STATISTICAL PROPERTIES OF LORENZ LIKE FLOWS, RECENT
DEVELOPMENTS AND PERSPECTIVES

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Abstract. We comment on mathematical results about the statistical behavior of Lorenz equations an its attractor, and more generally to the class of singular hyperbolic systems. The mathematical theory of such kind of systems turned out to be surprisingly difficult. It is remarkable that a rigorous proof of the existence of the Lorenz attractor was presented only around the year 2000 with a computer assisted proof together with an extension of the hyperbolic theory developed to encompass attractors robustly containing equilibria.

We present some of the main results on the statistical behavior of such systems. We show that for attractors of three-dimensional flows, robust chaotic behavior is equivalent to the existence of certain hyperbolic structures, known as singular-hyperbolicity. These structures, in turn, are associated to the existence of physical measures: in low dimensions, robust chaotic behavior for flows ensures the existence of a physical measure.

We then give more details on recent results on the dynamics of singular-hyperbolic (Lorenz-like) attractors: (1) there exists an invariant foliation whose leaves are forward contracted by the flow (and further properties which are useful to understand the statistical properties of the dynamics); (2) there exists a positive Lyapunov exponent at every orbit; (3) there is a unique physical measure whose support is the whole attractor and which is the equilibrium state with respect to the center-unstable Jacobian; (4) this measure is exact dimensional; (5) the induced measure on a suitable family of cross-sections has exponential decay of correlations for Lipschitz observables with respect to a suitable Poincaré return time map; (6) the hitting time associated to Lorenz-like attractors satisfy a logarithm law; (7) the geometric Lorenz flow satisfies the Almost Sure Invariance Principle (ASIP) and the Central Limit Theorem (CLT); (8) the rate of decay of large deviations for the volume measure on the ergodic basin of a geometric Lorenz attractor is exponential; (9) a class of geometric Lorenz flows exhibits robust exponential decay of correlations; (10) all geometric Lorenz flows are rapidly mixing and their time-1 map satisfies both ASIP and CLT.

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1. Introduction

The development of the theory of dynamical systems has shown that many important dynamical models exhibit sensitive dependence on initial conditions\(^1\), a common feature of chaotic dynamics: small initial differences are rapidly augmented as time passes, causing two trajectories originally coming from practically indistinguishable points to behave in a completely different manner after a short while. Pointwise, long term predictions based on such models are unfeasible since it is not possible to both specify initial conditions with arbitrary accuracy and numerically calculate with arbitrary precision; for an introduction to these notions see [Devaney(1989), Robinson(2004)].

On the other hand in these systems, even if the pointwise description or forecasting of the system is forbidden by the initial condition sensitivity, the statistical behavior is often relatively simple and its properties are often (with a certain effort) predictable.

A theory which explain this statistical behavior is quite satisfactorily developed for systems having some uniformly hyperbolic behavior (see below for the definition), yet the rigorous description of the statistical behavior of relatively simple systems as: quadratic polynomials \(^2\), or autonomous ordinary differential equations with a hyperbolic equilibrium of saddle-type accumulated by regular orbits, as the Lorenz flow is still far from being complete. In this article we are going to describe some relatively recent developments in this second direction.

1.1. The Lorenz equations. In 1963 the meteorologist Edward Lorenz published in the Journal of Atmospheric Sciences [Lorenz(1963)] an example of a parametrized polynomial system of differential equations

\[
\begin{align*}
\dot{x} &= a(y - x) & a &= 10 \\
\dot{y} &= r x - y - xz & r &= 28 \\
\dot{z} &= xy - bz & b &= 8/3
\end{align*}
\]

(1)

as a very simplified model for thermal fluid convection, motivated by an attempt to understand the foundations of weather forecast.

\(^1\)Formally the definition of sensitivity for a flow \(X^t\) on some compact manifold \(M\) is as follows: an \(X^t\)-invariant subset \(\Lambda\) is sensitive to initial conditions or has sensitive dependence on initial conditions, or simply chaotic if, for every small enough \(r > 0\) and \(x \in \Lambda\), and for any neighborhood \(U\) of \(x\), there exists \(y \in U\) and \(t \neq 0\) such that \(X^t(y)\) and \(X^t(x)\) are \(r\)-apart from each other: \(\text{dist}(X^t(y), X^t(x)) \geq r\); see Figure 1 and Section 9.1. An analogous definition holds for maps \(f\) of some manifold or even metric spaces.

\(^2\)(as the logistic family or Hénon attractor, see e.g. [Devaney(1989)] for a gentle introduction)
Figure 1. Sensitive dependence on initial conditions.

The origin $\sigma = (0, 0, 0)$ is an equilibrium of saddle type for the vector field defined by equations (1) with real eigenvalues $\lambda_i$, $i \leq 3$ satisfying

$$\lambda_2 < \lambda_3 < 0 < -\lambda_3 < \lambda_1.$$ 

(in this case $\lambda_1 \approx 11.83$, $\lambda_2 \approx -22.83$, $\lambda_3 = -8/3$).

Numerical simulations performed by Lorenz for an open neighborhood of the chosen parameters suggested that almost all points in phase space tend to a chaotic attractor, that is, a bounded region in phase-space, invariant under time evolution, such that the forward trajectories of most or even all points nearby converge to it, and these trajectories are sensitive with respect to initial data. The well known picture of the Lorenz attractor is presented in Figure 2.

Figure 2. A view of the Lorenz attractor calculated numerically

Lorenz’s equations proved to be very resistant to rigorous mathematical analysis, and also presented serious difficulties to rigorous numerical study. Indeed, these two main difficulties are:

**conceptual:** the presence of an equilibrium point at the origin accumulated by regular orbits of the flow prevents this attractor from being hyperbolic [Araújo & Pacífico(2010)],

**numerical:** the presence of an equilibrium point at the origin, implying that solutions slow down as they pass near the origin, which means unbounded return times and, thus, unbounded integration errors.
Moreover the attractor is robust, that is, the features of the limit set persist for all nearby vector fields. More precisely, if $U$ is an isolating neighborhood of the attractor $\Lambda$ for a vector field $X$, then $\Lambda$ is robustly transitive if, for all vector fields $Y$ which are $C^1$ close to $X$, the corresponding $Y$-invariant set

$$\Lambda_Y(U) = \bigcap_{t>0} Y^t(U)$$

also admits a dense positive $Y$-orbit. The persistence of transitivity, that is, the fact that, for all nearby vector fields, the corresponding limit set is transitive, is quite remarkable and implies a dynamical characterization of the attractor, as we shall see.

1.2. Geometric Lorenz model, computer aided approach, singular hyperbolic attractors. The difficulties in the study of the Lorenz system led, in the seventies, to the construction of geometric flows presenting a similar behavior as the one generated by equations (1). Nowadays these models are known as geometric Lorenz flows. We describe this construction in Section 3.1; see [Afraimovich et al. (1995) Afraimovich, Chernov & Sataev, Guckenheimer & Williams (1979)] for full details.

These models are three-dimensional flows for which it is easy to rigorously prove the existence of an attractor containing an equilibrium point of the flow, together with regular solutions. They have a natural cross-section given by a two-dimensional square crossed by all orbits of the flow inside the attractor except the singularity. Several properties about the statistical behavior of these flows have been rigorously understood by the use of the properties of the Poincaré first return map to the cross-section.

A successful approach for the real Lorenz flow was through rigorous numerics. In this way, it could be proved [Hassard et al. (1994) Hassard, Hastings, Troy & Zhang, Hastings & Troy (1992), Mischaikow & Mrozek (1995), Mischaikow & Mrozek (1998)] that the Lorenz system of equations exhibits a suspended Smale horseshoe [Smale (1967)] which implies, in particular, the existence of infinitely many closed orbits. However, proving the existence of an attractor as in the geometric models is an even harder task, because one cannot avoid the fact that solutions slow down as they pass near the equilibrium, which, as said before means unbounded return times and so unbounded integration errors. This was finally settled by Tucker in [Tucker (2002)] around the turn of the century.

Using a combination of rigorous numerics and normal form theory, Tucker proved that the Lorenz equations (1) support a robust strange attractor $\Lambda$ and the flow admits a unique Sinai-Ruelle-Bowen measure $\mu$ with $\text{supp}(\mu) = \Lambda$; this notion will be presented in Section 2. Tucker’s proof uses a computer algorithm to estimate convenient solutions of (1), keeping rigorous bounds on the errors. Successful termination of this algorithm proves the presence of a robustly transitive attractor in (1). It is worth to remark that even in this approach a key step is the construction of a suitable Poincaré map on a cross-section and the understanding of its properties (see Section 3) which are similar to the ones found in the geometric model.

From robust transitivity it follows after [Morales et al. (1998) Morales, Pacifico & Pujals, Morales et al. (2004) Morales, Pujals & Sambarino] that the attractor supported by the equations (1) is a singular-hyperbolic attractor. We will see in the following that also for these attractors we can construct a suitable cross section and prove that they share all the fundamental features of the geometric Lorenz models (see Section 4) so that its geometry and its ergodic properties can be well understood.

Singular hyperbolicity plays the role of hyperbolicity for flows presenting equilibria accumulated by regular orbits. Hyperbolicity alone, a classical notion going back to Smale [Smale (1967)] means that the complementary direction to the flow can be further split into a pair of complementary invariant directions, one uniformly contracting and the other uniformly expanding by the tangent map to the flow. The theory of hyperbolic systems describing their geometric and ergodic properties is very rich: the interested reader should consult [Bowen & Ruelle (1975), Guckenheimer & Williams (1979), Palis & de Melo (1982), Shub (1987)] and references therein.

Singular hyperbolicity replaces the expanding direction by a two-dimensional direction containing the flow direction on regular orbits along which the flow should expand area. Singular hyperbolicity encompasses flows exhibiting equilibria attached to regular orbits, since by
an attractor robustly containing equilibria is singular hyperbolic.

Moreover singular hyperbolicity is a natural generalization of hyperbolicity, since compact invariant sets which are singular hyperbolic, but have no equilibria, can be proved to be hyperbolic in the usual sense. Remarkable dynamical properties can be proved for singular hyperbolic attractors extending in this way the hyperbolic theory to a wider class of systems. We present some of these dynamical and ergodic properties of singular-hyperbolic attractors in what follows.

1.3. Overview. After reviewing some main ideas from dynamical systems theory, using results on robustness of attractors from Mañé [Mañé(1982)] and Morales, Pacifico and Pujals [Morales et al.(2004)] together with observations on their proofs, we show that for attractors of three-dimensional flows, robust chaotic behavior (in the above sense of sensitiveness to initial conditions for all close enough flows) is equivalent to the existence of certain partially hyperbolic structures. These structures, in turn, allow the construction of suitable Poincaré sections with nice properties and to deduce several consequences about the statistical behavior of the dynamics.

In the following, we review the construction and several more or less recent results about the dynamics of singular-hyperbolic (or Lorenz-like) attractors:

- there exists an invariant foliation whose leaves are forward contracted by the flow and the dynamics satisfies a list of properties which is similar to the ones of geometric Lorenz attractors; see Section 5.1.
- there exists a positive Lyapunov exponent at every orbit; see Section 5.1.
- there is a unique physical measure whose support is the whole attractor and which is the equilibrium state with respect to the center-unstable Jacobian; see Section 5.
- this physical measure is exact dimensional and the hitting time associated to a Lorenz-like attractor satisfies a logarithm law; see Sections 5.3 and 5.5.
- the induced measure on a suitable family of cross-sections of a Lorenz-like flow has exponential decay of correlations with respect to the Poincaré first return map; see Section 5.4.
- the geometric Lorenz flow satisfies the Almost Sure Invariance Principle and the Central Limit Theorem; see Section 6.3.
- the rate of decay for large deviations with respect to the volume measure on the ergodic basin of a geometric Lorenz attractor is exponential; see Section 6.1.
- there are open sets of geometric Lorenz flows each of which exhibits exponential decay of correlations; see Section 6.2.
- In low dimensions, robust chaotic behavior ensures the existence of a physical measure; see Section 9.\footnote{We remark that this provides a partial answer to a conjecture of Viana: \textit{existence of positive Lyapunov exponent for a positive Lebesgue measure subset of orbits implies the existence of a physical measure}. This is proved in Section 9.1.}

We finish with a brief list of conjectures on the dynamics of singular-hyperbolic attractors, in Section 7.

2. Preliminary notions

Here and throughout the text we assume that $M$ is a three-dimensional compact connected manifold without boundary endowed with some Riemannian metric which induces a distance denoted by dist and a volume form $\text{Leb}$ which we name \textit{Lebesgue measure} or \textit{volume}. For any subset $A$ of $M$ we denote by $\bar{A}$ the (topological) closure of $A$.

We denote by $\mathfrak{X}^r(M)$, $r \geq 1$ the set of $C^r$ smooth vector fields $X$ on $M$ endowed with the $C^r$ topology. Given $X \in \mathfrak{X}^r(M)$ we denote by $X^t$, with $t \in \mathbb{R}$, the flow generated by the vector field $X$. Since we assume that $M$ is a compact manifold the flow is defined for all time. Recall that the flow $(X^t)_{t \in \mathbb{R}}$ is a family of $C^r$ diffeomorphisms satisfying the following properties:

\begin{enumerate}
  \item $X^0 = \text{Id} : M \to M$ is the identity map of $M$;
  \item $X^{t+s} = X^t \circ X^s$ for all $t, s \in \mathbb{R}$,
\end{enumerate}

and it is \textit{generated by the vector field $X$} if

\footnote{We remark that this provides a partial answer to a conjecture of Viana: \textit{existence of positive Lyapunov exponent for a positive Lebesgue measure subset of orbits implies the existence of a physical measure}. This is proved in Section 9.1.}
We say that a compact $X^t$-invariant set $\Lambda$ is isolated if there exists a neighborhood $U$ of $\Lambda$ such that

$$\Lambda = \bigcap_{t \in \mathbb{R}} X^t(U).$$

A compact invariant set $\Lambda$ is attracting if $\Lambda_X(U) := \bigcap_{t \geq 0} X^t(U)$ equals $\Lambda$ for some neighborhood $U$ of $\Lambda$ satisfying $X^t(U) \subset U$, for all $t > 0$. In this case the neighborhood $U$ is called an isolating neighborhood of $\Lambda$. Note that $\Lambda_X(U)$ is in general different from $\bigcap_{t \in \mathbb{R}} X^t(U)$, but for an attracting set the extra condition $X^t(U) \subset U$ for $t > 0$ ensures that every attracting set is also isolated. We say that $\Lambda$ is transitive if $\Lambda$ is the closure of both $\{ X^t(q) : t > 0 \}$ and $\{ X^t(q) : t < 0 \}$ for some $q \in \Lambda$. An attractor of $X$ is a transitive attracting set of $X$ and a repellor is an attractor for $-X$. We say that $\Lambda$ is a proper attractor or repellor if $\emptyset \neq \Lambda \neq M$.

An equilibrium (or singularity) for $X$ is a point $\sigma \in M$ such that $X^t(\sigma) = \sigma$ for all $t \in \mathbb{R}$, i.e. a fixed point of all the flow maps, which corresponds to a zero of the associated vector field $X$: $X(\sigma) = 0$. An orbit of $X$ is a set $O(q) = O_X(q) = \{ X^t(q) : t \in \mathbb{R} \}$ for some $q \in M$. A periodic orbit of $X$ is an orbit $O = O_X(p)$ such that $X^T(p) = p$ for some minimal $T > 0$. A critical element of a given vector field $X$ is either an equilibrium or a periodic orbit.

We say that a compact invariant subset is singular hyperbolic if all the singularities in $\Lambda$ are hyperbolic, and the tangent bundle $T\Lambda$ decomposes in two complementary $DX^t$-invariant bundles $E^s \oplus E^{cu}$, where: $E^s$ is one-dimensional and uniformly contracted by $DX^t$; $E^{cu}$ is bidimensional, contains the flow direction, $DX^t$ expands area along $E^{cu}$ and $DX^t \mid E^{cu}$ dominates $DX^t \mid E^s$ (i.e. any eventual contraction in $E^s$ is stronger than any possible contraction in $E^{cu}$), for all $t > 0$.

The notion of singular hyperbolicity was introduced in [Moraes et al.(1998)Morales, Pacífico & Pujals, Morales et al.(2004)Morales, Pacífico & Pujals] where it was proved that any $C^1$ robustly transitive set for a 3-flow is either a singular hyperbolic attractor or repellor.

We note that the presence of an equilibrium together with regular orbits accumulating on it prevents any invariant set from being hyperbolic, see e.g. [Bowen & Ruelle(1975)]. Indeed, in our 3-dimensional setting a compact invariant subset $\Lambda$ is hyperbolic if the tangent bundle $T\Lambda$ decomposes in three complementary $DX^t$-invariant bundles $E^s \oplus E^X \oplus E^u$, each one-dimensional, $E^X$ is the flow direction, $E^s$ is uniformly contracted and $E^u$ uniformly expanded by $DX^t$, $t > 0$. This implies the continuity of the splitting and the presence of a non-isolated equilibrium point in $\Lambda$ leads to a discontinuity in the splitting dimensions.

