Topology in Shallow-Water Waves: A Violation of Bulk-Edge Correspondence

Gian Michele Graf\textsuperscript{1}, Hansueli Jud\textsuperscript{1}, Clément Tauber\textsuperscript{1,2}

\textsuperscript{1} Institute for Theoretical Physics, ETH Zürich, Wolfgang-Pauli-Str. 27, 8093 Zürich, Switzerland
\textsuperscript{2} Institut de Recherche Mathématique Avancée, UMR 7501, Université de Strasbourg et CNRS, 7 rue René-Descartes, 67000 Strasbourg, France E-mail: clement.tauber@math.unistra.fr; clement.tauber.pro@gmail.com

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Abstract: We study the two-dimensional rotating shallow-water model describing Earth's oceanic layers. It is formally analogue to a Schrödinger equation where the tools from topological insulators are relevant. Once regularized at small scale by an odd-viscous term, such a model has a well-defined bulk topological index. However, in presence of a sharp boundary, the number of edge modes depends on the boundary condition, showing an explicit violation of the bulk-edge correspondence. We study a continuous family of boundary conditions with a rich phase diagram, and explain the origin of this mismatch. Our approach relies on scattering theory and Levinson's theorem. The latter does not apply at infinite momentum because of the analytic structure of the scattering amplitude there, ultimately responsible for the violation.

1. Introduction

Concepts developed to describe topological insulators can be applied far beyond their original context of the quantum Hall effect, or more generally, that of solid state physics. They are actually relevant to classical wave phenomena occurring in various fields such as optics [12,28], acoustics [27] or even fluid dynamics [11], as soon as the partial differential equations ruling the system are formally equivalent to a Schrödinger equation and to the extent that they both engender analogous geometric structures.

A central concept in topological insulators is the bulk-edge correspondence [19]. As a rule, it states that, when an infinite and gapped system—the bulk—admits a topological index, the latter predicts the number of chiral modes appearing at the edge of a sample with a boundary. More precisely such modes are counted by a topological edge index, which coincides with the bulk one. The correspondence was established in a wide range of settings, starting with seminal work [19] to be followed by a mathematical formulation [20,30]. Further extensions, a few among them being [1,14,17,26], included refinements, such as the inclusion of symmetries and the use of different methods. Even situations where physical space is a continuum have been covered [3,7,8,10].
In this paper we study a planar quasi two-dimensional, rotating and classical fluid called the shallow-water model. Such a model describes certain oceanic and atmospheric layers on Earth, and explains the presence of a large structure, called Kelvin equatorial wave, propagating near the equator in the Pacific ocean. Such a propagation is always from West to East with a remarkable stability. Ref. [11] first provided an interpretation of the Kelvin wave as a topological mode at the interface between the two hemispheres. By changing sign at the equator, the Coriolis force is analogue to a magnetic field in the quantum Hall effect as already noticed earlier in [16]. Later, it was also realized that each hemisphere has a well-defined bulk index—Chern number—after adding an odd-viscous term which provides a small-scale regularization for this continuous model [32,33].

The most striking feature of this model is a violation of the bulk-edge correspondence: The number of edge modes for a sample with a sharp boundary, like a coast, depends on the boundary condition and hence does not always match with the associated bulk index. Such a mismatch was conjectured in [34] for some boundary condition. The main result of this paper is to prove it for a continuous family of conditions and to explain the cause of such a violation.

Bulk-edge correspondence was proved in a very general setting for two-dimensional discrete systems with translation invariance [17]. One approach relies on scattering theory, that studies how plane waves that come from the bulk are reflected at the boundary. The associated scattering amplitude encodes for the number of edge modes merging with a band edge in accordance with a variant of Levinson’s theorem. Ultimately, it relies on the analytic continuation of the Bloch variety. The main difference here is that our model is continuous, so that the momentum as well as the Hamiltonian are not bounded. Even though the bulk picture is properly compactified, the analytic structure of the scattering amplitude at infinite momentum is exceptional and leads to two alternatives to Levinson’s scenario. In both of them the scattering amplitude fails to account for the asymptotic number of edge modes, which clarifies the anomaly in the bulk-edge correspondence. To our knowledge, this is one of the rare cases of Levinson’s theorem where the scattering amplitude is not trivial at infinity [22].

It is also worthwhile to mention another approach to deal with a topological index for continuous models. A way to soften the edge problem is to consider a thick boundary or interface potential, gluing two samples with different bulk indices. The bulk-edge correspondence is usually satisfied in that case [5,13,15,24]. We also mention a study of the edge conductivity for the massive Dirac operator via spectral flow arguments, revealing a nontrivial dependence on the boundary condition [18]. A compactified version of this model has recently been studied numerically via scattering theory [34]. A similar violation of bulk-edge correspondence was observed there, suggesting that our analysis applies beyond the shallow-water model.

The paper is organized as follows. Section 2 describes the model from its physical origin to its topological bulk and edge features, and states the main result in terms of scattering theory. Section 3 is devoted to the proofs and also provides further details about the mismatch. The appendices generalize some of the results beyond the particular choices that are made in the main text.

2. Shallow-Water Model and its Topology

2.1. The linearized, rotating and odd-viscous shallow-water model. The shallow-water model describes a thin layer of fluid between a flat bottom and a free surface [35]. It has
three degrees of freedom: the vertical height of the surface \( \eta(x, y, t) \) and a horizontal two-component velocity field \( u(x, y, t), v(x, y, t) \). They are ruled by a system of partial differential equations:

\[
\begin{align*}
\partial_t \eta &= -\partial_x u - \partial_y v, \\
\partial_t u &= -\partial_x \eta - \left( f + \nu \nabla^2 \right) v, \\
\partial_t v &= -\partial_y \eta + \left( f + \nu \nabla^2 \right) u.
\end{align*}
\]

This model is derived from the three-dimensional Euler equations for an incompressible and homogeneous fluid. Equation (2.1a) comes from mass conservation, whereas (2.1b) and (2.1c) come from horizontal momentum conservation. The main assumption is that the typical wavelength of the fluid is much larger than its height. This allows to neglect vertical acceleration and implies hydrostatic pressure, leading to the \(-\partial_x \eta \) and \(-\partial_y \eta \) terms (gravity \( g \) has been rescaled to 1). Moreover \( u \) and \( v \) are the depth-averaged horizontal components of the three dimensional velocity field. The system above is then obtained by linearizing the problem by looking at small fluctuations around a layer of fluid at rest.

When the fluid layer is the ocean, one takes into account Earth’s rotation through the Coriolis acceleration \( f(v, -u) \) where \( f \) depends on the latitude. It is positive (resp. negative) in the northern (resp. southern) hemisphere, and vanishes at the equator. Finally, the term \( \nu \nabla^2 (v, -u) \) is called odd viscosity and comes from the antisymmetric part of the viscosity tensor, meant as a map between symmetric tensors of rank 2 [2]. This exotic term is non dissipative and allowed in dimension two if time reversal symmetry is broken. In the context of geophysical fluids this effect is not manifest, but it appears in some active liquids [6,32], and also in the quantum Hall effect, where it is called Hall viscosity. In the following \( v \) is some positive and arbitrarily small parameter that regularizes the problem at small scales [33,34].

2.2. **Topology in the bulk.** The topology of shallow-water waves is revealed by studying their internal structure [11,32,33]. We approximate some local region on Earth by its tangent plane, so that \( (x, y) \in \mathbb{R}^2 \) and \( f > 0 \) is a constant. We also require

\[
4 \pi f < 1,
\]

so as to simplify matters, compatibly with \( v \) being a small regulator. The previous system (2.1) is analogous to a Schrödinger equation with

\[
i \partial_t \psi = \mathcal{H} \psi, \quad \psi = \begin{pmatrix} \eta \\ u \\ v \end{pmatrix}, \quad \mathcal{H} = \begin{pmatrix} 0 & p_x & -p_y \\ p_x & 0 & i(\nu - f p_x) \\ p_y & i(\nu - f p_y) & 0 \end{pmatrix},
\]

where \( p_x = -i \partial_x, p_y = -i \partial_y \) and \( p^2 = p_x^2 + p_y^2 \). \( \mathcal{H} \) is a self-adjoint operator on \( L^2(\mathbb{R}^2) \otimes \mathbb{C}^3 \) with domain \( H^1(\mathbb{R}^2) \otimes H^2(\mathbb{R}^2) \otimes H^2(\mathbb{R}^2) \). It is also translation invariant so that the stationary solutions are given by the normal modes \( \psi := \hat{\psi}(k_x, k_y, \omega) e^{i(k_x x + k_y y - \omega t)} \) with momentum \( k = (k_x, k_y) \in \mathbb{R}^2 \) and frequency \( \omega \in \mathbb{R} \), leading to the eigenvalue problem

\[
H \hat{\psi} = \omega \hat{\psi}, \quad \hat{\psi} = \begin{pmatrix} \hat{\eta} \\ \hat{u} \\ \hat{v} \end{pmatrix}, \quad H(\mathbf{k}) = \begin{pmatrix} 0 & k_x & k_y \\ k_x & 0 & -i(\nu k_x^2) \\ k_y & i(\nu k_y^2) & 0 \end{pmatrix},
\]
with $k^2 = k_x^2 + k_y^2$ and $H(k)$ a Hermitian matrix. The system (2.4) admits three frequency bands:

$$\omega_{\pm}(k) = \pm \sqrt{k^2 + (f - \nu k^2)^2}, \quad \omega_0(k) = 0,$$

which are separated by two gaps of size $f$. The only critical point for $\omega_{\pm}$ is $k = 0$ if $2\nu f < 1$ cf. (2.2). In contrast to models that are periodic with respect to a lattice, such as (discrete) tight binding models, here momentum space is unbounded. As we shall see shortly, it is however appropriate to compactify it. Each band may then carry a non-trivial topology, characterized by a Chern number. The latter encodes the obstruction of finding a global eigensection that is non-vanishing and regular for all $k \in \mathbb{R}^2$ [33].

The Hamiltonian (2.4) can be rewritten as $H = d \cdot \vec{S}$ where $d = (k_x, k_y, f - \nu k^2)$ and

$$S_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad S_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix},$$

is an irreducible spin 1 representation. $H$ shares its eigenprojection with the flat Hamiltonian $H' = \vec{e} \cdot \vec{S}$ where $\vec{e} = d/|d|$. It reads

$$P_{\pm} = \frac{1}{2}((\vec{e} \cdot \vec{S})^2 \pm \vec{e} \cdot \vec{S}), \quad P_0 = 1 - (\vec{e} \cdot \vec{S})^2.$$ 

We note that $\vec{e} = \vec{e}(k)$ is convergent for $k \to \infty$, and so are $P_{\pm}$ and $P_0$; in fact $\vec{e} \to (0, 0, -1)$ by $\nu > 0$. Consequently, the Chern number

$$C(P) = \frac{1}{2\pi i} \int_{\mathbb{R}^2} dk_x dk_y \, \text{tr}(P[\partial_{k_x} P, \partial_{k_y} P])$$

is a well-defined topological invariant. Indeed the momentum plane can be compactified to the 2-sphere $S^2$, so that the Berry curvature on the r.h.s is eventually computed on a closed manifold.

**Proposition 1.** Let $M$ be a compact two-dimensional manifold without boundary, $\vec{e} : M \to S^2$ and $H = \vec{e} \cdot \vec{S}$ with $\vec{S}$ an irreducible spin $s$ representation ($s \in \mathbb{N}/2$). Let $P_m$ be the eigenprojection of $H$ for the eigenvalue $m \in \{-s, -s+1, \ldots, s-1, s\}$. One has

$$C(P_m) = \frac{m}{2\pi} \int_M (\vec{e})^* w, \quad (\vec{e})^* w = \vec{e} \cdot (\partial_1 \vec{e} \wedge \partial_2 \vec{e}) \, dx_1 dx_2,$$

where $w$ is the volume form on $S^2$. In particular if $\vec{e}$ wraps exactly once around the sphere then $C(P_m) = 2m$.

The proof is given in “Appendix A”. In the case of (2.4) the map $\vec{e} : \mathbb{R}^2 \cong S^2 \to S^2$ wraps exactly once around the sphere when $f$ and $\nu$ have the same sign. Given that $s = 1$ we infer $C_{\pm} = \pm 2$ and $C_0 = 0$. In the rest of the article we will assume $f, \nu > 0$.

