3D mean Projective Shape Difference for Face Differentiation from Multiple Digital Camera Images

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Abstract
We give a nonparametric methodology for hypothesis testing for equality of extrinsic mean objects on a manifold embedded in a numerical spaces. The results obtained in the general setting are detailed further in the case of 3D projective shapes represented in a space of symmetric matrices via the quadratic Veronese-Whitney (VW) embedding. Large sample and nonparametric bootstrap confidence regions are derived for the common VW-mean of random projective shapes for finite 3D configurations. As an example, the VW MANOVA testing methodology is applied to the multi-sample mean problem for independent projective shapes of 3D facial configurations retrieved from digital images, via Agisoft PhotoScan technology.

1 Introduction
In this paper, we continue the Object Data Analysis program started by Patrangenaru and Ellingson (2015) [8]. In Section 2, we revisit the hypothesis testing for equality of mean vectors from g multivariate populations, in nonparametric setting based on the idea that the numbers in a finite set are all equal, if the squares of their differences add up to zero (see Bhattacharya and Bhattacharya (2012)). The main difference between our approach and classical MANOVA, is that we do not assume that all populations have a common covariance matrix Σ, and we do not make any distributional assumption. In Section 3, we extend this methodology to test for the equality of multiple extrinsic means, based on random samples of various sizes collected from g independent probability measures on a manifold. Our newly developed extrinsic MANOVA test is applied to the particular case of 3D projective shape data in Section 4, using the Veronese Whitney embedding of the projective shape space (see e.g. Mardia and Patrangenaru (2005)). This method builds upon previous results on one sample hypothesis testing methods, as developed in Patrangenaru et al. (2010, 2014). The space $PΣ^k_3$ of 3D projective shapes of k-ads including a projective frame at given landmark indices is isomorphic to $(\mathbb{R}P^3)^{k-5}$. Therefore a 3D projective shape face differentiation via VW-MANOVA testing is presented in Section 5. Note that behind the 3D Agisoft reconstruction software are results by Faugeras (1992) and Hartley et. al. (1992), showing that a 3D configuration of landmarks can be obtained from multiple noncalibrated camera images up to a projective transformation in 3D, thus allowing us to conduct without ambiguity a 3D projective shape analysis.

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Motivations for new MANOVA on manifolds

For \( a = 1, \ldots, g \), suppose \( X_{a,i} \sim N_p(\mu_a, \Sigma) \), \( i = 1, \ldots, n_a \) are \( p \)-dimensional i.i.d random vectors. To test if the mean vectors of the \( g \) groups are the same, one considers the hypothesis testing problem

\[
H_0 : \mu_1 = \mu_2 = \ldots = \mu_g = \mu
\]

\[
H_a : \text{at least one equation does not hold.}
\]

Assuming that the covariance matrix \( \Sigma \) is invertible, by the Central Limit Theorem, for large sample sizes \( n_a, a = 1, \ldots, g \), we have

\[
\sqrt{n_a} \Sigma^{-\frac{1}{2}} (\bar{X}_a - \mu) \sim N_p(0_p, I_p), \tag{2.2}
\]

\[
n_a (\bar{X}_a - \mu)^T \Sigma^{-1} (\bar{X}_a - \mu) \sim \chi^2_p. \tag{2.3}
\]

However, \( \Sigma \) is always unknown, so in practice, one has to use its unbiased estimator \( S_a, a = 1, \ldots, g \).

\[
n_a (\bar{X}_a - \mu)^T S_a^{-1} (\bar{X}_a - \mu) \sim \chi^2_p. \tag{2.4}
\]

Let us consider the pooled sample mean \( \bar{X} = \frac{1}{n} (n_1 \bar{X}_1 + \ldots + n_g \bar{X}_g), n = \sum_{a=1}^g n_a \).

**Lemma 2.1.** Under the null, \( \bar{X} \) is a consistent estimator of \( \mu \), provided \( \frac{n_a}{n} \to \lambda_a > 0 \), as \( n \to \infty \), \( a = 1, \ldots, g \).

**Proof.** Indeed, for any \( a \in \{1, 2, \ldots, g\} \), since \( \frac{n_a}{n} \to \lambda_a > 0 \), as \( n \to \infty \), and \( \bar{X}_a \) is the consistent estimator of \( \mu \), therefore,

\[
\bar{X} \to_p \lambda_1 \mu + \lambda_2 \mu + \ldots + \lambda_g \mu = \mu. \tag{2.5}
\]

**Theorem 2.1.** The statistic for the hypothesis in (2.1) is

\[
\sum_{a=1}^g n_a (\bar{X}_a - \bar{X})^T S_a^{-1} (\bar{X}_a - \bar{X}) \sim \chi^2_{gp}. \tag{2.6}
\]

So the rejection region for the test is

\[
\sum_{a=1}^g n_a (\bar{X}_a - \bar{X})^T S_a^{-1} (\bar{X}_a - \bar{X}) > \chi^2_{gp}(c). \tag{2.7}
\]

