Multi-centered black holes with a negative cosmological constant

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Abstract: We present a recipe that allows to construct multi-centered black holes embedded in an arbitrary FLRW universe. These solutions are completely determined by a function satisfying the conformal Laplace equation on the spatial slices $E^3$, $S^3$ or $H^3$. Since anti-de Sitter space can be written in FLRW coordinates, this includes as a special case multi-centered black holes in AdS, in the sense that, far away from the black holes, the energy density and the pressure approach the values given by a negative cosmological constant. We study in some detail the physical properties of the single-centered asymptotically AdS case, which does not coincide with the usual Reissner-Nordström-AdS black hole, but is highly dynamical. In particular, we determine the curvature singularities and trapping horizons of this solution, compute the surface gravity of the trapping horizons, and show that the generalized first law of black hole dynamics proposed by Hayward holds in this case. It turns out that the spurious big bang/big crunch singularities that appear when one writes AdS in FLRW form, become real in presence of these dynamical black holes. This implies that actually only one point of the usual conformal boundary of AdS survives in the solutions that we construct. Finally, a generalization to arbitrary dimension is also presented.

Keywords: Black Holes, Classical Theories of Gravity, Gauge-Gravity Correspondence.
1 Introduction

Composite objects formed by elementary constituents with mass to charge ratio equal to one have been studied for a long time in general relativity, and more recently in supergravity and string theory. While in the Newtonian theory of gravity it is clear that static equilibrium for a system of point charges can be achieved by fine-tuning the charge suitably with the particle mass, and thus balancing the mutual gravitational and electrostatic forces, the existence of such static composite configurations is far from obvious in general relativity, whose equations of motion are highly nonlinear, and therefore there is a priori no reason for a superposition principle to hold. Yet, indications that such a general relativistic analogue exists first emerged when Weyl [1] obtained a particular class of static electromagnetic vacuum fields, later generalized independently by Majumdar [2] and Papapetrou [3], who removed Weyl’s original restriction of axial symmetry. In vacuum, the stationary generalization of the Majumdar-Papapetrou (MP) solution was constructed by Israel, Wilson and Perjés [5, 6]. In the same year, Hartle and Hawking [7] showed that the vacuum MP spacetime can describe a system of multi-centered extremal Reissner-Nordström black holes.

These multi-black hole geometries admit supercovariantly constant spinors [8], a result that actually extends to all the solutions belonging to the MP class [9]. The MP solution can thus be seen as an early example of a BPS configuration that satisfies rather simple first-order equations, and this explains also why one can build arbitrary

\footnote{For higher-dimensional generalizations see [4].}
superpositions of the elementary constituents, in spite of the nonlinear nature of the Einstein-Maxwell equations. Nevertheless, supersymmetry does not seem to be necessary for the existence of these bound states, since by now we know many examples of multi-centered black holes that are not BPS, cf. e.g. [10].

The study of composite systems like ‘black hole molecules’ has played a crucial role in several recent developments of supergravity and string theory, especially in attempts to understand the quantum structure of black holes. Moreover, they are of interest in the field of holography, in particular for applications of the gauge/gravity correspondence to condensed matter phenomena (cf. e.g. [11] for a review). In this context, it was established recently in [12] that stable and metastable stationary bound states in four-dimensional anti-de Sitter space exist, and it was argued that their holographic duals represent structural glasses. The glassy feature of these black hole bound states is related to their rugged free energy landscape, which in turn is a consequence of the fact that the constituents can have a wide range of different possible charges [12].

Multi-centered black holes can also be generalized to dynamical situations. Kastor and Traschen (KT) [13] showed that the MP solution can be embedded in a de Sitter universe, where the no-force condition implies that the whole system is just comoving with the cosmological expansion. The KT solution was then used in [14] to study analytically black hole collisions. The embedding of composite black holes in higher-dimensional de Sitter spaces or in more general classes of FLRW universes was subject of [15–20].

Here we go one step further, and show how to construct multi-center solutions in any FLRW spacetime and for arbitrary dimension. These geometries are sourced by a U(1) gauge field and by a perfect fluid. Since anti-de Sitter space can be written in an FLRW form (with hyperbolic spatial slices and trigonometric scale factor), our recipe allows, as a particular subcase, to obtain multi-center solutions in AdS. Like the underlying FLRW universe, these are highly dynamical, and thus different in spirit from the bound states of [12]. Generically, the \((D+1)\)-dimensional black holes that we construct are determined by a function satisfying the conformal Laplace equation on the spatial slices \(E^D\), \(S^D\) or \(H^D\) of the FLRW background universe. This generalizes the well-known fact that asymptotically flat extremal black holes are characterized by harmonic functions. Unfortunately, the spurious big bang/big crunch singularities that appear when one writes AdS in FLRW coordinates, become real once such a dynamical black hole is present. We show that this implies that actually only one point of the conformal boundary of AdS survives. This makes it questionable if our solutions admit

\(^2\)Multi-black hole systems in Euclidean AdS were obtained in [21]. However, these have no Lorentzian analogue.
an AdS/CFT interpretation in the usual sense.

