Abstract. We establish an interaction Morawetz estimate for the magnetic Schrödinger equation for \( n \geq 3 \) under certain smallness conditions on the gauge potentials. As an application, we prove global wellposedness and scattering in \( H^1 \) for the cubic defocusing magnetic Schrödinger equation for \( n = 3 \).

1. Introduction

The purpose of this article is to initiate the study of the interaction Morawetz estimates for the magnetic Schrödinger equation. Morawetz type estimates have their origins in \([26]\) and \([22]\). The first interaction Morawetz estimate was established for the cubic defocusing NLS \([6]\), and it reads as follows

\[
\int_T^0 \int_{\mathbb{R}^3} |u(t,x)|^4 \, dx \, dt \lesssim \|u(0)\|_{L^2}^2 \sup_{[0,T]} \|u(t)\|_{H^\frac{1}{2}}^2 \cdot
\]

In particular, it allowed for a simpler proof of scattering obtained previously in \([15]\). The estimate was extended to \( n \geq 4 \) in \([31, 34]\) giving

\[
\left\| \nabla \left( |u(t,x)|^2 \right) \right\|_{L^2([0,T] \times \mathbb{R}^n)}^2 \lesssim \|u(0)\|_{L^2}^2 \sup_{[0,T]} \|u(t)\|_{H^\frac{1}{2}}^2 \cdot
\]

Then building on an idea of Hassell and other advances, a new proof was obtained in \([5]\) that applies to all dimensions \( n \geq 1 \). An independent proof was also achieved in \([28]\). For a more detailed background on Morawetz type estimates we refer to \([5, 17]\).

Now let \( n \geq 3 \) and consider the magnetic nonlinear Schrödinger equation

\[
(iD_t + \Delta_A)u = \mu g(|u|^2)u,
\]

\[(m\text{NLS})\]

where

\[
u : \mathbb{R}^{n+1} \mapsto \mathbb{C},
\]

\[
A_\alpha : \mathbb{R}^n \mapsto \mathbb{R}, \quad \alpha = 0, \cdots, n,
\]

\[
D_\alpha = \partial_\alpha + iA_\alpha, \quad \alpha = 0, \cdots, n, \quad D_t = D_0,
\]

\[
\Delta_A = D^2 = D_jD_j = (\partial_j + iA_j)(\partial_j + iA_j) = \Delta + iA \cdot \nabla + i\nabla \cdot A - |A|^2,
\]

\[
g(r) = r^p, \quad r \geq 0, p > 0.
\]

We use the standard notation, that the greek indices range from 0 to \( n \), and Roman indices range from 1 to \( n \). We also sum over repeated indices. The case of \( \mu = 1 \) is called defocusing.

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and $\mu = -1$ is called focusing. We suppose we are in the Coulomb gauge, $\nabla \cdot A = 0$. The main result is

**Theorem 1.1.** Let $n \geq 3$, and let $u$ solve the defocusing mNLS. Suppose $(A_0, A)$ satisfy (2.8)–(2.10) and (2.16)–(2.20). Then the following estimate holds

$$
\|\nabla |^{-\frac{n-3}{2}} (|u|^2)\|_{L^2([0,T] \times \mathbb{R}^n)}^2 \lesssim \|u_0\|_{L^2_x}^2 \sup_{[0,T]} \left\| (-\Delta + A)^{\frac{1}{4}} u \right\|_{L^2_t}^2.
$$

The conditions on the gauge potentials $(A_0, A)$ will soon be discussed in more detail in Section 2.1. As an application we show

**Theorem 1.2.** Let $(A_0, A)$ satisfy (2.8)–(2.10), (2.16)–(2.20) and (2.21)–(2.24). Then for given initial data in $H^1(\mathbb{R}^3)$, mNLS with a defocusing cubic nonlinearity is globally wellposed and scatters to the linear magnetic Schrödinger equation.

While the theory of existence and uniqueness has been considered before for the nonlinear mNLS (see [3, 9, 27, 25]) this is the first result (to our knowledge) on scattering for the nonlinear equation. Scattering for the one particle mNLS without the nonlinearity has been considered by many authors. We refer the reader to [23, 29, 2, 21, 19] and references therein.

The proof of Theorem 1.1 relies on the commutator vector operators method developed in [5]. The main ingredient comes from the local conservation laws. In the case of the classical NLS the momentum is conserved. In the case of mNLS, we obtain only a balance law (see (2.2) and (3.3)–(3.5) for precise definitions)

$$
\partial_t T_{0j} + \partial_k T_{jk} = F_{\alpha j} T_{\alpha 0},
$$

which eventually results in a need to control a term of the form

$$
B(t) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{x_j - y_j}{|x - y|} F_{\alpha j}(x, t) T_{\alpha 0}(x, t) |u(y, t)|^2 \, dx \, dy.
$$

If $B(t)$ were positive, we could just ignore it (see for example the proof of (3.12)). However as shown in the appendix, this cannot be expected in general. Another way to handle this term follows the path used by Fanelli and Vega [12] for the linear magnetic Schrödinger equation. Moreover, applying the results of [12], D’Ancona, Fanelli, Vega and Visciglia established a family of Strichartz estimates [8]. Their work motivates us to assume similar conditions on our gauge potentials. As a result, we can control the term $B(t)$ leading to interaction Morawetz estimates for mNLS.

To show Theorem 1.2 we need an inhomogeneous Strichartz estimate. This is a simple consequence of the Christ-Kiselev Lemma and Strichartz estimates from [8] (stated in Theorem 2.7 and Theorem 2.8), and we record it here for completeness.

**Theorem 1.3.** Consider an inhomogeneous linear magnetic Schrödinger equation with zero initial data on $\mathbb{R}^{1+n}, n \geq 3$.

\begin{align*}
 iD_t u + \Delta_A u &= N, \\
 u(0) &= 0,
\end{align*}

and suppose $u$ is a solution of (1.4) and that $(A_0, A)$ satisfy (2.8)–(2.10), (2.21)–(2.24). Then

$$
\|u\|_{L^6_t L^6_x} \lesssim \|N\|_{L^6_t L^6_x},
$$
for Schrödinger admissible Strichartz pairs \((q,r), (\tilde{q}, \tilde{r})\) such that both admissible pairs satisfy
\[
\frac{2}{q} + \frac{n}{r} = \frac{n}{2}, \quad 2 \leq q, q \neq 2 \text{ if } n = 3, \quad \text{and } \tilde{q} \neq 2 \text{ if } q = 2 \text{ for } n > 3,
\]
and where \(p'\) denotes the Hölder dual exponent of \(p\).

The dispersive properties of the magnetic Schrödinger equations have been studied also by \[7, 10, 11, 14, 24\]. We would like to investigate in the future if the interaction Morawetz estimates could be recaptured in the setting of these works. Also see \[1, 13\].

The organization of the paper is as follows. In Section 2 we gather some identities and known estimates. In Section 3 we derive conservation laws and the generalized magnetic virial identity, which are then applied in Section 4 to show Theorem 1.1. In Section 5 we prove the inhomogeneous Strichartz estimate. Section 6 is devoted to the proof of Theorem 1.2.

2. Preliminaries

We start by stating the following identities, which are easily verified by a direct computation
\[
(2.1) \quad \partial_\alpha (uv) = (D_\alpha u)v + u D_\alpha v,
\]
\[
(2.2) \quad D_\alpha D_\beta = iF_{\alpha\beta} + D_\beta D_\alpha, \quad \text{where } F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha,
\]
\[
(2.3) \quad D_\alpha (uv) = (D_\alpha u)v + u \partial_\alpha v.
\]

We recall the standard Schrödinger estimates \[16, 35, 20\]. If \((q,r)\) is Schrödinger admissible, i.e.,
\[
\frac{2}{q} + \frac{n}{r} = \frac{n}{2}, \quad q \geq 2, \quad q \neq 2 \text{ if } n = 2,
\]
then
\[
(2.4) \quad \left\| e^{it\Delta} \phi \right\|_{L_t^q L_x^r} \leq C \| \phi \|_{L_x^2},
\]
\[
(2.5) \quad \left\| \int_0^t e^{i(t-s)\Delta} N(s, \cdot) ds \right\|_{L_t^q L_x^r} \leq C \| N \|_{L_t^{q'} L_x^{r'}}.
\]

where \((q', r')\) are Hölder dual exponents of a Schrödinger admissible pair \((\tilde{q}, \tilde{r})\).

We also use the following local smoothing estimate (see \[8\] for historical remarks)

**Theorem 2.1.** \[30\] If \((q,r)\) is Schrödinger admissible, then
\[
(2.6) \quad \left\| \nabla^{\frac{1}{2}} \int_0^t e^{i(t-s)\Delta} N(s, \cdot) ds \right\|_{L_t^q L_x^r} \lesssim \sum_{j \in \mathbb{Z}} 2^{j^2} \| \chi_j N \|_{L_t^q L_x^r},
\]

where \(\chi_j = \chi(2^j |x| \leq 2^{j+1})\).

