Relay-mapper aided multi-user lattice coding for the multiple-access relay channel

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Abstract—This paper considers the multi-antenna multiple access relay channel (MARC), in which multiple users transmit messages to a common destination with the assistance of a relay. In a variety of MARC settings, the dynamic decode and forward (DDF) protocol is very useful due to its outstanding rate performance. However, the lack of good structured codebooks so far hinders practical applications of DDF for MARC. In this work, a new structured MARC code, one-to-one relay-mapper aided multiuser lattice coding (O-MLC), is proposed. It is shown that, in order to approach the rate performance obtained previously by using an unstructured codebook with maximum-likelihood decoding, it is crucial to use a new K-stage coset decoder instead of the one-stage decoder proposed in previous works. However, if O-MLC is decoded with the one-stage decoder only, it can still achieve the optimal DDF diversity-multiplexing gain tradeoff when the signal-to-noise ratio is high. Simulation results show that the proposed coding scheme outperforms existing schemes in terms of outage probabilities.

I. INTRODUCTION

In recent years, cooperative communication has drawn a significant amount of interest as a means of providing spatial diversity. Cooperative communication techniques for single-source networks have been extensively studied in terms of rate, outage probability or diversity-multiplexing tradeoff (DMT) perspectives [1] [2] [3]. However, practical communication networks usually involve more than one source (user). In this paper, we consider an important multi-user cooperative communication channel, that is, the multi-antenna multiple-access relay channel (MARC). The MARC is a multiple-access channel (MAC) with an additional shared half-duplex relay [4]. It has been shown that the MARC provides a much larger achievable rate region [4] and diversity gain per user [5], compared to those of the MAC. Also, since a single relay is shared by multiple users in the MARC, the extra cost of adding such a relay is acceptable.

The achievable rate region of the MARC has been characterized in [4] and [6]. The decode and forward protocol, which is a special case of the dynamic decode and forward (DDF) protocol [7], was shown to achieve the capacity region of the MARC when the source-relay link is good enough [6], thus having a larger achievable rate region than those of the multiple-access amplify and forward (MAF) [5] and compress and forward (CF) protocols [8]. However, the capacity region of the general MARC remains unknown. The DMT for the MARC with single antenna nodes was studied in [5] [7] and [8]. Although the MAF and CF are both DMT optimal in the high multiplexing gain regime, compared with the DDF strategy, they both achieve lower diversity gains in the low to medium multiplexing gain regimes [5] [8]. We focus on the DDF in this paper due to its good performance in a variety of operational settings.

Previous results in [4]–[8] are based on unstructured random codebooks and maximum likelihood (ML) decoders, and are very difficult to implement in practice. In this paper, we propose structured multiuser lattice coding aided by a relay mapper for the MARC under the DDF protocol, in which each node in the MARC has multiple antennas. To simplify the joint codebook design problem for the multiple users and the relay, we introduce a relay mapper which selects the codeword to be transmitted at the relay to aid the users’ transmissions. The relay mapper is a key new ingredient for our coding design, which can also help implement the unstructured codebooks in [4], [6] and [7] in practice, and does not appear in [4]–[8]. We will see that the one-stage coset decoding proposed in [9] fails to achieve the rate performance of the unstructured codebook with ML decoding demonstrated in [6]. Instead, we propose a new K-stage coset decoder that achieves the rate performance in [6] by successive cancellation on the multiuser decoding tree. With the K-stage coset decoder, the structured codes, referred to as one-to-one relay mapper aided multiuser lattice coding (O-MLC), can achieve the rate performance obtained by the unstructured codebook in [6]. If only one-stage coset decoding is used, we also show that O-MLC is DMT optimal for the DDF. Moreover, a naive application of the theoretical error analysis in [9] suffers from significant losses in prediction of the achievable rates of the proposed codes, due to the introduction of the relay mapper. We overcome this problem by introducing a new technique for bounding the error probability over the random relay-mapper codebook ensemble.
Some practical MARC code designs were proposed in [10] and [11], but these studies lack theoretical performance analysis. Through simulations, we show that our proposed lattice coding scheme outperforms the schemes in [5] [8] [10] and [11] in terms of outage probabilities.

