Sets of tetrahedra, 
defined by maxima of distance functions

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Abstract

We study tetrahedra and the space of tetrahedra from the viewpoint of local and global maxima for intrinsic distance functions.

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1 Introduction and main result

One direction in present differential geometry is the study of differentiable spaces with singularities, of which polyhedral convex surfaces are simple examples. Surprisingly, even the geometry of tetrahedra is not completely known, see for example the recent results of V. A. Zalgaller [19]. On the other hand, the geometry of convex polyhedra is an important part of computational geometry, see for example the book by E. D. Demaine and J. O’Rourke [7].

Consider a tetrahedron $T$ in the Euclidean space $\mathbb{R}^3$, endowed with the intrinsic distance $\rho$ induced by the distance in $\mathbb{R}^3$. The metric $\rho$ is defined, for any points $x, y$ in $T$, as the length of a segment (i.e., shortest path on $T$) from $x$ to $y$. For a given point $x \in T$, let $\rho_x$ be the distance function from $x$, defined by $\rho_x(y) = \rho(x, y)$. When necessary, we shall write $\rho^S$ to emphasize the surface $S$.

Denote by $M_x$ the set of local maxima of $\rho_x$, and by $F_x$ the set of global maxima of $\rho_x$ (or farthest points from $x$, or antipodes of $x$).

The study of farthest points and, more generally, of local maxima for distance functions on convex surfaces, has its origin in several questions of H. Steinhaus, see §A35 in [6]. The paper [17] surveys fundamental results on this subject; more recent work is due to T. Zamfirescu, the authors and their collaborators.

Define the following subsets of the space $\mathcal{T}$, of all the tetrahedra in $\mathbb{R}^3$ up to isometry and homothety:

$$\mathcal{M}_n = \{ T \in \mathcal{T} \mid \exists x \in T \ \# M_x \geq n \},$$

$$\mathcal{F}_n = \{ T \in \mathcal{T} \mid \exists x \in T \ \# F_x \geq n \},$$

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where \(#S\) denotes the cardinality of the set \(S\).

We have \(\mathcal{M}_k \supset \mathcal{M}_{k+1} \cup \mathcal{F}_k\), directly from the definitions, wherefrom the following diagram.

\[
\begin{array}{c}
\mathcal{T} \supset \mathcal{M}_2 \supset \mathcal{M}_3 \supset \mathcal{M}_4 \supset \mathcal{M}_5 \supset \mathcal{M}_6 \supset \ldots \\
\cup \cup \cup \cup \cup \\
\mathcal{F}_2 \supset \mathcal{F}_3 \supset \mathcal{F}_4 \supset \mathcal{F}_5 \supset \ldots
\end{array}
\]

The equality \(\mathcal{M}_2 = \mathcal{F}_2\) holds for polyhedral convex surfaces, see [18]. Moreover, for polyhedral convex surfaces with \(n\) vertices, we have \(\mathcal{F}_n \neq \emptyset\) [13] and \(\mathcal{F}_{n+1} = \emptyset\) [14]. In [13] it is also proven that \(\mathcal{M}_{n+2} = \emptyset\), though this is not emphasized as a theorem. For tetrahedra, it has been proven that \(\mathcal{F}_2 = \mathcal{T}\) [12].

We examine each inclusion in the above diagram and make precise the “relative size” of the subset. We write \(A \supset B\) whenever \(A \supset B\) and both \(B\) and \(A \setminus B\) have interior points. If \(B\) is nowhere dense in \(A\) (i.e., int(cl(B)) = \(\emptyset\)) we write \(A \nabla B\).

**Theorem** All sets \(\mathcal{M}_k\) are open in \(\mathcal{T}\) and we have the following diagram

\[
\begin{array}{c}
\mathcal{T} = \mathcal{M}_2 \supset \mathcal{M}_3 \supset \mathcal{M}_4 \supset \mathcal{M}_5 \supset \mathcal{M}_6 = \emptyset \\
\mathcal{F}_2 \cup \mathcal{F}_3 \cup \mathcal{F}_4 \cup \mathcal{F}_5 = \emptyset
\end{array}
\]

We do not know if the set \(\mathcal{M}_2 \setminus \mathcal{M}_3\) has interior points in \(\mathcal{T}\).

Parts of this result hold as well for general polyhedral convex surfaces. We choose to give them for tetrahedra, for the sake of a unified presentation.

The proof of our theorem is split into several lemmas, treating each inclusion separately. In Section 3 we study the chain of inclusions concerning the sets \(\mathcal{M}_k\), in Section 4 we treat the inclusions \(\mathcal{F}_k \subset \mathcal{M}_k\), while in Section 5 we consider the chain of inclusions concerning the sets \(\mathcal{F}_k\).

