Partially critical 2-structures

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Abstract

A 2-structure $\sigma$ consists of a vertex set $V(\sigma)$ and of an equivalence relation $\equiv_{\sigma}$ defined on $(V(\sigma) \times V(\sigma)) \setminus \{(v, v) : v \in V(\sigma)\}$. Given a 2-structure $\sigma$, a subset $M$ of $V(\sigma)$ is a module of $\sigma$ if for $x, y \in M$ and $v \in V(\sigma) \setminus M$, $(x, v) \equiv_{\sigma} (y, v)$ and $(v, x) \equiv_{\sigma} (v, y)$. For instance, $\emptyset, V(\sigma)$ and $\{v\}$, for $v \in V(\sigma)$, are modules of $\sigma$ called trivial modules of $\sigma$. A 2-structure $\sigma$ is prime if $v(\sigma) \geq 3$ and all the modules of $\sigma$ are trivial. A prime 2-structure $\sigma$ is critical if for each $v \in V(\sigma)$, $\sigma - v$ is not prime. A prime 2-structure $\sigma$ is partially critical if there exists $X \subset V(\sigma)$ such that $\sigma[X]$ is prime, and for each $v \in V(\sigma) \setminus X$, $\sigma - v$ is not prime. We characterize finite or infinite partially critical 2-structures.

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1 Introduction

The 2-structures were introduced by Ehrenfeucht et al. [8]. They are well adapted generalizations of binary combinatorial structures like graphs, tournaments,... within the framework of modular decomposition. We consider finite or infinite 2-structures.

A module (or a clan [8]) of a 2-structure is a subset such that each vertex outside is linked in the same way to all the vertices inside. A 2-structure is prime if all its modules are trivial. In a finite and prime 2-structure, we can remove one or two vertices in order to obtain a prime 2-substructure. This result is false for infinite and prime 2-structures. In fact, there exist infinite and prime 2-structures that become non-prime after removing any finitely many vertices. In the sequel, such prime 2-structures are called finitely critical.
vertex \( v \) of a prime 2-structure is critical (in terms of primality) when the 2-substructure obtained by removing \( v \) is not prime. Now, a prime 2-structure is critical if all its vertices are critical. The finite and critical 2-structures were characterized independently by Bonizzoni [2], and Schmerl and Trotter [19]. The problem of the characterization of infinite and critical 2-structures remains open. The central difficulty comes from the existence of finitely critical 2-structures. Nevertheless, Bondabbous and Ille [14] succeeded in characterizing infinite and prime digraphs that are critical, but not finitely critical.

A prime 2-structure is partially critical if every vertex outside a prime induced 2-substructure is critical. Finite and partially critical graphs were characterized by Breiner et al. [7]. Finite and partially critical tournaments were characterized by Sayar [17] who adapted the examination of partial criticality presented in [7] to tournaments.

Almost all finite and prime 2-structures are prime. Thus, it is impossible to characterize or to describe the finite and prime 2-structures of a given cardinality. Now, suppose that a finite and prime 2-structure admits a critical vertex. The withdrawal of this vertex creates a partial module, which imposes conditions on the 2-structure. When the 2-structure is critical, that is, when all its vertices are critical, we obtain so many conditions that it is possible to characterize the finite and critical 2-structures up to isomorphism (see [2] and [19]). For finite and partially critical 2-structures, we have less conditions, and we do not succeed in characterizing them up to isomorphism. Nevertheless, we can localize the created partial modules because of the prime induced 2-substructure, which leads us to a description by using an auxiliary graph.

In this paper, we characterize finite or infinite partially critical 2-structures. For the finite case, we follow the same approach as that of [7]. We associate with the prime induced 2-substructure its outside graph (see Definition [6]). For a finite and partially critical 2-structure, the components of its outside graph are critical and bipartite (see Theorem [17]), that is, are half graphs (see Proposition [57]). This result establishes an important structural link between partial criticality and (global) criticality via the outside graph. Furthermore, always in the finite case, if we add an odd number of vertices to the prime induced 2-substructure, we obtain a non-prime induced 2-substructure. This fact is false in the infinite case when we consider finitely critical 2-structures as particular partially critical 2-structures. Therefore, to study infinite and partially critical 2-structures, we suppose that the addition of 5 vertices to the prime induced 2-substructure gives a non-prime induced 2-substructure. Under this assumption, we can proceed by compactness. We obtain that the components of the outside graph are critical and \( P_5 \)-free bipartite graphs. It turns out that the critical and \( P_5 \)-free bipartite graphs are the half graphs defined from a discrete linear order (see Theorem [22]).

At present, we formalize our presentation. A 2-structure \( \sigma \) consists of a finite or infinite vertex set \( V(\sigma) \), and of an equivalence relation \( \equiv_\sigma \) defined on \((V(\sigma) \times V(\sigma)) \setminus \{(v,v) : v \in V(\sigma)\}\). The cardinality of \( V(\sigma) \) is denoted by \( v(\sigma) \). The set of the equivalence classes of \( \equiv_\sigma \) is denoted by \( E(\sigma) \). Given a 2-structure \( \sigma \), with each \( W \in V(\sigma) \) associate the 2-substructure \( \sigma[W] \) of \( \sigma \) induced by \( W \)
defined on \(V(\sigma[W]) = W\) such that
\[
(\equiv_{\sigma[W]}) = (\equiv_{\sigma})_{T(W \times W \setminus \{(w,w) : w \in W\}}.
\]
Given \(W \subseteq V(\sigma)\), \(\sigma[V(\sigma) \setminus W]\) is denoted by \(\sigma-W\), and by \(\sigma-w\) when \(W = \{w\}\).

A graph \(\Gamma = (V(\Gamma), E(\Gamma))\) is identified with the 2-structure \(\sigma_T\) defined on \(V(\sigma_T) = V(\Gamma)\) as follows. For \(u, v, x, y \in V(\Gamma)\) such that \(u \neq v\) and \(x \neq y\), \((u, v) \equiv_{\sigma_T} (x, y)\) if \(\{u, v\}, \{x, y\} \in E(\Gamma)\) or \(\{u, v\}, \{x, y\} \notin E(\Gamma)\). Similarly, a tournament \(T = (V(T), A(T))\) is identified with the 2-structure \(\sigma_T\) defined on \(V(\sigma_T) = V(T)\) as follows. For \(u, v, x, y \in V(T)\) such that \(u \neq v\) and \(x \neq y\), \((u, v) \equiv_{\sigma_T} (x, y)\) if \((u, v), (x, y) \in A(T)\) or \((u, v), (x, y) \notin A(T)\).

1.1 Prime 2-structures

We remind the important results on prime 2-structures.

Convention. Let \(\sigma\) be a 2-structure. For \(X \subseteq V(\sigma)\), \(\overline{X}\) denotes \(V(\sigma) \setminus X\).

Let \(\sigma\) be a 2-structure. A subset \(M\) of \(V(\sigma)\) is a module [18] of \(\sigma\) if for any \(x, y \in M\) and \(v \in M\), we have
\[
(x, v) \equiv_{\sigma} (y, v) \text{ and } (v, x) \equiv_{\sigma} (v, y).
\]
For instance, \(\emptyset, V(\sigma)\) and \(\{v\}\), for \(v \in V(\sigma)\), are modules of \(\sigma\) called trivial modules of \(\sigma\). A 2-structure \(\sigma\) is prime if \(v(\sigma) \geq 3\) and all the modules of \(\sigma\) are trivial. The main definitions follow.

Definition 1. Given a prime 2-structure \(\sigma\), a vertex \(v\) of \(\sigma\) is critical (in terms of primality) if \(\sigma-v\) is not prime. More generally, a subset \(W\) of \(V(\sigma)\) is critical if \(\sigma-W\) is not prime. A prime 2-structure is critical if all its vertices are critical.

Let \(\sigma\) be a prime 2-structure. Given \(W \subseteq V(\sigma)\), \(\sigma\) is \(W\)-critical if all the elements of \(W\) are critical vertices of \(\sigma\). Lastly, a prime 2-structure \(\sigma\) is partially critical if there exists \(X \subseteq V(\sigma)\) such that \(\sigma[X]\) is prime, and \(\sigma\) is \(\overline{X}\)-critical.

Notation 2. Let \(\sigma\) be a 2-structure. With \(X \subseteq V(\sigma)\) such that \(\sigma[X]\) is prime, associate the following subsets of \(\overline{X}\)

- \(\text{Ext}_{\sigma}(X)\) is the set of \(v \in \overline{X}\) such that \(\sigma[X \cup \{v\}]\) is prime;
- \(\langle X \rangle_{\sigma}\) is the set of \(v \in \overline{X}\) such that \(X\) is a module of \(\sigma[X \cup \{v\}]\);
- given \(\alpha \in X\), \(X_{\sigma}(\alpha)\) is the set of \(v \in \overline{X}\) such that \(\{\alpha, v\}\) is a module of \(\sigma[X \cup \{v\}]\).

The set \(\{\text{Ext}_{\sigma}(X), \langle X \rangle_{\sigma} \cup \{X_{\sigma}(\alpha) : \alpha \in X\}\}\) is denoted by \(p_{(\sigma, \overline{X})}\). It is called the outside partition.

The next result (see [3] Lemmas 6.3 and 6.4) is basic in the study of primality.
Lemma 3. Given a 2-structure $\sigma$, consider $X \subseteq V(\sigma)$ such that $\sigma[X]$ is prime. The set $p_{(\sigma,X)}$ is a partition of $\overline{X}$. Moreover, the three assertions below hold

1. for $v \in \langle X \rangle_{\sigma}$ and $w \in \overline{X} \setminus \langle X \rangle_{\sigma}$, if $\sigma[X \cup \{v, w\}]$ is not prime, then $X \cup \{w\}$ is a module of $\sigma[X \cup \{v, w\}]$;
2. given $\alpha \in X$, for $v \in X_{\sigma}(\alpha)$ and $w \in \overline{X} \setminus X_{\sigma}(\alpha)$, if $\sigma[X \cup \{v, w\}]$ is not prime, then $\{\alpha, v\}$ is a module of $\sigma[X \cup \{v, w\}]$;
3. for distinct $v, w \in \text{Ext}_{\sigma}(X)$, if $\sigma[X \cup \{v, w\}]$ is not prime, then $\{v, w\}$ is a module of $\sigma[X \cup \{v, w\}]$.

The classic parity theorem \cite[Theorem 6.5]{8} follows from Lemma 3.

Theorem 4. Given a 2-structure $\sigma$, consider $X \subseteq V(\sigma)$ such that $\sigma[X]$ is prime and $|X| \geq 2$. If $\sigma$ is prime, then there exist distinct $v, w \in \overline{X}$ such that $\sigma[X \cup \{v, w\}]$ is prime.

Theorem 4 leads us to introduce the outside graph as follows. We need the next notation.

Notation 5. Given a 2-structure $\sigma$, consider $X \subseteq V(\sigma)$ such that $\sigma[X]$ is prime. The set of the nonempty subsets $Y$ of $X$, such that $\sigma[X \cup Y]$ is prime, is denoted by $\epsilon(\sigma, X)$. Hence $\text{Ext}_{\sigma}(X) = \{v \in \overline{X} : \{v\} \in \epsilon(\sigma, X)\}$. Furthermore, suppose that $|\overline{X}| \geq 2$. By Theorem 4, $\epsilon(\sigma, X)$ contains an unordered pair.

Definition 6. Given a 2-structure $\sigma$, consider $X \subseteq V(\sigma)$ such that $\sigma[X]$ is prime. The outside graph $\Gamma(\sigma, X)$ is defined on $\overline{X}$ by

$$E(\Gamma(\sigma, X)) = \{Y \in \epsilon(\sigma, X) : |Y| = 2\}.$$ 

By Theorem 4, $\Gamma(\sigma, X)$ is nonempty when $|\overline{X}| \geq 2$. The outside graph is a common tool in the study of prime graphs \cite{12, 15}.

By applying Theorem 4 several times, we obtain the following result.

Corollary 7. Given a 2-structure $\sigma$, consider $X \subseteq V(\sigma)$ such that $\sigma[X]$ is prime. Suppose that $|\overline{X}|$ is finite, with $|\overline{X}| \geq 2$. If $\sigma$ is prime, then there exist $v, w \in \overline{X}$ such that $\sigma - \{v, w\}$ is prime.

Schmerl and Trotter \cite{19} characterized the finite and critical 2-structures (see Definition 1). Using their characterization, they obtained the following improvement of Corollary 7 which is an important result on the finite and prime 2-structures.

Theorem 8. Given a finite and prime 2-structure $\sigma$, if $v(\sigma) \geq 7$, then there exist distinct vertices $v$ and $w$ of $\sigma$ such that $\sigma - \{v, w\}$ is prime.

In the next theorem, Ille \cite{12} succeeded in localizing a non-critical unordered pair outside a prime 2-substructure. Initially, it was established for finite digraphs. The same proof holds for finite 2-structures.
Theorem 9. Given a prime 2-structure $\sigma$, consider $X \subseteq V(\sigma)$ such that $\sigma[X]$ is prime. If $X$ is finite and $|X| \geq 6$, then there exist distinct $v, w \in X$ such that $\sigma - \{v, w\}$ is prime.

Sayar [17] improved Theorem 9 for finite tournaments as follows.

Theorem 10. Given a prime tournament $T$, consider $X \subseteq V(T)$ such that $T[X]$ is prime. If $X$ is finite and $|X| \geq 4$, then there exist distinct $v, w \in X$ such that $T - \{v, w\}$ is prime.

We extend Theorem 10 to particular 2-structures in Appendix B (see Theorem 82).

Remark 11. Given a 2-structure $\sigma$, consider $X \subseteq V(\sigma)$ such that $\sigma[X]$ is prime, and $X$ is finite.

Suppose that $\sigma$ is $X$-critical. For a contradiction, suppose that $|X|$ is odd. By applying several times Theorem 4 from $\sigma[X]$, we obtain a non-critical vertex $v$ of $\sigma$ such that $v \in X$, which contradicts the fact that $\sigma$ is $X$-critical. It follows that $|X|$ is even.

Now, consider $Y \subseteq X$ such that $\sigma[X \cup Y]$ is prime. Since $\sigma$ is $X$-critical, $\sigma$ is $(X \cup Y)$-critical as well. Therefore $|X \cup Y|$ is even. Since $|X|$ is even, $|Y|$ is even too. Consequently, for each $k \in \{1, \ldots, |X| - 1\}$ such that $k$ is odd, we have the following statement

$$\{Y \in \varepsilon(\sigma, X) : |Y| = k\} = \emptyset.$$  \hspace{1cm} (Sk)

Clearly, Ext$_\sigma(X) = \emptyset$ means that Statement (S1) holds.

Lastly, consider $k \in \{1, \ldots, |X| - 1\}$ such that $k$ is odd. Suppose that Statement (Sk) holds. It follows from Theorem 4 that Statement (Sm) holds for every odd integer $m \in \{1, \ldots, k - 2\}$.

1.2 Infinite and prime 2-structures

Concerning infinite and prime 2-structures, Ille [11, 14] obtained the following two theorems. Initially, they were proved for digraphs. The same proofs hold for 2-structures.

Theorem 12. Given a prime 2-structure $\sigma$, consider $X \subseteq V(\sigma)$ such that $\sigma[X]$ is prime. For each $x \in X$, there exists $F \in \varepsilon(\sigma, X)$ such that $F$ is finite and $x \in F$.

The next result follows from Theorem 12.

Corollary 13. Given a 2-structure $\sigma$, consider $X \subseteq V(\sigma)$ such that $\sigma[X]$ is prime. The following two assertions are equivalent

1. $\sigma$ is prime;
2. for each finite subset $F$ of $X$, there exists $F' \in \varepsilon(\sigma, X)$ such that $F'$ is finite and $F \subseteq F'$. 

5
The next compactness result follows from Corollary 13.

**Theorem 14.** Given an infinite 2-structure \( \sigma \), the following two assertions are equivalent

1. \( \sigma \) is prime;
2. for each finite subset \( F \) of \( V(\sigma) \), there exists a finite subset \( F' \) of \( V(\sigma) \) satisfying \( F \subseteq F' \) and \( \sigma[F'] \) is prime.

**Definition 15.** Given an infinite and prime 2-structure \( \sigma \), \( \sigma \) is **finitely critical** if \( \sigma - F \) is not prime for every nonempty and finite subset \( F \) of \( V(\sigma) \). It follows from Theorem 4 that a prime 2-structure \( \sigma \) is finitely critical if and only if \( \sigma - \{v, w\} \) is not prime for any \( v, w \in V(\sigma) \).

Boudabbous and Ile [4] characterized the critical digraphs that are not finitely critical, that is, the infinite and prime digraphs \( D \) satisfying

- for each \( v \in V(D) \), \( D - v \) is not prime;
- there exist (distinct) \( v, w \in V(D) \) such that \( D - \{v, w\} \) is prime.

### 1.3 Main results

We begin with a hereditary property of primality through the components of the outside graph, which constitutes the central result of the paper.

**Theorem 16.** Given a 2-structure \( \sigma \), consider \( X \subset V(\sigma) \) such that \( \sigma[X] \) is prime. Suppose that Statement (S3) holds. The following three assertions are equivalent

1. \( \sigma \) is prime;
2. for each component \( C \) of \( \Gamma(\sigma, X) \), \( \sigma[X \cup V(C)] \) is prime;
3. for each component \( C \) of \( \Gamma(\sigma, X) \), \( v(C) = 2 \) or \( v(C) \geq 4 \) and \( C \) is prime.

Theorem 16 allows us to provide a simple and short proof of Theorem 9 (see Appendix [1]). Furthermore, Theorem 16 is proved for finite graphs in [15] (see [15, Theorem 17] and [15, Corollary 18]). We pursue with a hereditary property of partial criticality through the components of the outside graph.