The remainder of this paper is organized as follows: In the next section, starting from the charged generalization of the McVittie spacetime [22] found a long time ago by Shah and Vaidya [23], we show how to construct multi-center solutions in an arbitrary FLRW universe. In section 3, we discuss some physical properties of the single-centered nonextremal solution in AdS. In particular, we determine the curvature singularities and trapping horizons, compute the surface gravity of the latter, and show that the generalized first law of black hole dynamics proposed by Hayward [24] holds. In section 4, the higher-dimensional case is considered, and in 5 we present our conclusions.

2 Multi-centered maximally charged McVittie solutions

In [23], Shah and Vaidya presented a charged generalization of the McVittie solution [22], with metric and U(1) field strength given by

\[
ds^2 = \left[1 - (M^2 - Q^2)^{\frac{1 + kr^2}{4a^2r^2}}\right]^2 \frac{dt^2}{\left[1 + M \sqrt{1 + \frac{kr^2}{a^2}} + (M^2 - Q^2)^{\frac{1 + kr^2}{4a^2r^2}}\right]^2} - 4a^2 \left[1 + M \sqrt{1 + \frac{kr^2}{a^2}} + (M^2 - Q^2)^{\frac{1 + kr^2}{4a^2r^2}}\right]^2 \frac{dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2}{(1 + kr^2)^2},
\]

\[
F = \frac{Q}{ar^2} \frac{1}{\sqrt{1 + kr^2}} \left[1 - (M^2 - Q^2)^{\frac{1 + kr^2}{4a^2r^2}}\right] \left[1 + M \sqrt{1 + \frac{kr^2}{a^2}} + (M^2 - Q^2)^{\frac{1 + kr^2}{4a^2r^2}}\right]^2 dr \wedge dt.
\]

(2.1) satisfies the Einstein-Maxwell equations

\[
G_{\mu\nu} = 8\pi T_{\mu\nu}, \quad \nabla_\nu F^{\mu\nu} = 4\pi J^\mu,
\]

(2.2)

where the pressure, energy density, charge density and four-velocity of the charged perfect fluid source read respectively

\[
8\pi p = -2 \left(\frac{\dot{a}}{a} - \frac{\dot{a}^2}{a^2}\right) \frac{\left[1 + M \sqrt{1 + \frac{kr^2}{a^2}} + (M^2 - Q^2)^{\frac{1 + kr^2}{4a^2r^2}}\right]}{\left[1 - (M^2 - Q^2)^{\frac{1 + kr^2}{4a^2r^2}}\right]} - 3 \frac{\dot{a}^2}{a^2}
\]

\[-k \left\{ a^2 \left[1 + M \sqrt{1 + \frac{kr^2}{a^2}} + (M^2 - Q^2)^{\frac{1 + kr^2}{4a^2r^2}}\right]^2 \left[1 - (M^2 - Q^2)^{\frac{1 + kr^2}{4a^2r^2}}\right]\right\}^{-1},
\]
\[ 8\pi \rho = 3 \frac{a^2}{a^2} + \frac{3k}{2a^2} \left[ 1 + M \frac{\sqrt{1 + kr^2}}{ar} + (M^2 - Q^2) \frac{1 + kr^2}{4a^2 r^2} \right]^{-3} \left[ 2 + M \frac{\sqrt{1 + kr^2}}{ar} \right], \]
\[ 4\pi \sigma = - \frac{3kQ}{4a^3} \frac{\sqrt{1 + kr^2}}{r} \left[ 1 + M \frac{\sqrt{1 + kr^2}}{ar} + (M^2 - Q^2) \frac{1 + kr^2}{4a^2 r^2} \right]^{-3}, \] (2.3)
\[ u = \frac{1 - (M^2 - Q^2) \frac{1 + kr^2}{4a^2 r^2}}{1 + M \frac{\sqrt{1 + kr^2}}{ar} + (M^2 - Q^2) \frac{1 + kr^2}{4a^2 r^2}} dt. \] (2.4)

Moreover, \( k = 0, \pm 1 \) determines the geometry of the spatial slices. From (2.3) it is clear that the cosmic fluid is required to be charged if the spatial geometry of the underlying FLRW universe is curved.

In the maximally charged case \( M = |Q| \) (obtained in [25]), after the coordinate change \( r = \frac{1}{\sqrt{k}} \tan \frac{\sqrt{k} \psi}{2} \), (2.1) boils down to
\[ ds^2 = \frac{1}{\left[ 1 + M \frac{\sqrt{k}}{a \sin(\sqrt{k} \psi/2)} \right]^2} dt^2 \]
\[ - a^2 \left[ 1 + M \frac{\sqrt{k}}{a \sin(\sqrt{k} \psi/2)} \right]^2 \left[ d\psi^2 + \frac{\sin^2(\sqrt{k} \psi)}{k} (d\theta^2 + \sin^2 \theta d\phi^2) \right], \] (2.5)
\[ F = \frac{Mk}{2a} \cos(\sqrt{k} \psi/2) \frac{d\psi \wedge dt}{2a \sin^2(\sqrt{k} \psi/2) \left[ 1 + M \frac{\sqrt{k}}{a \sin(\sqrt{k} \psi/2)} \right]^2} = d \left[ \left( 1 + M \frac{\sqrt{k}}{a \sin(\sqrt{k} \psi/2)} \right)^{-1} dt \right], \]

