BOUNDARIES OF HYPERBOLIC METRIC SPACES

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Abstract. We investigate the relationship between the metric boundary and the Gromov boundary of a hyperbolic metric space. We show that the Gromov boundary is a quotient topological space of the metric boundary, and that therefore a word-hyperbolic group has an amenable action on the metric boundary of its Cayley graph. This result has significance for the study of Lip-norms on group C*-algebras.

1. Introduction

The Gromov boundary of a hyperbolic metric space has been extensively studied, but the Gromov boundary is not guaranteed to exist for non-hyperbolic metric spaces. Gromov [4] introduced another boundary which makes sense for any metric space, but this was little studied until Rieffel [8] showed that this second boundary, called the metric boundary in his papers, is important in the study of metrics on the state spaces of group C*-algebras.

If $G$ is a countable discrete group equipped with a length function $\ell$, and $C_r^*(G)$ is its reduced C*-algebra, then one has a seminorm $L_\ell(f) = \|[M_\ell, f]\|$ defined on a dense *-subalgebra of $C_r^*(G)$, where $M_\ell$ is multiplication by $\ell$ and $f$ operates by convolution on $\ell^2(G)$. This in turn gives a metric on the state space of $C_r^*(G)$ by

$$\rho_{L_\ell}(\varphi, \psi) = \sup\{|\varphi(f) - \psi(f)| : L_\ell(f) \leq 1\},$$

and a natural question to ask is whether the topology generated by this metric coincides with the weak-* topology on the state space, i.e. the seminorm is a Lip-norm [6, 7, 9]. Rieffel proves that this is in fact the case for $\mathbb{Z}^d$ with certain length functions, and a critical requirement in his proof is that the action of $\mathbb{Z}^d$ on its metric boundary is always amenable.

There is some interest, then, in knowing when the action of a group is amenable on its metric boundary. In the case of word-hyperbolic groups with the standard word-length metric, it is known that the action of a word-hyperbolic group on its Gromov boundary is amenable [2, 3], and as Rieffel points out in [8], if there is an equivariant, continuous surjection from the metric boundary onto the Gromov boundary, then the action of the group on the metric boundary must be amenable.

We show that this is in fact the case, and more: the Gromov boundary is a quotient topological space of the metric boundary in a completely natural way, and that the quotient map is therefore such an equivariant, continuous surjection from the metric boundary to the Gromov boundary.

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We note here that Ozawa and Rieffel [5] have shown that, for hyperbolic groups, $L$ is in fact a Lip-norm using techniques which do not use the notion of the metric boundary. However these methods do not work for $\mathbb{Z}^d$, and we hope that our result may lead to a unified way of showing that the seminorms for these groups are in fact Lip-norms.

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2. The Gromov Boundary

There are many different but equivalent definitions for a hyperbolic metric space, but for our purposes we are only interested in a couple. We follow Alon, Smith, et. al. [2], in our presentation, and a more complete discussion of hyperbolic spaces can be found there.

**Definition 2.1.** A metric space $(X, d)$ is geodesic if given any two points $x, y \in X$, there is an isometry $\gamma$ from the interval $[0, d(x, y)]$ into $X$.

If $(X, d)$ is a metric space, with a base-point $0$, we define an inner product by

$$(x \cdot y)_0 = \frac{1}{2}(d(x, 0) + d(y, 0) - d(x, y)).$$

Where the base point is implicit, we will just write $(x \cdot y)$.

The metric space $(X, d)$ is hyperbolic if it is geodesic and there is some $\delta \geq 0$ such that

$$(x \cdot y) \geq \min\{(x \cdot z), (y \cdot z)\} - \delta$$

for all $z \in X$.

One can show that although the constant $\delta$ may be different for different base-points, whether or not the space is hyperbolic does not depend on the choice of base-point.

We have a particular interest in groups whose Cayley graphs are hyperbolic, and there is an equivalent definition based on properties of generators and relations alone. We note that if $G$ is a group with a finite presentation $\langle S | R \rangle$, then given a reduced word $w$ in the generators, $S$, with $w = e$ in $G$, we can write $w$ as a product

$$w = \prod_{k=1}^{n} u_k^{-1} r_k u_k,$$

where $u_k$ is a word in $S \cup S^{-1}$, and $r_k \in R \cup R^{-1}$. For a given $w$, let $n_w$ be the smallest possible number of terms in such a product, and let $l(w)$ be the length of $w$.

