Abstract

In this paper the asymptotic behaviour of a critical 2-type Galton-Watson process with immigration is described when its offspring mean matrix is reducible, in other words, when the process is decomposable. It is proved that, under second or fourth order moment assumptions on the offspring and immigration distributions, a sequence of appropriately scaled random step processes formed from a critical decomposable 2-type Galton-Watson process with immigration converges weakly. The limit process can be described using one or two independent squared Bessel processes and possibly the unique stationary distribution of an appropriate single-type subcritical Galton-Watson process with immigration. Our results complete and extend the results of Foster and Ney (1978) for some strongly critical decomposable 2-type Galton-Watson processes with immigration.

1 Introduction

The study of the limit behaviour of critical branching processes has a long tradition and history. For critical branching processes without immigration, so-called conditioned limit theorems and for critical branching processes with immigration, unconditioned limit theorems are usually established. Investigation of asymptotic properties of critical multi-type Galton-Watson processes with or without immigration goes back at least to the 60’s and it is still an active area of research. Below we give a review of some results in this field, with our main focus on critical decomposable multi-type Galton-Watson processes without or with immigration. Then after
giving the precise definition and basic properties of multi-type Galton-Watson processes with immigration (see Section 2), we present our main results on the asymptotic behaviour of critical decomposable 2-type Galton-Watson processes with immigration (see Section 3). It is proved that, under second or fourth order moment assumptions on the offspring and immigration distributions, a sequence of appropriately scaled random step processes formed from a critical decomposable 2-type Galton-Watson process with immigration converges weakly. The limit process can be described using one or two independent squared Bessel processes and possibly the unique stationary distribution of an appropriate single-type subcritical Galton-Watson process with immigration. Our results complete and extend the results of Foster and Ney [13, Section 9] for some strongly critical decomposable 2-type Galton-Watson processes with immigration.

A multi-type Galton-Watson process with immigration is referred to as subcritical, critical, or supercritical if the spectral radius of its offspring mean matrix is less than 1, equal to 1 or greater than 1, respectively. A multi-type Galton-Watson process with immigration is called indecomposable and decomposable if its offspring mean matrix is irreducible and reducible, respectively. An indecomposable multi-type Galton-Watson process with immigration is called primitive (also called positively regular) if its offspring mean matrix is primitive. For more details on these concepts, see Section 2.

For a review of some of the results on the asymptotic behaviour of critical single-type Galton-Watson processes with immigration, see, e.g., the introduction of Barczy et al. [2]. Here we only mention the result of Wei and Winnicki [35, Theorem 2.1] who proved weak convergence of a sequence of random step processes \((n^{-1}X_{nt})_{t \geq 0}, n \geq 1, n \rightarrow \infty\), formed from a critical single-type Galton-Watson process with immigration \((X_k)_{k \geq 0}\) under second order moment assumptions and characterized the limit process as a squared Bessel process (for more details, see Theorem B.2).

Next, we make an overview of the existing results on the asymptotic behaviour of critical, decomposable multi-type Galton-Watson processes without immigration or with immigration, then those of critical, indecomposable or primitive branching processes, and finally we recall a result on the asymptotic behaviour of supercritical decomposable multi-type Galton-Watson processes without immigration, and on the weak convergence of a sequence of appropriately scaled (arbitrary) 2-type Galton-Watson processes with immigration towards a continuous state and continuous time branching process with immigration, respectively.

Under second order moment assumptions on the offspring distributions, Foster and Ney [12] described the asymptotic behaviour of the extinction probability of a critical decomposable multi-type Galton-Watson process without immigration and with a deterministic initial distribution.

Foster and Ney [13] proved conditioned limit theorems for some special strongly critical decomposable multi-type Galton-Watson processes without immigration (see [13, Theorem 2]), and unconditioned limit theorems for some special strongly critical decomposable multi-type Galton-Watson processes with immigration (see [13, Theorems 4 and 5]). They also specialized
their results in case of 2 types, see \cite{13} Section 6 and page 42. This special 2-type case in case of immigration corresponds to our case (2) (see (3.2)). For a special strongly critical decomposable 2-type Galton-Watson process with immigration \((X_k)_{k\geq 0} = ((X_{k,1}, X_{k,2}))_{k\geq 0}\), under second order moment assumptions on the offspring distributions and first order moment assumptions on the immigration distributions, Foster and Ney \cite{13} Theorems 4 and 5] showed that \((n^{-1}X_{n,1}, n^{-2}X_{n,2})\) converges in distribution as \(n \to \infty\), and they characterized the limit distribution by its Laplace transform containing an integral as well (see \cite{13} formula (9.11)). Note that here the right normalization is \(n^{-i}\) for the \(i\)-th-type, \(i = 1, 2\). In our Theorem 3.2 we extend this result of Foster and Ney by proving weak convergence of a sequence of appropriately scaled random step processes formed from \((X_k)_{k\geq 0}\), and we characterize the limit process as a solution of a system of stochastic differential equations. In Section 4 we compare our results in Theorem 3.2 and those of Foster and Ney \cite{13} Section 9] in detail, where we give a closed formula of the Laplace transform of the above mentioned limit distribution in Foster and Ney \cite{13} formula (9.11)] as well. Foster and Ney \cite{13} Section 8] also indicated some conjectures about the nature of possible results for critical decomposable multi-type Galton-Watson processes without immigration not supposing some of their restrictive hypotheses including strong criticality. Our results in Section 3 for critical decomposable 2-type Galton-Watson processes with immigration handle all the remaining cases not included in Foster and Ney \cite{13} Section 9], and instead of convergence of one-dimensional distributions we can prove weak convergence of a sequence of appropriate random step processes formed from the branching processes in question.

Sugitani \cite{30,31} extended the results of Foster and Ney \cite{13} on conditional limit theorems for critical decomposable multi-type Galton-Watson processes without immigration \((0.2 and 2.4)]\, and some unconditional limit theorems were established as well \cite{31} Theorems 2.1, 2.3 and 2.5]).

Zubkov \cite{37} proved some conditioned limit theorems for critical decomposable 2-type Galton-Watson processes without immigration such that the generating functions of the offspring distributions satisfy some regularity assumptions yielding that the offspring distributions do not have finite second moments.

Studying asymptotic properties of critical decomposable multi-type Galton-Watson processes with or without immigration is still attracting the attention of researchers. Smadi and Vatutin \cite{29} considered a critical decomposable 2-type Galton-Watson process \((X_k)_{k\geq 0}\) without immigration such that the variance of the offspring distributions may be infinite. Let \(X_{m,n,i}\) be the number of individuals of type \(i\) alive at time \(m\) and having descendants at time \(n\), where \(i = 1, 2\) and \(m < n\), in other words, \(X_{m,n,i}\) is the number of the type \(i\) ancestors alive at generation \(m\) of all the individuals of the population at generation \(n\). Smadi and Vatutin \cite{29} described the asymptotic behaviour of the conditional distribution of \((X_{m,n,1}, X_{m,n,2})\) given that \(X_n \neq 0\) as \(m,n \to \infty\). Here for each \(n \in \mathbb{N}\), \((X_{m,n,1}, X_{m,n,2})_{m \in \{0,1,\ldots,n-1\}}\) can be thought as the family tree relating the individuals alive at time \(n\), and \((X_{m,n,1}, X_{m,n,2}),\ 0 \leq m \leq n,\ m,n \in \mathbb{Z}_+\), is sometimes called a reduced branching process. For strongly critical decomposable multi-type Galton-Watson processes without immigration, similar problems were considered and solved by Vatutin \cite{33,34}.
Next, we recall some results on the asymptotic behaviour of critical, indecomposable or primitive multi-type Galton-Watson processes without immigration or with immigration.

For a certain class of critical, primitive (also called positively regular) multi-type branching processes \((Z_n)_{n \geq 0}\) without immigration Mullikin [26] Theorems 8 and 9] characterized the limit of the conditional expectation and distribution of \(n^{-1}Z_n\) given that \(Z_n \neq 0\) as \(n \to \infty\). Mullikin’s results [26] Theorems 8 and 9 are in fact much more general, a discrete time temporally homogeneous Markov process \((Z_n)_{n \geq 0}\) was considered, where the range of \(Z_n\) is a set of finitely additive, non-negative and integer-valued set functions on an abstract set (representing the set of possibly infinite number of types) furnished with a \(\sigma\)-algebra, and \(Z_0\) is a given non-random functional.

Joffe and Métivier [20] Theorem 4.3.1] studied a sequence \((X_k^{(n)})_{k \geq 0}, n \geq 1\), of critical multi-type Galton-Watson processes with the same offspring distributions having finite second moments, but without immigration and starting from a deterministic initial value \(X_0^{(n)}\), supposing that the offspring mean matrix is primitive and \(n^{-1}X_0^{(n)}\) converges to a non-zero (deterministic) limit as \(n \to \infty\). They determined the limiting behaviour of the martingale part \((n^{-1}\sum_{k=1}^{|n|}(X_k^{(n)} - \mathbb{E}(X_k^{(n)} | X_0^{(n)}, \ldots, X_{k-1}^{(n)})))_{t \geq 0}\) as \(n \to \infty\). Joffe and Métivier [20] Theorem 4.2.2] also studied a sequence of multi-type Galton-Watson processes without immigration \((X_k^{(n)})_{k \geq 0}, n \geq 1\), which is nearly critical of special type (see (i) of Theorem 4.2.2 in [20]), and, under second order moment assumptions and a Lindeberg-type condition, they proved that the sequence \((n^{-1}X_{|nt|}^{(n)})_{t \geq 0}\) converges in distribution towards a diffusion process as \(n \to \infty\).

Ispány and Pap [17] Theorem 3.1] described the asymptotic behaviour of a sequence of critical primitive (also called positively regular) multi-type Galton-Watson processes with immigration \((X_k^{(n)})_{k \geq 0}\) sharing the same offspring and immigration distributions, but having possibly different initial distributions such that \(n^{-1}X_0^{(n)}\) converges in distribution to \(\mathcal{X}u\) as \(n \to \infty\), where \(\mathcal{X}\) is a nonnegative random variable with distribution \(\mu\) and \(u\) is the Perron (right) eigenvector of the offspring mean matrix. Under fourth order moment assumptions on the offspring and immigration distributions, they showed that \((n^{-1}X_{|nt|}^{(n)})_{t \geq 0}\) converges in distribution as \(n \to \infty\). They characterized the limit process as \((\mathcal{X}_t u)_{t \geq 0}\), where \((\mathcal{X}_t)_{t \geq 0}\) is a squared Bessel process with initial distribution \(\mu\). Here it is interesting to point out the fact the limiting diffusion process \((\mathcal{X}_t u)_{t \geq 0}\) is always one-dimensional in the sense that for all \(t \geq 0\), the distribution of \(\mathcal{X}_t u\) is concentrated on the ray \([0, \infty) \cdot u\), while the original sequence of branching processes does not have this property.

For a critical indecomposable \(p\)-type Galton-Watson process \((X_k)_{k \geq 0}\) with immigration and starting from \(X_0 = 0\), Danka and Pap [10] obtained a generalization of Theorem 3.1 in Ispány and Pap [17]. In the indecomposable case the set of types \(\{1, \ldots, p\}\) can be partitioned according to communication of types, namely, into \(r\) nonempty mutually disjoint subsets \(D_1, \ldots, D_r\) such that an individual of type \(j\) may not have offsprings of type \(i\) unless there exists \(\ell \in \{1, \ldots, r\}\) with \(i \in D_{\ell-1}\) and \(j \in D_\ell\), where the subscripts are considered modulo \(r\) (for more details, see, e.g., Danka and Pap [10, Section 2]). This partitioning is unique up to
cyclic permutation of the subsets, and the number \( r \) is called the index of cyclicity (in other words, the index of imprimitivity) of the mean matrix \( A \). Note that \( r = 1 \) if and only if the matrix \( A \) is primitive, i.e., the branching process in question is primitive (in other words positively regular). Under second order moment assumptions on the offspring and immigration distributions for the given \( p \)-type Galton-Watson process \( (X_k)_{k \geq 0} \) with immigration, using Theorem C.2, Danko and Pap [10, Theorem 3.1] determined the joint asymptotic behaviour of the random step processes \( ((nr)^{-1}X_{r\lfloor nt\rfloor+i-1})_{t \geq 0}, \ n \in \mathbb{N}, \ i \in \{1, \ldots, r\} \) towards the limiting diffusion processes \( (A^{r+i-1}Y_t)_{t \geq 0}, \ i \in \{1, \ldots, r\} \) as \( n \to \infty \). Here the process \( (Y_t)_{t \geq 0} \) is 1-dimensional in the sense that for each \( t \geq 0, \) the distribution of \( Y_t \) is concentrated on the ray \( [0, \infty) \cdot u \), where \( u \) is the Perron (right) eigenvector of the offspring mean matrix \( A \).

To close the review of existing and connecting literature, we recall two more results that are somewhat connected. It is interesting to note that Kesten and Stigum [22, Theorems 2.1, 2.2 and 2.3] considered a supercritical decomposable multi-type Galton-Watson process \( (X_n)_{n \geq 0} \) without immigration and with a fixed deterministic initial distribution, and they proved that appropriately normalizing \( X_n \) (or its appropriate subsequence) it converges almost surely to a random limit vector as \( n \to \infty \). The normalizing factors in question always have the form \( n^{-\gamma}\lambda^{-n} \), where \( \gamma \) is a non-negative integer and \( \lambda \) is a positive real number greater than or equal to one. In some cases, they specialized their results to 2-type processes as well, see Kesten and Stigum [22, page 321].

Ma [25, Theorem 2.1, (i)] established sufficient conditions for the weak convergence of a sequence of (arbitrary, not necessarily critical or decomposable) 2-type Galton-Watson processes with immigration towards a continuous state and continuous time branching process with immigration using appropriate time and space scalings such that the time scaling in question depends on the immigration distributions (and being different from what we will consider in our limit theorems in Section 3). Ma [25] proved the convergence of the sequence of infinitesimal generators of the branching processes in question towards the infinitesimal generator of the limit process.

The paper is organized as follows. In Section 2 we recall the definition of multi-type Galton-Watson processes with immigration, their classification as subcritical, critical and supercritical ones, and the special classes of indecomposable, decomposable and primitive (also called positively regular) branching processes. Section 3 contains our main results on the asymptotic behaviour of critical decomposable 2-type Galton-Watson processes with immigration, see Theorems 3.1–3.7. The investigation of such processes can be reduced to five cases presented in (3.2) according to the form of the offspring mean matrix. We also explain how decomposable 2-type Galton-Watson processes may model the sizes of a geographically structured population divided into two parts. Under second or fourth order moment assumptions on the offspring and immigration distributions, in the above mentioned five cases, we describe the limit behaviour of a sequence of appropriately scaled random step processes formed from a critical decomposable 2-type Galton-Watson process with immigration. The limit process can be described using either one or two independent squared Bessel processes, and possibly the unique stationary distribution of an appropriate single-type subcritical Galton-Watson process with immigration.
This is a new phenomenon compared to the existing results on critical indecomposable (special-ly primitive) multi-type Galton-Watson processes with immigration. Concerning Theorem 3.4 we formulate a conjecture on the independence of the limit processes of the two coordinate processes. We note that Theorem 3.2 can be considered as a functional version of Theorems 4 and 5 in Foster and Ney [13] for some strongly critical decomposable 2-type Galton-Watson processes with immigration. For a detailed comparison of our results in Theorem 3.2 and those of Foster and Ney [13, Section 9], see Section 4 where, as a consequence of Theorem 3.2 we also give a functional generalization of the Corollary on page 42 in Foster and Ney [13], regarding the joint convergence of the appropriately normalized population size at time $n$ and total progeny up to time $n$ of a critical single-type Galton–Watson process with immigration as $n \to \infty$. In Corollary 3.3 we describe the asymptotic behaviour of the relative frequency of individuals of type 2 with respect to individuals of type 1 under the conditions of Theorem 3.2 together with that the mean of immigration distribution of type 1 individuals is positive and finite. For different models, one can find similar results, e.g., in Jagers [19, Corollary 1] and in Yakovlev and Yanev [36, Theorem 2] (for more details, see the paragraph before Corollary 3.3). Remark 3.8 is devoted to a discussion on the moment conditions in Theorems 3.1, 3.5 and 3.7 we explain why we suppose the finiteness of the fourth order moments of the offspring and immigration distributions in these theorems. Section 5 contains some preliminaries for the proofs such as a formula for the powers of the offspring mean matrix and a useful decomposition of the process using martingale differences (see (5.4)). Sections 6–10 are devoted to the proofs of Theorems 3.1, 3.7 and Corollary 3.3. We close the paper with four appendices. Appendix A contains some formulae and estimates for the first, second and fourth order moments of the coordinates of the branching process in question and those of the derived martingale differences, respectively. These estimates are extensively used in the proofs. In Appendix B we present a result on the asymptotic behaviour of finite dimensional distributions of a single-type subcritical Galton-Watson process with immigration satisfying first order moment conditions, which may be known, but we could not address any reference for it, so we provided a proof as well. We also recall the asymptotic behaviour of a single-type critical Galton-Watson process with immigration due to Wei and Winnicki [35, Theorem 2.1]. Appendix C contains a version of the continuous mapping theorem. In Appendix D we recall a result about the convergence of random step processes towards a diffusion process due to Ispány and Pap [16], this result is heavily used in our proofs.

2 Multi-type Galton-Watson processes with immigration

Let $\mathbb{Z}_+, \mathbb{N}, \mathbb{R}, \mathbb{R}_+, \mathbb{R}_{++}$ and $\mathbb{C}$ denote the set of non-negative integers, positive integers, real numbers, non-negative real numbers, positive real numbers and complex numbers, respectively. For $x, y \in \mathbb{R}$, the minimum of $x$ and $y$ is denoted by $x \wedge y$. The Euclidean norm on $\mathbb{R}^d$ is denoted by $\| \cdot \|$, where $d \in \mathbb{N}$. The $d \times d$ identity matrix is denoted by $I_d$. For
a function \( f : \mathbb{R} \to \mathbb{R} \), its positive part is denoted by \( f^+ \). Every random variable will be defined on a fixed probability space \((\Omega, \mathcal{A}, \mathbb{P})\). Convergence in probability, convergence in \( L_1 \), convergence almost surely, equality in distribution and almost sure equality is denoted by \( \xrightarrow{p} \), \( \xrightarrow{L_1} \), \( \xrightarrow{a.s.} \), \( \xrightarrow{D} \) and \( \xrightarrow{a.s.} \), respectively. We will use \( \xrightarrow{D} \) for the weak convergence of the finite dimensional distributions, and \( \xrightarrow{D} \) for the weak convergence of \( \mathbb{R}^d \)-valued stochastic processes with sample paths in \( D(\mathbb{R}_+, \mathbb{R}^d) \), where \( d \in \mathbb{N} \) and \( D(\mathbb{R}_+, \mathbb{R}^d) \) denotes the space of \( \mathbb{R}^d \)-valued càdlàg functions defined on \( \mathbb{R}_+ \) (for more details and notations, e.g., for \( \xrightarrow{u} \), see Appendix [C]). Given a non-empty set \( I \), a stochastic process \((Y_t)_{t \in I}\) is called an i.i.d. process if the random variables \( \{Y_t : t \in I\} \) are independent (i.e., for each \( m \in \mathbb{N} \) and each subset \( \{t_1, \ldots, t_m\} \subset I \), the random variables \( Y_{t_1}, \ldots, Y_{t_m} \) are independent) and identically distributed. We note that in time series analysis, by a white noise (process), one usually means an uncorrelated, zero mean process having finite second moments, so according to our definition, an i.i.d. process is not necessarily a white noise (process).

We will investigate a certain 2-type Galton-Watson process with immigration. First we recall the definition and first order moment formulae of \( p \)-type Galton-Watson processes with immigration, where \( p \in \mathbb{N} \).

For each \( k \in \mathbb{Z}_+ \) and \( i \in \{1, \ldots, p\} \), the number of individuals of type \( i \) in the \( k \)th generation is denoted by \( X_{k,i} \). For simplicity, we suppose that the initial values are \( X_{0,i} = 0 \), \( i \in \{1, \ldots, p\} \). By \( \xi_{k,j,i,\ell} \) we denote the number of type \( \ell \) offsprings produced by the \( j \)th individual who is of type \( i \) belonging to the \( (k-1) \)th generation. The number of type \( i \) immigrants in the \( k \)th generation is denoted by \( \varepsilon_{k,i} \). Consider the random vectors

\[
X_k := \begin{bmatrix} X_{k,1} \\ \vdots \\ X_{k,p} \end{bmatrix}, \quad \xi_{k,j,i} := \begin{bmatrix} \xi_{k,j,i,1} \\ \vdots \\ \xi_{k,j,i,p} \end{bmatrix}, \quad \varepsilon_k := \begin{bmatrix} \varepsilon_{k,1} \\ \vdots \\ \varepsilon_{k,p} \end{bmatrix}.
\]

Then we have

\[
X_k = \sum_{i=1}^{p} \sum_{j=1}^{X_{k-1,i}} \xi_{k,j,i} + \varepsilon_k, \quad k \in \mathbb{N},
\]

with \( X_0 = 0 \) (and using the convention \( \sum_{j=1}^{0} := 0 \)). Here \( \{\xi_{k,j,i,\ell} : k, j \in \mathbb{N}, i \in \{1, \ldots, p\}\} \) are supposed to be independent. Moreover, \( \{\xi_{k,j,i} : k, j \in \mathbb{N}\} \) for each \( i \in \{1, \ldots, p\} \), and \( \{\varepsilon_k : k \in \mathbb{N}\} \) are supposed to consist of identically distributed \( \mathbb{Z}_+^p \) valued random vectors. For notational convenience, let \( \{\xi_i : i \in \{1, \ldots, p\}\} \) and \( \varepsilon \) be random vectors such that \( \xi_i \overset{D}{=} \xi_{1,1,i} \) for all \( i \in \{1, \ldots, p\} \) and \( \varepsilon \overset{D}{=} \varepsilon_1 \).

In all what follows we will suppose

\[
\mathbb{E}(\|\xi_i\|^2) < \infty, \quad i = 1, \ldots, p, \quad \text{and} \quad \mathbb{E}(\|\varepsilon\|^2) < \infty.
\]

7
Introduce the notations
\[
A := \begin{bmatrix} \mathbb{E}(\xi_1) & \cdots & \mathbb{E}(\xi_p) \end{bmatrix} \in \mathbb{R}_+^{p \times p}, \quad b := \mathbb{E}(\varepsilon) \in \mathbb{R}^p,
\]
\[
V^{(i)} := \text{Var}(\xi_i) \in \mathbb{R}_+^{p \times p}, \quad i \in \{1, \ldots, p\}, \quad V^{(0)} := \text{Var}(\varepsilon) \in \mathbb{R}_+^{p \times p}.
\]

The matrix \( A \) and the vector \( b \) is called the offspring mean matrix and immigration mean vector of \((X_k)_{k \in \mathbb{Z}_+}\), respectively. Note that some authors define the offspring mean matrix as \( A^\top \).

For \( k \in \mathbb{Z}_+ \), let \( F_k^X := \sigma(X_0, X_1, \ldots, X_k) \), where \( F_0^X = \{\emptyset, \Omega\} \) (due to \( X_0 = 0 \)). By (2.1), we get
\[
(2.3) \quad \mathbb{E}(X_k | F_{k-1}^X) = \sum_{i=1}^p X_{k-1,i} \mathbb{E}(\xi_i) + b = AX_{k-1} + b, \quad k \in \mathbb{N}.
\]
Consequently,
\[
\mathbb{E}(X_k) = A \mathbb{E}(X_{k-1}) + b, \quad k \in \mathbb{N},
\]
and, since \( X_0 = 0 \), we have
\[
\mathbb{E}(X_k) = \sum_{j=0}^{k-1} A^j b, \quad k \in \mathbb{N}.
\]

Hence the offspring mean matrix \( A \) plays a crucial role in the asymptotic behaviour of the sequence \((\mathbb{E}(X_k))_{k \in \mathbb{Z}_+}\). A \( p \)-type Galton-Watson process \((X_k)_{k \in \mathbb{Z}_+}\) with immigration is referred to respectively as subcritical, critical or supercritical if \( \rho(A) < 1 \), \( \rho(A) = 1 \) or \( \rho(A) > 1 \), where \( \rho(A) \) denotes the spectral radius of the matrix \( A \), i.e., the maximum of the absolute values of the eigenvalues of \( A \) (see, e.g., Athreya and Ney [1, V.3] or Quine [27]).

