The end-parameters of a Leonard pair

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Abstract

Fix an algebraically closed field \( \mathbb{F} \) and an integer \( d \geq 3 \). Let \( V \) be a vector space over \( \mathbb{F} \) with dimension \( d + 1 \). A Leonard pair on \( V \) is a pair of diagonalizable linear transformations \( A : V \to V \) and \( A^* : V \to V \), each acting in an irreducible tridiagonal fashion on an eigenbasis for the other one. There is an object related to a Leonard pair called a Leonard system. It is known that a Leonard system is determined up to isomorphism by a sequence of scalars \( \{ \theta_i \}_{i=0}^{d} \), \( \{ \phi_i \}_{i=1}^{d} \), \( \{ \varphi_i \}_{i=1}^{d} \), called its parameter array. The scalars \( \{ \theta_i \}_{i=0}^{d} \) (resp. \( \{ \phi_i \}_{i=0}^{d} \)) are mutually distinct, and the expressions \( (\theta_{i-2} - \theta_{i+1})/(\theta_{i-1} - \theta_i) \), \( (\theta_{i-2}^* - \theta_{i+1}^*)/(\theta_{i-1}^* - \theta_i^*) \) are equal and independent of \( i \) for \( 2 \leq i \leq d - 1 \). Write this common value as \( \beta + 1 \). In the present paper, we consider the “end-parameters” \( \theta_0, \theta_d, \phi_0, \phi_d \), \( \varphi_1, \varphi_d \) of the parameter array. We show that a Leonard system is determined up to isomorphism by the end-parameters and \( \beta \). We display a relation between the end-parameters and \( \beta \). Using this relation, we show that there are up to isomorphism at most \( \lfloor (d - 1)/2 \rfloor \) Leonard systems that have specified end-parameters. The upper bound \( \lfloor (d - 1)/2 \rfloor \) is best possible.

1 Introduction

Throughout the paper \( \mathbb{F} \) denotes an algebraically closed field.

We begin by recalling the notion of a Leonard pair. We use the following terms. A square matrix is said to be tridiagonal whenever each nonzero entry lies on either the diagonal, the subdiagonal, or the superdiagonal. A tridiagonal matrix is said to be irreducible whenever each entry on the subdiagonal is nonzero and each entry on the superdiagonal is nonzero.

Definition 1.1 (See [5 Definition 1.1].) Let \( V \) be a vector space over \( \mathbb{F} \) with finite positive dimension. By a Leonard pair on \( V \) we mean an ordered pair of linear transformations \( A : V \to V \) and \( A^* : V \to V \) that satisfy (i) and (ii) below:

(i) There exists a basis for \( V \) with respect to which the matrix representing \( A \) is irreducible tridiagonal and the matrix representing \( A^* \) is diagonal.

(ii) There exists a basis for \( V \) with respect to which the matrix representing \( A^* \) is irreducible tridiagonal and the matrix representing \( A \) is diagonal.

Note 1.2 According to a common notational convention, \( A^* \) denotes the conjugate transpose of \( A \). We are not using this convention. In a Leonard pair \( A, A^* \) the matrices \( A \) and \( A^* \) are arbitrary subject to the conditions (i) and (ii) above.

We refer the reader to [3, 5, 8] for background on Leonard pairs.

For the rest of this section, fix an integer \( d \geq 0 \) and a vector space \( V \) over \( \mathbb{F} \) with dimension \( d + 1 \). Consider a Leonard pair \( A, A^* \) on \( V \). By [5] Lemma 1.3] each of \( A, A^* \)
has mutually distinct \( d + 1 \) eigenvalues. Let \( \{ \theta_i \}_{i=0}^d \) be an ordering of the eigenvalues of \( A \), and let \( \{ V_i \}_{i=0}^d \) be the corresponding eigenspaces. For \( 0 \leq i \leq d \) define \( E_i : V \to V \) such that \( (E_i - I)V_i = 0 \) and \( E_iV_j = 0 \) for \( j \neq i \) \((0 \leq j \leq d)\). Here \( I \) denotes the identity. We call \( E_i \) the primitive idempotent of \( A \) associated with \( \theta_i \). The primitive idempotent \( E_i^* \) of \( A^* \) associated with \( \theta_i^* \) is similarly defined. For \( 0 \leq i \leq d \) pick a nonzero \( v_i \in V_i \). Note that \( \{ v_i \}_{i=0}^d \) is a basis for \( V \). We say the ordering \( \{ E_i \}_{i=0}^d \) is standard whenever the basis \( \{ v_i \}_{i=0}^d \) satisfies Definition 1.1(ii). A standard ordering of the primitive idempotents of \( A^* \) is similarly defined. For a standard ordering \( \{ E_i \}_{i=0}^d \), the ordering \( \{ E_{d-i} \}_{i=0}^d \) is also standard and no further ordering is standard. Similar result applies to a standard ordering of the primitive idempotents of \( A^* \).

**Definition 1.3** (See [5] Definition 1.4.) By a Leonard system on \( V \) we mean a sequence

\[
\Phi = (A, \{ E_i \}_{i=0}^d, A^*, \{ E_i^* \}_{i=0}^d),
\]  

(1)

where \( A, A^* \) is a Leonard pair on \( V \), and \( \{ E_i \}_{i=0}^d \) (resp. \( \{ E_i^* \}_{i=0}^d \)) is a standard ordering of the primitive idempotents of \( A \) (resp. \( A^* \)). We say \( \Phi \) is over \( \mathbb{F} \). We call \( d \) the diameter of \( \Phi \).

We recall the notion of an isomorphism of Leonard systems. Consider a Leonard system (1) on \( V \) and a Leonard system \( \Phi' = (A', \{ E'_i \}_{i=0}^d, A'^*, \{ E'_i^* \}_{i=0}^d) \) on a vector space \( V' \) with dimension \( d + 1 \). By an isomorphism of Leonard systems from \( \Phi \) to \( \Phi' \) we mean a linear bijection \( \sigma : V \to V' \) such that \( \sigma A = A', \sigma A^* = A'^*, \sigma E_i = E'_i, \sigma E_i^* = E'^*_i \) for \( 0 \leq i \leq d \). Leonard systems \( \Phi \) and \( \Phi' \) are said to be isomorphic whenever there exists an isomorphism of Leonard systems from \( \Phi \) to \( \Phi' \).

For a Leonard system (1) over \( \mathbb{F} \), each of the following is a Leonard system over \( \mathbb{F} \):

\[
\Phi^* := (A^*, \{ E_i^* \}_{i=0}^d, A, \{ E_i \}_{i=0}^d),
\]
\[
\Phi^↓ := (A, \{ E_i \}_{i=0}^d, A^*, \{ E_i^* \}_{i=0}^d),
\]
\[
\Phi^\downarrow := (A, \{ E_{d-i} \}_{i=0}^d, A^*, \{ E_i^* \}_{i=0}^d).
\]

Viewing \( *, \downarrow, \downarrow \) as permutations on the set of all the Leonard systems,

\[
*^2 = \downarrow^2 = \downarrow\downarrow = 1, \quad \downarrow * = \downarrow*, \quad \downarrow * = \downarrow, \quad \downarrow \downarrow = \downarrow\downarrow. \tag{2}
\]

The group generated by symbols \( *, \downarrow, \downarrow \) subject to the relations (2) is the dihedral group \( D_4 \). We recall \( D_4 \) is the group of symmetries of a square, and has 8 elements. For an element \( g \in D_4 \) and for an object \( f \) associated with \( \Phi \), let \( f^g \) denote the corresponding object associated with \( \Phi^g^{-1} \).

We recall the notion of a parameter array.

**Definition 1.4** (See [7] Section 2, [2] Theorem 4.6.) Consider a Leonard system (1) over \( \mathbb{F} \). By the parameter array of \( \Phi \) we mean the sequence

\[
(\{ \theta_i \}_{i=0}^d, \{ \theta_i^* \}_{i=0}^d, \{ \phi_i \}_{i=1}^d, \{ \psi_i \}_{i=1}^d), \tag{3}
\]
where \( \theta_i \) is the eigenvalue of \( A \) associated with \( E_i \), \( \theta_i^* \) is the eigenvalue of \( A^* \) associate with \( E_i^* \), and

\[
\varphi_i = (\theta_0^* - \theta_i^*) \frac{\text{tr}(E_0^* \prod_{h=0}^{i-1}(A - \theta_h I))}{\text{tr}(E_0^* \prod_{h=0}^{i-2}(A - \theta_h I))},
\]
\[
\phi_i = (\theta_0^* - \theta_i^*) \frac{\text{tr}(E_0^* \prod_{h=0}^{i-1}(A - \theta_d - h I))}{\text{tr}(E_0^* \prod_{h=0}^{i-2}(A - \theta_d - h I))},
\]

where \( \text{tr} \) means trace. In the above expressions, the denominators are nonzero by [2, Corollary 4.5].

The following two results are fundamental in the theory of Leonard pairs.

