TENSORS WITH MAXIMAL SYMMETRIES

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Abstract. We classify tensors in $\mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$ for $m \geq 7$ with maximal and next to maximal dimensional symmetry groups under a natural genericity assumption, called 1-genericity. In other words, we classify minimal dimensional $GL_3 \times GL_3$-orbits in $\mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$ assuming 1-genericity. Our study uncovers new tensors with striking geometry. This paper was motivated by Strassen’s laser method for bounding the exponent of matrix multiplication. The best known tensor for the laser method is the large Coppersmith-Winograd tensor, and our study began with the observation that it has a large symmetry group, of dimension $(m+1)^2/2$. We show that in odd dimensions, this is the largest possible dimension for a 1-generic tensor, but in even dimensions we exhibit a tensor with a larger dimensional symmetry group. In the course of the proof, we classify nondegenerate bilinear forms with large dimensional stabilizers, which is of interest in its own right.

1. Introduction

Linear algebra has been predominantly concerned with two-fold tensor products: linear maps from a vector space $A$ to a vector space $B$ (the space $A^* \otimes B$), bilinear forms (the space $A^* \otimes B^*$), linear endomorphisms (the space $A^* \otimes A$), and bilinear forms on a single space (the space $A^* \otimes A^*$). In contrast, remarkably little is known about three-fold tensor products, despite their relevance to numerous important topics such as signal processing, complexity theory, classification of algebras, etc. See, e.g., [26] for a discussion.

The orbit structures in $A^* \otimes A$ under the action of $GL(A)$ and in $A^* \otimes B$ and $A^* \otimes B^*$ under $GL(A) \times GL(B)$ have been known since Jordan. Already the orbit structure in $A \otimes A$ under $GL(A)$ is not completely understood and there is a vast literature on the subject, see, e.g., [21] and the numerous references therein. Lemma 1.2 below is an important addition to the literature, characterizing the small nondegenerate orbits. More precisely, it determines which nondegenerate bilinear forms on $\mathbb{C}^k$ have stabilizers of dimension at least $k^2/2$.

The corresponding problems for tensors, i.e., orbits in $A \otimes B \otimes C$ under $GL(A) \times GL(B) \times GL(C)$, is a wild problem in the sense of Gabriel’s theorem [18] (see [16] for an exposition in English), except in small dimensions and unbalanced cases such as when $dim C = dim A \cdot dim B$. In this paper we classify tensors $T \in A \otimes B \otimes C$ with large symmetry groups, or equivalently small $GL(A) \times GL(B) \times GL(C)$-orbits, in the case $dim A = dim B = dim C$. The smallest orbits in such a tensor space under this action are classically known. Our study is primarily motivated by the complexity of matrix multiplication, and in this context one imposes a natural genericity...
condition on the tensors of interest. This brings into play new small orbits with unexpectedly rich geometric structure.

Besides their relevance for computer science, our results are connected to a classical question in algebraic geometry and representation theory: given a representation \( V \) of a group \( G \), what are the vectors \( v \in V \) whose orbit closures are of small dimension, i.e., with large stabilizers? Our main result (Theorem 1.1) fits into a long tradition of studying small orbits; see for instance [24, 13, 25, 36, 23].

**Motivation from matrix multiplication complexity.** The exponent of matrix multiplication \( \omega \) is a fundamental constant that controls the complexity of basic operations in linear algebra. It is generally conjectured that \( \omega = 2 \), which would imply that one could multiply \( n \times n \) matrices using \( O(n^{2+\epsilon}) \) arithmetic operations for any \( \epsilon > 0 \). The current state of knowledge is \( 2 \leq \omega < 2.3728596 \) [3], but it has been known since 1989 that \( \omega \leq 2.3755 \) [15].

One motivation for this paper is the Ambainis-Filmus-Le Gall challenge: find new tensors that give good upper bounds on \( \omega \) via Strassen’s laser method [32]. See [15, 9, 4] for expositions of the method. This challenge is motivated by the results of [4], where the authors showed that the main tool used so far to obtain upper bounds, Strassen’s laser method applied to the Coppersmith-Winograd tensor using coordinate restrictions, can never prove \( \omega < 2.3 \). Further limitations are proved in [1, 2, 11].

Tensors with continuous symmetry are central to the implementation of the laser method. Advancing ideas in [28], we isolate geometric features of the Coppersmith-Winograd tensors and find other tensors with similar features, in the hope they will be useful for the laser method. The point of departure of this paper was the observation that Coppersmith-Winograd tensors have very large symmetry groups. This led us to the classification problem. Our main theorem, while uncovering new geometry, fails to produce new tensors good for the laser method, as none of the tensors in Theorem 1.1 is better than the big Coppersmith-Winograd tensor for the laser method. However, in [14], guided by the results in this paper, we introduce a skew cousin of the small Coppersmith-Winograd tensor \( T_{cw,q} \), analyze its utility for the laser method, and show it is potentially better for the laser method than existing tensors. In particular, \( T_{skewcw,2} \), like its cousin \( T_{cw,2} \), potentially could be used to prove \( \omega = 2 \). See Corollary 7.5 and the discussion below it for an important consequence for the laser method.

**1-generic tensors.** A tensor \( T \in A \otimes B \otimes C \) is concise if the induced maps \( T_A : A^* \to B \otimes C \), \( T_B : B^* \to A \otimes C \), \( T_C : C^* \to A \otimes B \) are injective. In our main theorem, we will require an additional natural genericity condition that dates back to [35] and has been recently studied in [8, 28, 12]. A tensor is \( 1_A \)-generic if the subspace \( T_A(A^*) \subset B \otimes C \) contains an element of maximal rank; \( 1_A, 1_B \) or \( 1_C \)-generic tensors are essentially those for which Strassen’s equations [31] are most useful. A tensor is binding if it is at least two of \( 1_A, 1_B, 1_C \)-generic. As observed in [8], binding tensors are exactly the structure tensors of unital (not necessarily associative) algebras. Binding tensors are automatically concise. A tensor is \( 1 \)-generic if it is \( 1_A, 1_B \) and \( 1_C \) generic. (1-genericty is called bequem in [35] and comfortable in [12].) Propositions 3.1 and 3.2 respectively determine the maximum possible dimension of the symmetry group of a \( 1_A \)-generic tensor and a binding tensor and show in each case that there is a unique such tensor with maximal dimensional symmetry.

Theorem 1.1 classifies 1-generic tensors with symmetry group of maximal and next to maximal dimension. In particular, when \( m \) is even, there is a striking gap in that the second largest symmetry group has dimension \( m - 2 \) less than the largest.
Notations and conventions. Let \( a_1, \ldots, a_m \) be a basis of the vector space \( A \), and \( \alpha^1, \ldots, \alpha^m \) its dual basis of \( A^* \). Similarly \( b_1, \ldots, b_m \) and \( c_1, \ldots, c_m \) are bases of \( B \) and \( C \) respectively, with corresponding dual bases \( \beta^1, \ldots, \beta^m \) and \( \gamma^1, \ldots, \gamma^m \). Informally, the symmetry group of a tensor \( T \in A \otimes B \otimes C \) is its stabilizer under the natural action of \( GL(A) \times GL(B) \times GL(C) \). For a tensor \( T \in A \otimes B \otimes C \), let \( G_T \) denote its symmetry group. We say \( T' \) is isomorphic to \( T \) if they are in the same \( GL(A) \times GL(B) \times GL(C) \)-orbit. We identify isomorphic tensors. Since the action of \( GL(A) \times GL(B) \times GL(C) \) on \( A \otimes B \otimes C \) is not faithful, we work modulo the kernel of its inclusion into \( GL(A \otimes B \otimes C) \), which is a 2-dimensional abelian subgroup. See Section 2 for details. The transpose of a matrix \( M \) is denoted \( M^t \). For a set \( X \), \( \overline{X} \) denotes its Zariski closure. For a subset \( Y \subset \mathbb{C}^N \), we let \( \langle Y \rangle \subset \mathbb{C}^N \) denote its linear span.

Throughout we use the summation convention that if a free index appears as a subscript and a superscript, then it is summed over its range.

Main Theorem.

Theorem 1.1. Let \( m \geq 7 \) and let \( \dim A = \dim B = \dim C = m \). Let \( T \in A \otimes B \otimes C \) be a 1-generic tensor. Then

\[
\dim G_T < \frac{m^2}{2} + \frac{m}{2}
\]

except when \( T \) is isomorphic to

(2) \( S_B := a_1 \otimes b_1 \otimes c_m + a_1 \otimes b_m \otimes c_1 + a_m \otimes b_1 \otimes c_1 + \sum_{p=2}^{m-1} a_1 \otimes b_p \otimes c_p + \sum_{p=2}^{m-1} a_p \otimes b_1 \otimes c_p + B \otimes c_1 \),

where \( B \in A \otimes B \) is one of the four following bilinear forms

(3) \( \sum_{\xi=2}^{p+1} a_\xi \otimes b_{\xi+p} - a_{\xi+p} \otimes b_\xi \) \( m = 2p \) even \( (T_{skew\; CW,m-2}) \)

(4) \( \sum_{\rho=2}^{m-1} a_\rho \otimes b_\rho \) all \( m \) \( (T_{CW,m-2}) \)

(5) \( a_{m-1} \otimes b_{m-1} + \sum_{\xi=2}^{p} (a_\xi \otimes b_{\xi+p-1} - a_{\xi+p-1} \otimes b_\eta) \) \( m = 2p \) even \( (T_{s+skew\; CW,m-2}) \)

(6) \( a_{m-1} \otimes b_{m-1} + \sum_{\xi=2}^{p} (a_\xi \otimes b_{\xi+p-1} - a_{\xi+p-1} \otimes b_\eta) \) \( m = 2p + 1 \) odd \( (T_{s\otimes skew\; CW,m-2}) \)

All these except \( T_{skew\; CW,m-2} \) have \( \dim G_{SB} = \frac{m^2}{2} + \frac{m}{2} \), and \( \dim G_{T_{skew\; CW,m-2}} = \frac{m^2}{2} + \frac{3m}{2} - 2 \).

Theorem 1.1 implies: When \( m \) is even, there is a unique, up to isomorphism, 1-generic tensor \( T \) with maximal dimensional symmetry group, namely \( T_{skew\; CW,m-2} \), and there are exactly two, up to isomorphism, additional 1-generic tensors \( T \) such that \( \dim G_T \geq \frac{m^2}{2} + \frac{m}{2} \), which are \( T_{CW,m-2} \) and \( T_{s+skew\; CW,m-2} \), where equality holds. When \( m \) is odd, there are exactly two 1-generic tensors \( T \) up to isomorphism with maximal dimensional symmetry group \( \frac{m^2}{2} + \frac{m}{2} \), which are \( T_{CW,m-2} \) and \( T_{s\otimes skew\; CW,m-2} \).

Call a 1-generic tensor skeletal if it is of the form (2) for some nondegenerate bilinear form \( B \) (not necessarily one appearing in the theorem).

The Lie algebras of skeletal tensors are given explicitly in Proposition 5.4.

The statement of the theorem hints at the method of proof. First, we observe that any 1-generic tensor may be degenerated to a skeletal tensor with some bilinear form \( B \). We then classify bilinear forms with large stabilizer, and show that the corresponding skeletal tensor has symmetry group of dimension lower than \( \binom{m+1}{2} \), except for the four cases mentioned in the
Nondegenerate bilinear forms. An important step in the proof of Theorem [11] requires us to classify nondegenerate bilinear forms with large symmetry groups. This is done in Lemma [12], where we classify bilinear forms $B \in \mathbb{C}^k \otimes \mathbb{C}^k$ with large stabilizer under the action of $GL_k$ given by $g \cdot B = gBg^t$ for $g \in GL_k$.

Let $H_B \subset GL_k$ be this stabilizer and let $h_B$ be its Lie algebra. Elements $X \in h_B$ are characterized by the condition

$$X B + B X^t = 0.$$  

The dimension of the solution space of (7) is computed in [33], via the Kronecker normal forms of pencil of matrices discussed in [34, 21].

In the particular case where the symmetric part or the skew-symmetric part of the bilinear form $B$ is non-degenerate, the problem of determining the dimension of $h_B$ is equivalent to the one of determining the dimension of certain orbits in the adjoint representations of $so_k$ and $sp_k$. This is addressed extensively in the literature; see, e.g., [37, 10, 19, 27].

However, the characterization of [33], and the results of the related references, are not suitable for explicitly determining the maximal and next to maximal dimension of $h_B$. We characterize these cases explicitly in Lemma [12].

Before stating Lemma [12] we set some notation. Write $W = \mathbb{C}^k$. As a $gl(W)$-module, $W^* \otimes W^* = S^2 W^* \oplus \Lambda^2 W^*$. Write $B = Q + \Lambda$ with $Q \in S^2 W^*$ symmetric and $\Lambda \in \Lambda^2 W^*$ skew-symmetric. Write $E = \ker(\Lambda)$, $F = \ker(Q)$, which are subspaces of $W$; let $L^* = E^\perp \cap F^\perp \subseteq W^*$. Choose a complement $L \subset W$ of $E \oplus F$ so that we have a direct sum decomposition $W = E \oplus L \oplus F$.

and we may identify $L$ with the dual space of $L^*$. We may also identify $E^*$ with $(L \oplus F)^\perp$ and $F^*$ with $(E \oplus L)^\perp$, regarded as subspaces of $W^*$.

