\) performs the same steps as \((B_n)_{n \geq 0}\) and when \( F_n = 0 \), \( S_{n+1} \) takes value \( S_n + 1 \) if \( \beta_n = 1 \), and does the same transition as \((B_n)_{n \geq 0}\) otherwise. Using Lemma 1.2 and the fact that \( c_n = 0 \) when \( F_n = 0 \), we get that \((S_n)_{n \geq 0}\) is a random walk started from \( B_0 \) with steps distributed as the random variable \( X \), as wanted.

Let us also define the set \( \Xi_n := \{0 \leq k < n : F_n = 0\} \) of zeros of \((F_n)_{n \geq 0}\) before time \( n \geq 0 \), and set \( \xi_n := \#\Xi_n \) for every \( n \geq 0 \). The steps of \((S_n)_{n \geq 0}\) and \((B_n)_{n \geq 0}\) are the same except when \( F_n = 0 \), in which case, since \( S_{n+1} - S_n \) for every \( n \geq 0 \), we have \( S_{n+1} - S_n \leq 1 - (B_{n+1} - B_n) \). Hence, we get that almost surely for every \( n \geq 0 \),

\[
S_n - R_n \leq B_n \leq S_n,
\]

where for every \( n \geq 0 \),

\[
R_n := \sum_{k \in \Xi_n} (1 - (B_{k+1} - B_k)).
\]

Then, if \((r_n)_{n}\) is a sequence of i.i.d. random variables with the same law as \( R_r \), we have for every \( n \geq 0 \) that

\[
R_n \xrightarrow{\text{(d)}} \sum_{k=1}^{\xi_n} (1 + r_k).
\]
In order to prove that \((B_n)_{n \geq 0}\) has the same scaling limit as \((S_n)_{n \geq 0}\), the idea is to establish that the process \((R_n)_{n \geq 0}\) is small compared to \((S_n)_{n \geq 0}\). Let us introduce some notation. We let \(\sigma_0 = 0\) and recursively for every \(k \geq 1\),

\[
\sigma_k := \inf\{n \geq \sigma_{k-1} + 1 : F_n = 0\}.
\]

The stopping time \(\sigma_k\) is the time of the \(k\)th return of \((F_n)_{n \geq 0}\) to zero, and \(\sigma_{k-1} + 1\) is the first time after \(\sigma_{k-1}\) when \((F_n)_{n \geq 0}\) reaches \(\mathbb{Z}_+\) by construction. Notice that the total number of “returns” to 0 before time \(n\) equals \(\xi_n = \#\{k \geq 0 : \sigma_k < n\}\). Recall that using the previous discussion and the strong Markov property, for every \(k \geq 0\), between the stopping times \(\sigma_{k-1} + 1\) and \(\sigma_k\), the process \((F_n)_{n \geq 0}\) behaves as a random walk with steps distributed as \(\hat{X}\) (killed at the first entrance in \(\mathbb{Z}_-\)). The random variable \(\hat{X}\) being centered, starting from any integer greater than 1, \((F_n)_{n \geq 0}\) reaches 0 in finite time almost surely, so that all the \((\sigma_k)_{k \geq 0}\) are almost surely finite. The following lemma gives a bound on the asymptotic behaviour of the quantity \((\xi_n)_{n \geq 0}\).

**Lemma 4.4.** For every \(\alpha > 0\), we have the convergence in probability

\[
\frac{\xi_n}{n^{1/3+\alpha}} \xrightarrow{\mathbb{P} \quad n \to +\infty} 0.
\]

**Proof.** Clearly, the strong Markov property applied to the stopping times \((\sigma_k)_{k \geq 0}\) yields that for every \(k \geq 0\) and \(n \geq 0\),

\[
\mathbb{P}(\xi_n > k) \leq \prod_{j=1}^{k} \mathbb{P}(\sigma_j - \sigma_{j-1} \leq n).
\]

Now, we may simplify the computation with the following remark: for every \(k \geq 1\), \(\sigma_k - \sigma_{k-1}\) stochastically dominates the time needed by the random walk \((\hat{S}_n)_{n \geq 0}\) started from 0 with steps distributed as \(\hat{X}\) to reach \(\mathbb{Z}_- \setminus \{0\}\) (because \(F_{\sigma_{k-1}+1} \geq 1\) almost surely and by the previous arguments). In other words, we have for every \(k \geq 1\) that

\[
\sigma_k - \sigma_{k-1} \geq \inf\{n \geq \sigma_{k-1} + 1 : F_n - F_{\sigma_{k-1}+1} + 1 \leq 0\} - \sigma_{k-1},
\]

so that for every \(k \geq 1\) and \(n \geq 0\),

\[
\mathbb{P}(\sigma_k - \sigma_{k-1} \leq n) \leq \mathbb{P}(\hat{\sigma} \leq n),
\]

where \(\hat{\sigma} := \inf\{n \geq 0 : \hat{S}_n + 1 \leq 0\}\). Thus, we get that \(\mathbb{P}(\xi_n > k) \leq (\mathbb{P}(\hat{\sigma} \leq n))^k\), and

\[
\mathbb{E}(\xi_n) \leq \frac{1}{1 - \mathbb{P}(\hat{\sigma} \leq n)} = \frac{1}{\mathbb{P}(\hat{\sigma} > n)}.
\]

We now want to apply a result from Rogozin, which can be found in [14], Theorem 0. Let us check the assumptions of this result. Clearly, the random variable \(-\hat{X}\) is bounded from below. Using the fact that \(\mathbb{E}(\hat{X}) = 0\), and the convergence \([3]\), we also get

\[
\mathbb{P}(\hat{S}_n > 0) = \mathbb{P}\left(\frac{\hat{S}_n}{n^{2/3}} > 0\right) \xrightarrow{n \to +\infty} \mathbb{P}(\kappa\mathbb{S}_1 > 0) = \rho = \frac{2}{3},
\]

where \(\mathbb{S}\) is a 3/2-stable process. Now, Césaro’s Lemma ensures that
Thus, the theorem applies and yields
\[
P(\tilde{\sigma} \geq n) \sim \frac{1}{n^{1/3}} L_+(n),
\]
where \( L_+ \) is a slowly varying function at \(+\infty\). In particular, for every \( \alpha > 0 \) and \( \varepsilon > 0 \), Markov’s inequality gives that
\[
P(\xi_n > n^{1/3+\alpha}\varepsilon) \leq \frac{1}{n^{1/3+\alpha/2}} \sum_{k=1}^{n} \frac{n^{1/3+\alpha/2}}{\xi_n} (r_k - \mathbb{E}(R_r)).
\]

Using the previous lemma and the law of large numbers, we get that
\[
R_n \xrightarrow{p} 0,
\]
and then
\[
\left( \frac{R_{\lambda^{\alpha/2}t}}{\lambda} \right)_{t \geq 0} \xrightarrow{p} 0
\]
for the topology of uniform convergence on compact sets, and thus for Skorokhod’s topology. Recalling the inequality (5) and the convergence (4), we immediately have
\[
\left( \frac{B_{\lambda^{\alpha/2}t}}{\lambda^{2/3}} \right)_{t \geq 0} \xrightarrow{d} \kappa(S_t)_{t \geq 0},
\]
and thus the proof of Proposition 4.3 with an obvious substitution.

4.1.2 Stopped peeling process

We now focus on the possible situations when the process \((B_n)_{n \geq 0}\) reaches \(Z_-\), which happens almost surely since \((B_n)_{n \geq 0}\) is bounded from above by the recurrent random walk \((S_n)_{n \geq 0}\) we previously introduced. Throughout this section, we denote by \(T\) the first time when \((B_n)_{n \geq 0}\) reaches nonpositive values:
\[
T := \inf\{n \geq 0 : B_n \leq 0\}.
\]
The quantity \(|B_T|\) is the so-called overshoot of the process \((B_n)_{n \geq 0}\) at its first entrance into \(Z_-\). We now split our study into three cases, depending on the value of \(|B_T|\).
**Case 1:** \(|B_T| < |\lambda b|\). In this case, only a fraction of the white segment on the boundary is swallowed, see Figure 10. Notice that the swallowed black finite segment has size \(B_{T-1}\), since it is the black segment’s size just before it is completely swallowed by a revealed face.

