Model of a relativistic oscillator in a generalized Schrödinger picture

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Abstract

We consider the motion of a relativistic particle in an external field like the harmonic oscillator potential in the picture in which the analogue of Schrödinger operators of the particle are independent of both the time and the space coordinates. The spacetime independent operators in the equations of states induce operators which are related to Killing vectors of the AdS space. We also consider the nonrelativistic limit.

Key words Lorentz group, relativistic wave equations, quantum mechanics, extended objects

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1 Introduction

In this paper we present a model of one-dimensional relativistic harmonic oscillator which is based on the expansions of the Lorentz group and a generalized Schrödinger picture. The functions that realize the unitary representation of the one-dimensional Lorentz group ($p =$ momentum, $m =$ mass, $p_0^2 - c^2 p^2 = m^2 c^4$) have the form

$$\xi(p, \alpha) = e^{i \alpha \ln(p_0 - cp)/mc^2},$$

(1)

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and are the eigenfunctions of the boost generator $N(p) = ip_0 \partial_{p^0}$ ($N \Rightarrow \alpha$).
The following expansion (Shapiro transformation) of the wave function of a particle in the momentum representation ($\rho = \alpha \hbar / mc$)

$$\psi(p) = \frac{1}{\sqrt{2\pi \hbar}} \int_{-\infty}^{\infty} \xi(p, \rho) \psi(\rho) d\rho, \quad (2)$$

leads to the functions $\psi(\rho)$ in the spacetime-independent $\rho$ representation. In [4], in the framework of a two-particle equation of the quasipotential type the variable $\rho$ was interpreted as the relativistic generalization of a relative coordinate. Quasipotential models of a relativistic oscillator were first considered in [6, 7, 8].

The $\rho$-representation may also be used in the generalized Schrödinger (GS) picture in which the analogue of Schrödinger operators of a particle are independent of both the time and the space coordinates $t, x$ in different representations [13]. In the $\rho$-representation the motion of a free particle is described by the equations

$$i\hbar \frac{\partial}{\partial t} \psi(\rho, t, x) = H(\rho) \psi(\rho, t, x), \quad -i\hbar \frac{\partial}{\partial x} \psi(\rho, t, x) = P(\rho) \psi(\rho, t, x), \quad (3)$$

where the Hamilton operator $H(\rho)$ and the momentum operator $P(\rho)$ have the form

$$H(\rho) = mc^2 \cosh(-\frac{i\hbar}{mc} \partial_{\rho}), \quad P(\rho) = mc \sinh(-\frac{i\hbar}{mc} \partial_{\rho}). \quad (4)$$

The operators $H(\rho)$, $P(\rho)$ and the generator of the Lorentz group $N(\rho)$ ($N(\rho) = \rho$) satisfy the commutation relations of the Poincaré algebra,

$$[N, P] = i\frac{\hbar}{mc^2} H, \quad [P, H] = 0, \quad [H, N] = -i\frac{\hbar}{m} P. \quad (5)$$

In the recent paper [14], it was shown that in the GS picture the propagators with different $\rho$ in the relativistic region and nonrelativistic limit may be used to describe the motion of extended objects like strings. The main aim of the present paper is to describe in this picture the motion of a relativistic particle in an external field like the harmonic oscillator potential.

Our treatment is based on the following assumptions: We will introduce the operators $\hat{P}_0 = H + H'$ and $\hat{P}_1 = P + P'$ instead of $H$ and $P$, where $H'$ and $P'$ represent the external field in the $p$ or in the $\rho$ representations.
We make the assumption that only the operators \( \hat{P}_0 \) and \( \hat{P}_1 \) contain the interaction parts. We assume that the equations of the motion of a particle in an external field may be written in the form of a direct generalization of the equations (3),

\[
K_0(t,x)\Psi = \hat{P}_0 \Psi, \quad K_1(t,x)\Psi = \hat{P}_1 \Psi. \tag{6}
\]

