Birationally rigid complete intersections of quadrics and cubics

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Abstract. We prove the birational superrigidity of generic Fano complete intersections $V$ of type $2^{k_1} \cdot 3^{k_2}$ in the projective space $\mathbb{P}^{2k_1+3k_2}$ provided that $k_2 \geq 2$ and $k_1 + 2k_2 = \dim V \geq 12$, and of certain families of Fano complete intersections of dimensions 10 and 11.

Keywords: Fano variety, complete intersection, birational rigidity, maximal singularity, multiplicity.

Introduction

0.1. Statement of main result. Let $k_1, k_2 \in \mathbb{Z}_+$ be a pair of non-negative integers with $k_1 + 2k_2 \geq 4$. A Fano complete intersection of type $2^{k_1} \cdot 3^{k_2}$ is a smooth $(k_1 + 2k_2)$-dimensional complete intersection

$$V = F_1 \cap \cdots \cap F_{k_1+k_2} \subset \mathbb{P} = \mathbb{P}^{2k_1+3k_2}$$

of codimension $k = k_1 + k_2$ in the projective space $\mathbb{P}$ over the field $\mathbb{C}$ of complex numbers, where $F_i \subset \mathbb{P}$ are hypersurfaces of degree 2 for $i = 1, \ldots, k_1$ and degree 3 for $i = k_1 + 1, \ldots, k$, $k = k_1 + k_2$. We put $M = k_1 + 2k_2 = \dim V$. Clearly, $V$ is a Fano variety of index 1, $\text{Pic} V = \mathbb{Z} H$ and $K_V = -H$, where $H$ is the class of a hyperplane section.

The main result of this paper is the following theorem.

Theorem 0.1. Assume that $k_2 \geq 2$ and either $M = k_1 + 2k_2 \geq 12$ or the pair $(k_1, k_2)$ is one of the five pairs

$$(5, 3), \ (3, 4), \ (1, 5), \ (2, 4), \ (0, 5).$$

Then a generic (in the sense of the Zariski topology on the space of coefficients of the polynomials that determine the hypersurfaces $F_1, \ldots, F_{k_1+k_2}$) Fano complete intersection $V$ of type $2^{k_1} \cdot 3^{k_2}$ is birationally superrigid. In particular, the following assertions hold.

(i) There is no rational dominant map $\gamma : V \dashrightarrow S$ onto a variety $S$ of positive dimension with generic fibre $\gamma^{-1}(s)$ of negative Kodaira dimension.

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(ii) Any birational map $\chi: V \dasharrow V'$ onto a Fano variety $V'$ with $\mathbb{Q}$-factorial terminal singularities and Picard number $\text{rk} \text{Pic} V' = 1$ is a biregular isomorphism.

(iii) The group $\text{Bir} V$ of birational automorphisms coincides with the group $\text{Aut} V$ of biregular (projective) automorphisms and is therefore trivial.

Assertions (i)–(iii) of Theorem 0.1 follow immediately from the birational superrigidity (see [1]). In its turn, birational superrigidity (understood as equality of the virtual and actual thresholds of canonical adjunction: $c_{\text{virt}}(\Sigma) = c(\Sigma) = n$ for every movable linear system $\Sigma \subset |-nK_V|$, see [1] for a definition) follows immediately from the canonicity of every pair $(V, \frac{1}{n} \Sigma)$, where $\Sigma \subset |-nK_V|$ is a movable system or, equivalently, from the absence of maximal singularities of the linear system $\Sigma$.

Birational superrigidity has so far been proved for Fano complete intersections $F_1 \cap \cdots \cap F_k$ of index 1 provided that at least one degree $\deg F_i$ satisfies $\deg F_i \geq 4$ (see [2], [3]). Theorem 0.1 establishes the superrigidity of complete intersections $V$ of quadrics and cubics provided that $\dim V \geq 12$ and there are at least two cubics, as well as for five families of complete intersections of dimensions 10 and 11. Thus the problem of birational superrigidity (of a generic variety) remains open for several families of Fano complete intersections of dimension $\leq 11$ and for the three following infinite series

$$2 \cdot 2 \cdot \ldots \cdot 2, \quad 2 \cdot 2 \cdot \ldots \cdot 2 \cdot 3, \quad 2 \cdot 2 \cdot \ldots \cdot 2 \cdot 4$$

in arbitrary dimension. These varieties require further improvement of the technique of the proof.

In the next subsection we attach a precise meaning to the condition of general position of a variety $V$ in its family. The main difficulty in the proof of Theorem 0.1 is in showing that this condition is realizable: the space of parameters contains a non-empty Zariski-open subset corresponding to those varieties $V$ that satisfy this condition. The actual proof of birational superrigidity takes a few pages (see §1) and the rest of the paper is almost wholly devoted to proving the conditions of general position.

In §0.3 we give a more detailed description of the structure of the paper, the main idea of the proof of Theorem 0.1, and the position of this theorem among other results on birational rigidity.

0.2. Conditions of general position. As usual, the conditions of general position (or the regularity conditions) are local and must hold at every point of $V$. Let $o \in V \subset \mathbb{P}$ be a point and $\mathbb{C}^{M+k} \subset \mathbb{P}$ a standard affine set with coordinates $(z_1, \ldots, z_{M+k})$ such that $o = (0, \ldots, 0)$. Let

$$f_i = q_{i,1} + q_{i,2}, \quad i = 1, \ldots, k_1,$$

be the equations of the quadrics $F_1, \ldots, F_{k_1}$, and

$$f_i = q_{i,1} + q_{i,2} + q_{i,3}, \quad i = k_1 + 1, \ldots, k_1 + k_2,$$
the equations of the cubics $F_{k_1+1}, \ldots, F_k$, decomposed into components homogeneous in $z_*$. The tangent space $T_o V \subset \mathbb{C}^{M+k}$ is given by the system of equations
\[
q_{1,1} = \cdots = q_{k,1} = 0.
\]
We denote its projectivization $\mathbb{P}(T_o V) \cong \mathbb{P}^{M-1}$ by $T$, put $\bar{q}_{i,j} = q_{i,j}|_T$ and introduce the following sets of pairs of indices:
\[
J_1 = \{(i,2) \mid 1 \leq i \leq k\}, \quad J_2 = \{(i,3) \mid k_1 + 1 \leq i \leq k_1 + k_2\}
\]
and $J = J_1 \cup J_2$. The first (traditional) regularity condition is stated as follows.

(R1) The system of homogeneous equations
\[
\{\bar{q}_{i,2} = 0 \mid 1 \leq i \leq k\} \quad (0.1)
\]
determines a closed subset of pure codimension $k$ in $T$, while the system of equations
\[
\{\bar{q}_{i,j} = 0 \mid (i,j) \in J\} \quad (0.2)
\]
determines either the empty subset of $T$ or finitely many points which are linearly independent in $T$.

The last condition (linear independence) is a new addition to the usual regularity condition [2], [3], which only requires the set (0.2) to be finite. Clearly, if condition (R1) holds at the point $o$, then $V$ contains at most $M$ lines through $o$.

The second regularity condition was introduced in [3].

(R2) None of the irreducible components of the closed algebraic set (0.1) is contained in a linear subspace of codimension 2 in $T$.

In [3], condition (R2) was called correctness in the quadratic terms.

To state the third regularity condition, we need some additional constructions. Assume that the set (0.2) is non-empty and denote it by $\Delta$. We define a linear system
\[
\Sigma_1^T = \langle \bar{q}_{i,2} \mid 1 \leq i \leq k\rangle
\]
of quadrics on $T$ and a linear system $\Sigma_2^T$ of cubics given by all linear combinations of the form
\[
\sum_{i=1}^k \bar{s}_i(z)\bar{q}_{i,2} + \sum_{i=k_1+1}^{k_1+k_2} \lambda_i \bar{q}_{i,3},
\]
where $\lambda_i$ are constants and $\bar{s}_i(z) = s_i(z)|_T$ are linear polynomials on $T$. Let
\[
\mathcal{P} = \Sigma_1^T \times \cdots \times \Sigma_1^T \times \Sigma_2^T \times \cdots \times \Sigma_2^T
\]
be the space of all tuples of polynomials $(g_1, \ldots, g_{M-4})$, where the first $k-2$ (resp. the last $k_2-2$) polynomials lie in the space $\Sigma_1^T$ (resp. $\Sigma_2^T$).

We now state the third (and last) regularity condition.

(R3) For every irreducible subvariety $R \subset T$ of degree $d_R \geq 1$ and codimension 3 there is a non-empty Zariski-open subset $U_R \subset \mathcal{P}$ such that for all $(g_1, \ldots, g_{M-4}) \in U_R$ we have
\[
\sum_{p \in \Delta} \dim \mathcal{O}_{p,R}/(g_1, \ldots, g_{M-4}) \leq 2^{k-4}3^{k_2-1}d_R. \quad (0.3)
\]
We know from condition (R1) that for generic tuples \((g_1, \ldots, g_{M-4}) \in \mathcal{P}\) the scheme
\[
(R \circ \{g_1 = 0\} \circ \cdots \circ \{g_{M-4} = 0\})
\]
is zero-dimensional and hence has degree \(2^{k_1-2}3^{k_2-2}d_R\). Condition (R3) means that at most \(3/4\) of this full degree is concentrated at the points of \(\Delta\). This is a very strong condition since it must hold for an arbitrary subvariety \(R\) of the projective space \(\mathbb{T}\).

Let
\[
\mathcal{F}_{\text{sm}} \subset \prod_{i=1}^{k_1} H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(2)) \times \prod_{i=k_1+1}^{k} H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(3))
\]
be the space of tuples \((f_1, \ldots, f_k)\) that determine smooth complete intersections of type \(2^{k_1} \cdot 3^{k_2}\) in \(\mathbb{P}\).

**Theorem 0.2.** There is a non-empty Zariski-open subset \(\mathcal{F}_{\text{reg}} \subset \mathcal{F}_{\text{sm}}\) of tuples \((f_1, \ldots, f_k)\) for which the corresponding complete intersection \(V(f_1, \ldots, f_k) = \{f_1 = \cdots = f_k = 0\} \subset \mathbb{P}\) satisfies the regularity conditions (R1)–(R3) at every point \(o \in V\).

We can now state the main result in more precise terms.

**Theorem 0.3.** Let \(V\) be a Fano complete intersection of type \(2^{k_1} \cdot 3^{k_2}\) in \(\mathbb{P}\) such that the regularity conditions (R1)–(R3) hold at every point \(o \in V\) and the numbers \(k_1, k_2\) satisfy the hypotheses of Theorem 0.1. Then \(V\) is birationally superrigid. In particular, assertions (i)–(iii) of Theorem 0.1 hold.

Clearly, Theorem 0.1 follows from Theorems 0.2, 0.3, which are independent of each other and will be proved separately.

**0.3. The structure of the paper and historical remarks.** Theorem 0.3 is proved in §1. Its proof takes a few pages and uses the technique of hypertangent linear systems and the \((8n^2)\)-inequality. The idea of the proof can be briefly explained as follows. In previous papers (see [1]–[5] and the bibliography in [1]), the birational superrigidity of Fano complete intersections was proved by constructing (departing from the self-intersection \(Z\) of a movable linear system \(\Sigma\) on \(V\) with a maximal singularity) an effective 1-cycle \(Y \subset V\) satisfying the inequality
\[
\text{mult}_o Y > \deg Y,
\]
which is certainly impossible. The resulting contradiction shows that no such \(\Sigma\) can exist and proves the birational superrigidity of \(V\). This construction does not work in a straightforward way for complete intersections of quadrics and cubics: the technique of hypertangent systems yields a curve \(Y \subset V\) of high multiplicity \(\text{mult}_o Y\), which nevertheless does not exceed \(\deg Y\), thus giving no contradiction.

An easy analysis shows that irreducible curves \(C \subset V\) which are not lines (that is, \(\deg C \geq 2\)) satisfy an inequality much stronger than the negation of (0.4):
\[
\text{mult}_o C \leq \frac{2}{3} \deg C.
\]
Breaking the effective 1-cycle $Y$ into two subcycles

$$Y = Y_1 + Y_{\geq 2},$$

where $Y_1$ is concentrated on the lines through $o$ and $Y_{\geq 2}$ is concentrated on the curves of degree $\geq 2$, we see that the above strategy of proof is successful whenever the subcycle $Y_1$ is not too large. In other words, if the ratio $\deg Y_1 / \deg Y$ is sufficiently small, then using the sharper estimate (0.5) for $Y_{\geq 2}$ enables us to get a contradiction and prove birational superrigidity. This is possible provided that the technique of hypertangent divisors applied to the self-intersection $Z$ of the linear system $\Sigma$ (or to an irreducible component of $Z$) yields an effective 1-cycle $Y$ containing 'not too many lines'. A proof of Theorem 0.3 realizing this strategy is given in §1.

The main part of the paper (§§2–5) deals with estimating the number of lines (counting multiplicities) on the intersection of an arbitrary irreducible subvariety with hypertangent divisors. In §§2–4 we consider the local problem: estimating the multiplicity of intersection of a generic tuple of polynomials in a given linear system with an arbitrary effective cycle at a fixed point. This problem is globalized in §5, where we estimate the sum of the local intersection multiplicities over the base points of a linear system. The result is used to prove Theorem 0.2.

The theory of the birational rigidity of Fano complete intersections has a long history. It begins with a paper of Fano [6], who studied complete intersections $V_{2,3} \subset P^5$ of a quadric and a cubic, and complete intersections $V_{2,2,2} \subset P^6$ of three quadrics. He gave a description (which later appeared to be incomplete [7]) of generators in the group $\text{Bir} V_{2,3}$ of birational self-maps and stated a theorem on the non-rationality of both classes of varieties. His arguments were later seen [8] to contain numerous mistakes and gaps, so that they cannot even be regarded as a first approximation to a rigorous proof. At the same time, Fano outlined many important ideas and constructions (including the ‘double projection’ and Noether–Fano inequality) that were further developed in birational geometry.

The pioneering work of Iskovskikh and Manin [8] in 1970 gives the first rigorous proof of birational superrigidity (in the modern terminology) for certain three-dimensional Fano varieties: smooth three-dimensional quartics $V_4 \subset P^4$. This proof works with minimal changes (in the easier part of the proof concerning the exclusion or untwisting of those maximal curves that lie on the variety itself) for the Fano double spaces $V_2 \to P^3$ of index 1 and double quadrics $V_4 \to Q_2 \subset P^4$ of index 1 (see [7]). However, the proof of the birational superrigidity of the variety $V_{2,3}$ and the description of the group of its birational self-maps (announced in [9]) turned out to be much harder. The test class method (developed in [8]) worked successfully only for varieties of degree at most 4, and already the case of degree $\deg V_{2,3} = 6$ met a serious obstruction. The arguments used in [7] to exclude the infinitely near maximal singularity ([7], §4.8, step 3) were erroneous and this most important step of the proof remained an open problem until 1987 [10] (see [11] for a complete proof of the birational superrigidity of a generic complete intersection of a quadric and a cubic of dimension 3). The problem of describing the structures of a rationally connected fibre space and the group of birational self-maps
of $V_{2,2,2}$ is still open. (Note that the birational geometry of Fano complete intersections has also been studied by transcendental methods. There are not many such papers, see [12]–[14].)

Work on extending the test class method of Iskovskikh and Manin to arbitrary dimensions was begun in [15], [16]. The ideas developed in [15] are sufficient for proving the birational superrigidity of generic complete intersections of a quadric and a quartic $V_{2,4} \subset \mathbb{P}^6$ of dimension 4 and index 1. This result was announced in 1985, but a complete proof was not published. (The birational superrigidity of complete intersections of a quadric and a quartic containing no planes was later proved in [17] using the $(8n^2)$-inequality, but the proof of this inequality that was known at that time turned out to be erroneous and a complete proof was not obtained until later; see the history of this problem in [18].)

Replacing the test class method by the method of counting multiplicities, introducing the technique of hypertangent divisors and using some other ideas [4], one can prove the birational superrigidity of Fano hypersurfaces of index 1 and then of generic complete intersections $V_{d_1,\ldots,d_k} \subset \mathbb{P}^{M+k}$ of index 1 for $M \geq 2k + 1$, $\dim V \geq 4$ [2]. The next step in the development of this area was a systematic use of the connectedness principle of Shokurov and Kollár [19] and the proof of the $(8n^2)$-inequality. This formed a basis for proving birational superrigidity for a wider class of complete intersections with $M \geq k + 3$, $d_k \geq 4$ [3]. The last paper is an immediate precursor of the present one, where we remove the restriction $d_k \geq 4$.

As already said above, the problem of birational superrigidity now remains open only for very few classes of Fano complete intersections of index 1. The conjecture on the birational superrigidity of all smooth Fano complete intersections of index 1 has been made several times by the author (see, for example, [20]).

The method of maximal singularities also enables one to prove the birational superrigidity of varieties fibred into Fano complete intersections over the projective line $\mathbb{P}^1$ (see [21], [22]), but this is a different topic.

The work on the problem whose solution forms the content of this paper was completed in spring 2012. To the author’s knowledge, the problem of estimating the multiplicity for a generic tuple of polynomials in a subvariety of given codimension has never been considered before. However, in June 2012, Khovanskii informed the author that another (but related) problem was studied in [23]: that of estimating the multiplicity of an isolated solution of a system of $n$ equations

$$\phi_1 = \cdots = \phi_n = 0$$

in $n$ complex variables for a tuple $(\phi_1, \ldots, \phi_n)$ in a ring of Noetherian functions $K \supset \mathbb{C}[z_1, \ldots, z_n]$ which is finitely generated over $\mathbb{C}[z_1]$ and closed under differentiation. The estimate in [23] is obtained in terms of invariants of $K$ by methods that are essentially different from the algebro-geometric technique in §§2–4: reduction to the one-dimensional problem of restricting a polynomial to a trajectory of a polynomial vector field by estimating the complexity of an integral manifold of an analytic vector-valued function and using deformations. By studying a less general (and in fact somewhat different) problem, we obtain a considerably stronger and (which is especially important for applications to birational geometry) more effective estimate of the multiplicity, which does not follow from those in [23].
The method of solution developed in this paper (that is, the method of proof of Theorem 0.2) is new. A simplified version (for systems of $N$ polynomial equations in $N$ variables, without reference to the effective cycle $R$ of given degree) was published in [24].

It is remarkable that the problem of estimating the multiplicity of an (isolated) solution of $n$ equations in $n$ variables arises in different contexts, requires different types of technique, and has diverse applications. The author is grateful to A. G. Khovanskii for pointing out the paper [23].

§ 1. Proof of birational superrigidity

In this section we prove Theorem 0.3.

1.1. Beginning of the proof. The $(8n^2)$-inequality. The proof of Theorem 0.3 starts in a standard way. We fix a complete intersection $V \subset \mathbb{P}$ satisfying the hypotheses of the theorem and a movable linear system $\Sigma \subset |nH|$ (where $H$ is the class of a hyperplane section of $V$) with a maximal singularity [1]. This means that there is a birational morphism $\varphi: V^\sharp \to V$, where $V^\sharp$ is a non-singular projective variety, and an exceptional divisor $E^\sharp \subset V^\sharp$ (the maximal singularity) satisfying the Noether–Fano inequality

$$\text{ord}_{E^\sharp} \varphi^* \Sigma > na(E^\sharp, V).$$

The irreducible subvariety $B = \varphi(E^\sharp) \subset V$ is called the centre of $E^\sharp$. We have $\text{mult}_B \Sigma > n$. There are three options for the codimension of $B$:

(i) $\text{codim} B = 2$,

(ii) $\text{codim} B = 3$,

(iii) $\text{codim} B \geq 4$.

It was shown, for example in [2] (the argument can be found in many papers on birational rigidity), that option (i) never holds since the numerical Chow group satisfies $A^2V = \mathbb{Z}H^2$. Option (ii) was excluded, for example in [3]. Therefore we shall assume that $\text{codim} B \geq 4$.