II. SYSTEM MODEL

We consider the K-user multiple-antenna MARC as shown in Fig. 1, in which a relay node is assigned to assist the multiple-access users in transmitting data to a common destination. Each user and the relay is equipped with \( M_u \) and \( M_r \) antennas, respectively, and the destination has \( N \) antennas. In DDF for the MARC, each codeword spans \( L \) slots each consisting of \( T \) vector symbols, and the block of \( LT \) vector symbols is split into two phases due to the half-duplex constraint at the relay node. In Phase 1, the relay receives the signals from the users, then it decodes the users’ messages at decision time \( \ell_1 T \). Following [7], \( \ell_1 T \) is chosen to be the earliest time index such that after \( \ell_1 T \) symbols, the relay can decode the users’ messages without error. If there is no such \( \ell_1 \in \{1, \ldots, L-1\} \), the relay remains silent. Denote the \( M_r \times M_u \) and \( N \times M_r \) channel matrices from user \( i \) to the relay and the destination by \( \mathbf{H}_{ri} \) and \( \mathbf{H}_{di} \), respectively, and assume they are perfectly known at the corresponding receivers. For Phase 1, the received signal at the relay is

\[
y_{rl} = \sqrt{\frac{\rho_r}{M_r}} \sum_{i=1}^{K} \mathbf{H}_{ri} \mathbf{x}_{r,i} + \mathbf{v}_l, \quad l = 1, 2, \ldots, \ell_1 T
\]

where \( \rho_r \) is the received SNR at the relay, \( \mathbf{x}_{r,i} \) is the vector signal transmitted by user \( i \) at time index \( l \), and the noise at the relay \( \mathbf{v}_l \sim \mathcal{CN}(0, I_M) \) is a Gaussian vector. Similar to (1), the received signal at the destination in Phase 1 is

\[
y_{dl,j} = \sqrt{\frac{\rho_d}{M_d}} \sum_{i=1}^{K} \mathbf{H}_{di} \mathbf{x}_{d,i} + \mathbf{v}_l, \quad l = 1, 2, \ldots, \ell_1 T
\]

where \( \rho_d \) is the received SNR and \( \mathbf{v}_l \sim \mathcal{CN}(0, I_N) \) is the noise vector at the destination. In Phase 2, based on the decoded messages obtained at the decision time \( \ell_1 T \), the relay transmits the corresponding coded vector symbols to the destination. The signal received by the destination is then

\[
y_{dl,j} = \sqrt{\frac{\rho_d}{M_d}} \sum_{i=1}^{K} \mathbf{H}_{dl,i} \mathbf{x}_{d,i} + \sqrt{\frac{\rho_d}{M_r}} \mathbf{H}_{dl,K+1} \mathbf{x}_{d,K+1} + \mathbf{v}_l, \tag{3}
\]

for \( l = \ell_1 T + 1, \ell_1 T + 2, \ldots, LT \), where \( \mathbf{x}_{d,K+1} \) denotes the signal transmitted by the relay and \( \mathbf{H}_{dl,K+1} \) is the channel matrix from the relay to the destination. As for the (normalized) MARC input power constraint, it is imposed on each user and the relay as

\[
E \left[ \frac{1}{LT} \sum_{l=1}^{LT} \mathbf{x}_{r,i}^H \mathbf{x}_{r,i} \right] \leq M_r, \quad E \left[ \frac{1}{LT} \sum_{l=1}^{LT} \mathbf{x}_{d,i}^H \mathbf{x}_{d,i} \right] \leq M_d, \tag{4}
\]

for \( i = 1, \ldots, K \), where the expectation \( E[\cdot] \) is taken over all codewords in the codebook.

To simplify the presentation for the proposed lattice coding scheme, it is useful to transform our received signal model (1), (2) and (3) into the equivalent real channel model form as in (5) and (6), for the relay and the destination, respectively,

\[
y_{relay} = \mathbf{H}_{rel} \mathbf{y}_{relay} + \mathbf{n}_{relay}, \tag{5}
\]

\[
y_{dst} = \mathbf{H}_{dst} \mathbf{y}_{dst} + \mathbf{n}_{dst}. \tag{6}
\]

