Asymmetric Choi–Davis inequalities

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ABSTRACT
Let \( \Phi \) be a unital positive linear map and let \( A \) be a positive invertible operator. We prove that there exist partial isometries \( U \) and \( V \) such that

\[
|\Phi(f(A))\Phi(A)\Phi(g(A))| \leq U^* \Phi(f(A)Ag(A))U
\]

and

\[
|\Phi(f(A))^{-f}\Phi(A)^f\Phi(g(A))^{-f}| \leq V^* \Phi(f(A)^{-f}Ag(A)^{-f})V
\]

hold under some mild operator convex conditions and some positive numbers \( r \). Further, we show that if \( f^2 \) is operator concave, then

\[
|\Phi(f(A))\Phi(A)| \leq \Phi(Af(A)).
\]

In addition, we give some counterparts to the asymmetric Choi–Davis inequality and asymmetric Kadison inequality. Our results extend some inequalities due to Bourin–Ricard and Furuta.

1. Introduction

Throughout the paper, let \( \mathcal{B}(\mathcal{H}) \) stand for the \( C^* \)-algebra of all bounded linear operators on a Hilbert space \( \mathcal{H} \) with the identity \( I \). It is identified by the full matrix algebra \( \mathbb{M}_n \) when \( \dim \mathcal{H} = n \). A capital letter displays an operator in \( \mathcal{B}(\mathcal{H}) \). The usual (Löwner) order on the real space of all self-adjoint operators is denoted by \( \leq \); in particular, we write \( A \geq 0 \) when \( A \) is a positive operator (positive semidefinite matrix). When \( mI \leq A \leq MI \), we write \( m \leq A \leq M \) for simplicity. A map \( \Phi \) defined on \( \mathcal{B}(\mathcal{H}) \) is called positive whenever it takes positive operators to positive operators.

In the classical probability theory, the variance of a random variable \( X \) is defined by \( \text{Var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2 \), where \( \mathbb{E} \) is the expectation value. One of the basic properties of this quantity is its positivity. As a noncommutative extension, the operator valued map \( \text{Var}(A) = \Phi(A^2) - \Phi(A)^2 \) is said to be the variance of the self-adjoint operator \( A \), where \( \Phi \) is a unital positive linear map. The celebrated Kadison inequality asserts that \( \text{Var}(A) \) is...
a positive operator, that is,
\[ \Phi(A)^2 \leq \Phi(A)^2. \]

Throughout the paper, we assume that all real-valued functions defined on some intervals of \( \mathbb{R} \) are continuous. A real-valued function \( f \) defined on an interval \( J \subseteq \mathbb{R} \) is called operator convex if \( f(\lambda A + (1 - \lambda)B) \leq \lambda f(A) + (1 - \lambda)f(B) \) for all self-adjoint operators \( A \) and \( B \) with spectra in \( J \) and all \( \lambda \in [0,1] \). It is called operator concave whenever \( -f \) is operator convex. It can be shown that a function \( f \) defined on an interval \( J \) is operator convex if and only if the so-called Choi–Davis inequality (or sometimes Choi–Davis–Jensen’s inequality; see [1])
\[ f(\Phi(A)) \leq \Phi(f(A)) \quad (1) \]
holds for all self-adjoint operators \( A \) with spectrum in \( J \) and for all unital positive linear maps \( \Phi \). In fact, Davis [2] proved that (1) holds when \( f \) is an operator convex function and \( \Phi \) is a completely positive linear map. Choi [3] showed that inequality (1) remains true for all positive unital linear maps \( \Phi \) and all operator convex functions \( f \).

If \( f \) is convex but not operator convex, it is shown in [4] that the Choi–Davis inequality remains valid for every \( 2 \times 2 \) Hermitian matrix \( A \). Bourin and Lee in the nice survey [5] gave a variety of Choi–Davis type inequalities for general convex or concave functions. Niezgoda [6] utilized generalized inverses of some linear operators and presented a refinement of the Choi–Davis inequality. The following inequalities are special cases of the Choi–Davis inequality:
\[ \Phi(A)^p \leq \Phi(A^p) \quad (1 \leq p \leq 2 \text{ or } -1 \leq p \leq 0) \quad \text{and} \quad \Phi(A)^p \geq \Phi(A^p) \quad (0 \leq p \leq 1). \quad (2) \]

Sharma et al. [7] gave a generalization of the Kadison inequality by showing the positivity of the operator matrix
\[
\begin{bmatrix}
I & \Phi(A) & \cdots & \Phi(A^r) \\
\Phi(A) & \Phi(A^2) & \cdots & \Phi(A^{r+1}) \\
\vdots & \vdots & \ddots & \vdots \\
\Phi(A^r) & \Phi(A^{r+1}) & \cdots & \Phi(A^{2r})
\end{bmatrix}
\]

