On the linear convergence of distributed Nash equilibrium seeking for multi-cluster games under partial-decision information

Min Meng, Xiuxian Li

School of Electrical and Electronic Engineering, Nanyang Technological University, 50 Nanyang Avenue, Singapore 639798

Abstract

This paper considers the distributed strategy design for Nash equilibrium (NE) seeking in multi-cluster games under a partial-decision information scenario. In the considered game, there are multiple clusters and each cluster consists of a group of agents. A cluster is viewed as a virtual noncooperative player that aims to minimize its local payoff function and the agents in a cluster are the actual players that cooperate within the cluster to optimize the payoff function of the cluster through communication via a connected graph. In our setting, agents have only partial-decision information, that is, they only know local information and cannot have full access to opponents’ decisions.

To solve the NE seeking problem of this formulated game, a discrete-time distributed algorithm, called distributed gradient tracking algorithm (DGT), is devised based on the inter- and intra-communication of clusters. In the designed algorithm, each agent is equipped with strategy variables including its own strategy and estimates of other clusters’ strategies. With the help of a weighted Frobenius norm and a weighted Euclidean norm, theoretical analysis is presented to rigorously show the linear convergence of the algorithm. Finally, a numerical example is given to illustrate the proposed algorithm.

Key words: Nash equilibrium seeking, multi-cluster games, partial-decision information, distributed gradient tracking algorithm.

1 Introduction

Game theory, which has been found to be a powerful tool to deal with optimization problems arising in multi-agent systems with the objective functions being coupled through decision variables of agents, has various applications including competitive markets [1], smart grids [2], transport systems [3], to name just a few. A challenging issue in games is to design strategies to find a Nash equilibrium (NE) corresponding to the desirable and stable state, from which no agents want to deviate. Some references, such as [4–6], made an assumption that each agent can access all the competitors’ decisions, which is impractical since a central node with bidirectional communication with all the players must exit in such case.

Therefore, in recent years, most scholars have focused on distributed algorithms for seeking Nash equilibria of noncooperative games composed of selfish decision-makers. For example, a payoff-based scheme was proposed in [7, 8], where each player is required to measure its cost function but not to communicate with others. In most circumstances, a player may not be aware of other players’ strategies, i.e., in a partial-decision information scenario. To handle such kind of partial-decision information scenarios, many results on the NE seeking problems were obtained both in continuous-time [9, 10] and in discrete-time [11–14], in which gradient and consensus based algorithms were designed to estimate other players’ strategies relying on local information. The algorithms in [11, 12] equipped with vanishing step sizes may have a slower convergence than those in [13, 14] where fixed-step schemes were applied. Another technique used in [15, 16] was to apply the operator theoretic theory since an NE can be characterized as a zero point of a monotone operator.

In contrast to noncooperative games, distributed optimization concerns a network of agents that collaborate to minimize the global cost function [17–22]. This problem is also an active research topic and has wide applications in resource allocation, machine learning, sensor networks, and energy systems [23]. Competition and cooperation among agents always coexist in many practical situations, such as healthcare networks [24], transportation networks [25] and so on. These practical situations may not be well modeled by only noncooperative games or distributed optimization problems. Inspired by the coexistence of competition and cooperation among agents, a multi-cluster (or multi-coalition) game was formulated in [26]. This game is conducted by multiple clusters (or coalitions), each of which is regarded as a virtual selfish player and aims to minimize its local payoff func-
tion. The agents in the same cluster are the actual players that cooperate within the cluster to optimize the payoff function of the cluster through communication via a connected graph. Then, a new NE seeking strategy was designed in [27] to reduce the communication and computation costs compared with that in [26]. In [28], the authors investigated the NE seeking problem of multi-cluster games with nmsn explicit expressions of the agents’ local objective functions by an extremum seeking-based approach. Furthermore, a generalized NE seeking strategy was given for multi-cluster games with nonsmooth payoff functions, a coupled of nonlinear inequality constraint and set constraints [29].

However, all the above existing distributed algorithms for seeking Nash equilibria of multi-cluster games are in continuous-time and under full-decision information, i.e., each agent has access to all agents’ decisions that influence its cost. As discrete-time algorithms are easily implemented in practical applications, in this paper, we aim to design a distributed discrete-time algorithm for seeking an NE of multi-cluster games under partial-decision information. In the studied multi-cluster game, the payoff function of a cluster is defined as the average sum of local payoff functions of its agents and every cluster designates a representative agent to interact with other representative agents from other clusters through an arbitrary connected network. With the aid of the available local information, each agent makes estimations of other clusters’ strategies and the gradient of its cluster’s payoff function at every iteration. Based on the inter- and intra-communication, a distributed gradient tracking algorithm (DGT) is devised to find the NE of the studied multi-cluster game. Under some mild conditions, by introducing a weighted Frobenius norm and a weighted Euclidean norm, the algorithm is rigorously proved to converge to the NE at a linear rate. Finally, we present a numerical example of Cournot Competition games to illustrate the developed algorithm.

The main contributions of this paper can be summarized as follows:

1) This paper investigates distributed NE seeking for multi-cluster games under partial-decision information. Compared with related works [26–29], where continuous-time algorithms were designed under full-decision information, this paper gives a discrete-time algorithm based on local information only.

2) The investigated problem includes distributed NE seeking of noncooperative games and distributed optimization as special cases. The designed algorithm is consistent with that in [14] on distributed NE seeking for noncooperative games when there is only one agent in every cluster, and with that in [19] on distributed optimization when only one cluster is evolved.

3) Rigorous convergence analysis is presented by defining a weighted Frobenius norm and a weighted Euclidean norm induced by a new Frobenius inner product and a new dot product, respectively. Moreover, the convergence of the devised algorithm can be achieved at a linear rate. The rest of this paper is organized as follows. In Section 2, the problem formulation is introduced. Section 3 presents the main result and the proof of main result is given in Section 4. Section 5 uses a numerical example to show the effectiveness of the proposed algorithm. Section 6 makes a brief conclusion.

**Notations.** Let \( \mathbb{R}, \mathbb{R}^n \) and \( \mathbb{R}^{m \times n} \) be the sets of real numbers, \( n \)-dimensional real column vectors and \( m \times n \) real matrices, respectively. For an integer \( n > 0 \), denote \( [n] := \{1, 2, \ldots, n\} \). \( I_n \) is the identity matrix of dimension \( n \). \( I_n \) (resp. \( 0_n \)) represents an \( n \)-dimensional vector with all of its elements being 1 (resp. 0).

For a vector or matrix \( A, A^\top \) denotes the transpose of \( A \) and \( \text{Row}_i(A) \) is the \( i \)-th row of \( A \). \( \rho(A) \) represents the spectral radius of \( A \) and \( \text{det}(A) \) is the determinant of \( A \). For real symmetric matrices \( P \) and \( Q \), \( P \succ (\succeq, \succeq, \preceq) Q \) means that \( P - Q \) is positive (positive semi-, negative, negative semi-) definite, while for two vectors/matrices \( w, v \) of the same dimension, \( w \preceq v \) means that each entry of \( w \) is no greater than the corresponding one of \( v \). \( A \otimes B \) denotes the Kronecker product of matrices \( A \) and \( B \). \( \text{diag}\{a_1, a_2, \ldots, a_n\} \) represents a diagonal matrix with \( a_i, i \in [n] \), on its diagonal. For a vector \( v \), we use \( \text{diag}(v) \) to represent the diagonal matrix with the vector \( v \) on its diagonal. Denote by \( \text{col}(z_1, \ldots, z_n) \) the column vector or matrix by piling up \( z_i, i \in [n] \). A matrix is consensual if its row vectors are the same. For any real vector space \( \mathcal{F} \), denote by \( \mathcal{F}^* \) the dual space of \( \mathcal{F} \). \( \langle w, v \rangle \) represents the inner product of \( w, v \) in \( \mathcal{F}^* \). If \( \mathcal{F} = \mathbb{R}^n \), the inner product is defined as \( \langle w, v \rangle := \sqrt{\langle Qv, v \rangle} \). A mapping \( g : \mathcal{F} \to \mathcal{F}^* \) is said to be strongly monotone with a constant \( \mu \) on \( \mathcal{F} \subseteq \mathcal{F}^* \) if \( \langle g(w) - g(v), w - v \rangle \geq \mu \|w - v\|^2 \) for any \( w, v \in \mathcal{F} \).

