EQUIVARIANT DERIVED CATEGORIES ASSOCIATED TO A SUM OF TWO POTENTIALS

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Abstract. Suppose $f, g$ are homogeneous polynomials of degree $d$ defining smooth hypersurfaces $X_f = V(f) \subset \mathbb{P}^{m-1}$ and $X_g = V(g) \subset \mathbb{P}^{n-1}$. Then the sum of $f$ and $g$ defines a smooth hypersurface $X = V(f \oplus g) \subset \mathbb{P}^{m+n-1}$ with an action of $\mu_d$ scaling the $g$ variables. Motivated by the work of Orlov, we construct a semi-orthogonal decomposition of the derived category of coherent sheaves on $[X/\mu_d]$ provided $d \geq \max\{m, n\}$.

1. Introduction

1.1. Semi-orthogonal decompositions in algebraic geometry. To a space $X$, i.e. a smooth and projective variety or more generally a smooth and proper Deligne-Mumford stack, we can associate the bounded derived category of coherent sheaves on the space, denoted $\mathcal{D}(X)$. The category $\mathcal{D}(X)$ lives in the intersection between homological algebra and algebraic geometry and has proved to be a useful tool when applied to algebro-geometric problems.

Of particular interest is when $\mathcal{D}(X)$ admits a semi-orthogonal decomposition (see Section 2.2 for the definition). Roughly, a semi-orthogonal decomposition is the analogue of a group extension for triangulated categories. If $\mathcal{D}(X)$ admits a semi-orthogonal decomposition, one can hope to further understand $\mathcal{D}(X)$, or sometimes $X$, using the components of the decomposition.

See the surveys [BO02] and [Bri06] for examples of semi-orthogonal decompositions and their uses.

1.2. Orlov’s Theorem. Let $k$ be an algebraically closed field of characteristic zero and $V$ a vector space over $k$ of dimension $n$. Assume $f \in k[V]_d$ defines a smooth hypersurface, say $X_f = V(f) \subset \mathbb{P}(V)$. We call $f$ a potential. Let $\text{HMF}^{gr}(f)$ denote the homotopy category of graded matrix factorizations of the potential $f$. Recall, objects of $\text{HMF}^{gr}(f)$ are $\mathbb{Z}/2$-graded, curved complexes of $G_m$-equivariant vector bundles on $V$ with curvature $f$. There is a natural differential on the space of morphisms between two matrix factorizations. The category $\text{HMF}^{gr}(f)$ is the corresponding homotopy category.

A relationship between $\text{HMF}^{gr}(f)$ and $\mathcal{D}(X_f)$ was discovered by Orlov in [Orl09]. Orlov constructs two $\mathbb{Z}$-indexed families of exact functors $\Psi_i : \mathcal{D}(X_f) \to \text{HMF}^{gr}(f)$ and $\Phi_i : \text{HMF}^{gr}(f) \to \mathcal{D}(X_f)$. If $X_f$ is Fano or Calabi-Yau, then $\Phi_i$ is a full embedding. If $X_f$ is general type or Calabi-Yau, then $\Psi_i$ is a full embedding. Moreover, the semi-orthogonal complement is determined:

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Orlov’s Theorem. [Orl09] Theorem 3.11] Let $f$ be as above. For each $i \in \mathbb{Z}$, we have the following semi-orthogonal decompositions:

Fano : $\mathcal{D}(X_f) = \langle \mathcal{O}_{X_f}(-i - n + d + 1), \ldots, \mathcal{O}_{X_f}(-i), \Phi_i \text{HMF}^{gr}(f) \rangle$;

General Type : $\text{HMF}^{gr}(f) = \langle k^{\text{stab}}(-i), \ldots, k^{\text{stab}}(-i + n - d + 1), \Psi_i \mathcal{D}(X_f) \rangle$;

Calabi-Yau : $\Phi_i, \Psi_i$ induce mutual inverse equivalences $\mathcal{D}(X_f) \cong \text{HMF}^{gr}(f)$.

Here $k^{\text{stab}}$ is a certain matrix factorization associated to the residue field of $k[V]$ at the origin.

1.3. Adding two potentials. Let $f, g$ be homogeneous polynomials of degree $d$ defining smooth hypersurfaces $X_f \subset \mathbb{P}^{m-1}$ and $X_g \subset \mathbb{P}^{n-1}$. Let $X = V(f \oplus g) \subset \mathbb{P}^{m+n-1}$. Then $X$ is smooth since $X_f$ and $X_g$ are smooth.

Suppose $d \geq \max\{m, n\}$. Then there is a $\mathbb{Z}$-indexed family of embeddings $\Psi_i : \mathcal{D}(X_f) \to \text{HMF}^{gr}(f)$ and similarly $\Psi_j : \mathcal{D}(X_g) \to \text{HMF}^{gr}(g)$. By tensoring, we can consider the family of embeddings:

$\Psi_{i,j} : \mathcal{D}(X_f \times X_g) \cong \mathcal{D}(X_f) \otimes \mathcal{D}(X_g) \to \text{HMF}^{gr}(f) \otimes \text{HMF}^{gr}(g)$.

where $\Psi_{i,j} = \Psi_i \otimes \Psi_j$ and the tensor product is understood to be taken in suitable dg-enhancement. We further have an identification, see [BFK14a Corollary 5.18]

$\text{HMF}^{gr}(f) \otimes \text{HMF}^{gr}(g) \cong \text{HMF}^{gr, \mu_d}(f \oplus g)$.

where $\mu_d$ acts on the $g$ variables.

If in addition $d \leq n + m$, it was noticed in [BFK14b Example 3.10] that we can then embed $\text{HMF}^{gr, \mu_d}(f \oplus g)$ into $\mathcal{D}[X/\mu_d]$ using Orlov’s Theorem a second time. Fix one such embedding to get a doubly indexed family of fully-faithful functors $\Xi_{i,j} : \mathcal{D}(X_f \times X_g) \to \mathcal{D}[X/\mu_d]$. The complement consists of $mn$ exceptional objects, $d - m$ copies of $\mathcal{D}(X_g)$, and $d - n$ copies of $\mathcal{D}(X_f)$. Specifically, we have

$\mathcal{D}[X/\mu_d] = \langle \mathcal{A}, \mathcal{K}, \mathcal{D}_f, \mathcal{D}_g, \mathcal{D}_{fg} \rangle$

where $\mathcal{A}$ consists of $(m+n-d)d$ line bundles, $\mathcal{K} \cong \langle k^{\text{stab}}(-i), \ldots, k^{\text{stab}}(-i + m - d + 1) \rangle \otimes \langle k^{\text{stab}}(-j), \ldots, k^{\text{stab}}(-j + n - d + 1) \rangle$, $\mathcal{D}_f = \Phi_i \mathcal{D}(X_f) \otimes \langle k^{\text{stab}}(-j), \ldots, k^{\text{stab}}(-j + n - d + 1) \rangle$, $\mathcal{D}_g = \langle k^{\text{stab}}(-i), \ldots, k^{\text{stab}}(-i + m - d + 1) \rangle \otimes \Phi_j \mathcal{D}(X_g)$, and $\mathcal{D}_{fg} = \Xi_{i,j} \mathcal{D}(X_f \times X_g)$.

These functors are not easy to compute with however and, with the exception of $\mathcal{A}$, explicitly understanding the left and right semi-orthogonal complements to the image of $\Xi_{i,j}$ as $\mu_d$-equivariant complexes of sheaves on $X$ is not easy.

1.4. Main result. In this paper, we give a more geometric definition of the functors $\Xi_{i,j}$ and show that they, miraculously, remain embeddings even if $d > n + m$. Moreover, we explicitly determine the other components in the associated semi-orthogonal decomposition.

Main Theorem. There is a semi-orthogonal decomposition

$\mathcal{D}[X/\mu_d] = \langle \mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3, \mathcal{D}_4, \mathcal{A} \rangle$.

Here $\mathcal{D}_1$ and $\mathcal{D}_2$ collectively consist of $d - m$ twists of $\mathcal{D}(X_g)$ (Section 1.3.2), $\mathcal{D}_3$ consists of $d - n$ twists of $\mathcal{D}(X_f)$ (Section 1.3.1), $\mathcal{D}_4$ is the image of $\Xi_{-m,-n}$ (Section 4.1), and $\mathcal{A}$ consists of an exceptional collection of line bundles (Section 4.2).
To align with the picture given by Orlov’s theorem we can mutate the decomposition; however, it gets complicated quickly. As stated, each of the components has a simple description given by explicit Fourier-Mukai functors.

1.5. **Outline of paper.** In Section 2 we recall facts about (equivariant) triangulated categories. Section 3 is devoted to understanding the derived category of the quotient stack $[\mathbb{P}^{m+n-1}/\mu_d]$. In Section 4 we define all of the terms in the above decomposition. In Section 5 we prove semi-orthogonality. In Section 6 we discuss sheaves constructed by taking Koszul complexes. In Section 7 we prove fullness. We end the paper with Section 9 which is devoted to the special case when $m = 1$.

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2. **Preliminaries on (Equivariant) Triangulated Categories**

Throughout $k$ is an algebraically closed field of characteristic zero. For an overview of triangulated categories in algebraic geometry see [Huy06].

2.1. **Triangulated categories.** Recall, a triangulated category $\mathcal{T}$ is a $k$-linear category together with an autoequivalence $[1] : \mathcal{T} \to \mathcal{T}$ and a class of exact triangles $t \to u \to v \to t[1]$ satisfying certain axioms, see [GM03].

**Example 2.1.** If $X$ is a scheme, or more generally an algebraic stack, we can associate a triangulated category called the bounded derived category of coherent sheaves on $X$, denoted $\mathcal{D}(X)$. This category is the Drinfeld-Verdier localization of the category of chain complexes of coherent sheaves with bounded cohomology with respect to the class of quasi-isomorphisms.

2.2. **Semi-orthogonal decompositions.** Let $\mathcal{T}$ be a triangulated category. A semi-orthogonal decomposition of $\mathcal{T}$, written

$$\mathcal{T} = \langle A_1, \ldots, A_n \rangle$$

is a sequence of full triangulated subcategories $A_1, \ldots, A_n$ of $\mathcal{T}$ such that:

- $\text{Hom}_\mathcal{T}(a_i, a_j) = 0$ for $a_i \in A_i$, $a_j \in A_j$, and $i > j$;
- For any $t \in \mathcal{T}$, there is a sequence of morphisms $0 = a_n \to a_{n-1} \to \cdots \to a_1 \to a_0 = t$

where $\text{Cone}(a_i \to a_{i-1}) \in A_i$.

**Example 2.2.** The most common examples of semi-orthogonal decompositions occur when $\mathcal{T} = \mathcal{D}(X)$ for some smooth projective scheme over $k$. In this case, there is often a vector bundle $\mathcal{E}$ such that $\text{Ext}^*_X(\mathcal{E}, \mathcal{E}) \cong k[0]$. Such an object in $\mathcal{D}(X)$ is called exceptional. The subcategory generated by $\mathcal{E}$ is also abusively denoted by $\mathcal{E}$ and there are two semi-orthogonal decompositions (that it exists follows from Example 2.4):

$$\mathcal{D}(X) = \langle \mathcal{E}^\perp, \mathcal{E} \rangle = \langle \mathcal{E}, \mathcal{E}^\perp \rangle$$
where $\mathcal{E}^\perp = \{ \mathcal{F} \in \mathcal{D}(X) \mid \text{Ext}_X^*(\mathcal{E}, \mathcal{F}) = 0 \}$ and $\perp \mathcal{E}$ is defined similarly.

2.3. Spanning classes. Let $\mathcal{T}$ be a triangulated category. A subclass of objects $\Omega \subset \mathcal{T}$ is called a **spanning class** if for every $t \in \mathcal{T}$ the following two conditions hold:

- $\text{Hom}_\mathcal{T}(t, \omega[i]) = 0$ for all $\omega \in \Omega$ and all $i \in \mathbb{Z}$ implies $t = 0$;
- $\text{Hom}_\mathcal{T}(\omega[i], t) = 0$ for all $\omega \in \Omega$ and all $i \in \mathbb{Z}$ implies $t = 0$.

**Example 2.3.** If $X$ is a smooth projective variety over $k$, then a spanning class is furnished by the structure sheaves of closed points:

$$\Omega = \{ \mathcal{O}_x \mid x \in X \text{ is a closed point} \}.$$  

More generally, if $\mathcal{X}$ is a smooth and proper Deligne-Mumford stack over $k$, then we can consider the coarse moduli space $\pi : \mathcal{X} \to X$ and take the following collection as a spanning class:

$$\Omega = \{ \mathcal{O}_Z \mid Z \text{ is a closed substack of } \mathcal{X} \text{ and } \pi(Z) \text{ is a closed point of } X \}.$$  

2.4. Admissible triangulated subcategories. Let $A \subset \mathcal{T}$ be a full triangulated subcategory of a triangulated category. We say $A$ is **admissible** if the embedding functor $\iota : A \to \mathcal{T}$ has a left and right adjoint.

If $A$ is admissible, then it follows formally that $\mathcal{T}$ admits two semi-orthogonal decompositions

$$\mathcal{T} = \langle A^\perp, A \rangle = \langle A, \perp A \rangle.$$

where

- $A^\perp := \{ t \in \mathcal{T} \mid \text{Hom}_\mathcal{T}(t, a[i]) = 0 \text{ for all } a \in A, i \in \mathbb{Z} \}$;
- $\perp A := \{ t \in \mathcal{T} \mid \text{Hom}_\mathcal{T}(a[i], t) = 0 \text{ for all } a \in A, i \in \mathbb{Z} \}$.

We have the following useful lemma regarding admissible subcategories.

**Proposition 2.4.1.** Suppose $\Omega$ is a spanning class for $\mathcal{T}$ and $A$ is a full, admissible, triangulated subcategory containing $\Omega$, then $A = \mathcal{T}$.

**Proof.** Since $A$ is admissible, there is a semi-orthogonal decomposition of $\mathcal{T}$ of the form:

$$\mathcal{T} = \langle A^\perp, A \rangle$$

The condition that $A$ contains a spanning class implies that $A^\perp$ must be trivial. \qed

2.5. Saturated triangulated subcategories. A triangulated category $\mathcal{T}$ is called **saturated** if every cohomological functor (contravariant or covariant) $H : \mathcal{T} \to \text{Vect}_k$ of finite type is representable. We have the following important proposition regarding saturated subcategories, see [BK89, Proposition 2.6].

**Proposition 2.5.1.** Let $A$ be a saturated triangulated category and $\iota : A \to \mathcal{T}$ is a full embedding. Then $A$ is an admissible subcategory of $\mathcal{T}$.

