A NOTE ON ENVELOPES OF HOLOMORPHY

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Abstract. Let \( p : X \to M \) be a Riemann domain over a connected \( n \)-dimensional complex submanifold \( M \) of \( \mathbb{C}^N \) and let \( \mathcal{F} \subset \mathcal{O}(X) \) be such that \( p \in \mathcal{F}^N \). Our aim is to discuss relations between the \( \mathcal{F} \)-envelope of holomorphy of \((X, p)\) in the sense of Riemann domains over \( M \) and the \( \mathcal{F} \)-envelope of holomorphy of \( X \) in the sense of complex manifolds.

1. Introduction

Let \( M \) be a connected \( n \)-dimensional complex submanifold of \( \mathbb{C}^N \), let \((X, p) \) (\( p : X \to M \)) be a Riemann domain over \( M \), and let \( \mathcal{F} \subset \mathcal{O}(X) \) be such that \( p \in \mathcal{F}^N \) (cf. §2). Our aim is to discuss the notion of the \( \mathcal{F} \)-envelope of holomorphy of \( X \). More precisely, we like to discuss relations between the \( \mathcal{F} \)-envelope of holomorphy of \((X, p)\) in the sense of Riemann domains over \( M \) and the \( \mathcal{F} \)-envelope of holomorphy of \( X \) in the sense of complex manifolds. We will see that, even in the case of domains in \( \mathbb{C}^n \), both approaches lead to some fundamental difficulties. Notice that the problem does not appear in the case where \( \mathcal{F} = \mathcal{O}(X) \).

Let \( \varphi : (X, \mathcal{F}) \to (\bar{X}, \bar{\mathcal{F}}) \) be the \( \mathcal{F} \)-envelope of holomorphy of \( X \) in the sense of complex manifolds (cf. [Vig 1982], see also §3) and let \( \bar{p} \in \bar{\mathcal{F}}^N \) be such that \( \bar{p} \circ \bar{\varphi} \equiv p \). Observe that \( \bar{p} : \bar{X} \to M \) (cf. the proof of Theorem 3.4). Put \( Z_p := \{ a \in \bar{X} : \bar{p} \text{ is not biholomorphic near } a \} \); the set \( Z_p \) is an analytic subset of \( \bar{X} \) with \( \dim Z_p \leq n - 1 \). The following result characterizes relations between the \( \mathcal{F} \)-envelopes of holomorphy of \( X \).

Theorem 1.1 (cf. Theorem 3.1). Under the above notation, \( \varphi : (X, p) \to (\bar{X} \setminus Z_p, \bar{p}) \) is the \( \mathcal{F} \)-envelope of holomorphy in the sense of Riemann domains over \( M \). Moreover, if \( \mathcal{F} = \mathcal{O}(X) \), then \( Z_p = \emptyset \). In particular, if \( \mathcal{F} = \mathcal{O}(X) \), then \((\bar{X}, \bar{p})\) must be a Stein Riemann domain over \( M \).

The case where \( \mathcal{F} = \mathcal{O}(X) \) has been discussed in [Ker 1959]. The proof will be given in §6.

Using the above result we get the following examples (details will be given in §§6, 7).

Example 1.2. Let \( \mathbb{D} \subset \mathbb{C} \) be the unit disc and let \( \mathbb{D}_* := \mathbb{D} \setminus \{0\} \). Take \( M := \mathbb{C} \), \( X := \mathbb{D}_* \), \( \mathcal{F} := \mathcal{H}^\infty(\mathbb{D}_*) \). Then \( \text{id} : (\mathbb{D}_*, \mathcal{H}^\infty(\mathbb{D}_*)) \to (\mathbb{D}, \mathcal{H}^\infty(\mathbb{D})) \) is the \( \mathcal{H}^\infty(\mathbb{D}_*) \)-envelope of holomorphy of \( \mathbb{D}_* \) in the sense of complex manifolds.

• If \( p := \text{id} \), then \( Z_p = \emptyset \).
• If \( p := \text{id}^2 \), then \( Z_p = \{0\} \).

Consequently, the \( \mathcal{H}^\infty(\mathbb{D}_*) \)-envelope of holomorphy of \( (\mathbb{D}_*, p) \) depends on a particular choice of the projection \( p \) (notice that \( \mathbb{D} \) and \( \mathbb{D}_* \) are not homeomorphic).

Example 1.3. Let \( M := \mathbb{C} \), \( X := \mathbb{C} \setminus \{(-\infty, 1] \cup [1, +\infty)\} \) (note that \( X \) is simply connected), \( \mathcal{F} = \{f_1, f_2\} := \{\text{id}, X \ni \lambda \mapsto \sqrt{1 - \lambda^2}\} \) (the branch of the square root is arbitrarily fixed),

\[ \varphi := (f_1, f_2) : X \to \bar{X} := \{(z_1, z_2) \in \mathbb{C}^2 : z_1^2 + z_2^2 = 1\} \]

(\( \bar{X} \) is a connected one dimensional complex manifold), \( \bar{\mathcal{F}} = \{\bar{f}_1, \bar{f}_2\} := \{z_1|\bar{x}, z_2|\bar{x}\} \). Then

\[ \varphi : (X, \mathcal{F}) \to (\bar{X}, \bar{\mathcal{F}}) \]

is the \( \mathcal{F} \)-envelope of holomorphy in the sense of complex manifolds.

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Taking \( p := f_1 = \text{id} \), we get \( \tilde{p} = \tilde{f}_1 \), which implies that \( Z_p = \{(0, -1), (0, +1)\} \). Consequently, 
\[
\varphi : (X, \text{id}) \longrightarrow (\tilde{X} \setminus \{(0, -1), (0, +1)\}, z_1)
\]
is the \( \mathcal{F} \)-envelope of holomorphy in the sense of Riemann domains.

The above examples might suggest that the \( \mathcal{F} \)-envelope of holomorphy in the sense of complex manifolds is perhaps better. Unfortunately, the following result shows that this is not so.