while the pressure, energy- and current density become
\[ 8\pi p = -3 \frac{\dot{a}^2}{a^2} - \frac{k}{a^2} \left[ 1 + M \frac{\sqrt{k}}{a \sin(\sqrt{k} \psi/2)} \right]^2 - 2 \left( \frac{\dot{a}}{a} - \frac{\dot{a}^2}{a^2} \right) \left[ 1 + M \frac{\sqrt{k}}{a \sin(\sqrt{k} \psi/2)} \right], \]
\[ 8\pi \rho = 3 \frac{\dot{a}^2}{a^2} + \frac{3k}{2a^2} \left[ 1 + M \frac{\sqrt{k}}{a \sin(\sqrt{k} \psi/2)} \right]^3 \left[ 2 + M \frac{\sqrt{k}}{a \sin(\sqrt{k} \psi/2)} \right], \]
\[ 4\pi J = -3 \frac{kM}{4a^3} \frac{\sqrt{k}}{\sin(\sqrt{k} \psi/2)} \left[ 1 + M \frac{\sqrt{k}}{a \sin(\sqrt{k} \psi/2)} \right]^4 dt. \] (2.6)

This solution appears to be characterized by the function \( H = \frac{M \sqrt{\psi}}{\sin(\sqrt{k} \psi/2)} \), which happens to satisfy the conformal Laplace equation on \( E^3, S^3 \) or \( H^3 \),
\[ \nabla^2 H = \frac{1}{8} RH, \] (2.7)
where $R = 6k$ is the corresponding scalar curvature. It is straightforward to verify that one can take any function $\mathcal{H}$ solving (2.7), and the resulting fields still satisfy the Einstein-Maxwell equations (2.2). This allows to generalize (2.5) to a multi-centered solution by choosing $\mathcal{H}$ to be a linear combination of terms obtained by acting on $H$ with the isometries of the three-dimensional base space metric. Alternatively, one can use the conformal invariance of (2.7), which implies

$$\tilde{\nabla}^2 \tilde{H} = \frac{1}{8} \tilde{R} \tilde{H}, \tag{2.8}$$

where $\tilde{\nabla}^2$ and $\tilde{R}$ denote the Laplacian and scalar curvature of the conformally related metric $\tilde{g}_{ij} = \Omega^2 g_{ij}$ respectively, and $\tilde{H} = \Omega^{-1/2} H$. Now let $g_{ij}$ be the flat metric, $g_{ij} dx^i dx^j = d\vec{x}^2$, and

$$\tilde{g}_{ij} dx^i dx^j = \frac{4 d\vec{x}^2}{[1 + k\vec{x}^2]^2}. $$

Starting from the usual one-center solution for a flat base, $H = \sqrt{2}M/|\vec{x}|$, one gets

$$\tilde{H} = \frac{M}{|\vec{x}|} \sqrt{1 + k\vec{x}^2} = \frac{M \sqrt{k}}{\sin(\sqrt{k}\psi/2)},$$

which is the function appearing in (2.5). Taking instead

$$\mathcal{H} = \sum_{I=1}^{N} \frac{Q_I}{|\vec{x} - \vec{x}_I|}$$

leads to

$$\tilde{\mathcal{H}} = \frac{1}{\sqrt{2}} \left[1 + k\vec{x}^2\right]^{1/2} \sum_{I=1}^{N} \frac{Q_I}{|\vec{x} - \vec{x}_I|}. \tag{2.9}$$

It would be interesting to understand whether there is a deeper reason for the appearance of this conformal structure.

Notice that the existence of this multi-centered generalization of (2.5) is also suggested by considering a charged probe particle in the geometry (2.5), whose equation of motion is

$$\nabla_{\nu} p^\nu = -q F^\mu_{\nu} v^\nu. \tag{2.10}$$

If the particle is BPS, $m = q$, and we take $v = v^i \partial_i$ for its four-velocity, it is easy to show that the attractive gravitational force encoded in the Christoffel connection exactly cancels the repulsive Lorentz force, such that the particle can stay at rest at fixed $\psi, \theta, \phi$. 


3 Singularities and horizons in the single-centered asymptotically AdS case

In this section, we shall discuss some physical properties of the single-centered (non necessarily maximally charged) solution in AdS, which does not coincide with the well-known Reissner-Nordström-AdS black hole, but is highly dynamical.

