Synergetic theory for jamming transition in traffic flow

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The theory of a jamming transition is proposed for the homogeneous car-following model within the framework of Lorenz scheme. We represent a jamming transition as a result of the spontaneous deviations of headway and velocity that is caused by the acceleration/braking rate to be higher than the critical value. The stationary values of headway and velocity deviations, and time of acceleration/braking are derived as functions of control parameter (time needed for car to take the characteristic velocity).

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1 Introduction

In recent years considerable study has been given to the traffic problems [1]. It is shown, in particular, that the jamming transition is similar to the conventional gas-liquid phase transition, where the freely moving traffic and the jammed traffic correspond to the gas and liquid phases respectively. The transition between them is caused by growth of car density above a critical value. The congested traffic flow with unstable uniform part leads to the formation of traffic jams where the freely moving traffic and jammed traffic coexists. Within the framework of Ref. [2], the jamming transition
is represented as a first-order phase transition, whose behavior is defined by
headway (car density) that acts as the volume (density) and by the inverted
delay time (sensitivity parameter) that reduces to temperature.

Our approach is to take into consideration the complete set of freedom
degrees as equivalent variables. We obtain the self-consistent analytical de-
scription of the jamming transition as result of the self-organization caused
by the positive feedback of the headway deviation and acceleration/braking
time – on the one hand, as well as the negative feedback of the deviations of
headway and velocity – on the other one.

The paper is organized as follows. In Section 2 the self-consistent Lorenz
system of the governed equations for the headway and velocity deviations
as well as for the acceleration/braking time is obtained. The jamming tran-
sition is shown to be supercritical in character (has the second order) if a
relaxation time for the first of pointed out quantities does not depend on its
value; it transforms to subcritical regime with this dependence appearance.
Section 3 deals with the determination of steady-state values for the headway
deviation and the acceleration/braking time within adiabatic approximation.
Out of the latter limit, the time dependencies for the headway and velocity
deviations are studied on the basis of the phase-portrait method. Section 4
contains a short discussion of used assumptions.

2 Basic equations

Within framework of the simplest car-following model the acceleration $\dot{V}$
of a given vehicle as a function of its distance $\Delta x$ to the front vehicle is
defined by equality $\dot{V} = [v_{opt}(\Delta x) - V]/\tau$, where $v_{opt}(\Delta x) = \Delta x/t_0$
is the optimal velocity function ($t_0$ being a characteristic time interval), $h = V t_0$
is the optimal headway and $\tau$ is the time of acceleration/braking needed for
car to take the optimal velocity. It is convenient to introduce deviations
$\eta \equiv \Delta x - h$ and $v \equiv \Delta \dot{x} - h/t_0 + V$ of headway $\Delta x$ and its velocity $\Delta \dot{x}$
from corresponding optimal values $h$ and $h/t_0 - V$. Then, the flow of cars
can be described in terms of the pointed out quantities $\eta$, $v$, and $\tau$. The key
point of our approach is that the above degrees of freedom are assumed to
be of dissipative type, so that, when they are not coupled, their relaxation to
the steady state is governed by the Debye-type equations with corresponding
relaxation times $t_\eta$, $t_v$, $t_\tau$. Within the simplest approach, equations for the
time dependencies $\eta(t)$, $v(t)$, and $\tau(t)$ are supposed to coincide formally with the Lorenz system that describes the self-organization process. 

The first of the stated equations has the form

$$\dot{\eta} = -\eta/t_\eta + v,$$

where the dot stands for a derivative with respect to time $t$. The first term in the right-hand side describes the Debye relaxation during time $t_\eta$, the second one is the usual addition. In a stationary state, when $\dot{\eta} = 0$, solution of Eq. (1) defines conventional linear relationship $\eta = t_\eta v$, so that the headway deviation is proportional to the velocity deviation.

The equation for the rate of quantity $v$ variation is supposed to have the nonlinear form

$$\dot{v} = -v/t_v + g_v \eta \tau,$$

where $t_v$, $g_v$ are positive constants. As in Eq. (1), the first term in the right-hand side of Eq. (2) describes the relaxation process of velocity deviation $v$ to the stationary value $v = 0$ determined by a time $t_v$. The second term describes the positive feedback of the headway deviation $\eta$ and the time $\tau$ of acceleration/braking on the velocity deviation $v$ that results in the increase of value $v$ and, thus, causes the self-organization process.

The kinetic equation for the acceleration/braking time $\tau$

$$\dot{\tau} = (\tau_0 - \tau)/t_\tau - g_\tau \eta v$$

differs from Eqs. (1), (2) as follows: the relaxation of quantity $\tau$ occurs not to the zero but to the finite value $\tau_0$, representing the stationary time needed for car to take the characteristic velocity (in other words, $\tau_0$ is the car characteristic); $t_\tau$ is a corresponding relaxation time. In Eq. (3) the negative feedback of the quantities $\eta$ and $v$ on $\tau$ is introduced to imply the decrease of acceleration/braking time $\tau$ with the growth of the headway and velocity deviations ($g_\tau > 0$ is a corresponding constant).

