Thermal vacuum, cosmic microwave radiation, neutrino masses and fractal-like self-similar structure

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Abstract. The behavior of thermal vacuum condensates of scalar and fermion fields is analyzed and it is shown that the condensate of Maxwell fields reproduces the characteristics of the cosmic microwave radiation. By studying fermion thermal states with the temperature of the cosmic neutrino background, we derive a value of the sum of the active neutrino masses which is compatible with its estimated lower bound. Moreover, we reveal the fractal self-similar structure of the thermal radiation and we relate it to the coherent structure of the thermal vacuum.

1. Introduction
In the context of the Thermo Field Dynamic [1, 2], we analyze the possible links existing among the thermal vacuum states, the cosmic microwave background (CMB) radiation and the neutrino masses [3].

The CMB radiation is a thermal radiation, left over from an early stage in the expansion of the universe [4], filling almost uniformly the observable universe. Such a radiation has a thermal black body spectrum corresponding to the temperature of $T = 2.72548 \pm 0.00057$ K. The anisotropy is very small [5]-[7].

Together with the CMB, there is an indirect evidence of the universe’s background particle radiation composed of neutrinos, called, cosmic neutrino background (CNB). The today estimated temperature for the CNB is roughly $1.95K$ [7]-[12].

Neutrinos play a significant role in the understanding of many phenomena of high energy physics and astrophysics [13]-[24]. The discovery of neutrino oscillations has shown that neutrinos have mass. Oscillations do not yield a value for the mass, but do set a lower limit of order of 0.06eV on the sum of the three neutrino masses $\sum m_\nu$ [25]. The upper limit of $\sum m_\nu$ is 2eV.

In this paper we report the results of Ref.[3], in which we have shown that the thermal condensate derived in thermal field theories, behave as a perfect fluid, gives a contribution to the radiation of the universe and reproduces the behavior of CMB and CNB. Moreover, by exploiting the CMB and CNB energy contribution, we derive an upper bound on the absolute mass of the lightest neutrino. The absolute masses of the other neutrinos and the sum of the active neutrino masses are obtained by assuming the hierarchical neutrino model and using the experimental differences between the squared masses $\Delta m^2_{12}$ and $\Delta m^2_{23}$. We find a value of $\sum m_\nu$ of the order of 0.06eV, which is in agreement with the estimated lower bound on the neutrino
masses. A result which does not depend on whether the neutrino is a Dirac or Majorana particle. Moreover, we study the fractal self-similar structure of the thermal states. It might be interesting to investigate about a possible relation between such fractal self-similarity properties and some aspects of holographic cosmology (see e.g. [26]).

The structure of the paper is the following, in Section II, we present the Thermo Field Dynamic formalism and derive the general expressions of energy density and pressure of the vacuum condensate. In Section III, we show the explicit results for Maxwell and scalar field. The results for neutrinos are reported in Section IV. The fractal structure of the thermal vacuum is studied in Section V and the conclusions are in Section VI.

2. Thermal vacuum condensate

We describe the observed black body spectrum of the CMB at $T = T_\gamma$, by using the Thermo Field Dynamics (TFD) formalism [1, 2] in quantum field theory (QFT). Thus we introduce the thermal vacuum state $|0(\theta)\rangle$, with $\theta = \theta(\beta)$, $\beta = 1/(k_BT)$ and $k_B$ the Boltzmann constant. The thermal vacuum $|0(\theta)\rangle$ is defined in such a way that the thermal statistical average $N_{a_k}(\theta)$ is given by the expectation value of the number operator $N_{a_k} = a_k^\dagger a_k$, with $a_k$ denoting the bosonic CMB modes, on $|0(\theta)\rangle$, i.e. $N_{a_k}(\theta) = \langle 0(\theta)|N_{a_k}|0(\theta)\rangle$.

