POLYNOMIAL TAU-FUNCTIONS OF THE SYMPLECTIC KP, ORTHOGONAL KP AND BUC HIERARCHIES

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Abstract. This paper is concerned with the construction of the polynomial tau-functions of the symplectic KP (SKP), orthogonal KP (OKP) hierarchies and universal character hierarchy of B-type (BUC hierarchy), which are proved as zero modes of certain combinations of the generating functions. By applying the strategy of carrying out the action of the quantum fields on vacuum vector, the generating functions for symplectic Schur function, orthogonal Schur function and generalized $Q$-function have been presented. The remarkable feature is that polynomial tau-functions are the coefficients of certain family of generating functions. Furthermore, in terms of the Vandermonde-like identity and properties of Pfaffian, it is showed that the polynomial tau-functions of the SKP, OKP and BUC hierarchies can be written as determinant and Pfaffian forms, respectively. In addition, the soliton solutions of the SKP and OKP hierarchies have been discussed.

Keywords: polynomial tau-functions; symplectic/orthogonal Schur function; generalized $Q$-function; generating functions; soliton solutions

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1. Introduction

Symmetric functions are the characters of the irreducible highest weight representation of the classical groups [1]-[3], which play a significant role in mathematical physics especially in the theory of integrable systems [4]-[6]. Schur functions and Schur $Q$-functions are the basic symmetric functions which are the solutions of differential equations in Kadomtsev-Petviashvili (KP) and Kadomtsev-Petviashvili sub-hierarchy of B-type (BKP) hierarchies, respectively. In the famous work of the Kyoto School, the authors [7]-[13] investigated the core connection of the infinite dimensional Lie algebra and their highest weight vectors to the integrable hierarchies involving KP, BKP, discrete KP (DKP), modified KP (MKP) and $s$-component KP hierarchies.

Koike [14] introduced a polynomial with a pair of partitions called the universal character, which is a generalization of Schur function. The universal character (UC) hierarchy, proposed by Tsuda [15], as an infinite-dimensional integrable system satisfied by the universal character. It can be regarded as a extension of the KP hierarchy. Furthermore, in the subsequent paper [16]-[19], Tsuda presented the relations between the $q$-Painlevé equations and the lattice $q$-UC hierarchy which are the extended $q$-KP and $q$-UC hierarchies. Based upon this, the structure and properties of the Painlevé equations and their higher order analogues have been developed, such as rational solutions, Lax formalism, bilinear relations for $\tau$-functions and Weyl group symmetry. Wang et.al [20] discussed the algebra of universal characters and the phase model of strongly correlated bosons. Recently, by means of the vertex operator realization of symplectic and orthogonal Schur functions [21], the authors [22][23] established generalizations of symplectic KP (SKP) and orthogonal KP (OKP) hierarchies called the symplectic and orthogonal universal character hierarchies corresponding to symplectic and orthogonal universal characters, respectively. Ogawa [24] defined a generalized $Q$-function expressed by the Pfaffian and constructed an integrable UC hierarchy of B-type (BUC hierarchy) characterized by the generalized $Q$-function. Lately, in the paper [25][26], the author generalized the theory of BUC hierarchy to a coupled and plethystic cases, which can derive coupled and plethystic infinite order nonlinear PDEs.

All the polynomial tau-functions of the KP hierarchy can be expressed as a disjoint union of Schubert cells and the Schur polynomial is the center of the Schubert cell. You [27][28] showed polynomial tau-functions of the BKP, DKP and modified DKP (MDKP) hierarchies all include the $Q$-Schur polynomials which are the centers of Schubert cells of the infinite-dimensional orthogonal Grassmann manifold. Kac et al. [29] constructed all the polynomial tau-functions of the KP and MKP hierarchies from Schur polynomials by some shift of arguments. Moreover, the polynomial tau-functions of the
BKP, DKP, and MDKP hierarchies have been well discussed in terms of boson-fermion correspondence [30]. Then by means of the s-component boson-fermion correspondence, Kac et al. [31] have studied the polynomial tau-functions of the multi-component KP hierarchy. Based on the quantum fields, they also develop the twisted quantum fields presentation of Hall-Littlewood polynomials and derived a novel deformed boson-fermion correspondence [32]. Rozhkovskaya [33] proved multiparameter Schur $Q$-functions are tau-functions of the BKP hierarchy. Besides, in the frame work of quantum fields presentation and generating functions of the symmetric functions, recent study has shown that the polynomial tau-functions of the KP, BKP and s-component KP hierarchies can be expressed as the zero-modes of certain combinations of generating functions [34]. Recently, the polynomial tau-functions of the UC and multi-component UC hierarchies have also been analyzed [35]. To our best knowledge, there is no existing references on the study on the polynomial tau-functions of the SKP, OKP and BUC hierarchies. Based upon the facts, in the view point of the quantum field presentation of the symmetric functions, we will concentrate on the construction of the exact solutions of these integrable hierarchies including polynomial-type and soliton-type solutions. We shall prove the polynomial tau-functions of SKP, OKP and BUC hierarchies can be regarded as zero modes of certain combinatorial generating functions.

The present paper is organized as follows. In Section 2, we begin with a review of the elementary, complete symmetric functions, power sums and Schur polynomial. Section 3 is devoted to construction of quantum fields of symplectic Schur functions and the polynomial tau-functions of the SKP hierarchy. Meanwhile, the $n$-soliton solutions of this integrable system are derived. In section 4, the generating functions for the orthogonal Schur functions are investigated by the action of the operators on the vacuum vector $1$. In terms of quantum fields presentation, we also study the polynomial tau-functions and $n$-soliton solutions of the SKP hierarchy. The fact that the polynomial tau-functions of the BUC hierarchy are the coefficients of certain family generating functions is described in Section 5. The last Section are conclusions and discussions.

2. Preliminaries on symmetric functions

In this section, we mainly retrospect some basic facts and properties about symmetric functions.

Let $\Lambda(x)$ be the ring of symmetric functions in variables $x = (x_1, x_2 \ldots)$. The $r$th elementary symmetric function $e_r$, complete symmetric function $h_r$ and power sum $p_r$ are defined by (cf. [3])

$$e_r(x) = \sum_{i_1 < i_2 < \ldots < i_r < \infty} x_{i_1} x_{i_2} \ldots x_{i_r}, \quad \text{for} \quad r \geq 1,$$

$$h_r(x) = \sum_{i_1 \leq i_2 \leq \ldots \leq i_r \leq \infty} x_{i_1} x_{i_2} \ldots x_{i_r}, \quad \text{for} \quad r \geq 1,$$

$$p_r(x) = \sum_i x_i^r. \quad (2.1)$$
It is universally known that $h_r(x) = e_r(x) = p_r(x) = 0$ for $r < 0$ and $h_0 = e_0 = p_0 = 1$. The generating functions for these symmetric functions are

$$E(u) = \sum_{k \geq 0} e_k(x)u^k = \prod_{i \geq 1} (1 + x_i u) = \exp \left( -\sum_{n \geq 1} \frac{(-1)^n p_n}{n} u^n \right),$$

$$H(u) = \sum_{k \geq 0} h_k(x)u^k = \prod_{i \geq 1} \frac{1}{1 - x_i u} = \exp \left( \sum_{n \geq 1} \frac{p_n}{n} u^n \right),$$

$$P(u) = \sum_{k \geq 1} p_k(x)u^{k-1} = \frac{H'(u)}{H(u)}. \tag{2.2}$$

It is easy to obtain that $H(u)E(-u) = 1$.

The polynomials $S_k(t_1, t_2, \ldots), k \in \mathbb{Z}$, is determined by the generating function

$$\sum_{k \in \mathbb{Z}} S_k(t_1, t_2, \ldots)u^k = \exp \left( \sum_{j=1}^{\infty} t_j u^j \right). \tag{2.3}$$

The Schur polynomial $S_\lambda(t_1, t_2, \ldots)$ can be expressed as (cf. [3])

$$S_\lambda(t_1, t_2, \ldots) = \det[S_{\lambda_i-i+j}(t_1, t_2, \ldots)]_{1 \leq i, j \leq l}, \tag{2.4}$$

where $\lambda = (\lambda_1, \ldots, \lambda_l)$ is a partition.

For each symmetric function $f \in \Lambda$, let $f^\perp : \Lambda \to \Lambda$ be the adjoint of multiplication by $f$

$$\langle f^\perp g, \omega \rangle = \langle g, f \omega \rangle, \quad g, f, \omega \in \Lambda. \tag{2.5}$$

Let us consider generating functions of the adjoint operators

$$E^\perp(u) = \sum_{k \geq 0} \frac{e_k^\perp}{u^k}, \quad H^\perp(u) = \sum_{k \geq 0} \frac{h_k^\perp}{u^k}. \tag{2.6}$$

It is straightforward to show that

$$E^\perp(u) = \exp \left( -\sum_{k \geq 1} (-1)^k \frac{\partial}{\partial p_k} \frac{1}{u^k} \right), \quad H^\perp(u) = \exp \left( \sum_{k \geq 1} \frac{\partial}{\partial p_k} \frac{1}{u^k} \right). \tag{2.7}$$

**Proposition 2.1.** The generating functions satisfy the following relations (cf. [3])

$$\left(1 - \frac{v}{u}\right) E^\perp(u)E(v) = E(v)E^\perp(u),$$

$$\left(1 - \frac{v}{u}\right) H^\perp(u)H(v) = H(v)H^\perp(u),$$

$$H^\perp(u)E(v) = \left(1 + \frac{v}{u}\right) E(v)H^\perp(u),$$

$$E^\perp(u)H(v) = \left(1 + \frac{v}{u}\right) H(v)E^\perp(u). \tag{2.8}$$
3. Polynomial tau-functions and \( n \)-soliton solutions of the SKP hierarchy

By means of charged free fermions, we devote to discussing structures and properties of polynomial tau-functions for the SKP hierarchy. Furthermore, the generating functions for polynomial tau-functions of the SKP hierarchy can be obtained by acting the quantum fields of symplectic Schur functions on the bosonic Fock space \( B^m \). There is an interesting conclusion that the polynomial tau-functions of the SKP hierarchy are the coefficients of certain family of generating functions. Finally, the soliton-type solutions of the SKP hierarchy have been derived.

3.1. Quantum fields presentation of symplectic Schur functions and the SKP hierarchy.

The symmetric polynomial ring \( \Lambda : \Lambda = \mathbb{C}[h_1, h_2, \ldots] = \mathbb{C}[p_1, p_2, \ldots] \) can be generated by elementary, complete symmetric functions and power sums, respectively. Introduce the bosonic Fock space \( B = \mathbb{C}[z, z^{-1}] \otimes \Lambda \), it is decomposed to obtain the charged graded space

\[
B = \bigoplus_{m \in \mathbb{Z}} B^m, \quad \text{where } B^m = \mathbb{C}[p_1, p_2, \ldots] = z^m \Lambda. \tag{3.1}
\]

Let \( R(u) \) act on the elements of the form \( z^m f, \ f \in \Lambda, \ m \in \mathbb{Z} \), \( R(u) : B \rightarrow B \) is defined as (cf. [36])

\[
R(u)(z^m f(x)) = z^{m+1} u^{m+1} f. \tag{3.2}
\]

Then it leads to

\[
R^{-1}(u)(z^m f(x)) = z^{m-1} u^{-m} f. \tag{3.3}
\]

Operators \( R^{\pm 1}(u) \) map the grading of the boson Fock space \( B^{(m)} \) into \( B^{(m \pm 1)} \).

