BOUNDARY LAYERS AND THE VANISHING VISCOSITY LIMIT FOR INCOMPRESSIBLE 2D FLOW

MILTON C. LOPES FILHO

Abstract. This manuscript is a survey on results related to boundary layers and the vanishing viscosity limit for incompressible flow. It is the lecture notes for a 10 hour minicourse given at the Morningside Center, Academia Sinica, Beijing, PRC from 11/28 to 12/07, 2007. The main topics covered are: a derivation of Prandtl’s boundary layer equation; an outline of the rigorous theory of Prandtl’s equation, without proofs; Kato’s criterion for the vanishing viscosity limit; the vanishing viscosity limit with Navier friction condition; rigorous boundary layer theory for the Navier friction condition and boundary layers for flows in a rotating cylinder.

1. Introduction

In 1904, the issue of heavier-than-air, self propelled flight by human-made machines was at the very edge of both science and technology. The first such flight, by the Wright brothers, occurred at December 14th, 1903. A Brazilian author is honor bound to remark that a more satisfying, and better publicized “first flight” was achieved by Santos Dumont, a Brazilian living in France, in September, 1906. The flight of a fixed-wing airplane could, at least in principle, be described by near-steady, zero-viscosity, irrotational theory of airfoils, which was already available at the beginning of the twentieth century.

Classical airfoil theory explained satisfactorily the balance of forces in a wing in steady flight. In short, the force that air exerts on the wing is divided into two standard components: the lift (vertical force) and the drag (horizontal force), where horizontal means the direction of steady motion. In steady flight, these forces are balanced by the weight and the propulsion force. The theory predicts that the lift and the drag are proportional to the circulation of air velocity around the airfoil, and it was in agreement with experiments, see [1] for details. Trouble occurs when one wants to change the lift, as one should do when attempting to take off or land in a fixed wing aircraft. A theorem, due to Lord Kelvin, states that the circulation around a material curve, such as the boundary of an airfoil, is a constant of motion in ideal (i.e. non-viscous) flow - or maybe, nearly constant in slightly viscous flow. So, changing propeller speed and moving control surfaces does not change the circulation. Since airplanes start out at rest, with zero circulation around the wings, no airplane could, on theoretical ground, develop a lift, and therefore fly. Something was clearly wrong with the theory.

The correction was due to the young theoretical mechanician Ludwig Prandtl (1875 - 1953), who published a short paper in the Proceedings of the Third ICM (Heidelberg 1904) whose German title roughly translates as “fluid flow in very...
little friction”. In this article, Prandtl established a perfectly satisfactory, and revolutionary, explanation of the following observation:

(O) The interaction of incompressible flow with a material boundary is completely different if the flow has very small viscosity or none at all.

This observation, the associated explanation, called boundary layer theory and some of what mathematicians made of this subject in the following century and a bit, make up the subject of these lectures. For a thorough account of the development and understanding of the physics of boundary layers, we would like to refer the reader to the classical book [29]. It can be argued that this short paper by Prandtl marks the birth of modern applied mathematics.

The theory of boundary layers is a cornerstone of modern fluid mechanics, but, as in much of this field, it lacks a rational framework, i.e. a rigorously established connection with first principles. Although substantial mathematical work has been done in this direction, some basic questions remains unanswered. The purpose of these lectures is to probe the boundaries in the mathematical understanding of the interaction between nearly ideal flow and solid objects, perhaps to bring what is not known about this question more sharply into focus. The choice of material covered is strongly slanted towards recent work by the author and his collaborators, and it includes detailed consideration of Kato’s criterion for the vanishing viscosity limit in a bounded domain, a long discussion on the vanishing viscosity limit for incompressible flows with Navier boundary condition and the detailed behavior of circularly symmetric flow inside a rotating cylinder. The choice of working with two dimensional flow is both a reasonable pedagogical choice and a comfort zone for the author - in the issue of boundary layers, the sharp distinction in behavior between 2D and 3D flows is not yet apparent, and much of the work we will discuss here generalizes readily to 3D. Finally, we mention that these notes are written thinking of a reasonably mature audience - we assume, not only familiarity with standard PDE theory, but some familiarity with the basics of mathematical fluid dynamics as well.

The remainder of these notes is divided in seven sections as follows: Section 2 contains a derivation of Prandtl’s equation; Section 3 contains a broad overview of rigorous results on Prandtl’s equation, including some of O. Oleinik’s work, and more recent progress; Section 4 introduces and proves Kato’s criterion and some related results; Section 5 is concerned with vanishing viscosity under Navier friction conditions and a proof of $L^p$ vorticity estimates in this case; Section 6 contains an exposition on a rigorous method to treat boundary layer expansions based on ideas of geometric optics, applied to the Navier boundary condition; Section 7 explores a nearly explicit example of the behavior of the boundary layer for the no-slip condition; Section 8 contains some conclusions and open problems.

2. Prandtl’s theory

In this Section, we present an asymptotic derivation of Prandtl’s boundary layer equation. Our point of departure is the Navier-Stokes equations, which are an expression of Newton’s second law applied to the motion of a fluid, subject to an incompressibility constraint. We write

\[
\begin{align*}
\partial_t u + u \cdot \nabla u &= -\nabla p + \mu \Delta u \\
\text{div } u &= 0,
\end{align*}
\]

(1)
where \( u \) is the fluid velocity, \( p \) is the scalar pressure and \( \mu \) is the kinematic viscosity of the fluid. We have assumed that mass units have been chosen so that fluid density is one.

For this derivation, and for most of the discussion in these lectures, we will assume that we are discussing a two-dimensional fluid occupying a half plane \( H \equiv \{ x_2 > 0 \} \). Two-dimensional fluids really occur either in a computer simulation, or as three-dimensional fluids which are translation-invariant along some direction (often an unstable state of affairs) or as an approximate model for thin fluid layers. Much of the discussion extends very naturally to a fairly general fluid domain in three dimensions, but we will stay with the simplest possible situation for pedagogical reasons.

As we all know, problem (1) requires a boundary condition at \( \{ x_2 = 0 \} \), and the condition usually deemed appropriate is the no slip condition \( u(x_1, 0, t) = 0 \). This condition expresses an assumption that viscous fluids adhere to material objects, something that is neither physically nor mathematically obvious and was subject for heated debate until the mid nineteenth century, when it became clear that it gave good agreement with experiments.

Ideal, or inviscid flow is represented by solutions of Euler’s equations, which is system (1) with \( \mu = 0 \). For ideal flow, the correct boundary condition is the non-penetration condition \( u_2(x_1, 0, t) = 0 \). The Navier-Stokes system is a singular perturbation of the Euler system, because the small constant \( \mu \) appears in front of the highest order term of (1). One consequence of this singular perturbation is the disparity in boundary conditions between Euler and Navier-Stokes flows - namely that \( u_1 \) at the boundary goes from being identically zero for any positive viscosity to some (in principle) nonzero function when \( \mu = 0 \). This disparity is the root cause of the boundary layer trouble.

Our objective here is to derive Prandtl’s boundary layer equations. This is a way to quantify nearly ideal fluid behavior near the boundary by means of an appropriate set of limit equations. Let us begin with a simplifying assumption: we assume that the disparity between ideal and viscous flow is concentrated on a thin layer near \( x_2 = 0 \).

Our next step is to non-dimensionalize equation (1), using the time scale \( T \), the length scale \( L \) for horizontal lengths, and a reference vertical length scale \( h \). We introduce the non-dimensional constant \( \nu = \mu T / L^2 \), which is a measurement of the quotient of viscous by inertial forces in our flow, and measures in a physically appropriate manner how far from ideal our flow really is. The non-dimensionalization procedure simply means introducing new variables

\[
\tilde{u}^1(y, s) = \frac{T}{L} u^1(y_1, hy_2, Ts),
\]

\[
\tilde{u}^2(y, s) = \frac{T}{h} u^2(y_1, hy_2, Ts),
\]

and

\[
\tilde{p}(y, s) = \frac{T^2}{L^2} p(y_1, hy_2, Ts).
\]

which results in the system
We introduce \( \varepsilon \equiv h/L \), which is assumed to be a small, non-dimensional, parameter because we are focusing in a thin layer. We also want to consider \( \nu \) small. A key issue in boundary layer theory is that the magnitude of the small parameters \( \varepsilon \) and \( \nu \) are naturally related. Indeed, if \( \nu \ll \varepsilon^2 \) then matched asymptotics indicates that, to leading order, \( \tilde{u} \) satisfies the Euler system (with pressure independent of the vertical variable) and with no-slip conditions at the boundary. These boundary conditions are inconsistent with the fact that the Euler system is of first order. On the other hand, if \( \nu \gg \varepsilon^2 \) then, to leading order, \( \tilde{u} \) is such that \( \partial_{y_2}^2 \tilde{u} = 0 \), which, together with the no-slip boundary condition implies that \( \tilde{u} = c(y_1)y_2 \), for some vector \( c \). Now, if \( \tilde{u} \) is to represent the behavior of the flow in a thin layer near the boundary, then the velocity \( \tilde{u} \) should match the inviscid velocity in the limit \( y_2 \to \infty \), and not just blow-up. The only regime that appears to yield a consistent asymptotic regime is

\[
\nu/\varepsilon^2 = O(1).
\]

From another perspective, condition (3) highlights the region near the boundary where the vertical viscous stress balances the inertial terms in the Navier-Stokes system. Assuming \( \nu = \varepsilon^2 \) and implementing matching asymptotics for \( \nu \) small we obtain the following system for the leading order approximation, denoted \( v = (v^1, v^2) \),

\[
\begin{aligned}
\partial_y v^1 + v \cdot \nabla_y v^1 &= -\partial_{y_1} q + \partial_{y_2}^2 v^1 \\
\partial_{y_2} q &= 0 \\
\text{div}_y v &= 0.
\end{aligned}
\]

These are the unsteady Prandtl equations for the boundary layer profile \( v \). They represent the behavior of the flow near the boundary. To obtain a complete problem, these equations must be supplemented with boundary conditions. First we impose the no-slip condition

\[
v = 0 \text{ at } y_2 = 0.
\]

An additional condition must be imposed in order to capture the assumption that far from the boundary layer the small viscosity Navier-Stokes solutions match the Euler solutions. Let \( u^\nu \) a family of solutions of the non-dimensional Navier-Stokes equations with non-dimensional viscosity \( \nu \) and \( u^E \) be a solution of the incompressible Euler equations. For example, can assume that both the family \( u^\nu \) and \( u^E \) are defined by solving the Navier-Stokes equations and the Euler equations with the same initial data \( u^\nu(x, 0) = u^E(x, 0) = u_0(x) \).

