What is the least number of moves needed to solve the 4-peg Towers of Hanoi problem?

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Abstract

We prove that the solutions to the 4-peg Tower of Hanoi problem given by Frame and Stewart are minimal. The proof can be easily generalized to the case of more than 4 pegs.

This paper aims at solving the problem of finding the least number of moves needed to transfer a Tower of Hanoi of \( n \) disks, from an initial peg to another one of 4 available pegs. This problem generalizes a well known puzzle proposed and solved in [5] for the case \( k = 3 \). The generalization to the case of 4 pegs was proposed in [6], and can be phrased as follows: “Given 4 pegs and \( n \) disks initially stacked on one peg in decreasing order of size (i.e. no larger disk can be on top of a smaller one), how many moves are needed to transfer the stack of disks from the initial peg to another peg, assuming that you can move only one disk at a time and it is not allowed to place a larger disk on top of a smaller disk?”

Two solutions to this problem were proposed in [2] and [7] using methods that have been shown to be equivalent in [4]. However, as already observed in [1], both presumed solutions make use of a special assumption, which up to date is still unproven, and restricts their proofs of optimality only to algorithms of a certain scheme. These two models of solution can be however regarded as empirically optimal, as verified in [3]. Rigorous lower bounds estimates are in any case available, as e.g. in [3].

In this paper we prove that the solutions to the 4-peg Tower of Hanoi problem given in [2] and [7] are minimal.

We introduce now some preliminary terminology and notations. We indicate \( n \) disks with the number 1 \( \ldots \) \( n \), with the convention that the disk \( j \) is larger than the disk \( i \) if and only if \( j > i \). We use the symbol \( \infty \) to indicate a free peg. The triple \((j,i,t)\), with \( 1 \leq j < i \leq \infty \) and \( j < t \leq \infty \), denotes that \( j \) moves from being on \( i \) to be placed on \( t \). We say that \( j \leq n \) is freed on a peg, when it moves for the first time (i.e. when we find for the first time the move \((j,j+1,\infty)\)) or, in the case \( j = n \) the move \((j,\infty,\infty)\).

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Definition 1 A sequence of moves is said **demolishing sequence** if:

1. it ends its moves with the triple \((n, \infty, \infty)\),
2. the triple \((n, \infty, \infty)\) appears exactly once

We can always split a no-demolishing sequence \(S\) solving the Tower oh Hanoi problem in a **demolishing phase** and in a **reconstructing phase**, we call demolishing phase the moves until the first triple \((n, \infty, \infty)\), and reconstructing phase the remaining moves.

**Definition 2** Given a sequence of moves \(S\) of \(n\) disks, we denote with \(C_S(j)\) the number of moves of a disk \(j\) in \(S\). \(C_S(j)\) is said cost of \(j\). For simplicity, when it is obvious, we will delete \(S\) from \(C_S(j)\).

Let \(H_4(n)\) be the minimum number of moves needed to transfer a Tower of Hanoi of \(n\) disks with 4 pegs. We have:

**Lemma 1** Let \(S\) be a minimal sequence solving Tower of Hanoi problem, then its demolishing phase is composed by \(\frac{H_4(n)+1}{2}\) moves.

**Proof.** We observe the following: suppose we have made the move \((n, \infty, \infty)\), and let this move be the \(l\)-th move; for every \(u < l\), let \((r, s, y)\) be the \(l-u\)-th move; then continuing with the sequence of moves in which \((r, y, s)\) the \(l+u\)-th move for every \(u < l\), we can reconstruct the Tower on a peg different from the original one. Thus we may conclude that in a minimal sequence the triple \((n, \infty, \infty)\) appears exactly once. As a consequence, a minimal sequence is known if its demolishing phase or its reconstructing phase is known. (We note that in general it is not always the case that all minimal sequences have symmetric demolishing and reconstructing phases, but in any case there are minimal sequences with this feature). It means that if \(S\) is minimal the number of moves of its demolishing phase minus 1 (the move \((n, \infty, \infty)\)) must be the same of the number of moves of its reconstructing phase, otherwise we could use the shortest phase to build a sequence shorter than \(S\). As a consequence we have that a demolishing phase of a minimal sequence is composed by \(\frac{H_4(n)+1}{2}\) moves.

Now we give an example: taking \(n = 5\), it can be empirically verified that \(H_4(5) = 13\), hence the following are all minimal demolishing phases:

\[
(1, 2, \infty)(2, 3, \infty)(1, \infty, 2)(3, 4, \infty)(4, 5, \infty)(3, \infty, 4)(5, \infty, \infty),
(1, 2, \infty)(2, 3, \infty)(3, 4, \infty)(1, \infty, 3)(4, 5, \infty)(2, \infty, 4)(5, \infty, \infty),
(1, 2, \infty)(2, 3, \infty)(3, 4, \infty)(2, \infty, 3)(4, 5, \infty)(1, \infty, 4)(5, \infty, \infty),
(1, 2, \infty)(2, 3, \infty)(3, 4, \infty)(2, \infty, 3)(1, \infty, 3)(4, 5, \infty)(5, \infty, \infty).
\]
As a consequence of Lemma 1, if we want to look for a minimal solution of the 4-peg Tower of Hanoi problem, we can focus on minimal demolishing phases. Moreover, as we proved that it is always possible to build a minimal sequence having symmetric demolishing and reconstructing phases, we can imagine to study only minimal sequences having this feature, that we will call minimal symmetric sequences.

