UNKNOTTING SURFACE LINKS WHICH ARE COVERINGS OF A TRIVIAL TORUS KNOT

INASA NAKAMURA

Abstract. We consider surface links in the 4-sphere or 4-space which can be deformed to simple branched coverings of a trivial torus knot, which we call torus-covering-links. Torus-covering-links contain spun $T^2$-knots, turned spun $T^2$-knots, symmetry-spun tori and torus $T^2$-knots. In this paper we study unknotting numbers of torus-covering-links. In particular we give examples of torus-covering-knots whose unknotting number is an arbitrary positive integer.

0. Introduction

Locally flatly embedded closed 2-manifolds in the 4-space $\mathbb{R}^4$ are called surface links. It is known that any oriented surface link can be deformed to the closure of a simple surface braid, that is, a simple branched covering of the 2-sphere ([14]).

As surface knots of genus one which can be made from classical knots, there are spun $T^2$-knots, turned spun $T^2$-knots, symmetry-spun tori and torus $T^2$-knots. Consider $\mathbb{R}^4$ as obtained by rotating $\mathbb{R}^3_1$ around the boundary $\mathbb{R}^2$. Then a spun $T^2$-knot is obtained by rotating a classical knot ([2]), a turned spun $T^2$-knot by turning it once while rotating ([2]), a symmetry-spun torus by turning a classical knot with periodicity rationally while rotating ([21]), and a torus $T^2$-knot is a surface knot on the boundary of a neighborhood of a solid torus in $\mathbb{R}^4$ ([9]). Symmetry-spun tori include spun $T^2$-knots, turned spun $T^2$-knots and torus $T^2$-knots. We call the link version of a symmetry-spun torus, a spun $T^2$-link, and a turned spun $T^2$-link respectively.

Now we consider surface links in the 4-sphere or the 4-space which can be deformed to simple branched coverings of a trivial torus knot, which we define as torus-covering-links ([21]). By definition, a torus-covering-link is described by a torus-covering-chart, which is a chart on the trivial torus knot. Torus-covering-links include symmetry $T^2$-links (and spun $T^2$-links, turned spun $T^2$-links, and torus $T^2$-links). A torus-covering-link has no 2-knot component. Each component of a torus-covering-link is of genus at least one.

In this paper we study unknotting numbers of torus-covering-links.

In Section 2 we give the definitions of torus-covering-links (Definition 2.1), and the turned torus-covering-links (Definition 2.3) (cf. [21]).

In Section 3 we show that if a torus-covering-link is unknotted, then its turned torus-covering-link is also unknotted (Theorem 3.1). We also give

Key words and phrases. surface link, 2-dimensional braid, quandle, unknotting number.
examples of torus-covering-knots whose unknotting number is an arbitrary positive integer (Theorem 3.3).

The author would like to thank professors Takashi Tsuboi and Elmar Vogt for suggesting this problem, and professor Akio Kawauchi for advising about titles.

1. Definitions and Preliminaries

**Definition 1.1.** A locally flatly embedded closed 2-manifold in $S^4$ or $\mathbb{R}^4$ is called a **surface link**. A surface link with one component is called a **surface knot**. A surface link whose each component is of genus zero (resp. one) is called a **2-link** (resp. **$T^2$-link**). In particular a surface knot of genus zero (resp. one) is called a **2-knot** (resp. **$T^2$-knot**).

An orientable surface link $F$ is **trivial** (or **unknotted**) if there is an embedded 3-manifold $M$ with $\partial M = F$ such that each component of $M$ is a handlebody.

An oriented surface link $F$ is called **pseudo-ribbon** if there is a surface link diagram of $F$ whose singularity set consists of double points and **ribbon** if $F$ is obtained from a trivial 2-link $F_0$ by 1-handle surgeries along a finite number of mutually disjoint 1-handles attaching to $F_0$. By definition, a ribbon surface link is pseudo-ribbon.