The boson modes $a_k$ have usual canonical commutation relations (CCR) and $|0(\theta)\rangle$ is given by

$$|0(\theta)\rangle = \prod_k \frac{1}{\cosh \theta_k} \exp (\tanh \theta_k a_k^\dagger b_k)|0\rangle, \quad \theta = \theta(\beta).$$

$|0(\theta)\rangle$ is a two-mode generalized $SU(1, 1)$ squeezed coherent state [27, 28] at finite temperature, condensate of pairs of $a_k$ and $b_k$ quanta, with $b_k$ auxiliary boson mode, commuting with $a_k$. The quantum $b_k$ is introduced in order to produce the trace operation in the computing thermal averages of the observable quantities. In Eq.(1), $|0\rangle$ is the vacuum for $a_k$ and $b_k$. One has that $\langle 0(\theta)|0(\theta)\rangle = 1$, $\forall \theta$, and in the infinite volume limit (for $\int d^3k \, \theta_k$ finite and positive) $\langle 0(\theta)|0(\theta)\rangle \to 0$ as $V \to \infty$, $\forall \beta$. Moreover, the thermal vacuum $|0(\theta)\rangle$ provides a representation of the CCR defined at each $\beta$ and unitarily inequivalent to any other representation $\{0(\beta')\}, \forall \beta' \neq \beta$ in the infinite volume limit. Indeed one has $\langle 0(\theta(\beta'))|0(\theta(\beta'))\rangle \to 0$ as $V \to \infty$, $\forall \beta$ and $\beta', \beta' \neq \beta$. This means that in thermal equilibrium ($\beta$ constant in time $t$) at a given temperature $T$, the system sits in the representation of the CCR corresponding to such a $T$. In thermal non-equilibrium conditions, with temperature changing in time, i.e. $\beta = \beta(t)$, the system evolves in time through unitarily inequivalent representations of the CCR and it is known that such a time evolution is controlled by the entropy operator [1, 2].

It is crucial to note that the annihilation operators for the state $|0(\theta)\rangle$, denoted with $A_k(\theta_k)$ and $B_k(\theta_k)$, are different from $a_k$ and $b_k$ which annihilate the state $|0\rangle$. The relation among the two sets of annihilators is provided by the Bogoliubov transformation

$$A_k(\theta_k) = a_k \cosh \theta_k - b_k^\dagger \sinh \theta_k, \quad B_k(\theta_k) = b_k \cosh \theta_k - a_k^\dagger \sinh \theta_k .$$

The condensate density is given by the expectation value of the number operator $a_k^\dagger a_k$ on the thermal vacuum, $N_{a_k}(\theta) = \langle 0(\theta)|a_k^\dagger a_k|0(\theta)\rangle = \sinh^2 \theta_k$. In the boson case, minimization of the free energy leads to the Bose-Einstein distribution function for $a_k$

$$N_{a_k}^B(\theta) = \sinh^2 \theta_k = \frac{1}{e^{\beta \theta_k} - 1} .$$

(2)
Thus, we can reproduce the observed black body spectrum of the CMB by computing the expectation value $\langle 0(\theta, z) | T^{ij}_\gamma(0(\theta, z)) = 0 \rangle$, for $i = j$. Then, the energy density and pressure of the vacuum condensates (2), (3) at a given time (we consider the red shift $z$), are represented by the $(0, 0)$ and $(j, j)$ components of $T^{\mu\nu}_\gamma$ of the vacuum $|0(\theta, z)\rangle$,

$\rho(z) = g_{00} \langle 0(\theta, z) | T^{00}(x) : |0(\theta, z)\rangle$, \hspace{1cm} (4)

$p(z) = g_{jj} \langle 0(\theta, z) | T^{jj}(x) : |0(\theta, z)\rangle$, \hspace{1cm} (5)

with $: \cdots :$, normal ordering with respect to $|0\rangle$.

