Tidal mass loss in star clusters and treatment of escapers in Fokker–Planck models

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ABSTRACT
This paper presents a new scheme to treat escaping stars in the orbit-averaged Fokker–Planck models of globular star clusters in a galactic tidal field. The existence of a large number of potential escapers, which have energies above the escape energy but are still within the tidal radius, is taken into account in the models. The models allow potential escapers to experience gravitational scatterings before they leave clusters and thus some of them may lose enough energy to be bound again. It is shown that the mass evolution of the Fokker–Planck models is in good agreement with that of N-body models including the full tidal-force field. The mass-loss time does not simply scale with the relaxation time due to the existence of potential escapers; it increases with the number of stars more slowly than the relaxation time, though it tends to be proportional to the relaxation time in the limit of a weak tidal field. The Fokker–Planck models include two parameters: the coefficient γ in the Coulomb logarithm ln (γN) and the coefficient νe controlling the efficiency of the mass loss. The values of these parameters are determined by comparing the Fokker–Planck models with the N-body models. It is found that the parameter set (γ, νe) = (0.11, 7) works well for both single-mass and multimass clusters, but that the parameter set (γ, νe) = (0.02, 40) is another possible choice for multimass clusters.

Key words: methods: numerical – globular clusters: general – galaxies: star clusters: general.

1 INTRODUCTION
The numerical integration scheme of the orbit-averaged Fokker–Planck (FP) equation developed by Cohn (1979) has been one of the most useful tools for simulating the dynamical evolution of globular star clusters. In addition to two-body relaxation, many physical processes have been incorporated into FP models to achieve realistic modelling of the globular cluster evolution; these processes include tidal cut-off, binary heating, disc and bulge shocks, mass loss via stellar evolution, etc. (see Shin, Kim & Takahashi 2008, for a recent example of detailed FP modelling).

In this paper we consider the dynamical evolution of globular clusters in a steady galactic tidal field. Our main purpose is to investigate what boundary condition can give a better description of escape of stars from clusters in the tidal field. This study has been motivated by the studies of Fukushige & Heggie (2000) and Baumgardt (2001).

Fukushige & Heggie (2000) found that a large fraction of stars with energies above the escape energy (i.e. potential escapers) take much longer escape time than the dynamical time. Until their study it had been generally thought that the escape time-scale is of the order of the dynamical time and that the mass-loss times of the clusters essentially scale with the relaxation time, which is much longer than the dynamical time. The findings of Fukushige & Heggie (2000) indicate that this simple scaling may be spoiled by potential escapers with long escape times.

In fact Baumgardt (2001) performed N-body simulations and showed that the mass-loss times (lifetimes) of clusters do not scale with the relaxation time t_th but scale with t_th^{3/4}. He concluded that the reason is that some of potential escapers are scattered back to lower energies before they leave the cluster. More recently Tanikawa & Fukushige (2005) showed that the dependence on the relaxation time changes with the strength of the tidal filed. These two studies have revealed that the behaviour of potential escapers greatly influences the rate of mass loss from clusters in the tidal field.

The effects of long escape times and re-scattering of potential escapers have never been considered in previous FP models in the literature, but it was assumed that escapers leave a cluster on the
dynamical time-scale, as is described in detail in Section 2. Since the effect of the galactic tidal field is essentially important to the cluster evolution, it is necessary to find a way to include the effect into FP models as precisely as possible.

We should mention that Takahashi & Portegies Zwart (1998, 2000) compared FP and N-body models of star clusters in the tidal field and found good agreement between these two theoretical models over a wide range of initial conditions. They showed that the use of anisotropic FP models with the apocentre escape criterion (Takahashi, Lee & Inagaki 1997) and the dynamical-time removal of escapers (Lee & Ostriker 1987) is necessary to obtain such good agreement. However, note that in their N-body models the tidal force field is not included but the tidal cut-off is applied. Takahashi & Portegies Zwart (2000) confirmed that the difference between tidal cut-off and self-consistent tidal field N-body models is small for a particular set of initial conditions, but did not do systematic investigations on this problem.

In this study we have devised a new scheme to treat escapers in FP models. The scheme defines a region of potential escapers in phase space and allows them to be scattered again. Comparing the results of FP models calculated with the new scheme with the results of N-body models, we examine the accuracy of the FP models.

2 FOKKER–PLANCK MODELS OF STAR CLUSTERS IN A STEADY TIDAL FIELD

2.1 Basic assumptions

The orbit-averaged FP equation is derived under the assumption of spherical symmetry of star clusters (Cohn 1979). Therefore the tidal field, which is not spherically symmetric, cannot be directly incorporated into orbit-averaged FP models. In FP models the effect of the tidal field is taken into account by imposing a tidal cut-off radius $r_t$ on the cluster, which is treated as an isolated system in other respects. Under these assumptions the distribution function $f$ of stars at time $t$ depends only on the energy of a star per unit mass, $E$, and the angular momentum per unit mass, $J$.

2.2 Classical treatments of escapers

First, we summarize classical treatments of escapers used in FP models of previous studies.

2.2.1 Escape criteria in phase space

In previous studies, two kinds of criteria were adopted to define an escape region in $(E, J)$ space:

(i) energy criterion,

$$E > E_i = -\frac{GM}{r_t},$$

(ii) apocentre criterion,

$$r_a(E, J) > r_t,$$

where $M$ is the cluster mass and $r_a(E, J)$ is the apocentre radius of a star having energy $E$ and angular momentum $J$. It is assumed that a star is destined to escape once it enters into the escape region.