Bourin and Ricard [8] utilized the celebrated Furuta inequality and presented an asymmetric Kadison inequality by showing that if \( \gamma \in [0,1] \), then
\[ \left| \Phi(X^\gamma)\Phi(X) \right| \leq \Phi(X)^{1+\gamma} \quad (3) \]
holds for every positive operator \( X \). This further implies a noncommutative version of Chebychev’s inequality as follows:
\[ \left| \Phi(X^\alpha)\Phi(X^\beta) \right| \leq \Phi(X^{\alpha+\beta}) \quad (4) \]
for all \( 0 \leq \alpha \leq \beta \). Sharma and Thakur [9] proved that a unital positive linear map \( \Phi \) on \( \mathbb{M}_2 \) preserves the commutativity of operators and used this fact to establish some results analogue to (4) with \( \Phi(X^\beta)\Phi(X^\alpha) \) instead of \( \left| \Phi(X^\alpha)\Phi(X^\beta) \right| \).
An extension of (4) was presented by Furuta [10] as follows:

\[ \left| \Phi(X^\alpha)^\gamma \Phi(X^\beta)^\gamma \right| \leq \Phi(X^{(\alpha+\beta)^\gamma}), \tag{5} \]

when \(0 \leq \alpha \leq \beta\) and \(\frac{\beta}{\alpha+\beta} \leq \gamma \leq \frac{2\beta}{\alpha+\beta}\). In fact, he gave a result interpolating (4) and the first inequality in (2). Furthermore, he showed that under the same conditions as above, the inequality

\[ \left| \Phi(X^{-\alpha})^{-\gamma} \Phi(X^\beta)^{-\gamma} \right| \leq \Phi(X^{(\alpha+\beta)^\gamma}) \tag{6} \]

is true.

Some further extensions of (5) have been discussed in [11].

The organization of the paper is as follows. In Section 2, we examine possible extensions of the classical Chebyshev inequality and then present some asymmetric Choi–Davis inequalities, which extend inequalities (4) and (6) in some certain directions. More precisely, we prove that if \(\Phi\) is a unital positive linear map and \(A\) is a positive invertible operator, then under some mild operator convex conditions and some positive numbers \(r\), there exist partial isometries \(U\) and \(V\) such that

\[ |\Phi(f(A))\Phi(A)\Phi(g(A))| \leq U^*\Phi(f(A)Ag(A))U \]

and

\[ \left| \Phi \left( f(A) \right)^{-r} \Phi(A)^{-r} \Phi \left( g(A) \right)^{-r} \right| \leq V^* \Phi \left( f(A)^{-r}A^r g(A)^{-r} \right) V. \]

In Section 3, we give some counterparts to the asymmetric Choi–Davis inequality and asymmetric Kadison inequality (4). Among other things, we show that, for every positive invertible operator \(A\) and certain real numbers \(\alpha, \beta, \) and \(\gamma\), there exists a partial isometry \(W\) such that

\[ \Phi(A^{\alpha+\beta+\gamma}) \leq KW \left| \Phi(A^\alpha)\Phi(A^\beta)\Phi(A^\gamma) \right| W^* \tag{7} \]

for some Kantorovich type constant \(K\).

## 2. Asymmetric Choi–Davis inequality

Assume that \(\{a_i\}\) and \(\{b_i\}\) \((i = 1, \ldots, k)\) are increasing sequences of positive real numbers. The classical Chebyshev inequality asserts that

\[ \left( \frac{1}{k} \sum_{i=1}^{k} a_i \right) \left( \frac{1}{k} \sum_{i=1}^{k} b_i \right) \leq \frac{1}{k} \sum_{i=1}^{k} a_i b_i. \]

If one of the sequences is decreasing, then the reverse inequality holds.
Assume that \( \Phi \) is a unital positive linear map. The operator extension

\[
|\Phi(B)\Phi(A)| \leq \Phi(A^{1/2}BA^{1/2})
\]  

(8)
does not hold in general. To see this, assume that the unital positive linear map \( \Phi : \mathbb{M}_3 \rightarrow \mathbb{M}_2 \) is defined by \( \Phi([a_{ij}]_{1 \leq i,j \leq 3}) = [a_{ij}]_{1 \leq i,j \leq 2} \) and consider the positive matrices

\[
B = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 3 \end{bmatrix}.
\]

Then \( |\Phi(B)\Phi(A)| = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} \leq \begin{bmatrix} 4 & 2.4 \cdots \\ 2 & 3.89 \cdots \end{bmatrix} = \Phi(A^{1/2}BA^{1/2}) \).

Another possible extension is

\[
\Phi(A)\Phi(B)\Phi(A) \leq \Phi(ABA).
\]  

(9)

This is not true in general, too. Using the same unital positive linear map \( \Phi \) and positive matrices \( A \) and \( B \) as above, we get

\[
\Phi(A)\Phi(B)\Phi(A) = \begin{bmatrix} 8 & 4 \\ 4 & 8 \end{bmatrix} \preceq \begin{bmatrix} 8 & 6 \\ 6 & 9 \end{bmatrix} = \Phi(ABA).
\]

Bourin and Ricard [8] showed that in the case when \( B := A^\gamma \) and \( \gamma \in [0,1] \), inequality (8) is valid. They also presented a variant of (9) in the setting of complex matrices \( \mathbb{M}_n \) as

\[
\Phi(A)\Phi(B)\Phi(A) \leq V\Phi(ABA)V^*
\]

for some unitary matrix \( V \), where \((A,B)\) is a pair of matrices with the property that \( A = h_1(C) \) and \( B = h_2(C) \) for some nonnegative, nondecreasing, and functions \( h_1 \) and \( h_2 \).