The argument begins with an almost verbatim repetition of the proof of Theorem 3 in [3], § 3. We assume that the codimension of $B$ is minimal among all centres of maximal singularities of $\Sigma$. In particular, $B$ is not strictly contained in the centre of another maximal singularity (if there are any). Take a point $o \in B$ of general position. Let $\lambda: V^+ \to V$ be its blow up, $E = \lambda^{-1}(o) \cong \mathbb{P}^{M-1}$ the exceptional divisor, $\Sigma^+$ the strict transform of the movable system $\Sigma$ on $V^+$, $Z = (D_1 \circ D_2)$ the self-intersection of $\Sigma$, and $Z^+$ the strict transform of $Z$ on $V^+$.

**Proposition 1.1** (the $(8n^2)$-inequality). There is a linear subspace $P \subset E$ of codimension 2 such that

$$\text{mult}_o Z + \text{mult}_P Z^+ > 8n^2.$$

If $\text{mult}_o Z \leq 8n^2$, then $P$ is uniquely determined by the system $\Sigma$.

**Proof.** See [18], § 4.1. □
Let $\Lambda \supset P$ be a generic hyperplane in $E \cong \mathbb{P}^{M-1}$ containing the subspace $P$, and let $L \in |H|$ be a generic hyperplane section through $o$ such that $L^+ \cap E = \Lambda$, where $L^+$ is the strict transform of the divisor $L$ on $V^+$. By the generic choice of $\Lambda$ and $L$, none of the irreducible components of $Z$ lies in $L$. Hence the scheme-theoretic intersection $Z_L = (Z \circ L)$ is well defined. The effective cycle $Z_L$ of codimension 3 satisfies the inequality

$$\text{mult}_o Z_L \geq \text{mult}_o Z + \text{mult}_P Z^+ > 8n^2$$

and the equations $\deg Z_L = \deg Z = dn^2$, where $d = 2^{k_1}3^{k_2}$ is the degree of the complete intersection $V$. Hence there is an irreducible subvariety $Q \subset V$ of codimension 3 (an irreducible component of $Z_L$) such that

$$\frac{\text{mult}_o Q}{\deg Q} > \frac{8}{d}.$$ (1.1)

Let $Q_E = (Q^+ \circ E) = \sum_{i \in I} m_i R_i$ be the projectivized tangent cone to $Q$ at $o$ (here $Q^+ \subset V^+$ is the strict transform of $Q$, the subvarieties $R_i \subset E$ are irreducible components of the effective cycle $Q_E$, $m_i \geq 1$, and all the constructions of elementary intersection theory are understood in the sense of [25] throughout the paper). We apply to $Q \subset V$ the technique of hypertangent divisors as in [3], but with a small modification. Namely, at every step we remove those irreducible components of the intersection with a tangent (hypertangent) divisor that do not contain $o$.

### 1.2. The technique of hypertangent divisors.

Let

$$\Sigma_1 = \{\lambda_1 q_{1,1}|_V + \cdots + \lambda_k q_{k,1}|_V\}$$

be the $k$-dimensional space of equations of the tangent hyperplanes at $o$. By condition (R1), for every non-zero tuple $(\lambda_1, \ldots, \lambda_k)$ the tangent divisor

$$T(\lambda) = \{\lambda_1 q_{1,1}|_V + \cdots + \lambda_k q_{k,1}|_V = 0\}$$

satisfies the equality

$$T^E(\lambda) = (T^+(\lambda) \circ E) = \{\lambda_1 \bar{q}_{1,2} + \cdots + \lambda_k \bar{q}_{k,2} = 0\}.$$ In particular, we put $T_i = \{q_{i,1}|_V = 0\}$. By condition (R2), none of the irreducible components of the effective cycle $(T_1^E \circ \cdots \circ T_k^E)$ of codimension $k$ is contained in a linear subspace of codimension 2 in $E$. Therefore we have

$$\text{codim}_E(T_1^E \cap \cdots \cap T_k^E \cap \Lambda) = k + 1.$$ (1.2)

The cycle $Q_E$ constructed above is supported in the hyperplane $\Lambda$. By (1.2), the following equality holds for any $k - 2$ generic divisors $D_{1,1}, \ldots, D_{1,k-2}$ in the linear system $\Sigma_1$:

$$\text{codim}_E(Q_E \cap D_{1,1}^E \cap \cdots \cap D_{1,k-2}^E) = k + 1.$$
where $D_{1,i}^{E} = (D_{1,i}^{+} \circ E)$. Hence in a neighbourhood of $o$ we also have
\[
\text{codim}_o(Q \cap D_{1,1} \cap \cdots \cap D_{1,k-2}) = k + 1. \tag{1.3}
\]

We now construct the following sequence of effective cycles $Q_i$, $i = 0, 1, \ldots, k-2$: 
(i) $Q_0 = Q$, 
(ii) $Q_{i+1}$ is obtained from the effective cycle $(Q_i \circ D_{1,i+1})$ by removing all irreducible components that do not contain $o$.

This procedure is well defined. Indeed, since all components of every cycle $Q_i$ contain $o$, equality (1.3) yields that none of the components of $Q_i$ is contained in the divisor $D_{1,i+1}$. In particular, codim $Q_i = i + 3$ for $i = 0, \ldots, k - 2$. The degree $\deg Q_i$ does not increase (and even decreases when some components are indeed removed). Hence $\deg Q_{k-2} \leq \deg Q$, and the multiplicity at $o$ satisfies
\[
\text{mult}_o Q_{k-2} = 2^{k-2} \text{mult}_o Q.
\]

We now consider the hypertangent linear system
\[
\Sigma_2 = \left\{ h(z)|_V = \sum_{i=1}^{k} s_i(z)q_{i,1}|_V + \sum_{i=k_1+1}^{k_1+k_2} \lambda_i(q_{i,1} + q_{i,2})|_V \right\}.
\]

Clearly, $\Sigma_2^+ \subset |2H - 3E|$: the projectivized tangent cone of the divisor $h|_V = 0$ at $o$ is given by the equation
\[
-\left( \sum_{i=1}^{k} \bar{s}_i \bar{q}_{i,2} + \sum_{i=k_1+1}^{k_1+k_2} \lambda_i \bar{q}_{i,3} \right) \in \Sigma_2^T.
\]

Let $(D_{2,1}, \ldots, D_{2,k_2-2}) \in \Sigma_2^{x(k_2-2)}$ be a generic tuple of $k_2 - 2$ hypertangent divisors. By condition (R1), the closed set
\[
Q_E \cap \left( \bigcap_{i=1}^{k_2-2} D_{1,i}^{E} \right) \cap \left( \bigcap_{i=1}^{k_2-2} D_{2,i}^{E} \right)
\]
is zero-dimensional, whence the closed set
\[
Q \cap \left( \bigcap_{i=1}^{k_2-2} D_{1,i} \right) \cap \left( \bigcap_{i=1}^{k_2-2} D_{2,i} \right)
\]
is one-dimensional in a neighbourhood of $o$. We continue to construct the chain of effective cycles $Q_i$, $i = k - 2, \ldots, M - 4$, where $Q_{k-2}$ has already been constructed and $Q_{i+1}$ is obtained from the effective cycle $(Q_i \circ D_{2,i+3-k})$ by removing all irreducible components that do not contain $o$.

Put $C = Q_{M-4}$. This is an effective 1-cycle all of whose components contain $o$. We have $\deg C \leq 2^{k_2-2} \deg Q$, and the multiplicity at $o$ satisfies
\[
\text{mult}_o C = 2^{k-2} 3^{k_2-2} \text{mult}_o Q.
\]
By the choice of the subvariety $Q$, we get
\[
\frac{\text{mult}_o C}{\deg C} > \frac{2^k - 2 3^{k_2 - 2} 8}{2^{k_2 - 2}} \frac{8}{d} = \frac{8}{9}.
\]

Although this estimate is obviously very strong, it does not yet give a contradiction: all lines through $o$ satisfy this inequality. We now use condition (R3) to estimate the contribution of lines to the effective cycle $C$.

**1.3. The effective 1-cycle free from lines.** We write $C = C_1 + C_{\geq 2}$, where the support of the effective 1-cycle $C_1$ (resp. $C_{\geq 2}$) consists of lines (resp. curves of degree $\geq 2$). Clearly, $\text{mult}_o C_1 = \deg C_1$.

**Lemma 1.1.** For every irreducible curve $\Gamma \subset V$ of degree $\deg \Gamma \geq 2$ we have
\[
\text{mult}_o \Gamma \leq \frac{2}{3} \deg \Gamma. \tag{1.4}
\]

**Proof.** Consider the system of homogeneous equations in $\mathbb{C}^{M+k}$ of the form
\[
q_{*,*}(z) = 0
\]
given by all irreducible components of all $k$ equations $f_i$. By condition (R1), it determines either the origin $o \in \mathbb{C}^{M+k}$ or finitely many lines through $o$. Clearly, a line $t(a_1, \ldots, a_{M+k})$ lies on $V$ if and only if $q_{i,j}(a_*) = 0$ for all $i, j$. Assume that a curve $\Gamma \ni o$ is not a line and satisfies the inequality
\[
\text{mult}_o \Gamma > \frac{2}{3} \deg \Gamma.
\]

Clearly, $\Gamma$ is contained in the support of any tangent divisor $D \in \Sigma_1$ (that is, a divisor with $D^+ \in |H - 2E|$) and any hypertangent divisor $D \in \Sigma_2$ (that is, $D^+ \in |2H - 3E|$). Hence the following polynomials vanish identically on $\Gamma$:
\[
q_{i,1}, \quad i = 1, \ldots, k,
q_{i,1} + q_{i,2}, \quad i = k_1 + 1, \ldots, k.
\]

Moreover, since $\Gamma \subset V$, the following polynomials also vanish identically on $\Gamma$:
\[
f_i = q_{i,1} + q_{i,2}, \quad i = 1, \ldots, k_1,
f_i = q_{i,1} + q_{i,2} + q_{i,3}, \quad i = k_1 + 1, \ldots, k.
\]

We conclude that all the homogeneous polynomials $q_{i,j}$ vanish on $\Gamma$. But then $\Gamma$ is a line. This contradiction proves the lemma. □

**Corollary 1.1.** We have
\[
\text{mult}_o C_{\geq 2} \leq \frac{2}{3} \deg C_{\geq 2}.
\]
To complete the proof of Theorem 0.3, it remains to estimate from above the contribution of lines into the 1-cycle $C$, that is, the ratio $\deg C_1 / \deg C$.

Consider the generic tangent divisors $D_{1,i}$ which were used to construct the curve $C$. If

$$(\lambda_1 q_{1,1} + \lambda_2 q_{2,1} + \cdots + \lambda_k q_{k,1})|_V = 0$$

is the equation of the divisor $D_{1,i}$, then, clearly,

$$g_i := - (\lambda_1 \bar{q}_{1,2} + \lambda_2 \bar{q}_{2,2} + \cdots + \lambda_k \bar{q}_{k,2}) \in \Sigma^T_1$$

is the equation of its projectivized tangent cone. Therefore $(g_1, \ldots, g_{k-2})$ form a generic tuple, that is, a generic element of $(\Sigma^T_1)^{(k-2)}$. In a similar way, the projectivized tangent cone of the divisor $D_{2,i}$, $i = 1, \ldots, k_2 - 2$, has the equation

$$g_{k-2+i} = \sum_{j=1}^k \bar{s}_j \bar{q}_{j,2} + \sum_{j=k_1+1}^{k_1+k_2} \lambda_j \bar{q}_{j,3} \in \Sigma^T_2$$

(here we simplify the notation for $\bar{s}_j$ and $\lambda_j$ by omitting the second subscript, which indicates the dependence of the divisor $D_{2,i}$ on the number $i$), whereas $(g_1, \ldots, g_{M-4})$ is a generic element of $\mathcal{P}$. Since the intersection with the tangent (hypertangent) divisor $D_{i,j}$ is proper at every step of the construction of $C$, the zero-dimensional cycle $C_E = (C_1 + E)$ is the scheme-theoretic intersection

$$(Q_E \circ D_{1,1} \circ \cdots \circ D_{1,k-2} \circ D_{2,1} \circ \cdots \circ D_{2,k_2-2}),$$

that is, we have the following equality of 0-cycles:

$$(C_1^+ \circ E) + (C_{\geq 2}^+ \circ E) = \sum_{i \in I} m_i (R_i \circ D_{1,1} \circ \cdots \circ D_{2,k_2-2}),$$

whose right-hand side contains the scheme-theoretic intersections with all $M - 4$ tangent (hypertangent) divisors $D_{i,j}$ that participate in the construction of the 1-cycle $C$. For every line $L \subset V$ passing through the point $o$, we have

$$(L^+ \cap E) \in \Delta$$

in the notation of §0.2 (the set $\Delta$ consists precisely of the tangent directions of all lines through $o$ in $V$). Hence the support of the 0-cycle $(C_1^+ \circ E)$ is a subset of the finite set $\Delta$. Using condition (R3), that is, the estimate (0.3), we have

$$\sum_{p \in \Delta} \sum_{i \in I} m_i \dim \mathcal{O}_{p,R_i}/(g_1, \ldots, g_{M-4}) \leq 2^{k-4} 3^{k_2-1} \mult_o Q$$

since the degree of the effective cycle $Q_E$ of codimension 3 on $E$ is clearly equal to $\mult_o Q$. Then, $a \text{ fortiori}$,

$$\delta = \deg(C_1^+ \circ E) = \deg C_1 \leq 2^{k-4} 3^{k_2-1} \mult_o Q.$$  \hfill (1.5)

Thus the 1-cycle $C_{\geq 2}$ satisfies the equality

$$\mult_o C_{\geq 2} = 2^{k-2} 3^{k_2-3} \mult_o Q - \delta$$
and the inequality
\[ \deg C_{\geq 2} \leq 2^{k_2-2} \deg Q - \delta. \]
By easy computation, we deduce from inequality (1.4) that
\[ \delta \geq 2^{k-2} 3^{k_2-1} \mult_o Q - 2^{k_2-1} \deg Q. \]
Combining this inequality with (1.5), we finally get
\[ 2^{k_2-1} \deg Q \geq 2^{k-4} 3^{k_2} \mult_o Q. \]
Since \( k = k_1 + k_2 \) and \( d = 2^{k_1} 3^{k_2} \), this yields the inequality
\[ \mult_o Q \leq \frac{8}{d} \deg Q, \]
which contradicts (1.1). This contradiction excludes the third (and last) option \( \text{codim } B \geq 4 \) and completes the proof of Theorem 0.3.

Remark 1.1. In all previous papers (see, for example, [1], [4], [5], [20]), the technique of hypertangent divisors was used in a somewhat different way: the irreducible component with maximal ratio \( \mult_o \deg \) was chosen at every step when intersecting with a hypertangent divisor, and all other components of the scheme-theoretic intersection were ignored. Our argument controls the whole cycle of intersection, not only one of its components, in order to estimate the contribution of lines at the last step. This type of argument can also be used in the previous problems. Instead of taking the component with maximal ratio \( \mult_o \deg \), one can consider the whole cycle of scheme-theoretic intersection, removing only those components that do not contain \( o \) (because the regularity conditions provide a well-defined intersection with hypertangent divisors only in the neighbourhood of \( o \)).

§ 2. Local multiplicities. I. Spaces of tuples of polynomials

In this section we give a precise formulation of the problem of estimating the local multiplicity. We introduce the space of tuples of polynomials, define the local effective multiplicities of intersection and state the main result (Theorem 2.1). The theory developed in this and the next two sections is independent of § 1 and is self-contained. The notation is also independent of § 1.

2.1. Tuples of polynomials and effective multiplicities. We fix a complex coordinate space \( \mathbb{C}^N(z_1, \ldots, z_N) \), \( N \geq 1 \), and regard it as being embedded in the projective space \( \mathbb{P}^N(x_0:x_1: \ldots : x_N) \) as the standard affine chart \( \{ x_0 \neq 0 \} \), that is, \( z_i = x_i/x_0 \).

The space of homogeneous polynomials of degree \( d \geq 1 \) in the variables \( z_* \) is denoted by \( \mathcal{P}_{d,N} \). For \( e \leq d \) we put
\[ \mathcal{P}_{[e,d],N} = \bigoplus_{i=e}^d \mathcal{P}_{i,N}. \]
For example, \( \mathcal{P}_{[1,d],N} \) is the space of polynomials of degree \( \leq d \) without constant term. The matrix group \( \text{GL}_N(\mathbb{C}) \) acts on each of these spaces by linear changes
of coordinates. Let

$$p_{n_1,n_2} = p_{n_1,0} \times p_{n_2,0} = \prod_{i=1}^{n_1} \mathcal{P}_{[i,2],n} \times \prod_{i=1}^{n_2} \mathcal{P}_{[i,3],n}$$

be the space of tuples \((f_1, \ldots, f_{n_1}, f_{n_1+1}, \ldots, f_{n_1+n_2})\) of polynomials, where the first \(n_1\) polynomials are of degree \(\leq 2\) and the other \(n_2\) polynomials are of degree at most 3. All polynomials vanish at the point \(o = (0, \ldots, 0)\). We assume that \(n_1 + n_2 = N\).

Given an effective cycle \(R\) of pure codimension \(l\) on \(\mathbb{P}^N\) and a tuple of polynomials \((f_1, \ldots, f_N) \in \mathcal{P}_{n_1,n_2}\), we define the \textit{effective multiplicity}

$$\mu_R(f_1, \ldots, f_N) \in \mathbb{Z}_+ \cup \{\infty\}$$
in the following way. If \(o \not\in \text{Supp} R\), then we put \(\mu_R(f_*) = 0\). If the closed set

$$Z_R(f_1, \ldots, f_N) = \{f_1 = \cdots = f_N = 0\} \cap R$$

has positive dimension at \(o\), then we put \(\mu_R(f_*) = \infty\). If neither of these two cases holds, then \(\mu_R(f_*)\) is a positive integer, whose definition requires some additional constructions. Let

$$\Sigma_1(f_*) = \langle f_1, \ldots, f_{n_1} \rangle \subset \mathcal{P}_{[1,2],n}$$

be the linear span of the quadratic polynomials. Let

$$\Sigma_{12}(f_*) = \langle f_1 \mathcal{P}_{[0,1],n}, \cdots, f_{n_1} \mathcal{P}_{[0,1],n} \rangle \subset \mathcal{P}_{[1,3],n}$$

be the linear space of all linear combinations of \(f_1, \ldots, f_{n_1}\) whose coefficients are polynomials of degree at most 1 in \(z_*\). In particular, we have an inclusion \(\Sigma_1 \subset \Sigma_{12}\). Let

$$\Sigma_{22}(f_*) = \langle f_{n_1+1}, \ldots, f_{n_1+n_2} \rangle \subset \mathcal{P}_{[1,3],n}$$

be the linear span of the cubic polynomials, and let

$$\Sigma_2(f_*) = \Sigma_{12}(f_*) + \Sigma_{22}(f_*) \subset \mathcal{P}_{[1,3],n}$$

be the sum of these subspaces. We define the \textit{polynomial span}

$$[f_1, \ldots, f_N] \subset \mathcal{P}_{n_1,n_2}$$

of the tuple \((f_*)\) as the set of all tuples \((f_1^*, \ldots, f_N^*)\) such that \(f_1^* \in \Sigma_1(f_*)\) and \(f_{n_1+1}^*, \cdots, f_{n_1+n_2}^* \in \Sigma_2(f_*)\). Clearly, \([f_*]\) is a closed irreducible subset of \(\mathcal{P}_{n_1,n_2}\).

