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On some extension of Paley Wiener theorem

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Abstract: Paley Wiener theorem characterizes the class of functions which are Fourier transforms of $C^\infty$ functions of compact support on $\mathbb{R}^n$ by relating decay properties of those functions or distributions at infinity with analyticity of their Fourier transform. The theorem is already proved in classical case: the real case with holomorphic Fourier transform on $L^2(\mathbb{R})$, the case of functions with compact support on $\mathbb{R}^n$ from Hörmander and the spherical transform on semi simple Lie groups with Gangolli theorem.

Let $G$ be a locally compact unimodular group, $K$ a compact subgroup of $G$, and $\delta$ an element of unitary dual $\hat{K}$ of $K$. In this work, we’ll give an extension of Paley-Wiener theorem with respect to $\delta$, a class of unitary irreducible representation of $K$, where $G$ is either a semi-simple Lie group or a reductive Lie group with non-empty discrete series after introducing a notion of $\delta$-orbital integral. If $\delta$ is trivial and one dimensional, we obtain the classical Paley-Wiener theorem.

Keywords: delta-orbital integral; reductive Lie group; spherical Fourier transform of type delta

MSC: 43, 22D, 22E, 46Em, 46H, 47L

1 Introduction

In this section, we shall study Paley-Wiener theorem in classical case.

Let $f \in C^\infty$, then the following conditions are equivalent:

i) $f \in \text{Hol}(C^\infty)$ and $\sup_{y\in\mathbb{R}} \int_{0}^{+\infty} |f(x + iy)|^2 \, dx = C < \infty$

ii) $f \in L^2(\mathbb{R})$ and $F \equiv 0$ on $]-\infty; 0[$ and $f(z) = \int_{0}^{+\infty} F(t)e^{-itz} \, dt$, $\forall z \in \Pi^+$. 

The classical Paley–Wiener theorem uses the holomorphic Fourier transform on classes of square-integrable functions supported on $L^2(\mathbb{R})$.

Schwartz’s Paley–Wiener theorem asserts that the Fourier transform of a distribution of compact support on $\mathbb{R}^n$ is an entire function on $C^n$ and gives estimates on its growth at infinity. This theorem says that: An entire function $F$ is the Fourier–Laplace transform of a distribution of compact support if and only if $\forall z \in C^n$, there exists constants $\alpha$ and $\beta$ such that

$|F(z)| \leq \alpha(1 + |z|)^{-\beta} e^{R|Imz|}$.

Let $G$ be a connected semi-simple Lie group with finite center, $G = KAN$ an Iwasawa decomposition for $G$, where $A$ is an abelian subgroup of $G$, $N$ a nilpotent subgroup of $G$, $K$ a compact subgroup of $G$, and $K^\sharp(G)$, the convolution algebra of continuous complex functions with compact support which are bi-invariant under $K$.

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Paley-Wiener Theorem exists for spherical functions, and is called Gangolli Theorem. This theorem says that for all \( f \in K^c(G) \) which vanishes outside the ball of radius \( R \), the spherical Fourier-Laplace transform \( \hat{f} \) of \( f \) defined by the rule

\[
\hat{f}(v) = \int_G f(x) \phi_v(x) d_G(x), \quad (\phi_v \in S(G \setminus K), \text{ the set of spherical functions on } G \text{ with respect to } K),
\]

exists for all \( v = \xi + i\eta \in \mathfrak{g}_c \) and is a \( W \)-invariant entire holomorphic function of \( v (v \in \mathfrak{g}_c) \), where \( W \) is the Weyl group, \( \mathfrak{g}_c \) the complexification of \( \mathfrak{g} \), and \( \mathfrak{g} \) the real dual of \( A \). \( A \) is the Lie algebra of \( A \) in the Iwasawa decomposition. Moreover, given any integer \( \beta \geq 0 \), there exists a constant \( C_\beta > 0 \) such that

\[
|\hat{f}(\xi + i\eta)| \leq C_\beta (1 + ||\xi + i\eta||)^{-\beta} e^{R||\eta||}, \quad \xi, \eta \in \mathfrak{g}.
\]

Conversely, if \( F \) is a \( W \)-invariant entire holomorphic function on \( \mathfrak{g}_c \) with the property that there exists an \( R > 0 \) such that for any integer \( \beta > 0 \), there exists a constant \( C_\beta > 0 \) for which

\[
|F(\xi + i\eta)| \leq C_\beta (1 + ||\xi + i\eta||)^{-\beta} e^{R||\eta||}, \quad \xi, \eta \in \mathfrak{g},
\]

then there exists a unique function \( f \in K(G) \), the algebra of continuous functions with compact support, such that \( \hat{f} = F \); \( f \) vanishes outside the ball of radius \( R \) in \( A \) and is given by the formula

\[
f(x) = [W]^{-1} \int_{\mathfrak{g}} F(v) \phi_v(x) c(v)^{-1} d_{\mathfrak{g}}(v), \quad x \in G,
\]

where \( c(v) \) is the Harish-Chandra’s constant.