A multi-type Galton-Watson process \((X_k)_{k \in \mathbb{Z}_+}\) with immigration is called indecomposable and decomposable if its offspring mean matrix \( A \) is irreducible and reducible, respectively. We recall that the matrix \( A \) is called reducible if there exist a permutation matrix \( P \in \mathbb{R}^{p \times p} \) and \( q \in \{1, \ldots, p-1\} \) such that
\[
A = P \begin{bmatrix} R & 0 \\ S & T \end{bmatrix} P^\top,
\]
where \( R \in \mathbb{R}^{q \times q} \), \( S \in \mathbb{R}^{(p-q) \times q} \), \( T \in \mathbb{R}^{(p-q) \times (p-q)} \) and \( 0 \in \mathbb{R}^{q \times (p-q)} \) is a null matrix. The matrix \( A \) is called irreducible if it is not reducible; see, e.g., Horn and Johnson [14, Definitions 6.2.21 and 6.2.22]. We do emphasize that no 1-by-1 matrix is reducible. It is known that the matrix \( A \) is irreducible if and only if for all \( i, j \in \{1, \ldots, p\} \) there exists \( n_{i,j} \in \mathbb{N} \) such that the matrix entry \((A^{n_{i,j}})_{i,j}\) is positive. An indecomposable multi-type Galton-Watson process \((X_k)_{k \in \mathbb{Z}_+}\) with immigration is called primitive (also called positively regular) if its offspring mean matrix \( A \) is primitive, i.e., there exists an \( n \in \mathbb{N} \) such that the matrix entry \((A^n)_{i,j}\) is positive for each \( i, j \in \{1, \ldots, p\} \).
3 Convergence of random step processes

In what follows we consider a critical decomposable 2-type Galton-Watson process \((X_k)_{k \in \mathbb{Z}^+}\) with immigration starting from \(X_0 = 0\), and we suppose that the moment conditions (2.2) hold. Since \(p = 2\) and \(A = (a_{i,j})_{i,j=1}^{2,2} \in \mathbb{R}^{2 \times 2}\) is reducible, we have \(a_{1,2} = 0\) or \(a_{2,1} = 0\). Note also that if \((X_{k,1}, X_{k,2})_{k \in \mathbb{Z}^+}\) is a decomposable 2-type Galton-Watson process with immigration having an offspring mean matrix with \((1,2)\)-entry 0, then \((X_{k,2}, X_{k,1})_{k \in \mathbb{Z}^+}\) is also a decomposable 2-type Galton-Watson process with immigration having an offspring mean matrix with \((2,1)\)-entry 0. Because of this, when dealing with decomposable 2-type Galton-Watson processes with immigration it is enough to focus on those ones which have an offspring mean matrix with \((1,2)\)-entry 0. So we may assume that the offspring mean matrix \(A\) and the immigration mean vector \(b\) take the following forms:

\[
A = \begin{bmatrix}
\mathbb{E}(\xi_1) & \mathbb{E}(\xi_2)
\end{bmatrix} = \begin{bmatrix}
a_{1,1} & 0 \\
a_{2,1} & a_{2,2}
\end{bmatrix}
\quad \text{and} \quad
b = \begin{bmatrix}
b_1 \\
b_2
\end{bmatrix},
\]

respectively, with \(\rho(A) = \max \{a_{1,1}, a_{2,2}\} = 1\). Taking into account that \(a_{1,2} = 0\) implies \(\mathbb{P}(\xi_{1,1,2,1} = 0) = 1\), Equation (2.1) with \(p = 2\) takes the form

\[
\begin{bmatrix}
X_{k,1} \\
X_{k,2}
\end{bmatrix} = \sum_{j=1}^{X_{k-1,1}} \begin{bmatrix}
\xi_{k,j,1,1} \\
\xi_{k,j,1,2}
\end{bmatrix} + \sum_{j=1}^{X_{k-1,2}} \begin{bmatrix}
0 \\
\xi_{k,j,2,2}
\end{bmatrix} + \begin{bmatrix}
\varepsilon_{k,1} \\
\varepsilon_{k,2}
\end{bmatrix}, \quad k \in \mathbb{N},
\]

with \([X_{0,1}, X_{0,2}] = [0,0]\). For a decomposable 2-type Galton-Watson process \((X_k)_{k \in \mathbb{Z}^+}\) with immigration given by (3.1), the individuals of type 1 may produce individuals of types 1 or 2, and the individuals of type 2 may produce individuals of type 2 only. This process may be viewed as a stochastic model of the sizes of a geographically structured population divided into two parts such that

- the individuals are located at one of the two parts, and the location of an individual is considered as its type,
- the newborn individuals of the part 1 either stay at the part 1 or migrate, just after their birth, to the part 2,
- the newborn individuals of the part 2 stay at the part 2 (they do not migrate),
- at each step immigrants (newcomers) may join the part \(i \in \{1,2\}\) and they become individuals of the part \(i\),
- the offspring and immigration distributions depend on the parts on which the individuals are located, and the immigrants join, respectively.

Jagers [19] also pointed out that the reproduction of biological populations consisting of two types of individuals often displays the irreversibility property described above in the sense that
individuals of one type might give birth to descendants of both kinds, whereas those of the other type can have descendants only of their own kind. For example, if human diploid cells in a tumour are considered the first type in the cell population, and cells of higher diploidity are considered the second type, then, provided that endomitosis (a process where chromosomes duplicate but the cell does not subsequently divide, causing higher ploidy) is possible, the population of cells in this tumour has the irreversibility property in question.

We can distinguish the following 5 cases \((a_{1,2} = 0\) for each case):

\[
\begin{array}{ccc}
(1) & a_{1,1} = 1 & a_{2,1} = 0 & a_{2,2} = 1 \\
(2) & a_{1,1} = 1 & a_{2,1} \in \mathbb{R}_{++} & a_{2,2} = 1 \\
(3) & a_{1,1} = 1 & a_{2,1} = 0 & a_{2,2} \in [0,1) \\
(4) & a_{1,1} = 1 & a_{2,1} \in \mathbb{R}_{++} & a_{2,2} \in [0,1) \\
(5) & a_{1,1} \in [0,1) & a_{2,1} \in \mathbb{R}_+ & a_{2,2} = 1 \\
\end{array}
\]

(3.2)

For abbreviation, we can write the above five cases in matrix form as follows:

\[
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}_1, \quad \begin{pmatrix}
1 & 0 \\
++ & 1
\end{pmatrix}_2, \quad \begin{pmatrix}
1 & 0 \\
0 & 1-
\end{pmatrix}_3, \quad \begin{pmatrix}
1 & 0 \\
++ & 1-
\end{pmatrix}_4, \quad \begin{pmatrix}
1 & 0 \\
+ & 1
\end{pmatrix}_5.
\]

We remark that in the literature the cases (1) and (2) are called strongly critical due to \(a_{1,1} = a_{2,2} = 1\), the other cases (3), (4) and (5) are critical, but not strongly critical, see, e.g., Foster and Ney [13, page 13].

Note that the first coordinate process \((X_{k,1})_{k \in \mathbb{Z}_+}\) of \((X_k)_{k \in \mathbb{Z}_+}\) satisfies

\[
X_{k,1} = \sum_{j=1}^{X_{k-1,1}} \xi_{k,j,1,1} + \varepsilon_{k,1}, \quad k \in \mathbb{N},
\]

(3.3)

hence \((X_{k,1})_{k \in \mathbb{Z}_+}\) is a single-type Galton-Watson process with immigration, which is critical in cases (1)–(4) and is subcritical in case (5) (due to \(E(\xi_{1,1,1,1}) = a_{1,1}\)).

If the process \((X_{k,1})_{k \in \mathbb{Z}_+}\) given in (3.3) is critical, i.e., \(E(\xi_{1,1,1,1}) = a_{1,1} = 1\), then, by a result of Wei and Winnicki [35] (see also Theorem B.2), we have

\[
(n^{-1} X_{[nt],1})_{t \in \mathbb{R}_+} \xrightarrow{D} (X_{t,1})_{t \in \mathbb{R}_+} \quad \text{as} \quad n \to \infty,
\]

(3.4)

where the limit process \((X_{t,1})_{t \in \mathbb{R}_+}\) is the pathwise unique strong solution of the stochastic differential equation (SDE)

\[
dX_{t,1} = b_1 \, dt + \sqrt{v^{(1)}_{1,1}} \, dW_{t,1}, \quad t \in \mathbb{R}_+, \quad X_{0,1} = 0,
\]

where \((W_{t,1})_{t \in \mathbb{R}_+}\) is a standard Wiener process, \(b_1 = E(\varepsilon_{1,1})\) and \(v^{(1)}_{1,1} := Var(\xi_{1,1,1,1})\). The process \((X_{t,1})_{t \in \mathbb{R}_+}\) is called a squared Bessel process.
If the process \((X_{k,1})_{k \in \mathbb{Z}_+}\) given in (3.3) is subcritical, i.e., \(\mathbb{E}(\xi_{1,1,1}) = a_{1,1} \in [0,1)\), then the Markov chain \((X_{k,1})_{k \in \mathbb{Z}_+}\) admits a unique stationary distribution \(\mu_1\) (for its existence and generator function, see Appendix B) and, by Lemma B.1, we have
\[
(X_{[nt],1})_{t \in \mathbb{R}^+} \xrightarrow{D} (\mathcal{X}_{t,1})_{t \in \mathbb{R}^+} \quad \text{as } n \to \infty,
\]
where \((\mathcal{X}_{t,1})_{t \in \mathbb{R}^+}\) is an i.i.d. process (see the first paragraph of Section 2) such that for each \(t \in \mathbb{R}^+\), the distribution of \(\mathcal{X}_{t,1}\) is \(\mu_1\). We note that the index set for the weak convergence of finite dimensional distributions above is \(\mathbb{R}^+\) and not \(\mathbb{R}_+\), since \(X_{0,1} = 0\) not having the stationary distribution \(\mu_1\) unless \(\mathbb{P}(\varepsilon_{1,1} = 0) = 1\) (for more details, see Appendix B). If \(\mathbb{P}(\varepsilon_{1,1} = 0) = 1\), then \(\mathbb{P}(X_{n,1} = 0) = 1, n \in \mathbb{Z}_+\) (due to \(X_{0,1} = 0\)), and in this case the index set in question can be chosen as \(\mathbb{R}_+\) as well.

If \(a_{2,1} = 0\) holds as well, then \(\mathbb{P}(\xi_{1,1,1,2} = 0) = 1\) and (3.1) yields that in this case the second coordinate process \((X_{k,2})_{k \in \mathbb{Z}_+}\) satisfies
\[
X_{k,2} = \sum_{j=1}^{X_{k-1,2}} \xi_{k,j,2,2} + \varepsilon_{k,2}, \quad k \in \mathbb{N}.
\]
Hence if \(a_{2,1} = 0\) holds as well, then \((X_{k,2})_{k \in \mathbb{Z}_+}\) is a single-type Galton-Watson process with immigration, which is critical in cases (1) and (5), and is subcritical in case (3) due to \(\mathbb{E}(\xi_{1,1,2,2}) = a_{2,2}\).

Next we present our results on the asymptotic behaviour of \((X_k)_{k \in \mathbb{Z}_+}\) in the five cases (1)-(5) of its offspring mean matrix \(A\). The matrices \(V^{(1)}\) and \(V^{(2)}\) (introduced in Section 2) in case of \(p = 2\) will be written in the form \(V^{(1)} = (v_{i,j}^{(1)})_{i,j=1}^{2}\) and \(V^{(2)} = (v_{i,j}^{(2)})_{i,j=1}^{2}\), respectively.

3.1 Theorem. Let \((X_k)_{k \in \mathbb{Z}_+}\) be a critical decomposable 2-type Galton-Watson process with immigration such that \(X_0 = 0\), the moment conditions \(\mathbb{E}(|\xi_i|^4) < \infty, i = 1,2,\) and \(\mathbb{E}(|\varepsilon|^4) < \infty\) hold and its offspring mean matrix \(A\) satisfies (1) of (3.2). Then we have
\[
\left(\begin{pmatrix}
\frac{1}{n} X_{[nt],1} \\
\frac{1}{n} X_{[nt],2}
\end{pmatrix}\right)_{t \in \mathbb{R}^+} \xrightarrow{D} \left(\begin{pmatrix}
\mathcal{X}_{t,1} \\
\mathcal{X}_{t,2}
\end{pmatrix}\right)_{t \in \mathbb{R}^+} \quad \text{as } n \to \infty,
\]
where the limit process is the pathwise unique strong solution of the SDE
\[
\begin{align*}
\frac{d\mathcal{X}_{t,1}}{dt} &= b_1 dt + \sqrt{v_{1,1}^{(1)}} \mathcal{X}_{t,1}^+ d\mathcal{W}_{t,1}, & \quad t \in \mathbb{R}_+, \\
\frac{d\mathcal{X}_{t,2}}{dt} &= b_2 dt + \sqrt{v_{2,2}^{(2)}} \mathcal{X}_{t,2}^+ d\mathcal{W}_{t,2},
\end{align*}
\]
with initial value \((\mathcal{X}_{0,1}, \mathcal{X}_{0,2}) = (0,0)\), where \((\mathcal{W}_{t,1})_{t \in \mathbb{R}_+}\) and \((\mathcal{W}_{t,2})_{t \in \mathbb{R}_+}\) are independent standard Wiener processes yielding the independence of \((\mathcal{X}_{t,1})_{t \in \mathbb{R}_+}\) and \((\mathcal{X}_{t,2})_{t \in \mathbb{R}_+}\) as well.
3.2 Theorem. Let \((X_k)_{k \in \mathbb{Z}^+}\) be a critical decomposable 2-type Galton-Watson process with immigration such that \(X_0 = 0\), the moment condition (2.2) holds and its offspring mean matrix \(A\) satisfies (2) of (3.2). Then we have
\[
\left(\begin{array}{c}
 n^{-1}X_{[nt],1} \\
 n^{-2}X_{[nt],2}
\end{array}\right)_{t \in \mathbb{R}^+} \overset{D}{\to} \left(\begin{array}{c}
 X_{t,1} \\
 X_{t,2}
\end{array}\right)_{t \in \mathbb{R}^+}
\]
as \(n \to \infty\),
where the limit process is the pathwise unique strong solution of the SDE
\[
\left\{\begin{array}{ll}
dX_{t,1} &= b_1 \, dt + \sqrt{v_{1,1}^{(1)}} \, X_{t,1}^+ \, dW_{t,1}, \\
& \quad t \in \mathbb{R}^+,
\end{array}\right.
\]
with initial value \((X_{0,1}, X_{0,2}) = (0, 0)\), where \((W_{t,1})_{t \in \mathbb{R}^+}\) is a standard Wiener process.

In Section 4, we will compare our results in Theorem 3.2 and those of Foster and Ney [13, Section 9] in detail. Here we only note that Theorem 3.2 can be considered as a functional version of Theorems 4 and 5 in Foster and Ney [13] for some strongly critical decomposable 2-type Galton-Watson processes with immigration. In Section 4 as a consequence of Theorem 3.2 we also give a functional generalization of the Corollary on page 42 in Foster and Ney [13], which concerns the joint convergence of the appropriately normalized population size and total progeny of a critical single-type Galton–Watson process with immigration as \(n \to \infty\). We also give a stochastic representation of the limit process.

In the next corollary we describe the asymptotic behaviour of the relative frequency of individuals of type 2 with respect to individuals of type 1 under the conditions of Theorem 3.2 together with \(b_1 \in \mathbb{R}^{++}\). For different models, one can find similar results, e.g., in Jagers [19, Corollary 1] for supercritical decomposable age-dependent 2-type Galton-Watson processes without immigration, and in Yakovlev and Yanev [36, Theorem 2] for some primitive multi-type Galton-Watson processes without immigration.

3.3 Corollary. Let us suppose that the conditions of Theorem 3.2 together with \(b_1 \in \mathbb{R}^{++}\) hold. Then for all \(t \in \mathbb{R}^{++}\), we have
\[
n^{-1}1_{\{X_{[nt],1} \neq 0\}} \frac{X_{[nt],2}}{X_{[nt],1}} \overset{D}{\to} a_{2,1} \frac{\int_0^t X_{s,1} \, ds}{X_{t,1}}
\]
as \(n \to \infty\),
where \((X_{t,1})_{t \in \mathbb{R}^+}\) is the pathwise unique strong solution of the first SDE in (3.7) with initial value \(X_{0,1} = 0\).

3.4 Theorem. Let \((X_k)_{k \in \mathbb{Z}^+}\) be a critical decomposable 2-type Galton-Watson process with immigration such that \(X_0 = 0\), the moment condition (2.2) holds and its offspring mean matrix \(A\) satisfies (3) of (3.2). Then we have
\[
(n^{-1}X_{[nt],1})_{t \in \mathbb{R}^+} \overset{D}{\to} (X_{t,1})_{t \in \mathbb{R}^+}
\]
as \(n \to \infty\),
\[
\text{In Section 4, we will compare our results in Theorem 3.2 and those of Foster and Ney [13, Section 9] in detail. Here we only note that Theorem 3.2 can be considered as a functional...}
\]
where the limit process is the pathwise unique strong solution of the SDE

\( (3.8) \quad dX_{t,1} = b_t \, dt + \sqrt{v^{(1)}_{1,1} X_{t,1}^+} \, dW_{t,1}, \quad t \in \mathbb{R}_+, \)

with initial value \( X_{0,1} = 0 \), where \((W_{t,1})_{t \in \mathbb{R}_+}\) is a standard Wiener process. Further, the Markov chain \((X_{k,2})_{k \in \mathbb{Z}^+_0}\) admits a unique stationary distribution \( \mu_2 \) (for its existence and generator function, see the beginning of Appendix B) and

\( (3.9) \quad (X_{[nt],2})_{t \in \mathbb{R}_+} \xrightarrow{D_t} (X_{t,2})_{t \in \mathbb{R}_+} \quad \text{as} \quad n \to \infty, \)

where \((X_{t,2})_{t \in \mathbb{R}_+}\) is an i.i.d. process such that for each \( t \in \mathbb{R}_+ \), the distribution of \( X_{t,2} \) is \( \mu_2 \). Moreover,

\( (3.10) \quad \lim_{n_1 \to \infty} \sup_{t_1,t_2 \in \mathbb{R}_+} \sup_{n_2 \in \mathbb{N}} \left| \text{Cov}(n_1^{-1}X_{[nt_1],1},X_{[nt_2],2}) \right| = 0, \)

and

\( (3.11) \quad \lim_{n_2 \to \infty} \text{Cov}(n_1^{-1}X_{[nt_1],1},X_{[nt_2],2}) = 0, \quad t_1,t_2 \in \mathbb{R}_+, \quad n_1 \in \mathbb{N}. \)

We note that the index set for the weak convergence of finite dimensional distributions in (3.9) is \( \mathbb{R}_+ \) and not \( \mathbb{R}_+ \), since \( X_{0,2} = 0 \) not having the stationary distribution \( \mu_2 \) unless \( \mathbb{P}(\varepsilon_{1,2} = 0) = 1 \) (for more details, see Appendix B).

Note that, under the conditions of Theorem 3.4, if the two coordinates \( \varepsilon_{1,1} \) and \( \varepsilon_{1,2} \) of \( \varepsilon_1 \) are independent, then \((X_{t,1})_{t \in \mathbb{R}_+}\) and \((X_{t,2})_{t \in \mathbb{R}_+}\) are independent in Theorem 3.4, since in this special case the two coordinate processes \((X_{k,1})_{k \in \mathbb{Z}_0^+}\) and \((X_{k,2})_{k \in \mathbb{Z}_0^+}\) of \((X_k)_{k \in \mathbb{Z}_0^+}\) are independent. Motivated by this, (3.10) and (3.11), under the conditions of Theorem 3.4, we conjecture that

\[
\left( \begin{array}{c}
\frac{n^{-1}X_{[nt],1}}{n^{-1}X_{[nt],2}}
\end{array} \right)_{t \in \mathbb{R}_+} \xrightarrow{D} \left( \begin{array}{c}
X_{t,1}^\prime
X_{t,2}^\prime
\end{array} \right)_{t \in \mathbb{R}_+} \quad \text{as} \quad n \to \infty,
\]

where the driving process \((W_{t,1})_{t \in \mathbb{R}_+}\) of \((X_{t,1})_{t \in \mathbb{R}_+}\) is independent of \((X_{t,2})_{t \in \mathbb{R}_+}\), yielding the independence of \((X_{t,1})_{t \in \mathbb{R}_+}\) and \((X_{t,2})_{t \in \mathbb{R}_+}\) as well.

3.5 Theorem. Let \((X_k)_{k \in \mathbb{Z}_0^+}\) be a critical decomposable 2-type Galton-Watson process with immigration such that \( X_0 = 0 \), the moment conditions \( \mathbb{E}(|\xi_i|^4) < \infty, \quad i = 1,2, \quad \text{and} \quad \mathbb{E}(|\varepsilon|^4) < \infty \) hold and its offspring mean matrix \( A \) satisfies (4) of (3.2). Then we have

\[
\left( \begin{array}{c}
\frac{n^{-1}X_{[nt],1}}{n^{-1}X_{[nt],2}}
\end{array} \right)_{t \in \mathbb{R}_+} \xrightarrow{D} \left( \begin{array}{c}
X_{t,1}^\prime
X_{t,2}^\prime
\end{array} \right)_{t \in \mathbb{R}_+} \quad \text{as} \quad n \to \infty,
\]

where the limit process is the pathwise unique strong solution of the SDE

\( (3.12) \quad \left\{ \begin{array}{l}
dX_{t,1} = b_t \, dt + \sqrt{v^{(1)}_{1,1} X_{t,1}^+} \, dW_{t,1},
\quad t \in \mathbb{R}_+, \\
dX_{t,2} = \frac{a_{2,1}}{1-a_{2,2}} \, dX_{t,1},
\end{array} \right. \)

with initial value \((X_{0,1},X_{0,2}) = (0,0)\), where \((W_{t,1})_{t \in \mathbb{R}_+}\) is a standard Wiener process.
3.6 Remark. If the conditions of Theorem 3.3 hold together with \( a_{2,1} = 1 \) and \( a_{2,2} = 0 \), then
\[
X_{[nt],2}^{(1)} = \sum_{j=1}^{[nt]-1} (M_{j,1} + b_1) = \sum_{j=1}^{[nt]-1} (X_{j,1} - X_{j-1,1}) = X_{[nt]-1,1}^{(1)} \text{ and } X_{[nt],2}^{(2)} = M_{[nt],2} + b_2
\]
for \( n \in \mathbb{N} \) and \( t \in \mathbb{R}_+ \) (see (5.1) and (5.4)). So (3.1) and Theorem 3.5 yield that
\[
\left( \begin{array}{c}
\left[ n^{-1}X_{[nt],1} \\
n^{-1}X_{[nt],2}
\end{array} \right]_{t \in \mathbb{R}_+}
\right) \overset{D}{\rightarrow} \left( \begin{array}{c}
\left[ n^{-1}X_{[nt],1} \\
n^{-1}X_{[nt]-1,1} + n^{-1}(M_{[nt],2} + b_2)
\end{array} \right]_{t \in \mathbb{R}_+}
\right)
\]
as \( n \to \infty \), where \((X_{t,1})_{t \in \mathbb{R}_+}\) is the pathwise unique strong solution of the first SDE in (3.12) with initial value \( X_{0,1} = 0 \). \( \square \)

3.7 Theorem. Let \((X_k)_{k \in \mathbb{Z}_+}\) be a critical decomposable 2-type Galton-Watson process with immigration such that \( X_0 = 0 \), the moment conditions \( \mathbb{E}(\|\xi_i\|^4) < \infty \), \( i = 1, 2 \), and \( \mathbb{E}(\|\varepsilon\|^4) < \infty \) hold and its offspring mean matrix \( \mathbf{A} \) satisfies (5) of (3.2). Then the Markov chain \((X_k)_{k \in \mathbb{Z}_+}\) admits a unique stationary distribution \( \mu_1 \) (for its existence and generator function, see the beginning of Appendix B) and
\[
(X_{[nt],1})_{t \in \mathbb{R}_+} \overset{D}{\rightarrow} (X_{t,1})_{t \in \mathbb{R}_+} \text{ as } n \to \infty,
\]
where \((X_{t,1})_{t \in \mathbb{R}_+}\) is an i.i.d. process such that for each \( t \in \mathbb{R}_+ \), the distribution of \( X_{t,1} \) is \( \mu_1 \). Further, we have
\[
(n^{-1}X_{[nt],2})_{t \in \mathbb{R}_+} \overset{D}{\rightarrow} (X_{t,2})_{t \in \mathbb{R}_+} \text{ as } n \to \infty,
\]
where the limit process is the pathwise unique strong solution of the SDE
\[
dX_{t,2} = \left( \frac{a_{2,1}}{1-a_{1,1}} b_1 + b_2 \right) dt + \sqrt{\nu_{2,2}^{(2)}} dW_{t,2}, \quad t \in \mathbb{R}_+,
\]
with initial value \( X_{0,2} = 0 \), where \((W_{t,2})_{t \in \mathbb{R}_+}\) is a standard Wiener process. Moreover,
\[
\lim_{n_2 \to \infty} \sup_{t_1, t_2 \in \mathbb{R}_+} \sup_{n_1 \in \mathbb{N}} \left| \text{Cov}(X_{[nt_1],1}, n_2^{-1}X_{[nt_2],2}) \right| = 0,
\]
and
\[
\lim_{n_1 \to \infty} \text{Cov}(X_{[nt_1],1}, n_2^{-1}X_{[nt_2],2}) = 0, \quad t_1, t_2 \in \mathbb{R}_+, \quad n_2 \in \mathbb{N}.
\]

We remark that the index set for the weak convergence of finite dimensional distributions in (3.13) is \( \mathbb{R}_+ \) and not \( \mathbb{R}_+ \), since \( X_{0,1} = 0 \) not having the stationary distribution \( \mu_1 \) unless \( \mathbb{P}(\varepsilon_{1,1} = 0) = 1 \) (for more details, see Appendix B). Further, note that the parameters \( a_{1,1} \) and \( b_1 \) related to the first coordinate process \((X_k)_{k \in \mathbb{Z}_+}\) appear in the drift coefficient of the SDE (3.14) for \((X_{t,2})_{t \in \mathbb{R}_+}\), which is the limit process corresponding to the second coordinate process \((X_k)_{k \in \mathbb{Z}_+}\). It can be considered as a consequence of the decomposition \( X_{k,2} = \sum_{t=1}^k (M_{t,2} + a_{2,1} X_{t-1,2} + b_2) \), \( k \in \mathbb{N} \) (see (10.1)), where \( k^{-1} \sum_{t=1}^k X_{t-1,2} \) converges in probability to \( b_1/(1-a_{1,1}) \) as \( k \to \infty \) (see (10.6)). Moreover, note that if \( a_{2,1} = 0 \) in Theorem 3.7 and if we switch the two coordinate processes, then we get back Theorem 3.3.
under fourth order moment assumptions on the offspring and immigration distributions. The question of joint convergence of the two coordinate processes in Theorem 3.7 remains open.

