Some properties of amplitudes at multi boson thresholds in spontaneously broken scalar theory

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Abstract

It is shown that in a $\lambda\phi^4$ theory of one real scalar field with spontaneous breaking of symmetry a calculation of the amplitudes of production by a virtual field $\phi$ of $n$ on-mass-shell bosons all being exactly at rest is equivalent in any order of the loop expansion to a Euclidean space calculation of the mean field of a kink-type configuration. Using this equivalence it is found that all the $1 \to n$ amplitudes have no absorptive part at the thresholds to any order of perturbation theory. This implies non-trivial relations between multi-boson threshold production amplitudes. In particular the on-mass-shell amplitude of the process $2 \to 3$ should vanish at the threshold in all loops. It is also shown that the factor $n!$ in the $1 \to n$ amplitudes at the threshold is not eliminated by loop effects.
The tree-level amplitudes of the process in which one virtual scalar field $\phi$ produces $n$ on-mass-shell bosons of this field in a $\lambda\phi^4$ theory displays a growth proportional to $n!$ at large $n$. If this growth is not eliminated by the loop effects this would imply that a strong interaction develops in a theory with weak coupling $\lambda$ at energies where production of $n > 1/\lambda$ interacting bosons is kinematically possible. It was recently found that in a $\lambda\phi^4$ theory of a scalar field $\phi$ one can explicitly sum all tree graphs for the process $1 \rightarrow n$ at the threshold, i.e. when all the produced particles have zero spatial momenta, both in the theory without spontaneous symmetry breaking and in the case of the spontaneous breaking of the reflection symmetry $\phi \rightarrow -\phi$. An elegant development of these calculations was subsequently suggested by Brown who had pointed out that the threshold amplitudes of the $1 \rightarrow n$ processes are related to a complex vacuum solution of the field equations. An extension of Brown’s technique has enabled explicit calculation of these amplitudes at the one-loop level. In particular it was found that the relative magnitude of the loop corrections is described by the parameter $n^2\lambda$ so that the quantum effects may indeed completely modify the behavior of the amplitudes at $n \geq \lambda$.

A surprising result of this study was that the on-mass-shell scattering amplitudes for the processes $2 \rightarrow n$ at the tree level are vanishing at the $n$ particle threshold for all $n$ greater than 4 in unbroken $\lambda\phi^4$ theory and for $n > 2$ in the theory with spontaneously broken symmetry. One of the consequences of the latter behavior is that the threshold amplitudes for the $1 \rightarrow n$ processes do not develop absorptive part at the one-loop level in the theory with the spontaneous symmetry breaking.

In this paper we consider the threshold production of particles in the theory of one real scalar field with spontaneously broken symmetry beyond the one-loop approximation. This is possible due to the fact that the loop expansion around the Brown’s solution can be mapped into the loop expansion in the Euclidean space around a kink-type configuration of the field. As a result it will be shown that the threshold amplitudes of the $1 \rightarrow n$ processes have no absorptive part to any order in this expansion. Through the unitarity relation this implies that the on-mass-shell amplitude of the process $2 \rightarrow 3$ should vanish at the threshold in all orders of perturbation theory, while for higher final state multiplicities in the on-mass-shell processes $k \rightarrow n$ this implies existence of non-trivial relations between amplitudes with different $k$. Also we will use the analytical properties of the time dependence of the Brown type solution to prove that the generating function for the amplitudes of $1 \rightarrow n$ has a singularity at a finite distance in the complex plane, so that the $n!$ growth survives all the loop effects.
The theory to be discussed in this paper is described in the Minkowski space by the Lagrangian density

\[ L = \frac{1}{2} (\partial_\mu \phi)^2 + \frac{m^2}{4} \phi^2 - \frac{\lambda}{4} \phi^4. \]  

