A free boundary problem in biological selection models

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Abstract
We prove local existence for classical solutions of a free boundary problem which arises in one of the biological selection models proposed by Brunet and Derrida. The problem we consider describes the limit evolution of branching brownian particles on the line with death of the leftmost particle at each creation time as studied in [7]. We use extensively results in [1] and [2].

1 Introduction
Brunet and Derrida [3], have proposed a class of models for biological selection processes including the one we consider here. This is a system of $N$ brownian particles on the line which branch independently at rate 1 creating a new brownian particle on the same position of the father; simultaneously the leftmost particle (which is the less fit) disappears. Thus the total number of particles does not change.

In [7] it is proved that the particles density has a limit as $N$ diverges (under suitable assumptions on the initial datum). It is also proved that if the following FBP has a “classical solution” then this is the same as the limit density of the branching brownians, similar results have been obtained for other FBP, see [6].

Let $b \in \mathbb{R}$, say $b > 0$, and $\rho_0 \in C_c^2([b, \infty))$ such that $\rho_0(b) = 0$, $\frac{d}{dx}\rho_0(b) = 2$, $\int_b^\infty \rho_0(x)dx = 1$. The free boundary problem FBP (∗) that we consider is: fixed $b$ and $\rho_0$ as above find a continuous curve $L_t$, $t \geq 0$, starting from $L_0 = b$, and $\rho(x, t)$, $x \geq L_t; t \geq 0$, so that

\[
\begin{cases}
\rho_t(x, t) = \frac{1}{2}\rho_{xx}(x, t) + \rho(x, t), & \text{if } L_t < x, \ t > 0, \\
\rho(L_t, t) = 0, & \text{if } t \geq 0, \\
\rho(x, 0) = \rho_0(x), & \text{if } b \leq x, \\
\int_{L_t}^\infty \rho(x, t)dx = \int_b^\infty \rho_0(x)dx = 1, & \text{if } t > 0.
\end{cases}
\]

Classical solutions of the FBP (∗) are defined in the next section. In this paper we prove the local in time existence of a classical solution of the FBP (∗), observing that uniqueness follows from the results in [7]. Traveling wave solutions and the fine asymptotics of $L_t$ are studied in [8] for a large class of FBP including our FBP (∗).

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2 Main results

By classical solutions we mean the following:

Definition. The pair \((L, \rho)\) is a classical solution of the FBP \((*)\) in the time interval \([0, T]\), \(T > 0\), if:

- \(L \in C^1([0, T]), \; L_0 = b\)
- \(\rho \in C(D_{L,T}) \cap C^{2,1}(D_{L,T})\), where \(D_{L,T} = \{ (x, t) : L_t < x, \; 0 < t < T \}\)
- \((L, \rho)\) satisfies the equations in \((*)\).

Theorem 1 There is \(T > 0\) so that the FBP \((*)\) has a classical solution in the time interval \([0, T]\) (denoted by \((L, \rho)\)).

We will prove Theorem 1 in the next sections. We will also prove that the derivative \(\rho_x(x, t)\) has a limit when \(x \to L_t\) denoted by \(\rho_x(L_t, t)\) and that it vanishes as \(x \to \infty\). This together with the conservation law implies \(\rho_x(L_t, t) = 2\) for all \(t \in [0, T]\). Furthermore if also \(\rho_{xx}(L_t, t)\) exists, by differentiating \(\rho(L_t, t) = 0\) we get \(\dot{L}_t = -\frac{1}{4} \rho_{xx}(L_t, t)\).

Given \((L, \rho)\) as above we define \(v(x, t) := e^{-t} \rho_x(x, t)\) and observe that \((L, v)\) satisfies the following free boundary problem FBP \((**):\)

\[
\begin{cases}
    v_t(x, t) = \frac{1}{2} v_{xx}(x, t), & \text{if } L_t < x, \; t > 0, \\
    v(L_t, t) = 2e^{-t}, & \text{if } t > 0, \\
    v(x, 0) = h(x), & h \in C_c^1([b, \infty)) \\
    \dot{L}_t = -\frac{1}{4} e^t v_x(L_t, t), & \text{if } t > 0,
\end{cases}
\]

To prove Theorem 1 we will first prove the existence of a classical solution of the FBP \((**)\) in \([0, T]\):

Theorem 2 There is \(T > 0\) and a pair \((L, v)\) which satisfies the FBP \((**)\) in \([0, T]\) with: \(L \in C^1([0, T]), \; L_0 = b\), and \(v \in C^{2,1}(D_{L,T})\), where \(D_{L,T} = \{ (x, t) : L_t < x, \; 0 < t \leq T \}\).