In the next remark we discuss the role of fourth order moment conditions in Theorems 3.1, 3.5 and 3.7.

3.8 Remark. We suspect that the moment conditions in Theorems 3.1 and 3.7 might be relaxed to $E(\|\xi_i\|^2) < \infty$, $i = 1, 2$, and $E(\|\varepsilon\|^2) < \infty$ using the method of the proof of Theorem 3.1 in Barczy et al. [3]. For Theorem 3.7 in the special case $a_{2,1} = 0$, it follows by Theorem 3.4 (by switching the two coordinate processes). In fact, the fourth order moment assumptions in the proofs of Theorems 3.1 and 3.7 are used only for checking the conditional Lindeberg condition, namely, condition (iii) of Theorem D.1 in order to prove convergence of some random step processes towards a diffusion process. For single-type critical Galton-Watson processes with immigration, a detailed exposition of a proof of the conditional Lindeberg condition in question under second order moment assumptions can be found, e.g., in Barczy et al. [2]. The fourth order moment conditions in Theorem 3.5 come into play in another way, namely, via the estimation of tail probabilities of the maximum of a stable AR(1) process with heteroscedastic innovations $(M_{k,i})_{k\in\mathbb{N}}, i = 1, 2$, that are martingale differences formed from the coordinate processes of the branching process in question. Our technique is not suitable for relaxing them to second order ones, and we do not know any other technique (for more details, see the proof of Theorem 3.5). 

4 Comparison of Theorem 3.2 and some results of Foster and Ney [13]

Under the conditions of Theorem 3.2 together with $v_{1,1}^{(1)} \in \mathbb{R}^{+}$, Foster and Ney [13] Theorems 4 and 5, and formula (9.11)) proved that

$$\left(n^{-1}X_{n,1}, n^{-2}X_{n,2}\right) \overset{D}{\longrightarrow} (Y_1, Y_2) \text{ as } n \to \infty, \tag{4.1}$$

where the Laplace transform of $(Y_1, Y_2)$ takes the form

$$E(e^{-s_1Y_1-s_2Y_2}) = \exp\left\{-b_1\int_0^1 \sqrt{2a_{1,2}s_1} \frac{1}{2}v_{1,1}^{(1)}s_1 + \sqrt{\frac{1}{2}v_{1,1}^{(1)}a_{1,2}s_2} \tanh \left(\tau \sqrt{\frac{1}{2}v_{1,1}^{(1)}a_{1,2}s_2}\right) d\tau\right\} \tag{4.2}$$

for $s_1 \in \mathbb{R}^{+}$ and $s_2 \in \mathbb{R}^{++}$, where we recall $b_1 = E(\varepsilon_{1,1})$ and $v_{1,1}^{(1)} = \text{Var}(\xi_{1,1,1,1})$. They also derived the Laplace transforms of the marginal distributions $Y_1$ and $Y_2$, respectively. Namely,

$$E(e^{-s_1Y_1}) = \left(1 + \frac{1}{2}v_{1,1}^{(1)}s_1\right)^{-\frac{2b_1}{v_{1,1}^{(1)}}}, \quad s_1 \in \mathbb{R}^{+},$$
yielding that $Y_1$ is Gamma-distributed with parameters $\frac{v_{1,1}^{(1)}}{2}$ and $\frac{2b_1}{v_{1,1}^{(1)}}$, and

$$
\mathbb{E}(e^{-s_2 Y_2}) = \left( \cosh \left( \sqrt{\frac{1}{2} v_{1,1}^{(1)} a_{1,2} s_2} \right) \right)^{-\frac{2b_1}{v_{1,1}^{(1)}}}, \quad s_2 \in \mathbb{R}_+.
$$

As a consequence of Theorem 3.2, the distribution of $(Y_1, Y_2)$ coincides with that of $(X_{1,1}, X_{1,2}) = (X_{1,1}, a_{2,1} \int_0^1 X_{1,2} \, du)$, where $(X_{1,1})_{t \in \mathbb{R}_+}$ is given as the pathwise unique strong solution of the first SDE in (3.7) with initial value $X_{0,1} = 0$. We check that

$$
\mathbb{E}(e^{-s_1 Y_1 - s_2 Y_2}) = \mathbb{E}(e^{-s_1 X_{1,1} - s_2 a_{2,1} f_0^1 X_{1,1} \, du})
$$

(4.3)

$$
= \left( \cosh \left( \sqrt{\frac{v_{1,1}^{(1)} a_{2,1} s_2}{2}} \right) + \frac{s_1 \sqrt{v_{1,1}^{(1)}}}{\sqrt{2a_{2,1}s_2}} \sinh \left( \sqrt{\frac{v_{1,1}^{(1)} a_{2,1} s_2}{2}} \right) \right)^{-\frac{2b_1}{v_{1,1}^{(1)}}},
$$

where $s_1 \in \mathbb{R}_+$ and $s_2 \in \mathbb{R}_{++}$, by giving a closed formula for the Laplace transform $\mathbb{E}(e^{-s_1 Y_1 - s_2 Y_2})$ given in (4.2). First, recall that for all $\nu \in [-1, \infty)$, the pathwise unique strong solution of the SDE

$$
\begin{aligned}
\frac{dX_t}{dt} &= (2\nu + 2) \, dt + \sqrt{4\nu^2 + 1} \, dW_t, \quad t \in \mathbb{R}_+, \\
X_0 &= 0,
\end{aligned}
$$

is called a squared Bessel process with parameter $\nu$, where $(W_t)_{t \in \mathbb{R}_+}$ is a standard Wiener process, and for $\nu \in (-1, 0)$, we have

$$
\mathbb{E}(e^{-\alpha X_t - \frac{\beta^2}{2} f_0^t X_t \, ds}) = \left( \cosh(\beta t) + \frac{2\alpha}{\beta} \sinh(\beta t) \right)^{-\nu-1}, \quad t \in \mathbb{R}_+, 
$$

(4.4)

where $\alpha \in \mathbb{R}_+$ and $\beta \neq 0$, $\beta \in \mathbb{R}$, see, e.g., Borodin and Salminen [8, pages 76 and 135]. Since $v_{1,1}^{(1)} \in \mathbb{R}_{++}$, we can introduce $\tilde{X}_{t,1} := \frac{4}{v_{1,1}^{(1)}} X_{t,1}$, $t \in \mathbb{R}_+$, where $(\tilde{X}_{t,1})_{t \in \mathbb{R}_+}$ is given as the pathwise unique strong solution of the first SDE in (3.7) with initial value $\tilde{X}_{0,1} = 0$. Then $(\tilde{X}_{t,1})_{t \in \mathbb{R}_+}$ is the pathwise unique strong solution of the SDE

$$
\begin{aligned}
\frac{d\tilde{X}_{t,1}}{dt} &= \frac{4}{v_{1,1}^{(1)}} b_1 \, dt + \sqrt{4\tilde{X}_{t,1}^2} \, dW_t, \quad t \in \mathbb{R}_+, \\
\tilde{X}_{0,1} &= 0,
\end{aligned}
$$

so $(\tilde{X}_{t,1})_{t \in \mathbb{R}_+}$ is a squared Bessel process with parameter $\frac{2b_1}{v_{1,1}^{(1)}} - 1$, and using (4.4) we have

$$
\mathbb{E}(e^{-\tilde{s}_1 \tilde{X}_{t,1} - \tilde{s}_2 \int_0^t \tilde{X}_{1,1} \, du}) = \left( \cosh(\sqrt{2\tilde{s}_2} t) + \frac{2\tilde{s}_1}{\sqrt{2}\tilde{s}_2} \sinh(\sqrt{2\tilde{s}_2} t) \right)^{-\frac{2b_1}{v_{1,1}^{(1)}}}, \quad t \in \mathbb{R}_+, 
$$

where $t \in \mathbb{R}_+$, $\tilde{s}_1 \in \mathbb{R}_+$ and $\tilde{s}_2 \in \mathbb{R}_{++}$. This readily implies (4.3).
For historical fidelity, we remark that Foster and Ney [13] proved (4.1) using the finiteness of $\mathbb{E}(\|\varepsilon\|)$ instead of that of $\mathbb{E}(\|\varepsilon\|^2)$ as we supposed in Theorem 3.2.

Finally, we give a functional generalization of the Corollary on page 42 in Foster and Ney [13]. Let us suppose that the conditions of Theorem 3.2 hold. Then, by the proof of this theorem (see (7.2)),

\[(4.5) \quad \left( \begin{bmatrix} n^{-1}X_{[nt],1} \\ n^{-2}\sum_{j=1}^{[nt]-1}X_{j,1} \end{bmatrix} \right)_{t \in \mathbb{R}_+} \xrightarrow{\mathcal{D}} \left( \begin{bmatrix} \mathcal{X}_{t,1} \\ \int_0^t \mathcal{X}_{s,1}\, ds \end{bmatrix} \right)_{t \in \mathbb{R}_+} \]

as $n \to \infty$, where $(\mathcal{X}_{t,1})_{t \in \mathbb{R}_+}$ is the pathwise unique strong solution of the first SDE in (3.7) with initial value $X_{0,1} = 0$, since $X^{(1)}_{[nt],2} = \sum_{j=1}^{[nt]-1}X_{j,1}$, $n \in \mathbb{N}$, $t \in \mathbb{R}_+$, where $X^{(1)}_{[nt],2}$ is defined by (5.4) and the equality in question follows by (7.1). Note that, by (3.3), $(X_{k,1})_{k \in \mathbb{Z}_+}$ is a single-type critical Galton-Watson process with immigration due to $a_{1,1} = 1$, and \(\sum_{j=1}^{[nt]-1}X_{j,1}\) is the total progeny of individuals of type 1 up to time $[nt] - 1$. So (4.5) gives us a functional generalization of the Corollary on page 42 in Foster and Ney [13] together with a stochastic representation of the limit process as well. We mention that in the considered special case, $\sum_{j=1}^{[nt]-1}X_{j,1} = a_{1,1}^{(1)}(X_{[nt],2} - X^{(2)}_{[nt],2})$, $t \in \mathbb{R}_+$, where $X^{(2)}_{[nt],2}$ is given by (5.4) and, by (7.3), $\sup_{t \in [0,T]}|n^{-2}X^{(2)}_{[nt],2}| \xrightarrow{p} 0$ as $n \to \infty$ for all $T \in \mathbb{R}_+$. In the end, we remark that one can alternatively derive (4.5) directly from the statement of Theorem 3.2. Namely, if $\mathbb{P}(\xi_{k,j,1,1} = 1) = 1$, $\mathbb{P}(\xi_{k,j,2,2} = 1) = 1$ and $\mathbb{P}(\varepsilon_{k,2} = 0) = 1$ for each $k, j \in \mathbb{N}$, then $X_{k,2} = X_{k-1,1} + X_{k-1,2} = \sum_{j=1}^{k-1}X_{j,1}$ for each $k \in \mathbb{N}$ almost surely, where $(X_{k,1})_{k \in \mathbb{Z}_+}$ is a single-type Galton-Watson process with immigration and $a_{2,1} = 1$. Consequently, if, in addition, $\mathbb{E}(\xi_{1,1,1,1}) = a_{1,1} = 1$, then Theorem 3.2 directly yields (4.5).

5 Preliminaries for the proofs: decompositions

Let us introduce the sequence

\[
\begin{bmatrix} M_{k,1} \\ M_{k,2} \end{bmatrix} := M_k := X_k - \mathbb{E}(X_k | \mathcal{F}_{k-1}^X) = X_k - AX_{k-1} - b
\]

\[(5.1) = \begin{bmatrix} X_{k,1} - a_{1,1}X_{k-1,1} - b_1 \\ X_{k,2} - a_{2,1}X_{k-1,1} - a_{2,2}X_{k-1,2} - b_2 \end{bmatrix}, \quad k \in \mathbb{N},
\]

of martingale differences with respect to the filtration $(\mathcal{F}_k^X)_{k \in \mathbb{Z}_+}$, where we used (2.3). From (5.1), we obtain the recursion

\[
X_k = AX_{k-1} + M_k + b, \quad k \in \mathbb{N},
\]

which together with $X_0 = 0$ implies

\[
X_k = \sum_{j=1}^{k} A^{k-j}(M_j + b), \quad k \in \mathbb{N}.
\]
Indeed, since $X_0 = 0$, we have $X_1 = M_1 + b$, and, by induction, for all $k \in \mathbb{N}$,

$$X_{k+1} = AX_k + M_{k+1} + b = A \sum_{j=1}^{k} A^{k-j}(M_j + b) + M_{k+1} + b$$

$$= \sum_{j=1}^{k} A^{k+1-j}(M_j + b) + M_{k+1} + b = \sum_{j=1}^{k+1} A^{k+1-j}(M_j + b).$$

For each $\ell \in \mathbb{N}$, we have

$$A^\ell = \begin{bmatrix} a_{1,1}^\ell & 0 \\ a_{2,1} \sum_{i=1}^{\ell} a_{2,2}^{-i} a_{1,1}^{-i} & a_{2,2}^\ell \end{bmatrix},$$

where

$$\sum_{i=1}^{\ell} a_{2,2}^{-i} a_{1,1}^{-i} = \begin{cases} \frac{a_{1,1}^\ell - a_{2,2}^\ell}{a_{1,1} - a_{2,2}} & \text{if } a_{1,1} \neq a_{2,2}, \\ \ell a_{1,1}^{\ell-1} & \text{if } a_{1,1} = a_{2,2}. \end{cases}$$

Indeed, by induction, for all $\ell \in \mathbb{N}$, we have

$$A^{\ell+1} = A^\ell A = \begin{bmatrix} a_{1,1}^\ell & 0 \\ a_{2,1} \sum_{i=1}^{\ell} a_{2,2}^{-i} a_{1,1}^{-i} & a_{2,2}^\ell \end{bmatrix} \begin{bmatrix} a_{1,1} & 0 \\ a_{2,1} & a_{2,2} \end{bmatrix} = \begin{bmatrix} a_{1,1}^{\ell+1} & 0 \\ a_{2,1} \sum_{i=1}^{\ell} a_{2,2}^{-i} a_{1,1}^{-i} + a_{2,1} a_{2,2}^\ell & a_{2,2}^{\ell+1} \end{bmatrix},$$

and, if $a_{1,1} \neq a_{2,2}$, then

$$\sum_{i=1}^{\ell} a_{2,2}^{-i} a_{1,1}^{-i} = a_{2,2} \sum_{i=1}^{\ell} \frac{a_{2,2}}{a_{1,1}} = a_{2,2} a_{1,1}^{\ell-1} = \frac{a_{1,1}^\ell - a_{2,2}^\ell}{a_{1,1} - a_{2,2}}.$$

Note that (5.3) holds for $\ell = 0$ as well with the convention $\sum_{i=1}^{0} := 0$. Consequently, by (5.2), we get a decomposition

$$\begin{bmatrix} X_{k,1} \\ X_{k,2} \end{bmatrix} = \begin{bmatrix} X_{k,1} \\ a_{2,1} X_{k,2}^{(1)} + X_{k,2}^{(2)} \end{bmatrix}, \quad k \in \mathbb{N},$$

where

$$X_{k,1} = \sum_{j=1}^{k} a_{1,1}^{k-j}(M_{j,1} + b_1),$$

18
\[ X_{k,2}^{(1)} := \begin{cases} \sum_{j=1}^{k} a_{1,1}^{k-j} - a_{2,2}^{k-j} (M_{j,1} + b_1) & \text{if } a_{1,1} \neq a_{2,2}, \\ \sum_{j=1}^{k} (k-j) a_{1,1}^{k-j-1} (M_{j,1} + b_1) & \text{if } a_{1,1} = a_{2,2}, \end{cases} \]

\[ X_{k,2}^{(2)} := \sum_{j=1}^{k} a_{2,2}^{k-j} (M_{j,2} + b_2). \]

6 Proof of Theorem 3.1

The SDE (3.6) has a pathwise unique strong solution \( (\mathcal{X}_t := (X_{t,1}, X_{t,2})^\top)_{t \in \mathbb{R}_+} \) for all initial values \( \mathcal{X}_0 = x \in \mathbb{R}^2 \), and if \( x \in \mathbb{R}_+^2 \), then \( \mathcal{X}_t \in \mathbb{R}_+^2 \) almost surely for all \( t \in \mathbb{R}_+ \) since \( b_1, b_2, v_{1,1}^{(1)}, v_{2,2}^{(2)} \in \mathbb{R}_+ \), see, e.g., Ikeda and Watanabe [15, Chapter IV, Example 8.2]. Since \( a_{1,1} = 1 \), \( a_{2,1} = 0 \) and \( a_{2,2} = 1 \), as it was explained in Section 3, the second coordinate process \( (X_{k,2})_{k \in \mathbb{Z}_+} \) is a critical single-type Galton-Watson process with immigration, so \( (n^{-1} X_{[nt],2})_{t \in \mathbb{R}_+} \xrightarrow{d} (X_{t,2})_{t \in \mathbb{R}_+} \) as \( n \to \infty \), where \( (\mathcal{X}_{t,1})_{t \in \mathbb{R}_+} \) satisfies the first equation of the SDE (3.6) with initial value \( \mathcal{X}_{0,1} = 0 \). Similarly, since \( a_{2,1} = 0 \) and \( a_{2,2} = 1 \), we need to prove joint convergence of \( (n^{-1} X_{[nt],1})_{t \in \mathbb{R}_+} \) and \( (n^{-1} X_{[nt],2})_{t \in \mathbb{R}_+} \) as \( n \to \infty \).

Using \( a_{1,1} = 1, a_{2,1} = 0, a_{2,2} = 1 \) and (5.1), we obtain that the sequence \( (M_k)_{k \in \mathbb{N}} \) of martingale differences with respect to the filtration \( (\mathcal{F}_k^X)_{k \in \mathbb{Z}_+} \) takes the form

\[ M_k = X_k - X_{k-1} - b, \quad k \in \mathbb{N}. \]

Consider the random step processes

\[ \mathcal{M}_t^{(n)} := \begin{bmatrix} M_{t,1}^{(n)} \\ M_{t,2}^{(n)} \end{bmatrix} := \frac{1}{n} \sum_{k=1}^{[nt]} M_k = \frac{1}{n} X_{[nt]} - \frac{[nt]}{n} b, \quad t \in \mathbb{R}_+, \quad n \in \mathbb{N}, \]

where we used that \( X_0 = 0 \). We show that

\[ (\mathcal{M}_t^{(n)})_{t \in \mathbb{R}_+} \xrightarrow{d} (\mathcal{M}_t)_{t \in \mathbb{R}_+} \quad \text{as } n \to \infty, \]

where the limit process \( \mathcal{M}_t = (\mathcal{M}_{t,1}, \mathcal{M}_{t,2})^\top, t \in \mathbb{R}_+ \), is the pathwise unique strong solution of the SDE

\[ \begin{cases} d \mathcal{M}_{t,1} = \sqrt{v_{1,1}^{(1)}(\mathcal{M}_{t,1} + b_1 t)} + d \mathcal{W}_{t,1}, & t \in \mathbb{R}_+, \\ d \mathcal{M}_{t,2} = \sqrt{v_{2,2}^{(2)}(\mathcal{M}_{t,2} + b_2 t)} + d \mathcal{W}_{t,2}, & t \in \mathbb{R}_+ \end{cases} \]

with initial value \( (\mathcal{M}_{0,1}, \mathcal{M}_{0,2}) = (0,0) \). In order to prove (6.2), we want to apply Theorem D.1 for \( d = r = 2, \mathcal{U} = \mathcal{M}, U_k^{(n)} = n^{-1} M_k, n, k \in \mathbb{N}, U_0^{(n)} = 0, n \in \mathbb{N}, \mathcal{F}_k^{(n)} = \mathcal{F}_k^X, n \in \mathbb{N}, \)
for \( t \in \mathbb{R}_+ \) and \( x = (x_1, x_2)^T \in \mathbb{R}^2 \). First we check that the SDE (6.3) has a pathwise unique strong solution \((M_t^{(x)})_{t \in \mathbb{R}_+}\) for all initial values \(M_0^{(x)} = x \in \mathbb{R}^2\). Observe that if \((M_t^{(x)})_{t \in \mathbb{R}_+}\) is a strong solution of the SDE (6.3) with initial value \(M_0^{(x)} = x \in \mathbb{R}^2\), then, by Itô’s formula, the process \((P_{t,1}, P_{t,2}) := (M_t^{(x)} + bt, t \in \mathbb{R}_+,\) is a pathwise unique strong solution of the SDE

\[
\begin{align*}
\begin{cases}
\frac{dP_{t,1}}{dt} &= b_1 dt + \sqrt{v_{1,1}^{(1)}} P_{t,1}^{+} dW_{t,1}, & t \in \mathbb{R}_+,
\frac{dP_{t,2}}{dt} &= b_2 dt + \sqrt{v_{2,2}^{(2)}} P_{t,2}^{+} dW_{t,2}, & t \in \mathbb{R}_+,
\end{cases}
\end{align*}
\]

with initial value \((P_{0,1}, P_{0,2})^T = x\). Conversely, if \((P_{t,1}^{(p)}, P_{t,2}^{(p)})^T, t \in \mathbb{R}_+,\) is a strong solution of the SDE (6.4) with initial value \((P_{0,1}^{(p)}, P_{0,2}^{(p)})^T = p \in \mathbb{R}^2\), then, by Itô’s formula, the process \(M_t := (P_{t,1}^{(p)}, P_{t,2}^{(p)})^T - bt, t \in \mathbb{R}_+,\) is a strong solution of the SDE (6.3) with initial value \(M_0 = p\). The equations in (6.4) are the same as in (3.6). Consequently, as it was explained at the beginning of the proof, the SDE (6.4) and hence the SDE (6.3) as well admits a unique strong solution with arbitrary initial value in \(\mathbb{R}^2\), and \((M_t + bt)_{t \in \mathbb{R}_+} \overset{P}{=} (X_t)_{t \in \mathbb{R}_+}\).

