Laplace Beltrami Operator in the Baran Metric and Pluripotential Equilibrium Measure: The Ball, the Simplex, and the Sphere

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Abstract

The Baran metric $\delta_E$ is a Finsler metric on the interior of $E \subset \mathbb{R}^n$ arising from pluripotential theory. When $E$ is an Euclidean ball, a simplex, or a sphere, $\delta_E$ is Riemannian. No further examples of such property are known. We prove that in these three cases, the eigenfunctions of the Laplace Beltrami operator associated with $\delta_E$ are the orthogonal polynomials with respect to the pluripotential equilibrium measure $\mu_E$ of $E$. We conjecture that this may hold in wider generality. The differential operators that we consider were introduced in the framework of orthogonal polynomials and studied in connection with certain symmetry groups. In this work, we highlight the connections between orthogonal polynomials with respect to $\mu_E$ and the Riemannian structure naturally arising from pluripotential theory.

Keywords Pluripotential theory · Orthogonal polynomials · Spherical harmonics · Harmonic analysis on manifolds

Mathematics Subject Classification 65E05 · 41A10 · 32U35 · 42C05

1 Introduction

1.1 Potential Theory and Polynomials

The study of approximation theory in the complex plane and on the real line (by polynomials and rational functions) is intimately connected with logarithmic potential theory (i.e., the study of subharmonic functions and the Laplace operator). The
connections of logarithmic potential theory with approximation theory are manifested in Markov, Bernstein, and Nikolski type polynomial inequalities, the asymptotics of optimal polynomial interpolation arrays and Fekete points, overconvergence phenomena (i.e., uniformly convergent sequence of polynomials defining a holomorphic function in a larger open set) and its quantitative version, the Bernstein Walsh Theorem, and the asymptotics of orthogonal polynomials, random polynomials, and random matrices. Moreover, most of such connections extend to the more general case of weighted polynomials and Logarithmic potential theory in the presence of an external field. We refer to [54,56,59,60] and the references therein for extensive treatment.

More recently, a non linear potential theory in multidimensional complex spaces has been introduced and many analogies with the linear case have been shown, provided that there is a suitable “translation” of the quantities that come into play. Pluripotential theory (see, for instance, [37,38]) is the study of plurisubharmonic functions (i.e., functions which are subharmonic along each complex line) and the complex Monge Ampere operator [10].

Though the lack of linearity makes this new theory much more difficult and requires working with different tools, many connections with polynomial approximation have been extended to this multidimensional framework; see [17,20,40]. Indeed, polynomial inequalities in $\mathbb{C}^n$ are usually obtained by means of pluripotential theory, see, for instance, [3,7], the Bernstein Walsh Theorem has been extended by Siciak to $\mathbb{C}^n$ [58], to more general complex spaces by Zeriahi [68], and very recently to different polynomial spaces by Bos and Levenberg [23]. In his seminal work [61–63], Zaharjuta extended the equivalence between (a suitably re-defined version of) the Chebyshev constant (i.e., the asymptotics of the uniform norms of monic polynomials) and the transfinite diameter (i.e., the asymptotics of the maximum of the Vandermonde determinant). Very recently, Berman Boucksom and Nystrom [12,13] showed that Fekete points converge weak* to the pluripotential equilibrium measure of the considered set in $\mathbb{C}^n$ and in much more general settings. This is a deep extension of the one-dimensional case which can be obtained only by means of the weighted theory. The work of Berman and Boucksom stimulated different lines of research such as $L^2$ theory and general orthogonal polynomials [16], the study of multivariate random polynomials and holomorphic sections [21,53,64], the theory of sampling and interpolation arrays [14,42,43], and the study of Bernstein Markov measures [22,47]. From the point of view of approximation theory, the widely used heuristic that the equilibrium measure is the “best” measure for producing uniform polynomial approximations by $L^2$ projection has been motivated and theoretically explained in [13] also in its multivariate setting.

For a wide class of compact subsets $E$ in $\mathbb{R}^n \subset \mathbb{C}^n$, there is a natural Finsler metric $\delta_E$ associated with $E$ called the Baran metric [see (9)]. In particular, for a convex body $E$ (i.e., $E \subset \mathbb{R}^n$ is compact and convex, and has non-empty interior), this metric, arising from pluripotential theory, has been well studied [3,4,25–27,29]. Baran metric is closely related to polynomial approximation and interpolation. Indeed, the Baran inequality (see [2, Thm. 1.1.4] and [4–6])
\[
\left| \frac{d}{dt} p(x_0 + tv) \right|_{t=0} \leq (\deg p) \delta_E(x_0, v),
\]
\[
\forall x_0 \in \text{int } E, v \in S^{n-1}, p \in \mathcal{P}(\mathbb{C}^n), \|p\|_E \leq 1,
\]
(1)
can be understood as a generalization of the classical Bernstein inequality and has applications in polynomial sampling. For instance, let us assume that \( E \subset \mathbb{C}^n \) is a compact and polynomial determining set and \( N \subset E \). Let us denote by \( d_E \) the Finsler distance induced by \( \delta_E \) (see (10) below). If we have
\[
\sup_{x \in E} \min_{x_0 \in N} d_E(x, x_0) \leq \frac{1}{ck}
\]
for some \( k \in \mathbb{N} \) and \( c > 1 \), then \( N \) is a norming set for \( E \) and constant \( c/(c - 1) \) for the space of polynomials of degree not greater than \( k \), that is
\[
\|p\|_E \leq \frac{c}{c - 1} \|p\|_N, \quad \text{for all } p \in \mathcal{P}^k(\mathbb{C}^n).
\]
This essentially follows by the Baran inequality (1). Even more importantly in connection with the present work, in [26], it is shown that, if \( E \) is the simplex or the ball or the sphere, then Fekete points of degree \( k \) of \( E \) (arrays of points maximizing the modulus of the Vandermonde determinant and thus near optimal for polynomial interpolation) have spacing of order \( 1/k \) on \( E \). These results may be used to construct good sampling sets for polynomials, namely admissible meshes, see [28,30,39,46,50,51], that have applications in polynomial approximation and optimization [52].

In what follows we will focus only on the case when the Baran metric turns out to be Riemannian.

The present work attempts on one hand to (partially) extend to the \( \mathbb{C}^n \) case another connection between polynomials and potential theory, and on the other hand, to highlight how polynomial \( L^2 \) approximation with respect to the equilibrium measure may be regarded as Fourier analysis on a suitable Riemannian manifold. These ideas rest upon the relation between the Laplace Beltrami operator relative to the Baran metric and the orthogonal polynomials with respect to the pluripotential equilibrium measure.

We would like to introduce such relations starting by some examples that treat the instances of the interval \([-1, 1]\) and the unit sphere.

### 1.2 Two Motivational Examples

#### 1.2.1 Chebyshev Polynomials

The Chebyshev polynomials \( T_n(x) := \arccos(n \cos x) \) are the orthogonal polynomials with respect to \( \frac{1}{\pi \sqrt{1-x^2}} dx \), the equilibrium measure of the interval \([-1, 1]\) as a subset of \( \mathbb{C} \), i.e., the unique minimizer of the logarithmic potential \( -\int \log |z - w| d\mu(z) d\mu(w) \) among all Borel probability measures \( \mu \) on the interval \([-1, 1]\). Another classical characterization of Chebyshev polynomials is given by the eigenfunctions of the Sturm–Liouville eigenvalue problem:
The set of eigenvalues turns out to be \( \{ n^2 : n \in \mathbb{N} \} \) and \( \mathcal{S}[T_n] = n^2 T_n \).

Instead, we re-write this eigenvalue problem as

\[
\frac{1}{\sqrt{1-x^2}} \frac{d}{dx} \left( \frac{1}{\sqrt{1-x^2}} (1-x^2) \varphi'(x) \right) = -n^2 \varphi(x), \quad x \in ]-1,1[.
\]

(3)

This apparently useless manipulation actually illustrates another property of Chebyshev polynomials. To explain this property, we first recall that the Laplace Beltrami operator relative to a metric \( g \) can be written in local coordinates

\[
\Delta_{LB} f = \frac{1}{\sqrt{\det g}} \sum_{i=1}^{n} \partial_{x_i} \left( \sqrt{\det g} \sum_{j} g^{i,j} \partial_{x_j} f \right),
\]

where \( g^{i,j} \) are the components of the inverse of the matrix representing \( g \).

If we endow \( ]-1,1[ \) with the Riemannian metric \( g(x) := \frac{1}{1-x^2} \), then we canonically obtain the Riemannian distance \( d(x_0,x_1) = \int_{x_0}^{x_1} \frac{1}{\sqrt{1-x^2}} \, dx \). Note that, up to a renormalization, the resulting volume form is precisely the equilibrium measure of \([-1,1]\). If we plug \( g(x) := \frac{1}{1-x^2} \) in the expression (4) for the Laplace Beltrami operator, we obtain precisely the left-hand side of (3). In other words, we observe that:

(†) **Chebyshev polynomials are eigenfunctions of the Laplace Beltrami operator with respect to the density of the equilibrium measure of the interval.**

It is relevant to note that the density of the equilibrium measure on \([-1,1]\) at \( x \) is obtained as the normal (i.e., purely complex) derivative of the Green function of \( \mathbb{C} \setminus [-1,1] \) with pole at infinity; see [57, Ch II.1]. This operation has a multidimensional counterpart (see [3]) that, under some assumptions, leads to the so-called Baran metric ([4,18]), see Eq. (9) below.

**Remark 1** The observation (†) above can also be understood in a more general framework. To this aim, let us recall that a weighted Riemannian manifold is a triple \( (M, g, \rho) \), where \( (M, g) \) is a Riemannian manifold and \( \rho \) is a positive smooth function on \( M \). In such a setting, one defines the weighted Laplace Beltrami operator \( \Delta_{\rho} \) acting on smooth functions by setting

\[
\Delta_{\rho} u := \frac{1}{\rho \sqrt{\det g}} \sum_{i=1}^{n} \partial_{x_i} \left( \rho \sqrt{\det g} \sum_{j} g^{i,j} \partial_{x_j} u \right).
\]

It turns out that the classical orthogonal polynomials on the interval \([-1,1]\), e.g., Legendre and Gegenbauer orthogonal polynomials, are indeed eigenfunctions of \( \Delta_{\rho} \).
on \((-1,1], \frac{1}{1-x^2}, (1-x^2)^{\beta}\) with an appropriate choice of \(\beta\); this has already been shown in [31].

### 1.2.2 Spherical Harmonics

We mention another relevant example of this relation between eigenfunctions of the Laplace Beltrami operator with respect to the metric defined by (pluri-)potential theory and the (pluripotential) equilibrium measure. In contrast to case of Chebyshev polynomials, we now work in a multidimensional setting and the flat Euclidean space \(\mathbb{C}^n\) is replaced by a complex manifold. A more detailed account of this example requires some preliminary notions in addition to Sect. 2.1. The explicit computations are reported in Appendix A. At this stage, we only sketch the results to underline the analogy with the case of Chebyshev polynomials.

Let us consider the unit sphere \(S^{n-1} \subset \mathbb{R}^n\) endowed with the round metric \(g\) induced by the flat metric on \(\mathbb{R}^n\) and denote by \(\Delta\) the Laplace Beltrami operator on \(S^{n-1}\). It is well known that spherical harmonics are a dense orthogonal system of \(L^2(S^{n-1})\) which consists of polynomials that are eigenfunctions of \(\Delta\).

Let us look at \(S^{n-1}\) as a compact subset of the complexified sphere \(S^{n-1} := \{z \in \mathbb{C}^n : \sum z_i^2 = 1\}\). By a fundamental result due to Sadullaev [55], since \(S^{n-1}\) is an irreducible algebraic variety, one can relate (see Appendix A) the traces of polynomials on \(S^{n-1}\) to pluripotential theory on the complex manifold of \(S^{n-1}\). On the other hand, due to Lemma 3 below, we can define a smooth Riemannian metric \(g_{S^{n-1}}\) on \(S^{n-1}\) suitably modifying the construction [see Eq. (9)] of the Baran metric of convex real bodies. In particular, such a definition is given by the generalization of the case of the real interval \([-1,1]\). Indeed, it turns out that \(g_{S^{n-1}} = g\) and its volume form is, up to a constant scaling factor (chosen to make it a probability measure), the pluripotential equilibrium measure of \(S^{n-1},\) as a compact subset of \(S^{n-1}\). In other words:

(‡) **The eigenfunctions of the Laplace Beltrami operator of \((S^{n-1}, g)\) are the orthogonal polynomials with respect to the pluripotential equilibrium measure of \(S^{n-1},\)** seen as a compact subset of \(S^{n-1}\); see Corollary 1.

