Joint distribution of two local times for diffusion processes with the application to the construction of various conditioned processes

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Abstract
For a diffusion process \(X(t)\) of drift \(\mu(x)\) and of diffusion coefficient \(D = 1/2\), we study the joint distribution of the two local times \(A(t) = \int_0^t d\tau \delta(X(\tau))\) and \(B(t) = \int_0^t d\tau \delta(X(\tau) - L)\) at positions \(x = 0\) and \(x = L\), as well as the simpler statistics of their sum \(\Sigma(t) = A(t) + B(t)\). Their asymptotic statistics for large time \(t \to +\infty\) involves two very different cases: (i) when the diffusion process \(X(t)\) is transient, the two local times \([A(t), B(t)]\) remain finite random variables \([A^*(\infty), B^*(\infty)]\) and we analyze their limiting joint distribution; (ii) when the diffusion process \(X(t)\) is recurrent, we describe the large deviations properties of the two intensive local times \(a = A(t)/t\) and \(b = B(t)/t\) and of their intensive sum \(\sigma = \Sigma(t)/t = a + b\). These properties are then used to construct various conditioned processes \([X^*(t), A^*(t), B^*(t)]\) satisfying certain constraints involving the two local times, thereby generalizing our previous work (Mazzolo and Monthus 2022 J. Stat. Mech. 103207) concerning the conditioning with respect to a single local time \(A(t)\). In particular for the infinite time horizon \(T \to +\infty\), we consider the conditioning towards the finite asymptotic values \([A^*(\infty), B^*(\infty)]\) or \(\Sigma^*(\infty)\), as well as the conditioning towards the intensive values \([a^*, b^*]\) or \(\sigma^*\), that can be compared with the appropriate ‘canonical conditioning’ based on the generating function of the local times.
in the regime of large deviations. This general construction is then applied to
the simplest case where the unconditioned diffusion is the Brownian motion of
uniform drift $\mu$.

Keywords: Brownian motion, local time, conditioned diffusion

(Some figures may appear in colour only in the online journal)

1. Introduction

1.1. Local times as the basic time-additive observables for diffusion processes

For a one-dimensional diffusion process $X(t)$, two essential time-additive observables of the
stochastic trajectory $X(0 \leq \tau \leq t)$ are:

(i) the occupation time $O_{[x_L,x_R]}(t)$ of the space interval $[x_L,x_R]$ during the time window $[0,t]$

$$O_{[x_L,x_R]}(t) = \int_0^t d\tau \theta(x_L \leq X(\tau) \leq x_R)$$  \hspace{1cm} (1)

that belongs to the interval $0 \leq O_{[x_L,x_R]}(t) \leq t$.

(ii) the local time $A_x(t)$ at the position $x$ during the time window $[0,t]$ (see the mathematical
review [1] and references therein)

$$A_x(t) = \int_0^t d\tau \delta(X(\tau) - x)$$  \hspace{1cm} (2)

that has for physical dimension $\text{Time} \times \text{Length}$ (in contrast to the physical dimension $\text{Time}$ for
equation (1)) and that belongs to $[0, +\infty]$ with no upper bound. The direct links with the
occupation times of equation (1) can be summarized as follows. On one hand, the local time
$A_x(t)$ of equation (2) can be constructed from the occupation time $O_{[x-\epsilon,x+\epsilon]}(t)$ of the space
interval $[x-\epsilon,x+\epsilon]$ of size $2\epsilon > 0$ centered at the position $x$ in the limit $\epsilon \to 0^+$

$$A_x(t) = \int_0^t d\tau \lim_{\epsilon \to 0^+} \left( \frac{\theta(x-\epsilon \leq X(\tau) \leq x+\epsilon)}{2\epsilon} \right) = \lim_{\epsilon \to 0^+} \left( \frac{O_{[x-\epsilon,x+\epsilon]}(t)}{2\epsilon} \right).$$  \hspace{1cm} (3)

On the other hand, the occupation time $O_{[x_L,x_R]}(t)$ can be reconstructed from the local time
$A_x(t)$ for all the internal positions $x \in [x_L,x_R]$

$$O_{[x_L,x_R]}(t) = \int_0^t d\tau \int_{x_L}^{x_R} dx \delta(X(\tau) - x) = \int_{x_L}^{x_R} dx A_x(t).$$  \hspace{1cm} (4)

As a consequence, both the occupation times and the local times have attracted a lot of
interest recently in the physics literature for many different contexts (see [2–15] and references therein).

More generally, the local times $A_x(t)$ of equation (2) allow to reconstruct any time-additive
observable involving an arbitrary function $f(.)$
\[ \int_0^t dx f(X(\tau)) = \int_{-\infty}^{+\infty} dx f(x) A_x(t) \]  

and can be thus considered as the basic time-additive observables. Time-additive observables like equation (5) are of course interesting for their own statistical properties, but they can also be used to construct conditioned processes as we now recall.

1.2. Reminder on the conditioning of stochastic processes with respect to time-additive observables

Since its introduction by Doob \cite{16, 17}, the conditioning of stochastic processes (see the mathematical books \cite{18–20} and the physics recent review \cite{21}) have found many applications in various fields like ecology \cite{22}, finance \cite{23} or nuclear engineering \cite{24, 25}. Over the years, many different conditioning constraints have been considered, including the Brownian excursion \cite{26, 27}, the Brownian meander \cite{28}, the taboo processes \cite{29–34}, or non-intersecting Brownian bridges \cite{35}. Let us also mention the conditioning in the presence of killing rates \cite{18, 36–43} or when the killing occurs only via an absorbing boundary condition \cite{44–47}. Stochastic bridges have also been studied for many other Markov processes, including various diffusions processes \cite{48–50}, discrete-time random walks and Lévy flights \cite{51–53}, continuous-time Markov jump processes \cite{53}, run-and-tumble trajectories \cite{54}, or processes with resetting \cite{55}.

For the present work, the recent important generalization concerns the conditioning with respect to global dynamical constraints involving time-additive observables of the stochastic trajectories. In particular, the conditioning on the area has been studied via various methods for Brownian processes or bridges \cite{56} and for Ornstein–Uhlenbeck bridges \cite{57}. The conditioning on the area and on other time-additive observables has been then analyzed both for the Brownian motion and for discrete-time random walks \cite{58}. This approach has been generalized \cite{59} to various types of discrete-time or continuous-time Markov processes, while the time-additive observable can involve both the time spent in each configuration and the increments of the Markov process. This general reformulation of the ‘microcanonical conditioning’, where the time-additive observable is constrained to reach a given value after the finite time window \( T \), allows to make the link \cite{59} with the ‘canonical conditioning’ based on generating functions of additive observables that has been much studied recently in the field of dynamical large deviations of Markov processes over a large time-window \( T \) \cite{60–105}. The equivalence between the ‘microcanonical conditioning’ and the ‘canonical conditioning’ at the level of the large deviations for large time \( T \) is explained in detail in the two complementary papers \cite{83, 84} and in the HDR thesis \cite{85}.

1.3. Goals and organization of the present paper

As recalled in the previous subsection, the methods to construct conditioned processes with respect to time-additive observables are now well-established, both in the ‘microcanonical’ and in the ‘canonical’ perspectives. However, the concrete application to specific time-additive observables like equation (5) for given Markov processes remains often challenging. It is thus important to identify the cases where conditioned processes can be explicitly constructed and what level of technicality is required to perform the computations.

At first sight, the delta function that enters the definition of the local time in equation (2) might appear as very singular. However, as in quantum mechanics where delta impurities are
well-known to be much simpler than smoother potentials, the delta function in equation (2) is actually a huge technical simplification with respect to the arbitrary general additive observable of equation (5). Indeed, the exact Dyson equation associated to a single delta impurity allows to analyze the statistics of a single local time in terms of the properties of the propagator $G_{l}(x|x_{0})$ of the diffusion process $X(t)$ alone, as recalled in detail in [106]. In the present paper, it will be thus interesting to use similarly the exact Dyson equation associated to two single delta impurities in order to characterize the joint statistics of the two Local Times $A(t) = A_{x=0}(t)$ and $B(t) = A_{x=L}(t)$ at positions $x = 0$ and $x = L$

$$A(t) \equiv \int_{0}^{t} d\tau \delta(X(\tau))$$

$$B(t) \equiv \int_{0}^{t} d\tau \delta(X(\tau) - L).$$

(6)

for a diffusion process $X(t)$ of diffusion coefficient $D = 1/2$ with an arbitrary position-dependent drift $\mu(x)$, thereby generalizing the explicit joint statistics of the two local boundary times for the Brownian motion with no drift $\mu(x) = 0$ on the interval $[0, L]$ with reflecting boundary conditions at $x = 0$ and $x = L$ computed recently [6]. The paper is organized as follows.

In section 2, we focus on the joint dynamics of the position $X(t)$ and of the two local times $A(t)$ and $B(t)$ of equation (6) satisfying the Ito Stochastic Differential System involving the Wiener process $W(t)$

$$dX(t) = \mu(X(t))dt + dW(t)$$

$$dA(t) = \delta(X(t))dt$$

$$dB(t) = \delta(X(t) - L)dt.$$ (7)

We use the Feynman–Kac formula and the Dyson equation in order to compute explicitly the time-Laplace-transform of parameter $s$ of the joint distribution $P(x, A, B|x_{0})$ to see $[X(t) = x; A(t) = A, B(t) = B]$ when starting at $[X(0) = x_{0}; A(0) = 0, B(0) = 0]$

$$\hat{P}_{s}(x, A, B|x_{0}) \equiv \int_{0}^{+\infty} dt e^{-st} P_{s}(x, A, B|x_{0}).$$ (8)

In section 3, we focus on the joint distribution $\Pi_{s}(A, B|x_{0})$ of the two local times $[X(t) = x; A(t) = A, B(t) = B]$ at time $t$ when starting at $[X(0) = x_{0}; A(0) = 0, B(0) = 0]$

$$\Pi_{s}(A, B|x_{0}) \equiv \int_{-\infty}^{+\infty} dx P_{s}(x, A, B|x_{0})$$ (9)

and we compute explicitly the time-Laplace-transform of parameter $s$

$$\hat{\Pi}_{s}(A, B|x_{0}) \equiv \int_{0}^{+\infty} dt e^{-st} \Pi_{s}(A, B|x_{0}) = \int_{-\infty}^{+\infty} dx \hat{P}_{s}(x, A, B|x_{0}).$$ (10)

In section 4, we analyze the behavior for large time $t$ of the joint distribution $\Pi_{s}(A, B|x_{0})$. When the diffusion process $X(t)$ is transient, the two local times $(A, B)$ remain finite random variables for $t \rightarrow +\infty$ and we compute the limit joint distribution $\Pi_{\infty}(A, B|x_{0})$. When the diffusion process $X(t)$ is recurrent, then it is more appropriate to introduce the two intensive local times

4
\[ a \equiv \frac{A}{t} \]
\[ b \equiv \frac{B}{t} \]

and to analyze their joint large deviations properties

\[ \Pi_t(A = ta, B = tb|x_0) \sim \mathcal{N}(a, b, x_0)e^{-d(a, b)} \]

where the positive rate function \( I(a, b) \) governs the leading exponential decay in time, while the prefactor \( \mathcal{N}(a, b, x_0) \) that contains the dependence with respect to the initial position \( x_0 \) will be also computed explicitly. In section 5, we describe the corresponding simpler statistical properties of the sum \( \Sigma(t) = A(t) + B(t) \) of the two local times.

In section 6, we construct various conditioned joint processes \([X^*(t), A^*(t), B^*(t)]\) satisfying certain constraints involving the two local times, thereby generalizing our previous work [106] concerning the conditioning with respect to the single local time \( A(t) \). The conditioned Bridge towards the given local times \([A_T^*, B_T^*]\) at the finite time horizon \( T \) can be constructed as follows: the conditioned process \([X^*(t), A^*(t), B^*(t)]\) satisfies the following Itô SDE system analog to equation (7)

\[
\begin{align*}
\frac{dX^*(t)}{dt} & = \mu_{T}^{[A_T^*, B_T^*]}(X^*(t), A^*(t), B^*(t))dt + dW(t) \\
\frac{dA^*(t)}{dt} & = \delta(X^*(t))dt \\
\frac{dB^*(t)}{dt} & = \delta(X^*(t) - L)dt
\end{align*}
\]

where the conditioned drift \( \mu_{T}^{[A_T^*, B_T^*]}(x, A, B, t) \) involves the unconditioned drift \( \mu(x) \) and the logarithmic derivative of the unconditioned distribution \( \Pi_{T-}(A_T^* - A, B_T^* - B|x) \) with respect to its initial position \( x \)

\[ \mu_{T}^{[A_T^*, B_T^*]}(x, A, B, t) = \mu(x) + \partial_x \ln \Pi_{T-}(A_T^* - A, B_T^* - B|x). \]

In the limit of the infinite time horizon \( T \to +\infty \), we will analyze the conditioning towards the finite asymptotic values \([A^*(\infty), B^*(\infty)]\), as well as the conditioning towards the intensive values \([a^*, b^*]\).

In section 7, the general framework of the previous sections written for the diffusion process with an arbitrary unconditioned drift \( \mu(x) \) is applied to the simplest example of the Brownian motion with the uniform drift \( \mu(x) = \mu \) on the whole line \( \mathbb{R} \). In order to construct various conditioned processes involving its two local times. Our conclusions are summarized in section 8. The two appendices A and B are devoted to the canonical conditioned processes \( X_{p,q}^*(t) \) of parameters \((p, q)\) conjugated to the two local times \( A \) and \( B \), in order to compare with the microcanonical conditioning described in the main text. Two other appendices C and D contain more technical computations.

### 2. Propagator \( P_t(x, A, B|x_0) \) for the position \( x \) and the two local times \( A \) and \( B \)

In this section, we analyze the joint propagator \( P_t(x, A, B|x_0) \) associated to the Itô system of equation (7) that satisfies the Fokker–Planck dynamics

\[ dX^*(t) = \mu(X^*(t), A^*(t), B^*(t))dt + dW(t) \]
\[ \partial_t P_1(x, A, B | x_0) = -\delta(x) \partial_A P_1(x, A, B | x_0) - \delta(x - L) \partial_B P_1(x, A, B | x_0) - \partial_x [\mu(x) P_1(x, A, B | x_0)] + \frac{1}{2} \partial_x^2 P_1(x, A, B | x_0). \]  

(15)

For clarity, it will be convenient to denote by another letter the propagator \( G_1(x | x_0) \) of the position \( x \) alone

\[ G_1(x | x_0) = \int_{-\infty}^{+\infty} dA \int_{-\infty}^{+\infty} dB P_1(x, A, B | x_0) \]  

that satisfies

\[ \partial_t G_1(x | x_0) = -\partial_x [\mu(x) G_1(x | x_0)] + \frac{1}{2} \partial_x^2 G_1(x | x_0). \]  

(17)

2.1. Decomposition of the joint propagator \( P_1(x, A, B | x_0) \) into four contributions

The joint propagator \( P_1(x, A, B | x_0) \) can be decomposed into the four contributions based on the two delta functions \( \delta(A) \) and \( \delta(B) \) and on the two Heaviside functions \( \theta(A > 0) \) and \( \theta(B > 0) \)

\[ P_1(x, A, B | x_0) = \delta(A) \delta(B) G_t^{abs(0,L)}(x | x_0) + \theta(A > 0) \delta(B) A_t^{abs(L)}(x, A | x_0) \]

\[ + \delta(A) \theta(B > 0) B_t^{abs(0)}(x, B | x_0) + \theta(A > 0) \theta(B > 0) C_t(x, A, B | x_0) \]  

(18)

with the following meaning.

(1) The first contribution containing the two delta functions \( \delta(A) \delta(B) \) means that the two local times have kept their initial values \( A = 0 \) and \( B = 0 \), i.e. the diffusion process has not been able to visit the positions \( x = 0 \) and \( x = L \). As a consequence, the amplitude is given by the propagator \( G_t^{abs(0,L)}(x | x_0) \) in the presence of two absorbing boundary conditions at position 0 and at position \( L \), so that it is non-vanishing only if the final position \( x \) and the initial position \( x_0 \) belong both to the left region \( ] -\infty, 0 [ \), or belong both to the middle region \( ]0, L[ \) or belong both to the right region \( ]L, +\infty[ \), and in the two external regions, only one absorbing boundary condition is sufficient

\[ G_t^{abs(0,L)}(x | x_0) = \theta(x < 0) \theta(x_0 < 0) G_t^{abs(0)}(x | x_0) \]

\[ + \theta(0 < x < L) \theta(0 < x_0 < L) G_t^{abs(0,L)}(x | x_0) \]

\[ + \theta(x > L) \theta(x_0 > L) G_t^{abs(L)}(x | x_0). \]  

(19)

(2) The second contribution containing \( \theta(A > 0) \delta(B) \) means that the diffusion process has not been able to visit \( x = L \) but has visited \( x = 0 \), so that it is non-vanishing only for \( x < L \) and \( x_0 < L \)

\[ A_t^{abs(L)}(x, A | x_0) = \theta(x < L) \theta(x_0 < L) A_t^{abs(L)}(x, A | x_0). \]  

(20)

(3) The third contribution containing \( \delta(A) \theta(B > 0) \) means that the diffusion process has not been able to visit \( x = 0 \) but has visited \( x = L \), so that it is non-vanishing only for \( x > 0 \) and \( x_0 > 0 \)
Figure 1. Examples of simulated trajectories $x(t)$ of the Brownian bridge on the interval $t \in [0, 1]$ that illustrate the four contributions of the joint propagator $P_t(x, A, B|x_0)$ in equation (18). The four trajectories $x(t)$ start from $x_0 = 0.5$ at time $t = 0$ and end at $x = 0.8$ at time $t = 1$, but have different properties with respect to the two local times $A$ and $B$: the blue trajectory that has not been able to visit the position $x = 0$ and $x = L = 1$ contributes to the first term of equation (18) involving $\delta(A)\delta(B)$; the green trajectory that has not been able to visit $x = L = 1$ but has visited $x = 0$ contributes to the second term of equation (18) involving $\theta(A > 0)\delta(B)$; the red trajectory that has not been able to visit $x = 0$ but has visited $x = L = 1$ contributes to the third term of equation (18) involving $\delta(A)\theta(B > 0)$; the brown trajectory that has visited both positions $x = 0$ and $x = L = 1$ contributes to the fourth term of equation (18) involving $\theta(A > 0)\theta(B > 0)$. The time step used in the discretization is $dt = 10^{-4}$.

$B_{t,0}^{abs}(x, B|x_0) = \theta(x > 0)\theta(x_0 > 0)B_{t,0}^{abs}(x, B|x_0)$. \hspace{1cm} (21)

(4) The fourth contribution containing $\theta(A > 0)\theta(B > 0)$ means that the diffusion process has visited both positions $x = 0$ and $x = L$.

Figure 1 shows examples of trajectories of the Brownian bridge that illustrate these four contributions of the propagator $P_t(x, A, B|x_0)$ in equation (18).

2.2. Laplace transform $\hat{P}_{t,p,q}(x|x_0)$ with respect to the two local times $A$ and $B$: Feynman–Kac formula

For the Laplace transform $\hat{P}_{t,p,q}(x|x_0)$ of the propagator $P_t(x, A, B|x_0)$ of equation (18) with respect to the two local times $A$ and $B$
\[
\hat{P}_{t,p,q}(x|x_0) \equiv \int_0^{+\infty} dA e^{-\beta A} \int_0^{+\infty} dB e^{-\beta B} P_r(x,A,B|x_0) \\
= G_t^{ab,(0,L)}(x|x_0) + \hat{A}_{x,p}^{ab,(L)}(x|x_0) + \hat{G}_{x,q}^{ab,(0)}(x|x_0) + \hat{G}_{t,p,q}(x|x_0). \tag{22}
\]

Equation (15) translates into
\[
\partial_t \hat{P}_{t,p,q}(x|x_0) = -p \delta(x) \hat{P}_{t,p,q}(x|x_0) - q \delta(x - L) \hat{P}_{t,p,q}(x|x_0) - \partial_x \left[ \mu(x) \hat{P}_{t,p,q}(x|x_0) \right] \\
+ \frac{1}{2} \partial_x^2 \hat{P}_{t,p,q}(x|x_0). \tag{23}
\]

For \( p = 0 = q \), one recovers the propagator \( G_t(x|x_0) \) of the position alone
\[
\hat{P}_{t,0,0}(x|x_0) = G_t(x|x_0). \tag{24}
\]

Equation (23) is an example of the Feynman–Kac formula, where the initial Fokker–Planck dynamics of equation (17) is now supplemented by the two additional terms involving \( p \delta(x) \) and \( q \delta(x - L) \).

### 2.3. Explicit solution for the further time-Laplace-transform \( \hat{P}_{t,p,q}(x|x_0) \) via the Dyson equation

For the further Laplace transform \( \hat{P}_{t,p,q}(x|x_0) \) of equation (22) with respect to the time \( t \)
\[
\hat{\hat{P}}_{t,p,q}(x|x_0) \equiv \int_0^{+\infty} dt e^{-\beta t} \hat{P}_{t,p,q}(x|x_0) \\
= \hat{G}_t^{ab,(0,L)}(x|x_0) + \hat{A}_{x,p}^{ab,(L)}(x|x_0) + \hat{E}_{x,q}^{ab,(0)}(x|x_0) + \hat{C}_{t,p,q}(x|x_0). \tag{25}
\]

Equation (23) translates into
\[
-\delta(x - x_0) + \hat{\hat{P}}_{t,p,q}(x|x_0) = -p \delta(x) \hat{\hat{P}}_{t,p,q}(0|x_0) - q \delta(x - L) \hat{\hat{P}}_{t,p,q}(L|x_0) - \partial_x \left[ \mu(x) \hat{\hat{P}}_{t,p,q} \right] \\
+ \frac{1}{2} \partial_x^2 \hat{\hat{P}}_{t,p,q}. \tag{26}
\]

The knowledge of the solution for \( p = q = 0 \) of equation (24)
\[
\hat{\hat{P}}_{t,0,0}(x|x_0) = \hat{G}_t(x|x_0) \tag{27}
\]
allows to rewrite the solution for arbitrary Laplace parameters \((p, q)\) as
\[
\hat{\hat{P}}_{t,p,q}(x|x_0) = \hat{G}_t(x|x_0) - p \hat{G}_t(x|0) \hat{\hat{P}}_{t,p,q}(0|x_0) - q \hat{G}_t(x|L) \hat{\hat{P}}_{t,p,q}(L|x_0) \tag{28}
\]
where \( \hat{\hat{P}}_{t,0,0}(0|x_0) \) and \( \hat{\hat{P}}_{t,0,0}(L|x_0) \) should satisfy the following system, where equation (28) is written for \( x = 0 \) and for \( x = L \) respectively
\[
\left[ 1 + p \hat{G}_t(0|0) \right] \hat{P}_{t,p,q}(0|x_0) + q \hat{G}_t(0|L) \hat{\hat{P}}_{t,p,q}(L|x_0) = \hat{G}_t(0|x_0) \\
p \hat{G}_t(L|0) \hat{\hat{P}}_{t,p,q}(0|x_0) + \left[ 1 + q \hat{G}_t(L|L) \right] \hat{\hat{P}}_{t,p,q}(L|x_0) = \hat{G}_t(L|x_0). \tag{29}
\]
The solution of this system

\[
\hat{P}_{s,p,q}(0|x_0) = \frac{1 + q\hat{G}_s(L|x)}{1 + q\hat{G}_s(L|x_0)} - \frac{q\hat{G}_s(0|x_0)\hat{G}_s(L|x_0)}{1 + q\hat{G}_s(L|x_0)}
\]

\[
\hat{P}_{s,p,q}(L|x_0) = \frac{1 + p\hat{G}_s(0|x_0)}{1 + q\hat{G}_s(L|x_0)} - \frac{p\hat{G}_s(0|x_0)\hat{G}_s(L|x_0)}{1 + q\hat{G}_s(L|x_0)}
\]

(30)

can be plugged into equation (28) to obtain the general solution

\[
\hat{P}_{s,p,q}(x|x_0) = \hat{G}_s(x|x_0) - \frac{pq\hat{G}_s(0|x_0) + p\frac{\hat{G}_s(x|x_0)}{\Delta_i} + q\frac{\hat{G}_s(L|x_0)}{\Delta_i}}{pq + p\frac{\hat{G}_s(0|x_0)}{\Delta_i} + q\frac{\hat{G}_s(L|x_0)}{\Delta_i}}
\]

(31)

where we have introduced the two following notations

\[
\Delta_i = \hat{G}_s(0|x_0)\hat{G}_s(L|x) - \hat{G}_s(L|x_0)\hat{G}_s(0|x)
\]

\[
\Omega_i(x|x_0) = \frac{\hat{G}_s(L|x_0)\hat{G}_s(x|x_0)\hat{G}_s(0|x_0)\hat{G}_s(L|x_0) + \hat{G}_s(0|x_0)\hat{G}_s(x|x_0)\hat{G}_s(L|x_0) - \hat{G}_s(L|x_0)\hat{G}_s(0|x_0)}{\Delta_i}
\]

(32)

in order to see more clearly the rational fraction structure of equation (31) with respect to the two Laplace parameters \((p, q)\).

