Continuity in time of solutions of a phase-field model

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Abstract
A phase field model proposed by G. Caginalp for the description of phase changes in materials is under consideration. It is assumed that the medium is located in a container with heat conductive walls that are not subjected to phase changes. Therefore, the temperature variable is defined both in the medium and wall regions, whereas the phase variable is only considered in the medium part. The case of Lipschitz domains in two and three dimensions is studied. We show that the temperature and phase variables are continuous in time functions with values in $L^2$ and $H^1$, respectively, provided that the initial values of them are from $L^2$ and $H^1$, respectively. Moreover, continuous dependence of solutions on the initial data and boundary conditions is proved.

1. Introduction
Phase-field models have been introduced by G. Caginalp in \cite{1} (see also e.g. \cite{2} for generalizations) as a relaxation of Stefan problems where the development of the sharp interface between phases should be found exactly. In contrast to that, phase field models operate with phase functions that assume values from -1 (solid state) to 1 (liquid state) at each spatial point and change sharply but smoothly over the solidification fronts so that the phase interfaces become smoothed. Mathematically speaking, such models consist of two coupled parabolic equations describing the temperature and phase fields satisfying initial and boundary conditions.

Phase field models are frequently used to describe processes of melting, solidification, evaporation, and condensation, which is directly related to applications such as metal casting, design of cooling systems, cryopreservation of living
tissues etc. (see e.g. [3, 4]). Phase field models are also appropriate for the description of phase changes when modeling CO$_2$ sequestration. It is assumed that the supercritical carbon dioxide, CO$_2$ pressured to a phase between gas and liquid, is injected into a saline aquifer where it may either dissolve in the brine, react with the dissolved minerals and the surrounding rocks, or become trapped in the pore space of the aquifer. In this relation, phase field models are useful for the description of transitions between supercritical, ordinary, and dissolved CO$_2$ phases.

In this paper, we study the case where a material subjected to a phase change is located inside of a container with heat conductive walls that are not subjected to phase changes. For example, this can be an ampoule filled with a liquid containing living cells (see [4, Sec. 2.2]) and subjected to cooling applied to the outer surface but not immediately to the liquid to be frozen.

The paper is structured as follows. Section 2 presents the mathematical model. The definition of weak solutions and formulation of main results are given in Section 3. The existence of weak solutions is shown in Section 4. In Section 5, continuity in time of the phase function and the validity of an energy equality are established. The uniqueness of weak solutions and their continuous dependence on the initial and boundary data are proved in Section 6.

2. The model

First, the phase field model proposed by G. Caginalp in [1] will be recalled, then generalizations considered in this paper will be introduced.

2.1. Phase field model

Let us outline the model proposed by G. Caginalp in [1]. Assume that a material subjected to phase changes, e.g. liquid to solid and back, occupies a region $\Omega \subset \mathbb{R}^N$. The evolution of the system is described in terms of the temperature $u$ and a phase function $\phi$ satisfying the following system of non-linear parabolic equations:

$$
\begin{align*}
&u_t + \frac{l}{2} \phi_t - k \Delta u = 0 \\
&\tau \phi_t - 2u + \frac{1}{2} (\phi^3 - \phi) - \xi^2 \Delta \phi = 0
\end{align*}
$$

in $\Omega \times (0,T)$. (1)

The first equation of (1) expresses the balance of heat energy. Notice that this equation is scaled so that the term $u_t$ appears without any multiplier. Here, $k$ is the scaled heat conductivity coefficient which is assumed to be the same in the solid and liquid states, and $l$ is the scaled specific latent heat of the phase change. The second equation of (1) is derived from statistical physics, namely from the Landau-Ginzburg theory of phase transitions [5]. In equilibrium, $\phi$ minimizes the following free energy functional:

$$
F_u\{\phi\} = \int_{\mathbb{R}^N} \left[ \frac{\xi^2}{2} |\nabla \phi|^2 + \frac{1}{8} (\phi^2 - 1)^2 - 2u \phi \right],
$$

2
where \( \xi \) is a length scale, the thickness of the interfacial region between liquid and solid. In non-equilibrium, \( \phi \) does not minimize \( F_u \{ \phi \} \) but satisfies the gradient flow equation \( \tau \phi_t = -\delta F_u \{ \phi \} \), where the symbol “\( \delta \)” denotes the variation of \( F_u \) in \( \phi \), which yields the second equation of (1). It is clear that the constant \( \tau \) characterizes the relaxation time.

The following boundary conditions are usually imposed (see e.g. [4]):

\[
-k \partial_{\nu} u = \lambda (u - g), \quad -\partial_{\nu} \phi = 0 \quad \text{on} \quad \partial \Omega, \tag{2}
\]

where \( \nu \) denotes the outer normal to \( \Omega \), and \( \lambda \) is the scaled overall heat conductivity.

The specific of the case considered in this paper is that the temperature \( g \) is not directly applied to the boundary of the liquid but to the outer surface of the container (ampoule). To include the container into the model, still denote the inner part occupied by the medium subjected to the phase change by \( \Omega \), and the solid walls of the container by \( D \). Moreover, let \( U = D \cup \overline{\Omega} \) be the region occupied by the whole system. In this case, the phase function \( \phi \) is defined in \( \Omega \), the temperature \( u \) is defined in \( U \), and the boundary function \( g \) is defined on \( \partial U \). Thus, the modification of the model (1) and (2) reads as follows:

\[
\begin{aligned}
\frac{u_t}{\tau} + \frac{l}{2} \phi_t - k_\Omega \Delta u &= 0 & \text{in } \Omega \times (0, T), \\
\tau \phi_t - 2u + \frac{1}{2} (\phi^3 - \phi) - \xi^2 \Delta \phi &= 0 & \text{in } \Omega \times (0, T), \\
-k_D \partial_{\nu} u_D &= -k_\Omega \partial_{\nu} u_\Omega & \text{on } \partial \Omega \times (0, T), \\
u_D &= u_\Omega & \text{on } \partial \Omega \times (0, T), \\
-\partial_{\nu} \phi &= 0 & \text{on } \partial \Omega \times (0, T), \\
\end{aligned}
\]

\[
\begin{aligned}
\frac{u_t}{\tau} - k_D \Delta u &= 0 & \text{in } D \times (0, T), \\
-k_D \partial_{\nu} u &= \lambda (u - g) & \text{on } \partial U \times (0, T), \\
u(x, 0) &= u^0(x) & \text{in } U, \\
\phi(x, 0) &= \phi^0(x) & \text{in } \Omega.
\end{aligned}
\]

Here the indices \( D \) and \( \Omega \) denote restrictions to the domains \( D \) and \( \Omega \), respectively, and the symbol \( \nu \) is used to indicate outward normals as well to \( \Omega \) as to \( U \), whenever it is not ambiguous. Note that the matching conditions imposed on \( u \) in (3) mean, in particular, the continuity of the temperature and the heat flux across \( \partial \Omega \).

