Hausdorff dimensions and Hitting probabilities for some general Gaussian processes

FREDERI VIENS
Department of Statistics and Probability, Michigan State University, USA
e-mail: viens@msu.edu

MOHAMED ERRAOUI
Department of mathematics, Faculty of science El Jadida, Chouaib Doukkali University, Morocco
e-mail: erraoui@uca.ac.ma

YOUSSEF HAKIKI
Department of mathematics, Faculty of science Semlalia, Cadi Ayyad University, 2390 Marrakesh, Morocco
e-mail: youssef.hakiki@ced.uca.ma

Abstract
Let $B$ be a $d$-dimensional Gaussian process on $\mathbb{R}$, where the component are independents copies of a scalar Gaussian process $B_0$ on $\mathbb{R}_+$ with a given general variance function $\gamma^2(r) = \text{Var}(B_0(r))$ and a canonical metric $\delta(t,s) := (\mathbb{E}(B_0(t) - B_0(s))^2)^{1/2}$ which is commensurate with $\gamma(t - s)$. We provide some general condition on $\gamma$ so that for any Borel set $E \subset [0,1]$, the Hausdorff dimension of the image $B(E)$ is constant a.s., and we explicit this constant. Also, we derive under some mild assumptions on $\gamma$ an upper and lower bounds of $\mathbb{P}\{B(E) \cap F \neq \emptyset\}$ in terms of the corresponding Hausdorff measure and capacity of $E \times F$. Some upper and lower bounds for the essential supremum norm of the Hausdorff dimension of $B(E) \cap F$ and $E \cap B^{-1}(F)$ are also given in terms of $d$ and the corresponding Hausdorff dimensions of $E \times F$, $E$, and $F$.

Keywords: Gaussian process, Hitting probabilities, Hausdorff dimension, Capacity.

Mathematics Subject Classification 60J45, 60G17, 28A78, 60G15

1 Introduction
This paper aims to study some fractal properties for Gaussian processes that have a general covariance structure, such as the Hausdorff dimension of the image set, the hitting probabilities problem, and the Hausdorff dimension of the level sets and the pre-images. The motivation arises from the high irregularity presented by certain types of Gaussian processes, see, for example, the family of processes $B^\gamma$ defined

---

1Partially supported by the National Science Foundation award number DMS 1811779.
2Supported by National Center for Scientific and Technological Research (CNRST)
in [21], which associate to each appropriate function $\gamma$ its corresponding process $B^\gamma$, which is defined through the following Volterra representation

$$
B^\gamma(t) := \int_0^t \sqrt{\left( \frac{d\gamma^2}{dt} \right)} (t - s) dW(s),
$$

(1.1)

where $W$ is a standard Brownian motion. In the particular case $\gamma(r) := \log^{-\beta}(1/r)$, where $\beta > 1/2$, the process $B^\gamma$ is an element of a class of Gaussian processes called the class of logarithmic Brownian motions, which will be considered as the highly irregular class of continuous Gaussian processes for our study (there are some other highly irregular classes of Gaussian processes, but they are not continuous). In that case, considering the well known fact that the process $B^\gamma$ should have the function $h : r \mapsto \gamma(r) \log^{1/2}(1/r)$ as uniform modulus of continuity, up to a deterministic constant, we may deduce that $B^\gamma$ is no longer Hölder continuous for any order $\alpha \in (0, 1)$, which illustrate a high level of irregularity. So, most of the results in the literature about the fractal properties for Gaussian processes, for the hitting probabilities problem see for example [14, 25], and for the Hausdorff dimension of the image and the graph sets see [10], do not apply to the case of logarithmic Brownian motion. Because the conditions assumed in those previous works restrict the processes studied to the Hölder continuity scale type, i.e. when $\gamma(r) \lesssim r^\alpha$ for some $\alpha \in (0, 1)$. Since there are many regularity scales between the Hölder continuity scale and the logarithmic scale mentioned above, we need some precise quantitative results on the fractals properties for a class of Gaussian processes $B$ that are satisfying only the condition (2.1), those results hold only under some general conditions on the variance function $\gamma$ that are already satisfied by large class of processes within and/or beyond the Hölder scale.

We note that when $\gamma^2$ is of class $C^2$ on $(0, \infty)$, $\lim_{r \to 0} \gamma = 0$, and $\gamma^2$ is increasing and concave, the Gaussian processes $B^\gamma$ defined above in (1.1) satisfy (21) with $l = 2$ (see Proposition 1 in [21]). This model of concentrating only on the commensurability condition (2.1) has the power to relax the restriction of stationarity of increments (see Proposition 5 in [21]), and the Hölder continuity, as illustrated above by the logarithmic Brownian motion. Another interesting class of Gaussian processes with non-stationary increments, which satisfy (2.1), are the solutions of the linear stochastic heat equation, see those studied in [24].

In section 2, we give some general hypotheses on $\gamma$, that are important to ensure some interesting properties for process $B$, as the two-points local nondeterminism in (2.3), and the estimation (2.4), in order to provide the optimal upper and lower bounds for both of the Hausdorff dimension of the image $B(E)$ and the hitting probabilities in the sections 3 and 4, respectively.

The objective of section 3 in this paper is to find when the variance function $\gamma$ is strictly concave in a neighborhood of 0, an explicit formula for the Hausdorff dimension of the image $B(E)$ where $E \subset \mathbb{R}_+$ is a Borel set. Hawkes resolved this problem at first in his paper [10], but only in the case of stationary increments and under the strong condition $\text{ind}(\gamma) > 0$, where $\text{ind}(\cdot)$ is the lower index of $\gamma$, it will be defined in (3.12). Our aim in this section is to relax those two last conditions. Precisely, we will give a lower bound for the Hausdorff dimension of the image $\text{dim}_{Euc}(B(E))$, and under the condition (3.13), we show that the random variable $\text{dim}_{Euc}(B(E))$ is almost surely constant by proving that the lower bound is also an upper bound. This constant is expressed as the minimum between $d$ and $\text{dim}_\delta(E)$ where $\text{dim}_\delta(\cdot)$ denote the Hausdorff dimension associated to the canonical metric $\delta$. We note that Lemma 3.3 may illustrates that (3.13) is general than the condition of Hawkes ”$\text{ind}(\gamma) > 0$”, and is even satisfied by an important class of functions zero lower index as we will see bellow.

In section 4, our investigation will be focused on providing upper and lower bounds for the probabilities of the event $\{B(E) \cap F \neq \emptyset\}$ where $E \subset \mathbb{R}_+$ and $F \subset \mathbb{R}^d$ are Borel sets, in terms of $\mathcal{H}^d_{\rho_b} (E \times F)$ and $\mathcal{C}_{\rho_b,d} (E \times F)$, respectively. $\mathcal{H}^d_{\rho_b} (\cdot)$ and $\mathcal{C}_{\rho_b,d} (\cdot)$ denote $d$-dimensional Hausdorff measure and the Bessel-Riesz type capacity of order $d$ with respect to an appropriate metric $\rho_b$ depending on the canonical metric.
δ, which will be defined in the sequel. Similar to section 3, the lower bound is given just under the condition of strict concavity near 0 of γ. However, the upper bound can be obtained with the help of Lemmas 2.2 and 2.3 under the mild condition (2.5) on γ, which is stronger than (3.13), but it remains satisfied by almost all examples of interest. A less optimal upper bound for the probability of the above event is also given under the weaker condition (3.13) in terms of $\mathcal{H}^{d/\rho^c} (E \times F)$ for any $\varepsilon > 0$ small enough.

Our motivation for section 5 is the following: when the random intersections $B(F) \cap F$ and $B^{-1}(F) \cap E$ are non-empty with positive probability, it is natural to ask about their Hausdorff dimensions. In general, those Hausdorff dimensions are not necessarily a.s. constant, like it was illustrated by Khoshnivisan and Xiao through an example -due to Gregory Lawler- in the introduction of their paper [22]. For this reason we seek to give some upper and lower bounds of the $L^\infty$-norm of those Hausdorff dimensions. We note that when $B$ is a standard Brownian motion, Khoshnivisan and Xiao have obtained in [22] an explicit formula for the essential supremum of the Hausdorff dimension of $E \cap B^{-1}(F)$ and $B(E) \cap F$ in terms of $d$ and $\dim_{\rho^c} (E \times F)$. A generalization of their result to the fractional Brownian case was proven by Erraoui and Hakiki in [16], by giving only an upper and lower bound. Our goal is to generalize those results to the Gaussian processes satisfying (2.1) and under general conditions on $\gamma$.

## 2 Preliminaries

Let $\{B_0(t), t \in \mathbb{R}_+\}$ be a real valued centered continuous Gaussian process defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, defining the canonical metric $\delta$ of $B$ on $(\mathbb{R}_+)^2$ by

$$\delta(s, t) := (\mathbb{E}(B_0(s) - B_0(t))^2)^{1/2}.$$

Let $\gamma$ be continuous increasing function on $\mathbb{R}_+$ (or possibly only on a neighborhood of 0 in $\mathbb{R}_+$), such that $\lim_0 \gamma = 0$ and for some constant $t \geq 1$ we have, for all $s, t \in \mathbb{R}_+$

$$\begin{cases} 
\mathbb{E}(B_0(t))^2 = \gamma^2(t) \\
and

1/\sqrt{t} \gamma(|t-s|) \leq \delta(t, s) \leq \sqrt{t} \gamma(|t-s|).
\end{cases} \tag{2.1}$$

Now, we consider the $\mathbb{R}^d$-valued process $B = \{B(t) : t \in \mathbb{R}_+\}$ defined by

$$B(t) = (B_1(t), ..., B_d(t)), \quad t \in \mathbb{R}_+, \tag{2.2}$$

where $B_1, ..., B_d$ are independent copies of $B_0$. Let us consider the following hypotheses:

**Hypothesis 2.1.** The increasing function $\gamma$ is concave in a neighborhood of the origin, and for all $0 < a < b < \infty$, there exists $\varepsilon > 0$ such that $\gamma'(\varepsilon^+) > \sqrt{t} \gamma'(a^-)$.

**Hypothesis 2.2.** For all $0 < a < b < \infty$, there exists $\varepsilon > 0$ and $c_0 \in (0, 1/\sqrt{t})$, such that for all $s, t \in [a, b]$ with $0 < t - s \leq \varepsilon$,

$$\gamma(t) - \gamma(s) \leq c_0 \gamma(t-s).$$

It was proven in Lemma 2.3 in [9] that Hypothesis 2.1 implies Hypothesis 2.2 and that under the strong condition $\gamma'(0^+) = \infty$, the constant $c_0$ in Hypothesis 2.2 can be chosen arbitrary small. The following lemma establish, under Hypothesis 2.2, the so called two point local-nondeterminism. It was proven in Lemma 2.4 of [9].
Lemma 2.1. Assume Hypothesis 2.2. Then for all \(0 < a < b < \infty\), there exist \(\varepsilon > 0\) and positive constant \(c_1\) depending only on \(a, b\), such that for all \(s, t \in [a, b]\) with \(|t - s| \leq \varepsilon\),

\[
\text{Var} \left( \frac{B_0(t)}{B_0(s)} \right) \geq c_1 \gamma^2(|t - s|). \tag{2.3}
\]

We denote by \(B_0(t, r) = \{s \in \mathbb{R} : \delta(s, t) \leq r\}\) the closed ball of center \(t\) and radius \(r\). The following lemma is useful for the proof of the upper bounds for the Hausdorff dimension in \([3,4]\). It can be seen also as an improvement of both of Proposition 3.1. and Proposition 4.1. in \([9]\). The proof that we give here rely on the arguments of the classical Gaussian path regularity theory, like it was done in Lemma 3.1 in \([14]\) and Lemma 7.8 in \([28]\).

Lemma 2.2. Let \(0 < a < b < \infty\), and \(I := [a, b]\). Then for all \(M > 0\), there exist a positive constants \(c_2\) and \(r_0\) such that for all \(r \in (0, r_0)\), \(t \in I\) and \(z \in [-M, M]^d\) we have

\[
P \left\{ \inf_{s \in B_0(t, r) \cap I} \|B(s) - z\| \leq r \right\} \leq c_2 (r + f_\gamma(r))^d, \tag{2.4}
\]

where \(\| \cdot \|\) is the Euclidean metric, and \(f_\gamma\) is defined by

\[
f_\gamma(r) := r \sqrt{\log 2} + \int_0^{1/2} \gamma \left( \gamma^{-1}(r \sqrt{1/y}) \right) \frac{dy}{y \sqrt{\log(1/y)}}.
\]

Corollary 2.3. Assume that there exists \(x_0, c_3\) such that

\[
\int_0^{1/2} \gamma(xy) \frac{dy}{y \sqrt{\log(1/y)}} \leq c_3 \gamma(x) \tag{2.5}
\]

for all \(x \in [0, x_0]\). Then, there is some constant \(c_4\) depending on \(\gamma, I, r_0,\) and \(x_0\), such that for all \(z \in [-M, M]^d\) and for all \(r \in (0, r_0 \wedge \gamma(x_0))\) we have

\[
P \left\{ \inf_{s \in B_0(t, r) \cap I} \|B(s) - z\| \leq r \right\} \leq c_4 r^d. \tag{2.6}
\]

It is easy to check that any power function \(\gamma(x) = x^H\) with \(H \in (0, 1)\) satisfies (2.5). Moreover, we will show that (2.5) is satisfied by the class of regularly varying functions, which include all power functions. First of all, a function \(\gamma\) is said to be regularly function with index \(0 < \alpha < 1\) if it can be written as

\[
\gamma(x) = x^\alpha L(x),
\]

where \(L(x) : [0, x_0) \to [0, \infty)\) is a slowly varying function near 0 in the sense of Karamata, then it can be represented by

\[
L(x) = \exp \left( \eta(x) + \int_x^A \frac{\varepsilon(t)}{t} dt \right), \tag{2.7}
\]

where \(\eta : [0, x_0) \to \mathbb{R}, \varepsilon : [0, A) \to \mathbb{R}\) are Borel measurable and bounded functions, and there exists a finite constant \(c_0\) such that

\[
\lim_{x \to 0} \eta(x) = c_0, \quad \text{and} \quad \lim_{x \to 0} \varepsilon(x) = 0.
\]

For more properties of regularly varying functions see Seneta \([25]\) or Bingham et al. \([2]\).
Proposition 2.4. Let $\gamma$ be a regularly varying function near 0, with index $0 < \alpha < 1$. Then $\gamma$ satisfies (2.5).