**Theorem 17.** Given a 2-structure \( \sigma \), consider \( X \subset V(\sigma) \) such that \( \sigma[X] \) is prime. Suppose that Statement (S5) holds. The following three assertions are equivalent

1. \( \sigma \) is \( X \)-critical;
2. for each component \( C \) of \( \Gamma(\sigma, X) \), \( \sigma[X \cup V(C)] \) is \( V(C) \)-critical;
3. for each component \( C \) of \( \Gamma(\sigma, X) \), \( v(C) = 2 \) or \( v(C) \geq 4 \) and \( C \) is critical.
Given a 2-structure $\sigma$, consider $X \subseteq V(\sigma)$ such that $\sigma[X]$ is prime. Suppose that Statement (S5) holds. Suppose also that $\sigma$ is $\overline{X}$-critical. Consider a component $C$ of $C(\Gamma_{(\sigma,X)})$ such that $\nu(C) \geq 4$. It follows from Theorem 17 that $C$ is critical. Moreover, since Statement (S5) holds, $P_5 \nsubseteq C$ (see Lemma 53), where for $n \geq 2$, $P_n$ denotes the path on $n$ vertices. In Theorem 22 below, we characterize the bipartite graphs $\Gamma$ such that $P_5 \nsubseteq \Gamma$ and $\Gamma$ is critical. We need the following three definitions.

**Definition 18.** Given a bipartite graph $\Gamma$, with bipartition $\{X,Y\}$, $\Gamma$ is a *half graph* [9] if there exist a linear order $L$ defined on $X$, and a bijection $\varphi$ from $X$ onto $Y$ such that

$$E(\Gamma) = \{\{x,\varphi(x')\} : x \leq x' \mod L\}. \quad (1)$$

**Remark 19.** Given a bipartite graph $\Gamma$, with bipartition $\{X,Y\}$. Suppose that $\Gamma$ is a half graph. There exist a linear order $L$ defined on $X$, and a bijection $\varphi$ from $X$ onto $Y$ such that (1) holds. Given $x, y \in X$, we obtain that

$$x \leq y \mod L \text{ if and only if } N_\Gamma(x) \supseteq N_\Gamma(y).$$

Therefore, the linear order $L$ is unique.

**Definition 20.** A linear order $L$ is *discrete* [10] if the following two conditions are satisfied

1. for every $v \in V(L)$, if $v$ is not the smallest element of $L$, then $v$ admits a predecessor;
2. for every $v \in V(L)$, if $v$ is not the largest element of $L$, then $v$ admits a successor.

**Definition 21.** A half graph is *discrete* if the linear order $L$ in Definition 18 is discrete.

**Theorem 22.** Given a bipartite graph $\Gamma$, with $\nu(\Gamma) \geq 4$, the following assertions are equivalent

1. $\Gamma$ is a discrete half graph;
2. $P_5 \nsubseteq \Gamma$ and $\Gamma$ is critical.

We establish Theorem 22 in Section 5. The next result follows from Theorems 16 and 17, Proposition 57, and Lemma 53.

**Corollary 23.** Given a 2-structure $\sigma$, consider $X \subseteq V(\sigma)$ such that $\sigma[X]$ is prime. Suppose that $\overline{X}$ is finite.

The following two assertions are equivalent

1. Statement (S5) holds, and $\sigma$ is prime;
2. \( \sigma \) is \( \overline{X} \)-critical.

**Remark 24.** Consider the path \( P_Z = (Z, \{(p,q) : |p-q|=1\}) \). We show that \( P_Z \) is prime by using Theorem 14. Indeed, let \( F \) be a finite and nonempty subset of \( P_Z \). There exist \( p, q \in Z \) such that \( p \leq \min(F) \), \( q \geq \max(F) \) and \( q - p \geq 3 \). Clearly, \( F \subseteq \{p, \ldots , q\} \), and \( P_Z[p, \ldots , q] \cong P_{q-p+1} \). Since \( q - p + 1 \geq 4 \), \( P_{q-p+1} \) and hence \( P_Z[p, \ldots , q] \) are prime. By Theorem 14, \( P_Z \) is prime.

For every \( z \in Z \), \( P_Z - z \) is disconnected, and hence \( P_Z - z \) is not prime. Consequently, \( P_Z \) is critical. In fact, \( P_Z \) is finitely critical.

Set \( X = \{z \in Z : z \leq 0\} \). By Theorem 14, \( P_Z[X] \) is prime. Since \( P_Z \) is critical, \( P_Z \) is \( \overline{X} \)-critical. For every \( k > 0 \), \( P_Z[X \cup \{1, \ldots , k\}] \) is prime by Theorem 14. Consequently, for every \( k > 0 \), Statement (Sk) does not hold. Moreover, \( \{1, 2\} \) is the only edge of \( \Gamma_{(P_Z,X)} \). Hence, for every \( z \geq 3 \), \( z \) is an isolated vertex of \( \Gamma_{(P_Z,X)} \). It follows that Theorem 16 does not hold when Statement (S3) is not satisfied. Similarly, Theorem 17 does not hold when Statement (S5) is not satisfied.

Corollary 13 and the fact that Statement (S5) is supposed to be satisfied in Theorem 17 lead us to introduce the next definition. The next definition is a weakening of the partial criticality (see Theorem 20).

**Definition 25.** Given a 2-structure \( \sigma \), consider \( X \subseteq V(\sigma) \) such that \( \sigma[X] \) is prime. We say that \( \sigma \) is finitely \( \overline{X} \)-critical if for each finite subset \( F \) of \( \overline{X} \), there exists a finite subset \( F' \) of \( \overline{X} \) such that \( F \subseteq F' \) and \( \sigma[X \cup F'] \) is \( (F') \)-critical.

The next result follows from Corollaries 13 and 20.

**Theorem 26.** Given a 2-structure \( \sigma \), consider \( X \subseteq V(\sigma) \) such that \( \sigma[X] \) is prime. The following two assertions are equivalent

1. Statement (S5) holds, and \( \sigma \) is prime;

2. \( \sigma \) is finitely \( \overline{X} \)-critical.

Theorem 26 is discussed in Remark 20. Precisely, in Remark 20, we provide a prime 2-structure showing that we do not have a compactness theorem with partial criticality.

The last main result is an immediate consequence of Theorem 17 and Claim 14.

**Theorem 27.** Given a 2-structure \( \sigma \), consider \( X \subseteq V(\sigma) \) such that \( \sigma[X] \) is prime. Suppose that Statement (S5) holds. Suppose also that \( \sigma \) is \( \overline{X} \)-critical. For each \( x \in \overline{X} \), there exists \( y \in \overline{X} \setminus \{x\} \) such that \( \sigma - (x,y) \) is \( (\overline{X} \setminus \{x,y\}) \)-critical.

**Remark 28.** Given a 2-structure \( \sigma \), consider \( X \subseteq V(\sigma) \) such that \( \sigma[X] \) is prime. Suppose that Statement (S5) holds. Suppose also that \( \sigma \) is \( \overline{X} \)-critical. Lastly, suppose that \( \overline{X} \) is infinite. Consider a finite and nonempty subset \( F \) of \( \overline{X} \). By applying several times Theorem 27, we obtain a finite subset \( F' \) of \( \overline{X} \) such that \( F \subseteq F' \) and \( \sigma - F' \) is \( (\overline{X} \setminus F') \)-critical. Furthermore, it follows from Theorem 4 that \( |F'| \) is even.
In Appendix A we describe simply partially critical 2-structures. A nice presentation of finite and partially critical 2-structures is provided in [1].

**Warning.** As mentioned at the beginning of Section 1, we adopt the same approach as that of [7] to examine finite and partially critical 2-structures. In what follows, we omit the proof of a result when it is closed to that provided in [7].

# 2 Preliminaries

We use the following notation.

**Notation 29.** Let \( \sigma \) be a 2-structure. For \( W, W' \subseteq V(\sigma) \), with \( W \cap W' = \emptyset \), \( W \leftrightarrow_\sigma W' \) signifies that \((v, v') \equiv_\sigma (w, w') \) and \((v', v) \equiv_\sigma (w', w) \) for any \( v, w \in W \) and \( v', w' \in W' \). Given \( v \in V(\sigma) \) and \( W \subseteq V(\sigma) \setminus \{v\} \), \( v \leftrightarrow_\sigma W \) is also denoted by \( v \rightarrow_\sigma W \). The negation is denoted by \( v \leftarrow_\sigma W \).

Given distinct vertices \( v, w \) of \( \sigma \), the equivalence class of \((v, w)\) is denoted by \((v, w)_{\sigma}\). If we consider \( \sigma \) as the function from \((V(\sigma) \times V(\sigma)) \setminus \{(v, v) : v \in V(\sigma)\} \) to \( E(\sigma) \), which maps \((v, w)\) to \((v, w)_\sigma\), then \( \sigma \) becomes a 2-structure labeled by \( E(\sigma) \). Given distinct vertices \( v \) and \( w \) of \( \sigma \), set

\[
[v, w]_{\sigma} = (((v, w)_{\sigma}, (w, v)_{\sigma})).
\]

Given \( W, W' \subseteq V(\sigma) \) such that \( W \leftrightarrow_\sigma W' \), \((W, W')_{\sigma}\) denotes the equivalence class of \((w, w')\), where \( w \in W \) and \( w' \in W' \). Furthermore, set

\[
[W, W']_{\sigma} = (((W, W')_{\sigma}, (W', W)_{\sigma})).
\]

Lastly, given \( v \in V(\sigma) \) and \( W \subseteq V(\sigma) \setminus \{v\} \) such that \( v \leftrightarrow_\sigma W \), \((\{v\}, W)_{\sigma}\) is also denoted by \((v, W)_{\sigma}\), and \([\{v\}, W]_{\sigma}\) is also denoted by \([v, W]_{\sigma}\).

Let \( \sigma \) be a 2-structure. Using Notation 29, a subset \( M \) of \( V(\sigma) \) is a module of \( \sigma \) if and only if for each \( v \in M \), we have \( v \leftrightarrow_\sigma M \).

To continue, we examine the isolated vertices of an outside graph. We utilize the following remark.

**Remark 30.** Given a 2-structure \( \sigma \), consider \( X \subseteq V(\sigma) \) such that \( \sigma[X] \) is prime. Consider distinct \( x, y \in X \). If \( x, y \in (X)_{\sigma} \), then \( X \) is a module of \( \sigma[X \cup \{x, y\}] \).

Given \( \alpha \in X \), if \( x, y \in X_{\sigma}(\alpha) \), then \( \langle \alpha, x, y \rangle \) is a module of \( \sigma[X \cup \{x, y\}] \).

Consequently, for each \( B \in p(\sigma, X) \setminus \{\text{Ext}_\sigma(X)\} \), \( \Gamma(\sigma, X, B) \) is empty. In other words, if \( \text{Ext}_\sigma(X) = \emptyset \), then \( \Gamma(\sigma, X) \) is multipartite with partition \( p(\sigma, X) \) (see Lemma 3).

The proof of the next lemma is analogous to that of [7] Lemma 2.7.

**Lemma 31.** Given a 2-structure \( \sigma \), consider \( X \subseteq V(\sigma) \) such that \( \sigma[X] \) is prime.

1. If \( M \) is a module of \( \sigma \) such that \( X \subseteq M \), then the elements of \( \overline{M} \) are isolated vertices of \( \Gamma(\sigma, X) \).
2. Given \( \alpha \in X \), if \( M \) is a module of \( \sigma \) such that \( M \cap X = \{ \alpha \} \), then the elements of \( M \setminus \{ \alpha \} \) are isolated vertices of \( \Gamma_{(\sigma, \mathbb{N})} \).

The next result is an immediate consequence of Lemma 31.

**Corollary 32.** Given a 2-structure \( \sigma \), consider \( X \subseteq V(\sigma) \) such that \( \sigma[X] \) is prime. If \( \sigma \) admits a nontrivial module \( M \) such that \( M \cap X \neq \emptyset \), then \( \Gamma_{(\sigma, \mathbb{N})} \) possesses isolated vertices.

Now, we study the modules of the outside graph. We need the following refinement of the outside partition (see Notation 2).

**Notation 33.** Given a 2-structure \( \sigma \), consider \( X \subseteq V(\sigma) \) such that \( \sigma[X] \) is prime. We consider the following subsets of \( X \):

- for \( e, f \in E(\sigma) \), \( \langle X \rangle_{(\sigma, f)}^{(e, f)} \) is the set of \( v \in \langle X \rangle_\sigma \) such that \( (v, \alpha) \in e \) and \( (\alpha, v) \in f \), where \( \alpha \in X \);
- for \( e, f \in E(\sigma) \) and \( \alpha \in X \), \( X^{(e, f)}_{(\sigma, f)}(\alpha) \) is the set of \( v \in X_\sigma(\alpha) \) such that \( (v, \alpha) \in e \) and \( (\alpha, v) \in f \).

The set \( \{\text{Ext}_\sigma(X)\} \cup \{\langle X \rangle_{(\sigma, f)}^{(e, f)} : e, f \in E(\sigma)\} \cup \{X^{(e, f)}_{(\sigma, f)}(\alpha) : e, f \in E(\sigma), \alpha \in X\} \) is denoted by \( q_{(\sigma, \mathbb{N})} \).

**Lemma 34.** Given a 2-structure \( \sigma \), consider \( X \subseteq V(\sigma) \) such that \( \sigma[X] \) is prime. Suppose that Statement (S1) holds. Given \( M \subseteq \mathbb{N} \), if \( M \) is a module of \( \sigma \), then \( M \) is a module of \( \Gamma_{(\sigma, \mathbb{N})} \), and there exist \( B_q \in p_{(\sigma, \mathbb{N})} \) and \( B_q \in q_{(\sigma, \mathbb{N})} \) such that \( M \subseteq B_q \subseteq B_p \), and \( M \) is a module of \( \sigma[B_p] \).

**Proof.** Consider a module \( M \) of \( \sigma \) such that \( M \cap X = \emptyset \). Let \( x \in M \). Denote by \( B_q \) the unique block of \( q_{(\sigma, \mathbb{N})} \) containing \( x \). Consider \( y \in M \setminus \{x\} \). Since \( M \) is a module of \( \sigma \) such that \( M \cap X = \emptyset \), we have \( \alpha \leftrightarrow_{(\sigma, \mathbb{N})} \{x, y\} \) for every \( \alpha \in X \). It follows that \( y \in B_q \). Consequently \( M \subseteq B_q \). Denote by \( B_p \) the unique block of \( p_{(\sigma, \mathbb{N})} \) containing \( B_q \). We obtain

\[
M \subseteq B_q \subseteq B_p.
\]

Since \( M \) is a module of \( \sigma \), \( M \) is a module of \( \sigma[B_p] \).

Lastly, we prove that \( M \) is a module of \( \Gamma_{(\sigma, \mathbb{N})} \). Let \( v \in \mathbb{N} \setminus M \). Recall that \( \text{Ext}_\sigma(X) = \emptyset \) because Statement (S1) holds. If \( v \in B_p \), then it follows from Remark 30 that \( \{y, v\} \notin E(\Gamma_{(\sigma, \mathbb{N})}) \) for every \( y \in M \). Hence suppose that \( v \in \mathbb{N} \setminus B_p \). Since \( \text{Ext}_\sigma(X) = \emptyset \), we distinguish the following two cases.

- Suppose that \( B_p = \langle X \rangle_\sigma \). Let \( \alpha \in X \). Recall that \( x \in M \).

  First, suppose that \( x \leftrightarrow_\sigma \{\alpha, v\} \). Let \( y \in M \). Since \( M \) is a module of \( \sigma \), we obtain \( y \leftrightarrow_\sigma \{\alpha, v\} \). Since \( y \leftrightarrow_\sigma X \), we obtain \( y \leftrightarrow_\sigma X \cup \{v\} \). Hence \( X \cup \{v\} \) is a module of \( \sigma[X \cup \{y, v\}] \). It follows that \( \{y, v\} \notin E(\Gamma_{(\sigma, \mathbb{N})}) \) for every \( y \in M \).
Second, suppose that \( x \leftrightarrow_\sigma \{ \alpha, v \} \). Let \( y \in M \). Since \( M \) is a module of \( \sigma \), we obtain \( y \leftrightarrow_\sigma \{ \alpha, v \} \). Hence \( X \cup \{ v \} \) is not a module of \( \sigma[X \cup \{ y, v \}] \). It follows from the first assertion of Lemma 3 that \( \{ y, v \} \in E(\Gamma_{(\sigma, X)}\sigma) \) for every \( y \in M \).

- Suppose that \( B_p = X_\sigma(\alpha) \), where \( \alpha \in X \). Recall that \( x \in M \).
  
  First, suppose that \( v \leftrightarrow_\sigma \{ \alpha, x \} \). Let \( y \in M \). Since \( M \) is a module of \( \sigma \), we obtain \( v \leftrightarrow_\sigma \{ \alpha, y \} \). Since \( \{ \alpha, y \} \) is a module of \( \sigma[X \cup \{ y \}] \), \( \{ \alpha, y \} \) is a module of \( \sigma[X \cup \{ y, v \}] \). It follows that \( \{ y, v \} \notin E(\Gamma_{(\sigma, X)}\sigma) \) for every \( y \in M \).

  Second, suppose that \( v \leftrightarrow_\sigma \{ \alpha, x \} \). Let \( y \in M \). Since \( M \) is a module of \( \sigma \), we obtain \( v \leftrightarrow_\sigma \{ \alpha, y \} \). Thus \( \{ \alpha, y \} \) is not a module of \( \sigma[X \cup \{ y, v \}] \). It follows from the second assertion of Lemma 3 that \( \{ y, v \} \in E(\Gamma_{(\sigma, X)}\sigma) \) for every \( y \in M \).

The opposite direction in Lemma 34 is false. Nevertheless, it is true for (finite) graphs (see the second assertion of [7, Lemma 2.6]). Moreover, the opposite direction in Lemma 34 is true if we require that Statement (S3) holds (see Corollary 37 below).

### 3 The first results

The proof of the next fact is analogous to that of [7, Lemma 4.3].

**Fact 35.** Given a 2-structure \( \sigma \), consider \( X \notin V(\sigma) \) such that \( \sigma[X] \) is prime. Suppose that Statement (S3) holds. Given distinct elements \( x, y, z \) of \( \overline{X} \), if \( \{ x, y \}, \{ x, z \} \in E(\Gamma_{(\sigma, X)}\sigma) \), then \( \{ y, z \} \) is a module of \( \sigma[X \cup \{ x, y, z \}] \), and hence there exists \( B_q = q_{(\sigma, X)} \sigma \) such that \( y, z \in B_q \).

The proof of the next fact is analogous to that of [7, Lemma 4.4].