The main result of this paper is that the problem (3) admits a unique weak solution \((u, \phi)\), and the mapping

\[
(u_0^0, \phi_0^0, g) \rightarrow (u, \phi) : L^2(U) \times H^1(\Omega) \times L^2(\partial U \times (0, T)) \rightarrow C([0, T]; L^2(U)) \times C([0, T]; H^1(\Omega))
\]

is continuous provided that \( U \) and \( \Omega \) are bounded Lipschitz domains in \( \mathbb{R}^N \), \( N = 2, 3 \). A precise formulation of this result is given in the next section, see Definition 1 and Theorem 1.
3. Problem statement and main result

Problem (3) will be studied in a weak formulation. Denote \( Q_T := \Omega \times (0, T) \) and introduce the following spaces according to [6] and [7]:

\[
X_U := \{ u \in L^2(0, T; H^1(U)) : u_t \in L^2(0, T, (H^1(U))') \},
\]

\[
C_s([0, T]; H^1(\Omega)) := \{ \eta \in L^\infty(0, T, H^1(\Omega)) : t \to \langle \eta(t); \xi \rangle
\text{ is continuous on } [0, T] \text{ for each } \xi \in (H^1(\Omega))' \},
\]

where \( \langle \cdot; \cdot \rangle \) denotes the dual pairing between \( H^1(\Omega) \) and \( (H^1(\Omega))' \).

**Definition 1 (Weak solutions)** A pair \((u, \phi)\) of functions

\[
u \in X_U \quad \text{and} \quad \phi \in H^1(0, T; L^2(\Omega)) \cap L^\infty(0, T; H^1(\Omega))
\]

satisfying the initial conditions

\[
u(0) = u^0 \quad \text{and} \quad \phi(0) = \phi^0
\]

is a weak solution of problem (3), if the identities

\[
0 = \langle u_t; \psi \rangle_{X_U} + \int_0^T \int_\Omega \frac{1}{2} \phi_t \psi + \int_0^T \int_U k \nabla u \cdot \nabla \psi + \int_0^T \int_{\partial U} \lambda (u - g) \psi,
0 = \int_0^T \int_{\Omega} \left[ \left( \tau \phi_t - 2u + \frac{1}{2} (\phi^3 - \phi) \right) \eta + \xi^2 \nabla \phi \cdot \nabla \eta \right]
\]

hold for all test functions \( \psi \in L^2(0, T; H^1(U)) \) and \( \eta \in L^2(\Omega) \cap L^1(0, T; H^1(\Omega)) \).

**Remark 1** Notice that the initial conditions in Definition [4] have a sense because

\[
X_U \subset C([0, T]; L^2(U))
\]

and

\[
H^1(0, T; L^2(\Omega)) \cap L^\infty(0, T; H^1(\Omega)) \subset C_s([0, T]; H^1(\Omega)),
\]

see [3, Th. 25.5] for the first assertion and [6, Chap. 3, Lemma 8.1] for the second one.

The next theorem states the main result of this paper.

**Theorem 1 (Main result)** Let \( U \) and \( \Omega \) be bounded Lipschitz domains in \( \mathbb{R}^N, N = 2, 3 \), with \( \overline{\Omega} \subset U \), and \( T > 0 \) be finite. If \( u^0 \in L^2(U), \phi^0 \in H^1(\Omega) \), and \( g \in L^2(\partial U \times (0, T)) \), then

1. There exists a unique weak solution \((u, \phi)\) of problem (3) in the sense of Definition [4].
2. The phase function $\phi$ has the additional regularity $\phi \in C([0, T]; H^1(\Omega))$.

3. For all $s, t \in [0, T]$, the following energy equation holds:

$$\int_{\Omega} \left[ \frac{1}{8} \phi(t)^4 - \frac{1}{4} \phi(t)^2 + \frac{\xi^2}{2} |\nabla \phi(t)|^2 \right] = \int_{\Omega} \left[ \frac{1}{8} \phi(s)^4 - \frac{1}{4} \phi(s)^2 + \frac{\xi^2}{2} |\nabla \phi(s)|^2 \right] + \int_t^t \int_{\Omega} \phi_t [2u - \tau \phi_t].$$

(7)

4. Problem (3) is well-posed in the sense that the mapping

$$(u^0, \phi^0, g) \mapsto (u, \phi) : L^2(U) \times H^1(\Omega) \times L^2(\partial U \times (0, T))$$

$$\rightarrow X_U \times \left( H^1(0, T; L^2(\Omega)) \cap C([0, T]; H^1(\Omega)) \right)$$

is continuous.

The proof of Theorem 1 is given in the following sections.

4. Existence of weak solutions

The existence of weak solutions of problem (3) can be proved in the similar way as Theorem 3.1.2 in [3], and therefore we only discuss changes in the construction of approximate solutions and a priori estimates. For details related to the passage to the limit, we refer to [3].

The next lemma establishes some a priori estimates and the existence of weak solutions to problem (3).

Lemma 1 Let $U$ and $\Omega$ be bounded Lipschitz domains in $\mathbb{R}^N$, $N = 2, 3$, with $\overline{\Omega} \subset U$, and $T > 0$. If $u^0 \in L^2(U)$, $\phi^0 \in H^1(\Omega)$, and $g \in L^2(\partial U \times (0, T))$, then there exists a weak solution $(u, \phi)$ to problem (3) in the sense of Definition 1.

PROOF. Let $\{\zeta_i\}_{i=1}^\infty$ be a basis of $H^1(U)$ which is orthonormal in $L^2(U)$, and $\{\omega_i\}_{i=1}^\infty$ a basis of $H^1(\Omega)$ which is orthonormal in $L^2(\Omega)$. Consider Galerkin approximations of the form

$$u^m(\mathbf{x}, t) = \sum_{i=0}^m a_i^m(t) \zeta_i(\mathbf{x}), \quad \phi^m(\mathbf{x}, t) = \sum_{i=0}^m b_i^m(t) \omega_i(\mathbf{x}),$$

(8)

where the functions $a_i^m(t)$ and $b_i^m(t)$ are to be determined. Let $\{g^m\} \subset C([0, T]; L^2(\partial U))$ be a sequence such that $g^m \to g$ in $L^2(\partial U \times (0, T))$ as $m \to \infty$.

To determine the functions $a_i^m$ and $b_i^m$, we require that $u^m$ and $\phi^m$ satisfy
the relations

\[
0 = \int \frac{1}{2} \phi^m_i(t) \psi^m + \int_U \left[ u^m_i(t) \psi^m + k \nabla u^m(t) \cdot \nabla \psi^m \right] + \int_{\partial U} \lambda (u^m(t) - g^m(t)) \psi^m,
\]

\[
0 = \int \left[ \tau \phi^m_i(t) - 2u^m(t) + \frac{1}{2} \left( (\phi^m_i(t))^3 - \phi^m_i(t) \right) \right] \eta^m_i + \int \xi^2 \nabla \phi^m(t) \cdot \nabla \eta^m
\]

for all test functions \( \psi^m \in \text{span}\{\zeta_j : j = 1, \cdots, m\} \) and \( \eta^m \in \text{span}\{\omega_j : j = 1, \cdots, m\} \).