**Remark 2.** In absence of regularization ($\nu = 0$) the image of $\vec{e}$ has boundaries, as it covers only half the sphere. Therefore the r.h.s of (2.7) happens to be an integer, $\pm 1$, but not a topological invariant. Indeed, $\vec{e}$ can be continuously deformed to the constant map $\vec{e}_0 = (0, 0, 1)$ that has zero Chern number. The integer value obtained for $\nu = 0$ is due to the normalization, especially to $s = 1$. The analogue with a spin $s = 1/2$ (Dirac Hamiltonian) leads to $\pm 1/2$ without regularization.
2.3. Edge modes. The non-trivial topology in the bulk should be manifest by the presence of edge modes in a sample with a boundary, according to the bulk-edge correspondence [19,20]. We thus study the shallow-water problem in the upper half-plane \((x, y) \in \mathbb{R} \times \mathbb{R}_+\) with a horizontal boundary at \(y = 0\), where we impose the following condition:

\[
|v|_{y=0} = 0, \quad (\partial_x u + a \partial_y v)_{y=0} = 0,
\]

for some \(a \in \mathbb{R}\). The first constraint means that the velocity at the boundary has no normal component. The second one is less easy to interpret. It was studied in [34] for \(a = \pm 1\). When \(a = 1\) the constraint implies by (2.1a) that \(\eta\) is fixed to a constant value at the boundary. For \(a = -1\) the (odd-viscous) stress tensor \(\sigma_{xy} = \nu(-\partial_x u + \partial_y v)\) vanishes, so that there is no shear at the boundary. Here we shall consider the entire family of conditions for \(a \in \mathbb{R}\), in order to study the transition between different regimes.

Proposition 3. For any \(a \in \mathbb{R}\), the Hamiltonian \(H\) in (2.3) is self-adjoint on \(L^2(\mathbb{R} \times \mathbb{R}_+)^\otimes 3\) when equipped with boundary conditions (2.8); more precisely its domain is the subspace of the Sobolev space \(H^1 \oplus H^2 \oplus H^2\) (\(H^k = H^k(\mathbb{R} \times \mathbb{R}_+)\)) defined by them.

The proof is provided in “Appendix B” where we classify all local, translation-invariant and self-adjoint boundary conditions. A rough count would suggest that a second order system with three unknowns, such as (2.1), would require three boundary conditions so as to ensure self-adjointness. However (2.1a) is first order, lowering the count by one.

The problem is translation invariant in the \(x\)-direction, so that the stationary solutions are given by the normal modes \(\tilde{\psi} = \tilde{\psi} e^{i(k_x x - \omega t)}\) with momentum \(k_x \in \mathbb{R}\), frequency \(\omega \in \mathbb{R}\) and \(\tilde{\psi}(y; k_x, \omega) =: (\tilde{\eta}, \tilde{u}, \tilde{v})\). The system (2.1) becomes a system of ordinary differential equations

\[
\begin{align*}
i\omega \tilde{\eta} & = ik_x \tilde{u} + \partial_y \tilde{v}, \quad \text{(2.9a)} \\
i\omega \tilde{u} & = ik_x \tilde{\eta} + (f - \nu k_x^2) \tilde{v} + v \partial_{yy} \tilde{v}, \quad \text{(2.9b)} \\
i\omega \tilde{v} & = \partial_y \tilde{\eta} - (f - \nu k_x^2) \tilde{u} - v \partial_{yy} \tilde{u}, \quad \text{(2.9c)}
\end{align*}
\]

that is exactly solvable for each value of the parameters \(k_x, \omega\) and \(a\).

In Fig. 1 the edge spectrum is plotted for different values of \(a\), corresponding to the existence of solutions to (2.9) that satisfy the boundary condition (2.8) and stay bounded when \(y \to \infty\). The nature of the solutions depends on \(k_x\) and \(\omega\) and is of one of two types. For \(|\omega| \geq \omega_c(k_x, 0)\) or \(\omega = 0\) the solutions (in blue) are delocalized in the upper half plane. The same blue region also corresponds by the way to bounded solutions in the whole plane, and is nothing but the projection of the surface generated by (2.5). In the gaps between them, the yellow curves in the spectrum are edge modes that decay exponentially when \(y \to \infty\).

What is striking here is that the number of such modes changes with the choice of boundary condition. As we shall see this is in contradiction with the bulk-edge correspondence. Moreover, in each case there are edge modes that saturate at a finite frequency \(\omega\) as \(|k_x| \to \infty\). Such modes are perfectly allowed when \(k_x\) is unbounded and have the physical interpretation of inertial waves in classical fluids [21]. Because of such branches we have to specify a consistent way to count edge modes.
Definition 4. The number $n_b$ of edge modes below a bulk band is the signed number of edge mode branches emerging (+) or disappearing (−) at the lower band limit, as $k_x$
increases. The number $n_a$ of edge modes above a band is counted likewise up to a global sign change.

In the following, we focus on the upper band only since the lower one is its symmetric counterpart and the middle one is trivial. By taking the diagrams of Fig. 1 in the order of increasing $a$ one reads off $n_b = 2, 3, 1, 2$; moreover $n_a = 0$ in all cases, because the upper band has no upper edge. We defer any objections to this count and invoke bulk-edge correspondence, in the form of the Hatsugai relation. That principle, if accepted in the present context, would state

$$C_+ = n_b - n_a;$$

(2.10)

yet it is violated, at least for some $a$, because only the l.h.s. is independent of it.

**Proposition 5.** The phase diagram of the total number $n_b$ of edge modes below the upper band for $k_x$ in an arbitrary large but finite interval reads:

$$n_b = \begin{cases} 2 & a > \sqrt{2} \\ 3 & a = \sqrt{2} \\ 1 & a < \sqrt{2} \\ -\sqrt{2} & a = 0 \end{cases}$$

At the transition $a = \sqrt{2}$, an edge mode branch existing for $a > \sqrt{2}$ is repelled to $k_x = +\infty$ and vanishes from the spectrum for $a < \sqrt{2}$, and likewise at $a = -\sqrt{2}$ and $k_x = -\infty$. At the transition $a = 0$ an edge mode branch changes from emerging to disappearing at $k_x = 0$.

The main goal of this paper is to explain such a mismatch in the bulk-edge correspondence. The proof of this proposition is a direct consequence of Theorem 9 below.

A possible objection to the count is that the diagrams only cover a finite interval in $k_x$, thus missing some distant eigenvalue branches which, if included, could possibly yield $n_b = 2$ always. By the proposition this is explicitly not the case. Another objection is that the definitions of $n_b$ and $n_a$ ought to be modified in situations like in the first and last diagrams, and more generally for $|a| > \sqrt{2}$, since they feature one edge state that is asymptotic to the bulk spectrum at $k_x \to -\infty$ and $+\infty$, respectively. Since these cases are actually those for which (2.10) holds true (recalling $C_+ = 2$), such a modification would not help.

**Remark 6.** Another standard way to define the edge index is to count the number of signed crossings of edge mode branches with a fiducial line of constant frequency inside the spectral gap. In the case of a bounded spectrum the two definitions coincide [17], but here they do not. Beyond the fact that this prescription for counting would not take into account the extra modes appearing higher up in the spectrum, this quantity also jumps when varying $a$, and depends non-trivially on the height of the fiducial line inside the gap due to edge mode branches that saturate at finite $\omega$ in the spectral gap as $|k_x| \to \infty$ (See Proposition C.1). The issue is exemplified by considering a portion of the edge spectrum for three different values of $a$ depicted in Fig. 2. We observe that this way of counting would lead to an ill defined number of edge modes for $a = 0.75$ and $a = 1.25$, as the number would depend on the choice of fiducial line and is thus not unique. On the other hand, even though in the case where $a = 1$ such a counting would be unique, it shows an explicit mismatch with the bulk invariant.
Fig. 2. Instability of edge mode counting via signed crossings with a fiducial line of constant frequency. Near \( a = 1 \), such a number depends on \( a \) and also on the height of the line for fixed \( a \), except for \( a = 1 \) but the unique value there does not match with the bulk index.

Finally notice that, even though \( n_b \) depends on \( a \), there is a branch of edge states with (non-dispersive) dispersion relation

\[
\omega = -k_x, \quad (k_x^2 < f/\nu),
\]

regardless of \( a \). It is clearly seen in all images of Figs. 1 and 2. In a domain with boundary, such a branch is called coastal Kelvin wave, in contrast to its equatorial analogue [11]. We show in “Appendix C” the existence of such a wave for any \( a \). Special branches for specific values of \( a \) are also discussed there.

2.4. Scattering theory. The scattering approach, to be described below, is twofold: For one part, it is a way to compute the bulk index in terms of solutions of the edge problem, which are actually superpositions of bulk solutions and are interpreted as scattering of waves off the boundary. Up to that point, the approach falls short of establishing bulk-edge correspondence, because the scattering data still need to be linked to edge states; this second part being accomplished by a variant of Levinson’s theorem.

The scattering approach was used in [17] to establish the bulk-edge correspondence for discrete models in solid state physics. Here we adapt it to continuum models where \( k_x \) and \( \omega \) are unbounded. Notice that, apart from this approach, scattering theory has been used in other ways to study topological indices, see e.g. [4,9,31].

In the upper half-plane, \( k_y \) is not a good quantum number and the bulk normal mode

\[
\psi = \tilde{\psi} e^{i(k_x x + k_y y - \omega t)}
\]

from Sect. 2.2 is a solution of the eigenvalue Eq. (2.9), but does not satisfy the boundary condition (2.8). However for \( \kappa > 0 \), \( e^{i(k_x x - k_y y - \omega t)} \) and \( e^{i(k_x x + k_y y - \omega t)} \) can be seen as incoming and outgoing plane waves with respect to the boundary at \( y = 0 \). Moreover, they share the same frequency \( \omega_+(k_x, \kappa) = \omega_+(k_x, -\kappa) \). Actually, as will be seen later, \( \omega_+(k_x, k_y) = \omega_+(k_x, \kappa) \) admits two further solutions: \( k_y = \kappa_{ev}, \kappa_{div} = -\kappa_{ev} \) that are purely imaginary, \( \kappa_{ev}/\div \in \pm i\mathbb{R}^+ \). The normal mode \( \psi(k_x, \kappa_{ev}) \) is evanescent away from the boundary whereas \( \psi(k_x, \kappa_{div}) \) is divergent as \( y \to \infty \). The latter cannot be part of the solution to the boundary problem but the former must be taken into account.

**Definition 7.** For \( k_x \in \mathbb{R} \) and \( \kappa > 0 \) a scattering state is a solution \( \widetilde{\psi}_s = \tilde{\psi}_s(y; k_x, \kappa) e^{i(k_x x - \omega t)} \) taking values in \( \mathbb{C}^3 \) with \( \omega = \omega_+(k_x, \kappa) \) that is of the form

\[
\widetilde{\psi}_s = \psi_{in} + \psi_{out} + \psi_{ev}
\]
and satisfying the boundary condition (2.8), where the three terms correspond to bulk solutions of momenta \(k_y = -\kappa, \kappa,\) and \(\kappa_{ev}.\)

For any self-adjoint boundary condition, the scattering state exists and is unique up to multiples, see Lemma D.3. We denote by \(\mathcal{E}_{k_x, \kappa}\) the fiber of the eigenbundle with frequency \(\omega_+(k_x, \kappa),\) as discussed in Sect. 2.2. Then Eq. (2.12) defines the scattering map

\[
\mathcal{S} : \mathcal{E}_{k_x, -\kappa} \to \mathcal{E}_{k_x, \kappa}, \quad \psi_{in} \mapsto \psi_{out}
\]

for any \(\kappa > 0.\) Since any global section of the bundle would have to vanish somewhere, we proceed by patches. Given an open set \(U_{out} \subset \mathbb{R}^2,\) let \(U_{in} \subset \mathbb{R}^2\) and \(U_{ev} \subset \mathbb{R} \times i\mathbb{R}\) be the images under the maps \((k_x, \kappa) \mapsto (k_x, -\kappa)\) and \((k_x, \kappa) \mapsto (k_x, \kappa_{ev}).\) Let \(\hat{\psi}_{in} = \hat{\psi}_{in}(k_x, -\kappa)e^{-ik_y}, \quad \hat{\psi}_{out} = \hat{\psi}_{out}(k_x, \kappa)e^{ik_y}, \quad \hat{\psi}_{ev} = \hat{\psi}_{ev}(k_x, \kappa_{ev})e^{ik_{ev}y}\) be a choice of solutions on \(U_{in/out/ev},\) i.e. of sections that do not vanish anywhere in their domains. Then a scattering amplitude \(S = S(k_x, \kappa) \in \mathbb{C}\) is defined by

\[
\hat{\psi}_{in}(k_x, -\kappa) = S(k_x, \kappa)\hat{\psi}_{out}(k_x, \kappa),
\]

and likewise an evanescent amplitude \(T.\) In other words, a unique scattering state is singled out by the requirement that the amplitude of \(\hat{\psi}_{in}\) be 1:

\[
\tilde{\psi}_s = \hat{\psi}_{in} + S\hat{\psi}_{out} + T\hat{\psi}_{ev}. \quad (2.13)
\]

Remark 8. In our context, scattering theory should be understood as follows: The free asymptotic states are incoming and outgoing planes waves from the bulk, and the former are scattered to the latter at the wall and as imposed by the boundary condition. Transmission through the boundary is forbidden so that this is a fully reflective process. Because in and out states belong to one-dimensional Hilbert spaces there is a single reflection coefficient, called scattering amplitude.