The field has two degenerate vacuum states, which in the classical approximation correspond to \( \phi = \pm v \) with \( v = m/\sqrt{2\lambda} \) and \( m \) is the mass of bosons propagating in either of the two vacua. For definiteness we will discuss scattering processes in the “left” vacuum \( \phi = -v \). The amplitudes of the process \( 1 \to n \) are determined by the matrix elements

\[ a_n = \langle n | \phi(0) | 0 \rangle, \]  

where \( \langle n \rangle \) is the final state of the \( n \) bosons, which at the threshold all have exactly zero spatial momenta, \( p_a = 0 \). Using the standard reduction formula Brown[5] has noticed that the sum

\[ \Phi(t) = \sum_{n=0}^{\infty} \frac{z_0^n e^{i\omega t}}{n!} \langle n | \phi(0) | 0 \rangle = \sum_{n=0}^{\infty} \frac{z_0^n}{n!} \langle n | \phi(t) | 0 \rangle \]  

is given by the vacuum-to-vacuum matrix element of the field \( \phi(t) \)

\[ \Phi(t) = \langle 0_{\text{out}} | \phi(t) | 0_{\text{in}} \rangle, \]  

in the presence of the source \( \rho(t) \) in the following limit. The source is chosen in the form \( \rho(t) = \rho_\omega \exp(i\omega t) \) and then \( \omega \) is taken to the mass shell, \( \omega \to m \), simultaneously with tending \( \rho_\omega \) to zero in such a way that the quantity

\[ z(t) = \frac{\rho_\omega e^{i\omega t}}{m^2 - \omega^2 - i\epsilon} \]  

tends to a finite limit \( z(t) = z_0 e^{i\omega t} \). Since the source drives only the positive-frequency response of the field and the amplitude of the source is vanishing on the mass shell, this limiting procedure can be replaced by the requirement that \( \Phi(t) \) is a solution of the field equations which admits expansion in ascending powers of \( z(t) \), i.e. it has only the positive frequency part.

The solution of the classical Euler-Lagrange equation

\[ \frac{d^2}{dt^2} \Phi - \frac{m^2}{2} \Phi + \lambda \Phi^3 = 0, \]  

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subject to this condition, thus generates the expressions for the matrix elements (2) as
given by the sum of tree graphs. The explicit form of the classical solution is

$$\Phi_{cl}(t) = -v \frac{1 + z(t)/(2v)}{1 - z(t)/(2v)}$$

and thus the matrix elements (2) are given by

$$\langle n | \phi(0) | 0 \rangle = \left( \frac{\partial}{\partial z} \right)^n |\Phi|_{z=0} = -n!(2v)^{1-n}, \quad n \geq 1$$

which reproduces the previously known result [4].

The loop corrections in this approach arise from quantum corrections to the response
of the field to the source in the described above limit. In the one loop approximation
these corrections in the theory under discussion were calculated by Smith [7]:

$$\Phi_{one-loop}(t) = -\bar{v} \frac{1 + z(t)/(2\bar{v})}{1 - z(t)/(2\bar{v})} - v \frac{\sqrt{3} \lambda \ [z(t)/(2v)]^2}{2\pi \ [1 - z(t)/(2v)]^3}$$

where $\bar{v}$ is an appropriately renormalized vacuum expectation value. From this result one
easily finds the amplitudes with the one-loop correction [7]:

$$\langle n | \phi(0) | 0 \rangle = -n!(2\bar{v})^{1-n} \left( 1 + n(n-1)\frac{\sqrt{3} \lambda}{8\pi} \right)$$