2.4. Explicit Laplace inversion of \(\hat{P}_{s,p,q}(x|x_0)\) with respect to \(p\) and \(q\) to obtain \(\hat{P}_s(x,A,B|x_0)\)

The goal is now to obtain the time-Laplace-transform of parameter \(s\) of the propagator of equation (18) with its four contributions

\[
\hat{P}_s(x,A,B|x_0) \equiv \int_0^{+\infty} dt e^{-st} P_s(x,A,B|x_0)
\]

\[
= \delta(A)\delta(B)\hat{G}_s^{\mathrm{abs}(0,L)}(x|x_0) + \theta(A > 0)\delta(B)\hat{A}_s^{\mathrm{abs}(L)}(x,A|x_0)
\]

\[
+ \delta(A)\theta(B > 0)\hat{B}_s^{\mathrm{abs}(0)}(x,B|x_0) + \theta(A > 0)\theta(B > 0)\hat{C}_s(x,A,B|x_0)
\]

(33)

from the Laplace inversion with respect to the two Laplace parameters \((p, q)\) of

\[
\hat{P}_{s,p,q}(x|x_0) = \int_0^{+\infty} dA e^{-pA} \int_0^{+\infty} dB e^{-qB} \hat{P}_s(x,A,B|x_0)
\]

\[
= \hat{G}_s^{\mathrm{abs}(0,L)}(x|x_0) + \int_0^{+\infty} dA e^{-pA} \hat{A}_s^{\mathrm{abs}(L)}(x,A|x_0) + \int_0^{+\infty} dB e^{-qB} \hat{B}_s^{\mathrm{abs}(0)}(x,B|x_0)
\]

\[
+ \int_0^{+\infty} dA e^{-pA} \int_0^{+\infty} dB e^{-qB} \hat{C}_s(x,A,B|x_0)
\]

\[
= \hat{G}_s^{\mathrm{abs}(0,L)}(x|x_0) + \hat{A}_s^{\mathrm{abs}(L)}(x|x_0) + \hat{B}_s^{\mathrm{abs}(0)}(x|x_0) + \hat{C}_{s,p,q}(x|x_0)
\]

(34)

whose explicit expression was computed in equation (31).
2.4.1. First contribution of amplitude $\hat{G}_s^{\text{abs}(0,L)}(x|x_0)$. Equation (34) yields that the first contribution $\hat{G}_s^{\text{abs}(0,L)}(x|x_0)$ can be found by considering the double limit $(p \to +\infty, q \to +\infty)$ of $\hat{P}_{s,p,q}(x|x_0)$ given by equation (31)

$$\hat{G}_s^{\text{abs}(0,L)}(x|x_0) = \hat{P}_{s,p=+\infty,q=+\infty}(x|x_0) = \hat{G}_s(x|x_0) - \Omega_s(x|x_0).$$

(35)

2.4.2. Second contribution of amplitude $\hat{A}_s^{\text{abs}(L)}(x,A|x_0)$. Equation (34) yields that the second contribution $\hat{A}_s^{\text{abs}(L)}(x,x_0)$ can be found by considering the limit $q \to +\infty$ of $\hat{P}_{s,p,q}(x|x_0)$ given by equation (31)

$$\hat{G}_s^{\text{abs}(0,L)}(x|x_0) + \hat{A}_s^{\text{abs}(L)}(x|x_0) = \hat{P}_{s,p,q=+\infty}(x|x_0) = \frac{p\Omega_s(x|x_0) + \hat{G}_s(x|x_0) - \Omega_s(x|x_0)}{p + \frac{\hat{G}_s(x|x_0)}{\Delta_s}}.$$

(36)

The difference with the first contribution of equation (35) yields

$$\hat{A}_s^{\text{abs}(L)}(x|x_0) = \Omega_s(x|x_0) - \frac{p\Omega_s(x|x_0) + \hat{G}_s(x|x_0) - \Omega_s(x|x_0)}{p + \frac{\hat{G}_s(x|x_0)}{\Delta_s}} = \frac{\alpha_s^{[00]}(x|x_0)}{p + \frac{1}{\Delta_s}}$$

(37)

where we have introduced the Laplace transform $\hat{G}_s^{\text{abs}(L)}(x|x_0)$ of the propagator $G_t^{\text{abs}(L)}(x|x_0)$ in the presence of a single absorbing boundary at $x = L$

$$\hat{G}_s^{\text{abs}(L)}(x|x_0) \equiv \int_0^{+\infty} dt e^{-st} G_t^{\text{abs}(L)}(x|x_0) = \hat{G}_s(x|x_0) - \frac{\hat{G}_s(x|L) \hat{G}_s(L|x_0)}{\hat{G}_s(L|L)}$$

(38)

in order to simplify the denominator of equation (37) in terms of

$$\hat{G}_s^{\text{abs}(L)}(0|0) = \hat{G}_s(0|0) - \frac{\hat{G}_s(0|L) \hat{G}_s(L|0)}{\hat{G}_s(L|L)} = \frac{\Delta_s}{\hat{G}_s(L|L)}$$

(39)

and in order to simplify the numerator of equation (37)

$$\alpha_s^{[00]}(x|x_0) \equiv \Omega_s(x|x_0) \hat{G}_s(L|L) - \hat{G}_s(x|L) \hat{G}_s(L|x_0) = \left(\frac{\hat{G}_s^{\text{abs}(L)}(x|0)}{\hat{G}_s^{\text{abs}(L)}(0|0)}\right) \left(\frac{\hat{G}_s^{\text{abs}(L)}(0|x_0)}{\hat{G}_s^{\text{abs}(L)}(0|0)}\right)$$

(40)

and to make obvious that it is non-vanishing only for $x < L$ and $x_0 < L$ as equation (20)

$$\alpha_s^{[00]}(x|x_0) = \theta(x < L)\theta(x_0 < L)\alpha_s^{[00]}(x|x_0).$$

(41)

The Laplace inversion of $\hat{A}_s^{\text{abs}(L)}(x|x_0)$ in equation (37) with respect to $p$ gives the following exponential form with respect to the local time $A$
The fourth contribution given by equation (2.4.4. Fourth contribution of amplitude) similarly, equation (34) yields that the third contribution \( \tilde{B}_{s,q}^{(0)}(x|x_0) \) can be found by considering the limit \( p \to +\infty \) \( \tilde{P}_{x,p,q}(x|x_0) \) given by equation (31)

\[
\tilde{G}_s^{(0,L)}(x|x_0) + \tilde{B}_{s,q}^{(0)}(x|x_0) = \tilde{P}_{x,p=+\infty,q}(x|x_0) = \tilde{G}_s(x|x_0) - \frac{q\Omega(x|x_0) + \tilde{G}_s(x|0)\tilde{G}_s(0|0)}{q + \tilde{G}_s(0|0)\Delta}.
\]

(43)

The difference with the first contribution of equation (35) yields

\[
\tilde{B}_{s,q}^{(0)}(x|x_0) = \Omega_s(x|x_0) - \frac{q\Omega_s(x|x_0) + \tilde{G}_s(x|0)\tilde{G}_s(0|0)}{q + \tilde{G}_s(0|0)\Delta} = \beta_s^{(0)}(x|x_0)
\]

where we have introduced the Laplace transform of the propagator in the presence of a single absorbing boundary at \( x = 0 \)

\[
\tilde{G}_s^{(0)}(y|0) = \int_0^{+\infty} dt e^{-\Delta t} \tilde{G}_s^{(0,L)}(x|x_0) = \tilde{G}_s(y|0) - \frac{\tilde{G}_s(y|0)\tilde{G}_s(0|0)}{\tilde{G}_s(0|0)}
\]

(45)

in order to simplify the denominator of equation (44) in terms of

\[
\tilde{G}_s^{(0)}(L|L) = \tilde{G}_s(L|L) - \frac{\tilde{G}_s(L|0)\tilde{G}_s(0|0)}{\tilde{G}_s(0|0)} = \Delta_s
\]

(46)

and in order to simplify the numerator of equation (44)

\[
\beta_s^{(0)}(x|x_0) = \frac{\Omega_s(x|x_0)\tilde{G}_s(0|0) - \tilde{G}_s(x|0)\tilde{G}_s(0|x_0)}{\Delta_s} = \left( \frac{\tilde{G}_s^{(0)}(x|L)}{\tilde{G}_s^{(0)}(L|L)} \right) \left( \frac{\tilde{G}_s^{(0)}(L|0)}{\tilde{G}_s^{(0)}(L|L)} \right)
\]

(47)

and to make obvious that it is non-vanishing only for \( x > 0 \) and \( x_0 > 0 \) as in equation (21)

\[
\beta_s^{(0)}(x|x_0) = \theta(x > 0)\theta(x_0 > 0)\beta_s^{(0)}(x|x_0).
\]

(48)

The Laplace inversion of \( \tilde{B}_{s,q}^{(0)}(x|x_0) \) in equation (44) with respect to \( q \) gives the following exponential form with respect to the local time \( B \)

\[
\tilde{B}_{s}^{(0)}(x,B|x_0) = \beta_s^{(0)}(x|x_0) e^{-\frac{\Delta_s}{\tilde{G}_s^{(0)}(L|L)}}.
\]

(49)

2.4.4. Fourth contribution of amplitude \( \tilde{C}_s(x,A,B|x_0) \). The fourth contribution \( \tilde{C}_{s,p,q}(x|x_0) \) of equation (34) can be obtained from the difference between the full solution of equation (31) and the three previous contributions of equations (35), (37) and (44)
\[
\hat{C}_{s,p,q}(x|x_0) = \hat{P}_{s,p,q}(x|x_0) - \hat{G}_s^{\text{abs}(0,L)}(x|x_0) - \hat{A}_{s,p}(x|x_0) - \hat{B}_{s,q}(x|x_0)
\]

\[
= \Omega_s(x|x_0) - \frac{pq\Omega_s(x|x_0)}{p + \frac{1}{G_0^{\text{abs}(0,L)}}} \left( q + \frac{1}{G_0^{\text{abs}(0,L)}} \right) - \frac{\alpha_s^{[0]}(x|x_0)}{p + \frac{1}{G_0^{\text{abs}(0,L)}}} - \frac{\beta_p^{[00]}(x|x_0)}{q + \frac{1}{G_0^{\text{abs}(0,L)}}}.
\]

(50)

The Laplace inversion with respect to \(p\) and \(q\) is described in appendix C. The final result

\[
\hat{C}_s(x,A,B|x_0) = \hat{C}_s^{[0]}(x,A,B|x_0) + \hat{C}_s^{[\beta]}(x,A,B|x_0) + \hat{C}_s^{[\gamma]}(x,A,B|x_0)
\]

involves the three terms

\[
\hat{C}_s^{[0]}(x,A,B|x_0) = \alpha_s^{[00]}(x|x_0) e^{-\frac{c_0 - \Delta_s}{2\sqrt{\lambda_0}}} I_0 \left( 2c_0 \sqrt{AB} \right)
\]

\[
\hat{C}_s^{[\beta]}(x,A,B|x_0) = \beta_s^{[\beta]}(x|x_0) e^{-\frac{c_0 - \Delta_s}{2\sqrt{\lambda_0}}} I_0 \left( 2c_0 \sqrt{AB} \right)
\]

\[
\hat{C}_s^{[\gamma]}(x,A,B|x_0) = \gamma_s(x|x_0) e^{-\frac{c_0 - \Delta_s}{2\sqrt{\lambda_0}}} I_0 \left( 2c_0 \sqrt{AB} \right)
\]

(52)

where \(I_0(z)\) is the modified Bessel function of order zero

\[
I_0(z) = \sum_{k=0}^{+\infty} \frac{(\frac{z}{2})^{2k}}{k!^2}
\]

(53)

with its derivative

\[
I_0'(z) = I_1(z) = \sum_{k=1}^{+\infty} \frac{(\frac{z}{2})^{2k-1}}{k!(k-1)!}
\]

(54)

that both display the same asymptotic leading behavior for large \(z\)

\[
I_0(z) \sim e^\frac{z}{\sqrt{2\pi}} \quad \text{as} \quad z \to +\infty
\]

(55)

\[
I_1(z) \sim \frac{e^\frac{z}{\sqrt{2\pi}}}{\sqrt{2\pi}} \quad \text{as} \quad z \to +\infty
\]

while the amplitude \(\gamma_s(x|x_0)\) of the third term in equation (52)

\[
\gamma_s(x|x_0) = \gamma_s^{[0]}(x|x_0) + \gamma_s^{[\beta]}(x|x_0)
\]

(56)

can be decomposed into the two terms

\[
\gamma_s^{[0]}(x|x_0) \equiv \left( \frac{\hat{G}_s^{\text{abs}(0)}(x|L)}{\hat{G}_s^{\text{abs}(0,L)}(0,0)} \right) \frac{\hat{G}_s(L|0)}{\Delta_s} \left( \frac{\hat{G}_s^{\text{abs}(L)}(0|0)}{\hat{G}_s^{\text{abs}(L)}(0,L)} \right)
\]

\[
\gamma_s^{[\beta]}(x|x_0) \equiv \left( \frac{\hat{G}_s^{\text{abs}(L)}(x|0)}{\hat{G}_s^{\text{abs}(L)}(0,0)} \right) \frac{\hat{G}_s(L|0)}{\Delta_s} \left( \frac{\hat{G}_s^{\text{abs}(L)}(L|x)}{\hat{G}_s^{\text{abs}(L)}(0,L)} \right)
\]

(57)
in order to make obvious that the first term $\gamma^{|00\rangle}_{x}(x|x_{0})$ is non-vanishing only for $x > 0$ and $x_{0} < L$ and that the second term $\gamma^{|0L\rangle}_{x}(x|x_{0})$ is non-vanishing only for $x < L$ and $x_{0} > 0$

$$\gamma^{|00\rangle}_{x}(x|x_{0}) = \theta(x > 0)\theta(x_{0} < L)\gamma^{|00\rangle}_{x}(x|x_{0})$$

$$\gamma^{|0L\rangle}_{x}(x|x_{0}) = \theta(x < L)\theta(x_{0} > 0)\gamma^{|0L\rangle}_{x}(x|x_{0}). \tag{58}$$

In order to understand the physical meaning of the various terms, it is now useful to consider the following four special cases when $x$ and $x_{0}$ are either at 0 or at $L$.

### 2.4.4.1. Special cases when the final position $x$ and the initial position $x_{0}$ are either at 0 or at $L$

#### [00]
For $x = 0 = x_{0}$, equations (48) and (58) yield that only the contribution $|\alpha\rangle$ survives in equation (52) while the amplitude of equation (40) reduces to $\alpha_{x}^{(00)}(0|0) = 1$

$$C_{s}(x = 0, A, B|x_{0} = 0) = C_{s}^{(\alpha)}(x = 0, A, B|x_{0} = 0)$$

$$= c_{s}e^{-\frac{A}{2\gamma^{(00)}(0|0)}} e^{-\frac{B}{2\gamma^{(00)}(0|0)}} \frac{\sqrt{A}}{\sqrt{B}} I_{0} \left(2c_{s}\sqrt{AB}\right). \tag{59}$$

#### [LL]
For $x = L = x_{0}$, equations (41) and (58) yield that only the contribution $|\beta\rangle$ survives in equation (52) while the amplitude of equation (47) reduces to $\beta_{x}^{(LL)}(L|L) = 1$

$$C_{s}(x = L, A, B|x_{0} = L) = C_{s}^{(\beta)}(x = L, A, B|x_{0} = L)$$

$$= \beta_{x}^{(LL)}(L|L)c_{s}e^{-\frac{A}{2\gamma^{(00)}(0|0)}} e^{-\frac{B}{2\gamma^{(00)}(0|0)}} \frac{\sqrt{B}}{\sqrt{A}} I_{1} \left(2c_{s}\sqrt{AB}\right). \tag{60}$$

#### [0L]
For $x = 0$ and $x_{0} = L$, equations (41) and (48) yield that only the contribution $|\gamma\rangle$ survives in equation (52) while the amplitude of equation (57) reduces to

$$\gamma_{x}(0|L) = \gamma^{|0L\rangle}_{x}(0|L) = \frac{\tilde{G}_{x}(0|L)}{\Delta_{x}} \tag{61}$$

so that one obtains

$$C_{s}(x = 0, A, B|x_{0} = L) = C_{s}^{(\gamma)}(x = 0, A, B|x_{0} = L)$$

$$= \frac{\tilde{G}_{x}(0|L)}{\Delta_{x}} e^{-\frac{A}{2\gamma^{(00)}(0|0)}} e^{-\frac{B}{2\gamma^{(00)}(0|0)}} I_{0} \left(2c_{s}\sqrt{AB}\right). \tag{62}$$

#### [L0]
For $x = L$ and $x_{0} = 0$, equations (41) and (48) yield that only the contribution $|\gamma\rangle$ survives in equation (52) while the amplitude of equation (57) reduces to

$$\gamma_{x}(L|0) = \gamma^{|0L\rangle}_{x}(L|0) = \frac{\tilde{G}_{x}(L|0)}{\Delta_{x}} \tag{63}$$
so that one obtains

\[ C_s(x = L, A, B | x_0 = 0) = C_s^{(f)}(x = L, A, B | x_0 = 0) = \frac{\widetilde{G}_s(L(0))}{\Delta_s} \exp \left( \frac{A}{\Delta_s} \right) \frac{\Delta_s}{\Delta_s} I_0 \left( 2c_s \sqrt{AB} \right). \]  

\[ C_s^{(00)}(x, A, B | x_0) = \frac{\tilde{C}_s^{(00)}(x, A, B | x_0)}{\tilde{G}_s^{abs}(0)} \tilde{C}_s(x = 0, A, B | x_0 = 0) \frac{\tilde{G}_s^{abs}(0)}{\tilde{G}_s^{abs}(0)} I_0 \left( 2c_s \sqrt{AB} \right). \]  

2.4.4.2. Decomposition with respect to the first-passage at 0 or L and with respect to the last-passage at 0 or L  

The contribution \( \tilde{C}_s(x, A, B | x_0) \) of equation (52) involve the stochastic trajectories that visit the two positions 0 and \( L \), so

\[ (i) \text{ either } 0 \text{ or } L \text{ is visited first when starting at } x_0 \]

\[ (ii) \text{ either } 0 \text{ or } L \text{ is visited last when ending at } x. \]

It is thus interesting to decompose equation (52) according to the these last-passage and first-passage properties

\[ \tilde{C}_s(x, A, B | x_0) = \tilde{C}_s^{(0)}(x, A, B | x_0) + \tilde{C}_s^{(LL)}(x, A, B | x_0) + \tilde{C}_s^{(0L)}(x, A, B | x_0) + \tilde{C}_s^{(L0)}(x, A, B | x_0) \]  

where the four possibilities are directly related to the four special cases described above:

[00] If the last-passage is at \( x_{last} = 0 \) and the first-passage is at \( x_{first} = 0 \), the corresponding contribution \( C_s^{(00)}(x, A, B | x_0) \) can be rewritten as a product of three terms using the special case of equation (59)

\[ C_s^{(00)}(x, A, B | x_0) = \left( \frac{\tilde{C}_s^{abs}(0)}{\tilde{G}_s^{abs}(0)} \right) \tilde{C}_s(x = 0, A, B | x_0 = 0) \frac{\tilde{G}_s^{abs}(0)}{\tilde{G}_s^{abs}(0)} I_0 \left( 2c_s \sqrt{AB} \right). \]  

[LL] If the last-passage is at \( x_{last} = L \) and the first-passage is at \( x_{first} = L \), the corresponding contribution \( C_s^{(LL)}(x, A, B | x_0) \) can be rewritten as a product of three terms using the special case of equation (60)

\[ C_s^{(LL)}(x, A, B | x_0) = \left( \frac{\tilde{C}_s^{abs}(L)}{\tilde{G}_s^{abs}(L)} \right) \tilde{C}_s(x = L, A, B | x_0 = L) \frac{\tilde{G}_s^{abs}(L)}{\tilde{G}_s^{abs}(L)} I_0 \left( 2c_s \sqrt{AB} \right). \]  

[0L] If the last-passage is at \( x_{last} = 0 \) and the first-passage is at \( x_{first} = L \), the corresponding contribution \( C_s^{(0L)}(x, A, B | x_0) \) can be rewritten as a product of three terms using the special case of equation (62)

\[ C_s^{(0L)}(x, A, B | x_0) = \left( \frac{\tilde{C}_s^{abs}(0)}{\tilde{G}_s^{abs}(0)} \right) \tilde{C}_s(x = 0, A, B | x_0 = L) \frac{\tilde{G}_s^{abs}(0)}{\tilde{G}_s^{abs}(0)} I_0 \left( 2c_s \sqrt{AB} \right). \]  

[L0] If the last-passage is at \( x_{last} = L \) and the first-passage is at \( x_{first} = 0 \), the corresponding contribution \( C_s^{(L0)}(x, A, B | x_0) \) can be rewritten as a product of three terms using the special case of equation (64).
Figure 2. Examples of simulated trajectories $x(t)$ of the Brownian bridge on the interval $t \in [0, 1]$ that illustrate the four contributions of $\hat{C}_s(x, A, B| x_0)$ in equation (65). The four trajectories $x(t)$ start from $x_0 = 0.5$ at time $t = 0$ and end at $x = 0.8$ at time $t = 1$ with two non-vanishing local times $A > 0$ and $B > 0$, but have different last-passage and first-passage properties: the blue trajectory corresponding to $x_{\text{last}} = 0$ and $x_{\text{first}} = L$ contributes to the first term $[00]$ of equation (65) given by equation (66); the red trajectory corresponding to $x_{\text{last}} = L$ and $x_{\text{first}} = 0$ contributes to the second term $[LL]$ of equation (65) given by equation (67); the green trajectory corresponding to $x_{\text{last}} = 0$ and $x_{\text{first}} = L$ contributes to the third term $[0L]$ of equation (65) given by equation (68); the brown trajectory corresponding to $x_{\text{last}} = L$ and $x_{\text{first}} = 0$ contributes to the fourth term $[L0]$ of equation (65) given by equation (69). The time step used in the discretization is $dt = 10^{-4}$.

$$C_s^{[LL]}(x, A, B| x_0) = \left( \frac{\partial \hat{C}_{ab}(L)}{\partial x} \right) \left( \frac{\partial \hat{G}_{ab}(0)}{\partial x} \right) \hat{C}_{s}(x = L, A, B| x_0 = 0) \left( \frac{\partial \hat{G}_{ab}(0)}{\partial x} \right) \left( \frac{\partial \hat{C}_{ab}(L)}{\partial x} \right) \hat{G}_{ab}^{(0)}(0| x_0) \hat{G}_{ab}^{(0)}(0| 0). \tag{69}$$

Figure 2 shows examples of trajectories of the Brownian bridge that illustrate these four contributions of $\hat{C}_s(x, A, B| x_0)$ in equation (65).

3. Distribution $\Pi_t(A, B| x_0) = \int dx P_t(x, A, B| x_0)$ of the two local times $A$ and $B$ at time $t$

In this section, we focus on the joint distribution $\Pi_t(A, B| x_0)$ of the local times $A$ and $B$ at time $t$ if one starts at position $x_0$.

3.1. Decomposition of the distribution $\Pi_t(A, B| x_0) = \int dx P_t(x, A, B| x_0)$ into four contributions

The integration over the final position $x$ of the joint propagator $P_t(x, A, B| x_0)$ of equation (18)
The first contribution corresponding to the integration over $x$ of $G_t^{ab(0,L)}(x|x_0)$ of equation (19) represents the survival probability at time $t$ when starting at $x_0$ in the presence of absorbing boundaries at positions $x = 0$ and $x = L$.

\[ S_t^{ab(0,L)}(x_0) = \int_{-\infty}^{+\infty} dx G_t^{ab(0,L)}(x|x_0) \]
\[ = \theta(x_0 < 0) S_t^{1-\infty,0}(x_0) + \theta(0 < x_0 < L) S_t^{0,L}(x_0) + \theta(x_0 > L) S_t^{L,\infty}(x_0) \]

so that it can be decomposed into the survival probabilities in the three regions $]-\infty,0[\,$, $]0,L[\,$ and $]L,\infty[\,$.

(2) The second contribution corresponding to the integration over $x$ of $A_t^{ab(L)}(x,A|x_0)$ of equation (20) is non-vanishing only for $x_0 < L$.

\[ A_t^{1-\infty,L}(A|x_0) \equiv \int_{-\infty}^{+\infty} dx A_t^{ab(L)}(x,A|x_0) = \theta(x_0 < L) A_t^{1-\infty,L}(A|x_0). \]

The time-Laplace transform $\hat{\Pi}_s(A,B|x_0)$ with its four contributions

The time-Laplace transform of equation (70)

\[ \hat{\Pi}_s(A,B|x_0) \equiv \int_{0}^{+\infty} dt e^{-st} \Pi_t(A,B|x_0) \]
\[ = \delta(A) \delta(B) S_t^{ab(0,L)}(x_0) + \theta(A > 0) \delta(B) A_t^{1-\infty,L}(A|x_0) \]
\[ + \delta(A) \theta(B > 0) B_t^{0,+\infty}(B|x_0) + \theta(A > 0) \theta(B > 0) C_t(A,B|x_0) \]
can be computed via the integration over $x$ of $\tilde{P}_s(x,A,B|x_0)$ of equation (33)

$$\hat{P}_s(A,B|x_0) = \int_{-\infty}^{+\infty} dx \tilde{P}_s(x,A,B|x_0)$$

(76)

so that one needs to integrate over $x$ the various contributions computed in the previous section.

