EXISTENCE RESULTS FOR MEAN FIELD EQUATIONS

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ABSTRACT. Let \( \Omega \) be an annulus. We prove that the mean field equation

\[
-\Delta \psi = \frac{e^{-\beta \psi}}{\int_{\Omega} e^{-\beta \psi}} \quad \text{in } \Omega
\]

\[
\psi = 0 \quad \text{on } \partial \Omega
\]

admits a solution for \( \beta \in (-16\pi, -8\pi) \). This is a supercritical case for the Moser-Trudinger inequality.

1. INTRODUCTION

Let \( \Omega \) be a smooth bounded domain in \( \mathbb{R}^2 \). In this paper, we consider the following mean field equation

\[
(1.1) \quad -\Delta \psi = \frac{e^{-\beta \psi}}{\int_{\Omega} e^{-\beta \psi}}, \quad \text{in } \Omega,
\]

\[
\psi = 0, \quad \text{on } \partial \Omega,
\]

for \( \beta \in (-\infty, +\infty) \). (1.1) is the Euler-Lagrange equation of the following functional

\[
(1.2) \quad J_\beta(\psi) = \frac{1}{2} \int_{\Omega} |\nabla \psi|^2 + \frac{1}{\beta} \log \int_{\Omega} e^{-\beta \psi}
\]

in \( H_0^{1,2}(\Omega) \). This variational problem arises from Onsager’s vortex model for turbulent Euler flows. In that interpretation, \( \psi \) is the stream function in the infinite vortex limit, see [MP,p256ff]. The corresponding canonical Gibbs measure and partition function are finite precisely if \( \beta > -8\pi \). In that situation, Caglioti et al. [CLMP1] and Kiessling [K] showed the existence of a minimizer of \( J_\beta \). This is based on the Moser-Trudinger inequality

\[
(1.3) \quad \frac{1}{2} \int_{\Omega} |\nabla \psi|^2 \geq \frac{1}{8\pi} \log \int_{\Omega} e^{-8\pi \psi}, \quad \text{for any } \psi \in H_0^{1,2}(\Omega),
\]

which implies the relevant compactness and coercivity condition for \( J_\beta \) in case \( \beta > -8\pi \). For \( \beta \leq -8\pi \), the situation becomes different as described in [CLMP1]. On the unit disk, solutions blow up if one approaches \( \beta = -8\pi \) - the critical case for (1.3) - (see also [CLMP2] and [Su]), and more generally, on starshaped domains, the Pohozaev identity yields a lower bound on the possible values of \( \beta \) for which solutions exist. On the other hand, for an annulus, [CLMP1] constructed radially
symmetric solutions for any $\beta$, and the construction of Bahri-Coron [BC] makes it plausible that solutions on domains with non-trivial topology exist below $-8\pi$. Thus, for $\beta \leq -8\pi$, $J_\beta$ is no longer compact and coercive in general, and the existence of solution depends on the geometry of the domain.

In the present paper, we thus consider the supercritical case $\beta < -8\pi$ on domains with non-trivial topology.

**Theorem 1.1.** Let $\Omega \subset \mathbb{R}^2$ be a smooth, bounded domain whose complement contains a bounded region, e.g. $\Omega$ an annulus. Then (1.1) has a solution for all $\beta \in (-16\pi, -8\pi)$.

The solutions we find, however, are not minimizers of $J_\beta$—those do not exist in case $\beta < 8\pi$, since $J_\beta$ has no lower bound—but unstable critical points. Thus, these solutions might not be relevant to the turbulence problem that was at the basis of [CLMP1] and [K].

Certainly we can generalize Theorem 1.1 to the following equation

$$-\Delta \psi = \frac{Ke^{-\beta \psi}}{\int_{\Omega} Ke^{-\beta \psi}}, \quad \text{in } \Omega,$$

$$\psi = 0, \quad \text{on } \partial \Omega,$$

which was studied in [CLMP2]. Here $K$ is a positive function on $\Omega$.