The apocentre criterion (Takahashi et al. 1997) is considered to be more realistic, at least as long as the tidal field is modelled as a radial cut-off, and in fact gives better agreement between FP and N-body models (Takahashi & Portegies Zwart 1998, 2000). For isotropic FP models, where the distribution function does not depend on $J$, only the energy criterion can be applied (e.g. Lee & Ostriker 1987).

2.2.2 Removal of escapers

In previous studies stars in the escape region are assumed to leave the cluster inevitably, as mentioned above. It is also assumed that the time required for this travel is of the order of the dynamical time at the tidal radius. Considering this travel time, Lee & Ostriker (1987) applied the following equation to the distribution function $f$ in the escape region:

$$\frac{\partial f}{\partial t} = -v_c f \left[1 - \frac{E}{E_i}\right]^{1/2} / t_{\text{tid}},$$

where $v_c$ is a dimensionless constant determining the efficiency of escape (see also Lee, Fahlman & Richer 1991). The time-scale $t_{\text{tid}}$ is an orbital time-scale at the tidal radius defined by

$$t_{\text{tid}} = \frac{2\pi}{\sqrt{(4\pi/3)G\rho t}},$$

where $\rho_t$ is the mean mass density within the tidal radius.

Since the dynamical time is generally much smaller than the relaxation time in globular clusters, we may assume that escapers leave the cluster immediately after they enter into the escape region, when we are interested only in the evolution on the relaxation time-scale. This assumption leads to the boundary condition

$$f = 0$$
on the tidal boundary (e.g. Chernoff & Weinberg 1990).

2.3 A new treatment of escapers

The boundary condition of equation (3) takes account of the fact that stars satisfying the escape criterion, i.e. potential escapers, need time to actually leave the cluster. However, the effect of re-scattering of potential escapers is not considered there. Here we propose a new scheme in which the re-scattering effect is taken into account.

First, we summarize basic assumptions and equations. Suppose that the cluster is on a circular orbit, with radius $R_0$ and angular velocity $\omega$, round the centre of a spherical galaxy. We consider the motion of a star in the rotating coordinate system moving with the cluster; the origin is at the cluster centre, the $x$-axis points to the galactic centre and the $y$-axis is in the cluster orbital plane. If the cluster and the galaxy are treated as point masses $M$ and $M_G (\gg M)$ and the size of the cluster is much smaller than $R_0$, there exists a conserved quantity known as the Jacobi integral given by

$$E_j = \frac{v^2}{2} - \frac{GM}{r} - \frac{1}{2} \omega^2 (3x^2 - z^2)$$

(cf. Spitzer 1987, chap. 5). Here $v$ is the velocity of the star measured in the rotating frame, $r$ is the distance from the star to the cluster centre and the angular velocity $\omega$ is given by

$$\omega = \sqrt{\frac{GM_G}{R_0^3}}.$$  

The third term on the right-hand side in equation (6) is a combination of the centrifugal and tidal potentials.

The effective potential is defined as

$$\phi_{\text{eff}}(x, y, z) = \phi - \frac{1}{2} \omega^2 (3x^2 - z^2).$$

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A contour plot of $\phi_{\text{eff}}$ is shown e.g. in Fig. 5.1 of Spitzer (1987). The effective potential has the saddle points at $(\pm x_e, 0, 0)$, where

$$x_e = \left(\frac{M}{3G}\right)^{1/3} R_G$$

(9)

and

$$\phi_{\text{eff}}(\pm x_e, 0, 0) = -\frac{3 GM}{2 x_e}.$$  (10)

The equipotential surface passing through these saddle points intersects with the y-axis at $y = \pm y_e$, where

$$y_e = \frac{2}{3} x_e.$$  (11)

The necessary condition for escape of a star from the cluster is given by

$$E_1 > E_{1,\text{crit}} \equiv -\frac{3 GM}{2 x_e}.$$  (12)

Note that equations (10)–(12) are valid for any spherical galactic potential. Fukuhige & Heggie (2000) found that the time-scale for escape of stars with $E_1 > E_{1,\text{crit}}$ varies as

$$t_e \propto (E_1 - E_{1,\text{crit}})^{2}.$$  (13)

With this relation in mind we have devised a new scheme to follow the evolution of potential escapers. In this scheme the evolution of the distribution function $f$ for potential escapers is described by

$$\frac{\partial f}{\partial t} = \left(\frac{\partial f}{\partial t}\right)_{\text{coll}} - \frac{\partial f}{\partial t}(E).$$  (14)

where the first term on the right-hand side is the FP collision term and the second term represents mass loss due to escape. Here the escape time-scale $t_e$ is given by

$$\frac{1}{t_e(E)} = \frac{y_e}{t_{\text{tid}}(E)} \left(1 - \frac{E}{E_{\text{crit}}(E)}\right)^2,$$  (15)

where $y_e$ is a dimensionless numerical constant. It should be noted that energy $E$, not the Jacobi integral $E_1$, is used in equations (14) and (15). Energy $E$ does not include the centrifugal and tidal potentials. Despite this difference, we use the same critical value of energy:

$$E_{\text{crit}} = \frac{3 GM}{2 r_t},$$  (16)

where the tidal radius $r_t$ is identified with $x_e$. One might think that using equation (15) with equation (16) is too crude a approximation, but it brings good agreement between FP and $N$-body models as is shown in Section 3.

