On Carnot algebra
with the growth vector \((2, 3, 5, 8)\)

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Abstract
We compute two vector field models of the Carnot algebra with the growth vector \((2, 3, 5, 8)\), and an infinitesimal symmetry of the corresponding sub-Riemannian structure.

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1 Introduction

Carnot groups provide a nilpotent approximation to generic sub-Riemannian manifolds [8, 11, 3, 2, 5]. The free nilpotent sub-Riemannian structures are a natural first subject of study in sub-Riemannian geometry starting from the growth vector (2, 3) — the left-invariant sub-Riemannian structure on the Heisenberg group [4, 16]. The next free rank 2 case — the growth vector (2, 3, 5) — was studied in [12, 13, 14, 15].

In this work we start to study the next free rank 2 case — the growth vector (2, 3, 5, 8). We compute two vector field models of the corresponding Carnot algebra, and an infinitesimal symmetry of the corresponding sub-Riemannian structure (see Sec. 3).

2 Free nilpotent and Carnot Lie algebras

2.1 Free nilpotent Lie algebras

Let $L_d$ be the real free Lie algebra with $d$ generators [6]; $L_d$ is the Lie algebra of commutators of $d$ variables. We have $L_d = \oplus_{i=1}^{\infty} L_d^i$, where $L_d^i$ is the space of commutator polynomials of degree $i$. Then

$$L_d^{(r)} := L_d / \oplus_{i=r+1}^{\infty} L_d^i$$

is the free nilpotent Lie algebra of step $r$ (or of length $r$).

Denote

$$l_d(i) := \dim L_d^i, \quad l_d^{(r)} := \dim L_d^{(r)} = \sum_{i=1}^{r} l_d(i).$$

The classical expression of $l_d(i)$ is

$$il_d(i) = d^i - \sum_{j|i, 1 \leq j < i} j l_d(j).$$

In this work we will be interested in free nilpotent Lie algebras with 2 generators. Dimensions of such Lie algebras for small step are given in Table 1.

2.2 Carnot algebras and groups

A Lie algebra $L$ is called a Carnot algebra if it admits a decomposition

$$L = \oplus_{i=1}^{r} L_i$$
Table 1: Dimensions of free nilpotent Lie algebras with 2 generators

| i  | 1   | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|----|-----|---|---|---|---|---|---|---|---|----|
| $l_2(i)$ | 2   | 1 | 2 | 3 | 6 | 9 | 18 | 30 | 56 | 99 |
| $l_2^{(i)}$ | 2   | 3 | 5 | 8 | 14 | 23 | 41 | 71 | 127 | 226 |

as a vector space, such that

$[L_i, L_j] \subset L_{i+j}$,

$L_s = 0$ for $s > r$,

$L_{i+1} = [L_1, L_i]$.

A free nilpotent Lie algebra $\mathcal{L}_d^{(r)}$ is a Carnot algebra with $L_i = \mathcal{L}_d^i$.

A Carnot group $G$ is a connected, simply connected Lie group whose Lie algebra $L$ is a Carnot algebra. If $L$ is realized as the Lie algebra of left-invariant vector fields on $G$, then the degree 1 component $L_1$ can be thought of as a completely nonholonomic (bracket-generating) distribution on $G$. If moreover $L_1$ is endowed with a left-invariant inner product $\langle \cdot, \cdot \rangle$, then $(G, L_1, \langle \cdot, \cdot \rangle)$ becomes a nilpotent left-invariant sub-Riemannian manifold [5]. Such sub-Riemannian structures are nilpotent approximations of generic sub-Riemannian structures [8, 11, 3, 2].

The sequence of numbers

$(\dim L_1, \dim L_1 + \dim L_2, \ldots, \dim L_1 + \cdots + \dim L_r = \dim L)$

is called the growth vector of the distribution $L_1$ [13].

For free nilpotent Lie algebras, the growth vector is maximal compared with all Carnot algebras with the bidimension $(\dim L_1, \dim L)$.