3. Thermal vacuum, CMB radiation and boson fields

In the photon fields case, Eqs.(4) and (5) become

$\rho_\gamma(z) = \frac{\pi^2 k_B^4 (1 + z)^4 T_\gamma^4}{15 h^3 c^3}$, \hspace{1cm} (6)

$p_\gamma(z) = \frac{\pi^2 k_B^2 (1 + z)^3 T_\gamma^4}{45 h^3 c^3}$, \hspace{1cm} (7)

and the state equation is the one of the radiation $w_\gamma(z) = p_\gamma(z)/\rho_\gamma(z) = 1/3$. These results coincides with the ones obtained by solving the Boltzmann equation for the distribution function of photons in thermal equilibrium [12].

By considering the present CMB temperature and the present red shift, $z = 0$, one obtains a value of the thermal vacuum energy density $\rho_\gamma = 2 \times 10^{-51} GeV^4$, coinciding with the energy density of the CMB [12].

Let us now study the energy momentum tensor of the thermal vacuum $|0(\theta, z)\rangle$ for massive boson and fermion fields. For bosons, at any epoch, one has [3]

$\rho_B(z) = \frac{1}{2\pi^2} \int_0^\infty dk k^2 \frac{\Omega_k}{\exp\left(\frac{\Omega_k}{k_B T_\gamma(1+z)}\right) - 1}$, \hspace{1cm} (8)

$p_B(z) = \frac{1}{6\pi^2} \int_0^\infty dk k^2 \left[ \frac{k^2}{\Omega_k} \frac{1}{\exp\left(\frac{\Omega_k}{k_B T_\gamma(1+z)}\right) - 1} \right.$

$\left. - \left( \frac{k^2}{\Omega_k} + \frac{3m^2}{2\Omega_k} \right) \frac{1}{\exp\left(\frac{\Omega_k}{k_B T_\gamma(1+z)}\right) - 1} \cos(\Omega_k t) \right]$. \hspace{1cm} (9)
In particular, at the present epoch, \( z = 0 \) one obtains \( \rho_B \simeq 9 \times 10^{-52} \text{GeV}^4 \) for masses less or equal than the CMB temperature \( m \leq T_\gamma \), i.e. \( m \leq 2.3 \times 10^{-4} \text{eV} \). The maximum value of \( \rho_B \) is achieved for masses \( m \ll 10^{-4} \text{eV} \). In this case, the energy density is \( \rho_B \simeq 10^{-51} \text{GeV}^4 \). Negligible values of \( \rho_B \) are obtained for boson masses \( m \gg 10^{-3} \text{eV} \). Therefore, the thermal vacuum contribution of bosons to the energy of the universe is completely negligible with respect to \( \rho_\gamma \) and with respect to the critical density of the present universe \( \rho_{cr} \simeq 4.5 \times 10^{-47} \text{GeV}^4 \). Only hypothetical particles as axions with masses \( m_a \in (10^{-3} - 10^{-6}) \text{eV} \), could generate non-trivial contributions [3].

4. Thermal vacuum and neutrino masses
The thermal vacuum contribution of fermion fields to the vacuum energy density and to the pressure, are

\[
\rho_F(z) = \frac{1}{\pi^2} \int_0^\infty dk \frac{k^2 \Omega_k}{\exp\left(\frac{\Omega_k}{k_B T_\gamma(1+z)}\right)} + 1, \tag{10}
\]

\[
p_F(z) = \frac{1}{3\pi^2} \int_0^\infty dk k^4 \frac{1}{\Omega_k \exp\left(\frac{\Omega_k}{k_B T_\gamma(1+z)}\right)} + 1. \tag{11}
\]

At the present epoch, \( z = 0 \), and for masses \( m \leq T_\gamma \), one has the maximum value of \( \rho_F \) at \( T = T_\gamma \), which is \( \rho_F \sim 1.6 \times 10^{-54} \text{GeV}^4 \). In this case, the state equation is \( w_F \sim 1/3 \). We note that only the thermal vacuum condensate of sub-eV massive particles, like the neutrinos, is relevant, since the condensate of more heavy fermions give negligible contributions to the energy of universe.