**Example 2.4.** The derived category of coherent sheaves on a smooth projective variety, $X$, is saturated, [BK89, Theorem 2.14]. If $\mathcal{E}$ is an exceptional object of $\mathcal{D}(X)$, then there is a full embedding $\iota_\mathcal{E} : \mathcal{D}(\text{Spec}(k)) \to \mathcal{D}(X)$ given by $\iota_\mathcal{E}(V) = \mathcal{E} \otimes V$. This justifies Example 2.2.\footnote{We will not need the more general notions of left and right admissibility in this paper.}
We will use the following proposition in conjunction with Proposition \[2.4.1\] in Section \[7\].

**Proposition 2.5.2.** \[BK89\] Theorem 2.10] If $\mathcal{A}, \mathcal{B} \subset \mathcal{T}$ are full, saturated, triangulated subcategories such that $(\mathcal{A}, \mathcal{B})$ is semi-orthogonal, then $(\mathcal{A}, \mathcal{B})$ is saturated.

We will also need the following theorem in Section \[7\].

**Theorem 2.5.1.** Suppose $F : \mathcal{D}(X) \to \mathcal{T}$ is a full embedding where $X$ is smooth and projective over $k$. Further suppose there exists a saturated subcategory $\mathcal{A}$ containing $F(\Omega)$, where $\Omega$ is a spanning class for $\mathcal{D}(X)$. Then $F(\mathcal{D}(X)) \subset \mathcal{A}$.

**Proof.** It is sufficient to prove that the right adjoint $G : \mathcal{T} \to \mathcal{D}(X)$ is zero on $\mathcal{A}^\perp$. Suppose $b \in \mathcal{A}^\perp$. For every $\omega \in \Omega$ and $i \in \mathbb{Z}$ we have

$$\text{Ext}^i_X(\omega, G(b)) \cong \text{Hom}_\mathcal{A}(F(\omega), b[i]) = 0.$$ 

Since $\Omega$ is a spanning class, we conclude $G(b) = 0$ for all $b \in \mathcal{A}^\perp$. \[\square\]

### 2.6. Equivariant triangulated categories.

**Remark 2.1.** It is not true that $\mathcal{T}^G$ is always triangulated, i.e. that the triangulated structure on $\mathcal{T}$ descends to $\mathcal{T}^G$, see \[Ela14\] Example 8.4].

**Example 2.5.** Suppose a finite group $G$ acts on a scheme $X$, then there is an exact equivalence $\mathcal{D}[X/G] \cong \mathcal{D}(X)^G$, see [Viss03, Section 3.8]. So in this case there is a natural triangulated structure on $\mathcal{D}(X)^G$. A good reference for equivariant derived categories of coherent sheaves is \[BK01\] Section 4).

**Example 2.6.** Let $(\mathcal{F}, \theta)$ be an equivariant object in $\mathcal{D}(X)$. Further, let $\chi : G \to \mathbb{G}_m$ be a multiplicative character of $G$. Define a new equivariant object $(\mathcal{F} \otimes \chi, \theta \otimes \chi)$ where $\mathcal{F} \otimes \chi = \mathcal{F}$ as an object of $\mathcal{D}(X)$ but the maps $\theta_g \otimes \chi : \mathcal{F} \to g^* \mathcal{F}$ are twisted by $\chi$. Thus if $\mathcal{F}$ admits one linearization it can admit several distinct linearizations.

### 2.7. Bondal-Orlov fully-faithfulness criterion.

We shall need the well known fully-faithfulness criterion of Bondal and Orlov.

**Theorem 2.7.1** (Bondal, Orlov). Let $X$ be a smooth projective variety over $k$ and $\mathcal{T}$ be a triangulated category. Suppose $F : \mathcal{D}(X) \to \mathcal{T}$ is an exact functor with a right adjoint $G$. Then $F$ is fully-faithful if and only if for any two closed points $x, y \in X$ we have

$$\text{Hom}_\mathcal{T}(F(O_x), F(O_y)[i]) = \begin{cases} k & \text{if } x = y \text{ and } i = 0 \\ 0 & \text{if } x \neq y \text{ and } i \notin [0, \dim(X)]. \end{cases}$$
Proof. The proof in [Huy06 Proposition 7.1] only requires that the functor $F$ has a right adjoint. 

Remark 2.2. We will use Theorem 2.7.1 when $T$ is the derived category of a smooth and proper Deligne-Mumford stack over $k$. In this case, the existence of a (left and) right adjoint is guaranteed as the relative dualizing sheaf exists.

3. Derived Category of $[\mathbb{P}^{m+n-1}/\mu_d]$  

For the rest of this paper, we let $\chi : \mu_d \to \mathbb{G}_m$ denote the standard primitive character $\chi(\lambda) = \lambda$.

There is an action of $\mu_d$ on $\mathbb{P}^{m+n-1}$, where $\mathbb{P}^{m+n-1}$ has coordinates $[x_1 : \ldots : x_m : y_1 : \ldots : y_n]$ and $\mu_d$ acts by scaling the variables $y_1, \ldots, y_n$. In terms of the homogeneous coordinate algebra $k[x_1, \ldots, x_m, y_1, \ldots, y_n]$, the variables $y_i$ have weight $\chi^{-1}$ and the variables $x_i$ have trivial weight.

In this section, we will study the corresponding quotient stack $[\mathbb{P}^{m+n-1}/\mu_d]$ and its derived category $\mathcal{D}[\mathbb{P}^{m+n-1}/\mu_d]$.

3.1. Equivariant objects. Let $H_y = V(x_1, \ldots, x_m)$ and $H_x = V(y_1, \ldots, y_n)$. The fixed locus of the $\mu_d$ action is $(\mathbb{P}^{m+n-1})^{\mu_d} = H_x \sqcup H_y$. Therefore the sheaves $\mathcal{O}_{H_x}$ and $\mathcal{O}_{H_y}$ have a natural equivariant structure given by the identity morphism. As in Example 2.7.1, we can form the equivariant sheaves $\mathcal{O}_{H_x} \otimes \chi^i$ and $\mathcal{O}_{H_y} \otimes \chi^i$ for $i = 0, \ldots, d - 1$.

We equip $\mathcal{O}(-1)$ with the $\mu_d$-linearization $\theta_{\lambda} : \mathcal{O}(-1) \to \lambda^* \mathcal{O}(-1)$ given by fiberwise multiplication by $\lambda$ and consider $\mathcal{O}(i)$ with the induced $\mu_d$-linearizations. We can also twist these sheaves by characters to get the equivariant line bundles $\mathcal{O}_{\mathbb{P}^{m+n-1}}(i) \otimes \chi^j$ for $i \in \mathbb{Z}$ and $j = 0, \ldots, d - 1$.

3.2. Serre duality. The canonical bundle on $\mathbb{P}^{m+n-1}$ is $\mathcal{O}(-m - n)$. It is locally trivial as a $\mu_d$-equivariant bundle; however, the identification $\omega_{\mathbb{P}^{m+n-1}} \cong \mathcal{O}(-m - n)$ may involve twisting by a character. To determine the twist, we recall the Euler exact sequence on $\mathbb{P}^{m+n-1}$

$$0 \to \Omega^1 \to \mathcal{O}(-1)^{\oplus m+n} \xrightarrow{\alpha} \mathcal{O} \to 0$$

where $\alpha = (x_1, \ldots, x_m, y_1, \ldots, y_n)$. Since the sections $y_i$ have weight $-1$, the above Euler exact sequence admits the following $\mu_d$-linearization:

$$0 \to \Omega^1 \to (\oplus_{i=1}^m \mathcal{O}(-1)) \oplus (\oplus_{j=1}^n \mathcal{O}(-1) \otimes \chi^{-1}) \xrightarrow{\alpha} \mathcal{O} \to 0$$

Now taking determinants yields $\omega_{[\mathbb{P}^{m+n-1}/\mu_d]} \cong \mathcal{O}_{[\mathbb{P}^{m+n-1}/\mu_d]}(-m - n) \otimes \chi^{-n}$ as $\mu_d$-equivariant sheaves. Serre duality therefore takes the following form:

**Proposition 3.2.1 (Serre Duality).** For any $\mathcal{F}, \mathcal{G} \in \mathcal{D}[\mathbb{P}^{m+n-1}/\mu_d]$ there is a natural isomorphism

$$\text{Ext}_{[\mathbb{P}^{m+n-1}/\mu_d]}^*(\mathcal{F}, \mathcal{G}) \cong \text{Ext}_{[\mathbb{P}^{m+n-1}/\mu_d]}^{m+n-1-*}(\mathcal{G}, \mathcal{F}(-m - n) \otimes \chi^{-n}).$$
3.3. Semi-orthogonal decomposition of \([\mathbb{P}^1/\mu_d]\). Let us consider the case \(m = n = 1\). In this case, we can describe a useful semi-orthogonal decomposition of \([\mathbb{P}^1/\mu_d]\). Recall, we have Beilinson’s exceptional collection in \(D(\mathbb{P}^1)\), \cite{Beilinson78}:

\[
D(\mathbb{P}^1) = \langle \mathcal{O}(-1), \mathcal{O} \rangle.
\]

The projective line is a coarse moduli space for \([\mathbb{P}^1/\mu_d]\) and the mapping \(\pi : [\mathbb{P}^1/\mu_d] \to \mathbb{P}^1\) is defined by the \(\mu_d\)-equivariant morphism \(\tilde{\pi} : \mathbb{P}^1 \to \mathbb{P}^1\) given by \([x : y] \mapsto [x^d : y^d]\). Since \(\tilde{\pi}\) can be described as as the \(d\)-uple embedding \(\mathbb{P}^1 \to \mathbb{P}^{d(d+1)}\) followed by the linear projection onto the \(x^d, y^d\) variables, we have \(\pi^*\mathcal{O}_{\mathbb{P}^1}(-1) \cong \mathcal{O}_{[\mathbb{P}^1/\mu_d]}(-d)\).

The fixed orbit consists of two points \(\{p, q\}\). In the notation before, we have \(H_x = \{p\}\) and \(H_y = \{q\}\). The following semi-orthogonal decomposition is used in Section \cite{7}:

**Theorem 3.3.1.** There is a semi-orthogonal decomposition

\[
D([\mathbb{P}^1/\mu_d]) = \langle \mathcal{O}_p \otimes \chi^{d-1}, \ldots, \mathcal{O}_p \otimes \chi, \mathcal{O}_q \otimes \chi^{-(d-1)}, \ldots, \mathcal{O}_q \otimes \chi^{-1}, \pi^*\mathcal{D}(\mathbb{P}^1) \rangle
\]

\[
= \langle \mathcal{O}_p \otimes \chi^{d-1}, \ldots, \mathcal{O}_p \otimes \chi, \mathcal{O}_q \otimes \chi^{-(d-1)}, \ldots, \mathcal{O}_q \otimes \chi^{-1}, \mathcal{O}(-d), \mathcal{O} \rangle.
\]

We prove something slightly more general than Theorem 3.3.1 regarding \(\mu_d\)-actions.

**Theorem 3.3.2.** Let \(\mu_d\) act on a smooth projective variety \(X\) of dimension \(n\). Suppose the geometric quotient \(\pi : X \to X/\mu_d\) is smooth and the fixed locus \(X^{\mu_d} = Z\) is a smooth divisor such that \(\mu_d\) acts freely on \(X \setminus Z\) such that \(\mathcal{N}_{Z/X} \cong \mathcal{L} \otimes \chi^{-1}\) for some fixed line bundle \(\mathcal{L}\) on \(Z\), where \(\mathcal{N}_{Z/X}\) is the normal bundle. Let \(\iota : Z \rightarrow X\) denote the inclusion. Then there is a semi-orthogonal decomposition of \(D[X/\mu_d]\):

\[
D[X/\mu_d] = \langle \iota_* (\mathcal{D}(Z)) \otimes \chi, \ldots, \iota_* (\mathcal{D}(Z)) \otimes \chi^{d-1}, \pi^*\mathcal{D}(X/\mu_d) \rangle.
\]

**Proof.** We first show \(\iota_* : \mathcal{D}(Z) \to D[X/\mu_d]\) is fully-faithful using Theorem 2.7.1. Pick \(z \in Z\). Since \(\mathcal{N}_{Z/X,z} \cong \chi^{-1}\), we have an isomorphism of \(\mu_d\)-representations:

\[
\text{Ext}^1_\mathcal{O}_Z(\mathcal{O}_z, \mathcal{O}_z) \cong \wedge^1(T_z X) \cong \wedge^1(1 \oplus \chi^{-1}).
\]

It follows that \(\iota_*\) is fully-faithful. Semi-orthogonality follows from this identification as well.

To see fullness, take an object \(\mathcal{F} \in D[X/\mu_d]\) such that \(\mathcal{F}\) is left orthogonal to

\[
\langle \iota_* (\mathcal{D}(Z)) \otimes \chi, \ldots, \iota_* (\mathcal{D}(Z)) \otimes \chi^{d-1} \rangle.
\]

As the action of \(\mu_d\) on \(X \setminus Z\) is free, we can apply \cite{Terasoma03} Theorem 2.4 to see \(\mathcal{F} \in \pi^*\mathcal{D}(X/\mu_d)\). \(\square\)

**Remark 3.1.** Of course the theorem can be adapted to the case where \(\mathcal{N}_{Z/X}\) has different weights. But the components \(\iota_* \mathcal{D}(Z) \otimes \chi^i\) may need to be reordered to ensure semi-orthogonality. The correct reordering for \([\mathbb{P}^1/\mu_d]\) is provided in the statement and so Theorem 3.3.2 proves Theorem 3.3.1.

**Remark 3.2.** The sheaves \(\mathcal{O}(n)\) have a natural \(\mu_d\)-equivariant structure and so we could equally as well have considered an equivariantized Beilinson’s exceptional collection:

\[
D[\mathbb{P}^1/\mu_d] = \langle \mathcal{O}(-1), \ldots, \mathcal{O}(-1) \otimes \chi^{d-1}, \mathcal{O}, \ldots, \mathcal{O} \otimes \chi^{d-1} \rangle.
\]
We could then tediously argue that the decomposition of Theorem 3.3.1 is a mutation of Beilinson’s collection. The above argument is more pleasant and Theorem 3.3.2 will be needed in Section 9.

3.4. Grothendieck Splitting Theorem. The classical Grothendieck splitting theorem decomposes any vector bundle on $P^3$. The above argument is more pleasant and Theorem 3.3.1 is a mutation of Beilinson’s collection. The above argument is more pleasant and Theorem 3.3.2 will be needed in Section 9.