Let \( \delta \) be a class of unitary irreducible representations of a compact subgroup \( K \) of \( G \). Our goal is to establish and to prove the Paley-Wiener theorem with respect to the unitary dual of a compact group, in other words, to characterize functions which are Fourier transforms of type \( \delta \) of some class of functions. As seen above, this result exists where \( \delta \) is trivial and one dimensional. We’ll start by defining generalized Abel according to \( \delta \).

### 2 Preliminaries

Let \( G \) be a locally compact unimodular group, \( K \) a compact subgroup of \( G \), \( \hat{K} \) the set of all equivalence classes of irreducible unitary representations of \( K \).

For all class \( \delta \) of \( \hat{K} \), we denote by \( \xi_\delta \) the character of \( \delta \), \( d(\delta) \) the degree of \( \delta \) and \( \chi_\delta = \chi(\delta) \xi_\delta \).

If \( \hat{\delta} \) is the class of contragredient representations of \( \delta \) in \( \hat{K} \), we have \( \chi_\delta = \chi_{\hat{\delta}} \) and thanks to the Schur Orthogonality relations, we can check that \( \chi_\delta * \chi_{\hat{\delta}} = 1 \). For all function \( f \in K(G) \), the algebra of continuous functions with compact support, we set

\[
\delta f(x) = \chi_\delta * f(x) = \int_K \chi_\delta(k) f(kx) dk
\]

\[
f_\delta(x) = f * \chi_{\hat{\delta}}(x) = \int_K \chi_{\hat{\delta}}(k^{-1} f(kx) dk,
\]

(\text{where } dk \text{ is the normalized Haar measure on } K), \text{ and}

\[
K_\delta(G) = \{ f \in K(G), f = \delta f = f_\delta \}.
\]

\( K_\delta(G) \) is a subalgebra of \( K(G) \) and the map \( \chi_{\hat{\delta}} * f * \chi_{\hat{\delta}} \) is a projection of \( K(G) \) onto \( K_\delta(G) \).
Consider a Banach representation $U$ of $G$ on a Banach space $E$. Set $P(\delta) = U(\bar{\delta})$ et $E(\delta) = P(\delta)E$, $E(\delta)$ the closed subspace of $E$ consisting of those vectors in $E$ which transform under $K$ according to $\delta$. If $g = \bar{\delta} * f * \delta$, we have $P(\delta)U(f)P(\delta) = U(g)$, $\forall f \in K(G)$; Thus, $E(\delta)$ is $U(f)$-stable, $f \in K_0(G)$. Let $U_\delta(f)$ denote the restriction of $U(f)$ to $E(\delta)$; then we obtain a representation $f \mapsto U_\delta(f)$ of $K_0(G)$ onto $E(\delta)$. Let $\mathcal{J}_c(G)$ the set of all functions $f$ of $K(G)$ which are $K$-central (i.e. $f(kx) = f(xk)$ $\forall k \in K$ and $x \in G$).

$\mathcal{J}_c(G)$ is a subalgebra of $K(G)$ and the map $f \mapsto f_K$, with $f_K(x) = \int f(kx^{-1})dk$, is a projection of $K(G)$ onto $\mathcal{J}_c(G)$. For all $f, g \in K(G)$, we have the following properties

$$(fK * g)_K = f_K * g_K = (f * g_K)_K \text{ and } (\bar{\delta} * f)_K = \bar{\delta} * f_K = (f * \bar{\delta})_K = f_K * \bar{\delta}.$$  

We set

$$K_0^1(G) = K_0(G) \cap \mathcal{J}_c(G).$$

If $K$ is a compact subgroup of $G$ and $U$ a topological completely irreducible Banach representation of $G$ on $E$, then the set of all operators $U_\delta(f)$, $(f \in K_0^1(G))$ is the centralizer of representation $k \mapsto U_\delta(k)$ of $K$ onto $E(\delta)$.

