CONE TOPOLOGIES OF PARATOPOLOGICAL GROUPS

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Abstract. We introduce so-called cone topologies of paratopological groups, which are a wide way to construct counterexamples, especially of compact-like paratopological groups with discontinuous inversion. We found a simple interplay between the algebraic properties of a basic cone subsemigroup $S$ of a group $G$ and compact-like properties of two basic semigroup topologies generated by $S$ on the group $G$.

1. Introduction

Let $G$ be a group endowed with a topology $\tau$. A pair $(G, \tau)$ is called a paratopological group provided the multiplication map $\cdot : G \times G \to G$ is continuous. In this case the topology $\tau$ is called a semigroup topology on $G$. Moreover, if the inversion $(\cdot)^{-1} : G \to G$ is continuous with respect to the topology $\tau$, then $(G, \tau)$ is a topological group.

Properties of paratopological groups, their similarity and difference to properties of topological groups, are described in book [ArhTka] by Alexander V. Arhangel’skii and Mikhail G. Tkachenko, in author’s PhD thesis [Rav3] and papers [Rav], [Rav2]. New Mikhail Tkachenko’s survey [Tka2] exposes recent advances in this area.

A standard example of a paratopological group failing to be a topological group is the Sorgenfrey line, that is the real line endowed with the Sorgenfrey topology (generated by the base consisting of half-intervals $[a, b)$, $a < b$).

Author’s experience suggests that counterexamples play an essential role in the development of the theory of paratopological groups. In topology and topological algebra have already been used a so-called cone topology or its variations to produce various pathological counterexamples, (see, for instance [Ban], [BanRav] [Rav], [Rav2] or [Tod]).

One of ways to obtain the examples is to take a topological group $(G, \tau)$ and a sub-semigroup $S \ni e$ of the group $G$ such that $g^{-1}Sg \subset S$ for each element $g \in G$. Then we can endow the group $G$ with cone semigroup topology $\tau_S$ consisting of sets $U \subset G$ such that for each point $x \in U$ there is a neighborhood $W \in \tau$ of $e$ such that $x(W \cap S) \subset U$.

A particular case of this construction is when $G$ is a real linear space and $S$ is a cone of $G$, that is a subset of $G$ such that $\lambda x + \mu y \in S$ for each elements $x, y \in S$ and each non-negative reals $\lambda, \mu$, which explains the term ”cone topology”. Sorgenfrey topology on the real line $\mathbb{R}$ is a cone topology $\tau_S$ on $\mathbb{R}$ for a cone $S$ of non-negative reals.

When the topology $\tau$ is antidiscrete, we shall denote the group $G$ endowed with the topology $\tau_S$ by $G_S$.

Let $\tau$ be a semigroup topology on the group $G$. The identity map from $(G, \tau_S)$ to $G_S$ is continuous. So if a topological property $P$ is preserved by continuous isomorphisms (for instance, compactness, countably compactness, pseudocompactness) and we wish the
group \((G, \tau_S)\) to have the property \(\mathcal{P}\), then we must to assure the group \(G_S\) to have the property \(\mathcal{P}\) too.

So we start from the simple case when the topology \(\tau\) is antidiscrete. As expected, in this basic case cone topologies has very weak separation properties, and even \(T_1\) or \(T_2\) cases are degenerated, see Section 3. But non-Hausdorffness of these constructions will be compensated by Hausdorff topology \(\tau\) in Section 4.

It turned out that there is a simple interplay between the algebraic properties of the semigroup \(S\) and properties of two basic semigroup topologies generated by \(S\) on the group \(G\), see Section 3.

The choice of investigated by as topological properties \(\mathcal{P}\) has the following motivation.

It turned out that the inversion on a paratopological group automatically is continuous under some topological conditions. The search and the investigation of these conditions is one of main branches in the theory of paratopological groups and it has a long history, especially related with compact-like properties (see the next section). This branch still is growing by efforts of people throughout the world. An interested reader can find known results on this subject, for instance, in [AlaSan, Introduction], in the survey [Rav3, Section 5.1], in Section 3 of the survey [Tka2] and in a recent paper [Rav6].

The present paper is the second, somewhat updated, part of the authors preprint [Rav7] which grew too large and will be split into several articles.

2. Definitions

Now it is the time and the place to recall the definitions of the used notions.

In this paper the word ”space” means ”topological space”.

Conditions which are close to compactness. In general topology an essential role play compact spaces which have nice properties. These properties are so nice that even spaces satisfying slightly weaker conditions are still useful. We recall that a space \(X\) is called

- **sequentially compact** if each sequence of \(X\) contains a convergent subsequence,
- **\(\omega\)-bounded** if each countable subset of \(X\) has the compact closure,
- **totally countably compact** if each sequence of \(X\) contains a subsequence with the compact closure,
- **countably compact at a subset** \(A\) of \(X\) if each infinite subset \(B\) of \(A\) has an accumulation point \(x\) in the space \(X\) (the latter means that each neighborhood of \(x\) contains infinitely many points of the set \(B\)),
- **countably compact** if \(X\) is countably compact at itself,
- **countably pracompact** if \(X\) is countably compact at a dense subset of \(X\),
- **pseudocompact** if each locally finite family of nonempty open subsets of the space \(X\) is finite,
- **finally compact** if each open cover of \(X\) has a countable subcover.

The following inclusions hold.

- Each compact space is \(\omega\)-bounded.
- Each \(\omega\)-bounded space is totally countably compact.
- Each totally countably compact space is countably compact.
- Each sequentially compact space is countably compact.
- Each countably compact space is countably pracompact.
- Each countably pracompact space is pseudocompact.
In these terms, a space $X$ is compact if and only if $X$ is countably compact and finally compact. A Tychonoff space $X$ is pseudocompact if and only if each continuous real-valued function on $X$ is bounded. A paratopological group $G$ is left (resp. right) precompact if for each neighborhood $U$ of the unit of $G$ there exists a finite subset $F$ of $G$ such that $FU = G$ (resp. $UF = G$). A paratopological group is left precompact if and only if it is right precompact. So we shall call left precompact paratopological groups precompact. A sequence $\{U_n : n \in \omega\}$ of subsets of a space $X$ is non-increasing if $U_n \supset U_{n+1}$ for each $n \in \omega$. A paratopological group $G$ is 2-pseudocompact if $\bigcap U_n^{-1} \neq \emptyset$ for each non-increasing sequence $\{U_n : n \in \omega\}$ of nonempty open subsets of $G$. Clearly, each countably compact paratopological group is 2-pseudocompact. A paratopological group $G$ is left $\omega$-precompact if for each neighborhood $U$ of the unit of $G$ there exists a countable subset $F$ of $G$ such that $FU = G$. A paratopological group $G$ is saturated if for each nonempty open subset $U$ of $G$ there exists a nonempty open subset $V$ of $G$ such that $V^{-1} \subset U$. Each precompact paratopological group is saturated and each saturated paratopological group is quasiregular.