Two surface links are **equivalent** if there is an ambient isotopy or an orientation-preserving diffeomorphism of $S^4$ or $\mathbb{R}^4$ which deforms one to the other.

**Definition 1.2.** A compact and oriented 2-manifold $S$ embedded properly and locally flatly in $D_2^1 \times D_2^2$ is called a **braided surface** of degree $m$ if $S$ satisfies the following conditions:

(i) $\text{pr}_2|_S : S \to D_2^2$ is a branched covering map of degree $m$,
(ii) $\partial S$ is a closed $m$-braid in $D_2^1 \times \partial D_2^2$, where $D_2^1, D_2^2$ are 2-disks, and $\text{pr}_2 : D_2^1 \times D_2^2 \to D_2^2$ is the projection to the second factor.

A braided surface $S$ is called a **surface braid** if $\partial S$ is the trivial closed braid. Moreover, $S$ is called **simple** if every singular index is two.

Two braided surfaces are **equivalent** if there is a fiber-preserving ambient isotopy of $D_2^1 \times D_2^2$ rel $D_2^1 \times \partial D_2^2$ which carries one to the other.

There is a theorem which corresponds to Alexander’s theorem for classical oriented links.

**Theorem 1.3** (Kamada [14]). Any oriented surface link can be deformed by an ambient isotopy of $\mathbb{R}^4$ to the closure of a simple surface braid.

There is a **chart** which represents a simple surface braid.

**Definition 1.4.** Let $m$ be a positive integer, and $\Gamma$ be a graph on a 2-disk $D_2^2$. Then $\Gamma$ is called a **surface link chart** of degree $m$ if it satisfies the following conditions:

(i) $\Gamma \cap \partial D_2^2 = \emptyset$.
(ii) Every edge is oriented and labeled, and the label is in $\{1, \ldots, m-1\}$.
(iii) Every vertex has degree 1, 4, or 6.
(iv) At each vertex of degree 6, there are six edges adhering to which, three consecutive arcs oriented inward and the other three outward, and those six edges are labeled \(i\) and \(i + 1\) alternately for some \(i\).

(v) At each vertex of degree 4, the diagonal edges have the same label and are oriented coherently, and the labels \(i\) and \(j\) of the diagonals satisfy \(|i - j| > 1\).

Vertices of degree 1 and 6 are called a \textit{black vertex} and a \textit{white vertex}.

A black vertex (resp. white vertex) of a chart corresponds to a branch point (resp. triple point) of the simple surface braid associated with the chart.

An edge without end points is called a \textit{loop}. An edge whose end points are black vertices is called a \textit{free edge}, and a configuration consisting of a free edge and a finite number of concentric simple loops such that the loops are surrounding the free edge is called an \textit{oval nest}.

An \textit{unknotted chart} is a chart presented by a configuration consisting of free edges. A trivial oriented surface link is presented by an unknotted chart\([14]\).

A \textit{ribbon chart} is a chart presented by a configuration consisting of oval nests. A ribbon surface link is presented by a ribbon chart\([14]\).

A chart with a boundary represents a simple braided surface.

There is a notion of \textit{C-move equivalence} \([14]\) between two charts of the same degree. The following theorem is well-known:

\textbf{Theorem 1.5} \([14]\). \textit{Two charts of the same degree are C-move equivalent if and only if their associated simple braided surfaces are equivalent.}

2. Torus-covering-links

In this section, we give the definitions of torus-covering-links and turned torus-covering-links (cf. \[21\]).

