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ON THE DISCRETIZATION OF BACKWARD DOUBLY STOCHASTIC DIFFERENTIAL EQUATIONS

OMAR ABOURA

Abstract. In this paper, we are dealing with the approximation of the process \((X_t, Y_t, Z_t)\) solution to the backward doubly stochastic differential equation (BDSDE)

\[
X_s = x + \int_0^s b(X_r) \, dr + \int_0^s \sigma(X_r) \, dW_r,
\]

\[
Y_s = \phi(X_T) + \int_s^T f(r, X_r, Y_r, Z_r) \, dr + \int_s^T g(r, X_r, Y_r, Z_r) \, d\widetilde{B}_r - \int_s^T Z_r \, dW_r.
\]

After proving the \(L^2\)-regularity of \(Z\), we use the Euler scheme to discretize \(X\) and the Zhang approach in order to give a discretization scheme of the process \((Y, Z)\).

1. Introduction

Since the pioneering work of E. Pardoux and S. Peng [PP92], backward stochastic differential equations (BSDEs) have been intensively studied during the two last decades. Indeed, this notion has been a very useful tool to study problems in many areas, such as mathematical finance, stochastic control, partial differential equations; see e.g. [MY99] where many applications are described. Discretization schemes for BSDEs have been introduced and studied by several authors. The first papers on this topic are that of V. Bally [Ba97] and D. Chevance [Ch97]. In his thesis, Zhang made an interesting contribution which was the starting point of intense study among which the works of B. Bouchard and N. Touzi [BT04], E. Gobet, J. P. Le mor and X. W arin [GLW05], ... The notion of BSDE has been generalized by E. Pardoux and S. Peng [PP94] to that of Backward Doubly Stochastic Differential Equation (BDSDE) as follows. Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space, \(T\) denote some fixed terminal time which will be used throughout the paper, \((W_t)_{0 \leq t \leq T}\) and \((B_t)_{0 \leq t \leq T}\) be two independent standard Brownian motions defined on \((\Omega, \mathcal{F}, \mathbb{P})\) and with values in \(\mathbb{R}^d\), and \(\mathbb{R}\) respectively. On this space we will deal with two families of \(\sigma\)-algebras:

\[
\mathcal{F}_t := \mathcal{F}^W_{0,t} \vee \mathcal{F}^B_{0,t} \vee \mathcal{N}, \quad \hat{\mathcal{F}}_t := \mathcal{F}^W_{0,t} \vee \mathcal{F}^B_{0,t} \vee \mathcal{N}, \quad \mathcal{H}_t = \mathcal{F}^W_{0,t} \vee \mathcal{H}^B_{t,T} \vee \mathcal{N},
\]

where \(\mathcal{F}^B_{t,T} := \sigma(B_r - B_t; t \leq r \leq T)\), \(\mathcal{F}^W_{0,t} := \sigma(W_r - W_0; 0 \leq r \leq t)\) and \(\mathcal{N}\) denotes the class of \(\mathbb{P}\) null sets. We remark that \(\hat{\mathcal{F}}_t\) is a filtration, \(\mathcal{H}_t\) is a decreasing family of \(\sigma\)-algebras, while \(\mathcal{F}_t\) is neither increasing nor decreasing. Given an initial condition \(x \in \mathbb{R}^d\), let \((X_t)\) be the \(d\)-dimensional diffusion process defined by

\[
X_t = x + \int_0^t b(X_r) \, dr + \int_0^t \sigma(X_r) \, dW_r.
\]  

Let \(\xi \in L^2(\Omega)\) be an \(\mathbb{R}^d\)-valued, \(\mathcal{F}_T\)-measurable random variable, \(f\) and \(g\) be regular enough coefficients; consider the BDSDE defined as follows:

\[
Y_s = \xi + \int_s^T f(r, X_r, Y_r, Z_r) \, dr
\]

\[
+ \int_s^T g(r, X_r, Y_r, Z_r) \, d\widetilde{B}_r - \int_s^T Z_r \, dW_r.
\]

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In this equation, $dW$ is the forward integral and $dB$ is the backward integral (we send the reader to [NP88] for more details on backward integration). A solution to \( (1.3) \) is a pair of real-valued process \((Y_t, Z_t)\), such that \( X_t \) and \( Y_t \) are \((\mathcal{F}_t)\) for every \( t \in [0, T] \), such that \((1.3)\) is satisfied and
\[
E \left( \sup_{0 \leq s \leq T} |Y_s|^2 \right) + \int_0^T |Z_s|^2 ds < +\infty.
\]

In [PP94] Pardoux and Peng have proved that under some Lipschitz property on \( f \) and \( g \) which will be stated more precisely in section 2, \((1.3)\) has a unique solution. The following backward stochastic partial differential equation when \( \xi = \phi(X_T) \) for a regular function \( \phi \):
\[
u(t, x) = \phi(x) + \int_t^T \left( Lu(s, x) + f(s, x, u(s, x), \nabla u(s, x)\sigma(x)) \right) ds
+ \int_t^T g(s, x, u(s, x), \nabla u(s, x)\sigma(x)) dB_s,
\]
where \( L \) is the differential operator defined by:
\[
Lu(t, x) = \frac{1}{2} \sum_{i,j=1}^d (\sigma\sigma^\tau)_{i,j}(x) \frac{\partial^2}{\partial x_i \partial x_j} u(t, x) + \sum_{i=1}^d b_i(x) \frac{\partial}{\partial x_i} u(t, x).
\]

The paper is organized as follows: first we prove the \( L^2 \)-regularity of \( Z \) in section 2. This is a crucial step in order to the scheme using Zhang’s method, which is done in section 3. Finally, a numerical scheme is described in the last section. To ease notations, we set \( \Theta := (X_r, Y_r, Z_r) \) for \( r \in [0, T] \).

As usual, we denote by \( C_p \) a constant which depends on some parameter \( p \), and which can change from one line to the next one. Finally, for some function \( h(t, x, y, z) \) defined on \([0, T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R} \), we let \( \partial_x h(t, x, y, z) \) (resp. \( \partial_y h(t, x, y, z) \)) the partial derivatives of \( h \) with respect to the real variable \( y \) (resp. \( z \)), while \( \partial_x h(t, x, y, z) \) will denote the vector \( (\partial_x h(t, x, y, z), i = 1, \cdots, d) \).

