PROOF OF A DOUBLE INEQUALITY IN TRIANGLES

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Abstract. A double geometric inequality involving the side lengths, medians, angle bisectors and exradius of a triangle is proved by applying the “$R-r-s$” method in the theory of triangle inequalities. Several corollaries are obtained by using the main result and the other known inequalities.

1. Introduction

Let $P$ be a point inside triangle $ABC$, let $PA = R_1$, $PB = R_2$, $PC = R_3$ and let $r_1$, $r_2$, $r_3$ denote the distances from $P$ to the sides $BC$, $CA$, $AB$, respectively. Then the following beautiful linear inequality holds:

$$R_1 + R_2 + R_3 \geq 2(r_1 + r_2 + r_3). \quad (1)$$

This is the famous Erdőss-Mordell inequality. Some recent results on this subject can be found in [6, 12, 13, 14, 15, 16, 20]. In fact, the inequality (1) can be extended to

$$R_a + R_b + R_c \geq R_1 + R_2 + R_3 \geq 2(r_1 + r_2 + r_3), \quad (2)$$

where $R_a$, $R_b$, $R_c$ are the circumradius of the triangles $BPC$, $CPA$, $APB$, respectively. It is interesting that the first inequality in (2) is actually equivalent to the Erdőss-Mordell inequality (see the fourth chapter of the author’s monograph [17]).

In [9], the author and Chu established the following inequality for the sum of $R_1$, $R_2$, and $R_3$:

$$R_1 + R_2 + R_3 \geq \frac{1}{2}(m_a + m_b + m_c + 3r), \quad (3)$$

where $m_a$, $m_b$, $m_c$ are the medians of the triangle $ABC$ and $r$ is the inradius. We also further showed, via inequality (3), that

$$R_1 + R_2 + R_3 \geq \frac{1}{3}(m_a + m_b + m_c + h_a + h_b + h_c), \quad (4)$$

where $h_a$, $h_b$, $h_c$ are the altitudes of the triangle $ABC$.

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Inspired and motivated by the first inequality in (2) and inequality (4), the author conjectures that the following inequality holds:

\[ R_a + R_b + R_c \geq \frac{1}{3}(m_a + m_b + m_c + w_a + w_b + w_c), \]  

(5)

where \( w_a, w_b, w_c \) are the lengths of angle bisectors of the triangle \( ABC \). On the other hand, noticing that the following known result (see [17, p. 252] and [22]):

\[ R_a + R_b + R_c \geq \sqrt{bc + ca + ab}, \]  

(6)

where \( a, b, c \) are the lengths of the side of triangle \( ABC \), the author also further conjectures that the following inequality holds:

\[ m_a + m_b + m_c + w_a + w_b + w_c \leq 3\sqrt{bc + ca + ab}. \]  

(7)

In addition, for any triangle \( ABC \) we have known that

\[ m_a + m_b + m_c \leq r_a + r_b + r_c, \]  

(8)

where \( r_a, r_b, r_c \) are the radii of excircles of the triangle \( ABC \) (see [1, inequality 8.20]). With the help of the computer, we find that if replacing \( m_a + m_b + m_c \) by \( r_a + r_b + r_c \) in (7), then we have the reverse inequality, i.e.,

\[ r_a + r_b + r_c + w_a + w_b + w_c \geq 3\sqrt{bc + ca + ab}. \]  

(9)

Of course, the above inequality can only be regarded as a conjecture before it is proved.

The aim of this article is to prove inequality (7) and inequality (9), i.e., the following double inequality which was given in [17, p. 253] as a conjecture.

**Theorem.** For any triangle \( ABC \), the following inequality holds:

\[ m_a + m_b + m_c + w_a + w_b + w_c \leq 3\sqrt{bc + ca + ab} \leq w_a + w_b + w_c + r_a + r_b + r_c. \]  

(10)

Both equalities in (10) hold if and only if triangle \( ABC \) is equilateral.

Clearly, the double inequality (10) can be written as

\[ m_a + m_b + m_c \leq 3\sqrt{bc + ca + ab} - (w_a + w_b + w_c) \leq r_a + r_b + r_c, \]  

(11)

which gives a refinement of the known inequality (8).

## 2. Lemmas

We shall apply the “\( R - r - s \)” method to prove the double inequality (10). This method has been proved to be effective for a number of symmetric triangle inequalities (cf. [2], [4], [5], [7], [10], [11], [23], [24]).