**Theorem 1.4.** For every \( n \geq 2 \) there exists a domain \( X \subset \mathbb{C}^n \) and a family \( \mathcal{F} \subset \mathcal{O}(X) \) such that the \( \mathcal{F} \)-envelope of holomorphy of \( X \) in the sense of complex manifolds is neither Stein nor a Riemann domain over \( \mathbb{C}^n \).

The proof will be given in §8

### 2. Riemann domains over complex manifolds

The aim of this section is to recall basic terminology related to Riemann domains over complex manifolds — cf. [Jar-Pfl 2000]. Let \( M \) be a connected \( n \)-dimensional complex manifold (e.g. \( M = \mathbb{C}^n \)). Denote by \( \mathcal{R}(M) \) the family of all Riemann regions over \( M \), i.e. the family of all pairs \((X, p)\), where \( X \) is a Hausdorff topological space and \( p : X \longrightarrow M \) is locally homeomorphic (each point \( a \in X \) has an open neighborhood \( U \) such that \( p(U) \) is open in \( M \) and \( p|_U : U \longrightarrow p(U) \) is homeomorphic). The projection \( p \) introduces on \( X \) a structure \( \text{Str}(X, p) \) of an \( n \)-dimensional complex manifold (such that \( p \) is locally biholomorphic). Let \( \mathcal{R}_c(M) \) be the subfamily of those \((X, p) \in \mathcal{R}(M)\) for which \( X \) is connected.

Let \((X, p), (Y, q) \in \mathcal{R}(M)\). We say that a continuous mapping \( \varphi : X \longrightarrow Y \) is a morphism if \( q \circ \varphi \equiv p \). We shortly write \( \varphi : (X, p) \longrightarrow (Y, q) \) is a morphism. Each morphism is locally biholomorphic. We say that \( \varphi : X \longrightarrow Y \) is an isomorphism if \( \varphi \) is bijective and \( \varphi^{-1} : (Y, q) \longrightarrow (X, p) \) is also a morphism. Notice that a morphism \( \varphi : (X, p) \longrightarrow (Y, q) \) is an isomorphism iff \( \varphi \) is bijective. Each locally biholomorphic mapping \( \varphi : X \longrightarrow Y \) induces a homomorphism \( \mathcal{O}(Y) \ni g \mapsto g^\varphi = g \circ \varphi \in \mathcal{O}(X) \). Observe that \( \varphi^\ast \) is injective iff each connected component of \( Y \) intersects \( \varphi(X) \). If \( \varphi^\ast \) is injective and \( g \circ \varphi = f \), then we write \( g = f^\varphi \).

Let \((X, p), (Y, q) \in \mathcal{R}_c(M)\) and \( \emptyset \neq \mathcal{F} \subset \mathcal{O}(X) \). We say that a morphism \( \varphi : (X, p) \longrightarrow (Y, q) \) is an \( \mathcal{F} \)-extension if \( \mathcal{F} \subset \varphi^\ast(\mathcal{O}(Y)) \). We put \( \mathcal{F}^\varphi := \{f^\varphi : f \in \mathcal{F}\} \). Note that \( \varphi^\ast|_{\mathcal{F}^\varphi} : \mathcal{F}^\varphi \longrightarrow \mathcal{F} \) is bijective. We say that an \( \mathcal{F} \)-extension \( \varphi : (X, p) \longrightarrow (X, \tilde{p}) \) is an \( \mathcal{F} \)-envelope of holomorphy if for every \( \mathcal{F} \)-extension \( \psi : (X, p) \longrightarrow (Y, q) \) there exists a morphism \( \sigma : (Y, q) \longrightarrow (X, \tilde{p}) \) such that \( \sigma \circ \psi \equiv \varphi \). Observe that in fact \( \sigma : (Y, q) \longrightarrow (X, \tilde{p}) \) is an \( \mathcal{F} \)-extension. Such an \( \mathcal{F} \)-envelope of holomorphy is uniquely determined up to an isomorphism. By the Thullen theorem the \( \mathcal{F} \)-envelope of holomorphy always exists — cf. [Jar-Pfl 2000], Theorem 1.8.4 for the case \( M = \mathbb{C}^n \) (the general case goes in the same way). Moreover, if \( M \) is Stein, then the \( \mathcal{F} \)-envelope of holomorphy is also Stein — cf. [Jar-Pfl 2000], Cartan–Thullen Theorem 1.10.4 (the case \( M = \mathbb{C}^n \)) and Theorem [X3] (the general case). We say that \((X, p) \) is an \( \mathcal{F} \)-domain of holomorphy if \( \text{id} : (X, p) \longrightarrow (X, p) \) is the \( \mathcal{F} \)-envelope of holomorphy.

Our main problem is to discuss the following situation. Suppose that \((X, p), (X, q) \in \mathcal{R}_c(M)\) are such that \( \text{Str}(X, p) = \text{Str}(X, q) \) (equivalently: \( q \) is holomorphic in the sense of \( \text{Str}(X, p) \) and \( p \) is holomorphic in \( \text{Str}(X, q) \)). Let \( \varphi_p : (X, p) \longrightarrow (\tilde{X}, \tilde{p}), \varphi_q : (X, q) \longrightarrow (\tilde{X}, \tilde{q}) \) be the \( \mathcal{F} \)-envelopes of holomorphy. We are interested in the situation when there exists a biholomorphic mapping \( \tau : \tilde{X}_p \longrightarrow \tilde{X}_q \) such that \( \tau \circ \varphi_p \equiv \varphi_q \). It is known that in the case where \( \mathcal{M} = \mathbb{C}^n, \mathcal{F} = \mathcal{O}(X) \) such a biholomorphic mapping \( \tau \) exists (cf. [Jar-Pfl 2000], Theorem 2.12.1). Consequently, if \( M = \mathbb{C}^n \) and \( \mathcal{F} = \mathcal{O}(X) \), then the \( \mathcal{O}(X) \)-envelope of holomorphy depends only on the complex structure \( \text{Str}(X, p) \).