Here, by analogy with (3) for the left-hand side we have introduced operators \( K_0(t,x) \) and \( K_1(t,x) \) in terms of the spacetime coordinates \( t, x \). In the present paper, taking the practically important example of the harmonic oscillator, we shall show that the use of the equations (6) makes it possible to give a description of the motion of a relativistic particle in an external field. The external field do violate the commutation relations of the Poincaré algebra (\( [\hat{P}_1, \hat{P}_0] \neq 0 \)). We have the problem of determining observables in the GS picture. In the equations (6) the operators on the right-hand side are spacetime-independent operators and therefore correspond to constants of motion. For the harmonic oscillator we will show that the equations (6) induce operators \( K_0(x) \), \( K_1(x) \) which are related to Killing vectors of the AdS space. First we set up the rules which may be used to derive the explicit form of the operators \( K_0(x) \), \( K_1(x) \). Then we study the model of relativistic oscillator. We will consider the nonrelativistic limit.

## 2 The relativistic problem

We consider the motion of a relativistic particle in external field which is presented by the operators \( (\omega=\text{frequency}) \)

\[
H'(\rho) = \frac{m\omega^2}{2}\rho(\rho - i\frac{\hbar}{mc})e^{-i\frac{\hbar}{mc}\partial_\rho}, \quad P'_1(\rho) = \frac{m\omega^2}{2c}\rho(\rho - i\frac{\hbar}{mc})e^{-i\frac{\hbar}{mc}\partial_\rho}. \tag{7}
\]

For the operators on the right-hand side of the Eqs. (6) we have

\[
\hat{P}_0(\rho) = mc^2 \cosh(-\frac{i\hbar}{mc}\partial_\rho) + \frac{m\omega^2}{2}\rho(\rho - i\frac{\hbar}{mc})e^{-i\frac{\hbar}{mc}\partial_\rho}, \tag{8}
\]

\[
\hat{P}_1(\rho) = mc \sinh(-\frac{i\hbar}{mc}\partial_\rho) + \frac{m\omega^2}{2c}\rho(\rho - i\frac{\hbar}{mc})e^{-i\frac{\hbar}{mc}\partial_\rho}. \tag{9}
\]

The realization of the operators \( \hat{P}_0 \) and \( \hat{P}_1 \) in the momentum representation given in appendix A.
In the nonrelativistic limit the operators \( \hat{P}_0(\rho) - mc^2 \) and \( \hat{P}_1(\rho) \) assume the form
\[
\hat{P}_{0nr} = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial \rho^2} + \frac{m\omega^2}{2}\rho^2, \quad \hat{P}_{1nr} = -i\hbar \frac{\partial}{\partial \rho}.
\] (10)
The operators \( \hat{P}_0(\rho), \hat{P}_1(\rho) \) and \( \rho \) satisfy the commutation relations
\[
[\rho, \hat{P}_1] = i\frac{\hbar}{mc^2} \hat{P}_0, \quad [\hat{P}_1, \hat{P}_0] = -i\hbar m\omega^2 \rho, \quad [\hat{P}_0, \rho] = -i\frac{\hbar}{m} \hat{P}_1.
\] (11)
The Casimir operator is a multiple of the identity operator \( I \),
\[
C(\rho) = \frac{1}{(\hbar\omega)^2} \{ \hat{P}_0^2 - c^2 \hat{P}_1^2 \} - \frac{mc^2}{\hbar^2} \rho^2 = \left( \frac{mc^2}{\hbar\omega} \right)^2 I.
\] (12)
Let us examine the restrictions imposed on the operators \( K_0(t, x) \) and \( K_1(t, x) \) by the equations
\[
K_0(t, x)\Psi(\rho; t, x) = \hat{P}_0(\rho)\Psi(\rho; t, x),
\] (13)
\[
K_1(t, x)\Psi(\rho; t, x) = \hat{P}_1(\rho)\Psi(\rho; t, x),
\] (14)
and the commutations rules (11). We multiply the equation (13) from the left by the operator \( \hat{P}_1(\rho) \), the equation (14) from the left by \( \hat{P}_0(\rho) \) and substract the two resulting equations one from the other. Bearing in mind that
\[
[\hat{P}_1(\rho), K_0(t, x)] = 0, \quad [\hat{P}_0(\rho), K_1(t, x)] = 0,
\] (15)
we find
\[
[K_0(t, x), K_1(t, x)]\Psi = -[\hat{P}_0(\rho), \hat{P}_1(\rho)]\Psi,
\] (16)
or
\[
[K_1(t, x), K_0(t, x)]\Psi = i\hbar m\omega^2 \rho \Psi.
\] (17)
In the following we introduce the operator
\[
K(t, x) = \frac{1}{i\hbar m\omega^2} [K_1(t, x), K_0(t, x)]
\] (18)
for which we have
\[
K(t, x)\Psi(\rho; t, x) = \rho \Psi(\rho; t, x).
\] (19)
Eqs. (16) and
\[
[K_1, K]\Psi = -[\hat{P}_1, \rho]\Psi, \quad [K, K_0]\Psi = -[\rho, \hat{P}_0]\Psi
\] (20)
states that the equations (13), (14) and (19) induce operators $K_0(t, x)$, $K_1(t, x)$ and $K(t, x)$ with the same commutation rules as $\hat{P}_0(\rho)$, $\hat{P}_1(\rho)$ and $\rho$, except for the minus sign on the right-hand side