In this paper, the real vector spaces \( \mathcal{F} \) and \( \mathcal{F}^* \) are \( \mathcal{F} = \mathbb{R}^n \) or \( \mathcal{F} = \mathbb{R}^{n \times q} \) (or \( \mathbb{R}^{n \times \mathcal{F}^*} \)). If \( \mathcal{F} = \mathcal{F}^* = \mathbb{R}^n \), the inner product \( \langle w, v \rangle \) is the standard dot product in \( \mathbb{R}^n \). Denote by \( \| \cdot \|_F \) and \( \| \cdot \| \) the Frobenius norm and the Euclidean norm induced by the Frobenius inner product and the standard dot product, respectively, i.e., \( \|w\|_F := \sqrt{\langle w, w \rangle} \) and \( \|w\| := \sqrt{w^\top w} \) for \( v \in \mathbb{R}^{n \times q} \) (or \( \mathbb{R}^{n \times \mathcal{F}^*} \)) and \( w \in \mathbb{R}^n \).

An undirected graph, denoted as \( \mathcal{G} = (\mathcal{V}, \mathcal{E}, A) \), where \( \mathcal{V} = \{1, 2, \ldots, N\}, \mathcal{E} \subseteq \mathcal{V} \times \mathcal{V} \) and \( A = (a_{ij}) \in \mathbb{R}^{N \times N} \) represent the vertex set, the edge set and the weighted adjacency matrix of \( \mathcal{G} \), respectively. The weights are defined as \( a_{ij} > 0 \) if \( (i, j) \in \mathcal{E} \) and \( a_{ij} = 0 \) otherwise. \( a_{ii} > 0 \) for all \( i \in [N] \) in this paper. For an edge \( (i, j) \), \( i \) is called a neighbor of \( j \). Denote by \( \mathcal{N}_i \) the sets of the neighbors of node \( i \), i.e., \( \mathcal{N}_i = \{j : (j, i) \in \mathcal{E}\} \). A path from node \( i_1 \) to node \( i_\ell \) is composed of a sequence of edges \( (i_h, i_{h+1}), h = 1, 2, \ldots, \ell - 1 \). An undirected graph \( \mathcal{G} \) is said to be connected if for any vertices \( i, j \), there is a path from node \( i \) to node \( j \).
2 Problem formulation

This paper is concerned with the multi-cluster noncooperative game, which is conducted by m clusters. Each cluster \( i \in [m] \) is a virtual self-interested player and contains \( n_i \) agents communicating via an undirected graph \( \mathcal{G}_i = ([n_i], \mathcal{E}_i, A_i) \). In the meantime, each cluster designates a representative agent to interact with other representative agents from other clusters through an undirected communication topology \( \mathcal{G}_0 = ([m], \mathcal{E}_0, A_0) \). Without loss of generality, it is supposed that the representative agent in every cluster is the first agent. The number of all the agents in this game is \( n := \sum_{i=1}^{m} n_i \). The concepts of strategy variables and payoff functions of the multi-cluster game are given as follows.

- The strategy variable of agent \( j \) in cluster \( i \) is denoted as \( x_{ij} \in \mathbb{R}^{q_i} \). Let \( x_i := col(x_{i1}, \ldots, x_{in_i}) \in \mathbb{R}^{n_i q_i} \) be the strategy variable of cluster \( i \) and \( x_{-i} \) is the joint action of all the other clusters except that of \( i \), i.e., \( x_{-i} := col(x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_m) \). The strategy variable of this game is defined as \( x := col(x_1, \ldots, x_m) \in \mathbb{R}^N \), where \( N := \sum_{i=1}^{m} n_i q_i \). The agents in the same cluster cooperate to reach a common strategy that minimizes the payoff function of the cluster. However, all the agents only know local information from their neighbors through communications, i.e., under a partial-decision information setting.

- The payoff function of cluster \( i \), \( f_i : \mathbb{R}^N \to \mathbb{R} \), is defined as

\[
    f_i(x_i, x_{-i}) = \frac{1}{n_{i}} \sum_{j=1}^{n_i} f_{ij}(x_{ij}, \Gamma_i(x_{-i})),
\]

where \( f_{ij}(x_{ij}, \Gamma_i(x_{-i})) \) is only available to agent \( j \) in cluster \( i \) and \( \Gamma_i(x_{-i}) \in \mathbb{R}^{q_{-i}} \) with \( q := \sum_{i=1}^{m} q_i \) is the stacked strategies of the representative agents of all the clusters except that of cluster \( i \), i.e., \( \Gamma_i(x_{-i}) = col(x_{1,-i}, x_{2,-i}, \ldots, x_{m,-i}) \). Cluster \( i \in [m] \) aims to choose a strategy \( x_i := col(x_{i1}, \ldots, x_{in_i}) \) with \( x_{ij} = x_{il} \) for \( j, l \in [n_i] \) that minimize its own payoff function \( f_i(x_i, x_{-i}) \) under \( x_{-i} \).

Note that the strategies of agents in the same cluster are ensured to reach an agreement. A strategy profile \( (x^*_i, x^*_j) \) is called an NE of the formulated cluster game if \( x_{ij} = x_{jl} \) for all \( j, l \in [n_i] \), and for all \( i \in [m] \),

\[
    f_i(x^*_i, x^*_j) \leq f_i(x_i, x_{-i}^*), \quad x_i = 1_{n_i} \otimes y_i, \quad \forall y_i \in \mathbb{R}^{q_i}.
\]

The objective of this paper is to design a distributed discrete-time algorithm to find an NE of the studied multi-cluster game.

Remark 1 The formulated cluster game can model the coexistence of competition and cooperation simultaneously and subsume noncooperative games and distributed optimization as special cases. Specifically, if \( n_i = 1 \) for all \( i \in [m] \), the multi-cluster game is a noncooperative game among \( m \) players [11–16]. If \( m = 1 \), the considered problem is reduced to the distributed optimization problem, which has been investigated such as in [17–22].

To proceed, some standard assumptions are listed below.

Assumption 1 Graphs \( \mathcal{G}_0, \mathcal{G}_1, \ldots, \mathcal{G}_m \) are undirected and connected. All adjacency matrices \( A_0, A_1, \ldots, A_m \) are row and column stochastic, i.e., \( A_0 1_m = 1_m, \quad 1_m^T A_0 = 1_m^T, \quad A_i 1_{n_i} = 1_{n_i}, \quad 1_{n_i}^T A_i = 1_{n_i}^T, \quad i \in [m] \).

This assumption is standard in distributed discrete-time algorithms, such as distributed optimization, consensus, and NE seeking in noncooperative games [11, 14, 15, 19].

Assumption 2 For every \( j \in [n_i], i \in [m], \) local payoff function \( f_{ij}(x_{ij}, \Gamma_i(x_{-i})) \) is continuously differentiable and the gradient \( \nabla_i f_{ij}(x_{ij}, \Gamma_i(x_{-i})) := \frac{\partial f_{ij}(x_{ij}, \Gamma_i(x_{-i}))}{\partial x_{ij}} \) is Lipschitz continuous on \( \mathbb{R}^{q_i} \), i.e., for some constant \( L_{ij} > 0 \),

\[
    \| \nabla_i f_{ij}(x_{ij}, \Gamma_i(x_{-i})) - \nabla_i f_{ij}(\tilde{x}_{ij}, \Gamma_i(\tilde{x}_{-i})) \| \leq L_{ij} \| x_{ij} - \tilde{x}_{ij} \|, \quad \forall x_{ij}, \tilde{x}_{ij} \in \mathbb{R}^{q_i}.
\]

The condition in Assumption 2 is the \( L_{ij} \)-smooth condition of the local payoff function \( f_{ij} \), which is commonly used in distributed optimization, consensus, and NE seeking in noncooperative games [11, 14, 15, 19].

Denote for \( i \in [m] \),

\[
    g_i(x_i, x_{-i}) := col(\nabla_i f_{i1}(x_{i1}, \Gamma_i(x_{-i})), \ldots, \nabla_i f_{im}(x_{im}, \Gamma_i(x_{-i}))) \in \mathbb{R}^{n_i q_i}.
\]

The game mapping \( F : \mathbb{R}^N \to \mathbb{R}^N \) is defined as follows:

\[
    F(x) := col(g_1(x_1, x_{-1}), \ldots, g_m(x_m, x_{-m})).
\]

Assumption 3 The mappings \( \text{diag}(\frac{l_{11}}{\mu_1}, \ldots, \frac{l_{mm}}{\mu_2}) F(x) \) and \( F(x) \) are strongly monotone on the set \( \Omega := \{\text{col}(1_{n_1} \otimes y_1, \ldots, 1_{n_m} \otimes y_m) : \text{col}(y_1, \ldots, y_m) \in \mathbb{R}^{q_i}, i \in [m] \} \) with constants \( \mu_1 > 0 \) and \( \mu_2 > 0 \), respectively.

