Tomograms in the Quantum-Classical transition

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Abstract

The quantum-classical limits for quantum tomograms are studied and compared with the corresponding classical tomograms, using two different definitions for the limit. One is the Planck limit where $\hbar \to 0$ in all $\hbar$–dependent physical observables, and the other is the Ehrenfest limit where $\hbar \to 0$ while keeping constant the mean value of the energy. The Ehrenfest limit of eigenstate tomograms for a particle in a box and a harmonic oscillator is shown to agree with the corresponding classical tomograms of phase-space distributions, after a time averaging. The Planck limit of superposition state tomograms of the harmonic oscillator demonstrates the decreasing contribution of interference terms as $\hbar \to 0$.

Key words: Tomograms, Symplectic tomograms, Radon transform, Schrödinger cat, Ehrenfest limit
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1 Introduction

The quantum-classical transition has been the subject of numerous studies, [1,2,3] for the classical limit of bi-Hamiltonian systems see e.g. [4]. In the

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standard formulation of quantum mechanics the states of a system are associated with wave functions[5] or density operators,[6,7] and the WKB procedure can be employed to take the classical limit. Further insight in the connection to classical mechanics can be gained from other formulations of quantum mechanics,[8] e.g., the Wigner-Moyal[9,10] and Feynman path integral formulations.[11] In the Moyal approach, the Wigner function satisfies the quantum evolution equation which is similar to the kinetic equation for classical probability distributions in phase-space. In the limit $\hbar \to 0$ the Moyal equation yields the Liouville equation of classical statistical mechanics. The classical action plays an important role in the Feynman path integral representation for the Green functions (propagators) of the Schrödinger equation, and the only contributions that appear in the complex probability in the $\hbar \to 0$ limit are associated with classical paths. Also the classical action satisfying the classical Hamilton-Jacobi equation appears in the wave function after using the WKB decomposition.

Despite all the utility of these well known approaches, because objects such as wave functions, Wigner functions or other representations of density operators (e.g., Sudarshan-Glauber singular P-quasidistribution[12,13] and Husimi-Kano Q-quasidistribution[14,15]) differ in an essential way from classical phase-space probability distributions, it is not easy to make a precise comparison between the two systems. This can be rectified by going to the tomographic approach where both the classical and quantum theory offer a description in terms of tomographic probabilities.[16,17,18] The quantum tomogram realizes a specific version of star-product quantization[19] and allows for a comparison with the classical tomogram, both being written on the same domain. The advantage of a tomogram is connected with its property to be a standard positive probability distribution function describing the quantum state and this property was found useful to study the entanglement criterion for continuous variables,[20] as well as to formulate a separability condition for spin degrees of freedom. [21]

The aim of this work is to consider the quantum-classical transition within the framework of the tomographic approach. We shall mainly be concerned with examples and postpone more general considerations to a future paper. In particular, we shall consider the tomographic approach for the description of quantum states for a particle in a box, as well as the quantum tomograms of harmonic oscillator coherent states and stationary states. We study two kinds of quantum-classical transitions, the Planck limit and Ehrenfest limit of the quantum tomograms. In Planck limit we obtain the quantum tomograms of harmonic oscillator coherent states and stationary states for $\hbar \to 0$. In the Ehrenfest limit we obtain the limit of the quantum tomogram of harmonic oscillator stationary states and of the states of a particle in a box for $\hbar \to 0$ with fixed energy. We will show that the Ehrenfest limit provides the time-average of classical motion (both for the harmonic oscillator and the particle in a box).
Also we study the behaviour of tomograms for superposition states of the harmonic oscillator in the Planck limit. Namely, we show that the tomogram of the superposition of two arbitrary eigenstates (the Schroedinger cat[22]), goes to a mixture of classical tomograms of these states. The interference contribution to the tomogram goes to zero in the Planck limit. Another type of Schroedinger cat system (consisting of even and odd coherent states[23]) will be considered in both the Planck and Ehrenfest limits.

The article is organized as follows: in section 2 we review known properties of symplectic tomograms for classical and quantum systems. In section 3 we introduce the quantum-classical transition. In subsection 3.1 the Planck limits are computed for few systems. In subsection 3.2 we study the Ehrenfest classical limit for the stationary states of the harmonic oscillator and the infinite square well potential. Concluding remarks and perspectives are given in the final section. Some relevant formulae for generalized functions and their limits are considered in the appendix.

2 Symplectic tomograms in classical and quantum mechanics

Following references[16,17,24] we review the tomographic description of particle states in classical and quantum mechanics. For simplicity we restrict to one-dimensional particle systems.

2.1 Classical tomograms

The standard description of classical states with fluctuations is given by a non-negative joint probability distribution function \( f(p,q) \) on the phase space of the particle with one degree of freedom. The function is normalized, i.e.

\[
\int f(p,q)dpdq = 1.
\] (1)

The tomogram is the probability distribution function in a rotated and scaled reference frame on the phase-space. The classical tomogram is constructed from \( f(p,q) \) and is a function of a coordinate \( X \in \mathbb{R} \), which is related to the position \( q \) and momentum \( p \) of a canonical reference frame on phase-space by

\[
X = \mu q + \nu p,
\] (2)

where \( \mu \) and \( \nu \) are real parameters. It can be expressed in terms of a scaling parameter \( s \) and a rotation parameter \( \theta \):

\[
\mu = s \cos \theta , \quad \nu = s^{-1} \sin \theta .
\] (3)
For fixed $\mu$ and $\nu$ one then gets a line on the commutative plane $(q, p)$ with an orientation $\theta$ from the position axis. Thus the physical meaning of the variable $X$ is that it is the “position” of the particle measured in the reference frame of the phase-space whose axes are rotated by an angle $\theta$ with respect to the old reference frame, after preliminary canonical scaling of the initial position $q \rightarrow sq$ and momentum $p \rightarrow s^{-1}p$. The coordinate $X$ of equation (2) together with

$$Y = -s^2\nu q + s^{-2}\mu p \tag{4}$$

provides a canonical transformation preserving the symplectic form in the phase-space. For that reason the classical tomogram is called “symplectic”.

