The maximal singular fibres of elliptic K3 surfaces

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Abstract. We prove that the maximal singular fibres of elliptic K3 surfaces have type $I_{19}$ and $I^*_{14}$ unless the characteristic of the ground field is 2. In characteristic 2, the maximal singular fibres are $I_{18}$ and $I^*_{13}$. The paper supplements work of Shioda in [9] and [10].

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Key words: elliptic surface, (supersingular) K3 surface, singular fibre, Artin invariant.

1 Introduction

This note investigates the maximal singular fibres of elliptic K3 surfaces

$$\pi : X \to \mathbb{P}^1$$

in arbitrary characteristic. Here, maximality should be understood in terms of the number of components. Throughout we will assume the fibration to have a section (since otherwise we can consider its Jacobian).

In characteristic 0, the maximal fibres are known to have type $I_{19}$ and $I^*_{14}$ in Kodaira’s notation. This follows from the Lefschetz bound

$$\rho(X) \leq h^{1,1}(X) = 20$$

by way of the Shioda-Tate formula [8, Cor. 5.3]. Here, $\rho(X) = \text{rk } NS(X)$ denotes the Picard number, i.e. the rank of the Néron-Severi group. The existence of such K3 fibrations follows from the classifications of [4] and [5].

In positive characteristic $p$, however, we only have the trivial weaker bound

$$\rho(X) \leq b_2(X) = 22$$

(2)

involving the second Betti number $b_2(X)$. Note that here $\rho(X) = 21$ is impossible by the work of M. Artin [1]. Hence, the existence of a larger singular fibre (than $I_{19}$ or $I^*_{14}$) of the elliptic K3 surface $\pi : X \to \mathbb{P}^1$ already implies

$$\rho(X) = 22,$$

again due to the Shioda-Tate formula. In other words, $X$ is a supersingular K3 surface.

This property enabled Shioda in [9, Rem. (4.3)] to exclude the fibres of type $I_{21}$ and $I^*_{16}$ which are a priori maximal possible by equation (2). In [10], he reproved existence and maximality of $I_{19}$ and $I^*_{14}$ over $\mathbb{C}$ by using Davenport-Stothers triples. The remark in [10, §3] states that the arguments for the maximality stay valid in characteristic $p$ if $p$ is sufficiently large. In this note, we will employ a different approach to extend these results to arbitrary characteristic.
2 Multiplicative case

Theorem 1.1

In every characteristic $p \neq 2$, the maximal singular fibres of elliptic K3 fibrations are of type $I_{19}$ and $I_{14}^*$. In characteristic 2, they are $I_{18}$ and $I_{13}^*$. 

The main claim of the theorem will follow from Theorems 2.2 and 6.1. For odd characteristic, the proofs rely on a strong property of supersingular K3 surfaces, encoded in the Artin invariant. We then only use elementary congruences involving the height pairing.

In characteristic 2, we will perform explicit calculations involving the Weierstrass equation. In the additive case, we will also make extensive use of Tate’s algorithm to determine the explicit type of the special fibre. It then turns out that the methods already apply to the subsequent fibre types. The corresponding results can be found in Propositions 2.3 and 6.2.

The paper is organized as follows: We first consider multiplicative fibres and give a complete proof for these. This is followed by an application to the reduction of the $[1,1,1,1,1,19]$ fibration (Cor. 2.4), i.e. the elliptic K3 surface with singular fibres of types $I_1, \ldots, I_{19}$. The note is concluded by the proofs for additive fibres, but these will only be sketched roughly.

2 Multiplicative case

The following three sections are devoted to the proof that in arbitrary characteristic there is no elliptic K3 surface

$$\pi : X \rightarrow \mathbb{P}^1$$

with a fibre of type $I_{20}$. To achieve this, we will assume on the contrary that there is such a fibration, and establish a contradiction. In particular, we require that the characteristic $p$ is positive due to the Lefschetz bound (1) in zero characteristic.

