A PATH-INTEGRAL APPROACH TO POLYNOMIAL INVARIANTS OF LINKS*

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A brief review of a self-contained genuinely three-dimensional monodromy-matrix based non-perturbative covariant path-integral approach to polynomial invariants of knots and links in the framework of (topological) quantum Chern-Simons field theory is given. An idea of “physical” observables represented by an auxiliary topological quantum-mechanics model in an external gauge field is introduced substituting rather a limited notion of the Wilson loop. Thus, the possibility of using various generalizations of the Chern-Simons action (also higher-dimensional ones) as well as a purely functional language becomes open. The theory is quantized in the framework of the best suited in this case antibracket-antifield formalism of Batalin and Vilkovisky. Using the Stokes theorem and formal translational invariance of the path-integral measure a monodromy matrix corresponding to an arbitrary pair of irreducible representations of an arbitrary semi-simple Lie group is derived.

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1. Introduction

Topological quantum field theory\(^1\) (TQFT) has recently become a fascinating and fashionable subject in mathematical physics. Interestingly, the main source of applications of TQFT is coming from mathematics (topology of low-dimensional manifolds) rather than from physics itself. The issue of classification of knots and links is one of the most interesting ones in low-dimensional topology. To approach this problem one usually tries to encode topology of a knot/link into some algebraic structure. As was firstly indicated by Witten,\(^2\) the problem can be attacked by means of some standard theoretical physics techniques of quantum field theory. In particular, in the framework of three-dimensional TQFT (Chern-Simons theory) not only can all well-known polynomial invariants of knots and links be derived but a lot of their generalizations as well.

Most of authors working on TQFT description of polynomial invariants follow the original Witten’s approach, which heavily bases upon the underlying conformal field theory. There is also a genuinely three-dimensional covariant approach advocated in its perturbative version in Ref. 3. A non-perturbative “Hamiltonian” version has been proposed in Ref. 4. A self-contained genuinely three-dimensional non-perturbative covariant path-integral approach has been firstly introduced in Ref. 5. The aim of our brief review is to give a concise account of the developed form of the proposed idea, taking as an example the simplest Chern-Simons model.

One should emphasize that the proposed approach has a number of advantages: It is self-contained, i.e. no notions of conformal field theory are used explicitly or implicitly; it is genuinely three-dimensional, i.e. there is no “2 + 1” decomposition; the approach is not limited to the Chern-Simons description,\(^6\) and it can easily be extended to dimensions greater than three.\(^7\)

In Sect. 2 we introduce the classical Chern-Simons action, which is next quantized in Sect. 3. In Sect. 4, using the Stokes theorem, we derive the monodromy matrix, whereas skein relations are dealt with in Sect. 5. Finishing remarks (Sect. 6) concern a relation of the proposed approach to the (quasi-)quantum group one.

2. Classical Action

To begin with, we introduce, for an arbitrary semi-simple compact Lie group \(G\), the classical topological Chern-Simons action\(^2\) on the three-dimensional sphere \(S^3\)

\[
S_{CS} = \frac{k}{4\pi} \int_{S^3} \text{Tr} \left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right)
\]

\[
= \frac{k}{4\pi} \int_{S^3} d^3x \varepsilon^{\mu\nu\lambda} \text{Tr} \left( A_\mu \partial_\nu A_\lambda + \frac{2}{3} A_\mu A_\nu A_\lambda \right), \quad \mu, \nu, \lambda = 1, 2, 3,
\]  

(2.1)

