Zero-viscosity limit of the Navier-Stokes equations with the Navier friction boundary condition

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Abstract

In this paper, we consider the zero-viscosity limit of the Navier-Stokes equations in a half space with the Navier friction boundary condition

$$\left((\beta u^\varepsilon - \varepsilon \gamma \partial_y u^\varepsilon)\right)|_{y=0} = 0,$$

where $\beta$ is a constant and $\gamma \in (0,1]$. In the case of $\gamma = 1$, the convergence to the Euler equations and the Prandtl equation with the Robin boundary condition is justified for the analytic data. In the case of $\gamma \in (0,1)$, the convergence to the Euler equations and the linearized Prandtl equation is justified for the data in the Gevrey class $\frac{1}{\gamma}$.

1 Introduction

In this paper, we consider the Navier-Stokes equations in the half space $\mathbb{R}^2_+$:

$$\begin{cases}
\partial_t u^\varepsilon + u^\varepsilon \partial_x u^\varepsilon + v^\varepsilon \partial_y u^\varepsilon + \partial_x p^\varepsilon = \varepsilon^2 \Delta u^\varepsilon, \\
\partial_t v^\varepsilon + u^\varepsilon \partial_x v^\varepsilon + v^\varepsilon \partial_y v^\varepsilon + \partial_y p^\varepsilon = \varepsilon^2 \Delta v^\varepsilon, \\
\partial_x u^\varepsilon + \partial_y v^\varepsilon = 0,
\end{cases}
$$

with the Navier friction boundary condition

$$v^\varepsilon|_{y=0} = 0, \quad \eta u^\varepsilon + \partial_y u^\varepsilon|_{y=0} = 0,$$

which was first proposed by Navier and derived for gases by Maxwell. Here $(u^\varepsilon, v^\varepsilon)$ is the velocity field, $p^\varepsilon$ is a scalar pressure, $\varepsilon^2$ is the viscosity coefficient and $\eta$ is the slip length.

As $v^\varepsilon|_{y=0} = 0$, the boundary condition $\eta u^\varepsilon + \partial_y u^\varepsilon|_{y=0} = 0$ can be written as

$$\eta u^\varepsilon + (\partial_y u^\varepsilon + \partial_x v^\varepsilon)|_{y=0} = 0,$$
which means that the rate of strain on the boundary is proportional to the tangential slip velocity. For $\eta = 0$, the Navier friction boundary condition is just the Navier slip boundary condition; Letting $\eta \to +\infty$, we derive the non-slip boundary condition. As mentioned in [28], the slip length should depend on the viscosity coefficient. For simplicity, we consider the slip length $\eta$ of the form $\eta = -\beta \varepsilon^{-\gamma}$ for some $\gamma \geq 0$, where $\beta$ is a constant independent of $\varepsilon$. Thus, the Navier friction boundary condition (1.2) is reduced to
\[
v^\varepsilon|_{y=0} = 0, \quad \beta u^\varepsilon - \varepsilon^\gamma \partial_y v^\varepsilon|_{y=0} = 0.
\]

We are concerned with the behaviour of the solution as $\varepsilon \to 0$, i.e., the zero viscosity limit. Formally, as $\varepsilon \to 0$, the solution of (1.1) will be approximated by the Euler equations:
\[
\begin{aligned}
\frac{\partial u^\varepsilon}{\partial t} + u^\varepsilon \frac{\partial u^\varepsilon}{\partial x} + v^\varepsilon \frac{\partial u^\varepsilon}{\partial y} + \partial_x p^\varepsilon &= 0, \\
\frac{\partial v^\varepsilon}{\partial t} + u^\varepsilon \frac{\partial v^\varepsilon}{\partial x} + v^\varepsilon \frac{\partial v^\varepsilon}{\partial y} + \partial_y p^\varepsilon &= 0, \\
\partial_x u^\varepsilon + \partial_y v^\varepsilon &= 0, \\
v^\varepsilon(t,x,0) &= 0.
\end{aligned}
\]

In the absence of physical boundary, it has been proved that the Navier-Stokes equations indeed converge to the Euler equations in various functional settings [12, 4, 1, 21]. However, in the presence of physical boundaries, this is a challenging problem due to the possible formation of boundary layer. As the boundary layer is weak for the Navier slip boundary condition, the limit from the Navier-Stokes equations to the Euler equations has been justified in [10, 36, 11, 22, 35]. While, the boundary layer is strong for the non-slip boundary condition. Prandtl developed the boundary layer theory in [27], where the Prandtl equation was derived by the following asymptotic boundary layer expansion:
\[
\begin{aligned}
\frac{\partial u^\varepsilon}{\partial t} + u^\varepsilon \frac{\partial u^\varepsilon}{\partial x} + v^\varepsilon \frac{\partial u^\varepsilon}{\partial y} + \partial_x p^\varepsilon &= 0, \\
\frac{\partial v^\varepsilon}{\partial t} + u^\varepsilon \frac{\partial v^\varepsilon}{\partial x} + v^\varepsilon \frac{\partial v^\varepsilon}{\partial y} + \partial_y p^\varepsilon &= 0, \\
\partial_x u^\varepsilon + \partial_y v^\varepsilon &= 0, \\
v^\varepsilon(t,x,0) &= 0.
\end{aligned}
\]

where $(u^p, v^p)$ satisfies a Prandtl type equation. Roughly speaking, it was expected that the Navier-Stokes equations when $\varepsilon$ is small can be approximated by the Euler equations away from the boundary, and by the Prandtl equation near the boundary.

To justify the Prandtl boundary layer expansion (1.5), one of the key steps is to establish the well-posedness of the Prandtl equation. Up to now, the well-posedness of Prandtl equation was only established in some special functional space. Under a monotonic assumption on the velocity of the outflow, Oleinik and Samokhin [26] established the local existence and uniqueness of classical solutions in 2-D. The global existence of weak solution was established for the favorable pressure by Xin and Zhang [37]. Recently, Alexandre et. al. [2] and Masmoudi and Wong [23] independently proved the local well-posedness in Sobolev space by a direct energy method. Sammartino and Caffiisch [29] obtained the local existence and uniqueness of analytic solution for full analytic data, see [17, 38] for tangential analytic data. On the other hand, Gerard-Vare and Dormy [7] proved the ill-posedness in Sobolev space for the linearized Prandtl equation around non-monotonic shear flows.

Although one has a good understanding for the Prandtl equation, there are few results on the rigorous verification of the Prandtl boundary layer expansion. In [30], Sammartino and Caffiisch achieved this in the analytic setting, and Wang, Wang and Zhang [34] present...
a new proof based on a direct energy method. Recently, Maekawa \cite{19} justified the Prandtl boundary layer expansion for the initial vorticity supported away from the boundary. Fei, Tao and Zhang \cite{6} generalized Maekawa’s result to $\mathbb{R}^2_+$ by using a direct energy method. Very recently, Gerard-Varet, Maekawa and Masmoudi \cite{8} proved its stability of a class of shear flows of Prandtl type in the Gevrey class. Let us also mention some conditional convergence results \cite{31, 32, 14, 15, 31, 32} initiated by Kato \cite{13} and some convergence results for special flows of Prandtl type in the Gevrey class. Let us also mention some conditional convergence results \cite{18, 24, 25}. We refer to the review paper \cite{20} for more results.

For the Navier friction boundary condition \cite{14, 13}, Wang, Wang and Xin \cite{33} formally derived the boundary layer expansion by using the multi-scale analysis. The asymptotic behaviour of the solution depends on the slip length. For $\gamma > 1$, the behaviour is the same as the case of the non-slip boundary condition; For $\gamma = 1$, the boundary layer equation is the Prandtl equation with the Robin boundary condition; For $\gamma \in (0, 1)$, the boundary layer equation is the linearized Prandtl-type equation.

The goal of this paper is to justify the boundary layer expansion derived by Wang, Wang and Xin in the Gevrey class with the regularity exponent depending on $\gamma$.

2 Error equations, functional spaces and main result

In this section, we will derive the error equations, introduce the Gevrey functional spaces and state our main result.

2.1 The error equations

Let us first assume that the approximate solution $(u^a, v^a, p^a)$ satisfies

$$
\begin{cases}
\partial_t u^a + u^a \partial_x u^a + (v^a - \varepsilon^2 f(t, x)e^{-y})\partial_y u^a + \partial_x p^a - \varepsilon^2 \Delta u^a = -R_1, \\
\partial_t v^a + u^a \partial_x v^a + (v^a - \varepsilon^2 f(t, x)e^{-y})\partial_y v^a + \partial_y p^a - \varepsilon^2 \Delta v^a = -R_2, \\
\partial_x u^a + \partial_y v^a = 0, \\
v^a(t, x, y)|_{y=0} = \varepsilon^2 f(t, x), \\
\partial_y u^a(t, x, y)|_{y=0} = \beta \varepsilon^{-\gamma} u^a(t, x, 0) - \varepsilon g_0(t, x), \\
(u^a, v^a)(t, x, y)|_{t=0} = (u_0(x, y), v_0(x, y)),
\end{cases}
$$

(2.1)

where $g_0(t, x)$ satisfies $g_0(0, x) = 0$ and $(R_1, R_2)$ are remainders which are small in some functional space.

We introduce the error between the solution and the approximate solution

$$u = u^\varepsilon - u^a, \quad v = v^\varepsilon - v^a, \quad p = p^\varepsilon - p^a.$$

Thanks to \cite{11, 13} and \cite{21}, we deduce that $(u, v, p)$ satisfies

$$
\begin{cases}
\partial_t u + u\partial_x u + u^\varepsilon \partial_x u + (v + \varepsilon^2 f(t, x)e^{-y})\partial_y u^\varepsilon + (v^a - \varepsilon^2 f(t, x)e^{-y})\partial_y u \\
\quad + \partial_x p - \varepsilon^2 \Delta u = R_1, \\
\partial_t v + u\partial_x v + u^\varepsilon \partial_x v + (v + \varepsilon^2 f(t, x)e^{-y})\partial_y v^\varepsilon + (v^a - \varepsilon^2 f(t, x)e^{-y})\partial_y v \\
\quad + \partial_y p - \varepsilon^2 \Delta v = R_2, \\
\partial_x u + \partial_y v = 0,
\end{cases}
$$

(2.2)
It follows from (2.2), (2.3) and (2.5) that

\[ v|_{y=0} = -\varepsilon^2 f(t, x), \quad \partial_y u|_{y=0} = \beta\varepsilon^{-\gamma}u(t, x, 0) + \varepsilon g_0(t, x), \] (2.3)

and zero initial data.

For simplicity, we write

\[ U^a = (u^a, v^a), \quad U = (u, v), \quad \tilde{U}^a = (u^a, v^a - \varepsilon^2 f(t, x)e^{-y}) = (u^a, \tilde{v}^a), \]
\[ \tilde{U} = (u, v + \varepsilon^2 f(t, x)e^{-y}) = (u, \tilde{v}), \quad \tilde{R} = (R_1, R_2). \]

Then we have

\[
\begin{cases}
\partial_t U + \tilde{U} \cdot \nabla (U + U^a) + \tilde{U}^a \cdot \nabla U + \nabla p - \varepsilon^2 \Delta U = \tilde{R}, \\
\operatorname{div} U = 0, \\
\partial_y u|_{y=0} = \beta\varepsilon^{-\gamma}u(t, x, 0) + \varepsilon g_0(t, x), \\
v|_{y=0} = -\varepsilon^2 f, \\
U|_{t=0} = 0.
\end{cases}
\] (2.4)

We introduce the vorticity \( \omega, w^a \) defined by

\[ \omega = \partial_y u - \partial_x v, \quad \omega^a = \partial_y u^a - \partial_x v^a. \]

It is easy to see from (2.4) that

\[
\begin{cases}
\partial_t \omega - \varepsilon^2 \Delta \omega + \tilde{U} \cdot \nabla (\omega + \omega^a) + \tilde{U}^a \cdot \nabla \omega \\
\quad = \partial_y R_1 - \partial_x R_2 + \varepsilon^2 f e^{-y} \partial_y u^a + \varepsilon^2 \partial_x f e^{-y} \partial_y v^a, \\
w|_{y=0} = \beta\varepsilon^{-\gamma}u(t, x, 0) + \varepsilon g_0(t, x) + \varepsilon^2 \partial_x f, \\
w|_{t=0} = 0.
\end{cases}
\] (2.5)

Let \( g(t, x) = g_0(t, x) + \varepsilon \partial_x f(t, x) \) and

\[ \eta = \omega - \beta\varepsilon^{-\gamma}u - \varepsilon g(t, x)e^{-y}, \quad \eta^a = \omega^a - \beta\varepsilon^{-\gamma}u^a. \] (2.6)

It follows from (2.2), (2.4) and (2.5) that

\[
\begin{cases}
\partial_t \eta - \varepsilon^2 \Delta \eta + u^a \partial_x \eta + \tilde{v}^a \partial_x \eta + u \partial_x \eta^a + \tilde{v} \partial_x \eta^a + u \partial_x \eta \\
\quad + \tilde{v} \partial_y \eta - \beta \frac{\partial \partial_x p}{\varepsilon^2} = \partial_y R_1 - \partial_x R_2 + \varepsilon^2 f e^{-y} \partial_y u^a + \varepsilon^2 \partial_x f e^{-y} \partial_y v^a - \frac{\beta R_1}{\varepsilon^2} + h, \\
\eta(0, x, z) = 0, \\
\eta|_{y=0} = 0,
\end{cases}
\] (2.7)

where

\[ h = -\varepsilon e^{-y} \left[ \partial_t g + u^a \partial_x g - \tilde{v}^a g + u \partial_x g - \tilde{v} g - \varepsilon^2 g - \varepsilon^2 \partial_x g \right]. \]
2.2 The Gevrey functional spaces

As in [22, 23], we introduce the conormal operator \( Z = \psi(y)\partial_y \), where \( \psi(y) \) is a smooth function defined by

\[
\psi(y) = \begin{cases} 
\delta y, & \text{for } y \leq 1, \\
\frac{\delta y}{1 + y}, & \text{for } y \geq 2,
\end{cases}
\]

where \( \delta > 0 \) is to be decided later. We denote

\[
Z^k = (\psi(y))^k \partial_y^k, \quad \tilde{Z}^k = (\delta z)^k \partial_z^k.
\]

The conormal Sobolev space \( \overline{H}^s(\mathbb{R}^2_+) \) for \( s \in N \) is defined by

\[
\overline{H}^s(\mathbb{R}^2_+) = \left\{ u; \| \partial^\alpha u \|_{L^2_{t,x}(\mathbb{R}^2_+)} < \infty \right\}.
\]

We also denote \( \| f \|_{L^p_{t,x}(\mathbb{R}^2_+)} \) by \( \| f \|_p \) for \( 1 \leq p \leq \infty \).

Furthermore, for \( \alpha \in N^2 \) with \( \alpha = (\alpha_1, \alpha_2) \), let \( \partial^\alpha = \partial_x^{\alpha_1} Z^{\alpha_2} \) and

\[
\| \partial^k u \|_\infty = \left( \sum_{|\alpha| = k} \| \partial^\alpha u \|_\infty^2 \right)^{\frac{1}{2}}, \quad k \in N_+.
\]

For \( k \in N_+ \) and \( \gamma \in (0, 1] \), we define the conormal Gevrey norm as

\[
\| u \|_{X^k}^2 = \sum_{m=k}^{\infty} \frac{\rho(t)^{2(m-k)}}{((m-k)!)^\frac{2}{4}} \sum_{|\alpha|=m} \| \partial^\alpha u \|_2^2 + \| u \|_2^2,
\]

\[
\| u \|_{Y^k}^2 = \sum_{m=k+1}^{\infty} \frac{\rho(t)^{2(m-k)}(m-k)}{((m-k)!)^\frac{2}{4}} \sum_{|\alpha|=m} \| \partial^\alpha u \|_2^2 + \| u \|_2^2.
\]

Here \( \rho(t) = \rho - \lambda t \) and \( \rho(t) \in [1, 2] \). Especially, it is reduced to the analytic semi-norm as in [31] when \( \gamma = 1 \). For simplicity, we set

\[
|u|_{\tilde{X}^k}^2 = \frac{\rho(t)^{2(j-k)}}{((j-k)!)^2} \sum_{|\alpha|=j} \| \partial^\alpha u \|_2^2.
\]

Then we have

\[
\| u \|_{\tilde{X}^k}^2 = \sum_{m=k}^{\infty} |u|_{m,k}^2 + \| u \|_2^2, \quad \| u \|_{\tilde{Y}^k}^2 = \sum_{m=k+1}^{\infty} (m-k)|u|_{m,k}^2 + \| u \|_2^2.
\]

We also introduce the Gevrey norm

\[
\| u \|_{X_k}^2 = \sum_{m=k}^{\infty} \frac{\rho(t)^{2(m-k)}}{((m-k)!)^\frac{2}{4}} \sum_{|\alpha|=m} \| \partial^\alpha_{x,y} u \|_2^2 + \| u \|_2^2,
\]

\[
\| u \|_{Y_k}^2 = \sum_{m=k}^{\infty} \frac{\rho(t)^{2(m-k)}}{((m-k)!)^\frac{2}{4}} \| \partial^\alpha_{x,y} u \|_2^2 + \| u \|_2^2 \quad \text{for } u = u(x).
\]

Let us say \( \beta \leq \alpha \) in \( N^2 \) if \( \alpha = (\alpha_1, \alpha_2) \), \( \beta = (\beta_1, \beta_2) \) satisfy \( \beta_1 \leq \alpha_1, \beta_2 \leq \alpha_2 \). We denote \( C_\alpha^\beta = C^\beta_\alpha C_{\alpha_1}^\beta C_{\alpha_2}^{\beta_2} \). Let us state a useful lemma, which will be used frequently.
Lemma 2.1  
(a) Let \( \{x_\alpha\}_{\alpha \in \mathbb{N}^2} \) and \( \{y_\beta\}_{\beta \in \mathbb{N}^2} \) be real numbers. Then we have
\[
\sum_{|\alpha|=m} \sum_{|\beta|=j, \beta \leq \alpha} x_\beta y_{\alpha-\beta} = 
\left( \sum_{|\alpha|=j} x_\alpha \right) \left( \sum_{|\beta|=m-j} y_\beta \right).
\]
(b) For \( \alpha, \beta \in \mathbb{N}^2, |\alpha| = m \) and \( j \leq m \), there holds
\[
\sum_{|\beta|=j, \beta \leq \alpha} C^\beta_\alpha = C^j_m.
\]
Part (a) can be found in [16], and (b) can be obtained by computing the coefficient of \( x^j \) in the binomial expansion of \( (1 + x)^{\alpha_1}(1 + x)^{\alpha_2} \) and \( (1 + x)^m \), where \( \alpha = (\alpha_1, \alpha_2) \) and \( |\alpha| = m \). Here we omit the details.

2.3 Main result

Let us first introduce some assumptions (H) on approximate solutions and remainders.

(H1) Formulation and uniform bounds of approximate solutions: let \( z = \frac{y}{\varepsilon} \) and
\[
u^\alpha(t, x, y) = u^\varepsilon(t, x, y) + \varepsilon^{1-\gamma} u^p(t, x, z), \\
v^\alpha(t, x, y) = v^\varepsilon(t, x, y) + \varepsilon^{2-\gamma} v^p(t, x, z),
\]
and there exist \( a_0 > 0 \) and \( T_a > 0 \) such that for any \( t \in [0, T_a] \),
\[
\sum_{m=3}^{\infty} \rho(t)^{2(m-3)} \sum_{m-3 \leq |\alpha| \leq m+6} \left\| \partial_{x, y}^\alpha (u^\varepsilon, v^\varepsilon)(t, \cdot) \right\|_2^2 \leq C_0,
\]
\[
\sum_{m=3}^{\infty} \frac{\rho(t)^{2(m-3)}}{((m-3)!)^2} \sum_{m-3 \leq |\alpha| \leq m+6} \sum_{k=0}^{2} \left\| \varepsilon^{a_0 z^2} \partial^\alpha_x \partial^k_z (u^p, v^p)(t, \cdot) \right\|_2 \leq C_0, 
\tag{2.8}
\]
where \( \partial^\alpha_x = \partial x_1^{\alpha_1} \cdot z^{\alpha_2}, \partial^k_z = (\delta z)^k \partial z^k \).

(H2) Uniform bounds of the remainders: for \( T_a > 0 \) as in (H1), there holds
\[
\left\| (R_1, R_2)(t, \cdot) \right\|_{X^4}^2 \leq C_0 \varepsilon^4, \quad \left\| \nabla (R_1, R_2)(t, \cdot) \right\|_{X^2}^2 \leq C_0 \varepsilon^2, 
\tag{2.9}
\]
for any \( t \in [0, T_a] \).

(H3) Formulation and uniform bounds of \( (f, g_0) \): there exists \( \bar{f}(t, x) \) such that \( f(t, x) = \partial_x \bar{f} \) with
\[
\left\| \partial_t f(t, \cdot) \right\|_{X_3^4} + \left\| f(t, \cdot) \right\|_{X_3^4} + \left\| g_0(t, \cdot) \right\|_{X_3^4} + \left\| \partial_t g_0(t, \cdot) \right\|_{X_3^4} + \left\| \partial_t \bar{f}(t, \cdot) \right\|_{L_2^3} \leq C_0 
\tag{2.10}
\]
for any \( t \in [0, T_a] \).

Our main result is stated as follows.
Theorem 2.2 Assume (H1)-(H3). There exist $T > 0$ and $C_0 > 0$ independent of $\varepsilon$ such that for any sufficiently small $\varepsilon > 0$, the error equation (2.4) admits a unique solution $(u,v)(t,\cdot) \in X^3$, which satisfies
\[
\sup_{0 \leq t \leq T} \| (u,v)(t,\cdot) \|_{X^3} \leq C_0 \varepsilon^2, \quad \sup_{0 \leq t \leq T} \| w(t,\cdot) \|_{X^2} \leq C_0 \varepsilon.
\]
In particular, we have
\[
\sup_{0 \leq t \leq T} \| (u,v)(t,\cdot) \|_{L^2 \cap L^\infty(R^3_+)} \leq C_0 \varepsilon.
\]

Remark 2.3 For every rational number $\gamma \in (0,1]$, we can construct the approximate solution of $(u^a,v^a)$ of (1.1) by using the matched asymptotic expansion, which satisfies the assumption (H1)-(H3) if the initial data $(u_0,v_0) \in X_{\varepsilon}^{k(\gamma)}$ for some $k(\gamma) \in \mathbb{N}_+$ depending on $\gamma$. Thus, for any rational number $\gamma \in (0,1]$ and $(u_0,v_0) \in X_{\varepsilon}^{k(\gamma)}$, there exist $T > 0$ and $C_0 > 0$ independent of $\varepsilon$ such that the Navier-Stokes equations (1.1) admit a unique solution $(u^\varepsilon, v^\varepsilon)$ on $[0,T]$, which satisfies
\[
\sup_{0 \leq t \leq T} \| (u^\varepsilon, v^\varepsilon) - (u^a, v^a) \|_{X^3} \leq C_0 \varepsilon^2, \quad \sup_{0 \leq t \leq T} \| w - w^0 \|_{X^2} \leq C_0 \varepsilon.
\]
In particular,
\[
\sup_{0 \leq t \leq T} \| (u^\varepsilon, v^\varepsilon) - (u^a, v^a) \|_{L^2 \cap L^\infty(R^3_+)} \leq C_0 \varepsilon,
\]
where $(u^a,v^a)$ has the form as (H1), and $(u^\varepsilon, v^\varepsilon)$ is the sum of solutions of the Euler equations and linearized Euler equations, $(w^\varepsilon, p^\varepsilon)$ is the sum of solutions of the linearized Prandtl-type equations (for $\gamma = 1$, it is the sum of solutions of nonlinear Prandtl equation with Robin boundary condition and some linearized Prandtl equation).

Remark 2.4 When $\gamma = 0$, (1.1) is just the Navier-Stokes equations with the Navier-slip boundary condition. In this case, the boundary layer is weak so that the convergence from the Navier-Stokes equations to the Euler equations can be proved in Sobolev space. In fact, we can give an explanation from the viewpoint of asymptotic expansion. Observe that the approximate solutions $u^\varepsilon(t,x,y) = u^\varepsilon(t,x,y) + \varepsilon w^\varepsilon(t,x,z), v^\varepsilon(t,x,y) = v^\varepsilon(t,x,y) + \varepsilon^2 p^\varepsilon(t,x,z)$ for $\gamma = 0$, and so $\partial_y u^\varepsilon(t,x,y) = \partial_y u^\varepsilon(t,x,y) + \partial_y w^\varepsilon(t,x,z), \partial_y v^\varepsilon(t,x,y) = \partial_y v^\varepsilon(t,x,y) + \varepsilon \partial_y p^\varepsilon(t,x,z)$ are not singular terms, hence the error equation (2.4) is not a singular equation and we can make the energy estimate in conormal Sobolev space.

2.4 Proof of main result

The key point is to prove that the solution of error equation (2.4) is uniformly bounded in the suitable functional spaces. However, from the assumption (H1) and the error equation (2.4), we note that there are some singular terms such as
\[
\tilde{u} \partial_y (\varepsilon^{1-\gamma} w^\varepsilon(t,x,y)) = \frac{\tilde{u}}{\varepsilon^{\gamma}} \partial_2 w^\varepsilon(t,x,y).
\]
This singular term is handled as follows: near boundary, using \( \tilde{v}|_{y=0} = 0 \), it can be transformed to a term which loses “\( \gamma \)”—order derivative
\[
\frac{\tilde{v}}{\varepsilon^\gamma} \partial_2 u^p(t, x, \frac{y}{\varepsilon}) = \frac{\tilde{v}}{y^\gamma} \varepsilon^\gamma \partial_2 u^p(t, x, \frac{y}{\varepsilon}) \sim y^{1-\gamma} \partial_y \tilde{v} \partial_2 u^p(t, x, \frac{y}{\varepsilon});
\]
Away from the boundary, this term is good and has a decay factor \( y^{-\gamma} \). Hence, it is natural to work in the Gevrey setting and we estimate this term near boundary and away from boundary in a different way respectively. On the other hand, if we directly take \( \partial_y \) derivative to the error equation, the Prandtl part \( u^p, v^p \) of the approximate solution will give rise to a bad factor \( \varepsilon^{-1} \). To avoid this singularity, we use the conormal derivative motivated by [22, 34]. However, we can not obtain a control in the normal derivative near the boundary by using the conormal derivative. Motivated by [19, 34], we will use the vorticity equation to gain one order normal derivative. Finally, we complete our argument by combining the velocity estimate and vorticity estimate together.

Let us complete the proof of main result by admitting Proposition 4.1, Proposition 4.5, Proposition 5.1 and Proposition 5.3 which will be proved in the later sections.