The next problem appears when \( p : X \longrightarrow M \) and \( q : X \longrightarrow M' \) are Riemann domains over different connected \( n \)-dimensional Stein manifolds such that \( \text{Str}(X, p) = \text{Str}(X, q) \). Observe that \( \text{id} : (M, \text{id}) \longrightarrow (M, \text{id}) \) is the \( \mathcal{F} \)-envelope of holomorphy over \( M \) for any \( \mathcal{F} \). This may lead to some pathological situations, e.g. \( \text{id} : (\mathbb{D}_a, \text{id}) \longrightarrow (\mathbb{D}_a, \text{id}) \) is the \( H^\infty(\mathbb{D}_a) \)-envelope of holomorphy over \( M = \mathbb{D}_a \) but not over \( M = \mathbb{C} \). If \( M \) and \( M' \) are biholomorphic, then the situation is simple: if \( \Phi : M \longrightarrow M' \) is biholomorphic, then the mapping \( \mathcal{R}(M) \ni (X, p) \longrightarrow (X, \Phi \circ p) \in \mathcal{R}(M') \) is bijective and \( \text{Str}(X, p) = \text{Str}(X, \Phi \circ p) \); moreover, \( \varphi : (X, p) \longrightarrow (Y, q) \) is an \( \mathcal{F} \)-extension (resp. \( \mathcal{F} \)-envelope of holomorphy) over \( M \) iff \( \varphi : (X, \Phi \circ p) \longrightarrow (Y, \Phi \circ q) \) is an
$\mathcal{F}$-extension (resp. $\mathcal{F}$-envelope of holomorphy) over $M'$. In particular, by the Remmert embedding theorem, we may always assume that our Stein manifold $M$ is a connected $n$-dimensional complex submanifold of $\mathbb{C}^N$.

3. Domains over Stein manifolds vs. domains over $\mathbb{C}^N$

The aim of this section (inspired by [Ros 1963]) is to show that in fact many properties of the Riemann domains over Stein manifolds may be easily deduced from the corresponding properties of Riemann domains over $\mathbb{C}^N$. Let $M$ be a connected $n$-dimensional complex submanifold of $\mathbb{C}^N$ and let $\sigma : S \rightarrow M$ be a holomorphic retraction, where $S \subset \mathbb{C}^N$ is a domain (cf. [Gun-Ros 1965], Ch. VIII, C, Theorem 8).

**Lemma 3.1.** Let $(X, p) \in \mathcal{R}(M)$. Put

$$\tilde{X}_0 := \{(x, v) \in X \times \mathbb{C}^N : p(x) + v \in S, \sigma(p(x) + v) = p(x)\}, \quad p_0 : \tilde{X}_0 \rightarrow S, \quad p_0(x, v) := p(x) + v.$$ 

Then $(\tilde{X}_0, p_0) \in \mathcal{R}(\mathbb{C}^N)$.

Proof. Fix a point $(x_0, v_0) \in \tilde{X}_0$. Let $U \subset X$ be an open neighborhood of $x_0$ such that $p|_U : U \rightarrow p(U)$ is biholomorphic. Define $V := \tilde{X}_0 \cap (U \times \mathbb{C}^N)$, $V' := \sigma^{-1}(p(U))$. It suffices to show that $p_0|_V : V \rightarrow V'$ is homeomorphic. Clearly, $p_0(V) \subset V'$. Define $g : V' \rightarrow X \times \mathbb{C}^N$, $g(z) := ((p|_U)^{-1}(\sigma(z)), z - \sigma(z))$. Observe that $g(V') \subset V$. Moreover, $p_0 \circ g = \text{id}_V$ and $g \circ p_0 = \text{id}_V$. □

As a direct corollary we get the following proposition.

**Proposition 3.2.** Let $(X, p) \in \mathcal{R}_c(M)$ and let $(\tilde{X}_0, p_0)$ be as in Lemma 3.1. Let $X_0$ be the connected component of $\tilde{X}_0$ that contains $X \times \{0\}$. Then:

- $(X_0, p_0) \in \mathcal{R}_c(\mathbb{C}^N)$,
- $X \times \{0\} \simeq X_0$ is a submanifold of $X_0$,
- the mapping $X_0 \ni (x, v) \mapsto x \in X \simeq X \times \{0\}$ is a holomorphic retraction.

**Lemma 3.3.** Let $(X, p), (Y, q) \in \mathcal{R}_c(M)$, $\emptyset \neq \mathcal{F} \subset \mathcal{O}(X)$, and let $\varphi : (X, p) \rightarrow (Y, q)$ be an $\mathcal{F}$-extension (over $M$). Assume that $(X_0, p_0)$ and $(Y_0, q_0)$ are constructed according to Proposition 3.2. Put $X_0 \ni (x, v), (\varphi(x), v) \in Y \times \mathbb{C}^N$, $F_0 := \{f \circ \sigma_X : f \in \mathcal{F}\} \subset \mathcal{O}(X_0)$. Then $\varphi_0 : (X_0, p_0) \rightarrow (Y_0, q_0)$ is an $F_0$-extension (over $\mathbb{C}^N$).

Proof. First observe that $\varphi_0 : X_0 \rightarrow Y_0$ is well defined: $q(\varphi(x)) + v = p(x) + v \in S$ and $\sigma(q(\varphi(x)) + v) = \sigma(p(x) + v) = p(x) = q(\varphi(x))$. It is clear that $q_0 \circ \varphi_0 = p_0$ and $\varphi_0(X \times \{0\}) \subset Y \times \{0\}$. Thus $\varphi_0(X_0) \subset Y_0$. Moreover, for each $f \in F$ we have $(f \circ \sigma_Y) \circ \varphi_0 = f \circ \sigma_X$. □

Let $\psi : (X_0, p_0) \rightarrow (Z, r)$ be an $F_0$-extension (over $\mathbb{C}^N$). Put $Z^M := r^{-1}(M)$. Observe that $(Z^M, r) \in \mathcal{R}(M)$ and $\psi(X \times \{0\}) \subset Z^M$. Let $Z^{M,\psi}$ be the connected component of $Z^M$ that contains $\psi(X \times \{0\})$. Then $\psi : (X, p) \rightarrow (Z^{M,\psi}, r)$ is an $\mathcal{F}$-extension (over $M$); recall that $X \simeq X \times \{0\}$.