We now discuss the needed conditions on the gauge potentials.
2.1. Conditions on the gauge potentials. The curvature, $F$, of the gauge potential $(A, A_0)$ is a two-form given by

$$F = \frac{1}{2} F_{\alpha \beta} dx^\alpha \wedge dx^\beta,$$

where $F_{\alpha \beta}$ is given in (2.2). From $F$ we can extract the magnetic field $dA$ by only considering the spatial coordinates of $F$ in (2.7). Similarly we can extract the electric field from the temporal-spatial components.

In 3 dimensions the magnetic field is often identified with a vector field curl$A$. It was observed in [12] that the trapping component, $B_\tau$, of the magnetic field given by

$$B_\tau = \frac{x}{|x|} \wedge \text{curl} A,$$

was an obstruction to the dispersion. This can be thought of as the tangential component of the magnetic field with respect to the unit sphere. In higher dimensions the trapping component can be rephrased as

$$B^T_\tau = \frac{x^T}{|x|} (F_{jk}),$$

where $(F_{jk})$ is a matrix with the $(j,k)$ entry given by $F_{jk}$. Thus the $k$’th entry of the vector $B_\tau$ is $\frac{x_j}{|x|} F_{jk}$.

Next, if we take the radial derivative of $A_0$ and decompose it into positive and negative parts

$$\partial_r A_0 = \left( \nabla A_0 \cdot \frac{x}{|x|} \right)_+ - \left( \nabla A_0 \cdot \frac{x}{|x|} \right)_-, $$

then the positive part can also affect dispersion [12]. The conditions that were used in [12] are

(2.8) $(A_0, A) \in C^1_{\text{loc}}(\mathbb{R}^n \setminus \{0\})$, $\Delta A$, $H = -\Delta A + A_0$ are self adjoint and positive on $L^2$, (2.9) $\text{div} A = 0$,

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(2.10) if $n = 3$, \[ \left( \frac{M + \frac{1}{2}}{M} \right)^2 \left\| x^{3/2} B_\tau \right\|_{L^2(S_r)}^2 + (2M + 1) \left\| x^2 (\partial_r A_0)_+ \right\|_{L^2(S_r)} < \frac{1}{2}, \]

if $n \geq 4$, $\left\| x^2 B_\tau \right\|_{L^\infty(S_r)}^2 + 2 \left\| x^3 (\partial_r A_0)_+ \right\|_{L^\infty(S_r)} < \frac{2}{3} (n-1) (n-3)$,

for some $M > 0$ (see [8, Remark 1.3]), and where

$$\|f\|_{L^p L^\infty(S_r)}^p = \int_0^\infty \sup_{|x|=r} |f|^p \, dr.$$  

With those assumptions Fanelli and Vega were able to show some weak dispersion properties of the solutions of the linear mNLS [12, Theorems 1.9 and 1.10]. The following is a part of Theorems 1.9 and 1.10 [12].

Theorem 2.2. [12] Let $\phi \in L^2, \Delta A \phi \in L^2$, $(A_0, A)$ satisfy (2.8)-(2.10), and let $\nabla_A u$ denote the projection of $\nabla A$ on the tangent space to the unit sphere $|x| = 1$, $\nabla_A^u = \nabla_A u -$
\[
\frac{x}{|x|} \left( \frac{x}{|x|} \cdot \nabla_A u \right),
\]
then

\[
\begin{align*}
\text{if } n = 3, & \quad \int_0^T \int_{\mathbb{R}^3} \left( \frac{\nabla_A e^{it\phi}}{|x|} \right)^2 dxdt + \sup_{R > 0} \frac{1}{R} \int_0^\infty \int_{|x| \leq R} \left| \nabla_A e^{it\phi} \right|^2 dxdt \\
& \quad + \int_0^\infty \sup_{R > 0} \frac{1}{R^2} \int_{|x| = R} \left| e^{it\phi} \right|^2 d\sigma dt \leq C \left\| (-\Delta)_{\frac{1}{2}} \phi \right\|_{L^2}^2,
\end{align*}
\]

(2.11)

and

\[
\begin{align*}
\text{if } n \geq 4, & \quad \int_0^T \int_{\mathbb{R}^n} \left( \frac{\nabla_A e^{it\phi}}{|x|} \right)^2 dxdt + \sup_{R > 0} \frac{1}{R} \int_0^\infty \int_{|x| \leq R} \left| \nabla_A e^{it\phi} \right|^2 dxdt \\
& \quad + \int_0^\infty \int_{\mathbb{R}^n} \frac{\left| e^{it\phi} \right|^2}{|x|^3} dxdt \leq C \left\| (-\Delta)_{\frac{1}{2}} \phi \right\|_{L^2}^2.
\end{align*}
\]

(2.12)

Following the proof of Theorem 2.2, we can establish analogs of these estimates for the nonlinear, defocusing mNLS.

**Corollary 2.3.** With the same assumptions as in Theorem 2.2 we have

\[
\begin{align*}
\text{if } n = 3, & \quad \int_0^T \int_{\mathbb{R}^3} \left( \frac{\nabla_A u}{|x|} \right)^2 dxdt + \sup_{R > 0} \frac{1}{R} \int_0^T \int_{|x| \leq R} \left( |\nabla_A u|^2 + G(|u|^2) \right) dxdt \\
& \quad \quad + \sup_{R > 0} \frac{1}{R^2} \int_0^T \int_{|x| = R} |u|^2 d\sigma dt \leq C \sup_{t \in [0,T]} \left\| (-\Delta)_{\frac{1}{4}} u(t) \right\|_{L^2}^2,
\end{align*}
\]

(2.13)

and

\[
\begin{align*}
\text{if } n \geq 4, & \quad \int_0^T \int_{\mathbb{R}^n} \left( \frac{\nabla_A u}{|x|} \right)^2 dxdt + \sup_{R > 0} \frac{1}{R} \int_0^T \int_{|x| \leq R} \left( |\nabla_A u|^2 + \frac{n-1}{2} G(|u|^2) \right) dxdt \\
& \quad \quad + \int_0^T \int_{\mathbb{R}^n} \frac{|u|^2}{|x|^3} dxdt \leq C \sup_{t \in [0,T]} \left\| (-\Delta)_{\frac{1}{4}} u(t) \right\|_{L^2}^2,
\end{align*}
\]

(2.14)

for any \( T \in (0, \infty) \), where \( u \) solves mNLS with a defocusing nonlinearity \( g(|u|^2)u \), \( G \geq 0 \) and satisfies \( G''(x) = xg''(x) \), and \( M \) is as in (2.10).

This follows immediately from Theorems 1.9 and 1.10 in [12] once we observe that the proofs of these theorems rely on the generalized virial identity. The virial identity [12, Theorem 1.2] is for the homogeneous equation, but the addition of the defocusing nonlinearity leads to an addition of a term (see Lemma 3.1 and Corollary 3.2) that is positive with \( a \) as in [12] and results in an identical proof as before.

### 2.1.1. Interaction Morawetz: curvature conditions.

In order to establish Theorem 1.1, in addition to conditions (2.8)-(2.10) we impose the following (compare with (2.10) and (2.24) below). Let

\[
C_j = \{ x: 2^j \leq |x| \leq 2^{j+1} \}.
\]

(2.15)

and we assume there is \( 0 < b < 1 \) satisfying the following:

\[
\sum_{j \in \mathbb{Z}} 2^j \sup_{C_j} |dA|^{2-2b} < \infty.
\]

(2.16)
For \( n = 3 \),
\[
\| |dA|^b| x| \|_{L^2_x L^\infty(S_r)} = \int_0^\infty \sup_{|x|=r} |x|^2 |dA|^2b \, dr < \infty,
\]
(2.17)
\[
\| |x|^2 \nabla A_0 \|_{L^1_x L^\infty(S_r)} = \int_0^\infty \sup_{|x|=r} |x|^2 |\nabla A_0| \, dr < \infty,
\]
(2.18)
and for \( n \geq 4 \)
\[
\| |x|^3 |dA|^b \|_{L^\infty_x} < \infty,
\]
(2.19)
\[
\| |x|^3 \nabla A_0 \|_{L^\infty_x} < \infty.
\]
(2.20)

Remark 2.4. Note that in comparison to (2.10), the assumptions are made on the whole curvature and not just the projected components. On the other hand, we do not require the curvature to be small in these norms as in (2.10), but merely to be bounded. In addition, the norms for the temporal component \( F_{0j} = -\partial_j A_0 \) are the same as (2.10) whereas the magnetic field \( dA \) is using now a slightly stronger norm.

Finally, we observe that for example \( |dA| \sim \langle x \rangle^{2+\epsilon} \) satisfies the conditions with \( b = \frac{3}{4} \).

