HYPERBOLIC TUNNEL-NUMBER-ONE KNOTS WITH
SEIFERT-FIBERED DEHN SURGERIES

SUNGMO KANG

ABSTRACT. Suppose \( \alpha \) and \( R \) are disjoint simple closed curves in the boundary
of a genus two handlebody \( H \) such that \( H[R] \) embeds in \( S^3 \) as the exterior of
a hyperbolic knot \( k \) (thus, \( k \) is a tunnel-number-one knot), and \( \alpha \) is Seifert in
\( H \) (i.e., a 2-handle addition \( H[\alpha] \) is a Seifert-fibered space) and not the meridian
of \( H[R] \). Then for a slope \( \gamma \) of \( k \) represented by \( \alpha \), \( \gamma \)-Dehn surgery \( k(\gamma) \) is
a Seifert-fibered space. Such a construction of Seifert-fibered Dehn surgeries
generalizes that of Seifert-fibered Dehn surgeries arising from primitive/Seifert
positions of a knot, which was introduced in [D03].

In this paper, we show that there exists a meridional curve \( M \) of \( k \) (or \( H[R] \))
in \( \partial H \) such that \( \alpha \) intersects \( M \) transversely in exactly one point. It follows
that such a construction of a Seifert-fibered Dehn surgery \( k(\gamma) \) can arise from
a primitive/Seifert position of \( k \) with \( \gamma \) its surface-slope. This result supports
partially the two conjectures: (1) any Seifert-fibered surgery on a hyperbolic
knot in \( S^3 \) is integral, and (2) any Seifert-fibered surgery on a hyperbolic
tunnel-number-one knot arises from a primitive/Seifert position whose surface
slope corresponds to the surgery slope.

1. Introduction

A primitive/Seifert knot \( k \), which was introduced by Dean [D03], is represented
by a simple closed curve \( \alpha \) lying a genus two Heegaard surface \( \Sigma \) of \( S^3 \) bound-
ing handlebodies \( H \) and \( H' \) such that \( \alpha \) is primitive in one handlebody, say \( H' \),
and is Seifert in \( H \), that is to say, a 2-handle addition \( H'[\alpha] \) is a solid torus and
\( H[\alpha] \) is a Seifert-fibered space and not a solid torus. Such a pair \( (\alpha, \Sigma) \) is called
a primitive/Seifert position of \( k \). Note that a knot may have more than one primitive
/Seifert position. Also note that since \( H \) is a genus two handlebody, the Seifert
condition of \( \alpha \) in \( H \) indicates that \( H[\alpha] \) is either a Seifert-fibered space over the
disk with at most two exceptional fibers or a Seifert-fibered space over the Möbius
band with at most one exceptional fiber. The curve \( \alpha \) in the former (the latter, resp.) is said to be Seifert-d(Seifert-m, resp.).

To perform Dehn surgeries on \( k \), we consider a surface-slope \( \gamma \), which is defined
to be an isotopy class of \( \partial N(k) \cap \Sigma \), where \( N(k) \) is a tubular neighborhood of \( k \)
in \( S^3 \). Note that \( \alpha \) is isotopic to a component of \( \partial N(k) \cap \Sigma \) in \( \Sigma \) and thus \( \alpha \)
can represent the surface-slope \( \gamma \). Also since \( \alpha \) intersects a meridional curve of
\( k \) once, the surface-slope \( \gamma \) is integral. Then Lemma 2.3 of [D03] implies that \( \gamma \)-
Dehn surgery \( k(\gamma) \) on \( k \) is either a Seifert-fibered space over \( S^2 \) with at most three
exceptional fibers or a Seifert-fibered space over \( \mathbb{R}P^2 \) with at most two exceptional
fibers. Note that a connected sum of lens spaces may arise as a Dehn surgery \( k(\gamma) \)
but due to [EM92] it can be excluded if a primitive/Seifert knot \( k \) is hyperbolic.

Primitive/Seifert knots have some properties. Since \( \alpha \) is primitive in \( H' \), there
exists a complete set of cutting disks \( \{D_M, D_R\} \) of \( H' \) such that \( \alpha \) intersects the
boundary $M$ of $D_M$ once transversely and is disjoint from the boundary $R$ of $D_R$. Note that such a cutting disk $D_R$ is unique up to isotopy in $H'$. Then it follows that $M$ can be considered as a meridional curve of $k$ and $H[R]$ is homeomorphic to the exterior of $k$ in $S^3$, which indicates that such a knot $k$ is a tunnel-number-one knot in $S^3$ such that the curve $R$ is the boundary of a cocore of the 1-handle regular neighborhood of a tunnel. Therefore, if $k$ is a primitive/Seifert knot, then there exist three simple closed curves $\alpha$, $R$, and $M$ in the boundary of a genus two handlebody $H$ satisfying:

1. $\alpha$ is Seifert in $H$.
2. $R$ is disjoint from $\alpha$ such that $H[R]$ is homeomorphic to the exterior of $k$ implying that $k$ is a tunnel-number-one knot.
3. $M$ is a meridional curve of $k$ such that $M$ is disjoint from $R$ and $M$ intersects $\alpha$ once transversely implying that the surface-slope is integral and $\alpha$ represents $k$.

In this paper, by taking only the conditions (1) and (2) we generalize a construction of Seifert-fibered Dehn surgeries arising from primitive/Seifert knots. We will show that the conditions (1) and (2) imply the condition (3), and thus this generalization constructing Seifert-fibered Dehn surgeries narrows down to the construction of Seifert-fibered Dehn surgeries arising from primitive/Seifert knots.

More explicitly, we suppose $\alpha$ and $R$ are disjoint simple closed curves in the boundary of a genus two handlebody $H$ such that $H[R]$ embeds in $S^3$ as the exterior of a hyperbolic knot $k$, and $\alpha$ is Seifert in $H$ and not the meridian of $H[R]$. Since $\alpha$ is disjoint from $R$, we can consider $\alpha$ as a curve representing a slope $\gamma$ in $\partial N(k)$ of $k$ in $S^3$. Then note that since $\alpha$ is Seifert in $H$, it follows that the $\gamma$-Dehn surgery $k(\gamma)$ is either a Seifert-fibered space over $S^2$ with at most three exceptional fibers or a Seifert-fibered space over $\mathbb{R}P^2$ with at most two exceptional fibers.