The most important difference between equations (3) and (14) is that the latter includes the collision term. Thus equation (14) allows potential escapers to be scattered back to lower energies. The effect of mass loss is included in both equations in a similar way, though the functional forms of the escape time-scale $t_e$ are different.

In this new treatment of the tidal field, the escape criteria described in Section 2.2.1 are modified as follows:

(i) energy criterion,

$$E > E_{\text{crit}} = -\frac{3 GM}{2 r_t},$$  (17)

(ii) apocentre criterion,

$$r_a(E, J) > \frac{2}{3} r_t.$$  (18)

Note that $\phi_{\text{eff}}(0, \pm 2r/3, 0) = \phi(0, \pm 2r/3, 0) = -3GM/2r_a$. Equation (14) is applied only in the region where an adopted criterion is satisfied.

2.4 The Fokker–Planck code

The FP code used in the present study is essentially the same as that used by Takahashi & Portegies Zwart (2000), but adopts the new scheme for treating escapers described above. The code calculates the evolution of the distribution function $f(E, J, t)$. Unlike Takahashi & Portegies Zwart (2000), stellar evolution is not considered in the models presented in this paper. Instead the effect of heating by three-body binaries is considered in the manner described in Takahashi (1997).

For all the models presented in this paper, 201 energy mesh points, 51 angular momentum mesh points and 101 radial mesh points are used. The meshes are constructed as described in Takahashi (1995). When calculating the evolution of multimass clusters, 10 discrete mass components are used to represent a continuous mass function.

Our FP models have two free parameters: one is $v_\ast$ in equation (15) and the other is $\gamma$ in the Coulomb logarithm $\ln(\gamma N)$ appearing in the FP collision term. How the value of $v_\ast$ is determined is described in Section 3. We set $\gamma = 0.11$ (Giersz & Heggie 1994a) in most of our runs and $\gamma = 0.02$ (Giersz & Heggie 1996) in a part of runs for multimass clusters.

3 RESULTS

3.1 Comparison with $N$-body models: single-mass clusters

First, we compare FP models with the full tidal field models of Baumgardt (2001) and additional $N$-body runs performed for this comparison. All the model clusters are composed of equal-mass stars and move on circular orbits round a point-mass galaxy. The initial distribution of stars is given by King models (King 1966). Results are presented in $N$-body units, where the initial total mass and energy of a cluster are equal to 1 and $-0.25$, respectively, and the gravitational constant $G = 1$. The same units are used throughout this paper.

Here we will refer to FP models with the boundary condition of equation (14) as ‘FPf’ models, which aim to model clusters in a self-consistent full tidal field. FP models with equation (3) will be called ‘FPd’ models, where stars beyond the tidal cut-off radius are removed on the dynamical time-scale.

Fig. 1 compares FPf and $N$-body models concerning the evolution of the total mass of bound stars. The initial model stars are $W_0 = 3$ King models with the number of stars $N = 1024, 4096, 16384$ and 65 536. The new treatment of escapers described by equation (14) with the apocentre criterion of equation (18) is employed in the FPf models. The agreement between the FPf and $N$-body models is good in all the cases. In fact the value of the parameter $v_\ast$ in equation (14) has been determined so that good agreement is obtained by performing test runs with different values of $v_\ast$ as was done by Takahashi & Portegies Zwart (2000). We have finally chosen the value of $v_\ast = 7$. All the FPf models shown in Fig. 1 are calculated with this value.

Fig. 2 shows the evolution of the ratio of the mass of potential escapers $M_{\text{pe}}$ to the total cluster mass $M$ for the runs shown in Fig. 1. The agreement between the FPf and $N$-body models is fairly good also in this comparison. Note that here $M_{\text{pe}}$ for the FPf models is defined as the mass of stars with $E > E_{\text{crit}}$, although the apocentre criterion is used in the simulations. The mass of stars satisfying the apocentre criterion is smaller than that of stars with $E > E_{\text{crit}}$, but shows a similar trend in time variation.

Fig. 3 shows the half-mass time $T_{\text{half}}$, which is the time required for a cluster to lose a half of its initial mass, as a function of the
The initial half-mass relaxation time $t_{\text{rh},i}$. Here the half-mass relaxation time is defined by

$$t_{\text{rh}} = 0.138 \frac{N^{1/4}}{G^{1/2} m^{1/2} \ln(\gamma N)}$$

(Spitzer 1987, chap. 2) with $\gamma = 0.11$ (Giersz & Heggie 1994a). The results are summarized also in Table 1. The FPf and N-body models show good agreement over the whole range of $N$ where the comparison is made. The scaling $T_{\text{half}} \propto t_{\text{rh}}^{3/4}$ gives a reasonable fit to the results of these models as Baumgardt (2001) found.

The results of FPd models are also shown in Fig. 3. In these models the parameter $v_e = 2.5$ is used for equation (3) (Takahashi & Portegies Zwart 2000). The FPd models show clearly a different scaling from the other models; $T_{\text{half}} \propto t_{\text{rh}}$ expect for models with very short $t_{\text{rh}}$ (i.e. small $N$).

In Fig. 4 FPf models with the energy criterion are compared with those with the apocentre criterion as well as the N-body models. We have set $v_e = 5$ in the energy-criterion models so that their mass evolution reasonably agrees with that of the N-body models for small $N$. There is no significant difference between the energy-criterion models and the other models for $t_{\text{rh}} \lesssim 100$, but the energy-criterion models tend to lose mass much faster as $t_{\text{rh}}$ increases. This indicates that the apocentre criterion is a better escape criterion for FPf models.