In this work we consider the Carnot algebra with the growth vector $(2, 3, 5, 8)$.

### 3 Lie algebra with the growth vector $(2, 3, 5, 8)$

The Carnot algebra with the growth vector $(2, 3, 5, 8)$

$\mathcal{L}_2^{(4)} = \text{span}(X_1, \ldots, X_8)$

is determined by the following multiplication table:

$[X_1, X_2] = X_3, \quad (1)$

$[X_1, X_3] = X_4, \quad [X_2, X_3] = X_5, \quad (2)$

$[X_1, X_4] = X_6, \quad [X_1, X_5] = [X_2, X_4] = X_7, \quad [X_2, X_5] = X_8, \quad (3)$
with all the rest brackets equal to zero.

### 3.1 Hall basis

Free nilpotent Lie algebras have a convenient basis introduced by M. Hall \[9\]. We describe it using the exposition of \[7\].

The Hall basis of the free Lie algebra $L_d$ with $d$ generators $X_1, \ldots, X_d$ is the subset $\text{Hall} \subset L_d$ that has a decomposition into homogeneous components

$$\text{Hall} = \bigcup_{i=1}^{\infty} \text{Hall}_i$$

defined as follows.

Each element $H_j, j = 1, 2, \ldots$, of the Hall basis is a monomial in the generators $X_i$ and is defined recursively as follows. The generators satisfy the inclusion

$$X_i \in \text{Hall}_1, \quad i = 1, \ldots, d,$$

and we denote

$$H_i = X_i, \quad i = 1, \ldots, d.$$

If we have defined basis elements

$$H_1, \ldots, H_{N_p-1} \in \bigoplus_{j=1}^{p-1} \text{Hall}_j,$$

they are simply ordered so that $E < F$ if $E \in \text{Hall}_k, F \in \text{Hall}_l, k < l$:

$$H_1 < H_2 < \cdots < H_{N_p-1}.$$

Also if $E \in \text{Hall}_s, F \in \text{Hall}_t$ and $p = s + t$, then

$$[E, F] \in \text{Hall}_p$$

if:

1. $E > F$, and
2. if $E = [G, K]$, hen $K \in \text{Hall}_q$ and $t \geq q$.

By this definition, one easily computes recursively the first components $\text{Hall}_i$ of the Hall basis for $d = 2$:

- $\text{Hall}_1 = \{H_1, H_2\}$, $H_1 = X_1$, $H_2 = X_2$,
- $\text{Hall}_2 = \{H_3\}$, $H_3 = [X_2, X_1]$,
- $\text{Hall}_3 = \{H_4, H_5\}$, $H_4 = [[X_2, X_1], X_1]$, $H_5 = [[X_2, X_1], X_2]$,
- $\text{Hall}_4 = \{H_6, H_7, H_8\}$,
- $H_6 = [[[X_2, X_1], X_1], X_1]$, $H_7 = [[[X_2, X_1], X_1], X_2]$, $H_8 = [[[X_2, X_1], X_2], X_2]$. 

\[4\]
Consequently,
\[ \mathcal{L}_2^{(4)} = \text{span}\{H_1, \ldots, H_8\}. \]

In the sequel we use a more convenient basis
\[ \mathcal{L}_2^{(4)} = \text{span}\{X_1, \ldots, X_8\} \]
with the multiplication table (1)–(3).

3.2 Asymmetric vector field model for \( \mathcal{L}_2^{(4)} \)

Here we recall an algorithm for construction of a vector field model for the Lie algebra \( \mathcal{L}_2^{(r)} \) due to Grayson and Grossman [7]. For a given \( r \geq 1 \), the algorithm evaluates two polynomial vector fields \( H_1, H_2 \in \text{Vec}(\mathbb{R}^N) \), \( N = \dim \mathcal{L}_2^{(r)} \), which generate the Lie algebra \( \mathcal{L}_2^{(r)} \).

