WEAK APPROXIMATION OVER FUNCTION FIELDS

by
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Abstract. — We prove that rationally connected varieties over the function field of a complex curve satisfy weak approximation for places of good reduction.

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1. Introduction

Let $F$ be a number field and $X$ an algebraic variety over $F$. Does there exist an $F$-rational point on $X$? If so, are they ubiquitous on $X$? For many classes of varieties, such problems are analyzed using local-to-global principles. The Hasse principle says that $X$ has an $F$-rational point provided it has a rational point over each completion of $F$. The principle of weak approximation says that, given a finite collection of places of $F$ together with a point of $X$ over each of the corresponding completions, there exists an $F$-rational point approximating these arbitrarily closely.
The impetus for this paper was the following result by Graber, Harris and Starr:

**Theorem 1 (8, Theorem 1.2).** — *Let $F$ be the function field of a smooth curve over $\mathbb{C}$. Every proper rationally connected variety $X$ over $F$ has an $F$-rational point.*

An algebraic variety is rationally connected if any two points can be joined by a rational curve (see Section 3.1 for more details). Rational and unirational varieties are rationally connected. We refer the reader to [9] for related results in positive characteristic.

From an arithmetic viewpoint, proving such a theorem entails surmounting two obstacles: First one needs to show that there are no obstructions to the existence of a local point, i.e., $X(F_\nu) \neq \emptyset$ for all completions $F_\nu$ of $F$. If $B$ is the smooth projective curve with $F = \mathbb{C}(B)$ and $\pi : \mathcal{X} \to B$ is a model for $X$ over $B$, one has to show there exist local analytic sections of $\pi$ at each point of $B$. Secondly, one has to prove the Hasse principle for $X$ over $F$, which entails constructing a global section of $\pi$. Note that over a number field, the Hasse principle may fail even for cubic surfaces.

Theorem 1 naturally leads one to ask whether rationally connected varieties over $\mathbb{C}(B)$ satisfy weak approximation as well. In this paper we prove this away from singular fibers of $\pi$, i.e., away from the places of bad reduction:

**Theorem 2.** — *Let $X$ be a smooth, proper, rationally connected variety over the function field of a curve over $\mathbb{C}$. Then $X$ satisfies weak approximation for places of good reduction.*

Theorem 1 and Theorem 2.13 of [11] give the zeroth-order case: There exists a section of $\pi : \mathcal{X} \to B$ passing through arbitrary points of smooth fibers.

For varieties over function fields of curves, weak approximation is satisfied in the following cases [7]:

– stably rational varieties;
– connected linear algebraic groups and homogeneous spaces for these groups;
– homogeneous space fibrations over varieties that satisfy weak approximation, for example, conic bundles over rational varieties;
– Del Pezzo surfaces of degree at least four.

Weak approximation is not known for general cubic surfaces. Madore has a manuscript addressing weak approximation for cubic surfaces away from places of bad reduction.

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2. Basic properties of weak approximation

2.1. Definition. — Let \( F \) be a number field or a function field of a smooth projective curve \( B \) over an algebraically closed ground field \( k \) of characteristic zero. For each place \( \nu \) of \( F \), let \( F_\nu \) denote the \( \nu \)-adic completion of \( F \). Let \( X \) be an algebraic variety of dimension \( d \) over \( F \); in this paper, all varieties are assumed to be geometrically integral. Let \( X(F) \) denote the set of \( F \)-rational points of \( X \). One says that rational points on \( X \) satisfy weak approximation if, for any finite set of places \( \{\nu_i\}_{i \in I} \) of \( F \) and \( \nu_i \)-adic open subsets \( U_i \subset X(F_{\nu_i}) \), there is a rational point \( x \in X(F) \) such that its image in each \( X(F_{\nu_i}) \) is contained in \( U_i \). In particular, for any collection of \( x_i \in X(F_{\nu_i}), i \in I \), there exists an \( x \in X(F) \) arbitrarily close to each \( x_i \).

It is well known that weak approximation is a birational property: If \( X_1 \) and \( X_2 \) are smooth varieties birational over \( F \) then \( X_1 \) satisfies weak approximation if and only if \( X_2 \) satisfies weak approximation. Given a smooth proper variety \( X_1 \), after applying Chow’s lemma and resolution of singularities we obtain a smooth projective variety \( X_2 \) birational to \( X_1 \). Thus in proving weak approximation, it usually suffices to consider projective varieties. In particular, Theorem 2 reduces to this case.
For the rest of this paper we restrict our attention to the function field case. Places $\nu$ of $F$ correspond to points $b$ on $B$. We also assume that $X$ is projective, so it admits a projective model

$$\pi : \mathcal{X} \to B,$$

i.e., a flat projective morphism with generic fiber $X$; for each $b \in B$, the fiber over $b$ is denoted

$$X_b = \mathcal{X} \times_B \text{Spec}(O_{B,b}/m_{B,b}).$$

Sections of $\pi$ yield $F$-valued points of $X$ and conversely, each $F$-valued point of $X$ extends to a section of $\pi$. Let $\hat{B}_b$ denote the completion of $B$ at $b$ and $(\hat{O}_{B,b}, \hat{m}_{B,b})$ the associated complete local ring, which has fraction field $F_\nu$. Restricting to this formal neighborhood of $b$ gives

$$\xymatrix{ \mathcal{X} \times_B \hat{B}_b \ar[r] \ar[d]_{\hat{\pi}_b} & \mathcal{X} \ar[d]_{\pi} \\
\hat{B}_b \ar[r] & B },$$

Sections of $\hat{\pi}_b$ restrict to $F_\nu$-valued points of $X$ and conversely, each $F_\nu$-valued point of $X$ extends to a section of $\hat{\pi}_b$. Basic $\nu$-adic open subsets of $X(F_\nu)$ consist of those sections of $\hat{\pi}_b$ which agree with a given section over $\text{Spec}(\hat{O}_{B,b}/\hat{m}_{B,b}^{N+1}) \subset \hat{B}_b$, for some $N \in \mathbb{N}$. Weak approximation means that for any finite set of points $\{b_i\} \in I$ in $B$, sections $\hat{s}_i$ of $\hat{\pi}_b$, and $N \in \mathbb{N}$, there exists a section $s$ of $\pi$ agreeing with $\hat{s}_i$ modulo $\hat{m}_{B,b}^{N+1}$ for each $i$.