The tomogram of the classical statistical density $f(p, q)$ is defined by the Radon transform

$$W(X, \mu, \nu) = \int f(p, q)\delta(X - \mu q - \nu p)dpdq. \tag{5}$$

It can be written in the form

$$W(X, \mu, \nu) = \langle \delta(X - \mu q - \nu p) \rangle_f \tag{6}$$

where the average is done using the probability distribution $f(p, q)$ in the phase space. Its inverse is

$$f(p, q) = \frac{1}{(2\pi)^2} \int W(X, \mu, \nu) \exp[i(X - \mu q - \nu p)]dXd\mu d\nu. \tag{7}$$

Since the classical probability distribution $f(p, q)$ is normalized, the classical tomogram is also normalized for any values of the parameters $\mu$ and $\nu$, i.e.

$$\int W(X, \mu, \nu)dX = 1. \tag{8}$$

For $\mu = 1, \nu = 0$ the tomogram provides the marginal distribution of the position $(X = q)$

$$W(X, 1, 0) = \int f(p, q)dp. \tag{9}$$

For $\mu = 0, \nu = 1$ the tomogram provides the marginal probability distribution of the momentum $(X = p)$

$$W(X, 0, 1) = \int f(p, q)dq. \tag{10}$$

The expression (5) is written for the case of stationary distribution function in phase space. But it is easily generalized to time-dependent functions $f(p, q; t)$

$$W(X, \mu, \nu; t) = \int f(p, q; t)\delta(X - \mu q - \nu p)dpdq. \tag{11}$$
For distribution functions associated with particle motion along a given trajectory in phase space \(q(t), p(t)\)

\[
\tilde{f}(p, q; t) = \delta(p - p(t))\delta(q - q(t))
\] (12)

the tomogram given by Eq.(11) reads

\[
\tilde{W}_f(X, \mu, \nu; t) = \delta(X - \mu q(t) - \nu p(t)).
\] (13)

For example, classical free motion with initial position \(q_0\) and initial momentum \(p_0\), i.e. \(q(t) = q_0 + p_0 t; p(t) = p_0\), is described by the tomogram

\[
\tilde{W}_f(X, \mu, \nu; t) = \delta(X - \mu(q_0 + p_0 t) - \nu p_0).
\] (14)

Below we will compare tomograms associated with motion along classical trajectories with classical limits of their corresponding quantum tomogram. When the trajectory is periodic it is possible to introduce the time-averaged classical tomogram

\[
\langle W \rangle(X, \mu, \nu) = \frac{1}{T} \int_0^T W(X, \mu, \nu; t) dt.
\] (15)

The time-averaged tomogram corresponds to the time-averaged distribution function on phase-space

\[
\langle f \rangle(p, q) = \frac{1}{T} \int_0^T f(p, q; t) dt.
\] (16)

### 2.2 Quantum tomograms

Given a density matrix \(\rho(x, x')\), one constructs the corresponding Wigner function \(W\) as:

\[
W(p, q) = \int \rho(q + \frac{u}{2}, q - \frac{u}{2}) \exp(-i\frac{pu}{\hbar}) du,
\] (17)

and the Radon transform of Wigner function \(W\) is the quantum tomogram \(\mathcal{W}\) of \(\rho\), which is a positive function[18] of three variables,

\[
\mathcal{W}(X, \mu, \nu) = \frac{1}{2\pi} \int W(p, q)\delta(X - \mu q - \nu p) \frac{dp dq}{\hbar}
\] (18)

\[
= \frac{1}{2\pi \hbar |\nu|} \int \rho(q + \frac{u}{2}, q - \frac{u}{2}) \exp \left[-i\frac{X - \mu q}{\hbar \nu} u \right] dq du
\] (19)

Equation (18) can be written in a form similar to Eq.(6),

\[
\mathcal{W}(X, \mu, \nu) = \langle \delta(X - \mu \hat{q} - \nu \hat{p}) \rangle.
\] (20)

The difference with Eq.(6) is that here the position and momentum are quantum operators \(\hat{q}\) and \(\hat{p}\), i.e. we use non-commutative geometry of phase-space
plane, taking into account uncertainty relations. The averaging in Eq.(20) is done using density operators $\hat{\rho}$, i.e.

$$\langle \hat{A} \rangle := Tr\hat{\rho}\hat{A}.$$  \hspace{1cm} (21)

The inverse formula of Eq.(18) is readily written as:

$$W(p,q) = \frac{\hbar}{2\pi} \int W(X,\mu,\nu) \exp \left[ i(X - \mu q - \nu p) \right] dX d\mu d\nu.$$  \hspace{1cm} (22)

The density matrix $\rho(x,x')$ can be obtained from the Wigner function as

$$\rho(x,x') = \frac{1}{2\pi\hbar} \int W(p,\frac{x + x'}{\hbar}) \exp \left[ i\left(\frac{p(x - x')}{\hbar}\right) \right] dp.$$  \hspace{1cm} (23)

In terms of tomogram we get:

$$\rho(x,x') = \frac{1}{2\pi} \left| \int \hat{\psi}(p) e^{i\frac{\mu}{2\pi\hbar}p^2} \right|^2, \quad \mu \neq 0.$$  \hspace{1cm} (24)

For pure states $\rho(x,x') = \psi(x)\psi^*(x')$ and $W$ can be computed from Eq.(19):

$$W(X,\mu,\nu) = \frac{1}{2\pi|\nu|} \left| \int \psi(q + \frac{\nu}{2})\psi^*(q - \frac{\nu}{2}) e^{-i(X - \mu q)\frac{\nu}{\hbar}} dq du \right|^2, \quad \nu \neq 0.$$  \hspace{1cm} (25)

Thus, apart from the pre-factor $1/2\pi\hbar|\nu|$, the tomogram $W$ of the wave function $\psi$ is the square modulus of the tomogram amplitude $A_\psi$:

$$A_\psi(X,\mu,\nu) := \int \psi(y) e^{i\frac{\mu}{2\pi\hbar}\frac{y^2}{\hbar^2}} e^{-i\frac{X}{\hbar}y} dy.$$  \hspace{1cm} (26)

When $\nu = 0$ and $\mu \neq 0$ one can instead use

$$W(X,\mu,\nu) = \frac{1}{2\pi\hbar|\mu|} \left| \int \hat{\psi}(p) e^{-i\frac{\mu}{2\pi\hbar}p^2 + i\frac{X}{\hbar}p} dp \right|^2, \quad \mu \neq 0,$$  \hspace{1cm} (27)

where the Fourier transform of the wave function has been introduced

$$\hat{\psi}(p) = \frac{1}{\sqrt{2\pi\hbar}} \int \psi(y) e^{-i\frac{\nu}{\hbar}y} dy.$$  \hspace{1cm} (28)

If both $\mu = \nu = 0$, we get $W(X,0,0) = \delta(X)$, after using $\int \rho(x,x)dx = 1$.

