Weyl conformastatic perihelion advance

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ABSTRACT

In this paper, we examine a static gravitational field with axial symmetry over probe particles in the Solar system. Using the Weyl conformastatic solution as a model, we find a non-standard expression to perihelion advance due to the constraints imposed by the topology of the local gravitational field. We show that the application of the slow motion condition to the geodesic equations without altering Einstein’s equations does not necessarily lead to the Newtonian limit; rather it leads to an intermediate nearly Newtonian gravitational stage, which can be applied to astrophysical problems in Solar system scale. We apply the model to the perihelion advance of inner planets and minor objects (NEO asteroids and the comets). As a result, we obtain an expression of a non-standard relativistic precession that reveals a close agreement to observational data calibrated with the Ephemerides of the Planets and the Moon (EPM2011) (Pitjeva & Pitjev; Pitjev & Pitjeva). The study of perihelion advance of eight small celestial bodies (asteroids and comets) is also considered.

Key words: gravitation – relativistic processes.

1 INTRODUCTION

Usually in general relativity (GR) when slow motions are described, the first logical approach to a classical gravitational theory is to reduce Einstein’s equations to the Newtonian limit. That limit can be reached by many methods, such as the Parametrized Post-Newtonian (PPN) approximation which can be seen more as a stage of the Newtonian approximation. This method is basically constituted of terms with superior orders in the metric of a fallen test particle. In the PPN approximation, the metric tensor is generated through matter distribution and hypothesized under conditions of weak gravitational field and low light speed \( (v \ll c) \). The arbitrary potential’s coefficients are the well-known PPN parameters.

In order to avoid such parameters, in a different approach, we study the possibility of an application of GR to slow motion focusing on the geodesic equations alone. This paper aims at showing that the application of the slow motion condition to the geodesic equations without altering Einstein’s equations will produce a different dynamics. As a result, it will not necessarily lead to the Newtonian limit. This solution has to do with shape, topology or aspects of symmetry of the gravitational field. In this respect, we use the Weyl metric that describes a cylindrical symmetry, motivated mainly on the fact that the Solar system has an axial symmetry. From the geometrical point of view the Weyl cylindrically symmetric solution is diffeomorphic to the Schwarzschild solution and also is asymptotically flat as shown in (Weyl 1917; Rosen 1949; Zipoy 1966; Gautreau, Hoffman & Armenti 1972; Stephani et al. 2003).

For applications in astrophysics, the Weyl metric is fundamentally relevant as well as several non-asymptotically flat metrics, for instance, in the study of physics of black holes, stellar evolution and Galaxies as well as several studies of relativistic effects on Solar system scale (Sitarski 1983; Roberts 1987; Katz, Bicák & Lynden-Bell 1999; Ujevic & Letelier 2004, 2007; González, Gutiérrez-Piñeres & Osipina 2008; Vogt & Letelier 2008; Gutiérrez-Piñeres, González & Quevedo 2013).

In the following section, we make a brief review of the nearly Newtonian limit. In the third section, we study the gravitational field produced by the Weyl’s metric defining the coefficients of the metric. In the fourth section, the orbit equations and calculations of a non-standard expression for the perihelion advance are shown. To this end, we study the Weyl conformastatic solution (in the sense that the potential \( \lambda(r, z) = 0 \)) and obtain an expression for an orbit equation. Moreover, we study the perihelion advance for inner planets and the precession of asteroids and comets. Particularly, the relativistic effects in asteroids and comets have been explored in the literature (Sitarski 1983; Shahid & Yeomans 1994) and advances have been made in monitoring near-Earth objects (NEOs) using radar astronomy techniques (Yeomans, Ostro & Chodas 1987; Yeomans et al. 1992; Margot & Giorgini 2010).

In this work, we study NEOs such as 1566 Icarus (Shapiro, Ash & Smith 1968), 1862 Apollo, 2101 Adonis asteroids, 433 Eros and 3200 Phaethon. In addition, two Jupiter family comets (26P/Grigg-Skjellerup and 22p/Kopff) and an Encke-type comet.
(2p/Encke comet) are also studied. For the perihelion of inner planets we compare with the observational data (Nambuya 2010; Wilhelm & Dwivedi 2014) calibrated with the Ephemerides of the Planets and the Moon (EPM2011) (Pitjeva & Pitjev 2013; Pitjev & Pitjeva 2013). For the perihelion of asteroids, we compare our results with the observational data from both optical and radar observations (Shahid & Yeomans 1994). For the comets, we compare the results with the numerical results based on Painlevé coordinates to one-body Schwarzschild problem (Sitarski 1983). Finally, we make the final remarks in the conclusion section.

2 THE NEARLY NEWTONIAN REGIME

The most classical usage of the PPN approximation was done to explain the Mercury’s perihelion advance (Nobili & Will 1986). However, the PPN formalism is not the only tool available that describes slow motions, there is another stage that also involves space–time curvature. Infeld and Plebanski (Infeld & Plebanski 1960) made a very interesting and consistent demonstration of Einstein’s equations through Newton’s equations. They showed how geodesic equations are contained in Einstein’s equations. Starting from Newton’s equations, they took successive approximations of the metric with the parameter \( \nu / c \ll 1 \). As a result, once the geodesic equations are built, Newton’s equations postulate of motion could be dispensed. Even though Einstein conjectured this idea in 1915, this procedure was not clear at that time. Today it is known that a specific choice of parameters has to be made for a specific ending. There are other parameters, not velocity related, as for example, the weak field limit that gives the linear gravitational wave equation or the Schwarzschild weak field with the parameter \( \frac{1}{c} \). Thus, the Newtonian equation of motion appears in the limit of GR as an option which we need to impose the weak field condition to obtain Einstein’s equations.