We note that a weaker version of (9) as

\[
\Phi(A)\Phi(B)^{-1}\Phi(A) \leq \Phi(AB^{-1}A)
\]

is valid in general for all positive operators \( A \) and \( B \). This is, a special case of the inequality \( g(\Phi(A), \Phi(B)) \leq \Phi(g(A,B)) \), which holds for every operator perspective function \( g \) defined by \( g(A,B) = A^{1/2}f(A^{-1/2}BA^{-1/2})A^{1/2} \), where \( f : [0, \infty) \rightarrow [0, \infty) \) is an operator convex function; see [12].

We need some known properties of operator concave functions. The next lemma can be found in [13]; see [13, Theorem 1.13 and Corollary 1.14].

**Lemma 2.1:** Suppose that \( f : (0, \infty) \rightarrow (0, \infty) \) is a function. The followings assertions are equivalent:

(i) \( f(t) \) is operator concave;
(ii) \( f(t) \) is operator monotone;
(iii) \( \frac{t}{f(t)} \) is operator monotone;
(iv) \( \frac{\partial}{\partial t} f(t) \) is operator convex.

Our first main result presents a variant of (9).
Theorem 2.2: Assume that $\Phi$ is a unital positive linear map and that $0 \leq r \leq 1/2$. If $f, g : (0, \infty) \to (0, \infty)$ are operator convex functions, then, for every positive invertible operator $A$, there exists a partial isometry $V$ such that

$$
\left| \Phi \left( f(A) \right)^{-r} \Phi(A)^r \Phi \left( g(A) \right)^{-r} \right| \leq V^* \Phi \left( f(A)^{-r} A^r g(A)^{-r} \right) V \tag{10}
$$

provided that $\frac{f(t)}{t}$ is operator concave. If $f$, $g$, and $t \mapsto \frac{t}{f(t)}$ are operator concave, then the reverse inequality holds.

As a special case of Theorem 2.2, assume that $f$ is operator convex and put $g(t) = 1$. In this case, taking account of Lemma 2.1, the operator concavity of $t \mapsto t^r$ is automatically satisfied. Note that the following equivalence assertions are derived from Lemma 2.1 and the operator convexity of $t \mapsto t^r$:

$$
f \text{ is operator convex} \iff \frac{f(t)}{t} \text{ is operator concave} \iff \frac{f(t)^r}{t^r} \text{ is operator concave.}
$$

The last one further implies the operator convexity of the function $\frac{t^r}{f(t)}$. Hence, we obtain the next result.

**Corollary 2.3:** Let $\Phi$ be a unital positive linear map, let $0 \leq r \leq 1/2$, and let $f : (0, \infty) \to (0, \infty)$ be an operator convex function. Then

$$
\left| \Phi \left( f(A) \right)^{-r} \Phi(A)^r \Phi \left( g(A) \right)^{-r} \right| \leq \Phi \left( A f(A)^{-r} \right)
$$

for every positive invertible operator $A$.

If $\gamma \in [0, 1]$, then with $f(t) = t^{-\gamma}$ and $g(t) = 1$, Theorem 2.2 concludes a variant of [10, Theorem 2.1].

**Corollary 2.4:** If $\gamma \in [0, 1]$ and $0 \leq r \leq 1/2$, then

$$
\left| \Phi \left( A^{-\gamma} \right)^{-r} \Phi(A)^r \right| \leq \Phi \left( A^{(1+\gamma)r} \right)
$$

for every unital positive linear map $\Phi$ and every positive invertible operator $A$.

**Proof of Theorem 2.2:** Since $f$ and $g$ are operator convex, the Choi–Davis inequality together with the operator monotonicity of $t \mapsto t^{2r}$ imply that

$$
\Phi(f(A))^{-2r} \leq f(\Phi(A))^{-2r} \quad \text{and} \quad \Phi(g(A))^{-2r} \leq g(\Phi(A))^{-2r}. \tag{11}
$$

Therefore

$$
\left| \Phi(f(A))^{-r} \Phi(A)^r \Phi(g(A))^{-r} \right| = \left\{ \Phi(g(A))^{-r} \Phi(A)^r f(\Phi(A))^{-2r} \Phi(A)^r \Phi(g(A))^{-r} \right\}^{1/2}
$$

$$
\leq \left\{ \Phi(g(A))^{-r} \Phi(A)^r f(\Phi(A))^{-2r} \Phi(A)^r \Phi(g(A))^{-r} \right\}^{1/2}
$$

$$
= \left| f(\Phi(A))^{-r} \Phi(A)^r \Phi(g(A))^{-r} \right| \cdot \tag{12}
$$
There exists a partial isometry $V$ such that

$$|f(\Phi(A))^{-r} \Phi(A)^r \Phi(g(A))^{-r}| = V^* |\Phi(g(A))^{-r} \Phi(A)^r f(\Phi(A))^{-r}| V.$$  \hspace{1cm} (13)

Moreover, by employing (11), we get

$$|\Phi(g(A))^{-r} \Phi(A)^r f(\Phi(A))^{-r}| = \left\{ f(\Phi(A))^{-r} \Phi(A)^r g(\Phi(A))^{-2r} \Phi(A)^r f(\Phi(A))^{-r} \right\}^{1/2} \leq \left\{ f(\Phi(A))^{-r} \Phi(A)^r g(\Phi(A))^{-2r} \Phi(A)^r f(\Phi(A))^{-r} \right\}^{1/2} = (f(\Phi(A))^{-1} \Phi(A)g(\Phi(A))^{-1})^r.$$  \hspace{1cm} (14)