### 1.3 Our Results and Conjecture

The aim of the present paper is to present a conjecture on the extension to the \(\mathbb{C}^n\) case of the relation between potential theory and certain Riemannian structure that holds in the examples above. We support it by full proofs of all the few known instances, see Theorems 1 and 2 below, fulfilling the required hypothesis, i.e., their Baran metrics are Riemannian.

**Conjecture 1** Let \(C\) denote either \(\mathbb{C}^n\) or any irreducible algebraic subvariety of it. Let \(E \subset C\) be a fat real compact set. Assume that the Baran metric \(\delta_E\) of \(E\) is a Riemannian metric on \(\text{int} S^{n-1} \cap C\), and then, the orthonormal polynomials with respect to the pluripotential equilibrium measure \(\mu_{E,C}\) of \(E\) in \(C\) are eigenfunctions of the Laplace Beltrami operator relative to the metric \(\delta_E\).
Remark 2 We stress that the orthogonal bases used in our proofs as well as most of their properties are already known in the framework of orthogonal polynomials (see [32,33,35] and the references therein). Moreover, our differential operators (i.e., Laplace Beltrami operators with respect to the metrics arising from pluripotential theory) have already been studied in relation to certain symmetry groups [35, Ch. 8], but they have not been connected to any potential theoretic aspect before. More precisely, the Laplace Beltrami operator on the ball endowed with its Baran metric turns out to be the operator \( \mathcal{D}_\mu \) in [35, p. 142] with the parameter choice \( \mu = 0 \). Instead, in the simplex case, \( \Delta \) is precisely the operator defined in [35, eq. 5.3.4] (see Eq. 29 4 below) if we set (in the authors notation) \( \kappa = (0, \ldots, 0) \in \mathbb{R}^{n+1} \).

Our goal is precisely to relate such families of functions and their properties to the Riemannian structure that comes from pluripotential theory.

Remark 3 Note that we will deal with the open sets \( S^n \) and \( B^n \), but we will refer, by a slight abuse of notation and nomenclature that aims to an easier notation, to their Baran metrics \( \delta S^n \) and \( \delta B^n \) instead of \( \delta S^n \) and \( \delta B^n \).

Theorem 1 (Laplace Beltrami on the Baran ball) Let us denote by \( \Delta \) the Laplace Beltrami operator of the Riemannian manifold \((B^n, \delta B^n)\) acting on

\[
C^2_b(B^n) := \left\{ u \in C^2(B^n) : \max_{|\alpha| \leq 2} \sup_{x \in B^n} |\partial^\alpha u(x)| < \infty \right\},
\]

where \( B^n := \{x \in \mathbb{R}^n : |x| < 1 \} \) and the Baran metric \( \delta B^n(x) \) of the ball, see (12), is represented by the matrix

\[
G_{B^n}(x) := \begin{bmatrix}
1 + \frac{x_1^2}{1 - \sum_{i=1}^n x_i^2} & \frac{x_1 x_2}{1 - \sum_{i=1}^n x_i^2} & \cdots & \frac{x_1 x_n}{1 - \sum_{i=1}^n x_i^2} \\
\frac{x_1 x_2}{1 - \sum_{i=1}^n x_i^2} & 1 + \frac{x_2^2}{1 - \sum_{i=1}^n x_i^2} & \cdots & \frac{x_2 x_n}{1 - \sum_{i=1}^n x_i^2} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{x_n x_1}{1 - \sum_{i=1}^n x_i^2} & \frac{x_n x_2}{1 - \sum_{i=1}^n x_i^2} & \cdots & 1 + \frac{x_n^2}{1 - \sum_{i=1}^n x_i^2}
\end{bmatrix}.
\]

The operator \( \Delta \) is symmetric and unbounded, it has discrete spectrum

\[
\sigma(\Delta) = \{ \lambda_s := s(s + n - 1) : s \in \mathbb{N} \},
\]

and the eigenspace of \( \lambda_s \) is span\{\( \varphi_\alpha, |\alpha| = s \)\}, where \( \varphi_\alpha \) (see Proposition 4) are orthonormal polynomials with respect to the pluripotential equilibrium measure

\[
\mu_{B^n} := \frac{1}{\sqrt{1 - |x|^2}} \chi_{B^n} \text{Vol}_{\mathbb{R}^n} = \text{Vol}_{\delta B^n}.
\]
Moreover, $\Delta$ can be closed to a self-adjoint operator $\mathcal{D}(\Delta) \to L^2(B^n, \delta_{B^n})$ (having the same spectrum), where

$$\mathcal{D}(\Delta) := \left\{ u \in L^2(B^n, \delta_{B^n}) : \sum_{s=0}^{\infty} \lambda_s^2 \sum_{|\alpha|=s} |\hat{u}_\alpha|^2 < \infty \right\} \subset H^1(B^n, \delta_{B^n})$$

and $\hat{u}_\alpha$ is the Fourier coefficient $\int_{B^n} u \frac{\psi_\alpha}{\|\psi_\alpha\|_{L^2(\mu_{B^n})}} d\mu_{B^n}$.

The operator $\Delta^{1/2}$ has domain

$$\mathcal{D}(\Delta^{1/2}) := \left\{ u \in L^2(B^n, \delta_{B^n}) : \sum_{s=0}^{\infty} \lambda_s \sum_{|\alpha|=s} |\hat{u}_\alpha|^2 < \infty \right\} = H^1(B^n, \delta_{B^n}).$$

Here and from now on, we denote by $\alpha$ an integer multiindex and by $|\alpha|$ its length. For a precise definition of the Sobolev space $H^1(B^n, \delta_{B^n})$, see Sect. 2.2.2 below.

**Theorem 2** (Laplace Beltrami on the Baran simplex) Let us denote by $\Delta$ the Laplace Beltrami operator on the Riemannian manifold $(S^n, \delta_{S^n})$, acting on

$$\mathcal{C}^2_b(S^n) := \left\{ u \in \mathcal{C}^2(S^n) : \max_{|\alpha| \leq 2} |\partial^\alpha u(x)| < \infty \right\},$$

where $S^n := \{ x \in \mathbb{R}^n : \sum_{i=1}^{n} x_i < 1, x_i > 0 \forall i = 1, 2, \ldots, n \}$ and the Baran metric $\delta_{S^n}(x)$ of the simplex, see Eq. (13), is represented by the matrix

$$G_{S^n}(x) := \begin{bmatrix} x_1^{n-1} & 0 & \ldots & 0 \\ 0 & x_2^{n-1} & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & \ldots & x_n^{n-1} \end{bmatrix} + \frac{1}{1 - \sum_{i=1}^{n} x_i} \begin{bmatrix} 1 & 1 & \ldots & 1 \\ 1 & 1 & \ldots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \ldots & 1 \end{bmatrix}.$$

The operator $\Delta$ is symmetric and unbounded, it has discrete spectrum

$$\sigma(\Delta) = \left\{ \lambda_s := s \left( s + \frac{n-1}{2} \right) : s \in \mathbb{N} \right\}$$

and the eigenspace of $\lambda_s$ is $\text{span}\{\psi_\alpha, |\alpha| = s\}$, where $\psi_\alpha$ (see Proposition 5) are orthonormal polynomials with respect to the pluripotential equilibrium measure of the simplex

$$\mu_{S^n} := \frac{1}{\sqrt{(1 - \sum_{i=1}^{n} x_i) \prod_{i=1}^{n} x_i}} \chi_{S^n}(x) \text{Vol}_{\mathbb{R}^n} = \text{Vol}_{\delta_{S^n}}.$$
Moreover, $\Delta$ can be closed to a self-adjoint operator (still denoted by $\Delta$) $\mathcal{D}(\Delta) \to L^2(S^n, \delta_{S^n})$ (having the same spectrum), where

$$\mathcal{D}(\Delta) := \left\{ u \in L^2(S^n, \delta_{S^n}) : \sum_{s=0}^{\infty} \lambda_s^2 \sum_{|\alpha|=s} |\hat{u}_\alpha|^2 < \infty \right\} \subset H^1(S^n, \delta_{S^n})$$

and $\hat{u}_\alpha$ is the Fourier coefficient $\int_{B^n} u \frac{\psi_\alpha}{\|\psi_\alpha\|_{L^2(\mu_{S^n})}} d\mu_{S^n}$.

The operator $\Delta^{1/2}$ has domain

$$\mathcal{D}(\Delta^{1/2}) := \left\{ u \in L^2(S^n, \delta_{S^n}) : \sum_{s=0}^{\infty} \lambda_s \sum_{|\alpha|=s} |\hat{u}_\alpha|^2 < \infty \right\} = H^1(S^n, \delta_{S^n}).$$

Remark 4 In view of Remark 1, one may ask if Theorems 1, 2 have counterparts if the Riemannian manifold we consider is endowed with a weight $\rho$. It is possible to prove that this is indeed the case, provided that $\rho$ satisfies $\rho = \eta \text{Vol}_{\mathbb{R}^n}$, where $\eta$ is a positive power of the density of $\mu_{B^n}$ (or $\mu_{S^n}$) with respect to $\text{Vol}_{\mathbb{R}^n}$ (or $\text{Vol}_{S^n}$, respectively).

Remark 5 To better understand how the Baran metrics of the ball and the simplex look like, it is worth recalling their special relation with a certain portion of the sphere.

Let us denote by $(\mathbb{H}^n_+, g_{\mathbb{H}^n_+})$ the upper unit hemisphere, i.e., the Riemannian manifold which can be obtained by intersecting the unit sphere $S^n$ (thought as a submanifold of $\mathbb{R}^{n+1}$ endowed with the Euclidean metric) with the positive half space $\{ \xi \in \mathbb{R}^{n+1} : \xi_{n+1} > 0 \}$. The map $\pi : \mathbb{H}^n_+ \to B^n$, $\pi(\xi) := (\xi_1, \ldots, \xi_n)$ clearly is a one-to-one $\mathcal{C}^\infty$ map of manifolds. Therefore, we can define a metric $g$ on $B^n$ by means of the pull-back operator with respect to $F := \pi^{-1}$:

$$g(v, w) := F^*g_{\mathbb{H}^n_+}(v, w) = g_{\mathbb{H}^n_+}(dFv, dFw), \forall v, w \in TB^n.$$  

One can verify by direct computations that indeed $g \equiv \delta_{B^n}$.

Similarly, we can define the map Sqrt : $S^n = \{ x \in \mathbb{R}^n_+ : \sum_{i=1}^{n} x_i \leq 1 \} \to B^n \cap \{ x \in \mathbb{R}^n : x_i > 0, \forall i = 1, \ldots, n \}$, Sqrt$(x) := (\sqrt{x_1}, \sqrt{x_2}, \ldots, \sqrt{x_n})$, and pull back by Sqrt on $S^n$ the Baran metric of the ball, see (12) and (13) below. Again, this new metric indeed coincides with the Baran metric of the simplex; [25].

Question 1 Since the property of a compact set of having a Riemannian Baran metric is stable under taking the pre-image of such a set via certain polynomial maps, we expect that it is possible to prove the equivalence of Theorems 1 and 2.

Note that since the manifolds $(B^n, \delta_{B^n})$ and $(S^n, \delta_{S^n})$ are isometric to certain portions of $S^n$, the local differential and metric properties of these manifolds are the same as those of $S^n$. We recall that a Riemannian manifold $(M, g)$ is termed Einstein when its metric tensor is a solution of the Einstein vacuum field equation

$$\text{Ric} = kg.$$  

(5)
Here

\[ \text{Ric}_{i,j} := \sum_{l=1}^{n} \left( \partial_l \Gamma^l_{j,i} - \partial_j \Gamma^l_{i,l} \right) + \sum_{l,k=1}^{n} \left( \Gamma^l_{l,k} \Gamma^k_{j,i} - \Gamma^l_{j,k} \Gamma^k_{i,l} \right) \]

is the Ricci tensor (written by means of the Christoffel symbols \( \Gamma^i_{j,k} \)) and \( k > 0 \). Since it is a well known fact that \((\mathbb{S}^n, g_{\mathbb{S}^n})\) is Einstein, we get the following proposition as a consequence of Remark 5.

**Proposition 1** The unit ball and the unit simplex, endowed with their Baran metric respectively, are Einstein manifolds.

Since for all cases where the Baran metric is known to be Riemannian it solves Eq. (5), the following question naturally arises.

**Question 2** Assume that \( E \) is a Baran body in the sense of Definition 1 below. Is it necessary for its Baran metric tensor to solve the Einstein vacuum field equation (5)?