Tomograms of quantum states and classical statistical density are positive functions on the same space $\mathbb{R}^3$ and therefore can be compared. Moreover, the same dimensional relations hold in both cases:

$$[X] = [\mu][q] = [\nu][p] \quad ; \quad [W(X,\mu,\nu)] = [X]^{-1}.$$  \hspace{1cm} (29)
where \( [\cdot] \) indicates the units. \( X \) and \( W \) can be made dimensionless after assuming

\[
[\mu] = [q]^{-1} \ ; \ [\nu] = [p]^{-1}.
\]  

(30)

The operator

\[
\hat{X} = \mu \hat{q} + \nu \hat{p}
\]

(31)

together with its conjugate

\[
\hat{Y} = -s^2 \nu \hat{q} + s^{-2} \mu \hat{p}
\]

(32)

preserves the canonical commutation relations: \( [\hat{X}, \hat{Y}] = [\hat{q}, \hat{p}] \). The observable \( \hat{X} \) is a new position operator, i.e. the position after a symplectic (linear canonical) transformation in the quantum phase-space \( (\hat{q}, \hat{p}) \) of the particle. The real variable \( X \) gives the possible results of a measure of \( \hat{X} \) and runs over the spectrum of \( \hat{X} \). In this way a description of quantum tomograms is recovered in complete analogy with the classical case. So the tomogram is also “symplectic” in the quantum case.

In classical mechanics, the transition from the distribution function of two canonically conjugate variables (position \( q \) and momentum \( p \)) to the distribution function of one variable (position \( X \)) does not play a crucial role due to absence of quantum mechanical constraints like the uncertainty relations of Heisenberg\(^\text{[26]}\) and Robertson-Schroedinger\(\text{[1,27,28,29,30]}\). On the contrary, the use of tomograms in quantum mechanics provides the possibility of describing a quantum state with a probability distribution which does not violate the uncertainty relations. Comparing the Eqs.\(\text{(6)} \) and \(\text{(20)}\) we see that the classical limit will be found starting from a non-commutative plane \( (\hat{q}, \hat{p}) \) and going to a commutative one \( (q, p) \). This means that in the limit process we take into full account the behaviour of the quantum uncertainty relations.

3 The quantum-classical transition

The classical limit of quantum states can be obtained using different procedures. One procedure is to take the limit \( \hbar \rightarrow 0 \) of certain quantum mechanical expressions. The wave function defined on the configuration space is not suitable for this purpose, since there is no physical interpretation of the resulting limit. It is more useful to associate an "observable" with any state vector, namely the associated rank-one projector. Following the Weyl-Wigner approach observables can be mapped to functions on phase-space and finally to tomograms via Radon transform. The classical limit \( \hbar \rightarrow 0 \) then can be done on tomograms in two different ways.
(1) At a kinematic level (Planck limit). Here no reference is made to any specific dynamics.
(2) At a dynamical level (Ehrenfest limit). Here the limit $\hbar \to 0$ on tomo-
grams is performed keeping constant the mean value of a given observable (usually the Hamiltonian).

3.1 Planck limits

We can anticipate some general features of the Planck limit for the tomogram of a quantum state. The quantum fluctuations of position and momentum, depending on Planck’s constant, are washed out when $\hbar \to 0$ and in general a Dirac $\delta(X)$ results, corresponding to the tomogram of the classical probability distribution $\delta(p)\delta(q)$. The fact that that Dirac delta function is centered at the origin has no particular physical significance, as it is only due to the arbitrary choice of the origin of the affine phase space.

The Planck limit can be predicted from a scaling argument for a large class of wave functions. Assume the dependence on $\hbar$ in the wave function $\psi$ is ($\gamma$ a real number)

$$\psi(x) = \hbar^{\frac{\gamma}{2}} \Psi(\hbar^\gamma x).$$

This assures that

$$\int |\psi(x)|^2 dx = 1$$

independent of the value of $\hbar$. Computing the tomogram in this case we get

$$W(X, \mu, \nu) = \frac{1}{2\pi \hbar} \left| \int \Psi(\hbar^\gamma y) \hbar^{\frac{\gamma}{2}} e^{i \frac{\mu}{\hbar^\gamma} y^2 - i \frac{X}{\hbar^\gamma} y} dy \right|^2
= \frac{1}{2\pi |\nu| \hbar^{\gamma+1}} \left| \int \Psi(t) e^{i \frac{\mu}{\hbar^{\gamma+1}} t^2 - i \frac{X}{\hbar^{\gamma+1}} t} dt \right|^2.

(35)$$

When $\gamma = -1/2$ it gives

$$W(X, \mu, \nu) = \frac{1}{2\pi |\nu|} \left| \int \Psi(t) e^{i \frac{\mu}{\hbar^{1/2}} t^2 - i \frac{X}{\hbar^{1/2}} t} dt \right|^2 = \frac{1}{\sqrt{\hbar}} F\left(\frac{X}{\sqrt{\hbar}}, \mu, \nu\right).

(36)$$

Since in general $\int W(X, \mu, \nu) dX = 1$, the Dirac delta theorem of the Appendix, with $n = 1/\sqrt{\hbar}$, applies and yields the Planck limit of the tomogram as

$$\lim_{\hbar \to 0} W(X, \mu, \nu) = \delta(X).

(37)$$

This is the case of the harmonic oscillator eigenstates and their superpositions.

When $\gamma = 0$ for the wave function in position space, its Fourier transform $\hat{\psi}$ of Eq.(28) scales with $\gamma = -1/2$ and using Eq.(27) we recover the same Planck
limit of Eq.(37). In general this happens when a quantum particle of mass \( m \) is confined in a potential with a characteristic length \( L \), then \( V_0 = \frac{\hbar^2}{2mL^2} \) is the scale factor of the potential energy so that the eigenvalues are proportional to \( V_0 \) while the eigenstates do not depend on \( \hbar \). This is the case of the Poeschl-Teller potential[31] and of its limit, the infinite square well.