The normalization of the propagator $G_i(x|x_0)$

$$\int_{-\infty}^{+\infty} dx G_i(x|x_0) = 1$$

(77)

translates for its Laplace transform into

$$\int_{-\infty}^{+\infty} dx \hat{G}_i(x|x_0) = \int_{0}^{+\infty} dt e^{-st} \int_{-\infty}^{+\infty} dx G_i(x|x_0) = \int_{0}^{+\infty} dt e^{-st} = \frac{1}{s}$$

(78)

3.3. First contribution $S_{x}^{\text{abs}(0,L)}(x_0)$

Using equation (78), the integration over $x$ of equation (35) corresponding to the Laplace transform $S_{x}^{\text{abs}(0,L)}(x_0)$ of the survival probability $S_{x}^{\text{abs}(0,L)}(x_0)$ of equation (71) reads

$$S_{x}^{\text{abs}(0,L)}(x_0) \equiv \int_{-\infty}^{+\infty} dx \hat{G}_{x}^{\text{abs}(0,L)}(x|x_0)$$

$$= \frac{1}{s} \left[ 1 - \hat{G}_{x}^{\text{abs}(0,L)}(0|x_0) - \hat{G}_{x}^{\text{abs}(0,L)}(L|x_0) + \hat{G}_{x}^{\text{abs}(0,L)}(L|x_0) - \hat{G}_{x}^{\text{abs}(0,L)}(0|x_0) \right]$$

$$= \frac{1}{s} \left[ 1 - \frac{\hat{G}_{x}^{\text{abs}(0,L)}(0|x_0)}{\hat{G}_{x}^{\text{abs}(0,L)}(L|x_0)} \right]$$

(79)

with its simplifications of equation (71) in the two extreme regions $]-\infty,0[$ and $]L,\infty[$

$$\hat{S}_{x}^{L-\infty,0}(x_0) = \frac{1}{s} \left[ 1 - \frac{\hat{G}_{x}^{\text{abs}(0,L)}(0|x_0)}{\hat{G}_{x}^{\text{abs}(0,L)}(0|0)} \right]$$

$$\hat{S}_{x}^{L+\infty,L}(x_0) = \frac{1}{s} \left[ 1 - \frac{\hat{G}_{x}^{\text{abs}(0,L)}(L|x_0)}{\hat{G}_{x}^{\text{abs}(0,L)}(L|L)} \right]$$

(80)

3.4. Second contribution $A_{x}^{L-\infty,0}(A|x_0)$

Using equation (78), the integration over $x$ of $\hat{\alpha}_{x}^{[0]}(x|x_0)$ of equation (40)

$$\alpha_{x}^{L-\infty,0}(x_0) \equiv \int_{-\infty}^{+\infty} dx \alpha_{x}^{[0]}(x|x_0) = \int_{-\infty}^{+\infty} dx \left[ \hat{G}_{x}^{\text{abs}(0)}(x|0) - \hat{G}_{x}^{\text{abs}(0)}(L|0) \right] \left[ \hat{G}_{x}^{\text{abs}(0)}(0|x_0) \right]$$

$$= \frac{1}{s} \left[ 1 - \frac{\hat{G}_{x}^{\text{abs}(0,L)}(0|x_0)}{\hat{G}_{x}^{\text{abs}(0,L)}(0|0)} \right]$$

(81)
that is non-vanishing only for \( x_0 \in [-\infty, 0] \) allows to obtain the integration over \( x \) of the contribution \( A_s^{\text{abs}}(x, A|x_0) \) of equation (42)

\[
A_s^{\text{abs}}(x, A|x_0) = \int_{-\infty}^{+\infty} dx A_s^{\text{abs}}(x, A|x_0) = e^{-A_s^{\text{abs}}(x_0)}.
\]

### 3.5. Third contribution \( B_s^{[0, +\infty]}(B|x_0) \)

Using equation (78), the integration over \( x \) of \( \beta_s^{[LL]}(x|x_0) \) of equation (47)

\[
\beta_s^{[0, +\infty]}(x_0) \equiv \int_{-\infty}^{+\infty} dx \beta_s^{[LL]}(x|x_0) = \int_{-\infty}^{+\infty} dx \left[ \frac{G_s(x|L) - \tilde{G}_i(x|0)\tilde{G}_j(x|0)\Gamma}{g_s^{(0,0)|L}(x|x_0)} \right] \frac{G_s^{(0,0)}(L|x_0)}{G_s^{(0,0)}(L|L)}
\]

that is non-vanishing only for \( x_0 \in [0, +\infty] \) allows to obtain the integration over \( x \) of the contribution \( B_s^{(0, +\infty)}(x, B|x_0) \) of equation (49)

\[
B_s^{[0, +\infty]}(B|x_0) \equiv \int_{-\infty}^{+\infty} dx B_s^{(0, +\infty)}(x, B|x_0) = e^{-B_s^{(0, +\infty)}(x_0)}.
\]

### 3.6. Fourth contribution \( \gamma_s(A, B|x_0) \)

The integration over \( x \) of \( \gamma_s(x|x_0) \) of equation (57)

\[
\gamma_s(x_0) \equiv \int_{-\infty}^{+\infty} dx \gamma_s(x|x_0) = \gamma_s^{[-\infty, L]}(x_0) + \gamma_s^{[0, +\infty]}(x_0)
\]

involves the two following terms using equation (78)

\[
\gamma_s^{[-\infty, L]}(x_0) \equiv \int_{-\infty}^{+\infty} dx \gamma_s^{[0, L]}(x|x_0) = \int_{-\infty}^{+\infty} dx \left( \frac{G_s(x|L) - \tilde{G}_i(x|0)\tilde{G}_j(x|0)\Gamma}{G_s^{(0,0)}(L|L)} \right) \frac{G_s(x|0)}{\Delta_s} \frac{G_s^{(0,0)}(L|x_0)}{G_s^{(0,0)}(L|0)}
\]

\[
= \left( \frac{G_s(x|0)}{G_s^{(0,0)}(L|L)} \right) G_s(x|0) \left( \frac{G_s^{(0,0)}(L|0)}{G_s^{(0,0)}(0|0)} \right)
\]

\[
\gamma_s^{[0, +\infty]}(x_0) \equiv \int_{-\infty}^{+\infty} dx \gamma_s^{[0, 1]}(x|x_0) = \int_{-\infty}^{+\infty} dx \left( \frac{G_s(x|0) - \tilde{G}_i(x|0)\tilde{G}_j(x|0)\Gamma}{G_s^{(0,0)}(L|0)} \right) \frac{G_s(x|0)}{\Delta_s} \frac{G_s^{(0,0)}(L|x_0)}{G_s^{(0,0)}(L|L)}
\]

\[
= \left( \frac{G_s(x|0)}{G_s^{(0,0)}(L|0)} \right) G_s(x|0) \left( \frac{G_s^{(0,0)}(L|x_0)}{G_s^{(0,0)}(L|L)} \right).
\]
The integration over $x$ of the three terms of equation (52)
\[ C^\alpha_t(A,B|x_0) = \int_{-\infty}^{+\infty} dx C^\alpha_t(x,A,B|x_0) = \alpha^1_{-\infty,\infty}x_0 \left( \hat{A}^{\alpha} e^{-\frac{n}{\alpha}} \right) e^{-\frac{\hat{B}^{\alpha}}{\alpha}} I_0(2c_t \sqrt{AB}) \]
\[ C^\beta_t(A,B|x_0) = \int_{-\infty}^{+\infty} dx C^\beta_t(x,A,B|x_0) = \beta^0_{0,\infty}x_0 \left( \hat{A}^{\beta} e^{-\frac{n}{\beta}} \right) e^{-\frac{\hat{B}^{\beta}}{\beta}} I_0(2c_t \sqrt{AB}) \]
\[ C^\gamma_t(A,B|x_0) = \int_{-\infty}^{+\infty} dx C^\gamma_t(x,A,B|x_0) = \gamma_s(x_0) e^{-\frac{\hat{A}^{\gamma}}{\gamma}} e^{-\frac{\hat{B}^{\gamma}}{\gamma}} I_0(2c_t \sqrt{AB}) \]

yields that the total contribution $C_t(A,B|x_0)$ of equation (51) reads
\[ C_t(A,B|x_0) = C^\alpha_t(A,B|x_0) + C^\beta_t(A,B|x_0) + C^\gamma_t(A,B|x_0) \]
\[ = e^{-\frac{\hat{A}^{\alpha}}{\alpha}} e^{-\frac{\hat{B}^{\alpha}}{\beta}} \left[ \left( \alpha^1_{-\infty,\infty}x_0 \frac{\sqrt{A}}{B} + \beta^0_{0,\infty}x_0 \frac{\sqrt{B}}{A} \right) c_t I_0(2c_t \sqrt{AB}) \right] + \gamma_s(x_0) I_0(2c_t \sqrt{AB}) \]
\[ (87) \]

In particular, the asymptotic behaviors of equation (55) for the modified Bessel functions $I_0(z)$ and $J_1(z)$ for large argument $z \to +\infty$ yield the following asymptotic behaviors for large $(AB)$ of the three terms
\[ C^\alpha_t(A,B|x_0) \sim \frac{\alpha^1_{-\infty,\infty}x_0}{2\sqrt{\pi} B^{\frac{3}{2}}} e^{\frac{A}{\alpha}} e^{-\frac{B}{\beta} + 2c_t \sqrt{AB}} \]
\[ C^\beta_t(A,B|x_0) \sim \frac{\beta^0_{0,\infty}x_0}{2\sqrt{\pi} A^{\frac{3}{2}}} e^{\frac{B}{\beta}} e^{-\frac{A}{\alpha} + 2c_t \sqrt{AB}} \]
\[ C^\gamma_t(A,B|x_0) \sim \frac{\gamma_s(x_0)}{2\sqrt{\pi} c_t A^{\frac{3}{2}} B^{\frac{3}{2}}} e^{\frac{A}{\alpha}} e^{-\frac{B}{\beta} + 2c_t \sqrt{AB}} \]
\[ (89) \]
so that the global asymptotic behavior for large $(AB)$ is given by
\[ C_t(A,B|x_0) \sim \frac{1}{2\sqrt{\pi}} \left[ \sqrt{C_t} \left( \frac{\alpha^1_{-\infty,\infty}x_0 A^{\frac{3}{2}}}{B^{\frac{3}{2}}} + \frac{\beta^0_{0,\infty}x_0 B^{\frac{3}{2}}}{A^{\frac{3}{2}}} \right) + \gamma_s(x_0) \right] \times e^{\frac{A}{\alpha}} e^{-\frac{B}{\beta} + 2c_t \sqrt{AB}} \]
\[ (90) \]

4. Joint distribution $\Pi_t(A,B|x_0)$ of the local times $A(t)$ and $B(t)$ for large time $t$

In this section, we analyze the joint distribution $\Pi_t(A,B|x_0)$ of the two local times $A(t)$ and $B(t)$ in the limit of large time $t \to +\infty$.

4.1. When $X(t)$ is transient: the two local times $(A, B)$ remain finite random variables for $t \to +\infty$

When the diffusion process $X(t)$ is transient, then the Laplace transform $\tilde{G}_t(x|x_0)$ of the propagator $G_t(x|x_0)$ remains finite for $s = 0$
\[ \hat{G}_{s=0}(t|x_0) < +\infty. \]  

(91)

Then the two local times \( A \) and \( B \) will remain finite random variables for \( t \to +\infty \) with the following notation for their limit distribution obtained from equation (70)

\[
\Pi_\infty(A, B|x_0) = \lim_{t \to +\infty} \Pi_t(A, B|x_0) \\
= \delta(A)\delta(B)S^{hB(0, L)}_\infty(x_0) + \theta(A > 0)\delta(B)A^{L}_\infty(A|x_0) \\
+ \delta(A)\theta(B > 0)B^{L}_\infty(B|x_0) + \theta(A > 0)\theta(B > 0)C_\infty(A, B|x_0). \tag{92}
\]

These contributions involving the infinite-time limit \( t \to +\infty \) can be obtained from their Laplace transforms by considering the limit \( s \to 0 \) as follows: the first contribution can be computed from equation (79)

\[
S^{hB(0, L)}_\infty(x_0) = \lim_{s \to 0} \left[ sS^{hB(0, L)}_s(x_0) \right] = 1 - \frac{\hat{G}_0^{0}(0)L(x_0)}{\hat{G}_0^{0}(0)L(0)} \frac{\hat{G}_0^{ab(0)}(0)L(x_0)}{\hat{G}_0^{ab(0)}(0)L(0)} \tag{93}
\]

while the second contribution can be computed from equation (82)

\[
A^{\infty,L}(A|x_0) = \lim_{s \to 0} \left[ sA^{\infty,L}_s(A|x_0) \right] = 1 - \frac{\hat{G}_0^{0}(0)L}{\hat{G}_0(0)L} \left( \frac{\hat{G}_0^{ab(0)}(0)L(x_0)}{\hat{G}_0^{ab(0)}(0)L(0)} \right)^2 e^{-\frac{\Delta}{\sqrt{C_0^0(\sqrt{AB})}}}, \tag{94}
\]

and the third contribution from equation (84)

\[
B^{0,\infty}(B|x_0) = \lim_{s \to 0} \left[ sB^{0,\infty}_s(B|x_0) \right] = 1 - \frac{\hat{G}_0(0)L}{\hat{G}_0(0)L} \left( \frac{\hat{G}_0^{ab(0)}(0)L(x_0)}{\hat{G}_0^{ab(0)}(0)L(0)} \right)^2 e^{-\frac{\Delta}{\sqrt{C_0^0(\sqrt{AB})}}}. \tag{95}
\]

Finally the three terms of equation (87) allow to compute

\[
C^{[\omega]}_\infty(A, B|x_0) = \lim_{s \to 0} \left[ sC^{[\omega]}_s(A, B|x_0) \right] = 1 - \frac{\hat{G}_0(0)L}{\hat{G}_0(0)L} \left( \frac{\hat{G}_0^{ab(0)}(0)L(x_0)}{\hat{G}_0^{ab(0)}(0)L(0)} \right)^2 e^{-\frac{\Delta}{\sqrt{C_0^0(\sqrt{AB})}}} e^{-\frac{\Delta}{\sqrt{C_0^0(\sqrt{AB})}}}, \tag{96}
\]

\[
\times \frac{\sqrt{A}}{\sqrt{B}} \left( 2c_0 \sqrt{AB} \right) \]

\[
C^{[\sigma]}_\infty(A, B|x_0) = \lim_{s \to 0} \left[ sC^{[\sigma]}_s(A, B|x_0) \right] = 1 - \frac{\hat{G}_0(0)L}{\hat{G}_0(0)L} \left( \frac{\hat{G}_0^{ab(0)}(0)L(x_0)}{\hat{G}_0^{ab(0)}(0)L(0)} \right)^2 e^{-\frac{\Delta}{\sqrt{C_0^0(\sqrt{AB})}}} e^{-\frac{\Delta}{\sqrt{C_0^0(\sqrt{AB})}}}, \tag{96}
\]

\[
\times \frac{\sqrt{B}}{\sqrt{A}} \left( 2c_0 \sqrt{AB} \right) \]

\[
C^{[\gamma]}_\infty(A, B|x_0) = \lim_{s \to 0} \left[ sC^{[\gamma]}_s(A, B|x_0) \right] = e^{-\frac{\Delta}{\sqrt{C_0^0(\sqrt{AB})}}} e^{-\frac{\Delta}{\sqrt{C_0^0(\sqrt{AB})}}} I_0 \left( 2c_0 \sqrt{AB} \right) \]

\[
\times \left[ \frac{\left( 1 - \frac{\hat{G}_0(0)L}{\hat{G}_0^{ab(0)}(0)L} \right)}{\hat{G}_0^{ab(0)}(0)L(x_0)} \right] \left( \frac{\hat{G}_0^{ab(0)}(0)L(x_0)}{\hat{G}_0^{ab(0)}(0)L(0)} \right) + \frac{1 - \frac{\hat{G}_0(0)L}{\hat{G}_0(0)L} \left( \frac{\hat{G}_0^{ab(0)}(0)L(x_0)}{\hat{G}_0^{ab(0)}(0)L(0)} \right)}{\hat{G}_0(0)L} \right) \Delta_0 \]

\[
\times \left( \frac{\hat{G}_0^{ab(0)}(0)L(x_0)}{\hat{G}_0^{ab(0)}(0)L(0)} \right) \right]. \tag{96}
\]
so that the fourth contribution of equation (92) reads
\[
C_\infty(A,B|x_0) = C_\infty^{[a]}(A,B|x_0) + C_\infty^{[b]}(A,B|x_0) + C_\infty^{[c]}(A,B|x_0)
\]
\[
= e^{-\frac{\int_{0}^{t} A_s \, ds}{\alpha}} e^{-\frac{\int_{0}^{t} B_s \, ds}{\beta_{\L}}}
\left(\frac{G_{0}^{\text{abs}}(L)(0,x_0)}{G_{0}^{\text{abs}}(L)(0,0)}\right)
\times \left[c_0 \left(1 - \frac{\hat{G}_0(L)|0\rangle}{\hat{G}_0(L)|L\rangle}\right) \sqrt{A} \left(2c_0\sqrt{AB}\right) + \frac{\hat{G}_0(L)|0\rangle}{\Delta_0 G_0^{\text{abs}}(0)|L\rangle} I_0 \left(2c_0\sqrt{AB}\right)\right]
\times \left[e^{-\frac{\int_{0}^{t} A_s \, ds}{\alpha}} e^{-\frac{\int_{0}^{t} B_s \, ds}{\beta_{\L}}}
\left(\frac{G_{0}^{\text{abs}}(L)(0,x_0)}{G_{0}^{\text{abs}}(L)(0,0)}\right)
\times \left[c_0 \left(1 - \frac{\hat{G}_0(L)|0\rangle}{\hat{G}_0(L)|L\rangle}\right) \sqrt{B} \left(2c_0\sqrt{AB}\right) + \frac{\hat{G}_0(L)|0\rangle}{\Delta_0 G_0^{\text{abs}}(0)|L\rangle} I_0 \left(2c_0\sqrt{AB}\right)\right].
\]
(97)

4.2. When \(X(t)\) is recurrent: large deviations properties of the two intensive local times \((a, b)\) at large time

When the diffusion process \(X(t)\) is recurrent, it is more appropriate to introduce the two intensive local times of equation (11) and to analyze their large deviations properties thereby generalizing the previous works concerning the single intensive local time \(a\) [2, 106].

4.2.1. Rate function \(I(a,b)\) for the two intensive local times \((a, b)\). Let us plug \(A = ta\) and \(B = tb\) into equation (70) with its four contributions
\[
\Pi_t(A = ta, B = tb|x_0) = \delta(a)\delta(b) \frac{1}{t} S_t^{\text{abs}}(0,L)(x_0) + \theta(a > 0)\delta(b) \frac{1}{t} A_t^{-\infty,L}(ta|x_0)
\]
\[
+ \delta(a)\theta(b > 0) \frac{1}{t} B_t^{0,\infty}(tb|x_0) + \theta(a > 0)\theta(b > 0) C_t(ta, tb|x_0).
\]
(98)

Since the leading behavior is the exponential decay with respect to the time \(t\) of equation (12) that involves the positive rate function \(I(a, b) \geq 0\) defined for \(a \in [0, +\infty]\) and \(b \in [0, +\infty]\), the four contributions of equation (98) are then governed by the four different exponential factors
\[
S_t^{\text{abs}}(0,L)(x_0) \sim e^{-tI(a=0,b=0)}
\]
\[
A_t^{-\infty,L}(A = ta|x_0) \sim e^{-tI(a,b=0)}
\]
\[
B_t^{0,\infty}(B = tb|x_0) \sim e^{-tI(a=0,b)}
\]
\[
C_t(A = ta, B = tb|x_0) \sim e^{-tI(a,b)}.
\]
(99)

In the following subsections, the goal is to evaluate at leading order the various contributions including the prefactors in order to have the dependence with respect to the initial position \(x_0\)
that will be needed later to construct conditioned processes. It is more convenient to begin with
the fourth contribution \(\mathcal{C}_t(A = ta, B = tb|x_0)\) since it is the only one that involves the full joint
rate function \(I(a,b)\) of the two variables \(a\) and \(b\).

4.2.2. Evaluation of the fourth contribution \(\mathcal{C}_t(A = ta, B = tb|x_0)\) at leading order for large time \(t\).

The contribution \(\mathcal{C}_t(A = ta, B = tb|x_0)\) of equation (88) displays the asymptotic behavior
given by equation (90) for large time \(t\)

\[
\mathcal{C}_t(A = ta, B = tb|x_0) \approx \frac{1}{2\sqrt{\pi t}} \left[ \frac{a^{1-\infty}(x_0)}{b^\frac{1}{2}} + \frac{b^{0,++}(x_0)}{a^\frac{1}{2}} \frac{\gamma_c(x_0)}{\sqrt{c^\frac{1}{2} b^\frac{1}{2}}} \right] \\
\times e^{-\left[ \frac{a^\frac{1}{2}}{c^\frac{1}{2}} \phi(0) + \frac{b^\frac{1}{2}}{c^\frac{1}{2}} (tt) - 2c \sqrt{ab} \right]}.
\]

(100)

The Laplace inversion of this asymptotic behavior allows to compute the fourth contribution
\(\mathcal{C}_t(A = ta, B = tb|x_0)\) at leading order for large \(t\)

\[
\mathcal{C}_t(A = ta, B = tb|x_0) = \int_{c-i\infty}^{c+i\infty} ds \frac{e^{s}(A = ta, B = tb|x_0)}{2\pi i}
\]

\[
\approx \frac{1}{2\sqrt{\pi t}} \left[ \frac{a^{1-\infty}(x_0)}{b^\frac{1}{2}} + \frac{b^{0,++}(x_0)}{a^\frac{1}{2}} \frac{\gamma_c(x_0)}{\sqrt{c^\frac{1}{2} b^\frac{1}{2}}} \right] \\
\times e^{-\left[ \frac{a^\frac{1}{2}}{c^\frac{1}{2}} \phi(0) + \frac{b^\frac{1}{2}}{c^\frac{1}{2}} (tt) - 2c \sqrt{ab} \right]}.
\]

(101)

For large \(t\), the saddle-point evaluation of this integral over \(s\) involves the solution \(s_{a,b}\) of

\[
0 = \partial_s \left[ \frac{a}{G_{a}^{(L)}(0|0)} + \frac{b}{G_{b}^{(L)}(0|L|L)} - 2c \sqrt{ab} - s \right].
\]

(102)

One then needs to make the change of variable

\[
s = s_{a,b} + i\omega
\]

(103)

and to use the Taylor expansion at second order in \(\omega\)

\[
\left[ \frac{a}{G_{a}^{(L)}(0|0)} + \frac{b}{G_{b}^{(L)}(0|L|L)} - 2c \sqrt{ab} - s \right]_{s = s_{a,b} + i\omega} = I(a,b) + 0 + \frac{\omega^2}{2} K(a,b) + o(\omega^2)
\]

(104)

that involves the two functions

\[
I(a,b) = \left[ \frac{a}{G_{a}^{(L)}(0|0)} + \frac{b}{G_{b}^{(L)}(0|L|L)} - 2c \sqrt{ab} - s \right]_{s = s_{a,b}}
\]

\[
K(a,b) = - \left[ \partial_s^2 \left( \frac{a}{G_{a}^{(L)}(0|0)} + \frac{b}{G_{b}^{(L)}(0|L|L)} - 2c \sqrt{ab} - s \right) \right]_{s = s_{a,b}}.
\]

(105)
The final result for the leading order of equation (101) based on the remaining Gaussian integral over $\omega$ reads

$$
C_\omega(A = ta, B = tb|x_0) \approx \frac{e^{-|t-a|}}{2\sqrt{2\pi}} e^{t^4/2} \left[\int_{-\infty}^{+\infty} d\omega e^{-\frac{\omega}{2}} \right] F(t, \omega, a, b)
$$

\[\times \int_{-\infty}^{+\infty} d\omega e^{-\frac{\omega}{2}} \left[\int_{-\infty}^{+\infty} d\omega e^{-\frac{\omega}{2}} \right] F(t, \omega, a, b)

4.2.3. Evaluation of the second contribution $A_t^{-\infty, L_1}(A = ta|x_0)$ at leading order for large time $t$. The second contribution $A_t^{-\infty, L_1}(A = ta|x_0)$ can be obtained from the Laplace inverse of $A_t^{-\infty, L_1}(A = ta|x_0)$ of equation (82)

$$
A_t^{-\infty, L_1}(A = ta|x_0) = \int_{c-i\infty}^{c+i\infty} \frac{ds}{2\pi i} e^{st} A_t^{-\infty, L_1}(A = ta|x_0)
$$

(107)

For large $t$, the saddle-point evaluation of this integral involves the solution $s_{a,0}$ of

$$
0 = \partial_s \left[ \frac{a}{G_{s_{a,0}}(0,0)} - s \right]
$$

that corresponds to the special case $b = 0$ in equation (102). So the Taylor expansion of equation (104) for special case $b = 0$

$$
\left[ \frac{a}{G_{s_{a,0}}(0,0)} - s \right]_{s_{a,0}+i\omega} = I(a,0) + \omega^2 K(a,0) + o(\omega^2)
$$

(109)

yields that equation (107) reads the leading order

$$
A_t^{-\infty, L_1}(A = ta|x_0) \approx e^{-|t-a|} \alpha_{s_{a,0}}^{-\infty, L_1}(x_0) \int_{-\infty}^{+\infty} \frac{da}{2\pi} e^{-\frac{t-a}{K(a,0)}}
$$

(110)

4.2.4. Evaluation of the third contribution $B_t^{-[0, \infty]}(B = tb|x_0)$ at leading order for large time $t$. The third contribution $B_t^{-[0, \infty]}(B = tb|x_0)$ can be obtained from the Laplace inverse of $B_t^{-[0, \infty]}(B = tb|x_0)$ of equation (84)

$$
B_t^{-[0, \infty]}(B = tb|x_0) = \int_{c-i\infty}^{c+i\infty} \frac{ds}{2\pi i} e^{st} B_t^{-[0, \infty]}(B = tb|x_0)
$$

(111)
For large \( t \), the saddle-point evaluation of this integral involves the solution \( s_{0,b} \) of

\[
0 = \partial_s \left[ \frac{b}{G(t)^{ab(0)(L|L)}} - s \right] \tag{112}
\]

that corresponds to the special case \( a = 0 \) in equation (102). So the Taylor expansion of equation (104) for special case \( a = 0 \)

\[
\left[ \frac{b}{G(t)^{ab(0)(L|L)}} - s \right] \bigg|_{s=s_{0,b}+i\omega} = I(0, b) + \frac{\omega^2}{2} K(0, b) + o(\omega^2) \tag{113}
\]

yields that equation (111) reads the leading order

\[
B_t^{ab(0)}(B = tb|x_0) \sim e^{-\delta(0,b)} \frac{\beta_{0+,\infty}^{[0]}(x_0)}{\sqrt{2\pi K(0,b)t}}
\]

\[
\tag{114}
\sim e^{-\delta(0,b)} \frac{\beta_{0+,\infty}^{[0]}(x_0)}{\sqrt{2\pi K(0,b)t}}
\]

4.2.5. Conclusion. Putting everything together, one obtains the asymptotic behavior at leading order for large time \( t \) of equation (98)

\[
\Pi_t(A = ta, B = tb|x_0) \sim e^{-\delta(0,b)} \frac{\beta_{0+,\infty}^{[0]}(x_0)e^{-\delta(0,b)}}{2\pi t^{1/2} K(a,0)t^{1/2}} + \delta(a)\theta(b>0) \frac{e^{-\delta(0,b)}}{2\pi t^{1/2} K(a,b)}
\]

\[\times \left[ \sqrt{G_0} \left( \frac{\alpha_{s}^{[1]}(x_0)x^1}{b^1} + \frac{\beta_{0+,\infty}^{[0]}(x_0)x^1}{a^1} \right) + \gamma_{s}(x_0) \right] \bigg|_{s=s_{0,b}}. \tag{115}\]

5. Statistics of the sum \( \Sigma(t) = A(t) + B(t) \) of the two local times \( A(t) \) and \( B(t) \)

In this section, we focus on the sum \( \Sigma(t) \) of the two local times

\[
\Sigma(t) \equiv A(t) + B(t) = \int_0^t d\tau \left[ \delta(X(\tau)) + \delta(X(\tau) - L) \right] \tag{116}
\]

whose statistical properties are simpler.

5.1. Propagator \( P_t(x, \Sigma|x_0) \) for the position \( x \) and the sum \( \Sigma \) of the two local times

5.1.1. Singular contribution in \( \delta(\Sigma) \) and regular contribution in \( \theta(\Sigma > 0) \). The joint propagator \( P_t(x, \Sigma|x_0) \) can be computed from the joint propagator \( P_t(x,A,B|x_0) \) of equation (18) with its four contributions

\[
P_t(x, \Sigma|x_0) = \int_0^{t+\infty} dA \int_0^{t+\infty} dB P_t(x,A,B|x_0) \delta(\Sigma - (A + B))
\]

\[= \delta(\Sigma) G_t^{ab(0,L)}(x|x_0) + \theta(\Sigma > 0) T_t(x,\Sigma|x_0) \tag{117}\]
where the regular contribution \( \Upsilon_t(x, \Sigma|x_0) \) reads
\[
\Upsilon_t(x, \Sigma|x_0) = A_t^{(0)}(x, \Sigma|x_0) + B_t^{(0)}(x, \Sigma|x_0) + \int_0^\Sigma dA_t(x, A, \Sigma - A|x_0)
\]
(118)

or equivalently for the time Laplace transform
\[
\hat{P}_s(x, \Sigma|x_0) \equiv \int_0^{+\infty} dt e^{-st} P_t(x, \Sigma|x_0) = \delta(\Sigma) \hat{G}_s^{(0,0)}(x|x_0) + \theta(\Sigma > 0) \hat{\Upsilon}_s(x, \Sigma|x_0)
\]
(119)

with
\[
\hat{\Upsilon}_s(x, \Sigma|x_0) = \hat{A}_s(x, \Sigma|x_0) + \hat{B}_s(x, \Sigma|x_0) + \int_0^\Sigma dA_t(x, A, \Sigma - A|x_0).
\]
(120)

However, instead of computing the convolution of the last term, it is simpler to consider the further Laplace transform with respect to \( \Sigma \) of parameter \( p \) as we now describe.

