Boundedness and volume of generalised pairs

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Abstract. In this paper we investigate boundedness and volumes of generalised pairs, and give applications to usual pairs especially to a class of pairs that we call stable log minimal models.

Fixing the dimension and a DCC set controlling coefficients, we will show that the set of volumes of all projective generalised lc pairs \((X, B + M)\) under the given data, satisfies the DCC. Furthermore, we will show that in the klt case, the set of such pairs with ample \(K_X + B + M\) and fixed volume, forms a bounded family.

We prove a result about descent of nef divisors to bounded families. This is the key to proving the above and various other results.

We will then apply the above to study projective lc pairs \((X, B)\) with abundant \(K_X + B\) of arbitrary Kodaira dimension. In particular, we show that the set of Iitaka volumes of such pairs satisfies DCC under some natural boundedness assumptions on the fibres of the Iitaka fibration.

We define stable log minimal models which consist of a projective lc pair \((X, B)\) with semi-ample \(K_X + B\) together with a divisor \(A \geq 0\) so that \(K_X + B + A\) is ample and \(A\) does not contain any non-klt centre of \((X, B)\). This is a generalisation of both usual stable pairs of general type and stable log Calabi-Yau pairs. Fixing appropriate invariants we show that stable log minimal models form a bounded family. Then we discuss connection with moduli spaces.

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### 1. Introduction

We work over an algebraically closed field of characteristic zero.

The classification theory of higher dimensional algebraic varieties proceeds by finding special models in each birational equivalence class and then classifying such special models. In practice such special models are good minimal models and Mori fibre spaces, that is, normal projective varieties $X$ with mild singularities such that either $K_X$ is semi-ample or $X$ admits a Mori-Fano fibration $X \to T$. Classification of such $X$ in practice means construction of their possible moduli spaces. Usually the first step of this moduli theory is to fix certain invariants of $X$ and then show that the $X$ with these invariants form a bounded family. A similar philosophy is applicable to pairs.

A particular class of such $X$ consists of those with ample $K_X$. In dimension one these correspond to curves of genus $\geq 2$ which form a bounded family for each fixed genus. It turns out that in higher dimension the right invariant to fix is the volume $\langle K_X \rangle^{\dim X}$. More generally, pairs $(X,B)$ with mild singularities of fixed dimension such that $K_X + B$ is ample with fixed volume $(K_X + B)^{\dim X}$ and such that the coefficients of $B$ are in some fixed finite set, form a bounded family [18].

When $K_X$ is semi-ample but not ample, we get a contraction $f : X \to Z$ onto a normal variety. The canonical bundle formula

$$K_X \sim_\mathbb{Q} f^*(K_Z + B_Z + M_Z)$$

expresses $K_X$ in terms of $K_Z$, the discriminant divisor $B_Z$ and the moduli divisor $M_Z$. To understand $X$ one then needs to understand not only $Z$ but the structure
(Z, B_Z + M_Z) which is a generalised pair with ample \( K_Z + B_Z + M_Z \). A similar construction applies when \((X, B)\) is a pair and \( K_X + B \) is semi-ample.

Generalised pairs, introduced in [12], have found many applications in higher dimensional algebraic geometry in recent years; see [4] for a survey. A projective generalised pair \((X, B + M)\) consists of a normal projective variety \(X\), an \(\mathbb{R}\)-divisor \(B\) with non-negative coefficients, and an \(\mathbb{R}\)-divisor \(M\) which is the pushdown of a nef \(\mathbb{R}\)-divisor \(M'\) on some birational model \(X' \to X\). We also assume \(K_X + B + M\) is \(\mathbb{R}\)-Cartier. When \(M' = 0\), we get a pair in the traditional sense.

The purpose of this paper is then twofold. On the one hand, we want to investigate the geometry of generalised pairs. On the other hand, we want to apply the results on generalised pairs to understand the geometry of usual pairs.

In the case of generalised pairs, we aim to study volumes and boundedness of such pairs. Let’s fix numbers \(d, p \in \mathbb{N}\) and \(v \in \mathbb{Q}_{>0}\). Consider the set of projective generalised pairs \((X, B + M)\) with data \(X' \to X, M'\) such that

- \((X, B + M)\) is generalised lc of dimension \(d\),
- \(pB\) is integral,
- \(pM'\) is Cartier, and
- \(K_X + B + M\) is big.

The first question is whether the set of the volumes \(\text{vol}(K_X + B + M)\) of such pairs satisfies the DCC.

The second question concerns boundedness of such generalised pairs. Obviously, boundedness does not hold in such a generality. We need to impose further restrictions. Consider the set of generalised pairs as above with the additional property that

- \(K_X + B + M\) is ample with \(\text{vol}(K_X + B + M) = v\).

The question then is whether such pairs form a bounded family, more precisely, whether there is \(m \in \mathbb{N}\) depending only on \(d, p, v\) such that \(m(K_X + B + M)\) is very ample.

These questions are important as they naturally appear in the context of varieties with semi-ample canonical divisor discussed above. They are also important for understanding the geometry of generalised pairs regardless of applications to usual varieties and pairs.

For usual pairs, that is when \(M' = 0\), the answer to the above two questions are affirmative as confirmed by Hacon-McKernan-Xu [19][18]. Unfortunately, as is usually the case, the case of generalised pairs cannot simply be reduced to the case of usual pairs. The nef divisor \(M'\) creates serious complications that do not arise in the context of usual pairs.

In dimension 2, Filipazzi [14] answered the above questions affirmatively. He took advantage of special features of the geometry of surfaces.

The main strategy of the proofs in [19][18] for usual pairs is to show first that the \((X, B)\) with bounded volume \(\text{vol}(K_X + B)\) are birational to models \((\overline{X}, \overline{B})\) which are log smooth and which belong to a bounded family. This is achieved by finding a fixed number \(m\) such that \(|m(K_X + B)|\) defines a birational map: this was already established in [19]. They then use boundedness of \((\overline{X}, \overline{B})\) to reduce the questions to log fibres of some fixed fibration.

One can try to apply the same strategy in the context of generalised pairs by first finding a bounded birational model \((\overline{X}, \overline{B})\). Such models exist in any dimension, by the results of [12]. However, unlike in the case of usual pairs, we also need to control
the nef divisor $M'$. In other words, we need to choose $\overline{X}$ so that $M'$ descends to it, that is, assuming $X' \to \overline{X}$ is a morphism, we want $M'$ to be the pullback of a divisor $\overline{M}$ on $\overline{X}$. Moreover, we need $\overline{M}$ to be bounded in some sense. In dimension two, to achieve these properties, [14] uses special features of surfaces such as the well-known fact that the intersection matrix of exceptional divisors of a birational morphism of surfaces is negative definite. Unfortunately the surface arguments do not work in higher dimension.

In this paper we devise completely new techniques to ensure that in higher dimension the nef divisor $M'$ can be controlled, that is, it descends to a bounded model, in a quite general context (see Theorems 1.2, 3.14). This is one of the key results of this paper which opens the door to studying many problems about generalised pairs.

We then give affirmative answers to the first question on DCC of volumes (Theorem 1.3), and to the second question on boundedness in the generalised klt case (Theorem 1.4).

We will then apply the above to study projective lc pairs $(X, B)$ with abundant $K_X + B$ of arbitrary Kodaira dimension. To be precise, we assume $K_X + B \sim_\mathbb{Q} 0/\mathbb{Z}$ for some contraction $f: X \to Z$ where the Kodaira dimension $\kappa(K_X + B) = \dim Z$. We show that the set of Iitaka volumes of such pairs satisfies DCC under some natural boundedness assumptions on the fibres of $f$ (Theorem 1.7). Conjecturally, the DCC holds without such boundedness assumptions but this is currently out of reach.

We then define stable log minimal models which consist of a projective lc pair $(X, B)$ with semi-ample $K_X + B$ together with a divisor $A \geq 0$ so that $K_X + B + A$ is ample and $A$ does not contain any non-klt centre of $(X, B)$. This is a generalisation of both usual KSBA-stable pairs of general type and stable log Calabi-Yau pairs. Indeed, when $A = 0$ we get exactly the KSBA-stable pairs of general type, and when $K_X + B \sim_\mathbb{Q} 0$ we get exactly the stable log Calabi-Yau pairs.

Fixing appropriate invariants we show that stable log minimal models form a bounded family (Theorems 1.10 and 1.9). More precisely, we fix the dimension, the Iitaka volume of $K_X + B$ which defines a contraction $f: X \to Z$, the volume of $A|_F$ for general fibres $F$ of $f$, and the function $\text{vol}(K_X + B + tA)$ where $t$ varies in $[0, 1]$. We will explain that these coditions are natural from the point of view of moduli of stable log minimal models.

Families of generalised pairs. Before we go any further we fix some notation to simplify statement of results.

**Definition 1.1.** Let $d \in \mathbb{N}$, $\Phi \subset \mathbb{R}^{\geq 0}$, and $v \in \mathbb{R}^{>0}$.

1. Let $\mathcal{G}_{glc}(d, \Phi)$ be the set of projective generalised pairs $(X, B + M)$ with data $X' \overset{\phi}{\to} X$ and $M'$ where
   - $(X, B + M)$ is generalised lc of dimension $d$,
   - the coefficients of $B$ are in $\Phi$,
   - $M' = \sum \mu_i M'_i$ where $M'_i$ are nef Cartier and $0 < \mu_i \in \Phi$, and
   - $K_X + B + M$ is big.

   Note that $M'_i = 0$ is allowed in which case the corresponding $\mu_i$ does not play any role.

2. Let

   $$\mathcal{G}_{glc}(d, \Phi, v) \subset \mathcal{G}_{glc}(d, \Phi)$$

   by setting $v = \text{vol}(K_X + B + tA)$ for $t = 0, 1$. This is a bounded family of projective generalised pairs $(X, B + M)$ with given invariants as above.
Boundedness and volume of generalised pairs

consist of those \((X, B + M)\) that have \(\text{vol}(K_X + B + M) = v\). Similarly, let

\[ G_{\text{glc}}(d, \Phi, \leq v) \subset G_{\text{glc}}(d, \Phi) \]

consist of those \((X, B + M)\) such that the volume \(\text{vol}(K_X + B + M) < v\).

(3) Let

\[ F_{\text{glc}}(d, \Phi) \subset G_{\text{glc}}(d, \Phi) \]

and

\[ F_{\text{glc}}(d, \Phi, v) \subset G_{\text{glc}}(d, \Phi, v) \]

and

\[ F_{\text{glc}}(d, \Phi, \leq v) \subset G_{\text{glc}}(d, \Phi, \leq v) \]

in each case consist of those \((X, B + M)\) with ample \(K_X + B + M\).

(4) Define

\[ G_{\text{klt}}(d, \Phi), \quad G_{\text{klt}}(d, \Phi, v), \quad G_{\text{klt}}(d, \Phi, \leq v), \]

and

\[ F_{\text{klt}}(d, \Phi), \quad F_{\text{klt}}(d, \Phi, v), \quad F_{\text{klt}}(d, \Phi, \leq v) \]

similarly by requiring the pairs \((X, B + M)\) to be generalised klt.

(5) When 0 is not an accumulation point of \(\Phi\), e.g. when \(\Phi\) is DCC, we say that a subset \(E \subset G_{\text{glc}}(d, \Phi)\) forms a bounded family if there is \(r \in \mathbb{N}\) such that for each \((X, B + M) \in E\) there is a very ample divisor \(A\) on \(X\) with \(A^d \leq r\) and

\[ (K_X + B + M) \cdot A^{d-1} \leq r. \]

**Descent of nef divisors to bounded models.** Our first main result is about descent of nef divisors. It essentially says that the nef parts of generalised pairs with volume bounded from above descend to bounded birational models.

**Theorem 1.2.** Let \(d \in \mathbb{N}, \Phi \subset \mathbb{R}^{\geq 0}\) be a DCC set, and \(v \in \mathbb{R}^{>0}\). Then there is a bounded set of couples \(\mathcal{P}\) depending only on \(d, \Phi, v\) satisfying the following. Assume that

\[ (X, B + M) \in G_{\text{glc}}(d, \Phi, \leq v) \]

with data \(X' \xrightarrow{\phi} X\) and \(M' = \sum \mu_i M'_i\) as in Definition 1.1. Then there exist a log smooth couple \((\overline{X}, \Sigma) \in \mathcal{P}\) and a birational map \(\overline{X} \dashrightarrow X\) such that

- \(\Sigma\) contains the exceptional divisors of \(\overline{X} \dashrightarrow X\) and the support of the birational transform of \(B\), and
- every \(M'_i\) descends to \(\overline{X}\).

Recall that a couple consists of a projective variety and a reduced divisor.

We will prove another result (Theorem 3.14) on descent of nef divisors that may be more useful in certain situations.

**DCC of volume of generalised pairs.** A consequence of Theorem 1.2 is the following DCC property of volumes of generalised pairs.
Theorem 1.3. Let $d \in \mathbb{N}$ and let $\Phi \subset \mathbb{R}_{\geq 0}$ be a DCC set. Then
\[ \{ \text{vol}(K_X + B + M) \mid (X, B + M) \in G_{glc}(d, \Phi) \} \]
satisfies the DCC.

This was proved in [19, Theorem 1.3][20] for usual pairs, that is, when the nef part $M' = 0$ (the dimension 2 case was proved in [1]).

For generalised pairs of dimension 2, it was proved in [14] (assuming all $\mu_i = \frac{1}{p}$ for a fixed number $p \in \mathbb{N}$); he also treated the case in higher dimension under the assumption that $X$ is birational to a fixed variety $Z$ and the nef part $M'$ descends to $Z$.

Boundedness of generalised pairs. Another application of Theorem 1.2 is to boundedness of generalised pairs under mild conditions.

Theorem 1.4. Let $d \in \mathbb{N}$, $\Phi \subset \mathbb{R}_{\geq 0}$ be a DCC set, and $v \in \mathbb{R}_{> 0}$. Then the set $\mathcal{F}_{glc}(d, \Phi, v)$ forms a bounded family.

For now we do not prove boundedness in the generalised lc case as it requires certain ingredients that are currently being developed. Instead we prove the following crucial result which is a key ingredient of the proof of 1.4 and also allows us to treat generalised lc pairs that appear in canonical bundle formulae, hence in particular treat semi-ample pairs discussed below.

Theorem 1.5. Let $d \in \mathbb{N}$, $\Phi \subset \mathbb{R}_{\geq 0}$ be a DCC set, and $v \in \mathbb{R}_{> 0}$. Then the set
\[ \{ a(D, X, B + M) \leq 1 \mid (X, B + M) \in \mathcal{F}_{glc}(d, \Phi, v), \ D \text{ prime divisor over } X \} \]
is finite.

Here $a(D, X, B + M)$ denotes the generalised log discrepancy of $D$ with respect to $(X, B + M)$. In particular, the theorem says that the coefficients of $B$ belong to a fixed finite set. Moreover, the proof shows that the $\mu_i$ in $M' = \sum \mu_i M'_i$ for which $M'_i \neq 0$ also belong to a fixed finite set.

DCC of Iitaka volumes. We would now like to go beyond the general type case and consider pairs of any Kodaira dimension. For the definition of Iitaka volumes, see 2.15. Let $d \in \mathbb{N}$ and let $\Phi \subset \mathbb{Q}_{\geq 0}$ be a DCC set. Consider the set of projective klt pairs $(X, B)$ of dimension $d$ where the coefficients of $B$ are in $\Phi$. It is conjectured in [31] that the set of Iitaka volumes of all such pairs satisfies the DCC. Using the canonical bundle formula [16], we can find a model $Z$ of the base of the Iitaka fibration of $K_X + B$ together with divisors $B_Z, M_Z$ so that $(Z, B_Z + M_Z)$ is a generalised pair, say with nef part $M'_Z$, so that
\[ I\text{vol}(K_X + B) = \text{vol}(K_Z + B_Z + M_Z). \]
The coefficients of $B_Z$ belong to a DCC set depending only on $d, \Phi$ by the ACC for lc thresholds [19]. Thus if we can ensure that $pM'_Z$ is Cartier for some $p \in \mathbb{N}$ depending only on $d, \Phi$, then the conjecture would follow from Theorem 1.3. However, existence of such $p$ in general is currently out of reach even when dim $Z = 1$. So we need to make some kind of boundedness assumption on the fibres of the Iitaka fibration.

An option is to impose boundedness on the fibres similar to [12]. This has the advantage that it does not require the fibres to actually belong to a bounded family, e.g. the fibres can be K3 surfaces.
Another option is to impose conditions that imply actual boundedness of the fibres. For now we take this route because it is more in line with the context of this paper. First we fix some notation which is inspired by the point of view and the results of [5] and by moduli of stable log minimal models discussed below.

**Definition 1.6.** Let \( d \in \mathbb{N} \), \( \Phi \subset \mathbb{Q}^{\geq 0} \), and \( u \in \mathbb{Q}^{>0} \). Let \( I_{lc}(d, \Phi, u) \) be the set of projective pairs \((X, B)\) such that

- \((X, B)\) is lc of dimension \(d\),
- the coefficients of \(B\) are in \(\Phi\),
- \(f: X \to Z\) is a contraction with \(K_X + B \sim_{\mathbb{Q}} 0/Z\),
- \(\kappa(K_X + B) = \dim Z\), and
- there is an integral divisor \(A \geq 0\) on \(X\) such that over some non-empty open subset of \(Z\): \((X, B + tA)\) is lc for some \(t > 0\) and \(A\) is ample,
- \(\text{vol}(A|_F) = u\) for the general fibres \(F\) of \(f\).

Define \(I_{lc}(d, \Phi, <u)\) similarly by replacing the condition \(\text{vol}(A|_F) = u\) with \(\text{vol}(A|_F) < u\). And define \(I_{klt}(d, \Phi, u)\) and \(I_{klt}(d, \Phi, <u)\) similarly by replacing the lc condition of \((X, B)\) with klt.

When \(\Phi\) is DCC, the divisor \(A\) ensures that if \((X, B)\) is in any of the above sets, then the log general fibres \((F, B_F)\) belong to a bounded family. This follows from [5, Corollaries 1.6 and 1.8].

**Theorem 1.7.** Let \(d \in \mathbb{N} \), \(\Phi \subset \mathbb{Q}^{\geq 0}\) be a DCC set, and \(u \in \mathbb{R}^{>0}\). Then the sets

\[ \{ \text{Ivol}(K_X + B) \mid (X, B) \in I_{klt}(d, \Phi, <u) \} \]

and

\[ \{ \text{Ivol}(K_X + B) \mid (X, B) \in I_{lc}(d, \Phi, u) \} \]

satisfy the DCC.

We will see that the assumptions of the theorem ensure that we get the generalised pair \((Z, B_Z + M_Z)\) from the adjunction

\[ K_X + B \sim_{\mathbb{Q}} f^*(K_Z + B_Z + M_Z) \]

where the coefficient of \(B_Z\) belong to a fixed DCC set and that \(pM_Z\) is Cartier for some fixed \(p\) on some high resolution \(Z' \to Z\), so we can apply Theorem 1.3. Moreover, fixing the Iitaka volume in the klt case, Theorem 1.4 shows that the log canonical model of \((X, B)\) which is given by the Proj of the log canonical ring of \((X, B)\) belongs to a bounded family.

The special case of the theorem when \((X, B)\) is \(\epsilon\)-lc for fixed \(\epsilon > 0\) and \(B\) is big over \(Z\) was proved in Li [31] using quite different arguments. In this case we can assume \(A = -K_X\) where \(\text{vol}(A|_F)\) is automatically bounded from above as \(F\) belongs to a bounded family by BAB [7].

Much more recently, Jiao [22] independently proved DCC of Iitaka volumes for pairs that are similar to those in \(I_{klt}(d, \Phi, <u)\), using methods that seem quite different from ours.

**Boundedness of stable log minimal models.** We now define stable minimal models. The class of such pairs includes both usual KSBA-stable pairs of general type and stable Calabi-Yau pairs (i.e. polarised Calabi-Yau pairs) as special cases. It is designed so that we can address moduli of minimal models of arbitrary Kodaira dimension.
In the general type case, say when \((X, B)\) is a projective lc pair with ample \(K_X + B\), to ensure boundedness of \((X, B)\) it is enough to fix the dimension of \(X\), let coefficients of \(B\) be in a fixed DCC set, and let the volume of \(K_X + B\) be fixed. In the stable Calabi-Yau case, say when \((X, B)\) is a projective lc log Calabi-Yau pair, to ensure boundedness we need to fix the dimension, let the coefficients of \(B\) be in a fixed DCC set, and in addition have an ample integral divisor \(A \geq 0\) on \(X\) with fixed volume so that \((X, B + tA)\) is lc for some \(t > 0\). In the intermediate Kodaira dimension case, we need more delicate conditions that are roughly a mixture of the conditions of the general type and the Calabi-Yau cases.

**Definition 1.8.** Let \(d \in \mathbb{N}\), \(\Phi \subset \mathbb{Q}^\geq 0\), and \(u, v, w \in \mathbb{Q}^> 0\).

1. A stable log minimal model \((X, B, A)\) consists of a projective pair \((X, B)\) and an \(\mathbb{R}\)-divisor \(A \geq 0\) such that
   - \((X, B)\) is lc,
   - \(K_X + B\) is semi-ample,
   - \(K_X + B + A\) is ample, and
   - \((X, B + tA)\) is lc for some \(t > 0\).

2. A \((d, \Phi, u, v)\)-stable log minimal model is a stable log minimal model \((X, B, A)\) such that
   - \(\dim X = d\),
   - the coefficients of \(B\) and \(A\) are in \(\Phi\),
   - \(\text{vol}(A|_F) = u\) where \(F\) is a general fibre of the contraction \(f : X \to Z\) defined by \(K_X + B\), and
   - \(\text{Ivol}(K_X + B) = v\).

3. Let \(S_{lc}(d, \Phi, u, v)\) consist of all the \((d, \Phi, u, v)\)-stable log minimal models. Similarly define \(S_{klt}(d, \Phi, u, v)\) by replacing the lc condition of \((X, B)\) with klt.

4. Let \(S_{klt}(d, \Phi, u, v, < w)\) consist of those \((X, B), A\) in \(S_{klt}(d, \Phi, u, v)\) such that \(\text{vol}(K_X + B + A) < w\).

5. Now let \(\sigma \in \mathbb{Q}[s]\) be a polynomial. Let \(S_{lc}(d, \Phi, u, v, \sigma)\) consist of those \((X, B), A\) in \(S_{lc}(d, \Phi, u, v)\) such that
   \[
   \text{vol}(K_X + B + tA) = \sigma(t), \quad \forall t \in [0, 1].
   \]

6. When 0 is not an accumulation point of \(\Phi\), e.g. when it is a DCC set, a subset \(E \subset S_{lc}(d, \Phi, u, v)\) is said to be a bounded family if there is a fixed \(r \in \mathbb{N}\) such that for any \((X, B), A\) in the family we can find a very ample divisor \(H\) on \(X\) so that \(H^d \leq r\) and
   \[
   (K_X + B + A) \cdot H^{d-1} \leq r.
   \]

Note that in (5), \(K_X + B + tA\) is nef for any \(t \in [0, 1]\), so its volume is just \((K_X + B + tA)^d\) which is a polynomial in \(t\) with rational coefficients, hence it makes sense to take \(\sigma\) to be a polynomial. We will see below the motivation for including such functions \(\sigma\) when we discuss moduli spaces. It turns out that the Iitaka volume \(v\) is determined by \(\sigma\).
Theorem 1.9. Let \( d \in \mathbb{N}, \Phi \subset \mathbb{Q}^\geq 0 \) be a DCC set, and \( u, v, w \in \mathbb{Q}^> 0 \). Then
\[
S_{klt}(d, \Phi, u, v, <w)
\]
is a bounded family.

Boundedness in the lc case is more delicate. We include a polynomial \( \sigma \) to get boundedness.

Theorem 1.10. Let \( d \in \mathbb{N}, \Phi \subset \mathbb{Q}^\geq 0 \) be a DCC set, \( u, v \in \mathbb{Q}^> 0 \), and \( \sigma \in \mathbb{Q}[s] \) be a polynomial. Then \( S_{lc}(d, \Phi, u, v, \sigma) \) is a bounded family.

There seems to be very little related results in the literature except for stable log minimal models \((X, B), A\) when \( A = 0 \) \([18]\) or when \( K_X + B \sim_\mathbb{Q} 0 \) \([5]\).

Applications to moduli. Recall that lc KSBA-stable pairs of general type are just stable log minimal models \((X, B), A\) with \( A = 0 \). Fixing appropriate invariants, projective coarse moduli spaces exist for slc KSBA-stable pairs; see Kollár \([26]\) and the references therein. Also recall that lc stable log Calabi-Yau pairs are just stable log minimal models \((X, B), A\) with \( K_X + B \sim_\mathbb{Q} 0 \). Again fixing appropriate invariants, projective coarse moduli spaces exist for stable log Calabi-Yau pairs; see Birkar \([5]\) and the references therein.

On the other hand, Viehweg used geometric invariant theory to construct quasi-projective coarse moduli spaces for polarised smooth good minimal models \( X, A \) where \( X \) is a smooth projective variety with semi-ample \( K_X \) and \( A \) is an ample Cartier divisor considered up to linear equivalence (so \( A \) is not a fixed Weil divisor); see \([34]\) and the references therein. So the moduli space parametrises \( X, \mathcal{O}(A) \). But this comes at a price: in general we cannot compactify the moduli space in a meaningful way because a family of such \( X, \mathcal{O}(A) \) over a punctured curve does not have a unique limit. This is one of the main reasons \( A \) is assumed to be an effective Weil divisor in the definition of stable minimal models.

To get a finite type moduli space, Viehweg fixes a polynomial \( g \in \mathbb{Q}[x, y] \) among other things, and then considers the family of polarised smooth good minimal models \( X, A \) with
\[
g(m, n) = \chi(mK_X + nA)
\]
for \( m, n \in \mathbb{N} \) where \( \chi \) stands for the Euler characteristic. In particular, \( g \) determines the polynomial \( \sigma \in \mathbb{Q}[s] \) satisfying \( \sigma(t) := \text{vol}(K_X + tA) \) for \( t \in [0, 1] \). Indeed, when \( t \in (0, 1] \) is a fixed rational number, for sufficiently divisible \( m \in \mathbb{N} \) we have
\[
h^0(m(K_X + tA)) = \chi(m(K_X + tA)) = g(m, mt)
\]
by the Kodaira vanishing theorem, so
\[
\sigma(t) = \limsup_{m \to \infty} \frac{d! h^0(m(K_X + tA))}{m^d} = \limsup_{m \to \infty} \frac{d! g(m, mt)}{m^d}.
\]
This explains the role of \( \sigma \) in the definition of stable log minimal models above.

In a forthcoming work we will treat moduli spaces of stable slc log minimal models where Theorem 1.10 plays a key role.

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2. Preliminaries

All the varieties in this paper are quasi-projective over a fixed algebraically closed field of characteristic zero unless stated otherwise. The set of natural numbers \( \mathbb{N} \) is the set of positive integers, so it does not contain 0.

2.1. Contraction. In this paper a contraction refers to a projective morphism \( f: X \to Y \) of varieties such that \( f_* \mathcal{O}_X = \mathcal{O}_Y \) (\( f \) is not necessarily birational). In particular, \( f \) has connected fibres and if \( X \to Z \to Y \) is the Stein factorisation of \( f \), then \( Z \to Y \) is an isomorphism.