**Lemma 1.5** (See [5, Theorem 1.9].) A Leonard system is determined up to isomorphism by its parameter array.

**Lemma 1.6** (See [5, Theorem 1.9].) Consider a sequence \((3)\) consisting of scalars taken from \( \mathbb{F} \). Then there exists a Leonard system \( \Phi \) over \( \mathbb{F} \) with parameter array \((3)\) if and only if \((i)-(v)\) hold below:

(i) \( \theta_i \neq \theta_j, \theta_i^* \neq \theta_j^* \) (0 \( \leq i < j \leq d \)).

(ii) \( \varphi_i \neq 0, \phi_i \neq 0 \) (1 \( \leq i \leq d \)).

(iii) \( \varphi_i = \phi_1 \sum_{\ell=0}^{i-1} \frac{\theta_\ell - \theta_d - \ell \theta_0}{\theta_0 - \theta_d} + (\theta_i^* - \theta_0^*)(\theta_{i-1} - \theta_d) \) (1 \( \leq i \leq d \)).

(iv) \( \phi_i = \varphi_1 \sum_{\ell=0}^{i-1} \frac{\theta_\ell - \theta_d - \ell \theta_0}{\theta_0 - \theta_d} + (\theta_i^* - \theta_0^*)(\theta_{d-i+1} - \theta_0) \) (1 \( \leq i \leq d \)).

(v) The expressions

\[
\frac{\theta_{i-2} - \theta_{i+1}}{\theta_{i-1} - \theta_i}, \quad \frac{\theta_i^* - \theta_{i+1}^*}{\theta_{i-1}^* - \theta_i^*}
\]

are equal and independent of \( i \) for 2 \( \leq i \leq d - 1 \).

**Definition 1.7** (See [7, Definition 1.1].) By a parameter array over \( \mathbb{F} \) we mean a sequence \((3)\) consisting of scalars taken from \( \mathbb{F} \) that satisfy conditions \((i)-(v)\) in Lemma 1.6.

**Definition 1.8** Let \( \Phi \) be a Leonard system over \( \mathbb{F} \) with parameter array \((3)\). By the fundamental parameter of \( \Phi \) (or \((3)\)) we mean one less than the common value of \((4)\).

The \( D_4 \) action affects the parameter array as follows:
Lemma 1.9 (See [5, Theorem 1.11].) Consider a Leonard system (1) over \( F \) with parameter array (3). Then for \( g \in \{ \downarrow, \downarrow, \ast \} \) the parameters \( \theta^g_i, \theta_i^*, \varphi^g_i, \phi_i^* \) are as follows:

| \( g \) | \( \theta^g_i \) | \( \theta_i^* \) | \( \varphi^g_i \) | \( \phi_i^* \) |
|-------|----------------|----------------|----------------|----------------|
| \( \downarrow \) | \( \theta_i \) | \( \theta_{d-i}^* \) | \( \phi_{d-i+1} \) | \( \varphi_{d-i+1} \) |
| \( \downarrow \) | \( \theta_{d-i} \) | \( \theta_i^* \) | \( \phi_i \) | \( \varphi_i \) |
| \( \ast \) | \( \theta_i^* \) | \( \theta_i \) | \( \varphi_i \) | \( \phi_{d-i+1} \) |

For the rest of this section, we assume \( d \geq 3 \). The present paper is motivated by the following result:

Proposition 1.10 (See [5, Corollary 14.1].) Consider a Leonard system (1) over \( F \) with parameter array (3). Then the isomorphism class of \( \Phi \) is determined by a sequence of 8 parameters consisting of \( \theta_0, \theta_1, \theta_2, \theta_0^*, \theta_2^* \), followed by one of \( \theta_3, \theta_3^* \), followed by one of \( \varphi_1, \varphi_d, \phi_1, \phi_d \).

Referring to Proposition 1.10 observe that the set of the 8 parameters is not invariant under the \( D_4 \) action. Our concern is to find a \( D_4 \)-invariant set of parameters that determines the isomorphism class of Leonard systems. In the present paper, we consider the end-parameters:

\[ \theta_0, \theta_d, \theta_0^*, \theta_d^*, \varphi_1, \varphi_d, \phi_1, \phi_d. \]

Apparently the set of the end-parameters is invariant under the \( D_4 \)-action. Note that the fundamental parameter is \( D_4 \)-invariant.

Theorem 1.11 A Leonard system is determined up to isomorphism by its end-parameters and its fundamental parameter.

The end-parameters are related to the fundamental parameter as follows:

Proposition 1.12 Consider a parameter array (3) over \( F \). Let \( \beta \) be the fundamental parameter of (3), and pick a nonzero \( q \in F \) such that \( \beta = q + q^{-1} \). Then the scalar

\[ \Omega = \frac{\phi_1 + \phi_d - \varphi_1 - \varphi_d}{(\theta_0 - \theta_d)(\theta_0^* - \theta_d^*)}, \]

is as follows:

| \( \beta \neq 2 \), \( \beta \neq -2 \) | \( \Omega \) |
|-----------------|-----------------|
| \( \beta = 2 \), \( \text{Char}(F) \neq 2 \) | \( \frac{q(q^{d-1} - 1)}{q^d - 1} \) |
| \( \beta = -2 \), \( \text{Char}(F) \neq 2 \), \( d \) is even | \( \frac{2(d - 1)}{d} \) |
| \( \beta = -2 \), \( \text{Char}(F) \neq 2 \), \( d \) is odd | \( 2 \) |
| \( \beta = 0 \), \( \text{Char}(F) = 2 \) | \( 0 \) |
Corollary 1.13 With reference to Proposition 1.12, \( \Omega \neq 1 \).

Theorem 1.14 There exist up to isomorphism at most \( \lfloor (d - 1)/2 \rfloor \) Leonard systems with diameter \( d \) that have specified end-parameters.

In Theorem 1.14 the upper bound \( \lfloor (d - 1)/2 \rfloor \) is best possible:

Theorem 1.15 Assume \( \text{Char}(\mathbb{F}) \neq 2 \) and \( d \) does not vanish in \( \mathbb{F} \). Then there exist mutually non-isomorphic \( \lfloor (d - 1)/2 \rfloor \) Leonard systems with diameter \( d \) that have common end-parameters.

The paper is organized as follows. In Section 2 we recall some formulas concerning the parameter array. In Section 3 we prove Theorem 1.11. In Section 4 we prove Proposition 1.12. In Section 5 we consider a certain polynomial which is used in the proof of Theorems 1.14 and 1.15. In Section 6 we prove Theorem 1.14. In Section 7 we try to construct a parameter array having specified end-parameters. In Section 8 we prove Theorem 1.15. In Appendix we display formulas that represent \( \{\varphi_i\}_{i=1}^d \) and \( \{\phi_i\}_{i=1}^d \) in terms of the end-parameters and the fundamental parameter.

2 Parameter arrays in closed form

Fix an integer \( d \geq 3 \). Let \( \mathcal{A} \) be a parameter array over \( \mathbb{F} \) with fundamental parameter \( \beta \). We consider the following types of the parameter array:

| Type | Description |
|------|-------------|
| I    | \( \beta \neq 2, \ \beta \neq -2 \) |
| II   | \( \beta = 2, \ \text{Char}(\mathbb{F}) \neq 2 \) |
| III+ | \( \beta = -2, \ \text{Char}(\mathbb{F}) \neq 2, \ d \) is even |
| III- | \( \beta = -2, \ \text{Char}(\mathbb{F}) \neq 2, \ d \) is odd |
| IV   | \( \beta = 0, \ \text{Char}(\mathbb{F}) = 2 \) |

For each type we display formulas that represent the parameter array in closed form.

Lemma 2.1 (See [4, Lemma 13.1].) Assume the parameter array \( \mathcal{A} \) has type I. Pick a nonzero \( q \in \mathbb{F} \) such that \( \beta = q + q^{-1} \). Then there exist scalars \( \eta, h, \mu, \eta^*, h^*, \mu^*, \tau \) in \( \mathbb{F} \) such that

\[
\theta_i = \eta + \mu q^i + h q^{d-i},
\]
\[
\theta_i^* = \eta^* + \mu^* q^i + h^* q^{d-i},
\]

for \( 0 \leq i \leq d \), and

\[
\varphi_i = (q^i - 1)(q^{d-i+1} - 1)(\tau - \mu \mu^* q^{i-1} - h h^* q^{d-i}),
\]
\[
\phi_i = (q^i - 1)(q^{d-i+1} - 1)(\tau - h \mu^* q^{i-1} - \mu h^* q^{d-i})
\]

for \( 1 \leq i \leq d \).
Note 2.2 With reference to Lemma 2.1, for 1 ≤ i ≤ d we have q^i ≠ 1; otherwise ϕ_i = 0.

Lemma 2.3 (See [4, Lemma 14.1].) Assume the parameter array (3) has type II. Then there exist scalars η, h, μ, η*, h*, μ*, τ in ℱ such that

\[ θ_i = η + μ(i - d/2) + hi(d - i), \]
\[ θ^*_i = η* + μ*(i - d/2) + h*i(d - i) \]

for 0 ≤ i ≤ d, and

\[ ϕ_i = i(d - i + 1)(τ - μμ*/2 + (hμ* + μh*)(i - (d + 1)/2) + hh*(i - 1)(d - i)), \]
\[ φ_i = i(d - i + 1)(τ + μμ*/2 + (hμ* - μh*)(i - (d + 1)/2) + hh*(i - 1)(d - i)) \]

for 1 ≤ i ≤ d.