Going back to the Prandtl system, we expect that \( v(y_1, y_2, t) \to u^E(y_1, 0, t) \equiv U(y_1, t) \), as \( y_2 \to \infty \). Let \( p^E \) be the pressure associated with the Euler solution \( u^E \). Since \( q \) does not depend on \( y_2 \), looking at \( y_2 \to \infty \) makes it also natural to assume
that \( q(y_1, t) = p^E(y_1, 0, t) \). If we look at the Euler equations and evaluate them at \( x_2 = 0 \) we obtain the following relation:

\[
U_t + UU_{y_1} = -p^E_{y_1} = -q_{y_1},
\]

which is called Bernoulli’s Law. This means that, for the Prandtl equation (4), the condition at infinity \( U \) determines, up to an irrelevant constant, the pressure \( q \).

With this construction, we hope that, when \( \nu \) is small,

\[
u'(x_1, x_2, t) = \begin{cases} (v_1, \varepsilon v_2)(x_1, x_2/\varepsilon, t) & \text{for } x_2 < \lambda(\nu) \\
u^E(x_1, x_2, t) & \text{for } x_2 > \lambda(\nu) + o(1), \end{cases}
\]

where \( \lambda(\nu) \) is any cutoff distance such that \( \varepsilon \ll \lambda(\nu) \ll 1 \).

In addition to the time-dependent Prandtl equation, this derivation also yields the steady Prandtl equation, given by

\[
\begin{align*}
v \cdot \nabla_y v_1 &= -\partial_{y_1} q + \partial^2_{y_2} v_1 \\
\partial_{y_2} q &= 0 \\
\text{div}_y v &= 0.
\end{align*}
\]

The typical problem associated with this equation is a quarter-plane BVP, where a profile \( v(0, y_2) \) is given and one attempts to find \( v(y_1, y_2) \) for \( y_1 > 0 \) as the induced boundary layer profile over a half-plane plate.

The derivation above is a nice example of multiscale asymptotic analysis, and from the complicated issue surrounding the interaction of nearly inviscid flow with material boundaries it derives a new equation, (4), and a simplified asymptotic model for the behavior of Navier-Stokes solutions near a material boundary, given by (5). The key issue that such a model raises is its validity (mathematical) and applicability (physical).

This model has been found useful in applications, specially where it concerns laminar boundary layers, and when its usefulness begins to break down, suitable extensions of the model have been obtained, notably the so called “triple deck” expansions, where an intermediate thin layer is added between the viscosity-dominated internal layer and the free irrotational Euler flow. In this intermediate layer, the flow is ideal, but not necessarily irrotational. One important situation where the boundary layer ansatz breaks down is when boundary layer separation occurs. Recall that one of the assumptions in deriving the Prandtl equation was that the disparity between ideal and viscous flow be concentrates in a thin layer near the boundary. It is quite common, even well within laminar flow regimes, that the boundary layer detaches itself from the boundary and affects the bulk of the flow. In that case, Prandtl’s theory and its extensions break down as models. In the next section we will consider what is known regarding the rigorous validation of the asymptotic approximation described here.

3. Prandtl’s equation

The purpose of this Section is to present the known theory for Prandtl’s equation, without much detail. The first question one must address regarding an approximate model is: can I solve it? The initial and boundary value problems for the Euler and Navier-Stokes equations are well-posed, globally in the case of the half-plane with reasonable initial data. So the issue of whether Euler + Prandtl is a good approximation for Navier-Stokes with \( \nu \) small, in the sense discussed in the previous section, depends first on understanding the well-posedness for Prandtl’s equation.
The mathematical theory of the Prandtl equation only got started in the sixties, by O. Oleinik. Over the years, Oleinik and her group made a large number of contributions to the theory of Prandtl’s equation and its many variants, collected and explained in the book [27]. For the present discussion, we would like to focus on one specific result, that first appeared in [26]. We also refer the reader to the survey [5], for the discussion of Oleinik’s result and its relation with the blow-up result of W. E and B. Engquist, [6].

Let \( v = (v^1, v^2) \) be a solution of the IBVP for Prandtl’s equation, which we write as

\[
\begin{align*}
\partial_t v^1 + v \cdot \nabla v^1 &= -\partial_{x_1} q + \partial_{x_2}^2 v^1, \\
\text{div} v &= 0, \\
v(x_1, 0, t) &= 0 \text{ and } \lim_{x_2 \to -\infty} v^1(x, t) = U(x_1, t) \\
v(x, 0) &= v_0(x),
\end{align*}
\]

(7)

where \(-\partial_{x_1} q = U_1 + UU_{x_1}\). The result we wish to discuss is the following

**Theorem 1.** (Oleinik 1967) Assume that both \( U \) and \( v_1^1 \) are positive and that, in addition, \( \partial_{x_2} v_1^0 \geq 0 \). Then there exists a unique global strong solution of (7).

We will not present a proof of this result, but we will discuss a key part of the proof, which is the recasting of this problem as a scalar, degenerate parabolic scalar equation, using Crocco’s transformation.

We begin by taking the derivative of Prandtl’s equation with respect to \( x_2 \) and introduce the new dependent variable \( \omega(x, t) \equiv \partial_{x_2} v^1 \). System (7) is equivalent to the following IBVP:

\[
\begin{align*}
\partial_t \omega + v \cdot \nabla \omega &= \partial_{x_2}^2 \omega \\
v &= K[\omega] \\
(\partial_{x_2} \omega)(x_1, 0, t) &= \partial_{x_1} q \text{ and } \lim_{x_2 \to -\infty} \omega(x, t) = 0 \\
\omega(x, 0) &= \partial_{x_2} v_0(x),
\end{align*}
\]

(8)

where the vector operator \( K \) reconstructs \((v^1, v^2)\) by first integrating in the vertical variable to obtain \( v_1 \) and then using the divergence-free condition and integrating again in the vertical variable to obtain \( v_2 \). The new equation (8) is, in a sense, the vorticity formulation of problem (7).

We assume that the solution \( v = v(x, t) \) we seek satisfies the condition \( \partial_{x_2} v^1(x, t) > 0 \) for all \( x \in \mathbb{R}, t > 0 \). In particular, this means that, for each fixed \((x_1, t)\), the map \( x_2 \mapsto v^1(x_1, x_2, t) \) is invertible. Let us denote this inverse by \( h = h(x_1, \xi, t) \). In other words, we have

\[
v^1(x_1, h(x_1, \xi, t), t) = \xi, \text{ for all } \xi > 0.
\]

(9)

The Crocco’s transform consists of introducing the new dependent variable:

\[
W = W(x_1, \xi, t) \equiv \omega(x_1, h(x_1, \xi, t), t),
\]

(10)

We verify that \( W \) is a solution of the following IBVP:

\[
\begin{align*}
\partial_t W + \xi W_{x_1} - (\partial_{x_1} q)W_\xi &= W^2 \partial_\xi^2 W \\
WW_\xi &= q_{x_1} \text{ for } \xi = 0 \\
W(x, 0) &= (\partial_{x_2} v_0)(x_1, h(x_1, \xi, 0)).
\end{align*}
\]

(11)

Indeed, we can compute directly to obtain:
\[ \partial_t W = -\frac{v_1}{\omega} \omega_{x_2} + \omega_t; \quad \partial_{x_1} W = -\frac{v_1}{\omega} \omega_{x_2} + \omega_{x_1}; \quad \partial_\xi W = \frac{\omega_{x_2}}{\omega}; \quad \partial^2_\xi W = \frac{\omega_{x_2}^2}{\omega^2} - \frac{\omega_{x_2}^3}{\omega^3}. \]

Substituting the corresponding equalities above into (11), and using (10), (9) and the evolution equations in (7) and (8) verifies the evolution equation in (11). In addition, we can check directly that

\[ \omega_{x_2}(x_1, x_2, t) = W(x_1, g^{-1}(x_1, x_2, t), t) W_\xi(x_1, g^{-1}(x_1, x_2, t), t), \]

which, together with the boundary condition in (8) gives the boundary condition in (11). Problem (11) is a scalar, degenerate parabolic equation, which is amenable to fairly standard treatment, using fixed point methods, and satisfies a comparison principle. In particular, the sign of \( W \) is retained in the evolution, and therefore, the monotonicity condition on \( v_1 \), necessary for the validity of Crocco's transform, is retained as well.

Finally, once a solution \( W \) is obtained for problem (11), one must reconstruct a solution to the original problem. Recall that

\[ v_1(x, g(x_1, \xi, t), t) = \xi, \]

and therefore, differentiating this identity with respect to \( \xi \) gives

\[ \omega(x_1, g(x_1, \xi, t), t) \frac{\partial g}{\partial \xi}(x_1, \xi, t) = 1. \]

Recalling (10), this implies that

\[ g(x_1, \xi, t) = \int_0^\xi (W(x_1, \eta, t))^{-1} d\eta, \]

which allows the reconstruction of \( \omega \) from \( W \) by means of (10). The interested reader may prove, as an exercise, that such an \( \omega \) is, in fact, a solution of (8).

This result, and others proved by Oleinik and her group, give useful, rigorously established descriptions of the vanishing viscosity asymptotics, but depend, to a greater or lesser extent, on monotonicity conditions such as \( \partial_{x_2} v_1^0 \geq 0 \). As we have seen, the monotonicity assumption is needed for the validity of the Crocco's transformation, but this assumption might just be a feature of the method, rather than an essential limitation of the theory. In 1997, E and Engquist produced a counterexample which showed that Prandtl's equation develops finite-time singularities if the monotonicity condition is not imposed, see [6]. In fact, E and Engquist's example suggests that the role of the monotonicity assumption is to prevent boundary layer separation, a phenomenon that actually occurs in real flows and corresponds to a breakdown of the Prandtl ansatz.

An alternative to the half-plane analysis described above is to study well-posedness of Prandtl's equation in bounded intervals in \( x_1 \), where the horizontal velocity of the boundary layer profile is specified in one side of the interval, and the length of the interval or the time horizon of the analysis are chosen small enough to prevent separation. Such a result was first proved by Oleinik in [24]. Recently, Z. Xin and L. Zhang improved Oleinik's result, showing existence of a global (in time) weak solution for Prandtl's equation on a finite horizontal interval, if the pressure is favorable, i.e., \( q_{x_1} < 0 \), see [34] and [35]. This condition is also known to discourage boundary layer separation.
A different approach to the theory of Prandtl's equation was taken, initially by A. Asano, in a couple of unpublished manuscripts, and later by R. Caffis and M. Sammartino, in a pair of articles, see [28], recently further improved by Lombardo, Cannone and Sammartino in [17]. The basic idea is that, without the monotonicity condition, or something analogous to it, one expects the initial-boundary value problem (7) to be ill-posed. As a result, it becomes natural to look for local (in time) solutions for Prandtl's equation in analytic function spaces, using results of Cauchy-Kowalewska type. The main results in [28] were well-posedness of the problem (7) if the data $v_0$ and $U$ are analytic, and compatible. In [17] the analyticity requirement on $v_0$ was imposed only in the horizontal variables.