![Diagram](image.png)

**Figure 10:** Configurations when \(|B_T| < |\lambda b|\).

In each case, the crossing event we are interested in implies the existence of an open path linking the rightmost revealed vertex on the boundary, denoted by \(v^*\), and the infinite black segment on the boundary (see the dashed path on the figures). The situation may again be simplified using the spatial Markov property: this event has the same probability as the event \(C_{\lambda}^1\) defined as follows. Consider a map which has the law of the UIHP-* and, \(C_{\lambda}^1\) the event that there exists a path linking the origin of the map and the infinite black segment of the boundary, where the boundary of the map has the colouring of Figure 11 (the crossing event \(C_{\lambda}^1\) is again represented with a dashed path). Notice that the free segment on the figure can have length 1 or 2 (depending on the configuration) but this plays no role in our discussion.
Figure 11: The crossing event $C_1^\lambda$.

Roughly speaking, the idea here is that when $\lambda$ becomes large, the length $d^\lambda := |\lambda b| - |B_T|$ of the white segment is also large and the event $C_1^\lambda$ becomes unlikely. We will need the following lemma in the sequel.

**Lemma 4.5.** Conditionally on $|B_T| < |\lambda b|$, we have

$$|\lambda b| - |B_T| \xrightarrow{P} +\infty.$$

*Proof.* Recall that $B_0 = |\lambda a|$ almost surely by definition and that if $\hat{T} := \inf\{t \geq 0 : \frac{1}{\lambda}B[\lambda^{3/2}] \leq 0\}$, then $[\lambda^{3/2}\hat{T}] = T$. Thus, using Proposition 4.3, we get the convergence in law

$$\lambda^{-1}B_T \mid |B_T| < |\lambda b| \xrightarrow{d} \kappa S_\tau \mid \kappa |S_\tau| < b,$$

where $\tau := \inf\{t \geq 0 : S_t \leq 0\}$ and $S$ is the spectrally negative Lévy $3/2$-stable process. As a consequence, we have

$$\lambda^{-1}(|\lambda b| - |B_T|) \mid |B_T| < |\lambda b| \xrightarrow{d} b - \kappa S_\tau \mid \kappa |S_\tau| < b,$$

Now, following Proposition 2.4, the distribution of the overshoot of the process $S$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}_+$, which yields the expected result. □

**Case 2:** $|B_T| > |\lambda b|$. In this situation, the whole white segment on the boundary is swallowed, which divides the problem in different subcases (see Figure 12).

In the first four cases, the crossing event we are interested in now depends on a crossing event in a submap $M^*$, which is a free Boltzmann $*$-angulation of the $(B_{T-1} + |B_T| + 1)$-gon (or $(B_{T-1} + |B_T| + 2)$-gon). Those events are represented by a dashed path on Figure 12. The idea here is that when $\lambda$ becomes large, this Boltzmann $*$-angulation converges in law towards the UIHP-$*$, so that the crossing event we consider occurs almost surely. The fifth case is slightly different, since the crossing occurs if the two free edges at the bottom of the revealed quadrangle are black, or if a crossing represented by the dashed path occurs, which seems unlikely. In fact, this fifth case does not occur when $\lambda$ goes to infinity in a sense we will make precise further. We will need the following equivalent of Lemma 4.5 in this setting.

**Lemma 4.6.** Conditionally on $|B_T| > |\lambda b|$, we have

$$|B_T| - |\lambda b| \xrightarrow{P} +\infty \quad \text{and} \quad B_{T-1} \xrightarrow{P} +\infty,$$
and for every $\beta > 1$,

$$\lambda^{-\beta}([|B_T| - [\lambda b]) \xrightarrow{\mathbb{P}} 0 \quad \text{and} \quad \lambda^{-\beta}B_{T^{-1}} \xrightarrow{\lambda \to +\infty} 0.$$ 

**Proof.** The proof is analogous to that of Lemma 4.3. Again, recall that $B_0 = [\lambda a]$ almost surely by definition and that if $\bar{T} := \inf\{t \geq 0 : \lambda^{3/2} B_{\lfloor \lambda^{3/2} t \rfloor} \leq 0\}$, then $[\lambda^{3/2} \bar{T}] = \bar{T}$. Applying Proposition 4.3, we get the convergence in law

$$\lambda^{-1}(B_{T^{-1}}, B_T)|B_T| > [\lambda b] \xrightarrow{(d)} \kappa(S_{\tau}, S_{\tau})|\kappa|S_{\tau}| > b,$$

where $\tau := \inf\{t \geq 0 : S_t \leq 0\}$ and $S$ is the spectrally negative Lévy $3/2$-stable process. As
a consequence, we have
\[
\lambda^{-1}(B_{T-1}, |B_T| - |\lambda b|) \mid |B_T| > |\lambda b| \xrightarrow{(d)} (\kappa S_{\tau -}, \kappa S_{\tau} - b) \mid \kappa|S_{\tau}| > b,
\]

We conclude the proof exactly as before, using the absolute continuity of the joint distribution of the undershoot and the overshoot of the process \(S\) with respect to the Lebesgue measure, given by Proposition 2.4. \(\square\)

We now treat the fifth configuration, denoted \(A_4^{\square}\) in Figure 12, corresponding to the case where the map has the law of the UIHPQ and the last revealed quadrangle has all of its four vertices lying on the boundary of the map, and two of them are at the right of the root edge. Recall that we work conditionally on \(|B_T| > |\lambda b|\), in other words the whole white segment has been swallowed by the revealed face. We thus work on the event \(A_4^{\square}\), and denote by \(K_\lambda^{(1)}\) and \(K_\lambda^{(2)}\) the (random) lengths of the two finite segments determined on the boundary by the last revealed quadrangle (see Figure 12).

**Lemma 4.7.** Conditionally on the event \(A_4^{\square}\), we have
\[
\lambda^{-1} \min \left( K_\lambda^{(1)}, K_\lambda^{(2)} \right) \xrightarrow{\lambda \to +\infty} 0.
\]
As a consequence,
\[
\max \left( B_{T-1} - K_\lambda^{(1)}, |B_T| - |\lambda b| - K_\lambda^{(2)} \right) \xrightarrow{\lambda \to +\infty} +\infty.
\]

**Proof.** Let us work conditionally on the event \(A_4^{\square}\). The first point is to determine the law of the pair \((K_\lambda^{(1)}, K_\lambda^{(2)})\) conditionally on \(B_{T-1}\) and \(|B_T|\). To do so, first observe that by construction, \(K_\lambda^{(1)} + K_\lambda^{(2)} = B_{T-1} + |B_T|\). Now, at each step, the process \((B_n)_{n \geq 0}\) depends only on the number of swallowed edges at the right of the root edge and not on the specific length of the segments that are determined on the boundary when all of the four vertices of the revealed face are lying on the boundary. This implies that for every \(x, x' \geq 0\) we have
\[
\mathbb{P} \left( K_\lambda^{(1)} = x, K_\lambda^{(2)} = x' \mid B_{T-1}, |B_T| \right) = \frac{1_{\{x+x'=B_{T-1}+|B_T|\}} q_{x,x'}}{\sum_{k_1+k_2=B_{T-1}+|B_T|} q_{k_1,k_2}}. \tag{7}
\]