As stated above, the results of the FPf models shown in Fig. 3 are reasonably well described by the scaling law $T_{\text{half}} \propto t_{\text{rh}}^{3/4}$. However, we should not expect this scaling continues to hold in the limit of...
large $N$. If this scaling continues, the half-mass time measured in the units of the half-mass relaxation time, $T_{\text{half}}/t_{\text{rh},i}$, would go to zero as $N \to \infty$. This must be impossible because the mass loss is driven by two-body relaxation. In order to see the scaling of $T_{\text{half}}$ in the limit of large $N$, we have calculated FPf models with very large $N$, $N = 2^{27} \approx 4.19 \times 10^8$ to $2^{30} \approx 1.07 \times 10^{10}$, which are much larger than typical numbers of stars in globular clusters. The results of these models are shown in Table 2 and Fig. 5. In Fig. 5 we see that $T_{\text{half}}$ is nearly proportional to $t_{\text{rh},i}$ for very large $N$ clusters, say, for $t_{\text{rh},i} \gtrsim 10^5$ or $N \gtrsim 10^7$. This trend is more qualitatively shown in Fig. 6, where the change in the logarithmic slope,

$$
\alpha = \frac{d \log T_{\text{half}}}{d \log t_{\text{rh},i}},
$$

is plotted. The slope $\alpha$ approaches one as $N$ increases. The ratio $T_{\text{half}}/t_{\text{rh},i} \approx 1.3$ for our largest $N$ models.

We have performed simulations also for the initial conditions of $W_0 = 5$ King models. The half-mass times of $N$-body and FPf models for $W_0 = 5$ are summarized in Table 3 and are plotted in Fig. 7. Here we find good agreement again. The same parameter $v_0 = 7$ is used for both the $W_0 = 3$ and 5 clusters. In Fig. 7 the slope of the log $t_{\text{rh},i}$–log $T_{\text{half}}$ relation seems to be in between 3/4 and 1. This point is further examined in Section 3.3.

### Table 2. Half-mass times $T_{\text{half}}$ given by FPf models for the initial conditions of $W_0 = 3$ King models with very large $N$.

| $N$       | $t_{\text{rh},i}$ | $T_{\text{half}}$ | $T_{\text{half}}/t_{\text{rh},i}$ |
|-----------|---------------------|-------------------|------------------------------------|
| $2^{27}$  | $4.19 \times 10^4$  | $5.77 \times 10^4$| 1.69                                |
| $2^{28}$  | $6.47 \times 10^4$  | $1.02 \times 10^5$| 1.57                                |
| $2^{29}$  | $1.23 \times 10^5$  | $1.82 \times 10^5$| 1.48                                |
| $2^{30}$  | $2.55 \times 10^5$  | $3.31 \times 10^5$| 1.41                                |
| $2^{31}$  | $4.50 \times 10^5$  | $6.09 \times 10^5$| 1.35                                |
| $2^{32}$  | $8.62 \times 10^5$  | $1.14 \times 10^6$| 1.32                                |
| $2^{33}$  | $1.65 \times 10^6$  | $2.14 \times 10^6$| 1.30                                |
| $2^{34}$  | $3.18 \times 10^6$  | $4.07 \times 10^6$| 1.28                                |
| $2^{35}$  | $6.12 \times 10^6$  | $7.77 \times 10^6$| 1.27                                |

### 3.2 Dependence on the escape-time function

Baumgardt (2001) argued that the scaling $T_{\text{half}} \propto t_{\text{rh}}^{3/4}$ can be explained by a steady state solution of a simple model for the evolution of potential escapers (see equation 12 of his paper). His model adopts the escape time-scale $t_e$ of equation (13). If a different function is assumed for $t_e$, his model predicts a different scaling law. It is shown that the scaling

$$
T_{\text{half}} \propto t_{\text{rh}}^{(\beta+1)/(\beta+2)}
$$

is obtained for $t_e \propto (E - E_{\text{crit}})^{-\beta}$ (see Appendix A). It is interesting to see if this prediction is confirmed by the results of our FPf models.
Table 3. Half-mass times $T_{\text{half}}$ given by $N$-body and FPf models for the initial conditions of King models with $W_0 = 5$.

| $N$    | $t_{\text{th},i}$ (N-body) | $T_{\text{half}}$ (FPf) |
|--------|-----------------------------|-------------------------|
| 1024   | $2.19 \times 10^1$          | $3.89 \times 10^2$      |
| 2048   | $3.82 \times 10^1$          | $5.78 \times 10^2$      |
| 4096   | $6.77 \times 10^1$          | $9.51 \times 10^2$      |
| 8192   | $1.22 \times 10^2$          | $1.51 \times 10^3$      |
| 16384  | $2.21 \times 10^2$          | $2.54 \times 10^3$      |
| 32768  | $4.04 \times 10^2$          | $4.14 \times 10^3$      |
| 65536  | $7.45 \times 10^2$          | $7.62 \times 10^3$      |
| 131072 | $1.38 \times 10^3$          | $1.31 \times 10^4$      |
| 262444 | $2.58 \times 10^3$          | $2.28 \times 10^4$      |
| 524888 | $4.83 \times 10^3$          | $4.03 \times 10^4$      |
| 1048576| $9.09 \times 10^3$          | $7.20 \times 10^4$      |
| 2097152| $1.72 \times 10^4$          | $1.30 \times 10^5$      |

Figure 7. Same as Fig. 3, but for the initial conditions of $W_0 = 5$ King models. FPf models with the apocentre criterion and $N$-body models are shown.