### 2.2. Fibers of good reduction.

We continue to assume that $X$ is a smooth projective variety over $F = k(B)$, $\nu$ a place of $F$, and $b \in B$ the corresponding point. A place $\nu$ is of good reduction for $X$ if there exists a scheme

$$\hat{\mathcal{X}}_b \to \hat{B}_b,$$

proper and smooth over $\hat{B}_b$, with generic fiber isomorphic to $X$ over $F_\nu$. Let $S$ denote the finite set of places of bad reduction.

**Definition 3.** — A variety $X$ satisfies weak approximation for places of good reduction if, for any finite set of places of good reduction $\{\nu_i\} \in I$
and \( \nu_i \)-adic open subsets \( U_i \subset X(F_{\nu_i}) \), there is a rational point \( x \in X(F) \) such that its image in each \( X(F_{\nu_i}) \) is contained in \( U_i \).

Suppose we have a model \( \pi : \mathcal{X} \to B \) smooth over \( B \setminus S \). Then we can express this in more geometric terms: For each finite set of points \( \{ b_i \}_{i \in I} \) in \( B \setminus S \), sections \( \hat{s}_i \) of \( \hat{\pi}_{b_i} \) and \( N \in \mathbb{N} \), there exists a section \( s \) of \( \pi \) agreeing with \( \hat{s}_i \) modulo \( \hat{m}_{B,b_i}^{N+1} \) for each \( i \).

**Proposition 4.** — Retain the notation introduced above. There exists an algebraic space

\[
\pi : \mathcal{X} \to B,
\]

proper and flat over \( B \), smooth over the places of good reduction, and with generic fiber \( X \). Such a space is called a good model of \( X \) over \( B \).

**Proof.** — Choose a projective model \( \pi' : \mathcal{X}' \to B \) for \( X \) over \( B \). If \( \pi' \) is smooth over \( B \setminus S \) there is nothing to prove. Otherwise, let \( \{ b_j \} \subset B \setminus S \) denote the points over which \( \mathcal{X}'_{b_j} \) is singular; let \( \hat{\mathcal{X}}'_{b_j} \) denote the completion of \( \mathcal{X}' \) along the central fiber \( \mathcal{X}'_{b_j} \). By assumption, there exists a proper smooth scheme

\[
\hat{\mathcal{X}}_{b_j} \to \hat{B}_{b_j},
\]

which is isomorphic to our original model over the generic point. Resolving the indeterminacy of the rational map

\[
\hat{\mathcal{X}}_{b_j} \dashrightarrow \hat{\mathcal{X}}'_{b_j},
\]

we find that these are related by a sequence of modifications in the central fiber. This gives a sequence of formal modifications to \( \mathcal{X}' \) along the singular fibers, in the sense of Artin \([3]\). Theorems 3.1 and 3.2 of \([3]\) give a unique proper algebraic space \( \pi : \mathcal{X} \to B \) realizing these formal modifications to \( \mathcal{X}' \).

**Example 5.** — There are simple examples justifying the introduction of algebraic spaces. Let \( \pi : \mathcal{X}' \to B \) be a flat projective morphism such that each fiber is a cubic surface with rational double points and the generic fiber \( X \) is smooth. Suppose that near each point \( b \in B \) the local monodromy representation

\[
\text{Gal}(\overline{F}_b/F_b) \to \text{Aut}(\text{Pic}(\overline{X}))
\]
is trivial. By a theorem of Brieskorn [4] [5], there exists a simultaneous resolution

\[ \pi : X \to X', \]

where \( X \) is a smooth proper algebraic space over \( B \) and \( \varphi_b : X_b \to X'_b \) is the minimal resolution of \( X'_b \) for each \( b \in B \). However, \( X \) is constructed by making modifications of \( X' \) in formal neighborhoods of the singular fibers, and hence is not necessarily a scheme. Note that blowing up the singularities of \( X' \) will usually introduce exceptional divisors in the fibers.

**Definition 6.** — Let \( b \in B \setminus S \) be a point of good reduction and \( \pi : X \to B \) a good model. An \( N \)-jet of \( \pi \) at \( b \) is a section of

\[ X \times_B \text{Spec}(\mathcal{O}_{B,b}/\mathfrak{m}_{B,b}^{N+1}) \to \text{Spec}(\mathcal{O}_{B,b}/\mathfrak{m}_{B,b}^{N+1}). \]

Hensel’s lemma guarantees that every \( N \)-jet is a restriction of a section of \( \hat{\pi}_b \). Let \( \{b_i\}_{i \in I} \) be a finite set of points of good reduction and \( j_i \) an \( N \)-jet of \( \pi \) at \( b_i \). We write \( J = \{j_i\}_{i \in I} \) for the corresponding collection of \( N \)-jets.

**2.3. Iterated blowups.** — Let \( \pi : X \to B \) be good model of \( X \) and \( J = \{j_i\}_{i \in I} \) a finite collection of \( N \)-jets at points of good reduction \( \{b_i\} \). The *iterated blowup* associated with \( J \)

\[ \beta(J) : X(J) \to X \]

is obtained by performing the following sequence of blowups: For each \( i \in I \) choose a section \( \hat{s}_i \) of \( \hat{\pi}_{b_i} \) with jet \( j_i \). Now blow up \( X \) successively \( N \) times, where at each stage the center is the point at which the proper transform of \( \hat{s}_i \) meets the fiber over \( b_i \). Observe that a blowup of \( X \) centered in the fibers of \( \pi \) is uniquely determined by the corresponding blowup of the completions along those fibers. Note that at each stage we blow up a smooth point of the fiber of the corresponding model and that the result does not depend on the order of the \( b_i \) or on the choice of \( \hat{s}_i \).

The fiber \( X(J)_{b_i} \) decomposes into irreducible components

\[ X(J)_{b_i} = E_{i,0} \cup \ldots \cup E_{i,N} \]

where
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- $E_{i,0}$ is the proper transform of $X_{b_i}$, isomorphic to the blowup of $X_{b_i}$ at $r_{i,0} := \hat{s}_i(b_i)$;
- $E_{i,n}$, $n = 1, \ldots, N-1$, is the blowup of $\mathbb{P}^d$ at $r_{i,n}$, the point where the proper transform of $\hat{s}_i$ meets the fiber over $b_i$ of the $n$th blowup;
- $E_{i,N} \cong \mathbb{P}^d$.

