Confinement from Correlated Instanton-Dyon Ensemble in SU(2) Yang-Mills Theory

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We study the confinement phase transition in SU(2) Yang-Mills theory, based on a statistical ensemble model of correlated instanton-dyons. We show for the first time that such a model provides a quantitative description, in light of the lattice data, for the temperature dependence of the order parameter. We characterize the short-range interaction which plays a crucial role for the properties of such ensemble. The chromo-magnetic charge density as well as the spatial correlations is found to be consistent with known lattice and phenomenological information.

Introduction. — Sixty-five years after the advent of Yang-Mills theory [1] and more than forty-five years after the discovery of Quantum Chromodynamics (QCD) [2, 3], an understanding of the mechanism for confinement phenomenon in such theories remains a significant challenge [4, 5]. First-principle lattice simulations have proven that confinement is indeed a consequence of the underlying gluon fields in the strongly coupled regime and provided ample detailed information about the transition between confined and deconfined phases [4, 6, 7]. Heavy ion collision experiments at the Relativistic Heavy Ion Collider (RHIC) and the Large Hadron Collider (LHC) have also allowed phenomenologically extracting many properties of hot matter in the vicinity of the confinement transition [8–11]. Nevertheless, we do not have a precise picture of how confinement occurs and what are the relevant degrees of freedom driving this phenomenon.

Recently, a promising approach has emerged; based on a new class of gluon topological configurations known as the instanton-dyons [12–24]. This paper aims to provide, for the first time, a quantitative description of the confinement transition in SU(2) Yang-Mills theory based on this approach. In the following, we will first formulate the confinement problem and introduce the instanton-dyon ensemble model in an accessible way. We will then present detailed results to be compared with lattice data as well as discuss the phenomenological implications.

Holonomy Potential. — Let us start by formulating the confinement problem in pure Yang-Mills theory in terms of the holonomy potential. In the imaginary time formalism for finite temperature field theory, one can define the Polyakov loop for a gauge configuration $A_\mu$ as:

$$\mathcal{L}[A_\mu] = \mathcal{P} \exp \left( i \int_0^{1/T} d\vec{x}_4 A_4(\vec{x}, x_4) \right),$$

where $T$ is temperature and $\mathcal{P}$ is the usual path ordering. As is well known, the gauge invariant expectation value $L \equiv \langle \mathcal{L} \rangle / \text{Tr} \mathcal{L}$, often also simply referred to as the Polyakov loop, is a well-defined order parameter for confinement transition in pure Yang-Mills theories [4, 6]. The value of $L$ provides a measure of the “penalty”:

- $L = 0$ implies infinite free energy cost while $L = 1$ implies zero cost for creating a free color charge (in the fundamental representation). Therefore, in the confined phase below critical temperature $T_c$, one has $L = 0$, whereas at $T > T_c$ one has $L > 0$ which approaches unity with increasing temperature.

One can classify all the gauge configurations according to the boundary values of the Polyakov loop, $L_\infty \equiv \mathcal{L}[A_\mu]|_{\vec{x} \to \infty}$ [25, 26]. We focus on the SU(2) case. Up to a gauge transformation and owing to the traceless nature of gauge group generators, one can always characterize the boundary values with one parameter $h \in [0, 1]$:

$$L_\infty = \text{diag}(e^{-i\pi h}, e^{i\pi h}).$$

Correspondingly, for configurations with such boundary values, one has

$$L_\infty = \frac{1}{2} \text{Tr} \mathcal{L}_\infty = \cos (\pi h).$$

The above gauge invariant value is the holonomy, with $h$ being the holonomy parameter. For later convenience we also introduce $\bar{h} \equiv 1 - h$. The confining holonomy corresponds to $L_\infty = 0$ thus $h = 1/2$, while the trivial holonomy corresponds to $L_\infty = 1$ hence $h = 0$.