Let $e = \dim E$, $f = \dim F$, and $\ell = \dim L$. Notice that $\rk(\Lambda) = \ell + f$ is even.

For a subspace $U \subset W$, write $B|_U := B|_{U \times U}$; this is naturally an element of $(W^*/U^\perp) \otimes (W^*/U^\perp) = U^* \otimes U^*$.

**Lemma 1.2.** With notations as above, let $B \in \mathbb{C}^{k^*} \otimes \mathbb{C}^{k^*}$ be a full rank bilinear form. Then

$$\dim h_B < \binom{k}{2}$$

except in the following cases

(i) $(e, \ell, f) = (0, 0, k)$ (so $k$ is even): in this case $B = \Lambda$ is skew-symmetric and $h_B = sp(\Lambda)$ with $\dim h_B = \binom{k+1}{2}$;

(ii) $(e, \ell, f) = (k, 0, 0)$: in this case $B = Q$ is symmetric and $h_B = so(Q)$ with $\dim h_B = \binom{k}{2}$;

(iii) $(e, \ell, f) = (0, 1, k - 1)$ (so $k$ is even): in this case $\dim h_B = \binom{k}{2}$;

(iv) $(e, \ell, f) = (1, 0, k - 1)$ (so $k$ is odd): $h_B = sp(\Lambda|_E)$ with $\dim h_B = \binom{k}{2}$. 


1.1. Overview. In §2 we review basic facts about symmetry groups and algebras of tensors and their degenerations. In §3 we determine the $1_A$-generic and binding tensors with maximal dimensional symmetry groups. In §4 we prove Lemma 1.2 on annihilators of bilinear forms. In §5 we begin the proof of Theorem 1.1, which we finish in §6. In §7 we show that among skeletal tensors, the only one with minimal border rank is the Coppersmith-Winograd tensor. We observe that this implies a result of [22] that any 1-degenerate minimal border rank tensor degenerates to the Coppersmith-Winograd tensor. We briefly discuss symmetry algebras of other tensors relevant for the laser method in §8.

2. The symmetry group of a tensor

In this section, we define the symmetry group of a tensor and its Lie algebra.

Let $\Phi : GL(A) \times GL(B) \times GL(C) \to GL(A \otimes B \otimes C)$ denote the natural action of $GL(A) \times GL(B) \times GL(C)$ on $A \otimes B \otimes C$. The map $\Phi$ has a two dimensional kernel

$$\ker \Phi = \{ (\lambda \text{Id}_A, \mu \text{Id}_B, \nu \text{Id}_C) : \lambda \mu \nu = 1 \} \simeq (\mathbb{C}^*)^2.$$ 

Thus

$$G := (GL(A) \times GL(B) \times GL(C))/(\mathbb{C}^*)^2$$

is naturally a subgroup of $GL(A \otimes B \otimes C)$.

Definition 2.1. Let $T \in A \otimes B \otimes C$. The symmetry group of $T$, denoted $G_T$, is the stabilizer of $T$ in $G$:

$$G_T := \{ g \in (GL(A) \times GL(B) \times GL(C))/(\mathbb{C}^*)^2 \mid g \cdot T = T \}.$$ 

The symmetry group $G_T$ is a closed algebraic subgroup of $G \subseteq GL(A \otimes B \otimes C)$. If $T$ and $T'$ are isomorphic tensors, then $G_T$ and $G_{T'}$ are conjugate subgroups of $G$; in particular $\dim G_T = \dim G_{T'}$.

Moreover, $\dim G_T = \dim g_T$, where $g_T$ is the corresponding Lie subalgebra of the algebra $g = (gl(A) \oplus gl(B) \oplus gl(C))/\mathbb{C}^2$ annihilating $T$:

$$g_T := \{ L \in (gl(A) \oplus gl(B) \oplus gl(C))/\mathbb{C}^2 \mid L.T = 0 \}.$$ 

Here $LT$ denotes the Lie algebra action.

The algebra $(gl(A) \oplus gl(B) \oplus gl(C))/\mathbb{C}^2$ is the image of the differential $\Phi = d\tilde{\Phi}$ of the map $\tilde{\Phi}$ defined above, see, e.g., [29, Section 1.2].

It is more convenient to describe the annihilator $\tilde{g}_T = \Phi^{-1}(g_T)$ as a subalgebra of $gl(A) \oplus gl(B) \oplus gl(C)$, acting on $A \otimes B \otimes C$ via the Leibniz rule. Notice that $\tilde{g}_T$ always contains $\ker \Phi = \{ (\lambda \text{Id}_A, \mu \text{Id}_A, \nu \text{Id}_A) : \lambda + \mu + \nu = 0 \} \simeq \mathbb{C}^2$, and so $\dim G_T = \dim g_T = \dim \tilde{g}_T - 2$.

Explicitly, if $L = (U, V, W) \in gl(A) \oplus gl(B) \oplus gl(C)$, and we write $U = (u_{ij}^i), V = (v_{ij}^j), W = (w_{ij}^k)$, the condition $L.T = 0$ is equivalent to the following system of linear equations (we remind the reader of the summation convention):

$$u_{ij}^i T^{ij}k + v_{ij}^j T^{ij}k + w_{ij}^k T^{ij}k' = 0, \text{ for every } i, j, k.$$

In what follows, we regard $u_{ij}^i, v_{ij}^j, w_{ij}^k$ as linear coordinates on $gl(A) \oplus gl(B) \oplus gl(C)$, i.e., as basis vectors of the dual space $gl(A)^* \oplus gl(B)^* \oplus gl(C)^*$. There is an inclusion $\tilde{g}_T \subseteq gl(A) \oplus gl(B) \oplus gl(C)$
and the conditions in (11) are the relations placed on these linear functions when they are pulled back to $\tilde{\mathfrak{g}}_T$.

In the rest of the paper, we often display special instances of (11), marking them with the corresponding triplet of indices $(ijk)$.

The codimension of $\tilde{\mathfrak{g}}_T$ in $\mathfrak{gl}(A) \oplus \mathfrak{gl}(B) \oplus \mathfrak{gl}(C)$ equals the number of linearly independent equations in the system (11). In order to prove an upper bound on $\dim \mathfrak{g}_T$, we will prove lower bounds on the rank of the linear system (11). This will be achieved via subsequent normalizations of the tensor $T$, obtained exploiting the genericity hypotheses.

We recall two basic facts, which will be useful throughout the paper. We refer to [17, 29] for the general theory.

**Lemma 2.2.** Let $G$ be a reductive algebraic group and let $W$ be a $G$-module. Let $w, w' \in W$ with $w \in G \cdot w'$. Then

$$\dim G_w \geq \dim G_{w'}.$$  

Recall that every nonzero vector contains a highest weight vector in its orbit closure, giving a general upper bound on the dimensions of stabilizers.

Given $T, T' \in A \otimes B \otimes C$, we say that $T$ degenerates to $T'$ if

$$T' \in \text{GL}(A) \times \text{GL}(B) \times \text{GL}(C) \cdot T.$$  

Thus a special case of Lemma 2.2 is that if $T$ degenerates to $T'$, then $\dim G_{T'} \geq \dim G_T$ and the inequality is strict unless $T$ and $T'$ are isomorphic tensors.

**Lemma 2.3.** Let $G$ be a reductive algebraic group and let $W$ be a $G$-module. Let $W = W_1 \oplus \cdots \oplus W_k$ be the decomposition of $W$ into its isotypic components. For $w \in W$, write $w = w_1 + \cdots + w_k$ for its isotypic decomposition. Then

$$\mathfrak{g}_w = \mathfrak{g}_{w_1} \cap \cdots \cap \mathfrak{g}_{w_k}.$$  

The following is an immediate consequence of Theorem 1.1 and Lemma 2.2.

**Corollary 2.4.** Let $m \geq 7$ and let $\{1 - \text{generic}\} \subseteq \mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$ be the open set of 1-generic tensors.

If $m$ is even then

$$\text{GL}_m^3 \cdot T_{\text{skew}CW,m-2} \cap \{1 - \text{generic}\} = \text{GL}_m^3 \cdot T_{\text{skew}CW,m-2}. $$

If $m$ is odd, then

$$\text{GL}_m^3 \cdot T_{s\oplus \text{skew}CW,m-2} \cap \{1 - \text{generic}\} = \text{GL}_m^3 \cdot T_{s\oplus \text{skew}CW,m-2};$$

$$\text{GL}_m^3 \cdot T_{CW,m-2} \cap \{1 - \text{generic}\} = \text{GL}_m^3 \cdot T_{CW,m-2}.$$  

3. **Symmetry groups of tensors: first results**

In this section, we review the classical result on the largest possible symmetry group of any tensor, and we characterize the maximal possible symmetry group for a $1_A$-generic tensor and for a binding tensor.
3.1. **Arbitrary tensors.** The unique tensor with largest symmetry group in $A \otimes B \otimes C$ is (up to change of bases) $a_1 \otimes b_1 \otimes c_1$. This follows immediately from Lemma 2.2 since any element of the form $a \otimes b \otimes c$ is highest weight vector in $A \otimes B \otimes C$ for the action of $G = GL(A) \times GL(B) \times GL(C)$, under some choice of Borel subgroup.

The annihilator of $a_1 \otimes b_1 \otimes c_1$, presented in $(1, m - 1) \times (1, m - 1)$ block form, is

$$\tilde{\mathfrak{g}}_{a_1 \otimes b_1 \otimes c_1} = \left\{ \begin{pmatrix} u_1^i & u \\ 0 & U \end{pmatrix}, \begin{pmatrix} v_1^i & v \\ 0 & V \end{pmatrix}, \begin{pmatrix} w_1^i & w \\ 0 & W \end{pmatrix} \right| u, v, w \in \mathbb{C}^{m-1}, U, V, W \in \mathfrak{gl}_{m-1} \right\}.$$  

Hence, $\dim G_T = [3(m - 1)^2 + 3(m - 1) + 2] - 2 = 3m^2 - 3m$. Indeed, the orbit of $a_1 \otimes b_1 \otimes c_1$ under the action of $G$ is the Segre variety of rank one tensors, which has dimension $3m - 2 = \dim G - \dim \mathfrak{g}_T$.

3.2. **1$_A$-generic tensors.**

**Proposition 3.1.** Let $T \in A \otimes B \otimes C$ be 1$_A$-generic. Then $\dim G_T \leq 2m^2 - m - 1$ and equality occurs uniquely for the tensor $T_0 = a_1 \otimes (\sum_{j=1}^m b_j \otimes c_j)$.

**Proof.** Let $T \in A \otimes B \otimes C$ be 1$_A$-generic, so there exists $\alpha \in A^*$ such that $T_A(\alpha) \in B \otimes C$ has rank $m$.

After a change of basis in $B$, we may assume that $T_A(\alpha) = \sum_{i=1}^m b_i \otimes c_i$, and after a change of basis in $A$, we may further assume that $\alpha = \alpha^1$. In other words, after a suitable choice of bases in $B$ and in $A$, $T^{ijk} = \delta_{jk}$.

We first observe that any such tensor degenerates to $T_0$ by the degeneration $a_\rho \mapsto 0$ for $\rho = 2, \ldots, m$. Thus, by Lemma 2.2 it suffices to compute $\mathfrak{g}_{T_0}$.

> From (11), we have

$$(1jk) \quad u_1^i \delta_{jk} + v_1^j k + \omega_1^k = 0.$$  

Set $2 \leq \rho, \sigma, \tau \leq m$ and use the summation convention. Setting $j = k$, we have

$$(\rho jj) \quad u_1^\rho = 0.$$  

Now (12) shows that the endomorphism $W \in \mathfrak{gl}(C)$ is completely determined by $u_1^1$ and $V \in \mathfrak{gl}(B)$. In summary, $u_1^1$, $u_1^\rho$ and $V$ completely determine $L \in \tilde{\mathfrak{g}}_{T_0}$. Thus

$$\tilde{\mathfrak{g}}_{T_0} = \left\{ \begin{pmatrix} -(\mu + \nu) & u \\ 0 & U \end{pmatrix}, (\mu I_d + V), (\nu I_d - V^t) \right| \mu, \nu \in \mathbb{C}, U \in \mathfrak{gl}_{m-1}, V \in \mathfrak{gl}_m, u \in \mathbb{C}^{m-1} \right\}.$$  

3.3. **Binding tensors.**

**Proposition 3.2.** Let $T \in A \otimes B \otimes C = \mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$ be a binding tensor. Then $\dim G_T \leq m^2 - 1$, and equality occurs uniquely (up to permutation of the three factors) for the tensor

$$T_{u-triv,m} := a_1 \otimes b_1 \otimes c_1 + \sum_{\rho=2}^m a_1 \otimes b_\rho \otimes c_\rho + \sum_{\rho=2}^m a_\rho \otimes b_1 \otimes c_\rho.$$
Proof. Assume $T$ is $1_A$-generic and $1_B$-generic. As in the proof of Proposition 3.1, we may assume $T(\alpha^1) \in B \otimes C$ has full rank and normalize it to $\sum_i b_i \otimes c_i$. Note that the action $GL_m \to GL(B) \times GL(C)$ defined by $g \mapsto g \otimes (g^t)^{-1}$ preserves this normalization.

