We propose a novel approach to manipulate two-dimensional bright matter-wave solitons by tuning the frequency of the trap which is different from Feshbach resonance technique. The exact bright soliton solutions for two-dimensional Gross-Pitaevskii (GP) equation with attractive interaction strength in a time-dependent trap are constructed analytically and its dynamics show no collapse while modulating the trap frequency. The two-soliton dynamics exhibits an interesting splitting and recombination phenomenon which generates interference pattern in the process. This type of behaviour in two-dimensional BECs has wider ramifications and our approach opens new avenues in stabilizing bright solitons in higher dimensional regime. We have also explored the experimental realization of this novel phenomenon.

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I. INTRODUCTION

A Bose-Einstein condensate (BEC) is a state of matter in which a large number of bosonic atoms occupy the lowest quantum state of the external potential at extremely low temperatures near absolute zero, allowing quantum effects to be observed on a macroscopic scale. The experimental realization of BECs in weakly interacting gases has opened the floodgates in the field of atom optics and condensed matter physics. The collective excitation of matter-waves in BECs like matter-wave solitons [1–4], periodic waves [5], shock waves [6], vortices [7] and necklaces [8] has generated a lot of interest on the exploration of the dynamics of BECs from both experimental and theoretical perspectives. Even though the experimental realization of Bose-Einstein condensation in different atomic systems has been observed, the stability of BECs imposes restriction on the observation of dynamical properties of such topological excitations. This is because of the higher dimensionality of the system. For example, attractive Li BECs have been shown to collapse in three dimensions [9, 10]. But, the one-dimensional attractive Li BECs in a standing light wave potential are stable [11]. This is primarily because of the fact that in higher dimensional BECs, a nonlinear excitation of atoms just bigger than the background is susceptible to even smaller perturbations. The dark solitons will decay into vortex pairs under small transverse perturbation [7]. Therefore, longtime dynamical behaviour of nonlinear excitations in higher dimensional BECs is hard to observe in real experiments.

However, it was identified that the bright solitons and vortex solitons can be stabilized without trap in higher dimensional BECs by rapidly oscillating atom-atom interaction strength using Feshbach resonance technique [12–14] where resonantly tuning uniform magnetic field produces an effective confinement creating stable self-confined condensates. Similarly, a stable shallow ring dark soliton has also been observed in two-dimensional BECs with tunable interaction [15]. Thus, to stabilize the solitons in BECs, one has to change the interaction strength between the atoms by tuning magnetic field near Feshbach resonance [16, 17].

Can one suitably modulate the trap frequency in a time-dependent trap and control the stability of the condensates instead of varying the interaction strength? This paper is aimed at the investigation of the (2+1) dimensional GP equation with time-dependent trapping potential describing the dynamics of matter waves in pancake-shaped BECs. The fact that the trap frequency varies with time means that one can suitably tune the time-dependent harmonic trap to stabilize the bright solitons and this is an alternative way of stabilizing the bright solitons in BECs rather than tuning the interaction strength. We first construct exact bright matter-wave soliton solutions using Hirota method [18] and then study the impact of modulating the trap frequency. We demonstrate split-recombination phenomenon by modulating the trap frequency. The frequency modulation technique is well known in controlling molecular BEC scattering length and condensate spilling [19]. Our results open a new window to manipulate atoms in higher dimensional BECs.

II. MODEL

At ultra low temperatures, the dynamics of BECs can be described by the three dimensional time-dependent GP equation of the following form [20]

$$i\hbar \frac{\partial \Psi(r,t)}{\partial t} = \left( -\frac{\hbar^2}{2m} \nabla^2 + V(r,t) + U|\Psi(r,t)|^2 \right) \Psi(r,t)$$

where $\Psi[r(x,y,t),t]$ represents the condensate wave function normalized by the particle number $N = \int dr|\Psi|^2$, $\nabla^2$ denotes the Laplacian operator and $V(r)$ is the trapping potential of the form $V(r) = m(\omega_r(t))^2 r^2 + \omega_z^2 z^2$, where $r^2 = x^2 + y^2$, $\omega_r, \omega_z$ are the confinement frequencies in the radial and axial directions, respectively. The coefficient
It should be mentioned that the choice of trapping potential strength depends on the solvability of Ricatti equation (Eq. (6)), but the solution of Ricatti equation is not unique. One could also get different physically realizable potential by properly choosing the time dependent function $\alpha(t)$.

