Quantum State Isomorphism

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Abstract

We consider a problem we call \textsc{StateIsomorphism}: given two quantum states of \(n\) qubits, can one be obtained from the other by rearranging the qubit subsystems? Our main goal is to study the complexity of this problem, which is a natural quantum generalisation of the problem \textsc{StringIsomorphism}. We show that \textsc{StateIsomorphism} is at least as hard as \textsc{GraphIsomorphism}, and show that these problems have a similar structure by presenting evidence to suggest that \textsc{StateIsomorphism} is an intermediate problem for QCMA. In particular, we show that the complement of the problem, \textsc{StateNonIsomorphism}, has a two message quantum interactive proof system, and that this proof system can be made statistical zero-knowledge. We consider also \textsc{StabilizerStateIsomorphism} (SSI) and \textsc{MixedStateIsomorphism} (MSI), showing that the complement of SSI has a quantum interactive proof system that uses classical communication only, and that MSI is QSZK-hard.

1 Introduction and statement of results

Ladner’s theorem \cite{Ladner} states that if \(P \neq \text{NP}\) then there exists \text{NP-intermediate problems}: \text{NP} problems that are neither \text{NP-hard}, nor in \(P\). While of course the \(P\) vs. \text{NP} problem is unresolved, the problem of testing if two graphs are isomorphic (\textsc{GraphIsomorphism}) has the characteristics of such an intermediate problem. \textsc{GraphIsomorphism} is trivially in \text{NP}, since isomorphism of two graphs can be certified by describing the permutation that maps one to the other, but as Boppana and Håstad show \cite{BoppanaHastad}, if it is \text{NP-complete} then the polynomial hierarchy collapses to the second level. Furthermore, while many instances of the problem are solvable efficiently in practice \cite{Babai}, it is still not known if there exists a polynomial time algorithm for the problem.

Recall that Quantum Merlin Arthur (QMA) is considered to be the quantum analogue of \text{NP}: the certificate is a quantum state, and the verifier has the ability to perform quantum computation. The class QCMA is defined in the same way but with certificates restricted to be classical bitstrings. In this paper, we show that there are problems that exhibit similar hallmarks of being intermediate for QCMA \cite{Oganov}. Succinctly: we formulate problems in QCMA that are not obviously in BQP, and which are unlikely to be QCMA-complete.

Babai’s recent quasi-polynomial time algorithm for \textsc{GraphIsomorphism} \cite{Babai} has revived a fruitful body of work that links the problem to algorithmic group theory \cite{BabaiLubotzky, BabaiZuk, BabaiViglas, BabaiOganov}. This literature deals with a closely related problem called \textsc{StringIsomorphism}: given bitstrings \(x, y \in \{0, 1\}^n\) and a permutation group \(G\), is there \(\sigma \in G\) such that \(\sigma(x) = y\) (where permutations act in the obvious way on the strings)? This problem has a number of similarities with \textsc{GraphIsomorphism}, and, as we show, can be recast in terms of quantum states.

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We study what is arguably the most direct quantum generalisation of this problem, a problem we call STATEISOMORPHISM. Such a generalisation is obtained by replacing the strings \( x \) and \( y \) by \( n \)-qubit pure states, and by considering the permutations in the group \( G \) to act as “reshufflings” of the qubits. The problem is obviously in QCMA: if there is a permutation mapping one state to the other then its permutation matrix acts as the certificate. Equality of two quantum states can be verified via an efficient quantum procedure known as the SWAP test \cite{11}. Also, if there is an efficient quantum algorithm then the same can be used as an algorithm for GRAPHISOMORPHISM: as we shall see later, there exists a polynomial time many-one reduction from GRAPHISOMORPHISM to STATEISOMORPHISM.

We first establish that in terms of interactive proof systems that solve the problem, STATEISOMORPHISM has a number of similarities with its classical counterpart. A central part of the Boppana-Håstad collapse result is that GRAPHISOMORPHISM belongs in co-IP(2): that is, that GRAPHNONISOMORPHISM has a two round interactive proof system. We show that STATEISOMORPHISM is in co-QIP(2): its complement has a two round quantum interactive proof system. GRAPHISOMORPHISM also admits a statistical zero knowledge proof system, and indeed, we prove that STATEISOMORPHISM has an honest verifier quantum statistical zero knowledge proof system. These results are summarised in the following theorem, where QSZK is the class of problems with (honest verifier) quantum statistical zero knowledge proof systems, defined by Watrous in \cite{27}. Note that since QIP(2) \( \supseteq \) QSZK = co-QSZK (see \cite{27}), inclusion in co-IP(2) follows as a corollary.

**Theorem 1.** STATEISOMORPHISM is in QSZK.

A corollary of this theorem provides evidence to suggest that STATEISOMORPHISM is not QCMA-complete. If it were, then every problem in QCMA would have an honest verifier quantum statistical zero knowledge proof system. Furthermore, this result is evidence against the problem being NP-hard; it is unlikely that NP \( \subseteq \) QSZK.

**Corollary 2.** If STATEISOMORPHISM is QCMA-complete then QCMA \( \subseteq \) QSZK.

In pursuit of stronger evidence against QCMA-hardness of STATEISOMORPHISM, we consider a quantum polynomial hierarchy in the same vein as those considered by Gharibian and Kempe \cite{26}, and Yamakami \cite{26}. This hierarchy is defined in terms of quantum \( \exists \) and \( \forall \) complexity class operators like those of \cite{25}, but from our definitions it is easy to verify that lower levels correspond to well known complexity classes. In particular, \( \Sigma_0 = \Pi_0 = \text{BQP} \), and \( \Sigma_1 = \text{QCMA} \) or \( \Sigma_1 = \text{QMA} \) depending on whether we take the certificates to be classical or quantum (see Section \[\text{?}\]). Also, from the definition we provide, it is clear that the class \( \text{co-} \Sigma_2 \) corresponds directly to the identically named class in \cite{26}.

We prove the following, where \( \text{QPH} = \cup_{i=1}^{\infty} \Sigma_i \), and QCAM is the quantum generalisation of the class AM where all communication between Arthur and Merlin is restricted to be classical \cite{33}.

**Theorem 3.** Let \( A \) be a promise problem in QCMA \( \cap \text{co-QCAM} \). If \( A \) is QCMA-complete, then QPH \( \subseteq \Sigma_2 \).

While the relationship between the levels of this hierarchy and the levels of the classical hierarchy remains an open research question \cite{19}, the fact that the lower levels of this quantum hierarchy coincide with well known classes gives weight to collapse results of this kind. We draw attention to the fact that the collapse implication in Theorem \[\text{?}\] is for the classical certificate classes QCMA and QCAM, rather than for the more well known QMA and QAM \cite{33}. While the problems we consider are in QCMA, meaning that the current statement of the theorem is all we need, already we have an interesting open question: is there a similar collapse theorem that relates QMA and QAM? The proof of Theorem \[\text{?}\] relies on the fact that QCMAM = QCAM (proved by Kobayashi et al. in \cite{30}), but it is unlikely that QMAM = QAM, since QMAM = QIP = PSPACE \cite{33} \cite{14}.

As we shall see in Section \[\text{?}\] there is a barrier that prevents us from applying Theorem \[\text{?}\] to STATEISOMORPHISM: our quantum interactive proof systems for STATENONISOMORPHISM require quantum communication between verifier and prover. This prevents us from proving inclusion in QCAM. It is not clear that the problem admits such a proof system.

However, if it is possible to produce an efficient classical description of the quantum states in the problem instance that is independent from how they are specified in the input, then it is possible to prove inclusion in QCAM. We show that this is the case for a restricted family of quantum states called **stabilizer states**, a fact which allows us to prove the following.
Corollary 4. If StabilizerStateIsomorphism is QCMA-complete, then QPH ⊆ Σ₂.

Furthermore, the fact that stabilizer states can be described classically also implies the following.

Theorem 5. StabilizerStateNonIsomorphism is in QCSZK.

Finally, we consider the state isomorphism problem for mixed quantum states. We show that this problem is QSZK-hard by reduction from the QSZK-complete problem of determining if a mixed state is product or separable.

Theorem 6. (ε, 1 − ε)-MixedStateIsomorphism is QSZK-hard.

While these state isomorphism problems all have classical certificates, we have been able to demonstrate that the complexity of each problem depends precisely on the inherent computational difficulty of working with the input states. Stabilizer states form one end of the spectrum: with a polynomial number of measurements a classical description can be produced. The other extreme is the mixed states, these are so computationally difficult to work with that it is not clear that MixedStateIsomorphism even belongs in QMA; even the problem of testing equivalence of two such states is QSZK-complete (see [27]). Between these two extremes we have StateIsomorphism. While such states can be efficiently processed by a quantum circuit, and isomorphism can be certified classically, the analysis in Section 3 uncovers an interesting caveat. It seems that the ability to communicate quantum states is still required when we wish to check non-isomorphism by interacting with a prover, or perhaps even to certify isomorphism with statistical zero knowledge. We thus draw attention to the following open question: can our protocols be modified to use exclusively classical communication?

The fact that an efficient quantum algorithm for StateIsomorphism would also yield one for GraphIsomorphism, combined with Corollary 2 gives weight to the idea that this problem can be thought of as a candidate for a QCMA-intermediate problem. The fact that there are problems “in between” BQP and QCMA, and furthermore, that such problems are obtained by generalising StringIsomorphism suggests an interesting parallel between the classical and quantum classes.

In Section 2 we give an overview of the tools and notation we will use for the rest of the paper. We also define the key problems and complexity classes we will be working with and prove some initial results that we build on later. In Section 3 we demonstrate quantum interactive proof systems for the StateIsomorphism problems. In Section 4 we define a notion of a quantum polynomial hierarchy, and prove the hierarchy collapse results.

## 2 Preliminaries and definitions

Recall that quantum states are represented by unit trace positive semi-definite operators ρ on a Hilbert space H called the state space of the system. A state is pure if ρ² = ρ. Otherwise, we say that the state is mixed. By definition then, for any pure state ρ on H we have that ρ = |ψ⟩⟨ψ| for some unit vector |ψ⟩ ∈ H, and we refer to pure states by their corresponding state vector |ψ⟩ (which is unique up to multiplication by a phase). Mixed states are convex combinations of the outer products of some set of state vectors ρ = ∑ᵢ pᵢ |ψᵢ⟩⟨ψᵢ|. In what follows we refer to the Hilbert space C² by H₂. Recall that an n-qubit pure state |ψ⟩ ∈ H₂⊗ⁿ is product if |ψ⟩ = |ψ₁⟩ ⊗ ... ⊗ |ψₙ⟩ where ⊗ denotes tensor product and for all i, |ψᵢ⟩ ∈ H₂. For any bitstring x₁...xₙ ∈ {0, 1}ⁿ, we say that |x⟩ = ⊗ᵢ₌₁ⁿ |xᵢ⟩ is a computational basis state.