Now, let \(\varepsilon, \lambda > 0\). Still conditionally on \(A_4^{\square}\), we have using the previous identity that
\[
\mathbb{P} \left( K_\lambda^{(1)} \geq \lambda \varepsilon, K_\lambda^{(2)} \geq \lambda \varepsilon \right) = \mathbb{E} \left( \mathbb{P} \left( K_\lambda^{(1)} \geq \lambda \varepsilon, K_\lambda^{(2)} \geq \lambda \varepsilon \mid B_{T-1}, |B_T| \right) \right) = \mathbb{E} \left( \frac{\sum_{k_1+k_2=B_{T-1}+|B_T|} 1_{\{k_1 \geq \lambda \varepsilon, k_2 \geq \lambda \varepsilon\}} q_{k_1,k_2}}{\sum_{k_1+k_2=B_{T-1}+|B_T|} q_{k_1,k_2}} \right).
\]

Let now \(\delta > 0\). We get from Lemma 4.6 that conditionally on \(A_4^{\square}\),
\[
\mathbb{P}(B_{T-1} \geq \lambda^{1+\delta}) \xrightarrow{\lambda \to +\infty} 0 \quad \text{and} \quad \mathbb{P}(|B_T| - |\lambda b| \geq \lambda^{1+\delta}) \xrightarrow{\lambda \to +\infty} 0.
\]

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Here, we use the fact that the law of $B_{T-1}$ and $|B_T| - |\lambda b|$ is the same conditionally on $|B_T| > |\lambda b|$ and $A_4^{\square}$ respectively, because of the same argument as for the identity \(7\). Now, observe that $q_{k_1,k_2}^{\square}$ is equivalent to $t_{\lambda}k_1^{-5/2}k_2^{-5/2}$ when $k_1, k_2$ go to infinity, using the estimation \(2\). Thus, if $\lambda$ is large enough, we have $q_{k_1,k_2}^{\square} \leq C(\lambda\varepsilon)^{-5}$ for every $k_1, k_2 \geq \lambda\varepsilon$, where $C$ is a positive constant. Then, we get

$$
\mathbb{E}\left( \frac{\sum_{k_1+k_2=|B_{T-1}|+|B_T|}q_{k_1,k_2}^{\square}1_{\{k_1,k_2 \geq \lambda\varepsilon, k_2 \geq \lambda|T|\}}1_{\{|B_{T-1}| \leq \lambda^{1+\delta}\}}1_{\{||B_T| - |\lambda b| \leq \lambda^{1+\delta}\}}}{\sum_{k_1+k_2=|B_{T-1}|+|B_T|}q_{k_1,k_2}^{\square}} \right) \leq \frac{6C(\lambda\varepsilon)^{-5}\lambda^{1+\delta}}{C'(3\lambda^{1+\delta})^{-5/2}}.
$$

Here, we dominated the denominator by a single term of the sum, and noticed that on the event \(\{B_{T-1} \leq \lambda^{1+\delta}\} \cap \{||B_T| - |\lambda b| \leq \lambda^{1+\delta}\}\), we have $q_{1,|B_{T-1}|+|B_T|}^\square \geq q_{1,3\lambda^{1+\delta}}^\square \geq C'(3\lambda^{1+\delta})^{-5/2}$ provided that $\lambda$ is large enough and for a positive constant $C'$, using again \(2\). Taking $\delta$ small enough, we see that this quantity vanishes when $\lambda$ goes to infinity which yields the first part of the result. For the second one, we use the arguments of the proof of Lemma \(4.6\) (and the previous remark on the law of $B_{T-1}$ and $|B_T| - |\lambda b|$ conditionally on $|B_T| > |\lambda b|$ and $A_4^{\square}$) to get that

$$
\lambda^{-1} \max \left( B_{T-1} - K^{(1)}_\lambda, |B_T| - |\lambda b| - K^{(2)}_\lambda \right) \\
\geq \lambda^{-1} \min \left( B_{T-1}, |B_T| - |\lambda b| \right) - \lambda^{-1} \min \left( K^{(1)}_\lambda, K^{(2)}_\lambda \right) \\
\xrightarrow{(d)} \min \left( \kappa \mathcal{S}_{\tau -}, \kappa \mathcal{S}_\tau - b \right) \bigg| \kappa |\mathcal{S}_{\tau}| > b
$$

and thus the expected result by the very same argument as in Lemma \(4.6\). \(\square\)

Intuitively, the consequence of this lemma is that only two situations may occur conditionally on the event $A_4^{\square}$, either $B_{T-1} - K^{(1)}_\lambda$ or $|B_T| - |\lambda b| - K^{(2)}_\lambda$ is large, or more precisely, respectively $K^{(1)}_\lambda$ or $K^{(2)}_\lambda$ is small with respect to $\lambda$. This corresponds to the subcases $A_{4,1}$ and $A_{4,2}$ of Figure \(13\). In some sense, this tells us that quadrangles “look like” triangles when $\lambda$ goes to infinity.

Figure 13: The subcase $A_{4,1}^{\square}$. 

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Let us be more precise here. Recall that we work conditionally on $|B_T| > |\lambda b|$ and denote by $A_{4,1}$, respectively $A_{4,2}$, the event $\{A_4^\square, K_\lambda^{(1)} < B_{T-1}\}$, respectively $\{A_4^\square, K_\lambda^{(2)} < |B_T| - |\lambda b|\}$. Lemma 4.7 ensures that the probability of the event $\{A_4^\square, K_\lambda^{(1)} \geq B_{T-1}, K_\lambda^{(2)} \geq |B_T| - |\lambda b|\}$ vanishes when $\lambda$ becomes large, so that we can restrict our attention to the events $A_{4,1}$ and $A_{4,2}$ instead of $A_4^\square$ in the sequel.

Looking at the dashed paths on Figures 12 and 13 one gets that the crossing event we focus on now depends on a simpler crossing event in a Boltzmann map (thanks to the spatial Markov property). We denote by $d^\lambda_l$ and $d^\lambda_r$ the length of the two open segments on the boundary of that map, which has thus a boundary coloured as in Figure 14. In the first four cases of Figure 12 we have $d^\lambda_l = B_T - 1$ and $d^\lambda_r = |B_T| - |\lambda b|$. The situation is a bit more complicated in the last case, but with the convention that $K_\lambda^{(1)} = K_\lambda^{(2)} = +\infty$ out of the event $A_4^\square$, we have that

$$d^\lambda_l = B_T - 1 \mathbf{1}_{\{K_\lambda^{(1)} > B_{T-1} + |\lambda b|\}} + \left(B_T - 1 - K_\lambda^{(1)}\right) \mathbf{1}_{\{K_\lambda^{(1)} < B_{T-1}\}},$$

and

$$d^\lambda_r = \left(|B_T| - |\lambda b|\right) \mathbf{1}_{\{K_\lambda^{(2)} > |B_T|\}} + \left(|B_T| - |\lambda b| - K_\lambda^{(2)}\right) \mathbf{1}_{\{K_\lambda^{(2)} < |B_T| - |\lambda b|\}}.$$

Using Lemmas 4.6 and 4.7 and the fact that $K_\lambda^{(1)} + K_\lambda^{(2)} = B_T + B_T$ on the event $A_4^\square$, we get that

$$d^\lambda_l \xrightarrow{\lambda \to +\infty} +\infty \quad \text{and} \quad d^\lambda_r \xrightarrow{\lambda \to +\infty} +\infty.$$

We thus proved that conditionally on $|B_T| > |\lambda b|$, the crossing event we are interested in occurs almost surely when there is an open path between the two black segments on the boundary of the map of Figure 14 which is a free Boltzmann $*$-angulation. In what follows, we denote by $C_2^\lambda$ this event. Note that the length of the free segment on the boundary can either be 1 or 2 depending on the situation, but this plays no role in the sequel.

![Figure 14: The crossing event $C_2^\lambda$.](image)
### 4.1.3 Asymptotic probabilities for submap crossings

Throughout this section, we discuss the asymptotic probabilities of the events \( C_1^λ \) and \( C_2^λ \) we previously introduced.

