Amenability and the bicrossed product construction

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Abstract
We study stability properties of amenable locally compact quantum groups under the bicrossed product construction. We obtain as our main result an equivalence between amenability of the bicrossed product and amenability of the matched quantum groups used as building ingredients of the bicrossed product. Finally, we give examples of non-amenable locally compact quantum groups obtained by a bicrossed product construction.

1 Introduction

The theory of locally compact quantum groups has been introduced by J. Kustermans and the third author in [9, 10], unifying compact quantum groups and Kac algebras. As the example of the quantum $SU_q(2)$-group, developed by Woronowicz, shows, the antipode of a compact quantum group need not be bounded and it need not respect the $\ast$-operation. For this reason, compact quantum groups are not always Kac algebras. The crucial difference between Kac algebras and locally compact quantum groups is the possible unboundedness of the antipode.

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Taking into account the importance of amenable locally compact groups within the category of all locally compact groups, it is natural to consider amenability of locally compact quantum groups. In fact, the main results on amenability of Kac algebras, have been developed by Enock and Schwartz [7], and their proofs can be repeated in the more general framework of locally compact quantum groups. However, there is still one open problem. Recall that there are many different characterizations of amenability of locally compact groups. A first characterization deals with the existence of an invariant mean on a suitable algebra of functions on the group $G$ ($L^\infty(G)$, or bounded continuous functions). Another characterization says that the trivial representation of $G$ is weakly contained in the left regular representation. Other characterizations are most of the time closely related to one of these two definitions. These two properties can be formulated for locally compact quantum groups and, in this way, one defines amenable and strongly amenable locally compact quantum groups. It is known that all strongly amenable quantum groups are amenable, but the converse has only been proven for locally compact groups, see e.g. [6], and for discrete Kac algebras [13].

Having defined amenability of locally compact quantum groups, one asks for examples. A systematic way of constructing examples of locally compact quantum groups has been developed by Majid [11], Baaj and Skandalis [2] and Vainerman and the third author [15], and in this paper, we precisely characterize when these locally compact quantum groups are amenable. We also give two examples of non-amenable locally compact quantum groups obtained by this so-called bicrossed product construction.

In [15], one also defines bicrossed products of quantum groups, and one makes the link with short exact sequences of locally compact quantum groups, called extensions. In this paper, we will characterize in this full generality, when the bicrossed product is amenable, and in fact, our result is a quantum version of the well known result that a locally compact group $G$ with normal closed subgroup $H$ is amenable if and only if $H$ and $G/H$ are amenable.

2 Preliminaries

We refer to [9] and [10] for the theory of locally compact quantum groups in the $C^*$-algebra, as well as in the von Neumann algebra language. For the non-specialists, [18] is a good starting point. We recall from [10] the definition of a von Neumann algebraic quantum group: $(M, \Delta)$ is called a (von Neumann algebraic) locally compact quantum group when

- $M$ is a von Neumann algebra and $\Delta : M \to M \otimes M$ is a normal and unital $*$-homomorphism satisfying the coassociativity relation: $(\Delta \otimes \iota)\Delta = (\iota \otimes \Delta)\Delta$;
there exist normal, semi-finite, faithful (n.s.f.) weights \( \varphi \) and \( \psi \) on \( M \) such that

- \( \varphi \) is left invariant, i.e. \( \varphi((\omega \otimes \iota)\Delta(x)) = \varphi(x)\omega(1) \) for all \( x \in M_\varphi^+ \) and \( \omega \in M_\varphi^+ \),

- \( \psi \) is right invariant, i.e. \( \psi((\iota \otimes \omega)\Delta(x)) = \psi(x)\omega(1) \) for all \( x \in M_\psi^+ \) and \( \omega \in M_\psi^+ \).

Here, we use the notation \( M_\varphi^+ = \{ x \in M^+ \mid \varphi(x) < +\infty \} \), and analogously for \( M_\psi^+ \).

Fix a left invariant n.s.f. weight \( \varphi \) on \( (M, \Delta) \) and represent \( M \) on the GNS-space of \( \varphi \) such that \( (H, \iota, \Lambda) \) is a GNS-construction for \( \varphi \). Then, we can define a unitary \( W \) on \( H \otimes H \) by

\[
W^*(\Lambda(a) \otimes \Lambda(b)) = (\Lambda \otimes \Lambda)((\Delta(b) (a \otimes 1)) \quad \text{for all } a, b \in \mathcal{N}_\varphi.
\]

Here, \( \Lambda \otimes \Lambda \) denotes the canonical GNS-map for the tensor product weight \( \varphi \otimes \varphi \).

One proves that \( W \) satisfies the pentagonal equation:

\[
W_{12}W_{13}W_{23} = W_{23}W_{12}.
\]

We say that \( W \) is a \textit{multiplicative unitary}. The comultiplication can be given in terms of \( W \) by the formula \( \Delta(x) = W^* (1 \otimes x) W \) for all \( x \in M \). Also the von Neumann algebra \( M \) can be written in terms of \( W \) as

\[
M = \{ (\iota \otimes \omega)(W) \mid \omega \in B(H)_* \} - \sigma\text{-strong}^*.
\]

Next, the locally compact quantum group \( (M, \Delta) \) has an \textit{antipode} \( S \), which is the unique \( \sigma\text{-strong}^* \) closed linear map from \( M \) to \( M \) satisfying \( (\iota \otimes \omega)(W) \in D(S) \) for all \( \omega \in B(H)_* \), \( S(\iota \otimes \omega)(W) = (\iota \otimes \omega)(W^*) \) and such that the elements \( (\iota \otimes \omega)(W) \) form a \( \sigma\text{-strong}^* \) core for \( S \). The antipode \( S \) has a polar decomposition \( S = R\tau_{-1/2} \), where \( R \) is an anti-automorphism of \( M \) and \( (\tau_t) \) is a strongly continuous one-parameter group of automorphisms of \( M \). We call \( R \) the \textit{unitary antipode} and \( (\tau_t) \) the \textit{scaling group} of \( (M, \Delta) \). It is known that \( \sigma(R \otimes R)\Delta = \Delta R \), where \( \sigma \) denotes the flip map on \( M \otimes M \).

We turn the predual \( M_* \) into a Banach algebra with product \( \omega * \mu = (\omega \otimes \mu)\Delta \), for all \( \omega, \mu \in M_* \).

We use the notation \( \Delta^\text{op} \) to denote the opposite comultiplication defined by \( \Delta^\text{op} := \sigma\Delta \).

The dual locally compact quantum group \( (\hat{M}, \hat{\Delta}) \) is defined as follows. Its von Neumann algebra \( \hat{M} \) is

\[
\hat{M} = \{ (\omega \otimes \iota)(W) \mid \omega \in B(H)_* \} - \sigma\text{-strong}^*.
\]

and the comultiplication is given by \( \hat{\Delta}(x) = \Sigma W(x \otimes 1)W^*\Sigma \) for all \( x \in \hat{M} \), where \( \Sigma \) denotes the flip map on the tensorproduct of Hilbert spaces.

Since \( (\hat{M}, \hat{\Delta}) \) is again a locally compact quantum group, we can introduce the antipode \( \hat{S} \), the unitary antipode \( \hat{R} \) and the scaling group \( (\hat{\tau}_t) \) exactly as we did.
it for \((M, \Delta)\). Also, we can again construct the dual of \((\hat{M}, \hat{\Delta})\), starting from the left invariant weight \(\hat{\varphi}\) with GNS-construction \((H, \iota, \hat{\Lambda})\). From the biduality theorem, we get that the bidual locally compact quantum group \((\hat{M}, \hat{\Delta})\) is isomorphic to \((M, \Delta)\).