A useful measure of the distinguishability of a pair of quantum states is the trace distance. Let ρ, σ be quantum states with the same state space. Their trace distance is the quantity ||M||₁ = tr(|M|), where ||M||₁ = tr(|M|) is the trace norm.

We say that a quantum circuit Q accepts a state |ψ⟩ if measuring the first qubit of the state Q|ψ⟩ in the computational basis yields outcome 1. We say that the circuit rejects the state otherwise. Let X be an index set. We say that a uniform family of quantum circuits {Qₓ : x ∈ X} is polynomial-time generated if there exists a polynomial-time Turing machine that takes as input x ∈ X and halts with an efficient description of the circuit Qₓ on its tape. Such a definition neatly captures the notion of an efficient quantum computation [16].

We make use of a quantum circuit known as the SWAP test [14], illustrated in Figure 1. This circuit takes as input pure states |ψ⟩, |φ⟩ and accepts (denoted T(|ψ⟩, |φ⟩) = 1) with probability (1 + |⟨ψ|φ⟩|²)/2.
Note that $T(|\psi\rangle, |\phi\rangle) = 1$ with probability 1 if $|\psi\rangle = e^{i\tau}|\phi\rangle$ for some $\tau \in [-2\pi, 2\pi]$, but is equal to 1 with probability $1/2$ if they are orthogonal. The SWAP test can be therefore be used as an efficient quantum algorithm for testing if two quantum states are equivalent. In what follows we use some notation from complexity theory and formal language theory. In particular, if a problem $A$ is polynomial-time many-one reducible to a problem $B$ we denote this by $A \leq_p B$. We denote by $\{0, 1\}^n$ the set of bitstrings of length $n$, furthermore, $\{0, 1\}^*$ denotes the set of all bitstrings. For a bitstring $x$, we denote by $|x|$ the length of the bitstring. We say that a function $f : \mathbb{N} \rightarrow [0, 1]$ is negligible if for every constant $c$ there exists $n_c$ such that for all $n \geq n_c$, $f(n) < 1/n^c$. We use the shorthand $f(n) = \text{poly}(n)$ (resp. $f(n) = \exp(n)$) to state that $f$ scales as a polynomially bounded (exponentially bounded) function in $n$.

A decision problem is a set of bitstrings $A \subseteq \{0, 1\}^*$. An algorithm is said to decide $A$ if for all $x \in \{0, 1\}^*$ it outputs YES if $x \in A$ and NO otherwise. In quantum computational complexity it is useful to use the less well known notion of a promise problem to allow for more control over problem instances. A promise problem is a pair of sets $(A_{YES}, A_{NO}) \subseteq \{0, 1\}^* \times \{0, 1\}^*$ such that $A_{YES} \cap A_{NO} = \emptyset$. An algorithm is said to decide $(A_{YES}, A_{NO})$ if for all $x \in A_{YES}$ it outputs YES and for all $x \in A_{NO}$ it outputs NO. Note that the algorithm is not required to do anything in the case where an input $x$ does not belong to $A_{YES}$ or $A_{NO}$.

### 2.1 Quantum Merlin-Arthur, Quantum Arthur-Merlin

For convenience, we give a number of definitions related to quantum generalisations of public coin proof systems. In particular, we focus on Quantum Arthur-Merlin (QAM) and Quantum Merlin-Arthur, the quantum versions of AM and MA respectively. We use the definitions in \cite{16, 33} as our guide.

**Definition 7 (QMA).** A promise problem $A = (A_{YES}, A_{NO})$ is in QMA($a, b$) for functions $a, b : \mathbb{N} \rightarrow [0, 1]$ if there exists a polynomial-time generated uniform family of quantum circuits $\{V_x : x \in \{0, 1\}^*\}$ and polynomially bounded $p : \mathbb{N} \rightarrow \mathbb{N}$ such that

- for all $x \in A_{YES}$ there exists $|\psi\rangle \in \mathcal{H}_2^{\otimes p(|x|)}$ such that
  $$\Pr[V_x \text{ accepts } |\psi\rangle] \geq a(|x|);$$

- for all $x \in A_{NO}$ and for all $|\psi\rangle \in \mathcal{H}_2^{\otimes p(|x|)}$,
  $$\Pr[V_x \text{ accepts } |\psi\rangle] \leq b(|x|).$$

The class QCMA is defined in the same way, but with the restriction that the certificate $|\psi\rangle$ must be a computational basis state $|x\rangle$.

A QAM verification procedure is a tuple $(V, m, s)$ where

$$V = \{V_{x,y} : x \in \{0, 1\}^*, y \in \{0, 1\}^{s(|x|)}\}$$

is a uniform family of polynomial time generated quantum circuits, and $m, s : \mathbb{N} \rightarrow \mathbb{N}$ are polynomially bounded functions. Each circuit acts on $m(|x|)$ qubits sent by Merlin, and $k(|x|)$ qubits which correspond to Arthur’s workspace. For all $x, y$, we say that $V_{x,y}$ accepts (resp. rejects) a state $|\psi\rangle \in \mathcal{H}_2^{\otimes m(|x|)}$ if, upon measuring the first qubit of the state

$$V_{x,y}(|\psi\rangle|0) \otimes k(|x|)$$

in the standard basis, the outcome is ‘1’ (resp. ‘0’).
Definition 8 (QAM). A promise problem $A = (A_{\text{YES}}, A_{\text{NO}})$ is in QAM($a, b$) for functions $a, b : \mathbb{N} \rightarrow \{0, 1\}$ if there exists a QAM verification procedure $(V, m, s)$ such that

- for all $x \in A_{\text{YES}}$, there exists a collection of $m(|x|)$-qubit quantum states $\{|\psi_y\rangle\}$ such that
  \[ \frac{1}{2^{s(|x|)}} \sum_{y \in \{0, 1\}^{s(|x|)}} \Pr[V_{x,y} \text{ accepts } |\psi_y\rangle] \geq a(|x|); \]

- for all $x \in A_{\text{NO}}$, and for all collections of $m(|x|)$-qubit quantum states $\{|\psi_y\rangle\}$, it holds that
  \[ \frac{1}{2^{s(|x|)}} \sum_{y \in \{0, 1\}^{s(|x|)}} \Pr[V_{x,y} \text{ accepts } |\psi_y\rangle] \leq b(|x|). \]

The class QCAM is defined in the same way but with the states $\{|\psi_y\rangle\}$ restricted to computational basis states. The class QCMAM is similar, but has an extra round of interaction.

Definition 9 (QCMAM). A promise problem $A = (A_{\text{YES}}, A_{\text{NO}})$ is in QCMAM($a, b$) for functions $a, b : \mathbb{N} \rightarrow \{0, 1\}$ if there exists a QAM verification procedure $(V, m, s)$ and a polynomially bounded function $p : \mathbb{N} \rightarrow \mathbb{N}$ such that

- for all $x \in A_{\text{YES}}$, there is a certificate bitstring $c \in \{0, 1\}^{p(|x|)}$ and a collection of length $m(|x|)$ bitstrings $\{z^c_y\}$ such that
  \[ \frac{1}{2^{s(|x|)}} \sum_{y \in \{0, 1\}^{s(|x|)}} \Pr[V_{x,y} \text{ accepts } |c \otimes z^c_y\rangle] \geq a(|x|); \]

- for all $x \in A_{\text{NO}}$, all certificate bitstrings $c \in \{0, 1\}^{p(|x|)}$ and all collections of length $m(|x|)$ bitstrings $\{z^c_y\}$, it holds that
  \[ \frac{1}{2^{s(|x|)}} \sum_{y \in \{0, 1\}^{s(|x|)}} \Pr[V_{x,y} \text{ accepts } |c \otimes z^c_y\rangle] \leq b(|x|). \]

2.2 Quantum interactive proofs and zero knowledge

An interactive proof system consists of a verifier and a prover. The computationally unbounded prover attempts to convince the computationally limited verifier that a particular statement is true. A quantum interactive proof system is where the verifier is equipped with a quantum computer, and quantum information can be transferred between verifier and prover. Our formal definitions will follow those of Watrous [27, 16].

A quantum verifier is a polynomial time computable function $V$, where for each $x \in \{0, 1\}^*$, $V(x)$ is an efficient classical description of a sequence of quantum circuits $V(x)_1, \ldots, V(x)_k(|x|)$. Each circuit in the sequence acts on $v(|x|)$ qubits that make up the verifier’s private workspace, and a buffer of $c(|x|)$ communication qubits that both verifier and prover have read/write access to.

A quantum prover is a function $P$ where for each $x \in \{0, 1\}^*$, $P(x)$ is a sequence of quantum circuits $P(x)_1, \ldots, P(x)_l(|x|)$. Each circuit in the sequence acts on $p(|x|)$ qubits that make up the prover’s private workspace, and the $c(|x|)$ communication qubits that are shared with each verifier circuit. Note that no restrictions are placed on the circuits $P(x)$, since we wish the prover to be computationally unbounded. We say that a verifier $V$ and a prover $P$ are compatible if all their circuits act on the same number of communication qubits, and if for all $x \in \{0, 1\}^*$, $k(|x|) = \lfloor m(|x|)/2 + 1 \rfloor$ and $l(|x|) = \lfloor m(|x|)/2 + 1/2 \rfloor$, for some $m(|x|)$ which is taken to be the number of messages exchanged between the prover and verifier. We say that $(P, V)$ are a compatible $m$-message prover-verifier pair.

Given some compatible $m$-message prover-verifier pair $(P, V)$, we define the quantum circuit

\[
(P(x), V(x)) := \begin{cases} 
V(x_1) \cdot P(x_1) \ldots P(x_{m(|x|)/2}) \cdot V(x_{m(|x|)/2+1}) & \text{if } m(|x|) \text{ is even}, \\
V(x_1) \cdot V(x_1) \ldots P(x_{(m(|x|)+1)/2}) \cdot V(x_{(m(|x|)+1)/2}) & \text{if } m(|x|) \text{ is odd}.
\end{cases}
\]
Let \( q(|x|) = p(|x|) + c(|x|) + v(|x|) \). We say that \((P, V)\) accepts an input \( x \in \{0, 1\}^* \) if the result of measuring the verifier’s first workspace qubit of the state

\[
(P(x), V(x))|0^{p(|x|)}
\]

in the computational basis is 1, and that it rejects the input if the measurement result is 0.