Separation axioms. All topological spaces considered in the present paper are not supposed to satisfy any of the separation axioms, if otherwise is not stated. The definitions and relations between separations axioms both for topological spaces and paratopological groups can be found in the first section of the paper [Rav2]. Each $T_0$ topological group is $T_{3\frac{1}{2}}$, but for paratopological groups neither of the implications

$$T_0 \Rightarrow T_1 \Rightarrow T_2 \Rightarrow T_3$$

holds (independently of the algebraic structure of the group). Mikhail Tkachenko’s survey [Tka2] contains a section devoted to relations between separations axioms for paratopological groups. Some problems about these relations are still open. In paper [Tka2] are investigated axioms of separation in paratopological groups and related functors. Therefore the separation axioms play essential role in the theory of paratopological groups.

In particular, these axioms essentially affect results about automatic continuity of the inversion. We recall some not very well known separation axioms which we shall use in our paper. A space $X$ is $T_1$ if for every distinct points $x, y \in X$ there exists an open set $x \in U \subset X \setminus \{y\}$. A space $X$ is $T_3$ if each closed set $F \subset X$ and every point $x \in X \setminus F$ have disjoint neighborhoods. A space $X$ is regular if it is $T_1$ and $T_3$. A space $X$ is quasiregular if each nonempty open subset $A$ of $X$ contains the closure of some nonempty open subset $B$ of $X$. A space $X$ is a CB, if $X$ has a base consisting of canonically open sets, that is such sets $U$ that $U = \text{int} \overline{U}$. Each $T_3$ regular space is quasiregular and CB.

Suppose $A$ is a subset of a group $G$. Denote by $\langle A \rangle \subset G$ the subgroup generated by the set $A$.

3. Basic cone topologies

In this section $G$ is an abelian group and $S \ni 0$ is a subsemigroup of $G$. We restrict our attention here by abelian groups, because author’s experience suggests that when we check an implication of topological properties for a paratopological groups then positive results are proved for all groups, but negative are illustrated by abelian counterexamples. In this section, we consider an interplay between the algebraic properties of the semigroup $S$ and properties of two semigroup topologies generated by $S$ on the group $G$. Finishing the section we construct Example 2 of two 2-pseudocompact paratopological groups which are not topological groups.
We need the following technical lemmas.

**Lemma 1.** If there is an element \( a \in S \) such that \( S \subset a - S \), then \( S \) is a group. \( \square \)

**Lemma 2.** The following conditions are equivalent:
1. For each countable subset \( C \) of \( S - S \) there is an element \( a \in S \) such that \( C \subset a - S \).
2. For each countable subset \( C \) of \( S \) there is an element \( a \in S \) such that \( C \subset a - S \).
3. For each countable infinite subset \( C \) of \( S \) there are elements \( a \in S \) and \( B \) of \( C \) such that \( B \subset a - S \).

**Proof.** (3 \( \Rightarrow \) 2). Let \( C \) be an arbitrary countable subset of \( S \). If \( C \) is finite, then \( C \subset (\sum_{c \in C} c) - S \). Now suppose the set \( C \) is infinite. Let \( C = \{c_n : n \in \omega\} \). Put \( D = \{d_n : n \in \omega\} \), where \( d_n = \sum_{i=0}^{n} c_i \) for each \( n \in \omega \). If the set \( D \) is finite, then \( C \subset (\sum_{d \in D} d) - S \).

Now suppose the set \( D \) is infinite. There are an element \( a \in S \) and an infinite subset \( B \) of \( D \) such that \( B \subset a - S \). Therefore for each number \( i \in \omega \) there exists a number \( n \in \omega \) such that \( i \leq n \) and \( d_n \in B \). Then \( c_i \in d_n - S \in a - S - S \in a - S \).

(2 \( \Rightarrow \) 1). Let \( \{a_n : n \in \omega\} \) and \( \{b_n : n \in \omega\} \) of \( S \) such that \( c_n = b_n - a_n \) for each \( n \). There exist an element \( a \in S \) such that \( b_n \in a - S \) for each \( n \). Then \( c_n \in a - S \) for each \( n \). \( \square \)

### 3.1. Cone topology

The one-element family \( \{S\} \) satisfies Pontrjagin conditions (see [Rav. Pr. 1]). Therefore there is a semigroup topology on \( G \) with the base \( \{S\} \) at the zero. This topology is called the cone topology generated by the semigroup \( S \) on \( G \). Denote by \( G_S \), the group \( G \) endowed with this topology.

It is easy to check that the closure \( \overline{S} \) of the semigroup \( S \) is equal to \( S - S \) and \( \overline{S} = \langle S \rangle \). Moreover, \( \overline{S} \) is clopen.

**Proposition 1.** The group \( G_S \) is \( T_0 \) if and only if \( S \cap (\overline{S}) = \{0\} \). \( \square \)

**Proposition 2.** The group \( G_S \) is \( T_1 \) if and only if \( S = \{0\} \). \( \square \)

**Lemma 3.** A subset \( K \) of \( G_S \) is compact if and only if there is a finite subset \( F \) of \( K \) such that \( F + S \supset K \). \( \square \)

**Lemma 4.** Let \( A \) be a subset of \( G_S \). Then \( \overline{A} = A - S \). \( \square \)

We remind that a paratopological group \( G \) is topologically periodic if for each \( x \in G \) and a neighborhood \( U \subset G \) of the unit there is a number \( n \geq 1 \) such that \( x^n \in U \), see [BokGur].

It is easy to check the following

**Proposition 3.** The group \( G_S \) is topologically periodic if and only if \( S \) is a subgroup of the group \( G \) and the quotient group \( G/S \) is periodic. \( \square \)

**Proposition 4.** The following conditions are equivalent:
1. The group \( G_S \) is compact
2. The group \( G_S \) is \( \omega \)-bounded.
3. The group \( G_S \) is totally countably compact.
4. The group \( G_S \) is precompact.
5. The semigroup \( S \) is a subgroup of \( G \) of finite index.

