1 Introduction

Throughout this paper $H$ is a finite dimensional Hopf algebra over a field $k$, and $A$ is a associative $k$-algebra.

Definition 1.1 It is said that $H$ acts on $A$, if $A$ is left $H$-module and for any $h \in H$, $a, b \in A$

$$h(ab) = \sum_h (h_{(1)} a)(h_{(2)} b), \quad h1 = \varepsilon(h),$$

where $\varepsilon : H \to k$ - counit and $\Delta$ - comultiplication:

$$\Delta(h) = \sum_h h_{(1)} \otimes h_{(2)} \in H \otimes H.$$

Often $A$ is called $H$-module algebra.

Definition 1.2 The invariants of $H$ in $A$ is the set $A^H$ of those $a \in A$, that $ha = \varepsilon(h)a$ for each $h \in H$.

Straightforward computations shows, that $A^H$ is the subalgebra of $A$. We refer reader to [3], [4] for the basic information concerning Hopf algebras and their actions on associative algebras.

As a generalization of the well-known fact for group actions the following question raised in [4] (Question 4.2.6.)

Question 1.3 If $A$ is a commutative $k$-algebra and $H$ any finite dimensional Hopf algebra such that $A$ is $H$-module algebra, is $A$ integral over $A^H$?

If $A$ is an affine algebra, then Artin-Tate lemma ensures that $A^H$ is also affine.

Some positive answers to question 1.3 are known.
Theorem 1.4 ([2]) Let $H$ be a finite dimensional cocommutative Hopf algebra and let $A$ be a commutative $H$-module algebra. Then $A$ is integral extension of $A^H$.

Some results on affine invariants were obtained without using integrality.

Definition 1.5 Element $t \in H$ is called left integral, if $ht = \varepsilon(h)t$ for all $h \in H$.

Note that $tA \subseteq A^H$.

Theorem 1.6 ([3], Theorem 4.3.7) Let $A$ be left Noetherian ring which is an affine $k$-algebra, and assume that $A$ is an $H$-module algebra, such that $tA = A^H$. Then $A^H$ is $k$-affine.

As mentioned in [3], p. 48, if $H$ is semisimple, then $tA = A^H$. By Maschke’s theorem, $H$ is semisimple if and only if $\varepsilon(t) \neq 0$ for some non-zero left integral $t$. Since the space $\int_H^l$ of left integrals in $H$ is one-dimensional ([3], §2.2), the semisimplicity of $H$ is equivalent to the fact that $\varepsilon(t) \neq 0$ for all non-zero left integrals.

The main result of the work is stated in the theorem 2.7. It gives positive answer to Question 1.3 in some partial cases. Let $H$ be a pointed Hopf algebra and let $A$ be an affine $H$-module algebra; if one of three conditions is satisfied, then $A$ is integral over $A^H$:

1. $H$ is commutative as an algebra,
2. $\text{char } k = p > 0$,
3. $A$ is integral domain.

We recall that Hopf algebra $H$ is called pointed if every simple subcoalgebra of $H$ is one-dimensional; pointed Hopf algebra is called connected if it has only one simple subcoalgebra (one-dimensional). The examples of pointed Hopf algebras are given by group algebras, universal enveloping algebras. In fact, if $G$ - group, then the only simple subcoalgebras of $kG$ are those of the form $kg$, $g \in G$. At the same time universal enveloping algebras are examples of connected Hopf algebras: the only simple subcoalgebra of the universal enveloping algebra is $k1_H$.

Another important example of pointed Hopf algebras is represented by series of Hopf algebras $A_{N,\xi}$, where $N \geq 2$ – integer number and $\xi$ – root
of unity of degree $N$, considered in [3] (see also section 3 of this paper). Note that with $N = 2$, $\xi = -1$, char $k \neq 2$ algebra $A_{2,-1}$ (sometimes called $H_4$) is the only Hopf algebra of minimal dimension neither cocommutative nor commutative ([3], example 1.5.6). In other words, algebra $A_{2,-1}$ is the minimal Hopf algebra not covered by theorem [1.4], but covered by theorem [2.7].

Notice that ideal generated by $x$ in $A_{N,\xi}$ is nilpotent, therefore it lies in radical of $A_{N,\xi}$, i.e., $A_{N,\xi}$ is not semisimple. The same fact can be verified using argument of left integrals of $H$. Thus, theorem [2.7] is not covered by theorem [1.6].