**Case 1: \( C_1^λ \).** Recall that the event \( C_1^λ \) is the crossing event given by the existence of a path linking the origin of a UIHP-∗ and the infinite open segment of the boundary, where the boundary of the map has the colouring of Figure [11]. We denote by \( v^∗ \) the origin of the map, and recall that the length of the white segment located at the right of the origin satisfies from Lemma 4.5:

\[
d^λ := |λb| - |B_T| \xrightarrow{λ \to +\infty} +\infty.
\]

For every \( x \geq 0 \), let \( \partial_∞ x \) be the set of vertices of the map that are located on the boundary, at least \( x \) vertices away on the right of \( v^∗ \). We denote by \( d_{gr} \) the graph distance on the set of vertices of the planar maps we consider. We are now able to compute the asymptotic probability we are interested in.

**Proposition 4.8.** We have

\[
P(C_1^λ) \xrightarrow{λ \to +\infty} 0.
\]

As a consequence, \( P(C_{λa,λb}^∗ | |B_T| < |λb|) \xrightarrow{λ \to +\infty} 0. \)

**Proof.** For every \( R \geq 0 \), we denote by \( B_R^∞ \) the ball of radius \( R \) of our map, and introduce the event \( C_R^∞ := \{ \text{There exists an open path leaving } B_R^∞ \} \). Notice that for every \( R \geq 0 \),

\[
P(C_1^λ) \leq P(d_{gr}(v^∗, \partial_∞) \leq R) + P(C_R^∞).
\]

Indeed, the open segment on the boundary of our map is at distance at least \( d_{gr}(v^∗, \partial_∞) \) from the origin, so that the crossing event cannot occur if no open path leaves \( B_R^∞ \) when \( d_{gr}(v^∗, \partial_∞) > R \). Now, we have that

\[
d_{gr}(v^∗, \partial_∞) \xrightarrow{λ \to +\infty} +\infty,
\]

since the balls \( (B_R^∞)_{R \geq 0} \) are almost surely finite and \( d^λ \) goes to infinity in probability when \( λ \) becomes large. As a consequence, we get for every \( R \geq 0 \) that

\[
\limsup_{λ \to +\infty} P(C_1^λ) \leq P(C_R^∞). \tag{8}
\]

Then, we have

\[
P(C_R^∞) \xrightarrow{R \to +\infty} Θ^∗_{bond}(p^∗_{c,bond}) = 0,
\]

which concludes the proof letting \( R \) go to infinity in \( (8) \). \qed

**Case 2: \( C_2^λ \).** We now focus on the crossing event \( C_2^λ \), which is defined by the existence of an open path linking the two open segments on the boundary of a Boltzmann ∗-angulation, whose boundary has the colouring of Figure [14]. Recall that the law of this map is denoted by \( µ^∗_{v(λ)} \), where \( v(λ) \) is the number of vertices on the boundary. The lengths of the open segments are given by \( d^λ_l \) and \( d^λ_r \). We also denote by \( v^∗ \) the origin of the map (which is on
the free segment with length 1 or 2), and, for every \( x \geq 0 \), by \( \partial_x \) the set of vertices of our map that are located at least \( x \) vertices on the right of \( v^* \) on the boundary. Finally, we set \( d^\lambda := \min(d_1^\lambda, d_2^\lambda) + 2 \). Using previous arguments, it is obvious that

\[
v(\lambda) \mathop{\longrightarrow}\limits_{\lambda \to +\infty} +\infty \quad \text{and} \quad d^\lambda \mathop{\longrightarrow}\limits_{\lambda \to +\infty} +\infty.
\]

We first establish a preliminary lemma.

**Lemma 4.9.** The following convergence holds:

\[
d_{\text{gr}}(v^*, \partial_{d^\lambda}) \mathop{\longrightarrow}\limits_{\lambda \to +\infty} +\infty.
\]

**Proof.** For every \( R \geq 0 \) and \( x \geq 0 \), the event \( \{d_{\text{gr}}(v^*, \partial_{d^\lambda}) \leq R\} \) is measurable with respect to the ball of radius \( \max(R, x) \) of the map (since this event is exactly saying that a vertex of the ball of radius \( R \) lies on the boundary at distance less than \( x \) of the origin). As a consequence, since \( \mu_{\nu(\lambda)} \mathop{\longrightarrow}\limits_{\lambda \to +\infty} \nu_\infty \) in the sense of narrow convergence for the local topology, we have

\[
\mathbb{P}(d_{\text{gr}}(v^*, \partial_{d^\lambda}) \leq R) \mathop{\longrightarrow}\limits_{\lambda \to +\infty} \mathbb{P}(d_{\text{gr}}(v^*, \partial_{d^\lambda}) \leq R).
\]

Therefore, since \( d^\lambda \) goes to infinity in probability when \( \lambda \) becomes large, we get for every \( R \geq 0 \) and \( x \geq 0 \) that

\[
\limsup_{\lambda \to +\infty} \mathbb{P}(d_{\text{gr}}(v^*, \partial_{d^\lambda}) \leq R) \leq \mathbb{P}(d_{\text{gr}}(v^*, \partial_{d^\lambda}) \leq R),
\]

and we conclude by letting \( x \) go to infinity, using that the balls \( (B_R^\lambda)_{R \geq 0} \) of the UIHP-* are almost surely finite. \( \square \)

**Proposition 4.10.** We have

\[
\mathbb{P}(C_\lambda^2) \mathop{\longrightarrow}\limits_{\lambda \to +\infty} 1.
\]

As a consequence, \( \mathbb{P}(C_{x\lambda,\lambda b}^* \mid |B_T| > |\lambda b|) \mathop{\longrightarrow}\limits_{\lambda \to +\infty} 1 \).

**Proof.** First, notice that the complementary event \( (C_\lambda^2)^c \) is exactly the existence of a dual closed crossing between the finite free and white segment in our map (see Figure 14). Then, two cases may occur: either \( d_{\text{gr}}(v^*, \partial_{d^\lambda}) \leq R \), or \( d_{\text{gr}}(v^*, \partial_{d^\lambda}) > R \) and in that case, \( (C_\lambda^2)^c \) occurs only if a closed dual path leaves the ball of radius \( R \) of the map, \( B_R^\lambda \). We denote by \( C'_{R,\lambda} \) this event, and then for every \( R \geq 0 \) we have that

\[
\mathbb{P}((C_\lambda^2)^c) \leq \mathbb{P}(d_{\text{gr}}(v^*, \partial_{d^\lambda}) \leq R) + \mathbb{P}(C'_{R,\lambda}).
\]

Now, the event \( C'_{R,\lambda} \) is measurable with respect to the ball of radius \( R \) of the map, so that the convergence in law \( \mu_{\nu(\lambda)} \mathop{\longrightarrow}\limits_{\lambda \to +\infty} \nu_{\infty} \) for the local topology implies that for every fixed \( R \geq 0 \):

\[
\mathbb{P}(C'_{R,\lambda}) \mathop{\longrightarrow}\limits_{\lambda \to +\infty} \mathbb{P}(C'_{R,\infty}),
\]

where \( C'_{R,\infty} \) is the event that a closed dual path leaves the ball of radius \( R \), denoted by \( B_R^\infty \), in the UIHP-*E. As a consequence, using Lemma 4.9, we get for every \( R \geq 0 \) that

\[
\limsup_{\lambda \to +\infty} \mathbb{P}((C_\lambda^2)^c) \leq \mathbb{P}(C'_{R,\infty}). \tag{9}
\]
We now consider dual bond percolation on our map (i.e., bond percolation on the dual graph), with the convention that the colour of a dual edge is the colour of the unique primal edge it crosses. We denote by $\Theta^*_\text{bond}'$ and $p^*_{c,\text{bond}'}$ the dual bond percolation probability and the dual bond percolation threshold on the UIHP-. Now we use the fact (see Section 3.4.1 in [3] for details) that $p^*_{c,\text{bond}'} = 1 - p^*_{c,\text{bond}'}$ and that there is no percolation at critical point in this setting. Then,

$$\mathbb{P}(C'_{R,\infty}) \underset{R \to +\infty}{\longrightarrow} \Theta^*_\text{bond}'(1 - p^*_{c,\text{bond}}) = 0,$$

since we are interested in closed dual cluster and by construction, the parameter for dual bond percolation equals $1 - p^*_{c,\text{bond}'}$. One should notice here that this result holds in the UIHP- with a “Free-Black” boundary as proved in [3], Section 3.4.1, but the probability of our dual percolation event is bounded from above by the dual percolation event in this setting by the standard coupling argument. This yields the expected result letting $R$ go to infinity in (9).