**Fact 36.** Given a 2-structure \( \sigma \), consider \( X \notin V(\sigma) \) such that \( \sigma[X] \) is prime. Suppose that Statement (S3) holds. Given \( B_p, D_p \in p_{(\sigma, X)} \), consider \( x \in B_p \) and \( y, z \in D_p \) such that \( \{ x, y \} \in E(\Gamma_{(\sigma, X)}\sigma) \) and \( \{ x, z \} \notin E(\Gamma_{(\sigma, X)}\sigma) \).

1. If \( D_p = (X)_\sigma \), then \( X \cup \{ x, y \} \) is a module of \( \sigma[X \cup \{ x, y, z \}] \).

2. If \( D_p = X_\sigma(\alpha) \), where \( \alpha \in X \), then \( \{ \alpha, z \} \) is a module of \( \sigma[X \cup \{ x, y, z \}] \).

The next result follows from Fact 35.

**Corollary 37.** Given a 2-structure \( \sigma \), consider \( X \notin V(\sigma) \) such that \( \sigma[X] \) is prime. Suppose that Statement (S3) holds. Consider \( M \subseteq \overline{X} \) such that there exist \( B_p \in p_{(\sigma, X)} \) and \( B_q \in q_{(\sigma, X)} \) with \( M \subseteq B_q \subseteq B_p \). Suppose that \( M \) is a module of \( \sigma[B_p] \). If \( M \) is a module of \( \Gamma_{(\sigma, X)} \), then \( M \) is a module of \( \sigma \).
Proof. Consider \( x, y \in M \) and \( v \in M \). It suffices to verify that
\[
v \longrightarrow_\sigma \{ x, y \}.
\]
(2)

Since \( M \) is a module of \( \sigma[B_p] \), (2) holds when \( v \in B_p \setminus M \). Furthermore, since \( x \) and \( y \) belong to the same block of \( q[\sigma,X] \), (2) holds when \( v \in X \).

Now, suppose that \( v \in X \cup B_p \). Since \( M \) is a module of \( \Gamma(\sigma,X) \), we have
\[
\{ x, v \}, \{ y, v \} \in E(\Gamma(\sigma,X))
\]
or
\[
\{ x, v \}, \{ y, v \} \notin E(\Gamma(\sigma,X)).
\]
(3)

Suppose that \( \{ x, v \}, \{ y, v \} \in E(\Gamma(\sigma,X)) \). By Fact \ref{fact:module_of_sigmaX} \( \{ x, y \} \) is a module of \( \sigma[X \cup \{ x, y \}] \), so \( v \longrightarrow_\sigma \{ x, y \} \).

Lastly, suppose that \( \{ x, v \}, \{ y, v \} \notin E(\Gamma(\sigma,X)) \). Since Statement (S3) holds, Statement (S1) holds by Remark \ref{remark:module_of_sigmaX}. Hence \( \text{Ext}_\sigma(X) = \emptyset \), and we distinguish the following two cases.

1. Suppose that \( B_p = \{ X \} _\sigma \). Since \( \{ x, v \}, \{ y, v \} \notin E(\Gamma(\sigma,X)) \), it follows from the first assertion of Lemma \ref{lemma:module_of_sigmaX} that \( X \cup \{ v \} \) is a module of \( \sigma[X \cup \{ x, v \}] \) and \( \sigma[X \cup \{ y, v \}] \). Given \( \alpha \in X \), we obtain \( x \longleftrightarrow_\sigma \{ \alpha, v \} \) and \( y \longleftrightarrow_\sigma \{ \alpha, v \} \).

Since \( x, y \in B_q \) and \( B_q \subseteq \{ X \} _\sigma \), \( \alpha \longleftrightarrow_\sigma \{ x, y \} \). It follows that \( v \longleftrightarrow_\sigma \{ x, y \} \).

Consequently, (2) holds when \( v \in X \cup B_p \) and \( B_p = \{ X \} _\sigma \).

2. Suppose that \( B_p = X_\sigma(\alpha) \), where \( \alpha \in X \). Since \( \{ x, v \}, \{ y, v \} \notin E(\Gamma(\sigma,X)) \), it follows from the second assertion of Lemma \ref{lemma:module_of_sigmaX} that \( \{ \alpha, x \} \) is a module of \( \sigma[X \cup \{ x, v \}] \), and \( \{ \alpha, y \} \) is a module of \( \sigma[X \cup \{ y, v \}] \). Therefore \( v \longleftrightarrow_\sigma \{ \alpha, x \} \) and \( v \longleftrightarrow_\sigma \{ \alpha, y \} \).

Consequently, (2) holds when \( v \in X \cup B_p \) and \( B_p = X_\sigma(\alpha) \).

The next two results follow from Fact \ref{fact:module_of_sigmaX}.

**Corollary 38.** Given a 2-structure \( \sigma \), consider \( X \not\subseteq V(\sigma) \) such that \( \sigma[X] \) is prime. Suppose that Statement (S3) holds. Let \( B_q \in q[\sigma,X] \). For each \( v \in X \setminus B_q \), \( \{ x \in B_q : \{ x, v \} \in E(\Gamma(\sigma,X)) \} \) and \( \{ x \in B_q : \{ x, v \} \notin E(\Gamma(\sigma,X)) \} \) are modules of \( \sigma[B_q] \). Precisely, if \( \{ x \in B_q : \{ x, v \} \in E(\Gamma(\sigma,X)) \} \neq \emptyset \) and \( \{ x \in B_q : \{ x, v \} \notin E(\Gamma(\sigma,X)) \} \neq \emptyset \), then the following two assertions hold.

1. If \( B_q = \{ X \} _{\sigma(e,f)} \), where \( e, f \in E(\sigma) \), then
\[
[\{ x \in B_q : \{ x, v \} \notin E(\Gamma(\sigma,X)) \}, \{ x \in B_q : \{ x, v \} \in E(\Gamma(\sigma,X)) \}]_\sigma = (e,f).
\]

2. If \( B_q = X(\sigma(e,f))(\alpha) \), where \( \alpha \in X \) and \( e, f \in E(\sigma) \), then
\[
[\{ x \in B_q : \{ x, v \} \notin E(\Gamma(\sigma,X)) \}, \{ x \in B_q : \{ x, v \} \in E(\Gamma(\sigma,X)) \}]_\sigma = (f,e).
\]
Corollary 39. Consider Since \(x, x\) \(E\) (since \(\Gamma\) and \(\{B\}\) isolated vertices. \[\\\\\\\
\]

Proof. Let \(v \in X \setminus B_q\). Suppose that \(\{x \in B_q : \{x, v\} \in E(\Gamma_{(\sigma, X)})\} \neq \emptyset\) and \(\{x \in B_q : \{x, v\} \notin E(\Gamma_{(\sigma, X)})\} \neq \emptyset\). Consider \(x^+, z^- \in B_q\) such that \(\{x^+, v\} \in E(\Gamma_{(\sigma, X)})\) and \(\{z^-, v\} \notin E(\Gamma_{(\sigma, X)})\). We distinguish the following two cases.

1. Suppose that \(B_q = (X)_{\sigma}^{(e, f)}\), where \(e, f \in E(\sigma)\). By the first assertion of Fact 39 applied to \(x^+, z^-, v\), \(X \cup \{x^+, v\}\) is a module of \(\sigma[X \cup \{x^+, z^-, v\}]\).

Since \(z^- \in (X)_{\sigma}^{(e, f)}\), we obtain \([z^-, x^+]_{\sigma} = (e, f)\).

2. Suppose that \(B_q = X_{\sigma}^{(e, f)}(\alpha)\), where \(\alpha \in X\) and \(e, f \in E(\sigma)\). By the second assertion of Fact 39 applied to \(x^+, z^-, v\), \(\{\alpha, z^-\}\) is a module of \(\sigma[X \cup \{x^+, z^-, v\}]\). Hence \([z^-, x^+]_{\sigma} = [\alpha, x^+]_{\sigma}\). Since \(x^+ \in X_{\sigma}^{(e, f)}(\alpha)\), we obtain \([\alpha, x^+]_{\sigma} = (f, e)\), so \([z^-, x^+]_{\sigma} = (f, e)\). \(\Box\)

The proof of the next corollary is analogous to that of \(\text{[7, Corollary 4.5]}\). It follows from Lemma 3 and Fact 39.

Corollary 39. Given a 2-structure \(\sigma\), consider \(X \not\in V(\sigma)\) such that \(\sigma[X]\) is prime. Suppose that Statement (S3) holds. If \(\sigma\) is prime, then \(\Gamma_{(\sigma, X)}\) has no isolated vertices.

We examine the blocks of the partitions \(p_{(\sigma, X)}\) and \(q_{(\sigma, X)}\) in the next three lemmas.

Lemma 40. Given a 2-structure \(\sigma\), consider \(X \not\in V(\sigma)\) such that \(\sigma[X]\) is prime. Suppose that Statement (S3) holds. Consider \(e, f \in E(\sigma)\), and \(\alpha \in X\). If \(\Gamma_{(\sigma, X)}\) does not have isolated vertices, then the following two assertions hold.

1. if \((X)_{\sigma}^{(e, f)} \neq \emptyset\), then \((X)_{\sigma}^{(e', f')} = \emptyset\) for any \((e', f') \in E(\sigma)\) such that \(\{e', f'\} \neq \{e, f\}\);

2. if \(X_{\sigma}^{(e, f)}(\alpha) \neq \emptyset\), then \(X_{\sigma}^{(e', f')}(\alpha) = \emptyset\) for any \((e', f') \in E(\sigma)\) such that \(\{e', f'\} \neq \{e, f\}\).

Proof. Consider \(e, f, e', f' \in E(\sigma)\). For the first assertion, suppose that there exist \(x \in (X)_{\sigma}^{(e, f)}\) and \(x' \in (X)_{\sigma}^{(e', f')}\). We have to prove that

\[\{e, f\} = \{e', f'\}.\]  \hspace{1cm} (4)

Since \(x, x' \in (X)_{\sigma}\), we have \(\{x, x'\} \notin E(\Gamma_{(\sigma, X)})\) by Remark 39. Furthermore, since \(\Gamma_{(\sigma, X)}\) does not have isolated vertices, there exist \(y, y' \notin X \setminus \{x, x'\}\) such that \(\{x, y\}, \{x', y'\} \in E(\Gamma_{(\sigma, X)}\). Suppose that \(y = y'\). We obtain \(\{y, x\}, \{y, x'\} \in E(\Gamma_{(\sigma, X)})\). It follows from Fact 39 that \((e, f) = (e', f')\), so (4) holds. We obtain the same conclusion when \(\{x, y'\} \in E(\Gamma_{(\sigma, X)}\) or \(\{x', y\} \in E(\Gamma_{(\sigma, X)}\). Thus, suppose that \(y \neq y'\), and \(\{x, y\}, \{x', y'\} \notin E(\Gamma_{(\sigma, X)}\). It follows from the first assertion of Fact 39 applied to \(x, x', y'\) that \(X \cup \{x', y'\}\) is a module of \(\sigma[X \cup \{x, x', y'\}]\). Since \(x \in (X)_{\sigma}^{(e, f)}\), we obtain \((x, x') \in e\) and \((x', x) \in f\).
Similarly, it follows from the first assertion of Fact 36 applied to \(x, x'\) that 
\((x', x) \in e'\) and \((x, x') \in f'\). Therefore \(e = f'\) and \(e' = f\). Consequently 41 holds.

For the second assertion, suppose that there exist \(x \in X_{\sigma}^{(e', f)}(\alpha)\) and \(x' \in X_{\sigma}^{(e', f')}(\alpha)\), where \(\alpha \in X\). We have to prove that 41 holds. Since \(x, x' \in X_{\sigma}(\alpha)\), we have \(\{x, x'\} \notin E(\Gamma_{(\sigma, \underline{X})})\) by Remark 30. Furthermore, since \(\Gamma_{(\sigma, \underline{X})}\) does not have isolated vertices, there exist \(y, y' \in \underline{X} \setminus \{x, x'\}\) such that \(\{x, y\}, \{x', y'\} \in E(\Gamma_{(\sigma, \underline{X})})\). Suppose that \(y = y'\). We obtain \(\{y, x\}, \{y, x'\} \in E(\Gamma_{(\sigma, \underline{X})})\). By Fact 35 \((e, f) = (e', f')\), so 41 holds. We obtain the same conclusion when \(\{x, y\}' \in E(\Gamma_{(\sigma, \underline{X})})\) or \(\{x', y\}' \in E(\Gamma_{(\sigma, \underline{X})})\). Now, suppose that \(y \neq y'\), and \(\{x, y\}', \{x', y\}' \notin E(\Gamma_{(\sigma, \underline{X})})\). It follows from the second assertion of Fact 36 applied to \(x, x'\), \(\{x, y\}\)' that \(\{\alpha, x\}\) is a module of \(\sigma[X \cup \{x, x', y\}']\). Hence \((x', \alpha) \equiv_{\sigma} (x', x)\) and \(\{x, x'\} \equiv_{\sigma} (x, x')\). Since \(x' \in X_{\sigma}^{(e', f')}(\alpha)\), we obtain \((x', x) \in e'\) and \((x, x') \in f'\). Similarly, it follows from the second assertion of Fact 36 applied to \(x, x', y\) that \((x, x') \in e\) and \((x', x) \in f\). Thus \(e = f'\) and \(e' = f\). Consequently 41 holds.

**Lemma 41.** Given a 2-structure \(\sigma\), consider \(X \notin V(\sigma)\) such that \(\sigma[X]\) is prime. Suppose that Statement (S3) holds. Consider distinct \(e, f \in E(\sigma)\), and \(\alpha \in X\). If \(\Gamma_{(\sigma, \underline{X})}\) does not have isolated vertices, then the following two assertions hold

1. if \((X)_{\sigma}^{(e, f)}(\alpha) \neq \emptyset\) and \((X)_{\sigma}^{(f, e)}(\alpha) \neq \emptyset\), then \((X)_{\sigma}^{(e, f)} \iff_{\sigma} (X)_{\sigma}^{(f, e)}\), and

\[
[(X)_{\sigma}^{(e, f)}(\alpha), (X)_{\sigma}^{(f, e)}(\alpha)]_{\sigma} = (e, f);
\]

2. if \(X_{\sigma}^{(e, f)}(\alpha) \neq \emptyset\) and \(X_{\sigma}^{(f, e)}(\alpha) \neq \emptyset\), then \(X_{\sigma}^{(e, f)}(\alpha) \iff_{\sigma} X_{\sigma}^{(f, e)}(\alpha)\), and

\[
[X_{\sigma}^{(e, f)}(\alpha), X_{\sigma}^{(f, e)}(\alpha)]_{\sigma} = (e, f).
\]

**Proof.** For the first assertion, consider \(x \in (X)_{\sigma}^{(e, f)}(\alpha)\) and \(x' \in (X)_{\sigma}^{(f, e)}(\alpha)\). Since \(x, x' \in (X)_{\sigma}\), we have \(\{x, x'\} \notin E(\Gamma_{(\sigma, \underline{X})})\) by Remark 30. Furthermore, since \(\Gamma_{(\sigma, \underline{X})}\) does not have isolated vertices, there exists \(y' \in \underline{X} \setminus \{x, x'\}\) such that \(\{x', y'\} \in E(\Gamma_{(\sigma, \underline{X})})\). Suppose for a contradiction that \(\{x, y'\} \notin E(\Gamma_{(\sigma, \underline{X})})\). It follows from the first assertion of Fact 30 applied to \(x, x', y'\) that \(X \cup \{x, x', y'\}\) is a module of \(\sigma[X \cup \{x, x', y'\}']\). Since \(x \in (X)_{\sigma}^{(e, f)}(\alpha)\), we obtain \((x, x') \in e\) and \((x', x) \in f)\). Consequently \([x, x']_{\sigma} = (e, f)\).

For the second assertion, consider \(x \in X_{\sigma}^{(e, f)}(\alpha)\) and \(x' \in X_{\sigma}^{(f, e)}(\alpha)\). Since \(x, x' \in X_{\sigma}(\alpha)\), we have \(\{x, x'\} \notin E(\Gamma_{(\sigma, \underline{X})})\) by Remark 30. Furthermore, since \(\Gamma_{(\sigma, \underline{X})}\) does not have isolated vertices, there exists \(y' \in \underline{X} \setminus \{x, x'\}\) such that \(\{x', y'\} \in E(\Gamma_{(\sigma, \underline{X})})\). Suppose for a contradiction that \(\{x, y'\} \notin E(\Gamma_{(\sigma, \underline{X})})\). We obtain \(\{x, y'\}, \{x', y'\} \in E(\Gamma_{(\sigma, \underline{X})})\). It follows from Fact 35 that \(e = f\), which contradicts our assumption. Therefore \(\{x, y'\} \notin E(\Gamma_{(\sigma, \underline{X})})\). It follows from
the second assertion of Fact \[39\] applied to \(x, x', y'\) that \(\{\alpha, x\}\) is a module of \(\sigma[X \cup \{x, x', y'\}]\). Thus \([x, x']_\sigma = [\alpha, x']_\sigma\). Since \(x' \in X^{(f,e)}(\alpha)\), we obtain \([x, x']_\sigma = (e, f)\). \(\blacksquare\)

To state the next result, we use the following notation and definition.

**Notation 42.** Let \(\sigma\) be a 2-structure. For \(e \in E(\sigma)\) and \(W \subseteq V(\sigma)\), set
\[
e[W] = e \cap (W \times W).
\]
Given \(e \in E(\sigma)\) and \(W \subseteq V(\sigma)\), we do not have \(e \in E(\sigma[W])\), but we have \(e[W] \in E(\sigma[W])\) when \(e[W] \neq \emptyset\).

**Definition 43.** A 2-structure \(\sigma\) is constant if \(|E(\sigma)| = 1\). Besides, a 2-structure \(\sigma\) is linear if there exist distinct \(e, f \in E(\sigma)\) such that
\[
(V(\sigma), \{(v, w) : v, w \in V(\sigma), v \neq w, [v, w]_\sigma = (e, f)\})
\]
is a linear order (see Remark 44).