$$[K, K_1] = -i\frac{\hbar}{mc^2}K_0, \quad [K_1, K_0] = i\hbar m\omega^2K, \quad [K_0, K] = i\frac{\hbar}{m}K_1. \quad (21)$$

Note that if at first we introduce for the operators $\hat{P}_0(\rho)$, $\hat{P}_1(\rho)$ and $\rho$ commutation relations like (21), then for $K_0$, $K_1$ and $K$ we must introduce commutation relations like (11). This can be made by replacing $\hat{P}_1$ by $-\hat{P}_1$ and $K_1$ by $-K_1$, respectively.

For the Casimir operators (12) and

$$C(t, x) = \frac{1}{(\hbar\omega)^2}\{K_0^2 - c^2K_1^2\} - \frac{m^2c^2}{\hbar^2}K^2 \quad (22)$$

we have the equation

$$C(t, x)\Psi(\rho; t, x) = C(\rho)\Psi(\rho; t, x). \quad (23)$$

The operators $N_1 = mcK/\hbar$, $N_2 = cK_1/\hbar\omega$, $N_3 = K_0/\hbar\omega$ satisfy the commutation relations of the noncompact Lie algebra $SO(2, 1)$,

$$[N_1, N_2] = -iN_3, \quad [N_2, N_3] = iN_1, \quad [N_3, N_1] = iN_2. \quad (24)$$

In the representations in which $N_1$ or $N_2$ is diagonal, their eigenvalue spectrum is continuous. We are interested in a representation in which $N_3$ is diagonal. In this case the eigenvalue spectrum of $N_3$ is discrete for an irreducible representation and has the form of $N_3 \Rightarrow -a + n \quad (n = 0, 1, 2, ...)$, where the number $a$ is related to eigenvalues of the Casimir operator $C \Rightarrow a(a + 1)$. For the unitary representations ($D^+$ - series), $-a = 1/2, 1, 3/2, 2, 5/2, ...$ It follows from (12) and (23) that $-a = 1/2 + \sqrt{1/4 + (mc^2/\hbar\omega)^2}$. The cases $-a = 1/2$ and $-a = 1$ must be rejected. For the cases $-a = n_0 = 3/2, 2, 5/2, ...$ we have $K_0 \Rightarrow \hbar\omega(n + 1/2 + \mu)$, where $\mu = \sqrt{1/4 + (mc^2/\hbar\omega)^2}$.

Similarly, putting $N_1 = \frac{mc}{\hbar}\rho$, $N_2 = -c\hat{P}_1/\hbar\omega$, $N_3 = \hat{P}_0/\hbar\omega$, we find that in the representation in which the operator $\hat{P}_0(\rho)$ is diagonal its spectrum is $E_{n, \mu} = \hbar\omega(n + 1/2 + \mu)$. For $mc^2/\hbar\omega \gg 1/2$ we have $E_{n, \mu} \approx \hbar\omega(n + 1/2) + mc^2$.

