A NOTE ON HURWITZ SCHEMES OF COVERS OF A POSITIVE GENUS CURVE

JOE HARRIS, TOM GRABER, AND JASON STARR

Abstract. Let $B$ be a smooth, connected, projective complex curve of genus $h$. For $w \geq 2d$ we prove the irreducibility of the Hurwitz stack $\mathcal{H}^d,w_{S_d}(B)$ parametrizing degree $d$ covers of $B$ simply-branched over $w$ points, and with monodromy group $S_d$.

1. Introduction

Suppose that $B$ is a smooth, connected, projective complex curve of genus $h$. Let $d>0$ and $w \geq 0$ be integers such that $g := d(h-1) + \frac{w}{2} + 1$ is a nonnegative integer (in particular $w$ is even). We define $\mathcal{H}^d,w(B)$ to be the open substack of the Kontsevich moduli stack $\overline{M}_{g,0}(B,d)$ parametrizing stable maps $f : X \to B$ such that $X$ is smooth and $f$ is finite with only simple branching. Let $\text{br}(f) \subset B$ denote the branch divisor of $f$. If we choose a basepoint $b_0 \in B - \text{br}(f)$ and an identification $\phi : f^{-1}(b_0) \to \{1, \ldots, d\}$, there is an induced monodromy homomorphism $\tilde{\phi} : \pi_1(B - \text{br}(f), b_0) \to S_d$ which associates to any loop $\gamma : [0,1] \to B$ with $\gamma(0) = \gamma(1) = b_0$, the permutation of $f^{-1}(b_0)$ determined by analytic continuation along $\gamma$. In particular, the subgroup image($\tilde{\phi}$) $\subset S_d$ is well-defined up to conjugation independently of $\phi$. The corresponding conjugacy class of subgroups determines a locally constant function on $\mathcal{H}^d,w(B)$. Given a subgroup $G \subset S_d$, we define $\mathcal{H}^d,w_G(B)$ to be the open and closed substack of $\mathcal{H}^d,w(B)$ parametrizing stable maps $f : X \to B$ whose corresponding monodromy group is conjugate to $G$. We are particularly interested in $\mathcal{H}^d,w_{S_d}(B)$, the stack parametrizing Hurwitz covers of $B$ with full monodromy group.

Theorem 1.1. If $w \geq 2d$, then $\mathcal{H}^d,w_{S_d}(B)$ is a connected, smooth, finite-type Deligne-Mumford stack over $\mathbb{C}$.

The fact that $\mathcal{H}^d,w_{S_d}(B)$ is a finite-type Deligne-Mumford stack follows from the fact that $\overline{M}_{g,0}(B,d)$ is a finite-type Deligne-Mumford stack. The fact that $\mathcal{H}^d,w_{S_d}(B)$ is smooth follows from a trivial deformation theory computation. So the content of theorem 1.1 is that $\mathcal{H}^d,w_{S_d}(B)$ is connected.

This is a classical fact when $h = 0$, i.e. for branched covers of $\mathbb{P}^1$, (c.f. [1], [2], and for a modern account [3, prop. 1.5]). This fact is well-known to experts, but there seems to be no reference. We used theorem 1.1 in our paper [4], and so we present a proof below. We wish to thank Ravi Vakil for useful discussions.

Date: November 17, 2018.
2. Setup

Our eventual goal is to prove theorem \[1\] but for most of this paper, we shall work with schemes which admit \'{e}tale maps to \(H_{S_d}^{d,w}(B)\). Suppose \(\Sigma \subset B\) is a finite subset, and suppose \(b_0 \in \Sigma\) is a point. We define \(M^{d,w}(B, \Sigma, b_0)\) to be the fine moduli scheme parametrizing pairs \((f : X \to B, \phi)\) where \(f : X \to B\) is a stable map in \(\mathcal{M}_{g,0}(B, d)\) and where \(\phi : f^{-1}(b_0) \to \{1, \ldots, d\}\) are such that

1. \(f\) is finite,
2. \(f\) is unramified over \(\Sigma\), and
3. \(\phi\) is a bijection.

Using known results on the Kontsevich moduli space \(\mathcal{M}_{g,0}(B, d)\), it is easy to show that \(M^{d,w}(B, \Sigma, b_0)\) is a nonempty, smooth, quasi-projective scheme of dimension \(w\). By \([2]\), there is a branch morphism \(br : M^{d,w}(B, \Sigma, b_0) \to (B - \Sigma)_w\) where \((B - \Sigma)_w\) is the \(\Sigma\)-fold symmetric power parametrizing effective degree \(w\) divisors on \(B - \Sigma\). It is clear that \(br\) is quasi-finite, and thus \(br : M^{d,w}(B, \Sigma, b_0) \to (B - \Sigma)_w\) is dominant. We denote by \((B - \Sigma)_{w}^0 \subset (B - \Sigma)_w\) the Zariski open subset parametrizing reduced effective divisors of degree \(w\) in \(B - \Sigma\). We define \(H^{d,w}(B, \Sigma, b_0) \subset M^{d,w}(B, \Sigma, b_0)\) to be the preimage under \(br\) of \((B - \Sigma)_{w}^0\).

For each pair \((f : X \to B, \phi)\) in \(H^{d,w}(B, \Sigma, b_0)\) with branch divisor \(br(f)\), there is an induced monodromy homomorphism \(\tilde{\phi} : \pi_1(B - br(f), b_0) \to S_d\) where \(S_d\) is the symmetric group of permutations of \(\{1, \ldots, d\}\). The image of \(\tilde{\phi}\) determines a locally constant function on \(H^{d,w}(B, \Sigma, b_0)\). Because \(H^{d,w}(B, \Sigma, b_0)\) is smooth and \(H^{d,w}(B, \Sigma, b_0)\) is dense in \(M^{d,w}(B, \Sigma, b_0)\), this locally constant function extends to all of \(M^{d,w}(B, \Sigma, b_0)\). Given a subgroup \(G \subset S_d\) we define \(M_G^{d,w}(B, \Sigma, b_0)\) (resp. \(H_G^{d,w}(B, \Sigma, b_0)\)) to be the open and closed subscheme of \(M^{d,w}(B, \Sigma, b_0)\) (resp. \(H^{d,w}(B, \Sigma, b_0)\)) on which the image of \(\tilde{\phi}\) equals \(G\).

