Derived tame quadratic string algebras

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ABSTRACT

In this paper we determine the derived representation type of quadratic string algebras and we prove that every derived tame quadratic string algebra whose quiver has cycles is derived equivalent to some skewed-gentle algebra.

1. Introduction

Throughout the paper $K$ denotes a fixed algebraically closed field. By an algebra we mean a basic, connected and finite dimensional $K$-algebra (associative, with an identity). For an algebra $A$, let $D^b(A)$ be the bounded derived category of the category of finitely generated right modules mod $A$.

During the last years there has been an active study of derived categories. The notion of derived tameness was introduced in [19] and the tame-wild dichotomy for bounded derived categories of finite-dimensional algebras was established in [5]. In particular, the study of derived representation type of an algebra becomes an important topic in representation theory. This question is well-known for tree algebras [10, 17], for algebras with radical squared zero [4, 6] and for Nakayama algebras [7, 26], for example.

In this paper we consider the class of quadratic string algebra (see Definition 3.1). This class of algebras is a subclass of the special biserial algebras and a generalization of well-known gentle algebras. Moreover, the derived category of quadratic string algebras also were studied in [14] with the name of string almost gentle algebras.

In the order to classify the derived tame quadratic string algebras, we introduce a new class of algebras called generalized gentle algebras (see Definition 3.5), which generalize the gentle algebras. We will show that every generalized gentle algebra is derived tame.

Recall that if $A$ is an algebra of finite global dimension, then its Euler quadratic form is defined on the Grothendieck group of $A$ by

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for any $A$-module $M$. Recall also that an algebra $A$ is a tree if its (ordinary) quiver is a tree. In particular, if $A$ is a tree then the finite global dimension of $A$ is finite.

Our main results are the following theorems.

**Theorem A.** Every generalized gentle algebra is derived equivalent to a skewed-gentle algebra. In particular, every generalized gentle algebra is derived tame.

**Theorem B.** Let $A$ be a quadratic string algebra such that $A$ is not a tree. Then $A$ is derived tame if and only if $A$ is a generalized gentle algebra.

Note that by Theorems A and B we have the following result.

**Corollary C.** Let $A$ be a quadratic string algebra such that $A$ is not a tree. If $A$ is derived tame, then $A$ is derived equivalent to some skewed-gentle algebra.

**Theorem D.** Let $A$ be a quadratic string algebra. Then $A$ is derived tame if and only if one of the following conditions holds:

1. $A$ is tree and its Euler form is non-negative;
2. $A$ is a generalized gentle algebra.

Note that if $A$ is a tree then the **Theorem D** follows from [10, 17].

The structure of this paper is as follows. In Section 2 we review some preliminary results about derived representation type classification of algebras. Moreover, we also recall the definitions of Galois coverings, mutations and blowing-up of an algebra. In Section 3 we recall the definition of gentle and skewed-gentle algebras and prove the **Theorem A**. Finally, in Section 3 we prove the Theorems B and D.

2. Preliminaries

2.1. Algebras, categories and quivers

Following [16], a locally bounded category $C$ is a $K$-linear category satisfying the following conditions:

1. different objects in $C$ are not isomorphic;
2. $C(x, x)$ is a local ring for every $x \in C$;
3. $\dim_K \sum_{y \in C} C(x, y) + \dim_K \sum_{y \in C} C(y, x) < \infty$ for all $x \in C$.

Moreover, if the number of objects in $C$ is finite, then $C$ is called a bounded category.

For a locally bounded category $C$, denote by $\text{Mod} C$ the category of all contravariant functors from $C$ to the category of $K$-vector spaces, and denote by $\text{mod} C$ the full subcategory of $\text{Mod} C$ formed by the finite-dimensional contravariant functors $M$ (i.e. $\sum_{x \in C} \dim_K M(x) < \infty$).

**Remark 2.1**. It is clear that every bounded category $C$ defines a basic finite-dimensional algebra $\bigoplus C$ formed by quadratic matrices $a = (a_{yx})_{x, y \in C}$ such that $a_{yx} \in C(x, y)$. Conversely, for every basic finite-dimensional algebra $A$ (associative with unity) can be attached to bounded category $C_A$ whose objects are a complete set of primitive orthogonal idempotents of $A$, the space $yAx$ as the set $C_A(x, y)$ of morphisms from $x$ to $y$ and the composition is induced by the multiplication $\otimes$.
in A. It is easy to see that $A \cong \bigoplus C_A$ and the categories $\text{mod} C_A$ and $\text{mod} A$ are equivalents. By an abuse of notation, we shall identify an algebra $A$ with its bounded category $C_A$.

A quiver is a tuple $Q = (Q_0,Q_1,s,t)$, where $Q_0$ and $Q_1$ are sets and $s,t : Q_1 \to Q_0$ are maps. The elements of $Q_0$ and $Q_1$ are called vertices and arrows of $Q$, respectively. For every arrow $a \in Q_1$ the vertex $s(a)$ is its source, while $t(a)$ is its target. An arrow $a$ is a loop if $s(a) = t(a)$.

We say that $Q$ is finite if $Q_0$ and $Q_1$ are both finite sets, and we say that $Q$ is connected if its underlying graph is a connected graph. In this paper, all the quivers will be considered connected quivers.

A quiver $\Delta$ is a subquiver of $Q$ if $\Delta_i \subseteq Q_i$ for $i = 0, 1$. Moreover, $\Delta$ is full subquiver if $\Delta_1 = \{ x \in Q_1 | s(x), t(x) \in \Delta_0 \}$.

For every arrow $a : x \to y$ of $Q$, we denote by $a^{-1} : y \to x$ its formal inverse. By a walk $w$ of length $l(w) = n$ we mean a sequence $w = w_1w_2 \cdots w_n$ where each $w_i$ is either of form $a$ or $a^{-1}$, $a$ being an arrow in $Q$ and where $s(w_{i+1}) = t(w_i)$ for any $1 \leq i < n$. The source and the target of a walk are defined in the natural way. The concatenation $ww'$ of two walks $w$, $w'$ in $Q$ is defined in the natural way whenever $s(w') = t(w)$. A path in $Q$ is a walk $w = a_1a_2 \cdots a_n$ constituted only by arrows of $Q$. Moreover, to every vertex $x$, one associates a trivial path $e_x$ with $s(e_x) = t(e_x) = x$ which is of length 0 by convention. We denote by $Q_2$ the set of paths of length 2. A walk $w \in Q$ is called closed if $s(w) = t(w)$; reduced if $w$ is either a trivial path, or $w = w_1 \cdots w_n$ such that $w_{i+1} \neq w_i^{-1}$ for all $1 \leq i < n$; and a cycle if $w$ is non-trivial, reduced and closed. In particular, a quiver $Q$ is called a tree if it has no cycles.

Given a quiver $Q$, the path category $KQ$ is defined as follows: its object class is $Q_0$ and given $x, y \in Q_0$, the morphism set $KQ(x, y)$ is the $K$-vector space having as basis the set of paths from $y$ to $x$ and the composition is induced by concatenation of paths. Denote by $R_Q$ the two-sided ideal of $KQ$ generated by $Q_1$. An ideal $I$ of $KQ$ is said to be admissible if for every $x, y \in Q_0$ one has $I(x, y) \subseteq R_Q(x, y)$ and for any $x \in Q_0$ there exists some $n_x \geq 2$ such that $I$ contains all the paths with length at least $n_x$ and with source or target $x$. If $Q$ is a finite quiver and $I$ is an admissible ideal of $KQ$, the pair $(Q, I)$ is called bounded quiver. It is well-known that any finite-dimensional $K$-algebra is Morita equivalent to a quotient $KQ/I$ where $I$ is an admissible ideal. Moreover, if $KQ/I$ is a finite-dimensional $K$-algebra, then the radical (Jacobson) of $KQ/I$ is given by $\text{rad } KQ/I = R_Q/I$.

A quiver-morphism $\phi : Q' \to Q$ consists of two maps $\phi_0 : Q_0 \to Q_0$ and $\phi_1 : Q'_1 \to Q_1$ such that for every arrow $a : i \to j$ of $Q'$, we have that $\phi_1(a) : \phi_0(i) \to \phi_0(j)$. Moreover, if $I'$ and $I$ are admissible ideals of $KQ'$ and $KQ$, respectively, and $\phi(I') \subseteq I$, we say that $\phi : (Q', I') \to (Q, I)$ is a morphism of quiver with relations. In this case, $\phi$ induces a homomorphism of $K$-algebras $\bar{\phi} : KQ'/I' \to KQ/I$. By abuse of notation, we write $\phi = \bar{\phi}$.

For a vertex $x \in Q_0$, consider the sets

$$x^+ = \{ z \in Q_1 | s(z) = x \} \quad \text{and} \quad x^- = \{ z \in Q_1 | t(z) = x \}.$$

One says that $Q$ is locally finite if the sets $x^+$ and $x^-$ are both finite for all $x \in Q_0$. Note that if $Q$ is a locally finite quiver and $I$ is an admissible ideal of $KQ$, then $KQ/I$ is a locally bounded category. Moreover, if $Q$ is finite, then $KQ/I$ is a finite-dimensional algebra whose the unit is $\sum_{x \in Q_0} e_x$, where $e_x = e_{xA} + I$ is the idempotent corresponding to the vertex $x$.

Given an algebra $A = KQ/I$ and a vertex of $Q$, we denote by $P_x = e_{xA}$ the corresponding indecomposable projective $A$-module associate to $x$. It is well-known that any arrow $a : x \to y$ gives rise to a map $a : P_y \to P_x$ given by left multiplication by $a$. Moreover, this map gives a natural isomorphism $e_y A e_x \cong \text{Hom}_A(P_y, P_x)$.

### 2.2. Derived representation type

It is well-known that for an algebra $A$ the category $D^b(A)$ can be identified with the homotopy category $K^{-,b}(\text{proj} A)$ of right bounded complexes of finitely generated projective right
A-modules with bounded cohomologies (see [27, Proposition 3.5.43]). Recall that the cohomology dimension vector of a complex $X \in D^b(A)$ is the vector $h - \dim(X) = (\dim_K H^i(X))_{i \in \mathbb{Z}}$, where $H^i(X)$ is the $i$-th cohomology space of $X$.

**Definition 2.2.** Let $A$ be an algebra.

1) [19] $A$ is said to be derived tame if for any $n = (n_i)_{i \in \mathbb{N}}$ there exist a localization $R = K[x]_f$ with respect to some $f \in K[x]$ and a finite number of bounded complexes of $R$-$A$-bimodules $X_1, ..., X_k$ such that each $X_i$ is finitely generated free over $R$ and (up to isomorphism) all but finitely many indecomposable object of cohomology dimension $n$ in $D^b(A)$ are of the form $S \otimes_R X_i$ for some $i = 1, ..., n$ and some simple $R$-module $S$.

2) [5] $A$ is said to be derived wild if there exists a bounded complex $M$ of $K\langle x, y \rangle$-$A$-bimodules such that each $M'$ is free and of finite rank as left $K\langle x, y \rangle$-module and the functor $- \oplus_{K\langle x, y \rangle} M : \text{mod} K\langle x, y \rangle \to \text{mod} A$ preserves indecomposability and isomorphism classes.