**Theorem 3.4.** Under the above notation if $\psi : (X_0, p_0) \rightarrow (Z, r)$ is the $F_0$-envelope of holomorphy (over $\mathbb{C}^N$), then $\psi : (X, p) \rightarrow (Z^{M,\psi}, r)$ is the $F$-envelope of holomorphy (over $M$). Moreover, $Z^{M,\psi}$ is Stein.

Consequently, for any Riemann domain $(X, p) \in \mathcal{R}_c(M)$ and for any family of functions $\emptyset \neq \mathcal{F} \subset \mathcal{O}(X)$, $\psi : (X, p) \rightarrow (Z^{M,\psi}, r)$ is the $\mathcal{F}$-envelope of holomorphy (over $M$), then $\tilde{X}$ is Stein.

Proof. We only need to show that the extension $\psi : (X, p) \rightarrow (Z^{M,\psi}, r)$ is maximal. Suppose that $\varphi : (X, p) \rightarrow (Y, q)$ is an $\mathcal{F}$-extension (over $M$). Then, by Lemma 3.3, $\varphi_0 : (X_0, p_0) \rightarrow (Y_0, q_0)$ is an $F_0$-extension (over $\mathbb{C}^N$). Thus, there exists a morphism $\tau : (Y_0, q_0) \rightarrow (Z, r)$ such that $\tau \circ \varphi_0 = \psi$. It remains to observe that $r(Y \times \{0\}) \subset Z^{M,\psi}$.

We know that $Z$ is a Stein manifold (cf. Jar-Pfl 2000, Carstan–Thullen Theorem 1.10.4). Let $M = \{z \in \mathbb{C}^N : g_j(z) = 0, \; j = 1, \ldots, k\}$, where $g_1, \ldots, g_k \in \mathcal{O}(\mathbb{C}^N)$. Then $Z^M = \{z \in Z : g_j \circ r(z) = 0, \; j = 1, \ldots, k\}$. Hence $Z^M$ is a submanifold of $Z$ and, therefore, $Z^M$ is Stein. Consequently, $Z^{M,\psi}$ is Stein. □
4. \(F\)-ENVELOPES IN THE SENSE OF COMPLEX MANIFOLDS

The aim of this section is to recall a more general notion of the \(F\)-envelope of holomorphy (cf. [Ker 1959], [Vig 1982]). Let \(S\) be the family of all pairs \((Y, G)\) such that:

- \(Y\) is a connected \(n\)-dimensional complex manifold,
- \(G \subset \mathcal{O}(Y)\),
- for every \(a \in Y\) there exist a \(g \in \mathcal{G}^n\) and an open neighborhood \(U\) of \(a\) such that \(g(U)\) is open and \(g|U : U \rightarrow g(U)\) is biholomorphic.

One may prove that if \((Y, G)\) in \(S\), then \(Y\) is countable at infinity (cf. [Gra 1953]). Observe that if \(M\) is a connected submanifold of \(\mathbb{C}^N\), \((Y, q) \in \mathcal{R}_c(M)\), and \(G \subset \mathcal{O}(Y)\) is such that \(q \in \mathcal{G}^N\), then \((Y, G)\) is in \(S\). We fix a pair \((X, F) \in S\). Let \((Y, G), (Z, H) \in S\). We say that a holomorphic mapping \(\varphi : Y \rightarrow Z\) is a \(C\)-morphism if \(\varphi^*|_H : H \rightarrow G\) is biholomorphic. We write "\(\varphi : (Y, G) \rightarrow (Z, H)\) is a \(C\)-morphism". Note that if \(\varphi : (X, p) \rightarrow (Y, q)\) is an \(F\)-extension in the sense of Riemann domains over \(M\) with \(p \in F^N\), then \(\varphi : (X, F) \rightarrow (Y, F^\sigma)\) is a \(C\)-morphism.

We say that a \(C\)-morphism \(\varphi : (X, F) \rightarrow (\tilde{X}, F)\) is the \(F\)-extension of holomorphy if for every \(C\)-morphism \(\psi : (X, F) \rightarrow (Y, G)\) there exists a holomorphic mapping \(\sigma : Y \rightarrow \tilde{X}\) such that \(\sigma \circ \psi \equiv \varphi\). Notice that in fact \(\sigma : (Y, G) \rightarrow (\tilde{X}, F)\) is a \(C\)-morphism. Such an \(F\)-extension of holomorphy is uniquely determined up to a \(C\)-isomorphism. It is clear that the \(F\)-extension has no defects of the \(F\)-extension in the sense of Riemann domains, i.e. it depends only on \(F\).

Theorem 4.1 (cf. [Vig 1982]). For arbitrary \((X, F) \in S\) the \(F\)-extension of holomorphy exists.

5. MAIN RESULT

Let \((X, p) \in \mathcal{R}_c(M), \) where \(M\) is a connected submanifold of \(\mathbb{C}^N\). Let \(F \subset \mathcal{O}(X)\) be such that \(p \in F^N\). Assume that \(\varphi : (X, F) \rightarrow (\tilde{X}, F^\sigma)\) is the \(F\)-extension of holomorphy.

Let \(\tilde{p} \in \tilde{F}^N\) be such that \(\tilde{p} \circ \varphi \equiv p\); note that \(\tilde{p} : \tilde{X} \rightarrow M\). Put \(Z_p := \{a \in \tilde{X} : \tilde{p} \neq \text{biholomorphic near } a\}\). Notice that \(Z_p\) is an analytic subset of \(\tilde{X}\) with \(\dim Z_p \leq n - 1\).

Theorem 5.1. Under the above notation we have:

(a) \(\varphi : (X, p) \rightarrow (\tilde{X} \setminus Z_p, \tilde{p})\) is the \(F\)-extension of holomorphy (over \(M\)).