### 4.1.4 Proof of Theorem 1.2: bond percolation case

We now prove the main result of this section. Recall that $C^*_{\lambda a, \lambda b}$ denotes the crossing event between the two black segments on the boundary in a UIHP- coloured as in Figure 8. Let us split up the problem conditionally on the value of $|B_T|$. We have

$$\mathbb{P}(C^*_{\lambda a, \lambda b}) = \mathbb{P}(C^*_{\lambda a, \lambda b} \mid |B_T| < [\lambda b])\mathbb{P}(|B_T| < [\lambda b]) + \mathbb{P}(C^*_{\lambda a, \lambda b} \mid |B_T| \geq [\lambda b])\mathbb{P}(|B_T| \geq [\lambda b]).$$

Using again the stopping time $\bar{T} := \inf\{t \geq 0 : \frac{1}{\lambda}B_{\lfloor \lambda^3/2t \rfloor} \leq 0\}$, we get the convergence

$$\frac{1}{\lambda}B_{\lfloor \lambda^3/2\bar{T} \rfloor} \overset{(d)}{\underset{\lambda \to +\infty}{\longrightarrow}} \kappa S_\tau,$$

where $\tau := \inf\{t \geq 0 : S_t \leq 0\}$. Then, the distribution of the overshoot of the process $S$ given in Proposition 2.4, which is absolutely continuous with respect to the Lebesgue measure, yields

$$\mathbb{P}(|B_T| \geq [\lambda b]) \overset{\lambda \to +\infty}{\longrightarrow} P_\pi(\kappa |S_\tau| \geq b) = \frac{1}{\pi} \arccos \left( \frac{b-a}{a+b} \right).$$

The results of Propositions 4.8 and 4.10 thus yield that

$$\lim_{\lambda \to +\infty} \mathbb{P}(C^\Delta_{\text{bond}}(\lambda a, \lambda b)) = \lim_{\lambda \to +\infty} \mathbb{P}(C^*_{\text{bond}}(\lambda a, \lambda b)) = \frac{1}{\pi} \arccos \left( \frac{b-a}{a+b} \right),$$

which is exactly Theorem 1.2 in the bond percolation case on the UIHPT and the UIHPQ.

### 4.2 Crossing probabilities for face percolation

We now consider the crossing events in the case of face percolation. We will see that although the peeling process is slightly different, the same method as for bond percolation applies in order to compute the scaling limits we are interested in. Recall that two faces are adjacent in our model if and only if they share an edge. We again suppose that the boundary has
“White-Black-White-Black” shape. Here, this means that the crossing event we are interested in (denoted by \( C_{\text{face}}(\lambda a, \lambda b) \)) has to occur between the two black segments of the boundary, such that one can imagine faces lying on the other side of the boundary as in Figure 15.

![Figure 15: Initial colouring for face percolation problem.](image)

The arguments being very similar to the previous section, we do not give as many details, except for the remarkable differences.

### 4.2.1 Peeling process and scaling limit

The main argument is again to define the appropriate peeling process, which is here simpler because we do not have to deal with a free segment on the boundary. However, a subtlety arising in this context is that peeling white edges leads to a bias in the Markov chain giving the size of the finite black segment, so that it may escape to infinity. The right thing to do is to peel the leftmost black edge instead.

**Algorithm 4.** Consider a half-plane map which has the law of the UIHP-* and a “White-Black” boundary.

- **Step 1:** Reveal the face incident to the leftmost black edge of the boundary, including its colour.
- **Step 2:** Repeat the algorithm on the UIHP-* given by the unique infinite connected component of the current map, deprived of the revealed face.

At each step, the exposed edges of the map form a boundary, whose colour is inherited from the adjacent revealed faces.

The properties of this algorithm are the same as before, due to the spatial Markov property. In particular, the random variables \((\mathcal{E}_n, \mathcal{R}_{r,n}, c_n)_{n \geq 0}\), defined exactly as in Section 4.1, are i.i.d. and have the law of \(\mathcal{E}, \mathcal{R}_r\) and \(c\) (which has the Bernoulli law of parameter \(p_{c,\text{face}}^\ast\)) respectively, where \(c_n\) denotes the colour of the face revealed at step \(n\) of the process for every \(n \geq 0\) and the other notations are the same as before. We again need the process \((B_n)_{n \geq 0}\), defined as follows.

We let \(B_0 = \lfloor \lambda a \rfloor\) and for every \(n \geq 0\):

\[
B_{n+1} = B_n + 1_{\{c_n=1\}} \mathcal{E}_n - \mathcal{R}_{r,n} - 1.
\]

The process \((B_n)_{n \geq 0}\) is a Markov chain with respect to the canonical filtration of the exploration process. Moreover, if we denote by \(T := \inf\{n \geq 0 : B_n \leq 0\}\) the first entrance time of \((B_n)_{n \geq 0}\) in \(\mathbb{Z}_-\), then for every \(0 \leq n < T\), \(B_n\) is the length of the finite black segment at step \(n\) of the exploration process.

**Behaviour of \((B_n)_{n \geq 0}\).** The properties of the peeling process yields that \((B_n)_{n \geq 0}\) is a random walk with steps distributed as \(X\) defined by

\[X \sim \begin{cases} 1_{\{c_n=1\}} \mathcal{E}_n - \mathcal{R}_{r,n} - 1, \\ 0 \end{cases}\]
\[ X := 1_{c=1}E - R_r - 1. \]

In particular, using the same arguments as before, we get

\[
\mathbb{E}(X) = p_{c,\text{face}}^* \mathbb{E}(E) - \mathbb{E}(R_r) - 1 = \begin{cases} 
\frac{5}{3} p_{c,\text{face}}^* - \frac{4}{3} & \text{when } * = \triangle \\
2 p_{c,\text{face}}^* - \frac{3}{2} & \text{when } * = \square 
\end{cases}.
\]

Thus, we have \( \mathbb{E}(X) = 0 \) in both the UIHPT and the UIHPQ, and this yields the convergence \( \frac{(B_{\lambda, t})_{t \geq 0}}{\lambda} \xrightarrow{(d)} \kappa(S_t)_{t \geq 0} \) in the sense of Skorokhod, where \( S \) is the Lévy spectrally negative \( 3/2 \)-stable process.