**Remark 44.** Let \(\sigma\) be a linear 2-structure. There exist distinct \(e, f \in E(\sigma)\) such that \((V(\sigma), \{(v, w) : v, w \in V(\sigma), v \neq w, [v, w]_\sigma = (e, f)\})\) is a linear order. Therefore, \((V(\sigma), e)\) and \((V(\sigma), f)\) are total orders such that
\[
(V(\sigma), e)^* = (V(\sigma), f).
\]
Clearly, we have \(E(\sigma) = \{e, f\}\).

**Lemma 45.** Given a 2-structure \(\sigma\), consider \(X \subseteq V(\sigma)\) such that \(\sigma[X]\) is prime. Suppose that Statement (S3) holds. If \(\sigma\) is prime, then the next two assertions hold.

1. Let \(e \in E(\sigma)\). If \(|(X)_\sigma^{(e,c)}| \geq 2\), then \(\sigma[(X)_\sigma]\) is constant, and
\[
E(\sigma[(X)_\sigma]) = \{e[(X)_\sigma]\}.
\]
   Similarly, given \(\alpha \in X\), if \(|X^{(e,c)}_\sigma(\alpha)| \geq 2\), then \(\sigma[X_\sigma(\alpha)]\) is constant, and
\[
E(\sigma[X_\sigma(\alpha)]) = \{e[X_\sigma(\alpha)]\}.
\]

2. Consider distinct \(e, f \in E(\sigma)\). If \(|(X)_\sigma^{(e,f)}| \geq 2\), then \(\sigma[(X)_\sigma]\) is linear, and
\[
E(\sigma[(X)_\sigma]) = \{e[(X)_\sigma], f[(X)_\sigma]\}.
\]
   Similarly, given \(\alpha \in X\), if \(|X^{(e,f)}_\sigma(\alpha)| \geq 2\), then \(\sigma[X_\sigma(\alpha)]\) is linear, and
\[
E(\sigma[X_\sigma(\alpha)]) = \{e[X_\sigma(\alpha)], f[X_\sigma(\alpha)]\}.
\]

**Proof.** Consider \(B_\sigma \in q_{(\sigma,X)}\), with \(|B_\sigma| \geq 2\). There exist \(e, f \in E(\sigma)\) such that \(B_\sigma = (X)_\sigma^{(e,f)}\) or \(X^{(e,f)}_\sigma(\alpha)\). We define on \(B_\sigma\) the equivalence relation \(\equiv\) in the following way. Given \(c, d \in B_\sigma\), \(c \equiv d\) if either \(c = d\) or \(c \neq d\) and there exist sequences \((c_0, \ldots, c_p)\) and \((d_0, \ldots, d_q)\) of elements of \(B_\sigma\) satisfying
\[c_0 = c \text{ and } c_p = d;\]

- for \(0 \leq m \leq p - 1, [c_m, c_{m+1}]_\sigma \neq (e, f);\)
- \(d_0 = y \text{ and } d_q = x;\)
- for \(0 \leq m \leq q - 1, [d_m, d_{m+1}]_\sigma \neq (e, f).\)

Let us consider an equivalence class \(C\) of \(X\). We prove that \(C\) is a module of \(\sigma\).

We utilize Corollary \(37\) in the following manner. Since \(B_q \in q_{(\sigma, X)}\), there exists \(B_p \in p_{(\sigma, X)}\) such that \(B_q \subseteq B_p\).

First, we show that \(C\) is a module of \(\sigma[B_q]\). Let \(x \in B_q \setminus C\). By definition of \(X\), \([x, c]_\sigma = (e, f)\) or \((f, e)\) for every \(c \in C\). Hence, \(C\) is a module of \(\sigma[B_q]\) when \(e = f\). Suppose that \(e \neq f\). For a contradiction, suppose that there exist \(c, d \in C\) such that \([x, c]_\sigma = (e, f)\) and \([x, d]_\sigma = (f, e)\). Since \(c \neq d\), there exists a sequence \(c_0, \ldots, c_p\) of elements of \(B_q\) satisfying

- \(d_0 = d\) and \(d_q = c;\)
- for \(0 \leq m \leq q - 1, [d_m, d_{m+1}]_\sigma \neq (e, f).\)

By considering the sequences \((d = d_0, \ldots, d_q = c, x)\) and \((x, d)\), we obtain \(x \equiv d\), which contradicts the fact that \(C\) is an equivalence class of \(X\). It follows that \([x, C]_\sigma \neq (e, f)\) or \((f, e)\). Thus, \(C\) is a module of \(\sigma[B_q]\) when \(e \neq f\).

Second, we show that \(C\) is a module of \(\sigma[B_p]\). Suppose that \(e = f\). It follows from Lemma \(40\) that \(B_q = B_p\). Hence \(C\) is a module of \(\sigma[B_q]\). Suppose that \(e \neq f\). If \(B_q = B_p\), then we proceed as previously. Hence suppose that \(B_q \neq B_p\). It follows from Lemma \(40\) that \(B_p \setminus B_q \in q_{(\sigma, X)}\) and

\[
B_p \setminus B_q = \begin{cases} 
(X)^{(f, e)}_{\sigma} & \text{if } B_q = (X)^{(e, f)}_{\sigma} \\
X^{(f, e)}_{\sigma}(\alpha) & \text{if } B_q = X^{(e, f)}_{\sigma}(\alpha).
\end{cases}
\]

It follows from Lemma \(41\) that \(B_q\) is a module of \(\sigma[B_p]\). Since \(C\) is a module of \(\sigma[B_q]\), we obtain that \(C\) is a module of \(\sigma[B_p]\).

Third, we prove that \(C\) is a module of \(\Gamma_{(\sigma, X)}\). Since \(C \subseteq B_p\), we have \(\{c, x\} \notin E(\Gamma_{(\sigma, X)})\) for \(c \in C\) and \(x \in B_p \setminus C\) (see Remark \(39\)). Therefore, we have to verify that \(C\) is a module of \(\Gamma_{(\sigma, X)}[C \cup \{x\}]\) for each \(x \in X \setminus B_p\). Let \(x \in X \setminus B_p\). Set

\[
C^+ = \{c \in C : \{c, x\} \in E(\Gamma_{(\sigma, X)})\}
\]

and

\[
C^- = \{c \in C : \{c, x\} \notin E(\Gamma_{(\sigma, X)})\}.
\]

For a contradiction, suppose that \(C^- \neq \emptyset\) and \(C^+ \neq \emptyset\). It follows from Corollary \(38\) that \([C^-, C^+]_\sigma = (e, f)\) or \((f, e)\), which contradicts the fact that \(C\) is an equivalence class of \(X\). Therefore, \(C^- = \emptyset\) or \(C^+ = \emptyset\), that is, \(C\) is a module of \(\Gamma_{(\sigma, X)}[C \cup \{x\}]\) for each \(x \in X \setminus B_p\). Thus \(C\) is a module of \(\Gamma_{(\sigma, X)}\).

16
Consequently, \( C \) is a module of \( \sigma[B_p] \), and \( C \) is a module of \( \Gamma_{(\sigma,X)} \). It follows from Corollary 37 that \( C \) is a module of \( \sigma \). Since \( \sigma \) is prime, \( C \) is trivial. Hence \(|C| = 1 \) because \( C \neq \emptyset \), and \( C \cap X = \emptyset \). We conclude as follows by distinguishing the following two cases.

- Suppose that \( e = f \). Recall that \( B_q = B_p \) by Lemma 10. Since every equivalence class of \( \approx \) is reduced to a singleton, we obtain \((v, w)_\sigma = e \) for distinct elements \( v \) and \( w \) of \( B_p \). In other words, \( \sigma[B_p] \) is constant, and \( E(\sigma[B_p]) = \{e[B_p]\} \).

- Suppose that \( e \neq f \). For instance, suppose that \( B_q = \langle X \rangle^{(e,f)}_\sigma \). We verify that \( \sigma[\langle X \rangle^{(e,f)}_\sigma] \) is linear, and \( E(\sigma[\langle X \rangle^{(e,f)}_\sigma]) = \{\epsilon[\langle X \rangle^{(e,f)}_\sigma], f[\langle X \rangle^{(e,f)}_\sigma]\} \). Since every equivalence class of \( \approx \) is reduced to a singleton, we obtain \( [v, w]_\sigma = (e, f) \) or \( (f, e) \) for distinct elements \( v \) and \( w \) of \( \langle X \rangle^{(e,f)}_\sigma \). We consider the digraph \( \lambda \) defined on \( \langle X \rangle^{(e,f)}_\sigma \) as follows. Given distinct \( (u,v),(v,w) \in \lambda(\lambda) \) if \( [v, w]_\sigma = (e, f) \) or \( (f, e) \) for distinct elements \( v \) and \( w \) of \( \langle X \rangle^{(e,f)}_\sigma \), \( \lambda \) is a tournament. For a contradiction, suppose that there exist distinct \( u, v, w \in \langle X \rangle^{(e,f)}_\sigma \) such that \((u,v),(v,w),(w,u) \in \lambda(\lambda) \). By considering the sequences \((v, u)\) and \((u, v)\), we obtain \( v \approx u \), which contradicts the fact that every equivalence class of \( \approx \) is reduced to a singleton. It follows that for distinct elements \( u, v, w \in \langle X \rangle^{(e,f)}_\sigma \), if \((u,v),(v,w) \in \lambda(\lambda) \), then \((u,v) \in \lambda(\lambda) \). Therefore, \( \lambda \) is a linear order, that is, \( \sigma[\langle X \rangle^{(e,f)}_\sigma] \) is linear, and \( E(\sigma[\langle X \rangle^{(e,f)}_\sigma]) = \{\epsilon[\langle X \rangle^{(e,f)}_\sigma], f[\langle X \rangle^{(e,f)}_\sigma]\} \). Lastly, suppose that \( B_q \not\subseteq B_p \). It follows from Lemma 39 that \( B_p \setminus B_q = \langle X \rangle^{(f,e)}_\sigma \). Similarly, we have \( \sigma[\langle X \rangle^{(f,e)}_\sigma] \) is linear, and \( E(\sigma[\langle X \rangle^{(f,e)}_\sigma]) = \{\epsilon[\langle X \rangle^{(f,e)}_\sigma], f[\langle X \rangle^{(f,e)}_\sigma]\} \). Moreover, we have \[\epsilon[\langle X \rangle^{(e,f)}_\sigma], \epsilon[\langle X \rangle^{(f,e)}_\sigma] = (e, f)\] by the first assertion of Lemma 11. Consequently, \( \sigma[\langle X \rangle^\sigma] \) is linear, and \( E(\sigma[\langle X \rangle^\sigma]) = \{\epsilon[\langle X \rangle^\sigma], f[\langle X \rangle^\sigma]\} \).  

Lemma 15 ends the examination of blocks of the partitions \( p_{(\sigma,X)} \) and \( q_{(\sigma,X)} \). We complete Section 3 with a result on the components of the outside graph, which follows from Fact 35 and the following easy consequence of Fact 36. We use the following notation.

**Notation 46.** Given a 2-structure \( \sigma \), consider \( X \not\subseteq V(\sigma) \) such that \( \sigma[X] \) is prime. First, the set \( \{\{X \}^{(e)}_{\sigma} : e \in E(\sigma)\} \cup \{X^{(e\alpha)}_{\sigma} : e \in E(\sigma), \alpha \in X\} \) is denoted by \( q^*_d(\sigma,X) \). Second, the set \( q_{(\sigma,X)} \setminus (q^*_d(\sigma,X) \cup \{\text{Ext}_\sigma(X)\}) \) is denoted by \( q^*_q(\sigma,X) \).

**Fact 47.** Given a 2-structure \( \sigma \), consider \( X \not\subseteq V(\sigma) \) such that \( \sigma[X] \) is prime. Suppose that Statement (S3) holds. Consider distinct elements \( x, x', y, y' \) of \( X \).
such that \( \{x, y\}, \{x', y'\} \in E(\Gamma_{(\sigma, X)}) \) and \( \{x, y'\}, \{x', y\} \notin E(\Gamma_{(\sigma, X)}) \). If there exist \( B_q \in q_{(\sigma, X)} \) such that \( y, y' \in B_q \), then \( B_q \in q^*_{(\sigma, X)} \).

**Proof.** Since \( y \) and \( y' \) belong to the same block of \( p_{(\sigma, X)} \), we have \( \{y, y'\} \notin E(\Gamma_{(\sigma, X)}) \) by Remark 36. Besides, there exist \( e, f \in E(\sigma) \) such that \( B_q = (X)^{e,f}_{(\sigma)} \) or \( B_q = X_{(\sigma)}^{e,f}(\alpha) \), where \( \alpha \in X \).

First, suppose that \( B_q = (X)^{e,f}_{(\sigma)} \). By the first assertion of Fact 36 applied to \( x, y, y', X \cup \{x, y\} \) is a module of \( \sigma[X \cup \{x, y, y'\}] \). Since \( y' \in (X)^{e,f}_{(\sigma)} \), \( [y', y]_\sigma = (e, f) \). Similarly, it follows from the first assertion of Fact 36 applied to \( x', y, y' \) that \([y, y']_\sigma = (e, f) \). Thus \( e = f \), and hence \( B_q \in q^*_{(\sigma, X)} \).

Second, suppose that \( B_q = X_{(\sigma)}^{e,f}(\alpha) \), where \( \alpha \in X \). By the second assertion of Fact 36 applied to \( x, y, y', \{\alpha, y'\} \) is a module of \( \sigma[X \cup \{x, y, y'\}] \). Thus \([y, y']_\sigma = [y, \alpha]_\sigma \). Since \( y \in X_{(\sigma)}^{e,f}(\alpha) \), we obtain \([y, y']_\sigma = (e, f) \). Similarly, it follows from the second assertion of Fact 36 applied to \( x', y, y' \) that \([y', y]_\sigma = (e, f) \). Therefore \( e = f \), so \( B_q \in q^*_{(\sigma, X)} \). \( \square \)

**Proposition 48.** Given a 2-structure \( \sigma \), consider \( X \subseteq V(\sigma) \) such that \( \sigma[X] \) is prime. Suppose that Statement (S3) holds. If \( \Gamma_{(\sigma, X)} \) does not have isolated vertices, then the following two assertions hold.

1. For each component \( C \) of \( \Gamma_{(\sigma, X)} \), there exist distinct \( B_p, D_p \in p_{(\sigma, X)} \) and \( B_q, D_q \in q_{(\sigma, X)} \) such that \( B_q \in B_p \), \( D_q \in D_p \), and \( C \) is bipartite with bipartition \( \{V(C) \cap B_q, V(C) \cap D_q\} \).

2. For a component \( C \) of \( \Gamma_{(\sigma, X)} \) and for \( B_q \in q_{(\sigma, X)} \), if \( V(C) \cap B_q \neq \emptyset \), then \( B_q \in V(C) \).

**Proof.** For the first assertion, consider a component \( C \) of \( \Gamma_{(\sigma, X)} \). Since \( \Gamma_{(\sigma, X)} \) does not have isolated vertices, \( u(C) \geq 2 \). Hence, there exist distinct \( c, d \in V(C) \) such that \( \{c, d\} \in E(\Gamma_{(\sigma, X)}) \). There exist \( B_p, D_p \in p_{(\sigma, X)} \) and \( B_q, D_q \in q_{(\sigma, X)} \) such that \( c \in B_q \), \( d \in D_q \), \( B_q \in B_p \) and \( D_q \in D_p \). Since \( \{c, d\} \in E(\Gamma_{(\sigma, X)}) \), we have \( B_p \neq D_p \) by Remark 36. Let \( x \in V(C) \setminus \{c, d\} \). Since \( C \) is a component of \( \Gamma_{(\sigma, X)} \), there exist a path \( x_0, x_1, \ldots, x_n \) such that \( x_0 \in \{c, d\} \), \( x_n = x \), and \( \{x_1, \ldots, x_n\} \cap \{c, d\} = \emptyset \). We have \( n \geq 1 \). We distinguish the following two cases.

1. Suppose that \( n \) is even. It follows from Fact 36 that \( x_0, x_2, \ldots, x_n \) belong to the same block of \( q_{(\sigma, X)} \). Since \( x_0 \in \{c, d\} \) and \( x_n = x \), we obtain \( x \in B_q \cup D_q \).

2. Suppose that \( n \) is odd. Set

\[
x_{-1} = \begin{cases} 
d & \text{if } x_0 = c \\
c & \text{if } x_0 = d.
\end{cases}
\]
We have \( x_{-1} \in B_q \cup D_q \). By considering the path \( x_{-1}x_0 \ldots x_n \), it follows from Fact 48 that \( x \) and \( x_{-1} \) belong to the same block of \( q_{(\sigma,\mathcal{X})} \). Hence \( x \in B_q \cup D_q \).

Therefore \( V(C) \setminus \{c,d\} \subseteq B_q \cup D_q \), so \( V(C) \subseteq B_q \cup D_q \). By Remark 30 \( C \) is bipartite with bipartition \( \{V(C) \cap B_q, V(C) \cap D_q\} \).

For the second assertion, consider a component \( C \) of \( \Gamma_{(\sigma,\mathcal{X})} \), and an element \( B_q \) of \( q_{(\sigma,\mathcal{X})}^a \) such that \( V(C) \cap B_q \neq \emptyset \). Consider \( y \in V(C) \cap B_q \). For a contradiction, suppose that \( B_q \setminus V(C) \neq \emptyset \), and consider \( y' \in B_q \setminus V(C) \). Since \( \Gamma_{(\sigma,\mathcal{X})} \) does not have isolated vertices, there exist \( x \in \mathcal{X} \setminus \{y\} \) and \( x' \in \mathcal{X} \setminus \{y'\} \) such that \( \{x,y\}, \{x',y'\} \in E(\Gamma_{(\sigma,\mathcal{X})}) \). Furthermore, since \( C \) is a component of \( \Gamma_{(\sigma,\mathcal{X})} \), with \( y \in V(C) \) and \( y' \notin V(C) \), we obtain \( x \in V(C) \) and \( x' \notin V(C) \). Hence \( x \neq x' \), and \( \{x,y\}, \{x',y'\} \notin E(\Gamma_{(\sigma,\mathcal{X})}) \). It follows from Fact 44 that \( B_q \notin q_{(\sigma,\mathcal{X})}^a \), which contradicts \( B_q \in q_{(\sigma,\mathcal{X})}^a \). \( \square \)

Proposition 48 leads us to the following notation.