The convergence \(U_t^{(n)} \overset{P}{\to} 0\) as \(n \to \infty\), and condition (i) of Theorem [D.1] trivially holds (since \(\mathbb{E}(M_k | F_{k-1}) = 0, \ k \in \mathbb{N},\) and \(\beta(t, x) = 0, t \in \mathbb{R}_+, x \in \mathbb{R}^2\)). Now, we show that conditions (ii) and (iii) of Theorem [D.1] hold. We have to check that for each \(T \in \mathbb{R}_++,

\[
\begin{align*}
\sup_{t \in [0,T]} \left\| \frac{1}{n^2} \sum_{k=1}^{\lfloor nt \rfloor} \mathbb{E}(M_k M_k^T | F_{k-1}) - \int_0^t \mathcal{R}_s^{(n)} V \xi \ ds \right\|_2 \overset{P}{\to} 0 & \quad \text{as } n \to \infty,
\end{align*}
\]

\[
\begin{align*}
\sup_{t \in [0,T]} \left\| \frac{1}{n^2} \sum_{k=1}^{\lfloor nT \rfloor} \mathbb{E}(\|M_k\|^2 1_{\{|M_k| > \theta\}} | F_{k-1}) \right\|_2 \overset{P}{\to} 0 & \quad \text{as } n \to \infty \quad \text{for all } \theta \in \mathbb{R}_+,
\end{align*}
\]

where the process \((\mathcal{R}_s^{(n)})_{s \in \mathbb{R}_+}\) and the matrix \(V \xi\) are defined by

\[
\mathcal{R}_s^{(n)} := \begin{bmatrix}
(M_{s,1}^{(n)} + b_1 t)^+ & 0 \\
0 & (M_{s,2}^{(n)} + b_2 t)^+
\end{bmatrix}, \quad s \in \mathbb{R}_+, \quad n \in \mathbb{N},
\]

\[
V \xi := \begin{bmatrix}
v_{1,1}^{(1)} & 0 \\
0 & v_{2,2}^{(2)}
\end{bmatrix}.
\]
Indeed, for \( t \in \mathbb{R}_+ \) and \( x \in \mathbb{R}^2 \),

\[
\gamma(t, x)\gamma(t, x)^\top = \begin{bmatrix} \sqrt{(x_1 + b_1 t)^+} & 0 \\ 0 & \sqrt{(x_2 + b_2 t)^+} \end{bmatrix} \begin{bmatrix} v_{1,1}^{(1)} & v_{1,1}^{(2)} \\ 0 & v_{2,2}^{(2)} \end{bmatrix} \begin{bmatrix} \sqrt{(x_1 + b_1 t)^+} & 0 \\ 0 & \sqrt{(x_2 + b_2 t)^+} \end{bmatrix} = \begin{bmatrix} v_{1,1}^{(1)}(x_1 + b_1 t)^+ & 0 \\ 0 & v_{2,2}^{(2)}(x_2 + b_2 t)^+ \end{bmatrix} = V_\xi \begin{bmatrix} (x_1 + b_1 t)^+ & 0 \\ 0 & (x_2 + b_2 t)^+ \end{bmatrix}.
\]

For each \( s \in \mathbb{R}_+ \) and \( n \in \mathbb{N} \), we have

\[
\mathcal{M}_{s}^{(n)} + bs = \frac{1}{n}X_{[ns]} + \frac{ns - \lfloor ns \rfloor}{n}b,
\]

thus

\[
\mathcal{R}_{s}^{(n)} = \left[ \begin{array}{cc} \mathcal{M}_{s,1}^{(n)} + bt & 0 \\ 0 & \mathcal{M}_{s,2}^{(n)} + b_2t \end{array} \right], \quad s \in \mathbb{R}_+, \quad n \in \mathbb{N},
\]

and hence

\[
\int_0^t \mathcal{R}_{s}^{(n)} ds = \frac{1}{n^2} \sum_{k=0}^{\lfloor nt \rfloor - 1} \left[ X_{k,1} 0 \\ 0 X_{k,2} \right] + \frac{nt - \lfloor nt \rfloor}{n^2} \left[ X_{\lfloor nt \rfloor,1} 0 \\ 0 X_{\lfloor nt \rfloor,2} \right] + \frac{\lfloor nt \rfloor + (nt - \lfloor nt \rfloor)^2}{2n^2} \left[ b_1 0 \\ 0 b_2 \right], \quad t \in \mathbb{R}_+, \quad n \in \mathbb{N},
\]

as, e.g., in the proof of Theorem 1.1 in Barczy et al. [2]. By Lemma \( A.1 \)

\[
\frac{1}{n^2} \sum_{k=1}^{\lfloor nt \rfloor} \mathbb{E}(M_k M_k^\top | \mathcal{F}_{k-1}^X) = \frac{\lfloor nt \rfloor}{n^2} \mathbf{V}^{(0)} + \frac{1}{n^2} \sum_{k=1}^{\lfloor nt \rfloor} (X_{k-1,1} \mathbf{V}^{(1)} + X_{k-1,2} \mathbf{V}^{(2)})
\]

for all \( t \in \mathbb{R}_+ \) and \( n \in \mathbb{N} \). Since \( \xi_{1,1,2,1} \overset{a.s.}{=} 0 \) and \( \xi_{1,1,1,2} \overset{a.s.}{=} 0 \) (due to \( a_{1,2} = a_{2,1} = 0 \)), we have \( v_{1,1}^{(2)} = v_{1,2}^{(2)} = v_{2,1}^{(2)} = 0 \) and \( v_{1,1}^{(1)} = v_{1,2}^{(1)} = v_{2,1}^{(1)} = 0 \), and consequently

\[
X_{k-1,1} \mathbf{V}^{(1)} + X_{k-1,2} \mathbf{V}^{(2)} = X_{k-1,1} \begin{bmatrix} v_{1,1}^{(1)} & 0 \\ 0 & 0 \end{bmatrix} + X_{k-1,2} \begin{bmatrix} 0 & 0 \\ 0 & v_{2,2}^{(2)} \end{bmatrix} = \begin{bmatrix} X_{k-1,1} & 0 \\ 0 & X_{k-1,2} \end{bmatrix} \mathbf{V}_\xi, \quad k \in \mathbb{N}.
\]

So

\[
\frac{1}{n^2} \sum_{k=1}^{\lfloor nt \rfloor} \mathbb{E}(M_k M_k^\top | \mathcal{F}_{k-1}^X) - \int_0^t \mathcal{R}_{s}^{(n)} V_\xi ds = \frac{\lfloor nt \rfloor}{n^2} \mathbf{V}^{(0)} - \frac{nt - \lfloor nt \rfloor}{n^2} \left[ X_{\lfloor nt \rfloor,1} 0 \\ 0 X_{\lfloor nt \rfloor,2} \right] V_\xi - \frac{\lfloor nt \rfloor + (nt - \lfloor nt \rfloor)^2}{2n^2} \left[ b_1 0 \\ 0 b_2 \right] V_\xi
\]

21
for $t \in \mathbb{R}_+$ and $n \in \mathbb{N}$. Hence, in order to show (6.5), by Slutsky’s lemma and taking into account the facts that for each $T \in \mathbb{R}_{++}$,

$$\sup_{t \in [0, T]} \frac{|nt| + (nt - |nt|)^2}{n^2} \leq \sup_{t \in [0, T]} \frac{|nt| + 1}{n^2} \to 0 \quad \text{as } n \to \infty,$$

and $\sup_{t \in [0, T]} \frac{|nt|}{n} V^{(0)} \xrightarrow{a.s.} 0$ as $n \to \infty$, it suffices to prove that for each $T \in \mathbb{R}_{++}$, we have

$$\frac{1}{n^2} \sup_{t \in [0, T]} \| (nt - |nt|)X_{|nt|} \| \leq \frac{1}{n^2} \sup_{t \in [0, T]} \| X_{|nt|} \| \xrightarrow{p} 0 \quad \text{as } n \to \infty. \quad (6.7)$$

For each $k \in \mathbb{N}$, we have $X_k = X_{k-1} + M_k + b$, and thus, using $X_0 = 0$,

$$X_k = \sum_{j=1}^{k} M_j + kb,$$

hence, for each $t \in \mathbb{R}_+$ and $n \in \mathbb{N}$, we get

$$\| X_{|nt|} \| \leq \sum_{j=1}^{|nt|} \| M_j \| + |nt|\| b\|.$$

Consequently, in order to prove (6.7), it suffices to show that for each $T \in \mathbb{R}_{++}$,

$$\frac{1}{n^2} \sum_{j=1}^{|nT|} \| M_j \| \xrightarrow{p} 0 \quad \text{as } n \to \infty.$$

By Lemma A.2, $\mathbb{E}(X_{k,i}) = O(k)$ for $k \in \mathbb{N}$, $i = 1, 2$, hence, by Lemma A.1, we get

$$\mathbb{E}(\| M_j \|) \leq \sqrt{\mathbb{E}(\| M_j \|^2)} = \sqrt{\mathbb{E}(M_j^\top M_j)} = \sqrt{\mathbb{E}(\text{tr}(M_j^\top M_j))} = \sqrt{\text{tr}(\mathbb{E}(M_j^\top M_j))} = \sqrt{\text{tr}(V^{(0)} + \mathbb{E}(X_{j-1,1})V^{(1)} + \mathbb{E}(X_{j-1,2})V^{(2)})} = O(j^{1/2}), \ j \in \mathbb{N}.$$

Thus for each $T \in \mathbb{R}_{++}$,

$$\mathbb{E}\left(\frac{1}{n^2} \sum_{j=1}^{|nT|} \| M_j \|\right) = \frac{1}{n^2} \sum_{j=1}^{|nT|} O(j^{1/2}) = O(n^{-1/2}) \quad \text{for } n \in \mathbb{N},$$

and consequently we obtain (6.7), and hence (6.5).

Next, we check condition (6.6). We show that for each $T \in \mathbb{R}_+$ and $\theta \in \mathbb{R}_{++}$,

$$\frac{1}{n^2} \sum_{k=1}^{|nT|} \mathbb{E}(\| M_k \|^2 I_{\{\| M_k \| > n\theta\}} | \mathcal{F}_{k-1}^X) \xrightarrow{L_1} 0 \quad \text{as } n \to \infty.$$
By Markov’s inequality and Lemma A.2, for each $T \in \mathbb{R}_{++}$ and $\theta \in \mathbb{R}_{++}$, we have
\[
\mathbb{E} \left( \frac{1}{n^2} \sum_{k=1}^{\lfloor nT \rfloor} \mathbb{E} \left( \| M_k \|^2 \mathbb{1}_{\{ \| M_k \| > n\theta \}} \mid \mathcal{F}^X_{k-1} \right) \right) = \frac{1}{n^2} \sum_{k=1}^{\lfloor nT \rfloor} \mathbb{E} \left( \| M_k \|^2 \mathbb{1}_{\{ \| M_k \| > n\theta \}} \right)
\leq \frac{1}{n^2} \sum_{k=1}^{\lfloor nT \rfloor} \mathbb{E} \left( \frac{\| M_k \|^4}{n^2 \theta^2} \right) \leq \frac{2}{n^4 \theta^2} \sum_{k=1}^{\lfloor nT \rfloor} \mathbb{E} (M_{j,1}^4 + M_{j,2}^4) = \frac{1}{n^4 \theta^2} \sum_{k=1}^{\lfloor nT \rfloor} O(k^2) = O(n^{-1}) \to 0
\]
as $n \to \infty$.

Using (3.2) and Lemma C.1, we can prove (3.5). For each $n \in \mathbb{N}$, by (6.1), we have $(n^{-1} X_{\lfloor nt \rfloor})_{t \in \mathbb{R}_+} = \Psi^{(n)}(\mathcal{M}^{(n)})$, where the mapping $\Psi^{(n)} : \mathcal{D}(\mathbb{R}_+, \mathbb{R}^2) \to \mathcal{D}(\mathbb{R}_+, \mathbb{R}^2)$ is given by
\[
(\Psi^{(n)}(f))(t) := f \left( \frac{\lfloor nt \rfloor}{n} \right) + \frac{\lfloor nt \rfloor}{n} b
\]
for $f \in \mathcal{D}(\mathbb{R}_+, \mathbb{R}^2)$ and $t \in \mathbb{R}_+$. Further, using that $(\mathcal{M}_t + bt)_{t \in \mathbb{R}_+} \overset{D}{=} (\mathcal{X}_t)_{t \in \mathbb{R}_+}$, we have $\mathcal{X} \overset{D}{=} \Psi(\mathcal{M})$, where the mapping $\Psi : \mathcal{D}(\mathbb{R}_+, \mathbb{R}^2) \to \mathcal{D}(\mathbb{R}_+, \mathbb{R}^2)$ is given by
\[
(\Psi(f))(t) := f(t) + bt, \quad f \in \mathcal{D}(\mathbb{R}_+, \mathbb{R}^2), \quad t \in \mathbb{R}_+.
\]
The mappings $\Psi^{(n)}$, $n \in \mathbb{N}$, and $\Psi$ are measurable, which can be checked in the same way as in Step 4/(a) in Barczy et al. [2] replacing $\mathcal{D}(\mathbb{R}_+, \mathbb{R})$ by $\mathcal{D}(\mathbb{R}_+, \mathbb{R}^2)$ in the argument given there. One can also check that the set $C := C(\mathbb{R}_+, \mathbb{R}^2)$ satisfies $C \in \mathcal{B}(\mathcal{D}(\mathbb{R}_+, \mathbb{R}^2))$, $\mathbb{P}(\mathcal{M} \in C) = 1$, and $\Psi^{(n)}(f^{(n)}) \to \Psi(f)$ in $\mathcal{D}(\mathbb{R}_+, \mathbb{R}^2)$ as $n \to \infty$ if $f^{(n)} \to f$ in $\mathcal{D}(\mathbb{R}_+, \mathbb{R}^2)$ as $n \to \infty$ with $f \in C$, $f^{(n)} \in \mathcal{D}(\mathbb{R}_+, \mathbb{R}^2)$, $n \in \mathbb{N}$. Namely, one can follow the same argument as in Step 4/(b) in Barczy et al. [2] replacing $\mathcal{D}(\mathbb{R}_+, \mathbb{R})$ by $\mathcal{D}(\mathbb{R}_+, \mathbb{R}^2)$, and $C(\mathbb{R}_+, \mathbb{R})$ by $C(\mathbb{R}_+, \mathbb{R}^2)$, respectively, in the argument given there. So we can apply Lemma C.1 and we obtain $(n^{-1} X_{\lfloor nt \rfloor})_{t \in \mathbb{R}_+} \overset{D}{=} \Psi^{(n)}(\mathcal{M}^{(n)}) \overset{D}{=} \Psi(\mathcal{M})$ as $n \to \infty$, where $((\Psi(\mathcal{M}))(t))_{t \in \mathbb{R}_+} = (\mathcal{M}_t + bt)_{t \in \mathbb{R}_+} \overset{D}{=} (\mathcal{X}_t)_{t \in \mathbb{R}_+}$, as desired.

7 Proofs of Theorem 3.2 and Corollary 3.3

Since $a_{1,1} = 1$, by (3.4), we have $(n^{-1} X_{\lfloor nt \rfloor}, 1)_{t \in \mathbb{R}_+} \overset{D}{=} (\mathcal{X}_{t,1})_{t \in \mathbb{R}_+}$ as $n \to \infty$, where $(\mathcal{X}_{t,1})_{t \in \mathbb{R}_+}$ satisfies the first equation of the SDE (3.7) with $\mathcal{X}_{0,1} = 0$. By (5.4), we have the decomposition
\[
\mathcal{X}^{(n)}_t = \begin{bmatrix} \mathcal{X}^{(n)}_{t,1} \\ \mathcal{X}^{(n)}_{t,2} \end{bmatrix} := \begin{bmatrix} n^{-1} X_{\lfloor nt \rfloor, 1} \\ n^{-2} X_{\lfloor nt \rfloor, 2} \end{bmatrix} = \begin{bmatrix} n^{-1} X_{\lfloor nt \rfloor, 1} \\ a_{2,1} n^{-2} X_{\lfloor nt \rfloor, 2} + n^{-2} X_{\lfloor nt \rfloor, 2} \end{bmatrix}, \quad t \in \mathbb{R}_+, \quad n \in \mathbb{N},
\]
where, since $a_{1,1} = a_{2,2} = 1$,
\[
X_{\lfloor nt \rfloor, 2} = \sum_{j=1}^{\lfloor nt \rfloor} ((\lfloor nt \rfloor - j)(M_{j,1} + b_1)
\]

23
and

\[ X^{(2)}_{[nt], 2} = \sum_{j=1}^{\lfloor nt \rfloor} (M_{j,2} + b_2). \]

Since \( a_{1,1} = 1 \), by \((5.1)\), we obtain \( M_{j,1} + b_1 = X_{j,1} - X_{j-1,1} \) for all \( j \in \mathbb{N} \), and hence, since \( X_{0,1} = 0 \),

\[
\begin{align*}
\frac{n^{-2}X^{(1)}_{[nt], 2}}{n^{-2}X^{(1)}_{[nt], 2}} &= n^{-2} \sum_{j=1}^{\lfloor nt \rfloor} ((\lfloor nt \rfloor - j)(X_{j,1} - X_{j-1,1})) = n^{-2} \sum_{j=1}^{\lfloor nt \rfloor} \sum_{k=1}^{\lfloor nt \rfloor - j} (X_{j,1} - X_{j-1,1}) \\

&= n^{-2} \sum_{k=1}^{\lfloor nt \rfloor - 1} \sum_{j=1}^{\lfloor nt \rfloor - k} (X_{j,1} - X_{j-1,1}) = n^{-2} \sum_{k=1}^{\lfloor nt \rfloor - 1} (X_{[nt] - k,1} - X_{0,1}) \\

&= n^{-2} \sum_{j=1}^{\lfloor nt \rfloor - 1} X_{j,1} = \int_0^{\lfloor nt \rfloor / n} \mathcal{X}^{(n)}_{s,1} \, ds
\end{align*}
\]

for all \( t \in \mathbb{R}_+ \) and \( n \in \mathbb{N} \), where the last equality follows by

\[
\int_0^{\lfloor nt \rfloor / n} \mathcal{X}^{(n)}_{s,1} \, ds = n^{-1} \int_0^{\lfloor nt \rfloor / n} X_{[ns],1} \, ds = n^{-1} \sum_{j=1}^{\lfloor nt \rfloor} \int_{j/n}^{(j+1)/n} X_{[ns],1} \, ds = n^{-1} \sum_{j=1}^{\lfloor nt \rfloor} \int_{j/n}^{(j+1)/n} X_{j-1,1} \, ds
\]

\[
= n^{-1} \sum_{j=1}^{\lfloor nt \rfloor} \frac{1}{n} X_{j-1,1} = n^{-2} \sum_{j=1}^{\lfloor nt \rfloor} X_{j-1,1} = n^{-2} \sum_{j=1}^{\lfloor nt \rfloor} X_{j,1}.
\]

By the continuous mapping theorem, we have

\[
(7.2) \quad \left( \begin{array}{c} n^{-1}X_{[nt],1} \\ n^{-2}X^{(1)}_{[nt], 2} \end{array} \right)_{t \in \mathbb{R}_+} \to \mathcal{D}(\mathbb{R}_+, \mathbb{R})^{(n)} \quad \text{as} \quad n \to \infty.
\]

Indeed, \((7.2)\) follows by an application of Lemma \(C.2\) with the choices \( d = 1 \), \( q = 2 \) and \( U_t^{(n)} := \mathcal{X}^{(n)}_{s,1} \), \( U_t := \mathcal{X}_{s,1} \),

\[
(\Phi_n(f))(t) := \left[ \int_0^{\lfloor nt \rfloor / n} f(t) \, ds \right], \quad (\Phi(f))(t) := \left[ \int_0^t f(s) \, ds \right]
\]

for \( n \in \mathbb{N} \), \( t \in \mathbb{R}_+ \), and \( f \in \mathcal{D}(\mathbb{R}_+, \mathbb{R}) \). Next, we check that the conditions of Lemma \(C.2\) with the above choices are satisfied, so we have right to apply Lemma \(C.2\). The mapping \( \Phi \) is continuous and hence measurable, see, e.g., Ethier and Kurtz [13] Problem 3.11.8. For each \( n \in \mathbb{N} \), the mapping \( \Phi_n \) is measurable as well, since \( \Phi_n = \Phi \circ \Phi \), where \( \Phi_n : \mathcal{D}(\mathbb{R}_+, \mathbb{R}^2) \to \mathcal{D}(\mathbb{R}_+, \mathbb{R}^2) \), \( \Phi_n(g)(t) := \left( g_1(t), g_2\left(\frac{\lfloor nt \rfloor}{n}\right) \right) \), \( t \in \mathbb{R}_+, \ g = (g_1, g_2) \in \mathcal{D}(\mathbb{R}_+, \mathbb{R}^2) \), is measurable (checked below), and composition of measurable mappings is measurable. Using that the finite dimensional sets in \( \mathcal{D}(\mathbb{R}_+, \mathbb{R}^2) \) generate the Borel \( \sigma \)-algebra on \( \mathcal{D}(\mathbb{R}_+, \mathbb{R}^2) \) (see, e.g., Jacod and Shiryaev [18] Chapter VI, Theorem 1.14, part c)), to check the measurability
of $\tilde{\Phi}_n$ it is enough to verify that the mapping $\pi_t \circ \tilde{\Phi}_n$ is measurable for each $t \in \mathbb{R}_+$, where $\pi_t : D(\mathbb{R}_+, \mathbb{R}^2) \to \mathbb{R}^2$, $\pi_t(g) := g(t)$, $g \in D(\mathbb{R}_+, \mathbb{R}^2)$, is the natural projection onto $t$. Since for each $n \in \mathbb{N}$ and $t \in \mathbb{R}_+$, $\pi_t \circ \tilde{\Phi}_n : D(\mathbb{R}_+, \mathbb{R}^2) \to \mathbb{R}^2$, $(\pi_t \circ \tilde{\Phi}_n)(g) = \left(\frac{g_1(t)_n}{n}, g_2\left(\frac{nt}{n}\right)^\top\right)$, $g = (g_1, g_2) \in D(\mathbb{R}_+, \mathbb{R}^2)$, the first and second coordinate functions of $\pi_t \circ \tilde{\Phi}_n$ can be identified with the natural projections of $D(\mathbb{R}_+, \mathbb{R})$ onto the coordinate $t$ and $\frac{nt}{n}$, respectively, which are measurable (see, e.g., Billingsley [7, Theorem 16.6, part (i)]), yielding that $\pi_t \circ \tilde{\Phi}_n$ is measurable as well. We check that $C_{\Phi_n}(\Phi_n)_{n \in \mathbb{N}} = C(\mathbb{R}_+, \mathbb{R})$. For this we need to verify that $\Phi_n(f_n) \xrightarrow{lu} \Phi(f)$ as $n \to \infty$ whenever $f_n \xrightarrow{lu} f$ as $n \to \infty$ with $f \in C(\mathbb{R}_+, \mathbb{R})$ and $f_n \in D(\mathbb{R}_+, \mathbb{R})$, $n \in \mathbb{N}$. For each $t \in \mathbb{R}_+$, we have

$$\|\Phi_n(f_n)(t) - (\Phi(f))(t)\| \leq |f_n(t) - f(t)| + \left|\int_0^{nt/n} f_n(s) \, ds - \int_0^t f(s) \, ds\right|$$

and hence for each $T \in \mathbb{R}_+$,

$$\sup_{t \in [0, T]} \|\Phi_n(f_n)(t) - (\Phi(f))(t)\| \leq \sup_{t \in [0, T]} |f_n(t) - f(t)| + \left(\sup_{t \in [0, T]} |f_n(t) - f(t)|\right) \frac{|nT|}{n}$$

$$+ \sup_{t \in [0, T]} |f(t)| \sup_{t \in [0, T]} \left(t - \frac{nt}{n}\right) \to 0$$

as $n \to \infty$, since $f_n \xrightarrow{lu} f$ as $n \to \infty$ and $\sup_{t \in [0, T]} |f(t)| < \infty$ (due to $f \in C(\mathbb{R}_+, \mathbb{R})$). All in all, the conditions of Lemma C.2 are satisfied with our choices, and hence we get (7.2).

The aim of the following discussion is to show that

$$(7.3) \quad \sup_{t \in [0, T]} |n^{-2} X_{[nt], 2}^{(2)}| \xrightarrow{P} 0 \quad \text{as } n \to \infty \quad \text{for all } T \in \mathbb{R}_+.$$  

For each $t \in \mathbb{R}_+$,

$$|X_{[nt], 2}^{(2)}| = \left|\sum_{j=1}^{[nt]} (M_{j, 2} + b_2)\right| \leq \left|\sum_{j=1}^{[nt]} M_{j, 2}\right| + b_2 [nt],$$

hence, in order to check (7.3), it suffices to prove

$$n^{-2} \sup_{t \in [0, T]} \left|\sum_{j=1}^{[nt]} M_{j, 2}\right| = n^{-2} \max_{k \in \{1, \ldots, [nt]\}} \sum_{j=1}^{k} M_{j, 2} \xrightarrow{P} 0 \quad \text{as } n \to \infty \quad \text{for all } T \in \mathbb{R}_+,$$

which is equivalent to

$$(7.4) \quad n^{-4} \max_{k \in \{1, \ldots, [nt]\}} \left(\sum_{j=1}^{k} M_{j, 2}\right)^2 \xrightarrow{P} 0 \quad \text{as } n \to \infty \quad \text{for all } T \in \mathbb{R}_+.$$
Applying Doob’s maximal inequality (see, e.g., Revuz and Yor [28, Chapter II, Corollary (1.6)]) for the martingale \( \sum_{j=1}^{k} M_{j,2} \), \( k \in \mathbb{N} \) (with the filtration \((\mathcal{F}_{k})_{k \in \mathbb{N}}\)), we obtain

\[
\mathbb{E} \left( \max_{k \in \{1, \ldots, [nT]\}} \left( \sum_{j=1}^{k} M_{j,2} \right)^2 \right) \leq 4 \mathbb{E} \left( \left( \sum_{j=1}^{[nT]} M_{j,2} \right)^2 \right)
\]

\[
= 4 \mathbb{E} \left( \sum_{j=1}^{[nT]} M_{j,2}^2 + 2 \sum_{j=1}^{[nT]} \sum_{\ell=j+1}^{[nT]} M_{j,2} M_{\ell,2} \right)
\]

\[
= 4 \sum_{j=1}^{[nT]} \mathbb{E}(M_{j,2}^2) + 8 \sum_{j=1}^{[nT]-1} \sum_{\ell=j+1}^{[nT]} \mathbb{E}(M_{j,2} M_{\ell,2}) = 4 \sum_{j=1}^{[nT]} \mathbb{E}(M_{j,2}^2),
\]

since for each \( j = 1, \ldots, [nT] - 1 \) and \( \ell = j + 1, \ldots, [nT] \), we have

\[
\mathbb{E}(M_{j,2} M_{\ell,2}) = \mathbb{E}(\mathbb{E}(M_{j,2} M_{\ell,2} | \mathcal{F}_{\ell-1}^X)) = \mathbb{E}(M_{j,2} \mathbb{E}(M_{\ell,2} | \mathcal{F}_{\ell-1}^X))
\]

\[
= \mathbb{E}(M_{j,2} \mathbb{E}(X_{\ell,2} - \mathbb{E}(X_{\ell,2} | \mathcal{F}_{\ell-1}^X) | \mathcal{F}_{\ell-1}^X)) = \mathbb{E}(M_{j,2} \cdot 0) = 0.
\]

Using Lemma A.3 we get \( \sum_{j=1}^{[nT]} \mathbb{E}(M_{j,2}^2) = \sum_{j=1}^{[nT]} O(j) = O(n^2) \) for \( n \in \mathbb{N} \) and \( T \in \mathbb{R}^+ \), and consequently we obtain

\[
n^{-4} \max_{k \in \{1, \ldots, [nT]\}} \left( \sum_{j=1}^{k} M_{j,2} \right)^2 \xrightarrow{L_1} 0 \quad \text{as} \quad n \to \infty \quad \text{for all} \quad T \in \mathbb{R}^+,
\]

yielding (7.4), and hence (7.3). Now, by Lemma VI.3.31 in Jacod and Shiryaev [18] (a kind of Slutsky’s lemma for stochastic processes with trajectories in \( D(\mathbb{R}^+, \mathbb{R}^d) \)), convergences (7.2) and (7.3) yield convergence

\[
(\mathcal{X}_{t}^{(n)})_{t \in \mathbb{R}^+} \xrightarrow{D} (\mathcal{X}_{t})_{t \in \mathbb{R}^+} \quad \text{as} \quad n \to \infty,
\]

where the process \( (\mathcal{X}_{t})_{t \in \mathbb{R}^+} \) is given by

\[
\mathcal{X}_{t} = \left[ \begin{array}{c} \mathcal{X}_{t,1} \\ a_{2,1} \int_{0}^{t} \mathcal{X}_{s,1} \, ds \end{array} \right], \quad t \in \mathbb{R}^+.
\]

By Itô’s formula, we obtain that \( (\mathcal{X}_{t})_{t \in \mathbb{R}^+} \) satisfies the SDE (3.7) with initial value \( \mathcal{X}_{0} = (0, 0) \), thus we conclude the statement of Theorem 3.2. \( \Box \)

**Proof of Corollary 3.3.** Since \( b_1 \in \mathbb{R}^+ \), we have \( \mathbb{P}(\mathcal{X}_{t,1} \in \mathbb{R}^+) = 1 \) for each \( t \in \mathbb{R}^+ \). Indeed, if \( u_{1,1}^{(1)} = 0 \), then \( \mathcal{X}_{t,1} = b_1 t \), \( t \in \mathbb{R}^+ \), and if \( u_{1,1}^{(1)} \in \mathbb{R}^+ \), then \( \mathcal{X}_{t,1} \) is Gamma distributed with parameters \( 2/u_{1,1}^{(1)} \) and \( 2b_1/u_{1,1}^{(1)} \) for each \( t \in \mathbb{R}^+ \), see, e.g., Ikeda and Watanabe [15, Chapter IV, Example 8.2]. Let \( g : \mathbb{R}^2 \to \mathbb{R} \) be defined by

\[
g(x, y) := \mathbb{1}_{\{x \neq 0\}} \frac{y}{x} = \begin{cases} \frac{y}{x} & \text{if} \ x \neq 0 \ \text{and} \ y \in \mathbb{R}, \\ 0 & \text{if} \ x = 0 \ \text{and} \ y \in \mathbb{R}. \end{cases}
\]

26
Then $g$ is continuous on the set $(\mathbb{R} \setminus \{0\}) \times \mathbb{R}$ and the distribution of $(X_{t,1}, X_{t,2})$ is concentrated on this set, since $\mathbb{P}(X_{t,1} \in \mathbb{R}_+) = 1$. By Theorem 3.2 and the continuous mapping theorem (see, e.g., Billingsley [3, Theorem 5.1]),

$$n^{-1} 1_{\{X_{nt,1} \neq 0\}} \frac{X_{nt,2}}{X_{nt,1}} = g(n^{-1} X_{nt,1}, n^{-2} X_{nt,2})$$

$$\xrightarrow{D} g(X_{t,1}, X_{t,2}) = 1\{X_{t,1} \neq 0\} a_{2,1} \frac{\int_0^t X_{s,1} \, ds}{X_{t,1}} = a_{2,1} \frac{\int_0^t X_{s,1} \, ds}{X_{t,1}} \quad \text{as} \ n \to \infty,$$

as desired. \qed

8 Proof of Theorem 3.4

Since $a_{1,1} = 1$, by [3.4], we have $(n^{-1} X_{nt,1})_{t \in \mathbb{R}_+} \xrightarrow{D} (X_{t,1})_{t \in \mathbb{R}_+}$ as $n \to \infty$, where $(X_{t,1})_{t \in \mathbb{R}_+}$ satisfies the SDE (3.8) with $X_{0,1} = 0$.

