Heterogeneous pair approximation for voter models on networks

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Abstract – For models whose evolution takes place on a network it is often necessary to augment the mean-field approach by considering explicitly the degree dependence of average quantities (heterogeneous mean field). Here we introduce the degree dependence in the pair approximation (heterogeneous pair approximation) for analyzing voter models on uncorrelated networks. This approach gives an essentially exact description of the dynamics, correcting some inaccurate results of previous approaches. The heterogeneous pair approximation introduced here can be applied in full generality to many other processes on complex networks.

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Introduction. – The voter model is one of the simplest possible models of nonequilibrium dynamics, as witnessed by the number of different contexts where it has been considered, ranging from probability to theoretical ecology and heterogeneous catalysis [1–3].

Each node of a graph is endowed with a binary variable (a spin) $s = \pm 1$. At each time step a node and one of its neighbors are chosen at random and the first node becomes equal to the second. This microscopic dynamics gives rise to a nontrivial macroscopic phenomena [3] with the formation of correlated domains that tend to grow in time. In finite systems this eventually leads to one of the two possible ordered states (consensus) with either all spins up (+1) or down (−1). In the thermodynamic limit full order is reached only in Euclidean dimension $d \leq 2$.

From the point of view of statistical physics voter dynamics has the very interesting feature that its overly simple rules allow for analytical treatment in many cases. On Euclidean lattices many exact results are available [4,5]. In the past few years, in the framework of the huge interest around complex networks and dynamics on top of them [6–8], the effect of nontrivial topologies on voter dynamics has been explored [9–18]. One interesting effect of the presence of a disordered connectivity pattern is that slightly different definitions of the model, perfectly equivalent on lattices, become inequivalent on networks.

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In the direct voter case, one picks up a vertex at random and this imitates a randomly chosen neighbor. In the reverse voter dynamics [13] or invasion process the order of selection is opposite: it is the neighbor that imitates the node selected first. An intermediate case is the link-update dynamics [12], where one has to choose at random first a link and then which of the two nodes at the end of the link imitates the other. Only the link-update rule preserves a crucial property of voter dynamics on lattices, conservation on average of magnetization [12], whereas other quantities are conserved in the two other cases [11,13]. On regular geometries the three types of dynamics coincide, but when the topology is strongly heterogeneous important differences arise: on scale-free networks [19] the scaling of average consensus time $T$ with the system size $N$ changes depending on which of the three versions is considered.

Theoretical progress on the understanding of these issues has been made possible by the application of a heterogeneous mean-field approach [11,13,16]. The basic idea is to define the densities $\sigma_k$ of up spins restricted to nodes of degree $k$ and to write down evolution equations for them. In this way it is possible to predict that for the direct voter the average time $T$ to reach consensus grows sublinearly as a function of the system size $N$ if the exponent $\gamma$ of the tail of the degree distribution is smaller than 3, while it is linear for $\gamma > 3$. Instead, for the reverse and link-update versions of the dynamics the
growth is linear for any $\gamma > 2$. Very recently, the theoretical picture for the voter model has been further refined by the application of an improved mean-field approach, a homogeneous pair approximation [20], taking into account dynamical correlations among nearest neighbors. This approach allows to follow in detail the temporal evolution of the system and to compute with remarkable accuracy even the prefactor of the law that expresses the average consensus time as a power of $N$.

Despite these achievements, some open problems remain, calling for further improvement. The first has to do with the prediction of a transition for average degree $\mu$ equal to 2. According to ref. [20], while for $\mu > 2$ consensus is attained via a quasi-stationary intermediate state over a long temporal scale, when $\mu < 2$ order is reached exponentially fast. Contrary to this prediction, numerical simulations on an Erdős-Renyi graph do not display any singular behavior in correspondence of $\mu = 2$ [20]. Another issue that requires a clarification has to do with the use of the same approach for reverse voter dynamics. As it will be explained below, the application is straightforward, but the agreement with numerical simulations is not complete.