Define \(M_c\) to be the norm closure of the space
\[
\{(\iota \otimes \omega)(W) \mid \omega \in \mathcal{B}(H)\}
\]
and \(\Delta_c\) to be the restriction of \(\Delta\) to \(M_c\). It is proven in \([10]\) that the pair \((M_c, \Delta_c)\) is a reduced \(C^*\)-algebraic locally compact quantum group. We know that there is a bijective correspondence between reduced \(C^*\)-algebraic quantum groups and von Neumann algebraic quantum groups. So, the choice for the von Neumann algebra language is not a restriction.

A \(*\)-homomorphism \(\varepsilon : M_c \to \mathbb{C}\) is called a co-unit of \((M_c, \Delta_c)\), if
\[
(\varepsilon \otimes \iota)\Delta = (\iota \otimes \varepsilon)\Delta = \iota.
\]

Classical locally compact groups appear as \(M = L^\infty(G)\) with \(\Delta(f)(p,q) = f(pq)\). The invariant weights are defined by integrating with respect to the left or the right Haar measure. The dual \(M\) can be identified with the group von Neumann algebra \(\mathcal{L}(G)\).

Working with tensor products with more than two factors, we will sometimes use the leg-numbering notation. For example, if \(H, K\) and \(L\) are Hilbert spaces and \(X \in \mathcal{B}(H \otimes L)\), we denote by \(X_{13}\) (respectively, \(X_{12}, X_{23}\)) the operator \((1 \otimes \Sigma^*)(X \otimes 1)(1 \otimes \Sigma)\) (respectively, \(X \otimes 1, 1 \otimes X\)) defined on \(H \otimes K \otimes L\). If now \(H = H_1 \otimes H_2\) is itself a tensor product of two Hilbert spaces, then we sometimes switch from the leg-numbering notation with respect to \(H \otimes K \otimes L\) to the one with respect to the finer tensor product \(H_1 \otimes H_2 \otimes K \otimes L\), for example, from \(X_{13}\) to \(X_{124}\). There is no confusion here, because the number of legs changes. Weak and \(\sigma\)-weak convergence are denoted by \(\overset{w}{\longrightarrow}\), respectively \(\overset{\sigma w}{\longrightarrow}\).

### 3 Amenability

Let \((M, \Delta)\) be a von Neumann algebraic locally compact quantum group. A state \(m \in M^*\) is said to be a \textit{left invariant mean (LIM)} on \((M, \Delta)\) if
\[
m((\omega \otimes \iota)\Delta(x)) = m(x)\omega(1),
\]
for all \(\omega \in M_*\) and \(x \in M\). It is said to be a \textit{right invariant mean (RIM)} if
\[
m((\iota \otimes \omega)\Delta(x)) = m(x)\omega(1),
\]
for all \(\omega \in M_*\) and \(x \in M\). Finally, if \(m\) is both a LIM and a RIM, we call \(m\) an \textit{invariant mean (IM)}.
Definition 1. We call \((M, \Delta)\) amenable if there exists a left invariant mean (LIM) on \((M, \Delta)\). We say that \((M, \Delta)\) is coamenable if \((\hat{M}, \hat{\Delta})\) is amenable.

Definition 2. We call \((M, \Delta)\) strongly amenable if there exists a bounded co-unit on \((\hat{M}, \hat{\Delta})\).

In the Preliminaries section, we saw that classical locally compact groups appear as \(L^\infty(G)\) in the theory of locally compact quantum groups. We have defined amenability in such a way that, for every locally compact group \(G\), we have that \(G\) is amenable if and only if \((L^\infty(G), \Delta_G)\) is amenable. Other authors sometimes use a “dual” terminology. This difference originates from the choice which quantum group is associated with a locally compact group, \(L^\infty(G)\) or \(\mathcal{L}(G)\). Here, we adopt the point of view of Enock and Schwartz [7] and Ruan [14] (i.e. we take \(L^\infty(G)\) as the associated quantum group). The “dual” convention is used, amongst others, by Banica [8], Banaj and Skandalis [2], Ng [12] and Bedos, Murphy and Tuset [4]. They use coamenable where we use strongly amenable. So, our notion of “coamenability” disagrees with their notion. Whenever we cite a result of one of the papers mentioned with different terminology, it will be already translated to our setting.

M. Enock and J.M. Schwartz prove in [7] that, for Kac algebras, the following statements are equivalent with the fact that a Kac algebra is strongly amenable:

(i) There exists a net \((\xi_j)\) of normalized vectors in \(H\) such that
\[
(t \otimes \omega \xi_j)(W) \xrightarrow{w^*} 1,
\]
(ii) There exists a bounded left (resp., right) approximate unit on \(\hat{M}_*\).

It is proven in [7] that strong amenability implies amenability. They also claim that the opposite implication is true, but, as mentioned by Ruan [14], there is a gap in their proof. It is an important open question whether or not amenability implies strong amenability. Until now, this is only known to be true for locally compact groups, see for example [8], and for discrete Kac algebras [3].

Further they show that the following statements are equivalent:

(i) there exist a LIM on \((M, \Delta)\) (resp., RIM);

(ii) there exists a net \((\omega_i)\) of states in \(M_*\) such that \(\omega * \omega_i - \omega_i\) converges weakly to 0 (resp., \(\omega_i * \omega - \omega_i\)), for all \(\omega \in M_*\) with \(\omega(1) = 1\);

(iii) there exists a net \((\omega_i)\) of states in \(M_*\) such that \(||\omega * \omega_i - \omega_i||\) converges to 0 (resp., \(||\omega_i * \omega - \omega_i||\)), for all \(\omega \in M_*\) with \(\omega(1) = 1\).

All these results are also true for locally compact quantum groups. Not surprisingly, we can prove the following proposition.
Proposition 3. Let \((M, \Delta)\) be a locally compact quantum group. There exists a LIM on \((M, \Delta)\) if and only if there exists an invariant mean on \((M, \Delta)\).

Proof. One implication is immediate.

Conversely, suppose there exists a LIM on \((M, \Delta)\). From the result mentioned above, we know that there exists a net of states \((\omega_i)\) in \(M^*\) such that \(\|\omega\ast\omega_i - \omega_i\|\) converges to 0 for all \(\omega \in M^*\) with \(\omega(1) = 1\). It is obvious that this is equivalent with the existence of a net of states \((\omega_i^o)\) in \(M^*\) such that \(\|\omega^o \ast \omega - \omega_i^o\|\) converges to 0 for all \(\omega \in M^*\) with \(\omega(1) = 1\), take \(\omega_i^o = \omega_i \circ R\). It is easy to prove that \(\mu_k = (\omega_i \ast \omega^o_j)_{(i,j)}\) is a net of states such that

\[
\|\mu_k \ast \omega - \mu_k\| \to 0 \quad \text{and} \quad \|\omega \ast \mu_k - \mu_k\| \to 0
\]

for all \(\omega \in M^*\) with \(\omega(1) = 1\). Let \(m\) be a weak-* limit point of \((\mu_k)\) in the unit ball of \(M^*\). It is obvious that \(m\) will be an invariant mean.

\(\square\)

4 Bicrossed products

In this section, we collect some results and definitions treated in [17].

