Spatially Homogeneous Rotating Solution in $f(R)$ Gravity and Its Energy Contents

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Abstract

In this paper, the metric approach of $f(R)$ theory of gravity is used to investigate the exact vacuum solutions of spatially homogeneous rotating spacetimes. For this purpose, $R$ is replaced by $f(R)$ in the standard Einstein-Hilbert action and the set of modified Einstein field equations reduce to a single equation. We adopt the assumption of constant Ricci scalar which maybe zero or non-zero. Moreover, the energy density of the non-trivial solution has been evaluated by using the generalized Landau-Lifshitz energy-momentum complex in the perspective of $f(R)$ gravity for some appropriate $f(R)$ model, which turns out to be a constant quantity.

Keywords: $f(R)$ gravity, Spatially homogeneous rotating solutions, Generalized Landau-Lifshitz complex.

1 Introduction

In recent years, the attention towards modified theories of gravity is increasing rapidly. Although General Relativity(GR) has solved many problems but yet there are some important issues like, the accelerated expansion of

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the universe, cosmological constants or dark energy problem: the 120 orders of magnitude variation between the theoretical and observational values of the vacuum energy density, which have to be explored. Lovelock and \( f(R) \) theories are some efforts to modify the GR, in which the higher powers of \( R \) and its derivatives are used to solve the unresolved issues.

Eddington was the first who pondered the \( f(R) \) actions. Afterwards Buchdahl studied these \( f(R) \) actions in case of non-singular oscillating cosmologies. These theories are termed as higher order gravitational theories and proved that these models are equivalent to scalar-tensor models of gravity. It is palpable that firstly we must check their consistency with the solar system tests of Einstein gravity. It was observed that without bringing new degree of freedom, it is not possible to perform these tests and meanwhile to account for the accelerated expansion of the universe, in most of the models. Soon after, Sawicki and Starobinski have illustrated that without bringing new degrees of freedom one can account for both the solar system tests and the accelerated expansion of the universe at the same time.

The most spaciously discovered exact solutions in \( f(R) \) gravity are the spherically symmetric solutions which were found by Multamäki and Vilja. They proved that the whole set of field equations in \( f(R) \) gravity yields exactly the Schwarzschild de Sitter metric. Lukas and Francisco have analyzed the exact solutions of static spherically symmetric spacetimes in \( f(R) \) modified theories of gravity. Cylindrical symmetric solution in \( f(R) \) theory were explored by Momeni, Azadi and Nouri-Zonoz. This work was extended by Momeni and Gholizade to the general cylindrical symmetric solutions.

Sharif and Farasat studied plane symmetric static solution and vacuum solutions of Bianchi types I and V spacetimes in \( f(R) \) theory of gravity by using metric affine approach. They also extended this work to perfect fluid solutions and found the non-vacuum solutions of Bianchi types I and V in \( f(R) \) gravity by taking stiff matter. Farast also studied Bianchi type I, III and Kantowski- Sachs spacetimes in \( f(R) \) gravity and. Black hole solutions in \( F(R) \) gravity with conformal anomaly are found by Hendi and Momeni. Noether symmetry approach in \( f(R) \)-Tachyon model is discussed in. Recently Farasat et al. studied conserved quantities in \( f(R) \) gravity via Noether symmetry.

Since the Einstein era the localization of energy has been a complicated and controversial issue in GR which is still unanswerable. Einstein ini-
tiated the energy-momentum pseudo-tensor and gave the energy-momentum conservation laws as follows

\[
\frac{\partial}{\partial x^\nu}\{\sqrt{-g}(T^\nu_{\mu} + t^\nu_{\mu})\} = 0, \quad (\mu, \nu = 0, 1, 2, 3)
\]

where \(t^\nu_{\mu}\) is energy-momentum density of gravitation and \(T^\nu_{\mu}\) is energy-momentum density of matter. Fundamental nature of conservation laws was deeply explained by Bergmann [20]-[22].

