ON THE STRUCTURE OF FUNDAMENTAL GROUPS OF COMPLEMENTS OF CONIC–LINE ARRANGEMENTS

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Abstract. The fundamental group of the complement of a hyperplane arrangement plays an important role in studying these arrangements. In particular, for large families of these arrangements, this fundamental group, being isomorphic to the fundamental group of a complement of a line arrangement, has some remarkable properties: either it is a direct sum of free groups and a free abelian group, or it has a conjugation-free geometric presentation.

In this paper, we generalize these ideas to the case of conic-line arrangements. Explicitly, we prove that once the graph associated to conic-line arrangements (defined slightly different than the corresponding graph for line arrangements) has no cycles, then the fundamental group of its complement has a conjugation-free geometric presentation and in addition can be written as a direct sum of free groups and a free abelian group. Moreover, if there is a cycle in the graph but the conic does not pass through all the multiple points of the cycle, then the fundamental group has a conjugation-free geometric presentation as well. On the other hand, if the graph is a unique cycle and the conic passes through all the multiple points of the cycle, then the presentation is not conjugation-free anymore; but if the length of the cycle is odd and the multiplicities of all the multiple points are 3, then the fundamental group is abelian.

In the appendix, we extend the family of line arrangements having a conjugation-free geometric presentation (for the fundamental group) by defining the notion of a conjugation-free graph. We also expand this notion to certain families of conic-line arrangements.

1. Introduction

The fundamental group of the complement of a plane curve is a very important topological invariant. For example, it is used to distinguish between curves that form a Zariski pair, which is a pair of curves having the same combinatorics but non homeomorphic complements in $\mathbb{C}P^2$ (see [6] for the exact definition and [7] for a survey). Another example is that while the fundamental group of the complement of a nodal curve is
abelian, there are curves with non-abelian fundamental groups. Thus, it is interesting to explore finite non-abelian groups which arise that way, see for example [4, 5, 8, 30].

Moreover, the Zariski-Lefschetz hyperplane section theorem (see [22]) states that \( \pi_1(\mathbb{C}P^N - S) \cong \pi_1(H - (H \cap S)) \), where \( S \) is a hypersurface and \( H \) is a generic 2-plane. Since \( H \cap S \) is a plane curve, the fundamental groups of complements of plane curves can be used also for computing the fundamental groups of complements of hypersurfaces in \( \mathbb{C}P^N \). Note that when \( S \) is a hyperplane arrangement, \( H \cap S \) is a line arrangement in \( \mathbb{C}P^2 \). Thus, one of the main tools for investigating the topology of hyperplane arrangements is the fundamental groups \( \pi_1(\mathbb{C}P^2 - \mathcal{L}) \) and \( \pi_1(\mathbb{C}^2 - \mathcal{L}) \), where \( \mathcal{L} \) is an arrangement of lines.

These groups, for line arrangements, have very interesting properties (see e.g. [27, Section 5.3]). They are abelian if and only if \( \mathcal{L} \) has only nodes as intersection points, see for example [9, Example 1.6(a)]. Moreover, Fan [14] and Eliyahu et al. [12] proved that this group is a direct sum of a free abelian group and free groups if and only if a certain graph, associated to the intersection points of \( \mathcal{L} \) (see Section [2]), has no cycles. Based on this, Eliyahu et al. [10, 11] showed that other properties hold for certain presentations of this group: conjugation-free and complemented presentations.

As conic–line arrangements are a natural generalization of line arrangements, an immediate question that arises is whether the above properties (e.g. conjugation–free or a direct sum of abelian and free groups) hold for these arrangements too. One should note, contrary to the situation for line arrangements, that this group can be abelian even if the conic–line arrangement has singular points which are not nodes (see [8] and Figure 3 below). Note that the fundamental groups \( \pi_1(\mathbb{C}P^2 - \mathcal{A}) \) and \( \pi_1(\mathbb{C}^2 - \mathcal{A}) \), for some families of conic-line arrangement \( \mathcal{A} \), were studied by Amram et al. (see e.g. [1, 2] and especially [3, Theorem 6]). Moreover, Zariski pairs consisting of conic-line arrangements were studied by, for example, Namba-Tsuchihashi [25] and Tokunaga [28], but a research in the spirit of the above questions has not been carried out yet.

In this paper, we generalize Fan’s result to the case of conic–line arrangements. After surveying the known results on line arrangements and the braid monodromy technique in Section [2] we give, in Section [3] a necessary condition when the fundamental group of some families of conic-line arrangements is a direct sum of a free abelian group and free groups. Explicitly, we generalize Fan’s concept of a graph associated to line arrangements to real conic-line arrangements and prove that once
this graph has no cycles, then the fundamental group has the desired structure (for an explicit formulation, see Theorem 2.5). In Sections 4 and 5 we prove a few unexpected propositions about the structure of a conic-line arrangement whose graph consists of a unique cycle. These surprising results indicate that if the conic passes through all the multiple points of the unique cycle, where all the multiple points are of multiplicity 3, then one has to distinguish between cycles of odd length, where in that case the fundamental group is abelian, and cycles of even length greater than 4, where the fundamental group is not isomorphic to a direct sum of free groups and a free abelian group.

We finish the paper with an appendix, which defines the notion of a conjugation-free graph, both for line arrangements and for conic-line arrangements. Explicitly, for every arrangement whose graph is a conjugation-free graph, the fundamental group of its complement has a conjugation-free geometric presentation.

Acknowledgements: We would like to thank Arkadius Kalka and Meital Eliyahu for stimulating talks. Also, we would like to thank an anonymous referee of a previous version of this paper for useful and important suggestions. The first author would like to thank the Max-Planck-Institute für Mathematik in Bonn for the warm hospitality and support.

2. ARRANGEMENTS AND BRAID MONODROMY

In this section, we give a short survey of the known results concerning the structure of the fundamental group of the complement of a line arrangement. After that, we present the family of conic-line arrangements that we work with and give a short explanation about the braid monodromy technique, for computing presentations of fundamental groups of complements of plane curves.

2.1. Line and Conic-Line arrangements. An affine line arrangement in $\mathbb{C}^2$ is a union of copies of $\mathbb{C}^1$ in $\mathbb{C}^2$. Such an arrangement is called real if the defining equations of all its lines can be written with real coefficients, and complex otherwise.

For real and complex line arrangements $\mathcal{L}$, Fan [14] defined a graph $G(\mathcal{L})$ which is associated to its multiple points (i.e. points where more than two lines are intersected). We give here its version for real arrangements (the general version is more delicate to explain and will be omitted): Given a real line arrangement $\mathcal{L}$, the graph $G(\mathcal{L})$ of multiple points lies on the real part of $\mathcal{L}$. It consists of the multiple points of
\( \mathcal{L} \), with the segments between the multiple points on lines which have at least two multiple points. Note that if the arrangement consists of three multiple points on the same line, then \( G(\mathcal{L}) \) has three vertices on the same edge (see Figure 1(a)). If two such lines happen to intersect in a simple point (i.e. a point where exactly two lines are intersected), it is ignored (i.e. there is no corresponding vertex in the graph). See another example in Figure 1(b) (note that Fan’s definition gives a graph slightly different from the graph defined in [20, 29]).

Fan [13, 14] proved the following result:

**Proposition 2.1** (Fan). Let \( \mathcal{L} \) be a complex arrangement of \( k \) lines and \( S = \{a_1, \ldots, a_p\} \) be the set of all multiple points of \( \mathcal{L} \). Suppose that \( \beta(\mathcal{L}) = 0 \), where \( \beta(\mathcal{L}) \) is the first Betti number of the graph \( G(\mathcal{L}) \) (hence \( \beta(\mathcal{L}) = 0 \) means that the graph \( G(\mathcal{L}) \) has no cycles). Then:

\[
\pi_1(\mathbb{CP}^2 - \mathcal{L}) \cong \mathbb{Z}^r \oplus \bigoplus_{i=1}^{p} \mathbb{F}_{m(a_i)-1},
\]

where \( m(a_i) \) is the multiplicity of the intersection point \( a_i \) and \( r = k + p - 1 - \sum_{i=1}^{p} m(a_i) \).

Eliyahu et al. [12] proved the inverse direction to Fan’s result (which was conjectured by Fan [14]), i.e. if the fundamental group of the arrangement is a direct sum of free groups and a free abelian group, then the associated graph has no cycles.

We will generalize Fan’s result to real conic-line arrangements.

**Definition 2.2.** A real conic-line (CL) arrangement \( \mathcal{A} \) is a collection of conics and lines in \( \mathbb{C}^2 \), where all the conics and the lines are defined
over $\mathbb{R}$ and every singular point of the arrangement is in $\mathbb{R}^2$. In addition, for every conic $C$, $C \cap \mathbb{R}^2$ is not an empty set, neither a point nor a (double) line.

Moreover, we assume from now on the following assumption:

**Assumption 2.3.** Let $A$ be a real CL arrangement. Then, for each pair of components $\ell_1, \ell_2$ of $A$, $\ell_1$ and $\ell_2$ intersect transversally (i.e. the intersection multiplicity of $\ell_1, \ell_2$ is 2 at each intersection point).

For example, a tangency point is not permitted.

**Remark 2.4.** As we will consider generic projections of CL arrangements from a point (not on the arrangement) to a generic line, we can assume that no line passes through the branch points of the conics with respect to this projection.

Similar to Fan’s graph for line arrangements, one can define the following graph $G(A)$ for a real CL arrangement $A$: its vertices will be the multiple points (with multiplicity larger than 2), and its edges will be the segments on the lines connecting these points if two such points are located on the same line, see an example in Figure 2.

**Figure 2.** An example for $G(A)$ for a CL arrangement $A$.

Then, one of the main results of this paper is:

**Theorem 2.5.** Let $A$ be a real CL arrangement with one conic and $k$ lines, and $S = \{a_1, \ldots, a_p, b_1, \ldots, b_q\}$ be the set of all multiple points of $A$, where the conic is passing through the intersection points $a_1, \ldots, a_p$. Suppose that $\beta(A) = 0$, where $\beta(A)$ is the first Betti number of the graph $G(A)$ (hence $\beta(A) = 0$ means that the graph $G(A)$ has no cycles). Then:

$$
\pi_1(\mathbb{C}P^2 - A) \cong \mathbb{Z}^r \oplus \bigoplus_{i=1}^p \mathbb{F}_{m(a_i)-2} \oplus \bigoplus_{i=1}^q \mathbb{F}_{m(b_i)-1},
$$
where \( m(x) \) is the multiplicity of the singular point \( x \) and
\[
r = k + 2p + q - \sum_{i=1}^{p} m(a_i) - \sum_{i=1}^{q} m(b_i).
\]

Note that while for line arrangements the inverse direction (i.e. such
a structure of the fundamental group implies that the associated graph
has no cycles) is correct [12], for CL arrangements it is not true any-
more. For example, take three generic lines and a circle passing through
the three intersection points (see Figure 3(a)). Then the fundamental
group of the complement of this arrangement is abelian [8], although
the first Betti number of the graph is 1. We generalize this phenomena
in Section 5 showing that for a CL arrangement whose graph is a cycle
of odd length and the conic passes through all the points of the cycle
(where all the multiple points have multiplicity 3), the corresponding
fundamental group is abelian (see Theorem 5.5).

Note also that there are CL arrangements with a tangent point
(which is excluded by our restrictions, see Assumption 2.3) with an
abelian fundamental group of the complement. For example, three
lines and a conic tangent to only one of the lines (see Figure 3(b)) has
an abelian fundamental group (see [8]).

![Figure 3. CL arrangements with an abelian affine fundamental group \( \mathbb{Z}^4 \): arrangement (a) has \( \beta(A) > 0 \), and arrangement (b) has a tangency point.](image)

2.2. The braid monodromy of plane curves and the Zariski-van
Kampen theorem. The reader who is familiar with the definition of
the braid monodromy, its computation and the relevant Zariski-van
Kampen theorem, can skip this section.

We start by defining the braid monodromy associated to a curve.

Let \( D \) be a closed disk in \( \mathbb{R}^2 \), \( K \subset \text{Int}(D) \), \( K \) finite, \( n = \#K \).
Recall that the braid group \( B_n[D,K] \) can be defined as the group of
all equivalent diffeomorphisms \( \beta \) of \( \overline{D} \) such that \( \beta(K) = K \), \( \beta|_{\partial D} = \)
Let \( x = \psi(D,K) \). This induces a diffeomorphism between the models \((D,K)\) path. This induces a diffeomorphism between the models \((\psi(D,K),\psi(U))\) at \( \partial D \).

Definition 2.6. The braid monodromy with respect to \( C,\pi,u \).