The equations (1), (2), (3) constitute the basis for self-consistent description of the car-following model driven by the control parameter $\tau_0$. The distinguishing feature of these equations is that nonlinear terms that enter Eqs. (2), (3) are of opposite signs, while Eq. (1) is linear. Physically, the latter means just that the velocity deviation is the derivative of headway deviation with respect to time. The negative sign of the last term in Eq. (3)
can be regarded as a manifestation of Le Chatelier principle, i.e. since an
decrease in the acceleration/braking time promote to the formation of a stable
car flow, the headway and velocity deviations $\eta$ and $v$ tend to impede
the growth of the acceleration/braking time and, as a consequence, the jamming.
The positive feedback of $\eta$ and $\tau$ on $v$ in Eq. (2) plays an important part
in the problem. As we will see later, it is precisely the reason behind the
self-organization that brings about the traffic jam.

To explain the relaxation transition to the stable jamming state, we will
show further that it is quite enough to use the adiabatic approxima-
tion: $t_v = 0, t_\tau = 0$. Therefore we could proceed not from Eqs. (2), (3) but from
much simple expressions

$$v = a_v \eta \tau, \quad a_v \equiv t_v g_v; \quad \tau = \tau_0 - a_\tau \eta v, \quad a_\tau \equiv t_\tau g_\tau,$$  

which are related to the stationary case $\dot{\eta} = 0, \dot{\tau} = 0$ in Eqs. (4), (5)
respectively. The equalities (4) have absolutely clear physical meaning: the
increase of the headway deviation $\eta$ or acceleration/braking time $\tau$ leads to
growth of the velocity deviation $v$, whereas the increase of the headway $\eta$
and velocity $v$ deviations should cause the decrease of acceleration/braking
time $\tau$ in comparison with characteristic time $\tau_0$ if the car flow is not broken.

After introducing the suitable scales for quantities $\eta$, $v$, $\tau$:

$$\eta_m \equiv (a_v a_\tau)^{-1/2}, \quad v_m \equiv \eta_m / t_\eta = t_\eta^{-1} (a_v a_\tau)^{-1/2}, \quad \tau_c \equiv (t_\eta a_v)^{-1},$$  

Eqs. (1), (2), (3) can be rewritten in the simplest form of the well–known
Lorenz system:

$$\dot{\eta} = -\eta + v, \quad (6)$$
$$\epsilon \dot{v} = -v + \eta \tau, \quad (7)$$
$$\delta \dot{\tau} = (\tau_0 - \tau) - \eta v, \quad (8)$$

where the relaxation times ratios $\epsilon \equiv t_v / t_\eta, \delta \equiv t_\tau / t_\eta$ are introduced and
the dot now stands for the derivative with respect to the dimensionless time
$t / \tau_\eta$. In general, the system (6) – (8) can not be solved analytically, but
in the simplest case $\epsilon \ll 1$ and $\delta \ll 1$, the left-hand sides of Eqs. (7), (8)
can be neglected. Then, the adiabatic approximation can be used to express
the velocity deviation $v$ and the acceleration/braking time $\tau$ in the form of
the equalities (4). As a result, the dependencies of $\tau$ and $v$ on the headway deviation $\eta$ are given by

$$
\tau = \frac{\tau_0}{1 + \eta^2}, \quad v = \frac{\tau_0 \eta}{1 + \eta^2}.
$$

(9)

Note that, under $\eta$ is in the physically meaningful range between 0 and 1, the acceleration/braking time is a monotonically decreasing function of $\eta$, whereas the velocity deviation $v$ increases with $\eta$ (at $\eta > 1$ we have $dv/d\eta < 0$ that has no physical meaning).

Substituting second equality (9) into Eq. (6) yields the Landau–Khalatnikov relation:

$$
\dot{\eta} = -\frac{\partial \Phi}{\partial \eta}
$$

(10)

with the effective potential given by

$$
\Phi = \frac{1}{2} \eta^2 - \frac{1}{2} \tau_0 \ln \left(1 + \eta^2\right).
$$

(11)

For $\tau_0 < 1$, the $\eta$–dependence of $\Phi$ is monotonically increasing and the only stationary value of $\eta$ equals zero, $\eta_c = 0$, so that there is no headway deviations in this case. If the parameter $\tau_0$ exceeds the critical value, $\tau_c = 1$, the effective potential assumes the minimum with non–zero steady state headway deviation $\eta_c = \sqrt{\tau_0 - 1}$ and the acceleration/braking time $\tau_e = 1$.