Proof. First, using the representation $\gamma(x) = x^\alpha L(x)$ for all $x \in (0, x_0)$, and thanks to the result of Adamović; see Proposition 1.3.4 in [2], we may assume without loss of generality that the slowly varying part $L(\cdot)$ is $C^\infty$. Since the constant $c_0$ is finite and we are interested only to the asymptotic behavior of $\gamma$ near 0, we restrict our attention to the case where the slowly varying part is given by

$$L(x) = M_0 \exp \left( \int x \frac{\varepsilon(t)}{t} dt \right),$$  \hspace{1cm} (2.8)

where $M_0 > 0$. Now, we check the condition (2.5);

$$\int_{x_0}^{x_0/2} \gamma(xy) \frac{dy}{y\sqrt{\log(1/y)}} = x^\alpha \int_{x_0}^{x_0/2} L(xy) \frac{dy}{y^{1-\alpha}\sqrt{\log(1/y)}} \leq \frac{x^\alpha}{\log^{1/2}(2)} \int_{0}^{1/2} L(xy) \frac{dy}{y^{1-\alpha}}$$  \hspace{1cm} (2.9)

Then, it suffice to show that $\limsup_{x \to 0} \int_{x_0}^{x} L(z)z^{\alpha-1}dz < \infty$. It is easy to check that $\gamma'(x) = x^{\alpha-1}L(x)(\alpha - \varepsilon(x))$. Thus we may apply the Hospital rule to get that

$$\lim_{x \to 0} \frac{\int_{0}^{x} L(z)z^{\alpha-1}dz}{\gamma(x)} = \lim_{x \to 0} \frac{x^{\alpha-1}L(x)}{\gamma'(x)} = 1/\alpha < \infty,$$

which finishes the proof.

Examples 2.1. Here is some families of regularly varying functions that are immediately satisfying (2.5) due to Proposition 2.4:

i) $\gamma_{\alpha, \beta}(r) := r^\alpha \log^\beta(1/r)$ for $\beta \in \mathbb{R}$ and $\alpha \in (0, 1)$,

ii) $\gamma_{\alpha, \beta}(x) := x^\alpha \exp \left( \log^\beta(1/x) \right)$ for $\beta \in (0, 1)$ and $\alpha \in (0, 1)$,

iii) $\gamma_{\alpha}(x) := x^\alpha \exp \left( \frac{\log(1/x)}{\log(\log(1/x))} \right)$ for $\alpha \in (0, 1)$.

Remark 2.5. Notice that condition (2.5) is also satisfied by the class of all "gauge" functions considered by Sanz-solé and Calleja in [15]. Indeed, they considered a "gauge" function $\gamma(\cdot)$ which satisfies that, for any $r, \eta \in [0, \varepsilon_0)$, with $\varepsilon_0$ sufficiently small, we have

$$\gamma(r\eta) \leq \varphi(\eta)\gamma(r) \quad \text{and} \quad \gamma'(r\eta) \leq \frac{1}{r}\Psi(\eta)\gamma(r\eta),$$  \hspace{1cm} (2.10)

where $\varphi$ and $\Psi$ are two Borel functions such that, for some $p \geq 1$ and $\alpha \geq 1$,

$$\int_{0}^{1} \log^p \left( 1 + \frac{c_0}{r^{2\alpha}} \right) \varphi(r)\Psi(r) dr < \infty.$$  \hspace{1cm} (2.11)

Then under these conditions, and by making use of the integration by parts we have
\[
\int_0^{\varepsilon_0} \frac{\gamma(x\eta)}{\eta \sqrt{\log(1/\eta)}} \, d\eta = -\gamma(\varepsilon_0 x) \sqrt{\log(1/\varepsilon_0)} + x \int_0^{\varepsilon_0} \sqrt{\log(1/\eta)} \gamma'(x\eta) \, d\eta \\
\leq \int_0^{\varepsilon_0} \sqrt{\log(1/\eta)} \Psi(\eta) \gamma(x\eta) \, d\eta \\
\leq \gamma(x) \int_0^{\varepsilon_0} \sqrt{\log(1/\eta)} \Psi(\eta) \, d\eta \\
\leq c_6 \gamma(x) \int_0^1 \log^b \left( 1 + \frac{c_5}{\eta^{2a}} \right) \varphi(\eta) \Psi(\eta) \, d\eta = c_7 \gamma(x).
\] 
Hence \( \gamma \) satisfies (2.5).

**proof of Lemma 2.2** Since the coordinate processes of \( B \) are independent copies of \( B_0 \), it is sufficient to prove (2.4) when \( d = 1 \). Note that for any \( s, t \in I \), we have

\[
\mathbb{E} (B_0(s) \mid B_0(t)) = \frac{\mathbb{E} (B_0(s)B_0(t))}{\mathbb{E} (B_0(t)^2)} B_0(t) := c(s, t)B_0(t). 
\] 
(2.13)

Then the Gaussian process \((R(s))_{s \in I}\) defined by

\[
R(s) := B_0(s) - c(s, t)B_0(t),
\] 
(2.14)
is independent of \( B_0(t) \). Let

\[
Z(t, r) = \sup_{s \in B_0(s) \cap I} |B_0(s) - c(s, t)B_0(t)|.
\]

Then

\[
\mathbb{P} \left\{ \inf_{s \in B_0(s) \cap I} |B_0(s) - z| \leq r \right\} \\
\leq \mathbb{P} \left\{ \inf_{s \in B_0(s) \cap I} |c(s, t)(B_0(t) - z)| \leq r + Z(t, r) + \sup_{s \in B_0(s) \cap I} |(1 - c(s, t))z| \right\} 
\] 
(2.15)

By the Cauchy-Schwarz inequality and (2.1), we have for all \( s, t \in I \),

\[
|1 - c(s, t)| = \frac{|\mathbb{E} [B_0(t)(B_0(t) - B_0(s))]|}{\mathbb{E} (B_0(t)^2)} \leq c_1 \delta(s, t).
\] 
(2.16)

Let \( r_0 := 1/2c_1 \), then (2.16) implies that for all \( 0 < r < r_0 \) and \( s \in B_0(t, r) \cap I \), we have \( 1/2 \leq c(s, t) \leq 3/2 \). Furthermore, for \( 0 < r \leq r_0 \), \( s \in B_0(t, r) \), and \( z \in [-M, M]^d \), we have

\[
|(1 - c(s, t))z| \leq c_1 Mr.
\]

Combining this inequality with (2.15), we obtain that for all \( z \in [-M, M]^d \)

\[
\mathbb{P} \left\{ \inf_{s \in B_0(s) \cap I} |B_0(s) - z| \leq r \right\} \leq \mathbb{P} \left\{ |B_0(t) - z| \leq 2(Mc_1 + 1) r + 2Z(t, r) \right\} \\
\leq c_2 (r + \mathbb{E} (Z(t, r))),
\] 
(2.17)

where the last inequality is due to the independence between \( B_0(t) \) and \( Z(t, r) \), we note also that the constant \( c_2 \) depends on \( M, a, b, \) and \( l \) only.
It remains the estimation of the term \( \mathbb{E}(Z(t,r)) \). Let us consider \( d(\cdot, \cdot) \) to be the canonical metric of the centered Gaussian process \( R(s) \) defined above in (2.14). From the Lipschitz property of the projection operator, we obtain that

\[
d(s, s') \leq \delta(s, s')
\]

for all \( s, s' \in B_\delta(t, r) \cap I \). Denote by \( D := \sup_{s, s' \in B_\delta(t, r) \cap I} d(s, s') \) the \( d \)-diameter, and \( N_d(B_\delta(t, r), \varepsilon) \) the smallest number of \( d \)-balls of radius \( \varepsilon \) by which we can cover \( B_\delta(t, r) \). For any fixed \( r > 0 \), we denote by \( \varepsilon_0(r) \) the quantity

\[
\varepsilon_0(r) := \inf\{\varepsilon > 0 : N_d(B_\delta(t, r), \varepsilon) < 2\},
\]

which is immediately smaller than \( r \). From the assumption (2.1) and the inequality (2.18), we obtain

\[
D \leq r \quad \text{and} \quad N_d(B_\delta(t, r), \varepsilon) \leq 2 \times 1_{\{\varepsilon_0(r) \leq \varepsilon \leq r\}} + \frac{\gamma^{-1}(r \sqrt{I})}{\gamma^{-1}(\varepsilon/\sqrt{I})} \times 1_{\{0 < \varepsilon < \varepsilon_0(r)\}}.
\]

It follows from (2.19) and the classical entropy upper bound of R. Dudley (see Corollary 4.15 [1]) that, for some universal constant \( c_3 \),

\[
\mathbb{E}(Z(t,r)) \leq c_3 \int_0^D \sqrt{\log N_d(B_\delta(t, r), \varepsilon)} d\varepsilon
\]

\[
\leq c_3 \left( \int_0^{\varepsilon_0(r)} \sqrt{\log \frac{\gamma^{-1}(r \sqrt{I})}{\gamma^{-1}(\varepsilon/\sqrt{I})}} d\varepsilon + \sqrt{\log 2} \int_{\varepsilon_0(r)}^r d\varepsilon \right)
\]

\[
\leq c_3 \left( \sqrt{r} \int_0^{\varepsilon_0(r)} \sqrt{\log \frac{\gamma^{-1}(r \sqrt{I})}{\gamma^{-1}(\varepsilon \sqrt{I})}} d\varepsilon + \sqrt{\log 2} r \right)
\]

\[
\leq c_3 \left( \sqrt{\log 2} r + \sqrt{r} \left[ \int_0^{1/2} \gamma^{-1}(r \sqrt{I}) \sqrt{\log \frac{\gamma^{-1}(r \sqrt{I})}{\gamma^{-1}(\varepsilon \sqrt{I})}} d\varepsilon + \sqrt{\log 2} (r \sqrt{I} - \gamma^{-1}(r \sqrt{I}/2)) \right] \right)
\]

(2.20)

where we used only (2.19) and a change of variable. Thanks to the continuity of the process \( B_0 \), which imply that \( \lim_{\eta \to \infty} \gamma(\eta)(\log 1/\eta)^{1/2} = 0 \) (see for example [20]). Then by using the integration by parts and another change of variables, we get

\[
\int_0^{1/2} \gamma^{-1}(r \sqrt{I}) \sqrt{\log \frac{\gamma^{-1}(r \sqrt{I})}{\gamma^{-1}(\varepsilon \sqrt{I})}} d\varepsilon = \gamma \left( \frac{\gamma^{-1}(r \sqrt{I})}{2} \right) \sqrt{\log 2} + \int_0^{1/2} \gamma \left( \gamma^{-1}(r \sqrt{I}) y \right) \frac{dy}{y \sqrt{\log(1/y)}}
\]

(2.21)

Then by combining (2.21) and (2.20), we get

\[
\mathbb{E}(Z(t,r)) \leq c_3 (l + 1)f_\gamma(r),
\]

which give that

\[
\mathbb{P} \left\{ \inf_{s \in B_\delta(t, r) \cap I} \|B(s) - z\| \leq r \right\} \leq c_4 (r + f_\gamma(r))^d,
\]

(2.22)

where \( c_4 \) depends on \( a, b, l \) and \( M \). This finishes the proof. \( \square \)
3 Hausdorff dimension of the image set $B(E)$

3.1 Hausdorff measure and dimension associated to the canonical metric

Before giving an explicit formula for the Hausdorff dimension of the image $B(E)$ under some general conditions on $\gamma$, we need first to define the Hausdorff measure and dimension associated to the canonical metric $\delta$ of the process $B$. For $\beta > 0$ and $E \subset \mathbb{R}^+$, the $\beta$-dimensional Hausdorff measure of $E$ with respect to the metric $\delta$ is defined by

$$H^\beta_\delta(E) = \lim_{\eta \to 0} \inf \left\{ \sum_{n=1}^{\infty} (2r_n)^\beta : E \subseteq \bigcup_{n=1}^{\infty} B_\delta(r_n), r_n \leq \eta \right\}.$$  \hfill (3.1)

The Bessel-Riesz type capacity of order $\beta$ on the metric space $(\mathbb{R}^+, \delta)$ is defined by

$$C_{\delta,\beta}(E) = \left[ \inf_{\nu \in \mathcal{P}(E)} E_{\delta,\beta}(\nu) \right]^{-1},$$  \hfill (3.2)

where $\mathcal{P}(E)$ is the family of probability measures carried by $E$, and $E_{\delta,\beta}(\nu)$ denote the $\beta$-energy of a measure $\nu \in \mathcal{P}(E)$ in the metric space $(\mathbb{R}^+, \delta)$, which is defined as

$$E_{\delta,\beta}(\nu) := \int_{\mathbb{R}^+} \int_{\mathbb{R}^+} \varphi_\beta(\delta(t,s))\nu(dt)\nu(ds),$$

and the function $\varphi_\beta : (0, \infty) \to (0, \infty)$ is defined by

$$\varphi_\beta(r) = \begin{cases} r^{-\beta} & \text{if } \beta > 0 \\ \log \left( \frac{r}{\pi^A} \right) & \text{if } \beta = 0 \\ 1 & \text{if } \beta < 0 \end{cases}.$$  \hfill (3.3)

The $\delta$-Hausdorff dimension associated to the Hausdorff measure $H^\beta_\delta(\cdot)$ is defined as

$$\dim_\delta(E) := \sup \left\{ \beta : H^\beta_\delta(E) > 0 \right\}.$$  \hfill (3.4)

