Skeleton Ideals of Certain Graphs, Standard Monomials and Spherical Parking Functions

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Abstract

Let $G$ be a graph on the vertex set $V = \{0, 1, \ldots, n\}$ with root 0. Postnikov and Shapiro were the first to consider a monomial ideal $M_G$, called the $G$-parking function ideal, in the polynomial ring $R = \mathbb{K}[x_1, \ldots, x_n]$ over a field $\mathbb{K}$ and explained its connection to the chip-firing game on graphs. The standard monomials of the Artinian quotient $\frac{R}{M_G}$ correspond bijectively to $G$-parking functions. Dochtermann introduced and studied skeleton ideals of the graph $G$, which are subideals of the $G$-parking function ideal with an additional parameter $k$ ($0 \leq k \leq n - 1$). A $k$-skeleton ideal $M_G^{(k)}$ of the graph $G$ is generated by monomials corresponding to non-empty subsets of the set of non-root vertices $[n]$ of size at most $k + 1$. Dochtermann obtained many interesting homological and combinatorial properties of these skeleton ideals. In this paper, we study the $k$-skeleton ideals of graphs and for certain classes of graphs provide explicit formulas and combinatorial interpretation of standard monomials and the Betti numbers.

Mathematics Subject Classifications: 05E40, 13D02

1 Introduction

Let $G$ be a graph on the vertex set $V = \{0, 1, \ldots, n\}$ with a root 0. The graph $G$ is completely determined by a symmetric $(n + 1) \times (n + 1)$ matrix $A(G) = [a_{ij}]_{0 \leq i, j \leq n}$, called its adjacency matrix, where $a_{ij}$ is the number of edges from $i$ to $j$. Let $R = \mathbb{K}[x_1, \ldots, x_n]$ be the standard polynomial ring in $n$ variables over a field $\mathbb{K}$. The $G$-parking function ideal $M_G$ of $G$ is a monomial ideal in $R$ given by the generating set

$$M_G = \langle m_A : \emptyset \neq A \subseteq [n] = \{1, \ldots, n\} \rangle,$$

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where \( m_A = \prod_{i \in A} x_i^{d_A(i)} \) and \( d_A(i) = \sum_{j \in V \setminus A} a_{ij} \) is the number of edges from \( i \) to a vertex outside the set \( A \) in \( G \). The standard monomial basis \( \{ x^b = \prod_{i=1}^n x_i^{b_i} \} \) of the Artinian quotient \( \frac{R}{\mathcal{M}_G} \) is determined by the set

\[
P\mathbb{F}(G) = \{ b = (b_1, \ldots, b_n) \in \mathbb{N}^n : x^b \notin \mathcal{M}_G \}
\]

of \( G \)-parking functions. Further, \( \dim_k \left( \frac{R}{\mathcal{M}_G} \right) \) is the number of spanning trees of \( G \), given by the determinant \( \det(L_G) \) of the reduced Laplacian matrix \( L_G \) of \( G \). Let \( \text{SPT}(G) \) be the set of spanning trees of \( G \). The edges of a spanning tree of \( G \) are given orientation so that all paths in the spanning tree are directed away from the root. As \( |P\mathbb{F}(G)| = |\text{SPT}(G)| \), one would like to construct an explicit bijection \( \phi : P\mathbb{F}(G) \to \text{SPT}(G) \). Using the Depth-First-Search version of burning algorithm, an algorithmic bijection \( \phi : P\mathbb{F}(G) \to \text{SPT}(G) \) for simple graphs \( G \), preserving reverse sum \( rsum(\mathcal{P}) \) of \( G \)-parking function \( \mathcal{P} \) and the number \( \kappa(G, \phi(\mathcal{P})) \) of \( \kappa \)-inversions of the spanning tree \( \phi(\mathcal{P}) \), is constructed by Perkinson, Yang and Yu [13]. A similar bijection for multigraphs \( G \) is constructed by Gaydarov and Hopkins [5].

Postnikov and Shapiro [15] introduced the \( G \)-parking function ideal \( \mathcal{M}_G \) and derived many of its combinatorial and homological properties. In particular, they showed that the cellular free complex supported on the first barycentric subdivision \( \mathcal{B}(\Delta_{n-1}) \) of an \( (n-1) \)-simplex \( \Delta_{n-1} \) is a free resolution of \( \mathcal{M}_G \). Further, the cellular resolution of \( \mathcal{M}_G \) is minimal, provided the graph \( G \) is saturated (i.e., \( a_{ij} > 0 \) for \( i \neq j \)). The minimal resolution of the parking function ideal \( \mathcal{M}_G \) for any graph \( G \) is described in [2, 10, 12].

In a series of papers, Dochtermann [3, 4] introduced and studied subideals of the \( G \)-parking function ideal \( \mathcal{M}_G \) described by \( k \)-dimensional ‘skeleta’. For an integer \( k \) \((0 \leq k \leq n-1)\), the \( k \)-skeleton ideal \( \mathcal{M}_G^{(k)} \) of the graph \( G \) is defined as the subideal

\[
\mathcal{M}_G^{(k)} = \langle m_A : \emptyset \neq A \subseteq [n]; |A| \leq k + 1 \rangle
\]

of the monomial ideal \( \mathcal{M}_G \). For \( k = 0 \), the ideal \( \mathcal{M}_G^{(0)} \) is generated by powers of variables \( x_1, \ldots, x_n \). Hence, its minimal free resolution and the number of standard monomials can be easily determined. For \( k = 1 \) and \( G = K_{n+1} \), the minimal resolution of the one-skeleton ideal \( \mathcal{M}_{K_{n+1}}^{(1)} \) is a cocellular resolution supported on the labelled polyhedral complex induced by any generic arrangement of two tropical hyperplanes in \( \mathbb{R}^n \) and the \( i^{th} \) Betti number

\[
\beta_i \left( \frac{R}{\mathcal{M}_{K_{n+1}}^{(1)}} \right) = \sum_{j=1}^{n} j \binom{j-1}{i-1} \quad \text{for} \quad 1 \leq i \leq n-1
\]

(see [3]). Also, the number of standard monomials of \( \frac{R}{\mathcal{M}_{K_{n+1}}^{(1)}} \) is given by

\[
\dim_k \left( \frac{R}{\mathcal{M}_{K_{n+1}}^{(1)}} \right) = (2n-1)(n-1)^{n-1} = \det(Q_{K_{n+1}}),
\]
where $Q_{K_{n+1}}$ is the reduced signless Laplacian matrix of $K_{n+1}$.

In this paper, we determine all the Betti numbers of the $k$-skeleton ideal $M_{K_{n+1}}^{(k)}$ of the complete graph $K_{n+1}$. The crucial observation is an identification of the ideal $M_{K_{n+1}}^{(k)}$ with an Alexander dual of some multipermutohedron ideal. We first describe a permutohedron and an associated permutohedron ideal. Let $u = (u_1, u_2, \ldots, u_n) \in \mathbb{N}^n$ such that $u_1 < u_2 < \cdots < u_n$ and let $\mathcal{S}_n$ be the set of permutations of $[n]$. For a permutation $\sigma$ of $[n]$, let $\sigma u = (u_{\sigma(1)}, \ldots, u_{\sigma(n)})$ and $x^\sigma = \prod_{i=1}^n x_i^{u_{\sigma(i)}}$. The convex hull of all permutations $\sigma u$ of $u$ in $\mathbb{R}^n$ is an $(n-1)$-dimensional polytope $P(u)$, called a permutohedron.

Also, the monomial ideal $I(u)$ is called a permutohedron ideal. If some coordinates of $u = (u_1, u_2, \ldots, u_n)$ are allowed to be equal, then the polytope $P(u)$ is called a multipermutohedron and the monomial ideal $I(u)$ is called a multipermutohedron ideal.

The multigraded Betti numbers of multipermutohedron ideals are described in [7]. Also, a combinatorial description of multigraded Betti numbers of Alexander duals of multipermutohedron ideals is given in [8]. Now from the identification of $M_{K_{n+1}}^{(k)}$ with an Alexander dual of some multipermutohedron ideal, we obtain a combinatorial expression for the $(i-1)^{th}$ Betti number $\beta_{i-1} \left( M_{K_{n+1}}^{(1)} \right)$ (Theorem 12). In particular, for $n \geq 3$, we show that $\beta_{i-1} \left( M_{K_{n+1}}^{(1)} \right) = \binom{n+1}{i+1}$ and $\beta_{i-1} \left( M_{K_{n+1}}^{(n-2)} \right)$ as in Corollary 13.

The main object of study in this paper are spherical $G$-parking functions. A finite sequence $P = (p_1, \ldots, p_n) \in \mathbb{N}^n$ is called a $G$-parking function if $x^P = \prod_{i=1}^n x_i^{p_i} \notin \mathcal{M}_G$, on the other hand, the sequence $P = (p_1, \ldots, p_n)$ is called a spherical $G$-parking function if $x^P \in \mathcal{M}_G \setminus \mathcal{M}_G^{(n-2)}$. A $G$-parking or a spherical $G$-parking function $P = (p_1, \ldots, p_n) \in \mathbb{N}^n$ can be equivalently thought of as a function $P : [n] \to \mathbb{N}$ with $P(i) = p_i$ ($1 \leq i \leq n$). The sum (or degree) of $P$ is given by $\text{sum}(P) = \sum_{i \in [n]} P(i)$. Let

$$\text{PF}(G) = \{ P \in \mathbb{N}^n : x^P \notin \mathcal{M}_G \} \text{ and } \text{SPF}(G) = \{ P \in \mathbb{N}^n : x^P \in \mathcal{M}_G \setminus \mathcal{M}_G^{(n-2)} \}$$

be the sets of $G$-parking functions and spherical $G$-parking functions, respectively. The standard monomials of $R_{\mathcal{M}_G^{(n-2)}}$ are of the form $x^P$ for $P \in \text{PF}(G)$ or $P \in \text{SPF}(G)$. Thus,

$$\dim_k \left( R_{\mathcal{M}_G^{(n-2)}} \right) = \dim_k \left( R_{\mathcal{M}_G} \right) + \dim_k \left( \mathcal{M}_G \setminus \mathcal{M}_G^{(n-2)} \right) = |\text{PF}(G)| + |\text{SPF}(G)|.$$

A notion of spherical $K_{n+1}$-parking functions is introduced in [4]. We recall that a $K_{n+1}$-parking function $P = (p_1, \ldots, p_n) \in \mathbb{N}^n$ is an ordinary parking function of length $n$, i.e., a non-decreasing rearrangement $p_1 \leq p_2 \leq \cdots \leq p_n$ of $P = (p_1, \ldots, p_n)$ satisfies $p_{i+1} < j$, for all $j$. It can be easily checked that $P = (p_1, \ldots, p_n) \in \mathbb{N}^n$ is a spherical $K_{n+1}$-parking function if a non-decreasing rearrangement $p_1 \leq p_2 \leq \cdots \leq p_n$ of $P = (p_1, \ldots, p_n)$ satisfies $p_1 = 1$ and $p_{i+1} < j$ for $2 \leq j \leq n$. The notion of spherical $K_{n+1}$-parking function has appeared earlier in the literature (see [16]) as prime parking functions of length $n$. Prime parking functions were defined and enumerated by Ira Gessel. The number of spherical $K_{n+1}$-parking functions is $(n-1)^{n-1}$, which is same as the number of
uprooted trees on the vertex set \([n]\). A (labelled) rooted tree \(T\) on the vertex set \([n]\) is called uprooted if the root is bigger than all its children. Let \(U_n\) be the set of uprooted trees on the vertex set \([n]\). Dochtermann conjectured existence of a bijection \(\phi_n : \text{sPF}(K_{n+1}) \rightarrow U_n\) such that \(\sum(P) = \genus(G)\) for all uprooted trees \(P\) in the complete graph \(K_n = K_{n+1} - \{0\}\) on the vertex set \([n]\).

For a simple graph \(G\) on the vertex set \(V\) whose root 0 is connected to all other vertices, we construct an injective map \(\phi_G : \text{sPF}(G) \rightarrow U(G')\), where \(G' = G - \{0\}\) and \(U(G')\) is the set of uprooted spanning trees of \(G'\). Moreover, the injective map \(\phi_G\) satisfies

\[
\sum(P) = g(G) - \kappa(G, \phi_G(P)) + 1 \quad \text{for all} \quad P \in \text{sPF}(G),
\]

where \(g(G)\) is the genus of the graph \(G\) (Theorem 20). We have determined the image of \(\phi_G\) for many simple graphs \(G\). In particular, we show that the map \(\phi_{K_{n+1}} = \phi_n : \text{sPF}(K_{n+1}) \rightarrow U_n\) is a bijection and establish a conjecture of Dochtermann on spherical \(K_{n+1}\)-parking functions.

If \(e\) is an edge of \(G\), then \(G - \{e\}\) is the graph obtained from \(G\) by deleting the edge \(e\). We show that \(|\text{sPF}(G)| = |\text{sPF}(G - \{e_0\})|\) (Lemma 17), where \(e_0\) is an edge from the root to another vertex. As an application, we observe that \(|\text{sPF}(K_{m+1,n})| = |\text{sPF}(K_{n+1,m})|\) for complete bipartite graphs (Proposition 33). If \(e_1\) is an edge in the complete graph \(K_{n+1}\), not through the root, we show that \(|\text{sPF}(K_{n+1} - \{e_1\})| = (n-1)^{a-3}(n-2)^2\) (Theorem 31). In this case, spherical \((K_{n+1} - \{e_1\})\)-parking functions correspond bijectively with some specified subset of uprooted trees on the vertex set \([n]\) (Theorem 23).