Our framework is the space of tetrahedra, and our main tools are properties of cut loci, gluings and unfoldings; they are all presented in Section 2.

For the reader’s convenience we recall below a few definitions and give additional notation. A **very acute vertex** of a polyhedral surface is a vertex \(v\) the total angle \(\theta_v\) of which is less than \(\pi\). The **curvature** \(\omega_v\) at the vertex \(v\) of a polyhedral surface is defined as \(\omega_v = 2\pi - \theta_v\). A tetrahedron is called **isosceles** if any opposite edges are equal or, equivalently, if the total angle at every vertex is equal to \(\pi\). The closure, interior, and boundary of any subset \(N\) of a topological space are respectively denoted by \(\text{cl}(N)\), \(\text{int}(N)\), and \(\text{bd}(N)\). In any metric space, \(B(x, r)\) stands for the closed ball centered at \(x\) of radius \(r > 0\), and \(S(x, r)\) for its boundary. As usual, \(\| \cdot \|\) denotes the standard norm in \(\mathbb{R}^3\).
2 Preliminaries

In this section we briefly present some basic tools employed in the intrinsic geometry of polyhedra. We will use them most often implicitly.

2.1 Space of tetrahedra

By a tetrahedron, we mean the boundary of the convex hull of four non-coplanar points. Let $\mathcal{T}$ denote the space of all tetrahedra. We endow it, as usual in convex geometry, with the Pompeiu-Hausdorff distance $d_{PH}$. Since a tetrahedron is determined by its vertex set, $\mathcal{T}$ is a quotient of an open set of $\mathbb{R}^{12}$ by the free and properly discontinuous action of the group $S_4$ of permutations.

By $\tilde{T}$ we denote the space of all tetrahedra, up to isometry and homothety; it is endowed with the induced topology from $\mathcal{T}$.

Lemma 2.1 There exists an open and dense set $\mathcal{O}$ in $\mathcal{T}$, which carries the structure of a 5-dimensional differentiable manifold.

Proof. Notice that a tetrahedron $T$ in $\mathcal{T}$ which is invariant under an isometry of $\mathbb{R}^3$ has two congruent faces.

Denote by $\mathcal{I}$ the subset of $\mathcal{T}$ consisting of all tetrahedra with two congruent faces, and put $\mathcal{O} = \mathcal{T} \setminus \mathcal{I}$. Clearly, $\mathcal{I}$ is closed and $\mathcal{O}$ is open in $\mathcal{T}$. Moreover, each tetrahedron $T \in \mathcal{I}$ can easily be approximated with tetrahedra $T_n$, each of which has distinct edge lengths; therefore, $\mathcal{O}$ is dense in $\mathcal{T}$.

The group $G$ generated by affine homotheties and affine isometries of $\mathbb{R}^3$ acts on $\mathcal{O}$ in a natural way. The action is clearly smooth, proper, and free by the definition of $\mathcal{O}$. Hence $\mathcal{O} = \mathcal{O}/G$ is a smooth 5-dimensional manifold (see, e.g., Theorem 5.119 in [10]). The density of $\mathcal{O}$ in $\mathcal{T}$ follows from the density of $\mathcal{O}$ in $\mathcal{T}$.

With some abuse of notation, we say that the dimension of $\mathcal{T}$ is 5.

The completions $\overline{\mathcal{T}}$ of $\mathcal{T}$ and $\overline{\mathcal{T}}$ of $\mathcal{T}$ include, beside tetrahedra, doubles of planar convex quadrilaterals or triangles. Notice that a doubly covered triangle appears in these complete spaces infinitely many times, once for each position of its fourth, flat vertex.

We introduce $\overline{\mathcal{T}}$ and $\overline{\mathcal{T}}$ for practical reasons: we aim to construct suitable doubly covered quadrilaterals or triangles, and afterwards to perturb their vertices to get non-degenerate tetrahedra. Clearly, the convergence in $\overline{\mathcal{T}}$ implies convergence in $\overline{\mathcal{T}}$.

Define the following subsets of the space $\overline{\mathcal{T}}$:

$$\hat{\mathcal{M}}_n = \{ T \in \overline{\mathcal{T}} | \exists x \in T \# M_x \geq n \},$$

$$\hat{\mathcal{F}}_n = \{ T \in \overline{\mathcal{T}} | \exists x \in T \# F_x \geq n \},$$

and notice that $\hat{\mathcal{M}}_k \supset \hat{\mathcal{M}}_{k+1} \cup \hat{\mathcal{F}}_k$ holds in $\overline{\mathcal{T}}$. 
The next classical lemma will implicitly be used for the convergence of tetrahedra.