We have performed FP runs using a generalized form of equation (15),

$$\frac{1}{t_c(E)} = \frac{v_c}{t_{\text{th}}(1 - \frac{E}{E_{\text{crit}}})^\beta},$$

with $\beta = 1$ and 3. Fig. 8 plots the half-mass time against the initial half-mass relaxation time for these runs as well as for the standard runs, where King models with $W_0 = 3$ are used as initial conditions.

The results of the FPf models actually depend on $\beta$, but the degree of the dependence is weaker than predicted by equation (21). While this equation predicts the slopes 2/3, 3/4 and 4/5 for $\beta = 1$, 2 and 3, respectively, linear least-squares fitting of the data in Fig. 8 gives the slopes 0.69, 0.72 and 0.75. When the fitting is done only for $N \geq 16384$, the slopes are 0.75, 0.75 and 0.77. Thus the scaling law $T_{\text{half}} \propto t_{\text{th},i}^{3/4}$ is not a bad approximation in all the cases investigated here. This is not consistent with equation (21).

3.3 Dependence on the strength of the tidal field

Tanikawa & Fukushige (2005) found that the dependence of $T_{\text{half}}$ on $t_{\text{th},i}$ is affected by the strength of the tidal field and that the logarithmic slope $\alpha$, defined by equation (20), approaches unity as the strength of the tidal field decreases. In order to confirm their findings, we have calculated FPf models for the initial conditions where the initial tidal radius $r_{i,t}$ is greater than the King cut-off radius $r_K$ (i.e. the radius at which the density drops to zero) for each value of $W_0$. On the other hand, all the models presented above are calculated for the initial conditions with $r_{i,t} = r_K$.

Table 4 lists the half-mass times for $W_0 = 3$ King models with $r_{i,t}/r_K = 1.4$, 2, 4 and 6, and Fig. 9 illustrates these results. In this

Table 4. Half-mass times $T_{\text{half}}$ given by FPf models for the initial conditions of King models with $W_0 = 3$ and $r_{i,t} > r_K$.

| $N$    | $T_{\text{half}}$ ($r_{i,t}/r_K = 1.4$) | $T_{\text{half}}$ ($r_{i,t}/r_K = 2$) | $T_{\text{half}}$ ($r_{i,t}/r_K = 4$) | $T_{\text{half}}$ ($r_{i,t}/r_K = 6$) |
|--------|----------------------------------------|--------------------------------------|--------------------------------------|--------------------------------------|
| 128    | $1.62 \times 10^2$                     | $2.71 \times 10^2$                   | $6.86 \times 10^2$                   | $1.11 \times 10^3$                   |
| 256    | $2.22 \times 10^2$                     | $3.48 \times 10^2$                   | $7.61 \times 10^2$                   | $1.22 \times 10^3$                   |
| 512    | $3.02 \times 10^2$                     | $4.52 \times 10^2$                   | $8.95 \times 10^2$                   | $1.25 \times 10^3$                   |
| 1024   | $4.44 \times 10^2$                     | $6.84 \times 10^2$                   | $1.24 \times 10^3$                   | $1.73 \times 10^3$                   |
| 2048   | $6.93 \times 10^2$                     | $1.01 \times 10^3$                   | $1.97 \times 10^3$                   | $2.78 \times 10^3$                   |
| 4096   | $1.12 \times 10^3$                     | $1.68 \times 10^3$                   | $3.33 \times 10^3$                   | $4.83 \times 10^3$                   |
| 8192   | $1.87 \times 10^3$                     | $2.84 \times 10^3$                   | $5.82 \times 10^3$                   | $8.64 \times 10^3$                   |
| 16384  | $3.14 \times 10^3$                     | $4.89 \times 10^3$                   | $1.02 \times 10^4$                   | $1.54 \times 10^4$                   |
| 32768  | $5.33 \times 10^3$                     | $8.49 \times 10^3$                   | $1.80 \times 10^4$                   | $2.73 \times 10^4$                   |
| 65536  | $9.16 \times 10^3$                     | $1.49 \times 10^4$                   | $3.16 \times 10^4$                   | $4.77 \times 10^4$                   |
| 131072 | $1.59 \times 10^4$                     | $2.64 \times 10^4$                   | $5.56 \times 10^4$                   | $8.34 \times 10^4$                   |
| 262144 | $2.80 \times 10^4$                     | $4.72 \times 10^4$                   | $9.89 \times 10^4$                   | $1.48 \times 10^5$                   |
| 524288 | $4.99 \times 10^4$                     | $8.54 \times 10^4$                   | $1.78 \times 10^5$                   | $2.64 \times 10^5$                   |
| 1048576| $8.98 \times 10^4$                     | $1.56 \times 10^5$                   | $3.24 \times 10^5$                   | $4.79 \times 10^5$                   |
| 2097152| $1.63 \times 10^5$                     | $2.87 \times 10^5$                   | $6.02 \times 10^5$                   | $8.99 \times 10^5$                   |
3.4 Comparison with N-body models: multimass clusters

So far we have concentrated on single-mass clusters. Here we consider the evolution of multimass clusters comparing our FP models with the N-body models of Gieles & Baumgardt (2008). They performed N-body simulations of clusters on circular orbits around a point-mass galaxy. In their simulations the initial mass function (IMF) is given by $dN/dm \propto m^{-2.35}$ with the ratio $m_{\text{max}}/m_{\text{min}} = 30$. Stellar evolution is not considered in their simulations. The clusters initially have the density distribution of King models with $W_0 = 5$. The ratio of the initial tidal radius to the King radius $r_{\text{t}}/r_K$ is varied from 1 to 8. The results of the simulations of Gieles & Baumgardt (2008) are summarized in their table 1. Note that they use different notations from ours: $r_{\text{t}}$ is for the tidal (Jacobi) radius and $r_i$ is for the King radius.

