On the exact sequences for the ideal class group, Picard group and group of units of the Grothendieck ring

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Abstract. For any extension of commutative rings $A \subseteq B$ we first define a group $\text{Cl}(A, B)$, that we call the ideal class group of this extension (the classical ideal class group is easily recovered from this construction), then as a main result we obtain the following exact sequence of Abelian groups

$$0 \rightarrow \text{Cl}(A, B) \rightarrow \text{Pic}(A) \rightarrow \text{Pic}(B).$$

In particular, we have the exact sequence of groups: $0 \rightarrow \text{Cl}(A) \rightarrow \text{Pic}(A) \rightarrow \text{Pic}(T(A))$ where $T(A)$ is the total ring of fractions of $A$. As another application, if $B$ has finitely many maximal ideals then we have the canonical isomorphism of groups $\text{Cl}(A, B) \simeq \text{Pic}(A)$. For any extensions of commutative rings $A \subseteq B \subseteq C$, we also obtain the following exact sequence of Abelian groups:

$$0 \rightarrow \text{Cl}(A, B) \rightarrow \text{Cl}(A, C) \rightarrow \text{Cl}(B, C).$$

We show that the above sequence is also right exact if and only if the canonical map $\text{Cl}(A_{\text{red}}, C_{\text{red}}) \rightarrow \text{Cl}(B_{\text{red}}, C_{\text{red}})$ is surjective. Next, we show that if $e$ and $e'$ are idempotents of a ring $A$, then we have the canonical isomorphism of $A$-modules: $Ae \oplus Ae' \simeq Ae/Ae(1-e') \oplus Ae'/Ae'(1-e) \oplus A(e + e' - 2ee')$. This result plays an important role in proving several results on the Grothendieck ring $K_0(A)$. Especially, for any ring $A$ we obtain the canonical isomorphisms of groups $\mathcal{B}(A) \simeq \mathcal{B}(K_0(A)) \simeq H_0(A)^*$. We show that a morphism of rings $A \rightarrow B$ lifts idempotents if and only if the induced ring map $K_0(A) \rightarrow K_0(B)$ lifts idempotents. If moreover, $\text{Max}(B)$ is a finite set then $K_0(A) \rightarrow K_0(B)$ is surjective. Finally, if a ring $A$ has the line bundle property (e.g. a Dedekind domain or more generally a Noetherian one-dimensional ring), then we obtain the following split exact sequence of groups

$$0 \rightarrow \text{Pic}(A) \rightarrow K_0(A)^* \rightarrow \mathcal{B}(A) \rightarrow 0.$$

1. Introduction

The first main goal of this article is to generalize naturally the classical notion of the ideal class group to the more general case of an arbitrary extension of commutative rings (it goes without saying that the classical ideal class group is defined for integral domains and is of special importance in algebraic number theory). We will then examine the relationship of this group to the Picard groups of the corresponding rings in the extension. In fact, in Theorem 5.6 for any extension of commutative rings...
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rings $A \subseteq B$ we obtain the following exact sequence of Abelian groups:

$$0 \longrightarrow \text{Cl}(A, B) \longrightarrow \text{Pic}(A) \longrightarrow \text{Pic}(B).$$

Then some applications of this result are given. In particular, if $B$ has finitely many maximal ideals then we have the canonical isomorphism of groups $\text{Cl}(A, B) \simeq \text{Pic}(A)$. As another application, for any commutative ring $A$, we obtain the following exact sequence of Abelian groups:

$$0 \longrightarrow \text{Cl}(A) \longrightarrow \text{Pic}(A) \longrightarrow \text{Pic}(T(A)).$$

In particular, if $A$ is a reduced ring with finitely many minimal primes, then we have the canonical isomorphism of groups $\text{Cl}(A, B) \simeq \text{Pic}(A)$. The classical ideal class group is also easily recovered from this construction.

We were informed that some exact sequences similar to the exact sequence we obtained in Theorem 3.5 can be found in the literature [1, Chap IX, §3, Theorem 3.3] and [10, Theorem 2.4]. Later, we realized that there are fewer connections between the exact sequence obtained in [1, Chap IX, §3, Theorem 3.3] and Theorem 3.5 (in this regard see also Remark 3.3), but our result has close connections with the Roberts-Singh Theorem [10, Theorem 2.4]. In fact, our result can be seen as an improved version of the Roberts-Singh Theorem, to which the idea of the ideal class group of an extension has been added.

Next, we will study the behaviour of the ideal class group for a tower of extensions of rings. In fact, in Theorem 3.14, for any extensions of commutative rings $A \subseteq B \subseteq C$, we obtain the following exact sequence of Abelian groups:

$$0 \longrightarrow \text{Cl}(A, B) \longrightarrow \text{Cl}(A, C) \longrightarrow \text{Cl}(B, C).$$

We then show that the above sequence is also right exact if and only if the canonical map $\text{Cl}(A_{\text{red}}, C_{\text{red}}) \rightarrow \text{Cl}(B_{\text{red}}, C_{\text{red}})$ is surjective.

The next goal of this article is to investigate the Grothendieck ring $K_0(A)$ of a given commutative ring $A$. There are various (but almost all equivalent) constructions of Grothendieck groups and rings in the literature. In this article we are interested in studying the version which is constructed in the most standard and canonical way. In this context, we first obtain an interesting formula regarding idempotents. In fact, we show that if $e$ and $e'$ are idempotents of a commutative ring $A$, then we have the canonical isomorphism of $A$-modules:

$$Ae \oplus Ae' \simeq Ae/Ae(1-e') \oplus Ae'/Ae'(1-e) \oplus A(e+e' - 2ee').$$

It is worth noting that during our e-mail correspondence with Pierre Deligne, he at first somewhat doubted the correctness of this formula, but in the end we managed to prove it.

Grothendieck discovered algebraic K-theory in the late 1950s during his proof of the Grothendieck–Riemann–Roch theorem. In particular, he proved a fundamental result in this field of mathematics which asserts that for any commutative ring $A$, then the ring $K_0(A)$ modulo its nil-radical is canonically isomorphic to $H_0(A)$, the ring of all continuous functions $\text{Spec}(A) \rightarrow \mathbb{Z}$ (for the proof see [1, Chap. IX, §3, Proposition 4.6] or [13, Corollary 10.7] or [18, Chap. II, §4, Corollary 4.6.1]). In this article, we will use the whole strength of Grothendieck’s theorem and the above result that we obtained on idempotents, to prove several results on the Grothendieck ring $K_0(A)$. In particular, we obtain the following canonical isomorphisms of groups $\mathcal{B}(A) \simeq \mathcal{B}(K_0(A)) \simeq H_0(A)^*$ where $\mathcal{B}(A)$ denotes the
additive group of idempotents of \( A \). Next, we show that a morphism of rings \( A \to B \) lifts idempotents if and only if the induced ring map \( K_0(A) \to K_0(B) \) lifts idempotents. If moreover, \( B \) has finitely many maximal ideals, then we show that \( K_0(A) \to K_0(B) \) is surjective. We also show that for any nonzero ring \( A \), then \( K_0(A) \) is always an infinite ring of characteristic zero. It seems that in the literature there is no known general result to compute the group of units of the Grothendieck ring \( K_0(A) \). In this article, we also compute this group for a certain class of rings (including Dedekind domains and more generally one dimensional Noetherian rings). In fact, in Corollary 2.11 we show that for such a ring \( A \), then we have the canonical isomorphism of groups \( K_0(A)^* \simeq \text{Pic}(A) \oplus \mathcal{B}(A) \).

2. Preliminaries

In this section, we recall some basic background for the convenience of the reader.

In this article, all monoids, groups, semirings and rings are assumed to be commutative. The group of units (invertible elements) of a ring \( A \) is denoted by \( A^* \).

Recall that if \( F = \bigoplus_{k \in S} A \) is a free module over a ring \( A \) and \( M \) is an \( A \)-module, then we have the canonical isomorphism of \( A \)-modules \( F \otimes_A M \to \bigoplus_{k \in S} M \) which sends each pure tensor \( (r_k) \otimes x \) into \( (r_k x) \). The inverse map \( \bigoplus_{k \in S} M \to F \otimes_A M \) is given by \( (x_k) \mapsto \sum_{k \in S} \epsilon_k \otimes x_k \) where \( \epsilon_k = (\delta_{i,k})_{k \in S} \) and \( \delta_{i,k} \) is the Kronecker delta.

If \( M \) is a finitely generated flat module over a ring \( A \), then for each \( p \in \text{Spec}(A) \), there exists a (unique) natural number \( n_p \geq 0 \) such that \( M_p \simeq (A_p)^{n_p} \) as \( A_p \)-modules, because it is well known that every finitely generated flat module over a local ring is a free module (see [3] Theorem 7.10)). In fact, this number \( n_p \) is the dimension of \( \kappa(p) \)-vector space \( M \otimes_A \kappa(p) \) where \( \kappa(p) = A_p/pA_p \) is the residue field of \( A \) at \( p \). Hence, we obtain a function \( r_M : \text{Spec}(A) \to \mathbb{Z} \) given by \( p \mapsto \text{rank}_{K_0}(M_p) = n_p \). This function is called the rank map of \( M \). It is well known that the rank map of a finitely generated flat \( A \)-module is continuous if and only if it is a projective \( A \)-module.