For convenience, in the text below, we omit explicit mention of the number $n_0$ (except of Eqs. (13)-(14)). In the functions $\Psi(\rho; t, x)$ the index $n_0$ is implied.
The explicit form of the operators $K_0(t, x)$, $K(t, x)$ and $K_1(t, x)$ depends on the realisation in terms of the spacetime coordinates. In order to interpret the operator $\hat{P}_0(\rho)$ as Hamilton operator, we choose the following realisation

$$K_0 = i\hbar \frac{\partial}{\partial t},$$

(25)

$$K(t, x) = \frac{x \cos \omega t}{mc^2 \sqrt{1 + (\omega x/c)^2}} i\hbar \partial_t + \frac{1}{m \omega} \sqrt{1 + (\omega x/c)^2} \sin \omega t (i\hbar \partial_x),$$

(26)

$$K_1(t, x) = \sqrt{1 + (\omega x/c)^2} \cos \omega t (i\hbar \partial_x) - \frac{(\omega x/c)^2}{\sqrt{1 + (\omega x/c)^2}} \sin \omega t i\hbar \partial_t.$$  

(27)

We have the equation

$$i\hbar \frac{\partial}{\partial t} \Psi(\rho; t, x) = \hat{P}_0(\rho) \Psi(\rho; t, x)$$  

(28)

which defines the operator $\hat{P}_0(\rho)$ as Hamilton operator. The operators (25)-(27) are related to Killing vectors of the AdS space with metric

$$ds^2 = (1 + \frac{\omega^2 x^2}{c^2}) c^2 dt^2 - \frac{1}{1 + \frac{\omega^2 x^2}{c^2}} dx^2.$$  

(29)

A general solution of $\Psi(\rho; t, x)$ can be written as a sum of separated solutions

$$\Psi(\rho; t, x) = \sum_{n=0}^{\infty} v_n(\rho) f_n(t, x),$$

(30)

$$i\hbar \frac{\partial}{\partial t} v_n(\rho) f_n(t, x) = \hat{P}_0(\rho) v_n(\rho) f_n(t, x),$$

(31)

where $v_n(\rho)$ are the eigenfunctions of the Hamilton operator $\hat{P}_0(\rho)$ and $f_n(t, x)$ are the eigenfunctions of the operator $K_0$ and the Casimir operator $C(t, x)$. In the following we express the functions $v_n(\rho) f_n(t, x)$ in terms of the operators ($\tau = \omega t$, $y = \omega x/c$, $l = c/\omega$)

$$A^+(\rho) = \frac{1}{\sqrt{2}} \left\{ \sqrt{\frac{m \omega}{\hbar}} \rho - \frac{i}{\sqrt{m \hbar \omega}} \hat{P}_1 \right\}, \quad K^+(\tau, y) = \frac{1}{\sqrt{2}} \left\{ \sqrt{\frac{m \omega}{\hbar}} K + \frac{i}{\sqrt{m \hbar \omega}} K_1 \right\}$$

(32)

and the functions

$$v_0(\rho) = 2^\mu \sqrt{\frac{mc}{\pi \hbar \Gamma(2\mu + 1)}} \left( \frac{\hbar \omega}{mc^2} \right)^{-i \frac{m \hbar}{\hbar} \rho} \Gamma(\mu + 1/2 - i \frac{mc}{\hbar} \rho),$$

(33)
\[ f_0(\tau, y) = \sqrt{\frac{\Gamma(\mu + \frac{1}{2})}{l\sqrt{\pi \Gamma(\mu)}}} e^{-i(\mu + \frac{1}{2})\tau} (1 + y^2)^{-\frac{\mu + \frac{1}{2}}{2}}. \] (34)

