QUASI-MODULAR FORMS FROM MIXED NOETHER-LEFSCHETZ THEORY

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Abstract. The Gromov-Witten theory of threefolds admitting a smooth K3 fibration can be solved in terms of the Noether-Lefschetz intersection numbers of the fibration and the reduced invariants of a K3 surface. Toward a generalization of this result to families with singular fibers, we introduce completed Noether-Lefschetz numbers using toroidal compactifications of the period space of elliptic K3 surfaces. As an application, we prove quasi-modularity for some genus 0 partition functions of Weierstrass fibrations over ruled surfaces, and show that they satisfy a holomorphic anomaly equation.

1. Introduction

The Gromov-Witten (GW) invariants of a smooth projective variety \(X\) over \(\mathbb{C}\) are virtual counts of curves on \(X\) in each genus \(g\) and homology class \(\alpha \in H_2(X, \mathbb{Z})\). For the counts to be well-defined (without insertions), we require that the virtual dimension of the Kontsevich moduli space to be zero:

\[
\text{vdim} \overline{M}_g(X, \alpha) = c_1(X) \cdot \alpha + (1 - g)(\dim X - 3) = 0.
\]

It is convenient to assemble the GW invariants of \(X\) into a potential function. At fixed genus \(g \geq 0\), we set

\[
N^X_{g, \alpha} = \deg \overline{M}_g(X, \alpha)^\text{vir},
\]

\[
F^X_g(q) = \sum_{\alpha \in H_2(X, \mathbb{Z})} N^X_{g, \alpha} q^\alpha.
\]

A striking phenomenon is the dependence of \(F^X_g(q)\) on a finite amount of data: the series lies in some finite-dimensional vector space dictated by the discrete invariants of \(X\). For example, when \(X\) is a quintic threefold, \(F^X_g(q)\) lies in the finitely generated graded algebra of Yau-Yamaguchi. In this paper, we study the case where \(X\) is elliptic, and \(\dim(X) \geq 3\).

Definition 1. A smooth projective variety \(X\) is elliptic if it admits a flat proper surjection \(\pi\) onto a smooth variety \(B\), with generic fiber a smooth curve of genus one, and a regular section \(z : B \to X\).

The existence of the section endows the smooth fibers of \(\pi\) with an elliptic curve group law. It also gives a splitting of the short exact sequence

\[
0 \to \ker(\pi_*) \to H_2(X, \mathbb{Z}) \to H_2(B, \mathbb{Z}) \to 0.
\]

The following modularity conjecture describes the potential functions of (2) in terms of the fibration structure \(\pi\).
Conjecture 2. If $\beta \in H_2(B, \mathbb{Z})$ is a primitive base class, then the relative potential function

$$F_{X,\beta}(q) = \sum_{\pi \ast \alpha = \beta} N_{g,\alpha}^X q^{k + \alpha \ast [\beta]} = \varphi(q) \cdot \Delta(q)^{-k},$$

where $k = \frac{1}{2} \beta \cdot c_1(\pi \ast \omega_X/B)$, $\varphi(q)$ is a quasi-modular form for $SL_2(\mathbb{Z})$, and

$$\Delta(q) = q \prod_{m \geq 1} (1 - q^m)^{24}.$$

This conjecture originated in the physics literature [11]. See [20] for a more general conjecture at the level of cycles with insertions.

In Section 2, we present a program for proving Conjecture 2 in general using Noether-Lefschetz cycles. This paper represents the first step in that program for genus 0. We describe modular completions of period maps for families of elliptic K3 surfaces with Type II degenerations. By performing intersections with completed Noether-Lefschetz divisors on a toroidal compactification, we find that the non-holomorphic behavior is explained by the boundary contribution.

Definition 3. A ruled surface $B$ is a smooth projective surface (other than $\mathbb{P}^2$) birational to $M \times \mathbb{P}^1$, with $M$ a smooth curve.

Proposition 4. Any ruled surface $B$ is isomorphic to an iterated blow up of a $\mathbb{P}^1$-bundle $\mathbb{P}E \to M$ ($b$ times).

Proof. See for instance Ch. III of [3].

Our main theorem concerns Weierstrass elliptic fibrations over ruled surfaces:

Theorem 5. Let $B$ be a ruled surface, and $X$ a smooth Weierstrass model over $B$ with fundamental line bundle $\pi \ast \omega_X/B \simeq \omega_B^{-1} \otimes L_M$, where $L_M$ is the pullback of a line bundle on $M$. If $\ell \in H_2(B, \mathbb{Z})$ is the ruling class, and $f \in H_2(X, \mathbb{Z})$ is the elliptic fiber class, then

$$F_{0,\ell}^X(q) = \sum_{n \geq 0} N_{0,\ell+n}^X q^n = \varphi(q) \cdot \frac{q}{\Delta(q)}.$$

Furthermore, $\varphi(q) \in \mathbb{Q}[E_2, E_4, E_6]_{10}$ satisfies a holomorphic anomaly equation:

$$\frac{\partial \varphi}{\partial E_2} = -\frac{b}{12} \cdot E_8,$$

where $b$ is the number of broken fibers of the ruled surface $B$.

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2. Sketeh of a Program

An old approach to computing Gromov-Witten invariants on $X$ is to embed $\iota: X \hookrightarrow P$ as a complete intersection in a more positive variety $P$, often with a torus action. Composing with $\iota$ induces an embedding of Kontsevich spaces $\overline{M}_g(X, \beta) \hookrightarrow \overline{M}_g(P, \iota_* \beta)$. 
When the latter is smooth, the invariant \( N^X_{g, \beta} \) can be realized as the integral of a top class on \( \overline{M}_g(P, \iota_* \beta) \).

Here we explore a dual construction; suppose instead that we are given a proper surjection \( \pi : X \rightarrow B \) onto a more positive variety \( B \) with \( \dim(X) > \dim B \geq 2 \). For \( \alpha \notin \ker(\pi_* : H_2(X) \rightarrow H_2(B)) \), composing with \( \pi \) induces a morphism of Kontsevich spaces

\[
\overline{M}_g(X, \alpha) \rightarrow \overline{M}_g(B, \pi_* \alpha).
\]

The fibers of this morphism are Kontsevich spaces with smaller dimensional target, and the image of this morphism is a proper subvariety of \( \overline{M}_g(B, \pi_* \alpha) \) whose class can be computed using topological methods.

**Proposition 6.** If the fibers of \( \pi \) are Gorenstein and \( K \)-trivial, then

\[
vdim \overline{M}_g(X, \alpha) = vdim \overline{M}_g(X, \alpha')
\]

for \( \alpha - \alpha' \in \ker(\pi_*) \).

**Proof.** By the \( K \)-triviality condition, \( \omega_{X/B} \) is the pullback of a line bundle on \( B \), so \( \omega_X \simeq \pi^* \omega_B \otimes \omega_{X/B} \) is also a pullback. By the projection formula, \( K_X \cdot (\alpha - \alpha') = 0 \). The statement now follows from the virtual dimension formula (1). \( \square \)

Proposition 6 implies that the virtual dimension is unchanged by the addition of fiber classes. This leads us to define generating series of Gromov-Witten invariants by summing in the fiber direction, as in Conjecture 2. We hope to implement the strategy below for general \( K \)-trivial fibrations. For now, we restrict to the case where the fibers are elliptic curves, but \( B \) can have any dimension \( \geq 2 \).

Let \( \pi : X \rightarrow B \) be an elliptic fibration, and let \( Z = z(B) \subset X \) the image of the section. The fundamental line bundle is defined by

\[
L = (z^* N_{Z/X})^\vee.
\]

The line bundle \( \mathcal{O}_{X}(3Z) \) defines a birational morphism \( X \rightarrow X^W \), where \( X^W \subset \mathbb{P}(L^2 \oplus L^3 + \mathcal{O}_B) \) is cut out by a global Weierstrass equation

\[
y^2z = x^3 + a_4xz^2 + b_6z^3
\]

with \( a_4 \in H^0(L^4) \) and \( b_6 \in H^0(L^6) \). The new variety \( X^W \) is called the Weierstrass model. If \( X^W \) is smooth, then its GW invariants are well-defined and can be related to those of \( X \) via blow up formulae [10]. If \( X^W \) is singular, then one can consider a smooth deformation, whose GW invariants are related to those of \( X \) via transition formulae [14].