Lemma 2.2 [5] Let $S_n$ be a sequence of convex surfaces, converging to $S$ with respect to the Pompeiu-Hausdorff metric. Let $x_n, y_n$ be points in $S_n$, converging to $x, y \in S$, and $\gamma_n$ segments on $S_n$ joining $x_n$ to $y_n$, converging to $\gamma \subset S$. Then $\gamma$ is a segment on $S$ joining $x$ to $y$, and $\rho^S(x,y) = \lim_n \rho^{S_n}(x_n,y_n)$.

2.2 Gluing and unfolding

One of our main tools is the Alexandrov’s gluing theorem, given next (see [3], p.100).

Lemma 2.3 (Alexandrov’s gluing theorem) Consider a topological sphere $S$ obtained by gluing planar polygons (i.e., naturally identifying pairs of sides of the same length) such that at most $2\pi$ angle is glued at each point. Then $S$, endowed with the intrinsic metric induced by the distance in $\mathbb{R}^2$, is isometric to a polyhedral convex surface $P \subset \mathbb{R}^3$, possibly degenerated. Moreover, $P$ is unique up to rigid motion and reflection in $\mathbb{R}^3$.

The double $D$ of the planar convex body $K$ is obtained by gluing two isometric copies of $K$ along their boundary, by identifying the naturally corresponding points. The two copies of $K$ are called the faces of $D$. With some abuse of notation, we shall identify $K$ with one face of $D$. $D$ can be seen as limit of convex surfaces in $\mathbb{R}^3$.

Opposite to Alexandrov’s gluing theorem are the results on unfoldings. For this subject, see the excellent book [7] by E. Demaine and J. O’Rourke.

Definition 2.4 Let $P$ be a convex polyhedron. The cut locus $C(x)$ of the point $x$ on the polyhedral convex surface $P$ is defined as the set of endpoints, different to $x$, of all nonextendable shortest paths (on the surface $P$) starting at $x$.

We postpone the presentation of the properties of cut loci to the next subsection.

Lemma 2.5 [10] (Source unfolding) Consider a polyhedral convex surface $P$ and a point $x$ in $P$. Cutting along $C(x)$ produces a surface $U$ with boundary which can be unfolded in the plane without overlappings. Moreover, $U$ is star-shaped with respect to $x$.

Lemma 2.6 [4] (Star unfolding) Consider a polyhedral convex surface $P$ and a point $x$ in $P$ which is joined to each vertex of $P$ by a unique segment. Cutting along the union of those segments produces a surface $U$ with boundary which can be unfolded in the plane without overlappings. Assume, with some abuse of notation, that $U \subset \mathbb{R}^2$; then the image of $C(x)$ in $U$ after unfolding is precisely the restriction to $U$ of the Voronoi diagram of the images of $x$. 

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Roughly speaking, the following lemma shows that close unfoldings must fold to close tetrahedra.

**Lemma 2.7** Let \( \{P^n\}_{n \in \mathbb{N}} = \{a_1^n a_2^n \ldots a_p^n\}_{n \in \mathbb{N}} \) be a sequence of \( p \)-gons in \( \mathbb{R}^2 \) converging to the \( p \)-gon \( P = a_1 \ldots a_p \), such that \( a_j^n \to a_j \) for \( j = 1, \ldots, p \). Endow each \( P_n \) and \( P \) with compatible sets \( R_n \) and \( R \) of gluing rules, meaning that \( a_ia_i' \) is glued (i.e., identified preserving the length) to \( a_i a_i' \) according to \( R_n \) if and only if \( a_i a_j \) is glued to \( a_k a_l \) according to \( R \). Assume that, for each \( n \), the gluing of \( P_n \) according to \( R_n \) yields a tetrahedron \( T_n \). Then the gluing of \( P \) according to \( R \) yields a tetrahedron \( T \) (possibly degenerated), and \( T_n \to T \).

**Proof.** By hypothesis, the gluing of \( P_n \) according to \( R_n \) yields a polyhedron \( T_n \) with only four vertices of positive curvature, say \( v_1^n, v_2^n, v_3^n, v_4^n \), corresponding to the points \( a^a_{i_1}, a^a_{i_2}, a^a_{i_3}, a^a_{i_4} \) respectively.

By continuity, the limits of the zero curvature vertices of \( T_n \) also have zero curvature, whence \( T \) has at most 4 vertices, \( v_i = \lim_n v_i^n, i = 1, \ldots, 4 \).

