THE MEMBERSHIP PROBLEM FOR POLYNOMIAL IDEALS IN TERMS OF RESIDUE CURRENTS

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Abstract. We find a relation between the vanishing of a globally defined residue current on $\mathbb{P}^n$ and solution of the membership problem with control of the polynomial degrees. Several classical results appear as special cases, such as Max Nöther’s theorem, and we also obtain a generalization of that theorem. There are also connections to effective versions of the Nullstellensatz. We also provide explicit integral representations of the solutions.

1. Introduction

Let $F_1, \ldots, F_m$ be polynomials in $\mathbb{C}^n$ and let $\Phi$ be a polynomial that vanishes on the common zero set of the $F_j$. By Hilbert’s Nullstellensatz, for some power $\Phi^\nu$ of $\Phi$, one can find polynomials $Q_j$ such that

$$\sum_j F_j Q_j = \Phi^\nu. \quad (1.1)$$

A lot of attention has been paid to find effective versions, i.e., control of $\nu$ and the degrees of $Q_j$ in terms of the degrees of $F_j$. The breakthrough was in [13] where Brownawell obtained bounds on $\nu$ and $\deg Q_j$ not too far from the best possible, using a combination of algebraic and analytic methods, cf., Remark 3 below. Soon after that Kollár [22] obtained by purely algebraic methods the following optimal result.

**Theorem (Kollár).** Let $F_1, \ldots, F_n$ and $\Phi$ be polynomials in $\mathbb{C}^n$ of degrees $d_j$, and $r$, respectively, and assume that $\Phi$ vanishes on the common zero set of $F_j$. Then (if $d_j \neq 2$), one can find polynomials $Q_j$ and a natural number $s$ such that $\sum F_j Q_j = \Phi^\nu$, and such that $\nu \leq N(d_1 \cdots d_m)$ and $\deg F_j Q_j \leq (1 + r)N(d_1 \cdots d_m)$; here $N(d_1 \cdots d_m) = d_1 \cdots d_m$ if $m \leq n$; for the case when $m > n$, see [22].

In particular, if $F_j$ have no common zeros in $\mathbb{C}^n$, then there are polynomials $Q_j$ such that

$$\sum_j F_j Q_j = 1, \quad (1.2)$$

*Date: November 13, 2018.*

The author was partially supported by the Swedish Natural Science Research Council.
with
\[ \deg F_j Q_j \leq N(d_1 \cdots d_m). \]
The restriction \( d_j \neq 2 \) has recently been removed by Jalonek, [21], in
the case when \( m = n \).

In [14] Brownawell gave a prime power version of Kollár’s theorem
which shed more geometric light on these questions, and there is a
generalization to smooth algebraic manifolds in [17].

Kollár’s result is optimal as long as one only makes assumptions of
the degrees of \( F_j \). However, if one imposes geometric conditions on
the zero set one can get sharper results. For instance, assuming that
\( m = n + 1 \) and \( F_j \) have no common zero set even at infinity, then a
classical theorem of Macaulay, [23], states that (1.2) has a solution such
that \( \deg F_j Q_j \leq \sum d_j - n \).

There is a related result due to Max Nöther, [24]; see also [19].

**Theorem (Max Nöther, 1873).** Assume that the zero set of \( F_1, \ldots, F_n \)
is discrete and contained in \( \mathbb{C}^n \) and that \( \Phi \) belongs to the ideal \( (F) \).
Then there are polynomials \( Q_j \) such that
\[ \Phi = \sum_{j=1}^{n} F_j Q_j \]
and \( \deg F_j Q_j \leq \deg \Phi \).

In this paper we present a more general result about solutions to the
equation
\[ (1.3) \quad \Phi = \sum_{j=1}^{m} F_j Q_j, \]
where \( F_1, \ldots, F_m \) are given polynomials in \( \mathbb{C}^n \), with control of the
degrees of \( F_j Q_j \). It is formulated in terms of a residue current associated
with \( F_j \) with support on their common zero set on \( \mathbb{P}^n \), and the theo-
rems of Macaulay and Max Nöther are simple consequences. We also
provide explicit representation formulas of solutions.

If \( f_j \) denote homogenizations of \( F_j \), i.e., \( f_j(z) = z_0^{d_j} F_j(z'/z_0) \), where
\( d_j \geq \deg F_j \), (here \( z = (z_0, z_1, \ldots, z_n) \) and \( z' = (z_1, \ldots, z_n) \)), then each
\( f_j \) defines a global holomorphic section of the line bundle \( L^{d_j} \to \mathbb{P}^n \),
and hence \( f = (f_1, \ldots, f_m) \) is a section of the rank \( m \) bundle \( E^* = L^{d_1} \oplus \cdots \oplus L^{d_m} \) over \( \mathbb{P}^n \) (here \( L^s \) denotes the line bundle \( O(s) \)). If
\( z \in \mathbb{C}^{n+1} \setminus \{0\} \) we let \( [z] \) denote the corresponding point in \( \mathbb{P}^n \) under
the natural projection; however, we write \( f(z) \) rather than \( f([z]) \). If
\( E^* \) is equipped with the natural Hermitian structure, then
\[ (1.4) \quad \|f(z)\|^2 = \sum_{j=1}^{m} \frac{|f_j(z)|^2}{|z|^{2d_j}}. \]
Following [2], we can define the residue current $R^f$ which is an element in $\oplus_l D_{0,l}(\mathbb{P}^n, \Lambda^l E)$ and with support on the zero set

$$Z^f = \{[z] \in \mathbb{P}^n; \ f(z) = 0\}.$$ 

If we assume that the polynomials $F_j$ have no common zeros in $\mathbb{C}^n$, then of course $Z^f$ is a subset of the hyperplane at infinity. If $\text{codim} Z^f = m$, i.e., $f$ is locally a complete intersection, then $R^f$ is a $(0,m)$-current with values in $\det E = L^{-\sum d_j}$; more precisely it can be identified with the Coleff-Herrera current

$$[\bar{\partial} \frac{1}{f_1} \wedge \ldots \wedge \bar{\partial} \frac{1}{f_m}],$$

in $\mathbb{C}^{n+1} \setminus \{0\}$, see Section 2. We can now formulate our main result in this paper.