By considering the geodesic coordinate system, Landau-Lifshitz [23] derived the energy-momentum complex (EMC) at some particular point of a spacetime. Various authors, like, Tolman [24], Papapetrou [25], Bergmann [26], Goldberg [27], Möller [28] and Weinberg [29] added their own EMCs. Misner et al. [30] proved that energy can only be localized in spherical systems. But soon after, Cooperstock and Sarracino [31] proved that if energy is localizable for spherical systems, then it can be localized (to express it as a unique tensor quantity) in any system.

Multamäk et al. [32] generalized the Landau-Lifshitz EMC in the context of \(f(R)\) theory of gravity. In the framework of metric \(f(R)\) gravity, the energy-momentum distribution (EMD) may be evaluated by using the generalized Landau-Lifshitz prescription. They calculated the energy density for the Schwarzschild de Sitter spacetime. Recently, Sharif and Farasat [33] calculated the energy density of plane symmetric static solutions and cosmic string spacetime by using generalized Landau-Lifshitz prescription.

Valerio Faraoni and Shahn Nadeau [34] have discussed some important \(f(R)\) models along with their stability conditions. We are now extending this work by exploring the spatially homogeneous rotating solutions in \(f(R)\) gravity and the energy contents of the obtained non-trivial solution for particular \(f(R)\) model.

The paper is arranged as follows: In section 2, we have briefly discussed the modified field equations in metric approach of \(f(R)\) gravity. Section 3 contains the derivation of generalized Landau-Lifshitz EMC. The exact vacuum solution, by applying the constant curvature condition, of spatially homogeneous rotating spacetimes is described in section 4. In section 5, we have calculated the energy density of the solution obtained in previous section by using Landau-Lifshitz EMC. We summarized the results in the last section.
2 Field Equations in $f(R)$ Theory of Gravity

In this section, we derive the field equations. For this purpose, we used the metric approach of $f(R)$ theory of gravity. In this approach, the variation of the action is done with respect to the metric tensor only. The action of $f(R)$ theory is stated as

$$S = \int \sqrt{-g} \left( \frac{1}{16\pi G} f(R) + L_m \right) d^4 x,$$  

(1)

where $f(R)$ is a general function of Ricci scalar $R$ and $L_m$ is the matter Lagrangian. The replacement of $R$ by $f(R)$ in the standard Einstein-Hilbert action gives us this action. The corresponding field equations can be obtained by varying this action with respect to metric tensor $g_{\mu\nu}$ and are given by

$$F(R)R_{\mu\nu} - \frac{1}{2} f(R)g_{\mu\nu} - \nabla_\mu \nabla_\nu F(R) + g_{\mu\nu} \Box F(R) = \kappa T_{\mu\nu},$$  

(2)

where

$$F(R) = \frac{df(R)}{dR}, \quad \Box = \nabla^\mu \nabla_\mu.$$  

(3)

Here, $\nabla_\mu$ represents the covariant derivative, $\Box = \nabla^\mu \nabla_\mu$ is called D’Alember operator and $T_{\mu\nu}$ is the standard matter energy-momentum tensor derived from $L_m$. These are the fourth order partial differential equations in the metric tensor. For $f(R) = R$ these equations reduce to the famous Einstein field equations of GR. After contraction the field equations turn out to be

$$F(R)R - 2f(R) + 3\Box F(R) = \kappa T$$  

(4)

and, in vacuum, i.e., when $T = 0$, the last equation takes the form

$$F(R)R - 2f(R) + 3\Box F(R) = 0.$$  

(5)

This gives an important relationship between $f(R)$ and $F(R)$ which will be helpful to simplify the field equations and to determine the $f(R)$. It is to be noted that any metric with constant scalar curvature, say $R = R_0$, is a solution of the Eq.(5) if the following equation holds

$$F(R_0)R_0 - 2f(R_0) = 0.$$  

(6)
This is the constant scalar curvature condition in vacuum and for non-vacuum case, it takes the form

\[ F(R_0)R_0 - 2f(R_0) = \kappa T. \quad (7) \]

These conditions play an important role to find the acceptability of \( f(R) \) models.