Definition 4. We call a pair \((\alpha, \mathcal{U})\) a cocycle action of a locally compact quantum group \((M, \Delta)\) on a von Neumann algebra \(N\) if

\[
\alpha : N \to M \otimes N
\]

is a normal, injective and unital \(*\)-homomorphism,

\[
\mathcal{U} \in M \otimes M \otimes N
\]

is a unitary, and if \(\alpha\) and \(\mathcal{U}\) satisfy

\[
(\iota \otimes \alpha)\alpha(x) = \mathcal{U} (\Delta \otimes \iota)\alpha(x)\mathcal{U}^* \quad \text{for all} \quad x \in N,
\]

\[
(\iota \otimes \iota \otimes \alpha)(\mathcal{U})(\Delta \otimes \iota \otimes \iota)(\mathcal{U}) = (1 \otimes \mathcal{U})(\iota \otimes \Delta \otimes \iota)(\mathcal{U}).
\]

If \(\mathcal{U}\) is trivial, i.e. \(\mathcal{U} = 1\), we call \(\alpha\) an action.

Notation 5. If \((\alpha, \mathcal{U})\) is a cocycle action of \((M, \Delta)\) on a von Neumann algebra \(N\), we introduce the notation

\[
\tilde{W} = (W \otimes 1) \mathcal{U}^*
\]

and then, \(\tilde{W}\) is a unitary in \(M \otimes \mathcal{B}(H) \otimes N\).
Given a cocycle action \((\alpha, \mathcal{U})\) of a locally compact quantum group \((M, \Delta)\) on a von Neumann algebra \(N\), we construct the crossed product \(M_{\alpha, \mathcal{U} \ltimes} N\). This is the von Neumann subalgebra of \(B(H) \otimes N\) generated by
\[
\alpha(N) \text{ and } \{(\omega \otimes \iota \otimes \iota)(W \otimes 1) \mathcal{U}^* | \omega \in M_\bullet\}.
\]
When \(\mathcal{U}\) is trivial, the crossed product is denoted by \(M_{\alpha, \mathcal{U} \ltimes} N\). There is a unique action \(\hat{\alpha}\) of \((M, \Delta^{\text{op}})\) on \(M_{\alpha, \mathcal{U} \ltimes} N\) such that, for all \(x \in N\),
\[
\hat{\alpha}(\alpha(x)) = 1 \otimes \alpha(x) \text{ and } (\iota \otimes \hat{\alpha})(W \otimes 1) \mathcal{U}^* = W_{12}W_{13}\mathcal{U}_{34}^*. \tag{1}
\]
We call this action \(\hat{\alpha}\) the dual action. It is proven in [15] that the fixed point algebra
\[
(M_{\alpha, \mathcal{U} \ltimes} N)^{\hat{\alpha}} = \{x \in M_{\alpha, \mathcal{U} \ltimes} N | \hat{\alpha}(x) = 1 \otimes x\} = \alpha(N).
\]

**Definition 6.** A pair \((M_1, \Delta_1), (M_2, \Delta_2)\) is said to be a matched pair of locally compact quantum groups if there exists a triple \((\tau, \mathcal{U}, \mathcal{V})\) (called a cocycle matching) satisfying the following conditions:

- \(\mathcal{U} \in M_1 \otimes M_1 \otimes M_2\) and \(\mathcal{V} \in M_1 \otimes M_2 \otimes M_2\) are both unitaries,
- \(\tau : M_1 \otimes M_2 \to M_1 \otimes M_2\) is a faithful \(*\)-homomorphism,
- defining \(\alpha(y) = \tau(1 \otimes y)\) and \(\beta(x) = \tau(x \otimes 1)\) we have
  - \((\alpha, \mathcal{U})\) is a cocycle action of \((M_1, \Delta_1)\) on \(M_2\),
  - \((\sigma, \mathcal{V})\) is a cocycle action of \((M_2, \Delta_2)\) on \(M_1\),
  - \((\alpha, \mathcal{U})\) and \((\beta, \mathcal{V})\) are matched in the following sense:
\[
\begin{align*}
\tau_{13}(\alpha \otimes \iota)\Delta_2(y) &= \mathcal{V}_{132}(\iota \otimes \Delta_2)\alpha(y)\mathcal{V}^*_{132}, \\
\tau_{23}\sigma_{23}(\beta \otimes \iota)\Delta_1(x) &= \mathcal{U}(\Delta_1 \otimes \iota)\beta(x)\mathcal{U}^*, \\
(\Delta_1 \otimes \iota \otimes \iota)(\mathcal{U}^* \otimes 1)(\iota \otimes \tau \sigma \otimes \iota)(\iota \otimes \Delta_2^{\text{op}})(\iota \otimes \alpha)(\mathcal{V}) &= (\mathcal{U}^* \otimes 1)(\iota \otimes \tau \sigma \otimes \iota)(\iota \otimes \Delta_2^{\text{op}})(\iota \otimes \alpha)(\mathcal{V})(1 \otimes \mathcal{V}).
\end{align*}
\]

Given a cocycle matching \((\tau, \mathcal{U}, \mathcal{V})\) of \((M_1, \Delta_1)\) and \((M_2, \Delta_2)\), one is able to construct the cocycle bicrossed product \((M, \Delta)\). By definition \(M = M_1_{\alpha, \mathcal{U} \ltimes} M_2\) and \(\Delta(x) = W^*(1 \otimes x)W\) with \(W = \Sigma W^*\Sigma\) and
\[
\hat{W} = (\beta \otimes \iota \otimes \iota)((W_1 \otimes 1) \mathcal{U}^*)((\iota \otimes \iota \otimes \iota)\mathcal{V}(1 \otimes \hat{W}_2)) \in M_1 \otimes \mathcal{B}(H_2) \otimes \mathcal{B}(H_1) \otimes M_2.
\]
It is proven in [13] that \((M, \Delta)\) is a locally compact quantum group and that \(W\) is its multiplicative unitary.

In Section 5, \((M_1, \Delta_1)\) and \((M_2, \Delta_2)\) will always be two locally compact quantum groups matched by \((\tau, \mathcal{U}, \mathcal{V})\) and their cocycle bicrossed product locally compact.
quantum group will be denoted by \((M, \Delta)\). All the objects associated with a quantum group (e.g. \(W, \Delta, \ldots\)) will be denoted with an index, when they refer to \((M_1, \Delta_1)\) and \((M_2, \Delta_2)\) respectively and without an index when they refer to \((M, \Delta)\). So we have that \(W_1\) (resp., \(W_2\)) is the multiplicative unitary of \((M_1, \Delta_1)\) (resp., \((M_2, \Delta_2)\)) and
\[
\tilde{W}_1 = (W_1 \otimes 1)U^*.
\]

From propositions 2.4 and 2.5 of \([15]\), we know how the comultiplication \(\Delta\) works on the generators \(\alpha(x)\) and \((\omega \otimes \iota \otimes \iota)(\tilde{W})\) of \((M, \Delta)\).

\[
\Delta(\alpha(x)) = (\alpha \otimes \alpha)\Delta_2(x)
\]
\[
(\iota \otimes \Delta^{\text{op}})(\tilde{W}_1) = (\tilde{W}_1 \otimes 1 \otimes 1)((\iota \otimes \alpha)\beta \otimes \iota \otimes \iota)(\tilde{W}_1)(\iota \otimes \alpha \otimes \alpha)(V) \quad (2)
\]

Define \(\hat{M}\) as the von Neumann subalgebra of \(M_1 \otimes B(H_2)\) generated by \(\beta(M_1)\) and \\{\((\iota \otimes \iota \otimes \omega)(V(1 \otimes \tilde{W}_2)) \mid \omega \in M_{2*}\}\). We define \(\Delta(z) = \tilde{W}^*(1 \otimes z)\tilde{W}\), for all \(z \in M\). It is proven in \([15]\) that \((M, \Delta)\) is the dual locally compact quantum group of \((M_1, \Delta_1)\). So, if we interchange the roles of \(\alpha\) and \(\beta\), \(M_1\) and \(M_2\) respectively, then we find, as the cocycle bicrossed product, the dual of the original cocycle bicrossed product.