In what follows, we shall continue to use the previous symbols. Also, denote the semi-perimeter, the circumradius and the inradius of the triangle \( ABC \) by \( s, R, r \), respectively. For the sake of simplicity, we shall occasionally use \( \Sigma \) and \( \Pi \) to express cycle sums and products respectively.
LEMMA 1. For any triangle ABC, the following inequality holds:

\[ \frac{(m_a + m_b + m_c)^2}{a^2 + b^2 + c^2} \leq 2 + \frac{r^2}{R^2}. \]  \hspace{1cm} (12)

Equality holds if and only if triangle ABC is equilateral.

The author has proved inequality (12) in [10]. Later, the author also gave two simpler proofs in [11].

Next, we apply inequality (12) to prove the following inequality, which was first given in [4] without proof.

LEMMA 2. For any triangle ABC, the following inequality holds:

\[ m_a + m_b + m_c \leq \frac{2s^2}{\sqrt{bc + ca + ab}}. \]  \hspace{1cm} (13)

Equality holds if and only if the triangle ABC is equilateral.

Proof. By inequality (12), to prove inequality (13) we need to prove

\[ (a^2 + b^2 + c^2) \left(2 + \frac{r^2}{R^2}\right) \leq \frac{4s^4}{bc + ca + ab}, \]

i.e.,

\[ 4s^4R^2 - (2R^2 + r^2)(bc + ca + ab)(a^2 + b^2 + c^2) \geq 0. \]

Then, using the following known identities:

\[ bc + ca + ab = s^2 + 4Rr + r^2, \]  \hspace{1cm} (14)

\[ a^2 + b^2 + c^2 = 2(s^2 - 4Rr - r^2), \]  \hspace{1cm} (15)

the proof becomes

\[ -2s^4r^2 + 2r^2(2R^2 + r^2)(4R + r)^2 \geq 0. \]

Since we have the following known inequality (see [1, inequality 5.5]):

\[ (4R + r)^2 \geq 3s^2, \]  \hspace{1cm} (16)

thus we only need to prove that

\[ -s^2 + 3(2R^2 + r^2) \geq 0, \]

which can be written as

\[ 4R^2 + 4Rr + 3r^2 - s^2 + 2R(R - 2r) \geq 0. \]

By Euler’s inequality \( R \geq 2r \) and Gerretsen’s inequality (see [1, inequality 5.8])

\[ g_2 \equiv 4R^2 + 4Rr + 3r^2 - s^2 \geq 0 \]  \hspace{1cm} (17)

with equality iff \( ABC \) is equilateral, one sees that the claimed inequality is valid. Thus, inequality (13) is proved. It is easy to know that the equality in (13) holds if and only if the triangle is equilateral. This completes the proof of Lemma 2. \( \blacksquare \)
**Lemma 3.** For any triangle ABC, the following inequality holds:

\[
\frac{1}{w_a} + \frac{1}{w_b} + \frac{1}{w_c} \leq \frac{1}{2R} + \frac{3}{4r}.
\]

(18)

Equality holds if and only if the triangle ABC is equilateral.

Inequality (18) was first proposed by the author in a Chinese paper [8]. A simple proof was given by Chu and the author in [4].

**Lemma 4.** For any triangle ABC, the following inequality holds:

\[
\frac{(w_a + w_b + w_c)^2}{bc + ca + ab} \leq \frac{s^4 + 2r(10R + 9r)s^2 + (4R + r)r^3}{(s^2 + 2Rr + r^2)^2}.
\]

(19)

Equality holds if and only if the triangle ABC is equilateral.

**Proof.** By inequality (18), we have

\[(w_a + w_b + w_c)^2 \leq w_a^2 + w_b^2 + w_c^2 + w_aw_bw_c \left(\frac{1}{R} + \frac{3}{2r}\right).\]

Then using the following known identities:

\[
w_aw_bw_c = \frac{16R^2 s^2}{s^2 + 2Rr + r^2},
\]

(20)

\[
w_a^2 + w_b^2 + w_c^2 = \frac{s^6 + 3r^2s^4 + (32R^2 + 40Rr + 3r^2)r^2s^2 + (4R + r)^2r^4}{s^4 + 2R(2R + r)s^2 + r^2(2R + r)^2},
\]

(21)

we easily obtain

\[(w_a + w_b + w_c)^2 \leq \left(\frac{s^2 + 4Rr + r^2}{(s^2 + 2Rr + r^2)^2}\right) \left[\frac{s^4 + 2r(10R + 9r)s^2 + (4R + r)r^3}{s^2 + 2R(2R + r)s^2 + r^2(2R + r)^2}\right].\]