We introduce the energy functional
\[
E(t) := \varepsilon^{-2} \left( \|U\|_{X^3}^2 + \varepsilon^4 \right) + \|\eta\|_{X^2}^2, \quad F(t) := \varepsilon^{-2} \|U\|_{Y^3}^2 + \|\eta\|_{Y^2}^2, \\
G(t) := \|\nabla U\|_{X^3}^2 + \varepsilon^2 \|\nabla \eta\|_{X^2}^2.
\]
Using that the fact \( \gamma \leq 1 \) and [2, 6], we have
\[
\|\omega\|_{Y^2}^2 \leq C_0 \left( \|\eta\|_{Y^2}^2 + \varepsilon^{-2} \|u\|_{Y^2}^2 + \varepsilon^2 \|\omega e^{-\gamma}\|_{Y^2}^2 \right).
\]
By integration by parts, we have
\[
\|\psi(y) \partial_y u\|_2 \leq C_0 \left( \|u\|_2 + \|\psi(y)^2 \partial_y^2 u\|_2 \right),
\]
and it follows from (2.10) that for \( \delta < \frac{1}{2} \)
\[
\|\omega e^{-\gamma}\|_{Y^2}^2(t) = \sum_{m=3}^{\infty} \rho(t)^{2(m-2)}(m-2) \sum_{|\alpha| = m} \|\partial^{\alpha}(\omega e^{-\gamma})\|_2^2 + \|\omega e^{-\gamma}\|_2^2 \\
\leq C_0 \sum_{\alpha_1 = 0}^{2} \|\partial^{\alpha_1}_y g\|_2^2 \delta^{-2\alpha_1} \sum_{m=3}^{\infty} \rho(t)^{2(m-2)}(m-2) \delta^{2m} \\
+ C_0 \sum_{\alpha_1 = 3}^{\infty} \|\partial^{\alpha_1}_y g\|_2^2 \delta^{-2\alpha_1} \sum_{m=3}^{\infty} \rho(t)^{2(m-2)}(m-2) \delta^{2m} + C_0 \\
\leq C_0 \|g\|_{X^2}(t) + C_0 \leq C_0,
\]
and similarly
\[
\|\omega\|_{X^2}^2 \leq C_0 E(t).
\]
Thus, Proposition 4.1 and Proposition 4.5 ensure that
\[
\varepsilon^{-2} \frac{1}{2} \frac{d}{dt} \|U\|_{X^3}^2 + \lambda \varepsilon^{-2} \|U\|_{Y^3}^2 + \frac{1}{2} \|\nabla U\|_{X^3}^2 \\
\leq C_0 \varepsilon \frac{1}{2} E(t)^{\frac{1}{2}} (E(t) + F(t)) + C_0 \varepsilon^{\frac{1}{2}} E(t)^{\frac{3}{2}} G(t)^{\frac{1}{2}} (\varepsilon^2 + \varepsilon E(t)^{\frac{1}{2}} + G(t)^{\frac{1}{2}}) \\
+C_0 (E(t) + F(t))(1 + E(t)^{\frac{1}{2}} + \varepsilon E(t)) + C_0 [E(t)G(t) + \varepsilon^2],
\]
where we used \(\partial_y u = w + \partial_x v\)
\(X^3 \hookrightarrow X^2 \hookrightarrow H^5\), \(Y^3 \hookrightarrow Y^2\).

Similarly, Proposition 5.1 and Proposition 5.3 ensure that
\[
\frac{1}{2} \frac{d}{dt} \|\eta\|_{X^2}^2 + \lambda \|\eta\|_{Y^2}^2 + \frac{\varepsilon^2}{2} \|\nabla \eta\|_{X^2}^2 \\
\leq C_0 (E(t) + F(t))(1 + E(t)) + C_0 \delta \varepsilon^{4-4\gamma}(G(t) + \varepsilon^2) \\
+C_0 E(t)G(t)^{\frac{1}{2}} + C_0 \varepsilon^{2} E(t)^{\frac{3}{2}} G(t)^{\frac{1}{2}} + C_0 E(t)G(t) + \frac{1}{10} G(t).
\]

Combining (2.11)-(2.12) and taking \(\delta\) small enough, we arrive at
\[
\frac{d}{dt} E(t) + \lambda F(t) + \frac{1}{2} G(t) \leq C_0 E(t)(1 + E(t) + G(t)) + C_0 (1 + E(t)) F(t) + C_0 \frac{E^3(t)}{\varepsilon^2}.
\]

Then the standard continuity argument yields that there exists \(T\) independent of \(\varepsilon\) and \(\varepsilon_0 > 0\) such that
\[
\sup_{0 \leq t \leq T} E(t) \leq C_0 \varepsilon^2 \quad \text{for all } \varepsilon \in (0, \varepsilon_0).
\]

Therefore, there holds
\[
\sup_{0 \leq t \leq T} \left( \varepsilon^{-1} \|U\|_{X^3} + \|\eta\|_{X^2} \right) \leq C_0 \varepsilon \quad \text{for all } \varepsilon \in (0, \varepsilon_0).
\]

Recalling (2.6), we arrive at
\[
\sup_{0 \leq t \leq T} \left( \varepsilon^{-1} \|U\|_{X^3} + \|\omega\|_{X^2} \right) \leq C_0 \varepsilon,
\]
from which and Sobolev embedding, it follows that
\[
\|U\|_{L^\infty(R^3_t)} \leq C_0 \varepsilon.
\]

The proof is completed. \(\Box\)
3 Nonlinear estimates in Gevrey type spaces

Our main goal is to obtain the uniform estimates of the solution for the error equations \(2.4\) and \(2.7\) in the Gevrey norms. The key point is to estimate some linear or nonlinear terms, for example \(u \partial_x u + (v + \varepsilon^2 f(t, x)e^{-y}) \partial_y u^\varepsilon + (v^a - \varepsilon^2 f(t, x)e^{-y}) \partial_y u\). Generally speaking, there are four different types:

1) \(u \partial_x u\), or \(u^a \partial_x u\);
2) \((v + \varepsilon^2 f(t, x)e^{-y}) \partial_y u\), or \((v^a - \varepsilon^2 f(t, x)e^{-y}) \partial_y u\);
3) \(u \partial_x u^a\); 4) \(v \partial_y u^a\).

In this section, we will deal with these terms. For simplicity, \(\langle \cdot, \cdot \rangle\) means the inner product in \(L^2_{x,y}(\mathbb{R}_+^2)\), and we denote

\[
\|f\|_{X \cap Y} = \|f\|_X + \|f\|_Y, \quad \|(f, g)\|_X = \|f\|_X + \|g\|_X.
\]

First of all, we deal with the terms in 1), for example, \(u \partial_x v\).

**Lemma 3.1** Let

\[
A = \sum_{m=3}^{\infty} \frac{\rho(t)^{2(m-3)}}{((m-3)!)^{\frac{3}{2}}} \sum_{|\alpha|=m} \left| \left\langle \partial^\alpha (u \partial_x v), \partial^\alpha v \right\rangle \right|.
\]

Then there holds

(a) \[
A \leq C_0 \|(u, v)\|_{X^3} \|\partial_y u, \partial_y v\|_{X^3} \left( \|v\|_{Y^3}^{3} + \|(u, v)\|_{X^3}^{3} \right);
\]

(b) \[
A \leq C_0 \|u\|_{X^3} \|\partial_y u\|_{X^3} \|v\|_{Y^3}^{2} \|X^3 \cap Y^3.
\]

**Proof.** (a) Note that the Sobolev inequality implies that

\[
\|\partial_x u\|_\infty \leq C_0 \|u\|_{H^2} \|\partial_y u\|_{H^2}^{\frac{1}{2}},
\]

thus, by integration by parts we have

\[
\sum_{m=3}^{\infty} \frac{\rho(t)^{2(m-3)}}{((m-3)!)^{\frac{3}{2}}} \sum_{|\alpha|=m} \left| \left\langle u \partial_x \partial^\alpha v, \partial^\alpha v \right\rangle \right| \leq C_0 \|u\|_{H^2}^{\frac{1}{2}} \|\partial_y u\|_{H^2}^{\frac{1}{2}} \|v\|_{X^3}^{2},
\]

then

\[
\sum_{m=3}^{\infty} \frac{\rho(t)^{2(m-3)}}{((m-3)!)^{\frac{3}{2}}} \sum_{|\alpha|=m} \left| \left\langle \partial^\alpha (u \partial_x v), \partial^\alpha v \right\rangle \right| \leq I + C_0 \|u\|_{H^2}^{\frac{1}{2}} \|\partial_y u\|_{H^2}^{\frac{1}{2}} \|v\|_{X^3}^{2},
\]

where

\[
I = \sum_{m=3}^{\infty} \frac{\rho(t)^{2(m-3)}}{((m-3)!)^{\frac{3}{2}}} \sum_{|\alpha|=m} \sum_{0<\beta \leq \alpha} C_0^{\beta} \left| \left\langle \partial^\alpha u \partial_x \partial^{\alpha-\beta} v, \partial^\alpha v \right\rangle \right|,
\]
and $I$ could be decomposed as follows

$$I \leq \sum_{m=3}^{\infty} \frac{\rho(t)^{2(m-3)}}{((m-3)!)^{\frac{3}{2}}} \sum_{|\alpha|=m \atop |\beta|=1} \sum_{1 \leq |\beta| \leq 2} C_{\alpha}^{\beta} \|\partial^{\beta} u \partial_x \partial^{\alpha-\beta} v\|_2 \|\partial^\alpha v\|_2$$

$$+ \sum_{m=5}^{\infty} \frac{\rho(t)^{2(m-3)}}{((m-3)!)^{\frac{3}{2}}} \sum_{|\alpha|=m} \sum_{j=3}^{m-2} \sum_{|\beta|=j} C_{\alpha}^{\beta} \|\partial^{\beta} u \partial_x \partial^{\alpha-\beta} v\|_2 \|\partial^\alpha v\|_2$$

$$+ \sum_{m=3}^{\infty} \frac{\rho(t)^{2(m-3)}}{((m-3)!)^{\frac{3}{2}}} \sum_{|\alpha|=m} \sum_{|\beta|=m} C_{\alpha}^{\beta} \|\partial^{\beta} u \partial_x \partial^{\alpha-\beta} v\|_2 \|\partial^\alpha v\|_2 = \sum_{i=1}^{3} I_i$$

Next we handle them term by term.

**Step 1. Estimate of $I_1$.** Let $I_1 = I_{11} + I_{12}$ according to the value of $|\beta| = 1, 2$.

Firstly, applying H"older inequality twice, we get

$$\sum_{m=4}^{\infty} \frac{\rho(t)^{2(m-3)}}{((m-3)!)^{\frac{3}{2}}} \sum_{|\alpha|=m} \sum_{|\beta|=1} C_{\alpha}^{\beta} \|\partial^{\beta} u \partial_x \partial^{\alpha-\beta} v\|_2 \|\partial^\alpha v\|_2$$

$$\leq \sum_{m=4}^{\infty} \frac{\rho(t)^{2(m-3)}}{((m-3)!)^{\frac{3}{2}}} \left( \sum_{|\alpha|=m} \left( \sum_{|\beta|=1} C_{\alpha}^{\beta} \|\partial^{\beta} u\|_{\infty} \|\partial_x \partial^{\alpha-\beta} v\|_2 \right)^2 \right)^{\frac{1}{2}} |v|_{m,3}$$

$$\leq \sum_{m=4}^{\infty} \frac{\rho(t)^{2(m-3)}}{((m-3)!)^{\frac{3}{2}}} \left[ \sum_{|\alpha|=m} \left( \sum_{|\beta|=1} \left( C_{\alpha}^{\beta} \right)^2 \sum_{|\beta|=1} \|\partial^{\beta} u\|_{\infty} \|\partial_x \partial^{\alpha-\beta} v\|_2 \right) \right]^{\frac{1}{2}} |v|_{m,3}$$

$$\leq C_0 \|\partial u\|_{\infty} \sum_{m=4}^{\infty} m |v|_{m,3}^2 \leq C_0 \|\partial u\|_{\infty} \|v\|_3^2,$$  \hspace{1cm} (3.1)

where we used Lemma 2.1 in the last step, that is, for $j = 1, \cdots, m$

$$\sum_{|\beta|=j, \beta \leq \alpha} (C_{\alpha}^{\beta})^2 \leq \left( \sum_{|\beta|=j} C_{\alpha}^{\beta} \right)^2 \leq \left( C_m^{\nu} \right)^2,$$  \hspace{1cm} (3.2)

and

$$\sum_{|\alpha|=m} \sum_{|\beta|=|\alpha|-1} \|\partial^{\beta} f\|_{\infty} \|\partial^{\alpha-\beta} \partial_x g\|_2 \leq \sum_{|\beta|=j} \|\partial^{\beta} f\|_{\infty} \sum_{|\alpha|=m-j+1} \|\partial^\alpha g\|_2.$$  \hspace{1cm} (3.3)

Hence, we obtain

$$I_{11} \leq C_0 \|\partial u\|_{\infty} \|v\|_3^2 + \sum_{|\alpha|=3} \sum_{|\beta|=1} C_{\alpha}^{\beta} \|\partial^{\beta} u \partial_x \partial^{\alpha-\beta} v\|_2 \|\partial^\alpha v\|_2$$

$$\leq C_0 \|u\|^{\frac{1}{p'}} \|\partial_u\|^{\frac{1}{p'}} \left( \|v\|_3^2 + \|v\|_{X^3} \right).$$

We remark here that the technique of (3.1) includes H"older inequality(twice), (3.2) and (3.3), which will be used frequently, and we just mention (3.1) later.

Secondly, for the term $I_{12}$, similar computations as (3.1) yield that

$$\sum_{m=5}^{\infty} \frac{\rho(t)^{2(m-3)}}{((m-3)!)^{\frac{3}{2}}} \sum_{|\alpha|=m} \sum_{|\beta|=2} C_{\alpha}^{\beta} \|\partial^{\beta} u \partial_x \partial^{\alpha-\beta} v\|_2 \|\partial^\alpha v\|_2$$
\[
    \leq C_0 \|\partial^2 u\|_\infty \sum_{m=5}^{\infty} \frac{\rho(t)^{2(m-3)} C_m^2}{((m-3)!)^{\frac{1}{2}}} \left( \sum_{|\alpha|=m} \|\partial^\alpha v\|^2 \right)^{\frac{1}{2}} \left( \sum_{|\alpha|=m-1} \|\partial^\alpha v\|^2 \right)^{\frac{1}{2}}
\]

\[
    \leq C_0 \|\partial^2 u\|_\infty \|v\|_{Y^3}^2,
\]

where we used \( \gamma \leq 1 \) and \( \frac{m}{(m-3)!} \leq 3 \) for \( m \geq 5 \). Hence,

\[
    I_{12} \leq C_0 \|\partial^2 u\|_\infty \|v\|_{Y^3}^2 + \frac{4}{(m-3)!^2} \sum_{m=3}^{\infty} \sum_{|\alpha|=m} \sum_{|\beta|=2,3} C_2^3 \|\partial^\beta u \partial_x \| \|\partial^\alpha - \beta v\|_{L^2} \|\partial^\alpha v\|_2;
\]

which can be controlled by

\[
    C_0 \|u\|_{H^1} \|\partial_y u\|_{H^1} \left( \|v\|_{Y^3}^2 + \|v\|_{X^3}^2 \right).
\]

Finally, collecting the estimates of \( I_{11} \) and \( I_{12} \), we obtain

\[
    I_1 \leq C_0 \|u\|_{H^1} \|\partial_y u\|_{H^1} \left( \|v\|_{Y^3}^2 + \|v\|_{X^3}^2 \right).
\]

**Step 2. Estimate of \( I_2 \).** Firstly, according to the different values of \( |\beta| \), \( I_2 \) is divided into two terms

\[
    I_2 = \sum_{m=6}^{\infty} \frac{\rho(t)^{2(m-3)}}{((m-3)!)^{\frac{1}{2}}} \sum_{|\alpha|=m} \sum_{|\beta|=j} \sum_{|\gamma|=k} C_2^\beta \|\partial^\beta u \partial_x \| \|\partial^\alpha - \beta v\|_{L^2} \|\partial^\alpha v\|_2
\]

\[
    + \sum_{m=5}^{\infty} \frac{\rho(t)^{2(m-3)}}{((m-3)!)^{\frac{1}{2}}} \sum_{|\alpha|=m} \sum_{|\beta|=j} \sum_{|\gamma|=k} C_2^\beta \|\partial^\beta u \partial_x \| \|\partial^\alpha - \beta v\|_{L^2} \|\partial^\alpha v\|_2 = I_{21} + I_{22}.
\]

By Sobolev embedding, we have

\[
    \|\partial^\beta u\|_\infty \leq C_0 \|\partial^\beta u\|_{L^2} \|\partial_y \partial^\beta u\|_{L^2} + C_0 \|\partial_x \partial^\beta u\|_{L^2} \|\partial_x \partial_y \partial^\beta u\|_{L^2},
\]

which gives

\[
    I_{21} \leq \sum_{m=6}^{\infty} \frac{\rho(t)^{2(m-3)}}{((m-3)!)^{\frac{1}{2}}} \sum_{|\alpha|=m} \sum_{|\beta|=j} \sum_{|\gamma|=k} C_2^\beta \|\partial^\beta u\|_{L^2} \|\partial_y \partial^\beta u\|_{L^2} \|\partial_x \partial^\alpha - \beta v\|_{L^2} \|\partial^\alpha v\|_2
\]

\[
    + \sum_{m=6}^{\infty} \frac{\rho(t)^{2(m-3)}}{((m-3)!)^{\frac{1}{2}}} \sum_{|\alpha|=m} \sum_{|\beta|=j} \sum_{|\gamma|=k} C_2^\beta \|\partial_x \partial^\beta u\|_{L^2} \|\partial_x \partial_y \partial^\beta u\|_{L^2} \|\partial_x \partial^\alpha - \beta v\|_{L^2} \|\partial^\alpha v\|_2.
\]

Obviously, it suffices to estimate the second term, since the order of derivative in the first term is more lower. Using the same argument as in \[341\] and discrete Young inequality, it can be controlled by

\[
    \sum_{m=6}^{\infty} \frac{\rho(t)^{m-3}}{((m-3)!)^{\frac{1}{2}}} \sum_{j=3}^{\infty} C_m^j \left( \sum_{|\beta|=j} \|\partial^\beta \partial_x u\|_{L^2} \|\partial_y \partial^\beta \partial_x u\|_2 \right)^{\frac{1}{2}} \left( \sum_{|\alpha|=m+1-j} \|\partial^\alpha v\|_2 \right)^{\frac{1}{2}} |v|_{m,3}
\]
\[
\leq C_0 \sum_{m=6}^{\infty} \sum_{j=3}^{[\frac{m}{2}]} \frac{C_m^j ((m-j-2)!)^{\frac{1}{7}} ((j-2)!)^{\frac{1}{7}} ((j-1)!)^{\frac{2}{7}}}{((m-3)!)^{\frac{2}{7}} \sqrt{m-3} \sqrt{m-j-2}} \\
\times \rho(t) \left( \frac{(j-1)!(m-3)!}{(j-1)!} \right)^{\frac{1}{7}} \left( \sum_{|\alpha|=j+1} \|\partial_\gamma \partial^\alpha u\|_2^2 \right)^{\frac{4}{7}} |u|^\frac{1}{7} |u|^\frac{1}{7} |u|^\frac{1}{7} \sqrt{m-j-2} |v|_m \sqrt{m-j-2} |v|_{m-3} \\
\leq C_0 \sum_{m=6}^{\infty} \sum_{j=3}^{[\frac{m}{2}]} \frac{1}{j} \left( \sum_{|\alpha|=j+1} \|\partial_\gamma \partial^\alpha u\|_2^2 \right)^{\frac{4}{7}} |u|^\frac{1}{7} |u|^\frac{1}{7} |u|^\frac{1}{7} \sqrt{m-j-2} |v|_m \sqrt{m-j-2} |v|_{m-3} \\
\leq C_0 \left( \sum_{m=6}^{\infty} \frac{1}{j} \left( \sum_{|\alpha|=j+1} \|\partial_\gamma \partial^\alpha u\|_2^2 \right)^{\frac{4}{7}} |u|^\frac{1}{7} |u|^\frac{1}{7} |u|^\frac{1}{7} \sqrt{m-j-2} |v|_m \sqrt{m-j-2} |v|_{m-3} \right) \frac{1}{7} \|v\|_{Y^3} \\
\leq C_0 \|u\|_{X^3} \|\partial_\gamma u\|_{X^2} \|v\|_{Y^3}, \tag{3.4}
\]
where we used i) the estimate:
\[
\frac{C_m^j ((m-j-2)!)^{\frac{1}{7}} ((j-2)!)^{\frac{1}{7}} ((j-1)!)^{\frac{2}{7}}}{((m-3)!)^{\frac{2}{7}} \sqrt{m-3} \sqrt{m-j-2}} \leq C_0 \left( \frac{C_m^{j-1}}{(m-3)!} \right)^{\frac{1}{7}} \leq C_0 j^{-1}
\]
for \( \gamma \leq 1 \) and \( j \in \{3, \ldots, [\frac{m}{2}]\} \); ii) discrete Young inequality to estimate
\[
\left( \sum_{m=6}^{\infty} \frac{1}{j} \left( \sum_{|\alpha|=j+1} \|\partial_\gamma \partial^\alpha u\|_2^2 \right)^{\frac{4}{7}} |u|^\frac{1}{7} |u|^\frac{1}{7} |u|^\frac{1}{7} \sqrt{m-j-2} |v|_m \sqrt{m-j-2} |v|_{m-3} \right)^{\frac{1}{7}} \\
\leq C \|v\|_{Y^3} \sum_{m=3}^{\infty} \frac{1}{m} \left( \sum_{|\alpha|=m+1} \|\partial_\gamma \partial^\alpha u\|_2^2 \right)^{\frac{1}{7}} |u|_m^{\frac{1}{7}} |u|_m^{\frac{1}{7}} |u|_m^{\frac{1}{7}} \\
\leq C \|v\|_{Y^3} \|u\|_{X^3} \left( \sum_{m=3}^{\infty} \frac{1}{m} \left( \frac{m-1}{(m-1)!} \right)^{\frac{1}{7}} \sum_{|\alpha|=m+1} \|\partial_\gamma \partial^\alpha u\|_2^2 \right)^{\frac{1}{7}},
\]
iii) the commutator estimate: as
\[
\|\partial_\gamma \partial^\alpha u\|_2 \leq \|\partial^\alpha \partial_\gamma u\|_2 + \|[\partial_\gamma, \partial^\alpha]u\|_2,
\]
thus,
\[
\sum_{m=4}^{\infty} \frac{\rho(t)^{2(m-2)}}{(m-2)!^{\frac{1}{7}}} \sum_{|\alpha|=m} \|\partial_\gamma \partial^\alpha u\|_2^2 \\
\leq C_0 \|\partial_\gamma u\|_{X^2}^2 + \sum_{m=4}^{\infty} \frac{\rho(t)^{2(m-2)}}{(m-2)!^{\frac{1}{7}}} \sum_{|\alpha|=m-1} m^2 \|\partial^\alpha \partial_\gamma u\|_2^2 \leq C_0 \|\partial_\gamma u\|_{X^2}^2. \tag{3.5}
\]
Hence, we arrive at
\[
I_{21} \leq C_0 \|u\|_{X^3} \|\partial_\gamma u\|_{X^2} \|v\|_{Y^3}^2.
\]
Secondly, different from the estimate of $I_{21}$, for $I_{22}$, we estimate $L^\infty$ norm of $\partial_x \partial^{\alpha-\beta} v$, and it can be bounded by

$$
\sum_{m=5}^{\infty} \frac{\rho(t)^{2(m-3)}}{((m-3)!)^{\frac{m}{2}}} \sum_{|\alpha|=m} \sum_{j=\left[\frac{m}{2}\right]+1}^{m-2} C_0^2 \|\partial^\beta u\|_2 \|\partial_x \partial^{\alpha-\beta} v\|_2 \left( \sum_{|\alpha|=m-2} \|\partial^\beta u\|_2 \|\partial_x \partial^{\alpha-\beta} v\|_2 \right)^{\frac{1}{2}} \|\partial^\alpha v\|_2 
$$

$$
+ \sum_{m=5}^{\infty} \frac{\rho(t)^{2(m-3)}}{((m-3)!)^{\frac{m}{2}}} \sum_{|\alpha|=m} \sum_{j=\left[\frac{m}{2}\right]+1}^{m-2} C_0^2 \|\partial^\beta u\|_2 \|\partial_x \partial^{\alpha-\beta} v\|_2 \left( \sum_{|\alpha|=m-2} \|\partial^\beta u\|_2 \|\partial_x \partial^{\alpha-\beta} v\|_2 \right)^{\frac{1}{2}} \|\partial^\alpha v\|_2.
$$

We only estimate the second term. Using similar arguments as in (3.4), it be controlled by

$$
\sum_{m=5}^{\infty} \frac{\rho(t)^{(m-3)}}{((m-3)!)^{\frac{m}{2}}} \sum_{j=\left[\frac{m}{2}\right]+1}^{m-2} C_0^2 \left( \sum_{|\alpha|=m-2} \|\partial^\beta u\|_2 \|\partial_x \partial^{\alpha-\beta} v\|_2 \right)^{\frac{1}{2}} \|v\|_{m,3} 
$$

$$
+ C_0 \sum_{m=6}^{\infty} \sum_{j=\left[\frac{m}{2}\right]+1}^{m-2} \frac{C_0^2 ((m-j-1)!)^{\frac{1}{2}} ((m-j)!)\frac{1}{2} (j-3)!^{\frac{1}{2}}}{(m-3)!^{\frac{1}{2}} \sqrt{m-3}} \left( \sum_{|\alpha|=m-2} \|\partial^\beta u\|_2 \|\partial_x \partial^{\alpha-\beta} v\|_2 \right)^{\frac{1}{2}} \|v\|_{m,3} 
$$

$$
\leq C_0 \|v\|_{X^3}^{\frac{1}{2}} \|\partial_y v\|_{X^3}^{\frac{1}{2}} \|u\|_{X^3} \|v\|_{X^3} 
$$

$$
+ C_0 \|v\|_{X^3} \|\partial_y v\|_{X^3} \|u\|_{X^3} \|v\|_{X^3 \cap Y^3},
$$

where we used

$$
\frac{C_0^2 ((m-j-1)!)^{\frac{1}{2}} ((m-j)!)\frac{1}{2} (j-3)!^{\frac{1}{2}}}{(m-3)!^{\frac{1}{2}} \sqrt{m-3}} \leq C_0 \left( C_0^{j-3} \right)^{\frac{1}{2}} (m-j)^{-\frac{1}{2}}
$$

for $j \in \{\left[\frac{m}{2}\right]+1, \cdots, m-2\}$ and (3.5).