There exists also an alternative expression given through the Bessel-Riesz capacities by

$$\dim_\delta(E) = \sup \left\{ \beta : C_{\delta,\beta}(E) > 0 \right\},$$  \hfill (3.5)

Notice that the lower inequality (i.e. $\dim_\delta(E) \geq \sup \{ \beta : C_{\delta,\beta}(E) > 0 \}$) follows from an application of the energy method (see for example Theorem 4.27 in [13]), and the upper inequality holds from an application of the Frostman’s Lemma in the metric space $(\mathbb{R}^+, \delta)$. Indeed, It was proven in [12] that for any general metric space $(E, \delta)$, we have

$$\dim_\delta(E) = \sup \left\{ \beta / \exists r_0 > 0, c_0 > 0, \text{ and } \nu \in \mathcal{P}(E) \text{ s. t. } \nu(B_\delta(x,r)) \leq c_0 r^\beta \text{ for all } r < r_0, x \in E \right\}.$$  \hfill (3.6)

One can see for example Proposition 5 and Note 12 in [12] for a good understanding of this last formulation. Then considering this definition, we can prove now the remaining inequality in (3.5). Let $\alpha < \dim_\delta(E)$, and we fix some $\beta \in (\alpha, \dim_\delta(E))$. By using the equality (3.6), there exists $\nu \in \mathcal{P}(E)$, $0 < r_0 < 1$, and
0 < c_0 < \infty$ such that $\nu(B_\delta(x, r)) \leq c_0 r^\beta$ for all $r < r_0$. For a fixed $t \in E$, since $\nu$ has no atom, we make the following decomposition

$$\int_E \frac{\nu(ds)}{\delta(t, s)^\alpha} = \sum_{k=1}^\infty \int_{\delta(t, s) \in (2^{-k}, 2^{-k+1}]} \frac{\nu(ds)}{\delta(t, s)^\alpha} \leq \sum_{k=1}^\infty 2^{k\alpha} \nu(B_\delta(t, 2^{-k+1}))$$

$$\leq c_1 \sum_{k=1}^\infty 2^{-k(\beta - \alpha)},$$

with $c_1 = 2^\beta c_0$. Since $\alpha < \beta$ the last sum is finite and independent of $t \in E$. Hence

$$\mathcal{E}_{\delta, \alpha}(\nu) := \int \int \frac{\nu(ds)\nu(dt)}{\delta(t, s)^\alpha} < \infty,$$

which finishes the proof.

**Remark 3.1.** To illustrate some interesting cases that are covered by our study, we consider:

i) For $\beta > 0$ and $\gamma$ defined near 0 by

$$\gamma(r) := r (\log(1/r))^{\beta}. $$

First, we remark that under (2.1), for all $\eta > 0$ small enough and for all $s, t \in [0, 1]$ such that $|t - s| \leq \varepsilon_0$ we have

$$\frac{1}{\sqrt{t}} |t - s| \leq \delta(s, t) \leq \sqrt{t} |t - s|^{1-\eta},$$

which implies immediately that, $\dim_{\text{Euc}}(E) \leq \dim_{\delta}(E) \leq \dim_{\text{Euc}}(E)/1 - \eta$ for all $\eta > 0$. By making $\eta \downarrow 0$ implies that

$$\dim_{\delta}(E) = \dim_{\text{Euc}}(E),$$

where $\dim_{\text{Euc}}(\cdot)$ denote the Hausdorff dimension associated to the Euclidean metric on $\mathbb{R}_+$. 

ii) Hölder scale: For $\beta \in \mathbb{R}$ and $H \in (0, 1)$ and $\gamma$ defined near 0 by

$$\gamma(r) = r^H (\log(1/r))^{\beta}. $$

By the same argument as in (i), we can verify that (in both cases $\beta > 0$ and $\beta < 0$) $\dim_{\delta}(E) = \frac{\dim_{\text{Euc}}(E)}{H}$.

iii) Logarithmic scale: This is an interesting case, which need to be studied carefully, because it is the most irregular case. For $\beta > 1/2$ and $\gamma$ defined near 0 by

$$\gamma(r) = \frac{1}{(\log(1/r))^{\beta}}. $$

First, let us fix $E \subset [0, 1]$ to be a Borel set such that $\dim_{\delta}(E) < \infty$, then using the fact that $r^\alpha = o(\gamma(r))$ for any $\alpha > 0$, we get that

$$\dim_{\text{Euc}}(E) \leq \alpha \dim_{\delta}(E) \text{ for all } \alpha > 0. $$

Then by letting $\alpha \downarrow 0$ we obtain that $\dim_{\delta}(E) = 0$. Hence the Euclidean scale is not sufficient to describe the geometry of some Borel sets. and we will see later that in order to describe a lot of geometric properties of the logarithmic Brownian motion, we need to restrict the process to the class of subsets $E \subset [0, 1]$ with $\dim_{\text{Euc}}(E) = 0.$
In order to better understand that the size of sets of finite and positive $\delta$-Hausdorff dimension is totally dependent on the metric $\delta$, we construct, for some fixed $\zeta > 0$, a Borel subset $E \subset [0,1]$ such that $\dim_\delta(E) = \zeta$ with $0 < \mathcal{H}_\delta^\zeta(E) < \infty$. This will be also helpful to understand the difference between the different scales defined in the previous remark.

**Lemma 3.2.** Let $\zeta > 0$, Then there exists a compact subset $C_\zeta$ of $[0,1]$, such that $0 < \mathcal{H}_\delta^\zeta(C_\zeta) < \infty$.

**Proof.** Let $\delta^*$ be the metric defined as $\delta^*(t,s) = \gamma(|t-s|)$. We know from (2.1) that the metrics $\delta$ and $\delta^*$ are commensurate, then it will be sufficient to construct a compact set $C_\zeta$ such that $0 < \mathcal{H}_\delta^\zeta(C_\zeta) < \infty$. We will construct a $\delta^*$-generalised Cantor set of $\delta^*$-Hausdorff dimension equal to $\zeta$. Indeed, let $I_0 \subset [0,1]$ an interval of length $\varepsilon_0$. Let first $t_1 = \gamma^{-1}(2^{-1/\zeta})$ and $l_1 = t_1$, and let $I_{1,1}$ and $I_{1,2}$ two subintervals of $I_0$ with length $l_1 \varepsilon_0$. For $k \geq 2$, we construct $t_k$, $l_k$, and $\{I_{k,j} : j = 1,\ldots,2^k\}$ inductively, in the following way: $t_k = \gamma^{-1}(2^{-k/\zeta})$ and $l_k = t_k/t_{k-1}$, and the intervals $I_{k,1},\ldots,I_{k,2^k}$ are constructed by conserving only two intervals of length $l_k |I_{k-1,i}| = t_k \varepsilon_0$ from each interval $I_{k-1,i}$ of the previous iteration. We define $C_{\zeta,k}$ to be the union of the intervals $(I_{k,j})$ of each iteration. The compact set $C_\zeta$ is defined to be the limit set of this construction, namely we have

$$C_\zeta := \bigcap_{k=1}^{\infty} C_{\zeta,k}. \quad (3.8)$$

It remains to show that $0 < \mathcal{H}_\delta^\zeta(C_\zeta) < \infty$. For the upper bound we use the fact that, for all $k$ fixed, the family $(I_{k,j})_{j \leq 2^k}$ is a covering of $C_\zeta$ and each $I_{k,j}$ is contained in an open ball $B_{\delta^*}(s_i,2^{-k/\zeta})$. Then by definition of the $\zeta$-dimensional Hausdorff measure (3.1) we have

$$\mathcal{H}_\delta^\zeta(C_\zeta) \leq \sum_{j=1}^{2^k} \left(2 2^{-k/\zeta}\right)^\zeta = 2^\zeta. \quad (3.9)$$

For the lower bound, we define a measure $\nu$ on $C_\zeta$ by the mass distribution principle (11). For any $k \geq 1$, we define

$$\nu(I_{k,i}) = 2^{-k} \text{ for } i = 1,\ldots,2^k \quad (3.10)$$

and $\nu([0,1] \setminus C_{\zeta,k}) = 0$. Then by Proposition 1.7 in (11), $\nu$ can be extended to a probability measure on $C_\zeta$. For $t \in C_\zeta$ and $r > 0$ small enough, let $k \geq 1$ such that $2^{-(k+1)/\zeta} < r \leq 2^{-k/\zeta}$, then it is easy to check that, the ball $B_{\delta^*}(t,r)$ intersect at most 4 interval $I_{k,i}$, which by using (3.10) imply that

$$0 < \nu(B_{\delta^*}(t,r)) \leq 2^{-k+2} \leq r^\zeta. \quad (3.11)$$

Then by using the mass distribution principle (see (11) pg. 60), we deduce that $\mathcal{H}_\delta^\zeta(C_\zeta) \geq \nu(C_\zeta)/8 = 1/8$, which finishes the proof. $\square$

### 3.2 Hausdorff dimension for the range set $B(E)$

Now our aim is to give, under some weaker assumptions on $\gamma$, an upper and lower bounds for the Hausdorff dimension of the image of a Borel set $E$ by the Gaussian process $B$. We notice that when $B$ has stationary increments, an explicit formula for $\dim_{\gamma}B(E)$ was given in terms of $d$ and $\dim_\delta(E)$ by Hawkes in Theorem 2 in (10), under the strong condition $\text{ind}(\gamma) > 0$, where $\text{ind}(\gamma)$ denote the lower index of the function $\gamma$, which is defined as

$$\text{ind}(\gamma) := \sup\{\alpha : \gamma(x) = o(x^\alpha)\} = \left(\inf\{\beta : \gamma(x) = o\left(x^{1/\beta}\right)\}\right)^{-1}. \quad (3.12)$$

\footnote{Another approach was given in (10) in order to define the term $\dim_\delta(E)$, but this approach seems to be valid only when $B$ has stationary increments, instead of our approach which does not require the stationarity of the increments.}
Firstly, we seek condition on $\gamma$ that can weaken the condition \textit{ind}(\gamma) > 0 and which could help us to build an appropriate covering of $B(E)$. It is worth noting that the condition (2.5) is important for the construction of some covering for $B(E)$, in order to be able to provide an upper bound for its Hausdorff dimension $\dim_{Euc}(B(E))$. But to be on the right path of generalizing Theorem 2 in [10], we need a condition which should be satisfied by all functions $\gamma$ with \textit{ind}(\gamma) > 0. Even though (2.5) is already satisfied by all power functions, and some other important examples of interest (see Proposition 2.4, Examples 2.1, and Remark 2.5), we are not able to show that is satisfied by all continuous functions $\gamma$ with strictly positive lower index. Nevertheless, we can provide another condition which is weaker than (2.5), and we will show that it is satisfied by all functions $\gamma$ of strictly positive lower index that we might work with. It will be also useful to provide an optimal upper bound for $\dim_{Euc}(B(E))$. Indeed, we state following condition:

For all $0 < \varepsilon < 1$ sufficiently small, there exist two constants $c_{1,\varepsilon} > 0$ and $x_\varepsilon > 0$, such that

$$
\int_0^{1/2} \gamma(xy) \frac{dy}{y\sqrt{\log(1/y)}} \leq c_{1,\varepsilon} (\gamma(x))^{1-\varepsilon} \quad \text{for all } 0 < x < x_\varepsilon. \tag{3.13}
$$

Therefore, this last condition is immediately weaker than condition (2.5). Now we provide proof of the comforting fact that all functions $\gamma$ with a positive finite lower index that we might work with are satisfying (3.13).

\textbf{Lemma 3.3.} Let $\gamma$ be continuous, strictly increasing, and concave near the origin. If we assume that \textit{ind}(\gamma) \in (0, \infty)$, then $\gamma$ satisfies the condition (3.13).

\textbf{Proof.} Let $\alpha := \text{ind}(\gamma)$, and we fix $\varepsilon > 0$ small enough, then there exists a constant $c_{2,\varepsilon}$ such that $\gamma(x) \leq c_{2,\varepsilon} x^{\alpha-\varepsilon}$, for any $x \in [0, 1/2]$. We have also the existence of another constant $c_{3,\varepsilon}$ and a sequence $(x_n)_n$ decreasing to 0 such that $\gamma(x_n) \geq c_{3,\varepsilon} x_n^{\alpha+\varepsilon}$ for all $n$. We may assume without loss of generality that $\gamma(x) \geq c_{3,\varepsilon} x^{\alpha+\varepsilon}$ for all $x \in (0, 1/2]$. We now only need to show that for some $0 < x_\varepsilon < 1$ small enough, we have

$$
I := \frac{1}{\gamma(x)} \int_0^{1/2} \gamma(xy) \frac{dy}{y\sqrt{\log(1/y)}} \leq c_{4,\varepsilon} \gamma^{-\varepsilon}(x), \tag{3.14}
$$

for all $0 < x < x_\varepsilon$. For $x < 1/2$, we splite the above integral into intervals $[0, x]$ and $(x, 1/2]$, and using the fact that $\gamma$ is increasing as well as the bounds obtained above on $\gamma$, we have

$$
I = \int_0^x \frac{\gamma(xy)}{\gamma(x)} \frac{dy}{y\sqrt{\log(1/y)}} + \int_{x}^{1/2} \frac{\gamma(xy)}{\gamma(x)} \frac{dy}{y\sqrt{\log(1/y)}} \leq \frac{c_{2,\varepsilon}}{c_{3,\varepsilon}} x^{-2\varepsilon} \int_0^x y^{\alpha-\varepsilon-1} dy + \frac{\gamma(x/2)}{\gamma(x)} \int_{x}^{1/2} \frac{dy}{y\sqrt{\log(1/y)}} \leq c_{5,\varepsilon} \left( x^{\alpha-3\varepsilon} + \sqrt{\log(1/x)} \right).
$$

By choosing $\varepsilon < \alpha/3$, we get that $I \leq 2c_{5,\varepsilon} \sqrt{\log(1/x)}$. Using again the fact that $\gamma$ has a positive lower index, we get that $\sqrt{\log(1/x)} = o(\gamma^{-\varepsilon}(x))$, which gives the desired inequality in (3.14). \qed

Now we can state the main result of this section.