Since $\rho(X) \neq 21$, the existence of a fibre of type $I_{20}$ (or $I_{21}$) already implies

$$\rho(X) = 22$$

by the Shioda-Tate formula. Hence $X$ is a supersingular K3 surface. Then, we have the following result concerning the Néron-Severi lattice $NS(X)$ which goes back to M. Artin [1] (for $p = 2$, this is due to Rudakov and Šafarevič [3]):

Theorem 2.1 (Artin, Rudakov-Šafarevič)

Let $X$ be a supersingular K3 surface over a field of characteristic $p$. Then

$$\text{discr } NS(X) = -p^{2\sigma_0} \quad \text{for some } \sigma_0 \in \{1, \ldots, 10\}.$$ 

Here, $\sigma_0$ is called the Artin invariant.

As a direct application, we can rule out the Mordell-Weil group $MW(\pi)$ of the K3 surface $\pi : X \rightarrow \mathbb{P}^1$ to be finite, if there is a fibre of type $I_{20}$. This was pointed out by Shioda in [2] Rem. (4.3)]. Otherwise there would be exactly one further reducible fibre. Since both possible types $I_2$ and $III$ correspond to the root lattice $A_1$ (as $I_{20}$ corresponds to $A_{19}$), this would give

$$\text{discr } NS(X) = -\frac{(\text{discr } A_1)(\text{discr } A_{19})}{|MW(\pi)|^2} = -10 \text{ or } -40. \quad (3)$$

This is clearly impossible. The same argument directly excludes fibres of type $I_{21}$. Hence a fibre of type $I_{20}$ is a priori the maximal possible for an elliptic K3 surface in positive characteristic.
Theorem 2.2
There is no elliptic K3 fibration with a singular fibre of type $I_{20}$.

The proof of this theorem will require us to distinguish between even and odd characteristic. By our above considerations, an elliptic K3 fibration $\pi : X \to \mathbb{P}^1$ with an $I_{20}$ fibre would necessarily have Mordell-Weil group of rank 1. Hence, the discriminant of $NS(X)$ could be expressed in terms of the height of the generator $P$ of $MW(\pi)$ (up to torsion). In odd characteristic, it will then be an easy exercise in congruences using the height pairing to derive a contradiction to Theorem 2.1.

For characteristic 2, on the contrary, we will have to perform explicit calculations involving the Weierstrass equation to verify Theorem 2.2. It will then turn out that the same arguments apply to fibres of type $I_{19}$.

Proposition 2.3
There is no elliptic K3 fibration over a field of characteristic 2 with a singular fibre of type $I_{19}$.

This can be compared to all other characteristics where there is indeed such a fibration. This comes from the $[1,1,1,1,1,19]$-fibration over $\mathbb{Q}$ (unique up to isomorphism) which was given by Shioda in [9]. By way of reducing mod $p$, his equation gives rise to fibrations in question for all characteristics $p > 3$. Furthermore, a $\mathbb{Q}(\sqrt{-3})$-isomorphic model with good reduction at $p = 3$ was found in [7]. In particular, this shows that the maximal multiplicative fibre given in Theorem 1.1 does occur in all characteristics different from 2. On the other hand, Proposition 2.3 implies the following

Corollary 2.4
The $[1,1,1,1,1,19]$ fibration, considered over any number field, cannot have good reduction at a prime above 2.

This answers the question of [7] which in the first instance motivated this note.

3 Proof of Theorem 2.2 in odd characteristic

In this section we are going to give a proof of Theorem 2.2 for odd characteristic $p > 2$. For this purpose, we assume that there is an elliptic K3 fibration $\pi : X \to \mathbb{P}^1$ with an $I_{20}$ fibre and establish a contradiction. In this setting, we have already excluded finite $MW(\pi)$ by discussing equation (3). In other words, $NS(X)$ is generated by the components of the $I_{20}$ fibre (which is the only reducible fibre), the 0-section $O$ and the section $P$ which generates the rank one Mordell-Weil group.

More precisely, this generation is only up to the torsion in $MW(\pi)$. However, there can only be $p$-torsion. This is because any torsion section of order $q$ coprime to $p$ gives rise to another elliptic K3 surface by way of the quotient by its translation. Here, this is a priori impossible for $q \neq 5$ due to the resulting fibre types which contradict the Euler number.

To exclude 5-torsion as well, we further need that the moduli problem for $\Gamma_1(5)$ is representable. Hence $\pi$ would factor through the corresponding modular surface $Y_1(5)$. In characteristic $p \neq 5$, this has the configuration of singular fibres $[1,1,5,5]$ (conf. [3]). Thus, this fibration cannot be connected to a K3 surface with a fibre of type $I_{20}$ by way of pull-back. This rules out 5-torsion in $MW(\pi)$ for $p \neq 5$. 