In the study of the asymptotic behavior of orbits of a flow $X \in \mathcal{X}^1(M)$, a fundamental problem is to understand how the behavior of the tangent map $DX$ determines the dynamics of the flow $X^t$. The main achievement along this line is the uniform hyperbolic theory: we have a very good description of the dynamics assuming that the tangent map has a hyperbolic structure since the work of Bowen and Ruelle [Bowen & Ruelle(1975)].

We recall standard facts about hyperbolic flows from e.g. [Hirsch et al.(1977)Hirsch, Pugh & Shub]. An embedded disk $\gamma \subset M$ is a (local) strong-unstable manifold, or a strong-unstable disk, if $\text{dist}(X^{-t}(x), X^{-t}(y))$ tends to zero exponentially fast as $t \to +\infty$, for every $x, y \in \gamma$. Similarly, $\gamma$ is called a (local) strong-stable manifold, or a strong-stable disk, if $\text{dist}(X^t(x), X^t(y)) \to 0$ exponentially fast as $n \to +\infty$, for every $x, y \in \gamma$. It is well-known that every point in a hyperbolic set possesses a local strong-stable manifold $W^{ss}_{loc}(x)$ and a local strong-unstable manifold $W^{uu}_{loc}(x)$ which are disks tangent to $E_x$ and $G_x$ at $x$ with topological dimensions $d_s = \dim(E^s)$ and $d_u = \dim(E^u)$, respectively. These disks are $X^t$-invariant, meaning for $x \in \Lambda$ and $t > 0$

\[
X^t(W^{ss}_{loc}(x)) \subset W^{ss}_{loc}(X^t(x)) \quad \text{and} \quad X^{-t}(W^{uu}_{loc}(x)) \subset W^{uu}_{loc}(X^{-t}(x))
\]

and so we obtain the (global) strong-stable manifold

\[
W^{ss}(x) = \bigcup_{t > 0} X^{-t}(W^{ss}_{loc}(X^t(x)))
\]

and the (global) strong-unstable manifold

\[
W^{uu}(x) = \bigcup_{t > 0} X^t(W^{uu}_{loc}(X^{-t}(x)))
\]
for every point $x$ of a hyperbolic set. These are immersed submanifolds with the same differentiability of the flow. We also consider the stable manifold $W^s(x) = \bigcup_{t \in \mathbb{R}} X^t(W^s(x))$ and unstable manifold $W^u(x) = \bigcup_{t \in \mathbb{R}} X^t(W^u(x))$ for $x$ in a hyperbolic set, which are flow invariant.

In the same vein, under the assumption of singular hyperbolicity and also following the standard reference [Hirsch et al.(1977)Hirsch, Pugh & Shub], one can show that at each point there exists a strong stable manifold and that the whole set is foliated by leaves that are contracted by forward iteration.

In particular, this shows that any robust transitive attractor with singularities displays similar properties to those of the geometric Lorenz model. It is also possible to show the existence of local central manifolds tangent to the central unstable direction; see e.g. [Araújo et al.(2014)Araújo, Galatolo & Pacifico]

Although these central manifolds do not behave as unstable ones, in the sense that points on them are not necessarily asymptotic in the past. The expansion of volume along the central unstable two-dimensional direction enables us to deduce some remarkable properties.

We recall that a $X^t$-invariant probability measure $\mu$ is a probability measure satisfying $\mu(X^t(A)) = \mu(A)$ for all $t \in \mathbb{R}$ and measurable $A \subset M$. Given an invariant probability measure $\mu$ for a flow $X^t$, let $B(\mu)$ be the (ergodic) basin of $\mu$, i.e., the set of points $z \in M$ satisfying for all continuous functions $\varphi : M \to \mathbb{R}$

$$\lim_{T \to +\infty} \frac{1}{T} \int_0^T \varphi(X^t(z)) \, dt = \int \varphi \, d\mu.$$ 

We say that $\mu$ is a physical (or SRB) measure for $X$ if $B(\mu)$ has positive Lebesgue measure: $\text{Leb}(B(\mu)) > 0$.

The existence of a physical measures for an attractor shows that most points in a neighborhood of the attractor have well defined long term statistical behavior. So, in spite of chaotic behavior preventing the exact prediction of the time evolution of the system in practical terms, we gain some statistical knowledge of the long term behavior of the system near the chaotic attractor.

3. Geometric Lorenz system(s)

In this section we introduce a concrete flow which is in some sense the simplest example of singular-hyperbolic system. This model was also historically, the first one where some rigorous results on the dynamic of Lorenz like system were proved. The system has a fixed point at the origin, and a linear vector field around it. Its global behavior is similar to the original Lorenz system but the linearity near the origin allows to write explicit formulas for the trajectory near it, and an explicit form for the Poincaré map on a suitable section. This allow to obtain many properties of the flow, which can be used to deduce several statistical consequences for its dynamics. We remark that, sometimes in the literature for geometric Lorenz system is meant a system satisfying a list of properties as the ones we will see in the geometric Lorenz one:

1. there exists a Lorenz-like singularity at the origin for a $C^2$ smooth vector field in $\mathbb{R}^3$;
2. there is a suitable cross-section $\Sigma$ whose Poincaré first return map $P$ preserves a uniformly contracting fibration;
3. the one dimensional induced map $f$ on the quotient $X$ by this contracting fibration is piecewise $C^{1+\epsilon}$, for some $\epsilon > 0$, with two branches and has a singularity corresponding to a certain leaf $\xi_0$: if $d(x)$ denotes the distance of $x \in \Sigma$ to $\xi_0$, then there exists $\beta \in (0, 1)$ such that $|f'(x)| = d(x)\beta^{-1}g(x)$ with $g \in C^\epsilon(\Sigma)$;
4. the Poincaré first return time $r : \Sigma \to \mathbb{R}^+$ is integrable with respect to Lebesgue area measure on the cross-section and there exists a constant $c_0 > 0$ such that $r(x) = -c_0 \log d(x) + h(x)$ with $h \in C^\epsilon(\Sigma)$.
5. $f$ is uniformly piecewise expanding: there are constants $\sigma > 1$ and $c > 0$ such that $|(f^n)'(x)| \geq c\sigma^n$ for all $x \in X$ and $n > 1$.

The precise assumptions which are considered may slightly vary from paper to paper. To avoid confusion, in this paper we will refer to a systems considered in this approach as Axiomatic geometric Lorenz system. Systems of this kind however do behave in a similar way having the same asymptotic features.
3.1. The construction. Next we briefly recall the construction of a concrete example of a flow satisfying the above list of properties. We will refer this as the geometric Lorenz flow, that is the simpler example of a singular-hyperbolic attractor; see [Afraimovich et al. (1995)] for full details. As explained in the Introduction, the purpose was the construction of a geometric flow presenting a similar behavior as the one generated by equations (1). We start by observing that under some non-resonance conditions, by the results of Sternberg [Sternberg (1958)], in a neighborhood of the origin, which we assume to contain the cube $[-1,1]^3 \subset \mathbb{R}^3$, the Lorenz equations are equivalent to the linear system $(\dot{x}, \dot{y}, \dot{z}) = (\lambda_1 x, \lambda_2 y, \lambda_3 z)$ through smooth conjugation, thus

$$X^t(x_0, y_0, z_0) = (x_0 e^{\lambda_1 t}, y_0 e^{\lambda_2 t}, z_0 e^{\lambda_3 t}),$$

where $\lambda_1 \approx 11.83$, $\lambda_2 \approx -22.83$, $\lambda_3 = -8/3$ and $(x_0, y_0, z_0) \in \mathbb{R}^3$ is an arbitrary initial point near $(0, 0, 0)$.

Consider $S = \{(x, y, 1) : |x| \leq 1/2, \ |y| \leq 1/2\}$ and

$$S^- = \{(x, y, 1) \in S : x < 0\}, \quad S^+ = \{(x, y, 1) \in S : x > 0\} \quad \text{and} \quad S^* = S^- \cup S^+ = S \setminus \ell,$$

where $\ell = \{(x, y, 1) \in S : x = 0\}$.

Assume that $S$ is a global transverse section to the flow so that every trajectory eventually crosses $S$ in the direction of the negative $z$ axis.

Consider also $\Sigma = \{(x, y, z) : |x| = 1\} = \Sigma^- \cup \Sigma^+$ with $\Sigma^\pm = \{(x, y, z) : x = \pm 1\}$.

For each $(x_0, y_0, 1) \in S^*$ the time $\tau$ such that $X^\tau(x_0, y_0, 1) \in \Sigma$ is given by

$$\tau(x_0) = -\frac{1}{\lambda_1} \log |x_0|,$$

which depends on $x_0 \in S^*$ only and is such that $\tau(x_0) \to +\infty$ when $x_0 \to 0$. This is one of the reasons many standard step by step numerical integration algorithms were unsuited to tackle the Lorenz system of equations. Hence we get (where $\text{sgn}(x) = x/|x|$ for $x \neq 0$)

$$X^\tau(x_0, y_0, 1) = \left( \text{sgn}(x_0), y_0 e^{\lambda_2 \tau}, e^{\lambda_3 \tau} \right) = \left( \text{sgn}(x_0), y_0 |x_0|^{-\frac{\alpha}{\lambda_1}}, |x_0|^{-\frac{\beta}{\lambda_1}} \right).$$

Since $0 < -\lambda_3 < \lambda_1 < -\lambda_2$, we have $0 < \alpha = -\frac{\lambda_3}{\lambda_1} < 1 < \beta = -\frac{\lambda_2}{\lambda_1}$. Let $L : S^* \to \Sigma$ be such that $L(x, y) = (y|x|^{\beta}, |x|^\alpha)$ with the convention that $L(x, y) \in \Sigma^+$ if $x > 0$ and $L(x, y) \in \Sigma^-$ if $x < 0$.

It is easy to see that $L(S^\pm)$ has the shape of a cusp like triangle without the vertex $(\pm 1, 0, 0)$.

![Figure 3. Behavior near the origin.](image)

In fact the vertex $(\pm 1, 0, 0)$ are cusp points at the boundary of each of these sets. The fact that $0 < \alpha < 1 < \beta$ together with equation (4) imply that $L(\Sigma^\pm)$ are uniformly compressed in the $y$-direction.

From now on we denote by $\Sigma^\pm$ the closure of $L(S^\pm)$. Clearly each line segment $S^* \cap \{x = x_0\}$ is taken to another line segment $\Sigma \cap \{z = z_0\}$ as sketched in Figure 3.
The sets $\Sigma^\pm$ should return to the cross section $S$ through a composition of a family of translations $T_t$, a family of expansions $E_t$ only along the $x$-direction and a family of rotations $R_t$ around $W^s(\sigma_1)$ and $W^s(\sigma_2)$, where $\sigma_i$ are saddle-type singularities of $X^t$ that are outside the cube $[-1,1]^3$, see [Araújo & Pacifico(2010)]. We assume that this composition takes line segments $\Sigma \cap \{z = z_0\}$ into line segments $S \cap \{x = x_1\}$ as sketched in Figure 3. The composition $T_t \circ E_t \circ R_t$ of linear maps describes a vector field $V$ in a region $W$ outside $[-1,1]^3$. The geometric Lorenz flow $X^t$ is then defined in the following way: for each $t \in \mathbb{R}$ and each point $x \in S$, the orbit $X^t(x)$ will start following the linear field until $\tilde{\Sigma}^\pm$ and then it will follow $V$ coming back to $S$ and so on. Let us write $\mathcal{B} = \{X^t(x), x \in S, t \in \mathbb{R}^+\}$ the set where this flow acts. The geometric Lorenz flow is then the pair $(\mathcal{B}, X^t)$ defined in this way. The set

$$\Lambda = \cap_{t \geq 0} X^t(W \cup [0,1]^3)$$

is the geometric Lorenz attractor.

The combined effects of $T \circ E \circ R$ and the linear flow given by equation (4) on lines implies that the foliation $\mathcal{F}^s$ of $S$ given by the lines $S \cap \{x = x_0\}$ is invariant under the first return map $F : S \setminus \ell \to S$. In other words, we have for any given leaf $\gamma$ of $\mathcal{F}^s$, its image $F(\gamma)$ is contained in a leaf of $\mathcal{F}^s$.

The main features of the geometric Lorenz flow and its first return map can be seen at figures 4 and 5.

![Figure 4](image)

**Figure 4.** The global cross-section for the geometric Lorenz flow and the associated 1d quotient map, the Lorenz transformation.

![Figure 5](image)

**Figure 5.** The image $F(S^*)$. 

The invariance of the foliation $\mathcal{F}^s$ by lines $S \cap \{x = x_0\}$ ensures that the Poincaré first return map $F : S \setminus \ell \to S$ can be written as a skew-product $F(x,y) = (f(x), g(x,y))$ and the one-dimensional map $f : [-1/2,0) \cup (0,1/2] \to [1/2,1/2]$ is also the quotient map of $F$ over the leaves.
of the stable foliation $\mathcal{F}^s$ defined above. It is crucial that this map is *piecewise expanding* and has a singularity at zero. More precisely, $f$ is $C^1$ on each interval $[-1/2, 0), (0, 1/2]$, its derivative is Hölder continuous and satisfies

- $f'(x) = |x|^\beta h(x)$ for $\alpha = -\lambda_3/\lambda_1$ and $h : [-1/2, 1/2] \to \mathbb{R}$ Hölder continuous (note that $f'(0^+) = f'(0^-) = +\infty$).
- $f'$ is uniformly expanding, that is, there are $C > 0$ and $\lambda > 1$ such that $|f^n'| > C\lambda^n$ for all iterates $n \geq 1$ and at all points where it is defined.

In addition

- $g$ uniformly contracts in the $y$-direction: there exists $\mu < 1$ such that $|\partial_y g| < \mu$;
- indeed, $\partial_y g(x, y) \approx x^\beta$ for $x \approx 0$ where $\beta = -\lambda_2/\lambda_1$ and since $\beta > 1 > \alpha > 0$ we have another crucial relation: $\lim_{x \to 0} \frac{\partial_y g(x, y)}{f'(x)} = 0$;

this ensures that the contraction in the $y$-direction is much stronger than the expansion near the singular line $\ell$, which ensures that the foliation $\mathcal{F}^s$ is persistent for all nearby $C^1$ flows. These properties ensure that $\Lambda$ contains a dense regular orbit and that this property persists for all $C^1$ close enough flows, that is, $\Lambda$ is *robustly transitive*.

For a detailed construction of a geometric Lorenz attractor see [Araújo & Pacifico(2010), Galatolo & Pacifico(2010)]. As mentioned above, a geometric Lorenz attractor is the simplest example of a singular hyperbolic attractor [Morales et al.(1998)Morales, Pacifico & Pujals].

Tucker in its PhD thesis [Tucker(2002)] presented a computer assisted proof that showed the existence of a rectangular cross-section $S$ for the flow of the Lorenz equations whose first return map $F$ to this cross-section $S$ satisfies the same conditions outlined above for the geometric Lorenz flow:

- the image of the $S$ (except the singular line) is contained in the interior of $S$, which shows there exists an attracting set for the flow crossing $S$;
- on $S$ there exists a field $C^u$ of cones which are invariant under the derivative of $F$ and whose vectors are expanded by a uniform rate after finitely many iterates of $F$ (the algorithm showed that 29 iterations are enough to ensure this expansion);
- the fact that the volume is contracting (the divergence of the flow of the Lorenz equations is constant and negative) together with the previous item, ensures that $F$ contracts area in $S$ and thus the field of cones $C^s$, given by the complement of the field $C^u$, is invariant for the inverse of $F$ restricted to the attracting set. This ensures that there exists a contracting foliation just like $\mathcal{F}^s$ on $S$ (this is a two-dimensional argument);
• the one-dimensional quotient map over this foliation is piecewise expanding as stated for the geometrical Lorenz case.\textsuperscript{4}

Hence the attractor of the flow of the Lorenz equations satisfies the axiomatic conditions stated in the beginning of this section and, after a suitable non-linear change of coordinates that rectifies the contracting foliation $\mathcal{F}^s$ on $S$, we obtain a flow similar to a geometric Lorenz attractor (the concrete example described above).

We stress that, in general, two such flows are neither topologically conjugate nor topologically equivalent; see for example [Guckenheimer & Williams (1979)] and [Araújo & Pacifico (2010), Chapter 3, Section 3].

4. SINGULAR HYPERBOLIC SYSTEMS

In this section we define and describe singular hyperbolic attractors. We outline a construction (a suitable Poincaré return map) which allows to rigorously investigate several properties of its dynamics. We include a list of properties of the construction and of the induced map, which are useful to obtain several results on the statistical properties of the dynamics, but can be also useful to the reader for future applications. The properties we obtain, shows that the dynamics of this general class of flows is not very different from the ones of the geometric Lorenz one, and thus with some effort is possible to generalize some result obtained for this simple model to the general ones. The idea is to consider a suitable Poincaré section and a suitable induced return map. It turns out that if both are well chosen, the properties of the return map are similar to the ones of the geometric Lorenz system described before. Then by suitable extension of the ideas which have been used for the geometric case, it is possible to treat this more general case.

We say that an attracting set $\Lambda = \Lambda_X(U)$ for a 3-flow $X$ and some open subset $U$ is robust if there exists a $C^1$ neighborhood $\mathcal{U}$ of $X$ in $\mathcal{X}^3(M)$ such that $\Lambda_Y(U)$ is transitive for every $Y \in \mathcal{U}$.