By assumption, there is $\beta \in B^*$ such that $T(\beta) \in A \otimes C$ has full rank. Using the simultaneous action of $GL_m$ on $B$ and $C$, we may assume $T(\beta^1) \in A \otimes C$ has full rank and normalize it to $T(\beta^1) = \sum_i a_i \otimes c_i$, that is $T_{ik}^1 = \delta_{ik}$.

After this normalization $T = T_{u-triv, m} + T'$ where $T' \in \langle a_2, \ldots, a_m \rangle \otimes \langle b_2, \ldots, b_m \rangle \otimes C$. Apply the degeneration defined by $(X_\epsilon, Y_\epsilon, Z_\epsilon)$ with

$$
\begin{align*}
X_\epsilon &: \quad a_1 \mapsto \frac{1}{\epsilon} a_1 & Y_\epsilon &: \quad b_1 \mapsto \frac{1}{\epsilon} b_1 & Z_\epsilon &: \quad c_1 \mapsto \epsilon^2 c_1 \\
a_\rho \mapsto \epsilon a_\rho & & b_\sigma \mapsto \epsilon b_\sigma & & c_\tau \mapsto \epsilon^2 c_\tau,
\end{align*}
$$

where $\rho, \sigma, \tau \geq 2$.

Among the bases elements appearing in $T$, $a_i \otimes b_j \otimes c_k$ is fixed if and only if $(i, j, k) = (1, 1, 1)$ or $(i, j, k) = (1, \rho, \rho)$ or $(i, j, k) = (\rho, 1, \rho)$. All the others basis elements have coefficient $\epsilon$, $\epsilon^2$, or $\epsilon^4$. This shows that $\lim_{\epsilon \to 0}(X_\epsilon, Y_\epsilon, Z_\epsilon) \cdot T = T_{u-triv, m}$, therefore $T$ degenerates to $T_{u-triv, m}$.

We conclude that either $T$ and $T_{u-triv, m}$ are isomorphic tensors, or $\dim G_T < \dim G_{T_{u-triv, m}}$ by Lemma 2.2.

An explicit calculation gives

$$
\mathfrak{g}_{T_{u-triv, m}} = \left\{ \begin{pmatrix} \lambda & u \\
0 & -u^2 + \nu \end{pmatrix}, \begin{pmatrix} \mu & v \\
0 & -v^2 + \nu \end{pmatrix}, \begin{pmatrix} -\lambda - \mu & 0 \\
-\nu^2 - \nu u + \nu v & 0 \end{pmatrix} \right| \lambda, \mu, \nu \in \mathbb{C}, u, v \in \mathbb{C}^{m-1}, \overline{W} \in \mathfrak{sl}_{m-1} \right\},
$$

which has dimension $[(m - 1)^2 - 1] + 2(m - 1) + 3$. Hence $\dim \mathfrak{g}_{T_{u-triv, m}} = \dim G_{T_{u-triv, m}} = m^2 - 1$.

This concludes the proof. \qed

The normalization performed in the proof of Proposition 3.2 shows that via the identifications induced by $T(\alpha^1)$ and $T(\beta^1)$, every binding tensor $T$ defines a bilinear map $T : C \times C \to C$ such that $T(\alpha^1, \cdot) : C \to C$ and $T(\cdot, \beta^1) : C \to C$ are both the identity map. The bilinear map $T$ is the structure tensor of a (not necessarily associative) unitary algebra structure on $C$, and $\alpha^1 \simeq \beta^1$ defines the identity element. We refer to 3.2 for a discussion of this perspective.

Remark 3.3. The tensor $T_{u-triv, m}$ is the structure tensor of the trivial unitary algebra of dimension $m$. Explicitly, this algebra may be identified with the quotient $\mathbb{C}[x_1, \ldots, x_{m-1}]/m^2$ of the polynomial ring on $m - 1$ variables modulo the square of the ideal $m = (x_1, \ldots, x_{m-1})$ generated by the variables.

Remark 3.4. The tensor $T_{u-triv, m}$ is concise. It has the largest dimensional symmetry group of any concise tensor we are aware of. Note that the unique up to scale element of $\Lambda^3 \mathbb{C}^3 \subset \mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$ also has an $m^2 - 1$ dimensional symmetry group with $m = 3$ and this tensor is not $1_A$, $1_B$, or $1_C$-generic.

Problem 3.5. Determine the largest possible dimension of the symmetry group of a concise tensor. Furthermore, classify concise tensors with symmetry groups of maximal dimension.
4. **Proof of Lemma 1.2**

In this section we prove Lemma 1.2 which classifies bilinear forms on \( \mathbb{C}^k \) with symmetry group of dimension at least \( \binom{k}{2} \).

The annihilator \( \mathfrak{a}_B \) of \( B \) is characterized by equation (4), which, by Lemma 2.3 is equivalent to the two conditions

\[
\begin{align*}
XQ + QX^t &= 0, \\
XA + AX^t &= 0.
\end{align*}
\]

where \( B = Q + \Lambda \) is the decomposition of \( B \) in its symmetric and skew-symmetric component.

Recall the notation \( W = E \oplus L \oplus F \) with \( e = \dim E \), \( f = \dim F \), and \( \ell = \dim L \).

If \( e = 0 \), then \( \Lambda \) has full rank, and \( \mathfrak{a}_B \) is the annihilator of \( Q \in S^2W^* \) in \( \mathfrak{sp}(\Lambda) \subseteq \mathfrak{gl}(W) \). In particular \( \text{codim}_{\mathfrak{sp}(\Lambda)}(\mathfrak{a}_B) \) equals the dimension of the \( \mathcal{S}\mathcal{P}(\Lambda) \)-orbit of \( Q \).

- If \( \ell = 0 \), then \( Q = 0 \) and \( f = k \). In this case \( B = \Lambda \) and \( \mathfrak{a}_B = \mathfrak{sp}(\Lambda) \), with \( \dim \mathfrak{a}_B = \binom{k+1}{2} \).
  This is case (i).

- If \( \ell = 1 \), then \( \text{rk}(Q) = 1 \). Rank one elements in \( S^2W^* \) are equivalent under the action of \( \mathcal{S}\mathcal{P}(\Lambda) \): the \( \mathcal{S}\mathcal{P}(\Lambda) \)-orbit of \( Q \) is the affine cone over the Veronese variety \( \nu_2(\mathbb{P}W^*) \), which has dimension \( k \). Therefore \( \dim \mathfrak{a}_B = \dim \mathfrak{sp}(\Lambda) - k = \binom{k+1}{2} - k = \binom{k}{2} \). This is case (iii).

The Veronese variety \( \nu_2(\mathbb{P}W^*) \) is the unique closed \( \mathcal{S}\mathcal{P}(\Lambda) \)-orbit in \( \mathbb{P}S^2W^* \). As a consequence, if \( [Q] \in \mathbb{P}S^2W^* \) is not a point in the Veronese variety, then \( \dim(\mathcal{S}\mathcal{P}(\Lambda) \cdot Q) > \dim \nu_2(\mathbb{P}W^*) \) and therefore \( \dim \mathfrak{a}_B < \binom{k}{2} \). Thus if \( e = 0 \) and \( \ell > 1 \), \( B \) is eliminated.

If \( f = 0 \), then \( Q \) has full rank and \( \mathfrak{a}_B \) is the annihilator of \( \Lambda \in \Lambda^2W^* \) in \( \mathfrak{so}(Q) \subseteq \mathfrak{gl}(W) \). In particular \( \text{codim}_{\mathfrak{so}(Q)}(\mathfrak{a}_B) \) equals the dimension of the \( \mathcal{S}\mathcal{O}(Q) \)-orbit of \( \Lambda \).

- If \( \ell = 0 \), then \( \Lambda = 0 \) and \( e = k \). In this case \( B = Q \) and \( \mathfrak{a}_B = \mathfrak{so}(Q) \), with \( \dim \mathfrak{so}(Q) = \binom{k}{2} \), which is case (ii).

If \( \Lambda \neq 0 \), then \( \dim(\mathcal{S}\mathcal{O}(Q) \cdot \Lambda) > 0 \), therefore \( \dim \mathfrak{a}_B < \binom{k}{2} \). This shows that if \( f = 0 \) and \( \ell \geq 1 \) then \( B \) is eliminated.

Now suppose \( \ell = 0 \) and \( e, f > 0 \). In this case \( \mathfrak{a}_B = \mathfrak{so}(Q|_E) \oplus \mathfrak{sp}(\Lambda|_F) \), therefore \( \dim \mathfrak{a}_B = \binom{e}{2} + \binom{f+1}{2} \). Write \( f = k - e \) and consider \( \dim \mathfrak{a}_B \) as a function of \( e \):

\[
\dim \mathfrak{a}_B = \frac{1}{2} \left[ e(e-1) + (k-e+1)(k-e) \right] = \frac{1}{2} [2e^2 - (2 + 2k)e + (k^2 + k)].
\]

The cases \( e = 0 \) and \( e = k \) were considered above.

- If \( e = 1 \), then \( \dim \mathfrak{a}_B = \binom{k}{2} \): this is case (iv).

For \( e \in \{2, \ldots, k-1\} \), then \( \dim \mathfrak{a}_B < \binom{k}{2} \): indeed, the maximal value is attained at \( e = 3 \) and \( e = k - 1 \) and it is \( \dim \mathfrak{a}_B = \frac{1}{2} (k^2 - 3k + 4) = \binom{k-1}{2} + 2 < \binom{k}{2} \) whenever \( k > 3 \).

Finally, we show that if \( e, \ell, f > 0 \) then \( \dim \mathfrak{a}_B < \binom{k}{2} \), eliminating these cases.
Consider the bilinear form $Q$ restricted to $E \oplus L$. It can be fully normalized as

$$(15) \quad Q|_{E \oplus L} = \begin{pmatrix} \text{Id}_q & 0 & 0 & 0 \\ 0 & 0 & \text{Id}_{e-q} & 0 \\ 0 & \text{Id}_{e-q} & 0 & 0 \\ 0 & 0 & 0 & \text{Id}_{\ell+e+q} \end{pmatrix}$$

where $q = \text{rk}(Q|_E : E^* \to E)$.

Moreover, writing $Q, \Lambda$ and $X$ in block form according to the decomposition $E \oplus L \oplus F$, we have

$$B = Q + \Lambda = \begin{bmatrix} Q_{EE} & Q_{EL} & 0 \\ Q_{EL}^t & Q_{LL} & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & \Lambda_{LL} & \Lambda_{LF} \\ 0 & -\Lambda_{LF}^t & \Lambda_{FF} \end{bmatrix}.$$  Write $X = \begin{bmatrix} X_{EE} & X_{EL} & X_{EF} \\ X_{LE} & X_{LL} & X_{LF} \\ X_{FE} & X_{FL} & X_{FF} \end{bmatrix}$.

Then (14) implies

$$\left( X_{EE} X_{EL} \right) \in \mathfrak{so}(Q|_{E \oplus L}), \quad \left( X_{LL} X_{LF} \right) \in \mathfrak{sp}(\Lambda|_{L \oplus F}),$$

$$\left( X_{FE} X_{FL} \right) = 0, \quad \left( X_{EL} X_{EF} \right) = 0.$$

Consider the upper-left size $(e + \ell) \times (e + \ell)$ block of $X$. In blocking $(q, e - q, e - q, \ell - (e - q))$, we have

$$\left( \begin{array}{cc} X_{EE} & 0 \\ X_{LE} & X_{LL} \end{array} \right) = \begin{pmatrix} X_{11} & X_{12} & 0 & 0 \\ X_{21} & X_{22} & 0 & 0 \\ X_{31} & X_{32} & X_{33} & X_{34} \\ X_{41} & X_{42} & X_{43} & X_{44} \end{pmatrix}.$$  

The condition $XQ + QX^t = 0$ provides

$$X_{21} = 0, \quad X_{31} = -X_{12}^t, \quad X_{11} \in \mathfrak{so}_q, \quad X_{32} \in \mathfrak{so}_{e-q}.$$  

This gives the upper bound

$$\dim \mathfrak{h}_B \leq \dim \mathfrak{sp}(\Lambda|_{L \oplus F}) + \dim \mathfrak{so}_q + \dim \mathfrak{so}_{e-q} + q \cdot (e - q).$$

We conclude

$$(16) \quad \dim \mathfrak{h}_B - \binom{k}{2} \leq \binom{\ell + f + 1}{2} + \binom{q}{2} + \binom{e - q}{2} + q(e - q) - \binom{k}{2} = (1 - e)(\ell + f).$$

Thus the case $e \geq 2$ is excluded from consideration.