Now, given an irreducible subvariety \(R \subset \mathbb{P}^N\) of codimension \(l\) with \(R \ni o\), we put

$$\mu_R(f_1, \ldots, f_N) = \dim \mathcal{O}_{o,R}/(f_1^*, \ldots, f_N^*)$$

where \((f_1^*, \ldots, f_N^*) \in [f_1, \ldots, f_N]\) is a generic tuple. We extend the definition of \(\mu_R(f_*)\) by linearity to all effective cycles \(R = \Sigma_{j \in J} r_j R_j\) of pure codimension \(l\):

$$\mu_R(f_*) = \sum_{j \in J} r_j \mu_{R_j}(f_*)$$.
This definition is equivalent to the following one, which works for any effective cycle of codimension $l$:

$$
\mu_R(f_*) = \text{mult}_o(\{f_{i+1}^2 = 0\} \circ \cdots \circ \{f_N^2 = 0\} \circ R).
$$

Here the term in parentheses is the effective zero-dimensional cycle of scheme-theoretic intersection of the hypersurfaces $\{f_i^2 = 0\}$, $i = l + 1, \ldots, N$, and the cycle $R$ in a neighbourhood of $o$.

**Remark 2.1.** Taking the polynomial span $[f_*]$ is a necessary step in the definition of effective multiplicity because some components of the cycle $R$ and the projectivized tangent cone to $R$ at $o$ may be contained in the divisors $\{f_i = 0\}$. Therefore the numbers

$$
\dim \mathcal{O}_{o,R}/(f_{i_1}, \ldots, f_{i_{N-l}})
$$

may happen to be strictly larger than the correct effective multiplicity defined above, even when $R$ is an irreducible subvariety not contained in the hypersurfaces $\{f_i = 0\}$.

### 2.2. Chow varieties and local effective multiplicities

We write $\mathbb{H}_{l,N}(d)$ for the Chow variety that parametrizes the effective cycles of pure codimension $l$ and degree $d$ on $\mathbb{P}^N$. Consider the sets

$$
X_{l,N}(m, d) \subset \mathcal{P}_{[1,2],N} \times \mathcal{P}_{[1,3],N} \times \mathbb{H}_{l,N}(d)
$$

of all tuples $((f_1, \ldots, f_N), R)$ such that

$$
\mu_R(f_1, \ldots, f_N) \geq m \in \mathbb{Z}_+ \cup \{\infty\}.
$$

We easily see that $X_{l,N}(m, d)$ is a closed algebraic set. Denote the projection

$$
((f_1, \ldots, f_N), R) \mapsto (f_1, \ldots, f_N)
$$

by $\pi_P$. Since Chow varieties are projective, we see that

$$
X_{l,N}(m, d) = \pi_P(X_{l,N}(m, d)) \subset \mathcal{P}_{[1,2],N} \times \mathcal{P}_{[1,3],N}
$$

is a closed algebraic set. Explicitly, it consists of the tuples $(f_1, \ldots, f_N)$ for which there is an effective cycle $R \in \mathbb{H}_{l,N}(d)$ such that $\mu_R(f_1, \ldots, f_N) \geq m$.

Let $B \subset \mathcal{P}_{[1,2],N} \times \mathcal{P}_{[1,3],N}$ be an irreducible subvariety. We define the local effective multiplicity

$$
\mu_{l,N}^{\text{local}}(B, d) = \mu_{l,N}(B, d) = \max_{m \in \mathbb{Z}_+ \cup \{\infty\}} \{m \mid B \subset X_{l,N}(m, d)\}.
$$

Explicitly, the equation $\mu_{l,N}(B, d) = m$ means by definition that the inequality

$$
\mu_R(f_1, \ldots, f_N) \leq m
$$

holds for a generic tuple $(f_*) \in B$ and any effective cycle $R \in \mathbb{H}_{l,N}(d)$ and becomes an equality for at least one cycle $R \in \mathbb{H}_{l,N}(d)$. Clearly, if $\mu_{l,N}(B, d) = \infty$, then the zero set $Z(f_1, \ldots, f_N)$ of a generic (and hence every) tuple $(f_1, \ldots, f_N)$ has a positive-dimensional component containing $o$. The converse also holds. If there is such a component, one can take $R$ to be any subvariety with $\dim(R \cap Z(f_*)) \geq 1$. 
Proposition 2.1. For every \( d \geq 1 \) we have

\[
\text{codim} \, X_{l,N}(\infty, d) = n_1 + 2n_2 + 1.
\]

The proof of Proposition 2.1 is given in § 5.

In §§ 2–4 we consider only local multiplicities. Therefore, to simplify the notation, we do not indicate that they are local and write simply \( \mu \) instead of \( \mu_{\text{local}} \) or \( \mu_{l,\text{local}} \). In § 5 we shall define the global multiplicities \( \mu_{\text{total}} \) and return to the notation \( \mu_{l,\text{local}} \) or \( \mu_{l,\text{local}} \) for local multiplicities.

Finally, we write \( \mu_{l,N}(a, d) = m \) for some \( a \in \mathbb{Z}_+ \) if there is an irreducible subvariety \( B \subset \mathcal{P}^{(n_1, n_2)}_N \) of codimension at most \( a \) such that \( \mu_{l,N}(B, d) = m \) and \( \text{codim} \, X_{l,N}(m + 1, d) \geq a + 1 \).

Remark 2.2. Because of our definitions, the inequality \( \mu_{l,N}(a, d) \leq m \) means that \( \text{codim} \, X_{l,N}(m + 1, d) \geq a + 1 \), or explicitly, the set of tuples \( (f_*) \in \mathcal{P}^{(n_1, n_2)}_N \) such that there is an effective cycle \( R \) of codimension \( l \) and degree \( d \) with \( \mu_R(f_1, \ldots, f_N) \geq m + 1 \) has codimension at least \( a + 1 \) in the space \( \mathcal{P}^{(n_1, n_2)}_N \).

It is clear from Proposition 2.1 that \( \mu_{l,N}(a, d) = \infty \) for all \( a \geq n_1 + 2n_2 + 1 \) and \( \mu_{l,N}(a, d) < \infty \) for \( a \leq n_1 + 2n_2 \). We assume throughout that \( a \leq n_1 + 2n_2 \).

(In what follows we are mainly interested in the case when \( a = n_1 + n_2 = N \).)

By construction, the sets \( X_{l,N}(m, d) \) are invariant under linear changes of coordinates (the group \( \text{GL}_N(\mathbb{C}) \)). They are also invariant under the action of another group \( G(n_1, n_2) \) on \( \mathcal{P}^{(n_1, n_2)}_N \), which will now be defined. The group \( G(n_1, n_2) \) is the extension

\[
1 \longrightarrow \mathcal{P}^{(n_1, n_2)}_{[0,1],N} \longrightarrow G(n_1, n_2) \longrightarrow \text{GL}_{n_1}(\mathbb{C}) \times \text{GL}_{n_2}(\mathbb{C}) \longrightarrow 1
\]

More precisely, each element \( g \in G(n_1, n_2) \) is associated with a triple of matrices \( (A_{11}, A_{12}, A_{22}) \), where

\[
A_{11} \in \text{GL}_{n_1}(\mathbb{C}), \quad A_{22} \in \text{GL}_{n_2}(\mathbb{C}), \quad A_{12} \in \text{Mat}_{(n_1, n_2)}(\mathcal{P}_{[0,1],N}).
\]

Putting

\[
A_{11} = \|a_{ij}\|_{1 \leq i,j \leq n_1}, \quad A_{22} = \|b_{ij}\|_{1 \leq i,j \leq n_2},
\]

\[
A_{12} = \|c_{ij}(z_1, \ldots, z_N)\|_{1 \leq i \leq n_1, 1 \leq j \leq n_2},
\]

we define an action

\[
g : f = (f_1, \ldots, f_N) \mapsto f^g = (f^g_1, \ldots, f^g_N) \in \mathcal{P}^{(n_1, n_2)}_N
\]
by the formulae

\[ f^g_i = \sum_{j=1}^{n_1} \alpha_{ij} f_j, \quad i = 1, \ldots, n_1, \]

\[ f^g_{n_1+i} = \sum_{j=1}^{n_2} \beta_{ij} f_{n_1+j} + \sum_{i=1}^{n_1} \gamma_{ij}(z_*) f_i, \quad j = 1, \ldots, n_2. \]

A closed subset \( B \subset \mathcal{P}_N^{(n_1,n_2)} \) is said to be bi-invariant if it is invariant under the actions of \( \text{GL}_N(\mathbb{C}) \) (changes of coordinates) and \( \text{G}(n_1,n_2) \). In particular, the sets \( X_{i,N}(m,d) \) are bi-invariant (the multiplicities \( \mu_R(f_*) \) are obviously invariant under the actions of both groups). Note that \( \text{G}(n_1,n_2) \) contains a subgroup \( \overline{\text{G}}(n_1,n_2) \subset \text{GL}_N(\mathbb{C}) \) corresponding to the triples \((A_{11}, A_{12}, A_{22})\) with

\[ A_{12} \in \text{Mat}_{(n_1,n_2)}(\mathbb{C}) \subset \text{Mat}_{(n_1,n_2)}(\mathcal{P}_{[0,1],N}). \]

### 2.3. Reduction to standard form

We define the type

\[ \tau(B) = (((a_1,b_1),(a_2,b_2)) \in \mathbb{Z}_+^4 \]

of an irreducible subvariety \( B \subset \mathcal{P}_N^{(n_1,n_2)} \) by putting \( a_1 = \text{codim}(\text{pr}_1(B)) \subset \mathcal{P}_{[1,2],N}^{n_1} \), where \( \text{pr}_1: \mathcal{P}_N^{(n_1,n_2)} \to \mathcal{P}_{[1,2],N}^{n_1} \) is the projection onto the first \( n_1 \) direct factors, \( a_2 = a - a_1 \), where \( a = \text{codim}(B) \subset \mathcal{P}_N^{(n_1,n_2)} \),

\[ b_1 = n_1 - \text{rk}(df_1(o), \ldots, df_{n_1}(o)) \]

for a tuple \((f_1, \ldots, f_{n_1}) \in \text{pr}_1(B)\) of general position, and \( b_2 = b - b_1 \), where

\[ b = \varepsilon(B) = N - \text{rk}(df_1(o), \ldots, df_N(o)) \]

for a tuple \((f_1, \ldots, f_N) \in B\) of general position. If the subvariety \( B \) is \( \text{G}(n_1,n_2)\)-invariant (or at least \( \overline{\text{G}}(n_1,n_2)\)-invariant) and \((f_1, \ldots, f_N) \in B\) is a generic tuple of \( N \) polynomials, then the linear forms

\[ df_1(o), \ldots, df_{n_1-b_1}(o), df_{n_1+1}(o), \ldots, df_{N-b_2}(o) \]  

(2.1)

are linearly independent, the forms

\[ df_{n_1-b_1+1}(o), \ldots, df_{n_1}(o) \]

are linear combinations of \( df_1(o), \ldots, df_{n_1-b_1}(o) \), and the forms

\[ df_{N-b_2+1}(o), \ldots, df_N(o) \]

are linear combinations of the forms (2.1). Hence for every \( \text{G}(n_1,n_2)\)-invariant (or \( \overline{\text{G}}(n_1,n_2)\)-invariant) irreducible subvariety \( B^o \subset B \) and a well-defined map (of reduction to standard form)

\[ \rho: B^o \to \mathcal{P}_{[1,2],N}^{n_1-b_1} \times \mathcal{P}_{[1,3],N}^{b_1} \times \mathcal{P}_{[1,2],N}^{n_2-b_2} \times \mathcal{P}_{[2,3],N}^{b_2} \]  

(2.2)
sending a generic tuple \((f_1, \ldots, f_N)\) to a tuple of polynomials

\[
(f_1, \ldots, f_{n_1-b_1}, f_{n_1-b_1+1}^+, \ldots, f_{n_1}^+, f_{n_1+1}, \ldots, f_{N-b_2}, f_{N-b_2+1}^+, \ldots, f_N^+),
\]

where \(f_i^+\) for \(i \in \{n_1 - b_1 + 1, \ldots, n_1\}\) is obtained from \(f_i\) by subtracting a uniquely determined linear combination of \(f_1, \ldots, f_{n_1-b_1}\) in such a way that \(df_i^+(o) = 0\), and \(f_i^+\) for \(i \in \{N - b_2 + 1, \ldots, N\}\) is obtained from \(f_i\) by subtracting a uniquely determined linear combination of \(f_j, j \in \{1, \ldots, n_1 - b_1, n_1 + 1, \ldots, N - b_2\}\) in such a way that \(df_i^+(o) = 0\). We denote the closure of the image \(\rho(B^o)\) by \(\overline{B}\). If \(B\) is invariant under the action of \(G(n_1, n_2)\), then the coefficients of the linear combinations mentioned above may be arbitrary and, therefore,

\[
\dim B = \dim \overline{B} + b_1(n_1 - b_1) + b_2(N - b) = \dim \overline{B} + b(n_1 - b_1) + b_2(n_2 - b_2).
\]

On the other hand, the direct product in (2.2) has codimension \(bN\) in \(P^{(n_1,n_2)}\). Hence we have

\[
\text{codim } \overline{B} = \text{codim } B - b_1(n_2 + b_1) - b_2b, \tag{2.3}
\]

where the codimension of each subvariety is taken with respect to the relevant ambient space: \(P^{(n_1,n_2)}\) for \(B\) and the direct product in (2.2) for \(\overline{B}\). From now on, the codimension will always be understood with respect to the natural ambient space unless otherwise stated. We shall sometimes recall which ambient space is meant.

Clearly, every fibre of the map \(\rho: B^o \to \rho(B^o)\) is \(\mathbb{C}^{(b_1(n_1-b_1)+b_2(N-b))}\).

By construction we have

\[
\mu_{l,N}(a,d) = \max_B \mu_{l,N}(B,d),
\]

where the maximum is taken over all bi-invariant irreducible subvarieties \(B \subset P^{(n_1,n_2)}\) of codimension at most \(a\). Our method for estimating the numbers \(\mu_{l,N}(B,d)\) (and thus the numbers \(\mu_{l,N}(a,d)\)) is to control the type \(\tau(B)\) of these subvarieties. We easily see that the conditions

\[
\text{rk}(df_1(o), \ldots, df_{n_1}(o)) \leq n_1 - b_1, \\
\text{rk}(df_1(o), \ldots, df_N(o)) \leq N - b
\]

determine a bi-invariant irreducible subvariety of type \(((a_1^*, b_1), (a_2^*, b_2))\) in \(P^{(n_1,n_2)}\), where \(a_1^* = b_1(N + b_1 - n_1)\) and \(a_2^* = b^2 - a_1^*\). Hence the following inequalities are necessary conditions for the existence of a bi-invariant irreducible subvariety \(B\) of type \(\tau(B) = ((a_1, b_1), (a_2, b_2)):\)

\[
a_1 \geq b_1(N + b_1 - n_1), \quad a = a_1 + a_2 \geq b^2.
\]

**Proposition 2.2.** Assume that a (bi-invariant) subvariety \(B\) is of type \(\tau(B) = ((a_1, 0), (a_2, 0))\), that is, \(b = 0\). Then

\[
\mu_{l,N}(B,d) = d
\]

for every \(d \geq 1\).
Proof. Let $R \ni o$ be an irreducible subvariety in $\mathbb{P}^N$ and $(f_1, \ldots, f_N) \in B$ a generic tuple of polynomials. By hypothesis,
\[ \text{rk}(df_1(o), \ldots, df_N(o)) = N. \]
Put $\Pi = \{df_1(o) = \cdots = df_{n_1}(o) = 0\} \subset \mathbb{C}^N$. This is a vector subspace of codimension $n_1$. Let
\[ (f^2_1, \ldots, f^2_N) \in [f_1, \ldots, f_N] \subset \mathcal{P}_N^{(n_1, n_2)} \]
be a generic element of the polynomial span of $(f_i)$. By construction, $df^2_1(o), \ldots, df^2_{n_1}(o)$ are linear forms of general position vanishing on the space $\Pi$, and $df^2_{n_1+1}(o), \ldots, df^2_N(o)$ are arbitrary linear forms of general position.

If $n_1 \leq l$, then $\mu_R(f_*) = \dim \mathcal{O}_{o,R}/(df^2_{l+1}, \ldots, f^2_N)$ satisfies the equation
\[ \mu_R(f_*) = \text{mult}_o R \]
because the differentials $df^2_i(o), i = l+1, \ldots, N$, are linear forms of general position. If $n_1 > l$, then there is a (unique) constraint on the first $n_1 - l$ differentials $df^2_i(o)$, where $i = l+1, \ldots, n_1$:
\[ df^2_i(o)|_{\Pi} \equiv 0. \]

Let $T_oR \subset \mathbb{C}^N$ be the algebraic tangent cone to $R$ at $o$. Each of its components is of codimension $l$. Since $\text{codim}(\Pi \subset \mathbb{C}^N) = n_1$, we may assume for a generic $f^2_{l+1}$ that the linear form $df^2_{l+1}$ vanishes on no component of $T_oR$. Therefore,
\[ \text{mult}_o(R \circ \{f^2_{l+1} = 0\}) = \text{mult}_o R. \]

For generic $f^2_{l+1}, \ldots, f^2_{n_1}$ we similarly have
\[ \text{mult}_o(R \circ \{f^2_{l+1} = 0\} \circ \cdots \circ \{f^2_{n_1} = 0\}) = \text{mult}_o R. \]

We now obtain the equation $\mu_R(f_*) = \text{mult}_o R$ as in the case $n_1 \leq l$. Since $\text{mult}_o R \leq \deg R$ and equality is attained, the proposition is proved. □

Note that if the multiplicity $\mu_{l,N}(B, d)$ is finite, then we have
\[ \mu_{l,N}(B, d) \leq \begin{cases} 2^{a_1-l}3^{a_2}d & \text{for } a_1 \geq l, \\ 3^{a_1-l}d & \text{for } a_1 < l. \end{cases} \] (2.4)

However, these estimates are too weak for our purposes. The following key facts will be proved below.

**Theorem 2.1.** We have the inequalities
\[ \mu_{l,N}(B, d) \leq \frac{1}{2\pi b} \left( \frac{2a - b(b-1)\sqrt{e^2}}{b^2} \right)^b d < \frac{1}{2\pi b} \left( \frac{2a}{b^2 e^2} \right)^b d, \]
\[ \mu_{l,N}(a, d) \leq \frac{c^2}{2\pi \sqrt{|a|}} (2e^2)^{|\sqrt{a}|} d. \]
Theorem 2.2. For \( a \geq 17 \) we have

\[
\mu_{l,N}(a,d) \leq \frac{e^2}{2\pi\sqrt{a}} \left( \frac{5}{3} e^2 \right)^{|\sqrt{a}|} d.
\]

Thus the effective multiplicity with fixed \( l \) has exponential growth of order \( \sqrt{a} \) as \( a \to \infty \), and not of order \( a \), as in the \( a \) priori estimates (2.4).

\[\text{§ 3. Local multiplicities. II. The inductive method of estimation}\]

In this section we describe the key procedure for estimating the local multiplicity in terms of the local multiplicities of truncated tuples of polynomials.

3.1. Splitting off a direct factor. If \( b_2 \geq 1 \), then we write \( \pi_2 \) for the projection

\[
\mathcal{P}^{n_1-b_1}_{[1,2],N} \times \mathcal{P}^{b_1}_{[2,2],N} \times \mathcal{P}^{n_2-b_2}_{[1,3],N} \times \mathcal{P}^{b_2}_{[2,3],N}
\]

along the last direct factor \( \mathcal{P}_{[2,3],N} \). If \( b_1 \geq 1 \), then we write \( \pi_1 \) for the corresponding projection onto \( \mathcal{P}^{n_1-b_1}_{[1,2],N} \times \mathcal{P}^{b_1-1}_{[2,2],N} \times \mathcal{P}^{n_2}_{[1,3],N} \times \mathcal{P}^{b_2}_{[2,3],N} \) along the last quadratic direct factor \( \mathcal{P}_{2,N} \). Let \( \overline{B} \) be the closed set constructed above by reducing to standard form. We denote the closure of \( \pi_i(\overline{B}) \), \( i = 1, 2 \), in the corresponding ambient space by \( [\overline{B}]_{N-1}^{(i)} \). Furthermore, let \( \lambda_2 \) and \( \lambda_1 \) be the projections complementary to \( \pi_2 \) and \( \pi_1 \), that is, the projections onto the last direct factor and the last quadratic direct factor respectively. Then \( \pi_2 \times \lambda_2 \) and \( \pi_1 \times \lambda_1 \) are the identity maps on the direct product \( \mathcal{P}^{n_1-b_1}_{[1,2],N} \times \mathcal{P}^{b_1}_{[2,2],N} \times \mathcal{P}^{n_2-b_2}_{[1,3],N} \times \mathcal{P}^{b_2}_{[2,3],N} \).