With the same method, we may also handle the equation

$$\Delta u - c + cKe^u = 0, \quad \text{for } 0 \leq c < \infty$$

(1.4) on a compact Riemann surface $\Sigma$ of genus at least 1, where $K$ is a positive function. (1.4) can also be considered as a mean field equation because it is the Euler-Lagrange equation of the functional

$$J_c(u) = \frac{1}{2} \int_{\Sigma} |\nabla u|^2 + c \int_{\Sigma} u - c \log \int_{\Sigma} Ke^u.$$  

Because of the term $c \int_{\Sigma} u$, $J_c$ remains invariant under adding a constant to $u$, and therefore we may normalize $u$ by the condition

$$\int_{\Sigma} Ke^u = 1$$

which explains the absence of the factor $(\int K e^u)^{-1}$ in (1.4). $c < 8\pi$ again is a subcritical case that can easily be handled with the Moser-Trudinger inequality. The critical case $c = 8\pi$ yields the so-called Kazdan-Warner equation [KW] and was treated in [DJLW] and [NT] by giving sufficient conditions for the existence of a minimizer of $J_{8\pi}$. Here, we construct again saddle point type critical points to show

**Theorem 1.2.** Let $\Sigma$ be a compact Riemann surface of positive genus. Then (1.4) admits a non-minimal solution for $8\pi < c < 16\pi$.

Now we give a outline of the proof of the Theorems. First from the non-trivial topology of the domain, we can define a minimax value $\alpha_\beta$, which is bounded below
by an improved Moser-Trudinger inequality, for $\beta \in (-16\pi, -8\pi)$. Using a trick introduced by Struwe in [St1] and [St2], for a certain dense subset $\Lambda \subset (-16\pi, -8\pi)$ we can overcome the lack of a coercivity condition and show that $\alpha_\beta$ is achieved by some $u_\beta$ for $\beta \in \Lambda$. Next, for any fixed $\bar{\beta} \in (-16\pi, -8\pi)$, considering a sequence $\beta_k \subset \Lambda$ tending to $\bar{\beta}$, with the help of results in [BM] and [LS] we show that $u_{\beta_k}$ subconverges strongly to some $u_{\bar{\beta}}$ which achieves $\alpha_{\bar{\beta}}$.

After completing our paper, we were informed that Struwe and Tarantello [ST] obtained a non-constant solution of (1.4), when $\Sigma$ is a flat torus with fundamental cell domain $[-\frac{1}{2}, \frac{1}{2}] \times [-\frac{1}{2}, \frac{1}{2}]$, $K \equiv 1$ and $c \in (8\pi, 4\pi^2)$. In this case, it is easy to check that our solution obtained in Theorem 1.2 is non-constant.

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2. Minimax values

Let $\rho = -\beta$ and $u = -\beta \psi$. We rewrite (1.1) as

$$\begin{align*}
-\Delta u &= \rho \frac{e^u}{\int_{\Omega} e^u}, \quad \text{in } \Omega, \\
u &= 0, \quad \text{on } \partial \Omega,
\end{align*}$$

and (1.2) as

$$J_{\rho}(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 - \rho \log \int_{\Omega} e^u$$

for $u \in H^{1,2}_0(\Omega)$.

It is easy to see that $J_{\rho}$ has no lower bound for $\rho \in (8\pi, 16\pi)$. Hence, to get a solution of (1.1) for $\rho \in (8\pi, 16\pi)$, we have to use a minimax method. First, we define a center of mass of $u$ by

$$m_c(u) = \frac{\int_{\Omega} x e^u}{\int_{\Omega} e^u}.$$ 

Let $B$ be the bounded component of $\mathbb{R}^2 \setminus \Omega$. For simplicity, we assume that $B$ is the unit disk centered at the origin. Then we define a family of functions

$$h : D \to H^{1,2}_0(\Omega)$$

satisfying

$$\lim_{r \to 1} J_{\rho}(h(r, \theta)) \to -\infty$$

and

$$\lim_{r \to 1} m_c(h(r, \theta))$$

is a continuous curve enclosing $B$.

Here $D = \{(r, \theta) | 0 \leq r < 1, \theta \in [0, 2\pi)\}$ is the open unit disk. We denote the set of all such families by $D_{\rho}$. It is easy to check that $D_{\rho} \neq \emptyset$. Now we can define a minimax value

$$\alpha_{\rho} := \inf_{h \in D_{\rho}} \sup_{u \in h(D)} J_{\rho}(u).$$

The following lemma will make crucial use of the non-trivial topology of $\Omega$, more precisely of the fact that the complement of $\Omega$ has a bounded component.
Lemma 2.1. $\alpha_\rho > -\infty$ for any $\rho \in (8\pi, 16\pi)$.