We are left to analyze the case $e = 1$. In this case (16) already implies $\dim \mathfrak{h}_B \leq \binom{k}{2}$ and it remains to show that the bound is strict. We consider two cases:

If $q = 1$ or $q = 0$ and $\ell \geq 2$, then $Q|_L \neq 0$. In this case $\dim \mathfrak{h}_B$ equals the dimension of the annihilator of $Q|_L$ in $\mathfrak{sp}(\Lambda|_{L \oplus F})$, hence it is strictly smaller than $\dim \mathfrak{sp}(\Lambda|_{L \oplus F}) = \binom{k}{2}$.

If $q = 0$ and $\ell = 1$, then one concludes via a direct calculation showing that $\dim \mathfrak{h}_B$ equals the dimension of the annihilator in $\mathfrak{sp}(\Lambda|_{L \oplus F})$ of a generator of $L$.

This concludes the proof of the Lemma 1.2. \qed
5. Proof of Theorem 1.1. Part one

First, we prove a result which allows us to achieve a convenient normalization of a 1-generic tensor (Proposition 5.2), similar to the standard presentation of the Coppersmith-Winograd tensor:

**Lemma 5.1.** Let $T \in A \otimes B \otimes C$ be 1-generic. Then there exist $\alpha \in A^*, \beta \in B^*, \gamma \in C^*$ such that

- $T(\alpha, \beta, \gamma) = 0$
- $T(\alpha, -, -) \in B \otimes C$, $T(\beta, -, -) \in A \otimes C$, $T(-, -, \gamma) \in A \otimes B$ are full rank.

**Proof.** Let $\Omega_A := \{\alpha \in A^* : T(\alpha, -,-) \text{ is full rank}\}$ and similarly $\Omega_B, \Omega_C$; then $\Omega_A, \Omega_B, \Omega_C$ are Zariski open in $A^*, B^*, C^*$ respectively. Consider the regular map $T : \Omega_A \times \Omega_B \to C$. It has irreducible image which, by conciseness of $T$, is not contained in any hyperplane. Consequently, this image is the cone over a positive dimensional set, so the set

$$\Theta_C = \{\gamma \in C^* : \gamma^+ \cap T(\Omega_A \times \Omega_B) \neq \emptyset\}$$

contains a Zariski open set. In particular, $\Theta_C \cap \Omega_C \neq \emptyset$.

Let $\gamma \in \Theta_C \cap \Omega_C$, let $c \in \gamma^+ \cap T(\Omega_A \times \Omega_B)$ and let $(\alpha, \beta) \in \Omega_A \times \Omega_B$ be an element satisfying $T(\alpha, \beta) = c$. The triple $(\alpha, \beta, \gamma)$ satisfies the desired conditions. $\square$

As a consequence of Lemma 5.1, there exist bases $\{a_i\}_{i=1,...,m}$ of $A$, $\{b_i\}_{i=1,...,m}$ of $B$, $\{c_i\}_{i=1,...,m}$ of $C$, such that

$$T = a_1 \otimes B_A + \sigma_{12}(b_1 \otimes B_B) + B_C \otimes c_1 + T' : \sigma_{12} \in \mathfrak{S}_3$$

is the permutation which swaps the first and second factors and

- $B_A, B_B, B_C$ are full rank bilinear forms;
- the coefficient of $b_1 \otimes c_1$ in $B_A$ is 0, and similarly for the corresponding coefficient in $B_B$ and $B_C$;
- $T' \in \langle a_2, \ldots, a_m \rangle \otimes \langle b_2, \ldots, b_m \rangle \otimes \langle c_2, \ldots, c_m \rangle$.

Let $A' = \langle a_2, \ldots, a_m \rangle$, $A'' = \langle a_2, \ldots, a_{m-1} \rangle$ and similarly on the other factors. In this basis consider the coefficient of $b_1 \otimes c_1$ in $T$; by the above it lies in $A'$ and is nonzero. Thus we may further change basis in $A'$ so that this part of $T$ is exactly $a_m \otimes b_1 \otimes c_1$. Doing the same with $B'$ and $C'$, we may additionally assume

$$T = a_1 \otimes b_1 \otimes c_m + a_1 \otimes b_m \otimes c_1 + a_m \otimes b_1 \otimes c_1$$

$$+ a_1 \otimes B'_A + \sigma_{12}(b_1 \otimes B'_B) + B'_C \otimes c_1 + \tilde{T}$$

where $B''_A \in B'' \otimes C''$, $B''_B \in A'' \otimes C''$, $B''_C \in A'' \otimes B''$ are full rank bilinear forms, and

$$\tilde{T} \in A' \otimes B' \otimes C' \oplus a_1 \otimes b_m \otimes C' \oplus a_m \otimes b_1 \otimes C'$$

$$\oplus a_1 \otimes B' \otimes c_m \oplus a_m \otimes B' \otimes c_1$$

(17) $$\oplus A' \oplus b_1 \oplus c_m \oplus A' \otimes b_m \otimes c_1$$

Now, $B''_A$ defines an isomorphism $B'' \simeq (C'')^*$, and similarly $B''_B : A'' \simeq (C'')^*$. Changing bases in $B''$ and $C''$ again, we may suppose these bases are dual to the distinguished basis in $A''$ under these isomorphisms. In these bases $B''_B$ and $B''_C$ are each given by the identity matrix. Writing $B = B''_A$, we have shown
Proposition 5.2. After a change of bases, a 1-generic tensor $T \in A \otimes B \otimes C$ may be written as $T = S_B + \bar{T}$, where

$$S_B = a_1 \otimes b_1 \otimes c_m + a_1 \otimes b_m \otimes c_1 + a_m \otimes b_1 \otimes c_1 + \sum_{\rho=2}^{m-1} [a_1 \otimes b_1 \otimes c_1]$$

(18)

and $\bar{T}$ is as in Proposition 5.1.

We have called such tensors $S_B$ skeletal.

Proposition 5.3. A tensor $T = S_B + \bar{T}$ normalized as in Proposition 5.2 degenerates to $S_B$.

Proof. We determine a degeneration which fixes $S_B$ and degenerates $\bar{T}$ to 0. Define $f_\varepsilon \in GL(A)$, $g_\varepsilon \in GL(B)$, $h_\varepsilon \in GL(C)$ as follows:

$$f_\varepsilon : a_1 \mapsto \frac{1}{\varepsilon} a_1$$
$$a_j \mapsto \varepsilon a_j \quad j = 2, \ldots, m - 1$$
$$a_m \mapsto \varepsilon^2 a_m$$

$$g_\varepsilon : b_1 \mapsto \frac{1}{\varepsilon^3} b_1$$
$$b_j \mapsto \varepsilon b_j \quad j = 2, \ldots, m - 1$$
$$a_m \mapsto \varepsilon^4 a_m$$

$$h_\varepsilon : c_1 \mapsto \frac{1}{\varepsilon^5} c_1$$
$$c_j \mapsto \varepsilon c_j \quad j = 2, \ldots, m - 1$$
$$c_m \mapsto \varepsilon^6 c_m.$$

Then $(f_\varepsilon, g_\varepsilon, h_\varepsilon)(S_B) = S_B$ and $\lim_{\varepsilon \to 0}(f_\varepsilon, g_\varepsilon, h_\varepsilon)(\bar{T}) = 0$, therefore

$$\lim_{\varepsilon \to 0}(f_\varepsilon, g_\varepsilon, h_\varepsilon)(T) = S_B. \quad \Box$$

It will be convenient to consider 1-generic tensors normalized as in Proposition 5.2 more invariantly. In particular, as in the remarks preceding the proposition, there are natural identifications $A'' \leftrightarrow B'' \leftrightarrow (C'')''$. Denote these common spaces by $M^*$, so that we have identifications $A \leftrightarrow L_1^A \oplus M^* \oplus L_A$, $B \leftrightarrow L_1^B \oplus M^* \oplus L_B$ and $C \leftrightarrow L_1^C \oplus M \oplus L_C$. Then $S_B$ has the expression

$$S_B = a_1 \otimes b_1 \otimes c_m + a_1 \otimes b_m \otimes c_1 + a_m \otimes b_1 \otimes c_1 + a_1 \otimes \text{Id}_M + \sigma_{12}(b_1 \otimes \text{Id}_M) + B \otimes c_1,$$

where $B \in M^* \otimes M^*$. In this presentation it clear that $GL(M) \subset GL(A) \times GL(B) \times GL(C)$ acts on $S_B$ exactly as its action on bilinear forms $B$. More generally, we have

Proposition 5.4. Let $m \geq 5$. Let $h_B$ be the annihilator of $B$ under the action of $gl(M^*)$. Then the following is a convenient choice of lift of $g_{S_B}$ to $gl(A) \oplus gl(B) \oplus gl(C)$:

(19)

$$g_{S_B} = \left\{ \begin{pmatrix} 2t & v & u_m^1 \\ 0 & X - t\text{Id}_M & vB - u \\ 0 & 0 & -4t \end{pmatrix}, \begin{pmatrix} 2t & w & v_m^1 \\ 0 & X - t\text{Id}_M & Bw - u \\ 0 & 0 & -4t \end{pmatrix}, \begin{pmatrix} 2t & u - vB - Bw & u_m^1 \\ 0 & -X^t - t\text{Id}_M & -v - w \\ 0 & 0 & -4t \end{pmatrix} \right\}$$
where \( u \in M^* \), \( v, w \in M \), \( t \in \mathbb{C} \), \( u_m^1 + v_m^1 + w_m^1 = 0 \), \( X \in \mathfrak{h}_g \). In particular, \( \dim \mathfrak{g}_S = 3m - 3 + \dim \mathfrak{h}_g \).

**Proof.** Fix the index range \( \rho, \sigma, \tau = 2, \ldots, m - 1 \). Then (111) specializes to the following:

\[
\begin{align*}
(111) & \quad u_m^1 + v_m^1 + w_m^1 = 0 \\
(11\tau) & \quad u_\tau^1 + v_\tau^1 + w_\tau^1 = 0 \\
(1\sigma 1) & \quad u_\sigma^1 B^{\sigma \tau} + v_\sigma^1 + w_\sigma^1 = 0 \\
(\rho 11) & \quad u_{\rho m} + v_{\rho m} B^{\rho \rho} + w_{\rho m} = 0 \\
(1\sigma \tau) & \quad u_1^1 \delta^{\sigma \tau} + v_1^1 + w_1^1 = 0 \\
(\rho 1\tau) & \quad u_\rho^1 + v_1^1 \delta^{\rho \tau} + w_\rho^1 = 0 \\
(\rho \sigma 1) & \quad u_\rho^1 B^{\rho \sigma} + v_\sigma^1 B^{\rho \sigma} + w_1^1 B^{\rho \sigma} = 0 \\
(\rho \sigma \tau) & \quad u_1^1 \delta^{\sigma \tau} + v_1^1 \delta^{\rho \tau} + w_1^1 B^{\rho \sigma} = 0 \\
(11m) & \quad u_1^1 + v_1^1 + w_m^m = 0 \\
(1m1) & \quad u_1^1 + v_m^m + w_1^1 = 0 \\
(m11) & \quad u_m^m + v_1^1 + w_1^1 = 0 \\
(1mm) & \quad v_1^m + w_1^m = 0 \\
(m1m) & \quad v_m^m + w_1^1 = 0 \\
(mm1) & \quad u_1^m + v_1^m = 0 \\
(1\sigma m) & \quad u_1^\sigma + v_1^m = 0 \\
(m\sigma 1) & \quad u_m^m B^{\rho \sigma} + v_1^\sigma = 0 \\
(\rho m1) & \quad u_\rho^m + w_1^m = 0 \\
(pm1) & \quad u_\rho^m + v_1^m B^{\sigma \sigma} = 0 \\
(1m\tau) & \quad v_\tau^m + w_1^1 = 0 \\
(m1\tau) & \quad u_\tau^m + w_1^1 = 0
\end{align*}
\]

Equation (111) determines \( w_m^1 \). Write \( \Pi = (u_{2, \ldots, m - 1}^1) \), \( \hat{u} = (u_{m, \ldots, m - 1}^1)^t \) and similarly for \( \overline{v}, \overline{w} \) and \( \hat{v}, \hat{w} \). The next three equations imply

\[
\begin{align*}
\hat{u} & = -B v^t - \overline{w}^t \\
\hat{v} & = -\Pi B^t - \overline{v}^t \\
\hat{w} & = -\overline{v}^t - \overline{w}^t
\end{align*}
\]

Equations (1\sigma \tau), (\rho 1\tau) allow us to solve for \( v_1^\sigma, u_\rho^\sigma \), and plugging the solution into (\rho \sigma 1) gives

\[
\begin{align*}
\Pi B + B \Pi^t & = 0 \\
u_1^1 + v_1^1 + w_1^1 & = 0
\end{align*}
\]

where \( \Pi = (u_\rho^m)_{\rho, \sigma = 2, \ldots, m - 1} \). The first line is equivalent to the assertion \( \Pi \in \mathfrak{h}_g \).