**Theorem 1.1.** Let $F_1, \ldots, F_m$ be polynomials in $\mathbb{C}^n$, $\deg F_j \leq d_j$, let $f = (f_1, \ldots, f_m)$ be the corresponding section of $E^* = L^{d_1} \oplus \cdots \oplus L^{d_m}$ over $\mathbb{P}^n$, and let $R^f$ be the associated residue current. Moreover, assume that

$$m \leq n \quad \text{or} \quad r \geq \sum_{j=1}^{n+1} d_j - n,$$

where $d_1 \geq d_2 \geq \ldots \geq d_m$. Let $\Phi$ be a polynomial, $\deg \Phi \leq r$, and let $\phi \in \mathcal{O}(\mathbb{P}^n, L^r)$ denote its $r$-homogenization. If

$$\phi R^f = 0,$$

then there are polynomials $Q_j$ such that (1.3) holds and $\deg F_j Q_j \leq r$. If $f$ is a complete intersection (then the condition (1.5) is fulfilled) and there exist such polynomials $Q_j$, then (1.6) holds.

It is clear that the conclusion about $\deg F_j Q_j$ cannot be improved. If $\Phi = 1$ the condition (1.6) means that $F_j$ have no common zeros in $\mathbb{C}^n$ and that $z_0^r$ annihilates the residue $R^f$ at infinity. If $Z^f$ is empty and $m = n + 1$ (actually any $m \geq n + 1$ works) we can choose $r = \sum d_j - n$ and hence we get a solution to the Bézout equation (1.2) such that $\deg F_j Q_j \leq \sum d_j - n$; thus we have obtained the theorem of Macaulay mentioned above.

We have the following generalization of Nöther’s theorem.

**Theorem 1.2.** Assume that the projective zero set of $F_1, \ldots, F_m$ has codimension $m$ and that there is no irreducible component contained in the hyperplane at infinity. If $\Phi$ belongs to ideal $(F)$, then there are polynomials $Q_j$ such that (1.3) holds and $\deg F_j Q_j \leq \deg \Phi$.

**Proof.** Since $m \leq n$ the condition (1.5) is fulfilled so we can take $r = \deg \Phi$. Since $\Phi \in (F)$, $\phi$ is in the ideal $(f)$ locally in $\mathbb{C}^n$ and since $f$ is a complete intersection, $\phi R^f = 0$ in $\mathbb{C}^n$ by the duality theorem (see Section 2). If $m = n$, i.e., as in Nöther’s theorem, $Z^f$ is contained
in $\mathbb{C}^n \subset \mathbb{P}^n$, so $R^f$ has its support in $\mathbb{C}^n$ as well, and hence (1.6) holds in $\mathbb{P}^n$. Thus Theorem 1.1 provides the desired solution. In the general case the assumption means that the intersection of $Z^f$ and the hyperplane at infinity has codimension $m+1$, and then Proposition 2.1 in Section 2 implies that $\phi R^f = 0$ in $\mathbb{P}^n$. □

Remark 1. Although this theorem is probably known before, we have not found it in the literature. A proof of Nöther’s theorem by multivariable residue calculus has previously been obtained by Tsikh, [31]. In [32] is given an argument starting with a representation of $\Phi$ with the Cauchy-Weil formula. Making series expansion of the kernel and using Jacobi formulas (vanishing of certain residues as in [33]) and the duality theorem, one obtains Nöther’s theorem. It is possible that one can prove the general form of Theorem 1.2 in a similar way, following the idea of [4] to add $n - k$ linear forms $L$ such that $(F, L)$ has no zeros at infinity, but we have not checked the details.

However some results related to Theorem 1.2 have appeared before. In [28], Proposition 2, it is assumed that $f_j$ is a regular sequence in $\mathbb{P}^n$, but with no extra condition on the hyperplane at infinity. If $\Phi$ belongs to the ideal $(F)$ as above, then there are $Q_j$ solving (1.3) such that $\deg F_j \Psi_j \leq N + \deg \Phi$, where $N = \Pi_{m} \deg F_j$. To see this in our setting, recall that (see, e.g., [28] Lemma 2) if $F_j$ is a regular sequence in $\mathbb{O}_x$, then

$$
(1.7) \quad (\sqrt{(F)_x})^N \subset (F)_x.
$$

Thus $z_0^{N + \deg \Phi}(z'/z_0)$ annihilates $R^f$ in $\mathbb{P}^n$, and therefore the statement follows from Theorem 1.1.

Now let $F_j$ be as in Theorem 1.2 and assume that $\Phi$ vanishes on their common zero set in $\mathbb{C}^n$. Then by (1.7), $\Phi^N$ belongs to $(F)$. Therefore we get the following corollary of Theorem 1.2, which recently appeared in [18] under the slightly stronger assumption that $F_j$ is a strictly regular sequence in $\mathbb{C}^n$.

**Corollary 1.3.** Assume that the projective zero set of $F_1, \ldots, F_m$ has codimension $m$ and that there is no component contained in the hyperplane at infinity. If $\Phi$ vanishes on the zero set of $F$ in $\mathbb{C}^n$, then there are polynomials $Q_j$ such that $\deg F_j Q_j \leq N \deg \Phi$ and $\sum F_j Q_j = \Phi N$, where $N = (\deg F_1) \cdots (\deg F_m)$.

If

$$
(1.8) \quad \|\phi\| \leq C\|f\|,
$$

then, see Section 2, $\phi^{\min(m, n)} R^f = 0$, and hence Theorem 1.1 implies
Corollary 1.4. Let $F_j$ and $\Phi$ be as in Theorem 1.1, $r \geq \deg \Phi$, and assume that

$$m \leq n \quad \text{or} \quad r \min(m, n) \geq \sum_{1}^{n+1} d_j - n.$$ 

If (1.8) holds, then there are polynomials $Q_j$ such that

$$\sum F_j Q_j = \Phi^{\min(m, n)}$$

and $\deg F_j Q_j \leq r \min(m, n)$.

Since there are examples where $f$ is a complete intersection and the full power $\min(m, n)$ of $\phi$ is needed to kill $R^f$, this result is then sharp.

Example 1. Let $M$ be a given positive integer. Take $F_j(z') = z_j^{2Mm}$ in $\mathbb{C}^n$, $1 \leq j \leq m \leq n$, $\Phi(z') = (z_1 + \cdots + z_m)^{2Mm}$, and let $f_j$ and $\phi$ be the homogenizations as before ($d_j = Mm$). Then (1.8) holds and hence the corollary states that (1.9) has a solution such that $\deg F_j Q_j \leq r \min(m, n) = Mm^2$. This is obvious also by a direct inspection, and one also immediately sees that $\Phi^{m-1}$ is not in the ideal $(\Phi')$. Thus the corollary is sharp.