**Definition 7.** An elliptic fibration is called Weierstrass if it is isomorphic to its Weierstrass model.

**Proposition 8.** The fundamental line bundle \( L \) of a Weierstrass fibration \( \pi : X \rightarrow B \) is isomorphic to \( \pi_* \omega_{X/B} \), and it is nef.
Proof. For any curve $C \subset B$, the fibered product

$$X_C := X \times_B C$$

is a Weierstrass fibration over $C$ with fundamental line bundle $L|_C$. Thus, $L^4|_C$ and $L^6|_C$ have sections, so $\deg(L|_C) \geq 0$. For the first statement, apply the adjunction formula to the global Weierstrass equation \(^{[3]}\).

With these considerations in mind, we make the following simplifying assumption:

**Assumption A.** $\pi : X \to B$ is a Weierstrass fibration whose fundamental line bundle $L$ is ample. Furthermore, $L^4$ and $L^6$ are very ample.

Using the virtual dimension formula \([1]\), we observe that,

$$0 = \text{vdim} \overline{M}_g(X, \alpha) = -K_X \cdot \alpha + (1 - g)(\dim X - 3)$$

$$= -(K_B + c_1(L)) \cdot \beta + (1 - g)(\dim B - 2)$$

$$= \text{vdim} \overline{M}_g(B, \beta) - c_1(L) \cdot \beta + (1 - g).$$

Setting $k = c_1(L) \cdot \beta > 0$, the equation above becomes

$$\text{vdim} \overline{M}_g(B, \beta) = k + g - 1.$$  \((4)\)

**Assumption B.** The general element $[C \to B] \in \overline{M}_g(B, \beta)$ has smooth domain.

This can be thought of as a positivity assumption, and as a restriction to genus $g = 0$. Assumption B can be weakened, but we must treat the other components of $\overline{M}_g(B, \beta)$ separately in the period map construction below.

Let $p : C \to \overline{M}_g(B, \beta)$ be the universal curve, and $u : C \to B$ the universal map. Consider the family of elliptic surfaces $q : \mathcal{S} \to \overline{M}_g(B, \beta)$ given by

$$\mathcal{S} := X \times_B C \to \overline{M}_g(B, \beta)$$

The fiber of $q$ at a stable map $[C \to B] \in \overline{M}_g(B, \beta)$ is the elliptic surface

$$X_C = X \times_B C.$$

Assumption B makes $q$ is generically smooth, so we have a rational period map

$$\rho : \overline{M}_g(B, \beta) \dashrightarrow \Gamma \backslash \mathcal{D}$$

to the appropriate period space of polarized Hodge structures for elliptic surfaces. The map $\rho$ is only defined for $[C \to B]$ with smooth domain; otherwise the surface $S = X_C$ would have a normal crossing singularity. Each elliptic surface $S$ in the family is Weierstrass with fundamental line bundle $L|_C$ of degree $k$, and thus has

$$p_g(S) = k + g - 1.$$  \((5)\)

by a standard calculation \([19]\). This matches with the virtual dimension in \([4]\). Inside the period space $\Gamma \backslash \mathcal{D}$, we have a countable collection of codimension $p_g(S)$ Noether-Lefschetz (NL) cycles\(^{[4]}\), which parametrize Hodge structures with extra algebraic curve classes. Set theoretically, the intersection

$$\rho(\overline{M}_g(B, \beta)) \cap \text{NL}$$

\(^{[4]}\)See Section\([4]\) for a formal definition of the components $\text{NL}_n$ of the Noether-Lefschetz locus.
contains the image of $\overline{M}_g(X, \alpha)$ with smooth domain curves. The primary challenge is to make this intersection topological. The issue is that both the period space $\Gamma \setminus \mathcal{D}$ and the Noether-Lefschetz cycles are non-compact.

**Conjecture 9.** There exists a smooth completion $(\Gamma \setminus \mathcal{D})^\ast$ such that $p$ extends to a morphism

$$\overline{\rho} : \overline{M}_g(B, \beta) \to (\Gamma \setminus \mathcal{D})^\ast.$$ 

Furthermore, the series

$$\varphi(q) = \sum_{n \geq 0} (\overline{\rho}_* [M_g(B, \beta)]^\text{vir} \cap \overline{\mu}_n) q^n$$

is quasi-modular, and it agrees with the form $\varphi(q)$ in Conjecture 2.

Our Theorem 5 confirms Conjecture 9 in the case where $g = 0$, $k = 2$, and $\overline{M}_0(B, \beta)$ is smooth of the expected dimension. Indeed, consider the Cartesian square

$$\begin{array}{ccc}
\mathcal{S} & \xrightarrow{\tilde{u}} & X \\
\downarrow \tilde{\pi} & & \downarrow \pi \\
\mathcal{C} & \xrightarrow{u} & B \\
\downarrow p & & \downarrow \\
\overline{M}_0(B, \beta).
\end{array}$$

Assumption A implies that $\mathcal{S} \to C$ is a smooth Weierstrass fibration with fundamental line bundle $u^* L$, if $X \to B$ is chosen generically. Since $g = 0$, the surface $C$ is ruled:

**Theorem 10.** [3] (Tsen) Let $\mathcal{C}$ be a projective surface with a morphism $\mathcal{C} \to M$ to a smooth curve. If the generic fiber is geometrically integral of genus 0, then $\mathcal{C}$ is birational to $M \times \mathbb{P}^1$.

We have $\overline{M}_0(X, \beta + nf) \simeq \overline{M}_0(\mathcal{S}, \ell + nf)$, since any stable map to $X$ factors through $\mathcal{S}$, and in fact the virtual classes agree too.

**Proposition 11.** $[\overline{M}_0(X, \beta + nf)]^\text{vir} = [\overline{M}_0(\mathcal{S}, \ell + nf)]^\text{vir}$

**Proof.** The relative obstruction theory used to define the virtual class on $\overline{M}_0(X)$ is $R_{p_X} u_X^* T_X$, where $p_X$ is the universal curve, and $u_X$ is the universal map. The short exact sequence

$$0 \to T_{X/B} \to T_X \to \pi^* T_B \to 0$$

induces an exact triangle in the derived category of $\overline{M}_0(X)$:

$$R_{p_X} u_X^* T_{X/B} \to R_{p_X} u_X^* T_X \to R_{p_X} u_X^* \pi^* T_B.$$ 

The analogous construction for $\mathcal{S} \to \mathcal{C}$ yields an exact triangle in the derived category of $\overline{M}_0(\mathcal{S})$:

$$R_{p_{\mathcal{S}}} u_{\mathcal{S}}^* T_{\mathcal{S}/\mathcal{C}} \to R_{p_{\mathcal{S}}} u_{\mathcal{S}}^* T_{\mathcal{S}} \to R_{p_{\mathcal{S}}} u_{\mathcal{S}}^* \tilde{\pi}^* T_{\mathcal{C}}.$$ 

Since the square in (6) is Cartesian, we have $T_{\mathcal{S}/\mathcal{C}} \simeq \tilde{u}^* T_{X/B}$, which induces an isomorphism

$$R_{p_{\mathcal{S}}} u_{\mathcal{S}}^* T_{\mathcal{S}/\mathcal{C}} \simeq R_{p_X} u_X^* T_{X/B}.$$
The derivative map $T_C \to u^*T_B$ gives a map $\bar{\pi}^*T_C \to \bar{u}^*\pi^*T_B$. This induces a quasi-isomorphism

$$Rp_{\mathcal{F}}u^*\bar{\pi}^*T_C \simeq Rp_{\mathcal{X}}u^*\pi^*T_B,$$

since the horizontal component of each stable map is unobstructed in $B$. By the completion axiom of a triangulated category, we have an isomorphism between the middle terms of the exact triangles. \hfill \Box

An advantage of the $k = 2$ case is that the generic fiber of the family

$$q : \mathcal{F} \to \overline{M}_0(B, \beta)$$

is an elliptic K3 surface. Period spaces for K3 surfaces have smooth compactifications, which are used to produce the quasi-modularity result. If $k = 1$, the generic fiber is a rational elliptic surface, which has trivial periods, and its Gromov-Witten theory is described in [20]. If $k \geq 3$, the problem of finding a smooth completion of the period space is still open. See [9] for some modularity results for $k \geq 3$ when all the domain curves in $\overline{M}_0(B, \beta)$ are smooth.