Let \(F : \mathcal{X}(w) \to H_{S_d}^{d,w}(B, \Sigma, b_0) \times B\) be the pullback of the universal stable map, i.e. \(\mathcal{X}(w)\) parametrizes data \((f : X \to B, \phi, x)\) where \((f : X \to B, \phi) \in H_{S_d}^{d,w}(B, \Sigma, b_0)\) and \(x \in X\), and \(F(f : X \to B, \phi, x) = (f : X \to B, \phi, f(x))\). We denote by \(U \subset \mathcal{X}(w) \times_{F,F} \mathcal{X}(w)\) the open subscheme of the fiber product of \(\mathcal{X}(w)\) with itself over \(H_{S_d}^{d,w}(B, \Sigma, b_0) \times B\) parametrizing data \((f : X \to B, \phi, x_1, x_2)\) such that \(x_1 \neq x_2\), and such that \(f(x_1) = f(x_2)\) is neither in \(S\) nor equal to any branch point of \(f\). We define \(\mathcal{X}_2(w)\) to be the quotient of \(U\) by the obvious involution \((f : X \to B, \phi, x_1, x_2) \sim (f : X \to B, \phi, x_2, x_1)\). We denote by \(\mathcal{X}_e^2(w)\) the open subscheme of the \(e\)-fold fiber product of \(\mathcal{X}_2(w)\) with itself over \(H_{S_d}^{d,w}(B, \Sigma, b_0)\) parametrizing data \((f : X \to B, \phi, \{x_1^1, x_2^1\}, \ldots, \{x_1^e, x_2^e\})\) such that \(f(x_1^1), \ldots, f(x_2^e)\) are all distinct points in \(B - \Sigma\). Notice that the projection \(\mathcal{X}_2^e(w) \to H_{S_d}^{d,w}(B, \Sigma, b_0)\) is flat. The condition that the image of \(\tilde{\phi}\) be all of \(S_d\), and therefore doubly-transitive, implies that \(\mathcal{X}_2(w) \to H_{S_d}^{d,w}(B, \Sigma, b_0)\) has irreducible fibers. Therefore also \(\mathcal{X}_e^2(w) \to H_{S_d}^{d,w}(B, \Sigma, b_0)\) has irreducible fibers.

For each \((f : X \to B, \phi, \{x_1^1, x_2^1\}, \ldots, \{x_1^e, x_2^e\})\) in \(\mathcal{X}_2^e(w)\) we can associate a pair \((f_a : X_a \to B, \phi_a)\) in \(H_{S_d}^{d,w+2e}(B, \Sigma, b_0)\) as follows:
By the Riemann existence theorem, for each $t \in \Delta$ there is a pair $(f_t : X_t \rightarrow B, \phi_t)$.

1. We define $X_a$ to be the $\epsilon$-nodal curve whose normalization is of the form $u : X \rightarrow X_a$ such that $u(x_i^i) = u(x_j^j)$ for each $i = 1, \ldots, e$,

2. we define $f_a : X_a \rightarrow B$ to be the unique morphism such that $f = f_a \circ u$, and

3. we define $\phi_a$ to be the unique map such that $\phi = \phi_a \circ u$.

This association defines a regular morphism $G_{w,e} : \mathcal{X}^w_\Sigma(w) \rightarrow M^{d,w+2e}_S(B, \Sigma, b_0)$.

We give a topological application of the morphism $G_{w,e}$. Suppose we have a pair $(f : X \rightarrow B, \phi)$ in $H^{d,w+2}_S(B, \Sigma, b_0)$ and $D \subset B$ is a closed disk which is disjoint from $\Sigma$, such that $D \cap \text{br}(f)$ consists of two branch points $b_1, b_2$ which are contained in the interior of $D$. Define $U = B - D$ and suppose that $\phi : \pi_1(U - \text{br}(f), b_0) \rightarrow S_\Sigma$ is surjective. Suppose moreover that $f$ is trivial over the boundary $\partial D$ of $D$, i.e. $f^{-1}(\partial D)$ consists of $d$ disjoint circles each of which maps homeomorphically to $\partial D$. Choose simple closed loops $\gamma_1, \gamma_1$ around $b_1$ and $b_2$ as displayed in Figure 1. Then $\tilde{\phi}(\gamma_1)$ and $\tilde{\phi}(\gamma_2)$ both equal the same transposition $\tau = (j, k)$.

**Lemma 2.1.** With the notations and assumptions in the last paragraph, there exists a pair $(f_a : X_a \rightarrow B, \phi_a)$ in $H^{d,w}_S(B, \Sigma, b_0)$, a datum $(f_a : X_a \rightarrow B, \phi_a, \{x_j, x_k\})$ in $\mathcal{X}_\Sigma(w)$, and an analytic isomorphism $h : f^{-1}(U) \rightarrow f_a^{-1}(U)$ such that

1. $f|_{f^{-1}(U)} = (f_a)|_{f_a^{-1}(U)} \circ h$ and $\phi = \phi_a \circ h$, and

2. the image by $G_{w,e}$ of $(f_a : X_a \rightarrow B, \phi_a, \{x_j, x_k\})$ lies in the same connected component of $M^{d,w+2}_S(B, \Sigma, b_0)$ as $(f : X \rightarrow B, \phi)$.

**Proof.** We may choose an analytic isomorphism of the disk $D \subset B$ with the unit disk $\Delta \subset \mathbb{C}$ such that $b_1$ and $b_2$ map to the two roots of $x^2 = t_0$ for some $t_0 \in \Delta - \{0\}$. Let $x$ be the coordinate on $\Delta$. Consider the map $f^{-1}(D) \rightarrow D$. For each $i \neq j, k$ the connected component of $f^{-1}(D)$ corresponding to $i$ maps isomorphically to $D$. The connected component of $f^{-1}(D)$ corresponding to $j$ and $k$ is identified with the covering $C_{t_0}$ of $\Delta$ given by $C_{t_0} = \{(x, y) \in \mathbb{C}^2 : x \in \Delta, y^2 - (x^2 - t_0) = 0\}$. For $t \in \Delta$, consider the family of covers $C_t = \{(x, y) \in \mathbb{C}^2 : x \in \Delta, y^2 - (x^2 - t) = 0\}$. By the Riemann existence theorem, for each $t \in \Delta$ there is a pair $(f_t : X_t \rightarrow B, \phi_t)$.
in $M^{d,w+2}(B,\Sigma,b_0)$ such that the restriction of $f_t$ to $f_t^{-1}(U)$ is identified with the restriction of $f$ to $f^{-1}(U)$ and such that the restriction of $f_t$ to $D$ consists of $d-2$ copies of $D$ mapping isomorphically to $D$ (one copy for each $i \neq j,k$), and the connected component corresponding to $j$ and $k$ is identified with $C_t \to \Delta$. We will see that $(f_0 : X_0 \to B,\phi_0)$ is in the image of $G_{w,1} : \mathcal{X}_2(w) \to M^{d,w+2}(B,\Sigma,b_0)$.