**Definition 7.** A cocycle action \((\alpha, U)\) of \((M, \Delta)\) on a von Neumann algebra \(N\) is said to be **stabilizable** with a unitary \(X \in M \otimes N\) if
\[
(1 \otimes X)(\iota \otimes \alpha)(X) = (\Delta \otimes \iota)(X)U^*.
\]

**Proposition 8.** Let \((\alpha, U)\) be a cocycle action of \((M, \Delta)\) on \(N\) which is stabilizable with a unitary \(X \in M \otimes N\). Then the formulas
\[
\beta : N \to M \otimes N : \beta(x) = X\alpha(x)X^* \quad \text{and} \quad \Phi : z \mapsto X^*zX
\]
define, respectively, an action of \((M, \Delta)\) on \(N\) and a \(\ast\)-isomorphism from \(M_{\beta \prec} N\) onto \(M_{\alpha, U \prec} N\) satisfying
\[
\hat{\alpha} \circ \Phi = (\iota \otimes \Phi) \circ \hat{\beta}.
\]

The next proposition shows that many cocycle actions are stabilizable.

**Proposition 9.** Let \((\alpha, U)\) be a cocycle action of \((M, \Delta)\) on \(N\). Then \((\alpha \otimes \iota, U \otimes 1)\) is a cocycle action of \((M, \Delta)\) on \(N \otimes B(H)\) which is stabilizable.

### 5 Amenability and the bicrossed product construction

In this section, we investigate the relation between amenability of the cocycle bicrossed product quantum group \((M, \Delta)\) and of its building ingredients \((M_1, \Delta_1)\) and \((M_2, \Delta_2)\).
We start with a technical remark about slicing with non-normal functionals. Let $N$ and $L$ be von Neumann algebras, $n \in N^*$ and $X \in N \otimes L$.

If $n \in N_*$, then it is obvious that $(n \otimes \iota)(X) \in L$. This remains true for $n \in N^*$, even if $n$ is not normal. Indeed, consider the map $L_* \to \mathbb{C} : \omega \mapsto n((\iota \otimes \omega)(X))$.

It is obvious that this is a bounded linear functional and since $L = (L_*)^*$, we know that there exists a unique $Y \in L$ such that $\omega(Y) = n((\iota \otimes \omega)(X))$ for all $\omega \in L_*$. Denote $Y = (n \otimes \iota)(X)$.

Suppose that $\Phi : L \to K$ is a normal $*$-homomorphism of von Neumann algebras. Since for all $\omega \in K_*$

$$\omega(\Phi(n \otimes \iota)(X)) = n((\iota \otimes \omega \circ \Phi)(X)),$$

$$= n((\iota \otimes \omega)(\iota \otimes \Phi)(X)),$$

$$= \omega((n \otimes \iota)(\iota \otimes \Phi)(X)),$$

we may conclude that for all $\Phi$

$$\Phi((n \otimes \iota)(X)) = (n \otimes \iota)(\iota \otimes \Phi)(X).$$

This will be used several times in the sequel, where $n$ will be an invariant mean and $\Phi = \alpha, \Delta, \ldots$.

**Definition 10.** If $\alpha$ is an action of $(M, \Delta)$ on a von Neumann algebra $N$, we define an $\alpha$-invariant mean to be a state $m \in N^*$ such that

$$m((\omega \otimes \iota)(\alpha(x))) = m(x)\omega(1)$$

for all $\omega \in M_*$ and $x \in N$.

**Proposition 11.** Let $(\alpha, U)$ be a cocycle action of $(M, \Delta)$ on a von Neumann algebra $N$, $M_{\alpha, U} \ltimes N$ the cocycle crossed product and $\hat{\alpha}$ the dual action. Then, $(\hat{M}, \hat{\Delta})$ is amenable if and only if there exists a $\hat{\alpha}$-invariant mean on $M_{\alpha, U} \ltimes N$.

**Proof.** Suppose that $\hat{m}$ is a invariant mean on $(\hat{M}, \hat{\Delta})$. Then we argue that there exists an $\hat{\alpha}$-invariant mean on $M_{\alpha, U} \ltimes N$. This can be done by generalizing a result in [6] from the Kac algebra level to the general setting. However, their proof is based on a non-constructive argument. We construct explicitly an $\hat{\alpha}$-invariant mean on $M_{\alpha, U} \ltimes N$. The dual weight construction is the source of inspiration. Define $T : M_{\alpha, U} \ltimes N \to M_{\alpha, U} \ltimes N$ by $T(z) := (\hat{m} \otimes \iota)\hat{\alpha}(z)$. We prove that $T(z) \in \alpha(N)$ for all $z \in M_{\alpha, U} \ltimes N$. Since $\alpha(N)$ is the fixed point algebra of $\hat{\alpha}$, it is sufficient to show that $\hat{\alpha}(T(z)) = 1 \otimes T(z)$. Observe that, $\hat{\alpha}$ being an action, $\hat{\alpha}(T(z)) = (\hat{m} \otimes \iota \otimes \iota)(\hat{\Delta}^{op} \otimes \iota)\hat{\alpha}(z)$. So we have to prove that, for all $\omega \in (M \otimes M_{\alpha, U} \ltimes N)_*$,

$$\omega((\hat{m} \otimes \iota \otimes \iota)(\hat{\Delta}^{op} \otimes \iota)\hat{\alpha}(z)) = \omega(1 \otimes (\hat{m} \otimes \iota)\hat{\alpha}(z)).$$
But, it is sufficient to check this for normal functionals of the form $\mu \otimes \nu$ with $\mu \in \tilde{M}$ and $\nu \in (M_{\alpha,\Delta} \ltimes N)_\ast$. Using the fact that $\hat{m}$ is a LIM on $(\hat{M}, \hat{\Delta})$ and hence a RIM on $(\hat{M}, \hat{\Delta}^{op})$, we get

$$(\mu \otimes \nu)((\hat{m} \otimes \iota \otimes \iota)(\hat{\Delta}^{op} \otimes \iota)\hat{\alpha}(z)) = \hat{m}((\iota \otimes \mu \otimes \nu)(\hat{\Delta}^{op} \otimes \iota)\hat{\alpha}(z)),$$

$$= \mu(1)\hat{m}((\iota \otimes \nu)(\hat{\alpha}(z))),$$

$$= (\mu \otimes \nu)(1 \otimes (\hat{m} \otimes \iota)\hat{\alpha}(z)).$$

So we may conclude that $T(z) \in \alpha(N)$ for all $z \in M_{\alpha,\Delta} \ltimes N$.

