Modelling the granular medium dynamics on rough vibrating plane using method of large particles

I P Lyan¹ and G Y Panovko¹,²

¹ Mechanical Engineering Research Institute of the Russian Academy of Sciences (IMASH RAN), Moscow, Russia
² Bauman Moscow State Technical University (BMSTU), Moscow, Russia

LyanIlyaIMASH@yandex.ru

Abstract. The article represents the results of mathematical modelling of behaviour of the granular medium on an oscillating rough surface. Granular medium under the action of vibration is considered as a non-Newtonian (dilatant) fluid. The system of differential equations of continuum mechanics is solved using the modified large particles method. An algorithm for solving the problem for granular media with different properties under external influences is proposed.

1. Introduction

In modern technological operations of processing granular medium various methods of vibration processing, such as transportation, mixing, compaction/loosening, separation and etc., are widely used. Vibration allows to significantly intensify the course of such processes, and in some cases it turns out to be the only possibility for technological process implementation [1, 2].

Vibration in a granular medium causes various effects, in particular, it can flow like a liquid [3]. This state of granular medium is called fluidization or pseudo fluidity. As the intensity of the vibration increases, the connection between individual particles of the medium can be broken and its behavior becomes similar to a gas (vibro-boiling). The properties of the granular medium should be taken into account to choose rational vibration parameters and design of the oscillating machine’s working body that leads to incremental growth of technological process efficiency.

For embodiment of vibration technological processes the oscillating machines with two and more self-synchronized inertial exciters are widely used. The effect of self-synchronization of exiter in such machines depends on a wide spectrum of parameters, including the position of a dynamic system’s center of mass. Moving granular medium on the tray of such a machine leads to a significant shift of the system’s entire center of mass coordinates due to heterogeneity of material distribution on the working body. This paper sets the goal to determine the displacement of the system’s center of mass in time during the movement of the granular medium on the rough vibrating surface (tray).

2. Setting the objectives and the loading scheme

The paper discusses the dynamics of granular medium (quartz sand) flowing down on the tray of the vibrating conveyor (figure 1). Granular medium under the action of gravity precipitates from hopper 1, forming embankment 2. Tray 3 of the vibrating conveyor is turned to an angle $\vartheta$ from the horizontal, commits unidirectional harmonic oscillations $\xi(t)$ to an angle $\alpha$ from the tray, which is considered to be absolutely rigid. In order to avoid falling cargo to the other side of the vibrating conveyor, there is a
limiter 4 on the left side. Supposed distribution of the granular medium on the tray under the action of vibration is shown as a dotted line 5.

**Figure 1.** Loading scheme of the vibrating conveyor.

Description of granular medium behavior in a vibrating field is based on relevant computational models [3–9]. These models can be divided into two main types: discrete models of the granular medium regarded as set of material particles or solids with viscoelastic relations between them, and continual models using approaches of continuous medium mechanics in the whole volume of medium or in each of its elementary layer.

In this paper, the motion simulation of granular medium is based on the works [6-10] where moving granular medium under vibrational loads is assumed as a viscous dilatant fluid. Features of the flow of such continuous medium are laid in the rheological equation connecting strain rates with stresses. Together with the mass and momentum conservation equations they close the system of equations of continuum medium mechanics. Numerical solution of that system is based on the large particles method developed by O. M. Belotserkovsky and Yu. M. Davydov [10].

In a similar formulation, the problem has already been considered in [6]. However, analysis of these papers showed that the used finite difference numerical solution schemes of equations lead to the occurrence of asymmetry in characteristic medium movements, which, by nature, must possess symmetry.

### 3. Mathematical model

In the simulation of granular medium the following hypotheses and assumptions are accepted:

- the studying problem is considered to be plane in the gravity field;
- granular medium is considered to flow without separation from the tray surface;
- medium is considered to be single-phased (monogenic) and significant size of particle is considered to be many times smaller than calculation area.

Mathematical model of the studying process is described by the system consisting of three groups of equations: rheological equations connecting components of stress tensor with strain rates of granular medium, law of conservation of momentum and the law of conservation of mass of the system [6, 8, 10].