Note 2.4 With reference to Lemma 2.3, Char(ℱ) ≠ i for any prime i ≤ d; otherwise ϕ_i = 0.

Lemma 2.5 (See [4, Lemma 15.1].) Assume the parameter array (3) has type III+. Then there exist scalars η, h, s, η*, h*, s*, τ in ℱ such that

\[ θ_i = \begin{cases} η + s + h(i - d/2) & \text{if } i \text{ is even,} \\ η - s - h(i - d/2) & \text{if } i \text{ is odd,} \end{cases} \]
\[ θ^*_i = \begin{cases} η* + s* + h*(i - d/2) & \text{if } i \text{ is even,} \\ η* - s* - h*(i - d/2) & \text{if } i \text{ is odd} \end{cases} \]

for 0 ≤ i ≤ d, and

\[ ϕ_i = \begin{cases} i(τ - sh* - s*h - hh*(i - (d + 1)/2)) & \text{if } i \text{ is even,} \\ (d - i + 1)(τ + sh* + s*h + hh*(i - (d + 1)/2)) & \text{if } i \text{ is odd,} \end{cases} \]
\[ φ_i = \begin{cases} i(τ - sh* + s*h + hh*(i - (d + 1)/2)) & \text{if } i \text{ is even,} \\ (d - i + 1)(τ + sh* - s*h - hh*(i - (d + 1)/2)) & \text{if } i \text{ is odd} \end{cases} \]

for 1 ≤ i ≤ d.

Note 2.6 With reference to Lemma 2.5, Char(ℱ) ≠ i for any prime i ≤ d/2; otherwise ϕ_i = 0. By this and since Char(ℱ) ≠ 2 we find Char(ℱ) is either 0 or an odd prime greater than d/2. Observe that neither of d, d − 2 vanish in ℱ; otherwise Char(ℱ) must divide d/2 or (d − 2)/2.
Lemma 2.7 (See [4, Lemma 16.1].) Assume the parameter array (3) has type III*. Then there exist scalars $\eta, h, s, \eta^*, h^*, s^*, \tau$ in $F$ such that

$$
\theta_i = \begin{cases} 
\eta + s + h(i - d/2) & \text{if } i \text{ is even}, \\
\eta - s - h(i - d/2) & \text{if } i \text{ is odd}, 
\end{cases}
$$

$$
\theta_i^* = \begin{cases} 
\eta^* + s^* + h^*(i - d/2) & \text{if } i \text{ is even}, \\
\eta^* - s^* - h^*(i - d/2) & \text{if } i \text{ is odd}, 
\end{cases}
$$

for $0 \leq i \leq d$, and

$$
\varphi_i = \begin{cases} 
hh^*i(d - i + 1) & \text{if } i \text{ is even}, \\
\tau - 2ss^* + i(d - i + 1)hh^* - 2(hs^* + h^*s)(i - (d + 1)/2) & \text{if } i \text{ is odd}, 
\end{cases}
$$

$$
\phi_i = \begin{cases} 
hh^*i(d - i + 1) & \text{if } i \text{ is even}, \\
\tau + 2ss^* + i(d - i + 1)hh^* - 2(hs^* - h^*s)(i - (d + 1)/2) & \text{if } i \text{ is odd} 
\end{cases}
$$

for $1 \leq i \leq d$.

Note 2.8 With reference to Lemma 2.7, $\text{Char}(F) \neq i$ for any prime $i \leq d/2$; otherwise $\varphi_2 = 0$. By this and since $\text{Char}(F) \neq 2$ we find $\text{Char}(F)$ is either 0 or an odd prime greater than $d/2$. Observe $d - 1$ does not vanish in $F$; otherwise $\text{Char}(F)$ must divide $(d - 1)/2$.

Lemma 2.9 (See [4, Lemma 17.1].) Assume the parameter array (3) has type IV. Then $d = 3$, and there exist scalars $h, s, h^*, s^*$, $r$ in $F$ such that

$$
\theta_1 = \theta_0 + h(s + 1), \quad \theta_2 = \theta_0 + h, \quad \theta_3 = \theta_0 + hs,
$$

$$
\theta^*_1 = \theta^*_0 + h^*(s^* + 1), \quad \theta^*_2 = \theta^*_0 + h^*, \quad \theta^*_3 = \theta^*_0 + h^*s^*,
$$

$$
\varphi_1 = hh^*r, \quad \varphi_2 = hh^*, \quad \varphi_3 = hh^*(r + s + s^*),
$$

$$
\phi_1 = hh^*(r + s + ss^*), \quad \phi_2 = hh^*, \quad \phi_3 = hh^*(r + s^* + ss^*).$$

We mention a lemma for later use. Pick a nonzero $q \in F$ such that $\beta = q + q^{-1}$.
Lemma 2.10 (See [5, Lemma 10.2].) The following hold:

(i) Assume the parameter array (3) has type I. Then for $1 \leq i \leq d$

$$
\sum_{\ell=0}^{i-1} \frac{\theta_{\ell} - \theta_{d-\ell}}{\theta_0 - \theta_d} = \frac{(q^i - 1)(q^{d-i+1} - 1)}{(q - 1)(q^d - 1)}.
$$

(ii) Assume the parameter array (3) has type II. Then for $1 \leq i \leq d$

$$
\sum_{\ell=0}^{i-1} \frac{\theta_{\ell} - \theta_{d-\ell}}{\theta_0 - \theta_d} = \frac{i(d - i + 1)}{d}.
$$

(iii) Assume the parameter array (3) has type III$^+$. Then for $1 \leq i \leq d$

$$
\sum_{\ell=0}^{i-1} \frac{\theta_{\ell} - \theta_{d-\ell}}{\theta_0 - \theta_d} = \begin{cases} 
\frac{i}{d} & \text{if } i \text{ is even,} \\
(d - i + 1)/d & \text{if } i \text{ is odd.}
\end{cases}
$$

(iv) Assume the parameter array (3) has type III$^-$. Then for $1 \leq i \leq d$

$$
\sum_{\ell=0}^{i-1} \frac{\theta_{\ell} - \theta_{d-\ell}}{\theta_0 - \theta_d} = \begin{cases} 
0 & \text{if } i \text{ is even,} \\
1 & \text{if } i \text{ is odd.}
\end{cases}
$$

3 Proof of Theorem 1.11

In this section we prove Theorem 1.11. Fix an integer $d \geq 3$. Let (3) be a parameter array over $\mathbb{F}$ with fundamental parameter $\beta$. Pick a nonzero $q \in \mathbb{F}$ such that $\beta = q + q^{-1}$. In the following five lemmas, we display formulas that represent $\{\theta_i\}_{i=0}^d$ and $\{\theta^*_i\}_{i=0}^d$ in terms of the end-parameters and $q$. These formulas can be routinely verified using Lemmas 2.1, 2.3, 2.5, 2.7, 2.9.

Lemma 3.1 Assume the parameter array (3) has type I. Then for $0 \leq i \leq d$

$$
\theta_i = \theta_0 - \frac{(q^i - 1)(q^{2d-i-1} - 1)(\theta_0 - \theta_d)}{(q^d - 1)(q^d - 1)} + \frac{(q^i - 1)(q^{d-i} - 1)(\phi_1 - \varphi_d)}{(q - 1)(q^{d-1} - 1)(\theta_0^* - \theta_0^*)},
$$

$$
\theta_i^* = \theta_0^* - \frac{(q^i - 1)(q^{2d-i-1} - 1)(\theta_0^* - \theta_d^*)}{(q^d - 1)(q^d - 1)} + \frac{(q^i - 1)(q^{d-i} - 1)(\phi_d - \varphi_d)}{(q - 1)(q^{d-1} - 1)(\theta_0 - \theta_d)}. 
$$
Lemma 3.2 Assume the parameter array has type II. Then for $0 \leq i \leq d$

\[
\begin{align*}
\theta_i &= \theta_0 - \frac{i(2d-i-1)(\theta_0 - \theta_d)}{d(d-1)} + \frac{i(d-i)(\phi_1 - \varphi_d)}{(d-1)(\theta_0^* - \theta_d^*)}, \\
\theta_i^* &= \theta_0^* - \frac{i(2d-i-1)(\theta_0^* - \theta_d^*)}{d(d-1)} + \frac{i(d-i)(\phi_d - \varphi_d)}{(d-1)(\theta_0 - \theta_d)}.
\end{align*}
\]

Lemma 3.3 Assume the parameter array has type III+. Then for $0 \leq i \leq d$

\[
\begin{align*}
\theta_i &= \begin{cases} 
\theta_0 - \frac{i(\theta_0 - \theta_d)}{d} & \text{if } i \text{ is even,} \\
\theta_0 - \frac{(2d-i-1)(\theta_0 - \theta_d)}{d} + \frac{\phi_1 - \varphi_d}{\theta_0^* - \theta_d^*} & \text{if } i \text{ is odd,}
\end{cases} \\
\theta_i^* &= \begin{cases} 
\theta_0^* - \frac{i(\theta_0^* - \theta_d^*)}{d} & \text{if } i \text{ is even,} \\
\theta_0^* - \frac{(2d-i-1)(\theta_0^* - \theta_d^*)}{d} + \frac{\phi_d - \varphi_d}{\theta_0 - \theta_d} & \text{if } i \text{ is odd.}
\end{cases}
\end{align*}
\]

