Oblique undular hydraulic jumps in turbulent free-surface flows

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Steady turbulent near-critical free-surface flow over a slightly inclined bottom with an oblique roughness discontinuity is considered. An asymptotic analysis for Froude numbers close to the critical value 1 and large Reynolds numbers results in an extended Korteweg–de Vries (KdV) equation describing the free-surface elevation without the need of turbulence modeling. Two new integral relations can be derived from the extended KdV equation. For enlarged downstream bottom roughness, new type of undular jump solutions with fully developed flow both upstream and downstream exists. Numerical solutions of the extended KdV equation show that an increasing downstream bottom roughness enhances the development of undulations.

1 Introduction and problem formulation

In near-critical free-surface flow, the transition from supercritical to subcritical flow is accompanied by a wavy (undular) surface called the undular hydraulic jump, see Fig. 1. In the present paper, steady turbulent free-surface flow with a fully developed reference state far upstream is considered, cf. [1]. The bottom is assumed to be inclined by the slope α, with a roughness discontinuity that is oriented obliquely with respect to the free stream, see Fig. 2. Assuming a very small bottom slope is equivalent to a very large Reynolds number. Thus, the flow in the defect layer is governed by the continuity equation of incompressible flow and the Reynolds-averaged Navier–Stokes equations neglecting viscous stresses, see [2]. Conventional kinematic and dynamic boundary conditions are prescribed at the free surface. At the bottom, the defect layer is matched with the viscous wall layer. The different bottom roughness upstream and downstream is represented by $C_r^+$ and $C^+$, respectively, being the roughness dependent constant in the logarithmic friction law, see [3].

The reference Froude number $Fr := \dot{q}/\sqrt{gh_r}$ is defined in terms of the free-stream volumetric mean velocity $\dot{q}$, and the free-surface height $h_r$. The subscript $r$ refers to the fully developed reference state far upstream. The acceleration due to gravity is denoted by $g$. The analysis is performed in terms of the $x, y, z$-coordinate system, which is aligned with the roughness discontinuity, i.e. inclined to the $\xi, \eta, \zeta$-coordinate system by the yaw angle $0 \leq \phi < \pi/2$, see Fig. 2. The normal Froude number $Fr_n := \dot{u}_r/\sqrt{gh_r} = Fr \cos \phi$ is defined with the normal component of the reference velocity, $\dot{u}_r$. The normal slope shown in Fig. 1 is $\alpha_n = \alpha \cos \phi$.

![Fig. 1: The undular hydraulic jump over an area of enlarged roughness. Cross-section normal to the roughness discontinuity.](image1)

![Fig. 2: Roughness discontinuity oriented obliquely with respect to the free stream, coordinate systems and velocity components.](image2)

2 Asymptotic analysis: extended Korteweg–de Vries equation and integral relations

An asymptotic analysis is performed following the investigation of oblique solitary waves over a strip of enlarged roughness [2]. Therefore, the upstream normal Froude number is assumed to be close to the critical value 1, i.e. $\frac{2}{3}(Fr_n - 1) = \varepsilon \ll 1$. Turbulence modeling can be avoided by introducing the constant coupling parameter $A = \alpha_n / \varepsilon^2 = O(1)$. Non-dimensional variables are introduced by referring to the reference state. The non-dimensional longitudinal coordinate $X = \delta x / h_r$ is contracted with the small parameter $\delta = 3\sqrt{\varepsilon}$.

All dependent variables are expanded in terms of powers of $\varepsilon$, e.g. the non-dimensional free-surface height $\bar{H}(X) = \bar{h}(x)/h_1 = 1 + \varepsilon H_1 + \varepsilon^2 H_2 + \ldots$. Investigation of the first-order equations reveals a non-skewed velocity profile of the three-dimensional flow, which is independent of the $z$-coordinate. The free-surface elevation $H_1$ remains undetermined in the...
framework of first-order equations. The analysis of the second-order equations yields the solvability condition \[2\]:

\[
H^\prime\prime_1 + H_1(H_1 - 1) = \beta(H_1 - \Gamma),
\]

\[
\beta = \sqrt{2} A/\chi, \quad \Gamma = 2\sqrt{A}\cos(\phi(C^+_1 - C^+)/3\chi), \quad \chi = 1 - (\sin^2 \phi)/3.
\]