**Remark 6** Recently, Zelditch [66] studied the spectral theory of the Laplace Beltrami operator on a real analytic Riemannian manifold \( M \) in connection with the pluripotential theory of the so-called Bruhat–Whitney complexification \( M_C \) of \( M \). In particular, \([65,67]\) present asymptotic results on the zero distribution of the eigenfunctions and series of functions with random Fourier coefficients, working under the assumption of ergodicity of the geodesic flow. These results closely resemble the relation between the behaviour of zeros of orthogonal polynomials (or random polynomials) and the pluripotential equilibrium measure.

Even though our study is far from being as general as the context of the above references, our result may be cast within this framework and offer concrete examples where explicit computations are performed. Indeed, Appendix A exactly fits in the framework of [66].

The paper is structured as follows. In Sect. 2, we furnish all the necessary definitions from pluripotential theory, operator theory, and differential geometry. In Sect. 3, we prove Theorems 1 and 2, giving a precise spectral characterization of the considered Sobolev spaces. Finally, in Appendix A, it is shown how to define the Baran metric on the sphere and its equivalence with the standard round metric.

## 2 Preliminaries and Tools

### 2.1 The Pluripotential Theory Framework

Pluripotential Theory is the study of plurisubharmonic functions, i.e., any uppersemi-continuous function \( u : \Omega \to [-\infty, +\infty] \) being subharmonic along each affine variety of complex dimension 1 in \( \Omega \subseteq_{\text{open}} \mathbb{C}^n \). We use the operators \( d := \partial + \bar{\partial} \) and \( d^c := i(-\partial + \bar{\partial}) \), where
\[ \partial := \sum_{j=1}^{n} \frac{\partial}{\partial z_j} \cdot dz_j, \quad \bar{\partial} := \sum_{j=1}^{n} \frac{\partial}{\partial \bar{z}_j} \cdot d\bar{z}_j. \]

The operator \( d\bar{d}^c \) is sometimes referred to as the complex Laplacian and it corresponds to the usual Laplacian (up to a scaling factor) when \( n = 1 \).

Since \( d\bar{d}^c \) is a linear operator, one can consider \( d\bar{d}^c u \) for a \( L^1_{\text{loc}} \) function in the sense of currents (distributions on the space of differential forms) and it turns out that, for an uppersemicontinuous function \( u \), \( d\bar{d}^c u \geq 0 \) if and only if \( u \) is plurisubharmonic.

The complex Monge Ampere operator \( (d\bar{d}^c)^n \) is defined for \( C^2 \) functions as

\[ (d\bar{d}^c u)^n := d\bar{d}^c u \wedge d\bar{d}^c u \wedge \cdots \wedge d\bar{d}^c u = c_n \det (d\bar{d}^c u) d\text{Vol}_{\mathbb{C}^n}. \]

Clearly, trying to define wedge products of factors of the type \( d\bar{d}^c u \) for any plurisubharmonic function \( u \) leads to serious difficulties due to the lack of linearity. Bedford and Taylor [10] showed that the definition of Eq. (6) can be extended to any locally bounded plurisubharmonic function, with \( (d\bar{d}^c u)^n \) being a positive Borel measure.

One may think to plurisubharmonic functions in \( \mathbb{C}^n \) as the correct counterpart (see [37, Preface]) of subharmonic functions on \( \mathbb{C} \), while harmonic functions should be replaced in this multidimensional setting by maximal plurisubharmonic functions, i.e., functions \( u \) dominating on any subdomain \( \Omega' \) any plurisubharmonic function \( v \), such that \( u \geq v \) on \( \partial \Omega' \). Locally bounded maximal plurisubharmonic functions satisfy \( (d\bar{d}^c u)^n = 0 \).

The multidimensional counterpart of the Green function for the unbounded component of the complement of a compact set \( E \) is the pluricomplex Green function (also known as Siciak–Zaharjuta extremal function) \( V^*_E \). Let \( E \subset \mathbb{C}^n \) be a compact set, and then, we set

\[ V_E(\zeta) := \sup\{u(\zeta), u \in \mathcal{L}(\mathbb{C}^n), u|_E \leq 0\}, \]

\[ V^*_E(z) := \limsup_{\zeta \to z} V_E(\zeta). \]

Here, \( \mathcal{L}(\mathbb{C}^n) \) is the Lelong class of plurisubharmonic functions on \( \mathbb{C}^n \) of logarithmic growth, i.e., \( u(z) - \log |z| \) is bounded above at infinity.

It is worth recalling that, as in the one-dimensional case, due to [58] (see also [37]), we can express \( V^*_E \) by means of polynomials \( \mathcal{P}(\mathbb{C}^n) \). That is

\[ V_E(\zeta) = \sup \left\{ \frac{1}{\deg p} \log^+ |p(\zeta)|, p \in \mathcal{P}(\mathbb{C}^n), \|p\|_E \leq 1 \right\}. \]

Here, \( \log^+(x) := \max\{\log x, 0\} \).

The function \( V^*_E \) is either identically \(+\infty\) or a locally bounded plurisubharmonic function on \( \mathbb{C}^n \), maximal on \( \mathbb{C}^n \setminus E \) (i.e., \( (d\bar{d}^c V^*_E)^n \) is a positive Borel measure with support in \( E \)) having logarithmic growth at \( \infty \); if the latter case occurs we say that \( E \) is non pluripolar. In principle, \( V^*_E \) is only a uppersemicontinuous function. When \( V^*_E \) is continuous, the compact set \( E \) is said to be regular. It is worth recalling that it turns out that \( V^*_E \) is continuous if and only if \( V^*_E \) identically vanishes on \( E \). We will treat only such a case in what follows.
For any non pluripolar compact set $E \subset \mathbb{C}^n$, the \textit{pluripotential equilibrium measure} of $E$ is defined as
\[
\mu_E := (dd^c V_E^*)^n.
\] (8)
This is a Borel \textit{probability} measure supported on $E$.

It is worth saying that explicit formulas for $V_E^*$ and $\mu_E^*$ have been computed only for few instances $E$, see, e.g., [1–3,19,24,41,48]. Numerical approximation schemes were introduced in [49].

We stress that, since $\mu_E(E) = 1$ for any non pluripolar set [10], the total mass of the measures (and volume forms) that we are going to deal with is not important. We avoid introducing normalizing constants in the metrics to keep the notation simple.

Let $E$ be a real convex body, Baran showed that in this case
\[
\delta_E(x, v) := \limsup_{t \to 0^+} \frac{V_E^*(x + it v)}{t}
\] (9)
exists for any $x \in \text{int } E$, $v \in \mathbb{R}^n$. We refer to $\delta_E(x, v)$ as the \textit{Baran metric} of $E$. It is worth mentioning that in many cases, $\limsup_{t \to 0^+}$ can be replaced by $\lim_{t \to 0^+}$ in the above definition, [5,11]. We refer the reader to [27] for a study on the connections among this metric, polynomial inequalities, and polynomial sampling. The Baran metric defines in general a Finsler distance on $E$
\[
d_E(x, y) := \inf \left\{ \int_0^1 \delta_E(\gamma(s), \gamma'(s))ds \middle| \gamma \in Lip([0, 1], E), \gamma(0) = x, \gamma(1) = y \right\};
\] (10)
however, it may happen that $\delta_E(x, v)$ is indeed Riemannian, that is
\[
\delta_E(x, v) = \sqrt{v^T G_E(x) v}
\]
for a positive definite matrix $G_E(x)$. Note that $G_E(x)$ is then well defined by the \textit{parallelogram law}. More precisely, we have
\[
u^T G_E(x) v = \frac{\delta_E^2(x, u + v) - \delta_E^2(x, u - v)}{4}.
\]

**Definition 1** (Baran body) Let $C$ denote either $\mathbb{C}^n$ or an irreducible algebraic variety of pure dimension $n$, and let $C_{\mathbb{R}}$ denote the real points of $C$. Let $E \subset C_{\mathbb{R}}$ a compact fat\textsuperscript{1} non pluripolar set. If the Baran metric of $E$ is a Riemannian metric for the real interior of $E$, then we term $E$ a \textit{Baran body}.

In [27], the Baran metrics of the real ball, real simplex are computed (see Theorems 1 and 2 above), showing, in particular, that they are Baran bodies. For the sake of completeness, we recall how $\delta_{B^n}$ and $\delta_{S^n}$ can be computed. The pluricomplex Green function of the real unit ball is given by the Lundin Formula (see for instance [37, Thm. 5.4.6]):

\footnote{ This mean that the closure in $C_{\mathbb{R}}$ of the interior of $E$ in $C_{\mathbb{R}}$ coincides with $E$.}
\[ V_{B^n}(x) = \frac{1}{2} \log \left[ h \left( \sum_{i=1}^{n} |x_i|^2 + \left| \sum_{i=1}^{n} x_i^2 - 1 \right| \right) \right], \quad \forall x \in \mathbb{C}^n, \quad (11) \]

where \( h : \mathbb{C} \setminus [-1, 1] \rightarrow \mathbb{C} \setminus \{|z| \leq 1\} \) is the inverse Joukowsky map \( z \mapsto z + \sqrt{z^2 - 1} \). The function \( V_{B^n}(x) \) is differentiable at any \( x \notin B^n \), so that one can compute the Baran metric taking the limit

\[ \delta_{B^n}(x, v) := \limsup_{t \to 0^+} \frac{V_{B^n}(x + itv)}{t} = \lim_{t \to 0^+} \frac{\partial_i \cdot v}{\mathcal{V}^*_{B^n}(x + itv)}. \]

A direct computation leads to

\[ \delta_{B^n}(x, v) = \sqrt{v^T G_{B^n}(x)v}, \]

where

\[ G_{B^n}(x) := \begin{bmatrix}
1 + \frac{x_1^2}{1 - \sum_{i=1}^{n} x_i^2} & \frac{x_1 x_2}{1 - \sum_{i=1}^{n} x_i^2} & \ldots & \frac{x_1 x_n}{1 - \sum_{i=1}^{n} x_i^2} \\
\frac{x_2 x_1}{1 - \sum_{i=1}^{n} x_i^2} & 1 + \frac{x_2^2}{1 - \sum_{i=1}^{n} x_i^2} & \ldots & \frac{x_2 x_n}{1 - \sum_{i=1}^{n} x_i^2} \\
\vdots & \ddots & \ddots & \vdots \\
\frac{x_n x_1}{1 - \sum_{i=1}^{n} x_i^2} & \frac{x_n x_2}{1 - \sum_{i=1}^{n} x_i^2} & \ldots & 1 + \frac{x_n^2}{1 - \sum_{i=1}^{n} x_i^2}
\end{bmatrix}. \quad (12) \]

The square map \((x_1, x_2, \ldots, x_n) \mapsto (x_1^2, x_2^2, \ldots, x_n^2)\) is a polynomial map from the unit ball to the simplex satisfying the so-called Klimek condition, see [37, p. 196], therefore, applying [37, Thm. 5.3.1]; one has

\[ V_{S^n}(z_1, z_2, \ldots, z_n) = 2V_{B^n}((\sqrt{z_1}, \sqrt{z_2}, \ldots, \sqrt{z_n})]. \]

The chain rule leads to

\[ G_{S^n}(x) = diag \begin{pmatrix} \sqrt{x_1} \\ \sqrt{x_2} \\ \vdots \\ \sqrt{x_n} \end{pmatrix} T G_{B^n}(\sqrt{x_1}, \sqrt{x_2}, \ldots, \sqrt{x_n}) diag \begin{pmatrix} \sqrt{x_1} \\ \sqrt{x_2} \\ \vdots \\ \sqrt{x_n} \end{pmatrix}, \quad (13) \]

**Remark 7** To the best of the author’s knowledge, these are all the examples of Baran compact sets in \( \mathbb{C}^n \) known in the literature. Indeed, such a property seems to be very
rare: for instance in [27], some counterexamples are given as the regular hexagon and the coordinate square \([-1, 1]\). We offer a further instance of a Baran manifold in Appendix A, namely the real sphere as a subset of the complexified sphere.

**Remark 8** In [8], Barthleme introduces for the first time a natural Laplace operator on Finsler manifolds. It is an interesting question to investigate if it is possible to prove statements equivalent to Theorems 1 and 2 at least for particular cases of Baran manifolds arising from pluripotential theory. A natural starting point for this should be the case of the standard coordinate square \(K = [-1, 1]^n\) endowed with its Baran metric, see [15] and references therein,

\[
\delta_K(x, v) = \max_{i \in \{1, 2, \ldots, n\}} \frac{|v_i|}{\sqrt{1 - x_i^2}}.
\]

We postpone this study for a future work.