2.1.2. Inhomogeneous Strichartz estimate: gauge potential conditions. Now, to establish the inhomogeneous Strichartz estimate, besides (2.8)-(2.10) we need additional conditions found in [8]. (We do not require here (2.16)-(2.20).) They are
\[
|A|^2 - 2iA \cdot \nabla + A_0 \in L^{\frac{2}{3}, \infty}, \quad A \in L^{n, \infty},
\]
(2.21)
\[
\|(A_0)_{+}\|_K < \infty,
\]
(2.22)
\[
\|(A_0)_{-}\|_K < \frac{\pi^{n/2}}{\Gamma(\frac{n}{2} - 1)},
\]
(2.23)
\[
\sum_{j \in \mathbb{Z}} 2^j \sup_{x \in C_j} |A| + \sum_{j \in \mathbb{Z}} 2^{2j} \sum_{x \in C_j} |A_0| < \infty,
\]
(2.24)
where \( \| \cdot \|_K \) is the Kato norm defined by
\[
\| f \|_K = \sup_{x \in \mathbb{R}^n} \int |f(y)| \frac{dy}{|x-y|^{n-2}},
\]
and where \( C_j \) is as in (2.15).

2.2. Magnetic Schrödinger Strichartz and other estimates used.

Theorem 2.5. [8] Let \( n \geq 3 \) and \( H = -\Delta_A + A_0 \). Suppose \( (A_0, A) \) satisfy (2.8)-(2.10) and (2.21)-(2.23), then
\[
\| H^{\frac{3}{4}} \phi \|_{L^q_x} \leq C_q \| \nabla \left| H^{\frac{3}{4}} \phi \right| \|_{L^q_x}, \quad 1 < q < 2n,
\]
(2.25)
\[
\| H^{\frac{3}{4}} \phi \|_{L^q_x} \geq c_q \| \nabla \left| H^{\frac{3}{4}} \phi \right| \|_{L^q_x}, \quad \frac{4}{3} < q < 4.
\]
(2.26)

As one consequence we have a boundedness of \( H^{-\frac{1}{4}} (\Delta_A)^{\frac{1}{4}} \) on \( L^2_x \) as follows. First apply (2.25) for an operator with \( A_0 = 0 \), and then (2.26) to get
\[
H^{-\frac{1}{4}} L^2_x \hookrightarrow (\Delta_A)^{-\frac{1}{4}} L^2_x,
\]
from which by duality we have,

\[-\Delta_A^{1/2} L^2 \to H^{1/2} L^2_x,
\]

and hence

\[
\|H^{-\frac{1}{2}}(-\Delta_A)^{\frac{1}{4}}\phi\|_{L^2_x} \lesssim \|(-\Delta_A)^{-\frac{1}{4}}(-\Delta_A)^{\frac{1}{4}}\phi\|_{L^2_x} = \|\phi\|_{L^2_x}.
\]

For future reference, we remark

\[
\|\nabla^1/2\phi\|_{L^2_x} \sim \|H^{1/4}\phi\|_{L^2_x} \sim \|(-\Delta_A)^{1/4}\phi\|_{L^2_x}.
\]

Next, from the proof of Theorem 2.5 we have

**Corollary 2.6.** With the same assumptions as in Theorem 2.5 we have

\[
\|H^{1/2}\phi\|_{L^q_t L^r_x} \leq C \|\nabla\phi\|_{L^q_t L^r_x}, \quad 1 < q < n,
\]

\[
\|\nabla\phi\|_{L^2} \leq C \|H^{\frac{1}{2}}\phi\|_{L^2}.
\]

**Proof.** For (2.28) interpolate (2.5) and (2.7) in [8]. (2.29) is (2.12) in [8].

The homogenous Strichartz estimate was established in [8].

**Theorem 2.7** (magnetic Schrödinger Strichartz, [8]). Let \( n \geq 3 \). If \((A_0, A)\) satisfy (2.8)-(2.10), (2.21)-(2.24), then for any Schrödinger admissible pair \((q, r)\), the following Strichartz estimates hold:

\[
\|e^{itH}\phi\|_{L^q_t L^r_x} \leq C \|\phi\|_{L^q_t L^r_x}, \quad \frac{2}{q} + \frac{n}{r} = \frac{n}{2}, \quad q \geq 2, \quad q \neq 2 \text{ if } n = 3,
\]

and if \( n = 3 \), then at the endpoint we have

\[
\|\nabla^1/2 e^{itH}\phi\|_{L^2_t L^6_x} \leq \|H^{1/2}\phi\|_{L^2}.
\]

In the proof of the inhomogeneous Strichartz estimate we rely on the Christ-Kiselev Lemma.

**Theorem 2.8** (Christ-Kiselev Lemma [1] and see [18, 32, 33]). Let \( X, Y \) be Banach spaces and suppose

\[T : L^p([a, b]; X) \to L^q([a, b]; Y),\]

where \(-\infty \leq a < b \leq \infty\) is an operator given by

\[Tf(t) := \int_a^t K(t, s)f(s)ds,\]

for some operator-valued kernel \( K(t, s) \) from \( X \) to \( Y \), and let \( T \) satisfy

\[
\|Tf\|_{L^q([a, b]; Y)} \leq C\|f\|_{L^p([a, b]; X)},
\]

where \( 1 \leq p < q \leq \infty \) and \( C > 0 \) is independent of \( f \). Now define

\[
\tilde{T}f(t) = \int_a^b K(t, s)\chi_{(a, t)}(s)f(s)ds = \int_a^t K(t, s)f(s)ds.
\]

Then

\[
\|\tilde{T}f\|_{L^q([a, b]; Y)} \leq 2^{\frac{2}{q} - \frac{2}{p}} - \frac{2}{1 - 2^{\frac{1}{q} - \frac{1}{p}}} C\|f\|_{L^p([a, b]; X)}.
\]
3. Local Conservation Laws and Virial Identity

Recall
\[ iD_t u + \Delta_A u = \mu g(|u|^2)u, \]
\[ u(0) = u_0, \]
where \( \mu \in \mathbb{R} \) and \( g \) is a real valued \( C^1 \) function such that \( g(0) = 0 \). For the convenience of the computations we write down an equivalent form of mNLS as
\[ (3.1) \]
\[ D_t u = i\Delta_A u - i\mu g(|u|^2)u. \]

The virial identity for the linear magnetic Schrödinger equations was already established in [12] with a potential \( V \) (which is \( A_0 \) in the above equation) satisfying
\[ \|Vu\|_{L^2_x} \leq (1 - \epsilon) \|\Delta_A u\|_{L^2_x} + C \|u\|_{L^2_x}, \quad \epsilon > 0. \]

We discuss local conservation laws.

3.1. Local conservation laws. Let \( G \) be a real valued function such that
\[ (3.2) \]
\[ G'(x) = x g'(x). \]

Define pseudo-stress energy tensors as
\[ (3.3) \]
\[ T_{00} = \frac{1}{2} |u|^2, \]
\[ (3.4) \]
\[ T_{j0} = \mathcal{I}m \{ \bar{u} D_j u \}, \]
\[ (3.5) \]
\[ T_{jk} = 2 \mathcal{R}e \{ D_j u \bar{D_k} u \} - \frac{1}{2} \delta_{jk} \Delta |u|^2 + \mu \delta_{jk} G(|u|^2), \]
for \( 1 \leq j, k \leq n \). We have the first local conservation law
\[ (3.6) \]
\[ \partial_\alpha T_{\alpha 0} = 0, \]
which can be checked easily as follows.
\[
\partial_t T_{00} = \mathcal{R}e \{ \bar{u} D_t u \} \quad \text{by \ (2.1)}
\]
\[ = \mathcal{R}e \{ \bar{u} (i\Delta_A u - i\mu g(|u|^2)u) \} \quad \text{by \ (3.1)}
\]
\[ = -\mathcal{I}m \{ \bar{u} \Delta_A u \} + 2 \mathcal{I}m \{ \mu g(|u|^2) |u|^2 \} \quad \text{(since \( \mathcal{R}e \{ iz \} = -\mathcal{I}m z, \ z \in \mathbb{C} \)}
\]
\[ = -\mathcal{I}m \{ \bar{u} \Delta_A u \}. \]

Now we compute
\[
\partial_j T_{j0} = \mathcal{I}m \{ \bar{D_j u D_j u} \} + \mathcal{I}m \{ \bar{u} \Delta_A u \} \quad \text{by \ (2.1)}
\]
\[ = \mathcal{I}m \{ \bar{u} \Delta_A u \}. \]

Hence \( \partial_\alpha T_{\alpha 0} = 0 \) as needed.