\textbf{Definition 2.1}. [Torus-covering-links] First, embed \(D^2 \times S^1 \times S^1\) into \(S^4\) or \(\mathbb{R}^4\) naturally, or more precisely, consider as follows (cf. \[24\], \[2\], and \[18\]). Let \(S^1 \times S^1\) be a standardly embedded torus in \(S^4\) and let \(D^2 \times S^1 \times S^1\) be a tubular neighborhood of \(S^1 \times S^1\) in \(S^4\). We can assume that its framing is canonical, that is, the homomorphism induced by the inclusion map \(H_1(\{0\} \times S^1 \times S^1; \mathbb{Z}) \rightarrow H_1(\{p\} \times S^1 \times S^1; \mathbb{Z}) \rightarrow H_1(S^4 - S^1 \times S^1; \mathbb{Z})\) where \(p \in \partial D^2\), is zero. Let \(\bar{l} = \partial D^2 \times 0 \times 0, \bar{s} = 0 \times S^1 \times 0,\) and \(\bar{r} = 0 \times 0 \times S^1\) be curves on \(\partial D^2 \times S^1 \times S^1\).

Let \(E^4 = \text{cl}(S^4 - D^2 \times S^1 \times S^1)\). Let \(l, s\) and \(r\) be canonical curves on \(\partial E^4\), which are identified with \(\bar{l}, \bar{s}\) and \(\bar{r}\) under the natural identification.
map \( i = \partial D^2 \times S^1 \times S^1 \to \partial E^4 \). Then \( l, r \) and \( s \) represent a basis of \( H_1(\partial E^4; \mathbb{Z}) \).

Let \( f : \partial E^4 \to E^4 \) be a diffeomorphism with \( f_* (l \ s \ r) = (l \ s \ r) A^f \), where \( A^f \in GL(3, \mathbb{Z}) \cong \pi_0 \text{Diff}(\partial E^4) \). Then \( f \) can be extended to a diffeomorphism \( \tilde{f} : E^4 \to E^4 \) if and only if \( A^f \in H \), where

\[
H = \left\{ \begin{pmatrix} \pm 1 & 0 & 0 \\ * & \alpha & \gamma \\ * & \beta & \delta \end{pmatrix} \in GL(3, \mathbb{Z}) ; \quad \alpha + \beta + \gamma + \delta \equiv 0 \pmod{2} \right\}.
\]

Consider \( D^2 \times S^1 \times S^1 \) to be embedded in \( S^4 = E^4 \cup_1 D^2 \times S^1 \times S^1 \) or \( \mathbb{R}^4 = (E^4 - \{*\}) \cup_1 (D^2 \times S^1 \times S^1) \) for some point \(*\) in \( \text{Int}E^4 \). Then identify \( D^2 \times S^1 \times S^1 \) with \( D^2 \times I_3 \times I_4 \) \( \sim \), where \((x, 0, v) \sim (x, 1, v)\) and \((x, u, 0) \sim (x, u, 1)\) for \( x \in D^2, u \in I_3 = [0, 1] \) and \( v \in I_4 = [0, 1] \).

Let us consider a surface link \( S \) embedded in \( D^2 \times S^1 \times S^1 \) such that \( S \cap (D^2 \times I_3 \times I_4) \) is a simple braided surface. We will call such a surface link a torus-covering-link.

A torus-covering-link \( S \) can be described by a chart on the trivial torus knot, i.e. by a chart \( \Gamma_T \) on \( D^2_2 = I_2 \times I_4 \) with \( \Gamma_T \cap (I_3 \times \{0\}) = \Gamma_T \cap (I_3 \times \{1\}) \) and \( \Gamma_T \cap (\{0\} \times I_4) = \Gamma_T \cap (\{1\} \times I_4) \). Let us denote the classical braids described by \( \Gamma_T \cap (I_3 \times \{0\}) \) and \( \Gamma_T \cap (\{0\} \times I_4) \) by \( \Gamma^u_T \) and \( \Gamma^h_T \) respectively. We will call \( \Gamma_T \) a torus-covering-chart with boundary braids \( \Gamma^u_T \) and \( \Gamma^h_T \).