**Notation 49.** Given a 2-structure \( \sigma \), consider \( X \nsubseteq V(\sigma) \) such that \( \sigma[X] \) is prime. Suppose that Statement (S3) holds. To use Proposition 48 we have also to suppose that \( \Gamma_{(\sigma,\mathcal{X})} \) does not have isolated vertices. By Corollary 89 we can also suppose that \( \sigma \) is prime.

Consider a component \( C \) of \( \Gamma_{(\sigma,\mathcal{X})} \). By the first assertion of Proposition 48 there exist distinct \( B_p, D_p \in q_{(\sigma,\mathcal{X})} \) and \( B_q, D_q \in q_{(\sigma,\mathcal{X})} \) such that \( B_q \subseteq B_p \), \( D_q \subseteq D_p \), and \( C \) is bipartite with bipartition \( \{V(C) \cap B_q, V(C) \cap D_q\} \). In the sequel, \( V(C) \cap B_q \) and \( V(C) \cap D_q \) are respectively denoted by \( B_q^{C} \) and \( D_q^{C} \). (Note that we use the Axiom of Ultrafilter to introduce such a notation for each component of \( \Gamma_{(\sigma,\mathcal{X})} \), when \( q_{(\sigma,\mathcal{X})} \) has infinitely many blocks.)

### 4 Proofs of the main results

We use the following notation.

**Notation 50.** Given a graph \( \Gamma \), \( \mathcal{C}(\Gamma) \) denotes the set of the components of \( \Gamma \).

**Proof of Theorem 16.** To begin, suppose that \( \sigma \) is not prime. We prove that there exists \( C \in \mathcal{C}(\Gamma_{(\sigma,\mathcal{X})}) \) such that \( \sigma[X \cup V(C)] \) is not prime. First, suppose that \( \Gamma_{(\sigma,\mathcal{X})} \) admits isolated vertices. There exists \( v \in \mathcal{X} \) such that \( \{v\} \in \mathcal{C}(\Gamma_{(\sigma,\mathcal{X})}) \). Since Statement (S3) holds, \( \text{Ext}_\sigma(X) = \emptyset \) by Remark 11. Thus \( \sigma[X \cup \{v\}] \) is not prime. Second, suppose that \( \Gamma_{(\sigma,\mathcal{X})} \) does not have isolated vertices. Since \( \sigma \) is not prime, \( \sigma \) admits a nontrivial module \( M \). It follows from Corollary 52 that \( M \cap X = \emptyset \). By Lemma 54 there exists \( B_p \in q_{(\sigma,\mathcal{X})} \) such that \( M \subseteq B_p \), and \( M \) is a module of \( \Gamma_{(\sigma,\mathcal{X})} \). Let \( x \in M \). Since \( \Gamma_{(\sigma,\mathcal{X})} \) does not have isolated vertices, there exists \( y \in \mathcal{X} \setminus \{x\} \) such that \( \{x,y\} \in E(\Gamma_{(\sigma,\mathcal{X})}) \). Since \( M \subseteq B_p \), we have \( y \notin M \) by Remark 50. Denote by \( C \) the component of \( \Gamma_{(\sigma,\mathcal{X})} \) containing \( x \). Hence \( y \in V(C) \) because \( \{x,y\} \in E(\Gamma_{(\sigma,\mathcal{X})}) \). Since \( M \) is a module
of $\Gamma_{(\sigma, X)}$, we obtain $\{x', y\} \in E(\Gamma_{(\sigma, X)})$ for every $x' \in M$. Therefore $M \subseteq V(C)$.

It follows that $M$ is a nontrivial module of $\sigma[X \cup V(C)]$.

Now, we suppose that there exists $C \in \mathcal{C}(\Gamma_{(\sigma, X)})$ such that $\sigma[X \cup V(C)]$ is not prime. Since $\sigma[X \cup V(C)]$ is not prime, we have $v(C) \neq 2$. We assume that $v(C) \geq 4$, and we have to prove that $C$ is not prime. Consider a nontrivial module $M$ of $\sigma[X \cup V(C)]$. Clearly, $\sigma[X \cup V(C)]$ satisfies Statement (S3).

Moreover,

$$\Gamma_{(\sigma[X \cup V(C)]; V(C))} = C.$$ 

Since $v(C) \geq 4$, it follows from Corollary 32 applied to $\sigma[X \cup V(C)]$ that $M \subseteq V(C)$. By Lemma 34 applied to $\sigma[X \cup V(C)]$, there exists $B_p \in p(\sigma[X \cup V(C)])$ such that $M \subseteq B_p$, and $M$ is a module of $C$. We have to verify that $M \neq V(C)$. Let $x \in M$. Since $v(C) \geq 4$, there exists $y \in V(C) \setminus \{x\}$ such that $\{x, y\} \in E(C)$. Since $M \subseteq B_p$, we have $y \notin M$ by Remark 30 applied to $\sigma[X \cup V(C)]$. Hence $y \in V(C) \setminus M$.

Lastly, we suppose that there exists $C \in \mathcal{C}(\Gamma_{(\sigma, X)})$ such that $v(C) = 1$ or $v(C) \geq 3$ and $C$ is not prime. We have to prove that $\sigma$ is not prime. Therefore, by Corollary 34, we can assume that

$$\Gamma_{(\sigma, X)}$$

does not have isolated vertices. (5)

In particular, we obtain $v(C) \geq 3$. Consider a nontrivial module $M$ of $C$. Clearly, $M$ is a module of $\Gamma_{(\sigma, X)}$, because $C$ is a component of $\Gamma_{(\sigma, X)}$. Since $\Gamma_{(\sigma, X)}$ does not have isolated vertices (see 34), it follows from the first assertion of Proposition 45 that there exist distinct $B_p, D_p \in p(\sigma, X)$ and $B_q, D_q \in q(\sigma, X)$ such that $B_q \subseteq B_p$, $D_q \subseteq D_p$, and $C$ is bipartite with bipartition $\{V(C) \cap B_q, V(C) \cap D_q\}$. Since $C$ is connected, we have $M \subseteq V(C) \cap B_q$ or $M \subseteq V(C) \cap D_q$. For instance, assume that $M \subseteq V(C) \cap B_q$. To conclude, we distinguish the following two cases.

1. Suppose that $B_q \in q(\sigma, X)$. There exists $e \in E(\sigma)$ such that $B_q = \langle X \rangle^{(e, e)}$ or $X^{(e, e)}(\alpha)$, where $\alpha \in X$. If $\sigma[B_p]$ is not constant, then it follows from the first assertion of Lemma 14 that $\sigma$ is not prime. Thus, suppose that $\sigma[B_p]$ is constant. It follows that any subset of $B_p$ is a module of $\sigma[B_p]$. In particular, $M$ is a module of $\sigma[B_p]$. Since $M$ is a module of $\Gamma_{(\sigma, X)}$, it follows from Corollary 37 that $M$ is a module of $\sigma$.

2. Suppose that $B_q \in q(\sigma, X)$. Since $\Gamma_{(\sigma, X)}$ does not have isolated vertices (see 35), it follows from the second assertion of Proposition 45 that $B_q \subseteq V(C)$. In general, $M$ is not a module of $\sigma[B_q]$, and hence $M$ is not a module of $\sigma[B_p]$. Therefore, we cannot apply Corollary 37 to $M$. Nevertheless, we construct a superset of $M$, which is a module of $\Gamma_{(\sigma, X)}$, and a module of $\sigma[B_p]$. Consider the set $M$ of the nontrivial modules $M'$ of $C$ such that $M \subseteq M'$. Set

$$\overline{M} = \bigcup M.$$ 

20
Clearly, $M \in \mathcal{M}$. Since $M \neq \emptyset$ and all the elements of $\mathcal{M}$ contain $M$, $\tilde{M}$ is a module of $C$. Since $C$ is a component of $\Gamma_{(\sigma, \overline{X})}$, $\tilde{M}$ is a module of $\Gamma_{(\sigma, \overline{X})}$. As previously seen for $M$, $\tilde{M} \subseteq V(C) \cap B_q$, or $\tilde{M} \subseteq V(C) \cap D_q$. Since $M \subseteq \tilde{M}$ and $M \subseteq V(C) \cap B_q$, we have $\tilde{M} \subseteq V(C) \cap B_q$. Therefore $\tilde{M} \subseteq B_q$. Set $$N = \{v \in B_q \setminus \tilde{M} : v \not\sim_\sigma \tilde{M}\}.$$ We verify that $\tilde{M} \cup N$ is a module of $C$. It suffices to show that for any $v \in V(C) \cap B_q$, $x \in \tilde{M}$ and $y \in N$, we have $\{v, x\}, \{v, y\} \in E(\Gamma_{(\sigma, \overline{X})})$ or $\{v, x\}, \{v, y\} \notin E(\Gamma_{(\sigma, \overline{X})})$. Since $y \in N$, there exist $x', x'' \in \tilde{M}$ such that $y \not\sim_\sigma \{x', x''\}$. Furthermore, since $\tilde{M}$ is a module of $C$, we have $\{v, x\}, \{v, x'\}, \{v, x''\} \in E(\Gamma_{(\sigma, \overline{X})})$ or $\{v, x\}, \{v, x'\}, \{v, x''\} \notin E(\Gamma_{(\sigma, \overline{X})})$. For instance, suppose that $\{v, x\}, \{v, x'\}, \{v, x''\} \in E(\Gamma_{(\sigma, \overline{X})})$. By Corollary $38$, $\{z \in B_q : \{z, v\} \in E(\Gamma_{(\sigma, \overline{X})})\}$ is a module of $\sigma[B_q]$. Since $x, x', x'' \in \{z \in B_q : \{z, v\} \in E(\Gamma_{(\sigma, \overline{X})})\}$ and $y \not\sim_\sigma \{x', x''\}$, we obtain $y \in \{z \in B_q : \{z, v\} \in E(\Gamma_{(\sigma, \overline{X})})\}$. Hence $\{v, x\}, \{v, x'\}, \{v, x''\}, \{v, y\} \in E(\Gamma_{(\sigma, \overline{X})})$. Similarly, if $\{v, x\}, \{v, x'\}, \{v, x''\} \notin E(\Gamma_{(\sigma, \overline{X})})$, then it follows from Corollary $38$ that $\{v, x\}, \{v, x'\}, \{v, x''\}, \{v, y\} \notin E(\Gamma_{(\sigma, \overline{X})})$. Consequently, $\tilde{M} \cup N$ is a module of $C$. It follows from the definition of $\tilde{M}$ that $N \subseteq \tilde{M}$. Therefore $N = \emptyset$, and hence $\tilde{M}$ is a module of $\sigma[B_q]$. Since $\Gamma_{(\sigma, \overline{X})}$ does not have isolated vertices (see $5$), it follows from Lemmas $40$ and $41$ that $\tilde{M}$ is a module of $\sigma[B_p]$. Lastly, since $\tilde{M}$ is a module of $\Gamma_{(\sigma, \overline{X})}$, it follows from Corollary $57$ that $\tilde{M}$ is a module of $\sigma$.

We use the next notation to demonstrate Theorem $17$.

**Notation 51.** Given graphs $G$ and $H$, $G \leq H$ means that $G$ is isomorphic to an induced subgraph of $H$.

**Notation 52.** Let $G$ and $H$ be graphs such that $V(G) \cap V(H) = \emptyset$. The **disjoint union** of $G$ and $H$ is the graph $G \oplus H = (V(G) \cup V(H), E(G) \cup E(H))$. If $V(G) \cap V(H) \neq \emptyset$, then we can define $G \oplus H$ up to isomorphism by considering graphs $G'$ and $H'$ such that $G \cong G'$, $H \cong H'$, and $V(G') \cap V(H') = \emptyset$.

We use also the following two lemmas. The next result is a consequence of Theorem $16$.

**Lemma 53.** Given a $2$-structure $\sigma$, consider $X \subseteq V(\sigma)$ such that $\sigma[X]$ is prime. Suppose that Statement (S5) holds. For each component $C$ of $\Gamma_{(\sigma, \overline{X})}$, $P_5 \nmid C$.

**Proof.** For a contradiction, suppose that there exists a component $C$ of $\Gamma_{(\sigma, \overline{X})}$ such that $P_5 \nmid C$. Hence, there exists $Y \subseteq V(C)$ such that $C[Y] \cong P_5$. We have

$$\Gamma_{(\sigma[X \cup Y], Y)} = \Gamma_{(\sigma, \overline{X})}[Y].$$

Since $\Gamma_{(\sigma, \overline{X})}[Y] = C[Y]$, $\Gamma_{(\sigma[X \cup Y], Y)}$ is prime. It follows from Theorem $16$ applied to $\sigma[X \cup Y]$ that $\sigma[X \cup Y]$ is prime, which contradicts the fact that Statement (S5) holds.

\[21\]
Since the proof of the next lemma is easy, we omit it.

**Lemma 54.** Given a connected graph $\Gamma$, $K_2 \oplus K_2 \leq \Gamma$ if and only if $P_5 \leq \Gamma$.

**Proof of Theorem 17.** To begin, suppose that the first assertion holds, that is, $\sigma$ is $\overline{X}$-critical. We have to prove that the second assertion holds. Consider $C \in \mathcal{C}(\sigma, \overline{X})$. By Theorem 10 applied to $\sigma$, $\sigma[X \cup V(C)]$ is prime. We have to show that $\sigma[X \cup V(C)]$ is $V(C)$-critical. Let $c \in V(C)$. Since $\sigma$ is $\overline{X}$-critical, $\sigma - c$ is not prime. Consequently

$$\Gamma_{(\sigma-c, \overline{X}\setminus\{c\})} = \Gamma_{(\sigma, \overline{X})} - c.$$ 

Therefore

$$\mathcal{C}(\Gamma_{(\sigma-c, \overline{X}\setminus\{c\})}) = (\mathcal{C}(\Gamma_{(\sigma, \overline{X})}) \setminus \{C\}) \cup \mathcal{C}(C - c). \quad (6)$$

Since $\sigma - c$ is not prime, it follows from Theorem 10 applied to $\sigma - c$ that there exists $C' \in \mathcal{C}(\Gamma_{(\sigma-c, \overline{X}\setminus\{c\})})$ such that $\sigma[X \cup V(C')]$ is not prime. By (6), $C' \in \mathcal{C}(\Gamma_{(\sigma, \overline{X})}) \setminus \{C\}) \cup \mathcal{C}(C - c)$. By Theorem 10 applied to $\sigma$, $\sigma[X \cup V(D)]$ is prime for every $D \in \mathcal{C}(\Gamma_{(\sigma, \overline{X})}) \setminus \{C\})$. Thus $C' \in \mathcal{C}(C - c)$. Finally, since

$$\Gamma_{(\sigma[X\cup V(C)]-c, V(C)\setminus\{c\})} = C - c,$$

it follows from Theorem 10 applied to $\sigma[X \cup V(C)] - c$ that $\sigma[X \cup V(C)] - c$ is not prime. Consequently $\sigma[X \cup V(C)]$ is $V(C)$-critical.

To continue, suppose that the second assertion holds. We have to prove that the third assertion holds. Consider $C \in \mathcal{C}(\Gamma_{(\sigma, \overline{X})})$. By Theorem 10 applied to $\sigma$, $v(C) = 2$ or $v(C) \geq 4$ and $C$ is prime. Suppose that $v(C) \geq 4$ and $C$ is prime. We have to show that $\sigma$ is critical. If $v(C) = 4$, then $C$ is critical by Proposition 57. Hence suppose that $v(C) \geq 5$. Let $c \in V(C)$. If $C - c$ is disconnected, then $C - c$ is not prime. Thus, suppose that $C - c$ is connected. Since the second assertion holds, $\sigma[X \cup V(C)] - c$ is not prime. We have

$$\Gamma_{(\sigma[X\cup V(C)]-c, V(C)\setminus\{c\})} = C - c.$$ 

It follows from Theorem 10 applied to $\sigma[X \cup V(C)] - c$ that $C - c$ is not prime.

Lastly, suppose that the third assertion holds. Hence, for every $C \in \mathcal{C}(\Gamma_{(\sigma, \overline{X})})$,

$$v(C) = 2 \text{ or } v(C) \geq 4 \text{ and } C \text{ is critical.} \quad (7)$$

We have to prove that $\sigma$ is $\overline{X}$-critical. By Theorem 10 applied to $\sigma$, $\sigma$ is prime. Let $x \in \overline{X}$. We have to prove that $\sigma - x$ is not prime. Denote by $C$ the component of $\Gamma_{(\sigma, \overline{X})}$ containing $x$. As seen in (9),

$$\mathcal{C}(C - x) \subseteq \mathcal{C}(\Gamma_{(\sigma-x, \overline{X}\setminus\{x\})}). \quad (8)$$

Suppose that $C - x$ admits isolated vertices. By (8), $\Gamma_{(\sigma-x, \overline{X}\setminus\{x\})}$ admits isolated vertices as well. It follows from Corollary 39 that $\sigma - x$ is not prime. Finally,
suppose that \( C - x \) does not admit isolated vertices, that is, \( v(C') \geq 2 \) for each \( C' \in \mathcal{C}(C - x) \). In particular, we do not have \( v(C) = 2 \). It follows from (7) that

\[
v(C) \geq 4 \text{ and } C \text{ is critical.} \tag{9}
\]

By Lemma 53, \( P_5 \not\subseteq C \). Therefore \( K_2 \oplus K_2 \not\subseteq C \) by Lemma 54. Since \( v(C') \geq 2 \) for each \( C' \in \mathcal{C}(C - x) \), we obtain that \( C - x \) possesses a unique component, that is, \( C - x \) is connected. By (8), \( C - x \in \mathcal{C}(\Gamma_{(\sigma - x, \overline{X}\setminus \{x\})}) \). Furthermore, it follows from (9) that \( v(C - x) \geq 3 \) and \( C - x \) is not prime. By Theorem 16 applied to \( \sigma - x, \sigma - x \) is not prime. \( \square \)

**Proof of Corollary 23.** To begin, suppose that \( \sigma \) is \( \overline{X} \)-critical. As seen in Remark 11, Statement (S5) holds.