### 2. Regularity properties

In this section we give some regularity properties of the process \( X, Y \) and \( Z \).

The following assumptions which ensure existence and uniqueness of the solution will be in force throughout the paper. For every integer \( n \geq 1 \), let \( M^2([0, T], \mathbb{R}^n) \) denote the set of \( \mathbb{R}^n \)-valued jointly measurable processes \((\varphi_t, t \in [0, T])\) such that \( \varphi_t \) is \( \mathcal{F}_t \)-measurable for almost every \( t \) and \( \int_0^T |\varphi_t|^2 dt < +\infty \).

**Assumption 1** (for the forward process \( X \)). The maps \( b : \mathbb{R}^d \to \mathbb{R}^d \) and \( \sigma : \mathbb{R}^d \to \mathbb{R}^{d \times d} \) are of class \( C^3 \).

**Assumption 2** (for the backward process \((Y, Z)\)). Let \( f : \Omega \times [0, T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \), \( g : \Omega \times [0, T] \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R} \) be such that \( f \) and \( g \) are jointly measurable, for every \((x, y, z) \in \mathbb{R}^{d+2} \), \( f(., x, y, z) \) and \( g(., x, y, z) \) belong to \( M^2([0, T], \mathbb{R}) \), and such that:

(i) There exist some nonnegative constants \( L_f, L_g \) and a constant \( \alpha \in [0, 1] \) such that for every \( \omega \in \Omega \), \( t, t' \in [0, T] \), \( x, x' \in \mathbb{R}^d \), \( y, y' \in \mathbb{R} \) and \( z, z' \in \mathbb{R} \):
\[
|f(t, x, y) - f(t', x', y')|^2 \leq L_f \left( |t - t'| + |x - x'|^2 + |y - y'|^2 + |z - z'|^2 \right),
|g(t, x, y) - g(t', x', y', z')|^2 \leq L_g \left( |t - t'| + |x - x'|^2 + |y - y'|^2 + \alpha |z - z'|^2 \right),
\]

(ii) For all \( s \in [0, T] \) \( f(., s) \) and \( g(., s) \) are of class \( C^3 \) with bounded partial derivatives up to order 3, uniformly in time.
(iii) For a function \( h(t, x, y, z) \), set \( h(t, 0) := h(t, 0, 0, 0) \). Then
\[
\sup_{r \in [0, T]} |f(r, 0)| + \sup_{r \in [0, T]} |g(r, 0)| < \infty.
\]

**Assumption 3.** Suppose that \( \xi := \phi(X_T) \) for some function \( \phi : \mathbb{R}^d \to \mathbb{R} \) of class \( C^2 \) and that for every \( \omega \in \Omega \),
\[
\sup_{t, x, y, z} |\partial_x g(t, x, y, z)| < 1.
\]

### 2.1. Some classical properties of the forward process \( X \)

We at first recall without proof the following well known results on diffusion processes. Define the \( \mathbb{R}^d \times d \)-valued process \( (\nabla X_t)_{0 \leq t \leq T} \) by:
\[
\nabla X_t := \left( \frac{\partial}{\partial x_j} X_t^i, i, j = 1, \ldots, d \right).
\]

Then \( \nabla X_t \) is an invertible \( d \times d \) matrix, solution to a linear stochastic differential equation with coefficients depending on \( X_t \). Furthermore, the assumptions on the coefficients \( \sigma \) and \( b \) yield the following classical result:

**Proposition 2.1.** (i) For all \( p \geq 1 \), there exist a constant \( C_p > 0 \) such that for all \( t, s \in [0, T] \):
\[
\mathbb{E} |X_t - X_s|^{2p} + \mathbb{E} \left| (\nabla X_t)^{-1} - (\nabla X_s)^{-1} \right|^{2p} \leq C_p |t - s|^p.
\]

(ii) For all \( p \in [1, \infty) \), there exist a constant \( C_p > 0 \) such that
\[
\mathbb{E} \left( \sup_{t \in [0, T]} |X_t|^{2p} + \sup_{t \in [0, T]} \left| (\nabla X_t)^{-1} \right|^p \right) \leq C_p.
\]

### 2.2. Time increments of \( Y \) and \( L^2 \)-regularity of \( Z \)

The following lemma provides upper bounds for time increments of \( Y \).

**Lemma 2.2.** Set \( \xi = \phi(X_T) \) for some function \( \phi : \mathbb{R}^d \to \mathbb{R} \) of class \( C^1 \). Then we have
(i) For all \( p \geq 2 \), there exist a constant \( C_p > 0 \) depending on \( T \) such that for all \( t, s \in [0, T] \)
\[
\mathbb{E} |Y_t - Y_s|^p \leq C_p |t - s|^\frac{p}{2}.
\]  

(ii) For all \( p \geq 1 \), there exist a constant \( C > 0 \) such that
\[
\sup_{0 \leq r \leq T} \mathbb{E} |Z_r|^{2p} \leq C.
\]

Notice that the inequality (2.1) is different from equation (2.11) in [Z04].

**Proof.** We at first prove (ii). Let \( (\nabla Y_t)_{0 \leq t \leq T} = (\partial_x Y_t)_{0 \leq t \leq T} \) denote the real-valued process defined by differentiation of \( Y \) as function of the initial condition \( x \) of the diffusion process \( (X_t) \). We recall the following representation of \( Z \) (see [PP92], Proposition 2.3):
\[
Z_t = \nabla Y_t (\nabla X_t)^{-1} \sigma (X_t).
\]

where \( (\nabla Y_t, \nabla Z_t) \) satisfies the linear BDSDE with the forward process \( (X_t, \nabla X_t) \) and the evolution equation:
\[
\nabla Y_t = \phi'(X_T) \nabla X_T + \int_t^T \left( f_x(r, \Theta_r) \nabla X_r + f_y(r, \Theta_r) \nabla Y_r + f_z(r, \Theta_r) \nabla Z_r \right) dr
\]
\[
+ \int_t^T \left( g_x(r, \Theta_r) \nabla X_r + g_y(r, \Theta_r) \nabla Y_r + g_z(r, \Theta_r) \nabla Z_r \right) d\widetilde{B}_r - \int_t^T \nabla Z_r dW_r.
\]