Let \( C \) be a curve, \( C \subseteq \mathbb{C}^2 \). Choose \( O \in \mathbb{C}^2, O \not\subseteq C \) such that the projection \( f : \mathbb{C}^2 \to \mathbb{C}^1 = \ell \) with a center \( O \) (to a generic line \( \ell \)) will be generic when restricting it to \( C \). Denote \( \pi = f|_C \) and let \( m = \deg \pi = \deg C \). Let \( N = \{ x \in \ell \mid \#\pi^{-1}(x) < m \} \). Take \( u \in \ell - N \), and let \( \mathbb{C}^1_u = f^{-1}(u) \). There is a naturally defined homomorphism

\[
\varphi : \pi_1(\ell - N,u) \to B_m[\mathbb{C}^1_u,\mathbb{C}^1_u \cap C],
\]

which is called the braid monodromy with respect to \( C,\pi,u \), and describes the motion of the points in the fiber (see [23]).

In fact, if \( E \) is a big disk in \( \ell = \mathbb{C}^1 \) such that \( N \subseteq E \), we can also take the path in \( E - N \) not to be a loop, but just a non-selfintersecting path. This induces a diffeomorphism between the models \((D,K)\) at the two ends of the considered path, where \( D \) is a big disk in \( \mathbb{C}^1_u \), and \( K = \mathbb{C}^1_u \cap C \subseteq D \).

Definition 2.7. \( \psi_T \), the Lefschetz diffeomorphism induced by a path \( T \).

Let \( x_0, x_1 \in E - N \) be two different points, \( T \) a non-selfintersecting path in \( E - N \) connecting \( x_0 \) with \( x_1 \), \( T : [0,1] \to E - N \). There exists a continuous family of diffeomorphisms \( \psi(t) : D \to D, t \in [0,1], \) such that \( \psi(0) = \text{Id}, \psi(t)(K(x_0)) = K(T(t)) \) for all \( t \in [0,1] \), and \( \psi(t)(y) = y \) for all \( y \in \partial D \). For emphasis, we write \( \psi(t) : (D,K(x_0)) \to (D,K(T(t))) \).

The Lefschetz diffeomorphism induced by a path \( T \) is the diffeomorphism

\[
\psi_T = \psi(t) : (D,K(x_0)) \simto (D,K(x_1)).
\]

Since \( \psi(t)(K(x_0)) = K(T(t)) \) for all \( t \in [0,1] \), we have a family of canonical isomorphisms

\[
\psi(t)_* : B_m[D,K(x_0)] \simto B_m[D,K(T(t))], \quad \text{for all } t \in [0,1].
\]

Let \( \Gamma \) be a geometric (free) base (called a \( g \)-base) of \( \pi_1(\mathbb{C}^1 - N,u) \) (see [23] for the exact definition), \( \varphi \) the braid monodromy of \( C,\varphi : \pi_1(\mathbb{C}^1 - N,u) \to B_m \). In order to find out a presentation of the fundamental group of the complement of \( C \) in \( \mathbb{C}^2 \), we have to find out what are
\( \phi(\Gamma_i) \), for all \( i \). We refer the reader to the definition of a skeleton \( \lambda_{x_j} \), for all \( x_j \in N \) (see [24]), which is a model of a set of consecutive paths connecting points in the fiber, which coincide when approaching \( A_j = (x_j, y_j) \in C \) from the right. To describe this situation in more details, for \( x_j \in N \), let \( x'_j = x_j + \alpha \), for \( 0 < \alpha \ll 1 \). The skeleton in \( x_j \) is defined as a system of consecutive lines connecting the points in \( K(x'_j) \cap D(A_j, \varepsilon) \), when \( 0 < \alpha \ll \varepsilon \ll 1 \) and \( D(A_j, \varepsilon) \) is a disk centered in \( A_j \) with radius \( \varepsilon \).

For a given skeleton, we denote by \( \Delta(\lambda_{x_j}) \) the braid which rotates by \( 180^\circ \) counterclockwise a small neighborhood of the given skeleton. Note that if \( \lambda_{x_j} \) is a single path, then \( \Delta(\lambda_{x_j}) = H(\lambda_{x_j}) \).

We also refer the reader to the definition of \( \delta_{x_0} \), for \( x_0 \in N \) (see [24]), which describes the Lefschetz diffeomorphism induced by a path going below \( x_0 \), for different types of singular points (either a transversal intersection of several lines in a point or a branch point; for example, when going below a node, a half-twist of its corresponding skeleton occurs).

Thus, the Lefschetz diffeomorphism induced by a path going from \( x'_j \) to \( u \) below the points \( x_i \), \( 1 \leq i \leq j - 1 \), is the composition of the corresponding \( \delta_{x_i} \)'s, i.e. \( \prod_{m=j-1}^{1} \delta_{x_m} \) [23, 24]. We illustrate the action of a specific Lefschetz diffeomorphism (induced by a line arrangement) in the following example.

**Example 2.8.** We present here an example for computing a skeleton and the effect of applying a Lefschetz diffeomorphism on it (more examples can be found in [10, 23, 24]).

\( \lambda_{x_4} \), the initial skeleton of the point \( x_4 \) in Figure 4(a) (i.e. above \( x'_4 \)), is presented in Figure 4(b).

\( \delta \), which is the Lefschetz diffeomorphism induced by \( \gamma \) (a path going from \( x'_4 \) to \( u \)), is:

\[ \Delta(3, 4) \Delta(2, 3) \Delta(1, 2) \].

\( \lambda_{x_4} \), the final skeleton of the point \( x_4 \) in Figure 4(a) (i.e. above \( u \); after applying \( \delta \) on \( \lambda_{x_4} \)), is presented in Figure 4(c).

Based on the braid monodromy, we can compute presentations for the groups \( \pi_1(\mathbb{C}P^2 - C) \) and \( \pi_1(C^2 - C) \) (where \( C = C \cap C^2 \)).

Let \( \{ \Gamma_i \} \) be a \( g \)-base of \( G = \pi_1(\mathbb{C}u - (\mathbb{C}u \cap C), u) \), where \( \mathbb{C}u = \mathbb{C} \times \{ u \} \).

Then \( \pi_1(C^2 - C, u) \) is generated by the images of \( \{ \Gamma_i \} \) in \( \pi_1(C^2 - C, u) \).

We use now the Zariski-van Kampen theorem [21] in order to compute the relations between the generators of \( G \). The theorem essentially says
Figure 4. Figure (a) is an example of a line arrangement, figure (b) is the initial skeleton of the point \( x_4 \) (i.e. above \( x_4' \)) and figure (c) is its final skeleton (above \( u \)).

that every singular point (with respect to a projection from \( O \) to \( \ell \)) induces a relation in \( \pi_1(\mathbb{C}^2 - C) \), and these induced relations are all the relations in \( \pi_1(\mathbb{C}^2 - C) \).

Since we are dealing only with CL arrangements, we formulate the theorem only for branch points, nodes and multiple intersection points.

\textbf{Theorem 2.9} (Zariski-van Kampen [21]). Let \( \overline{C} \) be a CL arrangement in \( \mathbb{CP}^2 \), \( C = \mathbb{C}^2 \cap \overline{C} \) and let \( p \) be the number of singular points of \( C \) with respect to the projection from \( O \). For every \( j \in \{1, \ldots, p\} \), consider the skeleton

\[ \lambda'_{x_j} = \left\langle (\lambda_{x_j}) \left( \prod_{m=j-1}^{1} \delta_{x_m} \right) \right\rangle. \]

Then \( \pi_1(\mathbb{C}^2 - C, u) \) is generated by the images of \( \{\Gamma_i\} \) in \( \pi_1(\mathbb{C}^2 - C, u) \) and the only relations are those induced by the skeletons \( \lambda'_{x_j} \) in the following way:

If the point \( x_j \) is either a node or a branch point, then the skeleton \( \lambda'_{x_j} \) is a path connecting two points. In this case, the relation is either \( a_1a_2 = a_2a_1 \) (for a node) or \( a_1 = a_2 \) (for a branch point), where the definition of the \( a_i \)'s will be described after the theorem.

If the point \( x_j \) is a multiple intersection point of multiplicity \( k \), then the skeleton \( \lambda'_{x_j} \) is a set of \( k - 1 \) consecutive paths connecting \( k \) points.
In this case, the relations are:

\[ a_k a_{k-1} \cdots a_1 = a_1 a_k a_{k-1} \cdots a_2 = \cdots = a_{k-1} a_{k-2} \cdots a_1 a_k, \]

where the definition of the \( a_i \)'s will be described after the theorem.

We start by describing the \( a_i \)'s in the case that the skeleton is a path connecting two points, i.e. the singular point is either a node or a branch point. Let \( D \) be a disk circumscribing the skeleton, and let \( K \) be the set of points. Choose an arbitrary point on the path and ‘pull’ it down to \( \partial D \), splitting the path into two parts, which are connected in one end to \( u_0 \in \partial D \) and in the other to the two endpoints of the path in \( K \). The loops associated to these two paths are elements in the group \( \pi_1(D - K, u_0) \) and we call them \( a_1 \) and \( a_2 \). The corresponding elements commute (in the case of a node) or equal (in the case of a branch point) in the fundamental group of the arrangement’s complement. Figure 5 illustrates this procedure.

Figure 5. Computation of \( a_1, a_2 \) for a node or a branch point.

Now we show how to write \( a_1 \) and \( a_2 \) as words in the generators \( \{\Gamma_1, \ldots, \Gamma_\ell\} \) of \( \pi_1(D - K, u_0) \). We start with the generator corresponding to the endpoint of \( a_1 \) (or \( a_2 \)), and conjugate it as we move along \( a_1 \) (or \( a_2 \)) from its endpoint in \( K \) to \( u_0 \) as follows: for every point \( i \in K \) which we pass from above, we conjugate by \( \Gamma_i \) when moving from left to right, and by \( \Gamma_i^{-1} \) when moving from right to left.

For example, in Figure 5

\[ a_1 = \Gamma_3 \Gamma_2 \Gamma_1 \Gamma_2^{-1} \Gamma_3^{-1}, \quad a_2 = \Gamma_4^{-1} \Gamma_6 \Gamma_4. \]
Assuming that the singular point is a node, the induced relation is the following commutative relation:

$$\Gamma_3 \Gamma_2 \Gamma_1 \Gamma_3^{-1} \cdot \Gamma_4^{-1} \Gamma_6 \Gamma_4 = \Gamma_4^{-1} \Gamma_6 \Gamma_4 \cdot \Gamma_3 \Gamma_2 \Gamma_1 \Gamma_2^{-1} \Gamma_3^{-1}.$$  

One can check that the induced relation is independent of the point in which the path is split.

For a multiple intersection point of multiplicity $k$, we compute the elements in the group $\pi_1(D - K, u_0)$ in a similar way, but the induced relations are of the following cyclic type:

$$a_k a_{k-1} \cdots a_1 = a_1 a_k a_{k-1} \cdots a_2 = \cdots = a_{k-1} a_{k-2} \cdots a_1 a_k.$$  

We choose an arbitrary point on one of the paths and pull it down to $u_0$. For each of the $k$ points of the skeleton, we generate the loop associated to the path from $u_0$ to that point, and translate this path to a word in $\Gamma_1, \ldots, \Gamma_k$ by the procedure described above.

![Diagram](image)

**Figure 6.** Computation of $a_1, a_2, a_3$ for a multiple intersection point

In the example given in Figure 6, we have:

$$a_1 = \Gamma_3 \Gamma_1 \Gamma_3^{-1}, a_2 = \Gamma_3 \Gamma_2 \Gamma_3^{-1} \text{ and } a_3 = \Gamma_4^{-1} \Gamma_6 \Gamma_4,$$

so the relations are:

$$\Gamma_4^{-1} \Gamma_6 \Gamma_4 \cdot \Gamma_3 \Gamma_2 \Gamma_3^{-1} \cdot \Gamma_3 \Gamma_1 \Gamma_3^{-1} = \Gamma_3 \Gamma_1 \Gamma_3^{-1} \cdot \Gamma_4^{-1} \Gamma_6 \Gamma_4 \cdot \Gamma_3 \Gamma_2 \Gamma_3^{-1} = \Gamma_3 \Gamma_2 \Gamma_3^{-1} \cdot \Gamma_3 \Gamma_1 \Gamma_3^{-1} \cdot \Gamma_4^{-1} \Gamma_6 \Gamma_4.$$
3. THE CASE OF A GRAPH WITH NO CYCLES

In this section, we prove Theorem 2.5. We first prove that the fundamental groups of the complements of these CL arrangements (with one conic), whose graph have no cycle, have a conjugation-free geometric presentation (Section 3.1), and then we conclude the structure of these fundamental groups (Section 3.2).