Let us choose \( k = -1 \) and \( a(t) = l \sin(t/l) \), with \( l > 0 \) and \( 0 < t/l < \pi \). Then, far from the black hole (\( \psi \to \infty \) or \( r \to 1 \)), the energy density and pressure approach the values given by a negative cosmological constant \( \Lambda = -3/l^2 \), while the charge density (2.3) goes to zero. In this limit, the metric in (2.1) tends to AdS in FLRW coordinates, i.e.,

\[
ds^2 \to dt^2 - l^2 \sin^2 \frac{t}{l} \left( d\psi^2 + \sinh^2 \psi d\Omega^2 \right). \tag{3.1}\]

The FLRW form is related to global coordinates \( \tau, \hat{r} \) by

\[
\hat{r} = l \sin \frac{t}{l} \sinh \psi, \quad \cos \frac{t}{l} = \left( 1 + \frac{\hat{r}^2}{l^2} \right)^{1/2} \cos \tau, \tag{3.2}\]

which casts (3.1) into

\[
ds^2 = \left( 1 + \frac{\hat{r}^2}{l^2} \right) d\tau^2 - \left( 1 + \frac{\hat{r}^2}{l^2} \right)^{-1} d\hat{r}^2 - \hat{r}^2 d\Omega^2. \tag{3.3}\]

(3.1) has a lightlike big bang/big crunch singularity in \( t = 0 \) and \( t = l\pi \) respectively, that are of course artefacts of the coordinate system \( t, \psi \). In fact, by introducing \( \tau, \hat{r} \), one extends the spacetime beyond these singularities. The causal structure of AdS in FLRW coordinates is visualized in the Carter-Penrose diagram fig. 1.

Notice also that, due to \( \cos^2(t/l) \leq 1 \), the last eq. of (3.2) implies \( \tau/l \to \pi/2 \) for \( \hat{r} \to \infty \), so that actually only the point \( \tau = l\pi/2 \) (which is of course a two-sphere) of the conformal boundary of AdS is visible in FLRW coordinates.

Rewriting the metric (2.1) for brevity as

\[
ds^2 = \frac{g^2}{f^2} dt^2 - a^2 f^2 \left( d\psi^2 + \frac{\sin^2(\sqrt{k} \psi)}{k} d\Omega^2 \right),
\]

with

\[
f = 1 + \frac{\sqrt{k} M}{a \sin(\sqrt{k} \psi/2)} + k \frac{M^2 - Q^2}{4 a^2 \sin^2(\sqrt{k} \psi/2)}, \quad g = 1 - k \frac{M^2 - Q^2}{4 a^2 \sin^2(\sqrt{k} \psi/2)}, \tag{3.4}\]

the scalar curvature is

\[
R = -12 \frac{\dot{a}^2}{a^2} - 6 \frac{f}{g} \left( \frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} \right) - \frac{3}{2} k \frac{f (g + 2) + g^2}{a^2 f^3 g}. \tag{3.5}\]
The spacetime with $k = -1$ has thus curvature singularities in $a(t) = 0$, $\sinh(\psi/2) = \pm\sqrt{M^2 - Q^2}/2a$ and $\sinh(\psi/2) = \pm Q - M^2/2a$; however the only singularity that is connected with the asymptotic region $\psi \to +\infty$ is the hypersurface $\sinh(\psi/2) = \sqrt{M^2 - Q^2}/2a$. In the maximally charged case, $M = |Q|$, this singular hypersurface becomes the union of the hypersurfaces $t = 0$, $t = l\pi$ and $\psi = 0$.

To determine if the present spacetime describes a black hole, one can look for trapping horizons [26]. Introducing the Newman-Penrose null tetrads

$$l = \frac{1}{\sqrt{2}} \left( \frac{g}{f} dt - af d\psi \right), \quad n = \frac{1}{\sqrt{2}} \left( \frac{g}{f} dt + af d\psi \right),$$

$$m = \frac{af \sinh \psi}{\sqrt{2}} (d\theta + i \sin \theta d\varphi), \quad (3.6)$$

and the complex conjugate $\bar{m}$, the expansions of the outgoing and ingoing radial null geodesics are respectively

$$\theta_+ \equiv -2m^{(\mu} \bar{m}^{\nu)} \nabla_{\mu} l_{\nu}, \quad \theta_- \equiv -2m^{(\mu} \bar{m}^{\nu)} \nabla_{\mu} n_{\nu}, \quad (3.7)$$

and once evaluated read

$$\theta_\pm = \frac{\sqrt{2}}{a} \left[ \dot{a} \pm \frac{g + \sinh^2(\psi/2)(f + g)}{\sinh \psi f^2} \right]. \quad (3.8)$$

Marginal surfaces are defined as spacelike 2-surfaces on which $\theta_+ = 0$ ($\theta_- = 0$), and trapping horizons are defined as the closure of 3-surfaces foliated by marginal surfaces.
such that $\theta_+ \neq 0$ and $\mathcal{L}_-\theta_+ \neq 0$ ($\theta_+ \neq 0$ and $\mathcal{L}_+\theta_- \neq 0$) on the 3-surface, where $\mathcal{L}_\pm$ is the Lie derivative along the outgoing or ingoing radial null geodesics. From eq. (3.8) it is clear that if $t \neq l\pi/2$ the two expansions can’t both vanish at the same time, while in $t = l\pi/2$ they only vanish behind or on the singularity, since outside of the singularity both $f$ and $g$ are positive, so that no horizon can exist in any case for $t = l\pi/2$. Furthermore $\mathcal{L}_-\theta_+$ and $\mathcal{L}_+\theta_-$ are negative in the whole considered region; as a consequence the only condition necessary to locate the trapping horizons is the vanishing of $\theta_+$ or $\theta_-$. For $M \neq |Q|$ there are always solutions to $\theta_\pm = 0$ that lie on the singularity; this means that the horizons intersect the singularity and there is a time interval around $t = l\pi/2$ for which they are not defined. On the other hand, if $M = |Q|$ the horizons are defined for every $t \neq l\pi/2$, while for $t = l\pi/2$ they tend to coincide on the singularity $\psi = 0$. For $\psi \to +\infty$, $\theta_\pm = 0$ implies $\dot{a} \to \pm 1$ which means that the horizons tend to the axes $t = 0$ and $t = l\pi$.