The main result of this paper is the following theorems.

**Theorem 1.1.** Suppose $\alpha$ and $R$ are disjoint simple closed curves in the boundary of a genus two handlebody $H$ such that $H[R]$ embeds in $S^3$ as the exterior of a hyperbolic knot $k$, and $\alpha$ is Seifert in $H$ and not the meridian of $H[R]$. Then there exists a meridional curve $M$ of $k$ (or $H[R]$) in $\partial H$ such that $\alpha$ intersects $M$ transversely in exactly one point.

As a consequence of Theorem 1.1 we have the following.

**Theorem 1.2.** Suppose $\alpha$ and $R$ are disjoint simple closed curves in the boundary of a genus two handlebody $H$ such that $H[R]$ embeds in $S^3$ as the exterior of a hyperbolic knot $k$, and $\alpha$ is Seifert in $H$ and not the meridian of $H[R]$, whence for a slope $\gamma$ represented by $\alpha$, $k(\gamma)$ is a Seifert-fibered space. Then $(\alpha, \partial H)$ is a primitive/Seifert position of $k$ and its surface-slope is $\gamma$ so that the Seifert-fibered Dehn surgery $k(\gamma)$ arises from the primitive/Seifert position $(\alpha, \partial H)$.

**Proof.** Let $H'$ be the closure of the complement of $H$ in $S^3$. Since $H[R]$ embeds in $S^3$ as the exterior of $k$, $H'$ is a genus two handlebody such that $R$ bounds a cutting disk of $H'$. Since by Theorem 1.1 there exists a meridional curve $M$ of $k$ (or $H[R]$) in $\partial H$ such that $\alpha$ intersects $M$ transversely in exactly one point, $\alpha$ is primitive in $H'$. Therefore, $(\alpha, \partial H)$ is a primitive/Seifert position of $k$, its surface-slope is $\gamma$, and the Seifert-fibered Dehn surgery $k(\gamma)$ arises from the primitive/Seifert position $(\alpha, \partial H)$, as desired. \qed
These results support partially the following conjectures.

**Conjecture 1.** Any Seifert-fibered surgery on a hyperbolic knot in $S^3$ is integral.

**Conjecture 2.** Any Seifert-fibered surgery on a hyperbolic tunnel-number-one knot arises from a primitive/Seifert position whose surface slope corresponds to the surgery slope.

Conjecture 1 is known to be true for various Seifert-fibered Dehn surgeries on a hyperbolic knot. Due to the famous result of [CGLS87], if $k(\gamma)$ is a lens space, then $\gamma$ is integral. Boyer-Zhang [BZ98] proved that the conjecture is true for toroidal Seifert-fibered surgeries. If a Seifert-fibered surgery $k(\gamma)$ has a projective plane as the base surface, then it contains a Klein bottle, in which case by Gordon-Leucke [GL95], $\gamma$ is integral. Thus the only remaining case is when $\gamma$-Dehn surgery $k(\gamma)$ is a Seifert fibered space over the sphere with three exceptional fibers. Theorem 1.2 gives a partial answer for this case.

Regarding Conjecture 2, there are families of hyperbolic knots admitting Seifert-fibered surgeries which do not arise from primitive/Seifert positions. See [MMM05], [T07], [DMM12], [DMM14], and [EJMM15]. All of the knots in [MMM05], [T07], and [DMM12] are not strongly invertible. Meanwhile, the knots in [DMM14] and [EJMM15] are strongly invertible but do not have tunnel number one. All of the knots above are not primitive/Seifert knots because any primitive/Seifert knot has tunnel number one and thus are strongly invertible. However, it is still unknown that there are examples of Seifert-fibered surgeries on hyperbolic tunnel-number-one knots in $S^3$ which do not arise from primitive/Seifert positions.

The main idea of proving Theorem 1.1 is to use the main result of [B20], which is originally introduced in [B93], saying that a meridian of $H[R]$ can be obtained from $R$ by surgery along a distinguished wave, and the main result of [K20b] claiming that there are two types of R-R diagrams of Seifert-d curves in $H$: rectangular form and non-rectangular form, and there is one type of R-R diagram of Seifert-m curves in $H$.

Some related definitions and properties necessary to prove Theorem 1.1 are provided in Section 2. In Sections 3 and 4 we prove Theorem 1.1 when $\alpha$ is Seifert-d with rectangular form and with non-rectangular form respectively. Section 5 provides the proof of Theorem 1.1 when $\alpha$ is Seifert-m.

**Acknowledgement.** This paper is originated from the joint work with John Berge for the project of the classification of hyperbolic primitive/Seifert knots in $S^3$. I would like to express my gratitude to John Berge for his collaboration and support. I would also like to thank Cameron Gordon and John Luecke for their great hospitality while I stayed in the University of Texas at Austin.

2. Preliminaries

We start with the following lemma, which can be found in [HOT80] or [O79] and shows some possible types of graphs of Heegaard diagrams of simple closed curves in the boundary of a genus two handlebody.

**Lemma 2.1.** Let $H$ be a genus two handlebody with a set of cutting disks $\{D_A, D_B\}$ and let $\mathcal{C}$ be a finite set of pairwise disjoint nonparallel simple closed curves on $\partial H$ whose intersections with $\{D_A, D_B\}$ are essential and not both empty. Then, after
Figure 1. The three types of graphs of Heegaard diagrams of simple closed curves on the boundary of a genus two handlebody $H$ which has cutting disks $D_A$ and $D_B$, excluding diagrams in which simple closed curves are disjoint from both $\partial D_A$ and $\partial D_B$.

perhaps relabeling $D_A$ and $D_B$, the Heegaard diagram of $C$ with respect to $\{D_A, D_B\}$ has the form of one of the three graphs in Figure 1

**Definition 2.2 (cut-vertex).** If $v$ is a vertex of a connected graph $G$ such that deleting $v$ and the edges of $G$ meeting $v$ from $G$ disconnects $G$, we say $v$ is a cut-vertex of $G$.