Note that since \( B \) is bi-invariant, the closed set \( \overline{B} \) is invariant under the group \( \text{GL}_N(\mathbb{C}) \) of linear changes of coordinates and under the subgroup \( G(n_1, b_1; n_2, b_2) \subset G(n_1, n_2) \) whose elements correspond to the triples of matrices \( (A_{11}, A_{12}, A_{22}) \) with \( A_{11} \in \text{GL}_{n_1-b_1}(\mathbb{C}) \) (these matrices act on the polynomials \( f_1, \ldots, f_{n_1-b_1} \) and leave \( f_{n_1-b_1+1}, \ldots, f_{n_1} \) fixed), \( A_{22} \in \text{GL}_{n_2-b_2}(\mathbb{C}) \) (these matrices act on \( f_{n_1+1}, \ldots, f_{n_1+n_2-b_2} \)) and \( A_{12} \in \text{Mat}_{(n_1, n_2)}(\mathcal{P}_{[0,1],N}) \), where only the polynomials \( \gamma_{ij} \) with \( j = 1, \ldots, n_2 - b_2 \) have an arbitrary constant term while the polynomials with \( j \geq n_2 - b_2 + 1 \) are homogeneous: \( \gamma_{ij} \in \mathcal{P}_{1,N} \). In what follows we use this trivial remark without special comments in order to estimate the parameters of the variety \( B_h \subset \mathcal{P}^{(n_1,n_2-1)}_{N-1} \).

To simplify the notation, we assume that \( b_2 \geq 1 \), consider the projection \( \pi_2 \) and write \([\overline{B}]_{N-1}\) instead of \([\overline{B}]_{N-1}^{(2)}\). The modifications required in the case \( i = 1 \) are obvious and we only give the final result. The case \( i = 2 \) will be treated in full detail.

Given a generic tuple \( (f_1, \ldots, f_{N-1}) \in [\overline{B}]_{N-1} \), we write

\[
[\overline{B}]^N = [\overline{B}]^N(f_1, \ldots, f_{N-1}) \subset \mathcal{P}_{[2,3],N}
\]
for the fibre of the projection $\pi_2|_\mathcal{B} : \mathcal{B} \to [\mathcal{B}]_{N-1}$. Clearly,
\[
\text{codim} \mathcal{B} = \text{codim}[\mathcal{B}]_{N-1} + \text{codim}[\mathcal{B}]^N
\]
(we recall that the codimension is taken with respect to the relevant ambient space; for example, the ambient space for $[\mathcal{B}]^N$ is the fibre $\mathcal{P}_{[2,3],N}$ of the projection $\pi_2$).

Put $\gamma_N = \gamma_N(B) = \text{codim}[\mathcal{B}]^N$. Since $\text{codim} B \leq a$, we get the following estimate (see (2.3)):
\[
\text{codim}[\mathcal{B}]_{N-1} \leq a - b_1(n_2 + b_1) - b_2b - \gamma_N.
\]
In particular, it follows that
\[
0 \leq \gamma_N \leq a - b_1(n_2 + b_1) - b_2b.
\]

Let $h(z_*) \in \mathcal{P}_{1,N}$ be a non-zero linear form and $H = \{ h = 0 \} \subset \mathbb{C}^N$ the corresponding hyperplane. We write $\pi_h$ for the projection of all spaces $\mathcal{P}_{[i,j],N}$ to $\mathcal{P}_{[i,j],N-1}$ and for the corresponding projections of direct product of these spaces, where we assume that an isomorphism $H \cong \mathbb{C}^{N-1}$ is fixed and $\pi_h : f \mapsto f|_H$ is the restriction of a polynomial to $H$.

Let $B_h \subset \mathcal{P}_{N-1}^{(n_1,n_2-1)}$ be the smallest bi-invariant (in the sense of the latter space, that is, invariant under the actions of $\text{GL}_{N-1}(\mathbb{C})$ and $G(n_1, n_2 - 1)$) closed set containing $\pi_h([\mathcal{B}]_{N-1})$. Clearly, $B_h$ is a bi-invariant irreducible subvariety.

For every generic tuple $(f_1, \ldots, f_{N-1}) \in [\mathcal{B}]_{N-1}$ we consider the following tuple of linear forms:
\[
df_1(o)|_H, \ldots, df_{n_1-b_1}(o)|_H, df_{n_1+b_1+1}(o)|_H, \ldots, df_{N-b_2}(o)|_H.
\]
The rank of this tuple is either $N - b$ or $N - b - 1$. More precisely, there is an integer-valued vector
\[
\alpha = (\alpha_1, \alpha_2) \in \{(0, 1), (-1, 1), (0, 0)\}
\]
such that the parameters $(b_1^1, b_2^1)$ of the variety $B_h$ satisfy
\[
b_1^1 = b_1 - \alpha_1, \quad b_2^1 = b_2 - \alpha_2.
\]
Let $\tau(B_h) = ((a_1^h, b_1^h), (a_2^h, b_2^h))$ be the type of $B_h$. We also put $b^h = b_1^h + b_2^h$ and $a^h = a_1^h + a_2^h$. Clearly, $a^h$ is the codimension of $B_h$ in the ambient space $\mathcal{P}_{N-1}^{(n_1,n_2-1)}$.

To realize the inductive procedure that will be described in §3.3, we need another (besides $B_h$) family of subvarieties of $\mathcal{P}_{N-1}^{(n_1,n_2-1)}$, which will now be defined again for $i = 2$ (with an obvious modification in the case $i = 1$). Take an arbitrary polynomial $f \in \lambda_2(\mathcal{B}) \subset \mathcal{P}_{[2,3],N}$ and consider the closed set
\[
\mathcal{B}(f) = \lambda_2^{-1}(f) \cap \mathcal{B} \subset \mathcal{P}_{[1,2],N}^{n_1-b_1} \times \mathcal{P}_{[2,3],N}^{n_2-b_2} \times \mathcal{P}_{[2,3],N}^{b_2-1}.
\]
The product on the right-hand side may be regarded as a natural ambient space for $\mathcal{B}(f)$. In particular, the codimension of $\mathcal{B}(f)$ will be understood in the sense of this space. By construction, $\mathcal{B}(f) \neq \emptyset$ and
\[
\text{codim} \mathcal{B} \geq \text{codim} \mathcal{B}(f) + \text{codim} \lambda_2(\mathcal{B}).
\]
whence we get the estimate
\[ \text{codim } B(f) \leq a - b_1(n_2 + b_1) - 2b_2 - \gamma(f) \] (3.2)
for some \( \gamma(f) \in \mathbb{Z}_+ \), \( \gamma(f) \geq \text{codim } \lambda_2(B) \). We again take a non-zero linear form \( h(z_*) \in P_{1,N} \), denote the corresponding hyperplane by \( H = \{ h = 0 \} \), fix an isomorphism \( H \cong \mathbb{C}^{N-1} \) and define \( \pi_h \) as above.

Let \( B_h(f) \subset P_{n_1-n_2-1}^{(n_1,n_2-1)} \) be the smallest bi-invariant (in the sense of the latter space, that is, invariant under the actions of \( \text{GL}_{N-1}(\mathbb{C}) \) and \( G(n_1,n_2-1) \)) closed set containing \( \pi_h(B(f)) \). Clearly, \( B_h(f) \subset B_h \) (since \( B(f) \subset \overline{B}_{N-1} \)) is a bi-invariant closed subset, which may be assumed without loss of generality to be an irreducible subvariety. We postpone our discussion of invariants of the subvarieties \( B_h(f) \) until §3.3, where polynomials \( f \) of a special form will be used. We now study the subvariety \( B_h \) in more detail.

### 3.2. Estimating the codimension of \( B_h \).

The following proposition describes how the type of \( B \) changes under restriction to a hyperplane.

**Proposition 3.1.** (i) In the case \( \alpha = (0,1) \) we have

\[ a_1^h \leq a_1 - b_1, \quad a^h \leq a - (2b - 1) - \gamma_N. \]

(ii) In the case \( \alpha = (-1,1) \) we have

\[ a_1^h \leq a_1, \quad a^h \leq a - b - \gamma_N. \]

(iii) In the case \( \alpha = (0,0) \) we have

\[ a_1^h \leq a_1 - b_1, \quad a^h \leq a - b_1 - b - \gamma_N. \]

**Proof.** We study cases (i) and (ii) in full detail and only give calculations in case (iii).

Suppose that \( \alpha = (0,1) \). This means that the forms \( df_i(o), i \in \{ 1, \ldots, n_1 - b_1, n_1 + 1, \ldots, N - b_2 \} \), remain linearly independent after restriction to \( H \). We have

\[ \text{codim } \pi_h([B]_{N-1}) \leq \text{codim } [B]_{N-1} \leq a - b_1(n_2 + b_1) - 2b_2 - \gamma_N. \]

The codimension on the left-hand side is taken with respect to the ambient space

\[ P_{1,2,3,4}^{n_1-b_1} \times P_{2,3,4}^{b_1} \times P_{1,2,3,4}^{n_2-b_2} \times P_{2,3,4}^{b_2-1}, \] (3.3)

and the linear forms \( dg_i(o), i \in \{ 1, \ldots, n_1 - b_1, n_1 + 1, \ldots, N - b_2 \} \), are linearly independent for a generic tuple \( (g_1, \ldots, g_{N-1}) \in \pi_h([B]_{N-1}) \). We now invert the procedure of reduction to normal form: being bi-invariant, the set \( B_h \) certainly contains all tuples of the form \( (g_1^+, \ldots, g_{N-1}^+) \), where \( g_i^+ = g_i \) for \( 1 \leq i \leq n_1 = b_1 \),

\[ g_i^+ = g_i + \sum_{j=1}^{n_1-b_1} \lambda_{ij} g_j, \quad i = n_1 - b_1 + 1, \ldots, n_1, \]
where the coefficients $\lambda_{ij}$ are arbitrary. Furthermore, $g_{i}^{+} = g_{i}$ for $n_{1} + 1 \leq i \leq N - b_{2}$ and
\[ g_{i}^{+} = g_{i} + \sum_{j=1}^{n_{1} - b_{1}} \lambda_{ij}g_{j} + \sum_{j=n_{1}+1}^{N - b_{2}} \lambda_{ij}g_{j}, \quad i = N - b_{2} + 1, \ldots, N - 1, \]
where the coefficients $\lambda_{ij}$ are arbitrary. Hence we have
\[ \dim B_{h} \geq \dim \pi_{h}([B]_{N-1}) + b_{1}(n_{1} - b_{1}) + (b_{2} - 1)(N - b). \]

On the other hand, the dimension of $\mathcal{P}_{N-1}^{(n_{1}, n_{2}-1)}$ exceeds the dimension of (3.3) by $(b - 1)(N - 1)$. Thus we get
\[ a^{h} \leq a - b_{1}(n_{2} + b_{1}) - 2b_{2}b - \gamma N - b_{1}(n_{1} - b_{1}) - (b_{2} - 1)(N - b) + (b - 1)(N - 1) \]
\[ = a - (2b - 1) - \gamma N, \]
as required. The estimate for $a_{1}^{h}$ is obtained by similar (but simpler) arguments. Since the procedure of reduction to standard form is well defined on the quadratic components, we get the following estimate for the projection $pr_{1}([B]_{N-1})$:
\[ \operatorname{codim} pr_{1}([B]_{N-1}) \leq a_{1} - b_{1}(n_{2} + b_{1}). \]

Note that the ambient space of this closed set is $\mathcal{P}_{[1,2],N}^{n_{1} - b_{1}} \times \mathcal{P}_{2,N}^{b_{1}}$. Restricting to $H$ and again inverting the procedure of reduction to standard form, we get the desired estimate,
\[ a^{h} \leq a_{1} - b_{1}(n_{2} + b_{1}) - b_{1}(n_{1} - b_{1}) + b_{1}(N - 1) = a_{1} - b_{1}. \]

In case (ii) the rank of the system $(df_{*}(o))$ of linear forms for a generic tuple $(f_{1}, \ldots, f_{N-1}) \in [B]_{N-1}$ decreases by unity after restriction to $H$, and the forms
\[ df_{1}(o)|_{H}, \ldots, df_{n_{1} - b_{1}}(o)|_{H} \]
are already linearly independent. Since $[B]_{N-1}$ is $G(n_{1}, b_{1}; n_{2}, b_{2})$-invariant, we may assume that for a generic tuple $(g_{1}, \ldots, g_{N-1}) \in \pi_{h}([B]_{N-1})$ the first $n_{1} - b_{1} - 1$ forms $dg_{i}(o)$, $i = 1, \ldots, n_{1} - b_{1} - 1$, are linearly independent, $dg_{n_{1} - b_{1}}(o)$ is a linear combination of them and for all $\lambda_{1}, \ldots, \lambda_{n_{1} - b_{1} - 1}$ we have
\[ \left(g_{1}, \ldots, g_{n_{1} - b_{1} - 1}, g_{n_{1} - b_{1}} + \sum_{i=1}^{n_{1} - b_{1} - 1} \lambda_{i}g_{i}, g_{n_{1} - b_{1} + 1}, \ldots, g_{N-1}\right) \in \pi_{h}([B]_{N-1}). \]

Hence one can subject the Zariski-open subset
\[ \operatorname{rk}(dg_{1}(o), \ldots, dg_{n_{1} - b_{1}}(o)) = n_{1} - b_{1} - 1 \]
to the procedure of reduction to standard form in the component $g_{n_{1} - b_{1}}$. Taking the closure, we obtain an irreducible subset
\[ \pi_{h}([B]_{N-1}) \subset \mathcal{P}_{[1,2],N-1}^{n_{1} - b_{1} - 1} \times \mathcal{P}_{2,N-1}^{b_{1} + 1} \times \mathcal{P}_{[1,3],N-1}^{n_{2} - b_{2}} \times \mathcal{P}_{[2,3],N-1}^{b_{2} - 1}. \]
of codimension
\[ \text{codim } \pi_h([B]_{N-1}) = N + n_1 - b_1. \]

We can now invert the procedure of reduction to standard form, as was done in case (i). This yields the following estimate for the codimension of \( B_h \):

\[
\text{codim } B_h \leq a - b_1(n_2 + b_1) - b_2 b - \gamma_N - N + n_1 - b_1 + b(N - 1) - (b_1 + 1)(n_1 - b_1 - 1) - (b_2 - 1)(N - b - 1) = a - b - \gamma_N,
\]
as required. The proof for \( a_h^1 \) is similar but simpler since it deals with quadratic polynomials only. We again reduce a generic tuple \((g_1, \ldots, g_{n_1})\) to standard form in the component \( g_{n_1 - b_1} \) and then invert the procedure of reduction to standard form. This results in the inequality

\[
a_h^1 \leq a_1 - b_1(n_2 + b_1) - N + n_1 - b_1 - (b_1 + 1)(n_1 - b_1 - 1) + (b_1 + 1)(N - 1) = a_1,
\]
as required (recall that \( N = n_1 + n_2 \)).

We finally consider case (iii). As in case (ii), one must complete the reduction of the set \( \pi_h([B]_{N-1}) \) to standard form, now in the component \( g_{N - b_2} \) since for a generic tuple \((g_1, \ldots, g_{N-1})\) the linear forms

\[ dg_1(o), \ldots, dg_{n_1 - b_1}(o), dg_{n_1 + 1}(o), \ldots, dg_{N - b_2 - 1}(o) \]

are linearly independent and \( dg_{N - b_2}(o) \) is a linear combination of them. Then we invert the procedure of reduction to standard form. We get

\[
a^h \leq a - b_1(n_2 + b_1) - b_2 b - \gamma_N - (N - 1) + (N - b - 1) - b_1(n_1 - b_1) - b_2(N - b - 1) + b(N - 1),
\]

where the second line corresponds to the reduction to standard form and the third to the inverse procedure. Simplifying, we get the estimate

\[
a^h \leq a - b_1 - b - \gamma_N,
\]
as required. For codimension \( a_h^1 \) in the quadratic components, this case is very easy since there is no need to reduce to standard form. Inverting the reduction to standard form, we obtain

\[
a_1^h \leq a_1 - b_1(n_2 + b_1) - b_1(n_1 - b_1) + b_1(N - 1) = a_1 - b_1,
\]
as required. □

We now consider the problem of estimating the codimension in the process of splitting off a quadratic factor. Here \((b_1^h, b_2^h) = (b_1, b_2) - (\alpha_1, \alpha_2)\), where \( \alpha = (\alpha_1, \alpha_2) \) can take the values \((1, 0), (1, -1)\) and \((0, 0)\). The inequalities for \( a^h \) and \( a_1^h \) are obtained by verbatim repetition of the argument used when a cubic factor was split off. Here is the final result.
Proposition 3.2. (i) In the case \( \alpha = (1, 0) \) we have
\[
a_1^h \leq a_1 - n_2 - 2b_1 + 1 - \gamma_N, \quad a^h \leq a - n_2 - 2b_1 - b_2 + 1 - \gamma_N.
\]
(ii) In the case \( \alpha = (1, -1) \) we have
\[
a_1^h \leq a_1 - n_2 - 2b_1 + 1 - \gamma_N, \quad a^h \leq a - n_2 - 2b_1 + 1 - \gamma_N.
\]
(iii) In the case \( \alpha = (0, 0) \) we have
\[
a_1^h \leq a_1 - n_2 - b_1 - \gamma_N, \quad a^h \leq a - n_2 - b_1 - \gamma_N.
\]

To conclude, we recall that in the process of splitting off a quadratic factor, we regard \( \mathcal{P}_N^{(n_1-1,n_2)} \) as the ambient space for \( B_h \), and \( a^h \) is the codimension of \( B_h \) in this space.

3.3. The main inductive estimate. We return to the main problem of estimating the multiplicities \( \mu_{l,N}(B,d) \) for an irreducible bi-invariant subvariety \( B \subset \mathcal{P}_N^{(n_1,n_2)} \) of codimension \( a \geq 1 \). Let \( \tau(B) = ((a_1,b_1), (a_2,b_2)) \) be the type of \( B \), where \( b = b_1 + b_2 \geq 1 \) (otherwise there is nothing to estimate; see Proposition 2.2). The following key fact enables us to estimate the multiplicity \( \mu_{l,N} \) from above in terms of the corresponding multiplicities for \( \mathbb{P}^{N-1} \). This gives us a procedure for estimating the multiplicities by induction on \( N \).