Let \( b(\Gamma_T) \) be the number of black vertices in the torus-covering-chart \( \Gamma_T \). Then let us consider the case \( b(\Gamma_T) = 0 \). In this case the torus-covering-link associated with \( \Gamma_T \) is determined by the boundary braids \( \Gamma^u_T \) and \( \Gamma^h_T \). We will call such a \( \Gamma_T \) a torus-covering-chart without black vertices and with boundary braids \( \Gamma^u_T \) and \( \Gamma^h_T \).

Remark. In the case \( b(\Gamma_T) = 0 \), the boundary braids \( \Gamma^u_T \) and \( \Gamma^h_T \) are commutative.

By definition, torus-covering-links contain symmetry-spun tori (and spun \( T^2 \)-knots, turned spun \( T^2 \)-knots and torus \( T^2 \)-knots).

As we stated in Theorem 1.5 if there are two surface link charts of the same degree, their associated surface links are equivalent if and only if their charts are C-move equivalent. It follows that if two torus-covering-charts of

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**Figure 2.1.** A torus-covering-link
the same degree are C-move equivalent, their associated torus-covering-links are equivalent.

A torus-covering-link has no 2-knot component. In particular, if a torus-covering-chart has no black vertices, then each component of the associated torus-covering-link is of genus one.

Let $\Gamma_T$ be a torus-covering-chart of degree $m$ and with the trivial boundary braids. Let $F$ be the surface link associated with $\Gamma_T$ by assuming $\Gamma_T$ to be a surface link chart. Then the torus-covering-link associated with the torus-covering-chart $\Gamma_T$ is obtained from $F$ by applying $m$ trivial 1-handle surgeries.

Example 2.2. (Example 2.2 in [21])

(2.2.1) Let $\Gamma_T$ be a torus-covering-chart of degree 2 without black vertices and with boundary braids $\sigma_1^3$ and $\varnothing$ (the trivial braid). Then the torus-covering-knot associated with $\Gamma_T$ is the spun $T^2$-knot of a right-handed trefoil.

(2.2.2) Let $\Gamma_T$ be a torus-covering-chart of degree 2 without black vertices and with boundary braids $\sigma_1^3$ and $\sigma_2^{-1}$ (or $\sigma_2^{-1}$ and $\sigma_1$). Then the torus-covering-knot associated with $\Gamma_T$ is the turned spun $T^2$-knot of a right-handed trefoil.

(2.2.3) Let $\Gamma_T$ be a torus-covering-chart without black vertices and with boundary braids $\beta^2$ and $\beta$. Then the torus-covering-knot associated with $\Gamma_T$ is a symmetry-spun torus.

We will consider the turned torus-covering-links, which include the turned spun $T^2$-links (cf. [21], [2]).

Definition 2.3. Let us use the notation in Definition 2.1. Let $\sigma : \partial E^4 \to \partial E^4$ be a diffeomorphism of a matrix

$$
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{pmatrix}
$$

is diffeomorphic to $S^4$.

Let $\Gamma_T$ be a torus-covering-chart and let $S$ be the torus-covering-link associated with $\Gamma_T$ in $S^4 = E^4 \cup_i D^2 \times S^1 \times S^1$. Then we can consider the torus-covering-link obtained from $S$ by changing the identification map $i$ to $\sigma i$, which we will call the turned torus-covering-link associated with $(S, \Gamma_T)$ or $S$, and use the notation $\tau(S, \Gamma_T)$ or $\tau(S)$. Moreover, we will denote by $\Gamma_{\tau(T)}$ the torus-covering-chart associated with $\tau(T)$. That is, we define $\tau(S)$ as follows:

$$
(S, E^4 \cup_i D^2 \times S^1 \times S^1 \cong S^4) = (\tau(S), S^4 = E^4 \cup_i D^2 \times S^1 \times S^1).
$$

The turned torus-covering-link $S$ must be in the form associated with the torus-covering-chart $\Gamma_T$, and for two equivalent torus-covering-links $S$ and $S'$ with their associated torus-covering-charts $\Gamma_T$ and $\Gamma'_T$, their turned torus-covering-links $\tau(S, \Gamma_T)$ and $\tau(S', \Gamma'_T)$ may be different (cf. Proposition 2.7 in [21]).