Ignoring the (\rho \sigma \tau) equations for the moment, the next three equations determine \( w_m^m, v_m^m, v_m^m \) as in (19), and the next three form a system that implies \( u_1^m, v_1^m, w_1^m = 0 \).
For every fixed $\rho$, let $\overline{\rho}$ be such that $B^{\rho, \overline{\rho}} \neq 0$. Then the last six equations form a full rank system in six unknowns, giving $u^\rho_1, v^\rho_1, u^\rho_\overline{\rho}, v^\rho_\overline{\rho}, w^\rho_\overline{\rho} = 0$.

Finally at this point the equations $(\rho \sigma \tau)$ are trivial and we obtain (19) which has the asserted dimension. \[ \square \]

All of the claims about the exceptional tensors in the theorem follow at once from Proposition 5.4.

It remains to show the upper bound on the dimension of the symmetry group of an arbitrary 1-generic tensor $T$. Write $T = S_B + \widetilde{T}$ as in Proposition 5.2. By Lemma 2.2, Proposition 5.3, and Proposition 5.4, we have

\begin{equation}
\dim g_T \leq \dim g_{S_B} = 3m - 3 + \dim h_B,
\end{equation}

with equality if and only if $T \simeq S_B$. Thus, except in the case $B$ is a nondegenerate skew-symmetric bilinear form, Lemma 1.2 establishes the remaining claims of the theorem.

We must analyze the case when $B = \Lambda$ is a nondegenerate skew-symmetric form. In this case $m - 2$ is even, so $m = 2p$ is even as well. The analysis above gives

\[ \dim g_T \leq \dim g_{S_\Lambda} = 3m - 3 + \dim \mathfrak{sp}(\Lambda) = \frac{m^2}{2} + \frac{3m}{2} - 2 \]

so it remains to prove that if $\widetilde{T} \neq 0$, the dimension drops by at least $m - 1$.

6. END OF PROOF OF THEOREM 1.1: THE REMAINING CASE

First, we obtain a more specific normalization than that shown in Proposition 5.2. Write $T = S_B + \widetilde{T}$ as before, and now change bases such that

\begin{align*}
a_m &\mapsto a_m + u^1_m a_1 \\
b_m &\mapsto b_m + v^1_m b_1 \\
c_m &\mapsto c_m + w^1_m c_1 \\
a_1 &\mapsto a_1 + u^\rho_1 a_\rho \\
b_1 &\mapsto b_1 + v^\rho_1 b_\rho \\
c_1 &\mapsto c_1 + w^\rho_1 c_\rho
\end{align*}

Then

\begin{align*}
T^{1mm} &\mapsto T^{1mm} + v^1_m + w^1_m \\
T^{m1m} &\mapsto T^{m1m} + u^1_m + w^1_m \\
T^{mm1} &\mapsto T^{mm1} + u^1_m + v^1_m \\
T^{1m\rho} &\mapsto T^{1m\rho} + v^\rho_1 \\
T^{mp1} &\mapsto T^{mp1} + v^\rho_1 \\
T^{1mp} &\mapsto T^{1mp} + w^\rho_1 \\
T^{m1\rho} &\mapsto T^{m1\rho} + w^\rho_1 \\
T^{1m\rho} &\mapsto T^{1m\rho} + u^\rho_1 \\
T^{\rho1m} &\mapsto T^{\rho1m} + u^\rho_1 \\
T^{\rho m1} &\mapsto T^{\rho m1} + u^\rho_1
\end{align*}
and the only other entries of $T$ which change are $T^{ijk}$, where $i, j, k \geq 2$. In particular, after such a change of variables, $T$ is still of the form $S_{\mathfrak{g}} + \tilde{T}$. Choose $u^1_m, v^1_m, w^1_m$ such that $-T^{1mm} = u^1_m + w^1_m, -T^{m1m} = u^1_m + w^1_m$, and $-T^{m1m} = u^1_m + v^1_m$, to send these quantities to zero. Choose $u^1 = -T^{1m1}, v^1 = -T^{m1m}$ and $u^1 = -T^{1mp}$ to send these quantities to zero. We have shown that an arbitrary 1-generic tensor may be written as $S_{\mathfrak{g}} + \tilde{T}$, where now

\[
\tilde{T} \in A' \otimes B' \otimes C' \oplus a_m \otimes b_1 \otimes C'' \oplus a_m \otimes B'' \otimes c_1 \oplus A'' \oplus b_m \otimes c_1.
\]

In the skeletal case we have $u := \langle u^1, u^1, v^1, v^1, v^1, w^1, w^1 \rangle$ (which is of dimension $3m - 1$) independent and independent of elements of $\mathfrak{sp}(M)$. We need to show that after our normalizations, if $\tilde{T} \neq 0$, then there are at least $m - 1$ relations among the elements of $u, \mathfrak{sp}(M)$.

Fix index ranges $2 \leq \xi, \eta \leq p, 2 \leq \rho, \sigma, \tau \leq m - 1$. Write $n := \langle u^1, u^1, v^1, v^1, v^1, u^1 \rangle$, the nilpotent part of $u$.

We have $T^{111}, T^{11\tau}, T^{1\rho1}, T^{\rho11}, T^{\rho m\tau}, T^{m\tau m}, T^{m \tau m m} = 0, T^{11m} = T^{1m1} = T^{m11} = 1$, and by the normalization lemma $T^{1mp} = T^{1pm} = T^{\rho1m} = 0$.

Let $e^\rho = 1$ if $\rho \leq p$ and $e^\rho = -1$ if $\rho > p$. Let $\tilde{\rho} = \rho + p$ if $\rho \leq p$ and $\tilde{\rho} = \rho - p$ if $\rho > p$. We have $T^{\rho\sigma1} = e^\rho \delta^\sigma_\tau$. Write $A = L^A_1 \oplus M^A \oplus L^A_1$ etc...

6.1. Outline.

Step 1: Solve $u^\rho_m, v^\rho_m, w^\rho_m, w^1_m$ (i.e., elements of $L^A_1 \otimes (M^A)^*, L^B_1 \otimes (M^B)^*, L^C_1 \otimes (M^C)^*, L^C_1 \otimes (L^C_1)^*$) in terms of $u$, solve $v^\rho_m, v^\rho_m$ (i.e., elements of $\mathfrak{gl}(M)_A, \mathfrak{gl}(M)_B$) in terms of $u \oplus \mathfrak{sp}(M)_C \oplus \mathfrak{sp}(M)_C^c$, and solve $u^\rho_m, v^\rho_m, w^\rho_m$ in terms of $u$.

Step 2: Using $(\rho\sigma1)$ solve $\mathfrak{sp}(M)_C^c$ in terms of $u$.

Step 3: use $(\rho\sigma\tau)$ to severely restrict $T^{\rho\sigma\tau}$, namely among the irreducible modules in $M^{\otimes3}$ only the highest weight vector of $S^3M$ and the three copies of $M$ (which we denote $M_A, M_B, M_C$) can occur, i.e., we have $3(m - 2) + 1$ parameters instead of $(m - 2)^3$. Moreover, if a highest weight vector is nonzero, at most one more relation among elements of $u \oplus \mathfrak{sp}(M)_C$ is allowed.

Step 4: We observe that the $\mathfrak{sp}(M) \oplus \mathfrak{sp}(M)_C$ terms do not appear if we symmetrize the $(\rho\sigma1)$ equations. In the special case $\rho = \sigma = 2$, we obtain in particular $u^1 \mathcal{T}^{22(p + 2)} \equiv 0$ mod other basis elements of $u$, which shows that if $T^{22(p + 2)} \neq 0$, no more relations among elements of $u \oplus \mathfrak{sp}(M)_C$ are allowed. Assuming $T^{22(p + 2)} \neq 0$, we obtain three relations among six modules isomorphic to $M$ that must occur if no further relations among the $u$ are to occur. Explicitly the relations are $L^A_1 \otimes M^B \otimes L^C_1 = T^{m11} = 0, M^A_1 = v^\sigma_1 = e^\sigma \delta^\tau_\nu = M^B_1, M^A \otimes L^B_1 \otimes L^C_1 = L^A_1 \otimes M^B \otimes M^C + L^A_1 + L^C_1 \otimes M^B \otimes M^C$.

Step 5: Using $(\rho\sigma\tau) = (22(p + 2))$ modulo $u$ we obtain another relation which shows $T^{22(p + 2)} = 0$.

Step 6: We revisit the symmetrized $(\rho\sigma1)$ equations with the knowledge $T^{22(p + 2)} = 0$ and obtain 3 expressions involving the six modules $T^{m11}$ (i.e., $M^A \otimes L^B_1 \otimes L^C_1, T^{m11}, T^{m11}$ (recall that its other permutations have been normalized to zero), and $v^\rho, v^\rho, v^\rho$ which are defined to be the three copies of $M$ in $M^{\otimes3}$, and if any of these expressions is nonzero, we obtain $m - 2$ relations and if any two are nonzero we obtain more than $m - 1$ relations so at most one of the expressions may be nonzero.

Step 7: We explicitly solve for $u^\rho_1, v^\rho_1, w^\rho_1, v^\rho_1, v^\rho_1$ in terms of elements of $u \oplus \mathfrak{sp}(M) \oplus \mathfrak{sp}(M)_C$. 

\[
\begin{align*}
\tilde{T} &
\end{align*}
\]
Step 8: We consider the ($\rho\sigma\tau$) equations modulo $u$ and obtain 6 additional expressions. among the 6 quantities of Step 6, and observe that at most two of these 6 expressions can be nonzero, as each imposes $m-2$ conditions. Even if two of the expressions are nonzero, we have enough equations combined with the previous to show all the terms defined in Step 6 are zero.

Step 9: We show $T^{\rho\sigma\tau}, T^{\rho\sigma\tau}, T^{\mu\sigma\tau}$, each of which lie in $M \otimes M \sim M^* \otimes M$ (tensored with a line) are zero by first considering the ($\rho\sigma\tau$) equations and their permutations modulo $n$ to show they can only be the trivial representation plus the highest weight vector in $S^2M$ and if the highest weight vector in $S^2M$ is nonzero, there can be at most two more relations. Here we reduce from $3(m-2)^2$ parameters to 6. We then use the ($\rho\sigma\tau$) equations to first eliminate the highest weight vector and then to eliminate the trivial module.

Step 10: We show $T^{\rho\sigma\tau}, T^{\rho\sigma\tau}, T^{\mu\sigma\tau}$ are zero using the ($\rho\sigma\tau$) equations.

Step 11: We show the last unassigned $T^{ijk}$, namely $T^{mmm}$ is zero via the ($\rho\sigma\tau$), ($\rho\sigma\tau$), and ($\mu\nu\tau$) equations to complete the proof.

6.2. Preliminaries. We write out some of the equations:

(111) \[ u_1^m + v_1^m + w_1^m = 0 \]

(11\tau) \[ u_1^\tau + v_1^\tau + w_1^\tau + u_1^m T^{m1\tau} = 0 \]

(1\sigma1) \[ \epsilon^\sigma u_1^\sigma + v_1^\sigma + w_1^\sigma + u_1^m T^{m\sigma1} = 0 \]

(\rho11) \[ u_1^\rho + v_1^\rho + w_1^\rho + u_1^m T^{m\rho1} = 0 \]

(1\sigma\tau) \[ u_1^{\sigma\tau} + v_1^{\sigma\tau} + w_1^{\sigma\tau} + u_1^m T^{m\sigma\tau} + v_1^\sigma + w_1^\sigma = 0 \]

(\rho1\tau) \[ u_1^\rho + u_1^m T^{m1\tau} + v_1^{\sigma\tau} + v_1^\sigma T^{\sigma\tau} + v_1^\rho T^{\rho\sigma\tau} + w_1^\rho T^{\rho\rho\tau} + u_1^m T^{m\rho\sigma\tau} + w_1^m T^{m\rho\mu\tau} + u_1^m T^{m\rho\sigma1} + v_1^\rho T^{m\rho1} = 0 \]

(\rho\sigma\tau) \[ u_1^{\rho\sigma\tau} + v_1^{\rho\sigma\tau} + w_1^{\rho\sigma\tau} + u_1^\rho T^{\rho\rho\tau} + u_1^m T^{m\rho\sigma\tau} + v_1^\rho T^{\rho\sigma\tau} + v_1^m T^{m\rho\sigma\tau} + w_1^\rho T^{\rho\rho\tau} + w_1^m T^{m\rho\rho\tau} = 0 \]

(11m) \[ u_1^m + v_1^m + w_1^m = 0 \]

(1m1) \[ u_1^m + v_1^m + w_1^m + u_1^m T^{m1\tau} = 0 \]

(m11) \[ u_1^m + v_1^m + w_1^m + v_1^m T^{m1\tau} + w_1^m T^{m1\tau} = 0 \]