Substituting ansatz (8) and test functions \( \psi^m = \zeta_j \) and \( \eta^m = \omega_j \), \( j = 1, \ldots, m \), into equations (9) yields the following system of ordinary differential equations for determining \( a^m_j \) and \( b^m_j \):

\[
0 = \sum_{i=1}^m \left\{ \dot{a}^m_i(t) \int_U \zeta_i \zeta_j + k a^m_i(t) \int_U \nabla \zeta_i \cdot \nabla \zeta_j + \lambda \int_{\partial U} a^m_i(t) \zeta_i \zeta_j \right\} + \sum_{i=1}^m \dot{b}^m_j(t) \frac{1}{2} \int_{\Omega} \omega_i \zeta_j - \lambda \int_{\partial U} g_m \zeta_j,
\]

\[
0 = \sum_{i=1}^m \left\{ \tau \dot{b}^m_i(t) \int_{\Omega} \omega_i \omega_j - \frac{1}{2} b^m_i(t) \int_{\Omega} \omega_i \omega_j + \xi^2 b^m_i(t) \int_{\Omega} \nabla \omega_i \cdot \nabla \omega_j \right\} - \sum_{i=1}^m 2a^m_i(t) \int_{\Omega} \zeta_i \omega_j + \frac{1}{2} \int_{\Omega} \left( \sum_{i=1}^m b^m_i(t) \omega_i \right)^3 \omega_j.
\]

Since the set \( \{\zeta_i\} \) is orthonormal in \( L^2(U) \) and the set \( \{\omega_i\} \) is orthonormal in \( L^2(\Omega) \), equations (10) can be rewritten as the following system of ODEs:

\[
\begin{bmatrix}
I_m \\
(1/2)M
\end{bmatrix}
\begin{bmatrix}
\dot{a}^m(t) \\
\dot{b}^m(t)
\end{bmatrix}
+ \begin{bmatrix}
A^m(a^m(t), b^m(t)) \\
B^m(a^m(t), b^m(t))
\end{bmatrix}
= \begin{bmatrix}
0 \\
0
\end{bmatrix},
\]

where \( M := \int_{\Omega} \omega_i \zeta_j, i,j = 1, \ldots, m; I_m \) denotes the \( m \times m \) identity matrix; and \( A^m(a^m(t), b^m(t)) \) and \( B^m(a^m(t), b^m(t)) \) denote the terms of (10) which do not comprise the time derivatives of the unknown functions.

Initial conditions for \( a^m(t) \) and \( b^m(t) \) are specified as follows:

\[
a^m_j(0) = \left( P^m_{L^2(U)} u^0 \right)_j \quad \text{and} \quad b^m_j(0) = \left( P^m_{H^1(\Omega)} \phi^0 \right)_j,
\]

where \( P^m_{L^2(U)} \) and \( P^m_{H^1(\Omega)} \) denote the projectors onto the subspaces \( \text{span}\{\zeta_j : j = 1, \cdots, m\} \) and \( \text{span}\{\omega_j : j = 1, \cdots, m\} \) in \( L^2(U) \) and \( H^1(\Omega) \), respectively.

Notice that, for each fixed \( m \), the functions \( A^m_i \) and \( B^m_i \) are analytic with respect to all their variables. Thus, the theory of ordinary differential equations
provides that, for each \( m \), there exist a nonempty time interval \([0, T_m]\) on which the initial value problem \((11)\) and \((12)\) admits a unique solution \([a^m(t), b^m(t)]\).

To continue these solutions to any time interval, establish some independent on \( m \) a priori estimates on \( u^m \) and \( \phi^m \). First, substitute \( \psi^m = u^m(t) \) into the first equation of \((9)\), integrate it over \((0, t)\) for some \( t \in (0, T_m] \), and use Young’s inequality to obtain the estimate

\[
\frac{1}{2} \int_U |u^m(t)|^2 + k \int_0^t \int_{\partial U} |\nabla u^m|^2 + \frac{\lambda}{2} \int_0^t \int_{\partial U} |u^m|^2 \leq \frac{1}{2} \int_U |u^0|^2 + \frac{\lambda}{2} \int_0^t \int_{\partial U} |g^m|^2 + \epsilon \int_0^t \int_{\partial U} |\phi^m|^2 + \frac{t^2}{8\epsilon} \int_0^t \int_U |u^m|^2
\]

\[(13)\]

for \( \epsilon > 0 \). Next, substitute \( \eta^m = \phi^m(t) \) into the second equation of \((9)\) and proceed as before to obtain

\[
\frac{\tau}{2} \int_{\Omega} |\phi^m(t)|^2 + \frac{\tau}{2} \int_{\Omega} \left[ \frac{1}{2} |\phi^m|^4 + \xi^2 |\nabla \phi^m|^2 \right] \leq \frac{\tau}{2} \int_{\Omega} |\phi^0|^2 + \frac{\tau}{2} \int_{\Omega} |\phi^m|^2 + 2 \int_0^t \int_U |u^m|^2.
\]

\[(14)\]

Now, substitute \( \eta^m = \phi^m(t) \) into the second equation of \((9)\) and proceed as before to obtain

\[
(\tau - \epsilon) \int_0^t \int_{\Omega} |\phi^m(t)|^2 + \frac{\tau}{4} \int_{\Omega} \left[ \frac{1}{8} |\phi^m(t)|^4 + \frac{\xi^2}{2} |\nabla \phi^m|^2 \right] \leq \int_0^t \left[ \frac{1}{8} |\phi^0|^4 + \frac{\xi^2}{2} |\nabla \phi^0|^2 \right] + \frac{1}{4} \int_{\Omega} |\phi^m(t)|^2 + \frac{1}{\epsilon} \int_0^t \int_U |u^m|^2
\]

\[(15)\]

for \( \epsilon > 0 \).