Therefore, we get

$$
I_{22} \leq C_0 \|v\|_{X^3}^{\frac{1}{2}} \|\partial_y v\|_{X^3} \|u\|_{X^3} \|v\|_{X^3 \cap Y^3}.
$$

Finally, collecting the estimates of $I_{21}$ and $I_{22}$, we obtain

$$
I_2 \leq C_0 \left[ \|u\|_{X^3}^{\frac{1}{2}} \|\partial_y u\|_{X^3}^{\frac{1}{2}} \|v\|_{X^3 \cap Y^3} + \|v\|_{X^3} \|\partial_y v\|_{X^3} \|u\|_{X^3} \|v\|_{X^3 \cap Y^3} \right].
$$

Step 3. Estimate of $I_3$. This is similar to $I_2$, but more easier. We rewrite $I_{31}$ as the term of $|\beta| = m - 1$ in $I_3$, and $I_{32}$ for the term of $|\beta| = m$. 




Again using the technique as in (5.1), we get
\[
\sum_{m=4}^{\infty} \frac{\rho(t)^{2(m-3)}}{((m-3)!)^2} \sum_{|\alpha|=m} \sum_{|\beta|=m-1, \beta \leq \alpha} C_{\alpha}^2 \left| \partial^\alpha u \| \partial_x \partial^{\alpha-\beta} v \|_\infty \| \partial^\alpha v \|_2 \right.
\]
\[
\leq C_0 \| \partial^2 v \|_\infty \sum_{m=4}^{\infty} \frac{\rho(t)^{2(m-3)} m}{((m-3)!)^2} \left( \sum_{|\alpha|=m-1} \| \partial^\alpha u \|_2^\beta \right) \left( \sum_{|\alpha|=m} \| \partial^\alpha v \|_2^\beta \right)^{\frac{1}{2}}
\]
\[
\leq C_0 \| \partial^2 v \|_\infty \| u \|_{X^3} \| v \|_{X^3}
\]

and
\[
\sum_{|\alpha|=3} \sum_{|\beta|=2, \beta \leq \alpha} C_{\alpha}^\beta \left| \partial^\beta u \| \partial_x \partial^{\alpha-\beta} v \|_\infty \| \partial^\alpha v \|_2 \right.
\]
\[
\leq C_0 \| \partial^2 v \|_\infty \| \partial^2 u \|_2 \| v \|_{X^3},
\]

which give
\[
I_{31} \leq C_0 \| v \|_T \| \partial_y v \|_T \| u \|_{X^3} \| v \|_{X^3}.
\]

Similarly, there holds
\[
I_{32} \leq \sum_{m=3}^{\infty} \frac{\rho(t)^{2(m-3)}}{((m-3)!)^2} \sum_{|\alpha|=m} \sum_{|\beta|=m, \beta \leq \alpha} C_{\alpha}^\beta \left| \partial^\beta u \| \partial_x \partial^{\alpha-\beta} v \|_\infty \| \partial^\alpha v \|_2 \right.
\]
\[
\leq C_0 \| v \|_T \| \partial_y v \|_T \| u \|_{X^3} \| v \|_{X^3},
\]

which along with the estimate of $I_{31}$ implies that
\[
I_3 \leq C_0 \| v \|_T \| \partial_y v \|_T \| u \|_{X^3} \| v \|_{X^3}.
\]

Collecting the estimates in Step 1-Step 3, the proof of the inequality (a) is completed.

(b) This inequality is used to estimate the linear term like $u^a \partial_x v$. We use the same notations as in (a). First of all, we know that
\[
\sum_{m=3}^{\infty} \frac{\rho(t)^{2(m-3)}}{((m-3)!)^2} \sum_{|\alpha|=m} \left| \langle \partial^\alpha (u \partial_x v), \partial^\alpha v \rangle \right| \leq I + C_0 \| u \|_{X^3} \| \partial_y u \|_{X^3} \| v \|_{X^3}^2
\]

and $I = \sum_{i=1}^{3} I_i$ as in (a).

The estimate of $I_1$ is as follows
\[
I_1 \leq C_0 \| u \|_{X^3} \| \partial_y u \|_{X^3} \left( \| v \|_{X^3}^2 + \| v \|_{X^3}^2 \right).
\]

**Estimate of $I_2$.** By Sobolev embedding, $I_2$ can be bounded by
\[
\sum_{m=6}^{\infty} \frac{\rho(t)^{2(m-3)}}{((m-3)!)^2} \sum_{|\alpha|=m} \sum_{j=3}^{m-3} \sum_{|\beta|=j, \beta \leq \alpha} C_{\alpha}^\beta \left| \partial^\beta u \| \partial_y \partial^\beta u \|_2 \| \partial_x \partial^{\alpha-\beta} v \|_2 \| \partial^\alpha v \|_2 \right.
\]
Now we estimate the term \( I_{22} \). It follows from Lemma 2.1 and discrete convolution inequality as (3.4) that

\[
I_{22} \leq \sum_{m=6}^{\infty} \frac{\rho(t)^{(m-3)}}{(m-3)!^\frac{1}{7}} \sum_{j=3}^{m-3} C_m^j \left( \sum_{|\beta|=j+1} \| \partial^\beta u \|_2 \| \partial_y \partial^\beta u \|_2 \right)^{\frac{1}{2}} \left( \sum_{|\alpha|=m+1-j} \| \partial^\alpha v \|_2^2 \right)^{\frac{1}{2}} \| v \|_{m,3}
\]

\[
\leq \frac{C_0 \sum_{m=6}^{\infty} \sum_{j=3}^{m-3} C_m^j \rho(t)^{(j-3)}}{(m-3)!^\frac{1}{7}} \frac{\sqrt{m-3}}{\sqrt{m-j-2}} \frac{(j-3)!^\frac{1}{7}}{(j-3)!^\frac{1}{7}} \sum_{|\beta|=j+1} \| \partial^\beta u \|_2 \| \partial_y \partial^\beta u \|_2 \right)^{\frac{1}{2}} \sum_{|\beta|=j+1} \| \partial^\beta u \|_2 \| \partial_y \partial^\beta u \|_2 \right)^{\frac{1}{2}} \sqrt{m-j-2} \| v \|_{m+1-j,3} \sqrt{m-3} \| v \|_{m,3}
\]

\[
\leq \frac{C_0 \sum_{m=6}^{\infty} \sum_{j=3}^{m-3} \frac{C_m^j \rho(t)^{(j-3)}}{(m-3)!^\frac{1}{7}} \frac{1}{(m-j)^\frac{1}{7}} \sum_{|\beta|=j+1} \| \partial^\beta u \|_2 \| \partial_y \partial^\beta u \|_2 \right)^{\frac{1}{2}} \sqrt{m-j-2} \| v \|_{m+1-j,3} \sqrt{m-3} \| v \|_{m,3}
\]

where we used

\[
a_{m,j} = \begin{cases} \frac{1}{j}, & j \in \left\{ 3, \ldots, \left\lceil \frac{m}{2} \right\rceil \right\}, \\ \frac{1}{(m-j)^\frac{1}{7}}, & j \in \left\{ \left\lceil \frac{m}{2} \right\rceil, \ldots, m-3 \right\} \end{cases}
\]

satisfying

\[
\frac{C_m^j ((m-j-2)!)^\frac{1}{7} ((j-3)!)^\frac{1}{7}}{(m-3)!^\frac{1}{7} \sqrt{m-j-2} \sqrt{m-3}} \leq C_0 a_{m,j}
\]

and the commutator estimate as in (3.5).

The same argument gives

\[
I_{21} \leq C_0 \| u \|_{X^3} \| \partial_y u \|_{X^3} \| v \|_{Y^3}^2.
\]

Thus, we arrive at

\[
I_2 \leq C_0 \| u \|_{X^3} \| \partial_y u \|_{X^3} \| v \|_{Y^3}^2 \| v \|_{X^3}^2 + I_{23},
\]

and \( I_{23} \) will be estimated in the next step.

**Estimate of \( I_3 \) and \( I_{23} \).** We first decompose \( I_3 \) as in (a). By similar arguments as in (3.4), we get

\[
\sum_{m=1}^{\infty} \frac{\rho(t)^{2(m-3)}}{(m-3)!^\frac{1}{7}} \sum_{|\alpha|=m} \sum_{|\beta|=m-1, \beta \leq \alpha} C_0^\beta \| \partial^\beta u \|_{L_x^\infty L_t^2} \| \partial_x \partial^\alpha \beta v \|_{L_x^4 L_t^\infty} \| \partial^\alpha v \|_2
\]

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There holds

\[ \text{Proof.} \]

We only give a proof for \( \text{Hence, we have} \)

\[ \text{The same argument gives} \]

\[ \text{Recalling the estimates of} \]

\[ \text{Finally, there holds} \]

\[ \text{Hence, we have} \]

\[ I_{31} \leq C_0 \| u \|_{X^k} \| \partial_y u \|_{X^k} \| v \|_{X^2} \| v \|_{H^1} + C_0 \| \partial^2 u \|_\infty \| \partial^2 v \|_2 \| v \|_{X^2} \]

\[ \leq C_0 \| u \|_{X^k} \| \partial_y u \|_{X^k} \| v \|_{X^2} \| v \|_{H^1}. \]

The same argument gives

\[ I_{32} \leq C_0 \| u \|_{X^k} \| \partial_y u \|_{X^k} \| v \|_{X^2} \| v \|_{H^1}, \]

\[ I_{23} \leq C_0 \| u \|_{X^k} \| \partial_y u \|_{X^k} \| v \|_{X^2} \| v \|_{H^1}. \]

Finally, there holds

\[ I_3 + I_{23} \leq C_0 \| u \|_{X^k} \| \partial_y u \|_{X^k} \| v \|_{X^2} \| v \|_{H^1}. \]

Recalling the estimates of \( I_1 \) and \( I_2 \), the proof is completed. \( \Box \)

To estimate the term in 2) like \( v \partial_y u \), we need the following lemma.

**Lemma 3.2** For \( k = 2, 3 \) and the suitable functions \( u, v \) with \( u|_{y=0} = 0 \), let

\[ B_k = \sum_{m=k}^{\infty} \frac{\rho(t)^{2(m-k)}}{(m-k)!^2} \| \partial^m v \|_2 \sum_{|\alpha|=m} |\langle \partial^\alpha (u \partial_y v), \partial^\alpha v \rangle|. \]

There holds

\[ (a) \quad B \leq C_0(\delta) \| (u, \partial_y u) \|_{H^1} \| (\partial_y u, \partial_{yy} u) \|_{H^1} \| v \|_{X^k}^2 \]

\[ \quad \quad \quad \quad \quad \quad \quad + C_0 \| u \|_{X^k}^2 \| \partial_y u \|_{X^k} \| \partial_y v \|_{X^k} \| v \|_{X^k} + C_0(\delta) \| v \|_{X^k}^3 \| \partial_y v \|_{X^k} \| (u, \partial_y u) \|_{X^k}. \]

\[ (b) \quad B \leq C_0(\delta) \| (u, \partial_y u) \|_{X^{k+1}} \| (\partial_y u, \partial_{yy} u) \|_{X^{k+1}} \| v \|_{X^{k+1}}^2 \]

**Proof.** We only give a proof for \( k = 3 \). The estimate for \( k = 2 \) can be obtained by the same argument.

\( (a) \) Let

\[ \sum_{m=3}^{\infty} \frac{\rho(t)^{2(m-3)}}{(m-3)!^2} \sum_{|\alpha|=m} |\langle \partial^\alpha (u \partial_y v), \partial^\alpha v \rangle| \leq \tilde{I} + \tilde{I}, \]

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where
\[
\tilde{I} = \sum_{m=3}^{\infty} \frac{\rho(t)^{2(m-3)}}{(m-3)!} \sum_{|\alpha|=m} \frac{C_\alpha^\beta}{\beta!} \left| \langle \partial^\alpha u \partial^{\alpha-\beta} \partial_y v, \partial^\alpha v \rangle \right|
\]
\[
\tilde{\Pi} = \sum_{m=3}^{\infty} \frac{\rho(t)^{2(m-3)}}{(m-3)!} \sum_{|\alpha|=m} \left| \langle u \partial^\alpha \partial_y v, \partial^\alpha v \rangle \right|
\]

We first estimate the term \( \tilde{\Pi} \). Thanks to the definition of \( \psi \), we have
\[
\left\| \frac{u}{\psi} \right\|_\infty \leq C_0(\delta) (\| \partial_y u \|_\infty + \| u \|_\infty) \leq C_0(\delta) \| (u, \partial_y u) \|_{\frac{7}{H^1}} \| (\partial_y u, \partial_{yy} u) \|_{\frac{7}{H^1}}.
\]
Using \( u|_{y=0} = 0 \) and integration by parts, we get
\[
\tilde{\Pi} \leq C_0(\delta) \| (u, \partial_y u) \|_{\frac{7}{H^1}} \| (\partial_y u, \partial_{yy} u) \|_{\frac{7}{H^1}} \| v \|_{X^{3^{-1},1,3}}.
\]

Then, similar to the estimate of \( I \) in Lemma 3.1, we decompose \( \tilde{I} \) into three terms according to the value of \( |\beta| \):
\[
\tilde{I} \leq \sum_{m=3}^{\infty} \frac{\rho(t)^{2(m-3)}}{(m-3)!} \sum_{|\alpha|=m} \frac{C_\alpha^\beta}{\beta!} \| \partial^\beta u \partial^{\alpha-\beta} \partial_y v \|_2 \| \partial^\alpha v \|_2
\]
\[
+ \sum_{m=5}^{\infty} \frac{\rho(t)^{2(m-3)}}{(m-3)!} \sum_{|\alpha|=m} \sum_{j=3}^{m-2} C_\alpha^\beta \| \partial^\beta u \partial^{\alpha-\beta} \partial_y v \|_2 \| \partial^\alpha v \|_2
\]
\[
+ \sum_{m=3}^{\infty} \frac{\rho(t)^{2(m-3)}}{(m-3)!} \sum_{|\alpha|=m} \sum_{m-1 \leq |\beta| \leq m, \beta \leq \alpha} C_\alpha^\beta \| \partial^\beta u \partial^{\alpha-\beta} \partial_y v \|_2 \| \partial^\alpha v \|_2 = \sum_{i=1}^{3} \tilde{I}_i
\]

We handle them term by term.

**Step 1. Estimate of \( \tilde{I}_1 \).** We denote \( \tilde{I}_1 = \tilde{I}_{11} + \tilde{I}_{12} \) according to \( |\beta| = 1 \) or \( |\beta| = 2 \), where
\[
\tilde{I}_{11} = \sum_{m=4}^{\infty} \frac{\rho(t)^{2(m-3)}}{(m-3)!} \sum_{|\alpha|=m} \sum_{|\beta|=1, \beta \leq \alpha} C_\alpha^\beta \| \partial^\beta u \partial^{\alpha-\beta} \partial_y v \|_2 \| \partial^\alpha v \|_2
\]
\[
+ \sum_{|\alpha|=3} \sum_{|\beta|=1, \beta \leq \alpha} C_\alpha^\beta \| \partial^\beta u \partial^{\alpha-\beta} \partial_y v \|_2 \| \partial^\alpha v \|_2
\]

Using \( \partial^\beta u|_{y=0} = 0 \) and Sobolev embedding, the second term of \( \tilde{I}_{11} \) can be controlled by
\[
C_0 \sum_{|\alpha|=3} \sum_{|\beta|=1, \beta \leq \alpha} \left\| \frac{\partial^\beta u}{\psi} \partial^{\alpha-\beta} \partial_y v \right\|_2 \| \partial^\alpha v \|_2
\]

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This shows that

As in (3.1), the first term of $\tilde{I}_{11}$ can be controlled by

$$
\sum_{m=4}^{\infty} \rho(t)^{2(m-3)} \left( \frac{m}{((m-3)!)^2} \right) \sum_{|\alpha|=m} \sum_{|\beta|=2, \beta \leq \alpha} C_{\alpha}^2 \left\| \frac{\partial^\beta u}{\psi} \right\|_\infty \left\| \psi \partial^{\alpha-\beta} \partial_y v \right\|_2 \left\| \partial^\alpha v \right\|_2
\leq C_0(\delta) \|(u, \partial_y u)\|_{\tilde{H}^3} \left( (\partial_y u, \partial_{yy} u)\right|_{\tilde{H}^2} \left\| v \right\|_{Y^3}.
$$

This gives

$$
\tilde{I}_{11} \leq C_0(\delta) \|(u, \partial_y u)\|_{\tilde{H}^3} \left( (\partial_y u, \partial_{yy} u)\right|_{\tilde{H}^2} \left\| v \right\|_{Y^3}.
$$

For the case of $|\beta| = 2$, similar arguments imply

$$
\tilde{I}_{12} = \sum_{m=5}^{\infty} \rho(t)^{2(m-3)} \left( \frac{m}{((m-3)!)^2} \right) \sum_{|\alpha|=m} \sum_{|\beta|=2, \beta \leq \alpha} C_{\alpha}^2 \left\| \partial^\beta u \partial^{\alpha-\beta} \partial_y v \right\|_2 \left\| \partial^\alpha v \right\|_2
\leq C_0(\delta) \|(u, \partial_y u)\|_{\tilde{H}^3} \left( (\partial_y u, \partial_{yy} u)\right|_{\tilde{H}^2} \left\| v \right\|_{Y^3}.
$$

The first term is bounded by

$$
C_0 \sum_{m=5}^{\infty} \rho(t)^{2(m-3)} \left( \frac{m}{((m-3)!)^2} \right) \sum_{|\alpha|=m} \sum_{|\beta|=2, \beta \leq \alpha} C_{\alpha}^2 \left\| \frac{\partial^\beta u}{\psi} \right\|_\infty \left\| \psi \partial^{\alpha-\beta} Z v \right\|_2 \left\| \partial^\alpha v \right\|_2
\leq C_0 \left\| \frac{\partial^\beta u}{\psi} \right\|_\infty \sum_{m=5}^{\infty} \rho(t)^{2(m-3)} m(m-1) \left( \sum_{|\alpha|=m} \left\| \partial^\alpha v \right\|_2 \right)^2 \left( \sum_{|\alpha|=m-1} \left\| \partial^\alpha v \right\|_2 \right)^2
\leq C_0(\delta) \|(u, \partial_y u)\|_{\tilde{H}^3} \left( (\partial_y u, \partial_{yy} u)\right|_{\tilde{H}^2} \left\| v \right\|_{Y^3}.
$$

This gives

$$
\tilde{I}_{12} \leq C_0(\delta) \|(u, \partial_y u)\|_{\tilde{H}^3} \left( (\partial_y u, \partial_{yy} u)\right|_{\tilde{H}^2} \left\| v \right\|_{Y^3}.
$$

Collecting the results of $\tilde{I}_{11}$ and $\tilde{I}_{12}$ together, we obtain

$$
\tilde{I}_1 \leq C_0(\delta) \|(u, \partial_y u)\|_{\tilde{H}^3} \left( (\partial_y u, \partial_{yy} u)\right|_{\tilde{H}^2} \left\| v \right\|_{Y^3}.
$$

(3.7)
Step 2. Estimate of $\tilde{I}_2$. As in Lemma 3.1 we decompose $\tilde{I}_2$ into two terms $\tilde{I}_{21}$ and $\tilde{I}_{22}$, where

$$\tilde{I}_{21} = \sum_{m=6}^{\infty} \frac{\rho(t)^{2(m-3)}}{(m-3)!} \sum_{|\alpha|=m} \sum_{j=3}^{\left\lfloor \frac{m}{2} \right\rfloor} \sum_{|\beta|=j, \beta \leq \alpha} C_\alpha^\beta \| \partial^\alpha u \partial^\beta \partial_y v \|_2 \| \partial^\alpha v \|_2$$

and

$$\tilde{I}_{22} = \sum_{m=6}^{\infty} \frac{\rho(t)^{2(m-3)}}{(m-3)!} \sum_{|\alpha|=m} \sum_{j=3}^{m-2} \sum_{|\beta|=j, \beta \leq \alpha} C_\alpha^\beta \| \partial^\alpha u \partial^\beta \partial_y v \|_2 \| \partial^\alpha v \|_2.$$

By Sobolev embedding, we have

$$\tilde{I}_{21} \leq \sum_{m=6}^{\infty} \frac{\rho(t)^{2(m-3)}}{(m-3)!} \sum_{|\alpha|=m} \sum_{|\beta|=j, \beta \leq \alpha} C_\alpha^\beta \| \partial^\alpha u \|_2 \| \partial_y \partial^\beta u \|_2 \| \partial^\alpha \partial_y v \|_2 \| \partial^\alpha v \|_2$$

and

$$\tilde{I}_{22} \leq \sum_{m=6}^{\infty} \frac{\rho(t)^{2(m-3)}}{(m-3)!} \sum_{|\alpha|=m} \sum_{|\beta|=j, \beta \leq \alpha} C_\alpha^\beta \| \partial_x \partial^\beta u \|_2 \| \partial_y \partial_x \partial^\beta u \|_2 \| \partial^\alpha \partial_y v \|_2 \| \partial^\alpha v \|_2.$$
By Sobolev embedding and Hardy inequality, we have

$$\tilde{I}_{22} \leq C_0(\delta) \sum_{m=5}^{\infty} \frac{\rho(t)^{(m-3)}}{((m-3)!)^{\frac{1}{2}}} \sum_{|\alpha|=m+j=\lfloor \frac{m}{2} \rfloor+1}^{m-2} \sum_{|\beta|=j, \beta \leq \alpha} C_0^\beta(\|\partial^\beta u\|_2 + \|\partial_y \partial^\beta u\|_2) \|\partial^\alpha v\|_2 \times \left( \|\psi \partial^\alpha \partial_y v\|_2^\frac{3}{2} \|\partial_y (\psi \partial^\alpha \partial_y v)\|_2^\frac{1}{2} + \|\partial_x (\psi \partial^\alpha \partial_y v)\|_2^\frac{1}{2} \|\partial_x \partial_y (\psi \partial^\alpha \partial_y v)\|_2^\frac{1}{2} \right).$$

We estimate the term \( \|\psi \partial^\alpha \partial_y v\|_2^\frac{3}{2} \|\partial_y (\psi \partial^\alpha \partial_y v)\|_2^\frac{1}{2} \). By similar arguments as in (3.4), it can be bounded by

$$\sum_{m=5}^{\infty} \frac{\rho(t)^{(m-3)}}{((m-3)!)^{\frac{1}{2}}} \sum_{j=\lfloor \frac{m}{2} \rfloor+1}^{m-2} C_0^j \left( \sum_{|\alpha|=m+j} \|\partial^\alpha v\|_2 \|\partial_y \partial^\alpha v\|_2 \right)^\frac{1}{2} \left( \sum_{|\alpha|=j} \|\partial^\alpha u\|_2 + \|\partial_y \partial^\alpha u\|_2 \right)^\frac{1}{2} \|v\|_{m,3}$$

$$\leq C_0(\delta) \sum_{m=6}^{\infty} \frac{\rho(t)^{m-j-2}}{((m-j-2)!)^{\frac{1}{2}}} \left( \sum_{|\alpha|=m-j+1} \|\partial_y \partial^\alpha v\|_2 \right)^\frac{1}{2} \frac{\rho(t)^{j-3}}{((j-3)!)^{\frac{1}{2}}} \left( \sum_{|\alpha|=j} \|\partial^\alpha u\|_2 + \|\partial_y \partial^\alpha u\|_2 \right)^\frac{1}{2} \|v\|_{m-1,3} \|v\|_{m,3}$$

$$\leq C_0(\delta) \|v\|_X^\frac{3}{4} \|\partial_y v\|_X^\frac{3}{4} \|(u, \partial_y u)\|_X^3,$$

where we used (4.8) and

$$\frac{C_0^j((m-j-2)!)^{\frac{1}{2}}((j-3)!)^{\frac{1}{2}}}{((m-j)!)^{\frac{1}{2}}} \leq C_0(\delta)^{j-1-\frac{1}{2}} \frac{1}{(m-j)^2}$$

for \( j \in \{\lfloor \frac{m}{2} \rfloor + 1, \ldots, m-2\} \).

The other terms can also be bounded by

$$C_0(\delta) \|v\|_X^\frac{3}{4} \|\partial_y v\|_X^\frac{3}{4} \|(u, \partial_y u)\|_X^3.$$

This shows that

$$\tilde{I}_{22} \leq C_0(\delta) \|v\|_X^\frac{3}{4} \|\partial_y v\|_X^\frac{3}{4} \|(u, \partial_y u)\|_X^3.$$ 

Collecting \( \tilde{I}_{21} \) and \( \tilde{I}_{22} \) together, we arrive at

$$\hat{I}_2 \leq C_0 \|u\|_X^\frac{3}{4} \|\partial_y u\|_X^\frac{3}{4} \|\partial_y v\|_X^\frac{3}{2} \|v\|_X^\frac{3}{2} + C_0(\delta) \|v\|_X^\frac{3}{4} \|\partial_y v\|_X^\frac{3}{4} \|(u, \partial_y u)\|_X^3.$$
**Step 3. Estimate of \( \tilde{I}_3 \).** We first decompose \( \tilde{I}_3 = \tilde{I}_{31} + \tilde{I}_{32} \) according to the value of \(|\beta|\) as in Lemma 3.1. By similar computations as in (3.1) and Hardy inequality, we get

\[
\tilde{I}_{31} \leq C_0(\delta) \sum_{m=4}^{\infty} \frac{\rho(t)^{2(m-3)}}{((m-3)!)^\frac{1}{2}} \sum_{|\alpha|=m} C_\alpha^\beta (\|\partial^\beta u\|_2 + \|\partial_y \partial^\beta u\|_2) \|\psi \partial^{\alpha-\beta} \partial_y v\|_\infty \|\partial^\alpha v\|_2
\]

\[
+ C_0 \sum_{|\alpha|=3, |\beta|=2} \|\partial^\beta u \partial^{\alpha-\beta} \partial_y v\|_2 \|\partial^\alpha v\|_2
\]

\[
\leq C_0(\delta) \|\partial^2 v\|_\infty \sum_{m=4}^{\infty} \frac{\rho(t)^{2(m-3)}}{((m-3)!)^\frac{1}{2}} \left( \sum_{|\alpha|=m} \|\partial^\alpha v\|_2 \right)^\frac{1}{2} \left( \sum_{|\beta|=m-1} (\|\partial^\beta u\|_2 + \|\partial_y \partial^\beta u\|_2^2) \right)^\frac{1}{2}
\]

\[
+ C_0(\delta) \|\partial^2 v\|_\infty \|v\|_{L^3} \sum_{|\beta|=2} (\|\partial^\beta u\|_2 + \|\partial_y \partial^\beta u\|_2)
\]

\[
\leq C_0(\delta) \|v\|_{L^3} \|\partial_y v\|_{L^3} \|u, \partial_y u\|_{L^3} \|v\|_{L^3}.
\]

Similarly, for the term \( \tilde{I}_{32} \), there holds

\[
\tilde{I}_{32} \leq C_0(\delta) \sum_{m=4}^{\infty} \frac{\rho(t)^{2(m-3)}}{((m-3)!)^\frac{1}{2}} \sum_{|\alpha|=m} \sum_{|\beta|=m, \beta \leq \alpha} C_\alpha^\beta (\|\partial^\beta u\|_2 + \|\partial_y \partial^\beta u\|_2) \|\psi \partial^{\alpha-\beta} \partial_y v\|_\infty \|\partial^\alpha v\|_2
\]

\[
+ C_0 \sum_{|\alpha|=3} \|\partial^\alpha u \partial \partial_y v\|_2 \|\partial^\alpha v\|_2
\]

\[
\leq C_0(\delta) \|Z v\|_\infty \sum_{m=4}^{\infty} \frac{\rho(t)^{2(m-3)}}{((m-3)!)^\frac{1}{2}} \left( \sum_{|\alpha|=m} (\|\partial^\alpha u\|_2 + \|\partial_y \partial^\alpha u\|_2^2) \right)^\frac{1}{2} \left( \sum_{|\beta|=m} \|\partial^\beta v\|_2^2 \right)^\frac{1}{2}
\]

\[
+ C_0 \sum_{|\alpha|=3} \|\partial^\alpha u \psi\|_2 \|Z v\|_\infty \|\partial^\alpha v\|_2
\]

\[
\leq C_0(\delta) \|v\|_{L^3} \|\partial_y v\|_{L^3} \|u, \partial_y u\|_{L^3} \|v\|_{L^3}.
\]

Finally, collecting the \( \tilde{I}_{31} \) and \( \tilde{I}_{32} \) together, we obtain

\[
\tilde{I}_3 \leq C_0(\delta) \|v\|_{L^3} \|\partial_y v\|_{L^3} \|u, \partial_y u\|_{L^3} \|v\|_{L^3}.
\]

Collecting the estimates in Step 1-Step 3, we complete the proof of the first inequality.