FPf models are calculated for the same initial conditions as those of Gieles & Baumgardt (2008). The results for $r_{\text{t}}/r_K = 1$ are summarized in Table 5 and Fig. 11. There the results of the FPf models with three different sets of parameters $\gamma$ and $v_e$ are reported. Giersz & Heggie (1994a) estimated the best value of $\gamma = 0.11$ for single-mass clusters by comparing N-body models with FP and gas models.


![Figure 9](image1.png)

**Figure 9.** Same as Fig. 3, but FPf models for the initial conditions of King models with $r_{\text{t}}/r_K > 1$ are compared with the cases of $r_{\text{t}}/r_K = 1$.

![Figure 10](image2.png)

**Figure 10.** Logarithmic slope $\alpha = d\log T_{\text{half}}/d\log t_{\text{rh,i}}$ as a function of the initial half-mass relaxation time $t_{\text{rh,i}}$. The models are the same as those shown in Fig. 9.

figure the results for $W_0 = 3$ and 5 King models with $r_{\text{t}}/r_K = 1$ are also plotted. Note that the ratio $r_K(W_0 = 5)/r_K(W_0 = 3) \approx 1.4$. Fig. 10 shows the variation of $\alpha$ with $t_{\text{rh,i}}$.

The results shown in Figs 9 and 10 confirm the findings of Tanikawa & Fukushige (2005). The dependence of $T_{\text{half}}$ on $t_{\text{rh,i}}$ does not depend on the strength of the tidal field. In the limit of $r_{\text{t}}/r_K \rightarrow \infty$ and $N \rightarrow \infty$, it is expected that $\alpha \rightarrow 1$.

Note that the curve for $W_0 = 3$ King models with $r_{\text{t}}/r_K = 1.4$ lies very close to that for $W_0 = 5$ King models with $r_{\text{t}}/r_K = 1$ in each of Figs 9 and 10. This indicates that the mass-loss time-scale does not depend very much on the initial concentration of the cluster but is mainly determined by the strength of the tidal field, as was found by Tanikawa & Fukushige (2005).

![Table 5](image3.png)

**Table 5.** Half-mass times $T_{\text{half}}$ given by N-body (Gieles & Baumgardt 2008) and FPf models for the initial conditions of multimass King models with $W_0 = 5$ and $r_{\text{t}}/r_K = 1$. Three sets of the parameters ($\gamma$, $v_e$) are used for the FPf models.

![Figure 11](image4.png)

**Figure 11.** Half-mass time $T_{\text{half}}$ as a function of the initial number of stars $N$ for $W_0 = 5$ King models with the IMF $dN/dm \propto m^{-2.35}$ ($m_{\text{max}}/m_{\text{min}} = 30$). FPf models with three different sets of the parameters $\gamma$ and $v_e$ are compared with the N-body models of Gieles & Baumgardt (2008).
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1.89. We have calculated FPF models for multimass clusters using these two values of \( \gamma \). Fig. 11 shows that the parameter set \((\gamma, \nu_e) = (0.11, 7)\) adopted for single-mass clusters gives good fit to the N-body models also for multimass clusters. On the other hand the parameter set \((\gamma, \nu_e) = (0.02, 40)\) results in a clear deviation from the N-body models. If we stick to \( \gamma = 0.02 \), the value of \( \nu_e \) needs to be increased to about 40 in order to obtain good agreement with the N-body models. We will discuss in more detail what values of the parameters we should choose in the next section.

The results for the initial conditions with \( r_{\text{li}}>r_K \) are shown in Tables 6 and 7 and Fig. 12. The results of Gieles & Baumgardt (2008) are not shown in these tables (see their Table 1). Fig. 12 shows that the FPF models with \((\gamma, \nu_e) = (0.11, 7)\) are in good agreement with the N-body models for \( r_{\text{li}}/r_K = 2 \) and 4. The FPF models with \((\gamma, \nu_e) = (0.02, 40)\) are a little farther to the N-body models but still follow them rather well. However, for \( r_{\text{li}}/r_K = 8 \), a noticeable difference is observed between the FPF and N-body models; in Fig. 12 the curve for the FPF models is approximately linear but the slopes of the curves for the FPF models apparently change with \( N \). Neither parameter set reproduces the results of the N-body models as well as in the cases of \( r_{\text{li}}/r_K < 8 \). The reason for this discrepancy is not clear at present, but there is a possibility that very early core collapse in the models with \( r_{\text{li}}/r_K = 8 \) is, at least partially, responsible for it. The FPF model with \((\gamma, \nu_e) = (0.11, 7)\) and \( r_{\text{li}}/r_K = 8 \) experiences core collapse (bounce) at \( t = 0.006T_{\text{half}} \) for \( N = 1024 \), and at \( t = 0.037T_{\text{half}} \) for \( N = 32768 \). The Coulomb logarithm may take different values for pre-collapse and post-collapse stages (see the next section), which affects the time-scale of the evolution of FPF models.