Next, we show we have
\[ (3.7) \]
\[ \partial_\alpha T_{j\alpha} = 2F_{0j} T_{00} + 2F_{kj} T_{k0} = 2F_{\alpha j} T_{\alpha 0}. \]
To establish (3.7) we compute
\[ \partial_0 T_{j0} = \text{Im} \{ D_i \bar{u} D_j u + \bar{u} D_i D_j u \} \quad \text{by (2.1)} \]
\[ = \text{Im} \{ (i \bar{\Delta} u - \imath \mu g(|u|^2)u) D_j u + \bar{u} F_{0j} u + \bar{u} D_j D_i u \} \quad \text{by (2.2)} \]
\[ = \text{Im} \{ (i \bar{\Delta} u - \imath \mu g(|u|^2)u) D_j u + \text{Im} \{ \bar{u} D_j (i \bar{\Delta} u - \imath \mu g(|u|^2)u) \} \quad \text{by (3.1)} \]
\[ = F_{0j} |u|^2 - \Re \{ \bar{\Delta} u D_j u - \bar{u} D_j (\Delta_A u) \} + \Re \{ \mu g(|u|^2) \bar{u} D_j u - \bar{u} D_j (\mu g(|u|^2)u) \}. \]

Since by (2.3)
\[ \Re \{ \mu g(|u|^2) \bar{u} D_j u - \bar{u} D_j (\mu g(|u|^2)u) \} = \Re \{ \mu g(|u|^2) \bar{u} D_j u - \bar{u} \mu \partial_j (g(|u|^2)u) - \bar{u} \mu g(|u|^2) D_j u \}
\[ = -\mu g'(|u|^2) |u|^2 \partial_j |u|^2, \]
we have
\[ \partial_0 T_{j0} = F_{0j} |u|^2 - \Re \{ \bar{\Delta} u D_j u - \bar{u} D_j (\Delta_A u) \} - \mu g'(|u|^2) |u|^2 \partial_j |u|^2. \]

Next observe
\[ \Delta |u|^2 = 2\partial_k \Re \{ \bar{u} D_k u \}
\[ = 2|\nabla_A u|^2 + 2\Re \{ \bar{u} \Delta_A u \}. \]

Hence
\[ \partial_k T_{jk} = 2\Re \{ D_k D_j u \bar{D}_k u + D_j u \bar{\Delta} u \} - \frac{1}{2} \partial_j \Delta |u|^2 + \mu \partial_j G(|u|^2) \]
\[ = 2\Re \{ D_k D_j u \bar{D}_k u + D_j u \bar{\Delta} u - \frac{1}{2} \partial_j (\bar{u} \Delta_A u) \} - \partial_j |\nabla_A u|^2 + \mu G'(|u|^2) \partial_j |u|^2. \]

Now
\[ \Re \{ D_k D_j u \bar{D}_k u \} = \Re \{ i F_{kj} u \bar{D}_k u + D_j u \bar{\Delta} u \}
\[ = \Re \{ i F_{kj} u \bar{D}_k u \} + \frac{1}{2} \partial_j |\nabla_A u|^2. \]

It follows
\[ \partial_k T_{jk} = 2\Re \{ i F_{kj} u \bar{D}_k u + D_j u \bar{\Delta} u - \frac{1}{2} \partial_j (\bar{u} \Delta_A u) \} + \mu G'(|u|^2) \partial_j |u|^2. \]

Combining and using (3.2) we have
\[ \partial_0 T_{j0} + \partial_k T_{jk} = F_{0j} |u|^2 - \Re \{ \bar{\Delta} u D_j u - \bar{u} D_j (\Delta_A u) \}
\[ + 2\Re \{ i F_{kj} u \bar{D}_k u + D_j u \bar{\Delta} u - \frac{1}{2} \partial_j (\bar{u} \Delta_A u) \}. \]

Since \( \partial_j (\bar{u} \Delta_A u) = D_j \bar{\Delta} A + \bar{u} D_j \Delta_A u \) from (2.1),
\[ \Re \{ -\bar{\Delta} u D_j u + \bar{u} D_j \Delta_A u + 2 D_j u \bar{\Delta} u - \partial_j (\bar{u} \Delta_A u) \} = 0, \]
and
\[ \partial_0 T_{j0} + \partial_k T_{jk} = F_{0j} |u|^2 - 2\text{Im} \{ F_{kj} u \bar{D}_k u \}
\[ = F_{0j} |u|^2 + 2 F_{kj} \text{Im} \{ \bar{u} D_k u \}
\[ = 2 F_{0j} T_{a0}, \]
as needed. We are now ready to proceed to the virial identity.
3.2. **Virial identity for mNLS.** Let \( a : \mathbb{R}^n \to \mathbb{R} \). Define (gauged) Morawetz action by

\[
M_a(t) = \int_{\mathbb{R}^n} \partial_j a T_{j0} dx.
\]

Note from Hölder’s inequality and the definition of \( T_{j0} \), we immediately have

\[
\sup_{[0,T]} M_a(t) \leq \| \nabla a \|_{L^\infty} \| u \|_{L^2_x} \| \nabla A u \|_{L^2_x}.
\]

This can be refined just like it was in the classical case in [6]. Using [12, Lemma 3.1] we have (we note the statement of the lemma gives \( \| u \|_{L^2_x} \), but the following can be deduced from the proof)

\[
\sup_{[0,T]} M_a(t) \leq C \| (-\Delta_A)^{\frac{1}{4}} u \|_{L^2_x}^2,
\]

if we assume \(|\nabla a|, |x| \Delta a\) to be bounded, which they always are in our case. Next, following [5] we obtain the following lemma.

**Lemma 3.1 (Generalized virial identity).** Let \( a : \mathbb{R}^n \to \mathbb{R} \), and \( u \) be a solution of (mNLS). Then

\[
M_a(T) - M_a(0) = \int_0^T \int_{\mathbb{R}^n} \left( 2\partial_j \partial_k a \Re(D_j u \overline{D_k u}) - \frac{\Delta^2 a}{2} |u|^2 + \mu \Delta a G(|u|^2) + 2\partial_j a F_{\alpha j} T_{\alpha 0} \right) dx dt.
\]

**Proof.** By (3.9), (3.7) and integration by parts,

\[
\partial_t M_a(t) = \int_{\mathbb{R}^n} \left( 2\partial_k \partial_j a T_{j0} + 2\partial_j a F_{\alpha j} T_{\alpha 0} \right) dx
\]

\[
= \int_{\mathbb{R}^n} \left( 2\partial_j \partial_k a \Re(D_j u \overline{D_k u}) - \frac{\Delta^2 a}{2} |u|^2 + \mu \Delta a G(|u|^2) + 2\partial_j a F_{\alpha j} T_{\alpha 0} \right) dx.
\]

(3.11) now follows by the fundamental theorem of calculus. \( \square \)

**Corollary 3.2.** If \( a \) is convex and \( \mu G(|u|^2) \geq 0 \) we can further conclude

\[
\int_0^T \int_{\mathbb{R}^n} 2\partial_j a F_{\alpha j} T_{\alpha 0} - \frac{\Delta^2 a}{2} |u|^2 \ dx dt \lesssim \sup_{[0,T]} |M_a(t)|.
\]

**Proof.** This is easy to see since if \( a \) is convex we can first show that

\[
\partial_j \partial_k a \Re(D_j u \overline{D_k u}) \geq 0.
\]

Indeed, we know if a function \( a : \mathbb{R}^n \to \mathbb{R} \) is convex then for \( X \in \mathbb{R}^n \),

\[
\partial_j \partial_k a X^j X^k \geq 0.
\]

We apply this twice to conclude (3.13). Define vectors \( X, Y \) by

\[
X^i = \Re D_i u \quad \text{for} \quad 1 \leq i \leq n,
\]

\[
Y^i = \Im D_i u \quad \text{for} \quad 1 \leq i \leq n.
\]

Next since for general \( z, w \in \mathbb{C} \),

\[
\Re(z \bar{w}) = \Re z \Re w + \Im z \Im w,
\]
we have
\[ \partial_j \partial_k a \Re(D_j u D_k u) = \partial_j \partial_k a X^j X^k + \partial_j \partial_k a Y^j Y^k \geq 0, \]
by (3.14). Finally since \( a \) is convex and the Hessian, \((H_{jk}) = (\partial_j \partial_k a)\) is positive-semidefinite, the trace, \( \text{tr}(H_{jk}) = \Delta a \geq 0 \), which implies
\[ \mu \int_{\mathbb{R}^n} \Delta a G(|u|^2) dx \geq 0, \]
and the result follows.