By E.Pardoux and S.Peng [PP94] page 217, we deduce
\[
\mathbb{E} \left( \sup_{0 \leq t \leq T} |\nabla Y_t|^p \right) < \infty.
\]

Then Hölder’s inequality and Proposition 2.1 yield
\[
\mathbb{E} |Z_t|^{2p} \leq \left( \mathbb{E} |\nabla Y_t|^{6p} \right)^{\frac{1}{3}} \left( \mathbb{E} \left| (\nabla X_t)^{-1} \right|^{6p} \right)^{\frac{1}{3}} \left( \mathbb{E} |\sigma (X_t)|^{6p} \right)^{\frac{1}{6}}.
\]
Hence, the inequalities (2.6)-(2.8) imply
\[
E |Y_t - Y_s|^p \leq C_p |t - s|^{p-1} \int_s^t |f (r, X_r, Y_r, 0)|^p dr + C_p \left( \int_s^t |Z_r|^p dr \right)^{\frac{p}{2}} + C_p \left( \int_s^t |Z_r|^2 dr \right)^{\frac{p}{2}}.
\]

Recall that \( \hat{F}_t \) and \( \mathcal{H}_t \) have been defined in (1.1). The process \( (\int_0^t Z_r dW_r, 0 \leq t \leq T) \) is a \((\hat{F}_t)\)-martingale, while the process \( (\int_0^T g(r, \Theta_r) d\bar{B}_r, 0 \leq t \leq T) \) is a backward martingale for \((\mathcal{H}_t)\). Hence, the Burkholder-Davies-Gundy and Hölder inequalities yield
\[
E |Y_t - Y_s|^p \leq C_p |t - s|^{p-1} \int_s^t |f (r, X_r, Y_r, 0)|^p dr + C_p \left( \int_s^t |Z_r|^p dr \right)^{\frac{p}{2}} + C_p \left( \int_s^t |Z_r|^2 dr \right)^{\frac{p}{2}}.
\]

Assumption (i) and (ii), Proposition 2.1 and (1.4) yield
\[
E \left( \int_s^t |g(r, \Theta_r)|^2 dr \right)^{\frac{p}{2}} \leq C_p \left( \int_s^t |g(r, \Theta_r) - g(r, 0)|^2 dr \right)^{\frac{p}{2}} + C_p \left( \int_s^t |g(r, 0)|^2 dr \right)^{\frac{p}{2}}
\leq C_p |t - s|^{\frac{p}{2}} + C_p \left( \int_s^t (|X_r|^2 + |Y_r|^2) dr \right)^{\frac{p}{2}} + C_p \left( \int_s^t |Z_r|^2 dr \right)^{\frac{p}{2}}
\leq C_p |t - s|^{\frac{p}{2}} + C_p \left( \int_s^t |Z_r|^2 dr \right)^{\frac{p}{2}}.
\]

Similarly,
\[
E \left( \int_s^t |f (r, X_r, Y_r, 0)|^p dr \right)^{\frac{p}{2}} \leq C_p \left( \int_s^t |f (r, 0)|^p dr \right)^{\frac{p}{2}} + C_p \left( \int_s^t |f (r, X_r, Y_r, 0) - f (r, 0)|^p dr \right)^{\frac{p}{2}}
\leq C_p \left( \int_s^t |f (r, 0)|^p dr \right)^{\frac{p}{2}} + C_p \left( \int_s^t (|X_r|^p + |Y_r|^p) dr \right) \leq C_p |t - s|.
\]

Hence, the inequalities (2.6)-(2.8) imply
\[
E |Y_t - Y_s|^p \leq C_p |t - s|^{\frac{p}{2}} + C_p \left( \int_s^t |Z_r|^2 dr \right)^{\frac{p}{2}}.
\]

Using Hölder’s inequality and (2.2) we conclude the proof of (2.1). \( \square \)

Since equation (2.4) proves that the pair \((\nabla Y, \nabla Z)\) is the solution of a BDSDE with forward process \((X, \nabla X) \in L^p\) for every \(p \in [1, +\infty]\), we deduce from (2.3) that for every function \(\phi : \mathbb{R}^d \to \mathbb{R}\) of class \(C_0^2\), we have for \(0 \leq s \leq t \leq T\) and \(p \in [1, +\infty]\):
\[
E |\nabla Y_t - \nabla Y_s|^p \leq C_p |t - s|^{\frac{p}{2}},
\]
for some constant \(C_p > 0\). We now establish some control of time increments of the process \(Z\), following the idea of J. Zhang 2014.

**Theorem 2.3** \((L^2\)-regularity of \(Z\)). There exists a non negative constant \(C\) such that for every subdivision \(\pi = \{t_0 = 0 < t_1 \cdots < t_n = T\}\) with mesh \(|\pi|\), one has
\[
\sum_{1 \leq i \leq n} E \int_{t_{i-1}}^{t_i} \left( |Z_t - Z_{t_{i-1}}|^2 + |Z_t - Z_{t_i}|^2 \right) dt \leq C |\pi|.
\]

**Proof.** Using the representation of \(Z\) as a product, we deduce (2.3),
\[
Z_t - Z_{t_i} = \nabla Y_t (\nabla X_t)^{-1} \sigma (X_t) - \nabla Y_{t_i} (\nabla X_{t_i})^{-1} \sigma (X_{t_i}).
\]
Then,
\[
|Z_t - Z_{t_i}|^2 \leq 3|\nabla Y_t - \nabla Y_{t_i}|^2 (\nabla X_t)^{-1} |\sigma(X_t)|^2 \\
+ 3|\nabla Y_{t_i}|^2 (\nabla X_t^{-1} - (\nabla X_{t_i})^{-1}) |\sigma(X_t)|^2 \\
+ 3|\nabla Y_{t_i}|^2 (\nabla X_{t_i}^{-1})^2 |\sigma(X_t) - \sigma(X_{t_i})|^2.
\]

To conclude the proof, we use Hölder’s inequality, Proposition 2.1 and (2.9).

