Fully Faithful Functors and Dimension

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Abstract

We define the countable Rouquier dimension of a triangulated category and use this notion together with Theorem 2 of [Ola21] to prove that if there is a fully faithful embedding $D^b_{coh}(X) \subset D^b_{coh}(Y)$ with $X,Y$ smooth proper varieties, then $\dim(X) \leq \dim(Y)$.

We will show how Theorem 2 follows from [Ola21, Theorem 2]. Theorem 2 was expected to be true by Conjecture 10 of [Orl09], but to the best of the author’s knowledge, it was unknown until now. The author is very grateful to Dmitri Pirozhkov for suggesting this proof almost immediately after the paper [Ola21] was posted. The author also thanks Dmitri Orlov for a helpful email. The key is the following definition due to Pirozhkov:

Definition 1. Let $\mathcal{T}$ be a triangulated category. The countable Rouquier dimension of $\mathcal{T}$, denoted $\text{CRdim}(\mathcal{T})$, is the smallest $n$ such that there exists a countable set $\{E_i\}_{i \in I}$ of objects of $\mathcal{T}$ such that $\mathcal{T} = \langle \{E_i\}_{i \in I} \rangle_{n+1}$.

We allow the countable Rouquier dimension to be infinity. We have an easy lemma:

Lemma 1. Let $\mathcal{A} \subset \mathcal{T}$ be an admissible subcategory in a triangulated category. Then $\text{CRdim}(\mathcal{A}) \leq \text{CRdim}(\mathcal{T})$.

Proof. Let $R$ be the right adjoint of the inclusion. If $\mathcal{T} = \langle \{E_i\}_{i \in I} \rangle_{n+1}$ then since $R$ is essentially surjective, $\mathcal{A} = \langle \{R(E_i)\}_{i \in I} \rangle_{n+1}$. □

The remark following the proof of [Ola21] directly implies:

Theorem 1. Let $X$ be a Noetherian regular scheme with affine diagonal. Then $\text{CRdim}(D^b_{coh}(X)) \leq \dim(X)$.

In fact Theorem 1 is true without the assumption on the diagonal of $X$ but we don’t need it here. The reverse inequality is not true in general. For example, if $X$ is a variety over a countable field $k$, then $D^b_{coh}(X)$ has only countably many objects up to isomorphism, hence in fact $\text{CRdim}(D^b_{coh}(X)) = 0$. However for varieties over $\mathbb{C}$ we do get the reverse inequality:

Proposition 1. Let $k$ be an uncountable field. Let $X$ be a reduced scheme of finite type over $k$. Then $\text{CRdim}(D^b_{coh}(X)) \geq \dim(X)$.

Proof. Compare to the proof of [Rou08, Proposition 7.17]. Let $n = \text{CRdim}(D^b_{coh}(X))$ and let $\{E_i\}_{i \in I}$ be a countable family of objects such that $D^b_{coh}(X) = \langle \{E_i\}_{i \in I} \rangle_{n+1}$. Consider the set of closed points $x \in X$ such that for every $i \in I$, the cohomology modules of $(E_i)_x$ are free $\mathcal{O}_{X,x}$-modules. Since a variety over $k$ is not a countable union of proper closed subsets ([Liu02, Exercise 2.5.10]), the set contains a closed point $x$ such that $\dim(\mathcal{O}_{X,x}) = \dim(X)$. We have $(E_i)_x \in \langle \mathcal{O}_{X,x} \rangle_1$ for each $i$ since a complex with projective cohomology modules is decomposable, hence

$$\kappa(x) \in \langle \{(E_i)_x\}_{i \in I} \rangle_{n+1} \subset \langle \mathcal{O}_{X,x} \rangle_{n+1},$$

hence $n \geq \dim(X)$ by [Rou08, Proposition 7.14]. □

Remark. Since the countable Rouquier dimension only gives the expected answer for varieties over a sufficiently large field, Orlov suggests an alternative notion of countable Rouquier dimension which makes sense for a $k$-linear dg-category $\mathcal{A}$ with $k$ a field: Take the smallest $n$ such that there exists a countable family $\{E_i\}_i$ of objects of $\mathcal{A}$ such that $\mathcal{A}_K = \langle \{(E_i)_K\}_i \rangle_{n+1}$ for every field extension $K/k$.

Finally, we prove our main result:

Theorem 2. Let $k$ be a field. Let $X,Y$ be smooth proper varieties over $k$. Assume there exists a fully faithful, exact, $k$-linear functor $F : D^b_{coh}(X) \rightarrow D^b_{coh}(Y)$. Then $\dim(X) \leq \dim(Y)$. 

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Proof. Choose any uncountable extension field $K/k$. By [Ola20, Theorem 1], $F$ is the Fourier–Mukai transform with respect to a kernel $E \in D_{coh}^b(X \times_k Y)$. Then $E_K$ gives rise to a functor $F_K : D_{coh}^b(X_K) \to D_{coh}^b(Y_K)$ which remains fully faithful: The fact that $F$ is fully faithful may be rephrased as $R \circ F \cong \text{id}$ where $R$ is the right adjoint of $F$. By the calculus of kernels, this may be rephrased as the existence of an isomorphism

$$Rpr_{13*}(Lpr_{12}^*(E) \otimes_{\mathcal{O}_{X \times V \times X}} Lpr_{23}^*(E')) \cong \mathcal{O}_\Delta$$

(1)

where $E' = \mathcal{R}Hom_{O_{X \times Y}}(E, Lpr_1^*(\omega_X)[\dim(X)])$, viewed as an object of $D_{coh}^b(Y \times_k X)$, is the kernel of $R$, and (1) remains valid upon base change to $K$.

Now $F_K$ is the inclusion of an admissible subcategory since $X, Y$ are smooth and proper ([Sta18, Tag 0FYN]). We have $\text{CRdim}(D_{coh}^b(X_K)) = \dim(X_K) = \dim(X)$ by Theorem 1 and Proposition 1 and similarly for $Y$. Thus by Lemma 1, we have

$$\dim(X) = \text{CRdim}(D_{coh}^b(X_K)) \leq \text{CRdim}(D_{coh}^b(Y_K)) = \dim(Y),$$

as needed. \hfill \Box

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