TWISTED LINNIK IMPLIES OPTIMAL COVERING EXponent FOR $S^3$

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Abstract. We show that a twisted variant of Linnik’s conjecture on sums of Kloosterman sums leads to an optimal covering exponent for $S^3$.

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1. Introduction

For any $r > 0$, let $S^3(r) \subset \mathbb{R}^4$ denote the hypersphere

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = r^2,$$

of radius $r$. (We set $S^3 = S^3(1)$ for the unit hypersphere.) In his letter [7] about the efficiency of a universal set of quantum gates, Sarnak has raised the question of how well one can approximate points on $S^3$ by rational and $S$-integral points of small height.

Consider the ball $B_\varepsilon(\xi) = \{x \in \mathbb{R}^4 : \|x - \xi\| < \varepsilon\}$, for any $\varepsilon > 0$ and any $\xi \in S^3$, where $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^4$. The spherical cap $S^3 \cap B_\varepsilon(\xi)$ has volume $\frac{4\pi}{3}\varepsilon^3 + O(\varepsilon^5)$. Given $r > 0$ such that $r^2 \in \mathbb{Z}$, we let $\lambda(r)$ denote the maximal volume of any cap $S^3 \cap B_\varepsilon(\xi)$, for $\xi \in S^3$, which contains no points of the form $x/r$, for $x \in \mathbb{Z}^4$. Sarnak then defines the covering exponent to be

$$K(S^3) = \limsup_{r \to \infty} \frac{\log(#S^3(r) \cap \mathbb{Z}^4)}{\log((\text{vol } S^3)/\lambda(r))}. \tag{1.1}$$

Date: September 21, 2016.

2010 Mathematics Subject Classification. 11E25 (11D09, 11P55, 81P68).
As is well-known, we have vol $S^3 = 2\pi^2$ and $\# S^3(r) \cap \mathbb{Z}^4 = c_r r^2 (1 + o(1))$, as $r \to \infty$, for an appropriate (slowly growing) function $c_r$ of $r$. According to [5, Thm. 20.9], we have $\log r \gg c_r \gg \epsilon r^{-\epsilon}$, for any $\epsilon > 0$, as long as the largest power of $2$ dividing $r^2$ is bounded absolutely. In particular, the limit in (1.1) should be understood as running over such $r$'s.

The “big holes” phenomenon, which is described in [7, Appendix 2], shows that $K(S^3) \geq \frac{4}{3}$. Sarnak conjectures that this lower bound should be the truth, before using automorphic forms for $\text{PGL}_2$ to show that $K(S^3) \leq 2$ in [7, Appendix 1]. This upper bound was recovered by Sardari [6] by incorporating Kloosterman’s method into a smooth $\delta$-function variant of the Hardy–Littlewood method due to Duke, Friedlander and Iwaniec [2], and later developed extensively by Heath-Brown [3]. (Sardari’s work is actually much more general and, in fact, he obtains the optimal covering exponent $K(S^{n-1}) = 2 - \frac{2}{n-1}$ for any $n > 4$.)

Our main result establishes Sarnak’s conjecture for $S^3$, under the assumption of a natural variant of the Linnik conjecture about sums of Kloosterman sums.

For any $m, n \in \mathbb{Z}$ and any $c \in \mathbb{N}$, recall the definition

$$S(m, n; c) = \sum_{x \mod c \atop (x, c) = 1} e_c(mx + nx),$$

(1.2)

of the Kloosterman sum, where $\overline{x}$ denotes the multiplicative inverse of $x$ modulo $c$. We propose the following conjecture.

**Conjecture 1.1** (Twisted Linnik). Let $\alpha \in [-2, 2]$ and let $m, n \in \mathbb{Z}$ be non-zero. Let $k \in \mathbb{N}$ and let $a \in \mathbb{Z}/k\mathbb{Z}$. Then

$$\sum_{c \equiv a \mod k} \frac{S(m, n; c)}{c} e \left( \frac{2\sqrt{mn}}{c} \alpha \right) \ll_{\epsilon, k} (|mn|X)^{\epsilon},$$

for any $\epsilon > 0$.

For comparison, on invoking the triangle inequality, it follows from Weil’s bound for the Kloosterman sum (see (3.1)) that the left hand side has size $O_{\epsilon}(|mn|^{\epsilon} X^{\frac{1}{2} + \epsilon})$. The usual Linnik conjecture corresponds to taking $\alpha = 0$ in Conjecture 1.1. The state of play concerning the case $\alpha = 0$ is discussed in work of Sarnak and Tsimerman [8]. As evidence for Conjecture 1.1, Steiner [10] has shown that the unconditional estimates achieved in [8] for $\alpha = 0$ continue to hold for any $\alpha \in \mathbb{R}$ such that $|\alpha| \leq 1 - \delta$, for a fixed $\delta > 0$. Unfortunately, these estimates are not sharp enough to prove that $K(S^3) < 2$ unconditionally.

Using Sardari’s work [6] as a base, we shall establish the following result.

**Theorem 1.2.** Assume the twisted Linnik conjecture. Then $K(S^3) = \frac{4}{3}$.
The proof of this theorem is founded on exploiting extra cancellation in sums of the form
\[
\sum_{q \equiv 1 \mod 2, q \leq Q} q^{-2} S(r^2, c_1^2 + c_2^2 + c_3^2 + c_4^2; q)e_q(-2rc_\xi)K_q(c),
\]
for non-zero vectors \(c \in \mathbb{Z}^4\), where \(K_q(c)\) is a certain 4-dimensional oscillatory integral that is revealed through an examination of (1.1) and (1.2). (There are similar expressions for \(q \equiv \{0, 2\} \mod 4\).) Whereas Sardari brings the modulus sign inside, before invoking Weil’s bound to estimate the Kloosterman sum, our goal is take advantage of sign changes in it. There are three key problems in carrying out this plan.

The first two problems arise when using partial summation to remove the factor \(q^{-1}e_q(-2rc_\xi)K_q(c)\). For typical vectors \(c\), the derivative of \(e_q(-2rc_\xi)\) with respect to \(q\) is very large. This deficiency is what lies behind our need to study sums of Kloosterman sums twisted by an exponential factor, as in Conjecture\(\ref{conj:main}\). Similarly, the derivative \(\frac{\partial}{\partial q}K_q(c)\) is also too large, unless \(q\) has exact order of magnitude \(Q\). This presents our second problem. To circumvent this difficulty we shall use stationary phase to get an asymptotic expansion of \(K_q(c)\), to arbitrary precision, before using partial summation to rid ourselves of each term in the asymptotic expansion separately.

Finally, consider the expression in the left hand side of Conjecture\(\ref{conj:main}\). The third problem comes from a need for complete uniformity in \(m\) and \(n\) in any unconditional treatment of this sum. In fact, in the present situation, we are faced with the harder Selberg range, where \(\sqrt{|mn|} > X\). Although Steiner \[10\] has achieved unconditional bounds that go beyond the Weil bound, these fall short of yielding an unconditional proof that \(K(S^3) < 2\). Thus, in this note, we content ourselves with showing that the optimal covering exponent is a consequence of our twisted version of Linnik’s conjecture.

**Remark 1.3.** As outlined by Sarnak \[7\], the study of \(K(S^3)\) has its roots in the Solovay–Kitaev theorem in theoretical quantum computing. Consider the single qubit gate set \(S = \{s_1, s_2, s_3\} \subset SU(2)\), where
\[
s_1 = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 + 2i & 0 \\ 0 & 1 - 2i \end{pmatrix}, \quad s_2 = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 2i \\ 2i & 1 \end{pmatrix}, \quad s_3 = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}.
\]
This set is symmetric and topologically dense in \(SU(2)\). Sarnak defines a covering exponent \(K(S)\), which measures how efficiently the free group \(\langle S \rangle\) generated by \(S\) covers \(SU(2)\). It follows from Theorem \[1.2\] that \(K(S) = \frac{4}{3}\) under the assumption of the twisted Linnik conjecture.
Acknowledgements. The authors are grateful to Peter Sarnak for his encouragement. While working on this paper the first author was supported by ERC grant 306457.