4. Interaction Morawetz Estimates

We end this section by a brief discussion of the conservation of mass and energy for the mNLS. From [12] we have
\[ \|e^{itH} \phi\|_{H^s} = \|f\|_{H^s}, \quad s \geq 0, \]
where \( \|f\|_{H^s} = \left\| H^{\frac{s}{2}} f \right\|_{L^2} \). This in particular implies conservation of mass and energy for the linear magnetic Schrödinger equations. In case of mNLS we have

**Lemma 3.3 (Conservation of mass and energy).** Let \( H = -\Delta + A_0 \) be self-adjoint and positive on \( L^2 \), \( F' = g \) and let \( u \) solve mNLS. Then for every \( t > 0 \)
\begin{align*}
(3.15) & \quad \|u(t)\|_{L^2} = \|u_0\|_{L^2}, \\
(3.16) & \quad \int_{\mathbb{R}^n} \left| H^{\frac{1}{2}} u(t) \right|^2 dx + \frac{\mu}{2} F(|u|^2) dx = \int_{\mathbb{R}^n} \left| H^{\frac{1}{2}} u(0) \right|^2 dx + \frac{\mu}{2} F(|u(0)|^2) dx.
\end{align*}

**Proof.** (3.15) follows by integrating in space \( \partial_t T_{00} + \partial_j T_{0j} = 0 \), and (3.16) by a direct computation using the equation. \( \Box \)

4.1. Proof of Theorem [1.1] using the commutator vector operators. The Morawetz action (3.9) for a tensor product of two solutions \( u_1 = u_2 = u \) with \( a = |x - y| \) can be
rewritten as
$$M(t) = \int_{\mathbb{R}^n \otimes \mathbb{R}^n} \partial_j a T_{j0} dxdy$$
$$= \int_{\mathbb{R}^n \otimes \mathbb{R}^n} \frac{x - y}{|x - y|} \cdot \mathcal{I}m \{ \bar{u}(x, t) \nabla u(x, t) \} |u(y, t)|^2 dxdy$$
$$- \int_{\mathbb{R}^n \otimes \mathbb{R}^n} \frac{x - y}{|x - y|} \cdot \mathcal{I}m \{ \bar{u}(y, t) \nabla u(y, t) \} |u(x, t)|^2 dxdy$$
$$= 2 \int_{\mathbb{R}^n \otimes \mathbb{R}^n} \frac{x - y}{|x - y|} \cdot \bar{p}(t, x) \rho(t, y) dxdy$$
$$= 4 \int_{\mathbb{R}^n \otimes \mathbb{R}^n} \frac{x - y}{|x - y|} \cdot \bar{p}(t, x) \rho(t, y) dxdy.$$  

Following [5] we use operators $|\nabla|^{-(n-1)}$ and $\bar{X}$ defined by
$$|\nabla|^{-(n-1)} f(x) = \int_{\mathbb{R}^n} \frac{1}{|x-y|} f(y) dy, \quad \bar{X} = [x; |\nabla|^{-(n-1)}],$$
so
$$\bar{X} f(x) = \int_{\mathbb{R}^n} \frac{x - y}{|x - y|} f(y) dy,$$
$$\langle \bar{F} \mid \bar{X} g \rangle = \int_{\mathbb{R}^n} \bar{F}(x) \cdot \bar{X} g(x) dx = -\langle \bar{X} \cdot \bar{F} \mid g \rangle.$$

Further, a computation shows
$$(\partial_j \bar{X}^k) f(x) = \int_{\mathbb{R}^n} \eta_{kj}(x, y) f(y) dy,$$
where
$$\eta_{kj}(x, y) = \delta_{k} |x - y|^2 - (x_j - y_j)(x_k - y_k),$$
and
$$\partial_j \bar{X}^j = n |\nabla|^{-(n-1)} + [x_j; R_j] = (n-1) |\nabla|^{-(n-1)}.$$  

The crucial observation made in [5] was that the derivatives of $\bar{X}$ are positive definite. Using the above operators we write
$$M(t) = 4 \langle x; |\nabla|^{-(n-1)} \rangle \rho(t) \mid \bar{p}(t) \rangle = 4 \langle \bar{X} \rho(t) \mid \bar{p}(t) \rangle.$$  

Then
$$\partial_t M(t) = 4 \langle \bar{X} \partial_t \rho(t) \mid \bar{p}(t) \rangle + 4 \langle \bar{X} \rho(t) \mid \partial_t \bar{p}(t) \rangle = I + II.$$  

By (4.4) and (4.1)
$$I = -4 \langle \partial_t \rho(t) \mid \bar{X} \cdot \bar{p}(t) \rangle = 4 \langle \partial_j p_j(t) \mid \bar{X} \cdot \bar{p}(t) \rangle = -4 \langle p_j(t) \mid \partial_j \bar{X}^k p_k(t) \rangle.$$  

And by (4.4) and (4.2)
$$II = 4 \langle \partial_k \bar{X}^j \rho(t) \mid \sigma_{jk} - \delta_{jk} \Delta \rho + \mu \delta_{jk} G(\rho) \rangle + 4 \langle X^j \rho(t) \mid 2 F_{o j} T_{o 0} \rangle$$
$$= 4 \langle \partial_k \bar{X}^j \rho(t) \mid \frac{1}{\rho} (p_j p_k + \partial_j \rho \partial_k \rho) - \delta_{jk} \Delta \rho + \mu \delta_{jk} G(\rho) \rangle + 4 \langle X^j \rho(t) \mid 2 F_{o j} T_{o 0} \rangle.$$  

It follows
$$\partial_t M(t) = P_1 + P_2 + P_3 + P_4 + P_5,$$
where
\[
P_1 = 4 \langle \frac{1}{\rho} \partial_j \rho \partial_k \rho \mid \partial_k X^j \rho(t) \rangle, \\
P_2 = 4 \langle \frac{1}{\rho} p_j p_k \mid \partial_k X^j \rho(t) \rangle - 4 \langle p_j \mid \partial_j X^k p_k(t) \rangle, \\
P_3 = 4 \langle (-\Delta \rho) \mid \partial_j X^j \rho(t) \rangle, \\
P_4 = 4 \langle \mu G(\rho) \mid \partial_j X^j \rho(t) \rangle, \\
P_5 = 8 \langle X^j \rho(t) \mid F_{\alpha j} T_{\alpha 0} \rangle.
\]

We discuss positivity of each term. This analysis is also the same as in [5], but the difference is that the momentum vector \( \vec{p} \) is now covariant, and we also have to address \( P_5 \). We sketch the details for \( P_1 \) through \( P_4 \) for completeness. Since \( \partial_j X^k \) is positive definite, \( P_1 \geq 0 \). For \( P_2 \) define the two point momentum vector
\[
\vec{J}(x,y) = \sqrt{\rho(y)} \frac{\vec{p}(x)}{\rho(x)} - \sqrt{\rho(x)} \frac{\vec{p}(y)}{\rho(y)}.
\]
Then (see [5] for details)
\[
P_2 = 2 \langle J^j J^k \mid \partial_j X^k \rangle \geq 0,
\]
since again \( \partial_j X^k \) is positive definite. For \( P_3 \) using \( -\Delta = |\nabla|^2 \),
\[
P_3 = 4(n-1) \langle (|\nabla|^2 \rho)(t) \mid |\nabla|^{-n-1} \rho(t) \rangle = (n-1) \| |\nabla|^{-\frac{n-3}{2}} |u|^2 \|^2_{L^2},
\]
and
\[
P_4 = \langle \mu G(\rho) \mid (\partial_j X^j) \rho(t) \rangle = (n-1) \langle \mu G(\rho) \mid |\nabla|^{-n-1} \rho(t) \rangle \geq 0
\]
as long as \( \mu G(\rho) \geq 0 \). Integrating in time we have,
\[
\int_0^T P_3 dt + \int_0^T P_5 dt \leq M(T) - M(0),
\]
so the estimate follows by (3.10) if we can handle the last term \( P_5 \).

We cannot expect \( P_5 \) to be positive (see the appendix). Examples when \( B_\tau = 0 \) were given in \([12]\) (note this still leaves the term involving \( F_{0j} \)). In general, as shown below, we can control \( P_5 \) by imposing the conditions (2.8)-(2.10) as they allow us to take advantage of the smoothing estimates proved in \([12]\). In addition, we also require (2.16)-(2.20).

4.2. \( P_5 \): Replacement of positivity condition by bounds on \( F \). Suppose (2.8)-(2.10) hold. Then
\[
\int_0^T P_5 dt = 8 \int_0^T \int_{\mathbb{R}^{2n}} \partial_j a F_{\alpha j} T_{\alpha 0} dx dy = 8 \int_0^T \int_{\mathbb{R}^{2n}} \frac{x_j - y_j}{|x - y|} F_{kj}(x) p_k(x) |u(y)|^2 dx dy dt \\
+ 8 \int_0^T \int_{\mathbb{R}^{2n}} \frac{x_j - y_j}{|x - y|} F_{0j}(x) |u(x)|^2 |u(y)|^2 dx dy dt = I + II.
\]
4.2.1. Estimates for \( n = 3 \). Choose \( 0 < b < 1 \) and let \( a = 1 - b \). Impose (2.16) - (2.18). Since \( \vec{p} = \text{Im} \{ \vec{u} \nabla_A u \} \) we get

\[
I \lesssim \int_0^T \int_{\mathbb{R}^6} |dA(x)| |u(x)| |\nabla_A u(x)| |u(y)|^2 \, dx \, dy \, dt
\]

\[
= \|u(0)\|^2_{L^2_6} \int_0^T \int_{\mathbb{R}^3} |dA(x)| |u(x)| |\nabla_A u(x)| \, dx \, dt
\]

\[
\lesssim \|u(0)\|^2_{L^2_6} \int_0^T \int_{\mathbb{R}^3} |dA(x)|^{2a} |\nabla_A u(x)|^2 \, dx \, dt + \|u(0)\|^2_{L^2_6} \int_0^T \int_{\mathbb{R}^3} |dA(x)|^{2b} |u(x)|^2 \, dx \, dt
\]

\[
= Ia + Ib.
\]