In this very work the rheological equations are obtained by using approaches described in [6]. In contrast to the work [6], the rates of angular deformations are taken into account in the form

$$\frac{1}{2} \left( \frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} \right),$$

which provides more correct description of the movement medium under the action of vibration. As a result, differential equations connecting strain rates and stresses in the projections on the axis $x, y$ associating with the tray are written as:
\[ \sigma_x = \alpha_0 + \alpha_1 \frac{\partial v_y}{\partial x} + \left( \alpha_2 + \alpha_3 \right) \frac{\partial v_x}{\partial x} \left[ \frac{\partial v_y}{\partial y} \frac{\partial v_x}{\partial y} - \frac{1}{4} \left( \frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y} \right)^2 \right] + \left( \alpha_4 + \alpha_5 \right) \frac{\partial v_x}{\partial x} \left( \frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y} \right); \]

\[ \sigma_y = \alpha_0 + \alpha_1 \frac{\partial v_x}{\partial y} + \left( \alpha_2 + \alpha_3 \right) \frac{\partial v_y}{\partial y} \left[ \frac{\partial v_x}{\partial x} \frac{\partial v_y}{\partial x} - \frac{1}{4} \left( \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} \right)^2 \right] + \left( \alpha_4 + \alpha_5 \right) \frac{\partial v_y}{\partial y} \left( \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} \right); \]

\[ \tau_{xy} = \frac{1}{2} \left( \alpha_4 + \alpha_3 \left( \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} \right) + \alpha_5 \left[ \frac{\partial v_x}{\partial y} \frac{\partial v_y}{\partial x} - \frac{1}{4} \left( \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} \right)^2 \right] \right), \]

where \( \sigma_x, \sigma_y, \tau_{xy} \) – components of the two-dimensional stress tensor; \( \frac{\partial v_x}{\partial x}, \frac{\partial v_y}{\partial y} \) – components of the strain rate tensor; \( \alpha_i = \alpha_i(\upsilon) \), \( i = 0, \ldots, 5 \) – coefficients describing the dissipative properties of the medium, \( \upsilon \) – volume concentration of the medium.

The equations of conservation of momentum are:

\[ \frac{\partial v_x}{\partial t} + v_y \frac{\partial v_x}{\partial x} + \frac{v_x}{\partial y} = \frac{1}{\upsilon \rho_0} \left( \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} \right) - g_x + \ddot{x}_x + R_x; \]

\[ \frac{\partial v_y}{\partial t} + v_x \frac{\partial v_y}{\partial x} + \frac{v_y}{\partial y} = \frac{1}{\upsilon \rho_0} \left( \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} \right) - g_y + \ddot{x}_y + R_y, \]

where \( t \) - current time; \( \rho_0 \) - average density of solid particles; \( g_x, g_y \) – gravitational acceleration projections on the \( x, y \) axis; \( \ddot{x}_x, \ddot{x}_y \) - projections on the \( x, y \) axes of external acceleration vibration exposure; \( R_x, R_y \) – projections on the \( x, y \) axis of aerodynamic drag forces.

The equations of conservation of momentum are:

\[ \frac{\partial \upsilon}{\partial t} + \frac{\partial (\upsilon v_x)}{\partial x} + \frac{\partial (\upsilon v_y)}{\partial y} = 0. \]

As a result, there is a complete system of nonlinear partial differential equations with variable coefficients describing flow of a granular medium along a vibrating surface. The principal feature of this system of equations is that a free boundary of granular medium is unknown in advance. To solve such problems, large particles method can be used as well proven in accounting practice [6-10]. When using this method, the free boundary of medium is determined at each integration step.

4. Solution

The large particle method is a variation of the Harlow’s particles method in cells, which uses the Lagrange and Euler approaches in numerical solving of continuum mechanics problems [10]. This method combines the advantages of Lagrangian and Eulerian approaches. The calculation area is
divided by a fixed (Eulerian) grid, but the continuous medium is treated as a discrete model, i.e. it is considered to be a set of particles of a fixed mass (Lagrangian grid of particles) that moves through the Eulerian grid. The motion of such system of particles during the time $\Delta t$ is carried out first in the form of a change in the internal state, assuming that the grid is stationary - the Eulerian stage. Then the displacement of all particles without changing the internal state is considered - the Lagrangian stage. At the final stage, the grid is recalculated to its original state.

In accordance with the basic idea of the method of large particles, the computational area is divided into a rectangular grid with the cell size $\Delta x \times \Delta y$ (figure 2 (a)). Each cell has four boundaries (figure 2, (b)) on which normal and tangential stresses are determined, and the velocity vectors and the value of the volume concentration are calculated inside each cell.