The intersection $E_{i,n} \cap E_{i,n+1}$ is the exceptional divisor $\mathbb{P}^{d-1} \subset E_{i,n}$ and a proper transform of a hyperplane in $E_{i,n+1}$, for $n = 0, \ldots, N-1$. Let $r_i \in E_{i,N} \setminus E_{i,N-1}$ denote the intersection of $\hat{s}_i$ with $E_{i,N}$.

FIGURE 1. One fiber of the iterated blowup

For each section $s' : \mathcal{X}(J) \to B$ the composition $\beta(J) \circ s'$ is a section of $\pi : \mathcal{X} \to B$. Conversely, given a section $s$ of $\pi : \mathcal{X} \to B$, its proper transform $s'$ is the unique section of $\mathcal{X}(J) \to B$ lifting $s$. Sections $s'$ of $\mathcal{X}(J) \to B$ with $s'(b_i) = r_i$ yield sections of $\mathcal{X} \to B$ with $N$-jet $j_i$ at $b_i$. We therefore have the following criterion for weak approximation in fibers of good reduction:

**Proposition 7.** — $X$ satisfies weak approximation in fibers of good reduction if and only if there exists a good model $\pi : \mathcal{X} \to B$ with the
following property: For each \( N \in \mathbb{N} \) and collection of \( N \)-jets \( J \) with corresponding iterated blowup \( \mathcal{X}(J) \), and for any choice of points \( r_i \in E_{i,N} \setminus E_{i,N-1}, i \in I \), there exists a section \( s \) of \( \mathcal{X}(J) \to B \) with \( s(b_i) = r_i \) for each \( i \in I \).

3. Rationally connected varieties

We retain the notation introduced in Section 2. In particular, the ground field \( k \) is algebraically closed of characteristic zero.

3.1. Terminology and fundamental results. — Rational connectedness was introduced in the classification of Fano varieties \[ \text{[6]} \] \[ \text{[11]} \]. However, rationally connected varieties are now of independent interest:

Definition 8 ([10] IV.3.2). — A variety \( Y \) is rationally chain connected (resp. rationally connected) if there is a family of proper and connected curves \( g : U \to Z \) whose geometric fibers have only rational components (resp. are irreducible rational curves) and a cycle morphism \( g : U \to Y \) such that

\[
u^{(2)} : U \times_Z U \to Y \times Y.
\]
is dominant.

Our definition of ‘rationally chain connected’ makes sense for reducible schemes \( Y \).

Example 9. — The class of rationally connected varieties includes unirational varieties and smooth Fano varieties, [10] IV.3.2.6, V.2.13. In particular, smooth hypersurfaces of degree \( \leq m \) in \( \mathbb{P}^m \) are rationally connected.

Definition 10. — Let \( Y \) be a smooth algebraic space of dimension \( d \) and \( f : \mathbb{P}^1 \to Y \) a morphism, so we have an isomorphism

\[ f^*T_Y \simeq \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \ldots \oplus \mathcal{O}_{\mathbb{P}^1}(a_d) \]
for suitable integers \( a_1, \ldots, a_d \). Then \( f \) is free (resp. very free) if each \( a_i \geq 0 \) (resp. \( a_i \geq 1 \)).

We recall some key properties:
(1) Let $Y$ be a proper rationally chain connected variety. Then any two closed points are contained in a connected curve with rational irreducible components, \cite{10} IV.3.5.1.

(2) If the ground field $k$ is uncountable then $Y$ is rationally connected (resp. rationally chain connected) if any two very general closed points $y_1$ and $y_2$ are contained in an irreducible rational curve (resp. connected curve with rational irreducible components), \cite{10} IV.3.6.

(3) Let $Y$ be a smooth proper rationally connected variety and $y_1, \ldots, y_m$ points in $Y$. Then there exists a very free morphism $f : \mathbb{P}^1 \to Y$ such that $y_1, \ldots, y_m \in f(\mathbb{P}^1)$. We may take $f$ to be an immersion if $\dim(Y) = 2$ and an embedding if $\dim(Y) \geq 3$, \cite{10} IV.3.9.

(4) A smooth variety $Y$ is rationally connected if it is rationally chain connected, \cite{10} IV.3.10.3.

(5) Let $\pi : \mathcal{Y} \to B$ be a proper equidimensional morphism over an irreducible base. If the generic fiber of $\pi$ is rationally chain connected then every fiber is rationally chain connected, \cite{10} IV.3.5.2.

(6) If $\pi : \mathcal{Y} \to B$ is a smooth morphism then the locus

$$\{b \in B : \mathcal{Y}_b \text{ is rationally connected}\}$$

is open, \cite{10} IV.3.11.

Since Properties (5) and (6) are local on the base, they also hold for good models (which are only assumed to be algebraic spaces over $B$).

**Example 11.** — Property (5) does not guarantee that every fiber is rationally connected: Consider the family of cubic surfaces

$$\mathcal{X} := \{(w, x, y, z; t) : x^3 + y^3 + z^3 = tw^3\} \to \mathbb{A}^1_t.$$  

The generic fiber is rationally connected but the fiber $\mathcal{X}_0$ is a cone over an elliptic curve, which is not rationally connected.

### 3.2. Producing sections through prescribed points.

— Theorem \cite{11} when combined with the machinery of Section \cite{12} has the following important consequence:

**Theorem 12** (\cite{10} IV.6.10, \cite{11} 2.13). — Let $X$ be a smooth projective rationally connected variety over the function field of a curve. Given a projective model $\pi : \mathcal{X} \to B$, a finite collection of points $\{b_i\}_{i \in I}$ such that
each $X_{b_i}$ is smooth, and points $x_i \in X_{b_i}$, there exists a section $s : B \to X$ such that $s(b_i) = x_i$, for each $i \in I$.

It is natural to wonder whether we can relax the hypothesis that the fibers $X_{b_i}$ be smooth. For simplicity, assume that the total space $X$ of our model is regular; this can always be achieved by resolving singularities. Then for each section $s$, $s(b) \in X_{b_i}$ is necessarily a smooth point; otherwise, the intersection multiplicity of the section with $X_{b_i}$ would be $> 1$. In light of this, the most optimistic generalization of Theorem 12 would be:

**Conjecture 13.** — Let $X$ be a smooth projective rationally connected variety over the function field of a curve. Given a regular model $\pi : \mathcal{X} \to B$, a finite collection of points $\{b_i\}_{i \in I} \subset B$ and smooth points $r_i \in X_{b_i}$, there exists a section $s : B \to X$ such that $s(b_i) = r_i$ for each $i \in I$.

Applying this to the iterated blowups as described in Proposition 7, we obtain:

**Conjecture 14.** — A smooth rationally connected variety over the function field of a curve satisfies weak approximation.