**Definition 10 (QIP(\(k\))).** Let \( M = (M_{\text{YES}}, M_{\text{NO}}) \) be a promise problem, let \( a, b : \mathbb{N} \to [0, 1] \) be functions and \( k \in \mathbb{N} \). Then \( M \in \text{QIP}(k)(a, b) \) if and only if there exists a \( k \)-message verifier \( V \) such that

- if \( x \in M_{\text{YES}} \) then
  \[
  \max_P (\Pr[ (P, V) \text{ accepts } x ]) \geq a(|x|),
  \]
- if \( x \in M_{\text{NO}} \) then
  \[
  \max_P (\Pr[ (P, V) \text{ accepts } x ]) \leq b(|x|),
  \]

where the maximisation is performed over all compatible \( k \)-message provers. We say that the pair \((P, V)\) is an interactive proof system for \( M \).

Let us now define what it means for a quantum interactive proof system to be statistical zero-knowledge. Define the function

\[
\text{view}_{P,V}(x, j) = \text{tr}_P[(P(x), V(x))_j|0^{p(|x|)}\langle 0^{p(|x|)}|(P(x), V(x))_j^\dagger],
\]

where \((P(x), V(x))_j\) is the circuit obtained from running \((P(x), V(x))\) up to the \( j \)-th message. For some index set \( X \), we say that a set of density operators \( \{\rho_x : x \in X\} \) is polynomial-time preparable if there exists a polynomial-time uniformly generated family of quantum circuits \( \{Q_x : x \in X\} \), each with a designated set of output qubits, such that for all \( x \in X \), the state of the output qubits after running \( Q_x \) on a canonical initial state \( |0\rangle^\otimes n \) is equal to \( \rho_x \).

**Definition 11 (Honest Verifier Quantum Statistical Zero-Knowledge (HVQSZK)).** Let \( M = (M_{\text{YES}}, M_{\text{NO}}) \) be a promise problem, let \( a, b : \mathbb{N} \to [0, 1] \) and \( k : \mathbb{N} \to \mathbb{N} \) be functions. Then \( M \in \text{HVQSZK}(k)(a, b) \) if and only if \( M \in \text{QIP}(k)(a, b) \) with quantum interactive proof system \((P, V)\) such that there exists a polynomial-time preparable set of density operators \( \{\sigma_{x,i}\} \) such that for all \( x \in \{0, 1\}^* \), if \( x \in M_{\text{YES}} \) then

\[
D(\sigma_{x,j}, \text{view}_{P,V}(x, i)) \leq \delta(|x|)
\]

for some negligible function \( \delta \).

It is known that the class of problems that have quantum statistical zero knowledge proof systems (QSZK) is equivalent to the class of problems that have honest verifier quantum statistical zero knowledge proof systems (HVQSZK) [27]. Therefore, we refer to HVQSZK as QSZK, and only consider honest verifiers.

In the next section we give a formal definition of STRING\text{ISOMORPHISM}.

### 2.3 Permutations and STRING\text{ISOMORPHISM}

Let \( \Omega \) be a finite set. A bijection \( \sigma : \Omega \to \Omega \) is called a permutation of the set \( \Omega \). The set of all permutations of a finite set \( \Omega \) forms a group under composition. This group is called the symmetric group, and we denote it by \( \mathfrak{S}(\Omega) \). For \( x \in \Omega \) and \( \sigma \in \mathfrak{S}(\Omega) \), we denote the image of \( x \) under \( \sigma \) by \( \sigma(x) \).

A string \( s : \Omega \to \Sigma \) is an assignment of letters from a finite set \( \Sigma \) called an alphabet to the elements of a finite index set \( \Omega \). Let \( s : \Omega \to \Sigma \) be a string. The letters of \( s \) are indexed by elements of the index set \( \Omega \). The letter corresponding to \( i \in \Omega \) is thus denoted by \( s_i \). Let \( \sigma \in \mathfrak{S}(\Omega) \) be a permutation. Then the action of \( \sigma \) on \( s \) is denoted by \( \sigma(s) \), and is a string such that for all \( i \in \Omega \), \( \sigma(s)_i = s_{\sigma(i)} \). In this paper we often deal with permutations of strings indexed by natural numbers. Hence, we denote the symmetric group \( \mathfrak{S}([n]) \) by \( \mathfrak{S}_n \), where \([n] := \{1, \ldots, n\} \). In what follows we denote the fact that a group \( G \) is a subgroup of a group \( H \) by \( G \leq H \). The following decision problem is related to GRAPH\text{ISOMORPHISM} [7, 12], and forms the basis of our work.
**Problem 12. **STRINGISOMORPHISM  

*Input:* Finite sets $\Omega, \Sigma$, a permutation group $G \leq \mathfrak{S}(\Omega)$ specified by a set of generators, and strings $s, t : \Omega \to \Sigma$.  

*Output:* Yes if and only if there exists $\sigma \in G$ such that $\sigma(s) = t$.

It is clear that STRINGISOMORPHISM is at least as hard as GRAPHISOMORPHISM: a polynomial time many-one reduction can be obtained from GRAPHISOMORPHISM by “flattening” the adjacency matrices of the graphs in question into bitstrings. The set of string permutations that correspond to graph isomorphisms form a proper subgroup of the full symmetric group. Indeed the algorithm in [12] is actually an algorithm for STRINGISOMORPHISM, which solves GRAPHISOMORPHISM as a special case.

### 2.4 Stabilizer states

The Gottesmann-Knill theorem [13] states that any quantum circuit made up of CNOT, Hadamard and phase gates along with single qubit measurements can be simulated in polynomial time by a classical algorithm. Such circuits are called stabilizer circuits, and any $n$-qubit stabilizer group $P_n$ for $\text{StringIsomorphism}$ form a proper subgroup of the full symmetric group. Indeed the algorithm in [12] is actually an algorithm for STRINGISOMORPHISM, which solves GRAPHISOMORPHISM as a special case.

#### 2.5 Permutations of quantum states and isomorphism

Let $\sigma \in \mathfrak{S}_n$ be a permutation. Then the following is a unitary map acting on $n$-partite states that implements it as a permutation of the subsystems (see e.g. [17])

$$P_\sigma := \sum_{i_1, \ldots, i_n \in \mathcal{P}} |i_{\sigma(1)} \cdots i_{\sigma(n)}\rangle \langle i_1, \ldots, i_n|.$$  

(1)

Note that $P_\sigma$ depends on the dimensions of the subsystems of the $n$-partite states on which it acts. Nevertheless, here we will only consider quantum states where each subsystem is a qubit.

The focus of this work is on a number of variations on the following promise problem, STATEISOMORPHISM.
In what follows, let $Q_{m,n}$ for $m \geq n$ denote the set of all quantum circuits with $m$ input qubits and $n$ output qubits. In particular, $Q_{n,n}$ is the set of all pure state quantum circuits on $n$ qubits. Then, for $m > n$, $Q_{m,n}$ is the set of all mixed state circuits that can be obtained by discarding the last $m - n$ output qubits of the circuits in $Q_{m,m}$.

When we specify a circuit with a subscript label, such as $Q_{\psi} \in Q_{m,n}$, we do so to easily refer to the state of the output qubits when the circuit is applied to the state $|0^m\rangle$. In particular, when $m = n$ this is the pure state $|\psi\rangle \in \mathbb{C}^2^n$, and the mixed state $\psi$ acting on $\mathbb{C}^2^n$ otherwise.

**Problem 14. StateIsomorphism (SI)**

*Input:* Efficient descriptions of quantum circuits $Q_{\psi_0}$ and $Q_{\psi_1}$ in $Q_{n,n}$, a set of permutations $\{\tau_1, \ldots, \tau_k\}$ generating some permutation group $\langle \tau_1 \ldots \tau_k \rangle =: G \leq \mathfrak{S}_n$, and a function $\epsilon : \mathbb{N} \rightarrow [0,1]$ such that $\epsilon(n) \geq 1/\text{poly}(n)$ for all $n$.

**YES:** There exists a permutation $\sigma \in G$ such that $|\langle \psi_1 | P_{\sigma} | \psi_0 \rangle| = 1$.

**NO:** For all permutations $\sigma \in G$, $|\langle \psi_1 | P_{\sigma} | \psi_0 \rangle| \leq \epsilon(n)$.

The next problem is a special case of the above, defined in terms of stabilizer states.

**Problem 15. StabilizerStateIsomorphism (SSI)**

*Input:* Efficient descriptions of quantum circuits $Q_{\rho_0}$ and $Q_{\rho_1}$ in $Q_{n,n}$ such that $|\rho_0\rangle$ and $|\rho_1\rangle$ are stabilizer states, a set of permutations $\{\tau_1, \ldots, \tau_k\}$ generating some permutation group $\langle \tau_1 \ldots \tau_k \rangle =: G \leq \mathfrak{S}_n$, and $\epsilon : \mathbb{N} \rightarrow [0,1]$ such that $\epsilon(n) \geq 1/\text{poly}(n)$ for all $n$.

**YES:** There exists a permutation $\sigma \in G$ such that $|\langle \psi_1 | P_{\sigma} | \rho_0 \rangle| = 1$.

**NO:** For all permutations $\sigma \in G$, $|\langle \psi_1 | P_{\sigma} | \rho_0 \rangle| \leq \epsilon(n)$.

Finally, we consider the state isomorphism problem for mixed states.

**Problem 16. $(\epsilon, 1 - \epsilon)$-MixedStateIsomorphism (MSI)**

*Input:* Efficient descriptions of quantum circuits $Q_{\rho_0}$ and $Q_{\rho_1}$ in $Q_{2n,n}$, a set of permutations $\{\tau_1, \ldots, \tau_k\}$ generating some permutation group $\langle \tau_1 \ldots \tau_k \rangle =: G \leq \mathfrak{S}_n$, and $\epsilon : \mathbb{N} \rightarrow [0,1]$.

**YES:** There exists a permutation $\sigma \in G$ such that $D(P_{\sigma} \rho_0 P_{\sigma}^{\dagger}, \rho_1) \leq \epsilon(n)$.