\textbf{Theorem 3.4.} Let $B$ the continuous $\mathbb{R}^d$-valued centered Gaussian process defined above such that the canonical metric $\delta$ satisfies (2.1) with a function $\gamma$ that satisfies Hypothesis 2.1. For any Borel set $E \subset [0, 1]$, we have

\begin{align*}
\end{align*}
\[ \dim_{Eu}(B(E)) \geq \min(d, \dim_{H}(E)) \ a.s. \quad (3.15) \]

\[ \dim_{Eu}(B(E)) = \min(d, \dim_{H}(E)) \ a.s. \quad (3.16) \]

\( \dim_{Eu}() \) denote the Hausdorff dimension associated to the Euclidean metric.

Before proving this theorem let us introduce some notations. Let \( C = \bigcup_{n=0}^{\infty} C_n \) be the class of all \( \gamma \)-dyadic intervals such that every \( C \in C_n \) has the form
\[
[(j-1)\gamma^{-1}(2^{-n}), j\gamma^{-1}(2^{-n})],
\]
for \( k, n \in \mathbb{N} \). By using (2.1) and substituting \( \delta \)-balls by \( \gamma \)-dyadic intervals in the definition of Hausdorff measure, we obtain another family of outer measures \( \tilde{H}^\beta(\cdot) \). Fortunately by making use of (2.1) we can check that for all fixed \( \beta \), the measures \( H^\beta(\cdot) \) and \( \tilde{H}^\beta(\cdot) \) still equivalent. The proof follows from the same lines as in Taylor and Watson [3] p. 326. A necessary condition is that, \( \gamma(2s) \leq c \gamma(s) \) for all \( 0 < s < \varepsilon_0 \) with some constant \( c > 0 \), which is an immediate consequence of the concavity of \( \gamma \) near zero.

**proof of Theorem 3.4** We begin by proving (i). First, by the countable stability of Hausdorff dimension we can suppose without loss of generality that \( \text{diam}(E) \leq \varepsilon \). Let \( \zeta < d \wedge \dim_{H}(E) \), then (3.5) implies that there is a probability measure \( \nu \) supported on \( E \) such that
\[
\int_{E} \int_{E} \frac{\nu(ds)\nu(dt)}{\delta(s,t)^{\zeta}} < \infty. \quad (3.17)
\]
Let \( \mu := \nu \circ B^{-1} \) be the image of \( \nu \) by the process \( B \), then
\[
\mathbb{E} \left( \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{\mu(dx)\mu(dy)}{||x-y||^{\zeta}} \right) = \int_{E} \int_{E} \mathbb{E} \left( \frac{1}{||B(t) - B(s)||^{\zeta}} \right) \nu(ds)\nu(dt) = c_{\zeta} \int_{E} \int_{E} \frac{\nu(ds)\nu(dt)}{\delta(t,s)^{\zeta}} < \infty, \quad (3.18)
\]
where \( c_{\zeta} = \mathbb{E} \left( 1/||X||^{\zeta} \right) \) with \( X \sim \mathcal{N}(0, I_d) \), which is finite because \( \zeta < d \). Then Frostman’s theorem, on \( \mathbb{R}^d \) endowed with the Euclidean metric, implies that \( C^\zeta(B(E)) > 0 \) a.s. Hence \( \dim_{Eu}(B(E)) \geq \zeta \), and by making \( \zeta \uparrow d \wedge \dim_{H}(E) \) we get the desired inequality. Let us now prove the upper bound part (ii). Indeed, we suppose that \( d > \dim_{H}(E) \) otherwise there is nothing to prove. Let \( \zeta > \dim_{H}(E) \), by definition of Hausdorff dimension we have \( \tilde{H}^\zeta_{\zeta}(E) = 0 \). Let \( \eta > 0 \), so that there is a family of \( \gamma \)-dyadic interval \( (C_k)_{k \geq 1} \) such that for every \( k \geq 1 \) there is \( n_k, j_k \in \mathbb{N} \) and \( C_k \) has the form \( [(j_k - 1)\gamma^{-1}(2^{-n_k}), j_k\gamma^{-1}(2^{-n_k})] \) and we have
\[
E \subset \bigcup_{k=1}^{\infty} C_k \quad \text{and} \quad \sum_{k=1}^{\infty} |C_k|_{\delta}^{\zeta} < \eta, \quad (3.19)
\]
where \( |\cdot|_{\delta} \) denote the diameter associated to the metric \( \delta \). By using (2.1) it is easy to verify that \( c_1 2^{-n_k} \leq |C_k|_{\delta} \leq c_2 2^{-n_k} \), where \( c_1 \) and \( c_2 \) depend on \( l \) only. For all fixed \( n \geq 1 \), let \( M_n \) be the number of indices \( k \) for which \( n_k = n \), implies that
\[
\sum_{n=1}^{\infty} M_n 2^{-n} \zeta < \frac{\eta}{c_1}. \]
Let $K \subset \mathbb{R}^d$ an arbitrary compact set, we will construct an adequate covering of $B(E) \cap K$. To simplify we suppose that $K = [0,1]^d$. For every $n \geq 1$ let $\mathcal{I}_n$ be the collection of Euclidean dyadic subcubes of $[0,1]^d$ of side length $2^{-n}$, and for all $i = 1, \ldots, M_n$ let $\mathcal{G}_{n,i}$ be the collection of cubes $I \in \mathcal{I}_n$ such that $B(C_i^n) \cap I \neq \emptyset$. Then we have

$$B(E) \cap [0,1]^d \subseteq \bigcup_{n=1}^{\infty} \bigcup_{i=1}^{M_n} \bigcap_{I \in \mathcal{G}_{n,i}} B(C_i^n) \cap I.$$  
(3.20)

For all $n \geq 1$, $i \in \{1, \ldots, M_n\}$, and $I \in \mathcal{I}_n$. Let $\varepsilon > 0$ small enough, by using \[24\] and condition \[3.13\], we get that

$$\mathbb{P}\{I \in \mathcal{G}_{n,i}\} \leq c_3 2^{-n(1-\varepsilon)d},$$

where $c_3$ may depends on $\varepsilon$, but not on $n$. We denote by $H_\zeta^c(\cdot)$ the $\zeta$-Hausdorff content. It is known that the Hausdorff dimension is defined also through Hausdorff contents in the same way as Hausdorff measures, one can see Proposition 4.9 in [13] for the proof of this fact. Then we obtain

$$\mathbb{E}\left(\mathcal{H}_\infty^{\zeta+\varepsilon d} \left(B(E) \cap [0,1]^d\right)\right) \leq c_4 \sum_{n=1}^{\infty} \sum_{i=1}^{M_n} \sum_{I \in \mathcal{I}_n} 2^{-n(\zeta+\varepsilon d)} \mathbb{P}\{I \in \mathcal{G}_{n,i}\}$$

$$\leq c_5 \sum_{n=1}^{\infty} M_n \text{card}(\mathcal{I}_n) \times 2^{-n(d+\zeta)}$$

$$= c_5 \sum_{n=1}^{\infty} M_n 2^{-n\zeta} < c_6 \eta,$$

(3.21)

where the constants $c_4$, $c_5$, and $c_6$ depend on $\varepsilon$ only. Since $\eta > 0$ is arbitrary we get that $\mathcal{H}_\infty^{\zeta+\varepsilon d}(B(E) \cap K) = 0$ a.s. and then $\dim_{Euc}(B(E) \cap [0,1]^d) \leq \zeta + \varepsilon d$ a.s. Hence by making $\zeta \downarrow \dim_\delta(E)$ and $\varepsilon \downarrow 0$, we obtain that $\dim_{Euc}(B(E) \cap K) \leq \dim_\delta(E) + \varepsilon d$, and the desired inequality follows by making $\varepsilon \downarrow 0$. So, the using the countable stability property of Hausdorff dimension ensures that $\dim(B(E)) \leq \dim_\delta(E)$, which finishes the proof of (ii). \hfill \square

**Remark 3.5.** Notice that condition \[3.13\] fails to holds in the logarithmic scale. But we still get a $\sqrt{\log}$ correction, precisely we get

$$\int_0^{1/2} \gamma(xy) \frac{dy}{y \sqrt{\log(1/y)}} \leq \gamma(x) \sqrt{\log(1/x)}.$$

Hence, by using Lemma [2.2] we obtain that

$$\mathbb{P}\left\{\inf_{s \in B_\delta(t,r) \cap I} \|B(s) - z\| \leq r\right\} \leq c_2 r^{(1-\frac{d}{2\beta})d},$$

(3.22)

and by following the same lines from \[3.19\] to \[3.21\] we get that (the lower bound does not change)

$$\dim_\delta(E) \wedge d \leq \dim_{Euc}(B(E)) \leq \left(\dim_\delta(E) + \frac{d}{2\beta}\right) \wedge d.$$

It is quite remarkable that the irregularity of the process $B$ increase when the lower index $\text{ind}(\gamma)$ decrease. When $\text{ind}(\gamma) := \alpha \in (0, 1)$, all trajectories of $B$ are $\beta$-Hölder continuous for all $\beta < \alpha$, and obviously by Lemma \[3.3\], we get that $\gamma$ satisfies \[3.13\]. Hence, the optimal upper bound in \[3.16\] holds immediately. In the other case, when $\text{ind}(\gamma) = 0$, the trajectories of the Gaussian process $B$ are never being Hölder continuous, and no one can be sure if condition \[3.13\] is satisfied in general or not. Since it
was shown in the previous remark that the logarithmic scale (i.e. when \( \gamma(x) := \log^{-\beta}(1/x) \) for \( \beta > 1/2 \) does not satisfy (3.13), and thinking of this scale as the irregular one, there are several other regularity scales which interpolate between Hölder-continuity and the aforementioned logarithmic scale, this compels us to ask the following question:

- Is there a continuous and strictly increasing function \( \gamma \) with zero index (\( \text{ind}(\gamma) = 0 \)) and satisfying the condition (3.13)?

A positive answer for the above question will be given by the following example, where we will give a class of functions \( (\gamma_\alpha)_{\alpha \in (0,1)} \) with zero indexes, such that the weaker condition (3.13) is satisfied. Therefore, by Theorem their associated Gaussian processes would satisfy the identity \( \text{dim}_E(\gamma_\alpha(x)) = \text{dim}_B(E) + d \) a.s.

**Examples 3.1.** To give an example of \( \gamma \) which satisfies the condition (3.13), we consider the family of functions \( (\gamma_\alpha)_{\alpha \in (0,1)} \) defined by \( \gamma_\alpha(x) := \exp\left(-\log^\alpha(1/x)\right) \). It is easy to see that, for any fixed \( \alpha \in (0,1) \), \( \gamma_\alpha(x) \) is less irregular than the logarithmic scale (i.e. \( \gamma_\alpha(x) = o\left(\log^{-\beta}(1/x)\right) \) for all \( \beta > 0 \)), but still more irregular than the Hölder scale (i.e. \( x^H = o\left(\gamma_\alpha(x)\right) \) for all \( H > 0 \)). It remains to show that \( \gamma_\alpha \) satisfies (3.13). Indeed, we have

\[
\int_0^{1/2} \gamma_\alpha(xy) \frac{dy}{y\sqrt{\log(1/y)}} = \int_0^{1/2} \exp\left(-\left(\log(1/x) + \log(1/y)\right)^\alpha\right) \frac{dy}{y\sqrt{\log(1/y)}}
\]

\[
= \int_{\log 2}^\infty \exp\left(-\left(\log(1/x) + z\right)^\alpha\right) \frac{dz}{\sqrt{z}}.
\]

where we used the change of variable \( z = \log(1/y) \). Using the fact that, for all \( \epsilon \in (0,1) \) there is some \( N := N(\epsilon) > 0 \) large enough, so that

\[
(1 + u)^\alpha \geq 1 + \epsilon u^\alpha \quad \text{for all } u \geq N,
\]

we may fix \( \epsilon \in (0,1) \), and its corresponding \( N(\epsilon) \). Then we break the integral in (3.23) into the intervals \([\log(2), N\log(1/x)]\) and \([N\log(1/x), +\infty)\) and denote them by \( I_1 \) and \( I_2 \), respectively. We write \( (\log(1/x) + z)^\alpha = \log^\alpha(1/x) \times (1 + z/\log(1/x))^\alpha \), and we note that the second term is bounded from below by \( 1 + \epsilon \left(\frac{z}{\log(1/x)}\right)^\alpha \) when \( z \geq N\log(1/x) \) (Thanks to (3.23), and by 1 when \( z < N\log(1/x) \). We first have

\[
I_1 \leq \exp\left(-\log^\alpha(1/x)\right) \int_0^{N\log(1/x)} \frac{dz}{\sqrt{z}} = 2\gamma_\alpha(x)\sqrt{N\log(1/x)}.
\]

On the other hand, we have

\[
I_2 \leq \exp\left(-\log^\alpha(1/x)\right) \int_0^\infty e^{-\epsilon z^\alpha} \sqrt{z} \frac{dz}{\sqrt{z}} = c_{1,\alpha}\gamma_\alpha(x).
\]

By combining (3.25) and (3.26), and the fact that \( \sqrt{\log(1/x)} = o\left(\gamma^{-\epsilon}(x)\right) \) for all \( \epsilon > 0 \), we obtain the estimation of (3.13).