**Proof.** Implications (1 \( \Rightarrow \) 4), (1 \( \Rightarrow \) 2) and (2 \( \Rightarrow \) 3) are trivial.

(4 \( \Rightarrow \) 5). Choose a finite subset \( F \) of \( G \) such that \( F + S = G \). Since \( f + S \subset f + \overline{S} \) for every \( f \in F \), \( S \subset (F \cap \overline{S}) + S \). Chose a number \( n \) and points \( x_1, \ldots, x_n, y_1, \ldots, y_n \in S \) such that \( F \cap \overline{S} = \{x_1 - y_1, \ldots, x_n - y_n\} \). Then \( \overline{S} = \bigcup x_i - y_i + S \subset \bigcup -y_i + S \). Put
\[ y = \sum y_i. \] Then \( S \subset \sum \). Therefore there is an element \( z \in S \) such that \(-2y = -y + z\). Then \( z = -y \) and hence \(-y_i \in S\) for every \( i \). Therefore \( S = S + S \) and \( \overline{S} \). Consequently \( S \) is a group. Since \( F + S = G \), we see that the index \( |G : S| \) is finite.

(5 \( \Rightarrow \) 1). Let \( U \) be a cover of the group \( G \) by its open subsets. Since \( S \) is a subgroup of \( G \) of finite index, we see that there exists a finite subset \( F \) of \( G \) such that \( F + S = G \).

For each point \( f \in F \) find set \( U_f \in U \) such that \( f \in U_f \). Let \( f \in F \) be an arbitrary point. Since the set \( U_f \) is open, \( f + S \subset U_f \). Therefore \( \{U_f : f \in F\} \) is a finite subcover of \( U \) and \( \bigcup \{U_f : f \in F\} = G \).

(3 \( \Rightarrow \) 5). The set \(-S = \{0\}\) is compact. Therefore there is a finite subset \( F \) of \( S \) such that \(-F + S \supset -S\). Let \( F = \{f_1, \ldots , f_n\} \). Put \( f = \sum f_i \). Then \(-f + S \supset F + S \supset -S\).

Moreover, \(-f + S \supset -f + S + S \supset S - S\). Hence there is an element \( g \in S \) such that \(-2f = -f + g\). Then \( g = -f \) and \(-f + S \supset S\). Therefore \( S = \overline{S} \). Suppose the index \( |G : S| \) is infinite. Then there is a countable set \( A \subset G \) that contains at most one point of each coset of the group \( S \) in \( G \). Let \( B \) be an arbitrary infinite subset of \( A \). The set \( \overline{B} = B - S \) intersects an infinite number of the cosets. Therefore \( \overline{B} \not\subset F + S \) for each finite subset \( F \) of \( G \). This contradiction proves that the index \( |G : S| \) is finite.

\[ \Box \]

**Proposition 5.** The following conditions are equivalent:

1. The group \( G_S \) is sequentially compact.
2. The group \( G_S \) is countably compact.
3. The group \( G_S \) is 2-pseudocompact.
4. The index \( |G : S| \) is finite and for each countable subset \( C \) of \( S \) there is an element \( a \in S \) such that \( C \subset a - S \).

Moreover, in this case each power of the space \( G_S \) is countably compact.

**Proof.** Implications (1 \( \Rightarrow \) 2) and (2 \( \Rightarrow \) 3) are trivial.

(3 \( \Rightarrow \) 4). Let \( C \) be an arbitrary countable infinite subset of \( S \). Let \( \{x_n : n \in \omega\} \) be an enumeration of elements of the set \( C \). For each \( n \in \omega \) put \( U_n = \bigcup_{i \geq n} x_i + S \).

Then \( \{U_n : n \in \omega\} \) is a non-increasing sequence of nonempty open subsets of \( G_S \). Since the group \( G_S \) is 2-pseudocompact then there is a point \( b \in G \) such that \((b + S) \cap -U_n \neq \emptyset\) for each \( n \in \omega \).

Then for each \( n \in \omega \) there is a number \( i \geq n \) such that \((b + S) \cap -(x_i - S) \neq \emptyset\) and hence \(-x_i \in b + S\).

Let \( B \) be the set of all such elements \( x_i \). Then \( B \) is an infinite subset of \(-b - S\). Since \( B \subset S \), we see that \(-b \in B + S \subset \emptyset \).

Now Lemma 2 implies that for each countable subset \( C \) of \( S \) there is an element \( a \in S \) such that \( C \subset a - S \).

Suppose the index \( |G : S| \) is infinite. Then there is a countable infinite set \( C \subset G \) that contains at most one point of each coset of the group \( S \) in \( G \). Let \( \{x_n : n \in \omega\} \) be an enumeration of the elements of the set \( C \). For each \( n \in \omega \) put \( U_n = \bigcup_{i \geq n} x_i + S \).

Then \( \{U_n\} \) is a non-increasing sequence of nonempty open subsets of \( G_S \). But for each point \( x \in G \) an open set \( x + S \) intersects only finitely many sets of the family \( \{-U_n : n \in \omega\} \).

This contradiction proves that the index \( |G : S| \) is finite.

(4 \( \Rightarrow \) 1). Let \( \{c_n : n \in \omega\} \) be a sequence of elements of the group \( G_S \). There is a point \( x \in G \) such that the set \( I = \{n \in \omega : c_n \in x + S \} \) is infinite.

By Lemma 2 there is a point \( a \in S \) such that \( x - c_n \in a - S \) for each \( n \in I \). Thus the sequence \( \{c_n : n \in I\} \) converges to the point \( x - a \).

Now let \( \kappa \) be an arbitrary cardinal. Condition 1 and Lemma 2 imply that for each countable subset \( C \) of \( S^\kappa \) there is a point \( a \in S^\kappa \) such that \( C \subset a - S^\kappa \). Consequently, the space \( S^\kappa \) is sequentially compact. Since \( S \) is a clopen subgroup of \( G_S \), we see that the
space $G_S$ is homeomorphic to $S \times D$, where $D$ is a finite discrete space. Then a space $G_S^\kappa$ is homeomorphic to the countably compact space $S^\kappa \times D^\kappa$.

The proof of the following lemma is straightforward.

**Lemma 5.** If $X$ is a countably pracompact space and $Y$ is a compact space then the space $X \times Y$ is countably pracompact.

**Proposition 6.** The following conditions are equivalent:

1. The group $G_S$ is countably pracompact.
2. The group $G_S$ is pseudocompact.
3. The index $|G : S|$ is finite.

Moreover, in this case each power of the space $G_S$ is countably pracompact.

**Proof.** Implication (1 $\Rightarrow$ 2) is trivial.

$(2 \Rightarrow 3)$. Suppose the index $|G : S|$ is infinite. Then there is a countable infinite set $C \subset G$ that contains at most one point of each coset of the group $S$ in $G$. Let $\{x_n : n \in \omega\}$ be an enumeration of the elements of the set $C$. For each $n \in \omega$ put $U_n = \bigcup_{i \geq n} x_i + S$. Then $\{U_n\}$ is a non-increasing sequence of nonempty open subsets of $G_S$. But for each point $a \in G$ an open set $x + S$ intersects only finitely many sets of the family $\{U_n : n \in \omega\}$. This contradiction proves that the index $|G : S|$ is finite.