Consider the Hall basis elements
\[ \text{span}\{H_1, \ldots, H_N\} = \mathcal{L}_2^{(r)}. \]

Each element \( H_i \in \text{Hall}_j \) is a Lie bracket of length \( j \):
\[ H_i = \ldots [[H_2, H_{k_j}], H_{k_{j-1}}], \ldots, H_{k_1}], \quad k_j = 1, \quad k_{n+1} \leq k_n \text{ for } 1 \leq n \leq j - 1. \]

This defines a partial ordering of the basis elements. We say that \( H_i \) is a direct descendant of \( H_2 \) and of each \( H_{k_l} \) and write \( i \succ 2, i \succ k_l, l = 1, \ldots, j \).

Define monomials \( P_{2,k} \) in \( x_1, \ldots, x_N \) inductively by
\[ P_{2,k} = -x_j P_{2,i}/(\deg_j P_{2,i} + 1), \]
whenever \( H_k = [H_i, H_j] \) is a basis Hall element, and where \( \deg_j P \) is the highest power of \( x_j \) which divides \( P \).

The following theorem gives the properties of the generators.

**Theorem 3.1** (Th. 3.1 [7]). Let \( r \geq 1 \) and let \( N = \dim \mathcal{L}_2^{(r)} \). Then the vector fields
\[ H_1 = \frac{\partial}{\partial x_1}, \quad H_2 = \frac{\partial}{\partial x_2} + \sum_{i>2} P_{2,i} \frac{\partial}{\partial x_i} \]
have the following properties:

1. they are homogeneous of weight one with respect to the grading
\[ \mathbb{R}^N = \text{Hall}_1 \oplus \cdots \oplus \text{Hall}_r; \]
2. Lie($H_1, H_2$) = $\mathcal{L}_2^{(r)}$.

The algorithm described before Theorem 3.1 produces the following vector field basis of $\mathcal{L}_2^{(4)}$:

$$H_1 = \frac{\partial}{\partial x_1},$$
$$H_2 = \frac{\partial}{\partial x_2} - x_1 \frac{\partial}{\partial x_3} - \frac{x_1^2}{2} \frac{\partial}{\partial x_4} - x_1x_2 \frac{\partial}{\partial x_5} + \frac{x_1^3}{6} \frac{\partial}{\partial x_6} + \frac{x_1^2x_2}{2} \frac{\partial}{\partial x_7} + \frac{x_1x_2^2}{2} \frac{\partial}{\partial x_8},$$
$$H_3 = \frac{\partial}{\partial x_3} + x_1 \frac{\partial}{\partial x_4} + x_2 \frac{\partial}{\partial x_5} - \frac{x_1^2}{2} \frac{\partial}{\partial x_6} - x_1x_2 \frac{\partial}{\partial x_7} - \frac{x_2^2}{2} \frac{\partial}{\partial x_8},$$
$$H_4 = -\frac{\partial}{\partial x_4} + x_1 \frac{\partial}{\partial x_6} + x_2 \frac{\partial}{\partial x_7},$$
$$H_5 = -\frac{\partial}{\partial x_5} + x_1 \frac{\partial}{\partial x_7} + x_2 \frac{\partial}{\partial x_8},$$
$$H_6 = -\frac{\partial}{\partial x_6},$$
$$H_7 = -\frac{\partial}{\partial x_7},$$
$$H_8 = -\frac{\partial}{\partial x_8},$$

with the multiplication table

$$[H_2, H_1] = H_3, \quad [H_3, H_1] = H_4, \quad [H_3, H_2] = H_5,$$
$$[H_4, H_1] = H_6, \quad [H_4, H_2] = H_7, \quad [H_5, H_2] = H_8.$$  

3.3 Symmetric vector field model of $\mathcal{L}_2^{(4)}$

The vector field model of the Lie algebra $\mathcal{L}_2^{(4)}$ via the fields $H_1, \ldots, H_8$ obtained in the previous subsection is asymmetric in the sense that there is no visible symmetry between the vector fields $H_1$ and $H_2$. Moreover, no continuous symmetries of the sub-Riemannian structure generated by the orthonormal frame $\{H_1, H_2\}$ are visible, although the Lie brackets (4)–(6) suggest that this sub-Riemannian structure should be preserved by a one-parameter group of rotations in the plane span$\{H_1, H_2\}$.