**Figure 2.** Calculation area (a) and parameters of one cell (b) of large particles method.

### 4.1. Euler stage.

At the first (Eulerian) stage, it is assumed that the change of velocities, stresses and energy are calculated without mass transfer effects. For the subsequent numerical implementation, equations (1) are written in a finite-difference form at the current integration step:

$$\sigma_{x,i,j} = \alpha_0 + \alpha_1 V_{xx} + \left(\alpha_0' + \alpha_1' V_{xx}\right)\left(V_{xx} + V_{yy}\right) + \left(\alpha_0'' + \alpha_1'' V_{xx}\right)\left(V_{xx} V_{yy} - \frac{1}{4}\left[V_{xy} + V_{yx}\right]^2\right);$$

$$\sigma_{y,i,j} = \alpha_0 + \alpha_1 V_{yy} + \left(\alpha_0' + \alpha_1' V_{yy}\right)\left(V_{yy} + V_{xx}\right) + \left(\alpha_0'' + \alpha_1'' V_{yy}\right)\left(V_{yy} V_{xx} - \frac{1}{4}\left[V_{yx} + V_{xy}\right]^2\right);$$

$$\tau_{xy,i,j} = \frac{1}{2}\left(V_{xy} + V_{yx}\right)\left(\alpha_1 + \alpha_1' \left[V_{xx} + V_{yy}\right] + \alpha_1'' \left[V_{xx} V_{yy} - \frac{1}{4}\left[V_{xy} + V_{yx}\right]^2\right]\right),$$

$$V_{xx} = \frac{v_{x,i+1,j} - v_{x,i-1,j}}{2\Delta x}, \quad V_{yy} = \frac{v_{y,i,j+1} - v_{y,i,j-1}}{2\Delta y}, \quad V_{xy} = \frac{v_{x,i,j+1} - v_{x,i,j-1}}{2\Delta y};$$

where $i, j$ - row and column numbers of grid respectively in which the cell of the computational area is located.

The equations for finding the intermediate velocities for each time step in a finite-difference form are the following ones:
At this stage, according to the idea of the large particles method, the convective terms of equations (2) are omitted [6-8]. It should be emphasized that in equations (3) there is a division by concentration, so that in order to avoid the occurrence of singular terms in cells with zero concentrations, the velocities and stresses of the medium are considered to be equal zero.

4.2. Lagrange stage.

At this stage, the velocity of the element of the granular material volume on the cell boundaries at each integration step is averaged with the velocity in the adjacent cell on this boundary. For the cell with the number \( i, j \), the velocity on the boundaries is determined by the following formula:

\[
U_{k,i,j} = \frac{v_{k,i,j} + v_{k,i,j-1}}{2},
\]

(5)

here \( k = \{1, 2, 3, 4\} \) - index of the cell boundary \( k = 1 \) corresponds to the left boundary, \( k = 2 \) - the lower boundary, \( k = 3 \) - the right boundary, \( k = 4 \) - the upper boundary), \( q = \{x, y\} \) - direction of the velocity, \( s = \{-1, 0, 1\} \) - coefficient determining the number of cell adjacent with the boundary \( k \). For example, for cell \( i,j \) the flow velocity of material through the left boundary:

\[
\frac{\Delta x U_{1,i,j} v_{1,i,j} \Delta t}{2} < \Delta x U_{1,i,j} v_{1,i,j} \Delta t \leq \frac{\Delta x U_{1,i,j} v_{1,i,j} \Delta t}{2}
\]

(6)

Then the change in mass and the new volume concentration in the cell are:

\[
\Delta M = \Delta M_{1,i,j} + \Delta M_{2,i,j} - \Delta M_{3,i,j} - \Delta M_{4,i,j},
\]

\[
\Delta v^{[n+1]}_{i,j} = \Delta v^{[n]}_{i,j} + \frac{\Delta M}{\Delta x \Delta y}
\]

where \( \{n\} \) is current and \( \{n+1\} \) is next integration step.

4.3. Final stage.

To describe the cell state, the function \( D_{k,i,j} \) is introduced, which is defined as following: \( D_{k,i,j} = 1 \) if the material flows into the cell \((i,j)\) through the boundary \(k\); \( D_{k,i,j} = 0 \) if the material flows from cell \((i,j)\) through the border \(k\).

Then the values of the cell velocities at the next integration step \( \{n + 1\} \) are determined by the formulas:
where $\Delta M_{k,i,j}^{(n)}$ play the role of weights of velocities. In contrast to [6], the values $\Delta M_{k,i,j}^{(n)}$ are taken in absolute value, which ensures the symmetry of the solutions obtained.