Since $a_{2,1} = 0$, as it was explained in Section 3 the second coordinate process $(X_{k,2})_{k \in \mathbb{Z}_+}$ satisfies

$$X_{k,2} = \sum_{j=1}^{X_{k-1,2}} \xi_{k,2, j} + \varepsilon_{k,2}, \quad k \in \mathbb{N}.$$ 

Since $a_{2,2} \in [0,1)$, the Markov chain $(X_{k,2})_{k \in \mathbb{Z}_+}$ admits a unique stationary distribution $\mu_2$ (for its existence and generator function, see the beginning of Appendix B), and, by Lemma B.1, we have $(X_{nt,2})_{t \in \mathbb{R}_+} \xrightarrow{D} (X_{t,2})_{t \in \mathbb{R}_+}$ as $n \to \infty$, where $(X_{t,2})_{t \in \mathbb{R}_+}$ is an i.i.d. process such that for each $t \in \mathbb{R}_+$, the distribution of $X_{t,2}$ is $\mu_2$.

Further, using [3.4] with $a_{1,1} = 1$, [A.4], and $\mathbb{E}(M_k | \mathcal{F}_{k-1}^X) = 0$, $k \in \mathbb{N}$ (yielding $\text{Cov}(M_{j,1}, M_{k,2}) = 0$, $j \neq k$, $j, k \in \mathbb{N}$), we have for all $t_1, t_2 \in \mathbb{R}_+$,

$$\text{Cov}\left(n^{-1} X_{nt_1,1}, X_{nt_2,2}\right)$$

$$= \text{Cov}\left(n^{-1} \sum_{j=1}^{n t_1} M_{j,1} \left[ M_{j,1} + n t_1 \right] b_1, \sum_{j=1}^{n t_2} a_{2,2}^{n t_2 - j} M_{j,2} + \frac{a_{2,2}^{n t_2} - 1}{a_{2,2} - 1} b_2\right)$$

$$= n^{-1} \text{Cov}\left(\sum_{j=1}^{n t_1} M_{j,1} \sum_{j=1}^{n t_2} a_{2,2}^{n t_2 - j} M_{j,2}\right) = n^{-1} \sum_{j=1}^{n t_1 \wedge n t_2} a_{2,2}^{n t_2 - j} \text{Cov}(M_{j,1}, M_{j,2})$$

$$= n^{-1} \sum_{j=1}^{n t_1 \wedge n t_2} a_{2,2}^{n t_2 - j} \left( v_{1,2}^{(0)} + \mathbb{E}(X_{j-1,1}) v_{1,2}^{(1)} + \mathbb{E}(X_{j-1,2}) v_{1,2}^{(2)}\right).$$

Since $a_{1,2} = a_{2,1} = 0$, we have $\mathbb{P}(\xi_{1,1,2,1} = 0) = 1$ and $\mathbb{P}(\xi_{1,1,1,2} = 0) = 1$, and hence $v_{1,2}^{(2)} = 0$.
and $v_{1,2}^{(1)} = 0$, yielding that for all $t_1, t_2 \in \mathbb{R}_+$,

$$\text{Cov} \left( n_1^{-1} X_{[nt_1],1}, X_{[nt_2],2} \right) = \frac{v_{1,2}^{(0)} |_{[nt_1] \cap [nt_2]} |_{[nt_2]}^{\cdot j} a_{2,2}^{\cdot n_2} - j}{n_1 a_{2,2}^{\cdot n_2} - |_{[nt_1] \cap [nt_2]}^{\cdot j} \frac{1 - a_{2,2}^{\cdot n_2}}{1 - a_{2,2}^{\cdot n_2}}},$$

which yields (3.11) (due to $a_{2,2} \in [0,1)$ and $X_{0,2} = 0$). Using again $a_{2,2} \in [0,1)$, we have

$$\sup_{t_1, t_2 \in \mathbb{R}_+} \sup_{n_2 \in \mathbb{N}} \left| \text{Cov} \left( n_1^{-1} X_{[nt_1],1}, X_{[nt_2],2} \right) \right| \leq \frac{v_{1,2}^{(0)}}{1 - a_{2,2}^{\cdot n_2}} \sup_{t_1, t_2 \in \mathbb{R}_+} \sup_{n_2 \in \mathbb{N}} a_{2,2}^{\cdot n_2} - |_{[nt_1] \cap [nt_2]}^{\cdot j} \frac{1}{1 - a_{2,2}^{\cdot n_2}} \rightarrow 0 \quad \text{as} \quad n_1 \rightarrow \infty,$$

which yields (3.10).

### 9 Proof of Theorem 3.5

Since $a_{1,1} = 1$, by (3.4), we have $(n^{-1}X_{[nt],1})_{t \in \mathbb{R}_+} \xrightarrow{D} (X_{t,1})_{t \in \mathbb{R}_+}$ as $n \rightarrow \infty$, where $(X_{t,1})_{t \in \mathbb{R}_+}$ satisfies the first equation of the SDE (3.12) with $X_{0,1} = 0$. By (3.4), we have the decomposition

$$X^{(n)}_t = \begin{bmatrix} X^{(n)}_{t,1} \\ X^{(n)}_{t,2} \end{bmatrix} = \begin{bmatrix} \sum_{j=1}^{[nt]} \frac{1 - a_{2,2}^{\cdot n}}{1 - a_{2,2}^{\cdot n}} (M_{j,1} + b_1) \\ n^{-1} X^{(1)}_{[nt],1} + n^{-1} X^{(2)}_{[nt],2} \end{bmatrix}, \quad t \in \mathbb{R}_+ \quad n \in \mathbb{N},$$

where, since $a_{1,1} = 1$ and $a_{2,2} \in [0,1),$

$$X^{(1)}_{[nt],2} := \sum_{j=1}^{[nt]} \frac{1 - a_{2,2}^{\cdot n}}{1 - a_{2,2}^{\cdot n}} (M_{j,1} + b_1) \quad \text{and} \quad X^{(2)}_{[nt],2} := \sum_{j=1}^{[nt]} a_{2,2}^{\cdot n} (M_{j,2} + b_2).$$

The aim of the following discussion is to show that

$$(9.2) \quad \sup_{t \in [0,T]} |n^{-1}X^{(2)}_{[nt],2}| \xrightarrow{p} 0 \quad \text{as} \quad n \rightarrow \infty \quad \text{for all} \quad T \in \mathbb{R}_+.$$  

We have

$$|n^{-1}X^{(2)}_{[nt],2}| \leq \frac{1}{n} \sum_{j=1}^{[nt]} a_{2,2}^{\cdot n} M_{j,2} + \frac{b_2}{n(1 - a_{2,2}^{\cdot n})} \leq \frac{1}{n} \sum_{j=1}^{[nt]} a_{2,2}^{\cdot n} M_{j,2} + \frac{b_2}{n(1 - a_{2,2}^{\cdot n})},$$

hence, in order to check (9.2), it suffices to prove

$$(9.3) \quad n^{-1} \sup_{t \in [0,T]} \left| \sum_{j=1}^{[nt]} a_{2,2}^{\cdot n} M_{j,2} \right| = n^{-1} \max_{k \in \{1, \ldots, [nt]\}} |V_{k,2}| \xrightarrow{p} 0.$$  

28
as \( n \to \infty \) for all \( T \in \mathbb{R}_{++} \), where

\[
V_{k,2} := \sum_{j=1}^{k} a_{2,2}^{k-j} M_{j,2}, \quad k \in \mathbb{N}.
\]

Note that

(9.4)

\[
V_{k,2} = a_{2,2} V_{k-1,2} + M_{k,2}, \quad k \in \mathbb{N},
\]

where \( V_{0,2} := 0 \), hence \( (V_{k,2})_{k \in \mathbb{Z}_+} \) is a stable AR(1) process with heteroscedastic innovation \( (M_{k,2})_{k \in \mathbb{N}} \). For all \( \delta > 0 \), by Markov’s inequality, we have

\[
\mathbb{P}\left(n^{-1} \max_{k \in \{1, \ldots, [nT]\}} |V_{k,2}| > \delta \right) = \mathbb{P}\left(\max_{k \in \{1, \ldots, [nT]\}} V_{k,2}^4 > \delta^4 n^4 \right) \leq \sum_{k=1}^{[nT]} \mathbb{P}(V_{k,2}^4 > \delta^4 n^4)
\]

\[
\leq \delta^{-4} n^{-4} \sum_{k=1}^{[nT]} \mathbb{E}(V_{k,2}^4) = \delta^{-4} n^{-4} \sum_{k=1}^{[nT]} O(k^2) = O(n^{-1}), \quad n \in \mathbb{N},
\]

where we applied \( \mathbb{E}(V_{k,2}^4) = O(k^2), \quad k \in \mathbb{N} \) (see Lemma [A.5]), thus we obtain (9.3), and hence (9.2). Here we note that in the previous application of Markov’s inequality, we took the fourth moment of \( |V_{k,2}| \), since even if we took its second moment, then we could not argue that \( n^{-2} \sum_{k=1}^{[nT]} \mathbb{E}(V_{k,2}^2) \rightarrow 0 \) as \( n \to \infty \) due to the fact that \( \mathbb{E}(V_{k,2}^2) = O(k), \quad k \in \mathbb{N} \) (which can be checked similarly as \( \mathbb{E}(V_{k,2}^4) = O(k^2), \quad k \in \mathbb{N} \), checked in the proof of Lemma [A.5]).

Recall that

\[
n^{-1}X_{[nt],2}^{(1)} = n^{-1} \sum_{j=1}^{[nt]} \frac{1 - a_{2,2}^{[nt]-j}}{1 - a_{2,2}} (M_{j,1} + b_1), \quad t \in \mathbb{R}_+, \quad n \in \mathbb{N}.
\]

Since \( a_{1,1} = 1 \), by (3.1), we obtain \( M_{j,1} + b_1 = X_{j,1} - X_{j-1,1} \) for all \( j \in \mathbb{N} \), and hence, using that \( X_{0,1} = 0 \), we have \( \sum_{j=1}^{k} (M_{j,1} + b_1) = \sum_{j=1}^{k} (X_{j,1} - X_{j-1,1}) = X_{k,1} \) for all \( k \in \mathbb{N} \). Consequently, we get

\[
n^{-1}X_{[nt],2}^{(1)} = n^{-1}X_{[nt],1}^{(1)} \left( 1 - \frac{a_{2,2}^{[nt]-j}}{1 - a_{2,2}} (M_{j,1} + b_1) \right)
\]

for all \( t \in \mathbb{R}_+ \) and \( n \in \mathbb{N} \). In a similar way as above for (9.3), using Lemma [A.5] we have

(9.5)

\[
\sup_{t \in [0,T]} \left| \sum_{j=1}^{[nt]} a_{2,2}^{[nt]-j} M_{j,1} \right| = n^{-1} \max_{k \in \{1, \ldots, [nT]\}} |V_{k,1}| \overset{P}{\to} 0
\]

as \( n \to \infty \) for all \( T \in \mathbb{R}_{++} \), where

\[
V_{k,1} := \sum_{j=1}^{k} a_{2,2}^{k-j} M_{j,1}, \quad k \in \mathbb{N},
\]

29
which satisfies
\[ V_{k,1} = a_{2,2} V_{k-1,1} + M_{k,1}, \quad k \in \mathbb{N}, \]
where \( V_{0,1} := 0 \). Moreover, since \( a_{2,2} \in [0, 1) \), we have
\[
n^{-1} \sup_{t \in [0, T]} \left| \sum_{j=1}^{\lfloor nt \rfloor} a_{2,2}^{\lfloor nt \rfloor - j} \right| = n^{-1} \sup_{t \in [0, T]} \left| \sum_{j=0}^{\lfloor nt \rfloor - 1} a_{2,2}^{j} \right| = n^{-1} \sum_{j=0}^{\lfloor nT \rfloor - 1} a_{2,2}^{j} \\
\leq n^{-1} \sum_{\ell=0}^{\infty} a_{2,2}^{\ell} = \frac{n^{-1}}{1 - a_{2,2}} \rightarrow 0 \quad \text{as} \ n \rightarrow \infty.
\]

By Lemma VI.3.31 in Jacod and Shiryaev [18] (a kind of Slutsky’s lemma for stochastic processes with trajectories in \( D(\mathbb{R}^+, \mathbb{R}^d) \)) and (9.1), the above convergences yield
\[
(\mathcal{X}_t^{(n)}), t \in \mathbb{R}_+ \xrightarrow{D} (\mathcal{X}_t), t \in \mathbb{R}_+ \quad \text{as} \ n \rightarrow \infty,
\]
where the process \( (\mathcal{X}_t), t \in \mathbb{R}_+ \) is given by
\[
\mathcal{X}_t = \begin{bmatrix}
\mathcal{X}_{t,1} \\
\frac{a_{2,1}}{1-a_{2,2}} \mathcal{X}_{t,1}
\end{bmatrix}, \quad t \in \mathbb{R}_+.
\]

10 Proof of Theorem 3.7

Since \( a_{1,1} \in [0, 1) \), the Markov chain \( (X_{k,1})_{k \in \mathbb{Z}_+} \) admits a unique stationary distribution \( \mu_1 \) (for its existence and generator function, see the beginning of Appendix B), and, by Theorem B.2, \( (X_{\lfloor nt \rfloor,1}), t \in \mathbb{R}_+ \xrightarrow{D} (\mathcal{X}_t), t \in \mathbb{R}_+ \) as \( n \rightarrow \infty \), where \( (\mathcal{X}_t), t \in \mathbb{R}_+ \) is an i.i.d. process such that for each \( t \in \mathbb{R}_+ \), the distribution of \( \mathcal{X}_{t,1} \) is \( \mu_1 \).

Since \( a_{2,2} = 1 \), by (5.1), we have
\[
M_{k,2} = X_{k,2} - a_{2,1} X_{k-1,1} - X_{k-1,2} - b_2, \quad k \in \mathbb{N},
\]
hence, using that \( X_{0,2} = 0 \),
\[
(10.1) \quad \mathcal{X}_{t,2}^{(n)} := n^{-1} X_{\lfloor nt \rfloor,2} = \frac{1}{n} \sum_{k=1}^{\lfloor nt \rfloor} (X_{k,2} - X_{k-1,2}) = \frac{1}{n} \sum_{k=1}^{\lfloor nt \rfloor} U_{k,2}, \quad t \in \mathbb{R}_+, \quad n \in \mathbb{N},
\]
where
\[
U_{k,2} := M_{k,2} + a_{2,1} X_{k-1,1} + b_2, \quad k \in \mathbb{N}.
\]

We show that
\[
(10.2) \quad (\mathcal{X}_{t,2}^{(n)}), t \in \mathbb{R}_+ \xrightarrow{D} (\mathcal{X}_{t,2}), t \in \mathbb{R}_+ \quad \text{as} \ n \rightarrow \infty,
\]

30
where the limit process \((\mathcal{X}_{t,2})_{t \in \mathbb{R}_+}\) is the unique strong solution of the SDE (3.14) with \(X_{0,2} = 0\). In order to prove (10.2), we want to apply Theorem D.1 for \(d = r = 1\), \((U_t)_{t \in \mathbb{R}_+} = (\mathcal{X}_{t,2})_{t \in \mathbb{R}_+}\), \(U_k^{(n)} = n^{-1}U_{k,2}\), \(n, k \in \mathbb{N}\), \(U_0^{(n)} = 0\), \(n \in \mathbb{N}\), \(\mathcal{F}_k^{(n)} = \mathcal{F}_k^X\) for \(n \in \mathbb{N}\) and \(k \in \mathbb{Z}_+\), and with coefficient functions \(\beta : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}\) and \(\gamma : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}\) of the SDE (3.14) given by

\[
\beta(t, x) = \frac{a_{2,1}}{1 - a_{1,1}} b_1 + b_2, \quad \gamma(t, x) = \sqrt{v_{2,2}^{(2)}} x^+, \quad t \in \mathbb{R}_+, \quad x \in \mathbb{R}.
\]

The SDE (3.14) has a pathwise unique strong solution \((\mathcal{X}_{t,2}^{(x)})_{t \in \mathbb{R}_+}\) for all initial values \(X_{0,2}^{(x)} = x \in \mathbb{R}\), and if \(x \in \mathbb{R}_+\), then \(\mathcal{X}_{t,2}^{(x)} \in \mathbb{R}_+\) almost surely for all \(t \in \mathbb{R}_+\) since

\[
\frac{a_{2,1}}{1 - a_{1,1}} b_1 + b_2 \in \mathbb{R}_+, \quad v_{2,2}^{(2)} \in \mathbb{R}_+,
\]

see, e.g., Ikeda and Watanabe [15, Chapter IV, Example 8.2].

Now, we show that conditions (i), (ii) and (iii) of Theorem D.1 hold. We have to check that for each \(T \in \mathbb{R}_++\),

(10.3) \[\sup_{t \in [0,T]} \left| \frac{1}{n} \sum_{k=1}^{[nt]} \mathbb{E}(U_{k,2} | \mathcal{F}_{k-1}^X) - \left( \frac{a_{2,1}}{1 - a_{1,1}} b_1 + b_2 \right) t \right| \overset{p}{\to} 0 \quad \text{as } n \to \infty,\]

(10.4) \[\sup_{t \in [0,T]} \left| \frac{1}{n^2} \sum_{k=1}^{[nt]} \text{Var}(U_{k,2} | \mathcal{F}_{k-1}^X) - \int_0^t (\mathcal{X}_{s,2}^{(n)}) + v_{2,2}^{(2)} ds \right| \overset{p}{\to} 0 \quad \text{as } n \to \infty,\]

(10.5) \[\frac{1}{n^2} \sum_{k=1}^{[nT]} \mathbb{E}(U_{k,2}^2 1_{\{U_{k,2}^{(n)} > n\theta\}} | \mathcal{F}_{k-1}^X) \overset{p}{\to} 0 \quad \text{as } n \to \infty \quad \text{for all } \theta \in \mathbb{R}_+.\]

For each \(k \in \mathbb{N}\), we have \(\mathbb{E}(U_{k,2} | \mathcal{F}_{k-1}^X) = a_{2,1} X_{k-1,1} + b_2\), and \(\sup_{t \in [0,T]} \left| \frac{[nt]}{n} - t \right| \to 0 \) as \(n \to \infty\) for each \(T \in \mathbb{R}_++\), hence, in order to show (10.3), it suffices to prove that for each \(T \in \mathbb{R}_++\),

(10.6) \[\sup_{t \in [0,T]} \left| \frac{1}{n} \sum_{k=1}^{[nt]} X_{k-1,1} - \frac{b_1}{1 - a_{1,1}} t \right| \overset{p}{\to} 0 \quad \text{as } n \to \infty.\]

By (5.1), we have

\[M_{k,1} = X_{k,1} - a_{1,1} X_{k-1,1} - b_1, \quad \text{i.e.,} \quad X_{k,1} = a_{1,1} X_{k-1,1} + M_{k,1} + b_1, \quad k \in \mathbb{N},\]

hence, using that \(X_{0,1} = 0\), we have

\[X_{k,1} = \sum_{j=1}^{k} a_{1,1}^{k-j} (M_{j,1} + b_1), \quad k \in \mathbb{N},\]
Thus
\[
\sum_{k=1}^{nt} X_{k-1,1} = \sum_{k=1}^{nt} \sum_{j=1}^{k-1} a_{1,1}^{k-1-j} (M_{j,1} + b_1) = \sum_{j=1}^{nt} \sum_{k=j+1}^{nt} a_{1,1}^{k-1-j} (M_{j,1} + b_1) = \sum_{j=1}^{nt-1} \frac{1 - a_{1,1}^{nt-j}}{1 - a_{1,1}} (M_{j,1} + b_1)
\]
\[
= \frac{1}{1 - a_{1,1}} \sum_{j=1}^{nt-1} (1 - a_{1,1}^{nt-j}) M_{j,1} + \frac{b_1}{1 - a_{1,1}} \sum_{j=1}^{nt-1} (1 - a_{1,1}^{nt-j})
\]
\[
= \frac{1}{1 - a_{1,1}} \sum_{j=1}^{nt-1} (1 - a_{1,1}^{nt-j}) M_{j,1} + \frac{b_1}{1 - a_{1,1}} \left( [nt] - 1 - a_{1,1}^{nt} \right).
\]
Hence for each \( t \in \mathbb{R}_+ \) and \( n \in \mathbb{N} \),
\[
\frac{1}{n} \sum_{k=1}^{nt} X_{k-1,1} - \frac{b_1}{1 - a_{1,1}} t = \frac{1}{1 - a_{1,1}} \cdot \frac{1}{n} \sum_{j=1}^{nt-1} M_{j,1} - \frac{1}{1 - a_{1,1}} \cdot \frac{1}{n} \sum_{j=1}^{nt-1} a_{1,1}^{nt-j} M_{j,1} + \frac{b_1}{1 - a_{1,1}} \cdot \frac{1}{n} \left( [nt] - nt - 1 - a_{1,1}^{nt} \right).
\]
Here, since \( a_{1,1} \in [0, 1) \) and \( |[nt] - nt| \leq 1 \), \( t \in \mathbb{R}_+ \), we have for each \( T \in \mathbb{R}_{++} \),
\[
\sup_{t \in [0,T]} n^{-1} \left( [nt] - nt - 1 - a_{1,1}^{nt} \right) \to 0 \quad \text{as } n \to \infty.
\]
Next we check that
\[
(10.7) \quad n^{-1} \sup_{t \in [0,T]} \left| \sum_{j=1}^{[nt]-1} M_{j,1} \right| \xrightarrow{p} 0 \quad \text{as } n \to \infty
\]
for each \( T \in \mathbb{R}_{++} \), which is equivalent to
\[
n^{-2} \sup_{t \in [0,T]} \left( \sum_{j=1}^{[nt]-1} M_{j,1} \right)^2 = n^{-2} \max_{k \in \{1, \ldots, [nT]-1\}} \left( \sum_{j=1}^{k} M_{j,1} \right)^2 \xrightarrow{p} 0 \quad \text{as } n \to \infty
\]
for each \( T \in \mathbb{R}_{++} \). Applying Doob’s maximal inequality (see, e.g., Revuz and Yor [28] Chapter II, Corollary (1.6)]) for the martingale \( \sum_{j=1}^{k} M_{j,1}, \ k \in \mathbb{N} \) (with the filtration \( (\mathcal{F}_k^X)_{k \in \mathbb{N}} \)), we obtain
\[
\mathbb{E} \left[ \max_{k \in \{1, \ldots, [nT]-1\}} \left( \sum_{j=1}^{k} M_{j,1} \right)^2 \right] \leq 4 \mathbb{E} \left[ \left( \sum_{j=1}^{[nT]-1} M_{j,1} \right)^2 \right] = 4 \sum_{j=1}^{[nT]-1} \mathbb{E}(M_{j,1}^2)
\]
\[
= 4 \sum_{j=1}^{[nT]-1} O(1) = O(n),
\]
where we used Lemma A.6, thus we obtain
\[ n^{-2} \max_{k \in \{1, \ldots, |nT| - 1\}} \left( \sum_{j=1}^{k} M_{j,1} \right)^{2} \xrightarrow{L_{1}} 0 \quad \text{as } n \to \infty \text{ for all } T \in \mathbb{R}_{++}, \]
yielding (10.7).