Some extensions of these results for the complete multigraph \(K_{n+1}^{a,b}\) and the complete bipartite multigraph \(K_{m+1,n}^{a,b}\) \((a, b \geq 1)\) are also obtained.

Remark 1. This paper is motivated by [3] and an earlier version of [4] posted on the arXiv. In the new version of [4], Dochtermann and King identify the standard monomials of \(k\)-skeleton ideals \(\mathcal{M}_{K_{n+1}}^{(k)}\) with the vector parking functions and using a Breadth-First-Search burning algorithm, they construct a bijection from spherical \(K_{n+1}\)-parking functions to uprooted spanning trees of \(K_n\) that takes degree to an inversion statistic. In this paper, we obtain the standard monomials and the Betti numbers of \(\mathcal{M}_{K_{n+1}}^{(k)}\) by identifying it with an Alexander dual of some multipermutohedron ideal. For constructing bijection, we use a Depth-First-Search variant of burning algorithm.

\section{Parking functions and Depth-First-Search algorithms}

In this section, we briefly describe some known results on parking functions and the Depth-First-Search algorithms. Most of the known results are stated without proof. These results and notions will be used in the subsequent sections of this paper.

\subsection{Parking functions}

A sequence \(P = (p_1, \ldots, p_n) \in \mathbb{N}^n\) is called an ordinary parking function of length \(n\), if a non-decreasing rearrangement \(p_1 \leq p_2 \leq \cdots \leq p_n\) of \(P\) satisfies \(p_j < j\) for \(1 \leq j \leq n\).
We denote the set of ordinary parking functions of length \( n \) by \( \text{PF}(n) \). The notion of ordinary parking function has a nice generalization.

**Definition 2.** Let \( \lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{N}^n \) with \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 1 \). A finite sequence \( \mathcal{P} = (p_1, \ldots, p_n) \in \mathbb{N}^n \) is called a \( \lambda \)-parking function of length \( n \), if a non-decreasing rearrangement \( p_{i_1} \leq p_{i_2} \leq \cdots \leq p_n \) of \( \mathcal{P} \) satisfies \( p_{i_j} < \lambda_{n-j+1} \) for \( 1 \leq j \leq n \). Let \( \text{PF}(\lambda) \) be the set of \( \lambda \)-parking functions.

Clearly, the ordinary parking functions of length \( n \) are precisely \( \lambda \)-parking functions of length \( n \) for \( \lambda = (n, n-1, \ldots, 2, 1) \in \mathbb{N}^n \). The number of \( \lambda \)-parking functions is given by the ‘so-called’ Steck determinantal formula (see [14]). Let

\[
\Lambda(\lambda_1, \ldots, \lambda_n) = \begin{bmatrix}
\frac{\lambda_j^{i-1+j}}{(j-i+1)!} \\
\end{bmatrix}_{1 \leq i, j \leq n}.
\]

In other words, the \((i, j)^{th}\) entry of the \( n \times n \) matrix \( \Lambda(\lambda_1, \ldots, \lambda_n) \) is \( \frac{\lambda_j^{i-1+j}}{(j-i+1)!} \), where by convention, \( \frac{1}{(j-i+1)!} = 0 \) for \( i > j + 1 \). The determinant \( \det(\Lambda(\lambda_1, \ldots, \lambda_n)) \) is called a Steck determinant.

**Theorem 3** (Pitman-Stanley). The number of \( \lambda \)-parking functions is given by

\[
|\text{PF}(\lambda)| = (n!) \det(\Lambda(\lambda_1, \ldots, \lambda_n)) = n! \det \begin{bmatrix}
\frac{\lambda_j^{i-1+j}}{(j-i+1)!} \\
\end{bmatrix}_{1 \leq i, j \leq n}.
\]

For \( \lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{N}^n \) with \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 1 \), Postnikov and Shapiro [15] considered the monomial ideal

\[
\mathcal{M}_\lambda = \langle \prod_{j \in A} x_j^{\lambda_{|A|}} : \emptyset \neq A \subseteq [n] \rangle
\]

in the polynomial ring \( R = \mathbb{K}[x_1, \ldots, x_n] \). A monomial \( x^b = \prod_{j=1}^n x_j^{b_j} \notin \mathcal{M}_\lambda \) is called a standard monomial of \( \frac{R}{\mathcal{M}_\lambda} \) or \( \mathcal{M}_\lambda \). Clearly, \( x^b = \prod_{j=1}^n x_j^{b_j} \) is a standard monomial of \( \mathcal{M}_\lambda \) if and only if \( b = (b_1, \ldots, b_n) \in \text{PF}(\lambda) \). In other words, a monomial basis of the \( \mathbb{K} \)-vector space \( \frac{R}{\mathcal{M}_\lambda} \) correspond bijectively with the \( \lambda \)-parking functions.

**Theorem 4** (Pitman-Stanley, Postnikov-Shapiro). The dimension of \( \frac{R}{\mathcal{M}_\lambda} \) is given by

\[
\dim_{\mathbb{K}} \left( \frac{R}{\mathcal{M}_\lambda} \right) = |\text{PF}(\lambda)| = \sum_{(b_1, \ldots, b_n) \in \text{PF}(n)} \prod_{i=1}^n (\lambda_{n-b_i} - \lambda_{n-b_i+1}),
\]

where the summation runs over ordinary parking functions of length \( n \) and \( \lambda_{n+1} = 0 \).

A closed formula for the number of \( \lambda \)-parking functions for various specific values of \( \lambda \) is given in [14, 17]. For more on parking functions, we refer to an excellent survey article by Yan [18].
2.2 Graph theoretic notions and G-parking functions

Let $G$ be a connected graph on the vertex set $V(G) = V = \{0, 1, \ldots, n\}$. Suppose $A(G) = [a_{ij}]_{0 \leq i,j \leq n}$ is the (symmetric) adjacency matrix of $G$. We assume that $G$ is a loopless graph, i.e., $a_{ii} = 0$ for all $i$. Let $E(i,j) = E(j,i)$ be the set of edges between distinct pair of vertices $i,j \in V$. If $E(i,j) \neq \emptyset$, then $i$ and $j$ are called adjacent vertices and we write $i \sim j$. On the other hand, if $i$ and $j$ are non-adjacent, we write $i \sim j$. We have $|E(i,j)| = a_{ij}$. The graph $G$ is called a simple graph if $|E(i,j)| = a_{ij} \leq 1$ for $i,j \in V$. Otherwise, $G$ is called a multigraph. The set $E(G) = \bigcup_{i,j \in V} E(i,j)$ is the set of edges of $G$.

If $v \in V$, then $G - \{v\}$ denotes the graph on the vertex set $V - \{v\}$ obtained from $G$ by deleting the vertex $v$ and all the edges through $v$. If $e \in E(G)$ is an edge of $G$, then $G - \{e\}$ denotes the graph on the vertex set $V$ obtained from $G$ by deleting the edge $e$. If $E(i,j) \neq \emptyset$, then $G - E(i,j)$ denotes the graph on vertex set $V$ obtained from $G$ on deleting all the edges between $i$ and $j$.

Fix a root $r \in V$ of $G$ (usually, we take $r = 0$). Set $\tilde{V} = V \setminus \{r\}$. Let $\text{SPT}(G)$ be the set of spanning trees of $G$ rooted at $r$. We orient spanning tree $T \in \text{SPT}(G)$ so that all paths in $T$ are directed away from the root $r$. For every $j \in \tilde{V}$, there is a unique oriented path in $T$ from the root $r$ to $j$. An $i \in \tilde{V}$ lying on this unique path in $T$ is called an ancestor of $j$ in $T$. Equivalently, we say that $j$ is a descendent of $i$ in $T$. If in addition, $i$ and $j$ are adjacent in $T$, then we say that $i$ is a parent of its child $j$. Every child $j$ has a unique parent $\text{par}_T(j)$ in $T$.

**Definition 5.** By an inversion of $T \in \text{SPT}(G)$, we mean an ordered pair $(i,j)$ of vertices such that $i$ is an ancestor of $j$ in $T$ with $i > j$. The total number of inversions of a spanning tree $T$ is denoted by $\text{inv}(T)$. An inversion $(i,j)$ of $T$ is called a $\kappa$-inversion of $T$ if $i$ is not the root $r$ and $\text{par}_T(i)$ is adjacent to $j$ in $G$.

The invariant $g(G) = |E(G)| - |V(G)| + 1$ is called the genus of the graph $G$. The $\kappa$-number $\kappa(G,T)$ of $T$ in $G$ is given by

$$\kappa(G,T) = \sum_{i,j \in \tilde{V}; i > j} |E(\text{par}_T(i),j)|.$$

For a simple graph $G$, the total number of $\kappa$-inversions of $T$ is $\kappa(G,T)$. If $G = K_{n+1}$ with root $0$, then $\kappa(K_{n+1},T) = \text{inv}(T)$ for every $T \in \text{SPT}(K_{n+1})$.

**Definition 6.** Let $G$ be a graph on the vertex set $V = \{0, 1, \ldots, n\}$ with the adjacency matrix $A(G) = [a_{ij}]_{0 \leq i,j \leq n}$. Let $r \in V$ be the root of $G$ and $\tilde{V} = V \setminus \{r\}$. A function $\mathcal{P} : \tilde{V} \rightarrow \mathbb{N}$ is called a $G$-parking function (with respect to the root $r$) if for every non-empty set $A \subseteq \tilde{V}$, there exists $i \in A$ such that $\mathcal{P}(i) < d_A(i) = \sum_{j \in V \setminus A} a_{ij}$.

Note that, if root $r = 0$, then $\mathcal{P}$ is a $G$-parking function if and only if $x^\mathcal{P} \notin \mathcal{M}_G$, i.e., $x^\mathcal{P}$ is a standard monomial of the $G$-parking function ideal $\mathcal{M}_G$. For a $G$-parking
function \( P : \tilde{V} \to \mathbb{N} \), the sum \( \text{sum}(P) \) and the reverse sum \( \text{rsum}(P) \) of \( P \) are respectively given by

\[
\text{sum}(P) = \sum_{i \in \tilde{V}} P(i) \quad \text{and} \quad \text{rsum}(P) = g(G) - \sum_{i \in \tilde{V}} P(i).
\]

**Definition 7.** A rooted tree on the vertex set \([n]\) is called an **uprooted tree** if the root is bigger than all its children.

Let \( U_n \) be the set of uprooted trees on the vertex set \([n]\). Then it is well known that \( |U_n| = (n - 1)^{n-1} \). For certain graphs \( G \) on the vertex set \( V \), we shall show that the spherical \( G \)-parking functions correspond to uprooted spanning trees of \( G' = G - \{0\} \).

### 2.3 Depth-First-Search Algorithms

We now describe the Depth-First-Search burning algorithm of Perkinson-Yang-Yu [13] for simple graphs. Let \( G \) be a simple graph on the vertex set \( V \) with a root \( r \in V \). Applied to an input function \( P : V \setminus \{r\} \to \mathbb{N} \), the Depth-First-Search algorithm of Perkinson-Yang-Yu [13] gives a subset \( \text{burnt vertices} \) of burnt vertices and a subset \( \text{tree edges} \) of tree edges as an output. We imagine that a fire starts at the root \( r \) and spreads to other vertices of \( G \) according to the depth-first rule. The value \( P(j) \) of the input function \( P \) can be considered as the number of water droplets available at vertex \( j \) that prevents spread of fire to \( j \). If \( i \) is a burnt vertex, then consider the largest non-burnt vertex \( j \) adjacent to \( i \). If \( P(j) = 0 \), then fire from \( i \) will spread to \( j \). In this case, add \( j \) in \( \text{burnt vertices} \) and include the edge \((i, j)\) in \( \text{tree edges} \). Now the fire spreads from the burnt vertex \( j \). On the other hand, if \( P(j) > 0 \), then one water droplet available at \( j \) will be used to prevent fire from reaching \( j \) through the edge \((i, j)\). In this case, the dampened edge \((i, j)\) is removed from \( G \), number of water droplets available at \( j \) is reduced to \( P(j) - 1 \) and the fire continue to spread from the burnt vertex \( i \) through non-dampened edges. If all the edges from \( i \) to unburnt vertices get dampened, then the search backtracks. At the start, \( \text{burnt vertices} = \{r\} \) and \( \text{tree edges} = \{\} \).

Perkinson, Yang and Yu [13] constructed a bijection \( \phi : \text{PF}(G) \to \text{SPT}(G) \) using their Depth-First-Search algorithm.

**Theorem 8** (Perkinson-Yang-Yu). Let \( G \) be a simple graph on the vertex set \( V \) with root \( r \). After applying Depth-First-Search burning algorithm to \( P : V \setminus \{r\} \to \mathbb{N} \), if \( \text{burnt vertices} = V \), then \( P \) is a \( G \)-parking function and tree edges in the set \( \text{tree edges} \) form a spanning tree \( \phi(P) \) of \( G \). If \( \text{burnt vertices} \neq V \), then \( P \) is not a \( G \)-parking function. Further, the mapping \( P \mapsto \phi(P) \) given by the Depth-First-Search algorithm induces a bijection \( \phi : \text{PF}(G) \to \text{SPT}(G) \) such that

\[
\text{rsum}(P) = g(G) - \text{sum}(P) = \kappa(G, \phi(P)) \quad \text{for all} \quad P \in \text{PF}(G).
\]
Let $\sum_{P \in PF(G)} q^{rsum(P)}$ be the reversed sum enumerator for $G$-parking functions. Theorem 8 establishes the identity

$$\sum_{P \in PF(G)} q^{rsum(P)} = \sum_{T \in SPT(G)} q^{\kappa(G,T)},$$

that extends a similar identity obtained by Kreweras [6] for the complete graph $K_{n+1}$.