Define the quantum fields \( \psi^{Sp,\pm}(u) \) [37]

\[
\psi^{Sp,+}(u) = u^{-1} R(u) H(u) E^\dagger(-u) E^\dagger(-\frac{1}{u}) = \sum_{k \in \mathbb{Z} + \frac{1}{2}} \psi^{Sp,+}_{k} u^{-k-\frac{1}{2}},
\]

\[
\psi^{Sp,-}(u) = (1-u^2) R^{-1}(u) E(-u) H^\dagger(u) H^\dagger(\frac{1}{u}) = \sum_{k \in \mathbb{Z} + \frac{1}{2}} \psi^{Sp,-}_{k} u^{-k-\frac{1}{2}}. \tag{3.4}
\]

**Proposition 3.1.** Quantum fields \( \psi^{Sp,+}(u), \psi^{Sp,-}(u) \) satisfy the anticommutation relations

\[
\psi^{Sp,\pm}(u)\psi^{Sp,\pm}(v) + \psi^{Sp,\pm}(v)\psi^{Sp,\pm}(u) = 0,
\]

\[
\psi^{Sp,+}(u)\psi^{Sp,-}(v) + \psi^{Sp,-}(v)\psi^{Sp,+}(u) = \delta(u, v), \tag{3.5}
\]

where \( \delta(u, v) = \sum_{k, m \in \mathbb{Z}, k+m=-1} u^k v^m \) is the delta-distribution.

From Eq. (3.4), Eq. (3.5) is equivalent to the relations with charged free fermions

\[
\psi^{Sp,\pm}_k \psi^{Sp,\pm}_l + \psi^{Sp,\pm}_l \psi^{Sp,\pm}_k = 0,
\]

\[
\psi^{Sp,+}_k \psi^{Sp,-}_l + \psi^{Sp,-}_l \psi^{Sp,+}_k = \delta_{k,-l}. \tag{3.6}
\]
Remark 3.2. From Eqs. (2.2) and (2.7), we easily get the bosonic form of the quantum fields $\psi^{Sp,\pm}(u)$:

$$
\psi^{Sp,+}(u) = u^{-1}R(u) \exp \left( \sum_{n \geq 1} \frac{p_n u^n}{n} \right) \exp \left( - \sum_{n \geq 1} \frac{\partial}{\partial p_n} \frac{1}{u^n} \right) \exp \left( - \sum_{n \geq 1} \frac{\partial}{\partial p_n} u^n \right),
$$

$$
\psi^{Sp,-}(u) = (1 - u^2)R^{-1}(u) \exp \left( - \sum_{n \geq 1} \frac{p_n u^n}{n} \right) \exp \left( \sum_{n \geq 1} \frac{\partial}{\partial p_n} \frac{1}{u^n} \right) \exp \left( \sum_{n \geq 1} \frac{\partial}{\partial p_n} u^n \right). \quad (3.7)
$$

Hence one has $\psi^{Sp,+}_i(z^m) = 0$ if $i > -m - \frac{1}{2}$ and $\psi^{Sp,-}_i(z^m) = 0$ if $i > m - \frac{1}{2}$.

Definition 3.3. For an unknown function $\tau = \tau(x)$, the bilinear equation

$$
\hat{\Omega}(\tau \otimes \tau) = 0,
$$

is called the SKP hierarchy, where

$$
\hat{\Omega} = \sum_{i \in \mathbb{Z} + \frac{1}{2}} \psi_{i}^{Sp,+} \otimes \psi_{-i}^{Sp,-}. \quad (3.9)
$$

Lemma 3.4. Let $\hat{X} = \sum_{i > N} C_i \psi_i^{Sp,+}$, where $C_i \in \mathbb{C}, N \in \mathbb{Z}$. Then $\hat{X}^2 = 0$.

Proof. Due to $\psi_k^{Sp,+} \psi_i^{Sp,+} + \psi_i^{Sp,+} \psi_k^{Sp,+} = 0, i, k \in \mathbb{Z} + \frac{1}{2}$, one immediately has

$$
\hat{X}^2 = \hat{X} \cdot \hat{X} = \sum_{l > N} C_l \psi_l^{Sp,+} \cdot \sum_{k > N} C_k \psi_k^{Sp,+} = \sum_{l > N} \sum_{k > N} C_l C_k \psi_l^{Sp,+} \psi_k^{Sp,+} = 0. \quad (3.10)
$$

Lemma 3.5. Let $\hat{X} = \sum_{i > N} C_i \psi_i^{Sp,+}$, where $C_i \in \mathbb{C}, N \in \mathbb{Z}$. Then $\hat{\Omega}(\hat{X} \otimes \hat{X}) = (\hat{X} \otimes \hat{X})\hat{\Omega}$.

Proof. Based on $\psi_{-l}^{Sp,-} \hat{X} = -\hat{X} \psi_{-l}^{Sp,-} + C_l$, we have

$$
\hat{\Omega}(\hat{X} \otimes \hat{X}) = \sum_{l \in \mathbb{Z} + \frac{1}{2}} \psi_{l}^{Sp,+} \hat{X} \otimes \psi_{-l}^{Sp,-} \hat{X} = \sum_{l \in \mathbb{Z} + \frac{1}{2}} (\hat{X} \psi_{l}^{Sp,+}) \otimes (-\hat{X} \psi_{-l}^{Sp,-} + C_l)
$$

$$
= (\hat{X} \otimes \hat{X})\hat{\Omega} - \hat{X} \sum_{l \in \mathbb{Z} + \frac{1}{2}} C_l \psi_{l}^{Sp,+} \otimes 1 = (\hat{X} \otimes \hat{X})\hat{\Omega} - \hat{X}^2 \otimes 1 = (\hat{X} \otimes \hat{X})\hat{\Omega}. \quad (3.11)
$$

Corollary 3.6. Let $\tau \in \mathcal{B}^m$ be a tau-function of the SKP hierarchy, and let $\hat{X} = \sum_{i > N} C_i \psi_i^{Sp,+}$, where $C_i \in \mathbb{C}, N \in \mathbb{Z}$. Then $\hat{\tau} = \hat{X} \tau \in \mathcal{B}^{m+1}$ is also a tau-functions of the SKP hierarchy.

Proof. Multiplying $\hat{X} \otimes \hat{X}$ left on both sides of $\hat{\Omega}(\tau \otimes \tau) = 0$, we get $(\hat{X} \otimes \hat{X})\hat{\Omega}(\tau \otimes \tau) = 0$. According to $\hat{\Omega}(\hat{X} \otimes \hat{X}) = (\hat{X} \otimes \hat{X})\hat{\Omega}$, it follows that $(\hat{X} \otimes \hat{X})\hat{\Omega}(\tau \otimes \tau) = (\hat{X} \otimes \hat{X})(\tau \otimes \tau) = \hat{\Omega}(\hat{X} \otimes \hat{X} \tau) = 0$. Therefore, $\hat{X} \tau$ is the solution of the SKP hierarchy. □
It is known that symplectic Schur functions are tau-functions of the SKP hierarchy \[22\]. If the non-zero solution of (3.8) is a polynomial function of variables \((p_1, p_2, \ldots)\), we call the non-zero solution a polynomial tau-function. It follows from Remark 3.2 that \(z^m\) is a solution of the SKP hierarchy. We now turn our attention to the polynomial tau-function of the SKP hierarchy.

### 3.2. Generating functions and the polynomial tau-functions of the SKP hierarchy.

Let \(\hat{G}(u_1, \ldots, u_l)\) be a generating function of the symplectic Schur function in \(u = (u_1, \ldots, u_l)\) defined by

\[
\hat{G}(u_1, \ldots, u_l) = \prod_{1 \leq i < j \leq l} (u_i - u_j) \prod_{i=1}^l H(u_i). \tag{3.12}
\]

From Proposition 2.1, we have

\[
\psi^{Sp.}+(u_1)\psi^{Sp.}+(u_2) \cdots \psi^{Sp.}+(u_l)(z^k f) = \sum_{1 \leq i,j \leq l} \prod_{1 \leq i < j \leq l} (1 - u_i u_j) \prod_{i=1}^l H(u_i) E^k(-u_i) E^k(-\frac{1}{u_i})(f) = z^{k+1} u_1^{l+1-k} \cdots u_{l-1}^{k+1} \prod_{1 \leq i<j \leq l} (1 - u_i u_j) \prod_{i=1}^l H(u_i) E^k(-u_i) E^k(-\frac{1}{u_i})(f). \tag{3.13}
\]

Let \(\hat{A}_i(u), \ldots, \hat{A}_l(u)\) be a set of formal Laurent series, define the formal Laurent series \(\hat{T}(u) = \sum_{p \in \mathbb{Z}} \hat{T}_i u^p, i = 1, \ldots, l\). Besides, let \(\hat{T}(u_1, \ldots, u_l)\) be a formal Laurent series in \((u_1, \ldots, u_l)\) defined by

\[
\hat{T}(u_1, \ldots, u_l) = \prod_{1 \leq i < j \leq l} (u_i - u_j) \prod_{i=1}^l \hat{A}_i(u_i) H(u_i). \tag{3.14}
\]

For any vector \(\xi = (\xi_1, \ldots, \xi_l)\) \(\in \mathbb{Z}^l\), \(\hat{T}_\xi\) is the coefficient of the following expansion

\[
\hat{T}(u_1, \ldots, u_l) = \sum_{\xi \in \mathbb{Z}^l} \hat{T}_\xi u_1^{\xi_1} \cdots u_l^{\xi_l}. \tag{3.15}
\]

**Theorem 3.7.**

1) Formal Laurent series \(\hat{T}(u_1, \ldots, u_l)\) can be expressed as

\[
\hat{T}(u_1, \ldots, u_l) = \frac{1}{2} \det \left[ \left( u_i^{l-j} + u_i^{l-j-2} \right) \hat{T}_i(u_i) \right]_{1 \leq i,j \leq l}. \tag{3.16}
\]

2) The coefficient \(\hat{T}_\xi\) of \(u_1^{\xi_1} \cdots u_l^{\xi_l}\) in (3.15) can be written as follows

\[
\hat{T}_\xi = \frac{1}{2} \det \left[ \hat{T}_{i,\xi_i-j} + \hat{T}_{i,\xi_i+j-2} \right]_{1 \leq i,j \leq l}, \tag{3.17}
\]

where \(\xi = (\xi_1, \ldots, \xi_l)\).

3) \(\hat{T}_\xi\) is a polynomial tau-function of the SKP hierarchy.