Remark. For a spun $T^2$-knot $S$, $\tau(S)$ is the turned spun $T^2$-knot. For more about turned torus-covering-links, see Section 2 in [21].

From now on throughout this paper we consider torus-covering-links in $\mathbb{R}^4$. 5
3. Unknotting numbers

It is known that any surface link can be deformed to a trivial surface link by applying a finite number of 1-handle surgeries.

For an oriented surface link, its unknotting number is the minimum number of oriented 1-handle surgeries necessary to deform it to trivial.

For an oriented surface link, adding a free edge to the surface link chart corresponds to a nice 1-handle surgery, which is a kind of an oriented 1-handle surgery.

In this section we study unknotting numbers of torus-covering-links.

Theorem 3.1. Let $\tau(S) = \tau(S,G_T)$ be the turned torus-covering-link obtained from the torus-covering-link $S$ associated with a torus-covering-chart $G_T$. Then if $S$ is unknotted, $\tau(S)$ is also unknotted.

Proof. (Case 1) Suppose that $S$ consists of one component. We consider that the torus-covering-knot $S$ is embedded in $B := B^3 \times S^1$ and $B^3 \times S^1$ is embedded in $\mathbb{R}^4$ by $\mathbb{R}^4 = S^4 - \{\ast\} = B^3 \times S^1 \cup_i S^2 \times B^2 - \{\ast\}$, where $\partial B^3 = S^2$ and $\partial B^2 = S^1$ and $i : S^2 \times S^1 \to S^2 \times S^1$ is the identity map. There is a handlebody whose boundary is $S$, for $S$ is unknotted. Let us denote this handlebody by $H$. Then $\partial H \subset B = B^3 \times S^1$. We can deform $S$ such that the handlebody $H$ is in $\mathbb{R}^3 \times \{t_0\}$ and $\partial B = S^2 \times S^1$ is embedded in $\mathbb{R}^4$. Let $f : S^2 \times S^1 \to \mathbb{R}^4$ be the embedding. For every $p \in S^1$, $f(S^2 \times \{p\}) \subset D^3$, where $D^3$ is a 3-ball. In other words, $S^2 \times \{p\}$ is trivially embedded. If for some $p \in S^1$, $f(S^2 \times \{p\})$ is not trivially embedded in $\mathbb{R}^4$, then $S^2 \times I \subset S^2 \times S^1$ cannot be embedded in $\mathbb{R}^4$, where $I \ni p$ is a sufficiently small interval.

Hence $f(S^2 \times S^1) = S^2 \times f(S^1)$, where $S^2$ is trivially embedded and $f(S^1)$ may be knotted. This knotted $f(S^1)$ can be deformed to the trivially embedded $S^1$ by an ambient isotopy of $\mathbb{R}^4$, but we must consider the following situation: $H \subset \mathbb{R}^3 \times \{t_0\}$, and $\partial H \subset B^3 \times f(S^1)$, where $B^3$ is the 3-ball with $\partial B^3 = S^2$.

Since $\partial H$ is a trivially embedded closed 2-manifold in $B^3 \times f(S^1)$, we see that $H \subset B^3 \times f(S^1)$. Therefore, $H \subset B$. From the construction of $\tau(S,G_T)$, the handlebody $H$ also represents a handlebody whose boundary is $\tau(S,G_T)$, which means $\tau(S,G_T)$ is also unknotted.

(Case 2) If $S$ has plural components, let $n > 1$ be the number of the components. By changing $H$ in Case 1 to an embedded $n$-component 3-manifold $M$ with $\partial M = S$ such that each component of $M$ is a handlebody, we can show that $\tau(S)$ is also unknotted.