\[\text{Notation: } |A| \text{ denotes the cardinality of a set } A. \text{ For a matrix } \mathbf{M}, \mathbf{M}^* \text{ is the conjugate transpose and } |\mathbf{M}| \text{ is the determinant. We use } \log(\cdot) \text{ for the logarithm with base 2, and } \times \text{ for the direct product. An } n\text{-dimensional real lattice } \Lambda \text{ is a discrete additive subgroup of } \mathbb{R}^n. \text{ The lattice quantization function is defined as } \mathcal{Q}_\Lambda(y) = \text{argmin}_{\mathbf{y} \in \Lambda} |y - \mathbf{y}| \text{ for } \mathbf{y} \in \mathbb{R}^n, \text{ and the modulo-lattice operation } \mathbf{y} = \mathbf{y} \mod \Lambda \equiv \mathbf{y} - \mathcal{Q}_\Lambda(y) [12]. \text{ Some other frequently used notation is summarized in Table I.}\]
The equivalent channel for the destination (6) is formed by concatenating the received signal (2) and (3), and the 2(KM + M + LT)×1 super signal vector \( \mathbf{x}_{\text{ds}} \) in (6) is \( \mathbf{x}_{\text{ds}} \triangleq [x_1^{K}, \ldots, x_{K+1}^{K}]^T \), where \( x_i = \left[ \left( R_{H_i} \right)^T, \ldots, \left( R_{H_{i+1}} \right)^T \right]^T \) for \( i = 1, \ldots, K + 1 \), with \( x_i^j = \left[ \text{Re}(x_i^j), \text{Im}(x_i^j) \right] \); while the 2NLT × 1 super received signal and noise at the destination \( \mathbf{y}_{\text{ds}} \) and \( \mathbf{n}_{\text{ds}} \) in (6) are similarly defined. The 2NLT × 2(KM + M + LT) super-channel matrix \( \mathbf{H}_{\text{ds}} \) in (6) is \( \mathbf{H}_{\text{ds}} \triangleq \left[ H_{1}^{\text{f}}, \ldots, H_{2NLT+1}^{\text{f}} \right] \), where the 2NLT × 2MNLT equivalent channel matrix \( \mathbf{H}_{c}^j \) for user \( i \) comes from (2) as

\[
\mathbf{H}_{c}^j = \sqrt{\frac{\rho_d}{M_u}} \mathbf{I}_T \otimes \left[ \begin{array}{cc}
\text{Re}\{\mathbf{H}_{d_i}\} & -\text{Im}\{\mathbf{H}_{d_i}\} \\
\text{Im}\{\mathbf{H}_{d_i}\} & \text{Re}\{\mathbf{H}_{d_i}\}
\end{array} \right],
\]

(7)

where \( \otimes \) denotes the Kronecker product and \( i = 1, \ldots, K \) while the \( \mathbf{H}_{K+1}^j \) for the relay comes from (3) as

\[
\mathbf{H}_{K+1}^j = \text{diag} \left( \mathbf{1}_T \otimes 0_{2N\times 2M}, \ldots, \mathbf{1}_T \otimes 0_{2N\times 2M} \right).
\]

if \( 1 \leq t_1 \leq L - 1 \), where the first \( 2N(t_1) \times 2M_{t_1} T \) is a zero matrix because the relay is listening in Phase 1. The equivalent channel for the relay (5) can be similarly obtained

\[
2NLT \times 2(KM + M + LT) \text{ super-channel matrix } \mathbf{H}_{\text{rel}} \text{ in (6) is } \mathbf{H}_{\text{rel}} \triangleq \left[ \mathbf{H}_1^j \ldots, \mathbf{H}_{2NLT+1}^j \right],
\]

for the relay's codebooks; and

\[
\text{the fixed channel and the slow fading channel. In the fixed }
\]

channel, \( \mathbf{H}_{\text{rel}} \) and \( \mathbf{H}_{\text{fas}} \) are random but remain constant over time. We consider two kinds of channel settings, the fixed channel and the slow fading channel. In the fixed channel setting, the channels are deterministic and we use the achievable rate as a performance metric. For the slow fading channel, \( \mathbf{H}_{\text{fas}} \) is random but remain constant over the whole code block. We assume Rayleigh fading with entries of the channel matrices being independent and identically distributed (i.i.d.) \( CN(0,1) \), and use the DMT or the outage probabilities as performance metrics.

III. PROPOSED RELAY-MAPPER AIDED MULTIUSER LATTICE CODING SCHEMES

In this section, we specify the proposed O-MLC for the MARC, which consists of three building blocks: 1) the relay mapper which decides which codeword to be transmitted at the relay; 2) Loeliger-type nested lattices for the users’ and the relay’s codebooks; and 3) a K-stage coset decoder, which generalizes the one-stage decoder of [9]. We first briefly introduce the adopted lattice codebooks. Tailored for them, the encoders and the relay one-to-one mapper \( \psi_{\text{con}} \) for O-MLC are shown in Section III-B. Then the K-stage decoders are introduced in Section III-C.

A. Loeliger-type Nested Lattice Codebooks

In our code construction, codebooks of the i-th user \((1 \leq i \leq K)\) and the relay \((i = K + 1)\) are generated from nested lattices. To be specific, we introduce the following definitions.

Definition 1 (Self-similar nested lattice code): For user \( i \), let \( \Lambda_c \) be a 2MLT-dimensional coding lattice and \( \Lambda_{\text{S}} \subset \Lambda_c \) be the shaping lattice. The nested lattice codebook is defined as \( \Lambda_{\text{nest}} \triangleq \{ \mathbf{c} : \mathbf{c} = \mathbf{c}_0 \mod \Lambda_{\text{S}}, \mathbf{c}_0 \in \Lambda_c \} \), where \( \mathbf{c}_0 \) are the coset leaders [12] of the partition \( \Lambda_c / \Lambda_{\text{S}} \) (the set of cosets of \( \Lambda_{\text{S}} \) relative to \( \Lambda_c \)). The codebook size is \( |\Lambda_{\text{nest}}| = 2^{|L|T} \), where the code rate is \( R_i \) bits per channel use (BPCU).