**Proof.**

1) According to Vandermonde-like identity [38]

\[
\det \left[ u_i^{k-j} + u_i^{k+j-2} \right] = 2 \prod_{1 \leq i < j \leq k} (u_i - u_j) (1 - u_i u_j) \tag{3.18}
\]
it is easy to verify that

\[ \hat{T}(u_1, \ldots, u_l) = \prod_{1 \leq i < j \leq l} \left( u_i - u_j \right) (1 - u_i u_j) \prod_{i=1}^l \hat{A}_i(u_i) H(u_i) \]

\[ = \frac{1}{2} \det \left[ u_i^{l-j} + u_i^{l+j-2} \right] \prod_{i=1}^l \hat{T}_i(u_i) \]

\[ = \frac{1}{2} \det \left[ \left( u_i^{l-j} + u_i^{l+j-2} \right) \hat{T}_i(u_i) \right]_{1 \leq i, j \leq l}. \]  

(3.19)

2) Observe that

\[ \hat{T}(u_1, \ldots, u_l) = \frac{1}{2} \det \left[ \sum_{p_i \in \mathbb{Z}} \hat{T}_{i,p_i} \left( u_i^{l+p_i-j} + u_i^{l+p_i+j-2} \right) \right] \]

\[ = \frac{1}{2} \sum_{p_i \in \mathbb{Z}} \sum_{\sigma \in \mathbb{S}_l \quad \epsilon_i = \pm 1} sgn(\sigma) u_1^{l+p_1-1} \cdots u_l^{l+p_l-1} \hat{T}_{1,p_1} u_1^{\epsilon_1(\sigma(1)-1)} \cdots \hat{T}_{l,p_l} u_l^{\epsilon_l(\sigma(l)-1)} \]

\[ = \sum_{\xi_i \in \mathbb{Z}} \frac{1}{2} \sum_{\sigma \in \mathbb{S}_l \quad \epsilon_i = \pm 1} sgn(\sigma) \hat{T}_{1,\xi_1-l+1-\epsilon_1(\sigma(1)-1)} \cdots \hat{T}_{l,\xi_l-l+1-\epsilon_l(\sigma(l)-1)} u_1^{\xi_1} \cdots u_l^{\xi_l} \]

\[ = \sum_{\xi_i \in \mathbb{Z}} \frac{1}{2} \det \left[ \hat{T}_{i,\xi_i-j} + \hat{T}_{i,\xi_i+j-2} \right]_{1 \leq i, j \leq l} u_1^{\xi_1} \cdots u_l^{\xi_l}, \]  

(3.20)

therefore, the coefficient \( \hat{T}_\xi \) of \( u_1^{\xi_1} \cdots u_l^{\xi_l} \) is \( \frac{1}{2} \det \left[ \hat{T}_{i,\xi_i-j} + \hat{T}_{i,\xi_i+j-2} \right]_{1 \leq i, j \leq l} \).

3) It is apparent from (3.12) that

\[ \hat{A}_1(u_1) \cdots \hat{A}_l(u_l) \psi^{Sp,+}(u_1) \cdots \psi^{Sp,+}(u_l)(z^k \cdot 1) = z^{l+k} u_1^{k} \cdots u_l^{k} \hat{T}(u_1, \ldots, u_l). \]  

(3.21)

Let \( \hat{A}_j(u) = \sum_{M_j \leq r \leq N_j} \hat{A}_{j,r-\frac{1}{2}} u^r(A_{j,r-\frac{1}{2}} \in \mathbb{C}, M_j, N_j, r \in \mathbb{Z}, j = 1, \ldots, l) \) be a power series expansion of the variable \( u \). Therefore, \( \hat{T}_\xi \) can be written as

\[ \hat{T}_\xi = z^{-l-k} \hat{X}_1 \cdots \hat{X}_l(z^k \cdot 1), \]  

(3.22)

where

\[ \hat{X}_j = \sum_{M_j-\xi_j-k-\frac{1}{2} \leq i_j \leq N_j-\xi_j-k-\frac{1}{2}} \hat{A}_{j,\xi_j+k+i_j} \psi^{Sp,+}_{i_j}, \quad j = 1, \ldots, l. \]  

(3.23)

Particularly, by Remark 3.12 and Corollary 3.6, the coefficient \( \hat{T}_\xi \) is a tau-function of the SKP hierarchy with \( k = 0 \). Since \( \hat{T}_\xi \) is a finite linear combination of \( \psi^{Sp,+}_{i_1} \cdots \psi^{Sp,+}_{i_l}(1) \), it is a polynomial tau-function.

\[ \square \]
By replacing $\widehat{A}_j(u)$ with $u^{c_j}\widehat{A}_j(u)$, we have

$$\widehat{A}_i(u) = u^{N_i}\widehat{h}_i \sum_{k=0}^{\infty} a_{i,k}u^k, \quad N_i \in \mathbb{Z}, \widehat{h}_i, a_{i,k} \in \mathbb{C}, a_{i,0} = 1, \widehat{h}_i \neq 0, i = 1, \ldots, l, \quad \text{(3.24)}$$

where $\widehat{A}_j(u)$ are non-zero Laurent series defined in the $\widehat{T}(u_1, \ldots, u_l)$.

According to (2.3), $\sum_{k=0}^{\infty} a_{i,k}u^k$ can be written as

$$\sum_{k=0}^{\infty} a_{i,k}u^k = \exp \left( \sum_{l=1}^{\infty} \widehat{c}_{i,l}u^l \right), \quad \text{and} \quad a_{i,k} = S_k(\widehat{c}_{i,1}, \widehat{c}_{i,2}, \ldots), \quad \text{(3.25)}$$

where $\{\widehat{c}_{i,l}\}$ is a set of constants in $\mathbb{C}$.

From Proposition 2.1, it is easy to check that

From (2.2) and setting $t_j = \frac{p_j}{\tau}$, we obtain

$$\widehat{T}_i(u) = \widehat{A}_i(u)H(u) = u^{N_i}\widehat{h}_i \exp \left( \sum_{l=1}^{\infty} \widehat{c}_{i,l}u^l \right) \exp \left( \sum_{l=1}^{\infty} \frac{p_l}{\tau}u^l \right)$$

$$= u^{N_i}\widehat{h}_i \exp \left( \sum_{l=1}^{\infty} (\widehat{c}_{i,l} + t_j)u^l \right) = u^{N_i}\widehat{h}_i \sum_{l=0}^{\infty} S_l(t_1 + \widehat{c}_{i,1}, t_2 + \widehat{c}_{i,2}, \ldots)u^l. \quad \text{(3.26)}$$

Hence $\widehat{T}_{i,p} = \widehat{h}_i S_{p-N_i}(t_1 + \widehat{c}_{i,1}, t_2 + \widehat{c}_{i,2}, \ldots) = 1$, $i = 1, \ldots, l$. From Theorem 3.7, polynomial tau-functions of the SKP hierarchy are given by

$$\widehat{T}_{\xi} = \frac{1}{2} \det \left[ \widehat{T}_{\xi,\xi-j} + \widehat{T}_{\xi,\xi+j-2} \right]$$

$$= \frac{1}{2} \det \left[ \widehat{h}_i S_{\xi-j-N_i}(t_1 + \widehat{c}_{i,1}, t_2 + \widehat{c}_{i,2}, \ldots) + \widehat{h}_i S_{\xi+j-2-N_i}(t_1 + \widehat{c}_{i,1}, t_2 + \widehat{c}_{i,2}, \ldots) \right]$$

$$= \prod_{i=1}^{l} \left[ \frac{1}{2} \det \left[ S_{\xi-j-N_i}(t_1 + \widehat{c}_{i,1}, t_2 + \widehat{c}_{i,2}, \ldots) + S_{\xi+j-2-N_i}(t_1 + \widehat{c}_{i,1}, t_2 + \widehat{c}_{i,2}, \ldots) \right] \right]_{j=1, \ldots, l}. \quad \text{(3.27)}$$

When $\widehat{c}_{i,l} = 0$, $\widehat{h}_i = 1$ and $N_i + 2 = i$, $\widehat{T}_{\xi}$ reduces to the symplectic Schur functions [22]. The polynomial tau-functions (3.27) of the SKP hierarchy are the generalization of the solution of the SKP hierarchy in [22], which are the zero mode of an appropriate combinatorial generating functions.

### 3.3. N-soliton solutions of the SKP hierarchy.

Now let us consider another extremely important exact solution of SKP hierarchy called the soliton solution.

Let

$$\Gamma^{Sp}(p, q) = p^{-1}(1 - q^2)R(p)R^{-1}(q)H(p)E(-q)E^\perp(-p)H^\perp(q)E^\perp(-\frac{1}{p})H^\perp\left(\frac{1}{q}\right). \quad \text{(3.28)}$$

From Proposition 2.1 it is easy to check that

$$\Gamma^{Sp}(p_i, q_i)\Gamma^{Sp}(p_j, q_j) = A_{ij} : \Gamma^{Sp}(p_i, q_i)\Gamma^{Sp}(p_j, q_j), \quad \text{(3.29)}$$
Lemma 3.8. If \( \Gamma \) is a solution of the SKP hierarchy, then \( \Gamma^{S_p}(u, v)\tau \) is also a solution.

Proof. Using Eq. (3.36), we obtain

\[
\hat{\Omega} \left( \psi^{S_p, +}(u)\psi^{S_p, -}(v) \otimes \psi^{S_p, +}(u)\psi^{S_p, -}(v) \right)
= \left( \sum_{l \in \mathbb{Z} + \frac{1}{2}} \psi_{l}^{S_p, +} \otimes \psi_{-l}^{S_p, -} \right) \left( \sum_{m, n \in \mathbb{Z} + \frac{1}{2}} \psi_{m}^{S_p, +} u^{-m - \frac{1}{2}} \psi_{n}^{S_p, -} v^{-n - \frac{1}{2}} \otimes \sum_{m, n \in \mathbb{Z} + \frac{1}{2}} \psi_{m}^{S_p, +} u^{-m - \frac{1}{2}} \psi_{n}^{S_p, -} v^{-n - \frac{1}{2}} \right)
= \sum_{l, m, n \in \mathbb{Z} + \frac{1}{2}} \left( \psi_{m}^{S_p, +} \psi_{n}^{S_p, -} - \psi_{l}^{S_p, +} \psi_{-l}^{S_p, -} \right) \delta_{m, n} \left( u^{-m - \frac{1}{2}} v^{-n - \frac{1}{2}} \right) \otimes \delta_{m, n} \left( u^{-m - \frac{1}{2}} v^{-n - \frac{1}{2}} \right)
= \left( \psi^{S_p, +}(u)\psi^{S_p, -}(v) \otimes \psi^{S_p, +}(u)\psi^{S_p, -}(v) \right) \hat{\Omega}.
\] (3.31)

It can easily be checked that

\[
\left( \psi^{S_p, +}(u)\psi^{S_p, -}(v) \otimes \psi^{S_p, +}(u)\psi^{S_p, -}(v) \right) \hat{\Omega}(\tau \otimes \tau)
= \hat{\Omega} \left( \psi^{S_p, +}(u)\psi^{S_p, -}(v) \otimes \psi^{S_p, +}(u)\psi^{S_p, -}(v) \right) (\tau \otimes \tau)
= \hat{\Omega} \left( \psi^{S_p, +}(u)\psi^{S_p, -}(v) \tau \otimes \psi^{S_p, +}(u)\psi^{S_p, -}(v) \tau \right) = 0.
\] (3.32)

Clearly, \( \psi^{S_p, +}(u)\psi^{S_p, -}(v)\tau \) is the solution of the SKP hierarchy. A routine computation gives rise to \( \psi^{S_p, +}(u)\psi^{S_p, -}(v) = \frac{1}{1 + v} \Gamma^{S_p}(u, v) \). Therefore, \( \Gamma^{S_p}(u, v)\tau \) is also a solution. \( \Box \)

Lemma 3.9. It holds that

\[
[\hat{\Omega}, 1 \otimes \Gamma^{S_p}(p, q) + \Gamma^{S_p}(p, q) \otimes 1] = 0,
\] (3.33)

where \([A, B] =_{\text{def}} AB - BA\).

