The number of perfect matchings, and the nesting properties, of random regular graphs

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Abstract
We prove that the number of perfect matchings in \( G(n, d) \) is asymptotically normal when \( n \) is even, \( d \to \infty \) as \( n \to \infty \), and \( d = O(n^{1/7}/\log n) \). This is the first distributional result of spanning subgraphs of \( G(n, d) \) when \( d \to \infty \). Moreover, we prove that \( G(n, d - 1) \) and \( G(n, d) \) can be coupled so that \( G(n, d - 1) \) is a subgraph of \( G(n, d) \) with high probability when \( d \to \infty \) and \( d = o(n^{1/3}) \). Furthermore, if \( d = o(\log^2 n) \), \( d = O(n^{1/7}/\log n) \), and \( d \leq d' \leq n - 1 \) then \( G(n, d) \) and \( G(n, d') \) can be coupled so that asymptotically almost surely (a.a.s.) \( G(n, d) \) is a subgraph of \( G(n, d') \).

KEYWORDS
the number of perfect matchings, sandwich conjectures, random regular graphs, normal distribution, linear regression

1 | INTRODUCTION

In this paper we address two problems regarding \( G(n, d) \), the random \( d \)-regular graph: the limiting distribution of the number of perfect matchings in \( G(n, d) \), and the sandwich conjectures for random regular graphs in terms of nesting \( G(n, d_1) \subseteq G(n, d_2) \) with high probability when \( d_1 \leq d_2 \).

1.1 | The number of perfect matchings

The study of subgraphs lies in the centre of random graph theory. The commonly studied examples include spanning subgraphs such as perfect matchings, Hamilton cycles, spanning trees, \( H \)-factors where \( H \) has a fixed size, as well as smaller subgraphs such as independent sets, cycles, and in general
subgraphs isomorphic to some given \( H \) of fixed size. Let \( Z_H \) denote the number of subgraphs isomorphic to \( H \). The phase transition for \( Z_H \) to turn from 0 to positive, and the distribution of \( Z_H \) have been well studied in \( \mathcal{G}(n, p) \) and \( \mathcal{G}(n, m) \) for both small and large \( H \). It is interesting that \( Z_H \) has different types of distributions for small and large \( H \). If \( H \) has fixed size and is balanced\(^1\) then \( Z_H \) is asymptotically normally distributed in \( \mathcal{G}(n, p) \) and \( \mathcal{G}(n, m) \) when \( \mathbb{E}Z_H \to \infty \) [22]. The distribution of \( Z_H \) for \( H \) with linear size becomes complicated. For \( p \gg n^{-1/2} \) and \( p \) not too close to 1, the numbers of perfect matchings, Hamilton cycles, and spanning trees are log-normally distributed in \( \mathcal{G}(n, p) \), but are normally distributed in \( \mathcal{G}(n, m) \) when \( m \gg n^{3/2} \) [12]. For \( m = \Theta(n^{3/2}) \), these random variables also become log-normally distributed in \( \mathcal{G}(n, m) \). It is not known if they remain log-normally distributed in \( \mathcal{G}(n, p) \) when \( p = O(n^{-1/2}) \) and in \( \mathcal{G}(n, m) \) when \( m = o(n^{3/2}) \), although it is conjectured so Reference [12]. The distributional phase transition of \( Z_H \) when the size of \( H \) grows from constant to linear size has been studied in Reference [8], where \( Z_H \) is the number of \( \ell \)-matchings (i.e., matchings of size \( \ell \)). Its distribution in \( \mathcal{G}(n, p) \) changes from normal to log-normal at the critical value \( \ell' = \ell'(p) \approx n\sqrt{p} \). Such distributional phase transition is also observed in \( \mathcal{G}(n, d) \) when \( d \) is a fixed constant. It is well known that the distributions of short cycles in \( \mathcal{G}(n, d) \) are asymptotically Poisson [1,23], whereas the distribution of the number of large subgraphs such as perfect matchings and Hamilton cycles in \( \mathcal{G}(n, d) \) is of an unusual type [14] as follows. Suppose that \( Z \) is the number of perfect matchings (or the number of Hamilton cycles) in \( \mathcal{G}(n, d) \). Then the limiting distribution of the logarithm of \( Z/\mathbb{E}Z \) is an infinite linear combination of independent Poisson variables. More precisely,

\[
\frac{Z}{\mathbb{E}Z} \to \prod_{i=1}^{\infty} \left(1 + \delta_i\right)^{X_i} e^{-\lambda_i}, \quad \text{as } n \to \infty, \tag{1}
\]

where \( X_1, X_2, \ldots \) are independent Poisson variables with mean \( \lambda_1, \lambda_2, \ldots, \) and \( \delta_1, \delta_2, \ldots \) are real numbers whose values depend on which subgraphs (i.e., perfect matchings or Hamilton cycles) \( Z \) counts. The distribution of \( Z \) is determined by using the small subgraph conditioning method, originally developed by Robinson and Wormald [20,21] to prove Hamiltonicity of \( \mathcal{G}(n, d) \). The argument is then tuned to produce the distribution result of \( Z \) by Janson [14]. Recently, Greenhill, Isaev and Liang [11] proved that the number of spanning trees in \( \mathcal{G}(n, d) \) has the same type of distribution as (1). On the other hand, Garmo [10] studied the distributional phase transition of the number of \( \ell \)-cycles in \( \mathcal{G}(n, d) \) as \( \ell \) grows from constant to linear in \( n \). Its limiting distribution changes from a linear combination of independent Poisson variables to the exponential of that form, and the critical phase transition occurs when \( \ell \) becomes linear in \( n \).

It is natural to ask, in the case \( d \to \infty \), whether the distribution type of these subgraphs (e.g., perfect matchings, Hamilton cycles, spanning trees) are the same as, or analogous to, that for constant \( d \), and whether the distributional phase transitions occur when the size of the subgraphs (e.g., \( \ell \)-matchings and \( \ell \)-cycles) grows from constant to linear in \( n \), as for constant \( d \). We give a negative answer to this question. There have been few distributional results that are known for the number of subgraphs of \( \mathcal{G}(n, d) \) when \( d \to \infty \), even for small subgraphs. The limiting distribution of the number of \( \ell \)-cycles was extended from constant \( d \) and \( \ell \) to those such that \( (d-1)^{\ell-1} = o(n) \) by McKay, Wormald and Wysocka [19]. Gao and Wormald [9] determined the limiting distributions of strictly balanced graphs of fixed sizes for \( d \) that grows sufficiently slowly with \( n \). There has been no result on the distribution of the number of subgraphs whose size is beyond \( \log n \) when \( d \to \infty \). In particular, the analysis for

\(^1\)A graph \( G \) is balanced if \( \max_{\mathcal{H}(G)} |E(H)|/|V(H)| = \frac{|E(G)|}{|V(G)|} \). Graph \( G \) is said strictly balanced if \( \max_{\mathcal{H}(G),\mathcal{H}(\bar{G})} |E(H)|/|V(H)| < \frac{|E(G)|}{|V(G)|} \).
the number of perfect matchings, Hamilton cycles, and the spanning trees when \( d = O(1) \), based on
the configuration model, cannot be extended easily to \( d \to \infty \).

One may expect that the number of large subgraphs such as perfect matchings or Hamilton cycles
would be of log-normal type in \( \mathcal{G}(n,d) \) as \( d \to \infty \), which can be viewed as an analog of (1). It is also
reasonable to believe that the number of \( \ell' \)-matchings may exhibit a distributional phase transition as
\( \ell' \) grows from constant to linear in \( n \), as that is what happens for the \( \ell' \)-matchings in \( \mathcal{G}(n,p) \) and for the
\( \ell' \)-cycles in \( \mathcal{G}(n,d) \) for constant \( d \). In contrast with the intuition, we show in this paper that the number
of perfect matchings is asymptotically normally distributed in \( \mathcal{G}(n,d) \) when \( d \to \infty \) as \( n \to \infty \) and
\( d = O(n^{1/7}/\log n) \). The power of the logarithmic term is not optimized.

**Theorem 1.** Let \( Y \) denote the number of perfect matchings in \( \mathcal{G}(n,d) \) where \( n \) is even. Then, \( Y \) is
asymptotically normally distributed if \( d \to \infty \) as \( n \to \infty \) and \( d = O(n^{1/7}/\log n) \). More formally,

\[
\frac{Y - \mathbb{E}Y}{\sqrt{\text{Var}Y}} \xrightarrow{d} \mathcal{N}(0,1), \quad \text{as } n \to \infty.
\]

This result suggests that there is likely no distributional phase transition on the number of
\( \ell' \)-matchings as \( \ell' \) grows. The condition \( d = O(n^{1/7}/\log n) \) in the result is imposed only for technical
reasons and we believe that the same distribution holds for all \( d \to \infty \) until \( d \) is too close to \( n - 1 \); see
Conjecture 2 below.

To our knowledge, this is the first result on the limiting distribution of the number of spanning
subgraphs in \( \mathcal{G}(n,d) \) when \( d \to \infty \). The main contribution of Theorem 1 is the discovery of the
distribution type of the number of perfect matchings, and we believe that this phenomenon is ubiquitous
among other spanning subgraphs such as the number of Hamilton cycles. For future research, it would
be interesting to determine the limiting distributions of the number of \( \ell' \)-matchings and \( \ell' \)-cycles in
\( \mathcal{G}(n,d) \) for all \( \ell' \).

**Conjecture 2.** The numbers of perfect matchings, Hamilton cycles, spanning trees, and \( k \)-factors,
where \( k \leq d - 1 \), are all asymptotically normally distributed in \( \mathcal{G}(n,d) \) for all \( d \) where \( dn \) is even and
\( \min\{d,n - d\} \to \infty \) as \( n \to \infty \) (and also \( n \) is even in the case of perfect matchings, and \( kn \) is even in the
case of \( k \)-factors).