Let \( A \) be a ring. It is well known that an \( A \)-module \( M \) is a finitely generated projective \( A \)-module of rank 1 if and only if the canonical morphism of \( A \)-modules \( M \otimes_A M^* \to A \) is an isomorphism where \( M^* = \text{Hom}_A(M, A) \) is the dual of \( M \). In this case, \( M^* \) is also a finitely generated projective \( A \)-module of rank 1.

For any ring \( A \), by \( \text{Pic}(A) \) we mean the set of isomorphism classes of finitely generated projective \( A \)-modules of rank 1. This set \( \text{Pic}(A) \) by the operation \( [M] \cdot [N] = [M \otimes_A N] \) is an Abelian group whose identity element is the isomorphism class of \( A \) and the inverse of each \( [M] \in \text{Pic}(A) \) is the isomorphism class of \( M^* \).

This group \( \text{Pic}(A) \) is called the Picard group of \( A \). If \( \varphi : A \to B \) is a morphism of rings, then the map \( \text{Pic}(A) \to \text{Pic}(B) \) given by \( [M] \mapsto [M \otimes_A B] \) is a morphism of groups.

If \( G = \{ [a, b] : a, b \in M \} \) is the Grothendieck group of a commutative monoid \( M \), then the canonical map \( f : M \to G \) given by \( m \mapsto [m, 0] \) is a morphism of monoids and the pair \((G, f)\) satisfies in the following universal property: for any such pair \((H, g)\), i.e., \( H \) is an Abelian group and \( g : M \to H \) is a morphism of monoids, then there exists a unique morphism of groups \( h : G \to H \) such that \( g = hf \). In fact, \( h([a, b]) = g(a) - g(b) \).
Let $S$ be a semiring. The Grothendieck group $G(S)$ of the additive monoid $(S, +)$ can be made into a ring by defining the multiplication on it as $[a, b] \cdot [c, d] = [ac + bd, ad + bc]$. The multiplicative identity of this ring is $[1, 0]$. The ring $G(S)$ is called the Grothendieck ring of the semiring $S$. The canonical map $f : S \to G(S)$ given by $s \mapsto [s, 0]$ is a morphism of semirings and the pair $(G(S), f)$ satisfies in the following universal property: For each such pair $(A, g)$, i.e. $g : S \to A$ is a morphism of semirings into a ring $A$, then there exists a unique morphism of rings $h : G(S) \to A$ such that $g = hf$.

Let $A$ be a ring. Then $S(A)$, the set of isomorphism classes of finitely generated projective $A$-modules, is a semiring with the following operations. If $M$ and $N$ are finitely generated projective $A$-modules, then the addition is defined as $[M] + [N] = [M \oplus N]$ and the multiplication is defined as $[M] \cdot [N] = [M \otimes_A N]$. The isomorphism class of the zero module is the additive identity of this semiring, and the isomorphism class of $A$ is the multiplicative identity of this semiring. In this semiring, we will often denote the isomorphism class $[M]$ simply by $M$ if there is no confusion. We denote the Grothendieck ring of the semiring $S(A)$ by $K_0(A)$. The ring $K_0(A)$ is of particular interest in mathematics, especially in algebraic K-theory.

Sometimes by abuse of the terminology, $K_0(A)$ is also called the Grothendieck ring of $A$. Every morphism of rings $\varphi : A \to B$ induces a morphism of semirings $S(A) \to S(B)$ that is given by $M \mapsto M \otimes_A B$. Then by the universal property of Grothendieck rings, we obtain a (unique) morphism of rings $K_0(\varphi) : K_0(A) \to K_0(B)$ which is given by $[M, N] \mapsto [M \otimes_A B, N \otimes_A B]$. It can be easily seen that every element of $K_0(A)$ is also of the form $[P, A^d]$ where $P$ is a finitely generated projective $A$-module and $d \geq 0$. Also note that in $K_0(A)$ we have $[M, A^m] = [N, A^n]$ if and only if $M \oplus A^{m+d} \cong N \oplus A^{m+d}$ as $A$-modules for some $d \geq 0$.

For any ring $A$, by $H_0(A)$ we mean the ring of all continuous functions $\text{Spec}(A) \to \mathbb{Z}$ where $\mathbb{Z}$ is equipped with the discrete topology. If $\varphi : A \to B$ is a morphism of rings then the map $H_0(\varphi) : H_0(A) \to H_0(B)$ given by $f \mapsto f \varphi^*$ is a morphism of rings where the map $\varphi^* : \text{Spec}(B) \to \text{Spec}(A)$ is induced by $\varphi$. For more information on this ring see e.g. [15 §5] or [18] or [13] or [1]. The map $S(A) \to H_0(A)$ given by $[M] \mapsto \tau_M$ is a morphism of semirings where $\tau_M$ denotes the rank map of $M$. Then by the universal property of Grothendieck rings, we obtain a (unique) morphism of rings $K_0(A) \to H_0(A)$ which is given by $[M, N] \mapsto \tau_M - \tau_N$.

3. Ideal class group of an extension vs Picard group

The classical ideal class group is defined for integral domains, more precisely, for such an extension $A \subseteq F$ where $A$ is an integral domain and $F$ is its field of fractions. In this section, we naturally generalize this notion to the more general setting of arbitrary extensions of rings.

Let $A \subseteq B$ be an extension of commutative rings. If $I$ and $J$ are $A$-submodules of $B$ then their multiplication $IJ$, the set of all finite sums of the form $\sum_{k=1}^n x_k y_k$ with $n \geq 1$, $x_k \in I$ and $y_k \in J$, is also an $A$-submodule of $B$. We call an $A$-submodule $I$ of $B$ an invertible ideal of this extension if $IJ = A$ for some $A$-submodule $J$ of $B$. In this case, it can be easily seen that $J = \{ b \in B : \exists a \in A \}$. Therefore such $J$ is unique and is denoted by $I^{-1}$. To prove the first main result of this section we need the following key lemmas:
Lemma 3.1. Let $A \subseteq B$ be an extension of rings and $I$ be an invertible ideal of this extension. Then for any $A$-submodule $J$ of $B$ we have the canonical isomorphism of $A$-modules $I \otimes_A J \cong IJ$.

Proof. We show that the canonical morphism of $A$-modules $I \otimes_A J \to IJ$ given by $x \otimes y \mapsto xy$ is an isomorphism. This map is clearly surjective. To see its injectivity, suppose $\sum_{k=1}^{n} x_k y_k = 0$ where $x_k \in I$ and $y_k \in J$ for all $k$. Since $II^{-1} = A$, we may write $1 = \sum_{i=1}^{d} b_i b_i'$ with $b_i \in I$ and $b_i' \in I^{-1}$ for all $i$. Then in $I \otimes_A J$ we have

\[
\sum_{k=1}^{n} x_k \otimes y_k = \sum_{k=1}^{n} x_k (\sum_{i=1}^{d} b_i b_i') \otimes y_k = \sum_{k=1}^{n} (\sum_{i=1}^{d} b_i b_i'(x_k) \otimes y_k = \sum_{k=1}^{n} (\sum_{i=1}^{d} b_i \otimes b'(x_ky_k) = \sum_{i=1}^{d} b_i \otimes (\sum_{k=1}^{n} b'(x_ky_k) = 0. \]

Lemma 3.2. Let $A \subseteq B$ be an extension of rings and $I$ be an $A$-submodule of $B$. Then the following assertions are equivalent:

(i) $I$ is an invertible ideal of the extension $A \subseteq B$.
(ii) $I$ is a finitely generated projective $A$-module of rank 1 with $IB = B$.

Proof. (i)$\Rightarrow$(ii): Since $II^{-1} = A$, we may write $1 = \sum_{k=1}^{n} x_k y_k$ where $x_k \in I$ and $y_k \in I^{-1}$ for all $k$. If $x \in I$ then $x = \sum_{k=1}^{n} (xy_k)x_k$ where $xy_k \in II^{-1} = A$ for all $k$. This shows that $I = \sum_{k=1}^{n} Ax_k$. Hence, $I$ is a finitely generated $A$-module. Then we show that $I$ is a projective $A$-module. Consider the surjective morphism of $A$-modules $\varphi : A^n \to I$ which sends each unit vector $e_k \in A^n$ into $x_k$. Then the map $\psi : I \to A^n$ given by $\psi(x) = (xy_1, \ldots, xy_n)$ is a morphism of $A$-modules and $\varphi \psi$ is the identity map of $I$. Thus the short exact sequence $0 \longrightarrow \text{Ker} \varphi \longrightarrow A^n \overset{\varphi}{\longrightarrow} I \longrightarrow 0$ splits and so $I$ is a projective $A$-module. Next, we show that $I$ is of rank 1. It suffices to show that for each $p \in \text{Spec}(A)$ then $I_p \cong A_p$ as $A_p$-modules. By Lemma 3.1 we have $I \otimes_A I^{-1} \cong A$ as $A$-modules. This yields that $I_p \otimes_{A_p} (I^{-1})_p \cong A_p$. But $I_p \cong (A_p)^m$ and $(I^{-1})_p \cong (A_p)^n$ for some natural numbers $m, n \geq 0$, because every finitely generated projective (even flat) module over a local ring is a free module. It follows that $mn = 1$ and so $m = n = 1$. Finally, it can be easily seen that $IB = B$.