We now describe the Depth-First-Search burning algorithm of Gaydarov-Hopkins [5] for multigraphs. Consider a connected multigraph $G$ on the vertex set $V$ with root $r$. Let $E(i,j) = E(j,i)$ be the set of edges between distinct pair of vertices $i$ and $j$. Fix a total order on $E(i,j)$ for all distinct pairs $\{i,j\}$ of vertices and write $E(i,j) = \{e^0_{ij}, e^1_{ij}, \ldots, e^{a_{ij}-1}_{ij}\}$, where $|E(i,j)| = a_{ij}$. Thus we assume that edges of the multigraph $G$ are labelled. Applied to an input function $P : V \setminus \{r\} \to \mathbb{N}$, the Depth-First-Search algorithm for multigraphs gives a subset burnt vertices of burnt vertices and a subset tree edges of tree edges with nonnegative labels on them as an output. As in the case of Depth-First-Search algorithm for simple graphs, we imagine that a fire starts at the root $r$ and spread to other vertices of $G$ according to the depth-first rule. If $i$ is a burnt vertex, then consider the largest non-burnt vertex $j$ adjacent to $i$. If $P(j) < a_{ij} = |E(i,j)|$, then $P(j)$ edges with higher labels, namely $e^{a_{ij}-P(j)}_{ij} - 1$ will be added to tree edges and $j$ included in burnt vertices. Now fire will spread from the burnt vertex $j$. On the other hand, if $P(j) \geq a_{ij}$, then all the edges in $E(i,j)$ get dampened and $P(j)$ reduced to $P(j) - a_{ij}$. The fire continue to spread from the burnt vertex $i$ through non-dampened edges. If all the edges from $i$ to unburnt vertices get dampened, then the search backtracks. At the start, burnt vertices = $\{r\}$ and tree edges = $\emptyset$. Gaydarov and Hopkins [5] extended Theorem 8 to multigraphs using the Depth-First-Search burning algorithm for multigraph.

**Theorem 9** (Gaydarov-Hopkins). Let $G$ be a multigraph on $V$ with root $r$. After applying Depth-First-Search burning algorithm to $P : V \setminus \{r\} \to \mathbb{N}$, if burnt vertices = $V$, then $P$ is a $G$-parking function and tree edges with labels in the set tree edges form a labelled spanning tree $\phi(P)$ of $G$. If burnt vertices $\neq V$, then $P$ is not a $G$-parking function. Suppose $\ell(e)$ is the label on an edge $e$ of $\phi(P)$. Then the mapping $P \mapsto \phi(P)$ given by Depth-First-Search burning algorithm induces a bijection $\phi : PF(G) \to SPT(G)$ such that

$$rsum(P) = \kappa(G,T) + \sum_{e \in E(T)} \ell(e) \quad \text{for all} \quad P \in PF(G), \quad \text{where} \quad T = \phi(P).$$

The bijective map induced by the Depth-First-Search algorithms is always denoted by $\phi$ in this paper ignoring its dependence on the graph and the root.
3 \(k\)-skeleton ideals of complete graphs

Let \(0 \leq k \leq n - 1\). Consider the \(k\)-skeleton ideal \( \mathcal{M}_{K_{n+1}}^{(k)} \) of the complete graph \(K_{n+1}\) on the vertex set \(V = \{0, 1, \ldots, n\}\). As stated in the Introduction, we have

\[
\mathcal{M}_{K_{n+1}}^{(k)} = \left\langle \prod_{j \in A} x_j^{n-|A|+1} \rightangle : \emptyset \neq A \subseteq [n]; |A| \leq k + 1 \right\rangle.
\]

For \(k = 0\), \( \mathcal{M}_{K_{n+1}}^{(0)} = \langle x_1^{n}, \ldots, x_n^{n} \rangle \) is a monomial ideal in \(R\) generated by \(n\)th power of variables. Thus, its minimal free resolution is given by the Koszul complex associated to the regular sequence \(x_1, \ldots, x_n\) in \(R\). Also, \( \dim_k \left( \frac{R}{\mathcal{M}_{K_{n+1}}^{(0)}} \right) = n^n \). For \(k = n - 1\), \( \mathcal{M}_{K_{n+1}}^{(n-1)} = \mathcal{M}_{K_{n+1}} \). The minimal free resolution of the \(K_{n+1}\)-parking function ideal \( \mathcal{M}_{K_{n+1}} \) is the cellular resolution supported on the first barycentric subdivision \( \text{Bd}(\Delta_{n-1}) \) of an \(n-1\)-simplex \(\Delta_{n-1}\) and

\[
\dim_k \left( \frac{R}{\mathcal{M}_{K_{n+1}}} \right) = |\text{PF}(K_{n+1})| = |\text{SPT}(K_{n+1})| = (n+1)^{n-1}.
\]

For \(k = 1\), the 1-skeleton ideal \( \mathcal{M}_{K_{n+1}}^{(1)} \) has a minimal cocellular resolution supported on the labelled polyhedral complex induced by any generic arrangement of two tropical hyperplanes in \(\mathbb{R}^{n-1}\) (see Theorem 4.6 of [3]) and \( \dim_k \left( \frac{R}{\mathcal{M}_{K_{n+1}}^{(1)}} \right) = (2n-1)(n-1)^{n-1} \).

3.1 Betti numbers of \( \mathcal{M}_{K_{n+1}}^{(k)} \)

We now express the \(k\)-skeleton ideal \( \mathcal{M}_{K_{n+1}}^{(k)} \) of \(K_{n+1}\) as an Alexander dual of a multipermutohedron ideal. Let \(u = (u_1, u_2, \ldots, u_n) \in \mathbb{N}^n\) such that \(u_1 \leq u_2 \leq \ldots \leq u_n\). For \(m = (m_1, \ldots, m_s)\) such that the smallest entry in \(u\) is repeated exactly \(m_1\) times, second smallest entry in \(u\) is repeated exactly \(m_2\) times, and so on. Then \(\sum_{j=1}^{s} m_j = n\) and \(m_j \geq 1\) for all \(j\). In this case, we write \(u(m)\) for \(u\). The monomial ideal \(I(u(m)) = \langle x_1^{su(m)} : \sigma \in \mathfrak{S}_n \rangle\) of \(R\) is called a multipermutohedron ideal. If \(m = (1, \ldots, 1) \in \mathbb{N}^n\), then \(I(u(m))\) is a permutohedron ideal.

Let \(u(m) = (1, 2, \ldots, k, k+1, \ldots, n) \in \mathbb{N}^n\), where \(m = (1, \ldots, 1, n-k) \in \mathbb{N}^{k+1}\). For \(k = 0\), \(u(m) = (1, \ldots, 1) \in \mathbb{N}^n\), while for \(k = n - 1\), \(u(m) = (1, 2, \ldots, n) \in \mathbb{N}^n\). Let \(I(u(m))[n]\) be the Alexander dual of the multipermutohedron ideal \(I(u(m))\) with respect to \(n = (n, \ldots, n) \in \mathbb{N}^n\).

**Theorem 10.** For \(0 \leq k \leq n - 1\), \( \mathcal{M}_{K_{n+1}}^{(k)} = I(u(m))[n] \).

**Proof.** Using Proposition 5.23 of [11], it follows from the Lemma 2.3 of [8]. \(\square\)

Let \(b = (b_1, \ldots, b_n) \in \mathbb{N}^n\). The \((i-1)\)th multigraded Betti number \(\beta_{i-1,b}(\mathcal{M}_{K_{n+1}}^{(k)})\) of \(\mathcal{M}_{K_{n+1}}^{(k)}\) in degree \(b\) is given by

\[
\beta_{i-1,b}(\mathcal{M}_{K_{n+1}}^{(k)}) = \dim_k \tilde{H}^{\text{Supp}(b)} -1 \left( K_b(\mathcal{M}_{K_{n+1}}^{(k)}); \mathbb{K} \right) \quad \text{for} \quad i \geq 1,
\]
where $K_b(M_{K_{n+1}}^{(k)})$ is the lower Koszul simplicial complex of $M_{K_{n+1}}^{(k)}$ in degree $b$ and $\text{Supp}(b) = \{j : b_j > 0\}$ (see Theorem 5.11 of [11]). Since $M_{K_{n+1}}^{(k)} = I(u(m))^{[n]}$, a combinatorial description of all multidegrees $b$ such that $\beta_{k-1,b}(M_{K_{n+1}}^{(k)}) \neq 0$ is given in terms of dual $m$-isolated subsets (see Definition 3.1 and Theorem 3.2 of [8]). For the particular case of $m = (1, \ldots, 1, n-k) \in \mathbb{N}^{k+1}$, the notion of dual $m$-isolated subsets can be easily described. Let $J = \{j_1, \ldots, j_t\} \subseteq [n]$ be a non-empty subset with $0 = j_0 < j_1 < \cdots < j_t$.

1. $J$ is a dual $m$-isolated subset of type-1 if $J \subseteq [k+1]$ and its dual weight $\text{dwt}(J) = t-1$.

Let $I_m^{(1)}$ be the set of dual $m$-isolated subsets of type-1 and let $I_m^{(1)}(i) = \{J \in I_m^{(1)} : \text{dwt}(J) = i\}$.

2. $J = \{j_1, \ldots, j_t\}$ is a dual $m$-isolated subset of type-2 if $J \setminus \{j_i\} \subseteq [k]$, $k+1 < j_i \leq n$ and its dual weight $\text{dwt}(J) = (t-2) + (j_i - k)$. Let $I_m^{(2)}$ be the set of dual $m$-isolated subsets of type-2 and let $I_m^{(2)}(i) = \{J \in I_m^{(2)} : \text{dwt}(J) = i\}$.

Let $I_m = I_m^{(1)} \bigcup I_m^{(2)}$ be the set of all dual $m$-isolated subsets and $I_m(i) = I_m^{(1)}(i) \bigcup I_m^{(2)}(i)$.

Consider $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ with $\lambda_i = n - i + 1$ for $1 \leq i \leq k$ and $\lambda_i = n - k$ for $k+1 \leq i \leq n$. Let $e_1, \ldots, e_n$ be the standard basis vectors of $\mathbb{R}^n$. For $0 < i < j \leq n$, we set $\varepsilon(i, j) = \sum_{i+1}^j e_i$. For any $J = \{j_1, \ldots, j_t\} \in I_m$, let $b(J) = \sum_{a=1}^t \lambda_{j_a} \varepsilon(j_{a-1}, j_a) \in \mathbb{N}^n$.

We illustrate the concept of dual $m$-isolated subsets and its relation with multigraded Betti numbers with an example.

**Example 11.** Let $n = 6$ and $k = 2$. Take $u(m) = (1, 2, 3, 3, 3, 3)$. Then $m = (1, 1, 4)$ and $\lambda = (6, 5, 4, 4, 4, 4)$. Consider the multipermutohedron ideal $I(u(m))$ and the 2-skeleton ideal $M_{K_{6+1}}^{(2)}$. Set $6 = (6, 6, 6, 6, 6)$. The Alexander dual $I(u(m))^{[6]} = M_{K_{6+1}}^{(2)}$.

A subset $J \subseteq [3]$ is a dual $m$-isolated subset of type-1. For example, $J = \{2\}$ and $\tilde{J} = \{1, 3\}$ are dual $m$-isolated subsets of type-1 with dual weights 0 and 1, respectively. Also, the associated multidegrees are $b(J) = (5, 5, 0, 0, 0, 0)$ and $b(\tilde{J}) = (6, 4, 4, 0, 0, 0)$. The lower Koszul simplicial complex $K_b(M_{K_{6+1}}^{(2)})$ for $b = b(J)$ or $b(\tilde{J})$ is isomorphic to the 0-dimensional simplicial complex consisting of two points. Thus $\beta_{0,b(J)}(M_{K_{6+1}}^{(2)}) = 1$ and $\beta_{1,b(\tilde{J})}(M_{K_{6+1}}^{(2)}) = 1$. Further, the subsets $J' = \{4\}$ and $J'' = \{1, 5\}$ are examples of dual $m$-isolated subsets of type-2 with dual weights 1 and 3, respectively. We have $b(J') = (4, 4, 4, 4, 4, 4)$ and $b(J'') = (6, 4, 4, 4, 4, 4)$. The lower Koszul simplicial complex $K_{b(J')}M_{K_{6+1}}^{(2)}$ is isomorphic to the 0-skeleton of a 3-simplex, while $K_{b(J'')}M_{K_{6+1}}^{(2)}$ is isomorphic to the 1-skeleton of a 3-simplex. Therefore $\beta_{1,b(J')}M_{K_{6+1}}^{(2)} = 3$ and $\beta_{3,b(J'')}M_{K_{6+1}}^{(2)} = 3$.

**Theorem 12.** For $b = (b_1, \ldots, b_n) \in \mathbb{N}^n$ and $1 \leq i \leq n$, let $\beta_{i-1,b}(M_{K_{n+1}}^{(k)})$ be the $(i - 1)^{th}$ multigraded Betti number of $M_{K_{n+1}}^{(k)}$ in degree $b$. Then the following statements hold.