### 3 Generalized Landau-Lifshitz Energy-Momentum Complex

The generalized Landau-Lifshits EMC is given by

\[ \tau_{\mu\nu} = f'(R_0)\tau_{\mu\nu}^{LL} + \frac{1}{6\kappa}\{f'(R_0)R_0 - f(R_0)\}\frac{\partial}{\partial x^\lambda}(g^{\mu\nu}x^\lambda - g^{\mu\lambda}x^\nu), \quad (8) \]

where \( \tau_{\mu\nu}^{LL} \) is the Landau-Lifshitz EMC in GR and \( \kappa = 8\pi G \). We can calculate EMD in the framework of \( f(R) \) theory of any metric tensor which has constant scalar curvature. Its 00-component represents the energy density and is given by the following equation

\[ \tau_{00} = f'(R_0)\tau_{00}^{LL} + \frac{1}{6\kappa}(f'(R_0)R_0 - f(R_0))\left(\frac{\partial}{\partial x^i}g^{00}x_i + 3g^{00}\right), \quad (9) \]

where \( \tau_{00}^{LL} \) represents the sum of energy-momentum tensor and the energy-momentum pseudo tensor and is given by

\[ \tau_{00}^{LL} = (-g)(T^{00} + t_{00}^{LL}) \quad (10) \]

and

\[ T^{00} = \frac{1}{\kappa}(R^{00} - \frac{1}{2}g^{00}R), \quad (11) \]

where \( R \) is the Ricci scalar and \( t_{00}^{LL} \) can be evaluated from the following expression

\[ t_{00}^{LL} = \frac{1}{2\kappa}[(2\Gamma^\gamma_{\alpha\beta}\Gamma^\delta_{\gamma\delta} - \Gamma^\gamma_{\alpha\delta}\Gamma^\delta_{\beta\gamma} - \Gamma^\gamma_{\alpha\gamma}\Gamma^\delta_{\beta\delta})\{g^{\mu\nu}g^{\mu\beta}g^{\nu\gamma} - g^{\mu\alpha}g^{\mu\beta}g^{\nu\gamma}\} + g^{\mu\alpha}g^{\nu\beta}(\Gamma^\alpha_{\alpha\delta}\Gamma^\beta_{\gamma\delta} + \Gamma^\mu_{\beta\gamma}\Gamma^\delta_{\alpha\delta} - \Gamma^\nu_{\gamma\delta}\Gamma^\delta_{\alpha\beta} - \Gamma^\mu_{\alpha\beta}\Gamma^\nu_{\gamma\delta}) + g^{\mu\alpha}g^{\nu\beta}(\Gamma^\mu_{\alpha\delta}\Gamma^\gamma_{\beta\delta} + \Gamma^\mu_{\beta\gamma}\Gamma^\gamma_{\alpha\delta} - \Gamma^\mu_{\gamma\delta}\Gamma^\gamma_{\alpha\beta} - \Gamma^\mu_{\alpha\beta}\Gamma^\gamma_{\gamma\delta}) + g^{\alpha\beta}g^{\gamma\delta}(\Gamma^\mu_{\alpha\gamma}\Gamma^\nu_{\beta\delta} - \Gamma^\mu_{\alpha\beta}\Gamma^\nu_{\gamma\delta})]. \quad (12) \]
4 Spatially Homogeneous Rotating Spacetimes

Solution

In this section, we solve the vacuum field equations of $f(R)$ theory for the metric representing the spatially homogeneous rotating spacetimes by using metric approach and the condition of constant scalar curvature, i.e., $(R = \text{constant})$. The line element representing the spatially homogeneous rotating spacetimes is given by

$$ds^2 = dt^2 - dr^2 - A(r)d\phi^2 - dz^2 + 2B(r)dt\,d\phi,$$

where $A(r)$ and $B(r)$ are arbitrary functions of $r$. This metric represents five spacetimes [35]-[36], which can be achieved by choosing particular values of the metric functions $A$ and $B$.