Let $<,>$ denote the height pairing as introduced by Shioda in [5 Sect. 8]. It induces the structure of a positive-definite lattice on $MW(\pi)/MW(\pi)_{tor}$. In our case, we have

$$< P, P > = 4 + 2(P,O) - \frac{i(20 - i)}{20} > 0. \quad (4)$$

Here, $(P,O)$ is the intersection number of the 0-section $O$ and the section $P$ in $NS(X)$. In particular, this is an integer, but we will not need any other information about it. Furthermore, $i$ denotes the component $C_i$ of the $I_{20}$ fibre which $P$ meets; that is, the components $C_j$ are numbered cyclically (up to orientation) such that $O$ meets $C_0$.

The height pairing uses the projection onto the orthogonal complement of the trivial sublattice generated by $O$ and the vertical divisors in $NS(X) \otimes \mathbb{Q}$. Hence we have, up to an even power of $p$ as explained,

$$\text{discr } NS(X) = -20 < P, P >. \quad (5)$$

Combining equations (4) and (5), we obtain

$$|\text{discr } NS(X)| = 80 + 40 (\varphi(P).O) - i(20 - i).$$

Comparing with Theorem 2.1 and reducing mod 8, this gives

$$i^2 - 4i \equiv p^{2x_0} \mod 8.$$

This congruence is compatible with the indeterminacy in the torsion of $MW(\pi)$. In particular, $i$ has to be odd. Inserting leads to the congruence

$$-3 \equiv 1 \mod 8.$$  

This gives the required contradiction. Thus, we have proved Theorem 2.2 in odd characteristic.

## 4 Proof of Theorem 2.2 in characteristic 2

The above argument breaks down in characteristic 2. Instead, we will perform an explicit calculation involving the general Weierstrass equation and the corresponding discriminant to prove Theorem 2.2 in this characteristic.

We start with the general Weierstrass equation

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6 \quad (6)$$

where the $a_i$ are polynomials in $\overline{\mathbb{F}}_2[t]$ of degree (at most) $2i$. In order to define an honest K3 surface (instead of a rational surface, the product of two elliptic curves or a singular surface), we further have to impose some conditions on the $a_i$ (e.g. there is an $i$ such that deg $a_i > i$), but we will not go into detail with this here.

In characteristic 2, the discriminant is given by

$$\Delta = a_1^4(a_4^2a_6 + a_1a_3a_4 + a_2a_3^2 + a_4^2) + a_3^3a_4^3 + a_3^4. \quad (7)$$

We emphasize that next to the vanishing order of $\Delta$, we also have to take the possibility of wild ramification into account. This can only occur at the additive fibres.

Our next aim is to normalize the Weierstrass equation (6). For this purpose we adopt the techniques of [8]. Since our analysis will depend on the quadratic polynomial $a_1$, we have to distinguish between three cases:
(i) \( a_1 \equiv 0 \),
(ii) \( a_1 \neq 0 \) is a square in \( \mathbb{F}_2[t] \),
(iii) \( a_1 \) has two distinct zeroes (possibly including \( \infty \)).

In case [i], all the singular fibres lie above the zeroes of \( a_3 \) and therefore are additive.
Thus there cannot be any singular fibre of type \( I_n \), \( n > 0 \), at all.

In case [ii], we normalize \( a_1 \) to become \( t^2 \). Then, using successive translations
\[ x \mapsto x + \alpha, \ y \mapsto y + \beta \quad \text{and} \quad y \mapsto y + \gamma x, \]
we can assume that
\[ a_3 = at + b, \ a_4 = ct + d \quad \text{and} \quad a_2 = t\tilde{a}_2, \]
respectively, where \( a, b, c, d \) are constants and \( \tilde{a}_2 \) has degree 3. Thus equation [8] becomes
\[ y^2 + t^2 xy + (at + b)y = x^3 + t\tilde{a}_2 x^2 + (ct + d)x + a_6 \quad \text{(8)} \]
with discriminant
\[ \Delta = t^6(t^4a_6 + t^2(at + b)(ct + d) + t\tilde{a}_2(at + b)^2 + (ct + d)^2) + t^8(at + b)^3 + (at + b)^4. \quad \text{(9)} \]

Note that we still have enough freedom to move a fibre above \( t_0 \neq 0 \) to \( \infty \) without changing the general shape of equation [8]. This comes from the translation \( s \mapsto s + \frac{1}{t} \) in the hidden parameter \( s = \frac{1}{t} \) at \( \infty \).