Proof. Lemma can be calculated directly from the \( \Gamma^{S_p}(p, q) = (1 - pq) \frac{1}{p} \psi^{S_p, +}(p)\psi^{S_p, -}(q) \). The specific calculation process is not listed here. \( \Box \)
Let us consider the function
\[
\tau(x,y) = \tau(x,y;p,q,c) = \prod_{i=1}^{n} e^{p_i c_i \Gamma^{Sp}(p_i,q_i)} \cdot 1, \quad p_i, q_i, c_i \in \mathbb{C}, p_i \neq q_j, p_i \neq \frac{1}{q_j} \text{ for } i \neq j,
\]
and set
\[
\eta_i = \sum_{k \geq 1} (p_i^k - q_i^k) \frac{p_k(x)}{k}.
\]
By (3.29), Eq. (3.34) can be rewritten as
\[
\tau(x,y;p,q,c) = \sum_{\mathcal{J} \subseteq I} \left( \prod_{i \in \mathcal{J}} c_i (1 - q_i^2) \right) \left( \prod_{i,j \in \mathcal{J}, i < j} A_{ij} \right) \exp \left( \sum_{i \in \mathcal{J}} \eta_i \right),
\]
where \( I = \{1, 2, \ldots, n\} \).

**Proposition 3.10.** The function \( \tau(x,y;p,q,c) \) in (3.34) is a solution of the SKP hierarchy, which we call the \( n \)-soliton solutions.

**Proof.** Suppose that \( \tau \) is a solution of the SKP hierarchy. We put \( \hat{\tau} = (1 + pc\Gamma^{Sp}(p,q)) \tau \). It follows from Lemma 3.8 and 3.9 that
\[
\hat{\Omega}(\hat{\tau} \otimes \hat{\tau}) = \hat{\Omega}(\tau \otimes \tau) + pc\hat{\Omega}(\tau \otimes \Gamma^{Sp}(p,q)\tau + \Gamma^{Sp}(p,q)\tau \otimes \tau) + p^2 c^2 \hat{\Omega}(\Gamma^{Sp}(p,q)\tau \otimes \Gamma^{Sp}(p,q)\tau)
\]
\[
= pc\hat{\Omega}(1 \otimes \Gamma^{Sp}(p,q))(\tau \otimes \tau) + (\Gamma^{Sp}(p,q) \otimes 1)(\tau \otimes \tau)
\]
\[
= pc\hat{\Omega}(1 \otimes \Gamma^{Sp}(p,q) + \Gamma^{Sp}(p,q) \otimes 1)(\tau \otimes \tau)
\]
\[
= pc(1 \otimes \Gamma^{Sp}(p,q) + \Gamma^{Sp}(p,q) \otimes 1)\hat{\Omega}(\tau \otimes \tau)
\]
\[
= 0.
\]
Hence \( \hat{\tau} \) is a solution of the SKP hierarchy. Note that \( \tau = 1 \) solves the SKP hierarchy, it is easy to see that the \( n \)-soliton solutions defined in (3.34) is really a solution of the SKP hierarchy.

\[\square\]

4. Polynomial tau-functions and \( n \)-soliton solutions of the OKP hierarchy

In this section, we firstly construct quantum fields of orthogonal Schur functions and deduce the relationship between these operators. Meanwhile, the generating functions of the orthogonal Schur functions have been investigated. Moreover, by applying the quantum field presentation of the OKP hierarchy, the polynomial tau-functions and the soliton solutions have been presented.
4.1. Quantum fields presentation of orthogonal Schur functions and the OKP hierarchy.

Introduce the quantum fields defined by

\[
\psi^{O,\pm}(u) = u^{-1}(1 - u^2)R(u)H(u)E^\perp(-u)E^\perp(-1/u) = \sum_{k \in \mathbb{Z} + \frac{1}{2}} \psi^{O,\pm}_k u^{-k-\frac{1}{2}},
\]

\[
\psi^{O,-}(u) = R^{-1}(u)E(-u)H^\perp(1/u) = \sum_{k \in \mathbb{Z} + \frac{1}{2}} \psi^{O,-}_k u^{-k-\frac{1}{2}}.
\] (4.1)

**Proposition 4.1.** It can be checked that \(\psi^{O,\pm}(u), \psi^{O,-}(u)\) satisfy the relations

\[
\psi^{O,\pm}(u)\psi^{O,\pm}(v) + \psi^{O,\pm}(v)\psi^{O,\pm}(u) = 0,
\]

\[
\psi^{O,\pm}(u)\psi^{O,-}(v) + \psi^{O,-}(v)\psi^{O,\pm}(u) = \delta(u,v).
\] (4.2)

Equivalently, Eq. (4.2) can be expressed as charged free fermions relation

\[
\psi_k^{O,\pm}\psi_l^{O,\pm} + \psi_l^{O,\pm}\psi_k^{O,\pm} = 0,
\]

\[
\psi_k^{O,\pm}\psi_l^{O,-} + \psi_l^{O,-}\psi_k^{O,\pm} = \delta_{k,-l}.
\] (4.3)

**Remark 4.2.** From the formula (2.2) and (2.7), we easily get the bosonic form of the fields \(\psi^{O,\pm}(u)\):

\[
\psi^{O,\pm}(u) = u^{-1}(1 - u^2)R(u)\exp\left(\sum_{n \geq 1} \frac{p_n}{n} u^n\right) \exp\left(-\sum_{n \geq 1} \frac{\partial}{\partial p_n} \frac{1}{n} \right) \exp\left(-\sum_{n \geq 1} \frac{\partial}{\partial p_n} u^n\right),
\]

\[
\psi^{O,-}(u) = R^{-1}(u) \exp\left(-\sum_{n \geq 1} \frac{p_n}{n} u^n\right) \exp\left(\sum_{n \geq 1} \frac{\partial}{\partial p_n} \frac{1}{n} \right) \exp\left(\sum_{n \geq 1} \frac{\partial}{\partial p_n} u^n\right).
\] (4.4)

Hence we obtain \(\psi_i^{O,\pm}(z^m) = 0\) if \(i > -m - \frac{1}{2}\) and \(\psi_i^{O,-}(z^m) = 0\) if \(i > m - \frac{1}{2}\).

**Definition 4.3.** For an unknown function \(\tau = \tau(x)\), the bilinear equation

\[
\tilde{\Omega}(\tau \otimes \tau) = 0,
\] (4.5)

is called the OKP hierarchy, where

\[
\tilde{\Omega} = \sum_{k \in \mathbb{Z} + \frac{1}{2}} \psi^{O,\pm}_k \otimes \psi^{O,-}_{-k}.
\] (4.6)

**Lemma 4.4.** Let \(\tilde{X} = \sum_{i \geq N} C_i \psi^{O,\pm}_i\), where \(C_i \in \mathbb{C}, N \in \mathbb{Z}\). Then \(\tilde{X}^2 = 0\).

**Lemma 4.5.** Let \(\tilde{X} = \sum_{i \geq N} C_i \psi^{O,\pm}_i\), where \(C_i \in \mathbb{C}, N \in \mathbb{Z}\). Then \(\tilde{\Omega}(\tilde{X} \otimes \tilde{X}) = (\tilde{X} \otimes \tilde{X})\tilde{\Omega}\).

**Corollary 4.6.** Let \(\tau \in \mathcal{B}^m\) be a tau-function of the OKP hierarchy, and let \(\tilde{X} = \sum_{i \geq N} C_i \psi^{O,\pm}_i\), where \(C_i \in \mathbb{C}, N \in \mathbb{Z}\). Then \(\tilde{\tau} = \tilde{X}\tau \in \mathcal{B}^{m+1}\) is also a tau-functions of the OKP hierarchy.

**Proof.** The proof of the Lemma 4.4 and Corollary 4.5 is quite similar to the Lemma 3.4 and Corollary 3.6, so is omitted. \(\square\)
4.2. Generating functions and the polynomial tau-functions of the OKP hierarchy. It is known that orthogonal Schur functions are tau-functions of the OKP hierarchy [23]. Let \( \tilde{G}(u_1, \ldots, u_l) \) be a generating function of the orthogonal Schur function in \( u = (u_1, \ldots, u_l) \) defined by

\[
\tilde{G}(u_1, \ldots, u_l) = \prod_{1 \leq i < j \leq l} (u_i - u_j) \prod_{1 \leq i \leq j \leq l} (1 - u_i u_j) \prod_{i=1}^{l} H(u_i).
\] (4.7)

By Proposition [21] we obtain

\[
\psi^O(u_1) \cdots \psi^O(u_l)(z^k f) = z^{k+l} u_1^k \cdots u_l^k \tilde{G}(u_1, \ldots, u_l). \] (4.8)

Consider the set of formal Laurent series \( \tilde{A}_1(u), \ldots, \tilde{A}_l(u) \), define the formal Laurent series \( \tilde{T}_i(u) = \tilde{A}_i(u) H(u) = \sum_{p \in \mathbb{Z}} \tilde{T}_{i,p} u^p \), \( i = 1, \ldots, l \). Besides, let \( \tilde{T}(u_1, \ldots, u_l) \) be a formal Laurent series in \( u_1, \ldots, u_l \) defined by

\[
\tilde{T}(u_1, \ldots, u_l) = \prod_{1 \leq i < j \leq l} (u_i - u_j) \prod_{1 \leq i \leq j \leq l} (1 - u_i u_j) \prod_{i=1}^{l} \tilde{A}_i(u_i) H(u_i).
\] (4.9)

For any vector \( \zeta = (\zeta_1, \ldots, \zeta_l) \in \mathbb{Z}^l \), \( \tilde{T}_\zeta \) is the coefficient of the following expansion

\[
\tilde{T}(u_1, \ldots, u_l) = \sum_{\zeta \in \mathbb{Z}^l} \tilde{T}_\zeta u_1^{\zeta_1} \cdots u_l^{\zeta_l}.
\] (4.10)

**Theorem 4.7.**

1) Formal Laurent series \( \tilde{T}(u_1, \ldots, u_l) \) can be written as

\[
\tilde{T}(u_1, \ldots, u_l) = \det \left[ \left( u_i^{l-j} - u_i^{l+j} \right) \tilde{T}_i(u_i) \right]_{1 \leq i \leq l}. \] (4.11)

2) For any vector \( \zeta = (\zeta_1, \ldots, \zeta_l) \in \mathbb{Z}^l \), the coefficient \( \tilde{T}_\zeta \) of \( u_1^{\zeta_1} \cdots u_l^{\zeta_l} \) in (4.10) is given by

\[
\tilde{T}_\zeta = \det \left[ \tilde{T}_{l-i-j} - \tilde{T}_{l-i-\zeta_j} \right]_{1 \leq i,j \leq l}. \] (4.12)

3) \( \tilde{T}_\zeta \) is a polynomial tau-function of the OKP hierarchy.