Assumption 3 is equivalent to that for any \( y = \text{col}(y_1, \ldots, y_m) \),

\[
    z = \text{col}(z_1, \ldots, z_m) \quad \text{with} \quad y_i, z_i \in \mathbb{R}^{q_i}, i \in [m],
\]

the following inequalities hold:

\[
    \sum_{i=1}^{m} \sum_{j=1}^{n_i} \frac{1}{n_{ji}} \langle \nabla_i f_{ij}(y) - \nabla_i f_{ij}(z), y_i - z_i \rangle \geq \mu_1 \sum_{i=1}^{m} \sum_{j=1}^{n_i} \|y_i - z_i\|^2 \geq \mu_1 \|y - z\|^2,
\]

(5)
representative agents, i.e., agents except that of cluster game, which is consistent with the condition ensuring the existence and uniqueness of the NE of the studied multi-noncooperative games in [14, 15].

Each agent $j$ in cluster $i$ maintains vector variables $x^t_{(ij)} = col(x^t_{(1)ij}, \ldots, x^t_{(m)ij}) \in \mathbb{R}^q$ and $v^t_{ij} \in \mathbb{R}^q$ at iteration $t$. **Initialization:** Initialize $x^0_{(ij)} \in \mathbb{R}^q$ arbitrarily and let $v^0_{ij} = \nabla_i f_{ij}(x^0_{(ij)})$.

**Iteration:** For $t \geq 0$, every agent $j$ in cluster $i$ processes the following update:

$$x^t_{ij} = \begin{cases} \frac{1}{2} \sum_{l=1}^{m} a^j_l x^t_{il} + \frac{1}{2} \sum_{h=1}^{m} a^h_j x^t_{(ij)h} - \alpha v^t_{ij}, & j = 1, \\
\sum_{l=1}^{m} a^j_l x^t_{il} - \alpha v^t_{ij}, & j \neq 1, \end{cases} \quad (10a)$$

$$x^t_{(ij)} = \begin{cases} \frac{1}{2} \sum_{l=1}^{m} a^j_l x^t_{(ij)l} + \frac{1}{2} \sum_{h=1}^{m} a^h_j x^t_{(ij)h}, & j = 1, s \neq i, \\
\sum_{l=1}^{m} a^j_l x^t_{il}, & j \neq 1, s \neq i, \end{cases} \quad (10b)$$

$$v^t_{ij} = \sum_{l=1}^{m} a^j_l v^t_{il} + \nabla_i f_{ij}(x^t_{ij}, x^t_{-(ij)}) - \nabla_i f_{ij}(x^t_{ij}, x^t_{-(ij)}), \quad (10c)$$

where $a^j_l$ is the $(i, h)$ element of $A_0$, $a^j_l$ is the $(j, l)$ element of $A_1$, $i \in [m]$, and $\alpha > 0$ is the step-size to be determined.

Algorithm 1 is designed relying on three parts: inter-cluster update mechanism, intra-cluster update mechanism and estimation of the gradients of local clusters’ payoff functions.

In Algorithm 1, the inter-cluster update mechanism is reflected at variables $x^t_{(ij)}, i \in [m]$, since only the first agents of clusters can communicate locally via communication topology $\mathcal{G}_0$. Meanwhile, the update of $x^t_{(ij)}, i \in [m]$ should also combine the intra-cluster communication via graph $\mathcal{G}_1, i \in [m]$. The non-representative agents only need to update their local variables by considering intra-cluster communication. That is, $x^t_{(ij)}$, for $j \neq 1, i \in [m]$ follow the intra-cluster update mechanism. $v^t_{ij}$ is an auxiliary variable to estimate the gradient of the payoff function of cluster $i$ at the estimated strategy of other representative agents, i.e., $\frac{1}{m} \sum_{j=1}^{m} \nabla_i f_{ij}(x^t_{ij}, x^t_{-(ij)})$. Every agent updates its local variables $x^t_{(ij)}$ and $v^t_{ij}$ only by local information. Thus this algorithm is distributed.

The convergence result of Algorithm 1 is presented in the following.

**Theorem** Under Assumptions 1–3, $x^t_{ij}$ generated by Algorithm 1 converges to $x^*_{ij}$ at a linear convergence rate, where $x^*$ is the NE of the multi-cluster game satisfying (7), if $0 < \alpha < \min\{\alpha^*, \frac{m-n}{2(m+1+n)}\}$, where $\alpha^* > 0$ is the smallest.
positive root of the equation \( \det(I_3 - \Phi(\alpha^*)) = 0 \) with \( \Phi(\alpha) \) being a matrix defined in (57).

**Proof.** The proof is postponed to Section 4.

**Remark 2** If there is only one cluster in the studied multi-cluster game, i.e., \( m = 1 \), then Algorithm 1 and Theorem 1 will reduce to the result for distributed optimization, which is consistent with that in [19]. If there is only one agent in each cluster, i.e., \( n_i = 1 \), \( i \in [m] \), then the studied game becomes a conventional noncooperative game in [11–14] and Algorithm 1 is the gradient-based algorithm in [15].

**Remark 3** In comparison, the existing algorithms for NE seeking of multi-cluster games in [26–29] were designed in continuous-time and under full-information, while Algorithm 1 here is a discrete-time algorithm under a partial-decision information setting, and meanwhile, an upper bound is provided for the step size \( \alpha \) to ensure a linear convergence rate.

### 4 Proof of the main result

In this section, the proof of Theorem 1 is given. For convenient analysis, the estimation matrix \( \mathbf{x}' \in \mathbb{R}^{n \times q} \) with \( n := \sum_{i=1}^{m} n_i \) is defined as

\[
\mathbf{x}' := \text{col}(\mathbf{x}'_1, \ldots, \mathbf{x}'_m),
\]

where

\[
\mathbf{x}'_i := \begin{bmatrix} (\mathbf{x}'_{i(1)})^\top \\ \vdots \\ (\mathbf{x}'_{i(n_i)})^\top \end{bmatrix} \in \mathbb{R}^{n_i \times q}, \ i \in [m].
\]

Similarly, denote \( \mathbf{v}'_i := \text{col}((v'_{i1})^\top, \ldots, (v'_{in_i})^\top) \in \mathbb{R}^{n_i \times q}, \ i \in [m] \), and \( V' := \text{diag}(\mathbf{v}'_1, \ldots, \mathbf{v}'_m) \in \mathbb{R}^{n \times n} \). Considering all the \( n \) agents in the multi-cluster game, the entire communication topology \( \mathcal{G} \) among the \( n \) agents is composed of graphs \( \mathcal{G}_0, \mathcal{G}_1, \ldots, \mathcal{G}_m \) with the vertex set being all the \( n \) agents and adjacency matrix \( \mathbf{A} \in \mathbb{R}^{n \times n} \) being given as

\[
\mathbf{A} = \begin{bmatrix}
\tilde{A}_1 + B_{11} & B_{12} & \cdots & B_{1m} \\
B_{21} & \tilde{A}_2 + B_{22} & \cdots & B_{2m} \\
\vdots & \vdots & \ddots & \vdots \\
B_{m1} & B_{m2} & \cdots & \tilde{A}_m + B_{mn}
\end{bmatrix},
\]

where \( \tilde{A}_i \in \mathbb{R}^{n_i \times n_i} \) is equal to the adjacency matrix \( A_i \) of \( \mathcal{G}_i \) except that the first row \( \text{Row}_1(\tilde{A}_i) \) is \( \frac{1}{2} \text{Row}_1(A_i) \) and \( B_{ij} \in \mathbb{R}^{n_i \times n_j} \) has the \((1,1)\) element as \( \frac{1}{2} a_{ij}^0 \) and other elements as \( 0, \ j \in [n_i], i \in [m] \). Then \( \mathbf{A} \) is a row stochastic matrix and has positive diagonal entries. By the above notations, (10) can be rewritten into a compact form as follows:

\[
\begin{aligned}
\mathbf{x}'_{i+1} &= \alpha \mathbf{x}'_i - \mathbf{A}\mathbf{v}', \\
\mathbf{v}'_{i+1} &= \mathbf{A}\mathbf{v}'_i + G_i(\mathbf{x}'_{i+1}) - G_i(\mathbf{x}'_i),
\end{aligned}
\]

where

\[
G_i(\mathbf{x}'_i) = \begin{bmatrix}
(\nabla_i f_{i1}(\mathbf{x}'_{i1}, \mathbf{x}'_{-(i1)})^\top \\
\vdots \\
(\nabla_i f_{in_i}(\mathbf{x}'_{in_i}, \mathbf{x}'_{-(in_i)})^\top
\end{bmatrix}, \ i \in [m].
\]