(b) \(Z_p = \emptyset\) iff the \(F\)- and \(F\)-extension of holomorphy coincide.

(c) If \(F = \mathcal{O}(X)\), then the \(\mathcal{O}(X)\)- and \(\mathcal{O}(X)\)-extension of holomorphy coincide.

Proof. (a) Let \(\tilde{\varphi} : (X, p) \rightarrow (\tilde{X}, \tilde{p})\) be the \(F\)-extension of holomorphy in the sense of Riemann domains over \(M\). Then \(\tilde{\varphi} : (X, F) \rightarrow (\tilde{X}, F^\sigma)\) is a \(C\)-morphism. Consequently, there exists a \(C\)-morphism \(\sigma : (\tilde{X}, F^\sigma) \rightarrow (\tilde{X}, F)\) such that \(\sigma \circ \tilde{\varphi} \equiv \varphi\).

On the other hand, \((\tilde{X} \setminus Z_p, \tilde{p})\) is a Riemann domain over \(M\). Since \(\tilde{p} \circ \varphi \equiv p\), we get \(\varphi(X) \subset \tilde{X} \setminus Z_p\). Consequently, \(\varphi : (X, p) \rightarrow (\tilde{X} \setminus Z_p, \tilde{p})\) is an \(F\)-extension. Thus, there exists a morphism \(\tau : (\tilde{X} \setminus Z_p, \tilde{p}) \rightarrow (\tilde{X}, \tilde{p})\) such that \(\tau \circ \varphi \equiv \tilde{\varphi}\). Then \((\sigma \circ \tau) \circ \varphi \equiv \sigma \circ \tilde{\varphi} \equiv \varphi\), which by the identity principle gives \(\sigma \circ \tau = \text{id}\). Moreover, \(\tilde{p} \circ \sigma \circ \varphi \equiv \tilde{p} \circ \varphi \equiv p \equiv \tilde{p} \circ \tilde{\varphi}\). Consequently, \(\tilde{p} \circ \sigma \equiv \tilde{p}\), which implies that \(\sigma(\tilde{X}) \subset \tilde{X} \setminus Z_p\). Hence \(\tau \circ \sigma = \text{id}\) and, therefore, \(\tau\) is an isomorphism.

(b) If \(\varphi : (X, F) \rightarrow (\tilde{X} \setminus Z_p, \tilde{F}^\sigma)\) is the \(F\)-extension of holomorphy, then there exists a holomorphic mapping \(\sigma : \tilde{X} \rightarrow \tilde{X} \setminus Z_p\) such that \(\sigma \circ \varphi \equiv \varphi\). Then \(\sigma = \text{id}\), which gives \(Z_p = \emptyset\).

(c) Put \(\tilde{X} := \tilde{X} \setminus Z_p\). We have \(\mathcal{O}(\tilde{X})|_{\tilde{X}} = \mathcal{O}(\tilde{X})\). In particular, the spaces \(\mathcal{O}(\tilde{X})\) and \(\mathcal{O}(\tilde{X})\) endowed with the Fréchet topologies of locally uniform convergence are isomorphic.

We know that \((\tilde{X}, \tilde{p})\) is Stein (cf. Theorem 4.4). In particular, \(\tilde{X}\) is holomorphically convex, i.e. for every compact \(K \subset \tilde{X}\) its holomorphically convex hull \(\tilde{K}\) is compact.

Suppose that \(Z_p \neq \emptyset\) and let \(a \in Z_p\). Let \(U\) be a relatively compact open neighborhood of \(a\). Then there exists a compact set \(K \subset \tilde{X}\) such that \(U \subset \tilde{K}\) (cf. [Jar-Pfl 2000], Remark 1.4.5(l)). Thus \(U \setminus Z_p \subset \tilde{K} \subset \tilde{X}\). Consequently, \(a \in U \subset U \setminus Z_p \subset \tilde{X}\) — a contradiction. \(\square\)
Theorem 5.2. Let \((X, p), (X, q) \in \mathfrak{R}_c(M)\) be such that \(\text{Str}(X, p) = \text{Str}(X, q)\) and let \(F \subset \mathcal{O}(X)\) be such that \(p, q \in F^N\). Let \(\varphi_p : (X, p) \rightarrow (\tilde{X}, \tilde{p})\) and \(\varphi_q : (X, p) \rightarrow (\tilde{X}, \tilde{q})\) be \(F\)-envelopes of holomorphy (over \(M\)). Let \(\varphi : (X, F) \rightarrow (\tilde{X}, \tilde{F})\) be the \(F\)-envelope of holomorphy and let \(Z_p\) and \(Z_q\) be as in Theorem 5.1. Then the following conditions are equivalent:

(i) there exists a biholomorphic mapping \(\tau : \tilde{X}_p \rightarrow \tilde{X}_q\) such that \(\tau \circ \varphi_p \equiv \varphi_q\);

(ii) \(Z_p = Z_q\).

Proof. By Theorem 5.1 we may assume that \(\varphi_p = \varphi_q = \varphi, (\tilde{X}, \tilde{p}) = (X \setminus Z_p, \tilde{p}),\) and \((\tilde{X}, \tilde{q}) = (X \setminus Z_q, \tilde{q})\). If \(Z_p = Z_q\), then we take \(\tau := \text{id}\). Conversely, if \(\tau : \tilde{X} \setminus Z_p \rightarrow \tilde{X} \setminus Z_q\) is such that \(\tau \circ \varphi \equiv \varphi\), then \(\tau = \text{id}\) and hence \(Z_p = Z_q\).