(mnm) \[ u_1^m T^{mnm} + v_1^m T^{mnm} + w_1^m + u_1^m = 0 \]

(m1m) \[ u_1^m + v_1^m T^{mnm} + v_1^m T^{mnm} + u_1^m + w_1^m T^{m1\tau} = 0 \]

(mm1) \[ u_1^m + v_1^m + w_1^m T^{mnm} + w_1^m T^{mnm} + u_1^m T^{m1\tau} + w_1^m T^{m1\tau} = 0 \]

(1\sigma\mu) \[ u_1^\rho T^{\rho\sigma\mu} + u_1^m T^{m\rho\sigma\mu} + v_1^\sigma + w_1^\sigma = 0 \]

(\rho1\mu) \[ u_1^\rho + v_1^\rho + v_1^m T^{m\rho\sigma\mu} + w_1^\rho T^{\rho\mu\sigma\tau} + v_1^m T^{m\rho\sigma\mu} + w_1^m T^{m\rho\sigma1} + v_1^m T^{m\rho1} = 0 \]

(\mu1\tau) \[ u_1^\mu + v_1^\mu T^{m\mu\tau} + v_1^\mu + w_1^\mu T^{m1\tau} + v_1^\mu T^{m1\tau} + w_1^\mu T^{m1\tau} = 0 \]

The asymmetry in some of these expressions is caused by our normalizations.
6.3. **Step 1.** We solve:

\[
\begin{align*}
(111) \quad w_m^1 &= -(u_m^1 + v_m^1) \\
(11\tau) \quad w^\tau_m &= -(u^\tau_m + v^\tau_m + u_m^1 T^{m1\tau}) \\
(\rho 11) \quad u^\sigma_m &= -(u^\sigma_m + \epsilon^\sigma v^\sigma_m + u_m^1 T^{m\sigma 1}) \\
(1\sigma 1) \quad v^\sigma_m &= -(\epsilon^\sigma u^\sigma_m + u^\sigma_m + v_m^1 T^{\sigma m1}) \\
(\rho 1\tau) \quad u^\rho_\tau &= -[w^\rho_\tau + v_1^1 \delta^\rho_\tau + v^\rho_1 T^{\rho \sigma \tau} + v_m^1 T^{\rho \sigma \tau} - (w^1_\rho + \epsilon^\rho v^1_\rho + u_m^1 T^{m \rho \tau 1})T^{m1\tau}] \\
(1\sigma \tau) \quad v^\sigma_\tau &= -[w^\sigma_\tau + u^1_\sigma \delta^\sigma_\tau + u^\rho_1 T^{\rho \sigma \tau} + v_m^1 T^{m \sigma \tau}] \\
(11m) \quad w_m^m &= -(u^1_m + v^1_m) \\
(1m1) \quad v_m^m &= -(u^1_m + u^1_\rho T^{m1m}) \\
(m11) \quad u_m^m &= -(v^1_m + u^1_\sigma T^{m \sigma m 1} + w^1_m T^{m1\tau}).
\end{align*}
\]

6.4. **Step 2.** Write \( w^\rho_\sigma = x^\rho_\sigma + y^\rho_\sigma \) where \( (x^\rho_\sigma) \in \mathfrak{sp}(M) \subset \mathfrak{gl}(M) \) and \( (y^\rho_\sigma) \in \mathfrak{sp}(M)^c \) where \( \mathfrak{sp}(M)^c \subset \mathfrak{gl}(M) \) is the complementary \( \mathfrak{sp}(M) \)-module to \( \mathfrak{sp}(M) \) in \( \mathfrak{gl}(M) \). We have \( (m-2) \) relations on the \( x^\rho_\sigma \): \( \epsilon^\rho x^\rho_\sigma + \epsilon^\sigma x^\sigma_\tau = 0 \), which may also be written \( \epsilon^\rho x^\rho_\sigma - \epsilon^\rho x^\sigma_\tau = 0 \). The \( (m-2)^2+1 \) relations on the \( y^\rho_\sigma \) are \( \epsilon^\sigma y^\rho_\sigma - \epsilon^\rho y^\sigma_\tau = 0 \). Thus if we consider

\[
(\rho \sigma 1) \quad 0 = \epsilon^\rho [w^\sigma_\rho + v_1^1 \delta^\sigma_\rho + v^\rho_1 T^{\rho \sigma \tau} + v_m^1 T^{m \sigma \tau} - (w^1_\rho + \epsilon^\sigma v^1_\rho + u_m^1 T^{m \rho \tau 1})T^{m1\tau}] \\
- \epsilon^\rho [w^\sigma_\rho + u^1_\sigma \delta^\sigma_\rho + u^\rho_1 T^{\rho \sigma \tau} + u_m^1 T^{m \sigma \tau}] + w^1_\rho \epsilon^\sigma \delta^\sigma_\rho + w^1_\rho T^{\rho \sigma \tau} + w^1_\sigma T^{\sigma \rho \tau} + (u^1_m + v^1_m)T^{m \sigma \tau} \\
- (w^1_\rho + \epsilon^\rho v^1_\rho + u^1_m T^{m \rho \tau 1})T^{m \sigma m 1} - (\epsilon^\sigma u^1_\sigma + v^1_m T^{m \sigma m 1} + w^1_\tau T^{m \sigma \tau}).
\]

the \( x^\rho_\sigma \) are eliminated and the \( (m-2) \) equations exactly allow us to solve for the \( (m-2) \) independent \( y^\rho_\sigma \) in terms of the elements of \( u \). We obtain

\[
\begin{align*}
\epsilon^\rho y^\sigma_\rho - \epsilon^\sigma y^\rho_\sigma &= \\
&= \epsilon^\rho [v^1_1 \delta^\sigma_\rho + v^1_\sigma T^{\rho \sigma \tau} + v_m^1 T^{m \sigma \tau} - (u^1_\rho + \epsilon^\sigma v^1_\rho + u_m^1 T^{m \rho \tau 1})T^{m1\tau}] \\
&\quad - \epsilon^\rho [u^1_1 \delta^\sigma_\rho + u^\rho_1 T^{\rho \sigma \tau} + u_m^1 T^{m \sigma \tau}] \\
&\quad + w^1_1 \epsilon^\rho \delta^\sigma_\rho + w^1_\sigma T^{\rho \sigma \tau} + u^1_m T^{m \sigma \tau} - (w^1_\rho + \epsilon^\rho v^1_\rho + u_m^1 T^{m \rho \tau 1})T^{m \sigma m 1} - (\epsilon^\sigma u^1_\sigma + v^1_m T^{m \sigma m 1} + w^1_\sigma T^{m \sigma \tau}) \\
&\quad + \epsilon^\rho [\epsilon^\sigma v^1_1 + \epsilon^\sigma u^1_1 - \epsilon^\rho u^1_\sigma] - \epsilon^\rho u^1_\rho T^{\rho \sigma \tau} - \epsilon^\rho u^1_\sigma T^{m \sigma m 1} + u^1_m (\epsilon^\sigma T^{m \rho \tau 1}T^{m1\tau} - \epsilon^\sigma T^{m \sigma m 1}T^{m \rho \tau} + T^{m \sigma m 1}T^{m \rho \tau}) \\
&\quad + v^1_\sigma \epsilon^\rho T^{\rho \sigma \tau} - v^1_\rho (\epsilon^\sigma T^{m1\sigma} + \epsilon^\sigma T^{m1\tau}) + v^1_m (\epsilon^\sigma T^{m1\sigma} + \epsilon^\sigma T^{m1\tau}) \\
&\quad + w^1_\rho (\epsilon^\sigma T^{m1\sigma} - T^{m \sigma m 1}) - w^1_\sigma T^{m \sigma m 1} + w^1_\tau T^{m \sigma \tau}.
\end{align*}
\]
6.5. **Step 3.** Consider \((\rho\sigma\tau)\)

\[
0 = u_1^0 \delta^{\sigma\tau} + v_1^0 \delta^{\rho\sigma} + w_1^0 e^\sigma \delta^\rho + \left[ (w_\rho^0 + v_\rho^1) \delta^{\rho\rho'} + v_\sigma^1 T^{\rho\sigma'} + v_1^m T^{\rho\sigma'} + \sum_\sigma [w_\sigma^0 + (w_\rho^1 + \delta^{\rho\rho'}) + u_1^m T^{\rho\sigma'} T^{\sigma\tau}] \right] - (w_\rho^0 + v_\rho^1 + u_1^m T^{\rho\sigma'}) \sum_{\rho\sigma\tau} \\
- (w_\rho^0 + v_\rho^1 + u_1^m T^{\rho\sigma'}) \sum_{\rho\sigma\tau} \\
+ u_1^m T^{\rho\sigma'} - (u_1^0 + v_1^1 + u_1^m T^{\rho\sigma'}) T^{\rho\sigma'} \\
+ w_\tau^1 T^{\rho\sigma'} - (u_\tau^1 + v_\tau^1 + u_1^m T^{\rho\sigma'}) T^{\rho\sigma'} \\
+ \ldots
\]

Specialize to where \(\rho \neq \overline{\sigma}, \rho \neq \tau, \sigma \neq \tau\) and, after writing \(w_\rho^0 = x_\rho^0 + y_\rho^0\), notice that modulo \(u\), we just have the action of \(\mathfrak{sp}(M)_C\) on \(M \otimes M \otimes M\). The stabilizers of a highest weight vector (and hence any vector) in the modules \(M_{\omega_3}\) and \(M_{\omega_2 + \omega_3}\) have codimension greater than \(m - 1\), so \(T^{\rho\sigma\tau} a_\rho \otimes b_\sigma \otimes c_\tau\) cannot have nonzero components in these modules. Moreover, for the \(S^3 M\) component, unless it is a highest weight vector, the case is similarly eliminated. The three copies corresponding to \(M\) are exactly the three cases \(\rho = \omega, \rho = \tau, \sigma = \tau\) and our analysis says nothing about them so far.

Thus the only components that are possibly nonzero in \(M \otimes M \otimes M\) are \(S^3 M\) and the three copies of \(M\), and we may assume the component in \(S^3 M\) is a highest weight vector which we take to be \(a_2 \otimes b_2 \otimes c_{p+2}\). That is

\[
T^{\rho\sigma\tau} a_\rho \otimes b_\sigma \otimes c_\tau \\
= T^{22p+2} a_2 \otimes b_2 \otimes c_{p+2} + e^\rho t_{C,0}^{\rho\rho'} a_\rho \otimes b_\rho' \otimes c_\tau' + t_\rho^{\rho\sigma} a_\rho \otimes b_\sigma \otimes c_\sigma + t_\rho^{\rho\rho'} a_\rho \otimes b_\rho' \otimes c_\rho \\
:= T^{22p+2} a_2 \otimes b_2 \otimes c_{p+2} + t_C^0 \sum_\rho e^\rho a_\rho \otimes b_\rho \otimes c_\tau' + t_A^0 \sum_\rho a_\rho \otimes b_\rho \otimes c_\sigma + t_B^0 \sum_\rho a_\rho \otimes b_\rho \otimes c_\rho
\]

where \(t_C^{\rho\rho'}\) is independent of \(\rho\), \(t_A^{\rho\sigma}\) is independent of \(\sigma\), \(t_B^{\rho\rho'}\) is independent of \(\rho\).

The parabolic stabilizing the highest weight vector in \(S^3 M\) has codimension \(m - 3\) in \(\mathfrak{sp}(M)\), so if there is a drop of dimension by two more, we must have \(T^{22p+2} = 0\).
6.6. Step 4. Consider $(\rho \sigma 1) + (\sigma p 1)$ to eliminate the $w^\tau_{\rho p}$:

$$
0 = \epsilon^\rho [v^1 \delta_{\sigma}^\rho + v^1 \delta_{\rho}^\sigma + v^1 m T_{\rho \sigma} - (w^1 + \epsilon^{\rho} v^1 \delta_{\rho}^p + u^1 m T_{\rho m p 1} T_{\rho m}^1)] \\
+ \epsilon^\rho [u^1 m T_{\rho \sigma} + u^1 m T_{\rho m p 1} T_{\rho m}^1 - (w^1 + \epsilon^p v^1 \sigma + u^1 m T_{\rho m 1} T_{\rho m}^1 - (w^1 + \epsilon^p v^1 \delta_{\rho}^p + u^1 m T_{\rho m 1} T_{\rho m}^1)] \\
+ \epsilon^\rho [u^1 m T_{\rho \sigma} + u^1 m T_{\rho m p 1} T_{\rho m}^1 - (w^1 + \epsilon^p v^1 \sigma + u^1 m T_{\rho m 1} T_{\rho m}^1)] \\
+ \epsilon^\rho [u^1 m T_{\rho \sigma} + u^1 m T_{\rho m p 1} T_{\rho m}^1 - (w^1 + \epsilon^p v^1 \sigma + u^1 m T_{\rho m 1} T_{\rho m}^1)]
$$

Note that when $\rho = \sigma, \epsilon^\rho + \epsilon^p = 0$, so all the $\delta_{\rho \sigma} (\epsilon^\rho + \epsilon^p)$ terms are zero.