2. Preliminaries

2.1. Overview. Let \( r \in \mathbb{N} \) such that the power of 2 dividing \( r \) is bounded absolutely. Let \( N = 4r^2 \). Fix a choice of \( \xi \in \mathbb{R}^4 \) such that \( F(\xi) = 1 \), where \( F \) henceforth denotes the non-singular quadratic form

\[
F(x) = x_1^2 + x_2^2 + x_3^2 + x_4^2.
\]

For any \( \varepsilon > 0 \), we let

\[
S_\varepsilon(N) = \left\{ x \in \mathbb{Z}^4 : F(x) = N, \frac{\|x/\sqrt{N} - \xi\|}{\varepsilon} < 1 \right\}.
\]

Our primary objective is to produce a lower bound on \( \varepsilon \), in terms of \( N \), which is sufficient to ensure that \( S_\varepsilon(N) \) is non-empty. Sardari’s work shows that \( S_\varepsilon(N) \neq \emptyset \) if \( \varepsilon \gg \delta \sqrt{N}^{-1} \), for any \( \delta > 0 \). This implies that

\[
\lambda(r) \ll \delta r^{-\frac{3}{2}} + 2^\delta,
\]

for any \( \delta > 0 \), whence \( K(S^3) \leq 2 \) in (1.1). Assuming Conjecture 1.1 we shall show that \( S_\varepsilon(N) \neq \emptyset \) if \( \varepsilon \gg \delta N^{-\frac{1}{4}} + 2, \) for any \( \delta > 0 \). This implies that \( \lambda(r) \ll \delta r^{-\frac{3}{2}} + 2^\delta, \) whence \( K(S^3) \leq 2 \), as required to complete the proof of Theorem 1.2.

2.2. Notation. We denote by \( \| \cdot \| \) the usual Euclidean norm, so that \( \|x\| = \sqrt{F(x)} \) on \( \mathbb{R}^4 \). Throughout our work we reserve \( \delta > 0 \) for a small positive parameter.

One of the key innovations in Sardari’s work [6] concerns the introduction of a new basis given by the tangent space of \( F \) at \( \xi \) and we proceed to recall the construction here. Let \( e_4 = \xi \). (This is the unit vector in the direction of \( \nabla F(\xi) = 2\xi \).) Choose an orthonormal basis \( e_1, e_2, e_3 \) for the tangent space \( T_\xi(F) = e_4 \). Recalling that \( F(\xi) = 1 \), it therefore follows that

\[
F(u_1e_1 + \cdots + u_4e_4) = F(u),
\]

for any \( u \in \mathbb{R}^4 \). Finally, any vector \( b \in \mathbb{R}^4 \) can be written \( b = \sum_{i=1}^4 \hat{b}_i e_i \), with \( \hat{b}_i = b.e_i \), for \( 1 \leq i \leq 4 \).

2.3. Activation of the circle method. We begin by choosing a smooth function \( w_0 : \mathbb{R} \to \mathbb{R}_{\geq 0} \) with unit mass, such that \( \text{supp}(w_0) = [-1, 1] \). We will work with the weight function \( w : \mathbb{R}^4 \to \mathbb{R}_{\geq 0} \), given by

\[
w(x) = w_0 \left( \frac{\|x - \xi\|}{\varepsilon} \right) w_0 \left( \frac{2\xi.(x - \xi)}{\varepsilon^2} \right).
\]
Let
\[ \Sigma(w) = \sum_{F(x) = N} w \left( \frac{x}{\sqrt{N}} \right), \]
for any \( N \in 4\mathbb{N} \). We want conditions on \( \varepsilon \), in terms of \( N \), under which \( \Sigma(w) > 0 \). Indeed, if \( \Sigma(w) > 0 \), then there exists a vector \( x \in \mathbb{Z}^4 \) such that \( F(x) = N \) and
\[ \| x/\sqrt{N} - \xi \| < \varepsilon, \quad |2\xi.(x/\sqrt{N} - \xi)| < \varepsilon^2. \]
It follows from Sardari’s argument that \( \Sigma(w) > 0 \) if \( \varepsilon \gg \delta N^{-\frac{1}{6}+\delta} \), for any \( \delta > 0 \). Our goal is to draw the same conclusion provided that \( \varepsilon \gg \delta N^{-\frac{1}{4}+\delta} \).

A few words are in order regarding the inequality \( |2\xi.(x/\sqrt{N} - \xi)| < \varepsilon^2 \) that is enshrined in our counting function \( \Sigma(w) \). Suppose that \( \| x/\sqrt{N} - \xi \| < \varepsilon \). Then we may write \( x/\sqrt{N} = \xi + \varepsilon z \), with \( \| z \| < 1 \). Under this change of variables, the inequality \( |2\xi.(x/\sqrt{N} - \xi)| < \varepsilon^2 \) is equivalent to \( |2\xi.z| < \varepsilon \), and
\[ F(x) - N = N (2\varepsilon \xi z + \varepsilon^2 F(z)). \]

Thus, we must have \( |2\xi.z| < \varepsilon \) when the left hand side vanishes. Moreover it is clear that \( F(x) - N \ll \varepsilon^2 N \) for any \( x \) such that \( w(x/\sqrt{N}) \neq 0 \).

One “level lowering” effect of this is that we are allowed to take
\[ Q = \varepsilon \sqrt{N} \]
in the version of the circle method recorded by Heath-Brown [3, Thm. 2], rather than \( Q = \sqrt{\varepsilon N} \), as might at first appear. We conclude that there exists a constant \( c_Q = 1 + O_A(Q^{-A}) \), for any \( A > 0 \), such that
\[ \Sigma(w) = c_Q \frac{Q}{2} \sum_{q=1}^{\infty} \sum_{c \in \mathbb{Z}^4} q^{-4} S_q(c) I_q(c), \quad (2.2) \]
where
\[ S_q(c) = \sum_{a \mod q} \sum_{b \mod q} e_q(a \{ F(b) - N \} + b.c), \]
\[ I_q(c) = \int_{\mathbb{R}^4} w \left( \frac{x}{\sqrt{N}} \right) h \left( \frac{q}{Q^2}, \frac{F(x) - N}{Q^2} \right) e_q(-c.x) \, dx. \quad (2.3) \]
Here \( h : (0, \infty) \times \mathbb{R} \to \mathbb{R} \) is a certain function such that \( h(x, y) \ll x^{-1} \) for all \( y \) and \( h(x, y) = 0 \) unless \( x \leq \max\{1, 2|y|\} \). In particular, only values of \( q \ll Q \) contribute to \( \Sigma(w) \) in \( (2.2) \). Thus, in all that follows, we may henceforth assume that \( Q \geq 1 \); viz. \( \varepsilon^{-1} \leq \sqrt{N} \).

We shall prove that Conjecture [11] implies \( \Sigma(w) > 0 \) if \( \varepsilon \gg \delta N^{-\frac{1}{6}+\delta} \), for any \( \delta > 0 \). In fact we shall establish an asymptotic formula for \( \Sigma(w) \), in which the main term involves a pair of constants \( \sigma_\infty \) and \( \mathcal{E} \). The constant \( \sigma_\infty \) is equal
to the weighted real density of points on $S^3$ and is given explicitly in (5.2). The constant $\mathcal{G}$ is the usual product of non-archimedean local densities, with value

$$\mathcal{G} = \prod_p \sigma_p, \quad \sigma_p = \lim_{k \to \infty} p^{-3k} \# \{ x \in (\mathbb{Z}/p^k \mathbb{Z})^4 : F(x) \equiv N \mod p^k \}. \quad (2.4)$$

We may now record our main result.

**Theorem 2.1.** Assume Conjecture 1.1. Then, for any $\delta > 0$, we have

$$\Sigma(w) = \frac{\varepsilon^3 N \sigma \mathcal{G}}{2} + O_{\delta} \left( \varepsilon^{4} N^{1+\delta} + \varepsilon^{2} N^{1+\delta} + \varepsilon N^{1+\delta} \right).$$

We shall see that $\sigma \gg 1$ in (5.2). Likewise, as remarked upon by Sardari [6, Remark 1.4], we have $\mathcal{G} \gg N^{-\delta}$ for any $\delta > 0$, if the power of 2 dividing $N$ is bounded. Thus Theorem 2.1 implies Theorem 1.2.

The remainder of the paper is as follows. In §3 we shall explicitly evaluate the sum $S_q(c)$ using Gauss sums. Next, in §4, we shall study the oscillatory integrals $I_q(c)$ using stationary phase. Finally, in §5, we shall combine the various estimates and complete the proof of Theorem 2.1.