Remark. It is an interesting question weather $\alpha_{16\pi} = -\infty$.

To prove Lemma 2.1, we use the improved Moser-Trudinger inequality of [CL] (see also [A]). Here we have to modify a little bit.

Lemma 2.2. Let $S_1$ and $S_2$ be two subsets of $\bar{\Omega}$ satisfying $\text{dist}(S_1, S_2) \geq \delta_0 > 0$ and $\gamma_0 \in (0, 1/2)$. For any $\epsilon > 0$, there exists a constant $c = c(\epsilon, \delta_0, \gamma_0) > 0$ such that
\[
\int_\Omega e^u \leq c \exp\left\{ \frac{1}{32\pi - \epsilon} \int_\Omega |\nabla u|^2 + c \right\}
\]
holds for all $u \in H^{1,2}_0(\Omega)$ satisfying
\[
\frac{\int_{S_1} e^u}{\int_\Omega e^u} \geq \gamma_0 \quad \text{and} \quad \frac{\int_{S_2} e^u}{\int_\Omega e^u} \geq \gamma_0.
\]

Proof. The Lemma follows from the argument in [CL] and the following Moser-Trudinger inequality
\[
(*) \quad \frac{1}{2} \int_\Omega |\nabla u|^2 - 8\pi \log \int_\Omega e^u \geq c
\]
for any $u \in H^{1,2}_0(\Omega)$, where $c$ is a constant independent of $u \in H^{1,2}_0(\Omega)$. □

We will discuss the inequality $(*)$ and its application in another paper.

Proof of Lemma 2.1. For fixed $\rho \in (8\pi, 16\pi)$ we claim that there exists a constant $c_\rho$ such that
\[
\sup_{u \in h(D)} J_\rho(u) \geq c_\rho, \quad \text{for any } h \in \mathcal{D}_\rho.
\]

Clearly (2.6) implies the Lemma. By the definition of $h$, for any $h \in \mathcal{D}_\rho$, there exists $u \in h(D)$ such that
\[
m_c(u) = 0.
\]

We choose $\epsilon > 0$ so small that $\rho < 16\pi - 2\epsilon$. Assume (2.6) does not hold. Then we have sequences $\{h_i\} \subset \mathcal{D}_\rho$ and $\{u_i\} \subset H^{1,2}_0(\Omega)$ such that $u_i \in h_i(D)$ and
\[
m_c(u_i) = 0
\]

(2.7)
\[
\lim_{i \to \infty} J(u_i) = -\infty.
\]

We have the following Lemma.
Lemma 2.3. There exists $x_0 \in \Omega$ such that

$$(2.9) \quad \lim_{i \to \infty} \frac{\int_{B_{1/2}(x_0) \cap \Omega} e^{u_i}}{\int_{\Omega} e^{u_i}} \to 1.$$ 

Proof. Set

$$A(x) := \lim_{i \to \infty} \frac{\int_{B_{1/4}(x) \cap \Omega} e^{u_i}}{\int_{\Omega} e^{u_i}}.$$ 

Assume that the Lemma were false, then there exists $x_0 \in \bar{\Omega}$ such that

$$A(x_0) < 1 \quad \text{and} \quad A(x_0) \geq A(x) \quad \text{for any} \ x \in \Omega.$$ 

It is easy to check $A(x_0) > 0$, since $\Omega$ can be covered by finite many balls of radius 1/4. Let $\gamma_0 = A(x_0)/2$. Recalling (2.8) and applying lemma 2.2, we obtain

$$\int_{\Omega \setminus B_{1/2}(x_0)} e^{u_i} \to 0$$

as $i \to \infty$, which implies (2.9). \qed

Now we continue to prove Lemma 2.1. (2.9) implies

$$\frac{\int_{\Omega} x e^{u_i}}{\int_{\Omega} e^{u_i}} - x_0 = \frac{\int_{\Omega} (x - x_0) e^{u_i}}{\int_{\Omega} e^{u_i}} = \frac{\int_{B_{1/2}(x_0)} (x - x_0) e^{u_i}}{\int_{\Omega} e^{u_i}} + o(1)$$

which, in turn, implies that $|m_c(u_i) - x_0| < 2/3$. This contradicts (2.7). \qed

Lemma 2.4. $\alpha_\rho/\rho$ is non-increasing in $(8\pi, 16\pi)$.