One can find a symmetric vector field model of $\mathcal{L}_2^{(4)}$ free of such shortages as in the following statement.
Theorem 3.2. (1) The vector fields

\[ X_1 = \frac{\partial}{\partial x_1} - \frac{x_2}{2} \frac{\partial}{\partial x_3} + \frac{x_2^2}{2} \frac{\partial}{\partial x_4} - \frac{x_1 x_2^2}{4} \frac{\partial}{\partial x_5} - \frac{x_2^3}{6} \frac{\partial}{\partial x_6} \]  

(7)

\[ X_2 = \frac{\partial}{\partial x_2} + \frac{x_1}{2} \frac{\partial}{\partial x_3} + \frac{x_2^2}{2} \frac{\partial}{\partial x_4} + \frac{x_1^3}{6} \frac{\partial}{\partial x_5} + \frac{x_1^2 x_2}{4} \frac{\partial}{\partial x_6} \]  

(8)

\[ X_3 = \frac{\partial}{\partial x_3} + x_1 \frac{\partial}{\partial x_4} + x_2 \frac{\partial}{\partial x_5} + \frac{x_1^2}{2} \frac{\partial}{\partial x_6} + x_1 x_2 \frac{\partial}{\partial x_7} + \frac{x_2^2}{2} \frac{\partial}{\partial x_8} \]  

(9)

\[ X_4 = \frac{\partial}{\partial x_4} + x_1 \frac{\partial}{\partial x_6} + x_2 \frac{\partial}{\partial x_7} \]  

(10)

\[ X_5 = \frac{\partial}{\partial x_5} + x_1 \frac{\partial}{\partial x_7} + x_2 \frac{\partial}{\partial x_8} \]  

(11)

\[ X_6 = \frac{\partial}{\partial x_6} \]  

(12)

\[ X_7 = \frac{\partial}{\partial x_7} \]  

(13)

\[ X_8 = \frac{\partial}{\partial x_8} \]  

(14)

satisfy the multiplication table (1)-(3). Thus the fields \( X_1, \ldots, X_8 \in \text{Vec}(\mathbb{R}^8) \) model the Lie algebra \( \mathcal{L}_2^{(4)} \).

(2) The vector field

\[ X_0 = x_2 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2} + x_5 \frac{\partial}{\partial x_3} - x_4 \frac{\partial}{\partial x_5} + P \frac{\partial}{\partial x_6} + Q \frac{\partial}{\partial x_7} + R \frac{\partial}{\partial x_8} \]  

(15)

\[ P = -\frac{x_4^4}{24} + \frac{x_2^2 x_2^2}{8} + x_7, \]  

(16)

\[ Q = \frac{x_1 x_2^2}{12} + \frac{x_1^3 x_2}{12} - 2x_6 + 2x_8, \]  

(17)

\[ R = \frac{x_2^4 x_2^2}{8} - \frac{x_4^4}{24} - x_7 \]  

(18)

satisfies the following relations:

\[ [X_0, X_1] = X_2, \quad [X_0, X_2] = -X_1, \quad [X_0, X_3] = 0, \quad (19) \]

\[ [X_0, X_4] = X_5, \quad [X_0, X_5] = -X_4, \quad (20) \]

\[ [X_0, X_6] = 2X_7, \quad [X_0, X_7] = X_8 - X_6, \quad [X_0, X_8] = -2X_7. \quad (21) \]

Thus the field \( X_0 \) is an infinitesimal symmetry of the sub-Riemannian structure generated by the orthonormal frame \{X_1, X_2\}.
Proof. In fact, the both statements of the proposition are verified by the
direct computation, but we prefer to describe a method of construction of
the vector fields \(X_1, \ldots, X_8\), and \(X_9\).