### 4  Hitting probabilities

#### 4.1 Preliminaries

Our aims now is to develop a criterion for hitting probabilities of a Gaussian process \( B \) with canonical metric \( \delta \) which satisfies the commensurability condition (2.11). We will establish lower and upper bounds
for hitting probabilities in terms of a capacity term and the Hausdorff measure term, respectively. Both of the capacity and Hausdorff measures terms would be constructed on $\mathbb{R}_+ \times \mathbb{R}^d$, and they would be associated to an appropriate metric $\rho_\delta$ on $\mathbb{R}_+ \times \mathbb{R}^d$, which will be defined below. First of all, we define the metric $\rho_\delta$ on $\mathbb{R}_+ \times \mathbb{R}^d$ by
\[
\rho_\delta ((s, x), (t, y)) = \max \{\delta(t, s), \|x - y\|\}, \quad \text{for all } (s, x), (t, y) \in \mathbb{R}_+ \times \mathbb{R}^d. \tag{4.1}
\]
For an arbitrary $\beta > 0$ and $G \subseteq \mathbb{R}_+ \times \mathbb{R}^d$, the $\beta$-dimensional Hausdorff measure of $G$ in the metric $\rho_\delta$ is defined by
\[
\mathcal{H}_\rho^\beta (E) = \lim_{\eta \to 0} \inf \left\{ \sum_{n=1}^{\infty} (2r_n)^\beta : E \subseteq \bigcup_{n=1}^{\infty} B_{\rho_\delta} (r_n), r_n \leq \eta \right\}. \tag{4.2}
\]
The corresponding Hausdorff dimension of $G$ is defined by
\[
\dim_{\rho_\delta} (G) = \inf \{\beta : \mathcal{H}_\rho^\beta (G) = 0\}. \tag{4.3}
\]
The $\alpha$-Bessel-Riesz type capacity of $G$ on the metric space $(\mathbb{R}_+ \times \mathbb{R}^d, \rho_\delta)$ is defined by
\[
\mathcal{C}_{\rho_\delta, \alpha} (G) = \left[ \inf_{\mu \in \mathbb{P}(E)} \int_{\mathbb{R}_+ \times \mathbb{R}^d} \int_{\mathbb{R}_+ \times \mathbb{R}^d} \varphi_\alpha (\rho_\delta (u, v)) \mu (du) \mu (dv) \right]^{-1}, \tag{4.4}
\]
where the kernel $\varphi_\alpha : (0, \infty) \to (0, \infty)$ is defined in (3.3).

It should be kept in mind that the strong condition (2.5) will be more beneficial than (3.13) to study the problem of Hitting probabilities. Specifically, this will be useful to derive an optimal upper bound for $\mathbb{P} \{B(E) \cap F \neq \emptyset\}$ in terms of $\mathcal{H}_\rho^d (E \times F)$. We also note that even though we can not provide a general result which may ensure that any function $\gamma$ with $\text{ind}(\gamma) > 0$ should satisfy (2.5), but at least it is known that all regularly varying functions with index $\alpha \in (0, 1)$ are satisfying (2.5), see Proposition 2.1. In the other hand, similarly to the case of the weaker condition (3.13), it is worth asking about the existence of an increasing function $\gamma$ with $\text{ind}(\gamma) = 0$ that satisfies (2.5)?

Taking into account the fact that all strictly increasing and continuous functions $\gamma$ with $\text{ind}(\gamma) = 0$ are highly irregular than any power function near zero. We may conjecture that there is a high possibility of the non-existence of a function $\gamma$ with $\text{ind}(\gamma) = 0$ which satisfies (2.5). Although we have not been able to prove this conjecture ultimately, we would like to observe its high probability to hold. So, we provide a mild sufficient condition for the non-existence of (2.5), which is already satisfied -under the Hypothesis 2.1- by a large class of functions with zero lower index as we will see. In order to provide a useful formula for the index of $\gamma$, we will assume that $\gamma$ is differentiable except perhaps at 0, and strictly increasing near 0. Then we have.

**Proposition 4.1.** Let $\gamma$ be a differentiable, strictly increasing near 0. We denote by $\Psi_\gamma (r) := \frac{r \gamma' (r)}{\gamma (r)}$. If we assume $\lim_{r \downarrow 0} \Psi_\gamma (r) \log^{1/2} (1/r) = 0$, then
\[
\lim_{x \downarrow 0} \left( \frac{1}{\gamma (x)} \int_0^{1/2} \gamma (xy) \frac{dy}{y \sqrt{\log (1/y)}} \right) = \infty. \tag{4.5}
\]

Before proving this last proposition, we give the following characterisation of $\text{ind}(\gamma)$, when $\gamma$ is a differentiable function.

**Lemma 4.2.** Let $\gamma$ be a differentiable, strictly increasing, and $\gamma'(0^+) = \infty$. Then we have
\[
\liminf_{r \downarrow 0} \Psi_\gamma (r) \leq \text{ind}(\gamma) \leq \limsup_{r \downarrow 0} \Psi_\gamma (r). \tag{4.6}
\]
Proof. We start by the lower inequality, we suppose that $\liminf_{r \downarrow 0} \Psi_\gamma(r) > 0$ otherwise there is nothing to prove. Let us fix $0 < \alpha' < \alpha < \liminf_{r \downarrow 0} \Psi_\gamma(r)$, then there is $r_0 > 0$ such that $\alpha/r \leq \gamma'(r)/\gamma(r)$ for any $r \in (0, r_0]$. Next, for $r_1 < r_2 \in (0, r_0]$ we integrate over $[r_1, r_2]$ both of elements of the last inequality, we obtain that $\log(r_2/r_1)^{\alpha\gamma} \leq \log(\gamma(r_2)/\gamma(r_1))$, this implies immediately that $r \mapsto \gamma(r)/r^{\alpha}$ is increasing on $(0, r_0]$, and then $\lim_{r \downarrow 0} \gamma(r)/r^{\alpha}$ exists and finite. Since $\alpha' < \alpha$, we get $\lim_{r \downarrow 0} \gamma(r)/r^{\alpha'} = 0$ and then $\alpha' \leq \text{ind} (\gamma)$. Considering the fact that $\alpha'$ and $\alpha$ are arbitrary, the desired inequality holds by letting $\alpha' \uparrow \alpha$ and $\alpha \downarrow \text{lim}_{r \downarrow 0} \Psi_\gamma(r)$. For the upper inequality, we assume that $\limsup_{r \downarrow 0} \Psi_\gamma(r) < \infty$, and we fix $\alpha' > \alpha > \limsup_{r \downarrow 0} \Psi_\gamma(r)$ in order to show by a similar argument as above that $r \mapsto \gamma(r)/r^{\alpha}$ is decreasing near 0, and $\lim_{r \downarrow 0} \gamma(r)/r^{\alpha} > 0$. Hence, $\lim_{r \downarrow 0} \gamma(r)/r^{\alpha} = \infty$ and then by letting $\alpha' \downarrow \alpha$ and $\alpha \downarrow \limsup_{r \downarrow 0} \Psi_\gamma(r)$ the desired inequality is obtained. 

Proof of Proposition \[4.1\] First, we note that the proposition’s assumption implies that $\lim_{r \to 0} \Psi_\gamma(r) = 0$, and thanks to Lemma \[4.2\] which ensures $\text{ind} (\gamma) = 0$. Now, using the change of variable $z = xy$, it’s easy to check that

$$\int_0^{1/2} \frac{\gamma(xy)}{y\sqrt{\log(1/y)}} \, dy \geq \int_0^{x/2} \frac{\gamma(z)}{z\sqrt{\log(1/z)}} \, dz. \quad (4.7)$$

We denote by $\Phi(x) := \int_0^x \frac{\gamma(z)}{z\sqrt{\log(1/z)}} \, dz$. Dividing both of the terms in (4.7) by $\gamma(x)$, and we use the fact that, for some constant $c_1 > 0$ we have $\gamma(x) \leq c_1 \gamma(x/2)$ for all $x > 0$ sufficiently small, we obtain

$$\frac{1}{\gamma(x)} \int_0^{1/2} \frac{\gamma(xy)}{y\sqrt{\log(1/y)}} \, dy \geq c_1^{-1} \frac{\Phi(x)}{\gamma(x/2)}. \quad (4.8)$$

Then it will be sufficient to show that $\lim_{x \downarrow 0} \Phi(x)/\gamma(x) = \infty$. Indeed, since $\Psi(0^+) = \gamma(0^+) = 0$, and the functions $\Phi$ and $\gamma$ are differentiable near 0, such that $\gamma'(x) > 0$ pour tout $0 < x < x_0$, for some small $x_0 > 0$, and by using the assumption, we derive that $\lim_{x \downarrow 0} \Phi'(x)/\gamma'(x) = \lim_{x \downarrow 0} 1/\Psi_\gamma(x) \log^{1/2}(1/x) = \infty$. Therefore, an application of the hospital rule yields that

$$\lim_{x \downarrow 0} \frac{\Phi(x)}{\gamma(x)} = \lim_{x \downarrow 0} \frac{\Phi'(x)}{\gamma'(x)} = +\infty,$$

which finishes the proof of \[4.1\].

To realize the usefulness of this sufficient condition, it is enough to check it on some examples

Examples 4.1. (i) $\gamma(x) = \log^{-\beta}(1/x) \times m(x)$, with $\beta \geq 1/2$, and the function $m(\cdot)$ admits slower variations than all of $\log^{-\alpha}(1/x)$ for any $\alpha > 0$, i.e. $m(r) = o(\log^\alpha(1/r))$, and such that $\Psi_m(r) = o \left( \log^{-1/2}(1/r) \right)$. After simple calculation we get $\Psi_\gamma(x) = \beta \log^{-1}(1/x) + \Psi_m(x)$. Hence, $\lim_{x \downarrow 0} \Psi_\gamma(x) \log^{1/2}(1/x) = 0$. We can consider $m(x) := \log^\alpha(\log(1/x))$ with $\alpha \in \mathbb{R}$ when $\beta > 1/2$. But when $\beta = 1/2$, we should choose $\alpha < 0$ in order to conserve the continuity of the Gaussian process $B$. In particular when $\alpha = 0$, it can be deduced that $\gamma(x) = \log^{-\beta}(1/x)$ satisfies \[4.1\] for any $\beta > 1/2$.

(ii) $\gamma(x) = \exp(-\log^\alpha(1/x))$ with $0 < \alpha < 1$. Therefore

$$\lim_{x \downarrow 0} \Psi_\gamma(x) \log^{1/2}(1/x) = 0 \quad \text{if and only if} \quad 0 < \alpha < 1/2.$$

So in that last case $\gamma$ satisfies \[4.1\]. Otherwise, the case $1/2 \leq \alpha < 1$ still without information!

Remark 4.3. Notice that the second example above combined with Example \[3.1\] ensure that the function $\gamma_\alpha(x) = \exp (-\log^\alpha(1/x))$ satisfies the weak condition \[3.13\] but does not satisfy the strong one \[2.5\], at least when $\alpha \in (0, 1/2)$.
4.2 Criteria for hitting probabilities

Now, we are ready to present the main results of this section.

Theorem 4.4. Assume that Hypothesis 2.4 holds. Then for all \(0 < a < b < \infty\) and \(M > 0\), and for \(E \subset [a, b]\) and \(F \subset [-M, M]^d\) are two Borel sets, we have

\[\text{i) If the diameter of } E \text{ is small enough, there exists a constant } C_1 > 0 \text{ depending only on } a, b, M \text{ and the law of } B, \text{ such that}\]
\[C_1 C_{\rho, d}(E \times F) \leq P \{B(E) \cap F \neq \emptyset\}. \tag{4.9}\]

Otherwise, if \(C_{\rho, d}(E \times F) > 0\) then \(P \{B(E) \cap F \neq \emptyset\} > 0\).

\[\text{ii) If in addition to the Hypothesis 2.4, the function } \gamma \text{ satisfies the condition (2.5), there exists a constant } C_2 > 0 \text{ also depending only on } a, b, M, \text{ and the law of } B, \text{ such that}\]
\[P \{B(E) \cap F \neq \emptyset\} \leq C_2 \mathcal{H}^d_{\rho, s}(E \times F). \tag{4.10}\]

Remark 4.5. We know that the integrability condition \(\int_0^1 \gamma^{-d}(r)dr < \infty\), which was stated in Remark 2.7. in [9] implied that the process \(B\)-restricted to the hull interval \([a, b]\)-hits points with positive probability, i.e. \(P\{B([a, b]) \ni x\} > 0\) for all \(x \in \mathbb{R}^d\). But this integrability condition would be largely sufficient in some irregular cases; for example in the logarithmic scale, the problem of hitting points might be studied for the process \(B\) restricted to some fractal set \(E \subset [a, b]\), which could be tiny in the sense that \(\dim_{Euc}(E) = 0\) but it still has the the capacity of ensuring the non-polarity of points, i.e. \(P\{B(E) \ni x\} > 0\). Indeed, the lower bound in (4.9) gives a sharp sufficient condition on \(E\) for \(B|_E\) to hit points. Namely, if \(C_{\delta, d}(E) > 0\), then for every \(x \in \mathbb{R}^d\) we have \(P\{B(E) \ni x\} > 0\). Then, based on the alternative expression of the Hausdorff dimension \(\dim_{Euc}(\cdot)\) in (4.5), we deduce that the condition \(\dim_{Euc}(E) > d\) is largely sufficient to ensure the non-polarity of points. This gives rise to a generalized integrability condition. Precisely, by using (3.6) and (3.7), it is easy to check that, for any \(0 < \varepsilon < \dim_{Euc}(E) - d\) there exists a probability measure \(\nu\) supported on \(E\) such that
\[\int_0^1 \gamma^{-d}(r)\nu(dr) < \infty.\]

Remark 4.6. It is worth noting that under the weaker upper condition (2.13), we lose the upper bound estimation of the probability \(P\{B(E) \cap F \neq \emptyset\}\) in terms of \(\mathcal{H}^d_{\rho, s}(E \times F)\). But we still have a weaker bound. Indeed, for all \(\varepsilon > 0\) small enough, there exists a positive and finite constant \(c_{\varepsilon}\) such that we have
\[P \{B(E) \cap F \neq \emptyset\} \leq c_{\varepsilon} \mathcal{H}^{d-\varepsilon}_{\rho, s}(E \times F). \tag{4.11}\]

Since for any compact sets \(E\) and \(F\), the condition \(C_{\rho, d}(E \times F) > 0\) ensures that \(P\{B(E) \cap F \neq \emptyset\} > 0\) (without assuming (3.13)), we note that the case \(\dim_{\rho, s}(E \times F) = d\) still a critical case even under the weaker condition (2.13), and this is the only critical case that exists. i.e.
\[P \{B(E) \cap F \neq \emptyset\} \left\{ \begin{array}{l} > 0 \quad \text{if } \dim_{\rho, s}(E \times F) > d \\ = 0 \quad \text{if } \dim_{\rho, s}(E \times F) < d \end{array} \right\},\]

Remark 4.7. Notice that when the condition (3.13) is not satisfied, there are many cases where the lack of information on the positivity of \(P\{B(E) \cap F \neq \emptyset\}\) holds, not only for one point \(\dim_{\rho, s}(E \times F) = d\), but for many values of \(\dim_{\rho, s}(E \times F)\). For example, in the logarithmic scale, the upper bound holds with the order \(d(1 - 1/2\beta)\) for the \(\rho, s\)-Hausdorff measure term, instead of the \(\rho, s\)-capacity in the lower bound,
which stills holds with the order $d$. Precisely, by using (3.22), and by the same covering arguments that will be used in (4.21) and (4.22), we get that

$$C_1 \mathcal{C}_{\rho_d}(E \times F) \leq \mathbb{P}\{B(E) \cap F \neq \emptyset\} \leq C_2 \mathcal{H}^{d(1-1/2\beta)}(E \times F),$$

(4.12)

which tell us that

$$\mathbb{P}\{B(E) \cap F \neq \emptyset\}\left\{\begin{array}{c} > 0 \quad \text{if } \dim_{\rho_d}(E \times F) > d \\
= 0 \quad \text{if } \dim_{\rho_d}(E \times F) < d - 1/2\beta) 
\end{array}\right.,$$

the critical case $d(1 - 1/2\beta) \leq \dim_{\rho_d}(E \times F) \leq d$ still without information.