**Conjecture 3.** Suppose \( \min\{d,n - d\} \to \infty \) as \( n \to \infty \). The number of \( \ell' \)-cycles in \( \mathcal{G}(n,d) \) is
asymptotically normal for all \( 3 \leq \ell' \leq n \). The number of \( \ell' \)-matchings of \( \mathcal{G}(n,d) \) is asymptotically
normal for all \( 3 \leq \ell \leq n/2 \).

**Remark 4.** The condition \( n - d \to \infty \) in the above conjectures is likely not necessary. Indeed, when
\( n - d = o(n^{1/3}) \) and \( k = o(n^{1/3}) \) the asymptotic number of \( k \)-factors of a \( d \)-regular graph \( G \) is
independent of \( G \) and can be obtained by Theorem 14 in Section 3.

1.2 | The sandwich conjectures of \( \mathcal{G}(n,d) \)

Analysis in \( \mathcal{G}(n,d) \) is highly nontrivial, especially when \( d \to \infty \). Kim and Vu initiated the study of
approximating \( \mathcal{G}(n,d) \) by \( \mathcal{G}(n,p) \), known as the sandwich conjecture [15]. Since then several groups
of authors [2,4,6,16] have worked on this conjecture, and it is close to being fully resolved. Along the
line of the research there has been new conjectures that are proposed, one of which is stated as follows
[6, Conjecture 1.2].
Conjecture 5. Let $0 \leq d_1 \leq d_2 \leq n - 1$ be integers, other than $(d_1, d_2) = (1, 2)$ or $(d_1, d_2) = (n - 3, n - 2)$. Assume that $d_1 n$ and $d_2 n$ are both even. Then, there exists a coupling $(G_1, G_2)$ such that $G_1 \sim \mathcal{G}(n, d_1)$, $G_2 \sim \mathcal{G}(n, d_2)$ and $\mathbb{P}(G_1 \subseteq G_2) = 1 - o(1)$.

The conjecture is only known to be true for $(d_1, d_2)$ where $d_1 = 1$ and $3 \leq d_2 \leq n - 1$, as well as for $(d_1, d_2)$ where $d_2 - d_1$ is larger than some function of $d_1$ (see [6, Corollary 1.7] for the precise statement). When $d_1$ and $d_2$ are both fixed constants and $(d_1, d_2) \neq (1, 2)$, it is known that $\mathcal{G}(n, d_2)$ is contiguous to the union of two independent copies of $\mathcal{G}(n, d_1)$ and $\mathcal{G}(n, d_2 - d_1)$ conditional on $\mathcal{G}(n, d_1)$ and $\mathcal{G}(n, d_2 - d_1)$ being disjoint (see more contiguity results in Reference [24, Section 4]). However, contiguity does not imply a coupling as in the conjecture. In this paper we prove Conjecture 5 for a certain range of $d_1$.

Theorem 6. Conjecture 5 holds for all integers $d_1 \leq d_2 \leq n - 1$ where $d_1 = o(\log^2 n)$ and $d_1 = O(n^{1/7} / \log n)$ if $n$ is even.

Theorem 6 follows as a corollary of [4, Theorem 2] and the following theorem that simultaneously couples a sequence of random regular graphs.

Theorem 7. Suppose $d \to \infty$ and $d = O(n^{1/7} / \log n)$. For any $\epsilon_n = o(1)$, there is a multiple coupling $(G_d, G_{d+1}, \ldots, G_{[1+\epsilon_n]d})$ such that marginally $G_i \sim \mathcal{G}(n, i)$ for all $d \leq i \leq [1 + \epsilon_n]d$ and jointly $G_d \subseteq G_{d+1} \subseteq \cdots \subseteq G_{[1+\epsilon_n]d}$ a.a.s.

If we only consider $(d_1, d_2)$ where $d_2 = d_1 + 1$ then we have the following coupling theorem which holds for a much larger range of $d_1$.

Theorem 8. Suppose $d \to \infty$ and $d = o(n^{1/3})$. There is a coupling $(G_d, G_{d+1})$ where marginally $G_d \sim \mathcal{G}(n, d)$ and $G_{d+1} \sim \mathcal{G}(n, d + 1)$, and jointly $G_d \subseteq G_{d+1}$ a.a.s.

Theorem 8 follows as a corollary of a more general version (Theorem 19) which we state in Section 4. Indeed, it is possible to prove that Theorem 8 holds for all $d \to \infty$ and $d = o(n^{1/2})$. However, for the sake of a simpler proof, we did not pursue that. See Remark 16 for more explanations.

The two problems studied in this paper seem unrelated. However, the key ingredient in the proof of Theorem 19 is the construction of a coupling procedure of $\mathcal{G}(n, d)$ and $\mathcal{G}(n, d + 1)$. The success of the coupling relies on the concentration of the number of perfect matchings in $\mathcal{G}(n, d + 1)$, which is one of the main results we obtain for the first problem under study.

All asymptotics in the paper refer to $n \to \infty$. Given two sequences of real numbers $a_n$ and $b_n$, we say $a_n = O(b_n)$ if there exists a constant $C > 0$ such that $|a_n| \leq C|b_n|$ for all $n$. We say $a_n = o(b_n)$ where $b_n > 0$ for all sufficiently large $n$, if $\lim_{n \to \infty} a_n / b_n = 0$. We say $a_n = o(b_n)$ if both $a_n$ and $b_n$ are positive for all sufficiently large $n$, and $b_n = o(a_n)$. We say $a_n = \Omega(b_n)$ if both $a_n$ and $b_n$ are positive for all sufficiently large $n$, and $b_n = O(a_n)$.

2 PROOF OF THEOREM 1

Recall that $Y$ denotes the number of perfect matchings in $\mathcal{G}(n, d)$. Throughout the paper we assume that $n$ is even. Let $X$ denote the number of triangles in $\mathcal{G}(n, d)$. We will approximate $Y$ by a linear function of $X$ using linear regression, and then study the distribution of $Y$ via analysing the distribution of $X$. This method is known as orthogonal decomposition and projection, developed by Janson [13]. Originally,
it is developed to determine the limiting distribution of the number of (large) subgraphs in $G(n,p)$, and Janson also applied the method to determine the distributions of the numbers of spanning trees, perfect matchings and Hamilton cycles in $G(n,m)$ [12]. We are not aware of any previous applications in $G(n,d)$. More specifically, we approximate $Y$ by $Y^* = aX + b$ where $a = \text{Cov}(X, Y)/\text{Var}X$ and $b = \mathbb{E}Y - a\mathbb{E}X$. The values of $a$ and $b$ are chosen so that $\mathbb{E}(Y - Y^*) = 0$ and $\mathbb{E}(Y - Y^*)^2$ is minimised. It follows immediately that $Y - Y^*$ and $Y^*$ are orthogonal random variables. We prove that $\mathbb{E}(Y - Y^*)^2$ is sufficiently small and thus the distribution of $Y$ is asymptotically determined by the distribution of $aX + b$. Since $X$ is asymptotically normally distributed, so is $Y$. The expectation $\mathbb{E}X$, the variance $\text{Var}X$ and the limiting distribution of $X$ have been studied in Reference [5, Theorems 8 and 10], which we state below.

**Theorem 9.** Suppose $d = o(n^{2/5})$ and $d \geq 2$. Then,

$$\mathbb{E}X = \frac{(d - 1)^3}{6} (1 + O(1/n)), \quad \text{Var}X \sim \mathbb{E}X,$$

$$\frac{X - \mathbb{E}X}{\sqrt{\text{Var}X}} \xrightarrow{d} \mathcal{N}(0,1) \text{ as } n \to \infty, \text{ provided } d \to \infty.$$ 

Next, we calculate the expectation $\mathbb{E}Y$, the second moment $\mathbb{E}Y^2$ and the covariance $\text{Cov}(X, Y)$, which allow us to estimate $a$ and $b$ and to bound $\mathbb{E}(Y - Y^*)^2$.

**Theorem 10.** Suppose $d = o(n^{1/3})$ and $d \geq 3$. Then,

$$\mathbb{E}Y = \frac{n!}{(n/2)!2^{n/2}} \left(\frac{e}{n}\right)^{n/2} \left(\frac{d - 1}{d}\right)^{\left(\frac{d-1}{2}\right)n} d^n \exp\left(\frac{1}{4} + O\left(\frac{d^3}{n}\right)\right)$$ \hspace{1cm} (2)

$$\mathbb{E}Y^2 = \left(1 + \frac{1}{6d^3} + O \left(d^{-4} + \frac{d^3}{n} + \sqrt{\frac{d}{n}\log^3 n}\right)\right) (\mathbb{E}Y)^2$$ \hspace{1cm} (3)

$$\text{Cov}(X,Y) = \left(-\frac{1}{d^3} + O \left(d^{-4} + \frac{d}{n}\right)\right) \mathbb{E}X \mathbb{E}Y. \hspace{1cm} (4)$$

**Remark 11.** Although Theorem 10 holds for constant $d$, the expressions in (3) and (4) do not provide any asymptotic information about $\mathbb{E}Y^2$ and $\text{Cov}(X, Y)$ since the error terms in $O(\cdot)$ are too big.