As we shall see, the scattering amplitude \(S\) is on the one hand a transition between bulk sections, and hence naturally related to the Chern number, and on the other hand by a variant of Levinson’s theorem it is sensitive to the presence of edge modes when approaching the limit of the bulk band. To explore the bottom of the upper band, including \(|k_x| = \infty,\) we define the following reciprocal variables everywhere for \((\lambda_x, \lambda_y) \in \mathbb{R}^2\) by

\[
k_x = \frac{\lambda_x}{\lambda_x^2 + \lambda_y^2}, \quad k_y = \frac{-\lambda_y}{\lambda_x^2 + \lambda_y^2}, \quad (2.14)
\]

or \(k_x + ik_y = (\lambda_x + i\lambda_y)^{-1}\) for short. This is an orientation preserving change of variable that exchanges 0 with \(\infty.\) For \(\epsilon > 0\) consider the following curve

\[
C_\epsilon = \left\{(k_x = \frac{\lambda_x}{\lambda_x^2 + \epsilon^2}, k_y = \frac{\epsilon}{\lambda_x^2 + \epsilon^2} + \epsilon) | \lambda_x \in \mathbb{R} := \mathbb{R} \cup \{\infty\}\right\}. \quad (2.15)
\]

This is a circle of center \((0, 1/2\epsilon + \epsilon)\) and radius \(1/2\epsilon\) in the \((k_x, \kappa)-plane,\) see Fig. 3a. One has \((k_x, \kappa) \to (0^+, \epsilon)\) as \(\lambda_x \to \pm \infty\) and conversely \((k_x, \kappa) \to (0^+, 1/\epsilon + \epsilon)\) as \(\lambda_x \to 0^+.\) We equip \(C_\epsilon\) with the reverse orientation of \(\lambda_x \in \mathbb{R},\) which implies that, in
Fig. 3. The curve $C_\epsilon = C_0^{\epsilon, \lambda_0} \cup C_\infty^{\epsilon, \lambda_0}$ in the $(k_x, \kappa)$-plane (a) and its image in the $(k_x, \omega)$-plane where $\omega = \omega_+(k_x, \kappa)$. As $\epsilon \to 0$, $C_\epsilon$ approaches the line $\kappa = 0$, whose image by $\omega_+$ is the bottom rim of the upper band.

The image of $C_\epsilon$ in the $(k_x, \omega)$ plane is plotted in Fig. 3b. As $\epsilon \to 0$, it explores the bottom of the upper band, including $|k_x| = \infty$. Finally, we separate as follows the contributions near $k_x = \pm \infty$ from the remaining ones. For $\lambda_0 > 0$ we define the arcs

$$C_{\epsilon, \lambda_0}^\infty = \{(k_x, \kappa) \in C_\epsilon | \lambda_0 \in [-\lambda_0, \lambda_0]\}, \quad C_{\epsilon, \lambda_0}^0 = C_\epsilon \setminus C_{\epsilon, \lambda_0}^\infty,$$

so that $(0, 0) \in C_{0, \lambda_0}^0$ and $(\pm \infty, 0) \in C_{0, \lambda_0}^{\infty}.$

**Theorem 9.** Let $a \in \mathbb{R} \setminus \{0, \pm \sqrt{2}\}$. The following statements hold:

- *(Bulk index and scattering amplitude)* For all $\epsilon > 0$

$$C_+ = \frac{1}{2\pi i} \int_{C_\epsilon} S^{-1} dS. \quad (2.16)$$

- *(Levinson’s theorem)* There exists $\lambda_0$ small enough such that for all $0 < \lambda < \lambda_0$

$$n_b = \lim_{\epsilon \to 0} \frac{1}{2\pi i} \int_{C_{\epsilon, \lambda}^0} S^{-1} dS. \quad (2.17)$$

- *(The anomaly)* There exists $\lambda_0$ small enough such that for all $0 < \lambda < \lambda_0$,

$$\lim_{\epsilon \to 0} \frac{1}{2\pi i} \int_{C_{\epsilon, \lambda}^\infty} S^{-1} dS = \begin{cases} 0, & |a| > \sqrt{2}, \\ \text{sign}(a), & 0 < |a| < \sqrt{2}. \end{cases} \quad (2.18)$$

Moreover, for $a > \sqrt{2}$ (resp. $< -\sqrt{2}$) there is an edge mode branch merging with the bulk band at $k_x = \infty$ (resp. $-\infty$). For $|a| < \sqrt{2}$ there are no edge modes in the neighborhood of the bulk band as $|k_x| \to \infty$.

Thus, the change of argument of the scattering amplitude along $C_\epsilon$ coincides with the Chern number for every $a$, cf. (2.16), but it is only the part of that change that occurs at finite momenta that matches the edge index, cf. (2.17). The mismatch observed in Sect. 2.3 is accounted for by the anomaly (2.18), thanks to a non-trivial contribution at $|k_x| = \infty$ that compensates the missing or excess edge modes. Usually a jump in the argument of $S$ is associated to an edge mode branch disappearing or emerging from the
bulk band limit [17]. This statement is valid as long as \( k_x \) is finite, leading to (2.17). However, the opposite occurs at \( |k_x| = \infty \) : the argument of \( S \) jumps while there is no edge mode branch merging in the spectrum, and conversely. This reversed association shows that Levinson’s theorem does not apply there, and also proves the mismatch in the number of edge modes stated in Proposition 5.

The proofs of (2.16) and (2.17) are done in Sect. 3.2, respectively 3.4. The ya r e adapted from [17] where \((k_x, k_y) \in \mathbb{T}^2\), the two-dimensional Brillouin torus. The key point is to study the poles and zeros of \( S \) after analytic continuation in \( \kappa \). The main result of the paper, Eq. (2.18), is proved in Sect. 3.5 by studying the singularity of such a complex continuation in \( \kappa \) as \(|k_x| \to \infty\), which is eventually responsible for the anomaly. Notice that \( S \) depends on the choice of section appearing in (2.13), but the theorem is true as long as such sections are regular in a neighborhood of \( \mathcal{C}_\epsilon \). Moreover we can even use sections that have singularities near \( \mathcal{C}_\epsilon \) (resp. \( \mathcal{C}_{\epsilon, \lambda}^0 \)) when dealing with the integrals (2.17) (resp. (2.18)) over the complementary arc. Such singular sections have extra symmetries that actually simplify the proof.

Remark 10. Such an anomaly does not always occur in the shallow-water model. Indeed, the standard Dirichlet boundary condition:

\[
\begin{align*}
  u|_{y=0} &= 0, & v|_{y=0} &= 0, \\
  u|_{x=0} &= 0, & v|_{x=0} &= 0,
\end{align*}
\]

(2.19) leads to \( n_b = 2 = C_+ \) with no jump in \( \text{arg} \ S \) at \( |k_x| = \infty \) and no asymptotic edge mode branch near the bottom of the upper bulk band as \( |k_x| \to \infty \). Thus, unbounded parameters \((k_x, \omega)\) do not necessarily lead to an anomaly. This non-anomalous case is detailed in “Appendix E”.

Remark 11. Besides of the shallow-water model, we expect the anomaly to occur for any Dirac-type operator in dimension 2, as long as it is compactified along the same lines as in the former model, cf. (2.6), so that the bulk index exists. One example is the massive Dirac Hamiltonian \( \mathcal{H} = p_x \sigma_x + p_y \sigma_y + (m - \eta p^2) \sigma_z \). See [34, Sect. VI].

3. Proofs

3.1. Evanescent modes. The frequency \( \omega_+(\mathbf{k}) \) is given, cf. (2.5), as the positive solution \( \omega \) of

\[
\omega^2 = X + (f - vX)^2,
\]

(3.1) \((X = k^2)\). As the r.h.s is an increasing function of \( X \geq 0 \), there is conversely only one such solution \( X \) for given \( \omega \geq f \). Plane waves arise that way. However, as we consider the half-plane problem, evanescent states are also of interest, and not just for \( \omega \geq f \).

The solutions of (3.1) are

\[
X_{\pm} = -\frac{(1 - 2vf) \pm \sqrt{\Delta}}{2v^2}, \quad (\Delta = 1 - 4vf + 4\omega^2 v^2).
\]

(3.2) We note that \( \Delta > 0 \) by (2.2) and Vieta’s formulae state

\[
X_+ + X_- = -\frac{1 - 2vf}{v^2}, \quad X_+ X_- = \frac{f^2 - \omega^2}{v^2}.
\]

(3.3) Evanescent and divergent modes, actually with \( k_y \) purely imaginary, arise from \( k_x^2 + k_y^2 = X_- \) by \( X_- < 0 \), but possibly even from \( k_x^2 + k_y^2 = X_+ \). In fact, if \( \omega^2 < \omega^2(k_x, 0) \), or
equivalently if \( X_+ < k_x^2 \) by the stated monotonicity, then \( k_y \) turns imaginary. That is definitely the case if \( \omega^2 < f^2 \).

The solutions having the same frequency as the propagating solutions \((k_x, \pm \kappa)\) have momenta

\[
\kappa_{\text{ev/div}}(k_x, \kappa) = \pm i \sqrt{\kappa^2 + 2k_x^2 + \frac{1 - 2v f}{\nu^2}} \in \pm i \mathbb{R}_+ ,
\]

(3.4)

because \( \kappa_{\text{ev/div}}^2 = k_x^2 - X_- = k_x^2 + X_+ - (X_+ + X_-) \).

3.2. Sections, scattering amplitudes and bulk-scattering correspondence. The proof of Theorem 9 Eq. (2.16) relies on the ambiguity in the definition of the scattering state and amplitude, due to the gauge freedom of bulk eigensections. The uniqueness of (2.13) is indeed conditioned on a choice of sections \( \hat{\psi}_{\text{in}}, \hat{\psi}_{\text{out}}, \hat{\psi}_{\text{ev}} \). The gauge freedom is to multiply any of them by a factor, \( \hat{\psi} \mapsto z \hat{\psi} \), where \( z \neq 0 \) depends on \( k_x, \kappa \).

Given a choice of section \( \hat{\psi}(k_x, -\kappa) \) on \( U_{\text{in}} \cap \{-\kappa < 0\} \) a section \( \hat{\psi}(k_x, \kappa) \) imposes itself naturally on \( U_{\text{out}} \cap \{\kappa > 0\} \):

\[
\tilde{\psi}_{\text{out}}(k_x, \kappa) = \mathcal{S} \hat{\psi}_{\text{in}}(k_x, -\kappa) .
\]

If \( U_{\text{in}} \) and \( U_{\text{out}} \) overlap then on \( U_{\text{in}} \cap U_{\text{out}} \) we may use a same section for in and out states, i.e. \( \hat{\psi}_{\text{out}} = \hat{\psi}_{\text{in}} \), while \( \hat{\psi}_{\text{ev}} \) may be different. As a result,

\[
\tilde{\psi}_{\text{out}}(k_x, \kappa) = S(k_x, \kappa) \hat{\psi}_{\text{in}}(k_x, \kappa) ,
\]

meaning that the scattering amplitude becomes the transition function between the sections on \( U_{\text{out}} \) and \( U_{\text{in}} \), fittingly defined on \( U_{\text{in}} \cap U_{\text{out}} \). Then (2.13) can be written more appropriately as

\[
\tilde{\psi}_{\text{s}} = \hat{\psi}_{\text{in}, -} + S \hat{\psi}_{\text{in}, +} + T \hat{\psi}_{\text{ev}} ,
\]

(3.5)

where \( \hat{\psi}_{\text{in}, \pm} = \hat{\psi}_{\text{in}}(k_x, \pm \kappa) \). We shall use patches \( U_{\text{in}}, U_{\text{out}} \) that cover \( \mathbb{R}^2 \), including the point at \( \infty \), as well as sections such that \(|S| = 1\) (see Proposition D.4). Therefore, the Chern number \( C_+ \) equals the winding of \( S \) along a loop winding once within \( U_{\text{in}} \cap U_{\text{out}} \). Its orientation must be such that its interior is contained in \( U_{\text{out}} \).