Of course, the well-posedness in analytic spaces, and the blow-up example by E and Engquist does not prove that (7) is ill-posed, which at this time remains an interesting open problem.

To conclude this section, it would make sense to mention the contribution of E. Grenier, which he describes as nonlinear instability of the Prandtl boundary layer. His result is not about Prandtl's equation per se, but about the vanishing viscosity limit of the Navier-Stokes equations. His result can be interpreted as mathematical evidence that the Prandtl ansatz is not always valid for solutions of the Navier-Stokes system in the half-plane with small viscosity, see [7]. In other words, although the theory of Prandtl's equation is relevant for understanding the vanishing viscosity limit, there is more to the original observation (O) than Prandtl's original explanation for it.

4. Kato's condition

In this section we move away from Prandtl's equation, and we begin a study of the vanishing viscosity limit from a broader point of view. Our first observation should be that, even in the absence of boundaries, all the mysteries of turbulence lurk in the background of the vanishing viscosity limit, see for example [18] for a small part of this story. However, under moderate regularity assumptions, for example, if the initial vorticity is bounded, explicit estimates for the difference between Euler and small viscosity Navier-Stokes solutions are known, see [2]. Also, and this distinction is a key point here, for very irregular flow, we still have the existence of subsequences of solutions of small viscosity Navier-Stokes converging to weak solutions of the Euler equations, up to and including initial vorticities which are measures, see [23] [22], but then no estimates on the difference are provided, or expected. Basically, in the absence of boundaries, as long as the underlying ideal flow has enough regularity so that uniqueness of weak solutions to the Euler equations is known, we have actual convergence of the vanishing viscosity limit. Furthermore, as long as existence of weak solutions is known we also have compactness of the vanishing viscosity sequence and weak continuity of the Euler/Navier-Stokes non-linearity. Nonuniqueness of weak solutions for Euler equations is also known, see the remarkable paper [3], and references therein, for the current knowledge on this nonuniqueness, but the behavior of the vanishing viscosity limit for these examples is a very interesting open problem.

As soon as we consider flows in the presence of boundaries, the story changes quite dramatically. Very little is actually known mathematically, and this very little is precisely the object under discussion in these notes. Physically, boundaries are the most natural source of small scales in incompressible flows, precisely through the
boundary layer mechanism, and these small scales are the source of the irregularities that justify considering irregular 2D flows in the first place. The point of departure in our discussion will be a classical open problem, which we formulate below.

∂ Layer Problem: Let \( u^\nu \) be a sequence of solutions of the incompressible Navier-Stokes equations in two space dimensions, in a smooth bounded domain \( \Omega \), satisfying the no-slip boundary condition on \( \partial \Omega \), with initial data \( u^\nu_0 \), bounded in \( L^2 \). Is there a subsequence \( u^{\nu_k} \) converging weakly in \( L^2 \) to a vector field \( u \), which is a weak solution of the incompressible Euler equations in \( \Omega \) with some initial data \( u_0 = \lim u^{\nu_k}_0 \)?

This problem is open even if \( \omega^\nu_0 = \omega_0 \in C^\infty_c(\Omega) \), with \( \omega_0 = \text{curl} u_0 \). Let us focus, for simplicity, in this case. Clearly, the Navier-Stokes equations have a unique smooth solution \( u^\nu \) with initial data \( u_0 \), and the Euler equations also have a unique smooth solution \( u \) with the same initial data. We will see that there are examples where \( u^\nu \rightarrow u \) in \( L^2 \), but the answer to the problem above may be positive even when \( u^{\nu} \) does not converge to \( u \), because there may be weak solutions of the incompressible Euler equations with initial velocity \( u_0 \) which are not \( u \).

In 1984, T. Kato wrote a short note where he proved a sharp criterion for the convergence of \( u^\nu \) to \( u \), see [11]. The observation by Kato is remarkable for at least two reasons. First, as we shall see, it is very natural from the analytical point of view. Second, it places the condition for convergence on the behavior of the small viscosity sequence at a distance \( O(\nu) \) of the boundary of \( \Omega \), hence in a much smaller region than what is the natural domain of the boundary layer. Next, we state and prove a simple version of Kato’s criterion.

We focus in the case \( \Omega = \{ |x| < 1 \} \) in \( \mathbb{R}^2 \). Let \( \omega_0 \in C^\infty_c(\Omega) \) and \( u_0 \equiv K[\omega_0] \), where \( K \) is the Biot-Savart operator in the unit disk. Let \( u^\nu \) be the unique classical solution of the Navier-Stokes equation in \( \Omega \) with no-slip boundary condition and initial velocity \( u_0 \), and \( u \) be the unique smooth solution of the Euler equations with \( u \cdot x = 0 \) for \( |x| = 1 \) and initial velocity \( u_0 \).

**Theorem 2. (Kato 1984)** Fix \( T > 0 \). There exists a constant \( c > 0 \) such that \( u^\nu \rightarrow u \) strongly in \( L^\infty((0, T); L^2(\Omega)) \) if and only if \( \nu \int_0^T \| \nabla u^\nu(\cdot, t) \|_{L^2(\Gamma_{c\nu})}^2 \, dt \rightarrow 0 \) as \( \nu \rightarrow 0 \), where \( \Gamma_{c\nu} \equiv \{ 1 - c\nu < |x| < 1 \} \).

**Proof.** First consider the energy identities for both \( u^\nu \) and \( u \). We have, for each \( t > 0 \),

\[
\| u^\nu(\cdot, t) \|_{L^2(\Omega)}^2 = \| u_0 \|_{L^2(\Omega)}^2 + \nu \int_0^t \int_\Omega |\nabla u^\nu|^2 \, dx \, dt,
\]

and

\[
\| u(\cdot, t) \|_{L^2(\Omega)}^2 = \| u_0 \|_{L^2(\Omega)}^2.
\]

Therefore, if \( u^\nu \rightarrow u \) strongly in \( L^\infty((0, T); L^2(\Omega)) \), then \( \| u^\nu(\cdot, t) \|_{L^2(\Omega)}^2 \rightarrow \| u(\cdot, t) \|_{L^2(\Omega)}^2 \) for almost all time, and therefore

\[
\nu \int_0^t \int_\Omega |\nabla u^\nu|^2 \, dx \, dt \rightarrow 0,
\]
for each $t > 0$, not almost everywhere anymore since the integral in time is increasing in time, and therefore,
\[
\nu \int_0^t \int_{\Gamma_{\nu}} |\nabla u^{\nu}|^2 \, dx \, dt \to 0,
\]
as we wished.

To prove the other implication, fix $\varepsilon > 0$ and let $\phi^\varepsilon \in C^\infty_0(\Omega)$ be such that $\phi^\varepsilon(x) = \varepsilon^{-1} |x|$ for $|x| < 1 - \varepsilon$, $\phi^\varepsilon(x) = 0$ for $1 - \varepsilon/2 < |x| \leq 1$, and $\phi^\varepsilon$ decreases monotonically from 1 to 0. Let $\omega = \text{curl } u$ be the vorticity and $\psi$ be the stream function associated with the Euler flow $u$. Define $u_\varepsilon = \nabla_\perp (\phi^\varepsilon \psi) = (-\partial x_2 (\phi^\varepsilon \psi), \partial x_1 (\phi^\varepsilon \psi))$.

Let $v_\varepsilon = u - u_\varepsilon = \nabla_\perp ((1 - \phi^\varepsilon)\psi)$. The stream function $\psi$ vanishes at $|x| = 1$, and it can be assumed to be uniformly bounded in $C^k$, for any $k = 1, 2, \ldots$, so that we can easily obtain the following estimates on $v_\varepsilon$:

\begin{align*}
(12) \quad \|v_\varepsilon\|_{L^\infty((0,T);L^2(\Omega))} &\leq C \varepsilon^{1/2} \\
(13) \quad \|\partial_t v_\varepsilon\|_{L^1((0,T);L^2(\Omega))} &\leq C \varepsilon^{1/2} \\
(14) \quad \|\nabla v_\varepsilon\|_{L^\infty((0,T);L^2(\Omega))} &\leq C \varepsilon^{-1/2} \\
(15) \quad \|v_\varepsilon\|_{L^\infty((0,T) \times \Omega)} &\leq C \\
(16) \quad \|\nabla v_\varepsilon\|_{L^\infty((0,T) \times \Omega)} &\leq C \varepsilon^{-1}
\end{align*}

In addition, we require certain estimates on $u^{\nu}$, uniform in $\nu$, which we collect below:

\begin{align*}
(17) \quad \|u^{\nu}\|_{L^\infty((0,T);L^2(\Omega))} &\leq C \\
(18) \quad \nu \|\nabla u^{\nu}\|_{L^2((0,T);L^2(\Omega))}^2 &\leq C.
\end{align*}

By using Cauchy-Schwarz in time, we also have

\begin{align*}
(19) \quad \nu^{1/2} \|\nabla u^{\nu}\|_{L^1((0,T);L^2(\Omega))} &\leq CT^{1/2} \left( \nu \int_0^T \|\nabla u^{\nu}\|_{L^2(\Omega)}^2 \, dt \right)^{1/2} \\
\text{Finally, a version of Poincaré’s Inequality, which reads}
\end{align*}

\begin{align*}
(20) \quad \|u^{\nu}\|_{L^2(\Gamma_\varepsilon)} &\leq C \varepsilon \|\nabla u^{\nu}\|_{L^2(\Gamma_\varepsilon)} \\
\text{Now we estimate, omitting the explicit dependence of } v \text{ on } \varepsilon:
\end{align*}

\[
\|u^{\nu} - u\|_{L^2(\Omega)}^2 = \|u^{\nu}\|_{L^2(\Omega)}^2 + \|u\|_{L^2(\Omega)}^2 - 2\langle u^{\nu}, u \rangle \\
\leq 2\|u_0\|_{L^2(\Omega)}^2 - 2\langle u^{\nu}, u - v \rangle - 2\langle u^{\nu}, v \rangle.
\]
We have that
\[
\|u''(u_v)\| \leq \|u''\|_{L^\infty(0,T); L^2(\Omega)} \|v\|_{L^\infty(0,T); L^2(\Omega)} \leq C\varepsilon^{1/2}.
\]
And therefore, taking \(\varepsilon = \nu\),
\[
(21) \quad \|u'' - u\|_{L^2(\Omega)}^2 \leq 2\|u_0\|_{L^2(\Omega)}^2 - 2(u'', u - v) + o(1).
\]