5.1.2. Explicit regular contribution \( \hat{\Upsilon}_{s,p}(x|x_0) \). The further Laplace transform of equation (119) with respect to \( \Sigma \) of parameter \( p \) corresponds to \( \hat{P}_{s,p}(x|x_0) \) computed in equation (31) for the special case \( q = p \)

\[
\int_0^{+\infty} d\Sigma e^{-p\Delta} \hat{P}_s(x, \Sigma|x_0) = \hat{G}_s^{(0,0)}(x|x_0) + \hat{\Upsilon}_{s,p}(x|x_0)
\]

\[
= \int_0^{+\infty} dA e^{-pA} \int_0^{+\infty} dBe^{-pB} \hat{P}_s(x, A, B|x_0)
\]

\[
= \hat{P}_{s,p}(x|x_0) = \hat{G}_s(x|x_0)
\]

\[
- \frac{p^2 \Delta_s \Omega_s(x|x_0) + p \left[ \hat{G}_s(x|0) \hat{G}_s(0|x_0) + \hat{G}_s(x|L) \hat{G}_s(L|x_0) \right]}{\Delta_s p^2 + p \hat{G}_s(0|0) + p \hat{G}_s(L|L) + 1}.
\]
(121)

Using equation (35), one obtains the explicit form of \( \hat{\Upsilon}_{s,p}(x|x_0) \)

\[
\hat{\Upsilon}_{s,p}(x|x_0) = \Omega_s(x|x_0) - \frac{p^2 \Delta_s \Omega_s(x|x_0) + p \left[ \hat{G}_s(x|0) \hat{G}_s(0|x_0) + \hat{G}_s(x|L) \hat{G}_s(L|x_0) \right]}{\Delta_s p^2 + p \hat{G}_s(0|0) + \hat{G}_s(L|L) + 1}
\]

\[
= \frac{\Omega_s(x|x_0) + p \Delta_s \left[ \alpha_s^{(0)}(x|x_0) + \beta_s^{(LL)}(x|x_0) \right]}{\Delta_s p^2 + p \hat{G}_s(0|0) + \hat{G}_s(L|L) + 1}
\]
(122)

in terms of the functions \( \alpha_s^{(0)}(x|x_0) \) and \( \beta_s^{(LL)}(x|x_0) \) introduced in equations (40) and (47).

5.1.3. Laplace inversion of \( \hat{\Upsilon}_{s,p}(x|x_0) \) with respect to \( p \) to obtain \( \Upsilon_s(x, \Sigma|x_0) \). The factorization of the denominator of equation (122)

\[
\Delta_s p^2 + p \left[ \hat{G}_s(0|0) + \hat{G}_s(L|L) \right] + 1 = \Delta_s (p + \lambda^+_s)(p + \lambda^-_s)
\]
(123)
involves the two positive functions $\lambda_\pm > 0$ given by

$$
\lambda_\pm = \frac{\left[ \hat{G}_s(0|0) + \hat{G}_s(L|L) \right] \pm \sqrt{\left[ \hat{G}_s(0|0) + \hat{G}_s(L|L) \right]^2 - 4\lambda_s}}{2\lambda_s} \tag{124}
$$

where we have introduced the following notation for the difference

$$
D_\pm = \lambda_+ - \lambda_-, \quad (p + \lambda_s)^2 \tag{125}
$$

So the partial fraction decomposition of equation (122) with respect to the variable $p$ reduces to

$$
\hat{Y}_{s,p}(x|x_0) = \frac{\Lambda_+(x|x_0)}{p + \lambda_+} + \frac{\Lambda_-(x|x_0)}{p + \lambda_-} \tag{126}
$$

where the identification with equation (122) yields that the two numerators $\Lambda_\pm(x|x_0)$ satisfy the following system using equation (C15)

$$
\lambda_- \Lambda_+ + \lambda_+ \Lambda_- = \frac{\Omega_s(x|x_0)}{\Delta_s} = \frac{\alpha_s^{[00]}(x|x_0)}{G_s^{\text{abs}(0)(L|L)}} + \frac{\beta_s^{[LL]}(x|x_0)}{G_s^{\text{abs}(L)(0|0)}} + \gamma_s(x|x_0)
$$

and thus read

$$
\Lambda_+(x|x_0) = \frac{\alpha_s^{[00]}(x|x_0) \left[ \lambda_+ - \frac{1}{G_s^{\text{abs}(0)(L|L)}} \right] + \beta_s^{[LL]}(x|x_0) \left[ \lambda_+ - \frac{1}{G_s^{\text{abs}(L)(0|0)}} \right] - \gamma_s(x|x_0)}{D_\pm}
$$

$$
\Lambda_-(x|x_0) = \frac{\alpha_s^{[00]}(x|x_0) \left[ \lambda_- - \frac{1}{G_s^{\text{abs}(0)(L|L)}} \right] + \beta_s^{[LL]}(x|x_0) \left[ \lambda_- - \frac{1}{G_s^{\text{abs}(L)(0|0)}} \right] + \gamma_s(x|x_0)}{D_\pm}. \tag{128}
$$

The Laplace inversion with respect to $p$ of equation (126) yields the following two exponential functions with respect to $\Sigma$ of parameters $\lambda_\pm$

$$
\hat{Y}_s(x, \Sigma|x_0) = \Lambda_+(x|x_0)e^{-\lambda_+ \Sigma} + \Lambda_-(x|x_0)e^{-\lambda_- \Sigma}. \tag{129}
$$

5.2. Distribution $\Pi_\Sigma(x|x_0)$ of the sum $\Sigma$ at time $t$

The distribution $\Pi_\Sigma(x|x_0)$ of the sum $\Sigma$ at time $t$ when starting at position $x_0$ can be obtained from the integration over the final position $x$ of the $P_r(x, \Sigma|x_0)$ of equation (117)
\[\Pi_{t}(\Sigma|x_0) \equiv \int_{-\infty}^{+\infty} dxP_{t}(x, \Sigma|x_0) = \delta(\Sigma)S_{t}^{ab(0,L)}(x_0) + \theta(\Sigma > 0)\Upsilon_{t}(\Sigma|x_0)\] (130)

where

\[\Upsilon_{t}(\Sigma|x_0) \equiv \int_{-\infty}^{+\infty} dx\Upsilon_{t}(x, \Sigma|x_0).\] (131)

Its time Laplace transform \(\tilde{\Upsilon}_{s}(\Sigma|x_0)\) can be obtained from the integration over \(x\) of \(\tilde{\Upsilon}_{s}(x, \Sigma|x_0)\) equation (129)

\[\tilde{\Upsilon}_{s}(\Sigma|x_0) = \int_{-\infty}^{+\infty} dx\tilde{\Upsilon}_{s}(x, \Sigma|x_0) = \Lambda_{+}^{s}(x_0)e^{-\lambda_{+}^{s}\Sigma} + \Lambda_{-}^{s}(x_0)e^{-\lambda_{-}^{s}\Sigma}\] (132)

where the amplitudes \(\Lambda_{+}^{s}(x_0)\) can be computed using the explicit expressions of the functions \(\alpha_{s}^{[-\infty,L]}(x_0), \beta_{s}^{[0,\infty]}(x_0)\) and \(\gamma_{s}(x_0)\) of equations (81), (83) and (85)

\[\Lambda_{+}^{s}(x_0) = \int_{-\infty}^{+\infty} dx\Lambda_{+}^{s}(x|x_0)\]

\[= \alpha_{s}^{[-\infty,L]}(x_0)\left[\lambda_{+}^{s} - \frac{1}{\xi_{s}^{ab(0,L)}}\right] + \beta_{s}^{[0,\infty]}(x_0)\left[\lambda_{+}^{s} - \frac{1}{\xi_{s}^{ab(0,0)}}\right] - \gamma_{s}(x_0)\]

\[= \lambda_{+}^{s}\left(\tilde{G}_{s}(L|L) - \tilde{G}_{s}(L|0) - \frac{1}{\lambda_{+}^{s}}\right)\frac{\tilde{\xi}_{s}^{ab(L)(0|x_0)}}{\tilde{\xi}_{s}^{ab(L)(0,0)}} + \left(\tilde{G}_{s}(0|0) - \tilde{G}_{s}(0|L) - \frac{1}{\lambda_{+}^{s}}\right)\frac{\tilde{\xi}_{s}^{ab(L)(0|x_0)}}{\tilde{\xi}_{s}^{ab(L)(0,0)}}\]

so that the normalization of equation (132) over \(\Sigma\)

\[\int_{0}^{+\infty} d\Sigma\tilde{\Upsilon}_{s}(\Sigma|x_0) \equiv \frac{\Lambda_{+}^{s}(x_0)}{\lambda_{+}^{s}} + \frac{\Lambda_{-}^{s}(x_0)}{\lambda_{-}^{s}} = \frac{1}{s} \left[\frac{\tilde{G}_{s}(0|x_0)}{\tilde{G}_{s}(0|0)} + \frac{\tilde{G}_{s}(0|L)}{\tilde{G}_{s}(0|L)}\right] + \frac{\tilde{G}_{s}(L|L)}{\tilde{G}_{s}(L|0)}(x_0)\] (133)

is complementary to \(\tilde{S}_{s}^{ab(0,L)}(x_0)\) of equation (79) as it should.

5.3. Moments \(m_{n}^{[k]}(x_{0}) = (\Sigma^{k}(t))_{x_0}\) of the sum \(\Sigma(t)\) when starting at \(x_{0}\)

The moments of order \(k > 0\) of \(\Sigma\) at time \(t\) can be computed from the distribution \(\Pi_{t}(\Sigma|x_0)\) of equation (130)
\[ m_i^{|k|}(x_0) = \langle \Sigma^k(t) \rangle_{x_0} = \int_0^{+\infty} d\Sigma \Sigma^k \Pi(\Sigma|x_0) = \int_0^{+\infty} d\Sigma \Sigma^k \tilde{Y}_x(\Sigma|x_0). \]  

Their time-Laplace transforms can be computed from the time Laplace transform \( \tilde{Y}_x(\Sigma|x_0) \) of equation (131)

\[
\tilde{m}_i^{|k|}(x_0) \equiv \int_0^{+\infty} dt e^{-t} m_i^{|k|}(x_0) = \int_0^{+\infty} d\Sigma \Sigma^k \tilde{Y}_x(\Sigma|x_0) \\
= \int_0^{+\infty} d\Sigma \Sigma^k \left[ \Lambda^+_x(x_0) e^{-\lambda^+_x \Sigma} + \Lambda^-_x(x_0) e^{-\lambda^-_x \Sigma} \right] \\
= k! \left( \frac{\Lambda^+_x(x_0)}{(\lambda^+_x)^{k+1}} + \frac{\Lambda^-_x(x_0)}{(\lambda^-_x)^{k+1}} \right).
\]

5.4. When \( X(t) \) is transient: the sum \( \Sigma(t) \) remain a finite random variable for \( t \to +\infty \)

As already explained in section 4.1, the two local times \( (A, B) \) remain finite random variables for \( t \to +\infty \). It is thus interesting to write the asymptotic distribution of the sum \( \Sigma \) obtained from equation (130)

\[
\Pi_{\infty}(\Sigma|x_0) = \lim_{t \to +\infty} \Pi_t(\Sigma|x_0) = \delta(\Sigma) S_{\infty}^{ab|0(L)}(x_0) + \theta(\Sigma > 0) \Upsilon_{\infty}(\Sigma|x_0)
\]

where the singular contribution \( S_{\infty}^{ab|0(L)}(x_0) \) has been already given in equation (93), while the regular contribution \( \Upsilon_{\infty}(\Sigma|x_0) \) involving the infinite-time limit \( t \to +\infty \) can be obtained from the Laplace transform \( \Upsilon_x(\Sigma|x_0) \) of equation (132) by considering the limit \( s \to 0 \) of

\[
\Upsilon_{\infty}(\Sigma|x_0) = \lim_{s \to 0} \left[ s \Upsilon_x(\Sigma|x_0) \right] = \lim_{s \to 0} \left[ s \Lambda^+_x(x_0) e^{-\lambda^+_x \Sigma} + s \Lambda^-_x(x_0) e^{-\lambda^-_x \Sigma} \right] \\
= \xi^+(x_0) e^{-\lambda^+_x \Sigma} + \xi^-(x_0) e^{-\lambda^-_x \Sigma}
\]

where the two amplitudes \( \xi^\pm(x_0) \) can be obtained from equation (133)

\[
\xi^+(x_0) = \lim_{s \to 0} \left[ i \Lambda^+_x(x_0) \right] \\
= \lambda^+_o \left( \frac{\hat{G}_0(L) - \hat{G}_0(0[L])}{\frac{G^{ab|0(L)}(0|x_0)}{G^{ab|0(L)}(0[L])}} + \frac{\hat{G}_0(0[0]) - \hat{G}_0(0[L])}{\frac{G^{ab|0(L)}(0|x_0)}{G^{ab|0(L)}(0[L])}} \right)
\]

\[
\xi^-(x_0) = \lim_{s \to 0} \left[ i \Lambda^-_x(x_0) \right] \\
= \lambda^-_o \left( \frac{\hat{G}_0(L)[L] - \hat{G}_0(0[L])}{\frac{G^{ab|0(L)}(0|x_0)}{G^{ab|0(L)}(0[L])}} + \frac{\hat{G}_0(0[0]) - \hat{G}_0(0[L])}{\frac{G^{ab|0(L)}(0|x_0)}{G^{ab|0(L)}(0[L])}} \right)
\]

so that the normalization of equation (138) over \( \Sigma \)

\[
\int_0^{+\infty} d\Sigma \Upsilon_{\infty}(\Sigma|x_0) = \frac{\xi^+(x_0)}{\lambda^+_o} + \frac{\xi^-(x_0)}{\lambda^-_o} = \frac{G^{ab|0(L)}(0|x_0)}{G^{ab|0(L)}(0[L])} + \frac{\hat{G}_0(0[L])}{\hat{G}_0(0[0])} = 1 - S_{\infty}^{ab|0(L)}(x_0)
\]

is complementary to \( S_{\infty}^{ab|0(L)}(x_0) \) of equation (93) as it should.
5.5. When $X(t)$ is recurrent: large deviations properties of the intensive sum $\sigma = \frac{\Sigma(t)}{t}$ at large time $t$

When $X(t)$ is recurrent, it is interesting to analyze the large deviations properties of the intensive sum of the two local times $A(t)$ and $B(t)$

$$\sigma = \frac{\Sigma(t)}{t} = \frac{A(t) + B(t)}{t}.$$ (141)

5.5.1. Evaluation of the distribution $\Pi_t(\Sigma = \sigma | x_0)$ at leading order in $t$. The distribution $\Pi_t(\Sigma = \sigma | x_0)$ of $\Sigma = \sigma t$ for $\sigma \in [0, +\infty]$ can be evaluated from the Laplace inversion of equation (132)

$$\Upsilon_t(\Sigma = \sigma | x_0) = \int_{-i\infty}^{+i\infty} ds \frac{e^{st}}{2\pi} \tilde{\Upsilon}_t(\Sigma | x_0)$$

$$= \int_{-i\infty}^{+i\infty} ds \frac{e^{st}}{2\pi} \left[ \Lambda_x^+ (x_0) e^{-t[|\sigma \lambda_x^+ - s|]} + \Lambda_x^- (x_0) e^{-t[|\sigma \lambda_x^- - s|]} \right].$$ (142)

For large $t$, the dominant exponential is the second term involving $\lambda_x^-$ (since $\lambda_x^- < \lambda_x^+$ in equation (124)), so that the saddle-point evaluation involves the solution $s_\sigma$ of

$$0 = \partial_s \left[ \sigma \lambda_x^- - s \right] = \sigma \partial_s \lambda_x^- - 1.$$ (143)

The change of variable

$$s = s_\sigma + i\omega$$ (144)

and the Taylor expansion at second order in $\omega$

$$\left[ \sigma \lambda_x^- - s \right]_{s = s_\sigma} = J(\sigma) + 0 + \frac{\omega^2}{2} \chi(\sigma) + o(\omega^2)$$ (145)

that involves the two functions

$$J(\sigma) = \left[ \sigma \lambda_x^- - s \right]_{s = s_\sigma}$$

$$\chi(\sigma) = -\sigma \left[ \partial_s^2 \lambda_x^- \right]_{s = s_\sigma}$$ (146)

leads to the final result for the leading order of equation (142)

$$\Pi_t(\Sigma = \sigma | x_0) \approx \frac{\Lambda_x^- (x_0) e^{-tJ(\sigma)}}{\sqrt{2\pi t \chi(\sigma)}}.$$ (147)

5.5.2. Link between the rate function $J(\sigma)$ of the intensive sum $\sigma = a + b$ and the joint rate function $l(a,b)$. The integration over $x$ of equation (118)

$$\Upsilon_t(\Sigma | x_0) = \mathcal{A}^{abs(L)}_t (\Sigma | x_0) + \mathcal{B}^{abs(0)}_t (\Sigma | x_0) + \int_0^\Sigma dA(\sigma, \Sigma - A | x_0)$$ (148)

reads for $\Sigma = \sigma t$

$$\Upsilon_t(\Sigma = \sigma | x_0) = \mathcal{A}^{abs(L)}_t (\sigma t | x_0) + \mathcal{B}^{abs(0)}_t (\sigma t | x_0) + \int_0^\sigma d\mathcal{C}(\sigma t, t | \sigma - a | x_0).$$ (149)
where the two first terms are governed by the rate functions $I(a = \sigma, b = 0)$ and $I(a = 0, b = \sigma)$ of equation (99), while the convolution of the last term involves the rate function $I(a, b = \sigma - a)$

$$
\int_0^{\sigma} da C_t(ta, I(\sigma - a)|x_0) \propto \int_0^{\sigma} da e^{\frac{c}{c_s}} e^{-tI(\sigma)}. \quad (150)
$$

As a consequence, the rate function $J(\sigma)$ should correspond to the optimization of the joint rate function $I(a, \sigma - a)$ over all possible values $a \in [0, \sigma]$

$$
J(\sigma) = \min_{0 \leq a \leq \sigma} I(a, \sigma - a). \quad (151)
$$

Since the rate function $I(a, b)$ is the minimum over $s$ of the function in the exponential in equation (101)

$$
I(a, b) = \min_s \left[ \frac{a}{G_x^{abi}(L)}(0|0) + \frac{b}{G_y^{abi}(L|L)} - 2c_s \sqrt{ab} - s \right] \quad (152)
$$

one can exchange the two minimizations to rewrite equation (151) as

$$
J(\sigma) = \min_{s} \min_{0 \leq a \leq \sigma} \left[ \frac{a}{G_x^{abi}(L)}(0|0) + \frac{\sigma - a}{G_y^{abi}(L|L)} - 2c_s \sqrt{a(\sigma - a)} - s \right]. \quad (153)
$$

The minimization over $a$

$$
0 = \partial_a \left[ \frac{a}{G_x^{abi}(L)}(0|0) + \frac{\sigma - a}{G_y^{abi}(L|L)} - 2c_s \sqrt{a(\sigma - a)} - s \right]
$$

$$
= \frac{1}{G_x^{abi}(L)}(0|0) - \frac{1}{G_y^{abi}(L|L)} - c_s \frac{\sigma - 2a}{\sqrt{a(\sigma - a)}}, \quad (154)
$$

yields the explicit optimal value

$$
a_{opt} = \frac{\sigma}{2} \left( 1 + \frac{\frac{1}{G_x^{abi}(L)}(0|0) - \frac{1}{G_y^{abi}(L|L)}}{\frac{1}{G_x^{abi}(L)}(0|0) - \frac{1}{G_y^{abi}(L|L)}} \right)
$$

$$
= \frac{\sigma}{2} \left( 1 + \frac{1}{D_s} \right), \quad (155)
$$

that can be plugged into equation (153) to obtain

$$
J(\sigma) = \min_s \left[ \frac{a_{opt}}{G_x^{abi}(L)}(0|0) + \frac{\sigma - a_{opt}}{G_y^{abi}(L|L)} - 2c_s \sqrt{a_{opt}(\sigma - a_{opt})} - s \right]
$$

$$
= \min_s \left[ \frac{\sigma}{2} \left( \frac{1}{G_x^{abi}(L|L)} + \frac{1}{G_y^{abi}(L)}(0|0) - D_s \right) - s \right]
$$

$$
= \min_s \left[ \sigma \lambda_s - s \right], \quad (156)
$$

so that one recovers the function $\lambda_s^-$ of equation (124), as it should for consistency with the direct calculation of equations (143) and (146) of the previous subsection.
6. Conditioned processes involving the two local times

In this section, the goal is to construct conditioned joint processes \([X^*(t), A^*(t), B^*(t)]\) satisfying certain constraints involving the two local times, thereby generalizing our previous work [106] concerning the conditioning with respect to the single local time \(A(t)\).

6.1. Conditioned Bridge towards the local times \([A^*_T, B^*_T]\) at the time horizon \(T\)

If the conditioning is towards the local times \([A^*_T, B^*_T]\) at the time horizon \(T\), without any condition on the final position \(x_T\), the conditioned distribution of the position \(x\) and the local times \((A, B)\) at some interior time \(t \in [0, T]\) involves the unconditioned joint distribution \(\Pi_{t \rightarrow 0}(A_2 - A_1, B_2 - B_1 | x_1)\) of the two increments \([A_2 - A_1, B_2 - B_1]\) during the time interval \((t_2 - t_1)\) described in section 3.

\[
P_{T}^{[A^*_T, B^*_T]}(x, A, B, t) = \frac{\Pi_{T-t}(A^*_T - A, B^*_T - B | x)P_{t}(x, A, B | x_0)}{\Pi_{T}(A^*_T, B^*_T | x_0)}. \tag{157}
\]

The corresponding conditioned process \([X^*(t), A^*(t), B^*(t)]\) then satisfies the SDE of equation (13) where the conditioned drift \(\mu_{T}^{[A^*_T, B^*_T]}(x, A, B, t)\) is given by equation (14). The decomposition of \(\Pi_{T-t}(A^*_T - A, B^*_T - B | x)\) into its four contributions of equation (70)

\[
\Pi_{T-t}(A^*_T - A, B^*_T - B | x) = \delta(A^*_T - A)\delta(B^*_T - B)\mathcal{S}_{T-t}^{[5, 0]}(x) + \theta(A^*_T > A)\delta(B^*_T - B)\mathcal{A}_{T-t}^{[7, -\infty, \infty]}(A^*_T - A | x) + \delta(A^*_T - A)\theta(B^*_T > B)\mathcal{B}_{T-t}^{[7, 0, -\infty]}(B^*_T - B | x) + \theta(A^*_T > A)\theta(B^*_T > B)\mathcal{C}_{T-t}(A^*_T - A, B^*_T - B | x) \tag{158}
\]
yields that the conditioned dynamics can be decomposed into the following regimes:

1. For \([0 \leq A < A^*_T, 0 \leq B < B^*_T]\) where the two local times \(A\) and \(B\) have not yet reached their conditioned final values \(A^*_T\) and \(B^*_T\), the conditioned drift of equation (14) involves the contribution \(\mathcal{C}_{T-t}(A^*_T - A, B^*_T - B | x)\)

\[
\mu_{T}^{[A^*_T, B^*_T]}(x, A < A^*_T, B < B^*_T, t) = \mu(x) + \partial_t \ln \mathcal{C}_{T-t}(A^*_T - A, B^*_T - B | x). \tag{159}
\]

2. The regime (1) ends when either \(A\) or \(B\) reaches its conditioned final value, so the next regime (2) contains two possibilities:

- \((2-A)\) If \(B\) has already reached its conditioned final value \(B = B^*_T\) while \(A\) has not yet reached its conditioned final value \(A^*_T\), then the conditioned drift of equation (14) involves the contribution \(\mathcal{A}_{T-t}^{[7, -\infty, \infty]}(A^*_T - A | x)\)

\[
\mu_{T}^{[A^*_T, B^*_T]}(x, A < A^*_T, B = B^*_T, t) = \mu(x) + \partial_t \ln \mathcal{A}_{T-t}^{[7, -\infty, \infty]}(A^*_T - A | x). \tag{160}
\]

- \((2-B)\) If \(A\) has already reached its conditioned final value \(A = A^*_T\) while \(B\) has not yet reached its conditioned final value \(B^*_T\), then the conditioned drift of equation (14) involves the contribution \(\mathcal{B}_{T-t}^{[7, 0, -\infty]}(B^*_T - B | x)\)

\[
\mu_{T}^{[A^*_T, B^*_T]}(x, A = A^*_T, B < B^*_T, t) = \mu(x) + \partial_t \ln \mathcal{B}_{T-t}^{[7, 0, -\infty]}(B^*_T - B | x). \tag{161}
\]
In the last regime \([A = A^*_+; B = B^*_T]\) where the two local times \(A\) and \(B\) have already reached their conditioned final values \(A^*_+\) and \(B^*_T\), the conditioned drift of equation (14) involves the contribution \(S^{abl(0;L)}_{T_{T-}}(x)\), since the process cannot visit 0 or \(L\) anymore

\[
\mu^*_{T_{T-}}(x, A = A^*_+, B = B^*_T, t) = \mu(x) + \partial_1 \ln B_{T_{T-}}^{0, +\infty}(B^*_T - B|x). \tag{161}
\]

(3) In the last regime \([A = A^*_+, B = B^*_T]\) where the two local times \(A\) and \(B\) have already reached their conditioned final values \(A^*_+\) and \(B^*_T\), the conditioned drift of equation (14) involves the contribution \(S^{abl(0;L)}_{T_{T-}}(x)\), since the process cannot visit 0 or \(L\) anymore

\[
\mu^*_{T_{T-}}(x, A = A^*_+, B = B^*_T, t) = \mu(x) + \partial_1 \ln S^{abl(0;L)}_{T_{T-}}(x). \tag{162}
\]

It is now interesting to consider two possibilities in the limit of the infinite horizon \(T \to +\infty\), as described in the two next subsections.