We will now verify the assumptions made for the general construction of the scattering amplitude \( S \) in Sect. 2.4 and its relation to \( C_+ \) explained above. In the following we identify the compactified \( k \)-plane with the Riemann sphere \( \mathbb{C} \cup \{\infty\} \cong S^2 \) via \( z \equiv k_x + i k_y \equiv (k_x, k_y) \). Since \( C_+ = 2 \), it is impossible to find a global bulk eigensection \( \hat{\psi} \) that is regular for all \( z \in S^2 \). We need at least two distinct ones, that are regular locally on two overlapping patches to cover the sphere. This leads to distinct scattering states and scattering amplitudes. It is readily verified that \( H \hat{\psi}^\infty = \omega_+ \hat{\psi}^\infty \), where

\[
\hat{\psi}^\infty(k) = \frac{1}{k_x - ik_y} \left( \frac{k_x^2/\omega_+}{k_x - ik_y q} \right) , \quad q(k) := \frac{f - \nu k_x^2}{\omega_+} , \quad \omega_+ = \omega_+(k) .
\]

(3.6)

Notice that \( q \to 1 \) (resp. \(-1\)) as \( k \to 0 \) (resp. \( \infty \)). Thus (3.6) defines a section of the eigenbundle of \( \omega_+ \) that is smooth for all \( z \in \mathbb{C} \), including \( z = 0 \), but not at \( \infty \), where it is singular and winds like \( z/\bar{z} \). However \( z = \infty \) belongs to the curve \( C_\epsilon \) as \( \epsilon \to 0 \), see
(2.15), so that \( \hat{\psi}^\infty \) cannot be used directly in the proof of Theorem 9. Instead we define for \( \zeta = \zeta_x + i\zeta_y \in \mathbb{C} \)

\[
\hat{\psi}^\zeta = t_\infty^\zeta \hat{\psi}^\infty, \quad t_\infty^\zeta(z) = \frac{z - \zeta}{\bar{z} - \bar{\zeta}}
\]

which is regular for all \( z \in S^2 \setminus \{\zeta, \infty\} \) and singular at the two omitted points. We shall mainly use \( \zeta = i\zeta_y \) with \( \zeta_y > 0 \) that is bounded away from \( C_\epsilon \) for \( \epsilon \) small enough, but on occasions we will pick \( \zeta = 0 \). According to (3.5), the scattering state for each section \( \hat{\psi}^\zeta, (\zeta \in S^2) \), reads

\[
\tilde{\psi}_\zeta := \psi_{in,-} + S_\zeta \psi_{in,+} + T_\zeta \psi_{ev}^\infty,
\]

with

\[
\psi_{in,-}^\zeta = \tilde{\psi}_\zeta (k_x, -\kappa) e^{-i\kappa y}, \quad \psi_{in,+}^\zeta = \tilde{\psi}_\zeta (k_x, \kappa) e^{i\kappa y}, \quad \psi_{ev}^\infty = \tilde{\psi}_\zeta (k_x, \kappa_{ev}) e^{i\kappa_{ev} y}.
\]

We recall that the momenta \((k_{in}, k_{out}, \kappa_{ev}) = (-\kappa, \kappa, \kappa_{ev}(k_x, \kappa))\) are parametrized by the outgoing momentum \( \kappa > 0 \). The section \( \psi_{in,-} \) is regular in the region \( U_{in} = S^2 \setminus \{\zeta\} \) containing the lower half-plane \( \{(k_x, -\kappa) | -\kappa < 0\} \). Correspondingly its mirror \( U_{out} \) contains the upper half-plane. Finally, we notice that \( \kappa \mapsto \tilde{\psi}_\zeta (k_x, \kappa_{ev}) \) is regular in the whole \( \kappa \)-plane since the denominator in (3.6) does not vanish anywhere, i.e. \( k_x - i\kappa_{ev} \neq 0 \), because of \( X_- \neq 0 \) there, which is in turn seen from \( \text{Re} X_- < 0 \), cf. (3.2). Moreover, by \( q(k_x, \kappa_{ev}) \rightarrow 1 \ ((k_x, \kappa) \rightarrow \infty) \), the section \( \tilde{\psi}_\zeta (k_x, \kappa_{ev}) \) is regular even at \( \kappa \rightarrow \infty \) with limit

\[
\tilde{\psi}^\infty (\pm \infty, i\infty) = \begin{pmatrix} 0 \\ 1 \\ i \end{pmatrix}.
\]

Among other things, this completes the proof of Theorem 9, Eq. (2.16).

3.3. More about the scattering amplitude. Additionally, we provide explicit expressions that are used in the next sections. The scattering amplitude \( S_\zeta \) can be computed as follows. As in (2.4), we write for \( \zeta \in S^2 \)

\[
\hat{\psi}^\zeta =: \begin{pmatrix} \eta_\zeta \\ u_\zeta \\ v_\zeta \end{pmatrix}.
\]

Inserting (3.7) into the boundary condition (2.8) we get

\[
f_{in} + S_\zeta f_{out} + T_\zeta f_{ev} = 0, \\
v_{in} + S_\zeta v_{out} + T_\zeta v_{ev} = 0,
\]

where

\[
f_\zeta (k_x, \kappa) = k_x u_\zeta (k_x, \kappa) + a \kappa v_\zeta (k_x, \kappa)
\]
and \( f_l = f_l^\xi(k_x, \kappa_l) \) for \( l = \text{in}, \text{out}, \text{ev} \) corresponds to \( \kappa_l = -\kappa, \kappa, \kappa_{\text{ev}} \) and \( \xi_l = \zeta, \zeta, \infty \). The solution of the linear inhomogeneous system is

\[
S_\xi(k_x, \kappa) = -\frac{g_\xi(k_x, -\kappa)}{g_\xi(k_x, \kappa)}, \quad T_\xi(k_x, \kappa) = -\frac{h_\xi(k_x, \kappa)}{g_\xi(k_x, \kappa)},
\]

where

\[
g_\xi(k_x, \kappa) = \left| \begin{array}{c} f_\xi(k_x, \kappa) \\ v_\xi(k_x, \kappa) \end{array} \right|, \quad h_\xi(k_x, \kappa) = \left| \begin{array}{c} f_\xi(k_x, \kappa) \\ v_\xi(k_x, \kappa) \end{array} \right|, (3.8)
\]

For \((k_x, \kappa) \in S^2 \setminus \{\zeta, \infty\}\), \( \hat{\psi}_\xi \) and \( \hat{\psi}_\infty \) are related through \( t_\infty^\xi \), and so is \( S \):

**Lemma 12.** Let \( \xi \in S^2, k_x \in \mathbb{R} \) and \( \kappa > 0 \) so that \( k_x + i\kappa \in S^2 \setminus \{\zeta, \infty\} \). Then

\[
S_\xi(k_x, \kappa) = \frac{t_\infty^\xi(k_x, -\kappa)}{t_\xi(k_x, \kappa)} S_{\infty}(k_x, \kappa).
\]

**Proof.** By direct inspection of (3.9). Alternatively, and as explained in connection with (2.12), the scattering solution remains the same up to a factor when changing sections. If the term \( \psi_{\text{in}}^\xi \) in (3.7) changes as

\[
\psi_{\text{in}}^\xi(k_x, -\kappa) = t_\infty^\xi(k_x, -\kappa) \psi_{\text{in}}^\infty(k_x, -\kappa)
\]

(and similarly \( \psi_{\text{out}}^\xi(k_x, \kappa) \)), then the middle term changes accordingly,

\[
S_\xi(k_x, \kappa) \psi_{\text{out}}^\xi(k_x, \kappa) = t_\infty^\xi(k_x, -\kappa) \cdot S_{\infty}(k_x, \kappa) \psi_{\text{out}}^\infty(k_x, \kappa),
\]

whence \( S_\xi(k_x, \kappa) t_{\infty}^\xi(k_x, \kappa) = t_{\infty}^\xi(k_x, -\kappa) S_{\infty}(k_x, \kappa). \) \( \square \)

An edge state amounts to a state like (2.12), but with \( \psi_{\text{in}} \) absent and \( \psi_{\text{out}} \) also evanescent. In terms of (3.7) this means that \( S(k_x, \kappa) \) has a pole with \( \text{Im} \kappa > 0 \), i.e. \( g(k_x, \kappa) = 0 \) by (3.8), and vice versa. It follows that at points \( k_x \in \mathbb{R} \) that are not merging points one has

\[
\lim_{\kappa \to 0} S_\xi(k_x, \kappa) = -1, \quad (3.10)
\]

because at these points \( g(k_x, \kappa = 0) \neq 0 \). The question of what happens at merging points is addressed in the following subsection.

**3.4. Relative Levinson’s theorem for finite \( k_x \)**. Usually Levinson’s theorem (see e.g. [29], Theorem XI.59) relates the number of bound states below the energy continuum with the scattering phase at threshold, provided it is normalized at infinite energy. This form of the theorem may be contrasted with a relative version thereof, for the case the Hamiltonian depends on some parameter. The quantities then being related are the changes of the number of bound states and of the scattering phase at threshold, respectively. An absolute normalization of the latter is no longer needed. In the present case, the parameter is the longitudinal momentum on which the edge Hamiltonian depends. To be more precise, the proof of Theorem 9, Eq. (2.17) is based on the following:
Relative Levinson’s theorem

Fig. 4. Relative Levinson’s theorem

**Theorem 13** ([17, Thm. 6.11]). Let \( \epsilon > 0 \) and \( \zeta = i\xi_y \) with \( \xi_y > \epsilon \). Let \( k_x^1 < k_x^2 \) be two points that do not correspond to a merger of an edge mode branch with the bulk region in the spectrum of (2.9). Then

\[
\lim_{\epsilon \to 0} \arg S_\xi(k_x, \epsilon) \Big|_{k_x^2}^{k_x^1} = 2\pi n(k_x^1, k_x^2)
\]

where \( \arg \) denotes a continuous argument and \( n(k_x^1, k_x^2) \) is the signed number of edge mode branches emerging (+) or disappearing (−) at the lower band limit between \( k_x^1 \) and \( k_x^2 \), as \( k_x \) increases.

By (2.16) and (2.18), which are proved independently, we deduce that for \( k_x, 1 < 0 \) and \( k_x, 2 > 0 \) large enough one has \( n(k_x^1, k_x^2) = n_b \). Indeed, all merging points at finite \( k_x \) lie in a compact subset of \( \mathbb{R} \), cf. the comment at the very end of Sect. 3.5. Moreover, \( C_{\epsilon, \lambda_0} = \{ (\lambda_x^{-1}, \epsilon) \lambda_x \in \mathbb{R} \setminus [-\lambda_0, \lambda_0] \} \) when \( \epsilon \to 0 \), so that for \( \lambda_0 \) small enough and given the orientation of \( C_{\epsilon, \lambda_0} \), (3.11) is equivalent to (2.17). We refer to [17] for the proof of Theorem 13, which is quite general and applies to our continuous model because (3.11) is valid as long as \( (k_x, \kappa) \) belong to a finite path that does not cross \( \infty \). This is the case for \( (k_x, \kappa) \in C_\epsilon \) as long as \( \lambda_x \neq 0 \) (see (2.15)). Below we briefly illustrate the main elements of the proof in our explicit model, see also Fig. 4, in order to compare with the anomalous case at \( \infty \) in the next section. Furthermore, this statement has also been checked numerically for the shallow-water model with boundary condition (2.8) and \( a = \pm 1 \) in [34].

As \( k_x \in \mathbb{R} \) will be fixed for a while we drop it from the notation. We also drop the singularity \( \xi \) and assume that the sections are regular in the region of interest. Up to a multiplication of (3.7) by \( g_\xi \equiv g \), an equivalent scattering state is

\[
\phi_s = g(\kappa)\hat{\psi}(-\kappa)e^{-i\xi_y} - g(-\kappa)\hat{\psi}(\kappa)e^{i\xi_y} - h(\kappa)\hat{\psi}(\kappa_{cv})e^{i\xi_{cv}}y,
\]

see (3.8) and (3.9). So far we focused on \( \kappa > 0 \), but it turns out that a neighborhood of the bulk region in the edge spectrum, and in particular edge mode branches, can be studied through the analytic continuation in \( \kappa \) of the scattering state. Indeed, assume that \( g(\kappa) = 0 \) for some \( \kappa \) with \( \Im \kappa > 0 \). The first term in (3.12) vanishes, whereas the second becomes exponentially decaying in \( y \), and the third one remains evanescent. In
that case \( \phi_s \) is a bound state of the edge spectrum and corresponds to a point of the edge mode branch below the bulk spectrum.

Then, as \( k_x \) varies, the zero of \( g \) might move from \( \text{Im} \kappa > 0 \) to \( \text{Im} \kappa < 0 \). In the latter case the second term in (3.12) would be exponentially diverging in \( y \) and \( \phi_s \) would not be a bound state anymore. Thus at \( k_x \) where \( \text{Im} \kappa = 0 \) the edge mode branch merges with the bulk continuum. Furthermore, a zero of \( g \) is a pole for \( S \), and such a sign change in \( \text{Im} \kappa \) induces a \( 2\pi \) shift in the argument of \( S \). This relative version of Levinson’s theorem is proved in a general framework in [17] via the argument principle. Here we illustrate it on a canonical form of \( S \): assume that \( g = k_x - i\kappa \), so that \( g = 0 \) for \( \kappa_0 = -ik_x \). For \( k_x < 0 \) one has a bound state, but not for \( k_x > 0 \). This means that an edge mode branch has disappeared at \( k^* \) by merging with the continuum. The \( S \) matrix reads

\[
S(k_x, \kappa) = \frac{k_x + i\kappa}{k_x - i\kappa}.
\]

Thus, for \( \kappa = \epsilon > 0 \), \( \arg S(k_x, \kappa) \) is shifted by \(-2\pi\) as \( k_x \) goes from \(-\infty\) to \(\infty\). Equivalently, the argument of \( S \) changes by \(-2\pi\) between some \( k_x < 0 \) and \( k_x > 0 \) finite when \( \epsilon \to 0 \).