2.2. Divisors. Let \( X \) be a normal variety, and let \( M \) be an \( \mathbb{R} \)-divisor on \( X \). We denote the coefficient of a prime divisor \( D \) in \( M \) by \( \mu_D M \).

We say \( M \) is \( b \)-Cartier if it is \( \mathbb{R} \)-Cartier and if there is a birational contraction \( \phi: W \to X \) from a normal variety such that \( \phi^* M \) is Cartier.

Now let \( f: X \to Z \) be a morphism to a normal variety. We say \( M \) is horizontal over \( Z \) if the induced map \( \text{Supp} \ M \to Z \) is dominant, otherwise we say \( M \) is vertical over \( Z \). If \( N \) is an \( \mathbb{R} \)-Cartier divisor on \( Z \), we sometimes denote \( f^* N \) by \( N|_X \).

Again let \( f: X \to Z \) be a morphism to a normal variety, and let \( M \) and \( L \) be \( \mathbb{R} \)-Cartier divisors on \( X \). We say \( M \sim L \) over \( Z \) (resp. \( M \sim_Q L \) over \( Z \))(resp. \( M \sim_R L \) over \( Z \)) if there is a Cartier (resp. \( \mathbb{Q} \)-Cartier)(resp. \( \mathbb{R} \)-Cartier) divisor \( N \) on \( Z \) such that \( M - L \sim f^* N \) (resp. \( M - L \sim_Q f^* N \))(resp. \( M - L \sim_R f^* N \)). For a point \( z \in Z \), we say \( M \sim L \) over \( z \) if \( M \sim L \) over \( Z \) perhaps after shrinking \( Z \) around \( z \).

The properties \( M \sim_Q L \) and \( M \sim_R L \) over \( z \) are similarly defined.

For a birational map \( X \dashrightarrow X' \) (resp. \( X \dashrightarrow X'' \))(resp. \( X \dashrightarrow X''' \)) whose inverse does not contract divisors, and for an \( \mathbb{R} \)-divisor \( M \) on \( X \) we usually denote the pushdown of \( M \) to \( X' \) (resp. \( X'' \))(resp. \( X''' \)) by \( M' \) (resp. \( M'' \))(resp. \( M''' \))(resp. \( M_Y \)).

2.3. \( b \)-divisors. We recall some definitions regarding \( b \)-divisors. Let \( X \) be a normal variety. A \( b \)-divisor \( M \) over \( X \) is a collection of \( \mathbb{R} \)-divisors \( M_Y \) on \( Y \) for each birational contraction \( Y \to X \) from a normal variety that are compatible with respect to pushdown, that is, \( Y' \to X \) is another birational contraction and \( \psi: Y' \to Y \) is a morphism, then \( \psi_* M_{Y'} = M_Y \).

A \( b \)-divisor \( M \) is \( b \)-\( \mathbb{R} \)-Cartier if there is a birational contraction \( Y \to X \) such that \( M_Y \) is \( \mathbb{R} \)-Cartier and such that for any birational contraction \( \psi: Y' \to Y/X \) we have \( M_{Y'} = \psi^* M_Y \). In other words, a \( b \)-\( \mathbb{R} \)-Cartier \( b \)-divisor over \( X \) is determined by the choice of a birational contraction \( Y \to X \) and an \( \mathbb{R} \)-Cartier \( \mathbb{R} \)-divisor \( M \) on \( Y \). But this choice is not unique, that is, another birational contraction \( Y' \to X \) and an \( \mathbb{R} \)-Cartier \( \mathbb{R} \)-divisor \( M' \) on \( Y' \) defines the same \( b \)-\( \mathbb{R} \)-Cartier \( b \)-divisor if there is a common resolution \( W \to Y \) and \( W \to Y' \) on which the pullbacks of \( M \) and \( M' \) coincide.

A \( b \)-\( \mathbb{R} \)-Cartier \( b \)-divisor represented by some \( Y \to X \) and \( M \) is \( b \)-Cartier if \( M \) is \( b \)-Cartier, i.e. its pullback to some resolution is Cartier.

2.4. Pairs. In this paper a sub-pair \((X, B)\) consists of a normal quasi-projective variety \( X \) and an \( \mathbb{R} \)-divisor \( B \) such that \( K_X + B \) is \( \mathbb{R} \)-Cartier. If the coefficients of \( B \) are at most 1 we say \( B \) is a sub-boundary, and if in addition \( B \geq 0 \), we say \( B \) is a boundary. A sub-pair \((X, B)\) is called a pair if \( B \geq 0 \) (we allow coefficients of \( B \) to be larger than 1 for practical reasons).
Let $\phi: W \to X$ be a log resolution of a sub-pair $(X, B)$. Let $K_W + B_W$ be the pullback of $K_X + B$. The log discrepancy of a prime divisor $D$ on $W$ with respect to $(X, B)$ is $1 - \mu_D B_W$ and it is denoted by $a(D, X, B)$. We say $(X, B)$ is sub-lc (resp. sub-$\epsilon$-lc) if $a(D, X, B) \geq 0$ (resp. $> 0$) (resp. $\geq \epsilon$) for every $D$. When $(X, B)$ is a pair we remove the sub and say the pair is lc, etc. Note that if $(X, B)$ is an lc pair, then the coefficients of $B$ necessarily belong to $[0, 1]$. Also if $(X, B)$ is $\epsilon$-lc, then automatically $\epsilon \leq 1$ because $a(D, X, B) = 1$ for most $D$.

Let $(X, B)$ be a sub-pair. A non-klt place of $(X, B)$ is a prime divisor $D$ on birational models of $X$ such that $a(D, X, B) \leq 0$. A non-klt centre is the image on $X$ of a non-klt place. When $(X, B)$ is lc, a non-klt centre is also called an lc centre.

2.5. Minimal model program (MMP). We will use standard results of the minimal model program (cf. [27][11]). Assume $(X, B)$ is a pair and $X \to Z$ is a projective morphism. Assume $H$ is an ample $\mathbb{R}$-divisor and that $K_X + B + H$ is nef$/\mathbb{R}$. Suppose $(X, B)$ is klt or that it is $\mathbb{Q}$-factorial dlt. We can run an MMP$/\mathbb{R}$ on $K_X + B$ with scaling of $H$. If $(X, B)$ is klt and if either $K_X + B$ or $B$ is big$/\mathbb{R}$, then the MMP terminates [11]. If $(X, B)$ is $\mathbb{Q}$-factorial dlt, then in general we do not know whether the MMP terminates but we know that in some step of the MMP we reach a model $Y$ on which $K_Y + B_Y$, the pushdown of $K_X + B$, is a limit of movable$/\mathbb{Z}$ $\mathbb{R}$-divisors: indeed, if the MMP terminates, then the claim is obvious; otherwise the MMP produces an infinite sequence $X_i \to X_{i+1}$ of flips and a decreasing sequence $\alpha_i$ of numbers in $(0, 1]$ such that $K_{X_i} + B_i + \alpha_i H_i$ is nef$/\mathbb{Z}$; by [11][10, Theorem 1.9], $\lim \alpha_i = 0$; in particular, if $Y := X_1$, then $K_Y + B_Y$ is the limit of the movable$/\mathbb{Z}$ $\mathbb{R}$-divisors $K_Y + B_Y + \alpha_i H_Y$.

2.6. Generalised pairs. For the basic theory of generalised pairs see [12, Section 4]. Below we recall some of the main notions and discuss some basic properties.

1. A generalised pair consists of

   - a normal variety $X$ equipped with a projective morphism $X \to Z$,
   - an $\mathbb{R}$-divisor $B \geq 0$ on $X$, and
   - a $b$-$\mathbb{R}$-Cartier b-divisor over $X$ represented by some birational contraction

   $X' \xrightarrow{\phi} X$ and $\mathbb{R}$-Cartier divisor $M'$ on $X'$

   such that $M'$ is nef$/\mathbb{Z}$ and $K_X + B + M$ is $\mathbb{R}$-Cartier, where $M := \phi_* M'$.

   We usually refer to the pair by saying $(X, B + M)$ is a generalised pair with data $X' \xrightarrow{\phi} X \to Z$ and $M'$. Since a $b$-$\mathbb{R}$-Cartier b-divisor is defined birationally, in practice we will often replace $X'$ with a resolution and replace $M'$ with its pullback. When $Z$ is not relevant we usually drop it and do not mention it: in this case one can just assume $X \to Z$ is the identity. When $Z$ is a point we also drop it but say the pair is projective.

   Now we define generalised singularities. Replacing $X'$ we can assume $\phi$ is a log resolution of $(X, B)$. We can write

   $K_{X'} + B' + M' = \phi_*(K_X + B + M)$

   for some uniquely determined $B'$. For a prime divisor $D$ on $X'$ the generalised log discrepancy $a(D, X, B + M)$ is defined to be $1 - \mu_D B'$.

   We say $(X, B + M)$ is generalised lc (resp. generalised klt)(resp. generalised $\epsilon$-lc) if for each $D$ the generalised log discrepancy $a(D, X, B + M)$ is $\geq 0$ (resp. $> 0$) (resp. $\geq \epsilon$). A generalised non-klt place of $(X, B + M)$ is a prime divisor $D$ on birational models of $X$ with $a(D, X, B + M) \leq 0$, and a generalised non-klt centre of $(X, B + M)$
is the image of a generalised non-klt place. The generalised non-klt locus of the generalised pair is the union of all the generalised non-klt centres.

We will also use similar definitions when $B$ is not necessarily effective in which case we have a generalised sub-pair.

(2) Let $(X, B+M)$ be a generalised pair as in (1). We say $(X, B+M)$ is generalised dlt if it is generalised lc and if $\eta$ is the generic point of any generalised non-klt centre of $(X, B+M)$, then $(X, B)$ is log smooth near $\eta$ and $M' = \phi^* M$ holds over a neighbourhood of $\eta$. Note that when $M' = 0$, then $(X, B)$ is generalised dlt if it is dlt in the usual sense.

The generalised dlt property is preserved under the MMP [8, 2.13(2)].

(3) Let $(X, B + M)$ be a generalised pair as in (1) and let $\psi : X'' \to X$ be a projective birational morphism from a normal variety. Replacing $\phi$ we can assume $\phi$ factors through $\psi$. We then let $B''$ and $M''$ be the pushdowns of $B'$ and $M'$ on $X''$ respectively. In particular,

$$K_{X''} + B'' + M'' = \psi^*(K_X + B + M).$$

If $B'' \geq 0$, then $(X'', B'' + M'')$ is also a generalised pair with data $X' \to X'' \to Z$ and $M'$.

Assume that we can write $B'' = \Delta'' + G''$ where $(X'', \Delta'' + M'')$ is $\mathbb{Q}$-factorial generalised dlt, $G'' \geq 0$ is supported in $[\Delta'']$, and every exceptional prime divisor of $\psi$ is a component of $[\Delta'']$. Then we say $(X'', \Delta'' + M'')$ is a $\mathbb{Q}$-factorial generalised dlt model of $(X, B + M)$. Such models exist by [3, Lemma 2.8] (also see [19, Proposition 3.3.1] and [12, Lemma 4.5]). If $(X, B + M)$ is generalised lc, then $G'' = 0$.

(4)

**Lemma 2.7.** Assume $(X, B + M)$ is a generalised pair with data $X' \xrightarrow{\phi} X$ and $M'$ and assume $K_X + B$ is $\mathbb{R}$-Cartier. Write $\phi^* M = M' + \sum e_i E_i$ where $E_i$ run through the exceptional divisors of $\phi$. Then for each $i$ we have

$$a(E_i, X, B) = a(X, B + M) + e_i.$$ 

**Proof.** Write

$$K_{X'} + C' = \phi^*(K_X + B)$$

and

$$K_{X'} + B' + M' = \phi^*(K_X + B + M).$$

Clearly $B' = C' + \sum e_i E_i$. Thus

$$a(E_i, X, B + M) = 1 - \mu_{E_i} B' = 1 - \mu_{E_i} C' - e_i = a(E_i, X, B) - e_i.$$

\[ \square \]

### 2.8. Generalised lc thresholds

Let $(X, B + M)$ be a generalised pair with data $X' \xrightarrow{\phi} X$ and $M'$. Assume that $D$ on $X$ is an effective $\mathbb{R}$-divisor and that $N'$ on $X'$ is an $\mathbb{R}$-divisor which is nef$/X$ and that $D + N$ is $\mathbb{R}$-Cartier where $N = \phi_* N'$. The generalised lc threshold of $D + N$ with respect to $(X, B + M)$ is defined as

$$\sup\{ s \in \mathbb{R} \mid (X, B + sD + M + sN) \text{ is generalised lc} \}$$

where the pair in the definition has boundary part $B + sD$ and nef part $M' + sN'$.

The next lemma says that the generalised lc threshold is bounded from below away from zero under some boundedness assumptions.
Lemma 2.9. Let \( d, r \in \mathbb{N} \) and \( \epsilon \in \mathbb{R}^{>0} \). Then there exists \( t \in \mathbb{R}^{>0} \) depending only on \( d, r, \epsilon \) satisfying the following. Assume that

- \((X, B + M)\) is a projective generalised \( \epsilon \)-lc pair of dimension \( d \) with data \( X' \xrightarrow{\phi} X \) and \( M' \),
- \( A \) is a very ample divisor on \( X \) with \( A^d \leq r \),
- \( N' \) is a nef \( \mathbb{R} \)-divisor on \( X' \) and \( D \) is an effective \( \mathbb{R} \)-divisor on \( X \),
- \( D + N \) is \( \mathbb{R} \)-Cartier where \( N = \phi_* N' \), and
- \( A - (B + M + N + D) \) is pseudo-effective.

Then

\[(X, B + tD + M + tN)\]

is generalised klt with nef part \( M' + tN' \).

Proof. Since \( N' \) is nef, by the negativity lemma, \( \phi^*(D + N) = D' + N' \) where \( D' \geq 0 \). Pick an ample \( \mathbb{Q} \)-divisor \( G \) on \( X \) and let \( G' = \phi_* G \). Write \( G' \sim Q H' + P' \) where \( H' \) is ample and \( P' \geq 0 \). Pick a general \( \mathbb{R} \)-divisor \( 0 \leq T' \sim \mathbb{R} N' + H' \).

Since

\[D' + T' + P' \sim \mathbb{R} D' + N' + H' + P' \sim \mathbb{R} 0/X,\]

we have

\[D' \leq D' + T' + P' = \phi^* \phi_*(D' + T' + P').\]

Moreover, picking \( G \) small enough and replacing \( A \) with \( 2A \) we can assume

\[A - B - M - \phi_*(D' + T' + P') \sim \mathbb{R} A - (B + M + D + N + G)\]

is pseudo-effective. Thus replacing \( N' \) with 0 and replacing \( D \) with \( \phi_*(D' + T' + P') \) we can assume \( N' = 0 \).

Applying a similar argument reduces the problem to the case \( M' = 0 \). Indeed, write

\[K_{X'} + B' + M' = \phi^*(K_X + B + M).\]

Pick a general \( \mathbb{R} \)-divisor

\[0 \leq S' \sim \mathbb{R} M' + H'\]

and let \( S = \phi_* S' \) and \( P = \phi_* P' \) where \( H', P' \) are as above. Then we can assume \( (X, B + S + P) \) is \( \frac{\epsilon}{2} \)-lc because

\[K_{X'} + B' + S' + P' \sim \mathbb{R} K_{X'} + B' + M' + H' + P' \sim \mathbb{R} 0/X\]

which means

\[K_{X'} + B' + S' + P' = \phi^*(K_X + B + S + P)\]

so \((X, B + S + P)\) is \( \frac{\epsilon}{2} \)-lc assuming \( P' \) is sufficiently small. If

\[(X, B + S + P + tD)\]

is klt for some \( t > 0 \), then \((X, B + tD + M)\) is generalised klt. Therefore, replacing \( B \) with \( B + S + P \) we can assume \( M' = 0 \).

Now by [7, Theorem 1.6], there is \( t > 0 \) depending only on \( d, r, \epsilon \) such that

\[(X, B + tD)\]

is klt.

\[\Box\]
2.10. **MMP for generalised pairs.** Given a generalised lc pair \((X, B + M)\) with data \(X' \to X \to Z\) and \(M'\), we can run the MMP on \(K_X + B + M\) over \(Z\) with scaling of an ample divisor if \((X, C)\) is klt for some boundary \(C\), for example, this is the case when \((X, B + M)\) is \(\mathbb{Q}\)-factorial generalised dlt or when it is generalised klt. However, we do not know in general if the MMP terminates. But the MMP terminates in the generalised klt case when \(K_X + B + M\) or \(B + M\) is big over \(Z\).

For more on MMP on generalised pairs, see [12, Lemma 4.4].

**Lemma 2.11.** Let \((X, B + M)\) be a generalised klt pair with data \(X' \to X \to Z\) and \(M'\). Assume that \(K_X + B\) is big over \(Z\), \(K_X + B + M\) is nef over \(Z\), and \(M\) is \(\mathbb{R}\)-Cartier. Then we can run the MMP on \(K_X + B\) over \(Z\) with scaling of \(M\) and it terminates.

**Proof.** Since \(K_X + B\) is big over \(Z\), we can write \(K_X + B \sim_{\mathbb{R}} D + A\) where \(D \geq 0\) and \(A \geq 0\) is ample over \(Z\). Assume \(t > 0\) is sufficiently small. Then

\[
K_X + B + tD + tA + (1 + t)M
\]

is nef over \(Z\) and

\[
(X, B + tD + tA + (1 + t)M)
\]

is generalised klt with nef part \(tA' + (1 + t)M'\) where \(A'\) on \(X'\) is the pullback of \(A\).

It is enough to run an MMP on

\[
K_X + B + tD + tA + (1 + t)M
\]

over \(Z\) with scaling of \(tD + tA + (1 + t)M\) as it induces an MMP on \(K_X + B\) with scaling of \(M\). Perturbing the nef and big/Z divisor \(A' + (1 + t)M'\) we can find

\[
0 \leq L \sim_{\mathbb{R}} tD + tA + (1 + t)M
\]

so that \((X, B + L)\) is klt. Now the MMP on \(K_X + B\) over \(Z\) with scaling of \(L\) terminates by [11].

\[\Box\]

2.12. **Toroidal pairs.** We recall a few remarks about toroidal pairs. A pair \((Y, \Delta)\) is toroidal if for each closed point \(y \in Y\) there is a toric pair \((Y', \Delta')\) and a closed point \(y' \in Y'\) such that \((Y, \Delta)\) and \((Y', \Delta')\) are formally isomorphic near \(y, y'\). In particular, \(\Delta\) is reduced, \((Y, \Delta)\) is lc, \(Y\) is smooth outside \(\Delta\), and \(K_Y + \Delta\) is Cartier (cf. [13, Exercise 7.7] for the Cartier property; in our setting in this paper this will be obvious). For the theory of toroidal pairs see [25]. A toroidal pair \((Y, \Delta = \sum_{i=1}^n S_i)\) is strict if the \(S_i\) are normal. In this case, \(\bigcap_{i \in I} S_i\) is normal and \(\bigcap_{i \in I} S_i \setminus \sum_{i \notin I} S_i\) is smooth for each subset \(I \subseteq \{1, \ldots, n\}\) [25, page 57]. The irreducible components of such \(\bigcap_{i \in I} S_i\) are the strata of \((Y, \Delta)\), hence are normal.

Let \((\hat{Y}, \Delta)\) be a strictly toroidal pair. The non-klt centres of this pair are exactly its strata. Assume that \(0 \leq C \leq \Delta\), \(K_Y + C\) is \(\mathbb{Q}\)-Cartier, and \(V\) is a non-klt centre of \((Y, C)\). Then there is an adjunction formula \(K_V + C|_V = (K_Y + C)|_V\): indeed, \(V\) is also a non-klt centre of \((Y, \Delta)\), hence it is an irreducible component of \(\bigcap_{i \in I} S_i\) for some \(I\); since \(\bigcap_{i \in I} S_i\) is normal, shrinking \(Y\) near \(V\) we can assume \(V = \bigcap_{i \in I} S_i\); also \(\sum_{i \in I} S_i \leq [C]\); picking \(i \in I\), we have divisorial adjunctions \(K_{S_i} + C|_{S_i} = (K_Y + C)|_{S_i}\) and \(K_{S_i} + \Delta|_{S_i} = (K_Y + \Delta)|_{S_i}\). Now \((S_i, \Delta|_{S_i})\) is strictly toroidal, so repeating the argument (or applying induction) proves the claim.

In the previous paragraph, if the coefficients of \(C\) belong to a DCC set \(\Phi\), then the coefficients of \(C_Y\) belong to a DCC set \(\Psi\) depending only on \(\Phi\) [33, Corollary 3.10][12, Proposition 4.9].
2.13. **Volume of divisors.** Recall that the volume of an $\mathbb{R}$-divisor $L$ on a normal projective variety $X$ of dimension $d$ is defined as

$$\text{vol}(L) = \limsup_{m \to \infty} \frac{h^0(mL)}{m^d/d!}.$$ 

**Lemma 2.14.** Let $\phi: W \to X$ be a birational morphism between normal projective varieties. Assume $H$ is an ample $\mathbb{R}$-divisor on $X$ and that $G$ is an $\mathbb{R}$-divisor on $W$ that is either

- effective and not exceptional over $X$, or
- nef but not numerically trivial.

Then $\text{vol}(\phi^*H + G) > \text{vol}(H)$.

**Proof.** Assume $\text{vol}(\phi^*H + G) = \text{vol}(H)$. First suppose $G$ is nef but not numerically trivial. Then

$$\text{vol}(\phi^*H + G) = (\phi^*H + G)^d \geq \text{vol}(H) + \phi^*H \cdot G > \text{vol}(H),$$

where $d = \dim X$, a contradiction.

Now assume $G$ is effective but not exceptional and let $D$ be a component of $G$ not contracted over $X$. We will argue as in the proof of [18, Lemma 2.2.2]. If $g$ is the coefficient of $D$ in $G$, then for $t \in [0, g]$ we have

$$\text{vol}(H) \leq v(t) := \text{vol}(\phi^*H + tD) \leq \text{vol}(\phi^*H + G),$$

so $v(t)$ is a constant function on $[0, g]$. But then by [30, 4.25(iii)] we have

$$0 = v'(0) = d\text{vol}_W(\phi^*H) \geq \phi^*H \cdot D \neq 0$$

where $v'(t)$ denotes the differentiation of $v(t)$, and $\text{vol}_W$ denotes restricted volume. This is not possible, a contradiction. \hfill $\square$

2.15. **Iitaka volumes.** Let $L$ be a $\mathbb{Q}$-divisor on a normal projective variety $X$ with Kodaira dimension $\kappa(L)$. Following [31], define the **Iitaka volume** of $L$ to be

$$\text{Ivol}(L) := \limsup_{m \to \infty} \frac{h^0(mL)}{m^{\kappa(L)}/\kappa(L)!}$$

when $\kappa(L) \geq 0$, and define it to be 0 when $\kappa(L) = -\infty$. When $\kappa(L) = 0$, by convention we let $\kappa(L)! = 1$, so in this case $\text{Ivol}(L) = 1$.

If $f: X \to Z$ is a contraction and $L \sim_{\mathbb{Q}} f^*A$ for some big $\mathbb{Q}$-divisor $A$, then $\text{Ivol}(L) = \text{vol}(A)$.

2.16. **Rational maps.** The following lemma is similar to [18, Lemma 2.2.2].

**Lemma 2.17.** Assume that

- $(X, B + M)$ is a projective generalised lc pair with data $X' \overset{\phi}{\to} X$ and $M'$,
- $K_X + B + M$ is ample,
- $(Y, B_Y + M_Y)$ is a projective generalised lc pair with data $X' \overset{\psi}{\to} Y$ and $M'$,
- $\rho: Y \to X$ is a birational map,
- that
  $$\text{vol}(K_Y + B_Y + M_Y) = \text{vol}(K_X + B + M),$$
- and
  $$a(D, X, B + M) \geq a(D, Y, B_Y + M_Y)$$
  for any prime divisor $D$ over $X$. 

Then \( \rho^{-1} \) does not contract any divisor and \((X, B + M)\) is the generalised lc model of \((Y, B_Y + M_Y)\).

Proof. Generalised lc models are analogues of lc models for usual pairs. Here \((X, B + M)\) being the generalised lc model of \((Y, B_Y + M_Y)\) means that \(X\) is the ample model of \(K_Y + B_Y + M_Y\) and that \(K_X + B + M\) is the pushdown of \(K_Y + B_Y + M_Y\).

By the last assumption,

\[
P' := \psi^*(K_Y + B_Y + M_Y) - \phi^*(K_X + B + M)
\]

is effective. Applying Lemma 2.14, we see that \(P'\) is exceptional over \(X\), hence \((X, B + M)\) is indeed the generalised lc model of \((Y, B_Y + M_Y)\), so \(\rho^{-1}\) does not contract any divisor.

\[\square\]

2.18. Adjunction for fibre spaces. Let \((X, B)\) be a projective sub-pair and let \(f: X \to Z\) be a contraction with \(\dim Z > 0\) such that \((X, B)\) is sub-lc near the generic fibre of \(f\), and \(K_X + B \sim_{\mathbb{R}} 0/Z\). Below we recall a construction giving a kind of canonical bundle formula based on [23] which we refer to as adjunction for fibre spaces. The klt case is done in [2] and the lc case is done in [15] by reducing it to the klt case. See also [16] for another variant.

For each prime divisor \(D\) on \(Z\) we let \(t_D\) be the lc threshold of \(f^*D\) with respect to \((X, B)\) over the generic point of \(D\), that is, \(t_D\) is the largest number so that \((X, B + t_D f^*D)\) is sub-lc over the generic point of \(D\). Next let \(b_D = 1 - t_D\), and then define \(B_Z = \sum b_D D\) where the sum runs over all the prime divisors on \(Z\).

By assumption, \(K_X + B \sim_{\mathbb{R}} f^*L_Z\) for some \(\mathbb{R}\)-Cartier \(\mathbb{R}\)-divisor \(L_Z\) on \(Z\). Letting \(M_Z = L_Z - (K_Z + B_Z)\) we get

\[K_X + B \sim_{\mathbb{R}} f^*(K_Z + B_Z + M_Z).\]

We call \(B_Z\) the discriminant divisor and \(M_Z\) the moduli divisor of adjunction. Obviously \(B_Z\) is uniquely determined but \(M_Z\) is determined only up to \(\mathbb{R}\)-linear equivalence because it depends on the choice of \(L_Z\).

Now let \(\phi: X' \to X\) and \(\psi: Z' \to Z\) be birational morphisms from normal projective varieties and assume the induced map \(f': X' \to Z'\) is a morphism. Let \(K_{X'} + B' = \phi^*(K_X + B)\). Similar to above we can define a discriminant divisor \(B_{Z'}\), and taking \(L_{Z'} = \psi^*L_Z\) gives a moduli divisor \(M_{Z'}\) so that

\[K_{X'} + B' \sim_{\mathbb{R}} f'^*(K_{Z'} + B_{Z'} + M_{Z'})\]

and \(B_{Z'} = \psi_*B_{Z'}\) and \(M_{Z'} = \psi_*M_{Z'}\).