$(3 \Rightarrow 1)$. Choose a finite set $A_0 \subset G$ such that the intersection of $A_0$ with each coset of the group $S$ in $G$ is a singleton. Put $A = A_0 + S$. Let $x \in G$ be an arbitrary point. Then there is an element $a \in A_0$ such that $x \in a + S$. Hence $(x + S) \cap (a + S) \neq \emptyset$. Therefore $(x + S) \cap A \neq \emptyset$. Thus $A$ is dense in $G_S$.

Let $C$ be an arbitrary countable infinite subset of $A$. There is a point $a \in A_0$ such that the set $B = C \cap (a + S)$ is infinite. But since $C \subset A_0 + S$, we see that $B \subset a + S$. Therefore $a$ is an accumulation point of the set $C$.

Now let $\kappa$ be an arbitrary cardinal. Since the set $S$ is dense in $S$, we see that the set $S^\kappa$ is dense in $S^\kappa$. Since each neighborhood of the zero of the group $S^\kappa$ contains the set $S^\kappa$, the space $S^\kappa$ is countably compact at $S^\kappa$. Hence the space $S^\kappa$ is countably pracompact. Since $S$ is a clopen subgroup of $G_S$, the space $G_S$ is homeomorphic to $S \times D$, where $D$ is a finite discrete space. Then a space $G_S^\kappa$ is homeomorphic to $S^\kappa \times D^\kappa$. By Lemma 5, the space $S^\kappa \times D^\kappa$ is countably pracompact.

**Proposition 7.** The group $G_S$ is finally compact if and only if $G_S$ is left $\omega$-precompact.

3.2. Cone* topology. The family $B_S = \{\{0\} \cup (x + S) : x \in S\}$ satisfies Pontrjagin conditions (see [Rav, Pr. 1]). Therefore there is a semigroup topology on $G$ with the base $B_S$ at the zero. This topology is called the a cone* topology generated by the semigroup $S$ and on $G$. Denote by $G_S^*$, the group $G$ endowed with this topology.

It is easy to check that the closure $\overline{S}$ of the semigroup $S$ is equal to $S - S$ and $\overline{S} = \langle S \rangle$. Moreover, $\overline{S}$ is clopen.

**Proposition 8.** The group $G_S^*$ is Hausdorff if and only if $S = \{0\}$.

It is easy to check the following

**Proposition 9.** The group $G_S^*$ is topologically periodic if and only if $S$ is a subgroup of the group $G$ and the quotient group $G/S$ is periodic.
Proposition 10. The following conditions are equivalent:
1. The group $G^*_S$ is $T_1$.
2. The group $G^*_S$ is $T_0$.
3. $S = \{0\}$ or $S$ is not a group.
4. $S = \{0\}$ or $\cap \{x + S : x \in S\} = \emptyset$.

Proof. Implications (1 $\Rightarrow$ 4), (4 $\Rightarrow$ 2), and (2 $\Rightarrow$ 3) are trivial. We show that (3 $\Rightarrow$ 1).
Suppose there is a point $y \in \cap \{x + S : x \in S\}$. Then $y \in 2y + S$. Hence there is an element $z \in S$ such that $y = 2y + z$. Therefore $-y = z \in S$. Now let $x$ be an arbitrary element of $S$. There is an element $s \in S$ such that $y = x + s$. Then $-x = -(x + s) + s \in S$. Therefore $S$ is a group. Thus $y = 0$. □

Proposition 11. The following conditions are equivalent:
1. The group $G^*_S$ is compact.
2. The group $G^*_S$ is $\omega$-bounded.
3. The group $G^*_S$ is totally countably compact.
4. The group $G^*_S$ is sequentially compact.
5. The group $G^*_S$ is countably compact.
6. The group $G^*_S$ is precompact.
7. The semigroup $S$ is a subgroup of $G$ of finite index.

Proof. Implications (1 $\Rightarrow$ 2), (2 $\Rightarrow$ 3), (3 $\Rightarrow$ 5), (4 $\Rightarrow$ 5) and (1 $\Rightarrow$ 6) are trivial. From (7) follows that $G^*_S = G_S$. Therefore by Proposition 11 (7 $\Rightarrow$ 1). From (6) follows that the group $G_S$ is precompact. Therefore by Proposition 11 (6 $\Rightarrow$ 7).

(7 $\Rightarrow$ 4). Let $\{c_n : n \in \omega\}$ be a sequence of elements of the group $G^*_S$. There is a point $x \in G$ such that the set $I = \{n \in \omega : c_n \in x + S\}$ is infinite. Then a subsequence $\{c_n : n \in I\}$ of the sequence $\{c_n : n \in \omega\}$ converges to the point $x$.

Finally we prove Implication (5 $\Rightarrow$ 7). Suppose (5). Let $x \in S$ be an arbitrary element. We claim that there is $n \geq 1$ such that $-nx \in S$. Assume the converse. Then the set $X = \{-nx : n \geq 1\}$ is infinite. There is a point $b \in G$ such that the set $(b + s + S) \cap X$ is infinite for each $s \in S$. Since $X \subset \overline{S}$, we see that $b \in \overline{S}$. Therefore there are elements $y, z \in S$ such that $b = y - z$. But since $S \supset (y - z) + z + S$, the set $S \cap X$ is infinite too. This contradiction proves that there is $n \geq 1$ such that $-nx \in S$. Therefore $S$ is a group. By Proposition 11 the index $|G : S|$ is finite. □

Proposition 12. The group $G^*_S$ is countably pracompact if and only if the index $|G : \overline{S}|$ is finite and there is a countable subset $C$ of $S$ such that $S \subset C - S$.

Moreover, in this case each power of the space $G^*_S$ is countably pracompact.

Proof. The necessity. Since the group $G_S$ is countably pracompact, by Proposition 11 the index $|G : \overline{S}|$ is finite. Let $A$ be a dense subset of $\overline{S}$ such that the space $\overline{S}$ is countably compact at $A$. Let $B$ be an arbitrary countable infinite subset of $A$. There are elements $x, y \in S$ such that the set $(x - y + s + S) \cap B$ is infinite for each $s \in S$. Since $s + S \supset (x - y) + y + s + S$, we see that the set $(s + S) \cap B$ is infinite for each $s \in S$. This condition holds for each countable infinite subset $B$ of $A$. Therefore the set $A \setminus (s + S)$ is finite for each $s \in S$. Since $A \subset S - S$, for each element $a \in A$ there is an element $s_a \in S$ such that $a + s_a \in S$. Put $A' = \{a + s_a : a \in A\}$. Then the set $A' \setminus (s + S)$ is finite for each $s \in S$. 