Conversely, suppose that Statement (S5) holds, and \( \sigma \) is prime. To prove that \( \sigma \) is \( \overline{X} \)-critical, we apply Theorem 17. Let \( C \) be a component of \( \Gamma_{(\sigma, \overline{X})} \). Since \( \sigma \) is prime, it follows from Theorem 16 that \( v(C) = 2 \) or \( v(C) \geq 4 \) and \( C \) is prime. Suppose that \( v(C) \geq 4 \) and \( C \) is prime. By Lemma 53, \( P_5 \not\subseteq C \). It follows from Proposition 57 that \( C \) is critical. Consequently, for each component \( C \) of \( \Gamma_{(\sigma, \overline{X})} \), we have \( v(C) = 2 \) or \( v(C) \geq 4 \) and \( C \) is critical. By Theorem 17, \( \sigma \) is \( \overline{X} \)-critical. \( \square \)

**Proof of Theorem 27.** To begin, suppose that Statement (S5) holds, and \( \sigma \) is prime. Let \( F \) be a finite subset of \( X \). By Corollary 13 there exist a finite subset \( F' \) of \( X \) such that \( F \subseteq F' \) and \( \sigma[X \cup F'] \) is prime. Since Statement (S5) holds, it follows from Corollary 24 that \( \sigma[X \cup F'] \) is \( (F') \)-critical. Consequently, \( \sigma \) is finitely \( X \)-critical.

Conversely, suppose that \( \sigma \) is finitely \( \overline{X} \)-critical. Hence, we obtain that for each finite subset \( F \) of \( \overline{X} \), there exist a finite subset \( F' \) of \( \overline{X} \) such that \( F \subseteq F' \) and \( \sigma[X \cup F'] \) is prime. By Corollary 13 \( \sigma \) is prime. Lastly, consider \( W \in \overline{X} \) such that \( |W| = 5 \). Since \( \sigma \) is finitely \( \overline{X} \)-critical, there exists \( W' \in \overline{X} \) such that \( W' \) is finite and \( \sigma[X \cup W'] \) is \( (W') \)-critical. As seen in Remark 11, Statement (S5) holds in \( \sigma[X \cup W'] \). Therefore Statement (S5) holds in \( \sigma \). \( \square \)

## 5 Half graphs

We begin with a remark on half graphs.

**Remark 55.** Consider a half graph \( \Gamma \), with bipartition \( \{X, Y\} \). There exist a linear order \( L \) defined on \( X \), and a bijection \( \varphi \) from \( X \) onto \( Y \) such that \( E(\Gamma) = \{\{x, \varphi(x')\} : x \leq x' \mod L\} \). Denote by \( \varphi(L) \) the unique linear order defined on \( Y \) such that \( \varphi \) is an isomorphism from \( L \) onto \( \varphi(L) \). We obtain

\[
E(\Gamma) = \{\{y, \varphi^{-1}(y')\} : y \leq y' \mod \varphi(L)^*\}.
\]

Consequently, \( \Gamma \) is also a half graph by considering the linear order \( \varphi(L)^* \) defined on \( Y \), and the bijection \( \varphi^{-1} : Y \rightarrow X \).
In the next remark, we explain how to decompose a discrete linear order (see Definition 20) into a lexicographic sum.

**Remark 56.** Given an infinite linear order \( L \), \( L \) is discrete if and only if \( L \) is decomposed into a lexicographic sum \( \sum l_v \) satisfying the following conditions.

1. If \( l \) admits a unique vertex \( v \), then \( L = l_v \), and \( L \cong \omega^* + \omega \) or \( \omega^* + \omega \).
2. For every \( v \in V(l) \), if \( v \) is neither the smallest nor the largest element of \( l \), then \( l_v \cong \omega^* + \omega \).
3. If \( l \) admits a smallest element denoted by \( \text{min} \), then \( l_{\text{min}} \cong \omega^* + \omega \).
4. If \( l \) admits a largest element denoted by \( \text{max} \), then \( l_{\text{max}} \cong \omega^* + \omega \).

**Hint.** For a linear order, both notions of an interval and a module coincide. Consider an infinite discrete linear order \( L \). We define on \( V(l) \) the binary relation \( \sim \) as follows. Given \( v, w \in V(l) \), \( v \sim w \) if the smallest interval of \( L \) containing \( v \) and \( w \) is finite. Clearly, \( \sim \) is an equivalence relation. Furthermore, the equivalence classes of \( \sim \) are intervals of \( L \). Thus, the set \( P \) of the vertex sets of the equivalence classes of \( \sim \) is an interval partition of \( L \). We consider for \( l \) the quotient \( L/P \) defined on \( P \) in the following manner. Given distinct \( I, J \in P \), \( I < J \mod L/P \) if \( i < j \mod L \) for \( i \in I \) and \( j \in J \). It is easy to verify that \( L/P \) is a linear order. Lastly, since \( L \) is discrete, \( L[I] \) is isomorphic to \( \omega \), \( \omega^* \) or \( \omega^* + \omega \) for each \( I \in P \).

Now, we examine Theorem 22 in the finite case. Given \( n \geq 1 \), we consider the graph \( H_{2n} \) defined on \( \{0, \ldots, 2n-1\} \) by

\[
E(H_{2n}) = \bigcup_{0 \leq p \leq n-1} \{ \{2p, 2q+1\} : p \leq q \leq n-1 \} \text{ (see Figure 1).}
\]

Clearly, the cardinality of a finite half graph is even. Up to isomorphism, \( H_{2n} \) is the unique finite half graph defined on \( 2n \) vertices.

![Figure 1: The half graph \( H_{2n} \).](image)

**Proposition 57.** For a finite and bipartite graph \( \Gamma \), with \( v(\Gamma) \geq 4 \), the following assertions are equivalent

1. \( P_5 \not\subseteq \Gamma \) and \( \Gamma \) is prime;
2. \( \Gamma \) is critical;
3. \( \Gamma \) is a half graph.

Proposition \[57\] is an immediate consequence of the following two facts. The next fact is due to Boudabbous et al. \[3\].

**Fact 58.** For a finite and prime graph \( \Gamma \), \( \Gamma \) is critical if and only if \( \Gamma \) does not admit a prime induced subgraphs of size 5.

A simple characterization of finite and critical digraphs is provided in \[5\] by using the primality graph (see Definition \[79\]). The next fact follows from it.

**Fact 59.** Given a finite and bipartite graph \( \Gamma \), with \( v(\Gamma) \geq 4 \), \( \Gamma \) is critical if and only if \( \Gamma \) is a half graph.

The next result is a consequence of Proposition \[57\] and Theorem \[14\].

**Corollary 60.** A half graph \( \Gamma \), with \( v(\Gamma) \geq 4 \), is prime.

**Proof.** There exists a bipartition \( \{X, Y\} \) of \( V(\Gamma) \), a linear order \( L \) defined on \( X \), and a bijection \( \varphi \) from \( X \) onto \( Y \) such that \( E(\Gamma) = \{\{x, \varphi(x')\} : x \leq x' \mod L\} \). By Proposition \[57\] we can suppose that \( \Gamma \) is infinite. Consider a finite subset \( F \) of \( V(\Gamma) \). Let \( X' \) be a finite subset of \( X \) such that \( F \cap X \subseteq X' \), \( \varphi^{-1}(F \cap Y) \subseteq X' \), and \( |X'| \geq 2 \). Set \( F' = X' \cup \varphi(X') \).

Clearly \( F \subseteq F' \). By considering \( Y' = \varphi(X') \), the linear order \( L' = L[X'] \), and the bijection \( \varphi_{|X'} : X' \to Y' \), we obtain that \( \Gamma[F'] \) is a half graph. By Proposition \[57\] \( \Gamma[F'''] \) is prime. To conclude, it suffices to use Theorem \[14\]. \( \square \)

Now, we are ready to demonstrate Theorem \[22\].

**Proof of Theorem \[22\]**. By Proposition \[57\] we can suppose that \( \Gamma \) is infinite.

To begin, suppose that \( \Gamma \) is a discrete half graph. There exists a bipartition \( \{X, Y\} \) of \( V(\Gamma) \), a discrete linear order \( L \) defined on \( X \), and a bijection \( \varphi \) from \( X \) onto \( Y \) such that \( E(\Gamma) = \{\{x, \varphi(x')\} : x \leq x' \mod L\} \). By Corollary \[60\] \( \Gamma \) is prime. Hence \( \Gamma \) is connected. Since \( \Gamma \) is a half graph, \( K_2 \oplus K_2 \not\subseteq \Gamma \). It follows from Lemma \[44\] that \( P_5 \not\subseteq \Gamma \). We verify that

\[
\text{for every } x \in X, \; \Gamma - x \text{ is not prime.} \tag{10}
\]

First, suppose that \( x \) is not the smallest element of \( L \). Since \( L \) is discrete, \( x \) admits a predecessor \( x^- \). It is easy to verify that \( \{\varphi(x^-), \varphi(x)\} \) is a module of \( \Gamma - x \). Second, suppose that \( x \) is the smallest element of \( \Gamma \). Clearly, \( \varphi(x) \) is an isolated vertex of \( \Gamma - x \), so \( \Gamma - x \) is not prime. Thus \( \text{(10)} \) holds. Similarly, it follows from Remark \[55\] that \( \Gamma - y \) is not prime for each \( y \in Y \). Consequently \( \Gamma \) is critical.

Conversely, suppose that \( P_5 \not\subseteq \Gamma \) and \( \Gamma \) is critical. Since \( \Gamma \) is bipartite, there exists a bipartition \( \{X, Y\} \) of \( V(\Gamma) \) such that \( X \) and \( Y \) are stable sets of \( \Gamma \). To complete the proof, we establish the next claims. \( \square \)
Definition 61. Since $\Gamma$ is prime, we have $N_{\Gamma}(x) \neq N_{\Gamma}(x')$ for distinct $x, x' \in X$. Moreover, since $P_5 \not\cong \Gamma$, $K_2 \oplus K_2 \not\cong \Gamma$ by Lemma 54. It follows that for distinct $x, x' \in X$, we have $N_{\Gamma}(x) \nsubseteq N_{\Gamma}(x')$ or $N_{\Gamma}(x') \nsubseteq N_{\Gamma}(x)$. Therefore, we can define on $X$ a linear order $L$ as follows. Given distinct $x, x' \in X$,

$$x < x' \mod L \text{ if } N_{\Gamma}(x) \nsubseteq N_{\Gamma}(x').$$

We show that $\Gamma$ is the half graph defined from the linear order $L$ (see Claim 69). We have also to define a suitable bijection from $X$ onto $Y$ (see Definition 65). We use the fact that $\Gamma$ is critical.

Claim 62. Given $x \in X$, if $\Gamma - x$ is disconnected, then the following assertions hold

1. $\Gamma - x$ admits a unique isolated vertex $i_x$, and $i_x \in Y$;
2. $N_{\Gamma}(x) = Y$, so $x$ is the smallest element of $L$;
3. $i_x$ is the unique element of $V(\Gamma) \setminus \{x\}$ such that $\Gamma - \{x, i_x\}$ is prime.

Proof. Since $\Gamma$ is connected, the set of the isolated vertices of $\Gamma - x$ is a module of $\Gamma$. Thus $|\{C \in \mathcal{C}(\Gamma - x) : v(C) = 1\}| \leq 1$. Furthermore, since $K_2 \oplus K_2 \not\cong \Gamma$, if $\Gamma - x$ admits at most one nontrivial component. Therefore $|\{C \in \mathcal{C}(\Gamma - x) : v(C) \geq 2\}| \leq 1$. It follows that $\Gamma - x$ admits a unique isolated vertex $i_x$, and $\Gamma - \{x, i_x\}$ is connected. Since $i_x$ is an isolated vertex of $\Gamma - x$, $\{x, i_x\} \subseteq E(\Gamma)$ because $\Gamma$ is connected. Hence $i_x \in Y$.

Now, we verify that $N_{\Gamma}(x) = Y$. Let $y \in Y \setminus \{i_x\}$. Since $\Gamma - \{x, i_x\}$ is connected, there exists $x' \in X \setminus \{x\}$ such that $\{x', y\} \in E(\Gamma)$. Since $\Gamma[\{x, x', y, i_x\}] \not\cong K_2 \oplus K_2$, we obtain $\{x, y\} \in E(\Gamma)$. It follows that $N_{\Gamma}(x) = Y$. Hence $x$ is the smallest element of $L$.

Lastly, we verify that $\Gamma - \{x, i_x\}$ is prime. Otherwise, $\Gamma - \{x, i_x\}$ admits a nontrivial module $M$. Since $\Gamma - \{x, i_x\}$ is connected and bipartite with bipartition $\{X \setminus \{x\}, Y \setminus \{i_x\}\}$, we have $M \subseteq X \setminus \{x\}$ or $M \subseteq Y \setminus \{i_x\}$. Since $N_{\Gamma}(x) = Y$ and $N_{\Gamma}(i_x) = \{x\}$, $M$ is a module of $\Gamma$, which contradicts the fact that $\Gamma$ is prime. Consequently $\Gamma - \{x, i_x\}$ is prime. Moreover, consider $v \in V(\Gamma) \setminus \{x, i_x\}$. Since $i_x$ is isolated in $\Gamma - x$, it is also isolated in $\Gamma - \{x, v\}$. Therefore $\Gamma - \{x, v\}$ is not prime. It follows that $i_x$ is the unique element of $V(\Gamma) \setminus \{x\}$ such that $\Gamma - \{x, i_x\}$ is prime. \qed

Claim 63. Let $x \in X$ such that $\Gamma - x$ is connected. For any nontrivial module $M$ of $\Gamma - x$, there exist $x^-, x^+ \in Y$ such that $M = \{x^-, x^+\}$, $\{x, x^-\} \notin E(\Gamma)$, and $\{x, x^+\} \in E(\Gamma)$.

Proof. Let $M$ be a nontrivial module of $\Gamma - x$. Since $\Gamma - x$ is connected, we have $M \subseteq X \setminus \{x\}$ or $M \subseteq Y$. In the first instance, $M$ is a module of $\Gamma$. Therefore $M \subseteq Y$. Set $M^- = \{y \in M : \{x, y\} \notin E(\Gamma)\}$ and $M^+ = \{y \in M : \{x, y\} \in E(\Gamma)\}$. Clearly, $M^-$ and $M^+$ are modules of $\Gamma$. Since $\Gamma$ is prime and $|M| \geq 2$, we obtain $|M^-| = 1$ and $|M^+| = 1$. Denote by $x^-$ the unique element of $M^-$, and denote by $x^+$ the unique element of $M^+$. We obtain $M = \{x^-, x^+\}$. Furthermore, $\{x, x^-\} \notin E(\Gamma)$ and $\{x, x^+\} \in E(\Gamma)$. \qed
Claim 64. Given \( x \in X \), if \( \Gamma - x \) is connected, then there exist \( x^-, x^+ \in Y \) satisfying the following assertions

1. \( \{ x^-, x^+ \} \) is the only nontrivial module of \( \Gamma - x \);
2. \( \{ x, x^- \} \notin E(\Gamma) \) and \( \{ x, x^+ \} \in E(\Gamma) \);
3. for every \( u \in X \), if \( u < x \mod L \), then \( \{ u, x^- \} \notin E(\Gamma) \);
4. for every \( u \in X \), if \( x < u \mod L \), then \( \{ u, x^+ \} \notin E(\Gamma) \);
5. \( \Gamma - \{ x, x^- \} \) and \( \Gamma - \{ x, x^+ \} \) are prime;
6. \( x^+ \) is the unique element of \( V(\Gamma) \setminus \{ x \} \) such that \( \{ x, x^+ \} \in E(\Gamma) \) and \( \Gamma - \{ x, x^+ \} \) is prime.

Proof. Since \( \Gamma \) is critical, \( \Gamma - x \) admits a nontrivial module \( M \). By Claim 63, there exist \( x^-, x^+ \in Y \) such that \( M = \{ x^-, x^+ \} \), \( \{ x, x^- \} \notin E(\Gamma) \), and \( \{ x, x^+ \} \in E(\Gamma) \). Hence \( \{ x^-, x^+ \} \) is a nontrivial module of \( \Gamma - x \).

For a contradiction, suppose that \( M \) is not the only nontrivial module of \( \Gamma - x \). Thus, there exists a nontrivial module \( N \) of \( \Gamma - x \) such that \( N \neq M \). By Claim 63, there exist \( z^-, z^+ \in Y \) such that \( N = \{ z^-, z^+ \} \), \( \{ x, z^- \} \notin E(\Gamma) \), and \( \{ x, z^+ \} \in E(\Gamma) \). If \( M \cap N \neq \emptyset \), then \( M \cup N \) is a nontrivial module of \( \Gamma - x \) of size 3, which contradicts Claim 63. Hence \( M \cap N = \emptyset \). We show that \( M \cup N \) is a module of \( \Gamma - x \). Let \( u \in (X \setminus \{ x \}) \setminus (M \cup N) \). It suffices to verify that \( M \cup N \) is a module of \( \Gamma \). Suppose that there exists \( v \in M \cup N \) such that \( \{ u, v \} \in E(\Gamma) \). For instance, suppose that \( v \in M \). Since \( M \) is a module of \( \Gamma - x \), we have \( \{ u, x^- \}, \{ u, x^+ \} \in E(\Gamma) \). We have \( \{ u, x^- \} \notin E(\Gamma) \), \( \{ x, x^- \} \notin E(\Gamma) \), and \( \{ x, x^+ \} \in E(\Gamma) \). Since \( K_2 \oplus K_2 \notin \Gamma \), we obtain \( \{ u, x^+ \} \in E(\Gamma) \). Since \( \{ z^-, z^+ \} \in E(\Gamma) \), it is a module of \( \Gamma - x \), we have \( \{ u, z^- \} \notin E(\Gamma) \). Therefore, \( \{ u, u \} \in E(\Gamma) \) for every \( u \in M \cup N \). It follows that \( M \cup N \) is a module of \( \Gamma - x \), which contradicts Claim 63 because \( |M \cup N| = 4 \). Consequently, \( \{ x^-, x^+ \} \) is the only nontrivial module of \( \Gamma - x \). It follows that \( \Gamma - \{ x, x^- \} \) and \( \Gamma - \{ x, x^+ \} \) are prime.