The following result obtained by Morales, Pacifico and Pujals in [Morales et al. (2004)] characterizes robust attractors for three-dimensional flows.

**Theorem 4.1.** Robust attractors for flows containing equilibria are singular-hyperbolic sets.

We remark that vector fields exhibiting robust attractors cannot be $C^1$ approximated by vector fields presenting either attracting or repelling periodic points in a neighborhood of the attractor. This implies that, on 3-manifolds, any periodic orbit inside a robust attractor is hyperbolic of saddle-type.

We now precisely define the concept of singular-hyperbolicity. A compact invariant set $\Lambda$ of $X$ is partially hyperbolic if there are a continuous invariant tangent bundle decomposition $T_x \Lambda = F^s \oplus E^c \oplus F^u$ and constants $\lambda, K > 0$ such that

1. $E^c$ is $(K, \lambda)$-dominates $E^c$, i.e. for all $x \in \Lambda$ and for all $t \geq 0$

\begin{equation}
\|DX^t(x) \mid E_x^c \| \leq \frac{e^{-\lambda t}}{K} \cdot m(DX^t(x) \mid E_x^c);
\end{equation}

where $m(L)$ for a linear map $L : (E, \| \cdot |_E) \to (F, \| \cdot |_F)$ between normed vector spaces denotes the conorm defined as $m(L) = \inf\{\|L(v)|_F : \|v|_E = 1\}$.

2. $E^c$ is $(K, \lambda)$-contracting: $\|DX^t \mid E_x^c \| \leq Ke^{-\lambda t}$ for all $x \in \Lambda$ and for all $t \geq 0$.

For $x \in \Lambda$ and $t \in \mathbb{R}$ we let $J^u_x(t)$ be the absolute value of the determinant of the linear map $DX^t(x) \mid E_x^c : E_x^c \to E_{X^t(x)}^c$. We say that the sub-bundle $E^c$ of the partial hyperbolic set $\Lambda$ is $(K, \lambda)$-volume expanding if

\[ J^u_x(t) = \left| \det(DX^t \mid E_x^c) \right| \geq Ke^{\lambda t}, \]

for every $x \in \Lambda$ and $t \geq 0$.

We say that a partially hyperbolic set is singular-hyperbolic if its singularities are hyperbolic and it has volume expanding central direction.

\textsuperscript{4}Note that we may not have that the inverse of the derivative of the one dimensional map is of bounded variation. This in general will be piecewise Hölder and of generalized bounded variation as explained in what follows.
A singular-hyperbolic attractor is a singular-hyperbolic set which is an attractor as well: an example is the geometric Lorenz attractor presented in Section 3.1; and also the attractor in the Lorenz equations (1) as a consequence of the work [Tucker(2002)] of Tucker (the work of Tucker indeed proves the existence of the attractor in that given ODE).

Any equilibrium \( \sigma \) of a singular-hyperbolic attractor for a vector field \( X \) is such that \( DX(\sigma) \) has only real eigenvalues \( \lambda_2 \leq \lambda_3 \leq \lambda_1 \) satisfying the same relations as in the Lorenz flow example:

\[
\lambda_2 < \lambda_3 < 0 < -\lambda_3 < \lambda_1,
\]

which we refer to as Lorenz-like equilibria; this is proved in [Morales et al.(2004)Morales, Pacifico & Pujals].

We recall that an compact \( X \)-invariant set \( \Lambda \) is hyperbolic if the tangent bundle over \( \Lambda \) splits \( T_\Lambda M = E^s_\Lambda \oplus E^c_\Lambda \oplus E^u_\Lambda \) into three \( DX(\cdot) \)-invariant subbundles, where \( E^u_\Lambda \) is uniformly contracted, \( E^c_\Lambda \) is uniformly expanded, and \( E^s_\Lambda \) is the direction of the flow at the points of \( \Lambda \). It is known, see [Morales et al.(2004)Morales, Pacifico & Pujals, Araújo & Pacifico(2010)], that a partially hyperbolic set for a three-dimensional flow, with volume expanding central direction and without equilibria, is hyperbolic. Hence the notion of singular-hyperbolicity is an extension of the notion of hyperbolicity.

### 4.1. Main properties.

Now we show how to work on this kind of systems, as mentioned in the introduction to this section we outline the main ideas and features of a construction, showing that there is a suitable section of the system, having a return map which preserves a contracting foliation, and has several other properties in common with the one of geometric Lorenz systems. We outline a construction which is useful to obtain the statistical properties we mention in the following sections. The construction has been modified in the literature to obtain slightly different properties (see [Araújo & Pacifico(2010)]) but the general strategy is the same.

The main idea is to obtain a family of adapted cross-sections and Poincaré maps between them which, under a suitable choice of coordinates, can be combined together to obtain a map \( F \) which has properties similar to the ones of the return map in the geometric Lorenz systems. More precisely, we will have the following.

**Theorem 4.2.** For an open and dense subset of \( C^2 \) vector fields \( X \) having a singular hyperbolic attractor \( \Lambda \) on a 3-manifold, there exists a finite family \( \Xi \) of cross-sections and a global (n-th return) Poincaré map \( \tau : \Xi_0 \rightarrow \Xi, R(x) = X_{\tau(x)}(x) \) such that

1. the domain \( \Xi_0 = \Xi \setminus \Gamma \) is the entire cross-sections with a family \( \Gamma \) of finitely many smooth arcs removed and \( \tau : \Xi_0 \rightarrow [\tau_0, +\infty) \) is a smooth function bounded away from zero by some uniform constant \( \tau_0 > 0 \).
2. We can choose coordinates on \( \Xi \) so that the map \( \tau \) can be written as \( R : \tilde{Q} \rightarrow Q, F(x,y) = (T(x),G(x,y)) \), where \( Q = \mathbb{I} \times \mathbb{C}, \mathbb{I} = [0,1] \) and \( \tilde{Q} = Q \setminus \Gamma_0 \) with \( \Gamma_0 = \mathbb{C} \times \mathbb{I} \) and \( C = \{e_1,\ldots,e_n\} \subset \mathbb{I} \) a finite set of points.
3. The map \( T : \mathbb{I} \setminus \mathbb{C} \rightarrow \mathbb{I} \) is \( C^{1+\alpha} \) piecewise monotonic with \( n+1 \) branches defined on the connected components of \( \mathbb{I} \setminus \mathbb{C} \) and has a finite set of a.c.i.m., \( \mu^T_\Gamma \). Also \( \int |T'| > 1 \) where it is defined, \( 1/|T'| \) has universal bounded \( p \)-variation and then \( d\mu^T_\Gamma/dm \) has bounded \( p \)-variation.
4. The map \( G : \tilde{Q} \rightarrow \mathbb{I} \) preserves and uniformly contracts the vertical foliation \( \mathcal{F} = \{\{x\} \times \mathbb{C}\}_{x \in \mathbb{I}} \) of \( Q \); there exists \( 0 < \lambda < 1 \) such that \( \dist(G(x,y_1),G(x,y_2)) \leq \lambda \cdot |y_1 - y_2| \) for each \( y_1, y_2 \in \mathbb{I} \). In addition, the map \( G \) satisfies \( \var^\Box(G) < \infty \).
5. The map \( F \) admits a finite family of physical probability measures \( \mu^F_\Gamma \) which are induced by \( \mu^T_\Gamma \) in a standard way. The Poincaré time \( \tau \) is integrable both with respect to each \( \mu^F_\Gamma \) and with respect to the two-dimensional Lebesgue area measure of \( Q \).
6. Moreover if, for all singularities \( \sigma \in \Lambda \), we have the eigenvalue relation \( -\lambda_2(\sigma) > \lambda_1(\sigma) \), then the second coordinate map \( G \) of \( F \) has a bounded partial derivative with respect to the first coordinate, i.e., there exists \( C > 0 \) such that \( |\partial_2 G(x,y)| < C \) for all \((x,y) \in (\mathbb{I} \setminus \{e_1,\ldots,e_n\}) \times \mathbb{I} \).

This result shows that the dynamics of singular hyperbolic attractors has several aspects in common with the one of geometric Lorenz attractors. This allows to extend some method of
proof, and obtain most of the few rigorous results on the statistical properties of the singular hyperbolic dynamics.

We now give some idea of the general construction that allows to obtain the result. Then we will show some technical details about the main steps.

The idea is to take a suitable Poincare section, which is made by a family of rectangles. Near each fixed point of the flow, we will put six rectangles like in figure 7. We know that the dynamics near the flow is similar to the one of the geometric Lorenz system. The two dynamics indeed can be identified by a local linearization of the flow near the fixed point. The linearization can have different regularity properties, according to the arithmetical properties of the eigenvalues associated to the fixed point. We will need a $C^2$ linearization. The precise requirements are outlined in theorem 4.3 below. This linearization allows to obtain precise informations on the behavior of the Poincaré maps near the fixed points, which will have (after linearization) the same form as the geometric Lorenz ones. What said allows to obtain information on the dynamics near the fixed points. Far from them we have an hyperbolic dynamics with expanding and contracting directions. We complete the set of sections we need by taking a suitable number of further sections, intercepting the flow near the attractor. We can take a finite number of them by a compactness argument. The hyperbolicity of the flow, implies that if we choose the Poincaré map in a way that the time which is necessary for an orbit to go from a rectangle to the next one is large enough, then the induced map is hyperbolic. The regularity of the flow, of the linearization, and of the sections reflects on the regularity of the induced Poincaré map, and of the preserved contracting foliation. The presence and regularity of the preserved one dimensional foliation imply that the map has the form given at item 2.

Now we present some of the main steps in the construction, see ([Araújo & Pacifico (2010)] and [Araújo et al. (2014) Araújo, Galatolo & Pacifico] ) for further details.

4.1.1. Linearization near the singularities. Here we present the assumption which are needed in the above theorem. One step which is important to obtain information on the return map, is the control of its behavior near the fixed point. This can be approached by linearization of the system near the fixed point.

We recall that, in general, hyperbolic singularities are linearizable by an Hölder homeomorphism according to the standard Hartman-Grobman Theorem [Palis & de Melo (1982), Robinson (1999)].

For our construction, we need a smoother linearization. A result in this direction is provided by Hartman (See [Hartman (2002), Theorem 12.1, p. 257]). In the absence of resonances, orbits of the flow in a small neighborhood $U_\sigma$ of the given equilibrium $\sigma$ are solutions of the linear system (3), modulo a smooth change of coordinates with $\lambda_2 < \lambda_3 < 0 < -\lambda_3 < \lambda_1$.

**Theorem 4.3.** Let $n \in \mathbb{Z}^+$ be given. Then there exists an integer $N = N(n) \geq 2$ such that: if $\Gamma$ is a real non-singular $d \times d$ matrix with eigenvalues $\gamma_1, \ldots, \gamma_d$ satisfying

$$ \sum_{i=1}^{d} m_i \gamma_i \neq \gamma_k \quad \text{for all} \quad k = 1, \ldots, d \quad \text{and} \quad 2 \leq \sum_{j=1}^{d} m_j \leq N $$

and if $\dot{\xi} = \Gamma \xi + \Xi(\xi)$ and $\dot{\zeta} = \Gamma \zeta$, where $\xi, \zeta \in \mathbb{R}^d$ and $\Xi$ is of class $C^N$ for small $\|\xi\|$ with $\Xi(0) = 0, \partial_\xi \Xi(0) = 0$; then there exists a $C^n$ diffeomorphism $R$ from a neighborhood of $\xi = 0$ to a neighborhood of $\zeta = 0$ such that $R\xi R^{-1} = \zeta_t$ for all $t \in \mathbb{R}$ and initial conditions for which the flows $\zeta_t$ and $\xi_t$ are defined in the corresponding neighborhood of the origin.

Hence it is enough for us to choose the eigenvalues $(\lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}^3$ of $\sigma$ satisfying a finite set of non-resonance relations (7) for a certain $N = N(2)$ and for each singularity $\sigma_k$ in $\Lambda$. For this condition defines an open and dense set in $\mathbb{R}^3$ and so all small $C^1$ perturbations $Y$ of the vector field $X$ will have a singularity whose eigenvalues $(\lambda_1(Y), \lambda_2(Y), \lambda_3(Y))$ are still in the $C^2$ linearizing region.

We note that in (3) $x_1$ corresponds to the strong-stable direction at $\sigma$, $x_2$ to the expanding direction and $x_3$ to the weak-stable direction.

Then for some $\delta > 0$ we may choose cross-sections contained in $U_\sigma$

- $\Sigma_{\sigma}^{\alpha, \pm}$ at points $y^\pm$ in different components of $W^{s, u}_{loc}(\sigma) \setminus \{\sigma\}$
• $\Sigma_{i,\pm}$ at points $x^\pm$ in different components of $W^s_{loc}(\sigma) \setminus W^{ss}_{loc}(\sigma)$ and Poincaré first hitting time maps $R^\pm : \Sigma_{\sigma,\pm} \setminus \ell^\pm \rightarrow \Sigma^o_{\sigma,-} \cup \Sigma^o_{\sigma,+}$, where $\ell^\pm = \Sigma_{\sigma,\pm} \cap W^s_{loc}(\sigma)$, satisfying (see Figure 7)

1. every orbit in the attractor passing through a small neighborhood of the equilibrium $\sigma$ intersects some of the incoming cross-sections $\Sigma_{i,\pm}$;
2. $R^\pm$ maps each connected component of $\Sigma_{i,\pm} \setminus \ell^\pm$ diffeomorphically inside a different outgoing cross-section $\Sigma^o_{\sigma,\pm}$, preserving the corresponding stable foliations.

Here we write $W^\ast_{loc}(\sigma), \ast = s, ss, u$ for the local invariant stable, strong-stable and unstable manifolds of the hyperbolic saddle-type singularity $\sigma$ (see e.g. [Palis & de Melo(1982)]), so that these invariant manifold extend up to the cross-sections $\Sigma_{i,\pm}$ and $\Sigma^o_{\sigma,\pm}$.

We note that at each flow-box near a singularity there are four cross-sections: two “ingoing” $\Sigma_{\sigma,\pm}$ and two “outgoing” $\Sigma^o_{\sigma,\pm}$.

Using $C^2$ linearizing coordinates in a flow-box near a singularity, with the appropriate rescaling, we can assume without loss of generality that, for a small $\delta > 0$, see Figure 7

\[
\Sigma^i_{\pm} = \{(x_1, x_2, \pm 1) : |x_1| \leq \delta, |x_2| \leq \delta\} \quad \text{and} \quad \Sigma^o_{\pm} = \{ (\pm 1, x_2, x_3) : |x_2| \leq \delta, |x_3| \leq \delta\}.
\]

Then from (3) we can determine the expression of the Poincaré maps between ingoing and outgoing cross-sections after the linearization, easily

\[
\Sigma^i_{\pm} \cap \{x_1 > 0\} \rightarrow \Sigma^0_{\pm}, \quad (x_1, x_2, 1) \mapsto (1, x_2 \cdot x_1^{-\lambda_2/\lambda_1}, x_1^{-\lambda_3/\lambda_1}).
\]

The cases corresponding to the other ingoing/outgoing pairs and signs of $x_1, x_2$ are similar. The possibility to have an explicit form (after change of variables) allows to understand several aspects of the regularity of the return map on the section.

4.1.2. Physical measure and basic properties. Here we would like to justify the existence of a physical measure for the Poincaré map, which in turn implies the existence of the physical measure for the flow. The starting point is that the induced one dimensional map being piecewise expanding has an absolutely continuous invariant measure. Outside this, there are only contracting directions, and this measure give rise to a physical measure for the two dimensional Poincaré map.

More precisely, since the one dimensional induced map is piecewise expanding, with generalized bounded variation (piecewise Holder) derivative, from [Keller(1985), Lemma 1.4]

\[\text{Lemma 4.4.} \quad \text{The one-dimensional map } T \text{ obtained above has finitely many ergodic physical measures } \mu_1^T, \ldots, \mu_l^T, \text{ whose density is a function of } p\text{-bounded variation, and whose ergodic basins cover Lebesgue almost all points of } I.\]
According to standard constructions described in [Araújo et al.(2009)] and [Araújo & Pacifico(2010)], each physical measure $\mu_T$ can be lifted to a physical measure $\nu_\Lambda$ for the flow of $X$ and supported on the attractor $\Lambda$: more on this in Proposition 4.6 of Subsection 4.1.3. Since a singular-hyperbolic attractor is transitive, that is, it has a dense orbit, it follows that there can be only one such physical measure for the flow in the basin of attraction of $\Lambda$: see [Araújo & Pacifico(2010)], Section 7.3.8, pp. 234-235. Therefore we have (see [Araújo et al.(2009)])

**Theorem 4.5.** Let $\Lambda = \Lambda_X(U)$ be a singular-hyperbolic attractor of a flow $X \in \mathcal{X}^2(M)$ on a three-dimensional manifold. Then $\Lambda$ supports a unique physical probability measure $\mu$ which is ergodic and its ergodic basin covers a full Lebesgue measure subset of the topological basin of attraction, i.e., $B(\mu) = W^s(\Lambda)$ Lebesgue mod 0. Moreover the support of $\mu$ is the whole attractor $\Lambda$.