**3 MANOVA on manifolds**

In this section we will focus on the asymptotic behavior of statistics related to means on a manifold \( \mathcal{M} \) based on samples of different sizes from different populations on \( \mathcal{M} \). Now let’s consider the set \( X_{a,1}, \ldots, X_{a,n_a} (a = 1, 2, \ldots, g) \) of iid random objects on \( \mathcal{M} \) with common probability measure \( Q_a \). We denote the extrinsic mean of the \( j \)-nonfocal probability measure \( Q_a \) on \( \mathcal{M} \) by \( \mu_{a,E} \) for ease of notation and because there is no ambiguity about the embedding used. The corresponding extrinsic sample means are written \( \bar{X}_{a,E} \) for \( a = 1, \ldots, g \). From this point on, we will assume that all the distributions are \( j \)-nonfocal.
3.1 Hypothesis testing and $T^2$ statistic

Assume $X_{a,1}, \ldots, X_{a,n_a}$ are iid random objects on $M$ a $p$-dimensional manifold, with probability measure $Q_a$ with $a = 1, 2, \ldots, g$. We are interested in comparing multiple extrinsic means.

We would like to develop a test similar to (2.7) designed to test the difference between the $g$ extrinsic means. One challenge that presents itself at the early stage is a proper definition of a pooled mean for random objects on a $p$-dimensional manifold $M$. Linearity becomes an issue when dealing with extrinsic means. For a proper definition we will focus on the equalities tied to the assumption

$$A_0 : \mu_{1,E} = \cdots = \mu_{g,E}$$

DEFINITION 3.1. Under the assumption $A_0$ and for any $a \in \{1, 2, \ldots, g\}$, with $\frac{n_a}{n} \rightarrow \lambda_a > 0$, as $n \rightarrow \infty$. We define

(i) The pooled extrinsic mean with weights $\lambda = (\lambda_1, \ldots, \lambda_g)$, denoted $\mu_E(\lambda)$ as the value in $M$ given by

$$j(\mu_E) = P_j(\lambda_1 j(\mu_{1,E}) + \cdots + \lambda_g j(\mu_{g,E}))$$

Where $\mu_{a,E}$ is the extrinsic mean of the random object $X_{a,1}$ and $\Sigma_{a=1}^g \lambda_a = 1$

(ii) The extrinsic pooled sample mean denoted $\bar{X}_E \in M$ given by:

$$j(\bar{X}_E) = P_j \left( \frac{n_1}{n} j(X_{1,E}) + \cdots + \frac{n_g}{n} j(X_{g,E}) \right)$$

Where $\bar{X}_{a,E}$ is the extrinsic sample mean for $X_{a,1}$ and $n = \Sigma_{a=1}^g n_a$

Note that since $A_0$ implies $j(\mu_{1,E}) = \cdots = j(\mu_{g,E})$, and with our definition of the extrinsic pooled mean we get $j(\mu_E) = j(\mu_{a,E})$ for each $a = 1, \ldots, g$. Furthermore, the linear combination $\lambda_1 j(\mu_{1,E}) + \cdots + \lambda_g j(\mu_{g,E}) \in j(M)$. Note that for $a = 1, \cdots, g$ $\bar{X}_{a,E}$ is a consistent estimator of $\mu_{a,E}$ and therefore we get that $j(\bar{X}_E) \rightarrow_p j(\mu_E)$. Since $j$ is a homeomorphism from $M$ to $j(M)$ we also have that $\bar{X}_E$ is a consistent estimator of $\mu_E$ the extrinsic pooled mean. With this definition at hand, we now express the following hypothesis test, designed to test the difference between extrinsic means and is given by;

$$H_0 : \mu_{1,E} = \mu_{2,E} = \cdots = \mu_{g,E}$$

$$H_a : \text{at least one equality } \mu_{a,E} = \mu_{b,E}, 1 \leq a < b \leq g \text{ does not hold.}$$

And since the embedding $j : M \rightarrow \mathbb{R}^N$ is one-to-one the hypothesis above can be interchangeably written

$$H_0^a : j(\mu_{1,E}) = j(\mu_{2,E}) = \cdots = j(\mu_{g,E}) = j(\mu_E)$$

$$H_a^a : \text{at least one equality } \mu_{a,E} = \mu_{b,E}, 1 \leq a < b \leq g \text{ does not hold.}$$

In order to test hypothesis (3.3) we will use a $T^2$ like statistic. The theorem below, gives us the asymptotic behavior needed to establish such a statistic. For $a = 1, \ldots, g$, we get, from Bhattacharya and Patrangenaru [2], the following:

(i) $S_n_a = (n_a)^{-1} \Sigma_{i=1}^{n_a} (j(X_{a,i}) - j(\bar{X}_E))(j(X_{a,i}) - j(\bar{X}_E))^T$ is a consistent estimator of $\Sigma_a$, the covariance matrix of $X_{a,1}$ and

(ii) $\tan(j(\bar{X}_E)) \nu$ is a consistent estimator of $\tan P_j(\mu) \nu$, where $\nu \in \mathbb{R}^N$.