Hence, inequality (19) follows immediately by using the previous identity (14). It is easy to know that the equality conditions of (19) is as mentioned in Lemma 4. □

**Lemma 5.** In the triangle ABC, if B − C, C − A, A − B ≠ 0, then

\[
\sum \frac{1}{\cos(B - C)} = \frac{2R \left[(R + 6r)s^2 - (2R + r)(2R^2 + 7Rr + 2r^2)\right]}{s^4 - (6R^2 + 8Rr - 2r^2)s^2 + (2R^2 + 4Rr + r^2)(2R + r)^2}.
\]

(22)

**Proof.** We first deduce the following identity (for any triangle ABC):

\[
\prod \cos(B - C) = \frac{s^4 - (6R^2 + 8Rr - 2r^2)s^2 + (2R^2 + 4Rr + r^2)(2R + r)^2}{8R^4}.
\]

(23)
Since \( \cos(B - C) = \cos B \cos C + \sin B \sin C \), it is easy to obtain
\[
\prod \cos(B - C) = \prod \sin^2 A + \prod \cos^2 A + \prod \cos A \sum \sin B \sin C \cos A
+ \prod \sin A \sum \sin A \cos B \cos C. \tag{24}
\]

Then using the following identities:
\[
\sum \sin B \sin C \cos A = \frac{1}{2} \sum \sin^2 A, \tag{25}
\]
\[
\sum \sin A \cos B \cos C = \prod \sin A, \tag{26}
\]
we get
\[
\prod \cos(B - C) = 2 \prod \sin^2 A + \prod \cos^2 A + \frac{1}{2} \prod \cos A \sum \sin^2 A. \tag{27}
\]

Further, using the following known identities (cf. [19, p. 55]):
\[
\prod \sin A = \frac{rs}{2R^2}, \tag{28}
\]
\[
\prod \cos A = \frac{s^2 - (2R + r)^2}{4R^2}, \tag{29}
\]
\[
\sum \sin^2 A = \frac{1}{2R^2} (s^2 - 4Rr - r^2), \tag{30}
\]
we obtain identity (23).

Now, we prove the following identity:
\[
\sum \cos(C - A) \cos(A - B) = \frac{(R + 6r)s^2 - (2R + r)(2R^2 + 7Rr + 2r^2)}{4R^3}. \tag{31}
\]

Firstly, it is easy to get
\[
\sum \cos(C - A) \cos(A - B)
= \prod \sin A \sum \sin A + \prod \cos A \sum \cos A + \sum \sin B \sin C \cos A (\cos B + \cos C). \tag{32}
\]

Using identity (25), we have
\[
\sum \sin B \sin C \cos A (\cos B + \cos C)
= \sum \sin B \sin C \cos A \sum \cos A - \sum \sin B \sin C \cos^2 A
= \frac{1}{2} \sum \sin^2 A \sum \cos A - \sum \sin B \sin C + \prod \sin A \sum \sin A,
\]
and then
\[
\sum \cos(C - A) \cos(A - B)
= 2 \prod \sin A \sum \sin A + \prod \cos A \sum \cos A - \sum \sin B \sin C
+ \frac{1}{2} \sum \cos A \sum \sin^2 A. \tag{33}
\]
Thus, by using identities (28), (30) and the following identities (cf. [19, p. 55]):

\[
\sum \sin A = \frac{s}{R}, \tag{34}
\]

\[
\sum \cos A = 1 + \frac{r}{R}, \tag{35}
\]

\[
\sum \sin B \sin C = \frac{s^2 + 4Rr + r^2}{4R^2}, \tag{36}
\]

we obtain identity (31).

When \(B - C, C - A, A - B \neq 0\), identity (22) follows from (23) and (31) immediately. Lemma 5 is proved.

\[\square\]

3. Proof of the Theorem

3.1. Proof of the first inequality of the double inequality (10)

In this section, we prove the first inequality in (10), i.e., inequality (7).