As a second proof, this implication is a special case of Lemma 3.2.

(ii)$\Rightarrow$(i): Since $I$ is a finitely generated $A$-module, we have a surjective morphism of $A$-modules $\varphi : A^n \to I$ for some $n \geq 1$. Since $I$ is $A$-projective, this map splits. So there exists a morphism of $A$-modules $\psi : I \to A^n$ such that $\varphi \psi : I \to I$ is the identity map. But $I \otimes_A B$ is a finitely generated projective $B$-module of rank 1. So the canonical surjective morphism of $B$-modules $I \otimes_A B \to IB = B$ is an isomorphism. Indeed, it can be readily verified that every surjective morphism of modules between finitely generated projective modules with the same rank maps is an isomorphism. So we may consider the map $\psi \otimes 1_B : B = I \otimes_A B \to A^n \otimes_A B = B^n$. Assume $(\psi \otimes 1_B)(1) = (b_1, \ldots, b_n)$ then $\sum_{k=1}^{n} b_k x_k = 1$ where $x_k = \varphi(e_k) \in I$ for
all $k$. Let $J = \sum_{k=1}^{n} Ab_k$. Then $A \subseteq IJ$. It is clear that $\psi \otimes 1_B$ is a morphism of $B$-modules. So for each $k$ we have $(x_k b_1, \ldots, x_k b_n) = x_k (\psi \otimes 1_B)(1) = (\psi \otimes 1_B)(x_k) = (\psi \otimes 1_B)(x_k \otimes 1) = \psi(x_k) \otimes 1 = \psi(x_k) \in A^n$. This shows that $x_k b_1 \in A$ for all $i$ and so $IJ \subseteq A$.

Remark 3.3. It is important to note that the notion of an invertible ideal of an extension of rings cannot be generalized to an arbitrary morphism of rings in a satisfactory way that the essential property of invertible ideals, namely projectivity, is preserved. More precisely, for a given morphism of rings $f : A \to B$ we may define an $A$-submodule $I$ of $B$ as an invertible ideal of this map if $IJ = f(A)$ for some $A$-submodule $J$ of $B$. Then in this case, it can be easily seen that $I$ is a finitely generated $A$-module, $IB = B$ and for any $A$-submodule $J$ of $B$, $I \otimes_A J$ is canonically isomorphic to $IJ$ as $A$-modules. But $I$ is not necessarily $A$-projective. As an example, consider the canonical ring map $f : Z \to Z/2$, then $f(Z) = Z/2$ is an invertible ideal of this map but it is not $Z$-projective (even it is not $Z$-flat).

If $A \subseteq B$ is an extension of rings and $b \in B$, then $I = Ab$ is an invertible ideal of this extension if and only if $b \in B^*$. In this case, $I^{-1} = Ab^{-1}$.

Definition 3.4. Let $A \subseteq B$ be an extension of rings. Then by $\mathcal{G}(A, B)$ we mean the set of all invertible ideals of this extension, which is an Abelian group under multiplication with the identity element $A$. This group $\mathcal{G}(A, B)$ modulo $H = \{Ax : x \in B^*\}$, the subgroup of principal invertible ideals of this extension, is called the ideal class group of this extension and is denoted by $\text{Cl}(A, B)$.

We are now ready to prove the first main result of this section:

Theorem 3.5. For any extension of rings $A \subseteq B$ we have the following exact sequence of Abelian groups:

$$0 \longrightarrow \text{Cl}(A, B) \longrightarrow \text{Pic}(A) \longrightarrow \text{Pic}(B).$$

Proof. By Lemma 3.1 the canonical map $\mathcal{G}(A, B) \to \text{Pic}(A)$ given by $I \mapsto [I]$ is a morphism of groups where $\mathcal{G}(A, B)$ is the group of invertible ideals of the extension $A \subseteq B$. It is not hard to see that $H = \{Ax : x \in B^*\}$ is the kernel of this map. Indeed, if $x \in B^*$ then the map $A \to Ax$ given by $a \mapsto ax$ is an isomorphism of $A$-modules, and so $[Ax] = [A]$ is the identity element of the group $\text{Pic}(A)$. Conversely, assume $I$ is an invertible ideal of the extension $A \subseteq B$ and $h : A \to I$ is an isomorphism of $A$-modules. Then $I = Ax$ where $x = h(1)$. Also $x$ is invertible in $B$. Indeed, we may write $1 = \sum_{k=1}^{n} x_k y_k$ with $x_k \in I$ and $y_k \in I^{-1} \subseteq B$ for all $k$. But $x_k = a_k x$ with $a_k \in A$ for all $k$. Thus $x(\sum_{k=1}^{n} a_k y_k) = 1$. So $I = Ax \in H$. Hence, the above map $\mathcal{G}(A, B) \to \text{Pic}(A)$ induces an injective morphism of groups $\varphi : \text{Cl}(A, B) \to \text{Pic}(A)$. The map $\psi : \text{Pic}(A) \to \text{Pic}(B)$ given by $[M] \mapsto [M \otimes_A B]$ is a morphism of groups. If $I$ is an invertible ideal of the extension $A \subseteq B$, then by Lemma 3.1 $I \otimes_A B \simeq IB = B$ as $A$-modules. It is indeed an isomorphism as $B$-modules and so $[I \otimes_A B] = [B]$ is the identity element of the group $\text{Pic}(B)$. This shows that $\text{Im}(\varphi) \subseteq \text{Ker}(\psi)$. Conversely, let $M$ be a finitely generated projective $A$-module of rank 1 such that $M \otimes_A B \simeq B$ as $B$-modules. Thus we have an isomorphism of $B$-modules $f : M \otimes_A B \to B$. Since $M$ is $A$-flat, the canonical
morphism of $A$-modules $g: M \to M \otimes_A B$ given by $x \mapsto x \otimes 1$ is injective. Then $M$ is isomorphic to $I = \text{Im}(fg)$ as $A$-modules. Thus the $A$-submodule $I$ of $B$ is a finitely generated projective $A$-module of rank 1. We also have $IB = B$. Indeed, if $b \in B$ then we may write $b = f(\sum_{k=1}^{n} x_k \otimes b_k) = \sum_{k=1}^{n} f(x_k \otimes b_k) = \sum_{k=1}^{n} f(b_k \cdot (x_k \otimes 1))$ where $x_k \in M$ and $b_k \in B$ for all $k$. But the map $f$ is a morphism of $B$-modules, thus $b = \sum_{k=1}^{n} b_k f(x_k \otimes 1) = \sum_{k=1}^{n} b_k fg(x_k) \in IB$. Then by Lemma 3.2, $I$ is an invertible ideal of the extension $A \subseteq B$. This shows that $\text{Ker}(\psi) \subseteq \text{Im}(\varphi)$. This completes the proof.

**Corollary 3.6.** Let $A \subseteq B$ be an extension of rings such that $B$ has finitely many maximal ideals. Then we have the canonical isomorphism of groups $\text{Cl}(A,B) \simeq \text{Pic}(A)$.

**Proof.** It is well known that every finitely generated projective (even flat) module of constant rank over a ring with finitely many maximal ideals is a free module (see [4, Tags 00NX, 00NZ, 02M9]). This fact, in particular, yields that the Picard group of every ring with finitely many maximal ideals is trivial. Thus $\text{Pic}(B) = 0$. Then the assertion is deduced from Theorem 3.5.

**Corollary 3.7.** Every invertible ideal of an extension of rings $A \subseteq B$ with $A$ has finitely many maximal ideals is principal.

**Proof.** The Picard group of $A$ is trivial. Then by Theorem 3.5, the ideal class group of the extension $A \subseteq B$ is trivial.

The following result tells us that invertible ideals have the avoidance property (and considerably generalizes [9, Theorem 1.5] and our result [2, Corollary 3.2]):

**Lemma 3.8.** Let $A \subseteq B$ be an extension of rings and $I$ be an invertible ideal of this extension. If $J_1, \ldots, J_n$ are finitely many $A$-submodules of $B$ with $I \subseteq \bigcup_{k=1}^{n} J_k$, then $I \subseteq J_k$ for some $k$.

**Proof.** We may write $I = \bigcup_{k=1}^{n} I_k$ where $I_k = I \cap J_k$ for all $k$. Since $II^{-1} = A$, each $I_k = AI_k = I(I^{-1}I_k)$ and $I^{-1}I_k \subseteq I^{-1}I = A$ is an ideal of $A$ (in the usual sense). Then $I = \bigcup_{k=1}^{n} (I^{-1}I_k)I$. We know that $I$ is a finitely generated faithful $A$-module (faithfulness means that $\text{Ann}(I) = 0$). Then by [2, Lemma 3.1], we have $I^{-1}I_k = A$ and so $I = I_k \subseteq J_k$ for some $k$.

**Corollary 3.9.** Let $A$ be a ring with an extension ring whose Picard group is trivial and $M$ be a finitely generated projective $A$-module of rank 1. If $M_1, \ldots, M_n$ are finitely many $A$-submodules of $M$ with $M = \bigcup_{k=1}^{n} M_k$, then $M = M_k$ for some $k$.