(b) We estimate the second inequality which is used to deal with the linear term like \( v_\alpha \partial_y u \). As in (a), we first have

\[
\sum_{m=3}^{\infty} \frac{\rho(t)^{2(m-3)}}{((m-3)!)^\frac{1}{2}} \sum_{|\alpha|=m} \left| \langle \partial^\alpha (u \partial_y v), \partial^\alpha v \rangle \right|
\]

\[
\leq \sum_{m=3}^{\infty} \frac{\rho(t)^{2(m-3)}}{((m-3)!)^\frac{1}{2}} \sum_{|\alpha|=m} \sum_{0<|\beta| \leq \alpha} C_\alpha^\beta \left| \langle \partial^\beta u \partial^{\alpha-\beta} \partial_y v, \partial^\alpha v \rangle \right|
\]

\[
+ C_0(\delta) \|(u, \partial_y u)\|_{L^3} \|(\partial_y u, \partial_y y u)\|_{L^3} \left( \|v\|_{L^3}^2 + \|v\|_{L^3}^2 \right),
\]

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where we used
\[ |\partial^\alpha \partial_y v - \partial_y \partial^\alpha v| \leq C_0 |\alpha||\partial^\beta \partial_y v|, \quad |\beta| = m - 1, \beta \leq \alpha. \]

Then, recall the definition of \( \tilde{I} \) and \( \tilde{I} \leq \sum_{i=1}^{3} \tilde{I}_i \). We will handle them term by term.

**Step 1: Estimate of \( \tilde{I}_1 \).** As in (a), we know that
\[ \tilde{I}_1 \leq C_0(\delta)\| (u, \partial_y u) \|_{\frac{1}{H^1}} \| (\partial_y u, \partial_y u) \|_{\frac{1}{H^1}} \| v \|_{X^3}^2 \| v \|_{X^3}. \]

**Step 2. Estimate of \( \tilde{I}_2 \).** By Sobolev embedding, we have
\[
\tilde{I}_2 \leq \sum_{m=6}^{\infty} \frac{\rho(t)^{2(m-3)}}{((m-3)!)^{\frac{3}{2}}} \sum_{|\alpha|=m, j=3}^{m-3} \sum_{|\beta|=j, \beta \leq \alpha} C_\alpha^\beta \| \partial^\beta u \|_{\psi} \| \psi \partial^\alpha \partial_y v \|_2 \| \partial^\alpha v \|_2
\]
\[
+ \sum_{m=6}^{\infty} \frac{\rho(t)^{2(m-3)}}{((m-3)!)^{\frac{3}{2}}} \sum_{|\alpha|=m, |\beta|=m-2, \beta \leq \alpha} C_\alpha^\beta \| \partial^\beta \partial^\alpha \partial_y v \|_2 \| \partial^\alpha v \|_2
\]
\[
+ C_0(\delta) \| (u, \partial_y u) \|_{\frac{1}{H^1}} \| (\partial_y u, \partial_y u) \|_{\frac{1}{H^1}} \| v \|_{X^3}^2
\]
\[
= \sum_{i=1}^{2} \tilde{I}_{2i} + C_0(\delta) \| (u, \partial_y u) \|_{\frac{1}{H^1}} \| (\partial_y u, \partial_y u) \|_{\frac{1}{H^1}} \| v \|_{X^3}^2.
\]

Applying the same arguments in (3.1) and discrete Young convolution inequality, we have
\[
\tilde{I}_{21} \leq C_0 \| (u, \partial_y u) \|_{X^4} \| (\partial_y u, \partial_y u) \|_{X^4} \| v \|_{Y^3},
\]
where we used the notation (3.6)
\[
\frac{C_m^j((m-j-2)!)^{\frac{1}{2}}((j-3)!)^{\frac{1}{2}}}{(m-3)!^{\frac{1}{2}}} \leq C_0 a_{m,j},
\]
and
\[
\sum_{m=3}^{\infty} \frac{\rho(t)^{2(m-3)}}{((m-3)!)^{\frac{3}{2}}} \sum_{|\alpha|=m} \| \partial^\beta u \|_{\psi} \| \psi \partial^\alpha \partial_y v \|_{X^4} \| (u, \partial_y u) \|_{X^4} \| (\partial_y u, \partial_y u) \|_{X^4} \| v \|_{Y^3}^2 + \tilde{I}_{22}.
\]

Summing up these estimate, we obtain
\[
\tilde{I}_2 \leq C_0(\delta) \| (u, \partial_y u) \|_{X^4} \| (\partial_y u, \partial_y u) \|_{X^4} \| v \|_{Y^3}^2 + \tilde{I}_{22}.
\]
Step 3. Estimate of $I_3$ and $I_{22}$. We first write $I_3 = I_{31} + I_{32}$ according to the value of $|\beta|$ as above. The argument for $I_{31}$ is similar to $I_{31}$:

$$\sum_{m=4}^{\infty} \frac{\rho(t)^{2(m-3)}}{(m-3)!^{\frac{3}{2}}} \sum_{|\alpha|=m} \sum_{|\beta|=m-1, \beta \leq \alpha} C_{\alpha}^\beta \| \partial^\beta u \|_{L^2_x} \| Z \partial^\alpha \psi \|_{L^2_y L^\infty_z} \| \partial^\alpha v \|_2$$

$$\leq C_0(\delta) \| v \|_{H^3} \sum_{m=4}^{\infty} \frac{\rho(t)^{2(m-3)}}{(m-3)!^{\frac{3}{2}}} \sum_{|\beta|=m-1} \| \partial^\beta u \|_{L^2_x} \| \partial^\alpha v \|_2 \| \psi \|_{L^2_x}$$

$$\leq C_0(\delta) \| v \|_{H^3} \sum_{m=4}^{\infty} \frac{\rho(t)^{2(m-4)}}{(m-4)!^{\frac{3}{2}}} \sum_{|\beta|=m-1} \| \partial^\beta u \|_{L^2_x} \| \partial^\alpha v \|_2 \| \psi \|_{L^2_x}$$

which gives

$$I_{31} \leq C_0(\delta) \| (u, \partial_y u) \|_{X^3} \| (\partial_y u, \partial_{yy} u) \|_{H^3} \| v \|_{H^3} \| v \|_{X^3}.$$

Then a proof similar to that used to treat $I_{32}$ and $I_{22}$ yields that

$$I_{32} \leq C_0(\delta) \| (u, \partial_y u) \|_{X^3} \| (\partial_y u, \partial_{yy} u) \|_{H^3} \| v \|_{H^3} \| v \|_{X^3},$$

$$I_{22} \leq C_0(\delta) \| (u, \partial_y u) \|_{X^3} \| (\partial_y u, \partial_{yy} u) \|_{H^3} \| v \|_{H^3} \| v \|_{X^3}.$$

Collecting $I_{31}, I_{32}, I_{22}$ together, we obtain

$$I_3 + I_{32} + I_{22} \leq C_0(\delta) \| (u, \partial_y u) \|_{X^3} \| (\partial_y u, \partial_{yy} u) \|_{H^3} \| v \|_{H^3} \| v \|_{X^3}.$$

Collecting the estimates in Step 1-Step 3, the proof is completed. \hfill \square

The goal of the following lemma is to deal with the terms in $3)$ like $u \partial_x u^a$.

**Lemma 3.3** For $k = 2, 3$, there holds

$$I_k' = \sum_{m=k}^{\infty} \frac{\rho(t)^{2(m-k)}}{(m-k)!^{\frac{3}{2}}} \sum_{|\alpha|=m} \| \langle \partial^\alpha (uv), \partial^\alpha w \rangle \|_{L^2}$$

$$\leq C_0 \| v \|_{X^{k+1}} \| \partial_y v \|_{X^{k+1}} \| u \|_{X^k} \| w \|_{X^k \cap Y^k}.$$
Thus, we have
\[ I_1' = 6 \rho(t)^{3(m-3)} \sum_{|\alpha|=m, |\beta|=3, \beta \leq \alpha} C_\alpha^\beta \| \partial^\beta u \partial^{\alpha-\beta} v \|_2^2 \| \partial^\alpha w \|_2. \]

**Step 1. Estimate of \( I_1' \).** Again, we denote \( I_1' = \sum_{i=0}^3 I_{1i}' \) according to the value of \(|\beta|\). It suffices to estimate \( I_{13}' \), since the others are similar. By Sobolev embedding, \( I_{13}' \) is clearly bounded by
\[
\sum_{m=6}^{\infty} \frac{\rho(t)^{2(m-3)}}{((m-3)!)^2} \sum_{|\alpha|=m, |\beta|=3, \beta \leq \alpha} C_\alpha^\beta \| \partial^\beta u \partial^{\alpha-\beta} v \|_2 \| \partial^\alpha w \|_2 + C_0 \| v \|_{H^m} \| \partial_y v \|_{H^1} \| u \|_{H^1} \| w \|_{X^3}.
\]

Applying the same technique as (3.21), we get
\[
\sum_{m=6}^{\infty} \frac{\rho(t)^{2(m-3)}}{((m-3)!)^2} \sum_{|\alpha|=m, |\beta|=3, \beta \leq \alpha} C_\alpha^\beta \| \partial^\beta u \partial^{\alpha-\beta} v \|_2 \| \partial^\alpha w \|_2 \\
\leq C_0 \| v \|_{H^1} \sum_{m=6}^{\infty} \frac{\rho(t)^{(m-3)}}{((m-3)!)^2} \left( \sum_{|\alpha|=m-3} \| \partial^\alpha v \|_{L^\infty L^2} \right)^2 \| w \|_{m,3}
\leq C_0 \| v \|_{X^3} \| \partial_y v \|_{X^3} \| u \|_{H^1} \| w \|_{X^3}.
\]

Thus, we have
\[ I_{13}' \leq C_0 \| v \|_{X^3} \| \partial_y v \|_{X^3} \| u \|_{H^1} \| w \|_{X^3}. \]

The same argument implies that
\[ I_{11}' + I_{12}' \leq C_0 \| v \|_{X^3} \| \partial_y v \|_{X^3} \| u \|_{H^1} \| w \|_{X^3}, \]
\[ I_{10}' \leq C_0 \| v \|_{X^3} \| \partial_y v \|_{X^3} \| u \|_{H^1} \| w \|_{X^3}. \]

Therefore, we obtain
\[ I_1' \leq C_0 \| v \|_{X^3} \| \partial_y v \|_{X^3} \| u \|_{H^1} \| w \|_{X^3}. \]

**Step 2. Estimate of \( I_2' \).** By lemma 2.1 and discrete Young inequality, we have
\[
I_2' \leq \sum_{m=7}^{\infty} \frac{\rho(t)^{(m-3)}}{((m-3)!)^2} \sum_{j=4}^{m-3} \sum_{|\beta|=j} C_j^m \left( \sum_{|\alpha|=m-j} \| \partial^\alpha v \|_2^2 \right)^{\frac{1}{2}} \left( \sum_{|\beta|=m-j} \| \partial^\beta v \|_\infty^2 \right)^{\frac{1}{2}} \| w \|_{m,3}
\leq C_0 \sum_{m=7}^{\infty} \sum_{j=4}^{m-3} \frac{C_j^m ((m-j-2)!)^{\frac{1}{2}} ((j-3)!)^{\frac{1}{2}}}{((m-3)!)^{\frac{1}{2}} ((m-j-2)!)^{\frac{1}{2}}} \rho(t)^{(m-j-2)} \left( \sum_{|\beta|=m-j} \| \partial^\beta v \|_\infty^2 \right)^{\frac{1}{2}} \| u \|_{j,3} \| w \|_{m,3}
\leq C_0 \sum_{m=7}^{\infty} \sum_{j=4}^{m-3} a_{m,j} \rho(t)^{(m-j-3)} \left( \sum_{|\beta|=m-j} \| \partial^\beta v \|_\infty^2 \right)^{\frac{1}{2}} \| u \|_{j,3} \sqrt{m-j-2} \| w \|_{m,3}
\leq C_0 \| v \|_{X^3} \| \partial_y v \|_{X^3} \| u \|_{X^3} \| w \|_{Y^3},
\]

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where we used (3.6)

\[
\frac{C_m((m-j-2)!)^{1/2}((j-3)!)^{1/2}}{((m-3)!)^{1/2} \sqrt{m-j-2} \sqrt{m-3}} \leq C_0 \alpha_{m,j}.
\]

**Step 3. Estimate of** $I_3'$. Similarly as above, we write $I_3'$ into three terms:

\[
I_{31}' + I_{32}' + I_{33}' = \sum_{m=3}^{\infty} \frac{\rho(t)2^{(m-3)}}{((m-3)!)^{1/2}} \sum_{|\alpha|=m} \sum_{|\beta|=m-2, \beta \leq \alpha} C_\alpha^2 \|\partial^\beta u \partial^\alpha v\|_2 \|\partial^\alpha w\|_2
\]

\[
+ \sum_{m=3}^{\infty} \frac{\rho(t)2^{(m-3)}}{((m-3)!)^{1/2}} \sum_{|\alpha|=m} \sum_{|\beta|=m-1, \beta \leq \alpha} C_\alpha^2 \|\partial^\beta u \partial^\alpha v\|_2 \|\partial^\alpha w\|_2
\]

\[
+ \sum_{m=3}^{\infty} \frac{\rho(t)2^{(m-3)}}{((m-3)!)^{1/2}} \sum_{|\alpha|=m} \sum_{|\beta|=m, \beta \leq \alpha} C_\alpha^2 \|\partial^\beta u \partial^\alpha v\|_2 \|\partial^\alpha w\|_2
\]

Similar to the estimate of (a) in Lemma 3.1, we know that

\[
I_{32}' \leq C_0 \|v\|_{\frac{2}{m-j}} \|\partial_y v\|_{\frac{2}{m-3}} \|u\|_{X^3} \|w\|_{X^3}.
\]

Similarly, there holds

\[
I_{31}' + I_{33}' \leq C_0 \|v\|_{\frac{2}{m-j}} \|\partial_y v\|_{\frac{2}{m-3}} \|u\|_{X^3} \|w\|_{X^3}.
\]

Finally, we obtain

\[
I_3' \leq C_0 \|v\|_{\frac{2}{m-j}} \|\partial_y v\|_{\frac{2}{m-3}} \|(u, w)\|_{X^3}^2.
\]

Collecting the estimates in Step 1-Step 3, the proof is completed. $\square$

Finally, we deal with the terms in 4). Recall the assumption (H1) and notice that $\varepsilon^{1-\gamma} \partial_y(u^\gamma(t, x, z)) = \varepsilon^{1-\gamma} \partial_x u^\gamma(t, x, z)$ is of $\varepsilon^{1-\gamma}$ order. Then we have the following estimate.

**Lemma 3.4** For $\gamma \leq 1$, there hold

\[
J = \sum_{m=3}^{\infty} \frac{\rho(t)2^{(m-3)}\varepsilon^{-\gamma}}{((m-3)!)^{1/2}} \sum_{|\alpha|=m} \langle \partial^\alpha (\bar{v} \partial_z u^\gamma), \partial^\alpha u \rangle \leq C_0 \|\|u, v\|\|_{X^3, Y^3}^2 + \varepsilon^4,
\]

and

\[
J' = \sum_{m=2}^{\infty} \frac{\rho(t)2^{(m-2)}\varepsilon^{-\gamma}}{((m-2)!)^{1/2}} \sum_{|\alpha|=m} \langle \partial^\alpha (\bar{v} \partial_z u^\gamma), \partial^\alpha \eta \rangle \leq C_0 \|\|\eta, U\|\|_{X^2, Y^2}^2 + \varepsilon^4.
\]

**Proof.** $J$ can be controlled by the sum of $J_1$ and $J_2$, where

\[
J_1 = \sum_{m=3}^{\infty} \frac{\rho(t)2^{(m-3)}\varepsilon^{-\gamma}}{((m-3)!)^{1/2}} \sum_{|\alpha|=m} \sum_{|\beta| \leq 3, \beta \leq \alpha} C_\alpha^2 \langle \partial^\beta \bar{v} \partial^\alpha \partial_z u^\gamma, \partial^\alpha u \rangle.
\]
\[ J_2 = \sum_{m=7}^{\infty} \frac{\rho(t)^{2(m-3)} \varepsilon^{-\gamma}}{((m - 3)!)^{\frac{1}{2}}} \sum_{|\alpha|=m} \sum_{|\beta| \geq 4} C_\alpha^\beta \left| \langle \partial^\beta \tilde{v} \partial^\alpha - \beta \partial_z u^p, \partial^\alpha u \rangle \right|. \]

For \( J_1 \), we have
\[
J_1 = \sum_{m=3}^{6} \frac{\rho(t)^{2(m-3)} \varepsilon^{-\gamma}}{((m - 3)!)^{\frac{1}{2}}} \sum_{|\alpha|=m} \sum_{|\beta| \leq 3, \beta \leq \alpha} C_\alpha^\beta \left| \langle \partial^\beta \tilde{v} \partial^\alpha - \beta \partial_z u^p, \partial^\alpha u \rangle \right|
+ \sum_{m=7}^{\infty} \frac{\rho(t)^{2(m-3)} \varepsilon^{-\gamma}}{((m - 3)!)^{\frac{1}{2}}} \sum_{|\alpha|=m} \sum_{|\beta| \leq 3, \beta \leq \alpha} C_\alpha^\beta \left| \langle \partial^\beta \tilde{v} \partial^\alpha - \beta \partial_z u^p, \partial^\alpha u \rangle \right|.
\]

Note that (2.8) implies that
\[
\| \partial^\beta \tilde{u} \partial^\alpha - \beta \partial_z u^p \|_\infty \leq C_0,
\]
thus the first term of the right hand side can be controlled by
\[
C_0((u, v))_H^2 + \varepsilon^4.
\]

Applying the technique from Lemma 2.1 and (2.8), the second term can be bounded by
\[
C_0((u, v))_H^2 + \varepsilon^2 \sum_{m=7}^{3} \sum_{j=0}^{3} C_m^j \frac{\rho(t)^{m-j-3}}{((m - 3)!)^{\frac{1}{2}}} \left| v \right|_{m, \beta} \left| \partial^\gamma \partial^\beta \partial_z u^p \right|_2^{\frac{1}{2}} \leq C_0((u, v))_H^2 + \varepsilon^2 \| v \|_{X^3}.
\]

This gives
\[
J_1 \leq C_0((u, v))_H^2 + \varepsilon^4.
\]

Now we turn to deal with the term \( J_2 \).
\[
J_2 = \sum_{m=7}^{\infty} \frac{\rho(t)^{2(m-3)} \varepsilon^{-\gamma}}{((m - 3)!)^{\frac{1}{2}}} \sum_{|\alpha|=m} \sum_{j=0}^{m-3} C_m^j \left| \langle \partial^\beta \tilde{v} \partial^\alpha - \beta \partial_z u^p, \partial^\alpha u \rangle \right|
+ \sum_{m=4}^{\infty} \frac{\rho(t)^{2(m-3)} \varepsilon^{-\gamma}}{((m - 3)!)^{\frac{1}{2}}} \sum_{|\alpha|=m} \sum_{j=m-2}^{m} C_m^j \left| \langle \partial^\beta \tilde{v} \partial^\alpha - \beta \partial_z u^p, \partial^\alpha u \rangle \right|
= J_{21} + J_{22}.
\]

For \( J_{21} \), it can be controlled by
\[
\sum_{m=7}^{\infty} \sum_{j=0}^{m-3} C_m^j \left| u \right|_{m, \beta} \rho(t)^{2(m-3)} \varepsilon^{-\gamma} \left| \partial^\gamma \partial^\beta \partial_z u^p \right|_2^{\frac{1}{2}} \left( \sum_{|\alpha|=m} \sum_{|\beta| = j} \left| \partial^\beta \tilde{v} \partial^\alpha - \beta \partial_z u^p \right|_2^{\frac{1}{2}} \right)^{-\frac{1}{2}}
\leq \sum_{m=7}^{\infty} \sum_{j=4}^{m-3} C_m^j \frac{((j - 2)!)^{\frac{1}{2}}}{((m - 3)!)^{\frac{1}{2}}} \sqrt{m - 3} \sqrt{j - 2} \frac{\rho(t)^{2(j-3)!} \rho(t)^{(j-2)!} \sqrt{j - 2}}{((j - 2)!)^{\frac{1}{2}}} \left( \sum_{|\alpha|=j} \left| \partial^\beta \tilde{v} \partial^\alpha \right|_2^{\frac{1}{2}} \right)^{\frac{1}{2}}
\]
\[27\]
Case ii) \( m \) 

Firstly, we deal with the term with where we used Hardy inequality and 

Here \( \chi \) is a cut-off function. Case i) 4

\[
\sum_{|\alpha|=m} \|\partial^\alpha \tilde{v}\|_2^2 = \int_{R^d} \left| \frac{\partial^\alpha \tilde{v}}{y^\gamma} \partial_x \partial^\alpha \partial_x u^p \right|^2 dx dy.
\]

Thus, by discrete convolution inequality and Lemma 2.8, we always have

\[
\frac{C_m^j((j-3)!)^\frac{1}{2}((m-j-3)!)^\frac{1}{2}|(j-2)!|^{\frac{1}{2}} \|\gamma_j(y) y^{1-\gamma}\|_\infty + \|(1-\chi_j(y)) y^{-\gamma}\|_\infty}{(m-3)!} \leq C_0 j^{-2}.
\]

Case ii) \( \left[ \frac{m}{2} \right] \leq j \leq m - 3 \): take \( \chi_j \) such that \( \|\gamma_j(y) y^{1-\gamma}\|_\infty \approx j^{1-\frac{2}{\gamma}} \), thus

\[
\frac{C_m^j((j-3)!)^\frac{1}{2}((m-j-3)!)^\frac{1}{2}|(j-2)!|^{\frac{1}{2}} \|\gamma_j(y) y^{1-\gamma}\|_\infty + \|(1-\chi_j(y)) y^{-\gamma}\|_\infty}{(m-3)!} \leq C_0.
\]

Thus, by discrete convolution inequality and Lemma 2.8, we always have

\[
J_{21} \leq C_0 \|v\|_{Y^3} \|u\|_{Y^3} \sum_{m=3} C_m^j((j-3)!)^\frac{1}{2}((m-j-3)!)^\frac{1}{2}|(j-2)!|^{\frac{1}{2}} \|\gamma_j(y) y^{1-\gamma}\|_\infty + \|(1-\chi_j(y)) y^{-\gamma}\|_\infty \leq C_0 \|u, v\|_{Y^3}^2 + \varepsilon^4.
\]

Next, we deal with the term \( J_{22} \). Recall that

\[
J_{22} = \sum_{m=4} \frac{\rho(t)^{(m-3)}}{(m-3)!} \sum_{|\alpha|=m} \sum_{|\beta|=m-2} \sum_{\beta \leq \alpha} C_\alpha^j \langle \partial^\beta \tilde{v} \partial^\alpha \partial_x u^p, \partial^\alpha u \rangle.
\]

Firstly, we deal with the term with \( |\beta| = m - 2 \). Due to \( \partial_y v = -\partial_x u \), we have

\[
\sum_{m=4} \frac{\rho(t)^{(m-3)}}{(m-3)!} \sum_{|\alpha|=m} \sum_{|\beta|=m-2} C_\alpha^j \langle \partial^\beta \tilde{v} \partial^\alpha \partial_x u^p, \partial^\alpha u \rangle 
\leq \sum_{m=4} \frac{\rho(t)^{(m-3)}}{(m-3)!} \sum_{|\alpha|=m} \sum_{|\beta|=m-2} C_\alpha^j \| \chi_{m-2}(y) y^{1-\gamma} \|_{\infty} \| \partial_y \partial^\beta \tilde{v} \|_2
\]

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where we take $\chi_{m-2}(y)$ to be a smooth function satisfying $0 \leq \chi_{m-2}(y) \leq 1$, 

$$
\chi_{m-2}(y) = \begin{cases} 1, & 0 \leq y \leq m^{-\frac{1}{4}}, \\ 0, & y \geq 2m^{-\frac{1}{4}}, \end{cases}
$$

and use the equality 

$$
\frac{\rho(t)^2(m-3)^{\frac{1}{2}}}{(m-3)!} \sum_{|\alpha| = m} C_\alpha^\beta (\partial^\beta \bar{v} \partial^\alpha \partial_x w^p, \partial^\alpha u)
$$

Secondly, for the case of $|\beta| = m$, there holds 

$$
\sum_{|\alpha| = m} \rho(t)^2(m-3)^{\frac{1}{2}} \sum_{|\alpha| = m} \sum_{|\beta| = m} C_\alpha^\beta (\partial^\beta \bar{v} \partial^\alpha \partial_x w^p, \partial^\alpha u)
$$

where we take $\chi_k(y)$ as above and use the inequality 

$$
\frac{(m - 2)^{\frac{1}{2}}}{\sqrt{m - 3}} \sum_{|\alpha| = m} \sum_{|\beta| = m} C_\alpha^\beta (\partial^\beta \bar{v} \partial^\alpha \partial_x w^p, \partial^\alpha u)
$$