Since $t \mapsto t^r$ is operator monotone, it follows from Lemma 2.1 that it is operator concave. Hence, if the function $t \mapsto \frac{f(t)g(t)}{t}$ is operator concave, then so is $t \mapsto \frac{t^r f(t)g(t)}{t^r}$. This guarantees the operator convexity of the function $t \mapsto \frac{t^r f(t)g(t)}{t^r}$. Hence, the Choi–Davis inequality yields that

$$f(\Phi(A))^{-r} \Phi(A)^r \Phi(g(A))^{-r} \leq \Phi \left( f(\Phi(A))^{-r} A^r g(\Phi(A))^{-r} \right).$$  \hspace{1cm} (15)

Inequality (10) is deduced by combining (12), (13), (14), and (15) together.

If $f$ and $g$ are operator concave, then

$$\Phi(f(A))^{-2r} \geq f(\Phi(A))^{-2r} \quad \text{and} \quad \Phi(g(A))^{-2r} \geq g(\Phi(A))^{-2r}.$$  

Furthermore, if $\frac{t}{f(t)g(t)}$ is operator concave, then the function $\frac{t^r}{f(t)g(t)}$ is also operator concave. A similar argument as in the proof of (10) shows that the reverse inequality of (10) holds. \hfill \blacksquare

The following theorem gives a Choi–Davis type asymmetric inequality. Further, it provides a generalization of the asymmetric Kadison inequality (4).

**Theorem 2.5:** Assume that $f, g : (0, \infty) \to (0, \infty)$ are functions and that $\Phi$ is a unital positive linear map. If $f^2$ and $g^2$ are operator concave, then for every positive invertible operator $A$, there exist partial isometries $U$ and $V$ such that

$$|\Phi(f(A))\Phi(A)\Phi(g(A))| \leq U^* \Phi(f(A)Ag(A))U$$  \hspace{1cm} (16)

holds provided that $tf(t)g(t)$ is operator convex, and

$$|\Phi(f(A))^{-1}A\Phi(g(A))^{-1}| \geq V^* \Phi(f(A)^{-1}A^{-1}g(A)^{-1})V$$  \hspace{1cm} (17)

holds provided that $t/f(t)g(t)$ is operator concave.

**Proof:** The proof is similar to that of Theorem 2.2 and so we omit its details. Just note that the Kadison inequality and the operator concavity of $f^2$ and $g^2$ imply that

$$\Phi(f(A))^2 \leq \Phi(f(A)^2) \leq f^2(\Phi(A)) \quad \text{and} \quad \Phi(g(A))^2 \leq \Phi(g(A)^2) \leq g^2(\Phi(A))$$

and

$$\Phi(f(A))^{-2} \geq \Phi(f^2(A))^{-1} \geq f(\Phi(A))^{-2} \quad \text{and} \quad \Phi(g(A))^{-2} \geq \Phi(g^2(A))^{-1} \geq g(\Phi(A))^{-2}.$$  \hfill \blacksquare
If \( f^2(t) \) is operator concave, then \( f(t) \) is also operator concave. Lemma 2.1 concludes that the functions \( h_1(t) = tf(t) \) and \( h_2(t) = \frac{t}{f(t)} \) are operator convex and operator concave, respectively. Therefore, if \( g(t) = 1 \), then the conditions of Theorem 2.5 are fulfilled automatically. Thus we arrive at the following asymmetric Choi–Davis inequality.

**Corollary 2.6:** Assume that \( f : (0, \infty) \to (0, \infty) \) is a function. If \( f^2 \) is operator concave, then

\[
|\Phi(f(A))\Phi(A)| \leq \Phi(Af(A)) \tag{18}
\]

and

\[
|\Phi(f(A))^{-1}\Phi(A)| \geq \Phi(Af(A)^{-1}) \tag{19}
\]

for every unital positive linear map \( \Phi \) and positive invertible operator \( A \). In addition, if \( f(t) \geq 1 \), then

\[
\Phi(Af(A)^{-1}) \leq |\Phi(f(A))^{-1}\Phi(A)| \leq |\Phi(f(A))\Phi(A)| \leq \Phi(Af(A)).
\]

Let \( f(t) = t^\gamma \), where \( 0 \leq \gamma \leq 1/2 \). Then \( f^2 \) is operator concave. Hence, as a consequence of Corollary 2.6, we obtain inequality (3):

**Corollary 2.7:** Inequalities

\[
|\Phi(A^\gamma)\Phi(A)| \leq \Phi(A^{\gamma+1})
\]

and

\[
|\Phi(A^\gamma)^{-1}\Phi(A)| \geq \Phi(A^{1-\gamma})
\]

hold for each unital positive linear map \( \Phi \), each positive invertible operator \( A \), and each \( 0 \leq \gamma \leq 1/2 \).

Our next result reads as follows.

**Proposition 2.8:** Let \( \Phi \) be a unital positive linear map, let \( f : (0, \infty) \to (0, \infty) \) be a function and let \( 0 \leq r \leq 1/2 \). If \( f^2 \) is operator concave, then

\[
\left| \Phi \left( f(A)^{-1} \right)^{-r} \Phi(A)^r \right| \leq \Phi(Af(A))^r
\]

for every positive invertible operator \( A \).