It follows that, under (3.4), $S_{E,a}(j, X_a)$, given by

$$S_{E,a}(j, X_a) = \left[ \sum_{a=1}^{m} \frac{d_{j \circ (X_a)}}{P_j(e_b)} P_j(e_b) \cdot e_i(j(\bar{X}_E)) e_i(j(\bar{X}_E)) \right]_{i=1, \ldots, p}^T S_n_a$$

$$\left[ \sum_{a=1}^{m} \frac{d_{j \circ (X_a)}}{P_j(e_b)} P_j(e_b) \cdot e_i(j(\bar{X}_E)) e_i(j(\bar{X}_E)) \right]_{i=1, \ldots, p}^T$$
where for $j^{(p)}(X) = \frac{1}{n} j(X_{1,E}) + \cdots + \frac{1}{n} j(X_{g,E})$ and $P_j(j^{(p)}(X))$ is a consistent estimator of $j(\mu_E)$. One must note that the extrinsic sample covariance matrix $S_{E,a}(j, X_a)$ is expressed in terms of $d_{j^{(p)}(X)} P_j(e_b) \in T_{j(X_a)}j(M)$ and not in terms of $d_{j^{(X_a,1)}} P_j(e_b) \in T_{j(X_a)}j(M)$.

**THEOREM 3.1.** Assume $j : M \to \mathbb{R}^N$ is a closed embedding of $M$. Let $\{X_{a,j}\}_{a=1}^n$ for $a = 1, \ldots, g$ be random samples from the $j$-nonfocal distributions $Q_a$. Let $\mu_a = E(j(X_{a,1}))$ and assume $j(X_{a,1})$’s have finite second-order moments and the extrinsic covariance matrices $\Sigma_{a,E}$ of $X_{a,1}$ are nonsingular. We also let $(e_1(p), \ldots, e_N(p))$, for $p \in M$ be an orthonormal frame field adapted to $j$. For $a = 1, \ldots, g$, assume $\lambda_a > 0$ are constants, such that $\sum_{a=1}^g \lambda_a = 1$. Furthermore, let $\frac{2n}{\lambda_a} \to \lambda_a > 0$, as $n \to \infty$, with $n = \sum_{a=1}^g n_a$. Then we have the following asymptotic behavior:

$$\sum_{a=1}^g n_a \tan_{j(\mu)}(j(\bar{X}_{a,E}) - j(\mu))^{T} \Sigma_{a,E}^{-1} \tan_{j(\mu)}(j(\bar{X}_{a,E}) - j(\mu)) \to_d \chi^2_{2p}.$$  

It follows that the statistics for hypothesis (3.3) have the following asymptotic results:

(a) the statistic

$$\sum_{a=1}^g n_a \tan_{j(\mu)}(j(\bar{X}_{a,E}) - j(\mu))^{T} S_{E,a}(j, X_a)^{-1} \tan_{j(\mu)}(j(\bar{X}_{a,E}) - j(\mu)) \to_d \chi^2_{2p}.$$  

(b) the statistic

$$\sum_{a=1}^g n_a \tan_{j(\bar{X})}(j(\bar{X}_{a,E}) - j(\bar{X}))^{T} S_{E,a}(j, X_a)^{-1} \tan_{j(\bar{X})}(j(\bar{X}_{a,E}) - j(\bar{X})) \to_d \chi^2_{2p}.$$  

**Proof.** Recall from the Bhattacharya and Patrangenaru (2005)\[2], from the consistency of the sample mean vector and from the continuity of the projection map $P_j$, that we have

$$\sqrt{n_a} \tan_{j(\mu)}(j(\bar{X}_{a,E}) - j(\mu)) \to_d N(0, \Sigma_{a,E}), \text{ for } a = 1, 2, \ldots, g$$

where

$$\Sigma_{a,E} = \left[ \sum_{k=1}^N d_{\mu}(e_b)^T e_k(P_j(\mu)) \right]_{k=1, \ldots, p} \Sigma_a \left[ \sum_{k=1}^N d_{\mu}(e_b)^T e_k(P_j(\mu)) \right]^{T}_{k=1, \ldots, p},$$

where $\mu = \lambda_1 j(\mu_{1,E}) + \cdots + \lambda_g j(\mu_{g,E})$ and $\Sigma_a$ is the covariance matrices of the $j(X_{a,1})$ with respect to the canonical basis $e_1, \ldots, e_N$. And under the null, from (3.3), the matrices $\Sigma_{a,E}$ are defined with respect to the basis $f_1(\mu_E), \ldots, f_p(\mu_E)$ of local frame fields, $f_r = d_{j^{-1}(P_j(\mu))}(e_r(P_j(\mu)))$. We then have for each $a = 1, \ldots, g$

$$n_a \tan_{j(\mu)}(j(\bar{X}_{a,E}) - j(\mu))^{T} \Sigma_{a,E}^{-1} \tan_{j(\mu)}(j(\bar{X}_{a,E}) - j(\mu)) \to_d \chi^2_{p},$$

and since the random samples are independent we have,

$$\sum_{a=1}^g n_a \tan_{j(\mu)}(j(\bar{X}_{a,E}) - j(\mu))^{T} \Sigma_{a,E}^{-1} \tan_{j(\mu)}(j(\bar{X}_{a,E}) - j(\mu)) \to_d \chi^2_{2p}.$$  

$\bar{X}_E$ is the consistent estimator of $\mu_E$, then the pooled sample mean

$$j(\bar{X}_E) = P_j \left( \frac{1}{n} \sum_{a=1}^g n_a j(\bar{X}_{a,E}) \right) \to_p j(\mu_E) \quad \text{(by Lemma 2.1)}$$