### 6.2. Conditioning towards the finite local times \([A_\infty, B_\infty]\) for the infinite horizon \(T \to +\infty\)

If one wishes to impose the finite asymptotic local times \(A_\infty < +\infty\) and \(B_\infty < +\infty\) at the infinite horizon \(T \to +\infty\), one needs to analyze the limit of the infinite horizon \(T \to +\infty\) for the conditioned drift of equation (14)

\[
\mu_{T_{T-}}(x, A, B) = \mu(x) + \lim_{\tau \to +\infty} \partial_1 \ln \Pi_{\tau}(A_\infty^* - A, B_\infty^* - B|x) \tag{163}
\]

with its different expressions for the various regimes of equations (159)–(162)

\[
\mu_{T_{T-}}(x, A < A^*_\infty, B < B^*_\infty) = \mu(x) + \lim_{\tau \to +\infty} \partial_1 \ln C_{\tau}(A^*_\infty - A, B^*_\infty - B|x)
\]

\[
\mu_{T_{T-}}(x, A < A^*_\infty, B = B^*_\infty) = \mu(x) + \lim_{\tau \to +\infty} \partial_1 \ln A_{\tau}^{-, L}(A^*_\infty - A|x)
\]

\[
\mu_{T_{T-}}(x, A = A^*_\infty, B < B^*_\infty) = \mu(x) + \lim_{\tau \to +\infty} \partial_1 \ln B_{\tau}^{0, +\infty}(B^*_\infty - B|x)
\]

\[
\mu_{T_{T-}}(x, A = A^*_\infty, B = B^*_\infty) = \mu(x) + \lim_{\tau \to +\infty} \partial_1 \ln S^{abl(0;L)}_{\tau}(x). \tag{164}
\]

### 6.3. Conditioning towards the intensive local times \(a_* = \frac{a^*_+}{T}\) and \(b_* = \frac{b^*_T}{T}\) for large time horizon \(T \to +\infty\)

If one wishes to impose instead the fixed intensive local times \(a_*>0\) and \(b_*>0\) for large time horizon \(T \to +\infty\), one needs to plug the values

\[
A^*_+ = Ta_*, \quad B^*_T = Tb_* \tag{165}
\]

into the conditioned drift of equation (14)

\[
\mu_{T_{T-}}^{[a_*, b_*]}(x, A, B, t) = \mu(x) + \partial_1 \ln \Pi_{T_{T-}}(Ta_* - A, Tb_* - B|x) \tag{166}
\]

and to use the large-time behavior of equation (115) that involves the corresponding intensive local times \(a_t\) and \(b_t\) on the time interval \((T - t)\)

\[
a_t = \frac{a_*}{T - t} \quad \text{and} \quad b_t = \frac{b_*}{T - t} \tag{167}
\]
that reduces to \( a_* \) and \( b_* \) at leading order when \( T \to +\infty \). Let us discuss two special cases before the general case \((a_*>0, b_*>0)\)

6.3.1. Case \( a_*>0 \) and \( b_*=0 \). For \( a_*>0 \) and \( b_*=0 \), the asymptotic behavior of equation (115)

\[
\Pi_{T \to \infty}(Ta_*=A,0|x) \approx \frac{a_* \L(x)e^{-d(a_*,0)}}{2\pi K(a_*,0)(T-t)^{z}}
\]  

(168)

can be plugged into equation (166) to obtain the limit \( \mu_{\infty}^{a_*,b_*=0}(x) \) of the conditioned drift using equation (81)

\[
\mu_{\infty}^{a_*,b_*=0}(x) \equiv \lim_{T \to +\infty} \mu_T^{[Ta_*,Tb_*]}(x,A,B,t) = \mu(x) + \frac{\partial_s}{\partial_s} \ln \left[G_{s_{a_*,b_*}}(0|x)\right]
\]

(169)

where \( s_{a_*,b_*} \) is the solution of equation (108)

\[
0 = \partial_s \left[ a_* \frac{1}{G_{s}^{ab}(0)} - s \right] = a_* \partial_s \left[ \frac{1}{G_{s}^{ab}(0)} \right] - 1.
\]  

(170)

So the conditioned drift of equation (169) can be also rewritten in the following parametric form of parameter \( s \)

\[
\mu_{\infty}^{a_*(s),b_*=0}(x) = \mu(x) + \partial_s \ln \left[G_{s}^{ab}(0|x)\right]
\]

\[
a^*(s) = \frac{1}{\partial_s \left[ \frac{1}{G_{s}^{ab}(0)} \right]}.
\]  

(171)

6.3.2. Case \( a_*=0 \) and \( b_*>0 \). For \( a_*=0 \) and \( b_*>0 \), the asymptotic behavior of equation (115)

\[
\Pi_{T \to \infty}(0,Tb_*-B|x) \approx \frac{b_* \L(x)e^{-d(0,b)}}{2\pi K(0,b)(T-t)^{z}}
\]  

(172)

can be plugged into equation (166) to obtain the limit \( \mu_{\infty}^{a_*,b_*=0}(x) \) of the conditioned drift using equation (83)

\[
\mu_{\infty}^{a_*,b_*=0}(x) = \lim_{T \to +\infty} \mu_T^{[Ta_*,Tb_*]}(x,A,B,t) = \mu(x) + \partial_s \ln \left[G_{s_{0,b_*}}^{n}(L|x)\right]
\]

(173)

where \( s_{0,b_*} \) is the solution of equation (112)

\[
0 = \partial_s \left[ b_* \frac{1}{G_{s}^{n}(L)\L} - s \right] = b_* \partial_s \left[ \frac{1}{G_{s}^{n}(L)\L} \right] - 1.
\]  

(174)

So the conditioned drift of equation (173) can be also written in the following parametric form of parameter \( s \)
\[
\mu_{\infty}^{[a_*, b_*]}(x) = \mu(x) + \partial_t \ln \left[ G_s^{\text{abs}(0)}(L|x) \right]
\]

\[
b_*(s) = \frac{1}{\partial_s \left[ C_{\infty}^{\text{abs}(0)}(L|x) \right]}. \tag{175}
\]

### 6.3.3. General case \(a_* > 0\) and \(b_* > 0\)

For \(a_* > 0\) and \(b_* > 0\), the asymptotic behavior of equation (115)

\[
\Pi_{T \to T}(Ta_* - A, Tb_* - B|x) \sim \frac{e^{-(T-t)\Pi(a_*, b_*)}}{2\pi (T-t) \sqrt{2 \Pi(a_*, b_*)}}
\]

\[
\times \left[ \sqrt{c_\Pi} \left( \frac{\alpha_0^{\infty}}{b_*^2} \right) + \frac{\beta_0^{0, +\infty} |x| b_*^2}{a_*^2} \right] \left( \frac{\gamma_0(x)}{\sqrt{\alpha_* b_*}} \right) \right]_{x = s_*, b_*} \tag{176}
\]

can be plugged into equation (166) to obtain the limit \(\mu_{\infty}^{[a_*, b_*]}(x)\) of the conditioned drift

\[
\mu_{\infty}^{[a_*, b_*]}(x) \equiv \lim_{T \to +\infty} \mu_{T}^{[a_*, b_*]}(x, A, B, t) = \mu(x) + \partial_t \ln
\]

\[
\times \left[ \sqrt{c_\Pi} \left( \frac{\alpha_0^{\infty}}{b_*^2} \right) + \frac{\beta_0^{0, +\infty} |x| b_*^2}{a_*^2} \right] \left( \frac{\gamma_0(x)}{\sqrt{\alpha_* b_*}} \right) \right]_{x = s_*, b_*} \tag{177}
\]

Using the explicit expressions of equations (81), (83), (85) and (C2), this conditioned drift becomes

\[
\mu_{\infty}^{[a_*, b_*]}(x) = \mu(x) + \partial_t \ln \left[ \sqrt{G_s(0)L} \frac{\alpha_*}{b_*} \left[ G_s(L|L) - G_s(L|0) \right] \right.
\]

\[
+ \left[ \frac{\alpha_*}{b_*} \left( G_s(0|0) - G_s(L|0) \right) G_s(L|L) - G_s(L|0) \right] G_s^{\text{abs}(0)}(L|x)
\]

\[
+ \left( \sqrt{G_s(0|0) L} \right) \frac{b_*}{a_*} \left[ \hat{G}_s(0|0) - \hat{G}_s(0|L) \right]
\]

\[
+ \left[ \hat{G}_s(L|L) - \hat{G}_s(L|0) \right] \hat{G}_s(0|0) G_s^{\text{abs}(0)}(L|x) \right] \tag{178}
\]

where \(s_{a_*, b_*}\) is the solution of equation (102)

\[
0 = \partial_t \left[ \frac{a_*}{G_s^{\text{abs}(0)}(0|0)} + \frac{b_*}{G_s^{\text{abs}(0)}(L|L)} - 2c_\Pi \sqrt{a_* b_*} - 8 \right]. \tag{179}
\]

The conditioned drift of equation (178) is much simpler in the two external regions:
(i) For \( x \in [-\infty, 0] \), the function \( \hat{G}_t^{ab}(0)(L|x) \) vanishes, so that equation (178) reduces to
\[
\mu^{[a, b]}_\infty(x < 0) = \mu(x) + \partial_t \ln \left[ \hat{G}_t^{ab}(0)(x) \right]_{x = x_a, b\_a}.
\] (180)

(ii) For \( x \in [L, +\infty[ \) the function \( \hat{G}_t^{ab}(L)(0|x) \) vanishes, so that equation (178) reduces to
\[
\mu^{[a, b]}_\infty(x > L) = \mu(x) + \partial_t \ln \left[ \hat{G}_t^{ab}(0)(L|x) \right]_{x = x_a, b\_a}.
\] (181)

Finally, if one replaces \((a\_s, b\_s)\) by the parametrization
\[
a\_s = \frac{\sigma\_s(1 + r\_s)}{2},
\]
\[
b\_s = \frac{\sigma\_s(1 - r\_s)}{2}
\] (182)

involving their sum \( \sigma\_s = a\_s + b\_s > 0 \) and the parameter \( r\_s \in ]-1, +1[ \) equation (179) becomes
\[
1 = \sigma\_s \partial_t \left[ \frac{1 + r\_s}{2G_t^{ab}(L)(0|0)} + \frac{1 - r\_s}{2G_t^{ab}(0)(L|L)} - cs \sqrt{1 - r\_s^2} \right]
\] (183)

then the conditioned drift of equation (178) can be written in the following parametric form
\[
\sigma\_s(x) = \partial_t \left[ \frac{1 + r\_s}{2G_t^{ab}(L)(0|0)} + \frac{1 - r\_s}{2G_t^{ab}(0)(L|L)} - cs \sqrt{1 - r\_s^2} \right]
\]
\[
\mu^{[\sigma\_s/(1+r\_s), \sigma\_s/(1-r\_s)]}_\infty(x) = \mu(x) + \partial_t \ln \left[ \sqrt{\hat{G}_t(x|L)\hat{G}_t(L|x)} \right]
\]
\[
\frac{\hat{G}_t(x|0) - \hat{G}_t(0|x)}{\hat{G}_t(x|0)} \frac{1 + r\_s}{1 - r\_s} \hat{G}_t(L|x) + \left[ \hat{G}_t(x|0) - \hat{G}_t(0|x) \right] \hat{G}_t(L|x) \hat{G}_t^{ab}(L)(0|x)
\]
\[
+ \left[ \hat{G}_t(x|0) - \hat{G}_t(0|x) \right] \hat{G}_t(L|x) \hat{G}_t^{ab}(0)(L|x)
\] (184)

### 6.4. Conditioned bridge towards the sum \( \Sigma_T^\pm \) at the time horizon \( T \)

If the conditioning is towards the sum \( \Sigma_T^\pm \) at time horizon \( T \), the conditioned distribution of the position \( x \) and the sum \( \Sigma \) at some interior time \( t \in [0, T] \) involves the unconditioned distribution \( \Pi_{x_2 - t}(\Sigma_2 - \Sigma_1, |x_1|) \) of the increment \( \Sigma_2 - \Sigma_1 \) during the time interval \( (t_2 - t_1) \) described in section 5.2
\[
P_T^{\Sigma_T^\pm}(x, \Sigma, t) = \frac{\Pi_{x_2 - t}(\Sigma_T^\pm - \Sigma|\Sigma_2 |x) P_T(x, \Sigma|\Sigma_0)}{\Pi_T(\Sigma_T^\pm |\Sigma_0)}.
\] (185)
The corresponding conditioned process $[X^*(t), \Sigma^*(t)]$ satisfies the following SDE system
\[ dX^*(t) = \mu_T^{[\Sigma^*_T]}(X^*(t), \Sigma^*(t), t) dt + dW(t) \]
\[ d\Sigma^*(t) = [\delta(X^*(t)) + \delta(X^*(t) - L)] dt \]
where the conditioned drift $\mu_T^{[\Sigma^*_T]}(x, \Sigma, t)$ reads
\[ \mu_T^{[\Sigma^*_T]}(x, \Sigma, t) = \mu(x) + \partial_t \ln \Pi_T(\Sigma^*_T - \Sigma|x). \]  
(187)
The decomposition of $\Pi_T(\Sigma^*_T - \Sigma|x)$ into its singular and regular contributions of equation (130)
\[ \Pi_{T\rightarrow}(\Sigma^*_T - \Sigma|x) = \delta(\Sigma^*_T - \Sigma)S^{\text{abs}(0,L)}_T(x) + \theta(\Sigma^*_T > \Sigma)T_{T\rightarrow}(\Sigma^*_T - \Sigma|x) \]  
(188) yields that the conditioned dynamics can be decomposed into the two regimes:

(i) For $0 \leq \Sigma < \Sigma^*_T$ where the sum $\Sigma$ has not yet reached its conditioned final value $\Sigma^*_T$, the conditioned drift of equation (187) involves the regular contribution $\Pi_T(\Sigma^*_T - \Sigma|x)$
\[ \mu_T^{[\Sigma^*_T]}(x, \Sigma < \Sigma^*_T, t) = \mu(x) + \partial_t \ln \Pi_T(\Sigma^*_T - \Sigma|x). \]  
(189)
(ii) For $\Sigma = \Sigma^*_T$ where the sum $\Sigma$ has already reached its conditioned final value $\Sigma^*_T$, the conditioned drift of equation (187) involves the survival probability $S^{\text{abs}(0,L)}_T(x)$, since the process cannot visit 0 or $L$ anymore
\[ \mu_T^{[\Sigma^*_T]}(x, \Sigma < \Sigma^*_T, t) = \mu(x) + \partial_t \ln S^{\text{abs}(0,L)}_T(x). \]  
(190)

It is now interesting to consider two possibilities in the limit of the infinite horizon $T \rightarrow +\infty$, as described in the two next subsections.

6.5. Conditioning towards the finite sum $\Sigma^*_T$ for the infinite horizon $T \rightarrow +\infty$

If one wishes to impose the finite asymptotic sum $\Sigma^*_T$ at the infinite horizon $T \rightarrow +\infty$, one needs to analyze the limit of the infinite horizon $T \rightarrow +\infty$ for the conditioned drift of equation (187)
\[ \mu_T^{[\Sigma^*_T]}(x, \Sigma) = \mu(x) + \lim_{T \rightarrow +\infty} \partial_t \ln \Pi_T(\Sigma^*_T - \Sigma|x) \]  
(191) with its two different expressions for the two regimes of equations (189) and (190)
\[ \mu_T^{[\Sigma^*_T]}(x, \Sigma < \Sigma^*_T) = \mu(x) + \lim_{T \rightarrow +\infty} \partial_t \ln \Pi_T(\Sigma^*_T - \Sigma|x) \]
\[ \mu_T^{[\Sigma^*_T]}(x, \Sigma = \Sigma^*_T) = \mu(x) + \lim_{T \rightarrow +\infty} \partial_t \ln S^{\text{abs}(0,L)}_T(x). \]  
(192)

6.6. Conditioning towards the intensive sum $\sigma_+ = \frac{\Sigma^*_T}{T}$ for large time horizon $T \rightarrow +\infty$

If one wishes to impose the fixed intensive sum $\sigma_+ > 0$ for large time horizon $T \rightarrow +\infty$, one needs to plug the values
\[ \Sigma^*_T = T\sigma_+ \]  
(193)
into the conditioned drift of equation (189)

$$
\mu_T^{[\sigma_s]}(x, \Sigma, t) = \mu(x) + \partial_s \ln \Pi_T(T_\sigma - \Sigma, |x|)
$$

(194)

and to use the large-time behavior of equation (147) that involves the corresponding intensive sum $\sigma_t$ on the time interval $(T - t)$

$$
\sigma_t \equiv \frac{T_\sigma - \Sigma}{T - t} \to +\infty \sigma_*
$$

(195)

that reduces to $\sigma_*$ at leading order when $T \to +\infty$, so that one can plug

$$
\Pi_{T-t}(T_\sigma - \Sigma, |x|) \to \frac{\Lambda_{-\sigma_+}^{-}(x)e^{-(T-t)\chi(\sigma_+)} \sqrt{2\pi(T-t)\chi(\sigma_+)} }{(T-t)}
$$

(196)

into equation (194) to obtain the limit $\mu_{[\sigma_*]}(x)$ of the conditioned drift of equation (194) using equation (133)

$$
\mu_{[\sigma_*]} \equiv \lim_{T \to +\infty} \mu_T^{[\sigma_s]}(x, \Sigma, t) \to \mu(x) + \partial_s \ln \Lambda_{-\sigma_+}^{-}(x)
$$

$$
= \mu(x) + \partial_s \ln \left(1 - \lambda_+^{-} \left[ \hat{G}_s(L|L) - \hat{G}_s(L|0) \right] \right) \frac{\tilde{G}_s^{\text{abs}(L)}(0|x)}{\tilde{G}_s^{\text{abs}(L)}(0|0)}
$$

$$
+ \left(1 - \lambda_+^{-} \left[ \hat{G}_s(0|0) - \hat{G}_s(0|L) \right] \right) \frac{\tilde{G}_s^{\text{abs}(0)}(L|x)}{\tilde{G}_s^{\text{abs}(0)}(L|L)} \bigg|_{s = \sigma_*}
$$

(197)

where $s_{\sigma_*}$ is the solution of equation (143)

$$
0 = \sigma_* \lambda_+^{-} - 1.
$$

(198)

So the conditioned drift of equation (178) can be written in the following parametric form of parameter $s$

$$
\mu_{[s]} \equiv \frac{1}{\partial_s \lambda_+^{-}}
$$

$$
= \mu(x) + \partial_s \ln \left(1 - \lambda_+^{-} \left[ \hat{G}_s(L|L) - \hat{G}_s(L|0) \right] \right) \frac{\tilde{G}_s^{\text{abs}(L)}(0|x)}{\tilde{G}_s^{\text{abs}(L)}(0|0)}
$$

$$
+ \left(1 - \lambda_+^{-} \left[ \hat{G}_s(0|0) - \hat{G}_s(0|L) \right] \right) \frac{\tilde{G}_s^{\text{abs}(0)}(L|x)}{\tilde{G}_s^{\text{abs}(0)}(L|L)} \bigg|_{s = \sigma_*}
$$

(199)

Again this conditioned drift is much simpler in the two external regions:

(i) For $x \in \to -\infty, 0]$, the function $\tilde{G}_s^{\text{abs}(0)}(L|x)$ vanishes, so that equation (199) reduces to

$$
\mu_{[s]} \equiv \frac{1}{\partial_s \lambda_+^{-}}
$$

$$
= \mu(x) + \partial_s \ln \left[ \tilde{G}_s^{\text{abs}(L)}(0|x) \right] \bigg|_{s = \sigma_*}
$$

(200)
(ii) For $x \in [L, +\infty[$ the function $G^\text{abs}(L)(0|x)$ vanishes, so that equation (199) reduces to

$$
\mu \equiv \left[ \frac{\sigma_x(t)}{b \kappa_x} \right] (x > L) = \mu(x) + \partial_t \ln \left[ G^\text{abs}(0)(L|x) \right]_{x=\sigma_x}.
$$

(201)

7. Application to the uniform drift $\mu(x) = \mu$ on the whole line $]-\infty, +\infty[$

In this section, the general framework described in the previous section is applied to the simplest example of the uniform drift $\mu(x) = \mu$ on the whole line $]-\infty, +\infty[$:

(i) For $\mu \neq 0$, the unconditioned diffusion $X(t)$ is transient, and the two local times $A(t)$ and $B(t)$ remain finite random variables for $t \to +\infty$ as described in section 4.1.

(ii) For $\mu = 0$, the unconditioned diffusion $X(t)$ is recurrent, but does not converge towards an equilibrium. The large deviations properties of the intensive local times $a = \frac{A(t)}{t}$ and $b = \frac{B(t)}{t}$ for large times are governed by some rate function $I(a, b)$, as described in section 4.2.

Here, the typical values where the rate function is vanishing and minimum are $a^\text{typ} = 0 = b^\text{typ}$

$$
0 = I(a = 0, b = 0) = \partial_a I(a = 0, b = 0) = \partial_b I(a = 0, b = 0).
$$

(202)

7.1. Useful propagators and notations

The unconditioned propagator $G(x, t|x_0, t_0)$ is Gaussian

$$
G(x, t|x_0, t_0) = \frac{1}{\sqrt{2\pi (t-t_0)}} e^{-\frac{|x-x_0|^2}{2(t-t_0)}} = \frac{1}{\sqrt{2\pi (t-t_0)}} e^{-\frac{(x-x_0)^2}{2(t-t_0)} + \mu(x-x_0) - \|x\|^2 / 2(t-t_0)}
$$

(203)

and its time Laplace transform reads

$$
\hat{G}_s(x|x_0) \equiv \int_{t_0}^{+\infty} dt e^{-(t-t_0)} G(x, t|x_0, t_0) = \frac{e^{\mu(x-x_0) - \sqrt{\mu^2 + 2\|x-x_0\|^2}}}{\sqrt{\mu^2 + 2s}} = \frac{e^{\mu(x-x_0) - \kappa_x |x-x_0|}}{\kappa_x}
$$

(204)

where we have introduced the notation

$$
\kappa_x \equiv \sqrt{\mu^2 + 2s}
$$

(205)

in order to see more clearly the structures of the other propagators of equation (38)

$$
\hat{G}^{\text{abs}(L)}_s(x|x_0) = \hat{G}_s(x|x_0) - \frac{\hat{G}_s(x|L) \hat{G}_s(L|x_0)}{\hat{G}_s(L|L)} = \frac{e^{\mu(x-x_0) - \kappa_x |x-x_0|}}{\kappa_x} \left[ e^{-\kappa_x |x-x_0|} - e^{-\kappa_x (|x|+|L|+|x-L|)} \right]
$$

(206)

and of equation (45)

$$
\hat{G}^{\text{abs}(0)}_s(x|x_0) = \hat{G}_s(x|x_0) - \frac{\hat{G}_s(x|0) \hat{G}_s(0|x_0)}{\hat{G}_s(0|0)} = \frac{e^{\mu(x-x_0) - \kappa_x |x-x_0|}}{\kappa_x} \left[ e^{-\kappa_x |x-x_0|} - e^{-\kappa_x (|x|+|0|)} \right]
$$

(207)

with the special cases

$$
\hat{G}^{\text{abs}(L)}_s(0|0) = \frac{1}{\kappa_x} [1 - e^{-2\kappa_x L}] = \hat{G}^{\text{abs}(0)}_s(L|L).
$$

(208)

These various propagators allow to evaluate the useful notations of equations (32), (C2) and (125)
\[
\Delta_s = \frac{1}{\kappa_s} \left[ 1 - e^{-2\kappa_s L} \right]
\]
\[
\epsilon_s = \frac{1}{\Delta_s} = \frac{\kappa_s}{2\sinh(\kappa_s L)}
\]
\[
D_s = \left( \frac{1}{G_s^{abhl}(0)\langle 0 \rangle} - \frac{1}{G_s^{abhl}(L)\langle 0 \rangle} \right)^2 + 4c_s^2 = \frac{\kappa_s}{\sinh(\kappa_s L)}
\]

as well as equation (210)
\[
\lambda_s^\pm = \frac{1}{2} \left[ \frac{1}{G_s^{abhl}(0)\langle 0 \rangle} + \frac{1}{G_s^{abhl}(L)\langle 0 \rangle} \right] \pm D_s = \frac{\kappa_s}{1 - e^{-2\kappa_s L}} = \frac{\kappa_s}{1 \pm e^{-\kappa_s L}}.
\]

7.2. Conditioning towards the intensive sum \( \sigma_s = \frac{\sqrt{T}}{T} \) for large time horizon \( T \to +\infty \)

The conditioned drift of equation (199) yields
\[
\mu_s^{[\sigma_s(x)]} = \mu + \partial_x \ln \left[ \left( 1 + e^{L} \right) G_s^{abl}(L)\langle 0 \rangle + \left( 1 + e^{-L} \right) G_s^{abhl}(0)\langle x \rangle \right]
\]

with the parametrization
\[
\sigma_s(x) = \frac{1}{\partial_x \lambda_s} = \frac{1}{\partial_x \left[ \frac{\kappa_s}{\kappa_s + \kappa_s e^{-x\kappa_s}} \right]} = \kappa_s \left[ 1 + e^{-\kappa_s L} \right]^2.
\]

In the three regions, the conditioned drift of equation (211) reduces to
\[
\mu_s^{[\sigma_s(x)]} = \begin{cases} 
\mu + \partial_x \ln \left[ G_s^{abl}(0)\langle 0 \rangle \right] = \kappa_s & \text{for } x \in (-\infty, 0] \\
\mu + \partial_x \ln \left[ e^{\kappa_s x} + e^{\kappa_s(L-x)} \right] = \kappa_s e^{\kappa_s x} e^{-\kappa_s L} = \kappa_s \frac{\sinh[\kappa_s(x - \frac{L}{2})]}{\cosh[\kappa_s(x - \frac{L}{2})]} & \text{for } x \in [0, L] \\
\mu + \partial_x \ln \left[ G_s^{abhl}(0)\langle x \rangle \right] = -\kappa_s & \text{for } x \in [L, +\infty]
\end{cases}
\]