**Proof.**

1) According to Vandermonde-like identity [35]

\[
\det \left[ u_i^{k-j} - u_i^{k+j} \right] = \prod_{1 \leq i < j \leq k} (u_i - u_j) \prod_{1 \leq i \leq j \leq k} (1 - u_i u_j)
\]

\[
= \sum_{\sigma \in S_k} \text{sgn}(\sigma) \varepsilon_1 \cdots \varepsilon_k u_1^{k-\varepsilon_1 \sigma(1)} \cdots u_k^{k-\varepsilon_k \sigma(k)},
\] (4.13)

we have

\[
\tilde{T}(u_1, \ldots, u_l) = \prod_{1 \leq i < j \leq l} (u_i - u_j) \prod_{1 \leq i \leq j \leq l} (1 - u_i u_j) \prod_{i=1}^{l} \tilde{A}_i(u_i) H(u_i)
\]

\[
= \det \left[ u_i^{l-j} - u_i^{l+j} \right] \prod_{i=1}^{l} \tilde{T}_i(u_i) = \det \left[ \left( u_i^{l-j} - u_i^{l+j} \right) \tilde{T}_i(u_i) \right]_{1 \leq i \leq l}. \] (4.14)
2) Noticing that

\[ \widetilde{T}(u_1, \ldots, u_l) = \det \left[ \sum_{p_i \in \mathbb{Z}} \widetilde{T}_{i,p_i} \left( u_i^{l+p_i-j} - u_i^{l+p_i+j} \right) \right] \]

\[ = \sum_{p_i \in \mathbb{Z}} \sum_{\sigma \in S_l} \text{sgn}(\sigma) \varepsilon_1 \cdots \varepsilon_l \widetilde{T}_{i,p_i} u_i^{l+p_1 - \varepsilon_1 \sigma(1)} \cdots \widetilde{T}_{i,p_i} u_i^{l+p_l - \varepsilon_l \sigma(l)} \]

\[ = \sum_{\zeta_i \in \mathbb{Z}} \sum_{\sigma \in S_l} \text{sgn}(\sigma) \varepsilon_1 \cdots \varepsilon_l \widetilde{T}_{i,\zeta_i - l + \varepsilon_1 \sigma(1)} \cdots \widetilde{T}_{i,\zeta_i - l + \varepsilon_l \sigma(l)} u_i^{\zeta_i \cdots \zeta_i} \]

\[ = \sum_{\zeta_i \in \mathbb{Z}} \det \left[ \widetilde{T}_{i,\zeta_i - l - j} - \widetilde{T}_{i,\zeta_i - l + j} \right]_{1 \leq i,j \leq l} u_i^{\zeta_i \cdots \zeta_i}. \quad (4.15) \]

Obviously, the coefficient \( \widetilde{T}_\zeta \) of \( u_1^{\zeta_1} \cdots u_l^{\zeta_l} \) is \( \det \left[ \widetilde{T}_{i,\zeta_i - l - j} - \widetilde{T}_{i,\zeta_i - l + j} \right]_{1 \leq i,j \leq l} \).

3) From (4.14), it is straightforward to show that

\[ \widetilde{A}_1(u_1) \cdots \widetilde{A}_l(u_l) \psi^{O,+}(u_1) \cdots \psi^{O,+}(u_l)(z^k \cdot 1) = z^{l+k} u_1^{l+k} \cdots u_l^{l+k} \widetilde{T}(u_1, \ldots, u_l). \quad (4.16) \]

Let \( \tilde{A}_j(u) = \sum_{M_j \leq r \leq N_j} \tilde{A}_{j,r-\frac{1}{2}} u^r \tilde{A}_{j,r-\frac{1}{2}} \in \mathbb{C}, M_j, N_j, r \in \mathbb{Z}, j = 1, \ldots, l \) be a power series expansion of the variable \( u \). Therefore, \( \widetilde{T}_\zeta \) can be written as

\[ \widetilde{T}_\zeta = z^{-l-k} \tilde{X}_1 \cdots \tilde{X}_l(z^k \cdot 1), \quad (4.17) \]

where

\[ \tilde{X}_j = \sum_{M_j - \zeta_j - k - \frac{1}{2} \leq i_j \leq N_j - \zeta_j - k - \frac{1}{2}} \tilde{A}_{j,\zeta_j + k + i_j} \psi^{O,+}_{i_j}, \quad j = 1, \ldots, l. \quad (4.18) \]

By Remark 4.2 and Corollary 4.6, it should be pointed out that the coefficient \( \widetilde{T}_\zeta \) is a tau-function of the OKP hierarchy with \( k = 0 \). Since \( \widetilde{T}_\zeta \) is a finite linear combination of \( \psi^{O,+}_{i_1} \cdots \psi^{O,+}_{i_l}(1) \), it is a polynomial tau-function.

\[ \square \]

By changing \( \tilde{A}_j(u) \to u^{\zeta_j} \tilde{A}_j(u) \), we obtain

\[ \tilde{A}_i(u) = u^{N_i} \tilde{h}_i \sum_{k=0}^{\infty} \tilde{a}_{i,k} u^k, \quad N_i \in \mathbb{Z}, \tilde{h}_i, \tilde{a}_{i,k} \in \mathbb{C}, \tilde{a}_{i,0} = 1, \tilde{h}_i \neq 0, i = 1, \ldots, l. \quad (4.19) \]

From (2.3), \( \sum_{k=0}^{\infty} \tilde{a}_{i,k} u^k \) can be expressed as

\[ \sum_{k=0}^{\infty} \tilde{a}_{i,k} u^k = \exp \left( \sum_{l=1}^{\infty} \tilde{c}_{i,l} u^l \right), \text{ and } \tilde{a}_{i,k} = S_k(\tilde{c}_{i,1}, \tilde{c}_{i,2}, \ldots), \quad (4.20) \]

where \( \{ \tilde{c}_{i,l} \} \) are constants in \( \mathbb{C} \). Then based on (2.2), we get

\[ \widetilde{T}_i(u) = \tilde{A}_i(u) H(u) = u^{N_i} \tilde{h}_i \sum_{l=0}^{\infty} S_l(\tilde{c}_{i,1} + \tilde{c}_{i,2} + \tilde{c}_{i,2}, \ldots) u^l. \quad (4.21) \]
Hence \( \bar{T}_{i,p} = \bar{h}_i S_{p-N_i}(t_1 + \bar{c}_{i,1}, t_2 + \bar{c}_{i,2}, \ldots) \), \( i = 1, \ldots, l \). From Theorem 4.7, polynomial tau-functions of the OKP hierarchy have the form

\[
\tilde{\tau} \quad = \quad \det \left[ \bar{T}_{i, \xi_j - l - j} - \bar{T}_{i, \xi_j - l + j} \right] 
= \det \left[ \bar{h}_i S_{\xi_j - l - N_i}(t_1 + \bar{c}_{i,1}, t_2 + \bar{c}_{i,2}, \ldots) - \bar{h}_i S_{\xi_j - l + j - N_i}(t_1 + \bar{c}_{i,1}, t_2 + \bar{c}_{i,2}, \ldots) \right] 
= \prod_{i=1}^{l} (\bar{h}_i) \det \left[ S_{\xi_j - l - N_i}(t_1 + \bar{c}_{i,1}, t_2 + \bar{c}_{i,2}, \ldots) - S_{\xi_j - l + j - N_i}(t_1 + \bar{c}_{i,1}, t_2 + \bar{c}_{i,2}, \ldots) \right]_{i,j=1, \ldots, l}.
\]  

(4.22)

Under the reduction \( \bar{c}_{i,l} = 0 \), \( \bar{h}_i = -1 \) and \( l + N_i = i \), \( \bar{T}_\xi \) lead to the orthogonal Schur functions [23]. Thus the polynomial tau-functions (4.22) of the OKP hierarchy can be reduced to the solution of the OKP hierarchy in [23], which are the zero mode of an appropriate combinatorial generating functions.

4.3. N-soliton solutions of the OKP hierarchy. Let

\[
\Gamma^O(p,q) = p^{-1}(1 - p^2)R(p)R^{-1}(q)H(p)E(-q)E^\perp(-p)H^\perp(q)E^\perp(-\frac{1}{p})H^\perp(\frac{1}{q}).
\]  

(4.23)

From Proposition 2.4, it is easy to check that

\[
\Gamma^O(p_i, q_i)\Gamma^O(p_j, q_j) = A_{ij} : \Gamma^O(p_i, q_i)\Gamma^O(p_j, q_j) ;
\]  

(4.24)

where

\[
A_{ij} = \frac{(1 - p_i p_j)(1 - q_i q_j)(p_i - p_j)(q_i - q_j)}{(1 - q_i p_j)(1 - p_i q_j)(q_i - p_j)(p_i - q_j)}.
\]  

(4.25)

In particular \( \Gamma^O(p,q)^2 = 0 \); therefore \( e^{pt\Gamma^O(p,q)} = 1 + p t \Gamma^O(p,q) \).

**Lemma 4.8.** If \( \tau \) is a solution of the OKP hierarchy, then \( \Gamma^O(u,v)\tau \) is also a solution.

**Lemma 4.9.** It holds that

\[
[\tilde{\Omega}, 1 \otimes \Gamma^O(p,q) + \Gamma^O(p,q) \otimes 1] = 0.
\]  

(4.26)

Considering the following function

\[
\tau(x,y) = \tau(x,y;p,q,c) = \prod_{i=1}^{n} e^{p_i c_i \Gamma^O(p_i, q_i)} \cdot 1, \quad p_i, q_i, c_i \in \mathbb{C}, p_i \neq q_j, p_i \neq \frac{1}{q_j} \text{ for } i \neq j.
\]  

(4.27)

Let us set

\[
\eta_i = \sum_{k \geq 1} (p_i^k - q_i^k) \frac{p_k(x)}{k}.
\]  

(4.28)

From Eq. (4.24), Eq. (4.27) can be rewritten as

\[
\tau(x,y;p,q,c) = \sum_{J \subseteq I} \left( \prod_{i \in J} c_i (1 - p_i^2) \right) \left( \prod_{i \in J} A_{ij} \right) \exp \left( \sum_{i \in J} \eta_i \right),
\]  

(4.29)
where $I = \{1, 2, \ldots, n\}$.

**Proposition 4.10.** The function $\tau(x, y; p, q, c)$ in (4.29) is a solution of the OKP hierarchy, which we call the $n$-soliton solutions.

**Proof.** Lemma 4.8, 4.9 and Proposition 4.10 can be proved with the similar procedure as in Lemma 3.8, 3.9 and Proposition 3.10. \qed

## 5. Polynomial tau-functions of the BUC hierarchy

In this section, the quantum fields of the generalized $Q$-functions shall be developed. By using neutral fermions, we construct an integrable BUC hierarchy characterized by the generalized $Q$-functions. Based upon the generating functions of the polynomial tau-functions of the BUC hierarchy, it is showed that the polynomial tau-function of the BUC hierarchy is a zero mode of certain generating functions.

### 5.1. Quantum fields presentation of the generalized $Q$-functions and the BUC hierarchy.

Introduce another class of symmetric functions $q_k(x_1, x_2, \ldots)$ by

$$Q(u) = \sum_{k \in \mathbb{Z}} q_k u^k = E(u) H(u), \quad (5.1)$$

where $q_k = \sum_{i=0}^{k} e_i h_{k-i}$ for $k > 0$, $q_0 = 1$ and $q_k = 0$ for $k < 0$.