It follows that $\phi^{m-1}R^f \neq 0$, and since $f$ is a complete intersection in $\mathbb{P}^n$, in fact $Z^f$ is the $n - m$-plane $\{[z] \in \mathbb{P}^n; z_1 = \cdots = z_m = 0\}$, it also follows that $\phi^{m}R^f = 0$. One can also verify these residue conditions directly. In fact, in the standard affine coordinates $z'$,

$$R^f = \left[ \bar{\partial} \frac{1}{z_1^{Mm}} \wedge \cdots \wedge \bar{\partial} \frac{1}{z_m^{Mm}} \right] \wedge \epsilon,$$

where $\epsilon$ is a non-vanishing section of the line bundle $\det E$, see Section 2. Since this residue current is a tensor product of one-variable currents, the residue conditions follow from the one-variable equality $z\bar{\partial}[1/z^{p+1}] = \bar{\partial}[1/z^p]$.

Let $F_j$ be polynomials with no common zeros in $\mathbb{C}^n$. Since the zero set of the section $f$ (take $d_j = \deg F_j$) is then contained in the hyperplane at infinity it follows from Lojasiewicz’ inequality that

$$\|z_0\|^M \leq C\|f\|$$

for some $M$, or equivalently,

$$\sum_{1}^{m} \frac{|F_j(z')|^2}{(1 + |z'|^2)^{d_j}} \geq \frac{1}{C(1 + |z'|^2)^M}.$$ 

Under this condition $z_0^M \min(m, n) R^f = 0$, so we have
Corollary 1.5. Let $F_1, \ldots, F_m$ be polynomials in $\mathbb{C}^n$ of degrees $d_j$ such that (1.11) (or equivalently (1.10)) holds for some number $M$, and assume that

$$m \leq n \text{ or } M \min(m, n) \geq \sum_1^{n+1} d_j - n.$$ 

Then there is a solution to $\sum F_j Q_j = 1$ with $\deg F_j Q_j \leq \min(m, n) M$.

Example 2. Also Corollary 1.5 is essentially sharp. Let $M$ be a given non-negative integer and take $F_j(z') = z_j^M$, $1 \leq j < m \leq n$, and $F_m(z') = (1 + z_1 + \cdots + z_{m-1})^M$. Then $f_j = z_j^M$ and $f_m = (z_0 + \cdots + z_{m-1})^M$, so (1.11) holds. The corollary thus gives a solution to (1.2) with $\deg F_j Q_j \leq mM$.

Writing $1 = (1 + z_1 + \cdots + z_{m-1}) - z_1 - \cdots - z_{m-1}$ and taking the power $Mm - m + 1$ we get a solution to (1.2) with $\deg F_j Q_j = Mm - m + 1$, and it is easily seen to be the best possible, cf., Example 1. However, for large $M$, $Mm - m + 1$ is close to $Mm$.

Remark 2. Given the estimate (1.11), one can obtain a solution to the Bézout equation (1.2) by a direct application of Skoda’s $L^2$-estimate from [29], as is done in [13]. If we for simplicity assume that all $d_j = d$, then one gets a solution with $\deg Q_j \leq \min(m + 1, n) M - d$, i.e., $\deg F_j Q_j \leq \min(m + 1, n) M$. For $m \leq n$ this is the same as in Corollary 1.5 but for $m > n$ it is strictly weaker. This phenomenon is the same as in the original proof of Briançon-Skoda’s theorem, [12]. Under the assumption $\|\phi\| \leq C\|f\|$, the (local) $L^2$-estimate immediately implies that $\phi^{\min(m, n+1)}$ belongs the the ideal $(f)$ locally; to obtain the correct result when $m > n$, that the power $n$ is enough, an additional argument is required. See also Section 2.

Remark 3. The main step in Brownawell’s paper [13] is to obtain good control of the power $M$ in (1.11) in terms of the degrees of $F_j$, assuming that they have no common zeros in $\mathbb{C}^n$, and this is done by means of Chow forms, see also [30].

Kollár’s theorem implies that the estimate (1.11) holds with $M = N(d_1, \ldots, d_m)$, see, [22], and this is in fact best possible. From this estimate one gets, via Corollary 1.5, a solution to (1.2) with $\deg F_j Q_j \leq \min(m, n) M$. In view of Kollár’s theorem one has then “lost” the factor $\min(m, n)$.

Remark 4. Kollár’s theorem holds for any field. Berenstein and Yger, [5], have obtained explicit solutions to the Bézout equation (1.2) in subfields of $\mathbb{C}$, by means of integral formulas; see also [8] and the more recent survey article [32] for a thorough discussion.
Remark 5. The condition (1.8) means that $\phi$ locally on $\mathbb{P}^n$ belongs to the integral closure of the ideal $(f)$. In [20], Hickel proves that if $\Phi$ is in the integral closure of $(F)$ in $\mathbb{C}^n$, then one can solve (assuming $m \leq n$ for simplicity) $\Phi^n = \sum F_j Q_j$ with $\deg (F_j Q_j) \leq m\deg \Phi + md_1 \cdots d_m$. This result would follow from Theorem 1.1 if one could prove that the current $z_0^{md_1 \cdots d_m} \phi^n R^f$ vanishes ($\phi$ is the deg $\Phi$ homogenization of $\Phi$). In $\mathbb{C}^n$ it vanishes since $|\Phi| \leq C|F|$ locally. If the zero set is contained in $\{z_0 = 0\}$, the current vanishes there by Kollár’s theorem. We do not know whether it vanishes in the general case.

Theorem 1.6. Let $f$ be holomorphic section of $E* = L^{d_1} \oplus \cdots \oplus L^{d_m}$ and assume that $\ell \geq 0$ is given and that

$$m - \ell \leq n \quad \text{or} \quad r \geq \sum_{j=1}^{n+\ell+1} d_j - n,$$

where $d_1 \geq d_2 \geq \cdots \geq d_m$. If $\phi \in \mathcal{O}(\mathbb{P}^n, \Lambda^\ell E \otimes L^r)$, then $\phi = \delta_f \psi$ for some $\psi \in \mathcal{O}(\mathbb{P}^n, \Lambda^{\ell+1} E \otimes L^r)$ if and only if

$$(1.12) \quad \nabla_f (w \wedge R^f) = \phi \wedge R^f$$

for some smooth $w$ defined in a neighborhood of $Z^f$.