Two algebras $A$ and $B$ are said to be derived equivalent if the respective derived categories $D^b(A)$ and $D^b(B)$ are equivalent as triangulated categories. Moreover, by an important result due to Rickard [25], this happens exactly when $B \cong (\text{End}(T))^{op}$ where $T \in K^b(\text{proj} A)$ is a complex (called tilting complex) satisfying:

(i) for all $i \neq 0$, $\text{Hom}_{K^b(\text{proj} A)}(T, T[i]) = 0$ (where $[-]$ denote the shift functor);
(ii) $T$ generates $K^b(\text{proj} A)$ as a triangulated category.

**Remark 2.3.** We recall from [19, Theorem A] that derived tameness is preserved under derived equivalence.

Let us recall from [6] the following result.

**Theorem 2.4.** Let $A$ be an algebra with radical squared zero, that is, $A \cong KQ/R_Q^2$. Then $A$ is derived tame if and only if $Q$ is either Dynkin or Euclidean graph.

### 2.3. Cleaving functors and Galois coverings

Let $F : B \to A$ be a $K$-linear functor between two locally bounded categories. It is well-known that the restriction functor $F_* : \text{Mod} A \to \text{Mod} B$, $F_*(M) = MF$ admits a left adjoint $F^* : \text{Mod} B \to \text{Mod} A$, called extension functor, which is up to natural isomorphism well-defined by requiring that $F^*$ is a right exact and coproduct preserving functor such that $F^*A(-, a) = B(-, Fa)$ for all objects $a$ of $A$. According to [3], the functor $F$ is called cleaving if the canonical natural transformation $\Phi_F : \text{Id}_{\text{Mod} B} \to F_* F^*$ admits a natural retraction, i.e., if there is a morphism $\Psi : F_* F^* \to \text{Id}_{\text{Mod} B}$ such that $\Psi(M) \Phi_F(M) = \text{Id}_M$ for each $M \in \text{Mod} B$.

The best known examples of cleaving functors are the Galois coverings. Following [12], a Galois covering of a bounded quiver $(Q, I)$ is a morphism of quivers with relations $\phi : (Q, \hat{I}) \to (Q, I)$ together with a group $G$ of automorphisms of $(Q, \hat{I})$ satisfying:

(i) $G$ acts freely on $\hat{Q}_0$;
(ii) $\phi g = \phi$ for every $g \in G$ and $\phi(x) = \phi(y)$ if, and only if, $y = gx$ for some $g \in G$;
(iii) $\phi$ induces bijections $x^+ \to p(x)^+$ and $x^- \to p(x)^-$, for all $x \in Q_0$;
(iv) $I$ is the ideal generated by the elements of the form $p(\rho)$, with $\rho \in \hat{I}$.

A Galois covering of bonded quiver $\phi : (Q, \hat{I}) \to (Q, I)$ induces naturally a Galois covering functor between $K$-categories $F : KQ/\hat{I} \to KQ/I$ in the sense of [11].

In this paper we have a particular interest on Galois coverings of monomial algebras, that is, algebras of type $KQ/I$ where $I$ is generated by paths. Following [20] (see also [22]), given $(Q, I)$ a
bounded quiver where $I$ is generated by paths, there is a Galois covering $F : (\tilde{Q}, \tilde{I}) \to (Q, I)$ defined by fundamental group $\pi_1(Q, I)$. More precisely, fixed $x_0 \in Q_0$, let $\pi_1(Q, I, x_0)$ be the set of reduce walks of $Q$ starting and ending at $x_0$. Of course the concatenation of walks induces a group structure to $\pi_1(Q, I, x_0)$. Moreover, since $Q$ is connected quiver, so $\pi_1(Q, I, x_0) \cong \pi_1(Q, I, x_1)$ for any $x_1 \in Q_0$. Thus $\pi_1(Q, I, x_0)$ will be denoted by $\pi_1(Q, I)$, and we say that it is the fundamental group of $(Q, I)$.

Now, the vertices of $\tilde{Q}$ are the reduced walks starting at a fixed vertex $x_0 \in Q_0$, and for all $w, w' \in \tilde{Q}_0$ there is an arrow $[x, w] \in \tilde{Q}_1$ from $w$ to $w'$ whenever $w' = w\alpha$ with $\alpha \in Q_1$. Note that there is at most one arrow $[x, w] : w \to w'$ since the arrow $\alpha$ is unique, if it exists. This provides a quiver-morphism $p : \tilde{Q} \to Q$ defined by $p(w) := t(w)$ and $p([x, w]) = x$. Finally, the ideal $\tilde{I}$ is defined to be generated by the inverse images under $p$ of the generators of $I$.

**Remark 2.5.**

(1) Note that if $(Q, I)$ is bounded quiver where $Q$ is connected and $I$ is generated by paths, then $\tilde{Q}$ is connected quiver without cycles quiver and the ideal $\tilde{I}$ is generated by paths. Moreover, if $Q$ has cycles, then $\tilde{Q}$ is a locally finite quiver with an infinite number of vertices.

(2) In this paper, all the Galois coverings will be considered Galois coverings defined by fundamental group.

(3) It is well-known that Galois coverings are cleaving functors (see [15, 3.2] for example).

The next result follows from [3, 3.8(b)].

**Lemma 2.6.**

(a) Let $C$ be a locally bounded category and $C'$ a full subcategory of $C$. The inclusion $C' \hookrightarrow C$ is a cleaving functor.

(b) The composition of two cleaving functors is cleaving.

Let us recall from [26] the following theorem.

**Theorem 2.7.** Let $F : B \to A$ be a cleaving functor between bounded categories with $\text{gl.dim} \; B < \infty$. Then $B$ derived wild implies $A$ derived wild.

**Lemma 2.8.** Let $A = KQ/I$ be a monomial algebra and $\tilde{A} = K\tilde{Q}/\tilde{I}$ its Galois covering.

(a) If $\tilde{A}$ has a full, bounded and derived wild subcategory $\tilde{A}$, then $A$ is also derived wild.

(b) If $\tilde{A}$ has a full and bounded subcategory $\tilde{A} \cong K\Delta/R_\Delta^2$ such that $\Delta$ is neither a Dynkin nor an Euclidean graph, then $A$ is derived wild.

**Proof.**

(a) Let $\tilde{F} : \tilde{A} \to A$ be the composition of the canonical inclusion $\tilde{A} \hookrightarrow \tilde{A}$ and the Galois covering $F : \tilde{A} \to A$, thus $\tilde{F}$ is a cleaving functor by Lemma 2.6. Since $\tilde{Q}$ is a tree and $\tilde{A}$ is bounded, then $\text{gl.dim} \; \tilde{A} < \infty$. Therefore $A$ is derived wild by Theorem 2.7.

(b) We get that $A$ is derived wild by Theorem 2.4, hence $A$ is derived wild by item (a).

**Example 2.9.** Let $A = KQ/I$ be the algebra where

$$Q : \begin{array}{c}
\bullet \\
\delta \quad \beta \\
\gamma
\end{array}$$

and $I = \langle \delta \gamma, \alpha \beta, \beta \gamma \rangle$. Let $h = \gamma z$, the Galois covering of $A$ has a full bounded subcategory $\tilde{A} = K\tilde{Q}/R_\tilde{Q}$ where...
Since $\tilde{A}$ is a tree, then $\text{gl.dim }\tilde{A} < \infty$. Moreover, it is derived wild by Theorem 2.4. Hence $A$ is derived wild by Lemma 2.8.

2.4. Mutations

We recall the notion of mutations of algebras from [21]. These are local operations on an algebra $A$ producing new algebras derived equivalent to $A$.

Let $A = KQ/I$ be an algebra and $x \in Q_0$ a fixed vertex without loops. For all vertex $i \neq x$ we denote by $R_i : 0 \longrightarrow P_i \longrightarrow 0$ the complex concentrated in degree 0. Moreover, we define the complexes

$$ R_x : \quad 0 \longrightarrow P_x \xrightarrow{f} \bigoplus_{j \rightarrow x} P_j \longrightarrow 0 \longrightarrow 0 $$

$$ L_x : \quad 0 \longrightarrow 0 \longrightarrow \bigoplus_{x \rightarrow j} P_j \xrightarrow{g} P_x \longrightarrow 0 $$

where the map $f$ is induced by all maps $P_x \rightarrow P_j$ corresponding to the arrows $j \rightarrow x$, the map $g$ is induced by the maps $P_j \rightarrow P_x$ corresponding to the arrows $x \rightarrow j$, the term $P_x$ lies in degree $-1$ in $R_x$ and in degree 1 in $L_x$, and all other terms are in degree 0.

Definition 2.10. Let $A = KQ/I$ be an algebra and let $x \in Q_0$ be a vertex without loops.

1) We say that the negative mutation of $A$ at $x$ is defined if $T_x^-(A) = \bigoplus_{i \in Q_0} R_i \in K^b(\text{proj } A)$ is a tilting complex. In this case, we call the algebra $\mu_x^-(A) = (\text{End}(T_x^-(A)))^{\text{op}}$ the negative mutation of $A$ at the vertex $x$.

2) We say that the positive mutation of $A$ at $x$ is defined if $T_x^+(A) = (\bigoplus_{i \neq x} R_i) \oplus L_x \in K^b(\text{proj } A)$ is a tilting complex. In this case, we call the algebra $\mu_x^+(A) = (\text{End}(T_x^+(A)))^{\text{op}}$ the positive mutation of $A$ at the vertex $x$.

By Rickard’s Theorem [25, Theorem 6.4], the negative and positive mutations of an algebra $A$ at a vertex, when defined, are always derived equivalent to $A$.

Remark 2.11. If $x^+ = \emptyset$, follows by [21, Proposition 2.3] that $\mu_x^-(A)$ is defined. Moreover, in this case $T_x^-(A)$ is isomorphic in $D^b(A)$ to the APR-tilting module corresponding to $x$ (see [2]). Similarly, if $x^- = \emptyset$ then $\mu_x^+(A)$ is defined.

2.5. Blowing-up

Let $A = KQ/I$ be an algebra and let $D$ be a set of vertices without loops of $Q$. Following [10], the blowing-up of $A$ at $D$ is the algebra $A[D] = KQ[D]/I[D]$, where $Q[D]$ and $I[D]$ are describe below.

The quiver $Q[D]$ is obtained from $Q$ replacing each vertex $d \in D$ by two vertices $d^-$ and $d^+$, each arrow $\alpha : x \rightarrow d$ with $d \in D$ by two arrows $\alpha^- : x \rightarrow d^-$ and $\alpha^+ : x \rightarrow d^+$ and dually for each arrow $\beta : d \rightarrow x$. 
There is an obvious quiver epimorphism $p : Q[D] \to Q$ which extends uniquely to an epimorphism of algebras $p : KQ[D] \to KQ$. The ideal $I[D]$ of $KQ[D]$ is defined as the ideal generated by the inverse image of the generators of $I$ under $p$.

**Example 2.12.** Let $A = KQ/I$ be given by the bounded quiver:

\[
\begin{array}{cccccc}
1 & \xrightarrow{\alpha} & 2 & \xrightarrow{\beta} & 3 & \xrightarrow{\gamma} & 4 & \xrightarrow{\delta} & 5 \\
1^- & & 2 & & 3^- & & 4 & & 5 \\
1 & & 2^- & & 3 & & 4^- & & 5^- \\
\end{array}
\]

and $I = \langle \alpha \beta \rangle$.