Proposition 3.3. (i) Assume that \( b_2 \geq 1 \). Then there are non-zero linear forms \( h_1, h_2 \in \mathcal{P}_{1,N} \) (depending only on \( B \)) and a set of non-negative integers
\[
\begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix} \in \text{Mat}_{2,2}(\mathbb{Z}_+)
\]
such that \( d_{11} + d_{12} = d_{21} + d_{22} = d \) and the subvarieties
\[
B_1 = B_{h_1} \subset \mathcal{P}_N^{(n_1,n_2-1)}, \quad B_2 = B_{h_2}(h_1 h_2) \subset \mathcal{P}_N^{(n_1,n_2-1)}
\]
satisfy the inequality
\[
\mu_{l,N}(B,d) \leq \mu_{l,N-1}(B_1,d_{11}) + \mu_{l-1,N-1}(B_1,d_{12}) + \mu_{l,N-1}(B_2,d_{21}) + \mu_{l-1,N-1}(B_2,d_{22}). \tag{3.4}
\]
Moreover, the subvariety \( B_1 \) is of type
\[
\tau(B_1) = ((a_{11},b_{11}), (a_{12},b_{12})), \quad (b_{11},b_{12}) = (b_1,b_2 - 1).
\]

(ii) Assume that \( b_1 \geq 1 \). Then there are non-zero linear forms \( h_1, h_2 \) (depending only on \( B \)) and a set \( (d_{ij})_{1 \leq i,j \leq 2} \) of non-negative integers such that \( d_{11} + d_{12} = d_{21} + d_{22} = d \) and the subvarieties
\[
B_1 = B_{h_1} \subset \mathcal{P}_N^{(n_1-1,n_2)}, \quad B_2 = B_{h_2}(h_1 h_2) \subset \mathcal{P}_N^{(n_1-1,n_2)}
\]
satisfy inequality (3.4). Moreover, the subvariety \( B_1 \) is of type
\[
\tau(B_1) = ((a_{11},b_{11}), (a_{12},b_{12})), \quad (b_{11},b_{12}) = (b_1 - 1,b_2).
Proof. We consider case (i) in full detail. Case (ii) is treated in a similar way. Fix a bi-invariant irreducible subvariety $B$. Let $h_1 \in P_{1,N}$ be a linear form of general position in the following sense. For a generic tuple $(f_1, \ldots, f_N) \in B$, the vector subspace
\[ \{ df_1(o) = \cdots = df_N(o) = 0 \} \]
(of dimension $b = b_1 + b_2$) is not contained in the hyperplane $\{ h_1 = 0 \}$. Let
\[ \Pi = \{ h_1(z_*)h(z_*) \mid h \in P_{1,N} \} \subset P_{2,N} \]
be the linear space of reducible homogeneous quadratic polynomials divisible by $h_1$. Since $P_{2,N} \subset P_{[2,3],N}$, we may regard $\Pi$ as a linear subspace of $P_{[2,3],N}$. Clearly, $\dim \Pi = N$. We put
\[ P_{\Pi} = P_{[1,2],N}^{n_1 - b_1} \times P_{2,N}^{b_1} \times P_{[1,3],N}^{n_2 - b_2} \times P_{[2,3],N}^{b_2 - 1} \times \Pi. \]
As explained above, $P_{\Pi}$ is a closed irreducible subset of
\[ P_{[1,2],N}^{n_1 - b_1} \times P_{2,N}^{b_1} \times P_{[1,3],N}^{n_2 - b_2} \times P_{[2,3],N}^{b_2}. \]

**Lemma 3.1.** (i) The intersection $\overline{B} \cap P_{\Pi}$ is non-empty and its codimension in $P_{\Pi}$ is not greater than $\text{codim} \overline{B}$.

(ii) The closure $\overline{\pi_2(\overline{B} \cap P_{\Pi})}$ coincides with $[\overline{B}]_{N-1}$.

**Proof.** Let us prove (i). Since the closed set $\overline{B}$ is bi-invariant, it contains the zero tuple $(0, \ldots, 0) \in P_N^{(n_1, n_2)}$. Therefore $\overline{B} \cap \mathcal{P} \neq \emptyset$. The rest is obvious.

To prove (ii), recall that $\pi_2$ is the projection onto the last direct factor $P_{[2,3],N}$. By (2.3), the codimension of $\overline{B}$ is strictly less than $N$. Hence, for a tuple $(f_1, \ldots, f_{N-1}) \in [\overline{B}]_{N-1}$ of general position, the intersection of the fibre $[\overline{B}]_{N}^{N}(f_1, \ldots, f_{N-1})$ with the subspace $\Pi$ in $P_{[2,3],N}$ is non-empty and, therefore, has positive dimension. (It is non-empty also by bi-invariance: $(f_1, \ldots, f_{N-1}, 0) \in \overline{B}$.) This proves (ii). □

**Remark 3.1.** There is no loss of generality in assuming that the irreducible subvariety $B$ is an irreducible component of the closed set $X_{l,N}(m, d)$.

The equation $\mu_{l,N}(B, d) = m$ means that $\mu_R(f_1, \ldots, f_N) \leq m$ for all generic tuples $(f_1, \ldots, f_N) \in \overline{B}$ and any effective cycle $R$ of codimension $l$ and degree $d$ and, moreover, equality holds for some cycle $R$ (depending on the tuple $(f_*)$). Hence, for a generic tuple
\[ (f_1, \ldots, f_{N-1}, h_1(z_*), h_2(z_*)) \in \overline{B} \cap P_{\Pi} \]
there is an effective cycle $R$ (depending on the tuple) such that
\[ \mu_R(f_1, \ldots, f_{N-1}, h_1h_2) \geq m. \]
As explained above, the form $h_2(z_*)$ is non-zero. We denote the corresponding hyperplanes by $H_i = \{ h_i = 0 \}, i = 1, 2$, and define effective cycles $R_{ij}, i, j = 1, 2$, of codimension $l$ by the following conditions:
3.2 Let we have and, for every 

\[ m \leq \mu_R(f_1, \ldots, f_{N-1}, h_1) + \mu_R(f_1, \ldots, f_{N-1}, h_2) \]  

(3.5) 

and, for every \( i \in \{1, 2\} \), 

\[ \mu_R(f_1, \ldots, f_{N-1}, h_i) \leq \mu_{R_1}(f_1, \ldots, f_{N-1}, h_i) + \mu_{R_2}(f_1, \ldots, f_{N-1}, h_i). \]  

(3.6) 

Since the irreducible components of \( R_{i1} \) are not contained in \( H_i \) and \( \text{Supp } R_{i2} \subset H_i \), we get the following estimates for the first and second multiplicities on the right-hand side of (3.6):

\[ \mu_{R_1}(f_1, \ldots, f_{N-1}, h_i) \leq \mu_{(R_1 \circ H_i)}(f_1|_{H_i}, \ldots, f_{N-1}|_{H_i}), \]  

(3.7) 

\[ \mu_{R_2}(f_1, \ldots, f_{N-1}, h_i) \leq \mu_{R_2}(f_1|_{H_i}, \ldots, f_{N-1}|_{H_i}). \]  

(3.8) 

Note that \( (R_1 \circ H_i) \) is an effective cycle of codimension \( l \) and degree \( d_{i1} \) on \( H_i \cong \mathbb{P}^{N-1} \), and \( R_{i2} \) is an effective cycle of codimension \( l-1 \) on \( H_i \cong \mathbb{P}^{N-1} \). Furthermore, the tuples 

\[(f_1|_{H_1}, \ldots, f_{N-1}|_{H_1}) \in \pi_{h_1}([B]_{N-1}), \quad (f_1|_{H_2}, \ldots, f_{N-1}|_{H_2}) \in \pi_{h_2}(B(h_1h_2))\] 

are generic and, therefore, inequalities (3.7) and (3.8) remain valid if we replace these tuples on their right-hand sides by any generic tuple of polynomials \((g_1, \ldots, g_{N-1}) \in B_i\). Inequality (3.4) now follows directly from (3.5)–(3.8).

Since \( h_1 \) is generic, we finally have 

\[ \text{rk}(dg_1(o), \ldots, dg_{N-1}(o)) = N - b, \] 

as required. \( \square \)

To complete the description of our inductive procedure, it remains to estimate the codimensions of the subvarieties \( B_2 \) and their type. We again give full details in the case of the projection onto the last factor (that is, \((f_s) \mapsto f_N\)), which corresponds to part (i) of Proposition 3.3. Obvious modifications are needed in the case of the projection onto the last quadratic factor (that is, \((f_s) \mapsto f_{n_1}\)), and we give only the final result. Put \( \gamma = \gamma(h_1h_2) \) (see §3.1).

**Proposition 3.4.** Let \( h_2 \in \mathcal{P}_{1,N} \) be a generic linear form such that \( h_1h_2 \in \lambda_2(B \cap \mathcal{P}_\Pi) \). Then there is an integer-valued vector 

\[ \alpha = (\alpha_1, \alpha_2) \in \{(0, 1), (-1, 1), (0, 0)\} \] 

such that the parameters \((\tilde{b}_1, \tilde{b}_2)\) of the subvariety \( B_{h_2}(h_1h_2) \) satisfy 

\[ \tilde{b}_1 = b_1 - \alpha_1, \quad \tilde{b}_2 = b_2 - \alpha_2. \]
The codimension $\bar{a}$ of the subvariety $B_{h_2}(h_1h_2)$ in the ambient space $\mathcal{P}_{N-1}^{(n_1,n_2-1)}$ satisfies the following estimates:

(i) $\bar{a} \leq a - (2b - 1) - \gamma$ in the case $\alpha = (0,1)$;
(ii) $\bar{a} \leq a - b - \gamma$ in the case $\alpha = (-1,1)$;
(iii) $\bar{a} \leq a - b_1 - b - \gamma$ in the case $\alpha = (0,0)$.

Proof. We repeat the proof of Proposition 3.1 almost verbatim. Although the set $\overline{B}(h_1h_2)$ may not be invariant under linear changes of coordinates, it is still invariant under the operation of taking linear combinations, in the same way as $[\overline{B}]_{N-1}$. By Lemma 3.1 we have

$$\bigcup_{h_1h_2 \in \lambda_2(\overline{B} \cap \mathcal{P}_\Pi)} B_{h_2}(h_1h_2) = [\overline{B}]_{N-1},$$

whence the linear forms

$$dg_1(o), \ldots, dg_{n_1-b_1}(o), dg_{n_1+1}(o), \ldots, dg_{N-b_2}(o)$$

are linearly independent for a generic tuple $(g_1, \ldots, g_{N-1}) \in \overline{B}(h_1h_2)$. Therefore the proof of Proposition 3.1 can be repeated using inequality (3.2) (with some modifications since nothing is claimed about the parameter $\bar{a}_1$ of the full type of $B_{h_2}(h_1h_2)$). □

We now state the result in the case when a quadratic factor is split off.

Proposition 3.5. Let $h_2 \in \mathcal{P}_{1,N}$ be a generic linear form such that $h_1h_2 \in \lambda_1(\overline{B} \cap \mathcal{P}_\Pi)$. Then there is an integer-valued vector

$$\alpha = (\alpha_1, \alpha_2) \in \{(1,0), (1,-1), (0,0)\}$$

such that the parameters $(\bar{b}_1, \bar{b}_2)$ of the subvariety $B_{h_2}(h_1h_2)$ satisfy

$$\bar{b}_1 = b_1 - \alpha_1, \quad \bar{b}_2 = b_2 - \alpha_2.$$

The codimension $\bar{a}$ of the subvariety $B_{h_2}(h_1h_2)$ in the ambient space $\mathcal{P}_{N-1}^{(n_1-1,n_2)}$ satisfies the following estimates:

(i) $\bar{a} \leq a - n_2 - 2b_1 - b_2 + 1 - \gamma$ in the case $\alpha = (1,0)$;
(ii) $\bar{a} \leq a - n_2 - 2b_1 + 1 - \gamma$ in the case $\alpha = (1,-1)$;
(iii) $\bar{a} \leq a - n_2 - b_1 - \gamma$ in the case $\alpha = (0,0)$.

Proof. This almost coincides with the proof of Proposition 3.2 and follows the same procedure as the proof of Proposition 3.1. □

§ 4. Local multiplicities. III. Explicit estimates

In this section we use the inductive procedure of §3 to obtain explicit estimates for the local multiplicity. We consider the cases of small values $b = 1, 2$ and small codimensions $a \leq 36$ separately. We prove Theorems 2.1 and 2.2.
4.1. Estimating the multiplicity in the case $b = b_1 + b_2 = 1$. We saw above that $\mu_{l, N}(B, d) = d$ for irreducible subvarieties $B \subseteq P_{N}^{(n_1, n_2)}$ with $b = 0$ (Proposition 2.2). We now consider the case $b = 1$, which is the next in complexity. To be definite, assume that $b_2 = 1$, $b_1 = 0$. By Proposition 3.3 we have

$$\mu_{l, N}(B, d) \leq d + \mu_{l, N-1}(B_2, d_{21}) + \mu_{l-1, N-1}(B_2, d_{22})$$

since $(b_{11}, b_{12}) = (0, 0)$. Let $\tau(B_2) = ((a_{21}, b_{21}), (a_{22}, b_{22}))$ be the type of $B_2$. If $b_{21} = b_{22} = 0$, then we get the estimate $\mu_{l, N}(B, d) \leq 2d$. Assume that $b_{21} + b_{22} = 1$. Putting $B_2 = B^{(1)}$, we apply Proposition 3.3 to $B_2$. Iterating this construction, we get a chain of subvarieties $B^{(1)}, \ldots, B^{(k)}$ of type $\tau(B^{(i)}) = ((a^{(i)}_1, b^{(i)}_1), (a^{(i)}_2, b^{(i)}_2))$ with $b^{(i)}_1 = b^{(i)}_2 = 1$. Here $B^{(i+1)} = B^{(i)}_2$ in the sense of Proposition 3.3. The varieties $B^{(i)}$ are irreducible subvarieties of $P_{N}^{(n_1^{(i)}, n_2^{(i)})}$, the corresponding subvarieties $B^{(i)}_1$ are of type $((a^{(i)}_{11}, 0), (a^{(i)}_{12}, 0))$ and their contribution to the estimate of $\mu_{l, N}$ is known. After $k$ steps we get the inequality

$$\mu_{l, N}(B, d) \leq kd + \sum_{j=0}^{\min\{k, l\}} \mu_{l-j, N-k}(B^{(k)}, d_{k,j}),$$

where $d_{k,0} + \cdots + d_{k,\min\{k, l\}} = d$. We have used the obvious inequality

$$\mu_{l, N}(B, d') + \mu_{l, N}(B, d'') \leq \mu_{l, N}(B, d' + d''). \quad (4.1)$$

By Propositions 3.1 and 3.2, the codimensions $a^{(i)} = \text{codim} B^{(i)}$ satisfy $a^{(i+1)} \leq a^{(i)} - 1$, and a necessary condition for the existence of $B^{(i)}$ is given by the inequality

$$a^{(i)} - b^{(i)}_1(k^{(i)}_2 + b^{(i)}_1) - b^{(i)}_2b^{(i)} \geq 0.$$

Therefore $a \geq a^{(k)} + k \geq k + 1$, whence after $k \leq a - 1$ steps we have $b_{21}^{(k)} = b_{22}^{(k)} = 0$ and the procedure terminates. As a result, we get

$$\mu_{l, N}(B, d) \leq (a + 1)d. \quad (4.2)$$

Note that the estimate (4.2) is exact for $l = 0$ (when all effective cycles are of the form $dP_{\mathbb{P}^N}$): the equalities $b_1 = 0$, $b_2 = 1$ mean that, for a generic tuple $(f_1, \ldots, f_N) \in B$, the complete intersection

$$\{f_1 = \cdots = f_{N-1} = 0\}$$

is a curve non-singular at $o$. The condition of being tangent of order $j \leq N$ to this curve imposes at most $j$ independent conditions on the polynomial $f_N$. Therefore we have

$$\mu_{0, N}(a, d) = (a + 1)d.$$
4.2. Estimating the multiplicity in the case $b = 2$. As in §2.2, we write $\mu_{l,N}(a,b;d) = m$ if the inequality $\mu_{l,N}(B,d) \leq m$ holds for all irreducible (bi-invariant) subvarieties $B \subset \mathcal{P}^{n_1,n_2}_N$ of codimension at most $a$ with $\varepsilon(B) = b$ and becomes an equation for at least one such subvariety $B$. The result of §4.1 says that

$$\mu_{l,N}(a,1;d) \leq (a + 1)d.$$ 

We now obtain an upper bound for $\mu_{l,N}(a,2;d)$. Let $B$ be a subvariety with $\varepsilon(B) = 2$. Applying the result of §4.1, we get

$$\mu_{l,N}(B, d) \leq (a - 2)d + \mu_{l,N-1}(B_2, d_{21}) + \mu_{l-1,N-1}(B_2, d_{22})$$

for some $d_{21}, d_{22} \in \mathbb{Z}_+$ with $d_{21} + d_{22} = d$. There are two options for the parameters of $B_2$:

1) codim $B_2 \leq a - 2$ and $\varepsilon(B_2) = 2$;
2) codim $B_2 \leq a - 3$ and $\varepsilon(B_2) = 1$.

In case 2) we get the estimate

$$\mu_{l,N}(B, d) \leq 2(a - 2)d.$$ 

In case 1), the process of reduction can be continued by applying Proposition 3.3 to $B_2 = B^{(1)}$. Arguing as in §4.1, we get the following final result (we omit the calculations, which are elementary and similar to those in §4.1). For even $a = 2u$ we have

$$\mu_{l,N}(a,2;d) \leq (2 + u(u - 1))d,$$

and for odd $a = 2u + 1$ we have

$$\mu_{l,N}(a,2;d) \leq (2 + u^2)d.$$ 

This procedure of obtaining explicit upper bounds for the numbers $\mu_{l,N}(a,b;d)$ can be continued by reducing the estimate for $b = 3$ to the already-known formulae for $b = 1, 2$. However, we have already seen in the case $b = 2$ that the number of cases requiring separate study starts to grow and the formulae get clumsier. Thus the numerical value of the upper bound for $\mu_{l,N}(a,b;d)$ is easily obtained for small codimensions, but there is a need for less precise but more manageable estimates for large values of $a$.

4.3. Small codimensions. We observe that our estimates for $b = 1$ and $b = 2$ are linear in the degree $d$ and actually independent of $l, N$. Let $U \subset \mathbb{Z}_+ \times \mathbb{Z}_+$ be the set $\{(a,b) \mid a \geq b^2\}$. We define a function

$$\overline{\mu} : U \to \mathbb{Z}_+$$

by induction: put $\overline{\mu}(a,0) \equiv 1$, $\overline{\mu}(a,1) \equiv a + 1$, and

$$\overline{\mu}(a,b) = 2\overline{\mu}(a - (2b - 1), b - 1)$$

for $a < b(b + 1)$, while

$$\overline{\mu}(a,b) = \overline{\mu}(a - (2b - 1), b - 1) + \max\{\overline{\mu}(a - (2b - 1), b - 1), \overline{\mu}(a - b, b)\}$$

for $a \geq b(b + 1)$. Here is a direct corollary of Propositions 3.1–3.3.
Proposition 4.1. We have

\[ \mu_{l,N}(a,b;d) \leq \bar{\mu}(a,b)d. \]

For small values of \( a \), the function \( \bar{\mu} \) can easily be computed by hand. It is also easy to write a computer program for this purpose. The values of \( \bar{\mu}(a,b) \) for \( a \leq 36 \) and \( b \leq 6 \) are shown in Table 1, where * means that \((a,b) \not\in U\) and hence the value of \( \bar{\mu} \) is not defined. The rate of growth of \( \bar{\mu}(a,b) \) can already be clearly seen for these small values of the codimension. The maximal value of \( \bar{\mu}(a,b) \) for a given \( a \) is shown in boldface.

Table 1

| \( a \) | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| \( b = 0 \) | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| \( b = 1 \) | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
| \( b = 2 \) | * | * | * | 4 | 6 | 8 | 11 | 14 | 18 | 22 | 27 | 32 | 38 | 44 | 51 |
| \( b = 3 \) | * | * | * | * | * | * | * | 8 | 12 | 16 | 22 | 28 | 36 | 44 | 51 |
| \( b = 4 \) | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * |

We now consider the problem of obtaining a simple effective upper bound for \( \mu_{l,N}(B,d) \). Technically, we must find a simple and visual formalization of the
procedure for estimating these numbers in terms of \( \mu_{\nu',N'}(B',d') \), where the varieties \( B' \subset \mathcal{P}_{n_1'(w),n_2'(w)}^N \) have a smaller value of \( b' = b_1' + b_2' \) and, therefore, a known upper bound for the multiplicities. This yields a procedure for estimating the multiplicity by induction on \( b \), as described below.

4.4. A general method for estimating the multiplicity. We consider a four-letter alphabet \( \{A, C_0, C_1, C_2\} \) and describe a procedure for constructing a set \( W \) of words in this alphabet and, for every word \( w \in W \), a certain irreducible bi-invariant subvariety

\[ B[w] \subset \mathcal{P}_{\nu(w),n_2(w)}^N \]

of type \( \tau(B[w]) = ((a_1(w), b_1(w)), (a_2(w), b_2(w))) \) and full codimension \( a(w) = a_1(w) + a_2(w) \). As usual, we put \( \varepsilon(B[w]) = b(w) = b_1(w) + b_2(w) \). The length of a word \( w \) is denoted by \( |w| \in \mathbb{Z}_+ \). The length of the empty word is equal to zero.