4.1.3. Integrability of $\tau$, $\log |T'|$ and $\log |\partial_y G|$. The global Poincaré time $\tau$ is integrable with respect to both the two-dimensional Lebesgue measure $m$ on $\mathbb{Q}$ and the $F$-invariant physical measure $\mu_F$ on $\mathbb{Q}$, which lifts to the physical measure $\nu$ for the flow on the singular-hyperbolic attractor and itself is a lift of the $T$-invariant absolutely continuous probability measure $\mu_T$ on $\mathbb{I}$.

**Proposition 4.6.** The global Poincaré time $\tau$ is integrable with respect to the $F$-invariant physical probability measure $\mu_F$ and with respect to $m$.

Some other integrability properties will be needed and can be obtained using the properties of the maps $T$ and $G$; see [Araújo et al.(2014)] for more details.

**Proposition 4.7.** We have the following properties:

1. $0 < \int \log |T'| \, d\mu_F < \infty$;
2. $\int - \log |\partial_y G(x,y)| \, d\mu_F < \infty$;
3. the maps $y \mapsto \partial_y G(x,y)$ are uniformly equicontinuous for $x \in \mathbb{I} \setminus \{c_1, \ldots, c_n\}$, i.e., outside the singularities of the map $T$.

4.2. Consequences of absence of sinks and sources nearby. The proof of Theorem 4.1 given in [Morales et al.(2004)] uses several tools from the theory of normal hyperbolicity developed first by Mañé in [Mañé(1982)] together with the low dimension of the flow.

Lorenz-like equilibria are the only ones contained in robust attractors naturally, since they are the only kind of equilibria in a 3-flow which cannot be perturbed into saddle-connections which generate sinks or sources when unfolded. We note that since this kind of hyperbolic fixed points $\sigma$ for vector fields $X$ has only real eigenvalues and a negative real eigenvalue $\lambda_2$ strictly smaller than the rest of the spectrum of the tangent map $DX(\sigma)$, then it is well-known that there exists an invariant strong-stable manifold through the fixed point and tangent to the one-dimensional eigenspace corresponding to $\lambda_2$.

**Proposition 4.8.** Let $\Lambda$ be a robustly transitive set of $X \in \mathcal{X}^1(M)$. Then, either for $Y = X$ or $Y = -X$, every singularity $\sigma \in \Lambda$ is Lorenz-like for $Y$ and satisfies $W^{ss}_Y(\sigma) \cap \Lambda = \{\sigma\}$.

The following shows in particular that the notion of singular hyperbolicity is an extension of the notion of hyperbolicity.

**Proposition 4.9.** Let $\Lambda$ be a singular hyperbolic compact set of $X \in \mathcal{X}^1(M)$. Then any invariant compact set $\Gamma \subset \Lambda$ without singularities is uniformly hyperbolic.

A consequence of Proposition 4.9 is that every periodic orbit of a singular hyperbolic set is hyperbolic. The existence of a periodic orbit in every singular-hyperbolic attractor was proved recently in [Bautista & Morales(2006)] and also a more general result was obtained in [Arroyo & Pujals(2007)].

**Proposition 4.10.** Every singular hyperbolic attractor $\Lambda$ has a dense subset of periodic orbits.

In the same work [Arroyo & Pujals(2007)] it was announced that every singular hyperbolic attractor is the homoclinic class associated to one of its periodic orbits. Recall that the *homoclinic*
class of a periodic orbit $O$ for $X$ is the closure of the set of transversal intersection points of it stable and unstable manifold: $H(O) = \overline{W^u(O) \cap W^s(O)}$. This result is well known for the elementary dynamical pieces of uniformly hyperbolic attractors. Moreover, in particular, the geometric Lorenz attractor is a homoclinic class as proved in [Bautista(2004)]. A proof of this property for every singular-hyperbolic attractor is given in [Araújo & Pacifico(2010)].

**Proposition 4.11.** A singular-hyperbolic attractor $\Lambda$ for a three-dimensional vector field $X$ is the homoclinic class $H(O)$ of a hyperbolic periodic orbit $O$ of $\Lambda$.

5. **More on the Ergodic Theory of Singular-hyperbolic attractors**

The ergodic theory of singular-hyperbolic attractors is incomplete. Many results still are proved only in the particular case of (axiomatic) geometric Lorenz flows (sometimes with additional assumptions) and several automatically extend to the original Lorenz flow after the work of Tucker [Tucker(2002)], but demand an extra effort to encompass the full singular-hyperbolic setting.

We note that a singular-hyperbolic attractor in general contains finitely many hyperbolic singularities and does not admit a single connected cross-section which is crossed by all orbits except the singularities, as is the case of the geometrical Lorenz attractor and the attractor of the Lorenz system of equations.

5.1. **The physical measure is a $u$-Gibbs state.** It follows from the proof of Theorem 4.5, in [Araújo et al.(2009)Araújo, Pujals, Pacifico & Viana] that the singular-hyperbolic attracting set $\Lambda_Y(U)$ for all $Y \in \mathcal{X}^2(M)$ which are $C^1$-close enough to $X$ admits finitely many physical measures whose ergodic basins cover $U$ except for a zero volume subset. We note that a singular-hyperbolic attractor is not necessarily robustly transitive: examples of this behavior are known; see [Araújo & Pacifico(2010), Example 5.7].

Theorem 4.5 shows that typical orbits in the basin of every singular-hyperbolic attractor, for a $C^2$ flow $X$ on a 3-manifold, have well-defined statistical behavior, i.e. for Lebesgue almost every point the forward Birkhoff time average converges, and it is given by a certain physical probability measure $\mu$. It was also obtained that this measure is hyperbolic and admits absolutely continuous conditional measures along the center-unstable directions on the attractor. As a consequence, it is a $u$-Gibbs state and an equilibrium state for the flow.

Here hyperbolicity of the invariant measure $\mu$ means non-uniform hyperbolicity of the probability measure $\mu$: the tangent bundle over $\Lambda$ splits into a sum $T_z\Lambda = E^u_z \oplus E^s_z \oplus F_z$ of three one-dimensional invariant subspaces defined for $\mu$-a.e. $z \in \Lambda$ and depending measurably on the base point $z$, where $\mu$ is the physical measure in the statement of Theorem 4.5, $E^u_z$ is the flow direction (with zero Lyapunov exponent) and $F_z$ is the direction with positive Lyapunov exponent, that is, for every non-zero vector $v \in F_z$, we have

$$\lim_{t \to +\infty} \frac{1}{t} \log \|DX^t(z) \cdot v\| > 0.$$ 

We note that the invariance of the splitting implies that $E^u_{z\cdot} = E^u_z \oplus F_z$ whenever $F_z$ is defined.

Theorem 4.5 is another statement of sensitiveness, this time applying to the whole essentially open set $B(\Lambda)$. Indeed, since non-zero Lyapunov exponents express that the orbits of infinitesimally close-by points tend to move apart from each other, this theorem means that most orbits in the basin of attraction separate under forward iteration. See Kifer [Kifer(1988)], and Metzger [Metzger(2000)], and references therein, for previous results about invariant measures and stochastic stability of the geometric Lorenz models.

The $u$-Gibbs property of $\mu$ is stated as follows.

**Theorem 5.1.** Let $\Lambda$ be a singular-hyperbolic attractor for a $C^2$ three-dimensional flow. Then the physical measure $\mu$ supported in $\Lambda$ has a disintegration into absolutely continuous conditional measures $\mu_{\gamma}$ along center-unstable surfaces $\gamma$ such that $\frac{d\mu_{\gamma}}{d\gamma}$ is uniformly bounded from above. Moreover $\text{supp}(\mu) = \Lambda$. 


5.2. Entropy formula. Here the existence of unstable manifolds is guaranteed by the hyperbolicity of the physical measure: the strong-unstable manifolds $W^{uu}(z)$ are the “integral manifolds” in the direction of the one-dimensional sub-bundle $F_z$, tangent to $F_z$ at almost every $z \in \Lambda$. The sets $W^{uu}(z)$ are embedded sub-manifolds in a neighborhood of $z$ which, in general, depend only measurably (including its size) on the base point $z \in \Lambda$.

We remark that since $\Lambda$ is an attracting set, then $W^{uu}(z) \subset \Lambda$ whenever defined. The central unstable surfaces mentioned in the statement of Theorem 5.1 are just small strong-unstable Mané (1987) measurable (including its size) on the base point.

The physical measure $\mu$ satisfies the conditions of Theorem 5.1 ensures that
\[
\int \log |\det(DX^1 | E^{cu})| \, d\mu,
\]
by the characterization of probability measures satisfying the Entropy Formula, obtained in [Ledrappier & Young (1985)]. The above integral is the sum of the positive Lyapunov exponents along the sub-bundle $E^{cu}$ by Oseledets Theorem [Mané (1987), Walters (1982)]. Since in the direction $E^{cu}$ there is only one Lyapunov exponent along the one-dimensional direction $F_z$, $\mu$-a.e. $z$, the ergodicity of $\mu$ then shows that the following is true.

**Corollary 5.2.** If $\Lambda$ is a singular-hyperbolic attractor for a $C^2$ three-dimensional flow $X^t$, then the physical measure $\mu$ supported in $\Lambda$ satisfies the Entropy Formula
\[
\int \log \|DX^1 | F_z\| \, d\mu(z).
\]

From the characterization of measures satisfying the Entropy Formula given in [Ledrappier & Young (1985)], we see that $\mu$ has absolutely continuous disintegration along the strong-unstable direction, along which the Lyapunov exponent is positive, thus $\mu$ is a $u$-Gibbs state [Pesin & Sinai (1982)]. This also shows that $\mu$ is an equilibrium state for the potential $-\log \|DX^1 | F_z\|$ with respect to the diffeomorphism $X^1$. We note that the entropy $h_\mu(X^1)$ of $X^1$ is the entropy of the flow $X^t$ with respect to the measure $\mu$ [Walters (1982)].

Hence we are able to extend a basic result on the ergodic theory of smooth hyperbolic attractors to the setting of smooth singular-hyperbolic attractors: a hyperbolic attractor of a $C^2$ diffeomorphism admits a physical measure which is the only equilibrium state with respect to the potential $\log \|\det(Df | E^u)\|$ given by the norm of the Jacobian of the map along the unstable directions at the attractor.

5.3. Exact dimensionality. We recall a result of Steinberger [Steinberger (2000)] about the local dimension of Lorenz like systems and prove that for the singular hyperbolic system the local dimension is defined at almost every point.

Let us consider a map $F : Q \to Q$, $F(x, y) = (T(x), G(x, y))$ where

1. $T : [0, 1] \to [0, 1]$ is piecewise monotonic: there are $c_i \in [0, 1]$ for $0 \leq i \leq N$ with $0 < c_0 < \cdots < c_N = 1$ such that $T(c_i, c_{i+1})$ is continuous and monotone for $0 \leq i < N$. Furthermore, for $0 \leq i < N$, $T(c_i, c_{i+1})$ is $C^1$ and that $\inf_{x \in \mathcal{P}} |T'(x)| > 0$ holds where
   \[
   \mathcal{P} = [0, 1] \setminus \bigcup_{0 \leq i < N} c_i.
   \]
2. $G : [0, 1] \to (0, 1)$ is $C^1$ on $\mathcal{P} \times [0, 1]$. Furthermore, $\sup |\partial G/\partial x| < \infty$, $\sup |\partial G/\partial y| < 1$ and $|\partial G/\partial y(x, y)| > 0$ for $(x, y) \in \mathcal{P} \times [0, 1]$.
3. $F((c_i, c_{i+1}) \times [0, 1]) \cap F((c_j, c_{j+1}) \times [0, 1]) = \emptyset$ for distinct $i, j$ with $0 \leq i, j < N$.

Now consider the projection $\pi_x : Q \to I$, set $\mathcal{V} = \{(c_i, c_{i+1}), 1 \leq i \leq N\}$ and $\mathcal{V}_k = \bigcap_{i=0}^{k} f^{-i} \mathcal{V}$. For $x \in E$ let $J_k(x)$ be the unique element of $\mathcal{V}_k$ which contains $x$. We say that $\mathcal{V}$ is a generator if the length of the intervals $J_k(x)$ tends to zero for $n \to \infty$ for any given $x$. In piecewise expanding maps it is easy to see that $\mathcal{V}$ is a generator. Set
\[
(9) \quad \psi(x, y) = \log |T'(x)| \quad \text{and} \quad \varphi(x, y) = -\log |\partial G/\partial y(x, y)|.
\]

The result of Steinberger that we shall use is the following
Theorem 5.3. [Steinberger(2000), Theorem 1] Let $F$ be a two-dimensional map as above and $\mu$ an ergodic, $F$-invariant probability measure on $\mathbb{I}$ with the entropy $h_\mu(F) > 0$. Suppose $\mathcal{V}$ is a generator, $\int \varphi \cdot d\mu_F < \infty$ and $0 < \int \psi d\mu_F < \infty$. If the maps $y \mapsto \varphi(x, y)$ are uniformly equicontinuous for $x \in \mathbb{I} \setminus \{0\}$ and $1/|T|$ has finite universal $p$-Bounded Variation, then

$$d_\mu(x, y) = h_\mu(F) \left( \frac{1}{\int \psi \cdot d\mu} + \frac{1}{\int \varphi \cdot d\mu} \right)$$

for $\mu$-almost all $(x, y) \in \mathbb{I}$.

Item (3) above is satisfied in our case because the map is induced by a first return Poincaré map induced by a flow. Moreover $\sup |\partial G/\partial x| < \infty$ in item (2) above is established at item (6) of Theorem 4.2, provided that for all equilibria $\sigma \in \Lambda$ we have the eigenvalue relation $-\lambda_2(\sigma) > \lambda_1(\sigma)$.

Let us also observe that, for the first return map $F: \mathbb{I} \setminus \Gamma \to \mathbb{I}$, associated to the singular-hyperbolic flow, the entropy is positive $h_\mu(F) > 0$. Indeed, since we know that $\pi \circ F = T \circ \pi$, where $\pi: \mathbb{I} \to \mathbb{I}$ is the projection on the first coordinate, and that $h_{\mu_F}(T) = \int \log |D\pi| \, d\mu_T > 0$, where $\mu_T$ is the unique absolutely continuous $T$-invariant probability measure, we see that $h_{\mu_F}(F) > 0$.

So, all we need to prove that $(\Xi, F, d\mu_F)$ is exact dimensional is to verify that $F(x, y)$ satisfies the hypothesis of Theorem 5.3, where $F: \Xi \to \Xi$ is the Poincaré return map to the family of cross-sections $\Xi$ described in Section 4; and $\mu_F$ is the $F$-invariant ergodic SRB measure induced on $\Xi$ by the physical measure of the attractor.

However, Proposition 4.7 provides precisely that for the functions $\varphi, \psi$ defined above in (9): we have

1. $\int \varphi d\mu_F < \infty$;
2. $0 < \int \psi d\mu_F < \infty$; and
3. the maps $y \mapsto \varphi(x, y)$ are uniformly equicontinuous for $x \in \mathbb{I} \setminus \{c_1, \ldots, c_n\}$.

This all together finishes the proof of Theorem 5.3 establishing that $\mu_F$ is exact dimensional.

The exact dimensionality of the measure on the section implies the exact dimensionality of the measure $\mu$ on the flow at almost each point, and the dimension satisfies $d_\mu(x) = d_{\mu_F}(x) + 1$ at almost every point $x$.

5.4. Decay of correlations for Poincaré maps on singular-hyperbolic attractors. After obtaining an interesting invariant probability measure for a dynamical system the next thing to do is to study the properties of this measure. Besides ergodicity there are various degrees of mixing (see e.g. [Walters(1982), Mané(1987)]).

5.4.1. Decay of correlations for maps versus flows. Given a flow $X$ and an invariant ergodic probability measure $\mu$, we say that the system $(X, \mu)$ is mixing if for any two measurable sets $A, B$

$$\mu(A \cap X^{-t}B) \xrightarrow{t \to \infty} \mu(A) \cdot \mu(B)$$

or equivalently

$$\int \varphi \cdot (\psi \circ X^t) \, d\mu \xrightarrow{t \to \infty} \int \varphi \, d\mu \int \psi \, d\mu$$

for any pair $\varphi, \psi : \mathbb{I} \to \mathbb{R}$ of continuous functions.