We outline the main issues in the proof of Theorem 2; details are given in Section 5. By Proposition 7, we are reduced to proving the existence of a section passing through specific smooth points $r_i$ of singular fibers of the iterated blow-up. Theorem 12 does not immediately imply this, but it does guarantee a section $\sigma$ passing through some point $x_i$ of each of these fibers. Property (5) of rationally connected varieties from Section 3.1 guarantees the existence of some chain $T_i$ of rational curves in the corresponding fiber joining $x_i$ and $r_i$. The difficulty is to choose these so that $C := \sigma(B) \cup_{i \in I} T_i$ deforms to a section containing $r_i$, for each $i \in I$. In particular, it is necessary that $C$ intersect the fibral exceptional divisor $E_{i,N}$ containing $r_i$ in one point and not intersect the other components of the corresponding fiber; this constrains the homology class of $T_i$. Furthermore, we must describe each $T_i$ explicitly so the deformation space of $C$ can be analyzed.
4. Deformation theory

We continue to work over an algebraically closed ground field of characteristic zero. In this section, a curve is a connected reduced scheme of dimension one.

4.1. Combs. — Recall the dual graph associated with a nodal curve $C$: Its vertices are indexed by the irreducible components of $C$ and its edges are indexed by the intersections of these components.

Definition 15. — A projective nodal curve $C$ is tree-like if

- each irreducible component of $C$ is smooth;
- the dual graph of $C$ is a tree.

We shall require a slight generalization of the standard notion of a comb (cf. [10]):

Definition 16. — A comb with $m$ broken teeth is a projective nodal curve $C$ with $m + 1$ subcurves $D, T_1, \ldots, T_m$ such that

- $D$ is smooth and irreducible;
- $T_\ell \cap T_{\ell'} = \emptyset$, for all $\ell \neq \ell'$;
- each $T_\ell$ meets $D$ transversally in a single point; and
- each $T_\ell$ is a chain of $\mathbb{P}^1$'s.

Here $D$ is called the handle and the $T_\ell$ the broken teeth.

Figure 2. A comb with five broken teeth
4.2. Vector bundle lemmas. — Let $C$ be a smooth curve and $\mathcal{V}$ a vector bundle on $C$. Given a collection of distinct points $q = \{q_1, \ldots, q_m\} \subset C$ and one-dimensional subspaces of the fibers $\xi_{q_\ell} \subset \mathcal{V}_{q_\ell}$, $\ell = 1, \ldots, m$, there exists a rank-one subbundle $\mathcal{L} \subset \mathcal{V}$ with fiber at $q_\ell$ equal to $\xi_{q_\ell}$, $\ell = 1, \ldots, m$. The extension

$$0 \rightarrow \mathcal{L} \rightarrow \mathcal{L} \otimes \mathcal{O}_C(q) \rightarrow \mathcal{L} \otimes (\mathcal{O}_C(q))_q \rightarrow 0$$

induces

$$0 \rightarrow \mathcal{V} \rightarrow \mathcal{V}' \rightarrow \mathcal{Q} \rightarrow 0,$$

where $Q := (\mathcal{L} \otimes \mathcal{O}_C(q))_q$ is supported on $q$ and has length one at each $q_\ell$. This extension depends on the $q_\ell$ and $\xi_{q_\ell}$ but not on $\mathcal{L}$. The saturation of $\mathcal{L}$ in $\mathcal{V}'$ is isomorphic to $\mathcal{L} \otimes \mathcal{O}_C(q)$.

**Lemma 17.** — Retain the notation introduced above. A subbundle $\mathcal{U} \subset \mathcal{V}$ is also a subbundle of $\mathcal{V}'$ if $\xi_{q_\ell} \cap U_{q_\ell} = 0$ for each $\ell$. 

**Proof.** — We have exact sequences

$$0 \rightarrow \text{Hom}(\mathcal{L} \otimes \mathcal{O}_C(q), \mathcal{V}) \rightarrow \text{Hom}(\mathcal{L}, \mathcal{V}) \rightarrow \text{Ext}^1(Q, \mathcal{V})$$

and the extension class $\eta_{\mathcal{V}'} \in \text{Ext}^1(Q, \mathcal{V})$ is the image of the inclusion $\mathcal{L} \hookrightarrow \mathcal{V}$ under the connecting homomorphism. Since $\mathcal{L}$ is saturated in $\mathcal{V}$ at $q_\ell$, $\eta_{\mathcal{V}'}$ localizes to a nonzero element of $\text{Ext}^1(Q, \mathcal{V})_{q_\ell}$ for each $\ell$.

Since $\mathcal{U} \subset \mathcal{V}$ is a subbundle, $\mathcal{V}/\mathcal{U}$ has no torsion and thus is locally free. The class $\eta_{\mathcal{V}'}$ naturally induces an extension

$$0 \rightarrow \mathcal{V}/\mathcal{U} \rightarrow \mathcal{V}'/\mathcal{U} \rightarrow \mathcal{Q} \rightarrow 0,$$

classified by $\eta_{\mathcal{V}'/\mathcal{U}} \in \text{Ext}^1(Q, \mathcal{V}/\mathcal{U})$, the image of the composition

$$\mathcal{L} \hookrightarrow \mathcal{V} \rightarrow \mathcal{V}/\mathcal{U}$$

under the connecting homomorphism. Our hypothesis guarantees that $\mathcal{L}$ is a subbundle of $\mathcal{V}/\mathcal{U}$ near $q_\ell$, hence $\eta_{\mathcal{V}'/\mathcal{U}}$ localizes to a nonzero element of $\text{Ext}^1(Q, \mathcal{V}/\mathcal{U})_{q_\ell}$ for each $\ell$. It follows that $\mathcal{V}'/\mathcal{U}$ is torsion-free, so $\mathcal{U}$ is a subbundle. 

**Lemma 18 ([8], Lemma 2.5).** — Retain the notation introduced above. Fix an integer $N$ and a vector bundle $\mathcal{V}$. Then there exist points $q_1, \ldots, q_m$
and one-dimensional subspaces \( \xi_{q_\ell} \subset V_{q_\ell}, \ \ell = 1, \ldots, m \), such that
\[
H^1(\mathcal{V}' \otimes \mathcal{O}_C(-w_1 - \ldots - w_N)) = 0
\]
for any points \( w_1, \ldots, w_N \in C \).