If $\ell > m - p$ ($p = \text{codim } Z^f$) then the condition on $\phi$ is void; if $\ell = m - p$, it means that $\phi \wedge R^f = 0$, see the remarks after Theorem 2.8 below. If $f$ is a complete intersection, then $m \leq n$ and therefore we have

Corollary 1.7. Let $f$ be a holomorphic section of $E* = L^{d_1} \oplus \cdots \oplus L^{d_m}$ that is a complete intersection, and assume that $r \geq 0$. If $\phi \in \mathcal{O}(\mathbb{P}^n, L^r)$, then $\phi = f \cdot q$ is solvable with $q \in \mathcal{O}(\mathbb{P}^n, E \otimes L^r)$ if and only if $\phi R^f = 0$.

Proof of Theorem 1.7. If the hypotheses in Theorem 1.1 are fulfilled, then Theorem 1.6 provides a section $q = (q_1, \ldots, q_m)$ of $E \otimes L^r$ such that $\sum f_j q_j = \delta_f q = \phi$; here $q_j$ are sections of $L^{-d_j+r}$. After dehomogenization this means that $Q_j$ are polynomials such that $\deg F_j Q_j \leq r$.

In Section 2 we recall the necessary background from [2] about the residue currents, and present a general result about the image of a holomorphic morphism $f$. Combined with well-known vanishing results for the line bundles $L^r \to \mathbb{P}^n$ it leads to a proof of Theorem 1.6.
In the last section we construct explicit integral representations of the solutions in Theorem 1.1. They give essentially the same results except for a small loss of precision. The construction is based on ideas in [1] and [2].

2. THE RESIDUE CURRENT OF A HOLOMORPHIC SECTION

Let $E \to X$ be a holomorphic Hermitian vector bundle of rank $m$ over the $n$-dimensional complex manifold $X$, and let $f$ be a holomorphic section of the dual bundle $E^*$, or in other words, a holomorphic morphism $f : E \to X \times \mathbb{C}$. Let

$$L^r = \bigoplus_\ell D^\ell_{0, l+r}(X, \Lambda^\ell E);$$

we consider $L^r$ as a subbundle to $\Lambda(T^{*}_{0,1} \oplus E)$, so that $\delta_f$ (i.e., interior multiplication with $f$) and $\bar{\partial}$ anticommutes. Then $\nabla_f = \delta_f - \bar{\partial}$ induces the complex $\to L^{r-1} \to L^r \to$. It is readily checked that $\nabla_f$ satisfies the Leibniz rule $\nabla_f(\alpha \wedge \beta) = \nabla_f \alpha \wedge \beta + (-1)^{\nu} \alpha \wedge \nabla_f \beta$, where $\nu$ is the total degree of $\beta$. Let $s$ be the dual section of $E$ of $f$ so that in particular $\delta_f s = \|f\|^2$. In [2] we defined the current

$$R^f = \bar{\partial}\|f\|^{2\lambda} \wedge \frac{s}{\nabla_f s}|_{\lambda=0};$$

for large $\text{Re} \lambda$ the right hand side is integrable and therefore a well defined current, and by a nontrivial argument based on Hironaka’s theorem one can make an analytic continuation to $\lambda = 0$. The resulting current is an element in $L^0$ with support on $Z^f = \{z; \ f(z) = 0\}$ and it satisfies the basic equality

(2.1) \[ \nabla_f U^f = 1 - R^f, \]

where $U^f \in L^{-1}$ is defined as

$$U^f = \|f\|^{2\lambda} \frac{s}{\nabla_f s}|_{\lambda=0};$$

Moreover,

(2.2) \[ R^f = R^f_{p,p} + \ldots + R^f_{m,m}, \]

where $p = \text{codim } Z^f$; here lower index $\ell, q$ means that the current has bidegree $(0, q)$-form and takes values in $\Lambda^\ell E$.

**Proposition 2.1.** Assume that $f$ defines a complete intersection and that $h$ is a holomorphic section of some line bundle such that $\{h = 0\} \cap Z^f$ has codimension $m + 1$. If $\phi$ is a holomorphic section such that $\phi R^f = 0$ in $X \setminus \{h = 0\}$, then $\phi R^f = 0$.

Notice that since $f$ is a complete intersection, $R^f = R^f_m$. The following lemma, which is the core of the proof, states that then $R^f$ is robust in a certain sense.
Lemma 2.2. The current $|h|^{2\lambda}R^f$ has an analytic continuation to $\operatorname{Re} \lambda > -\epsilon$ and

$$|h|^{2\lambda}R^f|_{\lambda=0} = R^f.$$ 

Proof. Clearly the statement is local. By Hironaka’s theorem and a toric resolution we may assume that $f = f_0f'$, where $f_0$ is a holomorphic function and $f'$ is a non-vanishing section. In this way we can write the action of $R^f$ on a test form $\xi$ as a finite sum of terms like

$$\int \bar{\partial} \left[ \frac{1}{f_0} \right] \wedge \alpha \wedge \tilde{\xi} \rho,$$

where $[1/f_0]$ is the principal value current, $\alpha$ is a $(0,m-1)$-form, $\tilde{\xi}$ is the pull-back of $\xi$ in the given resolution, and $\rho$ is a cut-off function. We may also assume that

$$f_0 = \tau_{k_1}^{\alpha_1} \cdots \tau_{k_\nu}^{\alpha_\nu},$$

in appropriate local coordinates $\tau_j$, and therefore the integral is a sum of terms like

$$(2.3) \quad \int \left[ \prod_{r \neq j} \frac{1}{\tau_{k_r}^{\alpha_r \ell}} \bar{\partial} \frac{1}{\tau_{k_j}^{\alpha_j \ell}} \right] \wedge \alpha \wedge \tilde{\xi} \rho.$$

We may also assume that $h = \tau_{m_1}^{\beta_1} \cdots \tau_{m_u}^{\beta_u} u$, where $u \neq 0$. Thus $|h|^{2\lambda}R^f \xi$ is a finite sum of terms like

$$(2.4) \quad \int |\tau_{m_1}^{2\lambda \beta_1} \cdots |\tau_{m_u}^{2\lambda \beta_u}| u |^{2\lambda} \left[ \prod_{r \neq j} \frac{1}{\tau_{k_r}^{\alpha_r \ell}} \bar{\partial} \frac{1}{\tau_{k_j}^{\alpha_j \ell}} \right] \wedge \alpha \wedge \tilde{\xi} \rho.$$