Hence, we only have to decide whether it is possible for an \( I_{20} \) fibre in equation [8] to sit above 0 or \( \infty \). We shall see that the same argument works for fibres of type \( I_{19} \). In fact, such a fibre above \( \infty \) is a priori impossible, since \( \Delta \) does not contain any term corresponding to \( t^5 \).

At first, let us assume that the vanishing order of \( \Delta \) at \( \infty \) is at least 20: \( v_\infty(\Delta) \geq 20 \).
That is, all terms in \( \Delta \) of order greater than 4 in \( t \) vanish. Writing \( \Delta = \sum d_it^i \), the first non-trivial equations read
\[ d_6 = b^3 = 0, \quad d_7 = ab^2, \quad d_8 = d^2 + a^2b = 0, \quad d_9 = a^3 + b^2\tilde{a}_2(0) = 0, \]
so \( b = d = a = 0 \). The further elimination process kills every coefficient except for \( \tilde{a}_2 \). But then the resulting fibration becomes singular in codimension 1 (\( \Delta \equiv 0 \)). This rules out \( v_\infty(\Delta) \geq 20 \).

Similarly, if \( v_0(\Delta) > 4 \), then \( d_0 = b^4 = 0 \) and \( d_4 = a^4 = 0 \). Hence the reduction at \( t = 0 \) becomes additive. This rules out the existence of an \( I_n \) fibre above 0 for \( n > 4 \).

For a brief description of the resulting additive fibre, we refer to the discussion in Section [022].

The remaining case which we have to consider is [iii]. Here we normalize \( a_1 = t \). In this setting, the changes of variables \( x \mapsto x + \alpha \) and \( y \mapsto y + \beta \) lead to the Weierstrass equation
\[ y^2 + txy + (at^6 + b)y = x^3 + a_2 x^2 + (ct^8 + d)x + a_6 \quad \text{(10)} \]
with discriminant
\[ \Delta = t^4(t^2a_6 + t(at^6 + b)(ct^8 + d) + a_2(at^6 + b)^2 + (ct^8 + d)^2) + t^3(at^6 + b)^3 + (at^6 + b)^4. \quad \text{(11)} \]
Note that the Weierstrass equation (10) is symmetric in 0 and \( \infty \) in the following sense: It takes the same general shape for the local parameters \( t \) and \( s = \frac{1}{t} \). On the other hand, we cannot move a fibre to 0 or \( \infty \) anymore while preserving the type of equation as given in (10). Hence we have to consider the problem of an \( I_{20} \) fibre above 0 (or equivalently \( \infty \)) and above \( t_0 \neq 0, \infty \). After rescaling, we will assume \( t_0 = 1 \).

At first, we shall assume that there is a singular fibre above 0 for the Weierstrass equation (10). Here \( v_0(\Delta) > 0 \) is equivalent to \( b = 0 \), but then the fibre has additive reduction. Hence multiplicative singular fibres above 0 are not compatible with equation (10).

On the other hand, let us investigate an \( I_{20} \) fibre above the fixed point \( t_0 = 1 \). In more generality, we set

\[ \Delta = (t + 1)^{20}g \]

with a non-zero polynomial \( g = \sum g_it^i \) of degree at most 4, i.e. \( v_1(\Delta) \geq 20 \). Here,

\[ (t + 1)^{20} \equiv t^{20} + t^{16} + t^4 + 1 \mod 2, \]

such that already \( g_1 = g_2 = 0 \) by the absence of \( t \) and \( t^2 \) terms in \( \Delta \). In total, an extensive comparison of coefficients again produces only zeroes except for \( a_2 \). But then, the fibre once more becomes singular. This rules out an \( I_{20} \) fibre in case (iii) and thereby concludes the proof of Theorem 2.2.

### 5 Application to \( I_{19} \) fibres

The methods used in the last section can directly be applied to the question of the existence of an elliptic K3 surface with an \( I_{19} \) fibre in characteristic 2. We will prove that there is no such (Prop. 2.3).