We multiply the Navier-Stokes equation by \(u - v\), integrate in space and time, and integrate by parts to obtain
\[
\langle u''(u_v), u - v \rangle - \langle u_0, u_0 - v \rangle = \int_0^t \langle u''(u_v) - \nu \nabla u'' \cdot \nabla (u - v) + \nu \nabla u'' \cdot \nabla (u - v) \rangle dt + \langle u'', \partial_t (u - v) \rangle dt,
\]
which, together with (21) implies
\[
(22) \quad \|u'' - u\|_{L^2(\Omega)}^2 \leq \int_0^t [-2\langle u'', u'' \cdot \nabla (u - v) \rangle + 2\nu \langle \nabla u'', \nabla (u - v) \rangle] dt + o(1).
\]

We write
\[
\int_0^t \langle u'', \partial_t (u - v) \rangle dt = - \int_0^t \langle u'', u \cdot \nabla u \rangle dt + \int_0^t \langle u'', \partial_t v \rangle dt,
\]
and we have
\[
\left| \int_0^t \langle u'', \partial_t v \rangle dt \right| \leq \int_0^t \|u''\| \|\partial_t v\| \leq C\nu^{1/2},
\]

hence,
\[
(23) \quad \|u'' - u\|_{L^2(\Omega)}^2 \leq \int_0^t [-2\langle u'', u'' \cdot \nabla (u - v) \rangle + 2\nu \langle \nabla u'', \nabla (u - v) \rangle + 2\langle u'', u \cdot \nabla u \rangle] dt + o(1).
\]

Note that
\[
\langle (u'' - u, (u'' - u) \cdot \nabla u) \rangle = \langle u'', u'' \cdot \nabla u \rangle - \langle u'', u \cdot \nabla u \rangle,
\]
and therefore, (23) becomes
\[
(24) \quad \|u'' - u\|_{L^2(\Omega)} \leq 2 \int_0^t [-\langle u'' - u, (u'' - u) \cdot \nabla u \rangle + \nu \langle \nabla u'' \cdot \nabla (u - v) \rangle + \{u''(u_v) \cdot \nabla v\}] dt + o(1).
\]

We analyze each term in the right hand side of (24) separately. First,
\[
\left| \int_0^t \langle u'' - u, (u'' - u) \cdot \nabla u \rangle \right| \leq \int_0^t \|u'' - u\|^2_{L^2(\Omega)} |\nabla u| dt \leq C \int_0^t \|u'' - u\|^2_{L^2(\Omega)} dt.
\]
Second,
\[
\nu \int_0^t \langle \nabla u'', \nabla (u - v) \rangle dt \leq \nu \int_0^t \langle \nabla u'', \nabla u \rangle dt + \nu \int_0^t \langle \nabla u'', \nabla v \rangle dt.
\]
\[
\leq \nu \int_0^t \|\nabla u''\|_{L^2(\Omega)} \|\nabla u\|_{L^2(\Omega)} dt + \nu \int_0^t \|\nabla u''\|_{L^2(\Gamma_{ov})} \|\nabla v\|_{L^2(\Gamma_{ov})} dt
\]
\[
\leq C\nu \|\nabla u''\|_{L^1(0,T; L^2(\Omega))} + C\nu^{1/2} \|\nabla u''\|_{L^1(0,T; L^2(\Gamma_{ov}))}
\]
\[ \leq C \nu^{1/2} t^{1/2} \| \nabla u'' \|_{L^2((0,T);L^2(\Gamma_{cv}))} + o(1) = C t^{1/2} \left( \nu \int_0^t \| \nabla u'' \|_{L^2(\Gamma_{cv})}^2 \right)^{1/2} = o(1), \]

by Kato’s criterion.

Third,

\[ \left| \int_0^t \langle u^\nu, u^\nu \cdot \nabla v^\nu \rangle \, dt \right| = \left| \int_0^t \langle v, u^\nu \cdot \nabla u^\nu \rangle \, dt \right| \leq 2 \| v \|_{L^\infty((0,T) \times \Omega)} \int_0^t \| u^\nu \|_{L^2(\Gamma_{cv})} \| \nabla u^\nu \|_{L^2(\Gamma_{cv})} \, dt \leq C \nu \int_0^t \| \nabla u^\nu \|_{L^2(\Gamma_{cv})}^2 \, dt = o(1), \]

where we used Poincaré’s Inequality and Kato’s criterion in the last line.

Therefore, we get back to (24) and use the estimates obtained above to conclude that

\[ \| u'' - u \|_{L^2(\Omega)}^2 \leq C \int_0^t \| u'' - u \|_{L^2(\Omega)}^2 \, dt + o(1), \]

which, by Gronwall, concludes the proof.

\[ \square \]

A number of observations are in order. First, the proof above can be adapted to general bounded domains in \( \mathbb{R}^n \), to Leray solutions of the Navier-Stokes equations and to problems with forcing and different initial data, as long as the forcing and the initial data converge to the corresponding ones when viscosity vanishes. The two equivalent conditions in the statement of the Theorem are also equivalent to weak convergence of \( u^\nu \) to \( u \) in \( L^2 \), pointwise in time. Clearly the strong convergence implies the weak convergence, and we weak convergence implies that

\[ \nu \int_0^t \| \nabla u'' \|_{L^2(\Omega)}^2 \, dt \to 0, \text{ as } \nu \to 0, \]

which can be seen using the energy identities and the weak lower semicontinuity of the \( L^2 \) norm. All these observations were contained in the original work by Kato, [11].

There have been several results modifying, extending and improving Kato’s criterion in the literature: work by R. Temam and Xiaoming Wang, [30, 31], where Kato’s condition is replaced by a condition on integrability of pressure near the boundary, work by Xiaoming Wang, [33] where the criterion is cast in terms of integrability of derivatives of tangential components of velocity, and work by James Kelliher, [13], where the integrability condition is placed on vorticity.

As we will see in the last lecture, there are some (trivial) examples where Kato’s criterion holds true, but this is not known in general, and it would not be surprising if it turns out to be false most of the time. Note that, if Kato’s criterion fails, this implies non-vanishing energy dissipation occurring in a thin neighborhood around the boundary, as viscosity vanishes.

Given the disparity of the understanding of the vanishing viscosity limit in the presence or absence of boundaries, it becomes natural to ask how much boundary it is required to obstruct the proof of convergence of the vanishing viscosity limit. The author has been involved with two recent results in this direction. First, with D. Iftimie and H. Nussenzveig Lopes, we have proved that if \( u^{\nu,\varepsilon} \) is the solution of the Navier-Stokes equations in the exterior of the domain \( \varepsilon \Omega \), with no-slip boundary conditions and fixed initial vorticity and \( u \) is the solution of the Euler equations in
the full space, with the same initial vorticity then there exists a constant $c > 0$ such that, if $\varepsilon < c\nu$, then $u^{\nu,\varepsilon} \to u$ strongly in $L^2$, see [8]. Second, with J. Kelliher and H. Nussenzveig Lopes, we have proved that if $u^{\nu,R}$ is the solution of the Navier-Stokes equations on a domain $\Omega_R$ which contains the ball of radius $R$, with fixed initial vorticity, and again $u$ is the solution of the Euler equation with the same initial vorticity. We have proved that, as long $R \to \infty$ as $\nu \to 0$, then $u^{\nu,R}$ converges strongly to $u$ as $\nu \to 0$, see [14]. Both proofs follow the basic idea and the structure of the proof of Kato’s criterion as we presented it here.

5. Vanishing viscosity with Navier friction $\partial$ condition

The main purpose of this Section is to highlight the role of vorticity production by the interaction of the flow with the boundary in the difficulty surrounding the Boundary Layer Problem. In particular, we briefly discuss the vanishing viscosity limit for flows in the full space, and also for flows in a bounded domain satisfying Navier friction condition at the boundary. Let us begin with the full space case.

A basic ingredient for the proof of compactness of the vanishing viscosity limit in the style of [23] is the set of estimates obtained through the vorticity formulation of the Navier-Stokes equations. Vorticity play a fundamental role in the analysis of incompressible fluid flow. This is specially true in two dimensions, because vorticity is conserved along particle trajectories for ideal flow, and it satisfies a nice convection-diffusion equation for viscous flow, namely:

\[
\begin{aligned}
\omega_t + u \cdot \nabla \omega &= \nu \Delta \omega \\
u &= K[\omega],
\end{aligned}
\]

where $K$ is the Biot-Savart law associated with the flow domain. In the absence of material boundaries, for example, in the case of the full plane, equation (26) implies a priori $L^p$ estimates for vorticity for any $1 \leq p \leq \infty$, as long as the initial vorticity is in $L^p$. The Biot-Savart operator is a pseudo-differential operator of order $-1$, which implies compactness of the sequence of velocities in $L^q$, for any $q < p^*$, $p^* = 2p/(2-p)$ (assuming $1 \leq p < 2$), and the nonlinearity of the Euler equations being quadratic, all that is required is compactness in $L^2$ in order to prove that the flows obtained through the limit of vanishing viscosity satisfy Euler equation. The case $p = 1$, which is what is involved in the vortex sheet initial data problem, is critical for the argument above, and requires a more involved analysis.

To highlight the importance of the added compactness provided by vorticity estimates, we consider briefly the situation without it. In that case, the only a priori estimate available comes from the energy estimate, which gives solely an $L^2$-bound on velocity, independent of viscosity. By Banach-Alaoglu, an uniform $L^2$-bound can provide us with a sequence of approximations which converge weakly in $L^2$ to a "candidate" for limit solution $u$. To prove that $u$ really is a solution of Euler’s equation, one must pass to the limit in the nonlinearity $u^n \cdot \nabla u^n$, which, after integration by parts on a weak formulation, can be written in terms of quadratic expressions in the components of velocity. Now, the point is: quadratic expressions are not continuous with respect to the weak topology of $L^2$. Two classical examples (in one space dimension) are:

\[
f^n = \begin{cases} 
\frac{1}{|x|^n}, & \text{for } |x| \leq \frac{1}{n} \\
0, & \text{for } |x| > \frac{1}{n}
\end{cases} \quad (\text{concentration}),
\]
and

\[ g^n = \sin(nx) \text{ (oscillation)}. \]

Both these sequences converge weakly to zero in \( L^2 \), but their squares converge to something else - we leave the details as an exercise to the reader. The bottom line is this: strong convergence in \( L^2 \) is required to pass to the limit in quadratic expressions, and this strong convergence has to come from some additional a priori estimate.

The presence of vorticity estimates breaks down in the presence of material boundaries because, although vorticity still satisfies equation (25) in a domain with boundary, there is no natural boundary condition for vorticity to complete equation (25), beyond the nonlocal boundary condition \( K[\omega] \cdot \hat{\tau} = 0 \) (only the vanishing of the tangential component of velocity is required to vanish because the vanishing of the normal component of velocity is implicit in the Biot-Savart law). In particular, one may argue that it is \textit{the lack of control on the production of vorticity at the boundary} which obstructs the solution of the \( \partial \) Layer Problem.