Our next lemma is well known (cf. [12, Section 2]) but we provide a proof for the convenience of the reader:

**Lemma 19.** — Let \( C \) be a tree-like curve and \( \mathcal{V} \) a vector bundle on \( C \). If for each irreducible component \( C_\ell \) of \( C \) the restriction \( \mathcal{V} \otimes \mathcal{O}_{C_\ell} \) is globally generated then \( \mathcal{V} \) is globally generated. Furthermore,
\[
H^1(C, \mathcal{V}) \to \bigoplus_\ell H^1(\mathcal{V} \otimes \mathcal{O}_{C_\ell})
\]
is an isomorphism.

**Proof.** — We do induction on the number of irreducible components; the case of one component is trivial. Otherwise, express \( C \) as a union \( D \cup D^c \), where \( D \) is irreducible with connected complement in \( C \) and \( D^c = C \setminus D \) is tree-like. Let \( q \) be the node of \( C \) joining \( D \) and \( D^c \),
\[
g : C' := D \amalg D^c \to C
\]
the partial normalization of \( C \) at \( q \), and \( r, r^c \) the points of \( C' \) with \( g(r) = g(r^c) = q \). The descent data for \( \mathcal{V} \) consist of the pullback \( g^* \mathcal{V} \) and an isomorphism
\[
\phi : (g^* \mathcal{V})_r \to (g^* \mathcal{V})_{r^c}
\]
induced by identifications of fibers
\[
(g^* \mathcal{V})_r \simeq \mathcal{V}_q \simeq (g^* \mathcal{V})_{r^c}.
\]
Recall the exact sequences relating the cohomology of \( \mathcal{V} \) and \( g^* \mathcal{V} \):
\[
0 \to H^0(C, \mathcal{V}) \to H^0(C', g^* \mathcal{V}) \to (g^* \mathcal{V})_r \oplus (g^* \mathcal{V})_{r^c} \xrightarrow{(-\phi, \text{Id})} (g^* \mathcal{V})_{r^c} \to 0
\]
\[
H^0(C', g^* \mathcal{V}) \to (g^* \mathcal{V})_r \oplus (g^* \mathcal{V})_{r^c} \to H^1(C, \mathcal{V}) \to H^1(C', g^* \mathcal{V}) \to 0.
\]

By the inductive hypothesis, \( g^* \mathcal{V} \) is globally generated on \( C' \). Since \( r \) and \( r^c \) are on different connected components of \( C' \), the second exact sequence guarantees that \( H^1(C, \mathcal{V}) \to H^1(C', g^* \mathcal{V}) \) is injective; the cohomology statement follows. Since \( g^* \mathcal{V} \) is globally generated, for each section over \( D \) there exists a section over \( D^c \) compatible under the isomorphism \( \phi \), and vice versa. These compatible pairs of sections descend
to elements of $H^0(C, \mathcal{V})$. Thus given $p \in D$ and $v \in \mathcal{V}_p$, a section $t \in H^0(D, \mathcal{V} \otimes \mathcal{O}_D)$ with $t(p) = v$ extends to a section over $C$. 

4.3. Analysis of normal bundles. — We describe the normal bundle of a nodal curve immersed in a smooth algebraic space. Our main references are Section 2 of [8] and Section 6 of [11]. See [2] and [10] I.5 for foundational results on Hilbert ‘schemes’ of algebraic spaces, [10] II.1 for applications to morphisms of curves into spaces, and [9] for an extension of Theorem [11] to positive characteristic using this machinery.

If $C$ is a nodal curve imbedded into a smooth space $Y$ then $\mathcal{N}_{C/Y}$ is defined as the dual to the kernel of the restriction homomorphism of Kähler differentials $\Omega^1_Y \otimes \mathcal{O}_C \to \Omega^1_C$; a local computation shows this is locally free. First order deformations of $C \subset Y$ are given by $H^0(C, \mathcal{N}_{C/Y})$; obstructions are given by $H^1(C, \mathcal{N}_{C/Y})$.

Let $D$ be the union of irreducible components of $C$ and $q = \{q_1, \ldots, q_m\}$ the locus where $D^c := C \setminus D$ meets $D$. At a node of $C$, the tangent cone is a union of two one-dimensional subspaces, the tangents to the transverse branches. The tangent to $D^c$ at $q_\ell$ yields a one-dimensional subspace $\xi_{q_\ell} \subset (\mathcal{N}_{D/Y})_{q_\ell}$. As in Section 4.2 these induce a natural extension $\mathcal{N}'_{D/Y}$ of $\mathcal{N}_{D/Y}$, which coincides with the restriction to $D$ of the normal bundle to $C$ in $Y$

$$0 \to \mathcal{N}_{D/Y} \to \mathcal{N}_{C/Y} \otimes \mathcal{O}_D \to Q \to 0.$$ 

Here $Q$ is a torsion sheaf supported on $q$, with length one at each point $q_\ell, \ell = 1, \ldots, m$. Sections of $\mathcal{N}_{C/Y} \otimes \mathcal{O}_D$ can be interpreted as sections of $\mathcal{N}_{D/Y}$ with poles at the $q_\ell$ in the directions $T_{q_\ell}D^c$.

We shall need a slight generalization: We continue to assume that $C$ is a nodal curve and $Y$ is nonsingular. Let $f : C \to Y$ denote a closed immersion whose image is a nodal curve. The restriction homomorphism $f^*\Omega^1_Y \to \Omega^1_C$ is surjective and the dual to its kernel is still locally free. This is denoted $\mathcal{N}_f$ and coincides with $\mathcal{N}_{C/Y}$ when $f$ is an embedding. First order deformations of $f : C \to Y$ are given by $H^0(C, \mathcal{N}_f)$; obstructions are given by $H^1(C, \mathcal{N}_f)$. This set-up differs from the standard deformation theory of morphisms in that we ignore reparametrizations of $C$. When $D$ is a union of irreducible components of $C$ as above, then
the analogous extension takes the form (cf. Lemma 2.6 of [8]):
\[ 0 \rightarrow N_{f|D} \rightarrow N_f \otimes O_D \rightarrow Q \rightarrow 0. \]

This analysis gives the following infinitesimal smoothing criterion: (cf. Lemma 2.6 of [8]):

**Lemma 20.** Retain the above notation. A first order deformation \( t \in H^0(C, N_f) \) smooths the node \( q_\ell \) if the restriction \( t|_D \in N_f \otimes O_D \) is nonzero in \( Q_{q_\ell} \).