Notice now that the distance in \( T_n \) between the vertices \( v_i^n, v_j^n \), depends continuously on the position of the vertices \( (a_1^n, \ldots, a_p^n) \). In other words, there exist continuous maps \( f_{ij} : \mathbb{R}^{2p} \to \mathbb{R} \), depending only on the gluing rules, such that \( \rho^n(v_i, v_j) = f_{ij}(a_1^n, \ldots, a_p^n) \) and \( \rho^n(v_i^n, v_j^n) = f_{ij}(a_1^n, \ldots, a_p^n) \), \( n \in \mathbb{N} \). Hence, the distance on \( T_n \) between two vertices converges to the distance on \( T \) between the limit vertices. Since the relative positions of a tetrahedron’s vertices depend continuously on its edge lengths, \( T_n \) converges to \( T \) in \( T \). 

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### 2.3 Cut loci and local maxima

A very important tool for the study of distance functions on convex polyhedra is the cut locus. See [15] for some of its applications in Riemannian geometry.

We have already presented the definition of cut loci in the previous subsection. Basic properties of cut loci are given in the following lemma.

**Lemma 2.8** Let \( x \) be a point on a polyhedral convex surface \( P \).

(i) \( C(x) \) is a tree whose leaves (endpoints) are vertices of \( P \), and all vertices of \( P \), excepting \( x \) (if the case), are included in \( C(x) \).

(ii) The junction points in \( C(x) \) are joined to \( x \) by as many segments as their degree in the tree, each leaf of \( C(x) \) is joined to \( x \) by one segment, and the other points in \( C(x) \) are joined to \( x \) by precisely two segments.

(iii) The edges of \( C(x) \) are shortest paths on \( P \).

(iv) Assume the shortest paths \( \Gamma \) and \( \Gamma' \) from \( x \) to \( y \in C(x) \) are bounding a domain \( D \) of \( P \), which intersects no other shortest path from \( x \) to \( y \); the case \( \Gamma = \Gamma' \) is also possible. Then the arc of \( C(x) \) at \( y \) towards \( D \) bisects the angle of \( D \) at \( y \).

The properties (i)-(ii) and (iv) are well known in a more general framework, while (iii) is Lemma 2.4 in [1].

Lemma 2.8 yields, for the particular case of polyhedra with 4 vertices, the following lemma.
Lemma 2.9 The cut locus of a point $x$ in a (possibly degenerated) tetrahedron $T$ is homeomorphic to one of the letters $H$, $X$, $Y$, or $I$.

The case of a Jordan arc cut locus is very particular, as stated by the next known result (see [8] for a proof).

Lemma 2.10 If the cut locus $C(x)$ of a point $x$ in a convex (possibly degenerated) polyhedron $P$ is an arc then $P$ is a doubly covered polygon.

We will give in the following some basic properties of local maxima.

Lemma 2.11 Let $P$ be a polyhedral convex surface and $x, y$ points in $P$.

(i) $M_x \subset C(x)$.

(ii) $y \in M_x$ if and only if the angle at $y$ between any two consecutive segments from $y$ to $x$ is less than $\pi$. Consequently, $M_x$ contains only strict local maxima.

(iii) Each point in $M_x$ is either a leaf or a junction point in $C(x)$ (a vertex of degree two in $C(x)$ is also considered as a junction point).

(iv) A leave $y$ of $C(x)$ belongs to $M_x$ if and only if $\omega_y > \pi$.

Proof. The inclusion $M_x \subset C(x)$ is clear from the definitions.

The second statement is straightforward, if one considers a local unfolding of $P$ along the segments from $x$ to $y$.

From (ii) in Lemma 2.8 and (ii) we obtain (iii) and (iv).

3 The inclusions $\mathcal{M}_k \supset \mathcal{M}_{k+1}$

In this section we treat the first chain of inclusions in our Theorem, concerning the sets $\mathcal{M}_k$. To each inclusion one or several lemmas are devoted.

Lemma 3.1 Let $X$ be a compact topological space, and $f : X \to \mathbb{R}$ a continuous function. Let $U$ be an open set in $X$, and $x \in X$ a point such that $f(x) > \max_{y \in \text{bd}(U)} f(y)$. Then $f$ admits a local maximum in $U$.

Proof. Let $z$ be a global maximum of $f$ on $\text{cl}(U)$. Clearly, $z \notin \text{bd}(U)$ for $f(z) < f(x)$, hence it is a global maximum for the restriction of $f$ to $U$ and, consequently, a local maximum for $f$.

Lemma 3.2 Let $S_n$ a sequence of convex surfaces, converging to $S$ with respect to the Pompeiu-Hausdorff metric. Let $x_n, y_n$ be two points of $S_n$ converging to $x \in S$ and $y \in S$ respectively. Let $y$ be a strict local maximum of $\rho^S_x$. Then for each $r > 0$, there exists an index $N$ such that for all $n > N$, the function $\rho^{S_n}_{x_n}$ admits a local maximum in $B(y_n, r)$.