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Theorem 3.4.1 (Equivariant Grothendieck Splitting). Let $E$ be a rank $r$ vector bundle on $[P^1/\mu_d]$, then there exists $n_1, \ldots, n_r$ and $j_1, \ldots, j_r$ such that

$$E \cong \bigoplus_{i=1}^r \mathcal{O}(n_1) \otimes \chi^{j_i}.$$ 

Proof. The proof is almost identical to the classical proof. The twists $\chi^{j_i}$ show up when looking for an equivariant global section of $E(n_i)$ with $n_i >> 0$. The Abelian condition guarantees that the irreducible representations are one-dimensional. □

4. The Hypersurface $[X/\mu_d]$ 

Let $X_f \subset \mathbb{P}^{m-1}$ and $X_g \subset \mathbb{P}^n$ be smooth degree $d$ hypersurfaces. Let $X = V(f \oplus g) \subset \mathbb{P}^{m+n-1}$ be the hypersurface associated to the sum of potentials. We impose the conditions $d \geq n \geq m \geq 2$, i.e. the hypersurfaces involved are Calabi-Yau or general type and are non-empty.

The action of $\mu_d$ on $\mathbb{P}^{m+n-1}$ descends to $X$ and we consider the quotient stack $[X/\mu_d]$. The fixed loci are given by the intersections with $(\mathbb{P}^{m+n-1})^{\mu_d} = H_x \sqcup H_y$:

$$X^{\mu_d} = X \cap (H_x \sqcup H_y) \cong X_f \sqcup X_g.$$ 

4.1. Equivariant geometry of $X$. Line bundles associated to hyperplane sections $\mathcal{O}_X(iH)$ have $d$ distinct equivariant structures. These equivariant line bundles are of the form $\mathcal{O}_X(iH) \otimes \chi^j$.

Proposition 4.1.1 (Serre Duality). The triangulated category $\mathcal{D}[X/\mu_d]$ has the Serre functor $(-) \otimes \mathcal{O}_X(d - m - n) \otimes \chi^{-n}[m + n - 2]$.

Proof. Since $[X/\mu_d]$ is a smooth substack of $[\mathbb{P}^{m+n-1}/\mu_d]$, we can use the adjunction formula

$$\omega_{[X/\mu_d]} \cong \omega_{[\mathbb{P}^{m+n-1}/\mu_d]} \otimes \mathcal{O}_{[X/\mu_d]}(d) \cong \mathcal{O}_X(d - m - n) \otimes \chi^{-n}.$$ 

□

For Fano hypersurfaces it is easy to see that line bundles are exceptional. With this extra $\mu_d$ action, all line bundles on $[X/\mu_d]$ are exceptional:

Proposition 4.1.2. Line bundles are exceptional objects of $\mathcal{D}[X/\mu_d]$.

Proof. It is sufficient to prove $H^*(\mathcal{O}_X)^{\mu_d} \cong k$. We have an equivariant exact sequence on $\mathbb{P}^{m+n-1}$:

$$0 \to \mathcal{O}_{\mathbb{P}^{m+n-1}}(-d) \xrightarrow{f \oplus g} \mathcal{O}_{\mathbb{P}^{m+n-1}} \to \mathcal{O}_X \to 0.$$ 

The only possible nonzero cohomology groups are $H^0, m + n - 2(\mathcal{O}_X)^{\mu_d}$. Further, we have an isomorphism

$$H^{m+n-2}(\mathcal{O}_X)^{\mu_d} \cong H^{m+n-1}(\mathcal{O}_{\mathbb{P}^{m+n-1}}(-d))^{\mu_d}.$$
If $d < m + n$, then the latter group is zero and we are finished. Suppose $d \geq m + n$.

Then

$$H^{m+n-1}((O_{R^{m+n-1}}(-d))^\mu_d) \cong H^0((O_{R^{m+n-1}}(d - m - n)\chi^{-n})^\mu_d).$$

The latter has a basis of monomials of the form $x^iy^j$ where $I = (i_1, \ldots, i_m), J = (j_1, \ldots, j_n)$ and $x^I = x_1^{i_1} \cdots x_m^{i_m}, y^J = y_1^{j_1} \cdots y_n^{j_n}$ such that $|I| + |J| = d - m - n$ and $|J| + n$ is a positive multiple of $d$. It follows that $|J| = d - n$ is the only possible option. Hence, $d - m - n = |I| + |J| = |I| + d - n$ from which we conclude $|I| = -m$, impossible.

**Proposition 4.1.3.** For $0 < i - j < m$, $H^*(O_X \otimes \chi^{j-i}) = 0$. 

**Proof.** Anagolus to the computation in Proposition 4.1.2. \qed

4.2. Subcategory of exceptional line bundles. Define subcategories $A_1, A_2, A_3$ of $D[X/\mu_d]$ as follows.

$$A_1 = (O_X(-(n-1)-(m-1)) \otimes \chi^{-(n-1)},$$

$$O_X(-(n-1)-(m-1)+1) \otimes \chi^{-(n-2)-(n-1)},$$

$$\ldots, O_X(-(n-1)-1) \otimes \chi^{-(n-m)-1,-(n-1)};$$

$$A_2 = (O_X(-(n-1)) \otimes \chi^{-(n-m),-(n-1)},$$

$$O_X(-(n-1)+1) \otimes \chi^{-(n-m)+1,-(n-1)+1},$$

$$\ldots, O_X(-(m-1)-1) \otimes \chi^{-(m-1)};$$

$$A_3 = (O_X(-(m-2)) \otimes \chi^{0,-(m-2), \ldots, O_X}.$$ It is understood that if $m = n$, then $A_2$ is zero. Further, the notation $O_X(i) \otimes \chi^{j_1, \ldots, j_k}$ means the subcategory generated by the exceptional objects $O_X(i) \otimes \chi^{j_1, \ldots, \chi^{j_k}}$. By Proposition 4.1.3 there is a semi-orthogonal decomposition:

$$O_X(i) \otimes \chi^{j_1, \ldots, j_k} = (O_X(i) \otimes \chi^{j_k}, \ldots, O_X(i) \otimes \chi^{j_1}),$$

where $j_1 > j_2 > \cdots > j_k$.

**Proposition 4.2.1.** The decomposition of the subcategories $A_i$ for $i = 1, 2, 3$ is semi-orthogonal. Moreover, $A = (A_1, A_2, A_3)$ is semi-orthogonal.

**Proof.** We only show the semi-orthogonality of the decomposition for $A_3$. The semi-orthogonality of the decomposition for $A_1$ and $A_2$ is similar.

The sheaves in $A_3$ are of the form $O(-i_1) \otimes \chi^{j_1}$ for $0 \leq i \leq m - 1$ and $0 \leq i_1 \leq i_2$. Pick two such sheaves with $i_1 \leq i_2$ and $j_1 \leq j_2$. By Serre Duality and the closed substack exact sequence:

$$H^{m+n-2}(O_X(i_1-i_2) \otimes \chi^{j_1-j_2}) \cong H^0((O_{R^{m+n-1}}(d+i_2-i_1-n-m) \otimes \chi^{j_2-j_1-n}).$$

We have the following inequalities:

$$m-1 \geq i_2 - i_1 \geq 0,$$

$$m-1 \geq j_2 - j_1 \geq 0.$$

An equivariant global section is of the form $x^iy^j$ where $|I| + |J| = d+i_2-i_1-n-m$ such that $|J| = j_2 - j_1 - n - d$. But

$$d+i_2-i_1-n-m = |I| + |J| = |I| + d + j_2 - j_1 - n.$$
Hence, \(|J| = i_2 - i_1 - (j_2 - j_1) - m \leq i_2 - i_1 - m \leq m - 1 - m = -1\), which is impossible. Thus \(H^*(O_X(i_1 - i_2) \otimes \chi^{j_1 - j_2}) = 0\) and the semi-orthogonal decomposition for \(A_3\) is verified.

We now check \(\langle A_2, A_3 \rangle\) the others are similar. Recall, for \(A_2\) to exist we require \(n > m\). The relevant group is

\[
H^{m+n-2}(O_X(i_1 - (m-1) - i_2) \otimes \chi^{j_1 - j_2})^{\mu_d}
\]

\[
\cong H^0(O_p \otimes (d + i_2 - i_1 - n - 1) \otimes \chi^{j_2 - j_1 - n})^{\mu_d}
\]

for \(i_1 = 0, \ldots, m - 1, j_1 = 0, \ldots, i_2, i_2 = 1, \ldots, n - m, j_2 = i_2, \ldots, (m - 1) + i_2\).

If \(d + i_2 - i_1 - n - 1 < 0\), there is nothing to prove. Assume \(d + i_2 - i_1 - n - 1 \geq 0\). Let \(x^I y^J\) be an equivariant global section. Since \(j_2 - j_1 - n < 0\) we require \(|J| = d + j_2 - j_1 - n\). Then

\[
d + i_2 - i_1 - n - 1 = |I| + |J| = |I| + j_2 - j_1 - n
\]

forces \(|I| = i_2 - j_2 + j_1 - i_1 - 1\). However, \(i_2 - j_2 \leq 0\) and \(j_1 - i_1 \leq 0\) so \(|I| \leq -1\) which is impossible. This finishes the proof. □

4.3. Geometric subcategories.

4.3.1. \(D_f\): Let \(\iota_f: X_f \to X\) be given by \(\iota_f([x_1 : \ldots : x_m]) = [x_1 : \ldots : x_m : 0 : \ldots : 0]\). Clearly \(\iota_f\) is \(\mu_d\)-equivariant as it coincides with a component of the fixed locus. Let \(\iota_{f*}\) denote the corresponding equivariant pushforward functor \(\iota_{f*}: D(X_f) \to D[X/\mu_d]\). This means first include \(D(X_f)\) into the trivial component of \(D[X/\mu_d]\), \(= \bigoplus_{i=0}^{d-1} D(X_f) \otimes \chi^i\), then use the equivariant pushforward.

**Proposition 4.3.1.** If \(d > n\), then \(\iota_{f*}\) is fully-faithful.

**Proof.** We use Theorem [2.7.1]. Let \(p \in X_f\) be a closed point and identify \(p\) with \(\iota_f(p)\). It is sufficient to show vanishing of \(\text{Ext}^*_X(\mathcal{O}_p, \mathcal{O}_p) \cong (\Lambda^*T_p X)^{\mu_d}\) for \(* > m - 2\). From the normal bundle exact sequence

\[
0 \to TX \to T \mathbb{P}^{m+n-1} |_X \to O_X(d) \to 0
\]

and the identification \(T_p \mathbb{P}^{m+n-1} \cong 1^{\oplus m-1} \oplus \chi^{\oplus n}\) coming from the \(\mu_d\)-linearized Euler exact sequence, we see

\[
T_p X \cong 1^{\oplus m-2} \oplus \chi^{\oplus n}
\]

If \(d > n\), then \((\Lambda^* T_p X)^{\mu_d} = 0\) for \(* > m - 2\). □

Define the following subcategories of \(D(X)^{\mu_d}\):

\[
D^i_f = \iota_{f*}(D(X_f)) \otimes \chi^i.
\]

**Proposition 4.3.2.** For \(0 < i_1 - i_2 < d - n\) we have the semi-orthogonality

\[
\langle D^i_f, D^j_f \rangle.
\]

**Proof.** This follows immediately from the isomorphism:

\[
\text{Ext}^*_X(\mathcal{O}_p \otimes \chi^{i_2}, \mathcal{O}_p \otimes \chi^{i_1})^{\mu_d} \cong \Lambda^*(1^{\oplus m-2} \oplus \chi^{\oplus n}) \otimes \chi^{i_2 - i_1}.
\]

Let \(D_f\) be the strictly full subcategory of \(D[X/\mu_d]\) generated by \(D_1^f, \ldots, D^d_f\).
Corollary 4.3.1. For \( d > n \), we have a semi-orthogonal decomposition
\[
\mathcal{D}_f = \langle \mathcal{D}^{d-n}_f, \mathcal{D}^{d-n-1}_f, \ldots, \mathcal{D}^1_f \rangle.
\]

4.3.2. \( \mathcal{D}_g \). Similarly to \( \mathcal{D}_f \), we have a closed embedding \( \iota_g : X_g \to X \) given by \( \iota_g([y_1 : \ldots : y_n]) = [0 : \ldots : 0 : y_1 : \ldots : y_n] \), which is the inclusion of the other component of the fixed locus and so is \( \mu_d \)-equivariant. Let \( \iota_g^* : \mathcal{D}(X_g) \to \mathcal{D}[X/\mu_d] \) be the associated equivariant pushforward. The following results are analogous to Propositions 4.3.1, 4.3.2 and Corollary 4.3.1.

Proposition 4.3.3. If \( d > m \), then \( \iota_g^* \) is fully-faithful.

Define the subcategories
\[
\mathcal{D}^i_g = \iota_g^*(\mathcal{D}(X_g)) \otimes \chi^i
\]
of \( \mathcal{D}([X/\mu_d]) \).

Proposition 4.3.4. For \( m - d < i_1 - i_2 < 0 \) we have
\[
\langle \mathcal{D}^{i_1}_g, \mathcal{D}^{i_2}_g \rangle.
\]

Corollary 4.3.2. For \( d > m \) we have a semi-orthogonal decomposition
\[
\mathcal{D}_g = \langle \mathcal{D}^{m-d}_g, \mathcal{D}^{m-d+1}_g, \ldots, \mathcal{D}^{-1}_g \rangle.
\]

In the case \( n > m \), it will be necessary to split \( \mathcal{D}_g \) into two subcategories. Define:
\[
\mathcal{D}_g1 = \langle \mathcal{D}^{m-d}_g, \mathcal{D}^{m-d+1}_g, \ldots, \mathcal{D}^{m-n-1}_g \rangle.
\]
and
\[
\mathcal{D}_g2 = \langle \mathcal{D}^{m-n}_g, \mathcal{D}^{m-n+1}_g, \ldots, \mathcal{D}^{-1}_g \rangle.
\]
We have \( \mathcal{D}_g = \langle \mathcal{D}_g1, \mathcal{D}_g2 \rangle \), where it is understood that if \( m = n \), then \( \mathcal{D}_g = \mathcal{D}_g1 \).

4.4. Embedding \( \mathcal{D}(X_f \times X_g) \). Let \( Y = \mathbb{P}(\mathcal{O}_{X_f}(-1) \oplus \mathcal{O}_{X_g}(-1)) \) and \( \pi : Y \to X_f \times X_g \) be the projection. Consider the commutative diagram:
\[
\begin{array}{ccc}
Y & \xrightarrow{i} & X_f \times X_g \\
\downarrow{\pi} & & \downarrow{\pi_X} \\
X_f \times X_g & \xrightarrow{\alpha} & X
\end{array}
\]
The cyclic group \( \mu_d \) acts on \( Y \) by scaling the second coordinate of the fiber. We endow \( X_f \times X_g \) with the trivial action rendering the diagram \( \mu_d \)-equivariant.