If we decompose the representation $k \mapsto U_\delta(k)$ of $K$ onto $E(\delta)$ to obtain $m$ irreducible equivalent representation, then the centralizer is isomorphic to the algebra $M_m(\mathbb{C})$ of square matrices of order $m$. Thus, there exists an isomorphism $U_\delta(f) \mapsto u_\delta(f)$ of algebra $(U_\delta(f), f \in K_0^1(G))$ onto $M_m(\mathbb{C})$ where $f \mapsto u_\delta(f)$ is an $m$-dimensional irreducible representation of $K_0^1(G)$ and

$$Tr(U_\delta(f)) = d(\delta)tr(u_\delta(f)) \forall f \in K_0^1(G).$$

The algebra $K_0^1(G)$ is isomorphic to algebra $U_{c,\delta}(G)$ of continuous functions $\psi$ with compact support of $G$ onto $F_\delta = Hom_C(E_\delta, E_\delta)$ and which verify the relation

$$\psi(k_1xk_2) = u_\delta(k_1)\psi(x)u_\delta(k_2).$$

A seminorm $\gamma$ on $G$ is a positive lower semicontinuous and bounded function on every compact of $G$ such that

$$\gamma(xy) \leq \gamma(x)\gamma(y) \quad (x, y \in G).$$

A function $f$ on $G$ with values in a Banach space is called quasi-bounded function if there exists a seminorm $\gamma$ on $G$ such that

$$\sup_{x \in G} \frac{||f(x)||}{\gamma(x)} < \infty.$$

A spherical function $\phi$ (on $G$) of type $\delta$ is a quasi-bounded continuous function on $G$ with values in $End_C(E)$, ($E$ is a finite dimensional vector space) such that

i) $\phi(kx^{-1}) = \phi(x)$

ii) $\bar{\delta} * \phi = \phi = \phi * \bar{\delta}$

iii) The map $u_\phi : f \mapsto \phi(f)$ is $\int f(x)\phi(x)dx$ is an irreducible representation of algebra $K_0^1(G)$.

If $\phi$ is a quasi-bounded continuous function on $G$ with values in $End_C(E)$ such that

$$\phi_K = \phi \text{ et } \bar{\delta} * \phi = \phi.$$
Then the function $\phi$ is spherical function of type $\delta$ iff

$$\int_K \phi(kxk^{-1}y)dk = \phi(x)\phi(y), \forall x, y \in G.$$ 

As well, if $U$ is an irreducible Banach representation of $G$ over a space $E$ such that $\delta$ occurs $m$ times in the restriction of $U$ to $K$, then there exists a function $\phi_U^\delta$ defined on $G$ which is spherical of type $\delta$. The function $\phi_U^\delta$ is said to be associated to the representation $U$.

Let $B$ be a commutative Banach algebra with identity element $e$. Let $B$ be an involutive normed complex algebra and $X_m(B)$ the set of all $m$-dimensional irreducible unitary representations of $B$.

For all $f$ in $B$, a generalized Gelfand transform of $f$ is a map denoted by $\mathcal{G}f$ of $X_m(B)$ onto the algebra $M_m(\mathbb{C})$ of square matrices of order $m$ which verifies

$$\mathcal{G}f: \ X_m(B) \rightarrow M_m(\mathbb{C}) \quad u \mapsto u(f)$$

The homomorphism $f \mapsto \mathcal{G}f$ of $B$ onto $M_m(\mathbb{C})^{X_m(B)}$ is called the generalized Gelfand transformation associated to $B$. If the algebra $B$ is commutative, then irreducible unitary representations of $B$ are one-dimensional and thus we identify them with characters of $B$. We get the usual definition of Gelfand transformation.

Let $\mathcal{S}_{m,\delta}(G)$ be the set of all spherical functions of type $\delta$ on $G$ and height $m$; thus, if $\phi$ is a function of $\mathcal{S}_{m,\delta}(G)$, there exists a representation $u_\phi^\delta \in X_m(K_\delta^\#(G))$ such that $u_\phi^\delta(f) = \int_G f(x)\phi(x)$ and conversely. This result allows us to identify spaces $\mathcal{S}_{m,\delta}(G)$ with $X_m(K_\delta^\#(G))$ and then we can define spherical Fourier transform of type $\delta$, $\mathcal{F}f$, accordingly the following diagram

![Diagram](image)