The operator $D_x$ and $D_t$ are defined as

$$D^m_x D^m_t (G \cdot F) = \left( \frac{\partial}{\partial t} - \frac{\partial}{\partial t'} \right)^m \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial x'} \right)^n \times$$

$$G(t, x) F(t', x') |_{t'=t, x'=x}. \quad (7)$$

The functions $G$ and $F$ can be extended as

$$G = \varepsilon \tilde{g}^{(1)} + \varepsilon^3 \tilde{g}^{(3)} + \varepsilon^5 \tilde{g}^{(5)} + \cdots, \quad (8)$$

$$F = 1 + \varepsilon^2 f^{(2)} + \varepsilon^4 f^{(4)} + \cdots. \quad (9)$$

Substituting $G$ and $F$ into the bilinear forms and collecting the coefficient of various powers of $\varepsilon$, one obtains the system of linear partial differential equations which can be recursively solved. The function $\tilde{g}^{(1)}$ can be represented as the plane wave solution of the form

$$\tilde{g}^{(1)} = \sum_{j=1}^{N} e^{\chi_j}, \quad \chi_j = h_1^{(j)}(t)x + h_2^{(j)}(t)y + h_3^{(j)}(t). \quad (10)$$

To generate one soliton solution, we take $N = 1$ to obtain $\tilde{g}^{(1)}$ and $f^{(2)}$ as

$$\tilde{g}^{(1)} = e^{\chi_1(x,y,t)}, \quad \chi_1 = h_1^{(1)}(t)x + h_2^{(1)}(t)y + h_3^{(1)}(t), \quad (11)$$

$$f^{(2)} = e^{\chi_1(x,y,t) + \chi_1^*(x,y,t) + \eta_1(t)}, \quad (12)$$

with

$$\chi_1(x,y,t) = h_1^{(1)}(t)x + h_2^{(1)}(t)y + h_3^{(1)}(t), \quad (13)$$

$$h_1^{(1)}(t) = a e^{-\frac{\alpha(t) dt}{2}}, \quad h_2^{(1)}(t) = b e^{-\frac{\alpha(t) dt}{2}}, \quad (14)$$

$$h_3^{(1)}(t) = \int \left( \frac{i}{2}(a^2 + b^2)e^{-\frac{\alpha(t) dt}{2}} - \alpha(t) \right) dt, \quad (15)$$

and

$$\eta_1(t) = g \frac{e^{\frac{\alpha(t) dt}{2}}}{(a + b^*)^2 + (b + b^*)^2}, \quad (16)$$

where $a$ and $b$ are complex constants, and $\chi_1^*$ denotes complex conjugate of $\chi_1$. $\alpha(t)$ is real function. Substituting $\tilde{g}^{(1)}$ and $f^{(2)}$ into the truncated series, we obtain $G = \tilde{g}^{(1)}$ and $F = 1 + f^{(2)}$. Thus, we get the one soliton solution of the following form

$$\psi_1 = \frac{1}{2} e^{-\eta_1/2} sech \left( Re(\chi_1) + \frac{\eta_1}{2} \right) e^{i \eta_1(t) x + \alpha(t) (t^2 + y^2)}. \quad (16)$$

From the above solution, we observe that the amplitude $(e^{-2f \alpha(t) dt} (a^2 + b^2))^{1/2}$, velocity$(Re(h_3(t)) + \eta_1(t)/2)$, position$(Re(\chi_1) + \eta_1/2)$ and phase$(arg(\psi))$ of the soliton depend on the frequency of the trapping potential (i.e. $2\omega_r(t)^2/\omega_z^2 = 2\Omega(t)^2 = \alpha'(t) + \alpha(t)^2$ and...
attractive interaction strength \( g \). Since the frequency of the trapping potential could be modulated with respect to time, one can easily control the physical properties of bright solitons and hence the condensates. We can choose the trapping potential to be either confining \((\Omega(t)^2 < 0)\) or expulsive \((\Omega(t)^2 > 0)\). And, we can also continuously change the nature of the trapping potential from confining to expulsive or expulsive to confining. It is interesting to note that this exact bright soliton solution involves constant attractive interaction strength which is independent of trapping potential strength unlike the quasi one-dimensional BECs [24–26], wherein one needs a delicate balance between time-dependent trap and interaction strength.