There are always two trapping horizons: One for $t > l\pi/2$ where $\theta_+ = 0$ and $\theta_- = 2\sqrt{2\dot{a}}_a < 0$, and the other for $t < l\pi/2$ where $\theta_- = 0$ and $\theta_+ = 2\sqrt{2\dot{a}}_a > 0$. Since $\mathcal{L}_-\theta_+$ and $\mathcal{L}_+\theta_-$ are negative these are respectively an outer future trapping horizon, which can be interpreted as the horizon of a black hole, and an outer past trapping horizon, which can be interpreted as the horizon of a white hole.

![Figure 2](image-url)

**Figure 2.** Plots of curvature singularity (red), trapping horizons (blue) and one pair of radial null geodesics (green) crossing in $t = l\pi/2$, in FLRW coordinates $(t,\psi)$ for $M \neq |Q|$ (left) and $M = |Q|$ (right). For $M = |Q|$ the curvature singularities coincide with the axes $\psi = 0$, $t = 0$ and $t = l\pi$.

In figures 2 and 3 we display, respectively in the cosmological (FLRW) coordinates
(t, ψ) and in the global coordinates (τ, ̂r) as defined in (3.2), the curvature singularity, the trapping horizons and the radial null geodesics intersecting in a point with t = lπ/2 or τ = lπ/2, for arbitrarily chosen parameters; the plots are obtained by numerical methods.

![Figure 3. Plots of curvature singularity (red), trapping horizons (blue) and one pair of radial null geodesics (green) crossing in τ = lπ/2, in the coordinate system (τ, ̂r) for M ≠ |Q| (left) and M = |Q| (right). The plot for M ≠ |Q| is zoomed in on the vertical axis to show its relevant features. For M = |Q|, the axis ̂r = 0 belongs to the curvature singularity.](image)

The radial null geodesics satisfy

\[
\frac{dt}{dψ} = \pm \frac{af^2}{g}.
\]  

(3.9)

For the case M = |Q| this means that the singularity at ψ = 0 is never reached, since for finite a the derivative tends to infinity. On the other hand the t-component of the geodesic equation for radial null or timelike geodesics in a ∼ 0 and finite ψ, using | ̇ψ| ≤ | ̇t|/(af^2) reads

\[
\ddot{t} + \frac{\cos(t/l)}{l \sin(t/l)} \dot{t}^2 \sim 0,
\]  

(3.10)

where a dot indicates a derivative with respect to the affine parameter. The solution, t ∼ ±l cos⁻¹ (c₁λ + c₂) shows that the singularities in t = 0, lπ are always reached for finite values of the affine parameter.

Taking advantage of the spherical symmetry, it is possible to define in a simple, geometrical way the surface gravity kₗ on the trapping horizons [24] and the associated
local Hawking temperature $T = \frac{k_i}{2\pi}$ [27], according to

$$ k_i = -\frac{1}{2} \left. \tilde{\nabla}_\mu \tilde{\nabla}^\mu R \right|_{\theta_{\pm} = 0} = -\frac{R}{2} \left[ \frac{f}{g} \left( \frac{\dot{a}}{a} - \frac{\dot{a}^2}{a^2} \right) + 2 \frac{\dot{a}^2}{a^2} \right] + \frac{1}{2} a^2 f^3 \sinh^2 (\psi/2) \left( \frac{3}{2} g - \cosh^2 (\psi/2) \right) \pm \frac{\dot{a}}{a^2 f} \cosh (\psi/2) \sinh (\psi/2) \left( \frac{1}{2} + \frac{1 - g}{f} \right) \right] , \quad (3.11) $$

where $R = a f \sinh \psi$ is the areal radius, $\tilde{\nabla}$ is the covariant derivative operator associated with the two-dimensional metric normal to the spheres of symmetry, and the vanishing of expression (3.8) was used. $k_i$ is in general not zero even in the maximally charged case, and is positive on the horizons, as is expected for outer trapping horizons. It is straightforward to verify that the generalized first law of black hole dynamics proposed by Hayward in [24],

$$ E' = \frac{k_i A'}{8\pi} + \frac{1}{2} TV' , \quad (3.12) $$

holds on the trapping horizons. Here a prime represents a derivative along a vector field tangent to the trapping horizon, $A = 4\pi R^2$ is the area of the spheres of symmetry, $V = \frac{4}{3} \pi R^3$ is the areal volume, $T$ is the trace of the total energy-momentum tensor $T$ with respect to the two-dimensional normal metric, and $E$ is the Misner-Sharp energy, defined as

$$ E = \frac{1}{2} R (1 + \nabla_\mu R \nabla^\mu R) . \quad (3.13) $$

Notice that $\nabla_\mu R \nabla^\mu R = \theta_+ \theta_- R^2 / 2$ is identically zero on the trapping horizons, implying $E' = \frac{1}{2} R'$.