Theorem 2.19. With the above notation and assumptions, suppose that \((X, B)\) is lc over the generic point of \(Z\) and that \(B\) is a \(\mathbb{Q}\)-divisor. If \(Z' \to Z\) is a high resolution, then \(M_{Z'}\) is nef and for any birational morphism \(Z'' \to Z'\) from a normal projective variety, \(M_{Z''}\) is the pullback of \(M_{Z'}\).

For a proof, see [2][15].

2.20. Bounded families. A couple \((X, D)\) consists of a normal projective variety \(X\) and a divisor \(D\) on \(X\) whose non-zero coefficients are all equal to 1, i.e. \(D\) is a reduced divisor. The reason we call \((X, D)\) a couple rather than a pair is that we are concerned with \(D\) rather than \(K_X + D\) and we do not want to assume \(K_X + D\) to be \(\mathbb{Q}\)-Cartier or with nice singularities. Two couples \((X, D)\) and \((X', D')\) are isomorphic if there is an isomorphism \(X \to X'\) mapping \(D\) onto \(D'\).
We say that a set $\mathcal{P}$ of couples of dimension $\leq d$ is bounded if there is $r \in \mathbb{N}$ such that for each $(X, D) \in \mathcal{P}$ we can find a very ample divisor $A$ on $X$ so that $A^{\dim X} \leq r$ and $D \cdot A^{\dim X-1} \leq r$. This is equivalent to say that there exist finitely many projective morphisms $V^i \to T^i$ of varieties and reduced divisors $C^i$ on $V^i$ such that for each $(X, D) \in \mathcal{P}$ there exist an $i$, a closed point $t \in T^i$, and an isomorphism $\phi: V^i_i \to X$ such that $(V^i_i, C^i_i)$ is a couple and $\phi^* C^i_i \geq D$.

A set $\mathcal{Q}$ of projective lc pairs $(X, B)$ is said to be bounded if the $(X, \text{Supp} B)$ form a bounded family of couples.

Recall that in the introduction we defined boundedness for generalised pairs and for stable log minimal models, in Definitions 1.1 and 1.8 when the coefficients involved are bounded from below away from zero.

### 3. Descent of nef divisors

#### 3.1. Descent of divisors.

1. Let $X,Y,Z$ be normal varieties and $X \to Y/Z$ be a birational map, and let $M$ be an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on $X$. We say that $M$ descends to $Y$ if there exist an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor $L$ on $Y$ and a commutative diagram of contractions of normal varieties

   \[
   \begin{array}{ccc}
   W & \xrightarrow{\phi} & X \\
   \downarrow{\psi} & & \downarrow{\psi} \\
   Y & \to & Z
   \end{array}
   \]

   compatible with the given $X \to Y$ such that $\phi^* M = \psi^* L$. We also refer to this situation by saying that $M$ descends to $Y$ as $L$.

2. In some special situations nef divisors automatically descend. Assume $X \to Y/Z$ is a birational contraction of normal varieties and that $X$ is of Fano type over $Y$. Assume that $(X, B)$ is lc for some $B$ and that $M$ is a Cartier divisor which is nef over $Y$. We claim that if $K_X + B + uM$ is anti-nef over $Y$ for some $u > 2d$, then $M$ descends to $Y$ as a Cartier divisor. First note that we can replace $Z$ with $Y$, hence assume $Y = Z$. Second note that $-(K_X + B + uM)$ is semi-ample over $Y$ as $X$ is of Fano type over $Y$, so adding a general divisor

   \[0 \leq L \sim_{\mathbb{R}} -(K_X + B + uM)/Y\]

   to $B$ it is enough to treat only the case $K_X + B + uM \equiv 0/Y$. Third since $X$ is of Fano type over $Y$, there is a boundary $C$ such that $(X, C)$ is klt and $K_X + C \equiv 0/Y$. Replacing $B$ with $tB + (1-t)C$ and $uM$ with $tuM$ for some $t < 1$ close to 1, we can assume that $(X, B)$ is klt.

   If $X \to Y$ is an isomorphism the claim is obvious. If not, then since $X$ is of Fano type over $Y$, there is an extremal contraction $X \to U/Y$. Assume $M$ is ample over $U$. Then $K_X + B$ is anti-ample over $U$. By boundedness of the length of extremal rays [24], there is a curve $\Gamma$ generating the extremal ray corresponding to $X \to U$ such that

   \[2d < u \leq uM \cdot \Gamma = -(K_X + B) \cdot \Gamma \leq 2d,\]

   a contradiction. Thus $M \equiv 0/U$, hence $M$ descends to $U$ as a Cartier divisor by the cone theorem [27, Theorem 3.7], so we can replace $X$ with $U$ and repeat the process until we reach $Y$. 

---

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Lemma 3.2. Let \( d, p \in \mathbb{N} \). Assume \((X, B + M)\) is a generalised pair of dimension \( d \) with data \( X' \xrightarrow{\phi} X \) and \( M' \) where \( pM' \) is \( b \)-Cartier. Assume that \((X, B)\) is klt and that \( \lambda \) is the generalised lc threshold of \( M \) with respect to \((X, B)\). Then the following are equivalent:

- \( \lambda \geq 3dp \),
- \( \lambda = \infty \),
- \( M' \) descends to \( X \) and \( pM \) is Cartier.

Proof. We can assume \( \phi \) is a log resolution of \((X, B)\) and that \( pM' \) is Cartier. Let \( \Gamma' \) be the birational transform of \( B \) plus the reduced exceptional divisor of \( \phi \). Let \( N = 3dp \) and run an MMP on \( K_{X'} + \Gamma' + nM' \) over \( X \) with scaling of some ample divisor. Then by (2) above, \( M' \) descends to every model appearing in the MMP. We reach a model \( X'' \) on which \( K_{X''} + \Gamma'' + nM'' \) is a limit of movable \( \mathbb{R} \)-divisors, relatively over \( X \).

Assume that \( \lambda \geq n \), i.e. \((X, B + nM)\) is generalised lc. Then we can write

\[
K_{X'} + \Gamma' + nM' = \phi^*(K_X + B + nM) + E'
\]

where \( E' \geq 0 \) is exceptional. Thus by the general negativity lemma [10, Lemma 3.3], the MMP terminates with a \( \mathbb{Q} \)-factorial dlt model \((X'', \Gamma'' + nM'')\) of \((X, B + M)\). In particular, \( K_{X''} + \Gamma'' + nM'' \sim_{\mathbb{R}} 0/X \). Moreover, \( X'' \) is of Fano type over \( X \) because \((X, B)\) is klt. Then \( M'' \) descends to \( X \) by (2) above, hence \( M' \) descends to \( X \) and that \( pM \) is Cartier.

Now assume that \( M' \) descends to \( X \). Then obviously \( \lambda = \infty \).

Finally if \( \lambda = \infty \), then clearly \( \lambda \geq n \).

\[ \square \]

3.3. Generalised lc modifications. Let \((X, B + M)\) be a generalised pair with data \( X' \rightarrow X \) and \( M' \). Let \( C \) be the divisor obtained from \( B \) by replacing every coefficient \( > 1 \) with 1. Assume that \((X'', B'' + M'')\) is a generalised pair with data \( X' \rightarrow X'' \) and \( M' \) equipped with a birational contraction \( X'' \rightarrow X \). We say that \((X'', B'' + M'')\) is the generalised lc modification of \((X, B + M)\) if the following hold:

- \( B'' = C'' + E \) where \( C'' \) is the birational transform of \( C \) and \( E \) is the reduced exceptional divisor of \( X'' \rightarrow X \),
- \((X'', B'' + M'')\) is generalised lc, and
- \( K_{X''} + B'' + M'' \) is ample over \( X \).

The generalised lc modification is unique if it exists. It is also obvious that if \((X, B + M)\) is generalised lc then it is the generalised lc modification of itself.

3.4. Bounded coefficients on generalised lc modifications. In this subsection we show that under certain assumptions the pullback of some divisors to a generalised lc modification have bounded coefficients. This is important for the subsequent results in this section.

Proposition 3.5. Let \( d, p, r \in \mathbb{N} \). Then there exists \( c \in \mathbb{N} \) depending only on \( d, p, r \) satisfying the following. Assume that

- \((X, B + M)\) is a projective generalised pair of dimension \( d \) with data \( X' \xrightarrow{\phi} X \) and \( M' \),
- \((X, B)\) is \( \mathbb{Q} \)-factorial klt,
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- $pB$ is integral and $pM'$ is b-Cartier,
- $A$ is a very ample divisor on $X$ with $A^d \leq r$, and
- $A - (B + M)$ is pseudo-effective.

Then there is a generalised lc modification $(X'', B'' + 3dpM'')$ of $(X, B + 3dpM)$ such that

- denoting $X'' \to X$ by $\pi$ we have
  \[ \pi^* M = M'' + \sum e_i E''_i \]
  where $E''_i$ are the exceptional divisors of $\pi$ and $\sum e_i \leq c$,
- $M'$ descends to $X''$ as $M''$, and
- $pM''$ is Cartier.

**Proof.** Note that we are not assuming $(X, B + M)$ to be generalised lc.

**Step 1.** We can assume that $\phi$ is a log resolution of $(X, B)$ and that $pM'$ is Cartier. Let $\Delta'$ be the sum of the birational transform $B'$ and the reduced exceptional divisor of $\phi$. Let $\Gamma' = \Delta' + \frac{1}{a}H'$ where $H'$ is a general element of $|6d\phi^* A|$. To ease notation put $n = 3dp$.

Run an MMP on $K_{X'} + \Gamma' + nM'$ with scaling of some ample divisor. Since
\[ \Gamma' = \Delta' + \frac{1}{a}H' \sim_\mathbb{Q} \Delta' + 3d\phi^* A, \]
any extremal ray in the course of the MMP intersects the pullback of $A$ trivially, by boundedness of the length of extremal rays [24]. Thus the MMP is automatically an MMP over $X$. In addition, since $n = 3dp$ and $pM'$ is Cartier, the MMP is $M'$-trivial, that is, in each step $M'$ intersects the contracted extremal ray trivially. In particular, by 3.1(2), $pM'$ descends as a Cartier divisor to every model appearing in the MMP.

**Step 2.** In the course of the MMP we reach a model $X''$ on which $K_{X''} + \Gamma'' + nM''$ is a limit of movable $\mathbb{R}$-divisors. In particular, for any exceptional $X$ prime divisor $S''$ on $X''$, $(K_{X''} + \Gamma'' + nM'')|_{S''}$ is pseudo-effective. Thus by the general negativity lemma [10, Lemma 3.3], we can write
\[ K_{X''} + \Gamma'' + nM'' + G'' = \pi^*(K_X + \Gamma + nM) \]
where $\pi$ denotes $X'' \to X$, $\Gamma = \pi_!\Gamma''$, and $G''$ is effective and exceptional.

Write
\[ \pi^* M = M'' + \sum e_i E''_i \]
where $E''_i$ are the exceptional divisors of $\pi$. Since $M''$ is a nef divisor, $e_i \geq 0$.

**Step 3.** We claim that $e_i > 0$ for every $i$. Assume not, say $e_1 = 0$. Write $\phi^* M = M' + \sum f_j F_j$ where $F_j$ runs through the exceptional divisors of $\phi$. Say $F_1$ is the birational transform of $E_1$, then $f_1 = 0$. Since $(X, B)$ is klt and $\Gamma = B + \frac{1}{a}H$ where $H = \phi_* H'$ is a general member of $|6dA|$, we see that $(X, \Gamma)$ is klt, so we have
\[ 0 < a(E_1, X, \Gamma) = a(F_1, X, \Gamma). \]
On the other hand, by Lemma 2.7 and the condition $f_1 = 0$,
\[ a(F_1, X, \Gamma) = a(F_1, X, \Gamma + nM) + nf_1 = a(F_1, X, \Gamma + nM). \]
In addition, since $G'' \geq 0$ and since $E_1$ is a component of $|\Gamma''|$,  
\[ a(F_1, X, \Gamma + nM) = a(E_1, X, \Gamma + nM) \leq a(E_1, X'', \Gamma'' + nM'') = 0. \]
To summarise we have proved
\[ 0 < a(E_1, X, \Gamma) = a(E_1, X, \Gamma + nM) \leq a(E_1, X'', \Gamma'' + nM'') = 0 \]
which is a contradiction.

**Step 4.** In this step we show that the MMP of Step 1 terminates with a good minimal model and then we replace \( X'' \) with the corresponding ample model. Assume \( t > 0 \) is a sufficiently small real number. Then \( \Gamma'' - t(\sum e_i E''_i) \geq 0 \). Moreover, the MMP from \( X'' \) onwards is also an MMP on
\[ K_{X''} + \Gamma'' - t(\sum e_i E''_i) + (n - t)M'' = K_{X''} + \Gamma'' + nM'' - t\pi^* M \]
over \( X \) with scaling of some divisor. Since all \( e_i > 0 \),
\[ \lfloor \Gamma'' \rfloor = \text{Supp} \sum e_i E''_i \]
and since \( M' \) descends to \( X'' \) as \( M'' \),
\[ (X'', \Gamma'' - t(\sum e_i E''_i) + (n - t)M'') \]
is generalised klt, hence the MMP terminates over \( X \): indeed, we can find a boundary \( \Theta'' \) such that over \( X \) we have
\[ \Theta'' \equiv \Gamma'' - t(\sum e_i E''_i) + (n - t)M'' \equiv \Gamma'' + nM'' \]
and \( (X'', \Theta'') \) is klt, so the termination follows from [11]. Therefore, the MMP of Step 1 terminates globally because as we observed the MMP is over \( X \). Moreover, the pushdown of \( K_{X''} + \Gamma'' + nM'' \) to the minimal model is semi-ample over \( X \). Replacing \( X'' \) with the end product of the MMP, we can assume \( K_{X''} + \Gamma'' + nM'' \) is nef and that it is semi-ample over \( X \).

Let \( X''' \) be the ample model of \( K_{X''} + \Gamma'' + nM'' \) over \( X \) which exists in view of the previous step. Since \( K_{X''} + \Theta'' \equiv 0/X''' \) and \( (X'', \Theta'') \) is klt, \( X'' \) is of Fano type over \( X''' \). Then by 3.1(2), \( pM'' \) descends to \( X''' \) as a nef Cartier divisor. Replacing \( X'' \) with \( X''' \), we can assume \( K_{X''} + \Gamma'' + nM'' \) is ample over \( X \). By our choice of \( H' \) and the definition of \( \Gamma' \), in fact \( K_{X''} + \Gamma'' + nM'' \) is globally ample. However, now \( (X'', \Gamma'' + nM'') \) may not be generalised dlt but it is generalised lc which is enough for our purposes.

By construction, \( (X'', B'' + nM'') \) is the generalised lc modification of \( (X, B + nM) \) where \( B'' \) is the sum of \( B_- \) and the reduced exceptional divisor of \( X'' \to X \) which we again denote by \( \pi \). Note that \( B'' \) is just the pushdown of \( \Delta' \). As before we will write
\[ \pi^* M = M'' + \sum e_i E''_i. \]

**Step 5.** In this step we show that the volume of the restriction of \( K_{X''} + \Gamma'' + nM'' \) to each component of \( \lfloor \Gamma'' \rfloor \) is bounded from below away from zero. Pick a component \( S'' \) of \( \lfloor \Gamma'' \rfloor = \text{Supp} \sum e_i E''_i \) and let \( T \) be its normalisation. Then by [8, 3.1(1),(2)], we have a generalised adjunction formula
\[ K_T + \Gamma''_T + nM''_T \sim_R (K_{X''} + \Gamma'' + nM'')|_T \]
where the coefficients of \( \Gamma''_T \) belong to a DCC set depending only on \( p \), and \( nM''_T \) is Cartier where \( nM''_T \) is the nef part of \( (T, \Gamma''_T + nM''_T) \).

Now by [12, Theorem 1.3], there is \( \alpha > 0 \) depending only on \( d, p \) such that
\[ \alpha < \text{vol}(K_T + \Gamma''_T + nM''_T) = \text{vol}((K_{X''} + \Gamma'' + nM'')|_{S''}) \]
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\[ (K_X + \Gamma + nM) = (K_X' + \Gamma' + nM')^{d-1} \cdot S' \cdot \frac{A}{A}. \]

**Step 6.** Since \( A \) is very ample and \( A^d \leq r \), \( X \) belongs to a bounded family of varieties, so there is \( q \in \mathbb{N} \) depending only on \( d, r \) such that \( qA - K_X \) is ample. Moreover, since \( A - (B + M) \) is pseudo-effective, both \( A - B \) and \( A - M \) are pseudo-effective, hence letting \( l = q + 1 + 3d + n \), we see that

\[ lA - (K_X + \Gamma + nM) = lA - (K_X + B + \frac{1}{2}H + nM) \]

\[ = qA - K_X + A - B + 3dA - \frac{1}{2}H + nA - nM \]

is pseudo-effective. Therefore,

\[ \text{vol}(K_X + \Gamma + nM + A) \leq \text{vol}((l + 1)A) \leq (l + 1)^d r. \]

In view of Step 2, this implies that

\[ \text{vol}(K_X + \Gamma + nM + \pi^*A) \leq (l + 1)^d r. \]

Since both \( K_X + \Gamma + nM \) and \( \pi^*A \) are nef,

\[ (K_X + \Gamma + nM)^{d-1} \cdot \pi^*A \leq \text{vol}(K_X + \Gamma + nM + \pi^*A) \leq (l + 1)^d r. \]

Thus since \( A - M \) is pseudo-effective, we get

\[ (K_X + \Gamma + nM)^{d-1} \cdot \pi^*M \leq (K_X + \Gamma + nM')^{d-1} \cdot \pi^*A \leq (l + 1)^d r, \]

hence by nefness of \( M' \) and the choice of \( \alpha \) we have

\[ \alpha(\sum e_i) \leq (K_X + \Gamma + nM')^{d-1} \cdot (\sum e_iE''_i) \]

\[ \leq (K_X' + \Gamma' + nM')^{d-1} \cdot \pi^*M \leq (l + 1)^d r. \]

Therefore, \( \sum e_i \) is bounded from above by some fixed number \( c \in \mathbb{N} \) depending only on \( d, l, r, \alpha \), hence depending only on \( d, p, r \).

\[ \square \]

**3.6. Bounded crepant models.** Given a generalised pair \((X, B + M)\) with data \( X' \xrightarrow{\phi} X \) and \( M' \) and a birational contraction \( \psi: Y \to X \) from a normal variety we can write

\[ K_Y + B_Y + M_Y = \psi^*(K_X + B + M) \]

in a natural way: we can assume \( X' \to Y \) is a morphism; we first write

\[ K_{X'} + B' + M' = \phi^*(K_X + B + M) \]

and then simply pushdown down \( B' + M' \) to \( Y \) to get \( B_Y + M_Y \). When \( B_Y \geq 0 \) we say that \((Y, B_Y + M_Y)\) is a crepant model of \((X, B + M)\). Note that \((Y, B_Y + M_Y)\) is a generalised pair with data \( X' \to Y \) and \( M' \).

When \((X, B + M)\) is projective generalised lc and \( X \) varies in a bounded family, in general, crepant models do not form a bounded family even for fixed \( X \), e.g. when \((X, B)\) is any projective toric pair of dimension \( \geq 2 \) and \( M' = 0 \). The crucial point of the next result is that we can sometimes ensure that certain crepant models are bounded.

In contrast when \((X, B + M)\) is projective generalised \( \epsilon \)-lc \((\epsilon > 0)\) and \( X \) varies in a bounded family and the “degree” of \( B + M \) is bounded from above, the crepant models are bounded by [6, Theorem 2.2]. This is an important ingredient of the proof of the next result.
Proof. Step 1. In this step we consider the generalised lc modification \((X, B + M)\) of \((X, B)\). By Proposition 3.5, there is \(c \in \mathbb{N}\) depending only on \(d, p, r\) such that, letting \(n = 3dp\), there is a generalised lc modification \((X''', B'' + nM''')\) of \((X, B + nM)\) so that

- denoting the given morphism \(X'' \to X\) by \(\pi\) we have
  \[
  \pi^*M = M''' + \sum e_i E_i'''
  \]
  where \(E_i'''\) are the exceptional divisors of \(\pi\) and \(\sum e_i \leq c\),
- \(M''\) descends to \(X''\) as \(M'''\), and
- \(\pi^*M'''\) is Cartier.

Note that \(\pi\) is not an isomorphism otherwise \(M''\) descends to \(X\) which means \(\lambda = \infty\) which is not the case by assumption.

Step 2. In this step we replace \(X''\) with a \(\mathbb{Q}\)-factorialisation and then show that we can run an MMP/\(X\) on \(K_{X'''} + B''\) with scaling of \(nM''\). By the proof of 3.5, for any small \(t > 0\),

\[
(X''', B'' - t(\sum e_i E_i''') + (n - t)M''')
\]

is generalised klt. Moreover,

\[
K_{X'''} + B'' - t(\sum e_i E_i''') + (n - t)M'''
= K_X + B'' + nM''' - t\pi^*M
\equiv K_X + B'' + nM''/X.
\]

In particular, there is a small \(\mathbb{Q}\)-factorialisation of \(X''\). Replace \(X''\) with this \(\mathbb{Q}\)-factorialisation. The nefness of \(K_{X'''} + B'' + nM'''\) over \(X\) is preserved and the property \(\sum e_i \leq c\) is also preserved since we do not get any new exceptional divisor.

By Lemma 2.11, we can run the MMP/\(X\) on \(K_{X'''} + B'' - t(\sum e_i E_i''')\) with scaling of \((n - t)M''\). The reduced exceptional divisor of \(X'' \to X\) is equal to \(|B''|\). Assuming \(t\) is small enough, each exceptional divisor appears in \(B'' - t(\sum e_i E_i''')\) with coefficient...
close to 1. But then since \((X, B)\) is klt and since \(B''\) is the sum of the birational transform \(B^\sim\) and the reduced exceptional divisor of \(\pi\), we can assume that
\[
K_{X''} + B'' - t(\sum e_i E_i'') = \pi^*(K_X + B) + L''
\]
where \(L'' \geq 0\) is exceptional whose support is equal to the reduced exceptional divisor of \(\pi\). Thus by the negativity lemma, the MMP contracts \(L''\) and it ends with \(X\) as \(X\) is \(\mathbb{Q}\)-factorial.

Adding \(t \pi^* M\) to \(K_{X''} + B'' - t(\sum e_i E_i'')\) we can see that the above MMP is also an MMP/\(X\) on \(K_{X''} + B'' + tM''\) with scaling of \((n-t)M''\) which in turn implies that the MMP is also an MMP/\(X\) on \(K_{X''} + B''\) with scaling of \(nM''\).

**Step 3.** In this step we define \((Y, B_Y + \lambda M_Y)\). Let
\[
X'' = X'_0 \to X'_1 \to \cdots \to X''_{l-1} \to X''_l = X
\]
be the steps of the MMP on \(K_{X''} + B''\) of the previous step. By definition of MMP with scaling, in each step we have a number \(\alpha_i\) which is the smallest number so that \(K_{X'_i} + B''_i + \alpha_i n M''_i\) is nef over \(X\). The last step \(X''_{l-1} \to X''_l = X\) of the MMP is an extremal divisorial contraction. Moreover,
\[
K_{X''_{l-1}} + B''_{l-1} + \alpha_{l-1} n M''_{l-1} \equiv 0/X
\]
and the coefficient of the exceptional divisor of \(X''_{l-1} \to X''_l\) in \(B''_{l-1}\) is equal to 1. Then \((X, B + \alpha_{l-1} n M)\) is generalised lc but not generalised klt. Therefore, we see that \(\alpha_{l-1} n = \lambda\).

Let \(i\) be the smallest number so that \(\alpha_i n \leq \lambda < \alpha_{i-1} n\). Then \(\alpha_j n = \alpha_{l-1} n = \lambda\), for \(i \leq j \leq l-1\), hence
\[
K_{X''_i} + B''_i + \lambda M''_i \equiv 0/X
\]
because in the steps \(X''_i \to X''_l = X\) the numbers \(\alpha_j n\) are all equal to \(\lambda\). In particular, this means that \((X''_i, B''_i + \lambda M''_i)\) is a crepant model of \((X, B + \lambda M)\).

Let
\[
(Y, B_Y + \lambda M_Y) := (X''_i, B''_i + \lambda M''_i)
\]
and let \(\rho\) denote \(Y \to X\).

**Step 4.** In this step we show that \((Y, \text{Supp } B_Y)\) belongs to a bounded family of pairs. By assumption, \(A^d \leq r\), \(A - B\) is pseudo-effective, \(pB\) is integral, and \((X, B)\) is klt. Then \((X, B)\) belongs to a bounded set of pairs, hence \((X, B)\) is \(\epsilon\)-lc for some \(\epsilon > 0\) depending only on \(d, p, r\). Thus by Lemma 2.9, \(\lambda\) is bounded from below away from zero depending only on \(d, p, r\).

By construction, \(\rho^* M = M_Y + \sum g_j G_j\) where \(G_j\) denote the exceptional divisors of \(\rho\). By Step 1, \(\sum g_j \leq \sum e_i \leq c\). Therefore, there is a real number \(u \in (0, \lambda)\) depending only on \(d, p, r\) such that \(B_Y - u \sum g_j G_j \geq 0\).

Now
\[
K_Y + B_Y - u \sum g_j G_j + (\lambda - u) M_Y = K_Y + B_Y + \lambda M_Y - u \rho^* M
\]

\[
= \rho^*(K_X + B + (\lambda - u) M).
\]

Thus
\[
(Y, B_Y - u \sum g_j G_j + (\lambda - u) M_Y)
\]
is a crepant model of \((X, B + (\lambda - u) M)\).
Since $\lambda < \infty$, we have $\lambda < n$ by Lemma 3.2. Let $b = \frac{1-\lambda}{\lambda}$. We can write
\[ K_X + B + (\lambda - u)M = (1 - b)(K_X + B) + b(K_X + B + \lambda M) \]
where
\[ 1 - b = \frac{u}{\lambda} \geq \frac{u}{n}. \]
Thus since $(X, B)$ is $\epsilon$-lc, we deduce that $(X, B + (\lambda - u)M)$ is $\delta$-lc where $\delta = \frac{\epsilon u}{n}$. But then by [6, Theorem 2.2], the $(Y, B_Y)$ form a bounded family of pairs. Thus we can find a very ample divisor $A_Y$ on $Y$ such that $A_Y^d$ is bounded, say by $s \in \mathbb{N}$. Moreover, we can ensure that $A_Y - \rho^*A$ is pseudo-effective, so perhaps after replacing $A_Y$ with a bounded multiple we can assume $A_Y - (B_Y + M_Y)$ is pseudo-effective.

Step 5. By construction, the reduced exceptional divisor of $Y \to X$ is equal to $[B_Y]$. On the other hand, by Step 2,
\[ (X''', B''' - t \sum e_iE_i'' + (n-t)M''') \]
is generalised klt for any small $t > 0$, and $M'$ descends to $X''$ as $M''$. Thus the generalised non-klt locus of
\[ (X'', B'' + (n-t)M'') \]
is equal to $\text{Supp} \sum e_iE_i'' = [B'']$. We then deduce that the generalised non-klt locus of $(X'', B'' + \lambda M'')$ is equal to $[B'']$ because $M'$ descends to $X''$ as $M''$.