Define $u : X_a \to X_0$ to be the normalization and define $\{x_j, x_k\}$ to be the preimage of the node $x_0 \in X_0$. We define $f_a : X_a \to B$ to be $f_a = f_0 \circ u$ and $\phi_a = \phi_0 \circ u$. Notice that $f_a : X_a \to B$ is unbranched over $D$. Define $x_j$ (resp. $x_k$) to be the preimage of $f_0(x_0)$ on the sheet of $f_0^{-1}(D)$ corresponding to $j \in \{1, \ldots, d\}$ (resp. to $k \in \{1, \ldots, d\}$). We have an identification of $u^{-1}(f_0^{-1}(U))$ with $f_0^{-1}(U)$. Therefore we have an identification $h : f^{-1}(U) \to f_a^{-1}(U)$ commuting with $f, f_a$ and with $\phi, \phi_a$. In particular, we conclude that $\phi_a : \gamma_1(U - \text{br}(f), b_0) \to S_d$ is identified with $\phi : \pi_1(U - \text{br}(f), b_0) \to S_d$ and so is surjective. So $(f_a : X_a \to B, \phi_a)$ is in $H_{S_d}^{d,w}(B,\Sigma,b_0)$. Clearly $(f_a : X_a \to B, \phi_a, \{x_j, x_k\})$ is in $\mathcal{X}_2(w)$ and, by construction, its image under $G_{w,1}$ is $(f_0 : X_0 \to B,\phi_0)$. Since $(f_0 : X_0 \to B,\phi_0)$ is in the same connected component of $H_{S_d}^{d,w+2}(B,\Sigma,b_0)$ as $(f : X \to B,\phi)$, this proves the lemma.

Lemma 2.2. With the notations and assumptions in lemma 2.1, suppose given a transposition $(j_h,k_b) \in S_d$. Then there exists a pair $(f_b : X_b \to B,\phi_b)$ in $H_{S_d}^{d,w+2}(B,\Sigma,b_0)$, and an analytic isomorphism $h : f^{-1}(U) \to f_b^{-1}(U)$ such that:

1. $\text{br}(f_b) = \text{br}(f)$,
2. $f|_{f^{-1}(U)} = f_b|_{f_b^{-1}(U)} \circ h$ and $\phi = \phi_b \circ h$,
3. $\phi_b(\gamma_1) = \phi_b(\gamma_2) = (j_h, k_b)$, and
4. $(f_b : X_b \to B, \phi_b)$ is in the same connected component of $H_{S_d}^{d,w+1}(B,\Sigma,b_0)$ as $(f : X \to B,\phi)$.

Proof. Let $(f_a : X_a \to B, \phi_a, \{x_j, x_k\})$, $h_a : f^{-1}(U) \to f_a^{-1}(U)$ be as constructed in the proof of lemma 2.1. Define $b_t$ to be $f_a(x_j) = f_0(x_k)$. Define $(X_a)_2 : B \to (\Sigma \cup \text{br}(f))$ to be the fiber of $\mathcal{X}_2(w) \to H_{S_d}^{d,w}(B,\Sigma,b_0)$ over $(f_a : X_a \to B,\phi_a)$. Notice that $(X_a)_2 : B \to (\Sigma \cup \text{br}(f))$ is an unbranched covering space. Define $x_{j_h}$ (resp. $x_{k_b}$) to be the elements of $f_0^{-1}(b_t)$ which lie on the sheets of $f_0^{-1}(D)$ corresponding to $j_h \in \{1, \ldots, d\}$ (resp. $k_b \in \{1, \ldots, d\}$). Because $\phi_a : \pi_1(B - \text{br}(f), b_t) \to S_d$ is surjective, in particular it is doubly transitive. Therefore $(X_a)_2$ is irreducible and $(f_a : X_a \to B, \phi_a, \{x_{j_h}, x_{k_b}\})$ is in the same connected component of $\mathcal{X}_2(w)$ as $(f_a : X_a \to B, \phi_a, \{x_j, x_k\})$. So the image $G_{w,1}(f_a : X_a \to B, \phi_a, \{x_{j_h}, x_{k_b}\})$ is in the same connected component of $M_{S_d}^{d,w+2}(B,\Sigma,b_0)$ as $(f : X \to B,\phi)$.

Consider the same family of covers $C_t \to \Delta$ as in the proof of lemma 2.1, with the roles of $j,k$ replaced by $j_h,k_b$. By the Riemann existence theorem there exists a pair $(f_b : X_b \to B,\phi_b)$ and an isomorphism $h_b : f_b^{-1}(U) \to f_b^{-1}(U)$ commuting with $f_a, f_b$ and $\phi_a, \phi_b$ such that $f_b^{-1}(D) \to D$ is identified with the covering $C_1 \to \Delta$. We define $h : f^{-1}(U) \to f_b^{-1}(U)$ to be $h_b \circ h_a$. Then $(f_b : X_b \to B,\phi_b)$ and $h$ satisfy items (1), (2), and (3) of the lemma. Moreover $(f_b : X_b \to B,\phi_b)$ is in the same connected component of $M_{S_d}^{d,w+2}(B,\Sigma,b_0)$.
as $G_{w,2}(f_a : X_a \to B, \phi_a, \{x_{j_k}, x_{k_l}\})$. Thus $(f_b : X_b \to B, \phi_b)$ is in the same connected component of $M_{S_d}^{d,w+2}(B, \Sigma, b_0)$ as $(f : X \to B, \phi)$. Since both pairs are in $H_{S_d}^{d,w+2}(B, \Sigma, b_0)$ and since $M_{S_d}^{d,w+2}(B, \Sigma, b_0)$ is smooth, we conclude that both pairs are in the same connected component of $H_{S_d}^{d,w+2}(B, \Sigma, b_0)$.