3.1. The conjugation-free property. For the first step, we define a conjugation-free geometric presentation of the fundamental group of CL arrangements, following the corresponding definition for line arrangements (see [10]):

**Definition 3.1.** Let $G$ be a fundamental group of the affine or projective complements of a real CL arrangement with $k$ lines and $n$ conics. We say that $G$ has a conjugation-free geometric presentation if $G$ has a presentation with the following properties:

- In the affine case, the generators $\{x_1, \ldots, x_{k+2n}\}$ are the meridians of lines and conics at some far side of the arrangement, and therefore the number of generators is equal to $k + 2n$.
- In the projective case, the generators are the meridians of lines and conics at some far side of the arrangement except for one, and therefore the number of generators is equal to $k + 2n - 1$.
- In both cases, the relations are of the following types:

\[ x_{i_1} x_{i_{t-1}} \cdots x_{i_1} = x_{i_{t-1}} \cdots x_{i_1} x_{i_t} = \cdots = x_{i_1} x_{i_t} \cdots x_{i_2} \]

or

\[ x_{i_1} = x_{i_2}, \]

where $\{i_1, i_2, \ldots, i_t\} \subseteq \{1, \ldots, m\}$ is an increasing subsequence of indices, where $m = k + 2n$ in the affine case and $m = k + 2n - 1$ in the projective case. Note that for $t = 2$ (in the first type) we get the usual commutator.

- In the projective case, we have an extra relation that a specific multiplication of all the generators is equal to the identity element.

**Remark 3.2.** The notion of a conjugation-free geometric presentation of the fundamental group can be generalized to any arrangement of plane curves (with the proper modifications with respect to the degrees of the curves and the types of singularities).

Note that the importance of the family of CL arrangements whose fundamental group has a conjugation-free geometric presentation is
that the fundamental group can be read directly from the arrangement.

We start with the following useful lemma, which is similar to [11, Proposition 2.2]:

**Lemma 3.3.** Let $A$ be a real CL arrangement with one conic such that $\pi_1(\mathbb{CP}^2 - A)$ has a conjugation-free geometric presentation. Let $L$ be a line that passes through a single intersection point of $A$. Then $\pi_1(\mathbb{CP}^2 - (A \cup L))$ has a conjugation-free geometric presentation.

**Proof.** As the proof is similar to the corresponding proof for the case of real line arrangements (see [11, Proposition 2.2]), we first outline the proof in that case and then we describe the major changes so that the proof will be correct for our case too.

Let $L$ be a real line arrangement, and $p \in L$ be an intersection point. The first step in the proof is to assume that the point $p$ can be placed as the leftmost and lowest point of the arrangement and the line having the maximal global number passes through $p$ (see Figure 7). This can be done using some results from [18] and a deformation argument, similar to the one used in Fan’s paper [14]. Then, the new line $L$ is drawn through $p = (x_p, y_p)$, such that the domain $A = \{ a \in \mathbb{R}^2 : x_a > x_p \}$ (where $x_a$ is the $x$-coordinate of $a$) contains all the intersection points of the arrangement $L$ (with respect to the projection $\mathbb{C}^2 \to \mathbb{C}$, $(x, y) \mapsto x$) and none of the points in $L \cap L$. Explicitly, the new line $L$ has a very negative slope.

![Figure 7. The setting for adding the line $L$ in the case of real line arrangements](image)

The second step is to notice that all the relations of $\pi_1(\mathbb{CP}^2 - (L \cup L))$ have no conjugations. For the simplification process of the relations which are induced by the intersection points in the domain $A$, the authors separate their treatment into two cases. In the first case, the simplification of these relations does not use the relation induced by
the point $p$, so this remains valid also in the arrangement $\mathcal{L} \cup L$ and hence we also have no conjugations in these relations. For the second case (where the simplification of these relations does use the relation induced by the point $p$ in $\mathcal{L}$), the authors show that by a conjugation with the new generator (that corresponds to $L$), one can get a similar simplification process as in the original arrangement, and so the simplified presentation has no conjugations.

The key point in the corresponding proof for a real CL arrangement $\mathcal{A}$ (with one conic) is modifying the first step of the proof described above. Indeed, after assuming that $p$ can be moved to the leftmost and lowest position without changing the projective fundamental group, then our proof follows exactly the same arguments as in the case of line arrangements, except for one case, which is treated explicitly.

The assumption that $p$ can be moved as described above relies on the following statements. First, note that if one rotates the arrangement, the fundamental group is preserved. This means that we can assume that the line $L$ does not pass through the branch point(s) of the conic with respect to the projection point (if it does pass, we can just rotate the arrangement or change the point of projection).

Proposition 4.13 of [18] implies that if, given a line that passes through only one multiple point, we rotate it around the multiple point it passes through, as long as it does not unite with a different line, then the fundamental group is also preserved. Explicitly, this proposition states that whether a line passes to the right of a multiple point or to the left of it, the fundamental group is not changed. Since the proof has a local nature (i.e. the computation is done in a neighborhood of the multiple point), one should only take a small enough neighborhood such that all the curves passing through the multiple point can be considered as lines, i.e. they do not have branch points with respect to the projection (if necessary, we change the point of projection).

Second, note that moving a line that participates in only one multiple point over a different line (i.e. the deformation takes place in $\mathbb{C}^2$ and not in $\mathbb{R}^2$) preserves the fundamental group, since this is an equisingular deformation, see Figure 8 and [14].

Third, in the case of line arrangements, the authors use [18, Theorem 4.11], which ensures that one can assume that the point $p$ is the leftmost point of the arrangement. Let us note that given a conic $C$ in $\mathbb{CP}^2$ (such that $C \cap \mathbb{RP}^2$ is neither a point nor an empty set), $C \cap \mathbb{R}^2$ can be either an ellipse, a parabola or a hyperbola. Now, possibly by choosing a different line as the line at infinity (explicitly, we change the coordinate system from $\mathbb{C}^2 \cong \mathbb{CP}^2 - \{z = 1\}$ to $\mathbb{CP}^2 - \{x = 1\}$ or $\mathbb{CP}^2 - \{y = 1\}$), we can
always assume that $C \cap \mathbb{R}^2$ is an ellipse. Note that this transformation (in $\mathbb{CP}^2$) sends real lines to real lines, and thus a real CL arrangement (with one conic) to a real CL arrangement. Now, let us consider three cases. The first case is that the point $p$ is outside the ellipse, the second case is that the point is on the ellipse and the third case is that the point is inside the ellipse.

Before treating these cases, let us note the following observation. Let $\mathcal{L} \subset \mathbb{C}^2$ be a real CL arrangement of degree $n$ with one conic, and let $\pi : \mathbb{C}^2 \to \mathbb{C} \cong \Pi$ defined by $(x, y) \mapsto x$ be a generic projection to a chosen complex line $\Pi$. Note that the images of all the singular points of $\mathcal{L}$ with respect to $\pi$ are real. Thus, there is a real positive number $E \gg 0$ such that for $B = \{x \in \Pi : |x| > E\}$ and for every $x_0 \in B$, the fiber above $x_0$: $\pi^{-1}(x_0)|_\mathcal{L}$ consists of $n$ points (possibly some of them are complex). Let $x_1 = -E, x_2 = E$ and consider the path $\gamma : [0, 1] \to \Pi, \gamma(t) = -Ee^{\pi it}$. Note that $\gamma(0) = x_1, \gamma(1) = x_2$. Then $\pi^{-1}(\gamma(t))$ induces a diffeomorphism of the $n$ points $K = \pi^{-1}(x_1)|_{\mathcal{L}}$, where $K$ is embedded in a big enough disk $D$ (where $\partial D$ is unchanged by the diffeomorphism). This is actually the Lefschetz diffeomorphism $\psi_\gamma$ induced from the path $\gamma$. Thus, $\pi^{-1}(\gamma(t))$ induces a braid acting on $(D, K)$. As is well-known, this braid is $\Delta_n$: the Dehn halftwist $\varphi$ which is a rotation of 180° counterclockwise of an open disk $U$ such that $K \subset U \subset D$, where $\varphi|_{\partial D} = \text{id}$.

Let us start by treating the second and third cases. For these cases, we first change the coordinate system as above such that the ellipse will be mapped to a hyperbola. We still get a real CL arrangement, and the point $p$ is inside or on the hyperbola. We now rotate the arrangement (if necessary) such that both branch points of the hyperbola will be to the right of $p$ and by moving some lines as described above, if necessary, we bring $p$ to be the lowest and the leftmost point, and the line having the maximal global number passes through $p$. Thus, we see that all the

Figure 8. Moving a line over another line (the 3’s in the figure indicates the multiplicity of the points).
new intersection points (which are now to the left of the point \( p \)) are real nodes, so we can proceed as in [11]. Note that the observation in the previous paragraph is used in [11, p. 780] in order to check that the product of all the half-twists (in our case, the Lefschetz diffeomorphism induced from the Lefschetz pairs) equals \( \Delta(2, n + 1)\Delta^{-1}(2, m + 1) \) (the action is performed from left to right). This observation shows us that this is also true in our case.

As for the first case, we first rotate the arrangement such that \( p \) is to the left of and below the two branch points of the ellipse. Indeed, as we possibly rotate some of the lines, there is a chance that some of the singular points (which are the intersections of the conic and these lines) will become complex; however, this does not change the induced relations in the fundamental group (see the next paragraphs for a detailed explanation regarding this). As we rotate the line \( L \) in \( \mathbb{R}^2 \) (around the point \( p \)) to be almost vertical (with respect to the line \( \Pi \cap \mathbb{R} \)), the arrangement \( A \cup L \) is no longer a real CL arrangement (since we have complex intersection points between the line \( L \) and the conic; see Figures 9(b) and 9(c)).

![Figure 9. Rotating a line over when passing to a complex CL arrangement.](image)

Moreover, the rotation of \( L \) in \( \mathbb{R}^2 \) is not an equisingular deformation, as at a particular moment during the rotation, \( L \) is tangent to the ellipse (and it can also happen that when \( L \) is tangent to the ellipse, \( L \) will also pass through another node, which is forbidden). We thus change the rotation in \( \mathbb{R}^2 \) to the following equisingular motion, according to the following local model (see Figure 10(a)).

Let \( C = \{(x, y) \in \mathbb{C}^2 : x^2 + y^2 = 1 \} \), \( p = (-1, 1) \), \( 0 < \varepsilon \ll 1 \), \( A = (-1 + \varepsilon, -1) \), \( B = (-1 - \varepsilon, -1) \) and let \( \gamma_0 : [0, 1] \rightarrow \mathbb{C}^2 \) be half a circle of radius \( \varepsilon \) on the plane \( \{y = -1\} \) in \( \mathbb{C}^2 \) which starts at \( A \) and
ends at $B$ (see Figure 10(b)). Let $L(t)$ be the family of lines passing through $p$ and $\gamma_0(t)$, $0 \leq t \leq 1$. Obviously, $C \cup L(t)$ is an equisingular deformation and thus $\pi_1(C^2 - (C \cup L(0))) \cong \pi_1(C^2 - (C \cup L(1))).$

Therefore, we can assume that $p$ is the leftmost and lowest point among all real singular points in the arrangement and the line having the maximal global number passes through $p$, e.g., as depicted in Figure 9(c).

At this stage, we cannot proceed exactly as in the former cases, as $L$ intersects the ellipse $C$ in two complex points $\alpha_0, \alpha'_0$. Note that since the slope of $L$ is very negative, we denote $0 < e \equiv d(\pi(\alpha), \pi(p)) \ll 1$ (when $\pi$ is the projection $\pi : C^2 \to C = \Pi$), and we choose now the domain $A$ to be $A = \{ a \in \mathbb{R}^2 : x_a > x_p + e \}$ to be the domain in $\mathbb{R}^2$ that contains all the singular points of the arrangement $A$ (and none of the points in $A \cap L$; see Figure 9(c)).

We now have to check what are the induced relations from the singular points. For the singular points in $A$, we can proceed as in [11]. However, for the new singular points (including the point $p$), we need to recompute the skeletons induced from them, as the existence of the two complex points changes the computations done in [11].

Let us start with the complex intersection points of $L$ and the ellipse $C$, which are nodes. The initial skeletons for these intersection points are presented in Figure 11(a).

By the observation above and the corresponding argument in [11], we have to apply on this skeleton first the diffeomorphism $\Delta^{-1}(2, m + 1)$, where $m$ is the multiplicity of the point $p$ (this does not affect the skeleton), and then $\Delta(U)$, a 180° rotation of an open disk $U$ surrounding the real points $2, \ldots, n + 1$ and the complex points $\alpha_0, \alpha'_0$. The results of the applications of these diffeomorphisms are presented in Figure 11(b). We now move the complex points to the right and rotate them in 90° clockwise (see Figure 11(c)), so we see that the induced relations
have no conjugations, i.e. the generator $x_1$ induced from the line $L$ in $\pi_1(\mathbb{CP}^2 - (A \cup L))$ commutes with the two generators induced from the ellipse $C$.