The corresponding Ricci scalar is given by

$$R = \frac{4ABB'' + 4B^3B'' - 4A'B'B + 3AB'' - B^2B' - 2AA'' + 2A''B^2 - A^2}{2(A + B^2)^2},$$

where prime denotes the derivative with respect to $r$. Eq.(3) implies that

$$f(R) = \frac{3\Box F(R) + F(R)R}{2}.\quad (15)$$

Making use of this value of $f(R)$ in Eq.(2) along with the substitution $T_{\mu\nu} = 0$ (for vacuum case), we reached at

$$F(R)R_{\mu\nu} - \nabla_{\mu}\nabla_{\nu}F(R) = \frac{F(R)R - \Box F(R)}{4}.\quad (16)$$

Since the metric Eq.(13) depends only upon $r$ so Eq.(16) can be viewed as the set of differential equations for $F(R)$, $A(r)$ and $B(r)$. Eq.(16) can be reduced to the following combination

$$A_{\mu} = \frac{F(R)R_{\mu\nu} - \nabla_{\mu}\nabla_{\nu}F(R)}{g_{\mu\nu}}.\quad (17)$$

This combination is independent of the index $\mu$ and hence $A_{\mu} - A_{\nu} = 0$ for all $\mu$ and $\nu$. Thus, $A_0 - A_1 = 0$ results that

$$(4B^2B'' - 4ABB'' - 4B''B^3 + 4A'B'B - 2AA'' - 2B^2A'' + A^2)F - 4(A + B^2)^2F'' = 0.$$
Similarly, for all other possible combinations, i.e., $A_0 - A_2 = 0$, $A_0 - A_3 = 0$, $A_1 - A_2 = 0$, $A_1 - A_3 = 0$ and $A_2 - A_3 = 0$, we have the following independent equations

\[
(2AA'' + 2B^2A'' + 2A'B'B - A'^2)F - 2A'(A + B^2)F' = 0, \quad (19)
\]
\[
\left(\frac{B^2}{2(A + B^2)}\right)F = 0, \quad (20)
\]
\[
(4A^2BB'' + 4B^3B'' - 2A'B'AB - 2AA''B^2 - 2A''B^4
+ 2A'B'B^3 + A'B^2)F - 2A'(A + B^2)^2F' + 4A(A + B^2)^2F'' = 0, \quad (21)
\]
\[
(2AB^2 + 4ABB'' + 4B^3B'' - 4A'B'B - 2B'^2B^2 + 2AA''
+ 2A''B^2 - A'^2)F - 4(A + B^2)^2F'' = 0, \quad (22)
\]
\[
(2AB^2 + 2AA'' - A'^2 - 2A'B'B + 2A''B^2)F
- 2\frac{A'}{A}(A + B^2)F' = 0. \quad (23)
\]

These are the six non-linear ordinary differential equations involving three unknown variables $A, B$ and $F$. We use the condition of constant curvature to solve these equations as given in the next subsection.

### 4.1 Constant Curvature Solution

For constant curvature solution, that is, $R = R_0$, it is obvious that the first and second derivatives of $F(R) = \frac{df(R)}{dR}$ will always vanish, that is,

\[
F'(R_0) = 0 = F''(R_0). \quad (24)
\]

In view of the Eq. (24), the Eqs. (18)-(23) reduce to

\[
4B^2B'^2 - 4ABB'' - 4B''B^3 + 4A'B'B - 2AA''
- 2B^2 A'' + A'^2 = 0, \quad (25)
\]
\[
2AA'' + 2B^2 A'' + 2A'B'B - A'^2 = 0, \quad (26)
\]
\[
B^2 = 0, \quad (27)
\]
\[
4A^2BB'' + 4B^3B'' - 2A'B'AB - 2AA''B^2 - 2A''B^4
+ 2A'B'B^3 + A'B^2 = 0, \quad (28)
\]
\[
2AB^2 + 4ABB'' + 4B^3B'' - 4A'B'B - 2B'^2B^2 + 2AA''
+ 2A''B^2 - A'^2 = 0, \quad (29)
\]
\[
2AB^2 + 2AA'' - A'^2 - 2A'B'B + 2A''B^2 = 0. \quad (30)
\]
Eq. (27) implies that
\[ B = \text{constant} = c_1 \quad (\text{say}). \] (31)

After inserting this value of \( B \), the Eqs. (25)-(26) and (28)-(30) reduce to a single equation
\[ 2AA'' + 2c_1^2 A'' - A'^2 = 0. \] (32)