Lemma 3.4 Assume the parameter array has type III−. Then for $0 \leq i \leq d$

\[
\begin{align*}
\theta_i &= \begin{cases} 
\theta_0 - \frac{i(\theta_0 - \theta_d)}{d-1} + \frac{i(\phi_1 - \varphi_d)}{(d-1)(\theta_0^* - \theta_d^*)} & \text{if } i \text{ is even,} \\
\theta_0 - \frac{(2d-i-1)(\theta_0 - \theta_d)}{d-1} + \frac{(d-i)(\phi_1 - \varphi_d)}{(d-1)(\theta_0^* - \theta_d^*)} & \text{if } i \text{ is odd,}
\end{cases} \\
\theta_i^* &= \begin{cases} 
\theta_0^* - \frac{i(\theta_0^* - \theta_d^*)}{d-1} + \frac{i(\phi_d - \varphi_d)}{(d-1)(\theta_0 - \theta_d)} & \text{if } i \text{ is even,} \\
\theta_0^* - \frac{(2d-i-1)(\theta_0^* - \theta_d^*)}{d-1} + \frac{(d-i)(\phi_d - \varphi_d)}{(d-1)(\theta_0 - \theta_d)} & \text{if } i \text{ is odd.}
\end{cases}
\end{align*}
\]

Lemma 3.5 Assume the parameter array has type IV. Then

\[
\begin{align*}
\theta_1 &= \theta_0 + \frac{\phi_1 - \varphi_3}{\theta_0^* - \theta_3}, & \quad \theta_2 &= \theta_3 + \frac{\phi_1 - \varphi_3}{\theta_0^* - \theta_3}, \\
\theta_1^* &= \theta_0^* + \frac{\phi_3 - \varphi_3}{\theta_0 - \theta_3}, & \quad \theta_2^* &= \theta_3^* + \frac{\phi_3 - \varphi_3}{\theta_0 - \theta_3}.
\end{align*}
\]

Proof of Theorem 1.11 By Lemmas 3.1–3.5 the scalars $\{\theta_i\}_{i=0}^d$, $\{\theta_i^*\}_{i=0}^d$ are determined by the end-parameters and $q$. By this and Lemma 1.6(iii), (iv) the scalars $\{\phi_i\}_{i=1}^d$, $\{\phi_i^*\}_{i=1}^d$ are determined by the end-parameters and $q$. The result follows from these comments and Lemma 1.5. ☐
4 Proof of Proposition 1.12

In this section we prove Proposition 1.12. Fix an integer \( d \geq 3 \).

**Proof of Proposition 1.12.** First assume the parameter array has type I. By Lemma 2.1,

\[
\begin{align*}
\theta_0 &= \eta + \mu + hq^d, & \theta_d &= \eta + \mu q^d + h, \\
\theta^*_0 &= \eta^* + \mu^* + h^* q^d, & \theta^*_d &= \eta^* + \mu^* q^d + h^*, \\
\phi_1 &= (q-1)(q^d-1)(\tau - \mu^* - h^* q^{d-1}), & \phi_d &= (q-1)(q^d-1)(\tau - \mu^* q^{d-1} - h^*), \\
\varphi_1 &= (q-1)(q^d-1)(\tau - \mu^* q^d - \mu h^*), & \varphi_d &= (q-1)(q^d-1)(\tau - h^* q^{d-1} - \mu h^*).
\end{align*}
\]

So,

\[
\begin{align*}
(\theta_0 - \theta_d)(\theta^*_0 - \theta^*_d) &= (q^d - 1)^2(\mu - h)(\mu^* - h^*), \\
\varphi_1 + \varphi_d - \varphi_1 - \varphi_d &= (q-1)(q^d-1)(q^{d-1} + 1)(\mu^* + h^* - h^* q^{d-1} - \mu h^*).
\end{align*}
\]

Thus

\[
\frac{\varphi_1 + \varphi_d - \varphi_1 - \varphi_d}{(\theta_0 - \theta_d)(\theta^*_0 - \theta^*_d)} = \frac{(q-1)(q^{d-1} + 1)}{q^d - 1}.
\]

We have shown the result for type I. The proof is similar for the remaining types. □

5 A polynomial

In this section we consider a polynomial which will be used in our proof of Theorems 1.14 and 1.15. This polynomial is related to Proposition 1.12 for type I. Fix an integer \( d \geq 3 \).

**Definition 5.1** For \( \omega \in F \) we define a polynomial in \( x \):

\[
f_\omega(x) = \omega(x^d - 1) - (x - 1)(x^{d-1} + 1).
\]

**Lemma 5.2** For \( \omega \in F \) the following hold:

(i) \( f_\omega(1) = 0 \).

(ii) Assume \( \omega \neq 1 \). Then \( f_\omega(x) \) has degree \( d \) and \( f_\omega(0) \neq 0 \).

(iii) Assume \( d \) is even. Then \( f_\omega(-1) = 0 \).

(iv) Assume \( \text{Char}(F) \neq 2 \), \( d \) is odd, and \( \omega \neq 2 \). Then \( f_\omega(-1) \neq 0 \).

(v) For \( 0 \neq q \in F \), if \( f_\omega(q) = 0 \) then \( f_\omega(q^{-1}) = 0 \).

**Proof.** Routine verification. □
Lemma 5.3 For \( \omega \in \mathbb{F} \) the following hold:

(i) We have \( f_\omega(x) = (x - 1)g_\omega(x) \), where

\[
g_\omega(x) = \omega \sum_{r=0}^{d-1} x^r - x^{d-1} - 1.
\]

(ii) Assume \( d \) is even. Then \( f_\omega(x) = (x - 1)(x + 1)g_\omega(x) \), where

\[
g_\omega(x) = \omega \sum_{r=0}^{(d-2)/2} x^{2r} - \sum_{r=0}^{d-2} (-1)^r x^r.
\]

(iii) Assume \( d \) is even and \( d \) does not vanish in \( \mathbb{F} \). Then for \( \omega = 2/d \) we have \( f_\omega(x) = -(2/d)(x - 1)^3(x + 1)g(x) \), where

\[
g(x) = \sum_{r=0}^{(d-4)/2} (r + 1)(d/2 - r - 1)x^{2r} + \sum_{r=1}^{(d-4)/2} r(d/2 - r - 1)x^{2r-1}.
\]

(iv) Assume \( d \) is odd and \( d \) does not vanish in \( \mathbb{F} \). Then for \( \omega = 2/d \) we have \( f_\omega(x) = -(1/d)(x - 1)^3g(x) \), where

\[
g(x) = \sum_{r=0}^{d-3} (r + 1)(d - r - 2)x^r.
\]

(v) Assume \( d \) is even and \( d \) does not vanish in \( \mathbb{F} \). Then for \( \omega = 2(d - 1)/d \) we have \( f_\omega(x) = (2/d)(x - 1)(x + 1)^3g(x) \), where

\[
g(x) = \sum_{r=0}^{(d-4)/2} (r + 1)(d/2 - r - 1)x^{2r} - \sum_{r=1}^{(d-4)/2} r(d/2 - r - 1)x^{2r-1}.
\]

(vi) Assume \( d \) is odd. Then for \( \omega = 2 \) we have \( f_\omega(x) = (x - 1)(x + 1)^2g(x) \), where

\[
g(x) = \sum_{r=0}^{(d-3)/2} x^{2r}.
\]

Proof. Routine verification. \( \square \)

Lemma 5.4 For \( \omega \in \mathbb{F} \) consider the equation \( f_\omega(x) = 0 \).

(i) Assume \( d \) is odd. Then there are at most \( d - 1 \) roots of \( f_\omega(x) = 0 \) other than \( \pm 1 \).

(ii) Assume \( d \) is even. Then there are at most \( d - 2 \) roots of \( f_\omega(x) = 0 \) other than \( \pm 1 \).
(iii) Assume $d$ is even and $d$ does not vanish in $F$. Then for $\omega = 2/d$ there are at most $d - 4$ roots of $f_\omega(x) = 0$ other than $\pm 1$.

(iv) Assume $d$ is odd and $d$ does not vanish in $F$. Then for $\omega = 2/d$ there are at most $d - 3$ roots of $f_\omega(x) = 0$ other than $\pm 1$.

(v) Assume $d$ is even and $d$ does not vanish in $F$. Then for $\omega = 2(d - 1)/d$ there are at most $d - 4$ roots of $f_\omega(x) = 0$ other than $\pm 1$.

(vi) Assume $d$ is odd and $\omega = 2$. Then there are at most $d - 3$ roots of $f_\omega(x) = 0$ other than $\pm 1$.

**Proof.** Immediate from Lemma 5.3. □

**Lemma 5.5** Assume $d$ does not vanish in $F$. Then the equation $f_\omega(x) = 0$ has a repeated root for at most $d$ values of $\omega$.