Proof. We first observe that if $J(u) \leq 0$, then $\log \int_{\Omega} e^u > 0$ which implies that

$$J_\rho(u) \geq J_{\rho'}(u) \quad \text{for} \ \rho' \geq \rho.$$ 

Hence $D_\rho \subset D_{\rho'}$ for any $16\pi > \rho' \geq \rho > 8\pi$. On the other hand, it is clear that

$$\frac{J_\rho}{\rho} - \frac{J_{\rho'}}{\rho'} = \frac{1}{2} \left( \frac{1}{\rho} - \frac{1}{\rho'} \right) \int_{\Omega} |\nabla u|^2 \geq 0,$$

if $\rho' \geq \rho$. Hence we have

$$\frac{\alpha_\rho}{\rho} \geq \frac{\alpha_{\rho'}}{\rho'},$$

for $16\pi > \rho' \geq \rho > 8\pi$. \qed
3. Existence for a dense set

In this section we show that $\alpha_\rho$ is achieved if $\rho$ belongs to a certain dense subset of $(8\pi, 16\pi)$ defined below.

The crucial problem for our functional is the lack of a coercivity condition, i.e. for a Palais-Smale sequence $u_i$ for $J_\rho$, we do not know whether $\int_\Omega |\nabla u_i|^2$ is bounded. We first have the following lemma.

**Lemma 3.1.** Let $u_i$ be a Palais-Smale sequence for $J_\rho$, i.e. $u_i$ satisfies

\begin{align}
(3.1) \quad |J_\rho(u_i)| &\leq c < \infty \\
(3.2) \quad dJ_\rho(u_i) &\to 0 \text{ strongly in } H^{-1,2}(\Omega)
\end{align}

If, in addition, we have

\begin{align}
(3.3) \quad \int_\Omega |\nabla u_i|^2 &\leq c_0, \quad \text{for } i = 1, 2, \ldots
\end{align}

for a constant $c_0$ independent of $i$, then $u_i$ subconverges to a critical point $u_0$ for $J_\rho$ strongly in $H_0^{1,2}(\Omega)$.

**Proof.** The proof is standard, but we provide it here for convenience of the reader.

Since $\int_\Omega |\nabla u_i|^2$ is bounded, there exists $u_0 \in H_0^{1,2}(\Omega)$ such that

(i) $u_i$ converges to $u_0$ weakly in $H_0^{1,2}(\Omega)$,

(ii) $u_i$ converges to $u_0$ strongly in $L^p(\Omega)$ for any $p > 1$ and almost everywhere,

(iii) $e^{u_i}$ converges to $e^{u_0}$ strongly in $L^p(\Omega)$ for any $p \geq 1$.

From (i)-(iii), we can show that $dJ(u_0) = 0$, i.e. $u_0$ satisfies

\[-\Delta u_0 = \rho \frac{e^{u_0}}{\int_\Omega e^{u_0}}.\]

Testing $dJ_\rho$ with $u_i - u_0$, we obtain

\[o(1) = \langle dJ_\rho(u_i) - dJ_\rho(u), u_i - u_0 \rangle \]

\[= \int_\Omega |\nabla(u_i - u_0)|^2 - \rho \int_\Omega \left( \frac{e^{u_i}}{\int_\Omega e^{u_i}} - \frac{e^{u_0}}{\int_\Omega e^{u_0}} \right)(u_i - u_0) \]

\[= \int_\Omega |\nabla(u_i - u_0)|^2 + o(1),\]

by (i)-(iii). Hence $u_i$ converges to $u_0$ strongly in $H_0^{1,2}(\Omega)$.  

Since by Lemma 2.4 $\rho \to \alpha_\rho/\rho$ is non-increasing in $(8\pi, 16\pi)$, $\rho \to \alpha_\rho/\rho$ is a.e. differentiable. Set

\begin{align}
(3.4) \quad &\Lambda := \{\rho \in (8\pi, 16\pi)| \alpha_\rho/\rho \text{ is differentiable at } \rho\} \\
&\bar{\Lambda} = [8\pi, 16\pi], \text{ see [St1]}. \text{ Let } \rho \in \Lambda \text{ and choose } \rho_k \nearrow \rho \text{ such that}
\end{align}

\[0 \leq \lim_{k \to \infty} -\frac{1}{(\rho - \rho_k)}(\frac{\alpha_\rho}{\rho} - \frac{\alpha_\rho_k}{\rho_k}) \leq c_1\]

for some constant $c_1$ independent of $k$. 