For a Loeliger-type nested-lattice ensemble, the coding lattice \( \Lambda_{\text{S}} \) for user \( i \) is randomly chosen from the Loeliger lattice ensemble which is generated from linear codes \( C_{\text{LIT}} \) [14]. The codebook for the relay is generated similarly as above but with dimension \( 2M_{\text{LT}} \).

B. Relay-mapper Aided Lattice Encoding

User \( i \) selects the codeword \( \mathbf{c}_i \) according to its message \( w_i \) from the codebook described in Section III-A, and sends signal \( \mathbf{x}_i \) into the MARC (5)-(6)

\[
\mathbf{x}_i = (\mathbf{c}_i - \mathbf{u}_i) \mod \Lambda_{\text{S}}(\mathbf{c}_i)
\]

where \( \mathbf{u}_i \) is a dither signal uniformly distributed over the Voronoi region \( V_{\Lambda_{\text{S}}} \) of the shaping lattice \( \Lambda_{\text{S}} \) ((T1.1) in Table I). From [15], due to the dither \( \mathbf{u}_i \), \( X_i \) is uniformly distributed over \( V_{\Lambda_{\text{S}}} \) and independent of \( \mathbf{c}_i \). To meet the input power constraints (4) as in [13], we let the second-order moment of the shaping lattice be \( \sigma^2(\Lambda_{\text{S}}) = 1/2 \) (see [16]). As for the relay (transmitter \( K + 1 \)), it will first decode the users’ messages, using the operation introduced below. Then the relay selects its codeword \( \mathbf{c}_{K+1} \) according to the decoded codewords \( \mathbf{c}_i \) using the relay mapper and then transmits \( X_{K+1} \) as in (9) with the power constraint (4).

The relay mapper \( \psi_{\text{con}} \) is used to select the codeword (coset leader) \( \mathbf{c}_{i+1} \) to be transmitted from the relay according to the codewords (coset leaders) \( \mathbf{c}_i \) for \( i = 1, \ldots, K \) of the K users. By concatenating the \( K + 1 \) codewords into a super one, we define \( \mathbf{c} = (\mathbf{c}_1, \ldots, \mathbf{c}_i, \mathbf{c}_i^+) \) for \( i = 1, \ldots, K \) in (T1.3) in Table I, then \( \psi_{\text{con}}(\mathbf{c}_i) = \mathbf{c}_i \). Now we introduce the mapper.

Definition 2 (One-to-one mapper): The one-to-one mapper \( \psi_{\text{con}} : C_{\text{nest}} \rightarrow C_{\text{nest}} \) for O-MLC is a one-to-one bijective mapping that maps coset leaders in the super-codebook of users \( C_{\text{nest}} \) to the relay codebook \( C_{\text{nest}} \). Here \( C_{\text{nest}} \triangleq \{ \mathbf{c}_0 : \mathbf{c}_0 = (c_0 \mod \Lambda_{\text{S}}), c_0 \in \Lambda_c \} \) and \( C_{\text{nest}} \triangleq \{ \mathbf{c}_0 : \mathbf{c}_0 = (c_0 \mod \Lambda_{\text{S}}), c_0 \in \Lambda_c \} \), where \( \Lambda_{\text{S}} \) and \( \Lambda_c \) are defined in (T1.4) while \( \Lambda_{\text{c}} \) and \( \Lambda_c \) are defined in (T1.5) in Table I. Note that \( |C_{\text{nest}}| = |C_{\text{nest}}| \) since the mapping is bijective.

C. K-stage Coset Decoding

We first introduce the decoder at the destination, which generalizes the single stage coset decoder in [9] to the multi-stage version. The coset decoder disregards the boundaries of the codewords and avoids the complicated boundary control [17]. Moreover, it facilitates the efficient sphere decoding algorithm [17]. To decode messages from the received signal \( \mathbf{y}_{\text{ds}} \) in (6), the proposed K-stage coset decoder works as in Table II with the detailed steps explained as follows.

According to Table II, the decoder first generates the decoding tree as in Step A. An example for \( K = 3 \) is given in Fig. 2. The decoder will traverse nodes from stage 1 to \( K \) in the tree, and produce the candidate codewords. We take...
Fig. 2. The multiuser decoding tree for the K-stage coset decoding procedure in Table II with K = 3. Here for each node, the label (k, j) denotes the j-th node from the left at the k-th stage (Node_stage in Table II), while the number t inside a circle denotes the index t of the user assumed to have been correctly decoded at the previous stage (Node_user in Table II). For example, when the coset decoding in Table II is performed at node (2, 1) (the leftmost child node of the root node), user 1 is assumed to have been correctly decoded. The path from root node (1, 1) to node (3, 3) is illustrated with bolded lines.