### 2.2 Differential Operators and Sobolev Spaces on a Riemannian Manifold

#### 2.2.1 Differential Operators

We recall that a linear connection on a vector bundle \(\pi : E \to M\) (built on the differentiable manifold \(M\)) is a mapping (here, \(\mathcal{E}(M)\) is the space of smooth sections of the vector bundle \(E\) and \(T(M)\) is the tangent bundle)

\[
\nabla : T(M) \times \mathcal{E}(M) \quad \longrightarrow \quad \mathcal{E}(M)
\]

\[
(X, V) \quad \longrightarrow \quad \nabla_X V,
\]

such that it is \(\mathcal{C}^\infty\)-linear in \(X\), \(\mathbb{R}\)-linear in \(V\), and for which the Liebniz Rule \(\nabla_X(fV) = VX(f) + f\nabla_X(V)\) holds for any \(f \in \mathcal{C}^\infty(M)\). In particular, we have \(\nabla_X f = X(f)\).

Let \((M, g)\) be a (possibly non compact) Riemannian manifold. It is well known that there exists a unique torsion-free linear connection on \(T(M)\) that is compatible with the metric \(g\); namely the Levi–Civita connection. Since we will deal only with such a connection, we will still denote it by \(\nabla\).

Note that, for a given \(u \in \mathcal{C}^\infty(M)\), \(\nabla u\) is a \((1, 0)\) tensor field (i.e., point-wise it is a linear form) having the property that \((X, \nabla u)_g = \nabla_X u = X(u)\), and thus, it can be written in local coordinates

\[

\nabla u = \sum_j g^{ij} \frac{\partial u}{\partial x_j} dx_j.
\]
Here, $(\cdot, \cdot)_g$ is the canonical duality induced by $g$ and $g^{ij}$ are the components of the matrix representing $g^{-1}$. Hence, it is convenient to define the tangent vector

$$(\text{grad } u)_i := \left( \sum_j g^{ij} \frac{\partial u}{\partial x_j} \right)_i,$$

namely the covariant gradient of $u$, having the property that $(X, \nabla u)_g = (X, \text{grad } u)_g$.

The divergence operator acting on $X \in T(M)$ is defined by

$$\text{div } X := \nabla \cdot X = \frac{1}{\sqrt{\det g}} \sum_i \partial_i (\sqrt{\det g} X^i).$$

Finally, we can recall the definition of the Laplace Beltrami operator $\Delta$

$$\Delta u := \text{div} (\text{grad } u) = \frac{1}{\sqrt{\det g}} \sum_i \partial_i (\sqrt{\det g} (\text{grad } u)_i). \quad (14)$$

2.2.2 Sobolev Spaces

Let $(M, g)$ be a Riemannian manifold. Let us introduce on $\mathcal{C}^\infty(M)$ the norm

$$||u||_{1,2} := \left( \int |u|^2 \text{Vol}_g \right)^{1/2} + \left( \int |\text{grad } u|^2 \text{Vol}_g \right)^{1/2},$$

where $|\text{grad } u|^2 = (\text{grad } u, \text{grad } u)_g$. Let us denote by $\mathcal{C}^\infty_{1,2}(M)$ the space $\{u \in \mathcal{C}^\infty(M), ||u||_{1,2} < \infty\}$.

The Sobolev space $H^1(M, g)$ is defined as the closure of $\mathcal{C}^\infty_{1,2}(M)$ with respect to $||\cdot||_{1,2}$ in the space of square integrable functions. We also introduce the space $H^1_0(M, g)$, the closure of $\mathcal{C}^\infty_c(M)$ in the same norm. Note that in general, $H^1_0(M, g) \subseteq H^1(M, g)$.

An important fact about Sobolev spaces and manifolds is that the above two spaces may coincide, that is

$$H^1_0(M, g) \equiv H^1(M, g). \quad (15)$$

Our interest in this phenomena is mainly due to the fact that the Laplace operator does not need to be complemented with boundary conditions in such a case.

Indeed, $H^1_0(M, g) \equiv H^1(M, g)$ for any complete Riemannian manifold $M$; see [36, Thm. 3.1]. We recall for the reader’s convenience that a Riemannian manifold $(M, g)$ is said to be complete if the metric space $(M, d_g)$ is complete, where

$$d_g(x, y) := \inf \left\{ \int_0^1 \sqrt{(\gamma'(s), \gamma'(s))_g(\gamma(s))} ds \mid \gamma \in \text{Lip}([0, 1], M), \gamma(0) = x, \gamma(1) = y \right\}. $$
The Hopf–Rinow Theorem asserts that the completeness of \((M, g)\) is equivalent to the fact that any relatively closed bounded subset of \(M\) is compact.

We denote by \(\mathcal{C}_c^\infty(M)\) the set uniformly bounded functions that have uniformly bounded partial derivatives of any order. Since for a complete manifold \(\mathcal{C}_c^\infty(M) \subseteq \mathcal{C}_b^\infty(M) \subset H^1(M, g)\), it follows that for any complete manifold \((M, g)\), \(\mathcal{C}_c^\infty(M)\) is dense in \(H^1(M, g)\).

Unfortunately, both \((\text{int } B^n, \delta_B^n)\) and \((\text{int } S^n, \delta_S^n)\) fail to be complete: it is very easy to construct a Cauchy sequence in \(B^n\) not converging in \(B^n\). For instance, take \(\{x_k\} = \cos(2^{-k})u\) for any unit vector \(u \in \mathbb{R}^n\). Since \(d(x_k, x_l) \leq 2^{-\min(k, l)}\), this is a Cauchy sequence; however, \(x_k \to u \notin B^n\). Nevertheless, one may wonder whether Eq. (15) holds in these instances. This fact indeed depends on finer properties of the manifolds than completeness. Namely, Masamune [44,45] showed that equality (15) holds if and only if the metric completion of \(M\) lies in the category of manifolds with almost polar boundary.

We recall that the Riemannian manifold \((M \cup \Gamma, g)\) with boundary \(\Gamma\) is said to have almost polar boundary if the outer capacity \(\text{cap}(\Gamma)\) of \(\Gamma\) vanishes. Here, we use the notation \(\text{cap}(A)\) for the Sobolev (outer) capacity of the Borel subset \(A\) of \(M \cup \Gamma\), where for any open subset \(O\) of \(M \cup \Gamma\), we set

\[
\text{cap}(O) := \inf\{|u|_{1.2}, u \in \mathcal{C}_c^\infty(M \cup \Gamma), 0 \leq u \leq 1, u|_O \equiv 1\},
\]

and for any Borel subset \(S\), we set

\[
\text{cap}(A) := \inf\{\text{cap}(O), A \subset O\}.
\]

It is clear that one can replace \(\mathcal{C}_c^\infty(M \cup \Gamma)\) by \(H^1_0(M \cup \Gamma, g)\) in the definition of \(\text{cap}(O)\) obtaining an equivalent definition.

At this stage, we can observe that \(\partial B^n\) fails the sufficient condition (see [45, Thm. 7]) to be polar

\[
\liminf_{\varepsilon \to 0^+} \frac{\log \text{Vol}(\{x \in B^n : d(x, \partial B^n) < \varepsilon\})}{\log \varepsilon} \geq 2. \tag{16}
\]

Here, the equality case is considered, since \(\partial B^n\) itself is a manifold (see [45, Thm. 7]).

Let us denote by \(N_\varepsilon\) the set \(\{x \in B^n : d(x, \partial B^n) < \varepsilon\}\), we have \(N_\varepsilon = B^n \setminus (\cos \varepsilon) \cdot B^n\); moreover

\[
\text{Vol}(N_\varepsilon) = \pi \beta(1/2, n/2, 1 - (\cos \varepsilon)^2).
\]

Here, \(\beta(a, b, z)\) denotes the Incomplete Beta Function \(\int_0^z t^{a-1}(1-t)^{b-1}dt\). Hence

\[
\frac{\text{Vol } N_\varepsilon}{\varepsilon^2} \sim \frac{\text{Vol } N_\varepsilon}{1 - (\cos \varepsilon)^2} \frac{1 - (\cos \varepsilon)^2}{\varepsilon^2} \sim 2 \frac{\text{Vol } N_\varepsilon}{1 - (\cos \varepsilon)^2}, \quad \text{as } \varepsilon \to 0^+.
\]
Note that
\[
\liminf_{\varepsilon \to 0^+} \frac{\text{Vol} \, N_{\varepsilon}}{1 - (\cos \varepsilon)^2} = \lim_{z \to 0^+} \frac{\beta(1/2, n/2, z)}{z} = \lim_{z \to 0^+} z^{-1/2}(1 - z)^{n/2 - 1} = +\infty.
\]
Thus, we have \(\liminf_{\varepsilon \to 0^+} \frac{\text{Vol} \, N_{\varepsilon}}{\varepsilon^2} = +\infty\) that in particular implies \(\frac{\log \text{Vol} \, N_{\varepsilon}}{\log \varepsilon} < 2\) for any \(\varepsilon < \varepsilon_0\).

Since the condition (16) is not fulfilled by \(\partial B^n\) or by \(\partial S^n\), we wonder if the ball and the simplex, endowed with their Baran metrics, are not manifolds with almost polar boundary. Indeed, this is the case, as stated in the following proposition. We stress that, since these conclusions are obtained as a consequence of Theorems 1 and 2 respectively, we cannot use them in the proof of these theorems.

**Proposition 2** The manifolds \((B^n, \delta_{B^n})\) and \((S^n, \delta_{S^n})\) are not manifolds with almost polar boundary and

\[
H^1(B^n, \delta_{B^n}) \neq H^1_0(B^n, \delta_{B^n}), \quad H^1(S^n, \delta_{S^n}) \neq H^1_0(S^n, \delta_{S^n}).
\]  

**Remark 9** We warn the reader that \(H^1(B^n, \delta_{B^n}) \neq H^1_0(B^n, \delta_{B^n})\) does not imply in general that the eigenvalue problem \(\Delta u = \lambda u\) is not well posed when we do not impose any boundary condition. The motivation depends on the following proposition which allows us to write the weak formulations (25) and (31) of the Laplace Beltrami operator used in the proofs of Theorems 1 and 2 which is based on \(C^\infty_b\) functions (for which the boundary terms appearing in the integration by parts formulas we use vanish).

**Proposition 3** Let \((M, g)\) be \((B^n, \delta_{B^n})\) or \((S^n, g_{S^n})\). The space \(C^\infty_b(M)\) is dense in \(C_{1,2}^\infty(M)\) with respect to the norm \(\| \cdot \|_{1,2}\). Thus, \(C^\infty_b(M)\) is dense in \(H^1(M, g)\).

Before proving Proposition 3, we need the following technical lemmas whose proofs are omitted, since it is sufficient to check the statements by easy direct computations.

**Lemma 1** (The inverse Baran metric of the ball) Let us denote by \(G_{B^n}^{-1}\) the inverse of the matrix \(G_{B^n}\) which represents the Baran metric of the \(n\)-dimensional ball. Then we have

\[
G_{B^n}^{-1}(x) := \begin{bmatrix}
1 - x_1^2 & -x_1 x_2 & \ldots & -x_1 x_n \\
-x_2 x_1 & 1 - x_2^2 & \ldots & -x_2 x_n \\
\vdots & \vdots & \ddots & \vdots \\
-x_n x_1 & \ldots & -x_n x_{n-1} & 1 - x_n^2
\end{bmatrix}.
\]

The matrix \(G_{B^n}^{-1}(x)\) has eigenvalues \(\{1, 1 - |x|^2\}\), where the eigenspace of 1 is the tangent space at \(x\) to the sphere of radius \(|x|\) and centred at zero, while the eigenspace of \(1 - |x|^2\) is the Euclidean normal to this sphere at \(x\).