On the other hand, if our concern is the connection, it is well known that geodesic equations are linear in terms of the connection and quadratic in terms of Einstein’s equations. This relation can affect the influence of the gravitational field in GR imprinting qualitative effects on solutions. For instance, if we only use the geodesic equation under the hypothesis of slow motion and weak gravitational field conditions, the field will act more smoothly, since the connection is linear in the geodesic equations. However, if now we focus our attention on Einstein’s equations, something different will occur, since the connection is of the fourth power on this particular set of equations. As a result, this means that a gravitational field originated only from geodesic equations will have different characteristics. This ‘intermediate’ gravitational regime will be located somewhere in between the gravitational filed created by Einstein’s equations, which is stronger, and the Newtonian field, which is weaker. This ‘new’ gravitational field is denominated nearly Newtonian field as stated in (Wheeler et al. 1973).

As an example, if we consider a free-falling slow moving particle its gravitational field will have additional increments, such as

\[
g_{\mu\nu} \approx \eta_{\mu\nu} + \delta h_{\mu\nu} + (\delta h_{\mu\nu})^2 + \cdots .
\]

Because of this additional increments, we recover the strength of the gravitational field. For this matter, the low-velocity hypothesis is the only valid one, since the field is stronger now. If this process is interrupted by an external force in a spherically symmetric matter distribution concurring with Einstein’s and geodesic equations, the Newtonian gravitational field can be restored through Poisson’s equation \( R^s_{\mu\nu} = \nabla^2 \phi = 4\pi G\rho \). Conversely, if we neglect the influence of any external action, this process goes on naturally, so we can sum all the metric’s \( g_{\mu\nu} \) perturbations from \( \delta h_{\mu\nu} = 0 \) to a finite value \( \delta h_{\mu\nu} \). Moreover, integrating from 0 to \( \delta h_{\mu\nu} \), we obtain

\[
\Phi_{hN} = -\frac{1}{2} \int_0^{\delta h_{44}} d(\delta h_{44}) = -\frac{1}{2} \delta h_{44} .
\]

Hence, we can find the nearly Newtonian gravitational potential \( \Phi_{hN} \) given by

\[
\Phi_{hN} = -\frac{1}{2} (1 + g_{44}) ,
\]

which is a similar equation for a spherically symmetric field but with a different qualitative interpretation. We stress that the \( g_{44} \) metric component is an exact and non-approximated solution.

One can insist that the same nearly Newtonian potential found in equation (1) cannot be obtained if we had just taken the limit \( \nu \ll c \) a priori. There is an error in this rationalization because of taking GR on its complete structure, the motion must not be given by Newton’s laws, but by the geodesic equation which is a non-linear equation of motion of the object at hand such that

\[
\frac{dv^i}{d\tau} + \Gamma^i_{jk} v^j v^k = 0 .
\]

At the end of the process if we apply the slow motion condition \( \nu \ll c \) to the system altogether with the metric tensor expansion, with the parameter \( \nu / c \), we simply obtain Newton’s theory and obtain the following potential

\[
\Phi_{hN} = -\frac{M}{r} ,
\]

which reduces to the Newtonian potential that does not explain the perihelion advance. In terms of comparison, we expect that the nearly Newtonian potential can be used to give a description of the perihelion precession rather than the PPN approximation and extending it also to non-standard systems.

3 WEYL’S GRAVITATIONAL FIELD

Besides of its historical relevance as one of the main tests for a gravitational theory candidate, the study of the perihelion advance plays an important role on the development of gravitational physics. In recent years, there has been a renewed interest in the perihelion advance and several proposals have been worked using it as one of the fundamental Solar system tests, such as, e.g. the modification of Newtonan dynamics (Schmidt 2008), azimuthally symmetric theory of gravitation based on the study of Poisson equation (Nambuya 2010), Kaluza-Klein five-dimensional gravity (Lim & Wesson 1992), Yukawa-like modified gravity (Iorio 2008a), Horava-Lifshitz gravity (Harko et al. 2011), brane-world models and variants (Mak & Harko 2004; Iorio 2009a,b; Jalalzadeh et al. 2009; Cheung & Xu 2013; Chakraborty & Sengupta 2014) and also in the PPN framework and outside it giving rise to extraperihelion precessions and approaches in the weak field/slow motion limits (Iorio 2005, 2006, 2008b, 2011, 2012a,b,c, 2013a,b, 2014a,b; Adkins & McDonnell 2007; Avalos-Vargas & Ares de Parga 2012; D’Eliseo 2012; Arakida 2013; Feldman 2013; Kalinowsk 2013; Deng & Xie 2014; Liang & Xie 2014; Li et al. 2014; Ruggiero 2014; Wilhelm & Dwivedi 2014; Xie & Deng 2014).