By construction, $Y = X''$ where $i$ is the smallest number so that $\alpha_i n \leq \lambda < \alpha_{i-1} n$. This implies that $X'' \dasharrow Y = X_i''$ is an MMP on $K_{X''} + B'' + \lambda M''$ over $X$. So we see that any generalised non-klt centre of $(Y, B_Y + \lambda M_Y)$ is the birational transform of a generalised non-klt centre of $(X'', B'' + \lambda M'')$, hence such centres are contained in $[B_Y]$. Therefore,
\[ (Y, B_Y - t [B_Y] + \lambda M_Y) \]
is generalised klt for any small $t > 0$.

\[ \square \]

3.8. Finiteness of generalised lc thresholds on bounded families.

Lemma 3.9. Let $d, p, r \in \mathbb{N}$. Then there is a finite set $\Lambda$ of rational numbers depending only on $d, p, r$ satisfying the following. Assume that
\begin{itemize}
  \item $(X, B + M)$ is a projective generalised pair of dimension $d$ with data $X' \phi \to X, M'$,
  \item $(X, B)$ is $\mathbb{Q}$-factorial klt,
  \item $pB$ is integral and $pM'$ is $b$-Cartier,
  \item $A$ is a very ample divisor on $X$ with $A^d \leq r$,
  \item $A - (B + M)$ is pseudo-effective, and
  \item $\lambda < \infty$ where $\lambda$ is the generalised lc threshold of $M$ with respect to $(X, B)$.
\end{itemize}
Then $\lambda \in \Lambda$.

Proof. By Proposition 3.7, there exist a number $s \in \mathbb{N}$ depending only on $d, p, r$ and a $\mathbb{Q}$-factorial crepant model $(Y, B_Y + \lambda M_Y)$ of $(X, B + \lambda M)$ and a very ample divisor $A_Y$ on $Y$ such that
\begin{itemize}
  \item the reduced exceptional divisor of $\pi: Y \to X$ is equal to $[B_Y]$,
\end{itemize}
• the pair
\[(Y, B_Y - t \lfloor B_Y \rfloor + \lambda M_Y)\]
is generalised klt for any small \(t > 0\),
• \(A^d_Y \leq s\) and \(A_Y - (B_Y + M_Y)\) is pseudo-effective.

It turns out that \(\pi\) is not an isomorphism: indeed, since \((X, B)\) is klt, for each \(t > 0\), \(B - t \lfloor B \rfloor = B\); so
\[(X, B - t \lfloor B \rfloor + \lambda M) = (X, B + \lambda M)\]
is not generalised klt; thus by the second property, \(\pi\) is not an isomorphism.

Assume that \(K_Y + B_Y\) is nef over \(X\). Then by the negativity lemma,
\[K_Y + B_Y + G_Y = \pi^* (K_X + B)\]
for some \(G_Y \geq 0\). But this is not possible because \(K_X + B\) is klt while \((Y, B_Y + G_Y)\) is not. Thus \(K_Y + B_Y\) is not nef over \(X\). Then there is a \(K_Y + B_Y\)-negative extremal ray over \(X\) generated by a curve \(C\) with \(0 < -(K_Y + B_Y) \cdot C \leq 2d\) [24]. Since \((Y, B_Y)\) belongs to a bounded family of pairs, there are finitely many possibilities for the number \(-(K_Y + B_Y) \cdot C\) as the Cartier index of \(K_Y + B_Y\) is bounded [8, Lemma 2.24].

On the other hand, since \(Y\) belongs to a bounded family, it is \(\epsilon\)-lc for some \(\epsilon > 0\) depending only on \(d, s\), hence only on \(d, p, r\). Thus by Lemma 2.9, there is a rational number \(u > 0\) depending only on \(d, s, \epsilon\), hence only on \(d, p, r\) such that \((Y, uM_Y)\) is generalised klt. Similarly we can choose \(u\) so that \((X, B + uM)\) is generalised klt.

By the previous paragraph and by boundedness of the length of extremal rays, \(K_Y + uM_Y + 2dA_Y\) is nef. Moreover,
\[(K_Y + uM_Y + 2dA_Y) \cdot A^{d-1}_Y\]
is bounded from above as \(A_Y - M_Y\) is pseudo-effective. Thus the Cartier index of \(K_Y + uM_Y + 2dA_Y\) is bounded from above by [8, Lemma 2.25] keeping in mind that \(l(K_Y + uM_Y + 2dA_Y)\) is integral for some \(l \in \mathbb{N}\) depending only on \(u, p\). This in turn implies that the Cartier index of \(M_Y\) is bounded from above.

Now with \(C\) as above, we have
\[(K_Y + B_Y + \lambda M_Y) \cdot C = 0,\]
so
\[\lambda = \frac{-(K_Y + B_Y) \cdot C}{M_Y \cdot C}.\]
Since \(\lambda > u\) and since \(-(K_Y + B_Y) \cdot C \leq 2d\), we see that \(M_Y \cdot C\) is bounded from above, hence it belongs to a fixed finite set as the Cartier index of \(M_Y\) is bounded. But then \(\lambda\) can take only finitely many possibilities.

\[\square\]

3.10. **Construction of towers of crepant models.** Let \(d, p, r \in \mathbb{N}\). Assume that
• \((X, B + M)\) is a projective generalised pair of dimension \(d\) with data \(X' \xrightarrow{\phi} X, M'\),
• \((X, B)\) is \(\mathbb{Q}\)-factorial klt,
• \(pB\) is integral and \(pM'\) is \(b\)-Cartier,
• \(A\) is a very ample divisor on \(X\) with \(A^d \leq r\), and
• \(A - (B + M)\) is pseudo-effective.
We will construct a finite sequence $X_i, B_i, M_i, A_i, \lambda_i, r_i$ of varieties, divisors and numbers where $i = 0, \ldots, l$ for some $l$. The sequence $r_0, \ldots, r_l$ is a sequence of natural numbers depending only on $d, p, r$, but the other data depend on $d, p, r, X, B, M, A$; a priori $l$ also depends on the latter data.

Let $X_0 = X, B_0 = B, M_0 = M, A_0 = A, r_0 = r$. Let $\lambda_0$ be the generalised lc threshold of $M_0$ with respect to $(X_0, B_0)$. The sequence to be constructed satisfies the following properties for $i > 0$, after replacing $X'$ with a high resolution:

1. $(X_i, B_i + M_i)$ is a projective generalised pair with data $X' \to X_i$ and $M'$,
2. $pB_i$ is integral,
3. $A_i$ is a very ample divisor on $X_i$ with $A_i^d \leq r_i$,
4. $A_i - (B_i + M_i)$ is pseudo-effective,
5. $\lambda_i$ is the generalised lc threshold of $M_i$ with respect to $(X_i, B_i)$,
6. $(X_i, \Gamma_i + \lambda_i M_i)$ is a $\mathbb{Q}$-factorial crepant model of $(X_{i-1}, B_i + \lambda_i M_{i-1})$ where $[\Gamma_i]$ is the reduced exceptional divisor of $X_i \to X_{i-1}$,
7. $B_i = \Gamma_i - \frac{1}{p} [\Gamma_i]$,
8. $(X_i, B_i + \lambda_i M_i)$ is generalised klt,
9. $\lambda_i > \lambda_{i-1}$, and
10. $M'$ descends to $X_i$.

Suppose that we have already inductively constructed $X_i, B_i, M_i, A_i, \lambda_i, r_i$ satisfying the properties (1)-(9). If $M'$ descends to $X_i$, we let $l = i$ and stop. Assume then that $M'$ does not descend to $X_i$. We will construct $X_{i+1}, B_{i+1}, M_{i+1}, A_{i+1}, \lambda_{i+1}, r_{i+1}$ as follows.

First note that $(X_i, B_i)$ is klt: if $i = 0$, then this holds by assumption; but if $i > 0$, then it follows from (8). Thus applying Lemma 3.2 to $(X_i, B_i + \lambda_i M_i)$ we see that $\lambda_i < 3dp$ otherwise the lemma implies $M'$ descends to $X_i$, a contradiction.

Now applying Proposition 3.7 to $(X_i, B_i + M_i)$, we see that:

- there is $r_{i+1} \in \mathbb{N}$ depending only on $d, p, r_i$,
- there is a $\mathbb{Q}$-factorial crepant model $(X_{i+1}, \Gamma_{i+1} + \lambda_i M_{i+1})$ of $(X_i, B_i + \lambda_i M_i)$ where $[\Gamma_{i+1}]$ is the reduced exceptional divisor of $X_{i+1} \to X_i$,
- there is a very ample divisor $A_{i+1}$ on $X_{i+1}$ with $A_{i+1}^d \leq r_{i+1}$,
- $A_{i+1} - (B_{i+1} + M_{i+1})$ is pseudo-effective, and
- $(X_{i+1}, B_{i+1} + \lambda_i M_{i+1})$ is generalised klt where $B_{i+1} = \Gamma_{i+1} - \frac{1}{p} [\Gamma_{i+1}]$.

In particular, $pB_{i+1}$ is integral because $p\Gamma_{i+1}$ is integral as $\Gamma_{i+1}$ is the sum of the birational transform of $B_i$ plus the reduced exceptional divisor of $X_{i+1} \to X_i$.

We let $\lambda_{i+1}$ be the generalised lc threshold of $M_{i+1}$ with respect to $(X_{i+1}, B_{i+1})$. Since $(X_{i+1}, B_{i+1} + \lambda_i M_{i+1})$ is generalised klt, $\lambda_{i+1} > \lambda_i$. Therefore, $X_{i+1}, B_{i+1}, M_{i+1}, A_{i+1}, \lambda_{i+1}, r_{i+1}$, satisfy the properties (1)-(9) listed above with $i + 1$ in place of $i$.

We show that the construction stops after finitely many steps, so property (10) holds for some $l$. By the ACC for generalised lc thresholds [12, Theorem 1.5], the numbers $\lambda_0 < \lambda_1 < \ldots$, cannot form an infinite sequence, hence there is a minimal $l$ such that $\lambda_l = \infty$, i.e. $M'$ descends to $X_l$, and the construction terminates with $X_l$.

By construction, $r_l$ depends only on $d, p, r_{l-1}$, and in turn $r_{l-1}$ depends only on $d, p, r_{l-2}$, and so on. Thus the sequence $r_0, r_1, \ldots$, depends only on $d, p, r$. But $l$ and the $X_i, B_i, M_i, A_i, \lambda_i$ depend on $d, p, r, X, B, M, A$. We will show in the next subsection that in fact $l$ is bounded from above depending only on $d, p, r$.

3.11. Boundedness of length of towers of crepant models.
Proposition 3.12. Let $d, p, r \in \mathbb{N}$. Then there is $m \in \mathbb{N}$ depending only on $d, p, r$ satisfying the following. Assume that

- $(X, B + M)$ is a projective generalised pair of dimension $d$ with data $X' \xrightarrow{\phi} X, M'$,
- $(X, B)$ is $\mathbb{Q}$-factorial klt,
- $pB$ is integral and $pM'$ is $b$-Cartier,
- $A$ is a very ample divisor on $X$ with $A^d \leq r$,
- $A - (B + M)$ is pseudo-effective.

Let $X_i, B_i, M_i, A_i, \lambda_i, r_i$, where $0 \leq i \leq l$, be the sequence constructed in 3.10 for $d, p, r, X, B, M, A$. Then $l \leq m$.

In particular,
- $\text{Supp } B_i$ contains the support of the birational transform of $B$ and the exceptional divisor of $X_i \to X$,
- $(X_i, B_i)$ belongs to a bounded set of pairs depending only on $d, p, r$, and
- $M'$ descends to $X_i$.

Proof. If the proposition is not true, then there is a sequence $X^j, B^j, M^j, A^j$ as in the proposition, $j \in \mathbb{N}$, so that if $X^j, B^j, M^j, A^j$ is the sequence constructed for $X^j, B^j, M^j, A^j$, where $0 \leq i \leq l$, then the $\nu^j$ form an infinite strictly increasing sequence of numbers. We will derive a contradiction.

By Lemma 3.9, for each $i$, there is a finite set of rational numbers $\Lambda_i$ depending only on $d, p, r, i$, hence depending only on $d, p, r, i$, such that the generalised lc thresholds $\lambda^j_i \in \Lambda_i$ whenever $\lambda^j_i \neq \infty$, that is, when $i < \nu^j$. Therefore, there is an infinite set $J_0 \subseteq \mathbb{N}$ such that $\alpha_0 := \lambda^0_0 \neq \infty$ is independent of $j$ for $j \in J_0$. Similarly, there is an infinite subset $J_1 \subseteq J_0$ such that $\alpha_1 := \lambda^1_1 \neq \infty$ is independent of $j$ for $j \in J_1$. In particular, since by construction, $\lambda^0_i < \lambda^1_i$ for every $j \in J_1$, we see that $\alpha_0 < \alpha_1$.

Continuing the above process gives a decreasing sequence

$$
\cdots \subseteq J_i \subseteq J_{i-1} \cdots \subseteq J_0 \subseteq \mathbb{N}
$$

of infinite subsets and thresholds $\alpha_i := \lambda^i_i \neq \infty$ independent of $j \in J_i$ so that we get an infinite strictly increasing sequence $\alpha_0 < \alpha_1 < \ldots$ of generalised lc thresholds. This contradicts the ACC for generalised lc thresholds [12, Theorem 1.5].

3.13. Descent of nef divisors to bounded models. We are now ready to prove our main result on the descent of nef divisors to bounded models.

Proof. (of Theorem 1.2) Step 1. Pick

$$(X, B + M) \in G_{lc}(d, \Phi_1 < v)$$

with data $X' \xrightarrow{\phi} X, M' = \sum \mu_i M'_i$. In this step we reduce the problem to the situation in which $(X, B + M)$ is generalised klt, $pB$ is integral, $pM_i$ are integral, and $K_X + B + (1 - u)M$ is big for some $p \in \mathbb{N}$ and $u \in \mathbb{Q}^{> 0}$ depending only on $d, \Phi$.

We can assume that $\phi$ is a log resolution of $(X, B)$. Let $\Gamma'$ be the sum of the birational transform of $B$ plus the reduced exceptional divisor of $\phi$. Then

$$K_{X'} + \Gamma' + M' = \phi^*(K_X + B + M) + G'$$

where $G'$ is effective and exceptional over $X$. By [12, Theorem 8.1], $K_{X'} + \alpha \Gamma' + \alpha M'$ is big for some real number $\alpha \in (0, 1)$ depending only on $d, \Phi$. Pick $\beta \in (\alpha, 1)$. Since
\( \Phi \) is DCC, there exists \( p \in \mathbb{N} \) such that for each \( 0 < \mu < \Phi \) there is a smallest natural number \( q \) such that \( \beta \mu < \frac{q}{p} < \mu \). Denote \( \bar{\mu} := \frac{q}{p} \). Define a boundary \( \Delta' \) by replacing each non-zero coefficient \( \mu \) of \( \Gamma' \) with \( \bar{\mu} \), and define \( N' := \sum \bar{\mu}_i M'_i \). This way we get a generalised klt pair \((X', \Delta' + N')\) where \( p\Delta' \) and \( pN' \) are integral.

Moreover, fixing \( u \in (0, 1 - \frac{\alpha}{\beta}) \), we have that \( K_{X'} + \Delta' + (1 - u)N' \) is big because \( \Delta' - \alpha \Gamma' \geq 0 \) and because

\[
(1 - u)N' - \alpha M' = \sum ((1 - u)\bar{\mu}_i - \alpha \mu_i) M'_i
\]

is nef as

\[
(1 - u)\bar{\mu}_i - \alpha \mu_i \geq (1 - u)\beta \mu_i - \alpha \mu_i \geq \frac{\alpha}{\beta} \beta \mu_i - \alpha \mu_i = 0.
\]

Replacing \((X, B + M)\) with \((X', \Delta' + N')\) we can assume that \((X, B + M)\) is generalised \( \frac{1}{p} \)-lc, \( pB \) is integral and \( p\mu_i \) are integral, and that \( K_X + B + (1 - u)M \) is big where \( p, u \) are chosen depending only on \( d, \Phi \).

**Step 2.** In this step we find a bounded birational model of \((X, B + M)\). By [12, Lemma 4.4], \((X, B + M)\) has a minimal model as \((X, B + M)\) is generalised klt and \( K_X + B + M \) is big. Replacing \((X, B + M)\) with the minimal model we can assume that \( K_X + B + M \) is nef and big. By [12, Theorem 1.3], there is a natural number \( m \) depending only on \( d, p, v \) such that \( pm \) and \( m(K_X + B + M) \) define a birational map. In particular, there is an integral divisor

\[
0 \leq L \sim m(K_X + B + M)
\]

with \( \text{vol}(L) < m^d v \). Applying [8, Proposition 4.4] to \( X, B, L \), we deduce that there exist a natural number \( c \) and a bounded set of couples \( Q \) depending only on \( d, p, v \) such that there is a projective log smooth \((X'', \Sigma'') \in Q \) and a birational map \( X'' \rightarrow X \) such that

- \( \Sigma'' \) contains the exceptional divisor of \( X'' \rightarrow X \) and the birational transform of \( \text{Supp}(B + L) \);
- replacing \( X' \) we can assume \( \psi : X' \rightarrow X'' \) is a morphism, and
- each coefficient of \( \psi_* \phi^* L \) is at most \( c \).

In particular, there is a very ample divisor \( A'' \) on \( X'' \) with \( A'' \) bounded from above, say by \( r \), and such that \( A'' - \Sigma'' \) and \( A'' - \psi_* \phi^* L \) are pseudo-effective.

**Step 3.** In this step we introduce divisors \( \Theta'', M'' \) on \( X'' \) and reduce the problem to the situation in which \( A'' - (\Theta'' + M'') \) is pseudo-effective. Let \( \Theta' \) on \( X' \) be the birational transform of \( B \) plus \((1 - \frac{1}{p})\) times the reduced exceptional divisor of \( \phi \). Let \( \Theta'', M'' \) be the pushdowns of \( \Theta', M' \). Then \( A'' - \Theta'' \) is pseudo-effective as \( \Theta'' \leq \Sigma'' \). Note that \( p\Theta'' \) is integral.

On the other hand, since \( K_X + B + (1 - u)M \) is big, we can write

\[
\frac{1}{m} L \sim_{\mathbb{Q}} K_X + B + M = K_X + B + (1 - u)M + uM \sim_{\mathbb{Q}} uM + R
\]

where \( R \geq 0 \). Thus

\[
\psi_* \phi^* \frac{1}{m} L \sim_{\mathbb{Q}} \psi_* \phi^* uM + \psi_* \phi^* R
\]

which implies that

\[
\frac{1}{m} A'' - \psi_* \phi^* uM \sim_{\mathbb{Q}} \frac{1}{m} A'' - \psi_* \phi^* \frac{1}{m} L + \psi_* \phi^* R
\]
boundness and volume of generalised pairs

is pseudo-effective. Therefore, replacing $A''$ with a bounded multiple we can assume that $A'' - \psi_* \phi^* M$ is pseudo-effective which in turn implies $A'' - M''$ is pseudo-effective as $\psi_* \phi^* M \geq M''$ because $\phi^* M \geq M'$ since $M'$ is nef. Since $A'' - \Theta''$ is pseudo-effective by the previous paragraph, replacing $A''$ with $2A''$ we can assume that $A'' - (\Theta'' + M'')$ is pseudo-effective.

**Step 4.** In this step we finish the proof by applying Proposition 3.12. By construction,

- $(X'', \Theta'' + M'')$ is a projective generalised pair of dimension $d$ with data $X' \to \overline{X}'$, $M'$.
- $(X'', \Theta'')$ is log smooth klt,
- $p\Theta''$ is integral and $pM'$ is Cartier,
- $A''$ is a very ample divisor on $X''$ with $A'' \delta \leq r$, and
- $A'' - (\Theta'' + M'')$ is pseudo-effective.

Applying Proposition 3.12 to $d, p, r, X'', \Theta'', M'', A''$, we deduce that there is a generalised klt pair $(\overline{X}, \overline{\Theta} + M)$ with data $\rho: X' \to \overline{X}$ and $M'$ such that

- $\pi: \overline{X} \to X''$ is a morphism,
- $\Sigma := \text{Supp} \overline{\Theta}$ contains the support of the birational transform of $\Theta''$ and the reduced exceptional divisor of $\pi$,
- $(\overline{X}, \Sigma)$ belongs to a bounded set of couples depending only on $d, p, r$, and
- $M'$ descends to $\overline{X}$.

Since $M'$ descends to $\overline{X}$, $M' = \rho^* \rho_* M'$. Since $M' = \sum \mu_i M'_i$ and $\mu_i > 0$ by definition, and since each $M'_i$ is nef, we get $M'_i = \rho^* \rho_* M'_i$ for each $i$, hence each $M'_i$ descends to $\overline{X}$.

By construction, $\Sigma$ contains the exceptional divisors of $\overline{X} \to X$ and the support of the birational transform of $B$: indeed, any prime exceptional divisor of $\overline{X} \to X$ is either an exceptional divisor of $\pi$ or a component of the birational transform of $\Theta''$ so it is a component of $\Sigma$; moreover, any component of the birational transform of $B$ is either exceptional over $X''$ or it is a component of the birational transform of $\Theta''$ so again it is a component of $\Sigma$.

There exist a bounded set of couples $\mathcal{P}$ depending only on $d, p, r$ such that $(\overline{X}, \Sigma)$ has a log resolution $Y \to \overline{X}$ so that if $\Sigma_Y$ is the sum of the birational transform of $\Sigma$ and the reduced exceptional divisor of $Y \to \overline{X}$, then $(Y, \Sigma_Y)$ belongs to $\mathcal{P}$. Now replacing $(\overline{X}, \Sigma)$ with $(Y, \Sigma_Y)$ we are done as obviously each $M'_i$ descends to $Y$.

We derive the following result on descent of nef divisors which is in some sense more general than 1.2 since we do not put any restriction on singularities. It essentially says that if $X$ varies in a bounded family and if $pM'$ is a nef Cartier divisor on some birational model whose image on $X$ has bounded “degree”, then we can descend $M'$ to a bounded model of $X$.

**Theorem 3.14.** Let $d, r \in \mathbb{N}$ and $\delta \in \mathbb{R}^{>0}$. Then there is a bounded set of couples $\mathcal{P}$ depending only on $d, p, \delta$ satisfying the following. Assume that

- $\phi: X' \to X$ is a birational morphism between normal projective varieties of dimension $d$,
- $B \geq 0$ is an $\mathbb{R}$-divisor on $X$ whose non-zero coefficients are $\geq \delta$,
- $M' = \sum \mu_i M'_i$ where $M'$ are nef and Cartier on $X'$ and $\mu_i \geq \delta$,
- $A$ is a very ample divisor on $X$ with $A^d \leq r$, and
• $A - (B + M)$ is pseudo-effective where $M = \phi_* M'$.

Then there exist a log smooth couple $(\overline{X}, \Sigma) \in \mathcal{P}$ and a birational morphism $\overline{X} \rightarrow X$ such that

• $\Sigma$ contains the exceptional divisors of $\overline{X} \rightarrow X$ and the support of the birational transform of $B$, and
• each $M'_i$ descends to $\overline{X}$.

Proof. Since $A^d \leq r$, $X$ belongs to a bounded family of varieties. Replacing $A$ we can assume $A - (K_X + B + M)$ is pseudo-effective. Replace $\delta$ with $\frac{\delta}{p} < \delta$ for some $p \in \mathbb{N}$. Decreasing the coefficients of $B$ and the $\mu_i$ we can assume that $pB$ is integral and the coefficients of $B$ are $\leq 1$, and that $p\mu_i$ are all integral.

Replacing $X'$ we can assume $\phi$ is a log resolution of $(X, B)$. Let $B'$ be the sum of the birational transform of $B$ and the reduced exceptional divisor of $\phi$. Let $N' = M' + 3dA'$ where $A' = \phi^* A$. Then $K_{X'} + B' + N'$ is big, $pB'$ is integral, and $pN'$ is Cartier. Moreover,

$$\text{vol}(K_{X'} + B' + N') \leq \text{vol}(K_X + B + M + 3dA)$$

$$\leq \text{vol}((1 + 3d)A) \leq (1 + 3d)^d \rho$$

because $A - (K_X + B + M)$ is pseudo-effective.

Considering $(X', B' + N')$ as a generalised pair with data $X' \rightarrow X'$ and $N'$,

$$(X', B' + N') \in \mathcal{G}_{lc}(d, p, v)$$

where $\Phi = \{\frac{1}{p}\}$ and $v = (1 + 3d)^d \rho + 1$. Thus by Theorem 1.2, there exists a bounded set of couples $\mathcal{P}$ depending only on $d, p, v$, hence depending only $d, r, \delta$, such that there exist a log smooth couple $(\overline{X}, \Sigma) \in \mathcal{P}$, and a birational map $\overline{X} \rightarrow X'$ such that

• $\overline{X}$ contains the exceptional divisors of $\overline{X} \rightarrow X'$ and the support of the birational transform of $B'$, and
• $N'$ descends to $\overline{X}$.

Thus there is a common resolution $X''$ of $X'$ and $\overline{X}$ such that if $N'' = M'' + 3dA''$ denotes the pullback of $N' = M' + 3dA'$, then $N''$ is the pullback of a divisor on $\overline{X}$. In particular, any curve $C$ contracted by $X'' \rightarrow \overline{X}$ is also contracted by $X'' \rightarrow X$ because from $N'' \cdot C = 0$ we get $A'' \cdot C = 0$. This implies that the induced map $\overline{X} \rightarrow X$ is actually a morphism.

Now $\Sigma$ contains the exceptional divisors of $\overline{X} \rightarrow X$ and the support of the birational transform of $B$: indeed any prime exceptional divisor of $\overline{X} \rightarrow X$ is either exceptional over $X'$ or else is a component of the birational transform of $B'$; moreover, the support of the birational transform of $B$ is contained in the support of the birational transform of $B'$.

On the other hand, since $N' = \sum \mu_i M'_i + 3dA'$ descends to $\overline{X}$ and since the $M'_i$ and $A'$ are all nef, each $M'_i$ descends to $\overline{X}$.

\[\square\]

4. DCC of volume of generalised pairs

In this section we prove Theorem 1.3. The proof consists of two main parts. One part treats the theorem for pairs birational to a fixed variety. This is 4.2 a variant of which was first proved for generalised pairs in [14, Theorem 1.10] where the proof is nearly identical to the proof of [20, Proposition 5.1] for usual pairs. Our proof here
also generally follows [20] but it is somewhat simpler than both [20][14]. The second part reduces the general statement to the case in part one which is again modelled on [20] but the nef divisors involved need extra care. For this part also we generally follow [14] although some of the details are different.

4.1. DCC of volumes of pairs with fixed birational model.

Proposition 4.2. Let $\Phi \subset \mathbb{R}^{\geq 0}$ be a DCC set. Assume that $(Y, \Delta)$ is a projective $\mathbb{Q}$-factorial strictly toroidal pair and $N_1, \ldots, N_q$ are pseudo-effective $\mathbb{R}$-Cartier $\mathbb{R}$-divisors on $Y$. Let $\mathcal{F}$ be the set of $(X, B + M)$ equipped with a birational morphism $\phi: X \to Y$ where

- $(X, B)$ is a projective lc pair,
- the coefficients of $B$ are in $\Phi$,
- $\phi_* B \leq \Delta$, and
- $M = \phi^* \sum \mu_j N_j$ where $\mu_j \in \Phi$.