The Heegaard diagram in Figure 1 either is not connected or has a cut-vertex.

**Definition 2.3 (Positive Heegaard Diagram).** A Heegaard diagram is positive if the curves of the diagram can be oriented so that all intersections of curves in the diagram are positive. Otherwise, the diagram is nonpositive.

Suppose $R$ is a nonseparating simple closed curve in the boundary of a genus two handlebody $H$ such that $H[R]$ embeds in $S^3$, i.e., $H[R]$ is an exterior of a knot $k$ in $S^3$. It is shown in [B20], which is essentially originated from [B93], that a meridian of $H[R]$ (or $k$) can be obtained from $R$ by surgery along a wave based at $R$. Recall that a wave on the curve $R$ in $\partial H$ is an arc $\omega$ whose endpoints lies on $R$ with the opposite signs. The following is one of the results of [B20], which shows how to get a meridian of $H[R]$.

**Theorem 2.4 (Waves provide meridians).** Let $H$ be a genus two handlebody with a set of cutting disks $\{D_A, D_B\}$ and let $R$ be a nonseparating simple closed curve on $\partial H$ such that the Heegaard diagram $\mathcal{D}_R$ of $R$ with respect to $\{D_A, D_B\}$ is connected and has no cut-vertex. Suppose, in addition, that the manifold $H[R]$ embeds in $S^3$. Then $\mathcal{D}_R$ determines a wave $\omega$ based at $R$ such that if $m$ is a boundary component of a regular neighborhood of $R \cup \omega$ in $\partial H$, with $m$ chosen so that it is not isotopic to $R$, then $m$ represents the meridian of $H[R]$. Furthermore, the wave $\omega$ determined by $R$ can be obtained as follows:

1. If $\mathcal{D}_R$ is nonpositive, then $\omega$ is a unique vertical wave $\omega_v$ which is isotopic to a subarc of the boundary of one of $D_A$ and $D_B$ with $R$ has both positive and negative signed intersections.
2. If $\mathcal{D}_R$ is positive, then $\omega$ is a horizontal wave $\omega_h$ such that one endpoint of $\omega_h$ lies on an edge of $\mathcal{D}_R$ connecting vertices $A^+$ and $A^-$, while the other endpoint of $\omega_h$ lies on an edge of $\mathcal{D}_R$ connecting vertices $B^+$ and $B^-$. 

Figure 2. A vertical wave $\omega_v$ in a nonpositive Heegaard diagram in a) and b) where $R$ has both positive and negative signed intersections with $D_B$, and a horizontal wave $\omega_h$ in a positive Heegaard diagram in c). They are said to be distinguished in the sense that they can be used in a surgery on $R$ to obtain a meridian of $H[R]$.

Figure 3. R-R diagrams of a simple closed curve $R$ in which $R$ has only one connection on one handle and at most two connections on the other handle. Here $a, b \geq 0$, gcd$(a, b) = 1$, and $m, n, s \in \mathbb{Z}$.

Figures 2a and 2b show vertical waves $\omega_v$ when $R$ has both positive and negative signed intersections with the cutting disk $D_B$ so that the Heegaard diagram $D_R$ is nonpositive. Figure 2c shows a horizon wave $\omega_h$ when $D_R$ is positive. Vertical waves and horizontal waves which are used to find a representative of a meridian of $H[R]$ as described in Theorem 2.4 are said to be distinguished.

Next proposition provides some special type of R-R diagrams of $R$ such that $H[R]$ is nonhyperbolic. For the definition and properties of R-R diagrams, see [K20c].

**Proposition 2.5.** Suppose $R$ is a simple closed curve in the boundary of a genus two handlebody $H$ with an R-R diagram of the form shown in Figure 3 with $a, b \geq 0$ and $m, n, s \in \mathbb{Z}$.

If $H[R]$ embeds in $S^3$, then $R$ is either a primitive curve, or a torus or cable knot relator on $H$. Therefore if $k$ is a knot whose exterior is homeomorphic to $H[R]$, then $k$ is either the unknot, a torus knot or a cable of a torus knot.

**Proof.** This is Theorem 2.4 of [K20a].
Figure 4. If $\alpha$ is a Seifert-d curve in the boundary of a genus two handlebody $H$, then $\alpha$ has an R-R diagram with the form of one of these figures with $P, S > 1, a, b > 1$, and $\gcd(a, b) = 1$. If $\alpha$ has an R-R diagram with the form of Figure 4a, we say $\alpha$ has rectangular form. Then $H[\alpha]$ is a Seifert-fibered space over $D^2$ with two exceptional fibers of index $P$ and $S$. If $\alpha$ has an R-R diagram with the form of Figure 4b, we say $\alpha$ has non-rectangular form. Then $H[\alpha]$ is a Seifert-fibered space over $D^2$ with two exceptional fibers of index $P(a + b) + b$ and $S$.

3. The case when $\alpha$ is Seifert-d with rectangular form

The classification theorem of Seifert-d curves in [K20b] says that if $\alpha$ is a Seifert-d curve, then $\alpha$ has an R-R diagram of the forms in Figure 4. If $\alpha$ has the R-R diagram of the form in Figure 4a (4b, resp.), then we say that $\alpha$ is of a rectangular form (a non-rectangular form, resp.). In this section and next section we prove Theorem 1.1 for the case when $\alpha$ is of a rectangular form and for the case when $\alpha$ is of a non-rectangular form respectively.

Suppose $\alpha$ is of a rectangular form, i.e., $\alpha$ has an R-R diagram of the form in Figure 4a.

Proposition 3.1. Theorem 1.1 holds if $\alpha$ is Seifert-d and has rectangular form.