Proof. According to Lemma 2, to prove inequality (7) we only need to prove

\[
\frac{2s^2}{\sqrt{bc + ca + ab}} + w_a + w_b + w_c \leq 3\sqrt{bc + ca + ab},
\]

that is

\[(w_a + w_b + w_c)\sqrt{bc + ca + ab} \leq 3(bc + ca + ab) - 2s^2.\]

Since \(bc + ca + ab > s^2\) holds for any triangle \(ABC\), we thus only consider to prove

\[(bc + ca + ab)(w_a + w_b + w_c)^2 \leq (3bc + 3ca + 3ab - 2s^2)^2.\]

By Lemma 4, it is sufficient to prove

\[
Q_0 \equiv (s^2 + 2Rr + r^2)^2 (3bc + 3ca + 3ab - 2s^2)^2
- (bc + ca + ab)^2 [s^4 + 2r(10R + 9r)s^2 + (4R + r)r^3] \geq 0. \tag{37}
\]

Substituting the previous identity (14) into \(Q_0\), we get

\[
Q_0 = 4r^2Q_1, \tag{38}
\]

where

\[
Q_1 = -3s^6 + (17R^2 - 12Rr - 4r^2)s^4 + r(4R + r)(22R^2 + 8Rr + r^2)s^2
+ (9R^2 + 8Rr + 2r^2)(4R + r)^2r^2.
\]

Hence, we have to prove \(Q_1 \geq 0\). Through analysis, we find the following identity (which is easily checked by expanding):

\[
Q_1 = Q_2 + Q_3 + rQ_4, \tag{39}
\]
where
\[ Q_2 = (5R^2 + 2r^2)s^2(s^2 - 16Rr + 5r^2), \]
\[ Q_3 = 3(s^2 + 24Rr)[-s^4 + (4R^2 + 20Rr - 2r^2)s^2 - r(4R + r)^3], \]
\[ Q_4 = (72R^3 - 1267R^2r + 224r^2 - 6r^3)s^2 + (297R^2 + 80Rr + 2r^2)(4R + r)^2r. \]

For any triangle \(ABC\), we have Gerretsen’s inequality (see [1, inequality 5.8])
\[ g_1 \equiv s^2 - 16Rr + 5r^2 \geq 0 \tag{40} \]
(with equality iff the triangle \(ABC\) is equilateral) and the fundamental Sondat’s inequality (see [1, inequality 13.8] and [19, p. 2]):
\[ t_0 \equiv -s^4 + (4R^2 + 20Rr - 2r^2)s^2 - r(4R + r)^3 \geq 0, \tag{41} \]
with equality if and only if the triangle \(ABC\) is isosceles. Hence, we have \(Q_2 \geq 0\) and \(Q_3 \geq 0\). It remains to prove that \(Q_4 \geq 0\) by (39). Obviously, if
\[ 72R^3 - 1267R^2r + 224r^2 - 6r^3 \geq 0, \]
then \(Q_4 > 0\). If the above inequality holds reversely, by the previous Gerretsen’s inequality (17) we need to prove that
\[ (72R^3 - 1267R^2r + 224r^2 - 6r^3)(4R^2 + 4Rr + 3r^2) + (297R^2 + 80Rr + 2r^2)(4R + r)^2r \geq 0, \]
i.e.,
\[ 4(R - 2r)(72R^4 + 137R^3r + 199R^2r^2 - 92Rr^3 + 2r^4) \geq 0, \]
which is true by Euler’s inequality \(R \geq 2r\). Hence, we conclude that \(Q_4 \geq 0\) holds for all triangles \(ABC\). This completes the proof of the inequality (7). Also, it is easily known that the equality in (7) holds iff the triangle \(ABC\) is equilateral. \(\square\)

### 3.2. Proof of the second inequality of the double inequality (10)

Next, we prove the second inequality in (10), i.e., inequality (9).

**Proof.** Firstly, we transform inequality (9) into an equivalent trigonometric inequality in the acute (non-obtuse) triangle \(ABC\).

Let \(I\) be the incenter of the triangle \(ABC\). Suppose that the line \(AI\) intersect \(BC\) at \(X\), then \(\angle AXC = B + \frac{1}{2}A = \frac{\pi + B - C}{2}\). Using \(w_a = AI + IX\), we have
\[ w_a = \frac{r}{\sin \frac{A}{2}} + \frac{r}{\cos \frac{B - C}{2}}, \tag{42} \]
so that
\[ \sum w_a = r \sum \frac{1}{\sin \frac{A}{2}} + r \sum \frac{1}{\cos \frac{B - C}{2}}. \tag{43} \]
Therefore, by the following two known identities:
\[ r_a + r_b + r_c = 4R + r, \]  
(44)  
\[ r = 4R \prod \sin \frac{A}{2}, \]  
(45)
we see that inequality (9) is equivalent to
\[ 4R \prod \sin \frac{A}{2} \left( \sum \frac{1}{\sin \frac{A}{2}} + \sum \frac{1}{\cos \frac{B - C}{2}} \right) + 4R \left( 1 + \prod \sin \frac{A}{2} \right) \geq 3 \sqrt{\sum bc}. \]