In spite of numerous partial positive results it turned out that hypothesis [1.3] isn't true in general, even for pointed Hopf algebras. The counterexamples is built in section 3 for series of pointed Hopf algebras $A_{N,\xi}$, $N \geq 2$, mentioned at previous paragraphs.

2 The main theorem

The proof of the theorem is based on properties of coradical filtration of an arbitrary coalgebra. We recall the basic facts.

Definition 2.1 Coradical $C_0$ of coalgebra $C$ is the (direct) sum of all simple subcoalgebras of $C$. Further by the induction for each $n \geq 1$ define

$$C_n = \Delta^{-1}(C \otimes C_{n-1} + C_0 \otimes C)$$

Theorem 2.2 ([3], Theorem 5.2.2) $\{C_n\}_{n \geq 0}$ is the family if subcoalgebras with the following properties:

1. $C_{n-1} \subseteq C_n$, $C = \bigcup_{n \geq 0} C_n$,

2. $\Delta C_n \subseteq \sum_{i=0}^n C_i \otimes C_{n-i}$.

Reader may find more detailed description of coradical filtration in [3], chapter 9.

\footnote{After the work was done authors became aware through e-mail by Susan Montgomery, that counterexample to hypothesis [1.3] was independently built by Zhu Shenglin. Also he obtained some positive results, solving problem [1.4]. His paper "Integrality of module algebras over its invariants" should have appeared at J.Algebra in 1996.}
Let $H$ be a pointed finite dimensional Hopf algebra over field $k$. Let $G = G(H)$ denote the set of grouplike elements of $H$, i.e.,

$$G = \{ g \in H \setminus \{0\} \mid \Delta g = g \otimes g \}.$$ 

It is known that elements of $G$ are linearly independent, $G$ is the group under multiplication arising from multiplication in $H$, and subalgebra generated by $G$ is group Hopf algebra $kG$. Also $kG$ is coradical of $H$.

By lemma 5.2.8 [5], coradical filtration $\{ H_n \}$ of Hopf algebra $H$ is Hopf filtration of Hopf algebra, i.e.,

$$\Delta H_n \subseteq \sum_{i=0}^{n} H_i \otimes H_{n-i}, \quad H_n H_m \subseteq H_{n+m}, \quad S H_n \subseteq H_n \text{ for all } n, m \geq 0,$$

if and only if $H_0$ is sub-Hopf algebra of $H$. If $H$ is a pointed finite dimensional Hopf algebra, then this condition is obviously satisfied. Moreover, coradical filtration is finite.

By theorem 5.4.1 from [5] (see also [6], [4] for reference), coradical filtration of pointed Hopf algebra $H$ have additional properties. If $x \in H_m$, $m \geq 1$, then

$$x = \sum_{g,h \in G(H)} c_{g,h},$$

where

$$\Delta(c_{g,h}) = c_{g,h} \otimes g + h \otimes c_{g,h} + w,$$

$$w \in H_{m-1} \otimes H_{m-1}. \quad (1)$$

Note that if $a, b, g, h \in G$ and $c = ac_{g,h}b$, then by (1)

$$\Delta(c) = c \otimes agb + abh \otimes c + w', \quad w' \in H_{n-1} \otimes H_{n-1}. \quad (2)$$

Define $H^+ = \ker \varepsilon$, $H^+_r = H_r \cap H^+$. Let $A^G$ denote subalgebra of $G$-invariants in $A$ ($A^H \subseteq A^G$). Extension $A/A^G$ is integral by Noether’s theorem for group actions ($H$ is finite dimensional, therefore $G$ is finite group).

Before we start to prove main theorem we are going to obtain few auxiliary results.

Let $I$ denote the ideal in $H$ generated by elements of form $g - 1$ that $g \in G$.

**Proposition 2.3** $I$ is Hopf ideal in $H$.

**Proof.** If $g \in G$, then

$$\Delta(g - 1) = g \otimes g - 1 \otimes 1 =$$

$$(g - 1) \otimes g + 1 \otimes (g - 1) \in I \otimes H + H \otimes I.$$

$S(I) \subseteq I$, because $S(g - 1) = g^{-1} - 1$ and $S$ is anti-homomorphism. This yields the proposition. $\square$
Proposition 2.4 If $J$ – Hopf ideal in $H$, then $H/J$ – pointed Hopf algebra. Moreover, natural epimorphism of Hopf algebras $\pi : H \to H/J$ induces epimorphism of groups of grouplike elements $G(H) \to G(H/J)$.