One can then classify all gauge configurations according to their holonomy values, and rewrite the path integral formulation of the theory’s partition function as:

$$Z = \int [DA_\mu] e^{-S_E} \to \int dh \left\{ \int [DA_\mu^h] e^{-S_E} \right\} = \int dh \ e^{-\mathcal{U}[h]/T}$$

where $A_\mu^h$ are all gauge configurations with holonomy value $h$, $V$ is the system volume, $T$ is temperature and the potential $\mathcal{U}[h]$ or $\mathcal{U}[L_\infty]$ is the holonomy potential.

In the thermodynamic equilibrium at a given temperature $T$, the expectation value of the Polyakov loop $\langle L_\infty \rangle$ must correspond to the minimum of the holonomy potential. Therefore, by computing this holonomy potential and examining its minimum, one would be able to determine $\langle L_\infty \rangle$ and its dependence on temperature. In this formulation of the confinement problem, the essential question is to reveal the shape of the potential $\mathcal{U}[L_\infty]$ and the holonomy value at its minimum.
TABLE I. Properties of the SU(2) instanton-dyons.

| M     | $\bar{M}$ | L     | $\bar{L}$ |
|-------|-----------|-------|-----------|
| Electric charge | 1 | 1 | -1 | -1 |
| Magnetic charge  | 1 | -1 | -1 | 1 |
| h-charge         | 1 | 1 | -1 | -1 |
| Action           | $\frac{8\pi^2}{g^2}$ | $\frac{8\pi^2}{g^2}$ | $\frac{8\pi^2}{g^2}$ | $\frac{8\pi^2}{g^2}$ |
| Size             | $(2\pi T h)^{-1}$ | $(2\pi T h)^{-1}$ | $(2\pi T h)^{-1}$ | $(2\pi T h)^{-1}$ |

As a famous example, one could compute the (one-loop) perturbative contributions from gluons to the holonomy potential. This neat result, from Gross-Pisarski-Yaffe (GPY) [25, 26], is given by:

$$ U_{GPY}^{I} = \frac{4\pi^2}{3} T^4 h^2 T^2. $$

It shall be obvious that the above perturbative potential has its minimum at $h = 0$ or $h = 1$, i.e. corresponding to trivial (non-confining) holonomy. That is, perturbative contributions can not lead to confinement. Contributions to holonomy potential that would be capable of changing its shape toward a minimum at confining holonomy (with $h = \bar{h} = 1/2$), therefore, must come from non-perturbative sectors, as we shall discuss next.

Correlated Instanton-Dyon Ensemble. It has been long suspected that an ensemble of gluonic topological configurations holds the key of confinement mechanism and their contributions to the holonomy potential should drive its minimum toward the non-trivial, confining value [27–31]. The hard question is what type of topological configurations would be the right degrees of freedom. They need to carry chromo-magnetic charges to be compatible with the “dual superconductor” picture for confining vacuum, which appears to be supported by extensive lattice studies [7, 32]. Their properties also need to be sensitive to holonomy in order to influence the behavior of the holonomy potential. The instanton-dyons, which are constituents of the KvBLL calorons, appear to be the promising candidate satisfying both requirements. Let us briefly discuss these objects in the following.

The KvBLL caloron, found relatively recently [12, 13], is a new type of finite-temperature instanton solution with non-trivial holonomy. See e.g. [16] for reviews. The most remarkable feature is that each such caloron of gauge group SU($N_c$) is made of $N_c$ constituents that are magnetically charged. These constituents are referred to as instanton-dyons. Specifically for the SU(2) case, there are four types of instanton-dyons: the $L$- and $M$-dyon together making a KvBLL caloron while the $L$- and $M$-dyon make an anti-caloron. The key properties of the instanton-dyons are summarized in Table I. While a caloron always has its action to be the familiar $8\pi^2/g^2$ (with $g$ the gauge coupling) independent of holonomy, the division of this action between the two constituents as well as the size of these objects do sensitively depend on the holonomy parameter $h$. Even though a caloron is both electrically and magnetically neutral, its constituent dyons do carry non-zero charges. These non-trivial features of instanton-dyons have generated hope that confinement could be explained by their contributions. A number of analytic and numerical studies have been performed, with the results in strong support of such a scenario [14–24].