Suppose the set $A'$ is finite. Put $C = A'$. Since $A$ is dense in $S$, we see that $A'$ is dense in $S$ too. Let $s \in S$ be an arbitrary point. Then the set $(s + S) \cap C$ is nonempty. Therefore $s \in C - S$.

Suppose the set $A'$ is infinite. Let $C$ be an arbitrary countable infinite subset of $A'$. Let $s \in S$ be an arbitrary point. There is a point $c \in (s + S) \cap C$. Therefore $s \in C - S$.

The sufficiency. Let $C = \{c_n : n \in \omega \}$. Put $C' = \{ \sum_{n=0}^{\omega} c_i : n \in \omega \}$. Then for each element $s \in S$ and each infinite subset $B'$ of $C'$ there is a finite subset $F$ of $B'$ such that $c - S \ni s$ for each $c \in B' \setminus F$.

Choose a finite set $A_0 \subset G$ such that the intersection of $A_0$ with each coset of the group $S$ in $G$ is a singleton. Put $A = A_0 + C'$. Let $x \in G$ be an arbitrary point. Then there is an element $a \in A_0$ such that $x \in a + S$. Therefore there are elements $s, s' \in S$ such that $x - a = s - s'$. There is an element $c \in C'$ such that $s \in c - S$. Then $a + c = x + s' - s + c \in x + s' + s - c + c \subset x + S$. Hence $A$ is dense in $G_S^\kappa$.

Let $A'$ be an arbitrary countable infinite subset of $A$. There is a point $a \in A_0$ such that the set $B' = A' \cap (a + S)$ is infinite. Let $s \in S$ be an arbitrary element. Since $B' - a$ is an infinite subset of $C'$, the set $(s + S) \cap (B' - a)$ is infinite. Then the set $(a + s + S) \cap B'$ is infinite too. Therefore $a$ is an accumulation point of the set $B' \subset A'$.

Let $\kappa$ be an arbitrary cardinal. Put $A = \{ \emptyset \} \cup C'$. Then $A$ is compact and dense in $S$. Therefore $A^\kappa$ is compact and dense in $S^\kappa$. Hence the space $S^\kappa$ is countably compact at its dense subset. Consequently $S^\kappa$ is countably pracompact. Since $S$ is a clopen subgroup of $G_S^\kappa$, we see that the space $G_S^\kappa$ is homeomorphic to $S \times D$, where $D$ is a finite discrete space. Then a space $G_S^{\kappa*}$ is homeomorphic to $S^\kappa \times D^\kappa$. By Lemma 13, the space $S^\kappa \times D^\kappa$ is countably pracompact.

\begin{proposition}
The group $G_S^\kappa$ is 2-pseudocompact if and only if the index $|G : S|$ is finite and for each countable subset $C$ of $S$ there is an element $a \in S$ such that $C \subset a - S$.

Moreover, in this case each finite power of the group $G_S^\kappa$ is 2-pseudocompact and each power of the group $S$ is 2-pseudocompact.
\end{proposition}

\begin{proof}
The necessity. If the group $G_S^\kappa$ is 2-pseudocompact then the group $G_S$ is 2-pseudocompact too. Thus Proposition 13 implies the necessity.

The sufficiency. Let $\{ U_n : n \in \omega \}$ be a non-increasing sequence of nonempty open subsets of $G$. For each $n$ choose a point $z_n \in U_n$ such that $z_n + S \subset U_n$. Since the index $|G : S|$ if finite, there are an infinite subset $I$ of $\omega$ and an element $z \in G$ such that $z_n \in z + S$ for each $n \in I$. For each $n \in I$ fix elements $x_n, y_n \in S$ such that $z_n - z = x_n - y_n$. There is an element $a \in S$ such that $x_n \in a - S$ for each $n \in I$. Let $n \in I$ be an arbitrary number. Then $z_n - z = x_n - y_n \in a - S$. Therefore $U_n \ni z_n + S \ni a + z$. Since the family $\{ U_n \}$ is non-increasing, we see that $a + z \in \bigcap \{ U_n : n \in \omega \}$.

Now let $\kappa$ be an arbitrary cardinal. Let $\{ U_n : n \in \omega \}$ be an arbitrary non-increasing sequence of nonempty open subsets of $S^\kappa$. For each $n$ choose a point $z_n \in U_n$ such that $z_n + S^\kappa \subset U_n$ and fix elements $x_n, y_n \in S^\kappa$ such that $z_n = x_n - y_n$. There is an element $a \in S^\kappa$ such that $x_n \in a - S^\kappa$ for each $n \in \omega$. Thus $U_n \ni z_n + S^\kappa \ni a$.

\begin{corollary}
The group $G_S^\kappa$ is compact if and only if the group $G_S^\kappa$ is countably pracompact and 2-pseudocompact.
\end{corollary}

\begin{proof}
The proof is implied from Proposition 13, Proposition 12 and Lemma 1.
\end{proof}

\begin{proposition}
The group $G_S^\kappa$ is pseudocompact if and only if the index $|G : S|$ is finite.
\end{proposition}

\begin{proof}
\end{proof}
Proof. The necessity. If the group $G^*_S$ is pseudocompact then the group $G_S$ is pseudo-
compact too. Thus Proposition 11 implies the necessity.

The sufficiency. Let $\{U_n : n \in \omega\}$ be a family of nonempty open subsets of $G$. For each $n$ choose a point $x_n \in U_n$ such that $x_n + S \subset U_n$. Since the index $|G : \mathcal{S}|$ if finite, 
there are a point $b \in G$ and an infinite set $I \subset \omega$ such that $x_n \in b + \mathcal{S}$ for each $n \in I$. 
Let $n \in I$ be an arbitrary number. Then $(x_n + S) \cap (b + s + S) \neq \emptyset$ for each $s \in S$. 
Therefore $b \in \overline{x_n + S} \subset \overline{U_n}$. Hence the family $\{U_n\}$ is not locally finite. Thus the space 
$G^*_S$ is pseudocompact. \hfill $\Box$

The counterpart of Proposition 11 does not hold for the group $G^*_S$.

Example 1. There are an abelian group $G$ and a subsemigroup $S$ of the group $G$ such that 
$G^*_S$ is $\omega$-precompact and not finally compact.