## 3. Reductions

In this section, we reduce Theorem 5 to the case where

- $L_M$ is sufficiently positive, and
- the birational morphism $B \to \mathbb{P}E$ is a blow up of $b$ points in distinct fibers of the $\mathbb{P}^1$ bundle $\mathbb{P}E \to M$.

We employ the degeneration formula of [15] to relate the invariants in Theorem 5 to invariants of a more flexible geometry. The strategy is to construct an elliptic fibration $\mathcal{X} \to B \to \text{Spec} \mathbb{C}[[t]]$ which realizes a normal crossings degeneration

$$\mathcal{X}_t \rightsquigarrow \mathcal{X}_0,$$

where $\mathcal{X}_0$ has star-shaped dual graph, and the gluing loci are all K3 surfaces. The general fiber $\mathcal{X}_t$ and the extremal components of the central fiber $\mathcal{X}_0$ will all satisfy the reduction conditions.

**Lemma 12.** Let $B$ be as in Theorem 5. If $\deg(L_M) \gg 0$, then $\omega_B^{-1} \otimes L_M$ is nef and big.

**Proof.** The line bundle $\omega_B^{-1} \otimes L_M$ has class $2\zeta + l\ell - \sum e_i$, and we assume that $l \gg 0$. Any irreducible vertical curve on $B$ has class $\ell, \ell - e_i, e_i$, or $e_i - e_j$. The degree of $\omega_B^{-1} \otimes L_M$ is nonnegative on each of these. Curves without vertical components will have classes of the form

$$m\zeta + d\ell - \sum m_ie_i,$$

where each $m_i \leq m$, and $d \geq sm$ for some slope $s \in \mathbb{Q}$ depending only on $\mathbb{P}E$. Now,

$$\left( m\zeta + d\ell - \sum_{i=1}^{b} m_ie_i \right) \cdot \left( 2\zeta + l\ell - \sum_{i=1}^{b} e_i \right) = 2m\zeta^2 + 2d + ml - \sum m_i$$

$$\geq 2m\zeta^2 + 2sm + ml - mb$$

$$= m(2\zeta^2 + 2s + l - b)$$

$$\geq 0,$$
since $\zeta^2$, $s$, and $b$ are constants independent of the test curve. To see that $\omega_B^{-1} \otimes L_M$ is big, one checks that its self-intersection is positive. □

Lemma 12 will allow us to lift Weierstrass equations, using the Kawamata-Viehweg vanishing theorem, which we state here for reference.

Theorem 13. (Kawamata-Viehweg) If $L$ is a nef and big line bundle on a smooth projective variety $X$, then for $i > 0$,\[ H^i(X, \omega_X \otimes L) = 0. \]

We are now ready to construct the degeneration. Start with $X \to B$ a Weierstrass fibration with fundamental line bundle $\omega_B^{-1} \otimes L_M$, as in Theorem 5, with no additional assumption on $L_M$.

Corollary 14. There exists some $m \gg 0$ such that for any divisor $D$ on $M$ of degree $m$, we have\[ H^1(B, \omega_B^{-4} \otimes L_M^4 \otimes \mathcal{O}_M(D)) = 0; \]
\[ H^1(B, \omega_B^{-6} \otimes L_M^6 \otimes \mathcal{O}_M(D)) = 0. \]

Proof. This follows easily from Lemma 12 and Theorem 13. □

Step 1. Construct a stable curve $M^{ct}$ of compact type by gluing $M$ to $m$ different smooth curves $M^{(i)}$ at general points $p_1, p_2, \ldots, p_m \in M$.

Step 2. Recall that $E \to M$ is a rank 2 vector bundle, and $PE$ its projectivization. Choose rank 2 vector bundles $E^{(i)}$ on each new component, and identify their fibers over the nodes to obtain a bundle $E^{ct} \to M^{ct}$.

Step 3. Let $\mathcal{M} \to \text{Spec}\, \mathbb{C}[[t]]$ be a smoothing deformation of $M^{ct}$, and let $\mathcal{E} \to \mathcal{M}$ be a rank 2 bundle extending $E^{ct}$. Taking the projectivization $\mathbb{P}\mathcal{E} \to \mathcal{M}$, we obtain a degeneration\[ \mathbb{P}E_t \sim \mathbb{P}E^{ct} = \mathbb{P}E \cup_{p_1} \mathbb{P}E^{(1)} \cup_{p_2} \mathbb{P}E^{(2)} \cup \cdots \cup_{p_m} \mathbb{P}E^{(m)}. \]

Step 4. Deform the blow up centers. Inductively, suppose we have a degeneration $B \to \text{Spec}\, \mathbb{C}[[t]]$ with central fiber\[ B \cup_{p_1} \mathbb{P}E^{(1)} \cup_{p_2} \mathbb{P}E^{(2)} \cup \cdots \cup_{p_m} \mathbb{P}E^{(m)}, \]
and let $y \in B$ be a point away from the glued $\mathbb{P}1$. Choose a general section $Y$ of $B \to \text{Spec}\, \mathbb{C}[[t]]$ specializing to $y$. The blow up $\text{Bl}_yB$ specializes to $\text{Bl}_yB \cup_{p_2} \mathbb{P}E^{(1)} \cup_{p_1} \cdots \cup_{p_m} \mathbb{P}E^{(m)}$. Hence, we can construct a degeneration whose generic fiber is $\mathbb{P}E_t$ blown up at $b$ points in distinct fibers of the projective bundle $\mathbb{P}E_t \to \mathcal{M}_t$.

Step 5. Since $\mathcal{M}$ is a compact type degeneration, it has a proper relative Jacobian. Let $\mathcal{L}_M$ be a line bundle on $\mathcal{M}$ such that $\mathcal{L}_M|_M = L_M$, and\[ \deg(\mathcal{L}_M|_M) \gg 0; \]
\[ \deg(\mathcal{L}_M|M^{(i)}) \gg 0. \]
Step 6. We construct a Weierstrass fibration \( \mathcal{X} \to B \) with fundamental line bundle \( \mathcal{L} := \omega_B^{-1} \otimes \mathcal{O}_B(-B) \otimes \mathcal{L}_M \), which restricts to \( \omega_B^{-1} \otimes L_M \) on \( B \) by adjunction. By the Leray spectral sequence,
\[
H^1(B, \mathcal{L}^4 \otimes \mathcal{O}_B(-B)) = H^1(B, \omega_B^{-4} \otimes L_M^4 \otimes \mathcal{O}_B(-B)); \\
H^1(B, \mathcal{L}^6 \otimes \mathcal{O}_B(-B)) = H^1(B, \omega_B^{-6} \otimes L_M^6 \otimes \mathcal{O}_B(-B)).
\]
Both of these vanish by Corollary 14. Hence, we get surjections of Weierstrass coefficient spaces:
\[
H^0(B, \mathcal{L}^4) \twoheadrightarrow H^0(B, \omega_B^{-4} \otimes L_M^4); \\
H^0(B, \mathcal{L}^6) \twoheadrightarrow H^0(B, \omega_B^{-6} \otimes L_M^6).
\]
This allows us to choose \( \mathcal{X} \to B \) extending \( X \to B \). Since \( \mathcal{L} \) restricts to \( \mathcal{O}_{P^1}(2) \) on each gluing locus in the central fiber, we have
\[
\mathcal{X}_0 = X \cup_{K3} X^{(1)} \cup_{K3} \cdots \cup_{K3} X^{(m)},
\]
the desired star-shaped degeneration. Note that \( X^{(i)} \to \mathbb{P}E^{(i)} \) satisfies the second reduction condition vacuously, since the projective bundle is not blown up.

Since the GW theory of a K3 surface is trivial, the degeneration formula of [15] implies that
\[
F_{0,\ell}^{\mathcal{X}_i}(q) = F_{0,\ell}^{X}(q) + \sum_{i=1}^{m} F_{0,\ell}^{X^{(i)}}(q).
\]
The properties stated in Theorem 5 are linear, so it suffices to prove them for \( \mathcal{X}_t \) and for the extremal components \( X^{(i)} \), all of which satisfy the reduction conditions.