4
And since $S_{E,a}(j, X_a)$ consistently estimates $\Sigma_a$ and $\tan(j(\bar{X}_E))$ is a consistent estimator of $\tan(j(\mu_E))$, we have the following
\[
\sum_{a=1}^{g} n_a \tan(j(\mu_E))(j(\bar{X}_{a,E}) - j(\bar{X}_E))' S_{E,a}(j, X_a)^{-1} \tan(j(\mu_E))(j(\bar{X}_{a,E}) - j(\bar{X}_E)) \rightarrow_d \chi^2_{gp}.
\]
\[
\sum_{a=1}^{g} n_a \tan(j(\bar{X}_E))(j(\bar{X}_{a,E}) - j(\bar{X}_E))' S_{E,a}(j, X_a)^{-1} \tan(j(\bar{X}_E))(j(\bar{X}_{a,E}) - j(\bar{X}_E)) \rightarrow_d \chi^2_{gp}.
\]

\[\square\]

### 3.2 Nonparametric bootstrap confidence regions for the common extrinsic mean

From Bhattacharya and Patrangenaru(2005)\(^{[2]}\) and from Corollary 3.2 in Bhattacharya and Bhattacharya(2012)\(^{[1]}\), under the hypothesis
\[
\begin{align*}
H_0 & : \mu_1, E = \mu_2, E = \ldots = \mu_g, E = \mu_E, \\
H_a & : \exists (i, j) 1 \leq i < j \text{ s.t. } \mu_i, E \neq \mu_j, E,
\end{align*}
\]
we have:

**COROLLARY 3.1.** Under the assumptions of Theorem (3.1), a confidence regions for $\mu_E$ of asymptotic level $1 - c$ is given by $C_{n,c}^{(g)}$ and $D_{n,c}^{(g)}$ which are defined below

(a) $C_{n,c}^{(g)} = j^{-1}(U_{n,c})$ where
\[
U_{n,c} = \{j(\nu) \in j(M) : \sum_{a=1}^{g} n_a \left\| S_{E,a}(j, X_a)^{-1/2} \tan(j(\nu))(j(\bar{X}_{a,E}) - j(\nu)) \right\|^2 \leq \chi^2_{gp,1-c} \}
\]

(b) $D_{n,c}^{(g)} = j^{-1}(V_{n,c})$ where
\[
V_{n,c} = \{j(\nu) \in j(M) : \sum_{a=1}^{g} n_a \left\| S_{E,a}(j, X_a)^{-1/2} \tan(j(\bar{X}_E))(j(\bar{X}_{a,E}) - j(\nu)) \right\|^2 \leq \chi^2_{gp,1-c} \}
\]

where $\bar{X}_E$ is the extrinsic pooled sample mean defined in Definition 3.7(ii)

Most of the data we will be focusing on will have value of $n$ relatively small. We will need to use resampling, in particular, bootstrap methods. For $a = 1, \ldots, g$, let $\{X_{a,i}\}_{i=1}^{n_a}$ be i.i.d.r.o’s from the $j$-nonfocal distributions $Q_a$. Let $\{X_{a,i}\}_{i=1}^{n_a}$ be random resamples with repetition from the empirical $Q_{n_a}$ conditionally given $\{X_{a,i}\}_{i=1}^{n_a}$. The confidence regions $C_{n,c}^{(g)}$ and $D_{n,c}^{(g)}$ described above have the corresponding bootstrap analogue $C_{n,c}^{* (g)}$ and $D_{n,c}^{* (g)}$ which are defined in the corollary below.

**COROLLARY 3.2.** The $(1 - c)100\%$ bootstrap confidence regions for $\mu_E$ with $d = gp$ are given by

(a) $C_{n,c}^{* (g)} = j^{-1}(U_{n,c}^{*})$ and
\[
(3.7) \quad U_{n,c}^{*} = \{j(\nu) \in j(M) : \sum_{a=1}^{g} n_a \left\| S_{E,a}(j, X_a)^{-1/2} \tan(j(\nu))(j(\bar{X}_{a,E}) - j(\nu)) \right\|^2 \leq c_{1-c}^{* (g)} \}
\]

where $c_{1-c}^{* (g)}$ is the upper $100(1 - c)\%$ point of the values
\[
(3.8) \quad \sum_{a=1}^{g} n_a \left\| S_{E,a}(j, X_{a}^{*})^{-1/2} \tan(j(\bar{X}_E))(j(\bar{X}_{a,E}) - j(\bar{X}_E)) \right\|^2
\]

among the bootstrap re samples.
regions. We will utilize the tests statistics expressed above to conduct our analysis with confidence regions

Using the bootstrap analogue in the previous Proposition 3.1 yields simpler method for finding

\[ X^* \]

\[ O \]

\[ \Sigma \]

\[ a \]

\[ E \]

\( n,c \)

\( g \)

\( a,E \)

\( m \)

\( E,g \)

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4 MANOVA on \((\mathbb{R}P^3)^q\)