If $D = \{1, 3\}$, then the blowing-up $A[D]$ is given by the quiver

\[
\begin{array}{cccccc}
1^+ & \xrightarrow{\alpha^+} & 2^+ & \xrightarrow{\beta^+} & 3^+ & \xrightarrow{\gamma^+} & 4 & \xrightarrow{\delta} & 5 \\
1^- & \xrightarrow{\alpha^-} & 2^- & \xrightarrow{\beta^-} & 3^- & \xrightarrow{\gamma^-} & 4 & & 5^- \\
\end{array}
\]

bounded by $I[D] = \langle \alpha^+ \beta^+, \alpha^- \beta^-, \alpha^+ \beta^-, \alpha^- \beta^+, \beta^+ \gamma^+ - \beta^- \gamma^- \rangle$.

**Remark 2.13.** Given $A = KQ/I$ and $D \subset Q_0$ a set of vertices without loops, let $p : Q[D] \to Q$ be the canonical quiver epimorphism. Since $p(I[D]) \subset I$, then $p$ induces a $K$-linear functor $\pi : A[D] \to A$ where $\pi(i) = p_0(i)$ for each $i \in Q[D]_0$ and $\pi_{i,j}(u) = p(u)$ for all $u \in A[D](i,j)$. Moreover, by construction of $A[D]$ we have that $\pi_{i,j}$ is a monomorphism for all $i, j \in Q[D]_0$ and

\[
\begin{cases}
\text{Im } \pi_{i,j} = A(p_0(i), p_0(j)), & \text{if } (i, j) \notin \mathfrak{D}; \\
\text{Im } \pi_{i,j} = \text{rad } A(p_0(i), p_0(j)), & \text{if } (i, j) \in \mathfrak{D};
\end{cases}
\]

where $\mathfrak{D} = \{(d^+, d^-), (d^-, d^+)|d \in D\}$.

**Lemma 2.14.** Let $A = KQ/I$ be an algebra and let $D \subset Q_0$ be a set of vertices without loops.

(a) The functor $\pi : A[D] \to A$ induces an exact functor

\[K(\Pi) : K^b(\text{proj } A[D]) \to K^b(\text{proj } A).\]

(b) Let $B$ be a locally bonded $K$-category such that there is a $K$-linear functor $\varphi : B \to A$ and a bijection $l : B_0 \to Q[D]_0$ such that:

- $\varphi(a) = p_0(l(a))$, for all $a \in B_0$;
- for all $a, b \in B_0$, the map $\varphi_{a,b} : B(a, b) \to A(\varphi(a), \varphi(b))$ is a monomorphism and $\varphi(l(a), l(b)) \notin \mathfrak{D}$.

Then $B \simeq A[D]$.

**Proof.** (a) For all $i \in Q[D]_0$ we denote by $\hat{P}_i = e_iA[D]$ the corresponding indecomposable projective $A[D]$-module. Thus $\pi$ induces, up to isomorphism, a coproduct preserving additive functor $\Pi : \text{proj } A[D] \to \text{proj } A$ such that $\Pi(\hat{P}_i) = P_{p_0(i)}$ for each $i \in Q[D]_0$, and

$$\Pi_{\hat{P}_i, \hat{P}_j} : \text{Hom}_{A[D]}(\hat{P}_i, \hat{P}_j) \to \text{Hom}_A(P_{p_0(i)}, P_{p_0(j)})$$

$$x \mapsto \pi(x)$$

for all $i, j \in Q[D]_0$. 

It follows from [23, Proposition 1.1.1, p.192] that \( \Pi \) induces an exact functor \( K(\Pi) : K^b(\text{proj } A[D]) \to K^b(\text{proj } A) \) defined as below.

Given a complex \( X \in K^b(\text{proj } A[D]) \)

\[
X : \cdots X^{n-1} \xrightarrow{d_X^{n-1}} X^n \xrightarrow{d_X^n} X^{n+1} \cdots,
\]

we get that

\[
K(\Pi)(X) : \cdots \Pi(X^{n-1}) \xrightarrow{\Pi(d_X^{n-1})} \Pi(X^n) \xrightarrow{\Pi(d_X^n)} \Pi(X^{n+1}) \cdots.
\]

If \( f \in K^b(\text{proj } A[D])(X, Y) \) is given by

\[
X : \cdots X^{n-1} \xrightarrow{f} X^n \xrightarrow{f^n} X^{n+1} \cdots,
\]

\[
Y : \cdots Y^{n-1} \xrightarrow{f} Y^n \xrightarrow{f^{n+1}} Y^{n+1} \cdots,
\]

then the morphism \( K(\Pi)(f) \in K^b(\text{proj } A)(K(\Pi)(X), K(\Pi)(Y)) \) is

\[
K(\Pi)(X) : \cdots \Pi(X^{n-1}) \xrightarrow{\Pi(f_{n-1})} \Pi(X^n) \xrightarrow{\Pi(f^n)} \Pi(X^{n+1}) \cdots,
\]

\[
K(\Pi)(Y) : \cdots \Pi(Y^{n-1}) \xrightarrow{\Pi(f_{n-1})} \Pi(Y^n) \xrightarrow{\Pi(f^n)} \Pi(Y^{n+1}) \cdots.
\]

(b) We define a functor \( F : B \to A[D] \) as follows. For all \( a \in B_0 \), define \( F(a) = l(a) \). Since \( \pi_{l(a), l(b)} \) is a monomorphism and \( \text{Im } \varphi_{a,b} = \text{Im } \pi_{l(a), l(b)} \) for every \( a, b \in B_0 \), then we can define:

\[
F_{a,b} : B(a, b) \to A[D](l(a), l(b))
\]

\[
x \mapsto \pi_{l(a), l(b)}^{-1}(\varphi_{a,b}(x)).
\]

Note that \( F_{a,b} \) is \( K \)-linear since \( \varphi_{a,b} \) and \( \pi_{l(a), l(b)} \) are \( K \)-linear. Moreover, for all \( a, b, c \in B_0 \), \( x \in B(a, b) \) and \( y \in B(b, c) \) we have that:

\[
\pi_{l(a), l(c)}(F_{b,c}(y)F_{a,b}(x)) = \pi_{l(a), l(c)} \left( (\pi_{l(b), l(c)}^{-1}(\varphi_{b,c}(y)))(\pi_{l(a), l(b)}^{-1}(\varphi_{a,b}(x))) \right)
\]

\[
= \left( \pi_{l(b), l(c)}^{-1}(\varphi_{b,c}(y)) \pi_{l(a), l(b)}^{-1}(\varphi_{a,b}(x)) \right)
\]

\[
= \varphi_{b,c}(y)\varphi_{a,b}(x)
\]

\[
= \pi_{l(a), l(c)}\pi_{l(a), l(b)}^{-1}(\varphi_{a,c}(yx))
\]

\[
= \pi_{l(a), l(c)}(F_{a,c}(yx)).
\]

Since \( \pi_{l(a), l(c)} \) is a monomorphism then \( F_{a,c}(yx) = F_{b,c}(y)F_{a,b}(x) \). Therefore \( F \) is a \( K \)-linear functor.

Finally, it is clear that \( F \) is an isomorphism since \( F \) is bijective on the objects and morphisms.

\[ \square \]

**Lemma 2.15.** Let \( A = KQ/I \) be an algebra, and let \( D \subset Q_0 \) be a set of vertices without loops and \( x \in Q_0 \setminus D \).
(a) If \(x^+ = \emptyset\) and for any \(d \in D\) there is no arrow \(d \to x\), then \(\mu^-_x(A[D]) \cong \mu^-_x(A)[D]\). In particular, \(A[D]\) is derived equivalent to \(\mu^-_x(A)[D]\).

(b) If \(x^- = \emptyset\) and for any \(d \in D\) there is no arrow \(x \to d\), then \(\mu^+_x(A[D]) \cong \mu^+_x(A)[D]\). In particular, \(A[D]\) is derived equivalent to \(\mu^+_x(A)[D]\).

**Proof.** We only prove (a), the proof of (b) is similar. To prove (a) we use induction on the number of elements of \(D\).

Suppose first that \(D = \{d\}\). Let \(T^-_x(A) = \bigoplus_{i \in Q_0} R_i \in \text{K}^b(\text{proj} A)\) and \(\mu^-_x(A) = (\text{End}(T^-_x(A)))^{op}\) be as in Definition 2.10. Since \(R_i\) is indecomposable for any \(i \in Q_0\), then \(\mu^-_x(A)\) is the \(K\)-category whose set of objects is \(Q_0\) and for all \(i,j \in Q_0\) we have \(\mu^-_x(A)(i,j) = \text{Hom}_{\text{K}^b(\text{proj} A)}(R_i, R_j)\). Moreover, suppose that \(\mu^-_x(A) \cong K\Lambda/J\) and let \(p : (Q[d], I[d]) \to (Q, I)\) and \(q : (\Delta[d], J[d]) \to (\Delta, J)\) be the canonical quiver-morphisms. Hence \(Q[d]_0 = \Delta[d]_0\) and the maps \(p_0 = Q[d]_0 \to Q_0\) and \(q_0 = \Delta[d]_0 \to \Delta_0\) are equal.

In the other hand, given \(\hat{P_i}\) the projective \(A[d]\)-module associated to \(i \in Q[d]_0\), we define the complexes of \(A[d]\)-modules:

\[
\hat{R}_i : \quad 0 \longrightarrow 0 \longrightarrow \hat{P}_i \longrightarrow 0 \cdots, \quad i \neq x,
\]

\[
\hat{R}_x : \quad 0 \longrightarrow \hat{P}_x \longrightarrow \bigoplus_{j \to x} \hat{P}_j \longrightarrow 0 \cdots,
\]

concentrated in degree \(-1\) and 0, where the map \(\hat{f}\) is induced by all the arrows \(x \in Q[d]_1\) such that \(\hat{f}(x) = x\). By definition \(T^-_x(A[d]) = \bigoplus_{i \in Q[d]} \hat{R}_i\), and \(\mu^-_x(A[d]) = (\text{End}(T^-_x(A[d])))^{op}\), thus \(\mu^-_x(A[d])\) is the \(K\)-category whose set of objects is \(Q[d]_0\) and for every \(i,j \in Q[d]_0\) we have \(\mu^-_x(A[d])(i,j) = \text{Hom}_{\text{K}^b(\text{proj} A)}(\hat{R}_i, \hat{R}_j)\). Hence there is a bijection, which we suppose to be the identity, between the objects of \(\mu^-_x(A[d])\) and the objects of \(\mu^-_x(A)[d]\).

Given the morphism \(p : (Q[d], I[d]) \to (Q, I)\), let \(\pi : A[d] \to A\) be the homomorphism of algebras induced by \(p\). By Lemma 2.14(a), \(\pi\) induces an additive functor \(K(\Pi) : \text{K}^b(\text{proj} A[d]) \to \text{K}^b(\text{proj } A)\). Since there is no arrow \(d \to x\), then \(P_d\) is not a summand of \(R^0_x\) and \(\hat{P}_d, \hat{P}_d\) are not summands of \(\hat{R}_x\), thus \(K(\Pi)(\hat{R}_i) = \hat{R}_{p_0(i)}\) for all \(i \in Q[d]_0\). Hence we can define a functor \(\pi^- : \mu^-_x(A[d]) \to \mu^-_x(A)\) where \(\pi^-(i) = p_0(i)\) for every \(i \in Q[d]_0\), and for \(g \in \mu^-_x(A[d])(i,j)\) we have \(\pi^-(g) = K(\Pi)(g) \in \mu^-_x(A)(p_0(i), p_0(j))\).