**Proof.** If the equation $f_\omega(x) = 0$ has a repeated root $q$, then both $f_\omega(q) = 0$ and $f'_\omega(q) = 0$, where $f'_\omega$ is the derivative of $f_\omega$. The equations $f_\omega(x) = 0$ and $f'_\omega(x) = 0$ have a common root if and only if the resultant of $f_\omega(x)$ and $f'_\omega(x)$ is zero (see [1, Chap. IV.8]). The resultant of $f_\omega(x)$ and $f'_\omega(x)$ is the determinant of the following matrix (we display the matrix for $d = 5$):

$$
M_\omega = \begin{pmatrix}
\omega - 1 & 1 & 0 & 0 & -1 & 1 - \omega & 0 & 0 & 0 \\
0 & \omega - 1 & 1 & 0 & 0 & -1 & 1 - \omega & 0 & 0 \\
0 & 0 & \omega - 1 & 1 & 0 & 0 & -1 & 1 - \omega & 0 \\
5(\omega - 1) & 4 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 5(\omega - 1) & 4 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 5(\omega - 1) & 4 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 5(\omega - 1) & 4 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 5(\omega - 1) & 4 & 0 & 0 & -1
\end{pmatrix}
$$

First assume $d$ is odd. Then

$$
\det(M_\omega) = (\omega - 1)(\omega - 2)(d\omega - 2)^3 \psi_1(\omega)^2,
$$

where $\psi_1(x)$ is a polynomial in $x$ with leading term $d^{(d-3)/2}x^{d-3}$. Thus there are at most $d$ values of $\omega$ such that $\det(M_\omega) = 0$. Next assume $d$ is even. Then

$$
\det(M_\omega) = (1 - \omega)(d\omega - 2)^3(d\omega - 2(d - 1))^3 \psi_2(\omega)^2,
$$

where $\psi_2(x)$ is a polynomials in $x$ with leading term $d^{(d-6)/2}x^{d-4}$. Thus there are at most $d - 1$ values of $\omega$ such that $\det(M_\omega) = 0$. The result follows. □
Lemma 5.6 For $3 \leq r \leq d$ let $\Gamma_r$ denote the set consisting of the $r$th roots of unity other than $\pm 1$:

$$\Gamma_r = \{ q \in \mathbb{F} \mid q^r = 1, \; q^2 \neq 1 \}.$$ 

Let $\Gamma$ be the union of $\Gamma_r$ for $3 \leq r \leq d$. Then there exist infinitely many $\omega \in \mathbb{F}$ such that the equation $f_\omega(x) = 0$ has no roots in $\Gamma$.

**Proof.** We claim that for any $\omega \in \mathbb{F}$ the equation $f_\omega(x) = 0$ has no roots in $\Gamma_d$. Suppose $f_\omega(q) = 0$ for some $q \in \Gamma_d$. Then $0 = f_\omega(q) = q^{d-1} - q$, so $q^{d-2} = 1$. By this and $q^d = 1$ we must have $q^2 = 1$, a contradiction. Thus the claim holds. For $q \in \Gamma \setminus \Gamma_d$ define

$$\omega_q = \frac{(q - 1)(q^{d-1} + 1)}{q^d - 1},$$ 

and consider the set

$$\Delta = \{ \omega_q \mid q \in \Gamma \setminus \Gamma_d \}.$$ 

Note that $\mathbb{F} \setminus \Delta$ has infinitely many elements, since $\mathbb{F}$ is infinite and $\Delta$ is finite. For $\omega \in \mathbb{F} \setminus \Delta$, the equation $f_\omega(x) = 0$ has no roots in $\Gamma \setminus \Gamma_d$. By this and the above claim, the equation $f_\omega(x) = 0$ has no roots in $\Gamma$. The result follows.

Corollary 5.7 Assume $d$ does not vanish in $\mathbb{F}$. Then there exist infinitely many $\omega \in \mathbb{F}$ that satisfy both (i) and (ii) below:

(i) The equation $f_\omega(x) = 0$ has no repeated roots.

(ii) The equation $f_\omega(x) = 0$ has no roots in $\Gamma$, where $\Gamma$ is from Lemma 5.6.

**Proof.** Follows from Lemmas 5.5 and 5.6.

Lemma 5.8 Let $\omega \in \mathbb{F}$ with $\omega \neq 1$, $\omega \neq 2$. Assume that the equation $f_\omega(x) = 0$ has no repeated roots.

(i) Assume $\text{Char}(\mathbb{F}) \neq 2$ and $d$ is odd. Then the equation $f_\omega(x) = 0$ has mutually distinct $d - 1$ nonzero roots other than $\pm 1$.

(ii) Assume $d$ is even. Then the equation $f_\omega(x) = 0$ has mutually distinct $d - 2$ nonzero roots other than $\pm 1$.

**Proof.** We claim that the equation $f_\omega(x) = 0$ has mutually distinct $d$ nonzero roots. By Lemma 5.2(ii) and since $\omega \neq 1$, the polynomial $f_\omega(x)$ has degree $d$ and $f_\omega(0) \neq 0$. Now the claim holds by this and since $f_\omega(x) = 0$ has no repeated roots.

(i): By Lemma 5.2(i) $f_\omega(1) = 0$. We have $f_\omega(-1) \neq 0$ by Lemma 5.2(iv) and since $\omega \neq 2$, $\text{Char}(\mathbb{F}) \neq 2$. By these comments and the claim, the equation $f_\omega(x) = 0$ has mutually distinct $d - 1$ nonzero roots other than $\pm 1$.

(ii): By Lemma 5.2(i), (iii) each of 1, $-1$ is a root of $f_\omega(x) = 0$. By this and the claim, the equation $f_\omega(x) = 0$ has mutually distinct $d - 2$ nonzero roots other than $\pm 1$. □
6 Proof of Theorem 1.14

Proof of Theorem 1.14. Suppose we are given a parameter array over $\mathbb{F}$:
\[
(\{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d, \{\varphi_i\}_{i=1}^d, \{\phi_i\}_{i=1}^d).
\]

Let $\tilde{P}$ denote the set of parameter arrays
\[
(\{\tilde{\theta}_i\}_{i=0}^d, \{\tilde{\theta}_i^*\}_{i=0}^d, \{\tilde{\varphi}_i\}_{i=1}^d, \{\tilde{\phi}_i\}_{i=1}^d)
\]
over $\mathbb{F}$ that satisfy
\[
\tilde{\theta}_0 = \theta_0, \quad \tilde{\theta}_d = \theta_d, \quad \tilde{\theta}_0^* = \theta_0^*, \quad \tilde{\theta}_d^* = \theta_d^*,
\]
\[
\tilde{\varphi}_1 = \varphi_1, \quad \tilde{\varphi}_d = \varphi_d, \quad \tilde{\phi}_1 = \phi_1, \quad \tilde{\phi}_d = \phi_d.
\]

We count the number of elements of $\tilde{P}$. By Theorem 1.11 a parameter array in $\tilde{P}$ is determined by its fundamental parameter. Let $\tilde{Q}$ denote the set of nonzero $\tilde{q} \in \mathbb{F}$ such that $\tilde{q} + \tilde{q}^{-1}$ is the fundamental parameter for some $\tilde{p} \in \tilde{P}$. Note that $\tilde{q}$ is determined up to inverse by the fundamental parameter. So we count the number of elements of $\tilde{Q}$ up to inverse. Define
\[
\Omega = \frac{\phi_1 + \phi_d - \varphi_1 - \varphi_d}{(\theta_0 - \theta_d)(\theta_0^* - \theta_d^*)}.
\]

By Proposition 1.12 for $\tilde{p} \in \tilde{P}$ we obtain the equation:

| Type of $\tilde{p}$ | Equation |
|---------------------|----------|
| I                   | $\frac{(\tilde{q} - 1)\tilde{q}^{d-1} + 1}{\tilde{q}^d - 1} = \Omega$ |
| II                  | $2/d = \Omega$ |
| III$^+$             | $2(d - 1)/d = \Omega$ |
| III$^-$             | $2 = \Omega$ |
| IV                  | $0 = \Omega$ |

where $\tilde{q} + \tilde{q}^{-1}$ is the fundamental parameter of $\tilde{p}$.