## 4 Higher-dimensional generalization

It is possible to construct higher-dimensional generalizations of the multi-centered solutions found in section 2. To this aim, inspired by previous results [28, 29], we use the ansatz

$$ ds^2 = \frac{g^2}{f^2} dt^2 - a^2 f^2 \pi^2 ds_D^2 , \quad (4.1) $$

$$ F = \sqrt{\frac{D - 1}{2(D - 2)}} \frac{g}{f} \left[ \left( 1 - \frac{g}{f} \right)^2 + 4 \frac{g}{f} \left( 1 - \frac{1}{g} \right) \right]^{1/2} \frac{dH}{H} \wedge dt , $$

with

$$ f = 1 + M \frac{H}{a^{D-2}} + \frac{M^2 - Q^2}{4} \frac{H^2}{a^{2(D-2)}} , \quad g = 1 - \frac{M^2 - Q^2}{4} \frac{H^2}{a^{2(D-2)}} , \quad (4.2) $$

$$ - 10 - $$
where $a(t)$ is a function of time, $H(\bar{x})$ is a function of the spatial coordinates, $D$ and $d s_D^2 \equiv h_{ij}dx^i dx^j$ are respectively the dimension and the metric of the spatial slices. Notice that the square bracket in the expression of $F$ is equal to $Q H / (a^{D-2} f)$ and is just a way to express the charge $Q$ in terms of the functions $f$ and $g$.

The nonvanishing components of the Einstein tensor for (4.1) are given by

$$
G_{tt} = \frac{D(D-1)\dot{a}^2 g^2}{2a^2 f^2} + \frac{\dot{R} g^2}{2a^2 f^{2\frac{b+1}{b-2}}} - \frac{D-1}{D-2} \left(1 - \frac{g}{f}\right) \frac{g^2}{f^{2\frac{b+1}{b-2}}} \hat{\nabla}^2 H
$$

$$
+ \left\{ \frac{2D-1}{D-2} f \left(1 - \frac{1}{g}\right) \left[g(D-1) + D - 3\right] + \frac{1}{2} \frac{D-1}{D-2} \left(1 - \frac{g}{f}\right)^2 \right\} \frac{g^2}{f^{2\frac{b+1}{b-2}}} \frac{\partial_t H h^{lm} \partial_m H}{a^2 H^2},
$$

$$
G_{ij} = (1-D) h_{ij} a^2 f^{n-2} \left(\frac{\dot{a}}{a} - \frac{\dot{a}^2}{a^2}\right) - \frac{D(D-1)}{2} \frac{\dot{a}^2 f^{n-2} h_{ij}}{g}
$$

$$
+ \frac{1}{2} \frac{\dot{R} h_{ij}}{g} + 2 \left(1 - \frac{1}{g}\right) \left(\frac{\nabla^2 H}{H} h_{ij} - \frac{\hat{\nabla}_i \hat{\nabla}_j H}{H}\right)
$$

$$
+ \left\{ \frac{D-1}{2} \left(1 - \frac{g}{f}\right)^2 - 2 \left(1 - \frac{1}{g}\right) \left[1 - (D-1) \frac{g}{f}\right] \right\} \frac{1}{D-2} \frac{1}{H^2} \frac{\partial_t H h^{lm} \partial_m H}{a^2 H^2} h_{ij}
$$

$$
- \left\{ (D-1) \left(1 - \frac{g}{f}\right)^2 - 2 \left(1 - \frac{1}{g}\right) \left[D + 2(D-1) \frac{g}{f}\right] \right\} \frac{1}{D-2} \frac{1}{H^2} \frac{\partial_t H \partial_j H}{a^2 H^2}, \quad (4.3)
$$

where $\hat{\nabla}$, $\hat{R}_{ij}$ and $\hat{R}$ represent respectively the covariant derivative, Ricci tensor and scalar curvature of the spatial metric $h_{ij}$. From the expression for $F$ one obtains for the electromagnetic energy-momentum tensor

$$
8\pi T_{tt}^{em} = \frac{1}{2} \frac{D-1}{D-2} \left[\left(1 - \frac{g}{f}\right)^2 + 4 \frac{g}{f} \left(1 - \frac{1}{g}\right)\right] \frac{g^2}{f^{2\frac{b+1}{b-2}}} \frac{\partial_t H h^{lm} \partial_m H}{a^2 H^2}, \quad (4.4)
$$

$$
8\pi T_{ij}^{em} = - \frac{1}{D-2} \left[\left(1 - \frac{g}{f}\right)^2 + 4 \frac{g}{f} \left(1 - \frac{1}{g}\right)\right] \frac{1}{H^2} \left(\partial_t H \partial_j H - \frac{1}{2} h_{ij} \partial_t H h^{lm} \partial_m H\right).
$$

The requirement to have a perfect fluid as matter source translates into the condition $G_{ij} - 8\pi T_{ij}^{em} \propto h_{ij}$. This implies that $\hat{R}_{ij} \propto h_{ij}$, that is, the spatial slices must be Einstein manifolds, and that the function $H$ must satisfy the condition

$$
- \frac{\hat{\nabla}_i \hat{\nabla}_j H}{H} + \frac{D}{D-2} \frac{\partial_t H \partial_j H}{H^2} \propto h_{ij}. \quad (4.5)
$$