### 3. Gauss sums and Kloosterman sums

In this section we explicitly evaluate the exponential sum $S_q(c)$ in (2.3), for $c \in \mathbb{Z}^4$ and relate it to the Kloosterman sum $S(m,n;c)$ in (1.2). The latter sum satisfies the well-known Weil bound

$$|S(m,n;c)| \leq \tau(c) \sqrt{(m,n,c) c}, \quad (3.1)$$

where $\tau$ is the divisor function.

Recalling that $N \in 4\mathbb{N}$, it will be convenient to write $N = 4N'$ for $N' \in \mathbb{N}$. We have

$$S_q(c) = \sum_{a \mod q}^* e_q(-4aN') \prod_{i=1}^4 G(a, c_i; q), \quad (3.2)$$

where

$$G(s,t; q) = \sum_{b \mod q} e_q \left( sb^2 + tb \right),$$

for given non-zero integers $s, t, q$ such that $q \geq 1$. The latter sum is classical and may be evaluated. Let

$$\delta_n = \begin{cases} 0, & \text{if } n \equiv 0 \mod 2, \\ 1 & \text{if } n \equiv 1 \mod 2, \end{cases} \quad \epsilon_n = \begin{cases} 1, & \text{if } n \equiv 1 \mod 4, \\ i, & \text{if } n \equiv 3 \mod 4. \end{cases}$$

The following result is recorded in [11, Lemma 3], but goes back to Gauss.
Lemma 3.1. Suppose that $(s, q) = 1$. Then

$$
\mathcal{G}(s, t; q) = \begin{cases} 
\delta_q e\left( -\frac{q}{q} \right) e\left( -\frac{q^2}{q} \right) & \text{if } q \text{ is odd,} \\
2\delta_t e\left( \frac{2a}{v} \right) e\left( -\frac{8at^2}{v} \right) & \text{if } q = 2v, \text{ with } v \text{ odd,} \\
(1 + i)^{-1}(1 - \delta_t)\sqrt{q} \left( \frac{2}{q} \right) e\left( -\frac{q^2}{4q} \right) & \text{if } 4 \mid q.
\end{cases}
$$

Our analysis of $S_q(c)$ now differs according to the 2-adic valuation of $q$. In each case we shall be led to an appearance of the Kloosterman sum (1.2).

Suppose first that $q \equiv 1 \pmod{2}$. Substituting Lemma 3.1 into (3.2) we directly obtain

$$
S_q(c) = q^2 \sum_{a \mod q}^{
\ast} e_q(-4aN' - 4aN''(c)) = q^2 S(N', F(c); q),
$$

since $S(A, tB; q) = S(tA, B; q)$ for any $t \in (\mathbb{Z}/q\mathbb{Z})^*$. If $q \equiv 2 \pmod{4}$ then we write $q = 2v$, for odd $v$. This time we obtain

$$
S_q(c) = 2^4 \delta_{1234} v^2 \sum_{a \mod q}^{
\ast} e_q(-4aN')e_v(-\overline{F}(c))
$$

$$
= 4\delta_{1234} q^2 S(N', F(c); q),
$$

since $e_v(-\overline{F}(c)) = e_q(-4aN'F(c))$.

If $q \equiv 0 \pmod{4},$ it follows from Lemma 3.1 that

$$
S_q(c) = -4(1 - \delta_{c1}) \ldots (1 - \delta_{c4}) \sum_{a \mod q}^{
\ast} e_q(-4aN')e_4(-\overline{F}(c)).
$$

Thus, in this case, we find that

$$
S_q(c) = \begin{cases} 
0 & \text{if } 2 \nmid c, \\
-4q^2 S(N, F(c'); q) & \text{if } c = 2c' \text{ for } c' \in \mathbb{Z}^4.
\end{cases}
$$

4. Oscillatory integrals

Recall the definition (2.3) of $I_q(c)$, in which $w$ is given by (2.1). We make the change of variables $x = \sqrt{N}x'$ and $x' = \xi + \varepsilon z$. This leads to the expression

$$
I_q(c) = N^2 \int_{\mathbb{R}^n} w(x') h\left( \frac{q}{Q}, \frac{F(x') - 1}{\varepsilon^2} \right) e_{\frac{1}{\sqrt{N}}}(-c.x') \, dx'
$$

$$
= \varepsilon^4 N^2 e_{\frac{1}{\sqrt{N}}}(-c, \xi) \int_{\mathbb{R}^n} w_0(|z|) \left( \frac{2\xi \cdot z}{\varepsilon} \right) h\left( \frac{q}{Q}, \frac{y(z)}{\varepsilon} \right) e_{\frac{1}{\sqrt{N}}}(-c.z) \, dz,
$$

where $y(z) = 2\xi \cdot z + \varepsilon F(z)$. Let $r = q/Q$ and $v = r^{-1}c$. Then we have

$$
I_q(c) = \varepsilon^4 N^2 e_r(-\varepsilon^{-1}c, \xi) I_r(v),
$$

(4.1)
where
\[ I^*_r(v) = \int_{\mathbb{R}^4} w_0(||x||)w_0\left(\frac{2\xi \cdot x}{\varepsilon}\right) h\left(\frac{y(x)}{\varepsilon}\right) e(-v \cdot x) \, dx. \] (4.2)

In particular, we have
\[ I^*_r(v) = O(\varepsilon/r), \]

since \( h(r,y) \ll r^{-1} \) and the region of integration has measure \( O(\varepsilon) \).

4.1. Easy estimates. Our attention now shifts to analysing \( I^*_r(v) \) for \( r \ll 1 \) and \( v \in \mathbb{R}^4 \). Let \( x \in \mathbb{R}^4 \) such that
\[ w_0(||x||)w_0(2\xi \cdot x/\varepsilon) \neq 0. \]

Then
\[ y(x) = 2\xi \cdot x + \varepsilon F(x) \]

where
\[ F(x) = \frac{w_0(||x||)w_0(2\xi \cdot x/\varepsilon)}{v(y(x)/\varepsilon)}. \] (4.3)

Let \( p(t) = \hat{f}(t) \) be the Fourier transform of \( f \). Then the proof of [3, Lemma 17] shows that
\[ p(t) \ll_j r(r|t|)^{-j}, \] (4.4)

for any \( j > 0 \). We may therefore write
\[ I^*_r(v) = \frac{1}{r} \int_{\mathbb{R}^4} w_3(x)f\left(\frac{y(x)}{\varepsilon}\right) e(-v \cdot x) \, dx, \]

where \( f(y) = v(y)rh(r,y) \) and
\[ w_3(x) = \frac{w_0(||x||)w_0(2\xi \cdot x/\varepsilon)}{v(y(x)/\varepsilon)}. \] (4.5)

Building on this, we proceed by establishing the following result.

\[ \textbf{Lemma 4.1.} \text{ Let } c \in \mathbb{Z}^4, \text{ with } c \neq 0. \text{ Then} \]
\[ I_q(c) \ll_j \frac{\varepsilon^5 N^2 Q}{q} \min_{i=1,2,3} \{|\hat{c}_i|^{-j}, (\varepsilon|\hat{c}_4|)^{-j}\}, \]

for any \( j > 0 \).

This result corresponds to [6, Lemma 5.1]. Since \( \max_i |\hat{c}_i| \gg ||c|| \), it follows that
\[ I_q(c) \ll_j \frac{\varepsilon^5 N^2 Q}{q} (\varepsilon||c||)^{-j}, \]

for any \( j > 0 \). In this way, for any \( \delta > 0 \), Lemma 4.1 implies that there is a negligible contribution to (2.2) from \( c \) such that either of the inequalities \( ||c|| > N^3/\varepsilon \) or \( \max_{i=1,2,3} \{|\hat{c}_i|, \varepsilon|\hat{c}_4|\} > N^3 \) hold. Thus, in (2.2), the summation over \( c \) can henceforth be restricted to the set \( \mathcal{C} \), which is defined to be the
Proof of Lemma 4.1. We make the change of variables $x = \sum_{i=1}^{4} u_i e_i$ in (4.3). In the notation of §2.2 let $v = \sum_{i=1}^{4} \hat{v}_i e_i$, where $\hat{v}_i = v \cdot e_i$. Then, on recalling (4.3), we find that

$$I_r(v) = \frac{1}{r} \int_{\mathbb{R}} p(t) \int_{\mathbb{R}^4} \left( \sum_{i=1}^{4} u_i e_i \right) e \left( \frac{ty(\sum_{i=1}^{4} u_i e_i)}{\varepsilon} - u \cdot \hat{v} \right) \, du \, dt$$

$$= \frac{1}{r} \int_{\mathbb{R}} p(t) \int_{\mathbb{R}^4} \frac{w_0(\|u\|)w_0(2u_4/\varepsilon)}{v((2u_4 + \varepsilon F(u))/\varepsilon)} e(H(u)) \, du \, dt,$$

where $H(u) = \frac{t}{\varepsilon} \{2u_4 + \varepsilon F(u)\} - u \cdot \hat{v}$. We have

$$\frac{\partial H(u)}{\partial u_i} = \begin{cases} 2tu_i - \hat{v}_i & \text{if } 1 \leq i \leq 3, \\ 2tu_4 - \hat{v}_4 + \frac{2u}{\varepsilon} & \text{if } i = 4. \end{cases}$$