### 4.2.2 Stopped peeling process and asymptotics

Let us focus on the first time when \( (B_n)_{n \geq 0} \) reaches \( \mathbb{Z}_- \). Again, two cases are likely to happen:

**Case 1:** \( |B_T| < \lfloor \lambda b \rfloor \). When \( |B_T| < \lfloor \lambda b \rfloor \), a finite part of the white segment on the boundary is swallowed. If the last revealed face is white, the crossing event cannot occur. If this face is black, the crossing event implies the existence of an open path linking the rightmost edge of the finite black segment and the infinite black segment on the boundary. This event, denoted by \( C_{1,\lambda} \), is similar as in the bond case. We again have that

\[ \lfloor \lambda b \rfloor - |B_T| \xrightarrow{\mathbb{P}} +\infty, \]

so that the crossing event \( C_{1,\lambda} \) has probability bounded by the percolation probability in this model and we get

\[ \lim_{\lambda \to +\infty} \mathbb{P}(C_{1,\lambda}) \leq \Theta_{\text{face}}^* (p_{c,\text{face}}^*) = 0. \]

**Case 2:** \( |B_T| > \lfloor \lambda b \rfloor \). When \( |B_T| > \lfloor \lambda b \rfloor \), the whole white segment on the boundary is swallowed. We have

\[ |B_T| - \lfloor \lambda b \rfloor \xrightarrow{\mathbb{P}} +\infty \text{ and } B_{T-1} \xrightarrow{\mathbb{P}} +\infty, \]

as in the bond percolation case. Whether the last revealed face is black or white, the crossing event is implied by a crossing event in a Boltzmann map, whose boundary has the colouring of Figure 14. We denote by \( C_{2,\lambda}^2 \) this event, and we have that

\[ d_{1,\lambda} \xrightarrow{\mathbb{P}} +\infty \text{ and } d_{r,\lambda} \xrightarrow{\mathbb{P}} +\infty \]

in every likely case. Then, when \( \lambda \) goes to infinity, we have that \( (C_{2,\lambda}^2)^c \) implies the closed dual face percolation event (dual face percolation being site percolation on the dual lattice where we add connections between sites whose dual faces share a vertex. This is analogous to the star-lattice in the case of \( \mathbb{Z}_d \), and we have that \( 1 - p_{c,\text{face}}^* = p_{c,\text{face}}^* \), with no percolation at critical point, see Section 3.4.2 in [3]). As a consequence, we get:
\[
\lim_{\lambda \to +\infty} \mathbb{P}(\overline{(C^2_\lambda)}) \leq \Theta^{*}_{\text{face'}}(1 - p^{*}_{\text{face}}) = 0.
\]

The statements we obtained up to this point yield, using the same arguments as in Section 4.1, that

\[
\lim_{\lambda \to +\infty} \mathbb{P} \left( C^{\triangle}_{\text{face}}(\lambda a, \lambda b) \right) = \lim_{\lambda \to +\infty} \mathbb{P} \left( C^{\square}_{\text{face}}(\lambda a, \lambda b) \right) = \frac{1}{\pi} \arccos \left( \frac{b - a}{a + b} \right),
\]

which is exactly Theorem 1.2 in the case of face percolation on the UIHPT and the UIHPQ.

### 4.3 Crossing probabilities for site percolation

The last case we study is site percolation on the UIHPQ. We make the same assumption on the colouring of the boundary, which is supposed to have “White-Black-White-Black” colouring as in Figure 16.

![Figure 16: The initial colouring for site percolation problem.](image)

We are still interested in the scaling limit of the probability of the crossing event that the two black segments are part of the same percolation cluster, denoted by \( C^{\square}_{\text{site}}(\lambda a, \lambda b) \), at the critical point \( p^{\square}_{\text{c,site}} \).

#### 4.3.1 Peeling process and scaling limit

Here, the appropriate peeling process is roughly the same as in the bond percolation case. The idea is to use the vertex-peeling process (Algorithm 1) introduced in Section 3 in order to peel vertices instead of edges.

**ALGORITHM 5.** Consider a half-plane map which has the law of the UIHPQ with a “White-(Free)” boundary.

- We reveal the colour of the rightmost free vertex, if any:
  - If it is black, repeat the algorithm.
  - If it is white, peel this vertex using the vertex-peeling process, without revealing the colour of the vertices.

- If there is no free vertex on the boundary of the map, as in the first step, peel the rightmost white vertex of the infinite white segment using the vertex-peeling process, without revealing the colour of the vertices.

After each step, repeat the algorithm on the UIHPQ given by the unique infinite connected component of the current map, deprived of the revealed faces. The map we obtain is again implicitly rooted at the next edge we have to peel.
Following the strategy of Section 4.1 we now define the processes \((F_n)_{n \geq 0}\) and \((B_n)_{n \geq 0}\). Let us first introduce some notation. For every \(n \geq 0\), we denote by \(c_n\) the color of the revealed vertex at step \(n\) of the exploration, with the convention that \(c_n = 0\) when there is no free vertex on the boundary. When \(c_n = 0\), the vertex-peeling process is executed and we let \(\sigma_n + 1\) be the number of steps of this process (so that \(\sigma_n \geq 0\)). Then, for every \(0 \leq i \leq \sigma_n\), we denote by \(E_i^{(n)}\), resp. \(R_i^{(l,n)}, R_i^{(r,n)}\) the number of exposed edges, resp. swallowed edges on the left and right of the root edge at step \(i\) of the vertex-peeling process. All of those variables are set to zero when \(c_n = 1\) by convention.

Thanks to the spatial Markov property and the definition of the vertex-peeling process, if we restrict ourselves to the consecutive stopping times when \(c_n = 0\), the random variables \((\sigma_n)_{n \geq 0}\) are i.i.d., with geometric law of parameter \(\mathbb{P}(R_r = 0)\) (i.e. they have the same law as \(\sigma\), where for every \(k \geq 0\), \(\mathbb{P}(\sigma = k) = (\mathbb{P}(R_r = 0))^k(1 - \mathbb{P}(R_r = 0))\)). Conditionally on \(\sigma_n\), the random variables \((E_i^{(n)})_{n \geq 0, 0 \leq i \leq \sigma_n-1}\) and \((R_i^{(l,n)})_{n \geq 0, 0 \leq i \leq \sigma_n-1}\) are i.i.d. and have the law of \(\mathcal{E}\) and \(\mathcal{R}_i\) conditionally on \(\mathcal{R}_r = 0\) respectively. Moreover, \((E_{\sigma_n}^{(n)})_{n \geq 0}, (R_{\sigma_n}^{(l,n)})_{n \geq 0}\) and \((R_{\sigma_n}^{(r,n)})_{n \geq 0}\) are i.i.d. and have the law of \(\mathcal{E}, \mathcal{R}_i\) and \(\mathcal{R}_r\) conditionally on \(\mathcal{R}_r > 0\) respectively. Finally, all those variables independent for different values of \(n\).

In the sequel, we let \((\tilde{E}_i)_{i \geq 0}\), \((\tilde{R}_i)_{i \geq 0}\), \(\tilde{E}\), \(\tilde{R}_l\) and \(\tilde{R}_r\) be independent random variables having the law of \(\mathcal{E}, \mathcal{R}_i\) and \(\mathcal{R}_r\) conditionally on \(\mathcal{R}_r = 0\) and \(\mathcal{R}_r > 0\) respectively. We also define \(\sigma\) and \(c\) that have the geometric law of parameter \(\mathbb{P}(\mathcal{R}_r = 0)\) and the Bernoulli law of parameter \(p^{(c,\text{site})}\) respectively. We again define the processes \((F_n)_{n \geq 0}\) and \((B_n)_{n \geq 0}\) as before.

First, for every \(n \geq 0\), we let \(F_n\) be the length of the free segment on the boundary at step \(n\) of the peeling process.

We now provide an alternative description of the process \((F_n)_{n \geq 0}\). We first have \(F_0 = 0\). Then, let \(F_0^{(0)} = 0\) and for every \(n \geq 0\), set recursively \(F_0^{(n)} := (F_n - 1)_{+}\). We let for every \(n \geq 0\) and \(0 \leq i \leq \sigma_n\),

\[
F_i^{(n)} := \begin{cases} 
F_i^{(n)} + E_i^{(n)} - R_i^{(l,n)} - 1 & \text{if } F_i^{(n)} - R_i^{(l,n)} - 1 \geq 0 \\
F_i^{(n)} + E_i^{(n)} - 1 & \text{otherwise}
\end{cases}
\]

Then, we have for every \(n \geq 0\),

\[
F_{n+1} = (F_n - 1)1_{\{c_n = 1\}} + F_{\sigma_n+1}^{(n)}1_{\{c_n = 0\}}.
\]

The process \((F_n)_{n \geq 0}\) is a Markov chain with respect to the canonical filtration of the exploration process, and \(c_n = 0\) when \(F_n = 0\).

