A New Knot Invariant

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Abstract

A polynomial is presented that models a topological knot in a unique manner. It distinguishes all types of knots including the orientation and has a group theory interpretation. The topologies may be labeled via a number, which upon a base 2 expansion generate the polynomial; the equivalent numbers via Reidemeister moves are grouped into a superset polynomial with coefficients labeling the equivalent knots.
1 Introduction

The classification of knots topologically has been of interest for many years, but a unique invariant appears to be lacking in the literature. In this paper a unique invariant is given.

There are several forms of knot invariants written in polynomial form, and they are of both mathematical and physical interest [1],[2]. An invariant that distinguishes all topologies from each other is relevant for many reasons.

The invariant presented here relies on labeling all intersections of the curve in three dimensions by two by two matrices. These two-by-two matrices are assembled into a larger matrix which could serve as an invariant; however, both for notational purposes and to make contact with previous forms this larger matrix is projected onto Sp(2n) adjoint generators into a polynomial form.

The knot is first labeled in the manner: (1) a starting point is chosen on the contour, (2) the knot is given a direction by attaching arrows one way through the contour, (3) a number is attached to every intersection along this direction post (or prior) to every intersection, and (4) each intersection of the contour with itself takes on only one of four forms and is labeled by two numbers generated in (3). Furthermore, the four types of oriented intersections are illustrated in the figure 1(a).

These four types of oriented intersections are labeled with a two by two matrix. These matrices are,

$$M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad M_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$M_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad M_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$  \hspace{1cm} (1.1)

There are a total of $n$ intersections in the knot configuration, which through a single closed contour are passed through twice each in traversing the loop. These matrices are assembled into a $2n$ by $2n$ matrix $M$ via block form by inserting at position $(i,j)$ the two by two matrix associated with the $(i,j)$ node along the contour; this fills up all but the diagonal elements. The diagonal entries along $(i,i)$ are given an empty two by two matrix. Also, via following the arrows, the lower triangular two by two matrices are the transpose of the upper triangular ones and the matrix satisfies $M = M^T$. (Up and then under to the right, $M_1$, is the transpose of passing through the intersection along the path of the other arrow, which is up and then over to the right, $M_4$).
Figure 1: (1) The four types of intersections. (2) A sample trefoil knot.
This matrix is a member of Sp$(2n)$ and allows a projection onto the adjoint representation, \( M = \sum_i a_i T^i \). Note that all entries are unity, which means that the knot matrix is associated with the homology of a (possibly degenerate) Riemann surface \( \Sigma_n \) of genus \( n \). Without loss of information, one could put minus signs in the upper triangular portion so that the final matrix satisfies \( M = -M^T \), i.e. a member of SO$(2n)$. The Sp$(2n)$ (or SO$(2n)$) generators could be given the standard form,

\[
(M_{ab})^{ij} = \delta^i_a \delta^j_b \pm \delta^i_b \delta^j_a .
\]  

(1.3)

The polynomial invariant is constructed from the topology of the knot, in \( M \), via the projection \( M = \sum a_i T^i \). The coefficients \( a_i \) from this explicit projection are assembled into the form \( P(z) \),

\[
P(z) = \sum_{i=1}^{2n} a_i z^i .
\]  

(1.4)

The invariant in (1.4) is unique and distinguishes all of the possible topologies, because the matrix uniquely reconstructs the knot and there is no loss of information between \( M \) and \( P(z) \). There is an ambiguity in mapping the coefficients in the matrix decomposition \( M = \sum a_i T^i \) to the polynomial invariant in (1.4). The ambiguity is removed via labeling one to one in order \( T^i \leftrightarrow z^i \).

As an illustration of the procedure, one of the two trefoil knots in figure 1(b) is analyzed. The trefoil knot has three intersections and so is dimension twelve. The associated matrix \( M_t \) written in block form with the \( M_j \) matrices is,

\[
M_t = \begin{pmatrix}
0 & 0 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 3 & 0 \\
0 & 0 & 0 & 0 & 0 & 2 \\
3 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 3 & 0 & 0 & 0
\end{pmatrix}
\]  

(1.5)

The decomposition of this trefoil’s \( M_t \) is \( a_{8,1} = 1 \), \( a_{9,4} = 1 \), and \( a_{12,5} = 1 \) (with symmetrization). The polynomial \( P_t(z) \) is, via the decomposition of the generators through \( z^{(j-1)\times2n+i} \),

\[
P_t(z) = z^8 + z^{40} + z^{60} .
\]  

(1.6)
Note that this labeling of the generators has vanishing elements for diagonal elements \( i = j \). This simple example describes the procedure for finding \( M \) and \( P(z) \). It is not clear if this polynomial can be given further number theoretic or geometric interpretation due to the appearance of the numbers \( 8 \times (1, 5, 12) \).