**Figure 1:** Transformée de Fourier Sphérique de type $\delta$

## 3 A Paley-Wiener theorem with respect to the unitary dual of a compact group

### 3.1 A Paley-Wiener on semi-simple Lie groups

Let $G$ be a connected semi-simple Lie group with finite center, $G = KAN$ an Iwasawa decomposition for $G$, where $A$ is an abelian subgroup of $G$, $N$ a nilpotent subgroup of $G$, $K$ a compact subgroup of $G$. Let’s normalize the Haar measure

$$d_G(x) = h^2 d_k(h)d_A(h)d_N(n), x \in G, x = khn \in KAN,$$
where $\rho$ is one-half the sum of the positive roots associated to Lie algebra of $G$.

Let $\delta \in \hat{K}$ and $\mu_\delta \in \delta$, an unitary irreducible representation of $K$ onto the hilbert space $E_\delta$. For every $f$ dans $K_\delta^G(G)$, let’s consider the integral defined by

$$F^\delta_f(h) = \frac{h^\delta}{h} \int_k \int_N f(khn)\mu_\delta(k^{-1})d_N(n)dk, \ h \in A.$$

We shall call the map $f \mapsto F^\delta_f$ the Abel transformation on $G$ of type $\delta$. We know that $K_\delta^G(G)$ is isomorphic to $U_{c,\delta}(G)$ under the map $f \mapsto \psi^\delta_f$ defined by $\psi^\delta_f(x) = \int_k \mu_\delta(k^{-1})f(kx)dk$. Then, for every $f$ in $K_\delta^G(G)$, we obtain that

$$F^\delta_f(h) = \frac{h^\delta}{h} \int_N \psi^\delta_f(hn)d_N(n), \ h \in A.$$

The Abel transformation is linear and one-to-one mapping of the algebra $K_\delta^G(G)$ onto $K_\delta^G(A)$. Let $N_K(A)$ and $Z_K(A)$, be respectively normalizer and centralizer of $A$ in $K$. The Weyl group of the pair $(G, A)$ can be identified with quotient $N_K(A)/Z_K(A)$. We say that a function $f$ is $W$-invariante if, $\forall w \in W, w f(x) = f(w^{-1}x) = f(x)$.

**Theorem 3.1.** Let $G$ be a connected semi-simple Lie group with finite center, and $K\Lambda N$ its Iwasawa decomposition. For all $f \in K_\delta^G(G, End_C(E))$ which vanishes outside the ball of radius $R$ in $A$, the spherical Fourier-Laplace transform of type $\delta$, $\hat{f}_\delta$ of $f$ defined by the rule

$$\hat{f}_\delta(v) = \int_G f(x)\phi_{\nu,\delta}(x)d_G(x), \ \phi_{\nu,\delta} \in S_{\nu,\delta}(G) \text{ with values in } End_C(E),$$

exists for all $v = \xi + i\eta \in \mathfrak{g}_c$ and is a $W$-invariant entire holomorphic function of $v$ ($v \in \mathfrak{g}_c$), where $W$ is the Weyl group, $\mathfrak{g}_c$ the complexification of $\mathfrak{g}$, and $\mathfrak{g}$ the real dual of $A$. Moreover, given any integer $\beta \geq 0$, there exists a constant $C_{\beta,\delta} > 0$ such that

$$\|\hat{f}_\delta(\xi + i\eta)\| \leq C_{\beta,\delta}(1 + ||\xi + i\eta||)^{-\beta}e^{\beta||\eta||}, \ \xi, \eta \in \mathfrak{g}.$$

Conversely, if $F$ is a $W$-invariant entire holomorphic function on $\mathfrak{g}_c$ with the property that there exists an $R > 0$ such that for any integer $\beta > 0$, there exists a constant $C_{\beta,\delta} > 0$ for which

$$\|F(\xi + i\eta)\| \leq C_{\beta,\delta}(1 + ||\xi + i\eta||)^{-\beta}e^{\beta||\eta||}, \ \xi, \eta \in \mathfrak{g}$$

then there exists a unique function $f \in K_\delta^G(G, End_C(E))$ such that $\hat{f}_\delta = F; f$ vanishes outside the ball of radius $R$ in $A$ and is given by the formula

$$f(x) = [W]^{-1} \int_{\mathfrak{g}} F(v)\overline{\phi_{\nu,\delta}(x)}c(v)|^2d_{\mathfrak{g}}(v), \ x \in G$$

where $c(v)$ is the Harish-Chandra’s constant.