3. Branching monodromy

Fix a closed disk $D \subset B$ disjoint from $\Sigma$. Fix a path from $b_0$ to the boundary $\partial D$. Denote by $U$ the open subset $B - D \subset B$. In most of this section we will restrict our attention to the analytic open subset $V \subset M_{S_d}^{d,w}(B, \Sigma, b_0)$ parametrizing $(f : X \to B, \phi)$ such that $\br(f)$ is contained in the interior of $D$. By assumption, the monodromy group of $f$, i.e. the image of $\tilde{\phi}$, is all of $S_d$. For each connected component of $H_{S_d}^{d,w}(B, \Sigma, b_0) \cap V$, the function which associates to each $(f : X \to B, \phi)$ the image of $\tilde{\phi} : \pi_1(D - \br(f), b_0) \to S_d$ is constant. We call this subgroup the branching monodromy group of $f$ (of course it depends on the choice of $D$ and the path from $b_0$ to $\partial D$).

Since $H_{S_d}^{d,w}(B, \Sigma, b_0) \cap V$ is the complement of proper analytic subvarieties of the complex manifold $V$, each connected component of $V$ is the closure of a unique connected component of $H_{S_d}^{d,w}(B, \Sigma, b_0) \cap V$. For each subgroup $G \subset D$ let us denote by $V_G \subset V$ the open and closed submanifold on which the image of $\tilde{\phi} : \pi_1(D - \br(f), b_0) \to S_d$ equals $G$. The goal of this section is to prove that when $w \geq 2d$, every connected component of $H_{S_d}^{d,w}(B, \Sigma, b_0)$ has nonempty intersection with $V_{S_d}$, i.e. there is a pair $(f : X \to B, \phi)$ in this connected component and in $V$ which has branching monodromy group equal to $S_d$.

Suppose that $(f : X \to B, \phi) \in H_{S_d}^{d,w}(B, \Sigma, b_0) \cap V_G$. Because $G$ is generated by transpositions, there is a partition $(A_1, \ldots, A_r)$ of $\{1, \ldots, d\}$ such that $G = S_{A_1} \times \cdots \times S_{A_r}$, where $S_{A_m} \subset S_d$ consists of those permutations which act as the identity on each subset $A_n \subset \{1, \ldots, d\}$ for which $n \neq m$. In other words, $G$ is the subgroup of permutations which stabilize each subset $A_m \subset \{1, \ldots, d\}$.

Choose a system of loops $\gamma_1, \ldots, \gamma_w$ as in Figure 3. Denote by $\tau_i$ the transposition $\tilde{\phi}(\gamma_i)$. Then each $\tau_i$ lies in one of the subgroups $S_{A_m(i)}$.

Suppose that $\gamma_i$ and $\gamma_{i+1}$ are adjacent loops such that $\tau_i$ lies in $S_{A_m}$ and $\tau_{i+1}$ lies in $S_{A_n}$ with $m \neq n$. Consider the element $\sigma_i$ of the braid group which interchanges the branch points $b_i$ and $b_{i+1}$ as shown in Figure 3. The result is to replace $\tau_i$ by $\tau_{i+1}$ and to replace $\tau_{i+1}$ by $\tau_i \tau_{i+1} \tau_i$. Since $A_m$ and $A_n$ are disjoint, we have $\tau_i \tau_{i+1} \tau_i = \tau_i$. In other words, the result is to interchange $\tau_i$ and $\tau_{i+1}$. Note that this operation does not change $G \subset S_d$. By repeating this process, we may arrange that there are integers $w_0 = 0, w_1, \ldots, w_r$ with the following property: for $m = 1, \ldots, r$, denote $v_m = w_0 + \cdots + w_{m-1}$; then for each $m = 1, \ldots, r$, each
Figure 2. Branch points contained in the disk $D$

Figure 3. Braid move exchanging two branch points
transpositions $\tau_i$ with $v_m + 1 \leq i \leq v_{m+1}$ is in $S_{A_m}$. Notice that since these transpositions generate $S_{A_m}$, we have $w_m \geq \#A_m - 1$. Stated more precisely, we have proved that given a pair $(f : X \to B, \phi)$ in $H^{d,w}_S(B, \Sigma, b_0) \cap V_G$, in the same connected component of $H^{d,w}_S(B, \Sigma, b_0) \cap V_G$ there is a pair $(f_a : X_a \to B, \phi_a)$ and an isomorphism $h : f^{-1}(U) \to (f_a)^{-1}(U)$ such that:

1. $\text{br}(f_a) = \text{br}(f)$,
2. $f|_{f^{-1}(U)} = (f_a)|_{f_a^{-1}(U)} \circ h$ and $\phi = \phi_a \circ h$, and
3. the transpositions $\tau_i = \tilde{\psi}(\gamma_i)$ satisfy $\gamma_i \in S_{A_m}$ for $v_m + 1 \leq i \leq v_{m+1}$.

We say that a pair $(f_a : X_a \to B, \phi_a)$ satisfying item (3) is in standard position. For each $m = 1, \ldots, r$, choose a subdisk $D_m \subset D$ as in Figure 4 which contains the loops $\gamma_i$ for $(w_0 + \cdots + w_{m-1}) + 1 \leq i \leq w_0 + \cdots + w_{m-1} + w_m$. Note that any braid move in $D_m$ has no effect on the branch points belonging to $D_n$ with $n \neq m$.

**Proposition 3.1.** Suppose that $(f : X \to B, \phi)$ in $H^{d,w}_S(B, \Sigma, b_0) \cap V_G$ is in standard position. Suppose $w_m \geq 2\#A_m$. Then there are braid moves in $D_m$ transforming $(\tau_{v_m+1}, \ldots, \tau_{v_m+w_m})$ into $(\tau'_1, \ldots, \tau'_{w_m-2}, \tau, \tau)$ such that $\tau'_1, \ldots, \tau'_{w_m-2}$ generate $S_{A_m}$.
Proof. Define $g = \tau_{m+1} \cdots \tau_{m+1}$. By [3, theorem 1], the braid group of $D_m$ acts transitively on the set

$$O_g := \{(\tau_1, \ldots, \tau_{w_m}) \in S_{A_m} \mid \text{each } \tau_i \text{ a transposition} \},$$

$$\langle \tau_1, \ldots, \tau_{w_m} \rangle = S_{A_m}, \tau_1 \cdots \tau_{w_m} = g \}.$$

Thus it suffices to find $(\tau'_1, \ldots, \tau'_{w_m-2}, \tau, \tau)$ as above which lies in $O_g$.

Suppose that $g$ has cycle type $(\lambda_1, \ldots, \lambda_s)$ for some partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_s)$ of $\#A_m$. Define $\lambda_0 = 0$ and for each $k = 1, \ldots, s$, define $\mu_k = \lambda_0 + \cdots + \lambda_{k-1}$. Then we may order the elements of $A_m$ so that $g$ is the permutation

$$g = (\mu_1 + 1, \mu_1 + 1, \mu_2 + 1, \ldots, \mu_2 + 1, \mu_3 + 1, \ldots, \mu_s + 1, \mu_s + 1, \mu_s + \lambda_2). \quad (1)$$

Of course this ordering has nothing to do with the ordering induced by $\phi$.