Table II

| Algorithm for K-stage coset decoding at the Destination |
|--------------------------------------------------------|
| A. Generation of the decoding tree:                    |
| Initialization: For the root node, Node_user = empty, Node_stage = 1 |
| for k = 1 to K - 1                                     |
| for each node with Node_stage = k                      |
| generate (K - k + 1) child nodes for the next stage    |
| (Node_stage = k + 1), for the child nodes from left to right. |
| Node_user values are assigned from the set \{1, ..., 2 \^ (K-1)\} in increasing order, where S = \{j: Node_user = j, for the ancestors** of the child node\}. |
| B. K-stage candidate generation via coset decoding:     |
| for k = 1 to K                                         |
| Step B.1 For the node (k, j), let \(Y_{ds}^{(k,j)} = y_{ds} - \sum_{i=k+1}^{\infty} w_{ki}^{(k,j)}\), where \(Y_{ds}^{(k,j)}\) is the set of previously-decoded user i's message, and the channel \(H_{ds}^{(k,j)}\) is formed from (7). For example, for the path starting from the root node to node (3, 1) in Fig. 2, the set \(S_{0,1}^{(3,1)}\) is \{(1,2)\}. |
| Step B.2 Decode the users' messages in the residual user set, defined as \(S^{(k)} = \{1, ..., K\} \setminus S_{0,k-1}^{(k)}\), by coset decoding: \(\hat{\mathbf{w}}^{(k,j)} = \arg\min_{\mathbf{w}} (C_{DS}(\mathbf{w})) M^{(k,j)}(\mathbf{w})\), where \(M^{(k,j)}(\mathbf{w}) = \sum_{i=k+1}^{\infty} \left( Y_{ds}^{(k,j)} - \sum_{l=k+1}^{\infty} w_{ki}^{(k,j)} \right)^2 \). Let \(\mathbf{w}_{0} = 2(K - k + 1)\mathbf{A}_{c} + N\mathbf{T}\), and \(N = 2KLT\), \(\mathbf{w}_{0} = 2\mathbf{v}_{i,k}^{(k,j)}\) is formed by collecting all \(c_{l}\) of transmitter i, where \(j \in S_{0,k-1}^{(k)}\), and the dither signal \(\mathbf{d}_{k,j}\) is defined similarly as \(v_{i,k}^{(k,j)}\). \(\mathbf{m}_{0} = N^{(k,j)} \mathbf{B}_{ds}^{(k,j)}\) and \(\mathbf{m}_{0} \times \mathbf{m}_{0}^{T} \mathbf{B}_{ds}^{(k,j)}\) are the corresponding MMSE-GDFE filters for \(\mathbf{w}_{i,k}^{(k,j)}\), and the searching cosets formed by previously-decoded users \(i < k\) of \(S_{0,k-1}^{(k)}\) is \(O_{DS}^{(k,j)} = \{ \mathbf{w}^{(k,j)} \in \mathcal{C} \mid \mathbf{w}^{(k,j)} \times w_{0}^{(k,j)} = 0 \} \), \(E_{DS}^{(k,j)} = \{ \mathbf{w}^{(k,j)} \in \mathcal{C} \mid \mathbf{w}^{(k,j)} \mod \mathcal{A}_{c} = \mathbf{c}^{(k,j)} \} \). The decoded message \(\mathbf{w}_{i,k}^{(k,j)}\) is defined as \(\mathbf{w}^{(k,j)}\), \(\mathbf{w}^{(k,j)} \in E_{DS}^{(k,j)}\). |
| C. Candidate elimination:                              |
| For node (k, j) at the final stage K, combine the decoded messages to produce the \(K \times 1\) super-message \(\hat{\mathbf{w}}^{(k,j)}\) as the candidate at node (k, j). The decoder searches for all \(K^{t}\) candidates \(\hat{\mathbf{w}}^{(k,j)}\) and declares the one such that \(H_{ds}^{(k,j)}\hat{\mathbf{w}}^{(k,j)}\) is nearest to the received signal \(y_{ds}\) as the final decoded message, where \(\hat{\mathbf{w}}^{(k,j)}\) is the transmitted signal according to message \(\mathbf{w}^{(k,j)}\). |

\*The algorithm for the relay can be identically obtained by ignoring the relay node. |
\**The ancestors of a node are all the nodes along the path from the root to that node (not included). |