Let \( B \subset \mathcal{P}_{n_1,n_2}^N \) be an irreducible bi-invariant subvariety of codimension \( a \) and type \( \tau(B) \). We put \( B[\emptyset] = B \). If \( b(\emptyset) = 0 \), then we put \( W = \{\emptyset\} \) and the procedure terminates. Assume that \( b(\emptyset) \geq 1 \).

The set \( W \) and the subvarieties \( B[w], w \in W \), will be constructed in elementary steps: we shall construct a sequence \( W_l, l = 0, 1, \ldots \) of finite sets of words. The set \( W_0 = \{\emptyset\} \) has already been constructed. Assume that \( W_0, \ldots, W_l \) have been constructed. If \( b(w) = 0 \) for all \( w \in W_l \), then we put \( W = W_l \) and the procedure terminates. Otherwise take any word \( w \in W_l \) with \( b(w) \geq 1 \). If \( b_2(w) \geq 1 \), then we apply part (i) of Proposition 3.3 to the subvariety \( B[w] \) (constructed at the previous step) and put \( w_1 = wA \) and \( w_2 = wC_\varepsilon \), where \( \varepsilon \in \{0,1,2\} \) is chosen in the following way.

Put \( B[w_1] = (B[w])_1 \) and \( B[w_2] = (B[w])_2 \) in the notation of Proposition 3.3, (i) and declare \( \varepsilon \) to be 0, 1 or 2 if the subvariety \( B[w_2] \) corresponds respectively to cases (i), (ii) or (iii) of Proposition 3.1. This uniquely determines the words \( w_1 \) and \( w_2 \). The set \( W_{l+1} \) is obtained from \( W_l \) by removing the word \( w \) and adding the words \( w_1, w_2 \) of length \( |w| + 1 \). The irreducible bi-invariant subvarieties \( B[w_i] \) are defined above, and this determines the values of all parameters \( (n_i(w) \text{ and so on}) \).

If \( b_2(w) = 0 \), then \( b_1(w) \geq 1 \) and we apply part (ii) of Proposition 3.3 to the subvariety \( B[w] \). Then the words \( w_1, w_2 \) and the subvarieties \( B[w_i] \) are constructed as in the case \( b_2(w) \geq 1 \), replacing part (i) of Proposition 3.3 by part (ii) and Proposition 3.1 by Proposition 3.2.

This procedure uniquely determines the set \( W_{l+1} \). Clearly,

\[ \#W_l = l + 1. \]

When we replace a word \( w \in W_l \) by the words \( w_1 \) and \( w_2 \), the codimension of the new subvarieties \( B[w_i] \) is strictly smaller than that of \( B[w] \) (Propositions 3.1, 3.2). Therefore our procedure for constructing the sequence \( \{W_l\} \) cannot be infinite. We easily see that \( \#W \leq 2^a \) for codim \( B = a \). Hence Proposition 3.3 yields the following proposition.

**Proposition 4.2.** We have

\[
\mu_{\nu,N}(B,d) \leq d(\#W),
\]

(4.3)
Proof. Inequality (4.3) follows from a more general fact:

\[
\mu_{l,N}(B, d) \leq \sum_{w \in W_i} \sum_{j=0}^{\min\{l,|w|\}} \mu_{l-j,N-|w|}(B[w], d_j(w))
\]

(4.4)

for any \(i = 0, 1, \ldots\) and certain partitions

\[
d = \sum_{j=0}^{\min\{l,|w|\}} d_j(w)
\]

(4.5)

for every word \(w \in W_i\). We prove (4.4) by induction on \(i = 0, 1, \ldots\). If \(i = 0\), then both sides of (4.4) coincide. Assume that (4.4) holds for \(i = 0, 1, \ldots\). If \(W_e = W_{e+1}\), then \(W_e = W\) and there is nothing to prove. If \(W_e \neq W_{e+1}\), then \(W_{e+1}\) is obtained from \(W_e\) by removing some word \(w \in W_e\) and adding two words \(w_1, w_2\). Therefore, to prove (4.4) for \(i = e + 1\), it suffices to show that

\[
\min\{l,|w|\} \sum_{j=0}^{\min\{l,|w|\}} \mu_{l-j,N-|w|}(B[w], d_j[w])
\]

does not exceed the sum of the corresponding expressions for \(w_1, w_2\). This is precisely the conclusion of Proposition 3.3 in view of (4.1). This proves (4.4) for all \(i\).

Finally, if \(W_i = W\), then \(b(w) = 0\) for all \(w \in W\). Hence every term on the right-hand side of (4.4) satisfies the equation

\[
\mu_{l-j,N-|w|}(B[w], d_j(w)) = d_j(w),
\]

and we obtain from (4.5) that

\[
\mu_{l,N}(B, d) \leq d \sum_{w \in W} 1,
\]

as required. \(\square\)

4.5. Estimating the cardinality of the set of words. We write down the words in the following way:

\[
w = \tau_1 \ldots \tau_K,
\]

where \(\tau_i \in \{A, C_0, C_1, C_2\}\). We define a map

\[
\nu: \{A, C_0, C_1, C_2\} \to \{A, C\}
\]

of the four-letter alphabet to the two-letter one by putting \(\nu(A) = A\) and \(\nu(C_i) = C\). Let

\[
\nu: w = \tau_1 \ldots \tau_K \mapsto \overline{w} = \nu(\tau_1) \ldots \nu(\tau_K)
\]

be the corresponding map of the set of words.

Lemma 4.1. The map \(\nu|_{W_i}\) is injective for every \(i = 0, 1, \ldots\). In particular, \(\nu|_W\) is injective.
A stronger assertion holds: none of the words \( \overline{w} = \nu(w), w \in W_i \), is a left segment of any other word in this set. (In particular, no two words coincide, whence \( \nu|_{W_i} \) is injective.) The last assertion is easily proved by induction. The set \( W_0 \) consists of one word, and our assertion is trivial for it. Assume that it holds for \( W_i \), where \( i = 0, \ldots, e \). If \( W_{e+1} = W_e \), there is nothing to prove. If \( W_{e+1} \neq W_e \), then \( W_{e+1} \) is obtained from \( W_e \) by removing some word \( w \in W_e \) and adding two words \( w_1 = wA \) and \( w_2 = wC_{\alpha} \), where \( \alpha \in \{0, 1, 2\} \). For these words we have \( \overline{w}_1 = \overline{w}A \) and \( \overline{w}_2 = \overline{w}C \). Clearly, \( \overline{w}_1 \) and \( \overline{w}_2 \) are not left segments of each other, and none of the words \( \overline{w} \) for \( w' \in W_e \setminus \{w\} \) is a left segment of \( \overline{w}_1 \) or \( \overline{w}_2 \). For otherwise \( \overline{w}' = \overline{w}_1 \) or \( \overline{w}' = \overline{w}_2 \) (since \( \overline{w}' \) is not a left segment of \( \overline{w} \) by the inductive hypothesis), but then \( \overline{w} \) is a left segment of \( \overline{w}' \) contrary to the inductive hypothesis. Clearly, \( \overline{w}_1 \) and \( \overline{w}_2 \) are not left segments of \( \overline{w}' \) since otherwise this would also hold for \( \overline{w} \) contrary to the inductive hypothesis. \( \Box \)

Thus we have reduced the problem of estimating \( \mu_{l, N}(B, d) \) to that of estimating the number of words in the set \( W \). As mentioned above, \( \sharp W \leq 2^a \), but this estimate is too coarse for our purposes. We shall control the lengths of words \( w \in W \) by the values of \( a(w') \) and \( b(w') = \varepsilon(B[w]) \) for left segments \( w' \) of \( w \).

**Lemma 4.2.** (i) If \( \tau = A \) or \( \tau = C_0 \), then

\[
a(w'\tau) \leq a(w') - (2b(w') - 1)
\]

and \( b(w'\tau) = b(w') - 1 \).

(ii) If \( \tau = C_1 \) or \( \tau = C_2 \), then

\[
a(w'\tau) \leq a(w') - b(w')
\]

and \( b(w'\tau) = b(w') \).

**Proof.** This follows from the inequalities in Propositions 3.1 and 3.2 in view of the obvious estimate \( b_i \leq n_i, i = 1, 2 \). \( \Box \)

Furthermore, inequality (3.1) yields the estimate

\[
a(w) \geq b_1(w)(n_2(w) + b_1(w)) + b_2(w)b(w) \geq b^2(w)
\]

for every word \( w \) and, in particular, for every word which is a left segment of a word in \( W \).

**Example 4.1.** Let us reconsider the case \( b = b(\emptyset) = 1 \) using the formalism developed above. Here the following alternative holds for every \( w \in W_i \): either \( b(w) = 0 \) (and then \( w \in W_i \)), or \( b(w) = 1 \) (and then \( a(w\tau) \leq a(w) - 1 \) for every letter \( \tau \)). Hence the set \( W \) is of the form

\[
A, \ C_{i_1}A, \ C_{i_1}C_{i_2}A, \ldots, \ C_{i_1}C_{i_2} \ldots C_{i_k}A, \ C_{i_1} \ldots C_{i_k}C_0,
\]

where \( i_\alpha \in \{1, 2\} \) and \( k + 1 \leq a \). Therefore \( \sharp W \leq a + 1 \), as already stated in § 4.1.

We now return to the general case.
Proposition 4.3. We have

\[ \#W \leq 2^b \frac{a^b}{(b!)^2}. \]

Proof. By construction we have \( b(w) = 0 \) for every word \( w \in W \). Since the letters \( C_1, C_2 \) do not change the value of the parameter \( b \) while the letters \( A \) and \( C_0 \) decrease it by 1, we conclude that precisely \( b \) positions in \( w \) are occupied by the letters \( A \) and \( C_0 \). Denote the numbers of these positions by

\[ m_1 + 1, \ m_1 + m_2 + 2, \ldots, m_1 + m_2 + \cdots + m_b + b, \]

where \( m_i \in \mathbb{Z}_+ \). By Lemma 4.2 we have the inequality

\[
0 \leq a(w) \leq a - m_1 b - (2b - 1) - m_2(b - 1) - (2(b - 1) - 1) - \cdots - m_i(b - (i - 1)) - (2(b - (i - 1)) - 1) - \cdots - m_b - 1 = a - b^2 - \sum_{i=1}^{b} m_i(b - (i - 1)),
\]

whence \((m_1, \ldots, m_b)\) is an arbitrary integer-valued point in the polytope

\[ \Delta = \{ x_1 \geq 0, \ldots, x_b \geq 0, bx_1 + (b - 1)x_2 + \cdots + x_b \leq a - b^2 \} \subset \mathbb{R}^b. \]

Thus, even if we assume that all possible arrangements of the letters \( A \) and \( C_0 \) in the selected positions are realized by words \( w \in W \) (which is actually not the case: the number of words in \( W \) is much smaller; see Remark 4.1), then we get

\[ \#W \leq 2^b \cdot \#(\Delta \cap \mathbb{Z}^b). \]

We now estimate the number of integer-valued points in \( \Delta \). To do this, we consider a larger polytope

\[ \Delta^+ = \left\{ x_1 \geq 0, \ldots, x_b \geq 0, bx_1 + \cdots + x_b \leq a - \frac{b(b - 1)}{2} \right\} \subset \mathbb{R}^b. \]

Clearly, \( \Delta \subset \Delta^+ \).

Lemma 4.3. We have

\[ \#(\Delta \cap \mathbb{Z}^b) \leq \text{vol}(\Delta^+). \quad (4.6) \]

Proof. With each point \( x = (x_1, \ldots, x_b) \in \mathbb{R}^b \) we associate a unit cube

\[ \Gamma(x) = [x_1, x_1 + 1] \times [x_2, x_2 + 1] \times \cdots \times [x_b, x_b + 1] \subset \mathbb{R}^b, \]
whose vertex with minimal value of the sum of coordinates \(x_1 + \cdots + x_b\) is the point \(x\). If \(x \in \Delta\), then \(\Gamma(x) \subset \Delta^+\) because

\[
b + (b - 1) + \cdots + 1 + a - b^2 = a - \frac{b(b - 1)}{2}.
\]

Therefore,

\[
\#(\Delta \cap \mathbb{Z}^b) = \sum_{x \in \Delta \cap \mathbb{Z}^b} \text{vol}(\Gamma(x)) = \text{vol} \left( \bigcup_{x \in \Delta \cap \mathbb{Z}^b} \Gamma(x) \right) \leq \text{vol}(\Delta^+),
\]

as required. \(\square\)

Calculating the volume of the polytope \(\Delta^+\) and using Stirling’s formula, we get the estimate

\[
\#W \leq 2^b \left( \frac{a - \frac{b(b - 1)}{2}}{(b!)^2} \right)^b = \frac{1}{2\pi b e^{\theta/(6b)}} \left( \frac{2a - b(b - 1)}{b^2 e^2} \right)^b
\]

for some \(0 < \theta < 1\) (here \(e\) is the base of the natural logarithms), whence a fortiori

\[
\#W \leq u_b = \frac{1}{2\pi b} \left( \frac{2a - b(b - 1)}{b^2 e^2} \right)^b.
\]

We recall that \(b \in \{1, \ldots, \lfloor \sqrt{a} \rfloor \}\). To obtain an effective bound for \(\#W\), we shall study the behaviour of the sequence \(u_b\) for these values of \(b\).

**Lemma 4.4.** The sequence \(u_b\) is increasing provided that

\[
2a - b(b - 1) \geq \frac{5}{2} b^2.
\]

**Proof.** We write

\[
\frac{u_{b+1}}{u_b} = \frac{1}{1 + \frac{1}{b}} \left( \frac{e^2}{(1 + \frac{1}{b})^{2b}} \right)^b \left( \frac{1}{1 + \frac{2b}{2a - b(b + 1)}} \right)^b \left( \frac{2a - b(b + 1)}{(b + 1)^2} \right)^b.
\]

First assume that \(b \geq 9\). If \(a\) and \(b\) satisfy \(2a - b(b + 1) \geq \frac{5}{2} (b + 1)^2\) (that is, inequality (4.9) for \(b + 1\)), then the denominator of the third factor on the right-hand side of (4.10) has an upper bound:

\[
\left( 1 + \frac{2b}{2a - b(b + 1)} \right)^b \leq \left( 1 + \frac{4}{5} \frac{1}{b} \right)^b < e^{4/5}.
\]

The second factor on the right-hand side of (4.10) is strictly larger than unity, and the fourth is not smaller than \(5/2\). Thus we get

\[
\frac{u_{b+1}}{u_b} > \frac{9}{10} \frac{5}{2} e^{-4/5} > 1,
\]

as required. For the smaller values \(b \leq 8\) one can estimate the second and third factors on the right-hand side of (4.10) more precisely, and elementary calculations using a computer to complete the proof. \(\square\)
Corollary 4.1. For \( a \geq 17 \) the value \( b_{\text{max}} \in \{1, \ldots, [\sqrt{a}]\} \) where the maximum of the sequence \( u_b \) is attained satisfies the inequality
\[
2a - b_{\text{max}}(b_{\text{max}} - 1) \leq \frac{5}{3}a.
\]

Proof. By the previous lemma, \( b_{\text{max}} \) satisfies the inequality
\[
2a - b_{\text{max}}(b_{\text{max}} + 1) \leq \frac{5}{2}b_{\text{max}}^2
\]
(otherwise the next term of the sequence \( u_b \) is larger). The proof is completed by elementary calculations. □

Corollary 4.2. (i) For \( a \geq 17 \) we have
\[
\#W \leq v_b = \frac{1}{2\pi b} \left( \frac{5a}{3b^2 e^2} \right)^b.
\]
(ii) For every \( a \) we have
\[
\#W \leq w_b = \frac{1}{2\pi b} \left( \frac{2a}{b^2 e^2} \right)^b.
\]

Proof. Part (ii) follows immediately from (4.8), and part (i) follows from (4.8) in view of Corollary 4.1. □

Corollary 4.3. (i) For \( a \geq 17 \) we have
\[
\mu_{l,N}(a, d) \leq \frac{e^2}{2\pi [\sqrt{a}]} \left( \frac{5}{3}e^2 \right)^{[\sqrt{a}]}.
\] (4.11)
(ii) For every \( a \) we have
\[
\mu_{l,N}(a, d) \leq \frac{e^2}{2\pi [\sqrt{a}]} (2e^2)^{[\sqrt{a}]}.
\] (4.12)

Proof. The same argument works in both cases, differing only in the parts of Corollary 4.2 that are used.

To prove part (i), we argue as in the proof of Lemma 4.4 and conclude that the sequence \( v_b \) is increasing. Hence its maximum is attained at \( b = [\sqrt{a}] \). Since
\[
a < (b + 1)^2 = b^2 + 2b + 1,
\]
we have
\[
\left( \frac{a}{b^2} \right)^b \leq \left( 1 + \frac{2}{b} \right)^b < e^2,
\]
whence part (i). Part (ii) is proved in the same way. □
It is easy to see that the assertions of Theorems 2.1, 2.2 are contained in the assertions of Corollaries 4.2, 4.3 in view of (4.8).

This completes the proofs of Theorems 2.1, 2.2. □

Remark 4.1. Our proof of Theorems 2.1, 2.2 shows that estimate (4.12) is not optimal and can be substantially improved. For $b \approx \sqrt{a}$ we have $2a - b(b - 1) \approx a$, whence the expression $(2e^2)$ in (4.12) can be replaced by $e^2$. Moreover, in the proof of Proposition 4.3 we took into account all possible tuples of positions $(m_1, \ldots, m_b)$ and all possible arrangements of $A$ and $C_0$ in $b$ positions. However, since no word in the set of words $W = \nu(W)$ in the two-letter alphabet $\{A, C\}$ is a left segment of any other and the map $\nu: W \to \overline{W}$ is one-to-one, we see that not all tuples $(m_1, \ldots, m_b) \in \Delta \cap \mathbb{Z}^b$ are realized for any fixed arrangement of the letters $A$ and $C_0$ in $b$ positions when at least two letters $C_0$ are neighbours: two distinct words $w_1 \neq w_2$, $\{w_1, w_2\} \subset W$, cannot differ only on a segment consisting of the letters $C_0, C_1, C_2$. Getting a precise (at least in the asymptotic sense) upper bound for $\mu_l,N(a, d)$ remains an open problem.

§ 5. Global multiplicities. Proof of Theorem 0.2

In this section we prove Theorem 0.2 using the theory developed in §§ 2–4. According to Theorem 2 in [3], two facts require a proof: the linear independence of the directions of the lines passing through $o \in V$ (the last requirement in condition (R1)), and the validity of condition (R3) at every point $o \in V$ on a Zariski-generic complete intersection $V$ of type $2k_1 \cdot 3k_2$. The linear independence is easy to prove: it suffices to estimate the codimension of the set of tuples of polynomials that have either a positive-dimensional set of solutions or a finite set of linearly dependent solutions. This is done in §5.1.

In §§ 5.2–5.4 we globalize the constructions and results of the local theory in §§ 2–4: define the global multiplicities, reduce the problem of their estimation to the corresponding problems for local multiplicities, and finally obtain the desired estimates for global multiplicities.

In §5.5 we complete the proof of Theorem 0.2.