For each $k = 1, \ldots, s$, consider the ordered sequence of transpositions, which is defined to be empty if $\lambda_k = 1$, and for $k > 1$ is defined to be

$$I_k = ((\mu_k + 1, \mu_k + 2), (\mu_k + 2, \mu_k + 3), \ldots, (\mu_k + \lambda_k - 1, \mu_k + \lambda_k)). \quad (2)$$

Thus $I_k$ contains $\lambda_k - 1$ transpositions. Next consider the sequence of transpositions

$$I_{s+1} = ((\mu_1, \mu_2), (\mu_1, \mu_2), \ldots, (\mu_3, \mu_3), \ldots, (\mu_{s-1}, \mu_{s-1}), (\mu_s, \mu_s)). \quad (3)$$

The concatenated sequence $I = I_1 \cup \cdots \cup I_s \cup I_{s+1}$ has length $L := \sum_k (\lambda_k - 1) + 2(s - 1) = \#A_m + s - 2 \leq 2\#A_m - 2$. The product of these transpositions is $g$, and these transpositions generate $S_{A_m}$.

Since the sign of $g$ is both $(-1)^{w_m}$ and $(-1)^L$, we have that $w_m - L$ is divisible by 2. And the assumption that $w_m \geq 2\#A_m$, implies that $w_m - L \geq 2$. If we choose any transposition $\tau \in S_{A_m}$, and let $J$ be the constant sequence of length $w_m - L$, $J = (\tau, \tau, \ldots, \tau)$, then we have that the concatenated sequence $I \cup J$ is an element of $O_g$ satisfying the hypotheses of the proposition.

Corollary 3.2. Given $w' \geq w$ with $w' \geq 2d, \ w \geq 2d - 2$, set $e = \frac{w' - w}{2}$. Suppose given a pair $(f : X \to B, \phi) \in H^{\omega}_{S_d}(B, \Sigma, b_0) \cap V_G$. Then there is a pair $(f_a : X_a \to B, \phi_a) \in H^{\omega}_{S_d}(B, \Sigma, b_0) \cap V_G$, an isomorphism $h : f^{-1}(U) \to (f_a)^{-1}(U)$ and a datum $(f_a : X_a \to B, \phi_a, \{x^e_1, x^e_2\})$ in $X_2^e(w)$ such that

1. $f_a(x^e_i) \in D$ for $i = 1, \ldots, e$ and for $j = 1, 2$,
2. $f|_{f^{-1}(U)} = (f_a)|_{f_a^{-1}(U)} \circ h$ and $\phi = \psi \circ h$, and
3. the image of $(f_a : X_a \to B, \phi_a, \{x^e_1, x^e_2\})$ under $G_{w,e}$ is contained in the same connected component of $H^{\omega}_{S_d}(B, \Sigma, b_0)$ as $(f : X \to B, \phi)$.

Proof. We prove this by induction on $w' - w$. For $w' = w$, there is nothing to prove. Suppose $w' - w > 0$ and suppose the proposition has been proved for all smaller values of $w' - w$. We note by proposition [3, 1] that there is a map $(f_c : X_c \to B, \phi_c)$ and $h_c : f^{-1}(U) \to f_c^{-1}(U)$ satisfying the conditions of that proposition. If we define $D' \to$ be a small disk containing the branch points of $f_c$ corresponding to the transposition $\tau$, then $(f_c : X_c \to B, \phi_c)$ and $D'$ satisfy the hypothesis of lemma [2, 1]. By that lemma, there is a datum $(f_b : X_b \to B, \phi_b, \{x_j, x_k\}) \in X_2(w' - 2)$ and an isomorphism $h_b : f_a^{-1}(B - D') \to f_b^{-1}(B - D')$ satisfying the conditions of that lemma.
If \( w' = w + 2 \), we are done by taking \((f_a : X_a \to B, \phi_a, \{x_1^a, x_2^a\}) = (f_b : X_b \to B, \phi_b, \{x_j, x_k\})\) and taking \( h = h_b \circ h_c \). Therefore suppose that \( w' > w + 2 \). Now \((f_b : X_b \to B, \phi_b)\) is in \( H^d_{S_d}(B, \Sigma, b_0) \cap V_G \). Since \((w' - 2) - w < w' - w \), by the induction assumption there exists a datum \((f_a : X_a \to B, \phi_a, \{x_1^a, x_2^a, \ldots, \{x_1^{e-1}, x_2^{e-1}\}\})\) in \( X_2^{e-1}(w) \) and \( h_a : f_a^{-1}(U) \to f_a^{-1}(U) \) satisfying the conditions of our corollary. Up to deforming this datum slightly, we may suppose that the isomorphism \( h \) satisfying the conditions of our corollary. Up to deforming this datum slightly, we may suppose that the isomorphism \( h_a \) extends to a larger open set which contains \( x_j, x_k \in X_b \), and, defining \( x_1' = h_a^{-1}(x_j) \) and \( x_2' = h_a^{-1}(x_k) \), the datum \((f_a : X_a \to B, \phi_a, \{x_1^a, x_2^a, \ldots, \{x_1^{e-1}, x_2^{e-1}\}\})\) in \( X_2^{e-1}(w) \). We define \( h = h_a \circ h_b \circ h_c \). The image of this datum under \( G_{w,c} \) is contained in the same connected component as the image of \((f_b : X_b \to B, \phi_b, \{x_j, x_k\})\) under \( G_{w'-2,1} \). So the corollary is proved by induction.

**Corollary 3.3.** If \( w \geq 2d \), then for any pair \((f : X \to B, \phi) \in H^d_{S_d}(B, \Sigma, b_0) \) in \( V \), there is a pair \((f_a : X_a \to B, \phi_a) \in H^d_{S_d}(B, \Sigma, b_0) \cap V_{S_d} \) and an isomorphism \( h : f^{-1}(U) \to (f_a)^{-1}(U) \) such that

1. \( f\big|_{f^{-1}(U)} = (f_a\big|_{f_a^{-1}(U)}) \circ h \) and \( \phi = \phi_a \circ h \), and
2. \((f_a : X_a \to B, \phi_a)\) is in the same connected component of \( H^d_{S_d}(B, \Sigma, b_0) \) as \((f : X \to B, \phi)\).