On the mechanism we are going to show, we consider the effects in a single plane of orbit. This consideration is compatible with
the observed movement of the planets around the Sun limited to the
plane of orbits. Thus, we can consider the Sun in the centre of
the circular basis of the cylinder and a planet (or a small celestial
object) as a particle with mass \( m \) orbiting by its edge. This cylinder
can be described by Weyl’s line element (Weyl 1917)
\[
dr^2 = e^{2(\nu - \sigma)} dr^2 + r^2 e^{-2\sigma} d\theta^2 + e^{2(\nu - \sigma)} dz^2 - e^{2\sigma} d\tau^2 ,
\]
where \( \lambda = \lambda(r, z) \) and \( \sigma = \sigma(r, z) \). The exterior gravitational field
in the cylinder outskirts is given by Einstein’s vacuum equations
\[
- \lambda_{,r} + r \sigma_{,r} - r \sigma_{,z} = 0 ,
\]
\[
\sigma_{,r} + r \sigma_{,rr} + r \sigma_{,zz} = 0 ,
\]
\[
2r \sigma_{,r} = \lambda_{,z} ,
\]
where the terms \((r), (z)\) and \((rr), (zz)\) denote, respectively, the
first and the second derivatives with respect to the variables \( r \) and \( z \).
It is worth noting that the original paper of Weyl showed that the
cylinder solution is diffeomorphic to a Schwarzschild’s solution. In
this process, the metric does not lose its asymptotes as shown in
(Weyl 1917; Rosen 1949; Zipoy 1966) and is also asymptotically
flat (Weyl 1917; Rosen 1949; Zipoy 1966; Gauthier et al. 1972;
Stephani et al. 2003). This is a fine example of the equivalence
problem in GR: How do we know that two solutions of Einstein’s
equations, written in different coordinates, do not describe the same
gravitational field? The answer is given by the application of Car-
tane’s equivalence problem (Cartan 1927) to GR. It was shown that
the Riemann tensors and their covariant derivatives up to the seventh
order must be equal.

It is interesting to note that regardless of velocity arguments,
the diffeomorphism invariance of GR is also broken down by the
condition that the cylinder thickness \( h_0 \) is smaller than its radius
\( R_0 \), i.e. \( h_0 < R_0 \) being reduced to its circular basis which can be
smoothly deformed by the expansion of the metric parameters.
In order to analyse effects of the lack of the diffeomorphism invariance,
we start with solving the non-linear system given by equations (3),
(4) and (5). Differently as proposed in (González et al. 2008) and
(Vogt & Letelier 2008) for a mass distribution with Weyl’s exact
solutions of Einstein equations, we study approximate solutions of
this metric by expanding its coefficient functions (or potentials).
To this end, we expand the coefficients \( \lambda(r, z) \) and \( \sigma(r, z) \) into a MacLaurin’s series such that
\[
\sigma(r, z) = \sigma(r, 0) + z \frac{\partial \sigma(r, z)}{\partial z} \bigg|_{z=0} + z^2 \frac{\partial^2 \sigma(r, z)}{\partial z^2} \bigg|_{z=0} + \cdots ,
\]
\[
\lambda(r, z) = \lambda(r, 0) + z \frac{\partial \lambda(r, z)}{\partial z} \bigg|_{z=0} + z^2 \frac{\partial^2 \lambda(r, z)}{\partial z^2} \bigg|_{z=0} + \cdots ,
\]
and considering the approximation up to the second order (to keep
aspects of the non-linearity of the system), we define
\[
\sigma(r, z) = A(r) + a(z)z + c(z)z^2 ,
\]
where we denote \( A(r) = \sigma(r, 0) \), \( a(r) = \frac{\partial \sigma(r, z)}{\partial z} \bigg|_{z=0} \) and \( c(r) = \frac{\partial^2 \sigma(r, z)}{\partial z^2} \bigg|_{z=0} \).
In addition, we use the same procedure as in equation (8) to the
coefficient \( \lambda(r, z) \) and define
\[
\lambda(r, z) = B(r) + b(r)z + d(r)z^2 ,
\]
where we denote \( B(r) = \lambda(r, 0) \), \( b(r) = \frac{\partial \lambda(r, z)}{\partial z} \bigg|_{z=0} \) and \( d(r) = \frac{\partial^2 \lambda(r, z)}{\partial z^2} \bigg|_{z=0} \).