Lemma 3.2. $\alpha_\rho$ is achieved by a critical point $u_\rho$ for $J_\rho$ provided that $\rho \in \Lambda$.

Proof. Assume, by contradiction, that the Lemma were false. From Lemma 3.1, there exists $\delta > 0$ such that

\begin{equation}
\|dJ_\rho(u)\|_{H^{-1,2}(\Omega)} \geq 2\delta
\end{equation}

in

\[ N_\delta := \{ u \in H^{1,2}_0(\Omega) | \int_\Omega |\nabla u|^2 \leq c_2, |J_\rho(u) - \alpha_\rho| < \delta \} \]

Here, $c_2$ is any fixed constant such that $N_\delta \neq \emptyset$. Let $X_\rho : N_\delta \rightarrow H^{1,2}_0(\Omega)$ be a pseudo-gradient vector field for $J_\rho$ in $N_\delta$, i.e. a locally Lipschitz vector field of norm $\|X_\rho\|_{H^{1,2}_0} \leq 1$ with

\begin{equation}
\langle dJ_\rho(u), X_\rho(u) \rangle < -\delta.
\end{equation}

See [P] for the construction of $X_\rho$.

Since

\[
\|dJ_\rho(u) - dJ_{\rho_k}(u)\| = \|dJ_\rho - \frac{\rho}{\rho_k} dJ_{\rho_k}(u)\| + \|(1 - \frac{\rho}{\rho_k})dJ_{\rho_k}(u)\|
\]

\[
\leq \frac{1}{2}(1 - \frac{\rho}{\rho_k}) \int_\Omega |\nabla u|^2 + c(1 - \frac{\rho}{\rho_k}) \int_\Omega |\nabla u|^2 \rightarrow 0
\]

uniformly in $\{ u | \int_\Omega |\nabla u|^2 \leq c_2 \}$, $X_\rho$ is also a pseudo-gradient vector field for $J_{\rho_k}$ in $N_\delta$ with

\begin{equation}
\langle dJ_{\rho_k}(u), X_\rho(u) \rangle < -\delta/2,
\end{equation}

for $u \in N_\delta$, provided that $k$ is sufficiently large.

For any sequence $\{h_k\}$, $h_k \in D_{\rho_k} \subset D_{\rho}$ such that

\begin{equation}
\sup_{u \in h_k(D)} J_{\rho_k}(u) \leq \alpha_{\rho_k} + \rho - \rho_k
\end{equation}

and all $u \in h_k(D)$ such that

\begin{equation}
J_\rho(u) \geq \alpha_\rho - (\rho - \rho_k),
\end{equation}

we have the following estimate

\begin{equation}
\frac{1}{2} \int_\Omega |\nabla u|^2 = \rho \cdot \rho_k \frac{J_{\rho_k}(u)}{\rho_k} - \frac{J_\rho(u)}{\rho} \leq \rho \cdot \rho_k \frac{\alpha_{\rho_k} - \alpha_\rho}{\rho - \rho_k} + (\rho + \rho_k)
\end{equation}

\[
\leq C
\]

by (3.5), (3.9) and (3.10), where $C = (16\pi)^2 c_1 + 32\pi$. 
Now we consider in $\mathcal{N}_\delta$ the following pseudo-gradient flow for $J_\rho$. First choose a Lipschitz continuous cut-off function $\eta$ such that $0 \leq \eta \leq 1$, $\eta = 0$ outside $N_\delta$, $\eta = 1$ in $N_{\delta/2}$. Then consider the following flow in $H^{1,2}_0(\Omega)$ generated by $\eta X_\rho$

$$
\frac{\partial \phi}{\partial t}(u, t) = \eta(\phi(u, t))X_\rho(\phi(u, t))
$$
$$
\phi(u, 0) = u.
$$

By (3.7) and (3.8), for $u \in N_{\delta/2}$, we have

$$
(3.12) \quad \frac{d}{dt}J_\rho(\phi(u, t))|_{t=0} \leq -\delta
$$

and

$$
(3.13) \quad \frac{d}{dt}J_{\rho_k}(\phi(u, t))|_{t=0} \leq -\delta/2
$$

for large $k$.