**Proof.** By corollary 2.2, it suffices to consider the case that \( w = 2d \). Suppose the branching monodromy group of \((f : X \to B, \phi)\) is \( G = S_{A_1} \times \cdots \times S_{A_r} \). We will prove the result by induction on \( r \). If \( r = 1 \), there is nothing to prove. So assume that \( r > 1 \) and assume the result is proved for all smaller values of \( r \).

Since \( \sum m(w_m - 2\#A_m) \) equals \( w - 2d = 0 \), there is some \( m \) such that \( w_m \geq 2\#A_m \). Without loss of generality, suppose \( w_1 \geq 2\#A_1 \). By proposition 2.3, we may suppose that the transpositions in \( D_1 \) are of the form \((\tau_1, \ldots, \tau_{w_1 - 2}, \gamma, \tau)\) such that \( \tau_1, \ldots, \tau_{w_1 - 2} \) generate \( S_{A_1} \). But then, choosing a small disk \( D' \) which contains only the branch points \( b_{2w_1 - 1} \) and \( b_{2w_1} \), we see that \((f : X \to B, \phi)\) and \( D' \) satisfy the hypothesis of lemma 2.2. Suppose that \( b_1 \in A_1 \) and \( k_1 \in A_2 \). By lemma 2.2, we can find a pair \((f_b : X_b \to B, \phi_b) \in H^d_{S_d}(B, \Sigma, b_0) \) and \( h_b : f_b^{-1}(B - D') \to (f_b)^{-1}(B - D') \) such that

1. \( f_b\big|_{f_b^{-1}(B - D')} = (f_b\big|_{f_b^{-1}(B - D')}) \circ h \) and \( \phi = \phi_b \circ h \),
2. \( \br(f_b) = \br(f) \),
3. the transposition of \((f_b : X_b \to B, \phi_b)\) corresponding to \( \gamma_{w_1 - 1} \) and \( \gamma_{w_1} \) is \((j_b, k_b)\), and
4. \((f_b : X_b \to B, \phi_b)\) is in the same connected component of \( H^d_{S_d}(B, \Sigma, b_0) \) as \((f : X \to B, \phi)\).

Since the branching monodromy group of \((f_b : X_b \to B, \phi_b) \in H^d_{S_d}(B, \Sigma, b_0) \cap V \) outside of \( D' \) already generates \( S_{A_1} \times S_{A_2} \), when we add the transposition \((j_b, k_b)\) we conclude the branching monodromy group of \((f_b : X_b \to B, \phi_b)\) is \( S_{A_1} \times A_2 \times S_{A_3} \times \cdots \times S_{A_r} \). By the induction assumption, there is \((f_a : X_a \to B, \phi_a)\) and \( h_a : f_a^{-1}(U) \to f_a^{-1}(U) \) satisfying the conditions of our corollary where \((f : X \to B, \phi)\) is replaced by \((f_b : X_b \to B, \phi_b)\). Then defining \( h = h_a \circ h_b \), we see that
4. Induction Argument

In this section we will prove that for \( w \geq 2d \), \( H_{d,w}^{d,w}(B, \Sigma, b_0) \) is connected. The basic strategy is as follows: If \( h = g(B) = 0 \), then this is a classical result due to Hurwitz (see the references in the introduction). Suppose given a disk \( D \subset B \) and two pairs \((f_1 : X_1 \to B, \phi_1)\) and \((f_2 : X_2 \to B, \phi_2)\) such that all branch points of \( f_1 \) and \( f_2 \) are contained in \( D \) and such that both \( f_1 \) and \( f_2 \) are trivial over \( B - D \), i.e. \( f_i^{-1}(B - D) \to B - D \) is just \( d \) isomorphic copies of \( B - D \) for \( i = 1, 2 \). Then the genus 0 argument shows that \((f_1 : X_1 \to B, \phi_1)\) and \((f_2 : X_2 \to B, \phi_2)\) are contained in the same connected component of \( H_{d,w}^{d,w}(B, \Sigma, b_0) \). So the argument is reduced to proving that given a general pair \((f : X \to B, \phi)\) with branch points in \( D \), we can perform braid moves such that \( f^{-1}(B - D) \to B - D \) is trivial.

Suppose \( g \geq 1 \) and choose a disk \( D \subset B_1 \subset B \) situated as in Figure 5 and which is disjoint from \( S \). Let \( V \) be as in section 3 with respect to this disk \( D \). Every connected component of \( H_{d,w}^{d,w}(B, \Sigma, b_0) \) clearly intersects \( V \). So to prove that \( H_{d,w}^{d,w}(B, \Sigma, b_0) \) is connected, it suffices to prove that for any two pairs \((f_1 : X_1 \to B, \phi_1)\) and \((f_2 : X_2 \to B, \phi_2)\) in \( H_{d,w}^{d,w}(B, \Sigma, b_0) \cap V \) both pairs in the same connected component. We prove this by induction on \( g \) through a sequence of intermediate steps (showing each pair is in the same connected component as a pair with some special properties, and finally linking up the resulting pairs).
Let \( D' \subset B_2 \subset B \) be as in Figure 6. Let \( V' \) be as in section 3 with respect to \( D' \).

We say that \((f : X \to B, \phi) \in H_{d,w}^d(B, \Sigma, b_0) \cap V' \) is \( B_1 \)-trivial if \( f^{-1}(B_1) \to B_1 \) is a trivial cover.

**Proposition 4.1.** Suppose \( w \geq 2d \). Any pair \((f : X \to B, \phi) \in H_{d,w}^d(B, \Sigma, b_0) \cap V \) is in the same connected component as a pair \((f_a : X_a \to B, \phi_a)\) which is \( B_1 \)-trivial.

**Proof.** We prove this in a number of steps. The idea is to apply braid moves to reduce \( \tilde{\phi}(\gamma_1) \) and \( \tilde{\phi}(\gamma_2) \) to the identity. Finally we will move all the branch points out of \( B_1 \) along a specified path to give a \( B_1 \)-trivial pair.

Our first braid move is displayed in Figure 7. It consists of choosing the final branch point \( b_w \), moving \( b_w \) across the loop \( \gamma_1 \), without crossing \( \gamma_2 \), and continuing along the loop “parallel” to \( \gamma_2 \) to return \( b_w \) into \( D \). If the resulting cover is \((f_b : X_b \to B, \phi_b)\), then we clearly have \( \phi_b(\gamma_1) = \tilde{\phi}(\gamma_1) \tilde{\phi}(\gamma) \) and \( \phi_b(\gamma_2) = \tilde{\phi}(\gamma_2) \). So the result is to multiply the permutation of \( \gamma_1 \) by the permutation of \( \gamma \) while leaving the permutation of \( \gamma_2 \) unchanged.