Theorem 2.3 immediately yields the following

**Corollary 2.4.**
\[
\sum_{1 \leq i \leq n-1} \mathbb{E} \int_{t_{i-1}}^{t_{i+1}} |Z_r - Z_{t_i}|^2 dr \leq C|\pi|.
\]

3. The discretization of \((X, Y, Z)\)

3.1. Discretization of the process \(X\): The Euler scheme. We briefly recall the Euler scheme and send the reader to [KP99] for more details. Let \(\pi := \{t_0 = 0 < t_1 < \ldots < t_n = T\}\) be a subdivision of \([0, T]\). We define the process \(X^n_\pi\), called the Euler scheme, by
\[
X^n_\pi = X^n_0 + \int_0^t b(X^n_s) ds + \int_0^t \sigma(X^n_s) dW_s,
\]
where \(s_\pi := \max\{t_i \leq s\}\). The following result is well known:

**Proposition 3.1.** There exists a constant \(C > 0\) such that for every subdivision \(\pi\),
\[
\max \mathbb{E} \left| X_{t_i} - X^n_{t_i} \right|^2 \leq C|\pi|, \quad \mathbb{E} \int_{t_{i-1}}^{t_i} \left| X_r - X^n_{t_i} \right|^2 dr \leq C|\pi|^2.
\]

3.2. Discretization of the process \((Y, Z)\): The step process. In this section, we construct an approximation of \((Y, Z)\) using Zhang’s approach.

Let \(\pi: t_0 = 0 < \ldots < t_n = T\) be any subdivision on \([0, T]\). Set \(G_t = G^n_t\) for \(t_{i-1} \leq t < t_i\), where we let
\[
G^n_t := \sigma(W_r - W_0; 0 \leq r \leq t) \vee \sigma(B_r - B_{t_{i-1}}; t_{i-1} \leq r \leq T), \quad t_{i-1} \leq t \leq t_i,
\]
and define the \((G^n_t)\)-adapted process \((Y^n_t, Z^n_t)_{0 \leq t \leq T}\) recursively (in a backward manner), as follows:

Set \(Y^n_{t_0} = \phi(X^n_{t_0}), Z^n_{t_0} = 0\); for \(n = 1, \ldots, 0\), let
\[
Z^n_{t_i} := \frac{1}{\Delta t_{i+1}} \mathbb{E} \left( \int_{t_i}^{t_{i+1}} Z^n_{r} dr \bigg| \mathcal{F}_{t_i} \right),
\]
and for \(i = n, \ldots, 1\), let
\[
\Delta t_i = t_i - t_{i-1}, \Delta B_{t_i} = B_{t_i} - B_{t_{i-1}}, \Theta^{\pi, 1}_{t_i} := \left( X^n_{t_i}, Y^n_{t_i}, Z^n_{t_i} \right),
\]
\[
Y^n_t = Y^n_{t_i} + f \left( t_i, \Theta^{\pi, 1}_{t_i} \right) \Delta t_i + g \left( t_i, \Theta^{\pi, 1}_{t_i} \right) \Delta B_{t_i} - \int_{t_i}^t Z^n_{s} dW_s, \quad \forall t \in [t_{i-1}, t_i).
\]

Note that the equation (3.1) is not a BDSDE in the sense of [PP94]; however, we have the following:

**Proposition 3.2.** For every \(i = 1, \ldots, n\), there exists a process \((Y^n_t, Z^n_t)_{t \in [t_{i-1}, t_i]}\) adapted to the filtration \((G^n_t, t_{i-1} \leq t < t_i)\), such that (3.1) holds. Furthermore, \(Y^n_{t_i} \in \mathcal{F}_{t_i}\).

**Proof.** The proof is similar to that in [PP94] page 212 and relies on the martingale representation theorem. Fix an integer \(i > 0\) and suppose that the processes \((Y^n_t)\) and \((Z^n_t)\) have been defined for \(t \geq t_i\), \((G^n_t)\)-adapted, and that \(Y^n_t\) is \(\mathcal{F}_{t_i}\)-measurable for \(k = i, \ldots, n\). We denote by \((M^n_t)_{t \in [t_{i-1}, t_i]}\) the process defined by
\[
M^n_t := \mathbb{E} \left( Y^n_t + f \left( t_i, \Theta^{\pi, 1}_{t_i} \right) \Delta t_i + g \left( t_i, \Theta^{\pi, 1}_{t_i} \right) \Delta B_{t_i} \bigg| G^n_t \right), \quad t_{i-1} \leq t \leq t_i.
\]
By the martingale representation theorem, there exists a \((\mathcal{G}_t^i, t_{i-1} \leq t \leq t_i)\)-adapted and square integrable process \((N_t^i, t_{i-1} \leq t \leq t_i)\) such that for \(t_{i-1} \leq t \leq t_i\), 
\[M_t^i = M_{t_{i-1}}^i + \int_{t_{i-1}}^t N_s^i dW_s.\]
Therefore, \(M_t^i = M_{t_{i-1}}^i - \int_{t_{i-1}}^t N_s^i dW_s\). Clearly, \(\mathcal{G}_t^i\) contains \(\mathcal{F}_{t_i}\), \(X_{t_i}^\pi\) is \(\mathcal{F}_{t_i}^W \subset \mathcal{F}_{t_i}\) measurable and \(\Theta_t^\pi_{t_i}\) is \(\mathcal{F}_{t_i}\)-measurable; hence

\[M_t^i = Y_{t_i}^\pi + f \left( t_i, \Theta_{t_i}^\pi \right) \Delta t_i + g \left( t_i, \Theta_{t_i}^\pi \right) \Delta B_{t_i}.\]

Furthermore, note that \(\mathcal{G}_{t_{i-1}}^i = \mathcal{F}_{t_{i-1}}\), so that \(M_{t_{i-1}}^i\) is \(\mathcal{F}_{t_i}\)-measurable. This completes the proof by setting: \(Y_{t_i}^\pi = M_{t_i}^i\), \(Z_{t_i}^\pi = N_{t_i}^i\) for \(t_{i-1} \leq t < t_i\).