In the last section, we indicated in all cases but the last that the vanishing order \( v(\Delta) \geq 19 \) also results either in additive reduction or in a singular fibration. Hence, the same arguments also rule out fibres of type \( I_{19} \).

Concerning the last situation, the same explicit elimination as above can be applied to the more general case of \( v_{t_0}(\Delta) \geq 19 \). Since this again gives rise to a singular fibration, we have thus also completed the proof of Proposition 2.3.

Together with Theorem 2.2 this concludes the investigation of the maximal multiplicative fibre for an elliptic K3 surface in positive characteristic \( p \). For \( p > 2 \), this is \( I_{19} \), appearing in the reduction of a suitable \( \mathbb{Q} \)-model of the \([1,1,1,1,1,19]\) fibration as indicated after Proposition 2.3. For \( p = 2 \), the maximal multiplicative fibre has type \( I_{18} \). For instance, this occurs in the purely inseparable base change of degree 2 of the (extremal) rational elliptic surface with fibres \([1,1,1,19]\). In other words, the maximal multiplicative fibres which Theorem 1.1 lists, do exist in some elliptic K3 fibration.

We shall now return to the \([1,1,1,1,1,19]\) fibration in characteristic 0, as analyzed in [7] and [9]. Since it has a model over \( \mathbb{Q} \), we can consider this surface over any number field \( K \). Then Proposition 2.3 implies that it has bad reduction at all the primes of \( K \) above 2 (Cor. 2.3).

More precisely, we will deduce that at any prime of \( K \) above 2 the fibration cannot have reduction with only isolated ADE-singularities. In order to distinguish from good reduction, one might compare two cases:
On the one hand, consider the $\mathbb{Q}$-model of the surface and its reduction mod 19. This has an additional $A_1$-singularity, so strictly speaking, the reduction at 19 is bad. The five $I_1$ fibres degenerate to two fibres of type $II$ and $III$. Meanwhile, the $I_{19}$ fibre is preserved. This is impossible in residue characteristic 2.

On the other hand, we can also consider a possible degeneration of the $I_{19}$ fibre itself. By way of reducing mod 2 and resolving singularities, it might become additive. A comparable case consists of the [1,1,1,1,4,18] fibration over $\mathbb{Q}$ (base change of degree 2 of [1,1,2,8]): Its reduction mod 2 has only 2 singular fibres where the $I_{18}$ fibre is preserved and the others degenerate to one singular fibre of type $I^*_1$.

We will see that such a degeneration of an $I_{19}$ fibre in an elliptic K3 fibration is impossible in any characteristic. Since the degenerate fibre has to contain a chain of 18 rational (-2)-curves, its type can only be $I^*_n$ for some $n > 14$. Hence, the claim will follow from Theorem 6.1.

6 Additive case

This section investigates the maximal additive fibre of elliptic K3 surfaces. In characteristic 0 or $p > 3$, we have an obvious model with an $I^*_14$ fibre. This comes from the $\mathbb{Q}$-model of the [1,1,1,1,14*] fibration given by Shioda in [9] by way of good reduction. For $p = 3$, we can produce a model with good reduction at $p$ by twisting over $\mathbb{Q}(\sqrt{3})$ and then translating $x \mapsto x - \frac{s - 2s^3}{3}$. The resulting Weierstrass equation is

$$y^2 = x^3 - s(1 + 2s^2) x^2 - 2s^6(1 + s^2) x - s^{11}.$$ 

Here, we can already see that the special fibre above 0 has type at least $I^*_9$. A closer exhibition of Tate’s algorithm gives the announced $I^*_14$ fibre which has all components defined over $\mathbb{Q}$.

In characteristic 0, this fibre is clearly maximal. In this section, we want to reprove this for odd characteristics and even give a stronger statement for characteristic 2:

**Theorem 6.1**

*There is no elliptic K3 surface with a fibre of type $I^*_n$ for $n > 14$.***

**Proposition 6.2**

*In characteristic 2, the maximal additive fibre for an elliptic K3 surface is $I^*_13$.***

We shall first prove the theorem in odd characteristic and for special cases. Again, this will rely on the Artin invariant for supersingular K3 surfaces. For the subsequent proof in characteristic 2, we will use Tate’s algorithm to determine the type of the special fibre, employing the approach of Section 4.