(i) For $J = \{j_1, \ldots, j_t\} \in I_m^{(1)}(i-1)$, $\beta_{i-1,b(J)}(M_{K_{n+1}}^{(k)}) = 1$, where $t = i$. 

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Corollary 13. Assume that $M$ be helpful in constructing a concrete minimal resolution of $\mathcal{M}^{(k)}_{K_{n+1}} = (j_{n-j_{i-1}}^{k-j_i-1})$, where $t + j_i - k = i + 1$.

(iii) If $b = \pi b(J)$ is a permutation of $b(J)$ for some $J \in \mathcal{I}^*_m(i-1)$ and some $\pi \in \mathcal{S}_n$, then $\beta_{i-1,b} \left( \mathcal{M}^{(k)}_{K_{n+1}} \right) = \beta_{i-1,b} \left( \mathcal{M}^{(k)}_{K_{n+1}} \right)$. Otherwise, $\beta_{i-1,b} \left( \mathcal{M}^{(k)}_{K_{n+1}} \right) = 0$.

(iv) The $(i-1)^{th}$ Betti number $\beta_{i-1} \left( \mathcal{M}^{(k)}_{K_{n+1}} \right) \cdot \mathcal{M}^{(k)}_{K_{n+1}}$ is given by,

$$
\beta_{i-1} \left( \mathcal{M}^{(k)}_{K_{n+1}} \right) = \beta_i \left( \frac{R}{\mathcal{M}^{(k)}_{K_{n+1}}} \right) = \sum_{J \in \mathcal{I}^{(i-1)}_m} \beta_{i-1}^J + \sum_{\bar{J} \in \mathcal{I}^{(i-1)}_m} \beta_{i-1}^{\bar{J}},
$$

where $\beta_{i-1}^J = \prod_{a=1}^i \left( j_{a+1}^{n+1} \right)$ and $\beta_{i-1}^{\bar{J}} = \left[ \prod_{a=1}^i \left( l_{a+1} \right) \right] \left( n - l_i - 1 \right)$ for $J = \{ j_1, \ldots, j_i \} \in \mathcal{I}^{(i-1)}_m$ and $\bar{J} = \{ \ell_1, \ldots, \ell_t \} \in \mathcal{I}^{(i-2)}_m \left( i - 1 \right)$. Here, $j_i = \ell_i = n$ and $l_0 = 0$.

Proof. Since $\mathcal{M}^{(k)}_{K_{n+1}} = \mathcal{I} \left( \mathcal{U} \left( \mathcal{M} \right) \right)^n$, theorem follows from Theorem 3.2 and Corollary 3.4 of [8].

Theorem 12 describes all multigraded Betti numbers of $\mathcal{M}^{(k)}_{K_{n+1}}$. We hope that it could be helpful in constructing a concrete minimal resolution of $\mathcal{M}^{(k)}_{K_{n+1}}$.

Corollary 13. Assume that $n \geq 3$ and $1 \leq i \leq n$. Then $\beta_{i-1} \left( \mathcal{M}^{(1)}_{K_{n+1}} \right) = i \left( n+1 \right)$ and

$$
\beta_{i-1} \left( \mathcal{M}^{(n-2)}_{K_{n+1}} \right) = \sum_{j} \frac{n!}{j_1! (j_2 - j_1)! \cdots (n - j_i)!} + \sum_{\ell} \frac{n!(n - l_{i-2} - 1)}{l_1! (l_2 - l_1)! \cdots (n - l_{i-2})!},
$$

where the first and second summations run over all sequences of integers $j = (j_1, \ldots, j_i)$ with $0 < j_1 < \cdots < j_i < n$ and $\ell = (l_0, l_1, \ldots, l_{i-2})$ with $0 = l_0 < l_1 < \cdots < l_{i-2} < n - 1$, respectively.

Proof. For $k = 1$, we have $\mathcal{M} = (1,n-1) \in \mathbb{N}^2$. We can easily see that $\mathcal{I}^*_m(i-1) = \{ \{ 1, i \}, \{ i+1 \} \}$ for $i \geq 2$ and $\mathcal{I}^*_m(0) = \{ \{ 1 \}, \{ 2 \} \}$. Thus, $\beta_0(\mathcal{M}^{(1)}_{K_{n+1}}) = \beta_0^{(1)} + \beta_0^{(2)} = \binom{n}{1} + \binom{n}{2} = \binom{n+1}{2}$. For $i \geq 2$,

$$
\beta_{i-1}(\mathcal{M}^{(1)}_{K_{n+1}}) = \beta_{i-1}^{(1,i)} + \beta_{i-1}^{(i+1)} = \binom{i}{1} \left( \binom{n}{i} \left( \binom{i-2}{0} + \frac{n}{i+1} \right) \right)\binom{i}{1} = \binom{i}{1},
$$

which is same as $\beta_i \left( \frac{R}{\mathcal{M}^{(1)}_{K_{n+1}}} \right) = \sum_{j=1}^n j(j-1) = \sum_{j=1}^n i(i) = (i) \sum_{j=1}^n (i) = \binom{n+1}{i}$ obtained in [3].
For $k = n - 2$, $J = \{j_1, \ldots, j_t\} \in \mathcal{I}^t_{kn+1}(i - 1)$ if and only if $J \subseteq [n - 1]$ and $\beta^J_{i-1} = \prod_{a=1}^i (\lambda^J_{a+1})$. Also, $\tilde{J} = \{l_1, \ldots, l_t\} \in \mathcal{I}^t_{kn+2}(i - 1)$ if and only if $l_{t-1} \leq n - 2$, $l_t = n$ and $t = i - 1$. Since, $\beta^J_{i-1} = \left[\prod_{a=1}^{i-2} (\lambda^J_{a+1})\right]_{(n-l_{t-2} - 1)}^{(n-l_{t-2} - 1)}$, we get the desired expression for $\beta_{i-1} \left(\mathcal{M}_{kn+2}^{(n-2)}\right)$.

Consider the first barycentric subdivision $\mathcal{Bd}^t(\Delta_{n-1})$ of an $n - 1$-simplex $\Delta_{n-1}$. We construct a polyhedral cell complex $\mathcal{Bd}^t(\Delta_{n-1})$ whose vertices are the vertices of $\mathcal{Bd}^t(\Delta_{n-1})$ corresponding to subsets $A \subseteq [n]$ with $|A| \leq k + 1$. An edge in $\mathcal{Bd}^t(\Delta_{n-1})$ corresponds either to a chain $A_1 \subseteq A_2 \subseteq [n]$ with $|A_2| \leq k + 1$ or a pair $\{A, B\}$ of subsets of $[n]$ with $|A| = |B| = k + 1$ and $|A \setminus B| = 1$. The higher dimensional faces of $\mathcal{Bd}^t(\Delta_{n-1})$ are polytopes spanned by its edges. A vertex of $\mathcal{Bd}^t(\Delta_{n-1})$ corresponding to $A$ with $|A| \leq k + 1$ has a natural label $\left(\prod_{j \in A} x_j\right)_{n - |A| + 1}$. The cellular resolution supported on the polyhedral cell complex $\mathcal{Bd}^t(\Delta_{n-1})$ is a non-minimal resolution of $\mathcal{M}_{Kn+1}^{(k)}$ if $1 \leq k \leq n - 2$. The minimal cellular resolution of $\mathcal{M}_{Kn+1}^{(1)}$ constructed in [3] can be obtained by deleting certain edges of the polyhedral cell complex $\mathcal{Bd}^1(\Delta_3)$.

### 3.2 Standard monomials of $\mathcal{M}_{Kn+1}^{(k)}$

A monomial $x^b = \prod_{j=1}^n x_j^{b_j} \notin \mathcal{M}_{Kn+1}^{(k)}$ is called a standard monomial of $\frac{R}{\mathcal{M}_{Kn+1}^{(k)}}$ or $\mathcal{M}_{Kn+1}^{(k)}$.

Let $\lambda = (\lambda_1, \ldots, \lambda_n)$, where $\lambda_i = n - i + 1$ for $1 \leq i \leq k$ and $\lambda_j = n - k$ for $k + 1 \leq j \leq n$. We have seen that $I(u(m))^{[n]} = \mathcal{M}_{Kn+1}^{(k)} = \mathcal{M}_\lambda$. In view of Theorem 4, the number of standard monomials of $\mathcal{M}_{Kn+1}^{(k)}$ is precisely the number of $\lambda$-parking functions and $\dim_{K} \left(\frac{R}{\mathcal{M}_{Kn+1}^{(k)}}\right) = |\text{PF}(\lambda)| = n! \det(A(n, n - 1, \ldots, n - k + 1, n - k, \ldots, n - k))$.

More generally, for $a, b \geq 1$, we consider the complete multigraph $K_{n+1}^{a,b}$ on the vertex set $V$ with adjacency matrix $A(K_{n+1}^{a,b}) = [a_{ij}]_{0 \leq i, j \leq n}$ given by $a_{0,i} = a_{i,0} = a$ and $a_{i,j} = b$ for $i, j \in V \setminus \{0\};\ i \neq j$. In other words, $K_{n+1}^{a,b}$ has exactly $b$ number of edges between the root $0$ and any other vertex $i$, while it has exactly $b$ number of edges between distinct non-root vertices $i$ and $j$. Clearly, $K_{n+1}^{a,1} = K_{n+1}$. The $k$-skeleton ideal $\mathcal{M}_{Kn+1}^{(k)}$ of $K_{n+1}^{a,b}$ is given by

$$\mathcal{M}_{Kn+1}^{(k)} = \left\{ \prod_{j \in A} x_j^{a + (n - |A|)b} : \emptyset \neq A \subseteq [n]; |A| \leq k + 1 \right\}.$$

Let $\lambda_i^{a,b} = (\lambda_1^{a,b}, \ldots, \lambda_n^{a,b})$, where $\lambda_i^{a,b} = a + (n - i)b$ for $1 \leq i \leq k$ and $\lambda_j^{a,b} = a + (n - k - 1)b$ for $k + 1 \leq j \leq n$. Then, $\mathcal{M}_{Kn+1}^{(k)} = \mathcal{M}_{\lambda^{a,b}}$ and from Theorem 4,

$$\dim_{K} \left(\frac{R}{\mathcal{M}_{Kn+1}^{(k)}}\right) = n! \det(A(\lambda_1^{a,b}, \ldots, \lambda_n^{a,b})).$$
We proceed to evaluate the Steck determinant and compute the number of standard monomials of $\mathcal{M}_{k_n+b}^{(k)}$. Consider the polynomial
\[ f_n(x) = \det(\Lambda(x + (n-1)b, x + (n-2)b, \ldots, x + b, x)) \]
in an indeterminate $x$. In other words, we have
\[ f_n(x) = \det \begin{bmatrix} \frac{x}{n!} & \frac{x^2}{2!} & \frac{x^3}{3!} & \cdots & \frac{x^{n-1}}{(n-1)!} & \frac{x^n}{n!} \\ 1 & \frac{x+b}{1!} & \frac{(x+b)^2}{2!} & \cdots & \frac{(x+b)^{n-2}}{(n-2)!} & \frac{(x+b)^{n-1}}{(n-1)!} \\ 0 & 1 & \frac{x+2b}{1!} & \cdots & \frac{(x+2b)^{n-3}}{(n-3)!} & \frac{(x+2b)^{n-2}}{(n-2)!} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{(x+(n-2)b)}{1!} & \frac{(x+(n-2)b)^2}{(n-1)!} \\ 0 & 0 & 0 & \cdots & 1 & \frac{(x+(n-1)b)}{n!} \end{bmatrix}. \]
The polynomial $f_n(x) = \frac{x(x+b)^{n-1}}{n!}$ and $\dim_k \left( \frac{R}{\mathcal{M}_{k_n+b}} \right) = a(a + nb)^{n-1}$ (see [14, 15]). Also, for $1 \leq k \leq n - 2$, consider another polynomial $g_{n,k}(x)$ in $x$ given by
\[ g_{n,k}(x) = \det(\Lambda(x + kb, x + (k-1)b, \ldots, x + b, x, \ldots, x)), \]
where the last $n - k$ coordinates in $(x + kb, x + (k-1)b, \ldots, x + b, x, \ldots, x)$ are $x$.

**Proposition 14.** The polynomial $g_{n,k}(x)$ is given by
\[ g_{n,k}(x) = \sum_{j=0}^{k} \frac{1}{j!(n-j)!} (x-j)(k+1)^{j-1}b^j. \]

**Proof.** We first give a simple proof of $f_n(x) = \frac{x(x+b)^{n-1}}{n!}$ as in [9]. Clearly, $f_1(x) = x$ and $f_2(x) = \frac{x(x+2b)}{2!}$. Proceeding by induction on $n$, we assume that $f_j(x) = \frac{x(x+jb)^{j-1}}{j!}$ for $1 \leq j \leq n - 1$. Further, using properties of determinants, we observe that the derivative $f'_n(x)$ of $f_n(x)$ satisfies $f'_n(x) = f_{n-1}(x+b)$. This shows that $f'_n(x) = \frac{(x+b)(x+nb)^{n-2}}{(n-1)!}$. As $f_n(0) = 0$, on integrating $f'_n(x)$ by parts, we get $f_n(x) = \frac{x(x+nb)^{n-1}}{n!}$.