**Proposition 21.** Let \( Y \) be a smooth algebraic space, \( E \subset Y \) a smooth subspace of codimension one, \( C \) a nodal curve, \( D \subset C \) a union of irreducible components of \( C \), \( D^c = D \setminus C \), and \( q = \{q_\ell\} = D \cap D^c \). Let \( f : C \rightarrow Y \) be an immersion with image a nodal curve so that \( f(D) \subset E \), \( f(D \setminus C) \subset Y \setminus E \), and \( f(D^c) \) is transverse to \( E \) at each point of \( f(q) \). If \( g : D \rightarrow E \) is the resulting immersion of \( D \) into \( E \) then \( N_g \) is saturated in \( N_f \otimes O_D \) and we have the following diagram:

\[
\begin{array}{ccccccccc}
0 & 
\rightarrow & N_g & 
\rightarrow & N_{f|D} & 
\rightarrow & g^*N_{E/Y} & 
\rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & 
\rightarrow & N_g & 
\rightarrow & N_f \otimes O_D & 
\rightarrow & g^*N_{E/Y} \otimes O_D(q) & 
\rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & 
\rightarrow & Q & 
\rightarrow & g^*N_{E/Y} \otimes (O_D(q))_q & 
\rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & 
\rightarrow & 0 & 
\rightarrow & 0 & 
\end{array}
\]

**Proof.** — For each \( \ell \) the composition
\[ T_{q_\ell}D^c \rightarrow T_{q_\ell}Y \rightarrow T_{q_\ell}Y/T_{q_\ell}D \]
determines a one-dimensional subspace \( \xi_\ell \subset (N_{f|D})_{q_\ell} \). The transversality hypothesis implies \( (N_g)_{q_\ell} \cap \xi_\ell = 0 \) in \( (N_{f|D})_{q_\ell} \). Using Lemma 17, we conclude that \( N_{f|D} \) is a subbundle in \( N_f \otimes O_D \), so the quotient
\[ R = (N_f \otimes O_D) / N_{f|D} \]
is locally free. This sheaf arises as an extension
\[(4.1) \quad 0 \to g^*\mathcal{N}_{E/Y} \to R \to Q \to 0,\]
where \(Q\) is supported on \(q\) with length one at each \(q_i\). We may therefore identify extension (4.1) with the tensor product of \(g^*\mathcal{N}_{E/Y}\) with
\[0 \to \mathcal{O}_D \to \mathcal{O}_D(q) \to (\mathcal{O}_D(q))_q \to 0.\]

**Proposition 22.** — Let \(C\) be a tree-like curve, \(Y\) a smooth algebraic space, and \(f : C \to Y\) an immersion with nodal image. Suppose that for each irreducible component \(C\) of \(C\), \(H^1(C, \mathcal{N}_f \otimes \mathcal{O}_{C_{i}}) = 0\) and \(\mathcal{N}_f \otimes \mathcal{O}_{C_{i}}\) is globally generated. Then \(f : C \to Y\) deforms to an immersion of a smooth curve into \(Y\).

Suppose furthermore that \(p = \{p_1, \ldots, p_w\} \subset C\) is a collection of smooth points such that for each component \(C\), \(H^1(C, \mathcal{N}_f \otimes \mathcal{O}_{C_{i}}(−p)) = 0\) and the sheaf \(\mathcal{N}_f \otimes \mathcal{O}_{C_{i}}(−p)\) is globally generated. Then \(f : C \to Y\) deforms to an immersion of a smooth curve into \(Y\) containing \(f(p)\).

**Proof.** — Our argument is similar to the constructions of Section 2 of [8] and Lemma 65 of [11]. Lemma 19 implies that \(H^1(C, \mathcal{N}_f) = 0\) and \(\mathcal{N}_f\) is globally generated. Hence the space of maps is unobstructed and every first-order deformation of \(f\) lifts to an actual deformation. Global generation implies the existence of \(t \in H^0(C, \mathcal{N}_f)\) so that, for each component \(C_{i}\), the image of \(t\) in
\[H^0(C_{i}, Q(C_{i})), \quad Q(C_{i}) := (\mathcal{N}_f \otimes \mathcal{O}_{C_{i}}) / \mathcal{N}_f|_{C_{i}}\]
is nonzero at each point of the support of \(Q(C_{i})\). The first-order deformation \(t\) smooths each node of \(C\) by Lemma 20.

For the second part, consider those maps with image containing \(f(p)\). Our cohomology assumption guarantees that this space is unobstructed; in addition, \(\mathcal{N}_f \otimes \mathcal{O}_{C}(−p)\) is globally generated. Hence there exists a \(u \in H^0(\mathcal{N}_f \otimes \mathcal{O}_{C}(−p))\) so that, for every component \(C_{i}\), the image of \(u\) in \(Q(C_{i})\) is nonzero at each point of its support. Note that
\[(\mathcal{N}_f \otimes \mathcal{O}_{C_{i}}(−p)) / (\mathcal{N}_f|_{C_{i}} \otimes \mathcal{O}_{C_{i}}(−p)) = (\mathcal{N}_f \otimes \mathcal{O}_{C_{i}}) / \mathcal{N}_f|_{C_{i}},\]
since the quotient is a torsion sheaf with support disjoint from \(p\). Hence the first-order deformation \(u\) smooths each node of \(C\) and contains \(f(p)\). 

\[\square\]
5. Proof of the main theorem

The theorem is well known when $d = \dim(X) = 1$. The only smooth proper rationally connected curve is $\mathbb{P}^1$, which satisfies weak approximation. We may therefore assume $d \geq 2$.

Recall the set-up of Proposition 7. It suffices to show that for each integer $N$, finite set $\{b_i\}_{i \in I} \subset B \setminus S$, and collection of $N$-jet data $J$ supported in the fibers over $\{b_i\}_{i \in I}$, there exists a section in the iterated blowup $X(J)$ passing through prescribed points $r_i \in E_{i,N} \setminus E_{i,N-1}$.

We proceed by induction on $N$; the base case $N = 0$ is essentially Theorem 12. However, our assumptions are slightly weaker: We are not assuming $X$ is a scheme.

The total space of $X$ is smooth along the fibers $X_{b_i}, i \in I$, so we may resolve the singularities of $X$ without altering these fibers. Theorem 11 gives a section $\sigma$ of $\pi$. Let $q_i = \sigma(b_i)$ for each $i \in I$; let $I' \subset I$ (resp. $I'' \subset I$) denote those indices with $q_i \neq r_i$ (resp. $q_i = r_i$).