Before stating the main theorem of this section, we introduce the following

**Definition 3.3.** Let \(\kappa \geq 1\) be a constant. The subdivision \(\pi\) is said to be \(\kappa\)-uniform if \(\kappa \Delta t_i \geq |\pi|\) for every \(i \in \{1, ..., n\}\).

The main example of a \(\kappa\)-uniform subdivision is a uniform subdivision (i.e. for all \(i\), \(\Delta t_i = |\pi|\)) where \(\kappa = 1\). The following lemma gives an upper estimate of \(Z_{t_i} - Z_{t_i}^{\pi,1}\).

**Lemma 3.4.** For any \(i = 0, ..., n - 1\), any \(\kappa\)-uniform subdivision \(\pi\) and \(\beta > 0\) we have:

\[\Delta t_i \mathbb{E} \left| Z_{t_i} - Z_{t_i}^{\pi,1} \right|^2 \leq \kappa(1 + \beta) \int_{t_i}^{t_{i+1}} |Z_r^\pi - Z_r|^2 dr + \kappa(1 + \beta^{-1}) \int_{t_i}^{t_{i+1}} |Z_r - Z_r|^2 dr.\]

**Proof.** For any \(i = 0, ..., n - 1\), \(Z_{t_i}\) is \(\mathcal{F}_{t_i}\)-measurable, and \(\Delta t_i \leq |\pi| \leq \kappa \Delta t_{i+1}\); thus

\[
\begin{align*}
\Delta t_i \mathbb{E} \left| Z_{t_i} - Z_{t_i}^{\pi,1} \right|^2 &= \Delta t_i \mathbb{E} \left| \frac{1}{\Delta t_{i+1}} \mathbb{E} \left( \int_{t_i}^{t_{i+1}} Z_r^\pi dr \bigg| \mathcal{F}_{t_i} \right) \right|^2 \\
&= \frac{\Delta t_i}{(\Delta t_{i+1})^2} \mathbb{E} \left( \int_{t_i}^{t_{i+1}} \mathbb{E} \left( (Z_r - Z_r^\pi) dr \bigg| \mathcal{F}_{t_i} \right) \right)^2 \\
&\leq \frac{\kappa}{\Delta t_{i+1}} \mathbb{E} \left( \int_{t_i}^{t_{i+1}} (Z_r - Z_r^\pi) dr \right)^2 \\
&\leq \kappa \mathbb{E} \int_{t_i}^{t_{i+1}} |Z_r - Z_r^\pi|^2 dr.
\end{align*}
\]

where the last step is deduced from Schwarz’s inequality. Using the usual estimate \(|Z_{t_i} - Z_r^\pi|^2 \leq (1 + \beta)|Z_r^\pi - Z_r|^2 + (1 + \beta^{-1})|Z_r - Z_r|^2\), we conclude the proof. \(\square\)

The following theorem is the main result of this section. It proves that as \(|\pi| \to 0\), \((Y^\pi, Z^\pi)\) converges to \((Y, Z)\).

**Theorem 3.5.** Let \(\pi\) be a \(\kappa\)-uniform subdivision with sufficiently small mesh \(|\pi|\), \(\alpha < \frac{1}{4}\), let \(\phi \in C^2\) and \(\xi = \phi(X_T)\). Then we have

\[\max_{0 \leq i \leq n} \mathbb{E} \left| Y_{t_i} - Y_{t_i}^\pi \right|^2 + \mathbb{E} \int_0^T |Z_r - Z_r^\pi|^2 dr \leq C|\pi|.\] (3.2)

**Proof.** Set \(I_n = \mathbb{E} |\phi(X_T) - \phi(X_T^\pi)|^2\) and for \(i = 1, ..., n\), let

\[I_{i-1} = \mathbb{E} \left| Y_{t_{i-1}} - Y_{t_{i-1}}^\pi \right|^2 + \mathbb{E} \int_{t_{i-1}}^{t_i} |Z_r - Z_r^\pi|^2 dr.\]

Using \(\left[ 3 \right]\) with \(\xi = \phi(X_T)\) and \(\left[ 3 \right]\), we deduce

\[
Y_{t_{i-1}} - Y_{t_{i-1}}^\pi + \int_{t_{i-1}}^{t_i} (Z_r - Z_r^\pi) dW_r = Y_{t_i} - Y_{t_i}^\pi + \int_{t_{i-1}}^{t_i} \left( f (r, \Theta_r) - f \left( t_i, \Theta_{t_i}^\pi \right) \right) dr \\
+ \int_{t_{i-1}}^{t_i} \left( g (r, \Theta_r) - g \left( t_i, \Theta_{t_i}^\pi \right) \right) d\overline{B}_r.
\] (3.3)
By construction, \( Y_{t_{i-1}} - Y_{t_i}^\pi \) is \( F_{t_{i-1}} \) measurable while for \( r \in [t_{i-1}, t_i) \), \( Z_r - Z_{t_{i-1}}^\pi \) is \((G_r)\)-adapted. Hence, \( Y_{t_{i-1}} - Y_{t_i}^\pi \) is orthogonal to \( \int_{t_{i-1}}^{t_i} (Z_r - Z_{t_{i-1}}^\pi) \, dW_r \). Therefore,

\[
I_{i-1} = \mathbb{E} \left| Y_{t_{i-1}} - Y_{t_i}^\pi + \int_{t_{i-1}}^{t_i} (Z_r - Z_{t_{i-1}}^\pi) \, dW_r \right|^2.
\]