**Proof.** (1) Suppose that $f$ is an element of $K_\delta^G(G, End_C(E))$ which vanishes outside the ball of radius $R$ in $A$. The spherical Fourier-Laplace transform of type $\delta$, $\hat{f}_\delta$ of $f$ defined by the rule

$$\hat{f}_\delta(v) = \int_G f(x)\phi_{\nu,\delta}(x)d_G(x), \ \phi_{\nu,\delta} \in S_{\nu,\delta}(G),$$

exists for all $v = \xi + i\eta \in \mathfrak{g}_c$ and is a $W$-invariant entire holomorphic function of $v$ ($v \in \mathfrak{g}_c$). Also, we see that the Abel transform $F^\delta_f$ of $f$ has support in the ball of radius $R$ in $A$; Then, classical Paley-Wiener Theorem tells
us that the Euclidean Fourier transform $\hat{F}_f$ of $F_f$ is holomorphic on $\mathfrak{g}$ and has the property that for any given integer $\beta \geq 0$, there exists a constant $C_{\beta, \delta} > 0$ such that

$$| \hat{F}_f(\xi + i\eta) | \leq C_{\beta, \delta}(1 + ||\xi + i\eta||)^\beta e^{R||\eta||}, \quad \xi, \eta \in \mathfrak{g}.$$

The result follows from the fact that $\hat{F}_f = \hat{\delta}$, indeed

$$\hat{\delta}(\nu) = \int_G f(x) \phi_{\nu, \delta}(x) d\nu(x) = \int_A \int_N \int_k (\frac{khn}{\nu}) \phi_{\nu, \delta}(hn)h^{2\nu} dk dh dn = \int_A \int f(hn) \phi_{\nu, \delta}(hn) h^{2\nu} dh dn$$

We have the following relation (1)

$$\int_G f(x) \phi_{\nu, \delta}(x) d\nu(x) = \int_A h^{\nu} F_{\delta}^\nu(h) d\nu(h), \quad \phi_{\nu, \delta} \in \mathcal{S}_{p, \delta}(G). \quad (1)$$

The algebra $K_0^\delta(G, \text{End}_C(E))$ is isomorphic to algebra $U_{C, \delta}(G)$ of continuous functions with compact support of $G$ in $F_\delta$ and which are $u_\delta$-spherical on $G$. The isomorphism is defined by $f \mapsto \psi_f^\delta$, with $\psi_f^\delta(x) = \int_K u_\delta(k^{-1}) f(kx) dk$. Moreover, if $\delta$ is trivial and one dimensional, we have $\phi(x) = \int_K e^{i(\nu - \rho)H(xk)} dk$, and the relation (1)

$$\int_G (\int_K u_\delta(k^{-1}) f(kx) dk) \int_K e^{i(\nu - \rho)H(xk)} dk dx = \int_A h^{\nu} F_{\delta}^\nu(h) d\nu(h)$$

$$= \int_A \int_K (\int_K u_\delta(k^{-1}) f(kx) e^{i(\nu - \rho)H(xk)} dk) dx = \int_A h^{\nu} F_{\delta}^\nu(h) d\nu(h)$$

$$= \int_G f(x) (\int_K u_\delta(k^{-1}) e^{i(\nu - \rho)H(xk)} dk) dx = \int_A h^{\nu} F_{\delta}^\nu(h) d\nu(h)$$

Thus, with identification, $\phi_{\nu, \delta}(x) = \int_K \psi_f^\delta(x) e^{i(\nu - \rho)H(xk)} dk$ so :

$$\hat{\phi}_\delta(v) = \int_A \int_N \int_K \int f(khn) u_\delta(k^{-1}) e^{i(\nu - \rho)H(xk)} h^{2\nu} dk dh dn$$

$$= \int_A \int_N \int_K \int f(khn) u_\delta(k^{-1}) e^{i(\nu)H(xk)} h^{2\nu} dk dh dn$$

$$= \int_A \int_N \int_K \int f(khn) u_\delta(k^{-1}) e^{i(\nu)H(xk)} h^{2\nu} h^{2\nu} dk dh dn$$

$$= \int_A \int_N \int_K e^{i(\nu)H(xk)} f(khn) u_\delta(k^{-1}) dk dh dn$$

But $\psi_f^\delta(hn) = \int_K u_\delta(k^{-1}) f(khn) dk$. Thus, We have

$$\hat{\phi}_\delta(v) = h^\rho \int_A \int_N e^{i(\nu)H(hk)} \psi_f^\delta(hn) dh dn$$

$$= \int_A e^{i(\nu)H(hk)} F_\delta^\nu(h) dh$$

$$= \hat{\delta}_f(h)$$
We have the result.