**Definition 2.2.** Let $G$ be a group with a finite presentation $\langle S | R \rangle$. We say that $G$ is word-hyperbolic if it satisfies a linear isoperimetric inequality: there is some $K \geq 0$ such that

$$n_w \leq Kl(w),$$

for all reduced words $w$ with $w = e$ in $G$.

One can show that the choice of generators and relations does not affect whether or not the group is word-hyperbolic and, moreover, a group is word-hyperbolic if
and only if its Cayley graph (regarded as a 1-complex with the graph metric) is hyperbolic.

Perhaps the simplest way to consider the Gromov boundary is as the limit points of geodesic rays, where two geodesic rays are considered equivalent if they are a finite distance apart. This definition highlights similarities between the Gromov boundary and the metric boundary discussed in the next section. However, the most useful definition of the Gromov boundary for our purposes is in terms of the inner product.

**Definition 2.3.** Let \((X, d)\) be a metric space. We say that a sequence \(x_k\) converges to infinity (in the Gromov sense) if

\[
\lim_{n,k \to \infty} (x_n \cdot x_k) = \infty.
\]

Given two sequences \(x = (x_n)_{n=1}^{\infty}\) and \(y = (y_n)_{n=1}^{\infty}\) which both converge to infinity, we define a relation \(\sim\) by

\[
x \sim y \iff \lim_{n \to \infty} (x_n \cdot y_n) = \infty.
\]

If \((X, d)\) is a hyperbolic metric space, then \(\sim\) is in fact an equivalence relation on sequences which converge to infinity. It is worthwhile noting that if \((X, d)\) is hyperbolic then

\[
x \sim y \iff \lim_{n,k \to \infty} (x_n \cdot y_k) = \infty.
\]

If \((X, d)\) is not hyperbolic, the relation \(\sim\) will not be an equivalence relation, in general:

**Example 2.1.** Consider the Cayley graph of \(\mathbb{Z}^2\) with the standard generators and relations. Let \(x_n = (n,0)\), \(y_n = (0,n)\) and \(z_n = (n,n)\). All three sequences converge to infinity, but although \(x \sim z\) and \(y \sim z\), \(x \not\sim y\).

We define the Gromov boundary \(\partial G X\) of a hyperbolic metric space \((X, d)\) to be the set of equivalence classes of sequences which converge to infinity. We will say that a sequence in \(X\) converges to an equivalence class in \(\partial G X\) if it is an element of the equivalence class.

We can topologise the boundary by extending the inner product to \(\overline{T}^G = X \cup \partial G X\).

**Definition 2.4.** Let \((X, d)\) be a hyperbolic metric space, and let \(x, y \in \overline{T}^G\). Then we define

\[
(x \cdot y) = \inf \{\liminf_n (x_n \cdot y_n) : x_n \to x, \ y_n \to y, \ \text{and} \ x_n, y_n \in X\}.
\]

One can show that if this inner product is restricted to \(X\), it is the same as the original inner product on \(X\). Indeed, if \(\omega \in \partial G X\), and \(y \in X\), we have

\[
(\omega \cdot y) = \inf \{\liminf_n (x_n \cdot y) : x_n \to \omega, \ \text{and} \ x_n \in X\}.
\]

It is also the case that if \((X, d)\) is hyperbolic, with

\[
(x \cdot y) \geq \min\{(x \cdot z), (y \cdot z)\} - \delta
\]

for all \(x, y\) and \(z \in X\), then the same identity holds for this extended inner product. We have

\[
(x \cdot y) \geq \min\{(x \cdot z), (y \cdot z)\} - \delta,
\]
for all \( x, y \) and \( z \in X^G \).

We then can say that a sequence \( x_n \in X^G \) converges to \( \omega \in \partial G \mathcal{X} \) if and only if
\[
(\omega \cdot x_n) \to \infty.
\]
With this definition, it can be shown that \( X^G \) is a compactification of \( X \).

3. The Metric Boundary

We now consider the metric compactification and the metric boundary. The most succinct definition is that the metric compactification \( X^d \) of a metric space \((X,d)\) corresponds to the pure states of the commutative, unital, C*-algebra \( \mathcal{G}(X,d) \) generated by the functions which vanish at infinity on \( X \), the constant functions, and the functions of the form
\[
\varphi_y(x) = d(x,0) - d(x,y),
\]
where 0 is some fixed base-point (which does not affect the resulting algebra). The metric boundary \( \partial_d X \) is simply \( X^d \setminus X \).