Since \( g(r, \Theta_r) \) (resp. \( g(t_i, \Theta_{t_i}^\pi) \)) is \( F_r \) (resp. \( F_{t_i} \))-measurable, the random variables \( Y_{t_{i-1}} - Y_{t_i}^\pi \) and \( \int_{t_{i-1}}^{t_i} (g(r, X_r, Y_r) - g(t_i, X_{t_i}^\pi, Y_{t_i}^\pi)) \, d\bar{B}_r \) are orthogonal. Hence for every \( \epsilon > 0 \), using assumption 2 the \( L^2 \)-isometry of backward stochastic integrals, Schwarz’s inequality and (3.3), we deduce

\[
I_{i-1} \leq \left( 1 + \frac{\Delta t_i}{\epsilon} \right) \mathbb{E} \left| Y_{t_{i-1}} - Y_{t_i}^\pi \right|^2 + \left( 1 + 2 \frac{\epsilon}{\Delta t_i} \right) \mathbb{E} \left| \int_{t_{i-1}}^{t_i} \left( f(r, \Theta_r) - f(t_i, \Theta_{t_i}^\pi) \right) \, dr \right|^2
\]

\[
+ \left( 1 + \frac{\Delta t_i}{\epsilon} \right) \mathbb{E} \left| \int_{t_{i-1}}^{t_i} \left( g(r, \Theta_r) - g(t_i, \Theta_{t_i}^\pi) \right) \, d\bar{B}_r \right|^2
\]

\[
\leq \left( 1 + \Delta t_i \epsilon^{-1} \right) \mathbb{E} \left| Y_{t_{i-1}} - Y_{t_i}^\pi \right|^2 + (\Delta t_i + 2\epsilon) \mathbb{E} \left| \int_{t_{i-1}}^{t_i} \left| f(r, \Theta_r) - f(t_i, \Theta_{t_i}^\pi) \right|^2 \, dr \right|
\]

\[
+ (1 + \Delta t_i \epsilon^{-1}) \mathbb{E} \left| \int_{t_{i-1}}^{t_i} \left| g(r, \Theta_r) - g(t_i, \Theta_{t_i}^\pi) \right|^2 \, dr \right|
\]

\[
\leq \left[ 1 + \Delta t_i \epsilon^{-1} + 2L_f (\Delta t_i^2 + 2\epsilon \Delta t_i) + 2L_g (\Delta t_i + \Delta t_i^2 \epsilon^{-1}) \right] \mathbb{E} \left| Y_{t_{i-1}} - Y_{t_i}^\pi \right|^2
\]

\[
+ \left[ L_f (\Delta t_i + 2\epsilon) + L_g (1 + \Delta t_i \epsilon^{-1}) \right] \mathbb{E} \left| \int_{t_{i-1}}^{t_i} \left( |r| + |X_r - X_{t_i}^\pi|^2 + 2 |Y_r - Y_{t_i}|^2 \right) \, dr \right|
\]

\[
+ \left[ L_f (\Delta t_i + 2\epsilon) + \alpha \left( 1 + \Delta t_i \epsilon^{-1} \right) \right] \mathbb{E} \left| \int_{t_{i-1}}^{t_i} \left| Z_r - Z_{t_i}^\pi \right|^2 \, dr \right|
\]

For \( |\pi| \leq 1, \Delta t_i^2 \leq \Delta t_i \); using Proposition 2.3 with \( p = 2 \) and Proposition 3.1, we deduce

\[
\mathbb{E} \left| \int_{t_{i-1}}^{t_i} \left( |r| + |X_r - X_{t_i}^\pi|^2 + 2 |Y_r - Y_{t_i}|^2 \right) \, dr \right| \leq C|\pi|^2,
\]

for some constant \( C > 0 \). Hence for any \( \gamma > 0 \)

\[
I_{i-1} \leq \left[ 1 + \left( \epsilon^{-1} + 2L_f (1 + 2\epsilon) + 2L_g (1 + \epsilon^{-1}) \right) \Delta t_i \right] \mathbb{E} \left| Y_{t_{i-1}} - Y_{t_i}^\pi \right|^2
\]

\[
+ C \left[ L_f (\Delta t_i + \epsilon) + L_g (1 + \Delta t_i \epsilon^{-1}) \right] |\pi|^2
\]

\[
+ (1 + \gamma^{-1}) \left[ L_f (\Delta t_i + 2\epsilon) + \alpha (1 + \Delta t_i \epsilon^{-1}) \right] \mathbb{E} \left| \int_{t_{i-1}}^{t_i} |Z_r - Z_{t_i}^\pi| \, dr \right|^2
\]

\[
+ (1 + \gamma) \left[ L_f (\Delta t_i + 2\epsilon) + \alpha (1 + \Delta t_i \epsilon^{-1}) \right] \Delta t_i \mathbb{E} \left| Z_{t_i} - Z_{t_i}^\pi \right|^2.
\]
Lemma 3.4 yields for some positive constants $C_e, C_{e, \gamma}$ and $C_{e, \gamma, \beta}$, we have:

$$I_{i-1} \leq (1 + C_e \Delta t_i) E |Y_{t_i} - Y_{t_i}^\pi|^2 + C_e |\pi|^2 + C_{e, \gamma, \beta} E \int_{t_{i-1}}^{t_i} |Z_r - Z_{t_i}|^2 dr$$

$$+ \kappa (1 + \gamma) (1 + \beta) \left[ L_f (\Delta t_i + 2 \epsilon) + \alpha (1 + \Delta t_i \epsilon^{-1}) \right] E \int_{t_{i-1}}^{t_i} |Z_r^\pi - Z_r|^2 dr$$

$$+ \kappa (1 + \gamma) (1 + \beta^{-1}) \left[ L_f (\Delta t_i + 2 \epsilon) + \alpha (1 + \Delta t_i \epsilon^{-1}) \right] E \int_{t_{i-1}}^{t_i} |Z_r - Z_{t_i}|^2 dr$$

$$\leq (1 + C_e \Delta t_i) E |Y_{t_i} - Y_{t_i}^\pi|^2 + C_e |\pi|^2 + C_{e, \gamma, \beta} E \int_{t_{i-1}}^{t_i} |Z_r - Z_{t_i}|^2 dr$$

$$+ \kappa (1 + \gamma) (1 + \beta) \left[ L_f (\Delta t_i + 2 \epsilon) + \alpha (1 + \Delta t_i \epsilon^{-1}) \right] \int_{t_i}^{t_{i+1}} |Z_r^\pi - Z_r|^2 dr. \tag{3.4}$$