Again using properties of determinants, we see that the $(n-k-1)^{th}$ derivative $g_{n,k}^{(n-k-1)}(x)$ of $g_{n,k}(x)$ satisfies
\[ g_{n,k}^{(n-k-1)}(x) = f_{k+1}(x) = \frac{x(x+(k+1)b)^k}{(k+1)!} = \sum_{j=0}^{k} \binom{k}{j} x^{k-j+1} \frac{(k+1)^j b^j}{(k+1)!}. \]
Since $g_{n,k}(0) = g'_{n,k}(0) = \cdots = g_{n,k}^{(n-k-1)}(0) = 0$ and the $(n-k-1)^{th}$ derivative of $\frac{x^n}{(n-j)(n-j-1) \cdots (n-j-k+2)}$ is $x^{k-j+1}$, we get $g_{n,k}(x) = \sum_{j=0}^{k} \binom{k}{j} x^{n-j} \frac{(k+1)^j b^j}{(n-j)(n-j-1) \cdots (k+1)!}$. \[\square\]
Theorem 15 (Yan). The number of standard monomials of \( \frac{R}{M^{(b)}_{K_{n+1}}^{a,b}} \) is given by

\[
\dim_K \left( \frac{R}{M^{(k)}_{K_{n+1}}^{a,b}} \right) = \sum_{j=0}^{k} \binom{n}{j} (a + (n - k - 1)b)^{n-j}(k - j + 1)(k + 1)^{j-1}b^j.
\]

In particular, we have \( \dim_K \left( \frac{R}{M^{(1)}_{K_{n+1}}^{a,b}} \right) = (a + (n - 2)b)^{n-1}(a + (2n - 2)b) \) for \( k = 1 \) and \( \dim_K \left( \frac{R}{M^{(n-2)}_{K_{n+1}}^{a,b}} \right) = a(a + nb)^{n-1} + (n - 1)^{n-1}b^n \) for \( k = n - 2 \).

Proof. The first part follows from \( \dim_K \left( \frac{R}{M^{(k)}_{K_{n+1}}^{a,b}} \right) = n! \ g_{n,k}(a + (n - k - 1)b) \) using Proposition 14.

For \( k = 1, g_{n,1}^{(n-2)}(x) = f_2(x) = \frac{x(x+2b)}{2!} = \frac{x^2}{2!} + bx \). As \( g_{n,1}^{(j)}(0) = 0 \) for \( 0 \leq j \leq n - 2 \), we obtain \( g_{n,1}(x) = \frac{x^n}{n!} + \frac{b x^{n-1}}{(n-1)!} = \frac{x^{n-1}(x + nb)}{n!} \).

Now \( \dim_K \left( \frac{R}{M^{(1)}_{K_{n+1}}^{a,b}} \right) = n! \ g_{n,1}(a + (n - 2)b) = (a + (n - 2)b)^{n-1}(a + (2n - 2)b) \).

Also, for \( k = n - 2 \), we have \( g_{n,n-2}^{(n-2)}(x) = f_{n-1}(x) = \frac{x(x+(n-1)b)^{n-1}}{(n-1)!} - \frac{(x+(n-1)b)^{n}}{n!(n-1)!} + C \), where \( C \) is a constant of integration. Since \( g_{n,n-2}(0) = 0 \), we get \( C = \frac{(n-1)^{n-1}b^n}{n!} \). Hence,

\[
g_{n,n-2}(x) = \frac{1}{n!} [(x - b)(x + (n - 1)b)^{n-1} + (n - 1)^{n-1}b^n].
\]

Again, from \( \dim_K \left( \frac{R}{M^{(n-2)}_{K_{n+1}}^{a,b}} \right) = n! \ g_{n,n-2}(a + b) \), we get the desired result.

Remark 16. The determinant \( \det(Q_{K_{n+1}}^{-}) \) of the reduced signless Laplacian matrix \( Q_{K_{n+1}}^{-} \) of \( K_{n+1}^{a,b} \) satisfies \( \dim_K \left( \frac{R}{M^{(n-2)}_{K_{n+1}}^{a,b}} \right) = (a + (n - 2)b)^{n-1}(a + (2n - 2)b) = \det(Q_{K_{n+1}}^{-}) \). Also, we have \( g_{n,n-2}^{(n-2)}(x) = f_{n-1}(x) = \frac{x(x+(n-1)b)^{n-2}}{(n-1)!} - \frac{(x+(n-1)b)^{n}}{n!(n-1)!} + C \), where \( C \) is a constant of integration. Since \( g_{n,n-2}(0) = 0 \), we get \( C = \frac{(n-1)^{n-1}b^n}{n!} \). Thus on integrating \( g_{n,n-2}(x) \) in two ways, we get \( g_{n,n-2}(x) \) and a polynomial identity

\[
\frac{(x - b)(x + (n - 1)b)^{n-1} + (n - 1)^{n-1}b^n}{n!} = \sum_{j=0}^{n-2} \binom{n}{j} x^{n-j}(n - j - 1)(n - 1)^{j-1}b^j.
\]
On substituting $x = a + b$, we get an identity
\[ \sum_{j=0}^{n-2} \binom{n}{j} (a + b)^{n-j}(n - j - 1)(n - 1)^{j-1}b^j = a(a + nb)^{n-1} + (n - 1)^{n-1}b^n \]
for positive integers $a$ and $b$. Taking $a = b = 1$, it justifies the equality
\[ \sum_{j=0}^{n-2} \binom{n}{j} 2^{n-j}(n - j - 1)(n - 1)^{j-1} = (n + 1)^{n-1} + (n - 1)^{n-1} \]
described in [4](Corollary 3.7).

### 4 Spherical G-parking functions

Let $G$ be a connected graph on the vertex set $V = \{0, 1, \ldots, n\}$ with root 0. As stated in the Introduction, $\mathcal{P} : [n] \to \mathbb{N}$ is a spherical $G$-parking function if $x^\mathcal{P} = \prod_{i \in [n]} x_i^{p(i)} \in \mathcal{M}_G \setminus \mathcal{M}_G^{(n-2)}$. Let $PF(G)$ (or $sPF(G)$) be the set of $G$-parking functions (respectively, spherical $G$-parking functions).

Let $e_0$ be an edge of $G$ joining the root 0 to another vertex. We shall compare $sPF(G)$ with $sPF(\tilde{G})$, where $\tilde{G} = G - \{e_0\}$. After renumbering vertices, we may assume that $e_0 = e_{0,n}$ is an edge joining the root 0 with $n$.

**Lemma 17.** Let $G$ be a connected graph on the vertex set $V$ and $\tilde{G} = G - \{e_0\}$. Then
\[ \mathcal{M}_G = (\mathcal{M}_G : x_n) = \{ z \in R : zx_n \in \mathcal{M}_G \}. \]
Further, the multiplication map $\mu_{x_n} : \{ x^\mathcal{P} : \mathcal{P} \in sPF(\tilde{G}) \} \to \{ x^\mathcal{P} : \mathcal{P} \in sPF(G) \}$ induced by $x_n$ is a bijection. In particular, $|sPF(G)| = |sPF(\tilde{G})|$.

**Proof.** For $\emptyset \neq A \subseteq [n]$, let $m_A$ and $m_A'$ be the generators of $\mathcal{M}_G$ and $\mathcal{M}_G$, respectively. Clearly, $m_A = m_A'$ if $n \notin A$ and $m_A = m_A x_n$ if $n \in A$. This shows that $\mathcal{M}_G = (\mathcal{M}_G : x_n)$. Also, $\mathcal{M}_G^{(n-2)} = (\mathcal{M}_G^{(n-2)} : x_n)$. Thus the natural sequences of $R$-modules (or $\mathbb{K}$-vectors spaces)
\[ 0 \to R/\mathcal{M}_G \xrightarrow{\mu_{x_n}} R/\mathcal{M}_G \to R/\mathcal{M}_G : x_n \to 0 \quad \text{and} \quad 0 \to R/\mathcal{M}_G^{(n-2)} \xrightarrow{\mu_{x_n}} R/\mathcal{M}_G^{(n-2)} \to R/\mathcal{M}_G^{(n-2)} : x_n \to 0 \]
are short exact. Let $\alpha : R/\mathcal{M}_G^{(n-2)} \to R/\mathcal{M}_G$ and $\beta : R/\mathcal{M}_G^{(n-2)} \to R/\mathcal{M}_G$ be the natural projections.

Since $\langle \mathcal{M}_G, x_n \rangle = \langle \mathcal{M}_G^{(n-2)}, x_n \rangle$, the multiplication map $\mu_{x_n}$ induces an isomorphism $\ker(\alpha) \xrightarrow{\sim} \ker(\beta)$ between kernels $\ker(\alpha)$ and $\ker(\beta)$. Also $\{ x^\mathcal{P} : \mathcal{P} \in sPF(\tilde{G}) \}$ and $\{ x^\mathcal{P} : \mathcal{P} \in sPF(G) \}$ are monomial basis of $\ker(\alpha)$ and $\ker(\beta)$, respectively. Thus $\mu_{x_n}$ induces a bijection between the bases. \qed

We now give a few applications of the Lemma 17.
Proposition 18. Let $E$ be the set of all edges of $K_{n+1}$ or $K_{n+1}^{a,b}$ through the root 0. Then

1. $|sPF(K_{n+1} - E)| = |sPF(K_{n+1})|.$
2. $|sPF(K_{n+1}^{a,b} - E)| = |sPF(K_{n+1}^{a,b})|.$
3. $|sPF(K_{n+1}^{a,b})| = b^n(n - 1)^{n-1}.$

Proof. By Lemma 17, we know that the number of spherical $G$-parking functions and the number of spherical $(G - \{e_0\})$-parking functions are the same for any edge $e_0$ of $G$ through the root 0. Now, repeatedly applying Lemma 17, we see that (1) and (2) hold.

Let $\lambda = ((n - 1)b, (n - 2)b, \ldots, 2b, b, b).$ Consider the graph $K_{n+1}^{a,b} - E$ and its $(n - 2)$-skeleton ideal $M_{K_{n+1}^{a,b} - E}^{(n-2)}.$ Clearly, $M_{K_{n+1}^{a,b} - E}^{(n-2)} = M_{\lambda}.$ As $K_{n+1}^{a,b} - E$ is disconnected, $PF(K_{n+1}^{a,b} - E) = \emptyset.$ Thus

$$|sPF(K_{n+1}^{a,b})| = |sPF(K_{n+1}^{a,b} - E)| = \dim_k \left( \frac{R}{M_{K_{n+1}^{a,b} - E}^{(n-2)}} \right) = |	ext{PF}(\lambda)| = (n!)g_{n,n-2}(b) = b^n(n - 1)^{n-1},$$

where the polynomial $g_{n,n-2}(x)$ is given in the Remark 16. \hfill \square

Note that the cardinality $|sPF(K_{n+1}^{a,b})|$ is independent of $a.$ As we have seen that $|\text{PF}(K_{n+1}^{a,b})| = a(a + bn)^{n-1},$ $|sPF(K_{n+1}^{a,b})| = b^n(n - 1)^{n-1}$ also follows from Theorem 15.

4.1 A modified Depth-First-Search burning algorithm

Let $G$ be a connected simple graph on the vertex set $V$ with a root 0. Let $\mathcal{M}_G = \langle m_A : \emptyset \neq A \subseteq [n] \rangle$ be the $G$-parking function ideal. For a spherical $G$-parking function $P \in sPF(G),$ define $\tilde{P} : [n] \rightarrow \mathbb{N}$ so that $x^\tilde{P} = \frac{x^P}{m_{[n]}},$ where $m_{[n]}$ is the generator of $\mathcal{M}_G$ corresponding to $[n].$ We say that $\tilde{P}$ is the reduced spherical $G$-parking function associated to $P \in sPF(G).$ Let $sPF(G) = \{\tilde{P} : P \in sPF(G)\}$ be the set of reduced spherical $G$-parking functions. We shall analyse the condition $sPF(G) \subseteq PF(G).$ Since removing (or adding) edges from the root 0 to another vertex in $G$ do not change the number of spherical $G$-parking functions (Lemma 17), we may assume that the root 0 is connected to all the other vertices in $G.$ In this case, $m_{[n]} = x_1x_2\cdots x_n$ and $\tilde{P}(i) = P(i) - 1$ for $i \in [n].$

Lemma 19. Let $G$ be a connected simple graph on the vertex set $V$ with a root 0. Suppose the root 0 is connected to all other vertices of $G.$ Then

1. $\tilde{PF}(G) \subseteq PF(G).$
2. Let $P \in sPF(G)$ and $r \in [n]$ be the unique vertex such that $\tilde{P}(r) = 0$ but $\tilde{P}(j) \geqslant 1$ for $j > r.$ Consider the graph $G' = G - \{0\}$ on the vertex set $[n]$ with root $r.$ Then $\tilde{P} = \tilde{P}_{[n]\{r\}}$ is a $G'$-parking function.
Let $\mathcal{P} \in \text{sPF}(G)$ such that $\tilde{\mathcal{P}} \notin \text{PF}(G)$. Then there exists $\emptyset \neq A \subseteq [n]$ such that $m_A \mid x^\mathcal{P}$, i.e., $m_A$ divides $x^\mathcal{P}$. Thus $m_A m_{[n]} \mid x^\mathcal{P}$. If $A \neq [n]$, then $m_A \mid x^\mathcal{P}$, a contradiction to $\mathcal{P} \in \text{sPF}(G)$. Also, if $A = [n]$, then $(m_{[n]})^2 \mid x^\mathcal{P}$. Since $G$ is a simple graph and the root 0 is connected to all other vertices of $G$, $m_B \mid (x_1 x_2 \cdots x_n)^2$ for any $B \subseteq [n]$ with $|B| = n - 1$. Again a contradiction. This proves the first part.