Proof. Since $y$ is a local maximum for $\rho^S_x$, we can assume (by making $r$ smaller) that

$$\varepsilon \overset{\text{def}}{=} \rho^S_x(y) - \max_{z \in S(y, r)} \rho^S_x(z) > 0.$$
Let \( z_n \) be a global maximum for \( \rho^S_n \) restricted to \( S(y_n, r) \). For \( n \) large enough, we have

\[
\left| \rho^S_n (x_n, y_n) - \rho^S (x, y) \right| < \frac{\varepsilon}{2} \tag{1}
\]

\[
\rho^S_n (x_n, z_n) \leq \limsup \rho^S_n (x_n, z_n) + \frac{\varepsilon}{2} = \rho^S (x, z') + \frac{\varepsilon}{2}
\]

for some limit point \( z' \) of \( \{z_n\} \). By Lemma 2.2, such a point \( z' \) must belong to \( S(x, r) \), whence

\[
\rho^S_n (x_n, z_n) \leq \limsup \rho^S_n (x_n, z_n) + \frac{\varepsilon}{2} = \rho^S (x, z') + \frac{\varepsilon}{2}
\]

for any point \( z \in S(y_n, r) \). Hence, by Lemma 3.1, \( \rho^S_n \) has a local maximum in \( B(y_n, r) \).

Lemmas 3.2 and 2.11 (ii) imply the following

**Corollary 3.3** For any polyhedral convex surface, each set \( M_k \) is open in \( T \) and each set \( \tilde{M}_k \) is open in \( T \), \( k \geq 2 \).

**Lemma 3.4** The isosceles tetrahedra belong to \( M_2 \setminus M_3 \).

**Proof.** Let \( T \) be an isosceles tetrahedron and \( x \) a point in \( T \).

Notice that a vertex \( v \) of \( T \) which is also a leaf of \( C(x) \) cannot be a local maximum for \( \rho_x \). Indeed, the total angle of \( T \) at \( v \) is \( \pi \), and there is only one segment from \( x \) to \( v \). Therefore (see Lemma 2.4), if \( C(x) \) is an \( H \)-tree then \( \#M_x \leq 2 \), if \( C(x) \) is an \( X \)-tree then \( \#M_x = 1 \), and if \( C(x) \) is an \( Y \)-tree then \( \#M_x \leq 2 \).

**Lemma 3.5** \( \text{int } (M_3 \setminus M_4) \neq \emptyset \).

**Proof.** Recall that all cut loci on tetrahedra have at most two flat junction points, by Lemma 2.4. This and Lemma 2.11 imply that each tetrahedron in \( M_4 \) has at least two very acute vertices. Therefore, it suffices to provide an open subset of \( M_3 \), each tetrahedron of which has at most one very acute vertex.

Let \( R \) be a non-square rectangle of center \( o \). The cut-locus \( C(o) \) on the double of \( R \) is a \( H \)-tree and its junctions points \( u, w \) clearly belong to \( M_o \). Consider a quadrilateral \( Q \) close to \( R \), with one vertex acute and other three obtuse, and a point \( x \) on the double \( D \) of \( Q \), close to \( o \). By Lemma 3.2 there is at least one point of \( M_x \) near \( u \), and another one near \( w \). Of course, the acute vertex of \( Q \) also belongs to \( M_x \), whence \( D \in M_3 \). Now, tetrahedra close enough to \( D \) cannot belong to \( M_4 \) because they have only one very acute vertex, and so they are interior to \( M_3 \setminus M_4 \), by Corollary 3.3.

**Lemma 3.6** \( \text{int } (M_4 \setminus M_5) \neq \emptyset \).
**Proof.** Lemmas 2.9 and 2.11 imply that each tetrahedron in \( M_5 \) has at least three very acute vertices. Therefore, it suffices to provide an open subset of \( M_4 \), each tetrahedron of which has at most two very acute vertices.

Consider a planar convex quadrilateral \( Q = v_1v_2v_3v_4 \) such that \( v_1, v_2, v_3, v_4 \) are consecutive vertices in a regular hexagon of centre \( x \).

Then \( M_x = \{v_1, v_2, v_3, v_4\} \) on the double \( R \) of \( Q \), hence \( R \in \tilde{M}_4 \). Since \( R \) has only two very acute vertices, we get \( R \not\in \tilde{M}_5 \).

To end the proof, notice that small perturbations of \( R \) in \( T \) provide tetrahedra with precisely two very acute vertices, which are therefore in \( M_4 \setminus M_5 \) by Corollary 3.3. \( \blacksquare \)

**Lemma 3.7** \( \text{int} (M_5) \neq \emptyset \).

**Proof.** By constructing an explicit example, we show next that \( \tilde{M}_5 \neq \emptyset \). This and Corollary 3.3 would then imply \( \text{int} (M_5) \neq \emptyset \).