From Theorem 10 it follows immediately that

$$\text{Var}Y = \left(\frac{1}{6d^3} + O \left(d^{-4} + \frac{d^3}{n} + \sqrt{\frac{d}{n}\log^3 n}\right)\right) (\mathbb{E}Y)^2.$$ 

**Proof.** (Proof of Theorem 1) Recalling that $a = \text{Cov}(X, Y)/\text{Var}X$, we make the following claim.

**Claim 12.** $\mathbb{E}((Y - Y^*)^2) = o(\text{Var}Y^*)$.

Since

$$Y = Y^* + (Y - Y^*), \quad \mathbb{E}Y^* = \mathbb{E}Y,$$

it follows then that

$$\frac{Y - \mathbb{E}Y}{\sqrt{\text{Var}Y^*}} = \frac{Y^* - \mathbb{E}Y^*}{\sqrt{\text{Var}Y^*}} + \frac{Y - Y^*}{\sqrt{\text{Var}Y^*}} \xrightarrow{d} \frac{X - \mathbb{E}X}{\sqrt{\text{Var}X}} + \frac{Y - Y^*}{\sqrt{\text{Var}Y^*}}.$$ \hspace{1cm} (5)
By Claim 12 and Markov’s inequality, a.a.s.

\[ |Y - Y*| = o(\sqrt{\text{Var}Y*}). \]  

(6)

Moreover, by the orthogonality of \( Y - Y^* \) and \( Y^* \), \( \text{Cov}(Y - Y^*, Y^*) = 0 \) and so \( \text{Var}Y = \text{Var}Y^* + \text{Var}(Y - Y^*) \sim \text{Var}Y^* \) by Claim 12. Thus, the left hand side of (5) is asymptotic to \((Y - \mathbb{E}Y)/\sqrt{\text{Var}Y}\) in probability, and the right hand side converges to a random variable whose limiting distribution is \( \mathcal{N}(0, 1) \) by (6) and Theorem 9. Consequently,

\[
\frac{Y - \mathbb{E}Y}{\sqrt{\text{Var}Y}} \xrightarrow{d} \mathcal{N}(0, 1), \quad \text{as } n \to \infty.
\]

\[
\boxdot
\]

Proof. (Proof of Claim 12.) Since \( \mathbb{E}Y^* = \mathbb{E}Y \), we know that \( \mathbb{E}((Y - Y^*)^2) = \text{Var}(Y - Y^*) \). Moreover, since \( \text{Cov}(Y^*, Y - Y^*) = 0 \), we have \( \text{Var}Y = \text{Var}Y^* + \text{Var}(Y - Y^*) \). Hence, it is sufficient to prove that

\[
\text{Var}Y \sim \text{Var}Y^* = \frac{\text{Cov}(X, Y)^2}{\text{Var}X}.
\]  

(7)

By (2) and (3),

\[
\text{Var}Y = \mathbb{E}Y^2 - (\mathbb{E}Y)^2 = \left(\frac{1}{6d^3} + O\left(d^{-4} + \frac{d^3}{n} + \sqrt{\frac{d}{n} \log^3 n}\right)\right)(\mathbb{E}Y)^2,
\]

and by (4) and Theorem 9,

\[
\frac{\text{Cov}(X, Y)^2}{\text{Var}X} \sim \frac{\left(-\frac{1}{d} + O(d^{-4} + d/n)\right)^2 (\mathbb{E}X \mathbb{E}Y)^2}{\mathbb{E}X} \sim \frac{1}{6d^3} (\mathbb{E}Y)^2.
\]

Now (7) follows since \( d \to \infty \) and \( d = O(n^{1/7}/\log n) \).

\[
\boxdot
\]

3 | PROOF OF THEOREM 10

We will use the tools from References [7, Theorem 1] and [18, Theorem 4.6] to estimate \( \mathbb{E}Y, \mathbb{E}Y^2 \) and \( \text{Cov}(X, Y) \).

3.1 | Edge and subgraph probabilities in \( G(n, d) \)

Let \( H \) be a graph on \([n]\) and let \( d_H^1, \ldots, d_H^d \) denote the degree sequence of \( H \). Suppose that \( d_H^i \leq d \) for every \( 1 \leq i \leq n \). Let \( |H| \) denote the number of edges in \( H \). The following result is a special case of [7, Theorem 1] for the conditional edge probability \( \mathbb{P}(uv \in G(n, d) | H \subseteq G(n, d)) \).

Theorem 13. Suppose \( d = o(n) \) and suppose that \( H \) is a graph on \([n]\) such that \( d_H^i \leq d \) for every \( 1 \leq i \leq n \) and \( dn - 2|H| = \Omega(dn) \). Then,

\[
\mathbb{P}(uv \in G(n, d) | H \subseteq G(n, d)) = \left(1 + O\left(\frac{d}{n}\right)\right) \frac{(d - d_H^1)(d - d_H^2)}{dn - 2|H|}.
\]
We will apply Theorem 13 to estimate the probabilities of small subgraphs of $G(n, d)$. For large subgraphs, we apply instead the following enumeration result of McKay [18, Theorem 4.6].

**Theorem 14.** Let $g = (g_1, \ldots, g_n)$ be a sequence of non-negative integers. Let $m = m(g) = \| g \|_1 / 2$. Let $X$ be a simple graph on $[n]$ with degree sequence $x$. Let $\Delta(g)$ and $\Delta(x)$ denote the maximum components of $g$ and $x$ respectively. Suppose $\Delta(g) \geq 1, \Delta(g) = o(m)$ where $\Delta(g) = \Delta(g)^2 + \Delta(g)\Delta(x)$.

Define

$$\lambda = \lambda(g) = \frac{1}{4m(g)} \sum_{i=1}^{n} g_i(g_i - 1), \quad \mu = \mu(g, X) = \frac{1}{2m(g)} \sum_{j \in X} g_i g_j.$$

Let $N(g, X)$ denote the number of simple graphs with degree sequence $g$ and with no edge in common with $X$. Then,

$$N(g, X) = \frac{(2m)!}{m! 2^m \prod_{j=1}^{g_i} g_i!} \exp\left( -\lambda(g) - \frac{\lambda(g)^2}{2} - \mu(g, X) + O(\Delta(g)^2 / m(g)) \right).$$

**Corollary 15.** Suppose $d \geq 3$ and $d = o(n)$. Let $0 \leq k \leq n/2$ be an integer. Let $H$ be a graph on $[n]$ containing $k$ isolated edges and a collection of disjoint cycles spanning the remaining $n - 2k$ vertices. Then, with $\alpha = 2k/n$,

$$\mathbb{P}(H \subseteq G(n, d)) = \frac{((d - 2)n + 2k)! d^n (d - 1)^{n-2} + \frac{1}{2} n k}{(d-2n) + k!(d!n)!} \exp\left( \phi(d, \alpha) + O\left( \frac{d^3}{n} \right) \right)$$

$$= \left( \frac{e}{n} \right)^{(1-\frac{\alpha}{n})n} \left( \frac{d - 2 + \alpha}{d} \right)^{\frac{d-2+\alpha}{d}} \exp\left( \phi(d, \alpha) + O\left( \frac{d^3}{n} \right) \right),$$

where

$$\phi(d, \alpha) = \frac{4(d - 2)^2 - (d^2 - 5)\alpha^2 - (2d^2 - 14d + 20)\alpha}{4(d - 2 + \alpha)^2}.$$  \hspace{1cm} (8)

**Remark 16.** (a) Theorem 13 can be deduced from an earlier work than [7], for example by McKay [17]. We cite [7, Theorem 1] as it is written in form of conditional edge probabilities, which is what we need in this paper.

(b) A stronger version of Theorem 13 is available in Reference [5, Theorem 6] which estimates the conditional edge probabilities up to a relative error $d^2/n^2$ instead of $d/n$. Using that result, we can deduce Corollary 15 with a smaller error $O(d^2/n)$ than $O(d^3/n)$. This will result in an improvement in the range of $d$ in several of theorems in the paper, for example in Theorems 8, 10, and 19. However, applying [5, Theorem 6] involves more intensive calculations, and for a simpler proof we deduce Corollary 15 from Theorem 14 instead.

(c) It might be useful to notice, in applications of Corollary 15, that $\phi(d, \alpha)$ is essentially $O(1)$. In particular, as $d \to \infty$, $\phi(d, \alpha) \to 1 - (\alpha^2 + 2\alpha)/4$.

**Proof.** (Proof of Corollary 15.) Let $d^H$ denote the degree sequence of $H$ and let $g = d - d^H$ where $d = (d, \ldots, d)$. Then, $g$ has exactly $2k$ components of value $d - 1$ and $n - 2k$ components of value $d - 2$. Hence,

$$2m(g) = (d - 2)n + 2k$$
\[ \lambda(g) = \frac{1}{2((d-2)n + 2k)} ((d-1)(d-2) \cdot 2k + (d-2)(d-3)(n-2k)) \]
\[ \mu(g, H) = \frac{1}{(d-2)n + 2k} ((d-1)^2 \cdot k + (d-2)^2(n-2k)) . \]

and

\[ 2m(d) = dn, \quad \lambda(d) = \frac{1}{2dn} (d(d-1)n), \quad \mu(d, \emptyset) = 0. \]

Moreover,

\[ \hat{\lambda}(g), \hat{\lambda}(d) = O(d^2) \quad \text{and} \quad m(g), m(d) = \Omega(dn). \]

Thus,

\[ \mathbb{P}(H \subseteq G(n, d)) = \frac{N(g, H)}{N(d, \emptyset)} \]
\[ = \frac{((d-2)n + 2k)!}{(dn)!} \left( \frac{(d-2)n + 2k}{2} \right)^{d-2+2k} \cdot (d^2 - 5) \alpha^2 - (2d^2 - 14d + 20) \alpha \]
\[ = \frac{4(d-2)^2 - (d^2 - 5) \alpha^2 - (2d^2 - 14d + 20) \alpha}{4(d-2 + \alpha)^2}. \]  

where

\[ \phi(d, \alpha) = -\lambda(g) - \lambda(g)^2 - \mu(g, H) + \lambda(d)^2 + \mu(d, \emptyset) \]
\[ = \frac{4(d-2)^2 - (d^2 - 5) \alpha^2 - (2d^2 - 14d + 20) \alpha}{4(d-2 + \alpha)^2}. \]

See Maple calculations of (10) in the Appendix. Now the corollary follows by applying the Stirling formula to the factorial in (9). The relative error \( O(1/dn) \) in the Stirling formula is absorbed by \( O(d^3/n) \).

