The Poisson Bracket of Length functions in the Hitchin Component

Martin Bridgeman

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Abstract

Wolpert’s cosine formula on Teichmüller space gives the Weil-Petersson Poisson bracket \{l_\alpha, l_\beta\} for geodesic length functions \(l_\alpha, l_\beta\) of closed curves \(\alpha, \beta\) as the sum of the cosines of the angle of intersection of the associated geodesics. This was recently generalized to Hitchin representations by Labourie. In this paper, we give a short proof of this generalization using Goldman’s formula for the Poisson bracket on representation varieties of surface groups into reductive Lie groups.

1 Introduction

Let \(S\) be a closed oriented surface of genus \(g \geq 2\). In [2], Hitchin considered the space

\[ \mathcal{R}_n(S) = \text{Hom}^{red}(\pi_1(S), \text{PSL}(n, \mathbb{R}))/\text{PSL}(n, \mathbb{R}) \]

of conjugacy classes of reducible representations of \(\pi_1(S)\) into \(\text{PSL}(n, \mathbb{R})\). The space \(\mathcal{R}_n(S)\) has the structure of algebraic variety.

In [4], Goldman showed that for \(n = 2\), \(\mathcal{R}_2(S)\) has \(4g - 3\) components, two of which are the Teichmüller components \(T(S), T(\bar{S})\) corresponding to the conformal structures on \(S\) and its complex conjugate \(\bar{S}\) respectively. The space \(\mathcal{R}_n(S)\) has a natural symplectic structure \(\omega\), called the Goldman symplectic form, discovered by Goldman (see [5]). This generalized the symplectic form discovered by Atiyah-Bott for the case of representations into the group \(U(n)\) (see [1]). For \(n = 2\) the form \(\omega\) restricts on \(\mathcal{R}_n(S)\) to (an integer multiple of) the well-known Weil-Petersson symplectic form \(\omega_{wp}\) on \(T(S)\).

The symplectic form \(\omega\) on \(\mathcal{R}_n(S)\) defines a dual Poisson structure on \(\mathcal{R}_n(S)\) given by \(\{f, g\} = \omega(Hf, Hg)\) where \(Hf, Hg\) are the Hamiltonian vector fields with respect to \(\omega\) of the smooth functions \(f, g : \mathcal{R}_n(S) \to \mathbb{R}\).

Given \(\alpha\) a homotopy class of a non-trivial closed curve on \(S\), we have the associated length function \(l_\alpha : T(S) \to \mathbb{R}\) which assigns the length of the geodesic representative of \(\alpha\) in the associated hyperbolic structure. In [11], Wolpert showed that for the Weil-Petersson symplectic form, then \(Hl_\alpha = -t_\alpha\) where \(t_\alpha\) is the twist vector field obtained by dehn twist about \(\alpha\) a simple non-trivial closed curve. Wolpert further proved the following cosine formula for the Poisson bracket of length functions.

**Theorem 1** (Wolpert, [11]) Let \(\{..\}_{wp}\) be the Poisson bracket on Teichmüller space \(T(S)\) given by the Weil-Petersson symplectic form. Let \(\alpha, \beta\) be homotopy classes of closed oriented curves in \(S\) with
unique closed geodesic representatives $\overline{\alpha}, \overline{\beta}$ in $X \in T(S)$. Then

$$\{l_\alpha, l_\beta\}_{wp}(X) = \sum_{p \in \overline{\alpha} \cap \overline{\beta}} \cos \theta_p$$

where $\theta_p$ is the angle of intersection of $\overline{\alpha}, \overline{\beta}$ at $p$ measured from $\overline{\alpha}$ to $\overline{\beta}$ counterclockwise.

As part of his proof of the Nielsen realization conjecture (see [6]), Kerkhoff also derived the above formula for the case when the curves are measured laminations.

In the recent preprint, Goldman algebra, opers and the swapping algebra, Labourie generalizes the above formula for Hitchin representations (see [7, Theorem 6.1.2]). In this note, we give another proof of this generalization using Goldman’s formula for the Poisson bracket of invariant functions (see [3]).