Proof. Let $G = \mathbb{R}$ and $S = [0, \infty)$. Then $G^*_S$ is $\omega$-precompact. Consider an open cover 
$\mathcal{U} = \{\{x\} \cup S : x < 0\}$ of the group $G^*_S$. Then $\mathcal{U}$ has no subcover of cardinality less than 
c. By Proposition 12 the group $G^*_S$ is countably pracom pact. By Proposition 13 the 
group $G^*_S$ is not 2-pseudocompact. \hfill $\Box$

Proposition 15. If the group $G^*_S$ is Baire then $S$ is a group or there is no countable 
subset $C$ of $S$ such that $S \subset C - S$.

Proof. Suppose that there is a countable subset $C$ of $S$ such that $S \subset C - S$. Then 
$\overline{S} \subset \bigcup_{c \in C} c - S$. Since the set $-S$ is closed in $G^*_S$, there are elements $c \in S$ and $x \in \overline{S}$ 
such that $c - S \supset x + S$. It follows easily that $-S \subset S$. Thus $S$ is a group. \hfill $\Box$

Corollary 2. The countably pracom pact group $G^*_S$ is Baire if and only if $S$ is a group. \hfill $\Box$

Corollary 3. The countable group $G^*_S$ is 2-pseudocompact if and only if the semigroup $S$ 
is a subgroup of $G$ of finite index.

Proof. If the semigroup $S$ is a subgroup of $G$ of finite index then from Proposition 11 
it follows that $G^*_S$ is compact. If the group $G^*_S$ is 2-pseudocompact then Lemma 5 from 
[Rav6] implies that $G^*_S$ is Baire. By Proposition 15 $S$ is a group. Proposition 13 implies 
that the index $|G : S| = |G : \mathcal{S}|$ is finite. \hfill $\Box$

A space $X$ is called a $P$-space if every $G^*_S$ subset of $X$ is open.

Proposition 16. The group $G^*_S$ is a $P$-space if and only if for each countable subset $C$ 
of $S$ there is an element $a \in S$ such that $C \subset a - S$.

Proof. The necessity. Let $C$ be an arbitrary countable subset of $G$. Then the set $\{0\} \cup 
\bigcap\{c + S : c \in C\}$ is a neighborhood of the zero. Therefore there is a point $b \in S$ such 
that $b + S \subset \{0\} \cup \bigcap\{c + S : c \in C\}$. If $b \neq 0$ then $C \subset b - S$. Suppose $b = 0$. If $S = \{0\}$ 
then $C \subset S = 0 - S$. If there is an element $a \in S \{0\}$ then $a \in \bigcap\{c + S : c \in C\}$ and 
$C \subset a - S$.

The sufficiency. Let $\{U_n : n \in \omega\}$ be an arbitrary sequence of nonempty open subsets 
of $G$, $U = \bigcap\{U_n : n \in \omega\}$ and $x \in U$. For each $n$ choose a point $c_n \in S$ such that 
x + $(\{0\} \cup (c_n + S)) \subset U_n$. There exists an element $a \in S$ such that $\{c_n : n \in \omega\} \subset a - S$. 
Then $a + S \subset c_n + S$ for each $n \in \omega$. Hence $x + (\{0\} \cup (a + S)) \subset U$. Thus the set $U$ is 
open. \hfill $\Box$

Corollary 4. The group $G^*_S$ is 2-pseudocompact if and only if the group $G^*_S$ is a pseudo-
docompact $P$-space.
Proof. The proof is implied from Proposition 13, Proposition 14, and Proposition 16. □

Corollary 5. The group $G^*_S$ is compact if and only if the group $G^*_S$ is a countably pracompact $P$-space.

Proof. The proof is implied from Corollary 1 and Corollary 4. □

Example 2. There are an abelian group $G$ and a subsemigroup $S$ of the group $G$ such that the paratopological group $G_S$ is $T_0$ sequentially compact, not totally countably compact, not precompact, and not a topological group and the paratopological group $G^*_S$ is $T_1$ 2-pseudocompact, not countably pracompact, not precompact and not a topological group.

Proof. Let $G = \bigoplus_{\alpha \in \omega_1} \mathbb{Z}$ be the direct sum of the groups $\mathbb{Z}$. Let

$$S = \{0\} \cup \{(x_\alpha) \in G : (\exists \beta \in \omega_1)((\forall \alpha > \beta)(x_\alpha = 0) \& (x_\beta > 0))\}.$$ 

Since $S \cap (-S) = \{0\}$ then by Proposition 1 the group $G_S$ is $T_0$ and by Proposition 10 the group $G^*_S$ is $T_1$. Since $G = S - S$ and for each countable infinite subset $C$ of $S$ there is an element $a \in S$ such that $C \subset a - S$, we see that by Proposition 5 the group $G_S$ is sequentially compact and by Proposition 13 the group $G^*_S$ is 2-pseudocompact. By Proposition 12 the group $G^*_S$ is not countably pracompact. By Proposition 4 the group $G^*_S$ is not totally countably compact and not precompact. Therefore the group $G^*_S$ is not precompact too. □

In [San], Iván Sánchez proved that the group $G^*_S$ from Example 2 is a $P$-space.

From Proposition 3 in [RavRez] it follows that each $T_1$ paratopological group $G$ such that $G \times G$ is countably compact is a topological group. Taras Banakh constructed an example (see, [Rav6, Ex.1]), which shows that in general this proposition cannot be generalized for $T_1$ countably pracompact groups. Example 2 shows that the proposition cannot be generalized for $T_0$ sequentially compact groups.

4. Cowide topologies

Topologies $\tau$ and $\sigma$ on the set $X$ we shall call cowide provided for each nonempty sets $U \in \tau$ and $V \in \sigma$ an intersection $U \cap V$ is nonempty too. If a topology $\sigma$ is cowide to itself then we shall call the topology $\sigma$ wide.

For two topologies $\tau$ and $\sigma$ on the set $X$ by $\tau \lor \sigma$ we denote the supremum of the topologies $\tau$ and $\sigma$, which has a base $\{U \cap V : U \in \tau, V \in \sigma\}$.

Lemma 6. Let $\tau$ and $\sigma$ be cowide topologies on the set $X$. Then $\overline{W}^{\tau \lor \sigma} = \overline{W}^\tau$ for each set $W \in \tau$. Moreover, if the topology $\sigma$ is wide, then $\overline{W}^{\tau \lor \sigma} = \overline{W}^\tau$ for each set $W \in \tau \lor \sigma$.

Proof. The inclusion $\tau \lor \sigma \supset \tau$ implies the inclusion $\overline{W}^{\tau \lor \sigma} \subset \overline{W}^\tau$.