Additionally, the following condition is introduced: as soon as the concentration in the cell has reached the limit value, the velocity on the boundary through which the material flows into the cell $(i,j)$ is zeroed:

\[
\text{if } \begin{cases} v_{i,j} > v_d \\ D_{i,j} = 1 \end{cases} \rightarrow U_{i,j} = 0, \quad \text{if } \begin{cases} v_{i,j+1} > v_d \\ D_{i,j+1} = 0 \end{cases} \rightarrow U_{i,j} = 0; \quad \text{if } \begin{cases} v_{i,j} > v_d \\ D_{2,i,j} = 1 \end{cases} \rightarrow U_{2,i,j} = 0, \quad \text{if } \begin{cases} v_{i,j+1} > v_d \\ D_{2,i,j+1} = 0 \end{cases} \rightarrow U_{2,i,j} = 0.
\]

Thus, the material in this cell does not flow into, but may leak. According to [6, 10], to obtain stable solutions, it is necessary that during the whole integration the following conditions are satisfied:

\[
\frac{q_{i,j}}{\Delta t} \leq \frac{\Delta q}{\Delta t}, \quad \Delta q = \{\Delta x, \Delta y\}.
\]

It should be highlighted that in formulas (3) - (8) the parameters of adjacent cells are used, which leads to the problem of their usage on the boundaries of the computational area. To overcome this problem, the computational area is surrounded by “fictitious” cells, which parameters are assumed to be equal (up to the signs of the speeds) to the parameters of adjacent cells of the computational area. To form the boundary conditions, the coefficients of “sticking” - $\beta$ and “leaking” - $\gamma$ [6, 8, 10] are introduced:

\[-1 \leq \beta \leq 1, \quad -1 \leq \gamma \leq 1.\]

For example, on the upper boundary of the computational area the boundary conditions for the integration domain of size $N \times M$ are written as following:
\[
\begin{align*}
\nu_{i,M} &= \nu_{i,M-1}, & \sigma_{y,i,M} &= \sigma_{y,i,M-1}, \\
\sigma_{x,i,M} &= \sigma_{x,i,M-1}, & \nu_{x,i,M} &= \beta \nu_{x,i,M-1}, \\
\tau_{xy,i,M} &= \tau_{xy,i,M-1}, & \nu_{y,i,M} &= \gamma \nu_{y,i,M-1}.
\end{align*}
\]

5. The results analysis

In the numerical implementation of the above algorithm, quartz sand was chosen as the medium continuously coming from the bunker to the inclined surface of the tray, harmonically vibrating in a direction parallel to its plane (see figure 1). The upper boundary of the computational area coincides with the lower edge of the bunker. The movement of material through the bottom and left boundaries of the computational area is forbidden and the right and the top boundaries are free. It is assumed that the characteristic size of the cross section of the bunker’s outlet is equal to four cells located in the direction of the \(x\) axis (figure 3, (a)).

Numerical calculations were carried out with the values of the coefficients \(\alpha_i, i = 0, ..., 5\) for quartz sand [6]. Euler mesh parameters: \(L = 5m, H = 0,5 m; \ M = 20, \ N = 100\). Initial concentration of cells with material - \(\nu_y = 0,53\). The plane is inclined at an angle \(\beta = 3^\circ\).

The calculation results are shown in figure 3 in the form of several successive discrete in time distributions of the granular medium (figure 3 (a-e)) and the graphic illustrating how the centre of mass moves in the computational area is shown as well in figure 3 (f). Under the action of gravity the medium falls onto the surface of the tray, which is illustrated by filling the cells in the direction of the \(y\) axis, see figure 3 (a, b). The free fall of material from the feeder is indicated by dots in figure 3 (f): the center of mass moves strictly vertically. After the first layers of the medium touch the surface of the tray, its spreading begins immediately (see figure 3 (c)), so the centre of mass starts to move along the surface of the tray (crosses in figure 3 (f)). With the further outflow of the medium from the bunker, the spreading zone increases (see figure 3 (d, e)). During the formation of the embankment and movement of the medium along the vibrating tray, the center of mass shifts in the direction of the \(x\) axis and by the time the tray is completely filled, it is in a position close to the center of the computational area (circles in figure 3, (e)).

\[\text{Figure 3. Calculational results. (a)-(e) - distributions of the granular medium, (f) – the trajectory of medium centre of mass.}\]
6. Conclusion

As a result of the research a system of differential equations based on the concepts of continuum mechanics describing the granular medium dynamics on a rough vibrating surface was formed. To solve this system of equations a numerical algorithm based on the method of large particles has been developed and programmatically implemented. The proposed algorithm allows both global (displacement and velocity of the centre of mass, average transport speed, distribution and shape of the medium on the tray, depending on the vibration parameters), and local parameters (concentration and velocity of the medium in an arbitrary zone) to be obtained at an arbitrary time.

The results obtained during the numerical experiment correspond to the physical nature of this process. The proposed algorithm is universal and can be used to simulate the movement of granular medium under other conditions of vibration effects, in particular, with a non-uniform field of vibration impact.

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