Considering $\varphi$ and $\psi \circ X^t : \mathbb{I} \to \mathbb{R}$ as random variables over the probability space $(\mathbb{I}, \mu)$, this definition just says that “the random variables $\varphi$ and $\psi \circ X^t$ are asymptotically independent” since the expected value $E(\varphi \cdot (\psi \circ X^t))$ tends to the product $E(\varphi) \cdot E(\psi)$ when $t$ goes to infinity. The correlation function

$$C_t(\varphi, \psi) = |E(\varphi \cdot (\psi \circ X^t)) - E(\varphi) \cdot E(\psi)|$$

(11)

satisfies $C_t(\varphi, \psi) \xrightarrow{t \to \infty} 0$ in this case. The rate of approach to zero of the correlation function is called the rate of decay of correlations for the observables $\varphi$ and $\psi$ of the system $(X, \mu)$. 
The study of decay of correlations for hyperbolic systems goes back to the work of Sinai [Sinai(1972)] and Ruelle [Ruelle(1976)]. Many results were obtained for transformations. For a diffeomorphism \( f \) the notion of decay of correlations is the same as above replacing \( X^1 \) by \( f^n \) and letting \( n \) go to infinity. Since [Bowen(1975), Ruelle(1976)] it is known that the physical (SRB) measures for Axiom A diffeomorphisms are mixing and have exponential decay of correlations, that is, there exists a constant \( \alpha \in (0, 1) \) such that given \( \varphi \) and \( \psi \) there exists \( C = C(\varphi, \psi) > 0 \) such that

\[
C_n(\varphi, \psi) \leq C \cdot e^{-\alpha n}
\]

for all \( n \geq 1 \), for a suitable class of continuous functions \( M \rightarrow \mathbb{R} \), in this case the Hölder continuous functions.

In more general cases for smooth endomorphisms (see e.g. [Holland(2005), Alves et al.(2005)] Alves, Luzzatto & Pinheiro and references therein) where the inverse in (10) is to be taken as the inverse image of \( f^n \), it is possible to have slower rates of decay.

In contrast to the results available in the case of discrete dynamical systems, obtaining the rate of decay of correlations for flows seems to be much more complex and some results have been established for Anosov flows only recently. Ergodicity and mixing for geodesic flows on manifolds of negative curvature is known since the early half of the XXth century [Hopf(1939), Anosov & Sinai(1967), Sinai(1960)].

The proof of exponential decay of correlations for geodesic flows on manifolds of constant negative curvature was first obtained in two [Collet et al.(1984)Collet, Epstein & Gallavotti, Moore(1987), Ratner(1987)] and three dimensions [Pollicott(1992)] through group theoretical arguments.

5.4.2. Decay of correlations for fiber contracting maps. In [Araújo et al.(2014)Araújo, Galatolo & Pacífico] we establish results on the decay of correlations and convergence to equilibrium for fiber contracting maps with a fastly converging to equilibrium base, which imply the following statement (and this in turn is then applied to singular hyperbolic attractors).

We recall that a measurable map \( h : [a, b] \rightarrow \mathbb{R} \) is of universal \( p \)-bounded variation if

\[
\sup_{a = a_0 < a_1 < \cdots < a_n = b} \left( \sum_{i=1}^{n} |h(a_i) - h(a_{i-1})|^{1/p} \right)^{p} < \infty,
\]

where the supremum is taken over all finite partitions of the interval \( I = [a, b] \).

We will need another definition of variation for maps with two variables. Similarly to the one dimensional case, if \( f : Q \rightarrow \mathbb{R} \) and \( x_1 \leq x_2 \leq \cdots \leq x_n \), let us define

\[
\varbar{\text{var}}(f, x_1, \ldots, x_n, y_1, \ldots, y_n) = \sum_{1 \leq i \leq n} |f(x_i, y_i) - f(x_{i+1}, y_i)|.
\]

We then consider the supremum \( \varbar{\text{var}}(f, x_1, \ldots, x_n, y_1, \ldots, y_n) \) over all subdivisions \( x_i \) and all choices of the \( y_i \)

\[
\varbar{\text{var}}(f) = \sup_n \left( \frac{\sup_{(x_i, \cdots, x_n) \in I, (y_i) \in I} \varbar{\text{var}}(f, x_1, \ldots, x_n, y_1, \ldots, y_n)}{n} \right).
\]

We recall that we denote by \( Q = \mathbb{I} \times \mathbb{I} \) the unit square, where \( \mathbb{I} = [0, 1] \). For a function \( g : Q \rightarrow \mathbb{R} \) we denote by \( L(g) \) the best Lipschitz constant of \( g \), that is, \( L(g) = \sup_{p, q \in Q} \frac{|g(p) - g(q)|}{|p - q|} \), where \( | \cdot | \) is the Euclidean distance. We define the Lipschitz norm by setting \( ||g||_{\text{lip}} = ||g||_{\infty} + L(g) \) where, as usual, \( ||g||_{\infty} = \text{ess sup}_{p \in Q} |g(p)| \) and set \( \text{Lip}(Q) = \{ g : Q \rightarrow \mathbb{R} : ||g||_{\text{lip}} < \infty \} \).

Theorem 5.4. Let us consider a map \( F : Q \setminus \Gamma \) from the unit square into itself such that:

1. \( F \) has the form \( F(x, y) = (T(x), G(x, y)) \) (is a skew-product and preserves the natural vertical foliation of the square);
2. \( F|_{\Gamma} \) is \( \lambda \)-Lipschitz with \( \lambda < 1 \) (hence is uniformly contracting) on each leaf \( \gamma \) of the vertical foliation of the square;
3. \( \varbar{\text{var}}(G) < \infty \);
4. \( T : \mathbb{I} \setminus \Gamma \) is piecewise monotonic, with \( n + 1 \), \( C^1 \) increasing branches on the intervals \( (0, c_1), \ldots, (c_n) \) and \( \inf_{x \in \mathbb{I}} |T'(x)| > 1 \).
5. \( \frac{1}{T'} \) has finite universal \( p \)-bounded variation (as defined above);
(6) \( T \) has only one absolutely continuous (w.r.t. Lebesgue on \( I \)) invariant probability measure (a.c.i.m.) for which it is weakly mixing.

Then the unique physical measure \( \mu_F \) of \( F \) has exponential decay of correlation with respect to Lipschitz observables, that is, there are \( C, \Lambda \in \mathbb{R}^+ \), \( \Lambda < 1 \), such that
\[
\left| \int f \cdot (g \circ F^n) \, d\mu_F - \int g \, d\mu_F \int f \, d\mu_F \right| \leq CA^n\|g\|_{\text{Lip}}\|f\|_{\text{Lip}}, \quad f, g \in \text{Lip}(Q).
\]

We remark that items (4) to (6) of the assumptions on the above theorem can be replaced by (more general) exponential convergence to equilibrium on the base map under suitable observables.

5.4.3. Decay of correlations for the Poincaré return map of singular-hyperbolic attractors. We now apply these results to singular-hyperbolic attractors for three-dimensional flows. We let \( SH^2(M^3) \) be the family of all \( C^2 \) vector fields \( X \) on a compact three-manifold having an open trapping region \( U \), i.e., \( \overline{X^t(U)} \subset U \) for all \( t > 0 \), such that its maximal invariant subset \( \Lambda = \cap_{t>0}X^t(U) \) is a compact transitive singular-hyperbolic set. We consider the \( C^2 \) topology of vector fields in \( SH^2(M^3) \) in what follows.

The proof of the existence of physical measure in Theorem 5.1 and Corollary 5.2 is based on a construction that we describe in the next subsection, and which suitably explored yields the assumptions on Theorem 5.4.

This construction shows that there exists a finite family \( \Xi \) of well-adapted cross-sections of the flow on the attractor where we can define a Poincaré return map \( F \) which satisfies the properties in the statement of Theorem 5.4 after a suitable choice of coordinates. We remark that, to take advantage of a result from [Steinberger(2000)] on exact dimensionality of certain classes of measures, we need that the Poincaré return map \( F \) be injective, which is not evident in the construction and choice of these cross-sections at [Araújo et al.(2009),Araújo, Pujals, Pacifico & Viana]. Because of this, the construction presented here is slightly different; see Subsection 4.1 for details.

We need some conditions on the eigenvalues of the equilibria of \( X \) inside \( \Lambda \) to reduce the dynamics on \( \Lambda \) to a map \( F \) as in Theorem 5.4, to obtain

**Corollary 5.5.** There exists an open dense set \( A \) of vector fields in \( SH^2(M^3) \) such that, for each \( X \in A \), we can find a finite family \( \Xi \) of cross-sections to the flow \( X_t \) of \( X \) whose Poincaré first return map \( F : \text{dom}(F) \subset \Xi \to \Xi \) has a unique SRB measure \( \mu_F \) which has exponential decay of correlations with respect to Lipschitz observables: there are \( C, \Lambda \in \mathbb{R}^+ \), \( \Lambda < 1 \) satisfying for every pair \( f, g : \Xi \to \mathbb{R} \) of Lipschitz functions
\[
\left| \int f \cdot (g \circ F^n) \, d\mu_F - \int g \, d\mu_F \int f \, d\mu_F \right| \leq CA^n\|g\|_{\text{Lip}}\|f\|_{\text{Lip}}, \quad n \geq 1.
\]

We remark that, since the works of Ruelle [Ruelle(1983)] and Pollicott [Pollicott(1999)] it is well-known that exponentially mixing for a base transformation of a suspension flow does not imply fast mixing for the suspension flow. In fact, in the suspension flow can be non-mixing! Hence we cannot deduce any kind of mixing results for the flow on a singular-hyperbolic attractor from Corollary 5.5. However, in a recent work of one of the authors with Varandas [Araújo & Varandas(2012)], described at Section 6.2.2, it has been proved the existence of a \( C^2 \) open subset of vector fields having a geometrical Lorenz attractor with exponential decay of correlations for the flow on \( C^1 \) observables, from which follows the exponential decay of correlations for the corresponding Poincaré map. But this \( C^2 \) open subset was obtained under very strong conditions which cannot hold in such generality as in Corollary 5.5. In another recent work [Araújo et al.(2013)]Araújo, Melbourne & Varandas by the same authors together with Melbourne, it was proved that all \( C^\infty \) geometric Lorenz attractors (including the attractor for the system of equations (1)) have superpolynomial decay of correlations, which provides rapid decay of correlations for the corresponding Poincaré map, but still slower than exponential.

5.5. Logarithm law for singular hyperbolic attractors. The decay of correlation for the return map on the section and the exact dimensionality, imply an estimation for the behavior of hitting times of shrinking targets which is called Logarithm law. This result essentially says that the time needed to hit a small target scales as the inverse of the measure of the target.
Let us recall the a result about discrete time systems we will use: let \((X, F, \mu)\) be an ergodic, measure preserving transformation on a metric space \(X\). Let us consider a family of target sets \(S_r\) indexed by a real parameter \(r\) and the time needed for the orbit of a point \(x\) to enter in \(S_r\)

\[\tau_F(x, S_r) = \min \{ n \in \mathbb{N}^+ : F^n(x) \in S_r \}.\]

We consider target sets of the form: \(S_r = \{ x \in X , f(x) \leq r \}\), where \(f : X \to \mathbb{R}^+\) is a Lipschitz function, together with the limits

\[\overline{d}(f) = \limsup_{r \to 0} \frac{\log \mu(S_r)}{-\log r}, \quad \underline{d}(f) = \liminf_{r \to 0} \frac{\log \mu(S_r)}{-\log r}\]

representing a sort of local dimension (the formula for the local dimension of \(\mu\) at a point \(x_0\) is obtained when \(f(x) = d(x, x_0)\)). When the above limits coincide, we set \(d(f) = \overline{d}(f) = \underline{d}(f)\). In this setting, the following result is proved in [Galatolo(2010)]; see also [Galatolo(2007)].

**Proposition 5.6.** Let \(f\) and \(S_r\) be as above. Then for \(\mu\)-almost every \(x\)

\[\limsup_{r \to 0} \frac{\log \tau_F(x, S_r)}{-\log r} \geq \overline{d}(f), \quad \liminf_{r \to 0} \frac{\log \tau_F(x, S_r)}{-\log r} \geq \underline{d}(f).\]

Moreover, if the system has super-polynomial decay of correlations under Lipschitz observables and \(d(f)\) exists, then for \(\mu\)-almost every \(x\) it holds

\[\lim_{r \to 0} \frac{\log \tau_F(x, S_r)}{-\log r} = d(f).\]

**Remark 5.7.** Since we are dealing with a ratio of logarithms, and 14 always hold, if we establish the Logarithm Law 15 for some iterate \(F^n\), then it will hold also for \(F\).

Let us now see how to extend the result to flows. Let \(X\) be a metric space, \(\Phi^t\) be a measure preserving flow and \(\Sigma\) be a section of \((X, \Phi^t)\). If the flow is ergodic and the return time is integrable, then the hitting time scaling behavior of the flow can be estimated by the one of the system induced on the section. Hence we can have a logarithm law for the flow if we can prove it on the section (with the induced return map).

Given any \(x \in X\) let us denote by \(t(x)\) the smallest strictly positive time such that \(\Phi^{t(x)}(x) \in \Sigma\). We also consider \(t'(x)\), the smallest non negative time such that \(\Phi^{t'(x)}(x) \in \Sigma\). We define \(\pi : X \to \Sigma\) as \(\pi(x) = \Phi^{t'(x)}(x)\), the projection on \(\Sigma\). We also denote by \(\mu_F\) the invariant measure for the Poincaré map \(F\) which is induced by the invariant measure \(\mu\) of the flow.

**Proposition 5.8** ([Galatolo & Nisoli(2011)]). Let us suppose that the flow \(\Phi^t\) is ergodic and has a section \(\Sigma\) with an induced map \(F\) and invariant measure \(\mu_F\) such that \(\int_\Sigma t(x) \, d\mu_F < \infty\). Let \(r \geq 0\) and \(S_r \subseteq \Sigma\) be a decreasing family of measurable subsets with \(\lim_{r \to 0} \mu_F(S_r) = 0\). Let us consider the hitting time relative to the Poincaré map

\[\tau^\Sigma(x, S_r) = \min \{ n \in \mathbb{N}^+ : F^n(x) \in S_r \}.\]

Then, there is a full measure set \(C \subseteq X\) such that if \(x \in C\)

\[\liminf_{r \to 0} \frac{\log \tau(x, S_r)}{-\log r} = \liminf_{r \to 0} \frac{\log \tau^\Sigma(\pi(x), S_r)}{-\log r},\]
\[\limsup_{r \to 0} \frac{\log \tau(x, S_r)}{-\log r} = \limsup_{r \to 0} \frac{\log \tau^\Sigma(\pi(x), S_r)}{-\log r}.\]

We proved in the previous subsections that the physical invariant measure is exact dimensional and has exponential decay of correlations. Hence by Proposition 5.6 a logarithm law must hold for an iterate of the first return map on the section. As remarked above, once we have the logarithm law for the iterate of the first return map, we obtain it for the first return map. By Remark 5.7 and Proposition 5.8 we have the possibility to extend the logarithm law to the flow.
Theorem 5.9. Let \( \Phi^t : X \rightarrow X \) be a flow having a singular hyperbolic attractor, and let us consider its physical invariant measure \( \mu \). Let us consider \( x_0 \) and the local dimension at \( x_0 \) (which was above proved to exist)

\[
\lim_{r \rightarrow 0} \frac{\log \mu(B_r(x_0))}{\log r},
\]

then for \( \mu \) almost every \( x \)

\[
\lim_{r \rightarrow 0} \frac{\log \tau(x, B_r(x_0))}{-\log r} = d_\mu(x_0) - 1
\]

where \( \tau(x, B_r(x_0)) \) is the time needed for the orbit of \( x \) to hit the ball \( B_r(x_0) \) as above.

Of course, noting that Proposition 5.6 holds for targets which are sublevels of Lipschitz functions, it is possible to give other statements, with the same methods, replacing balls with more general shrinking targets.

6. Ergodic theoretic results for the geometric Lorenz attractor

6.1. Large Deviations for Lebesgue measure on a neighborhood of a geometric Lorenz flow. Having shown that physical probability measures exist, it is natural to consider the rate of convergence of the time averages to the space average, measured by the volume of the subset of points whose time averages stay away from the space average by a prescribed amount up to some evolution time. More precisely, if we set \( \epsilon > 0 \) as an error margin and consider

\[
B_\epsilon = \{ z : \frac{1}{t} \int_0^t \psi(X^s(z)) - \int_0^t \psi \, d\mu > \epsilon \}
\]

then we search conditions under which the Lebesgue measure of \( B_\epsilon \) decays to zero exponentially fast, i.e. weather there are constants \( C, \xi > 0 \) such that

\[
\lambda(B_\epsilon) \leq Ce^{-\xi t} \quad \text{for all} \quad t > 0.
\]

The values of \( C, \xi > 0 \) above depend on \( \epsilon, \psi \) and on global invariants for dynamics.

An extension of part of the results on large deviation rates of Kifer [Kifer(1990)] from the hyperbolic setting to semiflows over non-uniformly expanding base dynamics and unbounded roof function was obtained in [Araújo(2007)]. These special flows model non-uniformly hyperbolic flows like the flow on singular-hyperbolic attractors. Related results (in fact, sharper) where obtained by Melbourne and Nicol [Melbourne & Nicol(2008)] for suspension flows over Markov towers assuming that the roof function is bounded with respect to the physical probability measure of these systems.

6.1.1. Suspension semiflows. We first present these flows, present the general strategy of the approach and then state the main assumptions related to the modelling of the geometric Lorenz attractor.

Given a Hölder-\( C^1 \) local diffeomorphism \( f : M \setminus S \rightarrow M \) outside a volume zero singular set \( S \), we say that \( S \) is non-flat if \( f \) behaves like a power of the distance to \( S : \| Df(x) \| \approx \text{dist}(x, S)^{-\beta} \) for some \( \beta > 0 \); see Alves-Araújo [Alves & Araújo(2004)] for a precise statement.