In this paper we go beyond the homogeneous pair approximation introduced in ref. [20] by allowing correlations between nearest-neighbor nodes to depend on their degree (heterogeneous pair approximation). In practice we lift the assumption of ref. [20] that the probability for a link between two nodes to be active (i.e. to connect nodes in different spin state) does not depend on the degree of the nodes. This is analogous to the introduction of the heterogeneous mean-field approach [8,21] necessary for dealing with the SIS model (and many others) on scale-free networks. With the heterogeneous pair approximation we find, for the direct voter dynamics, results in general very close to those obtained with the homogeneous pair approximation but we get rid of the spurious transition at $\mu = 2$, in agreement with computer simulations. Also for the reverse voter and link-update dynamics we obtain results in very good agreement with numerics, showing that the heterogeneous pair approximation fully captures the behavior of voter models. These successful results call for the application of the heterogeneous pair approach to other nontrivial dynamical processes on networks, including also epidemiological models or evolutionary games [22].

While the detailed implementation will differ depending on the case, the idea is completely general and may lead to substantial progress with respect to the heterogeneous mean-field approach.

**Problems with the homogeneous pair approximation.** – The homogeneous pair approximation introduced in ref. [20] is based on the derivation of the equation of motion for the quantity $\rho$, the fraction of active links, i.e. edges connecting nodes in opposite spin state. In the derivation, it is assumed that the probability for a link in the system to be active does not depend on the degree of the nodes it connects, and is equal to $\rho$. Under this hypothesis the equation of motion for $\rho$ is derived, involving the magnetization $m$ and the first moment of the degree distribution $\mu$. As mentioned before, the behavior predicted changes dramatically as a function of $\mu$. For $\mu < 2$, $\rho$ goes to zero exponentially fast in time, indicating that a fully ordered configuration (consensus) is quickly reached: the average time $T$ to reach consensus does not depend on the system size $N$ so that $T/N \to 0$. For $\mu > 2$ instead it is predicted that, over a very short temporal scale, a quasi-stationary state is reached, characterized by a density of active links

$$\rho^S = \frac{\mu - 2}{2(\mu - 1)} (1 - m^2). \quad (1)$$

The superscript $S$ indicates, throughout this letter, quantities in the quasi-stationary state. The eventual behavior of the system consists in the erratic variation of $m$ and $\rho^S$ that jointly fluctuate in time while obeying eq. (1), until a fluctuation leads to the absorbing state $m = 1$, $\rho = 0$, corresponding to full consensus. Other dynamical observables, describing the ordering process, are expressed using eq. (1), as for example the average consensus time that, for uncorrelated initial conditions with magnetization $m = 0$, is

$$T = \frac{\mu^2 N}{2\rho^S (m = 0) \mu_2} \ln(2) = \frac{(\mu - 1) \mu^2 N}{(\mu - 2) \mu_2} \ln(2), \quad (2)$$

where $\mu_2$ is the second moment of the degree distribution. While eq. (1) agrees with remarkable precision with the outcome of numerical simulations for large values of $\mu$, significant discrepancies arise on many types of networks when $\mu$ gets close to 2. For example, considering as a substrate an Erdős-Renyi graph, the divergence of $T/N$ for $\mu \to 2$ predicted by eq. (2) is not seen in numerical simulations [20] (fig. 1). This indicates that, despite the very good performance for large values of $\mu$, the homogeneous pair approximation of ref. [20] qualitatively fails in the limit $\mu \to 2$.

The same indication emerges when the approach is applied to the reverse voter dynamics. For the quasi-stationary value of the density of active links it is straightforward to obtain