Next

\[
Ia = \|u_0\|^2_{L^2_2} \int_0^T \sum_{j \in \mathbb{Z}} \int_{C_j} |dA(x)|^{2a} |\nabla_A u(x)|^2 \, dx \, dt
\]

\[
\leq \|u_0\|^2_{L^2_2} \sum_{j \in \mathbb{Z}} \sup_{x \in C_j} 2^{j+1} |dA(x)|^{2a} \int_0^T \int_{C_j} \frac{|\nabla_A u(x)|^2}{2^{j+1}} \, dx \, dt
\]

\[
\leq \|u_0\|^2_{L^2_2} \sum_{j \in \mathbb{Z}} \sup_{x \in C_j} 2^{j+1} |dA(x)|^{2a} \left( \sup_R \int_0^T \int_{|x| \leq R} |\nabla_A u(x)|^2 \, dx \, dt \right)
\]

\[
\leq C \|u(0)\|^2_{L^2_2} \sup_{t \in [0,T]} \left( -\Delta_A \right)^{\frac{1}{4}} u(t) \|^{2}_{L^2_x},
\]

by (2.13) and (2.16).

\[
Ib = \|u_0\|^2_{L^2_2} \int_0^T \int_0^\infty \int_{|x| = R} R^2 |dA(x)|^{2b} \frac{|u(x)|^2}{R^2} \, d\sigma \, dR \, dt
\]

\[
\leq \|u_0\|^2_{L^2_2} \left( \int_0^\infty \sup_{|x| = R} |x|^2 |dA(x)|^{2b} dR \right) \left( \int_0^T \sup_{R > 0} \int_{|x| = R} \frac{|u(x)|^2}{R^2} \, d\sigma \, dR \right)
\]

\[
\leq C \|u_0\|^2_{L^2_2} \sup_{t \in [0,T]} \left( -\Delta_A \right)^{\frac{1}{4}} u(t) \|^{2}_{L^2_x},
\]
by (2.17) and (2.13). To estimate $II$ note that $A$ is independent in time and $F_{0j} = -\partial_j A_0$. Then

$$II \lesssim \int_0^T \int_{\mathbb{R}^n} |\nabla A_0(x)||u(x)|^2 |u(y)|^2 \, dx \, dy \, dt$$

$$= \|u_0\|_{L^2}^2 \int_0^T \int_{|x|=r} |\nabla A_0(x)||u(x)|^2 \, d\sigma \, dt$$

$$\leq \|u_0\|_{L^2}^2 \sup_{t \in [0,T]} \left\{ |\Delta_A \frac{1}{2} u(t)|^2 \right\}_{L^2_x},$$

by (2.13) and (2.18). The estimates for $n \geq 4$ are analogous.

4.2.2. *Estimates for $n \geq 4$.* Just as before, we write

$$I \leq I_a + I_b,$$

where $I_a$ is estimated using (2.16) and (2.14). For $I_b$ we have

$$I_b \leq \|u_0\|_{L^2}^2 \left( \sup_{|x|} |x|^2 |dA(x)|^{2b} \right) \left( \int_0^T \int_{\mathbb{R}^n} \frac{|u(x)|^2}{|x|^3} \, dx \, dt \right)$$

$$\leq C \|u_0\|_{L^2}^2 \sup_{t \in [0,T]} \left\{ |\Delta_A \frac{1}{2} u(t)|^2 \right\}_{L^2_x},$$

by (2.19) and (2.14). Next,

$$II \lesssim \|u_0\|_{L^2}^2 \int_0^T \int_{\mathbb{R}^n} |\nabla A_0(x)||u(x)|^2 \, dx \, dt$$

$$= \|u_0\|_{L^2}^2 \int_0^T \int_{\mathbb{R}^n} |x|^3 |\nabla A_0(x)| \frac{|u(x)|^2}{|x|^3} \, dx \, dt$$

$$\leq C \|u_0\|_{L^2}^2 \sup_{t \in [0,T]} \left\{ |\Delta_A \frac{1}{2} u(t)|^2 \right\}_{L^2_x},$$

by (2.20) and (2.14).

5. **Proof of the Inhomogeneous Strichartz Estimate, Theorem 1.3**

Let $N(t, x)$, $t \geq 0$ be a space-time function which is sufficiently regular and $u$ be the solution of (1.4). Note that

$$u(t) = \int_0^t e^{iH(t-s)} N(s, \cdot) \, ds =: \int_0^t K(t, s) N(s, \cdot) \, ds =: T \tilde{N},$$

by Duhamel’s principle and define $Tf = \int_0^\infty K(t, s) N(s, \cdot) \, ds$. By the Christ–Kiselev lemma,

$$\|u\|_{L^q_t L^r_x} = \|T \tilde{N}\|_{L^q_t L^r_x} \leq c \|T\|_{L^q \to L^p} \|N\|_{L^q_t L^p_x}.$$
So it is enough to show
\[ \|Tg\|_{L^q_t L^r_x} \leq c \|g\|_{L^q_t L^r_x}, \]
for any \( g \in L^q_t L^r_x \), \( q' < q \). From the definition of \( Tg \), Strichartz estimate and self-adjointness of \( H \),
\[ \|Tg\|_{L^q_t L^r_x} = \left\| e^{itH} \int_0^\infty e^{-isH} g(s, \cdot) ds \right\|_{L^q_t L^r_x} \]
\[ \leq c \left\| \int_0^\infty e^{-isH} g(s, \cdot) ds \right\|_{L^q_t L^r_x} \]
\[ = \sup_{\|\phi\|_{L^2_t} = 1} \langle \phi, \int_0^\infty e^{-isH} g(s, \cdot) ds \rangle \]
\[ \leq \|e^{isH}\phi\|_{L^q_t L^r_x} \|g\|_{L^q_t L^r_x} \]
and proof is completed.

6. Application to Magnetic Nonlinear Schrödinger equations

In this section, we show applications of previous estimates to global existence and scattering. For simplicity, we consider magnetic NLS with defocusing cubic nonlinearity in \( \mathbb{R}^{1+3} \).

(6.1) \[
Du_t + Hu = |u|^2 u, \]
\[ u(0) = u_0 \in H^1(\mathbb{R}^3). \]

We note that by virtue of (2.28) and (2.29) we have

(6.2) \[
\|u\|_{H^1(\mathbb{R}^3)} \sim \|u\|_{L^2(\mathbb{R}^3)} + \|H^{1/2} u\|_{L^2(\mathbb{R}^3)}. \]

To establish Theorem 1.2 we begin with a local theory (see [3, 9, 27, 25] for related works).