The proof of the lemma now follows from repeated integration by parts in conjunction with (4.4), much as in the proof of [3, Lemma 19]. Thus, when $i \in \{1, 2, 3\}$, integration by parts with respect to $u_i$ readily yields

$$I_r(v) \ll_j \frac{\varepsilon}{r} \left\{ r|\hat{v}_i|^{1-j} + r^{1-j}|\hat{v}_j|^{1-j} \right\} \ll_j \varepsilon r^{-j}|\hat{v}_i|^{1-j},$$

for any $j > 0$, since $r \ll 1$. Likewise, integrating by parts with respect to $u_4$, we get

$$I_r(v) \ll_j \frac{\varepsilon}{r} \left\{ r|\hat{v}_4|^{1-j} + r^{1-j}(\varepsilon|\hat{v}_4|)^{1-j} \right\} \ll_j \varepsilon r^{-j}(\varepsilon|\hat{v}_4|)^{1-j}.$$
Proof. We follow the argument in Stein [9, §VIII.5.1]. Using the Fourier transform, we can write the integral as

$$\left(\frac{i\pi}{\lambda}\right)^{\frac{n}{2}} \int_{\mathbb{R}^n} e^{-i\pi^2\|\xi\|^2/\lambda} \hat{\varphi}(\xi) d\xi.$$  \hspace{1cm} (4.6)

Next, we split off the first $N$ terms in a Taylor expansion around 0, finding that

$$e^{-i\pi^2\|\xi\|^2/\lambda} = \sum_{j=0}^{N} \left(\frac{-i\pi^2\|\xi\|^2}{\lambda}\right)^j + R_N(\xi).$$

The main term now comes from integration by parts and Fourier inversion. We are left to deal with the integral involving $R_N(\xi)$. We have

$$R_N(\xi) \ll_N \left(\frac{\|\xi\|^2}{|\lambda|}\right)^{N+1}$$  \hspace{1cm} (4.7)

which follows from Taylor expansion when $\|\xi\|^2 \leq |\lambda|$ and trivially otherwise. Moreover,

$$\hat{\varphi}(\xi) = O_A \left(\|\xi\|^{-A}\|\varphi\|_{A,1}\right),$$

for any $A \geq 0$. We split up the remaining integral into two parts: $\|\xi\| \leq 1$ and $\|\xi\| > 1$. For the first part we use (4.7) and (4.8) with $A = 2N + 1 + n$. Recalling the additional factor $\lambda^{-\frac{3}{2}}$ from (4.6), we get an error term of size

$$O_{n,N} \left(|\lambda|^{-\frac{3}{2} - N-1}\|\varphi\|_{2N+1+n,1}\right).$$

For the second part we use (4.7) and (4.8), but this time with $A = 2N + 3 + n$. This leads to the same overall error term, but with the factor $\|\varphi\|_{2N+1+n,1}$ replaced by $\|\varphi\|_{2N+3+n,1}$.

4.3. Hard estimates. Having shown how to truncate the sum over $c$ in (2.2), we now return to (4.1) for $c \in \mathcal{C}$ and see what more can be said about the integral $I^*_r(v)$ in (4.2), with $r = q/Q$ and $v = r^{-1}c$. Our result relies on an asymptotic expansion of $I^*_r(v)$, but the form it takes depends on the size of $\varepsilon|\hat{v}_4|$.

It will be convenient to set $a = (\hat{v}_1, \hat{v}_2, \hat{v}_3)$, in what follows. To begin with, we make the change of variables $x = \sum_{i=1}^{4} u_i e_i$ in (4.2). This leads to the expression

$$I^*_r(v) = \int_{\mathbb{R}^4} w_0(\|u\|) w_0(2u_4/\varepsilon) h \left( r, \frac{2u_4}{\varepsilon} + F(u) \right) e(-u \cdot \hat{v}) \, du,$$

where $\hat{v}_i = v_4 e_i$ for $1 \leq i \leq 4$. We now write $y = 2u_4/\varepsilon + F(u)$, under which we have

$$u_4 = \frac{1}{\varepsilon} \left( -1 + \sqrt{1 + \varepsilon^2\{y - u^2_1 - u^2_2 - u^2_3\}} \right).$$  \hspace{1cm} (4.9)
Thus

\[ I_r^*(v) = \int_{\mathbb{R}} h(r,y) e\left(-\frac{\varepsilon \hat{v}_4 y}{2}\right) T(y) dy, \]  

(4.10)

where

\[ T(y) = e\left(\frac{\varepsilon \hat{v}_4 y}{2}\right) \int_{\mathbb{R}^3} w_0(\|u\|) w_0(2u_4/\varepsilon) e(-u.\hat{v}) \frac{du_1 du_2 du_3}{2/\varepsilon + 2u_4}, \]  

(4.11)

and \( u_4 \) is given in terms of \( y, u_1, u_2, u_3 \) by (4.9). In particular, on writing \( x = (u_1, u_2, u_3) \), we have \( w_0(\|u\|) w_0(2u_4/\varepsilon) = \psi_y(x) \), where \( \psi_y : \mathbb{R}^3 \to \mathbb{R}_{\geq 0} \) is the weight function

\[ \psi_y(x) = w_0(2\varepsilon^{-2}(1 + \varepsilon^2\{y - \|x\|^2\})) \times w_0(\sqrt{\|x\|^2 + \varepsilon^{-2}(1 - \sqrt{1 + \varepsilon^2\{y - \|x\|^2\}})^2}). \]  

(4.12)

We note, furthermore, that the integral in \( T(y) \) is supported on \([-1,1]^3\). Moreover, we have

\[ \frac{2u_4}{\varepsilon} = \frac{2}{\varepsilon^2} \left(-1 + \sqrt{1 + \varepsilon^2\{y - \|x\|^2\}}\right) = y - \|x\|^2 + O(\varepsilon^2), \]  

(4.13)

for any \( x \) such that \( \psi_y(x) \neq 0 \). In particular, it follows that

\[ \frac{1}{2/\varepsilon + 2u_4} = \frac{\varepsilon}{2} \left(1 + O(\varepsilon^2)\right) \]  

(4.14)

in (4.11).

Since \( e(z) = 1 + O(z) \), we invoke (4.9) and (4.13) to deduce that

\[ e(-u.\hat{v}) = e\left(-\frac{\varepsilon \hat{v}_4 y}{2}\right) e\left(\frac{\varepsilon \hat{v}_4}{2}\|x\|^2 - a.x\right) \left(1 + O(|\varepsilon \hat{v}_4|\varepsilon^2)\right), \]  

(4.15)

where we recall that \( a = (\hat{v}_1, \hat{v}_2, \hat{v}_3) \). Thus, it follows from (4.14) that

\[ T(y) = \frac{\varepsilon}{2} \left(1 + O(\varepsilon^2 + |\varepsilon \hat{v}_4|\varepsilon^2)\right) I(y), \]  

(4.16)

where

\[ I(y) = \int_{\mathbb{R}^3} \psi_y(x) e\left(\frac{\varepsilon \hat{v}_4}{2}\|x\|^2 - a.x\right) dx. \]  

(4.17)

In what follows it will be useful to record the estimate

\[ \int_{\mathbb{R}} \left| r^k y_\ell \frac{\partial^k h(r,y)}{\partial r^k} \right| dy \ll \ell r^k, \]  

(4.18)

for any \( \ell \geq 0 \) and \( k \in \{0,1\} \). This is a straightforward consequence of [3, Lemma 5]. The stage is now set to prove the following preliminary estimate for \( I_r^*(v) \) and its partial derivative with respect to \( r \).
Lemma 4.3. Let $k \in \{0, 1\}$. Then

$$r^{2k} \frac{\partial^k I^*_y(v)}{\partial r^k} \ll \frac{\varepsilon(1 + \varepsilon^3|\hat{v}_4|)}{\max\{1, (\varepsilon|\hat{v}_4|)\}^2} N^\delta.$$  

Proof. Suppose first that $k = 0$. An application of \cite{4}, Lemmas 3.1 and 3.2 shows that

$$I(y) \ll \frac{1}{\max\{1, (\varepsilon|\hat{v}_4|)\}^2},$$

since $\|\hat{\psi}_y\|_1 \ll 1$. The desired bound now follows on substituting this into (4.10) and (4.16), before using (4.18) with $k = \ell = 0$ to carry out the integration over $y$.