To investigate confinement, one needs to compute the contributions of instanton-dyons to the holonomy potential. To do that, we build a statistical ensemble of these objects for any given holonomy value as follows:

$$ Z_h^{dyon} = e^{-U_{GPY}^{I}(h)} V/T \sum_{N_{L}, N_{M}, N_{L}, N_{M}} \frac{1}{N_{L}!N_{M}!N_{L}!N_{M}!} \int \prod_{l=1}^{N_{L}} f_L T^3 d^3 r_{L_l} \prod_{m=1}^{N_{M}} f_M T^3 d^3 r_{M_m} \times \prod_{l=1}^{N_{L}} f_L T^3 d^3 r_{L_l} \prod_{m=1}^{N_{M}} f_M T^3 d^3 r_{M_m} \det(G_D) \det(G_D) e^{-V^{DD}}, $$

The above sums over various configurations with $N_{L}$, $N_{M}$, $N_{L}$, and $N_{M}$ numbers of $L$-, $M$-, $L$-, and $M$-dyons respectively. These objects are distributed over the spatial volume with their respective coordinates labeled by $r_{L_l}$, $r_{M_m}$, $r_{L_l}$, and $r_{M_m}$. The determinant terms $\det(G_D)$ and $\det(G_D)$ come from the quantum weight for dyons and antidyons by computing one-loop quantum fluctuations around background fields of the calorons; the detailed form of which can be found in e.g. [14, 18, 24]. The $f_L$, $f_M$, $f_{L}$, and $f_{M}$ are the fugacity factors for each type of dyons/anti-dyons, given by:

$$ f_M = f_{\bar{M}} = S^2 e^{-h S} \frac{\frac{h}{2} - 1}{h} , $$

$$ f_L = f_{\bar{L}} = S^2 e^{-h \bar{S}} \frac{\frac{h}{2} - 1}{h} . $$
Studies on instanton-dyon ensemble models of this sort were pioneered in [17, 18]. Various qualitative features of such models were investigated in [19, 20, 23, 24].

An important quantity in the partition function $Z_{dyon}$ is the caloron action $S$, which is essentially the “control parameter” of the ensemble. While classically one simply has $S = 8\pi^2/g^2$, quantum loop corrections render the coupling to run with temperature scale $T$. Here, next-to-leading order effects are considered by taking the two-loop correction to the gauge coupling $[\pi T]_c$, to-leading order effects are considered by taking the two-loop correction to the perturbative potential Eq. (4), thus defining the relation between the action and temperature via

$$S(T) \approx \frac{22}{3} \log \left( \frac{T}{\Lambda} \right) + \frac{34}{11} \log \left[ 2 \log \left( \frac{T}{\Lambda} \right) \right], \quad (7)$$

where $\Lambda$ is the non-perturbative scale. By varying $S$ from large to small values, the system changes from high to low temperature or equivalently from weak to strong coupling regime. In addition, we consistently include the two-loop correction to the perturbative potential $[\pi T]_c$, thus defining the relation between the action and temperature via

$$U = \left( 1 - \frac{5}{S} \right) U_{GPY}, \quad (8)$$

A crucial ingredient for the properties of the ensemble is the interaction among the instanton-dyons within the ensemble. This is implemented via the $V_{DD}$ term in Eq. (5). Such interaction has two features. At long distance, the interaction between a pair of constituents $i$ and $j$ at a spatial distance $r_{ij}$ should be a Coulomb force according to the objects’ $e, m, h$ charges as shown in Table I:

$$V_{long} = e_i e_j + m_i m_j - h_i h_j \left( \frac{S}{2\pi T} \right) \frac{e^{-MD_{ij}}}{r_{ij}}. \quad (9)$$