More concretely, we can understand the metric boundary as a limit of rays in much the same way as the simple definition of the Gromov boundary.

Definition 3.1. Let \((X,d)\) be a metric space, and \( T \) an unbounded subset of \( \mathbb{R}^+ \) containing 0, and let \( \gamma : T \to X \). We say that

1. \( \gamma \) is a geodesic ray if
\[
d(\gamma(s), \gamma(t)) = |s - t|
\]
for all \( s, t \in T \).

2. \( \gamma \) is an almost-geodesic ray if for every \( \varepsilon > 0 \), there is an integer \( N \) such that
\[
|d(\gamma(t), \gamma(s)) + d(\gamma(s), \gamma(0)) - t| < \varepsilon
\]
for all \( t, s \in T \) with \( t \geq s \geq N \).

3. \( \gamma \) is a weakly-geodesic ray if for every \( y \in X \) and every \( \varepsilon > 0 \), there is an integer \( N \) such that
\[
|d(\gamma(t), \gamma(0)) - t| < \varepsilon
\]
and
\[
|d(\gamma(t), y) - d(\gamma(s), y) - (t - s)| < \varepsilon
\]
for all \( t, s \in T \) with \( t, s \geq N \).

It is immediate that every geodesic ray is an almost-geodesic ray. Rieffel showed that every almost-geodesic ray is a weakly-geodesic ray. The significance of weakly geodesic rays is that they give the points on the metric boundary in reasonable metric spaces.

Theorem 3.1 (Rieffel). Let \((X,d)\) be a complete, locally compact metric space, and let \( \gamma : T \to X \) be a weakly geodesic ray in \( X \). Then
\[
\lim_{t \to \infty} f(\gamma(t))
\]
exists for every \( f \in \mathcal{G}(X,d) \), and defines an element of \( \partial_d X \). Conversely, if \( d \) is proper and if \((X,d)\) has a countable base, then every point of \( \partial_d X \) is determined as above by a weakly-geodesic ray.
This is similar in character to the definition of the Gromov boundary, although the reliance on weakly-geodesic rays is necessary in general. Rieffel defined any point $\partial dX$ which is the limit of an almost-geodesic ray to be a Busemann point, and it was shown in [10] that even for simple hyperbolic spaces the metric boundary may have non-Busemann points. It is an open question as to whether this phenomenon can occur with word-hyperbolic groups.

Unlike the Gromov boundary, the metric boundary is, in general, dependent upon the choice of metric. For example, different generating sets for an infinite discrete group generally give distinct metric boundaries for the corresponding word-length metrics.

From a practical viewpoint, the initial definition of the metric boundary means that a sequence $x_n \in X$ converges to a point on the metric boundary iff $x_n$ is eventually outside any compact subset of $X$, and $\varphi_y(x_n)$ converges for all $y \in X$. Two sequences converge to the same point on the metric boundary iff $\lim_{n \to \infty} \varphi_z(x_n) = \lim_{k \to \infty} \varphi_z(y_k)$ for every $z \in X$. We can extend the functions $\varphi_y$ to the boundary by letting $\varphi_y(\omega) = \lim_{n \to \infty} \varphi_y(x_n)$ for any sequence $x_n \to \omega \in \partial dX$. Then a sequence $x_n \in X$ converges to $x \in \partial dX$ iff $\varphi_y(x_n) \to \varphi_y(x)$ for all $y \in X$, and this is sufficient to determine the topology of the metric compactification.

4. The Gromov Boundary as a Quotient

We observe that the functions $\varphi_y$ and the inner product are closely related, since

$$(x \cdot y) = \frac{1}{2}(\varphi_y(x) + d(y, 0)),$$

and furthermore, they play similar roles in the definitions of Gromov and metric boundaries. It is natural, therefore, to ask what relationship there may be between the two different boundaries.

The key observation is that the triangle inequality implies that for any $z \in X$,

$$(x \cdot y) \geq \frac{1}{2}(d(x, 0) + d(y, 0) - d(x, z) - d(y, z)) = \frac{1}{2}(\varphi_z(x) + \varphi_z(y)),$$

with equality iff $z$ lies on a geodesic path $[x, y]$. We will want to show that that $(x \cdot y)$ gets large for elements from various sequences, and this implies that all we need do is find a $z$ so that $\varphi_z(x) + \varphi_z(y)$ is large.