**NO:** For all permutations $\sigma \in G$, $D(P_{\sigma} \rho_0 P_{\sigma}^{\dagger}, \rho_1) \geq 1 - \epsilon(n)$.

We also consider the above problems where the permutation group specified is equal to the symmetric group $G = \mathfrak{S}_n$. We denote these problems with the prefix $\mathfrak{S}_n$, for example, $\mathfrak{S}_n$-SI. It is clear that SSI $\leq_p$ SI $\leq_p$ MSI. We now show that SI is in QCMA.

**Proposition 17. StateIsomorphism $\in$ QCMA.**

*Proof:* In the case of a YES instance, there exists $\sigma \in G$ such that $|\langle \psi_1 | P_{\sigma} | \psi_0 \rangle| = 1$. The latter equality can be verified by means of a SWAP-test on the states $P_{\sigma} | \psi_0 \rangle$ and $| \psi_1 \rangle$, which by definition will accept with probability equal to 1. Since the states $|\psi_0\rangle$ and $|\psi_1\rangle$ are given as an efficient classical descriptions of quantum circuits that will prepare them, this verification can be performed in quantum polynomial time. Furthermore, there exists an efficient classical description of the permutation $\sigma$ in terms of the generators of the group specified in the input, each of which can be described via their permutation matrices. The unitary $P_{\sigma}$ can be implemented efficiently by Arthur given the description of $\sigma$.

Determining membership/non-membership of some permutation $\sigma \in \mathfrak{S}_n$ in the permutation group $G \leq \mathfrak{S}_n$ specified by the set of generators $\{\tau_1, \ldots, \tau_k\}$ can be verified in classical polynomial time by utilizing standard techniques from computational group theory. In particular, since we are considering permutation groups we can use the Schreier-Sims algorithm to obtain a base and a strong generating set for $G$ in polynomial time from $\{\tau_1, \ldots, \tau_k\}$. These new objects can then be used to efficiently verify membership in $G$.

In the case that the states are not isomorphic, we have by definition that for all permutations $\sigma \in G$, $|\langle \psi_1 | P_{\sigma} | \psi_0 \rangle| \leq \epsilon(n)$, which can again be verified by using the SWAP-test, which will accept the states with probability at most $1/2 + \epsilon(n)$. \qed
It is not clear if MSI is in QCMA, or even in QMA. While the isomorphism $\sigma$ can still be specified efficiently classically, it is not known if there exists an efficient quantum circuit for testing if two mixed states are close in trace distance. In fact, this problem is known as the StateDistinguishability problem, and is QSZK-complete [27].

There exists a polynomial-time many-one reduction from GraphIsomorphism to MSI, indeed it is identical to the reduction from GraphIsomorphism to StringIsomorphism. MSI is in turn trivially reducible to the isomorphism problems for pure and mixed states respectively. These problems are therefore at least as hard as GraphIsomorphism. Interestingly however, there also exists a reduction from GraphIsomorphism to a restricted form of SI where the permutation group $G$ is equal to the full symmetric group $\mathfrak{S}_n$ (as stated earlier, we refer to this problem as $\mathfrak{S}_n$-StateIsomorphism). In order to demonstrate this, we require a family of quantum states referred to as graph states [18]. Let $G = (V, E)$ be an $n$-vertex graph. For each vertex $v \in V$, define the observable $K^{(v)} := \sigma^{(v)}_z \prod_{w \in N(v)} \sigma^{(w)}_z$ where $N(v)$ is the neighborhood of $v$, and $\sigma^{(j)}_i$ denotes the $n$-qubit operator consisting of Pauli $\sigma_i$ applied to the $j$th qubit and identity on the rest. The graph state $|G\rangle$ is defined to be the state stabilized by the set $S_G := \{K^{(v)} : v \in V\}$, that is, $K^{(v)}|G\rangle = |G\rangle$ for all $v \in V$. Since the stabilizers of a graph state $|G\rangle$ are all elements of the $|V|$ qubit Pauli group, graph states are stabilizer states, and the following theorem provides an upper bound on the overlap of non-equal graph states.

**Theorem 18** (Aaronson-Gottesmann [23]. See also [24], Theorem 8). Let $|\psi\rangle, |\phi\rangle$ be non-orthogonal stabiliser states, and let $s$ be the minimum, taken over all sets of generators $\{P_1, \ldots, P_n\}$ for $S(|\psi\rangle)$ and $\{Q_1, \ldots, Q_n\}$ for $S(|\phi\rangle)$, of the number of $i$ values such that $P_i \neq Q_i$. Then $|\langle \psi | \phi \rangle| = 2^{-\frac{s}{2}}$.

We can now describe the reduction.

**Proposition 19.** GraphIsomorphism $\leq_p \mathfrak{S}_n$-StateIsomorphism.

*Proof.* Consider two $n$-vertex graphs $G$ and $H$. If $G = H$ then clearly $|\langle G | H \rangle|^2 = 1$ since $|G\rangle$ and $|H\rangle$ are the same state up to a global phase. Suppose $G \neq H$. Then necessarily $s > 0$, so by Theorem 18 we have that $|\langle G | H \rangle|^2 \leq \frac{1}{2^s}$. Consider a permutation $\sigma \in \mathfrak{S}_n$. Then for each $v \in V$, $K^{(\sigma(v))} = P_\sigma K^{(v)} P_\sigma^T$, so $|\langle \sigma(G) | P_\sigma(G) \rangle|^2 = 1$. Explicitly, if $G \cong H$ then there exists a permutation of the vertices $\sigma$ such that $\sigma(G) = H$ and so $|\langle \sigma(G) | H \rangle|^2 = |\langle G | P_\sigma^T | H \rangle|^2 = 1$. If $G \not\cong H$ then for all $\sigma$, $(G | P_\sigma^T | H \rangle|^2 \leq \frac{1}{2^s}$.

To complete the reduction we must show that for any graph $G = (V, E)$, a description of a quantum circuit that prepares $|G\rangle$ can be produced efficiently classically. This is trivial, an alternate definition of graph states [18] gives us that $|G\rangle = \Pi_{(i,j) \in E} CZ_{ij} |+\rangle \otimes |V\rangle$, where $CZ_{ij}$ is the controlled-$\sigma_z$ operator with qubit $i$ as control and $j$ as output.

Therefore, the StateIsomorphism problem where no restriction is placed on the permutations is at least as hard as GraphIsomorphism. This is in stark contrast to the complexity of the corresponding classical problem, which is trivially in P: two bitstrings are isomorphic under $\mathfrak{S}_n$ if and only if they have the same Hamming weight, which is easily determined.

### 3 Interactive proof systems

In this section we will prove Theorem 1. To do so, we will first demonstrate a quantum interactive proof system for StateNonIsomorphism (SNI) with two messages. We then show that this quantum interactive proof system can be made statistical zero knowledge. In order to prove the former, we will require the following lemma.

**Lemma 20** (Harrow-Lin-Montanaro [34], Lemma 12). Given access to a sequence of unitaries $U_1, \ldots, U_n$, along with their inverses $U_1^\dagger, \ldots, U_n^\dagger$ and controlled implementations $c-U_1, \ldots, c-U_n$, as well as the ability to produce copies of a state $|\psi\rangle$ promised that one of the following cases holds:

1. For some $i$, $U_i |\psi\rangle = |\psi\rangle$;
2. For all $i$, $|\langle \psi | U_i |\psi\rangle| \leq 1 - \delta$.
Then there exists a quantum algorithm which distinguishes between these cases using \(O(\log(n/\delta))\) copies of \(|\psi\rangle\), succeeding with probability at least 2/3.

We can now prove the following.

**Theorem 21.** StateNonIsomorphism is in QIP(2).

**Proof.** We will prove that the following constitutes a two message quantum interactive proof system for SNI.

1. (V) Uniformly at random, select \(\sigma \in G\) and \(j \in \{0,1\}\). Send the state \(|\Psi\rangle^k\) to the prover, where \(k = O(\log(|G|)/(1 - \epsilon(n)))\) and \(|\Psi\rangle = P_\sigma|\psi_j\rangle\).
2. (P) Send \(j' \in \{0,1\}\) to the verifier.
3. (V) Accept if and only if \(j' = j\).

Obtaining a uniformly random element from \(G\) as in step 1 can be achieved efficiently if the verifier is in possession of a base and a strong generating set for \(G\). These can be obtained in polynomial time from any generating set of \(G\) by using Schreier-Sims algorithm [8, 9, 10]. For a permutation \(\pi \in G\), we define the 2n qubit circuit \(U_\pi^{(j)} = \text{SWAP} \cdot (P_{\pi^{-1}} \otimes P_\pi)\), where the SWAP acts so as to swap the two \(n\) qubit states, that is, \(\text{SWAP}|\psi_0\rangle|\psi_1\rangle = |\psi_1\rangle|\psi_0\rangle\). Now consider the sets of quantum circuits \(C_\pi^{(j)} = \{U_\pi^{(j)} : \pi \in G\}\) for \(j \in \{0,1\}\), each of cardinality \(|G|\). Since each circuit in \(C_\pi^{(0)} \cup C_\pi^{(1)}\) is made up two permutations and a SWAP gate, each of their inverses can easily be obtained. Additionally, the controlled versions of these gates can be implemented via standard techniques.

Consider first the YES case. The \(k = O(\log(|G|)/(1 - \epsilon(n)))\) copies of \(|\Psi\rangle\) enables the prover to determine \(j\) with success probability at least 2/3 in the following manner.

1. Uniformly at random, select \(j' \in \{0,1\}\).
2. Prepare \(k\) copies of the state \(|\Psi\rangle|\psi_j\rangle\).
3. Use the HLM algorithm with the state \(|\Psi\rangle|\psi_j\rangle\) and the set of circuits \(C_\pi^{(j')}\) as input. If the algorithm reports case 1 then output \(j'\), otherwise output \(j' + 1\).