The same argument holds for the case of $|\beta| = m - 1$. Collecting these estimates, we obtain 

$$
J_{22} \leq C_0(\|u, v\|_{X_3 Y^3}^2 + \varepsilon^4).
$$

This together with the estimate of $J_{21}$ gives 

$$
J_2 \leq C_0(\|u, v\|_{X_3 Y^3}^2 + \varepsilon^4).
$$

Finally, collecting the estimates of $J_1$ and $J_2$, we obtain the estimate of $J$. By the same argument, we can obtain the estimate of $J'$. The proof is completed. \qed
4 Velocity estimates in Gevrey class and Sobolev space

4.1 Velocity estimate in Gevrey class

We introduce the following energy quantities:

\[ E_v(t) = \|U\|_{X^3}^2 + \varepsilon^4, \quad E_\omega(t) = \|\omega\|_{X^2}^2. \]

**Proposition 4.1 (Velocity estimate in Gevrey class)** There exist \( \delta > 0 \) and \( \varepsilon_0 > 0 \) such that for any \( \varepsilon \in (0, \varepsilon_0) \), there holds

\[
\frac{1}{2} \frac{d}{dt} (\|U\|_{X^3}^2 - \|U\|_{Y^3}^2) + \lambda (\|U\|_{Y^3}^2 - \|U\|_{X^3}^2) + \frac{\varepsilon^2}{2} (\|\nabla U\|_{X^3}^2 - \|\nabla U\|_{Y^3}^2) \\
\leq C_0 E_v(t) \left( (E_v(t) + E_v(t)) \right) \left( \|U\|_{Y^3}^2 + E_v(t) \right) + C_0 (\|U\|_{X^3}^2 + \|\partial_y U\|_{X^3}^2) + C_0 (1 + \|\nabla U\|_{X^3}^2 + \|\nabla U\|_{Y^3}^2) + C_0 (\|U\|_{X^3}^2 + \|\nabla U\|_{Y^3}^2 + 1) (\|U\|_{X^3}^2 + \varepsilon^4).
\]

Proposition 4.1 is a direct result of Lemma 4.2, Lemma 4.3 and Lemma 4.4

**Lemma 4.2** There exist \( \delta_0, \varepsilon_0 > 0 \) such that for any \( \delta \in (0, \delta_0) \) and \( \varepsilon \in (0, \varepsilon_0) \), there holds

\[
\frac{1}{2} \frac{d}{dt} (\|U\|_{X^3}^2 - \|U\|_{Y^3}^2) + \lambda (\|U\|_{Y^3}^2 - \|U\|_{X^3}^2) + \frac{\varepsilon^2}{2} (\|\nabla U\|_{X^3}^2 - \|\nabla U\|_{Y^3}^2) \\
\leq C_0 E_v(t) \left( (E_v(t) + E_v(t)) \right) \left( \|U\|_{Y^3}^2 + E_v(t) \right) + C_0 (\|U\|_{X^3}^2 + \|\partial_y U\|_{X^3}^2) + C_0 (1 + \|\nabla U\|_{X^3}^2 + \|\nabla U\|_{Y^3}^2) + C_0 (\|U\|_{X^3}^2 + \|\nabla U\|_{Y^3}^2 + 1) (\|U\|_{X^3}^2 + \varepsilon^4).
\]

**Proof.** Acting \( \partial^\alpha \) on both sides of (2.4), taking \( L^2 \) inner product with \( \frac{\rho(t)^{2(|\alpha|-3)}}{((|\alpha|-3)!)^{\frac{3}{2}}} \partial^\alpha U \), and summing up about \( |\alpha| = m \) for \( m = 3, 4, \cdots \), we have

\[
\frac{1}{2} \frac{d}{dt} (\|U\|_{X^3}^2 - \|U\|_{Y^3}^2) + \lambda (\|U\|_{Y^3}^2 - \|U\|_{X^3}^2) + \varepsilon^2 \sum_{m=3}^{\infty} \frac{\rho(t)^{2(m-3)}}{((m-3)!)^{\frac{3}{2}}} \sum_{|\alpha|=m} \langle \partial^\alpha \triangle U, \partial^\alpha U \rangle \\
\leq \sum_{m=3}^{\infty} \frac{\rho(t)^{2(m-3)}}{((m-3)!)^{\frac{3}{2}}} \sum_{|\alpha|=m} \langle \partial^\alpha ((\tilde{U} + \tilde{U}^\alpha) \cdot \nabla U), \partial^\alpha U \rangle \\
+ \sum_{m=3}^{\infty} \frac{\rho(t)^{2(m-3)}}{((m-3)!)^{\frac{3}{2}}} \sum_{|\alpha|=m} \langle \partial^\alpha (\tilde{U} \cdot \nabla U^\alpha), \partial^\alpha U \rangle + \sum_{m=3}^{\infty} \frac{\rho(t)^{2(m-3)}}{((m-3)!)^{\frac{3}{2}}} \sum_{|\alpha|=m} \langle \partial^\alpha (\nabla U), \partial^\alpha U \rangle \\
+ \sum_{m=3}^{\infty} \frac{\rho(t)^{2(m-3)}}{((m-3)!)^{\frac{3}{2}}} \sum_{|\alpha|=m} \langle \partial^\alpha \tilde{R}, \partial^\alpha U \rangle = \sum_{i=1}^{4} K_i.
\]
Step 1. Estimate of $K_1$.

\[
K_1 \leq \left| \sum_{m=3}^{\infty} \frac{\rho(t)^{2(m-3)}}{(m-3)!} \sum_{|\alpha|=m} \langle \partial^\alpha (u \partial_x u + \tilde{v} \partial_y u), \partial^\alpha u \rangle \right| \\
+ \left| \sum_{m=3}^{\infty} \frac{\rho(t)^{2(m-3)}}{(m-3)!} \sum_{|\alpha|=m} \langle \partial^\alpha (u \partial_x v + \tilde{v} \partial_y v), \partial^\alpha v \rangle \right| \\
+ \left| \sum_{m=3}^{\infty} \frac{\rho(t)^{2(m-3)}}{(m-3)!} \sum_{|\alpha|=m} \langle \partial^\alpha (u^a \partial_x u + \tilde{v}^a \partial_y u), \partial^\alpha u \rangle \right| \\
+ \left| \sum_{m=3}^{\infty} \frac{\rho(t)^{2(m-3)}}{(m-3)!} \sum_{|\alpha|=m} \langle \partial^\alpha (u^a \partial_x v + \tilde{v}^a \partial_y v), \partial^\alpha v \rangle \right| = \sum_{i=1}^{4} K_{1i}.
\]

Firstly, by (a) of Lemma 3.1 and (a) of Lemma 3.2 we obtain

\[
K_{11} \leq C_0 E_v(t)^{\frac{1}{2}} (E_w(t) + E_v(t))^{\frac{1}{2}} (\|u\|_{Y^3}^2 + E_v(t)) + C_0 (\|U\|_{\mathcal{P}^4} + \varepsilon^2) \|u\|_{X^3 \cap Y^3}^2 \\
+ C_0 \|u\|_{\mathcal{P}^4} \|v\|_{X^3} (\varepsilon^2 + \|\partial_y v\|_{X^3}) + (\|\partial_y v\|_{X^3} + \varepsilon) \|\partial_y u\|_{X^3}.
\]

\[
K_{12} \leq C_0 E_v(t)^{\frac{1}{2}} (E_w(t) + E_v(t))^{\frac{1}{2}} (\|u\|_{Y^3}^2 + E_v(t)) + C_0 (\|U\|_{\mathcal{P}^4} + \varepsilon^2) \|v\|_{X^3 \cap Y^3}^2 \\
+ C_0 \|v\|_{\mathcal{P}^4} \|\partial_y v\|_{X^3} (\varepsilon^2 + \|\partial_y v\|_{X^3}),
\]

where we used $\partial_y u = \omega + \partial_x v$ and $\partial_y v = -\partial_x u$.

Secondly, by (b) of Lemma 3.1, (b) of Lemma 3.2 and (2.8), we get

\[
K_{13} + K_{14} \leq C_0 \|U\|_{X^3 \cap Y^3}^2.
\]

It follows from the estimates on $K_{11} - K_{14}$ that

\[
K_1 \leq C_0 \|U\|_{X^3 \cap Y^3}^2 \left( \varepsilon^2 + \|\partial_y U\|_{X^3} + C_0 \|U\|_{X^3 \cap Y^3} \right) + C_0 \|U\|_{X^3 \cap Y^3}^2 \left[ 1 + \|\partial_y U\|_{\mathcal{P}^4} \right].
\]

Step 2. Estimate of $K_2$. By Lemma 3.3 and (2.8), we have

\[
\sum_{m=3}^{\infty} \frac{\rho(t)^{2(m-3)}}{(m-3)!} \sum_{|\alpha|=m} \langle \partial^\alpha (u \partial_x u^a), \partial^\alpha u \rangle \leq C_0 \|U\|_{X^3 \cap Y^3},
\]

\[
\sum_{m=3}^{\infty} \frac{\rho(t)^{2(m-3)}}{(m-3)!} \sum_{|\alpha|=m} \langle \partial^\alpha (u \partial_x v^a), \partial^\alpha v \rangle \leq C_0 \|U\|_{X^3 \cap Y^3},
\]

\[
\sum_{m=3}^{\infty} \frac{\rho(t)^{2(m-3)}}{(m-3)!} \sum_{|\alpha|=m} \langle \partial^\alpha (\tilde{v} \partial_y u^a), \partial^\alpha u \rangle \leq C_0 \left( \varepsilon^4 + \|U\|_{X^3 \cap Y^3}^2 \right),
\]

\[
\sum_{m=3}^{\infty} \frac{\rho(t)^{2(m-3)}}{(m-3)!} \sum_{|\alpha|=m} \langle \partial^\alpha (\tilde{v} \partial_y u^a), \partial^\alpha u \rangle \leq C_0 \left( \varepsilon^4 + \|U\|_{X^3 \cap Y^3}^2 \right).
\]
Thanks to \( u^\alpha = u^\varepsilon + \varepsilon^{1-\gamma}u^p \), it suffices to estimate the remaining term

\[
\sum_{m=3}^{\infty} \frac{\rho(t)^{2(m-3)}}{(m-3)!} \left| \langle \partial^\alpha ((v + \varepsilon^2 f(t,x)e^{-y})\partial_y u^p), \partial^\alpha u \rangle \right|,
\]

which is bounded from Lemma \ref{lem:boundedness} by

\[
C_0 \left( \varepsilon^4 + \|U\|^2_{X^3 \cap Y^3} \right).
\]

Consequently, we obtain

\[
K_2 \leq C_0 \left( \varepsilon^4 + \|U\|^2_{X^3 \cap Y^3} \right).
\]

**Step 3. Estimate of \( K_3 \).** Note that the following fact holds

\[
\sum_{|\alpha|=m} \left| [\partial^\alpha, \partial_y] f \right| \leq C_0 \delta m \sum_{|\alpha|=m-1} |\partial^\alpha \partial_y f|,
\]

and we have

\[
K_3 = \left| \sum_{m=3}^{\infty} \frac{\rho(t)^{2(m-3)}}{(m-3)!} \left( \sum_{|\alpha|=m} \langle \partial^\alpha (\nabla p), \partial^\alpha U \rangle \right) \right|
\]

\[
\leq \left| \sum_{m=3}^{\infty} \frac{\rho(t)^{2(m-3)}}{(m-3)!} \left( \sum_{|\alpha|=m} \langle \partial^\alpha p, \partial^\alpha (\nabla U) \rangle \right) + \sum_{m=3}^{\infty} \frac{\rho(t)^{2(m-3)}}{(m-3)!} \left( \sum_{|\alpha|=m} \langle [\partial^\alpha, \partial_y] p, \partial^\alpha v \rangle \right) \right|
\]

\[
+ \left| \sum_{m=3}^{\infty} \frac{\rho(t)^{2(m-3)}}{(m-3)!} \left( \sum_{|\alpha|=m} \langle \partial^\alpha p, [\partial^\alpha, \partial_y] v \rangle \right) \right| + \left| \sum_{m=3}^{\infty} \frac{\rho(t)^{2(m-3)}}{(m-3)!} \int_{\partial R^+} \partial_x^{m-1} p \partial_x^{m+1} v dx \right|
\]

\[
\leq C_0 \delta \sum_{m=3}^{\infty} \frac{\rho(t)^{2(m-3)} m}{(m-3)!} \sum_{|\alpha|=m-1} \left\| \partial^\alpha (\nabla p) \right\|_2^2 + C_0(\delta) \|v\|^2_{Y^3} + C_0(\delta) \varepsilon^4,
\]

where we estimate the boundary term as follows: due to \( v|_{y=0} = -\varepsilon^2 f(t,x) \) and the assumption \( (2.10) \), we get

\[
\varepsilon^2 \sum_{m=3}^{\infty} \frac{\rho(t)^{2(m-3)}}{(m-3)!} \int_{\partial R^+} \partial_x^{m-1} p \partial_x^{m+1} f dx \n\]

\[
= \varepsilon^2 \sum_{m=3}^{\infty} \frac{\rho(t)^{2(m-3)}}{(m-3)!} \int_{\partial R^+} \partial_y (\partial_x^{m-1} p \partial_x^{m+1} f e^{-y}) dx dy \n\]

\[
\leq C_0 \delta \sum_{m=3}^{\infty} \frac{\rho(t)^{2(m-3)} m}{(m-3)!} \sum_{|\alpha|=m-1} \left\| \partial^\alpha (\nabla p) \right\|_2^2 + C_0(\delta) \varepsilon^4.
\]

**Step 4. Estimate of \( K_4 \).** By \( (2.9) \), we obtain

\[
K_4 = \left| \sum_{m=3}^{\infty} \frac{\rho(t)^{2(m-3)}}{(m-3)!} \left( \sum_{|\alpha|=m} \langle \partial^\alpha \tilde{R}, \partial^\alpha U \rangle \right) \right| \leq C_0 \varepsilon^4 + C_0 \|U\|^2_{X^3}.
\]
Step 5. Estimate of dissipative term. Integration by parts gives

\[
E = -\sum_{m=3}^{\infty} \frac{\rho(t)^{2(m-3)}}{((m-3)!)^2} \sum_{|\alpha|=m} \langle \partial^\alpha \Delta U, \partial^\alpha U \rangle \\
\geq \left( \| \nabla U \|_{X^3}^2 - \| \nabla U \|_{L^2}^2 \right) - \left| \sum_{m=3}^{\infty} \frac{\rho(t)^{2(m-3)}}{((m-3)!)^2} \sum_{|\alpha|=m} \langle [\partial^\alpha \partial_y U, \partial^\alpha \partial_y U] \rangle \right|
\]

Recalling (4.9) and the boundary conditions of (2.4), we have

\[
E \geq (1 - C_0 \delta) \left( \| \nabla U \|_{X^3}^2 - \| \nabla U \|_{L^2}^2 \right) - \left| \sum_{m=3}^{\infty} \frac{\rho(t)^{2(m-3)}}{((m-3)!)^2} \int_{\partial R^2_+} \partial_x \partial_x^m \partial_x^m u \partial_x^m \left( \beta \varepsilon^{-\gamma} \partial_x^m u + \varepsilon \partial_x^m g_0 \right) dx \right|
\]

It follows from Sobolev inequality and Young’s inequality that

\[
E \geq \left( \frac{2}{3} - C_0 \delta \right) \left( \| \nabla U \|_{X^3}^2 - \| \nabla U \|_{L^2}^2 \right) - C_0 \varepsilon^{-2\gamma} \| u \|_{X^3}^2 - C_0 \varepsilon^4 - C_0 \| u \|_{X^3}^2
\]

Thus, by taking \( \delta \) small and using (2.10), we obtain

\[
\frac{\varepsilon^2}{2} \left( \| \nabla U \|_{X^3}^2 - \| \nabla U \|_{L^2}^2 \right) \leq -\varepsilon^2 \sum_{m=3}^{\infty} \frac{\rho(t)^{2(m-3)}}{((m-3)!)^2} \sum_{|\alpha|=m} \langle \partial^\alpha \Delta U, \partial^\alpha U \rangle + C_0 \varepsilon^4 + C_0 \| u \|_{X^3}^2.
\]

Finally, the proof is completed by collecting these estimates in Step 1-Step 5.

\[\square\]

4.2 Pressure estimate in Gevrey class and Sobolev space

In order to close the estimates of \((u, v)\) of the last subsection, we need to estimate the pressure in Gevrey class. The proof will show why our assumptions are made. We write

\[
F = u^a \partial_x u + \tilde{v}^a \partial_y u + u \partial_x u^a + \tilde{v} \partial_y u^a + u \partial_x u + \tilde{v} \partial_y u,
G = u^a \partial_x v + \tilde{v}^a \partial_y v + u \partial_x v^a + \tilde{v} \partial_y v^a + u \partial_x v + \tilde{v} \partial_y v.
\]

Thus, the system (2.4) can be expressed as follows

\[
\begin{align*}
\partial_t u - \varepsilon^2 \Delta u + F + \partial_x p &= R_1, \\
\partial_t v - \varepsilon^2 \Delta v + G + \partial_y p &= R_2, \\
\partial_x u + \partial_y v &= 0, \\
(\partial_y u, v)(t, x, 0) &= (\beta \varepsilon^{-\gamma} u + \varepsilon g_0(t, x), -\varepsilon^2 f(t, x)), \\
(u, v)(0, x, y) &= 0.
\end{align*}
\]
Taking the divergence operator on both sides, we obtain the equation of the pressure $p$:

\[
\begin{align*}
-\Delta p &= (\partial_x F + \partial_y G) - (\partial_x R_1 + \partial_y R_2), \\
\partial_y p &= -\beta \varepsilon^{2-\gamma} \partial_x u - \varepsilon^3 \partial_y g_0 + R_2 + \varepsilon^2 \partial_t f - \varepsilon^4 \partial_x f - u^0 \partial_x v \quad \text{on} \quad y = 0,
\end{align*}
\]

where we used $(v + v_0)|_{y=0} = 0$ and

\[
G(t, x, 0) = u^0 \partial_x v(t, x, 0), \quad v(t, x, 0) = -\varepsilon^2 f(t, x).
\]

**Lemma 4.3 (Pressure estimate in Gevrey class)** Under the assumptions (II), we have

\[
E' = \sum_{m=3}^{\infty} \frac{\rho(t)^{2(m-3)}m}{((m-3)!)^{\frac{5}{7}} |\alpha|=m-1} \|\partial^{\alpha} (\nabla p)\|_2^2 \\
\leq C_0 \varepsilon^{2(2-\gamma)} (\|u\|_{Y^3}^2 + \|\partial_y u\|_{Y^2}^2) + C_0 \|\nabla p\|_2 + C_0 (\|U\|_{X^3} \|\partial_y U\|_{X^2} + 1) (\varepsilon^4 + \|U\|_{X^3}^{\gamma} Y^3).
\]

**Proof.** Acting $\partial^\alpha$ on both sides of (2.10), taking $L^2$ inner product with $\frac{\rho(t)^{2(|\alpha|-3)}m}{((|\alpha|-3)!)^{\frac{5}{7}}} \partial^\alpha p$, and summing over $|\alpha| \geq 3$, we have

\[
\sum_{m=3}^{\infty} \frac{\rho(t)^{2(m-3)}m}{((m-3)!)^{\frac{5}{7}} |\alpha|=m-1} \langle -\partial^\alpha \Delta p, \partial^\alpha p \rangle \\
= \sum_{m=3}^{\infty} \frac{\rho(t)^{2(m-3)}m}{((m-3)!)^{\frac{5}{7}} |\alpha|=m-1} \langle \partial^\alpha (\partial_x F + \partial_y G) - \partial^\alpha (\partial_x R_1 + \partial_y R_2), \partial^\alpha p \rangle
\]

Especially, similar to the commutator estimate of dissipative term in Lemma 4.2 and by (4.11), the left side is bigger than

\[
\left( \frac{2}{3} - C_0 \partial \right) E' - C_0 \|\nabla p\|_2^2 + \sum_{m=3}^{\infty} \frac{\rho(t)^{2(m-3)}m}{((m-3)!)^{\frac{5}{7}} |\alpha|=m-1} \int_{\partial R^2_x} \partial^\alpha p \partial^\alpha p \right) \\
- \sum_{m=3}^{\infty} \frac{\rho(t)^{2(m-3)}m}{((m-3)!)^{\frac{5}{7}} |\alpha|=m-1} \int_{\partial R^2_x} \partial^\alpha p \partial^\alpha \partial^\alpha p \\
- \sum_{m=3}^{\infty} \frac{\rho(t)^{2(m-3)}m}{((m-3)!)^{\frac{5}{7}} |\alpha|=m-1} \int_{\partial R^2_x} \partial^\alpha p \partial^\alpha \partial^\alpha p - C_0 \varepsilon^4,
\]

where we used integration by parts and (2.10). Moreover, integration by parts, (4.9) and (2.9) yield that

\[
\sum_{m=3}^{\infty} \frac{\rho(t)^{2(m-3)}m}{((m-3)!)^{\frac{5}{7}} |\alpha|=m-1} \langle \partial^\alpha (\partial_x F + \partial_y G) - \partial^\alpha (\partial_x R_1 + \partial_y R_2), \partial^\alpha p \rangle \\
\leq \frac{1}{10} E' + C_0 \varepsilon^4 + C_0 \sum_{m=3}^{\infty} \frac{\rho(t)^{2(m-3)}m}{((m-3)!)^{\frac{5}{7}} |\alpha|=m-1} \|\partial^\alpha (F, G)\|_2^2 + C_0 \|\nabla p\|_2^2 \\
- \sum_{m=3}^{\infty} \frac{\rho(t)^{2(m-3)}m}{((m-3)!)^{\frac{5}{7}} |\alpha|=m-1} \int_{\partial R^2_x} \partial^\alpha p \partial^\alpha \partial^\alpha p + \sum_{m=3}^{\infty} \frac{\rho(t)^{2(m-3)}m}{((m-3)!)^{\frac{5}{7}} |\alpha|=m-1} \int_{\partial R^2_x} \partial^\alpha p \partial^\alpha \partial^\alpha p
\]

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Canceling the boundary term and by (2.9), we arrive at
\begin{align*}
\left( \frac{1}{2} - C_0 \delta \right) \sum_{m=3}^{\infty} \frac{\rho(t)^{2(m-3)m}}{(m-3)!} \sum_{|\alpha|=m-1} \| \partial^\alpha (\nabla p) \|_2^2 \\
\leq C_0 \varepsilon^4 + C_0 \sum_{m=3}^{\infty} \frac{\rho(t)^{2(m-3)m}}{(m-3)!} \sum_{|\alpha|=m-1} \| \partial^\alpha (F, G) \|_2^2 + C_0 \| \nabla p \|_{H^1}^2 \\
+ \sum_{m=3}^{\infty} \frac{\rho(t)^{2(m-3)m}}{(m-3)!} \int_{\partial R^2_+} \partial_x^{m-1}(\beta \varepsilon^{2-\gamma} \partial_x u) \partial_x^{m-1} p dx \right]. 
\end{align*}
(4.12)

Recalling (4.10), we have
\begin{align*}
\sum_{m=3}^{\infty} \frac{\rho(t)^{2(m-3)m}}{(m-3)!} \sum_{|\alpha|=m-1} \| \partial^\alpha (F, G) \|_2^2 & \leq C_0 \left( \| U \|_{X^3} \| \partial_y U \|_{X^2} + 1 \right) (\varepsilon^4 + \| U \|_{X^3 \cap Y^3})^2. 
\end{align*}
(4.13)

In fact, it is sufficient to handle the terms \( \tilde{v}^\alpha \partial_y u, \tilde{v} \partial_y u^\alpha \) and \( \tilde{v} \partial_y u \), and other terms are similar.

Firstly, by the assumption (H1), (2.8) and \( \tilde{v}_0 |_{y=0} = 0 \), and following the proof of Lemma 3.8, we have
\begin{align*}
\sum_{m=3}^{\infty} \frac{\rho(t)^{2(m-3)m}}{(m-3)!} \sum_{|\alpha|=m-1} \| \partial^\alpha (\tilde{v}^\alpha \partial_y u) \|_2^2 \\
& \leq \sum_{m=3}^{\infty} \frac{\rho(t)^{2(m-3)m}}{(m-3)!} \sum_{|\alpha|=m-1, \beta \leq \alpha} \left( C_0 \right)^2 \left( \| \psi \partial^\alpha \partial_y u \|_2 \right)^2 \leq C_0 \| U \|_{X^3 \cap Y^3}^2.
\end{align*}

Next, by the assumption (H1), (2.8), Hardy inequality and \( \tilde{v} |_{y=0} = 0 \), similar argument as in Lemma 3.8 gives
\begin{align*}
\sum_{m=3}^{\infty} \frac{\rho(t)^{2(m-3)m}}{(m-3)!} \sum_{|\alpha|=m-1} \| \partial^\alpha (\tilde{v} \partial_y u^\alpha) \|_2^2 \\
& \leq \sum_{m=3}^{\infty} \frac{\rho(t)^{2(m-3)m}}{(m-3)!} \sum_{|\alpha|=m-1, \beta \leq \alpha} \left( C_0 \right)^2 \left( \| \psi \partial^\alpha \partial_y u^\alpha \|_2 \right)^2 \leq C_0 \left( \varepsilon^4 + \| U \|_{X^3 \cap Y^3}^2 \right).
\end{align*}

Thirdly, by Sobolev inequality, Hardy inequality and \( \tilde{v} |_{y=0} = 0 \), similar argument as in Lemma 3.2 gives
\begin{align*}
\sum_{m=3}^{\infty} \frac{\rho(t)^{2(m-3)m}}{(m-3)!} \sum_{|\alpha|=m-1} \| \partial^\alpha (\tilde{v} \partial_y u) \|_2^2 \\
& \leq \sum_{m=3}^{\infty} \frac{\rho(t)^{2(m-3)m}}{(m-3)!} \sum_{|\alpha|=m-1, \beta \leq \alpha} \left( C_0 \right)^2 \left( \| \psi \partial^\alpha \partial_y u \|_2 \right)^2 \leq C_0 \| U \|_{X^3}^2 \| \partial_y U \|_{X^2}^2 \| U \|_{X^3 \cap Y^3}^2.
\end{align*}

Finally, there holds
\begin{align*}
\sum_{m=3}^{\infty} \frac{\rho(t)^{2(m-3)m}}{(m-3)!} \left| \int_{\partial R^2_+} \partial_x^{m-1}(\beta \varepsilon^{2-\gamma} \partial_x u) \partial_x^{m-1} p dx \right|
\end{align*}

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\[\leq \sum_{m=3}^{\infty} \rho(t)^{2(m-3)}m \left| \int_{R^2_+} \left( \beta \varepsilon^{2-\gamma} \partial_x^m \partial_y^{m-1}\partial_y p + \beta \varepsilon^{2-\gamma} \partial_x^m \partial_y \partial_y u \partial_y^{m-1} p \right) dxdy \right| \]
\[\leq \frac{1}{10} \sum_{m=3}^{\infty} \rho(t)^{2(m-3)}m \sum_{|a|=m-1} \|\partial^a \nabla p\|^2 \leq C_0 \varepsilon^{2(2-\gamma)} \left( \|u\|^2_H + \|\partial_y u\|^2_{H^2} \right). \quad (4.14)\]

Putting (4.13) and (4.14) into (4.12) and taking \( \delta \) small, the proof is completed. \( \square \)

Next we deal with the pressure estimate in \( L^2 \), which is similar to Lemma 4.3 but more delicate. Especially, it is necessary to use the condition \( f(t, x) = \partial_x f \) of the Assumption (H3).

