Universal uncertainty principle and quantum state control under conservation laws

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Abstract. Heisenberg’s uncertainty principle, exemplified by the γ ray thought experiment, suggests that any finite precision measurement disturbs any observables noncommuting with the measured observable. Here, it is shown that this statement contradicts the limit of the accuracy of measurements under conservation laws originally found by Wigner in 1950s, and should be modified to correctly derive the unavoidable noise caused by the conservation law induced decoherence. The obtained accuracy limit leads to an interesting conclusion that a widely accepted, but rather naive, physical encoding of qubits for quantum computing suffers significantly from the decoherence induced by the angular momentum conservation law.

1. INTRODUCTION

There has been a longstanding confusion on Heisenberg’s uncertainty principle. Many text books have incorrectly interpreted the mathematical relation between the standard deviations as the relation between noise and disturbance exemplified by the γ ray microscope. In fact, the purported reciprocal relation was not general enough to hold for all the possible measurements. Recently, a universally valid noise-disturbance relation was found by the present author [1, 2] and it has become clear that the new relation plays a role of the first principle to derive various quantum limits on quantum measurements and quantum information processing in a unified treatment. Here, we discuss a consequence of the universal uncertainty relation for limitations on measurements and quantum state controls in the presence of nondisturbed quantities such as under conservation laws or superselection rules.

Heisenberg’s uncertainty principle, in the naive formulation, suggests that any finite precision measurement disturbs any observables noncommuting with the measured observable. Interestingly, the above statement contradicts the limit on measurements under conservation laws known as the Wigner-Araki-Yanase (WAY) theorem, which allows the finite precision measurement without disturbing conserved quantities. Here, we shall resolve the conflict by deriving the correct limitation of measurement in the presence of nondisturbed quantities. Then, the obtained formula is shown to quantitatively generalize the WAY theorem [3]. The obtained accuracy limit leads to an interesting conclusion that a widely accepted, naive physical encoding of qubits suffers from the decoherence induced by the control system under the angular momentum conservation law [4]. Various numerical bounds are obtained for inevitable error probability in physical realizations of Hadamard gates.

2. UNIVERSAL UNCERTAINTY PRINCIPLE

The uncertainty relation for any observables A and B is usually formalized by the relation

\[ \sigma(A)\sigma(B) \geq \frac{1}{2}|\langle[A, B]\rangle| \]

proven first by Robertson [5] in 1929, where \(\sigma(A)\) and \(\sigma(B)\) are the standard deviations of A and B and \(\langle \cdot \cdot \cdot \rangle\) stands for the expectation value in the given state. For observables Q and P satisfying the canonical commutation relation \(QP - PQ = i\hbar\), we obtain the uncertainty relation

\[ \sigma(Q)\sigma(P) \geq \frac{\hbar}{2} \]
proven first by Kennard [6] in 1927. The notion of standard deviation in the above formulations depends only on the state of the system, but does not depend on the measuring apparatus to be used. However, it is often explained misleadingly that the physical content of the above formal relations is that if one measures observable \( Q \), the product of the noise in this measurement and the disturbance in observable \( P \) caused by this measurement is no less than \( \hbar/2 \) as claimed by Heisenberg [7] in 1927. If we introduce the apparatus dependent quantities, the root-mean-square noise \( \epsilon(A) \) in any \( A \) measurement and the root-mean-square disturbance \( \eta(B) \) of \( B \) caused by that measurement, the above interpretation is expressed by the relation

\[
\epsilon(A)\eta(B) \geq \frac{1}{2}|<[A,B]>|.
\]  

Heisenberg [7] claimed the above interpretation and demonstrated it by the famous \( \gamma \) ray microscope thought experiment. However, relation (3), usually called *Heisenberg’s uncertainty principle*, has been shown not universally valid [8], and a universally valid noise-disturbance uncertainty relation, the *universal uncertainty principle*, has been recently obtained by the present author [1,2] as

\[
\epsilon(A)\eta(B) + \sigma(A)\eta(B) + \epsilon(A)\sigma(B) \geq \frac{1}{2}|<[A,B]>|,
\]  

where the mean and the standard deviations are taken in the state just before the measurement.

### 3. UNCERTAINTY PRINCIPLE AND THE WIGNER-ARAKI-YANASE THEOREM

The uncertainty principle is a fundamental source of the noise in measurements, while the decoherence induced by conservation laws is another source of the noise. Every interaction brings an entanglement in the basis of a conserved quantity, so that measurements, and any other quantum controls such as quantum information processing, are subject to the decoherence induced by conservation laws. One of the earliest formulations of this fact was given by the Wigner-Araki-Yanase (WAY) theorem [9,10] stating that any observable which does not commute with an additively conserved quantity cannot be measured with absolute precision.