Note that it is required that all the agents in the same cluster take the same strategy when reaching the NE \( \mathbf{x}' \), namely, in cluster \( i \), the strategies \( x'_{ij} \) and \( x'_{j'i} \) of agent \( j \) and agent \( l \), respectively, should be equal. For ease of notations, we can denote the NE as \( \mathbf{x}' = \text{col}(\mathbf{x}'_{11}, \ldots, \mathbf{x}'_{m1}) \in \mathbb{R}^q \), where \( q = \sum_{i=1}^{m} q_i \). \( \mathbf{x}'_i \) is the strategy of agents in cluster \( i \) at the NE \( \mathbf{x}' \), i.e., \( \mathbf{x}'_i = x'_{ij} \) for \( j \in [n_i] \) and \( i \in [m] \). Then, the proof of Theorem 1 is equivalent to showing that the sequence \( \mathbf{x}' \) converges to a consensus matrix \( \mathbf{1}_n(\mathbf{x}')^{-1} \).

Before giving the detailed proof of Theorem 1, we first introduce some new concepts and lemmas which will be used in the sequel. Under Assumption 1, \( \mathbf{A} \) is a row stochastic matrix with a single eigenvalue 1. Then, \( \mathbf{A} \) has a left eigenvector \( \pi \in \mathbb{R}^n \) corresponding to eigenvalue 1 satisfying that \( \pi^\top \mathbf{A} = \pi^\top, \pi^\top \mathbf{1}_n = 1 \) and every element of \( \pi \) is positive. Indeed, by the structure of \( \mathbf{A} \) in (13), \( \pi \) can be obtained in the following lemma.

**Lemma 1** Under Assumption 1, the left eigenvector \( \pi \in \mathbb{R}^n \) of matrix \( \mathbf{A} \) in (13) corresponding to eigenvalue 1 such that \( \pi^\top \mathbf{A} = \pi^\top \) and \( \pi^\top \mathbf{1}_n = 1 \) is given as \( \pi = \text{col}(\pi^1, \ldots, \pi^m) \) with \( \pi^i \) being \( \pi^i = \text{col}(\frac{2}{n_i}, \frac{1}{n_i}, \ldots, \frac{1}{n_i}) \in \mathbb{R}^{n_i}, \ i \in [m] \). Moreover, \( \mathbf{1}_n, \pi^i \) for \( n_i \in [n] \). i.e., \( i \in [m] \).

**Proof.** By splitting the adjacency matrices \( A_i \) of \( \mathcal{G}_i \) as

\[
A_i = \begin{bmatrix}
A_{i1}^1 & A_{i1}^2 \\
A_{i2}^1 & A_{i2}^2
\end{bmatrix},
\]

where \( A_{i1}^1 \in \mathbb{R}, \ A_{i2}^1 \in \mathbb{R}^{1 \times (n_i - 1)}, \ A_{i2}^1 \in \mathbb{R}^{n_i - 1}, \ A_{i2}^2 \in \mathbb{R}^{(m-1) \times (n_i - 1)}, \) and \( i \in [m] \), one can rewrite \( A_i \) as

\[
\tilde{A}_i = \begin{bmatrix}
\frac{1}{2} A_{i1}^1 & \frac{1}{2} A_{i2}^1 \\
\frac{1}{2} A_{i1}^2 & A_{i2}^2
\end{bmatrix}, \ i \in [m].
\]

Accordingly, \( B_{hi} \) can be partitioned as

\[
B_{hi} = \begin{bmatrix}
\frac{1}{2} a_{ij}^0 & 0_{1 \times (n_i - 1)} \\
0_{n_i - 1} & 0_{(m-1) \times (n_i - 1)}
\end{bmatrix}, \ h, i \in [m].
\]
Let $\pi^i = (\pi^i_1, (\pi^i_2)^\top)^\top$, where $\pi^i_j \in \mathbb{R}$, $i \in [m]$. Then from $\pi^i :\mathcal{A} = \pi^i$, it can be determined that for $i \in [m]$,

$$(\pi^i)^\top A_i + \sum_{h=1}^{m} (\pi^h) B_{hi} = (\pi^i)^\top,$$

which is equivalent to

$$
\begin{bmatrix}
\pi^i_1, (\pi^i_2)^\top
\end{bmatrix} \begin{bmatrix}
\frac{1}{2} A_{i1} & \frac{1}{2} A_{i2} \\
\frac{1}{2} A_{i1}^\top & A_{i2}^\top
\end{bmatrix} + \sum_{h=1}^{m} \begin{bmatrix}
\pi^h_1, (\pi^h_2)^\top
\end{bmatrix} \begin{bmatrix}
\frac{1}{2} \sigma a^h_{00} & \frac{1}{2} \sigma a^h_{01} \\
\frac{1}{2} \sigma a^h_{10} & 0_{1 \times (n_1 - 1)}
\end{bmatrix} = \begin{bmatrix}
\pi^i_1, (\pi^i_2)^\top
\end{bmatrix}.
$$

(20)

Note that $A_i A_{n_i} = A_{n_i}$, then right multiplying $A_{n_i}$ on both sides of (20) yields that

$$
\begin{bmatrix}
\pi^i_1, (\pi^i_2)^\top
\end{bmatrix} \begin{bmatrix}
\frac{1}{2} I_{n_i-1} \\
\frac{1}{2} I_{n_i-1}
\end{bmatrix} + \sum_{h=1}^{m} \begin{bmatrix}
\pi^h_1, (\pi^h_2)^\top
\end{bmatrix} \begin{bmatrix}
\frac{1}{2} \sigma a^h_{00} \\
\frac{1}{2} \sigma a^h_{01} & 0_{1 \times (n_1 - 1)}
\end{bmatrix} = \begin{bmatrix}
\pi^i_1, (\pi^i_2)^\top
\end{bmatrix},
$$

that is,

$$\frac{1}{2} \pi^i + (\pi^i_2)^\top 1_{n_i-1} = \frac{1}{2} \sum_{h=1}^{m} \pi^h_1 a^h_{00} = \pi^i + (\pi^i_2)^\top 1_{n_i-1}, i \in [m].$$

Thus, $\sum_{h=1}^{m} \pi^h_1 a^h_{00} = \pi^i_1, i \in [m]$, from which one can see

$$\pi^1, \ldots, \pi^m)(A_0 = (\pi^1, \ldots, \pi^m).$$

(21)

Hence, for $i \in [m]$, $[\frac{1}{2} \sigma, (\pi^i_2)^\top]$ is a left eigenvector of $A_i$ corresponding to eigenvalue 1, thus, under Assumption 1, one has

$$\pi^i_2 = \frac{1}{2} \sigma 1_{n_i-1}, i \in [m].$$

(24)

Note that $1 = \pi \top 1$, then $1 = \sum_{h=1}^{m} (1 + \frac{1}{m}) \sigma = \frac{1}{m} \sigma$, indicating $\sigma = \frac{2}{m}$ and $\pi^i = col(\frac{2}{m}, \frac{1}{m}, \ldots, \frac{1}{m})$. The last claim can be easily verified and the proof is thus completed.