Notice that (4.5) is conformally invariant on Einstein manifolds, in the sense that under a conformal transformation that maps $h_{ij}$ to $\tilde{h}_{ij} = e^{2\omega} h_{ij}$, assuming that $H$ transforms
as $\tilde{H} = e^{2-\frac{D}{2}x}H$, one has

$$-rac{\tilde{\nabla}_i \tilde{\nabla}_j \tilde{H}}{\tilde{H}} + \frac{D}{D-2} \frac{\partial_i \tilde{H} \partial_j \tilde{H}}{\tilde{H}^2} = -\frac{\tilde{\nabla}_i \tilde{\nabla}_j H}{H} + \frac{D}{D-2} \frac{\partial_i H \partial_j H}{H^2} + \frac{\tilde{R}_{ij} - \tilde{R}_{ij}}{2}.$$  \hspace{1cm} (4.6)

For a metric, U(1) gauge field, fluid velocity and current density of the form

$$ds^2 = V(t, x^i) dt^2 - g_{ij} dx^i dx^j, \quad A = \phi dt, \quad u = \sqrt{V} dt, \quad J = \rho_e dt,$$  \hspace{1cm} (4.7)

(which is precisely what we have here), the conservation laws $\nabla_{\mu} T^\mu_{\nu} = 0$ imply

$$\partial_i p + \frac{p + \rho}{2} g^{ij} \partial_i g_{ij} = 0, \quad \partial_i p + \frac{p + \rho}{2V} \partial_i V - \frac{\rho_e}{\sqrt{V}} \partial_i \phi = 0.$$  \hspace{1cm} (4.8)

These equations carry information on how the pressure gradients balance the equilibrium of the system. In particular, the second one shows that the spatial gradient of the pressure cancels the gravitational and electromagnetic forces. Note that, due to the explicit time-dependence, there is one additional equation w.r.t. (17) of [4].

Let us now turn to (4.5). In the particular case of a conformally flat spatial metric, $h_{ij} = e^{2\alpha} \delta_{ij}$, for it to be Einstein it must also be of constant curvature, and one can always take $e^{-\omega} = 1 + \frac{k}{4} r^2$, with $r^2 \equiv \sum x^i x^i$. Then we have $H = (1 + \frac{k}{4} r^2)^{D-2} H_0$, with $H_0$ satisfying (4.5) on flat space, i.e.,

$$H_0 = (\alpha r^2 + \beta^i x^i + \gamma)^{\frac{2-D}{2}}.$$  \hspace{1cm} (4.9)

where $\alpha$ can always be set to 1 by rescaling the parameters $M$ and $Q$. In this case the energy density and pressure of the fluid are given by

$$8\pi \rho = \frac{f^2}{g^2} (G_{tt} - 8\pi T^\text{em}_{tt}) = \frac{D(D-1) \dot{a}^2}{2a^2} + \frac{k D(D-1)}{2a^2} \frac{1}{f \nu^2}$$

$$- \frac{D-1}{D-2} \left( 1 - \frac{g}{f} \right) \frac{\hat{V}^2 H}{a^2} + \frac{D-3}{D-2} \left( 1 - \frac{1}{g} \right) \frac{\partial_i H \partial^i H}{a^2 H^2}.$$  \hspace{1cm} (4.10)

while the current density reads

$$4\pi J = -\sqrt{\frac{D-1}{2(D-2)}} \left[ \left( 1 - \frac{g}{f} \right)^2 + 4 \frac{g}{f} \left( 1 - \frac{1}{g} \right) \right]^{1/2} \frac{g}{a^2 f \nu^2} \frac{\hat{V}^2 H}{H} dt.$$  \hspace{1cm} (4.11)
In the maximally charged case, \( |Q| = M \), the ansatz (4.1) reduces to

\[
ds^2 = \frac{1}{f^2} dt^2 - a^2 f^{\frac{2}{n-2}} ds_D^2, \quad F = \sqrt{\frac{D-1}{2(D-2)}} \frac{1}{f} \left( 1 - \frac{1}{f} \right) \frac{dH}{H} \wedge dt,
\]

with

\[
f = 1 + M \frac{H}{a^{b-2}}, \quad (4.12)
\]

and the Einstein tensor boils down to

\[
G_{tt} = \frac{D(D-1)}{2} \frac{\dot{a}^2}{a^2 f^2} + \frac{\dot{R}}{2a^2 f^{2\frac{n-2}{n-4}}} - \frac{D-1}{D-2} \left( 1 - \frac{1}{f} \right) \frac{1}{f^2} \frac{\tilde{\nabla}^2 H}{a^2 H^2}
\]

\[
+ \frac{1}{2} \frac{D-1}{D-2} \left( 1 - \frac{1}{f} \right)^2 \frac{1}{f^2} \frac{\partial_t H h^{lm} \partial_m H}{a^2 H^2},
\]

\[
G_{ij} = (1 - D) h_{ij} a^2 f^{\frac{2}{n-2}} \left( \frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} \right) - \frac{D(D-1)}{2} \frac{\dot{a}^2}{f^{\frac{2}{n-2}}} h_{ij}
\]

\[
+ \dot{R}_{ij} - \frac{1}{2} \dot{R} h_{ij} + \frac{D-1}{D-2} \left( 1 - \frac{1}{f} \right)^2 \left[ \frac{1}{2} \frac{\partial_t H h^{lm} \partial_m H}{H^2} h_{ij} - \frac{\partial_t H \partial_j H}{H^2} \right]. \quad (4.13)
\]