Proof. This statement is direct consequence of corollary 5.3.5 from [5].

Theorem 2.5 Let one of three following conditions be satisfied:

1. $\text{char } k = p > 0$.
2. $A$ is integral domain;
3. $H$ – connected and commutative;

Then there exists the chain of subalgebras $A = A_{-1} \supseteq A_0 \supseteq \ldots \supseteq A_n$ with following properties:

1. each extension $A_i \supseteq A_{i+1}$ is integral;
2. if $x \in H_i^+$ then $x(A_i) = 0$.

Proof. To construct this chain we start with defining $A_0 = A^G$. Both of necessary conditions are satisfied. Let the chain $A = A_{-1} \supseteq A_0 \supseteq \ldots \supseteq A_r$, $r \geq 0$, be already constructed and let $x \in H^+_{r+1}$. By (1) we may assume that $x = c_{g,h}$, where $g, h \in G$. Then

$$\Delta(x) = x \otimes g + h \otimes x + \sum u_j \otimes v_j,$$

where $v_j, u_j \in H_r$. As $x \in H^+$, then by (3)

$$x = (1 \otimes \varepsilon)\Delta(x) = x + \sum u_j\varepsilon(v_j),$$

$$x = (\varepsilon \otimes 1)\Delta(x) = x + \sum \varepsilon(u_j)v_j.$$ Therefore,

$$\sum \varepsilon(u_j)v_j = \sum u_j\varepsilon(v_j) = 0,$$

and finally

$$\sum (u_j - \varepsilon(u_j)) \otimes (v_j - \varepsilon(v_j)) = \sum u_j \otimes v_j,$$

i.e., we may assume that $u_j, v_j \in H_r^+$. 

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If char \( k = p > 0 \), then we define \( A_{m+1} = A_m^p \). Really, by (3)

\[
x(a^p) = h(a)^{p-1}x(a) + h(a)^{p-2}x(a)g(a) + \ldots \\
\ldots + x(a)g(a)^{p-1} = p a^{p-1}x(a) = 0.
\]

Suppose \( A \) is integral domain and char \( k = 0 \). If \( a \in A_{r}, b \in A \), then by (3)

\[
x(ab) = x(a)b + ax(b),
\]
i.e., \( x : A_r \to A \) is derivation. By normalization lemma (see \([1]\), chapter 5, §3, p.344), there exists subalgebra of polynomials \( k[T_1, \ldots, T_d] \) in \( A_r \) such that extension \( A_r/k[T_1, \ldots, T_d] \) is integral. In this case we have for each \( f \in k[T_1, \ldots, T_d] \):

\[
x(f) = \sum_{i=1}^{d} \frac{\partial f}{\partial T_i} a_i, \quad a_i \in A.
\]

Therefore, for each integer \( q \geq 1 \)

\[
x^q(f) = \sum_{i_1 + \ldots + i_d = q} \frac{\partial^q f}{\partial T_{1}^{i_1} \ldots \partial T_{d}^{i_d}} a_1^{i_1} \ldots a_d^{i_d} +
\]

\[
+ \sum_{j_1 + \ldots + j_d = l < q} \frac{\partial^l f}{\partial T_{1}^{j_1} \ldots T_{d}^{j_d}} \Psi_{j_1, \ldots, j_d},
\]

(4)

where \( \Psi_{j_1, \ldots, j_d} \) is the sum of monomials of form

\[
x^{\alpha_{j_1}}(a_{j_1}) \ldots x^{\alpha_{j_d}}(a_{j_d}), \quad \alpha_{j_1} + \ldots + \alpha_{j_d} = q - l.
\]

\( H \) is finite dimensional, that is why there exists such integer \( m \) that

\[
x^m = \sum_{i=1}^{m-1} \beta_i x^i, \quad \beta_i \in k.
\]

Thus, by (4) and (5) for each \( f \in k[T_1, \ldots, T_d] \) and each \( i = 1, \ldots, d \) we have:

\[
\frac{\partial^m f}{\partial T_i^m} a_i^{m} + \sum_{1 \leq l < m} \frac{\partial^l f}{\partial T_i^{l}} \Phi_l + \Lambda = 0,
\]