**Proof.** By hypothesis, there exists an extension ring $B$ of $A$ such that $\text{Pic}(B) = 0$. Then using Theorem 3.5, there exists an invertible ideal $I$ of the extension $A \subseteq B$ such that $M \simeq I$ as $A$-modules. Then by Lemma 3.8, $M = M_k$ for some $k$.

Concerning the above result, we should add that there are rings for which the Picard group of every extension ring is non-trivial.
Let $A$ be a ring. If we take $B = T(A)$, the total ring of fractions of $A$, then every invertible ideal of this extension is simply called an invertible ideal of $A$, and the group $\text{Cl}(A, B)$ is also called the ideal class group of $A$ and is denoted by $\text{Cl}(A)$. In particular, the classical ideal class group (i.e. when $A$ is an integral domain) is naturally recovered from this construction. In fact, if $I$ is an invertible ideal of $A$ then $I$ is clearly a fractional ideal of $A$, i.e. $I$ is an $A$-submodule of $T(A)$ such that $sI \subseteq A$ for some non-zero-divisor $s \in A$.

**Corollary 3.10.** For any ring $A$ we have the following exact sequence of Abelian groups: $0 \rightarrow \text{Cl}(A) \rightarrow \text{Pic}(A) \rightarrow \text{Pic}(T(A))$.

*Proof.* It is clear from Theorem 3.5. □

**Corollary 3.11.** If $A$ is a reduced ring with finitely many minimal primes, then we have the canonical isomorphism of groups $\text{Cl}(A) \simeq \text{Pic}(A)$.

*Proof.* Since $A$ is reduced, the set of its zero-divisors is the union of its minimal primes, i.e. $Z(A) = \bigcup_{p \in \text{Min}(A)} p$. Then using the prime avoidance lemma, we conclude that $T(A)$ has finitely many maximal ideals. Hence, the Picard group of $T(A)$ is trivial. Then the assertion is deduced from Corollary 3.10. □

As an immediate consequence of Corollary 3.11, for every integral domain $A$, we have the canonical isomorphism of groups $\text{Cl}(A) \simeq \text{Pic}(A)$.

**Example 3.12.** The sequence in Theorem 3.5 is not right exact in general. That is, for a given extension of rings $A \subseteq B$, the induced group map $\text{Pic}(A) \rightarrow \text{Pic}(B)$ is not necessarily surjective. As an example, take an integral domain $R$ whose ideal class group is non-trivial (recall that every Abelian group is the ideal class group of some Dedekind domain). Thus $\text{Pic}(R) \simeq \text{Cl}(R) \neq 0$. But we have an extension of rings $\mathbb{Z} \subseteq R$ or $\mathbb{Z}/p \subseteq R$ for some prime number $p$. But the Picard group of every UFD is trivial, and so $\text{Pic}(\mathbb{Z}) = \text{Pic}(\mathbb{Z}/p) = 0$.

In the rest of this section, we will need a category, which we will call the category of extensions of rings, whose objects are the pairs $(A, B)$ where $A \subseteq B$ is an extension of rings, and whose morphisms are the arrows $\varphi : (A, B) \rightarrow (A', B')$ where $\varphi : B \rightarrow B'$ is a morphism of rings such that $\varphi(A) \subseteq A'$. Then the group of invertible ideals and the ideal class group constructions are indeed covariant functors from this category to the category of Abelian group. More precisely, let $\varphi : (A, B) \rightarrow (A', B')$ be a morphism in the category of extensions of rings. If $I$ is an invertible ideal of the extension $A \subseteq B$ then $I_{\varphi} = \sum_{x \in I} A'\varphi(x)$ is an invertible ideal of the extension $A' \subseteq B'$ and the map $\mathcal{G}(A, B) \rightarrow \mathcal{G}(A', B')$ given by $I \mapsto I_{\varphi}$ is a morphism of groups. If $x \in B^*$ then $\varphi(x) \in (B')^*$ and $(Ax)_{\varphi} = A'\varphi(x)$. Hence, we obtain a group morphism $\text{Cl}(A, B) \rightarrow \text{Cl}(A', B')$.

**Remark 3.13.** Let $\varphi : (A, B) \rightarrow (A', B')$ be a morphism in the category of extensions of rings. If $I$ and $J$ are $A$-submodules of $B'$ then $IJ$, the set of all finite sums of the form $\sum x_ky_k$ with $x_k \in I$ and $y_k \in J$, is an $A$-submodule of $B'$ (if one of them is an $A'$-module then $IJ$ is an $A'$-module). If $I \in \mathcal{G}(A, B)$ and $J$ is an $A'$-submodule of $B'$, then exactly like the proof of Lemma 3.1 one can see that $I \otimes_A J$ and $\varphi(I)J$ are canonically isomorphic as $A'$-modules. In particular, if
$I \in \mathcal{G}(A,B)$ then $I \otimes_A A' \simeq \varphi(I)A' = I_\varphi$. This shows that the following diagram is commutative:

$$
\xymatrix{
\mathcal{G}(A,B) \ar[r] \ar[d] & \text{Pic}(A) \ar[d] \\
\mathcal{G}(A',B') \ar[r] & \text{Pic}(A').
}
$$

We now examine the transitivity aspect of invertible ideals. In fact, we are interested in knowing how the ideal class group behaves when a tower of ring extensions are involved. In this context, we first prove the following general result:

**Theorem 3.14.** Let $A \subseteq B \subseteq C$ be extensions of rings. Then we have the following exact sequence of Abelian groups:

$$
\xymatrix{0 \ar[r] & \text{Cl}(A,B) \ar[r] & \text{Cl}(A,C) \ar[r] & \text{Cl}(B,C).}
$$

**Proof.** It is clear that $\mathcal{G}(A,B)$ is a subgroup of $\mathcal{G}(A,C)$, see Definition 3.3. The map $f : \mathcal{G}(A,C) \to \mathcal{G}(B,C)$ given by $I \mapsto IB$ is a morphism of groups. If $I \in \mathcal{G}(A,B)$ then $IB = B$. This shows that $\mathcal{G}(A,B) \subseteq \text{Ker}(f)$. If $I \in \text{Ker}(f)$ then $IB = B$ and so $I \subseteq B$. This shows that $I \in \mathcal{G}(A,B)$. Hence, the following sequence is exact:

$$
\xymatrix{0 \ar[r] & \mathcal{G}(A,B) \ar[r] & \mathcal{G}(A,C) \ar[r]^f & \mathcal{G}(B,C).
}
$$

Since $B^* \subseteq C^*$, we have $H_1 = \{Ax : x \in B^*\} \subseteq H_2 = \{Ax : x \in C^*\}$. This induces a group map $g : \text{Cl}(A,B) \to \text{Cl}(A,C)$ that is given by $IH_1 \mapsto IH_2$. If $I \in H_2$ for some $I \in \mathcal{G}(A,B)$ then $I = Ax$ for some $x \in C^*$. But $I \subseteq B$ and $Ax^{-1} = I^{-1} \subseteq B$. It follows that $x \in B^*$ and so $I \in H_1$. Hence $g$ is injective. It is clear that $f(H_2) \subseteq H_3 = \{Bx : x \in C^*\}$. Then we obtain a group map $h : \text{Cl}(A,C) \to \text{Cl}(B,C)$ that is given by $IH_3 \mapsto (IB)H_3$. Then it is clear that $\text{Im}(g) \subseteq \text{Ker}(h)$. If $IH_2 \in \text{Ker}(h)$ with $I \in \mathcal{G}(A,C)$, then $IB = Bx$ for some $x \in C^*$. But $J = \{b \in B : bx \in I\}$ and $L = \{b \in B : bx^{-1} \in I^{-1}\}$ are $A$-submodules of $B$. We also have $JL = A$. Because if $b \in J$ and $b' \in L$ then $bb' = (bx)(b'x^{-1}) \in I^{-1} = A$. This shows that $JL \subseteq A$. To see the reverse inclusion, we may write $1 = \sum c_k c_k'$ where $c_k \in I$ and $c_k' \in I^{-1}$ for all $k$. But from $I \subseteq IB = Bx$ we get that each $c_k = b_k x$ for some $b_k \in B$. Then $b_k \in J$ for all $k$. We also have $I^{-1}B = Bx^{-1}$. Thus each $c_k' = b_k' x^{-1}$ for some $b_k' \in B$. Then $b_k' \in L$ for all $k$. It follows that $1 = \sum b_k b_k' \in JL$. This shows that $A \subseteq JL$. Hence, $J$ is an invertible ideal of the extension $A \subseteq B$, i.e. $J \in \mathcal{G}(A,B)$. We also have $I = Jx$. This shows that $g(JH_1) = JH_2 = IH_2$. Thus $\text{Ker}(h) \subseteq \text{Im}(g)$. This completes the proof. \[\square\]

**Remark 3.15.** Let $A \subseteq B \subseteq C$ and $A' \subseteq B' \subseteq C'$ be extensions of rings and let $\varphi : C \to C'$ be a ring map such that $\varphi(A) \subseteq A'$ and $\varphi(B) \subseteq B'$. If $I \in \mathcal{G}(A,C)$ then we have $(IB)_\varphi = I_\varphi B'$. In other words, the following diagram is commutative:

$$
\xymatrix{\mathcal{G}(A,C) \ar[r] \ar[d] & \mathcal{G}(B,C) \ar[d] \\
\mathcal{G}(A',C') \ar[r] & \mathcal{G}(B',C').}
$$
The sequence in Theorem 3.14 is not right exact in general. In other words, the canonical map $\text{Cl}(A, C) \to \text{Cl}(B, C)$ is not necessarily surjective. First note that for any extensions of rings $A \subseteq B \subseteq C$ the following diagram is commutative:

$$
\begin{array}{ccc}
\text{Cl}(A, C) & \longrightarrow & \text{Pic}(A) \\
\downarrow & & \downarrow \\
\text{Cl}(B, C) & \longrightarrow & \text{Pic}(B) \\
\end{array}
$$

Indeed, if $I \in \mathcal{G}(A, C)$ then by Lemma 3.1 or Remark 3.13, $I \otimes_A B \simeq IB$ as $B$-modules. In the above diagram, the bottom row is also exact (see Theorem 3.5). Now for instance, if the Picard groups of $A$ and $C$ are trivial but $\text{Pic}(B) \neq 0$ (see Example 3.12 or [12, Example 3.1]), then the map $\text{Cl}(A, C) \to \text{Cl}(B, C)$ is not surjective.