Assume the surface had a fibre of type $I^*_{15}$ (or $I^*_{16}$ which is a priori maximal). The Shioda-Tate formula predicts the supersingularity of the K3 surface as before. Hence we can use the Artin invariant in order to establish a contradiction. This works for odd characteristic and two special cases.

For instance, if there was a fibre of type $I^*_{16}$, then

$$\text{discr } NS(X) = -\text{discr } D_{20} = -4.$$ 

Hence, the surface could only live in characteristic 2 by virtue of Theorem 2.1. On the other hand, it would be extremal (i.e. the Picard number $\rho(X)$ is maximal and
the Mordell-Weil group finite). But this contradicts the classification of extremal elliptic K3 surfaces in characteristic 2 as given in \[2\] Table 1).

Similarly, if there was an $I^*_{15}$ fibre, but finite $MW(\pi)$, then

$$\text{discr } NS(X) = -\frac{(\text{discr } D_{19})(\text{discr } A_1)}{|MW(\pi)|^2} = -2 \text{ or } -8$$

gives a direct contradiction. These special cases were pointed out by Shioda in \[9\] Rem. (4.3).

The remaining case of Theorem 6.1 consists of an $I^*_{15}$ fibre together with Mordell-Weil group of rank 1. For the proof, we will distinguish between even (positive) and odd characteristic.

### 6.1 Odd characteristic

If the characteristic $p$ is odd, we can again use a congruence argument involving the Artin invariant and the height pairing to prove Theorem 6.1. Here we write $P$ for the generator of $MW(\pi)$ up to torsion. Note that, as before, there can possibly only be $p$-torsion.

Since all other fibres are irreducible, the height of $P$ is given by

$$< P, P > = 4 + 2(P.O) - \begin{cases} 
0 \\
1 \\
1 + \frac{15}{4}
\end{cases}$$

(12)

depending on the component of the $I^*_{15}$ fibre which $P$ meets (cf. \[8\] Sect. 8)).

Up to an even power of $p$ which comes from the possible $p$-torsion in $MW(\pi)$, the discriminant of the Néron-Severi lattice is given by

$$\text{discr } NS(X) = -4 < P, P > .$$

In the first two cases of (12), this is an even number. This gives the required contradiction to Theorem 2.1. In the third case of (12), we obtain

$$|\text{discr } NS(X)| = 16 + 8(P.O) - 4 - 15.$$  

Modulo 8, this becomes

$$|\text{discr } NS(X)| \equiv -3 \mod 8$$

which again contradicts Theorem 2.1. As this argument is compatible with squares of $p$ (possibly coming from torsion of $MW(\pi)$), it proves Theorem 6.1 in odd characteristic.

### 6.2 Characteristic 2

For the proof of Theorem 6.1 in characteristic $p = 2$, we return to the general Weierstrass form \[8\] and the corresponding discriminant \[7\]. In order to prove Proposition 6.2 as well, we will deal with $I^*_{14}$ and $I^*_{15}$ fibres simultaneously. Hence, choosing the fibre to sit above 0, we have the same condition

$$v_0(\Delta) \geq 20$$

as investigated in the multiplicative case. In view of the previous calculations, we have to consider the following three cases:
(i) \(a_1 \equiv 0\)
(ii) \(a_1 = t^2\),
(iii) \(a_1 = t\).

To determine the type of the special fibre at \(t = 0\), we apply Tate’s algorithm. Roughly speaking, this performs suitable changes of variables in order to obtain the greatest simultaneous vanishing orders of \(a_3, a_4\) and \(a_6\) possible at the fixed point. These respective orders then predict the type of the special fibre. We refer the reader to [11, IV.9] for the details and the notation employed.

In case (i), we have \(\Delta = a_4^3\), so \(a_3 = t^5\) or \(a_3 = t^6\) after normalizing. Note that a change of variables possibly involved in performing Tate’s algorithm, does not affect \(a_3\). Hence the algorithm definitely terminates when \(a_3, 5\) resp. \(a_3, 6\) enters (i.e. at vanishing order \(v_0(a_3) = 5\) resp. 6). The corresponding fibre types are \(I^*_8\) and \(I^*_9\).