For \(e \in \{+,-\}\), denote by \(A^e\) the full subcategory of \(A[d]\) defined by \(Q[d]_0 \setminus \{d^{\pm e}\}\) and denote by \(B^e\) the full subcategory of \(\mu^-_x(A[d])\) defined by \(Q[d]_0 \setminus \{d^{-e}\}\). Of course \(\pi^-|_{A^e} : A^e \to A\) is an isomorphism, thus \(\pi^-|_{B^e} : B^e \to \mu^-_x(A)\) is an isomorphism too.

Given \(i,j \in Q[d]_0\) such that \(\{i,j\} \neq \{d^+, d^-\}\), then \(i,j \in Q[d]_0 \setminus \{d^{-e}\}\) for any \(e \in \{+,-\}\), hence \(\pi^-|_{\{i,j\}} = \pi^-|_{\{i,j\}}\) is an isomorphism. Moreover, since \(\hat{R}_{d^+} = 0 \longrightarrow \hat{P}_{d^+} \longrightarrow 0\), so \(\pi^-|_{d^+} = \pi^-|_{d^{-e}}\) and therefore it is a monomorphism and \(\text{Im } \pi^-|_{d^+} = \text{Im } \pi^-|_{d^{-e}} = \text{rad } A(d,d) = \mu^-_x(A)(d,d)\).

Thus \(\mu^-_x(A[d]) \cong \mu^-_x(A)[d]\) by Lemma 2.14(b).

Now, suppose that \(|D| = n > 1\). Fixed \(d_0 \in D\), let \(D_0 = D \setminus \{d_0\}\). Then \(\mu^-_x(A[D]) = \mu^-_x((A[d_0])[D_0]) \cong \mu^-_x((A[d_0])[D_0])[D_0]\) by induction hypotheses. Moreover, since \(\mu^-_x(A[d_0]) \cong \mu^-_x(A)[d_0]\), then \(\mu^-_x(A[D]) \cong \mu^-_x((A[d_0])[D_0])D_0 \cong (\mu^-_x(A)[d_0])[D_0] = \mu^-_x(A)[D]\).

\(\Box\)

### 3. Generalized gentle algebras

Let us recall the definition of quadratic string algebras and gentle algebras.
Definition 3.1. An algebra \( A = KQ/I \) is called quadratic string algebra if the following conditions are satisfied:

(i) each vertex in \( Q \) is the source of at most two arrows and the target of at most two arrows;
(ii) for each arrow \( x \in Q_1 \) there is at most one arrow \( \beta \) such that \( t(x) = s(\beta) \) and \( x\beta \notin I \), and at most one arrow \( \gamma \) such that \( s(x) = t(\gamma) \) and \( \gamma x \notin I \);
(iii) \( I \) is generated by paths of length two.

Definition 3.2. Let \( A = KQ/I \) be a quadratic string algebra. A vertex \( x \in Q_0 \) is called gentle if the following conditions holds:

(i) for each arrow \( x \) in \( Q \) such that \( s(x) = x \), there is at most one arrow \( \beta \) such that \( t(\beta) = x \) and \( x\beta \in I \);
(ii) for each arrow \( x \) in \( Q \) such that \( t(x) = x \), there is at most one arrow \( \gamma \) such that \( s(\gamma) = x \) and \( \gamma x \in I \).

Following [1], \( A \) is said to be gentle if every vertex of \( Q \) is gentle.

Definition 3.3. Let \( A = KQ/I \) be a quadratic string algebra. For every \( i = 1, \ldots, 6 \) we define the sets \( E_i, O_i \subset Q_0 \) as follows.

1. A vertex \( x \) belongs to \( E_1 \) if there are exactly two arrows \( x = x_1 \) and \( \beta = \beta_1 \) ending at \( x \) and exactly two arrows \( \gamma = \gamma_1 \) and \( \delta = \delta_1 \) starting at \( x \) such that:
   - \( s(x_1) = \{x\}, s(x_1)^\ominus = \emptyset, t(\gamma_1)^+ = \emptyset \) and \( t(\gamma_1)^- = \{\gamma_1\} \);
   - \( s(\beta_1) \) and \( t(\delta_1) \) are pairwise distinct;
   - \( x_1, \beta_1, \gamma_1, \delta_1 \in I \), but \( \beta_1 \delta_1 \notin I \).

   Moreover, set \( O_1 = \left\{ s(x_1), t(\gamma_1) \middle| x \in E_1 \right\} \).

2. A vertex \( x \) belongs to \( E_2 \) if there are exactly two arrows \( x = x_2 \) and \( \beta = \beta_2 \) ending at \( x \) and exactly two arrows \( \gamma = \gamma_2 \) and \( \delta = \delta_2 \) starting at \( x \) such that:
   - \( s(x_2)^+ = \{x\}, s(x_2)^\ominus = \emptyset, t(\gamma_2)^+ = \emptyset \) and \( t(\gamma_2)^- = \{\gamma_2\} \);
   - \( s(\beta_2) = t(\delta_2) \);
   - \( x_2, \beta_2, x_2, \delta_2 \in I \), but \( \beta_2 \delta_2 \notin I \).

   Moreover, set \( O_2 = \left\{ s(x_2), t(\gamma_2) \middle| x \in E_2 \right\} \).

3. A vertex \( x \) belongs to \( E_3 \) if there are exactly two arrows \( x = x_3 \) and \( \beta = \beta_3 \) ending at \( x \) and exactly one arrow \( \gamma = \gamma_3 \) starting at \( x \) such that:
   - \( s(x_3)^+ = \{x\}, s(x_3)^\ominus = \emptyset, s(\beta)^+ = \{\beta\} \) and \( s(\beta)^\ominus = \emptyset \);
   - \( \gamma \), \( \beta \gamma \in I \).

   Moreover, set \( O_3 = \left\{ s(x_3), s(\beta_3) \middle| x \in E_3 \right\} \).

4. A vertex \( x \) belongs to \( E_4 \) if there are exactly two arrows \( x = x_4 \) and \( \beta = \beta_4 \) ending at \( x \) and exactly one arrow \( \gamma = \gamma_4 \) starting at \( x \) such that:
   - \( s(x_4)^+ = \{x\}, s(x_4)^\ominus = \emptyset, t(\gamma_4)^- = \{\gamma_4\} \) and \( t(\gamma_4)^+ = \emptyset \);
   - \( \gamma \), \( \beta \gamma \in I \).

   Moreover, set \( O_4 = \left\{ s(x_4), t(\gamma_4) \middle| x \in E_4 \right\} \).

5. A vertex \( x \) belongs to \( E_5 \) if there are exactly two arrows \( \gamma = \gamma_5 \) and \( \delta = \delta_5 \) starting at \( x \) and exactly one arrow \( x = x_5 \) ending at \( x \) such that:
   - \( t(\gamma_5)^- = \{\gamma_5\}, t(\gamma_5)^+ = \emptyset, t(\delta_5)^- = \{\delta_5\} \) and \( t(\delta_5)^+ = \emptyset \);
   - \( \gamma , \delta \gamma \in I \).

   Moreover, set \( O_5 = \left\{ t(\gamma_5), t(\delta_5) \middle| x \in E_5 \right\} \).
A vertex \( x \) belongs to \( E_6 \) if there are exactly two arrows \( c = c_x \) and \( d = d_x \) starting at \( x \) and exactly one arrow \( a = a_x \) ending at \( x \) such that:

- \( t(c)^- = \{c\}, t(c)^+ = \emptyset, s(x)^+ = \{x\} \) and \( s(x)^- = \emptyset \);
- \( x, y, c, d \in I \).

Moreover, set \( O_6 = \{t(c_x), s(x)|x \in E_6\} \).

Furthermore, the elements of \( \cup_{i=1}^{6} E_i \) and \( \cup_{i=1}^{6} O_i \) are called exceptional vertices and ordinary vertices, respectively.

**Remark 3.4.** Note that any exceptional vertex is not gentle. In the other hand, any ordinary vertex is a gentle vertex.

**Definition 3.5.** A quadratic string algebra \( A = KQ/I \) is said to be generalized gentle algebra if every vertex of \( Q_0 \) is a gentle or an exceptional vertex.

**Example 3.6.** Let \( A = KQ/I \) given by

\[
\begin{array}{cccccccc}
1 & \to & 3 & \to & 5 & & 8 & & 11 \\
\alpha & & \beta & & \delta & & \mu & & \eta \\
4 & \to & 7 & \to & 9 & & 10 & & 12 \\
\gamma & & \rho & & \kappa & & \sigma & & \tau \\
2 & \to & 6 & & & & & & \\
\end{array}
\]

and \( I = \langle x, y, \gamma, \delta, \lambda, \rho, \mu, \kappa, \eta, \tau, \sigma, \epsilon \rangle \). Then \( A \) is a generalized gentle algebra where \( E_1 = \{4\} \), \( E_1 = \{5, 6\} \), \( E_2 = \{10\} \), \( E_2 = \{11, 12\} \), \( E_4 = \{3\} \) and \( O_4 = \{1, 2\} \).

In the order to show that generalized gentle algebras are derived tame, let us recall the definition of skewed-gentle algebras.

**Definition 3.7.** Let \( A = KQ/I \) be a quadratic string algebra. A vertex \( x \in Q_0 \) is said to be special if there is at most one arrow \( a \) starting at \( x \), at most one arrow \( b \) ending at \( x \) and if both exist then \( ab \notin I \).

Following [13] (see also [8, 18]), a basic algebra \( B \) is said to be skewed-gentle if \( B \cong A[D] \) where \( A = KQ/I \) is a gentle algebra and \( D \subset Q_0 \) is set of special vertices.

It is well-known that gentle ([9, 24]) and skewed-gentle algebras ([8, 18]) are derived tame.

**Lemma 3.8.** Let \( A = KQ/I \) be a generalized gentle algebra with \( n > 0 \) exceptional vertices and let \( D \subset Q_0 \) be a set of special but not ordinary vertices. Then \( A[D] \) is derived equivalent to \( B[S] \) where \( B = KA/I \) is a generalized gentle algebra with \( n - 1 \) exceptional vertices and \( S \subset \Delta_0 \) is a set of special but not ordinary vertices.

**Proof.** Fix \( I \) a set of paths with length two such that \( I = \langle I \rangle \). If \( x \in Q_0 \) is an exceptional vertex, then \( x \in E_i \) for any \( i \in \{1, \ldots, 6\} \), by definition.

**Case 1:** If \( x \in E_1 \), let \( x, \alpha, \gamma, \beta \) be arrows as in Definition 3.3 (1). We can suppose that \( x : 1 \to 3, \beta : 2 \to 3, \gamma : 3 \to 4 \) and \( \delta : 3 \to 5 \). In this way, \( x = 3 \) and the quiver \( Q \) has the shape
where \( Q' \) is the full subquiver of \( Q \) with \( Q'_0 = Q_0 \setminus \{1, 3, 4\} \). By hypotheses, \( 1, 4 \not\in D \). Moreover, since \( 3 \) is an exceptional vertex, then it is not gentle, in particular, \( 3 \) is not a special vertex. Hence \( D \subset Q_0 \setminus \{1, 3, 4\} \).

Since \( 4^+ = \emptyset \) and for any \( d \in D \) there is no arrow \( d \to 4 \), then \( A[D] \) is derived equivalent to \( \mu^-_1(A)[D] \) by Lemma 2.15. Moreover, \( \mu^-_1(A) \cong K\Omega/\langle I^+ \rangle \) where

\[
\Omega: 1 \xrightarrow{\alpha^*} 4 \xleftarrow{\beta^*} 3 \xrightarrow{\gamma^*} 2 \xleftarrow{\delta^*} 5
\]

and \( I^+ = (\mathcal{I} \cap KQ') \cup \{\alpha^*\gamma^*\delta\} \cup \{\mu\beta^* | \mu \in Q'_1, \mu\beta \in \mathcal{I}\} \cup \{\delta\mu | \mu \in Q'_1, \delta\mu \in \mathcal{I}\} \).