$$\rho^S = \frac{\mu_2 - 2\mu}{2(\mu_2 - \mu)} (1 - m^2). \quad (3)$$

This formula coincides, as expected, with the result for the direct voter model (eq. (1)) when the degree distribution is a delta-function so that $\mu_2 = \mu^2$. However, a careful comparison with numerics shows that eq. (3) is not correct when $\mu_2 \gg \mu^2$ (see fig. 2). The parabolic dependence on magnetization remains (not shown), but the value for $m = 0$ predicted by eq. (3) (diamond symbols) is considerably larger than what simulations give (circles), unless $\gamma$ is very large. Here, as in the rest of the paper unless specified otherwise, we compare the results of the theoretical treatment with simulations.
Fig. 1: (Colour on-line) Average time to reach full consensus for direct voter dynamics on an Erdős-Rényi graph, as a function of the average degree \( \mu \), starting from a symmetric uncorrelated initial configuration \( \sigma(t=0) = 1/2 \). The curve for the heterogeneous mean-field approach is obtained using the formula \( T/N = \mu^2/\mu_2 \ln(2) \) from ref. [11]. The curve for the homogeneous pair approximation is obtained using eq. (2). The curve for the heterogeneous pair approximation is obtained using eq. (10). For finite \( N \) and \( \mu \) very close to 1, the network is separated into many nonextensive disconnected components. This is the origin of the discrepancy between numerical results and the theoretical prediction in an interval that shrinks to zero as the system size diverges. Each point is the average over 1000 realizations.

![Graph](image1.png)

Fig. 2: (Colour on-line) Reverse voter dynamics: quasi-stationary value \( \rho_{k}^{S}(m=0) \) as a function of the system size for three different values of \( \gamma \) (top to bottom, \( \gamma = 2.5, \gamma = 4, \gamma = 8 \)).

Fig. 3: (Colour on-line) Plot of the probability that a link connected to a node of degree \( k \) is active in the quasi-stationary state (and for \( m = 0 \)) for reverse voter dynamics taking place on an uncorrelated network with \( \gamma = 2.5 \) and size \( N = 10^6 \). The black symbols are the results of a numerical simulation. The dashed red line is the prediction of the homogeneous pair approximation, eq. (3), the blue solid line is the prediction of the heterogeneous pair approximation.

Fig. 4: (Colour on-line) Plot of the probability that a link connected to a node of degree \( k \) is active in the quasi-stationary state (and for \( m = 0 \)) for direct voter dynamics taking place on an uncorrelated network with \( \gamma = 2.5 \) and size \( N = 10^6 \). The black symbols are the results of a numerical simulation. The dashed red line is the prediction of the homogeneous pair approximation, eq. (1), the blue solid line is the prediction of the heterogeneous pair approximation.

The origin of the discrepancies in figs. 1 and 2 is not hard to identify. It is sufficient to evaluate numerically the quantity \( \rho_k^S \), defined as the fraction of neighbors of a node of degree \( k \) that are in a different spin state in the quasi-stationary state, and plot it as a function of \( k \) (fig. 3). While the approach of ref. [20] assumes this quantity to be the same for all nodes, a substantial dependence on \( k \) shows up. Interestingly, a numerical check indicates that also for direct voter dynamics \( \rho_k^S \) has a sizeable dependence on the degree (fig. 4). Based on this evidence, we now
generalize the pair approximation approach by considering \( \rho_{k,k'} \), i.e., allowing the density of active links to depend explicitly on the degree on the connected nodes. From \( \rho_{k,k'} \) it is possible to determine all other quantities of interest.

The density \( \rho_k \) of active links connected to a node of degree \( k \) is \( \rho_k = \sum_{k'} Q_{k,k'} \rho_{k,k'} \), where \( Q_{k,k'} = P_k/k/\mu \) is the degree distribution of the neighbors of a randomly chosen node. The total density of active links \( \rho \) is likewise given by \( \rho = \sum_k Q_k \rho_k = \sum_{k,k'} Q_{k,k'} \rho_{k,k'} \).

**Heterogeneous pair approximation for the direct voter model.** – The first goal is to determine the equation of motion for \( \rho_{k,k'} \). This quantity is modified if the flipping node has degree \( k \) and one of its neighbors has degree \( k' \) (or vice versa). Let us assume that the flipping (first selected) node has degree \( k \) and call \( k'' \) the degree of the copied (second selected) node. It is useful to consider separately the two cases where \( k'' \neq k' \) or \( k'' = k' \).

In the first case the variation \( \Delta \rho_{k,k'} \) for a single dynamical step (occurring over a time \( \Delta t = 1/N \)) is determined as follows. The probability that a node in state \( s \) and degree \( k \) flips is given by the probability \( P_k \) that the first node selected has degree \( k \) times the probability \( \sigma(s) \) that it is in state \( s \), times the probability \( Q_{k,k'} \) that the second has degree \( k' \) multiplied by the probability \( \rho_{k,k'}/[2\sigma(s)] \) that the link connecting the two is active.