Corollary 3.2. Let $S$ be the torus-covering-link associated with a torus-covering-chart $G_T$ and $\tau(S)$ be the turned torus-covering-link of $(S,G_T)$. Let $u(S)$ (resp. $u(\tau(S))$) be the unknotting number of $S$ (resp. $\tau(S)$), and $u_F(S)$ (resp. $u_F(\tau(S))$) be the minimal number of free edges necessary to deform $G_T$ (resp. $G_{\tau(T)}$) to represent an unknotted surface link. Then we have

$$u(\tau(S)) \leq u_F(S), \quad u(S) \leq u_F(\tau(S)), \quad u_F(S) = u_F(\tau(S)).$$

Proof. We show the last equation. By Theorem 3.1, $u_F(\tau(S)) \leq u_F(S)$. On the other hand, since $\tau^2(S) = S$ (Proposition 2.6 in [21]), we have
\[ u_F(S) \leq u_F(\tau(S)). \]

We give examples of torus-covering-knots whose unknotting number is an arbitrary positive integer:

**Theorem 3.3.** There is a torus-covering-knot whose unknotting number is \( n \), where \( n \) is any positive integer. Let \( S \) be the spun or turned spun \( T^2 \)-knot of a classical knot \( \beta = \text{cl}(\sigma_1^3 \sigma_2^3 \cdots \sigma_n^3) \) (degree \( n+1 \)) with \( n > 0 \). Then the unknotting number of \( S \) is \( n \).

Before the proof, we show two lemmas and give the definition of tri-coloring.

**Lemma 3.4.** Let \( \Gamma_T \) be a torus-covering-chart of degree \( m \) with only free edges. Then the torus-covering-link \( S \) associated with \( \Gamma_T \) is unknotted.

**proof.** Let \( S' \) be a surface link obtained by \( \Gamma_T \) by regarding it as a surface link chart. Then \( S \) is obtained from \( S' \) by adding \( m \) trivial 1-handles. Since \( S' \) is unknotted, \( S \) is also unknotted. \( \square \)

**Lemma 3.5.** Let \( \Gamma_T \) be a torus-covering-chart of degree \( m \) which has no white vertices. In other words, \( \Gamma_T \) has only free edges and loops. Then the unknotting number of the torus-covering-link \( S \) associated with \( \Gamma_T \) is at most \( m - 1 \).

**proof.** The torus-covering-chart \( \Gamma_T \) can be deformed to the form of disjoint union of free edges by adding \( (m - 1) \)-free edges, which have all the labels running from 1 to \( m - 1 \). Hence, by Lemma 3.4, the unknotting number of \( S \) is at most \( m - 1 \). \( \square \)

Now we give the definition of tri-coloring by the dihedral quandle of order three \( R_3 \).

A set \( X \) with a binary operation \( * : X \times X \to X \) is called a quandle if it satisfies the following conditions:

(i) \( a * a = a \), for every \( a \) in \( X \).
(ii) for any \( b, c \in X \), there exists a unique \( c \in X \) such that \( a = c * b \).
(iii) \( (a * b) * c = (a * c) * (b * c) \), for \( a, b, c \in X \).

The **dihedral quandle of order 3**, \( R_3 \), is the set \( \{0, 1, 2\} \) with the binary operation \( a * b = 2b - a \) (mod 3).

Let \( \pi : \mathbb{R}^4 \to \mathbb{R}^3 \) be a generic projection. Then a **surface diagram** of a surface link \( F \) is the image \( \pi(F) \) with additional crossing information at the singularity set. There are two intersecting sheets along each edge, one of which is higher than the other with respect to \( \pi \). They are called an over-sheet and an under sheet along the edge, respectively. In order to indicate crossing information, we break the under-sheet into two pieces missing the over sheet. This can be extended around a triple point. The sheets are called a top sheet, a middle sheet, and a bottom sheet from the higher one. Then the surface diagram is presented by a disjoint union of compact surfaces which are called broken sheets. For a surface diagram \( D \), we denote by \( B(D) \) the set of broken sheets of \( D \).