Suppose next that $k = 1$. Then, in view of (4.10), we have

$$r^2 \frac{\partial I^*_y(v)}{\partial r} = \int_{\mathbb{R}} r^2 \frac{\partial h(r, y)}{\partial r} e\left(-\frac{\varepsilon\hat{v}_4 y}{2}\right) T(y) dy$$

$$+ \int_{\mathbb{R}} h(r, y) e\left(-\frac{\varepsilon\hat{v}_4 y}{2}\right) \tilde{T}(y) dy,$$

where

$$\tilde{T}(y) = e\left(\frac{\varepsilon\hat{v}_4 y}{2}\right) \int_{\mathbb{R}^3} w_0(||u||) w_0(2u_4/\varepsilon) r^2 \frac{\partial}{\partial r} e(-u.\hat{v}) \frac{du_1 du_2 du_3}{2/\varepsilon + 2u_4}.$$

The contribution from the first integral in (4.19) is satisfactory, since $r \ll 1$, on reapplying our argument for $k = 0$ and using (4.18) with $k = 1$ and $\ell = 0$. Turning to the second integral in (4.19), we recall (4.14) and (4.15). These allow us to write

$$\tilde{T}(y) = \varepsilon \pi i \left(1 + O(\varepsilon^2 + |\varepsilon\hat{v}_4|\varepsilon^2)\right) \tilde{I}(y),$$

where

$$\tilde{I}(y) = \int_{\mathbb{R}^3} \tilde{\psi}_y(x) e\left(\frac{\varepsilon\hat{v}_4}{2}||x||^2 - a.x\right) dx$$

and

$$\tilde{\psi}_y(x) = \left(r a.x + \frac{\hat{c}_4}{\varepsilon} \left(-1 + \sqrt{1 + \varepsilon^2\{y - ||x||^2\}}\right)\right) \psi_y(x).$$

Here, the definition of $C$ implies that $r|a| = \max\{||\hat{c}_1||, ||\hat{c}_2||, ||\hat{c}_3||\} \ll N^\delta$ and $\varepsilon|\hat{c}_4| \ll N^\delta$. Thus the $L^1$-norm of the Fourier transform of $\tilde{\psi}_y$ is $O(N^\delta)$. Once combined with (4.18) with $k = \ell = 0$, we apply \cite{4} Lemmas 3.1 and 3.2 to estimate $\tilde{I}(y)$, which concludes our treatment of the case $k = 1$. \qed
The case $k = 0$ of Lemma 4.3 is already implicit in Sardari’s work (see [6, Lemma 5.2]). We shall also need the case $k = 1$, but it turns out that it is only effective when $r$ is essentially of size 1. For general $r$, we require a pair of asymptotic expansions for $I^*_r(\mathbf{v})$, that are relevant for small and large values of $\varepsilon|\hat{v}_4|$, respectively. This is the objective of the following pair of results.

**Lemma 4.4.** Let $A \geq 0$. Then

$$I^*_r(\mathbf{v}) = \frac{\varepsilon I(0)}{2} + O_A \left( \varepsilon^3(1 + \varepsilon|\hat{v}_4|) + \varepsilon(1 + \varepsilon|\hat{v}_4|)^A r^A \right).$$

**Proof.** Our first approach is founded on the Taylor expansion

$$e^{-\varepsilon\hat{y}_4 y} = \sum_{j=0}^{A-1} \frac{(-\pi i \varepsilon \hat{y}_4 y)^j}{j!} + R_A(y),$$

where $R_A(y) \ll_A (\varepsilon|\hat{y}_4|)^A$. Since $I(y) \ll 1$, we conclude from (4.10), (4.16) and (4.18) that

$$I^*_r(\mathbf{v}) = \frac{\varepsilon}{2} \sum_{j=0}^{A-1} \frac{(-\pi i \varepsilon \hat{y}_4)^j}{j!} \int_R y^j h(r, y) I(y) dy + O_A \left( \varepsilon^3(1 + \varepsilon|\hat{y}_4|) + \varepsilon(1 + \varepsilon|\hat{y}_4|)^A r^A \right).$$

Next, we claim that

$$\int_R y^j h(r, y) I(y) dy = O_A(r^A) + \begin{cases} I(0) & \text{if } j = 0, \\
0 & \text{if } j > 0. \end{cases}$$

(4.20)

To see this, note that $I(y)$ belongs to the class of weight functions considered in [3, Lemma 9]. This settles (4.20) when $j = 0$. When $j > 0$ we truncate the integral to $|y| \leq \sqrt{r}$ and expand $I(y)$ as a Taylor series, before invoking [3, Lemma 8], as in the proof of [3, Lemma 9]. This settles (4.20) when $j > 0$. The statement of the lemma is now obvious. □

**Lemma 4.5.** Assume that $\varepsilon|\hat{v}_4| > 1$. For each $j \geq 0$, we define

$$\varphi_j(y) = \Delta^j \psi_y \left( (\varepsilon \hat{y}_4)^{-1} \mathbf{a} \right) = \Delta^j \psi_y \left( (\varepsilon \hat{c}_4)^{-1} (\hat{c}_1, \hat{c}_2, \hat{c}_3) \right),$$

where $\psi_y$ is given by (1.12). Let $A \geq 0$. Then there exist constants $k_j$ that depend only on $j$ such that

$$I^*_r(\mathbf{v}) = \frac{\varepsilon \delta(\hat{c})}{(\varepsilon \hat{v}_4)^2} e^{-\|\mathbf{a}\|^2 / 2\varepsilon \hat{v}_4} \sum_{j=0}^{A} \frac{k_j}{(\varepsilon \hat{v}_4)^j} \int_R h(r, y) e \left( -\varepsilon \hat{y}_4 y^2 / 2 \right) \varphi_j(y) dy + O_A \left( \frac{\varepsilon^3}{|\varepsilon \hat{v}_4|^2} + \frac{\varepsilon}{|\varepsilon \hat{v}_4|^2 + A} \right),$$
where
\[
\delta(\hat{c}) = \begin{cases} 
1 & \text{if } \varepsilon|\hat{c}_4| \gg |(\hat{c}_1, \hat{c}_2, \hat{c}_3)|, \\
0 & \text{otherwise}.
\end{cases}
\]

Proof. It will be convenient to set \(\lambda = \varepsilon\hat{v}_4\) in the proof of this result, recalling our hypothesis that \(|\lambda| > 1\). Our starting point is the expression for \(T(y)\) in (4.16), in which \(I(y)\) is given by (4.17). By completing the square, we may write
\[
T(y) = \frac{\varepsilon}{2} \left(1 + O(|\varepsilon|^2)\right) e \left(-\frac{\|a\|^2}{2\lambda}\right) I^*(y),
\]
since \(|\lambda| > 1\), where
\[
I^*(y) = \int_{\mathbb{R}^3} \psi_y(x + \frac{a}{\lambda}) e \left(\frac{\lambda}{2} \|x\|^2\right) dx.
\]

If \(|a| \gg \varepsilon|\hat{v}_4|\), then it follows from [3, Lemma 10] that \(T(y) \ll_A \varepsilon|\lambda|^{-A}\), for any \(A \geq 0\). Alternatively, if \(|a| \ll \varepsilon|\hat{v}_4|\), which is equivalent to \(\delta(\hat{c}) = 1\), then all the hypotheses of Lemma 4.2 are met. Thus, for any \(A \geq 0\), there exist constants \(k_j\) that depend only on \(j\) such that
\[
I^*(y) = \frac{1}{\lambda^{2j}} \sum_{j=0}^{A} \frac{k_j \Delta^j \psi_y(\lambda^{-1}a)}{\lambda^j} + O_A \left(\frac{1}{|\lambda|^{2j+A}}\right).
\]