In order to see what are the induced relations from the other singular points, we recall the following setting from [15]. Let $C_0 = \{(y^2 - x)(y + x + 1) = 0\}$ (see its real part in Figure 12(a)). Define: $\pi_1, \pi_2 : C_0 \to \mathbb{C}, \pi_1(x, y) \mapsto x, \pi_2(x, y) \mapsto y$. Denote by $p_1, p_2$ the complex intersection points of $y^2 = x$ and $y = -x - 1$. Define:

$$x_0 = -\frac{1}{4}, A = \pi_2(\pi_1^{-1}(x_0)) = \left\{ \pm \frac{1}{2}i, -\frac{3}{4} \right\},$$

$$x_1 = -\frac{3}{4}, A' = \pi_2(\pi_1^{-1}(x_1)) = \left\{ \pm \frac{\sqrt{3}i}{2}, -\frac{1}{4} \right\}.$$

Let $\gamma(t), 0 \leq t \leq 1$, be a curve starting at $x_1$, ending at $x_0$ and surrounding $x_{p_2}$ from below (see Figure 12(b)). Let $D$ be a disk in the $y$-axis such that $A, A' \subset D$ and denote by $\psi_\gamma$ the Lefschetz diffeomorphism induced from $\gamma(t)$. Let $\sigma$ be the segment connecting $-\frac{1}{4}$ and $\frac{\sqrt{3}i}{2}$ in $A'$, see Figure 12(c). Then, by [15] Corollary 2.2, $\psi_\gamma = (H(\sigma))^2$, where $H(\sigma)$ is the halftwist induced from the path $\sigma$.

The initial skeleton of the multiple point $p$ is presented in Figure 13(a). We apply on this skeleton the braids $\Delta^2(1, \alpha_0)$ (induced by a path in $\Pi$ going below the image of the complex intersection points of the line and the conic, as explained in the former paragraph), then

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**Figure 11.** A model for the skeletons of the complex intersection points between the line $L$ and the ellipse:
(a) The initial skeletons; (b) The final skeletons; (c) After the moving and the rotation of $\alpha_0, \alpha'_0$ by 90° clockwise.
\[ \Delta^{-1}(2, m+1) \text{ and finally } \Delta(U). \] The resulting skeleton is depicted in Figure 13(b). We now move the complex points to the right and rotate them in 90° clockwise (see Figure 13(c)). We use now the fact that \( x_1 \) commutes with the generator \( x_{\alpha_0} \) (this is the relation induced by the complex intersection points \( \alpha_0, \alpha'_0 \), as described above), and we get that the relation induced from the point \( p \) has no conjugations.

As for the remaining intersection points (of the line \( L \) with the other lines), which are to the left of \( p \), the final skeleton will connect the points 1 and \( m_j = n - m + 2 - j \) (where \( 0 \leq j \leq n - m \)) below the real points between them (see Figure 14). Note that moving the complex points to the right and rotating them in 90° clockwise does not affect
the skeleton). This means that $x_1$ commutes with the generator associated to this line and hence the induced relation has no conjugations.

![Figure 14](attachment:image.png)

**Figure 14.** A model for the skeleton of an intersection point between $L$ and another line in the arrangement, where $m_j = n - m + 2 - j$.

To summarize, we have proven the lemma also in the first case, and we are done. □

Using Lemma 3.3 inductively, we have the following proposition:

**Proposition 3.4.** Let $\mathcal{A}$ be a real CL arrangement with one conic and $k$ lines. Suppose that $\beta(\mathcal{A}) = 0$. Then $\pi_1(\mathbb{CP}^2 - \mathcal{A})$ has a conjugation-free geometric presentation.

**Proof.** Note that $\beta(\mathcal{A}) = 0$ implies that the graph $G(\mathcal{A})$ is a forest. Hence, the arrangement can be constructed inductively according to the graph (see an example in Figure 15): first draw the conic and all the lines that do not contribute to the graph (i.e. the lines that do not pass through any multiple point). Obviously the fundamental group of this arrangement has a conjugation-free geometric presentation (since all the components are in general position, the fundamental group of the affine complement is abelian, due to [26], thus the fundamental group of the projective complement is abelian too). Now start from the root of one of the trees, i.e. draw all the lines that correspond to the edges connected to this root (see Figure 15(a)). By Lemma 3.3, the conjugation-free property is preserved. In the following steps, construct the rest of the arrangement by going to the direct successors of the root of the tree, and drawing the corresponding lines (see Figures 15(b) and 15(c)). Note that since at each step, we draw only one line passing through only one multiple point, the conjugation-free property is preserved. When comparing the resulting arrangement to the original one, the only lines that can be missing are lines that pass only through one multiple point. Thus adding these lines will again preserve the conjugation-free property.

Now, do the same to any other tree in the graph, if any. As this process is finite, we are done. □
Figure 15. An example for an inductive construction of the arrangement according to the graph: in step (a), we draw the conic and three more lines which induce a multiple point on the conic, which corresponds to the root of the tree. In step (b), we add three (dotted) lines inducing three new multiple points, thus the arrangement corresponds to a tree with a root and three successors. In step (c), we add another (dotted) line inducing a new multiple point, which corresponds to a new successor in the tree.

At this stage, we have that the fundamental group of the complement of the arrangement $\mathcal{A}$ has a conjugation-free geometric presentation. This means that whenever we have a relation in $\pi_1(\mathbb{C}P^2 - \mathcal{A})$ which involves conjugations of the geometric generators, these conjugations can be removed.

3.2. The structure of the fundamental groups of these CL arrangements. One important implication of the conjugation-free property is that while the conic induces two geometric generators $x_1, x_2$ in $\pi_1(\mathbb{C}P^2 - \mathcal{A})$, the conjugation-free property implies the relations, coming from both branch points, are $x_1 = x_2$. Thus, we can say that not only the conic contributes one generator in $\pi_1(\mathbb{C}P^2 - \mathcal{A})$, denoted by $x$, but that there are no other relations induced by the branch points (apart from $x_1 = x_2$). Note that if the presentation is not conjugation-free, we can get new relations from the branch points which are not $x_1 = x_2$, see Proposition 5.12(a), Theorem 5.5 and especially Remark 5.6(1).

Recall that $\beta(\mathcal{A}) = 0$. 
Proposition 3.5. Let $y_1, \ldots, y_k$ be the geometric generators associated to the $k$ lines. Then, for each $1 \leq i \leq k$,

$$[x, y_i] = xy_ix^{-1}y_i^{-1} = e.$$ 

Proof. First, note that if the conic $C$ intersects a line $L_\alpha$ transversally at two simple points, then $[x, y_\alpha] = e$, due to the conjugation-free property. Thus we assume that there is at least one multiple point in $\mathcal{A}$ such that the conic passes through it. We look at the forest $G(\mathcal{A})$ and we start from a leaf, assuming that the conic passes through the multiple point that corresponds to this leaf (if not, move to its direct ancestor, i.e. to the next step in the proof).

In this case, the induced relations by this point are (see Theorem 2.9 above):

$$y_{i_m} \cdots y_1x = y_{i_{m-1}} \cdots y_1x y_{i_m} = \cdots = y_{i_1}x y_{i_m} \cdots y_{i_2} = x y_{i_m} \cdots y_{i_1},$$

where $y_{i_j}, 1 \leq j \leq m$, are the generators of the lines $L_{i_j}$ which pass through this intersection point. Note that there are no conjugations in the relations, since the group has a conjugation-free geometric presentation. Since we are dealing with a leaf, all the lines (except maybe one, which corresponds to the edge connected to the direct ancestor) intersect the conic also in a simple point. Numerate the lines in such a way that $L_{i_1}$ is the line that possibly does not intersect the conic in a simple point. Again, since the group has a conjugation-free geometric presentation, we have the following relations, induced by these simple points:

$$[x, y_{i_j}] = e, \ 2 \leq j \leq m.$$ 

Therefore, from relations (1) and (2), one can easily get that

$$[x, y_{i_1}] = e.$$ 

Indeed, using relations (2) and the equality $y_{i_m} \cdots y_{i_1}x = x y_{i_m} \cdots y_{i_1}$ (left hand side and right hand side of Equation (1)), we have $[x, y_{i_1}] = e$ as needed.

With this data, we can proceed to the immediate upper level of the tree, which means that we proceed to the second multiple point that is on $L_{i_1} \cap C$ (if it exists).

Now, we do the same process as above to the new level, as this point can now be treated as a “leaf”, i.e. with the same properties regarding the relations in the fundamental group. In this way, we go over all the vertices of the graph that the conic passes through the corresponding points. \qed
Now, we can finish the proof of Theorem 2.5.  

Proof of Theorem 2.5: Based on Proposition 3.5, we can conclude that:  
$$\pi_1(\mathbb{CP}^2 - A) \cong \langle x \rangle \oplus \pi_1(\mathbb{CP}^2 - (A - C)),$$
where $x$ is the generator of the conic. Thus, it remains to prove that $\pi_1(\mathbb{CP}^2 - (A - C))$ is a direct sum of free groups and a free abelian group. However, this is straightforward, since $\beta(A) = 0$ implies that $\beta(A - C) = 0$. Now, since $A - C$ is an arrangement of lines, we can use Fan’s result [14] that the fundamental group of an arrangement of lines whose graph has no cycles is a direct sum of free groups and a free abelian group.

Explicitly, this means that:
$$\pi_1(\mathbb{CP}^2 - A) \cong \mathbb{Z}^r \oplus \bigoplus_{i=1}^p \mathbb{F}_{m(a_i) - 2} \oplus \bigoplus_{i=1}^q \mathbb{F}_{m(b_i) - 1},$$
where $r = k + 2p + q - \sum_{i=1}^p m(a_i) - \sum_{i=1}^q m(b_i)$.

By [17] and that $A$ contains at least one line, we have that:
$$\pi_1(\mathbb{C}^2 - A) \cong \mathbb{Z} \oplus \pi_1(\mathbb{CP}^2 - A).$$
Therefore, we have that the affine fundamental group of $A$ is also a direct sum of free groups and a free abelian group.

As a result, we have an immediate corollary stating when the fundamental group is abelian.

**Corollary 3.6.** Let $A$ be a real CL arrangement with one conic $C$ with only branch points, nodes and triple points as singularities. Assume that $\beta(A) = 0$ and all the triple points are on the conic. Then $\pi_1(\mathbb{CP}^2 - A)$ and $\pi_1(\mathbb{C}^2 - A)$ are abelian.

There are some immediate consequences of Theorem 2.5 using the following theorem of Oka and Sakamoto [26]:

**Theorem 3.7** (Oka-Sakamoto). Let $C_1$ and $C_2$ be algebraic plane curves in $\mathbb{C}^2$. Assume that the intersection $C_1 \cap C_2$ consists of distinct $d_1 \cdot d_2$ points, where $d_i$ ($i = 1, 2$) are the respective degrees of $C_1$ and $C_2$. Then:
$$\pi_1(\mathbb{C}^2 - (C_1 \cup C_2)) \cong \pi_1(\mathbb{C}^2 - C_1) \oplus \pi_1(\mathbb{C}^2 - C_2)$$

We state the consequence for the case of arrangements with two conics, but the general case is straightforward.
Corollary 3.8. Let $\mathcal{A}$ be a real CL arrangement with two conics and $k$ lines. Assume that the conics intersect each other transversally and $\beta(\mathcal{A}) = 0$. For $i = 1, 2$, let $V_i$ be the set of vertices of $G(\mathcal{A})$ which lie on the conic $C_i$. If $G(\mathcal{A})$ is a disjoint union of two graphs $G_1, G_2$ such that $V_i \subset G_i$, then $\pi_1(\mathbb{C}P^2 - \mathcal{A})$ and $\pi_1(\mathbb{C}^2 - \mathcal{A})$ are a direct sum of free groups and a free abelian group.

Remark 3.9. The simplest case of a CL arrangement $\mathcal{A}$ with two conics, where we cannot apply Corollary 3.8, is presented in Figure 16. However, a calculation shows that $\pi_1(\mathbb{C}P^2 - \mathcal{A})$ and $\pi_1(\mathbb{C}^2 - \mathcal{A})$ have a conjugation-free geometric presentation and thus abelian. Therefore, it is reasonable to conjecture the following:

Conjecture 3.10. Let $\mathcal{A}$ be a real CL arrangement with $n$ conics and $k$ lines, where for each pair of conics, the two conics intersect each other transversally and neither a line nor another conic passes through those intersection points. Suppose that $\beta(\mathcal{A}) = 0$. Then $\pi_1(\mathbb{C}P^2 - \mathcal{A})$ is a direct sum of free groups and a free abelian group.