Define a family of Fourier-Mukai functors
\[
\Xi_{i,j} : \mathcal{D}(X_f \times X_g) \to \mathcal{D}[X/\mu_d]
\]
using the kernel \( \iota_* \mathcal{O}_Y \otimes \pi^* \mathcal{O}_X(iH) \otimes \chi^i \), i.e.
\[
\Xi_{i,j}(\mathcal{F}) = R\pi_{X}^*(\pi_{X_f \times X_g}^*(\mathcal{F}) \otimes \iota_* \mathcal{O}_Y \otimes \mathcal{O}_X(iH) \otimes \chi^j),
\]
where it is understood that before applying \( \Xi_{i,j} \) we precompose with the embedding \( \mathcal{D}(X_f \times X_g) \hookrightarrow \mathcal{D}[X_f \times X_g/\mu_d] = \oplus_{i=0}^{d-1} \mathcal{D}(X_f \times X_g) \otimes \chi^i \) (into the trivial component). Then the derived push and pull functors are taken equivariantly.
We show Ξ_{i,j} is an embedding using Theorem 2.7.1. Since the kernel is flat over \( X_f \times X_g \) we see \( Ξ_{i,j}(\mathcal{O}(p,q)) \cong \mathcal{O}(l(p,q)) \otimes χ^i \), where \( l(p,q) \cong \mathbb{P}^1 \) is the line in \( X \) joining \( t_f(p) \) to \( t_g(q) \).

**Lemma 4.4.1.** Let \( \mathcal{N} \) denote the normal bundle to \( l(p,q) \) inside \( X \). Then

\[
\mathcal{N} \cong (\oplus_{i=1}^{m-2} \mathcal{O}(l(p,q))(1)) \oplus (\oplus_{j=1}^{n-2} \mathcal{O}(l(p,q))(1) \otimes χ) \oplus \mathcal{O}(l(p,q))(2 - d) \otimes χ.
\]

**Proof.** By the \( µ_d \)-equivariant Grothendieck splitting theorem (Theorem 3.4.1), we have an isomorphism

\[
\mathcal{N} \cong \oplus_{i=1}^{m+n-3} \mathcal{O}(l(p,q))(n_i) \otimes χ^j,
\]

for some \( n_i \in \mathbb{Z} \) and weights \( j \).

As \( X \) is a degree \( d \) hypersurface in \( \mathbb{P}^{m+n-1} \) and \( l(p,q) \) is a linear subvariety of \( \mathbb{P}^{m+n-1} \), the normal bundle \( \mathcal{N} \) fits into the following equivariant exact sequence:

\[
0 \rightarrow \mathcal{N} \rightarrow (\oplus_{i=1}^{m-1} \mathcal{O}(l(p,q))(1)) \oplus (\oplus_{j=1}^{n-1} \mathcal{O}(l(p,q))(1) \otimes χ) \rightarrow \mathcal{O}(l(p,q))(d) \rightarrow 0
\]

on \( l(p,q) \). The weights come from the description of the morphism

\[
\mathcal{O}(l(p,q))(1) \otimes \mathcal{O}(l(p,q))(d) \rightarrow \mathcal{O}(l(p,q))(d).
\]

It is given by multiplication by

\[
(\partial_{u_1} f|l(p,q)), \ldots, \partial_{u_{m-1}} f|l(p,q)), \partial_{v_1} g|l(p,q)), \ldots, \partial_{v_{n-1}} g|l(p,q))
\]

where \( u_1, \ldots, u_{m-1} \) are linear sections cutting out \( p \in \mathbb{P}^{m-1} \) and \( v_1, \ldots, v_{n-1} \) are linear sections cutting out \( q \in \mathbb{P}^{n-1} \).

Up to a linear change of coordinates, we can assume this mapping is

\[
(u_1^{d-1}, 0, \ldots, 0, v_1^{d-1}, 0, \ldots, 0).
\]

Hence,

\[
\mathcal{N} \cong (\oplus_{i=1}^{m-2} \mathcal{O}(l(p,q))(1)) \oplus (\oplus_{j=1}^{n-2} \mathcal{O}(l(p,q))(1) \otimes χ) \oplus \mathcal{O}(i) \otimes χ^j
\]

Since \( \text{deg}(\mathcal{N}) = m + n - 2 - d \) we must have \( i = 2 - d \). By checking the stalks of the normal bundle exact sequence, we must have \( j = 1 \). \[\square\]

**Lemma 4.4.2.** For \((p,q), (p',q') \in X_f \times X_g \). If \( p \neq p' \) or \( q \neq q' \), then

\[
\text{Ext}^*_{\mathcal{X}/µ_d}(\mathcal{O}(l(p,q)), \mathcal{O}(l(p',q'))) = 0.
\]

**Proof.** If \( p \neq p' \) and \( q \neq q' \), then the subvarieties \( l(p,q) \) and \( l(p',q') \) are disjoint. The vanishing follows. Without loss of generality, suppose \( p = p' \). We must compute

\[
\text{Ext}^*_{\mathcal{X}/µ_d}(\mathcal{O}(l(p,q)), \mathcal{O}(l(p',q'))) \cong \text{Ext}^*_{\mathcal{X},p}(\mathcal{O}(l(p,q)), \mathcal{O}(l(p',q')))^{µ_d}
\]

Let \( R = \widetilde{\mathcal{O}_{X,p}} \cong k[[x_1, \ldots, x_{m-2}, y_1, \ldots, y_n]] \). The action of \( µ_d \) on Spec(\( R \)) endows \( x_1, \ldots, x_{m-2} \) with trivial weight and \( y_1, \ldots, y_n \) with weight -1. The completions of \( \mathcal{O}(l(p,q)),p \) and \( \mathcal{O}(l(p',q')),p \) are isomorphic to the modules

\[
M_q = R/(x_1, \ldots, x_{m-2}, y_2, \ldots, y_n) \cong k[[y_1]]
\]

and

\[
M_{q'} = R/(x_1, \ldots, x_{m-2}, y_1, y_3, \ldots, y_n) \cong k[[y_2]],
\]

respectively.
Since $M_q$ is cut out by the regular sequence $x_1, \ldots, x_{m-2}, y_2, \ldots, y_n$, we have the following equivariant Koszul resolution
\[
\left( \bigotimes_{i=1}^{m-2} \text{Re}_{x_i} \xrightarrow{\lambda} R \right) \otimes \left( \bigotimes_{j=2}^{n} \text{Re}_{y_j} \otimes \chi^{-1} \text{Re}_{y_j} \right)
\]
of $M_q$. We apply $\text{Hom}_R(-, M_{q'})$:
\[
\left( \bigotimes_{i=1}^{m-2} M_q \bigotimes_{y_i}^\vee \bigotimes_{y_j}^\vee M_{q'} \right) \otimes \left( M_{q'} \bigotimes_{y_j}^\vee \right) \otimes \left( \bigotimes_{j=3}^{n} M_{q'} \right) \otimes k\chi
\]
It is easy to see know that since $d \geq n$, the terms appearing in $\text{Ext}_{k}^{n}(M_q, M_{q'})$ will all have nontrivial weight. Indeed, the weights will be between 1 and $n$. Hence, $\text{Ext}_{k}^{n}(M_q, M_{q'}) = 0$. Since completion is faithful, we have the desired vanishing. \hfill \Box

We can now prove $\Xi_{i,j}$ is fully-faithful.

**Theorem 4.4.1.** The functors $\Xi_{i,j}$ are fully-faithful for all $i, j$.

**Proof.** Using Theorem 2.7.1 and Lemma 4.3.2 we only need to show
\[
\text{Ext}_{X/\mu M}^{n}(\mathcal{O}_{l(p,q)}, \mathcal{O}_{l(p,q)}) \neq \begin{cases} k & \ast = 0 \\ 0 & \ast \notin [0, m+n-4] \end{cases}.
\]
That $\text{Hom}_{X/\mu M}(\mathcal{O}_{l(p,q)}, \mathcal{O}_{l(p,q)}) \cong k$ is clear, we now show vanishing.

For this we use the local-to-global spectral sequence. Since $l(p,q)$ and $X$ are smooth, this reduces to:
\[
H^r(X, \mathcal{N}) \Rightarrow \text{Ext}_{X/\mu M}^{n}(\mathcal{O}_{l(p,q)}, \mathcal{O}_{l(p,q)}).
\]
here $\mathcal{N}$ is the normal bundle from Lemma 3.3.1.

To establish the relevant vanishing, we must compute $H^r(X, \mathcal{N})$ for $(r, s) = (0, m+n-3), (1, m+n-4)$. We will compute separately.

For the case $(r, s) = (0, m+n-3)$, we have
\[
\Lambda^{m+n-3} \mathcal{N} \cong \mathcal{O}_{l(p,q)}(m+n-4 + 2 - d) \otimes \chi^{n-1} \cong \mathcal{O}(m+n-d-2) \otimes \chi^{n-1}.
\]
Suppose $m+n-d-2 \geq 0$ (otherwise there is nothing to check), then we would require a monomial of the form $x^ay^{n-1}$ with $a \geq 0$ and $a+n-1 = m+n-d-2$. Solving for $a$, we have $a = m-d-1 \leq -1$, but $d \geq m$, which is impossible.

Now for the case $(r, s) = (1, m+n-4)$. Since $l(p,q) \cong \mathbb{P}^1$, the only way $H^1(X/\mu M)$ can be nonvanishing is if $\mathcal{O}(2-d) \otimes \chi$ is involved in the product. In which case, the isotypical summands of $\Lambda^{m+n-4} \mathcal{N}$ involving $\mathcal{O}(2-d) \otimes \chi$ are:
\[
\mathcal{O}(m+n-3-d) \otimes \chi^{n-2}, \mathcal{O}(m+n-3-d) \otimes \chi^{n-1}.
\]
If $m+n-3-d \geq -1$, then the first cohomology group is zero without equivariance. Assume $m+n-3-d \leq -2$. By Serre duality, we have an isomorphism
\[
H^1(\mathcal{O}_{l(p,q)}(m+n-3-d)) \cong H^0(\mathcal{O}_{l(p,q)}(d+1-m-n) \otimes \chi^{n-1-n}) \cong k.
\]
We remark that $d > n$ here; otherwise, if $d = n$, then $m-3 \leq -2$ forces $m = 1$ and we assume $m \geq 2$. In particular, the weights $\chi^{n-1-n}$ are nontrivial above. We must find a monomial of the form $x^ay^{d-n}$ where $a \geq 0$ and $a+d-n = d+1-m-n$. This forces $a = 1 - m$, which is absurd. Similarly, we would need a monomial of the form $x^ay^{d-n+1}$ with $a \geq 0$ and $a+d-n+1 = d+1-m-n$ and hence $a = -m$,
which is still absurd. Thus there are no equivariant global sections and the group vanishes.

We conclude $\Xi_{0,0}$ is fully-faithful and so $\Xi_{i,j}$ is fully-faithful for all $i, j \in \mathbb{Z}$ as it differs from $\Xi_{0,0}$ by an autoequivalence. \hfill \square

Let $\mathcal{D}_{fg} = \Xi_{-m,-n}\mathcal{D}(X_f \times X_g)$. By Lemma \ref{lem:4.4.1} we have $\Xi_{-m,-n}$ is a full embedding. The main result can now be stated.

**Main Theorem.** In the above notation, we have a semi-orthogonal decomposition

$$\mathcal{D}[X/\mu_d] = \langle \mathcal{D}_{g1}, \mathcal{D}_{fg}, \mathcal{D}_{g2}, \mathcal{D}_f, \mathcal{A} \rangle.$$

The proof of this theorem will occupy §5. In §6 we finish proving that the decomposition is semi-orthogonal. We analyze other sheaves that we can construct from the components in §7. They will be necessary in using other kernels and proving fullness. In §8 we complete the proof of fullness.

It is worth noting that in the cases $(m, n) = (2, 2), (2, 3), (3, 3)$, there is an easier proof of this result. The idea of the proof is what is used in the subsequent sections and so we believe it does no harm in proving these special cases now.

**Pf. in the special cases.** We will only do the case $(m, n) = (2, 2)$ with the understanding that the other two are similar. The subcategories are as follows:

- $\mathcal{D}_{g1} = \mathcal{D}_g = \langle \mathcal{D}_g^{2-d}, \ldots, \mathcal{D}_g^{-1} \rangle$,
- $\mathcal{D}_{fg} = \Xi_{-2,-2}(\mathcal{D}(X_f \times X_g))$,
- $\mathcal{D}_f = \langle \mathcal{D}_f^{d-2}, \ldots, \mathcal{D}_f^1 \rangle$,
- $\mathcal{A} = \langle \mathcal{O}_X(-2) \otimes \chi^{-1}, \mathcal{O}_X(-1)\chi^{0,-1}, \mathcal{O}_X \rangle$.

Define $\mathcal{T} = \langle \mathcal{D}_g, \mathcal{D}_{fg}, \mathcal{D}_f, \mathcal{A} \rangle$. We will prove orthogonality in §4 the difficult part is fullness. To do this we show $\mathcal{T}$ has a spanning class. This will be sufficient to conclude $\mathcal{D}[X/\mu_d] = \mathcal{T}$.

Using Example \ref{ex:2.4}, we see that the collection of objects consisting of free orbits, say $\mathcal{O}_Z$ where $Z = \{ \lambda \cdot z \mid \lambda \in \mu_d \}$ and $\lambda \cdot z \neq z$, as well as the sheaves $\mathcal{O}_{\tau_f(p)} \otimes \chi^i$ and $\mathcal{O}_{\tau_g(q)} \otimes \chi^i$ for $i = 1, \ldots, d$ form a spanning class.

Let $J = J(X_f, X_g)$ inside of $X$ denote the join of $X_f$ and $X_g$. A free orbit $Z \subset X \setminus J$ is a complete intersection with respect to two sections $s_x \in \Gamma(\mathcal{O}_X(1))$ and $s_y \in \Gamma(\mathcal{O}_X(1) \otimes \chi)$. It follows that the corresponding resolution of $\mathcal{O}_Z$ given by

$$0 \to \mathcal{O}_X(-2) \otimes \chi^{-1} \to \mathcal{O}_X(-1) \otimes \mathcal{O}_X(-1) \otimes \chi^{-1} \to \mathcal{O}_X$$

is in the subcategory $\mathcal{A}$, hence the free orbits $\mathcal{O}_Z \subset \mathcal{T}$ provided we are away from $J$.