Hence we conclude from (4.16) that
\[
T(y) = \frac{\varepsilon \delta(\hat{c})}{2\lambda^2} e \left(-\frac{\|a\|^2}{2\lambda}\right) \sum_{j=0}^{A} \frac{k_j \Delta^j \psi_y(\lambda^{-1}a)}{\lambda^j} + O_A \left(\frac{\varepsilon^3}{|\lambda|^2} + \frac{\varepsilon}{|\lambda|^{2j+A}}\right).
\]

We now wish to substitute this into our expression (4.10) for \(I^*_r(v)\). In order to control the contribution from the error term, we apply (4.18) with \(\ell = 0\). We therefore arrive at the statement of the lemma on redefining \(k_j\) to be \(k_j/2\). \(\square\)

It remains to consider the integral
\[
J_{j,q}(c) = \int_{\mathbb{R}} h(r, y) e \left(-\frac{\varepsilon \hat{v}_4 y}{2}\right) \varphi_j(y) dy = \int_{\mathbb{R}} h \left(\frac{q}{Q}, y\right) e \left(-\frac{\varepsilon \hat{c}_4 y Q}{2q}\right) \varphi_j(y) dy,
\]
for \(j \geq 0\). Recollecting (4.12), all we shall need to know about \(\varphi_j\) is that it is a smooth compactly supported function with bounded derivatives, and that it is does not depend on \(q\). (Note that we may assume that \(|(\hat{c}_1, \hat{c}_2, \hat{c}_3)| \ll \varepsilon|\hat{c}_4|\) in what follows, since otherwise \(\delta(\hat{c}) = 0\).)
Lemma 4.6. Let $c \in \mathcal{C}$ and $k \in \{0, 1\}$. Then

$$q^k \frac{\partial^k J_{j,q}(c)}{\partial q^k} \ll_j N^\delta.$$  

Proof. When $k = 0$ the result follows immediately from (4.18). Suppose next that $k = 1$. Then (4.21) implies that

$$\frac{\partial J_{j,q}(c)}{\partial q} = \frac{1}{Q} \int_{\mathbb{R}} \frac{\partial h(r,y)}{\partial r} e\left(-\frac{\varepsilon c_4 y Q}{2q}\right) \varphi_j(y)dy
+ \int_{\mathbb{R}} \frac{\pi i \varepsilon c_4 y Q}{q^2} h(r,y) e\left(-\frac{\varepsilon c_4 y Q}{2q}\right) \varphi_j(y)dy = J_1 + J_2,$$

say. It follows from (4.18) that $J_1 \ll_j Q^{-1}r^{-1} = q^{-1},$ which is satisfactory. Next, a further application of (4.18) yields

$$J_2 \ll_j \frac{\varepsilon |c_4| Q}{q^2} \int_{\mathbb{R}} |yh(r,y)| dy \ll_j \frac{\varepsilon |c_4| Q}{q^2} \cdot r \leq N^\delta,$$

for $c \in \mathcal{C}$.

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5. Putting everything together

It is now time to return to (2.2), in order to conclude the proof of Theorem 2.1.

5.1. The main term. We begin by dealing with the main contribution, which comes from the term $c = 0$. Denoting this by $M(w)$, we see that

$$M(w) = \frac{1}{Q^2} \sum_{q \leq Q} q^{-A} S_q(0)I_q(0) + O_A(Q^{-A}),$$  

for any $A > 0$.

In view of (4.12), $\psi_0(x)$ is equal to

$$w_0 \left(2\varepsilon^{-2}(-1 + \varepsilon^2\|x\|^2)\right) w_0 \left(\sqrt{\|x\|^2 + \varepsilon^{-2}(1 - \sqrt{1 - \varepsilon^2\|x\|^2})^2}\right).$$

As in (4.13), when $\psi_0(x) \neq 0$ we must have

$$2\varepsilon^{-2} \left(-1 + \sqrt{1 - \varepsilon^2\|x\|^2}\right) = -\|x\|^2 + O(\varepsilon^2)$$

$$\|x\|^2 + \varepsilon^{-2}(1 - \sqrt{1 - \varepsilon^2\|x\|^2})^2 = \|x\|^2 + O(\varepsilon^2).$$

In particular it is clear that

$$\sigma_\infty = \int_{\mathbb{R}^3} \psi_0(x)dx \gg 1,$$

for an absolute implied constant. We now establish the following result.
Lemma 5.1. We have

\[ I_q(0) = \frac{1}{2} \varepsilon^5 N^2 \sigma_\infty (1 + O(\varepsilon^2) + O_A((q/Q)^A)) , \]

for any \( A > 0 \), where \( \sigma_\infty \) is given by (5.2).

Proof. Returning to (4.1), it follows from (4.10) and (4.11) that

\[ I_q(0) = \varepsilon \frac{N^2}{2} \int h(r, y) K(y) dy, \]

where

\[ K(y) = \int_{\mathbb{R}^3} w_0(\|u\|) u_0(2u_4/\varepsilon) \frac{du_1 du_2 du_3}{2/\varepsilon + 2u_4}, \]

and \( u_4 \) is given in terms of \( y, u_1, u_2, u_3 \) by (4.9). Using (4.14), we may write

\[ K(y) = \frac{\varepsilon}{2} (1 + O(\varepsilon^2)) K^*(y), \quad \text{with} \quad K^*(y) = \int_{\mathbb{R}^3} \psi_y(x) dx. \]

The integral \( K^*(y) \) is a smooth weight function belonging to the class of weight functions considered in [3, Lemma 9]. Noting from (5.2) that \( K^*(0) = \sigma_\infty \), it therefore follows from this result that

\[ \int h(r, y) K^*(y) dy = \sigma_\infty + O_A(r^A), \]

for any \( A > 0 \). We therefore deduce that

\[ I_q(0) = \frac{1}{2} \varepsilon^5 N^2 \sigma_\infty (1 + O(\varepsilon^2) + O_A(r^A)) , \]

which completes the proof of the lemma. \( \square \)

Now it is clear from [3] that \( q^{-4}|S_q(c)| \leq 4q^{-2}|S(m, n; q)| \), for any \( c \in \mathbb{Z}^4 \), where \( (m, n) \) is \((N, F(\hat{c})/4)\) or \((N/4, F(\hat{c}))\) depending on whether \( 4 \mid q \) or not, respectively. Hence it follows from (3.1), together with the standard estimate for the divisor function, that

\[
\sum_{t/2 < q \leq t} q^{-4}|S_q(c)| \ll \sum_{t/2 < q \leq t} q^{-2}|S(m, n; q)| \ll \delta t^{\delta/2} \sum_{t/2 < q \leq t} \frac{\sqrt{(q, N)}}{q^{3/2}} \ll \delta t^{-1/2+\delta/2}N^{\delta/2},
\]

for any \( t > 1 \) and any \( \delta > 0 \). Returning to (5.1), we may now conclude from Lemma 5.1 and (5.3) with \( c = 0 \), that the contribution to \( M(w) \) from \( q \leq Q^{1-\delta} \) is

\[
= \frac{1}{Q^2} \sum_{q \leq Q^{1-\delta}} q^{-4}S_q(0)I_q(0) + O_A(Q^{-A})
\]

\[
= \frac{\varepsilon^5 N^2}{2Q^2} \sigma_\infty S(Q^{1-\delta}) + O \left( \frac{\varepsilon^7 N^{2+\delta/2}}{Q^2} \right) + O_A(Q^{-A}),
\]
where

$$\mathcal{G}(t) = \sum_{q \leq t} q^{-4}S_q(0).$$

This sum is absolutely convergent and satisfies \( \mathcal{G}(t) = \mathcal{G} + O_\delta(t^{-1/2+\delta/2}N^{\delta/2}) \), for any \( \delta > 0 \), by (5.3). Here, in the usual way, \( \mathcal{G} \) is the Hardy–Littlewood product of local densities recorded in (2.4).

Next, on invoking (5.3), once more, the contribution from \( q > Q^{1-\delta} \) is

$$\ll_A \frac{\varepsilon^5N^2}{Q^2} \sum_{q > Q^{1-\delta}} q^{-4}|S_q(0)| + Q^{-A} \ll \frac{\varepsilon^5N^{2+\delta/2}Q^{\delta/2}}{Q^{5/2}}.$$ 

Hence we have established the following result, on recalling that \( Q = \varepsilon\sqrt{N} \), which shows that the main term is satisfactory for Theorem 2.1.