\(\square\)

6. Proof of Example 1.3

In view of Theorem 5.1(a) the only problem is to prove that \(\text{id} : (\mathbb{D}, \mathcal{H}_c(\mathbb{D})) \rightarrow (\mathbb{D}, \mathcal{H}_c(\mathbb{D}))\) is the \(\mathcal{H}_c(\mathbb{D})\)-envelope of holomorphy. It is clear that it is a \(C\)-morphism. Let \(\varphi : (\mathbb{D}, \mathcal{H}_c(\mathbb{D})) \rightarrow (\tilde{X}, \tilde{F})\) be the \(\mathcal{H}_c(\mathbb{D})\)-envelope of holomorphy. Then there exists a \(C\)-morphism \(\sigma : (\mathbb{D}, \mathcal{H}_c(\mathbb{D})) \rightarrow (\tilde{X}, \tilde{F})\) such that \(\varphi = \sigma\) on \(\mathbb{D}\). Let \(\tilde{p} \in \tilde{F}\) be such that \(\tilde{p} \circ \sigma \equiv \text{id}\). Observe that \(\tilde{p}(\tilde{X}) \subset \mathbb{D}\). In fact, since \(\tilde{p} \neq \text{const},\) we only need to show that \(\tilde{p}(\tilde{X}) \subset \mathbb{D}\). Suppose that \(z_0 \in \tilde{p}(\tilde{X}) \setminus \mathbb{D}\). Put \(f(z) := 1/(z - z_0)\). Then \(f \in \mathcal{H}_c(\mathbb{D})\).

Let \(f \in \tilde{F}\) be such that \(\tilde{f} \circ \sigma \equiv f\). Thus, by the identity principle, \(\tilde{f} \circ (\tilde{p} - z_0) \equiv 1\) — a contradiction.

Finally, \(\sigma : \mathbb{D} \rightarrow \tilde{X}\) is biholomorphic and \(\sigma^{-1} = \tilde{p}\).

7. Proof of Example 1.3

(1) It is clear that \(\varphi : (X, F) \rightarrow (\tilde{X}, \tilde{F})\) is a \(C\)-morphism. Suppose that \(\varphi^0 : (X, F) \rightarrow (X^0, F^0)\) is the \(F\)-envelope of holomorphy and let \(\sigma : (X, F) \rightarrow (X^0, F^0)\) be a \(C\)-morphism such that \(\sigma \circ \varphi \equiv \varphi^0\). Let \(f^0 \in F^0\) be such that \(f^0 \circ \varphi^0 \equiv f_j, j = 1, 2\). Then \(\|(f^0_1)^2 + (f^0_2)^2\| \circ \varphi^0 \equiv 1\), which shows that \(f^0 : (f^0_1, f^0_2) : X^0 \rightarrow \tilde{X}\). Moreover, \(\sigma \circ f^0 \circ \varphi^0 = \sigma \circ \varphi = \varphi^0\). Consequently, \(\sigma \circ f^0 \equiv \text{id}\), which implies that \(\sigma\) is a \(C\)-isomorphism.

8. Proof of Theorem 1.4

Let \(Y := \{(z_1, \ldots, z_{n+1}) \in \mathbb{C}^{n+1} : z_1^2 + \cdots + z_{n+1}^2 = 0\}, Y_0 := Y \setminus \{0\}\). Observe that \(Y_0\) is a connected \(n\)-dimensional complex manifold. Let \(X_0 := Y_0 \setminus M_{n+1}\), where \(M_{n+1} := \{(z_1, \ldots, z_{n+1}) \in Y_0 : z_{n+1} = 0\}, (p_0, p_0) \in \mathfrak{R}_c(\mathbb{C}^n)\). Put \(F_0 := \mathcal{H}_c(X_0) \cup \{z_j|X_0 : j = 1, \ldots, n+1\}\).

(a) First we will prove that \(\text{id} : (X_0, F_0) \rightarrow (Y_0, \mathcal{G})\) is the \((F_0)_c\)-envelope of holomorphy, where \(\mathcal{G} := \mathcal{H}_c(Y_0) \cup \{z_j|X_0 : j = 1, \ldots, n+1\}\).

By the Riemann removable singularities theorem we see that \(\text{id} : (X_0, F_0) \rightarrow (Y_0, \mathcal{G})\) is the \((F_0)_c\)-extension. Let \(\varphi_0 : (X_0, F_0) \rightarrow (\tilde{X}_0, \tilde{F}_0)\) be the \((F_0)_c\)-envelope of holomorphy. Observe that \(\tilde{F}_0 = \mathcal{H}_c(\tilde{X}_0) \cup \{F_1, \ldots, F_{n+1}\}\), where \(F_j \in \mathcal{O}(\tilde{X}_0)\) is such that \(F_j \circ \varphi_0 \equiv z_j|X_0, j = 1, \ldots, n+1\). Let \(\sigma : (Y_0, \mathcal{G}) \rightarrow (\tilde{X}, \tilde{F})\) be a \(C\)-morphism with \(\sigma = \varphi_0\) on \(X_0\). Clearly, \(F_j \circ \sigma \equiv z_j|X_0, j = 1, \ldots, n+1\). Let \(N := \{x \in \tilde{X}_0 : F_j(x) = 0, j = 1, \ldots, n+1\}\); \(N\) is an analytic subset of \(\tilde{X}_0\) with \(\text{dim} N \leq n\). Put \(F := (F_1, \ldots, F_{n+1}) : \tilde{X}_0 \rightarrow \mathbb{C}^{n+1}\). Observe that \(F : \tilde{X}_0 \rightarrow Y\). It is clear that \(\sigma : Y_0 \rightarrow \tilde{X}_0 \setminus N\) is

(1) To see that \(\tilde{X}\) is connected it suffices to observe that \(\mathbb{C} \setminus \{0\} \ni \lambda \rightarrow (\frac{1}{2}(\lambda + 1/\lambda), \frac{1}{2}(\lambda - 1/\lambda))\) in \(\tilde{X}\) is a global parametrization.