(2) Suppose now that $F$ is a $W$-invariant entire holomorphic function on $\mathfrak{g}_c$ with the property that there exists an $R > 0$, such that for any integer $\beta > 0$ there exists a constant $C_{R, \beta} > 0$ for which:

$$|F(\xi + i\eta)| \leq C_{R, \beta}(1 + ||\xi + i\eta||^2 e^{R||\eta||}, \xi, \eta \in \mathfrak{g}$$

Define a function $f \in K_0^2(G, \text{End}_C(E))$ by

$$f(x) = [W]^{-1} \int_{\mathfrak{g}} F(v)\phi_{x, \delta}(v) \mathcal{C}(v)^{-2} d\gamma(v), \ x \in G$$

$f \in K_0^2(G, \text{End}_C(E))$ and $f$ vanishes outside the ball of radius $R$ in $A$. Indeed:

$$f(kxk^{-1}) = [W]^{-1} \int_{\mathfrak{g}} F(v)\phi_{x, \delta}(kxk^{-1}) \mathcal{C}(v)^{-2} d\gamma(v) = f(x), \ x \in G,$$

because $\phi_{x, \delta} \in \mathfrak{z}_c(G)$. Moreover, $K$ is unimodular, we also have

$$\tilde{\chi}_\delta * f = f.$$

And, by analytic continuation $\tilde{f} = F$ on $\mathfrak{g}_c$.

It remains only to show that $f$ is unique; Thus, suppose that $g$ a function which verify the asserted properties above, then

$$\tilde{f} = \tilde{g} \Rightarrow \tilde{F}_f = \tilde{F}_g \Rightarrow F_f = F_g \Rightarrow f = g$$

Because the is one-to-one of $K_0^2(G)$ onto $K_0^2(A)$.

\[\square\]

### 3.2 A Paley-Wiener theorem on reductive Lie groups

Let $G$ be a reductive Lie group with non-empty discrete series, $\mathfrak{g}$ the Lie algebra of $G$; $J$ a Cartan compact subgroup of $G$; $\mathfrak{u}$ an open subset of $G$ which is completely invariant, $\gamma$ a regular element of $\mathfrak{u}$ ($\gamma \in \mathfrak{u}_{\text{reg}}$).

One denotes by $f^0$ the identity component of $J$ and $K = \gamma f^0$ is Cartan compact subgroup of $G$, $X$ the Lie algebra of $K$, and the Weyl group $W(G; K)$ acts on $G/K \times K_{\text{reg}}, dg$ a measure of $G$ which is invariant on $G/K$.

As seen previously, the algebra $K_0^2(\mathfrak{u})$ is isomorphic to the algebra $U_{c, \delta}(\mathfrak{u})$ of continuous functions with compact support $\psi$ of $\mathfrak{u}$ onto $F_\delta = \text{Hom}_C(E_\delta, E_\delta)$ for any double Banach representation $(u_\delta, u_\delta)$ of $K$.

Indeed, $\forall f \in K_0^2(\mathfrak{u})$, for $\psi_f^\delta(x) = \int_k u_\delta(k^{-1})f(kx)dk$, we have $\psi_f^\delta(x) \in U_{c, \delta}(\mathfrak{u})$. The isomorphism is defined by the $f \mapsto \psi_f^\delta$.

The classical orbital integral on a reductive Lie group is defined by

$$J_\mathfrak{g}(f)(\gamma) = |\text{det}(1 - Ad(\gamma^{-1}))|_{\mathfrak{g}/\mathfrak{x}}^{-1/2} \int_{G/K} \mathcal{F}(\mathfrak{g})d\gamma, \ f \in D(\mathfrak{u}_{\text{reg}})$$

Thanks to the isomorphism of $K_0^2(\mathfrak{u})$ onto $U_{c, \delta}(\mathfrak{u})$, we set

$$J_{\mathfrak{g}}^\delta(f)(\gamma) = |\text{det}(1 - Ad(\gamma^{-1}))|_{\mathfrak{g}/\mathfrak{x}}^{-1/2} \int_{G/K} \psi_f^\delta(x)\mathcal{F}(\mathfrak{g})d\gamma$$

The $\delta$-orbital integral $J_{\mathfrak{g}}^\delta(f)(\gamma)$ is defined by

$$J_{\mathfrak{g}}^\delta(f)(\gamma) = |\text{det}(1 - Ad(\gamma^{-1}))|_{\mathfrak{g}/\mathfrak{x}}^{-1/2} \int_{K} \mu_\delta(kf(k)^\delta)dk$$
If $\delta$ is trivial and one dimensional, we obtain the classical orbital integral.