One can readily notice that the explicit expressions as well as the initial definition
(3) define $\Phi(t)$ in terms of a Taylor series in the variable $z(t)$ with a finite radius of
convergence. The quantity $z_0$ in the equation (3) in fact serves as a “tag” of the variable
$z(t)$ and can be chosen arbitrarily small, but finite, within the radius of convergence of the
series. It can be also noticed that the problem of whether the $n!$ growth of the amplitudes
survives the loop effects is equivalent to the problem of whether the latter effects keep the
radius of convergence of the series finite, or whether they suppress the amplitudes to the
extent that this radius becomes infinite, in which case the $n!$ growth disappears. This is
one of the central questions to be addressed in this paper. Another essential point is that
the expansion is indeed of the Taylor series type. This behavior is ensured by the time
evolution of the Heisenberg operator $\phi(t)$:

$$\langle n | \phi(t) | 0 \rangle = e^{iE_n t} \langle n | \phi(0) | 0 \rangle$$

where $E_n$ is the energy of the state $\langle n |$ relative to the vacuum. In the standard setting
of the scattering problem the energy of $n$ particles all being at rest in the asymptotic out
state is always the sum of their energies, so that the expansion for \( \Phi(t) \) is polynomial in \( \exp(i mt) \) in a field theory. In the case of Quantum Mechanics the behavior is different: high levels in an anharmonic oscillator are not equidistant, hence the present analysis can not be directly applied in that case. This helps to resolve the conflict between the absence of the factorial growth of the amplitudes in Quantum Mechanics and the possibility of such growth in the field theory. We will return to discussion of this point in the concluding part of this paper.

If \( z_0 \) is chosen within the radius of convergence of the Taylor series, the generating function \( \Phi(t) \) does not have singularities in the upper half-plane of complex \( t \), where the Taylor series does converge. In the lower half-plane the classical solution (7) as well as the one loop correction (9) develops singularities, which are located with the period \( 2\pi/m \) parallel to the real axis of \( t \) (see Fig. 1). Higher loops produce mounting singularities at these points, near which the perturbation theory breaks down.

For the following we introduce the variable \( \tau \) in such way that \( z(t)/(2v) = -e^{m\tau} \), so that

\[
\tau = it + i\frac{\pi}{m} + \frac{1}{m} \ln \frac{z_0}{2v} .
\]  

(12)

In terms of the variable \( \tau \) the classical solution has the familiar simple form of the kink:

\[
\Phi_{cl}(\tau) = v \tanh \frac{m\tau}{2} .
\]  

(13)

The calculation of the first quantum correction to the classical field \( \Phi \) amounts to the calculation of the tadpole graph of Fig.2, where the propagator is the Green function in the background \( \Phi_{cl} \). In the course of this calculation it was found\[6,7\] that the appropriate Green function satisfies the condition that it vanishes, when either of its arguments corresponds to either \( \tau \to -\infty \) or \( \tau \to +\infty \).

The convergence of the Taylor series for \( \Phi(t) \) in the upper half-plane of \( t \) implies that the series is convergent for sufficiently large in absolute value negative \( \tau \). Therefore one can impose the boundary condition on the solution \( \Phi(t) \) to the full quantum problem at \( \tau \to -\infty \) by requiring that the asymptotic behavior there is of the form

\[
\Phi(t) \to -v + ve^{m\tau}, \quad \tau \to -\infty .
\]  

(14)

Now we have the following ingredients, which relate our problem to that of calculating the full quantum mean field of the kink configuration: the zeroth order approximation
(equation (13)), the condition on the Green function, which ensures that the quantum corrections vanish at both infinities in $\tau$, and, finally, the boundary condition (14), which fixes the normalization of the first term in the expansion in powers of $e^{m\tau}$ and thus fixes the translational zero mode in the kink background. These ingredients are sufficient to relate the problem of calculation of $\Phi(\tau)$ to that of calculating the mean field in the Euclidean-space field theory, i.e. the path integral

$$\Phi(\tau) = \frac{\int \phi(x) e^{-S[\phi]} \mathcal{D}\phi}{\int e^{-S[\phi]} \mathcal{D}\phi}$$

(15)

with

$$S[\phi] = \int \left( \frac{1}{2} (\partial_{\mu} \phi)^2 - \frac{m^2}{4} \phi^2 + \frac{\lambda}{4} \phi^4 \right) d^4x$$

(16)

and with the kink-type boundary conditions, i.e. the condition in the equation (14) and also $\phi(\tau \to +\infty) = v$, the latter is ensured by the asymptotic behavior at $\tau \to +\infty$ of the classical solution (13) and the condition that the Green function is vanishing at $\tau \to +\infty$.