One has then to multiply this quantity by the associated variation of the fraction of active links between \( k \) and \( k'' \) (Fig. 5). Among the \( k - 1 \) other links of the flipping node, the number of those connecting to a node of degree \( k' \) will be \( j \) distributed according to a binomial \( R(j,k-1) \) with probability of the single event equal to \( Q_{k,k'} \). In their turn, only \( n \) out of these \( j \) links will be active, with \( n \) binomially distributed with single event probability \( \rho_{k,k'}/[2\sigma(s)] \). Finally one has to multiply by the variation of \( \rho_{k,k'} \) when \( n \) out of \( j \) links go from active to inactive as a consequence of the flipping of the node in \( k \). This is given by the variation of the number of active links \( (j - n) - n \) divided the total number of links between nodes of degree \( k \) and \( k' \) (\( N\mu Q_{k,k'} \)). One has then to sum over \( k'' \neq k' \), \( s, j \) and \( n \), obtaining

\[
\Delta \rho_{k,k'} = P_k \sum_s \sigma(s) \sum_{k'' \neq k'} Q_{k,k'} \frac{\rho_{k,k'}}{2\sigma(s)}.
\]

By performing explicitly the summations (and using \( \sum_s 1/\sigma(s) = 4/(1-m^2) \)) the formula becomes

\[
\frac{\Delta \rho_{k,k'}}{\Delta t} = \sum_{k'' \neq k'} Q_{k,k'} \rho_{k,k'} \frac{(k-1)}{k} \left(1 - \frac{2}{1+\mu^2} \rho_{k,k'}\right).
\]

When \( k'' = k' \) the value of \( \Delta \rho_{k,k'} \) is similar to eq. (4) with (obviously) \( Q_{k,k'} \) instead of \( Q_{k,k''} \), no sum over \( k'' \), and in the numerator of the last factor \( j + 1 \) \( (n + 1) = j - 2n - 1 \), because there are \( j + 1 \) links to nodes of degree \( k \), \( n + 1 \) of which are active in the initial state and inactive in the final. Summing up the two contributions and adding the symmetric terms with \( k \) and \( k' \) swapped, we get

\[
\frac{d\rho_{k,k'}}{dt} = \rho_k \frac{k-1}{k} + \rho_{k'} \frac{k'-1}{k'} + \rho_{k''} \left(1 + \frac{1}{k} + \frac{2\rho_k}{1-m^2} \frac{k-1}{k} + \frac{2\rho_{k'}}{1-m^2} \frac{k'-1}{k'}\right).
\]

From this equation, by summing over \( k \) and \( k' \) one obtains the equation of motion for the total density \( \rho \)

\[
\frac{d\rho}{dt} = \sum_{k,k'} Q_{k,k'} \frac{d\rho_{k,k'}}{dt} = 2 \left(\rho - 2 \frac{\pi^2}{\mu} \right) - \frac{4}{1-m^2} \frac{\rho^2_2 - \pi^2}{\mu},
\]

where \( \pi = \sum_k P_{k} \rho_k \), \( \pi^2 = \sum_k P_{k} \rho_k^2 \) and \( \rho^2_2 = \sum_k Q_{k,k} \rho_k^2 \).

It is easy to see that, if one assumes \( \rho_k = \rho \), i.e., that the density does not depend on the degree (and as a consequence \( \pi = \rho \), \( \rho^2_2 = \pi^2 = \rho^2 \)), eq. (7) coincides with the analogous equation of ref. [20].

After a time scale of order unity the quasi-stationary state is established. From \( d\rho_{k,k'}/dt = 0 \) we obtain

\[
\rho_{k,k'}^S = \frac{\rho_k^{S-k-1} + \rho_{k'}^{S-k'-1}}{2} + \frac{1}{2}. \]

From this equation it is evident that all \( \rho_{k,k'}^S \) are proportional to \( 1-m^2 \). Imposing the consistency condition \( \rho_k = \sum_{k'} Q_{k,k'} \rho_{k,k'} \), we have a set of coupled equations for \( \rho_{k,k'}^S \) for all \( k \). The numerical iterative solution of this set of equations allows the determination of the quasi-stationary values of all \( \rho_{k,k'}^S \) and hence of all \( \rho_{k,k'}^S \).