Again, since \( 1^- = \emptyset \) and for any \( d \in D \) there is no arrows \( 1 \to d \), then \( \mu^-_1(A)[D] \) is derived equivalent to \( \mu^+_1(\mu^-_1(A))[D] \) by Lemma 2.15. Moreover, a straightforward calculation shows that \( \mu^+_1(\mu^-_1(A)) \cong K\Gamma/\langle \mathcal{I} \rangle \) where

\[
\Gamma: 2 \xrightarrow{\tilde{\beta}} 4 \xleftarrow{\tilde{\gamma}} 3 \xrightarrow{\tilde{\delta}} 5
\]

and \( \mathcal{I} = (\mathcal{I} \cap KQ') \cup \{\tilde{\alpha}\tilde{\lambda} - \tilde{\gamma}\tilde{\delta}\} \cup \{\tilde{\mu}\tilde{\delta} | \mu \in Q'_1, \mu\delta \in \mathcal{I}\} \cup \{\tilde{\delta}\mu, \tilde{\lambda}\mu | \mu \in Q'_1, \delta\mu \in \mathcal{I}\} \).

Now, let \( \Delta \) be the full subquiver of \( \Gamma \) with \( \Delta_0 = \Gamma_0 \setminus \{1\} \) and let \( J \) be the ideal of \( K\Gamma/\langle \mathcal{I} \rangle \) generated by \( K\Delta \cap \mathcal{I} \).

\[
\Delta: 2 \xrightarrow{\tilde{\beta}} 4 \xleftarrow{\tilde{\gamma}} 3 \xrightarrow{\tilde{\delta}} 5
\]

Clearly \( B = K\Delta/J \) is a generalized gentle algebra with \( n-1 \) exceptional vertices and \( S = \{3\} \cup D \subset \Delta_0 \) is a set of special but not ordinary vertices. Since \( K\Gamma/\langle \mathcal{I} \rangle \cong B[3] \), then \( \mu^+_1(\mu^-_1(A))[D] \cong B[S] \).

Case 2: If \( x \in E_2 \), let \( \alpha, \beta, \gamma, \delta \) be arrows as in Definition 3.3 (2). We can suppose that \( \alpha : 1 \to 3, \beta : 2 \to 3, \gamma : 3 \to 4 \) and \( \delta : 3 \to 2 \). Note that this case is completely analogous to the previous case if we identify the vertices 2 and 5. In this way, \( x = 3 \) and the quiver \( Q \) has the shape

\[
Q: 1 \xleftarrow{\beta} 3 \xrightarrow{\gamma} 4
\]

where \( Q' \) is the full subquiver of \( Q \) with \( Q'_0 = Q_0 \setminus \{1, 3, 4\} \). By hypotheses we have that \( D \subset Q_0 \setminus \{1, 3, 4\} \).
Proceeding as in the case (1), we have that $A[D]$ is derived equivalent to $\mu_1^+ (\mu_4^- (A)) \cong K\Gamma / (\bar{I})$ where

$$\bar{I} = (I \cap KQ') \cup \{ \bar{\alpha} \lambda - \gamma \bar{\delta}, \bar{\delta} \bar{\beta}, \bar{\delta} \bar{\beta}, \bar{\delta} \bar{\mu}, \bar{\lambda} \bar{\mu}, \bar{\mu} \bar{\mu} \in Q_1, \bar{\mu} \bar{\mu} \in \mathcal{I} \} \cup \{ \bar{\delta} \bar{\mu} \bar{\lambda} \bar{\mu} \bar{\mu} \in Q_1, \bar{\delta} \bar{\mu} \in \mathcal{I} \}.$$

Let $B = K\Delta / J$ be the algebra where $\Delta$ is the full subquiver of $\Gamma$ such that $\Delta_0 = Q_0 \setminus \{1\}$ and let $J$ be the ideal of $K\Delta$ generated by $\bar{I} \cap K\Delta$. It is clear that $B$ is a generalized gentle algebra with $n-1$ exceptional vertices, $S = \{3\} \cup D \subset \Delta_0$ is a set of special but not ordinary vertices and $K\Gamma / (\bar{I}) \cong B[3]$. Hence $\mu_1^+ (\mu_4^- (A))[D] \cong B[S].$ 

Case 3: If $x \in E_3$, let $\alpha, \beta, \gamma$ be arrows as in Definition 3.3 (3). We may assume without loss of generality that $x : 1 \to 3, \beta : 2 \to 3$ and $\gamma : 3 \to 4$. Thus, $x = 3$ and the quiver $Q$ has the shape

where $x = 3$ and $Q'$ is the full subquiver of $Q$ with $Q'_0 = Q_0 \setminus \{1, 2, 3\}$. Note that $D \subset Q_0 \setminus \{1, 2, 3\}$ by hypotheses.

Let $\Delta$ be the full subquiver of $Q$ such that $\Delta_0 = Q_0 \setminus \{2\}$ and let $J$ be the ideal of $K\Delta$ generated by $\bar{I} \setminus \{\beta\}$. Clearly $B = K\Delta / J$ is a generalized gentle algebra with $n-1$ exceptional vertices, $S = \{3\} \cup D \subset \Delta_0$ is a set of special but not ordinary vertices and $A = B[1]$. Therefore $A[D] \cong B[S].$

Case 4: If $x \in E_4$, let $\alpha, \beta, \gamma$ be arrows as in Definition 3.3 (4). We may assume without loss of generality that $x : 1 \to 3, \beta : 2 \to 3$ and $\gamma : 3 \to 4$. Thus $x = 3, \alpha', \beta, \gamma \in \mathcal{I}$ and the quiver $Q$ has the shape

where and $Q'$ is the full subquiver of $Q$ with $Q'_0 = Q_0 \setminus \{1, 3, 4\}$. Note that $D \subset Q_0 \setminus \{1, 3, 4\}$ by hypotheses.

Since $\Delta^+ = \emptyset$ and for any $d \in D$ there is no arrow $d \to 4$, then $A[D]$ is derived equivalent to $\mu_4^- (A)[D]$ by Lemma 2.15. Moreover, a straightforward calculation shows that $\mu_4^- (A) \cong K\Omega / (\mathcal{I}^*)$ where

and $\mathcal{I}^* = (I \cap KQ') \cup \{ \mu \beta \mu \in Q'_1, \mu \beta \in \mathcal{I} \}.$
Since \( 1^- = \emptyset \) and for any \( d \in D \) there is no arrow \( 1 \to d \), then \( \mu_q^{-}(A)[D] \) is derived equivalent to \( \mu_1^{-}(\mu_q^{-}(A))[D] \) by Lemma 2.15. Furthermore, \( \mu_1^+(\mu_q^{-}(A)) \cong K\Gamma/\langle I \rangle \) where \( \Gamma \) is the quiver \\
\[
\begin{array}{c}
\begin{array}{c}
1
\end{array}
\end{array}
\]

Let \( A \) be the full subquiver of \( \Gamma \) such that \( \Delta_0 = \Gamma_0 \backslash \{1\} \) and let \( f \) be the ideal of \( K\Delta \) generated by \( K\Delta \cap I^+ \). Obviously \( B = K\Delta/J \) is a generalized gentle algebra with \( n - 1 \) exceptional vertices, \( S = \{3\} \cup D \subset \Delta_0 \) is a set of special but not ordinary vertices and \( K\Gamma/L = B[3] \). Therefore \( \mu_1^+(\mu_q^{-}(A))[D] \cong B[S] \).

Finally, if \( x \in E_5 \) or \( x \in E_6 \), then we proceed in a similar way as in cases 3 and 4, respectively. Thus we have that \( A[D] \) is derived equivalent to \( B[S] \), where \( B \) is a generalized gentle algebra with \( n - 1 \) exceptional vertices and \( S \) is a set of special but not ordinary vertices of \( B \).

**Proposition 3.9.** Let \( A = KQ/I \) be a generalized gentle algebra and let \( D \subset Q_0 \) be a set of special but not ordinary vertices. Then \( A[D] \) is derived equivalent to some skewed-gentle algebra. In particular, \( A[D] \) is derived tame.

**Proof.** We proceed by induction on the number of exceptional vertices of \( A \). If \( A \) does not have exceptional vertices, then \( A \) is a gentle algebra and \( A[D] \) is a skewed-gentle algebra by definition.

If \( A \) has \( n > 0 \) exceptional vertices, then by Lemma 3.8 we have that \( A[D] \) is derived equivalent to \( B[S] \), where \( B \) is a generalized gentle algebra with \( n - 1 \) exceptional vertices and \( S \) is a set of special but not ordinary vertices of \( B \). Hence, \( B[S] \) is derived equivalent to some skewed-gentle algebra by induction hypotheses.

**Proof of Theorem A.** For any generalized gentle algebra \( A \), we have that \( A = A[D] \) where \( D = \emptyset \). Therefore, the result follows by Proposition 3.9.

### 4. Quadratic string algebras

The objective of this section is to prove the Theorems B and D.

**Lemma 4.1.** Let \( A = KQ/I \) be a monomial algebra and suppose that there are arrows \( x, \beta, \gamma \in Q_1, x \neq \beta, \) such that \( s(x) = t(\gamma), t(x) = s(\beta) = t(\gamma) \) and \( x\gamma, \gamma\beta, \beta\gamma \in I \) (resp. \( s(x) = s(\beta) = t(\gamma), t(x) = s(\gamma) \) and \( x\gamma, \gamma\beta, \beta\gamma \in I \)). Then \( A \) is derived wild.

**Proof.** We assume that \( s(x) = t(\gamma), t(x) = s(\beta) = t(\gamma) \) and \( x\gamma, \gamma\beta, \beta\gamma \in I \) (the other case is similar). By construction, the Galois covering of \( A \) (induced by fundamental group \( \pi_1(Q, I) \)) has a full subcategory \( A \cong KQ/RQ^2 \) where \( Q \) has the following shape:

\[
\begin{array}{c}
\begin{array}{c}
\cdot
\end{array}
\end{array}
\]

Therefore, \( A \) is derived wild by Lemma 2.8.

**Lemma 4.2.** Let \( A = KQ/I \) be a derived tame monomial algebra and suppose there are arrows \( x, \beta, \gamma \in Q_1, x \neq \beta, \) such that \( t(x) = t(\beta) = s(\gamma) \) and \( x\gamma, \gamma\beta, \beta\gamma \in I \) (resp. \( s(x) = s(\beta) = t(\gamma) \) and \( x\gamma, \gamma\beta, \beta\gamma \in I \)). Then \( x \neq \gamma, \beta \neq \gamma \) and the vertices \( s(x), s(\beta), t(x) \) and \( t(\gamma) \) are pairwise distinct.
Proof. We assume that \( t(x) = t(\beta) = s(\gamma) \) and \( x_0, \beta_0, \gamma_0 \in I \) (the other case is similar).

If \( x = \gamma \), then \( s(x) = t(x) = t(\beta) \) and \( x_0, \beta_0, \gamma_0 \in I \), and so \( A \) is derived wild by Lemma 4.1, which is a contradiction. Hence \( x \neq \gamma \). In the same way \( \beta \neq \gamma \).