3.5. The anomaly at infinity. Let \( \lambda_0 \) be small enough such that (2.17) is true. In that case the only possible singularity for the scattering amplitude along \( C_{\epsilon, \lambda_0} \) is at infinity. So far, we worked with \( S_\zeta \) with \( \zeta = i\zeta_y \) and \( \zeta_y > 0 \). In order to study the neighborhood of \( \infty \), it is rather convenient to work with \( \zeta = 0 \) instead. This has no influence on (2.18). Indeed, by Lemma 12 the two amplitudes are related by

\[
S_\zeta (k_x, \kappa) = \frac{t_\zeta (k_x, -\kappa)}{t_\zeta (k_x, \kappa)} \frac{t_0 (k_x, \kappa)}{t_0 (k_x, -\kappa)} S_0 (k_x, \kappa),
\]

and by

\[
\frac{t_0 (z)}{t_0 (\bar{z})} = \frac{1 - \zeta/z}{1 - \zeta/\bar{z}}, \quad (z = k_x \pm ik),
\]

(3.13)

such a transition function is regular at \( \infty \). We have that

\[
\lim_{\epsilon \to 0} \frac{1}{2\pi i} \int_{C_{\epsilon, \lambda_0}^\infty} S^{-1}_\zeta dS_\zeta = \lim_{\epsilon \to 0} \frac{1}{2\pi i} \int_{C_{\epsilon, \lambda_0}^\infty} S_0^{-1} dS_0,
\]

which follows from (3.10) and because merging points do not accumulate at \( \infty \). This allows to prove the equality just for \( \lambda_0 \) arbitrary small, so that on \( C_{\epsilon, \lambda_0}^\infty \) the winding of (3.13) is correspondingly small. The scattering state (3.7) reads \( \tilde{\psi}^0_s = \psi^0_{\text{in}} + S_0 \psi^0_{\text{out}} + T_0 \psi^\infty_{\text{ev}} \). It involves

\[
\tilde{\psi}^0 (k) = \frac{1}{k_x + ik_y} \begin{pmatrix} k^2/\omega_+ \\ k_x - ik_y q \\ k_y + ik_x q \end{pmatrix},
\]

(3.14)

which appears to be dual to \( \tilde{\psi}^\infty \), see (3.6) and has limit

\[
\tilde{\psi}^0 (\infty) = \begin{pmatrix} 0 \\ 1 \\ -i \end{pmatrix}.
\]
According to the previous section, the existence of edge modes near \( \infty \) is encoded in the poles of \( S_0(k_x, \kappa) \) for \( \text{Im} \kappa > 0 \), or equivalently in the zeros of \( g_0 \). Using (3.9) and (3.2, 3.5) we compute \( g_0 \) to leading order near \((k_x, \kappa) \to \infty\):

\[
g_0(k_x, \kappa) \approx i(2k_x + ia(\kappa_{ev}(k_x, \kappa) - \kappa)),
\]

with \( \kappa_{ev}(k_x, \kappa) \sim i\sqrt{2k_x^2 + \kappa^2} \), see (3.4).

**Winding number at infinity.** In terms of the reciprocal variables (2.14) that parametrize \( \mathcal{C}_{\epsilon, \lambda_0} \) one has for \( \lambda_x \) near 0

\[
g_0(k_x, \kappa) \approx \frac{i}{\lambda_x^2 + \epsilon^2} (2\lambda_x + a\sqrt{2\lambda_x^2 + \epsilon^2 - i\epsilon \epsilon}).
\]

By (3.8), the winding number of \( S_0 \) is, up to a sign, twice that of \( g_0 \). When \( \epsilon \to 0 \) and \( \lambda_x \) finite, such a winding can be inferred from the one of \( \lambda_x = -\infty \) to \( \lambda_x = +\infty \) with \( \epsilon \) finite. The prefactor on the r.h.s. can be ignored. As for the remaining real part,

\[
2\lambda_x + a\sqrt{2\lambda_x^2 + \epsilon^2} \to \left\{ \begin{array}{ll}
\sqrt{2} (\sqrt{2} + a) \cdot (\infty), & (\lambda_x \to \infty), \\
\sqrt{2} (\sqrt{2} - a) \cdot (-\infty), & (\lambda_x \to -\infty).
\end{array} \right.
\]

Thus, for \( |a| < \sqrt{2} \), the range covers the whole real line, just as \( \lambda_x \) does. We recall that \( \mathcal{C}_\epsilon \) is parametrized with reverse orientation for \( \lambda_x \). In view of the imaginary part, \( g_0 \) winds along \( \mathcal{C}_{\epsilon, \lambda_0}^\infty \) by \(-\pi \text{sign}(a)\) in counter-clockwise direction, and \( S_0 \) winds by \( 2\pi \text{sign}(a) \). For \( |a| > \sqrt{2} \), it does not cover the whole line so \( g_0 \) and \( S_0 \) do not wind. This proves (2.18). In particular we deduce that the transitions occur at \( a = \pm \sqrt{2} \), 0. Together with (2.16, 2.17), we infer the value of \( n_b \) claimed in Proposition 5.

**Analytic continuation.** According to Theorem 13, the existence of edge modes near the bulk continuum is related to the zeros of \( g_0(k_x, \kappa) \) for \( \text{Im} \kappa > 0 \) and \( k_x \) finite. Since the r.h.s. of (3.15) is homogeneous of degree 1 in \((k_x, \kappa)\), the zero locus of \( g_0(k_x, \kappa) \) is homogeneous near infinity, even for complex \( \kappa \). There however \( \kappa_{ev} \) exhibits a conical singularity, which makes it advisable to equivalently look for zeros of

\[
G_0(k_x, k_{y+}, k_{y-}) = i(2k_x + ia(k_{y-} - k_{y+}))
\]

under the constraint \( k_x^2 + k_{y\pm}^2 = X_\pm \), where \( X_\pm \) are the solutions of \( \omega^2 = X + (f - \nu X)^2 \) for a given \( \omega > 0 \) cf. (3.2). In particular, for \( k_{y+} = \sqrt{X - k_x^2} := \kappa \) then \( k_{y-} = i\sqrt{k_x^2 - X^-} = \kappa_{ev}(k_x, \kappa) \) so that \( G_0 \) and the leading order of \( g_0 \) coincide. However the definition of \( G_0 \) avoids specifying any branch for the square root. Moreover, at leading order in \( \omega \to \infty \), one has

\[
k_x^2 + k_{y\pm}^2 = \frac{\omega}{\nu},
\]

by (3.2), which implies

\[
2k_x^2 + k_{y+}^2 + k_{y-}^2 = 0.
\]

Together with (3.16), the zeros of \( G_0 \) prompt the ansatz

\[
k_{y\pm} = \pm ic_\pm k_x,
\]
Fig. 5. \((c_+, c_-)\)-diagram corresponding to a zero of \(G_0\) near \(\infty\) through the ansatz (3.19). A solution is given by the intersection of the circle with a straight line of slope \(-1\) and intercept \(-2/a\) with the axes. Furthermore, solutions are restricted to lie within the red areas, leading to a single pair \((c_+, c_-)\) for each \(a\). Such a solution corresponds to a bound state only in the upper left or lower right quadrant, depending on sign \(k_x \not\approx 0\) for \(c_\pm \in \mathbb{R}\). The different choice of sign for \(k_y \pm\) is conventional, but ensures a symmetry between \(c_\pm\) in expressions below, such as \(G_0\) becoming

\[
G_{00} = i(2 + a(c_+ + c_-))k_x.
\]  

Thus, a zero of \(G_0\) implies

\[
2 + a(c_+ + c_-) = 0,
\]

\[
c_+^2 + c_-^2 = 2.
\]

The solution of this system is represented diagrammatically in Fig. 5 as the intersection between a circle and a straight line that depends on \(a\). Moreover, due to (3.17),

\[
k_x^2(1 - c_\pm^2) = \pm \frac{\omega}{v},
\]

so that \(c_+^2 < 1\) and \(c_-^2 > 1\). Therefore, with the ansatz (3.19) there is a unique pair \((c_+, c_-)\) corresponding to zeros of \(G_0\), and hence of \(g_0\). Finally, such zeros are associated to a bound state only if \(\text{Im} \ k_y \pm > 0\). This requires \(c_+ > 0, c_- < 0\) for \(k_x \to \infty\) and \(c_+ < 0, c_- > 0\) for \(k_x \to -\infty\). Consequently, according to Fig. 5, there exists an edge mode in the neighborhood of \(\infty\) for \(a > \sqrt{2}\) and \(k_x \to \infty\), or \(a < -\sqrt{2}\) and \(k_x \to -\infty\) and there is no edge mode otherwise, as claimed below Eq. (2.18) in Theorem 9.

Two alternatives to Levinson scenario. Beyond the proof of Theorem 9 we provide an interpretation of the mismatch when compared with Levinson’s scenario described in Sect. 3.4. For clarity we focus on what happens near the transition \(a = \sqrt{2}\). A zero of \(G_0\) (and hence of \(g_0\)) with the ansatz (3.19) corresponds to a bound state only if both \(\text{Im} \ k_y^+ > 0\) and \(\text{Im} \ k_y^- > 0\). In the usual Levinson’s scenario described in Theorem 13, \(\kappa_{\text{ev}} = k_y^-\) always satisfies the second condition, so that the nature of the state only depends on \(\text{Im} \ k_y^+ = \text{Im} \ k_x\), as illustrated in Fig. 4b. However, near \(\infty\) this is not the case, due to the particular structure of \(g_0\) there. Two alternatives occur:
Fig. 6. Two alternatives to Levinson scenario at $k_x \to \pm \infty$: zeros of $G_0$ with ansatz (3.19) in terms of the reciprocal variables. A plain (resp. dotted) line corresponds to an evanescent (resp. divergent) mode. A bound state is a superposition of two evanescent modes. a In case 1, there is no bound state whereas $S_0$ winds by $2\pi$. b In case 2, no bound state exist for $\lambda_x < 0$ but one emerges at $\lambda_x = 0$, whereas $S_0$ does not wind. In both cases, Levinson’s theorem is violated.

1. For $a < \sqrt{2}$, one has $-1 < c_+ < 0$ and $c_- < -1$ so that $\text{Im} \ k_{y+} > 0$ and $\text{Im} \ k_{y-} < 0$ for $k_x \to -\infty$, and conversely for $k_x \to \infty$. Therefore, the scattering state is unbounded on either side of $k_x = \infty$, because it always contains some divergent part. Yet, the winding of $S_0$ is $2\pi$.

2. For $a > \sqrt{2}$, one has $0 < c_+ < 1$ and $c_- < -1$ so that $\text{Im} \ k_{y+} < 0$ and $\text{Im} \ k_{y-} < 0$ for $k_x \to -\infty$, and conversely for $k_x \to \infty$. Therefore, the scattering state is divergent for $k_x \to -\infty$ and a bound state for $k_x \to \infty$. A bound state emerges at $\infty$, and yet the winding of $S_0$ is 0.

The two scenarios are illustrated in Fig. 6. We use the reciprocal variables (2.14) so that $k_x \to \pm \infty$ is replaced by $\lambda_x \to 0^{\pm}$ which makes the comparison with Fig. 4b easier. Notice that the ansatz (3.19) becomes $\lambda_{y\pm} = \mp ic_{\pm} \lambda_x$.

**Second order computation.** Finally, we provide more details about the edge mode branch that exists near $|k_x| \to \infty$ for $|a| > \sqrt{2}$. As we shall see, this branch actually emerges from the bulk continuum at finite $k_x > 0$ ($a > \sqrt{2}$) or $k_x < 0$ ($a < -\sqrt{2}$), stays close to it when $k_x \to +\infty$ or $k_x \to -\infty$, respectively, and disappears there (cf. Fig. 1).

To compute the second order correction to the result obtained we use again (3.9), (3.6) and (3.14) and find

$$g_0 = G_0 + G_1 + \cdots,$$

with $G_0$ as in (3.16) and

$$G_1 = \frac{1}{2\nu \omega} \frac{ia k_x}{(k_x - ik_{y-})(k_x + ik_{y+})} (k_{y+}^2 - k_{y-}^2).$$

We extend the ansatz (3.19) by a term of the appropriate order in $k_x$

$$k_{y\pm} = \pm i(c_{\pm} k_x + d_{\pm} k_x^{-1}),$$

and obtain

$$G_0(k_x, k_{y+}, k_{y-}) = G_{00} + ia(d_+ + d_-) k_x^{-1}$$
with $G_{00}$ as in (3.20) and thus $G_{00} = 0$ by the earlier computation at leading order. Moreover,

$$G_1 = \frac{1}{2\nu \omega} \frac{ik_x}{(1-c_-)(1-c_+)} (c_+^2 - c_-^2).$$

Here it still suffices to retain Eq. (3.2) at leading order in $\omega$, whence $(c_+^2 - c_-^2)k_x^2 = 2\omega/\nu$. We find for the solutions $g_0 = 0$

$$d_+ + d_- = -\frac{1}{\nu^2(1-c_-)(1-c_+)}.$$

Furthermore, we have to amend the constraint (3.18) to

$$2k_x^2 + k_{yx+}^2 + k_{yx-}^2 = -\frac{1 - 2vf}{\nu^2},$$

as seen from (3.3). Plugging in the ansatz and using (3.21) we find that $d_{\pm}$ are determined by

$$c_+d_+ + c_-d_- = \frac{1 - 2vf}{2\nu^2},$$

besides of (3.23). We will from now on focus on the transition at $a = \sqrt{2}$ and explain the emergence of an edge mode branch for $a > \sqrt{2}$. The case of opposite values of $a$ works analogously. From Fig. 5 and its explanation we see that the first order solution for $a = \sqrt{2}$ is given by $(c_+, c_-) = (0, -\sqrt{2})$, whence (3.23, 3.5) become

$$d_+ + d_- = \nu^{-2}(1 - \sqrt{2}),$$

$$d_- = -\nu^{-2} \frac{1 - 2vf}{2\sqrt{2}},$$

their solution being

$$d_+ = -\nu^{-2} \frac{3 - 2\sqrt{2} + 2vf}{2\sqrt{2}}.$$

A further incipient state (besides of $k_x = -\infty$) occurs when $\text{Im} k_{yx} = 0$. Thus by the ansatz (3.22) $\text{Im} k_{yx}$ changes sign for $k_x$ such that

$$k_x^2 = -\frac{d_+}{c_+}.$$

For $a \searrow \sqrt{2}$ we have $c_+ \searrow 0$ (see Fig. 5). Thus and by $d_+ < 0$ there is one solution $k_x$ of (3.24) with $k_x \to \infty$ in agreement with Fig. 1. This also verifies that there is no accumulation of merging points at $\infty$, so that merging points at finite $k_x$ are contained in a compact subset of $\mathbb{R}$.