Mathematicians are not discouraged by difficulties - if we cannot solve a problem, we find another problem nearby which we can actually solve. In the book [16], J.-L. Lions used as an illustration the following problem:

\[
\left\{ \begin{array}{l}
\partial_t u + u \cdot \nabla u = -\nabla p + \mu \Delta u \\
\text{div} u = 0 \\
u \cdot \hat{n} = 0 \text{ and } \omega = \text{curl} u = 0 \text{ at } \partial \Omega,
\end{array} \right.
\tag{26}
\]

which is usually called the “free boundary problem” for the Navier-Stokes equations in the planar domain \( \Omega \). The standard theory for the Navier-Stokes equations with no-slip boundary condition can be easily adapted to this problem, but even more, the vanishing viscosity limit is quite well behaved in this case. Indeed, the vorticity equation (25) can be closed by adding the Dirichlet boundary condition \( \omega = 0 \) at \( \partial \Omega \) and initial data. By multiplying the equation by \( \omega^{p-1} \), integrating, and performing the usual integration by parts, one quickly arrives at the conclusion that the \( L^p \)-norms of vorticity are bounded independently of \( \nu \), and the compactness argument outlined in the beginning of the section leads to an affirmative answer to this variant of the \( \partial \) Layer Problem. The homogeneous Dirichlet condition is dissipative for the heat equation; in particular, it means that the \( L^p \) norms of vorticity must decay in time.

There is a generalization of the free boundary condition called \textit{Navier friction condition} which is both physically and mathematically interesting. The physically correct definition involves the rate-of-strain matrix of the flow at the boundary but, in two space dimensions, one can prove that this boundary condition can be written as \( \omega = (2\kappa - \alpha)u \cdot \hat{\tau} \), where \( \kappa \) is the curvature of the boundary, \( \alpha \) is the friction coefficient and \( \hat{\tau} \) is the counterclockwise unit tangent vector to \( \partial \Omega \), if \( \Omega \) is a bounded domain with connected smooth boundary. We refer the reader to [3] for a full discussion. This condition, also called Navier “slip” condition, still allows for the creation of vorticity at the boundary, but in a way that can be controlled.

In [3], T. Clopeau, A. Mikelić and R. Robert studied the initial-boundary value problem for the Navier-Stokes equations in 2D, with Navier friction condition. They proved global well-posedness for this problem for fixed viscosity, and they proved the convergence of the vanishing viscosity limit to the (unique) solution of the Euler
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Thus obtain equations if the initial vorticity is bounded. An estimate on the rate of convergence was obtained by J. Kelliher in [12].

In [21] the author, together with H. Nussenzveig Lopes and G. Planas, extended the vanishing viscosity result by considering flows with initial vorticity in $L^p$, $p > 2$. The key point of this result is the uniform $L^p$ estimate on vorticity, which is what we will do next. Let $\Omega$ be a bounded, connected and simply-connected smooth domain in $\mathbb{R}^2$ and $\alpha \in C^\infty(\partial\Omega)$. We consider $\omega^\nu = \omega^\nu(x,t)$ to be the unique solution of the IBVP:

\begin{align}
\omega_t^\nu + \nu \cdot \nabla \omega^\nu &= \nu \Delta \omega^\nu \\
\omega^\nu &= K_\Omega[\omega^\nu] \\
\omega^\nu(x,t) &= (2\kappa - \alpha)(u(x,t) \cdot \tau) \quad \text{for } x \in \partial\Omega \\
\omega(x,0) &= \omega_0
\end{align}

We assume that $\omega_0 \in H^1 \cap L^\infty$ is compatible, i.e. that $\omega_0 = (2\kappa - \alpha)(K_\Omega[\omega_0] \cdot \tau)$ in $\partial\Omega$. Existence of a solution to problem (27) with $\omega^\nu \in C([0,T]; H^1(\Omega))$ (and $u^\nu \in C([0,T]; H^2(\Omega))$) was proved in [3], and density of compatible vorticities in $L^p$ with respect to the strong norm was proved in [21]. Our objective is to prove the following

**Theorem 3.** (L., Nussenzveig Lopes and Planas, 2005) For each $p > 2$, there exists a constant $C > 0$, independent of $\nu$, such that

$$
\|\omega^\nu(\cdot,t)\|_{L^p(\Omega)} \leq C(\|\omega_0\|_{L^p(\Omega)} + \|u_0\|_{L^2(\Omega)}),
$$

with $u_0 = K_\Omega[\omega_0]$.

**Proof.** In this proof we will use some results of the theory of parabolic equations, which can be found in the book [15].

The proof involves applying a maximum principle to two auxiliary problems. First observe that $u \cdot \tau$ is $H^2$, and therefore bounded. Set

$$
\Lambda = \|(2\kappa - \alpha)u \cdot \tau\|_{L^\infty(\partial\Omega \times (0,T))}.
$$

Consider the initial-boundary value problem for the Fokker-Planck equation:

\begin{align}
\tilde{\omega}_t - \nu \Delta \tilde{\omega} + u \cdot \nabla \tilde{\omega} &= 0 \quad \text{in } \Omega \times (0,T), \\
\tilde{\omega}(\cdot,0) &= |\omega_0| \quad \text{in } \Omega, \\
\tilde{\omega} &= \Lambda \quad \text{on } \partial\Omega \times (0,T).
\end{align}

This problem has a unique weak solution $\tilde{\omega} \in L^2((0,T); H^1(\Omega))$. Then, $\omega_1 = \omega - \tilde{\omega}$ is a weak solution for the following initial-boundary value problem:

\begin{align}
(\omega_1)_t - \nu \Delta \omega_1 + u \cdot \nabla \omega_1 &= 0 \quad \text{in } \Omega \times (0,T), \\
\omega_1(\cdot,0) &= \omega_0 - |\omega_0| \quad \text{in } \Omega, \\
\omega_1 &= (2\kappa - \alpha)u \cdot \tau - \Lambda \quad \text{on } \partial\Omega \times (0,T).
\end{align}

The coefficients of the Fokker-Planck operator $\partial_t - \nu \Delta + u \cdot \nabla$ are such that the maximum principle for weak solutions of parabolic equations is valid. Therefore, as $\omega_1 \leq 0$ on the parabolic boundary $\partial\Omega \times (0,T) \cup \Omega \times \{t = 0\}$, it follows that $\omega_1 \leq 0$ a.e. in $\Omega \times [0,T)$. Analogously, we prove that $\omega_2 = -\omega - \tilde{\omega}$ is non-positive. We thus obtain
(30) \[ |\omega| \leq \tilde{\omega} \text{ a.e. in } \Omega \times [0, T). \]

Moreover, since \( \omega_0 \) is bounded, the maximum principle may also be applied to equation (28) yielding that \( \tilde{\omega} \in L^\infty((0, T) \times \Omega) \).

Next we obtain an estimate for \( \tilde{\omega} \). Let \( \tilde{\omega} = \omega - \Lambda \). This is a solution of the following problem:

\[
\begin{aligned}
\tilde{\omega}_t - \nu \Delta \tilde{\omega} + u \cdot \nabla \tilde{\omega} &= 0 \quad \text{in } \Omega \times (0, T), \\
\tilde{\omega}(\cdot, 0) &= |\omega_0| - \Lambda \quad \text{in } \Omega, \\
\tilde{\omega} &= 0 \quad \text{on } \partial \Omega \times (0, T).
\end{aligned}
\]  

(31)

We formally multiply (31) by \( \tilde{\omega} |\tilde{\omega}|^{p-2} \), where \( p > 2 \), we integrate by parts and use the incompressibility of the flow \( u \) to obtain:

\[ \frac{1}{p} \frac{d}{dt} \int_\Omega |\tilde{\omega}|^p + (p-1)\nu \int_\Omega |\nabla \tilde{\omega}| |\tilde{\omega}|^{(p-2)/2} \, dx = 0. \]  

Then,

\[ \|\tilde{\omega}(\cdot, t)\|_{L^p(\Omega)} \leq \|\tilde{\omega}(\cdot, 0)\|_{L^p(\Omega)} \leq \|\omega_0\|_{L^p(\Omega)} + \Lambda |\Omega|^{1/p}. \]

Therefore,

\[ \|\tilde{\omega}\|_{L^p(\Omega)} \leq \|\tilde{\omega}\|_{L^p(\Omega)} + \Lambda |\Omega|^{1/p} \leq \|\omega_0\|_{L^p(\Omega)} + 2\Lambda |\Omega|^{1/p}. \]  

(33)

This formal calculation can be made rigorous by using the weak formulation of (31) given in (35). We begin by observing that \( \tilde{\omega}_t \in L^2((0, T); H^{-1}(\Omega)) \) and \( \tilde{\omega} \in L^2((0, T); H^1_0(\Omega)) \). This implies that \( \tilde{\omega} |\tilde{\omega}|^{p-2} \in L^2((0, T); H^1_0(\Omega)) \). Therefore we can multiply (31) by \( \tilde{\omega} |\tilde{\omega}|^{p-2} \) if we understand the product with \( \tilde{\omega}_t \) and with \( \Delta \tilde{\omega} \) as duality pairings. Finally, in order to justify (32) one still needs to approximate \( \tilde{\omega} \) by suitable smooth functions and pass to the limit in each term of the weak formulation so as to obtain

\[ \frac{1}{p} \frac{d}{dt} \int_\Omega |\tilde{\omega}|^p = (\tilde{\omega}_t, \tilde{\omega} |\tilde{\omega}|^{p-2}) \]

\[ = \nu (\Delta \tilde{\omega}, \tilde{\omega} |\tilde{\omega}|^{p-2}) = -(p-1)\nu \int_\Omega |\nabla \tilde{\omega}| |\tilde{\omega}|^{(p-2)/2} \, dx. \]

This can be easily accomplished using mollification in time together with the Dirichlet heat semigroup for \( \Omega \), thus generating a family of smooth functions \( \tilde{\omega}_\varepsilon \) such that \( \partial_t \tilde{\omega}_\varepsilon \to \tilde{\omega}_t \) strongly in \( L^2((0, T); H^{-1}(\Omega)) \), while \( \tilde{\omega}_\varepsilon |\tilde{\omega}_\varepsilon|^{p-2} \to \tilde{\omega} |\tilde{\omega}|^{p-2} \) weakly in \( L^2((0, T); H^1_0(\Omega)) \) and \( \tilde{\omega}_\varepsilon \) is uniformly bounded in \( \Omega \times (0, T) \).

Given (33) we now turn to the estimate of \( \Lambda \). Using Sobolev imbedding and interpolating between \( W^{1,p} \) and \( L^2 \), we find:

\[ \|u^\nu(\cdot, t) \cdot \tau\|_{L^\infty(\Omega)} \leq C\|u^\nu(\cdot, t)\|_{C(\Omega)} \leq C\|u^\nu(\cdot, t)\|_{L^2(\Omega)}^\theta \|u^\nu(\cdot, t)\|_{W^{1,p}(\Omega)}^{1-\theta} \]

\[ \leq C\|u^\nu(\cdot, t)\|_{L^2(\Omega)}^{\theta} \|\omega^\nu(\cdot, t)\|_{L^p(\Omega)}^{1-\theta}, \]

where \( \theta = (p-2)/(2p-2) \).