We did not find a proof of Theorem 2 in the existing literature, see for instance [4], [5] and references therein. Our proof exploits the one dimensionality of the problem and uses extensively the Cannon estimates, [1], following the strategy proposed by Fasano in [2]. In the next sections we will prove Theorem 2 and then, taking \(h\) equal to the space derivative of \(\rho_0\), we will prove Theorem 1 as a corollary of Theorem 2.
3 Strategy of the proof of Theorem 2

The idea is to reduce the analysis of the FBP (**) to a fixed point problem:

- Take a curve \( L_t, t \in [0, T] \) and find \( v \) which solves from the first equation to the third one of (**) in the domain \([L_t, \infty), t \in [0, T]\)
- Construct a new curve \( K[L](t) = b - \frac{1}{4} \int_0^t e^\tau v_x(L_\tau, \tau) d\tau \) for \( 0 \leq t \leq T \)
- Find \( L \) so that \( K[L] = L \) and prove that the corresponding pair \((L, v)\) solves the FBP (**)

The first task is to prove existence and smoothness of \( v \), that we do in this section using the lemmas below, see [1]. Let \( A > 0 \) and

\[
\Sigma(A, T) := \left\{ L \in C[0, T] : L_0 = b, \left| \frac{L_{t_2} - L_{t_1}}{t_2 - t_1} \right| \leq A \text{ for } 0 \leq t_1 < t_2 \leq T \right\}.
\]

We define \( C^0_{(1)}((0, T]) \) as the subspace of \( C((0, T]) \) that consists of those functions \( \varphi \) such that

\[
\| \varphi \|^{(1)}_T = \sup_{0 < t \leq T} |\varphi(t)| < \infty. \tag{3.1}
\]

Thus \( C^0_{(1)}((0, T]) \) is a Banach space under the norm \( \| \cdot \|^{(1)}_T \).

Let us denote by \( G(x, t; \xi, \tau) \) the Gaussian kernel

\[
\frac{1}{\sqrt{2\pi(t-\tau)}} \exp \left\{ -\frac{|x-\xi|^2}{2(t-\tau)} \right\}
\]

which satisfies \( G_t = \frac{1}{2} G_{xx} \) and \( G_\tau = -\frac{1}{2} G_{\xi\xi} \).

Lemma 1 (jump relation) Let \( L \in \Sigma(A, T) \) and \( \varphi \in C^0_{(1)}((0, T]) \). Define

\[
w_\varphi(x, t; L) := \int_0^t G(x, t; L_\tau, \tau) \varphi(\tau) d\tau.
\]

Then

\[
\lim_{x \to L_t} \frac{\partial w_\varphi}{\partial x}(x, t; L) = -\varphi(t) + \int_0^t G_x(L_t, t; L_\tau, \tau) \varphi(\tau) d\tau. \tag{3.2}
\]

Proof See Lemma 14.2.5. of [1].

Lemma 2 Let \( L \in \Sigma(A, T) \), then there exists a unique \( \varphi \in C^0_{(1)}((0, T]) \) such that

\[
\int_b^\infty h(\xi) G(L_t, t; \xi, 0) d\xi - \varphi(t) + \int_0^t G_x(L_t, t; L_\tau, \tau) \varphi(\tau) d\tau = 2e^{-t} \text{ for all } t \in (0, T].
\]
Proof Since $|G_x(L_t; L_t, \tau)| \leq \frac{C}{\sqrt{t-\tau}}$ and $\int_0^\infty h(\xi)G(L_t; t; \xi, 0)d\xi \in C^1_{(1)}((0, T])$, we have a Volterra integral equation:

$$\varphi(t) = \psi(t) + \int_0^t G_x(L_t; t; L_t, \tau)\varphi(\tau)d\tau$$

(3.3) where $\psi(t) = -2e^{-t} + \int_0^\infty h(\xi)G(L_t; t; \xi, 0)d\xi \in C^1_{(1)}((0, T])$. As in [1] p.247 the right hand side of (3.3) defines a contraction map from $C^1_{(1)}((0, T])$ into itself for $T$ small. Thus there is a unique $\varphi \in C^1_{(1)}((0, T])$ which satisfies (3.3). •