Let \( u \in X \) such that \( u < x \mod L \). Since \( u < x \mod L \), we have \( N_{\Gamma}(u) \supseteq N_{\Gamma}(x) \). Hence \( \{ u, x^+ \} \in E(\Gamma) \) because \( \{ x, x^+ \} \in E(\Gamma) \). Since \( \{ x^-, x^+ \} \in E(\Gamma) \), we obtain \( \{ u, x^+ \} \notin E(\Gamma) \).

Let \( u \in X \) such that \( x < u \mod L \). Since \( x < u \mod L \), we have \( N_{\Gamma}(x) \supseteq N_{\Gamma}(u) \). Hence \( \{ u, x^- \} \notin E(\Gamma) \) because \( \{ x, x^- \} \notin E(\Gamma) \). Clearly, \( \{ u, x^- \} \notin E(\Gamma) \). Since \( \{ x^-, x^+ \} \in E(\Gamma) \), we obtain \( \{ u, x^- \} \notin E(\Gamma) \).

As previously seen, \( \Gamma - \{ x, x^- \} \) and \( \Gamma - \{ x, x^+ \} \) are prime. Now, consider \( v \in V(\Gamma) \setminus \{ x, x^-, x^+ \} \). Clearly, \( \{ x^-, x^+ \} \) is a nontrivial module of \( \Gamma - \{ v \} \), so \( \Gamma - \{ x, v \} \) is not prime. Since \( \{ x, x^- \} \notin E(\Gamma) \), \( x^+ \) is the unique element of \( V(\Gamma) \setminus \{ x \} \) such that \( \{ x, x^+ \} \in E(\Gamma) \) and \( \Gamma - \{ x, x^+ \} \) is prime.

Definition 65. We define a function \( \varphi : X \longrightarrow Y \) as follows. Given \( x \in X \),

\[
\varphi(x) = \begin{cases} 
  i_x & \text{if } \Gamma - x \text{ is disconnected (see Claim 62)}, \\
  x & \text{if } \Gamma - x \text{ is connected (see Claim 64)}.
\end{cases}
\]
The next claim follows easily from Claim 62 and 64.

Claim 66. For every \( x \in X \), \( \varphi(x) \) is the unique element of \( V(\Gamma) \setminus \{x\} \) such that \( \{x, \varphi(x)\} \in E(\Gamma) \) and \( \Gamma - \{x, \varphi(x)\} \) is prime.

In the next two claims, we verify that \( \varphi \) is bijective.

Claim 67. \( \varphi \) is injective.

Proof. Consider distinct \( u, v \in X \). For instance, suppose that \( u < v \) mod \( L \). In particular, \( v \) is not the smallest element of \( L \). It follows from Claim 62 that \( \Gamma - v \) is connected. By Claim 64, there exist \( v^-, v^+ \in Y \) such that \( \{v, v^-\} \notin E(\Gamma) \), \( \{v, v^+\} \in E(\Gamma) \), and \( \{v^-, v^+\} \) is the only nontrivial module of \( \Gamma - v \). We have \( \varphi(v) = v^+ \).

First, suppose that \( \Gamma - u \) is disconnected. We have \( \varphi(u) = i_u \), where \( i_u \) is the unique isolated vertex of \( \Gamma - u \) by Claim 62. We obtain \( \{v, \varphi(u)\} \notin E(\Gamma) \). Thus \( \varphi(u) \neq \varphi(v) \) because \( \{v, \varphi(v)\} \in E(\Gamma) \) (see Claim 66).

Second, suppose that \( \Gamma - u \) is connected. By Claim 64, there exist \( u^-, u^+ \in Y \) such that \( \{u, u^-\} \notin E(\Gamma) \), \( \{u, u^+\} \in E(\Gamma) \), and \( \{u^-, u^+\} \) is the only nontrivial module of \( \Gamma - u \). We have \( \varphi(u) = u^+ \). Since \( u < v \) mod \( L \), it follows from the fourth assertion of Claim 64 applied to \( u \) that \( \{v, \varphi(u)\} \notin E(\Gamma) \). Since \( \{v, \varphi(v)\} \in E(\Gamma) \) (see Claim 66), \( \varphi(u) \neq \varphi(v) \).

Claim 68. \( \varphi \) is surjective.

Proof. Let \( v \in Y \). Since \( \Gamma \) is critical, \( \Gamma - v \) is not prime.

First, suppose that \( \Gamma - v \) is disconnected. As in Claim 62, we obtain that \( \Gamma - v \) admits an isolated vertex \( i_v \). Thus \( N_\Gamma(i_v) = \{v\} \). Since \( \{i_v, \varphi(i_v)\} \in E(\Gamma) \), we obtain \( \varphi(i_v) = v \).

Second, suppose that \( \Gamma - v \) is connected. As in Claim 64, there exist \( v^-, v^+ \in X \) such that \( \{v^-, v^+\} \) is the only nontrivial module of \( \Gamma - v \), \( \{v, v^-\} \notin E(\Gamma) \), and \( \{v, v^+\} \in E(\Gamma) \). Furthermore, \( \Gamma - \{v, v^-\} \) and \( \Gamma - \{v, v^+\} \) are prime. Thus \( \Gamma - \{v, v^+\} \) is prime, and \( \{v, v^+\} \in E(\Gamma) \). It follows from Claim 66 that \( v = \varphi(v^+) \).

It follows from Claims 67 and 68 that \( \varphi \) is bijective.

Claim 69. \( \Gamma \) is the half graph defined from the linear order \( L \), and the bijection \( \varphi \).

Proof. Consider distinct \( u, x \in X \). We have to verify that

\[
\{u, \varphi(x)\} \in E(\Gamma) \text{ if and only if } u \leq x \text{ mod } L.
\]

Suppose that \( u \leq x \) mod \( L \). We obtain \( N_\Gamma(x) \subseteq N_\Gamma(u) \). By Claim 65, \( \varphi(x) \in N_\Gamma(x) \). Hence \( \varphi(x) \in N_\Gamma(u) \). Conversely, suppose that \( x < u \) mod \( L \). In particular, \( u \) is not the smallest element of \( L \). It follows from Claim 62 that \( \Gamma - u \) is connected. By the fourth assertion of Claim 64 applied to \( x \), \( \{u, x^+\} \notin E(\Gamma) \), that is, \( \{u, \varphi(x)\} \notin E(\Gamma) \).

\[\square\]
Claim 70. Given \( x \in X \), if \( x \) is not the smallest element of \( L \), then \( x \) admits a predecessor in \( L \).

Proof. Let \( x \in X \). Suppose that \( x \) is not the smallest element of \( L \). It follows from Claim 62 that \( \Gamma - x \) is connected. By Claim 64 there exist \( x^- \), \( x^+ \in Y \) such that \( \{x^-, x^+\} \) is the only nontrivial module of \( \Gamma - x \), \( \{x, x^-\} \notin E(\Gamma) \), and \( \{x, x^+\} \in E(\Gamma) \). Furthermore, for every \( u \in X \), we have

\[
\text{if } u < x \text{ mod } L, \text{ then } \{u, x^-\} \in E(\Gamma),
\]

by the third assertion of Claim 64 applied to \( x \). Set \( t = \varphi^{-1}(x^-) \).

By Claim 63 \( \{t, \varphi(t)\} \in E(\Gamma) \), that is, \( \{t, x^-\} \in E(\Gamma) \). We obtain \( x^- \in N_\Gamma(t) \setminus N_\Gamma(x) \). Hence \( N_\Gamma(t) \varnot\subseteq N_\Gamma(x) \), so \( t < x \) mod \( L \). We prove that \( t \) is the predecessor of \( x \). It suffices to verify that

\[
(t, x)_L = \emptyset.
\]

First, suppose that \( \Gamma - t \) is disconnected. By Claim 62 there exists \( i_t \in Y \) such that \( i_t \) is an isolated vertex of \( \Gamma - t \). Since \( \varphi(t) = i_t \), \( i_t = x^- \). We obtain that \( \{u, x^-\} \notin E(\Gamma) \) for every \( u \in V(\Gamma) \setminus \{t, x^-\} \). It follows from 11 that \( (t, x)_L = \emptyset \). Second, suppose that \( \Gamma - t \) is connected. By Claim 64 there exist \( t^-, t^+ \in Y \) such that \( \{t^-, t^+\} \) is the only nontrivial module of \( \Gamma - t \), \( \{t, t^-\} \notin E(\Gamma) \), and \( \{t, t^+\} \in E(\Gamma) \). Furthermore, for every \( u \in X \) such that \( t < u \) mod \( L \), we have \( \{u, t^+\} \notin E(\Gamma) \) by the fourth assertion of Claim 64 applied to \( t \). Recall that \( t^+ = \varphi(t) \). Since \( t = \varphi^{-1}(x^-) \), we obtain \( t^+ = x^- \). Therefore, for every \( u \in X \) such that \( t < u \) mod \( L \), we have \( \{u, x^-\} \notin E(\Gamma) \). It follows from 11 that \( (t, x)_L = \emptyset \). □

By Remark 55 \( \Gamma \) is also the half graph defined from the linear order \( \varphi(L)^* \) defined on \( Y \), and the bijection \( \varphi^{-1} : Y \rightarrow X \). The analogue of Claim 70 for \( \varphi(L)^* \) follows.

Claim 71. Given \( y \in Y \), if \( y \) is not the smallest element of \( \varphi(L)^* \), then \( y \) admits a predecessor in \( \varphi(L)^* \).

The next claim is an immediate consequence of Claims 71.

Claim 72. Given \( x \in X \), if \( x \) is not the largest element of \( L \), then \( x \) admits a successor in \( L \).

It follows from Claims 70 and 72 that \( L \) is discrete, which completes the proof of Theorem 22.

As announced in Subsection 1.3 we discuss Theorem 26 by using Theorems 17 and 22.
Remark 73. We denote by $\mathbb{Q}$ the set of rational numbers, and $L_\mathbb{Q}$ denotes the usual linear order on $\mathbb{Q}$. Obviously, $L_\mathbb{Q}$ is not discrete. We consider the graph $G$ defined on $\{0, 1, 2, 3\} \cup (\{0, 1\} \times \mathbb{Q})$ by

$$E(G) = \{(0, 1), (1, 2), (2, 3)\} \cup \{(1, (1, q)) : q \in \mathbb{Q}\} \cup (\bigcup_{q \in \mathbb{Q}} \{(0, q), (1, r) : r \geq q\}).$$

Set $X = \{0, 1, 2, 3\}$, $Y = \{0\} \times \mathbb{Q}$ and $Z = \{1\} \times \mathbb{Q}$. We have $G[X]$ is prime because $G[X] = P_4$. We consider the 2-structure $\sigma_G$ associated with $G$. Since $G[X]$ is prime, $\sigma_G[X]$ is prime too. We have $Y = \langle X \rangle_{\sigma_G}$, $Z = X_{\sigma_G}(0)$, and $p_{(\sigma_G, X)} = \{Y, Z\}$. Furthermore, it is not difficult to verify that

$$\Gamma_{(\sigma_G, X)} = G[Y \cup Z].$$

(12)

We verify that $\sigma_G$ is finitely $\overline{X}$-critical (see Definition 25), without being $\overline{X}$-critical.

We show that Statement (Sk) holds for every odd integer $k \geq 1$. Let $W$ be a finite and nonempty subset of $Y \cup Z$ such that $W \in \varepsilon_{(\sigma_G, X)}$ (see Notation 3). We have to show that $W$ is even. If $W \cap Y = \emptyset$, then $\{0\} \cup W$ is a module of $\sigma_G[X \cup W]$ because $Z = X_{\sigma_G}(0)$. Hence $W \cap Y = \emptyset$. We denote the elements of $W \cap Y$ by $(0, q_0), \ldots, (0, q_m)$, where $m \geq 0$, in such a way that $q_0 < \cdots < q_m$, when $m \geq 1$. Set $Z^- = \{j < q_0 : (1, j) \in W\}$. Since $Z = X_{\sigma_G}(0), \{0\} \cup \{(1) \times Z^\ast\}$ is a module of $\sigma_G[X \cup W]$. Hence $Z^- = \emptyset$. Set $Z^\ast = \{j \geq q_m : (1, j) \in W\}$. We obtain that $\{1\} \times Z^\ast$ is a module of $\sigma_G[X \cup W]$. Hence $|Z^\ast| \leq 1$. If $Z^\ast = \emptyset$, then $(X \cup W) \setminus \{(0, q_m)\}$ is a module of $\sigma_G[X \cup W]$ because $(0, q_m) \in X_{\sigma_G}$. Thus $|Z^\ast| = 1$. Therefore, $|W| = 2$ if $m = 0$. Suppose that $m \geq 1$, and set $Z_i = \{j \in [q_i, q_{i+1}) : (1, j) \in W\}$ for $i = 0, \ldots, m - 1$. Given $i = 0, \ldots, m - 1$, we have $\{1\} \times Z_i$ is a module of $\sigma_G[X \cup W]$. Hence $|Z_i| \leq 1$. Moreover, $\{(0, q_i), (0, q_{i+1})\}$ is a module of $\sigma_G[X \cup W]$ if $Z_i = \emptyset$. Therefore, $|Z_i| = 1$. Consequently, $Z^- = \emptyset$, $|Z^\ast| = 1$, and $|Z_i| = 1$ for $i = 0, \ldots, m - 1$. Thus, $|W \cap Z| = m + 1$, and hence $|W| = 2m + 2$.

We prove that $\sigma_G$ is finitely $\overline{X}$-critical. Let $F$ be a finite subset of $Y \cup Z$. There exists a finite subset $F'$ of $Q$ such that $|F'| \geq 2$ and $F \subseteq \{(0, 1) \times F'\}$. We have $G[\{(0, 1) \times F'\}] \cong H_{2\times|F'|}$ (see Figure 1). It follows from Proposition 67 that $G[\{(0, 1) \times F'\}]$ is critical. Set $\tilde{F} = \{(0, 1) \times F'\}$. We obtain that

$$F \subseteq \tilde{F} \text{ and } G[\tilde{F}] \text{ is critical.}$$

(13)

It follows from 12 and 13 that $\Gamma_{(\sigma_G[X \cup F], F)}$ is critical. Since Statement (S5) holds, it follows from Theorem 17 that $\sigma_G[X \cup \tilde{F}]$ is $\tilde{F}$-critical. Consequently, $\sigma_G$ is finitely $\overline{X}$-critical.

Since $\sigma_G$ is finitely $\overline{X}$-critical, it follows from Theorem 20 that $\sigma_G$ is prime. Lastly, we verify that $\sigma_G$ is not $\overline{X}$-critical. To begin, we verify that $G[Y \cup Z]$ is a non discrete half graph. Clearly, $G[Y \cup Z]$ is bipartite with bipartition $\{Y, Z\}$. Consider the bijection $\varphi : Y \rightarrow Z$, which maps $(0, q)$ to $(1, q)$ for each $q \in \mathbb{Q}$. 30
Moreover, consider the linear order $L_Y$ defined on $Y$ as follows. Given distinct $q, r \in \mathbb{Q}$, $(0, q) < (0, r) \mod L_Y$ if $q < r \mod L_Q$. Clearly, $G[Y \cup Z]$ is the half graph defined from $L_Y$ if $\varphi$. Since $L_Y \simeq L_Q$, $G[Y \cup Z]$ is not discrete.

Since Statement (S5) holds, $P_5 / \not \mid \Gamma(\sigma_G, X)$ by Lemma 53. Since $G[Y \cup Z]$ is a non-discrete half graph, $\Gamma(\sigma_G, X)$ is the half graph defined from $L_Y$ and $\varphi$. Clearly, $G[Y \cup Z]$ is connected. Therefore, $\Gamma(\sigma_G, X)$ is connected by (12). It follows from Theorem 22 that $\sigma_G$ is not critical. Clearly, $G[Y \cup Z]$ is connected.

Therefore, $\Gamma(\sigma_G, X)$ is connected by (12). It follows from Theorem 17 that $\sigma_G$ is not $X$-critical. Since $\sigma_G$ is prime, there exists $v \in X$ such that $\sigma_G - v$ is prime.

In fact, we have $\sigma_G - w$ is prime for every $w \in X$.

Appendices

A Description of partially critical 2-structures

We use the following notation.

**Notation 74.** Given a 2-structure $\sigma$, consider $X \subseteq V(\sigma)$ such that $\sigma[X]$ is prime. Suppose that Statement (S3) holds. Let $C$ be a component of $\Gamma(\sigma, X)$. Consider $x, x' \in B^C_q$ and $y, y' \in D^C_q$ (see Notation 49) such that $\{x, y\} \in E(\Gamma(\sigma, X))$ and $\{x', y'\} \in E(\Gamma(\sigma, X))$. Since $C$ is connected, it follows from Fact 35 that $[x, y] = [x', y']_\sigma$. We denote $[x, y]_\sigma$ by $s_C$.

**Fact 75.** Given a 2-structure $\sigma$, consider $X \subseteq V(\sigma)$ such that $\sigma[X]$ is prime. Suppose that Statement (S5) holds, and $\sigma$ is $X$-critical. Under these assumptions, $\sigma$ is entirely determined by $\sigma[X]$, $q(\sigma, X)$, $\Gamma(\sigma, X)$, and $\{s_C : C \in C(\Gamma(\sigma, X))\}$.

Proof. We make the following preliminary observation. Since Statement (S5) holds, Statement (S3) holds as well (see Remark 11). Since $\sigma$ is prime, it follows from Corollary 39 that $\Gamma(\sigma, X)$ has no isolated vertices.

We have to determine $[x, y]_\sigma$ for distinct vertices $x, y$ of $\sigma$ such that $\{x, y\} \setminus X \neq \emptyset$. To begin, consider $x \in X$ and $y \in \overline{X}$. Since $\text{Ext}_\sigma(X) = \emptyset$, $[x, y]_\sigma$ is determined by the block of $q(\sigma, X)$ containing $y$. For instance, if $y \in X_{\sigma,e,f}(\alpha)$, where $e, f \in E(\sigma)$ and $\alpha \in X$, we have

$$[x, y]_\sigma = \begin{cases} (f, e) & \text{if } x = \alpha \\ [x, \alpha]_\sigma & \text{if } x \neq \alpha. \end{cases}$$

Now, we consider distinct $x, y \in \overline{X}$. To begin, we suppose that $x$ and $y$ belong to the same block of $p(\sigma, X)$.
First, suppose that \( \{x, y\} \cap B^q_q \neq \emptyset \), where \( B^q_q \in q(\sigma, X) \). Since \( B^q_q \in q(\sigma, X) \), there exists \( e \in E(\sigma) \) such that \( B^q_q = \langle X \rangle^{(e, e)}_\sigma \) or \( B^q_q = X^{(e, e)}_\sigma(\alpha) \), where \( \alpha \in X \). Since \( \Gamma(\sigma, X) \) has no isolated vertices (see (14)), it follows from Lemma 40 that \( B^q_q \in p(\sigma, X) \). Hence \( x, y \in B^q_q \). Since \( \sigma \) is prime, it follows from the first assertion of Lemma 41 that \( \{x, y\}_\sigma = (e, e) \).