The field equation (5) can be written as
\[
\sigma_{,r} = y \Rightarrow \sigma_{,r} = y' \Rightarrow y + r y' + 2c(r) = 0 ,
\]
which is linear and can be solved by a factor integration. Thus, one can find
\[
\sigma(r, z) = - \int \frac{1}{r'} \left( \int 2r c(r') dr' \right) dr' + A_1(z) \ln(r) + A_2(z) .
\]
At this point, in order to make integrable equation (11), let us consider an \( n \)th power-law solution for \( c(r') \), such as,
\[
c(r') = \frac{c_0}{r'^n} , \quad c_0 = \text{const} , \quad n > 0 ,
\]
which substituting into equation (11) and after some calculations, we get
\[
\sigma(r, z) = k(z) \ln(r) + \frac{-2c_0}{(2 - n)r^n} r^{2-n} + A_2(z) , \quad n \neq 2 .
\]
On the other hand, taking the derivative of equation (13) with respect to \( z \), we obtain
\[
\sigma_{,z} = k(z) \ln(r) + A_2(z) ,
\]
which from equation (8), for the same derivative we get the result
\[
\sigma_{,z} = a(r) + 2c(r) z = a(r) + \frac{2c_0}{r^n} z .
\]
From equations (14) and (15) we see that
\[
A_2(z) = a(r) + \frac{2c_0}{r^n} z - k(z) \ln(r) .
\]
Noting that \( A_2(z) \) is a function of \( z \) only, then
\[
A_2(z) \Rightarrow a(r) = a_0 ,
\]
\[
k(z) = \frac{k_0}{2} , \quad n = 0 ,
\]
where \( a_0 \) and \( k_0 \) are constants. Therefore,
\[
A_2(z) = a_0 + 2c_0z \Rightarrow A_2(z) = a_0 + c_0 z^2 + c_1 .
\]

As the field equations (3) and (4) involve only derivatives of \( \sigma(r, z) \), we may set \( c_1 = 0 \). For the second-order approximation, the final
form of the coefficient \( \sigma(r, z) \) is then
\[
\sigma(r, z) = \frac{k_0}{2} \ln(r) - \frac{c_0 z^2}{2} + a_0 z + c_0 z^2 .
\]
As in the same fashion for the previous development to equation
(8) resulting in equation (19), we apply to equation (9). First, we must define a \( m \)-th power-law solution for the function \( d(r) \) as
\[
d(r) = \frac{d_0}{r^m} , \quad d_0 = \text{const} , \quad m > 0 .
\]
Considering equations (3) and (19), and after integrating by parts in
the variable \( r \), we obtain
\[
\lambda(r, z) = \frac{k_0}{4} \ln(r) - k_0 c_0 \frac{r^2}{2} + c_0 z^4 \frac{r^4}{4} - (a_0 + 2c_0 z)^2 \frac{r^2}{2} + B_1(z) .
\]
Moreover, to obtain a closed form for the coefficient \( \lambda(r, z) \), we need to find the function \( B_1(z) \). To do so, we take the derivative of
equation (21) with respect to \( z \) and obtain
\[
\lambda_z = \left(-2a_0 c_0 - 4c_0^2 z \right) r^2 + B_1(z) ,
\]
and doing the same for equation (9), we get
\[
\lambda_z = b(r) + 2d(r) z = b(r) + 2 \frac{d_0}{r^m} z .
\]
Thus, comparing equations (22) and (23), one can obtain
\[
B_t(z, z) = b(r) + 2 \frac{d_0}{r^2} + 2a_0c_0r^2 + 4c_0^3z^2.
\]  
(24)

Integrating equation (24) with respect to \( \lambda \), gives
\[
B_t(z) = \left( \frac{d_0}{r^2} + 2c_0^2r^2 \right) z^2 + (b(r) + 2a_0c_0r^2) z + b_1.
\]  
(25)

As \( B_t(z) \) is a function of \( z \) only, we need that
\[
C_1 = \frac{d_0}{r^2} + 2c_0^2r^2 = d(r) + 2c_0^2r^2,
\]
(26)
\[
C_2 = b(r) + 2a_0c_0r^2,
\]
(27)

where \( C_1 \) and \( C_2 \), are constants. Thus, \( \lambda(r, z) \) is expressed by
\[
\lambda(r, z) = \frac{\sigma}{4} \ln(r) - k_0 c_0^2 \frac{r^2}{2} - (a_0 + 2c_0z)^2 \frac{r^2}{2} + \frac{1}{4} c_0^2 r^4 + C_1 z + C_2 z^2,
\]
(28)

where it has been assumed that the constant \( b_1 = 0 \). Substituting equations (27) and (19) into field equations (3)–(5), we find that
\[
C_1 = k_0 c_0 \ ,
\]
(29)

and
\[
C_2 = k_0 c_0 \phi \ .
\]
(30)

Finally, the second-order approximation for the coefficient \( \lambda(r, z) \) can be written as
\[
\lambda(r, z) = \frac{\sigma}{8} \ln(r) - k_0 c_0^2 \frac{r^2}{2} + \frac{1}{4} c_0^2 r^4 - (a_0 + 2c_0z)^2 \frac{r^2}{2} + k_0 u_0 z + k_0 c_0 z^2.
\]
(31)

To complete the calculations, we need to know the relations between the parameters of the coefficients of expansion \( a(r), b(r), c(r) \) and \( d(r) \). It can be easily checked by using equations (26), (28) and (29), and we have the results
\[
b(r) = k_0 a_0 - 2a_0 c_0 \phi r^2,
\]
(32)

and also
\[
d(r) = k_0 c_0 - 2c_0^2 r^2,
\]
(33)

which we conclude that
\[
b(r) = \frac{a_0}{c_0} d(r) .
\]
(34)

It is worth nothing that for superior orders, the terms in the coefficients \( \sigma(r, z) \) and \( \lambda(r, z) \) turn to be redundant and can be reduced to the second order.