The above scenario represents supercritical regime of the traffic jam formation and corresponds to the second–order phase transition. The latter can be easily seen from the expansion of the effective potential (11) in power series of $\eta^2 \ll 1$:

$$
\Phi \approx \frac{1 - \tau_0}{2} \eta^2 + \frac{\tau_0}{4} \eta^4.
$$

(12)

So the critical exponents are identical to those obtained within the framework of the mean-field theory [4].

The drawback of the outlined approach is that it fails to account for the subcritical regime of the self–organization that is the reason for the appearance of traffic jam and analogous to the first–order phase transition, rather than the second–order one. So one has to modify the above theory by taking the assumption that the effective relaxation time $t_\eta(\eta)$ increases with headway deviation $\eta$ from initial value $t_\eta/(1 + m)$ fixed by a parameter $m > 0$ to
the final one $t_\eta$ \[\text{?}\]. The simplest two–parameter approximation is as follows:

$$\frac{t_\eta}{t_\eta(\eta)} = 1 + \frac{m}{1 + (\eta/\eta_0)^2}, \quad (13)$$

where $0 < \eta_0 < 1$. The expression for the effective potential \[\text{?}\] then changes by adding the term

$$\Delta \Phi = \frac{m^2}{2 \eta_0^2} \ln \left(1 + \frac{\eta^2}{\eta_0^2}\right) \quad (14)$$

and the stationary values of $\eta$ are

$$\eta_e^m = \eta_0 \left\{ 1 \mp \left[ 1 + \eta_0^2 \eta_0^{-4} (\tau_0 - \tau_e) \right]^{1/2} \right\}^{1/2}, \quad (15)$$

$$2\eta_0^2 \equiv (\tau_0 - 1) - \tau_e \eta_0^2, \quad \tau_e \equiv 1 + m.$$

The upper sign in the right-hand side of Eq. \[\text{?}\] is for the value at the unstable state $\eta^m$ where the effective potential $\Phi + \Delta \Phi$ has the maximum, the lower one corresponds to the stable state $\eta_e$. The corresponding value of the stationary acceleration/braking time

$$\tau^m = \frac{1 + \eta_0^2 + \sqrt{(1 + \eta_0^2)^2 - (1 - \eta_0^2) \tau_0}}{1 - \eta_0^2} \quad (16)$$

smoothly increases from the value

$$\tau_m = 1 + \eta_0 \sqrt{\frac{m}{1 - \eta_0^2}} \quad (17)$$

at the parameter $\tau_0 = \tau_{c0}$ with

$$\tau_{c0} = \left(1 - \eta_0^2\right) \tau_m^2 \quad (18)$$

to the marginal value $\tau_e = 1 + m$ at $\tau_0 = \tau_e$.

3 Results

The $\tau_0$–dependencies of $\eta_e$, $\eta^m$, and $\tau_e$ are depicted in Fig. 1. As is shown in Fig. 1a, under the adiabatic condition $t_\tau, t_v \ll t_\eta$ is met and the parameter
slowly increases being below $\tau_c$, no traffic jam can form. At the point $\tau_0 = \tau_c$ the stationary headway deviation $\eta_e$ jumps upward to the value $\sqrt{2} \eta_0$ and its further smooth increase is determined by Eq. (15). If the parameter $\tau_0$ then goes downward, the headway deviation $\eta_e$ continuously decreases up to the point, where $\tau_0 = \tau_{e0}$ and $\eta_e = \eta_0$. At this point the headway deviation jump-like goes down to zero. Referring to Fig. 1b, the stationary acceleration/braking time $\tau_e$ shows a linear increase from 0 to $\tau_e$ with the parameter $\tau_0$ being in the same interval. Then, after the jump down to the value $(1 - \eta_0^2)^{-1}$ at $\tau_0 = \tau_c$, the stationary time $\tau_e$ smoothly decays to 1 at $\tau_0 \gg \tau_c$. Under the parameter $\tau_0$ then decreases from above $\tau_c$ down to $\tau_{e0}$ the acceleration/braking time $\tau_e$ grows. When the point (18) is reached, the traffic becomes freely moving, so that the stationary acceleration/braking time undergoes the jump from the value (17) up to the one defined by Eq. (18). For $\tau_0 < \tau_{e0}$ again the parameter $\tau_e$ does not differ from $\tau_0$. Note that this subcritical regime is realized provided the parameter $m$, that enters the dispersion law (13), is greater than value