3.2 \quad \text{Cov}(X, Y)

Fix a perfect matching \( H \) of \( K_n \). There are \( (n/2) \cdot (n-2) \) ways to choose a triangle \( T \) such that \( |H \cap T| = 1 \). For any such \( T \), by Theorem 13 twice, the conditional probability of \( T \subseteq G(n, d) \) given \( H \subseteq G(n, d) \) is \( \frac{(d-1)^2(d-1)2}{(dn-n)^2} (1 + O(d/n)) \). There are \( \left( \frac{n}{3} \right) - (n/2 \cdot (n-2)) = (1 + O(1/n)) \) ways to choose a triangle \( T \) such that \( H \cap T = \emptyset \). For any such \( T \), again by Theorem 13 three times, the conditional probability of \( T \subseteq G(n, d) \) given \( H \subseteq G(n, d) \) is \( \frac{(d-1)^2}{(dn-n)^3} (1 + O(d/n)) \). Hence,

\[ \mathbb{E}XY = \sum_{H \in \Phi} \mathbb{P}(H \subseteq G(n, d)) \sum_{T \in \Psi} \mathbb{P}(T \subseteq G(n, d) | H \subseteq G(n, d)), \]

where \( \Phi \) is the set of all perfect matchings in \( K_n \), and \( \Psi \) is the set of all triangles in \( K_n \). By the discussions above, \( \sum_{T \in \Psi} \mathbb{P}(T \subseteq G(n, d) | H \subseteq G(n, d)) \) is the same for every perfect matching \( H \). Noting that \( \sum_{H \in \Phi} \mathbb{P}(H \subseteq G(n, d)) = \mathbb{E}Y \), we have, by setting \( x = 1/d \),

\[ \mathbb{E}XY = \mathbb{E}Y \left( \frac{n^2}{2} \frac{(d-1)(d-2)}{(dn-n)^2} + \frac{n^3}{6} \frac{(d-1)^3(d-2)}{(dn-n)^3} \right) (1 + O(d/n)) \]
where the last equation above is obtained by taking the product of $\frac{d^2}{2}(1-x)(1-2x) + \frac{d^3}{6}(1-2x)^3$ and $(\mathbb{E}X)^{-1} = (1 + O(1/n))\frac{6}{d^3(1-x)^3}$ from Theorem 9, and then taking the Taylor expansion of the product at $x = 0$. We include the Maple expansion formulae in the Appendix.

3.3 \( \mathbb{E}Y \)

Let $H$ be a perfect matching of $K_n$. By Corollary 15 with $k = n/2$ (i.e., $\alpha = 1$), we have $\phi(d, \alpha) = 1/4$ and thus,

$$\mathbb{P}(H \subseteq \mathcal{G}(n, d)) = (1 + O(d^3/n))\rho_1(n, d).$$

where

$$\rho_1(n, d) = \left(\frac{e}{n}\right)^{n/2} \left(\frac{d-1}{d}\right)^{(\frac{d+1}{2})n} d^\frac{n}{2} \exp\left(\frac{1}{4}\right).$$

Hence,

$$\mathbb{E}Y = (1 + O(d^3/n))\frac{n!}{(n/2)!2^{n/2}}\rho_1(n, d).$$

3.4 \( \mathbb{E}Y^2 \)

Let $0 \leq k \leq n/2$ be an integer. Fix two perfect matchings $H_1$ and $H_2$ of $K_n$ such that $|H_1 \cap H_2| = k$. Let $\alpha = \alpha(k) = 2k/n$. Then, by Corollary 15,

$$\mathbb{P}(H_1 \cup H_2 \subseteq \mathcal{G}(n, d)) = (1 + O(d^3/n))\rho_2(n, d, \alpha)$$

where

$$\rho_2(n, d, \alpha) = \frac{((d-2)n+2k)!dn!2^{n-k}d^n(d-1)^{n-2k}}{(d-2^n)^2+k!(dn)!} \exp(\phi(d, \alpha))$$

$$= \left(\frac{e}{n}\right)^{(1-\frac{2}{d})n} \left(\frac{d-2+\alpha}{d}\right)^{\frac{d-2+\alpha}{2}} d^{\frac{n}{2}} (d-1)^{(1-\alpha)n} \exp(\phi(d, \alpha) + O(1/dn)),$$

with $\phi(d, \alpha)$ defined in (8). Next we compute the number of pairs $(H_1, H_2)$ of perfect matchings of $K_n$ such that $|H_1 \cap H_2| = k$.

Lemma 17. \( \text{Let } 0 \leq k \leq n/2 - 2 \text{ be an integer. The number of pairs } (H_1, H_2) \text{ of perfect matchings of } K_n \text{ such that } |H_1 \cap H_2| = k \text{ is} \)

$$(1 + O((n-2k)^{-1})) \frac{n!}{2^k k! \sqrt{e\pi(n-2k)/2}}.$$
Proof. The exponential generating function\(^2\) for bi-coloured alternating cycles (i.e., edges along the cycle have alternating colours) of length at least 4 is

\[
F(z) = \sum_{n=2}^{\infty} \frac{(2n)!}{2 \cdot 2n} \cdot 2 \cdot \frac{z^{2n}}{(2n)!} = -\frac{1}{2} (\log(1 - z^2) + z^2).
\]

Thus, the number of pairs \((H_1, H_2)\) of disjoint perfect matchings of \(K_{2m}\) is

\[
(2m)! \cdot [z^{2m}] e^{F(z)} = (2m)! \cdot [z^{2m}] \frac{e^{-z^2/2}}{\sqrt{1 - z^2}}.
\]

We know \(e^{z^2/2} \cdot \frac{e^{-1/2}}{\sqrt{1 - z^2}} = O((1 - z)^{1/2})\) by expanding \(e^{-z^2/2}\) at \(z = 1\). Hence, by the transferring theorem [3, Theorems VI.3 and VI.4] and the binomial theorem,

\[
[z^{2m}] e^{F(z)} = [z^{2m}] \frac{e^{-1/2}}{\sqrt{1 - z^2}} + [z^{2m}] O((1 - z)^{1/2}) = \frac{e^{-1/2}}{4m} \left( \frac{2m}{m} \right) + O(m^{-3/2}) = \frac{1}{\sqrt{e \pi m}} (1 + O(m^{-1})).
\]

Thus, the number of pairs \((H_1, H_2)\) of perfect matchings of \(K_n\) such that \(|H_1 \cap H_2| = k\) is

\[
\binom{n}{2k} \cdot \frac{(2k)!}{2^{k}k!} \cdot \frac{(n - 2k)!}{\sqrt{e \pi (n - 2k)/2}} (1 + O((n - 2k)^{-1})) = (1 + O((n - 2k)^{-1})) \frac{n!}{2^{k}k! \sqrt{e \pi (n - 2k)/2}}.
\]

By Lemma 17 and recalling (14),

\[
\mathbb{E} Y^2 = \sum_{k=0}^{n/2 - 2} (1 + O((n - 2k)^{-1} + d^3/n)) \frac{n!}{2^{k}k! \sqrt{e \pi (n - 2k)/2}} \rho_2(n, d, \alpha(k)) + \mathbb{E} Y. \tag{17}
\]

Next, we show that the main contribution to \(\mathbb{E} Y^2\) comes from \(k\) near some specific value. The proof of the lemma is postponed till Section 3.4.2.

Lemma 18. Assume \(d = o(n^{1/3})\) and \(d \geq 3\). Let

\[
\bar{a} = \frac{1}{d}, \quad \bar{k} = \lfloor \bar{a}n/2 \rfloor, \quad \bar{d} = \frac{2d}{n} \frac{d(d - 2)}{(d - 1)^2}.
\]

Then

\[
\mathbb{E} Y^2 = \left( 1 + O\left( \sqrt{d} \log n + \frac{d^3}{n} \right) \right) \frac{2}{\sqrt{e \bar{d} \bar{k} \bar{d}} \sqrt{n - 2\bar{k}}} n! \rho_2(n, d, \bar{a}).
\]

3.4.1 Comparing \(\mathbb{E} Y^2\) with \((\mathbb{E} Y)^2\)

We complete the proof of Theorem 10 by verifying that

\[
\mathbb{E} Y^2 = \left( 1 + \frac{1}{6d^3} + O(\xi) \right) (\mathbb{E} Y)^2, \tag{18}
\]

\(^2\)We refer the readers to Reference [3, Part A] for enumeration by generating functions.
where $\xi = d^{-4} + \frac{d^3}{n} + \sqrt{\frac{d}{n}} \log^3 n$. By (13) and Lemma 18,

$$\left( \mathbb{E} Y \right)^2 = (1 + O(\xi)) 2\left( \frac{n}{e} \right)^n \rho_1(n,d)^2,$$

and

$$\mathbb{E} Y^2 = (1 + O(\xi)) \frac{2}{\sqrt{e \delta}} \sqrt{\frac{n}{n/e} \rho_2(n,d,\alpha)} (2k/e)^k \sqrt{n - 2k}.$$

Hence,

$$\frac{\mathbb{E} Y^2}{(\mathbb{E} Y)^2} = (1 + O(\xi)) \frac{n}{\sqrt{e \delta k(n - 2k)}} \frac{\rho_2(n,d,\alpha)}{(2k/e)^k \rho_1(n,d)^2}.$$

By straightforward but tedious calculations (see Appendix for more details)

$$\sqrt{\frac{n}{e \delta k(n - 2k)}} \frac{\rho_2(n,d,\alpha)}{(2k/e)^k \rho_1(n,d)^2} = 1 + \frac{1}{6d^3} + O(\xi), \quad (19)$$

and now (18) follows.