**Lemma 5.2.** For any \( \delta > 0 \) we have

$$M(w) = \frac{\varepsilon^3N\sigma_\infty\mathcal{G}}{2} + O_\delta \left( \varepsilon^5N^{1+\delta} + \varepsilon^2N^{\frac{3}{2}+\delta} \right).$$

### 5.2. The error term

It remains to analyse the contribution \( E(w) \), say, to \( \Sigma(w) \) from vectors \( c \neq 0 \) in (2.2). According to our work in [4] the value of \( S_q(c) \) differs according to the residue class of \( q \) modulo 4. We have

$$E(w) = \sum_{i \mod 4} E_i(w),$$

where \( E_i(w) \) denotes the contribution from \( q \equiv i \mod 4 \). Recall the definition of \( \mathcal{C} \) from after the statement of Lemma 4.1. In order to unify our treatment of the four cases, we write \( \mathcal{C}_1 = \mathcal{C}_2 = \mathcal{C} \) and we denote by \( \mathcal{C}_2 \) (resp. \( \mathcal{C}_4 \)) the set of \( c \in \mathcal{C} \) for which \( 2 \not| c_1 \ldots c_4 \) (resp. \( 2 \mid c \)). It will also be convenient to set

\((m_1, n_1) = (m_2, n_2) = (m_3, n_3) = (N/4, F(c)), \quad (m_4, n_4) = (N, F(c)/4).\)

In particular, \( m_3n_i = NF(c)/4 > 0 \) for \( 1 \leq i \leq 4 \), since \( F(c) = F(\hat{c}) \).

Let \( 1 \ll R \ll Q \). We denote by \( E_i(w, R) \) the overall contribution to \( E_i(w) \) from \( q \sim R \). (We write \( q \sim R \) to denote \( q \in (R/2, R] \).) On recalling (4.1), it follows from our work so far that

\begin{equation}
E_i(w, R) \ll \frac{1}{Q^2} \sum_{\substack{c \in \mathcal{C}_i \setminus \mathcal{C}_i \setminus 0 \atop c \not\equiv 0 \mod 4}} \left| \sum_{q \sim R \atop q \equiv i \mod 4} q^{-2}S(m_i, n_i; q)I_q(\hat{c}) \right| \ll \frac{\varepsilon^4N^2}{Q^2} \sum_{\substack{c \in \mathcal{C}_i \setminus \mathcal{C}_i \setminus 0 \atop c \not\equiv 0 \mod 4}} \left| \sum_{q \sim R \atop q \equiv i \mod 4} q^{-2}S(m_i, n_i; q)e_r(-\varepsilon^{-1}c, \xi)I^*_q(v) \right|. \tag{5.4}
\end{equation}
Contribution from large $q$. Suppose first that $R \geq Q^{1-\eta}$, for some small $\eta > 0$. (The choice $\eta = 2\delta$ is satisfactory.) We have

$$e_r(-\varepsilon^{-1} \mathbf{c}, \xi) = e \left( \frac{2\sqrt{m_in_i}}{q} \alpha \right),$$

with

$$|\alpha| = \varepsilon^{-1}|\hat{c}_4| \cdot \frac{Q}{q} \cdot \frac{q}{2\sqrt{m_in_i}} = \frac{|\hat{c}_4|}{\sqrt{F(c)}} \leq 1.$$  

It now follows from Conjecture 1.1 that

$$L(t) = \sum_{\substack{q \leq t \\ q \equiv i \mod 4}} S(m_i,n_i; q) e \left( \frac{2\sqrt{m_in_i}}{q} \alpha \right) \ll_{\delta} (tN)^{\delta}. \quad (5.5)$$

Applying partial summation, based on Lemma 4.3, we deduce that

$$E_i(w, R) \ll_{\delta} \frac{\varepsilon^5 N^{2+O(\delta)}}{Q^3} \cdot \frac{Q^2}{R^2} \cdot \sum_{\substack{c \neq 0 \\ \varepsilon \neq 0}} \frac{1}{\max \{1, |\hat{c}_4|Q/R \}^2}$$

$$\ll_{\delta} \frac{\varepsilon^5 N^{2+O(\delta)}}{QR^2} \cdot \frac{\varepsilon^{-1}R}{Q} = \frac{\varepsilon^4 N^{2+O(\delta)}}{Q^2R}.$$  

Since $R \geq Q^{1-\eta}$, we deduce that

$$E_i(w, R) \ll_{\delta} \frac{\varepsilon^4 N^{2+O(\delta)}Q_R}{Q^3} \ll \varepsilon N^{\frac{1}{2}+O(\delta)+\eta}.$$  

This is satisfactory for Theorem 2.11 on redefining the choice of $\delta$, provided that $\eta$ is small enough.

Contribution from small $q$ and small $\varepsilon |\hat{c}_4|$. For the rest of the proof we suppose that $R < Q^{1-\eta}$. Let us put

$$\mathbf{b} = (\hat{c}_1, \hat{c}_2, \hat{c}_3),$$

so that $\mathbf{a} = r^{-1} \mathbf{b}$ in Lemmas 4.4 and 4.5. Let $E_{i}^{(\text{small})}(w, R)$ denote the contribution to $E_i(w, R)$ from $\mathbf{c} \in C_i$ such that

$$\varepsilon |\hat{c}_4| \leq \frac{R^{1+\delta}}{Q}. \quad (5.6)$$

In this case it is advantageous to apply Lemma 4.4 to evaluate $I^*_r(\mathbf{v})$. To begin with, we consider the effect of substituting the main term from Lemma 4.4
Noting that \((\varepsilon \hat{\nu}_4)^{-1} \mathbf{a} = (\varepsilon \hat{\nu}_4)^{-1} \mathbf{b}\) does not depend on \(q\), we deduce from (4.17) that the only dependence on \(q\) in \(I(y)\) comes through the term

\[
e \left( \frac{\varepsilon \hat{\nu}_4}{2} \|\mathbf{x}\|^2 - \mathbf{a} \cdot \mathbf{x} \right) = e_r \left( \frac{\varepsilon \hat{\nu}_4}{2} \|\mathbf{x}\|^2 - \mathbf{b} \cdot \mathbf{x} \right),
\]

in the integrand. Thus, the main term in Lemma 4.4 makes the overall contribution

\[
\ll \frac{\varepsilon^5 N^2}{Q^2} \sum_{c \in \mathcal{C}_i} \sum_{q \sim R} \frac{S(m_i, n_i; q)}{q^2} e_r(-\epsilon^{-1} c \cdot \xi) I(0)
\]

(5.7) holds to \(E_i^{(small)}(w, R)\), where we recall from (4.17) that

\[
I(0) = \int_{\mathbb{R}^3} \psi_0(\mathbf{x}) e_r \left( \frac{\varepsilon \hat{\nu}_4}{2} \|\mathbf{x}\|^2 - \mathbf{b} \cdot \mathbf{x} \right) d\mathbf{x}.
\]

If \(c \neq 0\) and \(|\hat{\nu}_4| \leq \frac{1}{100}\) then

\[
\|\mathbf{b}\|^2 = F(\hat{c}) - \hat{c}_4^2 = F(c) - \hat{c}_4^2 \gg 1.
\]

It therefore follows from [4] Lemmas 3.1 and 3.2 that

\[
I(0) \ll_A \left( \frac{q}{|\mathbf{b}| Q} \right)^A \ll_A Q^{-\eta A},
\]

since \(q \leq Q^{1-\eta}\) in this case. The overall contribution to (5.7) from vectors \(c\) such that \(|\hat{\nu}_4| \leq \frac{1}{100}\) is therefore seen to be satisfactory.