Let $\mathcal{P} \in \text{sPF}(G)$. If $\tilde{\mathcal{P}}(i) \geq 1$ for all $i \in [n]$, then $\mathcal{P}(i) \geq 2$ for all $i$. Thus $(m_{[n]})^2 \mid x^\mathcal{P}$, which leads to a contradiction. Thus $\tilde{\mathcal{P}}(i) = 0$ for some $i$. Let $r = \max\{i \in [n] : \mathcal{P}(i) = 0\}$.

Now consider the graph $G' = G - \{0\}$ on the vertex set $[n]$ with root $r$. When we emphasize the root $r$ of $G'$, we denote this graph by $(G', r)$. Let $\mathcal{M}_{(G', r)} = \{\mathcal{P} : 0 \neq A \subseteq [n] \setminus \{r\}\}$ be the $G'$-parking function ideal in the polynomial ring $\mathbb{K}[x_1, \ldots, \hat{x}_r, \ldots, x_n]$. We see that $\tilde{\mathcal{P}} = \mathcal{P}|_{[n] \setminus \{r\}}$ is not a $G'$-parking function, then $\tilde{\mathcal{P}}(i) = 0$, $x^\mathcal{P} = \prod_{i \in [n] \setminus \{r\}}(x_i)^{\tilde{\mathcal{P}}(i)} = x^{\mathcal{P}}_{m_{[n]}}.$

Thus $m_A \mid x^\mathcal{P}$, a contradiction to $\mathcal{P} \in \text{sPF}(G)$. □

We now proceed to associate uprooted trees to spherical parking functions by modifying the Depth-First-Search burning algorithm. Let $G$ be a connected simple graph satisfying the hypothesis of Lemma 19. Let $\mathcal{P} \in \text{sPF}(G)$ and $\hat{\mathcal{P}}$ be the associated reduced spherical $G$-parking function. In the following three steps, an uprooted spanning tree of $G'$ is associated to each $\mathcal{P} \in \text{sPF}(G)$.

1. Set $r = \max\{i \in [n] : \hat{\mathcal{P}}(i) = 0\}$ and consider the graph $G' = G - \{0\}$ with root $r$.

2. Let $\phi : \text{PF}(G', r) \to \text{SPT}(G', r)$ be the bijective map induced by Depth-First-Search algorithm (Theorem 8). As $\bar{\mathcal{P}} = \hat{\mathcal{P}}|_{[n] \setminus \{r\}}$ is a $(G', r)$-parking function, $\phi(\hat{\mathcal{P}})$ is a spanning tree of $G'$. Also, $\sum(\bar{\mathcal{P}}) = g(G') - \kappa(G', \phi(\hat{\mathcal{P}}))$.

3. Since $\bar{\mathcal{P}} \in \text{PF}(G', r)$ and $\bar{\mathcal{P}}(j) \geq 1$ for all $j > r$, there exists $i < r$ such that $\bar{\mathcal{P}}(i) = 0$. On applying the Depth-First-Search algorithm to $\hat{\mathcal{P}}$, all the edges $(r, j)$ for $j > r$ get dampened. Thus the spanning tree $\phi(\hat{\mathcal{P}})$ is an uprooted spanning tree of $G'$.

Let $\mathcal{U}(G')$ be the set of uprooted spanning trees of the graph $G'$. We define a map $\phi_G : \text{sPF}(G) \to \mathcal{U}(G')$ given by $\phi_G(\mathcal{P}) = \phi(\hat{\mathcal{P}})$, where $\hat{\mathcal{P}} = \bar{\mathcal{P}}|_{[n] \setminus \{r\}}$. We say that the map $\phi_G$ is induced by a modified Depth-First-Search algorithm.

**Theorem 20.** Let $G$ be a simple graph on the vertex set $V$ with root 0 and $G' = G - \{0\}$. Suppose the root 0 is connected to all other vertices of $G$. Then there exists an injective map $\phi_G : \text{sPF}(G) \to \mathcal{U}(G')$ such that $\sum(\mathcal{P}) = g(G') - \kappa(G', \phi_G(\mathcal{P})) + 1$ for all $\mathcal{P} \in \text{sPF}(G)$.

**Proof.** We have already constructed the map $\phi_G$. Let $\mathcal{P}, \mathcal{P}' \in \text{sPF}(G)$ such that $\phi_G(\mathcal{P}) = \phi_G(\mathcal{P}') = T \in \mathcal{U}(G')$. Let $r$ be the root of $T$. Since $\phi : \text{PF}(G', r) \to \text{SPT}(G', r)$ is a bijection and $\phi(\hat{\mathcal{P}}) = \phi(\hat{\mathcal{P}}')$, we have $\hat{\mathcal{P}} = \hat{\mathcal{P}}'$ and hence $\mathcal{P} = \mathcal{P}'$. Note that $\sum(\mathcal{P}) = \sum(\hat{\mathcal{P}}) + n$ and $g(G) = g(G') + n - 1$. Thus $\sum(\mathcal{P}) = g(G') - \kappa(G', \phi_G(\mathcal{P})) + 1$ follows from $\sum(\hat{\mathcal{P}}) = g(G') - \kappa(G', \phi(\hat{\mathcal{P}}))$. □
Let \( \mathrm{Im}(\phi_G) = \{ \phi_G(\mathcal{P}) : \mathcal{P} \in \mathrm{sPF}(G) \} \) be the image of \( \phi_G \) in \( U(G') \). Theorem 20 shows that under some mild conditions on the simple graph \( G \), the spherical \( G \)-parking functions correspond bijectively with the uprooted trees in \( \mathrm{Im}(\phi_G) \). In general, it is not easy to give a combinatorial description for the image \( \mathrm{Im}(\phi_G) \).

Let \( T \in U(G') \) be an uprooted spanning tree of \( G' = G - \{0\} \). Suppose \( \mathrm{root}(T) = r \). Consider the bijective map \( \phi : \mathrm{PF}(G',r) \to \mathrm{SPT}(G',r) \). Then there exists a unique \((G',r)\)-parking function \( \mathcal{P}_T \) such that \( \phi(\mathcal{P}_T) = T \). Let

\[
\overline{U}(G') = \{ T \in U(G') : \mathcal{P}_T(j) \geq 1 \text{ for } j > r = \mathrm{root}(T) \},
\]

**Proposition 21.** \( \mathrm{Im}(\phi_G) \subseteq \overline{U}(G') = \{ T \in U(G') : \mathcal{P}_T(j) \geq 1 \text{ for } j > r = \mathrm{root}(T) \} \).

**Proof.** Let \( \phi_G(\mathcal{P}) = \hat{\mathcal{P}} \), where \( \hat{\mathcal{P}} = \overline{\mathcal{P}}|_{[n]\{r\}} \). As \( \mathcal{P}_T = \hat{\mathcal{P}} \) and the root is given by \( \mathrm{root}(T) = \max \{ i \in [n] : \hat{\mathcal{P}}(i) = 0 \} \), the result follows. \( \square \)

### 4.2 Spherical parking functions for complete graphs

Let \( K_{n+1} \) be the complete graph on the vertex set \( V \) and \( K_n = K_{n+1} - \{0\} \) be the complete graph on the vertex set \( [n] \). Let \( U_n = U(K_n) \) be the set of uprooted trees on the vertex set \( [n] \). From Theorem 20, there exists an injective map \( \phi_n = \phi_{K_{n+1}} : \mathrm{sPF}(K_{n+1}) \to U_n \). We show that \( \phi_n \) is a bijection and solve a conjecture of Dochtermann on spherical \( K_{n+1} \)-parking functions.

**Theorem 22.** There exists a bijection \( \phi_n : \mathrm{sPF}(K_{n+1}) \to U_n \) such that

\[
\sum(\mathcal{P}) = \left( \begin{array}{c} n \\ 2 \end{array} \right) - \kappa(K_n,\phi_n(\mathcal{P})) + 1 \quad \text{for all } \mathcal{P} \in \mathrm{sPF}(K_{n+1}).
\]

**Proof.** The existence of injective map \( \phi_n = \phi_{K_{n+1}} : \mathrm{sPF}(K_{n+1}) \to U_n \) with the desired property follows from the Theorem 20. We just need to show that \( \phi_n \) is surjective. Let \( T \in U_n \) and \( \mathrm{root}(T) = r \). Consider the bijective map \( \phi : \mathrm{PF}(K_n,r) \to \mathrm{SPT}(K_n,r) \) induced by Depth-First-Search algorithm and \( \mathcal{P}_T \) is the unique \((K_n,r)\)-parking function such that \( \phi(\mathcal{P}_T) = T \). Since \( T \) is uprooted, \( \mathcal{P}_T(j) \geq 1 \) for \( j > r \). Now consider ideals \( \mathcal{M}_{K_{n+1}} = \langle m_A : \emptyset \not\subseteq A \subseteq [n] \rangle \) and \( \mathcal{M}_{K_n,r} = \langle \bar{m}_B : \emptyset \not\subseteq B \subseteq [n] \setminus \{r\} \rangle \).

Suppose, if possible, \( \mathcal{P}_T \neq \hat{\mathcal{P}} \) for all \( \mathcal{P} \in \mathrm{sPF}(K_{n+1}) \). Then \( m|_{[n]\{r\}} \prod_{j \in [n]\{r\}} x_j^{\mathcal{P}_T(j)} \) is not a standard monomial of \( \mathcal{M}_{K_{n+1}}^{(n-2)} \). Thus there exists \( \emptyset \not\subseteq A \subseteq [n] \) such that \( m_A \) divides \( m|_{[n]\{r\}} \prod_{j \in [n]\{r\}} x_j^{\mathcal{P}_T(j)} \). If \( r \in A \), then \( x_r \) appearing in \( m_A = \left( \prod_{j \in A} x_j \right) x_r \) must have the multiplicity 1. This is possible, only if \( A = [n] \), a contradiction. If \( r \not\in A \), then \( \bar{m}_A = \frac{m_A}{\gcd(m_A,m|_{[n]\{r\}})} \) and \( \bar{m}_A \prod_{j \in [n]\{r\}} x_j^{\mathcal{P}_T(j)} \). This shows that \( \mathcal{P}_T \) is not a \((K_n,r)\)-parking function, again a contradiction. Hence \( \phi_n \) is surjective.

The surjectivity of \( \phi_n \) also follows from \( |\mathrm{sPF}(K_{n+1})| = |U_n| = (n-1)^{n-1} \). \( \square \)

We now study spherical \( G \)-parking functions for \( G = K_{n+1} - \{e\} \), where \( e \) is an edge not through the root 0. Let \( e = e_{p,q} = (p,q) \) be the edge in \( K_{n+1} \) joining \( p \) and...
have multiplicity 1. Thus $A \in \text{trees on the vertex set } [n]$ and $U(G)$ be the set of uprooted spanning trees of $G'$. In fact, $U_n^{(p \sim q)} = U(G')$ is the set of uprooted trees on the vertex set $[n]$ with no edge between $p$ and $q$ (i.e., $p \sim q$). Let $\overline{U}_n^{(p \sim q)} = U(G') = \{ T \in U(G') : P_T(j) \geq 1 \text{ for } j > r = \text{root}(T) \}$ as in Proposition 21 and set $U'_n = U_n^{(1 \sim n)}$. In view of Theorem 20 and Proposition 21, there exists an injective map $\phi_G : \text{sPF}(G) \to \overline{U}_n^{(p \sim q)}$.

**Theorem 23.** For $n \geq 3$ and $G = K_{n+1} - \{ e_{p,q} \}$, the map $\phi_G : \text{sPF}(G) \to \overline{U}_n^{(p \sim q)}$ is a bijection such that $\sum(\mathcal{P}) = \binom{n}{2} - \kappa(G', \phi_G(\mathcal{P}))$ for all $\mathcal{P} \in \text{sPF}(G)$, where $G' = G - \{ 0 \}$.

**Proof.** We only need to show that $\text{Im}(\phi_G) = \overline{U}_n^{(p \sim q)}$. This proof is similar to the proof of Theorem 22. Let $T \in \overline{U}(G') = \overline{U}_n^{(p \sim q)}$ and $\text{root}(T) = r$. Consider the bijective map $\phi : \text{PF}(G', r) \to \text{SPT}(G', r)$ induced by Depth-First-Search algorithm and $\mathcal{P}$ is the unique $(G', r)$-parking function such that $\phi(\mathcal{P}) = T$. Let $\mathcal{M}_G = \{ m_A : 0 \neq A \subseteq [n] \}$ and $\mathcal{M}_{(G',r)} = \{ m_A : 0 \neq A \subseteq [n] \setminus \{ r \} \}$ be the parking function ideals. Suppose, if possible, $\mathcal{P}_{T} \neq \mathcal{P}$ for all $\mathcal{P} \in \text{sPF}(G)$. Then $m_{[n]} \prod_{j \in [n] \setminus \{ r \}} x_j^{P_{j}(j)}$ is a standard monomial of $\mathcal{M}_G^{(n-2)}$. Thus there exists $0 \neq A \subseteq [n]$ such that $m_A$ divides $m_{[n]} \prod_{j \in [n] \setminus \{ r \}} x_j^{P_{j}(j)}$.