To see we have the remaining objects of the spanning class, we will use Theorem \ref{thm:3.3.1}. Let $l(p, q)$ denote the line joining $\tau_f(p)$ to $\tau_g(q)$. We will see in §4 that the objects $\mathcal{O}_{l(p,q)}(-d)$ and $\mathcal{O}_{l(p,q)}$ are in $\mathcal{T}$ for all $p, q$. It remains to see that the twists of the fixed orbits are in $\mathcal{T}$.

Using $\mathcal{D}_g$ and $\mathcal{D}_f$ we only need one additional twist, say $\mathcal{O}_p, \mathcal{O}_q$ (or in the case $(m,n) = (2,3),(3,3)$ we will also need $\mathcal{O}_p \otimes \chi^{-1}, \mathcal{O}_q \otimes \chi$). To do that, we notice that
and so we need a monomial of the form \( x^m - D \).

We then have the exact sequence

\[
0 \to \mathcal{O}_{J(p,X)} \to \bigoplus_{q \in X_q} \mathcal{O}_{l(p,q)} \to \mathcal{O}_p^{\otimes d-1} \to 0.
\]

Since \( T \) is saturated, it follows that \( \mathcal{O}_p \in T \). In the case \((m,n) = (2,3), (3,3)\) we can look at a similar sequence using \( \mathcal{O}_{J(p,X)}(-1) \otimes \chi^{-1} \in \mathcal{A} \). A similar argument shows \( \mathcal{O}_q \in T \) and Theorem 5.3.1 finishes the proof.

5. Semi-orthogonality

5.1. Geometric subcategories.

5.1.1. \( D_{fg} \). As before, we let \( D_{fg} \) be the image of the fully-faithful functor \( \Xi_{-m,-n} \).

Let us compute the semi-orthogonality \( \langle D_{fg}, \mathcal{A} \rangle \). We have the formula

\[
\text{Ext}^*_X(\mathcal{O}_X(-i) \otimes \chi^{-j}, \mathcal{O}_{l(p,q)}(-m) \otimes \chi^{-n}) \cong H^*(\mathcal{O}(i-m) \otimes \chi^{j-n}).
\]

**Lemma 5.1.1.** There is a semi-orthogonal decomposition \( \langle D_{fg}, \mathcal{A} \rangle \).

**Proof.** We only check the semi-orthogonality \( \langle D_{fg}, \mathcal{A}_3 \rangle \) as the other computations are similar. The objects in \( \mathcal{A}_3 \) are of the form \( \mathcal{O}(-i) \otimes \chi^{-j} \), where \( 0 \leq i \leq m-1 \) and \( 0 \leq j \leq i \). In this case we have \( \mathcal{O}(i-m) \otimes \chi^{j-n} \) is a negative line bundle so

\[
H^1(\mathcal{O}(i-m) \otimes \chi^{j-n}) \cong H^0(\mathcal{O}(m-i-2) \otimes \chi^{n-j-1})
\]

Since \( i \geq j \geq 0 \) we have

\[
n - 1 \geq n - j - 1 \geq n - i - 1
\]

and so we need a monomial of the form \( x^a y^{n-j-1} \) where \( a \geq 0 \) and \( a + n - j - 1 = m - i - 2 \). This is impossible because

\[
a + n - j - 1 \geq n - i - 1 \geq m - i - 1 > m - i - 2.
\]

\( \square \)

5.1.2. \( D_f \) and \( D_g \). For \( D_f \) to be present in the semi-orthogonal decomposition, we need \( d > n \) and for \( D_g \) to be present, we require \( d > m \).

**Lemma 5.1.2.** We have the semi-orthogonality \( \langle D_g, D_f, \mathcal{A} \rangle \).

**Proof.** That \( D_g \) and \( D_f \) are semi-orthogonal is clear. We only show \( D_f \) is right orthogonal to \( \mathcal{A} \), the claim that \( D_g \) is also right orthogonal is analogous.

Let \( p \in X_f \) and consider the sheaves \( \mathcal{O}_p \otimes \chi^{-j} \). We compute

\[
\text{Ext}^*_X(\mathcal{O}(-i_1) \otimes \chi^{-i_2}, \mathcal{O}_p \otimes \chi^{-j}) \cong \Gamma(\mathcal{O}_p \otimes \chi^{i_2-j})^{\mu_a}
\]

which is nonzero if and only if \( i_2 - j = 0 \). Since \( 0 \leq i_2 \leq n - 1 \), if we choose \( n \leq j \leq d - 1 \), then

\[
1 - d \leq i_2 - j \leq -1.
\]

These are precisely the weights in \( D_f \). This shows the semi-orthogonality \( \langle D_f, \mathcal{A} \rangle \).

\( \square \)

**Lemma 5.1.3.** We have the semi-orthogonality \( \langle D_{g1}, D_{fg}, D_{g2}, D_f \rangle \).
Proof. Again, we only prove the semi-orthogonality $\langle D_{fg}, D_f \rangle$ the other claims are analogous. The only possible nonzero extension group in the standard spanning class for $D_{fg}$ is

$$\text{Ext}^*_{[X/\mu_d]}(O_p \otimes \chi^{-j}, O_l(-m) \otimes \chi^{-n}) \cong \left(\text{Ext}^*_X(O_p, O_l) \otimes \chi^{j-n}\right)^{\mu_d},$$

where $l$ is the line between $p \in X_f$ and any point $q \in X_g$. Set $R = \widehat{O}_{X,p}$, then $R \cong k[[x_1, x_2, \ldots, x_{m-1}, y_1, \ldots, y_{n-1}]]$. Here the variables $x_i$ have weight 0 and the variables $y_j$ have weight -1. The sheaf $O_p$ corresponds to the graded module

$$k_p = R/(x_1, \ldots, x_{m-1}, y_1, \ldots, y_{n-1})$$

The sheaf $O_l$ corresponds to the graded module

$$M_q = R/(x_1, \ldots, x_{m-1}, y_2, \ldots, y_{n-1}).$$

We can take the Koszul resolution of $k_p$:

$$\left(\otimes_{i=1}^{m-1} \text{Re}_{x_i}, x \rightarrow R\right) \otimes \left(\otimes_{j=1}^{n-1} \text{Re}_{y_j}, y \rightarrow R\right) \rightarrow k_p$$

and apply $\text{Hom}(-, M_q)$. This will kill all of the maps except $y_1$. The resulting complex has general term $M_q e^\vee_{x_1} \wedge e^\vee_{y_1}$, where $0 \leq |I| \leq m-1$ and $0 \leq |J| \leq n-1$. It’s easy to see that the cohomology of this complex has general term $ke^\vee_{x_1} \wedge e^\vee_{y_1}$ where $0 \leq |I| \leq m-1$ and $1 \leq |J| \leq n-1$. The weights therefore vary between 1 and $n-1$. Thus the summands of the extension group are of the form:

$$k \otimes \chi, \ldots, k \otimes \chi^{n-1}.$$

Thus the general term of $\text{Ext}^*_{[X/\mu_d]}(O_p \otimes \chi^{-j}, O_l(-m) \otimes \chi^{-n})$ is of the form

$$k \otimes \chi^{1+j-n}, \ldots, k \otimes \chi^{n-1+j-n} = k \otimes \chi^{j-1}$$

Since $n \leq j \leq d-1$, these terms all have nonzero weight and so the equivariant extension group vanishes.

This completes the semi-orthogonal claim in the Main Theorem. It remains to see fullness.

6. Koszul Complexes of Joins and Orbits.

6.1. Koszul complexes. Let $T = \langle D_{g1}, D_{fg}, D_{g2}, D_f, A \rangle$. We prove fullness by showing $T$ has a spanning class. This is done in Section 7. To do so, we need to construct more sheaves in $T$ using the subcategories present. Essential to these constructions is the Koszul complex of a regular section of a vector bundle.

Let $E$ be a $\mu_d$-equivariant locally free sheaf of rank $r$ over $X$ and $s \in \Gamma(E)^{\mu_d}$ be an equivariant global section. Then we have the corresponding Koszul complex:

$$0 \rightarrow \Lambda^r E^\vee \rightarrow \Lambda^{r-1} E^\vee \rightarrow \cdots \rightarrow E^\vee \rightarrow O_X.$$
6.2. Free orbits away from $J(X_f, X_g)$. Let $Z \subset X$ be a free orbit of the $\mu_d$ action away from the join of $X_f$ and $X_g$ inside $X$, i.e., pick $p \notin X \setminus J(X_f, X_g)$ and let $Z = \{ \lambda p \mid \lambda \in \mu_d \}$. In this case, we notice that $Z$ is the intersection of $X$ with $m + n - 2$ hyperplanes. Moreover, we can pick sections $s_i \in \Gamma(\mathcal{O}_X(1))^{\mu_d}$ and section $t_j \in \Gamma(\mathcal{O}_X(1) \otimes \chi)^{\mu_d}$ where $i = 1, \ldots, m - 1$ and $j = 1, \ldots, n - 1$ such that $Z$ is the vanishing locus of $s = (s_1, \ldots, s_{m-1}, t_1, \ldots, t_{n-1}) \in \Gamma(E)$, where $E = \left( \bigoplus_{i=1}^{m-1} \mathcal{O}_X(1) \right) \oplus \left( \bigoplus_{j=1}^{n-1} \mathcal{O}_X(1) \otimes \chi \right)$.

The summands of $\mathcal{K}(E, s)$ are precisely the sheaves that occur in $\mathcal{A}$. We conclude for any free orbit $Z$ in $X \setminus J(X_f, X_g)$, we know $\mathcal{O}_Z \in \mathcal{A}$.

6.3. Lines in $J(X_f, X_g)$. Let $p \in X_f$ and $q \in X_g$. As before, denote by $l(p, q) \cong \mathbb{P}^1$ the line joining $\iota_f(p)$ to $\iota_g(q)$. Contrary to the case of free orbits outside of $J(X_f, X_g)$, the structure sheaves of both fixed orbits and free orbits in the join are not complete intersections. Moreover, the structure sheaf $\mathcal{O}_{l(p, q)}$ is not a complete intersection subvariety. We can still take the corresponding Koszul complex cutting it out. Indeed, there exists a section $s = (s_1, \ldots, s_{m-1}, t_1, \ldots, t_{n-1}) \in \Gamma(E)$, where $E$ is as before, such that $V(s) = l(p, q)$.

**Lemma 6.3.1.** As above, let $\mathcal{K}(E, s)$ be the Koszul complex cutting out $l(p, q)$. Then

$$\mathcal{H}'(\mathcal{K}(E, s)) = \begin{cases} \mathcal{O}_{l(p, q)} & * = 0 \\ \mathcal{O}_{l(p, q)}(-d) & * = -1 \\ 0 & * \neq 0, -1 \end{cases}$$

**Proof.** By Bezout’s theorem, we can assume the intersection $X_f \cap V(s_1, \ldots, s_{m-2})$ consists of $d$ points, say $p_1, \ldots, p_d$. The intersection $X_g \cap V(t_1, \ldots, t_{n-1})$ is $\{q\}$. Let $J$ denote the join in $X$ of $\{p_1, \ldots, p_d\}$ with $\{q\}$.

The Koszul complex associated to the sections $(s_1, \ldots, s_{m-2}, t_1, \ldots, t_{n-1})$ is quasi-isomorphic to the following complex

$$0 \to \mathcal{O}_J(1) \to \mathcal{O}_J \to 0.$$

To each point $\{p_1, \ldots, p_d\} \in X_f \subset \mathbb{P}^1$ there exists a linear section $s_i$ such that $V(s_i) = p_i$. We have the exact sequence

$$0 \to \mathcal{O}_{l(p, q)}(-d) \xrightarrow{\mathcal{O}_J} \mathcal{O}_J(-1) \xrightarrow{\mathcal{O}_f(-1)} \mathcal{O}_{l(p_1, q)} \to 0.$$

The claim follows. □

6.4. Projective cones. The subvariety $\iota_g : X_g \to X$ is a complete intersection. Indeed, it is the zero locus of the section $s_{X_g} = (x_1, \ldots, x_m)$ of the vector bundle $E_{X_g} = \mathcal{O}_X(1)^{\oplus m}$. The summands of the Koszul resolution, $\mathcal{K}(E_{X_g}, s_{X_g})$ are of the form $\mathcal{O}_X(-m + i)$ for $i = 0, \ldots, m$.

**Lemma 6.4.1.** For $i = 1, \ldots, d - m$. The components of $\mathcal{K}(E_{X_g}, s_{X_g})(-n - 1 + i + t) \otimes \chi^{(n-1)+t}$ are in $\mathcal{F}$ for $t = 0, \ldots, n - 1$.

**Remark 6.1.** The restriction of the equivariant structure on a hyperplane divisor of $X$ to $X_g$ is not the trivial structure. In particular, we have isomorphisms:

$$\mathcal{O}_X(iH)|_{X_g} \cong \mathcal{O}_{X_g}(ih) \otimes \chi^{-i}.$$
Proof. We check the base case $i = 1$. In this case, we have explicitly:

\[
\mathcal{K}(E_{X_g}, s_{X_g})(1) \to \mathcal{O}_{X_g}(1) \otimes \chi^{-1}
\]

\[
\mathcal{K}(E_{X_g}, s_{X_g}) \otimes \chi^{-1} \to \mathcal{O}_{X_g} \otimes \chi^{-1}
\]

\[
\vdots \to \vdots
\]

\[
\mathcal{K}(E_{X_g}, s_{X_g})(-(n-1) + 1) \otimes \chi^{-(n-1)} \to \mathcal{O}_{X_g}(-(n-1) + 1) \otimes \chi^{-1}
\]

All line bundles appearing in the resolution are already in $T$ except the line bundle appearing in degree zero. Since $\mathcal{O}_{X_g}(j) \otimes \chi^{-1} \in T$ for all $j$, we know that the line bundles in degree zero are also in $T$.