$$m_{\text{min}} = \frac{\eta_0^2}{1 - \eta_0^2}. \quad (19)$$

clearly, according to the picture described, the jamming generation is characterized by the well pronounced hysteresis: the cars initially being at motion with optimal headway between them, begin to deviate only if the acceleration/braking time $\tau_0$ of cars exceeds its limiting value $\tau_c = 1 + m$, whereas the acceleration/braking time $\tau_{e0}$ needed for uniform car flow is less than $\tau_c$ (see Eqs. (17), (18)). This is the case in the limit $t_r/t_\eta \to 0$ and the hysteresis loop shrinks with the growth of the adiabaticity parameter $\delta \equiv t_r/t_\eta$. In addition to the smallness of $\delta$, the adiabatic approximation implies the ratio $t_v/t_\eta \equiv \epsilon$ is also small. In contrast to the former, the latter does not seem to be realistic for the system under consideration, where, in general, $t_v \approx t_\eta$. So it is of interest to study to what extent the finite value of $\epsilon$ could change the results.

Owing to the condition $\delta \ll 1$, Eq. (8) is still algebraic and $\tau$ can be expressed in terms of $\eta$ and $v$. As a result, we derive the system of two nonlinear differential equations that can be studied by the phase portrait method [9]. The phase portraits for various values of $\epsilon$ are displayed in Fig. 2, where the center $O$ represents the stationary state and the saddle point $S$ is related to the maximum of the effective potential. As is seen from the
figure, independently of $\epsilon$, there is the universal section – the "mainstream", that attracts all phase trajectories and its structure is appeared to be almost insensitive to changes in $\epsilon$. Analysis of time dependencies $v(t)$ and $\eta(t)$ reveals that the headway and velocity deviations slow down appreciably on this section in comparison to the rest parts of trajectories that are almost rectilinear (it is not difficult to see that this effect is caused by the smallness of $\delta$). Since the most of time the system is in vicinity of the "mainstream", we arrive at the conclusion that finite values of $\epsilon$ do not affect qualitatively the above results obtained in the adiabatic approximation.

4 Discussion

According to the above consideration, the simplest picture of the dissipative dynamic of traffic flow in a homogeneous car-following model can be represented within the framework of Lorenz model, where the headway $\eta$ and velocity $v$ deviations play a role of an order parameter and its conjugate field, respectively, and the acceleration/braking time $\tau$ is a control parameter. The model is examined to show that a jam creates if the car characteristic $\tau_0$ is larger than the critical magnitude $\tau_c$. The above pointed out dissipative regime is inherent in the systems with small values of the relaxation time $t_{\tau}$ for acceleration/braking, being apparently a characteristic of a car-driver, and large ones $t_{\eta}$, $t_v$ for the headway and velocity deviations. According to Ref. [5], in the opposite case $t_{\tau} \geq t_{\eta}$, $t_v$, the system behaves in auto-oscillation or stochastic manners.

It is worth-while to note that the above synergetic scheme allows us to explain the collective phenomena of jamming transition in the $N$-body problem with $N \to \infty$. Then, the question arises: why exactly three variables (the headway and velocity deviations $\eta$, $v$, and acceleration/braking time $\tau$) permit to explain the nontrivial behaviour of $N$-body problem? The answer to this question gives the theorem by Ruelle and Takens: nontrivial collective behaviour of many-body system (type of the strange attractor) can be represented only in the case, when number of variables is not less than three [3]. The interpretation of this fact is the simplest: the first of the freedom degrees can be chosen as the way along the phase trajectory, the second one corresponds to the negative Lyapunov exponent, ensuring an attraction to this trajectory, the third one acts in opposite manner to give repulsion. In
our case of the self-organization process, the second $v$ and third $\tau$ freedom degrees provide the positive and negative feedbacks in Eqs. (2), (3).

The last question to our approach is why the just Lorenz scheme allows us to describes main peculiarities of the jamming transition? The answer is that this is the simplest approach, permitting to catch on the self-organization effects, just as the Landau phenomenological theory of phase transitions describes the great variety of thermodynamical transformations in the simplest way [3]. Let us note in this connection that the effective potential given by the sum of equalities (11), (14) acts the part of the Landau free energy. But the above stated synergetic scheme has the principle difference from the Landau-type theory [2] because the former takes into account a feedback of thermostat (the velocity deviation and the acceleration/braking time) with subsystem under consideration (the headway deviation), whereas the latter does not.

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CAPTIONS

Fig. 1. The $\tau_0$-dependencies of the stationary values of: (a) headway deviations $\eta_e, \eta^m$; (b) acceleration/braking time $\tau_e$. The arrows indicate the hysteresis loop.

Fig. 2. Phase portraits in the $\eta-v$ plane at $m = 1, \eta_0 = 0.1, \tau_0 = 1.25\tau_c$ for: (a) $\epsilon = 10^{-2}$; (b) $\epsilon = 1$; (c) $\epsilon = 10^2$. 
FIG. 1