Finally, the electromagnetic energy-momentum tensor becomes

\[
8\pi T^\text{em}_{tt} = \frac{1}{2} \frac{D-1}{D-2} \left( 1 - \frac{1}{f} \right)^2 \frac{1}{f^2} \frac{\partial_t H h^{lm} \partial_m H}{a^2 H^2},
\]

\[
8\pi T^\text{em}_{ij} = -\frac{D-1}{D-2} \left( 1 - \frac{1}{f} \right)^2 \frac{1}{H^2} \left( \partial_i H \partial_j H - \frac{1}{2} h_{ij} \partial_t H h^{lm} \partial_m H \right). \quad (4.14)
\]

In this case the condition to have a perfect fluid source, \( G_{ij} - 8\pi T^\text{em}_{ij} \propto h_{ij} \), simply reduces to the requirement that the spatial slices are Einstein manifolds, \( \dot{R}_{ij} \propto h_{ij} \), while \( H \) can now be any function of the spatial coordinates, i.e., (4.5) does not need to hold anymore. The maximally charged solution is thus less constrained. This is of course also true in the four-dimensional case, with suitable forms for the density, pressure and current of the fluid. For a spatial metric of constant curvature the energy density, pressure and current density of the fluid are respectively given by

\[
8\pi \rho = \frac{D(D-1)}{2} \frac{\dot{a}^2}{a^2} + \frac{k D(D-1)}{2a^2 f^{\frac{2}{n-2}}} - \frac{D-1}{D-2} \left( 1 - \frac{1}{f} \right) \frac{1}{f} \frac{\tilde{\nabla}^2 H}{a^2 H},
\]

\[
8\pi p = -\frac{k}{2a^2} \frac{(D-1)(D-2)}{f^{\frac{2}{n-2}}} + (1-D) f \left( \frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} \right) - \frac{D(D-1)}{2} \frac{\dot{a}^2}{a^2},
\]

\[
4\pi J = -\sqrt{\frac{D-1}{2(D-2)}} \left( 1 - \frac{1}{f} \right) \frac{1}{a^2 f^{\frac{2}{n-2}}} \frac{\tilde{\nabla}^2 H}{H} dt. \quad (4.15)
\]
Given that $H$ can be an arbitrary function in the extremal case, what was the reason for the appearance of the conformal Laplace equation in section 2? To answer this question, let us go back to the nonextremal solution, and consider the case where $h_{ij}$ is the metric on a space of constant curvature. As we already said, one has then (setting $\alpha = 1$)

$$H = \left(1 + \frac{k}{4}r^2\right)^{-\frac{D-2}{2}} H_0, \quad H_0 = (r^2 + \beta^i x^i + \gamma)^{\frac{2-D}{2}}.$$  \hspace{1cm} (4.16)

If the parameters in (4.16) satisfy the constraint $\gamma = \beta^i \beta^i / 4$, $H_0$ can be rewritten as

$$H_0 = \frac{1}{|\vec{x} - \vec{x}_0|^{D-2}}, \quad (x^i_0 \equiv -\beta^i / 2),$$  \hspace{1cm} (4.17)

which is harmonic on $D$-dimensional flat space. In this case, $H$ in (4.16) satisfies the conformal Laplace equation

$$\hat{\nabla}^2 H = \frac{D - 2}{4(D - 1)} \hat{R}H.$$  \hspace{1cm} (4.18)

(4.18) results thus from extrapolating the nonextremal case (where (4.5) must hold) to the maximally charged situation, under the additional assumption that $h_{ij}$ has constant curvature.

5 Final remarks

In this paper, we showed how to construct multi-center black hole bound states in an arbitrary FLRW universe, and for any dimension. It turned out that these solutions are characterized by a function satisfying the conformal Laplace equation on the spatial slices of the FLRW background. For the single-center solution, we discussed some of the physical properties in the case when the energy density and the pressure of the perfect fluid source approach the values given by a negative cosmological constant far away from the black hole.

It would be nice to mimic the perfect fluid with one or more scalar fields, and to embed (2.5) in some simple model of matter-coupled (genuine or fake) $N = 2$ supergravity, similar to what was done in [17–20]. Since the charge density $\sigma$ of the cosmic fluid is nonvanishing for $k \neq 0$, these scalars have to be charged under a U(1) gauge field. In such a scenario, the cosmological expansion would be driven by the scalar field while rolling down its potential.

\footnote{Note that, in [17–20], the stress tensor of the scalar did not assume exactly a perfect fluid form everywhere, but only far away from the black holes.}
Our results represent another example for a superposition principle without supersymmetry: Generically, the existence of a Killing spinor implies a timelike or null Killing vector, which we clearly don’t have here.

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