A representation $\rho : \pi_1(S) \to \text{PSL}(n, \mathbb{R})$ is Hitchin if there exists a Teichmüller representation $\rho_0 : \pi_1(S) \to \text{PSL}(2, \mathbb{R})$ such that $\rho = \tau_n \circ \rho_0$ where $\tau_n : \text{PSL}(2, \mathbb{R}) \to \text{PSL}(n, \mathbb{R})$ is the irreducible representation. As Teichmüller space is connected, Hitchin representations correspond to (at most two) connected components of $R_n(S)$ given by the images of $T(S), T(S)$ under $\tau_n$. Thus for $n = 2$ the Hitchin components are exactly the Teichmüller components $T(S), T(S)$. Hitchin proved the following;

**Theorem 2 (Hitchin, [2])** Each Hitchin component is homeomorphic to $\mathbb{R}^{\chi(S)}(n^2 - 1)$. If $n$ is even there are exactly two Hitchin components and if $n$ is odd, there is exactly one.

Using techniques from the dynamics of Anosov flows, Labourie showed the following;

**Theorem 3 (Labourie, [7])** If $\rho$ is a Hitchin representation then $\rho$ is discrete faithful and for every $g \neq e$, $\rho(g)$ is diagonalizable over $\mathbb{R}$ with eigenvalues distinct $\lambda_1(g), \ldots, \lambda_n(g)$ satisfying

$$|\lambda_1(g)| > |\lambda_2(g)| > \ldots > |\lambda_n(g)|.$$

Thus given $\alpha$ a homotopy class or closed oriented curve in $S$, we therefore have functions $l^i_\alpha : H_n(S) \to \mathbb{R}$ given by

$$l^i_\alpha(\rho) = \log |\lambda_i(\rho(\alpha))|.$$

In [9], Labourie introduced the following cross-ratio on quadruples of lines and planes. We let $\mathbb{RP}^{n-1}$ be the space of lines in $\mathbb{R}^n$ (considered as non-zero vectors in $\mathbb{R}^n$ up to multiplication by $\mathbb{R}^*$), and $\mathbb{RP}^{n-1}$ the space of planes (considered as the space of non-zero linear functionals on $\mathbb{R}^n$ up to multiplication by $\mathbb{R}^*$).

The cross-ratio is given by the map $b : \mathbb{RP}^{n-1} \times \mathbb{RP}^{n-1} \times \mathbb{RP}^{n-1} \times \mathbb{RP}^{n-1}$

$$b(x, y, z, w) = \frac{< y' | z'>}{< y' | x'} \frac{< w' | x'>}{< w' | z'>}$$

where $x' \in x, y' \in y, z' \in z, w' \in w$ are any choice of non-zero elements. By linearity $b$ is well defined as the above formula is independent of the choices made. The cross-ratio $b$ is obviously only defined when the quadruple $(x, y, z, w)$ is in general position.
Given a matrix with eigenvalues having distinct absolute values, we define $\xi^i(A) \in \mathbb{R}P^{n-1}$ to be the $i$-th eigenspace, and $\theta^i(A) \in \mathbb{R}P^{n-1}$ to be the plane spanned by $\{\xi^j\}_{j \neq i}$. We let $\xi(A) = (\xi^1(A), \xi^2(A), \ldots, \xi^n(A))$ and $\theta(A) = (\theta^1(A), \theta^2(A), \ldots, \theta^n(A))$. We define

$$b_{ij}(A, B) = b(\xi^i(A), \theta^j(A), \xi^j(B), \theta^i(B)).$$

If $\rho : \pi_1(S) \to \text{PSL}(n, \mathbb{R})$ is a Hitchin representation, and $\alpha, \beta \in \pi_1(S)$ then we define

$$b_{ij}^\rho(\alpha, \beta) = b_{ij}(\rho(\alpha), \rho(\beta)).$$

In [8], Labourie gives the following generalization of Wolpert’s cosine formula.