4. Hodge Theory
Recall that a (smooth) K3 surface \( S \) has middle cohomology lattice
\[
(7) \quad H^2(S, \mathbb{Z}) \simeq \text{II}_{3,19}.
\]
The K3 surfaces that arise in this paper are elliptic; their Neron-Severi group contains a zero section class \( z \) and a fiber class \( f \), which span a sublattice:
\[
U \simeq \begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix}.
\]
Any primitive embedding \( U \subset \text{II}_{3,19} \) has orthogonal complement \( \Lambda \simeq \text{II}_{2,18} \). Via the isomorphism (7), a holomorphic 2-form \( \Omega \) on \( S \) will lie in
\[
\Lambda \otimes \mathbb{C} \subset H^2(S, \mathbb{C}).
\]
This allows us to define a period domain for \( U \)-polarized K3 surfaces. The following general definition is standard:

**Definition 15.** Let \( \Lambda \) be an even unimodular lattice of signature \( (2, l) \). We set
\[
\mathcal{P}(\Lambda) := \{ \omega \in \mathbb{P}(\Lambda \otimes \mathbb{C}) : (\omega, \omega) = 0, (\omega, \overline{\omega}) > 0 \}^+.
\]
It is well known that $\mathcal{D}(\Lambda)$ is a Hermitian symmetric domain of Type IV and complex dimension $l$. To remove the ambiguity in the marking isomorphism $[7]$, we quotient $\mathcal{D}(\Lambda)$ by the automorphism group $\Gamma$ of $\Lambda$ to obtain the global period space $\Gamma \backslash \mathcal{D}(\Lambda)$, which is the analytification of a quasi-projective variety $[2]$. The tautological line bundle $\mathcal{O}(-1)$ on $\mathbb{P}(\Lambda \otimes \mathbb{C})$ descends to $\Gamma \backslash \mathcal{D}(\Lambda)$, and there it is called the Hodge line bundle.

**Definition 16.** For a positive integer $n$, the Noether-Lefschetz locus is given by:

$$NL_n := \Gamma \backslash \left( \bigcup_{v \in \Lambda, (v,v) = -2n} v^\perp \right) \subset \Gamma \backslash \mathcal{D}(\Lambda).$$

A component of $NL_n$ is called primitive if it is $\Gamma \backslash v^\perp$ for $v$ a primitive vector.

These are divisors in $\Gamma \backslash \mathcal{D}(\Lambda)$ whose components are abstractly isomorphic to the quotient of $\mathcal{D}(v^\perp)$ by the stabilizer of $v$ in $\Gamma$. They parametrize polarized Hodge structures of K3 type with an integral $(1,1)$-class $v$.

Both $\Gamma \backslash \mathcal{D}(\Lambda)$ and $NL_n$ are non-compact spaces. As such, we can define fundamental cycle classes in the Borel-Moore homology:

$$[NL_n] \in H_{BM}^{2l-2}(\Gamma \backslash \mathcal{D}(\Lambda), \mathbb{Q}) \cong H^2(\Gamma \backslash \mathcal{D}(\Lambda), \mathbb{Q}).$$

The theta correspondence techniques of Borcherds and Kudla-Millson produce the following modularity property:

**Theorem 17.** $[4] [12]$ Let $\lambda \in H^2(\Gamma \backslash \mathcal{D}(\Lambda), \mathbb{Q})$ denote the Chern class of the Hodge line bundle. The formal power series

$$\varphi(q) := -\lambda + \sum_{n \geq 1} [NL_n] q^n$$

is an element of $\text{Mod}(SL_2(\mathbb{Z}), \text{rk}(\Lambda)/2) \otimes H^2(\Gamma \backslash \mathcal{D}(\Lambda), \mathbb{Q})$.

We will drop the reference to $\Lambda$ in the notation $\mathcal{D}(\Lambda)$ from now on, as we are only concerned with the case $\Lambda \simeq H_{2,18}$. Since $\text{Mod}(SL_2(\mathbb{Z}),10)$ has dimension 1, Theorem 17 reduces to:

**Corollary 18.** For any homology class $\alpha \in H_2(\Gamma \backslash \mathcal{D}, \mathbb{Q})$,

$$\varphi_\alpha(q) := -\alpha \cdot \lambda + \sum_{n \geq 1} \alpha \cdot [NL_n] q^n \in \mathbb{Q} \cdot E_{10}(q),$$

where $E_{10}(q)$ is the Eisenstein series of weight 10.

In the context of Theorem 5 (with reductions), the composition $X \to B \to M$ may be viewed as a flat family of elliptic K3 surfaces, with singular members. The associated period map

$$\rho : M \to \Gamma \backslash \mathcal{D}$$

is indeterminate at the $b$ points of $M$ where the fiber of $B \to M$ is reducible. These correspond to degenerations

$$S_t \leadsto S_0 = R_1 \cup_E R_2$$
of a K3 surface to a normal crossing of two rational elliptic surfaces obtained by identifying a pair of smooth elliptic fibers $E_1 \subset R_1$ and $E_2 \subset R_2$. The limiting Hodge structure of this semistable degeneration is impure, so the rational period map $\mathbb{K}$ cannot be extended. Indeed, next we will analyze the mixed Hodge structures associated to $\mathbb{K}$, working over $\mathbb{Q}$ for convenience.

The Mayer-Vietoris sequence applied to $S_0 = R_1 \cup_E R_2$ yields
\begin{equation}
0 \to H^1(E) \to H^2(S_0) \to H^2(R_1) \oplus H^2(R_2) \to H^2(E) \to 0,
\end{equation}
so $H^2(S_0, \mathbb{Q}) = \mathbb{Q}^{21}$. The Clemens-Schmid sequence of the degeneration yields
\begin{equation}
0 \to \mathbb{Q} \cdot [E_1 - E_2] \to H^2(S_0) \to H^2_{\text{lim}}(S_t) \xrightarrow{N} H^2_{\text{lim}}(S_t).
\end{equation}
Here, $N$ is the logarithm of the unipotent monodromy operator. The image of $N$ is Poicaré dual to the vanishing cycles of the degeneration. These can be described geometrically: let $\gamma$ be a loop in the base $\mathbb{P}^1$ which gets pinched to a point $p$ in the degeneration of the base $\mathbb{P}^1 \dashrightarrow \mathbb{P}^1 \cup_p \mathbb{P}^1$. The elliptic fibration $S_t \to \mathbb{P}^1$ is trivial over $\gamma$, so the homological vanishing cycles can be described as
\[ \gamma \times H^1(E). \]

Now, the weight filtration $W_\bullet$ on $H^2_{\text{lim}}(S_t)$ is given by
\[ 0 \subset \text{im}(N) \subset \ker(N) \subset H^2(S_t) \quad 0 \subset \mathbb{Q}^2 \subset \mathbb{Q}^{20} \subset \mathbb{Q}^{22}. \]

Using (10) and (11), the associated graded groups may be identified as follows:
\[
\begin{align*}
\text{gr}_1^W H^2_{\text{lim}}(S_t) &\simeq H^1(E) \\
\text{gr}_2^W H^2_{\text{lim}}(S_t) &\simeq \{(r_1, r_2 \in H^2(R_1) \oplus H^2(R_2) : r_1|_{E_1} = r_2|_{E_2}) / \mathbb{Q} \cdot [E_1 - E_2].
\end{align*}
\]
These groups are endowed with pure Hodge structures: $H^1(E)$ has the Jacobian structure, and $\text{gr}_2^W$ has a weight 2 Hodge structure of Tate type.