3.4.2 Proof of Lemma 18

Recall from (17) that

$$\mathbb{E} Y^2 = \sum_{k=0}^{n/2-2} (1 + O((n - 2k)^{-1} + d^3/n)) \frac{n!}{2^k k! \sqrt{\pi(n - 2k)/2}} \rho_2(n,d,\alpha(k)) + \mathbb{E} Y$$

$$= \sum_{k=0}^{n/2-2} (1 + O((n - 2k)^{-1} + d^3/n)) \sqrt{\frac{2}{e \pi}} n! \varphi(k) + \mathbb{E} Y, \quad (20)$$

where

$$\varphi(k) = \frac{\rho_2(n,d,\alpha(k))}{2^k k! \sqrt{n - 2k}}.$$

The proof of Lemma 18 is standard. We prove that the summand in (20) is maximised at $\tilde{k}$. Then, we approximate the summation around $\tilde{k}$ by an integral of a function of the form $e^{-\xi^2}$. The contributions to (20) from $k$ far away from $\tilde{k}$ is negligible.

It is easy to see then that $\exp(\varphi(d,\alpha(k)) - \varphi(d,\alpha(k - 1))) = \exp(O(1/n))$. Hence, by (15),

$$\frac{\varphi(k)}{\varphi(k - 1)} = \begin{cases} \frac{(d - 2)n + 2k}{2(d - 1)^2 k} \left( 1 + O \left( \frac{1}{n} \right) \right) & \text{for all } k \leq n/3 \\ O \left( \frac{(d - 2)n + 2k}{2(d - 1)^2 k} \right) = O(1/d) & \text{for all } n/3 < k \leq n/2 - 2 \end{cases} \quad (21)$$

By equating $\frac{(d - 2)n + 2k}{2(d - 1)^2 k}$ to 1 we find that $k = n/2d$. It follows immediately that at $\tilde{k}$, the ratio $\frac{(d - 2)n + 2k}{2(d - 1)^2 k}$ is $1 + O(d/n)$, since rounding $n/2d$ to $\tilde{k}$ would change this ratio by $O(d/n)$. Moreover, noticing that
where we have that for every positive \( j = o(\bar{k}) \):

\[
\varphi(\bar{k} + j) / \varphi(\bar{k}) = \left(1 + O\left(\frac{j}{n}\right)\right) \prod_{i=1}^{j} \frac{(d-2)n + 2\bar{k} + 2i}{2(\bar{k} + i)(d-1)^2} \\
= \left(1 + O\left(\frac{j}{n}\right)\right) \prod_{i=1}^{j} \left(1 + i \left(\frac{2}{(d-2)n + 2\bar{k}} - \frac{1}{\bar{k}}\right) + O\left(\frac{i^2}{\bar{k}^2} + \frac{d}{n} \right)\right) \\
= \exp\left(-\tilde{\delta} \sum_{i=1}^{j} i + O\left(\frac{j^3}{\bar{k}^2} + \frac{dj}{n}\right)\right) \\
= \exp\left(-\tilde{\delta} j^2 + O\left(\tilde{\delta} j + \frac{j^3}{\bar{k}^2} + \frac{dj}{n}\right)\right)
\]

(22)

recalling that \( \tilde{\alpha} = 1/d, \bar{k} = |\tilde{\alpha} \cdot n/2| \) and

\[
\tilde{\delta} = \frac{2d \cdot d(d-2)}{n \cdot (d-1)^2} = \left(\frac{1}{\bar{k}} - \frac{2}{(d-2)n + 2\bar{k}}\right) \left(1 + O\left(\frac{1}{\bar{k}}\right)\right) \\
= \left(\frac{1}{\bar{k}} - \frac{2}{(d-2)n + 2\bar{k}}\right) \left(1 + O\left(\frac{d}{n}\right)\right).
\]

Symmetric calculations show that (22) holds also for every negative \( j = o(\bar{k}) \). It follows then that

\[
\sum_{\bar{k} - \tilde{\delta}^{-1/2} \log(1/\tilde{\delta}) \leq k \leq \bar{k} + \tilde{\delta}^{-1/2} \log(1/\tilde{\delta})} \varphi(k) = (1 + O(\tilde{\delta})) \varphi(\bar{k}) \sqrt{\frac{2\pi}{\tilde{\delta}}} \int_{-\infty}^{+\infty} e^{-x^2} dx \\
= (1 + O(\tilde{\delta})) \sqrt{\frac{2\pi}{\tilde{\delta}}} \varphi(\bar{k}),
\]

where

\[
\xi = \tilde{\delta}^{-1/2} \log(1/\tilde{\delta}) + \tilde{\delta}^{-3/2} \log^3(1/\tilde{\delta}) + \frac{d}{n} \tilde{\delta}^{-1/2} \log(1/\tilde{\delta}) + \tilde{\delta}^{1/2} = O\left(\sqrt{\frac{d}{n} \log^3 n}\right).
\]

Note that the first three terms in \( \xi \) come from the accumulative error \( O(\tilde{\delta} j + j^3/\bar{k}^2 + dj/n) \) in (22), and the last term comes from approximating the sum of \( \exp(-\tilde{\delta}j^2/2) \) by an integral. The contributions to (20) from \( k \) where \( |k - \bar{k}| > \tilde{\delta}^{-1/2} \log(1/\tilde{\delta}) \) is smaller than \( n^{-1} \) — indeed, at most \( \exp(-\Omega( \log^2 n)) \) — as a relative error by (21) and standard calculations on summing a geometrically bounded series. So

\[
\mathbb{E}Y^2 = \left(1 + O\left(\sqrt{\frac{d}{n} \log^3 n + \frac{d^3}{n}}\right)\right) \frac{2}{\sqrt{\pi} \bar{k} \sqrt{2\bar{k} \sqrt{n - 2\bar{k}}} + n \cdot \rho_2(n, d, \bar{\alpha})},
\]

where the error \( d^3/n \) is carried from (20).

\[\square\]
4 | PROOF OF THEOREMS 6–8

4.1 | Proof of Theorem 6

We prove Theorem 6 assuming Theorem 7. Given $0 \leq p \leq n$, recall that $G(n, p)$ denotes the Erdős-Rényi random graph with edge probability $p$. Let $d_1 = o(\log^2 n)$. By [4, Theorem 2], there exists $\delta_n = o(1)$ such that $G(n, d_1), G(n, p_1)$ and $G(n, d_2)$ can be coupled together, where $p_1 = (1 + \delta_n)d_1/n$ and $d_2 \geq (1 + 2\delta_n)d_1$, such that a.a.s. $G(n, d_1) \subseteq G(n, p_1) \subseteq G(n, d_2)$. It follows now that Conjecture 5 holds for any $(d_1, d_2)$ where $d_1 = \omega(\log^2 n)$ and $d_2 - d_1 \geq 2\delta_n d_1$. Now suppose $d_1 + 1 \leq d_2 < (1 + 2\delta_n)d_1$ and $d_1 = O(n^{1/2}/\log n)$. Then, there is a coupling where a.a.s. $G(n, d_1) \subseteq G(n, d_2)$ by Theorem 7. Now Theorem 6 follows.

4.2 | Couple $G(n, d)$ and $G(n, d + 1)$

Throughout this section we assume $d \to \infty$ and $d = o(n^{1/3})$. We also assume that $n$ is sufficiently large so that various $o(1)$ quantities can be assumed to be small enough. Our goal is to couple $G(n, d)$ with $G(n, d + 1)$ so that $G(n, d) \subseteq G(n, d + 1)$ with sufficiently high probability. In the next section, we “stitch” a sequence of such couplings together to obtain a simultaneous coupling as in Theorem 7.

Given $\alpha = \alpha_n = o(1)$, define $\eta = \eta(\alpha)$ where

$$\eta(\alpha) = 2\alpha + \frac{1}{d^3 \alpha^2} + \frac{C'd^3}{na^2} + \frac{C'\sqrt{d/n}\log^3 n}{\alpha^2},$$

where $C' > 0$ is a sufficiently large constant.

We prove the following stronger version of Theorem 8.

**Theorem 19.** Assume $\alpha = o(1)$ is such that $\eta(\alpha) = o(1)$. There is a coupling $(G_d, G_{d+1})$ where marginally $G_d \sim G(n, d)$ and $G_{d+1} \sim G(n, d + 1)$, and jointly $G_d \subseteq G_{d+1}$ with probability at least $1 - 5\eta$ for all sufficiently large $n$.

4.2.1 | The coupling procedure

In this section we assume that $\eta(\alpha) = o(1)$ for some $\alpha = o(1)$. This assumption, together with the assumption that $n$ is sufficiently large, ensures that various quantities in our procedure below are positive. This assumption also immediately implies that $d = o(n^{1/3})$ and $d \to \infty$, which satisfies the condition for Theorem 10. In this subsection, with slight abuse of notation, we also let $G(n, d)$ denote the set of $d$-regular graphs on $[n]$. From the context it is always clear whether we refer to a set of graphs or a random graph from the set.