7.3. Conditioning towards the intensive local times \((a^+, b^-)\) for the infinite horizon \( T \to +\infty \)

7.3.1. Special case \( a^+ > 0 \) and \( b^- = 0 \). The parametric form of equation (171) involves the conditioned drift
\[
\mu_s^{[a^+, b^- = 0]}(x) = \mu + \partial_x \ln \left[ G_s^{abhl}(0)\langle 0 | x \rangle \right] = \mu + \partial_x \ln \left[ \frac{e^{-\mu x}}{\kappa_s} \left( e^{-\kappa_s x} - e^{-\kappa_s(L+|x|)} \right) \right]
\]
\[
= \partial_x \ln \left( e^{-\kappa_s x} - e^{-\kappa_s(L+|x|)} \right)
\]

and the parametrization
\[
a^+(x) = \frac{1}{\partial_x \left[ \frac{1}{G_s^{abhl}(0)\langle 0 \rangle} \right]} = \frac{1}{\partial_x \left[ \frac{\kappa_s}{1 - e^{x\kappa_s}} \right]} = \frac{\kappa_s}{\partial_x \left[ \frac{1}{1 - e^{x\kappa_s}} \right]} = \kappa_s \left[ \frac{1 - e^{-2\kappa_s L}}{1 - \frac{2\kappa_s}{e^{2\kappa_s L} - 1}} \right].
\]
As a consequence, the parametric form of parameter $s$ actually only involves the parameter $\kappa = \kappa_s$ of equation (205) with $\kappa \in ]0, +\infty[$, where the initial drift $\mu$ does not appear anymore. In addition, the conditioned drift does not exist in the region $x \geq L$ (where it is impossible to obtain $b_s = 0$ and $\alpha_s > 0$), and can be simplified in the two remaining regions $\mu_{[\alpha_s, 0, b_s]}(x) = 0$:

$$\mu_{[\alpha_s, 0, b_s]}(x) = \left\{ \begin{array}{ll}
\partial_1 \ln \left( e^{\kappa_s x} - e^{\kappa_s (L-2L)} \right) &= \kappa_s & \text{for } x \in ]-\infty, 0[ \\
\partial_1 \ln \left( e^{-\kappa_s x} - e^{\kappa_s (L-2L)} \right) &= -\kappa_s & \text{for } x \in ]0, L[ \\
\end{array} \right. .$$

(216)

7.3.2. Special case $a_s = 0$ and $b_s > 0$. The parametric form of equation (175) involves the conditioned drift

$$\mu_{[a_s=0, b_s]}(x) = \mu + \partial_1 \ln \left[ G_{\kappa s}^{\text{abs}(0)}(L|x) \right] = \mu + \partial_1 \ln \left[ \frac{e^{\mu(L-x)}}{\kappa_s} \left( e^{\kappa_s |L-x|} - e^{\kappa_s (L+|x|)} \right) \right]$$

(217)

and the parametrization

$$b_s(x) = \frac{1}{\partial_1 \left[ G_{\kappa s}^{\text{abs}(0)}(L|x) \right]} = \frac{1}{\partial_2 \left[ \frac{\kappa_s}{1 - e^{\kappa_s x}} \right]} = \kappa_s \left[ \frac{1 - e^{-2\kappa_s L}}{1 - \frac{2\kappa_s}{\kappa_s x}} \right]$$

(218)

the parametric form of parameter $s$ actually only involves the parameter $\kappa = \kappa_s$ of equation (205) with $\kappa \in ]0, +\infty[$, where the initial drift $\mu$ does not appear anymore. In addition, the conditioned drift does not exist in the region $x \leq 0$ (where it is impossible to obtain $a_s = 0$ and $b_s > 0$), and can be simplified in the two remaining regions $\mu_{[a_s=0, b_s]}(x) = 0$:

$$\mu_{[a_s=0, b_s]}(x) = \left\{ \begin{array}{ll}
\partial_1 \ln \left( e^{-\kappa_s (L-x)} - e^{-\kappa_s (L+|x|)} \right) &= \kappa_s \left( e^{\kappa_s x} + e^{-\kappa_s x} \right) = \kappa_s \cosh(\kappa_s x) & \text{for } x \in ]0, L[ \\
\partial_1 \ln \left( e^{-\kappa_s (x-L)} - e^{-\kappa_s (L+|x|)} \right) &= -\kappa_s & \text{for } x \in ]L, +\infty[ . \\
\end{array} \right.$$  

(219)

7.3.3. General case $a_s > 0$ and $b_s > 0$. The conditioned drift of equation (184) reads

$$\mu_{[a_s(\mu_{-a_s} + b_s)](x)} = \mu + \partial_1 \ln \left[ \left( \sqrt{\frac{1 + x_s}{1 - x_s}} \left[ 1 - e^{\mu_{-a_s} x_s} \right] - \left[ e^{\mu_{-a_s} x_s} - e^{-\kappa_s L} \right] \right) G_{\kappa_s}^{\text{abs}(L)}(0|x) \right] + \left( \sqrt{\frac{1 - x_s}{1 + x_s}} \left[ 1 - e^{\mu_{-a_s} x_s} \right] + \left[ e^{\mu_{-a_s} x_s} - e^{-\kappa_s L} \right] \right) G_{\kappa_s}^{\text{abs}(0)}(L|x) \right]$$

(220)

with the parametrization

$$\sigma_s(x) = \frac{1}{\partial_1 \left[ \frac{\kappa_s}{1 - e^{\kappa_s x}} \right] - \sqrt{1 - x_s^2} \frac{\kappa_s}{2 \sinh(\kappa_s L)}} = \frac{\kappa_s}{\partial_2 \left[ \frac{\kappa_s}{1 - e^{\kappa_s x}} \right] - \sqrt{1 - x_s^2} \frac{\kappa_s}{2 \sinh(\kappa_s L)}}$$

(221)
The conditioned drift can be simplified in the three regions:

(i) In the left region $x \in ]-\infty,0]$ where the function $\tilde{G}_{x}^{\text{abs}(0)}(L|x)$ vanishes, the conditioned drift of equation (220) reduces to

\[
\mu_{\infty} \left[ \frac{\sigma(0)+(x+L)}{\sigma(0)(1-x)} \right] (x < 0) = \mu + \partial_{x} \ln \left[ \frac{e^{-\mu L}}{\kappa_{x}} \left( e^{\kappa_{x}(L-x)} - e^{\kappa_{x}(L-2L)} \right) \right] = \kappa_{x},
\]

(ii) In the right region $x \in [L, +\infty[$ where the function $\tilde{G}_{x}^{\text{abs}(0)}(0|x)$ vanishes, the conditioned drift of equation (220) reduces to

\[
\mu_{\infty} \left[ \frac{\sigma(0)+(x+L)}{\sigma(0)(1-x)} \right] (x > L) = \mu + \partial_{x} \ln \left[ \frac{e^{\mu(L-x)}}{\kappa_{x}} \left( e^{-\kappa_{x}(x-L)} - e^{-\kappa_{x}(L+x)} \right) \right] = -\kappa_{x}.
\]

(iii) In the middle region $x \in [0, L]$, the conditioned drift of equation (220) reads

\[
\mu_{\infty} \left[ \frac{\sigma(0)+(x+L)}{\sigma(0)(1-x)} \right] (x \in ]0, L[)
\]

\[
= \mu + \partial_{x} \ln \left[ \left( \frac{1+r_{x}}{1-r_{x}} \right) \left[ 1 - e^{(\mu-\kappa_{x})L} \right] + \left[ e^{\mu L} - e^{-\kappa_{x} L} \right] \right] \frac{e^{-\mu L}}{\kappa_{x}} \left( e^{-\kappa_{x}(L-x)} - e^{\kappa_{x}(L-2L)} \right) \]

\[
+ \left( \frac{1-r_{x}}{1+r_{x}} \right) \left[ 1 - e^{(-\mu-\kappa_{x})L} \right] + \left[ e^{-\mu L} - e^{-\kappa_{x} L} \right] \right] \frac{e^{\mu(L-x)}}{\kappa_{x}} \left( e^{\kappa_{x}(L-x)} - e^{\kappa_{x}(L+x)} \right) \]

\[
= \partial_{x} \ln \left[ K_{x}^{+}(r_{x}) e^{\kappa_{x} x} + K_{x}^{-}(r_{x}) e^{\kappa_{x} (L-x)} \right] = \frac{\kappa_{x}^{+}(r_{x}) e^{\kappa_{x} x} - \kappa_{x}^{-}(r_{x}) e^{\kappa_{x} (L-x)}}{K_{x}^{+}(r_{x}) e^{\kappa_{x} x} + K_{x}^{-}(r_{x}) e^{\kappa_{x} (L-x)}}
\]

(224)

where we have introduced the two constants

\[
K_{x}^{+}(r_{x}) = \frac{1-r_{x}}{1+r_{x}} \left[ e^{\mu L} - e^{-\kappa_{x} L} \right] - e^{-\kappa_{x} L} \sqrt{\frac{1-r_{x}}{1+r_{x}}} \left[ 1 - e^{(\mu-\kappa_{x})L} \right] + 1 - 2e^{(\mu-\kappa_{x})L} + e^{-2\kappa_{x} L}
\]

\[
K_{x}^{-}(r_{x}) = \frac{1+r_{x}}{1-r_{x}} \left[ 1 - e^{(\mu+\kappa_{x})L} \right] - e^{\kappa_{x} L} \sqrt{\frac{1-r_{x}}{1+r_{x}}} \left[ e^{\mu L} - e^{-\kappa_{x} L} \right] + e^{\kappa_{x} L} - 2e^{-\kappa_{x} L} + e^{(\mu-2\kappa_{x})L}
\]

(225)

even if it is only their ratio that determines the conditioned drift of equation (224).
7.4. Case $\mu > 0$: Conditioning towards the finite sum $\Sigma_{\infty}$ for the infinite horizon $T \to +\infty$

7.4.1. Unconditioned distribution $\Pi_{\infty}(\Sigma|x_0)$ of the sum $\Sigma$ for $t = +\infty$ when starting at $x_0$. For the transient case $\mu > 0$, the unconditioned distribution $\Pi_{\infty}(\Sigma|x_0)$ is given by equation (137)

$$\Pi_{\infty}(\Sigma|x_0) = \delta(\Sigma) S_{\infty}^{ab}(0,L)_{(x_0)} + \theta(\Sigma > 0) Y_{\infty}(\Sigma|x_0). \quad (226)$$

The forever-survival probability $S_{\infty}^{ab}(0,L)_{(x_0)}$ of equation (93)

$$S_{\infty}^{ab}(0,L)_{(x_0)} = 1 - \frac{G_{0}^{ab}(L)_{(0)}(0|x_0)}{G_{0}|x_0)_{(0)}^{ab}(L)_{(L)}} = 1 - e^{\mu(L-x_0)-L(x_0)} = \begin{cases} 0 & \text{for } x_0 \in ]-\infty,L[ \\ 1 - e^{-2\mu(x_0-L)} & \text{for } x_0 \in ]L,\infty[ \end{cases} \quad (227)$$

is non-vanishing only in the region $x_0 > L$, where the particle can escape towards $ (+\infty) $ without touching $L$. The regular contribution of equation (138)

$$Y_{\infty}(\Sigma|x_0) = \xi^{+}(x_0)e^{-\lambda^{+}\Sigma} + \xi^{-}(x_0)e^{-\lambda^{-}\Sigma} \quad (228)$$

with equation (210)

$$\lambda^{\pm}_{0} = \frac{\mu}{1 \mp e^{-\mu L}} \quad (229)$$

and equation (139)

$$\xi^{+}(x_0) = \frac{\mu e^{\mu(L-x_0)}}{2(1 - e^{-2\mu L})} \left[ e^{-\mu|x_0|} + e^{-\mu(L+|L-x_0|)} + e^{-\mu|L-x_0|} - e^{-\mu(L+|x_0|)} \right]$$

$$= \begin{cases} \mu e^{\mu(L-x_0)} \left[ -e^{\mu|x_0|} + e^{-\mu(L+|L-x_0|)} + e^{-\mu|L-x_0|} - e^{-\mu(L+|x_0|)} \right] \quad \text{for } x_0 \in ]-\infty,0[ \\ \mu e^{\mu(L-x_0)} \left[ -e^{\mu|x_0|} + e^{-\mu(L+|L-x_0|)} + e^{-\mu|L-x_0|} - e^{-\mu(L+|x_0|)} \right] \quad \text{for } x_0 \in ]0,L[ \\ \mu e^{\mu(L-x_0)} \left[ e^{-\mu|x_0|} - e^{-\mu(L+|x_0|)} \right] \quad \text{for } x_0 \in ]L,\infty[ \end{cases} \quad (230)$$

$$\xi^{-}(x_0) = \frac{\mu e^{\mu(L-x_0)}}{2(1 - e^{-2\mu L})} \left[ e^{-\mu|x_0|} - e^{-\mu(L+|L-x_0|)} + e^{-\mu|L-x_0|} - e^{-\mu(L+|x_0|)} \right]$$

$$= \begin{cases} \frac{\mu e^{\mu(L-x_0)}}{2(1 - e^{-2\mu L})} \left[ \mu e^{\mu|x_0|} - e^{-\mu(2L-x_0)} \right] \quad \text{for } x_0 \in ]-\infty,0[ \\ \frac{\mu e^{\mu(L-x_0)}}{2(1 - e^{-2\mu L})} \left[ \mu e^{\mu|x_0|} - e^{-\mu(2L-x_0)} + e^{-\mu(L-x_0)} - e^{-\mu(L+|x_0|)} \right] \quad \text{for } x_0 \in ]0,L[ \\ \frac{\mu e^{\mu(L-x_0)}}{2(1 - e^{-2\mu L})} \left[ e^{-\mu(x_0|L-L)} - e^{-\mu(L+|x_0|)} \right] \quad \text{for } x_0 \in ]L,\infty[ \end{cases} \quad (230)$$
In summary, the regular contribution of equation (228) reads in the three regions

\[
\mathcal{Y}_\infty(\Sigma|x_0) = \begin{cases} 
\frac{\mu e^{\mu L}}{2} \left( -e^{\lambda_\alpha^+ \Sigma} + e^{-\lambda_\alpha^- \Sigma} \right) & \text{for } x_0 \in ]-\infty,0[ \\
\frac{1-e^{\lambda_\alpha^- (x_0-L)}}{2} \lambda_\alpha^+ e^{-\lambda_\alpha^+ \Sigma} + \frac{1+e^{\lambda_\alpha^- (x_0-L)}}{2} \lambda_\alpha^- e^{-\lambda_\alpha^- \Sigma} & \text{for } x_0 \in [0,L[ . \\
\frac{\mu e^{-2\mu (x_0-L)}}{2} \left( e^{\lambda_\alpha^+ \Sigma} + e^{-\lambda_\alpha^- \Sigma} \right) & \text{for } x_0 \in ]L,+\infty[ 
\end{cases}
\]

(231)

7.4.2. Conditioned drift to obtain the finite sum \(\Sigma^*_\infty\) for the infinite horizon \(T \to +\infty\). If one takes directly the limit \(\tau = +\infty\) in the conditioned drift of equation (192), one obtains for \(\Sigma < \Sigma^*_\infty\) using equation (231)

\[
\mu^{[\Sigma^*_\infty]}(x,\Sigma < \Sigma^*_\infty) = \mu + \partial_1 \ln \mathcal{Y}_\infty(\Sigma^*_t - \Sigma|x)
\]

\[
= \begin{cases} 
\mu & \text{for } x \in ]-\infty,0[ \\
\mu + \partial_1 \ln \left[ \frac{1-e^{\mu(L-2\tau)}}{2} \lambda_\alpha^+ e^{-\lambda_\alpha^+ (\Sigma^*_\tau - \Sigma)} + \frac{1+e^{\mu(L-2\tau)}}{2} \lambda_\alpha^- e^{-\lambda_\alpha^- (\Sigma^*_\tau - \Sigma)} \right] & \text{for } x \in [0,L[ \\
-\mu & \text{for } x \in ]L,+\infty[ 
\end{cases}
\]

(232)

and for \(\Sigma = \Sigma^*_\infty\) using equation (227) in the region \(x \in ]L,+\infty[\)

\[
\mu^{[\Sigma^*_\infty]}(x,\Sigma = \Sigma^*_\infty) = \mu + \partial_1 \ln S^{[\Sigma^*_\infty]}_\infty(x) = \mu + \partial_1 \ln \left[ 1-e^{-2\mu(x-L)} \right] = \mu \coth[\mu(x-L)] \text{ for } x \in ]L,+\infty[ 
\]

(233)

while for the two other regions where \(S^{[\Sigma^*_\infty]}_\infty(x)\) vanishes in equation (227), one should use the asymptotic behavior of \(S^{[\Sigma^*_\infty]}_\infty(x)\) for large time \(\tau\) to obtain that the conditioned drift of equation (192) corresponds to the taboo process on \(]-\infty,0[\) and on the interval \([0,L[\) respectively. We postpone these calculations in the appendix D and state the results

\[
\mu^{[\Sigma^*_\infty]}(x,\Sigma = \Sigma^*_\infty) = \mu + \lim_{\tau \to +\infty} \partial_1 \ln S^{[\Sigma^*_\infty]}_\tau(x) = \begin{cases} 
\frac{1}{2} \cot \left( \frac{\pi x}{T} \right) & \text{for } x \in ]-\infty,0[ \\
\frac{1}{2} \cot \left( \frac{\pi x}{T} \right) & \text{for } x \in [0,L[ . 
\end{cases}
\]

(234)

Figure 3 shows some realizations of this conditioned process with a drift \(\mu = 1/2\).

7.5. Case \(\mu > 0\): Conditioning towards the finite local times \((A^\infty_\infty, B^\infty_\infty)\) for the infinite horizon \(T \to +\infty\)

7.5.1. Unconditioned distribution \(\Pi_\infty(A,B|x_0)\) of the two local times \((A,B)\) for \(t = +\infty\) when starting at \(x_0\). For the transient case \(\mu > 0\), the unconditioned distribution \(\Pi_\infty(A,B|x_0)\) of equation (92)

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Figure 3. Red lines: Examples of realizations of diffusions conditioned to have a finite sum of local times \( \Sigma_\infty^* = 2 \) at the infinite time horizon (see the conditioned drift of equations (232)-(234)). For each trajectory, the associated local time \( A(t) \) at \( x = 0 \) (blue dotted line), the associated local time \( B(t) \) at \( x = L = 1 \) (blue dashed line) as well as their sum \( \Sigma(t) = A(t) + B(t) \) (blue line) are shown as a function of the time \( t \). All processes start at \( x_0 = -0.5 \), the constant drift \( \mu \) is equal to 0.5 and the time step used in the discretization is \( dt = 10^{-4} \). Once the desired sum of local times \( \Sigma_\infty^* = 2 \) is reached, there are three possibilities (i) Top figure: the process lives in the region \( x \in [L = 1, +\infty[ \) (ii) Middle figure: the process lives in the interval \( x \in ]0, L = 1[ \) (iii) Bottom figure: the process lives in the region \( x \in ]-\infty, 0[ \).

\[
\Pi_\infty(A, B|x_0) = \delta(A)\delta(B)S_\infty^{abs([0,L]}(x_0) + \theta(A > 0)\delta(B)\mathcal{A}_\infty^{[-\infty, L]}(A|x_0) + \delta(A)\theta(B > 0)\mathcal{B}_\infty^{[0, +\infty]}(B|x_0) + \theta(A > 0)\theta(B > 0)\mathcal{C}_\infty(A, B|x_0)
\]

(235)

involves the four contributions:

1. The first contribution \( S_\infty^{abs([0,L]}(x_0) \) has been given in equation (227).
2. The second contribution \( \mathcal{A}_\infty^{[-\infty, L]}(A|x_0) \) of equation (94) vanishes
\[ A_{\infty}^{[-L]}(A|x_0) = 0. \]  

(3) The third contribution \( B_{\infty}^{[0, +\infty]}(B|x_0) \) of equation (95) reads

\[
B_{\infty}^{[0, +\infty]}(B|x_0) = e^{\mu(L-x_0)} \left[ e^{-\mu|L-x_0|} - e^{-\mu(L+|x_0|)} \right] \frac{\mu}{1 - e^{-2\mu L}} e^{-\frac{\mu}{1 - e^{-2\mu L}} B} 
\]

\[
= \begin{cases} 
0 & \text{for } x_0 \in ]-\infty, 0[ \\
\left[ 1 - e^{-2\mu x_0} \right] \frac{\mu}{1 - e^{-2\mu L}} e^{-\frac{\mu}{1 - e^{-2\mu L}} B} & \text{for } x_0 \in ]0, L[ \\
e^{-2\mu(x_0 - L)} e^{-\frac{\mu}{1 - e^{-2\mu L}} B} & \text{for } x_0 \in ]L, +\infty] 
\end{cases}
\]

with the normalization over \( B \in ]0, +\infty[ \)

\[
\int_{0}^{+\infty} dB B_{\infty}^{[0, +\infty]}(B|x_0) = \begin{cases} 
0 & \text{for } x_0 \in ]-\infty, 0[ \\
1 - e^{-2\mu x_0} & \text{for } x_0 \in ]0, L[ \\
e^{-2\mu(x_0 - L)} & \text{for } x_0 \in ]L, +\infty[ 
\end{cases}
\]

(4) The fourth contribution of equation (97) reads

\[
C_{\infty}(A, B|x_0) = \frac{\mu^2}{(1 - e^{-2\mu L})^2} e^{-\frac{\mu}{1 - e^{-2\mu L}} (A+B)} e^{-\mu x_0} 
\]

\[
\times \left[ e^{-\mu|a|} - e^{-\mu(L+|a|)} \right] I_0 \left( \frac{\mu}{\sinh(\mu L)} \sqrt{AB} \right) + \left( e^{-\mu|L-x_0|} - e^{-\mu(L+|x_0|)} \right) \times \frac{\sqrt{B}}{\sqrt{A}} I_1 \left( \frac{\mu}{\sinh(\mu L)} \sqrt{AB} \right) 
\]

\[
= \begin{cases} 
\mu^2 e^{-\frac{\mu}{1 - e^{-2\mu L}} (a+b)} I_0 \left( \frac{\mu}{\sinh(\mu L)} \sqrt{AB} \right) & \text{for } x_0 \in ]-\infty, 0[ \\
\mu^2 e^{-\frac{\mu}{1 - e^{-2\mu L}} (a+b)} \left[ e^{-2\mu x_0} e^{-2\mu L} I_0 \left( \frac{\mu}{\sinh(\mu L)} \sqrt{AB} \right) \\
+ e^{-\mu L} \left( 1 - e^{-2\mu x_0} \right) \frac{\mu^2}{\sinh(\mu L)} \sqrt{AB} \right] & \text{for } x_0 \in ]0, L[ \\
\mu^2 e^{-\frac{\mu}{1 - e^{-2\mu L}} (a+b)} e^{\mu L-2\mu x_0} \frac{\mu^2}{\sinh(\mu L)} \sqrt{AB} & \text{for } x_0 \in ]L, +\infty[ 
\end{cases}
\]
If one takes directly the limit some realizations of this conditioned process with a drift asymptotic behavior of \( T \rightarrow \infty \).

For \( \mu \) and for \( \mu \), one obtains for \( \tau(\infty) \equiv 0 \) using equation (239)

\[
\mu_{\infty}^{[A_n^\infty,B_n^\infty]}(x,A < A_n^\infty,B < B_n^\infty) = \mu + \partial_t \ln C_{\infty}(A_n^\infty - A, B_n^\infty - B) + \mu \] \[
= \begin{cases}
\mu & \text{for } x \in ]-\infty,0[ \\
\sqrt{B_n^\infty - 2 \cosh(\mu x)} \frac{\sinh(\mu x)}{2 \cosh(\mu x)} - \sqrt{A_n^\infty - 2 \cosh(\mu (L-x))} \frac{\sinh(\mu (L-x))}{2 \cosh(\mu (L-x))} & \text{for } x \in [0,L[ \\
\mu - \mu \ln \left[ 1 - e^{-2\mu x} \right] = \mu \coth(\mu x) & \text{for } x \in [L, +\infty[ \\
\mu - \mu \coth(\mu (L-x)) & \text{for } x \in ]-\infty,0[ \\
\end{cases}
\] (242)

For \( A < A_n^\infty, B = B_n^\infty \) where \( A_{-\infty,L}^\infty(A|x) \) vanished in equation (236), one should use the asymptotic behavior of \( A_{-\infty,L}^\infty(A|x) \) for large time \( \tau \) to obtain that the conditioned drift has a behavior similar to the previous case, as expected

\[
\mu_{\infty}^{[A_n^\infty,B_n^\infty]}(x,A < A_n^\infty,B < B_n^\infty) = \mu + \lim_{\tau \rightarrow +\infty} \partial_t \ln A_{-\infty,L}^\infty(A_n^\infty - A|x) \] \[
= \begin{cases}
\mu & \text{for } x \in ]-\infty,0[ \\
-\mu \coth(\mu (L-x)) & \text{for } x \in [0,L[ \\
\end{cases}
\] (243)

At last, for \( A = A_n^\infty, B = B_n^\infty \), the conditioned drift involving the survival probability \( S_{L\tau}(0,L) \) has already been discussed above in equations (233) and (234). Figure 4 shows some realizations of this conditioned process with a drift \( \mu = 1/2 \).
Figure 4. Red lines: Examples of realizations of diffusions conditioned to have finite local times \( A^\infty_\infty = 3, B^\infty_\infty = 1 \) at the infinite time horizon (see the conditioned drift of equations (241)–(243)). For each trajectory, the associated local time \( A_t \) at \( x = 0 \) (blue line) and the associated local time \( B_t \) at \( x = L = 1 \) (blue dashed line) are shown as a function of the time \( t \). All processes start at \( x_0 = 2 \), the constant drift \( \mu \) is equal to 0.5 and the time step used in the discretization is \( dt = 10^{-4} \). Once both the desired local times \( A^\infty_\infty = 3 \) and \( B^\infty_\infty = 1 \) are reached, there are three possibilities (i) Top figure: the process lives in the region \( x \in [L = 1, +\infty[ \) (ii) Middle figure: the process lives in the interval \( x \in ]0, L = 1[ \) (iii) Bottom figure: the process lives in the region \( x \in ]-\infty, 0[ \).

8. Conclusions

In this paper, we have considered a diffusion process \( X(t) \) of drift \( \mu(x) \) and of diffusion coefficient \( D = 1/2 \) in order to study the joint statistics of the two local times \( A(t) = \int_0^t d\tau \delta(X(\tau)) \) and \( B(t) = \int_0^t d\tau \delta(X(\tau) - L) \) at positions \( x = 0 \) and \( x = L \), as well as the simpler statistics of their sum \( \Sigma(t) = A(t) + B(t) \). We have discussed the asymptotic behavior for large time \( t \to +\infty \): (i) when the diffusion process \( X(t) \) is transient, the two local times \( [A(t); B(t)] \)
remain finite random variables \([A^*(\infty), B^*(\infty)]\) and we have analyzed their limiting joint distribution; (ii) when the diffusion process \(X(t)\) is recurrent, we have described the large deviations properties of the two intensive local times \(a = \frac{A(t)}{t}\) and \(b = \frac{B(t)}{t}\) and of the intensive sum \(\sigma = \frac{\Sigma(t)}{t} = a + b\). We have then used these properties to construct various conditioned processes \([X^*(t), A^*(t), B^*(t)]\) satisfying certain constraints involving the two local times, thereby generalizing our previous work \([106]\) concerning the conditioning with respect to the single local time \(A(t)\). In particular for the infinite time horizon \(T \to +\infty\), we have considered the conditioning towards the finite asymptotic values \([A^*(\infty), B^*(\infty)]\) or \(\Sigma^*(\infty)\), as well as the conditioning towards the intensive values \([a^*, b^*]\) or \(\sigma^*\), that we have compared in appendix A with the appropriate ‘canonical conditioning’ based on the generating function of the local times in the regime of large deviations. Finally, we have applied this general construction to the simplest case where the unconditioned diffusion is the Brownian motion of uniform drift \(\mu\), while the comparison with the canonical conditioning is described in appendix B.

Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

Appendix A. Canonical conditioned process \(X^*_{\mu,q}(t)\) of parameters \((p, q)\)

In this appendix, we describe the properties of the canonical conditioning of parameters \((p, q)\), in order to compare with the other conditioned processes described in section 6 of the main text.

A.1. Canonical conditioned process \(X^*_{\mu,q}(t)\) of parameters \((p, q)\) based on the Laplace transform \(\tilde{P}_{t,\mu,q}(x|x_0)\)

The canonical conditioning is based on the Laplace transform \(\tilde{P}_{t,\mu,q}(x|x_0)\) of equation (22) where the Laplace parameters \((p, q)\) conjugated to the two local times \(A\) and \(B\) are fixed. For the bridge conditioned to end at the position \(x_T^*\) at the time horizon \(T\), the conditioned distribution for the position \(x\) at an interior time \(t \in [0, T]\) reads

\[
P_T^{x^*_{\mu,q}}(x,t) = \frac{\tilde{P}_{t-t,\mu,q}(x_T^*|x)\tilde{P}_{t,\mu,q}(x|x_0)}{\tilde{P}_{t,\mu,q}(x_T^*|x_0)}. \tag{A1}
\]

The corresponding Ito dynamics for the conditioned process \(X^*_{\mu,q}(t)\) of parameters \((p, q)\)

\[
dX^*_{\mu,q}(t) = \mu^*_{\mu,q}(X^*_{\mu,q}(t),t)dt + dB(t) \tag{A2}
\]

involves the conditioned drift

\[
\mu^*_{\mu,q}(x,t) = \mu(x) + \partial_x \ln \tilde{P}_{t-t,\mu,q}(x_T^*|x). \tag{A3}
\]

A.2. Properties of the \((p, q)\)-deformed propagator \(\tilde{P}_{t,\mu,q}(x|x_0)\)

The forward dynamics of the \((p, q)\)-deformed propagator \(\tilde{P}_{t,\mu,q}(x|x_0)\) is given by the Feynman–Kac formula of equation (23)
\[
\frac{\partial}{\partial t} \tilde{P}_{t,p,q}(x|x_0) = -p \delta(x) \tilde{P}_{t,p,q}(x|x_0) - q \delta(x - L) \tilde{P}_{t,p,q}(x|x_0) \\
- \partial_x [\mu(x) \tilde{P}_{t,p,q}(x|x_0)] + \frac{1}{2} \partial^2_x \tilde{P}_{t,p,q}(x|x_0).
\] (A4)

A.2.1. Similarity transformation towards an hermitian quantum Hamiltonian \(H_{p,q}\). The potential \(U(x)\) defined via the following integration of the drift \(\mu(y)\)

\[
U(x) \equiv -2 \int_0^x dy \mu(y)
\] (A5)
can be used to make the similarity transformation

\[
\tilde{P}_{t,p,q}(x|x_0) = e^{-\frac{U(x)}{2}} \psi_{t,p,q}(x|x_0) e^{\frac{U(x_0)}{2}} = e^{\int_0^x dy \mu(y)} \psi_{t,p,q}(x|x_0)
\] (A6)
in order to transform the dynamics of equation (A4) for \(\tilde{P}_{t,p,q}(x|x_0)\) into the Euclidean Schrödinger equation for \(\psi_{t,p,q}(x|x_0)\)

\[
-\partial_t \psi_{t,p,q}(x|x_0) = \tilde{H}_{p,q} \psi_{t,p,q}(x|x_0)
\] (A7)
where the hermitian quantum Hamiltonian \(H_{p,q}\) involves two additional delta impurities of amplitudes \(p\) and \(q\) at positions \(x = 0\) and \(x = L\) respectively

\[
H_{p,q} = H_{0,0} + p \delta(x) + q \delta(x - L)
\] (A8)
with respect to the supersymmetric Hamiltonian

\[
H_{0,0} = \frac{1}{2} (\partial_x + \mu(x)) (-\partial_x + \mu(x)) = -\frac{1}{2} \partial^2_x + V(x)
\] (A9)
that involves the quantum potential

\[
V(x) \equiv \frac{\mu^2(x)}{2} + \frac{\mu'(x)}{2}.
\] (A10)

A.2.2. Physical meaning of the normalizable zero-energy ground state of the supersymmetric Hamiltonian \(H_{0,0}\) when it exists. Let us recall the well-known discussion:

(i) If the following integral involving the potential \(U(x)\) of equation (A5) converges

\[
\int_{-\infty}^{+\infty} dx e^{-U(x)} < +\infty
\] (A11)
then the quantum Hamiltonian \(H_{0,0}\) has the following normalizable ground state at zero-energy \(E = 0\)

\[
\phi_{0,0}^{GS}(x) = \frac{e^{-\frac{U(x)}{2}}}{\sqrt{\int_{-\infty}^{+\infty} dx e^{-U(x)}}}
\] (A12)
The propagator \(G_t(x|x_0) = \tilde{P}_{t,p=0,q=0}(x|x_0)\) obtained from the similarity transformation of equation (A6)
\[ G(x,t|x_0,t_0) = e^{-\frac{U(x)}{2} \psi_{t,p=0,q=0}(x|x_0) e^{\frac{U(x_0)}{2}}_{(t-t_0) \rightarrow +\infty}} e^{-\frac{U(x)}{2} \phi_{0,0}^{GS}(x) \phi_{0,0}^{GS}(x_0) e^{\frac{U(x)}{2}}} \]

converges towards the Boltzmann equilibrium \( G_{eq}(x) \) in the potential \( U(x) \).

(ii) If the integral of equation (A11) diverges

\[ \int_{-\infty}^{+\infty} dx e^{-U(x)} = +\infty \] (A14)

then the quantum Hamiltonian \( H_{0,0} \) has no bound state, and the process \( X(t) \) does not converge towards an equilibrium, but it can be either transient or recurrent.

### A.2.3. Propagator \( \hat{P}_{t,p,q}(x|x_0) \) for large time \( t \) when the quantum Hamiltonian \( H_{p,q} \) has a normalizable ground-state \( \phi_{p,q}^{GS}(x) \) of energy \( E_{p,q} \)

When the quantum Hamiltonian \( H_{p,q} \) has a normalizable ground-state \( \phi_{p,q}^{GS}(x) \) of energy \( E_{p,q} \)

\[ H_{p,q} \phi_{p,q}^{GS}(x) = E_{p,q} \phi_{p,q}^{GS}(x) \] (A15)

the ground state can be chosen real and positive \( \phi_{p,q}^{GS}(x) \geq 0 \) with the normalization

\[ \langle \phi_{p,q}^{GS} | \phi_{p,q}^{GS} \rangle = \int_{-\infty}^{+\infty} dx \left[ \phi_{p,q}^{GS}(x) \right]^2 = 1. \] (A16)

This ground-state \( \phi_{p,q}^{GS}(x) \) and its energy \( E_{p,q} \) determine the leading asymptotic behavior of the quantum propagator

\[ \psi_{t,p,q}(x|x_0) \overset{t \rightarrow +\infty}{\sim} e^{-i E_{p,q} t} \phi_{p,q}^{GS}(x) \phi_{p,q}^{GS}(x_0). \] (A17)

The corresponding asymptotic behavior of the propagator \( \hat{P}_{t,p,q}(x|x_0) \) is then given by the similarity transformation of equation (A6)

\[ \hat{P}_{t,p,q}(x|x_0) = e^{-\frac{U(x)}{2} \psi_{t,p,q}(x|x_0) e^{\frac{U(x_0)}{2}}_{(t-t_0) \rightarrow +\infty}} e^{-\frac{i U(x)}{2} \phi_{p,q}^{GS}(x) \phi_{p,q}^{GS}(x_0)} \] (A18)

For the further time Laplace transform \( \hat{P}_{s,p,q}(x|x_0) \) of equation (25), the asymptotic behavior of equation (A18) for large \( t \) means that \( \hat{P}_{s,p,q}(x|x_0) \) exists for \( s \in ]-E_{p,q}, +\infty[ \) with the following pole singularity for \( s \rightarrow (-E_{p,q})^+ \)

\[ \hat{P}_{s,p,q}(x|x_0) = \int_{0}^{+\infty} dt e^{-\frac{U(x)}{2} \psi_{t,p,q}(x|x_0) e^{\frac{U(x_0)}{2}}_{(t-t_0) \rightarrow +\infty}} e^{-\frac{i U(x)}{2} \phi_{p,q}^{GS}(x) \phi_{p,q}^{GS}(x_0)} \] (A19)

The comparison with the denominator of \( \hat{P}_{s,p,q}(x|x_0) \) in equation (31) shows that \( s = -E_{p,q} \) can be found as the solution of

\[ 0 = 1 + p \hat{G}_s(0|0) + q \hat{G}_s(L|L) + pq \left[ \hat{G}_s(0|0) \hat{G}_s(L|L) - \hat{G}_s(0|0) \hat{G}_s(L|L) \right] \] (A20)
A.3. Canonical conditioning for large horizon $T$ when the Hamiltonian $H_{p,q}$ has a normalizable ground-state

When the quantum Hamiltonian $H_{p,q}$ has a normalizable ground-state $\phi^{GS}_{p,q}(x)$, the asymptotic behavior of equation (A18) can be plugged into the three propagators of equation (A1) to obtain that the conditioned density at any interior time $0 \ll t \ll T$

$$p^{[\tau^{+},p,q]}_{T}(x,t) \sim \begin{cases} e^{-E_{p,q}(T-t)} & \text{if } 0 \ll t \ll T \\ e^{-E_{p,q}t} & \text{if } t \ll T \end{cases}$$

$$= \left[\phi^{GS}_{p,q}(x)\right]^{2} \equiv p^{*}_{p,q}(x) \quad \text{(A21)}$$

does not depend on the interior time $t$ anymore.

The corresponding conditioned drift of equation (A3) is also independent of the interior time $t$ and reduces to

$$\mu_{T}^{[\tau^{+},p,q]}(x,t) \sim \begin{cases} \mu(x) & \text{if } 0 \ll t \ll T \\ \partial_{t} \ln \left(e^{-E_{p,q}(T-t)} \left[\phi^{GS}_{p,q}(x_T) \phi^{GS}_{p,q}(x)\right] e^{E_{p,q}t}\right) & \text{if } t \ll T \end{cases}$$

$$= \partial_{t} \ln \left[\phi^{GS}_{p,q}(x)\right] \equiv \mu^{*}_{p,q}(x) \quad \text{(A22)}$$

where we have used the derivative $U'(x) = -2\mu(x)$ of the potential $U(x)$ of equation (A5).

**A.3.1. Physical meaning of this canonical conditioning when the unconditioned process $X(t)$ is recurrent.** When the unconditioned process $X(t)$ is recurrent, the ground-state energy $E_{p,q}$ of the Hamiltonian $H_{p,q}$ that governs the leading exponential behavior of equation (A18)

$$P_{T,p,q}(x|\chi_0) \sim \frac{1}{\sqrt{2\pi TE_{p,q}}} e^{-TE_{p,q}} \quad \text{(A23)}$$

is directly related to the rate function $I(a,b)$ that governs the large deviations properties of the intensive local times $(a,b)$ of equation (12) : indeed, the saddle-point evaluation of the generating function of the two intensive local times $(a,b)$

$$\langle e^{-pTa-qTb}\rangle_{\chi_0} = \int_{0}^{+\infty} \int_{0}^{+\infty} db \int_{0}^{+\infty} da e^{-pTa-qTb} \Pi(ta, tb|\chi_0)$$

$$\sim \int_{0}^{+\infty} \int_{0}^{+\infty} db \int_{0}^{+\infty} da e^{-Tpa+qTb+I(a,b)} \quad \text{(A24)}$$

yields that the ground-state energy $E_{p,q}$ corresponds to the two-dimensional Legendre transform of the rate function $I(a,b)$

$$pa + qb + I(a,b) = E_{p,q}$$

$$p + \partial_{a}I(a,b) = 0 \quad \text{(A25)}$$

$$q + \partial_{b}I(a,b) = 0$$
while the reciprocal Legendre transform reads
\[
I(a, b) = E_{p,q} - pa - qb
\]
\[
a = \partial_p E_{p,q}
\]
\[
b = \partial_q E_{p,q}.
\] (A26)

As a consequence, the canonical conditioning of parameters \((p, q)\) can be considered as asymptotically equivalent for large \(T\) to the microcanonical conditioning towards the two intensive local times \(a_{p,q}^* = \partial_p E_{p,q}\) and \(b_{p,q}^* = \partial_q E_{p,q}\) corresponding to the Legendre values of equation (A26).

These two relations have a very simple interpretation via the first-order perturbation theory for the energy \(E_{p,q}\) of the ground state \(\phi_{p,q}^{GS}(x)\) in quantum mechanics when the parameter \(p\) is changed into \((p + \epsilon)\) or when the parameter \(q\) is changed into \((q + \eta)\)
\[
A_{p,q}^* = \partial_p E_{p,q} = \lim_{\epsilon \to 0} \left( \frac{E_{p+\epsilon, q} - E_{p,q}}{\epsilon} \right) = \langle \phi_{p,q}^{GS} | \delta(x) | \phi_{p,q}^{GS} \rangle = [\phi_{p,q}^{GS}(x = 0)]^2 = P_{p,q}^*(x = 0)
\]
\[
B_{p,q}^* = \partial_q E_{p,q} = \lim_{\eta \to 0} \left( \frac{E_{p,q+\eta} - E_{p,q}}{\eta} \right) = \langle \phi_{p,q}^{GS} | \delta(x - L) | \phi_{p,q}^{GS} \rangle = [\phi_{p,q}^{GS}(x = L)]^2 = P_{p,q}^*(x = L).
\] (A27)

A.3.2. Emergence of a normalizable ground-state for \(H_{p,q}\) when \(H_{p=0,q=0}\) has no ground state. When the quantum Hamiltonian \(H_{p=0,q=0}\) has no bound state, the Hamiltonian \(H_{p,q}\) of equation (A8) can nevertheless have a normalizable bound state. Indeed, for the Laplace transform \(\tilde{P}_{s,p,q}(x|x_0)\) of equation (25), the result of equation (31) shows that a new singularity can appear in \(\tilde{P}_{s,p,q}(x|x_0)\) with respect to \(\tilde{G}_s(x|x_0)\) when the variable \(s = -E_{p,q}\) makes equation (A20) vanish
\[
0 = 1 + p\tilde{G}_s(0|0) + q\tilde{G}_s(L|L) + pq \left[ \tilde{G}_s(0|0)\tilde{G}_s(L|L) - \tilde{G}_s(L|0)\tilde{G}_s(0|L) \right]
\]
\[
= \left(1 + p\tilde{G}_s(0|0)\right) \left(1 + q\tilde{G}_s(L|L)\right) - pq\tilde{G}_s(0|L)\tilde{G}_s(L|0).
\] (A28)

Let us mention the two limiting cases for the distance \(L\) between the two delta impurities:

(i) If \(L = 0\), equation (A28) becomes
\[
0 = 1 + (p + q)\tilde{G}_s(0|0)
\] (A29)
and corresponds to the case of a single delta impurity of amplitude \((p + q)\) at the origin.
So when \(H_{0,0}\) has no bound state, a normalizable ground state will emerge for \(H_{p,q}\) if the global amplitude is strictly negative \((p + q) < 0\).

(ii) If \(L \to +\infty\), where it is expected that the vanishing limit \(\tilde{G}_s(0|L)\tilde{G}_s(L|0) \to 0\), equation (A28) becomes
\[
0 = \left(1 + p\tilde{G}_s(0|0)\right) \left(1 + q\tilde{G}_s(L|L)\right)
\] (A30)
and corresponds to two independent delta impurities of amplitude \( p \) and of amplitude \( q \) that become separated by the distance \( L \to +\infty \). Therefore, when \( H_{0,0} \) has no bound state, a normalizable ground state state will emerge for \( H_{p,q} \) if at least one of the two amplitudes \( p \) or \( q \) is strictly negative. If the two amplitudes are strictly negative \( p < 0 \) and \( q < 0 \), there will be two bound states, so the ground state will correspond to the lower energy.

Let us now discuss three special cases for the parameters \((p, q)\):

(a) In the limit \( q \to +\infty \) that amounts to impose the vanishing local time \( B = 0 \) at position \( x = L \), equation (A28) for for \( s = -E_{p,q,+\infty} \) becomes

\[
-p = \frac{1}{\hat{G}_s(0|0) - \frac{G_s(0)L|G_s(L|0)}{G_s(L|L)}} = \frac{1}{\hat{G}_s^{abs}(L|0|0)}
\]  
(A31)

in agreement with the pole of equation (37).

(b) In the limit \( p \to +\infty \) that amounts to impose the vanishing local time \( A = 0 \) at position \( x = L \), equation (A28) becomes for \( s = -E_{p,+\infty,q} \)

\[
-q = \frac{1}{\hat{G}_s(L|L) - \frac{G_s(L|0)G_s(0|0)}{G_s(0|0)}} = \frac{1}{\hat{G}_s^{abs}(0|L|L)}
\]  
(A32)

in agreement with the pole of equation (44).

(c) In the special case of equal amplitudes \( p = q \), equation (A28) becomes

\[
0 = \left(1 + p\hat{G}_s(0|0)\right)\left(1 + p\hat{G}_s(L|L)\right) - p^2\hat{G}_s(0|L)\hat{G}_s(L|0) = 1 + p\left[\hat{G}_s(0|0) + \hat{G}_s(L|L)\right] + p^2\Delta_s
\]  
(A33)

in agreement with equation (124) discussed during the analysis of the sum \( \Sigma = A + B \). The two roots \( p^{\pm} = -\lambda_s^{\pm} \) correspond to two bound states, while the energy \( E_{p,p} = -s \) of the ground state corresponds to \( p = -\lambda_s^{-} \) that governs the rate function \( \bar{J}(\sigma) \) as discussed in equation (146) of the main text.

**Appendix B. Canonical conditioning of parameters \((p, q)\) for the case of uniform drift \( \mu(x) = \mu \)**

In this appendix, the canonical conditioning described in the previous appendix is applied to the case of uniform drift \( \mu(x) = \mu \), in order to compare with the other conditioned processes described in section 7 of the main text.

The quantum Hamiltonian of equation (A8) reduces to

\[
H_{p,q} = -\frac{1}{2}\partial_x^2 + \frac{\mu^2}{2} + p\delta(x) + q\delta(x - L).
\]  
(B1)

The Hamiltonian \( H_{p=0,q=0} \) has no bound state, but we will be interested in the regions of parameters \((p, q)\) where \( H_{p,q} \) has a normalizable ground-state wave function \( \phi_{p,q}^{GS}(x) \) (see section A.3.2) in order to apply the framework described in section A.3.
B.1. Analysis the energy $E_{p,q}$ of the normalizable ground state of $H_{p,q}$

Using $G_s(x,0)$ of equations (204) and (A28) for $s = -E_{p,q}$ reads

$$0 = \left(1 + pG_s(0,0)\right)\left(1 + qG_s(L,L)\right) - pqG_s(0,L)G_s(L,0)$$

$$= \left(1 + \frac{p}{\kappa_s}\right)\left(1 + \frac{q}{\kappa_s}\right) - pq\frac{e^{-2\kappa_sL}}{\kappa_s^2}$$  \hspace{1cm} (B2)

in terms of the parameter of equation (205)

$$\kappa_s \equiv \sqrt{\mu^2 + 2s} = \sqrt{\mu^2 - 2E_{p,q}}$$ \hspace{1cm} (B3)

that now parametrizes the ground-state energy

$$E_{p,q} = -s = \frac{\mu^2 - \kappa_s^2}{2}. \hspace{1cm} (B4)$$

In summary, once the three parameters $(p, q, L)$ are given, there will be a normalizable ground state for $H_{p,q}$ equation (B2) that can be rewritten as

$$0 = \left(1 + \frac{\kappa}{p}\right)\left(1 + \frac{\kappa}{q}\right) - e^{-2\kappa L} \hspace{1cm} (B5)$$

that has a positive solution $\kappa_{p,q} > 0$.

B.2. Analysis of the normalizable ground state wave function $\phi_{p,q}^{GS}(x)$ of $H_{p,q}$

The eigenvalue equation for the ground state wave function $\phi_{p,q}^{GS}(x)$ of energy $E_{p,q}$

$$E_{p,q}\phi_{p,q}^{GS}(x) = H_{p,q}\phi_{p,q}^{GS}(x)$$

$$= -\frac{1}{2}\phi_{p,q}^{GS}(x) + \frac{\mu^2}{2}\phi_{p,q}^{GS}(x) + p\phi_{p,q}^{GS}(0) + q\phi_{p,q}^{GS}(L)$$ \hspace{1cm} (B6)

can be rewritten using the parametrization $E_{p,q} = \frac{\mu^2 - \kappa_s^2}{2}$ of equation (B4) as

$$0 = -\partial_x^2\phi_{p,q}^{GS}(x) + \kappa_{p,q}^2\phi_{p,q}^{GS}(x) + 2p\delta(x)\phi_{p,q}^{GS}(0) + 2q\delta(x - L)\phi_{p,q}^{GS}(L). \hspace{1cm} (B7)$$

The continuous solution can be decomposed into the three following regions:

(i) For the middle region $x \in [0, L]$, the wave function can be written as the linear combination involving two constants $(K^+, K^-)$

$$\phi_{p,q}^{GS}(x) = K^+ e^{\kappa_{p,q}x} + K^- e^{\kappa_{p,q}(L-x)} \hspace{1cm} \text{for} \hspace{1cm} x \in [0, L]. \hspace{1cm} (B8)$$

(ii) For the left region $x \in (-\infty, 0]$, the wave function should be normalizable at $x \to (-\infty)$ and should be continuous with equation (B8) at $x = 0$

$$\phi_{p,q}^{GS}(x) = \phi_{p,q}^{GS}(0)e^{\kappa_{p,q}x} = (K^+ + K^- e^{\kappa_{p,q}L})e^{\kappa_{p,q}x} \hspace{1cm} \text{for} \hspace{1cm} x \in (-\infty, 0]. \hspace{1cm} (B9)$$

(iii) For the right region $x \in [L, +\infty]$, the wave function should be normalizable at $x \to (+\infty)$ and should be continuous with equation (B8) at $x = L$

$$\phi_{p,q}^{GS}(x) = \phi_{p,q}^{GS}(L)e^{-\kappa_{p,q}(x-L)} = (K^+ e^{\kappa_{p,q}L} + K^-)e^{-\kappa_{p,q}(x-L)} \hspace{1cm} \text{for} \hspace{1cm} x \in [L, +\infty]. \hspace{1cm} (B10)$$
Now one needs to take into account the delta functions of equation (B7) that impose the following discontinuities of the derivative of the wave function at \( x = 0 \) and at \( x = L \):

\[
2\phi_{p,q}^{GS}(0) = \left. \frac{d\phi_{p,q}^{GS}(x)}{dx} \right|_{x=0+} - \left. \frac{d\phi_{p,q}^{GS}(x)}{dx} \right|_{x=0-} = \kappa_{p,q}(K^+ - K^- e^{\kappa_{p,q}L}) - \kappa_{p,q}(K^+ + K^- e^{\kappa_{p,q}L})
\]

\[
2q\phi_{p,q}^{GS}(L) = \left. \frac{d\phi_{p,q}^{GS}(x)}{dx} \right|_{x=L-} - \left. \frac{d\phi_{p,q}^{GS}(x)}{dx} \right|_{x=L+} = -\kappa_{p,q}(K^+ e^{\kappa_{p,q}L} + K^-) - \kappa_{p,q}(K^+ e^{\kappa_{p,q}L} - K^-)
\]

leading to the following homogeneous system for the two constants \((K^+, K^-)\):

\[
0 = p(K^+ + K^- e^{\kappa_{p,q}L}) + \kappa_{p,q}K^- e^{\kappa_{p,q}L} = pK^+ + (p + \kappa_{p,q})e^{\kappa_{p,q}L}K^-
\]

\[
0 = q(K^+ e^{\kappa_{p,q}L} + K^-) + \kappa_{p,q}K^+ e^{\kappa_{p,q}L} = (q + \kappa_{p,q})e^{\kappa_{p,q}L}K^+ + qK^-.
\]

In order to have a non-vanishing solution, the determinant of this system should vanish

\[
0 = \left| \begin{array}{c}
\frac{p}{q} (p + \kappa_{p,q}) e^{\kappa_{p,q}L} \\
(p + \kappa_{p,q}) e^{\kappa_{p,q}L}
\end{array} \right| = pq - (p + \kappa_{p,q})(q + \kappa_{p,q}) e^{2\kappa_{p,q}L}
\]

i.e. one recovers equation (B5) that determines the energy \( E_{p,q} \), as it should for consistency. Therefore, equation (B12) allows to compute the ratio

\[
\frac{K^-}{K^+} = \frac{e^{-\kappa_{p,q}L}}{1 + \frac{\kappa_{p,q}}{p}} = -\left( 1 + \frac{\kappa_{p,q}}{q} \right) e^{\kappa_{p,q}L}.
\]

Finally, the global normalization of the wave function \( \phi_{p,q}^{GS}(x) \)

\[
1 = \int_{-\infty}^{+\infty} dx [\phi_{p,q}^{GS}(x)]^2
\]

can be computed, but is not needed to evaluate the conditioned drift below.

### B.3. Analysis of the conditioned drift \( \mu^\ast_{p,q}(x) \)

The conditioned drift of equation (A22) can be computed from the wave function \( \phi_{p,q}^{GS}(x) \) given by equations (B8)–(B10) in the three regions

\[
\mu^\ast_{p,q}(x) = \partial_x \ln [\phi_{p,q}^{GS}(x)] = \begin{cases}
\kappa_{p,q} & \text{for } x \in ]-\infty,0[ \\
\kappa_{p,q} \left[ e^{\kappa_{p,q}L} + K^- e^{-\kappa_{p,q}L} \right] & \text{for } x \in ]0,L[ \\
-\kappa_{p,q} & \text{for } x \in ]L,\infty[ 
\end{cases}
\]

that should be compared to equations (222)–(224) of the main text.