The limiting mixed Hodge structure can be polarized by the sublattice $U = \langle z, f \rangle$. First, the log monodromy operator $N$ is skew-symmetric with respect to the cup product, so we have
\[ \ker(N) = \text{im}(N)\perp. \]
Since the zero section and fiber cycles extend over the central fiber of the degeneration $\mathbb{K}$, we have
\[ \langle z, f \rangle \subset \ker(N) \subset H^2_{\text{lim}}(S_t), \]
by the Invariant Cycle Theorem. This gives a polarized mixed Hodge structure on $\Lambda = U\perp$ with weight filtration:
\[ 0 \subset \text{im}(N) \subset \ker(N) \cap \Lambda \subset \Lambda. \]

**Remark 19.** The vanishing cycle group $\text{im}(N) \subset \Lambda$ is a rank 2 isotropic sublattice, which plays the role of $J$ in Section $4$. The quotient
\[ \text{gr}_2^W \Lambda \simeq \text{im}(N)\perp / \text{im}(N) \]
can be identified with the lattice
\[ H^2_{\text{prim}}(R_1) \oplus H^2_{\text{prim}}(R_2) \simeq (-E_8) \oplus (-E_8). \]
Here $H^2_{\text{prim}}(R_i)$ denotes the orthogonal complement of $\langle z_i, e_i \rangle$ in $H^2(R_i)$. 
5. Compactifying the Period Space

In this section, we recall the different compactifications of the period space $\Gamma \setminus \mathcal{D}$, following closely the exposition of [17]. We can then extend the period map $\rho$ from $\mathcal{S}$ over the boundary, and take its intersection product with the closures of the Noether-Lefschetz divisors.

The Satake-Baily-Borel compactification is a projective variety with singularities at the boundary. As a set, it can be described by adding to $\mathcal{D}$ a collection of boundary components corresponding to isotropic $\mathbb{Q}$-lines $I \subset \Lambda_{\mathbb{Q}}$ and isotropic $\mathbb{Q}$-planes $J \subset \Lambda_{\mathbb{Q}}$. The lines $I$ correspond to points $p_I$ in the boundary, and the planes $J$ correspond to curves; let $H_J$ denote the upper half-plane which occurs as the image of $\mathcal{D}$ under the linear projection $\pi_{J'}: \mathbb{P}(\Lambda_{\mathbb{C}}) / \mathbb{R} \rightarrow \mathbb{P}(\Lambda_{\mathbb{C}} / J_{\mathbb{C}})$. Taken together, these components admit an action of $\Gamma$ with quotient a locally compact Hausdorff space:

$$(\Gamma \setminus \mathcal{D})^{SBB} := \Gamma \setminus \left( \mathcal{D} \sqcup \bigsqcup_{\{I\}} \{p_I\} \sqcup \bigsqcup_{\{J\}} H_J \right)$$

$$= (\Gamma \setminus \mathcal{D}) \sqcup \left( \bigsqcup_{\Gamma \setminus \{I\}} \{p_I\} \right) \sqcup \left( \bigsqcup_{\Gamma \setminus \{J\}} \Gamma_J \setminus H_J \right).$$

Here, $\Gamma_J \subset \Gamma$ denotes the stabilizer of $J$ as a subspace of $\Lambda_{\mathbb{Q}}$, which acts on $H_J$ through an arithmetic quotient group (see below).

As explained in [5], the Satake-Baily-Borel compactification for our particular choice of $\Lambda$ is given by:

$$(\Gamma \setminus \mathcal{D})^{SBB} = (\Gamma \setminus \mathcal{D}) \sqcup (\mathbb{P}^1 \cup_p \mathbb{P}^1).$$

The $\mathbb{P}^1$ boundary components correspond to the two (up to $\Gamma$) isotropic planes $J \subset \Lambda_{\mathbb{Q}}$. The quotient lattices $J^{1,1}/J$ are isomorphic to $(-E_8)^{\otimes 2}$ and $-D_{16}^+$, respectively. Points in the boundary of $(\Gamma \setminus \mathcal{D})^{SBB}$ can be interpreted as associated graded pieces of limiting mixed Hodge structures. For example, the first boundary component is identified with the $j$-line $\Lambda_j^1 = PSL(2,\mathbb{Z}) \setminus \mathbb{H}$. In the degeneration [11], the $j$-invariant of $E$ determines the local extension of $\rho$ to the boundary.

Mumford et al. [11] have constructed smooth projective resolutions

$$\epsilon : (\Gamma \setminus \mathcal{D})^{\Sigma} \rightarrow (\Gamma \setminus \mathcal{D})^{SBB},$$

for a choice of fan decomposition $\Sigma$ of the nilpotent cone. For our purpose, it suffices to describe the local geometry of this construction away from the cusps $p_I$.

Fix an isotropic $\mathbb{Q}$-plane $J \subset \Lambda_{\mathbb{Q}}$. Let $G_J \subset O(\Lambda_{\mathbb{R}})$ be its stabilizer in the indefinite orthogonal group. By restriction, $G_J$ maps to $GL(J_{\mathbb{R}}) \times O(J_{\mathbb{R}})$, and the kernel $N_J$ is a real Heisenberg group. These facts are summarized in the
following diagram of real Lie groups:

\[
\begin{array}{c}
Z(N_J) \simeq \wedge^2 J_R \\
\downarrow \\
N_J \quad \rightarrow \quad G_J \quad \rightarrow \quad GL(J_R) \times O(J^+ / J)(\mathbb{R}) \\
\downarrow \\
(J^+ / J)^\vee \otimes J_R.
\end{array}
\]

Restricting to the discrete subgroup \( \Gamma_J = G_J \cap \Gamma \subset O(\Lambda_R) \), we obtain

\[
\begin{array}{c}
Z \\
\downarrow \\
\Gamma_N \quad \rightarrow \quad \Gamma_J \quad \rightarrow \quad \Gamma_J \subset GL(J) \times O(J^+ / J) \\
\downarrow \\
L,
\end{array}
\]

where \( L \) is a lattice in the real vector space \((J^+ / J)^\vee \otimes J_R\). Consider the composition

\[
\Gamma_N \backslash \mathcal{D} \rightarrow \Gamma_N \backslash \pi_J(\mathcal{D}) \rightarrow \pi_{J^+}(\mathcal{D}) = \mathbb{H}_J.
\]

The first map is a punctured disk \((\Delta^* \simeq \mathbb{Z} \backslash \mathbb{H})\) bundle, and the second map is principal real torus bundle whose structure group is the quotient of \((J^+ / J)^\vee \otimes J_R\) by \( L \). It inherits a complex structure such that the latter bundle is isogenous to a repeated fiber product of the tautological family of elliptic curves over \( \mathbb{H}_J \). If \( \Gamma_J \subset \Gamma_J \) is the subgroup fixing \( J \), we have the (orbifold) composition

\[
\Gamma_J \backslash \mathcal{D} \rightarrow \Gamma_J \backslash \pi_J(\mathcal{D}) \rightarrow \pi_{J^+}(\mathcal{D}) = \mathbb{H}_J.
\]

The fibers of the second map are now quotients of tori by the image of \( \mathbb{F}_J \) in \( O(J^+ / J, \mathbb{Z}) \). In the case where \( \Gamma = O(\Lambda) \), the image is all of \( O(J^+ / J, \mathbb{Z}) \), so the quotients are weighted projective spaces by the following remarkable theorem of Looijenga:

**Theorem 20.** [16] Let \( R \) be a root system, \( W(R) \) its Weyl group, and \( Q \) its dual root lattice. For any elliptic curve \( E \),

\[
W(R) \backslash (E \otimes \mathbb{Z} Q) \simeq \mathbb{P}(1, g_1, g_2, \ldots, g_N),
\]

with weights \( g_i \) given by the coefficients of the highest coroot. The hyperplane class pulls back to the sum of reflection hypertori in \( E \otimes \mathbb{Z} Q \).

In our case, the root system \( R = E_{8,2}^{\oplus 2} \) is self-dual. The local structure of the toroidal compactification \((\Gamma \backslash \mathcal{D})^\Sigma\) can be understood by completing \( \Gamma_J \backslash \mathcal{D} \) as follows. We have the (orbifold) composition

\[
\Gamma_J \backslash \mathcal{D} \rightarrow \Gamma_J \backslash \pi_J(\mathcal{D}) \rightarrow (\Gamma_J / \Gamma^J) \backslash \pi_{J^+}(\mathcal{D}) = (\Gamma_J / \Gamma^J) \backslash \mathbb{H}_J,
\]

where \((\Gamma_J / \Gamma^J) \subset GL(J)\) is arithmetic. The first map is a punctured disk bundle, whose filling gives the toroidal compactification. The Satake-Baily-Borel compactification is obtained by contracting the filled zero section. Hence, away from the

\[^2\text{In the case where } \Gamma \text{ is the full automorphism group of } \Lambda, \text{ the isogeny is an isomorphism.}\]
cusp $p_1$, the map $\epsilon : (\Gamma_\mathcal{D})^\Sigma \to (\Gamma_\mathcal{D})^{SBB}$ is the blow up of the boundary components $\Gamma_J \setminus \mathbb{H}_J$.