The nonlinear third-order ODE (1a) may be identified as a steady-state version of an extended Korteweg-de Vries (KdV) equation. The extension on the right-hand side consists of the dissipation parameter \(\beta = O(\sqrt{\varepsilon})\), and the parameter \(\Gamma = O(1)\) containing information about the roughness difference upstream and downstream. Both \(\beta\) and \(\Gamma\) depend on the yaw angle \(\phi\), amongst others, via the parameter \(\chi\). Note that the orthogonal undular hydraulic jump over an area of enlarged roughness is included in the present theory, i.e. if \(\phi = 0\), the parameters \(\beta\) and \(\Gamma\) are identical as in [4].

For the present case of enlarged downstream bottom roughness, a new type of solution of (1a) with a fully developed flow both upstream and downstream can be found. The corresponding boundary conditions for the extended KdV equation (1a) are \(H_1 = 0\) as \(X \to -\infty\) and \(H_1 = \Gamma\) as \(X \to \infty\). If \(\Gamma > 1\), undular solutions will be obtained. Using these boundary conditions for the integration of (1a) and (1a) after multiplying with \(H_1\), respectively, yields the integral relations:

\[
\int_{-\infty}^{\infty} (H_1 - \Gamma) \, dX = \frac{\Gamma}{2\beta}(\Gamma - 2),
\]

\[
\int_{-\infty}^{\infty} H_1(H_1 - \Gamma) \, dX = \frac{\Gamma^2}{6\beta}(2\Gamma - 3).
\]

It can be shown that the first integral relation expresses the overall conservation of momentum.

### 3 Undular hydraulic jumps with fully developed flow upstream and downstream

Figure 3 shows numerical solutions of (1a) by prescribing the boundary conditions for fully developed flow up- and downstream. With a dissipation parameter \(\beta = 0.1\), each solution in Fig. 3 corresponds to a different \(\Gamma\) value. For a fixed upstream roughness, a rising \(\Gamma\) value is equivalent to increasing downstream roughness. The free-surface elevation \(H_1\) and the local normal Froude number \(Fr_{n, loc}\) are related via \(Fr_{n, loc} = 1 + 3\varepsilon(1 - H_1)/2\), see left and right scales in Fig. 3, respectively.

For \(\Gamma = 0.8\), the dashed solution is neither undular nor is it a hydraulic jump as the flow remains supercritical even in the area of enlarged roughness at \(X > 0\). The blue solution is obtained for \(\Gamma = 1\), showing a single undulation before reaching the fully developed critical downstream state. Thus, for \(\Gamma \leq 1\), the roughness difference is not significant enough to trigger an undular transition to subcritical flow. However, the green solution for \(\Gamma = 1.5\) shows a distinct undular jump with strongly decaying amplitude. The larger downstream roughness associated with \(\Gamma = 2\) enhances the development of undulations such that the fully developed state is reached further downstream. The case of \(\Gamma = 2\) is of particular interest because the fully developed up- and downstream states correspond to a hydraulic jump as if the flow were inviscid (\(\beta = 0\)). However, the inviscid jump would be a discontinuity without undulations [5]. The solutions for inviscid 3D flow over a wedge-shaped strut in [6] resemble the curves in Fig. 3 but do not represent undular jumps as the flow remains supercritical.

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### References

[1] W. Grillhofer and W. Schneider, Phys. Fluids, 15, 730–735 (2003).
[2] W. Schneider, Phys. Fluids, 33 (2021), doi: 10.1063/5.0050755.
[3] H. Schlichting and K. Gersten, "Boundary-Layer Theory", 9th Ed. (Springer, Berlin/Heidelberg, 2017), pp. 523–524.
[4] W. Schneider, J. Fluid Mech., 726, 137-159 (2013).
[5] F. M. Henderson, "Open Channel Flow", (Macmillan Publishing, New York, 1966), pp. 66–81.
[6] X. Chen, Eur. J. Mech. B/Fluids, 18, 475–492 (1999).