We finish this section with the following conjecture for a general smooth plane curve:

Conjecture 3.11. Let $C$ be a smooth plane curve, $\mathcal{L}$ a real line arrangement, such that for each line $\ell \in \mathcal{L}$, $\ell$ intersects $C$ transversally in a real point. Define the graph $G(\mathcal{L} \cup C)$ as in the case of CL arrangements (see Definition 3.7). If $\beta(\mathcal{L} \cup C) = 0$, then $\pi_1(\mathbb{C}P^2 - (\mathcal{L} \cup C))$ is a direct sum of a free abelian group and free groups.
4. A graph with one cycle where the conic does not pass through all the vertices of the cycle

In the following sections, we want to examine the case of $\beta(A) = 1$ (for a CL arrangement with one conic). We have to separate our treatment into two cases, depending whether the conic does or does not pass through all the multiple points of the cycle. Moreover, we are especially interested to find the cases where the presentation and the structure of the affine fundamental group can be directly read from the graph (i.e. the lattice of the arrangement determines the fundamental group of its complement), so one of our aims in these sections is to find out for which arrangements, the fundamental group is either abelian or conjugation-free.

In this section, we assume that the conic does not pass through all the points of the cycle. We prove that not only the affine fundamental group of the complement has a conjugation-free geometric presentation, but that the generator of the conic commutes with all the other generators.

**Proposition 4.1.** Let $A$ be a real CL arrangement with one conic $C$ such that:

1. $\beta(A) = 1$.
2. There is vertex $y \in G(A)$ such that $y$ is a vertex of the cycle of $G(A)$, $y \notin C$ and the two different edges exiting from $y$, which compose the cycle, are associated to two different line in $A$ (see Figure 17).

Then $\pi_1(C^2 - A)$ has a conjugation-free geometric presentation.

**Proof.** Let $y \in A$ be an intersection point satisfying Condition (2). There are $k$ lines $L_1, \ldots, L_k$ which pass through $y$; assume that $L_1, L_2$ are edges of the cycle of $G(A)$ and $L_3, \ldots, L_k$ are not, otherwise there would be more than one cycle in the graph. We can also assume that there are no other edges in the graph $G$ exiting from $y$ (indeed, by Lemma 3.3 adding the corresponding lines will preserve the conjugation-free property).

Let us look at $A' = A \setminus \{L_3, \ldots, L_k\}$. Note that $\beta(A') = 0$ (since $y$ is a node in $A'$) and therefore $\pi_1(C^2 - A')$ has a conjugation-free geometric presentation (by Proposition 3.4). The line $L_3$ does not pass through any other intersection points except for $y$, and thus, by Lemma 3.3, as $\pi_1(C^2 - A')$ has a conjugation-free presentation, so does $\pi_1(C^2 - (A' \cup L_3))$. In the same way, we add the lines $L_4, \ldots, L_k$ inductively and get that $\pi_1(C^2 - A)$ has a conjugation-free presentation. $\square$
Figure 17. The point $y$ is a point on the cycle of the graph, which satisfies Condition (2). The $z$ is a point on the cycle which does not satisfy this condition, since the edges in $G(A)$ exiting from $z$ are associated to the same line.

As described in the beginning of Section 3.2, the above proof implies that the conic contributes only one generator to the fundamental group, denoted by $x$.

The above proposition has an important consequence:

**Proposition 4.2.** Let $A$ be a real CL arrangement with one conic $C$ such that $A$ satisfies the conditions of Proposition 4.1. Then:

$$\pi_1(C^2 - A) \cong \mathbb{Z} \oplus \pi_1(C^2 - (A - C)).$$

**Proof.** Note that if $\ell$ is a line in $A$ such that $\ell$ intersects $C$ in a node (i.e. this point does not correspond to a vertex in a graph), then the generator that corresponds to $\ell$ in $\pi_1(C^2 - A)$ commutes with $x$, due to the conjugation-free property, proven in Proposition 4.1 above. Hence, we only have to consider lines that intersect the conic in two multiple points.

We split our proof into two cases. In the first case, we assume that the graph consists of a unique cycle (i.e. without any trees exiting from the vertices of this cycle). Therefore, for any vertex of the graph such that $C$ passes through it, there are only one or two lines (through the corresponding point) that pass through other multiple points. Let $\ell_1$ be a line that intersects $C$ at two multiple points, $p_1, p_2$. Consider, without loss of generality, the multiple point $p_2$. If all the other lines that pass through $p_2$ (except for $\ell_1$) intersect the conic $C$ in nodes, then their corresponding generators in $\pi_1(C^2 - A)$ commute with $x$ (by the conjugation-free property); this means that we can proceed in this case as was done in the proof of Proposition 4.1 in order to show that the generator that corresponds to $\ell_1$ commutes with $x$. 
Figure 18.

Assume now that there is another line $\ell_2$, different from $\ell_1$, passing through a different multiple point $p_3$. Again, if all the other lines that pass through $p_3$ (except for $\ell_2$) intersect the conic $C$ in nodes, then their corresponding generators commute with $x$ and thus the generator that corresponds to $\ell_2$ commutes with $x$. Note that through $p_2$ there is no other line, different from $\ell_1, \ell_2$, which is an edge of the graph (i.e. passing through another multiple point, different from $p_1, p_2, p_3$) – otherwise this would imply that either the graph contains more than one cycle or that the graph has trees exiting from the vertices on the cycle. We got that the generators that correspond to the lines passing through $p_2$, except for $\ell_1$, commute with $x$, and thus also the generator that corresponds to $\ell_1$ commutes with $x$.

If there is another line (different from $\ell_2$) that pass through $p_3$ and another multiple point on $C$, then we proceed recursively as before. Note that eventually we must find a multiple point $p_n$ on $C$ where all the lines that pass through it (except maybe for a one) intersect $C$ also in nodes (see Figure 18), as otherwise the conic would pass through all the multiple points of the cycle of $G(\mathcal{A})$, contradicting Condition (2). This finishes the proof for the first case.

In the second case, we assume that there are trees exiting from the vertices of the cycle. Then one can first prove that all the generators that correspond to the corresponding lines (which are the edges of the trees) commute with $x$, as was done in Proposition 3.5, then we are left with the generators that correspond to the edges of the cycle, and we proceed as in the first case. □

Remark 4.3. (1) Note that if $\mathcal{A}$ is a CL arrangement satisfying the conditions of Proposition 4.1 and $G(\mathcal{A} - C)$ has no cycles, then we know that $\pi_1(\mathbb{C}^2 - (\mathcal{A} - C))$ is a direct sum of free groups and a free abelian group (see [13] and [17]), and thus, by Proposition 4.2, $\pi_1(\mathbb{C}^2 - \mathcal{A})$ is also a direct sum of free groups and a free abelian group.

(2) The class of CL arrangements with a conjugation-free geometric presentation is larger than stated in Proposition 4.1. Indeed, it is easy
to see that if the graph of a CL arrangement with one conic is a disjoint union of several cycles, then the arrangement has a conjugation-free geometric presentation as well. Other generalizations can be made, in the spirit of Corollary 3.8 using Oka-Sakamoto’s argument [26], see Theorem 3.7 above.

See also the appendix, where we define the notion of a conjugation-free graph, and we show that if the associated graph of this arrangement is a conjugation-free graph, then the fundamental group of the arrangement has a conjugation-free geometric presentation.

**Example 4.4.** (1) The affine fundamental group of the complement of the arrangement in Figure 18 is isomorphic to $\mathbb{Z}_4 \oplus \mathbb{F}_2$.

(2) The affine fundamental group of the complement of the arrangement in Figure 19 is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{F}_2 \oplus \mathbb{F}_2$.

![Figure 19.](image)

5. **A graph with one cycle where the conic passes through all the vertices of the cycle**

In this section, we concentrate in the case where the graph consists of one cycle, and the conic passes through all the vertices of the graph. This means that the arrangement cannot be built as in Section 4: eventually we would have to draw a line that passes through two intersection points, an operation which does not necessarily preserve the conjugation-free property.

We are interested whether the affine fundamental group is either abelian or has a conjugation-free presentation, as this would enable us to induce the structure of the fundamental group for more complicated arrangements. We therefore start with the simplest case for this situation to happen. We take a regular $n$-gon in $\mathbb{R}^2$ and we pass a circle through its $n$ vertices. We then extend the edges to be infinite straight lines, and we look at the resulting complexified arrangement in $\mathbb{C}^2$. Thus we get a real CL arrangement $\mathcal{A}_n$ whose graph is a cycle of length $n$. Denote $G_n \doteq \pi_1(\mathbb{C}^2 - \mathcal{A}_n)$. 
Figure 20. An arrangement with a cycle of length 3: the small numbers stand for the numeration of the components in the arrangement which correspond to the generators of $G_3$; the larger numbers stand for the numeration of the singular points.

**Notation 5.1.** The notation $[a,b,c] = e$ stands for the cyclic relations $cba = bac = acb$.

5.1. The cases of $n = 3$ and $n = 4$. As was already indicated in the introduction, for $n = 3$, the projective fundamental group (and the affine fundamental group as well) of this arrangement is abelian (see [8] and Figure 3 above). Let us prove the following proposition.

**Proposition 5.2.** The groups $G_3, G_4$ are not conjugation-free and

(a) $G_3$ is abelian.
(b) $G_4 \cong \mathbb{Z}^3 \oplus \mathbb{F}_2$.

**Proof.** (a) Though this was already proved by Degtyarev [8], we repeat the proof here, as its argument will be generalized in Proposition 5.5. We look at Figure 20 where the dashed line is the initial fiber.

The following table describes how we compute the skeletons from the singular points of the arrangement with respect to the projection: $j$ stands for the numeration of the images of the singular points ($j \in \{1, \ldots, 5\}$), $\lambda_x$ is the local Lefschetz skeleton describing which points, locally numerrated, coincide when approaching the singular point $j$, and $\delta_x$ describes the Lefschetz diffeomorphism induced from a path going below the point $j$. Figure 21 presents the final skeletons of the singular points, induced by the Moishezon-Teicher method, see Section 2.2 above.
By the Zariski-van Kampen theorem (see Theorem 2.9), we get that $G_3$ is generated by 5 generators (denoted by $x_1, x_2, \ldots, x_5$) with the following relations:

1. $x_1 = x_4$,
2. $[x_1, x_2, x_3] = e$,
3. $[x_3, x_4, x_5] = e$,
4. $[x_1, x_2, x_5] = e$,
5. $x_2x_1x_2^{-1} = x_4$.

By the first and the last relations, we get that $[x_1, x_2] = e$ and thus the relations in (4) are decomposed into three commutating relations:

$$[x_1, x_2] = [x_1, x_5] = [x_2, x_5] = e,$$

which in turn dissolve the other two cyclic relations into commutators. Thus $G_3$ is abelian and isomorphic to the abelian group $\mathbb{Z}^4$.

$G_3$ is not conjugation-free, since if it were, then relation (5) should have been $x_1 = x_4$ in the above presentation.

(b) We pass to the computation of $G_4$. Let us look at Figure 22.

Again, the following table describes how we compute the induced relations from the singular points of the arrangement with respect to the projection: $j$ stands for the numeration of the images of the singular points ($j \in \{1, \ldots, 8\}$), $\lambda_{x_j}$ is the local Lefschetz skeleton, describing

| $j$ | $\lambda_{x_j}$ | $\delta_{x_j}$ |
|-----|----------------|----------------|
| 1   | [3, 4]         | -              |
| 2   | [1, 3]         | $\Delta \langle 1, 3 \rangle$ |
| 3   | [3, 5]         | $\Delta \langle 3, 5 \rangle$ |
| 4   | [1, 3]         | $\Delta \langle 1, 3 \rangle$ |
| 5   | [3, 4]         | -              |

Figure 21. The skeletons for the singular points (1)–(5).
Figure 22. An arrangement with a cycle of length 4: the small numbers stand for the numeration of the components of the arrangement which correspond to the generators of $G_4$; the larger numbers stand for the numeration of the singular points.

which points, locally numerated, coincide when approaching the singular point $j$, and $\delta_{x_j}$ describes the Lefschetz diffeomorphism induced by a path going below the point $j$. Figure 23 presents the skeletons of the singular points (3)–(8), induced by the Moishezon-Teicher method (see Section 2.2 above).

| $j$ | $\lambda_{x_j}$ | $\delta_{x_j}$ |
|-----|-----------------|----------------|
| 1   | [3, 4]          | $-\phantom{\lambda}$ |
| 2   | [1, 3]          | $\Delta\langle 1, 3 \rangle$ |
| 3   | [3, 5]          | $\Delta\langle 3, 5 \rangle$ |
| 4   | [5, 6]          | $\Delta\langle 5, 6 \rangle$ |
| 5   | [3, 5]          | $\Delta\langle 5, 6 \rangle$ |
| 6   | [1, 3]          | $\Delta\langle 1, 3 \rangle$ |
| 7   | [3, 4]          | $\Delta^{\frac{1}{2}}\langle 2 \rangle$ |
| 8   | [2, 3]          | $-\phantom{\lambda}$ |

Again, by the Zariski-van Kampen theorem, the group $G_4$ is generated by 6 generators (denoted by $x_1, \ldots, x_6$) with the following relations:

1. $x_3 = x_4$,
2. $[x_1, x_2, x_3] = e$, 
Figure 23. The skeletons for singular points (3)–(8).