Under the specified boundary conditions the path integral in the equation (15) is well defined. Indeed, the classical solution (13) provides the absolute minimum of the action (16) under these boundary conditions. The four dimensional theory is free from infra-red divergencies and the ultra-violet ones are as usual compensated by the counterterms for the mass and the coupling constant renormalization. Thus we arrive at the conclusion that the mean field $\Phi(\tau)$ defined by the integral (15) is real and the same is true for the coefficients $c_n$ of its expansion in ascending powers of $e^{m\tau}$ at $\tau \to -\infty$:

$$\Phi(\tau) = \sum_{n=0}^{\infty} c_n e^{n m \tau}.$$  

(17)

Therefore we conclude that the matrix elements (2)

$$\langle n | \phi(0) | 0 \rangle = (-1)^n n! c_n$$

(18)

are real in all orders of perturbation theory. In the one-loop approximation this non-trivial behavior is seen from the explicit result (equation (10)), and we see that this behavior extends to all orders of the loop expansion.

Before proceeding to discussion of the implications of this result, we can mention an example of a theory where such behavior does not hold. This is the theory with unbroken symmetry, i.e. with positive mass term in the Lagrangian. In that theory the classical
solution $\Phi_{cl}(\tau)$ provides a saddle point of the action, rather than the minimum, hence the integration over the negative mode yields the mean field in the Euclidean space problem essentially complex already in the one-loop approximation$^3$.

The absence of an absorptive part of the amplitudes $a_n$ imposes through the unitarity relation non-trivial conditions on the on-shell amplitudes. At the one-loop level the condition is that the on-mass-shell scattering amplitudes for the processes $2 \rightarrow n$ vanish in the tree approximation for all $n$ greater than 2. This result follows from the fact that the only possible cuts of the one-loop graphs, are those across two lines, and those cuts split the graphs into tree diagrams. In higher loops unitary cuts across more than two lines are possible, so that the imaginary part can be zero as a result of cancellation between contributions of intermediate states with different number of particles. So in general one would find relations between the amplitudes of the processes $k \rightarrow n$ with different $k$.

There is however one special case in which one can find a well formulated conclusion. Namely, let us consider the amplitude of the process $1 \rightarrow 3$. According to the previous discussion its imaginary part is zero to all orders in perturbation theory. On the other hand, the only intermediate state which could give rise to an imaginary part is that with two particles, i.e. $2\text{Im}A(1 \rightarrow 3) = A(1 \rightarrow 2) \cdot A(2 \rightarrow 3)$. For three intermediate particles there is no phase space (recall that the final three bosons are produced at rest), while for larger number $k$ of particles in the intermediate state the on-mass-shell process $k \rightarrow 3$ is kinematically impossible, when the final particles are at rest. Furthermore for the scattering $2 \rightarrow 3$ at the threshold there is no angular dependence, i.e. only the $S$ wave can be present, therefore no cancellation between the partial waves can occur. Therefore, since the amplitude $A(1 \rightarrow 2)$ is manifestly non zero, one concludes that the on-mass-shell amplitude $A(2 \rightarrow 3)$ is vanishing at the threshold to all orders of the loop expansion.

We now proceed to the most intriguing question of whether the factorial behavior of the matrix elements (18) at large $n$ holds in all loops. This is certainly true if the Taylor expansion (17) in the variable $u = e^{m\tau}$ has a finite radius of convergence $R$. Then the coefficients $c_n$ behave at large $n$ as $|c_n| = \zeta(n) R^{-n}$, where $\zeta(n)$ behaves weaker than exponent of $n$ at large $n$, and thus the behavior of the coefficients $c_n$ would not compensate the $n!$ in eq.(18). We will assume the opposite, i.e. that the series in eq.(17) has infinite radius of convergence and show that this assumption is inconsistent.