Suppose that $W \in \tau, x \in \overline{W}^\tau$ and $U \cap V$ is an arbitrary neighborhood of the point $x$ such that $U \in \tau$ and $V \in \sigma$. Then $W \cap U$ is a nonempty set from the topology $\tau$. Since the topologies $\tau$ and $\sigma$ are cowide, the intersection $W \cap U \cap V$ is nonempty. Therefore $x \in \overline{W}^{\tau \lor \sigma}$.

Suppose that $W \in \tau \lor \sigma, x \in \overline{W}^\tau$ and $U \cap V$ is an arbitrary neighborhood of the point $x$ such that $U \in \tau$ and $V \in \sigma$. Then $W \cap U$ is a nonempty set from the topology $\tau \lor \sigma$. Therefore there exist sets $U' \in \tau$ and $V' \in \sigma$ such that $\emptyset \neq U' \cap V' \subset W \cap U$. Since the topology $\sigma$ is wide, the intersection $V \cap V'$ is nonempty. Since the topologies $\tau$ and $\sigma$ are cowide, the intersection $U' \cap V \cap V' \subset W \cap U \cap V$ is nonempty. Therefore $x \in \overline{W}^{\tau \lor \sigma}$. □
Regularization. Given a topological space \((X, \tau)\) Stone [Sto] and Katetov [Kat] consider the topology \(\tau_r\) on \(X\) generated by the base consisting of all canonically open sets of the space \((X, \tau)\). This topology is called the regularization or semiregularization of the topology \(\tau\). If \((X, \tau)\) is a paratopological group then \((X, \tau_r)\) is a \(T_3\) paratopological group [Rav2, Ex. 1.9], [Rav3, p. 31], and [Rav3, p. 28].

Lemma 7. Let \(\tau\) and \(\sigma\) be cowide topologies on the set \(X\) and the topology \(\sigma\) is wide. Then \((\tau \vee \sigma)_r = \tau_r\).

Proof. The inclusion \(\tau \vee \sigma \supset \tau\) implies the inclusion \((\tau \vee \sigma)_r \supset \tau_r\). So we show the opposite inclusion. Let \(W = \text{int}_{\tau \vee \sigma} \overline{W} \cap \tau\) be an arbitrary canonical open set. Lemma 6 implies that \(\overline{W} \cap \tau \cap \tau\). The inclusion \(\tau \vee \sigma \supset \tau\) implies the inclusion \(\text{int}_{\tau \vee \sigma} \overline{W} \supset \text{int}_\tau \overline{W}\). So we show the opposite inclusion. Let \(x \in \text{int}_{\tau \vee \sigma} \overline{W}\) be an arbitrary point. Therefore there exist sets \(U' \in \tau\) and \(V' \in \sigma\) such that \(x \in U' \cap \tau \cap \tau\). Then \(U' \cap V' \in \sigma\). Since the topologies \(\tau\) and \(\sigma\) are cowide, the set \(V'\) is \(\tau\)-dense in the set \(U'\). Therefore \(U' \cap V' = U'\). Hence \(x \in U' \subset \overline{U'} \subset \overline{W}\). Thus \(x \in \text{int}_\tau \overline{W}\). □

Pseudocompactness.

Lemma 8. [Rav6] Let \((X, \tau)\) be a topological space. Then \((X, \tau)\) is pseudocompact if and only if the regularization \((X, \tau_r)\) is pseudocompact.

Lemmas 7 and 8 imply the following

Proposition 17. Let \(\tau\) and \(\sigma\) be cowide topologies on the set \(X\) such that the space \((X, \tau)\) is pseudocompact and the topology \(\sigma\) is wide. Then the space \((X, \tau \vee \sigma)\) is pseudocompact too.

\(H\)-closedness. A Hausdorff space \(X\) is called \(H\)-closed if \(X\) is a closed subspace of every Hausdorff space in which it is contained [AleUry1, AleUry2]. It is well known (see, for instance, [Eng, Ex. 3.12.5 (a)]), that a Hausdorff space \(X\) is \(H\)-closed if and only if for each open cover \(\{U_\alpha : \alpha \in A\}\) of the space \(X\) there exists a finite subset \(F\) of \(A\) such that \(X = \bigcup_{\alpha \in F} U_\alpha\). It is easy to check, using this criterion, that a Hausdorff space \((X, \tau)\) is \(H\)-closed is if and only if the space \((X, \tau_r)\) is \(H\)-closed. Then Lemma 7 imply the following

Proposition 18. Let \(\tau\) and \(\sigma\) be cowide topologies on the set \(X\) such that the space \((X, \tau)\) is \(H\)-closed and the topology \(\sigma\) is wide. Then the space \((X, \tau \vee \sigma)\) is \(H\)-closed too.

Proposition 19. A paratopological group \((G, \tau)\) is an \(H\)-closed space if and only if the regularization \((G, \tau_r)\) is Hausdorff and compact.

Proof. The group \((G, \tau)\) is an \(H\)-closed space if and only if the regularization \((G, \tau_r)\) is an \(H\)-closed space. By [Rav2, Ex. 1.9], [Rav3, p. 31], and [Rav3, p. 28] \((G, \tau_r)\) is a regular paratopological group. By [Eng, Ex. 3.12.5 (a)], a regular \(H\)-closed space is Hausdorff and compact. □

Lemma 9. [RavRez] Suppose \((G, \tau)\) is a quasiregular paratopological group such that \((G, \tau_r)\) is a topological group; then \((G, \tau)\) is a topological group.

Proposition 20. For a quasiregular Hausdorff paratopological group \((G, \tau)\) the following conditions are equivalent:
1. The space \((G, \tau)\) is \(H\)-closed.
2. The space \((G, \tau_r)\) is \(H\)-closed.
3. The space \((G, \tau)\) is compact.
4. The space \((G, \tau_r)\) is compact.
Proof. (4⇒3) By [Rav6, Pr.3] a quasiregular paratopological group \((G, \tau_r)\) is topological. Then by Lemma 9 the group \((G, \tau)\) is topological too. Therefore the space \((G, \tau)\) is regular and hence \(\tau_r = \tau\) and the space \((G, \tau)\) is compact.

Implications (3⇒1) and (1⇒2) are obvious.

(2⇒4) It follows from Proposition 19. □

Application to cone topologies. Let \((G, \tau)\) be an abelian paratopological group and \(S \ni 0\) be a subsemigroup of the group \(G\). It is easy to check that the cone topology \(\tau_S\) defined at the beginning of the paper coincides with the supremum of topologies of the spaces \((G, \tau)\) and \(G_S\). Similarly, let cone\(^*\) topology \(\tau_S^*\) be the superemum of topologies of the spaces \((G, \tau)\) and \(G_S^*\). It is clear that the group \(G\) endowed with the topology \(\tau_S^*\) is a paratopological group.