**Theorem 4** (Labourie, [8]) Let $\alpha, \beta$ be homotopy classes of closed oriented curves in $S$ represented by immersed curves $\alpha, \beta$ in $S$ which are in general position, then

$$\{l_\alpha^1, l_\beta^1\}(\rho) = \sum_{p \in \alpha \cap \beta} \epsilon(p, \alpha, \beta) \left( b_{p_p}^{\rho}(\alpha_p^p, \beta_p^p) - \frac{1}{n} \right).$$

We will give an elementary proof of this theorem.

We note that for $n = 2$ there is a single cross-ratio $b$ and for $A, B \in \text{PSL}(2, \mathbb{R})$, $b(A, B) = \cos^2(\phi_p/2)$ where $\phi_p$ is the angle of intersection between the positive rays of the associated geodesics in $\alpha, \beta$ in $\mathbb{H}^2$ at the point of intersection $p = \alpha \cap \beta$. Thus

$$b(A, B) - \frac{1}{2} = \frac{1}{2}(2\cos^2(\phi_p/2) - 1) = \frac{1}{2}\cos(\phi_p).$$

The angle $\theta_p < \pi$ is the counterclockwise angle between $\alpha, \beta$ at their intersection point. Thus if $0 < \phi_p < \pi$, $p$ is positively oriented then $\phi_p = \theta_p$ and if $\pi < \phi_p < 2\pi$ then $p$ is negatively oriented and $\theta_p = \phi_p - \pi$. Thus the above formula for $n = 2$ is

$$\{l_\alpha^1, l_\beta^1\}(\rho) = \sum_{p \in \alpha \cap \beta} \epsilon(p, \alpha, \beta) \left( \frac{1}{2}\cos(\phi_p) \right) = \frac{1}{2} \sum_{p \in \alpha \cap \beta} \cos(\theta_p).$$

For $g \in \text{SL}(n, \mathbb{R})(2, \mathbb{R})$ we have $\lambda_1(g) = e^{l(g)/2}$ where $l(g)$ is the hyperbolic translation of $g$. Therefore it follows that if $l_\gamma$ is the length function for closed curve $\gamma$ then $l_\gamma = 2l_\gamma^1$. Also the classical Weil-Petersson symplectic form $\omega_{wp}$ satisfies $\omega = 2\omega_{wp}$ (see [3]). Therefore we recover Wolpert’s cosine formula for the Weil-Petersson Poisson structure

$$\{l_\alpha, l_\beta\}_{wp} = \sum_{p \in \alpha \cap \beta'} \cos(\theta_p).$$
2 Background

We now describe the background on Goldman’s formula for the Poisson bracket of invariant functions. Let \( G \) be a reductive matrix group and consider the non-degenerate symmetric form \( \mathcal{B} : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R} \) given by \( \mathcal{B}(X,Y) = \text{Tr}(XY) \). An invariant function for \( G \) is a smooth function \( f : G \to \mathbb{R} \) which is conjugacy invariant. In particular \( f = \text{Tr} \) is an invariant function. Given \( f \) there is a natural function \( F : G \to \mathfrak{g} \) given by

\[
\mathcal{B}(F(A), X) = \frac{d}{dt} f(\exp(tX)A) \quad \text{for all } X \in \mathfrak{g}
\]

Thus \( F(A) \) is dual to \( R^*_\mathfrak{g}(df(A)) \in \mathfrak{g}^* \) under the isomorphism \( \hat{\mathcal{B}} : \mathfrak{g} \to \mathfrak{g}^* \) given by \( \hat{\mathcal{B}}(X)(Y) = \mathcal{B}(X,Y) \).

Let \( S \) be a closed oriented surface of genus \( g \geq 2 \) and \( \pi = \pi_1(S,p) \) for some \( p \in S \). We consider the space \( \text{Hom}(\pi, G)/G \) of representations \( \rho : \pi \to G \) up to conjugacy and let \( \mathcal{R}(S,G) \) be the space of smooth points of \( \text{Hom}(\pi, G)/G \). If \( \alpha \) is a non-trivial homotopy class of closed oriented curve in \( S \) then \( \alpha \) defines a conjugacy class in \( \pi \). If \( f \) is an invariant function for \( G \) then we can define \( f_\alpha : \mathbb{R}(S,G) \to \mathbb{R} \) by

\[
f_\alpha([\rho]) = f(\rho(\alpha'))
\]

where \( \alpha' \in \alpha \).