In order to test this assumption, we study the gravitational field produced by geodesic equation alone which can induce to an intermediate gravitational regime between Einstein’s strong field and the Newtonian field, known as nearly Newtonian regime (Infeld & Plebanski 1960; Wheeler et al. 1973).

Since the relativistic effects generated by solar gravity is about \( 10^{-8} \) weaker that Newtonian ones (Yamada & Asada 2012), we assume that the second order of expansion of the coefficients must represent a small perturbation of the first order in such a way that \( c_0 \ll 1 \). We also use the Weyl conformastatic solution (in the sense that the potential \( \lambda(r, z) = 0 \)) to guarantee that the resulting gravitational field produced can be enough strong to give a proper correction for the perihelion advance.

4 ORBIT EQUATIONS AND THE PERIHELION ADVANCE

Besides of calculating the Einstein’s equations for Weyl metric, we must obtain an orbit equation in order to deal with the perihelion advance. To this end, we calculate the geodesics from Weyl metric and find the following components
\[
\frac{d^2 r}{d\tau^2} + (\sigma_{,r} - \lambda_{,\tau}) \left( \frac{d z}{d\tau} \right)^2 + (2\lambda_{,\tau} - 2\sigma_{,r}) \frac{d r}{d\tau} \frac{d z}{d\tau} + e^{-2\lambda} (r^2 \sigma_{,r} - r) \left( \frac{d \theta}{d\tau} \right)^2 + e^{4\sigma - 2\lambda} \sigma_{,r} \left( \frac{d r}{d\tau} \right)^2 - (\sigma_{,r} - \lambda_{,\tau}) \left( \frac{d r}{d\tau} \right)^2 = 0 ,
\]
(35)

and also the following set of equations
\[
2 r \sigma_{,\tau} \frac{d \theta}{d\tau} \frac{d z}{d\tau} - r \frac{d^2 \theta}{d\tau^2} + 2 \sigma_{,\tau} \frac{d r}{d\tau} \frac{d \theta}{d\tau} \frac{d z}{d\tau} - 2 \frac{d r}{d\tau} \frac{d \theta}{d\tau} \frac{d z}{d\tau} = 0 ,
\]
(36)

\[
2 \sigma_{,\tau} \frac{d r}{d\tau} \frac{d z}{d\tau} + \frac{dr}{d\tau} \frac{d \theta}{d\tau} = 2 \sigma_{,r} \frac{d r}{d\tau} \frac{d \theta}{d\tau} \frac{d z}{d\tau} - \frac{d r}{d\tau} \frac{d \theta}{d\tau} \frac{d z}{d\tau} = 0 .
\]
(37)

As a result, one can obtain the orbit equation
\[
\left( \frac{d r}{d\theta} \right)^2 = e^{-2\lambda} \left[ r^4 e^{-2\lambda} (\alpha_0 + \beta_0 e^{-2\sigma} - r^2) \right] ,
\]
(38)

where \( \alpha_0 = \frac{1}{k_0^2} \) and \( \beta_0 = \frac{k_0}{r^2} \) are integration constants.

Due to the structure of Weyl’s metric, we only need the coefficient \( \sigma \) to produce a nearly Newtonian gravitational regime from the component \( g_{\tau \tau} \). Thus, we can consider only studying a conformastatic solution (Gutiérrez-Piñeres et al. 2013) for equation (38) and obtain
\[
\left( \frac{d r}{d\theta} \right)^2 = \left[ r^4 e^{-2\lambda} (\alpha_0 + \beta_0 e^{-2\sigma} - r^2) \right] ,
\]
(39)

which can be transformed into the following conformastatic orbit equation
\[
\left( \frac{d u}{d\theta} \right)^2 + u^2 = e^{-2\lambda} \left( \alpha_0 + \beta_0 e^{-2\sigma} \right) ,
\]
(40)

where the new variable \( u \) stands for \( u = \frac{1}{r} \) and \( \sigma = \sigma(u) \).

4.1 First-order approximation

In order to make clear the physical differences between the first and second order of the coefficient \( \sigma \), we present orbit equations produced by each order. Using equation (19), we stress that in the first approximation we have \( c_0 = 0 \) and the coefficient \( \sigma(r, z) \) is simply reduced to
\[
\sigma(r, z) |_{c_0 = 0} = \frac{k_0}{2} \ln r .
\]
(41)