We start with the 3-dimensional real projective space \(\mathbb{R}P^3\), set of 1-dimensional linear subspaces of \(\mathbb{R}^4\). \(\mathbb{R}P^3\) has a 3D manifold structure (see Patrangenaru and Ellsworth(2015)[8] p.106). A point \(p = [x] \in \mathbb{R}P^3\) is the equivalence class of \(x = (x^1 x^2 x^3 x^4)^T \in \mathbb{R}^4 \setminus \{0\}\), where two nonzero vectors in \(\mathbb{R}^4\) are equivalent, if one is a scalar multiple of the other. The point \(p\) can be represented as \(p = [x^1 : x^2 : x^3 : x^4]\) (homogeneous coordinates notation). One may also represent \(\mathbb{R}P^3\) as the sphere \(S^3\) with the antipodal points identified. We will often refer to this identification as the spherical representation of the real projective space. \(\mathbb{R}P^3\) is an embedded manifold with the VW-embedding \(j : \mathbb{R}P^3 \to S(4, \mathbb{R})\), given by

\[
(4.1)\quad j([x]) = xx^T, x^T x = 1.
\]

Given a random object \(Y\) on \(\mathbb{R}P^3\), \(Y = [X], X^T X = 1\), such that \(E(XX^T)\) has a simple largest eigenvalue, one can show that the VW (extrinsic)-mean \(\mu_j = [\gamma]\), where \(\gamma\) is a unit eigenvector of \(E(XX^T)\) corresponding to this largest eigenvalue (see Bhattacharya and Patrangenaru [2]).

Our analysis will be conducted on \(PS_0^3\), the projective shape space of 3D k-ads in \(\mathbb{R}P^n\) for which \(\pi = ([u_1], \ldots, [u_5])\) is a projective frame in \(\mathbb{R}P^3\). \(PS_0^3\) is homeomorphic to the manifold \(\{\mathbb{R}P^3\}^{k-5}\) with \(k - 5 = q\) (see Patrangenaru et al (2010)[3]). The embedding on this space is the VW (Veronese-Whitney) embedding given by

\[
(4.2)\quad j_k : (\mathbb{R}P^3)^q \to (S(4, \mathbb{R}))^q
\]

with \(j : \mathbb{R}P^3 \to S_+(4, \mathbb{R})\) the embedding given in (4.1). Additionally \(j_k\) is an equivariant embedding w.r.t. the group \((S_+(4, \mathbb{R}))^q\) and has the corresponding projection

\[
(4.3)\quad P_{j_k} : (S_+(4, \mathbb{R}))^q \setminus \mathcal{F}_q \to j_k (\mathbb{R}P^3)^q
\]

where \(m_1, \ldots, m_q\) are unit eigenvectors of \(A_1, \ldots, A_q\) (respectively) corresponding to the respective highest eigenvalues of those nonnegative definite symmetric matrices. Let \(Y\) be a random object from a VW distribution \(Q\) on \((\mathbb{R}P^3)^q\), where \(Y = (Y^1, \ldots, Y^q)\), and \(Y^s = [X^s] \in \mathbb{R}P^3\) for all \(s = 1, q\). The VW mean is given by

\[
(4.4)\quad \mu_{j_k} = ([\gamma_1(4)], \ldots, [\gamma_q(4)]),
\]

where, for \(s = 1, q\), \(\lambda_s(r)\) and \(\gamma_s(r), r = 1, \ldots, 4\) are the eigenvalues in increasing order and the corresponding eigenvectors of \(E [X^s (X^s)^T]\).

In case of a VW-nonfocal random object \([X]\) on \(\mathbb{R}P^3\), we know that \(\mu_{E,j} = [\nu_4]\), where \(\lambda_r\) and \(\nu_r, r = 1, 2, 3, 4\), are eigenvalues in increasing order and corresponding unit eigenvectors of \(\mu = E[XX^T]\). Similarly, given i.i.d.r.o’s \(Y_i = [X_i], i = 1, \ldots, n\) from \(Q\) on \(\mathbb{R}P^3\), their VW sample mean, is given by \(X_{E,j} = [g(4)]\), where \(d(r)\) and \(g(r)\) in \(\mathbb{R}^4\), \(r = 1, 2, 3, 4\), are eigenvalues in increasing order and corresponding unit eigenvectors of \(J = \frac{1}{n} \sum_{i=1}^n X_i X_i^T\).

We now recall from Bhattacharya and Patrangenaru (2005)[4] that the statistic

\[
T([X], Q) = n \| S(j, X)^{-1/2} (\tan_{j_E,j} j(j_E,j) - j(\mu_{E,j})) \|^2,
\]

in case of a random sample from a distribution on \(\mathbb{R}P^3\), has the form \(T([X], Q) = T([X], [\nu_4])\) given by

\[
(4.5)\quad T([X], [\nu_4]) = ng(4)^T ([\nu_r])_{r=1,2,3} S(j, X)^{-1} ([\nu_r])_{r=1,2,3} g(4),
\]

where the entries of the sample VW-covariance matrix are

\[
(4.6)\quad S(j, X)_{ab} = n^{-1}(d(4) - d(a))^{-1}(d(4) - d(b))^{-1} \times \sum_{i=1}^n (g(a) \cdot X_i) (g(b) \cdot X_r) (g(4) \cdot X_i)^2,
\]

for \(a, b = 1, 2, 3\).
If we project on the tangent space to the VW-sample mean, we get the statistic
\[
T([X], \hat{Q}) = T([X], \{g(4)\}) = \left\| S(j, X)^{-1/2} \tan_{j([X], \hat{Q})} (j([X], \hat{Q}) - j(\mu_{E,J})) \right\|^2 = n^T \nu_{i=1,2,3} S(j, X)^{-1} [g(r)]^T = n^T \nu_{i=1,2,3} \mu_{E,J},
\]
where \( S(j, X) \) is also given in (4.6), and from the Slutsky’s theorem, asymptotically \( T([X], \mu_{E,J}) \) and \( T([X], \nu_{i=1,2,3}) \) both have a \( \chi^2 \) distribution (see Bhattacharyya and Patrangenaru (2005) [2]).