Our second braid move is exactly like our first braid move with the roles of \( \gamma_1 \) and \( \gamma_2 \) switched. It is illustrated in Figure 8. We choose the first branch point \( b_1 \), move \( b_1 \) across the loop \( \gamma_2 \), without crossing \( \gamma_1 \), and then continue along the loop “parallel” to \( \gamma_1 \) we return \( b_1 \) into \( D \). If the resulting cover if \((f_b : X_b \to B, \phi_b)\), then we clearly have \( \phi_b(\gamma_2) = \tilde{\phi}(\gamma_2) \tilde{\phi}(\gamma) \) and \( \phi_b(\gamma_1) = \tilde{\phi}(\gamma_1) \). So the result is to multiply the permutation of \( \gamma_2 \) by the permutation of \( \gamma \) while leaving the permutation of \( \gamma_1 \) unchanged. Notice that in both of these moves, we are not concerned about the effect of the braid move on the branching monodromy of \( D \) (we may always use corollary 3.3 to “repair” the branching monodromy of \( D \)).
The main claim is that these braid moves along with corollary 3.3 suffice to trivialize the permutations of $\gamma_1$ and $\gamma_2$. Suppose given $(f : X \to B, \phi) \in H_{S_d}^{d,w}(B, \Sigma, b_0) \cap V$. Suppose that $\tilde{\phi}(\gamma_1)$ has cycle type $\lambda = (\lambda_1 \geq \cdots \geq \lambda_s)$ and $\tilde{\phi}(\gamma_2)$ has cycle type $\mu = (\mu_1 \geq \cdots \geq \mu_t)$. Define $|\lambda| = \sum_m (\lambda_m - 1) = d - s$ and define $|\mu| = \sum_n (\mu_n - 1) = d - t$. We claim that there is a pair $(f_b : X_b \to B, \phi_b) \in H_{S_d}^{d,w}(B, \Sigma, b_0) \cap V$ such that:
1. \( \tilde{\phi}_b(\gamma_1) = \tilde{\phi}_b(\gamma_2) = 1 \), and
2. \( (f_b : X_b \to B, \phi_b) \) is contained in the same connected component of \( H_{S_d}^{d,w} \) as 
   \( (f : X \to B, \phi) \).

We will prove this by induction on \(|\lambda| + |\mu|\). If \(|\lambda| + |\mu| = 0\), i.e. \( \lambda = \mu = 1^d \),
we may simply take \((f_b : X_b \to B, \phi_b) = (f : X \to B, \phi)\). Therefore suppose that
\(|\lambda| + |\mu| > 0\) and, by way of induction, suppose the result is proved for all smaller
values of \(|\lambda| + |\mu|\). We make one reduction at the outset: by corollary 2.3, we may replace
\((f : X \to B, \phi)\) with a pair which is equivalent over \(B - D\), but whose
branching monodromy group is all of \(S_d\).

Suppose first that \(|\lambda| > 0\). Let \( \sigma \in S_d \) be the \(\lambda_1\)-cycle occurring in \(\hat{\phi}(\gamma_1)\) and
suppose \( \tau \in S_d \) is a transposition such that \(\sigma \tau\) is a \((\lambda_1 - 1)\)-cycle. By proposition 3.1,
we may replace \((f : X \to B, \phi)\) by a pair which is equivalent over \(B - D\), and whose
sequence of transpositions is of the form \((\tau, \tau, \tau, \ldots, \tau_{w-2}, \tau, \tau)\). If we apply our first
braid move, the resulting cover \((f_c : X_c \to B, \phi_c)\) is such that \(|\lambda_c| = |\lambda| - 1\) and
\(|\mu_c| = |\mu|\) so that \(|\lambda_c| + |\mu_c| < |\lambda| + |\mu|\). By the induction assumption applied to
\((f_c : X_c \to B, \phi_c)\), we conclude there exists a pair \((f_b : X_b \to B, \phi_b)\) with \(\hat{\phi}_b(\gamma_1) = \hat{\phi}_b(\gamma_2) = 1\) and which is in the same connected component of \(H_{S_d}^{d,w}(B, \Sigma, b_0)\) as
\((f_c : X_c \to B, \phi_c)\). By construction, \((f_c : X_c \to B, \phi_c)\) is in the same connected
component of \(H_{S_d}^{d,w}(B, \Sigma, b_0)\) as \((f : X \to B, \phi)\). Thus \((f_b : X_b \to B, \phi_b)\) satisfies
conditions (1) and (2) above.

The second possibility is that \(|\lambda| = 0\) but \(|\mu| > 0\). Let \( \sigma \in S_d \) be the \(\mu_1\)-cycle occurring in \(\hat{\phi}(\gamma_2)\) and
suppose \( \tau \in S_d \) is a transposition such that \(\sigma \tau\) is a \((\mu_1 - 1)\)-cycle. By an obvious generalization of proposition 3.1,
we may replace \((f : X \to B, \phi)\) by a pair which is equivalent over \(B - D\) and whose
sequence of transpositions is of the form \((\tau, \tau, \tau_1, \ldots, \tau_{w-2})\). If we apply our second braid
move, the resulting cover \((f_c : X_c \to B, \phi_c)\) is such that \(|\lambda_c| = |\lambda| = 0\) and
\(|\mu_c| = |\mu| - 1\) so that \(|\lambda_c| + |\mu_c| < |\lambda| + |\mu|\). By the induction assumption applied to
\((f_c : X_c \to B, \phi_c)\), we conclude there exists a pair \((f_b : X_b \to B, \phi_b)\) with \(\hat{\phi}_b(\gamma_1) = \hat{\phi}_b(\gamma_2) = 1\) and which is in the same connected component of \(H_{S_d}^{d,w}(B, \Sigma, b_0)\) as
\((f_c : X_c \to B, \phi_c)\). By construction, \((f_c : X_c \to B, \phi_c)\) is in the same connected
component of \(H_{S_d}^{d,w}(B, \Sigma, b_0)\) as \((f : X \to B, \phi)\). Thus \((f_b : X_b \to B, \phi_b)\) satisfies
conditions (1) and (2) above. So in both the first and second case, we conclude
that the claim is true for \((f : X \to B, \phi)\). So the claim is proved by induction.