Let us check that the HLM algorithm will work for our purposes. In the case that the prover’s guess is correct and \(j' = j\), we have that \(|\Psi\rangle|\psi_j\rangle = (P_\sigma \otimes I)|\psi_j\rangle|\psi_j\rangle\), and so

\[
U_\sigma(P_\sigma \otimes I)|\psi_j\rangle|\psi_j\rangle = \text{SWAP} \cdot (P_{\sigma^{-1}} \otimes P_\sigma) \cdot (P_\sigma \otimes I)|\psi_j\rangle|\psi_j\rangle = \text{SWAP} \cdot (I \otimes P_\sigma)|\psi_j\rangle|\psi_j\rangle = |\Psi\rangle|\psi_j\rangle.
\]

This corresponds to case 1 of Lemma [20] If the prover’s guess is incorrect \(j' \neq j\) then for all \(\pi \in G\)

\[
|\langle \psi_j | U_\pi | \Psi \rangle | |\langle \psi_j | \psi_j \rangle | = |\langle \psi_j | \text{SWAP} \cdot (P_{\pi^{-1}} \otimes P_\pi)(P_\sigma \otimes I)|\psi_j\rangle | |\langle \psi_j | \psi_j \rangle | \\
= |\langle \psi_j | \psi_j \rangle |^2 |\langle \psi_j | \psi_j \rangle | |\langle \psi_j | P_{\pi^{-1}}|\psi_j\rangle | |\langle \psi_j | P_{\pi^{-1}}|\psi_j\rangle | \\
\leq |\langle \psi_j | P_{\pi^{-1}}|\psi_j\rangle | |\langle \psi_j | P_{\pi^{-1}}|\psi_j\rangle | \\
\leq \epsilon(n)^2,
\]

with the last inequality following from the fact that we are in the YES case: for all \(\sigma \in G\), we have that \(|\langle \psi_j | P_\sigma | \psi_1 \rangle | \leq a(n)\). This corresponds to case 2 of Lemma 20 Therefore, the HLM algorithm allows the prover to determine if her guess was correct or not, with success probability at least 2/3.

Consider now the NO case, where we have that for some \(\sigma \in G\), \( |\langle \psi_1 | P_\sigma | \psi_2 \rangle | = 1\). To determine \(j\) correctly, a cheating prover must be able to distinguish the mixed states \(\rho_j = \frac{1}{|G|} \sum_{\pi \in G} (P_\pi|\psi_j\rangle|\psi_j \rangle |\psi_j \rangle^k\)}
correctly for \( j \in \{1, 2\} \), when given \( k \) copies. However,
\[
\|\rho_1 - \rho_2\|_1 = \frac{1}{|G|} \left\| \sum_{\pi \in G} P^\pi_k (|\psi_1\rangle\langle\psi_1|)_{\pi}^{\otimes k} P^\pi |_{\pi}^{\otimes k} - \sum_{\pi \in G} P^\pi_k (|\psi_2\rangle\langle\psi_2|)_{\pi}^{\otimes k} P^\pi |_{\pi}^{\otimes k} \right\|_1
\]
\[
= \frac{1}{|G|} \left\| \sum_{\pi \in G} P^\pi_k P_{\sigma}^k (|\psi_1\rangle\langle\psi_1|)_{\pi}^{\otimes k} P^\pi |_{\pi}^{\otimes k} - \sum_{\pi \in G} P^\pi_k (|\psi_2\rangle\langle\psi_2|)_{\pi}^{\otimes k} P^\pi |_{\pi}^{\otimes k} \right\|_1
\]
\[
= \frac{1}{|G|} \left\| \sum_{\pi \in G} P^\pi_k (|\psi_2\rangle\langle\psi_2|)_{\pi}^{\otimes k} P^\pi |_{\pi}^{\otimes k} - \sum_{\pi \in G} P^\pi_k (|\psi_2\rangle\langle\psi_2|)_{\pi}^{\otimes k} P^\pi |_{\pi}^{\otimes k} \right\|_1
\]
\[
= 0,
\]
so they are indistinguishable. Note that the fact that the prover has been given \( k \) copies does not help, as the overlap is 0. In this case, the probability that the prover can guess \( j \) correctly is therefore equal to 1/2. \( \square \)

We can use a standard amplification argument to modify the above protocol so that it has negligible completeness error, which means that it can be made statistical zero knowledge. We prove this now.

**Theorem 22.** STATENONISOMORPHISM is in QSZK.

**Proof.** We first show that the protocol above can be modified to have exponentially small completeness error. This allows us to show that the protocol is quantum statistical zero knowledge.

First, the verifier sends the prover \( k' = O(n \log(|G|)/(1 - \epsilon(n))) \) copies of the state \(|\Psi\rangle\). The prover can then use HLM \( n \) times to guess \( j \), responding with the value of \( j \) that appears in \( n/2 \) or more of the trials. Let \( X_1 \ldots X_n \in \{\text{‘T’, ‘F’}\} \) be the set of independent random variables corresponding to whether or not the prover guessed correctly on the \( i \)th repetition. By Lemma 20, we have that \( \Pr[X_i = \text{‘T’}] \geq 2/3 \) and so

\[
\Pr[\text{Prover guesses correctly}] = 1 - \Pr \left[ \frac{1}{n} \sum_{i=1}^{n} X_i < 1/2 \right]
\]
\[
= 1 - \Pr \left[ \frac{1}{n} \sum_{i=1}^{n} X_i - 2/3 < -1/6 \right]
\]
\[
\geq 1 - 2^{-\Omega(n)}
\]

via the Chernoff bound (explicitly, for \( p, q \in [0, 1] \), we have that \( \Pr[\sum_{i=1}^{n} (X_i - p)/n < -q] < e^{-q^2/n/2p(1-p)} \)). Clearly, sending \( k' \) copies of \(|\Psi\rangle\) rather than \( k \) gives no advantage to the prover, the trace distance between the mixed states \( \rho_0 \) and \( \rho_1 \) is still 0 in the NO case.

What remains is to show that the protocol is statistical zero knowledge. This is easily obtained, and follows by similar reasoning to the protocol in [27]: the view of the verifier after the first step can be obtained by the simulator by selecting \( \sigma \) and \( j \) then preparing \( k' \) copies of the state \(|\Psi\rangle\). The view of the verifier after the prover’s response can be obtained by tracing out the message qubits and supplying the verifier with the value \( j \). Since completeness error is exponentially small, the trace distance between the simulated view and the actual view is a negligible function. \( \square \)

If we change (relax) the condition for the two states to be non isomorphic (NO instance) to: ‘There exists \( \sigma \in G \) such that \(|\langle \psi_2 | P_\sigma | \psi_1 \rangle| \geq b(n)\)’ then the distance between the two states \( \rho_j = \frac{1}{|G|} \sum_{\pi \in G} (P_\pi |\psi_j\rangle\langle\psi_j| P_\pi |_{\pi}^{\otimes k} \]

\[
\]
for $j \in \{1, 2\}$ is upper bounded by

$$
\|\rho_1 - \rho_2\|_1 = \frac{1}{|G|} \left\| \sum_{\sigma \in G} (P_{\sigma})^{\otimes k} (P_{\sigma} |\psi_1\rangle \langle \psi_1| P_{\sigma}^\dagger - |\psi_2\rangle \langle \psi_2| P_{\sigma}^\dagger) (P_{\sigma})^{\otimes k} \right\|_1
\leq \frac{1}{|G|} \left\| \left(P_{\sigma}\right)^{\otimes k} (P_{\sigma} |\psi_1\rangle \langle \psi_1| P_{\sigma}^\dagger - |\psi_2\rangle \langle \psi_2| P_{\sigma}^\dagger) \right\|_1
= \left\| \left(P_{\sigma}\right)^{\otimes k} \right\|_1
= 2 \sqrt{1 - \langle \psi_1 | P_{\sigma}^\dagger | \psi_2 \rangle^{2k}} \leq 2 \sqrt{1 - \epsilon(n)^{2k}},
$$

where first inequality is just triangular inequality, last inequality follows from the promise and last equality is just rewriting the trace distance for pure states in terms of their scalar product. Now, putting the value of $k = \frac{\log n}{1 - a(n)}$, algebraic manipulations and using the fact that $\log(1 - x) > -2x$ for all $x \in (0, 1/2)$, we get, for any $b(n) \in (1/2, 1)$,

$$
\|\rho_1 - \rho_2\|_1 = 2 \sqrt{1 - b(n) \frac{2 \log n}{1 - a(n)}} = 2 \sqrt{1 - n \frac{2 \log(b(n))}{1 - a(n)}}
\leq 2 \sqrt{1 - n \frac{-4(1 - b(n))}{1 - a(n)}}.
$$

Then the maximal probability of distinguishing between these two states is upper bounded by

$$
p \leq 1/2 + \sqrt{1 - n \frac{-4(1 - b(n))}{1 - a(n)}}.
$$

We have thus proved Theorem [1] Corollary [2] follows easily: if SI was QCMA-complete then all QCMA problems would be reducible to it, and would belong in QSZK. While SI belongs in QCMA, the above protocol requires quantum communication. It is not clear if a similar protocol exists that uses classical communication only. In the next theorem we show that such a protocol exists for STABILIZERSTATE\textsc{NonIsomorphism}, since stabilizer states can be described efficiently classically.

**Theorem 23.** STABILIZERSTATE\textsc{NonIsomorphism} is in QCSZK.

**Proof.** It suffices to show that the state $|\Psi\rangle$ in the protocol above can be communicated to the prover using classical communication only. We know from Theorem [13] that a classical description can be obtained efficiently from $O(n)$ copies of $|\Psi\rangle$. These copies can be prepared efficiently, since they are specified in the problem instance by quantum circuits that prepare them. We now prove that MIXEDSTATE\textsc{Isomorphism} is QSZK-hard (Theorem [3]). We actually prove the following stronger result.

**Theorem 24.** $(\epsilon, 1 - \epsilon)$-$\mathcal{S}_n$-MIXEDSTATE\textsc{Isomorphism} is QSZK-hard for all $\epsilon(n) = 1/\exp(n)$.

We prove this by reduction from the following problem $(\alpha, \beta)$-PRODUCTSTATE, which as shown in [20] is QSZK-hard even when $\alpha = \epsilon, \beta = 1 - \epsilon$ and $\epsilon$ goes exponentially small in $n$.

**Problem 25.** $(\alpha, \beta)$-PRODUCTSTATE

*Input:* Efficient description of a quantum circuit $Q_\rho$ in $Q_{0,n}$.

*YES:* There exists an $n$-partite product state $\sigma_1 \otimes \cdots \otimes \sigma_n$ such that $D(\rho, \sigma_1 \otimes \cdots \otimes \sigma_n) \leq \alpha$

*NO:* For all $n$-partite product states $\sigma_1 \otimes \cdots \otimes \sigma_n$, $D(\rho, \sigma_1 \otimes \cdots \otimes \sigma_n) \geq \beta$.