The screening effect in such a many-body ensemble of charges has been implemented through a Debye mass parameter $M_D$ in the above. Note that between an $L$-$M$ pair (and similarly $\bar{L}$-$\bar{M}$ pair), which together can make a full caloron, all interactions cancel out by virtue of their BPS nature [12–14]. The correlations between these pairs are encoded in the determinant terms. In between an $L$-$\bar{M}$ or $\bar{L}$-$M$ pair, the Coulomb force is repulsive and prevents unphysical overlapping between them. For the other pair combinations (i.e. $L$-$L$, $\bar{L}$-$\bar{L}$, $L$-$\bar{L}$ as well as $M$-$\bar{M}$, $M$-$\bar{M}$, $M$-$M$), a repulsive force at short distance needs to occur and stabilize the ensemble [17]. This short-range core-like interaction is set as follows [18, 34]:

$$V_{short} = \frac{c_h V_c}{1 + e^{(2\pi T)r_{ij} - \zeta_c}}, \quad \text{for } r_{ij} < \frac{\zeta_c}{(2\pi T)c_h}, \quad (10)$$

where the coefficient $c_h = h$ for $M$-sector while $c_h = \bar{h}$ for $L$-sector, reflecting the different properties of the two sectors. $V_c$ is the strength parameter of this repulsive potential. $\zeta_c$ is the range parameter that separates the short and long-distance regions. The repulsive potential becomes very important when the ensemble becomes dense and it strongly influences the short-range correlations among constituents. The confining properties of such ensemble are sensitive to the two key parameters $V_c$ and $\zeta_c$ [24].

Our goal here is to investigate the viability of this effective description for confinement in light of first-principle lattice calculation results and to characterize the necessary parameters of such an ensemble in order to quantitatively describe the confinement transition in the $SU(2)$ case. We then examine the consistency of this ensemble with other lattice and phenomenological findings.

Confinement Phase Transition.—In this study, we have performed extensive numerical simulations for the statistical ensemble of instanton-dyons as described above. We scan a wide range of choices for the parameter set $(V_c, \zeta_c)$. For each choice, we simulate the ensemble for a variety of values for the action $S$ (which is basically varying temperatures).

A first quantity to examine is the aforementioned holonomy potential $U$ at different temperatures. These results for $U(L_\infty)$ are shown in Fig. 1. (For this plot the parameters are chosen as $(V_c = 5, \zeta_c = 2.4)$, but the observed behavior of the holonomy potential is generically true for other choices of parameters.) As can be seen, when the action $S$ decreases (i.e. the temperature decreases), the holonomy potential smoothly evolves from a hump-shape featuring minima away from $L_\infty = 0$ toward a valley-shape featuring a minimum at the confining holonomy of $L_\infty = 0$. This is characteristic for a second-
order phase transition. In fact, one can identify the critical action $S_c$ (with a corresponding temperature we call $T_c$) where the minimum just moves to $L \to 0$. This allows us to do the scaling of temperature via Eq. \((7)\).

Clearly, in the strongly coupled regime (corresponding to smaller $S$ at lower temperature), confinement occurs in the system. Intuitively this result can be understood as follows. With increasingly strong coupling, it costs less action to create these objects. As a result, the ensemble would eventually become dense enough so that the short-range repulsive force becomes important. In this regime, the holonomy parameter $h$ would prefer to stay at $1/2$ where the $L$- and $M$-sectors are balanced. To see this, imagine that $h$ would deviate from $1/2$, say $h < 1/2$. In this case the number of $M$-dyons would increase (as their action cost is $hS$) but their size $\sim 1/h$ would also increase thus causing a significant increase of energy cost due to the repulsive interaction. The same argument for $L$-sector applies for the case of $h > 1/2$. As a result, when the ensemble becomes dense, the $h = 1/2$ point becomes the optimal state of the system.