For $G \in G(n, d + 1)$ let $Y(G)$ denote the number of perfect matchings of $G$. For $G \in G(n, d)$ let $Z(G)$ denote the number of perfect matchings in $K_n \setminus G$. We say $G$ and $G'$ are related, denoted by $G \sim G'$, for $G \in G(n, d)$ and $G' \in G(n, d + 1)$, if $G \subseteq G'$. We can represent this relation using an auxiliary directed bipartite graph $\mathcal{X}$ where $V(\mathcal{X}) = G(n, d) \cup G(n, d + 1)$, and $(G, G')$ is an arc if $G \subseteq G'$. Thus, a $d$-regular graph $G$ in $\mathcal{X}$ has out-degree $Z(G)$, and a $(d + 1)$-regular graph $G'$ in $\mathcal{X}$ has in-degree $Y(G')$.

Let $\alpha = \alpha_n = o(1)$. By Theorem 10 and Chebyshev’s inequality,

$$\mathbb{P}_{G(n,d+1)}(|Y - \mathbb{E}Y| \geq \alpha \mathbb{E}Y) \leq \frac{\text{Var}Y}{\alpha^2(\mathbb{E}Y)^2} = \frac{1}{6d^3 \alpha^2} + O\left(\frac{1}{d^4 \alpha^2} + \frac{d^3}{na^2} + \frac{\sqrt{d/n}\log^3 n}{\alpha^2}\right). \quad (23)$$
By Theorem 14 (with \( g \) being the all one vector and \( X \) being a \( d \)-regular graph) and Theorem 10, there exists a sufficiently large constant \( C > 0 \) such that

\[
|Z(G) - Z^*| \leq C \frac{d^2}{n} \cdot Z^*, \quad \text{for all } d \text{-regular graph } G,
\]

\[
|\mathbb{E}Y(G) - Y^*| \leq C \frac{d^3}{n} \cdot Y^*, \quad \text{for } G \sim \mathcal{G}(n, d + 1),
\]

where

\[
Z^* = \frac{n!e^{-d/2}}{(n/2)!2^{n/2}}, \quad Y^* = \frac{n!e^{1/4}}{(n/2)!2^{n/2}} \left( \frac{e}{n} \right)^{n/2} \left( \frac{d}{d + 1} \right)^{\frac{d}{2}} (d + 1)^{\frac{d}{2}}.
\]

Let

\[
Y = (1 - \alpha - Cd^3/n) Y^*,
\]

\[
Z = \left(1 + \frac{C d^2}{n}\right) Z^*.
\]

Define

\[
B = \{G \in \mathcal{G}(n, d + 1) : Y(G) < Y\}
\]

\[
B' = \{G \in \mathcal{G}(n, d + 1) : Y(G) > (1 + \alpha + Cd^3/n) Y^*\}.
\]

Since \( \alpha = o(1) \) and \( \eta(\alpha) = o(1) \), it follows immediately that \( \alpha \gg d^3/n \). Thus, by (23) and (25),

\[
\mathbb{P}(B \cup B') \leq \frac{1}{6d^3\alpha^2} + O \left( \frac{1}{d^4\alpha^2} + \frac{d^3}{n\alpha^2} + \frac{\sqrt{d/n\log^3 n}}{\alpha^2} \right).
\]

Let \( D \) and \( \hat{D} \) denote the total in-degrees of \( \mathcal{G}(n, d + 1) \) and \( \mathcal{G}(n, d + 1) \setminus B \) respectively in \( \mathcal{X} \). That is,

\[
D = |\{(G, G') \in \mathcal{G}(n, d) \times \mathcal{G}(n, d + 1) : G \sim G'\}|
\]

(31)

\[
\hat{D} = |\{(G, G') \in \mathcal{G}(n, d) \times (\mathcal{G}(n, d + 1) \setminus B) : G \sim G'\}|
\]

(32)

Further, let \( d^- (B) \) and \( d^- (B') \) denote the total in-degrees of \( B \) and \( B' \) respectively in \( \mathcal{X} \). We prove the following bounds on \( d^- (B) \) and \( d^- (B') \). Recall that

\[
\eta(\alpha) = 2\alpha + \frac{1}{d^3\alpha^2} + \frac{C' d^3}{n\alpha^2} + \frac{C' \sqrt{d/n\log^3 n}}{\alpha^2},
\]

where \( C' > 0 \) is a sufficiently large constant.

**Lemma 20.** Assume \( \alpha = o(1) \) is such that \( \eta(\alpha) = o(1) \). Then, \( d^- (B) + d^- (B') \leq \eta D \) for all sufficiently large \( n \).

**Proof.** The number of edges in \( \mathcal{X} \) is \( D = |\mathcal{G}(n, d + 1)| \mathbb{E}Y \). This can be rewritten as

\[
d^- (B) + d^- (B') + \mathbb{E}(\mathcal{G}(n, d + 1) \setminus (B \cup B')) \cdot \mathbb{E}Y (1 + \xi),
\]
where $|\xi| \leq \alpha + O(d^3/n)$ since $|Y(G)/\mathbb{E}Y - 1| \leq \alpha + O(d^3/n)$ for all $G \notin B \cup B'$ by the definition of $B$ and $B'$. By (30), the above is equal to

$$d^-(B) + d^-(B') + |\mathcal{G}(n, d + 1)| \left(1 - \xi'\right) \cdot \mathbb{E}Y \left(1 + \xi\right)$$

where

$$0 \leq \xi' \leq \frac{1}{6d^3\alpha^2} + O\left(\frac{1}{d^4\alpha^2} + \frac{d^3}{na^2} + \frac{\sqrt{d/n\log^3 n}}{\alpha^2}\right).$$

Thus,

$$|\mathcal{G}(n, d + 1)| \mathbb{E}Y = d^-(B) + d^-(B') + |\mathcal{G}(n, d + 1)| \cdot \mathbb{E}Y \left(1 - \xi' + \xi - \xi'\xi\right),$$

which implies that

$$d^-(B) + d^-(B') = |\mathcal{G}(n, d + 1)| \cdot \mathbb{E}Y \left(\xi' - \xi + \xi'\xi\right) < \eta D,$$

where the last inequality holds because $\xi' - \xi + \xi'\xi < |\xi| + 2\xi' < \eta$ by the definition of $\eta$. \qed

Finally we are ready to define the coupling $(G_d, G_{d+1})$.

1. Let $G_d$ be a uniformly random graph in $\mathcal{G}(n, d)$ and let $\overline{G}$ be the graph obtained from $G_d$ by adding a uniformly random perfect matching of $K_n \setminus G_d$. Let $H$ be a uniformly random graph in $\mathcal{G}(n, d + 1)$ independent of $G_d$ and $\overline{G}$.

2. If $\overline{G} \in B$ then let $G_{d+1} = \overline{G}$ with probability $(1 - \eta)\frac{Z(G_d)}{Z}$ and let $G_{d+1} = H$ with the remaining probability.

3. If $\overline{G} \in \mathcal{G}(n, d + 1) \setminus B$, then

$$G_{d+1} = \begin{cases} 
\overline{G} & \text{with probability } (1 - \eta)\frac{Z(G_d)}{Z} \cdot \frac{Y}{Y(G')} \\
G'' & \text{with probability } (1 - \eta)\frac{Z(G_d)}{Z} \cdot \frac{Y - Y(G'')}{D} \text{ for every } G'' \in B \\
H & \text{with the remaining probability.}
\end{cases}$$

Note that $Z(G) \leq Z$ for every $G \in \mathcal{G}(n, d)$ by the definition of $Z$. The following lemma bounds a few quantities in the above probability terms, and in particular, it justifies that the coupling procedure is well defined (for all sufficiently large $n$).

**Lemma 21.** Assume $\alpha = o(1)$ is such that $\eta(\alpha) = o(1)$, and $n$ is sufficiently large. Then,

$$\frac{Z(G)}{Z} \geq 1 - 3C\frac{d^2}{n}, \quad \text{for every } d\text{-regular graph } G,$$

$$\frac{Y}{Y(G')} \geq 1 - 3\alpha - 3C\frac{d^3}{n}, \quad \text{for every } (d + 1)\text{-regular graph } G' \in \mathcal{G}(n, d + 1) \setminus (B \cup B'),$$

and for every $(G, G') \in \mathcal{G}(n, d) \times (\mathcal{G}(n, d + 1) \setminus B)$ where $G \sim G'$,

$$(1 - \eta)\frac{Z(G)}{Z} \cdot \frac{Y}{Y(G')} + \sum_{G'' \in B} (1 - \eta)\frac{Z(G)}{Z} \cdot \frac{Y - Y(G'')}{D} \leq 1.$$
Proof. Since $\alpha = o(1)$ and $\eta(\alpha) = o(1)$, it follows immediately that $d = o(n^{1/3})$ and $d \to \infty$. The first inequality in the lemma follows by (24) and (27). The second inequality in the lemma follows by (26), (28) and (29). For the last inequality, note that

$$\frac{Z(G)}{Z}, \frac{Y}{Y(G')} \leq 1$$

always by (24) and the definition of $B$. Thus it is sufficient to show that

$$\sum_{G' \in B} \frac{(Y - Y(G'))}{D} \leq \eta.$$ 

By (23),

$$|B| \leq \left(\frac{1}{6d^3 \alpha^2} + O\left(\frac{1}{d^4 \alpha^2} + \frac{d^3}{na^2} + \frac{\sqrt{d/n \log^3 n}}{\alpha^2}\right)\right) |G(n, d + 1)|.$$ 

By Lemma 20,

$$\hat{D} = (1 + O(\eta))D = (1 + O(\eta))|G(n, d + 1)| \geq \mathbb{E}Y = (1 + O(\eta))|G(n, d + 1)|Y.$$ 

Thus,

$$\sum_{G' \in B} (Y - Y(G'')) \leq \frac{Y}{B} \leq \left(\frac{1}{6d^3 \alpha^2} + O\left(\frac{1}{d^4 \alpha^2} + \frac{d^3}{na^2} + \frac{\sqrt{d/n \log^3 n}}{\alpha^2}\right)\right) |G(n, d + 1)| \leq \eta \hat{D},$$

by the definition of $\eta$. Thus, the last inequality of the lemma follows. 