Points in the boundary of $(\Gamma_\mathcal{D})^\Sigma$ can be interpreted as mixed Hodge structures of local degenerations, and the morphism $\epsilon$ forgets the extension data, leaving the associated graded. Extension data are the same as one-motifs in the sense of Deligne:

**Theorem 21.** [7] Let $H^i$ denote a pure integral Hodge structure of weight $i$ (where $i = 1, 2$). Extensions in the abelian category MHS of mixed Hodge structures are identified with homomorphisms:

$$\text{Ext}^1_{\text{MHS}}(H^2, H^1) \simeq J^0 \text{Hom}(H^2, H^1),$$

where $J^0 := H_C/(F^0 H + H_Z)$ is the 0-th intermediate Jacobian.

In the relevant case where $H^i = \text{gr}_i^W \Lambda$, $H^2$ is of Tate type, so the Jacobian above simplifies to

$$J^0 \text{Hom}(H^2, H^1) \simeq \text{Hom}_\mathbb{Z}(H^2, J^1(H^1)) = (H^2)^\vee \otimes_{\mathbb{Z}} E,$$

which is isomorphic to the abelian variety $E^{16}$. Geometrically, the group homomorphism corresponds to the restriction of Cartier divisors, followed by summation using the group law on $E$:

$$\text{gr}_2^W \Lambda \simeq H^2_{\text{prim}}(R_1) \oplus H^2_{\text{prim}}(R_2) \to E.$$ 

In conclusion, we now have a concrete description of the extension of the period map (8) to $(\Gamma_\mathcal{D})^\Sigma$ in the case of the normal crossing degeneration (9).

### 6. Completed Noether-Lefschetz Numbers

In this section, we define topological Noether-Lefschetz numbers for complete one-parameter families $X \to M$ of elliptic K3 surfaces. Given such a family, we have a period map

$$\rho : M \to \Gamma_\mathcal{D},$$

defined away from the singular fibers. By the valuative criterion of properness, such a map extends to a morphism

$$\overline{\rho} : M \to (\Gamma_\mathcal{D})^\Sigma.$$  

The Noether-Lefschetz divisors $\text{NL}_n$ have Zariski closures

$$\overline{\text{NL}}_n \subset (\Gamma_\mathcal{D})^\Sigma.$$ 

The Hodge bundle extends to a line bundle on $(\Gamma_\mathcal{D})^{SBB}$ by the original construction of [23], and then we pull it back via the resolution $\epsilon : (\Gamma_\mathcal{D})^\Sigma \to (\Gamma_\mathcal{D})^{SBB}$. In a slight abuse of notation, we denote the Chern class of this extension by

$$\lambda \in H^2((\Gamma_\mathcal{D})^\Sigma, \mathbb{Q}).$$

**Definition 22.** The completed Noether-Lefschetz series for $X \to M$ is defined by setting $\alpha = \overline{\rho}_* [M] \in H^2((\Gamma_\mathcal{D})^\Sigma, \mathbb{Q})$, and then

$$\varphi_\Sigma^\alpha(q) = -\alpha \cdot \lambda + \sum_{n \geq 1} \alpha \cdot [\text{NL}_n] q^n.$$
The intersection numbers depend on the choice of fan $\Sigma$, but for our application they will not. The key result of this section applies to families with Type II degenerations in the sense of [13], but we prove it only for the relevant example [9].

**Theorem 23.** Let $X \to M$ be a complete family of elliptic $K3$ surfaces whose singular fibers are normal crossing: $R_1 \cup_E R_2$ with $R_i$ rational elliptic surfaces. Then the $q$-series $\varphi^\Sigma_{\alpha}(q)$ is a quasi-modular form of weight 10 for $SL_2(\mathbb{Z})$, and is independent of $\Sigma$.

**Proof.** By the Hodge theoretic description of the boundary components in Section 5, the extension $\mathcal{P}$ meets the boundary of $(\Gamma \setminus \mathcal{D})^{BB}$ in a finite subset of $\mathbb{A}_1^8$, once for each normal crossing singular fiber $R_1 \cup_E R_2$ at the point $j(E)$. It meets the divisorial boundary of $(\Gamma \setminus \mathcal{D})^2$ at points corresponding to the restriction map $[H^2_{prim}(R_1) \oplus H^2_{prim}(R_2) \to E] \in O(E_8^{\oplus 2}) \setminus \Hom_2((-E_8)^{\oplus 2}, E) \simeq WP^{16}$.

Using Lemma 20 below, we have a splitting $\mathcal{P}_\Sigma[M] = \alpha_0 + \alpha_1$, where $\alpha_0$ (resp. $\alpha_1$) is supported on the interior (resp. boundary) of $(\Gamma \setminus \mathcal{D})^2$. This allows us to rewrite $\varphi^\Sigma_{\alpha}(q) = \varphi_{\alpha_0}(q) + \varphi^\Sigma_{\alpha_1}(q)$.

Corollary 18 tells us that $\varphi_{\alpha_0}(q)$ is a scalar multiple of $E_{10}(q)$, so it suffices to compute $\varphi^\Sigma_{\alpha_1}(q)$. Since $\alpha_1$ is supported on the fibers of $\epsilon$, the intersection products take place in weighted projective spaces $WP^{16} \subset (\Gamma \setminus \mathcal{D})^2$, which satisfy $H_2(WP^{16}, \mathbb{Q}) \simeq \mathbb{Q}$. From the local description of the toroidal compactification in Section 5, the boundary of the completed Noether-Lefschetz divisor $NL_n$ in the fibers of $\epsilon$ is given by

$$\partial NL_n \cap WP^{16} = O(E_8^{\oplus 2}) \setminus \bigcup_{v \in (-E_8)^{\oplus 2}} v^+ \subset O(E_8^{\oplus 2}) \setminus \Hom_2((-E_8)^{\oplus 2}, E).$$

The pull-back of the canonical generator $\tilde{\alpha} \in H_2(WP^{16}, \mathbb{Q})$ to $\Hom_2((-E_8)^{\oplus 2}, E) \simeq E^{16}$ is represented by a Weyl group invariant collection of elliptic curves. A system of linear equations represented by an integer matrix $M$ has $\det(M)^2$ solutions on an elliptic curve. Hence, $\tilde{\alpha} \cap v^+$ is a quadratic form in $v$. All Weyl group invariant quadratic forms on a root lattice are scalar multiples of $(v, v)$, so we have

$$\varphi^\Sigma_{\alpha_1}(q) = c \sum_{v \in E_8^{\oplus 2}} (v, v) q^{(v, v)/2} = cq \frac{d}{dq} \Theta_{E_8^{\oplus E_8}}(q) = cq \frac{d}{dq} E_8(q).$$

The last $E_8(q)$ refers to the Eisenstein series. The Hodge bundle $\lambda$ restricts to a trivial bundle on the fiber $WP^{16}$, so there is no constant term. \qed
Remark 24. By the results of [N], any Type II degeneration of K3 surfaces has a stable model with central fiber

\[ R_1 \cup_\mathbb{E} R_2, \]

a normal crossing of two rational surfaces along a smooth elliptic curve. In the case of elliptic K3 surfaces, [5] describes the other possibility as

\[ S_t \sim S_0 = \mathbb{F}_2 \cup_\mathbb{E} B_{16} \mathbb{F}_2 \]

which corresponds to the boundary component for \( J^1 / J \simeq D_{16}^+ \). We expect quasi-modularity of \( \varphi_{\Sigma}^\infty(q) \) to hold for such families as well, independent of \( \Sigma \).

Question 25. Let \( X \to M \) be a family of elliptic K3 surfaces with Type III degenerations. Is the completed Noether-Lefschetz series \( \varphi_{\Sigma}^\infty(q) \) quasi-modular? What is the dependence on the choice of fan \( \Sigma \)?