The following lemma will be used to prove the lower bound part of the hitting probabilities (4.9). Its proof follows from the same lines as in Lemma 3.2. in [14] by using Lemma 2.1.

**Lemma 4.8.** Assume that Hypothesis 2.1 holds. Then for all $x, y \in \mathbb{R}^d$, $s, t \in [a, b]$ such that $|t - s| \leq \varepsilon$, we have

$$\int_{\mathbb{R}^{2d}} e^{-i((\xi, x) + (\eta, y))} \exp\left(-\frac{1}{2}(\xi, \eta) - n^{-1}I_{2d} + \text{Cov}(B(s), B(t))\right)(\xi, \eta)^T d\xi d\eta$$

$$\leq c \varphi_d(\rho_\delta((s, x), (t, y))),$$

(4.13)

where $I_{2d}$ denote the $2d \times 2d$ identity matrix, $\text{Cov}(B(s), B(t))$ denote the $2d \times 2d$ covariance matrix of $(B(s), B(t))$, and $\varphi_d(\cdot)$ is the kernel defined in (3.3).

**Proof of Theorem 4.2.** We begin by proving the lower bound in (4.9). First, let us suppose that the diameter of $E$ is less than $\varepsilon$. Assume that $\mathcal{C}_{\rho_d}(E \times F) > 0$ otherwise there is nothing to prove. Which implies the existence of a probability measure $\mu \in \mathcal{P}(E \times F)$ such that

$$\mathcal{E}_{\rho_d}(\mu) := \int_{\mathbb{R}_+ \times \mathbb{R}_d} \int_{\mathbb{R}_+ \times \mathbb{R}_d} \varphi_d(\rho_\delta(u, v)) \mu(du) \mu(dv) \leq \frac{2}{C_{\rho_d}(E \times F)}.$$  

(4.14)

Consider the sequence of random measures $(m_n)_{n \geq 1}$ on $E \times F$ defined as

$$m_n(\text{d}t \text{d}x) = (2\pi n)^{d/2} \exp\left(-\frac{n\|B(t) - x\|^2}{2}\right) \mu(\text{d}t \text{d}x),$$

$$= \int_{\mathbb{R}^d} \exp\left(-\frac{\|\xi\|^2}{2n} + i\langle\xi, B(t) - x\rangle\right) d\xi \mu(\text{d}x \text{d}t).$$

Denote the total mass of $m_n$ by $\|m_n\| = m_n(E \times F)$. We want to verify the following claim

$$\mathbb{E}(\|m_n\|) \geq c_1, \quad \text{and} \quad \mathbb{E}\left(\|m_n\|^2\right) \leq c_2 \mathcal{E}_{\rho_d}(\mu),$$

(4.15)

where the constants $c_1$ and $c_2$ are independent of $n$ and $\mu$.

First, we have

$$\mathbb{E}(\|m_n\|) = \int_{E \times F} \int_{\mathbb{R}^d} \exp\left(-\frac{\|\xi\|^2}{2} + \frac{1}{2\gamma^2(t)} - i\langle\xi, x\rangle\right) d\xi \mu(\text{d}t \text{d}x)$$

$$\geq \int_{E \times F} \frac{(2\pi)^{d/2}}{(1 + \gamma^2(t))^{d/2}} \exp\left(-\frac{\|x\|^2}{2\gamma^2(t)}\right) \mu(\text{d}t \text{d}x)$$

$$\geq \frac{(2\pi)^{d/2}}{(1 + \gamma^2(b))^{d/2}} \exp\left(-\frac{dM^2}{2\gamma^2(a)}\right) \int_{E \times F} \mu(\text{d}t \text{d}x) =: c_1,$$
This proves the first inequality in (4.15). We have also
\[
E \left( \|m_n\|^2 \right) = \int_{(E \times F)^2} \int_{\mathbb{R}^{2d}} e^{-i(\langle \xi, x \rangle + \langle \eta, y \rangle)} \\
\times \exp \left( -\frac{1}{2} \langle \xi, \eta \rangle \left( n^{-1} I_{2d} + \text{Cov}(B(s), B(t)) \right) \langle \xi, \eta \rangle^T \right) \, d\xi \, d\eta \, \mu(dtdx) \mu(dsdy).
\]
(4.17)

We use Lemma 4.8 and the fact that the diameter of \(E\) is less than \(\varepsilon\), we get that \(E \left( \|m_n\|^2 \right) \leq \mathcal{E}_{\rho_5, d}(\mu)\), which proves the second inequality in (4.15).

Now, using the moment estimates in (4.15) and the Paley-Zygmund inequality (c.f. Kahane [6], p.8), one can check that \(\{m_n, n \geq 1\}\) has a subsequence that converges weakly to a finite random measure \(m_\infty\) supported on the set \(\{(s, x) \in E \times F : B(s) = x\}\), which is positive on an event of positive probability and also satisfying the moment estimates of (4.15). Therefore, using again the Paley-Zygmund inequality, we conclude that
\[
\mathbb{P} \{ B(E) \cap F \neq \emptyset \} \geq \mathbb{P} \{ \|m_\infty\| > 0 \} \geq \frac{E(\|m_\infty\|^2)}{E(\|m_\infty\|^2)} \geq \frac{c_1^2}{c_2 E_{\rho_5, d}(\mu)}.
\]
Hence, (4.14) finishes the proof of (4.9). For the general case. Let us cover \(E\) by a countable family of compact sets \((E_i)_{i \geq 1}\) of diameter less than \(\varepsilon\). We assume again that \(C_{\rho_5, d}(E \times F) > 0\), and let \(\mu\) be a probability measure supported on \(E \times F\) such that \(\mathcal{E}_{\rho_5, d}(\mu)\). This implies that, for all \(i \geq 1\)
\[
\int_{E_i \times F} \int_{E_i \times F} \varphi_d(\rho_5(u, v)) \mu(du) \mu(dv) < \infty.
\]
(4.18)

Since the family \((E_i \times F)_{i \geq 1}\) cover \(E \times F\), there exists \(i \geq 1\) such that \(\mu(E_i \times F) > 0\). Then the measure \(\mu_i(\mu_{\text{meas}}) = \frac{\mu_{\text{meas}}}{\mu(E_i \times F)}\) is a probability measure supported on \(E_i \times F\) and (4.18) implies that \(\mathcal{E}_{\rho_5, d}(\mu_i) < \infty\), which ensures that \(C_{\rho_5, d}(E_i \times F) > 0\), and by using (4.9) we get
\[
\mathbb{P} \{ B(E) \cap F \neq \emptyset \} \geq \mathbb{P} \{ B(E_i) \cap F \neq \emptyset \} \geq C_i C_{\rho_5, d}(E_i \times F).
\]
(4.19)

which finishes the proof of (i).

For the upper bound in (4.10), we use a simple covering argument. We choose an arbitrary constant \(\zeta > \mathcal{H}_{\rho_5}^d(E \times F)\). Then there is a covering of \(E \times F\) by balls \(B_{\rho_5}((t_i, x_i), r_i), i \geq 1\) in \((\mathbb{R}_+ \times \mathbb{R}^d, \rho_5)\) with small radii \(r_i\), such that
\[
E \times F \subseteq \bigcup_{i=1}^{\infty} B_{\rho_5}((t_i, x_i), r_i) \quad \text{with} \quad \sum_{i=1}^{\infty} (2r_i)^d \leq \zeta.
\]
(4.20)

It follows that
\[
\{ B(E) \cap F \neq \emptyset \} \subseteq \bigcup_{i=1}^{\infty} \{ \exists (t, x) \in B_{\theta}(t_i, r_i) \times B(x_i, r_i) \text{ s.t. } B(t) = x \}
\]
\[
\subseteq \bigcup_{i=1}^{\infty} \left\{ \inf_{t \in B_{\theta}(t_i, r_i)} \| B(t) - x_i \| \leq r_i \right\}.
\]
(4.21)

Since the condition (2.5) is satisfied, Corollary 2.3 combined with (4.21) imply that \(\mathbb{P} \{ B(E) \cap F \neq \emptyset \} \leq c_1 \zeta\). Let \(\zeta \downarrow \mathcal{H}_{\rho_5}^d(E \times F)\), the upper bound in (4.10) follows.
\[\square\]
Now we want to provide some optimal lower and upper bounds for the quantity $\mathbb{P}\{B(E) \cap F \neq \emptyset\}$ in terms of a capacity term of $F$ and a Hausdorff measure term of $F$ with an appropriate order, respectively. Let us assume that the Borel set $E$ take some particular form; for example, as a first restriction; we assume that $E$ is $\zeta$-regular set with respect to the metric $\delta$, for some fixed $\zeta > 0$, in the sense that there exists a Borel probability measure $\nu$ supported on $E$ such that for some constants $c_1$ and $c_2$, we have
\begin{equation}
  c_1 r^\zeta \leq \nu(B_\delta(t, r)) \leq c_2 r^\zeta \quad \text{for all } t \in E. 
\end{equation}
(4.22)

Then we have the following result

**Proposition 4.9.** Assume again that Hypothesis [2.4] holds. Let $0 < a < b < \infty$ and $M > 0$, and let $E \subset [a, b]$ be a $\zeta$-regular set. Then for all Borel set $F \subset [-M, M]^d$, we have

i) There exists a constant $c_1 > 0$ depending only on $a, b, M, \zeta$, and the law of $B$, such that
\begin{equation}
  c_1 C_{d-\zeta}(F) \leq \mathbb{P}\{B(E) \cap F \neq \emptyset\},
\end{equation}
(4.23)
where $C_\alpha(\cdot)$ is the $\alpha$-Bessel-Riesz type capacity associated to the Euclidean metric on $\mathbb{R}^d$.

ii) If in addition to the Hypothesis [2.4], the function $\gamma$ satisfies the condition [2.5], there exists a constant $c_2 > 0$ depending only on $a, b, M, \zeta$, and the law of $B$, such that
\begin{equation}
  \mathbb{P}\{B(E) \cap F \neq \emptyset\} \leq c_2 \mathcal{H}^{d-\zeta}(F),
\end{equation}
(4.24)
where $\mathcal{H}^\alpha(\cdot)$ is defined as the Hausdorff measure of order $\alpha$ associated to the Euclidean metric on $\mathbb{R}^d$ when $\alpha > 0$, and is supposed equal to one when $\alpha \leq 0$.

**Proof.** We start by proving $(i)$. By using [4.9], it suffice to show that
\begin{equation}
  C_{\rho,\delta}(E \times F) \geq c_3 C_{d-\zeta}(F).
\end{equation}
(4.25)

Let us suppose that $C_{d-\zeta}(F) > 0$, otherwise there is nothing to prove. Let $0 < \eta < C_{d-\zeta}(F)$, then there exists a probability measure $\mu$ supported on $F$ such that
\begin{equation}
  \mathcal{E}_{d-\zeta}(\mu) := \int \int \varphi_{d-\zeta}(\|x - y\|) \mu(dx)\mu(dy) \leq \eta^{-1}. 
\end{equation}
(4.26)

Since the measure $\nu$ satisfies [4.22], by applying Lemma [5.4] in the metric space $([0, 1], \delta)$ we get that for all $x, y \in F$ we have
\begin{equation}
  \int \int \frac{\nu(ds)\nu(dt)}{(\max \{\delta(s, t), \|x - y\|\})^{\delta}} \leq c_4 \varphi_{d-\zeta}(\|x - y\|).
\end{equation}
(4.27)

Now, since $\nu \otimes \mu$ is a probability measure on $E \times F$, by applying Fubini’s theorem and the last estimation [4.27], we obtain
\begin{equation}
  \mathcal{E}_{\rho,\delta}(\nu \otimes \mu) = \int_{E \times F} \int_{E \times F} \frac{\nu(ds)\mu(dx)\nu(dt)\mu(dy)}{(\max \{\delta(s, t), \|x - y\|\})^{\delta}} \leq c_5 \int \int \varphi_{d-\zeta}(\|x - y\|) \mu(dx)\mu(dy) \leq c_5 \eta^{-1}.
\end{equation}
(4.28)

Hence, we have $C_{\rho,\delta}(\nu \otimes \mu) \geq c_5^{-1} \eta$. By making $\eta \uparrow C_{d-\zeta}(F)$ we get the desired inequality in [4.25].