Choosing \( \epsilon \) sufficiently small; multiplying inequalities \((13)\), \((14)\), and \((15)\) by suitable constants; adding the resulting inequalities; and using the embedding \( H^1(\Omega) \hookrightarrow L^4(\Omega) \) yield the estimate

\[
\int_U |u^m(t)|^2 + \int_{\Omega} \left[ |\phi^m(t)|^2 + |\phi^m(t)|^4 + |\nabla \phi^m(t)|^2 \right] + \int_0^t \int_U |\nabla u^m|^2 + \int_0^t \int_{\partial U} |u^m|^2 + \int_0^t \int_{\partial U} |\phi^m(t)|^2
\]

\[
\leq C \left[ 1 + \int_0^t \int_{\Omega} |\phi^m|^2 + \int_0^t \int_U |u^m|^2 \right],
\]

\[(16)\]

where the constant \( C \) depends on \( \tau, \xi, l, \lambda, \|u^0\|_{L^2(\Omega)}, \|\phi^0\|_{H^1(\Omega)}, \|g\|_{L^2(\partial U \times (0, T))} \), and \( \epsilon \) but is independent on \( m \). By Gronwall’s inequality, the left-hand side of \((16)\) is bounded independently on \( m \), which along with the embedding \( H^1(\Omega) \hookrightarrow
\(L^0(\Omega), N \leq 3,\) implies the following assertions:

\[
\begin{align*}
\{ u^m \} & \text{ is bounded in } L^\infty(0, T_m; L^2(U)) \cap L^2(0, T_m; H^1(U)), \\
\{ u_t^m \} & \text{ is bounded in } L^2(0, T_m; (H^1(U))'), \\
\{ \phi^m \} & \text{ is bounded in } L^\infty(0, T_m; H^1(\Omega)), \\
\{ \phi_t^m \} & \text{ is bounded in } L^2(0, T_m; L^2(\Omega)), \\
\{ (\phi^m)^3 \} & \text{ is bounded in } L^\infty(0, T_m; L^2(\Omega)),
\end{align*}
\]

where the bounds are independent on \(m.\) This means that the approximate solutions can be continued to any interval \([0, T]\) keeping the above mentioned bounds. Now, the proof of the lemma can be completed analogously to that of \(\mathbb{R}, \text{ Th. 3.1.2}. \) □

5. Energy equalities

In this section, two lemmas are proved. Lemma 2 states a slightly weaker than (7) energy equality, which nevertheless implies the continuity in time of the phase function: \(\phi \in C([0, T]; H^1(\Omega)).\) Lemma 3 completes the proof of the energy equality (7). These results will be used in Section 6 to show the claimed uniqueness of weak solutions and their continuous dependence on the initial and boundary data.

The proof of the next lemma is based on the techniques of \(\mathbb{R}, \text{ Chap 3, Sec 8.4, Lemma 8.3}.\)

**Lemma 2 (An energy equality)** Let \((u, \phi)\) be a weak solution considered in Lemma 7. Then, for all \(s, t \in [0, T],\) the following energy equality holds:

\[
\int_\Omega \left( |\nabla \phi(t)|^2 - |\nabla \phi(s)|^2 \right) = \frac{2}{\xi^2} \int_s^t \int_\Omega \phi_t \left[ 2u - \tau \phi_t - \frac{1}{\xi^2} (\phi^3 - \phi) \right].
\]

Moreover, equality (18) implies that \(\phi \in C([0, T]; H^1(\Omega)).\)

**PROOF.** Assume that (18) holds. Notice that the assertions of (17) provide the integrability of the integrand in the right-hand side of (18) and therefore the convergence of the integral to zero as \(t \to s,\) which proves the continuity of the function \(t \to \int_\Omega |\nabla \phi(t)|^2.\) This, along with the inclusion \(\phi \in C([0, T]; L^2(\Omega))\) provided by the third and forth assertions of (17), implies the continuity of the function \(t \to \|\phi(t)\|_{H^1(\Omega)}^2.\) Let \(t \in [0, T]\) be fixed, and \(t_n \to t\) as \(n \to \infty.\) Denote \(\epsilon_n = \|\phi(t_n) - \phi(t)\|_{H^1(\Omega)}^2\) and compute \(\epsilon_n = \|\phi(t_n)\|_{H^1(\Omega)}^2 + \|\phi(t)\|_{H^1(\Omega)}^2 - 2\langle \phi(t_n), \phi(t) \rangle_{H^1(\Omega)}\). Utilizing that \(\phi \in C_s([0, T]; H^1(\Omega))\) (see (15) and remark (14)) implies the convergence \(\epsilon_n \to 0,\) which means that \(\phi(t_n) \to \phi(t)\) in \(H^1(\Omega).\)

Now go on to the proof of equality (18) and introduce some notation. Similar to (6), denote the inner product in \(L^2(\Omega)\) or the dual pairing in \(H^1(\Omega)' \times H^1(\Omega)\) by \(\langle \cdot, \cdot \rangle,\) and denote the inner product in \(L^2(\mathbb{R}; L^2(\Omega))\) or the dual pairing in \(L^2(\mathbb{R}; H^1(\Omega)') \times L^2(\mathbb{R}; H^1(\Omega))\) by \(\langle \cdot, \cdot \rangle.\) Assume that \(\phi(t)\) is defined on \(\mathbb{R}\) and
possesses the same properties as $\phi(t)$ on $[0, T]$. This can be achieved through continuation of $\phi(t)$ by means of reflections. In the following, set $s = 0$ and $t = t_0$ in equation (13).

Define a bilinear form $a : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ and an operator $A : H^1(\Omega) \to H^1(\Omega)'$ by

$$a(v, w) = \langle Av ; w \rangle = \int_\Omega \nabla v \cdot \nabla w \, dx, \quad v, w \in H^1(\Omega).$$

Then the second equation of (6) is equivalent to the relation

$$\int_0^T \langle A\phi(t) ; \eta(t) \rangle \, dt = \frac{1}{\xi^2} \int_0^\tau \int_\Omega \left[ 2u - \tau \phi_t - \frac{1}{2} (\phi^3 - \phi) \right] \eta.$$  (20)

For $t_0 \in (0, T]$ and $\delta \in (0, t_0/2)$, define the function $Q_\delta : \mathbb{R} \to [0, 1]$ by

$$Q_\delta(t) := \begin{cases} 1 & \text{for } t \in [\delta, t_0 - \delta] \\ 0 & \text{for } t \not\in [0, t_0] \\ \text{linear on } [0, \delta] \cup [t_0 - \delta, t_0] \text{ and continuous on } \mathbb{R}. \end{cases}$$  (21)

Let $\{\rho_n\}$ be a sequence of non-negative even regularizing functions with $\int_\mathbb{R} \rho_n(t) \, dt = 1$ and $\text{supp}\rho_n \subset (-1/n, 1/n)$. Denote $\sigma_n = \rho_n * \rho_n$.

Moreover, as it was already mentioned (see remark 1),

$$\phi \in C_a([0, T]; H^1(\Omega)).$$  (22)

Since $A$ is a symmetric operator, it holds

$$0 = \int_\mathbb{R} \frac{d}{dt} \langle A(\rho_n * Q_\delta \phi) ; \rho_n * Q_\delta \phi \rangle \, dt$$

$$= 2 \langle A(\rho_n * Q_\delta \phi) ; \rho_n * Q_\delta \phi_t \rangle + 2 \langle A(\rho_n * Q_\delta \phi) ; \rho_n * Q'_\delta \phi \rangle$$

$$= 2 \langle A(\rho_n * Q_\delta \phi) ; \rho_n * Q_\delta \phi_t \rangle + 2 \langle \rho_n * A(Q_\delta - Q_0) \phi ; \rho_n * Q'_\delta \phi \rangle$$

$$+ 2 \langle \rho_n * (A Q_0 \phi) ; \rho_n * Q'_\delta \phi \rangle.$$  (23)

Notice that $A\phi \in L^2(Q_T)$ because of the second equation of (8) and the assertions of (17). This implies that

$$A(\rho_n * Q_\delta \phi) \in L^2(\mathbb{R}; L^2(\Omega)),$$

since $\rho_n$ and $Q_\delta$ depend only on time, and $A$ is time independent. Therefore, all duality brackets on the right-hand side of equation (23) present the inner product of $L^2(\mathbb{R}; L^2(\Omega))$.