If one of the $m_i$ is equal to $k_j$, then clearly this integral vanishes for $\operatorname{Re} \lambda >> 0$, and trivially therefore it has an analytic continuation to $\lambda > -\epsilon$, with the value $0$ at $\lambda = 0$. However, since $\tau_{k_j}$ is a factor in both $h$ and $f_0$, and codim $\{h = 0\} \cap Z = m - 1$, for degree reasons it follows that $\xi$ vanishes on this set, and therefore, cf. e.g., [7], [27] or [2], each term in $\tilde{\xi}$ contains either a factor $\bar{\tau}_{k_j}$ or $\partial \bar{\tau}_{k_j}$. In any case, this implies that already the integral $(2.3)$ vanishes. On the other hand, if no $m_i$ is equal to $k_j$, it is easy to see that $(2.4)$ has an analytic continuation to $\operatorname{Re} \lambda > -\epsilon$ and takes the value $(2.3)$ at $\lambda = 0$. In fact, this follows easily since if $[1/s^{\ell}]$ is the usual principal value distribution in $\mathbb{C}$ and $v > 0$ is smooth and strictly positive, then

$$|s|^{2\lambda} v^{\lambda} [1/s^{\ell}]$$

has an analytic continuation to $\operatorname{Re} \lambda > -\epsilon$ and takes the value $[1/s^{\ell}]$ at $\lambda = 0$. Thus the proposition is proved. \[\square\]

Proof of Proposition 2.1. By assumption $\phi R^f$ is a current with support on $\{h = 0\}$, and hence (locally) $|h|^{2\lambda} \phi R^f = 0$ if $\operatorname{Re} \lambda >> 0$. From Lemma 2.2 it follows that

$$\phi R^f = |h|^{2\lambda} \phi R^f|_{\lambda=0} = 0.$$
Let $L \to X$ be a holomorphic line bundle and let $\phi$ be a holomorphic section of $\Lambda^k E \otimes L$.

**Theorem 2.3.** Let $\ell \geq 0$ and suppose that $H^{0,s}(X, \Lambda^{s+\ell+1} E \otimes L) = 0$ for all $1 \leq s \leq m - \ell - 1$. Moreover, let $\phi \in \mathcal{O}(X, \Lambda^\ell E \otimes L)$. Then $\delta_f \psi = \phi$ has a solution $\psi \in \mathcal{O}((X, \Lambda^{\ell+1} E \otimes L))$ if and only if there is a smooth solution $w$, defined in a neighborhood of $Z^f$, to
\[
(2.5) \quad \nabla_f (w \wedge R^f) = \phi \wedge R^f.
\]

In view of (2.2), the condition on $\phi$ is void if $\ell > m - p$. Moreover, since $w = w_{\ell+1,0} + w_{\ell+2,1} + \cdots$ the condition means precisely that $\phi \wedge R^f = 0$ if $\ell = m - p$. In the case $\ell = 0$ and $p = m$, i.e., $f$ defines a complete intersection, we get back the well-known duality theorem, first proved in [16] and [25].

It was also proved in [2] that $h_{\min(m,n)} R^f = 0$ if $h$ is holomorphic and $\|h\| \leq C\|f\|$. The local version of Theorem 2.3 therefore immediately implies the Briançon-Skoda theorem, [12]: If $\|\phi\| \leq C\|f\|$, then locally $\phi_{\min(m,n)}$ belongs to the ideal $(f)$. There is also an explicit representation formula in [2].

**Proof of Theorem 2.3.** First suppose that the holomorphic solution $\psi$ exists. Then $\nabla_f \psi = \phi$ and hence $\nabla_f (\psi \wedge R^f) = \phi \wedge R^f$ since $\nabla_f R^f = 0$. Conversely, if (2.5) holds for some smooth $w$, we claim that $\nabla_f \psi = \phi$, if
\[
v = (-1)^\ell \phi \wedge U^f + w \wedge R^f.
\]
In fact, since $\nabla_f \phi = 0$,
\[
\nabla v = \phi \wedge \nabla_f U^f + \nabla_f (w \wedge R^f) = \phi \wedge (1 - R^f) + \phi \wedge R^f = \phi.
\]
This means that
\[
\bar{\partial} v_{m,m-\ell-1} = 0 \quad \text{and} \quad \delta_f v_{k+1,k-\ell} = \bar{\partial} v_{k,k-\ell-1}.
\]
By the assumption on the Dolbeault cohomology, we can successively solve the equations
\[
\bar{\partial} \eta_{m,m-\ell-2} = v_{m,m-\ell-1}, \quad \bar{\partial} \eta_{k,k-\ell-2} = v_{k,k-\ell-1} + \delta_f \eta_{k+1,k-\ell-1}, \quad k \geq \ell,
\]
and then finally $\psi = v_{\ell,0} + \delta_f \eta_{\ell+1,0}$ is the desired holomorphic solution. \(\square\)