However, we will observe that the canonical map $\text{Cl}(A, C) \to \text{Cl}(B, C)$ is surjective in some cases. To this end, let us recall that a ring $A$ modulo its nil-radical is denoted by $A_{\text{red}}$. Also recall from [14, §2] that a ring extension $A \subseteq B$ is called subintegral if it is integral, the induced map between the corresponding prime spectra is bijective and the extensions between the residue fields are isomorphisms. We then arrive at the following interesting result:

**Theorem 3.16.** Let $A \subseteq B \subseteq C$ be extensions of rings such that one of the following conditions hold:

(i) the canonical map $\text{Cl}(A', C') \to \text{Cl}(B', C')$ is surjective where $R' = R_{\text{red}}$.

(ii) the extension $A \subseteq C$ is subintegral.

Then we have the following short exact sequence of groups:

$$
0 \longrightarrow \text{Cl}(A, B) \longrightarrow \text{Cl}(A, C) \longrightarrow \text{Cl}(B, C) \longrightarrow 0.
$$

**Proof.** First, suppose that (i) holds. To prove the assertion, by Theorem 3.14 it suffices to show that the canonical map $\text{Cl}(A, C) \to \text{Cl}(B, C)$ is surjective. By Remark 3.15 the following diagram is commutative:

$$
\begin{array}{ccc}
\text{Cl}(A, C) & \longrightarrow & \text{Cl}(B, C) \\
\downarrow & & \downarrow \\
\text{Cl}(A', C') & \longrightarrow & \text{Cl}(B', C').
\end{array}
$$

We claim that in the above diagram, the vertical arrows are isomorphisms. To see this, consider the following diagram:

$$
\begin{array}{ccc}
0 & \longrightarrow & \text{Cl}(A, C) \longrightarrow \text{Pic}(A) \longrightarrow \text{Pic}(C) \\
\downarrow & & \downarrow \simeq \downarrow \simeq \\
0 & \longrightarrow & \text{Cl}(A', C') \longrightarrow \text{Pic}(A') \longrightarrow \text{Pic}(C').
\end{array}
$$

where the rows are exact (see Theorem 3.5) and it is well known that the second and third vertical arrows are the canonical isomorphisms. The above diagram is also commutative, since the commutativity of the left-hand side follows from Remark 3.13 and the right-hand side is clear. Hence, the vertical map $\text{Cl}(A, C) \to \text{Cl}(A', C')$ is an isomorphism. By the same argument, the canonical map $\text{Cl}(B, C) \to \text{Cl}(B', C')$ is also an isomorphism. This completes the proof.
of the claim. Then using hypothesis (i), we see that the map $\text{Cl}(A, C) \to \text{Cl}(B, C)$ is surjective.

Now assume that (ii) holds. Then by [12, Theorem 3.3], the canonical map $\mathcal{I}(A, C) \to \mathcal{I}(B, C)$ is surjective. Hence, the induced map $\text{Cl}(A, C) \to \text{Cl}(B, C)$ is surjective. Then apply Theorem 3.14. □

Remark 3.17. Note that Theorem 3.16 also holds (with the same proof) if we replace the condition (i) with the weaker condition that “the canonical map $\text{Cl}(A/\;(A \cap I), C/I) \to \text{Cl}(B/(B \cap I), C/I)$ is surjective where $I$ is an ideal of $C$ contained in the nil-radical.” Also note that the canonical map $\text{Cl}(A, C) \to \text{Cl}(B, C)$ is surjective if and only if the canonical map $\text{Cl}(A_{\text{red}}, C_{\text{red}}) \to \text{Cl}(B_{\text{red}}, C_{\text{red}})$ is surjective, or equivalently, the above map is surjective.

4. PICARD GROUP VS THE GROUP OF UNITS OF THE GROTHENDIECK RING

The first main result of this section asserts that there is a short complex (quasi-exact sequence) of Abelian groups:

$$0 \longrightarrow \text{Pic}(A) \longrightarrow K_0(A)^* \longrightarrow \mathcal{B}(A) \longrightarrow 0.$$  

for which the main factors involved are the Picard group $\text{Pic}(A)$ and the group of units of the Grothendieck ring $K_0(A)$ and the other factor of this sequence is $\mathcal{B}(A)$, the additive group of idempotents of $A$ (and that this sequence is split exact in some special cases). Next, we prove several results on the Grothendieck ring $K_0(A)$. To achieve these goals, we need the following results, which are interesting in their own right.

Lemma 4.1. If $e$ and $e'$ are idempotents of a ring $A$, then we have the canonical isomorphism of $A$-modules:

$$Ae \simeq Ae(1 - e') \oplus Ae/Ae(1 - e').$$

Proof. Consider the following canonical short exact sequence of $A$-modules:

$$0 \longrightarrow Ae(1 - e') \longrightarrow Ae \longrightarrow Ae/Ae(1 - e') \longrightarrow 0.$$  

But for each idempotent $e \in A$, we have $Ae \cap A(1-e) = 0$ thus $Ae \oplus A(1-e) \simeq A$ and so $Ae$ is a (finitely generated) projective $A$-module. It follows that $Ae/Ae(1-e')$ is also a projective $A$-module, because $Ae/A(1-e') \simeq Ae \otimes_A Ae'$. Hence, the above sequence splits. So, $Ae \simeq Ae(1-e') \oplus Ae/Ae(1-e')$. □

Lemma 4.2. If $e$ and $e'$ are orthogonal idempotents of a ring $A$, then we have the canonical isomorphism of $A$-modules $A(e+e') \simeq Ae \oplus Ae'$.

Proof. The map $f : A \to Ae \oplus Ae'$ given by $f(r) = (re, re')$ is a morphism of $A$-modules. Clearly $\text{Ker}(f) = A(1-e-e')$, because if $f(r) = 0$, then $re = re' = 0$ and so $r = r(1-e-e') \in A(1-e-e')$. The map $f$ is also surjective, because if $(a,b) \in A^2$ then $f(\alpha e + \beta e') = \alpha e \oplus \beta e'$. Thus $f$ induces an isomorphism of $A$-modules $A/A(1-e-e') \simeq Ae \oplus Ae'$. We also have $\text{Ann}(e+e') = A(1-e-e')$, because $e+e'$ is idempotent. Thus $A(e+e') \simeq A/\text{Ann}(e+e') \simeq Ae \oplus Ae'$. □

The above two lemmas gives us the following useful formula:
Corollary 4.3. If $e$ and $e'$ are idempotents of a ring $A$, then we have the canonical isomorphism of $A$-modules:

$$Ae \oplus Ae' \cong Ae/Ae(1-e') \oplus Ae'/Ae'(1-e) \oplus A(e+e'-2e'e').$$

Proof. We may write $e+e'-2e'e' = e(1-e') + e'(1-e)$. Then setting $a := e(1-e')$ and $b := e'(1-e)$. Clearly $ab = 0$. So by Lemma 4.2, $A(a+b) \cong Aa \oplus Ab$. Then by applying Lemma 4.1, the assertion is easily deduced.

By $\mathcal{B}(A) = \{e \in A : e = e^2\}$ we mean the set of all idempotent elements of a ring $A$ which is an Abelian group under the operation $e \oplus e' := e + e' - 2ee'$ (the calligraphy letter $\mathcal{B}$ stands for George Boole). For more information on this group we refer the interested reader to [17].

Lemma 4.4. For any ring $A$, we have the canonical isomorphism of groups $\mathcal{B}(A) \cong H_0(A)^*$. 