\(O^{\text{order}} \triangleq \{ \mathbf{c} \in \mathcal{A}_{L} \mid \mathbf{c} \mod \mathcal{A}_{c} = \hat{\mathbf{c}}(\mathbf{w}) \}, \mathbf{w} \in W\), with \(W\) being the set of all possible messages:
\[ \text{Order} \triangleq \{ \mathbf{c} \in \mathcal{A}_{L} \mid \mathbf{c} \mod \mathcal{A}_{c} = \hat{\mathbf{c}}(\mathbf{w}) \}, \mathbf{w} \in W\],

where the super-lattice of users and the relay \(\mathcal{A}_{L} \) is defined in (T1.6) of Table I. The decoded message \(\mathbf{w}\) is declared if \(\hat{\mathbf{c}}(\mathbf{w})\) and the decoded \(\hat{\mathbf{c}}\) from (10) belong to the same coset, \(\hat{\mathbf{c}} \mod \mathcal{A}_{c} = \hat{\mathbf{c}}(\mathbf{w})\). For the node (k, j) in the decoding tree (the j-th node from the left at the k-th stage) we consider a path from the root node to node (k, j). An example for (k, j) = (3, 1) is given in Fig. 2. In Step B.1 of Table II, the decoder assumes that all the users at the nodes along the path (users 1 and 2 for the example path in Fig. 2), have already been successively decoded (not necessarily correctly), and subtracts the corresponding transmitted signals from the received signal \(y_{ds}\). Then the decoder decodes the remaining messages by (T2.1) in Step B.2 of Table II (which corresponds to (10)). Finally, as in Step C, the decoder searches for all \(K^{t}\) candidates produced at the nodes at the K-th stage (instead of all \(2LT^{K-1}R_{L}\) codewords) to choose the final decoded message.

The decoder at the relay also uses (10) as the criterion to decode messages from \(y_{ds}^{(r)}\) in (5) with the corresponding MMSE-GDFE forward and feedback filters. The main difference is that now the decoding does not make use of the relay codebook, and the decoder searches in the super-lattice of users \(\mathcal{A}_{L} \) instead of the coset \(O^{\text{order}}\) in (11). The complexity of the decoder in Table II is about \(O(LT)^{c}\). This is much smaller compared with the complexity of the ML decoder \(O(2^{LT}\frac{K}{L-1}R_{L})\), which grows exponentially with the block length \(LT\).

Note that since the super-codewords have to satisfy the relay mapping rule in Section III-B, the set \(O^{\text{order}}\) is not necessary a sublattice of \(\mathcal{A}_{L} \). This makes (10) different from the decoder in [9]. Without the algebraic structure of a lattice, the error probability analysis in the next section will be much more difficult than that in [9].

\text{As shown in [18], one can use a linear mapper to implement O-MLC.}
IV. PERFORMANCE ANALYSIS OF THE CODING SCHEME

In this section, we establish the achievable rate region for the MARC defined in (5) and (6), using the proposed O-MLC. We show that the rate performance, which was originally achieved by using an unstructured random codebook in [6], is now achieved by our structured O-MLC. The key is using the proposed K-stage coset decoder which successively cancels the previously decoded messages, thus avoiding the rate loss incurred by the one-stage coset decoder in [9]. The rate loss due to use of a one-stage coset decoder is derived in Corollary 1. However, in Corollary 2, we show that the rate loss is relatively small in the high SNR regime, and O-MLC with the one-stage coset decoder achieves the optimal DMT for the MARC in (5) and (6). Note that the DMT was achieved by an unstructured random codebook and ML decoding in [7].

In the error analysis of the proposed scheme, the conventional approach tailored for ML decoding [5][7] fails in predicting the performance of the coset decoder in (10) due to the infinite number of points $c \in O^{\text{unw}}$. To solve this problem, from (11), we define the differential ambiguity cosets for the event that the transmitted message $w_i$ is erroneously decoded as $w$ as

$$O^{\text{unw}}_{\text{w}} \triangleq \{ d \in \Lambda_{\text{w}} : d \equiv \bar{d}(w), w \in W, w \neq w_i \},$$

where the differential codeword $\bar{d}(w) \equiv (\bar{c}(w) - c(w)) \mod \Lambda_{\text{w}}$ as (T1.6) of Table I and the vector after modulo operation is defined in (T1.7). Moreover, $O^{\text{unw}}_{\text{w}}$ is not a direct product of $K + 1$ lattices (i.e., $\Lambda_{\text{w}}$), and thus the techniques in [9] fail to predict the error probability of O-MLC. We propose a new error probability upper-bound which avoids directly counting points of $O^{\text{unw}}_{\text{w}}$ in the decision region of the decoder as this kind of evaluation is intractable. The achievable rate region of (5) and (6), using O-MLC is given as follows.