The functions \( v_0(\rho) \) and \( f_0(\tau, y) \) satisfy the equations
\[ A^{-}(\rho)v_0(\rho) = 0, \quad K^{-}(\tau, y)f_0(\tau, y) = 0, \] (35)
where
\[ A^{-}(\rho) = \frac{1}{\sqrt{2}} \left\{ \frac{m\omega}{\hbar} \rho + \frac{i}{\sqrt{m\hbar\omega}} \hat{P}_1 \right\}, \quad K^{-}(\tau, y) = \frac{1}{\sqrt{2}} \left\{ \frac{m\omega}{\hbar} \right\} K - \frac{i}{\sqrt{m\hbar\omega}}K_1. \] (36)

Using the operators \( A^{+}(\rho) \) and \( K^{+}(\tau, y) \) we can construct a system of normalized functions \( (b = \sqrt{\frac{\hbar\omega}{2mc^2}}) \)
\[ v_n(\rho) = \beta_n(A^{+})^nv_0(\rho), \quad f_n(\tau, y) = \beta_n(K^{+})^nf_0(\tau, y) \] (37)
where
\[ \beta_n = b^{-n} \sqrt{\frac{\Gamma(2\mu + 1)}{n!\Gamma(n + 2\mu + 1)}}. \] (38)

The explicit form of the functions \( v_n(\rho) \) and \( f_n(\tau, y) \), \( (n=1,2,3... \) are given in appendix B.

For the functions \( \phi_n(\rho, \tau, y) = v_n(\rho)f_n(\tau, y) \) we have the relations
\[ A^{-}K^{-}\phi_n(\rho, \tau, y) = g_n^2\phi_{n-1}(\rho, \tau, y), \] (39)
\[ A^{+}K^{+}\phi_n(\rho, \tau, y) = g_{n+1}^2\phi_{n+1}(\rho, \tau, y), \] (40)
where \( g_n = b\sqrt{n(n + 2\mu)}. \)

Additionally,
\[ A^{+}\Psi(\rho; t, x) = K^{-}\Psi(\rho; t, x), \quad A^{-}\Psi(\rho; t, x) = K^{+}\Psi(\rho; t, x). \] (41)

For the propagator we have the expression
\[ \mathcal{K}(2, 1) = \sum_{n=0}^{\infty} \phi_n(\rho_2, t_2, x_2)\phi_n^*(\rho_1, t_1, x_1). \] (42)

To conclude this section we make the following remarks. The expression for the energy levels \( E_{n,\mu} = \hbar\omega(n + 1/2 + \mu) \) can be written in the form
\[ E_{n, n_0} = \hbar\omega(n + n_0) = \frac{mc^2}{\sqrt{n_0(n_0 - 1)}}(n + n_0). \] (43)
This shows that the oscillator frequency is discrete and for higher \( n_0 \) decreases

\[
\omega = \frac{mc^2}{\hbar \sqrt{n_0(n_0-1)}}. \tag{44}
\]

The set of numbers \( n_0 \) is infinite. For \( n_0 \to \infty \) one has \( \omega \to 0 \). For the energy levels \( E_{0,n_0\to\infty} \to mc^2 \). There is no zero-point energy. We cannot introduce the notion of a ground state like the ground state of nonrelativistic oscillator in quantum mechanics.

For the AdS radius \( l = c/\omega \) we have

\[
l = \frac{\hbar}{mc} \sqrt{n_0(n_0-1)}. \tag{45}
\]

3 The nonrelativistic problem

Let us now consider the nonrelativistic limit. In this limit the operators (10) and \( \rho \) satisfy the commutation relations

\[
[\rho, \hat{P}_{1nr}] = i\hbar, \quad [\hat{P}_{1nr}, \hat{P}_{0nr}] = -i\hbar \omega^2 \rho, \quad [\hat{P}_{0nr}, \rho] = -i\frac{\hbar}{m} \hat{P}_{1nr}. \tag{45}
\]