### Table 6. Half-mass times \( T_{\text{half}} \) given by FPF models for the initial conditions of multimass King models with \( W_0 = 5 \) and \( r_{\text{li}}/r_K = 2, 4, 8 \). The adopted parameter set is \((\gamma, \nu_e) = (0.11, 7)\).

| \( N \) | \( T_{\text{half}} \) \((r_{\text{li}}/r_K = 2)\) | \( T_{\text{half}} \) \((r_{\text{li}}/r_K = 4)\) | \( T_{\text{half}} \) \((r_{\text{li}}/r_K = 8)\) |
|---|---|---|---|
| 1024 | \(3.34 \times 10^2\) | \(8.03 \times 10^2\) | \(1.81 \times 10^3\) |
| 2048 | \(5.12 \times 10^2\) | \(1.17 \times 10^3\) | \(2.41 \times 10^3\) |
| 4096 | \(7.79 \times 10^2\) | \(1.72 \times 10^3\) | \(3.35 \times 10^3\) |
| 8192 | \(1.21 \times 10^3\) | \(2.65 \times 10^3\) | \(5.08 \times 10^3\) |
| 16384 | \(1.95 \times 10^3\) | \(4.30 \times 10^3\) | \(8.33 \times 10^3\) |
| 32768 | \(3.26 \times 10^3\) | \(7.32 \times 10^3\) | \(1.45 \times 10^4\) |

### Table 7. Same as Table 6, but the results of FPF models with the parameter set \((\gamma, \nu_e) = (0.02, 40)\) are listed.

| \( N \) | \( T_{\text{half}} \) \((r_{\text{li}}/r_K = 2)\) | \( T_{\text{half}} \) \((r_{\text{li}}/r_K = 4)\) | \( T_{\text{half}} \) \((r_{\text{li}}/r_K = 8)\) |
|---|---|---|---|
| 1024 | \(3.68 \times 10^2\) | \(9.28 \times 10^2\) | \(2.20 \times 10^3\) |
| 2048 | \(5.53 \times 10^2\) | \(1.32 \times 10^3\) | \(2.87 \times 10^3\) |
| 4096 | \(8.25 \times 10^2\) | \(1.89 \times 10^3\) | \(3.86 \times 10^3\) |
| 8192 | \(1.26 \times 10^3\) | \(2.84 \times 10^3\) | \(5.63 \times 10^3\) |
| 16384 | \(2.01 \times 10^3\) | \(4.53 \times 10^3\) | \(8.92 \times 10^3\) |
| 32768 | \(3.55 \times 10^3\) | \(7.62 \times 10^3\) | \(1.52 \times 10^4\) |

potential escapers is implemented. This is the first time the effect of re-scattering of potential escapers has been taken into account in FPF models. Although Takahashi & Portegies Zwart (1998, 2000) showed that anisotropic FPF models are in good agreement with N-body models for the mass evolution of star clusters in a galaxy, the tidal field is treated as a tidal cut-off rather than an actual force field. In the present study we have found that our new FPF models are in good agreement with N-body models calculated with the inclusion of the tidal force field. Thus the new scheme has improved the accuracy of FPF models.

Baumgardt (2001) argued that some potential escapers are scattered back to lower energies before they leave the cluster and that this complicates the scaling of the mass-loss time. The success of our models is consistent with his argument. Actually our equation for potential escapers, equation (14), can be regarded as a generalization of the equation of his toy model, his equation (12), used for explaining the scaling \( T_{\text{half}} \propto t_{\text{in}}^{3/4} \).

The toy model of Baumgardt (2001) is useful for giving us insight into the effect of potential escapers on the cluster evolution. On the other hand, the results presented in Section 3.2 have revealed the limitation of the model. When the energy dependence of the escape time is artificially changed from the true one, the toy model does not correctly explain the results of our FPF models. This failure of the toy model is not a big surprise, because it is only a simplified model based on many assumptions, some of which are not very realistic. For example, our simulations show that an exact steady state is never established, but the toy model assumes a steady state. In addition, the scaling of the cluster lifetime depends on the strength of the tidal field, as found by Tanikawa & Fukushima (2005) and confirmed by the present study, but the toy model does not take account of the strength of the tidal field.

Our FPF models show good agreement with N-body models not only for single-mass clusters but also for multimass clusters. However, we have encountered a difficulty in determining proper values of the two parameters, \( \gamma \) and \( \nu_e \), in the FPF models. As shown in Section 3.4, the parameter set \((\gamma, \nu_e) = (0.11, 7)\) brings good agreement for both single-mass and multimass clusters. Since the escape
time-scale $t_e$ given by equation (15) is expected to be independent of stellar mass, it is natural that the same value of the parameter $v_c$ is applicable to both single-mass and multimass clusters.

On the other hand, the value of $\gamma$ is expected to depend on the stellar mass function. Hénon (1975) argued theoretically that the value of $\gamma$ is generally smaller in multimass clusters than in single-mass clusters. Based on the results of $N$-body simulations, Giersz & Heggie (1994a) obtained a value of $\gamma = 0.11$ for isolated single-mass clusters, and Giersz & Heggie (1996) obtained a much smaller value, $\gamma = 0.02$, for isolated multimass clusters having an IMF similar to the IMF used in our simulations.