In order to obtain the resulting perihelion advance, first of all, in the same way as in Harko et al. (2011), we evaluate the first
approximation of Weyl’s conformastatic metric and find
\[
\left(\frac{du}{d\tau}\right)^2 + u^2 = G(u),
\]
(41)
where \(G(u)\) is given by \((\alpha_0 u^6 + \beta_0 u^{26})\) with the constants \(\alpha_0\) and \(\beta_0\). Deriving equation (41), we can write
\[
u + u = F(u),
\]
(42)
where \(F(u)\) is defined
\[
F(u) = \frac{1}{2} \frac{dG(u)}{du} = \frac{1}{2} \left[ \alpha_0 k_0 u^{6-1} + 2 \beta_0 k_0 u^{26-1} \right].
\]
The perihelion advance is given by
\[
\delta \phi = \pi \frac{dF(u)}{du} \bigg|_{u=0}.
\]
(43)
A nearly circular orbit is given by the roots of the equation \(F(u_0) = u_0\). In order to get the correct decaying law for the gravitational field (in Solar system no stronger than \(\sim 1/r^3\)), we set \(k_0 = 2\) and find
\[
u_0 = \frac{1 - \alpha_0}{2 \beta_0}.
\]
(43)
Thus, after using the proper equivalence with the standard gravity and neglecting the redundant terms, one can find the following expression
\[
\delta \phi \approx \frac{4 \pi G M}{c^2 \gamma (1 - \epsilon^2)},
\]
(44)
where \(\gamma\) is the semimajor axis and \(\epsilon\) is the eccentricity. Clearly, the expression in equation (44) generates an angular deviation smaller than observed. Interestingly, equation (44) resembles the gravitational lens equation.

4.2 Second-order approximation

For the second order, we have that the constant \(c_0\) plays the role of the correction term. Essentially, we use the same procedure as describe in the first-order approximation. Using equation (19) calculated in \(z = 0\), we obtain the general expression for \(F(u)\) and find
\[
F(u) = \frac{1}{2} \left[ \alpha_0 k_0 u^{6-1} + 2 \beta_0 k_0 u^{26-1} \right] + \frac{1}{2} \alpha_0 c_0 (k_0 - 2) u^{6-3} + 4(k_0 - 1) \beta_0 c_0 u^{26-3}.
\]
(45)

Interestingly, we find that the values for the \(k_0\) parameter can be very constrained based on the fact that it must provide a correct decaying law in the Solar system scale, which is ruled essentially by Newtonian potentials (\(\sim \frac{1}{r}\)) and smooth deviations in decaying (\(\sim \frac{1}{r^2}, \sim \frac{1}{r^3}\)) and growing not large as \(\sim r^2\). The increasing of gravitational potentials of order of \(\sim r^3\), or quadratically on, should be quite worrisome in the Solar system scale.

For \(k_0 = 0\), we can reproduce the same orbit equation as shown in Roberts (1987) in the study of the perihelion of Pluto. From \(k_0 \leq -1\) in orbit equation, we find a very fast growing terms that reproduces an incorrect secular shift for the perihelion. For the case when \(k_0 > 2\), it produces orbit equations with high-decaying terms of order of \(O(u^7)\). On the other hand, for the case \(k_0 = 2\) we find that it reproduces the correct decaying law and one can obtain the following terms
\[
F(u) = \frac{1}{2} \left[ (2 \alpha_0 + 4 \beta_0 c_0) u + 4 \beta_0 u^3 \right].
\]
(46)

For \(F(u_0) = u_0\), one can find the roots
\[
u_0 = \frac{1 - (\alpha_0 + 2 \beta_0 c_0)}{4 \beta_0},
\]
(47)
and using (43), (46) and (47), one can obtain the shifted angle \(\delta \phi\) as
\[
\delta \phi = \frac{(3 - \alpha_0) \pi}{2} + 4 \beta_0 c_0 \pi.
\]
(48)
In order to relate to the standard gravity, we set \((3 - \alpha_0) = \frac{c^3 G M}{\gamma^3 (1 - \epsilon^2)}\) and find the correction for the perihelion advance
\[
\delta \phi = \delta \phi_{sch} \pm 4 \pi \beta_0 c_0.
\]
(49)
where we denote \(\delta \phi_{sch} = \frac{c^3 G M}{\gamma^3 (1 - \epsilon^2)}\), with \(\gamma\) being the semimajor axis and \(\epsilon\) denotes the eccentricity of the orbits. In addition, based on the fact that \(c_0 \ll 1\), it can be constrained setting \(c_0 = \pm \frac{v}{\sqrt{\gamma}}\) where \(v\) is the keplerian mean motion given by \(\sqrt{\frac{G M}{\gamma^3}}\). It is worth noting that two solutions are possible with the sign \(\pm\) related to the disc expansion (the variation of the orbit with reference to the plane). If we conveniently set the parameter \(\beta_0 \ll 1\), the equation (49) reproduces the same classical GR results. On the other hand, we are interested in the case that the parameter \(\beta_0\) can induce a correction to the standard \(\delta \phi_{sch}\) generated by PPN parameters, so the equation (49) in principle can be applied to model non-trivial systems with non-standard perihelion deviations with only one parameter. To this end, the values of \(\beta_0\) must carry some information of each studied problem and be constrained by the assumption
\[
\beta_0 = \epsilon^4 \sqrt{1 - \epsilon^2},
\]
(50)
with \(\beta_0\) restricted to the interval \([0, 1]\).

Thus, one can rewrite equation (49) as
\[
\delta \phi = \delta \phi_{sch} \pm \epsilon^4 \sqrt{\frac{G M (1 - \epsilon^2)}{\gamma^3}}.
\]
(51)
If the orbit is nearly circular, the previous equation is reduced to the standard Schwarzschild one.