We consider now the relic neutrino temperature \( T_\nu = 1.95K \) and we study the energy density of the thermal vacuum for cosmic neutrino background (CNB). We use massive Dirac neutrinos. Let \( |0(\theta,z)\rangle \) denote the vacuum condensate of the neutrino modes with temperature \( T_\nu \). We derive the absolute neutrino masses from the plausible value of the CNB energy.

In Eqs.(10) and (11), if one considers \( T = T_\nu \) and the sum on the three neutrino fields with masses \( m_\nu \), one has only small variations from the above results. Indeed, for \( z = 0 \) and \( m_\nu \sim 10^{-4} \text{eV} \), we find the maximum value of the energy density \( \rho_\nu \sim 10^{-51} \text{GeV}^4 \), and the state equation is \( w_\nu \sim 1/3 \). We assume \( \rho_\nu \leq \rho_\gamma \), and the mass of the lighter neutrino \( m_{\nu,1} \sim 10^{-3} \text{eV} \) [3]. Considering the hierarchical neutrino model, according to which \( \Delta m_{12}^2 = 8 \times 10^{-5} \text{eV}^2 \) and \( \Delta m_{23}^2 = 2.7 \times 10^{-3} \text{eV}^2 \), one can derive the following values of the other neutrino masses: \( m_{\nu,2} = 9 \times 10^{-3} \text{eV} \) and \( m_{\nu,3} = 5.3 \times 10^{-2} \text{eV} \). Then, the sum of the three neutrino masses is \( \sum m_\nu \sim 6 \times 10^{-2} \text{eV} \), which is in agreement with its estimated lower bound. Notice that the contributions to the energy density \( \rho_\nu \) given by \( \nu_2 \) and \( \nu_3 \) are negligible.

5. Fractal structure of the thermal vacuum
In this Section we show the fractal self-similar structure of the thermal vacuum \( |0(\theta)\rangle \) considered above for the CMB and CNB [3]. Consider the boson case. In full generality, we consider time-dependent \( \theta : \theta = \theta(t) \). We use the notation \( |0(t)\rangle \equiv |0(\theta(t))\rangle \). As shown in [29], \( |0(t)\rangle \) provides the quantum representation of the system of couples of damped/amplified oscillators

\[
m\ddot{x} + \gamma \dot{x} + kx = 0, \tag{12}
\]

\[
m\ddot{y} - \gamma \dot{y} + ky = 0, \tag{13}
\]

\[
L = m\ddot{x} + \frac{\gamma}{2}(x\ddot{y} - \dot{x}\dot{y}) - kx y, \tag{14}
\]

where “dot” denotes time derivative, \( m \), \( \gamma \) and \( \kappa \) are positive real constants and \( L \) is the Lagrangian from which Eqs. (12) and (13) are derived. The state \( |0(t)\rangle \) is indeed obtained
by quantizing the system represented by Eqs. (12) - (14). Therefore, we take into account the canonical commutation relations $[x, p_x] = i\hbar = [y, p_y]$, $[x, y] = 0 = [p_x, p_y]$ and the corresponding annihilators

$$a \equiv \left(\frac{1}{2\hbar\Omega}\right)^{\frac{1}{2}} \left(\frac{p_x}{\sqrt{m}} - i\sqrt{m}\Omega x\right);$$

$$\tilde{a} \equiv \left(\frac{1}{2\hbar\Omega}\right)^{\frac{1}{2}} \left(\frac{p_y}{\sqrt{m}} - i\sqrt{m}\Omega y\right);$$