Theorem 1: For the MARC in (5) and (6), the DDF rate region in (13) and (14), which is achieved by unstructured Gaussian codebooks and ML decoding in [6], is achievable by the structured O-MLC and the K-stage coset decoder in Table II, where the rate constraints at the relay and destination are

$$\sum_{n \in S} R_n \leq \frac{1}{L_T} R_{\text{relay}}^{\text{delay}}(H_{\text{delay}}^{\text{relay}}),$$

$$\sum_{n \in S} R_n \leq \frac{1}{L_T} R_{\text{unw}}^{\text{delay}}(H_{\text{unw}}^{\text{delay}}), \forall S \subseteq \{1, \ldots, K\} \leq \frac{1}{L_T} R_{\text{unw}}^{\text{delay}}(H_{\text{unw}}^{\text{delay}}), \forall S \subseteq \{1, \ldots, K\}$$

respectively, with $R_{\text{delay}}^{\text{relay}}(H_{\text{delay}}^{\text{relay}})$ and $R_{\text{unw}}^{\text{delay}}(H_{\text{unw}}^{\text{delay}})$ given in (T1.9) and (T1.10) in Table I. The channel matrix from the users in the set $S$ and the relay to the destination $H_{\text{relay}}^{\text{delay}}(S, K + 1)$ is formed from $H_{\text{delay}} = [H_1, \ldots, H_{K+1}]$ as in (T1.8) with $H_i$ given in (7) and (8), and the channel matrix from the users in the set $S$ to the relay $H_{\text{unw}}^{\text{delay}}$ is defined similarly to $H_{\text{relay}}^{\text{delay}}$.

Sketch of proof: We prove only (14) here since (13) follows similarly. We show that there exists at least one path at each stage (Fig. 2) of Step B of Table II on which the previously decoded messages are correct. Then we can at least obtain a better decoder for the remaining users in the next stage to improve the error performance. Finally, we show that the correct messages can always be chosen from the candidates at the final stage. Due to space limitations, we sketch only the proof for the first stage as stated in the upcoming Lemma 1. For the first stage ($k = 1$ in Fig. 2) of the candidate generation process in Step B of Table II, we show that at least one of the users’ messages is correctly decoded in the generated “super”-message $\bar{w}_i^{(1)}$ of all users (with probability 1) as $T \to \infty$. We first define the following error event.

Definition 3 (set-S error): A decoded super-message $w$ is with set-S error if the message in $w$ for every user $i$, where $i \in S$, is different from the corresponding transmitted message. That is, $w_i \neq (w_i)_t$, $\forall i \in S$, while $w_i = (w_i)_t$, otherwise.

Let $P_e(S|H_{\text{delay}})$ be the probability that there exists $w$ with set-S error with fixed $H_{\text{delay}}$, and $\min_{c \in \Lambda_{\text{w}}(w)} M(c) \leq \min_{c \in \Lambda_{\text{w}}(w)} M(c)$, with $M(c)$ defined in (10) and $o(w)$ being the coset of $w$.

For the first stage, we consider the erroneous user set $S(1) = \{1, \ldots, K\}$ and prove that $P_e(S(1)|H_{\text{delay}}) \to 0$, if the transmission rates $R_i$ satisfy (14) and the lattice codes are good as defined in the upcoming Lemma 1. Here $P_e(S(1)|H_{\text{delay}})$ is averaged over the random relay-mapper and linear-code ensemble $E_{\text{delay}}^{\text{unw}} = \{O^{\text{unw}}_{\text{w}}, c_{\text{w}}\}$ consisting of all possible one-to-one mappers $w$, and Loeliger linear codes $C_{\text{w}} \triangleq C_{\text{w}}^{(1)} \times \cdots \times C_{\text{w}}^{(K+1)}$, of the users and relay, where $C_{\text{w}}^{(k)}$ is the linear code corresponding to the lattice $A_{\text{w}}$ in Definition 1.

Lemma 1: For O-MLC, let $R_i$, $i = 1, \ldots, K + 1$, be the code rates for the users and the relay, and $\{A_{\text{w}}\}$ belong to the Loeliger lattice ensembles [14] with $\{A_{\text{w}}\}$ good for mean square error quantization [16]. For stage $k = 1$ of Step B in Table II, as $LT \to \infty$, the set-S(1) error probability (cf. Definition 3), where $S(1) = \{1, \ldots, K\}$, satisfies

$$P_e(S(1)|H_{\text{delay}}) \leq \frac{1}{\text{exp} \left[ \frac{1}{L_T} \log \left( 1 + \sum_{i \in S} R_i \right) \right] + \frac{1}{L_T} \log \left( 2^{K+1 LT - 1} \right)}$$

where $O^{\text{unw}}_{\text{w}}$ consists of points belonging to the differential ambiguity cosets $O^{\text{unw}}_{\text{w}}$ given in (12), with corresponding messages having set-S(1) errors; the decision region $R_D \triangleq \{ v : \|B_{\text{delay}}v\|^2 \leq (KM_0 + M_0)LT (1 + \beta) \}$ with filter $B_{\text{delay}}$ defined as in (10) and $\beta > 0$.