Next, we apply the model to the inner planets, once we have a more reliable data in this range. The relevant proper elements (semi-major, mean eccentricity and period) for computing the perihelion precession of inner planets and 1566 Icarus asteroid were extracted from (Wilhelm & Dwivedi 2014). The values of constant considered in the calculations are the following: the Newtonian constant of gravitation \(G = 6.673 \times 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2}\) (Wilhelm & Dwivedi 2014), 1 yr = 365.256 d, the speed of light \(c = 299,792,458 \text{ m s}^{-1}\) (Bureau International des Poids et Mesures 2006; Wilhelm & Dwivedi 2014) and the mass of sun \(M_\odot = 1.9885 \times 10^{30} \text{ kg}\).

In Table 1, we present the relevant quantities to calculation of the angular deviation for selected asteroids and comets. Due to the fact of its importance of study, we have selected NEO’s of Apollo group which consists asteroid orbits that cross the Earth’s orbit close to that of 1862 Apollo (with semimajor axis of the order of \(a > 1.0\) au and perihelion distance \(q < 1.017\) au) being also considered potentially hazardous asteroids. For the second-largest NEOs 433 Eros is located in Amor group (it crosses the Earth’s orbit close to that of 1221 Amor with \(1.017 < q < 1.3\) au). For the comets, we have selected a NEO 2p/Encke comet and two Jupiter-family comets 26P/Grigg-Skjellerup and 22P/Kopff which properties are being currently investigated (Levison et al. 2006; Busemann et al. 2009; Moreno et al. 2012).

As a result, we obtain Table 2 for the values of \(\beta_0\) (using equation 50) and a comparison of the different angular deviation of both observational and PPN results.
As Tables 2 and 3 indicate, the calculated angular deviation $\delta \phi$ from conformastatic solution is consistent with observations. For inner planets, it is clear that the planets with low eccentricity generates a low value for $\beta_0$ and essentially reproduce the same results as PPN approximation does, except for Mercury which presents a larger eccentricity among the planets and the correction for perihelion advance appears in good agreement with observations and other works in the literature Anderson et al. (1992), Shapiro, Counselman & King (1976), Nambuya (2010), Harko et al. (2011). Moreover, we find a very small $\beta_0$, i.e. $\beta_0 \ll 1$ for the cases of Venus, Earth and Mars, and we do not observe any difference in $\delta \phi^+$ and $\delta \phi^-$ solutions. In this scale, the values of $\beta_0$ seem to suggest that the larger is $\beta_0$, the larger is the perihelion advance. This similar pattern also appears in the asteroid/comet scale. We point out that $\beta_0$ essentially depends on the eccentricity of the orbit.

On the other hand, in the asteroids and comets scale, we find a tighter constraint for $\beta_0$ in the range $0 < \beta_0 < 0.3$ providing solutions compatible with the observed perihelion for the selected asteroids and in the case of comets the solutions are also compatible with the numerical results using Painlevé coordinates (Sitarski 1983). It is interesting to point out that the case of 1566 Icarus and 3200 Phaethon have a curious fact, they have similar eccentricities and perihelion advances. We conjecture if orbiting bodies with roughly same dynamical and cinematical characteristics with similar values of $\beta_0$ could have roughly the same perihelion advance. Nevertheless, doing such affirmation is still premature and must be investigated further.

It is worth noting that the $\delta \phi_{\text{sch}}$ deviation from the standard Einstein percession can be improved mainly on the small celestial bodies, where the relativistic effects can also be observed and their eccentricity are larger. Conversely, we show that a simply analysis of the non-linearities and qualitative effects of the gravitational field, we obtain non-standard results without proposing a different theory of gravitation just taking into account the shape, the topology and the symmetry aspects of the gravitational field.

### 5 FINAL REMARKS

In the nearly Newtonian potential, which is originated from the impositions made on geodesic equations, the $\beta_{\text{sch}}$ metric component carries the non-linearity of Einstein’s equations. This approximation is essentially an application of GR to slow motion. Note that the equivalence principle remains valid but the same does not

| Object        | Semimajor   | Mean eccentricity | Period (yr) |
|---------------|-------------|------------------|-------------|
| 1566 Icarus   | 1.077 92    | 0.826 95         | 1.119 95    |
| 1862 Apollo   | 1.470 11    | 0.559 93         | 1.78        |
| 2101 Adonis   | 1.874 47    | 0.763 81         | 2.57        |
| 433 Eros      | 1.457 97    | 0.222 63         | 1.76        |
| 3200 Phaethon | 1.271 12    | 0.889 85         | 1.43        |
| 26P/Grigg-Skjellerup | 3.017 21 | 0.640 16 | 5.24         |
| 22p/Kopff     | 3.439 68    | 0.547 34         | 6.41        |
| 2p/Encke      | 2.214 73    | 0.848 23         | 3.3         |

*Observational data extracted from (Wilhelm & Dwivedi 2014).