The tangent space at \([\rho] \in \mathcal{R}(S,G)\) can be identified with the group cohomology \( H^1(\pi, \mathfrak{g}_{Ad\rho}) \). Using \( \mathcal{B} \) to pair coefficients, we use the cup-product and cap-product for group cohomology to define the map

\[
H^1(\pi, \mathfrak{g}_{Ad\rho}) \times H^1(\pi, \mathfrak{g}_{Ad\rho}) \xrightarrow{\mathcal{B}(\cdot, \cdot)} H^2(\pi, \mathbb{R}) \xrightarrow{\cap[\pi]} H_0(\pi, \mathbb{R}) = \mathbb{R}
\]

This map defines the Goldman symplectic form \( \omega \) on \( \mathcal{R}(S,G) \) (see [5]). Specifically we have

\[
\omega|_{[\rho]}(X, Y) = \mathcal{B}(X \cup Y) \cap [\pi].
\]

Given a smooth function \( f : \mathcal{R}(S,G) \to \mathbb{R} \) the Hamiltonian vector field of \( f \) is the vector field \( Hf \) defined by \( \omega(Hf, Y) = df(Y) \). For \( f, g \) two smooth functions the associated Poisson bracket on smooth functions is the pairing \( \{.,.\} : C^\infty(\mathcal{R}(S,G), \mathbb{R}) \times C^\infty(\mathcal{R}(S,G), \mathbb{R}) \to C^\infty(\mathcal{R}(S,G), \mathbb{R}) \) given by

\[
\{f,g\}([\rho]) = \omega_{[\rho]}(Hf, Hg).
\]

Given \( \alpha \) an oriented curve in \( S \), if \( p \in \alpha \), we let \( \alpha_p \) be the oriented curve given by traversing \( \alpha \) starting at \( p \). If \( \alpha, \beta \) are two oriented closed curves, then \( \alpha, \beta \) are in general position if their intersections are transverse. If \( \alpha, \beta \) are in general position, then for \( p \in \alpha \cap \beta \) we define \( \epsilon(p, \alpha, \beta) = \pm 1 \) given by if the orientation of the point of intersection agrees or not with the orientation of the surface.

Also for \([\rho] \in \mathcal{R}(S,G)\) we let \( \rho_p : \pi_1(S,p) \to G \) be a representation defined by change of base point of \( \rho \). This is well-defined up to conjugacy.

Goldman gave the following description of the Poisson bracket for invariant functions.

**Theorem 5** (Goldman, [3]) Let \( f, f' : G \to \mathbb{R} \) be invariant functions for \( G \) with associated functions \( F, F' : G \to \mathfrak{g} \). Let \( \alpha, \beta \) be homotopy classes of closed oriented curves represented by immersed curves
\( \alpha, \beta \) in \( S \) which are in general position. Then
\[
\{ f_\alpha, f_\beta \} [p] = \sum_{p \in \alpha \cap \beta} \epsilon(p, \alpha, \beta) B(F(p_\rho(\alpha)), F'(p_\rho(\beta)))
\]

### 3 Length Functions

As Hitchin representations can be lifted to representations into \( \text{SL}(n, \mathbb{R})(n, \mathbb{R}) \) (see [7]), we can restrict to representations into \( \text{SL}(n, \mathbb{R})(n, \mathbb{R}) \). We define the hyperbolic elements \( H_{\text{yp}} \subseteq \text{SL}(n, \mathbb{R}) \) to be the open subset of diagonalizable matrices over \( \mathbb{R} \) with eigenvalues having distinct absolute values. For \( A \in H_{\text{yp}} \), \( A \) has eigenvalues \( \lambda_1(A), \ldots, \lambda_n(A) \) with \( |\lambda_1(A)| > |\lambda_2(A)| > \ldots > |\lambda_n(A)| \). We define the functions \( l^i : H_{\text{yp}} \to \mathbb{R} \) by letting \( L^i(A) = \log |\lambda_i(A)| \). We define the function \( L^i : H_{\text{yp}} \to \mathfrak{s}(n, \mathbb{R}) \) by
\[
B(L^i(A), X) = \frac{d}{dt} l^i(\exp(tX)A).
\]