The time-dependent trapping frequency allows us to choose different physically realizable potentials and study the effect of trap on the dynamics of solitons. For example, in the case of a constant trapping potential \((\Omega(t)^2 = \Omega_0 = 0.4)\), the amplitude and width of the soliton varies with time as shown in Fig. 1(a), in which the amplitude of the soliton increases and reach the maximum value which depends on the strength of trapping potential and then decreases. When the trapping potential is suddenly switched off, the amplitude decreases and the width increases as shown in Fig. 1(b). From these two cases, we observe that the time-independent trap and zero-trapping potential cannot keep the soliton as stable. As an alternative, one can look to stabilize the bright solitons by modulating the trapping frequency with time. For example, choosing \( \Omega(t)^2 = -2 - \cos(2t) \), the \( \alpha(t) \) takes the following form

\[
\alpha(t) = \frac{\text{mathieu}C(4,-1,t) + \text{mathieu}S(4,-1,t)}{\text{mathieu}C(4,-1,t) + \text{mathieu}S(4,-1,t)}.
\]

The mathieuC\((l, m, t)\) and mathieuS\((l, m, t)\) are even and odd functions, respectively. When \( m = 0 \), they are simply \( \cos(\sqrt{lt}) \) and \( \sin(\sqrt{lt}) \) where the prime functions are \( t \) derivatives of mathieu functions. When we modulate the trap strength periodically by tuning the radial trapping frequency \((\omega_r(t))\) within the two-dimensional confining regime, the amplitude and width of the soliton oscillates and gets amplified as shown in Fig. 1(c), but does not collapse or dilute even if one waits long enough. In this case, the stability of the soliton is partial. One can choose the other possible time-dependent potentials to stabilize the soliton where the amplitude and width of the soliton remains constant or oscillates without amplification.

When the confining trapping potential strength decreases exponentially \((\Omega(t)^2 = e^{-4t}(0.1352 - 0.52e^{2t}))\) as shown in Fig. 1(d), the amplitude and width becomes constant for large \( t \). This means that the stability of the soliton is ensured. If we choose the trapping strength to oscillate between confining and expulsive regime \((\Omega(t)^2 = -0.5 \cos(2t) + 0.125 \sin(2t)^2)\) as shown in Fig. 1(e), both the amplitude and width of the soliton oscillate but no amplification occurs. When we increase the frequency of oscillations of trapping frequency \((\Omega(t)^2 = -2 \cos(8t) + 0.125 \sin(8t)^2)\) as shown in Fig. 1(f), the amplitude and width oscillates but the oscillation is very small. In other words, the frequency of oscillation of the amplitude is proportional to frequency of oscillation of time-dependent frequency. This is similar to the Feshbach resonance [12] technique in which the amplitude of oscillating soliton is nearly a constant while increasing the frequency of interaction strength. From the Figs. 1(a-f), we observe that the longevity of soliton can be controlled by fine tuning the trapping potential strength. One can realize the above phenomenon in experiments as well. For example, in the case of \(^7\)Li, keeping BEC the attractive interaction strength \( g = 0.25 \) and \( \omega_r = 2\pi \times 710\text{Hz} \), one can oscillate the radial trapping frequency approximately between \( 2\pi \times 80\text{Hz} \) and \( 2\pi \times 200\text{Hz} \). Invoking this ideas in the two-soliton solution, one observes an interesting splitting and recombination phenomenon.