Let $r \in A$ but $r \notin \{ p, q \}$. As $m_A | m_{[n]} \prod_{j \in [n] \setminus \{ r \}} x_j^{P_{j}(j)}$, $x_r$ appearing in $m_A$ must have multiplicity 1. Thus $A = [n]$, a contradiction. Now suppose $r = q \in A$ (or $r = p \in A$). Then $A \neq [n]$ implies that $A = [n] \setminus \{ p \}$ (respectively, $A = [n] \setminus \{ q \}$). In fact, $m_{[n] \setminus \{ p \}} = \prod_{j \in [n] \setminus \{ p,q \}} x_j^{2}$ and $m_{[n] \setminus \{ q \}} = \prod_{j \in [n] \setminus \{ p,q \}} x_j^{2} x_p$. Clearly, in either of the cases, $\overline{m}_{[n] \setminus \{ p,q \}} = \prod_{j \in [n] \setminus \{ p,q \}} x_j$ divides $\prod_{j \in [n] \setminus \{ r \}} x_j^{P_{j}(j)}$, a contradiction to $\mathcal{P}_{T}$ being $(G', r)$-parking function.

Finally, if $r \notin A$, then $\overline{m}_A = \frac{m_A}{\gcd(m_A, m_{[n]})}$ and $\overline{m}_A$ divides $\prod_{j \in [n] \setminus \{ r \}} x_j^{P_{j}(j)}$. This shows that $\mathcal{P}_{T}$ is not a $(G', r)$-parking function, again a contradiction. This completes the proof. \hfill \Box

We now determine conditions so that $\overline{U}_n^{(p \sim q)} = \overline{U}_n^{(p \sim q)}$.

**Proposition 24.** $\overline{U}_n^{(p \sim q)} \setminus \overline{U}_n^{(p \sim q)} = \{ T \in U_n^{(p \sim q)} : \text{root}(T) = p \text{ and } \mathcal{P}_{T}(q) = 0 \}$.

**Proof.** Let $T \in U_n^{(p \sim q)}$ such that $\text{root}(T) = r \neq p$. Consider the unique $(G', r)$-parking function $\mathcal{P}_{T}$ such that $\phi(\mathcal{P}_{T}) = T$. As $T$ is uprooted, all the edges $(r, j)$ in $G'$ for $j > r$ must get dampened. Thus $\mathcal{P}_{T}(j) \geq 1$ for all $j > r$ such that $r \sim j$ in $G'$ or $G$. Since $G = K_{n+1} - \{ e_{p,q} \}, T \in \overline{U}_n^{(p \sim q)}$. \hfill \Box

Since there are no uprooted tree $T$ on the vertex set $[n]$ with root(T) = 1, it follows from Proposition 24 that $\overline{U}_n^{(p \sim q)} = \overline{U}_n^{(p \sim q)}$ if and only if $p = 1$. The following corollary is immediate.

**Corollary 25.** For $n \geq 3$ and $G = K_{n+1} - \{ e_{1,n} \}$, the map $\phi_G : \text{sPF}(G) \to U_n^{(1 \sim n)}$ induces a bijection between the set of spherical $G$-parking functions and the set of uprooted trees on the vertex set $[n]$ with $1 \sim n$. \hfill \Box

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Remark 26. By renumbering vertices of \( G \), we easily see that
\[
|sPF(K_{n+1} - \{e_{p,q}\})| = |sPF(K_{n+1} - \{e_{1,n}\})| = |\mathcal{U}_n|,
\]
for any edge \( e_{p,q} \) between vertices \( p, q \in [n] \) with \( p < q \). Thus, \(|\mathcal{U}_n^{(p,q)}| = |\mathcal{U}_n^e|\).

The bijection \( \phi_n : sPF(K_{n+1}) \rightarrow \mathcal{U}_n \) constructed in Theorem 22 can be extended to the case of the complete multigraph \( K_{n+1}^{a,b} \) on the vertex set \( V \).

Let \( sPF(K_{n+1}^{a,b}) \) be the set of spherical \( K_{n+1}^{a,b} \)-parking functions. Let \( \mathcal{U}_n^b \) be the set of uprooted tree \( T \) on the vertex set \([n]\) with label \( \ell : E(T) \rightarrow \{0, 1, \ldots, b - 1\} \) on the edges of \( T \) and a weight \( \omega(r) \in \{0, 1, \ldots, b - 1\} \) assigned to the root \( r \) of \( T \). Clearly, \(|\mathcal{U}_n^b| = b^n|\mathcal{U}_n| = b^n(n-1)^{n-1} \). Also, \(|sPF(K_{n+1}^{a,b})| = b^n(n-1)^{n-1} \) is independent of \( a \). We may assume that \( a \geq b \). As an application of the Depth-First-Search algorithm for multigraph (Theorem 9), we construct a bijection

\[
\phi_n^b : sPF(K_{n+1}^{a,b}) \rightarrow \mathcal{U}_n^b.
\]

The reduced spherical \( K_{n+1}^{a,b} \)-parking function \( \hat{P} \) associated to \( P \in sPF(K_{n+1}^{a,b}) \) is given by \( \hat{P}(i) = P(i) - a \) for all \( i \in [n] \). Let \( \hat{sPF}(K_{n+1}^{a,b}) = \{\hat{P} : P \in sPF(K_{n+1}^{a,b})\} \). Then as \( a \geq b \), we can verify that \( sPF(K_{n+1}^{a,b}) \subseteq PF(K_{n+1}^{a,b}) \). Let \( K_n^b = K_{n+1}^{a,b} - \{0\} \) be the complete multigraph on the vertex set \([n]\) such that \(|E(i,j)| = b\) for every distinct pair \( \{i,j\} \) of vertices.

Theorem 27. There exists a bijection \( \phi_n^b : sPF(K_{n+1}^{a,b}) \rightarrow \mathcal{U}_n^b \) such that
\[
rsum(P) + \omega(r) + 1 = \kappa(K_n^b, T) + \sum_{e \in E(T)} \ell(e) \text{ for all } P \in sPF(K_{n+1}^{a,b}),
\]
where \( T = \phi_n^b(P) \) and weight \( \omega(r) \in \{0, 1, \ldots, b - 1\} \) at the root \((T) = r\).

Proof. Let \( P \in sPF(K_{n+1}^{a,b}) \). Then \( \hat{P} \in PF(K_{n+1}^{a,b}) \). Choose the largest vertex \( r \) of \( K_n^b = K_{n+1}^{a,b} - \{0\} \) such that \( \hat{P}(r) < b \). We claim that \( \hat{P}(j) < b \) for some \( j < r \). Otherwise, \( \hat{P}(i) \geq a + b \), for all \( i \in [n] \setminus \{r\} \), a contradiction to \( P \in sPF(K_{n+1}^{a,b}) \). Now consider \( r \) to be the root of the complete multigraph \( K_n^b \) on the vertex set \([n]\). Then \( \hat{P} = \hat{P} \mid_{[n]\setminus\{r\}} \) is a \((K_n^b, r)\)-parking function. On applying the Depth-First-Search algorithm for multigraph (Theorem 9), we get \( \phi(\hat{P}) \in \mathcal{U}_n^b \) with root \( r \) and weight \( \omega(r) = \hat{P}(r) \). The mapping \( \phi_n^b : sPF(K_{n+1}^{a,b}) \rightarrow \mathcal{U}_n^b \) given by \( \phi_n^b(P) = \phi(\hat{P}) \) is clearly injective. Since \(|sPF(K_{n+1}^{a,b})| = |\mathcal{U}_n^b| = b^n(n-1)^{n-1} \), the map \( \phi_n^b \) is a bijection. Also,
\[
g(K_n^b) - \sum_{i \in [n]\setminus\{r\}} \hat{P}(i) = \text{rsum}(\hat{P}) = \kappa(K_n^b, \phi(\hat{P})) + \sum_{e \in E(\phi(\hat{P}))} \ell(e).
\]
Since \( \text{rsum}(P) = g(K_{n+1}^{a,b}) - \sum_{i \in [n]} P(i) \), we verify that \( \text{rsum}(\hat{P}) = \text{rsum}(P) + \omega(r) + 1 \). \( \square \)
4.3 Counting uprooted trees

In this subsection, we determine the number \(|\mathcal{U}'_n|\) of uprooted trees on the vertex set \([n]\) with \(1 \sim n\). Let \(\mathcal{T}_{n,0}\) be the set of labelled trees on the vertex set \([n]\) such that the root has no child (or son) with smaller labels. Let \(\mathcal{A}_n\) be the set of labelled rooted-trees on the vertex set \([n]\) with a non-rooted leaf \(n\). Chauve, Dulucq and Guibert [1] constructed a bijection \(\eta : \mathcal{T}_{n,0} \to \mathcal{A}_n\). As earlier, let \(\mathcal{U}_n\) be the set of uprooted trees on the vertex set \([n]\). Also, let \(\mathcal{B}_n\) be the set of labelled rooted-trees on the vertex set \([n]\) with a non-rooted leaf 1. We see that there are bijections \(\mathcal{U}_n \to \mathcal{T}_{n,0}\) and \(\mathcal{B}_n \to \mathcal{A}_n\) obtained by simply changing label \(i \to n - i + 1\) for all \(i\). The bijection \(\eta : \mathcal{T}_{n,0} \to \mathcal{A}_n\) induces a bijection \(\psi : \mathcal{U}_n \to \mathcal{B}_n\). For sake of completeness, we briefly describe construction of the bijection \(\psi\) essentially as in [1].

Let \(T \in \mathcal{U}_n\) with root \(r\). Note that \(r \neq 1\).

Step (1) : Consider a maximal increasing subtree \(T_0\) of \(T\) containing 1. Let \(T_1, \ldots, T_l\) be the subtrees (with at least one edge) of \(T\) obtained by deleting edges in \(T_0\). Let \(r_i\) be the root of \(T_i\) for \(1 \leq i \leq l\). The root \(r\) of \(T\) must be a root of one of the subtrees \(T_j\). Let \(r_j = r\). Then 1 is a leaf of \(T_j\).

Step (2) : If \(T_0\) has \(m\) vertices, then \(T_0\) is determined by an increasing tree \(\overline{T_0}\) on the vertex set \([m]\) and a set \(S_0\) of labels on \(T_0\). We write \(T_0 = (\overline{T_0}, S_0)\).

Step (3) : Let \(\overline{S_0} = (S_0 \setminus \{1\}) \cup \{r\}\). Then \((\overline{T_0}, \overline{S_0})\) determines an increasing subtree \(\overline{T_0}\) with root \(r' = \min \{\overline{S_0}\}\). Graft \(T_j\) on the increasing subtree \(\overline{T_0}\) at the root \(r\) and obtain a tree \(T'_j\). Now graft \(T_i (i \neq j)\) on \(T'_j\) at \(r_i\) and obtain a tree \(T''\) with root \(r'\). Also note that 1 is a non-rooted leaf of \(T''\).

All the above steps can be reversed, thus \(\psi(T) = T''\) defines a bijection \(\psi : \mathcal{U}_n \to \mathcal{B}_n\).

Lemma 28. \(|\mathcal{U}_n| = |\mathcal{B}_n| = (n - 1)^{n-1}\).

Proof. The bijection \(\psi : \mathcal{U}_n \to \mathcal{B}_n\) gives \(|\mathcal{U}_n| = |\mathcal{B}_n|\). The number of labelled rooted-trees on the vertex set \([2,3,\ldots,n]\) by Cayley’s formula is \((n - 1)^{n-2}\). Any tree in \(\mathcal{B}_n\) is obtained uniquely by attaching 1 to any node \(i\) of a labelled rooted tree on the vertex set \([2,3,\ldots,n]\). Since there are exactly \(n - 1\) possibilities for \(i\), we have \(|\mathcal{B}_n| = (n - 1)^{n-2}(n - 1) = (n - 1)^{n-1}\). \(\square\)

For \(n \geq 3\), let \(\mathcal{U}'_n = \{T \in \mathcal{U}_n : 1 \sim n \text{ in } T\}\). We shall determine the image \(\psi(\mathcal{U}'_n) \subseteq \mathcal{B}_n\) of \(\mathcal{U}'_n\) under the bijection \(\psi : \mathcal{U}_n \to \mathcal{B}_n\). Let \(\mathcal{B}'_n = \{T' \in \mathcal{B}_n : 1 \sim n \text{ in } T'\}\). Set

\[
\mathcal{A} = \{T' \in \mathcal{B}'_n : \text{root}(T') = r' = n\},
\]

\[
\mathcal{B}' = \{T' \in \mathcal{B}'_n : \text{root}(T') = r' \neq n \text{ with } r' \sim n \text{ and 1 is a descendent of } n\},
\]

\[
\mathcal{B}'' = \{T' \in \mathcal{B}'_n : \text{root}(T') = r' \neq n \text{ with } r' \sim n\}.
\]

Lemma 29. \(\psi(\mathcal{U}'_n) = \mathcal{A} \amalg \mathcal{B}' \amalg \mathcal{B}''\).