Based on $\pi = col(\pi_1, \ldots, \pi_n)$, two inner products, weighted dot product and weighted Frobenius inner product, are, respectively, defined as follows: for $x, y \in \mathbb{R}^n$ and $x, y \in \mathbb{R}^{n \times q}$,

$$\langle x, y \rangle := \langle \text{diag}(\pi)x, y \rangle, \quad \langle x, y \rangle := \langle \text{diag}(\pi)x, y \rangle.$$ 

(25)

(26)

Then the induced weighted Euclidean norm and Frobenius norm are, respectively,

$$\|x\|_\pi := \sqrt{\langle \text{diag}(\pi)x, x \rangle} = \|\text{diag}(\sqrt{\pi})x\|,$$

$$\|x\|_F := \|\langle \sqrt{\pi}x, x \rangle\|.$$ 

(27)

(28)

where $\sqrt{\pi} := col(\sqrt{\pi_1}, \ldots, \sqrt{\pi_n})$. We also denote by $\|B\|_\pi$ the matrix norm of matrix $B \in \mathbb{R}^{n \times n}$ induced by the weighted Euclidean norm. Then

$$\|B\|_\pi = \|\text{diag}(\sqrt{\pi})B\text{diag}(\sqrt{\pi})^{-1}\|.$$ 

(29)

The following results can be obtained.

**Lemma 2**

1. For two positive semi-definite matrices $P, Q \in \mathbb{R}^{n \times n}$, if $P \succeq Q$, then $\text{trace}(P) \geq \text{trace}(Q)$. Moreover, for any $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times q}$, we have $\|AB\|_F \leq \|A\|_F \|B\|_F$.

2. The Euclidean norm $\|\cdot\|$ and the weighted Euclidean norm $\|\cdot\|_\pi$ are equivalent. Specifically, for any $x \in \mathbb{R}^n$, there holds:

$$\sqrt{\pi_{\min}} \|x\| \leq \|x\|_\pi \leq \sqrt{\pi_{\max}} \|x\|,$$

(30)

where $\pi_{\min} := \min\{\pi_i, i \in [n]\} = \frac{1}{n}$ and $\pi_{\max} := \max\{\pi_i, i \in [n]\} = \frac{2}{m}$.  

3. The Frobenius norm $\|\cdot\|_F$ and the weighted Frobenius norm $\|\cdot\|_F$ are equivalent. Specifically, for any $x \in \mathbb{R}^{n \times q}$, the following inequality holds:

$$\sqrt{\pi_{\min}} \|x\|_F \leq \|x\|_F \leq \sqrt{\pi_{\max}} \|x\|_F.$$ 

(31)

4. Under Assumption 1, for any $x \in \mathbb{R}^n$ and $x \in \mathbb{R}^{n \times q}$, one has

$$\|\mathcal{A}x - \mathcal{A}x\|_\pi \leq \mathcal{A}\|x - \mathcal{A}x\|_\pi,$$

$$\|\mathcal{A}x - \mathcal{A}x\|_F \leq \mathcal{A}\|x - \mathcal{A}x\|_F,$$

(32)

(33)
where \( A_\infty := \mathbf{1}_n \pi^T \) and \( \sigma := \| A_\infty - A_\infty \|_\pi < 1 \).

**Proof.** 1) Note that for two positive semi-definite matrices \( P, Q \) if \( P \geq Q \), then \( P - Q \geq 0 \), implying that trace\( [P - Q] \geq 0 \). Therefore, trace\( [P] \geq \text{trace}[Q] \geq 0 \), that is, trace\( [P] \geq \text{trace}[Q] \). Moreover, for any \( A \in \mathbb{R}^{n \times n} \) and \( B \in \mathbb{R}^{n \times q} \), \( B^T A^T AB \leq \| A^T \| B^T B = \| A \| B^T B \), then

\[
\| AB \|_F = \sqrt{\text{trace}[B^T A^T AB]} \\
\leq \sqrt{\| A \|^2 \text{trace}[B^T B]} = \| A \| B^T B.
\]

2) By the notation in (27), for any \( x \in \mathbb{R}^n \),

\[
\sqrt{x_{\min}^2} \leq \| x \|_\infty = \sqrt{\sum_{i=1}^n x_i^2} \leq \sqrt{\sum_{i=1}^n x_i^2} = \sqrt{\text{trace} \left[ \text{diag}(\pi) x \right]}
\]

which implies that (30) holds.

3) Note that for any \( x \in \mathbb{R}^{n \times q} \),

\[
\pi_{\min} x^T x \preceq x^T \text{diag}(\pi) x \preceq \pi_{\max} x^T x.
\]

Then by the definition in (28) and 1) in this lemma, it holds that

\[
\sqrt{\text{trace} \left[ \pi_{\min} x^T x \right]} \leq \| x \|_2^2 = \sqrt{\text{trace} \left[ x^T \text{diag}(\pi) x \right]} \\
\leq \sqrt{\text{trace} \left[ \pi_{\max} x^T x \right]}.
\]

Then (31) is proved.

4) Referring to Lemma 1 in [30], one can obtain that (32) and \( \sigma = \| A_\infty - A_\infty \|_\pi < 1 \) hold. It suffices to prove (33). Since

\[
A_\infty x - A_\infty x = (A_\infty - A_\infty)(x - A_\infty x),
\]

where \( A_\infty A_\infty = A_\infty A_\infty = A_\infty A_\infty = A_\infty \) is used, one has that

\[
\| A_\infty x - A_\infty x \|_F^2 = \| \text{diag}(\sqrt{\pi}) (A_\infty - A_\infty)(x - A_\infty x) \|_F^2 \]

\[
= \| \text{diag}(\sqrt{\pi}) (A_\infty - A_\infty) \|_F^2 \| x - A_\infty x \|_F^2 \\
\leq \| \text{diag}(\sqrt{\pi}) (A_\infty - A_\infty) \|_F^2 \| x - A_\infty x \|_F^2 \\
= \sigma^2 \| x - A_\infty x \|_F^2,
\]

where the inequality is obtained based on 1) of this lemma. The proof is completed. \( \square \)

Several other necessary lemmas for proving Theorem 1 are listed as follows.

**Lemma 3** ([31], Theorem 1) For a complex matrix \( P_0 \) of size \( n \times n \), assume that \( P_0 \) has a simple eigenvalue \( \lambda_0 \). Denote by \( u, w \) the left and right eigenvectors of \( P_0 \) corresponding to the eigenvalue \( \lambda_0 \), respectively, which are normalized so that \( w^H u = 1 \), where \( w^H \) represents the conjugate transpose of \( u \). Let \( \tau_0 \) be a complex number and \( P(\tau) \) be a complex-valued function of a complex parameter \( \tau \) that is analytic in a neighborhood of \( \tau_0 \), satisfying \( P(\tau_0) = P_0 \). Then \( P(\tau) \) has a unique eigenvalue \( \lambda(\tau) \), which is analytic in a neighborhood of \( \tau_0 \) and satisfies

i) \( \lambda(\tau_0) = \lambda_0 \);

ii)

\[
\frac{d\lambda(\tau)}{d\tau} \bigg|_{\tau = \tau_0} = w^H \frac{dP(\tau)}{d\tau} \bigg|_{\tau = \tau_0} u,
\]

where \( \frac{d\lambda(\tau)}{d\tau} \bigg|_{\tau = \tau_0} \) and \( \frac{dP(\tau)}{d\tau} \bigg|_{\tau = \tau_0} \) are the derivatives of \( \lambda(\tau) \) and \( P(\tau) \) at \( \tau = \tau_0 \), respectively.

**Lemma 4** ([32]) Under Assumption 1, for any \( i \in [m] \),

\[
\sigma_i := \left\| A_i - \frac{1}{n_i} I_{n_i} \right\| < 1. \tag{34}
\]

**Lemma 5** ([33]) An irreducible nonnegative matrix \( M \in \mathbb{R}^{n \times n} \) is primitive if it has at least one non-zero diagonal entry.

**Lemma 6** ([33]) For an irreducible nonnegative matrix \( M \in \mathbb{R}^{n \times n} \), \( \rho(M) \) is an algebraically simple eigenvalue of \( M \).

Then the proof of Theorem 1 is given as follows.