We denote by $\mathcal{S}_{p,\delta}$ the set of spherical functions of type $\delta$ and height $p$. Let $H$ be the factor of Lagrange decomposition of $G$ which contains $K$, $(G = K.H)$ and $\mathcal{Y}$ the Lie algebra of $H$. For all $\phi \in \mathcal{S}_{p,\delta}$, $\exists u_\delta \in \hat{K}$ and $\nu \in \mathcal{H}_c^\infty$ such that

$$\phi(kh) = \int_K u_\delta(k_1 k)(k_1 h)^\nu dk_1$$

Thus, we can identify $\mathcal{S}_{p,\delta}$ with $\hat{K} \times \mathcal{H}_c^\infty$. For any $\phi \in \mathcal{S}_{p,\delta}(G)$, $\exists (u_\delta, \nu) \in \hat{K} \times \mathcal{H}_c^\infty$ and we set $||(u_\delta, \nu)|| = ||u_\delta|| + ||\nu||$.

We denote by $\rho_\Delta$ one-half the sum of the positive roots associated to Lie algebra of $G$, $\kappa$ the signature of the Weyl group $W$ of $(G, K)$, and $\Delta$ a positive root system.

If $m$ is a linear combination of roots of $K$, we denote by $\xi_\mu$ the corresponding character of $K$.

For all $r > 0$, we denote by $\mathcal{P}W_\delta(K)$, the set of all functions $F: \mathcal{S}_{p,\delta} \rightarrow End(E)$ such that

1. $\nu \rightarrow F(u_\delta, \nu)$ is holomorphic on $\hat{\mathcal{C}}$
2. $\forall \beta \in \mathbb{N}, \exists C_\beta$ such that $|F(u_\delta, \nu)| \leq C_\beta (1 + ||u_\delta|| + ||\nu||)^\beta e^{\rho_\Delta ||\nu||}$
3. $F(w.(u_\delta, \nu)\xi_{w,\rho_\Delta-\rho_\Delta}) = \kappa(w)F(u_\delta, \nu)$, $w \in W$.

Let put $\mathcal{P}W_\delta(K) = \bigcup_{r > 0} \mathcal{P}W_\delta(K)_r$. We equip $\mathcal{P}W_\delta(K)_r$ with the topology defined by norms $S_\delta(F) = \sup_{\delta, \nu} (1 + ||u_\delta|| + ||\nu||)^k |F(u_\delta, \nu)|$

We denote by $\mathcal{F}^\infty_c(G)_\delta$, the subspace of $K^2_\delta(G)$ of infinitely differentiable functions.

**Theorem 3.2.**

$$\mathcal{F}(\mathcal{F}^\infty_c(G)_\delta^A) = \mathcal{P}W_\delta(K)^A$$

where the map $f \mapsto \mathcal{F}f$ is the spherical Fourier transform of type $\delta$.

$\mathcal{F}^\infty_c(G)_\delta^A$ is the set of all functions of $\mathcal{F}^\infty_c(G)$ such that,

$$f(g^{-1}kg) = \kappa(g)\xi_{-\rho_\Delta-\rho_\Delta}f(k), \forall f \in \mathcal{F}^\infty_c(G), \forall k \in K_{reg}$$

**Proof.** $\mathcal{P}W_\delta(K)$ is a close subspace of $\mathcal{P}W(K, End(E))$, then $\mathcal{P}W_\delta(K)$ is a Frechet space. Let put $Z_{\delta, p}(K)$ the set of all continuous functions of positive type $\phi$ of $\mathcal{S}_{p,\delta}(K)$. There exists $z \in (End(E))^*$ such that $\forall c_1, c_2, \ldots, c_n \in \hat{\mathcal{C}}, x_i \in G$ and $i \in \{1, 2, \ldots, n\}$:

$$\langle \sum c_i \phi(x_i x_i^{-1}), z \rangle$$

Thanks to the Bochner theorem, there exist a measure $\hat{\mu}$ such that

$$f(kg) = \int_{Z_{\delta, p}(K)} F(u_\delta, \nu)(u_\delta, \nu)(kg) d\hat{\mu}(u_\delta, \nu),$$
∀ \mathcal{F} \in \mathcal{P} \mathcal{W}_\delta(K), we have \mathcal{F}f = F.