Let \( \varepsilon \) be an arbitrary positive number. We now use Young’s inequality together with the fact that \( \kappa \) and \( \alpha \) are bounded to conclude that:

\[ \Lambda \leq C_\varepsilon \|u^\nu\|_{L^\infty((0, T); L^2(\Omega))} + \varepsilon \|\omega^\nu\|_{L^\infty((0, T); L^p(\Omega))} \]

for some \( C_\varepsilon > 0 \). Taking \( \varepsilon \) small enough, from (30) - (34) we obtain:

\[ \|\omega^\nu\|_{L^\infty((0, T); L^p(\Omega))} \leq C(\|\omega_0\|_{L^p(\Omega)} + \|u^\nu\|_{L^\infty((0, T); L^2(\Omega))}) \]  

(35)
for any $p > 2$, where $C = C(p, \Omega, \|\kappa\|_{L^\infty(\partial\Omega)}, \|\alpha\|_{L^\infty(\partial\Omega)})$. Finally, a standard energy estimate, yields $\|u\|_{L^\infty(0,T;L^2(\Omega))} \leq \|u_0\|_{L^2(\Omega)}$, thereby concluding the proof. □

Let us first observe that the "standard energy estimate" used as last step of the proof above is far from obvious. The difficulty lies in the integration by parts of the viscous term in the energy estimate. This can actually be done by using the correct formulation of the Navier friction condition, namely $2(DU)_s \hat{n} \cdot \hat{\tau} + \alpha u \cdot \hat{\tau} = 0$, where $(DU)_s$ is the symmetric part of the Jacobian matrix $DU$. We leave the derivation of the energy estimate in this case as an exercise to the reader.

The estimate in Theorem 3 can be extended to any vorticity in $L^p$, $p > 2$ by density. It is not clear whether the restriction $p > 2$ is a consequence of the technique used above, or it is an essential restriction in the estimate above. This is an interesting open problem. One physical justification for the Navier friction boundary condition is that is approximates the interaction of incompressible flow with a rough boundary. This contention was rigorously justified by Jager and Mikelic, who derived the Navier friction condition as the effective behavior associated with the homogenization of an oscillatory boundary, see [10].

There is no physical interpretation, at present, for the vanishing viscosity limit associated with Navier friction condition. It is, however, a very natural question from the mathematical point of view, since it interpolates naturally between the trivial case of the free boundary condition and the Boundary Layer Problem. The Navier friction condition still gives rise to a boundary layer, albeit in a less singular way than Prandtl’s original theory. This is the subject of the next Section.

6. Boundary layer for the Navier friction condition

In this section we will explore a rigorous procedure to obtain a boundary layer equation, based on ideas from geometric optics, applied to the Navier friction condition. What we will present here is based on joint work of the author with D. Iftimie, H. Nussenzveig Lopes and F. Sueur.

We begin with the $\nu$-Navier-Stokes system

\[
\begin{aligned}
\partial_t u^\nu + u^\nu \cdot \nabla u^\nu &= -\nabla p^\nu + \nu \Delta u^\nu \\
\text{div } u^\nu &= 0 \\
u^\nu(t, x_1, 0) &= 0 \\
\omega^\nu(t, x_1, 0) &= \gamma u_1^\nu(t, x_1, 0) \\
\nu^\nu(0, x) &= u_0(x).
\end{aligned}
\]

We set

\[
u^\nu = u^E + \sqrt{\nu} \begin{pmatrix} v(t, x_1, x_2/\sqrt{\nu}) \\ 0 \end{pmatrix} + \nu u_R^\nu.
\]
Additionally, we write

\[ \omega^\nu = \omega^E + W(t, x_1, x_2/\sqrt{\nu}) + \sqrt{\nu} \phi^\nu, \]

where \( W = W(t, x_1, y) = (\partial_y \nu)(t, x_1, y) \) and \( \phi^\nu = \sqrt{\nu} \text{curl } u_R^\nu. \)

We are going to write an equation for \( v \), derive an equation for \( W \) and then use
the equations for \( W, \omega^\nu \) and \( \omega^E \) to deduce an equation for \( \phi^\nu. \) Our final objective
is to obtain an \( L^p \) estimate for \( \phi^\nu \) which is independent of \( \nu. \)

We first write an evolution equation with Neumann boundary condition for the
velocity profile of the boundary layer \( v \):

\[
\begin{aligned}
\partial_t v + u^E_1(t, x_1, 0) \partial_{x_1} v + \partial_{x_1} u^E_1(t, x_1, 0)v + y(\partial_{x_2} u^E_2)(t, x_1, 0) \partial_y v &= \partial_y^2 v \\
v(0, x_1, y) &= 0 \\
\partial_y v(t, x_1, 0) &= \gamma u^E_1(t, x_1, 0) - \omega^E(t, x_1, 0).
\end{aligned}
\]

(37)

Now we differentiate equation (37) with respect to \( y \) and use the div-free condition
on \( u^E \) to obtain an equation for the vorticity profile of the boundary layer \( W \). We find:

\[
\begin{aligned}
\partial_t W + u^E_1(t, x_1, 0) \partial_{x_1} W + y(\partial_{x_2} u^E_2)(t, x_1, 0) \partial_y W &= \partial_y^2 W \\
W(0, x_1, y) &= 0 \\
W(t, x_1, 0) &= \gamma u^E_1(t, x_1, 0) - \omega^E(t, x_1, 0).
\end{aligned}
\]

(38)

The equations for \( v \) and \( W \) are linear, with smooth coefficients and independent
of \( \nu. \) For the sake of these lectures, we assume well-posedness and exponential decay
of the solutions \( v \) and \( W \) as \( y \to \infty \). To be more precise, we assume existence of a
unique smooth solution \( v = v(t, x_1, y) \) of (37) and consequently of \( W = W(t, x_1, y) \),
solution of (38), such that:

1. \( W \in C^\infty_0 \cap L^p \), for some \( p > 2, \)
2. \( v \in C^\infty_0 \cap L^p \) and \( \partial_{x_1} v \in L^p, \)
3. \( \partial_{x_1} W \) and \( \partial_{x_1}^2 W \) both in \( L^p, \)
4. \( y \partial_y W \) bounded and \( y^2 \partial_y W \in L^p, \)
5. \( y \partial_y W \in L^p. \)

Next we write the equation for \( \phi^\nu. \) First set \( z = x_2/\sqrt{\nu} \). We introduce

\[
G^\nu = [u^E_1(t, x_1, x_2) - u^E_1(t, x_1, 0)](\partial_{x_1} W)(t, x_1, z) \\
+ [u^E_2(t, x_1, x_2) - x_2(\partial_{x_2} u^E_2)(t, x_1, 0)] \frac{1}{\sqrt{\nu}}(\partial_y W)(t, x_1, z),
\]

and

\[
F^\nu = \sqrt{\nu} \Delta \omega^E + \sqrt{\nu} \partial_{x_1}^2 W(t, x_1, z) - \left[ \begin{array}{c} v(t, x_1, z) \\ 0 \end{array} \right] + \sqrt{\nu} u^\nu_R \right) \cdot \nabla \omega^E \\
-(v(t, x_1, z) + \sqrt{\nu}(u^\nu_R)(\partial_{x_1} W)(t, x_1, z) - (u^\nu_R)_2(\partial_y W)(t, x_1, z) - \frac{G^\nu}{\sqrt{\nu}}.
\]

With this notation, the equation for \( \phi^\nu \) becomes
\[
\begin{align*}
\begin{cases}
\partial_t \phi^\nu + u^\nu \cdot \nabla \phi^\nu - \nu \Delta \phi^\nu = F^\nu \\
\phi^\nu(0, x) = 0 \\
\phi^\nu(t, x_1, 0) = \gamma(v + \sqrt{\nu}(u_R^\nu)_1)(t, x_1, 0).
\end{cases}
\end{align*}
\]
(39)

We break problem (39) into two problems, one with homogeneous boundary condition and nonzero forcing and another with nonhomogeneous boundary data and zero forcing. We write \( \phi^\nu \equiv \phi_1^\nu + \phi_2^\nu \), with
\[
\begin{align*}
\begin{cases}
\partial_t \phi_1^\nu + u^\nu \cdot \nabla \phi_1^\nu - \nu \Delta \phi_1^\nu = 0 \\
\phi_1^\nu(0, x) = 0 \\
\phi_1^\nu(t, x_1, 0) = \gamma(v + \sqrt{\nu}(u_R^\nu)_1)(t, x_1, 0),
\end{cases}
\end{align*}
\]
(40)

and
\[
\begin{align*}
\begin{cases}
\partial_t \phi_2^\nu + u^\nu \cdot \nabla \phi_2^\nu - \nu \Delta \phi_2^\nu = F^\nu \\
\phi_2^\nu(0, x) = 0 \\
\phi_2^\nu(t, x_1, 0) = 0.
\end{cases}
\end{align*}
\]
(41)

The basic idea is to treat (41) using the energy method and to treat (40) adapting the comparison principle argument used in [21]. In addition, it is necessary to obtain a priori estimates for the elliptic system which relates \( \phi^\nu \) with \( u_R^\nu \). We write this system as:
\[
\begin{align*}
\begin{cases}
\text{div} \sqrt{\nu}u_R^\nu = \partial_{x_1}v(t, x_1, x_2/\sqrt{\nu}) \\
\text{curl} \sqrt{\nu}u_R^\nu = \phi^\nu \\
(u_R^\nu)_2(t, x_1, 0) = 0.
\end{cases}
\end{align*}
\]
(42)

What we end up with is the following estimates
\[
\|\phi_1^\nu\|_{L^p} \leq C(1 + \|\phi_2^\nu\|_{L^p}),
\]
and
\[
d\frac{d}{dt}\|\phi_2^\nu\|_{L^p} \leq C(\|\phi^\nu\|_{L^p} + 1) \leq C(1 + \|\phi_1^\nu\|_{L^p} + \|\phi_2^\nu\|_{L^p}),
\]
the proofs are the technical core of this work, but for time limitations we omit them.

Just to conclude this story, the estimates above lead to
\[
d\frac{d}{dt}\|\phi_2^\nu\|_{L^p} \leq C(1 + \|\phi_2^\nu\|_{L^p}).
\]

It follows from Gronwall’s inequality that \( \|\phi_2^\nu\|_{L^p} \) is bounded independently of \( \nu \), for a fixed time interval \([0, T]\). Then, using the first estimate again concludes that there exists a constant \( C \) independent of \( \nu \) such that
\[
\|\phi^\nu\|_{L^p} \leq C.
\]
Remark: We observe that in [9], Iftimie and Sueur proved that $\phi$ is bounded in $L^2((0, T) \times \Omega)$, independently of $\nu$. We proved that there is a bound on $L^\infty((0, T); L^p)$, for $2 < p < \infty$. We do not expect to be able to extend this argument to $p = \infty$. Note also that the analysis in [9] is valid for general smooth domains in two or three space dimensions, whereas the analysis outlined above only works in two dimensions.