We claim that at least one of $1, -1$ is not contained in $\tilde{Q}$ when $\text{Char}(\mathbb{F}) \neq 2$. By way of contradiction, assume $\text{Char}(\mathbb{F}) \neq 2$ and $\{1, -1\} \subseteq \tilde{Q}$. Then there is a $\tilde{p}_1 \in \tilde{P}$ (resp. $\tilde{p}_2 \in \tilde{P}$) that has fundamental parameter 2 (resp. $-2$). Note that $\tilde{p}_1$ has type II and $\tilde{p}_2$ has type III$^+$ or III$^-$. So by (5) $d\Omega = 2$, and either $d\Omega = 2(d - 1)$ or $\Omega = 2$. If $d\Omega = 2$ and $d\Omega = 2(d - 1)$, then $d - 2$ vanishes in $\mathbb{F}$. If $d\Omega = 2$ and $\Omega = 2$, then $d - 1$ vanishes in $\mathbb{F}$. But, by Note 2.4 neither of $d - 1, d - 2$ vanishes in $\mathbb{F}$, a contradiction. We have shown the claim. Now we count the number of elements of $\tilde{Q}$ up to inverse. Note that $\Omega \neq 1$ by Corollary 1.13. First assume $\Omega \neq 2, d\Omega \neq 2$, and $d\Omega \neq 2(d - 1)$. By Lemma 5.4(i), (ii) there are up to inverse at most $((d - 1)/2)$ elements of $\tilde{Q}$. Next assume $d$ is even and $d\Omega = 2$. By Lemma 5.4(iii) there are up to inverse at most $(d - 4)/2$ elements of $\tilde{Q}$ other
than ±1. Next assume $d$ is odd and $d\Omega = 2$. By Lemma 5.4 (iv) there are up to inverse at most $(d - 3)/2$ elements of $\tilde{Q}$ other than ±1. Next assume $d$ is even and $d\Omega = 2(d - 1)$. By Lemma 5.4 (v) there are up to inverse at most $(d - 4)/2$ elements of $\tilde{Q}$ other than ±1. Next assume $d$ is odd and $\Omega = 2$. By Lemma 5.4 (vi) there are up to inverse at most $(d - 3)/2$ elements of $\tilde{Q}$ other than ±1. By these comments and the claim, there are up to inverse at most $\lfloor (d - 1)/2 \rfloor$ elements of $\tilde{Q}$. The result follows. \qed

7 How to construct a parameter array having specified end-parameters

In this section we try to construct a parameter array having specified end-parameters. To simplify our description, we restrict our attention to type I; we can proceed in a similar way for the other types. Fix an integer $d \geq 3$, and pick scalars

$$
\theta_0, \quad \theta_d, \quad \theta_0^*, \quad \theta_d^*, \quad \varphi_1, \quad \varphi_d, \quad \phi_1, \quad \phi_d
$$

in $\mathbb{F}$ such that $\theta_0 \neq \theta_d$ and $\theta_0^* \neq \theta_d^*$. We will try to construct a parameter array

$$
(\{\tilde{\theta}_i\}_{i=0}^{d}, \{\tilde{\Theta}_i^*\}_{i=0}^{d}, \{\tilde{\varphi}_i\}_{i=1}^{d}, \{\tilde{\phi}_i\}_{i=1}^{d})
$$

that satisfies

$$
\begin{align*}
\tilde{\theta}_0 &= \theta_0, & \tilde{\theta}_d &= \theta_d, & \tilde{\theta}_0^* &= \theta_0^*, & \tilde{\theta}_d^* &= \theta_d^*, \\
\tilde{\varphi}_1 &= \varphi_1, & \tilde{\varphi}_d &= \varphi_d, & \tilde{\phi}_1 &= \phi_1, & \tilde{\phi}_d &= \phi_d.
\end{align*}
$$

(6)

Define

$$
\Omega = \frac{\phi_1 + \phi_d - \varphi_1 - \varphi_d}{(\theta_0 - \theta_d)(\theta_0^* - \theta_d^*)}.
$$

In view of Note 2.2 and Proposition 1.12, we assume there exists a nonzero $q \in \mathbb{F}$ such that $q^i \neq 1$ for $1 \leq i \leq d$, and

$$
\Omega = \frac{(q - 1)(q^{d-1} - 1)}{q^d - 1}.
$$

(7)

In view of Lemma 3.1, we define scalars $\{\tilde{\theta}_i\}_{i=0}^{d}, \{\tilde{\Theta}_i^*\}_{i=0}^{d}$ as follows.

**Definition 7.1** For $0 \leq i \leq d$ define

$$
\begin{align*}
\tilde{\theta}_i &= \theta_0 - \frac{(q^i - 1)(q^{2d-i-1} - 1)(\theta_0 - \theta_d)}{(q^{d-1} - 1)(q^d - 1)} + \frac{(q^i - 1)(q^{d-i} - 1)(\varphi_1 - \varphi_d)}{(q - 1)(q^{d-1} - 1)(\theta_0^* - \theta_d^*)}, \\
\tilde{\Theta}_i^* &= \theta_0^* - \frac{(q^i - 1)(q^{2d-i-1} - 1)(\theta_0^* - \theta_d^*)}{(q^{d-1} - 1)(q^d - 1)} + \frac{(q^i - 1)(q^{d-i} - 1)(\phi_1 - \phi_d)}{(q - 1)(q^{d-1} - 1)(\theta_0 - \theta_d)}.
\end{align*}
$$

The following two lemmas can be routinely verified.

**Lemma 7.2** With reference to Definition 7.1

$$
\tilde{\theta}_0 = \theta_0, \quad \tilde{\theta}_d = \theta_d, \quad \tilde{\Theta}_0^* = \theta_0^*, \quad \tilde{\Theta}_d^* = \theta_d^*.
$$
Lemma 7.3 Assume \( \tilde{\theta}_i \neq \tilde{\theta}_j, \tilde{\theta}_i^* \neq \tilde{\theta}_j^* \) for \( 1 \leq i < j \leq d \). Then each of the expressions

\[
\frac{\tilde{\theta}_{i-2} - \tilde{\theta}_{i+1}}{\tilde{\theta}_{i-1} - \tilde{\theta}_i}, \quad \frac{\tilde{\theta}_{i-2}^* - \tilde{\theta}_{i+1}^*}{\tilde{\theta}_{i-1}^* - \tilde{\theta}_i^*}
\]

is equal to \( q + q^{-1} + 1 \) for \( 2 \leq i \leq d - 1 \).

In view of Lemma 2.10(i) we define scalars \( \{ \varphi_i \}_{i=1}^d \) as follows.

Definition 7.4 For \( 1 \leq i \leq d \) define

\[
\varphi_i = \frac{(q^i - 1)(q^{d-i+1} - 1)}{(q - 1)(q^d - 1)}.
\]

In view of Lemma 1.6(iii), (iv), we define scalars \( \{ \tilde{\varphi}_i \}_{i=1}^d, \{ \tilde{\phi}_i \}_{i=1}^d \) as follows.

Definition 7.5 For \( 1 \leq i \leq d \) define

\[
\tilde{\varphi}_i = \varphi_1 \tilde{\varphi}_i + (\tilde{\theta}_i^* - \tilde{\theta}_0^*) (\tilde{\theta}_i - \tilde{\theta}_0), \\
\tilde{\phi}_i = \varphi_1 \tilde{\phi}_i + (\tilde{\theta}_i^* - \tilde{\theta}_0^*) (\tilde{\theta}_{d-i+1} - \tilde{\theta}_0).
\]

Lemma 7.6 With reference to Definition 7.5

\[
\tilde{\varphi}_1 = \varphi_d, \quad \tilde{\varphi}_d = \varphi_d, \quad \tilde{\phi}_1 = \phi_1, \quad \tilde{\phi}_d = \phi_d.
\]

Proof. One routinely checks that

\[
\begin{align*}
\tilde{\varphi}_1 &= \phi_1 + \phi_d - \varphi_d - \frac{(q - 1)(q^{d-1} + 1)(\theta_0 - \theta_d)(\theta_0^* - \theta_d^*)}{q^d - 1}, \\
\tilde{\varphi}_d &= \varphi_d, \\
\tilde{\phi}_1 &= \varphi_1 + \varphi_d - \phi_d + \frac{(q - 1)(q^{d-1} + 1)(\theta_0 - \theta_d)(\theta_0^* - \theta_d^*)}{q^d - 1}, \\
\tilde{\phi}_d &= \varphi_1 + \varphi_d - \phi_d + \frac{(q - 1)(q^{d-1} + 1)(\theta_0 - \theta_d)(\theta_0^* - \theta_d^*)}{q^d - 1}.
\end{align*}
\]

Now the result follows from (7). \( \square \)

Proposition 7.7 The sequence \( \tilde{p} = (\{ \tilde{\theta}_i \}_{i=0}^d, \{ \tilde{\theta}_i^* \}_{i=0}^d, \{ \tilde{\varphi}_i \}_{i=1}^d, \{ \tilde{\phi}_i \}_{i=1}^d) \) is a parameter array over \( F \) if and only if

\[
\begin{align*}
\tilde{\theta}_i \neq \tilde{\theta}_j, & \quad \tilde{\theta}_i^* \neq \tilde{\theta}_j^* \quad (0 \leq i < j \leq d), \quad (8) \\
\tilde{\varphi}_i \neq 0, & \quad \tilde{\phi}_i \neq 0 \quad (1 \leq i \leq d). \quad (9)
\end{align*}
\]

In this case, the parameter array \( \tilde{p} \) satisfies (6).

Proof. The first assertion follows from Definition 1.7, Lemma 7.3, and Definition 7.5. The second assertion follows from Lemmas 7.2 and 7.6. \( \square \)
8 Proof of Theorem 1.15

In this section we prove Theorem 1.15. Fix an integer \( d \geq 3 \). Assume \( \text{Char}(F) \neq 2 \) and \( d \) does not vanish in \( F \). Recall the polynomial \( f_\omega(x) \) from Definition 5.1.