Choose a state $\eta \in N^\ast$. Define $m(z) = \eta(\alpha^{-1}(T(z)))$. We will prove that $m$ is $\hat{\alpha}$-invariant. For all $\omega \in \tilde{M}$ and $z \in M_{\alpha,\Delta} \ltimes N$, we get that

$$m((\omega \otimes \iota)\hat{\alpha}(z)) = \eta(\alpha^{-1}((\hat{m} \otimes \iota)\hat{\alpha}((\omega \otimes \iota)\hat{\alpha}(z)))),$$

$$= \eta(\alpha^{-1}((\hat{m} \otimes \iota)(\omega \otimes \iota \otimes \iota)((\iota \otimes \hat{\alpha})(\hat{\alpha}(z)))))$$

$$= \eta(\alpha^{-1}((\hat{m} \otimes \iota)(\omega \otimes \iota \otimes \iota)(\hat{\Delta}^{op} \otimes \iota)(\hat{\alpha}(z))))$$

$$= \eta(\alpha^{-1}((\hat{m} \otimes \iota)\hat{\alpha}(z)))\omega(1) = m(z)\omega(1).$$

Conversely, suppose that $m$ is a $\hat{\alpha}$-invariant mean on $M_{\alpha,\Delta} \ltimes N$. We have to prove that $(\hat{M}, \hat{\Delta})$ is amenable. The proof is cut into three cases.

**Case 1: $\mathcal{U}$ is trivial.**

We know that $M_{\alpha,\Delta} \ltimes N$ is generated by $\alpha(N)$ and $\hat{M} \otimes \mathbb{C}$. Define $\hat{m}(\hat{x}) := m(\hat{x} \otimes 1)$ for all $\hat{x} \in \hat{M}$. Using the formula $\hat{\alpha}(\hat{x} \otimes 1) = \hat{\Delta}^{op}(\hat{x}) \otimes 1$, we get that for all $\omega \in \tilde{M}$

$$\hat{m}((\omega \otimes \iota)\hat{\Delta}^{op}(\hat{x})) = m((\omega \otimes \iota \otimes \iota)(\hat{\Delta}^{op}(\hat{x}) \otimes 1)),$$

$$= m((\omega \otimes \iota \otimes \iota)\hat{\alpha}(\hat{x} \otimes 1))$$

$$= m(\hat{x} \otimes 1)\omega(1) = \hat{m}(\hat{x})\omega(1).$$

So, we may conclude that $\hat{m}$ is a left invariant mean on $(\hat{M}, \hat{\Delta}^{op})$.

**Case 2: $(\alpha, \mathcal{U})$ is stabilizable.**

We know from Proposition 8 that, in this case, there exists an action $\beta$ of $(M, \Delta)$ on $N$ and a $*$-isomorphism

$$\Phi : M_{\beta,\Delta} \ltimes N \rightarrow M_{\alpha,\Delta} \ltimes N,$$

such that $\hat{\alpha} \circ \Phi = (\iota \otimes \Phi) \circ \hat{\beta}$.

Define $\hat{m} := m \circ \Phi$, then it is easy to prove that $\hat{m}$ is $\hat{\beta}$-invariant. Using the first case, we may conclude that the restriction of $\hat{m}$ to $M$ will be a LIM on $(M, \hat{\Delta}^{op})$.

**General case: Arbitrary $(\alpha, \mathcal{U})$.**

In general, $(\alpha \otimes \iota, \mathcal{U} \otimes 1)$ will be a cocycle action of $(M, \Delta)$ on $N \otimes \mathcal{B}(H)$ and we know from Proposition 8 that it will be stabilizable. It is not too difficult to show that its corresponding cocycle crossed product factorizes as $(M_{\alpha,\Delta} \ltimes N) \otimes \mathcal{B}(H)$, as well as the dual action, which is given by $\hat{\alpha} \otimes \iota$.  

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Choose a normalized vector $\xi \in H$. Then, we have for all $z \in (M_{\alpha, \mathcal{U} \ltimes N}) \otimes B(H)$ that
\[
(\iota \otimes m \otimes \omega_\xi)((\hat{\alpha} \otimes \iota)(z)) = (\iota \otimes m)\hat{\alpha}((\iota \otimes \omega_\xi)(z)),
\]
\[
= m((\iota \otimes \omega_\xi)(z))1,
\]
\[
= (m \otimes \omega_\xi)(z)1.
\]
We find that $m \otimes \omega_\xi$ is $(\hat{\alpha} \otimes \iota)$-invariant and from the second case, we may conclude that $(\hat{M}, \hat{\Delta})$ is amenable.

With this theorem in mind, we are going to prove our main result, generalizing a result of Ng [13]. Ng proves in [13] that the bicrossed product with trivial cocycles of two locally compact groups $G_1$ and $G_2$, is amenable if $G_2$ is amenable. Notice that for any group $G_1$, $(L^\infty(G_1), \hat{\Delta}_1)$ is always coamenable.

To prove our main theorem, we need a lemma. We can get this result from propositions 3.1 and 3.4 of [15], but we have chosen to give a straightforward proof.

**Lemma 12.** Let $(\tau, \mathcal{U}, \mathcal{V})$ be a cocycle matching of $(M_1, \Delta_1)$ and $(M_2, \Delta_2)$ and let $(\hat{M}, \hat{\Delta})$ be the cocycle bicrossed product. Then
\[
(\iota \otimes \Delta^{\text{op}})\hat{\alpha}(z) = (\hat{\alpha} \otimes \iota)\Delta^{\text{op}}(z) \text{ for all } z \in M.
\]

**Proof.** It suffices to check it on the generators. Choose $x \in M_2$. Observe that
\[
(\iota \otimes \Delta^{\text{op}})\hat{\alpha}(\alpha(x)) = (\iota \otimes \Delta^{\text{op}})(1 \otimes \alpha(x)),
\]
\[
= 1 \otimes \Delta^{\text{op}}(\alpha(x)),
\]
\[
= 1 \otimes (\alpha \otimes \alpha)\Delta^{\text{op}}(x),
\]
\[
= (\hat{\alpha} \otimes \iota)((\alpha \otimes \alpha)\Delta^{\text{op}}(x)),
\]
\[
= (\hat{\alpha} \otimes \iota)\Delta^{\text{op}}(\alpha(x)).
\]

Now, we will prove that $(\iota \otimes \tau \otimes \Delta^{\text{op}})(\iota \otimes \hat{\alpha})(\hat{W}_1) = (\iota \otimes \hat{\alpha} \otimes \tau)((\iota \otimes \Delta^{\text{op}})(\hat{W}_1)).$

Using Equation (2), we get
\[
(\iota \otimes \tau \otimes \Delta^{\text{op}})(\iota \otimes \hat{\alpha})(\hat{W}_1) = (\iota \otimes \hat{\alpha} \otimes \tau)((\iota \otimes \Delta^{\text{op}})(\hat{W}_1)).
\]

Finally, observe that, as operators on $H_1 \otimes H_1 \otimes H_1 \otimes H_2 \otimes H_1 \otimes H_2$,
\[
(\iota \otimes \hat{\alpha} \otimes \tau)((\iota \otimes \Delta^{\text{op}})(\hat{W}_1))
\]
\[
= (\iota \otimes \hat{\alpha} \otimes \tau)((\hat{W}_1 \otimes 1 \otimes 1)((\iota \otimes \tau)(\hat{W}_1)(\iota \otimes \alpha \otimes \alpha)(\mathcal{V})),
\]
\[
= (W_1)_12(W_1)_134((\iota \otimes \alpha \otimes \alpha)(\mathcal{V}))_{13456},
\]
\[
= (W_1)_12(\iota \otimes \Delta^{\text{op}})(W_1)_13456,
\]
\[
= (W_1)_12(\iota \otimes \Delta^{\text{op}})(W_1)_134,
\]
\[
= (\iota \otimes \tau \otimes \Delta^{\text{op}})((W_1)_134),
\]
where we used Equation (2) in the first line.
Theorem 13. Let \((\tau, U, V)\) be a cocycle matching of \((M_1, \Delta_1)\) and \((M_2, \Delta_2)\) and let \((M, \Delta)\) be the cocycle bicrossed product. Then, \((M, \Delta)\) is amenable if and only if \((\hat{M}_1, \hat{\Delta}_1)\) and \((M_2, \Delta_2)\) are amenable.