Proof. In the R-R diagram of $\alpha$ of a rectangular form in Figure 4a, we add two arbitrary bands of connections in each handle, namely $Q$- and $R$-connections in the $A$-handle, and $U$- and $T$-connections in the $B$-handle as shown in Figure 5. Note that $P + R = Q$ and $S + U = T$. Here we overuse the letter $R$ meaning a simple closed curve as well as the label of the connection in the $A$-handle. However, the confusion will obviously be eliminated in the context.

Now we consider adding a simple closed curve $R$ disjoint from $\alpha$. We can observe that $R$ cannot have both $P$- and $S$-connections, otherwise the curve $R$ is forced to spiral endlessly and cannot be a simple closed curve. Therefore up to the symmetry of the R-R diagram of $\alpha$, without loss of generality we may assume that $R$ has no $P$-connections. There are two cases to consider: (1) $R$ has neither $P$-connections nor $S$-connections and (2) $R$ has no $P$-connections and has $S$-connections.

Case (1): The curve $R$ has neither $P$-connections nor $S$-connections.
A priori there are two possible R-R diagrams of such curves $R$. These appear in Figure 5. However, examination shows the R-R diagrams of Figure 5a and Figure 5b agree up to homeomorphism and relabeling parameters. So we may suppose $R$ has an R-R diagram of Figure 5a.

Note that the weights $a, b, c > 0$ in Figure 5, otherwise by Proposition 2.5 $H[R]$ is not hyperbolic.

First, suppose $R$ is nonpositive. Since $P,S > 1$, none of $R,Q,U,$ and $T$ is 0. This, when combined with nonpositivity, implies that the Heegaard diagram of $R$ underlying the R-R diagram is connected and has no cut vertex. Therefore there exists a distinguished vertical wave $\omega_v$ such that by Theorem 2.4 a meridian $M$ of $H[R]$ is obtained from $R$ by surgery along $\omega_v$. It follows immediately from the R-R diagram that the vertical wave $\omega_v$ intersects $\alpha$ transversely at a point. Therefore a meridian $M$ of $H[R]$ intersects $\alpha$ transversely at a point.

Now we assume that $R$ is positive. The conditions that $a,b,c > 0$ and $P,S > 1$ implies that $RQ > 0$ and $TU > 0$, and $\max\{|R|,|Q|\} > 1$ and $\max\{|T|,|U|\} > 1$. Therefore the Heegaard diagram of $R$ is connected and has no cut vertex and by Theorem 2.4 there exists a distinguished horizontal wave $\omega_h$ such that a meridian $M$ of $H[R]$ is obtained from $R$ by surgery along $\omega_h$.

Locating the horizontal wave $\omega_h$ in the R-R diagram of $R$ depends on which band of connections of $R$ has maximal labels. See [K20c] for the information on the location of horizontal waves in R-R diagrams. Therefore, it depends on the signs of $R,Q,T,$ and $U$. There are four cases to consider:

(a) $R, Q > 0$ and $T, U < 0$;
(b) $R, Q < 0$ and $T, U > 0$;
(c) $R, Q > 0$ and $T, U > 0$;
(d) $R, Q < 0$ and $T, U < 0$.

If $R, Q > 0$ and $T, U < 0$, then $Q$ and $U$ are the maximal labels in the $A$- and $B$-handles respectively and $\omega_h$ has an endpoint on a connection in one handle which borders the band of connections with maximal label. In order to locate $\omega_h$ in the R-R diagram, we isotope the outermost edge of the $b$ parallel edges entering the

---

**Figure 5.** The R-R diagrams of disjoint nonseparating curves $\alpha$ and $R$ in which $\alpha$ has rectangular form and $R$ has no $P$-connections and no $S$-connections.
Figure 6. R-R diagrams of horizontal waves $\omega_h$ based at the curve $R$ which show how $\omega_h$ depends on the signs of the parameters $R, Q, U,$ and $T$. In (a), $R, Q, P > 0$ and $T, U, -S < 0$. In (b), $-P, R, Q < 0$ and $S, T, U > 0$. In (c), $R, Q, P > 0$ and $S, T, U > 0$. In (d), $-P, R, Q < 0$ and $T, U, -S < 0$.

Case (2): $R$ has $S$-connections but no $P$-connections.

There are three possible R-R diagrams of $R$ as shown in Figure 7. However, using an orientation-reversing homeomorphism of $H$ and thus of the R-R diagram of $(\alpha, R)$, and relabelling the parameters, we observe that the R-R diagram in Figure 7a is equivalent to that of Figure 6a. Therefore, we consider the R-R diagrams of the forms in Figures 7b and 7d. Since $H[R]$ is hyperbolic, it follows by Proposition 2.5 that $c > 0$ in Figure 7b, and $b > 0$ in Figure 7d.
If $R$ is nonpositive, then since $P, S > 1$, it follows that the Heegaard diagram of $R$ is connected and has no cut vertex and thus there exists a distinguished vertical wave $\omega_v$ such that a meridian $M$ of $H[R]$ is obtained from $R$ by surgery along $\omega_v$. Since $c > 0 (b > 0, \text{ resp.})$ in Figure 7a (7b, resp.), $\omega_v$ does not intersect $\alpha$, a contradiction. Therefore, $R$ is positive.

First, suppose $R$ has the R-R diagram of Figure 7a. Since $R$ is positive, $Q, T, U > 0$ and $R < 0$.

Claim 3.2. In the diagram of $R$ of Figure 7a, we may assume that $R + Q \neq 0$.

Proof. Suppose $R + Q = 0$. Since $\gcd(|R|, |Q|) = 1$, $R = -1$, $Q = 1$ and $P = 2$. Then $\alpha = A^2B^S$ and the Heegaard diagram of $R$ has a cut vertex. Now we use the argument of the hybrid diagram. Its hybrid diagram of $\alpha$ and $R$ corresponding to the R-R diagram of Figure 7a is illustrated in Figure 8a. For the definition and properties of hybrid diagrams, see [K20c].