Recall that $\alpha < \frac{1}{\kappa}$ and let $0 < \delta < 1 - \kappa \alpha$. Then choose positive constants $\beta$ and $\gamma$ small enough to ensure $\kappa (1 + \gamma) (1 + \beta) \alpha < 1 - \frac{2 \delta}{\kappa}$. Finally, let $\epsilon > 0$ small enough to ensure that $2 \kappa (1 + \gamma) (1 + \beta \epsilon) L_f \epsilon < \frac{\delta}{\kappa}$. Then (3.4) implies the existence of $C > 0$ such that for every $i = 1, \ldots, n - 1$,

$$I_{i-1} + \frac{\delta}{3} E \int_{t_i}^{t_{i+1}} |Z_r^\pi - Z_r|^2 dr \leq (1 + C \Delta t_i) I_i + C |\pi|^2 + C E \int_{t_{i-1}}^{t_i} |Z_r - Z_{t_i}|^2 dr. \tag{3.5}$$

Using the discrete Gronwall lemma in [204] (Lemma 5.4 page 479), we deduce

$$\max_{0 \leq i \leq n} I_i \leq C e^{C T} E \left( I_n + \sum_{1 \leq i \leq n-1} \int_{t_{i-1}}^{t_i} |Z_r - Z_{t_i}|^2 dr + |\pi|^2 \right)$$

$$\leq C E \left( |\phi (X_T) - \phi (X_T^\pi)|^2 + \sum_{1 \leq i \leq n} \int_{t_{i-1}}^{t_i} \left( |Z_r - Z_{t_{i-1}}|^2 + |Z_r - Z_{t_i}|^2 \right) dr + |\pi| \right).$$

Since $\phi$ is Lipschitz, Proposition 3.1 implies that $E |\phi (X_T) - \phi (X_T^\pi)|^2 \leq C |\pi|$; thus Theorem 2.3 implies

$$\max_{0 \leq i \leq n} E |Y_{t_i} - Y_{t_i}^\pi|^2 \leq C |\pi|. \tag{3.6}$$

Moreover, summing both sides of (3.3) over $i$ from 1 to $n - 1$ and using Corollary 2.4 we obtain:

$$\sum_{0 \leq i \leq n-2} I_i + \frac{\delta}{3} E \int_{t_1}^{T} |Z_r^\pi - Z_r|^2 dr \leq \sum_{1 \leq i < n} (1 + C \Delta t_i) I_i + C |\pi|$$

$$+ C \sum_{1 \leq i < n} E \int_{t_{i-1}}^{t_i} |Z_r - Z_{t_i}|^2 dr,$$

$$\leq C |\pi| + \sum_{1 \leq i \leq n-1} (1 + C \Delta t_i) I_i.$$

Therefore,

$$I_0 + \frac{\delta}{3} E \int_{t_1}^{T} |Z_r^\pi - Z_r|^2 dr \leq C |\pi| + I_{n-1} + C \sum_{1 \leq i \leq n-1} \Delta t_i I_i$$

Since $\delta < 1 - \kappa \alpha < 3$, using (3.6) we deduce

$$\frac{\delta}{3} E \int_{0}^{T} |Z_r^\pi - Z_r|^2 dr \leq C |\pi| + E \int_{t_{n-1}}^{T} |Z_r^\pi - Z_r|^2 dr + C |\pi| E \int_{0}^{T} |Z_r^\pi - Z_r|^2 dr. \tag{3.7}$$
The equations (1.3) and (3.1) imply

\[
\int_{t_{n-1}}^{t_n} (Z^*_r - Z_r) dW_r = (Y^*_{t_n} - Y_{t_n}) - (Y^*_{t_{n-1}} - Y_{t_{n-1}}) \\
+ \int_{t_{n-1}}^{t_n} (f(t_n, X^*_{t_n}, Y^*_{t_n}, 0) - f(r, X_r, Y_r, Z_r)) \, dr \\
+ \int_{t_{n-1}}^{t_n} (g(t_n, X^*_{t_n}, Y^*_{t_n}, 0) - g(r, X_r, Y_r, Z_r)) \, dB_r.
\]

The \(L^2\)-isometry, Schwarz’s inequality, \((3.4)\), Lemma 2.2, Propositions 2.1 and 3.1, small enough, we have \(C \| \pi \| \leq \delta / 6\); thus \((3.7)\) and \((3.8)\) conclude the proof. \(\square\)

4. A NUMERICAL SCHEME

In this section we propose a numerical scheme based on the results of the previous sections. First of all, given \(x \in \mathbb{R}^d\), \(s < t\) we set:

\[X_t (s, x) := x + (t - s) b(x) + \sigma(x) (W_t - W_s),\]

we clearly have \(X^*_{t_i} = X_{t_i} (t_{i-1}, X^*_{t_{i-1}})\) for every \(i = 1, \ldots, n\). Then, given a vector \((x_0, \ldots, x_i; x_{i+1}, \ldots, x_n) \in \mathbb{R}^{(i+1)d} \times \mathbb{R}^{n-i}\), set \(x_{n+1} = \emptyset\) and for \(i = 0, \ldots, n-1\), let

\[x^i := (x_0, \ldots, x_i), \quad x^i_{i+1} := (x_{i+1}, \ldots, x_n),\]

Define by induction, the functions \(u^\pi_n, v^\pi_n : \mathbb{R}^{(i+1)d} \times \mathbb{R}^{n-i} \rightarrow \mathbb{R}\) (resp. the random variables \(U^\pi_n, V^\pi_n : \mathbb{R}^{(i+1)d} \times \Omega \times \mathbb{R}^{n-i} \rightarrow \mathbb{R}\)) as follows:

\[u^\pi_n (x_0, \ldots, x_n) := \phi(x_n), \quad v^\pi_n (x_0, \ldots, x_n) := 0,\]

and for \(i = 0, \ldots, n-1\) let

\[U^\pi_i (x^i, \omega, x_{i+2}) := u^\pi_{i+1} (x^i, X^i_{t_{i+1}} (t_i, x_i), x_{i+2}) \quad \text{(4.1)}\]

\[+ f(t_{i+1}, X^i_{t_{i+1}} (t_i, x_i), u^\pi_{i+1} (x^i, X^i_{t_{i+1}} (t_i, x_i), x_{i+2}), \]

\[v^\pi_{i+1} (x^i, X^i_{t_{i+1}} (t_i, x_i), x_{i+2}) \Delta t_{i+1}, \]