This is a second order ordinary differential equation and its non-trivial solution is given (by using the software Maple) as
\[ A(r) = \frac{1}{4} c_2^2 r^2 + \frac{1}{2} c_2 c_3 r + \frac{1}{4} c_3^2 - c_1, \] (33)

where \( c_2 \) and \( c_3 \) are constants. It is noticed that \( A = \text{constant} \) is a trivial solution of the Eq. (32). Now, renaming the constants, the Eq. (33) takes the form
\[ A(r) = c_4 r^2 + c_5 r + c_6, \] (34)

where \( c_4 = \frac{1}{4} c_2^2, \ c_5 = \frac{1}{2} c_2 c_3 \) and \( c_6 = \frac{1}{4} c_3^2 - c_1^2 \). Hence, the corresponding solution turns out to be
\[ ds^2 = dt^2 - dr^2 - (c_4 r^2 + c_5 r + c_6)d\phi^2 - dz^2 + 2c_1 dt d\phi \] (35)

It is mentioned here that the Ricci scalar evaluated for this solution vanishes identically, i.e.,
\[ R = 0 \] (36)

5 Energy Density of the Spatially Homogeneous Rotating Solution

In this section, we compute energy density of the spatially homogeneous rotating solution (35), which is obtained in the framework of \( f(R) \) theory of gravity in the last section. For this purpose, we use generalized Landau-Lifshitz EMC in the context of \( f(R) \) gravity. Substituting the value of \( g^{00} \), the Eq. (29), takes the form
\[ \tau^{00} = f'(R_0)\tau^{00}_{LL} + \frac{1}{2R_0}(f'(R_0)R_0 - f(R_0)). \] (37)
Now, by evaluating $T^{00}$ and $t_{00}^{LL}$ from Eqs. (11) and (12) respectively and then using in Eq. (10), $\tau_{00}^{LL}$ turns out to be

$$\tau_{00}^{LL} = \frac{1}{4\kappa} c^2. \quad (38)$$

Using this value of $\tau_{00}^{LL}$ in Eq. (37), the 00-component of the Landau-Lifshitz EMC takes the form

$$\tau^{00} = f'(R_0) \frac{1}{4\kappa} (c^2) + \frac{1}{2\kappa} (f'(R_0)R_0 - f(R_0)). \quad (39)$$

Finally, we will use suitable $f(R)$ model to calculate this component completely. It is revealed here that there must be some restrictions to choose the $f(R)$ model when $R = 0$ because if the model involves the logarithmic function of Ricci scalar $R$ or a linear superposition of $R^{-n}$, where $n$ is positive integer, then we can not evaluate this EMC. Thus, we consider the following model

$$f(R) = R + \epsilon R^2, \quad (40)$$

where $\epsilon$ is a positive real number. Accordingly, the 00-component of generalized Landau-Lifshitz EMC results as

$$\tau^{00} = \frac{1}{4\kappa} c^2. \quad (41)$$

Furthermore, the stability condition for this $f(R)$ model

$$\frac{1}{\epsilon(1 + 2\epsilon R_0)} = \frac{1}{\epsilon} > 0 \quad (42)$$

holds. The condition of constant scalar curvature is also satisfied for this $f(R)$ model.

6 Summary and Conclusion

This paper has mainly two parts: In first part, we have explored the spatially homogeneous rotating solution in $f(R)$ theory of gravity. For this purpose, we solve the field equations of $f(R)$ gravity for the metric representing the spatially homogeneous rotating spacetimes. It is found that there arise two
solutions, one trivial (when the metric functions $A(r)$ and $B(r)$ become constant) and other one is the non-trivial solution. It is interesting to mention here that the Ricci scalar $R = 0$ vanishes in both the cases.

The second part of the paper contains the energy density component of the non-trivial solution. For this purpose, we used the generalized Landau-Lifshitz EMC in the framework of $f(R)$ theory of gravity for a suitable $f(R)$ model. It has been shown that the particularly selected $f(R)$ model satisfies the condition of constant scalar curvature, which is the mandatory constraint for the validity of $f(R)$ models. Further, we have discussed the stability condition for this model. It is mentioned here that the energy density for this model turns out to be constant as given in Eq.(41). It is anticipated that such solutions may provide an entrance towards the solution of dark energy and dark matter problems.

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