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A. Chern Number for Spin $s$ Representations

Proposition 1 appears here and there in the literature for various values of $s$, see e.g. [25, Ex 11.4]. Here we provide a self-contained and general proof. It suffices to prove the case where $M = S^2$ and we do so by induction in $s$ in steps of 1/2, starting with $s = 0$ and $s = 1/2$. In the first case the bundle is trivial, $S^2 \times \mathbb{C}$, whence $C(P_{0,0}) = 0$, where the eigenprojection on the band with labels $(s, m)$ is denoted by $P_{s,m}$. In the case $s = 1/2$, the integrand of (2.6) is

$$\text{tr}(P_\pm[dP_\pm, dP_\pm]) = \pm \frac{i}{2} \vec{e} \cdot (d\vec{e} \wedge d\vec{e}) = \pm \frac{i}{2} w$$

with $P_\pm = P_{\frac{1}{2}, \frac{1}{2}}$. This result follows from

$$P_\pm = \frac{1}{2}(1 \pm \vec{e} \cdot \vec{a})$$

$$\text{tr} \vec{a} \cdot \vec{a} [\vec{b} \cdot \vec{a}, \vec{c} \cdot \vec{a}] = 4i\vec{a} \cdot (\vec{b} \wedge \vec{c})$$

where $\vec{a} = (\sigma_1, \sigma_2, \sigma_3)^T$ denotes the vector of Pauli matrices $\sigma_i$. This leads in turn to $C(P_{\pm}) = \pm 1$ by $\int_{S^2} w = 4\pi$.

We next assume the claim to be true up to $s$ and prove it for $s + 1/2$. Let $D_s$ be the irreducible representation of $SU(2)$ of spin $s$, equipped with the standard basis $|s, m\rangle_{m=-s}^{s=+s}$ with respect to the quantization axis $\vec{e}$, i.e., $\vec{S} \cdot \vec{e} |s, m\rangle = m|s, m\rangle$. We then have by the Clebsch-Gordan series

$$D_{s+\frac{1}{2}} \oplus D_{s-\frac{1}{2}} = D_s \otimes D_{\frac{1}{2}} \quad (A.1)$$

We first treat the case $m = s + 1/2$, for which we find

$$|s + \frac{1}{2}, s + \frac{1}{2}\rangle = |s, s\rangle \otimes |\frac{1}{2}, \frac{1}{2}\rangle$$

whence

$$P_{s+\frac{1}{2}, s+\frac{1}{2}} = P_{s,s} \otimes P_{\frac{1}{2}, \frac{1}{2}}$$

Since the vector bundles are line bundles, the Chern number is additive, meaning

$$C(P_{s+\frac{1}{2}, s+\frac{1}{2}}) = C(P_{s,s}) + C(P_{\frac{1}{2}, \frac{1}{2}}) = 2s + 1,$$

as claimed. The case $m = -(s + 1/2)$ is similar. Finally, we consider the intermediate cases $m = -(s - 1/2), ..., s - 1/2$. The eigenspace of (total) $\vec{S} \cdot \vec{e}$ acting on (A.1) for eigenvalue $m$ has dimension 2 and can be represented as a span in two ways:

$$\left[|s, m - \frac{1}{2}\rangle \otimes |\frac{1}{2}, \frac{1}{2}\rangle, |s, m + \frac{1}{2}\rangle \otimes |\frac{1}{2}, -\frac{1}{2}\rangle\right] = \left[|s - \frac{1}{2}, m\rangle, |s + \frac{1}{2}, m\rangle\right].$$

The bundle over $S^2 \ni \vec{e}$ having the eigenspaces as fibers is thus

$$\left( P_{s,m-\frac{1}{2}} \otimes P_{\frac{1}{2}, \frac{1}{2}} \right) \oplus \left( P_{s,m+\frac{1}{2}} \otimes P_{\frac{1}{2}, -\frac{1}{2}} \right) = P_{s-\frac{1}{2}, m} \oplus P_{s+\frac{1}{2}, m}.$$

Arguing as before we get for $c_{s,m} := C(P_{s,m})$

$$(c_{s,m-\frac{1}{2}} + c_{\frac{1}{2}, -\frac{1}{2}}) + (c_{s,m+\frac{1}{2}} + c_{\frac{1}{2}, \frac{1}{2}}) = c_{s-\frac{1}{2}, m} + c_{s+\frac{1}{2}, m},$$

i.e.

$$((2m - 1) + 1) + (2m + 1 - 1) = 2m + c_{s+\frac{1}{2}, m}$$

by induction assumption. Thus $c_{s+\frac{1}{2}, m} = 2m$ which proves Proposition 1.
B. Self-adjoint Boundary Conditions

In this appendix we will characterize self-adjoint boundary conditions for our model with domain \( \{(x, y) \mid y \geq 0\} \subset \mathbb{R}^2 \) which preserve translation invariance. After Fourier transformation along \( x \) with conjugate variable \( k_x \) (translation invariance in \( x \)-direction) the Hamiltonian is given by (cf. (2.9))

\[
\tilde{H}(k_x) = \begin{pmatrix}
0 & k_x & -i\partial_y \\
k_x & 0 & i(f - \nu(k_x^2 - \partial_y^2)) \\
-i\partial_y & i(f - \nu(k_x^2 - \partial_y^2)) & 0
\end{pmatrix},
\]

which is an operator on the half-line \( y > 0 \). We shall drop \( k_x \) from the notation till further notice, because it remains fixed. We also drop the \( \tilde{\cdot} \) and denote states by

\[
\psi = \psi(y) = \begin{pmatrix}
\eta \\
u \\
v
\end{pmatrix}.
\]

**Lemma B.1.** Self-adjoint realizations of the Hamiltonian \( H \) correspond to subspaces \( M \subset \mathbb{C}^6 \) with

\[
\Omega M = M^\perp, \tag{B.1}
\]

where

\[
\Omega = \begin{pmatrix}
0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\nu \\
-1 & 0 & 0 & 0 & \nu & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \nu & 0 & 0 & 0 \\
0 & -\nu & 0 & 0 & 0 & 0
\end{pmatrix}.
\]

The domain \( D(H) \) is a subspace of the Sobolev space \( H^1 \oplus H^2 \oplus H^2 \) characterized by:

\[
\psi \in D(H) \iff \Psi \in M, \tag{B.2}
\]

where the (stacked) column vector

\[
\Psi = \begin{pmatrix}
\psi \\
\psi'
\end{pmatrix}
\]

stands for the boundary values at \( y = 0 \).

**Proof.** Without yet imposing any boundary condition, the fact that \( \psi \in H^1 \oplus H^2 \oplus H^2 \) implies that \( \eta, u', v' \) are continuous and vanish at infinity. Given two such states \( \psi, \tilde{\psi}, \) a partial integration of \( \langle \tilde{\psi} \rangle \) thus yields boundary terms at \( y = 0 \) only:

\[
-i \left( \langle \tilde{\psi} \rangle - \langle H \tilde{\psi} \rangle \right) = -\tilde{\eta}v + \tilde{\nu} \eta + \nu(\tilde{v}u' - \tilde{\nu} u) - \nu(\tilde{\nu}v' - \tilde{uu}') \\
= \tilde{\Psi}^* \Omega \Psi \tag{B.3}
\]

with \( \Psi \) and \( \tilde{\Psi} \) as in (B.2), \( \Omega \) as in the Lemma to prove and \( = \partial_y \). Let the domain be \( \{ \psi \mid \Psi \in M \} \), where \( M \subset \mathbb{C}^6 \) is some subspace. Then \( M \) should have the properties

\[
\tilde{\Psi}^* \Omega \Psi = 0, (\Psi \in M) \Rightarrow \tilde{\Psi} \in M.
\]
\[ \Psi \in M \Rightarrow \tilde{\Psi}^* \Omega \Psi = 0 \ (\forall \tilde{\Psi} \in M). \]

In fact the first one implies \( H^* \subset H \) and the second \( H \subset H^* \), whence \( H = H^* \) as required. Because of \( \Omega^* = \Omega \) the two properties are summarized by

\[ \tilde{\Psi}^* \Omega \Psi = 0, \ (\Psi \in M) \iff \tilde{\Psi} \in M, \]

which is in turn equivalent to \( (\Omega M)^\perp = M \), i.e. to \( \Omega M = M^\perp \). \( \square \)

We note that \( \text{rk} \ \Omega = 4 \) and

\[ \ker \Omega \oplus \text{im} \ \Omega = \mathbb{C}^6 \tag{B.4} \]

(orthogonal direct sum) by \( \Omega = \Omega^* \). Dimensions are 2 and 4. Let \( \hat{\Omega} \) be a partial left inverse of \( \Omega \):

\[ \hat{\Omega} \Omega = P, \tag{B.5} \]

where \( P \) is the orthogonal projection on \( \text{im} \ \Omega \) associated to \( \Omega \). It follows

\[ \hat{\Omega} \tilde{\Omega} v = v, \ (v \in \text{im} \ \Omega). \tag{B.6} \]

**Lemma B.2.** Let \( M \subset \mathbb{C}^6 \). The following are equivalent:

(a) \( \Omega M = M^\perp \)

(b) There is a subspace \( \tilde{M} \subset \mathbb{C}^6 \) such that

1. \( M = \ker \Omega \oplus \tilde{M} \), orthogonal direct sum, (whence \( \tilde{M} \subset \text{im} \ \Omega \)),
2. \( \hat{\Omega} \tilde{M}^\perp = \tilde{M} \), where \( \tilde{M}^\perp \) is the orthogonal complement of \( \tilde{M} \) within \( \text{im} \ \Omega \).

(c) (1) \( \ker \Omega \subset M \),
(2) \( \dim M = 4 \),
(3) \( \hat{\Omega} M^\perp \subset M \).

**Proof.** \( (a) \Rightarrow (b) \): By \( (a) \), \( M^\perp \subset \text{im} \ \Omega \), and thus by \( \text{(B.4)} \) \( M \supset \ker \Omega \), proving \( (b1) \). Next we have \( M^\perp = \tilde{M}^\perp \) as we find by \( (b1) \)

\[ v \in M^\perp \iff v \perp \ker \Omega, \ v \perp \tilde{M} \]

\[ \iff v \in \tilde{M}^\perp \]

because \( (\ker \Omega)^\perp = \text{im} \ \Omega \). With \( \text{(B.5)} \) we get from \( (a) \) \( PM = \hat{\Omega} M^\perp \), i.e. \( \tilde{M} = \hat{\Omega} \tilde{M}^\perp \).

(b) \( \Rightarrow (c) \): First we see directly that \( (c1) \) follows from \( (b1) \). Furthermore, since \( \hat{\Omega} \) is regular as a map \( \text{im} \ \Omega \to \text{im} \ \Omega \), we have by \( (b2) \): \( 4 - \dim \tilde{M} = \dim M \), i.e. \( \dim \tilde{M} = 2 \), proving \( (c2) \). Property \( (c3) \) follows from \( (b2) \) and \( M^\perp = \tilde{M}^\perp \).

(c) \( \Rightarrow (a) \): By \( (c1) \), i.e. \( \text{im} \ \Omega \supset M^\perp \), and \( \text{(B.6)} \) we get from \( (c3) \)

\[ M^\perp \subset \Omega M. \tag{B.7} \]

By the rank-nullity theorem applied to \( \Omega : M \to \mathbb{C}^6 \), i.e.

\[ \dim M = \dim \ker(\Omega \upharpoonright M) + \dim \Omega M, \]

we get \( 4 = 2 + \dim \Omega M \) by \( (c1,c2) \). Hence equality in \( \text{(B.7)} \). \( \square \)
B.1. Boundary conditions in terms of equations. In the following the self-adjoint boundary conditions will be characterized more explicitly in terms of equations. For that we observe that ker $\Omega$ is spanned by the columns of the matrix

$$N = \begin{pmatrix} \nu & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}. \quad (B.8)$$

The partial left inverse $\hat{\Omega}$ is uniquely determined on $\text{im} \; \Omega$, but is arbitrary on ker $\Omega$. For definiteness, let us choose $\hat{\Omega}v = 0$, $(v \in \ker \Omega)$. By that we have explicitly

$$\hat{\Omega} = \begin{pmatrix} 0 & 0 & -\lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -v^{-1} \\ -\lambda & 0 & 0 & 0 & \lambda v & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda v & 0 & 0 & 0 \\ 0 & -v^{-1} & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \lambda = \frac{1}{1 + v^2}.$$  

In fact the so chosen partial left-inverse fulfills $\Omega \hat{\Omega} = P$ and $\hat{\Omega} N = 0$ (cf. (B.5)).