Before we express our statistics of interest, it will be important to note another result from Crane and Patrangenaru (2011) [3] concerning the statistics
\[
T(Y, \mu_{E,J}) = n^T |S_Y(j_k, Y)^{-1/2} \tan_{j_k ([Y], \hat{Q})} (j_k ([Y], \hat{Q}) - j_k (\mu_{E,J})) |^2
\]
and this Hotelling Type II statistic is given by
\[
(4.8) \quad T(Y, ([\gamma_1(4)], \cdots,[\gamma_q(4)])) = n \left( \gamma_1(4)^T D_1 \cdots \gamma_q(4)^T D_q \right) S_Y(j_k, Y)^{-1} \left( \gamma_1(4)^T D_1 \cdots \gamma_q(4)^T D_q \right)^T
\]
where for \( s = 1, \ldots, q \) we have \( D_s = (g_s(1) g_s(2) g_s(3)) \in M(4, 3, \mathbb{R}) \) and for a pair of indices \((s,a), s = 1, \ldots, q \) and \( a = 1, 2, 3 \) in their lexicographic order we have
\[
(4.9) \quad S_Y(j_k, Y)_{(s,a), (t,b)} = n^{-1} (d_s(a) - d_s(a))^{-1} (d_t(a) - d_t(b))^{-1} \sum_{i=1}^n \left( g_s(a) \cdot X_i^a \right) \left( g_t(b) \cdot X_i^a \right) (g_s(4) \cdot X_i^a) (g_t(4) \cdot X_i^a)
\]
In the next theorem we will take advantage of these results.

(4.10) \quad H_0 : \mu_{1,E} = \mu_{2,E} = \cdots = \mu_{g,E} = \mu_E,

\[
H_a : \text{at least one equality } \mu_{a,E} = \mu_{b,E}, 1 \leq a < b \leq g \text{ does not hold.}
\]

We aim to have an explicit representation of the expressions,
\[
(4.11) \quad T_c \left( Y(g), \mu_E^{(p)} \right) = n_a \sum_{\alpha=1}^g \left| S_Y(j_k, Y_a)^{-1/2} \tan_{j_k (\mu_E^{(p)})} (j_k (\mu_E^{(p)}) - j_k (\mu_E)) \right|^2
\]
\[
(4.12) \quad T_d \left( Y(g), \tilde{Y}_E^{(p)} \right) = n_a \sum_{\alpha=1}^g \left| S_Y(j_k, Y_a)^{-1/2} \tan_{j_k (\tilde{Y}_E^{(p)})} (j_k (\tilde{Y}_E^{(p)}) - j_k (\mu_E)) \right|^2
\]
where \( \mu_{a,E} = (\nu_{a}^{(4)}), \ldots, \nu_{a}^{(q)}(4) \) are the VW mean from distribution \( Q_a \) (of \( Y_{r,a} \)) and \( (\eta_{s}^{a}(r), \nu_{s}^{a}(r)) \), \( r = 1, \ldots, 4 \), are eigenvalues and corresponding unit eigenvectors of \( E(X_{a,i} X_{a,i}^T) \). The corresponding VW sample mean is given by
\[
\tilde{Y}_E^{(p)} = ([g_1^{(4)}, \ldots, g_q^{(4)}(4)]), \text{ where for each } s = 1, \ldots, q \text{ and } r = 1, \ldots, 4 \text{, (} d_s^{(r)}(r), g_s^{(r)}(r) \text{) are eigenvalues in increasing order and corresponding unit eigenvectors of } J_s^{n} = \frac{1}{n} \sum_{i=1}^n X_{a,i} X_{a,i}^T. \text{ Also } \mu_E^{(p)} \text{ is the VW pooled mean given by}
\]
\[
(4.13) \quad j_k (\mu_E^{(p)}) = P_{jk} \left( \sum_{\alpha=1}^g \lambda_{\alpha} jk(\mu_{\alpha,E}) \right)
\]
\[
(4.14) \quad \mu_E^{(p)} = ([\gamma_1^{(p)}(4)], \ldots, [\gamma_q^{(p)}(4)]),
\]
where for \( s = 1, \ldots, q \), \( \gamma_s^{(p)}(4) \) is the eigenvector corresponding to the largest eigenvalue of the \( s-\text{th} \) axial component of the pooled matrix with weights \( \lambda_{\alpha}, \alpha = 1, \ldots, g \):
\[
\sum_{\alpha=1}^g \frac{\lambda_{\alpha} X_{a,i} X_{a,i}^T. \right).
The pooled VW-sample mean \( \overline{Y}_E^{(p)} \) is given by