Let also \( X^r : M_r \rightarrow M_r \) be a semiflow with roof function \( r : M \setminus S \rightarrow \mathbb{R} \) over the base transformation \( f \), as follows.

Where \( M_r = \{(x, y) \in M \times [0, +\infty) : 0 \leq y < r(x)\} \) and \( X^0 \) is the identity on \( M_r \), where \( M \) is a compact Riemannian manifold. For \( x = x_0 \in M \) denote by \( x_n \) the \( n \)th iterate \( f^n(x_0) \) for \( n \geq 0 \). Denote \( S_n^r \varphi(x_0) = \sum_{j=0}^{n-1} \varphi(f^j(x)) \) for \( n \geq 1 \) and for any given real function \( \varphi \). Then for each pair \( (x_0, s_0) \in X_r \) and \( t > 0 \) there exists a unique \( n \geq 1 \) such that \( S_n^r r(x_0) \leq s_0 + t < S_{n+1}^r r(x_0) \) and define (see Figure 8)

\[
X^t(x_0, s_0) = (x_n, s_0 + t - S_n^r r(x_0)).
\]
The study of suspension (or special) flows is motivated by modeling a flow admitting a cross-section. Such flow is equivalent to a suspension semiflow over the Poincaré return map to the cross-section with roof function given by the return time function on the cross-section. This is a main tool in the ergodic theory of hyperbolic flows developed by Bowen and Ruelle [Bowen & Ruelle(1975)].

The general strategy of the approach is to consider a suspension semiflow over a piecewise expanding base transformation with a singular point just like the singularity of the one-dimensional Lorenz transformation. The result on large deviations for the Lebesgue measure (note that this measure is not an invariant measure for the flow!) is proved for this class of suspension flows under suitable conditions on the base map.

The uniformly contracting foliation on the global cross-section of a geometric Lorenz flow can be used to show that the estimates of time averages for points in the attractor essentially depend only on the stable leaves where the points lie. This enables a reduction of the estimates of time averages for the suspension semiflow over the one-dimensional Lorenz map. Then it must be checked that Lorenz one-dimensional map satisfies all the dynamical assumptions needed to obtain the large deviations bound for the suspension semiflow.

6.1.2. Conditions on the base dynamics. We assume that the singular set $S$ (containing the points where $f$ is either not defined, discontinuous or not differentiable) is regular, e.g. a submanifold of $M$, and that $f$ is non-uniformly expanding: there exists $c > 0$ such that for Lebesgue almost every $x \in M$

$$\limsup_{n \to +\infty} \frac{1}{n} S_n \psi(x) \leq -c \quad \text{where} \quad \psi(x) = \log \|Df(x)^{-1}\|.$$

Moreover we assume that $f$ has exponentially slow recurrence to the singular set $S$ i.e. for all $\epsilon > 0$ there is $\delta > 0$ s.t.

$$\limsup_{n \to +\infty} \frac{1}{n} \log \text{Leb} \left\{ x \in M : \frac{1}{n} S_n \log d_\delta(x,S) > \epsilon \right\} < 0,$$

where $d_\delta(x,y) = \text{dist}(x,y)$ if $\text{dist}(x,y) < \delta$ and $d_\delta(x,y) = 1$ otherwise.

These conditions ensure [Alves et al.(2000)Alves, Bonatti & Viana] in particular the existence of finitely many ergodic absolutely continuous (in particular physical) $f$-invariant probability measures $\mu_1, \ldots, \mu_k$ whose basins cover the manifold Lebesgue almost everywhere.

We say that an $f$-invariant measure $\mu$ is an equilibrium state with respect to the potential log $J$, where $J = |\det Df|$, if $h_\mu(f) = \mu(\log J)$, that is if $\mu$ satisfies the Entropy Formula. Denote by $E$ the family of all such equilibrium states. It is not difficult to see that each physical measure in our setting belongs to $E$.

We assume that $E$ is formed by a unique absolutely continuous probability measure.

6.1.3. Conditions on the roof function. We assume that $r : M \setminus S \to \mathbb{R}^+$ has logarithmic growth near $S$: there exists $K = K(\epsilon) > 0$ such that $r \cdot \chi_{B(S,\delta)} \leq K \cdot \log d_\delta(x,S)$ for all small enough $\delta > 0$, where $B(S,\delta)$ is the $\delta$-neighborhood of $S$. We also assume that $r$ is bounded from below by some $r_0 > 0$. 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure8.png}
\caption{The equivalence relation defining the suspension flow of $f$ over the roof function $r$.}
\end{figure}
Now we can state the result on large deviations.

**Theorem 6.1.** Let $X^t$ be a suspension semiflow over a non-uniformly expanding transformation $f$ on the base $M$, with roof function $r$, satisfying all the previously stated conditions.

Let $\psi : M_r \to \mathbb{R}$ be continuous and $\nu = \mu \times \text{Leb}^1$ be the induced invariant measure for the semiflow $X^t$, that is, for any $A \subset M_r$ we set $\nu(A) = \mu(r)^{-1} \int d\mu(x) f^{r(x)} ds \chi_A(x, s)$. Let also $\lambda = \text{Leb} \times \text{Leb}^1$ be the natural extension of volume to the space $M_r$. Then

$$\limsup_{T \to \infty} \frac{1}{T} \log \lambda \left\{ z \in M_r : \left| \frac{1}{T} \int_0^T \psi \left( X^t(z) \right) dt - \nu(\psi) \right| > \epsilon \right\} < 0.$$ 

### 6.1.4. Consequences for the geometric Lorenz flow.

Now consider a Lorenz geometric flow as constructed in Section 3.1 and let $F$ be the one-dimensional map associated, obtained quotienting over the leaves of the stable foliation, see Figure 4. This map has all the properties stated previously for the base transformation. The Poincaré return time gives also a roof function with logarithmic growth near the singularity line.

The exponentially slow recurrence property (21) depends on a delicate combinatorial argument for which the uniform expansion and the existence of a unique singular point for the one-dimensional map induced by the geometric Lorenz flow is technically important. The proof of this property for singular-hyperbolic attractors in general must deal with the simultaneous presence of several singularities in the one-dimensional map.

The uniform contraction along the stable leaves implies that the *time averages of two orbits on the same stable leaf under the first return map are uniformly close* for all big enough iterates. If $P : S \to [-1, 1]$ is the projection along stable leaves

**Lemma 6.2.** For $\varphi : U \supset \Lambda \to \mathbb{R}$ continuous and bounded, $\epsilon > 0$ and $\varphi(x) = \int_0^{r(x)} \psi(x, t) dt$, there exists $\zeta : [-1, 1] \setminus S \to \mathbb{R}$ with logarithmic growth near $S$ such that $\left\{ \left| \frac{1}{n} S^n \varphi - \mu(\varphi) \right| > 2\epsilon \right\}$ is contained in

$$P^{-1}\left( \left\{ \left| \frac{1}{n} S^n \zeta - \mu(\zeta) \right| > \epsilon \right\} \cup \left\{ \frac{1}{n} S^n \left| \log \text{dist}_S(y, S) \right| > \epsilon \right\} \right).$$

Hence in this setting it is enough to study the quotient map $f$ to get information about deviations for the Poincaré return map. Coupled with the main result we are then able to deduce

**Corollary 6.3.** Let $X^t$ be a flow on $\mathbb{R}^3$ exhibiting a geometric Lorenz attractor with trapping region $U$. Denoting by Leb the normalized restriction of the Lebesgue volume measure to $U$, $\psi : U \to \mathbb{R}$ a bounded continuous function and $\mu$ the unique physical measure for the attractor, then for any given $\epsilon > 0$

$$\limsup_{T \to \infty} \frac{1}{T} \log \text{Leb} \left\{ z \in U : \left| \frac{1}{T} \int_0^T \psi \left( X^t(z) \right) dt - \mu(\psi) \right| > \epsilon \right\} < 0.$$ 

Moreover for any compact $K \subset U$ such that $\mu(K) < 1$ we have

$$\limsup_{T \to +\infty} \frac{1}{T} \log \text{Leb} \left( \left\{ x \in K : X^t(x) \in K, 0 < t < T \right\} \right) < 0.$$ 

### 6.1.5. Idea of the proof.

We use properties of non-uniformly expanding transformations, especially a large deviation bound recently obtained [Araújo & Pacifico(2006)], to deduce a large deviation bound for the suspension semiflow reducing the estimate of the volume of the deviation set to the volume of a certain deviation set for the base transformation.

The initial step of the reduction is as follows. For a continuous and bounded $\psi : M_r \to \mathbb{R}$, $T > 0$ and $z = (x, s)$ with $x \in M$ and $0 \leq s < r(x) < \infty$, there exists the lap number $n = n(x, s, T) \in \mathbb{N}$
such that $S_n r(x) \leq s + T < S_{n+1} r(x)$, and we can write

$$
\int_0^T \psi (X^t(z)) dt = \int_s^{r(x)} \psi (X^t(x,0)) dt + \int_0^{T + r(x) - S_n r(x)} \psi (X^t(f^n(x)),0)) dt \\
+ \sum_{j=1}^{n-1} \int_0^{r(f^j(x))} \psi (X^t(f^j(x)),0)) dt.
$$

Setting $\varphi(x) = \int_0^{r(x)} \psi(x,0) dt$ we can rewrite the last summation above as $S_n \varphi(x)$. We get the following expression for the time average

$$
\frac{1}{T} \int_0^T \psi (X^t(z)) dt = \frac{1}{T} S_n \varphi(x) - \frac{1}{T} \int_0^r \psi (X^t(x,0)) dt \\
+ \frac{1}{T} \int_0^{T + s - S_{n+1} r(x)} \psi (X^t(f^n(x)),0)) dt.
$$

Writing $I = I(x, s, T)$ for the sum of the last two integral terms above, observe that for $\omega > 0$, $0 \leq s < r(x)$ and $n = n(x, s, T)$

$$\left\{ (x, s) \in M_r : \left| \frac{1}{T} S_n \varphi(x) + I(x, s, T) - \frac{\mu(\varphi)}{\mu(r)} \right| > \omega \right\}
$$

is contained in

$$\left\{ (x, s) \in M_r : \left| \frac{1}{T} S_n \varphi(x) - \frac{\mu(\varphi)}{\mu(r)} \right| > \omega \right\} \cup \left\{ (x, s) \in M_r : I(x, s, T) > \frac{\omega}{2} \right\}.$$

The left hand side above is a deviation set for the observable $\varphi$ over the base transformation, while the right hand side will be bounded by the geometric conditions on $S$ and by a deviations bound for the observable $r$ over the base transformation.

Analysing each set using the conditions on $f$ and $r$ and noting that for $\mu$- and Leb-almost every $x \in M$ and every $0 \leq s < r(x)$

$$\frac{S_{n+1} r(x)}{n} \leq \frac{T + s}{n} \leq \frac{S_n r(x)}{n}$$

we are able to obtain the asymptotic bound of the Main Theorem.

Full details of the proof are presented in [ Araújo(2007)].

6.2. Decay of correlations for flows.

6.2.1. Non-mixing flows and slow decay of correlations. Let $f : M \to M$ be a diffeomorphism with an invariant probability measure $\mu$ and consider the suspension flow $X_f$ over $f$ with constant roof function $r \equiv 1$. Then the probability measure $\nu = \mu \times \text{Leb}$ on $M \times [0, 1)$ defines in a straightforward way a $X_f$-invariant probability measure on $X_r$ which is NOT mixing, whatever $f$ may be.

Indeed, consider $A = \pi(M \times [0, 1/2))$ and $B = M_r \setminus A$ (recall that $\pi : M \times \mathbb{R} \to X_r$ is the projection defined in Section 6.1). Then the function $t \mapsto \nu(A \cap X^{-t} B)$ for $t > 0$ has the graph as in Figure 9 (here $X^{-t}$ is a shorthand for $(X^t)^{-1}$, the inverse image of the map $X^t$).

![Figure 9. A correlation function for a non-mixing flow.](image)

This system is clearly not mixing since the sawtooth pattern in Figure 9 goes on for all positive $t$. Moreover this shows in particular that this suspension flow is not even topologically mixing (see below for the definition).
If however if \((X, f, \mu)\) is ergodic, then \(\nu\) is \(X_f\)-ergodic also: indeed, given \(A \subset X\) such that \((X_f^t)^{-1}(A) = A\) for all \(t > 0\) (an \(X_f\)-invariant set), then \(A\) is saturated, i.e., \(p \in A\) if, and only if, \(O_{X_f}(p) \subset A\); thus we may find \(\hat{A} \subset X\) such that \(A \cap \pi(X \times \{0\}) = \pi(\hat{A})\) is \(X_f\)-invariant by construction (because \(r \equiv 1\), \(\hat{A}\) is \(f\)-invariant and \(\nu(A) = \mu(\hat{A}) \cdot \text{Leb}(\{0,1\})\). Hence \(\mu(\hat{A}) \cdot \mu(X \setminus \hat{A}) = 0\) by the ergodicity of \((f, \mu)\) which implies that \(\nu(A) \cdot \nu(X \setminus A) = 0\).

In addition to the examples of non-mixing suspension flows, which arguably can be characterized as very particular cases, not all Axiom A mixing flows have exponential decay of correlations: Ruelle [Ruelle(1983)] and Pollicott [Pollicott(1984)] exhibited suspensions semiflows with piecewise constant ceiling functions over uniformly expanding base dynamics, with arbitrarily slow decay rates of correlations.

The example from Ruelle is simple to describe: take the full shift on 2 symbols \(\sigma : \Sigma_2 \to \Sigma_2\) and the roof function \(r(\xi) = \lambda_0\) if \(\xi_0 = 0\) and \(r(\xi) = \lambda_1\) if \(\xi_0 = 1\); where \(\lambda_0, \lambda_1 > 0\) and \(\lambda_0/\lambda_1\) is not rational. Take any equilibrium state \(\mu\) for \(\sigma\) with respect to a Hölder continuous potential \(\phi : \Sigma_2 \to \mathbb{R}\) and consider the induced probability \(\nu = \mu \cdot \text{Leb}\) on \(\{(\xi, s), 0 \leq s < r(\xi)\}\). The suspension semiflow over \(\sigma\) with roof function \(r\) does not have exponential decay of correlations for \(\nu\).

Anosov [Anosov(1967)] showed that geodesic flows for negatively curved compact Riemannian manifolds are mixing and obtained the Anosov alternative: given a transitive volume preserving Anosov flow, either it is mixing (with respect to the volume measure), or it is a suspension of an Anosov diffeomorphism by a constant roof function. We note that Bowen [Bowen(1976)] showed that, if a mixing Anosov flow is the suspension of an Anosov diffeomorphism, then it is stably mixing, that is, the mixing property remains true for all nearby flows (which are Anosov also by the structural stability of Axiom A flows).

Bowen also showed [Bowen(1976)] that the class of \(C^r\) Axiom A flows, \(r \geq 1\), admits a residual subset \(\mathcal{R}\) such that for every \(X \in \mathcal{R}\) the spectral decomposition of \(\Omega(X)\) is formed by pairwise disjoint pieces \(\Omega_1 \cup \cdots \cup \Omega_k\) each of which is topologically mixing. That is, given any pair of open sets \(U, V\) in \(\Omega_i\), there exists \(T_0 = T_0(U, V) > 0\) such that \(U \cap X^t(V) \neq \emptyset\) for all \(t > T_0\).

6.2.2. Exponential decay of correlations for hyperbolic flows. Recently, a breakthrough was obtained by Dolgopyat [Dolgopyat(1998a), Dolgopyat(1998b), Dolgopyat(2000)]: smooth \((C^r\) with \(r \geq 7\) geodesic flows on manifolds of negative curvature, under a non-integrability condition exhibit exponential decay of correlations. Also Liverani [Liverani(2004)] building on the work [Dolgopyat(1998a)] obtained exponential decay of correlations for \(C^4\) contact Anosov flows.

Using these ideas, applied to the particular case of a suspension over uniformly expanding base dynamics, a conjecture of Ruelle was proved by Pollicott [Pollicott(1999)]: on a mild (cohomological) condition on the ceiling function, the decay of correlations for this type of suspension flows is exponential for observables not supported on the base. This was extended by Baladi-Vallée [Baladi & Vallée(2005)], clarifying the assumptions on the base and on the ceiling function which suffice to obtain exponential decay of correlations for suspension of one-dimensional expanding maps. All these ideas were used, in a more abstract setting, by Avila-Gouezel-Yoccoz [Avila et al.(2006)Avila, Gouëzel & Yoccoz] to obtain exponential decay of correlations for the Teichmüller flow on flat surfaces.

Recently Field-Melbourne-Török obtained [Field et al.(2007)Field, Melbourne & Török] what they call stability of rapid mixing among Axiom A flows, meaning that the correlation function \(C_t(\varphi, \psi)\) decays to zero faster than \(t^{-k}\) for all \(k \in \mathbb{N}\) when \(t \to \infty\), for a \(C^2\)-open and \(C^r\)-dense set of flows among the family of \(C^r\) Axiom A flows with \(r \geq 2\).

Liverani, Melbourne and Paccaut [Liverani(2005)Luzzatto, Melbourne & Paccaut] showed that the physical measure for the geometric Lorenz flow, as presented in Subsection 3.1, is mixing. The speed of mixing for the Lorenz flow has been an open problem.