Consider the planar polygon \( L \) drawn in Figure 1, where the line-segments marked with X are all equal, as are those marked with II. The two big rhombi have angles \( \pi/4 \) and \( 3\pi/4 \), and the smaller one to the right has angles \( \pi/8 \) and \( 7\pi/8 \). Our figure has horizontal symmetry.

![Figure 1: Unfolding of a tetrahedron \( T \) in \( \tilde{M}_5 \), obtained by cutting along the segments from \( x \in T \) to the vertices of \( T \).](image)

In order to obtain a polyhedral convex surface \( T \), glue the sides of \( L \) as follows: identify the two sides incident to \( v_i, i = 1, 3, 4 \), and identify the upper sides incident to \( v_2 \), as well as the lower sides incident to \( v_2 \). These identifications are all possible, due to the length equalities.

Notice that the total angle \( \theta_i \) at the point \( v_i \) after gluing verifies \( \theta_1 = 7\pi/8 \), \( \theta_2 = 13\pi/8 \), and \( \theta_3 = \theta_4 = 3\pi/4 \). Moreover, the total angle \( \theta_x \) after gluing at \( x \) verifies \( \theta_x = 2\pi \). By Alexandrov’s gluing theorem, the resulting surface \( T \) is a tetrahedron, and the point in \( T \) corresponding to \( x \in L \), also denoted by \( x \), is a flat point.

One can easily notice that the segments (on \( T \)) joining \( x \) to \( w, v_1, v_2, v_3 \), respectively \( v_4 \), correspond to marked line-segments in Figure 1. Therefore, on
\( T, \angle xv_2w < \pi/2, \angle xv_2v_1 < \pi/2; \) these inequalities and \( \theta_j < \pi, j = 1, 3, 4, \) imply \( M_x = \{ w, v_1, v_2, v_3, v_4 \} \) (by Lemma 2.11), whence \( T \in M_5 \) and the conclusion follows.

The next lemma follows from Lemma 3.4 and a more general result in [21].

**Lemma 3.8** Let \( S \) be a convex surface and \( x \) a point in \( S \). Then \( M_x \) is contained in a minimal (by inclusion) \( Y \)-subtree of \( C(x) \), possibly degenerated to an arc or a point.

**Lemma 3.9** \( M_6 = \emptyset \).

**Proof.** This follows from Lemmas 2.11 and 3.8.

\[ \theta_n \leq 4 \arcsin (2\varepsilon) = \frac{\pi}{4}. \]

\[ 0 \leq \rho(x, y)^2 - \rho(x, a)^2 = \rho(y, a) (\rho(y, a) - 2\rho(x, a) \cos \alpha). \]
Since $\alpha < \theta$, $\rho(y,a) \geq \sqrt{2}\rho(a,x)$, and similarly $\rho(y,b) \geq \sqrt{2}\rho(b,x)$, hence

$$\rho(y,b) + \rho(y,a) \geq \sqrt{2}.$$

In remains to prove that no point $y$ can satisfy this inequality. If $y$ belongs to $abc$ then

$$\rho(y,a) + \rho(y,b) \leq \rho(c,a) + \rho(c,b)$$

$$< \rho(a,m) + \rho(b,m) + 2\rho(c,m) \leq 1 + 2\varepsilon.$$

The proof is the same for $y \in abd$. If $y$ belongs $acd$, let $e$ be the intersection of the line $ay$ with the edge $cd$. Obviously,

$$\rho(y,b) + \rho(y,a) \leq \rho(e,b) + \rho(e,a)$$

$$\leq \max(\rho(c,b), \rho(d,b)) + \max(\rho(c,a), \rho(d,a))$$

$$< \rho(b,m) + \rho(a,m) + 2\max(\rho(c,m), \rho(d,m)) \leq 1 + 2\varepsilon.$$

The proof is similar for $bcd$. Since $1 + 2\varepsilon < \sqrt{2}$, we get a contradiction.

**Lemma 4.2** \[\text{int(} \mathcal{M}_3 \setminus \mathcal{F}_3\text{)} \neq \emptyset.\]

**Proof.** Consider an isosceles triangle $V = v_1v_2v_3$ such that $||v_1-v_2|| = ||v_1-v_3||$ and $||v_1 - \frac{v_2 + v_3}{2}|| < \frac{1}{2} \sin \frac{\pi}{16} ||v_1 - v_3||$, and denote by $D$ its double.

Denote by $u$ the foot of $v_1$ on $v_2v_3$, and notice that $M_u = \{v_1, v_2, v_3\}$. Therefore, $D \in \mathcal{M}_3 = \text{int}\left( \mathcal{M}_3 \right)$ (see Corollary 3.3).