**Definition 3** A set of disks arranged on the same peg is said a stack. As we have only 4 pegs, we can build at most 4 stacks.

For minimal demolishing sequences we have the following property:

**Proposition 1** Let us consider a Tower of Hanoi with \( n \) disks and one of its minimal demolishing phases \( S \). At the end of \( S \) we can find two different scenarios:

1. The disks have been arranged on two different stacks. \( n \) and \( n - 1 \) are the disks that lie at the bottom of the two stacks. Then during \( S \) \( n \) is never stacked on the peg on which the disk \( n - 1 \) ends its moves.

2. The disks have been arranged on three stacks. Let \( n, n - 1 \) and \( j_4 \) be the disks that lie at the bottom of the three stacks in \( S \). Then during the demolishing phase no disk \( y \), with \( y > j_4 \), is arranged on the peg on which the disk \( j_4 \) will be stacked when the move \((n, \infty, \infty)\) is performed.

**Proof.**

Let \( S \) be a minimal sequence of moves yielding a demolishing phase. The case 1 is trivial. We can just focus on the case 2. For the sake of a contradiction, let us assume that at some instant during the computation some disk \( y \), with \( y > j_4 \) is stacked on the peg on which \( j_4 \) will be stacked when the move \((n, \infty, \infty)\) is performed and we assume that \( y \) is the last disk bigger than \( j_4 \) that happens to be stacked on this peg. Notice that since \( y \) has to leave the peg to make room for \( j_4 \), there must be in \( S \) a move \((y, \infty, p)\). Since no disk larger than \( y \) will ever be stacked on that peg starting from this instant, we may avoid the move \((y, \infty, p)\) and build a sequence of moves \( S' \) as follows:

1. \( S' \) coincides with \( S \) up to, but not including, the last move of the form \((y, \infty, p)\).

2. Delete the move \((y, \infty, p)\) and all subsequent moves that concern the disk \( y \).

3. Correct in \( S' \) the computation, substituting the moves of the form \((x, z, \infty)\) going to peg on which \( j_4 \) will be stacked when the move \((n, \infty, \infty)\) in \( S \), with the move \((x, z, y)\).

4. Correct in \( S' \) the computation, substituting the moves of the form \((x, z, y)\) placing a disk on the disk \( y \), with the move \((x, z, y')\) (where \( y' \) is the disk, possibly \( y' = \infty \), on which lies \( y \) in the computation \( S \) when a move \((x, z, y)\) is performed).
The sequence $S'$ is shorter than $S$, since in $S'$ the move $(y, \infty, p)$ is missing. But this contradicts that $S$ is minimal. \hfill \Box

Thanks to the previous Proposition, the following sentences are true:

1. If we arrange the disks on three pegs, there is at least one peg on which no disk bigger than $j_4$ is ever stacked. In other words, moves relative to these disks may be simulated by moves using 3 pegs.

2. The cost of each disk lying at the bottom of the stacks, after performing the move $(n, \infty, \infty)$ in a demolishing phase, is one. To see this, let $x \in \{n - 1, j_4\}$ be one of these disks. We know that when a move $(x, x + 1, \infty)$ is made for the first time, $x$ transfers to a free peg; moreover, by the previous Proposition we know that there is a peg to which no peg bigger than $x$ is ever transferred. Therefore, suitably dealing with the disks that are smaller than $x$, we can arrange that $x$ is eventually freed just on a peg never used by disks larger than $x$: while being on such a peg, the disk is of no obstacle to any move and thus need not be moved.

3. If we arrange all the disks smaller than $n$ on the same stack (case 1 of the previous Proposition), we need at the least $1 + H_4(n - 1)$ moves.

**Definition 4** A sequence $S$ is said an ideal sequence of $n$ disks if and only if:

1. it is a minimal demolishing sequence or

2. it is possible to find a minimal demolishing sequence $S'$ such that the subsequence just until the move of the smallest disks $j_4$ that lie at the bottom of the three stacks in $S'$ is $S$.

Now we want to refer to Theorem [4.6] in [4] to recall the number of moves needed to solve a 4-pegs Tower of Hanoi problem using Stewart’s method ([7]):

**Theorem 1** Given $n$ disks and 4 pegs, the number of moves $X_4(n)$ achieved by Stewart’s method is $\sum_{j=1}^{n} 2^{f_4(j)-1}$ such that $f_4(j) - 1$ is the biggest $m$ for which $j > \frac{(m+1)m}{2}$.

$f_4(j)$ can be described as follows:
Now we will prove that $H_4(n) = X_4(n)$.