**Proposition B.3.** (i) Subspaces $M \subset \mathbb{C}^6$ as in Lemma B.1 of dimension 4 are determined by $2 \times 6$ matrices $A$ of maximal rank, i.e. $\text{rk} \; A = 2$, by means of

$$M = \{ \Psi \in \mathbb{C}^6 | A\Psi = 0 \} = \ker A, \quad (B.9)$$

and conversely. Two such matrices $A, \tilde{A}$ determine the same subspace if and only if $A = B \tilde{A}$ with $B \in \text{GL}(2)$.

(ii) Self-adjoint boundary conditions are determined precisely by matrices as in (i) with

$$AN = 0, \quad A\hat{\Omega}A^* = 0. \quad (B.10)$$

**Proof.** (i) Only the last sentence deserves proof, and in fact only the necessity of $A = B\tilde{A}$. For that consider a map $B : \mathbb{C}^2 \to \mathbb{C}^2$ which is well-defined by $Av \leftrightarrow \tilde{A}v$, $v \in \mathbb{C}^6$ because of ker $A = \ker \tilde{A}$.

(ii) By the Lemma B.2 and in particular by the equivalence between (a) and (c), $M$ is as in (B.9) by (c2). By (c1) and (B.8) the first equation (B.10) applies. Equation (B.9) states $M^\perp = \text{ran} \; A^* = \{ A^*v | v \in \mathbb{C}^2 \}$. Thus by (c3),

$$\langle A^*v_1 \rangle = 0, \quad (v_1, \; v_2 \in \mathbb{C}^2),$$

i.e. the second equation (B.10).

\[ \square \]

**Example B.4.** Dirichlet boundary conditions $u = 0, v = 0$ correspond to

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

which is seen to satisfy (B.10).
B.2. Local boundary conditions. In this section we will study local boundary conditions which contain the ones studied in the main text of the paper. For \( \psi(x, y) \), let us consider local boundary conditions at \( y = 0 \) of the form

\[
B_0 \psi + B_1 \partial_x \psi + B_2 \partial_y \psi = 0
\]

with \( l \times 3 \)-matrices \( B_i \) \( (l = 2 \) suffices). After using translation invariance

\[
\psi(x) = \psi(y) e^{ik_x x}
\]

they reduce to

\[
(B_0 + i k_x B_1) \psi + B_2 \psi' = 0,
\]

\( (\prime = \partial/\partial y) \), i.e. to \( A \Psi = 0 \) as in Proposition B.3 with

\[
A = (B_0 + i k_x B_1, B_2), \quad A_0 = (B_0, B_2), \quad A_1 = (B_1, 0).
\]

Remark B.5. Quite generally, the condition \( \text{rk} A \geq 2 \) is equivalent to \( A \wedge A \neq 0 \). So \( \text{rk} A(k_x) \geq 2 \) (and hence \( = 2 \), cf. Proposition B.3) for a.e. \( k_x \) means that at least one among

\[
A_0 \wedge A_0, \quad A_0 \wedge A_1 + A_1 \wedge A_0, \quad A_1 \wedge A_1
\]
does not vanish.

The conditions (B.10) now mean

\[
A_0 N = 0, \quad A_1 N = 0,
\]

\[
A_0 \hat{\Omega} A_0^* = 0, \quad A_0 \hat{\Omega} A_1^* - A_1 \hat{\Omega} A_0^* = 0, \quad A_1 \hat{\Omega} A_1^* = 0.
\]

Example B.6. The boundary conditions \( v = 0, \partial_x u + a \partial_y v = 0 \) correspond to

\[
A_0 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix},
\]

which are seen to fulfill the conditions stated above.

C. Remarkable Edge Mode Branches

Coastal Kelvin wave. In order to show the existence of a branch along (2.11), we note that such \((k_x, \omega)\) remain below the continuum, meaning that both solutions \( X_{\pm} \) of (3.1) correspond to evanescent waves of momenta \( \kappa_{\pm} \in i \mathbb{R} \). As explained at the end of Sect. 3.3, they may superpose to a bound state iff \( g(k_x, \kappa_{\pm}) = 0 \). A condition that is in turn sufficient, but not necessary, for that one is

\[
v_{\infty}(k_x, \kappa_{\pm}) = 0,
\]

cf. (3.9). We will confirm this in a few steps. The first one is that

\[
f - v X_{\pm} > 0.
\]
In fact $2v(f - vX_\pm) = 1 \mp \sqrt{\Delta}$ by (3.2) with $\Delta < 1$, because of $f > \omega^2 v$ by (2.11). In a second step we rewrite (3.1) as

$$\omega^2 = X_\pm + \omega^2 q_\pm^2, \quad X_\pm = k_x^2 + \kappa_\pm^2$$

with $\omega q_\pm := f - vX_\pm$, so as to conclude that $\kappa_\pm^2 + \omega^2 q_\pm^2 = 0$. By $\omega q_\pm > 0$ the solution with positive imaginary parts is $\kappa_\pm = \pm i\sqrt{\omega^2 - q_\pm^2}$, which implies (C.1) by (3.6).

**The case** $a = 1$. There is a branch of edge states with flat dispersion

$$\omega = 0, \quad (k_x \in \mathbb{R}),$$

as seen in the middle image of Fig. 2: As $a$ crosses 1, a line of edge modes changes the sign of its slope by going through a horizontal line along $k_x \in \mathbb{R}$. Just as for the Kelvin waves, both solutions $X_\pm$ of (3.1) correspond to evanescent modes; likewise we claim $g(k_x, \kappa) = 0$ on the basis of (3.9). Also in this case there is a sufficient condition, and it reads

$$f(\infty)(k_x, \kappa_\pm) = 0.$$  \hspace{1cm} (C.2)

Some caution is though to be taken first. Since the section $\Psi_\infty$, cf. (3.6), diverges at $\omega = 0$, we better multiply it by $\omega$ first, so that it remains finite there. Then, a short computation yields

$$f(\infty)(k_x, \kappa) = (k_x + i\kappa)\omega + (a - 1)\kappa v(\infty)(k_x, \kappa),$$  \hspace{1cm} (C.3)

where (3.1) is understood. Thus (C.2) holds for $a = 1$ at $\omega = 0$. A further feature of Fig. 2 is confirmed by the following claim.

**Proposition C.1.** Given any $\omega$ with $|\omega| < f$ there is a branch of edge eigenvalues taking values in $[-\omega, \omega]$ at all $k_x \in \mathbb{R}$, provided $|a - 1|$ is small enough.

We keep the normalization used for (C.3) and $g$ as in (3.9) with $\zeta = \infty$.

**Lemma C.2.** For $(k_x, \omega)$ in the gap, the function $g$ is purely imaginary. We have

$$g(k_x, \omega; a) = \omega g_1(k_x, \omega) + (a - 1)g_2(k_x, \omega)$$  \hspace{1cm} (C.4)

with asymptotic behavior at $k_x \to \pm \infty$

$$g_1(k_x, \omega) = ic(\omega)k_x^{-1} + O(|k_x|^{-3}),$$  \hspace{1cm} (C.5)

$$g_2(k_x, \omega) = O(|k_x|^{-1}),$$  \hspace{1cm} (C.6)

where $c(\omega) \neq 0$ for $\omega + f > 0$ and the estimates are locally uniform in $\omega$.

The proposition is immediate from the lemma. The statement follows by continuity in $a$ near $a = 1$ as long as $k_x$ ranges over a compact interval. To be thus proven is existence of eigenvalues in the stated interval for large $|k_x|$. Given $\omega > 0$, we have $g(k_x, \pm \omega; a) \neq 0$ for $|a - 1|$ small and $|k_x|$ large enough; moreover the two values have opposite (imaginary) sign. There thus is $\omega'$ in the interval with $g(k_x, \omega'; a) = 0$, i.e., an edge eigenvalue.

We next prove Lem. C.2. The function $g$ is imaginary because in the regime considered $u_\pm = u(\omega)(k_x, \kappa_\pm)$ and $v_\pm = v(\infty)(k_x, \kappa_\pm)$ are real and imaginary, respectively, cf. (3.6, 3.9). From the latter equation and (C.3) we get (C.4) with

$$g_1 = \begin{vmatrix} k_x + i\kappa_+ & k_x - i\kappa_- \\ v_+ & v_- \end{vmatrix}, \quad g_2 = \begin{vmatrix} \kappa_+v_+ & \kappa_-v_- \\ v_+ & v_- \end{vmatrix}.$$
We have the asymptotics for $k_x \to \pm \infty$

$$\kappa_{\pm} = i \sqrt{k_x^2 - X_{\pm}} = i k_x - i \frac{X_{\pm}}{2k_x} + O(|k_x|^{-3})$$

since $X_{\pm}$ are bounded here, just as $\omega$ is. Thus

$$k_x + i \kappa_{\pm} = \frac{X_{\pm}}{2k_x} + O(|k_x|^{-3})$$

and

$$v_{\pm} = \frac{1}{k_x - i \kappa_{\pm}} (\kappa_{\pm} \omega + i k_x (f - \nu X_{\pm})) = v_{\pm, \infty}(\omega) + O(k_x^{-2}),$$

$$v_{\pm, \infty}(\omega) = \frac{i}{2} (\omega + f - \nu X_{\pm}).$$

We then compute

$$g_1 = \frac{1}{2k_x} (X_+ v_- - X_- v_+) + O(|k_x|^{-3})$$

which fits (C.5) with

$$ic(\omega) = \frac{i}{4} (X_+ (\omega + f - \nu X_-) - X_- (\omega + f - \nu X_+)) = \frac{i}{4} (\omega + f)(X_+ - X_-),$$

which is $\neq 0$ by $\Delta \neq 0$, cf. (3.2). Likewise

$$g_2 = (\kappa_+ - \kappa_-) v_+ v_- = O(|k_x|^{-1}),$$

in line with (C.6).

**The case** $a = 0$. General results imply that $\mathcal{H}$ is a self-adjoint operator on the half-plane iff $H(k_x)$ is one on the half-line for a.e. $k_x$, yet not necessarily for all $k_x \in \mathbb{R}$. For $a = 0$ such an exception occurs, in that $A(k_x = 0) = A_0$, cf. (B.11), and $\text{rk} A_0 = 1$, which misses the required rank of 2. This lack of self-adjointness at $k_x = 0$ goes along with the transition at $a = 0$ seen in Proposition 5: As $a$ crosses 0, a line of edge modes changes the sign of its slope by going through a vertical line at $k_x = 0$.

**D. Scattering Theory for General Boundary Conditions**

The scattering states and the scattering amplitude can be defined for any self-adjoint boundary condition. In the main text we focused on the variables $(k_x, \kappa)$, so that $\omega = \omega_4(k_x, \kappa)$. Here, we work instead with $(k_x, \omega)$. Asymptotic states are given in terms of plane wave solutions

$$\psi(x, t) = \hat{\psi} e^{i(k_x x + k_y y - \omega t)}. \tag{D.1}$$

Let $k_x$ and $\omega > \sqrt{k_x^2 + (f - \nu k_y^2)^2}$ be given. There are two solutions $X_{\pm} \equiv k^2$ of

$$\omega^2 = X + (f - \nu X)^2;$$

they have $\pm X_{\pm} > 0$ because of $\nu^2 X_+ X_- = f^2 - \omega^2 < 0$; and hence 4 solutions $k_y$ of $X = k_x^2 + k_y^2$, namely two real ones, incoming ($\kappa_{\text{in}} < 0$), outgoing ($\kappa_{\text{out}} = -\kappa_{\text{in}} > 0$); and
two imaginary ones, decaying \((\kappa_{ev}, \imath \kappa_{ev} < 0)\) and diverging \((\kappa_{div} = -\kappa_{ev}, \imath \kappa_{div} > 0)\). The first three are, up to multiples, cf. \((3.6)\),

\[
\hat{\psi}_{in} = \left( \frac{k_x^2 + \kappa_{in}^2}{\kappa_{in} + i k_x q_+} \right), \quad \hat{\psi}_{out} = \left( \frac{k_x^2 + \kappa_{out}^2}{\kappa_{out} + i k_x q_+} \right), \quad \hat{\psi}_{ev} = \left( \frac{k_x^2 + \kappa_{ev}^2}{\kappa_{ev} + i k_x q_+} \right),
\]

with

\[
q_\pm = \frac{f - \imath \nu X_\pm}{\omega}.
\]

The solutions \((D.2)\) are to be seen in relation with \((D.1)\). They contribute boundary values

\[
\Psi_{in} = \left( \hat{\psi}_{in} \hat{\psi}_{in} \right), \quad \Psi_{out} = \left( \hat{\psi}_{out} \hat{\psi}_{out} \right), \quad \Psi_{ev} = \left( \hat{\psi}_{ev} \hat{\psi}_{ev} \right).
\]

We shall assume that there are no embedded eigenvalues. We conjecture this to be true for any self-adjoint boundary condition, and we show it for \((2.8)\), which is of relevance for the rest of this paper. We do so at the end of this section.