where \(A_\mu = A^a_\mu(x) t_F^a\) is the gauge potential, valued in the fundamental representation \(R_F\) of the Lie algebra \(\mathcal{G}\) (the Lie algebra of the Lie group \(G\)) with standardly normalized antihermitian generators, \(\text{Tr} \left( t_F^a t_F^b \right) = -\frac{1}{2} \delta^{ab}\), and \(k \in \mathbb{Z}^+\). For any irreducible representation \(R_i\) of \(\mathcal{G}\) we have \([t_i^a, t_i^b] = f^{abc} t_i^c\). The use of (2.1) does
not seem to be obligatory. One could as well pick out the action of the, so-called, BF-theory:
\[
S_{BF} = \frac{k}{4\pi} \int_{S^3} d^3x \, \varepsilon^{\mu\nu\lambda} \text{Tr}(B_\mu F_{\nu\lambda}),
\]  
(2.2)
where \( B_\mu = B^a_\mu(x) t^a_F \) is an auxiliary gauge field, \( F_{\mu\nu} \) is the strength of the gauge field, and \( k \in \mathbb{R}^\pm \). It appears that some generalization of (2.2) containing the term \( B^3 \) is related to the square of the modulus of (2.1).\(^9\) The action (2.2) possesses some advantages in comparison with (2.1): no longer need \( k \) be integer, and it can be easily generalized to higher dimensions.\(^7\) The action (2.1) itself can also be generalized to higher dimensions (the inhomogeneous case),\(^10\) but for the sake of simplicity we will confine ourselves to the standard homogeneous version.

3. BV-Quantization

It appears that the most suitable for our purposes method of quantization of gauge systems is the general antibracket-antifield technique of Batalin and Vilkovisky (BV).\(^11\) In spite of the fact that the gauge symmetry of Chern-Simons theory is irreducible, and obviously one could use the standard BRST method, from our point of view, the BV-quantized action is easier, for some technical reasons, to deal with.

In the framework of the BV approach one should find a proper non-degenerated solution of the master equation
\[
(S, S) = 0,
\]  
(3.1)
where “\((\cdot, \cdot)\)” denotes the (anti-)bracket in the “extended phase space”. Interestingly, the so-called, minimal part of the solution of (3.1) can be put in the following compact form, resembling the classical action (2.1),
\[
S_{\text{min}} = \frac{k}{4\pi} \int_{S^3} \text{Tr} \left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right),
\]  
(3.2)
where the inhomogeneous field \( A = A^a(x) t^a_F \) is defined by the formal sum of the fields (gauge potential, ghost) and their antifields (denoted with “\(*\)”) \( A = C + A + A^* + C^* \).

The form degrees and the ghost numbers of the components entering \( A \) are as follows:
\[
\begin{align*}
\text{deg} C &= 0, & \text{gh} C &= 1, \\
\text{deg} A &= 1, & \text{gh} A &= 0, \\
\text{deg} A^* &= 2, & \text{gh} A^* &= -1, \\
\text{deg} C^* &= 3, & \text{gh} C^* &= -2.
\end{align*}
\]  
(3.4)
Taking into account the fact that only zero-ghost number three-forms survive in the integrand of Eq. (3.2), we can rewrite it explicitly as

\[ S_{\text{min}} = S_{\text{CS}} + \frac{k}{4\pi} \int_{S^3} d^3x \, \varepsilon^{\mu \nu \lambda} \text{Tr} \left( A_{\mu \nu}^* D_\lambda C + \frac{1}{2} C_{\mu \nu \lambda}^* C^2 \right), \quad (3.5) \]

where

\[(D_\mu C)^a = \partial_\mu C^a + f^{abc} A_\mu^b C^c. \quad (3.6)\]

The auxiliary part of the quantum action has the following standard form

\[ S_{\text{aux}} = \frac{1}{6} \int_{S^3} d^3x \, \varepsilon^{\mu \nu \lambda} \text{Tr} \left( \bar{C}^* B_{\mu \nu \lambda} \right), \quad (3.7) \]

where \( \bar{C}^* \) is the antifield antighost corresponding to the antighost \( \bar{C} \), and \( B \) is the Lagrange multiplier field. The form degrees and the ghost numbers are as follows:

\[
\begin{align*}
\text{deg} \bar{C}^* &= 0, \quad \text{gh} \bar{C}^* = 0, \\
\text{deg} B &= 3, \quad \text{gh} B = 0, \\
\text{deg} \bar{C} &= 3, \quad \text{gh} \bar{C} = -1. \quad (3.8)
\end{align*}
\]

Thus the BV-quantized action is the sum

\[ S = S_{\text{min}} + S_{\text{aux}}. \quad (3.9) \]