A **tri-coloring** for a surface diagram \( D \) is a map \( C : B(D) \to R_3 \) such that \( C(H_1) * C(H_2) = C(H'_1) \).
along every edge of $D$, where $H_2$ is the over-sheet and $H_1$ (resp. $H_i'$) is the under-sheet such that the normal vector of $H_2$ points from (resp. toward) it. Then the color of the edge is the pair $(C(H_1), C(H_2))$. The number of all the possible tri-colorings is a knot invariant, i.e. it is independent of the choice of the surface diagrams. The notion of tri-coloring a classical knot diagram is similarly defined.

The proof of Theorem 3.3 The torus-covering-knot $S$ can be described by a torus-covering-chart $\Gamma_T$ of degree $n+1$ with neither black vertices nor white vertices. More precisely, $\Gamma_T$ has no black vertices and its boundary braids are $\beta$ and $e$ (the trivial braid of degree $n+1$), or $\beta$ and $\beta$. Then by Lemma 3.5 the unknotting number of $S$ is at most $n$. It suffices to show that we must apply at least $n$ 1-handle surgeries to deform $S$ to a trivial surface knot.

Let us tri-color $S$. It suffices to tri-color the classical knot $\beta$. The classical braid $b = \sigma_1^3 \sigma_2^3 \cdots \sigma_n^3$ can be divided into $n$ blocks $b_i = \sigma_i^3$ ($i = 1, 2, \ldots, n$).

Let us denote by $q_i$ (resp. $q_{i+1}$) the start point of the $i$-th (resp. $(i+1)$-th) string of $b_i$ and by $q''_i$ (resp. $q''_{i+1}$) the end point of the $i$-th (resp. $(i+1)$-th) string of $b_i$ for $i = 2, \ldots, n$. For some $q$, a point of the diagram of $b$, let $C(q)$ be the color of the edge with $q$ on it.

We consider tri-coloring $b$ with $C(q_i) = C(q''_i)$ for every $i$. The diagram of the braid $b_i$ consists of five edges. The four edges of them each contain one of the start points or end points. Let us denote by $p_i$ a point of the fifth edge.

The tri-coloring of the braid $b_i$ is determined by two of the three colors $C(q'_i) = C((q''_i), C(q_{i+1}) = C(q''_{i+1})$ and $C(p_i)$. Hence the tri-coloring of $b$ is determined by $C(q_1), C(q_2), \ldots, C(q_{n+1})$. Let $\Phi(S)$ be the number of all the possible tri-colorings. Then we have

$$\Phi(S) = 3^{n+1}.$$  

Adding one 1-handle to $S$ means identifying $C(e_1)$ and $C(e_2)$ for some edges $e_1$ and $e_2$ in the diagram of $b$. Let $e_k$ be on the braid $b_k$ ($k = 1, 2$). If $e_1 = e_2$ or $e_1, e_2 \in \{C(q'_i), C(q''_i)\}$ or $\{C(q_{i+1}), C(q''_{i+1})\}$, then the number of all the possible tri-colorings does not change. If not, then $C(q_{j_2}) = C(q''_{j_2})$ or $C(q_{j_2+1}) = C(q''_{j_2+1})$ is determined by the 1-handle, and the number of the tri-colorings becomes $3^n$. We have

$$\Phi(S') = \Phi(S)$$ or $\Phi(S)/3$,

where $S'$ is the resulting surface knot.

Hence we must apply at least $n$ 1-handle surgeries to deform $S$ to have just three tri-colorings, which is a necessary condition for a trivial surface knot.

\[\square\]

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\[\text{Graduate School of Mathematical Sciences, the University of Tokyo} \]

E-mail address: inasa@ms.u-tokyo.ac.jp