The following lemma tells us that as we get close to a metric boundary point, we can find $z$ such that $\varphi_z$ is large.

**Lemma 4.1.** Let $(X, d)$ be a proper geodesic metric space with a distinguished basepoint $0$. Then for any $\omega$ in the metric boundary of $X$, and any $N$, there is a point $z \in X$ such that $\varphi_z(\omega) > N$.

**Proof.** Let $x_n$ be any sequence which converges to $\omega$.

Let $r > 0$ and consider a collection of minimal paths $[0, x_n]$ for $n$ large enough that $d(0, x_n) > r$. Because $(X, d)$ is a geodesic metric space, there must be a unique point $y_n$ in each of these paths in the sphere $S(0, r)$ of radius $r$, centred at $0$. Since $(X, d)$ is proper the sphere $S(0, r)$ is compact, and so given any $\varepsilon > 0$ we must be
able to find at least one point \( z_r \in S(0, r) \) such that an infinite number of the \( y_n \) lie within \( \varepsilon/2 \) of \( z_r \). Let \( x_{n_j} \) be the subsequence of \( x_n \) corresponding to this infinite subset. Then we have, for \( r > \varepsilon \) and \( j \) sufficiently large,
\[
d(0, x_{n_j}) = d(0, y_{n_j}) + d(y_{n_j}, x_{n_j}) > d(0, z_r) + d(z_r, x_{n_j}) - \varepsilon,
\]
or, equivalently,
\[
\varphi_{z_r}(x_{n_j}) = d(0, x_{n_j}) - d(z_r, x_{n_j}) > d(0, z_r) - \varepsilon = r - \varepsilon.
\]
Taking limits, we conclude that
\[
\varphi_{z_r}(\omega) \geq r - \varepsilon.
\]
Hence, given any \( N \), we can choose \( r \) and \( \varepsilon \) such that \( r - \varepsilon > N \), and obtain a point \( z \) such that
\[
\varphi_z(\omega) > N.
\]
\[\square\]

This lemma has two immediate corollaries:

**Corollary 4.2.** Let \((X, d)\) be a proper geodesic metric space with a distinguished base-point \(0\), and let \( x_n \to \omega \in \partial dX \). Then \( x_n \) converges to infinity in the Gromov sense.

**Proof.** We know that for all \( z \), \( \varphi_z(x_n) \) eventually gets close to \( \varphi_z(\omega) \). Hence by the previous lemma, for any \( N \) can find a \( z \) such that \( \varphi_z(x_n) > N \) for all \( n \) sufficiently large.

However, we than have that if \( x_n \) and \( x_m \) are large enough that both \( \varphi_z(x_n) \) and \( \varphi_z(x_m) \) are greater than \( N \), then
\[
(x_n \cdot x_m) \geq \frac{1}{2} (\varphi_z(x_n) + \varphi_z(x_m)) > N
\]
Therefore
\[
\lim_{n,m \to \infty} (x_n \cdot x_m) = \infty
\]
and so \( x_n \) goes to infinity in the Gromov sense. \[\square\]

Let \((x_n)\) and \((y_k)\) be two sequences in \( X \) which converge to points on the metric boundary. We will say that \((x_n) \sim_d (y_k)\) if these two sequences converge to the same metric boundary point. Similarly, if these sequences converge to points on the Gromov boundary, we will say that \((x_n) \sim_G (y_k)\). Note that despite the notation \(\sim_G\) is not necessarily an equivalence relation.

**Corollary 4.3.** Let \((X, d)\) be a proper geodesic metric space. Then \((x_n) \sim_d (y_k)\) implies \((x_n) \sim_G (y_k)\).

**Proof.** Let \( x_n \) and \( y_n \) both converge to \( \omega \). Using the lemma, we can find a point \( z \) so that \( \varphi_z(\omega) \) is arbitrarily large, and since both \( \varphi_z(x_n) \) and \( \varphi_z(y_n) \) converge to \( \varphi_z(\omega) \), for any number \( N \) we can find \( z \) such that both \( \varphi_z(x_n) \) and \( \varphi_z(y_n) \) are greater then \( N \) for all \( n \) sufficiently large.