**Proof of Theorem 1.**

Consider (15). Note that \( A_i \in \mathbb{R}^{n_i \times n_i} \) is a row and column stochastic matrix, then

\[
1_{n_i} v_i^{t+1} = 1_{n_i} v_i^t + \sum_{j=1}^{n_i} \left[ \nabla_i f_j (x_{ij}^{t+1}, x_{-ij}^{t+1}) \right]^T - \sum_{j=1}^{n_i} \left[ \nabla_j f_i (x_{ij}^{t+1}, x_{-ij}^{t+1}) \right]^T,
\]

which indicates that

\[
1_{n_i} v_i^{t+1} - \sum_{j=1}^{n_i} \left[ \nabla_i f_j (x_{ij}^{t+1}, x_{-ij}^{t+1}) \right] = 0_{1 \times q_i},
\]

since the initial value \( v_i^0 = \nabla_i f_j (x_{ij}^{0}, x_{-ij}^{0}) \), \( j \in [n_i] \). Therefore, for any \( i \in [m] \),

\[
1_{n_i} v_i^t = \sum_{j=1}^{n_i} \left[ \nabla_i f_j (x_{ij}^t, x_{-ij}^t) \right]^T. \tag{35}
\]
Denote
\[ \tau_i^* := (\tau^i)^\top \in \mathbb{R}^q, \] (36)
\[ \nu_i^* := \frac{1}{n_i} \sum_{j=1}^{n_i} \nu_i^j, \quad i \in [n], \] (37)
then \( \| \tau_i^* - \alpha_i \nu_i^* \|_F \). In the following, we investigate the bounds of norms \( \| \tau_i^* - \alpha_i \nu_i^* \|_F \), \( \| \| \tau_i^* - \alpha_i \nu_i^* \|_F \), and \( \sum_{i=1}^{n} \| \nu_i^* - \nu_i^j \|_F \) in four steps.

**Step 1.** For \( \| \tau_i^* - \alpha_i \nu_i^* \|_F \), by iteration (14), we have that
\[
\| \tau_i^* - \alpha_i \nu_i^* \|_F = \| \| \tau_i^* - \alpha_i \nu_i^* \|_F + \alpha \| (\mathbf{I} - \alpha_i \nu_i^* V_i^\top) \|_F \\
\leq \| \| \tau_i^* - \alpha_i \nu_i^* \|_F + \alpha \| (\mathbf{I} - \alpha_i \nu_i^* V_i^\top) \|_F \\
\leq \| \| \tau_i^* - \alpha_i \nu_i^* \|_F + \alpha \sqrt{\max_i} \| (\mathbf{I} - \alpha_i \nu_i^* V_i^\top) \|_F \leq \sigma \| \tau_i^* - \alpha_i \nu_i^* \|_F, \]
(38)
where the first inequality is based on the norm property, the second inequality depends on \( \alpha_i \nu_i^* = \alpha_i \nu_i^* \) and 3) in Lemma 2, and the third inequality is by (33) and 1) in Lemma 2.

**Step 2.** For \( \| \tau_i^* - \alpha_i \nu_i^* \|_F \), denote
\[ \tau_i^* := \frac{1}{n_i} \sum_{j=1}^{n_i} \left( \nabla f_i (\tau_i^*) \right)^\top, \quad i \in [m], \]
(39)
where \( \tau_i^* \) is defined in (36), then by iteration (14), one can get that
\[
\| \tau_i^* - \alpha_i \nu_i^* \|_F = \| \| \tau_i^* - \alpha_i \nu_i^* \|_F + \alpha \| (\mathbf{I} - \alpha_i \nu_i^* V_i^\top) \|_F \\
\leq \| \| \tau_i^* - \alpha_i \nu_i^* \|_F + \alpha \| (\mathbf{I} - \alpha_i \nu_i^* V_i^\top) \|_F \leq \| \| \tau_i^* - \alpha_i \nu_i^* \|_F + \alpha \sqrt{\max_i} \| (\mathbf{I} - \alpha_i \nu_i^* V_i^\top) \|_F \leq \sigma \| \tau_i^* - \alpha_i \nu_i^* \|_F, \]
(40)
where \( \alpha_i \nu_i^* = \alpha_i \nu_i^* \) and the norm property are applied. For the first term on the right side of (40), holds
\[
\| \tau_i^* - \alpha_i \nu_i^* \|_F = \| \| \tau_i^* - \alpha_i \nu_i^* \|_F + \alpha \| (\mathbf{I} - \alpha_i \nu_i^* V_i^\top) \|_F \\
\leq \| \| \tau_i^* - \alpha_i \nu_i^* \|_F + \alpha \| (\mathbf{I} - \alpha_i \nu_i^* V_i^\top) \|_F \leq \| \| \tau_i^* - \alpha_i \nu_i^* \|_F + \alpha \sqrt{\max_i} \| (\mathbf{I} - \alpha_i \nu_i^* V_i^\top) \|_F \leq \sigma \| \tau_i^* - \alpha_i \nu_i^* \|_F, \]
(41)
Denote \( \tau_i^* = \text{col}(\tau_1^*, \ldots, \tau_m^*) \) with \( \tau_i^* \in \mathbb{R}^q \), then
\[ \| \tau_i^* - \alpha_i \nu_i^* \|_F = \| \| \tau_i^* - \alpha_i \nu_i^* \|_F + \alpha \| (\mathbf{I} - \alpha_i \nu_i^* V_i^\top) \|_F \\
\leq \| \| \tau_i^* - \alpha_i \nu_i^* \|_F + \alpha \| (\mathbf{I} - \alpha_i \nu_i^* V_i^\top) \|_F \leq \| \| \tau_i^* - \alpha_i \nu_i^* \|_F + \alpha \sqrt{\max_i} \| (\mathbf{I} - \alpha_i \nu_i^* V_i^\top) \|_F \leq \sigma \| \tau_i^* - \alpha_i \nu_i^* \|_F, \]
(42)
where the fourth equality is obtained based on Lemma 1 and \( \sum_{j=1}^{n_i} \nabla f_i (\tau_i^*) = 0, \quad i \in [m] \), then (7), the inequality is based on Assumption 3, and the last equality relies on \( \| \| \tau_i^* - \alpha_i \nu_i^* \|_F \). In addition,
\[
\| \| \tau_i^* - \alpha_i \nu_i^* \|_F = \| \| \tau_i^* - \alpha_i \nu_i^* \|_F + \alpha \| (\mathbf{I} - \alpha_i \nu_i^* V_i^\top) \|_F \\
\leq \| \| \tau_i^* - \alpha_i \nu_i^* \|_F + \alpha \| (\mathbf{I} - \alpha_i \nu_i^* V_i^\top) \|_F \leq \| \| \tau_i^* - \alpha_i \nu_i^* \|_F + \alpha \sqrt{\max_i} \| (\mathbf{I} - \alpha_i \nu_i^* V_i^\top) \|_F \leq \sigma \| \tau_i^* - \alpha_i \nu_i^* \|_F, \]
(43)
where \( L = \max\{L_i, i \in [m], j \in [n_i]\} \), the first inequality is based on the definition of the Frobenius norm and 1) in Lemma 2, the second equality is by \( \sum_{j=1}^{n_i} \nabla f_i (\tau_i^*) = 0, \) and the third inequality is from Assumption 2. Then substituting (42) and (43) into (41), it can be derived that
\[
\| \tau_i^* - \alpha_i \nu_i^* \|_F = \| \| \tau_i^* - \alpha_i \nu_i^* \|_F + \alpha \| (\mathbf{I} - \alpha_i \nu_i^* V_i^\top) \|_F \\
\leq \| \| \tau_i^* - \alpha_i \nu_i^* \|_F + \alpha \| (\mathbf{I} - \alpha_i \nu_i^* V_i^\top) \|_F \leq \| \| \tau_i^* - \alpha_i \nu_i^* \|_F + \alpha \sqrt{\max_i} \| (\mathbf{I} - \alpha_i \nu_i^* V_i^\top) \|_F \leq \sigma \| \tau_i^* - \alpha_i \nu_i^* \|_F, \]
(44)
by \( \alpha_i = 1, \) and \( \alpha_i \nu_i^* = 1 \), one has
\[
\alpha_i \| \tau_i^* - \alpha_i \nu_i^* \|_F = \| \| \tau_i^* - \alpha_i \nu_i^* \|_F + \alpha \| (\mathbf{I} - \alpha_i \nu_i^* V_i^\top) \|_F \\
\leq \| \| \tau_i^* - \alpha_i \nu_i^* \|_F + \alpha \| (\mathbf{I} - \alpha_i \nu_i^* V_i^\top) \|_F \leq \| \| \tau_i^* - \alpha_i \nu_i^* \|_F + \alpha \sqrt{\max_i} \| (\mathbf{I} - \alpha_i \nu_i^* V_i^\top) \|_F \leq \sigma \| \tau_i^* - \alpha_i \nu_i^* \|_F, \]
(45)
For the second term on the right side of (40), we have
\[
\alpha \| \mathcal{A} \| \| \text{diag} \{ \varphi_1(x^*_1) - \varphi_1, \ldots, \varphi_m(x^*_m) - \varphi_m \} \|_F \\
\leq \alpha \| \mathcal{A} \| \| \text{diag} \{ \varphi_1(x^*_1) - \varphi_1, \ldots, \varphi_m(x^*_m) - \varphi_m \} \|_F \\
= \alpha \| \mathcal{A} \| \\
\times \| \text{diag} \left\{ \frac{1}{n_i} \sum_{j=1}^{n_i} \left[ \nabla_1 f_{ij}(x^*_1) - \nabla_1 f_{ij}(x^*_1) \right] \right\}_F, \\
\leq \alpha \| \mathcal{A} \| \| \sum_{i=1}^m \sum_{j=1}^{n_i} \left[ \nabla_1 f_{ij}(x^*_1) - \nabla_1 f_{ij}(x^*_1) \right] \|_F^2 \\
\leq \alpha \| \mathcal{A} \| \| \sum_{i=1}^m \sum_{j=1}^{n_i} L_{ij}^2 \| \| x^*_1 - x_1 \|_2^2 \\
\leq \alpha L \| \mathcal{A} \| \| \sum_{i=1}^m \sum_{j=1}^{n_i} \| x^*_1 - x_1 \|_2^2 \\
= \alpha L \| \mathcal{A} \| \| x^*_1 - x_1 \|_F^2 \\
\leq \alpha L \| \mathcal{A} \| \| x^*_1 - \mathcal{A} x^*_1 \|_F^2 \\
\leq \alpha L \| \mathcal{A} \| \| x^*_1 - \mathcal{A} x^*_1 \|_F^2, \\
\] (45)
where the first inequality is based on 1) in Lemma 2, the first equality is from (35), the third inequality is by Assumption 2, and the last inequality is obtained by 3) in Lemma 2. For the third term on the right side of (40), we get
\[
\alpha \| \mathcal{A} \| \| \text{diag} \{ \varphi_1, \ldots, \varphi_m \} - \mathcal{A} \mathbf{V} \|_F \\
\leq \alpha \| \mathcal{A} \| \| \text{diag} \{ \varphi_1 - \varphi_1, \ldots, \varphi_m - \varphi_m \} \|_F \\
= \alpha \| \mathcal{A} \| \sqrt{\sum_{i=1}^m \| \varphi_i - \varphi_i \|_2^2} \\
= \alpha \| \mathcal{A} \| \sqrt{\sum_{i=1}^m \| \nabla_i - \nabla_i \|_F^2} \\
\leq \alpha \| \mathcal{A} \| \sum_{i=1}^m \| \nabla_i - \nabla_i \|_F, \] (46)