Lemma 26. The class \( \alpha = [\mathcal{M}] \in H_2(\Gamma \setminus \mathcal{D}^\Sigma, \mathbb{Q}) \) can be expressed as

\[ \alpha = \alpha_0 + \alpha_1, \]

where \( \alpha_0 \in H_2(\Gamma \setminus \mathcal{D}, \mathbb{Q}) \), and \( \alpha_1 \in H_2(\partial(\Gamma \setminus \mathcal{D})^\Sigma, \mathbb{Q}) \), pushed forward by inclusion.

Proof. This follows from the blow up description of \( (\Gamma \setminus \mathcal{D})^\Sigma \), but we a topological proof which will be easier to generalize to compactifications of non-algebraic period spaces. Consider the Mayer-Vietoris sequence for \( U = \Gamma \setminus \mathcal{D} \) and \( V \) a neighborhood of the fiber \( W_\mathbb{P} \) of the boundary component over \( \mathbb{A}^1 \):

\[ H_2(U) \oplus H_2(V) \to H_2(U \cup V) \to H_1(U \cap V). \]

We have \( \alpha \in H_2(U \cup V) \), so the obstruction to splitting lies in \( H_1(U \cap V) \), where \( U \cap V \) retracts to a circle bundle over \( W_\mathbb{P} \). The Serre spectral sequence computes the latter with \( E_2 \) page:

\[
\begin{align*}
H_0(W_\mathbb{P}, H_1(F)) &= \mathbb{Q} & H_1(W_\mathbb{P}, H_1(F)) &= 0 & H_2(W_\mathbb{P}, H_1(F)) &= \mathbb{Q} \\
H_0(W_\mathbb{P}, H_0(F)) &= \mathbb{Q} & H_1(W_\mathbb{P}, H_0(F)) &= 0 & H_2(W_\mathbb{P}, H_0(F)) &= \mathbb{Q}.
\end{align*}
\]

The transgression is given by the Chern class of the circle bundle, which kills \( H_1(U \cap V) \) since the normal bundle to \( \partial(\Gamma \setminus \mathcal{D})^\Sigma \) is negative on \( W_\mathbb{P} \). \( \square \)

7. Proof of Main Theorem

First, we use the reductions of Section 3 to prove that the completed period map \( \mathcal{P} : M \to (\Gamma \setminus \mathcal{D})^\Sigma \) meets the Noether-Lefschetz locus transversely.

Lemma 27. If \( \deg(L_M) \gg 0 \), then the Weierstrass model \( X \to B \) can be deformed to one such that

- The boundary \( \mathcal{P}(M) \cap \partial(\Gamma \setminus \mathcal{D})^\Sigma \) is disjoint from \( \overline{NL_n} \).
- \( \rho(M) \) intersects each \( NL_n \) transversely.
- \( \rho(M) \) is disjoint from pairwise intersections of distinct \( NL \) components.
Proof. Recall that \( \pi : X \to B \) is defined by a global Weierstrass equations with coefficients \( a_4 \in H^0(L^4), b_6 \in H^0(L^6) \), where \( L = \omega_B^{-1} \otimes L_M \). Varying these data gives deformations of \( X \). The restriction of \( \pi \) to a divisor \( D \) supported on the fibers \( B \simeq B_0 \mathbb{P}E \to M \) is a Weierstrass model with fundamental line bundle \( L|_D \). We have the restriction long exact sequence

\[
H^0(B, L^4) \to H^0(C, L^4|_D) \to H^1(B, L^4 \otimes \mathcal{O}(-D))
\]

\[
H^0(B, L^6) \to H^0(C, L^6|_D) \to H^1(B, L^6 \otimes \mathcal{O}(-D)),
\]

so any pair of Weierstrass coefficients on \( D \) have the restriction long exact sequence gives deformations of \( X \). By Lemma 27, \( \text{NS} \) has the restriction long exact sequence.

In light of Lemma 27, we can give a concrete description of the Kontsevich moduli space \( \mathcal{M}_6(X, \ell + nf) \), for \( X \to B \) very general. When \( n = 0 \), we have

\[
\mathcal{M}_6(X, \ell) \simeq \mathcal{M}_6(B, \ell) \simeq M,
\]

by pushing through the zero section \( z : B \to X \). For \( n \geq 2 \), the moduli space contains a finite reduced set corresponding to smooth rational curves. All other stable maps are obtained from these examples by adding vertical components which map to singular fibers of \( \pi : X \to B \).

Definition 28. For \( n \geq 2 \), let \( r_X(n) \) denote the number of smooth rational curves on a very general Weierstrass model \( X \) in class \( \ell + nf \).

Let \( S \to \mathbb{P}^1 \) be an elliptic surface. Its Mordell-Weil group \( \text{MW}(S/\mathbb{P}^1) \) is the finitely generated group of sections \( S \to \mathbb{P}^1 \), or equivalently the \( \mathbb{C}(\mathbb{P}^1) \)-rational points of the generic fiber. The short exact sequence of Shioda-Tate compares this group to the Néron-Severi group \( \text{NS}(S) \):

\[
0 \to V(S) \to \text{NS}(S) \to \text{MW}(S/\mathbb{P}^1) \to 0.
\]

The kernel \( V(S) \) is the subgroup spanned by vertical curve classes and the zero section class. The polarized version of the sequence is defined by taking the orthogonal complement of the sublattice \( U = \langle z, f \rangle \subset \text{NS}(S) \):

\[
0 \to V_{prim}(S) \to \text{NS}_{prim}(S) \to \text{MW}(S/\mathbb{P}^1) \to 0.
\]

By Lemma 27, \( \text{NS}_{prim}(S) \) has rank \( \leq 1 \), for the surfaces \( S \) in the family \( X \to M \). There are two cases:

- \( \text{MW}(S/\mathbb{P}^1) \simeq \mathbb{Z} \) so \( S \) contains nonzero section curves, or
- \( V_{prim}(S) \simeq \mathbb{Z} \) so \( S \) has an \( A_1 \) singularity.
Indeed, by Brieskorn’s simultaneous resolution, the limiting Hodge structure of a surface with an ADE singularity is pure, and it agrees with the Hodge structure of the resolved surface. Elliptic surfaces $S$ with $A_1$ singularities occur as $\pi^{-1}(\mathbb{P}^1)$, where $\mathbb{P}^1 \subset B$ is a line of the ruling tangent to the discriminant curve

$$\Delta = Z(4a_1^3 + 27b_2^2) \subset B.$$ 

The minimal resolution a Weierstrass elliptic surface with an $A_1$ singularity has a reducible fiber of Kodaira type $I_2$.

**Proposition 29.** If $MW(S/\mathbb{P}^1) = \mathbb{Z} \sigma$, then $\rho([S]) \in NL_{n,2}$ for $n = \sigma \cdot z + 2$ and $r \in \mathbb{N}$ arbitrary. If $S$ has an $A_1$ singularity, then $\rho([S]) \in NL_{r,2}$ for all $r \in \mathbb{N}$.

**Proof.** If $\sigma \in NS(S)$ is the class of a nonzero section curve, then its orthogonal projection to $NS_{prim}(S)$ has self-intersection $-2\sigma \cdot z - 4$ by a straightforward lattice calculation. If $S$ has an $A_1$ singularity, then the exceptional class on the resolved surface has self-intersection $-2$. In general, we have $NL_n \subset NL_{n,r}$ since $v \in NS(S)$ implies $rv \in NS(S)$.

**Proposition 30.** If $\sigma$ is the class of a nonzero section curve on $S = \pi^{-1}(\mathbb{P}^1)$, and $\iota : S \rightarrow X$ is the inclusion map, then $\iota_*\sigma = \ell + nf$ with $n = \sigma \cdot z + 2$.

**Proof.** The class $\iota_*\sigma$ can be determined by intersecting with two complementary divisors in $X$: $z_*[B]$ and $\pi^*$ of any section of $B \rightarrow M$. The shift by 2 occurs because the normal bundle to the section curve is $L^\vee \simeq \mathcal{O}_{\mathbb{P}^1}(-2)$.

This explains the lack of smooth rational curves in class $\ell + f$. The actual counts $r_X(n)$ can be obtained by intersecting with the Noether-Lefschetz cycles, and then subtracting off the contributions from $A_1$ singularities.