Then

$$\{\text{vol}(K_X + B + M) \mid (X, B + M) \in \mathcal{F}\}$$

satisfies the DCC.

Proof. Note that $(X, B + M)$ is not necessarily a generalised pair because the $N_i$ may not be nef. However, in practice when we apply the proposition, the $N_i$ would be nef.

Step 1. Since $\Phi$ is DCC, its closure $\Phi \subset [0, \infty)$ is also DCC. Replacing $\Phi$ with $\Phi \cup \{1\}$ we can assume $\Phi$ contains its limiting points and $1 \in \Phi$. If the proposition does not hold, then there is a sequence $(X_i, B_i + M_i)$, $\phi_i: X_i \to Y$, $M_i = \sum \mu_{i,j} N_j$ as in the proposition such that the volumes

$$v_i = \text{vol}(K_{X_i} + B_i + M_i)$$

form a strictly decreasing sequence of numbers.

Step 2. Replacing $(X_i, B_i)$ with a $\mathbb{Q}$-factorial dlt model we can assume $(X_i, B_i)$ is $\mathbb{Q}$-factorial dlt. Running an MMP on $K_{X_i} + B_i$ over $Y$ with scaling of some ample divisor, we reach a model on which the pushdown of $K_{X_i} + B_i$ is a limit of movable/Y $\mathbb{R}$-divisors. Replacing $X_i$ with that model we can assume that $K_{X_i} + B_i$ is a limit of movable/Y $\mathbb{R}$-divisors. By the general negativity lemma [10, Lemma 3.3],

$$K_{X_i} + B_i + G_i = \phi_i^*(K_Y + C_i)$$

for some $G_i \geq 0$ where $C_i = \phi_* B_j$. Since $C_i \leq \Delta$, for every component $D$ of $B_i + G_i$ the log discrepancies satisfy

$$a(D, Y, \Delta) \leq a(D, Y, C_i) = a(D, X_i, B_i + G_i) < 1,$$

hence $a(D, Y, \Delta) = 0$, so $D$ is a toroidal divisor with respect to $(Y, \Delta)$.

Step 3. For each $i$, let $M_i$ be the b-divisor with trace $M_i|_{X_i} = B_i$ and whose coefficient in any exceptional prime divisor over $X_i$ is 1. The set of all prime divisors $D$ over $Y$ such that $0 < \mu_D M_i < 1$ for some $i$, is a countable set because for each $i$ there are only finitely many $D$ with $0 < \mu_D M_i < 1$. Using the assumption that $\Phi$ is DCC, it is then not hard to reduce the problem to the situation in which for each prime divisor $D$ over $Y$, the sequence $\mu_D M_i$ is an increasing sequence for $i \gg 0$. Thus we can define a limiting b-divisor $C$ by setting $\mu_D C = \lim_i \mu_D M_i$ for each $D$. 
By Step 1, the coefficients of \( C \) are in \( \Phi \).

**Step 4.** Assume \( \pi : Y' \to Y \) is a birational contraction that is toroidal with respect to \((Y, \Delta)\), and let \( K_{Y'} + \Delta' \) be the pullback of \( K_Y + \Delta \). For each \( i \), taking a common log resolution \( W \) of \( X_i, Y' \) and then running an MMP on \( K_W + M_i, W \) over \( Y' \) as in Step 2, we can construct \((X'_i, B'_i)\) over \( Y' \) where \( B'_i \) is the birational transform of \( B_i \) plus the reduced exceptional divisor of \( X'_i \to X_i \), that is, \( B'_i = M_i, X'_i \). Since \( K_{X_i} + B_i \) is a limit of movable/\( Y \) \( \mathbb{R} \)-divisors, \( X_i \to X'_i \) does not contract any divisor. Note that

\[
v_i = \text{vol}(K_{X'_i} + B'_i + M'_i)
\]

where \( \phi'_i \) denotes \( X'_i \to Y' \) and \( M'_i = \phi'_i, \pi \sum \mu_{i,j} N_j \).

In the subsequent steps, when necessary, we will feel free to replace \((Y, \Delta)\) with \((Y', \Delta')\) and replace \((X_i, B_i + M_i)\) with \((X'_i, B'_i + M'_i)\). This preserves the b-divisors \( M_i \) and their limit \( C \) as \( X_i \to X'_i \) does not contract any divisor.

**Step 5.** Let \( C := C_Y = \lim_i C_i \). Let \( D_\leq (Y, C) \) (resp. \( D_\leq (Y, C) \)) be the set of exceptional prime divisors \( D \) over \( Y \), toroidal with respect to \((Y, \Delta)\), such that

\[
(*) \quad \mu_D C \leq 1 - a(D, Y, C) \quad \text{(resp. } \leq \text{ replaced by } <).\]

Assume \( D \in D_\leq (Y, C) \) and let \( \pi : Y' \to Y \) be the extremal birational contraction which extracts \( D \). Since \( D \) is toroidal with respect to \((Y, \Delta)\), there is a toroidal birational modification of \( Y \) extracting \( D \) (but no other divisors); this coincides with \( \pi \) since \( Y' \) is \( \mathbb{Q} \)-factorial and \( \pi \) is extremal. Moreover, if \( K_{Y'} + \Delta' \) is the pullback of \( K_Y + \Delta \), then \((Y', \Delta')\) is strictly toroidal [25, page 90, Theorem 6*].

Letting \( C' := C_{Y'} \), we have \( K_{Y'} + C' \leq \pi^*(K_Y + C) \) by \((*)\) where equality holds iff \( D \notin D_\leq (Y, C) \). In particular,

\[
D_\leq (Y', C') \subsetneq D_\leq (Y, C)
\]

as \( D \) does not belong to the former set. Moreover,

\[
D_\leq (Y', C') \subseteq D_\leq (Y, C)
\]

and \( \subseteq \) holds if \( D \in D_\leq (Y, C) \).

**Step 6.** We associate a weight \( w = (p, r, l, d) \) to \((Y, C)\) as follows. We first define the weight \( w_V \) for each non-klt centre \( V \) of \((Y, C)\). Let \( r \) be the codimension of \( V \). Since \((Y, \Delta)\) is strictly toroidal, we have an adjunction formula \( K_V + C_V = (K_Y + C)|_V \) (see 2.12). Let \( l \) be the number of prime exceptional divisors \( S \) over \( V \) mapping into the klt locus of \((V, C_V)\) and with \( a(S, V, C_V) < 1 \). And let \( d \) be the sum of the coefficients of \( C_V \). Put \( w_V = (r, l, d) \).

Now we define the weight \( w \) of \((Y, C)\). Let \( \mathcal{V}(Y, C) \) be the set of non-klt centres of \((Y, C)\) which intersect the centre of some \( D \in D_\leq (Y, C) \). Let \( p \) be the number of elements of \( \mathcal{V}(Y, C) \). If \( p = 0 \), then let \( r = 0 \), let \( l \) be the number of elements of \( D_\leq (Y, C) \), which is finite in this case, and let \( d \) be the sum of the coefficients of \( C \). If \( p > 0 \), then choose \( V \in \mathcal{V}(Y, C) \) with maximal \( w_V = (r, l, d) \) with respect to the lexicographic order on \( 3 \)-tuples. Then in any case let \( w := (p, r, l, d) \).

Note that \( p, r, l \in \mathbb{N} \cup \{0\} \) while \( d \) belongs to a DCC set \( \Theta \) depending only on \( \Phi \) because the coefficients of \( C \) belong to \( \Phi \) and the coefficients of \( C_V \) belong to a DCC set \( \Psi \) depending only on \( \Phi \). We consider the lexicographic order on the weights \( w \).
Step 7. Assume \( w = (p, r, l, d) \) is the weight of \((Y, C)\). Suppose \((p, r, l) \neq (0, 0, 0)\). If \( p = 0 \), then \( r = 0 \) but \( l \neq 0 \), so there is \( D \in \mathcal{D}_<(Y, C) \) with centre \( L \) contained in the klt locus of \((Y, C)\). But if \( p > 0 \), then \( r > 0 \) and there is \( D \in \mathcal{D}_<(Y, C) \) with centre \( L \) intersecting some element of \( \mathcal{V}(Y, C) \). In either case, let \( \pi: Y' \to Y \) and \( C' \) be as in Step 5 constructed for \( D \). Then each element of \( \mathcal{V}(Y', C') \) maps birationally onto an element of \( \mathcal{V}(Y, C) \).

Assume \( w' = (p', r', l', d') \) is the weight of \((Y', C')\) defined similarly as in Step 6. Then \( p' \leq p \) by the previous paragraph. If \( p = 0 \), then \( p' = r' = 0 \) but \( l' < l \), so \( w' < w \). If \( p' < p \), then again \( w' < w \); for example, if there is a non-klt centre of \((Y, C)\) contained in \( L \), then \( p' < p \) holds. In these cases, we replace \((Y, \Delta)\) with \((Y', \Delta')\) as in Step 4, so decrease the weight of \((Y, C)\).

Now assume \( p' > 0 \), in particular, no non-klt centre of \((Y, C)\) is contained in \( L \). Assume that \( V \in \mathcal{V}(Y, C) \) is minimal with respect to inclusion among the non-klt centres intersecting \( L \). Let \( V' \subset Y' \) be its birational transform. Assume \( w_V = (s, m, e) \) and \( w_{V'} = (s', m', e') \). We claim that \( w_{V'} < w_V \). Clearly \( s' = s \).

Define

\[
K_V + C_V = (K_Y + C)|_V \quad \text{and} \quad K_{V'} + C'_{V'} = (K_{Y'} + C')|_{V'}
\]

by adjunction. Then \((V, C_V)\) is klt near \( L \cap V \) by the minimality of \( V \). By construction, \( K_{V'} + C' \leq \rho^*(K_Y + C) \). Then \( K_{V'} + C'_{V'} \leq \rho^*(K_V + C_V) \) where \( \rho \) denotes \( V' \to V \), and equality holds over the non-klt locus of \((V, C_V)\) as this non-klt locus is disjoint from \( L \). Thus \( m' \leq m \). If \( \rho \) contracts a divisor, then \( m' < m \). But if \( \rho \) does not contract any divisor, then \( e' < e \). Thus the claim \( w_{V'} < w_V \) follows.

Assume \( U' \in \mathcal{V}(Y', C') \) and let \( U \) be its image on \( Y \). If \( U \) does not intersect \( L \), then \( w_{U'} < w_U \). But if \( U \) intersects \( L \), then \( w_{U'} < w_U \) if \( U \) is minimal among the non-klt centres intersecting \( L \), then \( w_{U'} < w_U \leq w \) by the previous paragraph; if \( U \) is not minimal, then there is \( T \in \mathcal{V}(Y, C) \) intersecting \( L \) and \( T \not\subset U \), so \( w_{U'} < w_T \leq w \).

Now replace \((Y, \Delta)\) with \((Y', \Delta')\) as in Step 4, and repeat this step. The above arguments show that eventually we can decrease the weight of \((Y, C)\) by choosing \( D \) appropriately in each step. Since we cannot decrease the weight infinitely many times we arrive at the case \( p = r = l = 0 \).

Step 8. From now we assume \( w = (p, r, l, d) = (0, 0, 0, d) \), equivalently, \( \mathcal{D}_<(Y, C) = \emptyset \). If there exists \( D \in \mathcal{D}_<(Y, C) \) with \( a(D, Y, C) < 1 \) and whose centre on \( Y \) is not contained in \( \text{Supp} \ C \), then we replace \((Y, C)\) with \((Y', C')\) where the latter is constructed for \( D \) as in Step 5. The condition \( \mathcal{D}_<(Y, C) = \emptyset \) is preserved. Since \( Y \) is klt outside \( \text{Supp} \ C \), repeating this finitely many times we can assume there is no such \( D \).

Pick \( t \in (0, 1) \). Then \((Y, tC)\) is klt, so there are finitely many prime divisors \( D \) over \( Y \) with \( a(D, Y, tC) < 1 \). We claim that, if \( i \gg 0 \), then \( K_{X_i} + B_i \geq \phi_i^*(Y + tC) \): it is enough to check this in prime divisors \( D \subset X_i \) as in the previous sentence which are automatically toroidal with respect to \((Y, \Delta)\); for such \( D \) we can assume \( \mu_D B_i \) is sufficiently close to \( \mu_D C_{X_i} \); if \( D \) is not exceptional over \( Y \), then the claim follows from \( t < 1 \); if it is exceptional, then since \( \mathcal{D}_<(Y, C) = \emptyset \), writing \( K_{X_i} + A_i = \phi_i^*(K_Y + C) \), we have \( \mu_D C_{X_i} \geq \mu_D A_i \); thus \( \mu_D C_{X_i} = \mu_D A_i \) by Step 2 as \( C_i = \phi_i*B_i \leq C \); then \( D \in \mathcal{D}_<(Y, C) \) and by the previous paragraph, the centre of \( D \) is contained in \( \text{Supp} \ C \); the claim then again follows from \( t < 1 \).

Now for \( i \gg 0 \), we get

\[
v_i = \text{vol}(K_{X_i} + B_i + M_i) \geq \text{vol}(K_Y + tC + \sum \mu_{i,j}N_j).
\]
We can assume that, for each \( j \), the \( \mu_{i,j} \) are increasing sequences with limit \( \mu_j \). Therefore,

\[
v = \lim_i v_i \geq \lim_i \vol(K_Y + tC + \sum \mu_{i,j}N_j) = \vol(K_Y + tC + \sum \mu_jN_j)
\]

because the volume is a continuous function on the numerical classes of divisors [29, Corollary 2.2.45]. Taking the limit when \( t \) approaches 1, we see that for each \( i \), we have

\[
\vol(K_Y + C + \sum \mu_{i,j}N_j) \geq \vol(K_{X_i} + B_i + M_i) \geq v \geq \vol(K_Y + C + \sum \mu_jN_j).
\]

Since \( N_j \) are pseudo-effective and \( \mu_{i,j} \leq \mu_j \), all the inequalities are equalities and \( v_i = v \), a contradiction.

\[\square\]

4.3. DCC of volumes for \( \mathcal{G}_{glc}(d, \Phi) \).

**Proof.** (of Theorem 1.3) **Step 1.** Assume the theorem does not hold. Then there is a sequence \(( X_i, B_i + M_i )\) of generalised pairs in \( \mathcal{G}_{glc}(d, \Phi) \) with data \( X'_i \to X_i \) and \( M' = \sum \mu_{i,j}M'_{i,j} \) so that the volumes

\[
v_i = \vol(K_{X_i} + B_i + M_i)\]

form a strictly decreasing sequence of numbers. Replacing \(( X, B + M )\) with a \( \mathbb{Q} \)-factorial dlt model, we can assume \( X \) is \( \mathbb{Q} \)-factorial. We can discard any \( M'_{i,j} \equiv 0 \).

**Step 2.** By Theorem 1.2 and its proof, there exists a bounded set of couples \( \mathcal{P} \) depending only on \( d, \Phi \), \( \lim v_i \) such that for each \( i \), there exist a log smooth couple \(( \overline{X}_i, \overline{\Sigma}_i ) \in \mathcal{P} \) and a birational map \( \psi_i : X_i \dashrightarrow \overline{X}_i \) satisfying the following:

- \( \Sigma_i \) contains the exceptional divisors of \( \psi_i^{-1} \) and the support of the birational transform of \( B_i \),
- each \( M'_{i,j} \) descends to \( \overline{X}_i \), say as \( \overline{M}_{i,j} \), and
- \( \overline{A}_i - \sum \mu_{i,j} \overline{M}_{i,j} \) is pseudo-effective for some very ample divisor \( \overline{A}_i \leq \overline{\Sigma}_i \).

In particular, the number of the \( M'_{i,j} \) is bounded from above.

Replacing \( X_i \) we can assume \( \psi_i \) are morphisms. Moreover, we can assume that \( X_i \) is obtained from \(( \overline{X}_i, \overline{\Sigma}_i ) \) by a sequence of toroidal smooth blowups; to achieve this we first let \( Y_i \to \overline{X}_i \) be a toroidal birational contraction which extracts exactly the prime divisors on \( X_i \) that are toroidal with respect to \(( \overline{X}_i, \overline{\Sigma}_i )\); denoting \( X_i \to \to Y_i \) by \( \lambda_i \) and letting \( B_{Y_i} = \lambda_i^*B_i \), we can see that \( K_{X_i} + B_i \geq \lambda_i^*(K_{Y_i} + B_{Y_i}) \), so we can replace \( ( X_i, B_i ) \) with \( ( Y_i, B_{Y_i} ) \) preserving the volume \( v_i \) in Step 1; next take a sequence of smooth blowups toroidal with respect to \(( \overline{X}_i, \overline{\Sigma}_i )\) to get a model \( Z_i \) so that \( X_i \to Z_i \), does not contract divisors, and then replace \( X_i \) with \( Z_i \) and replace \( B_i \) with the reduced exceptional divisor of \( Z_i \to X_i \) plus the birational transform of \( B_i \).

**Step 3.** By the effective base point free theorem [28], there is \( n \in \mathbb{N} \) depending only on \( d \) such that

\[
|n(K_{X_i} + 3dA_i + \overline{M}_{i,j})|
\]

and

\[
|n(K_{X_i} + 3dA_i + 2\overline{M}_{i,j})|
\]
are base point free for each \(i, j\). Therefore, we can write \(n\overline{M}_{i,j} \sim \overline{P}_{i,j} - \overline{Q}_{i,j}\) where \(\overline{Q}_{i,j}, \overline{P}_{i,j}\) are general members of the above linear systems, respectively. Since \(\deg \overline{Q}_{i,j}, \deg \overline{P}_{i,j}\) are bounded from above, extending \(\Sigma_i\), we can assume that \(\overline{P}_{i,j} + \overline{Q}_{i,j} \leq \Sigma_i\).

**Step 4.** Replacing the sequence in Step 1 with a subsequence, we can assume that there is a log smooth couple \((\overline{V}, \Sigma)\) and a smooth projective morphism \(\overline{V} \to T\) onto a smooth variety such that for each \(i\), there is a closed point \(t_i \in T\) so that we can identify \(\overline{X}_i\) with \(\overline{V}_{t_i}\) and that \(\Sigma_i \leq \Sigma_{t_i}\) where the subscript \(t_i\) means fibre over \(t_i\), and that \(\{t_i\}\) is dense in \(T\). Replacing \(\Sigma_i\) we can assume we have \(\overline{X}_i = \overline{X}_{t_i}\). After a finite base change and possibly shrinking \(T\) we can assume that \((\overline{V}, \Sigma)\) is log smooth over \(T\) with irreducible fibres i.e. \(I \to T\) is smooth with irreducible fibres if \(I = \overline{V}\) or if \(I\) is a stratum of \((\overline{V}, \Sigma)\). In particular, there is a 1-1 correspondence between the strata of \((\overline{V}, \Sigma)\) and the strata of \((\overline{X}_i, \Sigma_i)\), for each \(i\).

Thus, by the previous step, for each \(i, j\), there exist irreducible components \(E, F\) of \(\Sigma\) such that \(E|_{\overline{X}_i} = \overline{Q}_{i,j}\) and \(F|_{\overline{X}_i} = \overline{P}_{i,j}\), hence \((F - E)|_{\overline{X}_i} \sim n\overline{M}_{i,j}\). Therefore, passing to a subsequence, we can assume that there exist Cartier divisors \(n\overline{M}_j\) on \(\overline{V}\) such that \(n\overline{M}_j|_{X_i} \sim n\overline{M}_{i,j}\) for each \(i, j\).

**Step 5.** Fixing \(i\), we construct a model \(W_i\) over \(\overline{X}_i\). Recall that \(X_i \to \overline{X}_i\) is a sequence of smooth blowups toroidal with respect to \((\overline{X}_i, \Sigma_i)\). This induces a sequence \(\alpha: W \to \overline{V}\) of smooth blowups toroidal with respect to \((\overline{V}, \Sigma)\). Moreover, there is a boundary \(\Gamma\) on \(W\) supported in the birational transform of \(\overline{V}\) plus the exceptional divisors of \(\alpha\) and with coefficients in \(\Phi\) such that \(\Gamma|_{X_i} = B_i\). Let \(W_i\) be the fibre of \(W \to T\) over \(t_i\). Then we get an induced birational morphism \(\beta_i: W_i \to \overline{X}_i\).

Define \(\Gamma_i = \Gamma|_{W_i}\) and

\[
N_i = \beta_i^* \sum \mu_{i,j} \overline{M}_{1,j} \sim_{\mathbb{R}} (\alpha^* \sum \mu_{i,j} \overline{M}_j)|_{W_i}.
\]

**Step 6.** We claim that

\[
v_i = \text{vol}(K_{X_i} + B_i + M_i) = \text{vol}(K_{W_i} + \Gamma_i + N_i).
\]

Let \(H\) be an ample/T divisor on \(W\). Since both

\[
M_i = (\alpha^* \sum \mu_{i,j} \overline{M}_j)|_{X_i} \quad \text{and} \quad N_i = (\alpha^* \sum \mu_{i,j} \overline{M}_j)|_{W_i}
\]

are nef, for each \(l \in \mathbb{N}\) the divisors \(M_i + \frac{1}{l}H|_{X_i}\) and \(N_i + \frac{1}{l}H|_{W_i}\) are ample, so we can find a common open neighbourhood \(U_l\) of \(t_i, t_1\) such that \(\alpha^* \sum \mu_{i,j} \overline{M}_j + \frac{1}{l}H\) is ample over \(U_l\). Shrinking \(U_l\), we can assume that over \(U_l\) there is a boundary

\[
\Theta_l \sim_{\mathbb{R}} \Gamma + \alpha^* \sum \mu_{i,j} \overline{M}_j + \frac{1}{l}H
\]

so that \((W, \Theta_l)\) is lc and log smooth over \(U_l\). Then by [20, Theorem 1.8],

\[
\text{vol}(K_{X_i} + B_i + M_i + \frac{1}{l}H|_{X_i}) = \text{vol}((K_W + \Theta_l)|_{X_i})
\]

\[
= \text{vol}((K_W + \Theta_l)|_{W_i}) = \text{vol}(K_{W_i} + \Gamma_i + N_i + \frac{1}{l}H|_{W_i}).
\]

Note that \(\Theta_l\) may not be a \(\mathbb{Q}\)-divisor but we can still use [20, Theorem 1.8] by approximating with \(\mathbb{Q}\)-divisors as the volume is a continuous function on the numerical
classes of divisors. Taking the limit when \( l \to \infty \) we get the claim stated at the beginning of this step.

Finally, we get a contradiction by applying Proposition 4.2 to the generalised pairs \((W_i, \Gamma_i + N_i)\) which are equipped with the birational morphisms \(W_i \to X_1\).

\[\Box\]

5. Bounded birational models for \(G_{\text{gic}}(d, \Phi, v)\)

In this section we prove a stronger version of Theorem 1.2 for the generalised pairs in \(G_{\text{gic}}(d, \Phi, v)\), following [18, Lemmas 7.1 and 7.2]. This is used in later sections and its proof uses ideas similar to the proof of Theorem 1.3.

**Lemma 5.1.** Let \(d \in \mathbb{N} \), \(\Phi \subset \mathbb{R}^{\geq 0}\) be a DCC set, and \(v \in \mathbb{R}^{>0}\). Let \((Y, \Delta)\) be a projective log smooth toroidal pair and \(N_1, \ldots, N_q\) be nef Cartier divisors on \(Y\).

Consider the set \(H\) of \((X, B + M) \in G_{\text{gic}}(d, \Phi, v)\) with data \(X' \xrightarrow{\phi} X\) and \(M' = \sum \mu_i M'_i\), admitting a birational morphism \(\psi: X \to Y\) with \(\psi^* B \leq \Delta\) and \(M'_i = \psi^* N_i\).

Then there exists a sequence of smooth blowups \(\rho: U \to Y\) toroidal with respect to \((Y, \Delta)\) such that for each \((X, B + M) \in H\),

1. \(\Theta \geq \Gamma\) where \(K_U + \Theta = \rho^*(K_Y + \Delta)\), and \(\Gamma\) is the sum of the exceptional divisors of \(U \to X\) plus the birational transform of \(B\), and

2. letting \(N = \sum \mu_i \rho^* N_i\), we have

\[\text{vol}(K_U + \Gamma + N) = v.\]

**Proof.** Consider the set \(E\) of \((U, \Gamma + N) \in G_{\text{gic}}(d, \Phi)\) with data \(U' \xrightarrow{\phi} U\) and \(N' = \sum \mu_i N'_i\), admitting a birational morphism \(\rho: U \to Y\) with \(\rho^* \Gamma \leq \Delta\) and \(N'_i = \phi^* \rho^* N_i\). By Proposition 4.2, the set

\[\{\text{vol}(K_U + \Gamma + N) \mid (U, \Gamma + N) \in E\}\]

satisfies DCC, hence there is \(\delta > 0\) such that no member of this set belongs to \((v, v + \delta)\). On the other hand, by [12, Theorem 8.1], there is \(\epsilon \in (0, 1)\) such that \(K_U + e\Gamma + N\) is big for any \((U, \Gamma + N) \in E\).

Let \(\epsilon \in \mathbb{R}^{>0}\) so that \((1 - \epsilon)^d > \frac{v}{v + \delta}\) and let \(a = (1 - \epsilon) + \epsilon e\). Then for any \((U, \Gamma + N) \in E\), we have

\[\text{vol}(K_U + a\Gamma + N) = \text{vol}((1 - \epsilon)(K_U + \Gamma + N) + \epsilon(K_U + e\Gamma + N)) \geq (1 - \epsilon)^d \text{vol}(K_U + \Gamma + N).\]

Now since \(a < 1\), \((Y, a\Delta)\) is klt, so there is a sequence of smooth blowups \(\rho: U \to Y\) toroidal with respect to \((Y, \Delta)\) so that we can write

\[K_U + \Psi = \rho^*(K_Y + a\Delta) + E\]

where \(\Psi, E \geq 0\) have no common components and \((U, \Psi)\) has terminal singularities in codimension \(\geq 2\). Let \(K_U + \Theta\) be the pullback of \(K_Y + \Delta\). We will show that \((U, \Theta)\) satisfies the properties of the lemma.
Pick \((X, B + M) \in \mathcal{H}\). We can replace \(X\) hence assume \(\pi: X \dashrightarrow U\) is a morphism. Let \(\Gamma := \pi_* B\) and \(N := \pi_* M\). Let \(\Xi\) be the largest divisor satisfying \(\Xi \leq a\Gamma\) and \(\Xi \leq \Psi\). Then

\[
\text{vol}(K_U + a\Gamma + N) = \text{vol}(K_U + \Xi + N)
\]

where we use \(\rho_* a\Gamma \leq a\Delta\) together with either [20, Lemma 5.3(2)] adapted to our situation or running an MMP on \(K_U + a\Gamma\) over \(Y\) to get a minimal model \(U'\) and then pulling back \(K_U + a\Gamma\) to \(U\). On the other hand,

\[
\text{vol}(K_U + \Xi + N) \leq \text{vol}(K_X + \Xi^- + M) \leq v
\]

where we use the fact that \((U, \Xi)\) has terminal singularities in codimension \(\geq 2\) and that \(\Xi^- \leq aB\). Here \(\Xi^-\) denotes the birational transform of \(\Xi\).