**Proposition 1** For any $L \in \Sigma(A, T)$ let $\varphi$ as in Lemma 2. Define

$$v(x, t) := \int_0^\infty h(\xi)G(x, t; \xi, 0)d\xi + \int_0^t G_x(x, t; L_t, \tau)\varphi(\tau)d\tau$$

(3.4) Then $v \in C^{2,1}(D_{L,T})$ satisfies from the first equation to the third one of (**) and $\lim_{x \to \infty} \sup_{0 \leq t \leq T} |v(x, t)| = 0$. In addition, $v$ has the right derivative at the boundary $v_x(L_t, t) \in C^0_{(1)}((0, T])$.

Proof By (3.4) $v$ satisfies the heat equation with initial datum $h$ and boundary conditions $v(L_t, t) = 2e^{-t}$, $\lim_{x \to \infty} \sup_{0 \leq t \leq T} |v(x, t)| = 0$. Let us choose $R > \sup_{t \in [0, T]} L_t$. Then we have $v(R, t) \in C^0_{(1)}((0, T])$ such that by Lemma 14.4. and Theorem 14.4. of [1], we also obtain $v_x(L_t, t) \in C^0_{(1)}((0, T])$. •

4 Proof of Theorem 2

Theorem 2 is proved at the end of the section.

**Lemma 3** Let $L \in \Sigma(A, T)$ and $v$ as in (3.4), then there are positive constants $C_1, C_2, C_3$ such that

$$|v_x(L_t, t)| \leq C_1 \int_0^t \frac{|v_x(L_t, \tau)|}{\sqrt{t-\tau}}d\tau + (C_2\sqrt{t} + C_3).$$

(4.5) Moreover, we have $v_x(L_t, t) \in C([0, T])$.

Proof Let us fix $(x, t) \in D_{L,T}$ and let us define $D^{(i)}_{\epsilon,R} := \{ (\xi, \tau) : L_t + \epsilon < \xi < R, \epsilon < \tau < t - \epsilon \}$ for each $\epsilon > 0$ and $R \in \mathbb{R}$. By the Green’s identity, we have

$$\frac{1}{2}(v_xG - vG_x)\xi - (vG)_\tau = 0 \implies \int_{\partial D^{(i)}_{\epsilon,R}} \frac{1}{2}(v_xG - vG_x)d\tau + (vG)d\xi = 0.$$

(4.6)
Hence we obtain another representation of $v$ by letting $\epsilon \to 0$, $R \to \infty$,

$$v(x, t) = \int_b^\infty h(\xi)G(x, t; \xi, 0)d\xi - \frac{1}{2} \int_0^t G(x, t; L_\tau, \tau)v_\xi(L_\tau, \tau)d\tau + \int_0^t e^{-\tau}G_\xi(x, t; L_\tau, \tau)d\tau.$$ (4.7)

Differentiating both sides of (4.7) with respect to $x$ and integrating by parts, we get

$$v_x(x, t) = \int_b^\infty h_\xi(\xi)G(x, t; \xi, 0)d\xi - \frac{1}{2} \int_0^t G_x(x, t; L_\tau, \tau)v_x(L_\tau, \tau)d\tau + 2 \int_0^t e^{-\tau}G(x, t; L_\tau, \tau)d\tau,$$ (4.8)

$$\frac{1}{2}v_x(L_t, t) = \int_b^\infty h_\xi(\xi)G(L_t, t; \xi, 0)d\xi - \frac{1}{2} \int_0^t G_x(L_t, t; L_\tau, \tau)v_x(L_\tau, \tau)d\tau + 2 \int_0^t e^{-\tau}G(L_t, t; L_\tau, \tau)d\tau.$$ (4.9)

Since $|G_x(L_t, t; L_\tau, \tau)| \leq \frac{C}{\sqrt{t - \tau}}$ and there are constants $C_2$ and $C_3$ such that

$$\left| 2 \int_0^t e^{-\tau}G(x, t; L_\tau, \tau)d\tau \right| \leq C_2 \sqrt{t}, \quad \left| \int_b^\infty h_\xi(\xi)G(x, t; \xi, 0)d\xi \right| \leq \|h_\xi\|_\infty \leq C_3,$$ (4.10)

then

$$|v_x(L_t, t)| \leq (C_2 \sqrt{t} + C_3) + C_1 \int_0^t \frac{|v_x(L_\tau, \tau)|}{\sqrt{t - \tau}}d\tau.$$ (4.11)

Using (4.9), we have

$$v_x(L_t, t) \in C([0, T]).$$ (4.12)

**Lemma 4** Let $A > \frac{C_3}{4}$, $C_3$ as in Lemma 3. Then for all sufficiently small $T$, $K[L](t) := b - \frac{1}{4} \int_0^t e^\tau v_x(L_\tau, \tau)d\tau$, $0 \leq t \leq T$, maps $K : \Sigma(A, T) \to \Sigma(A, T)$.