It is clear that for any $h \in \mathcal{D}_{\rho_k}$ $h(r, \theta) \notin N_\delta$ for $r$ close to 1. Hence $\phi(h, t) \in \mathcal{D}_{\rho_k}$ for any $t > 0$. In particular, $\phi(\cdot, t)$ preserves the class of $h_k \in \mathcal{D}_{\rho_k}$ with condition (3.9). On the other hand, for any $h \in \mathcal{D}_\rho$ by definition

$$
\sup_{u \in h(D)} J_\rho(u) \geq \alpha_\rho.
$$

Hence for any $h_k \in \mathcal{D}_{\rho_k}$ with condition (3.9), $\sup_{u \in \phi(h(D), t)} J_\rho(u)$ is achieved in $N_{\delta/2}$, provided that $k$ is large enough. Consequently, by (3.12), we have

$$
\frac{d}{dt}\sup\{J_\rho(u)|u \in \phi(h(D), t)\} \leq -\delta
$$

for all $t \geq 0$, which is a contradiction. □

4. Proof of Theorem 1.1

From section 3, we know that for any $\bar{\rho} \in (8\pi, 16\pi)$ there exists a sequence $\rho_k \uparrow \bar{\rho}$ such that $\alpha_{\rho_k}$ is achieved by $u_k$. Consequently $u_k$ satisfies

$$
-\Delta u_k = \rho_k \frac{e^{u_k}}{\int_{\Omega} e^{u_k}}, \quad \text{in } \Omega,
$$
$$
u_k = 0, \quad \text{on } \partial \Omega.
$$

From Lemma 2.4, we have

$$
J_\rho(u_k) = \alpha_{\rho_k} \text{ is bounded.}
$$

for some constant $c_0 > 0$ which is independent of $k$. Let $v_k = u_k - \log \int_{\Omega} e^{u_k}$. Then $v_k$ satisfies

$$
-\Delta v_k = \rho_k e^{v_k}
$$

with

$$
\int_{\Omega} e^{v_k} = 1.
$$

By results of Brezis-Merle [BM] and Li-Shafir [LS] we have
**Lemma 4.1.** ([BM], [LS]) There exists a subsequence (also denoted by $v_k$) satisfying one of the following alternatives:

(i) $\{v_k\}$ is bounded in $L^\infty_{\text{loc}}(\Omega)$;
(ii) $v_k \to -\infty$ uniformly on any compact subset of $\Omega$;
(iii) there exists a finite blow-up set $\Sigma = \{a_1, \ldots, a_m\} \subset \Omega$ such that, for any $1 \leq i \leq m$, there exists $\{x_k\} \subset \Omega$, $x_k \to a_i$, $u_k(x_k) \to \infty$, and $v_k(x) \to -\infty$ uniformly on any compact subset of $\Omega \setminus \Sigma$. Moreover,

$$
\rho_k \int_\Omega e^{u_k} \to \sum_{i=1}^m 8\pi n_i \tag{4.5}
$$

where $n_i$ is positive integer.

For our special functions $v_k$, we can improve Lemma 4.1 as follows

**Lemma 4.2.** There exists a subsequence (also denoted by $v_k$) satisfying one of the following alternatives:

(i) $\{v_k\}$ is bounded in $L^\infty_{\text{loc}}(\Omega)$;
(ii) $v_k \to -\infty$ uniformly on $\bar\Omega$;
(iii) there exists a finite blow-up set $\Sigma = \{a_1, \ldots, a_m\} \subset \bar\Omega$ such that, for any $1 \leq i \leq m$, there exists $\{x_k\} \subset \Omega$, $x_k \to a_i$, $u_k(x_k) \to \infty$, and $v_k(x) \to -\infty$ uniformly on any compact subset of $\bar\Omega \setminus \Sigma$. Moreover, (4.5) holds.