Figure 4 shows that the heterogeneous pair approximation captures extremely well the detailed dependence of \( \rho_{k,k'}^S \).
on $k$. Also the comparison of the quasi-stationary value of $\rho_S^k$ (recovered by summing eq. (8) over $k$ and $k'$) with simulations for several values of $\gamma$ and $N$ indicates an excellent agreement (not shown).

The knowledge of the $\rho_k^S$ allows to determine also the consensus time $T$ to reach a fully ordered configuration. Following the formalism of ref. [16], using the backward Kolmogorov equation, one simply needs to express the transition probabilities $R_k$ and $L_k$ in terms of the $\rho_k$. In the quasi-stationary state the probability $R_k^S$ that the number of nodes of degree $k$ in the state $s = +1$ increases by 1 is

$$R_k^S = P_k(1 - \sigma_k) \sum_{k'} Q_{k'k} \rho_{k'k'}^S / [2(1 - \sigma_k)] = P_k \rho_k^S / 2.$$  \hspace{1cm} (9)

The probability $L_k$ that the number is reduced is the same, $L_k^S = R_k^S$. Using these expressions one obtains

$$T = -\tau(N) [(1 - \omega) \ln(1 - \omega) - \omega \ln \omega],$$ \hspace{1cm} (10)

where $\omega = \sum_k Q_k \sigma_k(t = 0)$ is conserved by the dynamics and

$$\tau(N) = N \mu^2 / 2 \sum_k P_k \kappa^2 \rho_k^S (m = 0)$$ \hspace{1cm} (11)

sets the temporal scale over which consensus is reached. Notice that with the mean-field assumption, $\rho^S_k (m = 0) = 1/2$, eq. (11) coincides with the result of Sood and Redner $\tau(N) = N \mu^2 / \mu_2$. With the homogeneous pair approximation $\rho^S_k (m = 0) = (\mu - 2) / [2(\mu - 1)]$ it returns the result of Vazquez and Eguíluz [20]. With eq. (10), valid for any degree distribution, we have computed the value of the ratio $T/N$ for an Erdős-Renyi graph of average degree $\mu$.

The plot in fig. 1 presents the comparison of this value with the prediction of previous approaches and with the outcome of numerical simulations. The singular behavior for $\mu = 2$ is an artifact of the homogeneous pair approximation, that is removed by the heterogeneous pair approximation, leading instead to a smooth behavior around $\mu = 2$ and a very good agreement between theory and simulations.

**Heterogeneous pair approximation for the reverse and link-update voter dynamics.** The application of the heterogeneous pair approximation to the reverse voter model proceeds along the same lines of the direct dynamics. The only difference is that the factor $P_k Q_{k'}$ is replaced by $P_{k'} Q_k$, because the role of the flipping and the copied nodes is swapped. As a consequence, the formula for the quasi-stationary value of $\rho_{k,k'}$ is slightly modified

$$\rho_{k,k'}^S = \frac{2}{1 - m^2} [\pi_k^S (k - 1) + \pi_{k'}^S (k' - 1)] / \mu / k + \mu / k',$$ \hspace{1cm} (12)

where $\pi_k = \sum_{k'} P_{k'} \rho_{k,k'}$. Much in the same way as for the direct voter dynamics a set of consistency equations for $\pi_k^S$ is solved iteratively. The values of $\pi_k^S$, inserted into eq. (12) provide $\rho_{k,k'}^S$, from which in turn $\rho_k^S$ and $\rho^S$ can be computed. Figure 3 shows that the agreement with the values of the $R_k^S$ obtained numerically is very good. Moreover, fig. 2 shows that the quasi-stationary value of $\rho^S$ is in fully satisfactory agreement with numerics for several values of $\gamma$ and $N$.