**Proof:** Note that the functions \( t \mapsto t^r \) and \( t \mapsto t^{-2r} \) are operator concave and operator convex, respectively. Therefore, the Choi–Davis inequality implies that

\[
\Phi \left( f(A)^{-1} \right)^{-2r} \leq \Phi \left( f(A)^{2r} \right) \leq \Phi \left( f(A)^{2} \right)^r \leq f(\Phi(A))^{2r},
\]

where the last inequality follows from the operator concavity of \( f^2 \). Hence

\[
\left| \Phi \left( f(A)^{-1} \right)^{-r} \Phi(A)^r \right| = \left( \Phi(A)^r \Phi \left( f(A)^{-1} \right)^{-2r} \Phi(A)^r \right)^{1/2}
\]
\[
\leq \left\{ \Phi(A) f(\Phi(A)) \right\}^{2r} \Phi(A)^r \leq \Phi(A)^r f(\Phi(A)) = \Phi(A)^r = h(\Phi(A))^r,
\]
where \( h(t) = tf(t) \) is operator convex by Lemma 2.1. The Choi–Davis inequality and the operator monotonicity of \( t \mapsto r(t) \) give
\[
h(\Phi(A))^r \leq \Phi(h(A))^r = \Phi(Af(A))^r.
\]

### 3. Counterparts to the Choi–Davis inequality

In this section, we present counterparts to some Choi–Davis inequalities. In particular, we give a converse to (18) in the next theorem. First, we recall a result from [14].

**Lemma 3.1** ([14, Corollary 2.4]): Let \( \Phi \) be a unital positive linear map, let \( f : [m, M] \to (0, \infty) \) be a function with \( 0 < m < M \), and let \( A \) be a positive operator with \( m \leq A \leq M \). If \( f \) is strictly concave, then
\[
\Phi(f(A)) \geq K_1(m, M, f)(\Phi(A)),
\]
where
\[
K_1(m, M, f) = \min_{t \in [m, M]} \left\{ \frac{(M-t)f(m) + (t-m)f(M)}{(M-m)f(t)} \right\}.
\]
If \( f \) is strictly convex, then
\[
\Phi(f(A)) \leq K_2(m, M, f)(\Phi(A)),
\]
where
\[
K_2(m, M, f) = \max_{t \in [m, M]} \left\{ \frac{(M-t)f(m) + (t-m)f(M)}{(M-m)f(t)} \right\}.
\]

The special case when \( f \) is the power function reads as follows. Recall that the generalized Kantorovich constant \( \kappa(h, p) \) is defined by
\[
\kappa(h, p) := \frac{h^p - 1}{(p-1)(h-1)} \left( \frac{p-1}{p} \frac{h^p - 1}{h^p - h} \right)^p.
\]

**Lemma 3.2** ([13, Lemma 4.3]): Let \( \Phi \) be a unital positive linear map and let \( A \) be a positive invertible operator with \( 0 < m \leq A \leq M \). Then,

(i) if \( p > 1 \), then \( \Phi(A^p) \leq K(m, M, p) \Phi(A)^p \);

(ii) if \( 0 < p < 1 \), then \( K(m, M, p) \Phi(A)^p \leq \Phi(A^p) \), where
\[
K(m, M, p) = \kappa(M/m, p) = \frac{mM^p - Mm^p}{(p-1)(M-m)} \left( \frac{p-1}{p} \frac{M^p - m^p}{M^p - Mm^p} \right)^p.
\]
The next theorem presents a reverse of inequality (18).

**Theorem 3.3:** Let $\Phi$ be a unital positive linear map, let $0 < m < M$, and let $f : [m, M] \to (0, \infty)$ be a function such that $f^2$ is strictly concave. If $A$ is a positive operator with $m \leq A \leq M$, then

$$\Phi(A^f(A)) \leq K|\Phi(f(A))\Phi(A)|,$$

where

$$K = \kappa(f(M)/f(m), 2)^{1/2}K_1(m, M, f^2)^{-1/2}K_2(m, M, tf(t)).$$

**Proof:** Assume that $f^2$ is a concave function. Using Lemma 3.1, we obtain $\Phi(f(A)^2) \geq K_1(m, M, f^2)f^2(\Phi(A))$. In addition, Lemma 3.2 gives

$$\Phi(f(A)^2) \leq \kappa(f(M)/f(m), 2)\Phi(f(A))^2.$$

Hence

$$f^2(\Phi(A)) \leq \kappa(f(M)/f(m), 2)K_1(m, M, f^2)^{-1}\Phi(f(A))^2. \quad (20)$$

Therefore

$$\Phi(A)f(\Phi(A)) = \left\{ \Phi(A)f(\Phi(A))^2\Phi(A) \right\}^{1/2}$$

$$\leq \kappa(f(M)/f(m), 2)^{1/2}K_1(m, M, f^2)^{-1/2}\left\{ \Phi(A)f(\Phi(A))^2\Phi(A) \right\}^{1/2}$$

$$= \kappa(f(M)/f(m), 2)^{1/2}K_1(m, M, f^2)^{-1/2}\Phi(f(A))\Phi(A). \quad (21)$$

From the hypothesis, we conclude that $f$ is concave and so, by Lemma 2.1, the function $tf(t)$ is convex. An application of Lemma 3.1 yields that

$$\Phi(A)f(\Phi(A)) \geq K_2(m, M, tf(t))^{-1}\Phi(A^f(A)). \quad (22)$$

Then (21) and (22) give the desired result. $\blacksquare$

In the special case when $f$ is the power function, Theorem 3.3 turns to the following corollary. It provides a counterpart to the asymmetric Kadison inequality [8, Theorem 1.1].