(6)

where \( \Lambda \) is the sum of all summands from (4) and (5) containing

\[
\frac{\partial^q f}{\partial T_{1}^{j_1} \ldots \partial T_{d}^{j_d}}
\]
as a multiplier, and besides one of the coefficients $j_s, s \neq i$, is not equal to zero. Substituting to (3) successively $1, T_i, T_i^2, \ldots, T_i^m$, we get that
\[
\Lambda = \Phi_1 = \ldots = \Phi_{m-1} = a_i^m = 0.
\]
We use here the fact that char $k = 0$. As $A$ – integral domain, so $a_i = 0$. Thus $x(k[T_1, \ldots, T_d]) = 0$ and we define $A_{r+1} = k[T_1, \ldots, T_d]$.

Suppose $H$ – connected, commutative Hopf algebra and char $k = 0$, i.e., $g = h = 1$. By (2),
\[
\Delta(x) = x \otimes 1 + 1 \otimes x + \sum u_j \otimes v_j,
\]
where $u_j, v_j \in H_r^+$. We consider ideal $HH_r^+$ in $H$.

**Lemma 2.6** $HH_r^+$ is coideal in $H$.

**Proof.** Let $u \in H, v \in H_r^+$. Since $H_r$ – coalgebra, $H_r^+$ – it’s coideal, i.e.,
\[
\Delta(v) \in H_r \otimes H_r^+ + H_r^+ \otimes H_r,
\]
and therefore,
\[
\Delta(uv) = \Delta(u)\Delta(v) \in H \otimes (HH_r^+) + (HH_r^+) \otimes H.
\]
Assume that $x(A_r) \neq 0$. $HH_r^+$ acts as zero on $A_r$, so $x \notin HH_r^+$. Suppose $1, x, \ldots, x^{m-1}$ are linearly independent modulo $HH_r^+$ and
\[
x^m = \sum_{j=0}^{m-1} \alpha_j x^j + w, \quad w \in HH_r^+.
\]
Choose $k$-basis $e_1, \ldots, e_d$ in $H$ such that first elements $e_1, \ldots, e_{t-1}$ form $k$-basis for $HH_r^+$, $e_t = 1$, and $e_{t+1} = x, \ldots, e_{t+m-1} = x^{m-1}, d \geq t + m - 1$. Use $\Delta$ on (8). By (7), lemma 2.6 and commutativity of $H$,
\[
(x \otimes 1 + 1 \otimes x)^m = \sum_{j=0}^{m-1} \alpha_j (x \otimes 1 + 1 \otimes x)^j + w',
\]
\[
w' \in H \otimes HH_r^+ + HH_r^+ \otimes H.
\]
Note that commutativity of $H$ is necessary only here to show, that $w'$ really lies in $H \otimes HH^+_r + HH^+_r \otimes H$. For each integer $q \geq 1$

\[(x \otimes 1 + 1 \otimes x)^q = \sum_{i=0}^{q} \binom{q}{i} x^i \otimes x^{q-i}.\]

Subtracting from (9) equation

\[1 \otimes x^m = \sum_{j=0}^{m-1} \alpha_j 1 \otimes x^j + 1 \otimes w,\]

by (8), (9) we get

\[
\sum_{i=1}^{m} \binom{m}{i} x^i \otimes x^{m-i} = \sum_{j=1}^{m-1} \alpha_j \sum_{i=1}^{j} \binom{j}{i} x^i \otimes x^{j-i} + w'', \tag{10}
\]

\[w'' \in H \otimes (HH^+_r) + (HH^+_r) \otimes H.\]

Since char $k = 0$, so $\binom{m}{i} = m \neq 0$ in $k$. From this and (10) it follows that element $x \otimes x^{m-1} = e_{t+1} \otimes e_{t+m-1}$ in $H \otimes H$ is linear combination of elements $e_s \otimes e_{s'}$, where either $s$ is less then $t + 1$ or $s'$ is less then $t + m - 1$. But it is impossible, because elements $e_q \otimes e_{q'}$, $q, q' = 1, \ldots, d$, form basis of $H \otimes H$. This contradiction shows that $x(A_r) = 0$. So in this case we define $A_{r+1} = A_r = \cdots = A_G$. \[\square\]

Notice that reasoning shown above is close to that used in [5], §5.5, §5.6.

We have come to main

**Theorem 2.7** Let $A$ be an affine $H$-module algebra, $H$ – finite dimensional pointed Hopf algebra and one of three conditions is satisfied:

1. char $k = p > 0$;
2. $H$ – commutative;
3. $A$ – integral domain.