In its hybrid diagram, we drag the vertex $A^-$ together with the edges of $R$ and $\alpha$ meeting the vertex $A^-$ over the $S$-connection on the $B$-handle. This performance corresponds to a change of cutting disks inducing an automorphism of $\pi_1(H)$ that takes $A \mapsto AB^{-S}$ and leaves $B$ fixed. With an orientation-reversing homeomorphism of $H$ applied, the resulting hybrid diagram of $\alpha$ and $R$ is depicted.
The hybrid diagram of Figure 8a corresponds to the R-R diagram of Figure 7a. The hybrid diagram in Figure 8b is obtained from the hybrid diagram in Figure 8a by dragging vertex \( A^- \) of Figure 8a, together with the edges of Figure 8a meeting the vertex \( A^- \) of Figure 8a, over the S-connection of the B-handle of Figure 8a. This induces an automorphism of \( \pi_1(H) \) which takes \( A \rightarrow AB^-S \).

It follows from Figure 8b that \( R \) has only two bands of connections labelled by \( U \) and \( U-S \) in the B-handle. For the labels of the bands of connections in the A-handle, by chasing the parallel arcs of weight \( c \) in the R-R diagram of \( R \) of Figure 7a, we observe that the subword \( ABS \) in \( R \) appears in the sequence \( \cdots AB^-SAB^-SAB \cdots \). This implies that after the automorphism taking \( A \rightarrow AB^-S \), \( A^2 \) does not appear as a single syllable in the word of \( R \) in \( \pi_1(H) \). On the other hand, \( \alpha \) is sent to \( A^2B^-S \) in \( \pi_1(H) \), which is still of a rectangular form. This implies that \( \alpha \) and \( R \) have no common single syllable, which means that \( \alpha \) and \( R \) have no common connections. Therefore this case belongs to Case (1) where \( R \) has neither \( P \)-connections nor \( S \)-connections.

By Claim 3.2, \( \max\{|Q|,|R|\} > 1 \). Since \( R \) has \( S \)-connections on the B-handle, the Heegaard diagram is connected and has no cut vertex. Therefore it has a distinguished horizontal wave \( \omega_h \) yielding a meridian of \( H[R] \). As in the case of (1), locating \( \omega_h \) in the R-R diagram of \( R \) depends on the sign of \( R+Q \) unless \( b = 0 \), in which case \( \omega_h \) also depends on the maximal label member of \( \{S,U\} \). Figure 9 where the \( P \)-connection of \( \alpha \) is isotoped to the \( Q \)- and \( -R \)-connection, shows \( \omega_h \) when \( b > 0 \). In either of the R-R diagrams \( \omega_h \) intersects \( \alpha \) transversely once. For the case where \( b = 0 \) and \( \alpha \neq 0 \), we insert \( (S-U) \)-connection in the B-handle to locate \( \omega_h \). Then it is easy to show that in this case \( \omega_h \) also intersects \( \alpha \) transversely. Note that at least one of \( a \) and \( b \) must be positive, otherwise \( R \) has only two bands of connections on the A-handle and only one band of connections so that by Proposition 2.5 \( H[R] \) is not hyperbolic.

Now, we assume that \( R \) has the R-R diagram in Figure 7b. Since \( R \) is positive, \( Q, T > 0 \) and \( R, U < 0 \).

**Claim 3.3.** In the diagram of \( R \) of Figure 7b, we may assume that \( R+Q \neq 0 \), or equivalently \( (P,Q,R) \neq (2,1,-1) \).
Proof. Suppose $R + Q = 0$. Since $\gcd(|Q|, |R|) = 1$, $(P, Q, R) = (2, 1, -1)$. Then the Heegaard diagram of $R$ has a cut vertex, and $R$ consists of the three types of two-syllable subwords $AB^S, AB^T,$ and $AB^{-U}$ with $|AB^S| = 2b, |AB^T| = a, |AB^{-U}| = c$. Here $|AB^S|$, for instance, denotes the total number of appearances of $AB^S$ in $R$ in $\pi_1(H)$. It follows that $|\alpha| = a + 2b + c$ in $R$. Furthermore $\alpha = A^2B^S$ in $\pi_1(H)$, and $\alpha$ and $R$ have no common connections in the $A$-handle.

As in the proof of Claim 3.2, since the Heegaard diagram of $R$ has a cut vertex, we perform a change of cutting disks that induces the automorphism of $\pi_1(H)$ taking $A \mapsto AB^{-S}$, and then an orientation-preserving homeomorphism of $H$ inducing the automorphism $(A, B) \mapsto (B, A^{-1})$ of $\pi_1(H)$. Then $\alpha$ is carried to $A^2B^S$ in $\pi_1(H)$, which implies that the R-R diagram of $\alpha$ is also of a rectangular form. The two-syllables $AB^S, AB^T,$ and $AB^{-U}$ of $R$ are sent to $B, A^{-U}B,$ and $A^T B$ respectively, which implies that there are only two exponents $-U$ and $T$ with base $A$ in $R$. Thus there are three bands of connections with the label set $(S, T, U)$ in the $A$-handle in the resulting R-R diagram of $\alpha$ and $R$ such that $\alpha$ and $R$ have no common connections in the $A$-handle. Also we can see that $|\alpha|$ is reduced strictly to $a + c$ in $R$.

Now the resulting R-R diagram of $\alpha$ and $R$ depends on the determination of the $B$-handle. However, since we have already proved Proposition 3.3 for all other types of the R-R diagrams of $R$ when $\alpha$ is of a rectangular form, we may assume that it has an R-R diagram of the form in Figure 7b with the three labels $(S, T, U)$ in the $A$-handle. If $(S, T, U) \neq (2, 1, -1)$ or equivalently $T + U \neq 0$, then the R-R diagram of $R$ satisfies the conclusion of this claim as desired. If $(S, T, U) = (2, 1, -1)$, then we continue to do the process above, which must eventually terminate since $|\alpha|$ in $R$ is strictly decreasing. \[\square\]

Now $R + Q \neq 0$ and thus $R$ is connected and has no cut vertex. We apply the similar argument as in the case of R-R diagram of $R$ in Figure 7b. Figure 10 shows $\omega_h$ when $b > 0$. In both of the R-R diagrams $\omega_h$ intersects $\alpha$ transversely once. Similarly when $b = 0$, we can show that $\omega_h$ intersects $\alpha$ transversely once.
Figure 10. Horizontal waves $\omega_h$ in the R-R diagram of $R$ when $R + Q > 0$ ($R + Q < 0$, resp.) in Figure 9a (9b, resp.) when $b > 0$.