The main difficulty in proving Lemma 1 is that the $O^{\text{unw}}_{\text{w}}$ is not a direct product of $K + 1$ lattices as in [9], so the methods in [9] cannot be directly applied to counting the number of points of $O^{\text{unw}}_{\text{w}}$ in $R_D$ in (15). We avoid explicitly counting points in $O^{\text{unw}}_{\text{w}}$, by developing new upper-bounds. Otherwise, naively applying the methods of [9] will result in rates as in (15) but without the factor $(2^{K+1 LT - 1} - 1)$ cancelling out $2^{K+1 LT}$, and lead to significant rate loss compared with (14) with $S = S(1)$ since $R_{K+1} = \sum_{i=1}^{K} R_i$ (bijective mapping).
It remains to show that among all “super”-messages $\mathbf{w}^{(K,l)}$ at stage $K$ (in Step B of Table II), there exists a correct message as $T \to \infty$. Moreover, for the Step C of Table II, we need to show that the correct message $\mathbf{w}$ will be chosen almost surely. These technical details are given in [18].

If only the one-stage coset decoder is used as in [9], we have the following. The details are given in [18].

**Corollary 1:** For the MARC in (5) and (6), the rate region constrained by (16) and (17), which is strictly smaller than that in Theorem 1, is achievable by O-MLC with the one-stage coset decoder in (10), where $\forall S \subseteq \{1,\ldots,K\}$,

$$\sum_{S \subseteq S} R_i < \frac{1}{L_T \ln(c)} (H_{\text{enc}}^{(S,K)}) - M_i |S| \log_2 \frac{K}{|S|}$$

and,

$$\sum_{S \subseteq S} R_i < \frac{1}{L_T \ln(c)} (H_{\text{dist}}^{(S,K+1)}) - (M_i |S| + M_r) \log_2 \frac{K M_i + M_r}{|S| M_i + M_r}$$

Clearly, compared to the rate region in (13) and (14), there are rate loss terms $M_i |S| \log_2 \frac{K_i}{|S|}$ and $(M_i |S| + M_r) \log_2 \frac{K M_i + M_r}{|S| M_i + M_r}$ in (16) and (17), respectively. These losses result from the fact that the MMSE-GDFE processing for the one-stage coset decoding in (10) is optimal only for the decoder to distinguish messages with set-Serror (zero rate losses when $|S| = K$).

Finally, for slow fading channels, we show that O-MLC with the one-stage coset decoder (10) is DMT optimal for the DDF MARC, as stated in the corollary below. The details are given in [18]. Despite the rate loss terms in (16) and (17) compared with (13) and (14), respectively, the losses become relatively negligible for the DMT analysis when the SNR is high.

**Corollary 2:** For the MARC in (5) and (6), with the one-stage coset decoder (10), the O-MLC achieves the optimal DDF DMT $d(r)$ of (5) and (6), respectively, where $d(r)$ is defined in (T1.11) of Table I.

V. SIMULATION RESULTS

In this section, we present numerical examples to illustrate our theoretical results. For simplicity, we consider the case in which there are two users with the same transmission rate, i.e., $R_1 = R_2 = R$. The number of slots is selected as $L = 2$, and the sum rate $(R_1 + R_2)$ is 4 BPCU. The relay forwards the message only when the users’ messages are correctly decoded. All the channel links are Rayleigh faded and the sources-to-relay (S-R) channel link is 10 dB better than the other channel links. In Fig. 3, for single-antenna nodes, we show that the O-MLC outperforms the protocols of [5], [8], [10] and [11] in terms of outage probability and achieves the optimal diversity $M_i (M_i + N_i) (M_r + M_r) |N| = 2$ as expected. In [18], we also design a practical O-MLC with finite code length which achieves the optimal diversity with the aid of a linear one-to-one relay mapper.

VI. CONCLUSION

In this work, we have proposed O-MLC for structured MARC coding. We have shown that with the new K-stage decoder instead of the one-stage decoder considered in previous works, the structured O-MLC can approach the rate performance of an unstructured codebook with ML decoding. When only the one-stage decoder is used, O-MLC can still achieve the optimal DMT of DDF. Simulation results have shown that our proposed coding scheme outperforms existing schemes in terms of outage probabilities.

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