We make use of the following lemma. For an $n$-partite mixed state $\rho$, let $\rho_i$ denote the state of the $i^{th}$ subsystem, obtained by tracing out the other subsystems.

**Lemma 26** (Gutoski et al. [20], Lemma 7.4). Let $\rho$ be an $n$ qubit state. If there exists a product state $\sigma_1 \otimes \cdots \otimes \sigma_n$ such that $\|\rho - \sigma_1 \otimes \cdots \otimes \sigma_n\|_1 \leq \alpha$, then $\|\rho - \rho_1 \otimes \cdots \otimes \rho_n\|_1 \leq (n + 1)\alpha$.
Figure 2: Constructing the state $\rho' = \rho_1 \otimes \cdots \otimes \rho_n$ from $n$ copies of the input circuit $Q_\rho$. 
Proof of Theorem 24. We now must show that every instance of $(\alpha, \beta)$-PRODUCTSTATE can be converted to an instance of $(\alpha', \beta')-\mathcal{S}_n$-MIXEDSTATEISOMORPHISM. In particular, consider an instance $\rho$ of $(\alpha, \beta)$-PRODUCTSTATE. Our reduction takes this to an instance $(\rho, \rho')$ of $((n+1)\alpha, \beta)-\mathcal{S}_n$-MIXEDSTATEISOMORPHISM, where $\rho' = p_1 \otimes \cdots \otimes p_n$ can be prepared in the following way from $n$ copies of the state $\rho$. Denote these $n$ copies as $\rho^{(1)}, \ldots, \rho^{(n)}$. The $i$th qubit line of $\rho'$ is the $i$th qubit line of $\rho^{(i)}$, all unused qubit lines are discarded (illustrated in Figure 2).

Let $\rho$ be an $n$-partite state. If $\rho$ is product then $D(\rho, p_1 \otimes \cdots \otimes p_n) \leq (n+1)\alpha/2$ and so $(\rho, \rho')$ correspond to a YES instance of $((n+1)\alpha, \beta)-\mathcal{S}_n$-MIXEDSTATEISOMORPHISM. If $\rho$ is a NO instance of $(\alpha, \beta)$-PRODUCTSTATE then $D(\rho, \theta) \geq \beta$ for all product states $\theta$. This means that $D(\rho, P_\sigma p_1 \otimes \cdots \otimes p_n P_\sigma) \geq \beta$ for all $\sigma \in \mathcal{S}_n$ since all such states are product.

In this section we have shown that STATEISOMORPHISM is in QSZK, and so is unlikely to be QCMA-complete unless all problems in QCMA have quantum statistical zero knowledge proof systems. We have also shown that STABILIZERSTATEISOMORPHISM has a quantum statistical zero knowledge proof system that uses classical communication only, and that MIXEDSTATEISOMORPHISM is QSZK-hard.

In the next section, we show that the quantum polynomial hierarchy collapses if STABILIZERSTATEISOMORPHISM is QCMA-complete.

4 A quantum polynomial hierarchy

Yamakami [25] considers a more general framework of quantum complexity theory, where computational problems are specified with quantum states as inputs, rather than just classical bitstrings. We find that using this more general view of computational problems makes it easier to define a very general quantum polynomial-time hierarchy, which can then be “pulled back” to a hierarchy that has more conventional complexity classes (e.g. BQP, QMA) as its lowest levels.

Following [24] we consider classes of quantum promise problems, where the YES and NO sets are made up of quantum states. We use the work’s notion of quantum $\exists$ and $\forall$ complexity class operators in our definitions. These yield classes that are more general than we need, so we use restricted versions where all instances are computational basis states.

Let $|\psi\rangle \in \mathcal{H}_2^\otimes n$ be an $n$-qubit state. Then in analogy to the length of a classical bitstring $|x_1 \ldots x_n| = n$, we define the length of the state $|\psi\rangle$ as $||\psi|| = n$. The set $\{0,1\}^* := \cup_{i=1}^\infty \{0,1\}^i$ is the set of all bitstrings. Analogously, the set $\mathcal{H}_2^\otimes := \cup_{i=1}^\infty \mathcal{H}_2^{\otimes i}$ is the set of all qubit states. A quantum promise problem is therefore a pair of sets $\mathcal{A}_{\text{YES}}, \mathcal{A}_{\text{NO}} \subseteq \mathcal{H}_2^\otimes$ with $\mathcal{A}_{\text{YES}} \cap \mathcal{A}_{\text{NO}} = \emptyset$. Note that to differentiate quantum promise problems from the traditional definition with bitstrings, we use the calligraphic font. We make use of the following complexity class, made up of quantum promise problems.

Definition 27 (BQP$^q$). A quantum promise problem $(\mathcal{A}_{\text{YES}}, \mathcal{A}_{\text{NO}})$ is in the class BQP$^q(a,b)$, for functions $a, b : \mathbb{N} \to [0,1]$ if there exists a polynomial-time generated uniform family of quantum circuits $\{Q_n : n \in \mathbb{N}\}$ such that for all $|\psi\rangle \in \mathcal{H}_2^\otimes$

- if $|\psi\rangle \in \mathcal{A}_{\text{YES}}$ then $\Pr[Q_l \text{ accepts } |\psi\rangle] \geq a(l)$;
- if $|\psi\rangle \in \mathcal{A}_{\text{NO}}$ then $\Pr[Q_l \text{ accepts } |\psi\rangle] \leq b(l)$,

where $l = ||\psi||$.

Classes made up of quantum promise problems will always be denoted with the ‘$q$’ superscript. It is clear that BQP $\subseteq$ BQP$^q$, because any classical promise problem can be converted to a quantum promise problem by considering bitstrings as computational basis states. There is nothing to be gained computationally by imposing that inputs are expressed as computational basis states rather than bitstrings, so we make no distinction between the “bitstring promise problems” and the “computational basis state” promise problems. Indeed let $\mathcal{C}^q$ be a quantum promise problem class. Then we define

$$C := \{A \in \mathcal{C}^q : \text{ all states in } \mathcal{A}_{\text{YES}} \text{ and } \mathcal{A}_{\text{NO}} \text{ are computational basis states.}\}$$

The classes BQP$^q$ and BQP are related in this way. For the remainder of this work we will assume that all complexity classes are made up of quantum promise problems. It will be convenient for us to consider even
conventional complexity classes such as QMA and QCMA to be defined with problem instances specified as computational basis states, rather than as bitstrings. Defining them in this way does not affect the classes in any meaningful way, but it is useful for our purposes. In particular, instead of referring to instances of a promise problem \( x \in A_{YES} \cup A_{NO} \), we will refer to computational basis states in a quantum promise problem \( |x⟩ \in A_{YES} \cup A_{NO} \).

The following operators are well known from classical complexity theory, and are adapted here for quantum promise problem classes.

**Definition 28** (\( \exists_3 / \forall_3 \) operator). Let \( C \) be a complexity class. A promise problem \((A_{YES}, A_{NO})\) is in \( \exists_3 C \) for \( s \in \{q, c\} \) if there exists a promise problem \((B_{YES}, B_{NO}) \subseteq C \) and a polynomially bounded function \( p : \mathbb{N} \to \mathbb{N} \) such that

\[
A_{YES} = \{ |ψ⟩ \in \mathcal{H}_2^s : ∃|y⟩ \in S \ |ψ⟩ \otimes |y⟩ \in B_{YES} \},
\]

and

\[
A_{NO} = \{ |ψ⟩ \in \mathcal{H}_2^s : ∀|y⟩ \in S \ |ψ⟩ \otimes |y⟩ \in B_{NO} \},
\]

where the set \( S \) is equal to \( \{|x⟩ : x \in \{0, 1\}^{p(|ψ⟩|)} \} \) if \( s = c \), and \( \mathcal{H}_2^{p(|ψ⟩|)} \) if \( s = q \). The class \( ∀_3 C \) is defined analogously, but with the quantifiers swapped.

We can now define the quantum polynomial hierarchy.

**Definition 29** (Quantum polynomial time hierarchy). Let \( Σ_0^q = Π_0^q = \text{BQP}^q \). For \( k \geq 1 \), let \( s_1 \ldots s_k \in \{c, q\}^k \). Then

\[
s_1 \ldots s_k \cdot Σ_k^q = \exists_{s_1} s_2 \cdots s_k \cdot Π_k^q
\]

and

\[
s_1 \ldots s_k \cdot Π_k^q = ∀_{s_1} s_2 \cdots s_k \cdot Σ_k^q
\]

This definition leads to complexity classes that include promise problems with quantum inputs. Such classes are not well understood, so we do not use this hierarchy in its full generality. Instead we take each level \( Σ_k^q \) or \( Π_k^q \), and strip out all problems except those defined in terms of computational basis states by using \( Σ_1 \) or \( Π_1 \). Doing so makes familiar classes emerge, indeed it is clear that \( Σ_0 = Π_0 = \text{BQP} \) and \( Π_1 = \text{QCMA} \). This provides a generalisation of the ideas of Gharibian and Kempe [26] into a full hierarchy: our definition of the class \( c_1Σ_2 \) corresponds directly to theirs. For our purposes we require the following technical lemma.

**Lemma 30.** For all \( k \), let \( C_k = s \cdot Σ_k^q \) or \( C_k = s \cdot Π_k^q \) for any \( s \in \{q, c\} \). Then

1. \( \exists_3 C_k = \exists C_k \)
2. \( ∀_3 C_k = ∀ C_k \)
3. \( ∃_c C_k \subseteq ∃_q C_k \)
4. \( ∀_c C_k \subseteq ∀_q C_k \)
5. \( ∀_3 C_k = ∀_q ∃_c C_k = ∃_q C_k \)
6. \( ∀_3 ∃_q C_k = ∀_q ∀_c C_k = ∀_c C_k \)

**Proof.** \([1] \) and \([2] \) are trivial. \([3] \) follows because a BQP verifier circuit can force all certificates to be classical by measuring each qubit in the standard basis before processing. \([4] \) follows because this class is complementary. \([5] \) follows by a similar argument: take \((A_{YES}, A_{NO}) \subseteq ∃_3 C_k \), where the classical certificate is of length \( p_1(|x|) \), and the quantum certificate is of length \( p_2(|x|) \). Clearly \((A_{YES}, A_{NO}) \) is in \( ∃_q C_k \) with certificate length \( p_1(|x|) + p_2(|x|) \), since the first \( p_1(|x|) \) qubits can be measured before processing, so that they are forced to be computational basis states. The other direction, \( ∃_q C_k \subseteq ∃_3 C_k \), follows trivially by setting the classical certificate length to 0. Then \([6] \) follows from \([5] \) because the classes are complementary. \( \square \)
### 4.1 Quantum hierarchy collapse

Our main focus in this paper is on problems in QCMA. Therefore, it is sufficient to adopt the definition of the hierarchy with all certificates classical. Let $\text{QPH}^q := \bigcup_{j=0}^{\infty} \text{cc} \cdot c \cdot \Sigma_j^q$. We consider the restricted hierarchy QPH, \((N.B., \text{without the ‘q’ superscript})\). Since each certificate is classical, when we refer to classes at each level we omit the certificate specification, referring to each level as simply $\Sigma_i$ or $\Pi_i$. Also, note that we are considering the computational basis state restriction of each level of the hierarchy so we omit the ‘q’ superscript. We make use of the following lemmas.