The statement that \( S \) satisfies the master equation (3.1) can be also reexpressed in a more traditional language as a closedness of \( S \) with respect to the nilpotent BRST operator \( s \), where the definition of \( s \) is, in a compact notation,

\[ sa = \mathcal{F} \equiv dA + A^2. \quad (3.10) \]

Expanding (3.10) with respect to (3.3), we explicitly obtain

\[
\begin{align*}
s C^a &= \frac{1}{2} f^{abc} C^b C^c, \\
s A_\mu^a &= (D_\mu C)^a, \\
s A^{*a}_{\mu \nu} &= F^{a}_{\mu \nu} + f^{abc} A^{b}_{\mu \nu} C^c, \\
s C^*_{\mu \nu \lambda} &= D_{[\mu} A_{\nu \lambda]}^* + f^{abc} C^{*b}_{\mu \nu \lambda} C^c,
\end{align*}
\]

where the first two BRST transformations correspond to the standard ones. The nilpotency of \( s, s^2 = 0 \), is equivalent to the generalized Bianchi identity, \( DF = 0 \). Additionally,

\[
\begin{align*}
s \bar{C}^a_{\mu \nu \lambda} &= B^a_{\mu \nu \lambda}, \\
s \bar{C}^*_{\mu \nu \lambda} &= s B^a_{\mu \nu \lambda} = 0. \quad (3.12)
\end{align*}
\]
The Landau gauge-fixing condition, \( \partial_{\mu}(\sqrt{g}g^{\mu\nu}A^a_\nu) = 0 \), is introduced to the theory with the gauge fermion \( \Psi \) of the form
\[
\Psi = \frac{1}{6} \int_{S^3} d^3x \varepsilon^{\mu\nu\lambda} \text{Tr} \left( g^{\rho\sigma} A_{\rho} \partial_{\sigma} \bar{C}_{\mu\nu\lambda} \right).
\]
(3.13)

According to the BV prescription each antifield should be substituted by the (dualized) derivative of \( \Psi \) with respect to the corresponding field. Thus, the partition function of Chern-Simons theory can be written in the following path-integral representation
\[
Z_{CS} = \int DA\, DB\, D\bar{C}\, DC \exp(iS) \equiv \int d\mu \exp(iS),
\]
(3.14)

where \( S \) is given by Eq. (3.9). In spite of the explicit dependence of the gauge fermion \( \Psi \) on the metric tensor \( g_{\mu\nu} \) the resulting theory is metric-independent.\(^{12}\)

4. Observables

To encode topology of a link \( \mathcal{L} = \{ C_i \} \) into a path integral we shall introduce physical observables in the form of some auxiliary one-dimensional topological field theory (topological quantum mechanics) in an external gauge field \( A_\mu \), living on the corresponding loops \( \{ C_i \} \). The classical action of this theory\(^5\) is picked out in the form of the sum (with respect to \( i \)) of the terms
\[
S_{C_i}(A) = \oint_{C_i} dt \bar{\eta}_i D_t^A \eta_i,
\]
(4.1)

where the covariant derivative \( D_t^A \equiv d_t + \dot{x}_i^\mu(t)A_\mu^a(x(t))\bar{t}_i^a \), \( \dot{x}_i^\mu(t) \) parametrizes \( C_i \), and the multiplet of the scalar fields \( \bar{\eta}_i, \eta_i \) is defined in the irreducible representation \( R_i \). The partition function corresponding to (4.1) assumes the following standard form
\[
Z_i(A) = \int D\bar{\eta}_i D\eta_i \exp \left( iS_{C_i}(A) \right).
\]
(4.2)

It can be demonstrated that our observables are essentially related to the Wilson ones,\(^{13}\) but this fact is not too important for our further analysis.