#### B.3.1. Special case \( q \to +\infty \) for \( p \in ]-\infty,0[ \) corresponding to corresponding to the conditioning towards \((a_\ast, b_\ast = 0)\). In the limit \( q \to +\infty \) for \( p \in ]-\infty,0[ \), the ground-state \( \phi_{p,q=\infty}^{GS}(x) \) should vanish at \( x = L \) and thus in the whole region \( x \geq L \) as a consequence of equation (B10), while the ratio of equation (B14) reduces to

\[
\frac{K^-}{K^+} \underset{q \to +\infty}{\sim} -e^{\kappa_{p,q}L}.
\]

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where \( \kappa_{p,\infty} > 0 \) is the solution of equation (B5)
\[
0 = \left(1 + \frac{\kappa_{p,\infty}}{p}\right) - e^{-2\kappa_{p,\infty}L}. \tag{B18}
\]

The conditioned drift of equation (B16) reduces to
\[
\mu^*_{p,q=+\infty}(x) = \begin{cases} 
\kappa_{p,\infty} e^{-2\kappa_{p,\infty}x} + e^{-\kappa_{p,\infty}(x-\frac{L}{2})} - \kappa_{p,\infty} e^{-\kappa_{p,\infty}(x+\frac{L}{2})} & \text{for } x \in (-\infty,0]\n\kappa_{p,\infty} e^{-2\kappa_{p,\infty}x} + e^{-\kappa_{p,\infty}(x-\frac{L}{2})} - \kappa_{p,\infty} e^{-\kappa_{p,\infty}(x+\frac{L}{2})} & \text{for } x \in [0,L[ 
\end{cases}
\tag{B19}
\]
that should be compared to equation (216) of the main text.

**B.3.2. Special case \( p \to +\infty \) for \( q \in ]-\infty,0[ \) corresponding to the conditioning towards \( a_* = 0, b_* > 0 \).** In the limit \( p \to +\infty \) for \( q \in ]-\infty,0[ \), the ground-state \( \phi_{p=\infty,q}^{GS}(x) \) should vanish at \( x = 0 \) and thus in the whole region \( x \leq 0 \) as a consequence of equation (B9), while the ratio of equation (B14) reduces to
\[
\frac{K^-}{K^+} \to \infty \quad \text{as } q \to 0^+ \tag{B20}
\]
where \( \kappa_{\infty,q} > 0 \) is the solution of equation (B5)
\[
0 = \left(1 + \frac{\kappa_{\infty,q}}{q}\right) - e^{-2\kappa_{\infty,q}L}. \tag{B21}
\]

The conditioned drift of equation (B16) reduces to
\[
\mu^*_{p=+\infty,q}(x) = \begin{cases} 
\kappa_{\infty,q} e^{-2\kappa_{\infty,q}x} + e^{-\kappa_{\infty,q}(x-\frac{L}{2})} - \kappa_{\infty,q} e^{-\kappa_{\infty,q}(x+\frac{L}{2})} & \text{for } x \in ]0,L[ 
-\kappa_{\infty,q} e^{-2\kappa_{\infty,q}x} + e^{-\kappa_{\infty,q}(x-\frac{L}{2})} - \kappa_{\infty,q} e^{-\kappa_{\infty,q}(x+\frac{L}{2})} & \text{for } x \in ]L,\infty[ 
\end{cases} \tag{B22}
\]
that should be compared to equation (216) of the main text.

**B.3.3. Special case \( p = q \in ]-\infty,0[ \) corresponding to the conditioning towards \( \sigma_* > 0 \).** In the case of equal negative amplitude \( p = q \in ]-\infty,0[ \), equation (B5)
\[
0 = \left(1 + \frac{\kappa}{p}\right)^2 - e^{-2\kappa L} \tag{B23}
\]
has two solutions
\[
\left(1 + \frac{\kappa}{p}\right) = \pm e^{-2\kappa L} \tag{B24}
\]
corresponding to two possible bound states, while the corresponding ratio of equation (B14) takes the two possible values
\[
\frac{K^-}{K^+} = \frac{-e^{-\kappa L}}{1 + \frac{\kappa}{p}} = \mp 1. \tag{B25}
\]
The positive ground-state wave function \( \phi_{p,p}(x) \) corresponds to the ratio \( \frac{K^-}{K^+} = +1 \) with \( \kappa_{p,p} \)
solution of
\[
\left(1 + \frac{\kappa_{p,p}}{p}\right) = -e^{-\kappa_{p,p}L} \tag{B26}
\]
corresponding to
\[
p = - \frac{\kappa_{p,p}}{1 + e^{-\kappa_{p,p}L}} = -\lambda^-_s
\]  
where one recognizes the form \( \lambda^-_s \) of equation (210). The other bound state associated to the negative ratio \( \frac{\mu}{\kappa} = -1 \) corresponds to the first excited state characterized by
\[
p = - \frac{\kappa_{p,p}}{1 - e^{-\kappa_{p,p}L}} = -\lambda^+_s
\]  
where one recognizes the form of \( \lambda^+_s \) of equation (210).

Plugging the ratio \( \frac{\mu}{\kappa} = +1 \) for the ground state into the conditioned drift of equation (B16) yields
\[
\mu^*_p(x) = \left\{ \begin{array}{ll}
\kappa_{p,p} & \text{for } x \in ]-\infty,0[ \\
\kappa_{p,p}e^{p(x-x_0)} - \kappa_{p,p}e^{-p(x_0-x)} & \text{for } x \in ]0,L[ \\
-\kappa_{p,p} & \text{for } x \in ]L,\infty[ 
\end{array} \right.
\]  
that should be compared to equation (213) of the main text.

**Appendix C. Explicit Laplace inversion with respect to \( p \) and \( q \) of the contribution \( \hat{C}_{l,p,q}(x|x_0) \) of equation (50)**

In this appendix, the goal is to compute the Laplace inversion with respect to \( p \) and \( q \) of the contribution \( \hat{C}_{l,p,q}(x|x_0) \) of equation (50).

**C.1. Partial fraction decomposition of \( \hat{C}_{l,p,q}(x|x_0) \)**

To see more clearly the structure of \( \hat{C}_{l,p,q}(x|x_0) \) of equation (50) with respect to \((p,q)\), it is convenient to introduce the shifted variables
\[
P \equiv p + \frac{\hat{G}_i(L|x)}{\Delta_i} = p + \frac{1}{G^{\text{abs}}_i(0)}
\]
\[
Q \equiv q + \frac{\hat{G}_i(0|x)}{\Delta_i} = q + \frac{1}{G^{\text{abs}}_i(L)}
\]  
and the notation
\[
c_i = \sqrt{G_i(0)LG_i(L)}
\]
\[
\Delta_i = G_i(0)G_i(L) - G_i(L)G_i(0)
\]  
in order to rewrite equation (50) as
\[
\hat{C}_{l,p,q}(x|x_0) = \frac{\gamma_i(x|x_0) + P\beta_i^{(LL)}(x|x_0) + Q\alpha_i^{(00)}(x|x_0)}{PQ - c_i^2} - \frac{\alpha_i^{(00)}(x|x_0)}{P} - \frac{\beta_i^{(LL)}(x|x_0)}{Q}
\]  
\[
= \frac{\gamma_i(x|x_0) + \alpha_i^{(00)}(x|x_0)}{PQ - c_i^2} + \frac{Q}{PQ - c_i^2} - \frac{1}{P}
\]  
\[
+ \frac{\beta_i^{(LL)}(x|x_0)}{PQ - c_i^2} - \frac{1}{Q}
\]  
(119)
The amplitude of the first fraction
\[ \gamma_s(x|\psi_0) = \frac{\dot{G}_s(L|L)\dot{G}_s(x|0)\dot{G}_s(0|0) + \dot{G}_s(0|0)\dot{G}_s(x|L)\dot{G}_s(L|0) - \dot{G}_s(0|0)\dot{G}_s(L|L) + \dot{G}_s(0|L)\dot{G}_s(L|0)}{\Delta_t^2} \]
\( = \gamma_s^{[0]}(x|\psi_0) + \gamma_s^{[0L]}(x|\psi_0) \) (C4)
can be decomposed into the two terms \( \gamma_s^{[0]}(x|\psi_0) \) and \( \gamma_s^{[0L]}(x|\psi_0) \) given in equation (57) of the main text.

C.2. Laplace inversion of \( \hat{G}_{s,p,q}(x|\psi_0) \) with respect to \( p \) and \( q \)

The Laplace inversion of the three terms of equation (C3) involves the modified Bessel function \( I_0(z) \) of equation (53) and its derivative \( I_0'(z) = I_1(z) \) of equation (54). The Laplace inversion of the first term of equation (C3) is based on the identity (already used in equation (39) of [6])

\[ \frac{1}{PQ - c^2} = \int_0^{+\infty} dx e^{-Px} \int_0^{+\infty} dy e^{-Qy} I_0(2c\sqrt{xy}) \]  
(55)

that can be checked by plugging the series representation of \( I_0(z) \) of equation (53) into the right-hand side of equation (C5)

\[ \int_0^{+\infty} dx e^{-Px} \int_0^{+\infty} dy e^{-Qy} I_0(2c\sqrt{xy}) = \int_0^{+\infty} dx e^{-Px} \int_0^{+\infty} dy e^{-Qy} \sum_{k=0}^{+\infty} \left( \sqrt{\frac{c}{xy}} \right)^{2k} \frac{1}{(k!)^2} \]
\[ = \sum_{k=0}^{+\infty} \frac{c^{2k}}{(k!)^2} \int_0^{+\infty} dx x^k e^{-Px} \int_0^{+\infty} dy y^k e^{-Qy} \]
\[ = \sum_{k=0}^{+\infty} \frac{c^{2k}}{(k!)^2} \frac{(k!)^2}{(PQ)^{k+1}} \]
\[ = \frac{1}{PQ} \sum_{k=0}^{+\infty} \left( \frac{c^2}{PQ} \right)^k \]
\[ = \frac{1}{PQ} \left( \frac{1}{1 - \frac{c^2}{PQ}} \right) = \frac{1}{PQ - c^2}. \]  
(66)

The multiplication of equation (C5) by \( P \) yields via integration by parts

\[ \frac{P}{PQ - c^2} = \int_0^{+\infty} dx (Pe^{-Px}) \int_0^{+\infty} dy e^{-Qy} I_0(2c\sqrt{xy}) \]
\[ = \int_0^{+\infty} dy e^{-Qy} \int_0^{+\infty} dx (-\partial_x e^{-Px}) I_0(2c\sqrt{xy}) \]
\[ = -\int_0^{+\infty} dy e^{-Qy} \left[ e^{-Px} I_0(2c\sqrt{xy}) \right]_{x=0}^{x=+\infty} \]
\[ + \int_0^{+\infty} dy e^{-Qy} \int_0^{+\infty} dx e^{-Px} \partial_x I_0(2c\sqrt{xy}) \]
\[ = \int_0^{+\infty} dy e^{-Qy} + \int_0^{+\infty} dy e^{-Qy} \int_0^{+\infty} dx e^{-Px} \frac{c\sqrt{y}}{\sqrt{x}} I_0'(2c\sqrt{xy}) \]
\[ = \frac{1}{Q} + \int_0^{+\infty} dy e^{-Qy} \int_0^{+\infty} dx e^{-Px} \frac{c\sqrt{y}}{\sqrt{x}} I_1(2c\sqrt{xy}) \]  
(77)
so one obtains the identity (already used in equation (47) of [6])
\[
\frac{P}{PQ - c^2} - \frac{1}{Q} = \int_0^{+\infty} dy e^{-Qy} \int_0^{+\infty} dx e^{-px} \frac{c\sqrt{x}}{\sqrt{y}} I_1 \left(2c\sqrt{xy} \right)
\]  
(C8)
and its analog (already used in equation (48) of [6])
\[
\frac{Q}{PQ - c^2} - \frac{1}{P} = \int_0^{+\infty} dy e^{-Qy} \int_0^{+\infty} dx e^{-px} \frac{c\sqrt{x}}{\sqrt{y}} I_1 \left(2c\sqrt{xy} \right)
\]  
(C9)
that will allow to write the Laplace inversions of the second and third terms of equation (C3).

If one replaces the shifted variables \((P, Q)\) of equation (C1) in terms of the original Laplace variables \((p, q)\), the three terms of the fourth contribution \(\tilde{C}_{s, p, q}(x|x_0)\) of equation (C3)
\[
\tilde{C}_{s, p, q}(x|x_0) = \tilde{C}_{s, p, q}^{[\alpha]}(x|x_0) + \tilde{C}_{s, p, q}^{[\beta]}(x|x_0) + \tilde{C}_{s, p, q}^{[\gamma]}(x|x_0)
\]  
(C10)
read
\[
\tilde{C}_{s, p, q}^{[\alpha]}(x|x_0) = \alpha_s^{[00]}(x|x_0) \left[ \frac{Q}{PQ - c^2} - \frac{1}{P} \right]
\]  
(C11)
\[
\tilde{C}_{s, p, q}^{[\beta]}(x|x_0) = \beta_s^{[LL]}(x|x_0) \left[ \frac{P}{PQ - c^2} - \frac{1}{Q} \right]
\]  
(C11)
\[
\tilde{C}_{s, p, q}^{[\gamma]}(x|x_0) = \gamma_s(x|x_0) \left[ \frac{p + \frac{1}{G_s^{(LL, (000))}}}{p + \frac{1}{G_s^{(LL, (000))}} \left( q + \frac{1}{G_s^{(LL, (000))}} \right) - c_s^2} \right] - \frac{1}{p + \frac{1}{G_s^{(LL, (000))}} \left( q + \frac{1}{G_s^{(LL, (000))}} \right) - c_s^2}
\]  
(C11)

So using equations (C5), (C8) and (C9), the Laplace inversion with respect to \(p\) and \(q\) of equation (C10) leads to equations (51) and (52) given in the main text.

C.3. Rewriting \(\Omega_s(x|x_0)\) in terms of \(\alpha_s^{[00]}(x|x_0), \beta_s^{[LL]}(x|x_0)\) and \(\gamma_s(x|x_0)\)

For another computation of the main text around equation (127), it is useful to rewrite \(\Omega_s(x|x_0)\) in terms of the three functions \(\alpha_s^{[00]}(x|x_0), \beta_s^{[LL]}(x|x_0)\) and \(\gamma_s(x|x_0)\) by considering the special case \(p = 0 = q\) of equation (50)
\[
\tilde{C}_{s, p=0, q=0}(x|x_0) = \tilde{\Omega}_s(x|x_0) - \alpha_s^{[00]}(x|x_0) \hat{G}_s^{\text{abs}(L)}(0|0) - \beta_s^{[LL]}(x|x_0) \hat{G}_s^{\text{abs}(0)}(L|L)
\]  
(C12)
and of equation (C10)
\[
\tilde{C}_{s, p=0, q=0}(x|x_0) = \tilde{C}_{s, p=0, q=0}^{[\alpha]}(x|x_0) + \tilde{C}_{s, p=0, q=0}^{[\beta]}(x|x_0) + \tilde{C}_{s, p=0, q=0}^{[\gamma]}(x|x_0)
\]  
(C13)
In this appendix, we provide technical details for the asymptotic analysis of the long-time equation (\(S\))

\[
\hat{\mathcal{C}}_{s,p=0,q=0}(x|x_0) = \alpha_s^{[0]}(x|x_0) \left[ \frac{\frac{1}{G_s^{\text{abs}(L)}(L|L)}}{\frac{1}{G_s^{\text{abs}(0)}(0|0)G_s^{\text{abs}(0)}(0|0)}} - \frac{1}{G_s^{\text{abs}(L)}(0|0)} \right]
\]

\[
= \alpha_s^{[0]}(x|x_0) \left[ \frac{\Delta_s}{G_s^{\text{abs}(L)}(L|L)} - \frac{\hat{\mathcal{C}}^{\text{abs}(L)}(0|0)}{G_s^{\text{abs}(0)}(0|0)} \right]
\]

\[
\hat{\mathcal{C}}_{s,p=0,q=0}(x|x_0) = \beta_s^{[L]}(x|x_0) \left[ \frac{\Delta_s}{G_s^{\text{abs}(L)}(L|L)} - \frac{\hat{\mathcal{C}}^{\text{abs}(L)}(0|0)}{G_s^{\text{abs}(0)}(0|0)} \right]
\]

\[
= \beta_s^{[L]}(x|x_0) \left[ \frac{\Delta_s}{G_s^{\text{abs}(L)}(0|0)} - \frac{\hat{\mathcal{C}}^{\text{abs}(0)}(L|L)}{G_s^{\text{abs}(0)}(0|0)} \right]
\]

\[
\hat{\mathcal{C}}_{s,p=0,q=0}(x|x_0) = \gamma_s(x|x_0) \left[ \frac{\Delta_s}{G_s^{\text{abs}(L)}(L|L)} - \frac{\hat{\mathcal{C}}^{\text{abs}(L)}(0|0)}{G_s^{\text{abs}(0)}(0|0)} \right]
\]

The comparison between equations (C12)–(C14) allows to rewrite

\[
\Omega_s(x|x_0) = \alpha_s^{[0]}(x|x_0) + \beta_s^{[L]}(x|x_0) + \gamma_s(x|x_0)
\]

\[
= \alpha_s^{[0]}(x|x_0) + \beta_s^{[L]}(x|x_0) + \gamma_s(x|x_0)
\]

**Appendix D. Calculation of the conditioned drift \(\mu^{[\Sigma_\infty]}(x, \Sigma = \Sigma^*_\infty)\) when \(S^{\text{abs}(0,L)}(x)\) vanishes for large time \(\tau\)**

In this appendix, we provide technical details for the asymptotic analysis of the long-time of \(S^{\text{abs}(0,L)}(x)\) when it vanishes. As described in the main text, when \(S^{\text{abs}(0,L)}(x)\) vanishes in equation (227), one should use the asymptotic behavior of \(S^{\text{abs}(0,L)}(x)\) for large time \(\tau\) to obtain the conditioned drift \(\mu^{[\Sigma_\infty]}(x, \Sigma = \Sigma^*_\infty)\) of equation (192). Recall that equation (192)

\[
\mu^{[\Sigma_\infty]}(x, \Sigma = \Sigma^*_\infty) = \mu(x) + \lim_{\tau \to \infty} \partial_x \ln S^{\text{abs}(0,L)}(x)
\]

where the Laplace transform of \(S^{\text{abs}(0,L)}(x)\) is given by equation (79)

\[
\hat{S}_{s}^{\text{abs}(0,L)}(x) = \frac{1}{s} \left[ 1 - \frac{\hat{G}_s^{\text{abs}(L)}(0|x)}{\hat{G}_s^{\text{abs}(0)}(0|0)} - \frac{\hat{G}_s^{\text{abs}(0)}(L|x)}{\hat{G}_s^{\text{abs}(0)}(L|L)} \right]
\]

where the expressions of \(\hat{G}_s^{\text{abs}(L)}(x|x_0)\) and \(\hat{G}_s^{\text{abs}(0)}(x|x_0)\) are given by equation (206) and equation (207) respectively. There are two cases depending on whether \(x \in [-\infty, 0]\) or \(x \in [0, L]\).
\[ D.1 \text{ Case } x \in ]-\infty, 0[ \]

When \( x < 0 \) the expression of equation (D2) reduces to

\[ \hat{S}_x^{ab}(0,L) (x) = \frac{1}{s} \left[ 1 - e^{i(\sqrt{2x+\mu^2} - \mu)} \right]. \]  

(D3)

From the inverse Laplace transform given in [107] (equation (21a))

\[ L^{-1} \left\{ \frac{1}{s} e^{\frac{s}{2} - i \sqrt{\frac{2}{D^2} + s}} \right\} = \frac{1}{2} \text{erfc} \left( \frac{x - \mu \tau}{2\sqrt{D\tau}} \right) + \frac{1}{2} e \pi \text{erfc} \left( \frac{x + \mu \tau}{2\sqrt{D\tau}} \right) \]

(D4)

where \( \text{erfc}(x) \) is the complementary error function, we immediately obtain

\[ \hat{S}_x^{ab}(0,L) (x) = \frac{1}{2} \left[ e^{-2\mu} \left( -2 + \text{erfc} \left( \frac{x - \mu \tau}{\sqrt{2\tau}} \right) \right) + \text{erfc} \left( \frac{x + \mu \tau}{\sqrt{2\tau}} \right) \right]. \]

(D5)

The asymptotic behavior at leading order for large time \( \tau \) is given by

\[ \hat{S}_x^{ab}(0,L) (x) \sim \frac{1}{\tau \to +\infty} \frac{1}{\mu^2 \tau^{3/2}} \sqrt{\frac{2}{\pi}} x e^{-x^2 \mu^2 \tau^2} \]

(D6)

and leads to

\[ \mu_s |_{\text{asym}} (x, \Sigma = \Sigma^s) = \mu + \lim_{\tau \to +\infty} \partial_x \ln \hat{S}_x^{ab}(0,L) (x) \]

(D7)

as stated in equation (234).

\[ D.2 \text{ Case } x \in ]0, L[ \]

When \( x \in ]0, L[ \) the expression of equation (D2) reads

\[ \hat{S}_x^{ab}(0,L) (x) = \frac{1}{s} \left[ 1 + \frac{e^{-\sqrt{2x+\mu^2(L+x) + \mu(L-x)}}(1 - e^{\sqrt{2x+\mu^2L}}) - e^{-\sqrt{2x+\mu^2(L-x)}}(1 - e^{\sqrt{2x+\mu^2(L-x)}})}{1 - e^{-2\sqrt{2x+\mu^2L}}} \right] \]

(D8)

or in a more symmetrical form

\[ \hat{S}_x^{ab}(0,L) (x) = \frac{1}{s} \left[ 1 - e^{\mu(L-x)} \frac{\sinh(\sqrt{2x+\mu^2L})}{\sinh(\sqrt{2x+\mu^2L})} + e^{-\mu x} \frac{\sinh(\sqrt{2x+\mu^2(L-x)})}{\sinh(\sqrt{2x+\mu^2L})} \right]. \]

(D9)

The inverse Laplace transform of \( \sinh(\sqrt{2x+\mu^2L})/(s\sinh(\sqrt{2x+\mu^2L})) \) can be computed exactly thanks to the residue theorem. First, there is a simple pole at \( s = 0 \) and the residue at this point is

\[ \lim_{s \to 0} \frac{se^{i\tau} \sinh(\sqrt{2x+\mu^2L})}{s\sinh(\sqrt{2x+\mu^2L})} = \frac{\sinh(\mu L)}{\sinh(\mu L)}. \]

(D10)

In addition, the denominator vanishes when \( \sinh(\sqrt{2x+\mu^2L}) = 0 \), i.e. when \( L\sqrt{2x+\mu^2} = n\pi i, n = 0, \pm 1, \pm 2, \ldots \) thus for
\[ s = a_n = -\frac{\mu^2}{2} - \frac{n^2\pi^2}{2L^2} \quad n = 0, 1, 2, \ldots \]  

(D11)

The residue at the point \( s = a_n \) is

\[
\lim_{s \to a_n} \frac{(s - a_n)e^{\tau s} \sinh(\sqrt{2s + \mu^2}L)}{s \sinh(\sqrt{2s + \mu^2}L)} = \lim_{s \to a_n} \left( s - a_n \right) \lim_{s \to a_n} \frac{e^{\tau s} \sinh(\sqrt{2s + \mu^2}L)}{s} = \frac{n\pi i (-1)^n}{L^2} e^{-\frac{\tau^2 a_n^2}{2}} \sinh(\frac{n\pi x}{L}) \sinh(\frac{\mu^2}{2})
\]

Adding all the residues, we obtain

\[
\mathcal{L}^{-1} \left\{ \frac{\sinh(\sqrt{2s + \mu^2}L)}{\sinh(\sqrt{2s + \mu^2}L)} \right\} = \frac{\sinh(\mu x)}{\sinh(\mu L)} + \sum_{n=1}^{\infty} \frac{n\pi (-1)^n}{L^2 (\frac{\mu^2}{2} + \frac{n^2\pi^2}{2L^2})} e^{-\left(\frac{\mu^2}{2} + \frac{n^2\pi^2}{2L^2}\right) \tau \sin \left(\frac{n\pi x}{L}\right)}
\]

(D13)

and then

\[
S^{\text{abs}}_{\tau}(0, L)(x) = \mathcal{L}^{-1} \left\{ \frac{1}{s} \left[ 1 - e^{\mu(L-x)} \frac{\sinh(\sqrt{2s + \mu^2}L)}{\sinh(\sqrt{2s + \mu^2}(L-x))} + e^{-\mu x} \sinh(\sqrt{2s + \mu^2}(L-x)) \right] \right\}
\]

\[
= 1 - e^{\mu(L-x)} \left[ \frac{\sinh(\mu x)}{\sinh(\mu L)} + \sum_{n=1}^{\infty} \frac{n\pi (-1)^n}{L^2 (\frac{\mu^2}{2} + \frac{n^2\pi^2}{2L^2})} e^{-\left(\frac{\mu^2}{2} + \frac{n^2\pi^2}{2L^2}\right) \tau \sin \left(\frac{n\pi x}{L}\right)} \right]
\]

\[
+ e^{-\mu x} \left[ \frac{\sinh(\mu x)}{\sinh(\mu L)} + \sum_{n=1}^{\infty} \frac{n\pi (-1)^n}{L^2 (\frac{\mu^2}{2} + \frac{n^2\pi^2}{2L^2})} e^{-\left(\frac{\mu^2}{2} + \frac{n^2\pi^2}{2L^2}\right) \tau \sin \left(\frac{n\pi (x-L)}{L}\right)} \right]
\]

\[
= e^{-\mu x} \sum_{n=1}^{\infty} \left( 1 - e^{\mu(L-x)} (-1)^n \right) \frac{2n\pi}{L^2 \mu^2 + n^2\pi^2} e^{-\left(\frac{\mu^2}{2} + \frac{n^2\pi^2}{2L^2}\right) \tau \sin \left(\frac{n\pi x}{L}\right)}.
\]

(D14)

Keeping only the term with \( n = 1 \), one gets the long-time asymptotic behavior

\[
S^{\text{abs}}_{\tau}(0, L)(x) \underset{\tau \to +\infty}{\simeq} \frac{2\pi \left( 1 + e^{\mu L} \right)}{L^2 \mu^2 + \pi^2} e^{-\left(\frac{\mu^2}{2} + \frac{\pi^2}{2L^2}\right) \tau} e^{-\mu x} \sin \left(\frac{\pi x}{L}\right).
\]

(D15)

And finally, we obtain

\[
\langle \Sigma_{\infty} \rangle_{\mu} = \lim_{\tau \to +\infty} \partial_{\mu} \ln S^{\text{abs}}_{\tau}(0, L)(x) = \frac{\pi}{L} \cot \left(\frac{\pi x}{L}\right)
\]

(D16)

as announced in equation (234).
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