**Lemma 31.** For all $i \geq 1$, $\exists \Sigma_i = \Sigma_i$ and $\forall \Pi_i = \Pi_i$.

Proof. Both follow as corollaries of Lemma 30 parts (1) and (2). \qed

**Lemma 32.** For all $i \geq 1$, if $\Sigma_i \subseteq \Pi_i$ or $\Pi_i \subseteq \Sigma_i$ then $\text{QPH} \subseteq \Sigma_i$.

Proof. We prove first that if the equality $\Sigma_i = \Pi_i$ held for some $i \geq 1$ then for all $j > i$, $\Sigma_j \subseteq \Sigma_i$. We prove this by induction on $j$. Consider the base case $j = i + 1$. By definition, if $A \in \Sigma_{i+1}$ then $A \in \exists \Pi_i = \exists \Sigma_i = \Sigma_i$. Assume for the induction hypothesis that if $\Sigma_i = \Pi_i$ then $\Sigma_j \subseteq \Sigma_i$. Let $k = j - i + 1$. For $k$ odd and $A \in \Sigma_{j+1}$ we have that $A \in \exists \forall \Sigma_k \cdot \exists \Sigma_i = \exists \forall \Sigma_k \cdot \exists \Sigma_i = \Sigma_j$. By the induction hypothesis this is a subclass of $\Sigma_i$. The case for even $k$ follows in the same way. Since for all $i \geq 0$, $\Sigma_i = \text{co-}\Pi_i$, we have that if $\Sigma_i \subseteq \Pi_i$ or $\Pi_i \subseteq \Sigma_i$ then $\Sigma_i = \Pi_i$, and so the hierarchy collapses. \qed

The following two propositions are important for our purposes, and can be proved using similar techniques to those used in the proofs of $\text{AM} = \text{BP} \cdot \text{NP}$ and $\text{AM} \subseteq \Pi_2^P$. We emphasise that the latter is in terms of the quantum polynomial hierarchy, indeed it would be remarkable if a similar result held for in terms of the classical hierarchy. The proofs follow in Sections 4.2 and 4.3.

**Proposition 33.** $\text{QCAM} \subseteq \text{BP} \cdot \text{QCMA}$, and $\text{QAM} \subseteq \text{BP} \cdot \text{QMA}$.

A corollary of this is the following.

**Proposition 34.** $\text{QCAM} \subseteq \text{cc-}\Pi_2$, and $\text{QAM} \subseteq \text{cq-}\Pi_2$.

In what follows, we will refer to the class $\text{QCMAM}$: a generalisation of QCAM which has an extra round of interaction between Arthur and Merlin. Kobayashi et al. [30] show that this class is equal to QCAM.

**Theorem 35** (Kobayashi-Le Gall-Nishimura [30], Theorem 7 (iv)). $\text{QCMAM} = \text{QCAM}$.

The next proposition uses this fact, and allows us to complete the proof of Theorem 3.

**Proposition 36.** If $\text{co-QCMA} \subseteq \text{QCAM}$ then $\text{QPH} \subseteq \text{QCAM} \subseteq \Pi_2$.

Proof. Let $A = (A_{\text{YES}}, A_{\text{NO}}) \in \Sigma_2$. Then by definition there exists a promise problem $B = (B_{\text{YES}}, B_{\text{NO}}) \in \Pi_1 = \text{co-QCMA}$ and a polynomially bounded function $p$ such that for all $|x\rangle \in A_{\text{YES}},$

$$\exists y \in \{0,1\}^{p(|x\rangle)} |x\rangle \otimes |y\rangle \in B_{\text{YES}}, \quad (2)$$

and for all $|x\rangle \in A_{\text{NO}},$

$$\forall y \in \{0,1\}^{p(|x\rangle)} |x\rangle \otimes |y\rangle \in B_{\text{NO}}. \quad (3)$$

If $\text{co-QCMA} \subseteq \text{QCAM}$ then $B \in \text{QCAM}$. The existentially (Eq. 2) and universally (Eq. 3) quantified $y$’s can be thought of as certificate strings, and so $A \in \text{QCMAM}$. By Theorem 35 $\text{QCMAM} = \text{QCAM}$, and so $A \in \Pi_2$. Hence, $\Sigma_2 \subseteq \Pi_2$, and the hierarchy collapses to the second level by Lemma 32. \qed

We now have the tools we need to prove 3.

**Proof of Theorem 3** Suppose $A \in \text{QCMA} \cap \text{co-QCAM}$. If $A$ is QCMA-complete then this implies that $\text{QCMA} \subseteq \text{co-QCAM}$, equivalently $\text{co-QCAM} \subseteq \text{QCAM}$. The hierarchy then collapses to the second level via Proposition 36. \qed

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We may now finish this section by providing evidence that StabilizerStateIsomorphism is not QCMA-complete, encapsulated in Corollary \[4\]. We do this by proving the following.

**Proposition 37.** StabilizerStateNonIsomorphism is in QCAM.

**Proof.** For a stabilizer state $|\psi\rangle$, denote by $s_\psi^{(1)}, \ldots, s_\psi^{(n)} \in \{\pm I, \pm X, \pm Y, \pm Z\}^n$ the classical strings that describe the stabilizer generators of $|\psi\rangle$ that we can obtain efficiently using the algorithm of Theorem \[13\]. We denote by $s_\psi$ the length $2n$ string that is obtained by concatenating these stabilizer strings, that is $s_\psi = s_\psi^{(1)} \ldots s_\psi^{(n)}$. Then for any permutation $\sigma \in S_n$, we take $\sigma(s_\psi) = s_\psi^{(\sigma(1))}, \ldots, s_\psi^{(\sigma(n))}$. For a permutation group $G \leq S_n$, consider the set

$$S_G := \bigcup_{j \in \{0,1\}, \sigma \in G} \left\{ (\sigma(s_\psi), \pi) : \pi \in G \land \sigma(s_\psi) = \sigma(s_\psi) \right\}.$$ 

If there exists $\sigma$ such that $|\langle \psi| P_\sigma |\psi\rangle| = 1$ then $\sigma(s_\psi) = s_\psi$, and so in this case $|S_G| = |G|$. If for all $\sigma \in G$ we have that $|\langle \psi| P_\sigma |\psi\rangle| \leq 1 - \epsilon(n)$ then likewise for all $\sigma \in G$, $\sigma(s_\psi) \neq s_\psi$, and therefore $|S_G| = 2|G|$. If we can show that membership in $S_G$ can be efficiently verified by Arthur then we can apply the Goldwasser-Sipser set lower bound protocol \[29\] to determine isomorphism of the states. To convince Arthur with high probability that $(\sigma(s_\psi), \pi) \in S_G$, Merlin sends the permutation $\sigma$ and the index $j \in \{0,1\}$. Arthur can then obtain the string $s_\psi$ with probability greater than $1 - 1/\exp(n)$ using Montanaro’s algorithm of Theorem \[17\] applied to $U_\psi |0\rangle$. He can then verify in polynomial time that the string he received is equal to $\sigma(s_\psi)$, that $\pi$ is an automorphism of $\sigma(s_\psi)$, and that the permutation $\sigma$ is in the group $G$. \qed

We have provided evidence that SSI can be thought of as an intermediate problem for QCMA. In particular, we have shown that if it were in BQP, then GRAPHISOMORPHISM would also be in BQP, and furthermore, that its QCMA-completeness would collapse the quantum polynomial hierarchy. Such evidence is unfortunately currently out of reach for STATEISOMORPHISM, because we have been unable to show that STATENONISOMORPHISM is in QCAM. Perhaps Arthur and Merlin must always use quantum communication if Arthur is to be convinced that two states are NOT isomorphic. This would be interesting, because he can be convinced that they are isomorphic using classical communication only (STATEISOMORPHISM $\in$ QCMA).

### 4.2 Proof of Proposition 33

We begin by giving a definition of the BP complexity class operator. Note that we are still working in terms of the quantum promise problems defined earlier, which is clear from the use of the calligraphic font $\mathcal{A}$. In the following we take $x \sim X$ to mean that $x$ is an element drawn uniformly at random from a finite set $X$.

**Definition 38 (BP operator).** Let $C$ be a complexity class. A promise problem $(\mathcal{A}_{\text{YES}}, \mathcal{A}_{\text{NO}})$ is in $\text{BP}(a,b) \cdot C$ for functions $a,b : \mathbb{N} \to [0,1]$ if there exists $(\mathcal{B}_{\text{YES}}, \mathcal{B}_{\text{NO}}) \in C$ and a polynomially bounded function $p : \mathbb{N} \to \mathbb{N}$ such that

- For all $|\psi\rangle \in \mathcal{A}_{\text{YES}},$

  $$\Pr_{y \sim \{0,1\}^p(x)}[|\psi\rangle \otimes |y\rangle \in \mathcal{B}_{\text{YES}} \geq a(|\psi\rangle)];$$

- For all $|\psi\rangle \in \mathcal{A}_{\text{NO}},$

  $$\Pr_{y \sim \{0,1\}^p(x)}[|\psi\rangle \otimes |y\rangle \notin \mathcal{B}_{\text{NO}} \leq b(|\psi\rangle)].$$

It is clear that the probabilities $a, b$ can be amplified in the usual way by repeating the protocol a sufficient number of times and taking a majority vote. Let $(\{V_{x,y}\}, m, s)$ be a QAM verification procedure. In what follows we make use of the functions

$$\mu(m, V_{x,y}) := \max_{|\psi\rangle \in H_2^m(x)} \left( \Pr[V_{x,y} \text{ accepts } |\psi\rangle] \right)$$

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and

\[ \nu(m, V_{x,y}) := \min_{|\psi\rangle \in H_2^{|m|}} (\Pr[V_{x,y} \text{ rejects } |\psi\rangle]). \]

The following results of Marriott and Watrous [33] are useful for our purposes.