\[
(4.15) \quad j_k \left( \overline{Y}_E^{(p)} \right) = P_{j_k} \left( \sum_{a=1}^{g} \frac{n_a}{n} j_k(\overline{Y}_{a,E}) \right)
\]

\[
(4.16) \quad \overline{Y}_E^{(p)} = \left( [\mathbf{g}_1^{(p)}(4)], \ldots, [\mathbf{g}_q^{(p)}(4)] \right)
\]

where for \( s = 1, \ldots, q \), \( \mathbf{d}_s^{(p)}(r) \) and \( \mathbf{g}_s^{(p)}(r) \) are non-focal probability measures \( Q_a \) on \( (\mathbb{R}P^m)^2 \), that have non degenerate \( j_k \)-extrinsic covariance matrices. Consider the statistics

\[
(4.17) \quad \mathbf{C}_s = (\gamma_s^{(p)}(1) \gamma_s^{(p)}(2) \gamma_s^{(p)}(3)) \in \mathcal{M}(4, 3 : \mathbb{R})
\]

\[
(4.18) \quad \mathbf{D}_s = (\mathbf{g}_s^{(p)}(1) \mathbf{g}_s^{(p)}(2) \mathbf{g}_s^{(p)}(3)) \in \mathcal{M}(4, 3 : \mathbb{R})
\]

**COROLLARY 4.1.** Assume \( j_k \) is the VW embedding of \( (\mathbb{R}P^3)^g \) and \( \{Y_{a,r_a}\}_{r_a=1}^{g} \), \( a = 1, \ldots, g \) are i.i.d.r. objects random from the \( j_k \)-nonfocal probability measures \( Q_a \) on \( (\mathbb{R}P^m)^2 \), that have non degenerate \( j_k \)-extrinsic covariance matrices. Consider the statistics

(i) \( T_c \left( Y^{(g)}, \mu_E^{(p)} \right) = \sum_{a=1}^{g} n_a \left( (g_1^{(p)}(4))^T \mathbf{C}_1 \cdots (g_q^{(p)}(4))^T \mathbf{C}_q \right) S_{Y_a}(j_k, Y_a)^{-1} (g_1^{(p)}(4))^T \mathbf{C}_1 \cdots g_q^{(p)}(4))^T \mathbf{C}_q \right)^T \]

(ii) \( T_d \left( Y^{(g)}, \overline{Y}_E^{(p)} \right) = \sum_{a=1}^{g} n_a \left[ (\gamma_1^{(p)}(4) - g_1^{(p)}(4))^T \mathbf{D}_1 \cdots (\gamma_q^{(p)}(4) - g_q^{(p)}(4))^T \mathbf{D}_q \right] \)

where

\[
S_{Y_a}(j_k, Y_a) = n_a^{-1} \left( (\mathbf{d}_s^{(p)}(4) - \mathbf{d}_s^{(p)}(c) - (\mathbf{d}_t^{(p)}(4) - \mathbf{d}_t^{(p)}(b))^{-1} \right.
\]

\[
\sum_i \left( \mathbf{g}_s^{(p)}(c) \cdot X_{a,i} \right) \left( \mathbf{g}_t^{(p)}(b) \cdot X_{a,i} \right) \left( \mathbf{g}_s^{(p)}(4) \cdot X_{a,i} \right) \left( \mathbf{g}_t^{(p)}(4) \cdot X_{a,i} \right)
\]

and \( s, t = 1, \ldots, q \) and \( c, b = 1, \ldots, m \). If \( \frac{n_a}{n} \rightarrow \lambda_a > 0 \), as \( n \rightarrow \infty \), then both \( T_c \left( Y^{(g)}, \mu_E^{(p)} \right) \) and \( T_d \left( Y^{(g)}, \overline{Y}_E^{(p)} \right) \)

have asymptotically a \( \chi^2_{3q} \) distribution.

**Proof.** For part (i) we note that for each \( a = 1, \ldots, g \) we get a natural extension of a result in Bhattacharya and Bhattacharya (2012) \([1]\) as shown in \((4.5)\). For part (ii) recall that

\[
T_d \left( Y^{(g)}, \overline{Y}_E^{(p)} \right) = n_a \sum_{a=1}^{g} \left\| S_{Y_a}(j_k, Y_a)^{-1/2} \tan j_k(\overline{Y}_{a,E}) \left( j_k(\overline{Y}_{a,E}) - j_k \left( \mu_E^{(p)} \right) \right) \right\|^2
\]

we start by rewriting the expression above and we have

\[
T_d \left( Y^{(g)}, \overline{Y}_E^{(p)} \right) = n_a \sum_{a=1}^{g} \left\| S_{Y_a}(j_k, Y_a)^{-1/2} \tan j_k(\overline{Y}_{a,E}) \left( j_k(\overline{Y}_{a,E}) - j_k \left( \mu_E^{(p)} \right) \right) \right\|^2
\]

\[
- S_{Y_a}(j_k, Y_a)^{-1/2} \left[ (\gamma_1^{(p)}(4))^T \mathbf{D}_1 \cdots (\gamma_q^{(p)}(4))^T \mathbf{D}_q \right]^T
\]