Now observe that \((U, \Gamma + N)\) is a generalised pair with data \(U' := X' \to U\) and \(N' := M'\), actually \((U, \Gamma + N) \in \mathcal{E}\). Then by the above arguments we have

\[
v = \text{vol}(K_X + B + M) \leq \text{vol}(K_U + \Gamma + N) \leq \frac{1}{(1 - \epsilon)^d} \text{vol}(K_U + a\Gamma + N) \leq \frac{v}{(1 - \epsilon)^d} < v + \delta,
\]

so we get \(\text{vol}(K_U + \Gamma + N) = v\) by our choice of \(\delta\).

\(\square\)

**Proposition 5.2.** Let \(d \in \mathbb{N}, \Phi \subset \mathbb{R}^{\geq 0}\) be a DCC set, and \(v \in \mathbb{R}^{\geq 0}\). Then there exists a bounded set of couples \(\mathcal{P}\) such that for each

\[(X, B + M) \in \mathcal{G}_{glc}(d, \Phi, v)\]

with data \(X' \xrightarrow{\phi} X\) and \(M' = \sum \mu_i M'_i\), there is a log smooth couple \((\overline{X}, \overline{\Sigma}) \in \mathcal{P}\) and a birational map \(X \dashrightarrow X\) such that

- \(\Sigma \geq \overline{B}\) where \(\overline{B}\) is the sum of the reduced exceptional divisor of \(X \dashrightarrow X\) plus the birational transform of \(B\),
- each \(M'_i\) descends to \(\overline{X}\), say as \(\overline{M}_i\), and
- letting \(\overline{M} = \sum \mu_i \overline{M}_i\), we have

\[
\text{vol}(K_{\overline{X}} + \overline{B} + \overline{M}) = v.
\]

**Proof.** Step 1. It is enough to prove the proposition for pairs in an arbitrary subset \(\mathcal{G} \subset \mathcal{G}_{glc}(d, \Phi, v)\). Applying Theorem 1.2, there exists a bounded set of couples \(\mathcal{P}\) such that for each

\[(X, B + M) \in \mathcal{G}\]

with data \(X' \xrightarrow{\phi} X\) and \(M' = \sum \mu_i M'_i\), there is a log smooth couple \((\overline{X}, \overline{\Sigma}) \in \mathcal{P}\) and a birational map \(X \dashrightarrow X\) such that

1. \(\overline{\Sigma} \geq \overline{B}\) where \(\overline{B}\) is the sum of the reduced exceptional divisor of \(X \dashrightarrow X\) plus the birational transform of \(B\), and
2. each \(M'_i\) descends to \(\overline{X}\), say as \(\overline{M}_i\).

As in the proof of Theorem 1.3, perhaps after shrinking \(\mathcal{G}\), we can assume that there is a couple \((\overline{V}, \overline{X})\) log smooth over a smooth base \(T\) such that for each \((X, B + M) \in \mathcal{G}\) there is a closed point \(t \in T\) so that we can identify the corresponding couple \((\overline{X}, \overline{\Sigma})\) (as above) with \((\overline{V}_t, \overline{X}_t)\) where the subscript \(t\) means fibre over \(t\). Moreover, we can assume that the strata of \((\overline{V}, \overline{X})\) are smooth over \(T\) with irreducible fibres. In particular, there is a 1-1 correspondence between the strata of \((\overline{V}, \overline{X})\) and the strata of \((\overline{X}, \overline{\Sigma})\). In addition, we can assume that there exist Cartier divisors \(n\overline{N}_i\) on \(\overline{V}\) and
$n \in \mathbb{N}$ depending only on $\mathcal{G}$ such that $n\overline{\mathcal{N}_i}|\overline{X} \sim n\overline{M}_i$ for each $i$.

Step 2. Fix a closed point $t \in T$ and assume $(\overline{Y}, \Delta) := (\overline{V}_t, \overline{\Sigma}_t)$ is in $\mathcal{P}$ and assume $\overline{M}_i|\overline{Y} := \overline{N}_i|\overline{Y}$ are nef Cartier divisors. Let $\mathcal{G}_t \subset \mathcal{G}$ consist of those pairs $(Y, B_Y + M_Y)$ with data $Y' \overset{\phi}{\rightarrow} Y$ and $M'_Y = \sum \mu_i M'_{i,Y}$ admitting a birational map $Y \dashrightarrow \overline{Y}$ such that $\Delta$ contains the reduced exceptional divisor of $\overline{Y} \dashrightarrow Y$ and the birational transform of $B_Y$, and such that $M'_i$, $Y$ descends to $\overline{Y}$ say as $\overline{M}_i|\overline{Y}$. Assume $\mathcal{G}_t \neq \emptyset$. Let $\mathcal{E}_t \subset \mathcal{G}_t$ consist of those pairs where $Y \dashrightarrow \overline{Y}$ is a morphism.

Note that for each $(Y, B_Y + M_Y)$ in $\mathcal{G}_t$, we can assume $Y' \dashrightarrow \overline{Y}$ is a morphism, hence $(Y', B_{Y'} + M_{Y'})$ is in $\mathcal{E}_t$ where $B_{Y'}$ is the sum of the reduced exceptional divisor of $Y' \rightarrow Y$ and the birational transform of $B_Y$, and $M_{Y'} = M'_Y$. Thus applying Lemma 5.1 to $\mathcal{E}_t$, there is a sequence of smooth blowups $\rho: \overline{U} \rightarrow \overline{Y}$ toroidal with respect to $(\overline{Y}, \Delta)$ so that if

$$K_{\overline{Y}} + \overline{\Theta} = \rho^*(K_{\overline{Y}} + \overline{\Delta}),$$

then we have the following: for any $(Y, B_Y + M_Y) \in \mathcal{G}_t$ with data $Y' \overset{\phi}{\rightarrow} Y$ and $M'_Y = \sum \mu_i M'_{i,Y}$,

- $\overline{\Theta} \geq \overline{\Gamma}$ where $\overline{\Gamma}$ is the sum of the reduced exceptional divisor of $\overline{U} \rightarrow Y$ plus the birational transform of $B_Y$, and
- letting $\overline{L} = \sum \mu_i \rho^* M'_i|\overline{Y}$, we have

$$\text{vol}(K_{\overline{Y}} + \overline{\Gamma} + \overline{L}) = v.$$

The morphism $\rho$ induces a sequence of smooth blowups $\overline{W} \rightarrow \overline{Y}$ toroidal with respect to $(\overline{V}, \overline{\Lambda})$. Let $K_{\overline{W}} + \overline{\Theta}$ be the pullback of $K_{\overline{Y}} + \overline{\Lambda}$. Replacing $(\overline{Y}, \Delta)$ with $(\overline{U}, \overline{\Theta})$ and $(\overline{V}, \overline{\Sigma})$ with $(\overline{W}, \overline{\Theta})$ and replacing $\mathcal{P}$ accordingly, we can assume that for the elements $(Y, B_Y + M_Y)$ of $\mathcal{G}_t$ as above, we have: if $\overline{\Gamma}$ is the sum of the reduced exceptional divisor of $\overline{Y} \rightarrow Y$ plus the birational transform of $B_Y$ and if $\overline{L} = \sum \mu_i \overline{M}_i|\overline{Y}$, then

$$\text{vol}(K_{\overline{Y}} + \overline{\Gamma} + \overline{L}) = v.$$

Step 3. Now pick any $(X, B + M) \in \mathcal{G}$ with data $X' \overset{\phi}{\rightarrow} X$ and $M' = \sum \mu_i M'_i$. Then there is a closed point $s \in T$ such that the couple in $\mathcal{P}$ corresponding to $(X, B + M)$ is $(\overline{X}, \overline{\Sigma}) := (\overline{V}_s, \overline{\Sigma}_s)$. Arguing as in Step 2 of the proof of Theorem 1.3, we can replace $X$ hence assume $\pi: X \rightarrow \overline{X}$ is a sequence of smooth blowups toroidal with respect to $(\overline{X}, \overline{\Sigma})$. This induces a similar sequence $Z \rightarrow \overline{Y}$ toroidal with respect to $(\overline{V}, \overline{\Lambda})$.

There is a boundary $C$ on $Z$ such that $(X, B) = (Z_t, C_t)$. Let $(Y, B_Y) := (Z_t, C_t)$ and $Y' = Y$, where $t$ is fixed and $\overline{Y} = \overline{V}_t$ as in Step 2. Let $\psi$ denote $Y \rightarrow \overline{Y}$ and let $M'_{i,Y} = \phi^* \overline{M'}_i|\overline{Y}$. Then $(Y, B_Y + M_Y) \in \mathcal{G}_t$ is a generalised pair with data $Y' = Y$ and $M_Y = \sum \mu_i M'_i|Y$ because the coefficients of $B_Y$ and the $\mu_i$ are in $\Phi$ and because

$$\text{vol}(K_Y + B_Y + M_Y) = \text{vol}(K_X + B + M) = v$$

where the first equality can be seen as in Step 6 of the proof of Theorem 1.3.

But then by Step 2, letting $\overline{L'} = \sum \mu_i \phi^* \overline{M}_i|\overline{Y} = \psi^* M_Y$ and letting $\overline{\Gamma'} = \psi^* B_Y$, we have

$$\text{vol}(K_{\overline{Y}} + \overline{\Gamma'} + \overline{L'}) = v.$$
Boundedness and volume of generalised pairs

This in turn implies that
\[ \text{vol}(K_X + B + M) = v \]
where \( B, M \) are the pushdowns of \( B, M \), arguing as in Step 6 of the proof of Theorem 1.3.

\[ \square \]

6. Boundedness of generalised pairs

In this section we prove the main results on boundedness of generalised pairs, that is, Theorems 1.4 and 1.5. First we discuss how volume changes after pulling back a generalised log divisor under a birational contraction and then removing a multiple of a prime divisor. Next we treat 1.5 on log discrepancies which is important for proving 1.4 and also for the boundedness results on stable log minimal models. In the end of the section we prove 1.4.

6.1. Decrease of volume.

**Lemma 6.2.** Let \( X \) be a normal projective variety and \( \phi: Y \to X \) be the blowup of \( X \) at a closed point \( x \in X \), and let \( E \geq 0 \) be a \( \mathbb{Q} \)-Cartier divisor whose support contains \( \phi^{-1}\{x\} \). Let \( A \) be an ample \( \mathbb{R} \)-divisor on \( X \). Then \( \text{vol}(\phi^*A - tE) < \text{vol}(A) \) for any \( t > 0 \).

**Proof.** First we treat the case when \( x \) is smooth and \( E \) is the exceptional divisor. It is enough to treat the case when \( t \) is sufficiently small. In this case, \( \phi^*A - tE \) is ample. Thus letting \( d = \dim X \) and using the fact that \( E \) is contracted to a point we have
\[ \text{vol}(\phi^*A - tE) = (\phi^*A - tE)^d = (\phi^*A)^d + (-tE)^d < A^d = \text{vol}(A) \]
because
\[ (\phi^*A)^i \cdot (-E)^{d-i} = (\phi^*A)|_E^i \cdot (-E|_E)^{d-i-1} = 0 \]
for each \( 0 < i < d \) and because \( -E|_E \) is ample which ensures
\[ (-E)^d = -(E|_E)^{d-1} < 0. \]

Now we prove the general case. There is a Cartier divisor \( G \geq 0 \) on \( Y \) mapping to \( x \) such that \( -G \) is ample over \( X \). Then \( \phi^*A - sG \) is ample for a sufficiently small \( s \). We claim that
\[ \text{vol}(\phi^*A - sG) < \text{vol}(A). \]
To see this let \( g: \tilde{Y} \to Y \) be the normalisation of \( Y \) and let \( \tilde{y} \) be a smooth point on the inverse image of \( G \). Next let \( \psi: U \to \tilde{Y} \) be the blowup of \( \tilde{Y} \) at \( \tilde{y} \) with exceptional divisor \( S \). Then by the above arguments,
\[ \text{vol}(\psi^* g^* (\phi^*A - s'G) - tS) < \text{vol}(g^*(\phi^*A - s'G)) \]
for any \( t > 0 \) where \( s' \in (0, s) \). But then there is \( t > 0 \) such that
\[ \text{vol}(\phi^*A - sG) = \text{vol}(g^*(\phi^*A - s'G - (s - s')G)) \leq \text{vol}(\psi^* g^* (\phi^*A - s'G) - tS) < \text{vol}(g^*(\phi^*A - s'G)) \leq \text{vol}(A). \]

Finally, since \( \text{Supp} E \) contains the fibre of \( \phi \) over \( x \), it contains \( G \), so \( lE \geq G \) for some \( l > 0 \). Therefore, \( \text{vol}(\phi^*A - tE) < \text{vol}(A) \) for any \( t > 0 \).

\[ \square \]

**Lemma 6.3.** Assume that
Then
$$\text{vol}(K_Y + \Gamma_Y + M_Y - tD) < \text{vol}(K_X + B + M)$$
for any $t > 0$.

Proof. Let $\Gamma'$ on $X'$ be the birational transform of $\Gamma_Y$ plus the exceptional divisors of $\psi$, and let $D'$ be the birational transform of $D$. Then we can write
$$K_{X'} + \Gamma' + M' = \psi^*(K_Y + \Gamma_Y + M_Y) + F'$$
where $F'$ is effective and exceptional over $Y$. Moreover, $G' := \psi^*D - D'$ is effective and exceptional over $Y$. Thus
$$\text{vol}(K_{X'} + \Gamma' + M' - tD') = \text{vol}(\psi^*(K_Y + \Gamma_Y + M_Y) + F' + tG' - t\psi^*D)
= \text{vol}(K_Y + \Gamma_Y + M_Y - tD)$$
for any $t \geq 0$.

Let $B'$ be the boundary on $X'$ such that its coefficient in $D'$ is $1 - a(D, X, B + M)$, its coefficient in any exceptional$/X$ prime divisor other than $D'$ is 1, and its coefficient in any non-exceptional prime divisor is the same as the coefficient in the birational transform $B^\sim$. Note that if $D'$ is not exceptional over $X$, then the coefficient of $D'$ in $B'$ is the same as its coefficient in the birational transform $B^\sim$. Also note that $\mu_{D'}B' > 0$, by (5).

Now $B' \geq \Gamma'$: indeed, let $T'$ be a prime divisor on $X'$; if $T'$ is not exceptional over $X'$, then $\mu_{T'}B' = \mu_{T'}B^\sim \geq \mu_{T'}\Gamma'$ by (4); if $T' = D'$, then $\mu_{T'}B' = \mu_{T'}\Gamma'$ by (5); otherwise $T'$ is exceptional over $X$ and $\mu_{T'}B' = 1 \geq \mu_{T'}\Gamma'$ clearly holds.

By definition of $B'$, we can write
$$K_{X'} + B' + M' = \phi^*(K_X + B + M) + E'$$
where $E'$ is effective and exceptional over $X$. Running an MMP on $K_{X'} + B' + M'$ over $X$ with scaling of some ample divisor contracts $E'$ and ends with a $Q$-factorial generalised dlt pair $(X'', B'' + M'')$ with $K_{X''} + B'' + M''$ being the pullback of $K_X + B + M$. Then
$$\text{vol}(K_{X'} + \Gamma' + M' - tD') \leq \text{vol}(K_{X''} + \Gamma'' + M'' - tD'')
\leq \text{vol}(K_{X''} + B'' + M'' - tD''),$$
hence it is enough to verify
$$\text{vol}(K_{X''} + B'' + M'' - tD'') < \text{vol}(K_X + B + M)$$
for $t > 0$.

Pick a closed point $x \in \gamma(D'')$ and let $V = \gamma^{-1}\{x\}$ where $\gamma$ denotes $X'' \to X$. Pick a small $t > 0$. Run an MMP on $K_{X''} + B'' + M'' - tD''$ over $X$ with scaling of some ample divisor. Since the MMP is a $-D''$-MMP over $X$, if $X''_i \to X''_{i+1}$ is a step of the MMP and if $S_{i+1} \subset X''_{i+1}$ is a subvariety not contained in $D''_{i+1}$, then $S_{i+1}$ has a birational transform $S_i \subset X''_i$ not contained in $D''_i$. Thus replacing $X''$
with some $X''_i$ for $i \gg 0$, we can assume that the irreducible components of $V$ not contained in $D''$ stabilise, that is, they are not contained in the exceptional locus of any step of the MMP beyond $X''$. Thus by definition of MMP with scaling, if $S$ is a component of $V$ not contained in $D''$, then $-D''|_S$ is pseudo-effective. This implies that $V \subset D''$ otherwise we can pick $S$ so that it intersects $D''$ in which case $-D''|_S$ cannot be pseudo-effective.

Now let $\pi: Z \to X$ be the blowup of $X$ at $x$. There is a Cartier divisor $P \geq 0$ on $Z$ mapping to $x$ such that $-P$ is ample over $X$. By Lemma 6.2,

$$\text{vol}(\pi^*(K_X + B + M) - sP) < \text{vol}(K_X + B + M)$$

for any $s > 0$.

Let $\alpha: W \to X''$ and $\beta: W \to Z$ be a common resolution. Since $D''$ contains the fibre of $X'' \to X$ over $x$ and since $P$ is contracted to $x$, we have $l\alpha^*D'' \geq \beta^*P$ for some $l \in \mathbb{N}$. But then

$$\text{vol}(K_{X''} + B'' + M'' - tD'') = \text{vol}(\alpha^*(K_{X''} + B'' + M'' - tD''))$$

$$\leq \text{vol}(\beta^*\pi^*(K_X + B + M) - \frac{t}{l}\beta^*P) = \text{vol}(\pi^*(K_X + B + M) - \frac{t}{l}P)$$

$$< \text{vol}(K_X + B + M)$$

for $t > 0$. □

6.4. Log discrepancies of $\mathcal{F}_{glc}(d, \Phi, v)$. In this subsection we prove Theorem 1.5 which says that the log discrepancies of the generalised pairs in $\mathcal{F}_{glc}(d, \Phi, v)$ not exceeding 1 form a finite set. In particular, this implies that such log discrepancies are zero if they are sufficiently small.

**Lemma 6.5.** Let $(X, B + M)$ be a generalised lc pair with data $X' \xrightarrow{\phi} X$ and $M' = \sum_i \mu_i M'_i$ where $M'_i$ are Cartier and nef$/X$ and $\mu_i \in \mathbb{R}^{\geq 0}$. Then for each $a \in \mathbb{R}^{\geq 0}$, the set

$$\{a(D, X, B + M) \leq a \mid D \text{ prime divisor over } X\}$$

is finite.

**Proof.** The lemma is obvious if $K_X + B + M$ is $\mathbb{Q}$-Cartier because in this the log discrepancies belong to $\frac{1}{l}\mathbb{Z}^{\geq 0}$ for some $l \in \mathbb{N}$. To treat the general case we use approximation. First replacing the pair with a $\mathbb{Q}$-factorial generalised dlt model, we can assume $(X, B + M)$ is $\mathbb{Q}$-factorial dlt. Replacing $p$, we can assume $B = \sum_i b_i B_i$ where $B_i$ are distinct prime divisors and $b_i \geq 0$. Consider the point

$$v = (b_1, \ldots, b_p, \mu_1, \ldots, \mu_p) \in \mathbb{R}^p \times \mathbb{R}^p.$$

Taking the smallest rational hyperplane (i.e. hyperplane generated by points with rational coordinates) passing through $v$, we can find $q \leq 2p + 1$ and positive real numbers $r_j$ and points

$$v_j = (b_{j,1}, \ldots, b_{j,p}, \mu_{j,1}, \ldots, \mu_{j,p}) \in \mathbb{R}^p \times \mathbb{R}^p$$

with non-negative rational coordinates sufficiently close to those of $v$, for $1 \leq j \leq q$, such that $\sum_j r_j = 1$ and $\sum_j r_j v_j = v$, and if the $k$-th coordinate of $v$ is rational, then all $v_j$ have the same $k$-th coordinate as $v$. 
Letting $B^j := \sum_{i=1}^q b_{j,i}B_i$ and $M'^j := \sum_{j=1}^p \mu_{j,i}M'_i$, we get
\[
\sum_{j=1}^q r_jB^j = \sum_{j=1}^p \left( \sum_{i=1}^p r_j b_{j,i}B_i \right) = \sum_{i=1}^p \left( \sum_{j=1}^q r_j b_{j,i}B_i \right) = \sum_{i=1}^p b_iB_i = B
\]
and similarly $M' = \sum_{j=1}^q r_jM'^j$. Thus
\[
(*) \sum_{j=1}^q r_j(K_X + B^j + M^j) = K_X + B + M.
\]

On the other hand, $(X, B + M)$ is $\mathbb{Q}$-factorial generalised dlt, so
\[(X, u(B - \lfloor B \rfloor) + \lfloor B \rfloor + uM)
\]
is generalised dlt for some $u > 1$. Therefore, letting $M^j = \phi_*M'^j$, we can assume that the generalised pairs $(X, B^j + M^j)$, which come with data $X' \to X$ and $M'^j$, are all generalised dlt.

Now let $D$ be a prime divisor over $X$ with $a(D, X, B + M) \leq a$. Then by $(*)$ and by $M' = \sum_{j=1}^q r_jM'^j$, we have
\[
\sum_{j=1}^q r_ja(D, X, B^j + M^j) = a(D, X, B + M) \leq a.
\]
Since $r_j$ are fixed and
\[
a(D, X, B^j + M^j) \leq \frac{a}{r_j},
\]
there are finitely many possibilities for the $a(D, X, B^j + M^j)$ as $K_X + B^j + M^j$ are $\mathbb{Q}$-Cartier. Thus there are also finitely many possibilities for the $a(D, X, B + M)$.

\[\square\]

**Proof.** (of Theorem 1.5) **Step 1.** Assume that the theorem does not holds. Then there is a sequence $(X_i, B_i + M_i), D_i$ of generalised pairs and prime divisors as in the theorem such that
\[
a(D_i, X_i, B_i + M_i) \leq 1
\]
and the set
\[
\{a(D_i, X_i, B_i + M_i)\}_{i \in \mathbb{N}}
\]
is not finite. Here $(X_i, B_i + M_i)$ comes with data $X'_i \xrightarrow{\phi_i} X_i$ and $M'_i = \sum_{j=1}^q \mu_{i,j}M'_{i,j}$.

**Step 2.** Applying Proposition 5.2, there exists a bounded set of couples $P$ such that for each $i$ there exist a log smooth couple $(\overline{X}_i, \overline{\Sigma}_i) \in P$ and a birational map $\overline{X}_i \dasharrow X_i$ such that
\[
\sum_i \geq 0 \text{ where } \overline{B}_i \text{ is the sum of the reduced exceptional divisor of } \overline{X}_i \dasharrow X_i
\]
plus the birational transform of $B_i$,
\[
each M'_{i,j} \text{ descends to } \overline{X}_i, \text{ say as } M_{i,j}, \text{ and}
\]
\[
\text{letting } M_i = \sum_{j=1}^q \mu_{i,j}M'_{i,j}, \text{ we have}
\]
\[
\text{vol}(K_{\overline{X}_i} + \overline{B}_i + \overline{M}_i) = v.
\]
We can assume that $\psi_i: X'_i \dasharrow \overline{X}_i$ is a morphism. Since $K_{X_i} + B_i + M_i$ is ample,
\[
\phi_i^*(K_{X_i} + B_i + M_i) \leq \psi_i^*(K_{\overline{X}_i} + \overline{B}_i + \overline{M}_i).
\]
Thus, by Lemma 2.17, $X_i \to \overline{X}_i$ does not contract any divisor.

**Step 3.** As in the proof of Theorem 1.3, we can assume that there is a couple $(\overline{V}, \Sigma)$ which is relatively log smooth over a smooth variety $T$ such that for each $i$ there is a closed point $t_i \in T$ so that we can identify $(\overline{X}_i, \Sigma_i)$ with $(\overline{V}_{t_i}, \Sigma_{t_i})$ where the subscript $t_i$ means fibre over $t_i$. Moreover, we can assume that the strata of $(\overline{V}, \Sigma)$ have irreducible fibres over $T$. In particular, we can assume that there is a 1-1 correspondence between the strata of $(\overline{V}, \Sigma)$ and the strata of $(\overline{X}_i, \Sigma_i)$ for each $i$. In addition, we can assume there exist $n, q \in \mathbb{N}$ such that for each $i$ and each $j > q$, we have $M_{i,j} \equiv 0$, and that there exist Cartier divisors $n\overline{N}_j$ on $\overline{V}$ such that $n\overline{N}_j|_{\overline{X}_i} \sim nM_{i,j}$ for each $i$ and each $j \leq q$.

For each $i$, write $\overline{B}_i = \sum b_{i,j} \overline{B}_{i,j}$ where $\overline{B}_{i,j}$ are distinct prime divisors. Replacing $q$ we can assume that $b_{i,j} = 0$ for $j > q$. Since $b_{i,j}, \mu_{i,j}$ are in the DCC set $\Phi$, replacing the sequence $(X_i, B_i + M_i)$ with a subsequence, we can assume that for each $j$ the numbers $b_{i,j}$ form an increasing sequence and the numbers $\mu_{i,j}$ also form an increasing sequence. Moreover, we can assume that for each $j$, there is a component of $\Sigma$ whose restriction to $\overline{X}_i$ is $\overline{B}_{i,j}$, for every $i$.

**Step 4.** For each $i$, there is $\Gamma_i \leq \Sigma$ on $\overline{V}$ such that $\Gamma_i|_{\overline{X}_i} = \overline{B}_i$. Moreover, for each $i$, we have

$$
(*) \ vol(K_{\overline{X}_i} + \overline{B}_i + \overline{1}) = \ vol(K_{\overline{X}_i} + \overline{B}_i + \sum_{j \leq q} \mu_{1,j} \overline{M}_{1,j})
$$

$$
= \ vol(K_{\overline{X}_i} + \overline{1}) = \ vol(K_{\overline{X}_i} + \overline{1} + \sum_{j \leq q} \mu_{1,j} \overline{M}_{1,j})
$$

$$
= \ vol(K_{\overline{X}_i} + \overline{1} + \sum_{j \leq q} \mu_{1,j} \overline{N}_j|_{\overline{X}_i})
$$

$$
= \ vol(K_{\overline{X}_i} + \overline{1} + \sum_{j \leq q} \mu_{1,j} \overline{M}_{1,j})
$$

where the fifth equality holds similar to Step 6 of the proof of 1.3.

We claim that $\Gamma_i|_{\overline{X}_i} = \overline{B}_i$: by the previous step,

$$
\overline{R} := \Gamma_i|_{\overline{X}_i} - \overline{B}_i = \sum (b_{i,j} - b_{1,j}) \overline{B}_{i,j} \geq 0
$$

as $b_{i,j} \geq b_{1,j}$; assume $\overline{R} \neq 0$ and let $\overline{S}$ be an irreducible component; then $\overline{S}$ is not exceptional over $X_1$ because the exceptional divisors have coefficient 1 in $\overline{B}_1$; but then the first and the last entry in the equation $(*)$ cannot be equal by Lemma 2.14 applied to a common resolution of $X_1$ and $\overline{X}_1$, a contradiction. Therefore, $\Gamma_i|_{\overline{X}_i} = \overline{B}_i$ and $b_{i,j} = b_{1,j}$ for every $i, j$. In particular, the coefficients of all the $\overline{B}_i$ put together is a finite set. Thus we can assume $D_i$ is exceptional over $X_i$ for every $i$.