5.1. Tuples of polynomials with a positive-dimensional set of solutions.

As above, we denote the space of homogeneous polynomials of degree $i$ in $N$ variables by $P_{i,N}$ and identify $P_{i,N+1}$ with $H^0(\mathbb{P}^N; \mathcal{O}_{\mathbb{P}^N}(i))$. Let

$$
\mathcal{H}_{N}^{(n_1,n_2)} = \prod_{i=1}^{n_1} P_{2,N} \times \prod_{i=1}^{n_2} P_{3,N}
$$

be the space of all tuples $(f_1, \ldots, f_{n_1}, f_{n_1+1}, \ldots, f_{n_1+n_2})$, where the first $n_1$ polynomials are quadratic and the last $n_2$ are cubic. (This is a global analogue of the space $P_{N}^{(n_1,n_2)}$ introduced in §2.) We shall consider the space $\mathcal{H}_{N+1}^{(n_1,n_2)}$ for $n_1 + n_2 = N, N + 1$. Let

$$
Z(f_1, \ldots, f_{n_1+n_2}) = Z(f_*)
$$


be the closed subscheme determined by the tuple \(f_1, \ldots, f_{n_1+n_2}\) of homogeneous polynomials, and let \(|Z(f_*)|\) be the closed set \(\{f_1 = \cdots = f_{n_1+n_2} = 0\} \subset \mathbb{P}^N\) of their common zeros. We write

\[
Y_\infty = \{(f_1, \ldots, f_{n_1+n_2}) \mid \dim Z(f_*) \geq 1\} \subset \mathcal{H}_{N+1}^{(n_1, n_2)}
\]

for the closed subset of those tuples whose zeros have ‘incorrect’ dimension (the set \(Z(f_*)\) is generically zero-dimensional for \(n_1+n_2 = N\) and empty for \(n_1+n_2 = N+1\)). We also define \(Y_{\text{line}} \subset Y_\infty\) as the set of tuples \((f_*)\) such that \(Z(f_*)\) contains a line in \(\mathbb{P}^N\). We easily see that

\[
\operatorname{codim} Y_{\text{line}} = n_1 + 2n_2 + 2
\]

for \(n_1 + n_2 = N\) and

\[
\operatorname{codim} Y_{\text{line}} = n_1 + 2n_2 + 4
\]

for \(n_1 + n_2 = N + 1\) (the codimensions are taken in the space \(\mathcal{H}_{N+1}^{(n_1, n_2)}\)). Let \(Y'_\infty = Y_\infty \setminus Y_{\text{line}}\) be the union of all the irreducible components of \(Y_\infty\) different from \(Y_{\text{line}}\).

**Proposition 5.1.** The irreducible closed set \(Y_{\text{line}}\) is a component of maximal dimension of the closed set \(Y_\infty\):

\[
\dim Y_{\text{line}} \geq \dim Y'_\infty.
\]

**Proof.** Clearly, \(Y_{\text{line}}\) is irreducible. We shall prove the proposition by a method developed in \([2]\). First consider the case \(n_1 + n_2 = N\). Let \(Y_i \subset \mathcal{H}_{N+1}^{(n_1, n_2)}\) be the set of tuples \((f_*)\) such that

1) \(\operatorname{codim}\{f_1 = \cdots = f_i = 0\} = i;\)

2) the set \(Z(f_1, \ldots, f_i)\) has a component \(B\) which is not a line in \(\mathbb{P}^N\) for \(i = N-1\) and on which the polynomial \(f_{i+1}\) vanishes identically.

It suffices to prove that

\[
\operatorname{codim} Y_i > n_1 + 2n_2 + 2, \quad i = 1, \ldots, N-1.
\]

Following \([2]\), §3, we represent \(Y_i\) as the union

\[
Y_i = Y_{i,0} \cup Y_{i,1} \cup \cdots \cup Y_{i,i}
\]

of smaller subsets \(Y_{i,e}, e \in \{0, \ldots, i\}\), determined by the condition that the codimension of \(B\) in its linear span \(\langle B \rangle\) is equal to \(e \in \mathbb{Z}_+\). First, we consider the case \(e = 0\). Then \(B \subset \mathbb{P}^N\) is a linear subspace, \(i \leq N - 2\). Since \(f_j|_B \equiv 0\) for \(j = 1, \ldots, i + 1\) and \(d_j = \deg f_j \geq 2\), we have

\[
\operatorname{codim} Y_{i,0} \geq (i+1)\frac{(N-i+1)(N-i+2)}{2} - i(N-i+1) = \frac{N-i+1}{2}(i(N-i)+N-i+2) \geq 3N,
\]

as required (the minimum of the right-hand side is attained at \(i = N-2\), which corresponds to the set of tuples \((f_*)\) such that \(Z(f_*)\) contains a plane).

Hence we may assume that \(e \geq 1\). Then we consider the restrictions \(f_j|_{\langle B \rangle}\), \(j = 1, \ldots, i\), and recall the following definition \([2]\).
Definition 5.1. Let $h_1, \ldots, h_m$ be homogeneous polynomials of degree $\geq 2$ on a projective space $\Pi$ of dimension $\dim \Pi \geq m + 1$. An irreducible subvariety $C \subset \Pi$ with $\langle C \rangle = \Pi$ and $\text{codim} C = m$ is called an associated subvariety of the sequence $(h_\ast)$ if there is a chain of irreducible subvarieties $C_j \subset \Pi, \ j = 0, \ldots, m$, with the following properties.

1) $C_0 = \Pi$.
2) For every $j = 0, \ldots, m - 1$ the subvariety $C_{j+1}$ is an irreducible component of the closed algebraic set $\{h_{j+1} = 0\} \cap C_j$, and we have $h_{j+1}|_{C_j} \not\equiv 0$, whence $\text{codim} \Pi C_j = j$ for all $j$.
3) $C_m = C$.

If a sequence $(h_\ast)$ has an associated subvariety, it is said to be good.

Lemma 5.1. (i) The good sequences $(h_\ast)$ form an open subset in the space of all sequences.

(ii) A good sequence $(h_\ast)$ cannot have more than

$$\left\lfloor \frac{1}{m+1} \prod_{j=1}^{m} \deg h_j \right\rfloor$$

associated subvarieties.

Proof. See [2], Lemma 4. □

As shown in [2], p. 73, the set of polynomials $f_j, \ j = 1, \ldots, i$, contains $e$ polynomials $f_{j_1}, \ldots, f_{j_e}$ such that the sequence $(f_{j_1}|_{\langle B \rangle}, \ldots, f_{j_e}|_{\langle B \rangle})$ is good and $B$ is one of its associated subvarieties. It was also shown in [2], p. 72, that the requirement $h|_{C} \equiv 0$ (where $C$ is any fixed irreducible subvariety in $\Pi$ with $\langle C \rangle = \Pi$) imposes at least $\deg h \cdot \dim \Pi + 1$ independent conditions on the polynomial $h$. Therefore, fixing the subspace $\langle B \rangle$ and the polynomials $f_{j_1}, \ldots, f_{j_e}$, we see that the requirement $f_j|_{B} \equiv 0$ imposes at least $(i - e + 1)(2(N - i + e) + 1)$ independent conditions on the polynomials $f_j$, where $j \in \{i + 1\} \cup (\{1, \ldots, i\} \setminus \{j_1, \ldots, j_e\})$. Subtracting the dimension of the Grassmanian of $(N - i + e)$-subspaces in $\mathbb{P}^N$, we get

$$\text{codim} Y_{i,e} \geq (i - e + 2)(N - i + e) + 1.$$  

We easily see that the minimum of the right-hand side is $2N + 1$. The inequality

$$2N + 1 \geq n_1 + 2n_2 + 2$$

holds in all cases except in the case $n_1 = 0, n_2 = N$. However, in our estimate for the number of independent conditions we assumed that $\deg f_j = 2$. When all the polynomials are cubic, the estimate becomes much stronger:

$$\text{codim} Y_{i,e} \geq (2i - 2e + 3)(N - i + e) + 1 \geq 3N + 1,$$

and this completes the proof of the proposition in the case $n_1 + n_2 = N$.

Consider the case $n_1 + n_2 = N + 1$. Let $(f_1, \ldots, f_{N+1})$ be a tuple of general position in some irreducible component $Q$ of the set $Y'_\ast$. Then there are two options: either $\dim Z(f_1, \ldots, f_N) = 1$ and $f_{N+1}$ vanishes on some irreducible component $C$
of $Z(f_1, \ldots, f_N)$ such that $C$ is not a line, or $\dim Z(f_1, \ldots, f_N) \geq 2$ and the polynomial $f_{N+1}$ is arbitrary.

Assume that the first option is realized. By what was proved above, the codimension of the set

$$\{(f_1, \ldots, f_N) \mid (f_1, \ldots, f_{N+1}) \in Q \} \subset \mathcal{H}^{(n_1,n_2-1)}_{N+1}$$

is not smaller than $n_1 + 2(n_2 - 1) + 2$ if $n_1 \leq N$ (that is, at least one of the polynomials is cubic) and not smaller than $n_1 + 1$ if $n_1 = N + 1$. We easily see that the condition $f_{N+1}|_C \equiv 0$, where $C$ is a curve with linear span of dimension at least 2, imposes at least 5 conditions on the polynomial $f_{N+1}$. Therefore $\operatorname{codim} Q > \operatorname{codim} Y_{\text{line}}$, as required.

Assume that the second option is realized. Then we may assume that $\dim Z(f_1, \ldots, f_N) = 2$. The polynomial $f_{N+1}$ is arbitrary. Interchanging it with some polynomial $f_i$, $i \leq N$, we get the situation already considered above. This proves Proposition 5.1. □

We now consider the set

$$Y_i^\Delta \subset \mathcal{H}^{(n_1,n_2)}_{N+1} \setminus Y_\infty$$

of all tuples $(f_1, \ldots, f_{N+1})$ for which one can find $i+1$ distinct points $p_1, \ldots, p_{i+1} \in Z(f_*)$ such that

$$\dim (p_1, \ldots, p_{i+1}) \leq i - 1.$$

**Proposition 5.2.** We have $\operatorname{codim} Y_i^\Delta = N + 2$.

*Proof.* For convenience, we put $Y_1^\Delta = \emptyset$. It suffices to show that $\operatorname{codim}(Y_i^\Delta \setminus Y_{i-1}^\Delta) = N + 2$. Hence we may assume that any $i$ points among the points $p_*$ are linearly independent and, therefore, the subspace $\Lambda = \langle p_* \rangle$ has dimension $i - 1$. Fix $\Lambda$. We easily see that the linear conditions $f_j(p_i) = 0$ are linearly independent. Hence the set

$$\{(f_1, \ldots, f_{N+1}) \mid \{p_1, \ldots, p_{i+1}\} \subset |Z(f_*)|\}$$

has codimension $(i+1)(N+1)$, $\dim Z(f_*)|$. The set of tuples $\{p_1, \ldots, p_{i+1}\} \subset \Lambda$ has dimension $(i+1)(i-1)$, and the projective Grassmanian has dimension $i(N - i + 1)$. Thus we get

$$\operatorname{codim}(Y_i^\Delta \setminus Y_{i-1}^\Delta) = (i + 1)(N - i + 2) - i(N - i + 1) = N + 2,$$

as required. □

The following proposition is a direct corollary of Propositions 5.1, 5.2.

**Proposition 5.3.** For $a \leq N + 1$ we have

$$\operatorname{codim}\{(f_1, \ldots, f_{N+1}) \mid |Z(f_*)| \geq a\} = a.$$

*Proof.* We may assume that the set $|Z(f_*)|$ is finite and consists of $a$ linearly independent points. The proposition is obvious in this case. □
5.2. Global multiplicities: definition and statement of the problem. We begin the globalization of the local theory developed in §§2–4. To distinguish between the local and global multiplicities, we denote the local multiplicities of type $\mu_{l,N}(a,b;d)$ by $\mu_{l,N}^{\text{local}}(a,b;d)$ and so on. Consider the space

$$\mathcal{H}^{(n_1,n_2)}_{N+1} = (\mathbb{P}_{2,N+1})^{\times n_1} \times (\mathbb{P}_{3,N+1})^{\times n_2}$$

of tuples $(f_1, \ldots, f_{n_1}, f_{n_1+1}, \ldots, f_{n_1+n_2})$ consisting of $n_1$ quadratic and $n_2$ cubic polynomials. We regard them as polynomials on $\mathbb{P}^N$. Assume that $n_1 + n_2 = N + 1$.

Let $\Sigma_1(f_*) = \langle f_1, \ldots, f_{n_1} \rangle$ be the linear system generated by the quadratic polynomials, and let $\Sigma_2(f_*) = \langle f_{n_1+1}, \ldots, f_{n_1+n_2} \rangle + \Sigma_1^P_{1,N+1}$ be the linear system of cubic polynomials generated by all polynomials $f_*$. We write $Y_\infty$ for the closed subset of $\mathcal{H}^{(n_1,n_2)}_{N+1}$ formed by the tuples $(f_*)$ with zero sets of positive dimension: $\text{dim} \ Z(f_*) \geq 1$. By Proposition 5.1 we have $\text{codim} \ Y_\infty = n_1 + 2n_2 + 4$. Let $(f_1, \ldots, f_{N+1}) \in \mathcal{H}^{(n_1,n_2)}_{N+1} \setminus Y_\infty$ be an arbitrary tuple. We write $\Sigma_1 = \Sigma_1(f_*)$ and $\Sigma_2 = \Sigma_2(f_*)$ when it is clear which tuple is meant. Consider the set

$$\Sigma_1^{n_1} \times \Sigma_2^{N-n_1} \subset (\mathbb{P}_{2,N+1})^{\times n_1} \times (\mathbb{P}_{3,N+1})^{\times n_2}.$$

Let $(f_1^*, \ldots, f_N^*) \in \Sigma_1^{n_1} \times \Sigma_2^{N-n_1}$ be a tuple of general position. For an arbitrary effective cycle $R$ of pure codimension $l \in \mathbb{Z}_+$ on $\mathbb{P}^N$ we define the global effective multiplicity by putting

$$\mu_{\text{total}}((f_*); R) = \sum_{p \in |Z(f_*)|} \mu_{\text{local}}((f_*); R, p)$$

(we recall that the set $|Z(f_*)|$ is finite since the tuple $(f_*)$ is chosen outside $Y_\infty$), where the local multiplicity at $p$ is understood in the sense of §2.1:

$$\mu_{\text{local}}((f_*); R, p) = \text{mult}_p(\{f_{l+1} = 0\} \circ \cdots \circ \{f_N = 0\} \circ R).$$

(The term in parentheses on the right-hand side is a zero-dimensional cycle, and $\text{mult}_p$ is the multiplicity of the point $p$ in this cycle.) If $R \subset \mathbb{P}^N$ is an irreducible subvariety of codimension $l$, then

$$\mu_{\text{local}}((f_*); R, p) = \text{dim} \mathcal{O}_{p,R}/(f_{l+1}^*, \ldots, f_N^*).$$

Furthermore, let $Y^\Delta \subset \mathcal{H}^{(n_1,n_2)}_{N+1} \setminus Y_\infty$ be the set of tuples $(f_*)$ such that the finite set $|Z(f_*)|$ contains a linearly dependent subset. By Proposition 5.2,

$$\text{codim} \ Y^\Delta = N + 2.$$
Thus we have a representation

$$\mathcal{H}_{N+1}^{(n_1, n_2)} \setminus (Y_\infty \cup \overline{Y^\Delta}) = Y_0 \sqcup Y_1 \sqcup \cdots \sqcup Y_{N+1},$$

where $Y_a$ is a constructive set such that the zero set $|Z(f_*)|$ of any $(f_*) \in Y_a$ consists of precisely $a \in \{0, \ldots, N + 1\}$ linearly independent points. The closures $\overline{Y_a}$ are irreducible, the sets $Y_a$ are open in their closures and, by Proposition 5.3, we have $\text{codim} \overline{Y_a} = a$. In particular, $Y_0 \subset \mathcal{H}_{N+1}^{(n_1, n_2)}$ is an open subset.

As in §2, we write $\mathbb{H}_{l, N}(d)$ for the Chow variety that parametrizes effective cycles of pure codimension $l \in \mathbb{Z}_+$ and degree $d \geq 1$ on the projective space $\mathbb{P}^N$.

We define a subset

$$\mathcal{Y}_{l, N, i}^{\text{total}}(m, d) \subset Y_i \times \mathbb{H}_{l, N}(d)$$

by the condition $\mu_{\text{total}}((f_*) ; R) \geq m \in \mathbb{Z}_+$. This is a closed algebraic subset (for a given value $i \in \{0, \ldots, N + 1\}$). We also denote the projection of the direct product $Y_i \times \mathbb{H}_{l, N}(d)$ onto $Y_i$ by $\pi_P$. Since Chow varieties are projective, the image

$$Y_{l, N, i}^{\text{total}}(m, d) = \pi_P(\mathcal{Y}_{l, N, i}^{\text{total}}(m, d)) \subset Y_i$$

is a closed subset. Clearly, $Y_i$ and $Y_{l, N, i}^{\text{total}}(m, d)$ are invariant under the action of $\text{GL}_{N+1}(\mathbb{C})$ by linear changes of coordinates. We consider the problem of estimating the codimension

$$\text{codim}(Y_{l, N, i}^{\text{total}}(m, d) \subset Y_i).$$

Besides $\text{GL}_{N+1}(\mathbb{C})$, the space $\mathcal{H}_{N+1}^{(n_1, n_2)}$ is acted upon by another group $G^*(n_1, n_2)$, which is analogous to the group $G(n_1, n_2)$ in the local situation. Let us describe this new group and its action. The elements $g \in G^*(n_1, n_2)$ are triples

$$g = (A_{11} \in \text{GL}_{n_1}(\mathbb{C}), A_{22} \in \text{GL}_{n_2}(\mathbb{C}), A_{12} \in \text{Mat}_{(n_1, n_2)}(\mathbb{P}_{1, N+1})), $$

and $g(f_1, \ldots, f_{N+1}) = (f_1^g, \ldots, f_{N+1}^g)$, where

$$(f_1^g, \ldots, f_{n_1}^g) = (f_1, \ldots, f_{n_1}) A_{11},$$

$$(f_{n_1+1}^g, \ldots, f_{n_1+n_2}^g) = (f_{n_1+1}, \ldots, f_{n_1+n_2}) A_{22} + (f_1, \ldots, f_{n_1}) A_{12}.$$

As in the local case, a subset is said to be bi-invariant if it is invariant under linear changes of coordinates and under the action of $G^*(n_1, n_2)$.

Clearly, the sets $Y_\infty, Y^\Delta, Y_i$ and $Y_{l, N, i}^{\text{total}}(m, d)$ are bi-invariant.

By construction, the set $Y_{l, N, i}^{\text{total}}(m, d)$ consists of the tuples of polynomials $(f_1, \ldots, f_{N+1}) \in Y_i$ for which there is an effective cycle $R$ of pure codimension $l$ and degree $d$ on $\mathbb{P}^N$ such that $\mu_{\text{total}}((f_*) ; R) \geq m$. As in the local case, we restate the problem of estimating the codimension $Y_{l, N, i}^{\text{total}}(m, d)$ in the following way: maximize the multiplicity $m$ for a fixed codimension. More precisely, let $B \subset Y_i$ be a bi-invariant irreducible subvariety. We put

$$\mu_{\text{total}}(B, d) = \max\{m \mid B \subset Y_{l, N, i}^{\text{total}}(m, d)\}$$
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\[ \mu_{\text{total}}(B, d) = \max_{R} \{ \mu_{\text{total}}((f_*) ; R) \}, \]

where the maximum is taken over all effective cycles of degree \( d \) and codimension \( l \) and, as usual, \((f_*) \in B\) is a generic tuple. Finally, we write \( \mu_{\text{total}}(i, a, d) = m \) if there is an irreducible bi-invariant subvariety \( B \subset Y_i \) of codimension at most \( a \) such that \( \mu_{\text{total}}(B, d) = m \), and there is no such variety for \( m + j, j \geq 1 \). We fix the codimension \( a \) and look for an upper bound for the multiplicity \( m \).