**Proof** By Lemma 17.7.1. in [1] applied to (4.11), we obtain

$$|v_x(L_t, t)| \leq [1 + 2C_1 \sqrt{T}]\exp\{\pi C_1^2 T\}(C_2 \sqrt{T} + C_3).$$ (4.13)

Then $\left| \frac{d}{dt}K[L](t) \right| = \frac{1}{4} |e^tv_x(L_t, t)| \leq \frac{1}{4} e^T[1 + 2C_1 \sqrt{T}]\exp\{\pi C_1^2 T\}(C_2 \sqrt{T} + C_3) \leq A$ for all sufficiently small $T$ so that the map $K$ is well defined.
Hereafter $A > \frac{C_3}{4}$ is fixed. We will show that $K$ is continuous, then, since $\Sigma(A, T)$ is convex and compact we can apply the Schauder fixed point theorem to conclude that $K$ has a fixed point.

**Lemma 5** For any sufficiently small $T > 0$ the map $K$ defined in Lemma 4 is continuous on $\Sigma(A, T)$ with sup norm.

**Proof**

Let $T > 0$ be such that $0 < b - AT$ and so small that $K$ is well defined. For $L \in \Sigma(A, T)$, we have $0 < b - AT \leq \inf L \leq \sup L \leq b + AT$. Let $v$ be the function determined by $L$ via Proposition 4. Then $v$ satisfies Green’s identity as follows; for $D_{\epsilon, R}$ defined in the proof of Lemma 3.

\[
\oint_{\partial D_{\epsilon, R}} [\xi v d\xi + \frac{1}{2} (\xi v_\xi - v)] d\tau = 0. \tag{4.14}
\]

Let $v^{(1)}$, $v^{(2)}$ be the functions which correspond to $L^{(1)}$, $L^{(2)} \in \Sigma(A, T)$. We denote $K[L^{(i)}] = \sigma^{(i)}$, $i = 1, 2$, we apply (4.14) and let $\epsilon \to 0$, $R \to \infty$. We get

\[
\int_0^t 2e^{-\tau} L^{(1)}_\tau \left[ \frac{d}{d\tau} (\sigma^{(1)}_\tau - \sigma^{(2)}_\tau) \right] d\tau + \int_0^t 2e^{-\tau} \left[ \frac{d}{d\tau} \sigma^{(2)}_\tau \right] (L^{(1)}_\tau - L^{(2)}_\tau) d\tau
\]

\[= \int_{L^{(1)}_\tau} \xi v^{(1)}(\xi, t) d\xi - \int_{L^{(2)}_\tau} \xi v^{(2)}(\xi, t) d\xi. \]

By integration by parts we have

\[
2e^{-t} L^{(1)}_t \left[ \sigma^{(1)}_t - \sigma^{(2)}_t \right] = \int_0^t (-2e^{-\tau} L^{(1)}_\tau + 2e^{-\tau} \frac{d}{d\tau} L^{(1)}_\tau)(\sigma^{(1)}_\tau - \sigma^{(2)}_\tau) d\tau
\]

\[ - \int_0^t 2e^{-\tau} \left[ \frac{d}{d\tau} \sigma^{(2)}_\tau \right] (L^{(1)}_\tau - L^{(2)}_\tau) d\tau + \left[ \int_{L^{(1)}_\tau} \xi v^{(1)}(\xi, t) d\xi - \int_{L^{(2)}_\tau} \xi v^{(2)}(\xi, t) d\xi \right]. \tag{4.15}
\]