satisfying the relations $[\alpha, \alpha^\dagger] = 1 = [\tilde{\alpha}, \tilde{\alpha}^\dagger]$, $[\alpha, \tilde{\alpha}] = 0 = [\alpha, \tilde{\alpha}^\dagger]$. Moreover, we consider the transformations $a \equiv (1/\sqrt{2})(\alpha + \tilde{\alpha})$, $b \equiv (1/\sqrt{2})(\alpha - \tilde{\alpha})$. As shown in [29], the proper quantization setting is the one of the quantum field theory (QFT) and the Hamiltonian $H$ of the system is $H = H_0 + H_I$, with $H_0 = \sum_k \hbar\Omega_k (\hat{a}_k^\dagger \hat{a}_k - \hat{b}_k^\dagger \hat{b}_k)$, and $H_I = i \sum_k \hbar \Gamma_k (\hat{a}_k^\dagger \hat{b}_k - \hat{a}_k \hat{b}_k)$. In this case, the parameter $\theta_k(t)$ of the thermal vacuum is $\theta_k(t) = \Gamma_k t \equiv (\gamma_k/2m) t$ for each $\kappa$-mode [29]. The time evolution of the vacuum $|0\rangle$ for $a_k$ and $b_k$ is driven by $H_I$ and given by $|0(\theta(t))\rangle = e^{-it\frac{\Gamma_t}{\hbar}}|0\rangle = e^{-it\frac{\Omega_t}{\hbar}}|0\rangle$, which leads to the explicit expression for $|0(\theta(t))\rangle$ similar to the one in Eq. (1) for time dependent $\beta$. The free energy for the $a$-modes is defined by

$$F_a \equiv \langle 0(t) | \left( H_a - \frac{1}{\beta} S_a \right) | 0(t) \rangle,$$

where $H_a$ is the free Hamiltonian for the $a$-modes, $H_a = \sum_k \hbar\Omega_k \hat{a}_k^\dagger \hat{a}_k$, and $S_a$ is the entropy:

$$S_a \equiv - \sum_k \left\{ a_k^\dagger a_k \ln \sinh^2(\theta) - a_k a_k^\dagger \ln \cosh^2(\theta) \right\}.$$

For $\beta(t)$ slowly varying in time, the Bose-Einstein distribution function Eq. (2) is derived by minimizing $F_a$. Therefore, $|0(t)\rangle$ provides a finite temperature representation of the CCR which is equivalent to the Thermo Field Dynamics representation $|\{0(\beta)\}\rangle$, with time dependent $\beta$.

We now remark that Eqs. (12) and (13) describe a system which has self-similarity properties. We indeed note that solutions of Eqs. (12) and (13) are the parametric time evolution of clockwise and anti-clockwise logarithmic spirals. Indeed, by defining $x(t) \equiv (z_1(t) + z_2^*(t))/2$ and $y(t) \equiv (z_1^*(t) - z_2(t))/2$, Eqs. (12) and (13) can be rewritten as [30]

$$m \ddot{z}_1 + \gamma \dot{z}_1 + \kappa z_1 = 0,$$

$$m \ddot{z}_2 - \gamma \dot{z}_2 + \kappa z_2 = 0,$$

whose solutions are the clockwise and anti-clockwise logarithmic spirals $z_1(t) = r_0 e^{-i\Gamma t} e^{-\Gamma t}$ and $z_2(t) = r_0 e^{+i\Omega t} e^{+\Omega t}$, respectively, with $\Gamma = \gamma/2m$ and $\Omega = (1/m)(\kappa - \gamma^2/4m)$, $\kappa > \gamma^2/4m$ [30, 31]. Therefore, Eqs. (12) and (13), whose quantum representation is given by $|0(t)\rangle$, describe the self-similar fractal structure of their logarithmic spiral solutions [32, 33]. The link between $SU(1,1)$ coherent states, fractal-like self-similarity and CMB is thus established [3]. A similar result may be obtained in the fermion case for CNB by considering that the $su(2)$ algebra contracts in the infinite volume limit to the $e(2)$ algebra (as it can be seen by considering the Holstein-Primakoff non-linear boson realization [34, 35, 36]), which is isomorphic to the Weyl-Heisenberg algebra.