\[V^\pi_i (x^i, \omega, x_{i+2}) := g(t_{i+1}, X^i_{t_{i+1}} (t_i, x_i), u^\pi_{i+1} (x^i, X^i_{t_{i+1}} (t_i, x_i), x_{i+2}), \]

\[v^\pi_{i+1} (x^i, X^i_{t_{i+1}} (t_i, x_i), x_{i+2}) \Delta t_{i+1}, \quad \text{(4.2)}\]

\[u^\pi_i (x^i; x_{i+1}) := \frac{1}{\Delta t_{i+1}} \mathbb{E} (U^\pi_i (x^i, X^i_{t_{i+1}} (t_i, x_i), x_{i+2}), \Delta W_{t_{i+1}}) \]

\[+ \frac{x_{i+1}}{\Delta t_{i+1}} \mathbb{E} (V^\pi_i (x^i, X^i_{t_{i+1}} (t_i, x_i), x_{i+2}) \Delta W_{t_{i+1}}), \quad \text{(4.4)}\]

\[u^\pi_i (x^i, x_{i+1}) := \frac{1}{\Delta t_{i+1}} \mathbb{E} (U^\pi_i (x^i, X^i_{t_{i+1}} (t_i, x_i), x_{i+2}) \Delta W_{t_{i+1}}) \]

\[+ \frac{x_{i+1}}{\Delta t_{i+1}} \mathbb{E} (V^\pi_i (x^i, X^i_{t_{i+1}} (t_i, x_i), x_{i+2}) \Delta W_{t_{i+1}}), \quad \text{(4.4)}\]
Theorem 4.1. We have for all $i = 0, \ldots, n$

$$Y_{t_i}^\pi = u_t^\pi \left( X_{t_0}^\pi, \ldots, X_{t_i}^\pi, \Delta B_{t_{i+1}}, \ldots, \Delta B_{t_n} \right),$$

(4.5)

$$Z_{t_i}^{\pi,1} = v_t^\pi \left( X_{t_0}^\pi, \ldots, X_{t_i}^\pi, \Delta B_{t_{i+1}}, \ldots, \Delta B_{t_n} \right).$$

(4.6)

Proof. We proceed by backward induction. For $i = n$, by definition $Y_{t_n}^\pi = \phi \left( X_{t_n}^\pi \right)$, so (4.5) and (4.6) hold trivially.

Suppose that the result is true for $j = n, n-1, \ldots, i$. The scheme described in (3.1) implies that

$$Y_{t_{i-1}}^\pi = Y_{t_i}^\pi + f \left( t_i, \Theta_{t_i}^{\pi,1} \right) \Delta t_i + g \left( t_i, \Theta_{t_i}^{\pi,1} \right) \Delta B_{t_i} - \int_{t_i}^{t_{i-1}} Z_{t_i}^{\pi} dW_r.$$  

(4.7)

To prove (4.5), we take the conditional expectation of (4.7) with respect to $\hat{F}_{t_i-1}$ this yields

$$E \left( Y_{t_i}^\pi \mid \hat{F}_{t_i-1} \right) = E \left( Y_{t_i}^\pi \mid \hat{F}_{t_i-1} \right) + E \left( f \left( t_i, \Theta_{t_i}^{\pi,1} \right) \Delta t_i \mid \hat{F}_{t_i-1} \right)$$

$$+ E \left( g \left( t_i, \Theta_{t_i}^{\pi,1} \right) \Delta B_{t_i} \mid \hat{F}_{t_i-1} \right) - \int_{t_i}^{t_{i-1}} Z_{t_i}^{\pi} dW_r \left( \hat{F}_{t_i-1} \right).$$

Using the fact that $\int_{t_i}^{t_{i-1}} Z_{t_i}^{\pi} dW_r$ is orthogonal to any $\hat{F}_{t_i-1}$-measurable random variable, and the induction hypothesis we deduce:

$$Y_{t_{i-1}}^\pi = E \left( Y_{t_i}^\pi \mid \hat{F}_{t_i-1} \right) + E \left( f \left( t_i, \Theta_{t_i}^{\pi,1} \right) \Delta t_i \mid \hat{F}_{t_i-1} \right)$$

$$+ E \left( g \left( t_i, \Theta_{t_i}^{\pi,1} \right) \Delta B_{t_i} \mid \hat{F}_{t_i-1} \right)$$

For (4.6), multiply (4.7) by $\Delta W_{t_i} = W_{t_i} - W_{t_{i-1}}$ and take the conditional expectation with respect to $\hat{F}_{t_i-1}$ this yields

$$E \left( Y_{t_i}^\pi \mid \Delta W_{t_i} \mid \hat{F}_{t_i-1} \right) = E \left( Y_{t_i}^\pi \Delta W_{t_i} \mid \hat{F}_{t_i-1} \right) + E \left( f \left( t_i, \Theta_{t_i}^{\pi,1} \right) \Delta t_i \Delta W_{t_i} \mid \hat{F}_{t_i-1} \right)$$

$$+ E \left( g \left( t_i, \Theta_{t_i}^{\pi,1} \right) \Delta B_{t_i} \Delta W_{t_i} \mid \hat{F}_{t_i-1} \right) - \int_{t_i}^{t_{i-1}} Z_{t_i}^{\pi} dW_r \left( \hat{F}_{t_i-1} \right).$$

Since $Y_{t_{i-1}}^\pi \in \hat{F}_{t_i-1}$ and $\Delta W_{t_i}$ is independent of $\hat{F}_{t_i-1}$ and centered we deduce

$$E \left( Y_{t_{i-1}}^\pi \Delta W_{t_i} \mid \hat{F}_{t_i-1} \right) = 0.$$  

Furthermore,

$$E \left( \Delta W_{t_i} \int_{t_{i-1}}^{t_i} Z_{t_i}^{\pi} dW_r \mid \hat{F}_{t_i-1} \right) = E \left( \int_{t_{i-1}}^{t_i} Z_{t_i}^{\pi} dW_r \mid \hat{F}_{t_i-1} \right)$$

$$= E \left( \int_{t_{i-1}}^{t_i} Z_{t_i}^{\pi} dW_r \mid \hat{F}_{t_i-1} \right) = \Delta t_i Z_{t_i}^{\pi,1}.$$  

This completes the proof of (4.6).

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