**Example 3.** Suppose that $X$ is a compact and $L$ is a strictly positive line bundle. Then there is an $r_0 > 0$ such that $H^{0,k}(X, \Lambda E^\omega \otimes L^r) = 0$ for all $k \geq 1$ if $r \geq r_0$. If $f$ is a holomorphic section of $E^\omega$, then a holomorphic section $\phi \in \mathcal{O}(\Lambda^\ell E \otimes L^r)$, $r \geq r_0$, is in the image of the morphism
\[
(2.6) \quad \mathcal{O}(X, \Lambda^{\ell+1} E \otimes L^r) \to \mathcal{O}(X, \Lambda^\ell E \otimes L^r)
\]
if $\phi \wedge R^f = 0$. If $\ell = m - p$ the condition is necessary.
We shall now focus on the case where $X = \mathbb{P}^n$ and $E$ is the Hermitian vector bundle from Section 1. Let $E_1, \ldots, E_m$ be trivial line bundles over $\mathbb{P}^n$ with basis elements $\epsilon_1, \ldots, \epsilon_m$, and let $E_j^*$ be the dual bundles, with bases $\epsilon_j^*$. Then we have that
\[
E^* = (L^{d_1} \otimes E_1^*) \oplus \cdots \oplus (L^{d_m} \otimes E_m^*),
\]
\[
E = (L^{-d_1} \otimes E_1) \oplus \cdots \oplus (L^{-d_m} \otimes E_m),
\]
and for instance our section $f$ can be written
\[
f = \sum_{j=1}^m f_j \epsilon_j^*.
\]
Its dual section $s$ is then, cf., (1.4),
\[
s = \sum_j \frac{f_j(z)}{|z|^{2d_j}} \epsilon_j,
\]
so
\[
R^f = \overline{\partial} \|f\|^{2\lambda} \wedge \sum_{\ell=1}^m s \wedge (\overline{\partial} s)^{\ell-1} \frac{1}{\|f\|^{2\ell}} \bigg|_{\lambda=0}.
\]
In $\mathbb{C}^n = \{z_0 \neq 0\} \subset \mathbb{P}^n$ we have the coordinates $z'$ and the natural holomorphic frame $e_j = z_0^{-d_j} \epsilon_j$ and its dual $e_j^* = z_0^{d_j} \epsilon_j^*$. If $f'_j(z') = f_j(1, z')$ then
\[
f = \sum_{j=1}^m f'_j \epsilon_j^*
\]
and
\[
s = \sum_{j=1}^m \frac{f'_j(z')}{(1 + |z'|)^{d_j}} \epsilon_j.
\]
When codim $Z^f = m$, the residue current $R^f$ is independent of the metric, it just contains the top degree term $R^f_{m,m}$, and in fact, see [2],
\[
R^f = \left[ \overline{\partial} \frac{1}{f'_m} \wedge \ldots \wedge \overline{\partial} \frac{1}{f'_1} \right] \wedge e_1 \wedge \ldots \wedge e_m,
\]
where the expression in brackets is a Coleff-Herrera residue current. Choosing the local coordinates $z_0, \zeta_1, \ldots, \zeta_n$ in $\mathbb{C}^{n+1} \setminus \{0\}$, where $\zeta_j = z_j/z_0$, it is easy to see that
\[
\pi^* \left[ \overline{\partial} \frac{1}{f'_m} \wedge \ldots \wedge \overline{\partial} \frac{1}{f'_1} \right] = z_0^{d_j} \left[ \overline{\partial} \frac{1}{f''_m} \wedge \ldots \wedge \overline{\partial} \frac{1}{f''_1} \right],
\]
and hence we can identify $R^f$ with the Coleff-Herrera current
\[
\left[ \overline{\partial} \frac{1}{f''_m} \wedge \ldots \wedge \overline{\partial} \frac{1}{f''_1} \right] \wedge \epsilon_1 \wedge \ldots \wedge \epsilon_m
\]
in $\mathbb{C}^{n+1} \setminus \{0\}$. 
Proof of Theorem 1.6. It is well-known, see, e.g., [15], that $H^{0,k}(\mathbb{P}^n, L^\nu) = 0$ for all $\nu$ if $1 \leq k \leq n - 1$ and that $H^{0,n}(\mathbb{P}^n, L^\nu) = 0$ if (and only if) $\nu \geq -n$. Since $E = L^{-d_1} \oplus \cdots \oplus L^{-d_m}$ we have that

$$\Lambda^\nu E \otimes L^r = \bigoplus_{|J| = \nu} L^{-d_{J_1}} \otimes \cdots \otimes L^{-d_{J_\nu}} \otimes L^r = \bigoplus_{|J| = \nu} L^{r-d_{J_1} - \cdots - d_{J_\nu}}.$$

Thus $H^{0,s}(\mathbb{P}^n, \Lambda^{s+\ell+1} E \otimes L^r) = 0$ for $1 \leq s \leq m - \ell - 1$ if either $m - \ell - 1 \leq n - 1$ or

$$r - \sum_{1}^{n+\ell+1} d_j \geq -n.$$

Now Theorem 1.6 follows from Theorem 2.3. □

3. Integral representation

The aim of this section is to present an explicit integral representation of the solution $Q_j$ to the division problem in Theorem 1.1. We have

**Theorem 3.1.** Let $F_1, \ldots, F_m, \Phi$ be polynomials in $\mathbb{C}^n$, let $f$ and $R^f$ be as before, and let $\phi$ be the $r$-homogenization of $\Phi$ ($\deg \Phi \leq r$). Then there is an explicit decomposition

$$\Phi(z') = \sum_{1}^{m} F_j(z') \int_{\mathbb{P}^n} T^j(\zeta, z') \phi(\zeta) + \int_{\mathbb{P}^n} S(\zeta, z') \wedge R^f(\zeta) \phi(\zeta),$$

where $T^j(\zeta, z'), S(\zeta, z')$ are smooth forms (in $[\zeta]$) on $\mathbb{P}^n$ and holomorphic polynomials in $z'$, such that

$$\deg z'(F_j(z')T^j(\zeta, z')) \leq d_1 + d_2 + \cdots + d_{\mu+1} + r,$$

if $\mu = \min(n, m - 1)$ and $d_1 \geq d_2 \geq \cdots \geq d_m$.

Thus, if $\phi R^f = 0$ we get back the conclusion of Theorem 1.1 but with the extra term $d_1 + \cdots + d_{\mu+1}$ in the estimate of the degree.

For fixed $z \in \mathbb{C}^n$,

$$\eta = 2\pi i \sum_{0}^{n} z_j \frac{\partial}{\partial \zeta_j}$$

is an $L_z \otimes L^{-1}_\zeta$-valued $(1, 0)$-form on $\mathbb{P}^n$, and if $\delta_\eta$ denotes interior multiplication with $\eta$, then

$$\delta_\eta : \mathcal{D}'_{\ell+1,0}(\mathbb{P}^n, L^{r+1}) \rightarrow \mathcal{D}'_{\ell,0}(\mathbb{P}^n, L^r).$$

**Remark 6.** When we say that $\eta$ is a section of $L_z \otimes L^{-1}_\zeta$ rather than $L^{-1}_z = L^{-1}_\zeta$, we just indicate that it is 1-homogeneous in $z$; it would be more correct, but less convenient, to consider $\eta$ as a section of the bundle $L_z \otimes L^{-1}_\zeta \otimes (T^*_\zeta)_{0,1}$ over $\mathbb{P}_z \times \mathbb{P}_\zeta$. 