5.3. The local and global types of a subvariety of tuples of polynomials. Fix an irreducible bi-invariant subvariety \( B \subset Y_r, r \geq 1 \). For a generic (and arbitrary) tuple \((f_*) \in B\), the set-theoretic intersection \(|Z(f_*)|\) consists of precisely \( r \) linearly independent points \( p_1, \ldots, p_r \in \mathbb{P}^N \) (depending on the tuple \((f_*)\), of course). For every point \( p \in |Z(f_*)| \) we put \( \varepsilon_p(f_*) = b \), where

\[ \text{rk}(df_1(p), \ldots, df_{N+1}(p)) = N - b. \]

Put \( b_i = \varepsilon_{p_i}(f_*) \in \mathbb{Z}_+, i = 1, \ldots, r \). Clearly, the tuple of integers \((b_1, \ldots, b_r) \in \mathbb{Z}_r^+\) is independent of the choice of \((f_*)\) and determines an invariant of the subvariety \( B \). We assume that the integers \( b_i \) are ordered: \( b_1 \geq b_2 \geq \cdots \geq b_r \).

**Definition 5.2.** The ordered (non-increasing) tuple of integers \((b_1, \ldots, b_r)\) is called the **global type** of a bi-invariant subvariety \( B \subset Y_r \) and is denoted by \( \varepsilon_{\text{total}}(B) = (b_*) \).

We put \( r_* = \max \{ j \mid b_j \geq j, 1 \leq j \leq r \} \)

if \( b_1 \geq 1 \). If \( b_1 = \cdots = b_r = 0 \), then we put \( r_* = 0 \). Define \( \Phi(0, \ldots, 0) = 0 \) and, for \( b_1 \geq 1 \),

\[ \Phi(b_1, \ldots, b_r) = \sum_{j=1}^{r_*} (b_j + 1)(b_j + 1 - j). \]

**Lemma 5.2.** We have

\[ \text{codim}(B \subset Y_r) \geq \Phi(b_1, \ldots, b_r). \]

**Proof.** If \( b_1 = \cdots = b_r = 0 \), there is nothing to prove. Therefore we assume that \( b_1 \geq 1 \). By bi-invariance, it suffices to establish the following estimate for any fixed linearly independent points \( p_1, \ldots, p_r \in \mathbb{P}^N \):

\[ \text{codim}(B(p_1, \ldots, p_r) \subset Y_r(p_1, \ldots, p_r)) \geq \Phi(b_*), \]

where \( Y_r(p_*) = \{(f_*) \in Y_r \mid \{p_1, \ldots, p_r\} = |Z(f_*)|\} \) and \( B(p_*) = B \cap Y_r(p_*) \).

Passing to an affine chart \( \mathbb{C}^N_{(z_1, \ldots, z_N)} \subset \mathbb{P}^N \), we may assume that \( p_1 \) is the origin and, for \( j \geq 2 \),

\[ p_j = (0, \ldots, 0, 1, 0, \ldots, 0) \]
with 1 in the \((j - 1)\)th position. The closed subset \(B(p_*) \subset Y_r(p_*)\) may consist of several irreducible components. Take any component \(B^+\) of this set such that a generic tuple \((f_*) \in B^+\) satisfies

\[
\varepsilon_{p_*}(f_*) = b_i.
\]

Clearly, the lemma will be proved as soon as we know that

\[
\text{codim}(B^+ \subset Y_r(p_*)) \geq \Phi(b_*).
\]

To simplify the argument, we assume that all the polynomials \(f_i\) are quadratic: if \(\deg f_i = 3\), then everything works without any modification and with even stronger estimates for codimensions (because there are more coefficients).

We write

\[
f_i = a_1^{(i)} z_1 + \cdots + a_N^{(i)} z_N + \sum_{j \leq k} a_{jk}^{(i)} z_j z_k,
\]

If \(r \geq 2\), then the condition \(f_i(p_j) = 0\) takes the form of the equations

\[
a_j^{(i)} + a_{jj}^{(i)} = 0
\]

for all \(i, j = 1, \ldots, r - 1\). By the bi-invariance we may assume that

\[
\text{rk}(df_1(o), \ldots, df_{N-b_1}(o)) = N - b_1
\]

and the linear forms \(df_1(o), i \geq N - b_1 + 1\), are linear combinations of the first \(N - b_1\) forms \(df_1(o), \ldots, df_{N-b_1}(o)\). This gives \(b_1(b_1 + 1)\) independent conditions on the coefficients \(a_j^{(i)}\) for \(i \geq N - b_1 + 1\), assuming the polynomials \(f_1, \ldots, f_{N-b}\) to be fixed. If \(r = 1\), then there is nothing to prove.

Assume that \(r \geq 2\) and consider the conditions associated with the point \(p_2 = (1, 0, \ldots, 0)\). Recall that \(b_2 \leq b_1\). If \(r_* = 1\), there is nothing to prove. Therefore we assume that \(b_2 \geq 2\). We again assume the polynomials \(f_1, \ldots, f_{N-b_2}\) to be fixed (which does not contradict the first step in the proof) in such a way that the linear forms \(df_1(p_2), \ldots, df_{N-b_2}(p_2)\) are linearly independent and \(df_i(p_2), i \geq N - b_2 + 1\), are linear combinations of them. Explicitly,

\[
df_i(p_2) = (a_1^{(i)} + 2a_{11}^{(i)}) z_1 + \sum_{j \geq 2} (a_j^{(i)} + a_{jj}^{(i)}) z_j.
\]

Since \(a_1^{(i)} = -a_{11}^{(i)}\), the coefficient of \(z_1\) is a linear combination of the coefficients \(a_j^{(i)}\). However the coefficients \(a_{1j}^{(i)}\) are not involved in the conditions associated with the point \(p_1\). Therefore, requiring that

\[
df_i(p_2)|_{\{z_1=0\}} \in \langle df_i(p_2)|_{\{z_1=0\}}, \ldots, df_{N-b_2}(p_2)|_{\{z_1=0\}} \rangle
\]

for \(i \geq N - b_2 + 1\) imposes at least \((b_2 + 1)(b_2 - 1)\) independent conditions on the coefficients \(a_{1j}^{(i)}\). (The precise number of conditions is determined by the dimension of the space on the right-hand side of (5.1). If this dimension is equal to \(N - b_2\), then we get \((b_2 + 1)(b_2 - 1)\) conditions. If it is one less, we get \((b_2 + 1)b_2\) conditions.)
This completes our treatment of the second component \((j = 2)\) of the function \(\Phi\). If \(r_* = 2\), the lemma is proved.

If \(r_* \geq 3\), then we continue the argument by induction. Suppose that we already know that the requirement

\[
\text{rk}(df_1(p_{\alpha}), \ldots, df_{N+1}(p_{\alpha})) = N - b_{\alpha}
\]

for \(\alpha = 1, \ldots, j\) imposes in total at least

\[
\sum_{\lambda=1}^{j} (b_\lambda + 1)(b_\lambda + 1 - \lambda)
\]

independent conditions on the coefficients \(a^{(i)}_{\lambda k}\), \(i \geq N - b_{\lambda + 1} + 1\), where \(\lambda = 1, \ldots, j - 1\), \(k = \lambda + 1, \ldots, N\) and \(i \geq N - b_{\lambda + 1} + 1\). If \(r_* = j\), then the proof is complete. Otherwise, assuming \(f_1, \ldots, f_{N-b_{j+1}}\) to be fixed, we obtain that the linear forms

\[
df_i(p_{j+1})\mid_{\{z_1 = \cdots = z_j = 0\}}, \quad i \geq N - b_{j+1} + 1,
\]

belong to the vector space

\[
\langle df_1(p_{j+1})\mid_{\{z_1 = \cdots = z_j = 0\}}, \ldots, df_{N-b_{j+1}}(p_{j+1})\mid_{\{z_1 = \cdots = z_j = 0\}} \rangle.
\]

This gives at least \((b_{j+1} + 1)(b_{j} - j)\) independent conditions on the \textit{new} coefficients (that is, those not involved in the previous considerations)

\[
a^{(i)}_{j,j+1}, \ldots, a^{(i)}_{j,N}, \quad i \geq N - b_{j+1} + 1.
\]

This establishes the inductive step from \(j\) to \(j + 1\) and completes the proof of the lemma. \(\Box\)

\textbf{Corollary 5.1.} We have

\[
\mu_{\text{total}}(i, a, d) \leq \sum_{\Phi(b_1, \ldots, b_1) + i \leq a} \mu_{\text{local}}(a - b_i, b_i; d).
\]

\textbf{Proof.} This follows from Lemma 5.2 and the equation \(\text{codim} Y_i = i\) since the global multiplicity \(\mu_{\text{total}}\) is calculated in terms of \((N + 1)\)-tuples of polynomials while the local multiplicity \(\mu_{\text{local}}\) is calculated in terms of \(N\)-tuples of polynomials. Indeed, if

\[
B \subset \mathcal{P}^{n_1}_{[1,2],N} \times \mathcal{P}^{n_2}_{[1,3],N}
\]

is a bi-invariant subvariety of codimension \(a\) (in the sense of the local theory in §§2–4) and \(\varepsilon(B) = b\), where \(n_1 + n_2 = N + 1\), then the projection \([B]_{N+1}\) of the set \(B\) along the last direct factor has codimension at most \(a - b\) in the space \(\mathcal{P}^{n_1}_{[1,2],N} \times \mathcal{P}^{n_2 - 1}_{[1,3],N}\) because, for a generic tuple \((f_1, \ldots, f_{N+1}) \in B\), the form \(df_{N+1}(o)\) vanishes on a subspace (of codimension \(b\)) depending only on \((f_1, \ldots, f_N)\). Then the claim of the corollary becomes obvious. \(\Box\)
To simplify the notation, we shall omit the parameters \( l \) and \( N \) (our estimates do not depend on them) and write \( \mu_{\text{local}}(a, b; d) \) instead of \( \mu_{l,N}^{\text{local}}(a, b; d) \). In any case, the values of \( l \) and \( N \) are fixed in all the arguments that follow.

### 5.4. An explicit estimate for the global multiplicity

We can now obtain an effective upper bound for the global multiplicity \( \mu_{\text{total}}(r, a, d) \). Let \( B \subset Y_r \) be an irreducible bi-invariant subvariety whose codimension in \( \mathcal{H}_{N+1}^{(n_1, n_2)} \) does not exceed \( a \). In particular, we have \( \Phi(b_1, \ldots, b_r) + r \leq a \), where \( (b_1, \ldots, b_r) = \varepsilon(B) \) is the global type of \( B \).

**Proposition 5.4.** We have

\[
\mu_{\text{total}}(B, d) \leq \sum_{i=1}^{r} \mu_{\text{local}}(a - b_i, b_i; d). \tag{5.2}
\]

**Corollary 5.2.** We have

\[
\mu_{\text{total}}(r, a, d) \leq \max_{\Phi(b_1, \ldots, b_r) + r \leq a} \left\{ \sum_{i=1}^{r} \mu_{\text{local}}(a - b_i, b_i; d) \right\}, \tag{5.3}
\]

\[
\mu_{\text{total}}(r, a, d) \leq d \left( \max_{\Phi(b_1, \ldots, b_r) + r \leq a} \left\{ \sum_{i=1}^{r} \tilde{\mu}_{\text{local}}(a - b_i, b_i) \right\} \right). \tag{5.4}
\]

**Proof.** Inequality (5.3) follows from Proposition 5.4 by the definition of the numbers \( \mu_{\text{total}}(r, a, d) \). Inequality (5.4) holds by Proposition 4.1. \( \square \)

**Proof of Proposition 5.4.** Let \( (f_*) \in B \) be a tuple of general position, and let \( \{p_1, \ldots, p_r\} = |Z(f_*)| \) be its common zeros, where \( b_i = \varepsilon_{p_i}(f_*), \ i = 1, \ldots, r \). Inequality (5.2) follows from the estimate

\[
\mu_{\text{local}}((f_*); R, p_i) \leq \mu_{\text{local}}(a - b_i, b_i; d) \tag{5.5}
\]

for every effective cycle \( R \) of pure codimension \( l \) and degree \( d \). We now prove this estimate.

Put \( p = p_i \) and let \( B(p) \subset B \) be the closed subset of \( B \) formed by the tuples \( (g_*) \) that vanish at \( p \). Since \( B \) is bi-invariant, the original tuple \( (f_*) \), which was fixed at the beginning of the proof, is generic in one of the irreducible components \( B^+ \) of the set \( B(p) \), and the codimension of \( B^+ \) in the space \( \mathcal{P}_{N}^{(n_1, n_2)} \) coincides with \( \text{codim}(B \subset \mathcal{H}_{N+1}^{(n_1, n_2)}) \) and, therefore, does not exceed \( a \) (we use the natural identification of the linear space of tuples \( (g_*) \in \mathcal{H}_{N+1}^{(n_1, n_2)} \) that vanish at \( p \) with the space \( \mathcal{P}_{N}^{(n_1, n_2)} \) defined in §2). Let

\[
\pi: \mathcal{P}_{N}^{(n_1, n_2)} \to \mathcal{P}_{N}^{(n_1, n_2-1)}
\]

be the projection along the last direct factor and let \( [B^+]_N \) be the closure of the image \( \pi(B^+) \). Using the bi-invariance and the condition \( \varepsilon_{p}(f_*) = b \), we see that the generic fibre of the projection

\[
\pi_B: B^+ \to [B^+]_N
\]
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has codimension at least $b$ in $\mathcal{P}_{13, N}$ (the form $df_{N+1}(p)$ vanishes on a $b$-dimensional vector subspace depending only on $df_1, \ldots, df_N$). Hence we have

$$\text{codim}([B^+]_N \subset \mathcal{P}^{(n_1, n_2-1)}_N) \leq a - b.$$ 

This proves (5.5). □

We put

$$\mu_{\text{total}}(a, d) = \max_{1 \leq r \leq \min\{a, N+1\}} \{\mu_{\text{total}}(r, a, d)\}.$$ 

**Proposition 5.5.** For $a \geq 12$ we have

$$\mu_{\text{total}}(a, d) \leq 3 \cdot 2^{a-6} d.$$ 

**Proof.** First assume that $a \geq 21$. Since $r \leq a$, we obtain from inequalities (5.3) and (4.11) that

$$\mu_{\text{total}}(a, d) \leq a \frac{e^2}{2\pi \sqrt{a}} \left( \left\lfloor \frac{\sqrt{a}}{3} \right\rfloor \right)^d.$$ 

It is easy to check that for $a \geq 21$ the right-hand side of this inequality is strictly smaller than $3 \cdot 2^{a-6}$. This method, however, is very coarse. Inequality (5.4) yields a much more precise estimate. We put

$$\bar{\mu}_{\text{total}}(a) = \max_{1 \leq r \leq a} \left( \max_{\Phi(b_1) + r \leq a} \left\{ \sum_{i=1}^r \bar{\mu}(a - b_i, b_i) \right\} \right).$$ 

By (5.4) and Proposition 4.1 we have

$$\mu_{\text{total}}(a, d) \leq d \bar{\mu}_{\text{total}}(a).$$ 

Thus, to complete the proof of the proposition, it suffices to verify that $\bar{\mu}_{\text{total}}(a) \leq 3 \cdot 2^{a-6}$ for $12 \leq a \leq 20$. The function $\bar{\mu}_{\text{total}}(a)$ is explicitly computable for small values of the codimension $a$. The results of this computation (for $1 \leq a \leq 36$) are given in Table 2, which is organized as follows. Each row corresponds to a certain value of $a$, which is written in the first column. The second column contains the values of $r$ (the number of points) and $b_1, \ldots, b_r$ at which the maximum in the definition of $\bar{\mu}_{\text{total}}$ is attained. The third column contains the value of $\bar{\mu}_{\text{total}}(a)$.

Here are the subsequent values of $\bar{\mu}_{\text{total}}$:

$$\bar{\mu}_{\text{total}}(28) = 3002, \quad \bar{\mu}_{\text{total}}(29) = 3420, \quad \bar{\mu}_{\text{total}}(30) = 3864,$$

$$\bar{\mu}_{\text{total}}(31) = 4356, \quad \bar{\mu}_{\text{total}}(32) = 4876, \quad \bar{\mu}_{\text{total}}(33) = 5448,$$

$$\bar{\mu}_{\text{total}}(34) = 6050, \quad \bar{\mu}_{\text{total}}(35) = 6708.$$ 

All of them are attained at $r = a - 9$, $b_1 = \cdots = b_r = 2$. The next jump occurs for $a = 36$: the maximum is attained at $r = 12$ and $b_1 = \cdots = b_{12} = 3$ and is equal to $\bar{\mu}_{\text{total}}(36) = 7980$.

By Table 2, the desired inequality $\bar{\mu}_{\text{total}}(a) \leq 3 \cdot 2^{a-6}$ holds for all $a$ starting with $a = 12$. □
The following proposition is also obtained by elementary calculations form the values of \( \bar{\mu}_{\text{total}} \) given in Table 2.

**Proposition 5.6.** If \((a, n_1, n_2)\) is one of the triples

\[
(11, 5, 3), \quad (11, 3, 4), \quad (11, 1, 5), \quad (10, 2, 4), \quad (10, 0, 5),
\]

then we have

\[
\bar{\mu}_{\text{total}}(a, d) \leq 2^{n_1 + n_2 - 4} 3^{n_2 - 1} d.
\]
5.5. Regular complete intersections. We finally prove Theorem 0.2. Let

\[ \mathcal{B} \subset \mathbb{P} \times (\mathcal{P}_{2,M+k+1}^{xk_1} \times \mathcal{P}_{3,M+k+1}^{xk_2}) \]

be the closed set of ‘bad’ pairs \((o, (f_*))\), where \(f_1(o) = \cdots = f_k(o) = 0\) and at least one of conditions (R1)–(R3) is violated at this point. Let \(\pi_1\) and \(\pi_2\) be the projections of the direct product onto the first (\(\mathbb{P}\)) and second (the space of \(k\)-tuples of polynomials) factors respectively. We put \(\mathcal{B}(o) = \pi_1^{-1}(o) \cap \mathcal{B}\). The closed subset \(\mathcal{B}(o)\) is contained in the subspace

\[ \mathcal{L}(o) = \mathcal{P}_{[1,2],M+k}^{xk_1} \times \mathcal{P}_{[1,3],M+k}^{xk_2} \subset \pi_1^{-1}(0) \]

(where we again identify homogeneous polynomials that vanish at \(o\) with non-homogeneous polynomials without constant term) and it suffices to prove that its codimension in this subspace is at least \(M + 1\). Indeed, if this is the case, then

\[ \text{codim} \mathcal{B} = \text{codim}(\mathcal{B}(o) \subset \pi_1^{-1}(o)) \geq M + k + 1, \]

whence the map \(\pi_2|_{\mathcal{B}}\) cannot be surjective, and Theorem 0.2 follows.

For conditions (R1) and (R2), the inequality

\[ \text{codim}(\mathcal{B}(o) \subset \mathcal{L}(o)) \geq M + 1 \] (5.6)

is proved in [2] in view of Proposition 5.2. Therefore it suffices to prove (5.6) for those tuples \((f_*) \in \mathcal{B}(o)\) which satisfy conditions (R1) and (R2), but not (R3). Fixing the linear parts of the polynomials \(f_i\) at the point \(o\) (and thus fixing the projectivized tangent space \(\mathbb{T} \cong \mathbb{P}^{M-1}\)), we reduce the problem to the estimation of the codimension of the set of those tuples of \(n_1 = k_1 + k_2 = k\) quadratic and \(n_2 = k_2\) cubic homogeneous polynomials

\[ (\bar{q}_{1,2}, \cdots, \bar{q}_{k,2}, \bar{q}_{k_1+1,3}, \cdots, \bar{q}_{k,3}) \] (5.7)

for which condition (R3) does not hold. However, by Propositions 5.5 and 5.6, the main inequality (0.3) of condition (R3) holds for a generic element of every subvariety of codimension at most \(M\) in the space of tuples \(\mathcal{P}_{2,M}^{xn_1} \times \mathcal{P}_{3,M}^{xn_2}\) and for every irreducible subvariety \(R \subset \mathbb{T}\) of codimension \(3\). Hence the closed set of tuples (5.7) for which condition (R3) does not hold, has codimension at least \(M + 1\).

This proves the estimate (5.6) and Theorem 0.2.

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