We finally observe that a more general scaling law could be

\[
\psi(x) = \hbar^{\frac{\gamma}{2}} \Psi(h^\gamma(x - x_0)).
\] (38)

In that case, when \( \gamma = -1/2 \), the Planck limit of the tomogram is \( \delta(X - \mu x_0) \) and corresponds to the classical distribution \( \delta(p)\delta(q - x_0) \). Nevertheless, by using the shifted position operator \( \hat{q} - x_0 \), the Planck limit turns out to be again \( \delta(X) \).

In the following we study the Planck limit for examples involving Hermite eigenfunctions. We start by reviewing the Hermite tomograms, whose expressions will also be useful in evaluating the Ehrenfest limit.

### 3.1.1 Harmonic oscillator

Recall that the eigenstates of a quantum harmonic oscillator of mass \( m \) and frequency \( \omega \), normalized with respect to the Lebesgue measure \( dy \), are

\[
\varphi_n(\sqrt{\frac{\omega}{\hbar}} y) = (\sqrt{\pi} 2^n n!)^{-\frac{1}{2}} \sqrt{\frac{\omega}{\hbar}} H_n(\sqrt{\frac{\omega}{\hbar}} y) \exp \left[ -\frac{\omega}{2\hbar} y^2 \right]
\] (39)

with \( \omega = m\omega \) and \( n = 0, 1, 2, \ldots \). Using Eq.(25) the tomogram \( W_0(X, \mu, \nu) \) for the ground state can be computed as

\[
\frac{1}{\sqrt{\pi}} \sqrt{\frac{\omega}{\hbar(\omega^2 \nu^2 + \mu^2)}} \exp \left[ -\frac{\omega}{\hbar(\omega^2 \nu^2 + \mu^2)} X^2 \right] = \varphi_0^2 \left( \sqrt{\frac{\omega}{\hbar(\omega^2 \nu^2 + \mu^2)}} X \right).
\] (40)

Similarly, the tomogram \( W_n \) for the \( n \)-th excited state is

\[
W_n(X, \mu, \nu) = \varphi_n^2 \left( \sqrt{\frac{\omega}{\hbar(\omega^2 \nu^2 + \mu^2)}} X \right)
\] (41)

A suitable way to obtain these results is to evaluate a generating function \( J \) for the tomogram amplitudes \( A_n \) of Hermite functions \( \varphi_n \)

\[
J(s) := \sum_{n=0}^{\infty} \frac{s^n}{\sqrt{n!}} A_n = \sum_{n=0}^{\infty} \frac{s^n}{\sqrt{n!}} \int \varphi_n(\sqrt{\frac{\omega}{\hbar}} y) \exp \left[ i \frac{\mu}{2\hbar \nu} y^2 - i \frac{X}{\hbar \nu} y \right] dy
\]

\[
= \frac{1}{\sqrt{2\pi \hbar}} \sqrt{\frac{2\pi \hbar \nu}{\zeta^*}} \exp \left[ \frac{\nu}{2\zeta^*} s^2 - i \sqrt{2\nu X s - \frac{1}{2\hbar \nu} \zeta^* X^2} \right]
\] (42)
where $\zeta = \omega \nu + i \mu$ and $\zeta^*$ is its complex conjugate. It is possible to reconstruct a generating function of Hermite functions $\varphi_n$. Defining

$$\tau = -i \sqrt{\frac{\zeta}{\zeta^*}} ; \quad Q = \sqrt{\frac{\omega}{\hbar \zeta^*}} X,$$

Equation (42) becomes

$$J(\tau) = \sqrt{2 \pi \hbar \nu} \sqrt{\frac{\zeta}{\zeta^*}} \exp \left[ -i \frac{\mu}{2 \omega \nu} Q^2 \right] \sum_{n=0}^{\infty} \frac{\tau^n}{\sqrt{n!}} \varphi_n(Q).$$

Eventually, the Hermite tomogram amplitudes can be written

$$A_n = \frac{1}{\sqrt{n!}} \frac{d^n J}{ds^n}(0) = \sqrt{2 \pi \hbar \nu} \exp \left[ -i \frac{\mu}{2 \omega \nu} Q^2 \right] \sqrt{\frac{\zeta}{\zeta^*}} \left( -i \sqrt{\frac{\zeta}{\zeta^*}} \right)^n \varphi_n(Q),$$

which yield Hermite tomograms $W_n$ of Eq.(41) for $\varphi_n$. Observing that $\varphi_n$ is normalized to unity, the Planck limit of $W_n$, evaluated by means of the Dirac delta theorem (see Appendix), is

$$\lim_{\hbar \to 0} W_n(X, \mu, \nu) = \delta(X)$$

as expected.

3.1.2 Superposition

The case of a superposition of two harmonic oscillator eigenstates

$$\psi = \frac{1}{\sqrt{2}} (\varphi_n + \varphi_m)$$

shows in the simplest manner the vanishing of quantum interference in the classical Planck limit. Writing the tomogram in terms of Hermite amplitudes of Eq.(45)

$$W_\psi(X, \mu, \nu) = \frac{1}{2 \pi \hbar |\nu|} \left| \frac{1}{\sqrt{2}} (A_n + A_m) \right|^2 = \frac{1}{2} W_n + \frac{1}{2} W_m + \text{Re} \frac{A_n A_m^*}{2 \pi \hbar |\nu|}.$$ 

Dropping a unimodular factor independent of $\hbar$, the quantum interference term reads

$$\frac{A_n A_m^*}{2 \pi \hbar |\nu|} \propto \varphi_n(Q) \varphi_m(Q).$$

Due to the orthogonality of harmonic oscillator eigenstates, this term vanishes when $\hbar \to 0$ as is shown in the Appendix. Moreover, it is possible to show that the quantum interference term vanishes like $\sqrt{\hbar}$. Again, by using Eq.(46), the
classical Planck limit of superposition state tomogram is
\[
\lim_{\hbar \to 0} W_\psi(X,\mu,\nu) = \delta(X). \tag{50}
\]