6. Conclusion
We have shown that the structure of CMB is characterized by the arrow of time and by coherent, thermal and fractal-like self-similar properties. The observed black body spectrum of CMB is
confirmed by the results here presented. Moreover, by analyzing the thermal vacuum energy for neutrinos and considering the neutrino hierarchical model, we have derived the value of the neutrino masses which are in agreement with estimated lower bounds. An interesting question to ask is on a possible relation between the fractal self-similarity structures considered in this paper and the recently reported observations on the holographic structure of the Universe [26].

Acknowledgments
Partial financial support from MIUR and INFN is acknowledged.

References
[1] Takahashi Y, Umezawa H 1975 Collect. Phenom. 2 55; reprinted 1996 in Int. J. Mod. Phys. B 10 1755
[2] Umezawa H 1993 Advanced field theory: Micro, macro, and thermal physics (New York: AIP)
[3] Capolupo A, Lambiase G and Vitiello G 2016 Adv. High Energy Phys. 2016 312797
[4] Gawiser E and Silk J 2000 Phys. Rep. 333 245
[5] Fixsen D J 2009 APJ 707 916
[6] Bennett C L et al. 1996 APJL 464 L1-L4
[7] de Bernardis P et al. 2000 Nature 404 955
[8] Weinberg S 2008 Cosmology (Oxford: OUP)
[9] Bashinsky S and Seljak U 2004 Phys. Rev. D 69 083002
[10] Mangano G et al. 2005 Nucl. Phys. B 729 221
[11] Komatsu E 2011 APJ, Supp. Series 192 18
[12] Kolb E and Turner M 1994 The Early Universe (Frontiers in Physics)
[13] Capolupo A, Capozziello S and Vitiello G 2007 Phys. Lett. A 363 53
[14] Capolupo A, Capozziello S and Vitiello G 2009 Phys. Lett. A 373 601
[15] Capolupo A, Capozziello S and Vitiello G 2008 Int. J. Mod. Phys. A 23 4979
[16] Blasone M, Capolupo A, Carloni S and Vitiello G 2004 Phys. Lett. A 323 182
[17] Blasone M, Capolupo A, Capozziello S and Vitiello G 2008 Nucl. Instrum. Meth. A 588 272
[18] Blasone M, Capolupo A and Vitiello G 2010 Prog. Part. Nucl. Phys. 64 451
[19] Capolupo A, Di Mauro M and Iorio A 2011 Phys. Lett. A 375 3415
[20] Capolupo A and Di Mauro M 2015 Adv. in High Energy Phys., 2015 929362
[21] Capolupo A and Vitiello G 2013 Adv. in High Energy Phys., 2013 850395
[22] Capolupo A and Di Mauro M 2012 Phys. Lett. A 376 2830
[23] Capolupo A and Di Mauro M 2013 Acta Phys. Polonica B 44 81
[24] Capolupo A 2016 Adv. High Energy Phys. 2016 8089142
[25] Gonzalez-Garcia M C, Maltoni M, Salvado J and Schwetz T 2012 J. High Energy Phys. 12 123
[26] Ashordi N et al. 2017 Phys. Rev. Lett. 118 041301
[27] Perelomov A 1986 Generalized Coherent States and Their Applications (Berlin: Springer-Verlag)
[28] Klauder J R and Slagerstam B 1985 Coherent States (Singapore: World Scientific)
[29] Celeghini E, Rasetti M and Vitiello G 1992 Ann. Phys. 215 156
[30] Vitiello G 2012 Phys. Lett. A 376 2527
[31] Vitiello G 2014 Systems 2 203
[32] Peitgen H O, Jürgens H and Saupe D 1986 Chaos and Fractals: New Frontiers of Science (Berlin: Springer-Verlag)
[33] Andronov A A, Vitt A A and Khaikin S E 1966 Theory of Oscillators (Mineola, NY, USA: Dover Publications, Inc.)
[34] Holstein T and Primakoff H 1940 Phys. Rev. 58 1098
[35] Shah M N, Umezawa H and Vitiello G 1974 Phys. Rev. B 10 4724
[36] De Concini C and Vitiello G 1976 Nuc. Phys. B 116 141