By Corollary 5.7 there exists \( \omega \in F \) such that

- \( \omega \neq 1, \omega \neq 2 \);
- the equation \( f_\omega(x) = 0 \) has no repeated roots;
- the equation \( f_\omega(x) = 0 \) has no roots in \( \Gamma \), where \( \Gamma \) is from Lemma 5.6.

Fix \( \omega \in F \) that satisfies the above conditions.

By Lemma 5.8 there are up to inverse precisely \( \lfloor (d-1)/2 \rfloor \) nonzero roots of \( f_\omega(x) = 0 \) other than \( \pm 1 \). For such a root \( q \) and for \( \zeta \in F \), we construct a sequence \( \tilde{p}(q, \zeta) \) as follows. Define scalars

\[
\begin{align*}
\theta_0 &= 0, & \theta_d &= 1, & \theta_0^* &= 0, & \theta_d^* &= 1, \\
\varphi_1 &= 1, & \varphi_d &= -1, & \phi_1 &= \zeta, & \phi_d &= \omega - \zeta.
\end{align*}
\]

Observe that

\[
\omega = \phi_1 + \phi_d - \varphi_1 - \varphi_d \frac{(\theta_0 - \theta_d)(\theta_0^* - \theta_d^*)}{(\theta_0^* - \theta_d^*)(\theta_0 - \theta_d^*)}.
\]

For \( 0 \leq i \leq d \) define scalars \( \tilde{\theta}_i = \tilde{\theta}_i(q, \zeta) \) and \( \tilde{\theta}_i^* = \tilde{\theta}_i^*(q, \zeta) \) as in Definition 7.1. For \( 1 \leq i \leq d \) define scalars \( \tilde{\varphi}_i = \tilde{\varphi}_i(q, \zeta) \) and \( \tilde{\phi}_i = \tilde{\phi}_i(q, \zeta) \) as in Definition 7.5. We have constructed a sequence

\[
\begin{align*}
\tilde{p}(q, \zeta) &= (\{\tilde{\theta}_i(q, \zeta)\}_{i=0}^{d}, \{\tilde{\theta}_i^*(q, \zeta)\}_{i=0}^{d}, \{\tilde{\varphi}_i(q, \zeta)\}_{i=1}^{d}, \{\tilde{\phi}_i(q, \zeta)\}_{i=1}^{d}).
\end{align*}
\]

The following two lemmas can be routinely verified.

**Lemma 8.1** For \( 0 \leq i, j \leq d \)

\[
\tilde{\theta}_i(q, \zeta) - \tilde{\theta}_j(q, \zeta) = \frac{(q^i-q^j)Z_1(q, \zeta)}{(q-1)(q^{d-1}-1)(q^d-1)},
\]

where

\[
Z_1(q, \zeta) = \zeta(q^d-1)(q^{d-i-j}-1) + q(q^{d-1}-1)(q^{d-i-j-1}-1).
\]

**Lemma 8.2** For \( 0 \leq i, j \leq d \)

\[
\tilde{\theta}_i^*(q, \zeta) - \tilde{\theta}_j^*(q, \zeta) = \frac{(q^i-q^j)Z_2(q, \zeta)}{(q-1)(q^{d-1}-1)(q^d-1)},
\]

where

\[
Z_2(q, \zeta) = \zeta(q^d-1)(q^{d-i-j}-1) - (q^d-1)(q^{d-i-j+1}+1) + 2q^{d-i-j}(q^{i+j}+1).
\]
Lemma 8.3 For $0 \leq i < j \leq d$ the following hold:

(i) $\tilde{\theta}_i(q, \zeta) = \tilde{\theta}_j(q, \zeta)$ holds for only one value of $\zeta$.

(ii) $\tilde{\theta}^*_i(q, \zeta) = \tilde{\theta}^*_j(q, \zeta)$ holds for only one value of $\zeta$.

Proof. (i): Observe by Lemma 8.1 that $\tilde{\theta}_i(q, \zeta) = \tilde{\theta}_j(q, \zeta)$ if and only if $Z_1(q, \zeta) = 0$. First assume $q^{d-i-j} - 1 = 0$. Then

$$Z_1(q, \zeta) = (1 - q)(q^{d-1} - 1) \neq 0.$$ 

Next assume $q^{d-i-j} - 1 \neq 0$. Then $Z_1(q, \zeta)$ is a polynomial in $\zeta$ with degree 1. So $Z_1(q, \zeta) = 0$ holds for only one value of $\zeta$. The result follows. 

(ii): Similar to the proof of (i).

The following two lemmas can be routinely verified.

Lemma 8.4 For $1 \leq i \leq d$

$$\tilde{\varphi}_i(q, \zeta) = -\frac{(q^i - 1)(q^{d-i+1} - 1) Z_3(q, \zeta)}{(q - 1)^2(q^{d-1} - 1)^2(q^d - 1)^2},$$

where

$$Z_3(q, \zeta) = \zeta^2(q^d - 1)^2(q^{i-1} - 1)(q^{d-i} - 1)$$

$$- \zeta(q - 1)(q^d - 1)(q^{d-1} + 1)(q^{i-1} - 1)(q^{d-i} - 1)$$

$$- (q^{d-1} - 1)(q^i - 1)((q^{d-1} + 1)(q^{d-i+1} + 1) - 2q^{d-i}(q^i + 1)).$$

Lemma 8.5 For $1 \leq i \leq d$

$$\tilde{\phi}_i(q, \zeta) = -\frac{(q^i - 1)(q^{d-i+1} - 1) Z_4(q, \zeta)}{(q - 1)^2(q^{d-1} - 1)^2(q^d - 1)^2},$$

where

$$Z_4(q, \zeta) = \zeta^2(q^d - 1)^2(q^{i-1} - 1)(q^{d-i} - 1)$$

$$- \zeta(q - 1)(q^d - 1)((q^{d-i} - 1)(q^{d+1-i-2} - 1) - q^{d-i}(q^{i-1} - 1)^2)$$

$$- (q^{d-1} - 1)(q^{i-1} - 1)((q^{d-1} + 1)(q^{d-i+2} + 1) - 2q^{d-i+1}(q^{i-1} + 1)).$$

Lemma 8.6 For $1 \leq i \leq d$ the following hold:

(i) $\tilde{\varphi}_i(q, \zeta) = 0$ holds for at most two values of $\zeta$.

(ii) $\tilde{\phi}_i(q, \zeta) = 0$ holds for at most two values of $\zeta$.

Proof. (i): Observe by Lemma 8.4 that $\tilde{\varphi}_i(q, \zeta) = 0$ if and only if $Z_3(q, \zeta) = 0$. First assume $i = 1$. Then

$$Z_3(q, \zeta) = (1 - q)(q^{d-1} - 1)^2(q^d - 1) \neq 0.$$
Next assume \( i = d \). Then
\[
Z_3(q, \zeta) = (q - 1)(q^{d-1} - 1)^2(q^d - 1) \neq 0.
\]
Next assume \( i \neq 1 \) and \( i \neq d \). Then \( Z_3(q, \zeta) \) is a quadratic polynomial in \( \zeta \). So \( Z_3(q, \zeta) = 0 \) holds for at most two values of \( \zeta \).

(ii): Observe by Lemma 8.5 that \( \tilde{\phi}_i(q, \zeta) = 0 \) if and only if \( Z_4(q, \zeta) = 0 \). First assume \( i = 1 \). Then
\[
Z_4(q, \zeta) = (1 - q)(q^{d-1} - 1)^2(q^d - 1).
\]
So \( Z_4(q, \zeta) \neq 0 \) unless \( \zeta = 0 \). Next assume \( i = d \). Then
\[
Z_4(q, \zeta) = (q - 1)(q^{d-1} - 1)^2(\zeta(q^d - 1) - (q - 1)(q^{d-1} + 1))\].
So \( Z_4(q, \zeta) = 0 \) for only one value of \( \zeta \). Next assume \( i \neq 1 \) and \( i \neq d \). Then \( Z_4(q, \zeta) \) is a quadratic polynomial in \( \zeta \). So \( Z_4(q, \zeta) = 0 \) for at most two values of \( \zeta \).

Proof of Theorem 1.15. By Lemma 5.8 there are up to inverse precisely \([(d - 1)/2]\) nonzero roots of \( f_\omega(x) = 0 \) other than \( \pm 1 \). Write these roots as \( q_1, q_2, \ldots, q_n \), where \( n = [(d - 1)/2] \). For \( 1 \leq r \leq n \), by Lemmas 8.3 and 8.6 there are only finitely many \( \zeta \) such that \( \tilde{\phi}(q_r, \zeta) \) conflicts \[S\] or \[T\]. Thus there exists \( \zeta \in \mathbb{F} \) such that \( \tilde{\phi}(q_r, \zeta) \) satisfies both \[S\] and \[T\] for \( 1 \leq r \leq n \). Then by Proposition 7.7, for \( 1 \leq r \leq n \) the sequence \( \tilde{\phi}(q_r, \zeta) \) is a parameter array over \( \mathbb{F} \) that satisfy
\[
\tilde{\theta}_0(q_r, \zeta) = \theta_0, \quad \tilde{\theta}_d(q_r, \zeta) = \theta_d, \quad \tilde{\theta}_0^*(q_r, \zeta) = \theta_0^*, \quad \tilde{\theta}_d^*(q_r, \zeta) = \theta_d^*,
\]
\[
\tilde{\varphi}_1(q_r, \zeta) = \varphi_1, \quad \tilde{\varphi}_d(q_r, \zeta) = \varphi_d, \quad \tilde{\phi}_1(q_r, \zeta) = \phi_1, \quad \tilde{\phi}_d(q_r, \zeta) = \phi_d.
\]
Now the result follows by Lemma 1.6.

\[
\Box
\]

9 Appendix

Fix an integer \( d \geq 3 \). Let \( \mathbf{3} \) be a parameter array over \( \mathbb{F} \) with fundamental parameter \( \beta \). Pick a nonzero \( q \in \mathbb{F} \) such that \( \beta = q + q^{-1} \). In this appendix, we display formulas that represent \( \varphi_i \) and \( \phi_i \) in terms of the end-parameters and \( q \).