### 3.1.3 Coherent states

We next consider the tomogram of coherent states. Recalling that the coherent state \(|\alpha\rangle\) labeled by the complex number \(\alpha\) is defined by
\[
|\alpha\rangle := e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle, \tag{51}
\]
the tomogram amplitude \(A_\alpha\) of the coherent state can be expressed through Hermite amplitudes of Eq.(45)
\[
A_\alpha = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} A_n = e^{-|\alpha|^2/2} J(\alpha) \tag{52}
\]
From the expression for the generating function of Hermite amplitudes \(J\) in Eq.(42), the previous equation becomes
\[
A_\alpha = \sqrt{\frac{\omega}{\pi \hbar}} \sqrt{\frac{2\pi \hbar \nu}{\zeta^*}} e^{-\frac{|\alpha|^2}{2}} e^{-\frac{\mu^2}{2\hbar}} e^{-\frac{\nu^2}{2\hbar}} \exp \left[ -\frac{Q^2}{2} - \frac{i}{\hbar} \frac{\zeta \Re \alpha - \zeta^* \Im \alpha}{\omega} \right] \tag{53}
\]
so that, eventually, the tomogram \(W_\alpha(X,\mu,\nu) = A_\alpha A_\alpha^* / 2\pi \hbar |\nu|\) is given by
\[
\sqrt{\frac{\omega}{\pi \hbar (\omega^2 \nu^2 + \mu^2)}} \exp \left[ -\frac{(\sqrt{\omega} X - \mu \sqrt{2\hbar} \Re \alpha - \omega \nu \sqrt{2\hbar} \Im \alpha)^2}{2 \pi \hbar (\omega^2 \nu^2 + \mu^2)} \right]. \tag{54}
\]
After shifting the integration variable
\[
\int W_\alpha(X,\mu,\nu) dX = \int \frac{1}{\sqrt{\pi}} \exp \left[ -\xi^2 \right] d\xi = 1, \tag{55}
\]
and by the Dirac delta theorem the Planck limit of the tomogram is
\[
W_\alpha(X,\mu,\nu) \xrightarrow[\hbar \to 0]{} \delta(X). \tag{56}
\]

### 3.1.4 Schroedinger cat states

Even and odd coherent states (Schroedinger cat states) are defined\[23\] as:
\[
|\alpha_\pm\rangle = N_\pm (|\alpha\rangle \pm |-\alpha\rangle) \tag{57}
\]
where

\[ \langle \beta | \alpha \rangle = \exp \left[ -\frac{|\alpha|^2}{2} - \frac{|\beta|^2}{2} + \alpha \beta^* \right] \Rightarrow N_\pm = \frac{1}{\sqrt{2(1 \pm e^{-2|\alpha|^2})}}. \quad (58) \]

The tomograms \( W_{\alpha\pm}(X, \mu, \nu) = N_{\pm}^2 |A_\alpha \pm A_{-\alpha}|^2 / 2\pi \hbar |\nu| \) contain three terms

\[ N_{\pm}^2 \{ W_{\alpha}(X, \mu, \nu) + W_{-\alpha}(X, \mu, \nu) \pm I(X, \mu, \nu) \}. \quad (59) \]

The interference term reads

\[ I(X, \mu, \nu) = 2 \text{Re} \frac{A_\alpha A_{-\alpha}^*}{2\pi \hbar |\nu|} \quad (60) \]

with tomogram amplitude \( A_\alpha \) given by Eq.(53). As shown in the Remark of Appendix, it results

\[ \int \frac{A_\alpha A_{-\alpha}^*}{2\pi \hbar |\nu|} dX = \langle -\alpha | \alpha \rangle = e^{-2|\alpha|^2} \quad (61) \]

and by Dirac delta theorem the Planck limit of the interference term \( I(X, \mu, \nu) \) is \( 2e^{-2|\alpha|^2} \delta(X) \). Then the Planck limit of the Schroedinger cat tomogram is

\[ \lim_{\hbar \to 0} W_{\alpha\pm}(X, \mu, \nu) = N_{\pm}^2 (2\delta(X) \pm 2e^{-2|\alpha|^2} \delta(X)) = \delta(X). \quad (62) \]

We observe that the Planck limit \( \delta(X) \) is gained thanks to the contribution of a non-zero interference term between non-orthogonal states.

### 3.2 Ehrenfest classical limits

The Ehrenfest classical limit takes place at the dynamical level, i.e. it requires a specific dynamical system and the limit is performed while keeping constant the value of some selected observable, usually the energy. [1] In this section some Ehrenfest classical limits are evaluated with the constraint of constant energy.

We observe that after calculating the Ehrenfest limit, it is possible to consider the limit when the energy goes to zero. This is not equivalent to the Planck limit because in the Ehrenfest case, if there is a potential with a minimum, the particle goes to its equilibrium position \( q_0 \) when the energy vanishes. Then the expected result is \( \delta(X - \mu q_0) \), rather than \( \delta(X) \), agreeing with the classical tomogram of a rest state \( \delta(p) \delta(X - q_0) \), as the next example of the coherent state shows.
3.2.1 Schrödinger cats

The Ehrenfest classical limit of the coherent state tomogram $W_\alpha$ is readily evaluated from Eq. (54). Put $\omega = 1$ and impose the constraint on the energy by taking constant the mean value of the harmonic oscillator Hamiltonian $\hbar(\hat{a}^\dagger\hat{a} + 1/2)$ on the coherent state $|\alpha\rangle$. The constraint is satisfied by choosing

$$\alpha\sqrt{\hbar} = \text{const} \Rightarrow \hbar|\alpha|^2 = \text{const}. \quad (63)$$

Now, bearing in mind that the mean value of the annihilation operator $\hat{a}$ on the coherent state $|\alpha\rangle$ is

$$\langle \hat{a} \rangle_\alpha = \langle \hat{q} + i\hat{p} \rangle_\alpha = \frac{q_\alpha + ip_\alpha}{\sqrt{2\hbar}} = \alpha, \quad (64)$$

the Ehrenfest classical limit $\hbar \to 0$, $\alpha \to \infty$ with $\alpha\sqrt{2\hbar} = \text{const} = q_\alpha + ip_\alpha$ turns out to be, just by inspection of Eq. (54) (with $\omega = 1$), the displaced delta function

$$\delta(X - \mu q_\alpha - \nu p_\alpha) \quad (65)$$

which coincides with the tomogram of the classical distribution function

$$f(p,q) = \delta(p - p_\alpha)\delta(q - q_\alpha). \quad (66)$$

As far as the Schrödinger cat states of Eq. (59) are concerned, the interference term $I(X, \mu, \nu)$ vanishes in the Ehrenfest limit, due to the presence of a rapidly oscillating factor ($\hbar \to 0$)

$$\cos \frac{2\sqrt{2\hbar}X(\nu \Re \alpha - \mu \Im \alpha)}{\hbar(\nu^2 + \mu^2)} = \cos \frac{2X(\nu q_\alpha - \mu p_\alpha)}{\hbar(\nu^2 + \mu^2)} \quad (67)$$

while the normalizations $N_2^\pm$ go to the same constant value

$$N_2^\pm = \frac{1}{2(1 \pm e^{-2|\alpha|^2})} \rightarrow \frac{1}{2} \quad (68)$$

and the tomograms become the same sum of delta functions

$$W_\infty(X, \mu, \nu) = \frac{1}{2}[\delta(X - \mu q_\alpha - \nu p_\alpha) + \delta(X + \mu q_\alpha + \nu p_\alpha)]. \quad (69)$$