![Figure 1](image-url)

**FIG. 1**: (Color online) The amplitude, width and trapping strength of one-soliton solution plotted for different case. (a) For constant time-independent trapping potential \((\Omega(t)^2 = 0.4 \text{ (i.e., } \omega_r = 2\pi \times 284\text{Hz}, \omega_z = 2\pi \times 710\text{Hz})\)) with \( g = 0.0005, a = 10 + 0.2i, b = 10 - 0.3i \). (b) The trapping potential is switched off when \( g = 0.05, a = 1.5 + 0.2i, b = 1.5 - 0.3i \). (c) Oscillating time-dependent trap within confining regime \((\Omega(t)^2 = -2 - \cos(2t))\), (d) exponentially decreasing confining trap approaching zero, (e) trapping frequency oscillates between confining and expulsive regime and (f) increasing oscillation frequency of the trap with \( g = 0.05, a = 0.15 + 0.2i, b = 0.15 - 0.3i \).
IV. SPLIT AND RECOMBINATION OF TWO-SOLITON

To generate two-soliton solution, we consider \( N = 2 \) and \( i = 1, 2 \) for \( \alpha \) and \( \beta \) in Eq. (10) and the same procedure as in one soliton case, to get \( \tilde{g}^{(3)} \), \( f^{(2)} \) and \( f^{(4)} \) while the series gets truncated at \( f^{(4)} \). The two-soliton solution can be explicitly written as

\[
\psi_2 = e^{i\alpha(t)(x^2 + y^2) / 2} \frac{\tilde{g}^{(1)} + \tilde{g}^{(3)}}{1 + f^{(2)} + f^{(4)}},
\]

where

\[
\begin{align*}
\tilde{g}^{(1)} &= e^{\chi_1} + e^{\chi_2}, \\
\tilde{g}^{(3)} &= e^{\chi_1 + \chi_1^* + \chi_2 + \chi_2^*}, \\
f^{(2)} &= e^{\chi_1 + \chi_2 + \chi_1^* + \chi_2^*} + e^{\chi_1^* + \chi_2^*}, \\
f^{(4)} &= m(t)e^{\chi_1 + \chi_1^* + \chi_2 + \chi_2^*},
\end{align*}
\]

and

\[
\begin{align*}
\chi_1 &= h_1^{(1)}(t)x + h_2^{(1)}(t)y + h_3^{(1)}(t); \quad i = 1, 2, \\
h_1^{(1)}(t) &= ae^{-f^{(1)}(t)dt}, \quad h_2^{(1)} = be^{-f^{(1)}(t)dt}, \\
h_1^{(2)}(t) &= ce^{-f^{(1)}(t)dt}, \quad h_2^{(2)} = de^{-f^{(1)}(t)dt}, \\
h_3^{(1)}(t) &= \int \left( \frac{i}{2}a^2 + b^2 \right)e^{-2f^{(1)}(t)dt - \alpha(t)dt}dt, \\
h_3^{(2)}(t) &= \int \left( \frac{i}{2}c^2 + d^2 \right)e^{-2f^{(1)}(t)dt - \alpha(t)dt}dt,
\end{align*}
\]

\[
\begin{align*}
e^{\Gamma_1} &= \delta_1 e^{\eta_1} + \delta_2 e^{\eta_0}, \quad e^{\Gamma_2} = \delta_3 e^{\eta_0} + \delta_4 e^{\eta_2}, \\
e^{\eta_1} &= ge^{2f^{(1)}(t)dt} / ((a + a^*)^2 + (b + b^*)^2), \\
e^{\eta_2} &= ge^{2f^{(1)}(t)dt} / ((c + c^*)^2 + (d + d^*)^2), \\
e^{\eta_0} &= ge^{2f^{(1)}(t)dt} / ((a + a^*)^2 + (b + b^*)^2), \\
e^{\eta_0^*} &= ge^{2f^{(1)}(t)dt} / ((c + c^*)^2 + (d + d^*)^2),
\end{align*}
\]