Hence
\[
(x_n \cdot y_n) \geq \frac{1}{2} (\varphi_z(x_n) + \varphi_z(y_n)) > N
\]
for all \( n \) sufficiently large, and so
\[
\lim_{n \to \infty} (x_n \cdot y_n) = \infty,
\]
and so \((x_n) \sim_G (y_k)\).

These two corollaries mean that we have a well-defined relation \(\sim\) on \(\partial_{d}X\) given by \(\omega_1 \sim \omega_2\) if and only if for any \(x_n \rightarrow \omega_1\) and \(y_k \rightarrow \omega_2\), we have \((x_n) \sim_G (y_k)\). Furthermore, if \(\sim_G\) is an equivalence relation (as it is for hyperbolic spaces), then \(\sim\) is also an equivalence relation on \(\partial_{d}X\), and moreover \(\partial_{G}X = \partial_{d}X/\sim\) as sets. As usual, we will denote the equivalence class of a point \(\omega\) in the metric boundary by \([\omega]\).

What we want is to show that we in fact have \(\partial_{G}X = \partial_{d}X/\sim\) as topological spaces. In other words, we need to show that the quotient map is continuous.

**Lemma 4.4.** Let \((X, d)\) be a proper hyperbolic metric space. If \(\omega_n \rightarrow \omega\) in \(\partial_{d}X\), then \([\omega_n] \rightarrow [\omega]\) in \(\partial_{G}X\).

**Proof.** Let \(\delta > 0\) be the hyperbolic constant from \(\Pi\), \(x_k \rightarrow \omega\) and \(x_{n,k} \rightarrow \omega_n\). We know that we can find \(z\) such that \(\varphi_z(\omega)\) is arbitrarily large, and since we have \(\varphi_z(\omega_n) \rightarrow \varphi_z(\omega)\), we can choose \(z\) such that \(\varphi_z(\omega_n)\) is also arbitrarily large, for all \(n\) sufficiently large. Indeed, as in the previous corollaries, we have, given any \(N > 0\) we can find a number \(M\) such that for all \(n > M\), there is a number \(K_n\) such that

\[
(x_{n,k} \cdot x_k) \geq \frac{1}{2}(\varphi_z(x_{n,k}) + \varphi_z(x_k)) > N + 2\delta
\]

for all \(k > K_n\).

Now if \(y_{n,k} \rightarrow [\omega_n]\) and \(y_n \rightarrow [\omega]\), we know that we can find a subsequence of each sequence such that

\[
\liminf_{k \rightarrow \infty}(y_{n,k} \cdot y_k) = \lim_{k \rightarrow \infty}(y_{n,k} \cdot y_{k_j}).
\]

Furthermore, since \(x_{n,j} \rightarrow [\omega_n]\) we conclude that for any \(N\) there is some \(J_n\) such that

\[
(y_{n,k_j} \cdot x_{n,j}) > N + 2\delta
\]

for all \(j > J_n\), and similarly that there is some \(J\) such that

\[
(y_{k_j} \cdot x_j) > N + \delta
\]

for all \(j > J\).

Hence, given any \(N\), and fixing some \(n > M\), then we have

\[
(y_{n,k_j} \cdot x_j) \geq \min\{(y_{n,k_j} \cdot x_{n,j}),(x_{n,j} \cdot x_j)\} - \delta > N + \delta
\]

for all \(j > \max\{J_n, K_n\}\). But then

\[
(y_{n,k_j} \cdot y_{k_j}) \geq \min\{(y_{n,k_j} \cdot x_j),(y_{k_j} \cdot x_j)\} - \delta > N
\]

for all \(j > \max\{J_n, K_n, J\}\). Therefore, for any \(n > M\),

\[
\lim_{k \rightarrow \infty}(y_{n,k_j} \cdot y_{k_j}) > N,
\]

and so

\[
\liminf_{k \rightarrow \infty}(y_{n,k} \cdot y_k) > N
\]

for all \(n > M\). And since \(M\) does not depend on the choice of sequences converging to \([\omega_n]\) and \([\omega]\), we therefore have that

\[
([\omega_n] \cdot [\omega]) = \inf\{\liminf_{k \rightarrow \infty}(y_{n,k} \cdot y_k) : y_{n,k} \rightarrow [\omega_n], y_n \rightarrow [\omega]\} > N
\]

for all \(n > M\).

Therefore

\[
\lim_{n \rightarrow \infty}([\omega_n] \cdot [\omega]) = \infty,
\]
and so $[\omega_n] \to [\omega]$ in $\partial G X$. 