Consider the passage to the limit in each term of the right-hand side of (23) as $\delta \to 0$ and then as $n \to \infty$. The objective is to show the following three relations:

$$2 \langle A(\rho_n * Q_\delta \phi) ; \rho_n * Q_\delta \phi_t \rangle \to -\frac{2}{\xi^2} \int_0^T \int_\Omega \phi_t \left[ \tau \phi_t - 2u + \frac{1}{2} (\phi^3 - \phi) \right],$$

$$|\langle \rho_n * A(Q_\delta - Q_0) \phi ; \rho_n * Q'_\delta \phi \rangle| \to 0,$$

$$2 \langle \rho_n * (A Q_0 \phi) ; \rho_n * Q'_\delta \phi \rangle \to \langle A(\phi(0)) ; \phi(0) \rangle - \langle A(\phi(t_0)) ; \phi(t_0) \rangle,$$  (24)  (25)  (26)
as first $\delta \to 0$, and then $n \to \infty$, which along with equation (23) completes the proof of the lemma.

**Proof of (24):** Consider the limit as $\delta \to 0$. By properties of convolutions, it holds

$$A(\rho_n * (Q_\delta \phi)) = \rho_n * (Q_\delta A \phi) \to \rho_n * (Q_0 A \phi) \quad \text{in } L^2(\mathbb{R}; L^2(\Omega))$$

(27)
because these functions have compact supports in $t$. Moreover, $\rho_n * (Q_\delta \phi_t) \to \rho_n * (Q_0 \phi_t)$ in $L^2(\mathbb{R}; L^2(\Omega))$, and therefore, the first term of the right-hand side of the last equation of (23) satisfies

$$2 (A(\rho_n * Q_\delta \phi) ; \rho_n * Q_\delta \phi_t) \longrightarrow 2 (A(\rho_n * Q_0 \phi) ; (\rho_n * Q_0 \phi_t))$$

(28)
as $\delta \to 0$. Relation (28), the second equation of (6), and time independence of $A$ imply that

$$2 (A(\rho_n * Q_\delta \phi) ; \rho_n * Q_\delta \phi_t)$$

$$\longrightarrow - \frac{2}{\xi^2} \int_0^T \int_\Omega (\sigma_n * (Q_0 \phi_t)) \left[ \tau \phi_t - 2u + \frac{1}{2} (\phi^3 - \phi) \right]$$

(29)
as $\delta \to 0$. Moreover, note that

$$2\sigma_n * (Q_0 \phi_t)(\cdot) = \int_{\mathbb{R}^+} 2\sigma_n(\cdot - s) Q_0(s) \phi_t(s) \longrightarrow Q_0 \phi_t$$

(30)
in $L^2(\mathbb{R}; L^2(\Omega))$ as $n \to \infty$, because $\int_{\mathbb{R}^+} \sigma_n(s) ds = 1/2$. Therefore, relations (29) and (30) yield (24).

**Proof of (25):** Using Hölder’s inequality yields

$$|(\rho_n * A(Q_\delta - Q_0) \phi ; \rho_n * Q_\delta' \phi)|$$

$$\leq \|\rho_n * A(Q_\delta - Q_0) \phi\|_{L^\infty(\mathbb{R}; L^2(\Omega))} \cdot \|\rho_n * Q_\delta' \phi\|_{L^1(\mathbb{R}; L^2(\Omega))}.$$ 

(31)

Using again the relation $A\phi \in L^2(Q_T)$ and the properties of the convolution operator, one can show that the first factor on the right-hand side of estimate (31) tends to zero. The second factor on the right-hand side of (31) can be estimated using Young’s inequality as follows:

$$\|\rho_n * Q_\delta' \phi\|_{L^1(\mathbb{R}; L^2(\Omega))} \leq \|\rho_n\|_{L^1(\mathbb{R})} \cdot \|Q_\delta' \phi\|_{L^1(\mathbb{R}; L^2(\Omega))}$$

$$\leq \|Q_\delta'\|_{L^1(\mathbb{R})} \cdot \|\phi\|_{L^\infty(0,T; L^2(\Omega))}.$$ 

The right-hand side is bounded, because $\|\rho_n\|_{L^1(\mathbb{R})} = 1$, $\|Q_\delta'\|_{L^1(\mathbb{R})} = 2$, and supp $Q_\delta \subset [0,T]$. Thus, the right-hand side of (31) tends to zero as $\delta \to 0$, and relation (25) is proved.

**Proof of (26):** Notice that $\rho_n * (A\phi) \in C([0,T]; L^2(\Omega))$ because $A\phi \in L^2(Q_T)$. Thus, the function

$$t \mapsto \langle (\sigma_n * A Q_0 \phi)(t) ; \phi(t) \rangle$$

as $\rho_n * (A\phi) \to A\phi$ in $L^2(Q_T)$. Relation (29) and (30) imply that

$$2 (\rho_n * (Q_\delta \phi_t) ; (\rho_n * Q_\delta \phi_t))$$

$$\longrightarrow - \frac{2}{\xi^2} \int_0^T \int_\Omega (\sigma_n * (Q_0 \phi_t)) \left[ \tau \phi_t - 2u + \frac{1}{2} (\phi^3 - \phi) \right]$$

(29)
as $\delta \to 0$. Moreover, note that

$$2\sigma_n * (Q_0 \phi_t)(\cdot) = \int_{\mathbb{R}^+} 2\sigma_n(\cdot - s) Q_0(s) \phi_t(s) \longrightarrow Q_0 \phi_t$$

(30)
in $L^2(\mathbb{R}; L^2(\Omega))$ as $n \to \infty$, because $\int_{\mathbb{R}^+} \sigma_n(s) ds = 1/2$. Therefore, relations (29) and (30) yield (24).