To control $\int_{L^{(1)}_\tau} \xi v^{(1)}(\xi, t) d\xi - \int_{L^{(2)}_\tau} \xi v^{(2)}(\xi, t) d\xi$, using (4.7) and Fubini’s theorem, we obtain for $i = 1, 2$:

\[
\int_{L^{(i)}_\tau} x v^{(i)}(x, t) dx = \int_0^\infty h(\xi) \int_{L^{(i)}_\tau} x G(x, t; \xi, 0) dx d\xi - \frac{1}{2} \int_0^t \int_{L^{(i)}_\tau} v^{(i)}(L^{(i)}_\tau, \tau) \int_{L^{(i)}_\tau} x G(x, t; L^{(i)}_\tau, \tau) dx d\tau
\]

\[+ \int_0^t e^{-\tau} \int_{L^{(i)}_\tau} x G(x, t; L^{(i)}_\tau, \tau) dx d\tau. \tag{4.16}
\]
Taking the difference for the first term of $[4.16]$, we have
\[
I_1 := \int_b^\infty h(\xi)t[G(L_t^{(1)}, t; \xi, 0) - G(L_t^{(2)}, t; \xi, 0)]d\xi + \int_{L_t^{(1)}}^{L_t^{(2)}} \int_b^\infty \xi h(\xi)G(x, t; \xi, 0)d\xi dx.
\]
so that \(|I_1| \leq C \sup_{0 \leq \tau \leq T} |L_\tau^{(1)} - L_\tau^{(2)}|.