**Lemma 2** (The inverse Baran metric of the simplex) Let us denote by \(G_{S^n}^{-1}\) the inverse of the matrix \(G_{S^n}\) which represents the Baran metric of the \(n\)-dimensional simplex.
Then we have
\[
G^{-1}_{S^n}(x) := \begin{bmatrix}
(1 - x_1)x_1 & -x_1x_2 & \ldots & \ldots & -x_1x_n \\
-x_2x_1 & (1 - x_2)x_2 & \ldots & \ldots & -x_2x_n \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
-x_nx_1 & \ldots & \ldots & -x_nx_{n-1} & (1 - x_n)x_n
\end{bmatrix}.
\]

Moreover, we have
\[
G^{-1}_{S^n}(x) = \text{diag} \left( \frac{\sqrt{x_1}}{\sqrt{2}}, \ldots, \frac{\sqrt{x_n}}{\sqrt{2}} \right) G^{-1}_{B^n} \text{diag} \left( \frac{\sqrt{x_1}}{\sqrt{2}}, \ldots, \frac{\sqrt{x_n}}{\sqrt{2}} \right).
\]

Proof of Proposition 3 Let us start by considering the case \( M = B^n \subset \mathbb{R}^n \). We denote by \( S^n \) the \( n \)-dimensional unit real sphere endowed with the standard round metric \( g_{S^n} \) and we introduce the embedding map
\[
E : \mathcal{C}^{\infty}_{1,2}(M) \rightarrow H^1_{\text{even}}(S^n, g_{S^n}),
\]
where
\[
E[f]\left( x_1, \ldots, x_n, \pm \sqrt{1 - \sum_{i=1}^{n} x_i^2} \right) := \frac{1}{\sqrt{2}} \left\{ f(x_1, \ldots, x_n), \sum_{i=1}^{n} x_i^2 \neq 1 \right\} \text{lim sup}_{B \ni x \rightarrow f(\xi_1, \ldots, \xi_n)}, \sum_{i=1}^{n} x_i^2 = 1
\]
and
\[
H^1_{\text{even}}(S^n, g_{S^n}) := \left\{ g \in H^1(S^n, g_{S^n}), g(x_1, \ldots, x_{n+1}) = g(x_1, \ldots, -x_{n+1}) \text{ Vol}_{S^n} -a.e. \right\}.
\]

We claim that \( E \) is an isometry of Hilbert spaces.

Before proving this claim, we stress that this would conclude the proof for the case of the ball. For, by standard mollifications, we can construct a sequence \( \{ \tilde{f}_k \} \) of functions in \( \mathcal{C}^{\infty}(S^n) \) converging to \( E[f] \) in \( H^1(S^n, g_{S^n}) \). To ensure that \( \tilde{f}_k \in H^1_{\text{even}}(S^n, g_{S^n}) \), we replace \( \tilde{f}_k \) by \( (\hat{f}_k(x_1, \ldots, x_{n+1}) + \hat{f}_k(x_1, \ldots, -x_{n+1}))/2 \). Finally define \( \{ f_k \} := \{ E^{-1}[\hat{f}_k] \} \) and note that the claim above implies that \( f_k \rightarrow f \) in \( H^1(B^n, \delta B^n) \).

We stress that, while the injectivity of \( E \) is trivial, one needs to notice that the global boundedness of \( \tilde{f}_k \) together with its derivatives ensure that \( E^{-1}[\hat{f}_k] \) is a well-defined element of \( \mathcal{C}^{\infty}_{1,2}(M) \) which in particular is in \( \mathcal{C}^\infty_b(M) \).
Let us go back to prove that $E$ is an isometric embedding. For simplicity, we work in the easy case of $n = 2$. The general case can be proved in a completely equivalent way. Consider spherical coordinates

$$ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \cos \theta \cos \varphi \\ \cos \theta \sin \varphi \\ \sin \theta \end{pmatrix}. $$

We recall that the round metric represented in these coordinates is

$$ g_{\mathbb{S}^2}(\theta, \varphi) := \begin{bmatrix} 1 & 0 \\ 0 & \cos^2 \theta \end{bmatrix}, $$

and the corresponding volume form can be written $d \text{Vol}_{\mathbb{S}^2} = \cos \theta d\theta d\varphi$. It follows that, for any $h \in H^1(\mathbb{S}^2, g_{\mathbb{S}^2})$, we have

$$ \|h\|^2_{H^1(\mathbb{S}^2)} = \int_0^{2\pi} \int_{-\pi/2}^{\pi/2} \left(|h|^2 + |\partial_\theta h|^2 + \frac{|\partial_\varphi h|^2}{\cos^2 \theta}\right) \cos \theta d\theta d\varphi. $$

To compute $\|E[f]\|^2_{H^1(\mathbb{S}^2)}$, we perform the change of variables suggested by the first two components of the spherical coordinates, that is

$$(x_1, x_2) \mapsto (\cos \theta \cos \varphi, \cos \theta \sin \varphi).$$

Then

$$ \|E[f]\|^2_{H^1(\mathbb{S}^2)} = \int_{B^2} \left( |f|^2 + (1 - x_1^2 - x_2^2) \frac{|\partial_n f|^2}{2} + \frac{|\partial_t f|^2}{2} \right) \frac{1}{\sqrt{1 - x_1^2 - x_2^2}} dx_1 dx_2 $$

$$ = \int_{B^2} \left( |f|^2 + |\text{grad} f|^2 \right) d \text{Vol}_{g_{B^2}} $$

$$ = \|f\|^2_{H^1(B, g_{B^2})}. $$

Let us now consider the case $M = S^n$. We introduce the embedding map

$$ F : C_\infty^{\infty}(S^n, \delta_{S^n}) \to V, $$

$\mathcal{S}$ Springer
where

\[ V := \{ f \in C^\infty_{1,2}(B^n, \delta_{B^n}), \]
\[ f(\xi_1, \ldots, \xi_j, \ldots, \xi_n) = f(\xi_1, \ldots, -\xi_j, \ldots, \xi_n), \forall j \in \{1, \ldots, d\} \}
\]

and

\[ F[h](\xi_1, \ldots, \xi_n) := \frac{1}{2} h(\xi_1^2, \ldots, \xi_n^2). \]

Again, if the closure of \( F \) to \( H^1(S^n, \delta_{S^n}) \) is an isometric embedding we are done, since, for any given target function \( h \in H^1(S^n, \delta_{S^n}) \), we can pull back to \( C^\infty_b(S^n) \) any sequence of \( C^\infty_b(B^n) \) approximations to \( F[h] \).

To this aim, we introduce the partition \( Q_1, \ldots, Q_{2^n} \) of \([-1, 1]^n\) given by the coordinate hyperplanes, we denote by \( T : S^n \to B^n \) the map \((\xi_1, \ldots, \xi_n) \mapsto (\xi_1^2, \ldots, \xi_n^2) = x\) and we notice that, for any \( f \in C^\infty(S^n) \), we have

\[ \int_{B^n \cap Q_j} f \circ T \, d\text{Vol}_{B^n} = \frac{1}{2^n} \int_{S^n} f \, d\text{Vol}_{S^n}. \]

Finally, we compute

\[
\|F[h]\|_{H^1(B^n, \delta_{B^n})} = \sum_{j=1}^{2^n} \int_{B^n \cap Q_j} \left( |F[h](\xi)|^2 + |\text{grad } F[h](\xi)|^2_{\delta_{B^n}} \right) \, d\text{Vol}_{B^n}(\xi)
= \frac{1}{4} \sum_{j=1}^{2^n} \int_{B^n \cap Q_j} \left( |h \circ T(\xi)|^2 + Dh^t \circ T \quad JT^t \quad \delta_{B^n}^{-1} \quad JT \quad D \circ T(\xi) \right) \, d\text{Vol}_{B^n}(\xi)
= \frac{1}{4 \cdot 2^n} \sum_{j=1}^{2^n} \int_{T^{-1}(B^n \cap Q_j)} \left( |h(x)|^2 + Dh^t (JT^t \delta_{B^n}^{-1} \quad JT) \circ T^{-1} \quad Dh(x) \right) \, d\text{Vol}_{S^n}(x)
= \frac{1}{4 \cdot 2^n \cdot 2^n} \int_{S^n} \left( |h(x)|^2 + Dh^t (JT^t \delta_{B^n}^{-1} \quad JT) \circ T^{-1} \quad Dh(x) \right) \, d\text{Vol}_{S^n}(x).
\]

Since, due to Eq. (20),

\[ (JT^t \delta_{B^n}^{-1} \quad JT) \circ T^{-1} = 4 \text{diag}(\xi) \delta_{B^n}^{-1} \text{diag}(\xi) \bigg|_{\xi = \sqrt{x}} = 4\delta_{S^n}(x), \]

we conclude that \( \|F[h]\|_{H^1(B^n, \delta_{B^n})} = \|h\|_{H^1(S^n, \delta_{S^n})} \). In view of the above reasoning, this concludes the proof. \( \square \)
2.3 Unbounded Linear Operators on Hilbert Spaces: Some Tools

We need to recall some concepts from operator theory that allow a more precise and compact formulation of our results. A linear operator on a Banach space \( \mathcal{B} \) is a couple \((\mathcal{D}(\mathcal{B}(T)), T)\), where \( \mathcal{D}(\mathcal{B}(T)) \) is a dense linear subspace of \( \mathcal{B} \) and \( T \) is a linear map \( \mathcal{D}(\mathcal{B}(T)) \to \mathcal{B} \).

Let \((\mathcal{D}(\mathcal{B}(T)), T)\) be a linear operator. If, for any sequence, \( \{f_n\} \) in \( \mathcal{D}(\mathcal{B}(T)) \), such that

- \( \|f_n\| \to 0 \) for some \( f \in \mathcal{B} \),
- there exists \( g \in \mathcal{B} \) with \( \|Tf_n - g\| \to 0 \),

it follows that \( f \in \mathcal{D}(\mathcal{B}(T)) \) and \( Tf = g \), and then, the operator \( T \) is said to be closed. If \( \mathcal{B} \) is not finite dimensional, the notion of spectrum and set of eigenvalues are not the same. More precisely, we denote by \( \sigma(T) \) the spectrum of \( T \)

\[ \sigma(T) := \{ z \in \mathbb{C} : T - zI \text{ is not invertible} \} \]

Instead, \( \lambda \) is an eigenvalue of \( T \) if there exists an element \( f \in \mathcal{B} \), such that \( Tf = \lambda f \).

If an operator \( T \) is not closed, we may try to find an extension of it, i.e., \((\tilde{T}, \mathcal{D}(\mathcal{B}(\tilde{T}))\)) such that \( \mathcal{D}(\mathcal{B}(\tilde{T})) \supset \mathcal{D}(\mathcal{B}(T)) \) and \( \tilde{T}f = Tf \) for any \( f \in \mathcal{D}(\mathcal{B}(T)) \). If we can find such an extension in the category of closed operators, then \( T \) is said to be closable and its minimal closed extension \( \tilde{T} \) is termed the closure of \( T \).

Now, we replace the Banach space \( \mathcal{B} \) by an Hilbert space \( \mathcal{H} \); clearly, the above terminologies are still well defined, since any Hilbert space is in particular Banach.

If for any \( f, g \in \mathcal{D}(\mathcal{H}(T)) \), we have \( \langle Tf, g \rangle_{\mathcal{H}} = \langle f, Tg \rangle_{\mathcal{H}} \), then the operator \( T \) is said to be symmetric. It is a very useful fact that any symmetric operator is closable to a symmetric operator. Again, if \( \mathcal{H} \) is infinite dimensional, one must pay attention to the difference between symmetric and self-adjoint operators.

The adjoint \( T^* \) of the operator \( T \) is defined by the relation

\[ \langle Tf, g \rangle_{\mathcal{H}} = \langle f, T^*g \rangle_{\mathcal{H}}, \forall f \in \mathcal{D}(\mathcal{H}(T)), g \in \mathcal{D}(\mathcal{H}(T^*)) \]

where

\[ \mathcal{D}(\mathcal{H}(T^*)) := \{ g \in \mathcal{H} : \exists h \in \mathcal{H} \text{ such that } \langle Tf, g \rangle_{\mathcal{H}} = \langle f, h \rangle_{\mathcal{H}}, \forall f \in \mathcal{D}(\mathcal{H}(T)) \} \]

The operator \( T \) is said to be self-adjoint when the two domains indeed coincide.

The proofs of our results, besides the explicit computations, rely on the following theorem which collects some classical results of operator theory; see for instance [34, Ch. 1 and Ch. 4].

**Theorem 3** Let \( T \) be a linear non-negative unbounded operator on the separable Hilbert space \( (\mathcal{H}, \|\cdot\|) \) with domain \( \mathcal{D}(T) \). Assume that

a) \( T \) is symmetric.

b) It has discrete real spectrum \( \sigma(T) = \{ \lambda_j \}_{j \in \mathbb{N}} \) diverging to \( +\infty \).