3.2.2 Infinite square well

We consider a particle of mass $m = 1$ in an infinite square well $0 \leq q \leq L$. To deal with wave functions of compact support it is convenient to regard them as elements of $L_2(\mathbb{R})$ suitably projected by means of the projector $\chi_{[0,L]}$, which
is the standard characteristic function of the set \([0, L]\). The eigenfunctions and the eigenvalues are

\[
\psi_n(q) = \sqrt{2} \frac{\sin(n\pi q)}{L} \chi_{[0,L]}(q) ; \quad E_n = \frac{1}{2} \left( \frac{hn\pi}{L} \right)^2, \quad n = 1, 2, \ldots .
\]  

(70)

Fixing the energy to be one for convenience, the Ehrenfest classical limit amounts to

\[
h = \frac{\sqrt{2}L}{n\pi}, \quad n \to \infty.
\]  

(71)

In general, for \(\nu \neq 0\),

\[
\mathcal{W}(X, \mu, \nu) = \frac{1}{L\pi \hbar |\nu|} \left| \int_0^L \sin \left( \frac{n\pi L}{L} y \right) e^{\frac{\hbar}{2\nu^2} y^2 - i \frac{\hbar}{\mu} y} \, dy \right|^2 = \frac{n |A_- - A_+|^2}{4L^2 \sqrt{2} |\nu|}
\]  

(72)

where the amplitudes \(A_\pm\) are respectively given by

\[
\int_0^L \exp \left[ i nF_\pm(y) \right] \, dy := \int_0^L \exp \left\{ i n \frac{\pi}{L} \left[ \frac{\mu}{\sqrt{2\nu}^2} y^2 - \frac{X}{\sqrt{2\nu}} \mp 1 \right] \right\} \, dy.
\]  

(73)

These amplitudes can be computed exactly in terms of Error function. However, in view of the large \(n\) limit, it is more useful to evaluate these integrals in the stationary phase approximation. It results

\[
A_\pm \simeq \sqrt{\frac{2L |\nu|}{n |\mu|}} \sqrt{2} \exp \left\{ i n \left[ F_\pm(Q_s^\mp) + \text{sign} \left( \frac{\mu}{\nu} \right) \frac{\pi}{4} \right] \right\} \chi_{[0,L]}(Q_s^\mp).
\]  

(74)

where the stationary points are

\[
Q_s^\mp = \frac{X}{\mu} \mp \sqrt{2}\nu.
\]  

(75)

Eventually, the asymptotic behaviour of \(\mathcal{W}(X, \mu, \nu)\) is

\[
\frac{\chi_{[0,L]}(Q_s^-) + \chi_{[0,L]}(Q_s^+) - 2\chi_{[0,L]}(Q_s^-)\chi_{[0,L]}(Q_s^+) \cos n[F_-(Q_s^-) - F_+(Q_s^+)]}{2 |\mu| L}
\]

and evaluating the Ehrenfest classical limit \(n \to \infty\) in the sense of a distribution:

\[
\mathcal{W}_\infty(X, \mu, \nu) = \frac{1}{2 |\mu| L} \left\{ \chi_{[0,L]}(Q_s^-) + \chi_{[0,L]}(Q_s^+) \right\}.
\]  

(76)

This result has to be compared with its classical analogue. In that case, bearing in mind that \(m = 1\) and \(p = \pm \sqrt{2E} = \pm \sqrt{2}L\), we have

\[
f(p, q; t) = \delta(p - p(t))\delta(q - q(t))
\]  

(77)

where \(q(t)\) is a zigzag line of height \(L\) and half-period \(L/\sqrt{2}\),

\[
q(t) = \sqrt{2}t \chi_{[0,L/\sqrt{2}]}(t) - \sqrt{2}(t - \sqrt{2}L) \chi_{[L/\sqrt{2},\sqrt{2}L]}(t),
\]  

(78)
while
\[ p(t) = \sqrt{2} \chi_{[0, L/\sqrt{2}]}(t) - \sqrt{2} \chi_{[L/\sqrt{2}, L]}(t). \]  

(79)

To make a comparison with the Ehrenfest limit we have to average the classical time dependent distribution tomogram over a period:
\[ W(X, \mu, \nu) = \frac{1}{\sqrt{2L}} \int_0^{\sqrt{2L}} dt \int f(p, q; t) \delta(X - \mu q - \nu p) dp dq \]
\[ = \frac{1}{\sqrt{2L} |\mu|} \left( \int_0^{\sqrt{2L}/2} \delta(Q_s^- - q(t)) dt + \int_{\sqrt{2L}/2}^{\sqrt{2L}} \delta(Q_s^+ - q(t)) dt \right), \]

(80)

where \( Q_s^\pm \) are again given by Eq.(75). Eventually,
\[ W(X, \mu, \nu) = \frac{1}{2 |\mu| L} \left\{ \chi_{[0, L]}(Q_s^-) + \chi_{[0, L]}(Q_s^+) \right\} = W_\infty(X, \mu, \nu). \]

(81)

This general result yields the marginal probability distribution of \( q \), in the limit \( \nu \to 0, \mu = 1 \)
\[ \lim_{\nu \to 0, \mu \to 1} \frac{1}{L} \left\{ \chi_{[0, L]}(Q_s^-) + \chi_{[0, L]}(Q_s^+) \right\} = \frac{1}{L} \chi_{[0, L]}(X). \]

(82)

As expected, this result shows that the particle position is likely distributed in the box.