**Corollary 3.4:** Let $\Phi$ be a unital positive linear map and let $\gamma \in [0, 1]$. If $A$ is a positive operator with $0 < m \leq A \leq M$, then

$$\Phi\left(A^{1+\gamma}\right) \leq \kappa(h, 1 + \gamma)\kappa(h^\gamma, 2)^{1/2}\kappa(h^2, \gamma)^{1/2}\Phi(A^\gamma)\Phi(A).$$

In particular, for every $0 \leq \alpha \leq \beta$, it follows that

$$\Phi(A^{\alpha+\beta}) \leq \kappa(h^\beta, 1 + \alpha/\beta)\kappa(h^\beta, 2\alpha/\beta)^{-1/2}\kappa(h^\alpha, 2)^{1/2}\Phi(A^\alpha)\Phi(A^\beta). \quad (23)$$

A version of (23) including three parameters, gives a counterpart to [8, Proposition 1.3].
Theorem 3.5: Let $\Phi$ be a unital positive linear map. If $\alpha, \beta, \gamma \geq 0$ with $\min\{\alpha, \beta\} \leq \frac{\gamma}{2}$ and $\max\{\alpha, \beta\} \leq \gamma$, then, for every positive invertible operator $A$ with $0 < m \leq A \leq M$, there exists a partial isometry $U$ such that

$$\Phi(A^{\alpha+\beta+\gamma}) \leq K \, U \left| \frac{\Phi(A)^{\alpha}}{\Phi} \Phi(A)^{\beta} \Phi(A)^{\gamma} \right| U^*, \quad (24)$$

where

$$K = \kappa(h^\alpha, 2)^{\frac{1}{2}} \kappa(h^\beta, 2)^{\frac{1}{2}} \kappa(h^\gamma, 2 \beta/\gamma)^{-\frac{1}{2}} \kappa(h^{2\gamma}, \alpha/\gamma)^{-\frac{1}{2}} \kappa(h^\gamma, 1 + (\alpha + \beta)/\gamma).$$

Proof: Without loss of generality, we may assume that $\beta \leq \alpha$. First, we assume that $\gamma = 1$. We then have from our hypotheses that $\beta \leq 1/2$ and $\beta \leq \alpha \leq 1$. By virtue of $2\beta \leq 1$ and $m \leq A \leq M$, Lemma 3.2 ensures that $\Phi(A)^{2\beta} \leq \kappa(h, 2\beta)^{-1} \Phi(A^{2\beta})$. Using Lemma 3.2 once more, we obtain $\Phi(A^{2\beta}) \leq \kappa(h^2, 2) \Phi(A^{\beta})^2$. Therefore, we get

$$\Phi(A)^{2\beta} \leq \kappa(h, 2\beta)^{-1} \kappa(h^2, 2) \Phi(A^{\beta})^2. \quad (25)$$

Utilizing the operator monotonicity of $t \mapsto t^{\frac{1}{2}}$ and (25), we can write

$$\left| \frac{\Phi(A)^{1+\beta} \Phi(A^{\alpha})}{\Phi} \right| = \left\{ \frac{\Phi(A)^{\alpha}}{\Phi} \Phi(A)^{2+2\beta} \Phi(A^{\alpha}) \right\}^{\frac{1}{2}}$$

$$\leq \kappa(h, 2\beta)^{-\frac{1}{2}} \kappa(h^2, 2)^{\frac{1}{2}} \left\{ \frac{\Phi(A)^{\alpha}}{\Phi} \Phi(A) \Phi(A^{\beta})^2 \Phi(A) \Phi(A^{\alpha}) \right\}^{\frac{1}{2}}$$

$$\leq \kappa(h, 2\beta)^{-\frac{1}{2}} \kappa(h^2, 2)^{\frac{1}{2}} \left| \Phi(A^{\beta}) \Phi(A) \Phi(A^{\alpha}) \right|. \quad (26)$$

From $\alpha \leq 1$ and Lemma 3.2, we conclude that $\Phi(A^{\alpha})^2 \geq \kappa(h^\alpha, 2)^{-1} \Phi(A^{2\alpha})$ and $\Phi(A^{2\alpha}) \geq \kappa(h^2, \alpha) \Phi(A^{2\alpha})$. Therefore,

$$\left\{ \frac{\Phi(A)^{1+\beta} \Phi(A^{\alpha})}{\Phi} \Phi(A)^{1+\beta} \right\}^{\frac{1}{2}} \geq \kappa(h^\alpha, 2)^{-\frac{1}{2}} \left\{ \Phi(A)^{1+\beta} \Phi(A^{2\alpha}) \Phi(A)^{1+\beta} \right\}^{\frac{1}{2}}$$

$$\geq \kappa(h^\alpha, 2)^{-\frac{1}{2}} \kappa(h^2, \alpha)^{\frac{1}{2}} \left\{ \Phi(A)^{1+\beta} \Phi(A^{2\alpha}) \Phi(A)^{1+\beta} \right\}^{\frac{1}{2}}$$

$$\geq \kappa(h^\alpha, 2)^{-\frac{1}{2}} \kappa(h^2, \alpha)^{\frac{1}{2}} \left\{ \Phi(A)^{1+\beta} \Phi(A^{2\alpha}) \Phi(A)^{1+\beta} \right\}^{\frac{1}{2}}$$

$$= \kappa(h^\alpha, 2)^{-\frac{1}{2}} \kappa(h^2, \alpha)^{\frac{1}{2}} \Phi(A)^{1+\alpha+\beta}. \quad (27)$$