Suppose that \( s(x) = t(\gamma) \). Since \( I \) is admissible, there is \( n \geq 2 \) such that \( x_0^{-1} \notin I \) but \( x_0 \in I \). Let \( h = x_0^{-1} \), the Galois covering of \( A \) has a full subcategory \( \hat{A} \cong KQ/RQ^2 \) where \( \hat{Q} \) has the following shape:

\[
\hat{Q}: \quad \bullet \rightarrow h \bullet \rightarrow h \bullet \rightarrow h \bullet
\]

\( \gamma \)

Hence \( A \) is derived wild by Lemma 2.8, a contradiction. Thus \( s(x) \neq t(\gamma) \). Similarly we have that \( s(\beta) \neq t(\beta) \) and \( s(\delta) \neq t(\delta) \).

If \( s(x) = t(\beta) \), then \( \gamma x \notin I \) by Lemma 4.1. Let \( h = \gamma x \), the Galois covering of \( A \) has a full subcategory \( \hat{A} \cong K\hat{Q}/\hat{R}Q^2 \) given by following quiver:

\[
\hat{Q}: \quad \bullet \rightarrow h \bullet \rightarrow h \bullet \rightarrow h \bullet
\]

\( \beta \)

Hence \( A \) is derived wild by Lemma 2.8, a contradiction. Therefore \( s(x) \neq t(\gamma) \). Similarly we have that \( s(\beta) \neq t(\gamma) \).

Finally, if \( s(x) = s(\beta) \), then the Galois covering of \( A \) has a full subcategory \( \hat{A} \cong K\hat{Q}/\hat{R}Q^2 \), where:

\[
\hat{Q}: \quad \bullet \rightarrow \beta \bullet \rightarrow \alpha \bullet \rightarrow \beta \bullet \rightarrow \alpha \bullet
\]

\( \gamma \)

Again \( A \) is derived wild by Lemma 2.8, a contradiction. Therefore \( s(x) \neq s(\beta) \).

Lemma 4.3. Let \( A = KQ/I \) be a derived tame quadratic string algebra and suppose that there are arrows \( x, \beta, \gamma, \delta \in Q_1 \), \( x \neq \beta, \gamma \neq \delta \) such that \( t(x) = t(\beta) = s(\gamma) = s(\delta) \) and \( x_0, x_0, \beta_0, \gamma_0, \beta_0, \gamma_0 \in I \). Then \( Q \) has the following shape:

\[
Q: \quad \bullet \rightarrow \beta \bullet \rightarrow \alpha \bullet \rightarrow \gamma \bullet \rightarrow \delta \bullet
\]

In particular, \( A \) is a tree.

Proof. Note that the arrows \( x, \beta, \gamma, \delta \) are pairwise distinct and the vertices \( s(x), s(\beta), t(x), t(\gamma) \) and \( t(\delta) \) are also pairwise distinct by Lemma 4.2.

Suppose that \( \{x, \beta, \gamma, \delta\} \neq Q_1 \). Since \( Q \) is a connected quiver, then there is an arrow \( \lambda \in Q_1 \setminus \{x, \beta, \gamma, \delta\} \) such that \( s(\lambda) \) or \( t(\lambda) \) belongs to \( \{s(x), s(\beta), t(\gamma), t(\delta)\} \).
If $s(\lambda) = s(\mu)$ or $t(\lambda) = s(\mu)$, then the Galois covering of $A$ has a full subcategory $\tilde{A}$ which is isomorphic to the path algebra of some following quivers with relations:

$$\begin{align*}
(1) \quad \Delta: & \quad \begin{array}{c}
\bullet \\
\searrow \alpha \\
\downarrow \beta \\
\downarrow \delta
\end{array} & \quad \text{and } J = \mathcal{R}_\lambda^2; \\
(2) \quad \Gamma: & \quad \begin{array}{c}
\bullet \\
\searrow \lambda \\
\downarrow \alpha \\
\downarrow \beta \\
\downarrow \delta
\end{array} & \quad \text{and } H = \mathcal{R}_\Gamma^2; \\
(3) \quad \Omega: & \quad \begin{array}{c}
\bullet \\
\searrow \lambda \\
\downarrow \alpha \\
\downarrow \beta \\
\downarrow \delta
\end{array} & \quad \text{and } L = \{\rho \in \Omega_2 | \rho \neq \lambda \alpha\}.
\end{align*}$$

In the cases (1) and (2), we have that $A$ is derived wild by Lemma 2.8, a contradiction. If $\tilde{A} \cong K\Omega/L$, then since $s(\lambda) \neq \emptyset$ and there is no relation $\rho \in \{\rho \in \Omega_2 | \rho \neq \lambda \alpha\}$ such that $s(\rho) = s(\lambda)$, so $\tilde{A}$ is derived equivalent to $\mu_{s(\lambda)}^+(K\Omega/L) \cong K\Delta/J$, thus $A$ is derived wild by Lemma 2.8, a contradiction. Therefore $s(\lambda) \neq s(\mu)$ and $t(\lambda) \neq s(\mu)$.

Similarly we have that $s(\lambda), t(\lambda) \notin \{s(\beta), t(\gamma), t(\delta)\}$, a contradiction. Thus $Q_1 = \{\lambda, \beta, \gamma, \delta\}$. \hfill $\square$

**Proposition 4.4.** Let $A = KQ/I$ be a derived tame quadratic string algebra where $Q$ is not a tree. If $x \in Q_0$ is not gentle, then one of the following conditions holds:

1. There are $\alpha, \beta, \gamma \in Q_1$ such that $t(\alpha) = t(\beta) = s(\gamma) = x$; $s(\alpha), s(\beta), x$ and $t(\gamma)$ are pairwise distinct; $x^+ = \{\gamma\}$; $x_\gamma, \beta_\gamma \in I$.
2. There are $\alpha, \gamma, \delta \in Q_1$ such that $t(\alpha) = s(\gamma) = s(\delta) = x$; $s(\alpha), x, t(\gamma)$ and $t(\delta)$ are pairwise distinct; $x^- = \{\alpha\}$; $x_\gamma, x_\delta \in I$.
3. There are $\alpha, \beta, \gamma, \delta \in Q_1$ such that $t(\alpha) = t(\beta) = s(\gamma) = s(\delta) = x$; $s(\alpha), s(\beta), x, t(\gamma)$ and $t(\delta)$ are pairwise distinct; $x_\gamma, x_\delta, \beta_\gamma, \beta_\delta \in I$; $\beta_\delta \notin I$.
4. There are $\alpha, \beta, \gamma, \delta \in Q_1$ such that $t(\alpha) = t(\beta) = s(\gamma) = s(\delta) = x$; $s(\beta) = t(\delta)$; $s(\alpha), s(\beta), x$ and $t(\gamma)$ are pairwise distinct; $x_\gamma, x_\delta, \beta_\gamma, \delta_\beta \in I$; $\beta_\delta \notin I$.

**Proof.** Since $x$ is not a gentle vertex, then at least one of the following situations holds:

(i) There are $\alpha, \beta, \gamma \in Q_1$, $\alpha \neq \beta$, such that $t(\alpha) = t(\beta) = s(\gamma) = x$ and $x_\gamma, \beta_\gamma \in I$.
(ii) There are $\alpha, \gamma, \delta \in Q_1$, $\gamma \neq \delta$ such that $s(\alpha) = t(\gamma) = t(\delta) = x$ and $x_\gamma, x_\delta \in I$.

Suppose that the situation (i) holds. By Lemma 4.2 we have that $\alpha \neq \gamma, \beta \neq \gamma$ and the vertices $s(\alpha), s(\beta), x$ and $t(\gamma)$ are pairwise distinct. If $x^+ = \{\gamma\}$, then the condition (1) holds. In the other
hand, if there is an arrow $\delta \neq \gamma$ such that $s(\delta) = x$, then we must have $x\delta \in I$ or $\beta \delta \in I$. But, since $Q$ is not a tree, follows from Lemma 4.3 that we cannot have $x\delta, \beta \delta \in I$. Hence we can assume, without loss of generality, that $x\delta \in I$ but $\beta \delta \notin I$, thus $t(\delta) \neq t(\gamma)$ and $t(\delta) \neq s(\alpha)$ by Lemma 4.2. If $t(\delta) \neq s(\beta)$, then the condition (3) holds. But if $s(\beta) = t(\delta)$, then $\delta \beta \in I$ because $A$ has finite dimensional and $I$ is generated by paths of length two. Therefore the condition (4) holds.

Finally, a similar analysis to the case (ii) finish the proof. $\square$

**Lemma 4.5.** Let $A = KQ/I$ be an algebra where

\[ Q: 10 \xrightarrow{\omega_5} 9 \xrightarrow{\omega_4} 8 \xrightarrow{\omega_3} 7 \xrightarrow{\omega_2} 6 \xrightarrow{\omega_1} 1 \xrightarrow{\alpha} 3 \xrightarrow{\gamma} 4 \xrightarrow{\beta} 5 \]

$I = \langle \rho \in Q_2 | \rho \neq \beta \delta \rangle$ and the straight lines in $Q$ denotes an arrow oriented in either direction. Then $A$ is derived wild.

**Proof.** We define a bounded complex $T := \bigoplus_{i=1}^{10} T^i$ of projective $A$-modules. For $i \in Q_0 \setminus \{3, 4, 5\}$, let $T^i : 0 \longrightarrow P_i \longrightarrow 0$ be the stalk complex concentrated in degree 0. Moreover, consider the following complexes

\[ T^3 : \cdots 0 \longrightarrow P_4 \oplus P_5 \xrightarrow{[\gamma \beta]} P_3 \longrightarrow 0 \cdots , \]

\[ T^4 : \cdots 0 \longrightarrow P_4 \xrightarrow{\gamma} P_3 \longrightarrow 0 \cdots , \]

\[ T^5 : \cdots 0 \longrightarrow P_5 \xrightarrow{\delta} P_3 \longrightarrow 0 \cdots , \]

concentrated in degree 0 and $-1$. It is straightforward to check that $T$ is a tilting complex and that $(\text{End}(T))^{\text{op}} \cong K\Delta/J$ where

\[ \Delta: 10 \xrightarrow{\omega_5} 9 \xrightarrow{\omega_4} 8 \xrightarrow{\omega_3} 7 \xrightarrow{\omega_2} 6 \xrightarrow{\omega_1} 1 \xrightarrow{\alpha^*} 3 \xrightarrow{\gamma^*} 4 \xrightarrow{\beta^*} 2 \xrightarrow{\delta^*} 5 \]

and $J = \langle \Delta_2 \setminus \{x^*\gamma^*, x^*\delta^*\} \rangle$.

Since $4^+ = \emptyset, 5^+ = \emptyset$ and there is no relation $\rho \in \Delta_2 \setminus \{x^*\gamma^*, x^*\delta^*\}$ such that $t(\rho) \in \{4, 5\}$, it follows that $K\Delta/J$ is derived equivalent to $\mu_5^-(\mu_4^- (K\Delta/J)) \cong K\Gamma/R^2_\Gamma$ where $\Gamma$ is the following quiver:

\[ \Gamma: 10 \xrightarrow{\omega_5} 9 \xrightarrow{\omega_4} 8 \xrightarrow{\omega_3} 7 \xrightarrow{\omega_2} 6 \xrightarrow{\omega_1} 1 \xrightarrow{\alpha^*} 3 \xleftarrow{\tilde{\beta}} 4 \xleftarrow{\tilde{\gamma}} 2 \xleftarrow{\delta^*} 5 \]

Note that $\mu_5^-(\mu_4^- (K\Delta/J))$ is derived wild by Theorem 2.4. Therefore $A$ is also derived wild. $\square$

**Lemma 4.6.** Let $Q$ be a connected and locally finite tree with an infinite number of vertices. For all $x \in Q_0$ and $n \geq 1$ there is a reduced walk $w$ such that $s(w) = x$ and $l(w) = n$. 