If the Taylor series (17) for the mean field $\Phi(\tau)$ has infinite radius of convergence, the same is true for the quantity
\[ \Delta(\tau) = v^2 - \Phi^2(\tau) \]  

(19)

(notice that it is the square of the mean field entering here and not the mean value of the square of the field). The asymptotic behavior of \( \Delta(\tau) \) at \( \tau \to -\infty \) is \( \Delta(\tau) \to 2v^2e^{m\tau} \). At \( \tau \to +\infty \) the asymptotic behavior is determined by the boundary condition on \( \Phi(\tau) \) and by the fact that \( m \) is the lowest energy in the spectrum: \( \Delta(\tau) \to be^{-m\tau} \) where \( b \) is a constant.

The series (17) defines \( \Phi(\tau) \) and thus also \( \Delta(\tau) \) as a periodic function of complex \( \tau \) with the period \( 2\pi/m \). Let us consider the contour \( C \) in the complex plane of \( \tau \) shown in Fig. 3. Two links of the contour run along the lines separated by one period: \( \text{Im} \tau = 0 \) and \( \text{Im} \tau = 2\pi/m \), and the horizontal links between the two lines are chosen sufficiently far at negative and positive \( \text{Re} \tau \), so that one can use the asymptotic expression for \( \Delta(\tau) \) on those links. One can now consider calculation of the index of the function \( \Delta(\tau) \) on the contour \( C \):

\[ I = \frac{1}{2\pi i} \oint_C \frac{d\Delta(\tau)/d\tau}{\Delta(\tau)} d\tau , \]

(20)

which gives the difference between the number of zeros and poles of the function \( \Delta(\tau) \) inside the contour: \( I = Z - P \), where \( Z \) is the number of zeros and \( P \) is the number of poles inside the contour. Since the function \( \Delta(\tau) \) is assumed to have no singularities at finite complex \( \tau \), the only obstacle to calculating the index integral (20) can be presence of isolated zeros of \( \Delta(\tau) \) at the contour \( C \). This obstacle however can be easily removed by slightly shifting the contour away from each of the isolated zeros. (Clearly, a condensation of zeros of \( \Delta(\tau) \) at a finite \( \tau \) would either contradict the assumption that \( \Delta(\tau) \) is entire function, or imply that the function is identically zero, which obviously is not the case.) One can readily see, that the boundary condition that \( \Delta(\tau) \) falls exponentially at both infinities in \( \text{Re} \tau \) completely determines the index (20) : \( I = -2 \). Therefore \( \Delta(\tau) \) necessarily has a non-zero number of poles inside the contour \( C \), which is inconsistent with the assumption that the series (17) has infinite radius of convergence. This completes the proof that the amplitudes (18) have a factorial growth in \( n \) in the full quantum theory.

In the context of the present discussion one can illustrate the fundamental difference between the field theory matrix elements \( \langle n|\phi(0)|0 \rangle \), which grow proportionally to \( n! \) and seemingly analogous matrix elements \( \langle n|x|0 \rangle \) in Quantum Mechanics of an anharmonic oscillator, described by the Hamiltonian

\[ 7 \]
\[ H = \frac{p^2}{2} + \frac{x^2}{2} + \frac{\lambda}{4}x^4, \]  

where the frequency of the oscillator is set to one for simplicity. The matrix element of \( x \) between the ground state \( |0\rangle \) and the \( n \)-th excited state \( |n\rangle \) can be evaluated by the Landau WKB method and has different behavior in the two regions: \( n\lambda \ll 1 \), \( \langle n|x|0\rangle \sim n!\lambda^{n/2} \) and \( n\lambda \gg 1 \), \( \langle n|x|0\rangle \sim e^{-n} \), and in both cases \( n \gg 1 \) is assumed (see e.g. in Ref.[10]). Naturally, in the Quantum Mechanics of one degree of freedom there is no possibility for an unlimited growth with \( n \) of these matrix elements. This can be seen e.g. from the sum rule