Proof. The proof is divided into three parts.

1) If \((M, \Delta)\) is amenable, then \((\hat{M}_1, \hat{\Delta}_1)\) is amenable.

Let \(m\) be an invariant mean on \((M, \Delta)\). From Proposition 11, we know that it is sufficient to show that \(m\) is \(\hat{\alpha}\)-invariant. If we apply \(\iota \otimes \iota \otimes m\) on the result in Lemma 12, we get that, for all \(z \in M\),

\[(\iota \otimes m)\hat{\alpha}(z) \otimes 1 = m(z)1 \otimes 1.\]

From this we conclude that \((\iota \otimes m)\hat{\alpha}(z) = m(z)1\) and therefore that \(m\) is \(\hat{\alpha}\)-invariant.

2) If \((M, \Delta)\) is amenable, then \((M_2, \Delta_2)\) is amenable.

Suppose that \(m\) is a LIM on \((M, \Delta)\). Define \(m_2 \in M_2^*\) by \(m_2(x) = m(\alpha(x))\).

Since \((\alpha \otimes \alpha)\Delta_2 = \Delta \circ \alpha\) and \(M_{2s} = \{\omega \circ \alpha \mid \omega \in M_s\}\) it is obvious that \(m_2\) will be a LIM on \((M_2, \Delta_2)\).

3) If \((\hat{M}_1, \hat{\Delta}_1)\) and \((M_2, \Delta_2)\) are amenable, then \((M, \Delta)\) is amenable.

Suppose that \(\hat{m}_1\) and \(m_2\) are left invariant means on \((\hat{M}_1, \hat{\Delta}_1)\) and \((M_2, \Delta_2)\) respectively.

Consider the dual action \(\hat{\alpha} : M \to \hat{M}_1 \otimes M\). Define \(T(z) := (\hat{m}_1 \otimes \iota)\hat{\alpha}(z)\) for all \(z \in M\). From the proof of Proposition 11, we know that \(T(z) \in \alpha(M_2)\).

Define \(m := m_2 \circ \alpha^{-1} \circ T\). We prove that \(m\) is a left invariant mean on \(M\).

Choose any \(z \in M\). Applying \(\hat{m}_1 \otimes \iota \otimes \iota\) on both sides of the result of Lemma 12 we get

\[\Delta^{\text{op}}(T(z)) = (T \otimes \iota)\Delta^{\text{op}}(z).\]

So, we have for all \(\mu \in M_s\)

\[(\iota \otimes \mu)\Delta^{\text{op}}(T(z)) = T((\iota \otimes \mu)\Delta^{\text{op}}(z)).\]

Take \(y \in M_2\) such that \(T(z) = \alpha(y)\).

Since \((\iota \otimes \mu)\Delta^{\text{op}}(\alpha(y)) = (\iota \otimes \mu)(\alpha \otimes \alpha)\Delta^{\text{op}}_2(y) = \alpha((\iota \otimes \mu \circ \alpha)\Delta^{\text{op}}_2(y))\) we get

\[T((\iota \otimes \mu)\Delta^{\text{op}}(z)) = \alpha((\iota \otimes \mu \circ \alpha)\Delta^{\text{op}}_2(y)).\]

When we apply \(m_2 \circ \alpha^{-1}\) on both sides of Equation 14, we get

\[m((\iota \otimes \mu)\Delta^{\text{op}}(z)) = m_2((\iota \otimes \mu \circ \alpha)\Delta^{\text{op}}_2(y)).\]
Now we can use left invariance of $m_2$ and we find for all $\mu \in M_*$
\[
m((\iota \otimes \mu)\Delta^\op(z)) = m_2((\iota \otimes \mu \circ \alpha)\Delta_2^\op(y))
= \mu(\alpha(1))m_2(y)
= \mu(1)m_2(\alpha^{-1}(T(z)))
= \mu(1)m(z).
\]

Therefore, $m$ is a left invariant mean on $(M, \Delta)$.

A natural question is whether or not the strong version of Theorem 13 is true, i.e., Theorem 13 with amenability replaced by strong amenability. We can only give a partial answer. First of all, it is not too difficult to see that $(\hat{M}_1^*, \hat{\Delta}_1^\op)$ is strongly amenable if $(M, \Delta)$ is. Just suppose that the net $(\hat{\mu}_k)$ is an approximate unit of $\hat{M}_1$. Define $\mu_{1k} := \hat{\mu}_k \circ \beta$, then $(\mu_{1k})$ is an approximate unit of $M_{1*}$. So we arrive at the following proposition.

**Proposition 14.** Let $(\tau, U, V)$ be a cocycle matching of $(M_1, \Delta_1)$ and $(M_2, \Delta_2)$ and let $(M, \Delta)$ be the cocycle bicrossed product. If $(M, \Delta)$ is strongly amenable, then $(\hat{M}_1, \hat{\Delta}_1)$ is strongly amenable.

Next, we can prove the strong version of Theorem 13 in the case where the cocycles are trivial: $U = V = 1$. We do not know whether or not the same result holds with non-trivial cocycles.

**Theorem 15.** Let $(M, \Delta)$ be the bicrossed product of $(M_1, \Delta_1)$ and $(M_2, \Delta_2)$ with trivial cocycles. Then, $(M, \Delta)$ is strongly amenable if and only if $(M_1, \Delta_1)$ and $(M_2, \Delta_2)$ are strongly amenable.

**Proof.** We will first prove that if $(\hat{M}_1, \hat{\Delta}_1)$ and $(M_2, \Delta_2)$ are strongly amenable, then $(M, \Delta)$ is strongly amenable.

Suppose that $(\omega_i)$ is a bounded two-sided approximate unit for $M_{1*}$. It is sufficient to show that
\[
((\omega_i \otimes \iota)\beta \otimes \iota)(W_1) \overset{\text{sa}}{\longrightarrow} 1.
\]
Indeed, by definition, $\hat{W} = (((\beta \otimes \iota)(W_1) \otimes 1)(\iota \otimes \alpha)(\hat{W}_2)) \in M_1 \otimes B(H_2) \otimes B(H_1) \otimes M_2$ and so
\[
(\omega_i \otimes \iota \otimes \iota \otimes \iota)(\hat{W}) = (((\omega_i \otimes \iota)\beta \otimes \iota)(W_1) \otimes 1)((\iota \otimes \alpha)(\hat{W}_2)).
\]
Using Equation (5), we get that
\[
(\omega_i \otimes \iota \otimes \iota \otimes \iota)(\hat{W}) \overset{\text{sa}}{\longrightarrow} (\iota \otimes \alpha)(\hat{W}_2).
\]
Because $(M_2, \Delta_2)$ is strongly amenable, we can take a net $(\xi_j)$ of normalized vectors in $H_2$ such that $(\mu_{\xi_j} \otimes \iota)(\hat{W}_2) \overset{\text{sa}}{\longrightarrow} 1$. 