Define

$$Q(u) = S(u)^2, \quad S(u) = \exp \left( \sum_{n \in N_{odd}} \frac{p_n}{n} u^n \right),$$

$$S^\perp(u) = \exp \left( \sum_{n \in N_{odd}} \frac{\partial}{\partial p_n} \frac{1}{n} u^n \right), \quad N_{odd} = \{1, 3, 5, \ldots\}. \quad (5.2)$$

**Proposition 5.1.** The following commutation relations about generating functions hold (cf. [33])

$$H^\perp(u) Q(v) = \frac{u + v}{u - v} Q(v) H^\perp(u),$$

$$E^\perp(u) Q(v) = \frac{u + v}{u - v} Q(v) E^\perp(u),$$

$$S^\perp(u) Q(v) = \frac{u + v}{u - v} Q(v) S^\perp(u). \quad (5.3)$$

Define the formal distributions $\varphi(u)$ and $\overline{\varphi}(u)$ of operators acting on the boson Fock space $B_{odd} = \mathbb{C}[p_1, p_3, p_5, \ldots]$

$$\varphi(u) = Q(u) S^\perp(-\frac{1}{u}) S^\perp(-u) = \sum_{j \in \mathbb{Z}} \varphi_j u^{-j},$$

$$\overline{\varphi}(u) = Q'(u) S^\perp(-\frac{1}{u}) S^\perp(-u) = \sum_{j \in \mathbb{Z}} \overline{\varphi}_j u^{-j}. \quad (5.4)$$
where $Q'(u)$ means the generating functions for the $q_k(y)$ and their adjoint operators hold for the variable $y$, the operators $\varphi_j$ and $\overline{\varphi}_j$ are the neutral fermions.

Let

$$f(u, v) = \frac{u-v}{u+v} = 1 + 2 \sum_{k \geq 1} (-1)^k \frac{v^k}{u^k}, \quad |u| > |v|, \quad (5.5)$$

then

$$f(u, v) + f(v, u) = (v-u) \delta(v, -u) = 2 \sum_{k \in \mathbb{Z}} \frac{v^k}{(-u)^k} = 2v \delta(v, -u). \quad (5.6)$$

**Proposition 5.2.** $\varphi(u)$ and $\overline{\varphi}(u)$ satisfy the following relations

$$\varphi(u)\varphi(v) + \varphi(v)\varphi(u) = 2v \delta(v, -u),$$

$$\overline{\varphi}(u)\overline{\varphi}(v) + \overline{\varphi}(v)\overline{\varphi}(u) = 2v \delta(v, -u),$$

$$\varphi(u)\overline{\varphi}(v) - \overline{\varphi}(v)\varphi(u) = 0. \quad (5.7)$$

Eq. (5.7) can also be expressed as neutral fermions relation

$$\varphi_m \varphi_n + \varphi_n \varphi_m = 2(-1)^m \delta_{m+n, 0},$$

$$\overline{\varphi}_m \overline{\varphi}_n + \overline{\varphi}_n \overline{\varphi}_m = 2(-1)^m \delta_{m+n, 0},$$

$$\varphi_m \overline{\varphi}_n - \overline{\varphi}_n \varphi_m = 0. \quad (5.8)$$

**Proof.** We only prove the first formula of (5.7) and (5.8), other formulas can be proved similarly. In terms of Proposition 5.1 and (5.6), we have

$$\varphi(u)\varphi(v) = \frac{u-v}{u+v} \frac{v-u}{v+u} Q(u)Q(v)S^\dagger(-\frac{1}{v})S^\dagger(-u)Q(v)S^\dagger(-\frac{1}{v})S^\dagger(-v)$$

$$= \frac{-u+v}{-u-v} Q(u)Q(v)S^\dagger(-\frac{1}{v})S^\dagger(-u)S^\dagger(-\frac{1}{v})S^\dagger(-v), \quad (5.9)$$

$$\varphi(v)\varphi(u) = \frac{v-u}{v-u} Q(v)Q(u)S^\dagger(-\frac{1}{v})S^\dagger(-v)Q(u)S^\dagger(-\frac{1}{v})S^\dagger(-u)$$

$$= \frac{v-u}{v-u} Q(v)Q(u)S^\dagger(-\frac{1}{v})S^\dagger(-v)S^\dagger(-\frac{1}{v})S^\dagger(-u), \quad (5.10)$$

therefore,

$$\varphi(u)\varphi(v) + \varphi(v)\varphi(u) = \left(\frac{u-v}{u+v} + \frac{v-u}{v+u}\right) Q(u)Q(v)S^\dagger(-\frac{1}{v})S^\dagger(-u)S^\dagger(-\frac{1}{v})S^\dagger(-v)$$

$$= (v-u) \delta(v, -u)Q(u)Q(v)S^\dagger(-\frac{1}{v})S^\dagger(-u)S^\dagger(-\frac{1}{v})S^\dagger(-v)$$

$$= 2v \delta(v, -u). \quad (5.11)$$

Expanding $\varphi(u)\varphi(v) + \varphi(v)\varphi(u) = 2v \delta(v, -u)$ into

$$\sum_{m \in \mathbb{Z}} \varphi_m u^{-m} \sum_{n \in \mathbb{Z}} \varphi_n v^{-n} + \sum_{n \in \mathbb{Z}} \varphi_n v^{-n} \sum_{m \in \mathbb{Z}} \varphi_m u^{-m} = 2v \sum_{k \in \mathbb{Z}} \frac{v^k}{(-u)^{k+1}}, \quad (5.12)$$
and taking the coefficient of $u^{-m}v^{-n}$ at both ends of the above formula, we derive the first formula of (5.8).

\[\square\]

**Remark 5.3.** From the formula (5.2), we derive the bosonic form of the quantum fields $\varphi(u)$ and $\overline{\varphi}(u)$ as follows

\[
\varphi(u) = \exp \left(\sum_{n \in \mathbb{N}_{\text{odd}}} \frac{2p_n}{n} u^n\right) \exp \left(-\sum_{n \in \mathbb{N}_{\text{odd}}} \frac{\partial}{\partial p_n} u^n\right) \exp \left(-\sum_{n \in \mathbb{N}_{\text{odd}}} \frac{1}{n} \frac{\partial}{\partial p_n} u^n\right),
\]

\[
\overline{\varphi}(u) = \exp \left(\sum_{n \in \mathbb{N}_{\text{odd}}} \frac{2p'_n}{n} u^n\right) \exp \left(-\sum_{n \in \mathbb{N}_{\text{odd}}} \frac{\partial}{\partial p_n} u^n\right) \exp \left(-\sum_{n \in \mathbb{N}_{\text{odd}}} \frac{1}{n} \frac{\partial}{\partial p'_n} u^n\right).
\] (5.13)

Hence one can check that $\varphi_m(1) = 0 (m > 0)$, $\varphi_0(1) = 1$ and $\overline{\varphi}_n(1) = 0 (n > 0)$, $\overline{\varphi}_0(1) = 1$.

**Definition 5.4.** The BUC hierarchy is the system of bilinear relations

\[
\Omega(\tau \otimes \tau) = \overline{\Omega}(\tau \otimes \tau) = (\tau \otimes \tau),
\] (5.14)

where

\[
\Omega = \sum_{n \in \mathbb{Z}} \varphi_n \otimes (-1)^n \varphi_{-n}, \quad \overline{\Omega} = \sum_{n \in \mathbb{Z}} \overline{\varphi}_n \otimes (-1)^n \overline{\varphi}_{-n}.
\] (5.15)

From Remark 5.3, it is easy to see that $\tau = 1$ is a tau-function of the BUC hierarchy. Similarly, if the solution of (5.14) is a polynomial function of the variables $(p_1, p_3, \ldots)$, we say it is a polynomial tau-function. Now we consider other forms of tau-functions of the BUC hierarchy.

**Lemma 5.5.** Let $X = \sum_{n \geq N} A_n \varphi_n$, $Y = \sum_{m \geq M} B_m \overline{\varphi}_m$, where $A_n, B_m \in \mathbb{C}$ and $N, M \in \mathbb{Z}$. Then

\[
X^2 = \begin{cases} 
\sum_{N \leq k \leq -N} (-1)^k A_k A_{-k}, & N < 0, \\
A_0^2, & N = 0, \\
0, & N > 0.
\end{cases}
\]

\[
Y^2 = \begin{cases} 
\sum_{M \leq l \leq -M} (-1)^l B_l B_{-l}, & M < 0, \\
B_0^2, & M = 0, \\
0, & M > 0.
\end{cases}
\] (5.16)

**Lemma 5.6.**

\[
\Omega(X \otimes X) = (X \otimes X)\Omega, \quad \Omega(Y \otimes Y) = (Y \otimes Y)\Omega,
\]

\[
\overline{\Omega}(X \otimes X) = (X \otimes X)\overline{\Omega}, \quad \overline{\Omega}(Y \otimes Y) = (Y \otimes Y)\overline{\Omega}.
\] (5.17)

**Proof.** Using the similar approach in [34], we can prove the Lemma. \(\square\)

**Corollary 5.7.** Let $\tau \in \mathcal{B}_{\text{odd}}$ be a tau-function of the BUC hierarchy, and let $X = \sum_{n \geq N} A_n \varphi_n$, $Y = \sum_{m \geq M} B_m \overline{\varphi}_m$, where $A_n, B_m \in \mathbb{C}$ and $N, M \in \mathbb{Z}$. Then $\tau' = X\tau$ and $\tau'' = Y\tau$ are also tau-functions of the BUC hierarchy.

**Proof.** The proof method of this Corollary is similar to that of Corollary 3.6, so it will not be described in detail here. \(\square\)
5.2. Generating functions and polynomial tau-functions of the BUC hierarchy. Let $Q(u, v)$ be a generating function of the BUC hierarchy in $(u, v) = (u_1, \ldots, u_r, v_1, \ldots, v_s)$ defined by

$$Q(u, v) = \prod_{1 \leq i < j \leq r} \frac{u_i - u_j}{u_i + u_j} \prod_{1 \leq i < j \leq s} \frac{v_i - v_j}{v_i + v_j} \prod_{1 \leq i \leq r, 1 \leq j \leq s} \frac{1 - u_i v_j}{1 + u_i v_j} \prod_{i=1}^{r} Q(u_i) \prod_{j=1}^{s} Q'(v_j). \quad (5.18)$$

From Proposition 5.1 and $S^⊥(u)(1) = 1$, we have

$$\varphi(u_1) \cdots \varphi(u_r) \varphi(v_1) \cdots \varphi(v_s)(1) = Q(u, v). \quad (5.19)$$

It is expanded into rational function form $Q(u, v) = \sum_{\alpha, \beta \in \mathbb{Z}^s} Q_{\alpha, \beta} u_1^{\alpha_1} \cdots u_r^{\alpha_r} v_1^{\beta_1} \cdots v_s^{\beta_s}$, then

$$Q_{\alpha, \beta} = \varphi - \alpha_1 \cdots \varphi - \alpha_r \varphi - \beta_1 \cdots \varphi - \beta_s(1). \quad (5.20)$$

In the following, in order to express the family of generation functions of the BUC hierarchy as a certain Pfaffian, we denote the variables as $\tilde{u} = deq (u_1, u_2, \ldots, u_r), \tilde{v} = deq (u_1^{-1}, u_2^{-1}, \ldots, u_s^{-1})$. Consider the set of Laurent polynomial $A_1(u), \ldots, A_l(u)$ and $B_{-s}(u^{-1}), \ldots, B_{-1}(u^{-1})$, define a formal distribution

$$T(\tilde{u}, \tilde{v}) = \prod_{j=1}^{r} A_j(u_j) \prod_{i=-s}^{1} B_i(u_i^{-1}) Q(\tilde{u}, \tilde{v})$$

$$= \prod_{j=1}^{r} A_j(u_j) \prod_{i=-s}^{1} B_i(u_i^{-1}) \prod_{-s \leq i, j \leq r, i \neq j} f(u_i, u_j) \prod_{i=-s}^{1} Q'(u_i^{-1}) \prod_{j=1}^{r} Q(u_j). \quad (5.21)$$