Now we prove the proposition. By the claim, we may suppose that \((f : X \to B, \phi)\) is such that
\(\tilde{\phi}(\gamma_1) = \tilde{\phi}(\gamma_2) = 1\). Finally we move the disk \(D\) and all its
branch points out of \(B_1\) to \(D'\) as shown in Figure 2. Let \((f_a : X_a \to B, \phi_a)\) be the
resulting pair. Notice that since the path of \(D\) never crosses \(\gamma_1\) or \(\gamma_2\), we still have
\(\hat{\phi}_a(\gamma_1) = \hat{\phi}_a(\gamma_2) = 1\). As the fundamental group \(\pi_1(B_1, b_0)\) is generated by \(\gamma_1\) and
\(\gamma_2\), we conclude that \((f_a : X_a \to B, \phi_a)\) is trivial over \(B_1\). Thus \((f_a : X_a \to B, \phi_a)\)
is \(B_1\)-trivial, and the proposition is proved.

Now we are ready to prove the theorem.

**Theorem 4.2.** If \(w \geq 2d\), then \(H_{S_d}^{d,w}(B, \Sigma, b_0)\) is connected.
Proof. The proof is by induction on the genus $h$ of $B$. If $h = 0$, the theorem is due to Hurwitz (see the references in the introduction). Thus suppose $h > 0$, and by way of induction suppose that the theorem is proved for all genera smaller than $h$. Suppose that $\Sigma \subset \Sigma' \subset B$. There is a natural map $H_{d,w}^d(B, \Sigma', b_0) \to H_{d,w}^d(B, \Sigma, b_0)$ whose image is a dense Zariski open set. So if $H_{d,w}^d(B, \Sigma', b_0)$ is connected, it follows that $H_{d,w}^d(B, \Sigma, b_0)$ is also connected. Therefore we may enlarge $S$, if need be, so that it contains a point $b'_0$ in the boundary circle $B_1 \cap B_2$ (and such that this is the only point of $S$ on the boundary circle).

Now by proposition 4.1, we see that every connected component of $H_{d,w}^d(B, \Sigma, b_0)$ contains a $B_1$-trivial pair. So to finish the proof, it suffices to prove that for two $B_1$-trivial pairs, say $(f_1 : X_1 \to B, \phi_1)$ and $(f_2 : X_2 \to B, \phi_2)$, there are braid moves which change the first pair to the second. Let $U \subset B_2$ denote the interior of $B_2$, i.e. the complement of the boundary circle. Choose a path $\gamma$ in $B_1$ from $b_0$ to $b'_0$ and in this way identify $\phi_i : f_i^{-1}(b_0) \to \{1, \ldots, d\}$ with $\phi'_i : f_i^{-1}(b'_0) \to \{1, \ldots, d\}$. Now $U$ is homeomorphic to $B' - \{b'_0\}$ for some Riemann surface $B'$ of genus $h - 1$ and for some point $b'_0 \in B'$. Let $\Sigma' \subset B'$ denote the union of the image of $\Sigma \cap U$ and $\{b'_0\}$. Then the restricted covers $(f_i : f_i^{-1}(U \to U, \phi_i))$ for $i = 1, 2$ are equivalent to covers $(f'_i : X'_i \to B', \phi'_i)$ in $H_{d,w}^d(B', \Sigma', b'_0)$. By the induction assumption, we know that $H_{d,w}^d(B', \Sigma', b'_0)$ is connected. Therefore there is a path $\alpha : [0, 1] \to (B' - \Sigma')_0$ such that

1. $\alpha(0) = \text{br}(f'_1)$,
2. $\alpha(1) = \text{br}(f'_2)$, and
3. if $\tilde{\alpha} : [0, 1] \to H_{d,w}^d(B', \Sigma', b'_0)$ is the lift with $\tilde{\alpha}(0) = (f'_1 : X'_1 \to B', \phi'_1)$, then $\tilde{\alpha}(1) = (f'_2 : X'_2 \to B', \phi'_2)$.
Using our homeomorphism, we may identify $\alpha$ with a path $\beta : [0, 1] \to (U - \Sigma \cap U)^0$. It follows that if $\tilde{\beta} : [0, 1] \to H^{d,w}_{S_d}(B, \Sigma, b_0) = \{ (f_1 : X_1 \to B, \phi_1), (f_2 : X_2 \to B, \phi_2) \}$ is the lift with $\tilde{\beta}(0) = (f_1 : X_1 \to B, \phi_1)$ and $\tilde{\beta}(1) = (f_2 : X_2 \to B, \phi_2)$, then $(f_1 : X_1 \to B, \phi_1)$ and $(f_2 : X_2 \to B, \phi_2)$ lie in the same connected component of $H^{d,w}_{S_d}(B, \Sigma, b_0)$. It follows that $H^{d,w}_{S_d}(B, \Sigma, b_0)$ is connected, and the theorem is proved by induction.

Now we can prove theorem 1.1. There is a forgetful map $H^{d,w}_{S_d}(B, \Sigma, b_0) \to H^{d,w}_{S_d}(B)$. This morphism is étale with dense image. Since $H^{d,w}_{S_d}(B, \Sigma, b_0)$ is connected, it follows that $H^{d,w}_{S_d}(B)$ is also connected, which proves theorem 1.1.

References

[1] A. Clebsch. Zur Theorie der Riemann’schen Flächen. *Mathematische Annalen*, 6:216–230, 1872.
[2] B. Fantechi and R. Pandharipande. Stable maps and branch divisors. preprint arXiv: math.AG/9601011.
[3] W. Fulton. Hurwitz schemes and moduli of curves. *Annals of Mathematics*, 90:542–575, 1969.
[4] T. Graber, J. Harris, and J. Starr. Families of rationally connected varieties. preprint arXiv: math.AG/0109220.
[5] A. Hurwitz. Ueber Riemann’sche Flächen mit gegebenen Verzweigungspunkten. *Mathematische Annalen*, 39:1–61, 1891.
[6] P. Kluitmann. Hurwitz actions and finite quotients of braid groups. In *Braids (Santa Cruz, CA, 1986)*, pages 299–325. American Mathematical Society, 1988.

Department of Mathematics, Harvard University, Cambridge MA 02138
*E-mail address:* harris@math.harvard.edu

Department of Mathematics, Harvard University, Cambridge MA 02138
*E-mail address:* graber@math.harvard.edu

Department of Mathematics, Massachusetts Institute of Technology, Cambridge MA 02139
*E-mail address:* jsstarr@math.mit.edu