**Lemma 4.4 (Pressure estimate in \( L^2 \))** Under the assumptions (H), we have

\[\| \nabla p \|_2^2 \leq C_0 \left( \|U\|^2_{H^4} + \|\nabla U\|^2_{H^4} + 1 \right) \left( \|U\|^2_{H^4} + \varepsilon^4 \right) + C_0 \varepsilon^{2(2-\gamma)} \| \nabla U \|^2_{H^4}. \]

**Proof:** As in Lemma 4.3, taking \( L^2 \) inner product with \( p \) in (4.11), we have

\[ -\langle \nabla p, p \rangle = \langle (\partial_x F + \partial_y G) - (\partial_x R_1 + \partial_y R_2), p \rangle \quad (4.15) \]

Integrating by parts and using the boundary value of \( \partial_y p \) in (4.11) yield that

\[ -\langle \nabla p, p \rangle = \| \nabla p \|_2^2 + \int_{\partial R^2_+} (-\beta \varepsilon^{2-\gamma} \partial_x u - \varepsilon^3 \partial_x g_0 + R_2 + \varepsilon^2 \partial_x f - \varepsilon^4 \partial_x f - u^a \partial_x v) p dx. \]

By the commutator estimate (4.9) and (2.10) of (H3), the left side of (4.15) is bigger than

\[ \left( \frac{2}{3} - C_0 \delta \right) \| \nabla p \|_2^2 + \int_{\partial R^2_+} p R_2 dx - \int_{\partial R^2_+} p (u^a \partial_x v) dx - C_0 \varepsilon^4 - C_0 \beta^2 \varepsilon^{2(2-\gamma)} \| \nabla U \|_2^2, \quad (4.16) \]

where we used that \( f(t, x) = \partial_x f \) and

\[ \int_{\partial R^2_+} \varepsilon^2 \partial_y f p dx = \int_{\partial R^2_+} \varepsilon^2 \partial_x \partial_x f p dx = - \int_{R^2_+} \varepsilon^2 \partial_y \partial_x f \partial_y (e^{-y} p) dxdy \]

\[ \leq C_0 \varepsilon^2 \| \partial_y \partial_x f \|_{L_2} \| \partial_y p \|_{L_2} + \| \partial_y f \|_{L_2} \| \partial_x p \|_2 \leq C_0 \varepsilon^4 + \frac{1}{10} \| \nabla p \|^2_2. \]

Recall that \( G(t, x, 0) = u^a \partial_x v(t, x, 0) \). By (2.9) and integration by parts, the right hand side of (4.15) can be controlled by

\[ \frac{1}{2} \| \nabla p \|^2_2 + C_0 \| (F, G) \|_2^2 + C_0 \varepsilon^4 - \int_{\partial R^2_+} p(u^a \partial_x v) dx + \int_{\partial R^2_+} p R_2 dx. \quad (4.17) \]

Thus, canceling the boundary term between (4.16) and (4.17), taking \( \delta \) small, we arrive at

\[ \| \nabla p \|^2_2 \leq C_0 \| (F, G) \|^2_2 + C_0 \varepsilon^4 + C_0 \varepsilon^{2(2-\gamma)} \| \nabla U \|^2_2. \]

At last, we are aimed to estimate the term \( \| (F, G) \|^2_2 \). Recalling (4.10), by (2.8), \( \tilde{v} \big|_{y=0} = 0 \) and \( (u^a - \varepsilon^2 f(t, x) e^{-y}) \big|_{y=0} = 0 \), we can obtain the following estimate directly

\[ \| (F, G) \|^2_2 \leq C_0 \left( \|U\|^2_H + \|\nabla U\|^2_H + 1 \right) \left( \|U\|^2_{H^4} + \varepsilon^4 \right). \]
For example, we only deal with the terms $\tilde{v}^a \partial_y u$, $\tilde{v} \partial_y u^a$ and $\tilde{v} \partial_y u$.

Firstly, by (H1) and (2.8), we have

$$\|\tilde{v}^a \partial_y u\|_2^2 = \left\|\frac{\tilde{v}}{\psi} Z u\right\|_2^2 \leq C_0 \|u\|_{H^l}.$$  

Moreover, by Hardy inequality and divergence-free property, we deduce that

$$\|\tilde{v} \partial_y u^a\|_2^2 = \left\|\frac{\tilde{v}}{\psi} Z u^a\right\|_2^2 \leq C_0 \left(\|U\|_{H^l} + \varepsilon^2\right).$$

Finally, by Sobolev embedding and divergence-free condition, we have

$$\|\tilde{v} \partial_y u\|_2^2 = \left\|\frac{\tilde{v}}{\psi} Z u\right\|_2^2 \leq C_0 \left(\left\|\partial_y \tilde{v}\right\|_{\infty} + \|\tilde{v}\|_{\infty}\right) \|u\|_{H^l} \leq C_0 \left(\|\nabla U\|_{H^l} + \|U\|_{H^l} + \varepsilon^2\right) \left(\|U\|_{H^l} + \varepsilon^2\right).$$

Hence, the proof is completed. □

4.3 Velocity estimate in Sobolev space

In this subsection, our goal is to prove the following proposition.

**Proposition 4.5 (Velocity estimate in $L^2$)** There exist $\delta > 0$ and $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$, we have

$$\frac{1}{2} \frac{d}{dt} \|U\|_2^2 + \frac{\varepsilon^2}{2} \|\nabla U\|_2^2 \leq C_0(\delta) \left(\|U\|_{H^l}^2 + \|\nabla U\|_{H^l}^2 + 1\right) \left(\|U\|_{H^l}^2 + \varepsilon^4\right) + C_0 \|U\|_{H^l}^2.$$

The proof of Proposition 4.5 is an immediate corollary of the following Lemma 4.6 and Lemma 4.3.

**Lemma 4.6** There exist $\delta_0, \varepsilon_0 > 0$ such that for any $\delta \in (0, \delta_0)$ and $\varepsilon \in (0, \varepsilon_0)$, we have

$$\frac{1}{2} \frac{d}{dt} \|U\|_2^2 + \frac{\varepsilon^2}{2} \|\nabla U\|_2^2 \leq C_0(\delta) \left(\|U\|_{H^l}^2 + \varepsilon^4\right) + C_0 \|U\|_{H^l}^2 + C_0 \varepsilon^2 \|\nabla p\|_2^2.$$

**Proof:** Taking $L^2(R^2_+)$ inner product with $U$ in (2.4), we arrive at

$$\frac{1}{2} \frac{d}{dt} \|U\|_2^2 \leq \langle u^a \partial_x U + \tilde{v}^a \partial_y U, U \rangle + \langle u \partial_x U^a + \tilde{v} \partial_y U^a, U \rangle + \langle u \partial_x U + \tilde{v} \partial_y U, U \rangle + \langle \nabla p, U \rangle + \langle \tilde{R}, U \rangle = \sum_{i=1}^5 N_i.$$

**Step 1. Estimate of $N_1$.** Note that $\tilde{v}^a|_{y=0} = 0$, it follows from integration by parts, divergence-free condition and (2.10) that

$$N_1 \leq \int_{R^2_+} \varepsilon^2 f(t, x) e^{-y} |U|^2 dx dy \leq C_0 \varepsilon^2 \|U\|_2^2.$$
Step 2. Estimate of $I_2$. Note the fact $v|_{y=0} = 0$. By (2.8), divergence free condition and Hardy inequality, we get

$$N_2 \leq C_0(\|U\|_2^2 + \varepsilon^4) + \varepsilon^{1-\gamma}\left|\frac{1}{y}\int_0^y \partial_y \tilde{v} dy \partial_y z \tilde{u}_p, u\right| \leq C_0(\|U\|_H^2 + \varepsilon^4),$$

where we used $\|z \partial_z u_p\|_{\infty} \leq C_0$.

Step 3. Estimate of $N_3$. By integration by parts, divergence free condition, (2.10) and $\tilde{v}|_{y=0} = 0$, we obtain

$$N_3 \leq \left|\int_{R_+^2} \varepsilon^2 f(t,x)e^{-y}|U|^2 dx dy\right| \leq C_0 \varepsilon^2 \|U\|_2^2.$$

Step 4. Estimate of $N_4$ and $N_5$. By integration by parts, divergence free condition, (2.10) and $v(t,x,0) = -\varepsilon^2 f(t,x)$, we have

$$N_4 \leq \varepsilon^2 \left|\int_{\partial R_+^2} p f dx\right| \leq C_0 \varepsilon^2 \|\nabla p\|_2.$$

By (2.9), it is easy to get

$$N_5 \leq C_0(\|U\|_2^2 + \varepsilon^4).$$

Step 5. Estimate of dissipative term. By integration by parts and the boundary condition (2.3), we have

$$-\langle \Delta U, U \rangle = \|\nabla U\|_2^2 + \beta \varepsilon^{-\gamma} \int_{\partial R_+^2} u^2 dx + \varepsilon \int_{\partial R_+^2} u g_0 dx - \varepsilon^2 \int_{\partial R_+^2} \partial_y v f dx \geq \left(\frac{2}{3} - C_0 \delta\right) \|\nabla U\|_2^2 - C_0 \beta \varepsilon^{-2\gamma} \|U\|_2^2 - C_0 \varepsilon^2.$$

Thus, choosing $\delta_0$ small such that when $\delta \leq \delta_0$, we arrive at

$$-\varepsilon^2 \langle \Delta U, U \rangle_2 \geq \frac{\varepsilon^2}{2} \|\nabla U\|_2^2 - C_0 \varepsilon^{2(1-\gamma)} \|U\|_2^2 - C_0 \varepsilon^4.$$

Collecting these estimates of Step 1-Step 5, we finish the proof. □

5 Vorticity estimates in Gevrey class and Sobolev space

5.1 Vorticity estimate in Gevrey class

Recall the equation (2.7) of $\eta$, and we are going to prove the following proposition.

Proposition 5.1 (Vorticity estimate in Gevrey class) There exist $\delta > 0$ and $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$, there holds

$$\frac{1}{2} \frac{d}{dt} \left(\|\eta\|_{X^2}^2 - \|\eta\|_{\tilde{X}_2}^2\right) + \lambda \left(\|\eta\|_{Y^2}^2 - \|\eta\|_{\tilde{Y}_2}^2\right) + \frac{\varepsilon^2}{2} \left(\|\nabla \eta\|_{X^2}^2 - \|\nabla \eta\|_{\tilde{X}_2}^2\right)$$

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Then there hold

Lemma 5.2 Let

\[ A' = \sum_{m=2}^{\infty} \frac{\rho(t)^{2(m-2)}}{(m-2)!} \sum_{|\alpha|=m} |\langle \partial^\alpha (u \partial_x v), \partial^\alpha v \rangle|. \]

Then there hold

(a)

\[ A' \leq C_0 \|u\|_{X^1} \|\partial_y u\|_{X^2} v\|_{X^2 \cap Y^2} + C_0 \|u\|_{X^1} \|v\|_{X^2} \|\partial_y v\|_{X^2}; \]

(b)

\[ A' \leq C_0 \|u\|_{X^1} \|\partial_y u\|_{X^2} v\|_{X^2 \cap Y^2}. \]

Proof. The proof of (b) is the same as that in Lemma 3.1 and we omit it here. The proof of (a) is also similar to that in Lemma 5.1 except the estimate of \( I_{22} \), and we sketch it.

Firstly, using Sobolev inequality and integrating by parts, we have

\[ \sum_{m=2}^{\infty} \frac{\rho(t)^{2(m-2)}}{(m-2)!} \sum_{|\alpha|=m} |\langle u \partial_\xi \partial^\alpha v, \partial^\alpha v \rangle| \leq C_0 \|u\|_{X^1} \|\partial_y u\|_{X^2} v\|_{X^2 \cap Y^2}, \]

which gives

\[ A' \leq I' + C_0 \|u\|_{X^1} \|\partial_y u\|_{X^2} v\|_{X^2}, \]

where

\[ I' = \sum_{m=2}^{\infty} \frac{\rho(t)^{2(m-2)}}{(m-2)!} \sum_{|\alpha|=m} \sum_{0<\beta<\alpha} C_{\alpha}^\beta |\langle \partial^\beta u_\partial_\xi \partial^\alpha-\beta v, \partial^\alpha v \rangle|, \]

and \( I' \) could be estimated as follows

\[ I' \leq \sum_{m=2}^{\infty} \frac{\rho(t)^{2(m-2)}}{(m-2)!} \sum_{|\alpha|=m} \sum_{1\leq|\beta|\leq2, \beta \leq \alpha} C_{\alpha}^\beta \|\partial^\beta u_\partial_\xi \partial^\alpha-\beta v\|_2 \|\partial^\alpha v\|_2. \]
\[ + \sum_{m=5}^{\infty} \frac{\rho(t)^{2(m-2)}}{(m-2)!^{\frac{1}{3}}} \sum_{|\alpha|=m} \sum_{j=3}^{m-2} \sum_{|\beta|=j, \beta \leq \alpha} C^\beta_\alpha \|\partial^\beta u \partial_x \partial^{\alpha-\beta} v\|_2 \|\partial^\alpha v\|_2 \]

Next we handle them term by term.

**Step 1. Estimate of \( I'_1 \).** The same argument as \( I_1 \) in Lemma 3.1 gives

\[ I'_1 \leq C_0 \|u\|_{\frac{1}{H^3}} \|\partial_y u\|_{\frac{1}{H^3}} \|v\|_{Y^2}^2. \]

**Step 2. Estimate of \( I'_2 \).** Similar to \( I_2 \) in Lemma 3.1, we decompose \( I'_2 \) into two parts

\[ I'_2 = \sum_{m=6}^{\infty} \frac{\rho(t)^{2(m-2)}}{(m-2)!^{\frac{1}{3}}} \sum_{|\alpha|=m} \sum_{j=3}^{m-2} \sum_{|\beta|=j, \beta \leq \alpha} C^\beta_\alpha \|\partial^\beta u \partial_x \partial^{\alpha-\beta} v\|_2 \|\partial^\alpha v\|_2 \]

\[ + \sum_{m=5}^{\infty} \frac{\rho(t)^{2(m-2)}}{(m-2)!^{\frac{1}{3}}} \sum_{|\alpha|=m} \sum_{j=3}^{m-2} \sum_{|\beta|=j, \beta \leq \alpha} C^\beta_\alpha \|\partial^\beta u \partial_x \partial^{\alpha-\beta} v\|_2 \|\partial^\alpha v\|_2 = I'_{21} + I'_{22}. \]

The same argument as \( I_{21} \) in Lemma 3.1 gives

\[ I'_{21} \leq C_0 \|u\|_{\frac{1}{X^2}} \|\partial_y u\|_{\frac{1}{X^2}} \|v\|_{X^2}^2. \]

Then \( I'_{22} \) can be bounded by

\[ \sum_{m=5}^{\infty} \frac{\rho(t)^{2(m-2)}}{(m-2)!^{\frac{1}{3}}} \sum_{|\alpha|=m} \sum_{j=3}^{m-2} \sum_{|\beta|=j, \beta \leq \alpha} C^\beta_\alpha \|\partial^\beta u\|_2 \|\partial_x \partial^{\alpha-\beta} v\|_2 \|\partial_x \partial_y \partial^{\alpha-\beta} v\|_2 \|\partial^\alpha v\|_2 \]

\[ + \sum_{m=5}^{\infty} \frac{\rho(t)^{2(m-2)}}{(m-2)!^{\frac{1}{3}}} \sum_{|\alpha|=m} \sum_{j=3}^{m-2} \sum_{|\beta|=j, \beta \leq \alpha} C^\beta_\alpha \|\partial^\beta u\|_2 \|\partial_x \partial^{\alpha-\beta} v\|_2 \|\partial_x \partial_y \partial^{\alpha-\beta} v\|_2 \|\partial^\alpha v\|_2. \]

We only estimate the first term, which can be controlled from similar arguments as in (3.3) by

\[ \sum_{m=5}^{\infty} \frac{\rho(t)^{m-2}}{(m-2)!^{\frac{1}{3}}} \sum_{j=3}^{m-2} C^\beta_\alpha \|\partial^\beta u\|_2 \|\partial_y \partial^\alpha v\|_2 \]

\[ \leq C_0 \sum_{m=6}^{\infty} \sum_{j=3}^{m-2} C^\beta_\alpha ((m-j-1)!)^{\frac{1}{3}} ((j-3)!)^{\frac{1}{3}} \]

\[ \times \left( \sum_{|\alpha|=m} \|\partial_y \partial^\alpha v\|_2 \right)^{\frac{1}{3}} \|v\|_{m-1,j+1}^\frac{1}{3} \|u\|_{j+1,j}^\frac{1}{3} \]

\[ + C_0 \|v\|_{X^2} \|\partial_y v\|_{X^2} \|u\|_{X^3} \|v\|_{X^2}. \]
\[ \leq C_0 \sum_{m=0}^{\infty} \sum_{j=-\infty}^{m-2} \frac{j^{-\frac{1}{2}} \rho(t)^{(m-j-1)!}}{(m-j-1)!} \left( \sum_{|\alpha|=m-j+1} \| \partial_y \partial^\alpha v \|_{L^2} \right)^{\frac{1}{2}} |v|_{m-j+1,2}^\frac{1}{2} |u|_{j,3} |v|_{m,2}^{\frac{1}{2}} \]

\[ + C_0 \|v\|_{T^2}^\frac{1}{2} \|\partial_y v\|_{T^2}^\frac{1}{2} \|u\|_{X^3} \|v\|_{X^2} \]

\[ \leq C_0 \|v\|_{X^2}^\frac{3}{2} \|\partial_y v\|_{X^2}^\frac{1}{2} \|u\|_{X^3}, \]

where we used

\[ \frac{C^j_m((m-j-1)!)^{\frac{1}{2}}((j-3)!)^{\frac{1}{2}}}{(m-2)!} \leq C_0 \left( \frac{C^{j-1}_m}{(m-2)!} \right)^{1-\frac{1}{2}} j^{-\frac{1}{2}} \]

for \( j \in \{\frac{m+1}{2}, \frac{m+3}{2}, \ldots, m-2\} \).

Finally, we obtain

\[ I_2 \leq C_0 \|v\|_{X^2}^\frac{3}{2} \|\partial_y v\|_{X^2}^\frac{1}{2} \|u\|_{X^3}. \]

**Step 3. Estimate of \( I_3 \).** Following the proof of \( I_3 \) in Lemma 3.1 line by line, we get

\[ I_3 \leq C_0 \|v\|_{T^2}^\frac{1}{2} \|\partial_y v\|_{T^2}^\frac{1}{2} \|u\|_{X^3} \|v\|_{X^2}. \]

**Proof of Proposition 5.1.** Acting \( \partial^\alpha \) on both sides of (2.7), multiplying \( \frac{\rho(t)^{2(|\alpha|-2)}}{(|\alpha|-2)!} \partial^\alpha \eta \) and summing over \( |\alpha| \geq 2 \), we arrive at

\[ \frac{1}{2} \frac{d}{dt} \left( \| \eta \|_{X^2}^2 - \| \eta \|_{Y^2}^2 \right) + \lambda \left( \| \eta \|_{Y^2}^2 - \| \eta \|_{X^2}^2 \right) - \epsilon^2 \sum_{m=2}^{\infty} \frac{\rho(t)^{2(m-2)}}{(m-2)!} \sum_{|\alpha|=m} \langle \partial^\alpha \Delta \eta, \partial^\alpha \eta \rangle \]

\[ \leq \left| \sum_{m=2}^{\infty} \frac{\rho(t)^{2(m-2)}}{(m-2)!} \sum_{|\alpha|=m} \langle \partial^\alpha ((u^a \partial_x \eta + \bar{v}^a \partial_y \eta), \partial^\alpha \eta \rangle \right| \]

\[ + \left| \sum_{m=2}^{\infty} \frac{\rho(t)^{2(m-2)}}{(m-2)!} \sum_{|\alpha|=m} \langle \partial^\alpha (u \partial_x \eta^a + \bar{v} \partial_y \eta^a), \partial^\alpha \eta \rangle \right| \]

\[ + \left| \sum_{m=2}^{\infty} \frac{\rho(t)^{2(m-2)}}{(m-2)!} \sum_{|\alpha|=m} \langle \partial^\alpha (u \partial_x \eta + \bar{v} \partial_y \eta), \partial^\alpha \eta \rangle \right| + \left| \sum_{m=2}^{\infty} \frac{\rho(t)^{2(m-2)}}{(m-2)!} \sum_{|\alpha|=m} \langle \beta \partial^\alpha \partial_x \bar{p}, \partial^\alpha \eta \rangle \right| \]

\[ + \left| \sum_{m=2}^{\infty} \frac{\rho(t)^{2(m-2)}}{(m-2)!} \sum_{|\alpha|=m} \langle \partial^\alpha (\partial_y R_1 - \partial_x R_2 + \epsilon^2 f e^{-\gamma} \partial_y u^a + \epsilon^2 \partial_x f e^{-\gamma} \partial_y v^a - \frac{\beta R_1}{\epsilon^2} + h), \partial^\alpha \eta \rangle \right| \]

\[ = \sum_{i=1}^{5} M_i. \]

**Step 1. Estimate of \( M_1 + M_3 \).** Using Lemma 5.2 and Lemma 3.2, we get

\[ M_1 + M_3 \]

\[ \leq C_0 \|\eta\|_{X^2}^\frac{3}{2} + C_0 \|u\|_{X^2}^\frac{1}{2} \|\partial_y u\|_{X^2}^\frac{1}{2} \|\eta\|_{X^2}^\frac{1}{2} + C_0 \|u\|_{X^2} \|\eta\|_{X^2}^\frac{3}{2} \|\partial_y v\|_{X^2}^\frac{1}{2} \]
where we used $k$ and $C_k$.

Meanwhile, a straightforward computation yields

$$
|M_{21}| \leq \sum_{m=2}^{\infty} \frac{\rho(t)^{2(m-2)}}{((m-2)!)^{\frac{3}{2}}} \sum_{|\alpha|=m} \left| \left\langle \partial^\alpha \left( u \partial_x u^a \frac{\eta}{\varepsilon^\gamma} + u \partial_y u^e - u \partial_x v^a \right), \partial^\alpha \eta \right\rangle \right|,
$$

where we used $u^a = u^e + \varepsilon^{1-\gamma} u^p$. Thus, the same argument as $M_{22}$ implies that

$$
|M_{21}| \leq C_0 \varepsilon^{-2\gamma} \| u \|_{X^2}^2 + C_0 \| \eta \|_{X^2 \cap Y^2}^2.
$$

Summing up $M_{21}$ and $M_{22}$, we arrive at

$$
\sum_{m=2}^{\infty} \frac{\rho(t)^{2(m-2)}}{((m-2)!)^{\frac{3}{2}}} \sum_{|\alpha|=m} \left| \left\langle \partial^\alpha \left( \tilde{v} \partial_y \eta^a \right), \partial^\alpha \eta \right\rangle \right| \leq C_0 \varepsilon^{-2\gamma} \| u \|_{X^2}^2 + C_0 \| \eta \|_{X^2 \cap Y^2}^2.
$$

On the other hand, we have

$$
\sum_{m=2}^{\infty} \frac{\rho(t)^{2(m-2)}}{((m-2)!)^{\frac{3}{2}}} \sum_{|\alpha|=m} \left| \left\langle \partial^\alpha \left( \tilde{v} \partial_y \eta^a \right), \partial^\alpha \eta \right\rangle \right| \leq C_0 \varepsilon^{-2\gamma} \| u \|_{X^2}^2 + C_0 \| \eta \|_{X^2 \cap Y^2}^2.
$$

Since the most singular term in (5.18) is

$$
\sum_{m=2}^{\infty} \frac{\rho(t)^{2(m-2)}}{((m-2)!)^{\frac{3}{2}}} \sum_{|\alpha|=m} \varepsilon^{1-\gamma} \left\langle \partial^\alpha \left( \tilde{v} \partial_y \left( u^p \right) \right), \partial^\alpha u \right\rangle,
$$

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and the others are similar, hence we only estimates this term. Note that \( \partial_{yy}(u^p) = \varepsilon^{-1} \partial_y(\partial_x u^p) \), and by \((2.8)\), \( \| \tilde{Z} \partial_x u^p \|_\infty \) is bounded for \( k \in N \). Hence, by Lemma 3.3, we obtain

\[
\sum_{m=2}^{\infty} \frac{\rho(t)^{2(m-2)}}{(m-2)!^{\frac{1}{2}}} \sum_{|\alpha|=m} \varepsilon^{1-\gamma} \langle \partial^\alpha (\tilde{v} \partial_{yy}(u^p)), \partial^\alpha \eta \rangle \\
\leq C_0 \| \eta \|^2_{X^2 \cap Y^2} + C_0 (\varepsilon^{-2} \| U \|^2_{X^2 \cap Y^2} + \varepsilon^2).
\]

Consequently,

\[
M_2 \leq C_0 \| \eta \|^2_{X^2 \cap Y^2} + C_0 (\varepsilon^{-2} \| U \|^2_{X^2 \cap Y^2} + \varepsilon^2).
\]

**Step 3. Estimate of \( M_4 \).** It is easy to get

\[
M_4 \leq C_0 \delta \varepsilon^{-2\gamma} \| \partial_x p \|^2_{X^2} + C_0 (\delta \| \eta \|^2_{X^2}.
\]

Thus, by Lemma 3.3, we get

\[
M_4 \leq C_0 (\delta \| \eta \|^2_{X^2} + C_0 \delta \varepsilon^{-2\gamma} (\| U \|^2_{Y^2} + \| \nabla U \|^2_{Y^2} + 1) (\| U \|^2_{Y^2} + \varepsilon^4) \\
+ C_0 \delta \varepsilon^{2(2-2\gamma)} \| \nabla U \|^2_{Y^2} + C_0 \delta \varepsilon^{2(2-2\gamma)} (\| u \|^2_{Y^2} + \| \partial_y u \|^2_{Y^2}) \\
+ C_0 \delta \varepsilon^{-2\gamma} (\| U \|^2_{X^2}) \| \partial_y U \|_{X^2} + 1) (\varepsilon^4 + \| U \|^2_{X^2 \cap Y^3} + 1).
\]