7. The rotating cylinder

Our purpose in this section is to provide a fairly explicit illustration of the boundary layer phenomena. The main point of this illustration is to show, that in a favorable scenario, the small viscosity limit has vorticity behaving like a vortex sheet. This explains, in part, the difficulty involved in the Boundary Layer Problem, since vortex sheet regularity is critical for the weak continuity of the nonlinearity present in Euler and Navier-Stokes. This material is based on joint work of the author with A. Mazzucato, H. Nussenzveig Lopes and M. Taylor, and it is contained in two articles, [19, 20].

The physical situation we wish to consider is that of an infinite circular cylinder filled with fluid. The boundary of the cylinder is a material shell which is assumed to rotate about its center of symmetry with angular velocity $\alpha = \alpha(t)$ (positive rotation is counterclockwise). We restrict our attention to planar and circularly symmetric motions, so that the fluid velocity $u$ is given by

$$u = u(x_1, x_2, x_3, t) = v(r, t) \left( \frac{-x_2, x_1, 0}{r} \right) = V(x_1, x_2, t) \left( \frac{-x_2, x_1, 0}{r} \right),$$

with $r = \sqrt{x_1^2 + x_2^2}$. This symmetry assumption is consistent with the Navier-Stokes equations as long as the initial velocity is planar and circularly symmetric, i.e., weak solutions satisfying the symmetry assumptions exist globally in time for symmetric initial data. It should be remarked that the assumption that such flows stay planar for all Reynolds number is unphysical - 3D turbulence is expected if the Reynolds number is large enough. The limit behavior we wish to present is motivated by mathematical considerations.

Let us first observe that, if velocity is circularly symmetric, and, in particular, tangent to concentric circles around the origin, vorticity is also circularly symmetric, and therefore constant in the same circles. This implies that $\nabla \omega$ is perpendicular to these circles, and therefore $u$ and $\nabla \omega$ are orthogonal everywhere. Therefore, the nonlinearity $u \cdot \nabla \omega$ is the vorticity equation vanishes identically, and the vorticity equation becomes the heat equation. This basic fact is what it makes possible to prove the following result.

**Theorem 4.** Let $u^\nu$ be the solution of the 2D Navier-Stokes equations in the unit disk, with no slip boundary data with respect to boundary rotation with prescribed angular velocity $\alpha \in BV(\mathbb{R})$. Assume that the initial velocity $u_0 \in L^2(D)$ has circular symmetry, i.e. $u_0(x) = v_0(|x|)x_1$. Then, $u_0$ is a steady solution of the 2D Euler equation and $u^\nu$ converges strongly in $L^\infty([0, T], L^2(D))$ to $u_0$ as $\nu \to 0$.

The proof of this result is based on a semigroup treatment of a symmetry-reduced problem. The important issue is that the symmetry assumption not only prevents boundary layer separation, but it also eliminates the nonlinearity of the problem. Thus, the symmetry-reduced problem is a linear, singular coefficient perturbation of the heat equation, and its treatment is classical.
Let $D \equiv \{|x| < 1\} \subset \mathbb{R}^2$. We begin by considering the following $2 \times 2$ system of uncoupled heat equations:

$$
\begin{cases}
\partial_t u^{\nu} = \nu \Delta u^{\nu}, & \text{in } (0, \infty) \times D; \\
u^{\nu}(x, 0) = u_0(x), & \text{in } D; \\
u^{\nu}(x, t) = \frac{\alpha(t)}{2\pi} x^\perp, & \text{on } |x| = 1.
\end{cases}
$$

(43)

Let us suppose that $u_0$ is covariant under rotations, i.e., $u_0(R_\theta x) = R_\theta u_0(x)$ for any rotation $R_\theta$. Then (43) is the symmetry-reduced 2D Navier-Stokes system of equations with initial velocity $u_0$.

Let us assume, to begin with, that $u_0 \in C^\infty(D)$ is covariant under rotations and that $u_0$ and $\alpha \in C^\infty(\mathbb{R})$ satisfy the following compatibility conditions:

$$
\begin{align*}
&u_0(x) = \alpha(0) \frac{x^\perp}{2\pi} \text{ on } \partial D; \\
&\Delta u_0 = \alpha'(0) \frac{x^\perp}{2\pi \nu} \text{ on } \partial D.
\end{align*}
$$

(44)

Then it follows that there exists a unique weak solution $u^{\nu} = u^{\nu}(x, t) \in L^2(\mathbb{R}; H^4(D)) \cap H^1(\mathbb{R}; H^2(D)) \cap H^2(\mathbb{R}; L^2(D))$ of (43). In particular we have $u^{\nu} \in C^0(\mathbb{R}; C^1(D))$.

With this we can now establish the following lemma.

**Lemma 5.** Let $\omega^{\nu} = \nabla^\perp \cdot u^{\nu} \equiv \partial_{x_1} u_2^{\nu} - \partial_{x_2} u_1^{\nu}$. Then $\omega^{\nu}$ is a classical solution of

$$
\begin{cases}
\partial_t \omega^{\nu} = \nu \Delta \omega^{\nu}, & \text{in } (0, \infty) \times D; \\
\frac{\partial \omega^{\nu}}{\partial \hat{n}}(x, t) = \frac{\alpha'(t)}{2\pi \nu} \text{ on } \{|x| = 1\} \times [0, \infty) \\
\omega^{\nu}(x, 0) = \nabla^\perp \cdot u_0(x), & \text{in } D \times \{t = 0\}.
\end{cases}
$$

(45)

Additionally, $\int_D \omega^{\nu}(x, t) \, dx = \alpha(t)$ at every $t > 0$.

**Proof.** Start by noting that, if $u_0$ is covariant under rotations about the origin, then $\omega_0$ is circularly symmetric and, hence, so is $\omega^{\nu}(\cdot, t)$; this is due to the rotational invariance of the heat equation.

The partial differential equation in (45) can be trivially deduced from (43), along with the initial condition, so that all that remains is to verify that the boundary condition is correct.

We integrate (45) in space and use the divergence theorem to obtain:

$$
\frac{d}{dt} \int_D \omega^{\nu}(x, t) \, dx = \nu \int_{\partial D} \nabla \omega^{\nu} \cdot x \, dS
$$

$$
= 2\pi \nu \frac{\partial \omega^{\nu}}{\partial \hat{n}} |\partial D|,
$$

where we used the circular symmetry of $\omega^{\nu}$ in the last step.

Next observe that $\omega^{\nu} = -\text{div} \, (u^{\nu})^\perp$, so that

$$
\int_D \omega^{\nu}(x, t) \, dx = -\int_{\partial D} (u^{\nu})^\perp \cdot n \, dS = \alpha(t),
$$

as $u^{\nu}$ is a solution of (43).

These two facts yield the desired conclusions.

□
Let us note in passing that the identity \( \int_D \omega^n(x, t) \, dx = \alpha(t) \) is valid (a.e. in time) under the much weaker assumption \( u_0 \in L^2(D) \) and \( \alpha \in BV(\mathbb{R}) \), without the compatibility conditions \( \text{4.1} \) of \([11]\), since then the solution \( u^n \in L^2((0, T); H^1(D)) \).

Next we state and prove our main theorem in which we examine the inviscid limit of \( \omega^n \).

**Theorem 6.** Let \( \alpha \in BV(\mathbb{R}) \) and assume that \( \alpha \) is compactly supported in \((0, \infty)\). Let \( u_0 \in L^2(D; \mathbb{R}^2) \) be covariant under rotations and assume that \( \omega_0 = \nabla^\perp \cdot u_0 \in L^1(D) \). Then we have:

1. There exists a constant \( C > 0 \) such that
   \[
   \int_D |\omega^n(x, t)| \, dx \leq C(\|\omega_0\|_{L^1} + \|\alpha\|_{BV}),
   \]
   for almost all time.
2. For almost every \( t \in \mathbb{R} \),
   \[
   \alpha(t) = \int_D \omega^n(x, t) \, dx.
   \]
3. We also have, for any \( 0 < a < 1 \),
   \[
   \|\omega^n - \omega_0\|_{L^a((|x| \leq a))} \to 0 \quad \text{as} \quad \nu \to 0,
   \]
   a.e. in time. If \( \omega_0 \in C^0(D) \) then the convergence is uniform in time.

Given \( \alpha \in BV(\mathbb{R}) \), we call the modified \( BV \) function \( \tilde{\alpha} \) the left-continuous “correction” of \( \alpha \).

**Proof.** We start by choosing appropriate regularizations of \( \alpha \) and of \( u_0 \). Let \( \alpha_n \) be the regularization obtained in Lemma 4.1 of \([19]\), satisfying the following properties:

1. \( \alpha_n \in C^\infty(\mathbb{R}) \);
2. \( \alpha_n \to \tilde{\alpha} \) pointwise and strongly in \( L^p_{loc}(\mathbb{R}) \), for any \( 1 \leq p < \infty \);
3. \( \alpha_n' \to \alpha' \) weak-* in \( BM(\mathbb{R}) \).

Above, the derivatives are understood in the sense of distributions. Let \( u_{0,n} \) be a sequence of vector fields in \( C^\infty_c(D) \), commuting with rotations, such that \( u_{0,n} \to u_0 \) strongly in \( L^2(D) \). Since \( \alpha \) is compactly supported in \((0, \infty)\) it follows immediately that the compatibility conditions \( \text{4.1} \) are satisfied for \( u_{0,n} \) and \( \alpha_n \). Hence, if \( \omega_n \) denotes the solution of the symmetry-reduced Navier-Stokes equations \( \text{4.1} \) with data \( u_{0,n} \) and \( \alpha_n \) then \( \omega_n = \nabla^\perp \cdot \nu^n \) is a solution of \( \text{15} \) such that

\[
\alpha_n(t) = \int_D \omega_n^\nu(\cdot; t),
\]

by Lemma \( \text{4} \).

Let us prove the estimate in item (1) for \( \omega_n^\nu \).

We will derive an energy-type estimate in \( L^1 \) after appropriately regularizing \( |\omega_n^\nu|(x, t) = sgn(\omega_n^\nu(x, t))|\omega_n^\nu(x, t)| \). We first mollify \( y \mapsto |y| \). For each \( \varepsilon > 0 \) there exists a \( C^1 \), convex function \( \phi_\varepsilon \), such that \( |\phi_\varepsilon| \) is bounded, uniformly in \( \varepsilon \), and \( \phi_\varepsilon \to |\cdot| \) as \( \varepsilon \to 0 \).