Assume \( \mathbf{3} \) has type I. Then for \( 1 \leq i \leq d \)
\[
\varphi_i = -\frac{q^{i-1}(q^i - 1)(q^{d-i} - 1)(q^{d-i+1} - 1)(q^{2d-i-1} - 1)(\theta_0 - \theta_d)(\theta_0^* - \theta_d^*)}{(q^{d-1} - 1)^2(q^d - 1)^2}
+ \frac{(q^i - 1)(q^{d-i+1} - 1)((q^{d-i} - 1)(q^{2d-i-1} - 1)\varphi_d + q^{i-1}(q^{d-i} - 1)^2(\phi_1 + \phi_d - \varphi_d))}{(q - 1)(q^{d-1} - 1)^2(q^d - 1)}
+ \frac{(q^{i-1} - 1)(q^i - 1)(q^{d-i} - 1)(q^{d-i+1} - 1)(\phi_1 - \varphi_d)(\phi_1 - \phi_d)}{(q - 1)^2(q^{d-1} - 1)^2(\theta_0 - \theta_d)(\theta_0^* - \theta_d^*)}
\]
\[
\phi_i = \frac{q^{i-1}(q^i - 1)(q^{d-i} - 1)(q^{d-i+1} - 1)(q^{2d-i-1} - 1)(\theta_0 - \theta_d)(\theta_0^* - \theta_d^*)}{(q^{d-1} - 1)^2(q^d - 1)^2}
+ \frac{(q^i - 1)(q^{d-i+1} - 1)((q^{d-i} - 1)(q^{2d-i-1} - 1)\phi_d + q^{i-1}(q^{d-i} - 1)^2(\varphi_1 + \varphi_d - \phi_d))}{(q - 1)(q^{d-1} - 1)^2(q^d - 1)}
- \frac{(q^{i-1} - 1)(q^i - 1)(q^{d-i} - 1)(q^{d-i+1} - 1)(\varphi_1 - \phi_d)(\varphi_1 - \phi_d)}{(q - 1)^2(q^{d-1} - 1)^2(\theta_0 - \theta_d)(\theta_0^* - \theta_d^*)}.
\]
Assume $i$ and if $i$.

Assume (3) has type III. Then for $1 \leq i \leq d$

\[
\varphi_i = \frac{i(d - i)(d - i + 1)(2d - i - 1)(\theta_0 - \theta_d)(\theta_0^* - \theta_d^*)}{d^2(d - 1)^2} + \frac{i(d - i + 1)((i - 1)(2d - i - 1)\phi_d + (d - i)^2(\phi_1 + \phi_d - \varphi_d))}{d(d - 1)^2} + \frac{i(i - 1)(d - i)(\phi_1 - \varphi_d)(\phi_d - \varphi_d)}{(d - 1)^2(\theta_0 - \theta_d)(\theta_0^* - \theta_d^*)}.
\]

\[
\phi_i = \frac{i(d - i)(d - i + 1)(2d - i - 1)(\theta_0 - \theta_d)(\theta_0^* - \theta_d^*)}{d^2(d - 1)^2} + \frac{i(d - i + 1)((i - 1)(2d - i - 1)\phi_d + (d - i)^2(\varphi_1 + \varphi_d - \phi_d))}{d(d - 1)^2} - \frac{i(i - 1)(d - i)(d - i + 1)(\varphi_1 - \phi_d)(\varphi_d - \phi_d)}{(d - 1)^2(\theta_0 - \theta_d)(\theta_0^* - \theta_d^*)}.
\]

Assume (3) has type III$^+$. Then for $1 \leq i \leq d$

\[
\varphi_i = \begin{cases} 
  \frac{i(d \varphi_d + (d - i)(\theta_0 - \theta_d)(\theta_0^* - \theta_d^*))}{(d - i + 1)(d(\phi_1 + \phi_d - \varphi_d) - (2d - i - 1)(\theta_0 - \theta_d)(\theta_0^* - \theta_d^*))} & \text{if } i \text{ is even,} \\
  \frac{i(d \phi_d - (d - i)(\theta_0 - \theta_d)(\theta_0^* - \theta_d^*))}{(d - i + 1)(d(\varphi_1 + \varphi_d - \phi_d) + (2d - i - 1)(\theta_0 - \theta_d)(\theta_0^* - \theta_d^*))} & \text{if } i \text{ is odd,}
\end{cases}
\]

\[
\phi_i = \begin{cases} 
  \frac{i(d \varphi_d + (d - i)(\theta_0 - \theta_d)(\theta_0^* - \theta_d^*))}{(d - i + 1)(d(\phi_1 + \phi_d - \varphi_d) - (2d - i - 1)(\theta_0 - \theta_d)(\theta_0^* - \theta_d^*))} & \text{if } i \text{ is even,} \\
  \frac{i(d \phi_d - (d - i)(\theta_0 - \theta_d)(\theta_0^* - \theta_d^*))}{(d - i + 1)(d(\varphi_1 + \varphi_d - \phi_d) + (2d - i - 1)(\theta_0 - \theta_d)(\theta_0^* - \theta_d^*))} & \text{if } i \text{ is odd.}
\end{cases}
\]

Assume (3) has type III$^-$. Then for $1 \leq i \leq d$ the following hold.

If $i$ is even,

\[
\varphi_i = \frac{i(i - 1)(\phi_1 - \varphi_d - (\theta_0 - \theta_d)(\theta_0^* - \theta_d^*))(\phi_d - \varphi_d - (\theta_0 - \theta_d)(\theta_0^* - \theta_d^*))}{(d - 1)^2(\theta_0 - \theta_d)(\theta_0^* - \theta_d^*)},
\]

and if $i$ is odd,

\[
\varphi_i = -\frac{(d - i)(2d - i - 1)(\theta_0 - \theta_d)(\theta_0^* - \theta_d^*)}{(d - 1)^2} + \frac{(i - 1)(2d - i - 1)\varphi_d + (d - i)^2(\phi_1 + \phi_d - \varphi_d)}{(d - 1)^2} + \frac{(i - 1)(d - i)(\phi_1 - \varphi_d)(\phi_d - \varphi_d)}{(d - 1)^2(\theta_0 - \theta_d)(\theta_0^* - \theta_d^*)}.
\]

Assume $\theta$.
If $i$ is even,
\[
\phi_i = -\frac{i(d-i+1)(\varphi_1 - \phi_d + (\theta_0 - \theta_d)(\theta_0^* - \theta_d^*))}{(d-1)^2(\theta_0 - \theta_d)(\theta_0^* - \theta_d^*)} (\varphi_d - \phi_d + (\theta_0 - \theta_d)(\theta_0^* - \theta_d^*))
\]
and if $i$ is odd,
\[
\phi_i = \frac{(d-i)(2d-i-1)(\theta_0 - \theta_d)(\theta_0^* - \theta_d^*)}{(d-1)^2}
+ \frac{(i-1)(2d-i-1)\phi_d + (d-i)^2(\varphi_1 + \varphi_d - \phi_d)}{(d-1)^2}
- \frac{(i-1)(d-i)(\varphi_1 - \phi_d)(\varphi_d - \phi_d)}{(d-1)^2(\theta_0 - \theta_d)(\theta_0^* - \theta_d^*)}.
\]

Assume (3) has type IV. Then
\[
\varphi_2 = \frac{\phi_1 - \varphi_1 + (\theta_0 - \theta_d)(\theta_0^* - \theta_d^*)}{(\theta_0 - \theta_d)(\theta_0^* - \theta_d^*)} (\varphi_1 - \varphi_3 + (\theta_0 - \theta_d)(\theta_0^* - \theta_d^*))
\]
\[
\phi_2 = \frac{\varphi_1 - \phi_1 + (\theta_0 - \theta_d)(\theta_0^* - \theta_d^*)}{(\theta_0 - \theta_d)(\theta_0^* - \theta_d^*)} (\varphi_1 - \phi_3 + (\theta_0 - \theta_d)(\theta_0^* - \theta_d^*)).
\]

10Acknowledgments

The author would like to thank Paul Terwilliger for giving this paper a close reading and offering many valuable suggestions.

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Keywords. Leonard pair, tridiagonal pair, tridiagonal matrix.
2010 Mathematics Subject Classification. 05E35, 05E30, 33C45, 33D45