**Proposition 21.** The following conditions are equivalent:
1. The topology of the space \(G_S\) is wide.
2. The topology of the space \(G_S^*\) is wide.
3. \(G = S - S\).

**Proof.** Since the topology of the space \(G_S\) is coarser than the topology of the space \(G_S^*\), we have implication (2⇒1).

(1⇒3) Let \(x \in G\) be an arbitrary element. Since an intersection of open subsets \(S\) and \(x + S\) of the group \(G_S\) is non-empty, we see that \(x \in S - S\).

(3⇒2) Each non-empty open subset of the group \(G_S^*\) contains a set \(x + S\) for some element \(x \in G\). Let \(U_1, U_2\) be arbitrary non-empty open subsets of the group \(G_S^*\). Then there are elements \(x_1, x_2 \in G\) such that \(x_1 + S \subset U_1\) and \(x_2 + S \subset U_2\). Since \(G = S - S\), there are elements \(y_1, y_2, z_1, z_2 \in S\) such that \(x_1 = y_1 - z_1\) and \(x_2 = y_2 - z_2\). Then a set \((x_1 + S) \cap (x_2 + S)\) contains an element \(y_1 + y_2\). □

**Proposition 22.** The following conditions are equivalent:
1. The topology of the space \(G_S\) and the topology \(\tau\) are cowide.
2. The topology of the space \(G_S^*\) and the topology \(\tau\) are cowide.
3. \(S\) is a dense subsemigroup of the group \((G, \tau)\).

**Proof.** Since the topology of the space \(G_S\) is coarser than the topology of the space \(G_S^*\), we have implication (2⇒1).

Implication (1⇒3) follows from opennes of the set \(S\) is the paratopological group \(G_S^*\).

Implication (3⇒2) follows from the fact that each non-empty open subset of the group \(G_S^*\) contains a dense in the space \((G, \tau)\) set \(x + S\) for some element \(x \in G\). □

**Proposition 23.** Let \(S \ni 0\) be a dense subgroup of an abelian paratopological group \((G, \tau)\) such that \(G = S - S\). Then the following conditions are equivalent:
1. The group \((G, \tau)\) is pseudocompact.
2. The group \((G, \tau_S)\) is pseudocompact.
3. The group \((G, \tau_S^*)\) is pseudocompact.

**Proof.** Implications (3⇒2) and (2⇒1) are obvious. Implication (1⇒3) follows from Proposition 22, Proposition 21 and Proposition 17. □

**Proposition 24.** Let \(S \ni 0\) be a dense subgroup of an abelian paratopological group \((G, \tau)\) such that \(G = S - S\). Then the following conditions are equivalent:
1. The group \((G, \tau)\) is an \(H\)-closed space.
2. The group \((G, \tau_S)\) is an \(H\)-closed space.
3. The group \((G, \tau^*_S)\) is an \(H\)-closed space.

Proof. Implications \((3 \Rightarrow 2)\) and \((2 \Rightarrow 1)\) are obvious. Implication \((1 \Rightarrow 3)\) follows from Proposition \[22\] Proposition \[21\] and Proposition \[18\].

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References

[AlaSan] Ofelia T. Alas, Manuel Sanchis, Countably Compact Paratopological Groups, Semigroup Forum, 74 (2007), 423–438.
[AleUry1] Paul Alexandroff, Paul Urysohn, Sur les espaces topologiques compacts, Bull. Intern. Acad. Pol. Sci. Sér. A (1923), 5–8.
[AleUry2] Paul Alexandroff, Paul Urysohn, Mémoire sur les espaces topologiques compacts, Vehr. Akad. Wetensch. Amsterdam 14 (1929), 1–93.
[AriTka] Alexander V. Arhangel'skii, Mikhail Tkachenko, Topological groups and related structures, Atlantis Press, Paris; World Sci. Publ., NJ, 2008.
[Ban] Taras O. Banakh, On cardinal invariants and metrizability of topological inverse Clifford semigroups, Topology Appl. 128:1 (2003), 13–48.
[Ban2] B. Banashewski, Minimal topological algebras, Math. Ann, 211 (1974), 107–114.
[BanRav] Taras O. Banakh, Alex V. Ravsky, The regularity of quotient paratopological groups. http://arxiv.org/abs/1003.5409
[BokGur] Bogdan M. Bokalo, Igor Y. Guran, Sequentially compact Hausdorff cancellative semigroup is a topological group, Matematychni Studii 6 (1996), 39–40.
[Eng] Ryszard Engelking, General topology, Moscow, 1986. (in Russian)
[Kat] M. Katétov. On \(H\)-closed extensions of topological spaces, Časopis Pěst. Mat. Fys. 72 (1947), 17–32.
[Rav] Alex V. Ravsky, Paratopological Groups I, Matematychni Studii 16, No. 1 (2001), 37–48. http://matstud.org.ua/texts/2001/16/37_48.pdf
[Rav2] Alex V. Ravsky, Paratopological Groups II, Matematychni Studii 17, No. 1 (2002), 93–101. http://matstud.org.ua/texts/2002/17/93_101.pdf
[Rav3] Alex V. Ravsky, The topological and algebraical properties of paratopological groups, Ph.D. Thesis. – Lviv University, 2002 (in Ukrainian).
[Rav6] Alex V. Ravsky, Pseudocompact paratopological groups that are topological. http://arxiv.org/abs/1406.2001
[Rav7] Alex V. Ravsky, Pseudocompact Paratopological Groups. http://arxiv.org/abs/1003.5343
[RavRez] Alex V. Ravsky, E. Reznichenko, The continuity of inverse in groups, Banach International Conference on Functional Analysis and Applications, Lviv, 2002, 170-172.
[San] Iván Sánchez. Subgroups of products of paratopological groups, Topology Appl. 163:1 (2014), 160–173.
[Sto] M. H. Stone. Applications of the theory of Boolean rings to general topology, Trans. Amer. Math. Soc. 41 (1937), 375–481.
[Tod] S. Todorcevic, Partition Problems in Topology, Contemp. Math. 84. Amer. Math. Soc., Providence, RI, 1989.
[Tka2] M. Tkachenko, Semitopological and paratopological groups vs topological groups, In: Recent Progress in General Topology III (K.P. Hart, J. van Mill, P. Simon, eds.), 2013, 803–859.
[Tka3] M. Tkachenko, Axioms of separation in semitopological groups and related functors, Topology Appl. 161, (2014), 364–376.

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