The limit \( \nu \to 1, \mu = 0 \) which yields the marginal probability distribution of \( p \) is more involved. However, observing that
\[ \int \frac{1}{|\mu| L} \chi_{[0, L]}(Q_s^\pm) dX = 1 \]

(83)

and remembering the explicit expression of \( Q_s^\pm \), by Dirac delta theorem it results
\[ \lim_{\nu \to 1, \mu \to 0} \frac{1}{|\mu| L} \left\{ \chi_{[0, L]}(Q_s^-) + \chi_{[0, L]}(Q_s^+) \right\} = \frac{1}{2} \delta(X - \sqrt{2}) + \frac{1}{2} \delta(X + \sqrt{2}) \]

(84)

As expected, the probability distribution of momentum is likely concentrated on the two allowed classical values of \( p \).

3.2.3 Harmonic oscillator

We assume \( m = 1, \omega = 1 \) and evaluate the Ehrenfest classical limit of the Hermite tomogram \( W_n \) of Eq.(41) when
\[ \hbar \to 0, E_n = \text{const} = 1 \Rightarrow \left( n + \frac{1}{2} \right) \hbar = 1. \]

(85)
In view of the large $n$ limit, we may assume $n + \frac{1}{2} \simeq n$, so that $\hbar = 1/n$. Besides, due to the spherical symmetry on the phase space, it is enough to consider the case $\mu = 1, \nu = 0$. Then the tomogram of the harmonic oscillator eigenstate $\varphi_n$ is written as

$$W_n(X, 1, 0) = \sqrt{\frac{n}{\pi}} \frac{e^{-nX^2}}{2^n n!} H_n^2(\sqrt{n}X)$$

(86)

Note that the tomogram is an even function of $X$, so we only need study positive values of $X$. We start by expressing Hermite polynomials $H_n$ in terms of parabolic cylinder functions $U$ (see [32])

$$H_n(y) = 2^{n/2} e^{y^2/2} U\left(-\left(n + \frac{1}{2}\right), \sqrt{2}y\right).$$

(87)

The main asymptotic formula is:

$$U(a, x) \simeq 2^{-1/4-a/2} \Gamma\left(\frac{1}{4} - \frac{a}{2}\right) \sqrt{\frac{\tau}{\xi^2 - 1}} Ai(\tau), \quad (x \geq 0, a \to -\infty),$$

(88)

where the Airy function $Ai$ and the Euler gamma function $\Gamma$ appear, with:

$$\xi = \frac{x}{2\sqrt{|a|}}, \quad \tau = (4|a|)^{2/3} \left[\mp \left(\frac{3}{2} \Theta_\leq\right)^{2/3}\right]$$

(89)

and correspondingly:

$$\Theta_\leq = \begin{cases} \frac{1}{4} \left[\arccos \xi - \xi \sqrt{1 - \xi^2}\right], & (\xi \leq 1) \\ \frac{1}{4} \left[\xi \sqrt{\xi^2 - 1} - \cosh^{-1} \xi\right], & (\xi \geq 1) \end{cases}.$$  

(90)

In our case, the asymptotic behaviour of $W_n(X, 1, 0)$ is

$$\sqrt{\frac{n}{\pi} n!} U^2\left(-(n + \frac{1}{2}), \sqrt{2n}X\right) \simeq \frac{n}{\pi} \frac{1}{n!} 2^n \Gamma^2\left(\frac{n+1}{2}\right) \sqrt{\frac{\tau}{\xi^2 - 1}} Ai^2(\tau)$$

(91)

Now, the numerical pre-factor can be readily estimated as:

$$\sqrt{\frac{n}{\pi} n!} 2^n \Gamma^2\left(\frac{n+1}{2}\right) = \sqrt{n} \frac{\Gamma(n+1)}{\Gamma(n+1/2)} \sim n^{n/2} \sqrt{\frac{2}{n}} = \sqrt{2}$$

(92)

Using definitions in equations (89), (90) we have

$$\xi = \frac{\sqrt{2n}}{2\sqrt{n + \frac{1}{2}}} X \implies \frac{X}{\sqrt{2}}; \quad \xi \leq 1 \Rightarrow \tau = \mp |\tau|, \quad \frac{2}{3} |\tau|^{3/2} = 4\left(n + \frac{1}{2}\right) \Theta_\leq$$

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and the asymptotic behaviour of the Airy function[32]:

\[
\begin{align*}
Ai(-|\tau|) & \simeq \pi^{-1/2}|\tau|^{-1/4}\sin \left[ \frac{2}{3}|\tau|^{3/2} + \frac{\pi}{4} \right] \\
Ai(|\tau|) & \simeq (4\pi)^{-1/2}|\tau|^{-1/4}\exp \left[ -\frac{2}{3}|\tau|^{3/2} \right]
\end{align*}
\] (93)

we obtain the Ehrenfest classical distribution limit of \( W_n(X, 1, 0) \) :

\[
\frac{\sqrt{2}}{\pi \sqrt{|1 - \xi^2|}} \left\{ \sin^2 \left[ (n + \frac{1}{2}) \left( \arccos \xi - \xi \sqrt{1 - \xi^2} \right) + \frac{\pi}{4} \right] \right\} \to \frac{1}{2} \quad (\xi \leq 1)
\]

\[
\frac{1}{4} \exp \left[ -2(n + \frac{1}{2}) \left( \xi \sqrt{\xi^2 - 1} - \cosh^{-1} \xi \right) \right] \to 0 \quad (\xi \geq 1)
\]

with \( \xi = X/\sqrt{2} \). Eventually, restoring the negative values of \( X \), we have

\[
W_\infty(X, 1, 0) = \frac{\sqrt{2}}{2\pi \sqrt{1 - \left( \frac{X}{\sqrt{2}} \right)^2}} \chi_{[-1,1]} \left( \frac{X}{\sqrt{2}} \right).
\] (94)