For proving $(ii)$, by using [4.10], it suffice to show that
\begin{equation}
  \mathcal{H}_{\rho,\delta}^d(E \times F) \leq c_6 \mathcal{H}^{d-\zeta}(F).
\end{equation}
(4.29)
We assume that $\mathcal{H}^{d-\zeta}(F) < \infty$ and $d > \zeta$, otherwise there is nothing to prove. Let $\eta > \mathcal{H}^{d-\zeta}(F)$ be arbitrary, then there is a covering $(B(x_n, r_n))_{n \geq 1}$ of $F$ such that

$$F \subset \bigcup_{n=1}^{\infty} B(x_n, r_n) \quad \text{and} \quad \sum_{n=1}^{\infty} (2r_n)^{d-\zeta} \leq \eta. \quad (4.30)$$

For all $n \geq 1$, let $N_{\delta}(E, r_n)$ be the smallest number of $\delta$-balls of radius $r_n$ required to cover $E$. Then the family $\{B_{\delta}(t_{n,j}, r_n) \times B(x_n, r_n) : 1 \leq j \leq N_{\delta}(E, r_n), n \geq 1\}$ form a covering of $E \times F$ by open balls of radius $r_n$ for the metric $\rho_{\delta}$. Let $0 < r < 1$, we define the so called packing number $P_{\delta}(E, r)$ which is defined to be the greatest number of disjoint $\delta$-balls $B_{\delta}(t_j, r)$ centered in $t_j \in E$ with radius $r$. The lower part in (4.22) implies that

$$c_1 P_{\delta}(E, \delta) r^\zeta \leq \sum_{j=1}^{P_{\delta}(E, r)} \nu(B_{\delta}(t_j, r)) \leq 1. \quad (4.31)$$

Using the well known fact that $N_{\delta}(E, 2r) \leq P_{\delta}(E, r)$, we obtain that

$$N_{\delta}(E, r) \leq c_7 r^{-\zeta}, \quad (4.32)$$

where the constant $c_7$ depends on $E$ only. Putting all those previous facts together, we have

$$\sum_{n=1}^{\infty} \sum_{j=1}^{N_{\delta}(E, r_n)} (2r_n)^d \leq c_8 \sum_{n=1}^{\infty} (2r_n)^{d-\zeta} \leq c_8 \eta. \quad (4.33)$$

Then by making $\eta \downarrow \mathcal{H}^{d-\zeta}(F)$, the inequality in (4.29) holds, which finishes the proof.

**Remark 4.10.** We note that when $E = [a, b]$, with $0 < a < b < 1$, lower and upper bounds have been obtained in Theorem 2.5 and Theorem 4.6 in [9] in terms of the capacity term $C_K(F)$ and the Hausdorff measure term $\mathcal{H}_\varphi(F)$, respectively, and under some reasonable conditions on $\gamma$, where the kernel $K$ and the function $\varphi$ are defined in terms of $\gamma$. Those bounds could be regained again by just proving that

$$C_{\rho_{\delta}, d}(E \times F) \geq c_9 C_K(F) \quad \text{and} \quad \mathcal{H}^{d}_{\rho_{\delta}}(E \times F) \leq c_{10} \mathcal{H}_\varphi(F). \quad (4.34)$$

The proof holds from the same reasoning of Proposition 4.9 and by using also some technics in the proof of Theorem 2.5 and Theorem 4.6 in [9]. We do not give it here for some reasons of the length of paper.

## 5 Hausdorff dimension of the random intersection $B(E) \cap F$ and $E \cap B^{-1}(F)$

Having known from the previous section that the random intersections $B(E) \cap F$ and $E \cap B^{-1}(F)$ are non-empty under some conditions on the Borel sets $E \subset (0, 1)$ and $F \subset \mathbb{R}^d$, it is worth interesting to ask how large are those random intersections. Therefore, the natural way to know more information about their size is to calculate their Hausdorff dimension. The following result may give an answer to this question.

**Theorem 5.1.** Assume again that the Hypothesis 2.7 holds. Then for all $0 < a < b < \infty$ and $M > 0$, let $E \subset [a, b]$ and $F \subset [-M, M]^d$ be two compact sets, then we have

(i) i-1) For all $\eta > 0$, $\mathbb{P}\{\dim_{\delta}(E \cap B^{-1}(F)) \geq \dim_{\delta}(E) + \dim_{Euc}(F) - d - \eta\} > 0$, and
\textit{i-2) If the function } \gamma \text{ satisfies the condition (3.13) then we have, a.s.}
\begin{equation}
\dim \delta \left( E \cap B^{-1}(F) \right) \leq \dim \rho_\delta \left( E \times F \right) - d
\end{equation}

\textit{ii) If } \dim \delta \left( E \right) \leq d, \text{ we have}
\begin{enumerate}
\item \textit{For all } \eta > 0, \mathbb{P} \left\{ \dim \text{Euc} \left( B(E) \cap F \right) \geq \dim \text{Euc} \left( F \right) + \dim \delta \left( E \right) - d - \eta \right\} > 0, \text{ and}
\item \textit{Again under (3.13) we have, a.s.}
\begin{equation}
\dim \text{Euc} \left( B(E) \cap F \right) \leq \dim \rho_\delta \left( E \times F \right) - d
\end{equation}
\end{enumerate}

\textit{iii) If } \dim \delta \left( E \right) > d, \text{ we have}
\begin{enumerate}
\item \textit{For all } \eta > 0, \mathbb{P} \left\{ \dim \text{Euc} \left( B(E) \cap F \right) \geq \dim \text{Euc} \left( F \right) - \eta \right\} > 0, \text{ and}
\item \textit{\dim \text{Euc} \left( B(E) \cap F \right) \leq \dim \text{Euc} \left( F \right) \text{ a.s.}}
\end{enumerate}

Let } Y : \Omega \rightarrow \mathbb{R}_+ \text{ be a positive random variable, the essential supremum norm } \| Y \|_{L^\infty(\mathbb{P})} \text{ is defined by}
\begin{equation}
\| Y \|_{L^\infty(\mathbb{P})} := \sup \{ \theta \geq 0 : \mathbb{P} \left( Y \geq \theta \right) > 0 \}.
\end{equation}

As an application of the previous theorem, we have the following corollary

\textbf{Corollary 5.2.} \textit{If we assume that } B \text{ is a } d\text{-dimensional Gaussian process such that for each component } B_i, \text{ the commenturability condition (2.4) holds with } \gamma(r) = r^H L(r), \text{ where } H \in (0, 1) \text{ and } L(\cdot) \text{ is a slowly varying function. Then for any Borel sets } E \subset [a, b] \text{ and } F \subset [-M, M]^d, \text{ we have}
\begin{enumerate}
\item \textit{If } \dim \text{Euc} \left( E \right) \leq H d, \text{ we have}
\begin{equation}
\frac{\dim \text{Euc} \left( E \right)}{H} + \dim \text{Euc} \left( F \right) - d \leq \| \dim \text{Euc} \left( B(E) \cap F \right) \|_{L^\infty(\mathbb{P})} \leq \dim \rho_\delta \left( E \times F \right) - d
\end{equation}
\item \textit{If } \dim \text{Euc} \left( E \right) > H d, \text{ we have}
\begin{equation}
\| \dim \text{Euc} \left( B(E) \cap F \right) \|_{L^\infty(\mathbb{P})} = \dim \text{Euc} \left( F \right).
\end{equation}
\end{enumerate}

\textbf{Remark 5.3.} \textit{We note that the equality between the upper and lower bounds in (5.1) and (5.2) occur when } \dim \text{Euc} \left( E \right) = \text{Dim} \text{Euc} \left( E \right) \text{ or } \dim \text{Euc} \left( F \right) = \text{Dim} \text{Euc} \left( F \right), \text{ because of the following comparison}
\begin{equation}
\frac{\dim \text{Euc} \left( E \right)}{H} + \dim \text{Euc} \left( F \right) \leq \dim \rho_\delta \left( E \times F \right) \leq \min \left\{ \frac{\text{Dim} \text{Euc} \left( E \right)}{H} + \dim \text{Euc} \left( F \right), \frac{\dim \text{Euc} \left( E \right)}{H} + \dim \text{Euc} \left( F \right) \right\},
\end{equation}

for any Borel sets } E \subseteq \mathbb{R}_+, F \subseteq \mathbb{R}^d.\]
proof of Corollary 5.2. For the lower bound it suffice to apply the previous theorem. Indeed, by Proposition 2.4 the process $B$ satisfies the condition (2.5). Since $L(\cdot)$ is slowly varying function, we have

$$L(r) = o\left(r^{-\varepsilon}\right) \text{ near } 0 \text{ for any } \varepsilon > 0 \text{ small enough,}$$

then we can repeat the same argument used in (i) of Remark 3.1 to show that $\dim_{\rho}(E) = \frac{\dim_{\rho}(E)}{H}$.

For the upper bound, using again the previous theorem, it suffice to check that $\dim_{\rho}H(\cdot) \equiv \dim_{\rho}H(\cdot)$. Indeed, thanks to the property (5.5) again; it is easy to check, for any $\varepsilon > 0$ small enough, that we have

$$0 \leq \dim_{\rho}(G) - \dim_{\rho}(G) \leq \dim_{\rho-H}(G) \leq \dim_{\rho-H}(G) - \dim_{\rho-H}(G) \leq 1 - \frac{1}{H - \varepsilon},$$

and by making $\varepsilon \downarrow 0$, the desired equality follows.

Before proving both of Theorem 5.1 we need the following lemma, which will be helpful to establish the lower bounds part.

Lemma 5.4. Let $(X, \rho)$ be a bounded metric space, such that there exists a probability measure $\mu$ supported on $X$ which satisfies

$$\mu(B_{\rho}(u, r)) \leq C_{1}r^{\kappa}, \quad (5.6)$$

for all $u \in X$, $r > 0$, where $C_{1} > 0$ and $\kappa > 0$ are two constants. Then for any $\theta > 0$, there exists $C_{2} > 0$ such that

$$\int_{X} \int_{X} \frac{\mu(du)\mu(dv)}{(\max\{\rho(u, v), r\})^{\theta}} \leq C_{2} \varphi_{\theta}(r), \quad (5.7)$$

for all $r \in (0, 1)$.

Proof. Since $\mu$ is a probability measure, it suffice to estimate the quantity $I := \sup_{v \in X} \int_{X} \frac{\mu(du)}{(\max\{\rho(u, v), r\})^{\theta}}$.

When $\theta < \kappa$, we have

$$I \leq \sup_{v \in X} \int_{X} \frac{\mu(du)}{\rho(u, v)^{\theta}} < \infty.$$

When $\theta \geq \kappa$, We decompose this last integral into two parts $I_{1}$ and $I_{2}$, where

$$I_{1} = \int_{\{u : \rho(u, v) \leq r\}} \frac{\mu(du)}{r^{\theta}} \quad \text{and} \quad I_{2} = \int_{\{u : \rho(u, v) \geq r\}} \frac{\mu(du)}{\rho(u, v)^{\theta}}.$$

By using (5.6) we get

$$I_{1} \leq C_{3}r^{\kappa-\theta}. \quad (5.8)$$

For estimating $I_{2}$, we assume that $\operatorname{diam}(X) \leq 1$, and we set $k(r) := \inf\{k : 2^{-k} \leq r\}$. Then we have

$$\{u : \rho(u, v) \geq r\} \subset \bigcup_{k=1}^{k(r)} \{u : 2^{-k} \leq \rho(u, v) < 2^{-k+1}\} \quad (5.9)$$
Using again (5.6)

\[
I_2 \leq \sum_{k=1}^{k(r)} 2^{k\theta} \mu \left( \{ u : 2^{-k} \leq \rho(u,v) < 2^{-k+1} \} \right)
\]

(5.10)

It follows from the definition of \(k(r)\) that \(2^{-k(r)} \leq r < 2^{-k(r)+1}\). Then, for \(\theta = \kappa\) we get easily that

\[
I_2 \leq C_4 \log(1/r),
\]

(5.11)

for the case \(\theta > \kappa\), we use a comparison with a geometric series to obtain

\[
I \leq C_5 r^{\kappa-\theta}.
\]

(5.12)

Putting (5.8), (5.10), (5.11), and (5.12) all together, we get the desired estimation.