The last inequality follows from the Kadison inequality and the operator monotonicity of $t \mapsto t^{\alpha}$. This implies that there exists a partial isometry $U$ such that

$$\left| \frac{\Phi(A)^{1+\beta} \Phi(A^{\alpha})}{\Phi} \right| = U \left| \frac{\Phi(A)^{\alpha}}{\Phi} \Phi(A)^{1+\beta} \right| U^* \geq \kappa(h^\alpha, 2)^{-\frac{1}{2}} \kappa(h^2, \alpha)^{\frac{1}{2}} U \Phi(A)^{1+\alpha+\beta} U^*. \quad (27)$$

Moreover, since $1 + \alpha + \beta > 1$, Lemma 3.2 gives $\Phi(A)^{1+\alpha+\beta} \geq \kappa(h, 1 + \alpha + \beta)^{-1} \Phi(A^{1+\alpha+\beta})$. Therefore, from (27), we infer that

$$\left| \frac{\Phi(A)^{1+\beta} \Phi(A^{\alpha})}{\Phi} \right| \geq \kappa(h^\alpha, 2)^{-\frac{1}{2}} \kappa(h^2, \alpha)^{\frac{1}{2}} \kappa(h, 1 + \alpha + \beta)^{-1} U \Phi(A^{1+\alpha+\beta}) U^*. \quad (28)$$

Combining (28) with (26), we deduce that

$$\Phi(A^{1+\alpha+\beta}) \leq K' \, U^* \left| \frac{\Phi(A)^{\beta}}{\Phi} \Phi(A) \Phi(A^{\alpha}) \right| U, \quad (29)$$
where

\[ K' = \kappa(h^\alpha, 2)^{1/2} \kappa(h^\beta, 2)^{1/2} \kappa(h, 2\beta)^{-1/2} \kappa(h^2, \alpha)^{-1/2} \kappa(h, 1 + \alpha + \beta). \]

This proves the desired inequality (24) in the case when \( \gamma = 1 \). If \( \gamma \neq 1 \), then replace \( \alpha \) and \( \beta \) by \( \alpha/\gamma \) and \( \beta/\gamma \), respectively, and put \( A' \gamma \) instead of \( A \) in (29) to get the result. ■

Next, we use a refinement of the Furuta inequality (see [15, 16]) to give a sharper inequality than (4). To this end, we need some lemmas.

**Lemma 3.6**: If \( \Phi \) is a unital positive linear map, \( A \) is a positive operator, and \( 1/2 \leq r < 1 \), then

\[ \Phi(A)^r - \Phi(A') \geq \omega(A, r), \quad (30) \]

in which

\[ \omega(A, r) = \|\Phi(A)\|^r - \left( \|\Phi(A)\| - \inf_{n \geq 1} \left\| \left( \Phi(A) + \frac{1}{n} - \Phi(A')^{1/2} \right)^{-1} \right\|^{-1} \right)^r. \quad (31) \]

**Proof**: Not that, for \( 1/2 \leq r < 1 \), the Choi–Davis inequality ensures that \( \Phi(A) \geq \Phi(A')^{1/2} \). Hence \( \Phi(A) + \frac{1}{n} > \Phi(A')^{1/2} \) for all positive integers \( n \). We use an extension of the Löwner–Heinz inequality presented in [17]: If \( A > B \geq 0 \) and \( r \in [0, 1] \), then

\[ A^r - B^r \geq \|A\|^r - \left( \|A\| - \|A - B\|^{-1} \right)^r. \quad (32) \]

Utilizing (32) with \( \Phi(A) + \frac{1}{n} \) and \( \Phi(A')^{1/2} \) instead of \( A \) and \( B \), respectively, we obtain

\[
\left( \Phi(A) + \frac{1}{n} \right)^r - \Phi(A') \geq \left\| \Phi(A) + \frac{1}{n} \right\|^r \\
- \left( \left\| \Phi(A) + \frac{1}{n} \right\| - \left\| \left( \Phi(A) + \frac{1}{n} - \Phi(A')^{1/2} \right)^{-1} \right\|^{-1} \right)^r.
\]

Taking the limits as \( n \to \infty \), we get

\[
\Phi(A)^r - \Phi(A') \geq \|\Phi(A)\|^r \\
- \left( \|\Phi(A)\| - \inf_{n \geq 1} \left\| \left( \Phi(A) + \frac{1}{n} - \Phi(A')^{1/2} \right)^{-1} \right\|^{-1} \right)^r.
\]

Note that the sequence \( \{ (\Phi(A) + \frac{1}{n} - \Phi(A')^{1/2})^{-1} \} \) is increasing, and hence the sequence \( \{ \|\Phi(A) + \frac{1}{n} - \Phi(A')^{1/2}\|^{-1} \} \) is decreasing. ■
To clarify Lemma 3.6, we give an example. Assume that the unital positive linear map \( \Phi : \mathbb{M}_2 \to \mathbb{M}_2 \) is defined by \( \Phi(A) = 1/2 \text{Tr}(A)I_2 \) and put
\[
A = \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}.
\]
Then, for \( r = 1/2 \), the infimum in (31) is approximately equal to 0.18 and \( \omega(A, r) \approx \sqrt{3} - \sqrt{3} - 0.18 \).