(1) In the previous work \([12]\) we constructed a similar symmetric vec-
tor field model for the Lie algebra \(L_2^{(3)}\), which has growth vector \((2, 3, 5)\):

\[
L_2^{(3)} = \text{span}\{X_1, \ldots, X_5\} \subset \text{Vec}(\mathbb{R}^5),
\]

\[
X_1 = \frac{\partial}{\partial x_1} - \frac{x_2}{2} \frac{\partial}{\partial x_4} - \frac{x_1^2 + x_2^2}{2} \frac{\partial}{\partial x_5},
\]

\[
X_2 = \frac{\partial}{\partial x_2} + \frac{x_1}{2} \frac{\partial}{\partial x_3} + \frac{x_1^2 + x_2^2}{2} \frac{\partial}{\partial x_4},
\]

\[
X_3 = \frac{\partial}{\partial x_3} + x_1 \frac{\partial}{\partial x_4} + x_2 \frac{\partial}{\partial x_5},
\]

\[
X_4 = \frac{\partial}{\partial x_4},
\]

\[
X_5 = \frac{\partial}{\partial x_5},
\]

with the Lie brackets \([1], [2]\). Now we aim to “continue” these relationships
to vector fields \(X_1, \ldots, X_8 \in \text{Vec}(\mathbb{R}^8)\) that span the Lie algebra \(L_2^{(4)}\). So we seek for vector fields of the form

\[
X_1 = \frac{\partial}{\partial x_1} - \frac{x_2}{2} \frac{\partial}{\partial x_3} - \frac{x_1^2 + x_2^2}{2} \frac{\partial}{\partial x_5} + \sum_{i=6}^{8} a_i^1 \frac{\partial}{\partial x_i},
\]

\[
X_2 = \frac{\partial}{\partial x_2} + \frac{x_1}{2} \frac{\partial}{\partial x_3} - \frac{x_1^2 + x_2^2}{2} \frac{\partial}{\partial x_4} + \sum_{i=6}^{8} a_i^2 \frac{\partial}{\partial x_i},
\]

\[
X_3 = \frac{\partial}{\partial x_3} + x_1 \frac{\partial}{\partial x_4} + x_2 \frac{\partial}{\partial x_5} + \sum_{i=6}^{8} a_i^3 \frac{\partial}{\partial x_i},
\]

\[
X_4 = \frac{\partial}{\partial x_4} + \sum_{i=6}^{8} a_i^4 \frac{\partial}{\partial x_i},
\]

\[
X_5 = \frac{\partial}{\partial x_5} + \sum_{i=6}^{8} a_i^5 \frac{\partial}{\partial x_i},
\]

\[
X_j = \sum_{i=j}^{8} a_i^j \frac{\partial}{\partial x_i}, \quad j = 6, 7, 8,
\]

such that \(\text{span}\{X_1, \ldots, X_8\} = L_2^{(4)}\).
Compute the required Lie brackets:

\[
\begin{align*}
[X_1, X_2] &= \frac{\partial}{\partial x_3} + x_1 \frac{\partial}{\partial x_4} + x_2 \frac{\partial}{\partial x_5} + \left( \frac{\partial a_6^6}{\partial x_1} - \frac{\partial a_1^6}{\partial x_2} \right) \frac{\partial}{\partial x_6} \\
&\quad + \left( \frac{\partial a_2^7}{\partial x_1} - \frac{\partial a_1^7}{\partial x_2} \right) \frac{\partial}{\partial x_7} + \left( \frac{\partial a_2^8}{\partial x_1} - \frac{\partial a_1^8}{\partial x_2} \right) \frac{\partial}{\partial x_8}, \\
[X_1, X_3] &= \frac{\partial}{\partial x_4} + \frac{\partial a_6^6}{\partial x_1} \frac{\partial}{\partial x_6} + \frac{\partial a_7^7}{\partial x_1} \frac{\partial}{\partial x_7} + \frac{\partial a_8^8}{\partial x_1} \frac{\partial}{\partial x_8}, \\
[X_2, X_3] &= \frac{\partial}{\partial x_5} + \frac{\partial a_6^6}{\partial x_2} \frac{\partial}{\partial x_6} + \frac{\partial a_7^7}{\partial x_2} \frac{\partial}{\partial x_7} + \frac{\partial a_8^8}{\partial x_2} \frac{\partial}{\partial x_8}, \\
[X_1, X_4] &= \frac{\partial}{\partial x_6} + \frac{\partial a_6^6}{\partial x_1} \frac{\partial}{\partial x_7} + \frac{\partial a_7^7}{\partial x_1} \frac{\partial}{\partial x_8} + \frac{\partial a_8^8}{\partial x_1} \frac{\partial}{\partial x_6}, \\
[X_1, X_5] &= \frac{\partial}{\partial x_7} + \frac{\partial a_6^6}{\partial x_2} \frac{\partial}{\partial x_8} + \frac{\partial a_7^7}{\partial x_2} \frac{\partial}{\partial x_6} + \frac{\partial a_8^8}{\partial x_2} \frac{\partial}{\partial x_7}, \\
[X_2, X_4] &= \frac{\partial}{\partial x_8} + \frac{\partial a_6^6}{\partial x_3} \frac{\partial}{\partial x_7} + \frac{\partial a_7^7}{\partial x_3} \frac{\partial}{\partial x_8} + \frac{\partial a_8^8}{\partial x_3} \frac{\partial}{\partial x_6}, \\
[X_2, X_5] &= \frac{\partial}{\partial x_7} + \frac{\partial a_6^6}{\partial x_3} \frac{\partial}{\partial x_8} + \frac{\partial a_7^7}{\partial x_3} \frac{\partial}{\partial x_6} + \frac{\partial a_8^8}{\partial x_3} \frac{\partial}{\partial x_7}.
\end{align*}
\]

The vector fields \(X_1, \ldots, X_8\) should be independent, thus the determinant constructed of these vectors as columns should satisfy the inequality

\[
D = \det (X_1, \ldots, X_8) = \begin{vmatrix}
a_6^6 & a_7^6 & a_8^6 \\
a_6^7 & a_7^7 & a_8^7 \\
a_6^8 & a_7^8 & a_8^8
\end{vmatrix} \neq 0.
\]

We will choose \(a_j^i\) such that \(D = 1\). It follows from the multiplication table for \(X_1, \ldots, X_8\) that

\[
D = \begin{vmatrix}
d^2 a_3^6 & d^2 a_4^6 & d^2 a_5^6 \\
d x_4^6 & d x_1 d x_2 & d x_3^6 \\
d x_4^7 & d x_1 d x_2 & d x_3^7 \\
d x_4^8 & d x_1 d x_2 & d x_3^8
\end{vmatrix}.
\]

In order to get \(D = 1\), define the entries of this matrix as following symmetric way:

\[
a_3^6 = \frac{x_1^2}{2}, \quad a_3^7 = x_1 x_2, \quad a_3^8 = \frac{x_2^2}{2}.
\]
Then we obtain from the multiplication table for $X_1, \ldots, X_8$ that
\[
\frac{\partial a^6_2}{\partial x_1} - \frac{\partial a^6_1}{\partial x_2} = a^6_3 = \frac{x_1^2}{2},
\]
\[
\frac{\partial a^7_2}{\partial x_1} - \frac{\partial a^7_1}{\partial x_2} = a^7_3 = x_1 x_2,
\]
\[
\frac{\partial a^8_2}{\partial x_1} - \frac{\partial a^8_1}{\partial x_2} = a^8_3 = \frac{x_2^2}{2}.
\]

We solve these equations in the following symmetric way:
\[
a^6_1 = 0, \quad a^6_2 = \frac{x_1^3}{6},
\]
\[
a^7_1 = -\frac{x_1 x_2^2}{4}, \quad a^7_2 = \frac{x_1^2 x_2}{4},
\]
\[
a^8_1 = -\frac{x_2^3}{6}, \quad a^8_2 = 0.
\]

Then we substitute these coefficients to (28), (29) and check item (1) of this theorem by direct computation.