Similarly, applying Lemma 2.14 shows that $\mu_{i,j} = \mu_{1,j}$ for every $i$ and $j \leq q$. 

Step 5. From here to the end of Step 8, fix $i > 1$. We can assume that $D_i$ is a divisor on $X'_i$. Moreover, we can assume
\[ a(D_i, X_i, B_i + M_i) < 1 \]
which ensures that
\[ a(D_i, X_i, \Sigma_i) \leq a(D_i, X_i, \bar{B}_i) = a(D_i, X_i, \bar{B}_i + \bar{M}_i) \leq a(D_i, X_i, B_i + M_i) < 1 \]
where the first inequality follows from $\bar{B}_i \leq \Sigma_i$, the equality follows from the fact that $M'_i$ descends to $\bar{X}_i$ as $\bar{M}_i$, and the second inequality follows from the fact that $K_{X_i} + B_i + M_i$ is ample and that $\bar{B}_i$ is the sum of the reduced exceptional divisor of $\bar{X}_i \to X_i$ plus the birational transform of $B_i$. Therefore, $D_i$ is toroidal with respect to $(\bar{X}_i, \Sigma_i)$.

Step 6. There is a sequence $Y_i \to \bar{X}_i$ of smooth blowups, toroidal with respect to $(\bar{X}_i, \Sigma_i)$, so that $D_i$ is not exceptional over $Y_i$. Abusing notation we denote the birational transform of $D_i$ on $Y_i$ again by $D_i$. Let $\Theta_i$ be the sum of the reduced exceptional divisor of $Y_i \to \bar{X}_i$ and the birational transform of $\bar{B}_i$. And then let $\Delta_i$ be the same as $\Theta_i$ except that we replace the coefficient of $D_i$ with $1 - a(D_i, X_i, B_i + M_i)$.

Let $M_{Y_i,j}$ be the pullback of $\bar{M}_{i,j}$, and let $M_{Y_i} = \sum_j \mu_{i,j} M_{Y_i,j}$. We can assume $X'_i \to Y_i$ is a morphism and then consider $(Y_i, \Gamma_i + M_{Y_i})$ as a generalised pair with data $X'_i \to Y_i$ and $M_{Y_i}$. Now we have
\[
v = \text{vol}(K_{X'_i} + B_i + M_i) \leq \text{vol}(K_{Y_i} + \Delta_i + M_{Y_i})
= \text{vol}(K_{Y_i} + \Delta_i + \sum_{j \leq q} \mu_{i,j} M_{Y_i,j}) \leq \text{vol}(K_{\bar{X}_i} + \bar{B}_i + \bar{M}_i) = v,
\]
where the first inequality follows from the definition of $\Delta_i$ and the last inequality follows from the fact that the pushdown of $\Delta_i$ to $\bar{X}_i$ is $\leq \bar{B}_i$. Therefore, all the inequalities are actually equalities.

On the other hand, the pushdown of $\Delta_i$ to $X_i$ is $B_i$ and $\mu_{D_i, \Delta_i}$ is just
\[ 1 - a(D_i, X_i, B_i + M_i). \]

Therefore,
\[
\text{vol}(K_{Y_i} + \Delta_i + \sum_{j \leq q} \mu_{i,j} M_{Y_i,j} - t D_i) = \text{vol}(K_{Y_i} + \Delta_i + M_{Y_i} - t D_i) < v
\]
for any $t > 0$, by Lemma 6.3.

Step 7. Now $Y_i \to \bar{X}_i$ induces a sequence $\alpha : W \to \bar{V}$ of smooth blowups, toroidal with respect to $(\bar{V}, \Sigma)$, and there is a boundary $\Delta$ on $W$ such that $\Delta|_{Y_i} = \Delta_i$. Also there is a prime divisor $J$ on $W$ such that $J|_{Y_i} = D_i$. Let $(W_i, \Lambda_i)$ and $S_i$ be the fibres of $(W, \Delta)$ and $J$ over $t_1$, respectively. Then we get a birational morphism $\beta_i : W_i \to \bar{X}_1$. Let $M_{W_i,j} = \beta_i^* \bar{M}_{1,j}$. Then recalling the fact that $\mu_{i,j} = \mu_{1,j}$ for
\[ j \leq q, \text{ we have} \]
\[
\text{vol}(K_{W_i} + \Lambda_i + \sum_{j \leq q} \mu_{i,j} M_{W_i,j} - tS_i) = \text{vol}(K_{W_i} + \Lambda_i + \sum_{j \leq q} \mu_{i,j} M_{W_i,j} - tS_i) \\
= \text{vol}((K_W + \Delta + \sum_{j \leq q} \mu_{i,j} \alpha^* N_j - tJ)|_{W_i}) \\
= \text{vol}((K_W + \Delta + \sum_{j \leq q} \mu_{i,j} \alpha^* N_j - tJ)|_{Y_i}) \\
= \text{vol}(K_{Y_i} + \Delta_i + \sum_{j \leq q} \mu_{i,j} M_{Y_i,j} - tD_i)
\]
for any small \( t \geq 0 \).

We can assume that \( X' \rightarrow W_i \) is a morphism. Then we get
\[
(**) \quad \text{vol}(K_{W_i} + \Lambda_i + M_{W_i} - tS_i) \begin{cases} = v & \text{if } t = 0, \\ < v & \text{if } t > 0 \end{cases}
\]
where \( M_{W_i} = \sum_j \mu_{i,j} M_{W_i,j} \) is the pushdown of \( M'_i \).

**Step 8.** By construction, \( \Delta \) on \( W \) is the sum of the reduced exceptional divisor of \( W \rightarrow Y \) plus the birational transform of \( \Gamma_i \) except that its coefficient at \( J \) is
\[ c_i := 1 - a(D_i, X_i, B_i + M_i). \]
Thus \( \Lambda_i \) is the sum of the reduced exceptional divisor of \( W_i \rightarrow \overline{X}_1 \) plus the birational transform of
\[ \sum b_{i,j} \overline{B}_{1,j} = \sum b_{1,j} \overline{B}_{1,j} = \overline{B}_1 \]
except that its coefficient at \( S_i \) is \( c_i \).

We claim that
\[ c_i = 1 - a(S_i, X_1, B_1 + M_1) =: e_i. \]
Let \( \Omega_i \) be the same as \( \Lambda_i \) except that we replace the coefficient of \( S_i \) with \( e_i \). Then \( \Omega_i \) is the sum of the reduced exceptional divisor of \( W_i \rightarrow X_1 \) plus the birational transform of \( B_1 \) except that its coefficient at \( S_i \) is \( e_i \). In particular, the pushdown of \( \Omega_i \) to \( X_i \) is just \( B_i \).

First assume \( c_i < e_i \). Then \( \Omega_i \) is a boundary, so applying Lemma 6.3 to the generalised pair \( (W_i, \Omega_i + M_{W_i}) \), we get
\[
\text{vol}(K_{W_i} + \Lambda_i + M_{W_i}) = \text{vol}(K_{W_i} + \Omega_i + M_{W_i} - (e_i - c_i)S_i) < v,
\]
which contradicts formula \((**\rangle\).

Next assume \( c_i > e_i \). Then
\[
v = \text{vol}(K_{W_i} + \Omega_i + M_{W_i}) = \text{vol}(K_{W_i} + \Lambda_i + M_{W_i} - (c_i - e_i)S_i) < v,
\]
where the first equality follows from the definition of \( \Omega_i \) and the inequality follows from \((**)\). This is a contradiction. Therefore, \( c_i = e_i \) as claimed.

**Step 9.** By the previous step and by assumptions,
\[ a(S_i, X_1, B_1 + M_1) = a(D_i, X_i, B_i + M_i) < 1 \]
for every \( i \). But by Lemma 6.5, the left hand of
\[
\{a(S_i, X_1, B_1 + M_1)\}_{i \in \mathbb{N}} = \{a(D_i, X_i, B_i + M_i)\}_{i \in \mathbb{N}}
\]
6.6. **Boundedness of** $\mathcal{F}_{gklt}(d, \Phi, v)$. In this subsection we prove Theorem 1.4. The key ingredients are descent of nef divisors to bounded models, more precisely, Proposition 5.2, together with Theorem 1.5.

**Proof.** (of Theorem 1.4) **Step 1.** By Theorem 1.5,

\[ \{a(D, X, B + M) \leq 1 | (X, B + M) \in \mathcal{F}_{gklt}(d, \Phi, v), D \text{ prime divisor over } X \} \]

is a finite set. Thus there is $\epsilon > 0$ depending only on $d, \Phi, v$ such that every

\[ (X, B + M) \in \mathcal{F}_{gklt}(d, \Phi, v) \]

is generalised $\epsilon$-lc.

**Step 2.** On the other hand, by Proposition 5.2, there exists a bounded set of couples $\mathcal{P}$ such that for each

\[ (X, B + M) \in \mathcal{F}_{gklt}(d, \Phi, v) \]

with data $X' \xrightarrow{\phi} X$ and $M' = \sum \mu_i M'_i$ there exist a log smooth couple $(\overline{X}, \Sigma) \in \mathcal{P}$ and a birational map $\overline{X} \dashrightarrow X$ such that

- $\Sigma \geq \overline{B}$ where $\overline{B}$ is the sum of the exceptional divisors of $\overline{X} \dashrightarrow X$ plus the birational transform of $B$,
- each $M'_i$ descends to $\overline{X}$, say as $\overline{M}_i$, and
- letting $\overline{M} = \sum \mu_i \overline{M}_i$, we have

\[ \text{vol}(K_{\overline{X}} + \overline{B} + \overline{M}) = v. \]

**Step 3.** Fix $(X, B + M) \in \mathcal{F}_{gklt}(d, \Phi, v)$ throughout the proof. Since $(X, B + M)$ is generalised klt, $\phi_* M'_i$ is $\mathbb{Q}$-Cartier if $M'_i \equiv 0$: this can be seen by considering a small $\mathbb{Q}$-factorialisation $Y \rightarrow X$ and then noting that $Y \rightarrow X$ can be decomposed into a finite number of extremal contractions, so we can use the cone theorem to ensure $\phi_* M'_i$ is $\mathbb{Q}$-Cartier. Thus we can remove any $M'_i \equiv 0$ from $M'$. In particular, we can assume that the number of the $\mu_i$ is bounded by some number $q \in \mathbb{N}$ otherwise $\text{vol}(K_{\overline{X}} + \overline{B} + \overline{M})$ would not be bounded. Moreover, by the proof Theorem 1.5, the coefficients of $\overline{B}$ and the $\mu_i$ all belong to a fixed finite set depending only on $d, \Phi, v$. In addition, there is a very ample divisor $\overline{A}$ on $\overline{X}$ such that $\overline{A}^q$ is bounded from above depending only on $d, \Phi, v$, and that

\[ \overline{A} - (\overline{B} + \overline{M}), \quad \overline{A} - (K_{\overline{X}} + \overline{B} + \overline{M}) \]

are pseudo-effective.

**Step 4.** By Lemma 2.17, $X \dashrightarrow \overline{X}$ does not contract any divisor and $(X, B + M)$ is the generalised lc model of $(\overline{X}, \overline{B} + \overline{M})$. Let $\Gamma$ be obtained from $\overline{B}$ by replacing each coefficient in $(1 - \epsilon, 1]$ with $1 - \epsilon$. Since $(X, B + M)$ is generalised $\epsilon$-lc, $(X, B + M)$ is also the generalised lc model of $(\overline{X}, \Gamma + \overline{M})$.

Now by [12, Theorem 1.8], we can find rational numbers $\nu_i \leq \mu_i$ belonging to a finite set depending only on $d, \Phi, v$ such that letting $N' := \sum \nu_i M'_i$ and letting...
\[ N = \sum \nu_i M_i, \] the divisor \( K_X + \Gamma + N \) is big. Then applying [12, Theorem 1.3], there is \( m \in \mathbb{N} \) depending only on \( d, \Phi, v \) such that there is
\[ 0 \leq L \sim m(K_X + \Gamma + N) \]
and \( mN \) is integral. In particular, the coefficients of \( L \) belong to a fixed DCC set: indeed, \( L = \overline{P} + m\overline{\Gamma} \) for some integral divisor \( \overline{P} \); the coefficients of \( m\overline{\Gamma} \) are in a DCC subset of \([0, m] \); so the coefficients of \( \overline{P} \) are \( \geq -m \), hence they are in a DCC set; this implies that the coefficients of \( L \) are in a DCC set.

In addition,
\[ L \cdot A^d = m(K_X + \Gamma + N) \cdot A^d \leq m(K_X + B + M) \cdot A^d, \]
so the left hand side is bounded from above. Thus \((X, \text{Supp}(\Gamma + L + A)) \) belongs to a bounded family and the coefficients of \( L \) are bounded from above. In particular, since \( L \) is big, we can replace it with a bounded multiple and change it linearly so that we can assume that \( L \geq aA \) for some \( a > 0 \).

**Step 5.** By construction,
\[ K_X + \Gamma + M \sim Q, \frac{1}{m} L + M - N. \]
On the other hand, \((X, \Gamma) \) is \( \epsilon \text{-lc} \) and \( A - \Gamma \) and \( A - \frac{1}{m} L \) are pseudo-effective, hence by [7, Theorem 1.8] (also see Lemma 2.9), there is a fixed \( t > 0 \) depending only on \( d, \epsilon, A^d \), hence only on \( d, \Phi, v \), such that
\[ (X, \Gamma + t\frac{1}{m} L) \]
is klt. Replacing \( t \) with \( \frac{t}{2} \), we can assume the pair is \( \frac{t}{2} \text{-lc} \).

Let
\[ \Delta := \Gamma + \frac{t}{m} L \]
and \( P' := \sum p_i M'_i := (1 + t)M' - tN' \)
and let \( \overline{P} \) be the pushdown of \( P' \) to \( X \). Then \((X, \Delta + \overline{P}) \) is generalised \( \frac{t}{2} \text{-lc} \) with nef part \( P' \). Note that \( \Delta \geq cA \) with \( c = \frac{ta}{m} \).

Moreover,
\[ K_X + \Delta + P = K_X + \Gamma + M + \frac{t}{m} L + tM - tN \]
\[ \sim Q(1 + t)(K_X + \Gamma + M). \]
Therefore, if \( \Delta + P \) is the pushdown of \( \Delta + \overline{P} \), then \((X, \Delta + P) \) is the generalised lc model of \((X, \Delta + \overline{P}) \).

**Step 6.** Perhaps after replacing \( A \) we can assume that \( A - K_X \) is ample, hence fixing a sufficiently large \( r \in \mathbb{N} \), there is \( n \in \mathbb{N} \) depending only on \( d \) such that
\[ n(A + rM_i) = n(K_X + A - K_X + rM_i) \]
is base point free, for each \( i \), by the effective base point free theorem [28]. Pick general
\[ 0 \leq R_i \sim n(A + rM_i) \]
and let
\[ \Theta := \Delta - \sum \frac{p_i}{r} A + \sum \frac{p_i}{rn} R_i, \]
Then recalling that $p_i = (1 + t)\mu_i - t\nu_i$, we get an $\frac{1}{2}$-lc pair $(X, \Theta)$ so that

$$K_X + \Theta \sim_{\mathbb{R}} K_X + \overline{\mathcal{M}} + \mathcal{P}.$$ 

**Step 7.** By construction, the coefficients of $\overline{\mathcal{M}}$ are bounded from below away from zero. Moreover, if $\Theta$ is the pushdown of $\overline{\mathcal{M}}$ to $X$, then

$$K_X + \Theta \sim_{\mathbb{R}} (1 + t)K_X + B + M$$

is ample. Thus $(X, \Theta)$ is the lc model of $(\overline{X}, \overline{\mathcal{M}})$, and $(X, \Theta)$ is $\frac{1}{2}$-lc. Now applying [19, Theorem 1.6] shows that $(X, \Theta)$ belongs to a bounded family. Therefore, $(X, B + M)$ also belongs to a bounded family in the sense that there is a very ample divisor $H$ on $X$ with bounded $H^d$ and bounded

$$(K_X + B + M) \cdot H^{d-1}.$$ 

$\square$

7. DCC of Iitaka volumes

In this section we prove our main result on the DCC of Iitaka volumes, that is, Theorem 1.7. Given $(X, B)$ in $I_{lc}(d, \Phi, u)$ or in $I_{klc}(d, \Phi, < u)$ with the corresponding contraction $X \to Z$, the key idea is to control coefficients of the discriminant and moduli divisors appearing in the adjunction formula for $(X, B) \to Z$. After that we can just apply DCC of volumes of generalised pairs, that is, Theorem 1.3. We start with some preparations.

7.1. Adjunction formula for $I_{lc}(d, \Phi, u)$. Given $q \in \mathbb{N}$ and two $\mathbb{R}$-divisors $C, D$ on a normal variety $X$, by $C \sim_q D$ we mean that $qC \sim qD$.

**Lemma 7.2.** Let $\Phi \subset \mathbb{Q}_{\geq 0}$ be a DCC set. Let $\mathcal{P}$ be a bounded set of projective lc pairs $(X, B)$ where the coefficients of $B$ are in $\Phi$ and $K_X + B \sim_{\mathbb{Q}} 0$. Then there exists $q \in \mathbb{N}$ depending only on $\Phi, \mathcal{P}$ such that $q(K_X + B) \sim 0$ for any $(X, B) \in \mathcal{P}$.

**Proof.** First applying [19, Theorem 1.5], we can assume that $\Phi$ is finite. Now assume the lemma does not hold. Then there is a sequence $(X_i, B_i) \in \mathcal{P}$ such that if $q_i \in \mathbb{N}$ is the smallest number satisfying $q_i(K_{X_i} + B_i) \sim 0$, then the $q_i$ form a strictly increasing sequence of numbers.

For each $i$, there is a log resolution $\phi_i : V_i \to X_i$ so that if $C_i$ is the sum of the exceptional divisors of $\phi_i$ plus the birational transform of $B_i$, then the $(V_i, C_i)$ belong to some bounded family of pairs depending only on $\mathcal{P}$. Since $K_{X_i} + B_i \sim_{\mathbb{Q}} 0$,

$$\kappa(K_{V_i} + C_i) = \kappa_{\sigma}(K_{V_i} + C_i) = 0.$$ 

Replacing the sequence $(X_i, B_i)$ with a subsequence, we can assume that there is a pair $(V, C)$ that is log smooth over some smooth variety $T$ such that each $(V_i, C_i)$ is isomorphic to the log fibre of $(V, C)$ over some closed point $t_i \in T$, and such that the set $\{t_i\}$ is dense in $T$. Here we are making use of the fact that $\Phi$ is finite.

Now by [20, Theorem 1.8],

$$\kappa_{\sigma}(K_G + C_G) = \kappa_{\sigma}(K_{V_i} + C_i) = 0$$

for each $i$ where $(G, C_G)$ is the generic log fibre of $(V, C)$ over $T$. Alternatively, we can argue that $K_G + C_G$ is pseudo-effective by showing that an MMP on $K_V + C$
over \( T \) cannot end with a Mori fibre space, so \( 0 \leq \kappa_0(K_G + C_G) \), and then use semi-continuity of cohomology to deduce \( \kappa_0(K_G + C_G) \leq \kappa_0(K_{V_i} + C_{i}) \), hence the equality above.

Applying [17] to \((G,C_G)\) after passing to the algebraic closure of \( k(T) \), we see that we have \( \kappa_0(K_G + C_G) = 0 \) and that there is a sufficiently divisible \( q \in \mathbb{N} \) such that \( h^0(q(K_G + C_G)) \neq 0 \). Applying semi-continuity of cohomology we deduce that \( h^0(q(K_{V_i} + C_{i})) \neq 0 \) for every \( i \). Therefore, for every \( i \), \( h^0(q(K_{X_i} + B_i)) \neq 0 \) and so \( q(K_{X_i} + B_i) \sim 0 \) as \( K_{X_i} + B_i \sim 0 \). This means \( q_i \leq q \) for every \( i \), a contradiction. \( \square \)

**Lemma 7.3.** Let \( d,q \in \mathbb{N} \) and \( u \in \mathbb{Q}^{>0} \). Then there exists \( p \in \mathbb{N} \) depending only on \( d,q,u \) satisfying the following. Assume that

- \((X,B)\) is a projective lc pair of dimension \( d \),
- \( f: X \to Z \) is a contraction onto a curve with \( K_X + B \sim_\mathbb{Q} 0/Z \),
- we have an adjunction formula

\[
(*) \quad K_X + B \sim_q f^*(K_Z + B_Z + M_Z),
\]

- \( A \geq 0 \) is an integral divisor on \( X \),
- over some non-empty open subset \( U \subset Z \): \((X,B + tA)\) is lc for some \( t > 0 \), and \( A \) is relatively semi-ample, and
- for a general fibre \( F \) of \( f \), we have \( 0 < \text{vol}(A|_F) \leq u \).

Then \( pM_Z \) is integral.

**Proof.** Step 1. Fix a closed point \( z_0 \in Z \). Shrinking \( U \) we can assume that \( z_0 \notin U \), so it is enough to show that \( \mu_z pM_Z \) is integral for any \( z \notin U \), for some \( p \in \mathbb{N} \) depending only on \( d,q,u \). Furthermore, by the formula \((*)\), \( q(K_F + B_F) \sim 0 \) for the general log fibres \((F,B_F)\), so shrinking \( U \) again we can assume that over \( U \), \( qB \) is an integral divisor without vertical components.

Take a log resolution \( W \to X \) of \((X,B + A)\) and let \( B_W \) be the sum of the reduced exceptional divisor of \( W \to X \) plus the birational transform of the horizontal/Z part of \( B \) plus the birational transform of the reduction of the fibres of \( f \) over \( Z \setminus U \). Then every component of the fibres of \( W \to Z \) over the points in \( Z \setminus U \) is a component of \([B_W]\). Let \( A_W \) be the birational transform of the horizontal/Z part of \( A \). Then \((W,B_W + tA_W)\) is lc for any small \( t > 0 \) because \([B_W]\) and \( A_W \) have no common components as \([B]\) and \( A \) have no common components over the generic point of \( Z \).

**Step 2.** Run an MMP on \( K_W + B_W + tA_W \) over \( X \) with scaling of some ample divisor for some sufficiently small \( t > 0 \). By construction, over \( f^{-1}U \), \( B_W + tA_W \) is the sum of the reduced exceptional divisor of \( W \to X \) and the birational transform of \( B + tA \). Thus since \((X,B + tA)\) is lc on \( f^{-1}U \), the MMP terminates over \( U \), so we reach a model \( Y \) so that \((Y,B_Y + tA_Y)\) is a \( \mathbb{Q} \)-factorial dlt model of \((X,B + tA)\) over \( U \). We can assume that \( A \) does not contain any non-klt centre of \((X,B + tA)\) over \( U \) as \( t \) is sufficiently small, so \((Y,B_Y)\) is a \( \mathbb{Q} \)-factorial dlt model of \((X,B)\) and \( A_Y \) is the pullback of \( A \) coinciding with birational transfrom of \( A \), over \( U \). In particular, over \( U \), \( K_Y + B_Y \sim_\mathbb{Q} 0 \) and \( K_Y + B_Y + tA_Y \) is semi-ample.

Next run an MMP on \( K_Y + B_Y + tA_Y \) over \( Z \) with scaling of some ample divisor. The MMP does not modify \( Y \) over \( U \). Moreover, the MMP is also an MMP on \( K_Y + B_Y + tA_Y - aF_Y \) where \( F_Y \) is the sum of the fibres of \( Y \to Z \) over the points in \( Z \setminus U \) and \( a > 0 \) is a small number. Thus the MMP terminates with a good minimal model \( V \), by [21], because \( K_Y + B_Y + tA_Y - aF_Y \) is semi-ample over \( U \) and every
non-klt centre of \((Y, B_Y + tA_Y - aF_Y)\) intersects \(f^{-1}U\).

**Step 3.** Decreasing \(t\) if necessary and running MMP as in the previous paragraph, we can assume that running MMP on \(K_Y + B_Y + sA_Y\) over \(Z\) with scaling of an ample divisor for any \(0 < s < t\) does not contract any divisor. Thus \(K_Y + B_Y\) is a limit of movable/\(\mathbb{R}\)-divisors. We claim that \(K_Y + B_Y \sim_{\mathbb{Q}} 0/Z\). Since \(K_Y + B_Y \sim_{\mathbb{Q}} 0\) over \(U\), we have \(K_Y + B_Y \sim_{\mathbb{Q}} P_Y/Z\) for some vertical/\(Z\) divisor \(P_Y\). Replacing \(P_Y\) with \(P_Y + \sum h_iF_i\) where \(F_i\) are the fibres of \(V \to Z\) with a common component with \(P_Y\) and \(h_i\) are appropriate rational numbers, we can assume that \(P_Y \leq 0\) and that \(\text{Supp} P_Y\) does not contain the support of any fibre of \(V \to Z\). Since \(P_Y\) is a limit of movable/\(\mathbb{Z}\)-divisors, \(P_Y|_S\) is pseudo-effective for any component \(S\) of any fibre of \(V \to Z\). This is possible only if \(P_Y = 0\) otherwise we can take \(S\) to be a component of a fibre intersecting \(P_Y\) but not a component of \(P_Y\) and get a contradiction. Therefore, \(K_Y + B_Y \sim_{\mathbb{Q}} 0/Z\) as claimed. In particular, \(A_Y\) is semi-ample and nef and big over \(Z\).

**Step 4.** Let \(\phi: N \to X\) and \(\psi: N \to V\) be a common resolution. By construction,

\[
L := \psi^*(K_Y + B_Y) - \phi^*(K_X + B)
\]

is zero over \(U\). Then since \(K_Y + B_Y \sim_{\mathbb{Q}} 0/Z\) and \(K_X + B \sim_{\mathbb{Q}} 0/Z\), we see that \(L\) is the pullback of a \(\mathbb{Q}\)-divisor \(R_Z\) supported in \(Z \setminus U\). Thus we have the adjunction formula

\[
K_Y + B_Y = \psi_*(\phi^*(K_X + B) + L) \sim_q g^*(K_X + B_Z + R_Z + M_Z)
\]

where \(g\) denotes \(V \to Z\) and \(B_Z + R_Z + M_Z\) are the discriminant and moduli divisors of \((V, B_V) \to Z\), respectively, and \(B_Z, M_Z\) are as in (*). Therefore, replacing \((X,B), A\) with \((V, B_V), A_V\) we can assume that \((X, B + tA)\) is \(\mathbb{Q}\)-factorial dlt for some \(t > 0\), \(A\) is semi-ample over \(Z\), that \(qB\) is integral, and that \([B]\) contains \(f^{-1}(Z \setminus U)\). In particular, this means \(\mu_2B_Z = 1\) for any \(z \in Z \setminus U\).