We define the topological invariant of the link \( \mathcal{L} \) as the (normalized) expectation value
\[
\left\langle \prod_i Z_i(A) \right\rangle \equiv \left[ \int d\mu \exp(iS) \right]^{-1} \int d\mu \exp(iS) \prod_i Z_i(A).
\]
(4.3)

We can calculate (4.3) recursively using the, so-called, skein relations. Thus, our present task reduces to the derivation of the corresponding skein relation. To this end, we should consider a special link \( \mathcal{L}_{2n} \), which contains a pair of loops, say \( C_1 \) and \( C_2 \), where a part of \( C_1 \), forming a small loop \( \ell (\ell = \partial \mathcal{N}) \), is wrapped round \( C_2 \) \( n \)-times. In other words, \( C_2 \) pierces \( \mathcal{N} \) in \( n \) points: \( \mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n \). Such an arrangement can be interpreted as a preliminary step towards finding the corresponding monodromy
matrix $\mathbf{M}$. Having given the loop $\ell$ we can utilize the Stokes theorem. Applying
the Stokes theorem to (4.1) ($i = 1$) we obtain

$$S_{C^1_1} = S_{C^1_1} \setminus \mathcal{\ell} (A) + \int_{\mathcal{N}} d^2 \sigma \varepsilon^{kl} \left( D^A_k \eta_1 D^A_l \eta_1 \right) + \frac{1}{2} \partial_k x_1^\nu (\sigma) \partial_l x_1^\nu (\sigma) F^a_{\nu \nu} (A(x(\sigma))) \eta_1 t_1^a \eta_1, \tag{4.4}$$

where the covariant derivative $D^A_k \equiv \partial_k + \partial_k x_1^\nu (\sigma) A^a_\nu (x(\sigma)) t_1^a$, and $x_1^\nu (\sigma^1, \sigma^2)$ parametrizes $\mathcal{N}$.

In general position, $\mathcal{N}$ and $C_2$ can intersect in a finite number of points, and the contribution to the path-integral coming from these points can be explicitly calculated. To this end, we should substitute the curvature in (4.4) for the functional derivative operator

$$F^a_{\mu \nu} (x) \rightarrow \frac{4 \pi}{i k} \varepsilon_{\mu \nu \lambda} \frac{\delta}{\delta A^a_\lambda (x)}, \tag{4.5}$$

The substitution (4.5) yields an equivalent expression in (4.4) provided the order of the terms in (4.3) is such that the functional derivative can act on $S_{C^1 S}$ producing $F$. Essentially, (4.5) is a translation operator in a function space of $A$. Using formal translational invariance of the product measure $DA$, and functionally integrating by parts in (4.3) with respect to $A$ we obtain, for each intersection $\mathcal{P}_m$ $(m = 1, 2, \ldots, n)$, the monodromy operator

$$M = \exp \left[ \frac{4 \pi}{i k} \varepsilon_{1 t_1^a \eta_1} (\bar{\eta}_1 t_1^a \eta_1) (\bar{\eta}_2 t_2^a \eta_2) (\mathcal{P}_m) \right]. \tag{4.6}$$

The expression (4.6) is a result of the translation of $A$ in $S_{C^2_2} (A)$. Strictly speaking the functional derivative acts on $S$ rather than on $S_{C^1 S}$, and we should check whether this does not yield some additional terms. Actually, the substitution (4.5) produces $F|_0$ rather than $F$, where “$|_0$” means that only zero ghost-number terms should be taken. But this difficulty can be easily solved, because we can substitute $A$ for $A$ in (4.1), and hence $F$ changes to $F|_0$ in (4.4). One should also note that $A$ entering the gauge fermion $\Psi$ (3.13) is as well subjected to the translation, but it is harmless due to the BV-theorem\textsuperscript{[11]} on the $\Psi$-independence of the partition function. Finally, we can observe that the functional derivative acts trivially, on geometrical grounds, on the “kinetic” term in (4.4).