Then, by substituting (44)–(46) into (40), we conclude that
\[
\| \mathcal{A} \| \| \mathbf{x}^{i+1} - \mathbf{x}_i \|_F \\
\leq \sqrt{1 - 2 \alpha (\mu_i + \mu_2) + \alpha^2 L \| \mathcal{A} \| \| \mathbf{x}^*_1 - \mathbf{x}_1 \|_F^2} \\
+ \alpha L \| \mathcal{A} \| \| \mathbf{x}^{i+1} - \mathbf{x}_i \|_F \\
+ \alpha \| \mathcal{A} \| \sum_{i=1}^m \| \nabla_i - \nabla_i \|_F. \] (47)

Step 3. For \( \sum_{i=1}^m \| \nabla_i - \nabla_i \|_F \), considering iteration (15), the iteration of \( \nabla_i \) is got as
\[
\nabla_i^{i+1} = \nabla_i + \frac{1}{n_i} \frac{1}{n_i} G_i(x^{i+1}_i) - \nabla_i - \frac{1}{n_i} \frac{1}{n_i} G_i(x^*_i), \] (48)
where \( \frac{1}{n_i} A_i = \frac{1}{n_i} \) is used. Then we obtain
\[
\| \nabla_i^{i+1} - \nabla_i \|_F \\
= \| A_i \nabla_i^{i+1} + G_i(x^{i+1}) - G_i(x^*_i) - \nabla_i - \frac{1}{n_i} \frac{1}{n_i} G_i(x^*_i) \|_F \\
+ \frac{1}{n_i} \frac{1}{n_i} G_i(x^*_i) \|_F \\
\leq \| A_i \nabla_i^{i+1} - \nabla_i \|_F + \frac{1}{n_i} \frac{1}{n_i} G_i(x^{i+1}) - G_i(x^*_i) \|_F \\
\leq \| A_i - \frac{1}{n_i} \frac{1}{n_i} G_i(x^*_i) \|_F \| \nabla_i^{i+1} - \nabla_i \|_F \\
\leq \| A_i - \frac{1}{n_i} \frac{1}{n_i} G_i(x^*_i) \|_F \| \nabla_i^{i+1} - \nabla_i \|_F \\
+ \| \sum_{j=1}^{n_i} \| \nabla_i f_{ij}(x^{i+1}_j) - \nabla_i f_{ij}(x^*_j) \|_F^2 \\
\leq \| \nabla_i^{i+1} - \nabla_i \|_F + \sum_{j=1}^{n_i} \| x^{i+1}_j - x^*_j \|_2 \\
\leq \| \nabla_i^{i+1} - \nabla_i \|_F + L \sum_{j=1}^{n_i} \| x^{i+1}_j - x^*_j \|_F, \] (49)
where the second equality is obtained based on \( A_i \frac{1}{n_i} = \frac{1}{n_i} \) and \( \frac{1}{n_i} \frac{1}{n_i} = 1 \), the second inequality is based on 1) in Lemma 2 and (16), the third inequality is from Lemma 4 and Assumption 2, and the last inequality hinges on the structure of \( x^*_j \) in (12). Therefore,
\[
\| \nabla_i^{i+1} - \nabla_i \|_F \\
\leq \sum_{i=1}^m \sigma_i \| \nabla_i^{i+1} - \nabla_i \|_F + L \sum_{i=1}^m \| x^{i+1}_j - x^*_j \|_F \\
\leq \sigma_{\max} \sum_{i=1}^m \| \nabla_i^{i+1} - \nabla_i \|_F + L \sqrt{m} \| x^{i+1} - x^*_2 \|_F, \] (50)
where \( \sigma_{\max} = \max \{ \sigma_i, i \in [m] \} \in (0, 1) \).

Step 4. In this step, we discuss the bounds of norms \( \| \nabla_i \|_F \) and \( \| x^{i+1} - x^*_2 \|_F \) appearing in (38) and (50), respectively.
By (14), we have

\[ \|x^{t+1} - x^t\|_F = \|Ax^t - \alpha V^t - x^t\|_F = \|((Ax - \alpha V) - x) - \alpha V^t\|_F \leq \|A - \alpha I\|_F \|x^t\|_F + \alpha \|V^t\|_F. \]  

(51)

where \(A = \alpha F\) and 1, 3) in Lemma 2 are used. Recalling \(V^t = \text{diag}(v_1^t, \ldots, v_m^t)\), it can be derived that

\[ \|V^t\|_F = \sqrt{\sum_{i=1}^m \text{trace}[(v_i^t)^\top v_i^t]} \]

\[ = \sqrt{\sum_{i=1}^m \|v_i^t\|_F^2} \leq \sum_{i=1}^m \|v_i^t\|_F. \]  

(52)

Note that

\[ \|v_i^t\|_F = \|v_i^t - v_i + v_i\|_F \leq \|v_i^t - v_i\|_F + \|v_i\|_F, \]  

(53)

and

\[ \|v_i^t\|_F = \sqrt{\text{trace}\left[\frac{1}{n_i^t}(v_i^t)^\top 1_n^t 1_n^t 1_n^t 1_n^t v_i^t\right]} \]

\[ = \frac{1}{\sqrt{n_i^t}} \|1_n^t v_i^t\| \]

\[ = \frac{1}{\sqrt{n_i^t}} \sum_{j=1}^{n_i} \left\|\nabla_i f_j(x_{ij}^t) - \nabla_i f_j(\bar{x})\right\| \]

\[ \leq \frac{1}{\sqrt{n_i^t}} \sum_{j=1}^{n_i} \left\|L_{ij} (x_{ij}^t - \bar{x})\right\| \]

\[ \leq \frac{1}{\sqrt{n_i^t}} L \sum_{j=1}^{n_i} \|x_{ij}^t - \bar{x}\| \]

\[ \leq L \|x_{ij}^t - 1_n(\bar{x})\|_F, \]  