## Table 2

Comparison between the values for secular percession of inner planets in units of arcsec century$^{-1}$ of the standard (Einstein) perihelion precession $\delta \phi_{\text{obs}}$ (Wilhelm & Dwivedi 2014) and the conformastatic solution $\delta \phi_{\text{model}}$. The $\delta \phi_{\text{obs}}$ stands for the secular observed perihelion precession in units of arcsec century$^{-1}$ adapted from (Nambuya 2010) by adding a supplementary precession corrections from EPM2011 (Pitjeva & Pitjev 2013; Pitjev & Pitjeva 2013).

| Object        | $\delta \phi_{\text{obs}}$ (arcsec century$^{-1}$) | $\delta \phi_{\text{ach}}$(arcsec century$^{-1}$) | $\delta \phi_{\text{model}}$(arcsec century$^{-1}$) | $\beta_0$ |
|---------------|-----------------------------------------------|-----------------------------------------------|-----------------------------------------------|----------|
| Mercury       | 43.098 ± 0.503                                | 42.978 17                                     | 43.1047                                      | 0.000    |
| Venus         | 8.026 ± 0.160                                 | 8.624 09                                      | 8.642 62                                      | 0.100    |
| Earth         | 5.000 19 ± 1.000 38                            | 3.838 73                                      | 3.838 73                                      | 0.000    |
| Mars          | 1.362 38 ± 0.000 537                           | 1.350 86                                      | 1.350 86                                      | 0.000    |

## Table 3

Comparison between the values for secular percession in units of arcsec century$^{-1}$ of the standard (Einstein) perihelion precession and the conformastatic solution $\delta \phi_{\text{model}}$ for selected NEO asteroids and comets. The $\delta \phi_{\text{obs}}$ stands for the secular observed perihelion precession in units of arcsec century$^{-1}$. For the comets, $\delta \phi_{\text{obs}}$ is the numerical results using Painlevé coordinates to one-body Schwarzschild problem (Sitarski 1983).

| Object        | $\delta \phi_{\text{obs}}$ (arcsec century$^{-1}$) | $\delta \phi_{\text{ach}}$(arcsec century$^{-1}$) | $\delta \phi_{\text{model}}$(arcsec century$^{-1}$) | $\beta_0$ |
|---------------|-----------------------------------------------|-----------------------------------------------|-----------------------------------------------|----------|
| 1566 Icarus   | 10.007                                        | 10.061 28$^a$                                 | 10.2920                                      | 0.262 964|
| 1862 Apollo   | 2.1239                                        | 2.133                                         | 2.238 63                                      | 0.081 442|
| 2101 Adonis   | 1.9079                                        | 1.912                                         | 2.051 92                                      | 0.219 684|
| 433 Eros      | 1.60                                          | 1.573 17                                      | 1.576 34                                      | 0.002 395|
| 3200 Phaethon | 10.1                                          | 10.1201                                       | 10.6921                                      | 0.286 071|
| 26P/Grigg-Skjellerup | 0.54 | 0.4106                          | 0.430 490                                    | 0.391 913| 0.129 018|
| 22p/Kopff     | 0.288                                         | 0.2474                                        | 0.255 405                                    | 0.240 378| 0.075 112|
| 2p/Encke      | 1.9079                                        | 1.868                                         | 1.977 88                                      | 0.274 174|

$^a$Extracted from (Wilhelm & Dwivedi 2014).
occur with the generalized covariance which is broken when the condition $v \ll c$ is postulated. This means that making the choice of an adequate geometry becomes a very important matter, since the diffeomorphism transformations are not valid anymore. In this respect, it should be noted that the Weyl cylindrical solution can be transformed to the Schwarzschild’s solution by a diffeomorphism (Rosen 1949). However, we cannot apply such transformation here because the diffeomorphism invariance has been lost.

Indeed, the nearly Newtonian limit is quite different from the PPN approximation in which a choice of parameters is necessary to define the arbitrary potential’s coefficients. Besides losing general covariance as a consequence of the slow motion condition in the nearly Newtonian domain, only one component of the metric has a direct contribution to the motion.

We know from the study of dynamical systems that the non-linearities imprint qualitative effects on the orbits of their solutions which was shown by the conformastatic coordinates used here. The second order of the expansion of the coefficients $\sigma$ and $\lambda$ was enough to obtain the appropriated gravitational field, once the superior orders in the course of this study revealed that they can be reduced to the second order considering the disc expansion in mind. In the perihelion case, we obtained a non-standard expression for the perihelion precession that can be extended to the analysis of extrasolar systems. Rather, the method itself can be applied to any problem that the strength of the gravitational field is constrained by the topology of the problem.

In order to test the model, we use the observational data adapted from (Nambuya 2010) with supplementary precession corrections from EPM2011 (Pitjeva & Pitjev 2013; Pitjev & Pitjeva 2013) to refine our results for the perihelion advance of inner planets. For asteroids and comets also have shown a good agreement with the data base (Sitarski 1983; Wilhelm & Dwivedi 2014) used for comparisons. In all cases, we have found that the value of $\beta_0$ is related to the perihelion advance in directly proportional form.

The main advantage of this process resides in its simplicity. In addition, the topological nature of the problem is now an important character which is not taken into account in the PPN approximation. As a result, we have obtained a model of only one parameter which was shown by the conformastatic coordinates used here. These objects in general are very difficult to model and we have obtained a good agreement to observations improving the results of the PPN approximation. All the results we have presented are essentially in the realm of GR. In this context, additional assumptions or modifications of the standard gravity are not needed. As future perspectives, an extended analysis of the deviation of light, radar echo and gravitational lens in spheroidal metrics are currently in progress.

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