**Theorem 39** (Marriott-Watrous [33], Theorem 4.2). Let \( a, b : \mathbb{N} \rightarrow [0, 1] \) and polynomially bounded \( q : \mathbb{N} \rightarrow [0, 1] \) satisfy

\[ a(n) - b(n) \geq \frac{1}{q(n)} \]

for all \( n \in \mathbb{N} \). Then \( \text{QAM}(a, b) \subseteq \text{QAM}(1 - 2^{-r}, 2^{-r}), \) for all polynomially bounded \( r : \mathbb{N} \rightarrow [0, 1] \).

**Proposition 40** (Marriott-Watrous [33], Proposition 4.3). Let

\[ \left\{ V_{x,y} : x \in \{0, 1\}^*, y \in \{0, 1\}^{s(|x|)} \right\}, m : \mathbb{N} \rightarrow \mathbb{N}, s : \mathbb{N} \rightarrow \mathbb{N} \]

be a QAM verification procedure for a promise problem \( A \) with completeness and soundness errors bounded by \( 1/9 \). Then for any \( x \in \{0, 1\}^* \) and for \( y \in \{0, 1\}^{s(|x|)} \) chosen uniformly at random,

- if \( |x\rangle \in \mathcal{A}_{\text{YES}} \) then \( \Pr[m(m, V_{x,y}) \geq 2/3] \geq 2/3; \)
- if \( |x\rangle \in \mathcal{A}_{\text{NO}} \) then \( \Pr[m(m, V_{x,y}) \leq 1/3] \geq 2/3. \)

We can use these tools to prove Proposition 33. We prove it for QAM, the result follows for QCAM by similar reasoning.

**Proof of Proposition 33**. Suppose \( \mathcal{A} = (\mathcal{A}_{\text{YES}}, \mathcal{A}_{\text{NO}}) \in \text{QAM}(a, b) \). By Theorem 39 there exists a QAM verification procedure \( (\{V_{x,y}\}, m, s) \) with completeness and soundness errors bounded by \( 1/9 \). Thus by Proposition 40 we know that for all \( x \in \{0, 1\}^* \), if \( |x\rangle \in \mathcal{A}_{\text{YES}} \) then

\[ \Pr_{y \sim \{0, 1\}^{s(|x|)}} [m(m, V_{x,y}) \geq 2/3] \geq 2/3, \]

which means that

\[ \frac{1}{2^{s(|x|)}} \left| \left\{ y \in \{0, 1\}^{s(|x|)} : \exists |z\rangle \in H_2^{2m(|x|)} \Pr[V_{x,y} \text{ accepts } |z\rangle] \geq 2/3 \right\} \right| \geq 2/3. \]

By similar reasoning, if \( |x\rangle \in \mathcal{A}_{\text{NO}} \) then

\[ \frac{1}{2^{s(|x|)}} \left| \left\{ y \in \{0, 1\}^{s(|x|)} : \forall |z\rangle \in H_2^{2m(|x|)} \Pr[V_{x,y} \text{ accepts } |z\rangle] \leq 1/3 \right\} \right| \geq 2/3. \]

These conditions are precisely the conditions for a promise problem to belong in QMA. This means we can fix some promise problem \( (\mathcal{B}_{\text{YES}}, \mathcal{B}_{\text{NO}}) \in \text{QMA}(2/3, 1/3) \) and re-express these statements in the following form:

- if \( |x\rangle \in \mathcal{A}_{\text{YES}} \) then
  \[ \Pr_{y \sim \{0, 1\}^{s(|x|)}} [|x\rangle \otimes |y\rangle \in \mathcal{B}_{\text{YES}}] \geq 2/3 \]

- if \( |x\rangle \in \mathcal{A}_{\text{NO}} \) then
  \[ \Pr_{y \sim \{0, 1\}^{s(|x|)}} [|x\rangle \otimes |y\rangle \in \mathcal{B}_{\text{NO}}] \geq 2/3, \]

and so \( \mathcal{A} \in \text{BP}(2/3, 1/3) \cdot \text{QMA}(2/3, 1/3). \)

\[ \square \]
4.3 Proof of Proposition 34

The following well known lemmas allow us to put BP· QMA (resp. BP· QCMA), and thus QAM (resp. QCAM), in the second level of the quantum polynomial-time hierarchy. We follow [35] but recast them in a more helpful form for our purposes. For a set of bitstrings \( S \subseteq \{0,1\}^m \) and \( x \in \{0,1\}^m \), we take \( S \oplus x = \{s \oplus x : s \in S\} \).

**Lemma 41.** Let \( S \subseteq \{0,1\}^m \) for \( m \geq 1 \) such that
\[
|S| \geq (1 - 2^{-k}) \cdot 2^m,
\]
for \( 2^k \geq m \). Then there exists \( t_1, \ldots, t_m \in \{0,1\}^m \) such that
\[
\bigcup_{i=1}^m S \oplus t_i = \{0,1\}^m.
\]

**Proof.** We prove this via the probabilistic method. Consider uniformly random \( t_1, \ldots, t_m \in \{0,1\}^m \). Then
\[
\Pr_{r \sim \{0,1\}^m} \left[ r \notin \bigcup_{i=1}^m S \oplus t_i \right] = \prod_{i=1}^m \Pr_{r \sim \{0,1\}^m} [r \notin S \oplus t_i] \leq 2^{-km}.
\]

Consider the probability that there exists some \( v \in \{0,1\}^m \) such that \( v \notin \bigcup_{i=1}^m S \oplus t_i \),
\[
\Pr[\exists v \in \{0,1\}^m. v \notin \bigcup_{i=1}^m S \oplus t_i] \leq \sum_{i=1}^{2^m} 2^{-k^m} = \frac{2^m}{2^{km}} < 1.
\]

Hence,
\[
\Pr \left[ \bigcup_{i=1}^m S \oplus t_i = \{0,1\}^m \right] > 0,
\]
and so there must exist \( t_1, \ldots, t_m \) as required. \( \square \)

This yields the following corollary.

**Corollary 42.** Let \( S \subseteq \{0,1\}^m \) for \( m \geq 1 \) such that
\[
|S| \geq (1 - 2^{-k}) \cdot 2^m,
\]
for \( 2^k \geq m \). Then there exists \( t_1, \ldots, t_m \) such that for all \( v \in \{0,1\}^m \), there exists \( i \in [m] \) such that \( t_i \oplus v \in S \).

We also require the following lemma, which comes from the opposite direction.

**Lemma 43.** Let \( S \subseteq \{0,1\}^m \) for \( m \geq 1 \) such that
\[
|S| \geq (1 - 2^{-k}) \cdot 2^m,
\]
for \( 2^k \geq m \). Then for all \( t_1, \ldots, t_m \in \{0,1\}^m \), there exists \( v \in \{0,1\}^m \) such that \( \bigwedge_{i \in [m]} (u_i \oplus v \in S) \).

**Proof.** Assume that there exists \( t_1 \ldots t_m \) such that for all \( v \in \{0,1\}^m \) there exists \( i \in [m] \) with \( t_i \oplus v \notin S \). This implies that there exists \( i \in \{1, \ldots, m\} \) such that, for at least \( 2^m/m \) elements \( v \in \{0,1\}^m \), we have that \( t_i \oplus v \notin S \). Then
\[
|S| < 2^m - 2^m/m = 2^m(1 - 1/m) \leq (1 - 2^{-k}) \cdot 2^m,
\]
contradicting our assumption about the cardinality of \( S \). \( \square \)
We can now prove the Proposition 33. We prove it for BP · QMA; the result for BP · QCMA follows in the same way.

Proof of Proposition 33. Let \((A_{YES}, A_{NO}) \in BP \cdot QMA\). Then by definition there exists \((B_{YES}, B_{NO}) \in QMA\) and polynomially bounded \(p, r : \mathbb{N} \rightarrow \mathbb{N}\) such that if \(|x\rangle \in A_{YES}\),

\[
\Pr_{y \sim \{0, 1\}^{p(|x|)}}\left([x] \otimes |y\rangle \in B_{YES}\right) \geq 1 - 2^{-r(|x|)}.
\]

Set \(S_x = \{ y \in \{0, 1\}^{p(|x|)} : |x \rangle \otimes |y\rangle \in B_{YES}\}\). Then \(|x\rangle \in B_{YES}\) implies that \(|S_x| \geq (1 - 2^{-r(|x|)}) \cdot 2^{p(|x|)}\).

By amplification of BP, we can choose \(r\) to be whatever we want, so we choose it such that \(2^{-r(|x|)} \geq p(|x|)\).

Then by Lemma 43

\[
x \in A_{YES} \implies \forall t_1 \ldots t_{p(|x|)} \in \{0, 1\}^{p(|x|)} \exists v \in \{0, 1\}^{p(|x|)} \exists i \in \{1 \ldots p(|x|)\} |x \rangle \otimes |t_i \oplus v\rangle \in B_{YES}.
\]

By definition of QMA, for any bitstring \(y\) such that \(|x \rangle \otimes |y\rangle \in B_{YES}\),

\[
\exists |\psi\rangle \in H_2^{|x|} \Pr[Q_{x, y} \text{ accepts } |\psi\rangle] \geq 2/3.
\]

From Lemma 30 we know we can collapse the classical \(\exists\) quantifiers into the quantum one, obtaining \(\forall_c \exists_q\).

This means that Eq. (4) is of the form required by a promise problem in cq-Π₂.

Set \(S'_x = \{ y \in \{0, 1\}^{p(|x|)} : |x \rangle \otimes |y\rangle \in B_{NO}\}\). For \(x \in A_{NO}\), \(|S'_x| \geq 1 - 2^{-r(|x|)} 2^{p(|x|)}\), for any \(r\) via amplification. Then by Corollary 12 we know that this can be written as a \(\exists_c \forall_q\) statement about belonging to \(B_{NO}\). By definition the membership condition for \(B_{NO}\) is a \(\forall_q\) statement. Again, the classical and quantum \(\forall\) statements can be collapsed so we obtain a \(\exists_c \forall_q\) statement for the NO instances, meaning that \((A_{YES}, A_{NO}) \in cq-\Pi_2\).

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