It is natural to expect that the WAY theorem can be derived by Heisenberg’s uncertainty principle for nondisturbing measurements. Suppose, for instance, that in order to measure the position \( Q_1 \) of particle 1 with momentum \( P_1 \), one measures the momentum \( P_2 \) of particle 2 that has been scattered from particle 1. According to the momentum conservation law, this measurement of \( Q_1 \) does not disturb \( P_1 + P_2 \). Thus Heisenberg’s uncertainty principle with \( \eta(P_1 + P_2) = 0 \) concludes that we cannot measure \( Q_1 \) even with finite error, i.e., \( \epsilon(Q_1) = \infty \).

However, the scenario is not that simple. In reality, we can measure the position \( Q_1 \) with finite or even arbitrarily small noise by the above method. Actually, the WAY theorem does not conclude unmeasurability of any observables even if they do not commute with the conserved quantity, but merely sets the accuracy limit of the measurement with size limited apparatus in the presence of bounded conserved quantities. In fact, the WAY theorem has a caveat that the noise decreases if the size of the apparatus increases and that if the apparatus is of macroscopic size, the noise can be negligible [11,12,13]. Thus, the WAY theorem allows an arbitrarily precise measurement with a large apparatus, but Heisenberg’s uncertainty principle for nondisturbing measurements does not allow any finite precision measurement with apparatus of any size.

Now, we abandon Heisenberg’s uncertainty principle (3) and consider the problem to find a correct lower bound for the noise of measurements in the presence of a nondisturbed quantity. For this purpose, we return to the universal uncertainty principle (4). We suppose that the measuring interaction does not disturb an observable \( B \). Then, we have \( \eta(B) = 0 \), so that by substituting this relation to Eq. (4) we obtain

\[
\epsilon(A)\sigma(B) \geq \frac{1}{2}|<[A,B]>|.
\]  

The above relation, the *universal uncertainty principle for nondisturbing measurements*, represents a correct lower bound for the noise in measuring observable \( A \) using any measuring apparatus that does not disturb observable \( B \).

The above new formulation of the uncertainty principle Eq. (5) can be used to derive the quantitative expression of the WAY theorem as follows. Suppose that the measuring interaction \( U \) satisfies the additive conservation law \([U, L_1 + L_2] = 0\), where \( L_1 \) belongs to the object and \( L_2 \) belongs to the apparatus. Also suppose that the meter observable \( M \) in the apparatus commutes with the conserved quantity, i.e., \([M, L_2] = 0\). Then, we can conclude that
this measurement does not disturb $L_1 + L_2$, thus Eq. (5) holds for $B = L_1 + L_2$. Since $A$ belongs to the object, we have $[A, L_1 + L_2] = [A, L_1]$. Since the object and the apparatus are statistically independent before the measurement, we have $|\sigma(L_1 + L_2)|^2 = \sigma(L_1)^2 + \sigma(L_2)^2$. Thus, we have derived a quantitative generalization of the W AY theorem \[\epsilon(A)^2 \geq \frac{|\langle A, L_1 | \rangle|^2}{4\sigma(L_1)^2 + 4\sigma(L_2)^2}, \tag{6}\]
as a straightforward consequence from Eq. (5). By the above, the lower bound of the noise decreases with the increase of the uncertainty of the conserved quantity in the apparatus.

### 4. OPERATIONAL DECOHERENCE IN QUANTUM LOGIC GATES

In most of current proposals for implementing quantum computing, a component of spin of a spin 1/2 system is chosen as the computational basis for the feasibility of initialization and read-out. For this choice of the computational basis, it has been shown that the angular momentum conservation law limits the accuracy of quantum logic operations based on estimating the unavoidable noise in CNOT gates; see also, Ref. [13, 14]. Here, we shall consider the accuracy of implementing Hadamard gates, which are essential components for quantum Fourier transforms in Shor’s algorithm, and show that Hadamard gates are no easier to implement under the angular momentum conservation law than CNOT gates.

Let $Q$ be a spin 1/2 system as a qubit with computational basis $\{\{0\}, \{1\}\}$ encoded by $S_z = (\hbar/2)(\{0\}\langle 0 | - \{1\}\langle 1 |)$, where $S_i$ is the $i$ component of spin for $i = x, y, z$. Let $H = 2^{-1/2}(\{0\}\langle 0 | + \{1\}\langle 1 |)$ be the Hadamard gate on $Q$. Let $U$ be a physical realization of $H$. We assume that $U$ is a unitary operator on the composite system of $Q$ and the ancilla $A$ included in the controller of the gate and that $U$ satisfies the angular momentum conservation law; see [4, 14] for general formulation. For simplicity, we only assume that the $x$ component of the total angular momentum is conserved, i.e., $[U, S_x + L_x] = 0$, where $L_x$ is the $x$ component of the total angular momentum of the ancilla.