**Proof of (25):** Using Hölder’s inequality yields

$$|(\rho_n * A(Q_\delta - Q_0) \phi ; \rho_n * Q_\delta' \phi)|$$

$$\leq \|\rho_n * A(Q_\delta - Q_0) \phi\|_{L^\infty(\mathbb{R}; L^2(\Omega))} \cdot \|\rho_n * Q_\delta' \phi\|_{L^1(\mathbb{R}; L^2(\Omega))}.$$ 

(31)

Using again the relation $A\phi \in L^2(Q_T)$ and the properties of the convolution operator, one can show that the first factor on the right-hand side of estimate (31) tends to zero. The second factor on the right-hand side of (31) can be estimated using Young’s inequality as follows:

$$\|\rho_n * Q_\delta' \phi\|_{L^1(\mathbb{R}; L^2(\Omega))} \leq \|\rho_n\|_{L^1(\mathbb{R})} \cdot \|Q_\delta' \phi\|_{L^1(\mathbb{R}; L^2(\Omega))}$$

$$\leq \|Q_\delta'\|_{L^1(\mathbb{R})} \cdot \|\phi\|_{L^\infty(0,T; L^2(\Omega))}.$$ 

The right-hand side is bounded, because $\|\rho_n\|_{L^1(\mathbb{R})} = 1$, $\|Q_\delta'\|_{L^1(\mathbb{R})} = 2$, and supp $Q_\delta \subset [0,T]$. Thus, the right-hand side of (31) tends to zero as $\delta \to 0$, and relation (25) is proved.

**Proof of (26):** Notice that $\rho_n * (A\phi) \in C([0,T]; L^2(\Omega))$ because $A\phi \in L^2(Q_T)$. Thus, the function

$$t \mapsto \langle (\sigma_n * A Q_0 \phi)(t) ; \phi(t) \rangle$$
We have

\[2 \langle \sigma_n * (A Q_0 \phi) \rangle (0) - 2 \langle \sigma_n * (A Q_0 \phi) \rangle (t_0)\]  

(32)

It remains to show that

\[2 \langle \sigma_n * (A Q_0 \phi) \rangle (t_0) \rightarrow \langle A \phi (t_0) ; \phi (t_0) \rangle, \]

\[2 \langle \sigma_n * (A Q_0 \phi) \rangle (0) \rightarrow \langle A \phi (0) ; \phi (0) \rangle\]  

(33)

as \(n \rightarrow \infty\). Consider the first relation of (33). By the definition of \(Q_0\) and \(\sigma_n\), we have

\[2 \langle \sigma_n * (A Q_0 \phi) \rangle (t_0) = 2 \int_0^{t_0} \sigma_n (t) \langle (A Q_0 \phi) (t_0 - t) ; \phi (t_0) \rangle \, dt\]

and \(\int_0^{t_0} \sigma_n (t) \, dt = \frac{t_0}{2}\) because \(\sigma_n\) is even. From the definition of the operator \(A\) (see (19)), it follows that

\[2 \langle \sigma_n * (A Q_0 \phi) \rangle (t_0) \rightarrow \langle A \phi (0) ; \phi (0) \rangle\]

(34)

because the function \(t \mapsto a (\phi (t_0 - t), \phi (t_0)) = \langle \phi (t_0 - t) ; A \phi (t_0) \rangle\) is continuous, see (22). Thus, Lemma 2 is proved.

The next lemma proves an integral form of the relation \((\phi^3 - \phi) \phi_t = (\phi^4 / 4 - \phi^2 / 2)\), the chain rule.

**Lemma 3** If \(\phi \in H^1 (0, T; L^2 (\Omega)) \cap C ([0, T]; H^1 (\Omega))\), then

\[\int_s^t \int_\Omega \phi_t [\phi^3 - \phi] = \frac{1}{4} \int_\Omega [(\phi (t))^4 - (\phi (s))^4] - \frac{1}{2} \int_\Omega [(\phi (t))^2 - (\phi (s))^2]\]  

(35)

for all \(s, t \in [0, T]\).

**Proof.** Consider the case \(s = 0\). Since \(\phi \in H^1 (0, T; L^2 (\Omega))\), it holds

\[\int_0^t \int_\Omega \phi_t \phi = \frac{1}{2} \int_\Omega [\phi (t)^2 - \phi (0)^2],\]  

(36)

see e.g. [8], Th. 1.67.

To show the relation

\[\int_0^t \int_\Omega \phi_t \phi^3 = \frac{1}{4} \int_\Omega [(\phi (t))^4 - (\phi (0))^4],\]  

(37)
notice that $H^1(0, T; L^2(\Omega)) \cap L^\infty(0, T; H^1(\Omega)) \hookrightarrow C([0, T]; L^4(\Omega))$, which follows from [8, Sec. 8, Cor 4] and the compact embedding $H^1(\Omega) \hookrightarrow L^4(\Omega)$ for $N \leq 3$. Let $\{\phi_\varepsilon\}$ be a sequence of smooth functions such that

$$
\phi_\varepsilon \to \phi \quad \text{in } H^1(0, T; L^2(\Omega)) \cap C([0, T]; H^1(\Omega)) \quad \text{as } \varepsilon \to 0.
$$

(38)
The chain rule yields

$$
\int_0^t \int_\Omega (\phi_\varepsilon)_t \phi_\varepsilon^2 = \frac{1}{4} \int_\Omega [\phi_\varepsilon(t)^4 - \phi_\varepsilon(0)^4].
$$

(39)
The passage to the limit on the both sides of equation (39), as $\varepsilon \to 0$, has to be done. The left-hand side is being processed as follows:

$$
\int_0^t \int_\Omega [\phi_\varepsilon^2 (\phi_\varepsilon)_t - \phi_\varepsilon^3 \phi_\varepsilon] = \int_0^t \int_\Omega \left[ (\phi_\varepsilon^2 - \phi^3) (\phi_\varepsilon)_t + \phi^3 ((\phi_\varepsilon)_t - \phi_\varepsilon) \right].
$$

(40)
The first summand in the integral on the right-hand side can be estimated using Hölder’s inequality and the formula $\phi_\varepsilon^3 - \phi^3 = (\phi_\varepsilon - \phi) \zeta$ with $\zeta = \phi_\varepsilon^2 + \phi^2 + \phi_\varepsilon \phi$. It holds

$$
\left| \int_0^t \int_\Omega (\phi_\varepsilon - \phi) \zeta \cdot (\phi_\varepsilon)_t \right|^2 \leq \int_0^t \int_\Omega (\phi_\varepsilon - \phi)^2 \zeta^2 \cdot \int_0^t \int_\Omega (|\phi_\varepsilon|)_t^2
\leq \int_0^t \left\{ \left( \int_\Omega |\phi_\varepsilon - \phi|^6 \right)^{1/3} \cdot \left[ \int_\Omega |\zeta|^3 \right]^{3/2} \right\} dt \cdot \int_0^t \int_\Omega (|\phi_\varepsilon|)_t^2
\leq C \|\phi_\varepsilon - \phi\|_{L^\infty(0, T; H^1(\Omega))} \cdot \|\zeta\|_{L^{6/2}(0, T; L^3(\Omega))} \cdot \|(\phi_\varepsilon)_t\|_{L^2(\Omega)}
$$