Since
\[
-\frac{1}{2} \int_0^t \dot{v}_t^{(i)}(L_t^{(i)}, \tau) \int_{L_t^{(i)}}^{\infty} xG(x, t; L_t^{(i)}, \tau)d\tau d\tau = 2 \int_0^t e^{-\tau} \left[ \frac{d}{d\tau}(\sigma_t^{(1)} - \sigma_t^{(2)}) \right] \int_{L_t^{(i)}}^{\infty} xG(x, t; L_t^{(i)}, \tau)d\tau d\tau,
\]
we also have by taking the difference for the second term of $[4.16]$,
\[
I_2 := 2 \int_0^t e^{-\tau} \frac{d}{d\tau}(\sigma_t^{(1)} - \sigma_t^{(2)}) \int_{L_t^{(i)}}^{\infty} xG(x, t; L_t^{(i)}, \tau)d\tau d\tau
\]
\[
+ 2 \int_0^t \left[ \int_{L_t^{(i)}}^{\infty} xG(x, t; L_t^{(i)}, \tau)d\tau - \int_{L_t^{(i)}}^{\infty} xG(x, t; L_t^{(i)}, \tau)d\tau \right] e^{-\tau} \frac{d}{d\tau} \sigma_t^{(2)} d\tau.
\]
By integration by parts, we get for the first term on right hand side of $[4.17]$
\[
2 \int_0^t e^{-\tau} \frac{d}{d\tau}(\sigma_t^{(1)} - \sigma_t^{(2)}) \int_{L_t^{(i)}}^{\infty} xG(x, t; L_t^{(i)}, \tau)d\tau d\tau
\]
\[
= e^{-\tau}[\sigma_t^{(1)} - \sigma_t^{(2)}]L_t^{(i)} - 2 \int_0^t (\sigma_t^{(1)} - \sigma_t^{(2)}) \frac{\partial}{\partial \tau} \left( e^{-\tau} \int_{L_t^{(i)}}^{\infty} xG(x, t; L_t^{(i)}, \tau)d\tau \right) d\tau.
\]
so that
\[
I_2 = e^{-\tau}[\sigma_t^{(1)} - \sigma_t^{(2)}]L_t^{(i)} - 2 \int_0^t (\sigma_t^{(1)} - \sigma_t^{(2)}) \frac{\partial}{\partial \tau} \left( e^{-\tau} \int_{L_t^{(i)}}^{\infty} xG(x, t; L_t^{(i)}, \tau)d\tau \right) d\tau
\]
\[
+ 2 \int_0^t \left[ \int_{L_t^{(i)}}^{\infty} xG(x, t; L_t^{(i)}, \tau)d\tau - \int_{L_t^{(i)}}^{\infty} xG(x, t; L_t^{(i)}, \tau)d\tau \right] e^{-\tau} \frac{d}{d\tau} \sigma_t^{(2)} d\tau.
\]
To estimate the last term of $[4.18]$, we use the following identity for i=1,2;
\[
\int_{L_t^{(i)}}^{\infty} xG(x, t; L_t^{(i)}, \tau)d\tau = -(t - \tau) \int_{L_t^{(i)}}^{\infty} G_x(x, t; L_t^{(i)}, \tau)d\tau + L_t^{(i)} \int_{L_t^{(i)}}^{\infty} G(x, t; L_t^{(i)}, \tau)d\tau
\]
\[
= (t - \tau)G(L_t^{(i)}, t; L_t^{(i)}, \tau) + L_t^{(i)} \int_{L_t^{(i)}}^{\infty} G(x, t; L_t^{(i)}, \tau)d\tau = (t - \tau)G(L_t^{(i)}, t; L_t^{(i)}, \tau) + L_t^{(i)} \Psi \left( \frac{L_t^{(i)} - L_t^{(i)}}{\sqrt{t - \tau}} \right),
\]
where \(\Psi(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}} \exp \left\{ -\frac{|z|^2}{2} \right\} dz\).
By applying the mean value theorem on $\Psi$, we have
\[
\left| \Psi\left( \frac{L_t^{(1)}(t) - L_t^{(2)}(t)}{\sqrt{t - \tau}} \right) - \Psi\left( \frac{L_t^{(2)}(t) - L_t^{(2)}(\tau)}{\sqrt{t - \tau}} \right) \right| \leq C_1 \left| \frac{L_t^{(1)}(1) - L_t^{(2)}(1) - L_t^{(2)}(2) + L_t^{(2)}}{\sqrt{t - \tau}} \right| \leq C_1 \sup_{0 \leq \eta \leq T} \left| L_\eta^{(1)}(1) - L_\eta^{(2)}(2) \right|
\]
By applying the mean value theorem on $\exp\left\{ -\frac{1}{2} \right\}$, for some $0 < \theta < 1$,
\[
\left| G(L_t^{(1)}, t; L_t^{(1)}, \tau) - G(L_t^{(2)}, t; L_t^{(2)}, \tau) \right|
\leq \frac{1}{\sqrt{2\pi(t - \tau)}} \left\| \theta L_t^{(1)}(1) - L_t^{(2)}(1) \right\| (1 - \theta) \left\| L_t^{(1)}(2) - L_t^{(2)}(2) \right\| \left\| L_t^{(1)}(1) - L_t^{(1)}(2) + L_t^{(2)}(2) \right\|
\leq C_2 \sup_{0 \leq \eta \leq T} \left| L_\eta^{(1)} - L_\eta^{(2)} \right|
\]
Thus we obtain
\[
\left| \int_{L_t^{(1)}}^\infty xG(x; t; L_t^{(1)}, \tau)dx - \int_{L_t^{(2)}}^\infty xG(x; t; L_t^{(2)}, \tau)dx \right| \leq C_3 \sup_{0 \leq \eta \leq T} \left| L_\eta^{(1)} - L_\eta^{(2)} \right| + C_4 \sup_{0 \leq \eta \leq T} \left| L_\eta^{(1)} - L_\eta^{(2)} \right|
\]
Also we observe
\[
\left| \frac{\partial}{\partial \tau} \left( e^{-\tau} \int_{L_t^{(1)}}^\infty xG(x; t; L_t^{(1)}, \tau)dx \right) \right|
\leq \left| \frac{\partial}{\partial \tau} \left( e^{-\tau} \left\{ (t - \tau)G(L_t^{(1)}, t; L_t^{(1)}, \tau) + L_t^{(1)}(1) \Psi\left( \frac{L_t^{(1)}(1) - L_t^{(1)}(2)}{\sqrt{t - \tau}} \right) \right\} \right| \right| \leq C_5 + \frac{C_6}{\sqrt{t - \tau}}.
\]
For the third term of (4.16), we have
\[
\int_0^t e^{-\tau} \int_{L_t^{(i)}}^\infty xG(x; t; L_t^{(i)}, \tau)d\tau = \int_0^t e^{-\tau} \left[ \int_{L_t^{(i)}}^\infty G(x; t; L_t^{(i)}, \tau)d\tau \right] d\tau.
\]
Then we have
\[
I_3 := \int_0^t e^{-\tau} \left[ L_t^{(1)}(1)G(L_t^{(1)}, t; L_t^{(1)}, \tau) + L_t^{(1)}G(L_t^{(1)}, t; L_t^{(1)}, \tau) - L_t^{(2)}G(L_t^{(2)}, t; L_t^{(2)}, \tau) - L_t^{(2)}G(L_t^{(2)}, t; L_t^{(2)}, \tau) \right] d\tau
\]
so that
\[
|I_3| \leq C_7 \sup_{0 \leq \eta \leq T} \left| L_\eta^{(1)} - L_\eta^{(2)} \right|
\]
Combining all the results and (4.15) finally we obtain for all sufficiently small $T > 0$,
\[
|\sigma_t^{(1)} - \sigma_t^{(2)}| \leq C_8 \int_0^t \left| \sigma_t^{(1)} - \sigma_t^{(2)} \right| d\tau + C_9 \int_0^t \left| \frac{\sigma_t^{(1)} - \sigma_t^{(2)}}{\sqrt{t - \tau}} \right| d\tau + C_{10} \sup_{0 \leq \tau \leq T} \left| L_\tau^{(1)} - L_\tau^{(2)} \right|
\leq (C_8 + C_9) \int_0^t \left| \frac{\sigma_t^{(1)} - \sigma_t^{(2)}}{\sqrt{t - \tau}} \right| d\tau + C_{10} \sup_{0 \leq \tau \leq T} \left| L_\tau^{(1)} - L_\tau^{(2)} \right|.
\]