For any $\gamma = (\gamma_1, \ldots, \gamma_r, \gamma_{-1}, \ldots, \gamma_{-s}) \in \mathbb{Z}^{r+s}$, $T_\gamma$ is the coefficient of the following expansion

$$T(\tilde{u}, \tilde{v}) = \sum_{\gamma \in \mathbb{Z}^{r+s}} T_\gamma u_1^{\gamma_1} \cdots u_r^{\gamma_r} u_{-1}^{\gamma_{-1}} \cdots u_{-s}^{\gamma_{-s}}. \quad (5.22)$$

We recall that if $A = [a_{ij}]$ is a skew symmetric matrix of even size $2n \times 2n$, its determinant is a perfect square: $\det[A] = Pf[A]^2$, where

$$Pf[A] = \sum_{\omega} sgn(\omega) a_{\omega(1)\omega(2)} \cdots a_{\omega(2n-1)\omega(2n)}, \quad (5.23)$$

summed over $\omega \in S_{2n}$ such that $\omega(2r - 1) < \omega(2r)$ for $1 \leq r \leq n$, and $\omega(2r - 1) < \omega(2r + 1)$ for $1 \leq r < n - 1$. In addition, it is well-known that

$$Pf \left[ \frac{u_i - u_j}{u_i + u_j} \right]_{1 \leq i, j \leq 2n, i \neq j} = \prod_{1 \leq i < j \leq 2n} \frac{u_i - u_j}{u_i + u_j}. \quad (5.24)$$

Introduce the skew symmetric matrix $F = [f_{i,j}]_{-2s \leq i, j \leq 2r}$, where

$$f_{i,j} = \begin{cases} f(u_i, u_j), & i < j, \\ 0, & i = j, \\ -f(u_j, u_i), & i > j. \end{cases} \quad (5.25)$$
Obviously,

\[ Pf[F] = \prod_{-2s \leq i < j \leq 2r, \ i, j \neq 0} f(u_i, u_j) = \prod_{-2s \leq i < j \leq 2r, \ i, j \neq 0} \frac{u_i - u_j}{u_i + u_j}. \quad (5.26) \]

Define the formal distributions

\[ T^{(i)}(u_i^{-1}) = B_i(u_i^{-1})Q'(u_i^{-1}), \quad i \in \{-2s, \ldots, -1\}, \]

\[ T^{(j)}(u_j) = A_j(u_j)Q(u_j), \quad j \in \{1, \ldots, 2r\}, \quad (5.27) \]

and

\[ T^{(i,j)} = f_{ij} T^{(i)}(u_i) T^{(j)}(u_j) = \sum_{m,n \in \mathbb{Z}} T^{(i,j)}_{m,n} u_i^m u_j^n, \quad (5.28) \]

where \( T^{(i)} \) denotes \( T^{(i)}(u_i^{-1}) \) for negative \( i \) and \( T^i(u_i) \) for positive \( i \).

**Theorem 5.8.**

1) The formal distribution \( T(u_1, \ldots, u_{2r}, u_{-1}^{-1}, \ldots, u_{-2s}^{-1}) \) can be expressed as

\[ T(u_1, \ldots, u_{2r}, u_{-1}^{-1}, \ldots, u_{-2s}^{-1}) = Pf[T^{(i,j)}]_{-2s \leq i, j \leq 2r}. \quad (5.29) \]

2) The coefficient \( T_\gamma \) about the expansion in Eq. (5.29) can be written as

\[ T_\gamma = Pf[T^{(i,j)}_{\gamma_1, \gamma_2}]_{-2s \leq i, j \leq 2r}, \quad (5.30) \]

where \( \gamma = (\gamma_1, \ldots, \gamma_{2r}, \gamma_{-1}, \ldots, \gamma_{-2s}) \in \mathbb{Z}^{2r+2s} \).

3) For any \( \gamma = (\gamma_1, \ldots, \gamma_r, \gamma_{-1}, \ldots, \gamma_{-s}) \in \mathbb{Z}^{r+s} \), the coefficient \( T_\gamma \) about \( u_1^{\gamma_1} \cdots u_r^{\gamma_r} u_{-1}^{\gamma_{-1}} \cdots u_{-s}^{\gamma_{-s}} \) in Eq. (5.22) is a polynomial tau-function of the BUC hierarchy.

4) There is a set of Laurent polynomials \( A_1(u), \ldots, A_r(u), B_{-s}(u^{-1}), \ldots, B_{-1}(u^{-1}) \) such that \( \tau \) is the zero-mode of the Eq. (5.22) if \( \tau \) is a polynomial tau-function of the BUC hierarchy.

**Proof.**

1) A direct calculation gives rise to

\[
T(u_1, u_2, \ldots, u_{2r}, u_{-1}^{-1}, u_{-2}^{-1}, \ldots, u_{-2s}^{-1}) \\
= \prod_{j=1}^{2r} A_j(u_j) \prod_{i=-2s}^{-1} B_i(u_i^{-1}) \prod_{-2s \leq i < j \leq 2r, \ i, j \neq 0} f(u_i, u_j) \prod_{i=-2s}^{-1} Q(u_i^{-1}) \prod_{j=1}^{2r} Q(u_j) \\
= Pf[F] \prod_{i=-2s}^{-1} T^{(i)}(u_i^{-1}) \prod_{j=1}^{2r} T^{(j)}(u_j) \\
= \sum_{\sigma \in S_{2s+2r}} sgn(\sigma) f_{\sigma(-2s)\sigma(-2s+1)} \cdots f_{\sigma(-2)\sigma(-1)} f_{\sigma(1)\sigma(2)} \cdots f_{\sigma(2r-1)\sigma(2r)} \cdot \prod_{i=-2s}^{-1} T^{(i)}(u_i^{-1}) \prod_{j=1}^{2r} T^{(j)}(u_j) \\
= \sum_{\sigma \in S_{2s+2r}} sgn(\sigma) f_{\sigma(-2s)\sigma(-2s+1)} T^{(\sigma(-2s))} T^{(\sigma(-2s+1))} \cdots f_{\sigma(-2)\sigma(-1)} T^{(\sigma(-2))} T^{(\sigma(-1))}
\]

20
\[
\begin{align*}
&f_{\sigma(1)} T^{(\sigma(1))} T^{(\sigma(2))} \ldots f_{\sigma(2r-1)} T^{(\sigma(2r-1))} T^{(\sigma(2r))} \\
&= Pf[f_{ij} T(i) T(j)]_{-2r \leq i, j \leq 2r} = Pf[T(ij)]_{-2r \leq i, j \leq 2r}.
\end{align*}
\]  
\text{(5.31)}

2) By the definition of \(T(i,j)\), after a straightforward calculation, we obtain
\[
\begin{align*}
&\text{Pf}[T(ij)]_{-2r \leq i, j \leq 2r} = \text{Pf} \left[ \sum_{r, \gamma} T_{\gamma r} u_1^{\gamma_1} u_2^{\gamma_2} \ldots u_r^{\gamma_r} \right]_{-2r \leq i, j \leq 2r}\\
&= \sum_{r, \gamma} \text{Pf}(T_{\gamma r} u_1^{\gamma_1} u_2^{\gamma_2} \ldots u_r^{\gamma_r})_{-2r \leq i, j \leq 2r}.
\end{align*}
\]
\text{(5.32)}

Clearly, the coefficient of \(u_1^{\gamma_1} \ldots u_r^{\gamma_r} u_{-1}^{\gamma_{-1}} \ldots u_{-r}^{\gamma_{-r}}\) in Eq. (5.29) is \(\text{Pf}[T(ij)]_{-2r \leq i, j \leq 2r}\).

3) Let \(A_j(u) = \sum_{M_j \leq k \leq N_j} A_{j,k} u^k\) and \(B_i(u^{-1}) = \sum_{U_i \leq m \leq V_i} B_{i,m} u^{-m}\) be power series expansions about variable \(u\), where \(j = 1, \ldots, l, i = -s, \ldots, -1\) and \(M_j, N_j, U_i, V_i \in \mathbb{Z}\). From (5.19), we can get
\[
T(\bar{u}, \bar{v}) = \prod_{j=1}^{r} A_{j}(u_j^{-1}) \varphi(u_1) \ldots \varphi(u_r) \bar{\varphi}(u_1^{-1}) \ldots \bar{\varphi}(u_r^{-1})(1)
\]
\[
= \sum_{M_1 \leq k_1 \leq N_1} A_{1,k_1} \varphi_{1} u_1^{k_1-1-l_1} \ldots \sum_{M_r \leq k_r \leq N_r} A_{r,k_r} \varphi_{r} u_r^{k_r-1-l_r} \sum_{U_{-1} \leq m_{-1} \leq V_{-1}} \sum_{n_{-1} \in \mathbb{Z}} B_{-1,m_{-1}} u_{-1}^{n_{-1}-m_{-1}} \ldots \sum_{U_{-s} \leq m_{-s} \leq V_{-s}} \sum_{n_{-s} \in \mathbb{Z}} B_{-s,m_{-s}} u_{-s}^{n_{-s}-m_{-s}}(1)
\]
\[
= \sum_{\gamma \in \mathbb{Z}^{2r}} \sum_{M_1-\gamma_1 \leq i_1 \leq N_1-\gamma_1} A_{1,\gamma_1+i_1} \varphi_{1} u_1^{\gamma_1} \ldots \sum_{M_r-\gamma_r \leq i_r \leq N_r-\gamma_r} A_{r,\gamma_r+i_r} \varphi_{r} u_r^{\gamma_r} \sum_{U_{-1}+\gamma_{-1} \leq n_{-1} \leq V_{-1}+\gamma_{-1}} \sum_{U_{-s}+\gamma_{-s} \leq n_{-s} \leq V_{-s}+\gamma_{-s}} B_{-s,n_{-s}} u_{-s}^{\gamma_{-s}}(1).
\]
\text{(5.33)}

Thus the coefficient \(T_{\gamma}\) of \(u_1^{\gamma_1} \ldots u_r^{\gamma_r} u_{-1}^{\gamma_{-1}} \ldots u_{-r}^{\gamma_{-r}}\) can be written as \(X_1 \ldots X_r Y_{-1} \ldots Y_{-r}(1)\), where
\[
X_i = \sum_{M_i-\gamma_i \leq i \leq N_i-\gamma_i} A_i_{,\gamma_i+i} \varphi_{i}, \quad i = 1, \ldots, r,
\]
\[
Y_i = \sum_{U_i+\gamma_i \leq n_i \leq V_i+\gamma_i} B_{i,n_i-\gamma_i} \varphi_{-i}, \quad i = -s, \ldots, -1.
\]
\text{(5.34)}