The Reidemeister moves may also be examined in this context. There are three moves that are analyzed. The first one involves an overlap with a twist, depicted in figure 1, and amounts to an expansion of the matrix \( M \) in the \( i \) row and \( i + 1 \) column with the matrix entry \( M_1 \),

\[
z^{2(i-1)n+2(i+1)+1},
\]

while changing the rest of the matrix with zeros in the \( i \)th row and and \( i + 1 \)th column, via a \( M_1 \). The second Reidemeister move involves the inclusion of two additional matrices \( M_4 \) and \( M_2 \), at the nodes \( i, j + 1 \) and \( i + 1, j \). This involves enlarging the matrix \( M \) by the terms

\[
z^{2m+2(j+1)} + z^{2m+2j},
\]

with zeros placed in the columns and rows of the entries at \( i, j + 1 \) and \( i + 1, j \). The third move involves the triple crossing, i.e. a slide of a bar, from the entries \( M_4 \) at \((i, j)\), \( M_2 \) at \((j + 1, k + 1)\) and \( M_4 \) and \((i + 1, k)\); to the entries \( M_3 \) at \((j, k)\), \( M_1 \) at \((k + 1, i)\), and \( M_1 \) at \((j + 1, k + 1)\). This involves the change of the entries from these nodes from,

\[
z^{2m+2j} + z^{2j+2(k+1)} + z^{2(i+1)n+k}
\]

(1.9)

to

\[
z^{jn+2(k-1)+1} + z^{kn+2(i-1)+1} + z^{jn+2k+1}.
\]

(1.10)

These Reidemeister moves may be incorporated directly at the level of the polynomials \( P(z) \) or in the matrices \( M \).

The polynomial form of the invariant \( P(z) = a_i z^i \) with the unit coefficients \( a_i \) may be given a base 2 interpretation via the expansion of a number
\[ N = a_i z^i \]  
(1.11)

with the expansion over the base 2 numbers \( z^0 = 1, \ z^1 = 2, \ z^2 = 4, \) etc. Not all numbers \( N \) may be reached via the expansion due to the expansion of the matrices \( M_1, \ M_2, \ M_3, \) and \( M_4.\) However, another interpretation is given in base 4 via the expansion of the matrix invariant with the labels 1 through 4. Considering the equivalence of the knots via the Reidemeister moves, a family of equivalences may be defined via a new polynomial \( Q_N(z), \)

\[ Q_N(z) = \sum b_i w^i, \]  
(1.12)

with the first coefficient \( b_0 \) defining the fundamental (minimal) knot. The coefficients \( b_i \) are numbers labeling further knots related to the minimal knot via Reidemeister moves. These numbers are base two (or base four), spanning the knot topology via the expansion,

\[ b_i = \sum a_j z^j, \]  
(1.13)

with the \( b_i \) essentially \( P(z). \) The tower of numbers \( b_i \) may be obtained by direct calculation or an iteration of the fundamental knot. There is potentially interesting group theory characteristics, e.g. representation dimensions, associated with the numbers \( b_i. \) For example, the individual equivalence classes form separate fields, subsets of the integers, which are closed under the Reidemeister moves.

The invariant \( P(z) \) is unique and completely characterizes the knot configuration; multiple disconnected but entangled contours are also described via the labeling of the intersections. Due to the construction this invariant has a group theoretic symplectic interpretation. The matrix forms \( M \) of the polynomials could be investigated further for more information (e.g. invariants of matrices, embeddings of one knot into another, quotients, ...). Furthermore, the matrix form has an interpretation in terms of the homology of a max genus \( n \) Riemann surface.

The polynomial form should have relations to other commonly used invariants such as the HOMFLY, Jones, Kauffman, or Vassiliev ones. Although these latter forms do not uniquely specify the knot configuration, the relation is relevant to physics models and mathematics.
Because the invariant $P(z)$ is unique, the classification and further development of associated three-dimensional Seifert manifolds, such as cohomology directly from $P(z)$, may be found in a more direct fashion. The algebraic nature of the knot further relates to geometry in $d = 2$ via the zero set $P(z) = 0$.

Last, the invariants $P(z)$ presented here always have unit coefficients. The information is encoded in the exponents $i$ in the expansion $P(z) = \sum a_i z^i$. Other invariants are typically of lower degree, but with non-unity in the (seemingly less sparse) coefficients; the $P(z)$ here contain more information in the exponents apparently. The information content is the same however, apart from the uniqueness issue. In comparison between the coefficients and exponents, it is not obvious how many bytes of information the different forms require to label a knot.

The equivalence classes of the knot numbers via the Reidemeister moves is found via the polynomial operations. These have an indirect number form $f_\sigma(i)(N)$ for the actions $\sigma(i)$ of the moves $i$ on the knot number $N$. 
References

[1] *Encyclopedic Dictionary of Mathematics*, Iwanamic Shoten Publishes, Tokyo, 3rd Ed., (1985), English Transl. MIT Press (1993).

[2] *Knots and Links*, AMS Chelsea Publishing, 2nd Ed., (1990).