Figure 11. An R-R diagram obtained from Figure 4b by adding a separating simple closed curve $\Gamma$, disjoint from $\alpha$, to Figure 4b so that $\Gamma$ represents $AB^SB^{-S}$ in $\pi_1(H)$. Here $P, S > 1$ with $a, b > 0$, and $\gcd(a, b) = 1$.

Thus, we have completed the proof of Proposition 3.1 and therefore Theorem 1.1 when $\alpha$ is Seifert-d and is of a rectangular form.

4. Cases in which $\alpha$ is Seifert-d and has non-rectangular form

The goal of this section is to prove Proposition 4.1 which shows that Theorem 1.1 holds in all cases in which $\alpha$ is Seifert-d and has non-rectangular form, i.e., those cases in which $\alpha$ has an R-R diagram with the form of Figure 4b.

Proposition 4.1. Theorem 1.1 holds if $\alpha$ is Seifert-d and has non-rectangular form.

Proof. Note it is possible to add a separating simple closed curve $\Gamma$ to the R-R diagram of $\alpha$ in Figure 4b so that the resulting R-R diagram of $\alpha$ and $\Gamma$ has the form of Figure 11. Then $\Gamma$ represents $AB^SA^{-1}B^{-S}$ in $\pi_1(H)$, and $\Gamma$ separates $\partial H$ into two once-punctured tori $F$ and $F'$ with $\alpha \subset F$. 
Claim 4.2. The curve $R$ in $\partial H$ has essential intersections with $\Gamma$.

Proof of Claim 4.2 Suppose $R$ has no essential intersections with $\Gamma$. Then $R$ lies completely in $F$ or completely in $F'$.

If $R$ lies completely in $F$, then $\alpha$ and $R$ are isotopic in $\partial H$, but this is impossible since $H[R]$ is hyperbolic, while $H[\alpha]$ is Seifert-fibered.

On the other hand, suppose $R$ lies completely in $F'$. If $R$ has no connections in the $A$-handle, then $R = B^S$ in $\pi_1(H)$, which is a contradiction to that $H[R]$ embeds as a knot exterior in $S^3$ and thus $H_1(H[R])$ is torsion-free. It follows that $R$ has a connection in the $A$-handle and Figure 11 implies that $R$ has only one band of connections labeled by 1 in the $A$-handle. If $R$ has a $S$-connection in the $B$-handle, then the Heegaard diagram of $R$ is nonpositive, is connected and has no cut vertex. So there exists a distinguished vertical wave $\omega_v$ yielding a meridian of $H[R]$. It is easy to see from the R-R diagram of $\alpha$ that $\omega_v$ does not intersect $\alpha$, a contradiction.

Therefore, $R$ has no $S$-connections and thus at most two bands of connections in the $B$-handle, implying by Proposition 2.5 that $H[R]$ is nonhyperbolic, a contradiction.

It follows $R$ has essential intersections with $\Gamma$. □

Next, consider Figure 12 which shows $F$ cut open along two properly embedded arcs in $F$ parallel to $\partial D_A \cap F$ and $\partial D_B \cap F$. Note that since $\partial D_A \cap F$ is a single connection in $F$, and $|a \cap \partial D_A| = (a+b)P + b$, one has $c = (a+b)(P-1)+b$. And therefore, since $a+b \geq 2$, and $P > 1$, one has $c > a+b > 2$.

The simple closed curve $\alpha$ together with the arcs of $\partial D_B \cap F$ and the arc of $\partial D_A \cap F$ cut $F$ into a number of faces, each of which is a rectangle, except for the pair of hexagonal faces $Hex_1$ and $Hex_2$, shown as shaded regions in Figure 12. Now it is easy to see that any connection in $F$ disjoint from $\alpha$ traverses each of the above rectangles. Since $R$ is disjoint from $\alpha$, and $R$ has essential intersections with $F$, $R \cap F$ contains such connections. Then, because $a+b \geq 2$ and $c > a+b$, we see that $A^m$ and $B^n$ appear in the cyclic word which $R$ represents in $\pi_1(H)$ with $|m|, |n| > 1$. It follows that the Heegaard diagram $D$ of $R$ with respect to \{\partial D_A, \partial D_B\} is connected and has no cut vertex. Therefore the invariant arc $\omega$ promised by [B20] appears in $D$ as a distinguished wave based at $R$.

Claim 4.3. $\omega$ intersects $\alpha$ transversely in one point.
Figure 13. The possible configurations of \( \alpha \), \( R \) and \( \omega \) in \( F \).

**Proof of Claim 4.3.** If \( \omega \) is disjoint from \( \alpha \), then \( \alpha \) is isotopic to a meridian of \( H[R] \), a contradiction. Therefore \( \omega \) intersects \( \alpha \).

Suppose \( p \) is a point of \( \omega \cap \alpha \). Then \( p \) lies in the boundary of a rectangular face, say \( R_p \), of \( F \). \( R_p \) is traversed by at least one connection of \( R \cap F \) which we may assume has the same orientation as \( \alpha \). But, since \( p \) is essential, one of the endpoints of \( \omega \), say \( p' \) must also lie in \( R_p \) on a subarc of a connection of \( R \cap F \). So we have the configuration shown in Figure 13.

Since \( \omega \) is an arc and has only two endpoints, it follows that \( \omega \cap \alpha \) consists of either one or two points, and if \( \omega \cap \alpha \) consists of two points, then these two intersections have opposite signs because of the definition of a wave. However this is impossible, because if \( \omega \cap \alpha \) consists of two points of intersection with opposite signs, then the algebraic intersection number of \( \omega \) and \( \alpha \) is equal to zero, which implies that the geometric intersection number of a meridian representative \( M \) and \( \alpha \) is equal to zero and thus \( \alpha \) is a meridian of \( H[R] \), a contradiction.