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Proof. Fix \( x \in Q_0 \). For every \( k \geq 0 \) we define the set
\[
U_k = \{ y \in Q_0 \mid w : x \to y \text{ a reduced walk, } l(w) = k \}.
\]
Since \( Q \) is connected and has no cycles, then for every vertex \( y \in Q_0 \) there is only one reduced walk \( w : x \to y \). In particular, \( Q_0 = \bigcup_{k \geq 0} U_k \) and \( U_k \cap U_j = \emptyset \), if \( k \neq j \). Hence \( U_k \) is a finite set for all \( k \geq 0 \).

We use induction on \( k \). Since \( U_0 = \{ x \} \), the statement holds for \( k = 0 \). Let \( k \geq 1 \) and suppose that \( U_{k-1} \) is finite. For every \( y \in Q_0 \) we define:
\[
V_y = \{ z \in Q_0 \mid z \in U_{k-1}, w : y \to z \text{ or } z \to y \}.
\]
It is clear that if \( z \in U_k \), then \( z \in V_y \) for some \( y \in U_{k-1} \). Thus \( U_k \subseteq \bigcup_{y \in U_{k-1}} V_y \). Moreover, since \( Q \) is locally finite then \( V_y \) is finite for all \( y \in Q_0 \). Hence \( \bigcup_{y \in U_{k-1}} V_y \) and \( U_k \) are finite sets. This proves the statement.

Now, suppose there is \( k_0 \geq 0 \) such that \( U_{k_0} = \emptyset \), so \( U_k = \emptyset \) if \( k \geq k_0 \). Since \( Q_0 = \bigcup_{k \geq 0} U_k = \bigcup_{0 \leq i < k_0} U_i \) is an infinite set, then there is \( 0 < k < k_0 \) such that \( U_k \) is an infinite set, which is a contradiction. Hence \( U_k \neq \emptyset \) for any \( k \geq 0 \).

Finally, given \( n \geq 1 \), take \( y \in U_n \). Thus, there is a reduced walk \( w : x \to y \) with \( l(w) = n \). \( \square \)

Lemma 4.7. Let \( Q \) be a connected and locally finite tree with infinite vertices, and let \( I \) be an admissible ideal of \( KQ \). Suppose that there is \( m \geq 2 \) such that \( R^n_m \subseteq I \). Then for all vertex \( x \) of \( Q \) and \( n \geq 1 \), there is a set of vertices \( \{ x = x_0, x_1, \ldots, x_n \} \subseteq Q_0 \) such that they define a full subcategory \( B \) of \( A = KQ/I \), such that \( B \cong K\Gamma/R^2_\Gamma \) where the underlying graph of \( \Gamma \) has the following shape:

\[
\Gamma: x_0 \overrightarrow{a_0} x_1 \overrightarrow{a_1} \cdots \overrightarrow{a_{n-1}} x_{n-1} \overrightarrow{a_n} x_n.
\]

Proof. Given \( x \in Q_0 \) and \( n \geq 1 \), then by Lemma 3.6 there is a reduced walk \( w \) such that \( s(w) = x \) and \( l(w) = mn \). Let \( \{ y_0 = x, y_1, \ldots, y_{mn} \} \subseteq Q_0 \) be the the set of vertices witch occur on \( w \), thus we can write \( w = z_0^0 z_1^1 \cdots z_{mn-1}^{mn-1} \), where \( x_i \in Q_1, e_i \in \{ 1, -1 \} \), \( s(z_i^e) = y_i \) and \( t(z_i^e) = y_{i+1} \) for all \( 0 \leq i \leq mn - 1 \).

Let \( l_0 = 0 \), we define \( l_{i+1} \) as follows:

- If \( e_i = 1 \), then \( x_i : y_i \to y_{i+1} \). In this case, \( l_{i+1} \) is the biggest integer such that the path \( x_i x_{i+1} \cdots x_{l_{i+1}-1} : y_i \to y_{l_{i+1}} \) is defined and it does not belong to \( I \);
- If \( e_i = -1 \), then \( x_i : y_{i+1} \to y_i \). In this case, \( l_{i+1} \) is the biggest integer such that the path \( x_{l_{i+1}-1} \cdots x_{i+1} x_i : y_{l_{i+1}} \to y_i \) is defined and it does not belong to \( I \).

Since every path of \( Q \) whose length is greater than \( m-1 \) belongs to \( I \), then for any \( i \) we have \( l_{i+1} - l_i < m \). Thus we have:

\[
l_n = l_n + (-l_{n-1} + l_{n-1}) + (-l_{n-2} + l_{n-2}) + \cdots + (-l_1 + l_1) + (-l_0 + l_0)
= (l_n - l_{n-1}) + (l_{n-1} - l_{n-2}) + \cdots + (l_1 - l_0) + l_0 < nm.
\]

Hence we can define \( x_i := y_i \) for all \( 0 \leq i \leq n \). Moreover, for \( 0 \leq i \leq n-1 \) we define the path \( w_i \) as follows:
\[
\begin{aligned}
    w_i = a_{l_i} \cdots a_{l_{i+1}} : x_i &\rightarrow x_{i+1}, & \text{if } \epsilon_{l_i} = 1; \\
    w_i = a_{l_{i+1}} \cdots a_{l_i} : x_{i+1} &\rightarrow x_i, & \text{if } \epsilon_{l_i} = -1.
\end{aligned}
\]

Note that for any \(0 \leq i \leq n-1\), the paths \(w_iw_{i+1}\) and \(w_{i+1}w_i\) are not defined or they belong to \(I\).

Let \(B\) be the full subcategory of \(A\) defined by \(\{x_0, \ldots, x_n\}\). Hence \(B \cong K\Gamma/R^2_\Gamma\) where the underline graph of \(\Gamma\) is given by

\[\Gamma : x = x_0 \xrightarrow{w_0} x_1 \xrightarrow{w_1} \cdots \xrightarrow{w_{n-1}} x_{n-1} \xrightarrow{w_n} x_n.\]

**Lemma 4.8.** Let \(A = KQ/I\) be a derived tame monomial algebra. Suppose that the Galois covering \(KQ/\tilde{I}\) of \(A\) has a full subcategory \(\tilde{KQ}/\tilde{I}\) where \(\tilde{Q}\) has the following shape

and for \(i = 1, 2, 3\), \(\tilde{Q}^{(i)}\) is a connected full subquiver of \(\tilde{Q}\) such that \(\tilde{Q}_0^{(i)} \cap \tilde{Q}_0^{(j)} = \emptyset\) if \(i \neq j\).

a. If \(\tilde{Q}^{(1)}\) has an infinite number of vertices, then \(s(\beta)^- = \emptyset, t(\gamma)^+ = \emptyset, s(\beta)^+ = \{\beta\}\) and \(t(\gamma)^- = \{\gamma\}\).

b. If \(\tilde{Q}^{(3)}\) has an infinite number of vertices, then \(s(\alpha)^- = \emptyset, s(\beta)^- = \emptyset, s(\alpha)^+ = \{\alpha\}\) and \(s(\beta)^+ = \{\beta\}\).

**Proof.** We will show only (a), the other is similar.

Since \(I\) is generated by monomial relations, then \(\tilde{Q}\) is a connected and locally finite tree. Moreover, since \(I\) is an admissible ideal of \(KQ\) and \(Q\) is finite, then there is an integer \(m \geq 2\) such that \(R_Q^m \subset \tilde{I}\).

If \(s(\beta)^+ \neq \{\beta\}\) or \(s(\beta)^- \neq \emptyset\), then since \(Q^{(1)}\) is an infinite set, it follows from **Lemma 4.7** that \(KQ/\tilde{I}\) has a full subcategory \(\tilde{A}\) such that \(\tilde{A}\) is isomorphic to path algebra of some of the following quivers with relations:

1. \(\Delta: \bullet \xrightarrow{o_5} \bullet \xrightarrow{o_4} \bullet \xrightarrow{o_3} \bullet \xrightarrow{o_2} \bullet \xrightarrow{o_1} \bullet \xrightarrow{o_0} \bullet \xrightarrow{\beta} \bullet \xrightarrow{\lambda} \bullet \xrightarrow{\gamma} \bullet \)

   and \(J = R^2_\Delta\);

2. \(\Omega: \bullet \xrightarrow{o_5} \bullet \xrightarrow{o_4} \bullet \xrightarrow{o_3} \bullet \xrightarrow{o_2} \bullet \xrightarrow{o_1} \bullet \xrightarrow{o_0} \bullet \xrightarrow{\beta} \bullet \xrightarrow{\lambda} \bullet \xrightarrow{\gamma} \bullet \)

   and \(H = R^2_\Omega\);

3. \(\Upsilon: \bullet \xrightarrow{o_5} \bullet \xrightarrow{o_4} \bullet \xrightarrow{o_3} \bullet \xrightarrow{o_2} \bullet \xrightarrow{o_1} \bullet \xrightarrow{o_0} \bullet \xrightarrow{\beta} \bullet \xrightarrow{\lambda} \bullet \xrightarrow{\gamma} \bullet \)

   and \(L = \langle \gamma \beta \rangle\).
In the cases (1) and (2), we have that $A$ is derived wild by Lemma 2.8, which is a contradiction. If $\bar{A} \cong KY/L$, then since $s(\lambda) = \emptyset$ and there is no relation $\rho \in \mathcal{Y}_2 \setminus \{\lambda, \beta\}$ such that $s(\rho) = s(\lambda)$, then $\bar{A}$ is derived equivalent to $\mu_{\bar{Y}(\lambda)}(KY/L) \cong K\Delta/J$. Hence $A$ is derived wild by Lemma 2.8, which is a contradiction. Therefore $s(\beta)^+ = \{\beta\}$ and $s(\beta)^- = \emptyset$.