When we adopt the value of $\gamma = 0.02$ for multimass clusters, we have to use a much larger value of $v_e, v_c = 40$, than the best value of $v_e = 7$ for single-mass clusters, in order to obtain good agreement with $N$-body models. Thus we have not found a parameter set satisfying both the independence of $v_e$ on the mass function and the dependence of $\gamma$ on it. It needs further investigation to solve this incompatibility, but even the determination of $\gamma$ itself is not a simple task. For example, Giersz & Heggie (1994b) obtained the best value of $\gamma = 0.035$ by examining the post-collapse evolution of $N$-body models of isolated single-mass clusters. This value is much smaller than the value of $\gamma = 0.11$ obtained for pre-collapse single-mass clusters. These results suggest that the value of $\gamma$ changes along with the evolution of clusters. It may also change with radius within a cluster (Giersz & Heggie 1994a).

Fukushige & Heggie (2000) theoretically estimated not only the energy dependence of the escape time-scale $t_e$ but also its numerical coefficient, which is given in their equation (9). If we ignore the difference between energy $E$ and the Jacobi integral $E_j$, their estimate for a $W_0 = 3$ King model leads to a value of $v_e = 29$. This is about four times larger than our best value of $v_e = 7$ for single-mass clusters. However, Fukushige & Heggie (2000) also did numerical experiments and found that their theoretical estimate of $t_e$ is too small; escape time-scales obtained from the numerical experiments are more than a few times larger than the theoretical one. Therefore our value $v_e = 7$ is not inconsistent with the result of Fukushige & Heggie (2000). On the other hand, our value of $v_e = 40$ for multimass clusters with $\gamma = 0.02$ is a little larger than their theoretical estimate.

Another issue not addressed in the present paper is how the mass profile of the parent galaxy affects the results. In all the simulations presented here we assume that the parent galaxy is represented by a point mass. On the other hand, Tanikawa & Fukushige (2010) showed that the mass-loss time-scale depends on the profile of the parent galaxy; the time-scale increases as the mass profile gets shallower. Therefore we expect that the parameter $v_e$ depends on the mass profile of the parent galaxy. This issue will be examined in a future study.

5 CONCLUSION

In this paper we have developed new FP models of globular clusters in a steady galactic tidal field. Our FP models are novel in the method of treating escapers: potential escapers are allowed to experience gravitational scattering with other stars before they really leave clusters. The new method has been devised in order to construct more realistic models of star clusters in a tidal field compared to simple tidal cut-off models as in previous studies. The mass evolution of clusters in a tidal field does not simply scale with the relaxation time, and our FP models are in good agreement with $N$-body models in this respect.

Our FP models include two parameters $\gamma$ and $v_e$: $\gamma$ is the numerical factor in the Coulomb logarithm ln ($\gamma N$) and $v_e$ adjusts the speed of the tidal mass loss. We have determined the best values of $v_e$ for given values of $\gamma$ by comparing FP results with $N$-body results. For single-mass clusters the best parameter set is $(\gamma, v_e) = (0.11, 7)$. This parameter set is applicable to multimass clusters as well, but another set $(\gamma, v_e) = (0.02, 40)$ does work equally well as long as multimass clusters are concerned. The parameter $v_e$ is expected to depend on the mass profile of the parent galaxy, though a point-mass galaxy is assumed in all the simulations of the present paper. Further investigation is required for the determination of the best values of the parameters $\gamma$ and $v_e$ under various conditions.

While FP models are generally thought to be less faithful models of globular clusters than $N$-body models, the present study has significantly improved the accuracy of FP models. An advantage of FP models is that they can be calculated much faster than $N$-body models. Therefore FP models are particularly useful when we need to calculate a huge number of models. For example, when we try to specify the initial conditions of individual clusters, we have to perform simulations for many sets of the initial conditions, because the parameter space to be searched is very large. We believe that our FP models are quite useful for such searching.

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APPENDIX A: ESTIMATION OF THE SCALING OF THE CLUSTER LIFETIME

We follow the arguments given by Baumgardt (2001) and Heggie (2001) in order to derive the scaling law of equation (21).

Let \( \dot{E} = (E - E_{\text{crit}})/|E_{\text{crit}}| \) and assume that the escape time-scale \( t_e \) has energy dependence such as

\[ t_e(\dot{E}) = t_{\text{esc}} \dot{E}^{-\beta} \quad (\beta > 0). \quad (A1) \]

Then Baumgardt’s toy model is modified as

\[ \frac{\partial n}{\partial t} = k_1 \frac{\partial^2 n}{\partial E^2} - \dot{E}^\beta n / t_{\text{esc}}, \quad (A2) \]

where \( n(\dot{E}, t) \) is the number of stars with energies in the range \( (E, E + dE) \) and \( k_1 \) is a constant. If we assume that the distribution of escapers is nearly in equilibrium, equation (A2) shows that the width of the distribution is approximately given by

\[ \Delta \dot{E} \sim \left( \frac{t_{\text{esc}}}{t_{\text{th}}} \right)^{1/(\beta+2)}, \quad (A3) \]

and the number of escapers \( N_{\text{esc}} \sim N \Delta \dot{E} \). The escape rate \( \dot{N}_{\text{esc}} \) is estimated to be

\[ \dot{N}_{\text{esc}} \sim \frac{N_{\text{esc}}}{t_e(\Delta \dot{E})} \sim \frac{N}{t_{\text{esc}}} \left( \frac{t_{\text{esc}}}{t_{\text{th}}} \right)^{(\beta+1)/(\beta+2)}. \quad (A4) \]

Therefore the scaling of the half-mass time is given by

\[ T_{\text{half}} \sim \frac{N}{\dot{N}_{\text{esc}}} \sim t_{\text{th}}^{(\beta+1)/(\beta+2)} t_{\text{esc}}^{1/(\beta+2)}. \quad (A5) \]

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