By Remark 5.3 and Lemma 5.6, we conclude that the coefficient \(T_{\gamma}\) is a tau-function of the BUC hierarchy. \(T_{\gamma}\) is a polynomial tau-function because it is a finite linear combination of \(\varphi_{l_1} \ldots \varphi_{l_r} \varphi_{-n_{-1}} \ldots \varphi_{-n_{-s}}(1)\).
4) Polynomial tau-function of the BUC hierarchy has the form
\[
\tau = X_1 \cdots X_r Y_{-1} \cdots Y_{-s}(1),
\]
where \([\lambda, \mu] = ([\lambda_1, \lambda_2, \ldots, \lambda_r], [\lambda_1, -\lambda_2, \ldots, -\lambda_s])\) is a pair of partitions, and
\[
X_i = \sum_{-\lambda_i \leq m \leq N_i} d_{m,i} \varphi^m, \quad d_{m,i} \in \mathbb{C}, b_{-\lambda_i,i} \neq 0, N_i \in \mathbb{Z}, i = 1, \ldots, r,
\]
\[
Y_i = \sum_{-\lambda_i \leq h \leq V_i} e_{h,i} \varphi^h, \quad e_{h,i} \in \mathbb{C}, b_{-\lambda_i,i} \neq 0, V_i \in \mathbb{Z}, i = -s, \ldots, -1.
\]

For a vector \(\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_r, \gamma_{-1}, \ldots, \gamma_{-s}) \in \mathbb{Z}^{r+s}\), we define \(A_i(u)\) and \(B_i(u^{-1})\) in the Eq. (5.21) to be the Laurent polynomial with the following form
\[
\begin{align*}
A_i(u) &= \sum_{\gamma_i - \lambda_i \leq t \leq N_i + \gamma_i} d_{t-\gamma_i,i} u^t, & i = 1, \ldots, r, \\
B_i(u^{-1}) &= \sum_{-\gamma_i - \lambda_i \leq w \leq V_i - \gamma_i} e_{w+h,i} u^{-w}, & i = -s, \ldots, -1.
\end{align*}
\]
By using \(A_1(u), \ldots, A_r(u)\) and \(B_{-s}(u^{-1}), \ldots, B_{-1}(u^{-1})\), it is easy to verify that Eq. (5.34) leads to (5.36). The coefficient \(T_\gamma\) corresponds to the polynomial tau-function (5.35). It is showed that \(\tau\) is the zero-mode of the series expansion of \(T(\tilde{u}, \tilde{v})\) with \(\gamma_1 = \cdots = \gamma_r = \gamma_{-1} = \cdots = \gamma_{-s} = 0\).

\(\square\)

**Corollary 5.9.**

1) We have proved that the polynomial tau-functions of the BUC hierarchy are zero-mode of certain generating functions \(T(\tilde{u}, \tilde{v})\). By replacing \(A_j(u)\) with \(u^{\gamma_j} A_j(u)\) \((j = 1, \ldots, r)\) and \(B_i(u^{-1})\) with \(u^{-\gamma_i} B_i(u^{-1})\) \((i = -s, \ldots, -1)\), we derive the any polynomial tau-function as a coefficient of a given monomial \(u_1^{\gamma_1} \cdots u_r^{\gamma_r} u_{-1}^{\gamma_{-1}} \cdots u_{-s}^{\gamma_{-s}}\).

2) Introducing
\[
\begin{align*}
A_j(u) &= h_j \sum_{i=0}^{M_j} a_{j,i} u^i, & M_j \in \mathbb{Z}, h_j, a_{j,i} \in \mathbb{C}, a_{j,0} = 1, j = 1, \ldots, r, \\
B_i(u^{-1}) &= g_i \sum_{m=0}^{M_i} b_{i,m} u^{-m}, & M_i \in \mathbb{Z}, g_i, b_{i,m} \in \mathbb{C}, b_{i,0} = 1, i = -s, \ldots, -1,
\end{align*}
\]
where \(A_1(u), \ldots, A_r(u), B_{-1}(u^{-1}), \ldots, B_{-s}(u^{-s})\) are non-zero Laurent series defined in the \(T(\tilde{u}, \tilde{v})\).

By means of (5.33), we have
\[
\sum_{i=0}^{M_j} a_{j,i} u^i = \exp \left( \sum_{s=1}^{\infty} c_{j,s} u^s \right), \quad \sum_{m=0}^{M_i} b_{i,m} u^{-m} = \exp \left( \sum_{l=1}^{\infty} c'_{i,l} u^{-l} \right),
\]
and
\[
a_{j,i} = S_i(c_{j,1}, c_{j,2}, \ldots), \quad b_{i,m} = S_m(c'_{i,1}, c'_{i,2}, \ldots),
\]
where \(c_{j,s}\) and \(c'_{i,t}\) are constants in \(\mathbb{C}\).

Setting \((\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \ldots) = (2p_1, 0, \frac{2}{3}p_3, 0, \ldots)\) and \((\tilde{x}'_1, \tilde{x}'_2, \tilde{x}'_3, \ldots) = (2p'_1, 0, \frac{2}{3}p'_3, 0, \ldots)\), we get

\[
T^{(j)}(u_j) = A_j(u_j)Q(u_j) = h_j \exp \left( \sum_{k \geq 1} c_{j,k}u_j^k \right) \exp \left( \sum_{k \geq 1} \tilde{x}_k u_j^k \right)
\]

\[
= h_j \sum_{k=0}^{\infty} S_k(\tilde{x}_1 + c_{j,1}, \tilde{x}_2 + c_{j,2}, \ldots)u_j^k, \quad j = 1, \ldots, r,
\]

\[
T^{(i)}(u_i^{-1}) = B_i(u_i^{-1})Q'(u_i^{-1}) = g_i \exp \left( \sum_{l \geq 1} c'_{i,l}u_i^{-l} \right) \exp \left( \sum_{l \geq 1} \tilde{x}'_l u_i^{-l} \right)
\]

\[
= g_i \sum_{l=0}^{\infty} S_l(\tilde{x}'_1 + c'_{i,1}, \tilde{x}'_2 + c'_{i,2}, \ldots)u_i^{-l}, \quad i = -s, \ldots, -1. \quad (5.41)
\]

Hence, \(T^{(i,j)}\) can be written as

\[
T^{(i,j)} = \begin{cases} 
    h_i h_j \left( 1 + 2 \sum_{k \geq 1} (-1)^k \frac{u_j^k}{u_i^k} \right) \sum_{m,n \in \mathbb{Z}} S_m(\tilde{x} + c_i) S_n(\tilde{x} + c_j) u_i^m u_j^n, & 0 < i < j \leq 2r, \\
    g_i g_j \left( 1 + 2 \sum_{k \geq 1} (-1)^k \frac{u_j^k}{u_i^k} \right) \sum_{m,n \in \mathbb{Z}} S_m(\tilde{x}' + c'_i) S_n(\tilde{x}' + c'_j) u_i^{-m} u_j^{-n}, & -2s \leq i < j < 0, \\
    h_i g_j \left( 1 + 2 \sum_{k \geq 1} (-1)^k \frac{u_j^k}{u_i^k} \right) \sum_{m,n \in \mathbb{Z}} S_m(\tilde{x}' + c'_i) S_n(\tilde{x} + c_j) u_i^{-m} u_j^n, & -2s \leq i < 0 < j \leq 2r.
\end{cases} \quad (5.42)
\]

Expanding \(T^{(i,j)}\), we can obtain the expression of the \(T^{(i,j)}_{m,n}\)

\[
T^{(i,j)}_{m,n} = \begin{cases} 
    2h_i h_j \chi^{(1)}_{m,n}(\tilde{x} + c_i, \tilde{x} + c_j), & 0 < i < j \leq 2r, \\
    2g_i g_j \chi^{(2)}_{m,n}(\tilde{x}' + c'_i, \tilde{x}' + c'_j), & -2s \leq i < j < 0, \\
    2h_i g_j \chi^{(3)}_{m,n}(\tilde{x}' + c'_i, \tilde{x} + c_j), & -2s \leq i < 0 < j \leq 2r,
\end{cases} \quad (5.43)
\]

where \(T^{(i,j)}_{m,n} = -T^{(j,i)}_{m,n}\) for \(i > j\), \(T^{(i,i)}_{m,n} = 0\), and

\[
\chi^{(1)}_{m,n}(\tilde{x} + c_i, \tilde{x} + c_j) = \frac{1}{2} S_m(\tilde{x} + c_i) S_n(\tilde{x} + c_j) + \sum_{k \geq 1} (-1)^k S_{m+k}(\tilde{x} + c_i) S_{n-k}(\tilde{x} + c_j),
\]

\[
\chi^{(2)}_{m,n}(\tilde{x}' + c'_i, \tilde{x}' + c'_j) = \frac{1}{2} S_m(\tilde{x}' + c'_i) S_n(\tilde{x}' + c'_j) + \sum_{k \geq 1} (-1)^k S_{m-k}(\tilde{x}' + c'_i) S_{n+k}(\tilde{x}' + c'_j),
\]

\[
\chi^{(3)}_{m,n}(\tilde{x}' + c'_i, \tilde{x} + c_j) = \frac{1}{2} S_m(\tilde{x}' + c'_i) S_n(\tilde{x} + c_j) + \sum_{k \geq 1} (-1)^k S_{m-k}(\tilde{x}' + c'_i) S_{n+k}(\tilde{x} + c_j). \quad (5.44)
\]

In order to facilitate expression, some new symbols will be introduced. Define the skew-symmetric matrix \(M = (m_{i,j})_{0 \leq i, j \leq 2r}\) by putting each \((i, j)\)-th as \(m_{i,j} = 2h_i h_j \chi^{(1)}_{m,n}(\tilde{x} + c_i, \tilde{x} + c_j)\).
for \( i < j \) and \( m_{i,j} = -m_{j,i} \) for \( i > j \), \( m_{i,i} = 0 \). Similarly, define another skew-symmetric matrix \( \mathbf{M} = (m_{i,j})_{-2s \leq i,j \leq 0} \), where \( m_{i,j} = 2g_i g_j \lambda^{(2)}_{m,n}(\tilde{x} + c'_i, \tilde{x}' + c'_j) \) for \( i < j \). Introduce the third matrix \( \mathbf{N} = (n_{i,j})_{-2s \leq i < 0, 0 < j \leq 2r} \), where \( n_{i,j} = 2h_i g_j \lambda^{(3)}_{m,n}(\tilde{x}' + c'_i, \tilde{x} + c_j) \).

From Theorem 5.8, polynomial tau-functions of the BUC hierarchy have the form

\[
T_\gamma = Pf \begin{bmatrix} \mathbf{M} & \mathbf{N} \\ -\mathbf{N}^T & \mathbf{M} \end{bmatrix}_{-2s \leq i,j \leq 2r}.
\]

**Remark 5.10.** It is noted that the polynomial tau-functions of the BUC hierarchy reduce to the solutions of the BKP hierarchy [37] with the reduction \( y = 0 \).

6. Conclusions and discussions

In this paper, we have discussed exact solutions of the SKP, OKP and BUC hierarchies including the polynomial-type and soliton-type solutions. It is showed that the generating functions play a vital role in establishing the polynomial tau-functions of the integrable systems. Furthermore, we expressed the polynomial tau-functions of the SKP, OKP and BUC hierarchies as determinant and Pfaffian forms, respectively. The results here are hoped to be helpful for better understanding the essential properties of the SKP, OKP and BUC hierarchies. It is known that symplectic universal character (SUC) and orthogonal universal character (OUC) hierarchies are the extensions of the SKP and OKP hierarchies. However, it should be pointed out that we have not expressed the polynomial tau-functions of the SUC and OUC hierarchies as a perfect determinant form due to the inappropriate quantum fields presentation of SUC and OUC. We will concentrate on studying this interesting question in the near future.

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