4.2.2 Proof of Theorem 19

By the construction, $G_d$ is obviously distributed as $G(n, d)$ marginally. We prove that the marginal distribution of $G_{d+1}$ is $G(n, d + 1)$. Let

$$\sigma_d = \frac{1}{|G(n, d)|}, \quad \sigma_{d+1} = \frac{1}{|G(n, d + 1)|}.$$ 

Let $\hat{G}$ be a $(d + 1)$-regular graph. If $\hat{G} \in G(n, d + 1) \setminus B$ then

$$\mathbb{P}(G_{d+1} = \hat{G}) = \sum_{G: G \sim \hat{G}} \frac{\sigma_d}{Z(G)} \cdot (1 - \eta) \frac{Z(G)}{Z} \cdot \frac{Y}{Y(\hat{G})} + \varphi,$$ 

where

$$\varphi = \sum_{G: G \sim \hat{G}, G \not\sim \hat{G}, G' \in B} \frac{\sigma_d}{Z(G)} \left(1 - (1 - \eta) \frac{Z(G)}{Z}\right) \sigma_{d+1}$$

$$+ \sum_{G: G \sim \hat{G}, G \not\sim \hat{G}, G' \in B} \frac{\sigma_d}{Z(G)} \left(1 - (1 - \eta) \frac{Z(G)}{Z}ight) \cdot \frac{Y}{Y(G')} - \sum_{G' \in B} \sum_{G: G \sim \hat{G}} (1 - \eta) \frac{Z(G)}{Z} \left(\frac{Y - Y(G'')}{D}\right) \sigma_{d+1}.$$
In the first summation in (33), $\sigma_d/Z(G)$ is the probability that $G_d = G$ and $\overline{G} = \hat{G}$. Conditioning on that, $(1 - \eta)\frac{Z(G)}{Z} \cdot \frac{Y}{Y(G')} \cdot \frac{Y}{Y(G)}$ is the probability that $G_{d+1}$ is set to be $\overline{G}$. Thus this summation gives the contribution to $\mathbb{P}(G_{d+1} = \hat{G})$ from the case that $\overline{G} = \hat{G}$ and $G_{d+1}$ is set to be $\overline{G}$. Similarly, it is easy to see that $\varphi$ is the probability that $H = \hat{G}$ and $G_{d+1}$ is set to be $H$. Note that the value of $\varphi$ is independent of $\hat{G}$. Hence, by noting that $Y(\hat{G}) = |\{G : G \sim \hat{G}\}|$, we obtain

$$\mathbb{P}(G_{d+1} = \hat{G}) = (1 - \eta)\sigma_d \frac{Y}{Z} + \varphi,$$

which is independent of $\hat{G}$ for all $\hat{G} \in \mathcal{G}(n, d + 1) \setminus B$.

Next, suppose $\hat{G} \in B$. Then,

$$\mathbb{P}(G_{d+1} = \hat{G}) = \sum_{G : \hat{G} \sim G} \frac{\sigma_d}{Z(G)} \cdot (1 - \eta) \frac{Z(G)}{Z} + \sum_{G' \sim \hat{G}, \overline{G} \not\in B} \frac{\sigma_d}{Z(G)} \cdot (1 - \eta) \frac{Z(G)}{Z} \frac{Y}{Y(\hat{G})} + \varphi,$$

where the second summation above is from the case where $G_d = G$, $\overline{G} = G' \not\in B$, and $G_{d+1}$ is set to be $G'' = \hat{G}$ which occurs with probability $(1 - \eta)(1 - \frac{Z(G)}{Z}) \frac{Y}{Y(\hat{G})}$, given $(G, G')$. The first summation above gives $(1 - \eta)\sigma_d Y(\hat{G})/Z$. The second summation above gives $(1 - \eta)\sigma_d(\frac{Y}{Y(\hat{G})})/Z$ by (32). Hence,

$$\mathbb{P}(G_{d+1} = \hat{G}) = (1 - \eta)\sigma_d \frac{Y}{Z} + \varphi,$$

which is independent of $\hat{G}$ for all $\hat{G} \in B$, and is the same for all $\hat{G} \in \mathcal{G}(n, d + 1) \setminus B$. This confirms that the marginal distribution of $G_{d+1}$ is uniform in $\mathcal{G}(n, d + 1)$.

Finally, we prove that $G_d \subseteq G_{d+1}$ with probability at least $1 - 5\eta$ for all sufficiently large $n$. We use $G_{d+1} \leftarrow \overline{G}$ and $G_{d+1} \leftarrow H$ to denote the events that the coupling procedure sets $G_{d+1}$ to be $\overline{G}$, and $H$, respectively. We use $G_{d+1} \leftarrow B$ to denote the event that $\overline{G} \in \mathcal{G}(n, d + 1) \setminus B$ but $G_{d+1}$ is set to be some graph $G'' \in B$. Note that $G_d \subseteq G_{d+1}$ if $G_{d+1} \leftarrow \overline{G}$. Thus, it is sufficient to show that the probability that $G_{d+1} \leftarrow H$ or $G_{d+1} \leftarrow B$ is at most $5\eta$.

First we see that

$$\mathbb{P}(G_{d+1} \leftarrow H \land \overline{G} \in B) = \sum_{G \in \mathcal{G}(n, d) : \overline{G} \sim G} \sum_{G' \sim G} \frac{\sigma_d}{Z(G)} \left(1 - (1 - \eta) \frac{Z(G)}{Z}\right),$$

$$\leq \sum_{G \in \mathcal{G}(n, d) : \overline{G} \sim G} \sum_{G' \sim G} \frac{\sigma_d}{Z(G)} \left(1 - (1 - \eta) \left(1 - 3\frac{\Delta^2}{n}\right)\right),$$

(by Lemma 21)

$$\leq 2\eta \sum_{G \in \mathcal{G}(n, d)} \sigma_d \sum_{G' : \overline{G} \sim G} \frac{1}{Z(G)} = 2\eta,$$

where the last inequality holds by the definition of $\eta$, the assumptions that $\eta = o(1)$ and $n$ is sufficiently large. Similarly,

$$\mathbb{P}\left((G_{d+1} \leftarrow H \text{ or } G_{d+1} \leftarrow B) \land \overline{G} \not\in B\right)$$

$$= \sum_{G \in \mathcal{G}(n, d) : \overline{G} \not\in B} \sum_{G' \sim G} \frac{\sigma_d}{Z(G)} \left(1 - (1 - \eta) \frac{Z(G)}{Z} \cdot \frac{Y}{Y(G')}\right),$$

$$\leq 3\eta \sum_{G \in \mathcal{G}(n, d) : \overline{G} \not\in B} \frac{\sigma_d}{Z(G)} + \sum_{G \in \mathcal{G}(n, d) : \overline{G} \not\in B} \frac{\sigma_d}{Z(G)}.$$
as for every \( G' \not\in B \cup B' \), \( 1 - (1 - \eta) \frac{Z(G)}{Z} \cdot \frac{\nu}{Y(G')} \leq \eta + 3\alpha + 6Cd^3/n \leq 3\eta \) by Lemma 21, the definition of \( \eta \), the assumptions that \( \eta = o(1) \) and \( n \) is sufficiently large; and for \( G' \in B' \) we use the trivial upper bound \( 1 - (1 - \eta) \frac{Z(G)}{Z} \cdot \frac{\nu}{Y(G')} \leq 1 \). Since \( \{(G' : G' \sim G, G' \not\in B \cup B')\} = Z(G) \), and \( \sigma_d \cdot |\mathcal{G}(n, d)| = 1 \), the first double summation above is at most \( 3\eta \). By (24), the second double summation above is equal to

\[
\left(1 + O(d^2/n)\right) \frac{\sigma_d}{Z^*} |\{(G, G') \in \mathcal{G}(n, d) \times B' : G \sim G'\}| = \left(1 + O(d^2/n)\right) \frac{\sigma_d}{Z^*} n^2 (B') \]

\[
\leq \left(1 + O(d^2/n)\right) \frac{\sigma_d}{Z^*} \eta \mathcal{D} \quad \text{(by Lemma 20)}
\]

\[
= \left(1 + O(d^2/n)\right) \frac{\sigma_d}{Z^*} \eta |\mathcal{G}(n, d)| |\mathbb{E}Z| \leq 2\eta.
\]

where the last equality above holds because \( Z^* \sim \mathbb{E}Z \) and \( \sigma_d |\mathcal{G}(n, d)| = 1 \). Combining all cases above, we know that the probability that \( G_{d+1} \leftarrow \overline{G} \) (and thus \( G_d \subseteq G_{d+1} \)) is at least \( 1 - 5\eta \).

\[\square\]

### 4.3 Proof of Theorem 8

Suppose \( d \to \infty \) and \( d = o(n^{1/3}) \). Then there exists \( \alpha = o(1) \) such that \( \eta(\alpha) = o(1) \). Theorem 8 follows by Theorem 19 with such a choice of \( \alpha \).

\[\square\]

### 4.4 Proof of Theorem 7

Suppose \( d \to \infty \), \( d = O(n^{1/7}/\log n) \) and \( n \) is sufficiently large. Let \( \alpha = 1/d \). It follows now that \( \eta = O(1/d) \). We prove that for each \( 1 \leq j \leq \lfloor \epsilon nd \rfloor \), there is a coupling \( (G_d, \ldots, G_{d+j}) \) where \( G_i \sim \mathcal{G}(n, i) \) for every \( d \leq i \leq d + j \) and with probability at least \( 1 - 5\eta \), \( G_d \subseteq G_{d+1} \subseteq \cdots \subseteq G_{d+j} \). Then Theorem 7 follows by taking \( j = \lfloor \epsilon nd \rfloor \).