**Definition 5** We denote with $X$ the following set: $x \in X$ if and only if we can find two numbers $n, j$ such that it is possible to build an ideal sequence of $n$ disks with $C(j) = x$.

We will consider the ordered set $(X, <)$ with the obvious convention that $x_r < x_{r+1}$ for each $r \geq 1$. For example it is easy to observe that $1, 2 \in X$.

**Definition 6** We define for all $x_i \in X$ the number $M(x_i)$. $M(x_i)$ is the maximum number for the cost $x_i$ if and only if:

1. for all ideal sequence $S$ the number of disks that has been moved $x_i$ times is less than $M(x_i) + 1$.

2. it is possible to find $n$ such that we can build an ideal sequence $S$ where the number of disks that has been moved $x_i$ times is $M(x_i)$.

Note that for every $n \in \mathbb{N}$ we can always find a $i \geq 1$ such that $n = M(1) + M(2) + \ldots + M(x_i) + k$, with $0 \leq k < M(x_{i+1})$

As we know by Lemma 1 that minimal simmetric sequences exist, we can claim:

$$H_4(n) \geq 1 + 4 + \sum_{t=2}^{i} M(x_t)2x_t + k2x_{i+1}. \quad (1)$$

**Definition 7** Let $L_4(x_i)$ be $\sum_{t=1}^{i} M(x_t)$.

We can imagine $L_4(x_i)$ as the maximum number of disks that we can move in a minimal demolishing sequence at a cost less or equal to $x_i$ considering all the possible costs in the set of all ideal sequences.
**Definition 8**  We denote with $I_4(x_i)$ the maximum number of disks that we can transfer from a peg to another peg at a cost smaller or equal to $x_i$ in minimal symmetric sequences. (In the same way we denote with $I_3(x_i)$ the maximum number of disks that we can transfer from a peg to another peg at a cost smaller or equal to $x_i$ only using three pegs). We denote with $m(x_i) = I_4(x_i) - I_4(x_{i-1})$.

For all $n$ we can find an $i$ and $k' \leq k' < m(x_i)$ such that $n = \sum_{t=1}^{i} m(x_i) + k'$ and we can write:

$$1 + 4 + \sum_{t=3}^{i} m(x_t) 2x_{t-1} + k'2x_i.$$  

(2)

As we want $H_4(n)$ bigger than (2), we can use formula (1) and compare it to the formula (2). If we want (2) equal to $1 + 4 + \sum_{t=1}^{i} M(x_t) 2x_t + k 2x_{i+1}$, we must have $m(x_t) = M(x_{t-1})$. We have to consequences:

1. for all $t$ $m(x_t)$ must be different to 0.
2. $I_4(x_i) = L_4(x_{i-1})$. It is in fact the maximum value that it could be.

We can deduce the following Lemmas:

**Lemma 2** If for all $t$ $m(x_t) \neq 0$, then $x_t \geq 2x_{t-1}$. As a result we can suppose $x_t = 2^{t-1}$.

*Proof.* By contradiction. Let $x_y \in X$ the smallest number such that $m(x_y) \neq 0$ and $x_y < 2x_{y-1}$. By Lemma 1 we know $I_4(x_y)$ is equal to the number of disks that can be freed at the maximum cost of $x_y/2$. As we are supposing that $x_y \in X$ the smallest number such that $m(x_y) \neq 0$ and $x_y < 2x_{y-1}$, we have $I_4(x_y) = I_4(x_{y-1})$, and so $m(x_y) = 0$. We find a contradiction and then we must have $x_t \geq 2x_{t-1}$ for all $t$. As we know $x_1 = 1$ we have that $x_t \geq 2^{t-1}$, and so, as we are looking for the minimum, we can set $x_t = 2^{t-1}$. \[\square\]

**Lemma 3** If $I_4(x_t) = L_4(x_{t-1})$ then $I_4(x_t) = t * (t + 1)/2$.

*Proof.*

Using Proposition 1, we know that in a minimal sequence $S$ we can always find a disk $j_4$ such that every disk bigger than $j_4$ has been moved only using three pegs.

As a result, if we are looking for $I_4(x_t)$, we can split this number of disks between $I_3(x_t)$ and the number of the set of the disks smaller than $j_4 + 1$. Moreover in a demolishing phase just until the first move of $j_4$ the disks must perform a number
of moves smaller than $x_{t-1}$. As a result we can say that this set of disks must be at most $L_4(x_{t-2})$. In other words:

$$I_4(x_t) = I_3(x_t) + L_4(x_{t-2}).$$

In conclusion, $I_4(x_t) = I_3(x_t) + I_4(x_{t-1})$, using that $I_4(x_1) = 1$, we have that $I_4(x_t) = (t) \ast (t + 1)/2$. □

As a result we proved $H_4(n) \geq 1 + 4 + \sum_{t=2}^{i} M(x_t) 2x_t + k^2x_{i+1} \geq \sum_{j=1}^{n} 2f_4(j-1) = X_4(n)$. As a consequence $H_4(n) = X_4(n)$.

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