Since \( a = 2R \sin A \) etc., the above inequality is equivalent to the following trigonometric inequality:
\[ 2 \prod \sin \frac{A}{2} \left( \sum \frac{1}{\sin \frac{A}{2}} + \sum \frac{1}{\cos \frac{B - C}{2}} \right) + 2 \left( 1 + \prod \sin \frac{A}{2} \right) \geq 3 \sqrt{\sum \sin B \sin C}. \]  
(46)

Also, it is easily known that inequality (46) is equivalent to the following inequality in the acute triangle \( ABC \):
\[ 2 \prod \cos A \left[ \sum \frac{1}{\cos A} + \sum \frac{1}{\cos (B - C)} \right] + 2 (1 + \prod \cos A) \geq 3 \sqrt{\sum \sin 2B \sin 2C}. \]  
(47)

Secondly, we prove that inequality (47) is valid for the acute triangle \( ABC \).

We denote by \( L_0 \) the left hand of (47). Using identity (22) of Lemma 5, identity (29) and the following known identity (see [19, p. 56]):
\[ \sum \frac{1}{\cos A} = \frac{4R^2 - r^2 - s^2}{4R^2 + 4Rr + r^2 - s^2}, \]  
(48)
we easily obtain
\[ L_0 = \frac{N_0}{M_0}, \]  
(49)
where
\[ M_0 = R^2 \left[ s^4 - (6R^2 + 8Rr - 2r^2)s^2 + (2R^2 + 4Rr + r^2)(2R + r)^2 \right], \]
\[ N_0 = s^6 - (7R^2 + 4Rr - 2r^2)s^4 + (12R^4 + 8R^3r - 2R^2r^2 - 4Rr^3 + r^4)s^2 \]
\[ + (4R + r)(2R + r)^2R^2r. \]

Also, it is easy to get
\[ \sum \sin 2B \sin 2C = \frac{K_0}{4R^4}, \]  
(50)
where
\[ K_0 = s^4 - (4R^2 + 8Rr - 2r^2)s^2 + r(4R + r)(2R + r)^2. \]
Now, in view of the identities (49) and (50), to prove inequality (47) we need to prove
\[ 4R^2N_0^2 - 9K_0M_0^2 \geq 0. \quad (51) \]

With the help Maple, one easily obtains
\[ 4R^2N_0^2 - 9K_0M_0^2 = R^4X_0, \quad (52) \]
where
\[ X_0 = -5s^{12} + (88R^2 + 184R - 38r^2)s^{10} - (608R^4 + 2592R^3r + 1792R^2r^2 \]
\[ -552R^3r + 111r^4)s^{8} + (2064R^6 + 13696R^5r + 22672R^4r^2 + 9696R^3r^3 \]
\[ -1392R^2r^4 + 336R^5 - 164r^6)s^6 - (3456R^8 + 32960R^7r + 90416R^6r^2 \]
\[ + 103296R^5r^3 + 47872R^4r^4 + 5536R^3r^5 - 640R^2r^6 + 464R^7r^7 + 131r^8)s^4 \]
\[ + (576R^8 + 7872R^7r + 26992R^6r^2 + 39744R^5r^3 + 26640R^4r^4 + 7200R^3r^5 \]
\[ -208R^2r^6 - 432R^7 - 54r^8)(2R + r)^2s^2 - r(4R + r)(144R^6 + 704R^5r \]
\[ + 1328R^4r^2 + 1152R^3r^3 + 504R^2r^4 + 108R^5r^5 + 9r^6)(2R + r)^4. \]

Hence, we have to prove that the inequality \( X_0 \geq 0 \) holds for the acute triangle \( ABC \).

In the acute (non-obtuse) triangle \( ABC \), we have the following two inequalities:
\[ s_0 \equiv s^2 - (2R + r)^2 \geq 0, \quad (53) \]
\[ s_1 \equiv s^2 - (2R^2 + 8Rr + 3r^2) \geq 0. \quad (54) \]

Inequality (53) is equivalent to Ciamberlini’s inequality ([3]):
\[ s \geq 2R + r, \quad (55) \]
which follows from the previous identity (29). Inequality (54) was first presented by Walker in [21] (the author obtained a general generalization of this inequality in the recent paper [18]). Also, we have known that the equality in (53) holds iff the triangle \( ABC \) is a right triangle and the equality in (54) holds iff the triangle \( ABC \) is an isosceles right triangle or an equilateral triangle.