By Lemma 4.1 $D \in \text{int}\left( \mathcal{F} \setminus \mathcal{F}_3 \right)$, and the conclusion follows.

**Lemma 4.3** \[\text{[20]} \text{For any convex surface } S \text{ and any point } x \in S, F_x \text{ is contained in a minimal (by inclusion) arc } J_x \subset C(x), \text{ possibly reduced to a point.}\]

**Lemma 4.4** $\text{dim } \mathcal{F}_4 = 4$.

**Proof.** Consider $T \in \mathcal{F}_4$, $x \in T$ with $\#F_x = 4$, and the arc $J_x$ given by Lemma 4.3. Then (see Lemma 2.11) $C(x)$ cannot be an $X$-tree, because this would imply $\#F_x \leq 3$. Neither can $C(x)$ be an arc, by Lemma 2.10 (The later case covers doubly covered quadrilaterals inscribed in a semi-circle centered at $x$, hence a subfamily of dimension 2 in $\mathcal{T}$.) Therefore, one of the following statements holds:

- $C(x)$ is a $Y$-tree. Then $F_x$, and consequently $J_x$, contains two leaves of $C(x)$, as well as its point of degree three and the vertex of $T$ of degree two in $C(x)$. Call such a tetrahedron of first type.

- $C(x)$ is a $H$-tree. Then $F_x$ contains the two junction points of $C(x)$ and two leaves, separated along $J_x$ by those junction points.
The last case is divided into two subcases. Let’s consider the directions at point \( x \) of the four segments between \( x \) and the leaves of its cut-locus. The two segments to the antipodes can be either consecutive (second type) or alternated with the other two (third type). The families of tetrahedra of the first and of the third type are explicitly described in [14]; they have respective dimensions 3 and 4. The family of tetrahedra of the second type is also 4-dimensional, the proof being similar to the one for the third case.

Lemma 4.5 \[ M_4 \supseteq F_4. \]

**Proof.** By Lemmas 4.4 and 2.1 \( \dim F_4 = 4 = \dim T - 1 \), and by Corollary 3.3 \( \dim M_4 = \dim T \).

5 The inclusions \( F_k \supseteq F_{k+1} \)

In this section we treat the second chain of inclusions in our theorem, concerning the sets \( F_k \).

Lemma 5.1 \( \text{int} (F_2 \setminus F_3) \neq \emptyset \).

**Proof.** The set of tetrahedra satisfying the hypothesis of Lemma 4.1 with a strict inequality is open and included in \( F_2 \setminus F_3 \).

Notice that \( \text{int} (F_2 \setminus F_3) \) doesn’t contain only “thin” tetrahedra, but also the regular one.

Lemma 5.2 \( \dim F_3 = \dim T \), hence \( \text{int} (F_3 \setminus F_4) \neq \emptyset \).

**Proof.** We construct a family of tetrahedra in \( F_3 \) depending on 5 independent parameters, and thus of maximal dimension in \( T \) (see Lemma 2.1). In fact, we construct next a family of planar and simple polygonal domains depending on 5 independent parameters, in such a way that they glue to tetrahedra. Moreover, the boundary of each such polygonal domain will correspond, after gluing, to the cut locus of a distinguished point \( x \), so we may think about these domains as being the source unfoldings of tetrahedra in a 5-dimensional family. Our construction directly shows that \#\( F_x = 3 \) on each resulting tetrahedron \( T \).

Consider the unit circle \( S(x, 1) \) and the points \( v_1, w, u, u', u'', w', w'' \in S(x, 1) \) having respectively the following circular coordinates: \( 0, \pi/6, 4\pi/6, 5\pi/6, 13\pi/12, 19\pi/12, 11\pi/6 \). A straightforward computation shows that

\[
\begin{align*}
|v_1 - w| &= |v_1' - w''| \\
|w - u| &= |w' - u''|
\end{align*}
\]

and

\[
\begin{align*}
\angle v_1wu + \angle u''w'u'' + \angle w'u''v_1 &= 2\pi + \pi/12, \\
\angle wwu' + \angle uu'u'' + \angle u'u''w' &= 2\pi + \pi/12.
\end{align*}
\]
Therefore, there exist points $v_2, v_3, v_4$ inside the polygon $v_1 w u u' w' w''$ (see Figure 2) such that

$$\begin{align*}
||u-v_4|| &= ||u'-v_4|| \\
||u'-v_3|| &= ||u''-v_3|| \\
||w'-v_2|| &= ||w''-v_2||
\end{align*}$$

and, moreover,

$$\begin{align*}
\angle v_1 w u + \angle u'' w' v_2 + \angle v_2 w'' v_1 &= 2\pi \\
\angle w u v_4 + \angle v_4 u' v_3 + \angle v_3 w'' w' &= 2\pi.
\end{align*}$$

Let $L$ denote the closed planar set bounded by $v_1 w u u' w' w'' v_3 u'' v_2 w''$.