If $t(\gamma)^- \neq \{\gamma\}$ or $t(\gamma)^+ \neq \emptyset$, then since $Q_{0(1)}$ is an infinite set, it follows from Lemma 4.7 that $KQ/I$ has a full subcategory $\bar{B}$ which is isomorphic to path algebra of some of the following quivers with relations:

\[
\begin{align*}
\{1\}' & \quad \Delta': \quad \bullet & \omega_5 & \omega_4 & \omega_3 & \omega_2 & \omega_1 & \omega_0 & \gamma & \lambda & \bullet \\
& & & & & & & & & & \beta & \\
& & & & & & & & & & \bullet \\
\text{and } & & \mathcal{J} = \mathcal{R}_{\Delta}^3; \\
\{2\}' & \quad \Omega': \quad \bullet & \omega_5 & \omega_4 & \omega_3 & \omega_2 & \omega_1 & \omega_0 & \gamma & \lambda & \bullet \\
& & & & & & & & & & \beta & \\
& & & & & & & & & & \bullet \\
\text{and } & & \mathcal{H} = \mathcal{R}_{\Omega}^3; \\
\{3\}' & \quad \Upsilon': \quad \bullet & \omega_5 & \omega_4 & \omega_3 & \omega_2 & \omega_1 & \omega_0 & \gamma & \lambda & \bullet \\
& & & & & & & & & & \beta & \\
& & & & & & & & & & \bullet & \\
\text{and } & & \mathcal{L} = \{\mathcal{Y}_2 \setminus \{\gamma, \lambda\}\}.
\end{align*}
\]

Again, this implies that $A$ is derived wild by Lemma 2.8, a contradiction. Therefore $t(\gamma)^- = \{\gamma\}$ and $t(\gamma)^+ = \emptyset$.

\[\square\]

**Lemma 4.9.** Let $A = KQ/I$ be a derived tame quadratic string algebra where $Q$ is not a tree. Suppose that there are $x, \beta, \gamma, \delta \in Q$, and $x \in Q_0$ such that $t(x) = t(\beta) = s(\gamma) = s(\delta) = x$.

a. If the vertices $s(x), s(\beta), x, t(\gamma)$ and $t(\delta)$ are pairwise distinct, $x, \beta, \gamma, \delta \in I$ but $\beta \delta \notin I$, then $s(x)^+ = \{x\}, s(x)^- = \emptyset$, $t(\gamma)^+ = \emptyset$ and $t(\gamma)^- = \{\gamma\}$. In particular, $x \in E_1$.

b. If the vertices $s(x), s(\beta), x$ and $t(\gamma)$ are pairwise distinct, $s(\beta) = t(\delta), x, \beta, \gamma, \delta \in I$ but $\beta \delta \notin I$, then $s(x)^+ = \{x\}, s(x)^- = \emptyset, t(\gamma)^+ = \emptyset$ and $t(\gamma)^- = \{\gamma\}$. In particular, $x \in E_2$.

**Proof.** We only prove (a), the proof of (b) is similar.

Let $KQ/I$ be the Galois covering of $A$. Since $Q$ has cycles and $I$ is generated by monomial relations, then $\bar{Q}$ is a connected and locally finite tree with infinite number of vertices. Moreover, since $Q$ is a finite quiver and $I$ is an admissible ideal of $KQ$, then there is $m \geq 2$ such that $\mathcal{R}_{\bar{Q}}^m \subset I$, hence $\mathcal{R}_{\bar{Q}}^m \subset I$.

By construction $\bar{Q}$ has the following shape:

\[
\begin{align*}
\bar{Q}: \quad & (\bar{Q}^{(1)}', \gamma), \\
& (\bar{Q}^{(2)}', \beta, \delta), \\
\end{align*}
\]

\[
\begin{align*}
\bar{Q}: \quad & (\bar{Q}^{(3)}), \\
& (\bar{Q}^{(4)}).
\end{align*}
\]
where for \(1 \leq i \leq 4\), \(\tilde{Q}^{(i)}\) is a full and connected subquiver of \(\tilde{Q}\) such that \(\tilde{Q}_0^{(i)} \cap \tilde{Q}_0^{(j)} = \emptyset\) if \(i \neq j\). Moreover, since \(\tilde{Q}\) has an infinite number of vertices, at least one of the subquivers \(\tilde{Q}^{(i)}\) has infinitely many vertices where \(i = 1, 2, 3, 4\).

If \(\tilde{Q}_0^{(1)}\) is an infinite set, then by Lemma 4.7 we can choose vertices of \(\tilde{Q}\) such that they define a full subcategory \(\tilde{A} = KQ/\tilde{I}\) where

\[
\begin{array}{c}
\tilde{Q}:
\begin{array}{c}
\bullet \circ \circ \circ \circ \circ \circ \\
\circ \circ \circ \circ \circ \circ \\
\circ \circ \circ \circ \circ \circ \\
\circ \circ \circ \circ \circ \circ \\
\circ \circ \circ \circ \circ \circ \\
\circ \circ \circ \circ \circ \circ
dots
\end{array}
\end{array}
\begin{array}{c}
\beta \\
\alpha \\
\delta \\
\gamma \\
\theta
\end{array}
\]

and \(\tilde{I} = (\tilde{Q}_2 \setminus \{\beta \delta\}\}. By Lemma 4.5 we get that \(KQ/\tilde{I}\) is derived wild, thus \(A\) is derived wild by Lemma 2.8, which is a contradiction. Hence \(\tilde{Q}_0^{(1)}\) is a finite set. Moreover, the same argument applied to \(A^{op}\) shows that \(\tilde{Q}_0^{(3)}\) is a finite set. Therefore \(\tilde{Q}_0^{(2)}\) or \(\tilde{Q}_0^{(4)}\) must be an infinite set.

If \(\tilde{Q}_0^{(2)}\) is an infinite set, then \(s(x)^\plus{} = \{x\}, s(x)^\minus{} = \emptyset, t(\gamma)^\minus{} = \{\gamma\}\) and \(t(\gamma)^\plus{} = \emptyset\) by Lemma 4.8(a). Moreover, the same argument applied to \(A^{op}\) shows that if \(\tilde{Q}_0^{(4)}\) is an infinite set, then \(s(x)^\plus{} = \{x\}, s(x)^\minus{} = \emptyset, t(\gamma)^\minus{} = \{\gamma\}\) and \(t(\gamma)^\plus{} = \emptyset\).

\begin{lemma}
Let \(A = KQ/\tilde{I}\) be a derived tame quadratic string algebra where \(Q\) is not a tree. Suppose that there are \(x, \beta, \gamma \in Q_1\) and \(x \in Q_0\) such that the vertices \(s(x), s(\beta), x\) and \(t(\gamma)\) are pairwise distinct, \(x^\gamma, \beta^\gamma \in \tilde{I}\), \(x^\gamma \neq t(\beta) = s(\gamma) = x\).

\begin{enumerate}
\item If \(t(\gamma)^\minus{} \neq \{\gamma\}\) or \(t(\gamma)^\plus{} = \emptyset\), then \(s(x)^\plus{} = \{x\}, s(x)^\minus{} = \emptyset, s(\beta)^\plus{} = \{\beta\}\) and \(s(\beta)^\minus{} = \emptyset\). In particular, \(x \in E_3\).
\item If \(s(x)^\plus{} \neq \{x\}\) or \(s(x)^\minus{} = \emptyset\), then \(s(\beta)^\plus{} = \{\beta\}, s(\beta)^\minus{} = \emptyset, t(\gamma)^\plus{} = \emptyset\) and \(t(\gamma)^\minus{} = \{\gamma\}\). In particular, \(x \in E_4\).
\end{enumerate}
\end{lemma}

\begin{proof}
Let \(KQ/\tilde{I}\) be the Galois covering of \(A\). Since \(Q\) has cycles and \(I\) is generated by monomial relations, then \(\tilde{Q}\) is a connected and locally finite tree with an infinite number of vertices. Moreover, since \(Q\) is a finite quiver and \(I\) is an admissible ideal of \(KQ\), then there is \(m \geq 2\) such that \(R_0^\circ \subset I\), hence \(R_0^\circ \subset \tilde{I}\).

By construction \(\tilde{Q}\) has the following shape

where for \(i = 1, 2, 3\), \(\tilde{Q}^{(i)}\) is a full and connected subquiver of \(\tilde{Q}\) such that \(\tilde{Q}_0^{(i)} \cap \tilde{Q}_0^{(j)} = \emptyset\) if \(i \neq j\). Moreover, since \(\tilde{Q}\) has an infinite many of vertices, at least one of the subquivers \(\tilde{Q}^{(i)}\) has infinitely many vertices where \(i = 1, 2, 3\).

If \(t(\gamma)^\minus{} \neq \{\gamma\}\) or \(t(\gamma)^\plus{} = \emptyset\), then \(\tilde{Q}_0^{(1)}\) and \(\tilde{Q}_0^{(2)}\) are finite sets by Lemma 4.8 (a). Hence \(\tilde{Q}_0^{(3)}\) is an infinite set, and so \(s(x)^\plus{} = \{x\}, s(x)^\minus{} = \emptyset, s(\beta)^\plus{} = \{\beta\}\) and \(s(\beta)^\minus{} = \emptyset\) by Lemma 4.8 (b). Thus the item (a) follows.

If \(s(x)^\plus{} \neq \{x\}\) or \(s(x)^\minus{} = \emptyset\), then \(\tilde{Q}_0^{(2)}\) and \(\tilde{Q}_0^{(3)}\) are finite sets by Lemma 4.8. Hence \(\tilde{Q}_0^{(1)}\) is an infinite set, and so \(s(\beta)^\plus{} = \{\beta\}, s(\beta)^\minus{} = \emptyset, t(\gamma)^\plus{} = \emptyset\) and \(t(\gamma)^\minus{} = \{\gamma\}\) by Lemma 4.8 (a). Thus, the item (b) is done.
\end{proof}
Lemma 4.11. Let $A = KQ/I$ be a derived tame quadratic string algebra where $Q$ is not a tree. Suppose that there are $x, \gamma, \delta \in Q_1$ and $x \in Q_0$ such that $t(x) = s(\delta) = s(\gamma) = x$, the vertices $s(x), x, t(\delta)$ and $t(\gamma)$ are pairwise distinct, $\gamma \delta, \alpha \delta \in I$ and $x^- = \{x\}$.

a. If $s(x) = \{x\}$ or $s(x) = \{x\} = \{x\}$, then $t(\delta) = \emptyset, t(\delta) = \{\delta\}, t(\gamma) = \{\gamma\}$, and $t(\gamma) = \{\gamma\}$, and $t(\gamma) = \{\gamma\}$. In particular, $x \in E_5$.

b. If $t(\gamma) = \emptyset$ or $t(\gamma) = \emptyset$, then $s(x) = \{x\}, s(x) = \emptyset$, $t(\delta) = \emptyset$ and $t(\delta) = \{\delta\}$. In particular, $x \in E_6$.

Proof. It is sufficient to apply the Lemma 4.10 to $A^{op}$.

Now we can prove the Theоремs B and D.

Proof of Theorем B. If $A$ is a generalized gentle algebra, then $A$ is derived tame by Theорem A.

In the other hand, if $A$ is a derived tame string quadratic algebra and let $x$ be the non gentle vertex of $A$, then one of the conditions (1), (2), (3) or (4) described in Proposition 4.4 holds.

If $x$ satisfy the condition (1), then there is arrows $x, \beta, \gamma \in Q_1$ such that $t(x) = t(\beta) = s(\gamma) = x$, $x^+ = \{x\}, x, s(\beta), x$ and $t(\gamma)$ are pairwise distinct, and $x \gamma, \beta \gamma \in I$. Since $Q$ is connected and it has cycles, then there is an arrow $\lambda \in Q_1 \{x, \beta, \gamma\}$ such that $s(\lambda) \in \{s(x), s(\beta), t(\gamma)\}$ or $t(\lambda) \in \{s(x), s(\beta), t(\gamma)\}$. Thus $x$ belongs to $E_5$ or $E_6$.

Finally, if $x$ satisfy the condition (2), we have a dual situation to the condition (1). Hence, it follows from Lemma 4.11 that $x$ belongs to $E_5$ or $E_6$.

Therefore, we have that every non gentle vertex of $A$ is an exceptional vertex. Hence $A$ is a generalized gentle algebra.

Proof of Theorем D. Let $A$ be a quadratic string algebra.

Suppose that $A$ is derived tame. If $A$ is a tree, then it follows by [10, Theorem 1.1] that its Euler form $\chi_A$ is non negative. But, if $A$ is not a tree, then $A$ is generalized gentle algebra by Theorем B.

Conversely, if $A$ is tree and $\chi_A$ is non negative, then $A$ is derived tame by Theorем 2.1 of [10]. Moreover, if $A$ is generalized gentle algebra, then $A$ is derived tame by Theorем A.

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