**Lemma D.1.** For \(k_y \neq 0\) (cf. \(\omega > \sqrt{k_x^2 + (f - \imath \nu k_x^2)^2}\) above) the three vectors in \((D.4)\) are linearly independent.

**Proof.** Inspection of \(H \psi = \omega \psi\) as a differential equation in \(y\), cf. \((2.9)\), shows that any initial values

\[
\Psi = \left( \begin{array}{c} \psi \\ \psi' \end{array} \right) = \left( \begin{array}{c} \eta \\ \vdots \\ v' \end{array} \right)
\]

determine an existing and unique solution provided

\[
k_x u - \imath v' = \omega \eta,
\]

which \((D.4)\) do. A linear combination

\[
f_{in} \Psi_{in} + f_{out} \Psi_{out} + f_{ev} \Psi_{ev} = 0
\]

then implies

\[
f_{in} \hat{\psi}_{in} e^{\imath k_{in} y} + f_{out} \hat{\psi}_{out} e^{\imath k_{out} y} + f_{ev} \hat{\psi}_{ev} e^{\imath k_{ev} y} = 0.
\]

Linear independence of the functions \(e^{\imath k_{in} y}, e^{\imath k_{out} y}, e^{\imath k_{ev} y}\) implies \(f_{in} = f_{out} = f_{ev} = 0\), as was to be shown. \(\square\)

**Lemma D.2.** Let \(\psi, \tilde{\psi}\) be two solutions of \(H \psi = \omega \psi\) that are bounded in \(y \geq 0\), but regardless of boundary conditions. If one of them vanishes at \(y \to +\infty\), then

\[
\tilde{\psi}^* \Omega \Psi = 0,
\]

where \(\Psi\) is given by \((D.3)\).

**Proof.** The terms on the r.h.s of \((B.3)\) are supposed to be evaluated at \(y = 0\). The equation itself was obtained because the same terms would vanish for \(y \to \infty\), and that is what they still do here, because in each of them one factor does while the other stays bounded. In fact, if a solution \(\psi\) is bounded (or even vanishes at infinity), i.e. \(\imath m_{\psi} \geq 0\) in \((D.1)\), then so does \(\psi'\). Moreover, the l.h.s. of \((B.3)\) vanishes by \(H \psi = \omega \psi\). \(\square\)
Let a boundary condition $M$ be determined by a matrix $A$ as in (B.9, B.10). It reads
\[ A\Psi = 0, \quad \text{for} \quad \Psi = f_{in}\Psi_{in} + f_{out}\Psi_{out} + f_{ev}\Psi_{ev}. \quad (D.6) \]

**Lemma D.3.** There is a unique solution $(f_{in}, f_{out}, f_{ev})$ up to multiples.

**Proof.** There is at least one solution, since the span of (D.4), which has dimension 3 by Lemma D.1, must intersect non-trivially $\ker A$ (of dimension 4) in view of $3 + 4 > 6$. On the other hand, there are no two linearly independent solutions. In fact, if so, there would be a solution of (D.6) with $f_{in} = 0$, i.e.
\[ \Psi = f_{out}\Psi_{out} + f_{ev}\Psi_{ev}, \]
which we shall rule out, unless trivial. Since $\Psi \in M$ we have $\Psi^*\Omega\Psi = 0$ by (B.1). Even though (D.5) does not allow to reach the same conclusion for $\Psi_{out}$, because $\psi_{out}$ is not vanishing at $y \to \infty$, it does for $f_{out}\Psi_{out} = \Psi - f_{ev}\Psi_{ev}$, because $\psi_{ev}$ does and so any contribution to (D.5) involving it:
\[ |f_{out}|^2 \cdot \Psi_{out}^*\Omega\Psi_{out} = 0. \]

We claim that the second factor does not vanish. In fact, a straightforward computation based on (B.3) gives
\[ \Psi_{out}^*\Omega\Psi_{out} = -\frac{2X_+}{\omega}k_{out} + 4\nu X_+k_{out}q_+ \]
\[ = -\frac{2X_+k_{out}}{\omega} (1 - 2\nu(f - \nu X_+)) < 0 \]
because of $1 - 2\nu f > 0$, $2\nu^2X_+ > 0$. □

**Proposition D.4.** For any self-adjoint boundary condition, $k_x \in \mathbb{R}$ and $\omega > \sqrt{k_x^2 + (f - \nu k_x^2)^2}$ the scattering amplitude $S(k_x, \omega) \equiv S = f_{out}/f_{in}$ is well-defined and satisfies $|S| = 1$.

**Proof.** A straightforward computation yields
\[ \Psi_{out}^*\Omega\Psi_{out} = -\Psi_{in}^*\Omega\Psi_{in} \neq 0, \]
\[ \Psi_{out}^*\Omega\Psi_{in} = 0 = \Psi_{in}^*\Omega\Psi_{out}, \]
where the last equality follows by $\Omega^* = \Omega$. Using $\Psi^*\Omega\Psi = 0$ for $\Psi$ as in (D.6) yields $|f_{out}|^2 - |f_{in}|^2 = 0$ by (D.5). □

**Remark D.5.** The boundary condition (2.8), and in particular $v = 0$, does not allow for embedded eigenvalues. In fact, in view of (D.2), that would amount to
\[ \kappa_{ev} + ik_xq_- = 0, \]
and thus to
\[ X_- = k_x^2 + \kappa_{ev}^2 = k_x^2(1 - q_-^2) \]
\[ = \frac{k_x^2}{\omega^2}(\omega^2 - (f - \nu X_-)^2) = \frac{k_x^2}{\omega^2}X_- \]
or equivalently $(\omega^2 - k_x^2)X_- = 0$. Since both factors are known to be non-zero this is impossible.
E. Dirichlet Boundary Condition

The computation of the scattering amplitude is similar to the one in Sect. 3.3, but with (2.8) replaced by (2.19) we end up with the simpler form

\[ S_\zeta(k_x, \kappa) = -\frac{g_\zeta(k_x, -\kappa)}{g_\zeta(k_x, \kappa)}, \quad g_\zeta(k_x, \kappa) = \begin{vmatrix} u_\zeta(k_x, \kappa) & u_\infty(k_x, \kappa_{\text{ev}}) \\ v_\zeta(k_x, \kappa) & v_\infty(k_x, \kappa_{\text{ev}}) \end{vmatrix}. \]

In particular, for \( \zeta = 0 \) one infers \( g_0(k_x, \kappa) \to 2i \) as \( (k_x, \kappa) \to \infty \), so that \( S_0 \to -1 \). Hence the scattering amplitude has no zero or pole in the neighborhood of \( \infty \) and does not wind either. This is the regular situation where Levinson’s theorem applies. Apart from (2.18), the rest of Theorem 9 applies indeed similarly for Dirichlet boundary condition. This implies \( C_+ = n_b \) so that the bulk-edge correspondence is satisfied.

References

1. Avila, J.C., Schulz-Baldes, H., Villegas-Blas, C.: Topological invariants of edge states for periodic two-dimensional models. Math. Phys. Anal. Geom 16(2), 137–170 (2013)
2. Avron, J.E.: Odd viscosity. J. Stat. Phys. 92(3–4), 543–557 (1998)
3. Avron, J.E., Seiler, R., Simon, B.: Charge deficiency, charge transport and comparison of dimensions. Commun. Math. Phys. 159(2), 399–422 (1994)
4. Bal, G.: Topological protection of perturbed edge states. (2017) arXiv:1709.00605
5. Bal, G.: Continuous bulk and interface description of topological insulators. J. Math. Phys. 60(8), 081506 (2019)
6. Banerjee, D., Souslov, A., Abanov, A.G., Vitelli, V.: Odd viscosity in chiral active fluids. Nat. Commun. 8(1), 1573 (2017)
7. Bellissard, J., van Elst, A., Schulz-Baldes, H.: The noncommutative geometry of the quantum Hall effect. J. Math. Phys. 35(10), 5373–5451 (1994)
8. Bourne, C., Rennie, A.: Chern numbers, localisation and the bulk-edge correspondence for continuous models of topological phases. Math. Phys. Anal. Geom. 21(3), 16 (2018)
9. Bräunlich, G., Graf, G.M., Ortelli, G.: Equivalence of topological and scattering approaches to quantum pumping. Commun. Math. Phys. 295(1), 243–259 (2010)
10. Combes, J.M., Germinet, F.: Edge and impurity effects on quantization of Hall currents. Commun. Math. Phys. 256(1), 159–180 (2005)
11. Delplace, P., Marston, J.B., Venaille, A.: Topological origin of equatorial waves. Science 358(6366), 1075–1077 (2017)
12. De Nittis, G., Lein, M.: Symmetry classification of topological photonic crystals. (2017) arXiv preprint arXiv:1710.08104
13. Drouot, A.: The bulk-edge correspondence for continuous honeycomb lattices. (2019) arXiv preprint arXiv:1901.06281
14. Essin, A.M., Gurarie, V.: Bulk-boundary correspondence of topological insulators from their Green’s functions. Phys. Rev. B 84, 125132 (2011)
15. Fefferman, C.L., Lee-Thorp, J.P., Weinstein, M.I.: Edge states in honeycomb structures. Ann. PDE 2(2), 12 (2016)
16. Fröhlich, J., Studer, U.M., Thiran, E.: Quantum theory of large systems of non-relativistic matter. (1995) arXiv preprint arXiv:cond-math/9508062
17. Graf, G.M., Porta, M.: Bulk-edge correspondence for two-dimensional topological insulators. Commun. Math. Phys. 324(3), 851–895 (2013)
18. Gruber, M.J., Leitner, M.: Spontaneous edge currents for the Dirac equation in two space dimensions. Lett. Math. Phys. 75(1), 25–37 (2006)
19. Halperin, B.I.: Quantized Hall conductance, current-carrying edge states, and the existence of extended states in a two-dimensional disordered potential. Phys. Rev. B 25(4), 2185 (1982)
20. Hatsugai, Y.: Chern number and edge states in the integer quantum Hall effect. Phys. Rev. Lett. 71(22), 3697 (1993)
21. Iga, K.: Transition modes of rotating shallow water waves in a channel. J. Fluid Mech. 294, 367–390 (1995)
22. Kellendonk, J., Pankrashkin, K., Richard, S.: Levinson’s theorem and higher degree traces for Aharonov-Bohm operators. J. Math. Phys. 52(5), (2011)
23. Kellendonk, J., Schulz-Baldes, H.: Quantization of edge currents for continuous magnetic operators. J. Funct. Anal. 209(2), 388–413 (2004)
24. Kotani, M., Schulz-Baldes, H., Villegas-Blas, C.: Quantization of interface currents. J. Math. Phys. 55(12), 121901 (2014)
25. Nakahara, M.: Geometry, Topology and Physics. CRC Press, Boca Raton (2003)
26. Prodan, E., Schulz-Baldes, H.: Bulk and boundary invariants for complex topological insulators. From K-theory to physics. Math. Phys. Stud. (2016)
27. Peri, V., Serra-Garcia, M., Ilan, R., Huber, S.D.: Axial-field-induced chiral channels in an acoustic Weyl system. Nat. Phys. 15(4), 357 (2019)
28. Raghu, S., Haldane, F.D.M.: Analogs of quantum-Hall-effect edge states in photonic crystals. Phys. Rev. A 78(3), 033834 (2008)
29. Reed, M., Barry, S.: III: Scattering Theory (Vol. 3). Elsevier (1979)
30. Schulz-Baldes, H., Kellendonk, J., Richter, T.: Simultaneous quantization of edge and bulk Hall conductivity. J. Phys. A: Math. Gen. 33, L27 (2000)
31. Schulz-Baldes, H., Toniolo, D.: Dimensional reduction and scattering formulation for even topological invariants. (2018) arXiv:1811.11781
32. Souslov, A., Dasbiswas, K., Fruchart, M., Vaikuntanathan, S., Vitelli, V.: Topological waves in fluids with odd viscosity. Phys. Rev. Lett. 122(12), 128001 (2019)
33. Tauber, C., Delplace, P., Venaille, A.: A bulk-interface correspondence for equatorial waves. J. Fluid Mech. 868 (2019)
34. Tauber, C., Delplace, P., Venaille, A.: Anomalous bulk-edge correspondence in continuous media. Phys. Rev. Res. 2(1), 013147 (2019)
35. Vallis, G.: Atmospheric and Oceanic Fluid Dynamics: Fundamentals and Large-Scale Circulation. Cambridge University Press, Cambridge (2017)

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