In the operators \( K(t,x) \) and \( K_1(t,x) \) we replace the operator \( i\hbar \partial_t \) by \( mc^2 \) and assume that \( (\omega x/c)^2 < 1 \). In this case the operators

\[
K_{0nr}(t,x) = K_0(t,x) = i\hbar \frac{\partial}{\partial t}, \quad K_{nr}(t,x) = x \cos \omega t + \frac{1}{m\omega} \sin \omega t(i\hbar \partial_x), \tag{46}
\]

\[
K_{1nr}(t,x) = -xmc \sin \omega t + \cos \omega t(i\hbar \partial_x), \tag{47}
\]

satisfy the same commutation rules as (45) except for the minus sign on the right-hand side

\[
[K_{nr}, K_{1nr}] = -i\hbar, \quad [K_{1nr}, K_0] = i\hbar \omega^2 K_{nr}, \quad [K_0, K_{nr}] = i\frac{\hbar}{m} K_{1nr}. \tag{48}
\]

For the Casimir relation we have

\[
C_{nr}(t,x) = K_0 = K_0 - \frac{1}{2m} K_{1nr}^2 - \frac{m\omega^2}{2} K_{nr}^2 = i\hbar \frac{\partial}{\partial t} - \tilde{H}(x), \tag{49}
\]

where the operator \( \tilde{H}(x) \) have the form of the harmonic-oscillator Hamilton operator in quantum mechanics,

\[
\tilde{H}(x) = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + \frac{m\omega^2 x^2}{2}. \tag{50}
\]
In the nonrelativistic limit the operators \( A^+(\rho), K^+(t, x) \) and \( A^-(\rho), K^-(t, x) \) go over into the operators \( \tilde{\rho} = \sqrt{m \omega / \hbar} \rho, \tilde{x} = \sqrt{m \omega / \hbar} x \)

\[
a^+(\tilde{\rho}) = \frac{1}{\sqrt{2}} \{ \tilde{\rho} - \partial_{\tilde{\rho}} \}, \quad k^+(t, \tilde{x}) = e^{-i \omega t} \frac{1}{\sqrt{2}} \{ \tilde{x} - \partial_{\tilde{x}} \}, \quad (51)
\]

and

\[
a^-(\tilde{\rho}) = \frac{1}{\sqrt{2}} \{ \tilde{\rho} + \partial_{\tilde{\rho}} \}, \quad k^-(t, \tilde{x}) = e^{i \omega t} \frac{1}{\sqrt{2}} \{ \tilde{x} + \partial_{\tilde{x}} \}, \quad (52)
\]

respectively. The Hamilton operator \( \hat{P}_{0nr}(\rho) \) in the GS picture have the same form as the operator \( \tilde{H}(x) \). Therefore, for the spectrum of \( \hat{P}_{0nr}(\rho) \) we have

\[
E_n = \bar{\hbar} \omega \left( n + \frac{1}{2} \right). \quad (53)
\]

The function \( \Psi(\rho; t, x)_{nr} \) in the equation of state

\[
i \hbar \frac{\partial}{\partial t} \Psi(\rho; t, x)_{nr} = \hat{P}_{0nr}(\rho) \Psi(\rho; t, x)_{nr},
\]

can be written as a sum of separated solutions \( w_n(\tilde{\rho}) u_n(t, \tilde{x}) \)

\[
i \hbar \frac{\partial}{\partial t} w_n(\tilde{\rho}) u_n(t, \tilde{x}) = \hat{P}_{0nr}(\tilde{\rho}) w_n(\tilde{\rho}) u_n(t, \tilde{x}), \quad (54)
\]

where

\[
w_n(\tilde{\rho}) u_n(t, \tilde{x}) = \frac{1}{n!} (a^+)^n (k^+)^n w_0(\tilde{\rho}) u_0(t, \tilde{x}). \quad (55)
\]

Here, the functions \((N_0 = (m\omega/\pi \hbar)^{1/4})\)

\[
w_0(\tilde{\rho}) = N_0 e^{-\tilde{\rho}^2/2}, \quad u_0(t, \tilde{x}) = N_0 e^{-i \frac{E_n \kappa}{\hbar} t} e^{-\tilde{x}^2/2} \quad (56)
\]

satisfy the equations

\[
a^- w_0(\tilde{\rho}) = 0, \quad k^- u_0(t, \tilde{x}) = 0. \quad (57)
\]