Then

\[ \text{Springer} \]
i) The closure $\overline{T}$ of $T$ is a self-adjoint unbounded operator (i.e., $T$ is essentially self-adjoint).

ii) $\sigma(\overline{T}) = \sigma(T)$.

iii) The domain of $\overline{T}$ is

$$
D(\overline{T}) = \left\{ u \in \mathcal{H} : \sum_{j=1}^{\infty} \lambda_j^2 |\hat{u}_j|^2 < \infty \right\}.
$$

(iv) The quadratic form

$$
Q(u) := \langle T^{1/2}u, T^{1/2}u \rangle_{\mathcal{H}}
$$

has domain

$$
D(Q) = \left\{ u \in \mathcal{H} : \sum_{j=1}^{\infty} \lambda_j |\hat{u}_j|^2 < \infty \right\},
$$

which is complete in the norm

$$
\|\|u\|\| := \sqrt{Q(u)} + \|u\|_{\mathcal{H}}.
$$

Here and throughout the paper, $\hat{u}_j$ denotes the $j$th Fourier coefficients of the function $u$.

### 3 Proofs

The strategy of the proofs of Theorems 1 and 2 is to show that the conditions $a)$ and $b)$ of Theorem 3 hold when $T$ is the Laplace Beltrami operator with respect to the Baran metric. Then, conclude the proofs applying Theorem 3. This will be done by considering the weak formulation of the Laplace Beltrami operator and performing explicit computations on a suitable orthogonal system.

#### 3.1 Orthogonal Polynomials in $L^2_{\mu_{bn}}$

The following family of orthogonal functions on the unit ball was first introduced in the approximation theory framework; indeed, the formula we will use is a special case of orthogonal polynomials for certain radial weight functions; see [35, Ch. 5].

**Proposition 4** [35] Let us set, for any $\alpha \in \mathbb{N}^n$,

$$
\varphi_\alpha := T_{a_n} \left( \frac{x_n}{\sqrt{1 - \sum_{k=1}^{n-1} x_k^2}} \right)^{n-1} \prod_{j=1}^{j-1} \left( 1 - \sum_{k=1}^{j-1} x_k^2 \right)^{\alpha_j/2} C_{\alpha_j} \left( \frac{x_j}{\sqrt{1 - \sum_{k=1}^{j-1} x_k^2}} \right),
$$

(23)
where $T_k$ is the Chebyshev polynomial of degree $k$, $\gamma_j := \binom{n}{2} + \sum_{k=j+1}^{n} \alpha_k$ and $C^s_t$ denote the monic Gegenbauer polynomials of degree $t$ (i.e., $C^s_t := J^{s_{-1/2}, s_{-1/2}}_t$ and $J^{\alpha, \beta}_t$ is the monic Jacobi polynomial orthogonal on $[-1, 1]$ with respect to the weight $(1-x)^{\alpha}(1+x)^{\beta}$).

The set $\{\varphi_\alpha : \alpha \in \mathbb{N}_n\}$ is a dense orthogonal system in $L^2(B^n, \delta_{B^n})$ and

$$\|\varphi_\alpha\|^2_{L^2(B^n, \delta_{B^n})} = \|T_{\alpha_0}\|^2_{-1/2, -1/2} \prod_{j=1}^{n-1} \|C_{\gamma_j}^{\alpha_j}\|^2_{\alpha_j-1/2, \alpha_j-1/2},$$

where $\|f\|_{a,b} := \left(\int_{-1}^{1} |f(t)|^2 (1-t)^a (1+t)^b \, dt\right)^{1/2}$.

Note that the density of the linear subspace span$\{\varphi_\alpha : \alpha \in \mathbb{N}_n\}$ in $H^1(B^n, \delta_{B^n})$ follows from Proposition 3.

### 3.2 Proof of Theorem 1

In this section, we will denote the Euclidean gradient of $f$ by $\nabla f$.

**Proof of Theorem 1** We start by showing that $\Delta$ acting on $C^2_B(B^n)$ is a symmetric operator. Namely, for any $u, v \in C^2_B(B^n)$, we have

$$\int_{B^n} u \Delta v \, \text{Vol}_{B^n} = - \int_{B^n} \langle \nabla u, \nabla v \rangle_{\delta_{B^n}} \, d \text{Vol}_{B^n} = \int_{B^n} v \Delta u \, \text{Vol}_{B^n}. \quad (25)$$

To prove this formula, we perform two integrations by parts on $B^n_r := \{x : |x| \leq r\}$ letting $r \to 1$.

$$- \int_{B^n} u \Delta v \, \text{Vol}_{B^n} = - \int_{B^n} \text{div}(\sqrt{\det \delta_{B^n} G_{B^n}^{-1} Du}) v \, dx$$

$$= \lim_{r \to 1} - \int_{B^n_r} \text{div}(\sqrt{\det \delta_{B^n} G_{B^n}^{-1} Du}) v \, dx$$

$$= \lim_{r \to 1} \left( \int_{B^n_r} Dv^T G_{B^n}^{-1} Du \sqrt{\det \delta_{B^n}} \, dx - \int_{\partial B^n_r} v^T G_{B^n}^{-1} Du \sqrt{\det \delta_{B^n}} \, d\sigma \right)$$

$$= \int_{B^n} \langle \nabla u, \nabla v \rangle_{\delta_{B^n}} \, d \text{Vol}_{B^n} - \lim_{r \to 1} \int_{\partial B^n_r} v^T G_{B^n}^{-1} Du \sqrt{\det \delta_{B^n}} \, d\sigma$$

$$= - \int_{B^n} \text{div}(\sqrt{\det \delta_{B^n} G_{B^n}^{-1} Du}) u \, dx +$$

$$\lim_{r \to 1} \int_{\partial B^n_r} uv^T G_{B^n}^{-1} Du \sqrt{\det \delta_{B^n}} \, d\sigma - \lim_{r \to 1} \int_{\partial B^n_r} uv^T G_{B^n}^{-1} Du \sqrt{\det \delta_{B^n}} \, d\sigma$$

$$= - \int_{B^n} u \Delta v \, \text{Vol}_{B^n} + \lim_{r \to 1} \int_{\partial B^n_r} uv^T G_{B^n}^{-1} Du \sqrt{\det \delta_{B^n}} \, d\sigma$$

$$- \lim_{r \to 1} \int_{\partial B^n_r} uv^T G_{B^n}^{-1} Du \sqrt{\det \delta_{B^n}} \, d\sigma.$$
Here, ψ is the (Euclidean) unit outward normal to ∂B^n | r := \{x ∈ R^n : |x| = r\}.

The proof of (25) is ended if we show that

$$\lim_{r \to 1} \int_{\partial B^n} \psi \nu G^{-1}_B Du \sqrt{\det \delta_B} d\sigma = 0$$

for any u, v ∈ \mathcal{C}^2_b(B^n). To see this, simply observe (see Lemma 1) that ψ is an eigenvector of $G^{-1}_B$ for the eigenvalue $(\det g)^{-1}|x| = r = (1 - r^2)$, and thus, we have

$$\lim_{r \to 1} \int_{\partial B_r} \psi \nu G^{-1}_B Du \sqrt{\det g} d\sigma = \lim_{r \to 1} \sqrt{1 - r^2} \int_{\partial B_r} \psi \partial_n u d\sigma$$

$$\leq \lim_{r \to 1} \sqrt{1 - r^2} \|u\|_{\mathcal{C}^2_b(B^n)} \|\psi\|_{\mathcal{C}^2_b(B^n)} = 0.$$  

This shows that condition a) of Theorem 3 holds for \(\Delta\). To conclude the proof, we need to show that b) holds as well, i.e., there exists a $L^2(B^n, \delta_B)$-orthogonal system in $\mathcal{C}^2_b(B^n)$ dense in $L^2(B^n, \delta_B)$ whose elements are of eigenfunctions of \(\Delta\), such that the corresponding eigenvalues are a positive diverging sequence. We claim that such an orthogonal system is indeed $\{\varphi_\alpha, \alpha \in \mathbb{N}^n\}$; see Proposition 4.

For the sake of readability, we present here the case $n = 2$, which allows a slightly easier notation and computations than the general case. However, all the elements of the proof of the general case are present in such a simplified exposition. To simplify the notation, we denote $B^n$ by $B$.

The orthogonal basis of Proposition 4 reads as

$$\varphi_{s,k}(x, y) := (1 - x^2)^{s/2} J_{s-k}^{1/k} (x) T_k \left( \frac{y}{\sqrt{1 - x^2}} \right), \quad 0 \leq k \leq s \in \mathbb{N},$$

where we denoted by $J_{m,\alpha,\beta}^{s,k}$ the $m$th Jacobi orthogonal polynomial with respect to $(1 - x)^{\alpha}(1 + x)^{\beta}$. We need to verify that

$$\langle -\Delta \varphi_{s,k}, \varphi_{m,l} \rangle_{L^2(B, \delta_B)} = \lambda_{s,k} \delta_{s,m} \delta_{k,l} = s(s + 1) \delta_{s,m} \delta_{k,l}.$$  

Since $\varphi_{s,k}$ are elements of $\mathcal{C}^\infty_b(B)$, we can use the above weak formulation (25) to get

$$\langle -\Delta \varphi_{s,k}, \varphi_{m,l} \rangle_{L^2(B, \delta_B)} = \int_B D\varphi_{s,k}^T G^{-1}_B D\varphi_{m,l} \sqrt{\det \delta_B} dx dy.$$  

Let us introduce a change of variables

$$(x, z) \mapsto \Psi(x, z) := (x, z\sqrt{1 - x^2}) = (x, y).$$
We denote by $J\psi$ the Jacobian matrix of $\Psi$, so we get

\[
\int_{B} Df_1^T G_B^{-1} Df_2 \sqrt{\det g} \, dx \, dy
= \int_{-1}^{1} \int_{-1}^{1} D(f_1 \circ \Psi)^T J\Psi^{-T} G_B^{-1} J\Psi^{-1} D(f_2 \circ \Psi) \, dx \frac{dz}{\sqrt{1 - z^2}}
= \int_{-1}^{1} \int_{-1}^{1} D(f_1 \circ \Psi)^T \begin{bmatrix} 1 - x^2 & 0 \\ 0 & \frac{1 - z^2}{1 - x^2} \end{bmatrix} D(f_2 \circ \Psi) \, dx \frac{dz}{\sqrt{1 - z^2}}.
\]

Note that not only $\Psi$ is a change of variables that diagonalize $G_B^{-1}$, it also has the property of giving to the basis functions $\varphi_{s,k}$ a tensor product structure. Indeed, we have $\varphi_{s,k} \circ \Psi(x, z) = (1 - x^2)^{k/2} J_{s-k}(x) T_k(z)$, and thus

\[
\int_{-1}^{1} \int_{-1}^{1} D(\varphi_{s,k} \circ \Psi)^T \begin{bmatrix} 1 - x^2 & 0 \\ 0 & \frac{1 - z^2}{1 - x^2} \end{bmatrix} D(\varphi_{m,l} \circ \Psi) \, dx \frac{dz}{\sqrt{1 - z^2}}
= \int_{-1}^{1} \partial_x [(1 - x^2)^{k/2} J_{s-k}(x)] \partial_x [(1 - x^2)^{l/2} J_{m-l}(x)] (1 - x^2) \, dx 
\times \int_{-1}^{1} T_k(z) T_l(z) \frac{dz}{\sqrt{1 - z^2}}
+ \int_{-1}^{1} (1 - x^2)^{(k+l)/2-1} J_{s-k}(x) J_{m-l}(x) \, dx 
\times \int_{-1}^{1} \partial_z T_k(z) \partial_z T_l(z) \sqrt{1 - z^2} \, dz.
\]

It is well known that

\[
\int_{-1}^{1} T_k(z) T_l(z) \frac{dz}{\sqrt{1 - z^2}} = 2^{k+l} \pi / 2 \delta_{k,l}.
\]

Here, we denoted by $\delta_k$ the Kroneker delta at 0, i.e., $\delta_k = 1$ if $k = 0$ and vanishes elsewhere. Also one has $T'_k = k U_{k-1}$, where $U_k$ are the orthogonal Chebyshev polynomials of the second kind, that is

\[
\int_{-1}^{1} U_k(z) U_l(z) \sqrt{1 - z^2} \, dz = \pi / 2 \delta_{k,l}.
\]

Using such orthogonality and differentiation relations in the above computation, we get

\[
\int_{-1}^{1} \partial_x [(1 - x^2)^{k/2} J_{s-k}(x)] \partial_x [(1 - x^2)^{l/2} J_{m-l}(x)] (1 - x^2) \, dx.
\]
\[
\int_{-1}^{1} T_k(z) T_l(z) \frac{dz}{\sqrt{1-z^2}} \\
+ \int_{-1}^{1} (1 - x^2)^{(k+l)/2-1} J_{s-k}^{k,k}(x) J_m^{l,l}(x) dx \cdot \\
\int_{-1}^{1} \partial_z T_k(z) \partial_z T_l(z) \sqrt{1-z^2} dz \\
= \frac{\pi}{2} \delta_{l,k} \left( \int_{-1}^{1} \partial_x [(1 - x^2)^{k/2} J_{s-k}^{k,k}(x)] \partial_x [(1 - x^2)^{k/2} J_m^{k,k}(x)](1 - x^2) dx \cdot 2 \delta_k \right) \\
+ k^2 \int_{-1}^{1} (1 - x^2)^{k-1} J_{s-k}^{k,k}(x) J_m^{k,k}(x) dx. \tag{26}
\]