\[
\begin{align*}
\delta_1 &= \frac{(c - a)(a + a^*) + (d - b)(b + b^*)}{(c + a^*)(a + a^*) + (d + b^*)(b + b^*)}, \\
\delta_2 &= \frac{(a - c)(c + a^*) + (b - d)(b + b^*)}{(a + c^*)(a + c^*) + (b + d^*)(b + d^*)}, \\
\delta_3 &= \frac{(c - a)(a + a^*) + (d - b)(b + b^*)}{(c + c^*)(c + c^*) + (d + d^*)(d + d^*)}, \\
\delta_4 &= \frac{(a - c)(c + c^*) + (b - d)(b + d^*)}{(a + c^*)(a + c^*) + (b + d^*)(b + d^*)}, \\
m(t) &= \frac{1}{4} e^{4f^{(1)}(t)dt} \frac{(a - c)^2(a^* - c^*)^2}{(a + a^*)^2(a + a^*)^2 + (c + c^*)^2(c + c^*)^2},
\end{align*}
\]

The constraint \( a = b \) and \( c = d \) ensures the two-soliton solution to be exact and the solitons are parallel to each other so called “parallel bright matter-wave solitons”.

From the two-soliton solution (Eq. (17)), one can observe the dynamics of parallel bright matter-wave solitons in the two-dimensional harmonic trap. We choose the function \( \alpha(t) = -0.001(t - 50) \) in which the trap strength \( (2\Omega(t)^2 = \alpha'(t) + \alpha(t)^2) \) varies within the confining regime \( (\Omega(t)^2 < 0) \) between \( t \approx 10 \) and \( t \approx 80 \).

First, we consider two parallel bright solitons merged in a single bound state with the initial conditions \( a = b = 0.3 + 0.1i, c = d = 0.3, \) \( \alpha(t) = -0.001(t - 50) \). The snapshots of two-soliton along z-axis at different instants of time. The trap strength \( \Omega(t)^2 \) parabolically varying between \(-0.0001 \) and \(-0.0005 \) in a span of 80 ms.

![Fig. 2: (Color online) (a) The split-recombination of two parallel bright matter-wave solitons with the following parameters \( a = b = 0.3 + 0.1i, c = d = 0.3, \alpha(t) = -0.001(t - 50) \). (b) The snapshots of two-soliton along z-axis at different instants of time. The trap strength \( \Omega(t)^2 \) parabolically varying between \(-0.0001 \) and \(-0.0005 \) in a span of 80 ms.](image)
paration depends on the strength of trapping potential as shown in Fig. 2(a). Experimentally, one can realize this phenomenon by parabolically modulating the trapping frequency within the two-dimensional confining regime.

In experiment, to generate parallel bright solitons in a two-dimensional harmonic trapping potential, initially the trapping frequencies in the case of $^7$Li BECs are $\omega_r = 2\pi \times 50$Hz and $\omega_z = 2\pi \times 71$0Hz with the effective attractive interaction being $g = 0.25$. This trap can be determined by a combination of spectroscopic observations, direct magnetic field measurement and the observed spatial cylindrical symmetry of the trapped atom cloud [28]. After making this set up to generate two-soliton, the condensates split into two parts using radio frequency (RF)-dressed potential [29]. This technique has the advantage of allowing a smooth transition from a single trap into a double well potential and hence coherent splitting is possible with splitting range from $3 - 80 \mu m$. Due to the constant attractive scattering length, the two-bright solitons are generated from two nearly separated condensates. We may consider this as a initial coherent state of two-soliton split-recombination phenomenon. Then, one starts to tune the radial trap frequency in a parabolic manner after switching off the RF. When the radial trapping frequency which after fine tuning reaches the initial value, the two-bright solitons recombine and produce fringe pattern.

V. CONCLUSION

We have found exact bright matter-wave solitons for a two-dimensional BEC with attractive interaction strength in a time-dependent harmonic trap. Even though attractive interaction strengths usually lead to the collapse of the condensates, one can sustain the stability of solitons by properly modulating the frequency of the radial trap. Our method is different from stabilizing solitons by Feshbach resonance technique. We also observe that the splitting and recombination of matter-wave solitons which generates interference pattern in the process is a new phenomenon in two dimensional BECs and may have wider ramifications in the manipulation of atoms in BECs. In particular, ultra-narrow two-dimensional BEC solitons is very useful in the field of nanolithography.

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