In a similar way as in the proof of (9.5) (replacing \(a_{2,2}\) by \(a_{1,1}\)), we prove that
\[ n^{-1} \sup_{t \in [0,T]} \left| \sum_{j=1}^{|nt|-1} a_{1,1,j}^{nT} M_{j,1} \right| \leq n^{-1} \sup_{t \in [0,T]} \left| \sum_{j=1}^{|nt|-1} a_{1,1}^{nT} M_{j,1} \right| = n^{-1} \max_{k \in \{1, \ldots, |nT| - 1\}} |\tilde{V}_{k,1}| \xrightarrow{p} 0 \]
as \(n \to \infty\) for all \(T \in \mathbb{R}_{++}\), where
\[ \tilde{V}_{k,1} := \sum_{j=1}^{k} a_{1,1}^{k-j} M_{j,1}, \quad k \in \mathbb{N}, \]
and we used that \(a_{1,1} \in [0,1)\). Note that
\[
(10.8) \quad \tilde{V}_{k,1} = a_{1,1} \tilde{V}_{k-1,1} + M_{k,1}, \quad k \in \mathbb{N},
\]
where \(\tilde{V}_{0,1} := 0\), hence \((\tilde{V}_{k,1})_{k \in \mathbb{Z}_{+}}\) is a stable AR(1) process with heteroscedastic innovation \((M_{k,1})_{k \in \mathbb{N}}\). For all \(\delta > 0\), by Markov’s inequality, we have
\[
\mathbb{P}\left(n^{-1} \max_{k \in \{1, \ldots, |nT| - 1\}} |\tilde{V}_{k,1}| > \delta \right) = \mathbb{P}\left( \max_{k \in \{1, \ldots, |nT| - 1\}} \tilde{V}_{k,1}^2 > \delta^2 n^2 \right) \leq \sum_{k=1}^{|nT|-1} \mathbb{P}(\tilde{V}_{k,1}^2 > \delta^2 n^2) \leq \delta^{-2} n^{-2} \sum_{k=1}^{|nT|-1} \mathbb{E}(\tilde{V}_{k,1}^2) = \delta^{-2} n^{-2} \sum_{k=1}^{|nT|-1} O(1) = O(n^{-1})
\]
for \(n \in \mathbb{N}\), where we applied \(\mathbb{E}(\tilde{V}_{k,1}^2) = O(1), k \in \mathbb{N}\) (see Lemma A.6). Thus we obtain (10.6), and hence (10.3).

Now we turn to prove (10.4). For each \(s \in \mathbb{R}_{+}\) and \(n \in \mathbb{N}\), we have \((X_{s,2}^{(n)})^+ = X_{s,2}^{(n)}\) (due to the fact that \(X_{k}\) is non-negative for each \(k \in \mathbb{Z}_{+}\)), and
\[
\int_{0}^{t} (X_{s,2}^{(n)})^+ ds = \int_{0}^{t} n^{-1} X_{[ns],2} ds = \sum_{k=0}^{|nt|-1} \int_{k/n}^{(k+1)/n} n^{-1} X_{k,2} ds + \int_{|nt|/n}^{t} n^{-1} X_{[nt],2} ds = \frac{1}{n^2} \sum_{k=0}^{|nt|-1} X_{k,2} + \frac{1}{n} \left( t - \frac{|nt|}{n} \right) X_{[nt],2} = \frac{1}{n^2} \sum_{k=0}^{|nt|-1} X_{k,2} + \frac{nt - |nt|}{n^2} X_{[nt],2}
\]
for all \(t \in \mathbb{R}_{+}\) and \(n \in \mathbb{N}\). Since
\[
U_{k,2} - \mathbb{E}(U_{k,2} | \mathcal{F}_{k-1}^{X}) = M_{k,2} + a_{2,1} X_{k-1,1} + b_{2} - a_{2,1} X_{k-1,1} - b_{2} = M_{k,2}, \quad k \in \mathbb{N},
\]
33
by Lemma A.1, we have for each \( t \in \mathbb{R}_+ \) and \( n \in \mathbb{N} \),
\[
\frac{1}{n^2} \sum_{k=1}^{\lfloor nt \rfloor} \text{Var}(U_{k,2} | F_{k-1}^X) = \frac{1}{n^2} \sum_{k=1}^{\lfloor nt \rfloor} \mathbb{E}(M_{k,2}^2 | F_{k-1}^X) = \frac{1}{n^2} \sum_{k=1}^{\lfloor nt \rfloor} (v_{2,1}^{(0)} + v_{2,2}^{(1)} X_{k-1,1} + v_{2,2}^{(2)} X_{k-1,2}).
\]
Hence
\[
\frac{1}{n^2} \sum_{k=1}^{\lfloor nt \rfloor} \text{Var}(U_{k,2} | F_{k-1}^X) - \int_0^t (X_{n}^{(n)})^+ v_{2,2}^{(2)} ds = \frac{1}{n^2} \sum_{k=1}^{\lfloor nt \rfloor} (v_{2,1}^{(0)} + v_{2,2}^{(1)} X_{k-1,1} - v_{2,2}^{(2)} nt - \lfloor nt \rfloor X_{\lfloor nt \rfloor,2}, t \in \mathbb{R}_+, n \in \mathbb{N}.
\]

By (10.6), we have for each \( T \in \mathbb{R}^{++}, (10.9) \)
\[
n^{-2} \sup_{t \in [0,T]} \sum_{k=1}^{\lfloor nT \rfloor} X_{k-1,1} \xrightarrow{p} 0 \quad \text{as} \quad n \to \infty,
\]
hence, in order to show (10.4), it suffices to prove
\[
n^{-2} \sup_{t \in [0,T]} |(nt - \lfloor nt \rfloor) X_{\lfloor nt \rfloor,2}| \leq n^{-2} \sup_{t \in [0,T]} X_{\lfloor nt \rfloor,2} \xrightarrow{p} 0 \quad \text{as} \quad n \to \infty \quad \text{for each} \quad T \in \mathbb{R}^{++}.
\]

For each \( t \in \mathbb{R}_+ \) and \( n \in \mathbb{N}, \)
\[
n^{-2} X_{\lfloor nt \rfloor,2} = \frac{1}{n^2} \sum_{k=1}^{\lfloor nt \rfloor} U_{k,2} = \frac{1}{n^2} \sum_{k=1}^{\lfloor nt \rfloor} (M_{k,2} + a_{2,1} X_{k-1,1} + b_2),
\]
hence
\[
n^{-2} X_{\lfloor nt \rfloor,2} = n^{-2} |X_{\lfloor nt \rfloor,2}| \leq \frac{1}{n^2} \sum_{k=1}^{\lfloor nt \rfloor} |M_{k,2}| + \frac{a_{2,1}}{n^2} \sum_{k=1}^{\lfloor nt \rfloor} X_{k-1,1} + \frac{|nt|}{n^2} b_2.
\]
Taking into account (10.9), in order to show (10.10), it suffices to show
\[
(10.10) \quad \frac{1}{n^2} \sum_{k=1}^{\lfloor nT \rfloor} |M_{k,2}| \xrightarrow{p} 0 \quad \text{as} \quad n \to \infty \quad \text{for each} \quad T \in \mathbb{R}^{++}.
\]

We have for each \( T \in \mathbb{R}^{++}, \)
\[
\mathbb{E}\left(\frac{1}{n^2} \sum_{k=1}^{\lfloor nT \rfloor} |M_{k,2}|\right) = \frac{1}{n^2} \sum_{k=1}^{\lfloor nT \rfloor} \mathbb{E}(|M_{k,2}|) \leq \frac{1}{n^2} \sum_{k=1}^{\lfloor nT \rfloor} \sqrt{\mathbb{E}(M_{k,2}^2)} \to 0 \quad \text{as} \quad n \to \infty
\]
since, by Lemma A.6 \( \mathbb{E}(M_{k,2}^2) = O(k), k \in \mathbb{N}. \) This yields (10.11) and hence (10.10), implying (10.4), as desired.
By Markov’s inequality, for each $T \in \mathbb{R}_{++}$ and $\theta \in \mathbb{R}_{++}$, we have

\begin{equation}
\frac{1}{n^2} \sum_{k=1}^{\lfloor nT \rfloor} \mathbb{E}(U_{k,2}^2 1_{\{U_{k,2} > n\theta\}} \mid \mathcal{F}_{k-1}^X) \xrightarrow{L_1} 0 \quad \text{as} \quad n \to \infty.
\end{equation}

(10.12)

By Markov’s inequality, for each $T \in \mathbb{R}_{++}$ and $\theta \in \mathbb{R}_{++}$, we have

\[
\mathbb{E} \left( \frac{1}{n^2} \sum_{k=1}^{\lfloor nT \rfloor} \mathbb{E}(U_{k,2}^4 1_{\{U_{k,2} > n\theta\}} \mid \mathcal{F}_{k-1}^X) \right) = \frac{1}{n^2} \sum_{k=1}^{\lfloor nT \rfloor} \mathbb{E}(U_{k,2}^4 1_{\{U_{k,2} > n\theta\}}) \leq \frac{3^3}{n^4 \theta^2} \sum_{k=1}^{\lfloor nT \rfloor} \mathbb{E}(M_{k,2}^4) + a_{2,1}^4 \mathbb{E}(X_{k-1,1}^4) + b_2^4.
\]

Here, by Lemma A.6

\[
\frac{1}{n^4} \sum_{k=1}^{\lfloor nT \rfloor} \mathbb{E}(M_{k,2}^4) = \frac{1}{n^4} \sum_{k=1}^{\lfloor nT \rfloor} O(k^2) = O(n^{-1}) \to 0 \quad \text{as} \quad n \to \infty.
\]

Further, since the first coordinate process $(X_{k,1})_{k \in \mathbb{Z}_+}$ is a subcritical Galton-Watson process with immigration having offspring and immigration distributions with finite fourth moments, by Szücs [32, Theorem 4] (for an alternative proof, see also Kevei and Wiandt [23]), the unique stationary distribution $\mu_1$ of $(X_{k,1})_{k \in \mathbb{Z}_+}$ has a finite fourth moment as well. Consequently, by Chung [9, Part I, Chapter 15, Theorem 3], we have

\[
\frac{1}{n} \sum_{k=1}^{n} \mathbb{E}(X_{k,1}^4) \to c_{4,\mu} \quad \text{as} \quad n \to \infty,
\]

where $c_{4,\mu} \in \mathbb{R}_+$ denotes the fourth moment of $\mu_1$. Hence

\[
\frac{1}{n^4} \sum_{k=1}^{n} \mathbb{E}(X_{k,1}^4) \to 0 \quad \text{as} \quad n \to \infty \quad \text{for each} \quad T \in \mathbb{R}_{++}.
\]

Putting parts together, we have (10.12), as desired.

Finally, using (5.4) with $a_{2,2} = 1$, (A.4), (A.5) (which holds in case of (5) as well, since only the fact that $a_{1,2} = 0$ was used for deriving it) and $\mathbb{E}(M_k \mid \mathcal{F}_{k-1}^X) = 0, \ k \in \mathbb{N}$ (yielding $\text{Cov}(M_{j,1}, M_{k,1}) = 0$ and $\text{Cov}(M_{j,1}, M_{k,2}) = 0$ for $j \neq k, j, k \in \mathbb{N}$), we have for all $t_1, t_2 \in \mathbb{R}_+$,

\[
\text{Cov} \left( X_{[n_1 t_1],1}, n_2^{-1} X_{[n_2 t_2],2} \right) = \text{Cov} \left( \sum_{j=1}^{[n_1 t_1]} a_{1,1}^{[n_1 t_1]-j} (M_{j,1} + b_1), a_{2,1} n_2^{-1} \sum_{j=1}^{[n_2 t_2]} \frac{1 - a_{1,1}^{[n_2 t_2]-j}}{1 - a_{1,1}} (M_{j,1} + b_1) + n_2^{-1} \sum_{j=1}^{[n_2 t_2]} (M_{j,2} + b_2) \right)
\]

35
\[= a_{2,1} n_2^{-1} \sum_{j=1}^{[n_1 t_1] \wedge [n_2 t_2]} a_{1,1}^{[n_1 t_1] - j} \left( 1 - a_{1,1}^{-[n_2 t_2] - j} \right) \frac{\mathbb{E}(M_j^2)}{1 - a_{1,1}} + n_2^{-1} \sum_{j=1}^{[n_1 t_1] \wedge [n_2 t_2]} a_{1,1}^{[n_1 t_1] - j} \text{Cov}(M_{j,1}, M_{j,2}) \]

\[= \frac{a_{2,1}}{1 - a_{1,1}} n_2^{-1} \sum_{j=1}^{[n_1 t_1] \wedge [n_2 t_2]} a_{1,1}^{[n_1 t_1] - j} (1 - a_{1,1}^{[n_2 t_2] - j}) \left( v_{1,1}^{(0)} + \mathbb{E}(X_{j-1,1}) v_{1,1}^{(1)} \right) \]

\[+ n_2^{-1} \sum_{j=1}^{[n_1 t_1] \wedge [n_2 t_2]} a_{1,1}^{[n_1 t_1] - j} \left( v_{1,2}^{(0)} + \mathbb{E}(X_{j-1,1}) v_{1,2}^{(1)} + \mathbb{E}(X_{j-1,2}) v_{1,2}^{(2)} \right). \]

Since \( a_{1,2} = 0 \), we have \( \mathbb{P}(\xi_{1,1,2,1} = 0) = 1 \), yielding \( v_{1,2}^{(2)} = 0 \), and using \([A.7]\), we have for all \( t_1, t_2 \in \mathbb{R}_+ \),

\[\text{Cov} \left( X_{[n_1 t_1],1}, n_2^{-1} X_{[n_2 t_2],2} \right)\]

\[= \frac{a_{2,1}}{1 - a_{1,1}} \left( v_{1,1}^{(0)} + \frac{v_{1,1}^{(1)} b_1}{1 - a_{1,1}} \right) n_2^{-1} \sum_{j=1}^{[n_1 t_1] \wedge [n_2 t_2]} a_{1,1}^{[n_1 t_1] - j} \]

\[+ \left( v_{1,2}^{(0)} + \frac{v_{1,2}^{(1)} b_1}{1 - a_{1,1}} \right) n_2^{-1} \sum_{j=1}^{[n_1 t_1] \wedge [n_2 t_2]} a_{1,1}^{[n_1 t_1] - j} \]

\[= \left( \frac{a_{2,1}}{1 - a_{1,1}} + 1 \right) \left( v_{1,2}^{(0)} + \frac{v_{1,2}^{(1)} b_1}{1 - a_{1,1}} \right) n_2^{-1} a_{1,1}^{[n_1 t_1] - [n_1 t_1] \wedge [n_2 t_2]} \frac{1 - a_{1,1}^{[n_2 t_2]}}{1 - a_{1,1}}, \]

which yields \([3.16]\) (due to \( a_{1,1} \in [0, 1) \) and \( X_{0,1} = 0 \)). Using again \( a_{1,1} \in [0, 1] \), we have

\[\sup_{t_1, t_2 \in \mathbb{R}_+} \sup_{n_1 \in \mathbb{N}} \left| \text{Cov}(X_{[n_1 t_1],1}, n_2^{-1} X_{[n_2 t_2],2}) \right| \leq \left( \frac{a_{2,1}}{1 - a_{1,1}} + 1 \right) \left( v_{1,2}^{(0)} + \frac{v_{1,2}^{(1)} b_1}{1 - a_{1,1}} \right) \frac{1}{n_2} \to 0 \]

as \( n_2 \to \infty \), which yields \([3.13]\), as desired. \(\square\)

**Appendices**

**A Moments**

In the proof of the results, we will use some formulae and estimates for the first, second and fourth order moments of the coordinates of the processes \((X_k)_{k \in \mathbb{Z}_+}\) and \((M_k)_{k \in \mathbb{Z}_+}\).
\textbf{A.1 Lemma.} Let \((X_k)_{k\in\mathbb{Z}_+}\) be a \(p\)-type Galton-Watson process with immigration such that \(X_0 = 0\) and the moment condition (2.2) holds. Then for all \(k \in \mathbb{N}\), we have \(\mathbb{E}(X_k | \mathcal{F}^X_{k-1}) = AX_{k-1} + b\) and

\begin{equation}
\mathbb{E}(X_k) = \sum_{j=0}^{k-1} A^j b,
\end{equation}

\begin{equation}
\text{Var}(X_k | \mathcal{F}^X_{k-1}) = \text{Var}(M_k | \mathcal{F}^X_{k-1}) = \mathbb{E}(M_k M_k^T | \mathcal{F}^X_{k-1}) = V^{(0)} + \sum_{i=1}^{p} X_{k-1,i} V^{(i)},
\end{equation}

\begin{equation}
\text{Var}(X_k) = \sum_{j=0}^{k-1} A^j \mathbb{E}(M_{k-j} M_{k-j}^T) (A^T)^j,
\end{equation}

\begin{equation}
\mathbb{E}(M_k M_k^T) = V^{(0)} + \sum_{i=1}^{p} \mathbb{E}(X_{k-1,i}) V^{(i)},
\end{equation}

where \(V^{(0)}\) and \(V^{(i)}, i = 1, \ldots, p\), are given in Section 2.

Lemma A.1 is a special case of Lemma A.1 in Ispány and Pap [17] for \(p\)-type Galton-Watson processes with immigration starting from 0. For completeness, we note that Lemma A.1 in Ispány and Pap [17] is stated only for critical \(p\)-type Galton-Watson processes with immigration, but its proof readily shows that it holds not only in the critical case.

\textbf{A.2 Lemma.} Let \((X_k)_{k\in\mathbb{Z}_+}\) be a critical decomposable 2-type Galton-Watson process with immigration such that \(X_0 = 0\), the moment conditions \(\mathbb{E}(\|\xi_i\|^4) < \infty, i = 1, 2\), and \(\mathbb{E}(\|e\|^4) < \infty\) hold and its offspring mean matrix \(A\) satisfies (1) of (3.2). Then we have

\[
\begin{align*}
\mathbb{E}(X_{k,1}) &= O(k), \quad \mathbb{E}(X_{k,2}) = O(k), \quad \mathbb{E}(|M_{k,1}|) = O(k^{1/2}), \quad \mathbb{E}(|M_{k,2}|) = O(k^{1/2}), \\
\mathbb{E}(M_{k,1}^2) &= O(k), \quad \mathbb{E}(M_{k,2}^2) = O(k), \quad \mathbb{E}(X_{k,1}^2) = O(k^2), \quad \mathbb{E}(X_{k,2}^2) = O(k^2), \\
\mathbb{E}(M_{k,1}^4) &= O(k^2), \quad \mathbb{E}(M_{k,2}^4) = O(k^2)
\end{align*}
\]

for \(k \in \mathbb{N}\).

\textbf{Proof.} By (A.1) and (5.3), we obtain

\[
\begin{bmatrix}
\mathbb{E}(X_{k,1}) \\
\mathbb{E}(X_{k,2})
\end{bmatrix} = \sum_{j=0}^{k-1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} b = \begin{bmatrix} b_1 k \\ b_2 k \end{bmatrix}, \quad k \in \mathbb{Z}_+,
\]

and we conclude the first two statements.

By (A.4),

\[
\begin{align*}
\mathbb{E}(M_{k,1}^2) &= v^{(0)}_{1,1} + \mathbb{E}(X_{k-1,1}) v^{(1)}_{1,1}, \quad k \in \mathbb{N}, \\
\mathbb{E}(M_{k,2}^2) &= v^{(0)}_{2,2} + \mathbb{E}(X_{k-1,2}) v^{(2)}_{2,2}, \quad k \in \mathbb{N},
\end{align*}
\]

37
since \( a_{1,2} = a_{2,1} = 0 \) implies \( \xi_{1,1,2,1} = 0 \) and \( \xi_{1,1,1,2} = 0 \), yielding \( v_{1,1}^{(2)} = v_{2,2}^{(1)} = 0 \). This together with the first two statements yield \( \mathbb{E}(M_{k,i}^2) = O(k) \) for \( k \in \mathbb{N} \) and \( i = 1, 2 \). Consequently, using the inequalities \( \mathbb{E}(|M_{k,i}|) \leq \sqrt{\mathbb{E}(M_{k,i}^2)} \), \( i = 1, 2 \), we also have \( \mathbb{E}(|M_{k,i}|) = O(k^{1/2}) \) for \( k \in \mathbb{N} \) and \( i = 1, 2 \).

Further, \( \mathbb{E}(X_{k,i}^2) = \text{Var}(X_{k,i}) + \mathbb{E}(X_{k,i})^2 \), \( i = 1, 2 \), and, by \( (A.3) \) and \( (A.4) \), we have

\[
\text{Var}(X_k) = \sum_{j=0}^{k-1} A^j \mathbb{E}(M_{k-j} \!\!\!\trans M_{k-j}^\trans) (A^\trans)^j = \sum_{j=0}^{k-1} I_j^2 \mathbb{E}(M_{k-j} \!\!\!\trans M_{k-j}^\trans) I_j^2
\]

\[
= \sum_{j=0}^{k-1} \left( V^{(0)} + \sum_{i=1}^{2} \mathbb{E}(X_{k-j-1,i})V^{(i)} \right), \quad k \in \mathbb{Z}_+.
\]

By the first two statements,

\[
\| \text{Var}(X_k) \| \leq \sum_{j=0}^{k-1} \left( \| V^{(0)} \| + \sum_{i=1}^{2} \mathbb{E}(X_{k-j-1,i}) \| V^{(i)} \| \right)
\]

\[
= \sum_{j=0}^{k-1} \left( \| V^{(0)} \| + \sum_{i=1}^{2} O(k-j-1) \| V^{(i)} \| \right) = O(k^2), \quad k \in \mathbb{N},
\]

where \( \| B \| \) denotes the operator norm of a matrix \( B \in \mathbb{R}^{2 \times 2} \) defined by \( \| B \| := \sup_{\| x \|=1, x \in \mathbb{R}^2} \| Bx \| \). This together with the first two statements yield \( \mathbb{E}(X_{k,i}^2) = O(k^2) \) for \( k \in \mathbb{N} \) and \( i = 1, 2 \). Finally, the relations \( \mathbb{E}(M_{k,i}^4) = O(k^2) \) for \( k \in \mathbb{N}, \ i = 1, 2 \), follow in the same way as in the proof of Lemma A.2 in Ispány and Pap [17] (at this part the authors do not use that the critical multi-type Galton-Watson process with immigration that they consider is primitive, they only use the fact that the second moments of the coordinates of the branching process in question at \( k \) is of \( O(k^2), \ k \in \mathbb{N} \)). \( \square \)

**A.3 Lemma.** Let \( (X_k)_{k \in \mathbb{Z}_+} \) be a critical decomposable 2-type Galton-Watson process with immigration such that \( X_0 = 0 \), the moment condition \((2.2)\) holds and its offspring mean matrix \( A \) satisfies \((2)\) of \((3.2)\). Then we have

\[
\mathbb{E}(X_{k,1}) = O(k), \quad \mathbb{E}(X_{k,2}) = O(k^2), \quad \mathbb{E}(|M_{k,1}|) = O(k^{1/2}), \quad \mathbb{E}(|M_{k,2}|) = O(k),
\]

\[
\mathbb{E}(M_{k,1}^2) = O(k), \quad \mathbb{E}(M_{k,2}^2) = O(k^2)
\]

for \( k \in \mathbb{N} \).