**Step 4. Estimate of \( M_5 \).** Noting that \( \gamma \leq 1 \), by \((2.8)-(2.10)\), it is easy to obtain

\[
M_5 \leq C_0 \varepsilon^2 + C_0 \| \eta \|^2_{X^2} + C_0 \| U \|^2_{X^2}.
\]

**Step 5. Estimate of dissipative term.** Noting that \( \eta \) vanishes on the boundary and by integrating by parts, we have

\[
- \sum_{m=2}^{\infty} \frac{\rho(t)^{2(m-2)}}{(m-2)!^{\frac{1}{2}}} \sum_{|\alpha|=m} \langle \partial^\alpha \Delta \eta, \partial^\alpha \eta \rangle \\
\geq \left( \| \nabla \eta \|^2_{X^2} - \| \nabla \eta \|^2_{Y^2} \right) - \left| \sum_{m=2}^{\infty} \frac{\rho(t)^{2(m-2)}}{(m-2)!^{\frac{1}{2}}} \sum_{|\alpha|=m} \langle [\partial^\alpha, \partial_y] \partial_y \eta, \partial^\alpha \eta \rangle \right| \\
- \left| \sum_{m=2}^{\infty} \frac{\rho(t)^{2(m-2)}}{(m-2)!^{\frac{1}{2}}} \sum_{|\alpha|=m} \langle \partial^\alpha \partial_y \eta, [\partial^\alpha, \partial_y] \eta \rangle \right| \geq (1 - C_0 \delta) \left( \| \nabla \eta \|^2_{X^2} - \| \nabla \eta \|^2_{Y^2} \right).
\]

Thus, fixed \( \delta \) small, we arrive at

\[
- \sum_{m=2}^{\infty} \frac{\rho(t)^{2(m-2)}}{(m-2)!^{\frac{1}{2}}} \sum_{|\alpha|=m} \langle \partial^\alpha \Delta \eta, \partial^\alpha \eta \rangle \geq \frac{1}{2} \left( \| \nabla \eta \|^2_{X^2} - \| \nabla \eta \|^2_{Y^2} \right).
\]

Collecting the estimates in Step 1-Step 5, the proof is completed. \( \square \)
5.2 Vorticity estimate in Sobolev space

Proposition 5.3 (Vorticity estimate in $L^2$) There exist $\delta > 0$ and $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$, we have

$$\frac{1}{2} \frac{d}{dt} \|\eta\|^2_2 + \varepsilon^2 \|\nabla \eta\|^2_2 \leq C_0 \varepsilon^{-2} \left( \|U\|^2_{H^1} + \|\nabla U\|^2_{H^1} + 1 \right) \left( \|\eta\|^2_{H^1} + \varepsilon^4 \right) + C_0 \delta \beta^2 \varepsilon^{4(1-\gamma)} \|\nabla U\|^2_{H^3} + C_0(\delta) \left( 1 + \|U\|^4_{H^1} + \|\nabla U\|^4_{H^1} \right) \|\eta\|^2_{H^3}.$$

Proof: Taking $L^2(R^3)$ inner product with $\eta$ in (2.7), we arrive at

$$\frac{1}{2} \frac{d}{dt} \|\eta\|^2_2 + \varepsilon^2 \|\nabla \eta\|^2_2 \leq \left| \langle u^a \partial_x \eta + \tilde{v}^a \partial_y \eta, \eta \rangle \right| + \left| \langle u \partial_x \eta^a + \tilde{v} \partial_y \eta^a, \eta \rangle \right| + \left| \langle u \partial_x \eta + \tilde{v} \partial_y, \eta \rangle \right| + \left| \langle \varepsilon^{-\gamma} \partial_x p, \eta \rangle \right|$$

$$+ \left| \langle \partial_y R_1 - \partial_x R_2 + \varepsilon^2 f e^{-y} \partial_y u^a + \varepsilon^2 \partial_y f e^{-y} \partial_y \eta^a + \beta \varepsilon^{-\gamma} R_1 + h, \eta \rangle \right| = \sum_{i=1}^5 \tilde{N}_i.$$

Step 1. Estimate of $\tilde{N}_1$. The same argument as $N_1$ gives

$$\tilde{N}_1 \leq C_0 \varepsilon^2 \|\eta\|^2_2.$$

Step 2. Estimate of $\tilde{N}_2$. Due to (2.6), we have

$$\tilde{N}_2 \leq \left| \left\langle u \frac{\beta \partial_x v^a}{\varepsilon^\gamma}, \eta \right\rangle \right| + \left| \left\langle u \partial_x w_a, \eta \right\rangle \right| + \left| \left\langle \tilde{v} \frac{\partial_x v^a}{\varepsilon^\gamma}, \eta \right\rangle \right| + \left| \left\langle \tilde{v} \partial_y w_a, \eta \right\rangle \right|.$$

In the above, one of the most singular terms is

$$\varepsilon^{1-\gamma} \left| \left\langle \tilde{v} \partial_y u^p, u \right\rangle \right|.$$

We only estimate this term. The same argument as $N_2$ yields

$$\varepsilon^{1-\gamma} \left| \left\langle \tilde{v} \partial_y u^p, u \right\rangle \right| = \varepsilon^{-\gamma} \left| \left\langle \frac{v + \varepsilon^2 f(t,x)e^{-y}}{\psi} Z \partial_x u, u \right\rangle \right| \leq C_0 \varepsilon^{-2\gamma} \|U\|^2_{H^1} + C_0 \varepsilon^2.$$

Step 3. Estimate of $\tilde{N}_3$. The same argument as $N_3$ gives

$$\tilde{N}_3 \leq C_0(1 + \|U\|^4_{H^1} + \|\nabla U\|^4_{H^1}) \|\eta\|^2_{H^1}.$$

Step 4. Estimate of $\tilde{N}_4$ and $\tilde{N}_5$. It is direct to get

$$\tilde{N}_4 \leq C_0 \delta \varepsilon^{-2\gamma} \|\partial_x p\|^2_2 + C_0(\delta) \|\eta\|^2_2.$$

By Lemma 4.4, we have

$$\tilde{N}_4 \leq C_0 \varepsilon^{-2\gamma} \left( \|U\|^2_{H^1} + \|\nabla U\|^2_{H^1} + 1 \right) \left( \|\eta\|^2_{H^1} + \varepsilon^4 \right) + C_0 \delta \varepsilon^{4(1-\gamma)} \|\nabla U\|^2_{H^3} + C_0(\delta) \|\eta\|^2_{H^3}.$$

By (2.9), we similarly have

$$\tilde{N}_5 \leq C_0 \|\eta\|^2_2 + C_0 \varepsilon^2.$$

Collecting these estimates in Step 1-Step 4, we complete the proof. \qed
6 Construction of the approximate solution

In this section, we will construct the approximate solutions of the Navier-Stokes equations (1.1) by using the matched asymptotic expansion method for \( \gamma = \frac{1}{2}, 1 \), and the presentation of reminders \( R_1, R_2 \) are shown. The same argument can be used to construct approximate solution for any rational number \( \gamma \in [0, 1] \).

6.1 The case of \( \gamma = \frac{1}{2} \)

In this subsection, we consider the case of \( \gamma = \frac{1}{2} \), and compute the specific presentation of the reminders \( R_1, R_2 \). Moreover, the Assumptions (H) will be verified to be reasonable in this case.

We make the following formal expansions

\[
\begin{align*}
    u^\varepsilon(t, x, y) &= \sum_{j \geq 0} \varepsilon^{\frac{j}{2}} u^{(j)}(t, x, y), \\
v^\varepsilon(t, x, y) &= \sum_{j \geq 0} \varepsilon^{\frac{j}{2}} v^{(j)}(t, x, y), \\
p^\varepsilon(t, x, y) &= \sum_{j \geq 0} \varepsilon^{\frac{j}{2}} p^{(j)}(t, x, y),
\end{align*}
\]

away from the boundary and

\[
\begin{align*}
    u^\varepsilon(t, x, y) &= \sum_{j \geq 0} \varepsilon^{\frac{j}{2}} [u^{(j)}(t, x, y) + u^{(j)}_p(t, x, z)], \\
v^\varepsilon(t, x, y) &= \sum_{j \geq 0} \varepsilon^{\frac{j}{2}} [v^{(j)}(t, x, y) + v^{(j)}_p(t, x, z)], \\
p^\varepsilon(t, x, y) &= \sum_{j \geq 0} \varepsilon^{\frac{j}{2}} [p^{(j)}(t, x, y) + p^{(j)}_p(t, x, z)]
\end{align*}
\]

near the boundary with \( z = \frac{2}{\varepsilon} \) and the following conclusions hold

\[
\begin{align*}
u^{(i)}_e(t, x, y) &= v^{(i)}_e(t, x, y) = p^{(i)}_e(t, x, y) = 0, \quad i = 1, 2 \\
u^{(0)}_p(t, x, y) &= 0, \quad v^{(i)}_p(t, x, y) = 0, \quad i = 0, 1, 2, \\
p^{(i)}_p(t, x, y) &= 0, \quad i = 0, 1, 2, 3, 4.
\end{align*}
\]

More generally, for \((u^{(j)}_e, v^{(j)}_e, p^{(j)}_e)\) with \( j \geq 0 \) we have

\[
\begin{cases}
    \partial_t u^{(j)}_e + \sum_{k=0}^{j} (u^{(k)}_e \partial_x + v^{(k)}_e \partial_y)u^{(j-k)}_e + \partial_x p^{(j)}_e = \Delta u^{(j-4)}_e, \\
    \partial_t v^{(j)}_e + \sum_{k=0}^{j} (u^{(k)}_e \partial_x + v^{(k)}_e \partial_y)v^{(j-k)}_e + \partial_y p^{(j)}_e = \Delta v^{(j-4)}_e, \\
    \partial_x u^{(j)}_e + \partial_y v^{(j)}_e = 0, \\
    v^{(j)}_e|_{y=0} = -v^{(j)}_p|_{y=0}, \\
    (u^{(j)}_e, v^{(j)}_e)|_{t=0} = (0, 0).
\end{cases}
\]
Here $u_k^e = v_k^e = 0$ if $k < 0$. Concluding the above analysis, we have $(u_p^{(1)}, v_p^{(3)})$ satisfies the following linear Prandtl-type equation

$$
\begin{align*}
\begin{cases}
\partial_t u_p^{(1)} + u_p^{(1)} \partial_x u_p^{(0)} + u_p^{(0)} \partial_x u_p^{(1)} + z \partial_y v_p^{(1)} \partial_z u_p^{(1)} = \partial^2 u_p^{(1)}, \\
\partial_x u_p^{(1)} + \partial_z v_p^{(3)} = 0, \\
\partial_z u_p^{(1)}(t, x, 0) = \beta u_p^{(0)}(t, x, 0), \\
u_p^{(1)}(t, x, \infty) = v_p^{(3)}(t, x, \infty) = 0, \\
u_p^{(1)}(0, x, z) = 0.
\end{cases}
\end{align*}
$$

(6.2)

Similarly, $(u_p^{(2)}, v_p^{(4)})$ satisfies

$$
\begin{align*}
\begin{cases}
\partial_t u_p^{(2)} + u_e^{(0)} \partial_x u_p^{(2)} + u_p^{(2)} \partial_x u_e^{(0)} + u_p^{(1)} \partial_x u_p^{(1)} + (v_p^{(3)} + v_p^{(3)}) \partial_z u_p^{(1)} \\
+ z \partial_y v_p^{(0)} \partial_z u_p^{(2)} = \partial^2 u_p^{(2)}, \\
\partial_x u_p^{(2)} + \partial_z v_p^{(4)} = 0, \\
\partial_z u_p^{(2)}(t, x, 0) = \beta (u_e^{(1)}(t, x, 0) + u_p^{(1)}(t, x, 0)) - \partial_y u_p^{(0)}(t, x, 0), \\
u_p^{(2)}(t, x, \infty) = v_p^{(4)}(t, x, \infty) = 0, \\
u_p^{(2)}(0, x, z) = 0.
\end{cases}
\end{align*}
$$

(6.3)

Moreover, $(u_p^{(3)}, v_p^{(5)})$ satisfies

$$
\begin{align*}
\begin{cases}
\partial_t u_p^{(3)} + u_e^{(0)} \partial_x u_p^{(3)} + u_p^{(3)} \partial_x u_e^{(0)} + u_p^{(1)} \partial_x u_p^{(2)} + u_p^{(2)} \partial_x u_p^{(1)} + v_p^{(3)} \partial_y u_e^{(0)} \\
+ 4 (v_p^{(3)} + v_p^{(5)}) \partial_z u_p^{(5-k)} + z \partial_y u_p^{(0)} \partial_x u_p^{(1)} \\
+ z \partial_y v_p^{(0)} \partial_z u_p^{(3)} + \frac{z^2}{2} \partial_{yy} v_p^{(0)} \partial_z u_p^{(1)} = \partial^2 u_p^{(3)}, \\
\partial_x u_p^{(3)} + \partial_z v_p^{(5)} = 0, \\
\partial_z u_p^{(3)}(t, x, 0) = \beta (v_e^{(2)}(t, x, 0) + u_p^{(2)}(t, x, 0)) - \partial_y u_p^{(0)}(t, x, 0), \\
u_p^{(3)}(t, x, \infty) = v_p^{(5)}(t, x, \infty) = 0, \\
u_p^{(3)}(0, x, z) = 0.
\end{cases}
\end{align*}
$$

(6.4)

Finally, the estimate of $p_p^{(5)}$ is needed, which can be solved by the equation of $v_p^{(3)}$:

$$
\begin{align*}
\partial_t v_p^{(3)} + u_e^{(0)} \partial_x v_p^{(3)} + u_p^{(3)} \partial_x v_e^{(0)} + v_p^{(3)} \partial_y v_e^{(0)} + z \partial_{xy} v_e^{(0)} u_p^{(1)} + z \partial_y v_e^{(0)} \partial_z v_p^{(3)} + \partial_z p_p^{(5)} = \partial^2 v_p^{(3)}
\end{align*}
$$

(6.5)

These equations can be solved by the following way

$$(u_e^{(0)}, v_e^{(0)}) \rightarrow (u_p^{(1)}, v_p^{(3)}) \rightarrow (u_e^{(3)}, v_e^{(3)}) \rightarrow (u_p^{(2)}, v_p^{(4)}) \rightarrow (u_p^{(4)}, v_p^{(4)}) \rightarrow (u_p^{(3)}, v_p^{(5)})$$
The approximate solutions are stated as follows

\[ u^a(t, x, y) = \sum_{j=0}^{3} \varepsilon^j u_e^{(j)}(t, x, y) + \sum_{j=1}^{3} \varepsilon^j v_p^{(j)}(t, x, y) = u^e(t, x, y) + \varepsilon u^p(t, x, z), \]

\[ v^a(t, x, y) = \sum_{j=0}^{3} \varepsilon^j v_e^{(j)}(t, x, y) + \sum_{j=3}^{5} \varepsilon^j v_p^{(j)}(t, x, y) = v^e(t, x, y) + \varepsilon v^p(t, x, z), \]

\[ p^a(t, x, y) = \sum_{j=0}^{3} \varepsilon^j p_e^{(j)}(t, x, y) + \varepsilon^2 p_p^{(5)}(t, x, z). \]

Let

\[ f(t, x) = v_p^{(4)} + \varepsilon v_p^{(5)}, \]

and

\[ g_0(t, x) = \beta u_e^{(3)}(t, x, 0) + \beta v_p^{(3)}(t, x, 0) - \varepsilon \partial_y u_e^{(3)}(t, x, 0), \]

then the approximate solution \((u^a, v^a, p^a)\) satisfies

\[
\begin{aligned}
\partial_t u^a + u^a \partial_x u^a + (v^a - \varepsilon^2 f(t, x) e^{-y}) \partial_y u^a + \partial_x p^a - \varepsilon^2 \Delta u^a &= -R_1, \\
\partial_t v^a + u^a \partial_x v^a + (v^a - \varepsilon^2 f(t, x) e^{-y}) \partial_y v^a + \partial_y p^a - \varepsilon^2 \Delta v^a &= -R_2, \\
\partial_x u^a + \partial_y v^a &= 0, \\
v^a(t, x, 0) &= \varepsilon^2 f(t, x), \\
\partial_y u^a(t, x, y)|_{y=0} &= \beta e^{-y} u^a(t, x, 0) - \varepsilon g_0(t, x), \\
(u^a, v^a)(0, x, y) &= (0, 0).
\end{aligned}
\]

Especially, direct computations show that the reminders \(R_1, R_2\) have the following representation:

\[ -R_1 = \varepsilon^3 (u_e^{(3)} \partial_x + v_e^{(3)} \partial_y) u_e^{(3)} - \varepsilon^2 \Delta u^e - \varepsilon^2 \partial_x u^p + \varepsilon^2 \partial_y u^p + \varepsilon \partial_x p_p^{(5)} + \varepsilon \partial_y p_p^{(5)} \]

\[ -\varepsilon^2 f e^{-y} \partial_y u^e - \varepsilon^2 f e^{-y} v_p^{(5)} \partial_x u^p - \varepsilon^2 e^{-y} v_p^{(4)} \partial_z (u_p^{(2)} + \varepsilon u_p^{(3)}) \]

\[ + \varepsilon^2 \left[ u_p^{(3)} \partial_x u^p + \partial_x u_p^{(3)} u^p + \varepsilon \left( \sum_{j=4}^{6} u_p^{(3)} \partial_x u_p^{(j-k)} + \varepsilon v_p^{(5)} \partial_z v_p^{(5)} + \varepsilon v_p^{(4)} \partial_y v_p^{(4)} \right) \right] \]

\[ + \varepsilon^2 \left[ \frac{1}{2} v_p^{(5)} \partial_y u^e + \varepsilon v_p^{(5)} \partial_y u_p^{(3)} + \varepsilon v_p^{(4)} \partial_z u_p^{(3)} + \varepsilon v_p^{(4)} \partial_y u^e + \varepsilon v_p^{(4)} \partial_y u^p + \varepsilon v_p^{(5)} \partial_z u_p^{(5)} \right] \]

\[ + \varepsilon^2 \left[ (u_e^{(0)} - u_e^{(0)}) \partial_x u_p^{(3)} + u_p^{(3)} (\partial_x u_e^{(0)} - \partial_x u_e^{(0)}) + (v_e^{(3)} - v_e^{(3)}) \partial_x u_p^{(2)} \right. \]

\[ + (v_e^{(4)} - v_e^{(4)}) \partial_z u_p^{(1)} + \partial_x u_p^{(0)} + (v_e^{(4)} - v_e^{(4)}) \partial_z u_p^{(0)} + \varepsilon (1 - e^{-y}) \partial_z u_p^{(1)} \]

\[ + \varepsilon \left[ (u_e^{(0)} - u_e^{(0)}) \partial_x u_p^{(2)} + u_p^{(2)} (\partial_x u_e^{(0)} - \partial_x u_e^{(0)}) + (v_e^{(3)} - v_e^{(3)}) \partial_z u_p^{(1)} \right] \]

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\[ + \varepsilon \frac{3}{2} \left[ (\partial_x u_{e}^{(0)} - \partial_x u_{e}^{(0)} - y \partial_y u_{e}^{(0)}) u_{p}^{(1)} + (u_{e}^{(0)} - u_{e}^{(0)} - y \partial_y u_{e}^{(0)}) \partial_x u_{p}^{(1)} \right. \\
\left. + (v_{e}^{(0)} - y \partial_y v_{e}^{(0)}) \partial_x u_{p}^{(2)} + \varepsilon^{-\frac{3}{2}} (v_{e}^{(0)} - y \partial_y v_{e}^{(0)}) - \frac{y^2}{2} \partial_y v_{e}^{(0)}) \partial_x u_{p}^{(1)} \right] \quad (6.9) \]

and
\[ -R_2 = \varepsilon^3 (u_{e}^{(3)} \partial_x + v_{e}^{(3)} \partial_y) v_{e}^{(3)} - \varepsilon^3 \Delta v - \varepsilon^3 \partial_x v e - \varepsilon^2 \partial_x v_p - \varepsilon^2 \partial_{zz} v_p - \varepsilon^2 f(t, x)e^{-y} \partial_y v e \\
+ \varepsilon^2 u_{e}^{(3)} \partial_x v_p + \varepsilon^2 u_{e}^{(3)} \partial_x v_p + \varepsilon^2 u_{e}^{(3)} \partial_x v_p + \varepsilon^2 v_{e}^{(3)} u_p + \varepsilon^2 \partial_x v_p + \varepsilon^2 \partial_x v_p + \varepsilon^2 \partial_x v_p \\
+ \varepsilon^3 \left[ (u_{e}^{(0)} - u_{e}^{(0)}) \partial_x v_p + \partial_x v_p + \partial_x v_p + v_{e}^{(0)} \partial_x v_p \right. \\
\left. + v_{e}^{(0)} \partial_x v_p - \partial_y v_{e}^{(0)} \right] \\
+ \varepsilon \left[ \partial_x v_p + \partial_x v_p + v_{e}^{(0)} \partial_x v_p + v_{e}^{(0)} \partial_x v_p - f(t, x)e^{-y} \partial_x v_p \right] \\
+ \varepsilon \left[ \partial_x v_p - \partial_x v_p \partial_x v_p \right]. \quad (6.10) \]

In fact, the approximate solutions and the remainders $R_1, R_2$ satisfy the Assumption (H) if the initial data $(u_0, v_0) \in X^{(2)}$. More precisely, we have the following three lemmas, whose proofs will be presented in the next section.

**Lemma 6.1** Let $\tilde{\partial} = \partial_x \tilde{Z}^{x} \partial_y \tilde{Z}^{y}$ with $\tilde{Z}^{k} = (\delta z)^{k} \partial_z^{k}$ for $k \in \mathbb{N}$. There exists $T_a > 0$ such that for any $t \in [0, T_a]$, there holds

\[ \sum_{m=3}^{\infty} \frac{\rho(t)^{2(m-3)}}{((m - 3)!)^{2}} \sum_{m-3 \leq |a| \leq m+6} \| \partial_x^{a}(u_{e}^{(j)} + v_{e}^{(j)}) \|_2^2 \leq C_0, \quad j = 0, 3; \]

\[ \sum_{a=0}^{2} \sum_{m=3}^{\infty} \frac{\rho(t)^{2(m-3)}}{((m - 3)!)^{2}} \sum_{m-3 \leq |a| \leq m+6} \| \varepsilon^2 \tilde{\partial}_x^{a} \tilde{\partial}_y^{a} (u_{e}^{(j)} + v_{e}^{(j)}) \|_2^2 \leq C_0, \quad j = 1, 2, 3. \]

**Lemma 6.2** There exists $T_a > 0$ such that for any $t \in [0, T_a]$, there holds

\[ \| (R_1, R_2) \|^2_{X^3} \leq C_0 \varepsilon^4, \quad \| \nabla (R_1, R_2) \|^2_{X^2} \leq C_0 \varepsilon^2. \]

**Lemma 6.3** There exist universal constant $C_0$ and $\overline{f}(t, x)$ such that $f(t, x) = \partial_x \overline{f}$ with

\[ \| \partial_t f \|_{X^3} + \| f \|_{X^3} + \| \partial_t \overline{f} \|_{L^2} + \| g_0 \|_{X^3} + \| \partial_t g_0 \|_{X^3} \leq C_0. \]

### 6.2 The case of $\gamma = 1$

In this subsection, we construct approximate solutions, derive the equations of approximate solutions and compute the remainders when $\gamma = 1$. 

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Approximate solutions are constructed similarly as above, but more simple, and the process can be found in \[34\] beside the boundary conditions of Prandtl equation. We only give a derivation of boundary condition for Prandtl equation.

Making the boundary layer (Prandtl layer) expansions

\[
\begin{align*}
    u^e(t, x, y) &= \sum_{j \geq 0} \varepsilon^j [u^{(j)}_e(t, x, y) + u^{(j)}_p(t, x, z)], \\
v^e(t, x, y) &= \sum_{j \geq 0} \varepsilon^j [v^{(j)}_e(t, x, y) + v^{(j)}_p(t, x, z)], \\
p^e(t, x, y) &= \sum_{j \geq 0} \varepsilon^j [p^{(j)}_e(t, x, y) + p^{(j)}_p(t, x, z)]
\end{align*}
\]

and matched boundary condition requires that

\[
    u^{(i)}_p(t, x, z) \to 0, \quad v^{(i)}_p(t, x, z) \to 0, \quad p^{(i)}_p(t, x, z) \to 0, \quad \text{as } z \to +\infty.
\]

While, the boundary condition of \((u^e, v^e)\) on \(y = 0\) requires that

\[
\begin{align*}
    \partial_x u^{(0)}_p(t, x, 0) &= \beta (u^{(0)}_e(t, x, 0) + u^{(0)}_p(t, x, 0)), \\
    \partial_x u^{(i)}_p(t, x, 0) &= \beta (u^{(i)}_e(t, x, 0) + u^{(i)}_p(t, x, 0)) - \partial_y u^{(i-1)}_e(t, x, 0), \quad i = 1, 2, \ldots \\
    v^{(i)}_e(t, x, 0) &= -v^{(i)}_p(t, x, 0), \quad i = 0, 1, \ldots.
\end{align*}
\]

The same argument as in \[34\] gives

\[
\begin{align*}
    \partial_t u^{(0)}_p - \partial_x u^{(0)}_p + u^{(0)}_p \partial_x u^{(0)}_e(t, x, 0) + (u^{(0)}_p + u^{(0)}_e(t, x, 0)) \partial_x u^{(0)}_p &+ (v^{(1)}_p + v^{(1)}_e(t, x, 0) + z \partial_y v^{(1)}_e(t, x, 0)) \partial_x u^{(0)}_p = 0, \\
    \partial_x u^{(0)}_p + \partial_x v^{(1)}_p &= 0, \\
    \lim_{z \to +\infty} (u^{(0)}_p, v^{(1)}_p)(t, x, z) &= 0, \\
    \partial_x u^{(0)}_p(t, x, 0) &= \beta (u^{(0)}_e(t, x, 0) + u^{(0)}_p(t, x, 0)) \\
    u^{(0)}_p(0, x, y) &= 0,
\end{align*}
\]

Remark 6.4 Let \(\tilde{u}^{(0)}_p(t, x, z) = u^{(0)}_p(t, x, z) + u^{(0)}_e(t, x, 0)\), and

\[
    \tilde{v}^{(1)}_p(t, x, z) = v^{(1)}_p(t, x, z) + v^{(1)}_e(t, x, 0) + z \partial_y v^{(0)}_e(t, x, 0).
\]

Then by Bernoulli law,

\[
\partial_t u^{(0)}_e(t, x, 0) + u^{(0)}_e(t, x, 0) \partial_x u^{(0)}_e(t, x, 0) + \partial_x p^{(0)}_e(t, x, 0) = 0,
\]

we arrive at

\[
\begin{align*}
    \partial_t \tilde{u}^{(0)}_p - \partial_x \tilde{u}^{(0)}_p + \tilde{u}^{(0)}_p \partial_x \tilde{u}^{(0)}_e(t, x, 0) + \tilde{v}^{(1)}_p \partial_x \tilde{u}^{(0)}_p + \partial_x p^{(0)}_e(t, x, 0) &= 0, \\
    \partial_x \tilde{u}^{(0)}_p + \partial_x \tilde{v}^{(1)}_p &= 0, \\
    \tilde{u}^{(0)}_p(0, x, z) &= u^{(0)}_e(0, x, 0), \\
    \lim_{z \to +\infty} \tilde{u}^{(0)}_p(t, x, z) &= u^{(0)}_e(t, x, 0), \\
    \partial_x \tilde{u}^{(0)}_p(t, x, 0) &= \beta \tilde{u}^{(0)}_p(t, x, 0), \\
    \tilde{v}^{(1)}_p(t, x, 0) &= 0,
\end{align*}
\]

which is a nonlinear Prandtl equation with Robin boundary condition.
Similarly, we can obtain the equation for \((u_p^{(1)}, v_p^{(2)})\).