(3) \[x_3x_2x_1^{-1}x_3^{-1}, x_4, x_5] = e,
(4) \[x_6, x_5x_4x_3x_2x_1x_2^{-1}x_3^{-1}x_4^{-1}x_5^{-1}] = e,
(5) \[x_5, x_5x_4x_5^{-1}, x_6] = e,
(6) \[x_3, x_3x_2x_3^{-1}, x_6] = e,
(7) \[x_3x_2x_3^{-1}x_3^{-1} = x_5x_4x_5^{-1},
(8) \[x_5, x_3x_2x_3^{-1}] = e.

By relation (2), we get that relation (3) is equivalent to
\[x_1, x_4, x_5] = [x_1, x_3, x_5] = e.

Using relations (2) and (3), we get that relation (4) is equivalent to 
\[x_1, x_6] = e.

Moreover, relation (5) is equivalent to the following two relations:
\[x_6x_5x_4x_5^{-1}x_5 = x_5x_4x_5^{-1}x_5x_6, x_5x_4x_5^{-1}x_5x_6 = x_5x_6x_5x_4x_5^{-1}
\]
or to
\[x_6x_5x_4 = x_5x_4x_6 = x_4x_6x_5,
\]
i.e., the relation is in fact equivalent to the cyclic relation \([x_4, x_5, x_6] = e\). Similarly, relation (6) can be reduced to \([x_2, x_3, x_6] = e\). Note that this is actually a general phenomena, see Remark 5.3 below.

Multiplying relation (7) by \(x_1\) from the left and using the fact that \(x_4 = x_3\), we get:
\[x_1x_3x_2x_3^{-1}x_3^{-1} = x_1x_5x_3x_5^{-1}.
\]

Now use relations (2) and (3) to get that relation (7) is redundant.

Hence, we get the following equivalent set of relations:

(1) \(x_3 = x_4,\)
(2) \([x_1, x_2, x_3] = e,\)
(3) \([x_1, x_4, x_5] = e,\)
(4) \([x_1, x_6] = e,\)
(5) \([x_4, x_5, x_6] = e,\)
(6) \([x_2, x_3, x_6] = e\),
(7) \([x_5, x_3x_2x_3^{-1}] = e\).

First, let us prove that \(G_4\) is not conjugation-free. If it were, then relation (7), which is the only relation which has conjugations, would have been \([x_5, x_2] = e\). Denote by \(G_4^{cf}\) the group generated by 6 generators \(x_1, \ldots, x_6\) with the relations (1)–(6) and the relation \([x_5, x_2] = e\). Using GAP [16], we find out that the number of epimorphisms of \(G_4^{cf}\) to the symmetric group \(S_3\) is 3, while the number of epimorphisms of \(G_4^{cf}\) to \(S_3\) is 1 (note that this fact already shows that \(G_4\) is not abelian). Since the two groups are not isomorphic, it means that \(G_4^{cf}\) is not conjugation-free.

Second, we prove that \(G_4 \cong \mathbb{Z}^3 \oplus \mathbb{F}_2\). Define: \(x_{1'} = x_3x_2x_1\). Using \([x_1, x_2, x_3] = e\) and \(x_1' = x_3^{-1}x_2^{-1}x_1\), we get that \([x_1', x_2] = [x_1', x_3] = e\).

Now, from \([x_1, x_3, x_5] = e\), we get \([x_2^{-1}x_3^{-1}x_5, x_3, x_5] = e\), which induces the relation: \(x_5x_3x_2^{-1}x_3^{-1}x_5 = x_3x_2^{-1}x_3^{-1}x_1x_5\). Using \([x_5, x_3x_2x_3^{-1}] = e\) (and thus \([x_5, x_3x_2^{-1}x_3^{-1}] = e\), we see that \([x_1', x_5] = e\). From the relation \([x_1, x_6] = e\), using \([x_2, x_3, x_6] = e\), we get that \([x_1', x_6] = e\).

Since \(x_3 = x_4\), we get that the new generator \(x_{1'}\) commutes with all the other generators, i.e. \(G_4\) is isomorphic to a group \(G_4'\) generated by 5 generators \(x_{1'}, x_2, x_3, x_5, x_6\) with the following relations:

1. \([x_{1'}, x_i] = e\), where \(i \in \{2, 3, 5, 6\}\),
2. \([x_3, x_2^{-1}x_3^{-1}x_5] = e\) (induced by the relation \([x_1, x_3, x_5] = e\)),
3. \([x_3, x_5, x_6] = [x_2, x_3, x_6] = [x_5, x_3x_2x_3^{-1}] = e\).

Now, define \(x_{2'} = x_3x_2x_3^{-1}\). Note that this substitution indicates that our group might not have a conjugation-free presentation (since we are using a different generator than the geometric meridian for the simplified presentation). This means that \([x_5, x_{2'}] = e\) and \([x_3, x_{2'}, x_6] = e\) (the last relation is in fact the second relation in relation (3) in the presentation of \(G_4'\)). The relation \([x_3, x_2^{-1}x_3^{-1}x_5] = e\) is turned into \([x_3, x_{2'}] = e\). Note that \([x_{2'}, x_2'] = e\). Now, let \(x_{2''} = x_2^{-1}x_5\), so \(x_{2''} = x_5x_{2'}^{-1}\). Thus \([x_3, x_{2''}] = e\) and since \([x_5, x_{2''}] = e\) we get that \([x_5, x_{2''}] = e\). Note that \([x_{1'}, x_{2''}] = e\). From the relation \([x_3, x_{2''}, x_6] = e\), we get \(x_6x_3x_2x_3 = x_2x_3x_6\) or \(x_6x_5x_{2''}x_3 = x_5x_{2''}x_3x_6\). Using \([x_3, x_{2''}] = e\), we get: \(x_6x_5x_3x_{2''} = x_5x_3x_{2''}x_6\). Since \([x_3, x_5, x_6] = e\) (i.e. \([x_6, x_5x_3] = e\)), we get that \([x_6, x_{2''}] = e\). Therefore, \(G_4'\) is isomorphic to a group \(G_4^2\) generated by 5 generators \(x_{1'}, x_{2''}, x_3, x_5, x_6\), where \(x_{1'}\) and \(x_{2''}\) commute with all the generators, and it has the additional cyclic relation \([x_3, x_5, x_6] = e\). Explicitly,

\[G_4 \cong \langle x_{1'} \rangle \oplus \langle x_{2''} \rangle \oplus \langle x_3, x_5, x_6 : [x_3, x_5, x_6] = e \rangle \cong \mathbb{Z}^3 \oplus \mathbb{F}_2.\]
Remark 5.3. As indicated in the proof above, note that whenever we have a cyclic relation induced by the set of paths appearing in Figures 24(a) or 24(b), then the cyclic relation is equivalent to the cyclic relation $[x_d, x_{d+1}, x_c] = e$.

![Figure 24.](image)

The proof of Proposition 5.2 suggests that there is a distinction between the case of even $n$ and the case of odd $n$. As can be seen, the relation induced by the second branch point of the conic (when we look from right to left) turns into a commutator relation in $G_3$, whereas in $G_4$ it becomes trivial, see Remark 5.7 below.

5.2. The general case for odd $n$. For proving that $G_n$ is abelian for every odd $n$, we use a lemma which helps us to analyze the braid monodromy of the arrangement $G_n$ for odd $n$. We recall the Artin presentation of the braid group on $n + 1$ strands:

$$B_{n+1} = \{\sigma_1, \ldots, \sigma_n : \langle \sigma_i, \sigma_i+1 \rangle = e, [\sigma_i, \sigma_j] = e \text{ for } |i - j| > 1 \},$$

where $\langle a, b \rangle = abab^{-1}a^{-1}b^{-1}$.

Lemma 5.4. Let $n \geq 2$, and

$$\Delta_{n+1} = \sigma_n(\sigma_{n-1}\sigma_n)(\sigma_{n-2}\sigma_{n-1}\sigma_n) \cdots (\sigma_1\sigma_2\sigma_3 \cdots \sigma_n)$$

be the Garside element in $B_{n+1}$ with respect to the Artin presentation. Define $\sigma_{n'} = \sigma_{n+1}\sigma_n\sigma_{n+1} \in B_{n+2}$, and let

$$\Delta'_{n+1} = \sigma_{n'}(\sigma_{n-1}\sigma_{n'})\sigma_{n-2}\sigma_{n-1}\sigma_{n'} \cdots (\sigma_1\sigma_2\sigma_3 \cdots \sigma_{n'}) \in B_{n+2}.$$

Then

$$\Delta'_{n+1} = \Delta_{n+2} \cdot \sigma_2\sigma_3 \cdots \sigma_n.$$

Proof. The proof is by induction. We start with $n + 1 = 3$:

$$\Delta_3 = (\sigma_3\sigma_2\sigma_3)(\sigma_1\sigma_3\sigma_2\sigma_3) = (\sigma_3\sigma_2\sigma_1)(\sigma_3\sigma_2)\sigma_3 \cdot \sigma_2 = \Delta_4 \cdot \sigma_2,$$

where in the second equality we used the fact that $[\sigma_1, \sigma_3] = e$, and in the third equality we used the fact that $\sigma_3\sigma_2\sigma_3 = \sigma_2\sigma_3\sigma_2$.

Denote:

$$\delta_{n+2} = \sigma_{n+1}\sigma_n\sigma_{n-1} \cdots \sigma_2\sigma_1 \in B_{n+2}.$$
Now, assume that $\Delta'_n = \Delta_{n+1} \cdot \sigma_2 \sigma_3 \cdots \sigma_{n-1}$, and compute:

$$\Delta'_{n+1} = \sigma_n'(\sigma_{n-1} \sigma_n') (\sigma_{n-2} \sigma_{n-1} \sigma_{n'}) \cdots (\sigma_1 \sigma_2 \sigma_3 \cdots \sigma_n') =$$

$$= (\sigma_{n+1} \sigma_n \sigma_{n+1}) (\sigma_{n-1} \sigma_{n+1} \sigma_n \sigma_{n+1}) (\sigma_{n-2} \sigma_{n-1} \sigma_{n+1} \sigma_n \sigma_{n+1}) \cdots (\sigma_1 \sigma_2 \sigma_3 \cdots \sigma_{n-1} \sigma_{n+1} \sigma_n \sigma_{n+1}) =$$

$$= \delta_{n+2} \cdot \sigma_{n+1} \cdot$$

$$\cdot [(\sigma_{n+1} \sigma_n \sigma_{n+1}) (\sigma_{n-1} \sigma_{n+1} \sigma_n \sigma_{n+1}) \cdots (\sigma_2 \sigma_3 \cdots \sigma_{n-1} \sigma_{n+1} \sigma_n \sigma_{n+1})],$$

where in the last equality we have used the fact that if $i > j + 1$, then $[\sigma_i, \sigma_j] = e$.

Observe that the braids in the squared brackets (in the right hand side of the last equation) do not affect the first strand, and thus the expression in the brackets can be presented as an element in the image of the homomorphism $\varphi : B_{n+1} \to B_{n+2}$, defined by $\varphi(\sigma_i) = \sigma_{i+1}$, for all $1 \leq i \leq n$. Therefore, the expression in these brackets is $\varphi(\Delta'_n)$. Using the induction hypothesis that

$$\Delta'_n = \Delta_{n+1} \cdot \sigma_2 \sigma_3 \cdots \sigma_{n-1},$$

we get:

$$\Delta'_{n+1} = \delta_{n+2} \sigma_{n+1} \cdot \varphi(\Delta_{n+1} : \sigma_2 \sigma_3 \cdots \sigma_{n-1}) =$$

$$= \delta_{n+2} \varphi(\sigma_n \cdot \Delta_{n+1}) \sigma_3 \sigma_{n-1} \sigma_{n-2} \sigma_3 \cdots \sigma_n =$$

$$= \delta_{n+2} \varphi(\Delta_{n+1} \cdot \sigma_1) \sigma_3 \sigma_{n-1} \sigma_{n-2} \sigma_3 \cdots \sigma_n =$$

$$= \delta_{n+2} \varphi(\Delta_{n+1}) \sigma_2 \sigma_3 \sigma_4 \cdots \sigma_n =$$

$$= \Delta_{n+1} \cdot \sigma_2 \sigma_3 \cdots \sigma_n.$$

Using this lemma, we can now show the following theorem:

**Theorem 5.5.** For odd $n = 2k + 1 > 1$, the group $G_n$ is abelian.

**Proof.** Let $P_n$ be a regular $n$-gon where $n = 2k + 1$, bounded by the circle $C = \{x^2 + y^2 = 1\}$ in $\mathbb{R}^2$, such that there is one edge parallel to the $x$-axis. Extend all the edges of $P_n$ to infinite lines, and rotate this arrangement by an $\varepsilon$ degrees ($0 < \varepsilon \ll 1$) clockwise, where the center of the rotation is at the point $(0, 0)$. Denote by $A_n$ the complexified arrangement.