The classical tomogram \( W(X, 1, 0) \) is the time average of the Radon transform of \( f(p, q; t) \):

\[
\frac{1}{T} \int_0^T dt \int \delta(p - p(t))\delta(q - q(t))\delta(X - q)dpdq = \frac{1}{T} \int_0^T \delta(X - q(t))dt
\] (95)

where \( T = 2\pi \) and \( q(t) = q_0 \cos t + p_0 \sin t \). Due to the rotational invariance of the harmonic oscillator Hamiltonian \( H = (p^2 + q^2)/2 \) we may assume

\[
q(t) = q_0 \cos t; \quad p(t) = -q_0 \sin t; \quad p_0^2 = 2 - q_0^2 = 0 \Rightarrow q_0 = \pm \sqrt{2}
\] (96)

so that the classical tomogram \( W(X, 1, 0) \) is

\[
\frac{1}{2\pi} \frac{2}{q_0 \sin(\arccos \frac{X}{q_0})} \chi_{[-1,1]} \left( \frac{X}{q_0} \right) = \frac{1}{\sqrt{2}\pi} \frac{1}{\sqrt{1 - \left( \frac{X}{\sqrt{2}} \right)^2}} \chi_{[-1,1]} \left( \frac{X}{\sqrt{2}} \right)
\] (97)

which coincides with the quantum Ehrenfest limit of Eq.(94). Furthermore,

\[
dq(t) = -q_0 \sin t \ dt \Rightarrow \left| \frac{dt}{dX} \right| = \left| \frac{1}{q_0 \sin t} \right| = \frac{1}{\sqrt{2} \sqrt{1 - \left( \frac{X}{\sqrt{2}} \right)^2}}
\] (98)

and \( dt/dX \) is the inverse of the classical velocity \( V \), so finally Eq.(97) can be written as:

\[
W(X, 1, 0) = \frac{2}{T} \frac{1}{V \left( \frac{X}{\sqrt{2}} \right)} \chi_{[-1,1]} \left( \frac{X}{\sqrt{2}} \right).
\] (99)

As expected, the probability distribution of the position depends on the inverse of the classical velocity \( V \) in that point and diverges at the turning points \( X/\sqrt{2} = \pm 1 \).
4 Conclusions

To conclude we found that the classical limit of quantum state tomograms is suitable for a comparison with classical state tomograms. From the Planck limit $\hbar \to 0$, the tomograms of stationary states of the harmonic oscillator, as well as of its coherent states, yield the localized state $\delta(X)$, agreeing with the classical tomogram of a state localized in the phase space. The Ehrenfest limit of quantum tomograms, where the energy is fixed while $\hbar \to 0$, gives the expected expressions of classical tomograms of classical states. When the energy vanishes, the result is a rest state $\delta(X - \mu q_0)$ with the particle sitting at the minimum of the potential.

The same results were shown to apply in the case of both the Planck and the Ehrenfest limits of the quantum tomogram for a particle in a box.

We have also found that, for the Schroedinger cat states, the interference term of a superposition of two orthogonal states vanishes in the Planck limit as $\sqrt{\hbar}$. In the Ehrenfest limit, even and odd coherent states yield the expected mixture of two classical delta distribution.

For composite systems, we believe that tomographic description of quantum and classical states can be a suitable tool to study the classical limit of entangled states both in the Planck and in Ehrenfest limit. This aspect will be considered in a forthcoming paper.

Finally we observe that the diverging inverse velocity factor in the Ehrenfest limit of the harmonic oscillator results from the presence of turning points of the harmonic oscillator motion along the $X$--axis. These divergences should disappear when describing the motion with a complex variable instead of a real one. To do that, the holomorphic (i.e. Bargmann-Fock) representation is suitable. However, this requires a preliminary discussion for dealing with tomograms associated with non-Hermitian operators, which will appear elsewhere.

5 Appendix

Many results of the present paper follow from a Theorem, valid under quite general assumptions, that here we recall:

**Dirac delta Theorem** Let $f$ be a summable function on the real line such that

\[ \int f(x)dx = N. \] (100)
Then \( nf(n(x - x')) \to N\delta(x - x') \) when \( n \to \infty \).

**Proof** Let \( \varphi \) be any test function. Then by Lebesgue Theorem

\[
\lim_{n \to \infty} n \int f(n(x - x')) \varphi(x) dx = \lim_{n \to \infty} \int f(s) \varphi\left(\frac{s}{n} + x'\right) ds
\]

\[
= \int \lim_{n \to \infty} f(s) \varphi\left(\frac{s}{n} + x'\right) ds = \varphi(x') \int f(s) ds = N\varphi(x'). \quad (101)
\]

**Remark** We can evaluate the Planck limit of the product of tomogram amplitudes \( A_\psi A_\phi^* \), when the state vectors \( \langle y|\psi \rangle \) and \( \langle y|\phi \rangle \) have scaling exponent \( \gamma = -1/2 \). In general, for any \( \gamma \), it is

\[
\int \frac{A_\psi A_\phi^*}{2\pi \hbar |\nu|} dX = \int \psi(y) \phi^*(u) e^{i \frac{\pi}{\hbar}(y^2 - u^2)} e^{-i \frac{\pi}{\hbar}(y - u)} dy du dX
\]

\[
= \int \psi(y) \phi^*(u) e^{i \frac{\pi}{\hbar}(y^2 - u^2)} \delta(y - u) dy du = \langle \phi|\psi \rangle \quad (102)
\]

In the case \( \gamma = -1/2 \), scaling the integration variables yields

\[
\frac{A_\psi A_\phi^*}{2\pi \hbar |\nu|} = \int \Psi(y) \Phi^*(u) e^{i \frac{\pi}{\hbar}(y^2 - u^2)} e^{-i \frac{\pi}{\hbar}(y - u)} \frac{dy du}{2\pi \sqrt{\hbar} |\nu|} \quad (103)
\]

So, as a consequence of the previous Theorem (with \( n = 1/\sqrt{\hbar} \)), it results

\[
\lim_{\hbar \to 0} \frac{A_\psi A_\phi^*}{2\pi \hbar |\nu|} = \langle \phi|\psi \rangle \delta(X). \quad (104)
\]

For the tomogram amplitudes \( A_n(X, \mu, \nu) \) of the eigenstates \( \psi_n \) of a quantum harmonic oscillator, it results

\[
\lim_{\hbar \to 0} \frac{A_n A_m^*}{2\pi \hbar |\nu|} = \delta_{n,m} \delta(X). \quad (105)
\]

where \( \delta_{n,m} \) is the Kronecker delta.

When the scaling exponent of \( \langle y|\psi \rangle \), \( \langle y|\phi \rangle \) is \( \gamma = 0 \), the same result can be obtained through their Fourier transforms which scale with \( \gamma = -1/2 \).

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