Next, we apply \( s_0 \geq 0, s_1 \geq 0 \), Gerretsen’s inequality \( g_2 \geq 0 \) (i.e., inequality (17)), and Sondat’s inequality \( t_0 \geq 0 \) (i.e., inequality (41)) to prove that inequality \( X_0 \geq 0 \) holds for the acute triangle \( ABC \).

Through analysis, we obtain the following identity:
\[ X_0 = s_0 \left[ g_2t_0(5g_2s^2 + s_0m_1 + m_2) + s_0s_1m_3 \right] + Y_0, \quad (56) \]
where
\[ m_1 = 8R^2 + 24Rr - 63r^2, \]
\[ m_2 = 2r(144R^3 - 14R^2r - 236Rr^2 - 145r^3), \]
\[ m_3 = 16R^6 - 256R^5r + 3280R^4r^2 + 4256R^3r^3 - 5944R^2r^4 - 2336Rr^5 - 800r^6, \]
\[ Y_0 = (32R^8 + 1088R^7r - 6848R^6r^2 + 15296R^5r^3 + 27504R^4r^4 - 21184R^3r^5 \]
\[ - 18640R^2r^6 - 8896Rr^7 - 4928r^8)s^2 - (256R^{10} + 8960R^9r - 47040R^8r^2 \]
\[ + 63616R^7r^3 + 326912R^6r^4 + 84736R^5r^5 - 250304R^4r^6 - 228928R^3r^7 \]
\[ - 110912R^2r^8 - 37888Rr^9 - 6144r^{10})s^2 + 16(8R^{10} + 280R^9r - 1502R^8r^2 \]
\[ + 1796R^7r^3 + 10224R^6r^4 + 2952R^5r^5 - 7711R^4r^6 - 7028R^3r^7 \]
\[ - 3063R^2r^8 - 836Rr^9 - 108r^{10})(2R + r)^2. \]

By Euler’s inequality \( R \geq 2r \), one sees that \( m_1 > 0 \). If we set
\[ e = R - 2r, \]
then \( e \geq 0 \) and it is easy to get
\[ m_2 = 2r(144e^3 + 850e^2r + 1436er^2 + 479r^3) > 0, \]
\[ m_3 = 16e^4(e^2 - 4er + 105r^2) + 22816e^3r^3 + 81672e^2r^4 + 112512er^5 + 50112r^6 > 0. \]

Therefore, according to identity (56) and inequalities \( g_2 \geq 0 \), \( t_0 \geq 0, \) \( s_0 \geq 0, \) \( s_1 \geq 0, \) to prove inequality \( X_0 \geq 0 \) it remains to prove that \( Y_0 \geq 0 \) holds for the acute triangle \( ABC \).

Putting
\[ Y_0 = f(s^2), \]
then
\[ f'(s^2) = 2(32R^8 + 1088R^7r - 6848R^6r^2 + 15296R^5r^3 + 27504R^4r^4 - 21184R^3r^5 \]
\[ - 18640R^2r^6 - 8896Rr^7 - 4928r^8)s^2 - (256R^{10} + 8960R^9r - 47040R^8r^2 \]
\[ + 63616R^7r^3 + 326912R^6r^4 + 84736R^5r^5 - 250304R^4r^6 - 228928R^3r^7 \]
\[ - 110912R^2r^8 - 37888Rr^9 - 6144r^{10}). \]

Since
\[ 32R^8 + 1088R^7r - 6848R^6r^2 + 15296R^5r^3 + 27504R^4r^4 - 21184R^3r^5 \]
\[ - 18640R^2r^6 - 8896Rr^7 - 4928r^8 \]
\[ = 32e^8 + 1600e^7r + 11968e^6r^2 + 38848e^5r^3 + 110064e^4r^4 \]
\[ + 381632e^3r^5 + 882992e^2r^6 + 971520er^7 + 371968r^8 > 0, \]
thus by inequality (55) we have for the acute triangle \( ABC \) that
\[ f'(s^2) \geq 2(32R^8 + 1088R^7r - 6848R^6r^2 + 15296R^5r^3 + 27504R^4r^4 - 21184R^3r^5 \]
\[ - 18640R^2r^6 - 8896Rr^7 - 4928r^8)(2R + r)^2 + (256R^{10} - 8960R^9r \]
\[ + 47040R^8r^2 - 63616R^7r^3 - 326912R^6r^4 - 84736R^5r^5 + 250304R^4r^6 \]
\[ + 228928R^3r^7 + 110912R^2r^8 + 37888Rr^9 + 6144r^{10}) \]
\[ = 32e^8 + 704e^7r + 6328e^6r^2 + 31024e^5r^3 + 91425e^4r^4 + 164970e^3r^5 \]
\[ + 173409e^2r^6 + 89424er^7 + 11752r^8 > 0. \]
Hence, \( f(s^2) \) is increasing. According to this conclusion, we next divide two cases to prove that \( Y_0 \geq 0 \) holds for the non-obtuse triangle \( ABC \).