\[
(4.19) \quad \overline{Y}_E^{(p)} = \left( [\mathbf{g}_1^{(p)}(4)], \ldots, [\mathbf{g}_q^{(p)}(4)] \right)
\]
If $Y_{a,r}$ are $j_k$-nonfocal distributions on $(\mathbb{R}P^3)^q$ with a nonzero absolutely continuous component (see Ferguson(1996)[5], p.30), one may obtain better coverage confidence regions, using nonparametric bootstrap. Consider the pivotal statistics $T_c \left(Y^{(g)}, \mu_E^{(p)}\right)$ and $T_d \left(Y^{(g)}, \bar{Y}_E^{(p)}\right)$, under the hypothesis

\[
H_0 : \mu_{1,E} = \mu_{2,E} = \cdots = \mu_{q,E} = \mu_E^{(p)} ,
\]

\[
H_a : \exists \ (i,j) 1 \leq i < j < q, \ \text{s.t.} \ \mu_{i,E} \neq \mu_{j,E}.
\]

**COROLLARY 4.2.** The $(1-c)100\%$ bootstrap confidence regions for $\mu_E$ with $d = gp$ are given by

(a) $C_{n,c}^{(g)}(\nu) = j^{-1}(U_{n,c}^{*}(\nu))$ and $U_{n,c}^{*} = \{ j_k(\nu) \in j_k((\mathbb{R}P^3)^q) : T_c \left(Y^{(g)}, \nu\right) \leq c^{(g)}_{1-c} \}$ where $c^{(g)}_{1-c}$ is the upper $100(1-c)\%$ point of the values

\[
T_c \left(Y^{(g)}, \bar{Y}_E^{(p)}\right) = \sum_{a=1}^{q} n_a \left( (g^{*(a)}(4)^T D_1 \cdots (g^{*(a)}(4)^T D_q) S_{\nu_a}(j_k, Y^{*})^{-1} (g^{*(a)}(4)^T D_1 \cdots g^{*(a)}(4)^T D_q)^T
\]

among the bootstrap resamples. The confidence regions given by (4.20) and (4.21) have both coverage error $O_p(n^{-2})$.

Note that here

\[
S_{\nu_a}(j_k, Y^{*})^{-1} = n_a^{-1} (d^{*(a)}(4) - d^{*(a)}(c))^{-1} (d^{*(a)}(4) - d^{*(a)}(b))^{-1} \sum_i (g^{*(a)}(c) \cdot X^{*(a)}(j_k, Y^{*}))(g^{*(a)}(b) \cdot X^{*(a)}(j_k, Y^{*}))(g^{*(a)}(4) \cdot X^{*(a)}(j_k, Y^{*})), b, c = 1, 2, 3.
\]

5 Application to face data analysis

A digital images data set was collected using a high resolution Panasonic-Lumix DMC-FZ200 camera. Our analysis will be conducted on $g = 5$ individuals. The images can be found at anl.stat.fsu.edu/ ~ vic/E – MANOVA. We tested for the existence of a 3D mean projective shape difference to differentiate between five faces which are represented in Fig[5]

The 3D surface reconstructions of these faces, with seven labeled landmarks, were obtained using the software Agisoft. These reconstructions (including texture) are displayed in Figure[5].

The 3D reconstruction was done using the AGISOFT software. The images in Fig[5] represent 19 facial reconstructions. Each of those reconstruction was created using mostly 4 to 5 digital camera images of a given individual. We placed seven anatomical landmarks as shown across the data in Figure[5].

Five of those landmarks (colored in red) are selected as the projective frame and the resulting two projective coordinates determine the 3D projective shape of the seven landmark configuration selected. Note that we used a different projective frame than the one in Yao (2016)[10], to insure that the landmarks are in general position.
We will compare these faces by conducting a MANOVA on manifold to compare $g = 5$ VW-means on $P\Sigma_3^7 = (\mathbb{R}^3)^2$. For $n = \sum_{a=1}^5 n_a = 31$ where $n_1 = n_2 = n_4 = n_5 = 6$ and $n_3 = 7$ our hypothesis problem is

$$H_0 : \mu_1, E = \mu_2, E = \mu_3, E = \mu_4, E = \mu_5, E = \mu_E,$$

$$H_a : \text{at least one of equalities above does not hold}.$$

Since the true pulled mean is unknown and our data set is relatively small we will reject the null hypothesis if

$$T_d \left( Y^{(3)}, Y^{(p)}_E \right) = \sum_{a=1}^5 n_a \left\| S_{Y_a} \left( jk, Y_a \right) \right\|^{1/2} \tan \left( \frac{1}{2} \left( jk \left( Y^{(p)}_a, E \right) - jk \left( Y^{(p)}_E \right) \right) \right)$$

is greater than $d^{* (3)}_{1-\alpha}$, where $d^{* (3)}_{1-\alpha}$ is the $(1 - \alpha)100\%$ cutoff of the corresponding bootstrap distribution in equation (4.21).

Using $\alpha = 0.05$, and 70543872 resamples we obtain a value $T_d \left( Y^{(3)}, Y^{(p)}_E \right) = 389860$ and $d^{* (3)}_{0.95} = 60616$, and we therefore reject the null hypothesis. We conclude that there exists a statistically significant VW-mean 3D-projective shape face difference between at least two of the individuals in our data set.

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Figure 2: Sample of Facial Reconstructions

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Figure 3: Projective Frame Shown in Red