(54)

seen from (52)–(54) that

\[ \|V^t\|_F \leq \sum_{i=1}^m \|v_i^t - v_i\|_F + L \sum_{i=1}^m \|x_{ij}^t - 1_n(\bar{x})\|_F \]

\[ \leq \sum_{i=1}^m \|v_i^t - v_i\|_F + L \sqrt{m} \|x^t - 1_n(\bar{x})\|_F \]

\[ \leq \sum_{i=1}^m \|v_i^t - v_i\|_F + L \sqrt{m} \pi_{\min}^{-0.5} \|x^t - A_\omega x^t\|_F \]

\[ + L \sqrt{m} \|A_\omega x^t - 1_n(\bar{x})\|_F. \]  

(55)

Denote \(\xi^t = (\|x^t - A_\omega x^t\|_F, \|A_\omega x^t - 1_n(\bar{x})\|_F, \sum_{i=1}^m \|v_i^t - v_i\|_F, \sum_{i=1}^m \|v_i^t\|_F)\top\), then by (38), (47), (50), (51), (55), the iteration of \(\xi^t\) satisfies

\[ \xi^{t+1} \leq \Phi(\alpha) \xi^t, \]  

(56)

where

\[ \Phi(\alpha) = \left[ \begin{array}{cccc} \sigma & \alpha a_{11} & \alpha a_{12} & \alpha a_{13} \\ \alpha a_{21} & \phi(\alpha) & \alpha a_{22} \\ a_1 + \alpha a_{31} & a_1 a_{32} & \sigma_{\max} + \alpha a_{33} \end{array} \right] \]  

(57)

with

\[ \phi(\alpha) = \sqrt{1 - \frac{2\alpha(\mu_1 + \mu_2)}{m + n} + \alpha^2 L^2} \]  

(58)

and \(a_{11} = \sqrt{\pi_{\max}} \|I_n - A_\omega\|_F \sqrt{m} \pi_{\min}^{-0.5}, a_{12} = \sqrt{\pi_{\max}} \|I_n - \alpha I\|_F \sqrt{m} \pi_{\min}^{-0.5}, a_{13} = \sqrt{\pi_{\max}} \|I_n - \alpha A_\omega\|_F \sqrt{m} \pi_{\min}^{-0.5}, a_{21} = L \|A_\omega\|_{\min}^{-0.5}, a_{22} = E^2 \frac{\pi_{\min}^{-0.5}}{m}, a_{31} = L_m \frac{\pi_{\min}^{-0.5}}{n} \), \(a_{32} = L^2 m, a_{33} = L \sqrt{m}\). The matrix \(\Phi(\alpha)\) is a continuous matrix function with respect to \(\alpha\), and when \(\alpha = 0\), \(\Phi(0)\) is

\[ \Phi(0) = \left[ \begin{array}{ccc} \sigma & 0 & 0 \\ 0 & 1 & 0 \\ a_1 & 0 & \sigma_{\max} \end{array} \right], \]  

(59)

whose spectral radius is 1 since \(\sigma, \sigma_{\max} \in (0, 1)\). Note that \(\Phi(0)\) has the left and right eigenvectors corresponding to eigenvalue 1 as \(u = (0, 1, 0)^\top\), then by Lemma 3, \(\Phi(\alpha)\) has a unique eigenvalue \(\lambda(\alpha)\) in the neighborhood of \(\alpha = 0\) satisfying

\[ \frac{d\lambda(\alpha)}{d\alpha} \bigg|_{\alpha=0} = \left[ u^\top \frac{d\Phi(\alpha)}{d\alpha} u \right] \bigg|_{\alpha=0} \]

\[ = \frac{-2(\mu_1 + \mu_2)}{m + n} + 2\alpha L^2 \|A_\omega\|_{\min}^{-2} \]

\[ = \frac{-2(\mu_1 + \mu_2)}{m + n} + 2\alpha^2 \|A_\omega\|_{\min}^{-2} \bigg|_{\alpha=0} \]

\[ = \frac{-\mu_1 + \mu_2}{m + n} < 0. \]  

(60)
Thus, the spectral radius of $\Phi(\alpha)$ is strictly smaller than 1 as $\alpha$ slightly increases from zero based on the continuity of the spectral radius. Then there must exist a positive number $\alpha^*$ such that $\rho(\Phi(\alpha^*)) < 1$ for all $\alpha \in (0, \alpha^*)$. To find $\alpha^*$, it can be seen that the graph associated with $\Phi(\alpha)$ when $\alpha > 0$ consisting of 3 agents is strongly connected, and then $\Phi(\alpha)$ is irreducible [34]. By Lemmas 5 and 6, $\rho(\Phi(\alpha))$ is a simple eigenvalue of $\Phi(\alpha)$ and all other eigenvalues have absolute values of less than $\rho(\Phi(\alpha))$. Therefore, $\alpha^*$ should be the smallest positive root of the equation $\det(1 - \Phi(\alpha^*)) = 0$. On the other hand, $1 - \frac{2m(m+n)}{m+n} > 0$, i.e., $\alpha \leq \frac{m+n}{2(m+n)}$. Therefore, for any $0 < \alpha < \max\{\alpha^*, \frac{m+n}{2(m+n)}\}$, the three entries of $\alpha^t$, $\|x' - \alpha x\|^2$, $\|\alpha x - x\|_F$, $\|\alpha x - x\|_F$, converge to zero in the order of $\rho(\Phi(\alpha)^t)$ as $t$ goes to infinity.

5 Example

In this section, we present a numerical example to illustrate our algorithm. To this end, a Cournot Competition game is considered as follows.

There are 5 father companies, which are regarded as clusters in the multi-cluster game, and each father company has 20 subsidiary companies, which are viewed as agents in clusters. The father companies compete with each other by adjusting the production quantity of goods. The subsidiary companies affiliated with the same father company produce components for this father company, cooperate to reach an agreement and meanwhile ensure that the profit of the father company is optimal. Assume that the subsidiary companies in the same father company $i$ can communicate through a connected graph $G_i$ and each father company appoints a subsidiary company to contact with other representative subsidiary companies from other father companies through another connected graph $G_0$. By regarding the quantities of goods as strategy variables, denote by $x_{ij}$ the quantity of goods of subsidiary company $j$ in father company $i$. The cost for producing components of goods and the price of components for per unit product by subsidiary company $j$ in father company $i$ are assumed to be $c_{ij} = 5x_{ij}^2 + 3x_{ij} + i$ and $p_{ij} = 6i - \sum_{h=1}^{5} A_{ih}^T x_{nj}$, respectively, where $A_0 = (a_{ij})$ is the adjacency matrix of graph $G_0$, $i, j \in \{1, 2, 3, 4, 5\}$. Then the payoff function of father company $i$ is $f_i = \sum_{j=1}^{20} f_{ij}$, where $f_{ij} = c_{ij} - \delta x_{ij} p_{ij}$.

This is a multi-cluster game studied in this paper. It can be verified that Assumptions 1–3 are satisfied. By (7) and a centralized method, the unique NE can be calculated as $x_{1j} = 3.9478$, $x_{2j} = 9.3400$, $x_{3j} = 14.7321$, $x_{4j} = 20.1243$, $x_{5j} = 25.5165$, where $j \in \{1, 2, 3, 4, 5\}$. By our proposed algorithm with $\alpha = 0.02$, the sequence $\{x_{ij}\}$, as well as the estimate sequence $\{x'_{(s)ij}\}$ of the strategy of cluster $s$ by agent $j$ in cluster $i$, converges to the unique NE, as shown in Figure 1.

![Fig. 1. Trajectories of $x'_{(s)ij}$, $s, i \in \{1, 2, 3, 4, 5\}$, $j \in \{1, 2, \ldots, 20\}$.

6 Conclusion

In this paper, we have studied the distributed NE seeking problem for a class of multi-cluster noncooperative games under a partial-decision information scenario. To design a distributed algorithm, every agent needs to make estimations of the strategies of other clusters at each iteration since each agent only has access to its local payoff function coupled with other clusters’ strategies and the neighbors’ information. Then based on the inter- and intra-communication of clusters, a distributed gradient tracking algorithm in discrete-time was devised to find the unique NE of the multi-cluster game. Rigorous convergence analysis with a linear convergence rate was provided by introducing a weighted Frobenius norm and a weighted Euclidean norm. To further study generalized NE seeking for the formulated multi-cluster noncooperative games with constrained action sets and inequality constraints under a partial-decision information scenario is an interesting future research direction.

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