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Choose $\mu \in M_*$. Observe that for all $i, j$

$$|\mu((\xi_i \otimes \iota \otimes \iota)(\omega_i \otimes \iota \otimes \iota)(\hat{W}))| \leq \|(\iota \otimes \iota \otimes \mu)(\hat{W})\|.$$ 

Taking first the limit over $i$ and then over $j$ we get

$$|\mu(1)| \leq \|(\iota \otimes \iota \otimes \mu)(\hat{W})\|.$$ 

Define $\hat{\epsilon}((\iota \otimes \mu)(\hat{W})) = \mu(1)$. Thus, $\hat{\epsilon}$ is a bounded co-unit for $(\hat{M_c}, \hat{\Delta_c})$.

It remains to prove (5). The definition of matched pairs implies that

$$\tau_{23}\sigma_{23}(\beta \otimes \iota)\Delta_1(x) = (\Delta_1 \otimes \iota)\beta(x).$$

When we apply $\omega_i \otimes \iota \otimes \iota$ on both sides we get

$$\tau\sigma(((\omega_i \otimes \iota)\beta \otimes \iota)\Delta_1(x)) = (\omega_i \otimes \iota \otimes \iota)((\Delta_1 \otimes \iota)\beta(x)).$$

For all $\omega \in M_{1*}$ and $\nu \in M_{2*}$, we have that

$$(\omega \otimes \nu)((\omega_i \otimes \iota \otimes \iota)((\Delta_1 \otimes \iota)\beta(x))) = (\omega_i \ast \omega \otimes \nu)\beta(x) \rightarrow (\omega \otimes \nu)(\beta(x)).$$

By linearity and the fact that $((\omega_i \otimes \iota \otimes \iota)((\Delta_1 \otimes \iota)\beta(x)))$ is uniformly bounded in $i$, we get that

$$((\omega \otimes \nu)(\omega_i \otimes \iota \otimes \iota)((\Delta_1 \otimes \iota)\beta(x))) \xrightarrow{\sigma_w} \beta(x).$$

Using Equation (6) and the normality of $\tau\sigma$ we find that

$$\tau\sigma(((\omega_i \otimes \iota)\beta \otimes \iota)\Delta_1(x)) \xrightarrow{\sigma_w} \beta(x) = \tau\sigma(1 \otimes x).$$

Now, $\tau\sigma$ is an injective and normal $*$-homomorphism and therefore it will be homeomorphic onto its image for the $\sigma$-weak topology ([3], p. 60). From this, we get

$$((\omega_i \otimes \iota)\beta \otimes \iota)\Delta_1(x) \xrightarrow{\sigma_w} 1 \otimes x.$$ 

When we apply $\beta \otimes \iota \otimes \iota$ on $(\Delta_1 \otimes \iota)(W_1) = W_{1,13}W_{1,23}$ we get

$$((\beta \otimes \iota)\Delta_1 \otimes \iota)(W_1) = ((\beta \otimes \iota)(W_1))_{124}W_{1,34}$$

and

$$((\omega_i \otimes \iota)\beta \otimes \iota)(W_1) = ((\omega_i \otimes \iota)\beta \otimes \iota)(W_1))_{13}W_{1,23}.$$ 

Using Equation (6), we may conclude that

$$((\omega_i \otimes \iota)\beta \otimes \iota)(W_1)_{13}W_{1,23} \xrightarrow{\sigma_w} 1 \otimes W_1 = W_{1,23}.$$ 

As $W_1$ is invertible, this implies that

$$(\omega_i \otimes \iota)\beta \otimes \iota)W_1 \xrightarrow{\sigma_w} 1.$$ 

This concludes the first part of the proof.
By taking trivial cocycles in Proposition 14, it is immediately clear that $(\hat{M}_1, \hat{\Delta}_1)$ is strongly amenable, if $(M, \Delta)$ is strongly amenable.

It remains to show that if $(M, \Delta)$ is strongly amenable, then $(M_2, \Delta_2)$ is strongly amenable. Using the biduality theorem, it is sufficient to prove that if $(\hat{M}, \hat{\Delta})$ is strongly amenable, then $(\hat{M}_1, \hat{\Delta}_1)$ is strongly amenable. Suppose that $(\omega_i)_i$ is a bounded two-sided approximate unit for $M^*$. We know that now

$$M = (\alpha(M_2) \cup \{ (\omega \otimes \iota)(W_1) \otimes 1 \mid \omega \in M_1^* \})^\sigma.$$

Using Equation (2), we get

$$(\iota \otimes \Delta^\text{op})(W_1 \otimes 1) = (W_1 \otimes 1 \otimes 1 \otimes 1)((\iota \otimes \alpha)\beta \otimes \iota \otimes \iota)(W_1 \otimes 1),$$

so

$$(\iota \otimes \iota \otimes \omega_i)(\iota \otimes \Delta^\text{op})(W_1 \otimes 1) = (W_1 \otimes 1)(\iota \otimes \alpha)\beta((\iota \otimes \omega_i)(W_1 \otimes 1)). \tag{9}$$

Using the fact that $(\omega_i)_i$ is an approximate unit of $M_*$, we have

$$(\iota \otimes \iota \otimes \omega_i)(\iota \otimes \Delta^\text{op})(W_1 \otimes 1) \xrightarrow{\text{w}} W_1 \otimes 1$$

and thus, by Equation (9)

$$(\iota \otimes \alpha)\beta((\iota \otimes \omega_i)(W_1 \otimes 1)) \xrightarrow{\text{w}} 1.$$

But $(\iota \otimes \alpha)\beta$ is a normal and injective $\ast$-homomorphism and therefore

$$(\iota \otimes \omega_i)(W_1 \otimes 1) \xrightarrow{\text{w}} 1. \tag{10}$$

Define $\mu_i \in \hat{M}_1^*$ such that $\mu_i(z) = \omega_i(z \otimes 1)$ for all $z \in \hat{M}_1$, so $(\iota \otimes \mu_i)(W_1) = (\iota \otimes \omega_i)(W_1 \otimes 1)$. Using Equation (10) we get that

$$(\iota \otimes \mu_i)(W_1) \xrightarrow{\text{w}} 1$$

and this concludes the proof.

\begin{flushright}
\Box
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6 Examples

In order to construct these examples, we rely on the extension procedure of locally compact quantum groups as developed in [2, 11, 15]. All the bicrossed product locally compact quantum groups in [15] are amenable. That is easily seen, since the groups from which one starts in the examples are both amenable. We give two examples of non-amenable locally compact quantum groups, obtained by a bicrossed product construction. From theorem 13, we know that, if we take,
as one of the ingredients, a non-amenable group, the bicrossed product locally compact quantum group will be not amenable. In the first we take $SL_2(\mathbb{R})$ as the non-amenable group and in the second (a double cover of) $SU(1, 1)$. It is a known that these groups are not amenable, since these are non-compact, almost connected, semi-simple Lie groups, see [6].

We briefly review what is needed from the extension procedure.