#### 3.1 Eigenvalue Perturbation

We now consider perturbation of eigenvalues in the space of hyperbolic matrices. Given \( A \in H_{\text{yp}} \) let \( p_i(A) : \mathbb{R}^n \to \mathbb{R}^n \) be projection onto the \( i \)-th eigenspace, parallel to the other eigenvectors.

**Lemma 1** The length function \( l^i : H_{\text{yp}} \to \mathbb{R} \) satisfies
\[
dl^i_A(X) = \frac{1}{\lambda^i(A)} Tr(p_i(A).X).
\]

**Proof:** We let \( A = A_0 \) and denote the eigenvalues and eigenvectors of \( A \) by \( \lambda^i, x^i \). We further let \( \dot{A} = \dot{A}_0 \). We have \( A_t \) has eigenvalues \( \lambda^i_t \) and unit length eigenvector \( x^i_t \). We have
\[
A_t.x^i_t = \lambda^i_t.x^i_t.
\]
Differentiating we get
\[
\dot{A}x^i + A.\dot{x}^i = \lambda^i.\dot{x}^i + \dot{\lambda}^i.x^i
\]
We let \( p_i(A) \) be linear projection onto the \( i \)-th eigenspace of \( A \) parallel to the other eigenspaces of \( A \). We apply to the above equation.
\[
p_i(A)\dot{A}x^i + p_i(A)A.\dot{x}^i = \lambda^i.p_i(A)\dot{x}^i + \dot{\lambda}^i.x^i
\]
As \( p_i(A)A = \lambda^i.p_i(A) \) we have \( p_i(A)A\dot{x}^i = \lambda^i.p_i(A)\dot{x}^i \) so after cancellation we get
\[
\dot{\lambda}^i.x^i = p_i(A).\dot{A}.x^i.
\]
Therefore we have
\[
\dot{\lambda}^i = tr(p_i(A).\dot{A}).
\]
As \( l^i(X) = \log |\lambda^i(X)| \) on \( Hyp \) we have
\[
dl^i \equiv \frac{d\lambda^i}{\lambda^i}.
\]
Therefore
\[
dl^i_A(X) = \frac{1}{\lambda^i(A)} Tr(p_i(A).X)
\]
\( \square \)

We now use the above lemma to calculate \( L^i \).

**Lemma 2**
\[
L^i(A) = p_i(A) - \frac{1}{n} I.
\]

**Proof:** By the above
\[
B(L^i(A), X) = \frac{d}{dt}l^i(\exp(tX)A) = dl^i_A(XA) = \frac{1}{\lambda^i(A)} Tr(p_i(A).XA).
\]
By definition \( A.p_i(A) = \lambda^i p_i(A) \). Therefore
\[
B(L^i(A), X) = \frac{1}{\lambda^i(A)} Tr(A.p_i(A).X) = \frac{1}{\lambda^i(A)} Tr(\lambda^i p_i(A).X) = Tr(p_i(A).X).
\]
We let \( \mathcal{B} : \mathfrak{gl}(n, \mathbb{R}) \times \mathfrak{gl}(n, \mathbb{R}) \rightarrow \mathbb{R} \) given by \( B(X, Y) = Tr(XY) \). Then \( \mathcal{B} \) is non-degenerate and restricts to \( B \) on \( g \). We let \( P : \mathfrak{gl}(n, \mathbb{R}) \rightarrow \mathfrak{sl}(n, \mathbb{R}) \) be orthogonal projection with respect to \( \mathcal{B} \). Then given \( A \in \mathfrak{gl}(n, \mathbb{R}) \), then for all \( X \in \mathfrak{g} \)
\[
\mathcal{B}(A, X) = \mathcal{B}(P(A), X) = B(P(A), X).
\]
Therefore we have
\[
B(L^i(A), X) = Tr(p_i(A).X) = \mathcal{B}(p_i(A), X) = B(P(p_i(A)), X).
\]
As \( B \) is non-degenerate on \( \mathfrak{sl}(n, \mathbb{R}) \), we have
\[
L^i(A) = P(p_i(A))
\]
The projection map \( P : \mathfrak{gl}(n, \mathbb{R}) \rightarrow \mathfrak{sl}(n, \mathbb{R}) \) is given by
\[
P(A) = A - \frac{1}{n} Tr(A).I
\]
Therefore as \( p_i(A) \) is projection onto a 1-dimensional eigenspace, \( Tr(p_i(A)) = 1 \) and we have
\[
L^i(A) = p_i(A) - \frac{1}{n} Tr(p_i(A)).I = p_i(A) - \frac{1}{n}.I
\]
\( \square \)
3.2 Poisson bracket