Let us prove now Theorem 5.1

**proof of Theorem 5.1** First, we note that we can assume without loss of generality that the diameter of \(E\) is smaller than \(\varepsilon\). Let us prove (i), we assume that \(\dim_{\text{Euc}}(F) + \dim_{\delta}(E) > d\) otherwise there nothing to prove. We may assume also that \(\dim_{\delta}(E) < \infty\). Let \(0 < \eta < \dim_{\text{Euc}}(F) + \dim_{\delta}(E) - d\), the definition of \(\delta\)-Hausdorff dimension ensures that there is \(\nu \in \mathcal{P}(E)\) such that \(E_{\delta,\alpha}(\nu) < \infty\), where \(\alpha := \dim_{\delta}(E) - \eta/2\), and the classical Frostman’s theorem ensures that there is \(\mu \in \mathcal{P}(F)\) such that

\[
\mu(B(x,r)) \leq C_1 r^\beta \quad \text{for all } x \in \mathbb{R}^d \text{ and } r > 0,
\]

(5.13)

where \(\beta := \dim_{\text{Euc}}(F) - \eta/2\), and \(C_1\) is a positive constant. Let us consider the sequence of random measures \((\nu_n)_{n \geq 1}\) on \(E\) defined as

\[
\nu_n(ds) = \left( \int_F (2\pi n)^{d/2} \exp \left( -\frac{n \| B(s) - x \|^2}{2} \right) \mu(dx) \right) \nu(ds)
\]

(5.14)

\[
= \left( \int_F \int_{\mathbb{R}^d} \exp \left( -\frac{\| \xi \|^2}{2n} + i \langle \xi, B(s) - x \rangle \right) d \xi \mu(dx) \right) \nu(ds)
\]

Denote the total mass of \(\nu_n\) by \(\|\nu_n\| = \nu_n(E)\). We want to verify the following claim

\[
\mathbb{E}(\|\nu_n\|) \geq C_2, \quad \mathbb{E}(\|\nu_n\|^2) \leq C_3
\]

(5.15)

and

\[
\mathbb{E}(\mathcal{E}_{\delta,\zeta}(\nu_n)) \leq C_4
\]

(5.16)

where \(\zeta := \beta + \alpha - d\), and the constants \(C_2, C_3,\) and \(C_4\) are independent of \(n\). For the first inequality in (5.15) we use the same technique as in (4.15). We prove only the estimation (5.16), and the second

\[\text{Because when } \dim_{\delta}(E) = \infty, \text{ the result will take the form } \mathbb{P}\{ \dim_{\delta}(E \cap B^{-1}(F)) \geq \eta \} > 0 \text{ for any } \eta > 0 \text{ and the proof follows from the same reasoning.}\]
estimation in (5.15) can be deduced from same technique of the last one. Indeed, we express the expected
energy in (5.16) as
\[
\mathbb{E} \left( \int_E \int_E \frac{\nu_n(ds)\nu_n(dt)}{\delta(t,s)^\zeta} \right) = \int_{E^2} \frac{\nu(ds)\nu(dt)}{\delta(t,s)^\zeta} \int_{F^2} \mu(dx)\mu(dy) \int_{\mathbb{R}^{2d}} e^{-\frac{1}{2}(\xi_x + (\eta_y))} \times \exp \left( -\frac{1}{2} (\xi, \eta) \left( n^{-1}I_{2d} + \text{Cov}(B(s), B(t)) \right) (\xi, \eta)^T \right) d\xi d\eta
\]
\[
\leq C_5 \int_{E^2} \frac{\nu(ds)\nu(dt)}{\delta(t,s)^\zeta} \int_{F^2} \mu(dx)\mu(dy) \left( \max \{ \gamma(t-s), \|x-y\| \} \right)^d \tag{5.17}
\]
\[
\leq C_6 \int_E \int_E \frac{\nu(ds)\nu(dt)}{\delta(t,s)^{\zeta+d-\beta}} = C_6 \mathcal{E}_{\delta,\alpha}(\nu) < \infty,
\]
where the first inequality follows from Lemma 4.8 and the second inequality follows by applying Lemma
5.4 to \(X = F\) and \(\rho\) is the euclidean metric.

Now, it is clear that (5.15) combined with the Paley Zygmund inequality (c.f. Kahane [6], p.8)
ensure that \(\{\nu_n : n \geq 1\}\) has a subsequence that converge weakly to a finite measure \(\nu_{\infty}\)
supported on \(E \cap B^{-1}(F)\) which is positive on an event of positive probability (larger than \(C_2^3/2C_3\)). We use in
(5.16) Fatou's lemma and the lower semicontinuity of the energy \(\mathcal{E}_{\delta,\alpha}(\cdot) : \nu \mapsto \int \delta(t,s)^{-\zeta}\nu(ds)\nu(dt)\)
on the space of positive measures \(\mathcal{M}^{+}([0,1])\) equipped with the weak topology (see for example [20], pg. 78), we can deduce that \(\mathcal{E}_{\delta,\alpha}(\nu_{\infty}) < \infty\) a.s., and by definition of \(\delta\)-Hausdorff dimension we deduce that
\(\mathbb{P} \left\{ \dim_{\delta}(E \cap B^{-1}(F)) \geq \zeta \right\} \geq C_2^3/2C_3\). This finishes the proof of the lower bound in (i-1).

For the upper bound in (i-2), let us fix an arbitrary \(\zeta > \dim_{\rho_{\delta}}(E \times F) - d\), and \(0 < \varepsilon < \zeta - \dim_{\rho_{\delta}}(E \times F) + d\), then \(\mathcal{H}_{\rho_{\delta}}^{d+\zeta-\varepsilon}(E \times F) = 0\). Let \(\eta > 0\) small enough, the definition of the \(\rho_{\delta}\)-Hausdorff measure ensures that there is a covering of \(E \times F\) by balls \(B_{\rho_{\delta}}((t_i, x_i), r_i), i \geq 1\) in \((\mathbb{R}^+ \times \mathbb{R}^d, \rho_{\delta})\) with small radii \(r_i\), such that
\[
E \times F \subseteq \bigcup_{i=1}^{\infty} B_{\rho_{\delta}}((t_i, x_i), r_i) \quad \text{with} \quad \sum_{i=1}^{\infty} (2r_i)^{d+\zeta-\varepsilon} \leq \eta. \tag{5.18}
\]
Since for any \(i \geq 1\), the ball \(B_{\rho_{\delta}}((t_i, x_i), r_i)\) is nothing but the Cartesian product of \(B_{\delta}(t_i, r_i)\) and \(B(x_i, r_i)\), it is not difficult to check that
\[
E \cap B^{-1}(F) \subseteq \bigcup_{\{ i : B_{\delta}(t_i, r_i) \cap B^{-1}(B(x_i, r_i)) \neq \emptyset \}} B_{\delta}(t_i, r_i).
\]
Hence, we have
\[
\mathbb{E} \left( \mathcal{H}_{\rho_{\delta}}^{\zeta}(E \cap B^{-1}(F)) \right) \leq \mathbb{E} \left( \sum_{i=1}^{\infty} (2r_i)^{\zeta} \mathbbm{1}_{\{ B_{\delta}(t_i, r_i) \cap B^{-1}(B(x_i, r_i)) \neq \emptyset \}} \right)
\leq \sum_{i=1}^{\infty} (2r_i)^{\zeta} \mathbb{P} \left\{ B_{\delta}(t_i, r_i) \cap B^{-1}(B(x_i, r_i)) \neq \emptyset \right\}
\leq C_1 \sum_{i=1}^{\infty} (2r_i)^{d+\zeta-\varepsilon} \leq C_1 \eta, \tag{5.19}
\]
where the last inequality follows from injecting the condition (3.13) within (2.4) for \(\varepsilon' = \varepsilon/d\). Since \(\eta > 0\) is arbitrary, we conclude that \(\mathcal{H}_{\rho_{\delta}}^{\zeta}(E \cap B^{-1}(F)) = 0\) a.s., and then \(\dim_{\delta}(E \cap B^{-1}(F)) \leq \zeta\). By making \(\zeta \downarrow \dim_{\rho_{\delta}}(E \times F) - d\), we get the desired inequality.
For (ii) we proceed by the same method, but just changing the roles between $E$ and $F$. For the lower bound part, let again $0 < \eta < \dim_{Euc}(F) + \dim_{Euc}(E) - d$ small enough, then there is $\mu \in \mathcal{P}(F)$ such that

$$E_{\beta}(\mu) < \infty,$$

where $\beta := \dim_{Euc}(F) - \eta/2$, and there is $\nu \in \mathcal{P}(E)$ such that

$$\nu(B_{\delta}(t, r)) \leq C_2 r^\alpha \text{ for all } t \in (0, 1) \text{ and } r > 0,$$

where $\alpha = \dim_{Euc}(E) - \eta/2$. We consider in this case $(\mu_n)_{n \geq 1}$ to be a sequence of random measures on $F$ defined as

$$\mu_n(dx) = \left( \int_F (2\pi n)^{d/2} \exp\left( -\frac{n \|B(s) - x\|^2}{2} \right) \nu(ds) \right) \mu(dx).$$

The estimation of $E(\|\mu_n\|)$ from below is easy to check. We just estimate the expectation of the energy $E(\xi(\mu_n))$, and the estimation of $E(\|\mu_n\|^2)$ follows from the same lines also. Indeed, for $\zeta = \alpha + \beta - d$, we have

$$E\left( \int_F \int_F \frac{\mu_n(dx)\mu_n(dy)}{\|x - y\|^{\zeta}} \right) \leq C_3 \int_F \int_F \frac{\mu(dx)\mu(dy)}{\|x - y\|^{\zeta}} \int_F \nu(ds)\nu(dt) \int_E (\max \{\gamma(|t - s|), \|x - y\|\})^d \leq C_4 \nu(\zeta(\mu)) < \infty,$$

where the second inequality come from an application of Lemma [5.4] to $X = E$ and $\rho := \delta$ (we note that $\theta := d$ and $\kappa := \alpha = \dim_{Euc}(E) - \eta/2$, and $\theta > \kappa$ because of the condition "$\dim_{Euc}(E) \leq d$"). Repeating the same argument as above, we deduce that $\mathbb{P} \{\dim_{Euc}(B(E) \cap F) \geq \zeta\} > 0$. Which finishes the proof of the lower bound in (ii-1). For the upper bound in (ii-2), we repeat the same covering techniques used above in [5.18] and [5.19], it suffice to remark that in this case, the random set $B(E) \cap F$ is covered by the family of balls $\{B(x_i, r_i) \text{ s.t. } B_{\delta}(t_i, r_i) \cap B^{-1}(B(x_i, r_i)) \neq \emptyset\}$.

For (iii), the upper bound is trivial (by monotonicity of the Hausdorff dimension). The lower bound can be deduced from the same argument in (ii), it suffice to take $\zeta = \dim_{Euc}(F) - \eta$, and the condition $\dim_{Euc}(E) > d$ ensure that the measure $\nu$ can be chosen such that $[5.20]$ is satisfied for some $d < \alpha < \dim_{Euc}(E)$.

\[\square\]

References

[1] R. J. Adler. \textit{An introduction to continuity, extrema, and related topics for general Gaussian processes}, IMS Lecture Notes-Monograph Series, Volume 12, (1990).

[2] N. H. Bingham, C. M. Goldie and J. L. Teugels. \textit{Regular Variation} Cambridge University Press Cambridge, (1987).

[3] C. J. Bishop and Y. Peres. \textit{Fractals in probability and analysis}. Cambridge Studies in Advanced Mathematics, 162. Cambridge University Press, Cambridge, (2017).

[4] Z. Chen and Y. Xiao. On intersections of independent anisotropic Gaussian random fields. Sci. China Math. 55, no. 11, 2217-2232, (2012).

[5] J. Cuzick and J. P. DuPreez. Joint continuity of Gaussian local times. Ann. Probab. 10, no. 3, 810-817, (1982).

[6] J. P. Kahane. \textit{Some random series of functions}, Second edition. Cambridge Studies in Advanced Mathematics, 5. Cambridge University Press, Cambridge, (1985).

[7] Hutchinson, J.E.: Fractals and self similarity, Indiana Univ. Math. J., 30, 713-747, (1981).
[8] S. J. Taylor and N. A. Watson. A Hausdorff measure classification of polar sets for the heat equation. Math. Proc. Cambridge Philos. Soc. 97, no. 2, 325–344, (1985).

[9] E. Nualart and F. Viens. Hitting probabilities for general Gaussian processes. (2013), available at https://arxiv.org/abs/1305.1758.

[10] J. Hawkes. Local Properties of Some Gaussian Processes. Z. Wahrscheinlichkeitstheorie und Verw. Gebiete, 40, 309–315, (1997).

[11] K. J. Falconer. Fractal geometry-mathematical foundations and applications. John Wiley Sons, Chichester, (1990).

[12] J.D. Howroyd. On dimension and on the existence of sets of finite positive Hausdorff measure. Proc. Lond. Math. Soc., III. 70, 581–604 (1995).

[13] P. Mörters and Y. Peres. Brownian motion. Cambridge Series in Statistical and Probabilistic Mathematics, 30. Cambridge University Press, Cambridge, (2010).

[14] H. Biermé and C. Lacaux and Y. Xiao. Hitting probabilities and the Hausdorff dimension of the inverse images of anisotropic Gaussian random fields. Bull London Math. Soc. 41, 253-273, (2009).

[15] M. sanz-solé and A. H. Calleja. Anisotropic Gaussian random fields: criteria for hitting probabilities and applications. J. Stoch. PDE Anal. Comp. 42 (2021).

[16] M. Erraoui, and Y. Hakiki. Fractional Brownian motion with deterministic drift: Hausdorff dimension of the intersection of the image with a non-random Borel set. Work in progress, (2021).

[17] M. Erraoui and Y. Hakiki. Images of fractional Brownian motion with deterministic drift: Positive Lebesgue measure and non-empty interior. Submitted paper, available at https://arxiv.org/abs/2112.02055, (2020).

[18] Y. Xiao. Hölder conditions for the local times and the Hausdorff measure of the level sets of Gaussian random fields. Probab. Theory Relat. Fields 109, 129–157, (1997).

[19] D. Geman and J. Horowitz. Occupation densities. Ann. Probab. 8, 1-67, (1980).

[20] L. S. Landkof. Foundations of modern potential theory, Springer-Verlag, Berlin (1972).

[21] O. Mocioalca, and F. Viens. Skorohod integration and stochastic calculus beyond the fractional Brownian scale; J. Functional Analysis 222, 385-434, (2005).

[22] D. Khoshnevisan and Y. Xiao. Brownian motion and thermal capacity. Ann. Probab. 43, no. 1, 405-434, (2015).

[23] D. Khoshnevisan. Multiparameter processes. An introduction to random fields. Springer-Verlag, New York, (2002).

[24] H. Ouahhabi and C.A. Tudor. Additive functionals of the solution to fractional stochastic heat equation. J. of Fourier Analysis and Application 19, 777-791, (2013).

[25] E. Seneta. Regularly varying functions, Lecture Notes in Math.508, Springer-Verlag, (1976).

[26] M. Talagrand. Regularity of gaussian processes. Acta Math. 159, 1–2, 99–149, (1987).

[27] R. Shieh. and Y. Xiao. Images of Gaussian random fields: Salem sets and interior points. Studia Math. 176, no. 1, 37-60, (2006).

[28] Y. Xiao. Sample path properties of anisotropic Gaussian random fields. A Minicourse on Stochastic Partial Differential Equations, (D Khoshnevisan and F Rassoul-Agha, editors), Lecture Notes in Math, 1962: 145-212. New York: Springer, (2009)