**Theorem 31.** For $X$ very general, the counts $r_X(n)$ have the following structure:

$$\sum_{n \geq 2} r_X(n)q^n = \varphi(q) - \frac{a_1}{2} \Theta_1(q) + \frac{a_1}{2} + \deg_M(\lambda);$$

$$\Theta_1(q) : = \sum_{r \in \mathbb{Z}} q^{r^2}.$$ 

Here $\varphi(q) \in \mathbb{Q}[E_2, E_4, E_6]^{10}$, and $a_1$ is the number of $A_1$ singular surfaces.

**Proof.** The completed period map $\overline{\calP} : M \rightarrow (\Gamma \setminus \mathcal{D})^\Sigma$ is defined as a map of varieties. To properly compute the $A_1$ contribution, we lift it to a map of stacks. Let $M' \rightarrow M$ be a double cover ramified at the $a_1$ points where the fiber of $X \rightarrow M$ has an $A_1$ singularity. The total space of the base change $X' \rightarrow M'$ contains $a_1$ threefold $A_1$ singularities, and it admits a small resolution $Y$. The new family $Y \rightarrow M'$ is a Brieskorn simultaneous resolution, and it satisfies the assumptions of Theorem 29, so we have a quasi-modularity statement for the period map $\overline{\calP} : M' \rightarrow (\Gamma \setminus \mathcal{D})^\Sigma$:

$$-\overline{\calP}_*[M'] \cdot \lambda + \sum_{n \geq 0} \overline{\calP}_*[M'] \cap [N_{T_n}] q^n \in \mathbb{Q}[E_2, E_4, E_6]^{10}.$$ 

Since the extra sections occur away from the $A_1$ singularities, we have

$$-\overline{\calP}_*[M'] \cdot \lambda + \sum_{n \geq 0} \overline{\calP}_*[M'] \cap [N_{T_n}] q^n = -2 \overline{\calP}_*[M] \cdot \lambda + 2 \sum_{n \geq 2} r_X(n)q^n + a_1 \Theta_1(q).$$

The result now follows by adjusting the constant term to 0. □
To pass from actual counts to Gromov-Witten invariants, we use the concrete description of $\overline{M}_0(X, \ell + nf)$ and the conifold transition formula of Li-Ruan:

**Theorem 32.** [14] Suppose that $Y$ and $Y_c$ are Calabi-Yau threefolds related by a conifold transition, that is a small contraction of disjoint $\mathbb{P}^1$’s to $A_1$ singularities, followed by a smoothing deformation. Then there is a surjective homomorphism $\phi: H_2(Y, \mathbb{Z}) \to H_2(Y_c, \mathbb{Z})$, and for any homology class $\alpha \in H_2(Y_c, \mathbb{Z})$,

$$N^Y_{g, \alpha} = \sum_{\phi(\gamma) = \alpha} N^Y_{g, \gamma}.$$  

This formula will cancel with the $a_1 \Theta_1(q)$ term above.

**Theorem 33.** The genus 0 Gromov-Witten invariants of $X$ in the classes $\ell + nf$ have the following structure:

$$F^X_{0, \ell}(q) = \sum_{n \geq 0} N^X_{0, \ell + nf} q^n = \varphi(q) \cdot \frac{q}{\Delta(q)}.$$  

**Proof.** Following [13], we deform $X'$ by moving the branch points of the double cover $B' \to B$ to general position. This gives a smoothing $Y_c$ of $X'$, related to $Y$ by a conifold transition. If we allow the branch points to collide in pairs, $B'$ degenerates to two copies of $B$ glued at $4g$ general points of $B$. The base change via $\pi : X \to B$ gives a normal crossing degeneration:

$$Y_c \sim X \cup_D X,$$

where $D$ is a disjoint union of smooth K3 surfaces fibers of $X \to M$. The degeneration formula of [15] gives $F^X_{0, \ell}(q) = 2 F^X_{0, \ell}(q)$. Theorem 32 in turn says that

$$N^Y_{0, \ell + nf} = \sum_{i=1}^{a_i} \sum_{r \in \mathbb{Z}} N^Y_{0, \ell + nf + r \gamma_i},$$

where the $\gamma_i = [\mathbb{P}^1]$ are exceptional curves of the small resolution. To compute $N^Y_{0, \ell + nf}$, observe that any stable map has a smooth horizontal component of class $\ell + jf$ and a collection of $(n-j)$ vertical components, all inside a K3 surface. The reduced invariant of the K3 surface is computed in [6] as $[q/\Delta(q)]_k$, where the self-intersection of the class in the K3 surface is $2k - 2$. This description decomposes the Kontsevich space $\overline{M}_0(X, \ell + nf)$ into closed-open substacks indexed by $j$, so the virtual class integrals sum:

$$N^Y_{0, \ell + nf} = \sum_{j=2}^{n} 2r_X(j) [q/\Delta(q)]_{n-j} - 2 \deg_M(\lambda) [q/\Delta(q)]_n.$$

Stable maps in class $\ell + nf + r \gamma_i$ ($r \neq 0$) are localized to the resolved K3 surface containing $\gamma_i$, where the curve class has self-intersection $2n - 2 - 2r^2$, so for $r \neq 0$,

$$N^Y_{0, \ell + nf + r \gamma_i} = [q/\Delta(q)]_{n-r^2}.$$  

Summing up the full potential function, we get non-trivial cancellation:

$$F^Y_{0, \ell}(q) = 2 \left( \varphi(q) - \frac{a_1}{2} \Theta_1(q) + \frac{a_1}{2} + \deg_M(\lambda) \right) \cdot \frac{q}{\Delta(q)} - 2 \deg_M(\lambda) \cdot \frac{q}{\Delta(q)}$$

$$= 2 \varphi(q) \cdot \frac{q}{\Delta(q)}.$$
Theorem 34. If \( \varphi(q) \) is the quasi-modular form from Theorems 31 and 33, then
\[
\frac{\partial \varphi}{\partial E_2} = -\frac{b}{12} \cdot E_8(q),
\]
where \( b \) is the number of broken fibers of \( B \to M \).

Proof. From the proof of Theorem 23, we know that
\[
\varphi(q) = -\deg_M(\lambda) E_{10}(q) + cq \frac{d}{dq} E_8(q).
\]
We compute \( \deg_M(\lambda) \) by expressing the Hodge bundle as
\[
q_*(\omega_{X/M}) = f_*(\pi_*\omega_{X/B} \otimes \omega_{B/M})
\]
and using the projection formula:
\[
q_*(\omega_{X/M}) = f_*(L \otimes \omega_B \otimes f^*\omega_M^{-1}) = L_M \otimes \omega_M^{-1}.
\]
Hence \( \deg_M(\lambda) = \deg(L_M) + 2(1 - g) \) for \( g = g(M) \). Next, we compute \( a_1 \) by counting vertical tangents to the discriminant curve \( \Delta \subset B \). The morphism \( \Delta \to M \) has degree 24, and if we compose with the normalization \( \nu : \tilde{\Delta} \to \Delta \), we obtain a morphism of smooth curves. The number \( r \) of ramification points is \( a_1 + \kappa(\Delta) \), since each cusp contributes once to the ramification. By the Riemann-Hurwitz formula,
\[
a_1 = r - \kappa = 2g(\Delta) - 2 + 48(1 - g) - \kappa
\]
\[
= 2p_a(\Delta) - 2 - 3\kappa + 48(1 - g)
\]
\[
= \Delta \cdot (K_B + \Delta) - 3 \cdot c_1(L^4) \cdot c_1(L^6) + 48(1 - g)
\]
\[
= 60K_B^2 - 132K_B \cdot c_1(L_M) + 48(1 - g).
\]
Now since \( K_B^2 = 8(1 - g) - b \) and \( K_B \cdot c_1(L_M) = -2 \deg(L_M) \), we are left with
\[
a_1 = 264 \deg(L_M) + 528(1 - g) - 60b.
\]
By Theorem 31 we have
\[
\sum_{n \geq 2} r_X(n)q^n = \varphi(q) - \frac{a_1}{2} \Theta_1(q) + \frac{a_1}{2} + \deg_M(\lambda).
\]
Extracting the coefficient of \( q^1 \), we find
\[
c = \frac{a_1 - 264 \deg_M(\lambda)}{480} = -\frac{b}{8}.
\]
The result now follows from this equality and the Ramanujan identity
\[
q \frac{d}{dq} E_4 = \frac{E_2 E_4 - E_6}{3}.
\]
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