Let $\ell = \{y = -2\}$ and consider the projection $\pi : \mathbb{C}^2 \to \ell, (x, y) \mapsto x$. Let $\text{Sing}$ be the images of the singular points of $A_n$ with respect to $\pi$. Numerate the images of the triple points from right to left and choose $p \in \ell$ such that $p$ is between the image of the first and the second triple point, see Figure 25 for the case of $n = 11$. Note that we used the same
approach for \( n = 3 \) in the proof of Proposition 5.2(a). The point \( p \) will be the base point for \( \pi_1(\ell - \text{Sing}, p) \),

In order to compute \( G_n \), we use the braid monodromy technique. We prove that \( G_n \) is abelian in four steps: first, we will compute the induced relation by the second branch point of the circle denoted by \( q \) (i.e. the branch point to the left of \( p \)). Second, we show that the triple points (except for the two leftmost triples) always induce a relation of the form \([a, b, c] = e\), where \( a, b, c \) are the geometric generators of \( G_n \), induced by the base of \( \pi_1(\pi^{-1}(p) - (\pi^{-1}(p) \cap A_n)) \). Third, we show that the second branch point of the circle induces a commutator relation \([x, y] = e\), where \( y \) is a generator corresponding to a line and \( x \) is a generator corresponding to the circle. The fourth step shows that the combination of the former two steps dissolves one of the cyclic relations into commutative relations, and thus “dissolving” the cycle in the graph. Finally, this yields that \( G_n \) is abelian.

**First step:** Let \( q \) be the branch point of the circle (with respect to \( \pi \)) to the left of \( p \). Then the initial skeleton of \( q \) is the path \([k + 2, k + 3]\) and the first two braids that are applied on it are \( \Delta\langle k, k + 2 \rangle \) and \( \Delta\langle k + 2, k + 4 \rangle \), corresponding to the two triple points \( t_1, t_2 \) that are located after \( q \) (with respect to the projection). Explicitly, \( t_1 \) is the first triple point to the right of \( q \) (below the \( x \)-axis), and \( t_2 \) is the second...
(above the $x$-axis); see Figure 25 for the case of $n = 11$. At this stage, the skeleton looks as in Figure 26(a).

![Figure 26](image)

Now we have to check what is the effect of the Lefschetz diffeomorphism (induced by the other nodes and triple points) on this skeleton (until the point $p$). Let us cut this skeleton into two paths, as depicted in Figure 26(b). We do that as all the braids that are induced by the singular points which are below the $x$-axis affect only the left path $\gamma_1$, while the braids induced by the singular points which are above the $x$-axis (i.e. the upper points on the circle) affect only the right path $\gamma_2$. Therefore, it remains to find out what are the braids that are being applied on the left and right paths.

Let us start by examining the braids induced by the singular points which are below the $x$-axis. Note that if we, for a moment, remove the circle $C$, then the sequence of the braids we apply is in fact the sequence of the braids we get by considering a generic line arrangement of $k$ lines, composed of the $k$ lines passing through the triple points on the circle below the $x$-axis, i.e. the composition of the sequence of braids is in fact $\Delta_k \in B_k$ (where $B_k$ is generated by $\sigma_i$, $1 \leq i \leq k - 1$). Adding the section of the circle from $t_1$ till the first branch point corresponds to replacing every instance of $\sigma_k \sigma_{k-1} \sigma_k$, since the circle only passes through the upper nodes.

Therefore, the sequence of braids being applied on $\gamma_1$ is $\Delta_k'$. This means that, by Lemma 5.4, that after applying $\Delta_k' = \Delta_{k+1} \cdot \sigma_2 \cdot \cdots \cdot \sigma_{k-1}$ on $\gamma_1$, we get the path presented in Figure 27(a) (recall that we apply the braids from right to left).

As for the braids that are being applied on the path $\gamma_2$: denote by $\ell'$ the line connecting the two triple points $t_1$ and $t_2$. Note that the braids induced by the nodes that are on $\ell'$ and to the right of $t_1$ are being applied on $\gamma_2$ before any braid induced by any other singular point above the $x$-axis. But these braids do not affect $\gamma_2$: they all affect only the points with index greater than $k + 3$. This means that we can ignore the line $\ell'$ when looking on the braids applied on $\gamma_2$. 
Figure 27. The final parts of the skeleton of the point $q$.

and hence we can apply a similar argument to the one used on $\gamma_1$. We have again a generic arrangement of $k$ lines and a half-circle now passing through the lowest nodes. A similar check shows that after applying the sequence of the appropriate braids on $\gamma_2$, we get the path as depicted in Figure 27(b). Thus, the final skeleton of the second branch point of the circle is presented in Figure 28.

Figure 28.

Therefore, the relation in $G_n$ induced by the second branch point is:

$$x_{k+1}x_k \cdots x_2x_1x_k^{-1}x_2^{-1} \cdots x_{k-1}^{-1}x_{k+1}^{-1} = x_{2k+2} \cdots x_{k+4}x_{k+3}x_{k+4}^{-1} \cdots x_{2k+2}^{-1},$$

(An explanation on how to induce a relation from a skeleton can be found at the end of Section 2.2).

Second step: We want to show that the relations induced by the triple points have no conjugations in $G_n$. Again, we consider two cases: where the triple points (to the left of $p$) are below the $x$-axis, and where they are above. Note that the relations induced by the triple point to the right of $p$ obviously have no conjugations, since this triple point is the closest point to the fiber (from its right side), and hence there is no braid that is applied on its initial skeleton.

Consider the case where the triple points (to the left of $p$) are below the $x$-axis. Excluding the point $t_1$ from the computation (the reason for this will be clear in the third step), we number them from right to left by $p_1, \ldots, p_{k-1}$. For the point $p_1$, the skeleton is $[k-1, k+1]$ and no braid is applied on it, so the induced relation is:

$$[x_{k-1}, x_k, x_{k+1}] = e.$$  

(3) 

Note that the initial skeleton of each point $p_i$, $2 \leq i \leq k - 1$, is also $[k-1, k+1]$ (see Figure 29 for an illustration of the case of $n = 11$).
For each $i$, let $\ell_{i1}, \ell_{i2}$ be the two lines passing through $p_i$. For each $2 \leq i \leq k - 1$, note that $\mathcal{L}_i = \bigcup_{j=1}^i \{\ell_{j1}, \ell_{j2}\}$ is a generic line arrangement with $i + 1$ lines. Let us fix an $i$.

![Figure 29. The numeration of the triple points for the arrangement $A_{11}$.](image)

If we consider only the line arrangement $\mathcal{L}_i$, then the initial skeleton of $p_i$ is $[i, i+1]$, and the composition of the sequence of braids (in $B_{i+1}$) which are applied on this skeleton (in order to get the final skeleton) is in fact $\Delta_{i+1} \cdot \sigma_i^{-1}$ (we apply the braids on the skeleton from right to left). Note that in fact, applying $\sigma_i$ on the skeleton does not change it – as this is a rotation of the points $i$ and $i + 1$ by $180^\circ$ counterclockwise, so the skeleton remains the same. Thus, we can say that the braid that is applied on $[i, i+1]$ is $\Delta_{i+1}$. Now, drawing again the circle $C$ through $p_i$ till $p_1$ corresponds to the fact that the initial skeleton (in the arrangement $\mathcal{L}_i \cup C$) is now $[i, i+2]$ and, similar to the argument used in the first step, the braid that acts on this skeleton (in order to get the final skeleton) is $\Delta'_{i+1}$. This means that the final skeleton (in the arrangement $\mathcal{L}_i \cup C$) is as depicted in Figure 30(a). Looking on the arrangement $\mathcal{A}_n$, let $n_i, n_i + 1$ be the global numeration of the lines (in the fiber over $p$) passing through $p_i$. Thus, the final skeleton in $\mathcal{A}_n$ is as depicted in Figure 30(b) (note that the point $k$ in the fiber over $p$ corresponds to the conic).
Figure 30.

Now, by Remark 5.3, the induced relation is $[x_{n_i}, x_{n_i+1}, x_k] = e$, as needed.

The argument for the triple points above the $x$-axis (not including $t_2$) is almost the same, and hence omitted.

**Third step:** Recall that the relation which was induced from the second branch point $q$ is:

$$ x_{k+1}x_k \cdots x_2x_1x_k^{-1}x_2^{-1} \cdots x_k^{-1}x_{k+1}^{-1} = x_{2k+2} \cdots x_{k+4}x_{k+3}x_{k+4}^{-1} \cdots x_{2k+2}. $$

Note also that the relation induced from the branch point of the circle to the right of $p$ is $x_k = x_{k+3}$.

We now split our treatment according to the remainder of $n$ modulo 4. Since $n$ is odd, the remainder can be either 1 or 3.

**Case 1:** Assume that $n \equiv 3 \pmod{4}$ (note that we can assume that $n > 3$, i.e. $k > 1$, since the case $n = 3$ was already investigated in Proposition 5.2). Then, the right hand side of Equation (4) is a conjugation of $x_{k+3}$ by the expression:

$$ x_{2k+2}x_{2k+1} \cdots x_{k+5}x_{k+4} = \prod_{m=2k+2, m \equiv 0 \pmod{2}}^{k+5} (x_mx_{m-1}). $$

Each pair of indices $m, m+1$ in the product corresponds to two consecutive lines (in the global numeration in the fiber above $p$), that intersects the circle in a triple point. By the second step, we know that $[x_{m-1}, x_m, x_{k+3}] = e$ and therefore $x_mx_{m-1}$ commutes with $x_{k+3}$. This means that the right hand side can be simplified to $x_{k+3}$. Note that we have not used the cyclic relation induced by the point $t_2$, which involves (possibly conjugations of) the generators $x_{k+3}, x_{2k+2}, x_{2k+3}$.

As for the left hand side of Equation (4), we can do the same procedure, until we get the following relation:

$$ x_{k+1}x_kx_{k-1}x_k^{-1}x_{k-2}x_k^{-1}x_{k-1}^{-1}x_k^{-1}x_{k+1}^{-1} = x_{k+3}, $$

or

$$ x_kx_{k-1}x_k^{-1}x_{k-2}x_k^{-1}x_{k-1}^{-1}x_k^{-1} = x_{k+1}^{-1}x_{k+3}x_{k+1}. $$
Using the relation \([x_{k-2}, x_{k-1}, x_k] = e\) induced by the point \(p_2\) (i.e. \(x_{k-1}x_{k-2}x_k = x_kx_{k-1}x_{k-2}\)), we get that:
\[
x_k = x_{k+1}^{-1}x_{k+3}x_{k+1} \text{ or } x_k = x_{k+1}^{-1}x_kx_{k+1} \implies [x_k, x_{k+1}] = e.
\]

**Case 2:** Assume now that \(n \equiv 1 \pmod{4}\). As in the previous case, the right hand side of Equation (4) is reduced to \(x_k\), the left hand side is reduced to \([x_{k-1}, x_{k-2}, x_k] = e\), see Equation (3)). Thus, we get:

\[
[x_{k-1}, x_{k-2}, x_k] = e.
\]

Therefore:
\[
x_k = x_{k+1}x_kx_{k-1}x_{k-2}^{-1}x_k^{-1} \implies x_k = x_{k+1}x_kx_{k-1}^{-1}x_k^{-1} \implies [x_k, x_{k+1}] = e.
\]

**Fourth step:** By the second step, \(G_n\) has a cyclic relation of the form \([x_k, x_{2k+1}, x_{2k+2}] = e\) and of the form \([x_{k-1}, x_k, x_{k+1}] = e\). In any case, the third step shows us that these relations are dissolved into commutative relations: either to \([x_k, x_{2k+1}] = [x_k, x_{2k+2}] = [x_{2k+1}, x_{2k+2}] = e\) or to \([x_{k-1}, x_k] = [x_{k-1}, x_{k+1}] = [x_k, x_{k+1}] = e\). Denote by \(t\) the triple point whose induced cyclic relation is dissolved and let \(\ell_{t_1}, \ell_{t_2}\) be the two lines that pass through it. Let \(t'\) be the other triple point that \(\ell_{t_1}\) passes through it. The fact that the cyclic relation turns into three commutative relations implies, from the perspective of the braid monodromy and the relations induced by it, that we can slightly rotate \(\ell_{t_1}\) around \(t'\) and still get an isomorphic fundamental group.

Let \(U\) be a small neighborhood of \(t\), i.e. \(U \cap \mathcal{A}_n\) is an intersection of two lines and a circle at \(t\). The slight rotation described above has the effect on \(U\) described in Figure 31.

However, this means that the graph of the revised arrangement has no cycles, and by Corollary 2.5, the fundamental group is abelian, and we are done. \(\square\)

**Remark 5.6.** (1) Theorem 5.5 implies that for a CL arrangement \(\mathcal{A}\) with \(\beta(\mathcal{A}) = 1\) having a cycle of odd length, then the affine fundamental
group is not conjugation-free: if it were, the second branch relation should have been $x_k = x_{k+3}$ after the simplification process.