Proof. If $e \in A$ is an idempotent then we have a continuous map $\varphi_e : \text{Spec}(A) \to \mathbb{Z}$ which is defined as $\varphi_e(p) = 1$ if $e \in p$ and otherwise $\varphi_e(p) = -1$. We show that the map $e \mapsto \varphi_e$ is an isomorphism of groups from the additive group $\mathcal{B}(A)$ onto $H_0(A)^*$. Clearly $\varphi_e \in H_0(A)^*$, because $\varphi_e^2 = 1$ (also note that $\varphi_e = -\varphi_{1-e}$). Then we show that the above map is a morphism of groups, i.e., $\varphi_{e \oplus e'} = \varphi_e \cdot \varphi_{e'}$ for any idempotents $e$ and $e'$ of $A$. Let $p$ be a prime ideal of $A$. If both $e, e' \in p$ then $e \oplus e' \in p$ and so $\varphi_{e \oplus e'}(p) = 1 = (\varphi_e \cdot \varphi_{e'})(p)$. Suppose $e \in p$ but $e' \notin p$ then $e \oplus e' \notin p$, because $e'(e+e') = e'(1-e) \notin p$, so in this case $\varphi_{e \oplus e'}(p) = -1 = (\varphi_e \cdot \varphi_{e'})(p)$. Finally, suppose $e, e' \notin p$ then $e \oplus e' \in p$, because we have $e \oplus e' = e(1-e') + e'(1-e) \in p$, thus in this case $(\varphi_e \cdot \varphi_{e'})(p) = \varphi_e(p) \cdot \varphi_{e'}(p) = (-1) \cdot (-1) = 1 = \varphi_{e \oplus e'}(p)$. Hence, the above map is a morphism of groups. If $\varphi_e = \varphi_{e'}$ for some idempotents $e, e' \in A$, then clearly $D(e) = D(e')$ and so $e = e'$. If $f \in H_0(A)^*$ then by [13] Theorem 1.1, there exists a (unique) idempotent $e \in A$ such that $f^{-1}(1) = V(e)$. Since $f^2 = 1$, so $\varphi_e = f$. This completes the proof.

It is well known that the Picard group $\text{Pic}(A)$ can be canonically embedded in the group of units of the Grothendieck ring $K_0(A)$. In the following result, not only it is proved by a new method, we also complete this observation by further involving the additive group of idempotents $\mathcal{B}(A)$. This result, in particular, paves the way to understand the structure of the group $K_0(A)^*$ for a certain class of rings (see Corollary 4.11).

Theorem 4.5. For a ring $A$, the following assertions hold:

(i) We have an injective morphism of groups $f : \text{Pic}(A) \to K_0(A)^*$.

(ii) We have a surjective morphism of groups $g : K_0(A)^* \to \mathcal{B}(A)$ and an injective morphism of groups $h : \mathcal{B}(A) \to K_0(A)^*$ such that $gh = \text{Id}$ is the identity map and $gf = 0$ for $f$ see (i).

Proof. (i): We show that the map $f : \text{Pic}(A) \to K_0(A)^*$ given by $[M] \mapsto [M,0]$ is an injective morphism of groups. If $[M] \in \text{Pic}(A)$ then $M \otimes_A M^* \cong A$. Thus $[M,0] \cdot [M^*,0] = [A,0]$. Hence, $[M,0]$ is invertible in $K_0(A)$ and so the above map is well-defined. This map is clearly a morphism of groups. For injectivity, suppose $[M,0] = [A,0]$. Then there exists some natural number $n \geq 0$ such that $M \oplus A^n \cong A^{n+1}$ as $A$-modules. It suffices to show that $M \cong A$ as $A$-modules. We will use exterior powers to establish this isomorphism. It is well known that for any two modules $M$ and $N$ over a ring $A$, we have the canonical isomorphism of
$A$-modules $\bigwedge^k(M \oplus N) \cong \bigoplus_{p+q=k} \bigwedge^p(M) \otimes_A \bigwedge^q(N)$. It is also well known that if $F$ is a free $A$-module of rank $d \geq 0$, then $\bigwedge^k(F)$ is a free $A$-module of rank $\binom{d}{k}$ for all $0 \leq k \leq d$ and $\bigwedge^k(F) = 0$ for all $k > d$. Finally, since $M$ is a finitely generated projective $A$-module of rank 1, thus $\bigwedge^k(M) = 0$ for all $k \geq 2$, because if $p \in \text{Spec}(A)$ then we have the canonical isomorphisms of $A_p$-modules $(\bigwedge^k_A(M))_p \cong \bigwedge^k_A(M) \otimes_A A_p \cong \bigwedge^k_A(M_p) \cong \bigwedge^k_A(M_p) = 0$ for all $k \geq 2$. Now using these observations, we have $A \cong \bigwedge^{n+1}(A^{n+1}) \cong \bigwedge^{n+1}(M \oplus A^n) \cong \bigoplus_{p+q=n+1} \bigwedge^p(M) \otimes_A \bigwedge^q(A^n) \cong \bigwedge^{1}(M) \otimes_A \bigwedge^n(A^n) \cong M \otimes_A A \cong M$ as $A$-modules.

(ii): The canonical ring map $K_0(A) \to H_0(A)$ given by $[M, N] \mapsto r_M - r_N$ induces a group map $K_0(A)^* \to H_0(A)^*$. Then using this and Lemma 4.4 we obtain a group map $g : K_0(A)^* \to \mathcal{B}(A)$ given by $[M, N] \mapsto e$ where $e \in A$ is an idempotent with $(r_M - r_N)^{-1}(\{1\}) = V(e)$. If $e \in A$ is an idempotent, then $1 - 2e$ is invertible in $A$, because $(1 - 2e)^2 = 1$. Also, the element $[Ae, 0]$ is an idempotent of $K_0(A)$, because $Ae \otimes_A Ae \cong Ae \otimes_A A/(1 - e) \cong Ae/Ae(1 - e) \cong Ae$ and so $[Ae, 0] \cdot [Ae, 0] = [Ae \otimes_A Ae, 0] = [Ae, 0]$. Using these observations, we obtain a function $h : \mathcal{B}(A) \to K_0(A)^*$ which is defined by $e \mapsto [Ae, 0 \oplus Ae]$. By Corollary 4.6 h is a group morphism. Its composition with $g$ gives us the identity map of $\mathcal{B}(A)$. Hence, $g$ is surjective and $h$ is injective. Finally, we show that $gf = 0$, i.e., $\text{Im}(f) \subseteq \text{Ker}(g)$. If $[M] \in \text{Pic}(A)$ then $M$ is a finitely generated projective $A$-module of rank 1. Hence, its rank map is the constant function 1. Thus we have $(r_M - r_0)^{-1}(\{1\}) = (r_M)^{-1}(\{1\}) = \text{Spec}(A) = V(0)$. This shows that $g([M, 0]) = 0$.

Remark 4.6. In the proof of Theorem 4.5 we observed that if $M$ is a finitely generated projective $A$-module of rank 1, then from $M \oplus A^n \cong A^{n+1}$ we obtained that $M \cong A$. But it is important to notice that this conclusion does not hold in general. More precisely, let $M$ be a module over a ring $A$ such that there exists natural numbers $d, n \geq 0$ for which $M \oplus A^d \cong A^n$ as $A$-modules (in this case, $M$ is called a stably free module). Then clearly $d \leq n$ and $M$ is a finitely generated projective $A$-module of constant rank $n - d$. If $n - d \geq 2$ then $M$ is not necessarily a free module. That is, there are stably free modules which are not free (see e.g. [5, Chap. 3], [6, p. 301] or [7, Chap. XXI, §2]).

Let $A$ be a ring. It is well known that the canonical rank map $K_0(A) \to H_0(A)$ given by $[M, N] \mapsto r_M - r_N$ is a surjective morphism of rings and its kernel is the nil-radical of $A$. Showing that the kernel of this map is contained in the nil-radical is the most difficult and technical part of the proof (for details see [11, Chap. IX, §3, Proposition 4.6] or [13, Corollary 10.7] or [18, Chap. II, §4, Corollary 4.6.1]). Then we have the canonical isomorphism of rings $K_0(A)_{\text{red}} \cong H_0(A)$. This fundamental identification leads us to the following nontrivial results.

Corollary 4.7. For any ring $A$, we have the canonical isomorphism of groups $\mathcal{B}(A) \cong \mathcal{B}(K_0(A))$.

Proof. We show that the map $\mathcal{B}(A) \to \mathcal{B}(K_0(A))$ given by $e \mapsto [Ae, 0]$ is an isomorphism of groups. In the proof of Theorem 4.5 we observed that $[Ae, 0]$ is idempotent. By Corollary 4.5 this map is a group morphism. Suppose $[Ae, 0] = [Ae', 0]$ for some idempotents $e, e' \in A$. To prove $e = e'$ it suffices to show that
\[ D(e) = D(e'). \] There exists some \( n \geq 0 \) such that \( Ae \oplus A^n \simeq Ae' \oplus A^n \) as \( A \)-modules. Thus the rank maps of \( Ae \oplus A^n \) and \( Ae' \oplus A^n \) are the same, and so \( r_{Ae} = r_{Ae'} \). Now if \( p \in D(e) \) then \( 1 = r_{Ae}(p) = r_{Ae'}(p) \). This shows that \( p \in D(e') \). Similarly, \( D(e') \subseteq D(e) \). Finally, we show that the above map is surjective. If \( z \in K_0(A) \) is an idempotent then its image \( g := \varphi(z) \) under the canonical ring map \( \varphi : K_0(A) \to H_0(A) \) is idempotent. It follows that \( g(p) \in \{0, 1\} \) for all \( p \in \text{Spec}(A) \). Since \( g : \text{Spec}(A) \to \mathbb{Z} \) is a continuous map, there exists an idempotent \( e \in A \) such that \( g^{-1}(\{1\}) = D(e) \). This shows that \( r_{Ae} = g \). Thus \([Ae, 0] - z \) is contained in \( \text{Ker}(\varphi) \). But \( \text{Ker}(\varphi) \) is the nil-radical of \( K_0(A) \). It can be easily seen that if \( e \) and \( e' \) are idempotents of a ring \( A \) such that \( e - e' \) is contained in the Jacobson radical of \( A \), then \( e = e' \). Therefore \( z = [Ae, 0] \). \( \square \)

We say that a morphism of rings \( f : A \to B \) lifts idempotents if \( e' \in B \) is an idempotent then there exists an idempotent \( e \in A \) such that \( f(e) = e' \).