Next, we multiply the PDE in \( \text{15} \) by \( \phi_\varepsilon'(\omega_n^\nu) \) and integrate over the disk to obtain:

\[
\frac{d}{dt} \int_{|x| \leq 1} \phi_\varepsilon(\omega_n^\nu) \, dx = \nu \int_{|x| \leq 1} (\Delta[\phi_\varepsilon(\omega_n^\nu)] - \phi_\varepsilon'(\omega_n^\nu)\nabla|\omega_n^\nu|^2) \, dx.
\]
Since $\phi_\varepsilon \in C^1$ and convex, $\phi_\varepsilon''(\omega_n^\nu) \geq 0$ so that, by the divergence theorem,
\[
\frac{d}{dt} \int_{|x| \leq 1} \phi_\varepsilon(\omega_n^\nu) \, dx \leq \nu \int_{|x| \leq 1} \Delta [\phi_\varepsilon(\omega_n^\nu)] \, dx = \nu \int_{|x| = 1} \nabla [\phi_\varepsilon(\omega_n^\nu)] \cdot x \, dS(x)
\]
\[
= \frac{\alpha_n'(t)}{2\pi} \int_{|x| = 1} \phi_\varepsilon'(\omega_n^\nu) \, dS(x) \leq C |\alpha_n'(t)|,
\]
where we used the boundary condition for $\omega_n^\nu$ in (45). Then, integrating in time gives:
\[
\int_{|x| \leq 1} \phi_\varepsilon(\omega_n^\nu) \, dx \leq \int_0^t |\alpha_n'(t)| \, dt + \int_{|x| \leq 1} \phi_\varepsilon(\omega_0,n) \, dx.
\]
Observe that we can choose $\phi_\varepsilon$ in such a way that $\phi_\varepsilon(y) \rightarrow |y|$ monotonically.
Consequently, by the Monotone Convergence Theorem, passing to the limit $\varepsilon \rightarrow 0$
in the expression above gives, for every $0 < t < \infty$:
\[
(47) \quad \int_{|x| \leq 1} |\omega_n^\nu(x,t)| \, dx \leq \int_0^t |\alpha_n'(t)| \, dt + \int_{|x| \leq 1} |\omega_0,n(x)| \, dx \leq \|\alpha\|_{BV} + \|\omega_0\|_{L^1},
\]
from which we obtain item (1) for $\omega_n^\nu$.

We have shown that, for each $\nu > 0$, $\{\omega_n^\nu\}_{n=1}^\infty$ is uniformly bounded in $L^\infty(\mathbb{R}; L^1(D))$.
Thus, passing to subsequences as needed, we find that $\{\omega_n^\nu\}_{n=1}^\infty$ is weak-* $L^\infty(\mathbb{R}; BM(D))$
convergent as $n \rightarrow \infty$.

On the other hand, the properties of $\alpha_n$, together with the proofs of Proposition 4.3 and Proposition 5.1 in [43] imply that $u_n^\nu \rightarrow u^\nu$ weakly in $L^2((0,T) \times D)$, where $u^\nu$ is the solution of (43), with data $u_0$ and $\alpha$. It follows immediately that
\[
\omega_n^\nu \rightarrow \omega^\nu = \nabla^\perp \cdot u^\nu,
\]
weakly in $L^2((0,T); H^{-1}(D))$. Uniqueness of limits implies that the convergence of $\omega_n^\nu$ to $\omega^\nu$ can be improved to weak-* $L^\infty(\mathbb{R}; BM(D))$. We use again an estimate obtained in the proof of Proposition 5.1 of [43], see (5.3) there, to find that $\{\omega_n^\nu\}_n$ is bounded uniformly in $L^2((0,T) \times D)$. Hence we obtain
\[
\omega_n^\nu \rightarrow \omega^\nu \text{ weakly in } L^2((0,T) \times D).
\]

Therefore, since $L^2((0,T) \times D) \subset L^2((0,T); L^1(D))$, it follows that $\omega^\nu(\cdot,t) \in L^1(D)$ a.e. $t > 0$. Weak lower-semicontinuity of the $L^1$-norm implies that item (1) follows from (47).

Since $\alpha_n \rightarrow \alpha$ strongly in $L^2_{\text{loc}}(\mathbb{R})$ it follows that, for any $\varphi = \varphi(x) \in L^\infty(D)$ and $\psi = \psi(t) \in L^2((0,T))$ we have
\[
\int_0^T \int_D \left( \frac{\alpha_n(t)}{\pi} - \omega_n^\nu(x,t) \right) \varphi(x) \psi(t) \, dx \, dt \rightarrow \int_0^T \int_D \left( \frac{\alpha(t)}{\pi} - \omega^\nu(x,t) \right) \varphi(x) \psi(t) \, dx \, dt.
\]
Take $\varphi(x) \equiv 1$. Then, using (46), we see that the left-hand-side above vanishes identically, so that
\[
\int_0^T \int_D \left( \frac{\alpha(t)}{\pi} - \omega^\nu(x,t) \right) \psi(t) \, dx \, dt = 0,
\]
i.e.
\[
\int_0^T \left( \alpha(t) - \int_D \omega^\nu(x,t) \, dx \right) \psi(t) \, dt = 0
\]
for any $\psi \in L^2((0,T))$. Take $\psi = \psi(t) = \alpha(t) - \int_D \omega^\nu(x,t) \, dx$; this gives item (2).
Note that $\omega^\nu$ is a solution of the heat equation with viscosity $\nu$ in $D$, with initial data $\omega_0$.

To prove (3), let $\tilde{\Omega}$ be an open, compactly contained subset of $D$ and let $\phi \in C_0^\infty(D)$ be such that $\phi \equiv 1$ in a neighborhood of $\tilde{\Omega}$. We consider $v^\nu \equiv \phi \omega^\nu$, extended to the full plane. Then $v^\nu$ satisfies:

$$\partial_t v^\nu = \nu \Delta v^\nu + F^\nu,$$

with

$$F^\nu = -\nu (2 \nabla \phi \cdot \nabla \omega^\nu + \omega^\nu \Delta \phi).$$

We apply Duhamel's formula in the whole plane to obtain:

$$v^\nu = e^{\nu t} \Delta (\phi \omega_0) + \int_0^t e^{\nu (t-s) \Delta} F^\nu(s) \, ds. \quad (48)$$

We wish to estimate $v^\nu = v^\nu(x, t)$ for $x \in \tilde{\Omega}$.

Clearly, the first term converges to $\omega_0$ in $\tilde{\Omega}$, as $\nu \to 0$, in whichever space $\omega_0$ lies in. What we are left to prove is that the other term in (48) converges to zero. This follows from the fact that the heat kernel is sharply localized when $\nu \to 0$, together with the fact that the support of $F^\nu$ is bounded away from $\tilde{\Omega}$, uniformly in $\nu$, as we are only interested in $x \in \tilde{\Omega}$.

From this result we derive three conclusions. The first is that, if $\alpha \in BV(\mathbb{R})$ then $\omega^\nu \rightharpoonup W$ weak-$*$ in $L^\infty_{\text{loc}}((0, \infty); BM(D))$ and hence $W = \omega_0 + \mu$, where $\mu$ is a measure supported on the boundary of the disk $D$. This implies that, in considering the classical open problem of the inviscid limit for the Navier-Stokes equations in domains with boundary, at the very least one has to deal analytically with regularity at the level of vortex sheets, without a priori sign conditions. The second observation is that the vorticity generated by a boundary layer appears to be proportional to the acceleration of the boundary with respect to the adjoining flow. Third, we have proved that, if $\alpha$ is not constant, then the vorticity $\omega^\nu$ does not converge in $L^1$ to the vorticity of the limit flow $\omega_0$. This shows that the $L^2$ convergence of velocity fields obtained in [19] cannot be improved to convergence in derivatives. Finally, we would like to mention that, in [20], we weaken the hypothesis that $\alpha \in BV$ to $\alpha \in L^p$, for $p > 1$, and we adjust the convergence obtained accordingly. However, the vorticity accounting we have shown here only works for $\alpha \in BV$.

8. Conclusion

As a conclusion for these notes, it would be interesting to highlight a few main points of the our discussion on the mathematical theory of boundary layers.

(1) We remark on the extent of the current lack of understanding of the vanishing viscosity limit in the presence of boundaries. At this stage, the Boundary Layer Problem formulated as formulated in Section 4 is wide open.

(2) The Prandtl equation and its refinements take a magnifying glass approach to the difficulty involved in boundary layers. This approach is very interesting and useful for specific problems, but the difficulties in the well-posedness
of Prandtl’s equation do not make it very promising as an avenue for the Boundary Layer Problem.

(3) The magnifying glass approach works well for the case of Navier friction condition mainly because the boundary layer equation in this case is linear and well-posed.

(4) At best, the Boundary Layer Problem involves treating a sequence of approximate solutions to the Euler equations with vortex-sheet level regularity, which is the critical case for the weak continuity of the nonlinearity.

(5) The strength of the boundary layer as a vortex sheet, and therefore some of the difficulty in the Boundary Layer Problem appears to be associated with the acceleration of fluid past the boundary.

Finally let us put together a few problems connected with what was presented here. First, establish rigorously the ill-posedness of Prandtl’s equation without monotonicity. Second, we mention extending the $L^p$ control on the vorticity in the case of Navier friction condition to $p \leq 2$. Third, in [22], the author, together with H. Nussenzveig Lopes and Zhouping Xin introduced the notion of boundary coupled weak solution of the incompressible Euler equations, and we proved the existence of such a weak solution in the case of the half-plane. It would be interesting to know if the vanishing viscosity limit, for example, for Navier friction condition in the half-plane, gives rise to such weak solutions. Also, if vorticity is $L^p$, $p \geq 2$ the theory of renormalized solutions of DiPerna-Lions implies that weak solutions of the Euler equations conserve $p$-norms exactly, for flows in domains without boundary. Does this remain true in the case of vanishing viscosity limits in domains with boundary? Would it be possible to find an example, in the spirit of the circularly symmetric flows in Section 7, but for which there is boundary layer separation? One possibility is to look for the solution of the Navier-Stokes system on the disk, with initial vorticity given by an odd eigenfunction of the Dirichlet Laplacian. Finally, is boundary layer separation possible for flows with Navier friction condition? A related problem is to study the vanishing viscosity limit with Navier friction condition in nonsmooth domains. Another pair of problems is to extend the boundary layer analysis done in Section 6 either to $p = \infty$ or $p \leq 2$.

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**Milton C. Lopes Filho**

DEPARTAMENTO DE MATEMATICA, IMECC-UNICAMP.
CAIXA POSTAL 6065, CAMINAS, SP 13083-970, BRASIL
*E-mail address*: mlopes@ime.unicamp.br