\[ f_K(x) = \int_{\mathcal{Z}_{\mathcal{A},\mathcal{P}}(K)} F(u_\delta, \nu)(u_\delta, \nu)_K(x) \, d\mu(u_\delta, \nu), \]

\[ f^* \chi_\delta(x) = \int_{\mathcal{Z}_{\mathcal{A},\mathcal{P}}(K)} F(u_\delta, \nu)(u_\delta, \nu)^* \chi_\delta(x) \, d\mu(u_\delta, \nu), \]

because \((u_\delta, \nu) \in \mathcal{Z}_{\mathcal{A},\mathcal{P}}(K)\) then \(f \in I^{\infty}_{c,\delta}(G)\).

If \(\mathcal{F} \in \mathcal{P} \mathcal{W}_\delta(K)^d\), for \(w \in \mathcal{W}_C\), we have \(F(g, (u_\delta, \nu)\xi_\delta, \rho_\mathcal{P} - \rho_\mathcal{A}) = \kappa(g)F(u_\delta, \nu), g \in G\).

\[ f(g, h) = \int_{\mathcal{Z}_{\mathcal{A},\mathcal{P}}(K)} F(u_\delta, \nu)(u_\delta, \nu, \nu)(g, h) \, d\mu(u_\delta, \nu), \]

\[ = \int_{\mathcal{Z}_{\mathcal{A},\mathcal{P}}(K)} F(g, (u_\delta, \nu))(u_\delta, \nu, \nu)(h) \, d\mu(u_\delta, \nu), \]

\[ = \int_{\mathcal{Z}_{\mathcal{A},\mathcal{P}}(K)} \kappa(g)\xi_\delta \rho_\mathcal{P} - \rho_\mathcal{A}(h)F(u_\delta, \nu)(u_\delta, \nu, \nu)(h) \, d\mu(u_\delta, \nu), \]

\[ = \kappa(g)\xi_\delta \rho_\mathcal{P} - \rho_\mathcal{A}(h)f(h), \]

Then \(f \in I^{\infty}_{c,\delta}(G)^d\). That prove that this map is surjective.

Let fix some conjugacy classes of Cartan subgroup of \(G, K_1, K_2, \ldots, K_k \in \text{Car}(G)\), and \(\Delta_1, \Delta_2, \ldots, \Delta_k\) the corresponding positive root system. Then \(\forall f \in I^{\infty}_{c,\delta}(G)\), we have

\[ \mathcal{F}(f)_{K_i} \in \mathcal{P} \mathcal{W}_\delta(K)^d. \]

**Theorem 3.3.** The map \(F\) given by

\[ F: I^{\infty}_{c,\delta}(G) \rightarrow \bigoplus_{1 \leq i \leq k} \mathcal{P} \mathcal{W}_\delta(K)^d, \]

\[ f \mapsto \sum_{i=1}^k \mathcal{F}f_{K_i}, \]

is surjective.

**Proof.** Let put \(F = \sum F_i \in \bigoplus_{1 \leq i \leq k} \mathcal{P} \mathcal{W}_\delta(K)^d\). Thanks to theorem 3.2, there exists \(\phi_i \in I^{\infty}_{c,\delta}(K_i)^d\) such that \(\phi_i = F_i\).

Let put \(c_{G, K_i}(\gamma) = |K_i/Z(G, \gamma K_i)|\). The function \(c_{G, K_i} \in I^{\infty}_{c,\delta}(K_i)^d\) and there exists \(f \in I^{\infty}_{c,\delta}(G)\) such that

\[ b_\Delta I_c(f)_{K_i} = c_{G, K_i}\phi_i, \]

where \(b_\Delta\) is the projection of \(I_{c,\delta}(G)\) onto \(I_{c,\delta}(G)^d\) defined by \(b_\Delta(\gamma) = \prod_{a \in \mathcal{A}} \frac{1 - \xi_a(\gamma^{-1})}{|1 - \xi_a(\gamma^{-1})|}\), and we have

\[ c_{G, K_i}^{-1}(b_\Delta I_c(f)_{K_i}) = \mathcal{F}f_{K_i}. \]

That prove that \(F\) is surjective.

\[ \square \]

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