In case (ii), a fibre of type \(I^*_n\) above 0 with \(v_0(\Delta) \geq 20\) is easily seen to require the Weierstrass equation

\[
y^2 + t^2xy = x^3 + t\alpha_2^2x^2 + t^8\bar{a}_6
\]

with \(\deg \bar{a}_2 \leq 3\), \(\deg \bar{a}_6 \leq 4\). Here, \(t \nmid \bar{a}_2\), since otherwise the surface would be rational. A careful analysis shows that the special fibre becomes maximal when \(\bar{a}_6 = et^4\). Here, \(e \neq 0\), since otherwise the fibration would be singular (\(\Delta \equiv 0\)). The change of variable \(y \mapsto y + \sqrt{e}t^6\) gives the equivalent Weierstrass equation

\[
y^2 + t^2xy = x^3 + t\bar{a}_2x^2 + t^8\sqrt{e}x.
\]

This shows that the algorithm terminates at type \(I^*_{12}\) where \(a_{4,8} = \sqrt{e}\) enters.

In case (iii), we start with Weierstrass equation (10) and discriminant (11). First of all, the vanishing order \(v_0(\Delta) \geq 20\) implies \(b = 0\). On the other hand, \(a \neq 0\), since otherwise \(\bar{a}_6 = 0\) and the fibration would be singular. Hence we can normalize such that \(a = 1\). We obtain \(d = 0\) and 14 further equations such that the Weierstrass equation (10) finally reads

\[
y^2 + txy + t^6y = x^3 + (et^4 + ct^3 + \bar{a}_6)x^2 + ct^8x + t^{10}\bar{a}_6.
\]

The special fibre can be seen to have type \(I^*_{12}\) unless \(c = \sqrt{e}\). Otherwise, it is \(I^*_{13}\). Note that \(v_0(\Delta) = 21\) if and only if \(c = e\). This completes the proofs of Theorem 6.1 and Proposition 6.2. In particular, the maximal additive fibre for an elliptic K3 fibration in characteristic 2 as given in Proposition 6.2 does exist.

As a corollary, we deduce that no reduction of the \([1,1,1,1,1,19]\) fibration gives an elliptic K3 surface with degenerate \(I_{19}\) fibre. This answers the question stated at the end of the previous section.

Furthermore, Proposition 6.2 implies that the \([1,1,1,1,14^*]\) fibration has bad reduction at 2. This, however, was clear a priori because the Néron-Severi group of this surface has discriminant -4.

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References

[1] Artin, M.: *Supersingular K3 surfaces*, Ann. scient. Éc. Norm. Sup. (4) **7** (1974), pp. 543–568.

[2] Ito, H.: *On extremal elliptic surfaces in characteristic 2 and 3*, Hiroshima Math. J. **32** (2002), pp. 179–188.

[3] Lang, W. E.: *Extremal rational elliptic surfaces in characteristic p. I: Beauville surfaces*, Math. Z. **207** (1991), pp. 429–438.

[4] Miranda, R., Persson, U.: *Configurations of $I_n$ Fibers on Elliptic K3 surfaces*, Math. Z. **201** (1989), pp. 339–361.

[5] Nishiyama, K.: *The Jacobian fibrations on some K3 surfaces and their Mordell-Weil groups*, Japan. J. Math. **22** (1996), pp. 293-347.

[6] Rudakov, A. N., Šafarevič, I. R.: *Supersingular K3 surfaces over fields of characteristic 2*, Math. USSR, Izv. **13**, No. 1 (1979), pp. 147–165.

[7] Schütt, M., Top, J.: *Arithmetic of the [19,1,1,1,1,1] fibration*, preprint (2005), arXiv:math.AG/0510063

[8] Shioda, T.: *On the Mordell-Weil lattices*, Comm. Math. Univ. St. Pauli **39** (1990), pp. 211–240.

[9] Shioda, T.: *The elliptic K3 surfaces with a maximal singular fibre*, C. R. Acad. Sci. Paris, Ser. I, **337** (2003), pp. 461–466.

[10] Shioda, T.: *Elliptic surfaces and Davenport-Stothers triples*, Comm. Math. Univ. St. Pauli **54** (2005), pp. 49–68.

[11] Silverman, J. H.: *Advanced Topics in the Arithmetic of Elliptic Curves*, Graduate Texts in Math. **151**, Springer (1994).

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