Also in this case the consensus time is computed using the backward Kolmogorov equation for $T$ [16]. The transition probabilities in the quasi-stationary state are obtained from those of the direct case with the replacement $P_k Q_{k'} \rightarrow P_{k'} Q_k$:

$$R_k^S = Q_k \pi_k^S / 2,$$ \hspace{1cm} (13)

$$L_k^S = R_k^S.$$  \hspace{1cm} (14)

Using these quantities and the fact [13,16] that the dynamics conserves the quantity $\omega_{-1} = \sum_k P_k \sigma_k k^{-1} / \mu_{-1}$, where $\mu_{-1} = \sum_k P_k / k$, one obtains

$$T = -\tau(N) [\omega_{-1} \ln(\omega_{-1}) + (1 - \omega_{-1}) \ln(1 - \omega_{-1})],$$ \hspace{1cm} (15)

and the temporal scale is

$$\tau(N) = \frac{N \mu^2}{2 \sum_k P_k \kappa^2 \pi_k^S (m = 0)}.$$ \hspace{1cm} (16)

For uncorrelated initial conditions $\sigma_k(t = 0) = 1/2$, one obtains

$$T = \tau(N) \ln(2).$$ \hspace{1cm} (17)

The comparison of the predictions of eq. (16) with numerical results is displayed in fig. 6 and again confirms the correctness of the theoretical treatment.
For link-update dynamics, all calculations proceed exactly as in the direct voter case, now with the replacement of $P_{k,k}$ with $Q_{k,k'}$. It is straightforward to obtain

$$\rho^S_{k,k'} = \frac{\rho^S_k(k-1) + \rho^S_{k'}(k'-1)}{1 - \frac{1}{2m} \rho^S_k(k-1) + \rho^S_{k'}(k'-1)} + 2. \quad (17)$$

The consistency condition $\rho = \sum_{k'} Q_{k,k'} \rho_{k,k'}$ is solved iteratively. Once again, the values of $\rho^S_k$ and $\rho^S_{k,k'}$ obtained in this way are in very good agreement with numerics (not shown). The transition probabilities for the Kolmogorov backward equation are $R^S_k = L^S_k = Q_k \rho^S_k / 2$, that yield, for the consensus time

$$T = \tau(N) \left[ \omega_0 \ln(\omega_0) + (1 - \omega_0) \ln(1 - \omega_0) \right], \quad (18)$$

where the conserved quantity $\omega_0 = \sigma(s+1)$, the total density of nodes in state $s = +1$ and $\tau(N) = N/[2\rho^S(m = 0)]$. Notice that using $\rho^S(m = 0) = 1/2$ one recovers the mean-field result $\tau(N) = N$, i.e. the independence of consensus time of network features [13]. However, since $\rho^S(m = 0)$ actually depends on the network, this dependence is just an artifact of the mean-field approach.

Conclusions. – In this letter, we have introduced a heterogeneous pair approximation to deal with the ordering dynamics of voter models on networks. This approach allows to determine, with great precision, not only the scaling but also the prefactors of the relevant dynamical observables (quasi-stationary density of active links, consensus time). The heterogeneous pair approximation thus constitutes a very good approximate theory for voter models on uncorrelated networks. The only point that remains to be understood is the origin of the very small discrepancy (at most of the order of 1%) that still exists between numerical and analytical results.

While the treatment presented here is valid for uncorrelated networks, generalization to correlated nets is straightforward. In particular, if $P(k'|k)$ is the probability that a neighbor of a randomly chosen node of degree $k$ has degree $k'$, the stationarity conditions for the three types of dynamics, eqs. (8), (12) and (17), remain the same, provided $P(k'|k)$ replaces its expression $Q_k$ valid in the uncorrelated case, in the definitions of $\rho_k$ and $\pi_k$: $\rho_k = \sum_{k'} P(k'|k) \rho_{k,k'}$, $\pi_k = \sum_{k'} \mu P(k'|k) / k' \rho_{k,k'}$. While the generalization of the analytical approach is immediate, an interesting question deserving further investigations is whether the agreement between theory and numerics remains good even in the correlated case.

The idea behind the heterogeneous pair approximation introduced here is fully general and its application to other dynamical models evolving on complex networks is indeed a promising avenue for future work. In particular, its role could be crucial for understanding cases where heterogeneous mean-field approaches do not work [24].

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