We shall construct a comb $C$ with handle $\sigma(B)$ and smooth teeth $T_1, \ldots, T_m$ and an immersion $f : C \to X$ with nodal image so that:

- the $T_\ell$ are free rational curves in distinct smooth fibers of $\pi$;
- for each $i \in I'$, there is a tooth $T_i$ containing $r_i$ as a smooth point;
- let $\tau$ denote the sum of the points of $C$ mapping to the $r_i, i \in I$;
  
  then the restriction of $N_f \otimes \mathcal{O}_C(-\tau)$ to each irreducible component of $C$ is globally generated and has no higher cohomology.

We emphasize that $f$ can be taken to be an embedding if $d > 2$.

Proposition 22 implies $f : C \to X$ admits a deformation $\tilde{f} : \tilde{C} \to X$, where $\tilde{C}$ is smooth and $\tilde{f}(\tilde{C})$ contains each of the $r_i$. Since all the teeth $T_1, \ldots, T_m$ are contained in fibers of $X \to B$, $C$ intersects the generic fiber in one point. Thus the deformed curve $\tilde{C}$ also meets the generic fiber in one point and hence is a section of $X \to B$.

Here are the details of the construction. For each $\ell$

$$f_\ell : T_\ell \to X_{b_\ell}$$

is a free rational curve with nodal image, so that $\sigma(b_\ell) \in f_\ell(T_\ell)$ as a smooth point. For $\ell = 1, \ldots, |I'|$ we choose these so that $r_i$ is contained in the image as a smooth point. For $\ell = |I'| + 1, \ldots, m$, we choose these in generic fibers of good reduction with generic tangent directions $\xi_\ell \subset$
\[ T_{\sigma(b_i)} X_{b_i} \text{ satisfying the hypotheses of Lemma 18, so that the extension} \]
\[ 0 \to N_{\sigma(B)} \to N_f \otimes O_{\sigma(B)} \to Q(\sigma(B)) \to 0 \]
\[ \text{is globally-generated and has no higher cohomology, even after twisting by} \]
\[ O_{\sigma(B)}(-\sum_{i\in I''} r_i). \]

We next address the inductive step. Let \( J' \) denote order-(\( N - 1 \)) truncation of \( J \), i.e., if
\[ j_i : \Spec(O_{B,b_i}/m_{B,b_i}^{N+1}) \to X \times_B \Spec(O_{B,b_i}/m_{B,b_i}^{N+1}) \]
then
\[ j'_i = j_i|_{\Spec(O_{B,b_i}/m_{B,b_i}^{N})}. \]
The inductive hypothesis applied to \( J' \) guarantees the existence of a section \( s' : B \to X(J') \) passing through arbitrary points
\[ r'_i \in (E_{i,N-1}' \setminus E_{i,N-2}') \subset X(J'). \]
Specifically, we choose \( s' \) so that it has jet data \( J' \) over the points \( \{b_i\}_{i \in I}. \)
Let \( \sigma : B \to X(J) \) denote the proper transform of \( s' \) in \( X(J) \). By construction, \( \sigma \) meets \( X(J)_{b_i} \) in a point \( q_i \in E_{i,N} \setminus E_{i,N-1} \) for each \( i \).
Our goal is to find a section \( s : B \to X(J) \) such that for each \( i \in I' \) and \( r_i \in E_{i,N} \setminus E_{i,N-1}, \) \( s(b_i) = r_i \).
Again, \( I' \subset I \) (resp. \( I'' \subset I \)) denotes those indices with \( q_i \neq r_i \) (resp. \( q_i = r_i \)).

Next, we construct a comb \( C \) with handle \( \sigma(B) \) and broken teeth \( T_1, \ldots, T_m \) and an immersion \( f : C \to X(J) \) with nodal image so that:
- for each \( i \in I' \), there is a broken tooth \( T_i \) mapped to \( X(J)_{b_i} \) and containing \( r_i \);
- \( C \) is smoothly embedded at \( r_i \) for each \( i \in I' \), so there is a unique component \( T_{i,N} \subset C \) containing \( r_i \);
- the remaining broken teeth \( T_{|I'|+1}, \ldots, T_m \) are free rational curves contained in generic fibers of \( X(J) \to B \) of good reduction;
- the restriction of \( N_f \otimes O_C(-\tau) \) to each irreducible component is globally generated and has no higher cohomology.
Again, \( f \) can be taken to be an embedding if \( d > 2 \).

Proposition 22 implies that \( f : C \to X(J) \) admits a deformation \( \tilde{f} : \tilde{C} \to X(J) \), where \( \tilde{C} \) is smooth and \( \tilde{f}(\tilde{C}) \) contains each of the \( r_i \). The image \( \tilde{f}(\tilde{C}) \) is the desired section of \( X(J) \to B \).
We start by describing the teeth $T_i$ with $i \in I'$. Recall from Section 2.3 that $E_{i,N} \simeq \mathbb{P}^d$ and $E_{i,N} \cap E_{i,N-1}$ is a hyperplane section in this $\mathbb{P}^d$.

Let $T_{i,N}$ denote the unique line joining $r_i$ to $q_i = \sigma(b_i)$; let $q_{i,N-1}$ denote the intersection of this line with $E_{i,N-1}$. We have

$$E_{i,N-1} \simeq \text{Bl}_{r_{i,N-1}} \mathbb{P}^d$$

with exceptional divisor $E_{i,N-1} \cap E_{i,N} \simeq \mathbb{P}^{d-1}$; there is a unique line in $\mathbb{P}^d$ containing $r_{i,N-1}$ whose proper transform $T_{i,N-1} \subset E_{i,N-1}$ meets $q_{i,N-1}$.

Let $q_{i,N-2}$ denote the intersection of this line with $E_{i,N-2}$. Continuing in this way, we obtain a sequence of embedded smooth rational curves

$$T_{i,n} \subset E_{i,n} \simeq \text{Bl}_{r_{i,n}} \mathbb{P}^d, \quad n > 0,$$

each the proper transform of a line meeting $r_{i,n}$. Let $q_{i,0}$ denote the intersection of $T_{i,1}$ with $E_{i,0}$, which is a point in the exceptional divisor of

$$E_{i,0} = \text{Bl}_{r_{i,0}} \rightarrow \mathcal{X}(J).$$

Let $g_{i,0} : T_{i,0} \rightarrow E_{i,0}$ be a free rational curve, immersed so that the image is a nodal curve, with $q_{i,0} \in g_{i,0}(T_{i,0})$ as a smooth point. Property (3) of rationally connected varieties gives such a curve; $g_{i,0}$ can be taken to be an embedding when $d > 2$. Let $f_{i,0}$ denote the composition of $g_{i,0}$ with the inclusion $E_{i,0} \subset \mathcal{X}(J)$, and

$$f_i : T_i = T_{i,0} \cup \ldots \cup T_{i,N} \longrightarrow \mathcal{X}(J)$$

the resulting map of the broken tooth into $\mathcal{X}(J)$.