**Lemma 3.7 ([15]):** Let \( A \) and \( B \) be positive invertible operators such that \( A - B \geq m > 0 \). Then
\[
A^{p+r} - (A^\frac{r}{q} B^p A^\frac{r}{q})^{\frac{1}{q}} \geq \|A\|^p (A^{1+r} - m A^{-1} - r m A^{-r})^\frac{p}{q(1+r)}
\]
holds for every \( p, r \geq 0 \) and \( q \geq 1 \) with \( (1+r)q \geq p+r \).

Our next result provides a refinement of the asymmetric Kadison inequality.

**Theorem 3.8:** Let \( \Phi \) be a unital positive linear map. If \( X \) is a positive invertible operator, then
\[
\Phi(X^\alpha)^{1+\frac{\beta}{\alpha}} \geq \left| \Phi(X^\beta) \Phi(X^\alpha) \right| + \left\| \Phi(X^\alpha)^{1+\frac{\beta}{\alpha}} \right\|
- \left\| \Phi(X^\alpha)^{2+\frac{\beta}{\alpha}} - \omega \left( X^\alpha, \frac{\beta}{\alpha} \right) \right\| \Phi(X^{-\frac{\beta}{\alpha}}) \right\|^{\frac{\beta}{\beta+2\alpha}}
\]
for all \( \alpha, \beta \geq 0 \) with \( \beta < \alpha \leq 2\beta \).

**Proof:** Assume that \( \beta < \alpha \leq 2\beta \) so that \( \frac{\beta}{\alpha} \in [1/2, 1] \). Lemma 3.6 then shows that the inequality
\[
\Phi(X)^{\frac{\beta}{\alpha}} - \Phi(X^{\frac{\beta}{\alpha}}) \geq \omega \left( X, \frac{\beta}{\alpha} \right)
\]
is valid for every positive invertible operator \( X \). Substituting \( X \) by \( X^\alpha \), we reach
\[
\Phi(X^\alpha)^{\frac{\beta}{\alpha}} - \Phi(X^{\frac{\beta}{\alpha}}) \geq \omega \left( X^\alpha, \frac{\beta}{\alpha} \right).
\]
Now assume that \( p = q = 2 \) and \( r = 2\frac{\alpha}{\beta} \) so that \( (1+r)q \geq p+r \). If \( A = \Phi(X^\alpha)^{\frac{\beta}{\alpha}} \) and \( B = \Phi(X^{\frac{\beta}{\alpha}}) \), the hypotheses of Lemma 3.7 are satisfied for \( m = \omega(X^\alpha, \frac{\beta}{\alpha}) \). Since
\[
A^{\frac{p+r}{q}} = \left( \Phi(X^\alpha)^{\frac{\beta}{\alpha}} \right)^{1+\frac{q}{\beta}} = \Phi(X^\alpha)^{1+\frac{\beta}{\alpha}}
\]
and
\[
\left( A^\frac{r}{q} B^p A^\frac{r}{q} \right)^{\frac{1}{q}} = \left( \Phi(X^\alpha)^{\frac{\beta}{\alpha}} \Phi(X^{\frac{\beta}{\alpha}})^2 \left( \Phi(X^\alpha)^{\frac{\beta}{\alpha}} \right)^{\frac{\beta}{\alpha}} \right)^{\frac{1}{q}} = \left| \Phi(X^{\beta}) \Phi(X^{\alpha}) \right|,
\]
applying Lemma 3.7, we get the desired result (33). \( \blacksquare \)
As a consequence, let \( \gamma = \frac{\beta}{\alpha} \). Employing \( X^{\frac{1}{2}} \) instead of \( X \), we get the following result.

**Corollary 3.9:** Let \( \Phi \) be a unital positive linear map. For every positive invertible operator \( X \) and every \( \gamma \in [1/2, 1] \), it holds that

\[
\Phi(X)^{1+\gamma} \geq \left| \Phi(X^\gamma) \Phi(X) \right| + \left\| \Phi(X)^{1+\gamma} \right\| - \left\| \Phi(X)^{2+\gamma} - \omega(X, \gamma) \Phi(X)^{-\gamma} \right\| \frac{2}{2+\gamma}.
\]

It is noted in [17] that inequality (32) is sharp in the sense that when \( A \) and \( B \) are positive scalars of the identity operator, and then (32) becomes equality. Following the proof of Lemma 3.6, we realize that if \( \Phi(A) = aI \) and \( \Phi(A^{1/2}) = bI \) with \( a > 0 \) and \( b > 0 \), then (30) turns into equality. Therefore, inequality (30) is sharp.

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