Now we prove item (2). We proceed exactly as for item (1): we start from an infinitesimal symmetry
\[
X_0 = x_2 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2} + x_5 \frac{\partial}{\partial x_4} - x_4 \frac{\partial}{\partial x_5} \in \text{Vec}(\mathbb{R}^5)
\]
of the sub-Riemannian structure on $\mathbb{R}^5$ determined by the orthonormal frame (23), (24) and “continue” symmetry (34) to the sub-Riemannian structure on $\mathbb{R}^8$ determined by the orthonormal frame (7), (8).

So we seek for a vector field $X_0 \in \text{Vec}(\mathbb{R}^8)$ of the form (15) for the functions $P, Q, R \in C^\infty(\mathbb{R}^8)$ to be determined so that the multiplication table (19)–(21) hold.

The first two equalities in (19) yield
\[
X_1 P = -\frac{x_1^3}{6}, \quad X_2 P = \frac{x_1^2 x_2}{2}.
\]

Further,
\[
X_3 P = [X_1, X_2] P = X_1 X_2 P - X_2 X_1 P = X_1 \frac{x_1^2 x_2}{2} + X_2 \frac{x_2^3}{6} = x_1 x_2.
\]

Similarly it follows that
\[
X_4 P = x_2, \quad X_5 P = x_1, \quad X_6 P = 0, \quad X_7 P = 1, \quad X_8 P = 0.
\]
Since $X_6 P = X_8 P = 0$, then $P = P(x_1, x_2, x_3, x_4, x_5, x_7)$. Moreover, since $X_7 P = 1$, then $P = x_7 + a(x_1, x_2, x_3, x_4, x_5)$. The equality $X_8 P = x_1$ implies that $\frac{\partial a}{\partial x_3} = 0$, i.e., $a = a(x_1, x_2, x_3, x_4)$. Similarly, since $X_4 P = x_2$, then $a = a(x_1, x_2, x_3)$. It follows from the equality $X_3 P = x_1 x_2$ that $\frac{\partial a}{\partial x_3} = x_1 x_2$, i.e., $a = x_1 x_2 x_3$. Moreover, the equality $X_2 P = \frac{x_2 x_3}{2}$ implies that $\frac{\partial b}{\partial x_2} = -x_1 x_3 - \frac{x_1^2 x_2}{4}$, i.e., $b = -x_1 x_2 x_3 - \frac{x_1^2 x_2}{8} + c(x_1)$. Finally, the equality $X_1 P = -\frac{x_1^3}{2}$ implies that $\frac{dc}{dx_1} = -\frac{x_1^3}{6} + \frac{x_1 x_2^2}{2}$ i.e., $c = -\frac{x_1^3}{24} + \frac{x_1^2 x_2^2}{4}$. Thus equality (16) follows. Similarly we get equalities (17), (18).

Then multiplication table (19)–(21) for the vector field (15)–(18) is verified by a direct computation.

4 Conclusion and future work

We plan to perform a further study of the nilpotent sub-Riemannian structure with the growth vector $(2, 3, 5, 8)$ using its model obtained in Th. 3.2:

• describe the multiplication rule on the corresponding Carnot group $\mathbb{R}^8$,
• characterize Casimir functions and orbits of the co-adjoint action,
• describe abnormal extremal trajectories and prove strict abnormality of some of them,
• study symmetries, integrals and integrability of the Hamiltonian system for normal extremals,
• describe normal extremal trajectories.

These results will be published elsewhere.

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