We verify the normal bundle conditions for the components of these teeth. First, observe that

$$\mathcal{N}_{T_{i,n}/E_{i,n}} \simeq \begin{cases} \mathcal{O}_{\mathbb{P}^1}(+1)^{d-1} & \text{for } n = N \\ \mathcal{O}_{\mathbb{P}^1}^{d-1} & \text{for } n = 1, \ldots, N-1 \end{cases}$$

$$\mathcal{N}_{g_{i,0}} \simeq \oplus_{w=1}^{d-1} \mathcal{O}_{\mathbb{P}^1}(a_w), \quad a_w \geq 0;$$

the $a_w$ are nonnegative because $\mathcal{N}_{g_{i,0}}$ is a quotient of $g_{i,0}^*\mathcal{O}_{E_{i,0}}$, which is nonnegative. Fibers of $\mathcal{X}(J) \rightarrow B$ restrict to the zero divisor on each $T_{i,n}$ and $\sum_{n=0}^N E_{i,n}$ is equivalent to the class of a fiber, hence

$$E_{i,n} |_{T_{i,n}} = (- \sum_{n' \neq n} E_{i,n'}) |_{T_{i,n}}.$$
Figure 3. Attaching broken teeth and moving the section

It follows that

\[ N_{E_i,n/X(J)} \otimes O_{T_i,n} = O_{X(J)}(E_{i,n}) \otimes O_{T_i,n} \simeq \begin{cases} O_{P_1}(-1) & \text{for } n = N \\ O_{P_1}(-2) & \text{for } n = 1, \ldots, N - 1 \\ O_{P_1}(-1) & \text{for } n = 0. \end{cases} \]

For \( n > 0 \) we have the exact sequence

\[ 0 \to N_{T_i,n/E_{i,n}} \to N_{T_i,n/X(J)} \to N_{E_i,n/X(J)} \otimes O_{T_i,n} \to 0, \]

which splits in our situation. Therefore, we find

\[ N_{T_i,n/X(J)} \simeq \begin{cases} O_{P_1}(+1)^{d-1} \oplus O_{P_1}(-1) & \text{for } n = N \\ O_{P_1}^{d-1} \oplus O_{P_1}(-2) & \text{for } n = 1, \ldots, N - 1. \end{cases} \]

For \( n = 0 \) we have

\[ 0 \to N_{g_{i,0}} \to N_{f_{i,0}} \to g_{i,0}^* N_{E_{i,0}/X(J)} \to 0, \]

which implies

\[ N_{f_{i,0}} \simeq \oplus_{w=1}^{d-1} O_{P_1}(a_w) \oplus O_{P_1}(-1). \]

On first examination, the negative summands would make it hard to satisfy the hypotheses of Proposition 22. However, the nodes in the
broken teeth give enough positivity to overcome the negative factors. We use Proposition 21 to analyze the relationship between the normal bundles to the $T_{i,n}$ and the restriction to the normal bundle of the broken comb to these components. When $n > 0$, we have an exact sequence

$$0 \rightarrow \mathcal{N}_{T_{i,n}/X(J)} \rightarrow \mathcal{N}_f \otimes \mathcal{O}_{T_{i,n}} \rightarrow Q(T_{i,n}) \rightarrow 0,$$

where $Q(T_{i,n})$ is a torsion sheaf, supported at the nodes of $C$ on $T_{i,n}$. However, the positive summands of $\mathcal{N}_{T_{i,n}/X(J)}$ are saturated in $\mathcal{N}_f$; only the negative summand fails to be saturated. When $n = 1, \ldots, N-1$, the negative summand is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(-2)$ and $Q(T_{i,n})$ has length two, so the saturation is $\mathcal{O}_{\mathbb{P}^1}$. When $n = N$, the negative summand is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(-1)$ and $Q(T_{i,N})$ has length two and support $\{r_i, q_{i,N-1}\}$, so the saturation is $\mathcal{O}_{\mathbb{P}^1}$. When $n = 0$ we have

$$0 \rightarrow \mathcal{N}_{f_{i,0}} \rightarrow \mathcal{N}_f \otimes \mathcal{O}_{r_{i,0}} \rightarrow Q(T_{i,0}) \rightarrow 0,$$

with $Q(T_{i,0})$ of length one and supported at $q_{i,0}$. The negative summand of $\mathcal{N}_{f_{i,0}}$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(-1)$, so the extension above induces

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^1}(-1) \rightarrow \mathcal{O}_{\mathbb{P}^1} \rightarrow Q(T_{i,0}) \rightarrow 0,$$

i.e., the saturation of the negative factor is $\mathcal{O}_{\mathbb{P}^1}$.

To summarize, we have shown

$$\mathcal{N}_f \otimes \mathcal{O}_{T_{i,n}} = \begin{cases} \mathcal{O}_{\mathbb{P}^1}(+1)^d & \text{for } n = N \\ \mathcal{O}_{\mathbb{P}^1}^d & \text{for } n = 1, \ldots, N-1 \\ \oplus_{w=1}^{d-1} \mathcal{O}_{\mathbb{P}^1}(a_w) \oplus \mathcal{O}_{\mathbb{P}^1}, & a_w \geq 0 \text{ for } n = 0 \end{cases},$$

so the hypotheses of Proposition 22 hold for the broken teeth $T_i$.

For $\ell = |I'| + 1, \ldots, m$ we take

$$f_\ell : T_\ell \rightarrow X_{b_\ell}$$

to be free rational curves, immersed in generic fibers of good reduction so that the images are nodal, with $\sigma(b_\ell) \in f_\ell(T_\ell)$ as a smooth point. We choose these with generic tangent directions $\xi_\ell \subset T_{\sigma(b_\ell)}X_{b_\ell}$ so that Lemma 18 guarantees $\mathcal{N}_f \otimes \mathcal{O}_{\sigma(B)}(-\sum_{i \in I''} r_i)$ is globally-generated and has no higher cohomology.
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