**Proof.** By \((A.1)\) and \((A.3)\), we obtain

\[
\begin{bmatrix}
\mathbb{E}(X_{k,1}) \\
\mathbb{E}(X_{k,2})
\end{bmatrix} = \sum_{j=0}^{k-1} \begin{bmatrix}
1 & 0 \\ a_{2,1,j} & 1
\end{bmatrix} b = \begin{bmatrix}
b_1k \\
\frac{1}{2}a_{2,1}b_1(k-1) + b_2k
\end{bmatrix}, \quad k \in \mathbb{Z}_+,
\]

and we conclude the first two statements.
By (A.4),

\[
(E(M_{k,1}^2) = v_{1,1}^{(0)} + E(X_{k-1,1})v_{1,1}^{(1)}, \quad k \in \mathbb{N},
\]

(A.5)

\[
E(M_{k,2}^2) = v_{2,2}^{(0)} + E(X_{k-1,1})v_{2,2}^{(1)} + E(X_{k-1,2})v_{2,2}^{(2)}, \quad k \in \mathbb{N},
\]

since \(a_{1,2} = 0\) implies \(\xi_{1,2,1} \overset{a.s.}{=} 0\), yielding \(v_{1,1}^{(2)} = 0\). Note that for deriving (A.5) we did not use that \(A\) satisfies (2), we only used \(a_{1,2} = 0\), so (A.5) holds if \(A\) has the form (1), (3), (4) or (5) as well. Using (A.5) together with the first two statements we have the last two statements. Finally, using the inequalities \(E(|M_{k,i}|) \leq \sqrt{E(M_{k,i}^2)}\), \(i = 1, 2\), and the last two statements, we have the third and fourth statements. \(\square\)

We note that the statements \(E(X_{k,1}) = O(k), \quad k \in \mathbb{N}\), and \(E(X_{k,2}) = O(k^2), \quad k \in \mathbb{N}\), in Lemma A.3 are in accordance with the corresponding ones in Theorem 3 in Foster and Ney [13].

A.4 Lemma. Let \((X_k)_{k \in \mathbb{Z}_+}\) be a critical decomposable 2-type Galton-Watson process with immigration such that \(X_0 = 0\), the moment condition (2.2) holds and its offspring mean matrix \(A\) satisfies (3) of (3.2). Then we have

\[
E(X_{k,1}) = O(k), \quad E(X_{k,2}) = O(1), \quad E(|M_{k,1}|) = O(k^{1/2}), \quad E(|M_{k,2}|) = O(1),
\]

\[
E(M_{k,1}^2) = O(k), \quad E(M_{k,2}^2) = O(1)
\]

for \(k \in \mathbb{N}\).

Proof. By (A.1) and (5.3), we obtain

\[
\begin{bmatrix}
E(X_{k,1}) \\
E(X_{k,2})
\end{bmatrix} = \sum_{j=0}^{k-1} \begin{bmatrix}
1 & 0 \\
0 & a_{2,2}^j
\end{bmatrix} b = \begin{bmatrix}
b_1k \\
\frac{b_1}{1-a_{2,2}} k
\end{bmatrix}, \quad k \in \mathbb{Z}_+,
\]

and we conclude the first two statements.

Using (A.5) (which holds in case of (3) as well, since only the fact that \(a_{1,2} = 0\) was used for deriving it), the first two statements, and that \(a_{2,1} = 0\) implies \(v_{2,2}^{(1)} = 0\), we have the last two statements. Finally, using the inequalities \(E(|M_{k,i}|) \leq \sqrt{E(M_{k,i}^2)}\), \(i = 1, 2\), and the last two statements, we have the third and fourth statements. \(\square\)

A.5 Lemma. Let \((X_k)_{k \in \mathbb{Z}_+}\) be a critical decomposable 2-type Galton-Watson process with immigration such that \(X_0 = 0\), the moment conditions \(E(||\xi_i||^4) < \infty, \quad i = 1, 2\), and \(E(||\xi||^4) < \infty\) hold and its offspring mean matrix \(A\) satisfies (4) of (3.2). Then we have

\[
E(X_{k,1}) = O(k), \quad E(X_{k,2}) = O(k), \quad E(|M_{k,1}|) = O(k^{1/2}), \quad E(|M_{k,2}|) = O(k^{1/2}),
\]

\[
E(M_{k,1}^2) = O(k), \quad E(M_{k,2}^2) = O(k), \quad E(X_{k,1}^2) = O(k^2), \quad E(X_{k,2}^2) = O(k^2),
\]

\[
E(M_{k,1}^4) = O(k^2), \quad E(M_{k,2}^4) = O(k^2), \quad E(V_{k,1}^4) = O(k^2), \quad E(V_{k,2}^4) = O(k^2)
\]

39
for \( k \in \mathbb{N} \), where

\[
V_{k,i} := \sum_{j=1}^{k} a_{2,2}^{k-j} M_{j,i}, \quad k \in \mathbb{N}, \quad i \in \{1, 2\}.
\]

Proof. By (A.1) and (5.3), we obtain

\[
\begin{bmatrix}
\mathbb{E}(X_{k,1}) \\
\mathbb{E}(X_{k,2})
\end{bmatrix} = \sum_{j=0}^{k-1} \begin{bmatrix}
1 & 0 \\
\frac{1-a_{2,2}^{j}}{1-a_{2,2}^{j}} & a_{2,2}^{j}
\end{bmatrix} \begin{bmatrix}
\frac{k}{a_{2,1}b_{1}} & \frac{k}{a_{2,1}b_{2}} \\
\frac{1-a_{2,2}^{j}}{1-a_{2,2}^{j}} & \frac{1-a_{2,2}^{j}}{1-a_{2,2}^{j}} + \frac{k}{a_{2,2}^{j}} \cdot b_{2}
\end{bmatrix}, \quad k \in \mathbb{Z}_{+},
\]

and we conclude the first two statements.

The next four statements follow from \([A.5]\) (which holds in case of (4) as well as it was explained earlier) using the inequalities \( \mathbb{E}(|M_{k,i}|) \leq \mathbb{E}(M_{k,i}^{2}), \quad i = 1, 2 \).

Further, \( \mathbb{E}(X_{k,i}^{2}) = \mathbb{V}(X_{k,i}) + (\mathbb{E}(X_{k,i}))^{2}, \quad i = 1, 2 \), and, by (A.3) and (A.4),

\[
\mathbb{V}(X_{k}) = \sum_{j=0}^{k-1} A^{j} \mathbb{E}(M_{k-j} M_{k-j}^{\top}) A^{j} = \sum_{j=0}^{k-1} A^{j} \left( V^{(0)} + \sum_{i=1}^{2} \mathbb{E}(X_{k-j-1,i}) V^{(i)} \right) A^{j}
\]

for \( k \in \mathbb{Z}_{+} \). Hence, using the first two statements, we have

\[
\| \mathbb{V}(X_{k}) \| \leq \sum_{j=0}^{k-1} \| A^{j} \|^{2} \left( \| V^{(0)} \| + \sum_{i=1}^{2} \mathbb{E}(X_{k-j-1,i}) \| V^{(i)} \| \right) = \sum_{j=0}^{k-1} \| A^{j} \|^{2} O(k-j-1)
\]

\[
= \left( \sum_{j=0}^{k-1} \| A^{j} \|^{2} \right) O(k), \quad k \in \mathbb{N}.
\]

Using the continuity of the operator norm function \( \| \cdot \| \), we have

\[
\lim_{j \to \infty} \| A^{j} \| = \lim_{j \to \infty} \left\| \begin{bmatrix}
\frac{1}{a_{2,1}b_{1}} & 0 \\
\frac{1-a_{2,2}^{j}}{1-a_{2,2}^{j}} & a_{2,2}^{j}
\end{bmatrix} \right\| = \left\| \begin{bmatrix}
\frac{1}{a_{2,1}b_{1}} & 0 \\
\frac{1-a_{2,2}^{j}}{1-a_{2,2}^{j}} & a_{2,2}^{j}
\end{bmatrix} \right\| < \infty,
\]

yielding that \( c_{A} := \sup_{j \in \mathbb{N}} \| A^{j} \| < \infty \). Hence \( \| \mathbb{V}(X_{k}) \| \leq c_{A}^{2} \left( \sum_{j=0}^{k-1} 1 \right) O(k) = O(k^{2}), \quad k \in \mathbb{N} \). This together with the first two statements yield \( \mathbb{E}(X_{k,i}^{2}) = O(k^{2}) \) for \( k \in \mathbb{N} \) and \( i = 1, 2 \). The relations \( \mathbb{E}(M_{k,i}^{4}) = O(k^{2}) \) for \( k \in \mathbb{N} \) and \( i = 1, 2 \), follow in the same way as in the proof of Lemma A.2 in Ispány and Pap [17] (at this part the authors do not use that the critical multi-type Galton-Watson process with immigration that they consider is primitive, they only use the fact that the second moments of the coordinates of the branching process in question at \( k \) is of \( O(k^{2}), \quad k \in \mathbb{N} \).

Finally, we prove \( \mathbb{E}(V_{k,i}^{4}) = O(k^{2}) \) for \( k \in \mathbb{N} \), \( i = 1, 2 \), using induction in \( k \). Since \( \mathbb{E}(M_{k,i}^{4}) = O(k^{2}) \) for \( k \in \mathbb{N} \) and \( i = 1, 2 \), there exists \( \widetilde{C} \in \mathbb{R}_{++} \) such that \( \mathbb{E}(M_{k,i}^{4}) \leq \widetilde{C} k^{2} \) for each \( k \in \mathbb{N} \) and \( i = 1, 2 \). For each \( k \in \mathbb{N} \), let

\[
C_{k} := \left( \sum_{j=0}^{k-1} a_{2,2}^{j} \right) ^{4} \widetilde{C} = \left( \frac{1-a_{2,2}^{k}}{1-a_{2,2}^{j}} \right) ^{4} \widetilde{C}.
\]
Then \( C_k \leq \frac{\tilde{c}}{(1-a_{2,2})} < \infty \), \( k \in \mathbb{N} \) (due to \( a_{2,2} \in [0,1) \)), and \( C_{k+1/4}^{1/4} = a_{2,2}C_k^{1/4} + \tilde{C}^{1/4} \), \( k \in \mathbb{N} \), since
\[
a_{2,2}C_k^{1/4} + \tilde{C}^{1/4} = a_{2,2} \frac{1-a_{2,2}^k}{1-a_{2,2}} \tilde{C}^{1/4} + \tilde{C}^{1/4} = \frac{1-a_{2,2}^{k+1}}{1-a_{2,2}} \tilde{C}^{1/4} = C_{k+1}^{1/4}.
\]

Since \( V_{1,i} = M_{1,i}, i = 1,2 \), we get \( \mathbb{E}(V_{1,i}^4) = \mathbb{E}(M_{1,i}^4) \leq \tilde{C} = C_1, i = 1,2 \). Now assume that for some \( k_0 \in \mathbb{N} \), the following inequalities hold
\[
(A.6) \quad \mathbb{E}(V_{k_0,i}^4) \leq C_{k_0} \ell^2, \quad \ell = 1, \ldots, k_0.
\]
By the decompositions \( V_{k,i} = a_{2,2}V_{k-1,i} + M_{k,i}, k \in \mathbb{N}, i = 1,2 \), where \( V_{0,i} := 0, i = 1,2 \) (see also (9.4) and (9.6)), and the triangular inequality for the \( L_4 \)-norm, we have that
\[
(\mathbb{E}(V_{k_0+1,i}^4))^{1/4} \leq a_{2,2}(\mathbb{E}(V_{k_0,i}^4))^{1/4} + (\mathbb{E}(M_{k_0,i}^4))^{1/4}, \quad i = 1,2.
\]
Consequently, using also the induction hypothesis \([A.6]\), we get that
\[
(\mathbb{E}(V_{k_0+1,i}^4))^{1/4} \leq a_{2,2}C_{k_0}^{1/4} \tilde{C}^{1/4} + \tilde{C}^{1/4} \tilde{C}^{1/4} \leq C^{1/4} \tilde{C}^{1/4} (k_0 + 1)^{1/4}, \quad i = 1,2
\]
yielding that \( \mathbb{E}(V_{k_0+1,i}^4) \leq C_{k_0+1} (k_0+1)^2 \), \( i = 1,2 \). Hence \( \mathbb{E}(V_{k,i}^4) \leq C_k k^2 \leq \frac{\tilde{c}}{(1-a_{2,2})} k^2 \) for each \( k \in \mathbb{N} \) and \( i = 1,2 \), which implies that \( \mathbb{E}(V_{k,i}^4) = O(k^2) \) for each \( k \in \mathbb{N} \), \( i = 1,2 \). \( \square \)

**A.6 Lemma.** Let \((X_k)_{k \in \mathbb{Z}_+}\) be a critical decomposable 2-type Galton-Watson process with immigration such that \( X_0 = 0 \), the moment conditions \( \mathbb{E}(|\xi_i|^4) < \infty, i = 1,2 \), and \( \mathbb{E}(\|\xi\|^4) < \infty \) hold and its offspring mean matrix \( A \) satisfies (5) of (3.2). Then we have
\[
\mathbb{E}(X_{k,1}) = O(1), \quad \mathbb{E}(X_{k,2}) = O(k), \quad \mathbb{E}(|M_{k,1}|) = O(1), \quad \mathbb{E}(|M_{k,2}|) = O(k^{1/2}),
\]
\[
\mathbb{E}(M_{k,1}^2) = O(1), \quad \mathbb{E}(M_{k,2}^2) = O(k), \quad \mathbb{E}(X_{k,1}^2) = O(1), \quad \mathbb{E}(X_{k,2}^2) = O(k^2),
\]
\[
\mathbb{E}(M_{k,1}^4) = O(1), \quad \mathbb{E}(M_{k,2}^4) = O(k^2), \quad \mathbb{E}(\tilde{V}_{k,1}^2) = O(1)
\]
for \( k \in \mathbb{N} \), where
\[
\tilde{V}_{k,1} := \sum_{j=1}^{k} a_{1,1}^{k-j} M_{j,1}, \quad k \in \mathbb{N}.
\]

**Proof.** By (A.6) and (5.3), we obtain
\[
(A.7) \quad \begin{bmatrix} \mathbb{E}(X_{k,1}) \\ \mathbb{E}(X_{k,2}) \end{bmatrix} = \sum_{j=0}^{k-1} \begin{bmatrix} a_{1,1}^{j+1} & 0 \\ a_{2,1}^{j+1} & 1 \end{bmatrix} \begin{bmatrix} b_1 \frac{1-a_{1,1}^{j+1}}{1-a_{1,1}} \\ b_2 \frac{1-a_{1,1}^{j+1}}{1-a_{1,1}} \end{bmatrix}, \quad k \in \mathbb{Z}_+,
\]
and we conclude the first two statements.

The next four statements follow from (A.6) (which holds in case of (5) as well as it was explained earlier) using the inequalities \( \mathbb{E}(|M_{k,i}|) \leq \sqrt{\mathbb{E}(M_{k,i}^2)}, i = 1,2 \).
Further, $\mathbb{E}(X^2_{k,1}) = \text{Var}(X_{k,1}) + (\mathbb{E}(X_{k,1}))^2$, and using that $(X_{k,1})_{k \in \mathbb{Z}_+}$ is a single-type Galton-Watson process with immigration starting from 0 (explained at the beginning of Section 3), by (A.3) and (A.4), we have

$$\text{Var}(X_{k,1}) = \sum_{j=0}^{k-1} a_j^{2j} \mathbb{E}(M^2_{k-j,1}) = \sum_{j=0}^{k-1} a_j^{2j} \left( v_{1,1}^{(0)} + \mathbb{E}(X_{k-j-1,1})v_{1,1}^{(1)} \right)$$

$$= \sum_{j=0}^{k-1} a_j^{2j} \left( v_{1,1}^{(0)} + \frac{1}{1-a_{1,1}} v_{1,1}^{(1)} \right) \left( \frac{b_1}{1-a_{1,1}} v_{1,1}^{(1)} \right) \sum_{j=0}^{k-1} a_j^{2j} \leq \left( \frac{v_{1,1}^{(0)}}{1-a_{1,1}} \right) \frac{1}{1-a_{1,1}} = O(1), \quad k \in \mathbb{N}.$$ 

This together with $\mathbb{E}(X_{k,1}) = O(1)$ for $k \in \mathbb{N}$ yield $\mathbb{E}(X^2_{k,1}) = O(1)$ for $k \in \mathbb{N}$. The relation $\mathbb{E}(M^4_{k,1}) = O(1)$ for $k \in \mathbb{N}$ follows in the same way as in the proof of Lemma A.2 in Ispány and Pap [17] taking into account the fact that $\mathbb{E}(X^2_{k,1}) = O(1)$ for $k \in \mathbb{N}$.

Next, we check that $\mathbb{E}(X^2_{k,2}) = O(k^2)$ for $k \in \mathbb{N}$. By (A.3), (A.4) and the first two statements, we have

$$\| \text{Var}(X_k) \| = \left\| \sum_{j=0}^{k-1} \mathbb{A}_j \left( V^{(0)} + \sum_{i=1}^2 \mathbb{E}(X_{k-j-1,i})V^{(i)} \right) (\mathbb{A}^\top)_j \right\|$$

$$\leq \sum_{j=0}^{k-1} \| \mathbb{A}_j \|^2 \left( \| V^{(0)} \| + \sum_{i=1}^2 \mathbb{E}(X_{k-j-1,i}) \| V^{(i)} \| \right)$$

$$= \sum_{j=0}^{k-1} \| \mathbb{A}_j \|^2 O(k-j-1) = \left( \sum_{j=0}^{k-1} \| \mathbb{A}_j \|^2 \right) O(k), \quad k \in \mathbb{N}.$$ 

Using the continuity of the norm function $\| \cdot \|$, we have

$$\lim_{j \to \infty} \| \mathbb{A}_j \| = \lim_{j \to \infty} \left\| \begin{array}{cc} a_{j,1}^{2j} & 0 \\ a_{2,1}^{2j} \frac{1-a_{1,1}}{1-a_{1,1}} & 1 \end{array} \right\| = \left\| \begin{array}{cc} 0 & 0 \\ a_{2,1}^{2j} \frac{1-a_{1,1}}{1-a_{1,1}} & 1 \end{array} \right\| < \infty,$$

yielding that $c_A := \sup_{j \in \mathbb{N}} \| \mathbb{A}_j \| < \infty$. Hence $\| \text{Var}(X_k) \| \leq c_A^2 \left( \sum_{j=0}^{k-1} 1 \right) O(k) = O(k^2),$ $k \in \mathbb{N}$. This together with $\mathbb{E}(X_{k,2}) = O(k)$ for $k \in \mathbb{N}$ yields $\mathbb{E}(X^2_{k,2}) = O(k^2)$ for $k \in \mathbb{N}$, as desired. Note that the above estimation for $\| \text{Var}(X_k) \|$ also yields the crude estimation $\mathbb{E}(X^2_{k,1}) = O(k^2)$ for $k \in \mathbb{N}$, but, as we already showed, $\mathbb{E}(X^2_{k,2}) = O(1)$ for $k \in \mathbb{N}$ holds.

The relation $\mathbb{E}(M^4_{k,2}) = O(k^2)$ for $k \in \mathbb{N}$ follows in the same way as in the proof of Lemma A.2 in Ispány and Pap [17] taking into account the fact that $\mathbb{E}(X^2_{k,2}) = O(k^2)$ for $k \in \mathbb{N}$.

Finally, we prove $\mathbb{E}(\tilde{V}^2_{k,1}) = O(1)$ for $k \in \mathbb{N}$. Note that, for each $k \in \mathbb{N}$, we have that $\mathbb{E}(M_{j,1}M_{\ell,1}) = 0$ for $j \neq \ell$, $j, \ell = 1, \ldots, k$. Consequently, using also that $\mathbb{E}(M^2_{k,1}) = O(1)$
for $k \in \mathbb{N}$ (which was proved before), we get that

$$
\mathbb{E}(\tilde{V}_{k,1}^2) = \mathbb{E}\left(\left(\sum_{j=1}^{k} a_{1,1}^{k-j} M_{j,1}\right)^2\right) = \sum_{j=1}^{k} a_{1,1}^{2(k-j)} \mathbb{E}(M_{j,1}^2) = \left(\sum_{j=1}^{k} a_{1,1}^{2(k-j)}\right) \mathcal{O}(1) = \mathcal{O}(1).
$$

for $k \in \mathbb{N}$, since $\sum_{j=1}^{k} a_{1,1}^{2(k-j)} \leq \sum_{j=1}^{\infty} a_{1,1}^{2j} = \frac{a_{1,1}^2}{1-a_{1,1}^2} < \infty$, $k \in \mathbb{N}$, due to $a_{1,1} \in [0, 1)$. \hfill \Box

## B  Asymptotic behaviour of a single-type Galton-Watson process with immigration in the subcritical and critical cases

Let $(X_k)_{k \in \mathbb{Z}_+}$ be a single-type Galton-Watson process with immigration, i.e., $X_k = \sum_{j=1}^{X_k} \xi_{k,j} + \varepsilon_k$, $k \in \mathbb{N}$, where $\{X_0, \xi_{k,j}, \varepsilon_k : k, j \in \mathbb{N}\}$ are supposed to be independent, $\{\xi_{k,j} : k, j \in \mathbb{N}\}$ and $\{\varepsilon_k : k \in \mathbb{N}\}$ are supposed to consist of identically distributed $\mathbb{Z}_+$-valued random variables. Let $\xi$ and $\varepsilon$ be random variables such that $\xi \overset{d}{=} \xi_{1,1}$ and $\varepsilon \overset{d}{=} \varepsilon_1$. If $a = \mathbb{E}(\varepsilon) \in [0, 1)$ and $\sum_{\ell=1}^{\infty} \log(\ell) \mathbb{P}(\varepsilon = \ell) < \infty$, then the Markov chain $(X_k)_{k \in \mathbb{Z}_+}$ admits a unique stationary distribution $\mu$ with a generator function

$$
\prod_{j=0}^{\infty} H(G(j)(z)), \quad z \in D := \{z \in \mathbb{C} : |z| \leq 1\}, \tag{B.1}
$$

where

$$
G(z) := \mathbb{E}(z^\xi), \quad z \in D, \quad \text{and} \quad H(z) := \mathbb{E}(z^\varepsilon), \quad z \in D,
$$

are the generator functions of $\xi$ and $\varepsilon$, respectively, $G(0)(z) := z$, $G(1)(z) := G(z)$, and $G(k+1)(z) := G(k)(G(z))$, $z \in D$, $k \in \mathbb{N}$, see, e.g., Quine [27]. Note also that if $a \in [0, 1)$ and $\mathbb{P}(\varepsilon = 0) = 1$, then $\sum_{\ell=1}^{\infty} \log(\ell) \mathbb{P}(\varepsilon = \ell) = 0$ and $\mu$ is the Dirac measure $\delta_0$ concentrated at the point 0. In fact, $\mu = \delta_0$ if and only if $\mathbb{P}(\varepsilon = 0) = 1$. Moreover, if $a = 0$ (which is equivalent to $\mathbb{P}(\xi = 0) = 1$), then $\mu$ is the distribution of $\varepsilon$.

The next result is about the asymptotic behaviour of single-type subcritical Galton-Watson processes with immigration satisfying first order moment conditions, which may be known, but we could not address any reference for it, so we provide a proof as well.