6.2.3. Robust exponential decay of correlations for a class of geometric Lorenz flows. The following result on exponential decay of correlations for a suspension semiflow over a piecewise expanding map with infinitely many branches is the basis for a ongoing work [Araújo & Varandas(2012)] to find the rate of decay of correlations for singular hyperbolic attractors.
Theorem 6.4 (Avila-Gouezel-Yoccoz [Avila et al.(2006) Avila, Gouëzel & Yoccoz]). Let $Y_t$ be a good hyperbolic skew-product semi-flow on a space $\hat{\Delta}$, preserving the probability measure $\hat{\eta}$. There exist constants $C > 0$ and $\delta > 0$ such that, for each pair of functions $\varphi, \psi \in C^1(\hat{\Delta})$, for all $t \geq 0$,

$$\left| \int \varphi \circ Y_t d\hat{\eta} - \left( \int \varphi d\hat{\eta} \right) \left( \int \psi d\hat{\eta} \right) \right| \leq C\|\varphi\|_1\|\psi\|_1 e^{-\delta t}.$$ 

The meaning of "good" here will be explained below.

We show in [Araujo & Varandas(2012)] that an open class of geometric Lorenz flows can be conjugated to semiflows in the above setting, concluding robust exponential decay of correlations for a wide class of singular flows.

Theorem 6.5 (V.A.-P. Varandas [Araujo & Varandas(2012)]). Given any compact 3-manifold $M$, we can find an open subset $\mathcal{U}$ of $X^3(M)$ such that each $X \in \mathcal{U}$ exhibits a geometric Lorenz flow which is smoothly semi-conjugated to a good hyperbolic skew-product semi-flow.

This result shows that the equilibrium point in the geometric Lorenz flow actually helps increase the speed of decay of correlations, since the exponential decay is in fact robust. There are no examples of robust exponential decay of correlations for Anosov flows!

Now we explain what good hyperbolic skew-product semiflow means, and how we relate a geometric Lorenz flow to these semiflows.

Good hyperbolic skew-product semiflow. We assume that $\bigcup_{t \in \mathbb{L}} \Delta^{(t)}$ is an at most countable partition (Lebesgue modulo zero) of an open domain $\Delta$ of some manifold by open subsets and let $F : \bigcup_{t \in \mathbb{L}} \Delta^{(t)} \to \Delta$ be a $C^r$ uniformly expanding Markov map, $r \geq 2$, that is

1. $F : \Delta^t \to \Delta$ is a $C^r$ diffeomorphism for every $t$;
2. there are $C > 0$ and $0 < \lambda < 1$ such that
   (a) for every inverse branch $h_n$ of $F^n$, with $n \geq 1$, $d(h_n(x), h_n(y)) \leq C\lambda^n d(x, y)$; and
   (b) if $JF$ is the Jacobian of $F$ with respect to the Lebesgue measure, then $\log JF$ is a $C^1$ function and $\|D((\log JF) \circ h)\|_0 \leq C$ for every inverse branch $h$ of $F$.

We denote by $h_n$ the family of inverse branches of $F^n$. It is well known that $F$ admits an invariant probability measure $\nu$ which is absolutely continuous with respect to Lebesgue.

We say that the roof function $r$ is good if

1. $r$ is bounded from below by some positive constant $r_0$;
2. there exists $C > 0$ such that $\sup_{h \in \mathcal{H}} \|D(r \circ h)\|_0 \leq C < \infty$;
3. it is not possible to write $r = v + u \circ F - u$ on $\Delta$, where $v : \Delta \to \mathbb{R}$ is constant on each $\Delta^t$ and $u : \Delta \to \mathbb{R}$ is a $C^1$-function.

The last cohomological condition corresponds to uniform non-integrability, or aperiodicity, as defined by Baladi-Vallée adapted from the work of Dolgoypat.

We say that the roof function $r : \Delta \to \mathbb{R}^+$ has exponential tail if there exists $\sigma_0 > 0$ such that $\int e^{\sigma_0 r} d\nu < \infty$.

Let $F : \bigcup \Delta^{(t)} \to \Delta$ be a uniformly expanding Markov map preserving a probability density $\nu$. An hyperbolic skew-product over $F$ is a map $\hat{F}$ from a dense open subset of an open domain $\hat{\Delta}$, to $\hat{\Delta}$, satisfying

1. there exists a continuous map $\pi : \hat{\Delta} \to \Delta$ such that $F \circ \pi = \pi \circ \hat{F}$ whenever both members of the equality are defined;
2. there is $\kappa > 1$ such that, for all $w_1, w_2 \in \hat{\Delta}$ in the same leaf, i.e. $\pi(w_1) = \pi(w_2)$, we have $d(\hat{F}w_1, \hat{F}w_2) \leq \kappa^{-1} d(w_1, w_2)$;
3. there is a $\hat{F}$-invariant probability measure $\eta$ on $\hat{\Delta}$, giving full mass to $\hat{\Delta}$;
4. there exists a smooth disintegration of $\eta$ along the stable leaves $\pi^{-1}(w)$, $w \in \Delta$, as follows.

There exists a family of probability measures $\{\eta_x\}_{x \in \Delta}$ on $\hat{\Delta}$ which is a disintegration of $\eta$ over $\nu$:

(a) $x \mapsto \eta_x$ is measurable;
(b) $\eta_x$ is supported on $\pi^{-1}(x)$, and
(c) for each measurable subset $A$ of $\hat{\Delta}$ we have $\eta(A) = \int \eta_x(A) d\nu(x)$.
Moreover, this disintegration is smooth: we can find a constant $C > 0$ such that, for any open subset $V \subset \bigcup \Delta^{(l)}$ and for each $u \in C^1(\pi^{-1}(V))$, the function $\tilde{u} : V \to \mathbb{R}, x \mapsto \tilde{u}(x) := \int u(y) \, d\nu_t(y)$ belongs to $C^1(V)$ and satisfies

$$\sup_{x \in V} \|D\tilde{u}(x)\| \leq C \sup_{y \in \pi^{-1}(V)} \|Du(y)\|.$$ 

Given $r : \cup_{t \in L} \Delta^{(t)} \to [r_0, +\infty)$ for some $r_0 > 0$ we define

$$\Delta_r = \{ (w, t) : w \in \Delta, 0 \leq t \leq r(\pi(w)) \} / \sim,$$

where $\sim$ is the equivalence relation $(w, r(\pi(w))) \sim (F(w), 0)$. Now we consider the suspension semiflow $Y_t(w, s) = (w, s + t)$.

If $Y_t$ is a semiflow over a hyperbolic skew-product with a good roof function which, moreover, has exponential tail, then we say that $Y_t$ is a good hyperbolic skew-product semiflow.

We remark that, if $\eta$ is an $F$-invariant probability measure so that $\int r \, d\eta < \infty$, then $(Y_t)_t$ preserves the probability measure $\bar{\eta} = (\eta \otimes \text{Leb})/ \int r \, d\eta$.

The following are the main steps of the proof of Theorem 6.5.

- obtain a robust $C^3$-smooth strong-stable foliation for the geometric Lorenz flow together with robust transitivity for the associated attractor;
- obtain a uniformly expanding $C^2$ Markov map $F$ as an induced map of the $C^2$ one-dimensional Lorenz transformation $f$;
- show that there is a related induced map $\tilde{F}$ from the Poincare return map $P$ to $S$ and a natural choice $r$ of roof function over $\tilde{F}$ such that the original flow is conjugated to the semiflow over $\tilde{F}$ with roof $r$;
- prove that $r$ has exponential tail, satisfies the aperiodicity condition and that the disintegration property holds for the right choice of measure on the semiflow over $\tilde{F}$;
- check that each of the above steps are robust for $C^3$ close flows.

The crucial first item above depends on the inequality $\lambda_3 > 2\lambda_1 + \lambda_2$ between the eigenvalues of the Lorenz-like singularity. This inequality does not hold in general and, in particular, does not hold for the flow of the Lorenz attractor.

6.2.4. Rapid mixing for geometric Lorenz flows. It follows from [Melbourne(2009)] that a $C^2$-open and $C^\infty$-dense set of geometric Lorenz flows have superpolynomial decay of correlations (in the sense of [Dolgopyat(1998b)]); we also say that these flows are rapidly mixing. It is likely, but unproven, that this open and dense set includes the classical Lorenz attractor.

In a recently work [Araujo et al.(2013)] it was proved that all $C^\infty$ geometric Lorenz attractors (including the concrete example of the attractor for the Lorenz system of equations (1)) satisfy this rapid mixing property. More precisely, let $U$ denote the open set of $C^\infty$ vector fields having a geometric Lorenz attractor; recall Section 3 for precise definitions. Given $X \in U$, let $X^t$ denote the flow generated by $X$ and let $\mu$ denote the unique physical measure supported on the geometric Lorenz attractor.

Theorem 6.6. [Araujo et al.(2013)] Let $X \in U$. Then for all $\beta > 0$, there exists $C > 0$ and $k \geq 1$ such that for all $C^k$ observables $\varphi, \psi : \mathbb{R}^3 \to \mathbb{R}$ and all $t > 0$,

$$\left| \int \varphi \psi \circ X^t \, d\mu - \int \varphi \, d\mu \int \psi \, d\mu \right| \leq C \|\varphi\|_{C^k} \|\psi\|_{C^k} t^{-\beta}.$$

This recent result encompasses all smooth geometric Lorenz flows, including in particular the Lorenz attractor given by (1).

6.3. Central limit theorem for the Lorenz flow. In [Holland & Melbourne(2007)] Holland and Melbourne, building on the work [Melbourne & Nicol(2005)] of Melbourne and Nicol, obtained the Almost Sure Invariance Principle (ASIP) for Axiomatic geometrical Lorenz attractors which, in turn, implies the Central Limit Theorem. More precisely, if $X^t$ is the axiomatic geometric Lorenz flow with physical probability measure $\mu$ and $\psi$ is a H"older continuous function (observable) on
the manifold with zero mean $\int \psi \, d\mu = 0$, then there exists a Brownian motion $W(t)$ with variance $\sigma^2 > 0$, and there is $\epsilon > 0$, such that

$$\int_0^t \psi \circ X^s \, ds = W(t) + O(t^{1/2 - \epsilon}) \quad \text{as} \quad t \to +\infty \quad \text{for} \ \mu\text{-almost all} \ x.$$  

This result, in turn, implies the Central Limit Theorem (CLT): in the same setting as above, for any interval $A \subset \mathbb{R}$

$$\mu \left\{ x : \frac{1}{\sqrt{t}} \left( \int_0^t \psi \circ X^s \, ds - \mu(\psi) \right) \in A \right\} \xrightarrow{t \to +\infty} \frac{1}{\sigma \sqrt{2\pi}} \int_A e^{-s^2/2} \, ds;$$

and the Law of the Iterated Logarithm

$$\limsup_{t \to +\infty} \frac{1}{\sqrt{2t \log \log t}} \int_0^t \psi \circ X^s \, ds = \sigma \quad \mu\text{-almost everywhere.}$$

In the recent work [Araujo et al. (2013)] a stronger property was obtained: the scalar ASIP holds for the time-1 map $X^1$ of all smooth geometric Lorenz flows. This information is used by the authors in [Araujo et al. (2013)] to prove the CLT for time-1 maps of geometric Lorenz flows.

We note that, even for hyperbolic flows, the time-1 map is only partially hyperbolic. In general, statistical results for the time-1 map are not straightforward consequences of the continuous versions.

**Theorem 6.7.** [Araujo et al. (2013)] Let $X \in \mathcal{U}$. Then there exists $k \geq 1$ such that for all $C^k$ observables $\varphi : \mathbb{R}^3 \to \mathbb{R}$ there exists $\sigma \geq 0$ such that

$$\frac{1}{\sqrt{n}} \left[ \sum_{j=0}^{n-1} \varphi \circ X^j - n \int \varphi \, d\mu \right] \xrightarrow{\mathbb{P}} \mathcal{N}(0, \sigma^2)$$

where the convergence is in distribution.

For the scalar ASIP we have, more precisely.

**Theorem 6.8.** [Araujo et al. (2013)] Let $X \in \mathcal{U}$. There exists $k \geq 1$ such that for all $C^k$ observables $\varphi : \mathbb{R}^3 \to \mathbb{R}$ the ASIP holds for the time-1 map: passing to an enriched probability space, there exists a sequence $X_0, X_1, \ldots$ of iid normal random variables with mean zero and variance $\sigma^2$ (as in Theorem 6.7), such that

$$\sum_{j=0}^{n-1} \varphi \circ X^j = n \int \varphi \, d\mu + \sum_{j=0}^{n-1} X_j + O(n^{1/4}(\log n)^{1/2}(\log \log n)^{1/4}), \ a.e.$$  

The ASIP implies the CLT and also the functional CLT (weak invariance principle), and the law of the iterated logarithm together with its functional version, together with many other results; see [Philipp & Stout (1975)] for a comprehensive list.

7. **Some conjectures**

Singular-hyperbolic theory can be seen as an extension of the theory of hyperbolicity, and so we may try to obtain the same properties of hyperbolic sets in the setting of singular-hyperbolic flows. Some of these results are already known, for instance, the fact that every singular-hyperbolic attractor is a homoclinic class, but their proof is, usually, rather different from the usual "hyperbolic proof". Other results are mostly wide open to research. We present some of them here.

7.1. **Dimension theory, ergodic and statistical properties.** Afraimovich and Pesin in [Afraimovich & Pesin (1987)] investigate the dimensional properties of “triangular maps” which are a class of maps generalizing the Poincaré first return map $P$ of the geometric Lorenz model.

Concerning fractal dimensions of Lorenz attractors we mention the results of Leonov [Leonov (1988), Leonov (2001)] together with Bouichenko [Boichenko & Leonov (1989)]. The first contains explicit formulas for the Lyapunov dimension of the Lorenz attractor and, in the second, a simple upper
bound on the Hausdorff dimension of Lorenz attractors is given in terms of the parameters of the Lorenz systems of equations (1).

**Conjecture 1.** As in a hyperbolic attractor on a surface, the Hausdorff dimension of any singular-hyperbolic attractor on a 3-manifold satisfies Bowen's formula: it is the value $2 + \gamma$ where $\gamma$ satisfies $P_{\text{top}}(\gamma \log |\text{det } DX^1| E^s) = 0$, $P_{\text{top}}$ is the topological pressure of the attractor, and $E^s$ is the one-dimensional stable bundle over the attractor.

In [Young(1981)] Young shows that the geometrical Lorenz attractor can be approximated by horseshoes with entropy close to that of the Lorenz attractor.

**Conjecture 2.** It is possible to approximate the topological entropy of a singular-hyperbolic attractor by the topological entropy of horseshoes contained in the attractor.

8. Large deviations for the Lorenz flow

As explained in Section 6.1, for a geometric Lorenz flow, the large deviations decay rate for the volume/physical measure is exponential. That is, if we set $\epsilon > 0$ as an error margin and consider

$$B_t = \left\{ z : \frac{1}{t} \int_0^t \psi(X^t(z)) - \int \psi d\mu > \epsilon \right\},$$

then sufficient conditions were found, in terms of the base transformation and the roof function, under which the Lebesgue measure of $B_t$ decays to zero exponentially fast, i.e., whether there are constants $C, \xi > 0$ such that

$$\text{Leb}(B_t) \leq C e^{-\xi t} \quad \text{for all} \quad t > 0.$$ 

We observe that in this setting Lebesgue measure or volume is not an invariant measure.

In [Melbourne & Nicol(2008)] Melbourne and Nicol and in [Rey-Bellet & Young(2008)] Rey-Bellet and Young obtained large deviations principles for invariant measures in the same setting, including subexponential or polynomial bounds on large deviations depending on the properties of the base transformation.

**Conjecture 3.** These results are also true for general singular-hyperbolic attractors and should be true, under some mild conditions, for singular-hyperbolic attracting sets as well.

8.1. Decay of correlations. As explained in Section 5.4.1, it has recently been obtained an example of a geometric Lorenz flow with robust exponential decay of correlations for all flows sufficiently $C^2$ close.

A main open question is to find the rate of decay of correlation for the original Lorenz flow, and more in general for singular hyperbolic attractors. In the direction of finding an appropriate functional analytic framework to face the problem, some avances was made recently in [Butterley(2014)].

Some other natural questions are as follows.

**Conjecture 4.** Non-hyperbolic robustly mixing flows in three-dimensional manifolds have robust exponential decay of correlations.

The rate of decay should depend continuously on the system considered.

**Conjecture 5.** Fixing the observable $\psi$, the rate of decay of correlations should depend continuously on the flow $X^t$ in a neighborhood of a singular-hyperbolic attractor.

Moreover, since the argument leading to robust exponential decay crucially depends on the smoothness of the stable foliation, which cannot be obtained robustly for globally hyperbolic flows, we conjecture the following.

**Conjecture 6.** There are no Anosov flows with robust exponential decay of correlations.

However, we should be able to obtain such smooth foliations for flows whose limit set is hyperbolic.
Conjecture 7. There are open sets of $C^2$ Axiom A flows on compact manifolds exhibiting robust exponential decay of correlations.