Then extension $A/A^H$ is integral.

**Proof.** Assume that $H$ is commutative, then $A^G$ is $H$-module algebra. It is sufficiently to show that $A^G$ is stable under $H$-action. In fact, for each $x \in H$, $a \in A$,

\[gx(a) = xg(a) = x(a),\]
i.e., \( x(a) \in A^G \). Let \( I \) denote the ideal in \( H \) generated by the elements of form \( g - 1 \ (g \in G) \). By proposition 2.3 \( I \) is Hopf ideal. Obviously, it acts as zero on \( A^G \). Hopf algebra \( H/I \) is pointed by proposition 2.4, moreover, it is connected. Thus the second case of this theorem is reduced to consideration of action of connected, commutative Hopf algebra \( H/I \) on algebra \( A^G \). Now we apply theorem 2.5 to all cases. Let

\[
A = A_{-1} \supseteq A_0 \supseteq \cdots \supseteq A_n
\]

be constructed chain of subalgebras. By condition 1) of theorem 2.5, extension \( A/A_n \) is integral; by condition 2) \( A_n \subseteq A^H \). \( \square \)

Note that if \( \text{char } k = p > 0 \), then \( (A^G)^{\text{dim } H} \subseteq A^H \). If \( H \) is commutative and \( \text{char } k = 0 \), then \( A^H = A^G \).

3 Counterexample to hypothesis

**Example 3.1** Hopf algebra \( H \) may be any from series \( A_{N, \xi} \), \( N \geq 2 \). Algebra \( A_{N, \xi} \) is generated by elements \( g, x \) with relations

\[
g^N = 1, \quad x^N = 0, \quad xg = \xi gx, \quad (11)
\]

where \( \xi \in k \) – root of unity of degree \( N \). Hopf algebra structure on \( A_{N, \xi} \) is given as follows:

\[
\Delta(g) = g \otimes g, \quad \varepsilon(g) = 1, \quad S(g) = g^{N-1} = g^{-1},
\]

\[
\Delta(x) = g \otimes x + x \otimes 1, \quad \varepsilon(x) = 0, \quad S(x) = -g^{N-1}x.
\]

We demand \( \xi \neq 1 \) and \( \text{char } k = 0 \). Algebra \( A_{N, \xi} \) is pointed,

\[
G(A_N) = \{1, g, g^2, \ldots, g^{N-1}\},
\]

it is non-commutative and non-cocommutative.

Let \( A \) be commutative algebra generated by elements \( y, z \) with relation \( z^2 = 0 \). Define action \( A_{N, \xi} \) on \( A \):

\[
g(y^n) = y^n, \quad g(y^n z) = \xi^{-1} y^n z, \quad x(y^n) = n y^{n-1} z, \quad x(y^n z) = 0.
\]

Straightforward computations shows the correctness of this action, i.e., \( A_{N, \xi}(I) \subseteq I \), where \( I \) is ideal of free algebra \( k < y, z > \) generated by elements \( yz - zy, z^2 \). We check that relations (11) in \( H \) are satisfied:

\[
\xi gx(y^n) = \xi^{-1} ny^{n-1} z = ny^{n-1} z = xg(y^n),
\]

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\[ \xi gx(y^n z) = 0 = xg(y^n z), \]
\[ g^N(y^n) = y^n, \quad g^N(y^n z) = \xi^{-N} y^n z = y^n z, \]
\[ x^2(y^n) = nx(y^{n-1} z) = 0, \quad x^2(y^n z) = x(0) = 0, \]
i.e., \( x^N(a) = x^2(a) = 0 \) for any \( a \in A \).

Obviously \( A^G = k[y] \) and \( A^H = k[y] \cap \text{ker} \ x = k \). But extension \( A/A^H \) is not integral, because \( A \) is not finite \( k \)-module ( \( \dim_k A = \infty \) ).

4 Conclusion

As it was shown in example 3.1, hypothesis 1.3 is not true in general. Nevertheless all known examples of Hopf algebra action shows that if \( A \) – affine, then \( A^H \) is also affine algebra, although extension \( A/A^H \) is not always integral. So we ask

**Question 4.1** Finite dimensional Hopf algebra \( H \) acts on commutative affine algebra \( A \). Is \( A^H \) affine?

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