With the holonomy potential obtained, one can then determine from its minimum the Polyakov loop expectation value $\langle L_\infty \rangle$ as a function of temperature. As is well known, this is the order parameter for confinement transition in pure Yang-Mills theories. In the SU(2) case, a second-order phase transition is expected with $\langle L_\infty \rangle = 0$ at low temperature while non-zero at high temperature. Such dependence $\langle L_\infty \rangle(T)$ for SU(2) Yang-Mills theories has been obtained from lattice simulations, as shown in Fig. 2 by the filled circle and diamond symbols from two recent lattice works \([35, 36]\). We use the grey band to indicate the lattice uncertainty as reflected by the minor difference of the two calculations. The results from instanton-dyon ensemble calculations for a few choices of parameters are shown in Fig. 2 as curves with open symbols. A second-order phase transition is clearly observed in all cases. We scan a wide range of parameter space and compare with lattice results with quantitative $\chi^2$ analysis to constrain the values of $V_c$ and $\zeta_c$. For the repulsive potential strength $V_c$, we find that the results are relatively insensitive to its value in the range from 5 to 20, with $V_c = 5$ giving the best agreement with lattice. The results are however quite sensitive to the range parameter $\zeta_c$, as can be seen from the visible variation of the curves with different $\zeta_c$ in Fig. 2. We see very good agreement with lattice for $\zeta_c \in [2.2, 2.6]$ and find the optimal value to be $\zeta_c = 2.4$ with $\chi^2$/d.o.f $\approx 1.21$. In passing, let us mention that the instanton-dyon ensemble results also reproduce well the critical scaling near $T_c$ as expected for a second-order transition \([36]\), $\langle L_\infty \rangle = b(T/T_c - 1)^{0.3265}[1 + d(T/T_c - 1)^{0.530}]$, as shown by the smooth curves in Fig. 2.

**Instanton-Dyon Density and Correlations.**—With the key parameters of the instanton-dyon ensemble being characterized above, we now examine its consistency with other relevant information. One such example is the density of chromo-magnetically charged objects. This has been studied on the lattice for SU(2) Yang-Mills theory \([37]\). In Fig. 3, we compare such density from our instanton-dyon ensemble with that from lattice calculation in \([37]\). As one can see, in the same parameter range for $\zeta_c$ where the confinement transition can be quantitatively described, we also see excellent agreement for the magnetic density with lattice results. It may be noted that recent phenomenological studies of jet energy loss observables and heavy flavor transport at the RHIC and LHC provide interesting evidence for the presence of a chromo-magnetic component in the near-$T_c$ region \([38-43]\). The density of magnetic charges extracted from those studies in the vicinity of $T_c$ \([43]\) is about $\rho T^{-3} \sim (N_c - 1) \cdot (0.4 \sim 0.6)$, which is also in consistency with the instanton-dyon ensemble results.

Finally, we have also computed the spatial density-
density correlations between dyons and anti-dyons in the ensemble. These correlations feature a typical liquid-like behavior in the near-\(T_c\) region, with a correlation length on the order of \((0.5 \sim 1) \cdot 1/T\). Such observations, again, appear to be viable with experimental observations of the quark-gluon plasma as a nearly perfect liquid at the RHIC and LHC [8–11] and with phenomenological studies that suggest the chromo-magnetic component to play a key role in such observed transport property [38, 44, 45].

**Conclusion.** — In summary, we have studied a model for describing the confinement transition in \(SU(2)\) Yang-Mills theory, based on a statistical ensemble of correlated instanton-dyons. This model has been shown for the first time to quantitatively describe the lattice data for the temperature dependence of the order parameter. The short-range interaction plays a crucial role and we have characterized the key parameters of this interaction. The chromo-magnetic charge density as well as the spatial correlations in such ensemble have also been found to be consistent with known lattice and phenomenological information. We conclude that the correlated instanton-dyon ensemble provides a successful explanation of the confinement mechanism in \(SU(2)\) Yang-Mills theory, and may hold the promise of a similar success for QCD.

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