**Case 1.** \( R \) and \( r \) satisfy the inequality \( R^2 - 2Rr - r^2 > 0 \).

In this case, to prove \( Y_0 > 0 \) we need to prove \( f((2R+r)^2) > 0 \) (since \( f(s^2) \) is increasing). It is easy to check that

\[
f((2R+r)^2) = 256(R^2-2Rr-r^2)(R^2+4Rr+2r^2)(2R+r)^4 r^4.
\]

(57)

Thus, by the hypothesis we have \( f((2R+r)^2) > 0 \) and inequality \( Y_0 > 0 \) is proved.

**Case 2.** \( R \) and \( r \) satisfy the inequality \( R^2 - 2Rr - r^2 \leq 0 \).

Since \( f(s^2) \) is increasing, by Walker’s inequality (54), to prove \( Y_0 \geq 0 \) in the above case we only need to prove that \( f(2R^2 + 8Rr + 3r^2) \geq 0 \). With the help of the Maple software, it is easy to obtain

\[
f(2R^2 + 8Rr + 3r^2) = -64(R - 2r)(R^2 - 2Rr - r^2)Z_0,
\]

(58)

where

\[
Z_0 = -2R^9 - 68R^8r + 462R^7r^2 - 628R^6r^3 - 1499R^5r^4 + 2224R^4r^5 + 3533R^3r^6
+ 2242R^2r^7 + 1036Rr^8 + 216r^9.
\]

Thus, by Euler’s inequality \( R \geq 2r \), we need to prove \( Z_0 > 0 \) under the hypothesis. Indeed, we can rewrite \( Z_0 \) as follows:

\[
Z_0 = -(R^2 - 2Rr - r^2)(2e^7 + 8122r^6e + 3372r^5e^2 + 1863r^4e^3 + 1788r^3e^4 + 716r^2e^5
+ 100re^6 + 1560r^7) + 6(1801R + 746r)r^8,
\]

(59)

where \( e = R - 2r \geq 0 \). Hence, by the hypothesis \( R^2 - 2Rr - r^2 \leq 0 \) we have \( Z_0 > 0 \) and \( Y_0 \geq 0 \) is proved. Also, in the second case, the equality in \( Y_0 \geq 0 \) holds if and only if \( (R - 2r)(r^2 + 2Rr - R^2) = 0 \), and we further know that the equality holds if and only if the triangle \( ABC \) is an equilateral triangle or an isosceles right triangle.

Combining the discussions of the above two cases, we complete the proof of \( Y_0 \geq 0 \) for the acute triangle \( ABC \) and conclude that the equality in \( Y_0 \geq 0 \) holds if and only if the triangle \( ABC \) is an equilateral triangle or an isosceles right triangle. Thus, we complete the proofs of inequality \( X_0 \geq 0 \) and inequality (47).

Finally, making the substitutions \( A \rightarrow (\pi - A)/2 \) etc., in (47), then inequality (46) follows immediately. Therefore, inequality (9) is proved. According to identity (56) and the equality conditions of (17), (41), (53) and (54), it is easy to conclude that the equality condition of \( X_0 \geq 0 \) is the same as that of \( Y_0 \geq 0 \). Furthermore, we easily know that both equalities of (46) and (9) hold if and only if the triangle \( ABC \) is equilateral. This completes the proof of the theorem. □

### 4. Corollaries and Conjectures

In this section, we give several corollaries of the theorem and present a few related interesting conjectures as open problems.