Proof. Let \(T' \in \mathcal{B}'_n\). Then there is a unique \(T \in \mathcal{U}_n\) such that \(T' = \psi(T)\). Let \(r\) and \(r'\) be the roots of \(T\) and \(T'\), respectively. Clearly, \(r \neq 1\). Let \(\text{Son}_T(1)\) be the set of sons of 1 in \(T\). Then from the construction of \(T' = \psi(T)\), \(r' = \min \{\{r\} \cup \text{Son}_T(1)\}\). Also, the leaf 1
in \( T' \) is adjacent to \( j \) if and only if \( j = \text{par}_T(1) \) is the parent of 1 in \( T \). This shows that \( 1 \sim n \) in \( T \) if and only if \( 1 \sim n \) in \( T' \). Hence, \( \psi(\mathcal{U}'_n) \subseteq \mathcal{B}'_n \). Further, we see that \( r' = n \) if and only if 1 is already a leaf in \( T \), and in this case, \( T' = \psi(T) = T \). In other words, \( \mathcal{A} \subseteq \mathcal{U}'_n \) and \( \psi(T) \) is for all \( T \in \mathcal{A} \).

If \( T' \in \mathcal{B}'', \) then the unique \( T \in \mathcal{U}_n \) with \( \psi(T) = T' \) must have \( 1 \sim n \) in \( T \), that is, \( T \in \mathcal{U}'_n \). Now we consider the remaining case. Let \( T' \in \mathcal{B}'_n \) with \( \text{root}(T') = r' \neq n \) and \( r' \sim n \) in \( T' \). We shall show that \( \psi(T) = T' \) for \( T \in \mathcal{U}'_n \) if and only if 1 is a descendant of \( n \) in \( T' \) (or equivalently, \( T' \in \mathcal{B}' \)). Consider the maximal increasing subtree \( T'_0 \) of \( T' \) containing the root \( r'. \) If 1 is a descendant of a leaf \( r'_j \) of \( T'_0 \), then the maximal increasing subtree \( T'_0 \) of \( T' \) containing 1 is obtained by replacing \( r'_j \) with 1 in the vertex set of \( T'_0 \) and labeling it as indicated in Step (2) of the construction of \( \psi \). Clearly, \( r'_j = r \) is the root of \( T. \) If \( r'_j = r \neq n \), then \( 1 \sim n \) in \( T \) as \( r' \sim n \) in \( T' \). Thus, if \( r'_j \neq n, \) i.e., 1 is not a descendant of \( n \) in \( T' \), then \( T' \notin \psi(\mathcal{U}'_n) \). On the other hand, if \( r'_j = n, \) i.e., 1 is a descendant of \( n \) in \( T' \) with \( 1 \sim n \), then \( \text{root}(T) = r = n \) and \( 1 \sim n \) in \( T. \)

**Proposition 30.** For \( n \geq 3, \) we have \( |\mathcal{U}'_n| = (n - 1)^{n-3}(n-2)^2. \)

**Proof.** By Lemma 29, we have \( |\mathcal{U}'_n| = |\psi(\mathcal{U}'_n)| = |\mathcal{A}| + |\mathcal{B}'| + |\mathcal{B}''. | \) First we enumerate the subset \( \mathcal{A} = \{ T' \in \mathcal{B}'_n : \text{root}(T') = r' \neq n \} \). The number of labelled trees on the vertex set \( \{2, 3, \ldots, n\} \) with root \( n \) is \( (n - 1)^{n-3} \). Since any tree in \( \mathcal{A} \) is uniquely obtained by attaching 1 to any node \( i \in \{2, \ldots, n - 1\} \) of a labelled tree on the vertex set \( \{2, \ldots, n\} \) with root \( n \), we have \( |\mathcal{A}| = (n - 1)^{n-3}(n-2). \)

Let us consider the subset \( \mathcal{C} = \{ T' \in \mathcal{B}'_n : \text{root}(T') = r' \neq n \} \subseteq \mathcal{B}'_n \). Clearly, \( \mathcal{B} = \mathcal{B}' \cup \mathcal{B}'' \supseteq \mathcal{C}. \) The enumeration of \( \mathcal{C} \) is similar to that of \( \mathcal{A} \), except now the root \( r' \in \{2, \ldots, n - 1\} \) can take any one of the \( n - 2 \) values. Thus \( |\mathcal{C}| = (n - 1)^{n-3}(n-2)^2. \) We can easily construct a bijective correspondence between \( \mathcal{A} \) and \( \mathcal{C} \setminus \mathcal{B} \). Let \( T' \in \mathcal{A}. \) Then \( 1 \sim n \) in \( T' \) and \( \text{root}(T') = n. \) Consider the unique path from the root \( n \) to the leaf 1 in \( T'. \) As \( 1 \sim n \) in \( T' \), the child \( \tilde{r} \) of \( n \) lying on this unique path is different from 1. Let \( T'' \) be rooted tree consisting of the tree \( T' \) with the new root \( \tilde{r}. \) As \( \text{root}(T'') = \tilde{r} \neq n, \tilde{r} \sim n \) and 1 is not a descendant of \( n \) in \( T'' \), we have \( T'' \in \mathcal{C} \setminus \mathcal{B}. \) The mapping \( T' \mapsto T'' \) from \( \mathcal{A} \) to \( \mathcal{C} \setminus \mathcal{B} \) is clearly a bijection. If \( T'' \in \mathcal{C} \setminus \mathcal{B}, \) then \( \text{root}(T'') = \tilde{r} \neq n, \tilde{r} \sim n \) and 1 is not a descendant of \( n \) in \( T'' \). Now unique \( T' \in \mathcal{A} \) that maps to \( T'' \) is the rooted tree obtained from \( T'' \) by taking \( n \) as the new root. Thus \( |\mathcal{A}| = |\mathcal{C} \setminus \mathcal{B}| \) and hence, \( |\mathcal{U}'_n| = |\mathcal{C}| = (n - 1)^{n-3}(n-2)^2. \)

**Theorem 31.** Let \( e_{p,q} \) be an edge of \( K_{n+1} \) joining distinct vertices \( p, q \in [n]. \) For \( n \geq 3, \) the number of spherical parking functions of \( K_{n+1} - \{e_{p,q}\} \) is given by

\[
|\text{sPF}(K_{n+1} - \{e_{p,q}\})| = |\mathcal{U}'_n| = (n - 1)^{n-3}(n-2)^2.
\]

**Proof.** In view of Theorem 23 and Remarks 26, the result follows.

Let \( F_l = \{e_{1,n}, e_{1,n-1}, \ldots, e_{1,n-l+1}\} \) be a set of \( l \)-edges through the vertex 1 in the complete graph \( K_{n+1}. \) We consider the graph \( K_{n+1} - F_l \) and ask the following question.

**Question 32.** What is the number of spherical \( (K_{n+1} - F_l) \)-parking functions?

Computations for smaller values of \( n \) and \( l \) indicate that

\[
|\text{sPF}(K_{n+1} - F_l)| = (n - 1)^{n-3}(n-l-1)^2.
\]
5 Spherical $K_{m+1,n}$-parking functions

Let $K_{m+1,n}$ be the complete bipartite graph on the vertex set $V' = [0,m] \bigsqcup [m+1,m+n]$, where $[0,m] = \{0,1,\ldots,m\}$ and $[m+1,m+n] = \{m+1,\ldots,m+n\}$. Let $K_{m+1,n}^{a,b}$ be the complete bipartite multigraph on $V'$. More precisely, there are $a$ number of edges in $K_{m+1,n}^{a,b}$ between the root 0 and $j$, while $b$ number of edges between $i$ and $j$, where $i \in [m]$ and $j \in [m+1,m+n]$.

**Proposition 33.** We have $|sPF(K_{m+1,n}^{a,b})| = |sPF(K_{n+1,m}^{a,b})|$.

**Proof.** Let $E$ and $E'$ be the set of all edges of $K_{m+1,n}^{a,b}$ and $K_{n+1,m}^{a,b}$ through the root 0, respectively. On repeatedly applying the Lemma 17, we see that

$$|sPF(K_{m+1,n}^{a,b})| = |sPF(K_{m+1,n}^{a,b} - E)| \quad \text{and} \quad |sPF(K_{n+1,m}^{a,b})| = |sPF(K_{n+1,m}^{a,b} - E')|.$$  

Since graphs $K_{m+1,n}^{a,b} - E$ and $K_{n+1,m}^{a,b} - E'$ are obtained from each other by interchanging vertices as $i \leftrightarrow n+i$ and $m+j \leftrightarrow j$ (for $i \in [m], j \in [n]$), $|sPF(K_{m+1,n}^{a,b} - E)| = |sPF(K_{n+1,m}^{a,b} - E')|$.

Although the root 0 is not connected to all the other vertices in the simple complete bipartite graph $K_{m+1,n}$, we can construct a map $\phi_{K_{m+1,n}} : sPF(K_{m+1,n}) \rightarrow \mathcal{U}(K_{m,n})$ as in Theorem 20, where $\mathcal{U}(K_{m,n})$ is the set of uprooted spanning trees of $K_{m,n} = K_{m+1,n} - \{0\}$.

The reduced spherical $K_{m+1,n}$-parking function $\widetilde{P}$ associated to $P \in sPF(K_{m+1,n})$ is given by $\widetilde{P}(j) = \mathcal{P}(j)$ for $1 \leq j \leq m$ and $\widetilde{P}(j) = \mathcal{P}(j) - 1$ for $m+1 \leq j \leq m+n$. We see that $K_{m,n} = K_{m+1,n} - \{0\}$ is the complete bipartite graph on the vertex set $[m] \bigsqcup [m+1,m+n]$. The following statements can be easily verified.

(i) $s\overline{\mathcal{PF}}(K_{m+1,n}) \subseteq \mathcal{PF}(K_{m+1,n})$.

(ii) Let $r = \max\{i \in [m+n] : \widetilde{P}(i) = 0\}$. Then $m+1 \leq r \leq m+n$.

(iii) $\widetilde{P} = \widetilde{P}|_{[m+n]\setminus\{r\}}$ is a $(K_{m,n},r)$-parking function.

(iv) If $\phi : \mathcal{PF}(K_{m,n},r) \rightarrow \mathcal{SPT}(K_{m,n},r)$ is the bijection induced by Depth-First-Search algorithm, then $\phi(\widetilde{P})$ is an uprooted spanning tree of $K_{m,n}$.

Now define a map $\phi_{K_{m+1,n}} : sPF(K_{m+1,n}) \rightarrow \mathcal{U}(K_{m,n})$ given by $\phi_{K_{m+1,n}}(\mathcal{P}) = \phi(\widetilde{P})$ for $\mathcal{P} \in sPF(K_{m+1,n})$. For each $T \in \mathcal{U}(K_{m,n})$, let $\mathcal{P}_T$ be the unique $(K_{m,n},r)$-parking function such that $\phi(\mathcal{P}_T) = T$. Let $\mathcal{U}(K_{m,n}) = \{T \in \mathcal{U}(K_{m,n}) : \mathcal{P}_T(j) \geq 1 \text{ for } j > \text{root}(T)\}$.

**Theorem 34.** The map $\phi_{K_{m+1,n}} : sPF(K_{m+1,n}) \rightarrow \mathcal{U}(K_{m,n})$ is injective with the image $\mathcal{U}(K_{m,n})$ and $\text{sum}(\mathcal{P}) = m(n-1) - \kappa(K_{m,n}, \phi_{K_{m+1,n}}(\mathcal{P})) + 1$ for all $\mathcal{P} \in sPF(K_{m+1,n})$.

**Proof.** Proceed as in the proof of Theorems 20 and 22. \qed
Remark 35. The following three statements can be easily verified.

1. \(|sPF(K_{m+1,1})| = 1 = |sPF(K_{1+1,n})|\).
2. Every spanning tree \(T\) of \(K_{m,n}\) with root\((T) = m + n\) lies in \(\overline{U}(K_{m,n})\). Thus
\[|\{P \in sPF(K_{m+1,n}) : \overline{P}(m + n) = 0\}| = |PF(K_{m,n})| = m^{n-1}n^{m-1}.
\]
3. We have \(|sPF(K_{m+1,n}^{a,b})| = b^{m+n}|sPF(K_{m+1,n})|\).

We could not enumerate \(sPF(K_{m+1,n})\) or \(U(K_{m,n})\). Thus we ask the following question.

**Question 36.** What is the number of spherical \(K_{m+1,n}\)-parking functions?

For \(n = 2\), this question has an easy answer.

**Proposition 37.** For \(m \geq 1\), \(|sPF(K_{m+1,2})| = (m - 1)2^m + 1\).

**Proof.** We know that \(|sPF(K_{m+1,2})| = |sPF(K_{m+1,2} - E)|\), where \(E\) is the set of all edges of \(K_{m+1,2}\) through the root 0. Now the \(m\)-skeleton ideal of the (disconnected) graph \(K_{m+1,2} - E\) is given by
\[\mathcal{M}_{K_{m+1,2} - E}^{(m)} = \left\langle x_i^2, y_j^m, y_1y_2, x_1x_2 \cdots x_isy_j^{m-s} : i \in [m]; j = 1, 2 \text{ and } \{i_1, \ldots, i_s\} \subseteq [m] \right\rangle,
\]
where \(y_j = x_{m+j}\) for \(j = 1, 2\). The standard monomials of \(\mathcal{M}_{K_{m+1,2} - E}^{(m)}\) are of the forms \(x_1x_2 \cdots x_isy_1^{\alpha}\) with \(0 \leq \alpha < m - s\) or \(x_1x_2 \cdots x_isy_2^{\beta}\) with \(1 \leq \beta < m - s\). Thus the number of standard monomials of the first type is \(\sum_{s=0}^{m-1} \binom{m}{s}(m-s) = m2^{m-1}\), while that of the second type is \(\sum_{s=0}^{m-1} \binom{m}{s}(m-s-1) = (m - 2)2^{m-1} + 1\).

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