We now use Goldman’s formula to give an alternative proof of Labourie’s generalization of the cosine formula.

**Theorem 4** (Labourie, \[8\])

\[
\{l^i_\alpha, l^j_\beta\}(\rho) = \sum_{p \in \alpha \cap \beta} \epsilon(p, \alpha, \beta) \left( b^{ij}_{\rho_p}(\alpha_p, \beta_p) - \frac{1}{n} \right).
\]

**Proof:** From the above we have

\[
B(L^i(A), L^j(B)) = Tr \left( \left( p_i(A) - \frac{1}{n} I \right) \cdot \left( p_j(B) - \frac{1}{n} I \right) \right)
\]

As \( Tr(p_i(A)) = Tr(p_j(B)) = 1 \)

\[
B(L^i(A), L^j(B)) = Tr(p_i(A)p_j(B)) - \frac{1}{n}
\]

Now applying Goldman’s formula from Theorem 5 we get

\[
\{l^i_\alpha, l^j_\beta\}(\rho) = \sum_{p \in \alpha \cap \beta} \epsilon(p, \alpha, \beta) B(\rho(L^i_p(\alpha_p)), \rho(L^j_p(\beta_p)))
\]

\[
= \sum_{p \in \alpha \cap \beta} \epsilon(p, \alpha, \beta) \left( Tr(p_i(\rho(\alpha_p))p_j(\rho(\beta_p))) - \frac{1}{n} \right).
\]

For any \( X \in Hyp \) and let \( \xi(X), \theta(X) \) be the \( n \)-tuples of eigenspaces and dual planes. We let \( A, B \in Hyp \) and we choose non-zero elements \( a^+_i \in \xi^i(A), a^-_i \in \theta^i(A), b^+_j \in \xi^j(B), b^-_j \in \theta^j(B) \). Then

\[
p_i(A)(v) = \frac{<a^-_i|v>}{<a^-_i|a^+_i>}, \quad p_j(B)(v) = \frac{<b^-_j|v>}{<b^-_j|b^+_j>}.
\]

Similarly for \( B \in Hyp \) with \( b^+_i, b^-_i \). Then if \( A, B \in Hyp \) we have

\[
p_i(A)p_j(B)v = \frac{<a^+_i|b^+_j> <b^+_j|v>}{<a^+_i|a^+_i> <b^+_j|b^+_j>}
\]

Thus

\[
Tr(p_i(A)p_j(B)) = \frac{<a^-_i|b^+_j> <b^+_j|a^+_i>}{<a^-_i|a^+_i> <b^-_j|b^+_j>} = b(\xi^i(A), \theta^i(A), \xi^j(B), \theta^j(B)) = b^{ij}(A, B).
\]

Therefore the Poisson bracket is

\[
\{l^i_\alpha, l^j_\beta\}(\rho) = \sum_{p \in \alpha \cap \beta} \epsilon(p, \alpha, \beta) \left( b^{ij}_{\rho_p}(\alpha_p, \beta_p) - \frac{1}{n} \right).
\]

□
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