Since \( 3(bc + ca + ab) \leq (a + b + c)^2 \), by inequality (7) we obtain the following linear inequality:

1. **Corollary**

Inequality (46) holds for the acute triangle \( ABC \). Hence, by identity (56) and the equality conditions of (17), (41), (53) and (54), it is easy to conclude that the equality condition of (46) holds if and only if the triangle \( ABC \) is equilateral. This completes the proof of the theorem.
COROLLARY 1. For any triangle ABC, the following inequality holds:
\[ m_a + m_b + m_c + w_a + w_b + w_c \leq 2\sqrt{3}s. \] (60)

REMARK 1. Inspired by the above inequality, the author has proved the following inequality:
\[ 3w_a + m_a + m_b + m_c \leq \frac{3\sqrt{3}}{2}(b + c). \] (61)

Clearly, we also have the other two similar relations. Adding up these three inequalities, we obtain inequality (60).

By the previous inequality (7) and the following equivalent form of the Gerretsen inequality (17) (see [1, inequality 5.17]), i.e.,
\[ bc + ca + ab \leq 4(R + r)^2, \] (62)
we obtain the following linear inequality.

COROLLARY 2. For any triangle ABC, the following inequality holds:
\[ m_a + m_b + m_c + w_a + w_b + w_c \leq 6(R + r). \] (63)

The above inequality is obviously stronger than the following known result (see [1, inequality 6.14]):
\[ w_a + w_b + w_c \leq 3(R + r). \] (64)

In the triangle ABC, we have known that
\[ m_a w_a \geq s(s - a). \] (65)
Thus, by inequality (7) and the simplest arithmetic-geometric mean inequality, we can obtain the following inequality involving the side lengths of the triangle ABC.

COROLLARY 3. For any triangle ABC, the following inequality holds:
\[ \sqrt{b + c - a} + \sqrt{c + a - b} + \sqrt{a + b - c} \leq 3\sqrt{\frac{bc + ca + ab}{a + b + c}}. \] (66)

In the acute triangle ABC, we have the following inequality (see [17, p. 252]):
\[ bc + ca + ab \geq \frac{9}{4}(R + 2r)^2. \] (67)
Thus, by the previous inequality (9) and identity (44) we can obtain the following inequality:

COROLLARY 4. For the acute triangle ABC, the following inequality holds:
\[ w_a + w_b + w_c \geq \frac{1}{2}R + 8r. \] (68)
In any triangle $ABC$, we have the known inequality $w_a \leq \sqrt{rbrc}$. Thus, using the Cauchy-Schwarz inequality we have

$$(w_a + r_a)^2 \leq (r_c + r_a)(r_a + r_b).$$

Consequently, by the previous inequality (9) one obtains the following corollary:

**COROLLARY 5.** For any triangle $ABC$, the following inequality holds:

$$\sqrt{(r_c + r_a)(r_a + r_b)} + \sqrt{(r_a + r_b)(r_b + r_c)} + \sqrt{(r_b + r_c)(r_c + r_a)} \geq 3\sqrt{bc + ca + ab}. \quad (70)$$

Next, we introduce a conjecture related to $\sqrt{bc + ca + ab}$:

**CONJECTURE 1.** For any triangle $ABC$, the following inequality holds:

$$m_a + w_b + w_c \leq \frac{3}{2}\sqrt{bc + ca + ab}. \quad (71)$$

**REMARK 2.** If the above inequality holds, then by the previous inequality (6) we can deduce the following geometric inequality:

$$R_a + R_b + R_c \geq \frac{2}{3}(m_a + w_b + w_c). \quad (72)$$

Also, if (71) is true then we can deduce the following three inequalities:

$$m_a + w_b + w_c \leq \sqrt{3}s, \quad (73)$$

$$m_a + w_b + w_c \leq 3(R + r), \quad (74)$$

$$2m_a + w_b + w_c \leq w_a + r_a + r_b + r_c. \quad (75)$$

However, the above four inequalities have not been proved at the present.

**CONJECTURE 2.** For any triangle $ABC$, the following inequality holds:

$$m_a + m_b + m_c \leq \frac{w_a^2}{m_b + m_c} + \frac{w_b^2}{m_c + m_a} + \frac{w_c^2}{m_a + m_b} \leq r_a + r_b + r_c. \quad (76)$$

Finally, for the Erdős-Mordell inequality we present a new sharpened version:

**CONJECTURE 3.** Let $k \geq 9/4$ be a real number, then for any point $P$ inside triangle $ABC$ it holds:

$$\frac{R_1 + R_2 + R_3}{r_1 + r_2 + r_3} \geq 2 \cdot \frac{ka^2 + m_a w_a}{ka^2 + r_b r_c}. \quad (77)$$

The equivalent form $m_a w_a \geq r_b r_c$ of inequality (65) shows that the value of the right hand of (77) is at least 2.
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