Thus, we have completed the proof of Proposition 4.1 and therefore Theorem 1.1 when \( \alpha \) is Seifert-d and is of a non-rectangular form.

\[ \square \]

5. The case when \( \alpha \) is Seifert-m

In this section, we prove Theorem 1.1 when \( \alpha \) is Seifert-m in a genus two handlebody \( H \). It follows from the classification theorem of Seifert-m curves in [K20b] that \( \alpha \) has an R-R diagram of the form in Figure 14 with \( S > 1 \).

**Proposition 5.1.** Theorem 1.1 holds if \( \alpha \) is Seifert-m.

**Proof.** We observe from Figure 14 that \( \alpha \) has two bands of connections labelled by 1 in the \( A \)-handle.

**Claim 5.2.** \( R \) must have only one band of connections labelled by 1 in the \( A \)-handle.

**Proof.** Suppose for a contradiction that \( R \) has no 1-connections in the \( A \)-handle or \( R \) has the two bands of connections labelled by 1 in the \( A \)-handle.

First, suppose \( R \) has no 1-connections in the \( A \)-handle. If \( R \) has either no connections or only 0-connections in the \( A \)-handle, then \( R \) should have only one \( S \)-connection in the \( B \)-handle, which implies that \( R = B^S \) in \( \pi_1(H) \). This is impossible since \( H[R] \) embeds as a knot exterior in \( S^3 \) and thus \( H_1(H[R]) \) is torsion-free. Thus
R has only 2-connections in the A-handle. If R has a S-connection in the B-handle, then it is easy to see that the Heegaard diagram of R is nonpositive, connected and has no cut-vertex. Thus there exists a distinguished vertical wave $\omega_v$ yielding a meridian of $H[R]$. It follows from the R-R diagram of $\alpha$ that $\omega_v$ does not intersect $\alpha$, which is a contradiction. Now R has only two bands of connections in the B-handle. However, this also cannot happen by Proposition 2.5 indicating that $H[R]$ is not hyperbolic.

Now we suppose that R has the two bands of connections labelled by 1 in the A-handle. Orient R so that the labels at the ends of the two bands where R enters are either both 1, or 1 and $-1$. If the two labels are 1 and $-1$, then an R-R diagram of $\alpha$ and R contains a subdiagram with the form of Figure 15 with both $a > 0$ and $b > 0$. Thus Heegaard diagram of R is nonpositive and also is connected and has no cut-vertex. It follows that a distinguished vertical wave $\omega_v$ yielding a meridian of $H[R]$ does not intersect $\alpha$, a contradiction.
The two R-R diagrams of disjoint curves $\alpha$ and $R$ in which $\alpha$ is Seifert-m and $R$ has no 2-connections on the A-handle.

If the two labels are both 1, then it follows from the R-R diagram of $\alpha$ that $R$ must have both $S$- and $(-S)$-connections. Note that in this case an R-R diagram of $\alpha$ and $R$ also contains a subdiagram with the form of Figure 15 with both $a > 0$ and $b > 0$ and with orientations of the b-weighted bands reversed. This implies that $R$ is nonpositive, is connected and has no cut-vertex. By the similar argument above, a distinguished vertical wave $\omega_v$ yielding a meridian of $H[R]$ does not intersect $\alpha$, a contradiction. □

By Claim 5.2, $R$ has only one band of connections with label 1 in the A-handle. There are two bands of connections with label 1 in the A-handle in the R-R diagram of $\alpha$: say, vertical and horizontal. Applying an orientation-reversing homeomorphism of $H$, if necessary, we may assume without loss of generality that $R$ has vertical 1-connections in the A-handle. Now we break the argument into two cases: (1) $R$ has no 2-connections and (2) $R$ has 2-connections in the A-handle.

**Case (1): R has no 2-connections in the A-handle.**

There are two possible R-R diagrams of $R$ as shown in Figure 16 depending on whether or not $R$ has 0-connections. Note that $a, b > 0$ in the R-R diagram of
Figure 16a and $a + b > 0$ in the R-R diagram of Figure 16b, because otherwise $H[R]$ is not hyperbolic.

If $R$ in Figure 16 is nonpositive, then it is easy to see from the R-R diagrams that a distinguished vertical wave $\omega_v$ yielding a meridian of $H[R]$ intersects $\alpha$ transversely at a point.

If $R$ in Figure 16 is positive, then the Heegaard diagram of $R$ has a cut-vertex. Since $a + b > 0$ in the R-R diagram of Figure 16b, either $a > 0$ or $b > 0$. Without loss of generality we may assume that $b > 0$. Therefore $b > 0$ in both of the R-R diagrams in Figure 16, which implies $R$ has a subword $\cdots B^S AB^T AB^S \cdots$. As we did in Claim 3.2 in Section 3, we perform a change of cutting disks of $H$ inducing an automorphism of $\pi_1(H)$ taking $A \mapsto AB^{-T}$. Using a hybrid diagram we can see that since $\alpha = AB^S A^{-1} B^S$ in $\pi_1(H)$, under this automorphism $\alpha$ remains same, i.e., $\alpha$ has the same form of R-R diagram in Figure 14 while since the subword $\cdots B^S AB^T AB^S \cdots$ of $R$ is sent to $\cdots B^{-U} A^2 B^{-U} \cdots$, $R$ is transformed into a simple closed curve whose word in $\pi_1(H)$ contains $A^2$. This implies that $R$ has 2-connections in the $A$-handle. So this case belongs to Case (2) where $R$ has 2-connections in the $A$-handle, which is handled next.
Figure 18. Horizontal waves $\omega_h$ in R-R diagrams of $\alpha$ and $R$ when $U > S$ in Figure 18a, and $U < S$ in Figure 18b.

Case (2): $R$ has 2-connections in the $A$-handle.