As a result we have

\[
w_n(\tilde{\rho}) u_n(t, \tilde{x}) = N_n e^{-\tilde{\rho}^2/2} H_n(\tilde{\rho}) N_n e^{-i \frac{E_n \kappa}{\hbar} t} e^{-\tilde{x}^2/2} H_n(\tilde{x}), \quad (58)
\]

\[
\Psi(\rho; t, x)_{nr} = \sum_{n=0}^{\infty} w_n(\tilde{\rho}) u_n(t, \tilde{x}), \quad (59)
\]

where

\[
u_n(t, \tilde{x}) = N_n e^{-i \frac{E_n \kappa}{\hbar} t} e^{-\tilde{x}^2/2} H_n(\tilde{x}) \quad (60)
\]
are the well-known harmonic-oscillator wave functions in quantum mechanics.

For the propagator we have the expression

$$\mathcal{K}(2, 1)_{nr} = \sum_{n=0}^{\infty} w_n(\rho_2)u_n(t_2, x_2)w^*_n(\rho_1)u^*_n(t_1, x_1).$$

(61)

In accordance with the interpretation of propagators in the GS picture which was proposed in [14] the propagators $\mathcal{K}(2, 1)$ and $\mathcal{K}(2, 1)_{nr}$ describe extended relativistic and nonrelativistic harmonic oscillator, respectively.

4 Conclusion

In this paper we have shown that the generalized Schrödinger (GS) picture may be used to describe the motion of a relativistic particle in an external field. For the harmonic oscillator potential we found that the spacetime independent operators in the equations of states induce in a natural way the operators which are related to Killing vectors of the AdS space. The problem of determining the Hamilton operator of the particle in the external field based on choosing the coordinate system in this space. We found that the oscillator frequency and the AdS radius are discrete. We have shown that in the relativistic region there is no zero-point energy. We have constructed propagators which describe extended relativistic and nonrelativistic harmonic oscillators in the GS picture.

5 Appendix A

In the momentum representation

$$p_0 = mc^2\cosh \chi, \quad p = mc\sinh \chi, \quad \chi = -\ln[(p_0 - cp)/mc^2],$$

(62)

the operators $\hat{P}_0$ and $\hat{P}_1$ are

$$\hat{P}_0(\chi) = mc^2[\cosh \chi - \frac{1}{2}\left(\frac{\hbar \omega}{mc^2}\right)^2e^\chi\{\frac{d^2}{d\chi^2} + \frac{d}{d\chi}\}],$$

(63)

$$\hat{P}_1(\chi) = mc[\sinh \chi - \frac{1}{2}\left(\frac{\hbar \omega}{mc^2}\right)^2e^\chi\{\frac{d^2}{d\chi^2} + \frac{d}{d\chi}\}].$$

(64)
These operators satisfy the commutation relations

\[
[N(\chi), \hat{P}_1(\chi)] = i \frac{\hbar}{mc^2} \hat{P}_0(\chi), \quad [\hat{P}_1(\chi), \hat{P}_0(\chi)] = -i \hbar m \omega^2 N(\chi),
\]

(65)

\[
[\hat{P}_0(\chi), N(\chi)] = -i \frac{\hbar}{m} \hat{P}_1(\chi),
\]

(66)

where

\[
N(\chi) = i \frac{\hbar}{mc} \frac{\partial}{\partial \chi}.
\]

(67)

The eigenfunctions of the operator \(\hat{P}_0(\chi)\) may be constructed with the help of the transformation of the Lorentz group (2).