**Remark 6.5** These equations can be solved in the following way

\((u_e^{(0)}, v_e^{(0)}) \rightarrow (u_p^{(1)}, v_p^{(1)}) \rightarrow (u_e^{(1)}, v_e^{(1)}) \rightarrow (u_p^{(1)}, v_p^{(2)}))\).

Now let us define the approximate solutions \((u^a, v^a, p^a)\) as following:

\[
\begin{align*}
  u^a(t, x, y) &:= \sum_{i=0}^{1} \varepsilon^i u_e^{(i)}(t, x, y) + \sum_{i=0}^{1} \varepsilon^i u_p^{(i)}(t, x, \frac{y}{\varepsilon}), \\
  v^a(t, x, y) &:= \sum_{i=0}^{1} \varepsilon^i v_e^{(i)}(t, x, y) + \sum_{i=0}^{2} \varepsilon^i v_p^{(i)}(t, x, \frac{y}{\varepsilon}), \\
  p^a(t, x, y) &:= \sum_{i=0}^{1} \varepsilon^i p_e^{(i)}(t, x, y) + \varepsilon^2 p_p^{(2)}(t, x, \frac{y}{\varepsilon}).
\end{align*}
\]

Set

\[
f(t, x) := \int_0^\infty \partial_x u_p^{(1)}(t, x, z) dz, \quad g_0(t, x) = -\partial_y u_e^{(1)}(t, x, 0).
\]

A straightforward computation gives that the approximate solution \((u^a, v^a, p^a)\) satisfies

\[
\begin{align*}
  \partial_t u^a + u^a \partial_x u^a + (v^a - \varepsilon^2 f(t, x)e^{-y}) \partial_y u^a + \partial_x p^a - \varepsilon^2 \Delta u^a &= -R_1, \\
  \partial_t v^a + u^a \partial_x v^a + (v^a - \varepsilon^2 f(t, x)e^{-y}) \partial_y v^a + \partial_y p^a - \varepsilon^2 \Delta v^a &= -R_2, \\
  \partial_x u^a + \partial_y v^a &= 0, \\
  (u^a, v^a)(0, x, y) &= (0, 0), \\
  v^a(t, x, 0) &= \varepsilon^2 f(t, x), \\
  \partial_y u^a(t, x, y)|_{y=0} &= \beta \varepsilon^{-1} u^a(t, x, 0) - \varepsilon g_0(t, x).
\end{align*}
\]

where the reminders \(R_1, R_2\) has the same form as in \([34]\).

**Remark 6.6**

i) In this case, the norms in the Assumptions (H) are just the analytical norms, and the above approximate solutions can also be verified to satisfy the Assumptions (H) with the initial analytic data similar to the case \(\gamma = \frac{1}{2}\).

ii) Note that at this time the equation of \(u_p^{(0)}\) is a nonlinear Prandtl equation with Robin boundary condition, which can be solved in the analytic setting (see also \([5]\)), and we omit the proof of local well-posedness result.

iii) Since the nonlinear Prandtl equations with Robin boundary condition occur in this case (the local well-posedness in analytic setting was established in \([3]\), the vanishing viscosity limit can only be verified in the analytic setting which is similar as in previous papers of dealing with no-slip conditions \([31, 34]\). In this paper, we handle this case and the cases \(\gamma < 1\) in Gevrey class together, and give a unified proof.
7 Appendix

In this section, our goal is to prove the local well-posedness of approximate solutions and verify the reasonability of the Assumptions (H). We consider the case \( \gamma = \frac{1}{2} \). To be specific, we will prove the well-posedness of the Euler system (1.4) and the linear Prandtl-type equation (6.2), and Lemma 6.1-6.3. Note that the proofs of well-posedness of the linearized equations \( (6.1) \) for \( j = 3 \) and \( (6.3), (6.4) \) is similar, and we omit them.

7.1 Well-posedness of the Euler system in Gevrey class

First, let us introduce some semi-norms. Set \( \partial_{x,y}^{\alpha} := \partial_{x}^{\alpha_{1}} \partial_{y}^{\alpha_{2}} \), \( k \in \mathbb{N} \) and define

\[
\|f\|_{X_{k}^{\gamma}}^{2} = \sum_{m=k}^{\infty} \rho(t)^{2(m-k)} \frac{1}{(m-k)!} \sum_{|\alpha|=m} \|\partial_{x,y}^{\alpha} f\|_{2}^{2} + \|f\|_{2}^{2},
\]

\[
\|f\|_{Y_{k}^{\gamma}}^{2} = \sum_{m=k+1}^{\infty} \rho(t)^{2(m-k)} \frac{1}{((m-k)!)} \sum_{|\alpha|=m} \|\partial_{x,y}^{\alpha} f\|_{2}^{2} + \|f\|_{2}^{2},
\]

where \( \rho(t) = 2 - \lambda \geq 1 \) and \( \lambda > 0 \), to be decided later. Moreover, \( X_{k}^{\gamma}(\mathbb{R}^{2}_{+}) \) states

\[
X_{k}^{\gamma}(\mathbb{R}^{2}_{+}) = \{ u; \|u\|_{X_{k}^{\gamma}} < \infty \}.
\]

**Lemma 7.1** Let \( u \cdot n|_{y=0} = 0 \), and

\[
A'' = \sum_{m=19}^{\infty} \frac{\rho(t)^{2(m-19)}}{((m-19)!)} \sum_{|\alpha|=m} |\langle \partial_{x}^{\alpha} (u \cdot \nabla v), \partial_{x}^{\alpha} v \rangle|.
\]

Then there holds

\[
A'' \leq C_{0} \| (u, v) \|_{X_{19}^{\gamma}} \left( \|v\|_{Y_{19}^{\gamma}}^{2} + \|(u, v)\|_{X_{19}^{\gamma}}^{2} \right).
\]

The proof is similar to Part (a) of Lemma 3.1 and we omit it here.

Since well-posedness of the Euler system in Sobolev space or Gevrey class is well-known, here we sketch it in our frame for completeness. The a priori energy estimates are stated as follows.

**Proposition 7.2** Let the initial data \((u_0, v_0)\) satisfy \( \partial_{x} u_0 + \partial_{y} v_0 = 0 \) and \( v_0(t, x, 0) = 0 \). Moreover,

\[
\|(u_0, v_0)\|_{X_{20}^{\gamma}}^{2} \leq M < \infty.
\]

Then there exists \( T_{e} > 0 \) such that the Euler system (1.4) has a unique solution \( U_{e} = (u_{e}, v_{e}) \) on \([0, T_{e}]\), which satisfies

\[
\sup_{0 \leq t \leq T_{e}} \left( \|U_{e}\|_{X_{20}^{\gamma}}^{2} + \|\partial_{t} U_{e}\|_{X_{20}^{\gamma}}^{2} \right) \leq C_{0}.
\]
Proof. At first, we consider the vorticity equation of the system (1.4):

\[
\begin{align*}
\partial_t w^e + U^e \cdot \nabla w^e &= 0, \\
U^e \cdot n|_{y=0} &= 0.
\end{align*}
\] (7.13)

Acting \(\partial_{x,y}^\alpha\) on both sides of (7.13), then taking \(L^2\) inner product with \(\rho_e(t)2^{(n-19)}\partial_{x,y}^\alpha w^e\), integrating by parts and summing over \(|\alpha| \geq 19\), we arrive at

\[
\frac{1}{2} \frac{d}{dt} \|w^e\|_{X_{19}^1}^2 + \lambda_e \|w^e\|_{Y_{19}^1}^2 \leq C_0 \sum_{m=19}^{\infty} \rho_e(t)2^{(m-19)} \left( \sum_{|\alpha|=m} \left| \left\langle \partial_{x,y}^\alpha (U^e \cdot \nabla w^e), \partial_{x,y}^\alpha w^e \right\rangle \right| \right).
\] (7.14)

By Lemma 7.1, the right side term can be controlled by

\[
C_0 \|(U^e, w^e)\|_{X_{19}^1} \|((U^e, w^e))\|_{X_{19}^1}^2 + \|w^e\|_{Y_{19}^1}^2.
\]

Thus, we arrive at

\[
\frac{1}{2} \frac{d}{dt} \|w^e\|_{X_{19}^1}^2 + \lambda_e \|w^e\|_{Y_{19}^1}^2 \leq C_0 \|(U^e, w^e)\|_{X_{19}^1} \|((U^e, w^e))\|_{X_{19}^1}^2 + \|w^e\|_{Y_{19}^1}^2.
\] (7.14)

Similarly, the energy estimate in Sobolev space \(H^{18}\) can be obtained as follows

\[
\frac{1}{2} \frac{d}{dt} \|w^e\|_{H^{18}}^2 \leq C_0 \|U^e\|_{H^{18}} \|w^e\|_{H^{18}}^2.
\] (7.15)

To close the estimates, we have to recover the estimates of the velocity. Due to

\[
\triangle v^e = -\partial_x w^e, \quad v^e|_{y=0} = 0,
\]

it is easy to obtain

\[
\|\nabla v^e\|_{X_{18}^2}^2 \leq C_0 \|w^e\|_{X_{18}^2}^2.
\]

Note that

\[
\partial_y u^e = w^e + \partial_x v^e, \quad \partial_x u^e = -\partial_y v^e,
\]

and we also have

\[
\|\nabla u^e\|_{X_{18}^2}^2 \leq C_0 \|w^e\|_{X_{18}^2}^2.
\]

Therefore,

\[
\|U^e\|_{X_{18}^1}^2 \leq C_0 \|w^e\|_{X_{18}^1}^2.
\] (7.16)

Similarly, a straightforward computation gives

\[
\|U^e\|_{H^{20}}^2 \leq C_0 \left( \|w^e\|_{H^{19}}^2 + \|U^e\|_{L^2}^2 \right), \quad \|w^e\|_{X_{18}^1}^2 \leq C_0 \|w^e\|_{X_{18}^1}^2.
\] (7.17)

On the other hand, it is direct to get from the velocity equation (1.4) that

\[
\frac{1}{2} \frac{d}{dt} \|U^e\|_{L^2}^2 = 0.
\] (7.18)
Finally, from (7.14)-(7.18) we deduce that
\[
\frac{1}{2} \frac{d}{dt} \left( \|w^e\|_{L^2}^2 + \|U^e\|_{L^2}^2 \right) + \lambda_e \|w^e\|_{L^2}^2 \leq C_0 \left( \|w^e\|_{X^0}^2 + \|U^e\|_{L^2} \right) + C_0 \left( \|w^e\|_{X^1}^2 + \|U^e\|_{L^2} \right)^3. \tag{7.19}
\]

Here we take \( \lambda_e = 4C_0M^\frac{1}{2} \) and \( T_e > 0 \) so that
\[
\rho_e(t) \geq 1 \quad \text{for} \quad t \in [0, T], \quad C_0T_eM \leq \frac{1}{2}.
\]

Thus, from (7.19), a continuity argument yields that
\[
\sup_{0 \leq t \leq T_e} \left( \|w^e\|_{X^0}^2 + \|U^e\|_{L^2}^2 \right) \leq C_0.
\]

Due to (7.16) and (7.17), there also holds
\[
\sup_{0 \leq t \leq T_e} \|U^e\|_{X^2}^2 \leq C_0.
\]

Moreover, from the equations (7.13), it is not difficult to get
\[
\sup_{0 \leq t \leq T_e} \|\partial_t w^e\|_{X^1}^2 \leq C_0.
\]

Consequently, similar arguments imply that
\[
\sup_{0 \leq t \leq T_e} \|\partial_t U^e\|_{X^1}^2 \leq C_0.
\]

### 7.2 Well-posedness of linear Prandtl-type equations in Gevrey class

To prove the well-posedness of linear Prandtl-type equations in Gevrey class, we introduce the following semi-norms and functional spaces:

\[
\|U^p\|_{X^k}^2 := \sum_{m=k}^{\infty} \rho_p(t)^2(m-k)^2 \sum_{|\alpha|=m} \int_{R^2} e^{\phi_p(t,z)} \partial^\alpha U^p \mid dxdz + \|U^p\|_{L^2}^2,
\]

\[
\|U^p\|_{Y^k}^2 := \sum_{m=k+1}^{\infty} \rho_p(t)^2(m-k)(m-k)! \sum_{|\alpha|=m} \int_{R^2} e^{\phi_p(t,z)} \partial^\alpha U^p \mid dxdz
\]

\[
+ \sum_{m=k}^{\infty} \rho_p(t)^2(m-k) \sum_{|\alpha|=m} \int_{R^2} e^{\phi_p(t,z)} \partial^\alpha U^p \mid dxdz + \|U^p\|_{L^2}^2,
\]

\[
\|U^p\|_{T^k}^2 := \int_{R^2} \left| e^{\phi_p(t,z)}U^p \right|^2 dxdz, \quad \|U^p\|_{L^2}^2 := \int_{R^2} \left| e^{\phi_p(t,z)}U^p \right|^2 dxdz
\]

where
\[
\rho_p(t) = 2 - \lambda_p t \geq 1, \quad \phi_p(t, z) = \rho_p(t)z^2, \quad \partial^\alpha = \partial_x^\alpha \partial_z^\alpha, \quad \partial^k = (\partial_z)^k \partial_x^k,
\]

and \( \lambda_p > 0 \) to be decided later. Moreover, we denote \( X^k_p \) by
\[
X^k_p = \{ u : \|u\|_{X^k_p}^2 < \infty \}.
\]
Lemma 7.3 Let $u$ be only the function of $x$, and

\[ A^m = \sum_{m=17}^{\infty} \rho_p(t)^{2(m-17)} \sum_{|\alpha|=m} \left| \langle \partial^{\alpha} (u \partial_x v), e^{2\phi_p} \tilde{\alpha} u^p \rangle \right|, \]
\[ B^m = \sum_{m=17}^{\infty} \rho_p(t)^{2(m-17)} \sum_{|\alpha|=m} \left| \langle \tilde{\alpha} u^p, e^{2\phi_p} \tilde{\alpha} u^p \rangle \right|. \]

Then there hold

\[ A^m \leq C_0 \left( \sum_{m=17}^{\infty} \rho_p(t)^{2(m-17)} \sum_{k=1}^{m+2} \| u \|^2_{H^5} \right)^{\frac{1}{2}} \| v \|^2_{X^{17}_p \cap Y^{17}_p}, \]
and

\[ B^m \leq C_0 \left( \sum_{m=17}^{\infty} \rho_p(t)^{2(m-17)} \sum_{k=1}^{m+2} \| u \|^2_{H^5} \right)^{\frac{1}{2}} \| v \|^2_{X^{17}_p \cap Y^{17}_p}. \]

The estimate of $A^m$ is similar as Part (b) of Lemma 3.1 and $B^m$ is similar to Lemma 3.3.

We omit the details here.

Proposition 7.4 Let $(u^e, v^c)$ be given in Proposition 7.2. There exists $T_p > 0$ such that the system (6.2) has a unique solution $U^p = (u^p, v^p)$ on $[0, T_p]$, which satisfies

\[ \sup_{0 \leq t \leq T_p} \left( \| U^p \|^2_{X^{17}_p} + \| \partial_{zz} U^p \|^2_{X^{14}_p} + \| \partial_t U^p \|^2_{X^{14}_p} \right) \leq C_0. \]

Proof. As in Proposition 7.2 we only derive a priori estimates. Recalling (6.2), $u^p = u_p^{(1)}$ satisfies

\[ \begin{align*}
& \partial_t u^p + u^p \partial_x u^p + \overline{w} \partial_x u^p + z \overline{\partial_y v_c \partial_x u^p} = \partial_x^2 u^p, \\
& \partial_x u^p(t, x, 0) = \beta u^e(t, x, 0), \quad u^p(t, x, \infty) = 0, \\
& u^p(0, x, z) = 0.
\end{align*} \tag{7.20} \]

Acting $\tilde{\alpha}$ on both sides of (7.20), then taking $L^2$ inner product with $\frac{\rho_p(t)^{2(|\alpha|-17)}}{((|\alpha|-17)!)^\frac{1}{2}} e^{2\phi_p(t,z)} \tilde{\alpha} u^p$ and summing over $|\alpha| \geq 17$, we arrive at

\[ \frac{1}{2} \frac{d}{dt} \left( \| u^p \|^2_{X^{17}_p} - \| u^p \|^2_{Z^2_p} \right) + \lambda_p \left( \| u^p \|^2_{Y^{17}_p} - \| U^p \|^2_{T^2_p} \right) \]
\[ - \sum_{m=17}^{\infty} \rho_p(t)^{2(m-17)} \sum_{|\alpha|=m} \langle \tilde{\alpha} \partial_{zz} u^p, e^{2\phi_p} \tilde{\alpha} u^p \rangle \]
\[ \leq C_0 \sum_{m=17}^{\infty} \rho_p(t)^{2(m-17)} \sum_{|\alpha|=m} \langle \tilde{\alpha} (u \partial_x \overline{w} + \overline{w} \partial_x u^p + z \overline{\partial_y v_c \partial_x u^p}), e^{2\phi_p} \tilde{\alpha} u^p \rangle. \tag{7.22} \]

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Similar to the dissipative term as in Lemma 4.2, we obtain

\[
\sum_{m=17}^{\infty} \frac{\rho_p(t)^{2(m-17)}}{((m-17)!)^{\frac{7}{2}}} \sum_{|\alpha|=m} \langle \partial^\alpha \partial_{zz} u^p, e^{2\phi_p} \partial_z u^p \rangle \\
\geq \left( \frac{3}{4} - C_0 \delta \right) \left( \|\partial_z u^p\|_{X_1^p}^2 - \|\partial_z u^p\|_{L_p^2}^2 \right) - C_0 \|U^p\|_{L_p^2}^2 \\
- \sum_{m=17}^{\infty} \frac{\rho_p(t)^{2(m-17)}}{((m-17)!)^{\frac{7}{2}}} \int \partial^n_x \partial_{zz} u^p \partial_z u^p dx \\
\geq \left( \frac{2}{3} - C_0 \delta \right) \left( \|\partial_z u^p\|_{X_1^p}^2 - \|\partial_z u^p\|_{L_p^2}^2 \right) - C_0 \|U^p\|_{L_p^2}^2 \\
- C_0 \sum_{m=17}^{\infty} \frac{\rho_p(t)^{2(m-17)}}{((m-17)!)^{\frac{7}{2}}} \|\partial_z u^p\|_{L_p^2}^2. \\
\tag{7.23}
\]

By Proposition 7.2, Sobolev embedding inequality, and Lemma 7.3 we have

\[
\left| \sum_{m=17}^{\infty} \frac{\rho_p(t)^{2(m-17)}}{((m-17)!)^{\frac{7}{2}}} \sum_{|\alpha|=m} \langle \partial^\alpha (u_x \partial_z u^p + u_{zz} \partial_x u^p), e^{2\phi_p} \partial_z u^p \rangle \right| \\
\leq C_0 \|u^p\|_{X_1^p \cap Y_1^p}^2 + \frac{1}{10} \|\partial_z u^p\|_{X_1^p}^2. \\
\tag{7.24}
\]

Note that for any \( m \in \mathbb{N}_+ \),

\[
[\tilde{Z}^m, z]f = m\delta_z \tilde{Z}^{m-1} f. \\
\tag{7.25}
\]

Using Proposition 7.2 and Sobolev embedding inequality again, it follows from a similar proof of Lemma 7.3 that

\[
\left| \sum_{m=17}^{\infty} \frac{\rho_p(t)^{2(m-17)}}{((m-17)!)^{\frac{7}{2}}} \sum_{|\alpha|=m} \langle \partial^\alpha (z \partial_{zz} u^p \partial_x u^p), e^{2\phi_p} \partial_z u^p \rangle \right| \\
\leq \sum_{m=17}^{\infty} \frac{\rho_p(t)^{2(m-17)}}{((m-17)!)^{\frac{7}{2}}} \sum_{|\alpha|=m} \sum_{\beta \leq \alpha} C_\beta^\alpha \langle \partial^\beta (\partial_y u^p) \partial_z u^p, e^{2\phi_p} z \partial_z u^p \rangle \\
+ \sum_{m=17}^{\infty} \frac{\rho_p(t)^{2(m-17)}}{((m-17)!)^{\frac{7}{2}}} \sum_{|\alpha|=m} \sum_{\beta \leq \alpha} C_\beta^\alpha \langle \partial^\beta (\partial_y u^p) \partial_z u^p, e^{2\phi_p} z \partial_z u^p \rangle \\
\leq C_0 \|u^p\|_{X_1^p \cap Y_1^p}^2 + \frac{1}{10} \|\partial_z u^p\|_{X_1^p}^2. \\
\tag{7.26}
\]

Taking \( \delta \) small, by (7.21)-(7.26), we arrive at

\[
\frac{1}{2} \frac{d}{dt} \left( \|u^p\|_{X_1^p}^2 + \|u^p\|_{L_p^2}^2 \right) + (\lambda_p - C_0) \|u^p\|_{X_1^p}^2 + \frac{1}{2} \|\partial_z u^p\|_{L_p^2}^2 \\
\leq C_0 \|u^p\|_{X_1^p}^2 + \frac{1}{10} \|\partial_z u^p\|_{H_1^p}^2.
\]

Moreover, it is easy to get

\[
\frac{1}{2} \frac{d}{dt} \|u^p\|_{L_p^2}^2 + (\lambda_p - C_0) \|u^p\|_{L_p^2}^2 + \frac{1}{2} \|\partial_z u^p\|_{L_p^2}^2 \leq C_0 (\|u^p\|_{L_p^2}^2 + 1).
\]

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Consequently, we arrive at
\[
\frac{1}{2} \frac{d}{dt} \| u^p \|^2_{X^p_{17}} + (\lambda_p - C_0) \| u^p \|^2_{Y^p_{17}} + \frac{1}{3} \| \partial_z u^p \|^2_{X^p_{17}} \leq C_0 \left( \| u^p \|^2_{X^p_{17}} + 1 \right).
\]

Then, by taking \( \lambda_p \geq C_0 \), there exists \( 0 < T_p \leq T_e \) such that
\[
\sup_{0 \leq t \leq T_p} \| u^p \|^2_{X^p_{17}} \leq C_0.
\]

Furthermore, due to
\[
\partial_{z} u^p + \partial_{z} v^p = 0, \quad \lim_{z \to \infty} v^p(t, x, z) = 0,
\]
we have
\[
\sup_{0 \leq t \leq T_p} \| v^p \|^2_{X^p_{16}} \leq C_0.
\]

Applying \( \partial_t \) to the equation (7.20), similar energy estimates yield that
\[
\sup_{0 \leq t \leq T_p} \| \partial_t (u^p, v^p) \|^2_{X^p_{14}} \leq C_0,
\]
and using the equation (7.20) again, we get
\[
\sup_{0 \leq t \leq T_p} \| \partial_{zz} (u^p, v^p) \|^2_{X^p_{14}} \leq C_0.
\]

\[\square\]

7.3 Verification of the Assumptions (H) for the case of \( \gamma = \frac{1}{2} \)

Making similar arguments as in Proposition 7.2 and Proposition 7.4, we can show that the linearized Euler equations (6.1) and linear Prandtl-type equations (6.3), (6.4) with the associated initial and boundary condition can also be solved, in \( X^p_{18}, X^p_{16} \) and \( X^p_{15} \), respectively. In conclusion, we obtain the following estimates for approximate solutions:

\[
\begin{align*}
&\| (u^{(0)}_e, v^{(0)}_e) \|_{X^p_{19}} + \| \partial_t (u^{(0)}_e, v^{(0)}_e) \|_{X^p_{18}} \leq C_0, \\
&\| (u^{(3)}_e, v^{(3)}_e) \|_{X^p_{18}} + \| \partial_t (u^{(3)}_e, v^{(3)}_e) \|_{X^p_{17}} \leq C_0, \\
&\| (u^{(1)}_p, v^{(3)}_p) \|_{X^p_{16}} + \| \partial_t (u^{(1)}_p, v^{(3)}_p) \|_{X^p_{14}} + \| \partial_{zz} (u^{(1)}_p, v^{(3)}_p) \|_{X^p_{14}} \leq C_0, \\
&\| (u^{(2)}_p, v^{(4)}_p) \|_{X^p_{15}} + \| \partial_t (u^{(2)}_p, v^{(4)}_p) \|_{X^p_{13}} + \| \partial_{zz} (u^{(2)}_p, v^{(4)}_p) \|_{X^p_{13}} \leq C_0, \\
&\| (u^{(3)}_p, v^{(5)}_p) \|_{X^p_{14}} + \| \partial_t (u^{(3)}_p, v^{(5)}_p) \|_{X^p_{12}} + \| \partial_{zz} (u^{(3)}_p, v^{(5)}_p) \|_{X^p_{12}} \leq C_0.
\end{align*}
\]

(7.27)

Then Lemma 6.1 follows easily.

Using (6.9) and (6.10), it is also easy to prove Lemma 6.2.

Note that (6.6) and
\[
\begin{align*}
&v^{(4)}_p = -\int_0^\infty \partial_z u^{(2)}_p(t, x, z) \, dz, \\
&v^{(5)}_p = -\int_0^\infty \partial_z u^{(3)}_p(t, x, z) \, dz,
\end{align*}
\]
and hence,
\[
f(t, x) = -\partial_x \left( \int_0^\infty u_0^{(2)}(t, x, z) \, dz + \varepsilon \frac{1}{2} \int_0^\infty u_0^{(3)}(t, x, z) \, dz \right) = \partial_x \mathcal{F}(t, x).
\]

By (7.27), a direct computation yields that
\[
\|f\|_{X^5} + \|\partial_t f\|_{X^3} + \|\partial_t \mathcal{F}\|_{L^2_x}(t) \leq C_0.
\]

Finally, due to (6.7) and (7.27), we also get
\[
\|g_0\|_{X^5} + \|\partial_t g_0\|_{X^3} \leq C_0.
\]

Accordingly, Lemma 6.3 is proved. Thus, the Assumptions (H) in Section 3 for \(\gamma = \frac{1}{2}\) is satisfied.

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