**Remark.** Contrary to the bond percolation case, the process \((F_n)_{n \geq 0}\) can reach zero even when \(c_n = 0\), and can also take value zero at consecutive times with positive probability.

Then, we also let \(B_0 = [\lambda a]\) and for every \(n \geq 0\):

\[
B_{n+1} = B_n + 1_{\{c_n = 1\}} - 1_{\{c_n = 0\}}(R_{\sigma_n}^{(r,n)} - 1).
\]

The process \((B_n)_{n \geq 0}\) is a Markov chain with respect to the canonical filtration of the exploration process. Moreover, if we denote by \(T := \inf\{n \geq 0 : B_n \leq 0\}\) the first entrance time of \((B_n)_{n \geq 0}\) in \(\mathbb{Z}_-\), then for every \(0 \leq n < T\), \(B_n\) is the length of the finite black segment at step \(n\) of the exploration process.
Behaviour of \((F_n)_{n \geq 0}\). Exactly as in Section 2.4, the process \((F_n)_{n \geq 0}\) is not a random walk, but has the same behaviour when it is far away from zero. More precisely, let us define 
\[ \hat{\sigma}_+ := \inf\{n \geq 0 : F_n > 0\} \] and 
\[ \hat{\sigma}_- := \inf\{n \geq \hat{\sigma}_+ : 3 \leq i \leq \sigma_n : F_i^{(n)} - R_i^{(l,n)} - 1 < 0\} \]. By construction, as long as \(\hat{\sigma}_+ \leq n < \hat{\sigma}_-\), we have
\[ F_{n+1} = F_n - 1_{\{c_n=1\}} + 1_{\{c_n=0\}} \left( \sum_{i=0}^{\sigma_n} (\hat{E}_i^n - \hat{R}_{i,n}^i) - 1 \right) \],
and \((F_{\hat{\sigma}_+ + n} - F_{\hat{\sigma}_+})_{0 \leq n \leq \hat{\sigma}_- - \hat{\sigma}_+}\) is a killed random walk with steps distributed as the random variable \(\hat{X}\) defined by
\[ \hat{X} := -1_{\{c=1\}} + 1_{\{c=0\}} \left( \sum_{i=1}^{\sigma} (\hat{E}_i - \hat{R}_{i} - 1) + E - R_{\tau} - 2 \right) \].
We get from the definitions that
\[ \mathbb{E}(\hat{X}) = -p_{\text{site}}^{(1)} + (1 - p_{\text{site}}^{(1)})[\mathbb{E}(\sigma)\mathbb{E}(\hat{E}_0 - \hat{R}_{\tau} - 1) + \mathbb{E}(\hat{E} - \hat{R}_{\tau} - 2)] 
\quad = -\frac{5}{9} + \frac{4}{9} \left( \frac{7}{2} [\mathbb{E}(\hat{E}_0) - \mathbb{E}(\hat{R}_{\tau}) - 1] + 9 - \frac{7}{2} \mathbb{E}(\hat{E}_0) - \frac{9}{4} + \frac{7}{2} \mathbb{E}(\hat{R}_{\tau}) - 2 \right) = 0 \].
It is again easy to check that the random variable \(\hat{X}\) satisfies the assumptions of Proposition 2.3 so that if we denote by \((\hat{S}_n)_{n \geq 0}\) a random walk with steps distributed as \(\hat{X}\), we have
\[ \left( \frac{\hat{S}_{[\lambda t]}}{\lambda^{2/3}} \right)_{t \geq 0} \xrightarrow{(d)} \kappa(S_t)_{t \geq 0}, \] in the sense of convergence in law for Skorokhod’s topology, where \(S\) is the Lévy 3/2-stable process of Section 2.4.

Behaviour of \((B_n)_{n \geq 0}\). Exactly as before, \((B_n)_{n \geq 0}\) is not a random walk but behaves that way when \(F_n\) is not equal to zero. More precisely, let \(\sigma_0^{(0)} = 0\), \(\sigma_0^{(1)} := \inf\{n \geq 0 : F_n > 0\}\) and recursively for every \(k \geq 1\),
\[ \sigma_k^{(0)} := \inf\{n \geq \sigma_{k-1}^{(1)} : F_n = 0\} \quad \text{and} \quad \sigma_k^{(1)} := \inf\{n \geq \sigma_{k}^{(0)} : F_n > 0\}. \] Using the strong Markov property, for every \(k \geq 0\), the process \((B_{\sigma_k^{(1)} + n} - B_{\sigma_k^{(0)}})_{0 \leq n \leq \sigma_k^{(0)} - \sigma_k^{(1)}}\) is a killed random walk with steps distributed as the random variable \(X\) defined by
\[ X := 1_{\{c=1\}} - 1_{\{c=0\}}(\hat{R}_{\tau} - 1). \] In particular, we have \(\mathbb{E}(X) = p_{\text{site}}^{(1)} - (1 - p_{\text{site}}^{(1)})[\mathbb{E}(R_{\tau} \mid R_{\tau} > 0) - 1] = 0\) and properties of the random variable \(X\) yields that if \((S_n)_{n \geq 0}\) is a random walk with steps distributed as \(X\), then
\[ \left( \frac{S^{[\lambda t]}}{\lambda^{2/3}} \right)_{t \geq 0} \xrightarrow{(d)} \kappa(S_t)_{t \geq 0} \] in the sense of Skorokhod, where \(S\) is a Lévy 3/2-stable process.
At this point, we are in the exact situation of Section 4.1, and the point is to get the following equivalent of Proposition 4.3

**Proposition 4.11.** We have, in the sense of convergence in law for Skorokhod’s topology

\[
\left( \frac{B_{t \lambda^{3/2} t}}{\lambda} \right)_{t \geq 0} \xrightarrow{(d)} \kappa(S_t)_{t \geq 0},
\]

where \( S \) is the Lévy spectrally negative \( 3/2 \)-stable process.

The proof of this result is analogous to that of Proposition 4.3, but some notable differences exist as we now explain. Let again \( S_0 = B_0 \) and introduce a sequence \((\beta_n)_{n \geq 0}\) of i.i.d. random variables with Bernoulli law of parameter \( p \). For every \( n \geq 0 \), we set

\[
S_{n+1} = S_n + \begin{cases} 
B_{n+1} - B_n = 1_{\{c_n = 1\}} - 1_{\{c_n = 0\}}(R_{\sigma_n}^{(r,n)} - 1) & \text{when } F_n > 0 \\
\beta_n + (1 - \beta_n)(B_{n+1} - B_n) = 1_{\{\beta_n = 1\}} - 1_{\{\beta_n = 0\}}(R_{\sigma_n}^{(r,n)} - 1) & \text{when } F_n = 0
\end{cases}
\]

Let also for every \( n \geq 0 \), \( \Xi_n := \{0 \leq k < n | F_n = 0\} \) and \( \xi_n := \# \Xi_n \). It is clear using arguments of Section 4.1 that \((S_n)_{n \geq 0}\) is a random walk started from \( B_0 \) with steps distributed as the random variable \( X \) we previously introduced, and that for every \( n \geq 0 \),

\[
S_n - R_n \leq B_n \leq S_n \quad \text{a.s.,}
\]

where \( R_n := \sum_{k \in \Xi_n} (1 - (B_{k+1} - B_k)) \). The point is now to prove that for every \( \alpha > 0 \),

\[
\frac{\xi_n}{n^{1/3+\alpha}} \xrightarrow{P} 0 \quad (12)
\]

We will then use the arguments of the proof of Proposition 4.3 to conclude that \((R_n)_{n \geq 0}\) is small with respect to \((S_n)_{n \geq 0}\) and does not affect the scaling limit of the process \((B_n)_{n \geq 0}\).