Now, letting \( k = l \), we note that

\[
\int_{-1}^{1} \partial_x [(1 - x^2)^{k/2} J_{s-k}^{k,k}(x)] \partial_x [(1 - x^2)^{k/2} J_m^{k,k}(x)](1 - x^2) dx \\
= \int_{-1}^{1} \partial_x [J_{s-k}^{k,k}(x)] \partial_x [J_m^{k,k}(x)](1 - x^2)^{k+1} dx \\
+ \int_{-1}^{1} -k x \partial_x [J_{s-k}^{k,k}(x)] J_m^{k,k}(x) dx \\
+ k^2 \int_{-1}^{1} x^2 J_{s-k}^{k,k}(x) J_m^{k,k}(x)(1 - x^2)^{k-1} dx.
\]

Integration by parts in the second term leads to

\[
\int_{-1}^{1} \partial_x [(1 - x^2)^{k/2} J_{s-k}^{k,k}(x)] \partial_x [(1 - x^2)^{k/2} J_m^{k,k}(x)](1 - x^2) dx \\
= \int_{-1}^{1} \partial_x [J_{s-k}^{k,k}(x)] \partial_x [J_m^{k,k}(x)](1 - x^2)^{k+1} dx \\
- 2k^2 \int_{-1}^{1} x^2 J_{s-k}^{k,k}(x) J_m^{k,k}(x)(1 - x^2)^{k-1} dx \\
+ k \int_{-1}^{1} J_{s-k}^{k,k}(x) J_m^{k,k}(x)(1 - x^2)^k dx \\
+ k^2 \int_{-1}^{1} x^2 J_{s-k}^{k,k}(x) J_m^{k,k}(x)(1 - x^2)^{k-1} dx.
\]

We plug this last identity to (26) with \( k = l \) to get

\[
\langle -\Delta_B \varphi_{s,k}, \varphi_{m,l} \rangle_{L^2(B,\delta_B^n)} \\
= \frac{\pi}{2} \delta_{l,k} 2 \delta_k \left( \int_{-1}^{1} \partial_x [J_{s-k}^{k,k}(x)] \partial_x [J_m^{k,k}(x)](1 - x^2)^{k+1} dx \right)
\]

\( \square \) Springer
Here, the last line is due to Proposition 4.
3.3 Orthogonal polynomials in $L^2_{\mu_{S^n}}$

**Proposition 5** [35] Let us set for any $\alpha \in \mathbb{N}^n$ and $x \in S^n$

\[
\psi_\alpha(x) := \prod_{j=1}^{n} \left(1 - \sum_{k=1}^{j-1} x_k\right)^{a_j} J_{a_j}^{a_j, -1/2} \left(-\frac{2x_j}{1 - \sum_{k=1}^{j-1} x_k} - 1\right), \tag{27}
\]

where $J_m^{a,b}$ is the $m$th Jacobi polynomial of parameters $a, b$ and

\[
a_j := 2 \sum_{k=1}^{\min(n,j+1)} \alpha_k + \frac{n - j - 1}{2}.
\]

The set $\{\psi_\alpha : \alpha \in \mathbb{N}^n\}$ is a dense orthogonal system in $L^2(S^n, \delta_{S^n})$.

This result (see [35, Thm. 8.2.2]) plays a key role in our proof.

**Theorem 4** [35] Let us introduce the differential operator

\[
Df := \sum_{i=1}^{n} x_i \partial^2_{i,i} f - 2 \sum_{1 \leq i < j \leq n} x_i x_j \partial_{i,j}^2 f + \frac{1}{2} \sum_{i=1}^{n} (1 - (n+1)x_i) \partial_i f. \tag{28}
\]

Then, we have

\[
D\psi_\alpha = |\alpha| \left( |\alpha| + \frac{n+1}{2} \right) \psi_\alpha. \tag{29}
\]

It will turn out in the proof of Theorem 2 that $D$ agrees on smooth functions with the aforementioned Laplace Beltrami operator with respect to the Baran metric of the simplex.

3.4 Proof of Theorem 2

**Proof of Theorem 2** Since $G_{S^n}$ blows up at the boundary of the simplex, our strategy is to carry out integration by parts on certain exhausting subsets of $S^n$ and then taking the limit approaching the boundary. To this aim, it is convenient to introduce, see Figure 1; the following notation for $\varepsilon > 0$

\[
S^n_\varepsilon := \left\{ x \in S^n : x_i > \varepsilon, \left(1 - \sum_{k=1}^{n} x_i\right) > \varepsilon \right\},
\]

\[
T^{n,0}_\varepsilon := \left\{ x \in \partial S^n_\varepsilon : \left(1 - \sum_{k=1}^{n} x_i\right) = \varepsilon \right\},
\]

\[
T^{n,i}_\varepsilon := \left\{ x \in \partial S^n_\varepsilon : x_i = \varepsilon \right\}, \quad i = 1, \ldots, n.
\]
Also let $\nu_i$ be the Euclidean unit normal to $T^n_{\varepsilon,i}$ (for any $\varepsilon > 0$). We note that

$$\partial S^n_\varepsilon = \bigcup_{i=0}^n T^n_{\varepsilon,i}.$$

Following the first part of the proof of Theorem 1, we show that $\Delta$ is a symmetric operator on the space $C^\infty_b(S^n)$ which is dense (see Proposition 3) in $H^1(S^n, \delta S^n)$. For, we perform integration by parts twice. Let $u, v \in C^\infty_b(S^n)$, and then

$$- \int_{S^n} v \Delta u d \text{Vol}_{S^n} = - \int_{S^n} \text{div}(\sqrt{\det \delta S^n} G^{-1}_{S^n} Du) v dx$$

$$= \lim_{\varepsilon \to 0^+} - \int_{S^n_\varepsilon} \text{div}(\sqrt{\det \delta S^n} G^{-1}_{S^n} Du) v dx$$

$$= \lim_{\varepsilon \to 0^+} \left( \int_{S^n_\varepsilon} Du^T G^{-1}_{S^n} Du \sqrt{\det \delta S^n} dx - \sum_{i=0}^n \int_{T^n_{\varepsilon,i}} v_{\nu_i}^T G^{-1}_{S^n} Du \sqrt{\det \delta S^n} d\sigma \right)$$

$$= \int_{S^n} \langle \text{grad} u, \text{grad} v \rangle \text{dVol}_{S^n} - \sum_{i=0}^n \lim_{\varepsilon \to 0^+} \int_{T^n_{\varepsilon,i}} v_{\nu_i}^T G^{-1}_{S^n} Du \sqrt{\det \delta S^n} d\sigma$$

$$= - \int_{S^n} v \Delta u d \text{Vol}_{S^n}$$

$$+ \lim_{\varepsilon \to 0^+} \int_{T^n_{\varepsilon,i}} (u v_{\nu_i}^T G^{-1}_{S^n} Du - v_{\nu_i}^T G^{-1}_{S^n} Du) \sqrt{\det \delta S^n} d\sigma.$$

Thus, we need to prove that for any $u, v \in C^\infty_b(S^n)$ and any $i \in \{0, 1, \ldots, n\}$, we have

$$\lim_{\varepsilon \to 0^+} \int_{T^n_{\varepsilon,i}} u v_{\nu_i}^T G^{-1}_{S^n} Du \sqrt{\det \delta S^n} d\sigma = 0. \quad (30)$$

To see this, it is sufficient to note (using Lemma 2) that for any $x \in T^n_{\varepsilon,0}$

$$v_0^T G_{S^n}^{-1} \sqrt{\det \delta S^n} = \frac{\varepsilon}{\prod_{k=1}^n x_k} (x_1, x_2, \ldots, x_n)^T$$

and for any $x \in T^n_{\varepsilon,i}$, $i = 1, 2, \ldots, n$,

$$v_i^T G_{S^n}^{-1} \sqrt{\det \delta S^n} = \frac{\varepsilon}{(1 - \varepsilon - \sum_{k=1, k \neq i}^n x_k) \prod_{k=1, k \neq i}^n x_k} (x_1, x_2, \ldots, x_{i-1}, 1 - \varepsilon, x_{i+1}, \ldots, x_n)^T.$$

Therefore, we have

$$\left| \int_{T^n_{\varepsilon,0}} u v_i^T G_{S^n}^{-1} Du \sqrt{\det \delta S^n} d\sigma \right| \leq \sqrt{\varepsilon} n \max_{S^n}(|Du|_\infty |u|) \left\| \prod_{k=1}^n \sqrt{x_k} \right\|_{L^1(T^n_{\varepsilon,0})} \to 0$$
and, for any \( i = 1, 2, \ldots, n \),

\[
\left| \int_{T_n^i} u v_i^T G_{S_n}^{-1} D v \sqrt{\det \delta_{S_n}} \, d\sigma \right|
\leq \sqrt{\varepsilon n} \max_{S_n}(|D v|_\infty |u|) \left\| \left( 1 - \varepsilon - \sum_{k=1, k \neq i}^n x_k \prod_{k=1, k \neq i}^n x_k \right)^{-1/2} \right\|_{L^1(T_n^i)} \to 0,
\]

and thus, (30) holds. This shows that \( \Delta \) is a symmetric operator on \( C^\infty(S^n) \), i.e., for any such \( u \) and \( v \),

\[
\int_{S^n} u \Delta v \, \text{Vol}_{S^n} = - \int_{S^n} \langle \text{grad} \, u, \text{grad} \, v \rangle_{\delta_{S^n}} \, d\text{Vol}_{S^n} = \int_{S^n} v \Delta u \, \text{Vol}_{S^n} \quad \text{. (31)}
\]

Now, we want to show that \( \Delta \) has discrete spectrum \( \sigma(\Delta_S) = \{ \lambda_s := s(s + \frac{n-1}{2}) : s \in \mathbb{N} \} \) and the eigenspace of \( \lambda_s \) is \( \text{span} \{ \psi_\alpha, |\alpha| = s \} \) (see Proposition 5).

Instead of proving this directly, we rely on the known properties of the basis \( \{ \psi_\alpha \} \), namely (29), and we simply show that for smooth functions

\[
\Delta f = D f \quad \text{, (32)}
\]

this allows us to characterize \( \sigma(\Delta) \) due to Theorem 4. Then, we apply Theorem 3 and the result follows.

We introduce the notation \( h(x) := (1 - \sum_{k=1}^n x_k) \prod_{k=1}^n x_k \). It is worth noting that

\[
\sqrt{h(x)} \frac{x_i}{\sqrt{h(x)}} = \frac{1 - \sum_{k \neq i} x_k}{2(1 - \sum_{k=1}^n x_k)} = \frac{1}{2} \left( 1 + \frac{x_i}{1 - \sum_{k=1}^n x_k} \right).
\]

For any smooth \( f \), we have

\[
\Delta_{S^n} f
\]

\[
= \sqrt{h(x)} \sum_{i=1}^n \frac{x_i}{\sqrt{h(x)}} \left( \partial_i f - \sum_{j=1}^n x_j \partial_j f \right)
\]

\[
= \sum_{i=1}^n \left\{ \sqrt{h(x)} \frac{x_i}{\sqrt{h(x)}} \left( \partial_i f - \sum_{j=1}^n x_j \partial_j f \right) + x_i \partial_i \left( \partial_i f - \sum_{j=1}^n x_j \partial_j f \right) \right\}
\]

\[
= \sum_{i=1}^n \left\{ \frac{1}{2} \left( 1 + \frac{x_i}{1 - \sum_{k=1}^n x_k} \right) \left( \partial_i f - \sum_{j=1}^n x_j \partial_j f \right) + x_i \partial_i \left( \partial_i f - \sum_{j=1}^n x_j \partial_j f \right) \right\}
\]

\[
= -\frac{1}{2} \sum_{j=1}^n x_j \partial_j f \cdot \sum_{i=1}^n \left( 1 + \frac{x_i}{1 - \sum_{k=1}^n x_k} \right)
\]