Suppose true for $1, \ldots, i$ we show true for $i + 1 \leq d - m$. In which case we have the following twists of the above diagram:

\[
\mathcal{K}(E_{X_g}, s_{X_g})(i + 1) \to \mathcal{O}_{X_g}(i + 1) \otimes \chi^{-i-1}
\]

\[
\mathcal{K}(E_{X_g}, s_{X_g})(i) \otimes \chi^{-1} \to \mathcal{O}_{X_g}(i) \otimes \chi^{-i-1}
\]

\[
\vdots \to \vdots
\]

\[
\mathcal{K}(E_{X_g}, s_{X_g})(-(n-1) + i + 1) \otimes \chi^{-(n-1)} \to \mathcal{O}_{X_g}(-(n-1) + i + 1) \otimes \chi^{-i-1}
\]

Again, all of the sheaves except the rightmost part of the resolution are already in $T$ by induction. That the line bundles in degree zero are in $T$ follows since $\mathcal{O}_{X_g}(j) \otimes \chi^{-i-1} \in T$ as $i + 1 \leq d - m$. □

Lemma 6.4.2. Let $J(X_f, q)$ denote the join of $X_f$ and $q$ inside $X$. Then

\[
\mathcal{O}_{J(X_f, q)}(i) \in T
\]

for $i = 0, \ldots, d - m$.

Proof. The subvariety $J(X_f, q)$ is a complete intersection:

\[
(\otimes_{i=1}^{n-1} \mathcal{O}_X(-1) \otimes \chi^{-1} \to \mathcal{O}_X) \to \mathcal{O}_{J(X_f, q)}.
\]

Twisting by $\mathcal{O}_X(i)$ for $i = 0, \ldots, d - m$ gives a general component of the Koszul resolution as

\[
\mathcal{O}_X(-(n-1) + i + t) \otimes \chi^{-(n-1)+t}.
\]

The statement now follows from Lemma 6.4.1. □

As $X_f$ is also a complete intersection subvariety, we have the following similar statement for $\mathcal{O}_{J(p, X_g)}$ with $p \in X_f$.

Lemma 6.4.3. Let $\mathcal{O}_{J(p, X_g)}$ denote the join of $p \in X_f$ with $X_g$. Then

\[
\mathcal{O}_{J(p, X_g)}(i) \otimes \chi^i \in T
\]

for $i = 0, \ldots, d - n$.

7. Fullness.

Our goal is to show $T$ has a spanning class. Recall, Example 2.3, if $X$ is a smooth DM stack with coarse moduli space $\pi: X \to \mathfrak{X}$, the sheaves

\[
\Omega = \{ \mathcal{Z} \subset X | \mathcal{Z} \text{ is a closed substack of } X \text{ and } \pi(\mathcal{Z}) \text{ is a closed point of } \mathfrak{X} \}
\]

form a spanning class.
For \( \mathcal{X} = [X/\mu_d] \), these sheaves are the structure sheaves of the free orbits and twists of the structure sheaves of fixed orbits by all characters. In \([6.2]\) we saw that the structure sheaves of the free orbits away from \( J(X_f, X_g) \) have Koszul resolutions using the sheaves in \( \mathcal{A} \). We will get the remaining sheaves by showing for all \( p \in X_f \) and \( q \in X_g \) we have \( \mathcal{D}[l(p,q)/\mu_d] \subset \mathcal{D}[X/\mu_d] \). For that we use Theorem \([3.3.1]\).

7.2.1. **Strategy.**

7.2. **Proof.**

By Proposition \([2.5.2]\), it follows from Lemma \([7.1.2]\) and Theorem \([2.5.1]\). We now compute \( \Xi_{d,m,0} \) and \( \Xi_{d,n,d-n} \) for various \( i = 0, \ldots, m-1 \) and \( j = 0, \ldots, n-1 \) to show that this gives us the missing sheaves: \( O_q \otimes \chi^{0,1,...,m-1}, \mathcal{O}_p \otimes \chi^{0,1,...,n-1}(n-1) \). In particular, we show there exists triangles

\[
\mathcal{O}_{j(X_f,q)}(d - m - i) \to \Xi_{d,m,0}(O_{X_f \times \{q\}}(-i)) \to T_q \to
\]

and

\[
\mathcal{O}_{j(p,X_g)}(d - n - j) \otimes \chi^{d-n} \to \Xi_{d,n,d-n}(\mathcal{O}_{p} \times X_g) \to T_p \to
\]

for every \( p \in X_f \) and \( q \in X_g \), where \( T_q \) and \( T_p \) are certain torsion sheaves supported at \( q \) and \( p \), respectively. Since the first two objects are in \( \mathcal{T} \) we have \( T_q, T_p \in \mathcal{T} \). We then build a filtration of \( T_q, T_p \) and argue then that \( \mathcal{T} \) has the remaining elements of the spanning class.

### 7.1. Other kernels.

It will be convenient to use the images of other Fourier-Mukai kernels from \( \Xi_{d-m,n} \) to \( \Xi_{d-m,0} \) and \( \Xi_{d-n,d-n} \). We justify their use in this subsection.

Using Theorem \([2.5.1]\), we must verify that the image of the spanning class \( \{\mathcal{O}_{(p,q)}\} \) under \( \Xi_{d-m,0} \) and \( \Xi_{d-n,d-n} \) factors through \( \mathcal{T} \). We will need to start by verifying the line bundles in Theorem \([3.3.1]\) are in \( \mathcal{T} \).

**Lemma 7.1.1.** For all \( p \in X_f \) and \( q \in X_g \) we have \( \mathcal{O}_{l(p.q)}(-d) \) and \( \mathcal{O}_{l(p.q)} \) in \( \mathcal{T} \).

**Proof.** By Lemma \([6.3.1]\), it suffices to show \( \mathcal{O}_{l(p,q)}(-d) \in \mathcal{T} \). We have the exact sequences

\[
0 \to \mathcal{O}_{l(p,q)}(-m - i - 1) \otimes \chi^{-n} \to \mathcal{O}_{l(p,q)}(-m - i) \otimes \chi^{-n} \to O_q \otimes \chi^{-(n-m)+i} \to 0.
\]

Since \( \mathcal{O}_{l(p,q)}(-m) \otimes \chi^{-n}, \mathcal{O}_q \otimes \chi^{-(n-m)}, \ldots, \mathcal{O}_q \otimes \chi^{-1} \in \mathcal{T} \), we have \( \mathcal{O}_{l(p,q)}(-n) \otimes \chi^{-n} \in \mathcal{T} \) by induction.

Now consider the sequences

\[
0 \to \mathcal{O}_{l(p,q)}(-n - i - 1) \otimes \chi^{-n-i+1} \to \mathcal{O}_{l(p,q)}(-n - i) \otimes \chi^{-n-i} \to \mathcal{O}_p \otimes \chi^{-n-i} \to 0.
\]

Since \( \mathcal{O}_{l(p,q)}(-n) \otimes \chi^{-n}, \mathcal{O}_p \otimes \chi^1, \ldots, \mathcal{O}_p \otimes \chi^{d-n} \in \mathcal{T} \), we have \( \mathcal{O}_{l(p,q)}(-d) \in \mathcal{T} \) by induction and this completes the proof.

**Lemma 7.1.2.** For all \( p \in X_f \) and \( q \in X_g \) we have \( \mathcal{O}_{l(p,q)}(d - m), \mathcal{O}_{l(p,q)}(d - n) \otimes \chi^{d-n} \in \mathcal{T} \).

**Proof.** Use the exact sequences in the proof of Lemma \([7.1.1]\). □

**Proposition 7.1.1.** The functors \( \Xi_{d-m,0} \) and \( \Xi_{d-n,d-n} \) factor through \( \mathcal{T} \).

**Proof.** By Proposition \([2.5.2]\), it follows from Lemma \([7.1.2]\) and Theorem \([3.3.1]\). □

### 7.2. Proof of Fullness.

#### 7.2.1. **Strategy.**

We now compute \( \Xi_{d-m,0}(O_{X_f \times \{q\}}(-i)) \) and \( \Xi_{d-n,d-n}(\mathcal{O}_{p} \times X_g)(-j) \) for various \( i = 0, \ldots, m-1 \) and \( j = 0, \ldots, n-1 \) to show that this gives us the missing sheaves: \( O_q \otimes \chi^{0,1,...,m-1}, \mathcal{O}_p \otimes \chi^{0,1,...,n-1}(n-1) \). In particular, we show there exists triangles

\[
\mathcal{O}_{j(X_f,q)}(d - m - i) \to \Xi_{d-m,0}(O_{X_f \times \{q\}}(-i)) \to T_q \to
\]

and

\[
\mathcal{O}_{j(p,X_g)}(d - n - j) \otimes \chi^{d-n} \to \Xi_{d-n,d-n}(\mathcal{O}_{p} \times X_g) \to T_p \to
\]

for every \( p \in X_f \) and \( q \in X_g \), where \( T_q \) and \( T_p \) are certain torsion sheaves supported at \( q \) and \( p \), respectively. Since the first two objects are in \( \mathcal{T} \) we have \( T_q, T_p \in \mathcal{T} \). We then build a filtration of \( T_q, T_p \) and argue then that \( \mathcal{T} \) has the remaining elements of the spanning class.
7.2.2. Factoring through the blowup. Let \( q \in X_g \) and let \( \mathbb{P}^m \) be the linear subspace spanned by \( x_1, \ldots, x_m, q \). Consider the following commutative diagram

\[
\begin{array}{c}
\mathbb{P}(\mathcal{O}_{X_f}(-1) \oplus \mathcal{O}_{X_f}(-1)) \\
\downarrow \\
X_f \times X_g \\
\downarrow j \\
X_f
\end{array}
\begin{array}{c}
\mathbb{P}(\mathcal{O}_{X_f}(-1) \oplus \mathcal{O}_{X_f}) = Z \\
\mathbb{P} \xrightarrow{\sigma} \mathbb{P}^m \xrightarrow{\beta} \text{Bl} = \text{Bl}_q \mathbb{P}^m
\end{array}
\]

where \( j \) includes \( X_f \) via \( j(x) = (x, q) \), \( Z \) is the fibered product, and \( \sigma \) also denotes the restriction of \( \sigma : Y \rightarrow \mathbb{P}^{m+n-1} \) which factors through \( \mathbb{P}^m \). The mapping \( \iota_q \) includes \( Y \) as the divisor \( dH - dE \) in \( \text{Bl} \). Here \( \mathbb{P}^m \) has coordinates \([x_1 : \cdots : x_m : y]\) and \( \mu_d \) acts by scaling the \( y \) coordinate. The \( \mu_d \)-action lifts to \( \text{Bl} \) fixing the exceptional divisor pointwise and thus rendering the enter diagram \( \mu_d \)-equivariant.

Recall, the canonical bundle of \([\mathbb{P}^m/\mu_d]\) is isomorphic to \( \mathcal{O}_{\mathbb{P}^m}(-m - 1) \chi^{-1} \). The usual formula for the canonical bundle of a blowup yields \( \omega_\text{Bl} \cong \beta^* \omega_{\mathbb{P}^m} \otimes \mathcal{O}((m - 1)E) \) which admits a \( \mu_d \)-linearization since the divisors involved are invariant under the \( \mu_d \)-action. It remains to determine if there is a twist by a character. Restricting to \( \text{Bl} \setminus E \) gives the isomorphism

\[
\omega_\text{Bl}|_{\text{Bl} \setminus E} \cong \beta^* \omega_{\mathbb{P}^m}|_{\text{Bl} \setminus E}
\]

and so it follows \( \omega_\text{Bl} \cong \beta^* \mathcal{O}_{\mathbb{P}^m}(-m - 1) \otimes \mathcal{O}_E((m - 1)E) \otimes \chi^{-1} \).

Let \( H_1 \) be a hyperplane section of \( X_f \) and \( H \) be a hyperplane section of \( \mathbb{P}^m \) which restricts to \( H_1 \) under the inclusion \( X_f \rightarrow \mathbb{P}^m \) where \([x] \mapsto [x : 0] \). Then \( H - E|_Y \cong \pi^* H_1 \).

7.2.3. Equivariant Grothendieck Duality.

**Theorem 7.2.1** (Equivariant Grothendieck Duality). There is a natural isomorphism:

\[
\mathbf{R} \beta_* \mathcal{O}_{[\text{Bl}]/\mu_d}(D) \cong \mathbf{R} \mathcal{H}om_{[\mathbb{P}^m/\mu_d]}(\mathbf{R} \beta_* (\omega_\text{Bl}|_{[\text{Bl}]/\mu_d}(-D)), \omega_{[\mathbb{P}^m/\mu_d]})
\]

for any \( \mu_d \)-invariant divisor \( D \) on \( \text{Bl} \).

**Proof.** This follows since \( \beta \) is \( \mu_d \)-equivariant and the usual Grothendieck duality is natural, hence commutes with automorphisms. \( \square \)

The divisors on \( \text{Bl} \) are, up to equivalence, well known to be of the form \( aH + bE \) for \( a, b \in \mathbb{Z} \). Using the projection formula, we have

\[
\mathbf{R} \beta_* \omega_\text{Bl}(-aH + bE) \cong (\mathbf{R} \beta_* \mathcal{O}_{[\text{Bl}]/\mu_d}((m - 1 - b)E)) \otimes \omega_{[\mathbb{P}^m/\mu_d]}(-aH)
\]

and by Grothendieck duality

\[
\mathbf{R} \beta_* (\mathcal{O}_{[\text{Bl}]/\mu_d}(aH + bE)) \cong \mathbf{R} \mathcal{H}om_{[\mathbb{P}^m/\mu_d]}(\mathbf{R} \beta_* (\mathcal{O}_{[\text{Bl}]/\mu_d}((m - 1 - b)E), \mathcal{O}_{\mathbb{P}^m})(-aH)).
\]

**Remark 7.1.** Since \( \{q\} \) is codimension \( m \), there is a canonical isomorphism

\[
\mathbf{R} \beta_* \mathcal{O}_{[\text{Bl}]/\mu_d}(kE) \cong \mathcal{O}_{X/\mu_d}
\]

for \( k = 0, \ldots, m - 1 \).

If \( k > 0 \), then \( \mathbf{R} \beta_* \mathcal{O}_{[\text{Bl}]/\mu_d}(-kE) \cong \mathcal{I}_q^k \), where \( \mathcal{I}_q \) is the ideal sheaf for the closed subscheme \( \{q\} \) in \( \mathbb{P}^m \). Moreover, \( \mathbf{R} \beta_* \mathcal{O}_{[\text{Bl}]/\mu_d}(kE) = 0 \) unless \( i = 0, m - 1 \).
7.2.4. Computing direct images. For $i = 0, \ldots, m-1$, we consider $\Xi_{d-m,0}(\mathcal{O}_{X_q}(-iH_1))$. On $\text{Bl}_I$ we have the divisor exact sequence

$$0 \to \mathcal{O}_{[\text{Bl}_I/\mu_d]}(-dH + dE) \to \mathcal{O}_{[\text{Bl}_I/\mu_d]} \to \mathcal{O}_Y \to 0.$$ 

Since $\tau_q\pi^*\mathcal{O}_{X_q}(-iH_1) = \tau_q\mathcal{O}_Y(-iH_1) \cong \tau_q\mathcal{O}_Y \otimes \mathcal{O}_{[\text{Bl}_I/\mu_d]}(-i(H - E))$, we can consider the twist of the divisor exact sequence

$$0 \to \mathcal{O}_{[\text{Bl}_I/\mu_d]}(-dH + dE - iH + iE) \to \mathcal{O}_{[\text{Bl}_I/\mu_d]}(-iH + iE) \to \mathcal{O}_Y(-iH_1) \to 0.$$ 

Using the long exact sequence of cohomology sheaves for $R\beta_b^*$, we see there is an isomorphism $R\sigma^m_!\mathcal{O}_Y(-iH_1) \cong R\beta_b^{d+1+m}O_{[\text{Bl}_I/\mu_d]}(-(d + i)H + (d + i)E)$. It follows from Remark 7.4.1 that the only possible nonzero higher direct image is $R\sigma^{m-2}_!\mathcal{O}_Y(-iH_1)$.