Here, the argument differs from the bond percolation case because the process \((F_n)_{n \geq 0}\) is slightly different from that of Section 4.1 in the sense that it can stay at value zero for consecutive steps with positive probability (see the previous Remark). However, if we introduce for \( n \geq 0 \) the quantity \( \chi_n := \# \{ k \geq 0 : \sigma_k^{(0)} \leq n \} \), then the definition of \( \hat{X} \) and the convergence (10) yield with the same proof as in Lemma 4.4 that

\[
\frac{\chi_n}{n^{1/3+\alpha}} \xrightarrow{P} 0 \quad (13)
\]

Now, thanks to the strong Markov property, the random variables \((\sigma_k^{(1)} - \sigma_k^{(0)})_{k \geq 0}\) are i.i.d. with geometric law of parameter \( \theta \in (0,1) \), where \( \theta \) is the probability that the vertex-peeling process expose a positive number of vertices (for instance bounded from below by \( P(E > 1 | R_r > 0) \)). Then, observe that for \( n \geq 0 \), we have by construction

\[
\xi_n \leq \sum_{k=0}^{\chi_n-1} (\sigma_k^{(1)} - \sigma_k^{(0)}),
\]

and we get the expected convergence (12) using the very same decomposition as in (10). This yields the proof of Proposition 4.11 with the same arguments as in Section 4.1.
4.3.2 Stopped peeling process and asymptotics

Let us now focus on the situation when \((B_n)_{n\geq 0}\) reaches \(\mathbb{Z}_-\). Two cases are likely to happen.

**Case 1:** \(|B_T| < [\lambda b]|. In that case, a finite part of the white segment on the boundary is swallowed, see Figure 17.

It is easy to see that for any possible configuration, \(|B_T| < [\lambda b]| implies that the percolation cluster from the finite black segment is confined in a finite region of the space, away from the infinite one. Thus, the crossing event does not occur: \(P(C_{\lambda_a,\lambda_b}^b | |B_T| < [\lambda b]) = 0.\)

**Case 2:** \(|B_T| > [\lambda b]|. In this situation, the whole white segment on the boundary is swallowed, see Figure 18.

![Figure 17: A possible configuration when \(|B_T| < [\lambda b]|.](image)

![Figure 18: The situation when \(|B_T| > [\lambda b]|.](image)

Here, we can treat the cases when four vertices of the last revealed face are lying on the boundary at the right of the origin vertex (the equivalent of case \(A_4^b\) in Section 4.1) exactly as before, and we get that the crossing event \(C_{\lambda_a,\lambda_b}^b\) is implied by the crossing event between the open segments on the boundary in a Boltzmann map whose boundary has the colouring of Figure 19. We denote by \(C_{\lambda}^2\) this crossing event. Recall that this map has law \(\mu_{\nu(\lambda)}\), where \(\nu(\lambda)\) is the number of vertices on its boundary.

Moreover, we can prove the same way as for bond percolation that the lengths \(d^\lambda_l, d^\lambda_r\) of the finite black segments satisfy

\[
d^\lambda := \min(d^\lambda_l, d^\lambda_r) \xrightarrow{\lambda \rightarrow +\infty} +\infty.
\] (14)

The main problem is now to prove that the event \(C_{\lambda}^2\) occurs almost surely asymptotically. The proof is here very different from the bond percolation case, because we have no information on the dual percolation model.
Proposition 4.12. We have
\[ \Pr(C^2_\lambda) \xrightarrow{\lambda \to +\infty} 1. \]

Proof. First, let \( R \geq 0 \) and notice that either \( d^\lambda \leq R \), or \( d^\lambda_r > R \) and in this situation, there are two open segments of length at least \( R \) on the boundary of the map (on each side of the origin vertex, see Figure 19). Let \( C^{\lambda}_R \) be the event that there exists a black crossing between those two finite segments of length \( R \) in the ball of radius \( R \), denoted by \( B^\lambda_R \). Then, for every \( R \geq 0 \), we get that
\[
\Pr((C^2_\lambda)^c) \leq \Pr(d^\lambda \leq R) + \Pr((C^{\lambda}_R)^c).
\]
Now, the event \( C^{\lambda}_R \) is obviously measurable with respect to the ball of radius \( R \) of the map, and the convergence in law \( \mu^{\square}_{v(\lambda)} \xrightarrow{\lambda \to +\infty} \nu^{\square}_\infty \) for the local topology implies that for every fixed \( R \geq 0 \),
\[
\Pr(C^{\lambda}_R) \xrightarrow{\lambda \to +\infty} \Pr(C^\infty_R),
\]
where \( C^\infty_R \) is the event that there exists an open crossing exists between the same segments of length \( R \) in the ball of radius \( R \), denoted by \( B^\infty_R \), in a map which has the law of the UIHPQ. As a consequence, using the convergence (14), we get for every \( R \geq 0 \) that
\[
\limsup_{\lambda \to +\infty} \Pr((C^2_\lambda)^c) \leq \Pr(C^\infty_R). \tag{15}
\]

Now, it is clear, since the events \( (C^\infty_R)_{R \geq 0} \) are increasing, that
\[
\Pr(C^\infty_R) \xrightarrow{R \to +\infty} \Pr(C^\infty),
\]
where the event \( C^\infty \) is the open crossing event between the two infinite black segments in a map which has the law of the UIHPQ and a boundary colored as in Figure 20.
It is easy to see, using the standard coupling argument, that the event $C^\infty$ has probability bounded from below by the probability of the very same event with an infinite free segment on the left, as in Figure 21.

The idea is now to use the very same peeling process as in Section 3, with reversed colours. It is defined as follows.

Reveal the colour of the rightmost free vertex on the boundary.

- If it is white, repeat the algorithm.
- If it is black, mark this vertex and execute the vertex-peeling process. Then, repeat the algorithm on the unique infinite connected component of the map deprived of the faces revealed by the vertex-peeling process.

The algorithm ends when the initial finite white segment has been completely swallowed.

We can check that the pattern of the boundary is preserved and that the steps of this process are i.i.d. exactly as before. For every $n \geq 0$, let $W_n$ be the length of the finite white segment at step $n$ of the exploration. Since $1 - p^{\natural}_{c,\text{site}} < p^{\natural}_{c,\text{site}}$, the “white” percolation is subcritical and we have that the stopping time $\tilde{T} := \inf\{n \geq 0 | W_n = 0\}$ is almost surely finite, using the arguments of Section 3. Then, two cases may occur at time $\tilde{T}$:

- There is an explicit black crossing, which occurs with positive probability, see Figure 22.

- There is no black crossing, see Figure 23.

Let us focus on the second case. Obviously, the total number $V$ of (free) vertices discovered by the peeling process up to time $\tilde{T}$ is almost surely finite (because $\tilde{T}$ is), so that the probability of the crossing event we are interested in is now bounded from below by the same probability in a planar map which has the law of the UIHPQ and a boundary coloured as in Figure 24.

Note that even though the $V$ vertices discovered by the peeling process are free, it is natural...
to suppose that they are closed vertices because they do not lie on the initial boundary of the map on which we want to exhibit a crossing event.

We now repeat the peeling process we previously introduced, with an initial white segment of size $V$. Since $V$ is almost surely finite, this process ends in finite time almost surely. The point is now to observe that the probability that a black crossing occurs at time $\tilde{T}$ is bounded from below by $p_{\text{site}}^V$, which is the probability that the free vertex between the two black vertices in the last revealed face (if any) is black. The successive executions of this process being independent thanks to the spatial Markov property, this implies that the black crossing is almost sure, i.e. that $P(C^\infty) = 1$, and ends the proof letting $R$ go to infinity in (15).

Using the same arguments as in Section 4.1, the statements we obtained yield that

$$P(C_{\text{site}}^\square(\lambda\alpha, \lambda\beta)) \xrightarrow{\lambda \to +\infty} \frac{1}{\pi} \arccos\left(\frac{b-a}{a+b}\right),$$

which is exactly Theorem 1.2 in the case of site percolation on the UIHPQ, and thus ends the proof of the main result.

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