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Fig. 1 Notation used in the proof of Theorem 2

\[
+ \frac{1}{2} \sum_{i=1}^{n} \partial_i f + \frac{1}{2} \left( 1 - \sum_{k=1}^{n} x_k \right) \sum_{i=1}^{n} x_i \partial_i f \\
+ \sum_{i=1}^{n} \left\{ x_i \partial_i^2 f - x_i \sum_{j \neq i} x_j \partial_{i,j}^2 f - x_i^2 \partial_i^2 f - x_i \partial_i f \right\} \\
= \sum_{i=1}^{n} x_i (1 - x_i) \partial_i^2 f - 2 \sum_{1 \leq j < i \leq n} x_i x_j \partial_{i,j}^2 f + \frac{1}{2} \sum_{i=1}^{n} \partial_i f \\
+ \left( \sum_{i=1}^{n} x_i \partial_i f \right) \cdot \left\{ -\sum_{i=1}^{n} \left( \frac{1}{2} + \frac{x_i}{2 \left( 1 - \sum_{k=1}^{n} x_k \right)} \right) + \frac{1}{2} \left( 1 - \sum_{k=1}^{n} x_k \right) - 1 \right\} \\
= \sum_{i=1}^{n} x_i (1 - x_i) \partial_i^2 f - 2 \sum_{1 \leq j < i \leq n} x_i x_j \partial_{i,j}^2 f + \frac{1}{2} \sum_{i=1}^{n} \partial_i f \\
+ \left( \sum_{i=1}^{n} x_i \partial_i f \right) \cdot \left\{ -\frac{n+2}{2} + \frac{-\sum_{i=1}^{n} x_i + 1}{2 \left( 1 - \sum_{k=1}^{n} x_k \right)} \right\} \\
= \sum_{i=1}^{n} x_i (1 - x_i) \partial_i^2 f - 2 \sum_{1 \leq j < i \leq n} x_i x_j \partial_{i,j}^2 f + \frac{1}{2} \sum_{i=1}^{n} \partial_i f \\
= \sum_{i=1}^{n} x_i \partial_{i,j}^2 f - 2 \sum_{1 \leq i < j \leq n} x_i x_j \partial_{i,j}^2 f + \frac{1}{2} \sum_{i=1}^{n} (1 - (n+1) x_i) \partial_i f \\
= D f.
\]
3.5 Proof of Proposition 2

Let us first recall a result of Masamune [44, Thm. 3] on which the proof of Proposition 2 relies. Assume \((M, g)\) to be a compact Riemannian manifold and let \(\Sigma\) be a submanifold of \(M\); let us define \(\Delta_M\) as the standard Laplace Beltrami operator acting on \(C^\infty_c(M \setminus \Sigma)\). Then

\[
\Delta_M \text{ is essentially self-adjoint if and only if } \dim(M) - \dim(\Sigma) > 3. \tag{33}
\]

**Proof of Proposition 2** Let \(M := S^n \subset \mathbb{R}^{n+1}\) and \(\Sigma := \{x \in M : x_{n+1} = 0\}\). Also introduce the notation \((x_1, x_2, \ldots, x_n, x_{n+1}) = (\xi, x_{n+1})\).

Let us assume for a contradiction that \(C^\infty_c(B^n)\) is dense in \(H^1(B^n, \delta_{B^n})\). In view of the proof of Proposition 3, we have

\[
\mathcal{H} := \left(C^\infty_c(B^n), \| \cdot \|_{L^2(\delta_{B^n})} \right) \subseteq \text{isometry} \left(C^\infty_c(M \setminus \Sigma), \| \cdot \|_{L^2(\delta_M)} \right) := \mathcal{E}_1.
\]

\[
\left(C^\infty_c(B^n), \| \cdot \|_{L^2(\delta_{B^n})} \right) \subseteq \text{isometry} \left(C^\infty_c(M \setminus \Sigma), \| \cdot \|_{L^2(\delta_M)} \right) := \mathcal{E}_2. \tag{34}
\]

Here, \(C^\infty_c(M \setminus \Sigma)\) denotes the subspace

\[
\left\{ u \in C^\infty_c(M \setminus \Sigma), \ g^{\otimes n}, \ u(\xi, x_{n+1}) = -u(\xi, -x_{n+1}) \forall (\xi, x_{n+1}) \in M \setminus \Sigma \right\}
\]

and \(C^\infty_{c, even}(M \setminus \Sigma)\) is defined similarly. Note that, given \(u \in C^\infty_c(M \setminus \Sigma)\), we can define \(u_{\text{even}} := \frac{1}{2}(u(\xi, x_{n+1}) + u(\xi, -x_{n+1})) \in \mathcal{E}_1\) and \(u_{\text{odd}} := \frac{1}{2}(u(\xi, x_{n+1}) - u(\xi, -x_{n+1})) \in \mathcal{E}_2\), such that \(u = u_{\text{even}} + u_{\text{odd}}\).

The assumption that \(C^\infty_c(B^n)\) is dense in \(H^1(B^n, \delta_{B^n})\) together with Theorem 1 and the isometry property of the map \(E\) in the proof of Proposition 3 implies that the Laplace Beltrami operator \(\Delta_1\) acting on \(\mathcal{E}_1\) and \(\Delta_2\) acting on \(\mathcal{E}_2\) are essentially self-adjoint. Moreover, since \(\Delta_M u = \Delta_1 u_{\text{even}} + \Delta_2 u_{\text{odd}}\) for any \(u \in C^\infty_c(M \setminus \Sigma)\), it follows that \(\Delta_M\) itself is essentially self-adjoint.

On the other hand, \(\dim \Sigma = n - 1\) and \(\dim M = n\); this is in contrast with Masamune’s result (33) and thus \(C^\infty_c(B^n)\) cannot be dense in \(H^1(B^n, \delta_{B^n})\) and thus \(H^1(B^n, \delta_{B^n}) \neq H^1_0(B^n, \delta_{B^n})\). Note that, in view of [45, Thm. 1], this is equivalent to the fact that \(B^n\) is not a manifold with almost polar boundary.

The proof for the simplex can be performed in an equivalent way but using the map \(F\) defined in the proof of Proposition 3 instead of the map \(E\).

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A Pluripotential Theory on the Complexified Sphere and Spherical Harmonics

In this section, we consider \( S^{n-1} \) as a compact subset of the complexified sphere \( S^{n-1} := \{ z \in \mathbb{C}^n : \sum z_i^2 = 1 \} \). We can consider the space \( \text{psh}(S^{n-1}) \) of plurisubharmonic functions on the complex manifold \( S^{n-1} \) and form the usual upper envelope

\[
V_{S^{n-1}}^*(z, S^{n-1}) := \limsup_{S^{n-1} \ni \zeta \to z} \sup_{u \in \mathcal{L}(S^{n-1})} \left\{ u(\zeta) : u \in \mathcal{L}(S^{n-1}), u|_{S^{n-1}} \leq 0 \right\},
\]

where \( \mathcal{L}(S^{n-1}) \) denotes the space of plurisubharmonic functions \( u \) on \( S^{n-1} \), such that \( u - \frac{1}{2} \log \sum_{i=1}^{n-1} |z_i|^2 \) is bounded above as \( \sum_{i=1}^{n-1} |z_i| \to \infty \) along \( S^{n-1} \), defining the extremal plurisubharmonic function; compare this definition with equation (7). This is a locally bounded plurisubharmonic function which is maximal on \( S^{n-1}\setminus S^{n-1} \); \([9, 68]\).

On the other hand, it is clear that \( S^{n-1} \) is an irreducible algebraic subvariety of \( \mathbb{C}^n \) of pure dimension \( n - 1 \), and hence, we can use the result of Sadullaev \([55]\) to get

\[
V_{S^{n-1}}^*(z, S^{n-1}) = \limsup_{S^{n-1} \ni \zeta \to z} \sup \left\{ \frac{1}{\deg p} \log^+ |p(\zeta)| : p \in \mathcal{P}(\mathbb{C}^n), \|p\|_{S^{n-1}} \leq 1 \right\}.
\]

Here, \( \mathcal{P} \) denotes the space of algebraic polynomials with complex coefficients. It is worth stressing that here \( \deg \) denotes the degree of a polynomial on \( \mathbb{C}^n \), not the degree over the coordinates ring of \( S^{n-1} \).

In \([24, \text{Prop. 4.1}]\), the authors proved the formula

\[
V_{S^{n-1}}^*(z, S^{n-1}) = \frac{1}{2} \log \left( |z|^2 + \sqrt{|z|^4 - 1} \right), \text{ for all } z \in S^{n-1}.
\]

(35)

We note that this function can be used to define the Baran metric on the sphere, due to the following differentiability property.

**Lemma 3** Let \( x \in S^{n-1} \), the function \( V_{S^{n-1}}(\cdot, S^{n-1}) \) has right tangent directional derivative at \( x \) in any direction \( i \cdot v \), for any \( v \in T_x S^{n-1} \); that is

\[
\partial_{i, v}^+ V_{S^{n-1}}(x, S^{n-1}) := \frac{d}{dt} V_{S^{n-1}}(\gamma(t), S^{n-1})|_{t=0} \in \mathbb{R},
\]

where \( \gamma : [0, 1] \mapsto S^{n-1} \) is any differentiable arc with \( \gamma(0) = x, \gamma'(0^+) = i \cdot v \).

Moreover, we have \( \partial_{i, v} V_{S^{n-1}}(x, S^{n-1}) = |v| \).

**Proof** The problem is clearly rotation independent. We can thus assume \( x = (1, 0, \ldots, 0) = e_1 \) and \( v = |v|(0, 1, 0, \ldots, 0) = |v|e_2 \) without loss of generality.

Let us introduce the arc

\[
z(t) := \sqrt{1 + |v|^2 \log^2 (1 + t)e_1 + |v| \log(1 + t)e_2}, \quad t \in [0, +\infty].
\]

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It is easy to verify that \( z \) enjoys the properties
\[
z(t) \in S^{n-1}, \quad \forall t \in [0, +\infty[,
\]
\[
z(0) = x,
\]
\[
\frac{d}{dt} z(0^+) = i \cdot v.
\]

Thus we are left to show that, setting \( u(t) := V_{S^{n-1}}^*(z(t), S^{n-1}) \), we have \( \frac{d}{dt} u(0^+) = |v| \).

Let us note first that \( |z(t)|^2 = 1 + 2|v|^2 \log^2(1 + t) \), then we can compute
\[
u(t) = \frac{1}{2} \log \left[ 1 + 2|v|^2 \log^2(1 + t) + \sqrt{4|v|^2 \log^2(1 + t)(1 + |v|^2 \log^2(1 + t))} \right]
\]
\[
= \frac{1}{2} \log \left[ 1 + 2|v|^2 \log^2(1 + t) + 2|v| \log(1 + t) \sqrt{1 + |v|^2 \log^2(1 + t)} \right]
\]
\[
\sim \frac{1}{2} \log \left[ 1 + 2|v|^2 t^2 + 2|v| t \sqrt{1 + |v|^2 t^2} \right]
\]
\[
\sim \frac{1}{2} \log(1 + 2|v| t) \sim |v| t, \quad \text{as } t \to 0^+.
\]

Therefore
\[
u'(0^+) = \lim_{t \to 0^+} \frac{u(t) - u(0)}{t} = \lim_{t \to 0^+} \frac{u(t)}{t} = |v|.
\]

\( \square \)

Due to Lemma 3, we can define the Baran metric on the real unit sphere by setting
\[
\delta_{S^{n-1}}(x, v) := \partial_{t,v} V_{S^{n-1}}^*(x, S^{n-1}) = |v|.
\]

Note the analogy with the partial derivative taken in the lemma with the definition of the Baran metric in the standard “flat” case.

Using the parallelogram identity, we can define for any \( x \in S^{n-1} \) and any \( u, v \in T_x S^{n-1} \) the scalar product related to the Baran metric as
\[
\langle u, v \rangle_{g_{S^{n-1}}(x)} := \frac{\delta^2_{S^{n-1}}(x, u + v) - \delta^2_{S^{n-1}}(x, u - v)}{4} = \frac{|u + v|^2 - |u - v|^2}{4} = \langle u, v \rangle_{\mathbb{R}^n},
\]

that turns out to coincide with the standard (round) metric.

It is very well known that the Laplace Beltrami operator on the real unit sphere (endowed with the round metric) has a discrete diverging set of eigenvalues and its eigenfunctions are polynomials: the spherical harmonics.
These observations lead automatically to the desired conclusion that we state as a corollary.

**Corollary 1** The eigenfunctions of the Laplace Beltrami operator with respect to the Baran metric on the real unit sphere are the orthogonal polynomials with respect to the pluripotential equilibrium measure $\mu_{S^{n-1},S^{n-1}}$ of the real unit sphere $S^{n-1}$ in the complexified sphere $S^{n-1}$.

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