\[ \langle 0|x^2|0\rangle = \sum_n |\langle n|x|0\rangle|^2, \]  

so that the sum on the right hand side has to be finite. The crucial difference from the field theory is that the analog of the sum (3)

\[ X(t) = \sum_{n=0}^{\infty} \frac{z_0^n}{n!} \langle n|x(t)|0\rangle = \sum_{n=0}^{\infty} \frac{z_0^n}{n!} e^{iE_n t} \langle n|x(0)|0\rangle \]  

for the Heisenberg operator \( x(t) \) is not an expansion in integer powers of \( e^{it} \), since the energies \( E_n \) of the anharmonic oscillator are not equidistant, when \( n\lambda > 1 \). Therefore one can not apply to \( X(t) \) the Abel’s theorems about power series expansion of an analytical function and the singularities of the function \( X(t) \) in the complex plane are not related to the asymptotic behavior of the matrix elements \( \langle n|x|0\rangle \) in the same simple way as the asymptotic growth of the matrix elements (2) is related to the existence of the singularities of \( \Phi(t) \) at finite complex \( t \).

Thus it has been shown that the problem of calculating the amplitude of the process \( 1 \to n \) at the threshold in the \( \lambda\phi^4 \) theory of one real scalar field \( \phi \) is equivalent to calculating in the corresponding Euclidean field theory the mean field \( \Phi(\tau) \) (equation (15)) with the kink-type boundary conditions. The fact that the classical solution provides the absolute minimum of the action in the Euclidean-space problem leads to the conclusion that all the \( 1 \to n \) amplitudes at the threshold have vanishing absorptive part despite the presence in higher loops of many intermediate states with on-mass-shell particles. This behavior in particular requires the on-mass-shell amplitude of the scattering \( 2 \to 3 \) to vanish at the threshold in all orders of perturbation theory. Also using the periodicity of the mean field \( \Phi(\tau) \) in complex \( \tau \) and the behavior of the field at large \( |\text{Re} \tau| \) it has been shown that the function \( \Phi(\tau) \) necessarily has singularities at finite complex
\( \tau \) and thus the threshold amplitudes \( 1 \rightarrow n \), which are related to the coefficients of the expansion of \( \Phi(\tau) \) in powers of \( e^{m\tau} \) (eqs. (17) and (18)), necessarily have a factorial growth in \( n \) in the full quantum theory. The properties of the specific theory, which were employed in the present paper, in general do not hold for other models. In the quantum-mechanical example of an anharmonic oscillator the spectrum of energies of highly excited states \( |n\rangle \) is not equally spaced, thus the generating function cannot be treated in terms of a power series expansion in \( e^{it} \), neither it is periodic in \( t \). In the \( \lambda \phi^4 \) theory without spontaneous symmetry breaking or in a theory with the spontaneous breaking of a continuous symmetry the Euclidean-space classical solution for the analog of \( \Phi(\tau) \) does not correspond to the minimum of the Euclidean action, so that a proper definition of the Euclidean-space path integral in the full quantum theory is somewhat subtle. It can be believed however, that the latter subtlety can eventually be sorted out and that the amplitudes in those theories also can be shown to display the \( n! \) growth in all orders of perturbation theory, unlike the case of Quantum Mechanics. However this is yet to be done.

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Figure captions

Figure 1. The structure of the mean field $\Phi(t)$ in the complex $t$ plane. $\Phi(t)$ is analytical in the upper half-plane. The heavy dots correspond to the poles of the classical mean field $\Phi_{cl}(t)$. The vertical line, running between two poles is the real axis for $\tau$.

Figure 2. The tadpole graph for calculating the one-loop correction to the mean field (the heavy dot). The propagator and the triple vertex are the Green function and the vertex in the background of the classical mean field $\Phi_{cl}(t)$.

Figure 3. The contour $C$ on which the index of the function $\Delta(\tau)$ is calculated (equation (20)).
Figure 1

Figure 2
Figure 3