We prove by induction on \( j \). The base case \( j = 1 \) follows directly by Theorem 19 with our choice of \( \alpha \). Suppose the statement holds for some \( 1 \leq j < \lfloor \epsilon nd \rfloor \). Let \( \pi_j \) be the joint probability distribution of such a coupling \( (G_d, \ldots, G_{d+j}) \). Again, by Theorem 19, there is a coupling \( (G_{d+j}, \ldots, G_{d+j+1}) \) where \( G_{d+j} \sim \mathcal{G}(n, d + j) \), \( G_{d+j+1} \sim \mathcal{G}(n, d + j + 1) \) and with probability at least \( 1 - 5\eta \), \( G_{d+j} \subseteq G_{d+j+1} \). Let \( \pi \) denote the joint probability of this coupling \( (G_{d+j}, G_{d+j+1}) \). We construct a coupling \( (G_d, \ldots, G_{d+j+1}) \) by first sampling \( (G_d, \ldots, G_{d+j}) \) according to the distribution \( \pi_j \), and then sampling \( G_{d+j+1} \) according to the conditional probability \( \pi(G_{d+j+1} | G_{d+j}) \). The resulting coupling satisfies the required marginal distribution conditions. Moreover, the probability that either \( G_d \subseteq \cdots \subseteq G_{d+j} \) fails or \( G_{d+j} \subseteq G_{d+j+1} \) fails is at most \( 5\eta + 5\eta = 5(j + 1)\eta \) by the union bound. The assertion follows by induction.

\[\square\]

### 5 Future Research

As mentioned in Remark 16, the error \( d^3/n \) in Corollary 15 can be improved to \( d^2/n \) if we apply [5, Theorem 6] and go through more involved calculations. Another approach is to improve the error in Theorem 14, which is of independent interest and can lead to improvements of many other existing results on subgraphs of \( \mathcal{G}(n, d) \).

We solved Conjecture 5 for a certain range of \( d_1 \) by simultaneously coupling a sequence of random regular graphs. This is certainly not necessary, and is the cause of the restrictions on \( d_1 \) in Theorem 6. A more direct approach would be to prove concentration of the number of \( k \)-factors in \( \mathcal{G}(n, d) \). This would significantly relax the restrictions on \( d_1 \), and is itself of independent interest.
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REFERENCES

1. B. Bollobás, A probabilistic proof of an asymptotic formula for the number of labelled regular graphs, Eur J Comb 1 (1980), no. 4, 311–316.
2. A. Dudek, A. Frieze, A. Ruciuński, and M. Šileikis, Embedding the Erdős-Rényi hypergraph into the random regular hypergraph and Hamiltonicity, J Comb Theory Ser B 122 (2017), 719–740.
3. P. Flajolet and R. Sedgewick, Analytic combinatorics, Cambridge University Press, Cambridge, England, 2009.
4. Pu Gao. Kim-Vu’s sandwich conjecture is true for all $d = \omega (\log^{7}n)$. arXiv preprint arXiv:2011.09449, 2020.
5. Pu Gao. Triangles and subgraph probabilities in random regular graphs. arXiv preprint arXiv:2012.01492, 2020.
6. Pu Gao, Mikhail Isaev and Brendan D McKay. Sandwiching random regular graphs between binomial random graphs. Proceedings of the Fourteenth Annual ACM-SIAM Symposium on Discrete Algorithms, SIAM, 2020, pp. 690–701.
7. Pu Gao and Yuval Oehapkin. Subgraph probability of random graphs with specified degrees and applications to chromatic number and connectivity. arXiv preprint arXiv:2007.02216, 2020.
8. P. Gao and C. M. Sato, A transition of limiting distributions of large matchings in random graphs, J Comb Theory Ser B 116 (2016), 57–86.
9. Z. Gao and N. C. Wormald, Distribution of subgraphs of random regular graphs, Random Struct Algorithms 32 (2008), no. 1, 38–48.
10. H. Garmo, The asymptotic distribution of long cycles in random regular graphs, Random Struct Algorithms 15 (1999), no. 1, 43–92.
11. Catherine Greenhill, Mikhail Isaev and Gary Liang. Spanning trees in random regular uniform hypergraphs. arXiv preprint arXiv:2005.07350, 2020.
12. S. Janson, The numbers of spanning trees, Hamilton cycles and perfect matchings in a random graph, Comb Probab Comput 3 (1994), no. 1, 97–126.
13. S. Janson, Orthogonal decompositions and functional limit theorems for random graph statistics, Mem Am Math Soc 111 (1994), no. 534, 1–78.
14. S. Janson, Random regular graphs: asymptotic distributions and contiguity, Comb Probab Comput 4 (1995), no. 4, 369–405.
15. J. H. Kim and H. V. Van, Sandwiching random graphs: universality between random graph models, Adv Math 188 (2004), no. 2, 444–469.
16. Tereza Klimošová, Christian Reiher, Andrzej Ruciuński and Matas Šileikis. Sandwiching biregular random graphs. arXiv preprint arXiv:2010.15751, 2020.
17. B. D. McKay, Subgraphs of random graphs with specified degrees, Congr Numer 33 (1981), 213–223.
18. B. D. McKay, Asymptotics for symmetric 0-1 matrices with prescribed row sums,Ars Com 19 (1985), 15–25.
19. B. D. McKay, N. C. Wormald, and B. Wysocka, Short cycles in random regular graphs, Electron J Comb R66 (2004), 1–12.
20. R. W. Robinson and N. C. Wormald, Almost all cubic graphs are Hamiltonian, Random Struct Algorithms 3 (1992), no. 2, 117–125.
21. R. W. Robinson and N. C. Wormald, Almost all regular graphs are Hamiltonian, Random Struct Algorithms 5 (1994), no. 2, 363–374.
22. A. Ruciński, When are small subgraphs of a random graph normally distributed? Probab Theory Relat Fields 78 (1988), no. 1, 1–10.
23. N. C. Wormald, The asymptotic distribution of short cycles in random regular graphs, J Comb Theory Ser B 31 (1981), no. 2, 168–182.
24. N. C. Wormald, Models of random regular graphs, London Mathematical Society Lecture Note Series, Cambridge University Press, Cambridge, England, 1999, 239–298.

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APPENDIX

- Justify (11)

\[ f = d - \frac{\left(\frac{d^2}{2} \left(1 - \frac{1}{d}\right) + \frac{d^2}{6} \left(1 - \frac{2}{d}\right)^3\right)}{\frac{d^2}{6} \left(1 - \frac{1}{d}\right)}; \]

simplify(f(d))

\[ \frac{(d-2) \left(d^2 - d + 1\right)}{(d-1)^3} \]

taylor\left(\frac{1 - x - x^3}{1 - x^3}, x = 0, d\right)

\[ 1 - x^3 + O(x^5) \]

- Justify (10)

Recall that \( \alpha = 1/d \) and \( \rho_1(n, d) = \left(\frac{e}{n}\right)^{n/2} \left(\frac{d-1}{d}\right)^{\frac{d+1}{2}} d^{-1} \exp\left(\frac{1}{4}\right), \)

and

\[ \rho_2(n, d, \alpha) = \left(\frac{e}{n}\right)^{(1-\frac{3}{2})n} \left(\frac{d-2 + \alpha}{d}\right)^{\frac{d+2}{2}} d^{-\frac{3}{2}} \exp\left(\phi(d, \alpha) + O(n^{-1})\right). \]

it is easy to check that all exponential terms cancel exactly from both sides of (A1). By Corollary 15 with \( \alpha = \bar{\alpha} \) (See Maple calculations and expansions below),

\[ \phi(d, \bar{\alpha}) = \frac{4d^2 - 10d + 5}{4(d-1)^2} = 1 - \frac{1}{2d} - \frac{3}{4d^2} - \frac{1}{d^3} + O(d^{-4}). \]
The polynomially bounded term on the left hand side of (A1) is

\[
\sqrt{\frac{n}{e^d(\alpha n/2)(n-\alpha n)}} \exp(\phi(d, a)) = \sqrt{\frac{d-1}{e(d-2)}} \exp\left(1 - \frac{1}{2d} - \frac{3}{4d^2} - \frac{1}{d^3} + O(d^{-4})\right) \\
= \exp(-1/2) \exp\left(\frac{1}{2d} + \frac{3}{4d^2} + \frac{7}{6d^3} + O(d^{-4})\right) \exp\left(1 - \frac{1}{2d} - \frac{3}{4d^2} - \frac{1}{d^3} + O(d^{-4})\right) \\
= \exp\left(\frac{1}{2} + \frac{1}{6d^3} + O(\xi)\right).
\]

See Maple expansion of \(\sqrt{(d - 1)/(d - 2)}\) below where \(x = 1/d\):

\[
\text{taylor}\left(\ln\left(1 + \frac{x}{1-2x}\right)^{\frac{1}{2}}, x = 0, 4\right) = \frac{1}{2} x + \frac{3}{4} x^2 + \frac{7}{6} x^3 + O(x^4)
\]

The polynomially bounded term on the right hand side of (A1) is

\[
\left(1 + \frac{1}{6d^3} + O(\xi)\right) \exp(1/2) = \exp\left(\frac{1}{2} + \frac{1}{6d^3} + O(\xi)\right).
\]