To calculate the matrix elements of (4.6) one should introduce the following scalar product

$$(f, g) = \frac{1}{2 \pi i} \int f g \exp (i \bar{\eta} \eta) d \bar{\eta} d \eta. \tag{4.7}$$

The definition of the scalar product, rather a standard one, follows from the form of the kinetic term in (4.1), and relates the operator version of the “evolution” to the (holomorphic) path-integral one. Expanding (4.6) in a power series, multiplying
with respect to the scalar product (4.7), and next resumming, we get the monodromy matrix
\[ M = (\bar{\eta}_1 \eta_2, M \eta_2 \eta_1) = \exp \left( \frac{4\pi i}{ik} t^a_1 \otimes t^a_2 \right). \] (4.8)
Thus, to the link \( \mathcal{L}_{2n} \), we can associate the monodromy matrix
\[ M_n = M^n. \] (4.9)

5. Skein Relations

The explicit form of the skein relation depends on the semi-simple Lie group \( G \), and on the pair of the irreducible representations \( R_1, R_2 \). A general method of the derivation of skein relations from the monodromy matrix\(^{14} \) \( M \) bases upon the spectral decomposition of \( M \), and it is given in terms of the Casimir operators \( C_1, C_2, C_\alpha \) and projectors \( P_\alpha \)
\[ M = \exp \left[ \frac{2\pi i}{k} (C_1 + C_2) \right] \sum_\alpha \exp \left( -\frac{2\pi i}{k} C_\alpha \right) P_\alpha, \] (5.1)
where \( \alpha \) numbers irreducible components in the Clebsch-Gordan expansion: \( R_1 \otimes R_2 = \bigoplus_\alpha R_\alpha \).

Geometrically, the skein relation consists of a collection of links \( \mathcal{L}_0, \mathcal{L}_2, \ldots, \mathcal{L}_{2N} \), and takes the form
\[ a_0 \mathcal{L}_0 + a_1 \mathcal{L}_2 + \cdots + a_N \mathcal{L}_{2N} = 0, \] (5.2)
where \( N \) is the number of different \( C_\alpha \), and all the \( \mathcal{L} \)'s are identical except a small neighborhood of \( \mathcal{N} \). The behavior of \( \mathcal{L}_{2n} \) in the neighborhood of \( \mathcal{N} \) has been described in Sect. 4. The coefficients of the skein relation (5.2) are given by the solution of the corresponding algebraic equation(s)
\[ a_0 M^0 + a_1 M^1 + \cdots + a_N M^N = 0. \] (5.3)
In fact, what we actually obtain is a monodromy or pure braid skein relation (even number of the twists) rather than the braid one (any number of the twists). In some cases, we can decompose the monodromy skein relation into the braid one.

For example, the simplest case of the fundamental representations of \( SU(N) \) group corresponds to the HOMFLY polynomial. The monodromy skein relation is given by
\[ \exp \left( \frac{2\pi i}{kN} \right) \mathcal{L}_0 - 2 \cos \left( \frac{2\pi}{k} \right) \mathcal{L}_2 + \exp \left( -\frac{2\pi i}{kN} \right) \mathcal{L}_4 = 0, \] (5.4)
whereas the standard one is
\[ \exp \left( \frac{\pi i}{kN} \right) \mathcal{L}_- - 2 \sin \left( \frac{\pi}{k} \right) \mathcal{L}_0 - \exp \left( -\frac{\pi i}{kN} \right) \mathcal{L}_+ = 0, \] (5.5)
in accordance with calculations given, for example, in Ref. 15. The specialization of this group to \( SU(2) \) corresponds to the Jones polynomial, whereas other representations correspond to the Akutsu-Wadati polynomial. For the fundamental representation of \( SO(N) \) group we obtain the Kauffman-Dubrovnik polynomial.
6. **Finishing Remarks**

There is also a very interesting algebraic side of our description connected with the notion of *(quasi-)*quantum groups. From that point of view one should emphasize that the *(quasi-)*braiding matrices, implicit in our construction, satisfy a *(quasi-)*Yang-Baxter equation, appearing in the context of quasi-triangular *quasi*-Hopf algebras introduced by Drinfeld,\textsuperscript{16} rather than the standard one. It confirms some recent observations,\textsuperscript{17} that it is quasi-quantum group structure that governs quantum symmetries of some low-dimensional field theories rather than quantum group one. Since the quasi-braiding matrix and braiding matrix are related due to the Drinfeld’s theorem both the approaches should yield equivalent results.

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