Let $G, G_1$ and $G_2$ be locally compact groups with fixed left invariant Haar measures. Let $i : G_1 \to G$ be a homomorphism and $j : G_2 \to G$ an antihomomorphism such that both have a closed image and are homeomorphisms onto these images. Suppose moreover that the mapping

$$\theta : G_1 \times G_2 \to \Omega \subset G : (g, s) \mapsto i(g)j(s)$$

is a homeomorphism of $G_1 \times G_2$ onto an open subset $\Omega$ of $G$ having a complement of measure zero. Then we call $G_1$ and $G_2$ a matched pair of locally compact groups. From this data, one constructs a cocycle matching of $(L^\infty(G_1), \Delta_1)$ and $(L^\infty(G_2), \Delta_2)$ with trivial cocycles as follows. Let $\rho : G_1 \times G_2 \to \Omega^{-1}$ be the homeomorphism given by $\rho(g, s) = j(s)i(g)$. Let $\mathcal{O} = \theta^{-1}(\Omega \cap \Omega^{-1})$ and for $(g, s) \in \mathcal{O}$ define $\beta_s(g) \in G_2$ and $\alpha_g(s) \in G_2$ by

$$\rho^{-1}(\theta(g, s)) = (\beta_s(g), \alpha_g(s)).$$

Finally, one can define a $*$-isomorphism

$$\tau : L^\infty(G_1) \otimes L^\infty(G_2) \to L^\infty(G_1) \otimes L^\infty(G_2)$$

by $\tau(f)(g, s) = f(\beta_s(g), \alpha_g(s))$. Then, $(\tau, 1, 1)$ gives a cocycle matching of $(L^\infty(G_1), \Delta_1)$ and $(L^\infty(G_2), \Delta_2)$ with trivial cocycles.

**Example 1.**

$$G = \left\{ \begin{pmatrix} a & b & x \\ c & d & y \\ 0 & 0 & 1 \end{pmatrix} \in SL_3\mathbb{R}, \ x, y \in \mathbb{R} \right\}$$

So, $G$ is a Lie-subgroup of $SL_3(\mathbb{R})$.

$$G_1 = \mathbb{R}^2, + \text{ and } G_2 = SL_2(\mathbb{R}).$$

Further, $i$ maps $G_1$ into $G$ in the canonical way and

$$i((x, y)) = \begin{pmatrix} 1 & 0 & -x \\ -x & 1 & -y + \frac{1}{2}x^2 \\ 0 & 0 & 1 \end{pmatrix} \quad j\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} d & -b & 0 \\ -c & a & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Suppose that

$$Q = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R}).$$
Then, the mutual actions are given by
\[
\alpha(x, y)(A) = \left( \begin{array}{cc} a + bx & b \\ c + dx - (a + bx)(ax + b(y + \frac{1}{2}x^2)) & d - b(ax + b(y + \frac{1}{2}x^2)) \end{array} \right)
\]
and
\[
\beta_A((x, y)) = (ax + by + \frac{b}{2}x^2, cx + d(y + \frac{1}{2}x^2) - \frac{1}{2}(ax + b(y + \frac{1}{2}x^2))^2)
\]
We take trivial cocycles and construct the bicrossed locally compact quantum group \((M, \Delta)\). It is not so difficult to show that \(\delta_1\) and \(\delta_2\) are trivial and \(\delta(A, (x, y)) = \det A = 1\). Therefore, the bicrossed product is a Kac algebra.

One might think that there is a hope to leave the Kac algebra ‘world’, if we would work with the general linear groups (GL) instead of the special linear groups (SL). Unfortunately, the determinant will be \(\alpha\)-invariant. So, we will also find that the bicrossed product is a Kac algebra.

Now, one can construct the infinitesimal Hopf algebra of the bicrossed product quantum group in the sense of [15]. It is an algebraic version of the same quantum group.

In this example the infinitesimal Hopf algebra has generators \(X, Y, A, B, C\) and \(D\) satisfying \(AD - BC = 1\) and the following relations

\[
\begin{align*}
[A, B] &= 0, \quad [A, C] = 0, \quad [A, D] = 0, \\
[B, C] &= 0, \quad [B, D] = 0, \quad [C, D] = 0, \\
[A, X] &= B, \quad [A, Y] = 0, \\
[B, X] &= 0, \quad [B, Y] = 0, \\
[C, X] &= D - A^2, \quad [C, Y] = -AB, \\
[D, X] &= -AB, \quad [D, Y] = -B^2,
\end{align*}
\]

\[
\begin{align*}
\Delta(A) &= A \otimes A + B \otimes C, \\
\Delta(B) &= B \otimes D + A \otimes B, \\
\Delta(C) &= C \otimes A + D \otimes C, \\
\Delta(D) &= D \otimes D + C \otimes B, \\
\Delta(X) &= 1 \otimes X + X \otimes A + Y \otimes C, \\
\Delta(Y) &= 1 \otimes Y + X \otimes B + Y \otimes D.
\end{align*}
\]

**Example 2.**

Now, we will construct a non-amenable locally compact quantum group that is not a Kac algebra.

\[G_1 = \{(x, z) | x \in \mathbb{R}, x \neq 0, z \in \mathbb{C}\} \quad \text{with} \quad (x, z)(y, u) = (xy, z + xu),\]
Define $Sq(x) = Sgn(x)\sqrt{|x|}$ for all $x \in \mathbb{R}$. Take embeddings $i$ and $j$ defined by

$i : (x, z) \mapsto \frac{1}{Sq(x)} \begin{pmatrix} x & -z \\ 0 & 1 \end{pmatrix}$ \quad $j : \begin{pmatrix} a & \bar{c} \\ c & \bar{a} \end{pmatrix} \mapsto \begin{pmatrix} a & \bar{c} \\ c & \bar{a} \end{pmatrix}^{-1}$.

The mutual actions are given by

$\alpha_{(x, z)}(a, c) = \frac{D}{Sq(|cz + \bar{a}x|^{2} - |c|^{2})} (\bar{c}z + ax, c)$

and

$\beta_{(a, c)}(x, z) = \frac{D}{|x|} (|cz + \bar{a}x|^{2} - |c|^{2}, (az + \bar{c}x)(\bar{c}z + ax) - a\bar{c})$

with $D = D(x, z, a, c) = \frac{x}{|x|}(|a|^{2} - |c|^{2})$.

Taking $U = V = 1$, we can construct the bicrossed product locally compact quantum group $(M, \Delta)$. Since $\delta$ and $\delta_{2}$ are trivial and $\delta_{1}(x, z) = \frac{x}{|x|}$, we conclude, using Propositions 2.17 and 4.16 of [15], that $(M, \Delta)$ is not a Kac algebra, is non-compact and non-discrete. As far as we know, there was, until now, no example of a non-discrete non-amenable quantum group that is not a group.

Now, the infinitesimal Hopf $\ast$-algebra is generated as a $\ast$-algebra by normal elements $A, C$ and $Y$, an antiselfadjoint element $X$ and a selfadjoint element $U$ satisfying the following commutation relations:

$[A, C] = [A, C^{\ast}] = 0, \quad A^{\ast}A - C^{\ast}C = U, \quad U^{2} = 1,$

$[X, Y] = Y,$

$[A, X] = -UACC^{\ast}, \quad [C, X] = -UAA^{\ast}C,$

$[A, Y] = 2C^{\ast} - UAA^{\ast}C^{\ast}, \quad [C, Y] = -UAA^{\ast}C^{\ast}C,$

$[A, Y^{\ast}] = UA^{2}C, \quad [C, Y^{\ast}] = UAC^{2}.$

Furthermore, the comultiplication is given by

$\Delta(A) = A \otimes A + C^{\ast} \otimes C,$

$\Delta(C) = C \otimes A + A^{\ast} \otimes C,$

$\Delta(X) = X \otimes U(A^{\ast}A + C^{\ast}C) + Y \otimes UAA^{\ast}C - Y^{\ast} \otimes UAC^{\ast} + 1 \otimes X,$

$\Delta(Y) = 1 \otimes Y + X \otimes 2UAA^{\ast}C^{\ast} + Y \otimes U(A^{\ast})^{2} - Y^{\ast} \otimes U(C^{\ast})^{2}.$
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