For the definition of Axiom A the reader should consult a standard reference on hyperbolic dynamics, e.g. [Smale(1967)].

8.2. Central Limit Theorem. We note that the “time-1” map $X_1$ of any flow $X_t$ on a hyperbolic or singular-hyperbolic attractor is a partially hyperbolic diffeomorphism. In general, limit theorems for diffeomorphisms given as time-$t$ maps of flows are harder to obtain. A very general result was obtained by Melbourne and Török [Melbourne & Török(2002)] under some assumptions on the decay of correlations for the flow. These ideas can be adapted to prove that the strong mixing properties for the $C^2$-open subset of geometric Lorenz attractors imply (robust) limit theorems for the corresponding time-one maps. More precisely we pose the following:

Conjecture 8. Let $U \subset \mathcal{X}^s(M)$ be the open family of vector fields for which exponential decay of correlations is verified, and denote by $(X_t)_t$ the flow generated by $X \in U$. For all but countably many values of $t \in \mathbb{R}$ the time-$t$ map $X_t$ the following Central Limit Theorem holds: for any $\varphi : \Delta_r \to \mathbb{R}$ in $L^\infty(\Delta_r)$ there exists $\sigma = \sigma(\varphi) > 0$ such that

$$\frac{1}{\sigma \sqrt{n}} \left[ \sum_{j=0}^{n-1} \varphi(X_{tn}) - \int \varphi \, d\mu \right] \overset{D}{\to} \mathcal{N}(0,1).$$

8.3. Thermodynamical formalism. The thermodynamical formalism was first developed for (uniformly) hyperbolic diffeomorphisms, borrowed from statistical mechanics by Bowen, Ruelle and Sinai (among others, see e.g. [Bowen(1975), Bowen & Ruelle(1975), Ruelle(1989), Ruelle(2004), Ellis(2006), Bonatti et al.(2005)Bonatti, Díaz & Viana]). This was extended to hyperbolic flows by Bowen and Ruelle in [Bowen & Ruelle(1975)]. The classical theory relies heavily on the coding of basic pieces of hyperbolic dynamics by subshifts of finite type, for which many tools are available to study in fine detail the relations among its invariant measures. Recently most of this theory was extended to countable shifts by Gurevich [Gurevich & Savchenko(1998)], Sarig [Sarig(1999), Sarig(2006)] and many others.

The extension of this theory for singular-hyperbolic attractors faces several difficulties: these attractors are modelled by a suspension semiflow whose base transformation is a Hölder-$C^1$ piecewise expanding but non-Markov map, and the roof function is unbounded. In the hyperbolic flow case, the corresponding suspension semiflow has a piecewise expanding Markov map as the base transformation and the roof function is continuous and bounded. In the singular-hyperbolic case, neither the thermodynamical formalism is complete for the base transformation, nor is it clear how to proceed with unbounded roof functions, which imply an extra restriction of integrability on the observables with respect to invariant measure for the base transformation.

Hope of solving this problem in the near future is provided by recent advances in the construction of a thermodynamical formalism for non-uniformly expanding transformations by Oliveira, Viana, Senti, Pesin, Varandas, Bruin, Todd, Pinheiro [Oliveira & Viana(2006), Pesin & Senti(2008), Varandas & Viana(2010), Bruin & Todd(2007), Bruin & Todd(2008), Pinheiro(2011)].

Recently, Leplaideur and Pinheiro in [Leplaideur & Pinheiro(2012)] obtain the first results of a thermodynamic formalism for a two-dimensional map representing the Poincaré first return map of an expanding geometric Lorenz attractor. Namely, they prove the existence of unique equilibrium state for any Hölder continuous potential on the attracting set of this two-dimensional map.

Conjecture 9. The same result on existence and uniqueness of equilibrium states can be extended to all singular-hyperbolic attractors in three-dimensional manifolds.

Ongoing work [Goncalvez(2009), Pacifico & Todd(2010)] on the study of equilibrium states for multiples of the logarithm of the derivative, for suspension flows over transformations resembling the Lorenz one-dimensional transformations, will enable the extension of this results from discrete dynamics to the flow of Lorenz-like attractors.

Conjecture 10. It is possible to build a thermodynamical formalism for Rovella-like and singular-hyperbolic attractors.
8.4. Higher dimensional singular flows. An example of a higher-dimensional invariant robust attractor with multidimensional expanding directions was given by Bonatti, Pumariño and Viana in [Bonatti et al. (1997) Bonatti, Pumariño & Viana], which we present below.

8.4.1. Singular-attractor with arbitrary number of expanding directions. Consider a “solenoid” constructed over a uniformly expanding map \( f : T^k \to T^k \) of the \( k \)-dimensional torus, for some \( k \geq 2 \). That is, let \( \mathbb{D} \) be the unit disk on \( \mathbb{R}^2 \) and consider a smooth embedding \( F : T^k \times \mathbb{D} \to T^k \times \mathbb{D} \) of \( N = T^k \times \mathbb{D} \) into itself, which preserves and contracts the foliation

\[
F^s = \{ \{ z \} \times \mathbb{D} : z \in T^k \},
\]

and moreover the natural projection \( \pi : N \to T^k \) on the first factor conjugates \( F \) to \( f : \pi \circ F = f \circ \pi \).

Now consider the linear flow over \( M = N \times (0,1] \) given by the vector field \( X = (0,1) \) on \( TN \times \mathbb{R} \) where we make the identification \( (x,0) \sim (x,1) \) for all \( x \in N \). Modify the flow on a cylinder \( U \times \mathbb{D} \times (0,1] \) around the orbit of a point \( p = (z,0) \in N \), where \( U \) is a neighborhood of \( z \) in \( T^k \), in such a way as to create a hyperbolic singularity \( \sigma \) of saddle-type with \( k \)-expanding and 3 contracting eigenvalues, as depicted in Figure 10.

![Figure 10. A sketch of the construction of a robust singular-attractor in higher dimensions](image)

This modified flow defines a transition map \( L \) from \( \Sigma_0 = T^k \times \{0\} \) to \( \Sigma_1 = T^k \times \{1\} \) which through the identification given by \( (w,1) \sim (F(w),0) \) defines the return map to the global cross-section \( \Sigma_0 \) of a flow \( Y \) on the space \( M^F = M/\sim_F \).

In [Bonatti et al. (1997) Bonatti, Pumariño & Viana] it is shown that, if the expanding rate of \( f \) is sufficiently big, then the set

\[
\Lambda = \bigcup_{T>0 \cap t>T} \bigcap_{t} Y_t(\Sigma_0)
\]

is a robust partially hyperbolic attractor with singularities.

8.4.2. The notion of sectionally hyperbolic sets. Metzger and Morales in [Metzger & Morales (2008)] introduced the notion of sectionally expanding or sectional-hyperbolic set in a manifold of arbitrary finite dimension. This notion encompasses that of singular-hyperbolic sets in 3-manifolds as a particular case.

We say that a compact invariant set \( \Lambda \) for a flow, generated by a vector field \( X \in \mathfrak{X}^1(M) \) on a compact finite dimensional manifold \( M \), is sectional-hyperbolic if it is partially hyperbolic and the central direction expands uniformly the area along any two-dimensional subspace. More precisely, the tangent bundle over \( \Lambda \) admits a \( DX^t \)-invariant and dominated splitting \( T_\Lambda M = E^s \oplus E^c \), such that there are \( C, \lambda > 0 \) satisfying for every \( x \in \Lambda \) and \( t > 0 \)

- \( E^s \) is uniformly contracted: \( \| DX^t \| E^s_x \| \leq Ce^{-\lambda t} \);
- \( E^c \) is sectionally expanded: for every bidimensional subspace \( F_x \) contained in \( E^c_x \) we have \( | \det(DX^t | F_x) | \geq Ce^{\lambda t} \).

Similarly to the notion of singular-hyperbolicity, robust attractors in higher dimensional manifolds need not be sectional-hyperbolic, as the example of Turaev and Shil’nikov in [Turaev & Shil’nikov (1998)] shows.

The results in Section 4.2 have a precise counterpart for sectional-hyperbolic attractors with very similar proofs. It is then natural to try to extend the three-dimensional results to this more general setting.
Lemma 9.1. If $\Lambda = \cap_{t \geq 0} X^t(U)$ is a compact isolated proper subset for $X \in \mathcal{X}^1(M)$ with isolating neighborhood $U$ and $\Lambda$ is not future chaotic (respective not past chaotic), then $\Lambda_X(U) := \cap_{t \geq 0} X^{-t}(U)$ (respective $\Lambda_X(U) := \cap_{t \geq 0} X^t(U)$) has non-empty interior.

Proof. If $\Lambda$ is not future chaotic, then for every $r > 0$ there exists some point $x \in \Lambda$ and a neighborhood $V$ of $x$ such that $\text{dist}(X^t(y), X^t(x)) < r$ for all $t > 0$ and each $y \in V$. If we choose $0 < r < \text{dist}(M \setminus U, \Lambda)$ (we note that if $\Lambda = U$ then $\Lambda$ would be open and closed, and so, by connectedness of $M$, $\Lambda$ would not be a proper subset), then we deduce that $X^t(y) \in U$, that is, $y \in X^{-t}(U)$ for all $t > 0$, hence $V \subset \Lambda_X(U)$. Analogously if $\Lambda$ is not past chaotic, just by reversing the time direction. $\square$

In particular if an invariant and isolated set $\Lambda$ with isolating neighborhood $U$ is given such that the volume of both $\Lambda_X(U)$ and $\Lambda_X(U)$ is zero, then $\Lambda$ is chaotic.

Sensitive dependence on initial conditions is part of many definitions of chaotic behavior in the literature, see e.g. [Devaney(1989)]. It is an interesting fact that sensitive dependence is a consequence of another two common features of most systems considered to be chaotic: existence of a dense orbit and existence of a dense subset of periodic orbits.

Proposition 9.2. A compact invariant subset $\Lambda$ for a flow $X^t$ with a dense subset of periodic orbits and a dense (regular and non-periodic) orbit is chaotic, in the sense defined above.

A short proof of this proposition can be found in [Banks et al.(1992)Banks, Brooks, Cairns, Davis & Stacey]. An extensive discussion of this and related topics can be found in [Glasner & Weiss(1993)].

9.1. Robust chaoticity, volume hyperbolicity and physical measure. Here we show that robust chaotic behavior of an attractor, under mild conditions, is equivalent to singular-hyperbolicity and ensures the existence of a physical measure. This is proved by noting a series of consequences of the previous results.

Here, by “robust chaotic attractor” $\Lambda = \Lambda_X(U)$ in a trapping region $U$ of a vector field $X$ we mean that, for all close enough vector fields $Y$ to $X$ in the $C^1$ topology, the corresponding maximal invariant subset $\Lambda_Y(U)$ is also chaotic.

Going through the proof of Theorem 4.1 in [Morales et al.(2004)Morales, Pacífico & Pujals] we can see that the arguments can be carried through assuming that

1. $\Lambda$ is an attractor for $X$ with isolating neighborhood $U$ such that every equilibria in $U$ is hyperbolic with no resonances;
2. there exists a $C^1$ neighborhood $\mathcal{U}$ of $X$ such that for all $Y \in \mathcal{U}$ every periodic orbit and equilibria in $U$ is hyperbolic of saddle-type.
The condition on the equilibria amounts to restricting the possible three-dimensional vector fields in the above statement to an open a dense subset of all $C^1$ vector fields. Indeed, the hyperbolic and no-resonance condition on a equilibrium $\sigma$ means that:

- either $\lambda \neq \Re(\omega)$ if the eigenvalues of $DX(\sigma)$ are $\lambda \in \mathbb{R}$ and $\omega, \overline{\omega} \in \mathbb{C}$;
- or $\sigma$ has only real eigenvalues with different norms.

Indeed, conditions (1) and (2) ensure that no bifurcations of periodic orbits or equilibria leading to sinks or sources are allowed for any nearby flow in $U$. This implies, by now standard arguments, that the flow on $\Lambda$ must have a dominated splitting which is *volume hyperbolic*: both subbundles of the splitting must contract/expand volume; see e.g. [Araújo & Pacifico(2010)].

For a 3-dimensional flow one of the subbundles is one-dimensional, and so we deduce singular-hyperbolicity either for $X$ or for $-X$. If $\Lambda$ has no equilibria, then $\Lambda$ is uniformly hyperbolic. Otherwise, it follows from the arguments in [Morales et al.(2004)Morales, Pacifico & Pujals] that all singularities of $\Lambda$ are Lorenz-like and this shows that $\Lambda$ must be singular-hyperbolic for $X$.

We note that condition (2) above is a consequence of any one of the following assumptions a neighborhood of $X$ and on the neighborhood $U$ in $M$:

- **robust chaoticity**: for every $Y \in \mathcal{U}$ the maximal invariant subset $\Lambda_Y(U)$ is chaotic;
- **zero volume and future chaoticity**: for every $Y \in \mathcal{U}$ the maximal invariant subset $\Lambda_Y(U)$ has zero volume and is future chaotic;
- **zero volume and robust positive Lyapunov exponent**: for every $Y \in \mathcal{U}$ the maximal invariant subset $\Lambda_Y(U)$ has zero volume and there exists a full Lebesgue measure subset $P_Y$ of $U$ such that

$$
\limsup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \log \|DY_x^n\| > 0, \quad x \in P_Y.
$$

The following result of Mañé analogous to Theorem 4.1 in [Mañé(1982)] also follows from the absence of sinks and sources for all $C^1$ close diffeomorphisms in a neighborhood of the attractor.

**Theorem 9.3.** Robust attractors for surface diffeomorphisms are hyperbolic.

Extensions of these results to higher dimensions for diffeomorphisms, by Bonatti, Díaz and Pujals in [Bonatti et al.(2003)Bonatti, Díaz & Pujals], show that robust transitive sets always admit a volume hyperbolic splitting of the tangent bundle. Vivier in [Vivier(2003)] extends previous results of Doering [Doering(1987)] for flows, showing that a $C^1$ robustly transitive vector field on a compact boundaryless $n$-manifold, with $n \geq 3$, admits a global dominated splitting. Metzger and Morales extend the arguments in [Morales et al.(2004)Morales, Pacifico & Pujals] to homogeneous vector fields (inducing flows allowing no bifurcation of critical elements, i.e. no modification of the index of periodic orbits or equilibria) in higher dimensions leading to the concept of 2-sectional expanding attractor in [Metzger & Morales(2008)].

The preceding observations allows us to deduce that robust chaoticity is a sufficient condition for singular-hyperbolicity of a generic attractor; see [Araújo & Pacifico(2010)].

**Corollary 9.4.** Let $\Lambda$ be an attractor for $X \in X^1(M^3)$ such that every equilibrium in its trapping region is hyperbolic with no resonances. Then $\Lambda$ is singular-hyperbolic if, and only if, $\Lambda$ is robustly chaotic.

This means that if we can show that arbitrarily close orbits, in an isolating neighborhood of an attractor, are driven apart, for the future as well as for the past, by the evolution of the system, and this behavior persists for all $C^1$ nearby vector fields, then the attractor is singular-hyperbolic.

To prove the necessary condition on Corollary 9.4 we use the concept of expansiveness for flows, and through it show that singular-hyperbolic attractors for 3-flows are robustly expansive and, as a consequence, robustly chaotic also. This is [Araújo et al.(2009)Araújo, Pujals, Pacifico & Viana, Theorem A] whose proof is presented also in [Araújo & Pacifico(2010), Chapter 7, Section 7.2].

We recall the following conjecture of Viana, presented in [Viana(1998)]
Conjecture 12. If an attracting set $\Lambda(U)$ of smooth map/flow has a non-zero Lyapunov exponent at Lebesgue almost every point of its isolated neighborhood $U$ (i.e. it satisfies (22) with $P_Y \subset U$), then it admits some physical measure.

From the preceding results and observations we can give a partial answer to this conjecture for 3-flows in the following form.

Corollary 9.5. Let $\Lambda_X(U)$ be an attractor for a flow $X \in \mathcal{X}^1(M)$ such that

- the divergence of $X$ is negative in $U$;
- the equilibria in $U$ are hyperbolic with no resonances;
- there exists a neighborhood $U$ of $X$ in $\mathcal{X}^1(M)$ such that for $Y \in U$ one has (22) almost everywhere in $U$.

Then there exists a neighborhood $V \subset U$ of $X$ in $\mathcal{X}^1(M)$ and a dense subset $D \subset V$ such that

1. $\Lambda_Y(U)$ is singular-hyperbolic for all $Y \in V$;
2. there exists a physical measure $\mu_Y$ supported in $\Lambda_Y(U)$ for all $Y \in D$.

Indeed, item (2) above is a consequence of item (1), the denseness of $\mathcal{X}^2(M)$ in $\mathcal{X}^1(M)$ in the $C^1$ topology, together with Theorem 4.5 and the observation following its statement.

Item (1) above is a consequence of Corollary 9.4 and the observations of Section 4.2, noting that negative divergence on the isolating neighborhood $U$ ensures that the volume of $\Lambda_Y(U)$ is zero for $Y$ in a $C^1$ neighborhood $V$ of $X$.

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\( \Lambda \) does not intersect a \( \delta \)-neighbd. of the cu-boundary