When $(\rho \sigma) = (2, 2)$, if $T^{22(p+2)} \neq 0$, we get a relation involving $u^1, v^1, w^1_{p+2}$ so there can be no further relations when $T^{22(p+2)} \neq 0$. Moreover, when $T^{22(p+2)} \neq 0$, we must also have for all $\sigma$

$$
0 = -T_{\sigma m 1} - t_{\sigma}^B + \epsilon^\sigma t_{\sigma}^C \\
0 = -\epsilon^\sigma T_{m 1 \sigma} - T_{m \sigma 1} + t_{\sigma}^B + t_{\sigma}^A \\
0 = -\epsilon^\sigma T_{m 1 \sigma} - T_{m \sigma 1} + \epsilon^\sigma t_{\sigma}^C + t_{\sigma}^A
$$

which we rewrite as

$$
T_{\sigma m 1} = 0 \\
t_{\sigma}^B = \epsilon^\sigma t_{\sigma}^C \\
t_{\sigma}^A = \epsilon^\sigma T_{m 1 \sigma} + T_{m \sigma 1} - \epsilon^\sigma t_{\sigma}^C
$$

This is because the terms appear inside coefficients that involve $\rho$ only, so even though they also appear in the relation involving $u^1, v^1, w^1_{p+2}$, we can peel that away from the others to get the relation $-u^1 + v^1 + w^1_{p+2} = 0$.

6.7. Step 5. Assume $T^{22(p+2)} \neq 0$ and reconsider $(\rho \sigma) = (22(p + 2))$ modulo $u$. Recall that when $T^{22(p+2)} \neq 0$, relations are imposed upon $x^2_3, ..., x^{n-2}$ but $x^2_2 \in sp(M)$ is still free.
Recall that so far there was no relation on $x^2$ so we obtain a new relation and thus $T^{22(p+2)} = 0$.

6.8. **Step 6.** We revisit $(\rho \sigma 1) + (\sigma \rho 1)$ which simplifies since $T^{22(p+2)} = 0$:

$$0 = - [w_2^{(p+2)}(1)_{p+2} - [w_2^2 + v_1]T^{22(p+2)}$$

$$- [w_2^{(p+2)}(1)_{p+2} - [w_2^{(p+2)}]_{p+2}$$

$$- [w_2^2 + u_1]T^{22(p+2)} - [w_2^{(p+2)}]_{p+2}$$

$$+ w_2^{(p+2)}t_B^2 + w_2^{(p+2)}T^{22(p+2)} + w_2^{(p+2)}t_A^2 \mod n$$

$$\equiv (-2w_2^2 + w_2^{(p+2)})T^{22(p+2)} \mod u$$

$$\equiv (3x_2^2 + y_2^2)T^{22(p+2)} \mod u$$

$$\equiv (3x_2^2)T^{22(p+2)} \mod u$$

If any of the quantities

$$- T^{\sigma m_1} - t_B^\rho + \epsilon^\tau t_C^\sigma$$

$$- \epsilon^\sigma T^{m_1 \tau} - T^{m_1 \sigma} + \epsilon^\tau t_C^\sigma + t_A^\rho$$

$$- \epsilon^\sigma T^{m_1 \tau} - T^{m_1 \sigma} - T^{\sigma m_1} + t_B^\rho + t_A^\rho$$

is not identically zero, we obtain $m - 2$ relations and can have no more. In particular at most one of these quantities is nonzero and if it is, there can be no further relations.

6.9. **Step 7.** Write the 6 equations $(1\sigma m), ..., (m1\tau)$ as

$$X_{1\sigma m} + v_1^\sigma + w_1^m = 0$$

$$X_{\sigma 1m} + u_1^\sigma + w_1^m = 0$$

$$X_{m\sigma 1} - \epsilon^\sigma v_1^m + v_1^\sigma = 0$$

$$X_{\sigma m 1} + u_1^\sigma + \epsilon^\tau v_1^m = 0$$

$$X_{1m \sigma} + v_1^m + w_1^\tau = 0$$

$$X_{m1 \sigma} + w_1^m + w_1^\tau = 0$$
Then

\[ u_1^\sigma = \frac{1}{2} [-X_{1\sigma m} + X_{\sigma 1m} - \epsilon^\sigma X_{m\sigma 1} + \epsilon^\sigma X_{\sigma m 1} + \epsilon^\sigma X_{1m\sigma} + X_{m1\sigma}] \]

\[ v_1^\sigma = \frac{1}{2} [X_{1\sigma m} - X_{\sigma 1m} + \epsilon^\sigma X_{m\sigma 1} - \epsilon^\sigma X_{\sigma m 1} - \epsilon^\sigma X_{1m\sigma} + X_{m1\sigma}] \]

\[ w_1^\sigma = \frac{1}{2} [X_{1\sigma m} + X_{\sigma 1m} + \epsilon^\sigma X_{m\sigma 1} - \epsilon^\sigma X_{\sigma m 1} - \epsilon^\sigma X_{1m\sigma} - X_{m1\sigma}] \]

\[ u_m^\sigma = \frac{1}{2} [X_{1\sigma m} + X_{\sigma 1m} - \epsilon^\sigma X_{m\sigma 1} + \epsilon^\sigma X_{\sigma m 1} + \epsilon^\sigma X_{1m\sigma} - X_{m1\sigma}] \]

\[ v_m^\sigma = \frac{1}{2} [-\epsilon^\sigma X_{1\sigma m} - \epsilon^\sigma X_{\sigma 1m} + X_{m\sigma 1} - X_{\sigma m 1} + X_{1m\sigma} + \epsilon^\sigma X_{m1\sigma}] \]

\[ w_m^\sigma = \frac{1}{2} [\epsilon^\sigma X_{1\sigma m} + \epsilon^\sigma X_{\sigma 1m} + X_{m\sigma 1} + X_{\sigma m 1} - X_{1m\sigma} - \epsilon^\sigma X_{m1\sigma}] \]

Observe that

\[ X_{1\sigma m} \equiv 0 \mod n \]

\[ X_{\sigma 1m} \equiv 0 \mod n \]

\[ X_{m\sigma 1} \equiv -x_\sigma^\sigma T^{m\sigma 1} + \frac{1}{2}\epsilon^\sigma (u_1^1 + v_1^1 + 2w_1^1)]T^{m\sigma 1} \mod n \]

\[ X_{\sigma m 1} \equiv -x_\sigma^\sigma T^{\sigma'm 1} + \frac{1}{2}\epsilon^\sigma (u_1^1 + v_1^1 + 2w_1^1)]T^{\sigma'm 1} \mod n \]

\[ X_{1m\sigma} \equiv 0 \mod n \]

\[ X_{m1\sigma} \equiv -\epsilon^\sigma x_\sigma^\sigma T^{m1\sigma'} + \frac{1}{2}\epsilon^\sigma (u_1^1 + v_1^1 - w_1^1)]T^{m1\sigma'} \mod n \]
6.10. **Step 8.** Now consider $(\rho \sigma \tau)$ mod $u$

\[
0 \equiv \frac{1}{2} [e^{\sigma \sigma'} T^{\rho \sigma'} - e^{\rho \sigma'} T^{\sigma \rho} + e^{\sigma' \sigma'} T^{\rho \sigma'}] \delta_{\sigma \tau}
\]

\[
+ \frac{1}{2} [e^{\sigma \sigma'} T^{\rho \sigma'} + e^{\sigma \sigma'} T^{\rho \sigma'} - e^{\sigma' \sigma'} T^{\rho \sigma'}] \delta_{\rho \tau}
\]

\[
+ \frac{1}{2} [e^{\tau \sigma'} T^{\rho \sigma'} + e^{\tau \sigma'} T^{\rho \sigma'} + e^{\sigma' \sigma'} T^{\rho \sigma'}] \delta_{\sigma \rho}
\]

\[
= x_{\rho}^{\sigma'} \delta_{\sigma \tau} [-\rho_{\sigma} + \frac{1}{2} (e^{\rho \sigma'} T^{\rho \sigma'} - e^{\sigma' \sigma'} T^{\rho \sigma'})]
\]

\[
+ x_{\rho}^{\rho'} \delta_{\rho \tau} [-\rho_{\rho} + \frac{1}{2} (-e^{\sigma \sigma'} + e^{\sigma \sigma'} - e^{\sigma' \sigma'})]
\]

\[
+ x_{\rho}^{\sigma'} \delta_{\sigma \rho} [-e^{\rho \sigma'} + \frac{1}{2} (-e^{\tau \sigma'} + e^{\tau \sigma'} + e^{\sigma' \sigma'})]
\]

\[
(u_{\rho}^{\sigma'} + v_{\rho}^{\rho'}) ([e^{\rho \sigma'} T^{\rho \sigma'} + (-e^{\rho \sigma'} T^{\rho \sigma'}) + \frac{1}{2} e^{\rho \sigma'} T^{\rho \sigma'}] \delta_{\sigma \tau}
\]

\[
+ w_{\rho}^{\sigma'} [(-2 e^{\rho \sigma'} - \frac{1}{2}) T^{\rho \sigma'} + (-2 e^{\rho \sigma'} - \frac{1}{2}) T^{\rho \sigma'} + (-2 - \frac{1}{2} e^{\rho \sigma'}) \delta_{\sigma \tau}
\]

\[
+ (u_{\rho}^{\sigma'} + v_{\rho}^{\rho'}) [(-e^{\sigma' \sigma'}) T^{\sigma \rho} + (e^{\rho \sigma'} - \frac{1}{2} e^{\sigma' \sigma'}) T^{\rho \sigma'} + \frac{1}{2} e^{\sigma' \sigma'} T^{\rho \sigma'}] \delta_{\rho \tau}
\]

\[
+ w_{\rho}^{\sigma'} [(-2 e^{\rho \sigma'} - \frac{1}{2} e^{\sigma' \sigma'}) T^{\rho \sigma'} + e^{\rho \sigma'} (2 + \frac{1}{2} e^{\sigma' \sigma'}) T^{\rho \sigma'} + (-2 - \frac{1}{2} e^{\rho \sigma'}) T^{\rho \sigma'}] \delta_{\rho \tau}
\]

\[
+ (u_{\rho}^{\sigma'} + v_{\rho}^{\rho'}) [(-e^{\tau \sigma'} - \frac{1}{2} e^{\sigma' \tau}) T^{\rho \sigma'} + e^{\rho \sigma'} (-1 + \frac{1}{2} e^{\sigma' \tau}) T^{\rho \sigma'} + \frac{1}{2} e^{\sigma' \sigma'} T^{\rho \sigma'}] \delta_{\sigma \rho}
\]

\[
+ w_{\rho}^{\sigma'} [(-2 e^{\rho \sigma'} - \frac{1}{2} e^{\sigma' \tau}) T^{\rho \sigma'} + (-2 e^{\rho \sigma'} + \frac{1}{2} e^{\sigma' \tau}) T^{\rho \sigma'} + (2 + \frac{1}{2} e^{\sigma' \tau}) T^{\rho \sigma'}] \delta_{\sigma \rho}
\]

We already know that at least two of the quantities from [21] must be zero. If exactly two are zero then all the terms in brackets above must be zero and we conclude all six quantities are zero (for all $\sigma$). If all three of the quantities are zero then we are allowed $m - 2$ relations among the elements of $\mathfrak{sp}(M) \oplus t$. Note each term in brackets with the $x$'s is two possible relations, depending on whether, e.g., in the first $\rho < p + 1$ or $\rho \geq p + 1$ and each non-vanishing imposes $m - 2$ relations. Thus two of the six can fail to be identities. Then the remaining four identities plus the previous three will be enough to set all quantities to zero.
We conclude $t^a_A, t^a_B, t^a_C, T^{m1}, T^{m\sigma1}, T^{\sigma m1}$ are all zero.

6.11. **Step 9.** We now show $T^\rho^\sigma\tau^m$ and permutations are zero. We may now solve the $(1m), (m1), (mm1)$ equations to obtain $u^m_1, v^m_1, w^m_1 \equiv 0 \mod n$. Using this we obtain from the $(\rho\sigma m)$ equations

$$-s^\rho_T T^\rho^\sigma^m - s^\sigma_T T^{\omega m} \equiv 0 \mod u.$$

Writing $M \otimes M = S^2 M \oplus \langle \Lambda \rangle \oplus \Lambda^2 M_0$, where $\Lambda^2 M_0 \oplus \langle \Lambda \rangle = \Lambda^2 M$, arguing as before, we immediately see that the component of $T^\rho^\sigma m a_\rho \otimes b_\sigma$ lying in $\Lambda^2 M_0$ must be zero and the component in $S^2 M$ must be a highest weight vector, i.e.,

$$T^\rho^\sigma m a_\rho \otimes b_\sigma = T^{22m} a_2 \otimes b_2 \otimes c_m \oplus \epsilon^\rho \epsilon^\sigma \epsilon^m a_\rho \otimes b_\sigma \otimes c_m$$

and if $T^{22m} \neq 0$, there can be at most two more relations imposed. Similarly, using the $(\rho m\tau)$ and $(m\sigma \tau)$ equations, and writing, as a $\mathfrak{sp}(M)$-module, $M^* \otimes M = M \otimes M = S^2 M \oplus \langle \Lambda \rangle \oplus \Lambda^2 M_0$, we obtain

$$T^{\rho m\tau} a_\rho \otimes c_\tau = T^{2m(p+2)} a_2 \otimes b_m \otimes c_{p+2} \oplus t^{m*} \sum_\rho a_\rho \otimes b_m \otimes c_\rho$$

$$T^{m\sigma \tau} b_\sigma \otimes c_\tau = T^{m2(p+2)} a_m \otimes b_2 \otimes c_{p+2} \oplus t^{m*} \sum_\sigma a_m \otimes b_\sigma \otimes c_\sigma$$

If any one of $T^{22m}, T^{2m(p+2)}, T^{m2(p+2)}$ is nonzero we immediately get $m - 3$ relations and just need two more.