Now, we consider the following process of measuring the operator $S_z$ of $Q$: (i) to operate $U$ on $Q + A$, and (ii) to measure $S_z$ of $Q$. Since $S_z = H^\dagger S_x H$, if $U = H$ the above process would measure $S_z$ precisely. Since each step does not disturb $S_x + L_x$, we can apply Eq. (6) to this measurement and obtain

$$\epsilon(S_z)^2 \geq \frac{|\langle S_z, S_z \rangle|^2}{4\sigma(S_z)^2 + 4\sigma(L_x)^2}. \tag{7}$$

In general, the squared-noise $\epsilon(S_z)^2$ amounts to the mean-square error of $U$ from the correct operation of $H$ in the given input state $\psi$. Since each error has the squared difference $\hbar^2$, the error probability $P_e$ is considered to be $P_e = \epsilon(S_z)^2/\hbar^2$. For the input state $\psi = (\{0\} + i\{1\})/\sqrt{2} = |S_y = \hbar/2\rangle$, the numerator is maximized as

$$P_e = \frac{\epsilon(S_z)^2}{\hbar^2} \geq \frac{1}{4 + 4(2\sigma(L_x)/\hbar)^2}. \tag{8}$$

In the following, we shall interpret the above relation for bosonic control systems and fermionic control systems separately. In current proposals, the external electromagnetic field prepared by laser beam is considered to be a feasible candidate for the controller $A$ to be coupled with the computational qubits $Q$ via the dipole interaction [17]. In this case, the ancilla state $|\xi\rangle$ is considered to be a coherent state, for which we have $\sigma(N) = |\langle N \rangle| = |\langle N \rangle|$, where $N$ is the number operator. We assume that the beam propagates to the $x$-direction with right-hand-circular polarization. Then, we have $L_x = \hbar N$, and hence $(2\sigma(L_x)/\hbar)^2 = (2\sigma(N))^2 = 4\langle N \rangle$. Thus, from Eq. (8) we have

$$P_e \geq \frac{1}{4 + 16\langle N \rangle}. \tag{9}$$

Enk and Kimble [18] and Gea-Banacloche [19] also showed that there is unavoidable error probability in this case inversely proportional to the average strength of the external field by calculations with the model obtained by rotating wave approximation. Here, we have shown the same result only from the angular momentum conservation law. If the field is in a number state $|n\rangle$, then $2\sigma(L_x)/\hbar)^2 = 2\sigma(N)^2 = 0$, so that we have

$$P_e = \frac{\epsilon(S_z)^2}{\hbar^2} \geq \frac{1}{4}. \tag{10}$$
Thus, if the field state is a mixture of number states such as the thermal state, i.e., \( \sigma = \sum_n p_n |n\rangle \langle n| \), we have also the lower bound \( \epsilon(S_z)^2/\hbar^2 \geq 1/4 \). Thus, it seriously matters whether the control field is really in a coherent state or a mixture of number states.

We now assume that the ancilla A comprises \( n \) spin 1/2 systems. Then, we have \( \sigma(L_x) \leq \|L_x\| = n\hbar^2 \). Thus, we have the following lower bound of the gate error probability

\[
P_e \geq \frac{1}{4 + 4n^2}. \tag{11}
\]

Thus, it has been proven that if the computational basis is represented by the \( z \)-component of spin, we cannot implement Hadamard gates within the error probability \( (4 + 4n^2)^{-1} \) with \( n \) qubit ancilla by rotationally invariant interactions such as the Heisenberg exchange interaction. In the above discussion, we have assumed that the control system can be prepared in an entangled state. However, it is also interesting to estimate the error in the case where we can prepare the control system only in a separable state. In this case, we have \( \sigma(L_x)^2 \leq \sum_{j=1}^n \sigma(S_x^{(j)})^2 \leq n\|S_x\|^2 = n\hbar^2 \), where \( S_x^{(j)} \) is the spin component of the \( j \)th ancilla qubit so that \( L_x = \sum_{j=1}^n S_x^{(j)} \). Thus, we have the following lower bound of the gate error probability

\[
P_e \geq \frac{\epsilon(S_z)^2}{\hbar^2} \geq \frac{1}{4 + 4n}. \tag{12}
\]

Thus, the error probability is lower bounded by \( (4 + 4n)^{-1} \), and hence the achievable error can be considered to be inversely proportional to \( 4n^2 \) for entangled control system but \( 4n \) for separable control system. Note that even if the ancilla is in a separable mixed state, the relation \( (4 + 4n)^{-1} \leq \epsilon(S_z)^2/\hbar^2 \) still holds, since \( \epsilon(S_z)^2 \) is an affine function of the ancilla state.

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