(2) Note that during the simplification process of the relation induced by the second branch point $q$, we did not use the relation induced by the unique triple point $t_0$ located to the right of the point $p$. This means that if we draw additional lines, passing only through $t_0$ (with a very negative slope), the simplification process of the resulting presentation will be identical, and therefore it implies that we can regard this new arrangement as an arrangement with $\beta(A) = 0$. Thus the fundamental group is a direct sum of a free group with $m(t_0) - 2$ generators (induced by $t_0$), and a free abelian group (recall that $m(t_0)$ is the multiplicity of $t_0$).

Remark 5.7. (1) Note that the third step of the proof of Theorem 5.5 fails in the case of a cycle of even length, as can be seen in the case of $G_4$.

(2) By a computation which is not presented here, we note that $G_6$ is neither abelian nor isomorphic to a direct sum of a free abelian group and free groups. The last statement is proved using the fact that $G_6$ does not have any epimorphism to $S_3$ (confirmed by GAP [16]): indeed, every group of the form $\mathbb{Z}^k \oplus \sum \mathbb{F}_{r_i}$ has an epimorphism to $S_3$.

6. Appendix: Graphs inducing conjugation-free presentations

Following Remark 4.3(2) and the methods presented in Sections 3, 4 and 5, this appendix examines a subclass of graphs, associated to a line arrangement or to a CL arrangement, which implies that the fundamental group of the arrangement has a conjugation-free geometric presentation. We start with the case of line arrangements.

6.1. A conjugation-free graph for line arrangements. Let us start with recalling the following result [11, Proposition 2.2]:

Lemma 6.1. Given a line arrangement $\mathcal{L}$ in $\mathbb{C}^2$ having a conjugation-free geometric presentation of the fundamental group, $\ell \not\in \mathcal{L}$ a line such
that ℓ passes through only one intersection point of L, then L∪ℓ is also conjugation-free.

We can use this lemma to reprove the main theorem of [10] in an inductive way:

**Proposition 6.2.** If G(L) is a cycle, then \( \pi_1(C^2-L) \) has a conjugation-free geometric presentation.

**Proof.** We build L inductively: at each step, we prove that the fundamental group is conjugation-free.

(1) First, draw only the lines that are the edges of the cycle G(L): denote by \( L_0 \) the resulting arrangement. Since G(L) is a cycle, then G(\( L_0 \)) is the empty graph, as the singular points of \( L_0 \) are only nodes. Thus \( \pi_1(C^2-L_0) \) is abelian and it is conjugation-free.

(2) Numerate the nodes x of \( L_0 \) such that \( x \in G(L) \) by \( x_1, \ldots, x_s \) (these are the vertices of G(L)). Looking at the arrangement L, there are \( i_k \) lines passing through the point \( x_i: L_{i_1}, \ldots, L_{i_k} \). Note that in \( L_0 \), for each \( x_i \) either one or two lines from the set \( \{L_{i_j}\}_{j=1}^k \) were drawn (see Figure 32).

\[
\begin{align*}
\text{Figure 32. For the point } p, \text{ the lines } L_1 \text{ and } L_2 \text{ are drawn; for the point } p', \text{ only the line } L_1 \text{ is drawn.}
\end{align*}
\]

Let us look at \( x_1 \) and assume that \( L_{1_1}, \text{ or } L_{1_1} \text{ and } L_{1_2} \) are already drawn while building \( L_0 \). Let \( L_1 = \{L_{1_1}\} \) (or \( L_1 = \{L_{1_1}, L_{1_2}\} \) resp.). Then \( L_{1}' = \{L_{1_j}\}_{j=1}^k - L_1 \) is a set of lines such that for each \( \ell \in L_{1}' \), \( \ell \cap \{x_i\}_{i=1}^s = x_1 \); otherwise (explicitly, if \( \ell \cap \{x_i\} = \{x_1, x_m\}, m \notin \{2, s\} \) we would get \( \beta(L) > 1 \). Note that if \( m = 2 \) or \( m = s \), then \( \ell \) would be already in \( L_1 \) (which is not possible, as \( \ell \in L_{1}' \)).

Therefore, each \( \ell \in L_{1}' \) passes through only one multiple point of \( L_0 \) and we can inductively add all the lines \( \ell \in L_{1}' \) to \( L_0 \) and preserve the conjugation-free property, by Lemma 6.1

(3) We continue as in step (2) for the points \( x_2, \ldots, x_s \), adding the missing lines. At each step the conjugation-free property is preserved.

\( \square \)

Let us recall the following proposition from [11]:
Proposition 6.3. Let $\mathcal{L}$ be a line arrangement such that $\beta(\mathcal{L}) \leq 1$. Then the fundamental group of $\mathcal{L}$ has a conjugation-free geometric presentation.

Proof. Indeed, this was already proved in [11, Corollary 2.5], and we review the proof shortly. For the case of $\beta(\mathcal{L}) = 0$ this is the content of Fan’s results [14]: in this case the fundamental group is a sum of a free abelian group and free groups and it is easy to see that it is conjugation-free. When $\beta(\mathcal{L}) = 1$, we can continue the construction above: all one needs to do is to add the lines corresponding to the trees whose roots lie on the unique cycle of $G(\mathcal{L})$. \hfill \Box

Note that it is shown in [11] that if the graph is a disjoint union of cycles, then the arrangement is also conjugation-free.

The above construction motivates the following definition:

Definition 6.4. Let $G$ be a planar graph and, for a vertex $v \in G$ denote by $\deg(v)$ the number of edges exiting from this vertex. $G$ is called a conjugation-free graph (CFG) if:

1. $\beta(\mathcal{L}) \leq 1$ or
2. Let $\{v_i\}_{i=1}^m$ be the set of vertices in $G$ satisfying $\deg(v_i) \leq 2$. Denote for each $v_i$, $1 \leq i \leq m$, the following subgraph $V_i$ of $G$, composed of $v_i$ and the edge(s) exiting from it. Let $X = X(G) = \bigcup_{i=1}^m V_i$. Then $G$ is a conjugation-free graph if $G - X$ is.

Theorem 6.5. Let $\mathcal{L}$ be a line arrangement. If $G(\mathcal{L})$ is a disjoint union of conjugation-free graphs, then the fundamental group of $\mathcal{L}$ has a conjugation-free geometric presentation.

Before proving the theorem, we give some examples of conjugation-free graphs.

Example 6.6. (1) Obviously, a forest is a CFG.
(2) Graphs (a) and (b) in Figure 33 are CFGs.
(3) The Ceva arrangement (also called the braid arrangement) induces a non conjugation-free graph (see Figure 33(c)).

Remark 6.7. By using the package TESTISOM (see [19]), one can show that there exist arrangements having a conjugation-free presentation for their fundamental group, but their associated graph is not a conjugation-free graph, e.g. the graph presented in Figure 34.

Proof of Theorem 6.5. We start with the case where $\mathcal{L}$ is a line arrangement whose graph $G(\mathcal{L})$ is a conjugation-free graph. If $\beta(\mathcal{L}) \leq 1$, then
the theorem is true due to Proposition 6.3. Assume now that $\beta(L) > 1$. Denote $G = G(L)$ and $X = X(G)$ (the set of edges and vertices defined in Definition 6.4). We split our treatment into two subcases.

**Case (1):** Assume that the graph $G - X$ has at most one cycle. Let $V = \{v_1, \ldots, v_s\} \subset X$ be the set of vertices (of $G$) which were removed. We now look at the following sub-arrangement $L_X$ of $L$, which is the line arrangement induced from $G - X$; i.e.

$$L_X = L - \{\text{lines that pass through } v_i\}.$$  

As $\beta(L_X) \leq 1$, $L_X$ is conjugation-free. Now, for each $v_i \in V \subset X$, there are either one or two lines in $L$, that pass through $v_i$, which correspond to removed edges in the graph $G$. Let us add these lines to the arrangement $L_X$ and call the new arrangement $L'_X$. Since these lines pass through at most one multiple point, then $L'_X$ is also conjugation-free, by Lemma 6.1. Note that if $p \in \text{Sing}(L)$ such that $p \in G$, then $p \in \text{Sing}(L'_X)$. This means that every “missing” line, i.e. every line in $L - L'_X$ passes through only one point in the set of vertices $V$, since otherwise there would be $j$ such that $\text{deg}(v_j) > 2$. This means that $L$ is conjugation-free too.

**Case (2):** Assume now that the graph $G - X$ has more than one cycle. Let $G_0 = G, X_0 = X, G_1 = G_0 - X_0, X_1 = X(G_1), \ldots, G_n = G_{n-1} - X_{n-1}$. Since $G_0$ is a CFG, then there is $k \in \mathbb{N}$ such that the
graph $G_k$ has at most one cycle. In this case, we proceed inductively: in the first step, starting from the graph $G_k$, we add to the arrangement corresponding to the graph $G_k$ (which is conjugation-free) the lines that were removed from it in the last step (i.e. the lines that pass through all the vertices with degree 2 in the graph $G_{k-1}$). As was proved in case (1), we get a arrangement whose fundamental group has a conjugation-free geometric presentation and its graph is $G_{k-1}$. Then we add to $G_{k-1}$ its missing lines and we continue till we rebuild the arrangement $\mathcal{L}$.

The above two cases finalize the proof for the case of a line arrangement $\mathcal{L}$, where $G(\mathcal{L})$ is a conjugation-free graph. The general case of a line arrangement $\mathcal{L}$, where $G(\mathcal{L})$ is a disjoint union of conjugation-free-graphs, can be deduced by Oka-Sakamoto’s theorem (see [26] and Theorem 3.7 above).

\section*{6.2. A conjugation-free graph for CL arrangements.}

In this subsection, we want to see which results from the former subsection can be adapted to the case of CL arrangements.

First, let us note that if $\mathcal{A}$ is a CL arrangement with only one conic, then the definition of a conjugation-free graph can be applied without any changes. Indeed, assume we are given a CL arrangement $\mathcal{A}$ with only one conic $C$ and the graph $G(\mathcal{A})$ is a conjugation-free graph. Since the graph of an arrangement which consists of only one conic is empty (and obviously this arrangement is conjugation-free), we can add the “missing” lines to this arrangement, i.e. all the lines in $\mathcal{A} - C$, in the same way described in the proof of Theorem 6.5, and the conjugation-free property of this arrangement is preserved (due to Lemma 3.3).

Second, the above observation leads us to the following lemma, which is, in a sense, parallel to Lemma 3.3.

**Lemma 6.8.** Let $\mathcal{L}$ be a real line arrangement whose graph is a conjugation-free graph (and thus $\pi_1(\mathbb{C}P^2 - \mathcal{L})$ has a conjugation-free geometric presentation). Let $C$ be a conic that passes through a unique intersection point of $\mathcal{L}$. Then $\pi_1(\mathbb{C}P^2 - (\mathcal{L} \cup C))$ has a conjugation-free geometric presentation.

**Proof.** Note that adding the conic $C$ to the arrangement $\mathcal{L}$ is equivalent, from the point of view of the associated graphs, to adding a line, passing through the same intersection point and not passing through any other intersection point. Thus if the graph $G(\mathcal{L})$ is a conjugation-free graph, then obviously $G(\mathcal{L} \cup C)$ is. Therefore, we can build the arrangement $\mathcal{L} \cup C$ by first drawing the conic $C$ and then start drawing the lines
in \( \mathcal{L} \) such that the conjugation-free property of this arrangement is preserved for every addition of a line, as described in Lemma 3.3. □

Note that adding a conic \( C_1 \), passing through a unique intersection point, to a CL arrangement \( \mathcal{A} \) with one conic \( C_2 \), which has a conjugation-free graph, also preserves the conjugation-free property, under the following condition: Let \( C_1, C_2 \) be the two conics of the arrangement \( \mathcal{A} \cup C_1 \). For \( i = 1, 2 \), let \( V_i \) be the set of vertices of \( G(\mathcal{A} \cup C_1) \) which lie on the conic \( C_i \). If \( G(\mathcal{A} \cup C_1) \) is a disjoint union of two conjugation-free graphs \( G_1, G_2 \) such that \( V_i \subset G_i \), then \( \pi_1(\mathbb{CP}^2 - (\mathcal{A} \cup C_1)) \) has a conjugation-free geometric presentation.

We finish with the following conjecture, related to arrangements with a fundamental group having a conjugation-free geometric presentation, when their associated graphs might not be conjugation-free graphs:

**Conjecture 6.9.** Let \( \mathcal{L} \) be a real line arrangement such that \( \pi_1(\mathbb{CP}^2 - \mathcal{L}) \) has a conjugation-free geometric presentation. Let \( C \) be a conic that passes through a single singular point of \( \mathcal{L} \). Then \( \pi_1(\mathbb{CP}^2 - (\mathcal{L} \cup C)) \) has a conjugation-free geometric presentation as well.

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