**Step 5.** Let \(X \to T/Z\) be the contraction defined by \(A\) over \(Z\). Let \(S\) be a vertical/\(Z\) component of \([B]\) that is not contracted over \(T\). Then \(A_S := A|_S\) is nef and big and \(\text{vol}(A_S) \leq \text{vol}(A|_F) \leq u\) where \(F\) is a general fibre of \(f\). By divisorial adjunction,

\[
K_S + B_S := (K_X + B)|_S \sim_{\mathbb{Q}} 0,
\]

so \((S, B_S)\) is a log Calabi-Yau pair and the coefficients of \(B_S\) are in a DCC set depending only on \(q\). In fact, by [19], they belong to a fixed finite set.

We claim that the Cartier index of \(A\) along any codimension two subvariety \(I\) of \(X\) contained in \(S\) is bounded from above. Indeed, since \(A\) does not contain any non-klt centre of \((X,B)\), \(I\) is not a non-klt centre of \((X,B)\). So, \((X,B)\) is plt near the generic point of \(I\). Then the Cartier index of \(K_X + S\) along \(I\) is bounded otherwise the coefficient of \(I\) in \(B_S\) would be 1 [33, Proposition 3.9] which would imply that \(I\) is a non-klt centre of \((X,B)\), a contradiction. But then the Cartier index along \(I\) of any Weil divisor on \(X\) would also be bounded by the same reference.

Replacing \(A\) with a bounded multiple, we can assume that \(A_S\) is integral. Moreover, \((S, B_S + tA_S)\) is lc for some \(t > 0\). Therefore, by [5, Theorem 1.7], \((S, B_S + \lambda A_S)\) is lc for some rational number \(\lambda > 0\) depending only on \(d, q, u\). In particular, \(\text{vol}(K_S + B_S + \lambda A_S)\) is bounded from below away from 0 by [19, Theorem 1.3].
Boundedness and volume of generalised pairs

Step 6. Pick \( z \in Z \setminus U \). Write the fibre of \( f \) over \( z \) as \( \sum m_i F_i \). Then
\[
\sum m_i \vol((K_X + B + \lambda A)|_{F_i}) = \vol((K_X + B + \lambda A)|_{F}) \leq \lambda^{d-1}u
\]
where \( F \) is a general fibre of \( f \). If \( F_i \) is contracted over \( T \), then \( A|_{F_i} \) is not big, so
\[\vol((K_X + B + \lambda A)|_{F_i}) = 0.\]
But if \( F_i \) is not contracted over \( T \), then \( \vol((K_X + B + \lambda A)|_{F_i}) \) is bounded from below away from 0, by the previous step, thus the corresponding \( m_i \) is bounded from above. Therefore, replacing \( X \) with \( T \), we can assume that the fibre \( \sum m_i F_i \) over \( z \) has bounded coefficients (the dlt property of \((X, B)\) may be lost but we do not need it anymore).

Now from the formula \( (*) \) and the facts that \( q(K_X + B) \) is integral and \( K_Z + B_Z \) is Cartier near \( z \), we see that \( qf^\ast M_Z \) is integral over \( z \). That is, \( q(\mu_z M_Z)(\sum m_i F_i) \) is integral, so \( q(\mu_z M_Z)m_i \) is integral for each \( i \). But then since the \( m_i \) are bounded, we see that \( p\mu_z M_Z \) is integral for some \( p \) depending only on \( q \) and the \( m_i \), hence depending only on \( d, q, u \).

\[\square\]

**Lemma 7.4.** Let \( d \in \mathbb{N}, \Phi \subset \mathbb{Q}^{\geq 0} \) be a DCC set, and \( u \in \mathbb{Q}^{>0} \). Then there exist \( p, q \in \mathbb{N} \) depending only on \( d, \Phi, u \) satisfying the following. Assume that
- \((X, B)\) is a projective lc pair of dimension \( d \),
- the coefficients of \( B \) are in \( \Phi \),
- \( f: X \to Z \) is a contraction with \( K_X + B \sim_0 \mathbb{Q}/Z \),
- \( A \geq 0 \) is an integral divisor on \( X \) such that over the generic point of \( Z \):
  - \((X, B + tA)\) is lc for some \( t > 0 \) and \( A \) is relatively ample,
- \( \vol(A|_F) = u \) for the general fibres \( F \) of \( f \).

Then we can write an adjunction formula
\[K_X + B \sim_q f^\ast(K_Z + B_Z + M_Z)\]
where \( pM_Z' \) is Cartier on some high resolution \( Z' \to Z \).

**Proof.** Let \((F, B_F)\) be a general log fibre of \((X, B)\) over \( Z \) and \( A_F = A|_F \). Then \( K_F + B_F \sim_\mathbb{Q} 0 \), the coefficients of \( B_F \) belong to \( \Phi \), \( A_F \) is ample with \( \vol(A_F) = u \), and \((F, B_F + tA_F)\) is lc for some \( t > 0 \). Then \((F, B_F), A_F\) is a stable log Calabi-Yau pair, that is, it is a polarised log Calabi-Yau pair in the language of [5]. Therefore, \((F, \Supp(B_F + A_F))\) belongs to a bounded family of couples, by [5, Corollary 1.8].

Thus by Lemma 7.2, there exists \( q \in \mathbb{N} \) depending only on \( d, \Phi, u \) such that \( q(K_F + B_F) \sim 0 \). Thus the same holds if \( F \) is the generic fibre of \( f \). This implies that we can find a rational function \( \alpha \) on \( X \) so that \( q(K_X + B) + \Div(\alpha) \) is vertical over \( Z \).

So since
\[q(K_X + B) + \Div(\alpha) \sim_\mathbb{Q} 0/Z,\]
we deduce that \( q(K_X + B) + \Div(\alpha) \) is the pullback of a \( \mathbb{Q} \)-Cartier \( \mathbb{Q} \)-divisor \( qL_Z \) on \( Z \), hence \( K_X + B \sim_q f^\ast L_Z \). Therefore, we get an adjunction formula
\[(*) \quad K_X + B \sim_q f^\ast(K_Z + B_Z + M_Z)\]
where \( B_Z \) is the discriminant divisor and \( M_Z := L_Z - (K_Z + B_Z) \) is the moduli divisor. The moduli divisor \( M_Z' \) is nef and commutes with taking higher resolutions, for some resolution \( Z' \to Z \).
On the other hand, the moduli part $M_Z$ depends only on the generic log fibre of $(X, B)$ over $Z$ (in a birational sense) and the chosen rational function $\alpha$, assuming we have fixed $K_X, K_Z$ as Weil divisors (cf. [32, §7][8, 3.4]).

Let $\phi: X' \to X$ be a log resolution so that $X' \to Z'$ is a morphism. Let $\Delta'$ be the horizontal part of the reduced exceptional divisor of $\phi$ plus the birational transform of the horizontal part of $B$. So every non-klt centre of $(X', \Delta')$ is horizontal over $Z'$. Run an MMP on $K_{X'} + \Delta'$ over $X$ with scaling of some ample divisor. It terminates over the generic point of $Z'$ [10, Theorem 1.8] because over this generic point, $K_{X'} + \Delta'$ is the sum of $\phi^*(K_X + B)$ plus an effective exceptional divisor. So we get a model $X''$ on which $K_{X''} + \Delta'' \sim_\mathbb{Q} 0$ over the generic point of $Z'$. Then applying [10, Theorem 1.5][19, Theorem 1.1] or applying [21] shows that we can run a further MMP on $K_{X''} + \Delta''$ over $Z'$ ending with a good minimal model. Replacing $X''$ with the minimal model we can assume $K_{X''} + \Delta''$ is semi-ample over $Z'$ defining a contraction $X'' \to Z''/Z'$. The moduli divisor of $(X'', \Delta'') \to Z''$ coincides with the moduli divisor of $(X, B) \to Z$ on $Z''$ because on a common resolution of $X, X''$ the pullbacks of $K_X + B$ and $K_{X''} + \Delta''$ are equal over the generic point of $Z''$. Also the adjunction formula $(\ast)$ induces an adjunction formula

$$
(**) \quad K_{X''} + \Delta'' \sim_\mathbb{Q} f''^* (K_{Z''} + B_{Z''} + M_{Z''})
$$

where $f''$ denotes $X'' \to Z''$.

It is enough to find $p \in \mathbb{N}$ depending only on $\Phi, \mathcal{P}$ so that $pM_{Z''}$ is an integral divisor because then $pM_{Z''}$ would be integral, hence Cartier. Cutting $Z''$ by hyperplane sections, we can find a curve $C \subset Z''$ and an lc pair $(S, \Gamma)$ over $C$ with $S \subset X''$ such that $(S, \Gamma)$ is lc, $K_S + \Gamma \sim_\mathbb{Q} 0/C$, and the coefficients of $\Gamma$ are in $\Phi$. Moreover, if $A''$ is the birational transform of the horizontal part of $A$ and $H = A''|_S$, then over the generic point of $C$: $(S, \Gamma + tH)$ is lc for some $t > 0$ and $H$ is big and semi-ample with $\text{vol}(H|_S) = u$ for the general fibres $G$ of $g: S \to C$. In addition, the adjunction formula $(\ast\ast)$ induces an adjunction formula

$$
(\ast\ast) \quad K_S + \Gamma \sim_\mathbb{Q} g^*(K_C + B_C + M_C)
$$

where $M_C = M_{Z''}|_C$ (cf. proof of [9, Lemma 3.2]).

Now by Lemma 7.3, $pM_C$ is integral for some $p \in \mathbb{N}$ depending only on $d, \Phi, u$. Then $pM_{Z''}$ is also integral as required because $C$ is a general curve on $Z''$.

\[\square\]

### 7.5. DCC of Iitaka volumes.

**Proof.** (of Theorem 1.7) First we treat the lc case. Pick

$$(X, B) \in \mathcal{I}_{lc}(d, \Phi, u)$$

and let $f: X \to Z$ be the given contraction, and let $(F, B_F)$ be a log general fibre of $(X, B)$ over $Z$. By assumption, there is an integral divisor $A \geq 0$ on $X$ such that $\text{vol}(A|_F) = u$ and over some non-empty open subset of $Z$: $(X, B + tA)$ is lc for some $t > 0$ and $A$ is ample.

By Lemma 7.4, there exist $p, q \in \mathbb{N}$ depending only on $d, \Phi, u$ such that we can write an adjunction formula

$$K_X + B \sim_\mathbb{Q} f^*(K_Z + B_Z + M_Z)$$

where $pM_{Z''}$ is Cartier on some high resolution $Z' \to Z$. Moreover, $(Z, B_Z + M_Z)$ is generalised lc and

$$\text{Ivol}(K_X + B) = \text{vol}(K_Z + B_Z + M_Z).$$
In addition, by the ACC for lc thresholds [19], the coefficients of $B_Z$ belong to a DCC set $\Psi$ depending only on $d, \Phi$. We can assume $\frac{1}{p} \in \Psi$. Thus

$$(Z, B_Z + M_Z) \in G_{\text{glc}}(\dim Z, \Psi),$$

hence $\text{vol}(K_Z + B_Z + M_Z)$ belongs to a DCC set depending only on $\dim Z, \Psi$, by Theorem 1.3. Therefore, $\text{Ivol}(K_X + B)$ belongs to a DCC set depending only on $d, \Phi, u$.

Now we treat the klt case. Pick $(X, B) \in I_{\text{klt}}(d, \Phi, <u)$, let $f: X \to Z$ be the given contraction, and let $(F, B_F)$ be a log general fibre of $(X, B)$ over $Z$. Then $(F, B_F)$ is a klt log Calabi-Yau pair. By assumption, there is an integral divisor $A \geq 0$ on $X$ such that $A_F = A|_F$ is ample with $\text{vol}(A|_F) < u$.

Now $(F, B_F + \lambda A_F)$ is klt for some fixed $\lambda > 0$, by [5, Theorem 1.7]. Moreover, $(F, \text{Supp}(B_F + \lambda A_F))$ belongs to a bounded family of couples, by [5, Corollary 1.6]. Therefore, there is a very ample divisor $H_F$ on $F$ with bounded $H_F^{\dim F}$ and $A_F \cdot H_F^{\dim F - 1}$. This implies that the Cartier index of $A_F$ is bounded, by [8, Lemma 2.25]. Thus $\text{vol}(A_F)$ takes finitely many possible values $u_1, \ldots, u_r$. Therefore,

$$(X, B) \in \bigcup_j I_{\text{lc}}(d, \Phi, u_j),$$

hence the DCC follows from the DCC in the lc case. \hfill \Box

8. Boundedness of stable log minimal models

In this section we prove our main results on boundedness of stable minimal models, that is, Theorems 1.9 and 1.10.

8.1. Log discrepancies on stable log minimal models. We prove a statement similar to Theorem 1.5 but for stable log minimal models.

Lemma 8.2. Let $d \in \mathbb{N}$, $\Phi \subset \mathbb{Q}^{\geq 0}$ be a DCC set, and $u, v \in \mathbb{Q}^{> 0}$. Then the set

$$\{a(D, X, B) \leq 1 \mid (X, B), A \in S_\text{lc}(d, \Phi, u, v), \text{ D prime divisor over } X\}$$

is finite.

Proof. Let $f: X \to Z$ be the given contraction defined by the semi-ample divisor $K_X + B$. If $f$ is birational, then $K_Z + B_Z = f_*(K_X + B)$ is ample with volume $v$ and $(Z, B_Z)$ is lc. So $(Z, B_Z)$, considered as a generalised pair with trivial nef part, belongs to $F_{\text{glc}}(d, \Phi, v)$. Thus the lemma follows from Theorem 1.5 in this case because $(X, B)$ and $(Z, B_Z)$ have the same log discrepancies.

From now on we assume $f$ is not birational. Let $(F, B_F)$ be a general log fibre of $f$ and $A_F = A|_F$. By assumption, $A_F$ is ample with $\text{vol}(A_F) = u$ and $(F, B_F + tA_F)$ is lc for some $t > 0$. Then there is a rational number $\lambda > 0$ depending only on $d, \Phi, u$ such that $(F, B_F + \lambda A_F)$ is lc, by [5, Theorem 6.4]. Since

$$\text{vol}(K_F + B_F + \lambda A_F) = \text{vol}(\lambda A_F) = \lambda^{\dim F} u,$$

we see that the pair $(F, B_F + \lambda A_F)$, considered as a generalised pair with trivial nef part, belongs to

$$F_{\text{glc}}(\dim F, \Phi + \lambda \Phi, \lambda^{\dim F} u),$$
so the coefficients of $B_F + \lambda A_F$ belong to a fixed finite set depending only on $d, \Phi, u$, by Theorem 1.5. This in turn implies that the coefficients of $A_F$ belong to a fixed finite set. Thus there is a bounded $l \in \mathbb{N}$ such that the horizontal/Z part $lA^h$ is integral. Therefore,

$$(X, B), lA^h \in \mathcal{T}_{loc}(d, \Phi, l^{\dim} F, u).$$

By Lemma 7.4, there exist $p, q \in \mathbb{N}$ depending only on $d, \Phi, u, l$, hence on $d, \Phi, u$ such that we can write an adjunction formula

$$K_X + B \sim_q f^*(K_Z + B_Z + M_Z)$$

where $pM_{Z'}$ is Cartier on some high resolution $Z' \to Z$. Moreover, $(Z, B_Z + M_Z)$ is generalised lc and

$$\text{Ivol}(K_X + B) = \text{vol}(K_Z + B_Z + M_Z) = v.$$ In addition, by the ACC for lc thresholds [19], the coefficients of $B_Z$ belong to a DCC set $\Psi$ depending only on $d, \Phi$. Adding $\lambda p$ we can assume $\lambda p \in \Psi$, hence

$$(Z, B_Z + M_Z) \in \mathcal{F}_{glc}(\dim Z, \Psi, v)$$

where now $\Psi$ depends only on $d, \Phi, u$.

By Theorem 1.5, all the generalised log discrepancies of $(Z, B_Z + M_Z)$ in the interval $[0, 1]$ belong to a fixed finite set $\Pi$ depending only on $d, \Phi, v$, hence depending only on $d, \Phi, u, v$.

Assume $D$ is a prime divisor over $X$ with $a(D, X, B) \leq 1$. If $D$ is horizontal over $Z$, then it determines a prime divisor $S$ over a general fibre $F$ of $f$ so that $a(S, F, B_F) \leq 1$. Then $a(S, F, B_F)$ belongs to a finite set because by the adjunction formula above, we have $q(K_F + B_F) \sim 0$ which in particular means that the Cartier index of $K_F + B_F$ is bounded.

We can then assume that $D$ is vertical over $Z$. There exist high log resolutions $X' \dashrightarrow X$ and $Z' \to Z$ such that $f': X' \dashrightarrow Z'$ is a morphism, $D$ is a divisor on $X'$, and its image on $Z'$ is a prime divisor, say $E$. Letting $K_{X'} + B'$ be the pullback of $K_X + B$ we have an adjunction formula

$$K_{X'} + B' \sim_q f'^*(K_{Z'} + B_{Z'} + M_{Z'}).$$

By definition of adjunction, $\mu_E B_{Z'} = 1 - t$ where $t$ is the lc threshold of $f'^*E$ with respect to $(X', B')$ over the generic point of $E$. Since

$$\mu_E B_{Z'} = 1 - a(E, Z, B_Z + M_Z),$$

we see that

$$t = a(E, Z, B_Z + M_Z).$$

On the other hand, since $f'^*E$ has integral coefficients, $t \leq 1 - \mu_D B'$, so we have

$$a(E, Z, B_Z + M_Z) = t \leq 1 - \mu_D B' = a(D, X, B) \leq 1.$$ Therefore,

$$a(E, Z, B_Z + M_Z) \in \Pi$$

which means $\mu_E B_{Z'}$ belongs to a fixed finite set of rational numbers. In particular, perhaps after replacing $p$ with a bounded multiple, we can assume that $p(K_{Z'} + B_{Z'} + M_{Z'})$ is Cartier near the generic point of $E$. But then from

$$pq(K_{X'} + B') \sim pq f'^*(K_{Z'} + B_{Z'} + M_{Z'})$$
we see that $pqB'$ is Cartier over the generic point of $E$, so it is Cartier near the generic point of $D$. Therefore,
\[ 1 - a(D, X, B) = \mu DB' \in \frac{1}{pq} \mathbb{Z} \]
which implies that
\[ a(D, X, B) \in \frac{1}{pq} \mathbb{Z} \cap [0, 1]. \]
\[ \square \]

Note that the proof uses $A$ only in the relative sense over $Z$. In particular, the condition of $K_X + B + A$ being ample is not used.

**Lemma 8.3.** Let $d \in \mathbb{N}, \Phi \subset \mathbb{Q}^{>0}$ be a DCC set, and $u, v, w \in \mathbb{Q}^{>0}$. Then there is $\lambda \in \mathbb{Q}^{>0}$ depending only on $d, \Phi, u, v, w$ such that for any
\[ (X, B), A \in S_{lc}(d, \Phi, u, v) \]
with $\text{vol}(K_X + B + A) < w$, the pair $(X, B + \lambda A)$ is lc.

**Proof.** Pick $(X, B), A$ as in the lemma and let $f: X \to Z$ be the given contraction defined by the semi-ample divisor $K_X + B$. By Lemma 8.2, the log discrepancies of $(X, B)$ in the interval $[0, 1]$ belong to a fixed finite set depending only on $(X, B, u, v)$. In particular, the coefficients of $B$ belong to a fixed finite set $\Phi_0 \subset \Phi$. Moreover, fixing a sufficiently small $\epsilon > 0$, if $D$ is a prime divisor over $X$ with $a(D, X, B) < \epsilon$, then $a(D, X, B) = 0$.

Let $(Y, B_Y)$ be a $\mathbb{Q}$-factorial dlt model of $(X, B)$ and let $A_Y$ be the pullback of $A$. Since $(X, B + tA)$ is lc for some $t > 0$, $A$ does not contain any non-klt centre of $(X, B)$, hence $A_Y$ is just the birational transform of $A$. In particular, the coefficients of $A_Y$ are in $\Phi$. Since $(Y, 0)$ is klt, there is a birational contraction $Y' \to Y$ which extracts exactly the prime divisors $D$ over $Y$ with $a(D, Y, 0) < \epsilon$. For such $D$ we have
\[ a(D, X, B) = a(D, Y, B_Y) \leq a(D, Y, 0) < \epsilon, \]
hence by the previous paragraph, $a(D, Y, B_Y) = 0$. Therefore, if $K_{Y'} + B_{Y'}$ is the pullback of $K_Y + B_Y$, then each exceptional divisor of $Y' \to Y$ is a component of $[B_{Y'}]$. Since $(Y, B_Y + tA_Y)$ is lc for some $t > 0$, we deduce that no exceptional divisor of $Y' \to Y$ maps into $A_Y$, hence if $A_{Y'}$ is the pullback of $A_Y$, then $A_{Y'}$ is the birational transform of $A_Y$.

Now replace $(Y, B_Y), A_Y$ with $(Y', B_{Y'}, A_{Y'})$ (the dlt property maybe lost but we will not need it). Then we have a crepant model $(Y, B_Y)$ of $(X, B)$ such that $(Y, 0)$ is $\epsilon$-lc, the exceptional divisors of $Y \to X$ are components of $[B_Y]$, and $A_Y$ is the pullback of $A$ which coincides with the birational transform of $A$.

By construction, $K_Y + B_Y + A_Y$ is nef and big, $K_Y + B_Y$ is nef, the coefficients of $B_Y$ are in a finite set of rational numbers, and the coefficients of $A_Y$ are in $\Phi$. Thus applying [5, Theorem 4.2] to an appropriate bounded multiple of $K_Y + B_Y + A_Y$, we see that $|m(K_Y + B_Y + A_Y)|$ defines a birational map for some $m \in \mathbb{N}$ depending only on $d, \Phi, \epsilon$, hence depending only on $d, \Phi, u, v$. In particular, $|m(K_X + B + A)|$ also defines a birational map.

Pick a member
\[ M \in |m(K_X + B + A)|. \]
Then
\[ \text{vol}(M) = \text{vol}(m(K_X + B + A)) < m^d w. \]
We can assume the non-zero coefficients of $mA$ are at least 1 and that $mB$ is integral. So for any component $D$ of $M$ we have $\mu_D(M+B+mA) \geq 1$ because any component of $M$ whose coefficient is not an integer is a component of $A$. Moreover, by construction, 

\[ M - (K_X + B + mA) \sim (m - 1)(K_X + B) \]

is pseudo-effective. Therefore, applying [8, Proposition 4.4] to $(X, B + mA)$, $M$ we deduce that $(X, B + mA + M)$ is log birationally bounded, that is, there is a bounded set of couples $\mathcal{P}$ and a number $c \in \mathbb{Q}^{>0}$ depending only on $d, \Phi, m^d w$, hence only on $d, \Phi, u, v, w$, such that there is a log smooth couple $(X, \Sigma) \in \mathcal{P}$ and a birational map $\overline{X} \dashrightarrow X$ such that

- $\Sigma$ contains the reduced exceptional divisor of $\overline{X} \dashrightarrow X$ and the support of the birational transform of $B + A + M$, and
- if $\overline{M}$ is the pullback of $M$ to $\overline{X}$, then $\overline{M} \cdot \overline{d} \leq c$ for some very ample divisor $\overline{H} \leq \Sigma$.

Here by pullback of $\overline{M}$ to $\overline{X}$ (and similarly for other divisors) we mean pulling back $\overline{M}$ to a common resolution of $X, \overline{X}$ and then pushing it down to $\overline{X}$. Since $M - A$ is pseudo-effective, if $\overline{A}$ is the pullback of $A$ to $\overline{X}$, then $\overline{A} \cdot \overline{d} \leq c$. In particular, the coefficients of $\overline{A}$ are bounded from above.

Let $K_{\overline{X}} + \overline{B}$ be the pullback of $K_X + B$ to $\overline{X}$. Since no log discrepancy of $(X, B)$ belongs to $(0, \epsilon)$, no coefficient of $\overline{B}$ is in $(b, 1)$ where $b = 1 - \epsilon$. On the other hand, since $A$ does not contain any non-klt centre of $(X, B)$, we see that $\overline{A}$ does not contain any component of $\overline{B}$ with coefficient 1. Therefore, there is $\lambda \in (0, 1)$ depending only on $d, \Phi, u, v, w$, such that no coefficient of $\overline{B} + \lambda \overline{A}$ exceeds 1, that is, $(X, \overline{B} + \lambda \overline{A})$ is sub-lc as it is log smooth. But then since

\[ K_X + B + \lambda A = (1 - \lambda)(K_X + B) + \lambda(K_X + B + A) \]

is ample, we deduce that $(X, B + \lambda A)$ is lc because the pullback of $K_X + B + \lambda A$ to a common resolution of $X, \overline{X}$ is less than or equal to the pullback of $K_{\overline{X}} + \overline{B} + \lambda \overline{A}$.

\[ \square \]

8.4. Boundedness of $S_{lc}(d, \Phi, u, v, \sigma)$ and $S_{klt}(d, \Phi, u, v, <w)$.

Proof. (of Theorem 1.10) Pick

\[ (X, B), A \in S_{lc}(d, \Phi, u, v, \sigma) \]

and let $f : X \rightarrow Z$ be the given contraction defined by the semi-ample divisor $K_X + B$. By assumption, $K_X + B + A$ is ample with volume $\sigma(1)$. By Lemma 8.3, there is a rational number $\lambda > 0$ depending only on $d, \Phi, u, v, \sigma(1)$, such that $(X, B + \lambda A)$ is lc.

The coefficients of $B + \lambda A$ belong to the DCC set $\Phi + \lambda \Phi$ and the volume

\[ \text{vol}(K_X + B + \lambda A) = \sigma(\lambda) \]

is fixed. Therefore, by [18], $(X, B + \lambda A)$ belongs to a bounded family of pairs. So there is a very ample divisor $L$ on $X$ with $L^d$ and

\[ (K_X + B + \lambda A) \cdot L^{d-1} \]

bounded from above. Then

\[ (K_X + B + A) \cdot L^{d-1} \]

is bounded from above, so the $(X, B), A$ form a bounded family.

\[ \square \]
Proof. (of Theorem 1.9) Pick 

\((X, B), A \in S_{\text{klt}}(d, \Phi, u, v, < w)\),

let \(f: X \to Z\) be the given contraction defined by the semi-ample divisor \(K_X + B\). By Lemma 8.2, the log discrepancies of \((X, B)\) in the interval \([0, 1]\) belong to a fixed finite set depending only on \(d, \Phi, u, v\). Therefore, \((X, B)\) is \(\epsilon\)-lc and \(l(K_X + B)\) is integral for some \(\epsilon > 0\) and \(l \in \mathbb{N}\) depending only on \(d, \Phi, u, v\).

By assumption, \(K_X + B + A\) is ample with volume \(< w\). By Lemma 8.3, there is a rational number \(\lambda \in (0, 1)\) depending only on \(d, \Phi, u, v, w\) such that \((X, B + \lambda A)\) is klt. Replacing \(\lambda\) with \(\frac{1}{2}\), we can assume \((X, B + \lambda A)\) is \(\frac{1}{2}\)-lc. Let 

\[ N := l(K_X + B + \lambda A). \]

Then

\[ \text{vol}(K_X + B + \lambda A + N) \leq \text{vol}((l + 1)(K_X + B + A)) < (l + 1)^d w. \]

Therefore, applying [5, Theorem 6.2] to \((X, B + \lambda A), N\) shows that \((X, B + \lambda A)\) belongs to a bounded family. This implies \((X, B), A\) belongs to a bounded family.

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