(2) To see that \(Y_0\) is connected we may argue as follows. Since \(Y_0\) is a \(C_*\)-cone \((C_* := \mathbb{C} \setminus \{0\})\), it suffices to show that the any two points from \(Q := Y_0 \cap \partial B_{n+1}\) may be joined in \(Y_0\) with a continuous curve, where \(B_k := \{z \in \mathbb{C}^k : \|z\| < 1\}\) is the unit Euclidean ball. We have \(Q = \{x + iy \in \mathbb{R}^{n+1} + i\mathbb{R}^{n+1} : \|x\| = \|y\| = 1/\sqrt{2}, \langle x, y \rangle = 0\}\). Any orthogonal operator \(A \in \mathcal{O}(n+1, \mathbb{R})\) acts on \(Q\) according to the formula \(x + iy \rightarrow Ax + iAy\). Since the special orthogonal group \(\text{SO}(n+1, \mathbb{R})\) is connected, each point \(a \in Q\) may be joined in \(Q\) with a point \(b\) of the form \((b', 0) + i(b', b')\) with \(\|b'\| = \|b\| = 1/\sqrt{2}\). Taking a suitable rotation \(\zeta \in \mathbb{R}^\mathbb{C}\) we may join \(b\) in \(Y_0\) with a point \(c = (c', c_n) := \zeta b\) such that \(\|c'\| = c_n = 1/\sqrt{2}\). It remains to use the fact that \(\partial B_n(1/\sqrt{2})\) is connected.
biholomorphic \((\sigma^{-1} = F)\). Thus, it remains to show that \(N = \emptyset\). Suppose that \(a \in N\). By the definition of the class \(S\) there exist an open neighborhood \(U \subset \mathbb{X}_0\) and a mapping \(f_0 = (\tilde{f}_0^1, \ldots, \tilde{f}_0^n) \in \tilde{F}^n\) such that 

\[ \tilde{f}_0^1|_U : U \rightarrow f_0^1(U) \text{ is biholomorphic. There are essentially the following three possibilities:} \]

- \(\tilde{f}_0^j \in \mathcal{H}^\infty(\mathbb{X}_0), j = 1, \ldots, n.\)

Since \(Y_0\) is a \(C_2\)-cone, we easily conclude that for every \(f \in \mathcal{H}^\infty(Y_0)\) we get \(f(\lambda b) = f(b), \lambda \in \mathbb{C}_+, b \in Y_0\). Consequently, if \(f \in \mathcal{H}^\infty(X_0)\), then \(f(\sigma(\lambda b)) = f(\sigma(b)), \lambda \in \mathbb{C}_+, b \in Y_0\). In particular, \(f^0(\sigma(\lambda b)) = f^0(\sigma(b)), \lambda \in \mathbb{C}_+, b \in Y_0\). Taking \(b \in \sigma^{-1}(U \setminus N)\) we get a contradiction with injectivity of \(f^0|_U\).

- \(\tilde{f}_0^1 = F_1, \ldots, \tilde{f}_0^n = F_n, \tilde{f}_{s+1}^j, \ldots, \tilde{f}_n^0 \in \mathcal{H}^\infty(\mathbb{X}_0)\) for some \(1 \leq s \leq n - 1\).

Using the above argument we see that if \(b \in Y_0\) is such that \(b_1 = \cdots = b_s = 0\), then \(\tilde{f}_0^0(\sigma(\lambda b)) = \tilde{f}_0^0(\sigma(b)), \lambda \in \mathbb{C}_+.\) Thus, it is enough to show that \((U \setminus N) \cap N_1 \cap \cdots \cap N_s \neq \emptyset\), where \(N_j := \{x \in \mathbb{X}_0 : F_j(x) = 0\}, j = 1, \ldots, n+1.\) Suppose the contrary. Then \(N_j \cap \cdots \cap N_s \cap U \subset N_{s+1} \cap \cdots \cap N_{n+1}\). Let \(W := \sigma^{-1}(U \setminus N) \neq \emptyset\). We have \(\{z \in W : z_1 = \cdots = z_s = 0\} \subset \{z \in W : z_{s+1} = \cdots = z_{n+1} = 0\}\). Since \(n+1 - s \geq 2\), we get a contradiction.

- \(\tilde{f}_0^j = F_j, j = 1, \ldots, n.\)

Using local complex coordinates \((\zeta_1, \ldots, \zeta_n)\) in a neighborhood of \(a\) we have \(\frac{\partial F}{\partial \zeta_k}(x), k = 1, \ldots, n \neq 0\). Since

\[ F_1^2 + \cdots + F_1^{n+1} = 0, \quad \text{we have} \quad F_1 \frac{\partial F}{\partial \zeta_k} + \cdots + F_n \frac{\partial F}{\partial \zeta_k} = -F_{n+1} \frac{\partial F}{\partial \zeta_k}, \quad k = 1, \ldots, n. \]

Thus, by Cramer’s formulas, \(F_j \equiv \Phi_j F_{n+1} \in \mathcal{H}^\infty(\mathbb{X}_0)\) in a neighborhood of \(a\), where \(\Phi_j\) is holomorphic, \(j = 1, \ldots, n.\) Consequently, \(\frac{\partial F_j}{\partial \zeta_k}(a) = \Phi_j(a) \frac{\partial F_{n+1}}{\partial \zeta_k}(a), j, k = 1, \ldots, n\), which gives a contradiction.

(b) Now we will prove that for every circular compact \(K \subset Y_0\) we have \(D_\ast : K \subset \tilde{K}_{\mathcal{O}(Y_0)}\), which directly implies that \(Y_0\) is not holomorphically convex. It suffices to prove that for any \(f \in \mathcal{O}(Y_0)\) and \(b \in Y_0\) the function \(\mathbb{C}_+ \ni \lambda \mapsto f(\lambda b)\) is bounded near \(\lambda = 0\) (and consequently extends holomorphically to \(\mathbb{C}\)). Indeed, then for every circular compact set \(K \subset Y_0\), \(f \in \mathcal{O}(Y_0), b \in K \land \lambda \in \mathbb{D}_\ast\), we have

\[ |f(\lambda b)| = |f_b(\lambda)| \leq \max_{\partial \bar{D}} |f_b| = \max_{\lambda \in \partial \bar{D}} |f(\lambda b)| \leq \max_K |f|. \]