6.1. Local existence. Let \( Q = \|u_0\|_{H^1(\mathbb{R}^3)} \). Note that by (2.26), Theorem 2.7 and (2.25),

(6.3) \[
\|e^{itH} u_0\|_{L^1_t W^{1,4}_x(\mathbb{R}^3)} \leq C \|u_0\|_{H^1(\mathbb{R}^3)} = \delta, \]
where \( \delta = CQ \). We show a solution of (6.1) uniquely exists locally in time in the space
\[ X_{a,b} := \{ u \in C^0_t([0,T_0];H^1_x) \cap L^3_{t,[0,T_0]} W^{1,18}_x \mid \|u\|_{L^1_t W^{1,18}_x} \leq a \} \cap \{ u \in L^3_{t,[0,T_0]} H^1_x \mid \|u\|_{L^3_{t,[0,T_0]} H^1_x} \leq b \}, \]
where \( a = 2\delta, b = 4CQ, \) and \( C \) is the maximum of the constant 1, and the constants \( C \) that appear in the estimates below. Define the sequence of Picard iterates by
\[ u^0(t) = e^{itH} u_0 \quad \text{and} \quad u^{k+1}(t) = \Phi(u^k)(t), \quad k \geq 0, \]
where
\[ \Phi(u)(t) = e^{itH} u_0 + \int_0^t e^{i(t-s)H} |u(s)|^2 u(s) ds. \]
By (6.3) and Theorem 2.7
\[ \|u^0\|_{L^2_x H^1_x}^2 \leq \frac{a}{2} \quad \text{and} \quad \|u^0\|_{L^\infty_x L^2_x} \leq \frac{b}{4}, \]
and again by (2.26), Theorem 2.7 and (2.25),
\[ \|u^0\|_{L^\infty_x [0,T_0] L^2_x} \leq \frac{b}{4}. \]
Now suppose that for \( k \geq 0 \), \( u^k \in X_{a,b} \). Then by Theorem 2.7, Theorem 1.3 and Sobolev embedding,
\[ \|u^{k+1}\|_{L^\infty_x H^1_x} \leq \|\nabla e^{itH} u_0\|_{L^\infty_x L^2_x} + \|\nabla \int_0^t e^{i(t-s)H} |u^k(s)|^2 u^k(s) ds\|_{L^\infty_x L^2_x} \]
\[ \leq C \|H^{\frac{1}{2}} e^{itH} u_0\|_{L^2_x} + C \|H^{\frac{1}{2}} \int_0^t e^{i(t-s)H} |u^k(s)|^2 u^k(s) ds\|_{L^\infty_x L^2_x} \]
\[ \leq C \|H^{\frac{1}{2}} u_0\|_{L^2_x} + C \|H^{\frac{1}{2}} (|u^k|^2 u^k)\|_{L^\frac{18}{14}_x L^2_x}^{\frac{18}{14}} \]
\[ \leq C \|\nabla u_0\|_{L^2_x} + C \|\nabla (|u^k|^2 u^k)\|_{L^\frac{18}{14}_x L^2_x} \]
\[ \leq C Q + C \|\nabla u^k\|_{L^\infty_x L^2_x} \|u^k\|_{L^2_x}^2 \]
\[ \leq \frac{b}{4} + C b a^2. \]
Hence, if \( a \) is small enough, i.e.,
\[ a^2 \leq \frac{1}{4C}, \]
and using conservation of mass (3.15), we have
\[ \|u^{k+1}\|_{L^\infty_x H^1_x} \leq b. \]
Finally,
\[ \|\nabla |\tfrac{1}{2} u^{k+1}|\|_{L^\frac{18}{14}_x L^\infty_x} \leq \|\nabla |\tfrac{1}{2} e^{itH} u_0|\|_{L^\frac{18}{14}_x L^\infty_x} + \|\nabla |\tfrac{1}{2} \int_0^t e^{i(t-s)H} |u^k(s)|^2 u^k(s) ds|\|_{L^\frac{18}{14}_x L^\infty_x} \]
\[ \leq \frac{a}{2} + C \|H^{\frac{1}{2}} \int_0^t e^{i(t-s)H} |u^k(s)|^2 u^k(s) ds\|_{L^\frac{18}{14}_x L^\infty_x} \]
\[ \leq \frac{a}{2} + C \|H^{\frac{1}{2}} (|u^k|^2 u^k)\|_{L^\frac{18}{14}_x L^2_x}^{\frac{18}{14}} \]
\[ \leq \frac{a}{2} + C \|(|\nabla |\tfrac{1}{2} u^k||u^k|^2\|_{L^\frac{18}{14}_x L^2_x} + \|u^k\|_{L^\infty_x L^2_x} \|u^k\|_{L^\infty_x L^2_x} \]
\[ \leq \frac{a}{2} + C T^{\frac{1}{2}} \|\nabla |\tfrac{1}{2} u^k|^2\|_{L^\frac{18}{14}_x L^2_x} \|u^k\|_{L^\infty_x L^2_x} \]
\[ \leq \frac{a}{2} + C T^{\frac{1}{2}} a^2 b. \]
If we require

\[ T^\frac{1}{4} \leq \frac{1}{2Ca^b}, \]

then

\[ \| |\nabla|^{\frac{1}{2}} \Phi(u) \|_{L_1^4 L_3^\infty} \leq a, \]

which shows the sequence \( u^k \) belongs to \( X_{a,b} \). To show the sequence converges, we need to consider the differences. The estimates are similar, and we only show some of the details.

Let

\[ F(u) = |u|^2 u, \]

then we can write

\[ F(u) - F(v) = (u - v) \int_0^1 F_z(\lambda u + (1 - \lambda)v) d\lambda + (u - v) \int_0^1 F_z(\lambda u + (1 - \lambda)v) d\lambda. \]

Now consider

\[ \| |\nabla|^{\frac{1}{2}} (\Phi(u) - \Phi(v)) \|_{L_1^4 L_3^\infty} \leq C \| |\nabla|^{\frac{1}{2}} (F(u) - F(v)) \|_{L_1^4 L_3^\infty} \]

\[ \leq C \| |\nabla|^{\frac{1}{2}} ((u - v) \int_0^1 F_z(\lambda u + (1 - \lambda)v) d\lambda) \|_{L_1^4 L_3^\infty} \]

\[ + C \| |\nabla|^{\frac{1}{2}} ((u - v) \int_0^1 F_z(\lambda u + (1 - \lambda)v) d\lambda) \|_{L_1^4 L_3^\infty} = I + II. \]

\[ I \leq C \sup_{\lambda \in [0,1]} CT^\frac{1}{4} \| |\nabla|^{\frac{1}{2}} (u - v) \|_{L_1^4 L_3^\infty} \| F_z(\lambda u + (1 - \lambda)v) \|_{L_1^4 L_3^\infty} \]

\[ + C \sup_{\lambda \in [0,1]} CT^\frac{1}{4} \| u - v \|_{L_1^4 L_3^\infty} \| |\nabla|^{\frac{1}{2}} F_z(\lambda u + (1 - \lambda)v) \|_{L_1^4 L_3^\infty} \]

\[ \leq CT^{\frac{1}{4}} ab \| |\nabla|^{\frac{1}{2}} (u - v) \|_{L_1^4 L_3^\infty} \]

\[ + C \sup_{\lambda \in [0,1]} CT^\frac{1}{4} \| u - v \|_{L_1^4 L_3^\infty} \| |\nabla|^{\frac{1}{2}} (\lambda u + (1 - \lambda)v) \|_{L_1^4 L_3^\infty} \| \lambda u + (1 - \lambda)v \|_{L_1^4 L_3^\infty} \]

\[ \leq CT^{\frac{1}{4}} ab \| |\nabla|^{\frac{1}{2}} (u - v) \|_{L_1^4 L_3^\infty} \]

\[ \leq \frac{1}{2} \| |\nabla|^{\frac{1}{2}} (u - v) \|_{X_{a,b}}, \]

by (6.4). We obtain the same bounds for term II, and for the other norms in \( X_{a,b} \), which show the sequence of the iterates is Cauchy and hence it converges as needed.

6.2. Global existence. Let \( u \) be the solution of (6.1) obtained from local existence. First note that by (6.2), (3.15) and (3.16), the \( H^1 \) norm of \( u \) is bounded. Let \( Q \) be the supremum of \( H^1 \) norm of \( u \). Now, let \( [0,T_*) \) be the maximal time interval that solution exists and suppose \( T_* < \infty \). Observe, if the global existence fails, then we have an increasing sequence
\{t_n\}_{n \in \mathbb{N}} \text{ such that}

\begin{align}
\lim_{n \to \infty} t_n &= T_*, \\
\|e^{itH}u(t_n)\|_{L_{[t_n,T_\ast]}^3 W_x^{\frac{1}{2}\frac{18}{5}}} &> \delta,
\end{align}

where \(\delta\) is defined as in the local existence proof. We shall prove that this gives a contradiction. From the conserved quantities of the equation, \(\|u\|_{H_x^1}\) and \(\|u\|_{L_x^2}\) are uniformly bounded on \([0,T_\ast]\). We consider an interval \([t_0,T_\ast]\) so that \(T_\ast - t_0 < \epsilon\) for some \(\epsilon > 0\) which will be chosen later. Assume \(u_0 = u(t_0)\). Then by the Strichartz estimate

\[ \|\nabla^{\frac{1}{2}} u\|_{L_t^4 L_x^6} \lesssim \|\nabla^{\frac{1}{2}} u_0\|_{L_x^2} + \|(|\nabla^{\frac{1}{2}} u|^2) u\|_{L_t^4 L_x^{18}} \lesssim \|\nabla^{\frac{1}{2}} u_0\|_{L_x^2} + \epsilon \frac{1}{2} \|\nabla^{\frac{1}{2}} u\|_{L_t^4 L_x^{18}} \|u\|_{L_t^\infty L_x^5} \lesssim \|\nabla^{\frac{1}{2}} u_0\|_{L_x^2} + \epsilon \frac{1}{2} \|\nabla^{\frac{1}{2}} u\|_{L_t^4 L_x^{18}} \|u\|_{L_t^\infty L_x^5}. \]

Since \(\|\nabla u\|_{L_t^\infty L_x^2}\) is uniformly bounded, if \(\epsilon\) is chosen to be sufficiently small, a standard argument of the method of continuity gives \(\|\nabla^{\frac{1}{2}} u\|_{L_{[t_0,T_\ast]}^3 W_x^{\frac{1}{2}\frac{18}{5}}} = \|u_0\|_{L_{[t_0,T_\ast]}^3 W_x^{\frac{1}{2}\frac{18}{5}}} < \infty\).

Now suppose \(\{t_n\}\) is a sequence satisfying (6.5) and (6.6). Then by the triangle inequality, Strichartz estimate and Sobolev embedding

\[ \|e^{itH}u(t_n)\|_{L_{[t_n,T_\ast]}^3 W_x^{\frac{1}{2}\frac{18}{5}}} \leq \|u_0\|_{L_{[t_n,T_\ast]}^3 W_x^{\frac{1}{2}\frac{18}{5}}} + \|\nabla^{\frac{1}{2}} \int_{t_n}^t e^{i(t-s)H} |u(s)|^2 u(s) ds\|_{L_{[t_n,T_\ast]}^3 W_x^{\frac{1}{2}\frac{18}{5}}} \leq \|u_0\|_{L_{[t_n,T_\ast]}^3 W_x^{\frac{1}{2}\frac{18}{5}}} + C(T_\ast - t_n) \|\nabla^{\frac{1}{2}} u\|_{L_{[t_n,T_\ast]}^3 W_x^{\frac{1}{2}\frac{18}{5}}} \|u\|_{L_{[t_n,T_\ast]}^\infty L_x^5} \lesssim \|u_0\|_{L_{[t_n,T_\ast]}^3 W_x^{\frac{1}{2}\frac{18}{5}}} \|u\|_{L_{[t_n,T_\ast]}^\infty L_x^5}. \]

The right-hand side is less than \(\delta\) for sufficiently large \(n\), which is a contradiction to the assumption of \(t_n\).