So we have proved the following result.

**Theorem 4.5.** Let $(X,d)$ be a proper, hyperbolic metric space. Then there is a natural continuous quotient map from $\partial dX$ onto $\partial G X$.

5. **Boundaries of Word-Hyperbolic Groups**

We observe that if $G$ is a hyperbolic group, then the group acts on either boundary by taking a sequence $x_k \to \omega$ and letting

$$\alpha_g(\omega) = \lim_{k \to \infty} gx_k.$$ 

This is a continuous action on either boundary. Clearly the quotient map is equivariant for these two actions, since if $\omega \sim \omega'$, we can easily see that $\alpha_g(\omega) \sim \alpha_g(\omega')$ by simply changing the base point of the inner product to $g$.

An action of a topological group $G$ on a topological space $X$ is amenable if there is a net of continuous maps $(m_\lambda: X \to M^+_1(G))_{\lambda \in \Lambda}$, where $M^+_1(G)$ is the set of Borel probability measures on $G$, such that

$$\lim_{\lambda \in \Lambda} \| g \cdot m_\lambda(x) - m_\lambda(g \cdot x) \| \to 0$$ 

uniformly on compact subsets of $G \times X$. Such a net of maps is called an approximate invariant continuous mean. It was shown by E. Germain (as discussed in \[2\] \[3\]) that the action of a word-hyperbolic group $G$ on its Gromov boundary is amenable. Rieffel pointed out that if there were a continuous, equivariant surjection from $\partial dG$ to the Gromov boundary, then the action of $G$ on the metric boundary must also be amenable. This is trivial given the above definition, since if $q : \partial dG \to \partial G G$ is the quotient map of Theorem \[4\] and $m_\lambda$ are the maps in an approximate invariant continuous mean for the action of $G$ on $\partial G G$, then $m_\lambda \circ q$ are an approximate invariant continuous mean for the action of $G$ on $\partial dG$.

**Corollary 5.1.** If $G$ is word-hyperbolic group with a finite generating set, and $d$ is the word-length metric, then the group action on the metric boundary is amenable.

This would seem to open the possibility of replicating Rieffel’s work on the metric boundary of $\mathbb{Z}^d$ in the setting of hyperbolic groups. However, Rieffel’s procedure relied on the fact that the action of $\mathbb{Z}^d$ on its metric boundary always has finite orbits, and it seems unlikely that this criterion holds with any frequency for general hyperbolic groups.

**References**

[1] J. M. Alonso, T. Brady, D. Cooper, V. Ferlini, M. Lustig, M. Mihalik, M. Shapiro, and H. Short (Editor). Notes on word hyperbolic groups. In Group theory from a geometrical viewpoint (Trieste, 1990), pages 3–63. World Sci. Publishing, 1991.

[2] C. Anantharaman-Delaroche and J. Renault. Amenable Groupoids, volume 36 of Monographies de L’Enseignement Mathmatique. L’Enseignement Mathmatique, Geneva, 2000.

[3] Claire Anantharaman-Delaroche. Amenability and exactness for dynamical systems and their C*-algebras. *Trans. Amer. Math. Soc.*, 354:4153–4178, 2002, arXiv:math.OA/0005014.

[4] M. Gromov. Hyperbolic manifolds, groups and actions. In *Riemann surfaces and related topics: Proceedings of the 1978 Stony Brook Conference*, pages 182–213. Princeton University Press, 1981.

[5] Narutaka Ozawa and Marc A. Rieffel. Hyperbolic group C*-algebras and free-product C*-algebras as compact quantum metric spaces. preprint, 2003, arXiv:math.OA/0302010.
[6] Marc A. Rieffel. Metrics on states from actions of compact groups. Doc. Math., 3:215–229, 1998, arXiv:math.OA/9807084.

[7] Marc A. Rieffel. Metrics on state spaces. Doc. Math., 4:559–600, 1999, arXiv:math.OA/9906151.

[8] Marc A. Rieffel. Group C*-algebras as compact quantum metric spaces. Doc. Math., 7:605–651, 2002, arXiv:math.OA/0205195.

[9] Marc A. Rieffel. Compact quantum metric spaces. preprint, 2003, arXiv:math.OA/9807084.

[10] Corran Webster and Adam Winchester. Busemann points of infinite graphs. preprint, 2003, arXiv:math.MG/0309291.

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