**Proof.** From Lemma 4.1, we only have to consider one more case in which blow-up points are in the boundary of $\Omega$. There are two possibilities: One is bubbling too fast such that after rescaling we obtain a solution of $\Delta u = e^u$ in a half plane; Another is bubbling slow such that after rescaling we obtain a solution of $\Delta u = e^u$ in $\mathbb{R}^2$. One can exclude the first case. In the second case, one can follow the idea in [LS] to show that (4.5) holds. See also [L]. □

**Proof of Theorem 1.1.** (4.4), (4.5) and $\bar{\rho} \in (8\pi, 16\pi)$ imply that cases (ii) and (iii) in Lemma 4.2 does not occur. Consequently $\{v_k\}$ is bounded in $L^\infty_{\text{loc}}(\Omega)$. Now we can again apply Lemma 2.2 as follows.

Let $S_1$ and $S_2$ be two disjoint compact subdomains of $\Omega$. Since $\{v_k\}$ is bounded in $L^\infty_{\text{loc}}(\Omega)$, we have

$$
\frac{\int_{S_i} e^{u_k}}{\int_\Omega e^{u_k}} = \frac{\int_{S_i} e^{v_k}}{\int_\Omega e^{v_k}} \geq c_0, \quad i = 1, 2
$$

for a constant $c_0 = c_0(S_1, S_2, \Omega) > 0$ independent of $k$. Choosing $\epsilon$ such that $16\pi - \bar{\rho} > 2\epsilon$ and applying Lemma 2.2, with the help of (4.2), we obtain

$$
c \geq J_{\rho_k}(u_k) = \frac{1}{2} \int_\Omega |\nabla u_k|^2 - \rho_k \log \int_\Omega e^{u_k}
\geq \frac{1}{2} \left(1 - \frac{\rho_k}{16\pi - \epsilon/2}\right) \int_\Omega |\nabla u|^2
\geq \frac{1}{3} \left(1 - \frac{\bar{\rho}}{16\pi - \epsilon/2}\right) \int_\Omega |\nabla u|^2
$$
which implies that \( \int_{\Omega} |\nabla u_k|^2 \) is bounded. Now by the same argument in the proof of Lemma 3.1, \( u_k \) subconverges to \( u_\rho \) strongly in \( H_0^{1,2}(\Omega) \) and \( u_\rho \) is a critical point of \( J_\rho \). Clearly, \( u_\rho \) achieves \( \alpha_\rho \). This finishes the proof of Theorem 1.1.

Proof of Theorem 1.2. Since the proof is very similar to one presented above, we only give a sketch of the proof of Theorem 1.2. Let \( \Sigma \) be a Riemann surface of positive genus. We embed \( X : \Sigma \to \mathbb{R}^N \) for some \( N \geq 3 \) and define the center of mass for a function \( u \in H^{1,2}(\Sigma) \) by

\[
m_c(u) = \frac{\int_{\Sigma} X e^u}{\int_{\Sigma} e^u}.
\]

Since \( \Sigma \) is of positive genus, we can choose a Jordan curve \( \Gamma^1 \) on \( \Sigma \) and a closed curve \( \Gamma^2 \) in \( \mathbb{R}^N \setminus \Sigma \) such that \( \Gamma^1 \) links \( \Gamma^2 \). We know that \( \inf_{u \in H^{1,2}(\Sigma)} J_c(u) \) is finite if and only if \( c \in [0, 8\pi] \) (see [DJLW]). Now define a family of functions \( h : D \to H^{1,2}(\Sigma) \) (as in section 2) satisfying

\[
\lim_{r \to 1} J_\rho(h(r, \theta)) \to -\infty
\]

and

\[
\lim_{r \to 1} m_c(h(r, \theta))
\]

as a map from \( S^1 \to \Gamma^1 \) is of degree 1.

Let \( D_c \) denote the set of all such families. It is also easy to check that \( D_c \neq \emptyset \). Set

\[
\alpha_c := \inf_{h \in D_c} \sup_{u \in h(D)} J_c(u).
\]

We first have

\[
\alpha_c > -\infty,
\]

using the fact that \( \Gamma^1 \) links \( \Gamma^2 \) and Lemma 2.2. Then by the same method as presented above, we can prove that \( \alpha_c \) is achieved by some \( u_c \in H^{1,2}(\Sigma) \), which is a solution of (1.4), for \( c \in (8\pi, 16\pi) \). □

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