Lemma 7.2.1. If $m > 2$, then for $i = 0, \ldots, m-1$, we have a distinguished triangle

$$\mathcal{O}_{J(X_q,q)}(d - m - i) \to \Xi_{d-m,0}(\mathcal{O}_{X_q}(-iH_1)) \to \mathcal{H}^{m-1}(I_q^{d-m+1+i})(-m-i)[2-m] \to$$

where $I_q$ is the ideal sheaf of $\{q\}$ in $\mathbb{P}^m$, and $(I_q^{d-m+1+i})^\vee$ is the derived dual. In particular,

$$\mathcal{H}^*(\Xi_{d-m,0}(\mathcal{O}_{X_q}(-iH_1))) \cong \begin{cases} 
\mathcal{O}_{J(X_q,q)}(d - m - i) & * = 0 \\
\mathcal{H}^{m-1}(I_q^{d-m+1+i})(-m-i) & * = m-2 \\
0 & * \neq 0, m-2
\end{cases}.$$ 

If $m = 2$, then for $i = 0, 1$ there is an exact sequence

$$0 \to \mathcal{O}_{J(X,q)}(d - 2 - i) \to \Xi_{d-2,0}(\mathcal{O}_{X_q}(-iH_1)) \to \mathcal{O}_q^{\otimes d-1} \otimes \chi^{2+i-d} \to 0.$$ 

Proof. The case $m = 2$ is easy to see directly and the vanishing statements for $m > 2$ follow from the preceeding discussion. It remains to show the isomorphisms. Since $d + i > 0$, the only possible higher direct image is in degree $m - 1$. We have an exact sequence

$$0 \to \mathcal{O}_{[\mathbb{P}_m/\mu_d]}(-d - i) \to \mathcal{O}_{[\mathbb{P}_m/\mu_d]}(-i) \to \mathcal{O}_{J(X,q)}(-i) \to 0.$$ 

Now $\mathcal{H}^0(\Xi_{d-m,0}(\mathcal{O}_{X_q}(-iH_1))) \cong \mathcal{H}^0(\Xi_{d-m,0}(\mathcal{O}_{X_q}(-iH_1)))(d - m) \cong \mathcal{O}_{J(X,q)}(d - m - i)$. This gives us the first arrow for $m > 2$. If $m = 2$, the first arrow is defined similarly but it is not surjective onto $\mathcal{H}^0(\Xi_{d-m,0}(\mathcal{O}_{X_q}(-iH_1)))$.

For the second we need to compute $R\beta_{b}^{m-1}\mathcal{O}_{[\text{Bl}_I/\mu_d]}(-(d + i)H + (d + i)E)$. By Grothendieck duality and the derived functor spectral sequence, we have an isomorphism

$$R\beta_b^{m-1}\mathcal{O}_{[\text{Bl}_I/\mu_d]}(-(d + i)H + (d + i)E)$$

$$\cong R\text{Hom}_{[\mathbb{P}_m/\mu_d]}(R\beta_b(\mathcal{O}_{[\text{Bl}_I/\mu_d]}(m - 1 - d - i)E), \mathcal{O}_{[\mathbb{P}_m/\mu_d]})(-d - i)$$

$$\cong R\text{Hom}_{[\mathbb{P}_m/\mu_d]}(I_q^{d-m+1+i}, \mathcal{O}_{[\mathbb{P}_m/\mu_d]})(-d - i).$$ 

and the second isomorphism follows by twisting by $(d - m)$. □

Corollary 7.2.1. For $i = 0, \ldots, d-m$, we have $\mathcal{H}^{m-1}(I_q^{d-m+1+i})(-m-i) \in \mathcal{T}$. □

Proof. This now follows by Lemmas 6.4.2 and 7.2.1.

It remains to compute $\mathcal{H}^{m-1}(I_q^{d-m+1+i})(-m-i)$ for $i = 0, \ldots, m-1$. □
7.2.5. Derived duals of powers of $\mathcal{I}_q$. We seek to understand the sheaves in Corollary 7.2.1. To that end, there is an exact sequence of sheaves on $\mathbb{P}^m$, where we have identified the conormal bundle of $\{q\}$ with $\Omega_{\mathbb{P}^m,q}$:

$$0 \to \mathcal{I}_q^{r+1} \to \mathcal{I}_q \to S^r(\Omega_{\mathbb{P}^m,q}) \cong \mathcal{O}_q^\oplus N(r) \otimes \chi^r \to 0.$$ 

where $N(r) = \binom{m+r-1}{r}$. Since $\mathcal{O}_q$ is a smooth closed subscheme of codimension $m$, we know

$$\mathcal{H}^m((\mathcal{O}_q \otimes \chi^r)^\vee) \cong \mathcal{O}_q \otimes \omega_{2m}^\vee \otimes \chi^{-r}$$

since $\omega_{[p^m/\mu_d]} \cong \mathcal{O}_{p^m}(-m-1) \otimes \chi^{-1}$, we see

$$\mathcal{H}^m((\mathcal{O}_q \otimes \chi^r)^\vee) \cong \mathcal{O}_q \otimes \chi^{-r-m}.$$ 

Taking the derived dual of the above sequence now yields the short exact sequence

$$0 \to \mathcal{H}^{m-1}((\mathcal{I}_q^r)^\vee) \to \mathcal{H}^{m-1}((\mathcal{I}_q^{r+1})^\vee) \to \mathcal{O}_q^\oplus N(r) \otimes \chi^{-r-m} \to 0.$$ 

**Lemma 7.2.2.** For $r > 0$, the sheaves $\mathcal{H}^{m-1}((\mathcal{I}_q^r)^\vee)$ have a filtration by sheaves

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_{r+1} = \mathcal{H}^{m-1}((\mathcal{I}_q^r)^\vee)$$

such that

$$\mathcal{F}_i/\mathcal{F}_{i+1} \cong \mathcal{O}_q^\oplus N(r) \otimes \chi^{-i-m}.$$ 

In particular, $\mathcal{H}^{m-1}((\mathcal{I}_q^{r+1})^\vee) \in \langle \mathcal{O}_q \otimes \chi^{-m}, \mathcal{O}_q \otimes \chi^{-1-m}, \ldots, \mathcal{O}_q \otimes \chi^{-r-m} \rangle$.

**Proof.** Immediate from the previous discussion and the observation $\mathcal{H}^{m-1}((\mathcal{I}_q^0)^\vee) \cong \mathcal{O}_q \otimes \chi^{-m}$. \hfill $\square$

Using Lemma 7.2.2, we have the following filtration of $\mathcal{H}^{m-1}((\mathcal{I}_q^{d-m+1+i})^\vee)$:

$$\begin{align*}
\mathcal{H}^{m-1}((\mathcal{I}_q^{d-m+1+i})^\vee)(d-m-i) & \to \mathcal{O}_q^\oplus N(d-m+i) \otimes \chi^{-(d-m)} \\
\mathcal{H}^{m-1}((\mathcal{I}_q^{d-m+i})^\vee)(d-m-i) & \to \mathcal{O}_q^\oplus N(d-m+i-1) \otimes \chi^{-(d-m)+1} \\
\vdots & \\
\mathcal{H}^{m-1}((\mathcal{I}_q^2)^\vee)(d-m-i) & \to \mathcal{O}_q^\oplus N(1) \otimes \chi^{-1+i} \\
\mathcal{H}^{m-1}((\mathcal{I}_q)^\vee)(d-m-i) & \to \mathcal{O}_q \otimes \chi^i
\end{align*}$$

**Lemma 7.2.3.** For all $i = 0, \ldots, d-1$ we have $\mathcal{O}_q \otimes \chi^i \in \mathcal{T}$.

**Proof.** We proceed by induction on $i$. When $i = 0$ we use the filtration given by Lemma 7.2.2 and the fact that $\mathcal{O}_q \otimes \chi^{-j} \in \mathcal{T}$ for $j = 1, \ldots, d-m$ to see $\mathcal{O}_q \in \mathcal{T}$. Suppose we have $\mathcal{O}_q, \ldots, \mathcal{O}_q \otimes \chi^i$, then we use the filtration again with $i+1$ to see $\mathcal{O}_q \otimes \chi^{i+1} \in \mathcal{T}$. \hfill $\square$

**Lemma 7.2.4.** For all $i = 0, \ldots, d-1$ we have $\mathcal{O}_p \otimes \chi^i \in \mathcal{T}$. 

Proof. This is analagous to the results proved in this section using the images $\Xi_d n.d n(O_{X_g}(-jH_2))$. One requires the extra twist because the hyperplane section that restricts to $H_2$ has nontrivial equivariant structure. □

Proof of Main Result. By Lemmas 7.1.1, 7.2.3, and 7.2.4 we have shown for all $p \in X_f$ and $q \in X_g$ we have $D[(p,q)/\mu_d] \subset T$. We have also shown in §6.2 that the structure sheaves of free orbits not in the join are in $T$. Thus $T$ has a spanning class. Since $T$ is saturated hence admissible, by Proposition 2.4.1 we conclude $T = D[X/\mu_d]$. □

8. Comparison with Orlov’s functors

In this section, we show that in the case of two Calabi-Yau hypersurfaces, i.e. $m = n = d$, our functor $\Xi_{0,0}$ agrees with Orlov’s up to a mutation and a twist by a line bundle. We first recall the relationship between matrix factorization and singularity categories. Then we discuss Orlov’s theorem in detail and use this discussion to show that the functors agree.

8.1. Graded Matrix Factorizations and Graded Singularity Categories. For a detailed account of the relationship between graded matrix factorization and graded singularity categories see [Orl09] and [BFK14a].

Let $V$ be a finite dimensional vector space over $k$ of dimension $n$. Set $R = \text{Sym}(V^\vee)$. Then $R$ is $\mathbb{Z}$-graded, where $V^\vee$ sits in degree 1. Let $f \in R_d$ define a smooth projective hypersurface $X := V(f) \subset \mathbb{P}(V)$. Set $A = R/(f)$ to be the hypersurface algebra.

Definition 8.1. A graded matrix factorization of $f$ is a pair of morphisms

$$\delta_0: P_{-1} \rightarrow P_0, \delta_{-1}: P_0 \rightarrow P_{-1}$$

between graded projective $R$-modules such that

$$\delta_0 \delta_{-1} = f = \delta_{-1} \delta_0.$$

Morphisms of graded matrix factorizations are morphisms of the underlying graded modules that make the relevant diagrams commute. There is a notion of homotopy between two morphisms and we set $\text{HMF}^{gr}(f)$ to be the corresponding homotopy category. This category is also triangulated.

Now let $\text{gr} - A$ denote the category of graded $A$-modules. Set $\text{grproj} - A$ to be the full subcategory generated by graded projective $A$-modules. This is a thick subcategory.

Definition 8.2. The Drinfield-Verdier quotient of $D(\text{gr} - A)$ by $D(\text{grproj} - A)$ is the graded singularity category of $A$, denoted $D_{Sg}^{gr}(A)$. In other words,

$$D(\text{grproj} - A) \rightarrow D(\text{gr} - A) \rightarrow D_{Sg}^{gr}(A)$$

is an exact sequence of triangulated categories.

There is a functor between the two categories we have introduced called the cokernel functor:

$$\text{cok}: \text{HMF}^{gr}(f) \rightarrow D_{Sg}^{gr}(A)$$

which sends a graded matrix factorization $(\delta_{-1}, \delta_0)$ to the graded $A$-module $\text{cok}(\delta_0)$. 
Theorem 8.1.1 ([Orl09]). The functor \( \text{cok} \) is well-defined and an equivalence of triangulated categories.

Definition 8.3. Let \( \text{stab}: \mathcal{D}_{Sg}^g(A) \to \text{HMF}^g(f) \) denote the quasi-inverse to \( \text{cok} \), call it the stabilization functor.

Remark 8.1. If the grading is given by some extension of \( \mathbb{G}_m \) by a finite group, everything still goes through as in the \( \mathbb{G}_m \) case, [BFK14a, BFK14b].

8.2. Orlov’s Theorem. Let \( \text{tr}_{\geq i} \) to be the truncation endofunctor on \( \text{gr} - A \). Set \( \text{gr} - A_{\geq i} \) to be the image of \( \text{gr} - A \) under \( \text{tr}_{\geq i} \).

Define \( S_{<i} \) to be the full triangulated subcategory of \( \mathcal{D}(\text{gr} - A) \) generated by the finite dimensional graded \( A \)-modules \( k(e) \) for \( e > -i \). Equivalently, this is the kernel of \( \text{tr}_{\geq i} \). Similarly define \( S_{\geq i} \).

Let \( P_{<i} \) be the full triangulated subcategory of \( \mathcal{D}(\text{gr} - A) \) generated by the projective \( A \)-modules \( A(e) \) for \( e > -i \). Similarly define \( P_{\geq i} \). Let \( \text{tors} - A \) be the full subcategory of \( \text{gr} - A \) generated by finite dimensional \( A \)-modules.

We also have the projection functor \( \pi_i: \mathcal{D}(\text{gr} - A_{\geq i}) \to \mathcal{D}_{Sg}^g(A) \) which has kernel \( P_{\geq i} \) and induces an exact equivalence

\[ \mathcal{D}(\text{gr} - A_{\geq i})/P_{\geq i} \simeq \mathcal{D}_{Sg}^g(A). \]

Moreover, there is a semi-orthogonal decomposition

\[ \mathcal{D}(\text{gr} - A_{\geq i}) = \langle P_{\geq i}, T_i \rangle \]

where \( T_i \) is equivalent to the graded singularity category via \( \pi_i \). Let \( \pi_i^{-1} \) be the quasi-inverse to \( \pi_i \) restricted to \( T_i \).