(41)
It is not hard to prove that $\zeta \in L^\infty(0, T; L^3(\Omega))$. Really, Young’s inequality and the embedding $H^1(\Omega) \hookrightarrow L^6(\Omega)$ yield the estimate

$$
\int_\Omega \zeta^3 = \int_\Omega [\phi_\varepsilon^2 + \phi_\varepsilon \phi + \phi^2]^3
\leq \int_\Omega \phi_\varepsilon^6 + \int_\Omega \phi^6 + C \sum_{k=1}^5 \int_\Omega |\phi_\varepsilon^k \phi^{6-k}|
\leq \int_\Omega \phi_\varepsilon^6 + \int_\Omega \phi^6 + C \sum_{k=1}^5 \left[ \frac{k}{6} \int_\Omega \phi_\varepsilon^6 + \frac{6-k}{6} \int_\Omega \phi^6 \right]
\leq C \left[ \|\phi\|_{L^\infty(0, T; H^1(\Omega))} + \|\phi_\varepsilon\|_{L^\infty(0, T; H^1(\Omega))} \right]
$$

(42)
Thus, estimates (41) and (42) yield

$$
\int_0^t \int_\Omega (\phi_\varepsilon - \phi) \zeta (\phi_\varepsilon)_t \to 0 \quad \text{for } \varepsilon \to 0.
$$

(43)
Consider the second summand in the integral on the right-hand side of equation (40). Processing it similarly to the first term yields

\[
\left| \int_0^t \int_\Omega \phi^3((\phi_\epsilon)_t - \phi_t) \right|^2 \leq \int_0^t \int_\Omega \phi^6 \cdot \int_0^t \int_\Omega |(\phi_\epsilon)_t - \phi_t|^2 \\
\leq CT \| \phi \|_{L^\infty((0,T);H^1(\Omega))}^6 \cdot \| (\phi_\epsilon)_t - \phi_t \|^2_{L^2(0,T;L^2(\Omega))} \\
\rightarrow 0 \quad \text{as } \epsilon \rightarrow 0.
\]

(44)

Now, estimates (43) and (44) along with equation (40) yield

\[
\int_0^t \int_\Omega \left[ \phi^3(\epsilon)_t - \phi^3 \phi_t \right] \rightarrow 0 \quad \text{as } \epsilon \rightarrow 0.
\]

(45)

Using the approximation property (38), and the embedding \( H^1(\Omega) \hookrightarrow L^6(\Omega) \) yields the estimate

\[
\int_\Omega \left[ \phi^4(t) - \phi^4_\epsilon(t) \right] = \int_\Omega \left[ (\phi(t) - \phi_\epsilon(t)) \zeta(t) \right] \\
\leq \| \phi(t) - \phi_\epsilon(t) \|_{L^2(\Omega)} \cdot \| \zeta(t) \|_{L^2(\Omega)} \\
\rightarrow 0
\]

for all \( t \in [0,T] \) as \( \epsilon \rightarrow 0 \). Here we have used the abbreviation \( \zeta = \phi^3 + \phi^2 \phi_\epsilon + \phi^2 \phi_\epsilon + \phi^3 \in C([0,T];L^2(\Omega)) \).

Thus, equation (37) follows from equation (39) and relations (45) and (46), which proves Lemma 3.

Finally, the energy equality (7) follows from Lemmas 2 and 3.

6. Uniqueness and continuous dependency on the data

The next lemma completes the proof of Theorem 1.

Lemma 4 Let \( U \) and \( \Omega \) be bounded Lipschitz domains in \( \mathbb{R}^N, N = 2, 3 \), with \( \Omega \subset U \), and \( T > 0 \). Let \( u^0_i \in L^2(U) \), \( \phi^0_i \in H^1(\Omega) \), and \( g_i \in L^2(\partial U \times (0,T)) \), \( i = 1, 2 \), be given initial and boundary data of problem (3); \( (u_i, \phi_i) \) weak solutions of problem (3) in the sense of Definition 7. If \( \bar{u} = u_1 - u_2 \) and \( \bar{\phi} = \phi_1 - \phi_2 \), then

\[
\begin{align*}
&\| \bar{u} \|_{C([0,T];L^2(U))} \\
&\| \bar{u} \|_{L^2(0,T;H^1(U))} \\
&\| \bar{\phi} \|_{C([0,T];H^1(\Omega))} \\
&\| \bar{\phi} \|_{H^1(0,T;L^2(\Omega))} \\
\end{align*}
\leq F\left( \| \bar{u}^0 \|_{L^2(U)}, \| \bar{\phi}^0 \|_{H^1(\Omega)}, \| g \|_{L^2(\partial U \times (0,T))} \right),
\]

(47)

where \( F : [0,\infty)^3 \rightarrow [0,\infty) \) is a continuous function with \( F(0,0,0) = 0 \).
PROOF. Denote \( \zeta = \phi_1^2 + \phi_2^2 + \phi_3^2 \) and bear in mind that \( \zeta \) is non-negative. Obviously, \( \bar{u} \) and \( \phi \) satisfy the equations

\[
0 = \langle \bar{u}; \psi \rangle_{X_U} + \int_0^T \int_\Omega \phi_t \psi + \int_0^T \int_U k \nabla \bar{u} \cdot \nabla \psi + \int_0^T \int_{\partial U} \lambda (\bar{u} - \bar{g}) \psi
\]

\[
0 = \int_0^T \int_\Omega \left[ \left( \tau \phi_t + -2\bar{u} + \frac{\zeta}{2} \bar{\phi} - \frac{1}{2} \bar{\phi} \right) \eta + \xi^2 \nabla \phi \cdot \nabla \eta \right]
\]

(48)

for all test functions \( \psi \in L^2(0, T; H^1(U)) \) and \( \eta \in L^2(0, T; H^1(\Omega)) \). In particular, test functions \( \psi = \chi_{(0, \tau)} \bar{u}, \eta = \chi_{(0, \tau)} \phi, \) and \( \eta = \chi_{(0, \tau)} \bar{\phi} \) will be considered to obtain three estimates. Here, \( \chi_{(0, \tau)} \) denotes the characteristic function of the interval \((0, t)\). Notice that the proofs of Lemmas 2 and 3 show that \( \eta = \chi_{(0, \tau)} \bar{\phi} \) is an admissible test function.