Let $\nabla_\eta = \delta_\eta - \bar{\partial}$. Notice that if
\[
\alpha = \alpha_0 + \alpha_1 = \frac{z \cdot \bar{\zeta}}{|\zeta|^2} - \frac{\bar{\partial} \bar{\zeta} \cdot d\zeta}{2\pi i |\zeta|^2},
\]
then the first term, $\alpha_0$, is a section of $L_z \otimes L^{-1}_\zeta$ and the second term, $\alpha_1$, is a projective form (since $\delta_\zeta \alpha_1 = 0$); moreover
\[\nabla(3.2) \nabla_\eta \alpha = 0.\]
We have the following basic integral representation of global holomorphic sections of $L^r$.

**Proposition 3.2.** Assume that $r \geq 0$ and that $\phi \in \mathcal{O}(\mathbb{P}^n, L^r)$. Then
\[
\phi(z) = \int_{\mathbb{P}^n} \alpha^{n+r} \phi.
\]
For degree reasons, actually
\[
\phi(z) = \frac{(n + r)!}{n!r!} \int_{\mathbb{P}^n} \alpha_0^n \wedge \alpha_1^r \phi;
\]
this formula appeared already in [11]; expressed in affine coordinates it is the well-known weighted Bergman representation formula for polynomials in $\mathbb{C}^n$. However, we prefer to supply a direct proof on $\mathbb{P}^n$, following the ideas in [1].

*Proof.* Let $\sigma$ be the $L^{-1}_z \otimes L_\zeta \otimes T^*_{1,0}(\mathbb{P}^n)$ valued $(1,0)$-form on $\mathbb{P}^n$ that is dual, with respect to the natural metric, to $\eta$. Then, since $\eta$ has a first order zero at $[z]$ (and no others), it follows (see [11]) that
\[
\nabla_\eta \frac{\sigma}{\nabla_\eta \sigma} = 1 - [[z]].
\]
The rightmost term is the $L^{-n}_z \otimes L^n_\zeta$-valued $(n,n)$-current point evaluation at $[z]$ for sections of $L^{-n}$. If $\phi$ is a global holomorphic section of $L^r$ it follows (see [1]) that
\[
\nabla(3.2) \nabla_\eta \left( \frac{\sigma}{\nabla_\eta \sigma} \wedge \alpha^{n+r} \phi \right) = \phi \alpha^{n+r} - \phi[[z]],
\]
where this time the last term is $\phi$ times the $L^n_\zeta \otimes L^{-r}_\zeta$-valued current point evaluation at $[z]$. If we integrate this equality over $\mathbb{P}^n$ we get the desired representation formula. \hfill $\Box$

Let $E_1, \ldots, E_m$ be the trivial line bundles over $\mathbb{P}^n$ with basis elements $e_1, \ldots, e_m$, so that $E = (L^{-d_1} \otimes E_1) \oplus \cdots \oplus (L^{-d_m} \otimes E_m)$ as in Section 2. We also introduce disjoint copies $\tilde{E}_j$ of $E_j$ with bases $\tilde{e}_j$ and the bundle
\[
\tilde{E} = (L^{-d_1} \otimes \tilde{E}_1) \oplus \cdots \oplus (L^{-d_m} \otimes \tilde{E}_m).
\]
Let $\Lambda$ be the exterior algebra bundle over the direct sum of all the bundles $E$, $\tilde{E}$, $E^*$, and $T^*(\mathbb{P}^n)$. Any form $\gamma$ with values in $\Lambda$ can be...
written uniquely as \( \gamma = \gamma' \wedge (\sum \epsilon_j^* \wedge \epsilon_j)^m/m! + \gamma'' \) where \( \gamma'' \) denotes terms that do not contain a factor \( (\sum \epsilon_j^* \wedge \epsilon_j)^m/m! \), and we define

\[
\int c \gamma = \gamma'.
\]

We have a globally defined form

\[
\tau = \sum_1^m \epsilon_j^* \wedge (\epsilon_j - \tilde{\epsilon}_j).
\]

From now on we consider \([z]\) as a fixed arbitrary point in \( \mathbb{C}^n \subset \mathbb{P}^n \), and let \( z = (1, z') \). We also introduce the section

\[
f_z = \sum_j \zeta_0^{d_j} f_j(1, z) \epsilon_j^* = \sum_j \zeta_0^{d_j} F_j(z') \epsilon_j^*
\]

of \( E^* \) and let \( \tilde{f}_z \) be the corresponding section of \( \tilde{E}^* \).

**Lemma 3.3.** There is a holomorphic section \( H = \sum H_j \wedge \epsilon_j \) of \( E^* \otimes L \otimes T^*_{1,0} \), thus \( H_j \) are sections of \( L^{d_j} \otimes L \otimes T^*_{1,0} \), such that

\[
\delta_\eta H = f - f_z,
\]

and such that the coefficients in \( H_j \) are polynomials in \( z'/z_0 \) of degrees (at most) \( d_j - 1 \).

**Proof.** For each \( F_j(z') \) we can find Hefer functions \( h_j^k(\zeta', z') \), polynomials of degree \( d_j - 1 \) in \( (\zeta', z') \), such that

\[
\sum_{k=1}^n h_j^k(\zeta', z')(\zeta_k - z_k) = F_j(\zeta') - F_j(z').
\]

If we then take

\[
H_j = \frac{\zeta_0^{d_j+1}}{2\pi i} \sum_1^m h_j^k(\zeta'/\zeta_0, z')d(\zeta_k/\zeta_0),
\]

then clearly \( H_j \) is a projective \((1, 0)\)-form, and moreover,

\[
\delta_\eta H_j = f_j(\zeta) - \zeta_0^{d_j} F_j(z')
\]
as wanted. \( \Box \)

Let \( \delta_F \) denote interior multiplication with the section \( F = f + \tilde{f}_z \) of \( E^* \oplus \tilde{E}^* \). Then \( \delta_F \tau = f - f_z = -\delta_\eta H \). If

\[
\nabla = \delta_F + \delta_\eta - \bar{\partial},
\]

thus

\[
(3.3) \quad \nabla(\tau + H) = 0.
\]

We are now ready to define the explicit division formula.
Proof of Theorem 3.1. From (3.3) it follows that
\[(3.4) \quad (\nabla_\eta + \delta_F)(e^{\tau + H} \wedge U^f) = e^{\tau + H} \wedge (1 - R^f). \]