6 Appendix B

The functions \(v_n(\rho)\) in (37) are \(\nu = \mu + 1/2\)

\[
v_n(\rho) = c_n \left( \frac{\hbar \omega}{mc^2} \right)^{-i \frac{mc\rho}{\hbar}} \Gamma(\nu - i \frac{mc}{\hbar} \rho) F_1(-n, \nu + i \frac{mc}{\hbar} \rho; 2\nu; 2),
\]

(68)

where

\[
c_n = i^n 2^{\nu} \sqrt{\frac{mc \Gamma(2\nu + n)}{2\pi n! \hbar [\Gamma(2\nu)]^2}}.
\]

(69)

The functions \(f_n(t, x)\) in (37) may be written in the form \(\lambda_n = \mu + 1/2 + n\)

\[
f_n(t, x) = e^{-i \frac{Et}{\hbar}} (1 + \frac{\omega^2 x^2}{c^2})^{-\frac{\lambda_n}{2}} \left[ \frac{\Gamma(2\mu + 1) \Gamma(\mu + \frac{1}{2})}{n! \Gamma(2\mu + 1 + n) \sqrt{n! \hbar} \Gamma(\mu)} \varrho_n(x) \right].
\]

(70)

Here, for the polynomials \(\varrho_n\) we have the relations \(l = c/\omega, y = x/l, \ varrho_0(y) = 1, n = 1, 2, 3...\)

\[
\varrho_n(y) = [2\lambda_{n-1}y - (1 + y^2) \partial_y] \varrho_{n-1}(y).
\]

(71)

For \(n = 1, 2, 3\) we have

\[
\varrho_1(y) = 2\lambda_0 y, \quad \varrho_2(y) = 2\lambda_0 [(2\lambda_0 + 1)y^2 - 1],
\]

(72)

\[
\varrho_3(y) = 4\lambda_0(\lambda_0 + 1)y[(2\lambda_0 + 1)y^2 - 3].
\]

(73)
References

[1] M. Moshinsky, The Harmonic Oscillator in Modern Physics: from Atoms to Quarks (New-York: Gordon and Breach), 1969.

[2] R. P. Feynman, M. Kislinger, and F. Ravndal, Phys. Rev. D 3, 2706 (1971).

[3] I. S. Shapiro, Sov. Phys. Doklady. 1, 91 (1956).

[4] V. G. Kadyshevsky, R. M. Mir-Kasimov and N. B. Skachkov, Nuovo Cimento 55, 233 (1968); Sov. J. Part. Nucl. 2, 69 (1973).

[5] Y. S. Kim and M. E. Noz, Amer. J. Phys. 46, 480 (1978); D. Han. Y. S. Kim and M. E. Noz, Phys. Rev. A, 41, 6233 (1990).

[6] A. D. Donkov, V. G. Kadyshevsky, M. D. Mateev, and R. M. Mir-Kasimov, Theor. Math. Phys., 8, 673 (1971).

[7] N. M. Atakishiev, R. M. Mir-Kasimov and Sh. M. Nagiev, Theor. Math. Phys., 44, 592 (1981)

[8] N. M. Atakishiev, Theor. Math. Phys., 58, 166 (1984).

[9] M. Moshinsky and Szczepaniak, J, Phys. A, 22, L817 (1989).

[10] V. Aldaya, J. A. de Azcarraga, J. Bisquert and J. M. Cervero, J. Phys. A 23 (1990) 707; V. Aldaya, J. Bisquert, R. Loll, and J. Navarro-Salas, J. Math. Phys., 33, 3087 (1992).

[11] V. Aldaya, J. Bisquert, and J. Navarro-Salas, Phys. Lett. A 156, 381 (1991).

[12] D. J. Navarro and J. Navarro-Salas, FTUV/94.27, IFIC/94-24; hep-th/9406001

[13] R. A. Frick, Sov. J. Nucl. Phys. 38, 481 (1983); JETP Lett. 39, 89 (1984); J. Math. Phys. 38, 3457 (1997); Eur. Phys. J. C 22, 581 (2001); Eur. Phys. J. C 28, 431 (2003).

[14] R. A. Frick, Annalen Phys. 17: 937, (2008).
[15] S. M. Nagiyev, E. I. Jafarov, R. M. Imanov, L. Homorodean, Phys. Lett. A334 260-266 (2005).

[16] O. Klein, Z. Phys, 1929, Bd 58, S. 730.

[17] A. O. Barut, and C. Fronsdal, Proc. Roy. Soc., A 287, 532 (1965).