### B.1 Lemma

Let $(X_k)_{k \in \mathbb{Z}_+}$ be a single-type Galton-Watson process with immigration such that $a = \mathbb{E}(\varepsilon) \in [0, 1)$, $\mathbb{E}(\varepsilon) < \infty$ and $\mathbb{E}(X_0) < \infty$. Then

$$(X_{[nt]})_{t \in \mathbb{R}_+} \overset{d_f}{\longrightarrow} (X_t)_{t \in \mathbb{R}_+} \quad \text{as} \quad n \to \infty,$$

where $(X_t)_{t \in \mathbb{R}_+}$ is an i.i.d. process such that for each $t \in \mathbb{R}_+$, the distribution of $X_t$ is $\mu$ having generator function given in (B.1).
Proof. Under the assumptions, the Markov chain \((X_k)_{k \in \mathbb{Z}_+}\) admits a unique stationary distribution \(\mu\) with expectation \(\frac{b}{1-b}\) (where \(b = \mathbb{E}(\varepsilon)\)) and a generator function given in (B.1). Let \((Y_k)_{k \in \mathbb{Z}_+}\) be a Galton-Watson processes with immigration with the same offspring and immigration variables as \((X_k)_{k \in \mathbb{Z}_+}\), but let the distribution of \(Y_0\) be \(\mu\). Hence the Markov chain \((Y_k)_{k \in \mathbb{Z}_+}\) is strongly stationary. By induction with respect to \(m \in \mathbb{N}\), first we check that for each \(t_1, \ldots, t_m \in \mathbb{R}_+\) with \(t_1 < \ldots < t_m\), we have

\[
(Y_{[nt_1]}, \ldots, Y_{[nt_m]}) \xrightarrow{D} (X_{t_1}, \ldots, X_{t_m}) \quad \text{as} \quad n \to \infty.
\]  

(B.2)

For \(m = 1\), we have \(Y_{[nt_1]} \xrightarrow{D} X_{t_1}\) as \(n \to \infty\), since for each \(n \in \mathbb{N}\), the distribution of \(Y_{[nt_1]}\) is \(\mu\) which coincides with the distribution of \(X_{t_1}\). Suppose that (B.2) holds for some \(m \in \mathbb{N}\), \(t_1, \ldots, t_m \in \mathbb{R}_+\) with \(t_1 < \ldots < t_m\), and let \(t_{m+1} \in \mathbb{R}_+\) such that \(t_m < t_{m+1}\). The strongly stationary Markov chain \((Y_k)_{k \in \mathbb{Z}_+}\) is strongly mixing, i.e.,

\[
\alpha_{\ell} := \sup_{i \in \mathbb{Z}_+} \sup_{A \in \mathcal{F}_{0,i}^Y, B \in \mathcal{F}_{i,\ell}^Y} \left| \mathbb{P}(A \cap B) - \mathbb{P}(A) \mathbb{P}(B) \right| \to 0 \quad \text{as} \quad \ell \to \infty,
\]

where \(\mathcal{F}_{0,i}^Y := \sigma(Y_0, \ldots, Y_i)\) and \(\mathcal{F}_{i,\ell}^Y := \sigma(Y_i, Y_{i+1}, \ldots)\) for \(i \in \mathbb{Z}_+\), see Barczy et al. [4] Lemma F.1] or Basrak et al. [5] Remark 3.1]. The strong stationarity of \((Y_k)_{k \in \mathbb{Z}_+}\) implies that

\[
|\mathbb{E}(UV) - \mathbb{E}(U) \mathbb{E}(V)| \leq 4C_1 C_2 \alpha_m
\]

for any \(\mathcal{F}_{0,j}^Y\)-measurable (real-valued) random variable \(U\) and any \(\mathcal{F}_{j+m,\infty}^Y\)-measurable random variable \(V\) with \(j, m \in \mathbb{N} \), \(|U| \leq C_1\) and \(|V| \leq C_2\) (see, e.g., Lemma 1.2.1 in Lin and Lu [24]). Hence, for any \(\mathcal{F}_{0,j}^Y\)-measurable complex-valued random variable \(U\) and any \(\mathcal{F}_{j+m,\infty}^Y\)-measurable complex-valued random variable \(V\) with \(j, m \in \mathbb{N} \), \(|U| \leq C_1\) and \(|V| \leq C_2\), we get

\[
|\mathbb{E}(UV) - \mathbb{E}(U) \mathbb{E}(V)|
\leq |\mathbb{E}(\Re(U)\Re(V)) - \mathbb{E}(\Re(U)) \mathbb{E}(\Re(V))| + |\mathbb{E}(\Im(U)\Im(V)) - \mathbb{E}(\Im(U)) \mathbb{E}(\Im(V))|
\]

\[
+ |\mathbb{E}(\Re(U)\Im(V)) - \mathbb{E}(\Re(U)) \mathbb{E}(\Im(V))| + |\mathbb{E}(\Im(U)\Re(V)) - \mathbb{E}(\Im(U)) \mathbb{E}(\Re(V))| \leq 16C_1 C_2 \alpha_m,
\]

where for a complex number \(z \in \mathbb{C}\), \(\Re(z)\) and \(\Im(z)\) denote the real and imaginary part of \(z\), respectively. Consequently, for each \(u_1, \ldots, u_m, u_{m+1} \in \mathbb{R}\), we obtain

\[
|\mathbb{E}(e^{iu_1Y_{[nt_1]}+\cdots+u_{m+1}Y_{[nt_{m+1}]+}}) - \mathbb{E}(e^{iu_1Y_{[nt_1]}+\cdots+u_mY_{[nt_m]}}) | \mathbb{E}(e^{iu_{m+1}Y_{[nt_{m+1}]+}})|
\]

\[
\leq 16\alpha_{[nt_{m+1]+-[nt_m]}}.
\]

(B.3)

We have \(\alpha_{[nt_{m+1]+-[nt_m]}} \to 0\) as \(n \to \infty\), since

\[
[nt_{m+1]+-[nt_m] \geq nt_{m+1} - 1 - nt_m = (t_{m+1} - t_m)n - 1 \to \infty \quad \text{as} \quad n \to \infty.
\]

Since (B.2) holds for \(m \in \mathbb{N}\), \(t_1, \ldots, t_m \in \mathbb{R}_+\) with \(t_1 < \ldots < t_m\), by the continuity theorem, we have

\[
\mathbb{E}(e^{iu_1Y_{[nt_1]}+\cdots+u_mY_{[nt_m]}}) \to \mathbb{E}(e^{iu_1X_{t_1}+\cdots+u_mX_m}) \quad \text{as} \quad n \to \infty.
\]
Moreover, by using (B.2) with \( m = 1 \), and the continuity theorem, we also have
\[
\mathbb{E}(e^{i(u_{n+1}Y_{nt_{n+1}})}) \to \mathbb{E}(e^{i(u_{n+1}X_{tm+1})}) \quad \text{as } n \to \infty,
\]
hence, by (B.3), we conclude for all \( u_1, \ldots, u_m, u_{m+1} \in \mathbb{R}, \)
\[
\mathbb{E}(e^{i(u_1Y_{nt_1} + \cdots + u_mY_{nt_m} + u_{m+1}X_{tm+1})}) \to \mathbb{E}(e^{i(u_1X_{t_1} + \cdots + u_mX_{tm} + u_{m+1}X_{tm+1})}) \quad \text{as } n \to \infty,
\]
by the independence of \( (X_t, \ldots, X_{tm}) \) and \( X_{tm+1} \). Again by the continuity theorem, we get
\[
(Y_{nt_1}, \ldots, Y_{nt_{m+1}}) \overset{D}{\to} (X_{t_1}, \ldots, X_{tm+1}) \quad \text{as } n \to \infty,
\]
which is (B.2) with \( m \) replaced by \( m + 1 \), as desired.

Next, using a coupling argument, we show that for each \( m \in \mathbb{N}, t_1, \ldots, t_m \in \mathbb{R}^+ \) with \( t_1 < \ldots < t_m \), we have
(B.4) \[
(X_{nt_1}, \ldots, X_{nt_m}) - (Y_{nt_1}, \ldots, Y_{nt_m}) \overset{P}{\to} (0, \ldots, 0) \quad \text{as } n \to \infty,
\]
which, together with (B.2) and Slutsky’s lemma, yields that for each \( m \in \mathbb{N}, t_1, \ldots, t_m \in \mathbb{R}^+ \) with \( t_1 < \ldots < t_m \), we have
\[
(X_{nt_1}, \ldots, X_{nt_m}) \overset{D}{\to} (X_{t_1}, \ldots, X_{tm}) \quad \text{as } n \to \infty,
\]
as desired. Observe that
\[
X_1 = \sum_{j=1}^{X_0} \xi_{1,j} + \varepsilon_1, \quad Y_1 = \sum_{j=1}^{Y_0} \xi_{1,j} + \varepsilon_1
\]
implies
\[
|X_1 - Y_1| = \sum_{j=X_0 \land Y_0}^{X_0 \lor Y_0} \xi_{1,j},
\]
where \( x \land y := \min(x, y) \) and \( x \lor y := \max(x, y) \) for \( x, y \in \mathbb{R} \). Thus
\[
\mathbb{E}(|X_1 - Y_1| \mid X_0, Y_0) = ((X_0 \lor Y_0) - (X_0 \land Y_0))a = a|X_0 - Y_0|,
\]
and hence
\[
\mathbb{E}(|X_1 - Y_1|) = a \mathbb{E}(|X_0 - Y_0|) \leq a(\mathbb{E}(X_0) + \mathbb{E}(Y_0)).
\]
In a similar way, by recursion, we obtain
\[
\mathbb{E}(|X_n - Y_n|) = a \mathbb{E}(|X_{n-1} - Y_{n-1}|) \leq a^n(\mathbb{E}(X_0) + \mathbb{E}(Y_0)), \quad n \in \mathbb{N}.
\]
Hence
\[
\mathbb{E}(|X_n - Y_n|) \to 0 \quad \text{as } n \to \infty,
\]
yielding
\[
|X_n - Y_n| \overset{P}{\to} 0 \quad \text{as } n \to \infty.
\]
Since $X_n$ and $Y_n$ are (nonnegative) integer-valued random variables, we conclude
\[ \mathbb{P}(X_n = Y_n) \to 1 \quad \text{as } n \to \infty. \]

If $X_N = Y_N$ is satisfied for some $N \in \mathbb{N}$, then, by the definition of $(Y_k)_{k \in \mathbb{Z}_+}$, we have $X_n = Y_n$ is satisfied for all $n \geq N$, thus
\[ \mathbb{P}(X_n = Y_n \text{ for all } n \geq N) \to 1 \quad \text{as } N \to \infty. \]

For each $N \in \mathbb{N}$, let $\Omega_N := \{X_n = Y_n \text{ for all } n \geq N\}$. For each $\delta \in \mathbb{R}_+$ and $n, N \in \mathbb{N}$ with $\lfloor nt_1 \rceil > N$, we have
\[
\mathbb{P}(\| (X_{\lfloor nt_1 \rceil}, \ldots, X_{\lfloor nt_m \rceil}) - (Y_{\lfloor nt_1 \rceil}, \ldots, Y_{\lfloor nt_m \rceil}) \| > \delta) \\
= \mathbb{P}(\{\| (X_{\lfloor nt_1 \rceil}, \ldots, X_{\lfloor nt_m \rceil}) - (Y_{\lfloor nt_1 \rceil}, \ldots, Y_{\lfloor nt_m \rceil}) \| > \delta \} \cap \Omega_N) \\
+ \mathbb{P}(\{\| (X_{\lfloor nt_1 \rceil}, \ldots, X_{\lfloor nt_m \rceil}) - (Y_{\lfloor nt_1 \rceil}, \ldots, Y_{\lfloor nt_m \rceil}) \| > \delta \} \cap \Omega_N^c) \\
\leq \mathbb{P}(\| (0, \ldots, 0) \| > \delta) + \mathbb{P}(\Omega_N^c) = \mathbb{P}(\Omega_N^c),
\]
where $\Omega_N^c$ denotes the complement of $\Omega_N$. Letting $N \to \infty$, we obtain (B.4), as desired. \(\square\)

The following result is about the asymptotic behaviour of a single-type critical Galton-Watson process with immigration due to Wei and Winnicki [35, Theorem 2.1].

**B.2 Theorem.** Let $(X_k)_{k \in \mathbb{Z}_+}$ be a single-type Galton-Watson process with immigration such that $\mathbb{E}(\xi^2) < \infty$, $\mathbb{E}(\varepsilon^2) < \infty$, $\mathbb{E}(\xi) = 1$ (critical case) and $\mathbb{E}(X_0^2) < \infty$. Then
\[ (n^{-1}X_{\lfloor nt \rceil})_{t \in \mathbb{R}_+} \overset{D}{\to} (\mathcal{X}_t)_{t \in \mathbb{R}_+} \quad \text{as } n \to \infty, \]
where the limit process $(\mathcal{X}_t)_{t \in \mathbb{R}_+}$ is the pathwise unique strong solution of the SDE
\[ d\mathcal{X}_t = \mathbb{E}(\varepsilon) dt + \sqrt{\text{Var}(\xi)} \mathcal{X}_t^+ d\mathcal{W}_t, \quad t \in \mathbb{R}_+, \quad \mathcal{X}_0 = 0, \]
where $(\mathcal{W}_t)_{t \in \mathbb{R}_+}$ is a standard Wiener process.

**C** A version of the continuous mapping theorem

A function $f : \mathbb{R}_+ \to \mathbb{R}^d$ is called càdlàg if it is right continuous with left limits. Let $D(\mathbb{R}_+, \mathbb{R}^d)$ and $C(\mathbb{R}_+, \mathbb{R}^d)$ denote the space of all $\mathbb{R}^d$-valued càdlàg and continuous functions on $\mathbb{R}_+$, respectively. Let $\mathcal{B}(D(\mathbb{R}_+, \mathbb{R}^d))$ denote the Borel $\sigma$-algebra on $D(\mathbb{R}_+, \mathbb{R}^d)$ for the metric defined in Jacod and Shiryaev [18, Chapter VI, (1.26)] (with this metric $D(\mathbb{R}_+, \mathbb{R}^d)$ is a complete and separable metric space and the topology induced by this metric is the so-called Skorokhod topology). Note that $C(\mathbb{R}_+, \mathbb{R}^d) \in \mathcal{B}(D(\mathbb{R}_+, \mathbb{R}^d))$, see, e.g., Ethier and Kurtz [11] Problem 3.11.25. For $\mathbb{R}^d$-valued stochastic processes $(\mathcal{Y}_t)_{t \in \mathbb{R}_+} \text{ and } (\mathcal{Y}_t^{(n)})_{t \in \mathbb{R}_+}$, $n \in \mathbb{N}$, with càdlàg paths we write $\mathcal{Y}^{(n)} \overset{D}{\to} \mathcal{Y}$ if the distribution of $\mathcal{Y}^{(n)}$ on the space $(D(\mathbb{R}_+, \mathbb{R}), \mathcal{B}(D(\mathbb{R}_+, \mathbb{R})))$
converges weakly to the distribution of $\mathbf{Y}$ on the space $(\mathcal{D}(\mathbb{R}_+,\mathbb{R}),\mathcal{B}(\mathcal{D}(\mathbb{R}_+,\mathbb{R}^d)))$ as $n \to \infty$. If $\xi$ and $\xi_n$, $n \in \mathbb{N}$, are random elements with values in a metric space $(E,d)$, then we denote by $\xi_n \xrightarrow{D} \xi$ the weak convergence of the distribution of $\xi_n$ on the space $(E,\mathcal{B}(E))$ towards the distribution of $\xi$ on the space $(E,\mathcal{B}(E))$ as $n \to \infty$, where $\mathcal{B}(E)$ denotes the Borel $\sigma$-algebra on $E$ induced by the given metric $d$.

The following version of the continuous mapping theorem can be found for example in Theorem 3.27 of Kallenberg [21].

**C.1 Lemma.** Let $(S,d_S)$ and $(T,d_T)$ be metric spaces and $(\xi_n)_{n \in \mathbb{N}}$, $\xi$ be random elements with values in $S$ such that $\xi_n \xrightarrow{D} \xi$ as $n \to \infty$. Let $f : S \to T$ and $f_n : S \to T$, $n \in \mathbb{N}$, be measurable mappings and $C \in \mathcal{B}(S)$ such that $\mathbb{P}(\xi \in C) = 1$ and $\lim_{n \to \infty} d_T(f_n(s_n), f(s)) = 0$ if $\lim_{n \to \infty} d_S(s_n, s) = 0$ and $s \in C$, $s_n \in S$, $n \in \mathbb{N}$. Then $f_n(\xi_n) \xrightarrow{D} f(\xi)$ as $n \to \infty$.

For functions $f$ and $f_n$, $n \in \mathbb{N}$, in $D(\mathbb{R}_+,\mathbb{R}^d)$, we write $f_n \xrightarrow{lu} f$ if $(f_n)_{n \in \mathbb{N}}$ converges to $f$ locally uniformly, i.e., if $\sup_{t \in [0,T]} \| f_n(t) - f(t) \| \to 0$ as $n \to \infty$ for all $T \in \mathbb{R}_+$. For measurable mappings $\Phi : D(\mathbb{R}_+,\mathbb{R}^d) \to D(\mathbb{R}_+,\mathbb{R}^q)$ and $\Phi_n : D(\mathbb{R}_+,\mathbb{R}^d) \to D(\mathbb{R}_+,\mathbb{R}^q)$, $n \in \mathbb{N}$, we will denote by $C_{\Phi,(\Phi_n)_{n \in \mathbb{N}}}$ the set of all functions $f \in C(\mathbb{R}_+,\mathbb{R}^d)$ for which $\Phi_n(f_n) \xrightarrow{lu} \Phi(f)$ whenever $f_n \xrightarrow{lu} f$ with $f_n \in D(\mathbb{R}_+,\mathbb{R}^d)$, $n \in \mathbb{N}$.

One can formulate the following consequence of Lemma C.1

**C.2 Lemma.** Let $d, q \in \mathbb{N}$. Let $(U_t)_{t \in \mathbb{R}_+}$ and $(U^n_t)_{t \in \mathbb{R}_+}$, $n \in \mathbb{N}$, be $\mathbb{R}^d$-valued stochastic processes with càdlàg paths such that $U^n \xrightarrow{D} U$ as $n \to \infty$. Let $\Phi : D(\mathbb{R}_+,\mathbb{R}^d) \to D(\mathbb{R}_+,\mathbb{R}^q)$ and $\Phi_n : D(\mathbb{R}_+,\mathbb{R}^d) \to D(\mathbb{R}_+,\mathbb{R}^q)$, $n \in \mathbb{N}$, be measurable mappings such that there exists $C \subset C_{\Phi,(\Phi_n)_{n \in \mathbb{N}}}$ with $C \in \mathcal{B}(D(\mathbb{R}_+,\mathbb{R}^d))$ and $\mathbb{P}(U \in C) = 1$. Then $\Phi_n(U^n) \xrightarrow{D} \Phi(U)$ as $n \to \infty$.

## D Convergence of random step processes

We recall a result about convergence of random step processes towards a diffusion process, see Ispány and Pap [16].

**D.1 Theorem.** Let $\beta : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ and $\gamma : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^{d \times r}$ be continuous functions. Assume that uniqueness in the sense of probability law holds for the SDE

$$
(D.1) \quad dU_t = \beta(t,U_t) \, dt + \gamma(t,U_t) \, dW_t, \quad t \in \mathbb{R}_+,
$$

with initial value $U_0 = u_0$ for all $u_0 \in \mathbb{R}^d$, where $(W_t)_{t \in \mathbb{R}_+}$ is an $r$-dimensional standard Wiener process. Let $(U_t)_{t \in \mathbb{R}_+}$ be a solution of (D.1) with initial value $U_0 = 0 \in \mathbb{R}^d$.

For each $n \in \mathbb{N}$, let $(U^{(n)}_k)_{k \in \mathbb{Z}_+}$ be a sequence of $d$-dimensional random vectors adapted to a filtration $(\mathcal{F}^{(n)}_k)_{k \in \mathbb{Z}_+}$ (i.e., $U^{(n)}_k$ is $\mathcal{F}^{(n)}_k$-measurable) such that $\mathbb{E}(\|U^{(n)}_k\|^2) < \infty$ for
each \( n, k \in \mathbb{N} \). Let
\[
U_t^{(n)} := \sum_{k=0}^{\lfloor nt \rfloor} U_k^{(n)}, \quad t \in \mathbb{R}_+, \quad n \in \mathbb{N}.
\]

Suppose that
\[
U_0^{(n)} = U_0^{(n)} \xrightarrow{D} 0 \quad \text{as} \quad n \to \infty \quad \text{and that for each} \quad T \in \mathbb{R}_{++},
\]
(i) \( \sup_{t \in [0, T]} \left\| \sum_{k=1}^{\lfloor nt \rfloor} \mathbb{E}(U_k^{(n)} \mid F_{k-1}^{(n)}) - \int_0^t \beta(s, U_s^{(n)}) \, ds \right\| \xrightarrow{P} 0 \quad \text{as} \quad n \to \infty, \)
(ii) \( \sup_{t \in [0, T]} \left\| \sum_{k=1}^{\lfloor nt \rfloor} \text{Var}(U_k^{(n)} \mid F_{k-1}^{(n)}) - \int_0^t \gamma(s, U_s^{(n)}) \gamma(s, U_s^{(n)})^\top \, ds \right\| \xrightarrow{P} 0 \quad \text{as} \quad n \to \infty, \)
(iii) \( \sum_{k=1}^{\lfloor nT \rfloor} \mathbb{E}(\| U_k^{(n)} \|^2 \mathbb{1}_{\{\| U_k^{(n)} \| > \theta \}} \mid F_{k-1}^{(n)}) \xrightarrow{P} 0 \quad \text{as} \quad n \to \infty \quad \text{for all} \quad \theta \in \mathbb{R}_{++}. \)

Then \( U^{(n)} \xrightarrow{D} U \) as \( n \to \infty. \)

Note that in (ii) of Theorem D.1 \( \| \cdot \| \) denotes an operator norm, while in (i) it denotes a vector norm.

Acknowledgements

This paper was finished after the sudden death of the third author Gyula Pap in October 2019. We would like to thank the referees for their comments that helped us improve the paper.

References

[1] ATHREYA, K. B. and NEY, P. E. (1972). Branching Processes, Springer-Verlag, New York, Heidelberg.

[2] BARCZY, M., BEZDÁNY, D. and PAP, G. (2021). A note on asymptotic behavior of critical Galton-Watson processes with immigration. Involve: A Journal of Mathematics 14(5) 871–891.

[3] BARCZY, M., ISPÁNY, M. and PAP, G. (2011). Asymptotic behavior of unstable INAR(p) processes. Stochastic Processes and their Applications 121(3) 583–608.

[4] BARCZY, M., NEDÉNYI, F. K. and PAP, G. (2022). Convergence of partial sum processes to stable processes with application for aggregation of branching processes. Brazilian Journal of Probability and Statistics 36(2) 315–348.
[5] Basrak, B., Kulik, R. and Palmowski, Z. (2013). Heavy-tailed branching process with immigration. *Stochastic Models* **29(4)** 413–434.

[6] Billingsley, P. (1968) *Convergence of Probability Measures*. John Wiley and Sons, Inc., New York.

[7] Billingsley, P. (1999) *Convergence of Probability Measures, 2nd ed*. Wiley-Interscience Publication.

[8] Borodin, A. N. and Salminen, P. (2002) *Handbook of Brownian motion–facts and formulae, 2nd ed*. Birkhäuser Verlag, Basel.

[9] Chung, K. L. (1960) *Markov Chains with Stationary Transition Probabilities*. Springer.

[10] Danka, T. and Pap, G. (2016). Asymptotic behavior of critical indecomposable multi-type branching processes with immigration. *ESAIM: Probability and Statistics* **20** 238–260.

[11] Ethier, S. N. and Kurtz, T. G. (1986). *Markov Processes. Characterization and Convergence*. Wiley, New York.

[12] Foster, J. and Ney, P. (1976). Decomposable critical multi-type branching processes. *Sankhyā: The Indian Journal of Statistics, Series A* **38(1)** 28–37.

[13] Foster, J. and Ney, P. (1978). Limit laws for decomposable critical branching processes. *Zeitschrift fü r Wahrscheinlichkeitstheorie und Verwandte Gebiete* **46** 13–43.

[14] Horn, R. A. and Johnson, Ch. R. (1985). *Matrix Analysis*. Cambridge University Press, Cambridge.

[15] Ikeda, N. and Watanabe, S. (1989). *Stochastic Differential Equations and Diffusion Processes, 2nd ed*. North-Holland, Kodansha, Amsterdam, Tokyo.

[16] Ispány, M. and Pap, G. (2010). A note on weak convergence of step processes. *Acta Mathematica Hungarica* **126(4)** 381–395.

[17] Ispány, M. and Pap, G. (2014). Asymptotic behavior of critical primitive multi-type branching processes with immigration. *Stochastic Analysis and Applications* **32(5)** 727–741.

[18] Jacod, J. and Shiryaev, A. N. (2003). *Limit Theorems for Stochastic Processes, 2nd ed*. Springer-Verlag, Berlin.

[19] Jagers, P. (1969). The proportions of individuals of different kinds in two-type populations. A branching process problem arising in biology. *Journal of Applied Probability* **6(2)** 249–260.
[20] Joffe, A. and Metivier, M. (1986). Weak convergence of sequences of semimartingales with applications to multitype branching processes. *Advances in Applied Probability* **18**(1) 20–65.

[21] Kallenberg, O. (1997). *Foundations of Modern Probability*. Springer, New York, Berlin, Heidelberg.

[22] Kesten, H. and Stigum, B. P. (1967). Limit theorems for decomposable multidimensional Galton-Watson processes. *Journal of Mathematical Analysis and Applications* **17** 309–338.

[23] Kevei, P. and Wiandt, P. (2021). Moments of the stationary distribution of subcritical multitype Galton-Watson processes with immigration. *Statistics and Probability Letters* **173** Article 109067.

[24] Lin, Z. and Lu, C. (1996). *Limit Theory for Mixing Dependent Random Variables*. Kluwer Academic Publishers, Dordrecht, Science Press Beijing, New York.

[25] Ma, C. (2009). A limit theorem of two-type Galton–Watson branching processes with immigration. *Statistics and Probability Letters* **79** 1710–1716.

[26] Mullikin, T. W. (1963). Limiting distributions for critical multitype branching processes with discrete time. *Transactions of the American Mathematical Society* **106** 469–494.

[27] Quine, M. P. (1970). The multi-type Galton-Watson process with immigration. *Journal of Applied Probability* **7**(2) 411–422.

[28] Revuz, D. and Yor, M. (2001). *Continuous Martingales and Brownian Motion*, 3rd ed., corrected 2nd printing. Springer-Verlag, Berlin.

[29] Smadi, C. and Vatutin, V. A. (2016). Reduced two-type decomposable critical branching processes with possibly infinite variance. *Markov Processes and Related Fields* **22**(2) 311–358.

[30] Sugitani, S. (1979). On the limit distributions of decomposable Galton-Watson processes. *Japan Academy. Proceedings. Series A. Mathematical Sciences* **55**(9) 334–336.

[31] Sugitani, S. (1981). On the limit distributions of decomposable Galton-Watson processes with the Perron-Frobenius root 1. *Osaka Journal of Mathematics* **18**(1) 175–224.

[32] Szűcs, G. (2014). Ergodic properties of subcritical multitype Galton-Watson processes. *arXiv*: 1402.5539.

[33] Vatutin, V. A. (2015). The structure of the decomposable reduced branching processes. I. Finite-dimensional distributions. *Theory of Probability and its Applications* **59**(4) 641–662.
[34] Vatutin, V. A. (2016). The structure of decomposable reduced branching processes. II. Functional limit theorems. Theory of Probability and its Applications 60(1) 103–119.

[35] Wei, C. Z. and Winnicki, J. (1989). Some asymptotic results for the branching process with immigration. Stochastic Processes and their Applications 31(2) 261–282.

[36] Yakovlev, A. Y. and Yanev, N. M. (2010). Limiting distributions for multitype branching processes. Stochastic Analysis and Applications 28(6) 1040–1060.

[37] Zubkov, A. M. (1982). The limit behavior of decomposable critical branching processes with two types of particles (Russian). Teoriya Veroyatnostei i ee Primeneniya 27(2) 228–238.