### 6.3. Scattering

In this section we consider the question of scattering (asymptotic completeness). We take a point of view analogous to the classical NLS. Hence we set out to show that given a solution of the nonlinear mNLS, \(u\), there exists a solution of the linear mNLS, \(e^{itH}u_+\) such that the \(H_x^1\) norm of the difference of the two solutions goes to 0 as \(t \to \infty\) (note, due to (6.2) this also gives convergence of \(\|H_x^1(u - e^{itH}u_+)\|_{L_x^2}\)).

Now, following the classical NLS setup for scattering, let \(u\) be the solution to the cubic defocusing mNLS with initial data \(u_0 \in H_x^1\). We define

\[ u_+ = u_0 + \int_0^\infty e^{-isH}|u(s)|^2 u(s) ds. \]

The convergence in \(H_x^1\) of the difference of \(u\) and \(e^{itH}u_+\) is then immediate if we can show

\[ \int_0^\infty e^{-isH}|u(s)|^2 u(s) ds, \]
We prove (6.7) for $\hat{H}^1$ separately. For $L^2$, we need to show
\[
\inf \limits_{\|f\|_{L_x^2} \leq 1} \left\{ \int_t^\infty e^{-isH} |u(s)|^2 u(s) ds \right\}_{L_x^2} \to 0,
\]
as $t \to \infty$. Note that
\[
\int_t^\infty e^{-isH} |u(s)|^2 u(s) ds = \int_t^\infty e^{isH} f, |u(s)|^2 u(s) ds \leq e^{isH} f, |u(s)|^2 u(s) ds \leq \|e^{isH} f\|_{L^3_{(t,\infty) \times \mathbb{R}^3}}^2 \|u\|_{L_x^4}^{18} \|\hat{u}\|_{L_x^6}^{54} \leq \|f\|_{L_x^2} \|u\|_{L^4_{(t,\infty) \times \mathbb{R}^3}} \|u\|_{L_x^6}^{\frac{1}{2}}.
\]
We used interpolation inequality $\|u\|_{L^2_{x}L^\infty_t((0,\infty) \times \mathbb{R}^3)} \leq \|u\|_{L^4_{x}L^4_t((0,\infty) \times \mathbb{R}^3)} \|u\|_{L^\infty_tL_x^6}$.

**Lemma 6.1.** For a solution $u$ of the given equation, $\|u\|_{L^4_{x}L^6_t((0,\infty) \times \mathbb{R}^3)}$ is finite.

**Proof.** By Sobolev embedding,
\[
\|u\|_{L^4_{x}L^6_t((0,\infty) \times \mathbb{R}^3)} \lesssim \|\nabla \frac{1}{2} u\|_{L^3_{x}L^\infty_t} \approx \|\nabla \frac{1}{2} u\|_{L^\infty_tL_x^3}.
\]
We now subdivide $[0,\infty)$ into finitely many disjoint intervals $I_1, I_2, \ldots, I_M$ so that
\[
\cup_{k=1}^M I_k = [0,\infty),
\]
for some $\epsilon > 0$ which will be chosen later. On each interval $I_k$, we have
\[
\|\nabla \frac{1}{2} u\|_{L^3_{x}L^\infty_t} \leq \|\nabla \frac{1}{2} u_0\|_{L^3_x} + \|\nabla \frac{1}{2} u\|_{L^3_{x}L^\infty_t} \leq \|\nabla \frac{1}{2} u_0\|_{L^3_x} + \|\nabla \frac{1}{2} u\|_{L^3_{x}L^\infty_t} \leq \|\nabla \frac{1}{2} u_0\|_{L^3_x} + \epsilon \|\nabla \frac{1}{2} u\|_{L^3_{x}L^\infty_t} \leq \|\nabla \frac{1}{2} u_0\|_{L^3_x} + \epsilon \|\nabla \frac{1}{2} u\|_{L^\infty_tL_x^3} \leq \|\nabla \frac{1}{2} u_0\|_{L^3_x} + \epsilon \|\nabla \frac{1}{2} u\|_{L^\infty_tL_x^3} \leq \|\nabla \frac{1}{2} u_0\|_{L^3_x} + \epsilon \|\nabla \frac{1}{2} u\|_{L^\infty_tL_x^3}.
\]
We take small enough $\epsilon$ to apply the continuity method. Note that $\epsilon$ only depends on the implicit constant of the Strichartz estimate and the size of the initial data. By the method of continuity, we conclude $\|\nabla \frac{1}{2} u\|_{L^3_{x}L^\infty_t}$ is finite on each interval $I_k$. Since we have only finitely many intervals, $\|\nabla \frac{1}{2} u\|_{L^3_{x}L^\infty_t}$ is finite on $[0,\infty) \times \mathbb{R}^3$, and the result follows by (6.8).
Now it is easy to show scattering as follows.
\[
\left\| \int_{t}^{\infty} e^{-i s H} |u(s)|^2 u(s) ds \right\|_{L^2} \leq \left\| H^{\frac{1}{2}} \int_{t}^{\infty} e^{-i s H} |u(s)|^2 u(s) ds \right\|, 
\]
and
\[
\langle f, H^{\frac{1}{2}} \rangle \left\| \int_{t}^{\infty} e^{-i s H} |u(s)|^2 u(s) ds \right\|_{L^2}^2 = \langle e^{i s H} f, H^{\frac{1}{2}} (|u(s)|^2 u(s)) \rangle_{L^2_x} \left( t, \infty \times \mathbb{R}^3 \right) \leq \|e^{i t H} f\|_{L^4_t L^4_x} \|H^{\frac{3}{2}} (|u|^2 u)\|_{L^2_t L^2_x} \leq \|f\|_{L^2_x} \|\nabla |u|\|_{L^\infty_t L^2_x}^2 \|u\|_{L^4_t L^4_x}^2 \leq \|f\|_{L^2_x} \|\nabla |u|\|_{L^\infty_t L^2_x} \|u\|_{L^4_t L^4_x}^2 \|u\|_{L^2_x} \|u\|_{L^\infty_t L^2_x}^2 \.
\]
We used interpolation inequality \( \|u\|_{L^4_t L^4_x} \leq \|u\|_{L^3_t L^3_x}^4 \|u\|_{L^\infty_t L^2_x} \|u\|_{L^4_t L^4_x}^2 \). Since \( \|u\|_{L^3_t L^3_x} \) is finite, the last quantity vanishes as \( t \to \infty \) which completes the proof of scattering.

APPENDIX A. Failure of pointwise nonnegativity of \( \partial_x aF_{\alpha j} T_{\alpha 0} \)

Let \( x, y \in \mathbb{R}^3 \) and \( A \) be time independent, divergence-free. Then
\[
\partial_x aF_{\alpha j} T_{\alpha 0} = \partial_x a(x, y) F_{k j}(x)p_k(x)|u(y)|^2 + \partial_x a(x, y) F_{0 j}(x) |u(x)|^2 |u(y)|^2.
\]
And since \( F = * \text{curl} A \), the above formula is
\[
- \text{curl} A(x) \cdot (\nabla_a a(x, y) \times \vec{p}(x)) |u(y)|^2 - \nabla_a a(x, y) \cdot \nabla A_0(x) |u(x)|^2 |u(y)|^2.
\]
Since \( \nabla_a a(x, y) \) is parallel to \( x - y \) as long as \( a(x, y) = a(|x - y|) \), for any given \( x = x_0 \), we can find \( y \) so that \( \text{curl} A(x_0) \cdot (\nabla_a a(x_0, y) \times \vec{p}(x_0)) > 0 \). Similarly, we can find \( y \) so that \( \nabla A_0 \) and \( x - y \) form an angle less than \( \frac{\pi}{2} \).

Alternatively, we can view \( \partial_x aF_{\alpha j} T_{\alpha 0} \) as
\[
\partial_x aF_{\alpha j} T_{\alpha 0} = -\vec{p}(x) \cdot (\text{curl} A \times \nabla a) - |u(y)|^2 - \nabla_a a(x, y) \cdot \nabla A_0(x) |u(x)|^2 |u(y)|^2,
\]
and again as long as \( a(x, y) = a(|x - y|) \), then this is a dot product of the momentum vector with a component of \( \text{curl} A \) tangent to the unit sphere centered at \( y \) and the second term is the radial component of \( \nabla A_0 \) with respect to the sphere centered at \( y \) (compare to the trapping component in [12, 8]). Therefore, as we move \( y \) around, pointwise nonnegativity is not possible.

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