**Corollary 4.8.** A morphism of rings \( f : A \to B \) lifts idempotents if and only if \( K_0(f) : K_0(A) \to K_0(B) \) lifts idempotents.

**Proof.** Assume \( f \) lifts idempotents. By Corollary 4.7 each idempotent of \( K_0(B) \) is of the form \([Be', 0]\) where \( e' \in B \) is idempotent. So there exists an idempotent \( e \in A \) such that \( f(e) = e' \). This yields that \( Ae \otimes_A B \simeq A/(1 - e) \otimes_A B \simeq B/(1 - e') \simeq Be' \). This shows that the image of the idempotent \([Ae, 0]\) under the map \( K_0(f) \) equals \([Be', 0]\). Hence, \( K_0(f) \) lifts idempotents. Conversely, if \( e' \in B \) is an idempotent then by hypothesis, there exists an idempotent \( e \in A \) such that \([Bf(e), 0] = [Be', 0]\). But in the proof of Corollary 4.7 we observe that in this case, \( f(e) = e' \). Hence, \( f \) lifts idempotents. \( \square \)

**Corollary 4.9.** If a morphism of rings \( f : A \to B \) lifts idempotents and \( B \) has finitely many maximal ideals, then \( K_0(f) : K_0(A) \to K_0(B) \) is surjective.

**Proof.** Take \( z' \in K_0(B) \). Then consider the following diagram of commutative rings:

\[
\begin{array}{ccc}
K_0(A) & \xrightarrow{K_0(f)} & K_0(B) \\
\downarrow{\varphi} & & \downarrow{\psi} \\
H_0(A) & \xrightarrow{H_0(f)} & H_0(B)
\end{array}
\]

where the vertical arrows are the canonical rank maps. Since \( f \) lifts idempotents, then by 13 Theorem 5.2, \( H_0(f) \) is surjective. The canonical map \( \varphi \) is also surjective. Thus there exists some \( z \in K_0(A) \) such that \( \psi(z') = H_0(f)\varphi(z) \). We know that if \( M \) is a finitely generated projective (resp. flat) \( A \)-module, then \( M \otimes_A B \) is a finitely generated projective (resp. flat) \( B \)-module and we have \( r_{M \otimes_A B} = r_M \circ f^* \). This shows that the above diagram is commutative. It follows that \( z' - z'' \in \text{Ker}(\psi) \) where \( z'' \) is the image of \( z \) under \( K_0(f) \). We know that \( \text{Ker}(\psi) \) is the nil-radical of \( K_0(B) \). But the nil-radical of \( K_0(B) \) is precisely the set of all elements of the form \([N, B^d]\) where \( N \) is a finitely generated projective \( B \)-module of rank \( d \geq 0 \). It is well known that every finitely generated projective (even flat) module of constant rank over a ring with finitely many maximal ideals is a free module (see 3 Tags 00NX, 00NZ, 02M9]). Therefore \( N \simeq B^d \) as \( B \)-modules and so \([N, B^d] = 0 \). This shows that \( K_0(B) \) is a reduced ring. Thus \( z' = z'' = K_0(f)(z) \). Hence, \( K_0(f) \) is surjective. \( \square \)
If $A$ is a local ring or a PID, then every projective module over $A$ is a free $A$-module. Thus in this case, the semiring of the isomorphism classes of finitely generated projective $A$-modules is isomorphic to the semiring of natural numbers $\mathbb{N}$. Hence, the ring $K_0(A)$ is isomorphic to the ring of integers $\mathbb{Z}$. Then using this observation, if $p$ is a prime ideal of a ring $A$ then the canonical ring map $\pi : A \to A_p$ induces a surjective ring map $K_0(\pi) : K_0(A) \to K_0(A_p) = \mathbb{Z}$. Similarly, if $m$ is a maximal ideal of $A$ then the canonical ring map $\pi : A \to A/m$ induces a surjective ring map $K_0(\pi) : K_0(A) \to K_0(A/m) = \mathbb{Z}$. Thus if $A$ is a nonzero ring, then the ring $K_0(A)$ is infinite. In other words, a ring $A$ is zero if and only if $K_0(A) = 0$, or equivalently, $K_0(A)$ is a finite ring.

**Corollary 4.10.** If $A$ is a nonzero ring, then $K_0(A)$ and $H_0(A)$ are infinite rings of characteristic zero.

**Proof.** In the above argument we observed that $K_0(A)$ is infinite. In fact, there exists a prime ideal $P$ in $K_0(A)$ such that the ring $K_0(A)/P$ is isomorphic to $\mathbb{Z}$. Then we obtain a surjective (ring) map $H_0(A) \simeq K_0(A)_{\text{red}} \to K_0(A)/P \simeq \mathbb{Z}$. Thus $H_0(A)$ is also infinite. The map $\mathbb{N} \to H_0(A)$ which assigns to each $n \in \mathbb{N}$ the constant function $c_n : \text{Spec}(A) \to \mathbb{Z}$ of $n$ is a morphism of semirings. We know that the ring of integers $\mathbb{Z}$ is indeed the Grothendieck ring of the semiring of natural numbers. Thus by the universal property of Grothendieck rings, there is a (unique) morphism of rings $\mathbb{Z} \to H_0(A)$ given by $[m, n] \mapsto c_m - c_n$. This map is injective, because $A$ is nonzero and so it has at least a prime ideal. Thus $H_0(A)$ is of characteristic zero. Then consider the map $H_0(A) \to K_0(A)$ given by $f \mapsto \sum_{n \in \mathbb{Z}} n[ Ae_n, 0]$ where $e_n \in A$ is an idempotent with $f^{-1}([n]) = D(e_n)$ for all $n \in \mathbb{Z}$. It can be seen that this map is an injective morphism of rings. Hence, $K_0(A)$ is also of characteristic zero. 

We say that a ring $A$ has the line bundle property if whenever $M$ is a finitely generated projective $A$-module of constant rank $d + 1$ with $d \geq 0$, then $A^n \oplus M \simeq A^{n+d} \oplus A^{d+1}(M)$ as $A$-modules for some $n \geq 0$. It is well known (due to J.P. Serre) that every Noetherian one dimensional ring has the line bundle property (with $n = 0$). Its proof can be found in [13, Chap I, Proposition 3.4].

**Corollary 4.11.** If a ring $A$ has the line bundle property, then we have the following split exact sequence of groups:

$$0 \longrightarrow \text{Pic}(A) \longrightarrow K_0(A)^* \longrightarrow \mathcal{B}(A) \longrightarrow 0.$$ 

**Proof.** By Theorem 4.6 it suffices to show that $\text{Ker}(g) \subseteq \text{Im}(f)$ where $f : \text{Pic}(A) \to K_0(A)^*$ and $g : K_0(A)^* \to \mathcal{B}(A)$. Take $[M, A^d] \in \text{Ker}(g)$ where $M$ is a finitely generated projective $A$-module and $d \geq 0$. It follows that $(r_M - r_{A^d})^{-1}([1]) = (r_M - d)^{-1}([1]) = V(0) = \text{Spec}(A)$. This shows that $M$ is of constant rank $d + 1$. Thus $N = A^{d+1}(M)$ is a finitely generated projective $A$-module of rank 1 and so $[N] \in \text{Pic}(A)$. Since $A$ has the line bundle property, $A^n \oplus M \simeq A^{n+d} \oplus N$ as $A$-modules for some $n \geq 0$. This shows that $[M, A^d] = [N, 0] \in \text{Im}(f)$. 

In particular, if $A$ has the line bundle property with no nontrivial idempotents, then $K_0(A)^* \simeq \text{Pic}(A) \oplus \mathbb{Z}/2$ where $\mathbb{Z}/2 = \{0, 1\}$ is the additive group of integers modulo 2.
Corollary 4.12. If a ring $A$ has the line bundle property, then the following assertions hold:

(i) If $\text{Min}(A)$ is a finite set, then $K_0(A)^* \simeq \text{Pic}(A) \oplus (\mathbb{Z}/2)^d$,

(ii) If $\text{Max}(A)$ is a finite set, then $K_0(A)^* \simeq (\mathbb{Z}/2)^d$,

where $d \geq 0$ is the number of connected components of $\text{Spec}(A)$.

Proof. By Corollary 4.11, we have $K_0(A)^* \simeq \text{Pic}(A) \oplus \mathcal{B}(A)$. If a ring has finitely many minimal primes or finitely many maximal ideals, then it has finitely many idempotents and so $\text{Spec}(A)$ has finitely many connected components. Then using the Chinese Remainder Theorem and [15, Lemma 4.7], we observe that the group $\mathcal{B}(A)$ is isomorphic to the additive group $(\mathbb{Z}/2)^d$. Note that if $A$ has finitely many maximal ideals, then its Picard group is trivial. □

Every Noetherian one dimensional ring satisfies in the hypothesis of the above result. In particular, if $A$ is a Dedekind domain then we have the canonical isomorphism of groups $K_0(A)^* \simeq \text{Cl}(A) \oplus \mathbb{Z}/2$.