Substituting \( \psi = \chi_{(0, \tau)} \bar{u} \) into the first equation of (48) and using Young’s inequality yield the estimate

\[
\frac{1}{2} \int_U |\bar{u}(t)|^2 + \int_0^t \int_U k|\nabla \bar{u}|^2 + \frac{\lambda}{2} \int_0^t \int_{\partial U} |\bar{u}|^2
\]

\[
\leq \frac{1}{2} \int_U |\bar{u}(t)|^2 + \frac{\lambda}{2} \int_0^t \int_{\partial U} |\bar{g}|^2 + \int_0^t \int_{\Omega} \left[ \xi |\phi_t|^2 + \frac{l^2}{16\epsilon} |\bar{u}|^2 \right]
\]

(49)

for any \( \epsilon > 0 \). Substituting \( \eta = \chi_{(0, \tau)} \bar{\phi} \) into the second equation of (48) and applying Young’s inequality yield

\[
\frac{\tau}{2} \int_\Omega |\bar{\phi}(t)|^2 + \int_0^t \int_\Omega \left[ \frac{\zeta}{2} |\bar{\phi}|^2 + \xi^2 |\nabla \bar{\phi}|^2 \right]
\]

\[
\leq \frac{\tau}{2} \int_\Omega |\bar{\phi}_0|^2 + \int_0^t \int_\Omega \left[ |\bar{u}|^2 + \frac{3}{2} |\bar{\phi}|^2 \right].
\]

(50)

Substituting \( \eta = \chi_{(0, t)} \bar{\phi}_t \) into the second equation of (48) and applying Young’s inequality yield

\[
\int_\Omega \left[ \frac{1}{4} |\bar{\phi}|^2 + \frac{\xi^2}{2} |\nabla \bar{\phi}(t)|^2 \right] + \int_0^t \int_\Omega \frac{\tau}{4} |\bar{\phi}_t|^2
\]

\[
\leq \int_\Omega \left[ \frac{1}{4} |\bar{\phi}(t)|^2 + \frac{\xi^2}{2} |\nabla \bar{\phi}_t|^2 \right] + \int_0^t \int_\Omega \left[ \frac{2}{\tau} |\bar{u}|^2 + \frac{1}{4\tau} \zeta^2 |\bar{\phi}|^2 \right].
\]

(51)

Using Hölder’s inequality and the embedding \( H^1(\Omega) \hookrightarrow L^6(\Omega) \) yield

\[
\int_0^t \int_\Omega [\zeta^2 |\bar{\phi}|^2] \leq \int_0^t \left\{ \left[ \int_\Omega [\zeta]^3 \right]^{2/3} \cdot \left[ \int_\Omega |\bar{\phi}|^6 \right]^{1/3} \right\}
\]

\[
\leq \| \zeta \|_{L^\infty(0,T;L^3(\Omega))} \| \bar{\phi}(t) \|_{L^6(\Omega)}^2
\]

\[
\leq C \| \zeta \|_{L^\infty(0,T;L^3(\Omega))} \cdot \int_0^t \int_\Omega \left[ |\bar{\phi}(t)|^2 + |\nabla \bar{\phi}(t)|^2 \right].
\]

(52)
Now, choose $\epsilon \in (0, \tau/4)$ in (49) and combine estimates (49), (50), (51), and (52) to obtain

$$\int_{U} |\bar{u}(t)|^2 + \int_{\Omega} |\bar{\phi}(t)|^2 + \int_{0}^{t} \int_{U} |\bar{u}|^2 + \int_{0}^{t} \int_{\Omega} |\bar{\phi}|^2 + \int_{0}^{t} \int_{\partial U} |\bar{g}|^2 \leq C \int_{\Omega} |\bar{\phi}(t)|^2 + C \left( \int_{0}^{t} \int_{U} |\bar{u}|^2 + \int_{0}^{t} \int_{\partial U} |\bar{\phi}|^2 \right)$$

where $C_\xi := \|\zeta\|_{L^\infty(0,T;L^3(\Omega))}$, and the constant $C$ depends on $k, l, \lambda, \xi, \zeta$ only. To eliminate the first term on the right-hand side of (53), multiply inequality (50) by a constant greater than $2C/\tau$ and add the resulting inequality to (53) (remember that $\zeta$ is non-negative). This yields

$$\int_{U} |\bar{u}(t)|^2 + \int_{\Omega} [ |\bar{\phi}(t)|^2 + |\nabla \bar{\phi}(t)|^2 ]$$

$$+ \int_{0}^{t} \int_{U} |\bar{\phi}|^2 + \int_{0}^{t} \int_{\Omega} |\bar{\phi}|^2 + \int_{0}^{t} \int_{\partial U} |\bar{g}|^2 \leq C \left( C_0 + (1 + C_\xi^2) \int_{0}^{t} \left( \int_{U} |\bar{u}|^2 + \int_{\Omega} [ |\bar{\phi}|^2 + |\nabla \bar{\phi}|^2 ] \right) \right),$$

where $C$ is sufficiently large, and

$$C_0 := \int_{U} |\bar{u}_0|^2 + \int_{\Omega} [ |\bar{\phi}_0|^2 + |\nabla \bar{\phi}_0|^2 ] + \int_{0}^{t} \int_{\partial U} |\bar{g}|^2.$$  

Applying Gronwall’s inequality to (54) yields

$$\int_{U} |\bar{u}(t)|^2 + \int_{\Omega} [ |\bar{\phi}(t)|^2 + |\nabla \bar{\phi}(t)|^2 ]$$

$$+ \int_{0}^{t} \int_{U} |\bar{\phi}|^2 + \int_{0}^{t} \int_{\Omega} |\bar{\phi}|^2 + \int_{0}^{t} \int_{\partial U} |\bar{g}|^2 \leq C_0 C [ 1 + TC(1 + C_\xi^2) \exp( TC(1 + C_\xi^2) ) ].$$

Due to the a priori estimates derived in Section 4 (see the assertions of (17)), the constant $C_\xi$ is bounded in terms of the problem data only. Thus, estimate (56) proves the continuous dependence of weak solutions on the problem data. In particular, inequality (56) and the definition of $C_0$ (see (55)) imply uniqueness of weak solutions. □

7. Simulation

The following simulation deals with a plastic ampoule used for freezing small tissue samples (see [4]). The diameter of the ampoule is equal to 1 cm, the height
equals 5 cm, and the wall thickness equals 0.1 cm. The ampoule is filled with water, and is being cooled with the rate of 1°C/s applied to the outer surface of the ampoule. The mass and thermal characteristics of plastic and water are easily available in reference books. The length scale $\xi$ and the relaxation time $\tau$ are equal to 0.03 and 0.005, respectively.

The first picture of Figure 1 shows an axial cut of the whole ampoule (region $U$). The others show the water part (region $\Omega$) only. The unfrozen part is shown in light, whereas the frozen one becomes dark.

![Figure 1](image)

**Figure 1.** Time development of the unfrozen (light) and frozen (dark) parts. Notice that $\phi \approx 1$ in the light region, and $\phi \approx -1$ in the dark one. There is a small transition zone between the unfrozen and frozen parts.

8. Conclusion

We have considered a phase field model describing phase changes of a medium located in a container with heat conductive walls that are free of phase changes. It is shown that the temperature and the phase variables are continuous functions with values in $L^2(U)$ and $H^1(\Omega)$, respectively, provided that the initial data are from $L^2(U)$ and $H^1(\Omega)$, respectively. Moreover, solutions depend continuously on the initial and boundary data of the problem.

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