Orlov defines two \( \mathbb{Z} \)-indexed families of functors

\[ \Psi_i: \mathcal{D}(X) \to \text{HMF}^g(f) \]

and

\[ \Phi_i: \text{HMF}^g(f) \to \mathcal{D}(X) \]

as follows.

The functors \( \Psi_i \) are the composite

\[ \Psi_i := \text{stab} \circ \pi_i \circ \text{tr}_{\geq i-n+d} \circ \Gamma_* \]

Here \( \Gamma_* \) is right derived graded global sections. The functors \( \Phi_i \) are the composite

\[ \Phi_i := \text{sh} \circ \pi_i^{-1} \circ \text{stab}. \]

Theorem 8.2.1 ([Orl09]). Set \( a = n - d \). The triangulated categories \( \mathcal{D}(X) \) and \( \mathcal{D}_{Sg}(A) \) are related as follows.

(i) If \( a > 0 \), \( \Phi_i \) is fully-faithful and there is a semi-orthogonal decomposition

\[ \mathcal{D}(X) = \langle \mathcal{O}_X(-i - a - 1), \ldots, \mathcal{O}_X(-1), \Phi_i \mathcal{D}_{Sg}^g(A) \rangle; \]

(ii) If \( a < 0 \), \( \Psi_i \) is fully-faithful and there is a semi-orthogonal decomposition

\[ \mathcal{D}_{Sg}^g(A) = \langle k^{\text{stab}}(-i), \ldots, k^{\text{stab}}(-i + a + 1), \Psi_i \mathcal{D}(X) \rangle; \]

(iii) If \( a = 0 \), \( \Psi_i \) and \( \Phi_i \) are mutually-inverse equivalence of categories.

Remark 8.2. The theorem still holds when \( A \) is graded by \( \mathbb{Z} \times \mu_d \). [BFK14b].
8.3. **Comparison with Orlov’s Functors.** Let $X_f$ and $X_g$ be smooth Calabi-Yau hypersurfaces, so $m = n = d$. Set $A_f$ and $A_g$ to be the corresponding hypersurface algebras, respectively, and $A$ to be the hypersurface algebra of $X = V(f + g) \subset \mathbb{P}^{2n-1}$. We consider the following functor defined using Orlov’s functors:
\[
\Omega = \Phi_0 \circ \Psi_{0,0} : \mathcal{D}(X_f \times X_g) \to \mathcal{D}[X/\mu_d]
\]
where $\Psi_{0,0}$ is the embedding:
\[
\Psi_{0,0} := \Psi_0 \otimes \Psi_0 : \mathcal{D}(X_f) \otimes \mathcal{D}(X_g) \to \text{HMF}^{gr}(f) \otimes \text{HMF}^{gr}(g) \cong \text{HMF}^{gr,\mu_d}(f \oplus g).
\]

Let $p = [p_0 : \cdots : p_{n-1}] \in X$ be such that $p_i \neq 0$. Let $l_p$ denote the graded $A$-module given by $\text{tr}_{\geq 0} \circ \Gamma_*(l_p)$. Then there is an isomorphism
\[
l_p \cong A/(x_0 - p_0 x_1, \ldots, x_n - p_n x_1).
\]
The stabilization of $l_p$ is given by a Koszul matrix factorization, see [PV16, §1.6] for the definition, say $l_p^{stab}$. Hence $\Psi_0(\mathcal{O}_p) \cong l_p^{stab}$.

**Lemma 8.3.1.** For $p \in X_f$ and $q \in X_g$, we have
\[
\Psi_{0,0}(\mathcal{O}_{(p,q)}) \cong S_{p,q}^{stab}
\]
where $S_{p,q}$ is the $\mathbb{Z} \times \mu_d$-graded $A$-module corresponding to the structure sheaf of the unique plane containing $p$ and $q$.

**Proof.** Immediate from the previous discussion and the isomorphisms
\[
\Psi_{0,0}(\mathcal{O}_{(p,q)}) \cong \Psi_0(\mathcal{O}_p) \otimes \Psi_0(\mathcal{O}_q) \cong l_p^{stab} \otimes l_q^{stab} \cong S_{p,q}^{stab},
\]
where the last isomorphism comes from the fact that the Koszul matrix factorizations are given by a regular sequence. \hfill \Box

We now need to compute $\Phi_0(S_{p,q}^{stab})$. To do this we first make a digression to note that the decomposition in Theorem [4.4]
\[
\mathcal{D}[X/\mu_n] = \langle \mathcal{O}_X(-2(n-1))(\chi^{-(n-1)}), \ldots, \mathcal{O}_X(-1)(\chi^{0,-1}), \mathcal{O}_X, \Xi_{0,0} \mathcal{D}(X_f \times X_g) \rangle
\]
doesn’t quite match Theorem [8.2.1]
\[
\mathcal{D}[X/\mu_n] = \langle \mathcal{O}_X(-(n-1))(\chi^{all}), \ldots, \mathcal{O}_X(\chi^{all}), \Omega \mathcal{D}(X_f \times X_g) \rangle.
\]

To that end, we set $\mathcal{A}_{\geq -(n-1)}$ to be the full subcategory generated of $\mathcal{D}[X/\mu_d]$ generated by
\[
\mathcal{O}_X(-(n-1))(\chi^{0,-(n-1)}), \ldots, \mathcal{O}_X(\chi^{0,-1}), \mathcal{O}_X
\]
and $\mathcal{A}_{\leq -(n-1)}$ to be the one generated by
\[
\mathcal{O}_X(-(n-1))(\chi^{-(n-1)}), \ldots, \mathcal{O}_X(-n)(\chi^{-1}, \ldots, -n-1)
\]
so that the decomposition from Theorem [8.2.1] is
\[
\mathcal{D}[X/\mu_n] = \langle \Xi_{0,0} \mathcal{D}(X_f \times X_g), \mathcal{A}_{\leq -(n-1)}, \mathcal{A}_{\geq -(n-1)} \rangle.
\]

To match the decomposition of Theorem [8.2.1] we perform some mutations and use the Serre functor to see that
\[
\mathcal{D}[X/\mu_n] = \langle \mathcal{O}_X(-(n-1))(\chi^{all}), \ldots, \mathcal{O}_X(\chi^{all}), R_{\mathcal{A}_{\leq -(n-1)}(n)}(\Xi_{0,0} \mathcal{D}(X_f \times X_g)) \rangle.
\]

Since $\text{cok}(S_{p,q}^{stab}) \cong S_{p,q}$ and the sheafification is $\mathcal{O}_{(p,q)} = \Xi_{0,0}(\mathcal{O}_{(p,q)})$, we will be done if we can see that the mutations that we have to do in $\mathcal{D}[X/\mu_d]$ agree with those that we need to do to mutate $S_{p,q}$ into $T_0$. Indeed, the object $S_{p,q}$ is already
left orthogonal to $S_{<0}$, so we just need to find a representative in $T_0$ mapping onto $S_{p,q}$.

**Lemma 8.3.2.** We have the vanishing

$$\text{Ext}^{*}_{gr-A}(S_{p,q}, A(e)(\chi^i))^{\mu_d} = 0$$

for $-n + 1 \geq e \geq 0$ and $e \leq i \leq 0$.

*Proof.* Using the isomorphism

$$\text{Ext}^{*}_{gr-A}(M, A) \cong \text{Ext}^{*+1}_{gr-R}(M, R(-n))$$

we have the following isomorphism computed by taking a Koszul complex

$$\text{Ext}^{*}_{A}(S_{p,q}, A(e)(\chi^i)) \cong S_{p,q}(n - 2 + e)(\chi^{i-1})[3 - 2n].$$

The statement follows from taken degree 0 pieces and $\mu_d$-invariants. $\square$

**Lemma 8.3.3.** There is an isomorphism

$$\text{Ext}^{*}_{gr-A}(S_{p,q}, A(e)(\chi^i)) \cong \text{Ext}^{*}_{X}(\mathcal{O}_{(p,q)}, \mathcal{O}_X(e)(\chi^i))$$

for $-n + 1 \leq e \leq 0$ and $-n \leq i < e$.

*Proof.* We compute:

$$\text{Ext}^{*}_{X}(\mathcal{O}_{(p,q)}, \mathcal{O}_X(e)(\chi^i)) \cong \text{Ext}^{2n-2-*}_{X}(\mathcal{O}_X(e)(\chi^i), \mathcal{O}_X(-n))$$

$$\cong H^{2n-2-*}(\mathcal{O}_{(p,q)}(-e - n)(\chi^{-i}))$$

$$\cong H^{3-2n+*}(\mathcal{O}_{(p,q)}(n + e - 2)(\chi^{i-1}))$$

$$\cong (S_{p,q}(n + e - 2)(\chi^{i-1}))_0[3 - 2n]$$

The claim follows from the computation in Lemma 8.3.2. $\square$

**Theorem 8.3.1.** For a point $(p, q) \in X_f \times X_g$, we have

$$\Omega(\mathcal{O}_{(p,q)}) \cong (R_{\mathcal{A}_{<-(n-1)(n)} \circ \Xi_{0,0}})(\mathcal{O}_{p,q}).$$

*Proof.* Follows from Lemmas 8.3.2 and 8.3.3. $\square$

**Corollary 8.3.1.** The functors $\Xi_{0,0}$ and $\Omega$ agree up to a twist by a line bundle.

*Proof.* We use [Huy06, Corollary 5.23], and notice that $f = \text{Id}$ since they agree on the structure sheaves of closed points. $\square$

## 9. Special Cases

For completeness, we devote this section to understanding $[X/\mu_d]$ when $m = 1$. We will independently study $n = 1$ and $n > 1$. In the case $n > 1$, the hypersurface $X$ is called a cyclic hypersurface. We also compare the decompositions to the work in [KP14].

**9.1. The case** $n = 1$. Let $m, n = 1$, then $D(X_f)$ and $D(X_g)$ will not appear. The associated quotient stack is, up to a change of coordinates, $X = V(x^d + y^d) \subset \mathbb{P}^1$. In particular, $|X| = d$ and the $\mu_d$ action permutes the linear factors. It follows that $[X/\mu_d]$ is a scheme and is represented by $\text{Spec}(k)$. There is a single exceptional object given by $\mathcal{O}_X$ and so

$$\mathcal{D}[X/\mu_d] = \langle \mathcal{O}_X \rangle.$$
9.2. **The case** $n > 1$. Let $n > 1$. Then $f(x) = x^d$ and $g(y_1, \ldots, y_n)$ be a degree $d$ polynomial defining a smooth hypersurface in $\mathbb{P}^{n-1}$. Let $\pi: X \to \mathbb{P}^{n-1}$ be the linear projection onto the $y$ variables. This is well defined since $[1:0:\ldots:0] \not\in X$. The map $\pi$ is a degree $d$ mapping ramified along the divisor $\iota_g: X_g \hookrightarrow X$. In particular, we have the following commutative diagram.

\[
\begin{array}{ccc}
X & \xrightarrow{\iota_g} & X_g \\
\downarrow{\pi} & & \downarrow{\pi} \\
\mathbb{P}^{n-1} & & 
\end{array}
\]

Endowing $\mathbb{P}^{n-1}$ with the trivial $\mu_d$ action renders the diagram commutative. Moreover, it is not hard to see that $\pi$ exhibits $\mathbb{P}^{n-1}$ as a coarse moduli space.

**Theorem 9.2.1.** There is a semi-orthogonal decomposition

\[
\mathcal{D}[X/\mu_d] = \langle \mathcal{D}^1_g, \ldots, \mathcal{D}^{d-1}_g, \pi^*\mathcal{D}([\mathbb{P}^{n-1}]) \rangle,
\]

where $\mathcal{D}^i_g = \iota_{g*}\mathcal{D}(X_g) \otimes \chi^i$.

**Proof.** The action of $\mu_d$ on $X \setminus X_g$ is easily seen to be free. Hence Theorem [3.3.2] applies. □

9.3. **Derived Categories of Cyclic Covers.** The case of a cyclic cover of a variety was investigated in [KPT14] §8.3. In particular, they discuss the equivariant derived category of cyclic hypersurfaces where $d \leq n$, here $X \subset \mathbb{P}^n$ and $X_g \subset \mathbb{P}^{n-1}$. For completeness, we recall their result. Since $d \leq n$, we have the standard semi-orthogonal decomposition of a hypersurface

\[
\mathcal{D}(X) = \langle \mathcal{A}_X, \mathcal{O}_X, \ldots, \mathcal{O}_X (d-n) \rangle,
\]

where $\mathcal{A}_X$ is characterized as the right orthogonal to $\langle \mathcal{O}_X, \ldots, \mathcal{O}_X(d-n) \rangle$. The category $\mathcal{A}_X$ is also quasi-equivalent to the homotopy category of graded matrix factorizations of the potential $f$, where $f$ is the defining equation for $X$.

**Theorem 9.3.1.** In the above notation, if $d \leq n$, then there is a decomposition

\[
\mathcal{D}[X/\mu_d] = \langle \mathcal{A}^{d_0}_X, \mathcal{O}_X \otimes \chi^0, \ldots, \mathcal{O}_X(d-n) \otimes \chi^{d-1} \rangle
\]

where

\[
\mathcal{A}^{d_0}_X = \langle \mathcal{A}_{X_g}, \mathcal{A}_{X_g} \otimes \chi, \ldots, \mathcal{A}_{X_g} \otimes \chi^{n-2} \rangle,
\]

where $\mathcal{A}_{X_g}$ is viewed as a subcategory of $\mathcal{D}[X/\mu_d]$ via $\iota_{g*}$.

**Remark 9.1.** This result does not apply when $d > n$ because $\pi: X \to \mathbb{P}^{n-1}$ is not a cyclic cover in the sense of [KPT14].

When $d = n$ the subcategory $\mathcal{A}_{X_g}$ is all of $\mathcal{D}(X_g)$. Using the notation $\mathcal{D}^i_g = \iota_{g*}(\mathcal{D}(X_g)) \otimes \chi^i$, the decomposition of Theorem 9.3.1 is

\[
\mathcal{D}[X/\mu_d] = \langle \mathcal{D}^0_g, \mathcal{D}^1_g, \ldots, \mathcal{D}^{n-2}_g, \mathcal{O}_X \otimes \chi^{0,\ldots,n-1} \rangle.
\]

The decomposition of Theorem 9.2.1 is

\[
\mathcal{D}[X/\mu_d] = \langle \mathcal{D}^1_g, \ldots, \mathcal{D}^{n-1}_g, \pi^*\mathcal{D}([\mathbb{P}^{n-1}]) \rangle.
\]

It follows that our decomposition agrees with theirs up to a twist by a character.
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