Corollary 4.13. Every Abelian group $G$ can be embedded in $K_0(A)^*$ for some Dedekind domain $A$ and $K_0(A)^* \simeq G \oplus \mathbb{Z}/2$.

Proof. It is well known that every Abelian group $G$ is isomorphic to the ideal class group of a Dedekind domain $A$ (see [3, Theorem 7]). By Corollary 3.11, $\text{Cl}(A) \simeq \text{Pic}(A)$. But every Dedekind domain has the line bundle property. Then the assertion follows from Corollary 4.11. □

5. Appendix

In this section, we prove several results on finitely generated projective modules that have close connections with some results of this article and also slightly improve related results in the literature.

Recall that if $M$ is a module over a ring $A$ then we have a canonical morphism of $A$-modules $M \otimes_A M^* \rightarrow A$ that is given by $x \otimes f \mapsto f(x)$ where $M^* = \text{Hom}_A(M, A)$. Then the image of this map is an ideal of $A$ which is called the trace ideal of $M$ and is denoted by $\text{tr}(M)$. Then the following result improves [11, Chap 3, Prop. 20].

Lemma 5.1. Let $M$ be a finitely generated projective module over a ring $A$. Then $\text{Supp}(M) = \text{Spec}(A)$ if and only if $\text{tr}(M) = A$.

Proof. First assume $\text{Supp}(M) = \text{Spec}(A)$. If $J = \text{tr}(M)$ is a proper ideal of $A$, then $J \subseteq p$ for some $p \in \text{Spec}(A)$. It is well known that $JM = M$. For its proof see e.g. [10, Theorem 3.1]. It follows that $(JR_p)M_p = M_p$. Thus by the Nakayama lemma, $M_p = 0$ which is a contradiction. Conversely, assume $\text{tr}(M) = A$. It is well known that (see e.g. [10, Corollary 3.2]) there exists an idempotent $e \in A$ such that $\text{tr}(M) = Ae$ and $I = \text{Ann}(M) = A(1 - e)$. Thus $e = 1$ and so $\text{Supp}(M) = V(I) = V(0) = \text{Spec}(A)$. □

There is a minor gap in the last lines of the proof of [1, Chap. III, Proposition 7.4]. In the following result, we fill it in.

Lemma 5.2. Let $M$ and $N$ be modules over a ring $A$. If $M \otimes_A N \simeq A^n$ as $A$-modules for some $n \geq 1$, then $M$ and $N$ are finitely generated projective $A$-modules.

Proof. Clearly $M \otimes_A N = (x_k \otimes y_k : k = 1, \ldots, d)$ is a finitely generated $A$-module where $d \geq n$. By the universal property of free modules, there is a (unique)
morphism of $A$-modules $h : A^d \to M$ such that $h(\epsilon_k) = x_k$ for all $k$. Since each $x_k \otimes \epsilon_k$ is in the image of the induced morphism $h \otimes 1_N : A^d \otimes_A N \to M \otimes_A N$, thus it is surjective. In fact, $h \otimes 1_N$ is a split epimorphism. That is, we have the following split exact sequence:

$$0 \to K \xrightarrow{inc} A^d \otimes_A N \xrightarrow{h \otimes 1_N} M \otimes_A N \to 0$$

where $K$ is the kernel of $h \otimes 1_N$. Note that split exact sequences are left split exact by additive functors. Hence, by applying the additive functor $\otimes_A -$ to the above sequence, we obtain the following split exact sequence:

$$0 \to M \otimes_R K \xrightarrow{} A^{nd} \xrightarrow{} M^n \to 0.$$

Since $n \geq 1$, so $M$ is a direct summand of the free $A$-module $A^{nd}$. Hence, $M$ is a finitely generated projective $A$-module. Similarly, $N$ is also a finitely generated projective $A$-module.

In Lemma 5.2 if $n = 1$ then $M$ and $N$ are finitely generated projective $A$-modules of rank 1. Also note that this lemma does not hold for $n = 0$. For instance, let $I$ and $J$ be coprime ideals of a ring $A$, then $A/I \otimes_A A/J = 0$ but $A/I$ and $A/J$ are not necessarily $A$-projective (nor $A$-flat). As a specific example, in the ring of integers $\mathbb{Z}$, take $I = 2\mathbb{Z}$ and $J = 3\mathbb{Z}$.

In the following result, we characterize finitely generated projective modules in terms of orthogonal idempotents.

**Lemma 5.3.** Let $M$ be a finitely generated module over a ring $A$. Then $M$ is a projective $A$-module if and only if there exists a finite sequence $e_0, \ldots, e_n$ of orthogonal idempotents of $A$ such that $\sum_{k=0}^{n} e_k = 1$ and $M_p \simeq (A_p)^k$ for all $p \in D(e_k)$. In this case, the annihilator of the $A$-module $\Lambda^k(M)$ is generated by the idempotent $\sum_{i=0}^{k-1} e_i$ for all $k \in \{1, \ldots, n, n+1\}$.

**Proof.** If $M$ is $A$-projective, then its rank map $r_M : \text{Spec}(A) \to \mathbb{Z}$ is continuous. Using the quasi-compactness of the prime spectrum, there exists a natural number $n \geq 0$ such that $\text{Spec}(A) = \bigcup_{k=0}^{n} r_M^{-1}(\{k\})$. Clearly each $r_M^{-1}(\{k\})$ is a clopen (both open and closed) subset of $\text{Spec}(A)$. Thus there exists an idempotent $e_k \in A$ such that $r_M^{-1}(\{k\}) = D(e_k)$. Now the desired assertion is easily deduced. Next, we prove the reverse implication. By the hypothesis, $M$ is a flat $A$-module, because flatness is a local property. Again by the hypothesis, the rank map of $M$ is continuous. Hence, $M$ is $A$-projective. Now we show that the annihilator of $\Lambda^k(M)$ is generated by $\sum_{i=0}^{k-1} e_i$. Indeed, we have the canonical isomorphism of $A_p$-modules $\Lambda^k_A(M) \otimes_A A_p \simeq \Lambda^k_{A_p}(M_p)$. Also remember that if $F$ is a free $A$-module of rank $d \geq 0$, then $\Lambda^k(F)$ is a free $A$-module of rank $\binom{d}{k}$ and hence $\Lambda^k(F) \neq 0$ for all $0 \leq k \leq d$ and $\Lambda^k(F) = 0$ for all $k > d$. Therefore $\text{Supp}(\Lambda^k(M)) = \bigcup_{i=k}^{n} D(e_i) = D(\sum_{i=k}^{n} e_i)$. Since $\Lambda^k(M)$ is a finitely generated projective $A$-module, so its annihilator is generated by an idempotent element $e \in A$, because the annihilator of every finitely generated projective module is generated by an idempotent element (see e.g. [10] Corollary.
Thus \( \text{Supp} \left( \Lambda_k^A(M) \right) = V(e) = D(1 - e) \). It follows that \( 1 - e = \sum_{i=k}^{n} e_i \), hence \( e = \sum_{i=0}^{k-1} e_i \). □

**Example 5.4.** We illustrate Lemma 5.3 with two examples. If \( A \) is a ring then for \( M = A^2 \) we have the sequence \( e_0 = e_1 = 0 \) and \( e_2 = 1 \). As another example, if \( e \in A \) is an idempotent then for the projective \( A \)-module \( M = Ae \) we have the sequence \( e_0 = 1 - e \) and \( e_1 = e \).

The following result shows that being of finite type in modules is a local property.

**Lemma 5.5.** Let \( M \) be a module over a ring \( A \) with the property that for each \( p \in \text{Spec}(A) \) there exists some \( f \in A \setminus p \) such that \( M_f \) is a finitely generated \( A_f \)-module. Then \( M \) is a finitely generated \( A \)-module.

**Proof.** Using the quasi-compactness of \( \text{Spec}(A) \), there exist finitely many elements \( f_1, \ldots, f_n \in A \) such that \( \text{Spec}(A) = \bigcup_{i=1}^{n} D(f_i) \) and \( M_{f_i} = (x_{i,1}/1, \ldots, x_{i,d_i}/1) \) is a finitely generated \( A_{f_i} \)-module for all \( i \in \{1, \ldots, n\} \). We show that \( M \) as \( A \)-module is generated by the elements \( x_{i,1}, \ldots, x_{i,d_i} \) with \( i = 1, \ldots, n \). If \( m \in M \) then for each \( i \in \{1, \ldots, n\} \) we may write \( m/1 = \sum_{k=1}^{d_i} (r_{i,k}/f_i^{n_k})(x_{i,k}/1) \). Thus there exists a natural number \( N \geq 1 \) such that \( f_i^{N_i} m = \sum_{k=1}^{d_i} r_{i,k}' x_{i,k} \) for all \( i \in \{1, \ldots, n\} \). We have \( \text{Spec}(A) = \bigcup_{i=1}^{n} D(f_i^N) \) and so \( 1 = \sum_{i=1}^{n} r_i'' f_i^N \). It follows that \( m = \sum_{i,k} r_{i,k}' r''_i x_{i,k} \). This completes the proof. □

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