On interchanging the sum and the integral we are left with the contribution

\[
\ll \frac{\varepsilon^5 N^2}{Q^2} \sum_{c \in \mathcal{C}_i} \int_{[-1,1]^3} |M_i(\mathbf{x})| d\mathbf{x},
\]

(5.8)

where

\[
M_i(\mathbf{x}) = \sum_{q \sim R} \sum_{q \equiv i \text{ mod } 4} \frac{S(m_i, n_i; q)}{q^2} e_r(-\epsilon^{-1} c \cdot \xi) e_r \left( \frac{\varepsilon \hat{\nu}_4}{2} \|\mathbf{x}\|^2 - \mathbf{b} \cdot \mathbf{x} \right).
\]

But

\[
e_r(-\epsilon^{-1} c \cdot \xi) e_r \left( \frac{\varepsilon \hat{\nu}_4}{2} \|\mathbf{x}\|^2 - \mathbf{b} \cdot \mathbf{x} \right) = e \left( \frac{2 \sqrt{m_i n_i}}{q} \alpha \right),
\]
\[ \alpha = \left( -\varepsilon^{-1} \hat{c}_4 + \frac{\varepsilon \hat{c}_4 \|x\|^2}{2} - b \cdot x \right) \cdot \frac{Q}{q} \cdot \frac{q}{2 \sqrt{m_i \bar{m}_i}} \]
\[ = -\frac{\hat{c}_4}{\sqrt{F('c)}} + \frac{\varepsilon \hat{c}_4 \|x\|^2}{2 \sqrt{F('c)}} - \frac{\varepsilon b \cdot x}{\sqrt{F('c)}}. \]

But the inequality \( \max\{\|b\|, |\hat{c}_4|\} \leq \sqrt{F('c)} \), implies that \( |\alpha| \leq 1 + O(\varepsilon) \), since \( x \in [-1,1]^3 \). Thus it follows from combining partial summation with Conjecture 1.1 that \( M_i(x) \ll \delta R^{-1} N^\delta \). (Recall that \( \varepsilon^{-1} \leq \sqrt{N} \) and \( R \leq Q^{1-\eta} \leq Q \).) Returning to (5.8), we conclude that the overall contribution to \( E_i^{(\text{small})}(w,R) \) from the main term in Lemma 4.4 is
\[ \ll \delta R^2 \# \left\{ c \in C_i : |\hat{c}_4| > \frac{1}{100} \text{ and (5.6) holds} \right\} \ll \delta \frac{\varepsilon^4 N^{2+4\delta} R^6}{Q^3} \ll \delta \varepsilon N^{\frac{1}{2}+5\delta}. \]

This is satisfactory for Theorem 2.1.

It remains to study the effect of substituting the error term from Lemma 4.4 into (5.4). Since \( r \leq R/Q \leq Q^{-\eta} \) and \( \varepsilon |\hat{v}_4| = r^{-1} \varepsilon |\hat{c}_4| \ll R \delta \), by (5.6), we see that the error term is
\[ \ll_A \varepsilon^3 (1 + \varepsilon |\hat{v}_4|) + \varepsilon (1 + \varepsilon |\hat{v}_4|)^A r^A \ll_A \varepsilon^3 R^\delta + \varepsilon R^A Q^{-\eta A} \leq \varepsilon^3 R^\delta + \varepsilon Q^A (\delta - \eta). \]

On ensuring that \( \delta < \eta \), we see that the second term is an arbitrary negative power of \( Q \) and so makes a satisfactory overall contribution to \( E_i^{(\text{small})}(w,R) \). In view of (5.3), the contribution from the term \( \varepsilon^3 N^\delta \) is found to be
\[ \ll \delta \frac{\varepsilon^7 N^{2+\delta}}{Q^2 R^\frac{3}{2}} \cdot \# C_i \ll \delta \frac{\varepsilon^7 N^{2+\delta}}{Q^2} \cdot \varepsilon^{-1} N^\delta = \frac{\varepsilon^6 N^{2+5\delta}}{Q^2}, \quad (5.9) \]

since \( R \gg 1 \). The right hand side is \( \varepsilon^4 N^{1+5\delta} \), which is also satisfactory for Theorem 2.1 on redefining \( \delta \).

**Contribution from small \( q \) and large \( \varepsilon |\hat{v}_4| \).** It remains to consider the case \( R < Q^{1-\eta} \) and
\[ \varepsilon |\hat{c}_4| > \frac{R^{1+\delta}}{Q}. \quad (5.10) \]

Let us write \( E_i^{(\text{big})}(w,R) \) for the overall contribution to \( E_i(w,R) \) from this final case. Our main tool is now Lemma 4.5. Let \( A \geq 0 \). We begin by considering
the effect of substituting the main term from this result into (5.4). This yields the contribution
\[ \ll \frac{\varepsilon^5 N^2}{Q^2} \sum_{c \in \mathcal{C}_i} \delta(c) \sum_{j=0}^{A} \frac{|k_j|}{(\varepsilon |\hat{c}_4|Q)^{\frac{3}{4}+j}} |M_{i,j}|, \tag{5.11} \]
where if \( J_{j,q}(c) \) is given by (4.21), then
\[ M_{i,j} = \sum_{q \sim R} S(m_i, n_i; q) \frac{q}{q} e_r(-\varepsilon^{-1} c, \xi) e_r \left( -\frac{\|b\|^2}{2 \varepsilon \hat{c}_4} \right) q^{\frac{1}{2}+j} J_{j,q}(c). \]

Our plan is to use partial summation to remove the factor \( q^{\frac{1}{2}+j} J_{j,q}(c) \).
First, as before, we note that
\[ e_r(-\varepsilon^{-1} c, \xi) e_r \left( -\frac{\|b\|^2}{2 \varepsilon \hat{c}_4} \right) = e \left( \frac{2 \sqrt{m_i n_i}}{\alpha} \right), \]
where
\[ \alpha = \left( -\varepsilon^{-1} \hat{c}_4 - \frac{\|b\|^2}{2 \varepsilon \hat{c}_4} \right) \cdot \frac{Q}{q} \cdot \frac{q}{2 \sqrt{m_i n_i}} \]
\[ = - \left( \frac{\hat{c}_4}{\sqrt{F(\hat{c})}} + \frac{\|b\|^2}{2 \hat{c}_4 \sqrt{F(\hat{c})}} \right). \]
We have \(|\alpha| \ll 1 + O(\varepsilon^2)\), since \( \|b\| \ll \varepsilon |\hat{c}_4| \) when \( \delta(\hat{c}) \neq 0 \). Applying partial summation, based on (5.5) and Lemma 4.6, we deduce that
\[ M_{i,j} = O_{j,\delta}(R^{\frac{1}{2}+j} N^{3\delta}). \]
Returning to (5.11), we conclude that the overall contribution to \( E_i^{(\mathrm{big})}(w, R) \) from the main term in Lemma 1.3 is
\[ \ll_{\delta,A} \frac{\varepsilon^5 N^{2+3\delta}}{Q^2} \sum_{j=0}^{A} \sum_{c \in \mathcal{C}_i} \frac{R^{\frac{1}{2}+j}}{(\varepsilon |\hat{c}_4|Q)^{\frac{3}{4}+j}} \ll_{\delta,A} \frac{\varepsilon^5 N^{2+3\delta}}{Q^2} \cdot \frac{N^{3\delta}}{\varepsilon Q} = \varepsilon N^{-\frac{1}{2}+6\delta}. \]
This is satisfactory for Theorem 2.1 on redefining \( \delta \).
We must now consider the effect of substituting the error term
\[ \ll_{A} \frac{\varepsilon^3}{|\varepsilon \hat{b}_4|^2} + \frac{\varepsilon}{|\varepsilon \hat{b}_4|^2+A} \]
from Lemma 1.3 into (5.4). Since \( q \sim R \), it follows from (5.10) that \( \varepsilon |\hat{b}_4| \gg R^\delta \). The first term is therefore \( O(\varepsilon^3) \), which makes a satisfactory overall contribution by (5.9). On the other hand, on invoking once more the argument in
the second term makes the overall contribution
\[
\ll_A \frac{\varepsilon^5 N^2}{Q^2} \sum_{c \in \mathbb{C}_i} \sum_{q \sim R} q^{-2} \left| S(m_i, n_i; q) \right| \left| \varepsilon \hat{v}_4 \right|^\frac{5}{2} + A
\]
\[
\ll_{A, \delta} \frac{\varepsilon^5 N^{2+\delta}}{R^2 Q^2} \left( \frac{R}{\varepsilon Q} \right)^{\frac{5}{2} + A} \sum_{c \in \mathbb{C}_i} \frac{1}{|\hat{c}|^{\delta + A}}
\]
\[
\ll_{A, \delta} \frac{\varepsilon^4 N^{2+4\delta} R_1^{\frac{1}{2} - A\delta}}{Q^3}.
\]
This is \(O_\delta(\varepsilon N^{\frac{1}{2} + 4\delta})\) on assuming that \(A\) is is chosen so that \(A\delta > \frac{1}{2}\). This is also satisfactory for Theorem 2.1 which thereby completes its proof.

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