By Lemma 17.7.1 in [1], we get $\sup_{0 \leq \tau \leq T} \left| \sigma_t^{(1)} - \sigma_t^{(2)} \right| \leq C_{11} \sup_{0 \leq \tau \leq T} \left| L_\tau^{(1)} - L_\tau^{(2)} \right|$ so that $K$ is a continuous map.

**Proof of Theorem 2**
Let $(L, v)$ as in Lemma 5 then by (4.12) $K[L] = L$ is in $C^1([0, T])$. This completes the proof.

**5 Proof of Theorem 1**

Suppose that $h(x) = \frac{d\rho_0(x)}{dx}$ and let $(L, v)$ be as in Theorem 2 with such $h$ as initial condition. Let $\rho(x, t) := \int_{L_t} e^{t}v(y, t)dy$. It can be readily shown that $\rho$ solves from the first to the third equations in (*), see Section 4. To prove the last equation in (*) we first remark that by (3.4) $v \in L^1([L_t, \infty))$. We then differentiate $\int_{L_t} \infty v(y, t)dy$ with respect to $t$:

$$
\frac{d}{dt} \left( \int_{L_t} \infty v(x, t)dx \right) = -L_t v(L_t, t) + \int_{L_t} \infty v(x, t)dx
$$

$$
= -2L_t e^{-t} + \int_{L_t} \infty v_{xx}(x, t) = -2L_t e^{-t} - \frac{1}{2} v_x(L_t, t) = 0.
$$

Since $\int_{b}^{\infty} h(x)dx = \int_{b}^{\infty} \frac{d\rho_0(x)}{dx}dx = 0$, we obtain that $\int_{L_t} \infty v(x, t)dx = 0$. By using this, (3.4), and Fubini’s theorem, we also have

$$
\rho(x, t) = e^t \left[ \int_{0}^{t} G(x, t; L_\tau, \tau) \varphi(\tau)d\tau - \int_{b}^{\infty} h(\xi) \int_{0}^{\infty} G(y, \tau; \xi, 0)dyd\xi \right]
$$

(5.19)

so that $\int_{L_t} \infty \rho(x, t)dx < \infty$. Similarly, if we differentiate $\int_{L_t} \infty \rho(x, t)dx$ with respect to $t$ and get

$$
\frac{d}{dt} \left( \int_{L_t} \infty \rho(x, t)dx \right) = \int_{L_t} \infty \rho_t(x, t)dx = \int_{L_t} \infty \left( \frac{1}{2} \rho_{xx}(x, t) + \rho(x, t) \right)dx = -\frac{1}{2} \rho_x(L_t, t) + \int_{L_t} \infty \rho(x, t)dx = -1 + \int_{L_t} \infty \rho(x, t)dx.
$$

Since $\int_{b}^{\infty} \rho_0(x)dx = 1$, we obtain that $\int_{L_t} \infty \rho(x, t)dx = 1$. The derivative $\rho_x(x, t)$ has a limit when $x \to L_t$ and $\rho_x(L_t, t) = 2$ and vanishes as $x \to \infty$ by Proposition 1. This completes the proof of Theorem 1; the statements below Theorem 1 also follow from what proved for $v(x, t)$.

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