Fix an \(f\). Let \(M := \{(w_1, \ldots, w_n) \in \mathbb{C}^n : w_1^2 + \cdots + w_n^2 = 0\}; M\) is a one-codimensional analytic subset of \(\mathbb{C}^n\). Define

\[ \mathbb{C} \times (\mathbb{C}^n \setminus M) \ni (\xi, w) \mapsto \left(\xi - f(w, \sqrt{-(w_1^2 + \cdots + w_n^2)})\right)\left(\xi - f(w, -\sqrt{-(w_1^2 + \cdots + w_n^2)})\right) =: \xi^2 + B(w)\xi + C(w), \]

where \(B, C \in \mathcal{O}(\mathbb{C}^n \setminus M)\). Observe that \(B, C\) are locally bounded in \(\mathbb{C}^n \setminus \{0\}\) — if \(w^0 \in M \setminus \{0\}\) and \(\mathbb{C}^n \setminus M \ni w^0 \rightarrow w^0\), then \(B(w^0) \rightarrow -2f(w^0, 0)\) and \(C(w^0) \rightarrow f^2(w^0, 0)\). Thus we may first assume that \(B, C \in \mathcal{O}(\mathbb{C}^n \setminus \{0\})\) and next, by the Hartogs theorem, that \(B, C \in \mathcal{O}(\mathbb{C}^n)\). In particular, the function \(\Delta := B^2 - 4C\) is holomorphic on \(\mathbb{C}^n\). This implies that the roots \(\xi_{\pm}(w) = \frac{1}{2}(B(w) \pm \sqrt{\Delta(w)})\) are locally bounded with respect to \(w \in \mathbb{C}^n\). Finally, \(f(\lambda b) \in \{\xi_{-}(\lambda b_1, \ldots, \lambda b_n), \xi_{+}(\lambda b_1, \ldots, \lambda b_n)\}\) is bounded near \(\lambda = 0\).

(c) Using Theorem 5.1 (a), we conclude that \(\tilde{p}_0 = (z_1|_{Y_0}, \ldots, z_n|_{Y_0})\). Hence \(Z_{p_0} = M_{n+1}\) and, consequently, \((X_0, p_0)\) is an \(F_\ast\)-domain of holomorphy. In particular, \(X_0\) is Stein.

(d) Observe that \((X_0, p_0)\) is a two-fold cover over \(\mathbb{C}^n\), i.e. every point \(a \in p_0(X_0)\) has an open neighborhood \(U\) such that \(p_0^{-1}(U) = U_1 \cup U_2, U_1 \cap U_2 = \emptyset\) and \(p_0|_{U_j} : U_j \rightarrow U\) is biholomorphic, \(j = 1, 2.\) Hence, there exists a domain \(X \subset \mathbb{C}^n\) such that \(\varphi : (X, \text{id}_X) \rightarrow (X_0, p_0)\) is the \(\mathcal{O}(X)\)-envelope of holomorphy in the sense of Riemann domains (cf. [For-Zam 1983], see also [Jar-Pit 2000], Theorem 4.5.18). Put \(F := \varphi^*(F_0)\). It remains to prove that \(\varphi : (X, F) \rightarrow (Y_0, \mathcal{G})\) is the \(F\)-envelope of holomorphy. It is obviously a \(C\)-morphism. Let \(\bar{\varphi} : (X, F) \rightarrow (\bar{X}, \bar{F})\) be the \(F\)-envelope of holomorphy. Then there exists a \(C\)-morphism \(\sigma : (Y_0, \mathcal{G}) \rightarrow (\bar{X}, \bar{F})\) such that \(\sigma \circ \varphi \equiv \bar{\varphi}\). Since \(\sigma|_{X_0} : (X_0, F_0) \rightarrow (\bar{X}, \bar{F})\) is a \(C\)-morphism, using (a) we conclude that there exists a \(C\)-morphism \(\tau : (\bar{X}, \bar{F}) \rightarrow (Y_0, \mathcal{G})\) such that \(\tau \circ \sigma|_{X_0} \equiv \text{id}_{X_0}\). Thus \(\tau \circ \sigma \equiv \text{id}\) and hence \(\tau\) is biholomorphic \((\tau^{-1} = \sigma)\).
We are going to prove that $\text{id} : (Y_0, \mathcal{O}(Y_0)) \rightarrow (Y_0, \mathcal{O}(Y_0))$ is the $\mathcal{O}(Y_0)_C$-envelope of holomorphy. Suppose that $\alpha : (Y_0, \mathcal{O}(Y_0)) \rightarrow (\tilde{Y}_0, \mathcal{O}(\tilde{Y}_0))$ is the $\mathcal{O}(Y_0)_C$-envelope of holomorphy. Then, by (a), $\alpha : (X_0, \mathcal{F}_0) \rightarrow (\tilde{Y}_0, \mathcal{F}_0')$ is a $C$-morphism. Consequently, there exists a $C$-morphism $\sigma : (\tilde{Y}_0, \mathcal{F}_0) \rightarrow (Y_0, \mathcal{G})$ such that $\sigma \circ \alpha \equiv \text{id}$.

Suppose that $Y_0$ is a Riemann domain over $\mathbb{C}^n$, i.e. there exists a locally biholomorphic mapping $q : Y_0 \rightarrow \mathbb{C}^n$. By Theorem 3.3 to get a contradiction it suffices to prove that $(Y_0, q)$ is a domain of holomorphy. Suppose that $\alpha : (Y_0, q) \rightarrow (Z, r)$ is an $\mathcal{O}(Y_0)$-extension. Then $\alpha : (Y_0, \mathcal{O}(Y_0)) \rightarrow (Z, \mathcal{O}(Z))$ is a $C$-morphism. Consequently, by (e), there exists a $C$-morphism $\sigma : (Z, \mathcal{O}(Z)) \rightarrow (Y_0, \mathcal{O}(Y_0))$ such that $\sigma \circ \alpha \equiv \text{id}$. In particular, $q \circ \sigma \circ \alpha \equiv q \equiv r \circ \alpha$. Thus $\sigma : (Z, r) \rightarrow (Y_0, q)$ is a morphism.

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