Glue the equal sides of $L$, according to the equalities (3) and (2). By Alexandrov's gluing theorem (Lemma 2.3), the result $T$ is (isometric to) a polyhedral convex surface. More precisely, the conditions (4) imply that $T$ is a tetrahedron, with vertices $v_1, v_2, v_3, v_4$.

Now fix $v_1 \in S(x,1)$, and freely perturb the positions of $w, u, u', u''$ on $S(x,1)$, wherefrom 4 independent parameters. These determine, through the above conditions (4), the positions of $w', w''$ on $S(x,1)$. The first equality in (4) determines then the position of $v_2$ on the bisector of the angle $\angle w' x w''$, but the second inequality does not determine the position of both $v_3$ and $v_4$. Hence we have a fifth degree of liberty.

By Alexandrov’s gluing theorem, each polygonal domain obtain as above yields a tetrahedron $T$, and Lemma 2.7 shows that the resulting tetrahedra are close in $\mathcal{T}$, because their unfoldings are close in the plane. Therefore, we have obtained a subfamily $\mathcal{L}$ of dimension 5 in $\mathcal{T}$. Of course, $\dim (\mathcal{L} \cap \mathcal{T}) = 5$.

We claim that the distance function $\rho_x^T$ coincides to the distance function from $x$ in the planar domain $L$ producing $T$. Assume this is not true, hence there exists a point $y$ in $L$ such that $\rho^T(x,y) < ||x - y||$. Let $\gamma$ be a segment on $T$ from $x$ to $y$, and denote by $\tilde{\gamma}$ the image of $\gamma$ in $L$, under the unfolding of
To $L$. Because $L$ is star-shaped with respect to $x$, $\bar{\gamma} \cap \bd(L) \neq \emptyset$. Consider $z \in \bar{\gamma} \cap \bd(L)$, say $z \in uv_4$, hence the point $z' \in u'v_4$ given by $||u-z|| = ||u'-z'||$ also belongs to $\bar{\gamma} \cap \bd(L)$. Assume, for the simplicity of the presentation, that $\bar{\gamma} \cap \bd(L) = \{ z, z' \}$. (The case $(\bar{\gamma} \cap \bd(L)) > 2$ follows by a straightforward induction.) The triangle inequality implies

$$||x-y|| < ||x-z'|| + ||z'-y|| = ||x-z|| + ||z'-y|| = \rho^T(x, y),$$

and a contradiction is obtained, proving the claim.

The claim shows that $\bd(L) \cap \bar{\gamma}$ yields $C(x)$ after gluing, so one can easily check that $\#F_x = 3$ on $T$, and thus $L \cap T \subset F_3$.

Lemmas 4.2, 5.1 and 5.2 imply the following

**Corollary 5.3** $M_3 \supset F_3$ and $F_2 \supset F_3$.

**Lemma 5.4** $F_4 \subset F_3$.

**Proof.** We first claim that any tetrahedron $T \in \cl(F_4)$ is limit of a sequence of tetrahedra of int($F_3$). Clearly it is sufficient to consider $T$ in a dense subset of $F_4$, so we can assume that $T$ is of second or third type (see Lemma 4.2 and 14). Let $x$ be a point on $T$ with 4 farthest points. The cut-locus of $x$ is an $H$-tree; let $v, w$ be the extremities of one of the vertical bars of the $H$. Exactly one of those vertices, say $v$, belongs to $F_x$. Denote by $p$ the point of degree three of $C(x)$ adjacent to $v$ and $w$. Consider the source unfolding of $T$ with respect to $x$. All points of $F_x$ lie on a circle centered at $x$ and of radius $\rho(x, F_x)$, and $w$ is inside this circle. Now, if one radially moves $v$ inside the circle, and radially moves $w$ in such a way that the total angle around $p$ remains $2\pi$, he obtains a new tetrahedron such that $\#F_x = 3$. This tetrahedron can be deformed with 5 degrees of liberty: four of them come from the fact that $F_4$ is 4-dimensional [14], and the fifth one is the radial position of $v$; see, e.g., the proof of Lemma 5.2. This proves the claim.

Assume now that there exists $T \in \int_{F_3}(\cl(F_4))$. In other words, there exists an open (in $T$) set $U$ containing $T$ such that $F_3 \cap U \subset \cl(F_3)$. By the claim, one can find a tetrahedron $T' \in \int(F_3) \cap U$, i.e., there exists an open set $V$ such that $T' \in V \subset F_3$. Now $V \cap U$ is open, nonempty, and included in $\cl(F_4)$, in contradiction with $\dim F_4 = 4$.

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