There are two possible R-R diagrams of $R$ as illustrated in Figure 17. Note that $a, b > 0$ in Figure 17a and $b, c > 0$ in Figure 17b. This is because for the R-R diagram of $R$ in Figure 17a, since $R$ has 2-connections in the $A$-handle, $a > 0$. If $b = 0$ there, then since $R$ is a simple closed curve, $c = 0$, which implies by Proposition 2.5 that $H[R]$ is not hyperbolic. For the R-R diagram of $R$ in Figure 17b, if $c = 0$, then Proposition 2.5 implies that $H[R]$ would not be hyperbolic. If $b = 0$ there, then $a = 0$ and thus $H[R]$ is not hyperbolic.

If $R$ in Figure 17 is nonpositive, as in the case (1), there exists a distinguished vertical wave $\omega_v$ yielding a meridian of $H[R]$ which intersects $\alpha$ transversely once.

We assume that $R$ in Figure 17 is positive. From the conditions that $a, b > 0$ in Figure 17a and $b, c > 0$ in Figure 17b, it follows that the Heegaard diagrams of $R$ are connected and has no cut-vertex. Therefore there exists a distinguished horizontal wave $\omega_h$ yielding a meridian of $H[R]$.

If $c > 0(a > 0, \text{resp.})$ in Figure 17b (17a, resp.), then $R$ has all of the three bands of connections of labels $U, T, S$ in the $B$-handle. Since $R$ is positive, all of $U, T,$ and $S$ are positive and thus $T$ is the maximal label of connections in the $B$-handle. Therefore, as in Figure 6 or in Figure 9 a horizontal wave $\omega_h$ can be located in the
R-R diagram of $R$ by isotoping the 2-connection and the $T$-connection in the $A$- and $B$-handle respectively. Then we can see that $\omega_h$ intersects $\alpha$ once. If $c = 0 (a = 0, \text{ resp.})$ in Figure 17a(17b, resp.), then the two R-R diagrams in Figure 17a(17b, resp.) have the same form. In other words, the R-R diagram of Figure 17a with $c = 0$ is the R-R diagram of Figure 17b with $c = 0$ by replacing $(b, c, T)$ by $(a, b, T)$. Therefore we focus only on the R-R diagram of Figure 17a with $c = 0$. Locating a horizontal wave $\omega_h$ in the R-R diagram of $R$ depends on the sizes of $U$ and $S$ in the $B$-handle. Figure 18a(18b, resp.) shows $\omega_h$ when $U > S$ ($U < S$, resp.). It follows that when $U > S$, $\omega_h$ intersects $\alpha$ at a point. On the other hand, when $U < S$, $\omega_h$ intersects $\alpha$ twice as shown in Figure 18b, where one $S$-connection of $\alpha$ is isotoped. However, it is easy to see from the R-R diagram that one meridian representative $M_1$ obtained from $R$ by surgery along $\omega_h$ represents $P(U)P(S)$ in $\pi_1(H)$. This is impossible because $H[M_1]$ also embeds in $S^3$ as a knot exterior and thus $H_1(H[M_1])$ is torsion-free.

Thus, we have completed the proof of Proposition 5.1 and therefore Theorem 1.1 when $\alpha$ is Seifert-m. □

References

[B20] Berge, J., Distinguished waves and slopes in genus two, preprint.
[B93] Berge, J., Embedding the Exteriors of One-Tunnel Knots and Links in the 3-Sphere, Unpublished transparencies of invited address at Cascade Topology Conf. Spring 1993.
[BZ98] Boyer, S., Zhang, X., On Culler-Shalen seminorms and Dehn filling, Ann. of Math. 148 (1998), 737–801.
[CGLS87] Culler, M., Gordon, C. McA., Luecke, J., Shalen, P.B., Dehn surgery on knots, Ann. of Math. 125 (1987), 237–300.
[D03] Dean, J., Small Seifert-fibered Dehn surgery on hyperbolic knots, Algebraic and Geometric Topology 3 (2003), 435–472.
[DMM12] Deruelle, A., Miyazaki, K., and Motegi, K., Networking Seifert Surgeries on Knots, Mem. Amer. Math. Soc. 217 (2012) viii + 130.
[DMM14] Deruelle, A., Miyazaki, K., and Motegi, K., Neighbors of Seifert surgeries on a trefoil knot in the Seifert Surgery Network, Boletin de la Sociedad Matemtica Mexicana 20 no. 2, (2014) 523–558.
[EJMM15] Eudave-Muñoz, M., Jasso, E., Miyazaki, K. and Motegi, K., Seifert fibered surgeries on strongly invertible knots without primitive/Seifert positions, Top. and its Appl. 196 Part B, (2015), 729–753.
[EM92] Eudave-Muñoz, M., Band sums of links which yield composite links. The cabling conjecture for strongly invertible knots, Trans. Amer. Math. Soc. 330 no. 2, (1992), 463–501.
[GL05] Gordon, C. McA., Luecke, J., Dehn surgeries on knots creating essential tori, I, Comm. Anal. Geom. 3 (1995), 597–644.
[HOT80] Homma, T., Ochiai, M. and Takahashi, M., An Algorithm for Recognizing $S^3$ in 3-Manifolds with Heegaard Splittings of Genus Two, Osaka J. Math. 17 (1980), 625–648.
[K20a] Kang, S. On nonhyperbolicity of $P/P$ and $P/SF$ knots in $S^3$, preprint.
[K20b] Kang, S. Primitive, proper power, and Seifert curves in the boundary of a genus two handlebody, preprint.
[K20c] Kang, S. Tunnel-number-one knot exteriors in $S^3$ disjoint from proper power curves, preprint.
[MMM05] Mattman, T., Miyazaki, K. and Motegi, K., Seifert-fibered surgeries which do not arise from primitive/Seifert-fibered constructions, Trans. Amer. Math. Soc. 358 no. 9, (2005), 4045–4055.
[O79] Ochiai, M., Heegaard-Diagrams and Whitehead-Graphs, Math. Sem. Notes of Kobe Univ. 7 (1979), 573–590.
[T07] Teragaito, M., A Seifert fibered manifold with infinitely many knot-surgery descriptions, Int. Math. Res. Not. 9 (2007), Art. ID rnm 028, 16 pp.

E-mail address: skang4450@chonnam.ac.kr