Reconsider $(\rho \sigma \tau)$

$$0 = u_1^\rho \delta^\sigma \tau + v_1^\rho \delta^\sigma \tau + w_1^\rho \epsilon^\rho \delta^\rho \tau$$

$$- (w_1^\rho + \epsilon^\rho v_1^\rho) \delta^\rho \tau t^{m*} - (v_1^\rho + \epsilon^\rho w_1^\rho) \delta^\rho \tau t^{m*} - (\epsilon^\rho u_1^\rho + \epsilon^\rho w_1^\rho) \delta^\rho \tau t^{m*} - (w_1^\rho + \epsilon^\rho v_1^\rho) \delta^\rho \tau t^{m*}$$

Respectively taking $(\rho \sigma \tau)$ to be $(\rho 2(p+2))$ with $\rho \neq p + 2$, $(2\sigma (p+2))$ with $\sigma \neq p + 2$, and $(22\tau)$ with $\tau \neq 2$, we see $T^{m2(p+2)} = 0, T^{2m(p+2)}$, and $T^{22m} = 0$.

At this point

$$u_1^\rho = \frac{1}{2} [\epsilon^\rho u_1^1 t^{m*} + \epsilon^\sigma v_1^1 t^{m*} + \epsilon^\rho w_1^1 t^{m*} + \epsilon^\sigma w_1^1 t^{m*} + \epsilon^\rho u_1^1 t^{m*} + \epsilon^\sigma u_1^1 t^{m*}]$$

$$v_1^\sigma = \frac{1}{2} [-\epsilon^\sigma u_1^1 t^{m*} + \epsilon^\rho v_1^1 t^{m*} + \epsilon^\sigma w_1^1 t^{m*} - \epsilon^\rho w_1^1 t^{m*} - \epsilon^\sigma u_1^1 t^{m*} + \epsilon^\rho u_1^1 t^{m*}]$$

$$w_1^\tau = \frac{1}{2} [\epsilon^\tau u_1^1 t^{m*} + \epsilon^\tau v_1^1 t^{m*} + \epsilon^\tau w_1^1 t^{m*} - \epsilon^\tau u_1^1 t^{m*} - \epsilon^\tau v_1^1 t^{m*} - \epsilon^\tau w_1^1 t^{m*}]$$

Subbing in to $(\rho \sigma \tau)$:
The rank of a tensor $T$ is the minimum number of rank one tensors that sum to $T$. In the context of matrix multiplication, the border rank is the minimum rank of tensors that approximate $T$ in the limit. We have discussed the border rank bounds and how they can be used to understand the complexity of matrix multiplication. The border rank of a tensor $M$ can be defined as $\text{BR}(M) = \sup \{ \text{rank}(T) : T \preceq M \}$, where $\preceq$ denotes the partial order of tensors.

Recall that if any of $t^{mss}, t^{sms}, t^{stm}$ are nonzero, we have $m - 3$ relations imposed on the elements of $\mathfrak{sp}(M)$ so we only need to show these equations imply at least two more. Say $t^{mss} \neq 0$, then for $\rho \neq \sigma, \bar{\sigma}$ and taking $\tau = \bar{\sigma}$, we get at least $\frac{1}{2}(m - 4)$ relations among the $u_\rho^I$ (in fact more unless $t^{mss} = t^{msr}$) and are done because $\frac{3}{2}m - 5 \geq m - 1$ as $m \geq 7$ and $m$ is even. The other two cases are similar.

### 6.12. Step 10.
Reconsider

\[
\begin{align*}
(\rho \sigma m) & \quad 0 = u_\rho^m T^{\sigma \rho m} + v_\rho^m T^{\rho \sigma m} \\
& \quad = -(w_\rho^1 + e^\rho v_\rho^1)T^{\sigma \rho m} - (e^\sigma u_\sigma^1 + w_\sigma^1)T^{\rho \sigma m} \\
(\rho \sigma m) & \quad 0 = w_\rho^m T^{\rho \sigma m} + w_\sigma^m T^{\rho \sigma m} \\
& \quad = -(w_\rho^1 + e^\rho v_\rho^1)T^{\rho \sigma m} - (u_\rho^1 + v_\rho^1)T^{\rho \sigma m} \\
(\rho m \bar{\sigma}) & \quad 0 = v_\rho^m T^{m \bar{\sigma}} + w_\rho^m T^{m \bar{\sigma}} \\
& \quad = -(w_\rho^1 + e^\rho v_\rho^1)T^{m \bar{\sigma}} - (u_\rho^1 + v_\rho^1)T^{m \bar{\sigma}} \\
(\rho m \bar{\sigma}) & \quad 0 = -w_\rho^m T^{m \bar{\sigma}} + w_\rho^m T^{m \bar{\sigma}} \\
& \quad = -(u_\rho^1 + v_\rho^1)T^{m \bar{\sigma}}
\end{align*}
\]

Note that if $T^{\rho \sigma m} \neq 0$ for some $\rho$, then the $u_\rho^1$ in the first line and the $u_\rho^1$ in the second give $2(m - 2)$ relations, eliminating this case. Similarly, if any of $T^{\rho \sigma m}, T^{m \sigma \rho}, T^{m \rho \sigma}$ are nonzero, the case is eliminated from consideration.

### 6.13. Step 11.

\[
\begin{align*}
(\rho m m) & \quad 0 = u_\rho^m T^{m m} \\
& \quad = -(w_\rho^1 + e^\rho v_\rho^1)T^{m m} \\
(\rho s \sigma) & \quad 0 = v_\rho^m T^{m m} \\
& \quad = -(e^\sigma u_\sigma^1 + w_\sigma^1)T^{m m} \\
(\rho m \bar{\sigma}) & \quad 0 = -(u_\rho^1 + v_\rho^1)T^{m m}
\end{align*}
\]

So if $T^{m m} \neq 0$ we are eliminated from consideration.

### 7. Border rank bounds

The rank of a tensor $T$, denoted $\text{R}(T)$, is the smallest $\tau$ such that $T$ may be written as a sum of $\tau$ rank one tensors, and the border rank, denoted $\text{BR}(T)$, is the smallest $\tau$ such that $T$ is a limit of rank $\tau$ tensors. Rank and border rank are standard measures of the complexity of a tensor. Strassen [33] showed that the exponent $\omega$ of matrix multiplication may be defined as the infimum over $\tau$ such that $\text{R}(M_{(n)}) = O(n^\tau)$, and Bini [7] showed one may use the border rank...
\( \textbf{R}(M_{(n)}) \) rather than the rank \( \textbf{R}(M_{(n)}) \) in the definition. The tensor \( T_{CW,m-2} \) has the minimal possible border rank \( m \) for any concise tensor, which is important for its use in proving upper bounds on \( \omega \).

Remark 7.1. The tensor of Proposition 3.1 satisfies \( \textbf{R}(a_1 \otimes (\sum_{j=1}^m b_j \otimes c_j)) = \textbf{R}(a_1 \otimes (\sum_{j=1}^m b_j \otimes c_j)) = m \).

Let \( T = S_B \) be a skeletal tensor. Change bases in \( B \) by permuting \( b_1 \) with \( b_m \). This will have the advantage that \( T(\alpha^1) = \text{Id} \). Explicitly

\[
T(A^*) = \begin{pmatrix}
\alpha^1 & \alpha^2 & \alpha^3 & \cdots & \alpha^m \\
\alpha^1 & \phi^2 & \cdots & \phi^3 & \cdots \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
\alpha^1 & \phi^{m-1} & \phi^m & \phi^{m-2} & \cdots
\end{pmatrix},
\]

where \( \phi = B(\alpha^2, \ldots, \alpha^{m-1})^t \), and \( B \) is the matrix of \( B \). Let \( Y = \sum_{s=2}^{m-2} y_s \alpha^s \) and \( Z = \sum_{s=2}^{m-2} z_s \alpha^s \) for constants \( \overline{y} = (y_2, \ldots, y_{m-1}) \) and \( \overline{z} = (z_2, \ldots, z_{m-1}) \). Applying Strassen’s commutation equations, the \((1, m)\) entry of \([ T(Y), T(Z) ]\) is the only potential nonzero entry and it is \( B(\overline{y}, \overline{z}) - B(\overline{z}, \overline{y}) \). Note that if \( B \) is symmetric, then \( T \) is isomorphic to \( T_{CW} \). We conclude:

Proposition 7.2. Let \( T \) be skeletal. Then \( \textbf{R}(T) \geq m + 1 \) unless \( T \) is isomorphic to the big Coppersmith-Winograd tensor.

Corollary 7.3. None of \( T_{skewCW,m-2} \), \( T_{s \oplus skewCW,m-2} \), \( T_{s \oplus skewCW,m} \) have minimal border rank \( m \).

Corollary 7.4. Let \( T \in \mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m \) be 1-generic and either symmetric or of minimal border rank. Then \( \dim G_T \leq \binom{m+1}{2} \) with equality holding only for \( T_{CW,m-2} \).

Numerical computations using ALS methods (see [5]) indicate, at least for \( m \leq 11 \), that \( \textbf{R}(T_{s \oplus skewCW,m-2}) \leq \frac{3m}{2} - \frac{1}{2} \) and for \( m \leq 14 \) that \( \textbf{R}(T_{skewCW,m-2}) \leq \frac{3m}{2} - 1 \).

The following Corollary first appeared in [22] with a proof at the level of deformations of algebras. It was proved, but not observed, in an earlier draft of this paper:

Corollary 7.5. Let \( T \) be 1-generic and of minimal border rank. Then \( T \) degenerates to \( T_{CW,m-1} \).

As remarked in a special case of Corollary 7.5 in [20], Corollary 7.5 indicates that there should be room for improvement with Strassen’s laser method. This is important for Strassen’s laser method, as it says 1-degenerate minimal border rank tensors that are not isomorphic to a Coppersmith-Winograd tensor are subject to barriers no worse than the Coppersmith-Winograd tensor.
8. Other tensors

We briefly describe the symmetry Lie algebras of other tensors used in the laser method and a related tensor.

**Example 8.1** (Strassen’s tensor). The following is the first tensor that was used in the laser method: 
\[ T_{\text{str},q} = \sum_{j=1}^{q} a_0 \otimes b_j \otimes c_j + a_j \otimes b_0 \otimes c_j + a_j \otimes b_j \otimes c_0 \in C^{q+1} \otimes C^{q+1} \otimes C^q. \]
Then, with blocking \((1,q) \times (1,q)\) in the first two matrices,
\[ \tilde{g}_{T_{\text{str},q}} = \left\{ \lambda \text{Id} + \begin{pmatrix} 0 & y \\ 0 & X \end{pmatrix}, \mu \text{Id} + \begin{pmatrix} 0 & y \\ 0 & X \end{pmatrix}, \nu \text{Id} + (-X^t) \mid X \in \mathfrak{gl}(q), x \in C^q, \lambda + \mu + \nu = 0 \right\}. \]
In particular, \(\text{dim}(g_{T_{\text{str},q}}) = q^2 + q\).

**Example 8.2** (The small Coppersmith-Winograd tensor). Another tensor used in the laser method is the small Coppersmith-Winograd tensor:
\[ T_{\text{cw},q} = \sum_{j=1}^{q} a_0 \otimes b_j \otimes c_j + a_j \otimes b_0 \otimes c_j + a_j \otimes b_j \otimes c_0 \in (C^{q+1})^3. \]
Then with blocking \((1,q) \times (1,q)\):
\[ \tilde{g}_{T_{\text{cw},q}} = \left\{ \left( \begin{pmatrix} -\mu - \nu & 0 \\ \lambda \text{Id} + X \end{pmatrix}, \begin{pmatrix} -\lambda - \nu & 0 \\ 0 \mu \text{Id} + X \end{pmatrix}, \begin{pmatrix} -\lambda - \mu & 0 \\ 0 \nu \text{Id} + X \end{pmatrix} \right) \mid \lambda, \mu, \nu \in \mathbb{C} \right\}. \]
In particular \(\text{dim}(g_{T_{\text{cw},q}}) = \binom{q}{2} + 1\).

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