We can rewrite this as
\[(3.5) \quad \delta_F(e^{\tau + H} \wedge U^f) + e^{\tau + H} \wedge R^f = e^{\tau + H} - \nabla_\eta(e^{\tau + H} \wedge U^f). \]

We claim that the component of full bidegree \((n, n)\) of
\[(3.6) \quad \int_\epsilon e^{\tau + H} - \nabla_\eta(e^{\tau + H} \wedge U^f)] \wedge \alpha^{n+r} \phi \]

is equal to
\[
\frac{(n + r)!}{n!r!} \alpha_1^n \alpha_0^r \phi + \bar{\partial}(\cdots) \]

where \((\cdots)\) is a scalar-valued \((n, n-1)\)-form. In fact, since \(\alpha^{n+r}\) has bidegree \((*, *)\) the factor \(U_{\ell, \ell-1}\) must be combined with \(H_{\ell}\), and then it follows that \(\tau\) can be replaced by \(\omega = \sum_j \epsilon_j^* \wedge \epsilon_j\). Observe that the component of \(U_{\ell, \ell-1}\) with basis element \(\epsilon_{J_1} \wedge \ldots \wedge \epsilon_{J_\ell}\) takes values in \(L^{-d_{J_1} - \cdots - d_{J_\ell}}\), whereas the component of \(H_{\ell}\) with basis element \(\epsilon_{J_1}^* \wedge \ldots \wedge \epsilon_{J_\ell}^*\) takes values in \(L^{d_{J_1} + \cdots + d_{J_\ell}} \otimes L^\ell\). The product of these two factors must be combined with \(\alpha_1^{n-\ell} \alpha_0^{\ell+r} \phi\) which gives a scalar-valued \((n, n)\)-form as claimed. Thus we can integrate (3.6) over \(\mathbb{P}^n\), and by Proposition 3.2 and Stokes’ theorem it is equal to \(\phi(z)\).

We now consider the left hand side of (3.5) multiplied with \(\alpha^{n+r}\phi\). To begin with,
\[
\int_{\mathbb{P}^n} \int_\epsilon e^{\tau + H} \wedge R^f \wedge \alpha^{n+r} \phi
\]
is well defined with the same argument as above, and again one can replace \(\tau\) by \(\omega\). Moreover, since \(\alpha^{n+r}\phi\) contains no \(\epsilon_j\),
\[
\int_\epsilon \delta_f(e^{\tau + H} \wedge U^f) \wedge \alpha^{n+r} \phi = \int_\epsilon \delta_f(e^{\tau + H} \wedge U^f \wedge \alpha^{n+r} \phi) = 0.
\]
Since
\[
\delta_f \sum_j \bar{\epsilon}_j \wedge \epsilon_j^* = \sum_j F(z') \zeta_0 \epsilon_j^* \epsilon_j = f_z,
\]

another computation shows that the component of bidegree \((n, n)\) of
\[
\int_\epsilon \delta_f(e^{\tau + H} \wedge U^f) \wedge \alpha^{n+r} \phi
\]
is equal to
\[
\int f_z \wedge \sum_{k=0}^{m-1} \omega_{m-k-1} \wedge H_k \wedge U_{k+1,k} \wedge \alpha_1^{n-k} \alpha_0^{k+r} \phi.
\]
Again one can check that this form is scalar valued. Summing up we have the desired decomposition (3.1) with

\[ S(\zeta, z') \wedge R^f(\zeta) = \int e^{\omega+H} \wedge R^f \wedge \alpha^{n+r} = \]

\[ \sum_{k=\text{codim} Z^f}^m \int_e \frac{(n+r)!}{(n-k)!(k+r)!} \omega_{m-k} \wedge H_k \wedge R^f_{k,k} \alpha_1^{n-k} \alpha_0^{k+r}, \]

and

\[ T^j(\zeta, z') = \]

\[ \int e^*_{\ell} \int_{\epsilon} \frac{(n+r)!}{(n-k)!(k+r)!} \tilde{t}_{n-k-1} \wedge H_k \wedge U_{k+1,k} \wedge \alpha_1^{n-k} \alpha_0^{k+r} \phi, \]

Both \( \alpha \) and \( H \) are polynomials in \( z' \) so it just remains to check the degrees of \( T^j \). The worst case occurs when \( k \) is as large as possible which is \( k = \mu = \min(m-1, n) \). Then the factor \( \alpha_0^{k+r} \) has degree \( k+r \). Recall that \( H = \sum H_\ell \wedge e^*_\ell \) and that \( \deg H_\ell = d_\ell - 1 \). The term \( H_j \) cannot occur, because of the presence of \( e^*_j \), and thus we get that \( d_j + \deg Q_j \) is at most \( d_1 - 1 + d_2 - 1 + \cdots + d_{\mu+1} - 1 + 1 + \mu + r = d_1 + \cdots + d_{\mu+1} + r \). Therefore, we have the desired decomposition.

The division formula constructed here, Theorem 3.1, is a generalization to \( \mathbb{P}^n \) of the formula in \cite{2}, which was used to give an explicit representation of the solutions in the local version of Theorem 2.3 in particular it provided the first known explicit proof of the Briançon-Skoda theorem. This division formula is based on the ideas in \cite{1} and it differs from Berndtsson’s classical formula, \cite{10}, in some respects. To begin with our formula works also for sections with values in \( \Lambda^\ell E \), although in this paper we have only generalized the scalar-valued part to \( \mathbb{P}^n \). The more interesting novelty with regard to this paper, is that the residue term contains precisely the factor \( \phi R^f \), so that our formula provides a solution of the division problem as soon as \( \phi R^f = 0 \) (or \( \phi R^f = \nabla_f(w \wedge R^f) \) for some smooth \( w \)). One can obtain a similar formula involving residues (but not precisely \( R^f \) except for the complete intersection case) from Berndtsson’s formula; this was first done by Passare in \cite{25}, and various variants have been used by several authors since then, see \cite{8} and the references given there. These formulas all go back to the construction of weighted integral formulas in \cite{9}. However, the division formula in \cite{2}, even in the simplest case, when \( f \) is nonvanishing, could not have been obtained from \cite{9}, because the required choice of weight, see formula (2.12) in Remark 3 in \cite{3}, is not encompassed by the method in \cite{9}, but the more general construction in \cite{11} is needed.

Acknowledgement I am indebted to the referee for his careful reading and for his many important remarks and constructive suggestions that have helped to clarify and improve the final version of this paper.
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