Second, suppose that \( x \in B^q_q \) and \( y \in D^q_q \), where \( B^q_q \) and \( D^q_q \) are distinct elements of \( q(\sigma, X) \). Recall that \( \Gamma(\sigma, X) \) has no isolated vertices (see (14)). Therefore, we can apply Lemmas 40 and 41 as follows. Since \( x \) and \( y \) belong to the same block of \( p(\sigma, X) \), it follows from Lemma 40 that \( B^q_q \cup D^q_q \in p(\sigma, X) \). We use Lemma 11 to determine \( \{x, y\}_\sigma \). For instance, if \( x \in \langle X \rangle^{(e, f)}_\sigma \) and \( y \in \langle X \rangle^{(f, e)}_\sigma \), where \( e, f \in E(\sigma) \) with \( e \neq f \), then \( \{x, y\}_\sigma = (e, f) \) by the first assertion of Lemma 11.

Third, suppose that \( x, y \in B^q_q \), where \( B^q_q \in q(\sigma, X) \). Since \( B^q_q \in q(\sigma, X) \), there exist distinct \( e, f \in E(\sigma) \) such that

\[
B^q_q = \begin{cases} 
\langle X \rangle^{(e, f)}_\sigma \\
X^{(e, f)}_\sigma(\alpha), \text{ where } \alpha \in X.
\end{cases}
\]

To determine \( \{x, y\}_\sigma \), we describe \( \sigma[B^q_q] \) in the following manner. Let \( C \) be the component of \( \Gamma(\sigma, X) \) containing \( x \). Since \( \Gamma(\sigma, X) \) has no isolated vertices (see (14)), it follows from the second assertion of Proposition 48 that \( B^q_q \in V(C) \). For distinct \( u, v \in B^q_q \), set

\[
u < v \mod l(B^q_q) \text{ if } \{u, v\}_\sigma = (e, f).
\]

Since \( \sigma \) is prime, it follows from the second assertion of Lemma 45 that \( l(B^q_q) \) is a linear order. For instance, suppose that \( B^q_q = B^q_q \) (see Notation 49). Recall that \( C \) is a bipartite graph, with bipartition \( \{B^q_q, D^q_q\} \). Since \( |B^q_q| \geq 2 \), it follows from Theorem 17 that \( v(C) \geq 4 \) and \( C \) is critical. Moreover, \( P_5 \notin C \) by Lemma 53. It follows from Theorem 22 that \( C \) is a discrete half graph. Precisely, for distinct \( u, v \in B^q_q \), set

\[
u < v \mod L(B^q_q) \text{ if } NC(u) \nsubseteq NC(v) \) (see Definition 61).

Furthermore, we define a function \( \varphi(B^q_q) : B^q_q \rightarrow D^q_q \) as in Definition 65. By Claims 67 and 68, \( \varphi(B^q_q) \) is bijective. Lastly, by Claim 69, \( C \) is the half graph defined from the linear order \( L(B^q_q) \), and the bijection \( \varphi(B^q_q) \). Consider distinct \( u, v \in B^q_q \) such that \( u < v \mod L(B^q_q) \). It follows that

\[
\varphi(B^q_q)(u) \in NC(u) \setminus NC(v).
\]

By Corollary 68,

\[
[u, v]_\sigma = \begin{cases} 
(f, e) \text{ if } B^q_q = \langle X \rangle^{(e, f)}_\sigma \\
(e, f) \text{ if } B^q_q = X^{(e, f)}_\sigma(\alpha).
\end{cases}
\]

32
Given distinct \( u, v \in B^a_q \), it follows that

\[
[u, v]_\sigma = (e, f) \quad \text{if and only if} \quad \begin{cases} N_C(v) \not\subseteq N_C(u) \text{ and } B^a_q = (X)_\sigma^{(e,f)} \\
\text{or} \\
N_C(u) \not\subseteq N_C(v) \quad \text{and} \quad B^a_q = X_\sigma^{(e,f)}(\alpha). \end{cases}
\]

Furthermore, observe that

\[
l(B^a_q) = \begin{cases} L(B^a_q)^* \quad \text{if} \quad B^a_q = (X)_\sigma^{(e,f)} \\
\text{and} \\
L(B^a_q) \quad \text{if} \quad B^a_q = X_\sigma^{(e,f)}(\alpha). \end{cases}
\]

Lastly, we suppose that \( x \in B_p \) and \( y \in D_p \), where \( B_p \) and \( D_p \) are distinct elements of \( p(\sigma, X) \).

- First, suppose that \( \{x, y\} \notin E(\Gamma(\sigma, X)) \). Suppose that \( B_p = (X)_\sigma \). There exist \( e, f \in E(\sigma) \) such that \( x \in (X)_\sigma^{(e,f)} \). By the first assertion of Lemma 8 [3] \( [x, y]_\sigma = (e, f) \). Suppose that there exist distinct \( \alpha, \beta \in X \) such that \( B_p = X_\sigma(\alpha) \) and \( B_q = X_\sigma(\beta) \). By the second assertion of Lemma 8 [3] \( [x, y]_\sigma = [\alpha, \beta]_\sigma \).

- Second, suppose that \( \{x, y\} \in E(\Gamma(\sigma, X)) \). There exist \( B_q, D_q \in q(\sigma, X) \) such that \( x \in B_q \) and \( y \in D_q \). Thus \( B_q \not\subseteq B_p \) and \( D_q \not\subseteq D_p \). Denote by \( C \) the component of \( \Gamma(\sigma, X) \) containing \( x \) and \( y \). We obtain \( x \in V(C) \cap B_q \) and \( y \in V(C) \cap D_q \). Therefore \( x \in B^C_q \) and \( y \in D^C_q \) (see Notation 49). Hence \( [x, y]_\sigma = s_C \) (see Notation 74).

\[\Box\]

**Remark 76.** Given a 2-structure \( \sigma \), consider \( X \not\subseteq V(\sigma) \) such that \( \sigma[X] \) is prime. Suppose that Statement (S3) holds, and \( \sigma \) is \( X \)-critical. Let \( C = C(\Gamma(\sigma, X)) \) such that \( v(C) > 2 \). Since \( C \) is prime by Theorem 16 we have \( \{x, y\}_\sigma \neq s_C \) for any \( x \in B^C_q \) and \( y \in D^C_q \) such that \( \{x, y\} \notin E(\Gamma(\sigma, X)) \).

We pursue by determining the modules created by partial criticality. We use the following notation.

**Notation 77.** Given a 2-structure \( \sigma \), consider \( X \not\subseteq V(\sigma) \) such that \( \sigma[X] \) is prime. Suppose that Statement (S5) holds, and \( \sigma \) is \( X \)-critical. Consider a component \( C \) of \( \Gamma(\sigma, X) \) such that \( v(C) \geq 4 \). By Theorem 17 \( C \) is critical. By Lemma 53 \( P_3 \notin C \). It follows from Theorem 22 that \( C \) is a half graph defined from a discrete linear order \( L \) defined on \( B^C_q \), and a bijection \( \varphi \) from \( B^C_q \) onto \( D^C_q \).

For distinct \( u, v \in B^C_q \), we have

\( \{u, \varphi(v)\} \in E(C) \) if and only if \( u \leq v \) mod \( L \).

It follows that for distinct \( u, v \in B^C_q \),

\( u < v \) mod \( L \) if and only if \( N_C(u) \not\subseteq N_C(v) \).
Thus, the linear order \( L \) is unique, it is denoted by \( L_C \).

Now, consider \( u \in B_C^q \). First, suppose that \( u \) is the largest element of \( L_C \). We obtain that \( \mathcal{N}_C(u) \notin \mathcal{N}_C(v) \) for each \( v \in B_C^q \setminus \{u\} \). It follows that \( \mathcal{N}_C(u) \) is a module of \( C \). Hence \( |\mathcal{N}_C(u)| = 1 \), and \( \varphi(u) \) is the unique element of \( \mathcal{N}_C(u) \).

Second, suppose that \( u \) is not the largest element of \( L_C \). Since \( L_C \) is discrete, \( u \) admits a successor \( u^* \) in \( L_C \). It follows that \( \mathcal{N}_C(u) \setminus \mathcal{N}_C(u^*) \) is a module of \( C \). Hence \( |\mathcal{N}_C(u) \setminus \mathcal{N}_C(u^*)| = 1 \), and \( \varphi(u) \) is the unique element of \( \mathcal{N}_C(u) \setminus \mathcal{N}_C(u^*) \).

Consequently, the bijection \( \varphi \) is unique, it is denoted by \( \varphi_C \).

Lastly, suppose that \( \Gamma_{(\sigma,\underline{X})} \) admits a component \( C \) such that \( v(C) \leq 3 \). By Theorem 17, \( v(C) = 2 \). Therefore \( |B_C^q| = |D_C^q| = 1 \) (see Notation 19). In this case, \( L_C \) denotes the unique linear order defined on \( B_C^q \), and \( \varphi_C \) denotes the unique function from \( B_C^q \) to \( D_C^q \).

The next fact follows from Theorems 10 and 22. We omit its proof.

**Fact 78.** Given a 2-structure \( \sigma \), consider \( X \in V(\sigma) \) such that \( \sigma[X] \) is prime. Suppose that Statement (S5) holds, and \( \sigma \) is \( \underline{X} \)-critical. Consider a component \( C \) of \( \Gamma_{(\sigma,\underline{X})} \). Let \( x \in B_C^q \). We have \( \sigma - \{x, \varphi_C(x)\} \) is prime. Set

\[
Y = V(\sigma) \setminus \{x, \varphi_C(x)\}.
\]

Then, one of the following assertions holds

- \( \varphi_C(x) \in (Y)_\sigma \), \( C \setminus x \) is disconnected, \( Y \) is the unique nontrivial module of \( \sigma - x \), and \( x \) is the smallest element of \( L_C \);
- \( \varphi_C(x) \in Y_\sigma(\alpha) \), where \( \alpha \in X \), \( C \setminus x \) is disconnected, \( \{\alpha, \varphi_C(x)\} \) is the unique nontrivial module of \( \sigma - x \), and \( x \) is the smallest element of \( L_C \);
- \( \varphi_C(x) \in Y_\sigma(\varphi_C(x^-)) \), where \( x^- \) is the predecessor of \( x \) in \( L_C \), \( C \setminus x \) is connected, and \( \{\varphi_C(x^-), \varphi_C(x)\} \) is the unique nontrivial module of \( \sigma - x \).

**Definition 79.** Given a prime 2-structure \( \sigma \), Theorem 8 leads Ille [10] to introduce the primality graph \( \mathbb{P}(\sigma) \) of \( \sigma \) as follows. It is defined on \( V(\sigma) \) as well, and its edges are exactly the non-critical unordered pairs of \( \sigma \) (see Definition 11). Hence, by Theorem 8, \( \mathbb{P}(\sigma) \) is nonempty when \( v(\sigma) \geq 7 \). The primality graph is an efficient tool to recognize primality in different contexts (see 10 and 6).

Given a 2-structure \( \sigma \), consider \( X \notin V(\sigma) \) such that \( \sigma[X] \) is prime. Suppose that Statement (S5) holds, and \( \sigma \) is \( \underline{X} \)-critical. Note that an element of \( \underline{X} \) is not isolated in \( \mathbb{P}(\sigma) \) by Theorem 24.

We end the section by determining the primality graph of a partially critical 2-structure outside the prime 2-substructure. We use the following lemma due to Ille [10].

**Lemma 80.** Consider a prime 2-structure \( \sigma \) such that \( v(\sigma) \geq 5 \). Given a critical vertex \( v \) of \( \sigma \) (see Definition 7), the following three assertions hold

1. \( d_{\mathbb{P}(\sigma)}(v) \leq 2 \);
2. if \( d_{\varphi(\sigma)}(v) = 1 \), then \( V(\sigma) \setminus (\{v\} \cup N_{\varphi(\sigma)}(v)) \) is a module of \( \sigma - v \);
3. if \( d_{\varphi(\sigma)}(v) = 2 \), then \( N_{\varphi(\sigma)}(v) \) is a module of \( \sigma - v \).

The next fact follows from Fact 71 and Lemma 80. We omit its proof.

**Fact 81.** Given a 2-structure \( \sigma \), consider \( X \subseteq V(\sigma) \) such that \( \sigma[X] \) is prime. Suppose that Statement (S5) holds, and \( \sigma \) is \( \overline{X} \)-critical. Consider a component \( C \) of \( \Gamma_{(\sigma,\overline{X})} \) such that \( v(C) \geq 6 \). Then, we have

\[
\mathbb{P}(\sigma)[V(C)] = \mathbb{P}(C).
\]

Moreover, the following two assertions hold.

1. For each \( x \in B^C_{\varphi} \), if \( \varphi_C(x) \in Y_\sigma(\alpha) \), where \( Y = V(\sigma) \setminus \{x, \varphi_C(x)\} \) and \( \alpha \in X \), then \( N_{\varphi(C)}(x) = \{\varphi_C(x)\} \) and \( N_{\varphi(\sigma)}(x) = \{\alpha, \varphi_C(x)\} \).
2. For each \( x \in B^C_{\varphi} \), \( N_{\varphi(C)}(x) \neq N_{\varphi(\sigma)}(x) \) if and only if \( \varphi_C(x) \in Y_\sigma(\alpha) \), where \( Y = V(\sigma) \setminus \{x, \varphi_C(x)\} \) and \( \alpha \in X \).

**B A new proof of Theorem 9**

**Proof of Theorem 9** Let \( \sigma \) be a prime 2-structure. Consider \( X \subseteq V(\sigma) \) such that \( \sigma[X] \) is prime. Suppose that \( \overline{X} \) is finite and \( |\overline{X}| \geq 6 \).

For a contradiction, suppose that for each proper subset \( Y \) of \( \overline{X} \), we have

\[
\text{if } \sigma[X \cup Y] \text{ is prime, then } |\overline{X} \cup Y| \text{ is odd.} \tag{16}
\]

For \( Y = \emptyset \) in (16), we obtain \( |\overline{X}| \) is odd. Hence \( |\overline{X}| \geq 7 \). For \( Y \neq \emptyset \), with \( |Y| = 1, 3 \) or 5, it follows from (16) that \( \sigma[X \cup Y] \) is not prime. Consequently Statement (S5) holds. Since \( |\overline{X}| \) is odd, there exists \( C \in \mathcal{C}(\Gamma_{(\sigma,\overline{X})}) \) such that \( v(C) \) is odd. Since \( \sigma \) is prime, it follows from Theorem 10 that \( \sigma[X \cup V(C)] \) is prime. We have \( \overline{X} = V(C) \cup \overline{X} \cup V(C) \). Since \( |\overline{X}| \) and \( v(C) \) are odd, we obtain that \( |\overline{X} \cup V(C)| \) is even. It follows from (16) that \( V(C) = \overline{X} \). Thus \( \mathcal{C}(\Gamma_{(\sigma,\overline{X})}) = \{\Gamma_{(\sigma,\overline{X})}\} \). Since \( \sigma \) is prime, it follows from Theorem 10 that \( \Gamma_{(\sigma,\overline{X})} \) is prime. By Proposition 11, \( \Gamma_{(\sigma,\overline{X})} \) is bipartite. Futhermore, \( P_5 \notin \Gamma_{(\sigma,\overline{X})} \) by Lemma 53. Therefore, it follows from Proposition 57 that \( \Gamma_{(\sigma,\overline{X})} \) is a half graph, which is impossible because \( v(\Gamma_{(\sigma,\overline{X})}) = |\overline{X}| \) and \( |\overline{X}| \) is odd.

Consequently (16) does not hold. Therefore, there exists \( Y \neq \overline{X} \) such that \( \sigma[X \cup Y] \) is prime, and \( |\overline{X} \cup Y| \) is even. Recall that \( \overline{X} \) is finite, so \( \overline{X} \cup Y \) is as well. Hence, by applying several times Theorem 4 from \( \sigma[X \cup Y] \), we obtain distinct \( v, w \in \overline{X} \cup Y \) such that \( \sigma - \{v, w\} \) is prime.

As announced in Subsection 1.1 we extend Theorem 10 as follows.
Theorem 82. Given a prime 2-structure \( \sigma \), consider \( X \subseteq V(\sigma) \) such that \( \sigma[X] \) is prime. Suppose that

\[
q^*(\sigma, X) \neq \emptyset \quad (\text{see Notation } 46).
\]

If \( X \) is finite and \( |X| \geq 4 \), then there exist distinct \( v, w \in X \) such that \( \sigma - \{v, w\} \) is prime.

Proof. By Theorem 9, we can assume that \( |X| = 4 \) or 5. If \( |X| = 4 \), then it suffices to apply Theorem 4. Hence suppose that \( |X| = 5 \). For a contradiction, suppose that Statement (S3) holds. It follows from Theorem 16 that for each component \( C \) of \( \Gamma(\sigma, X) \), we have \( v(C) = 2 \) or \( v(C) \geq 4 \) and \( C \) is prime. Since \( |X| = 5 \), we obtain that \( \Gamma(\sigma, X) \) is connected. Thus \( \Gamma(\sigma, X) \) is prime. Since \( \Gamma(\sigma, X) \) is connected, it follows from the first assertion of Proposition 18 that \( p(\sigma, X) = q(\sigma, X) \), and \( q(\sigma, X) \) has two elements, denoted by \( B_q \) and \( D_q \). Moreover, \( \Gamma(\sigma, X) \) is bipartite, with bipartition \( \{B_q, D_q\} \). Since \( \Gamma(\sigma, X) \) is prime and bipartite, we have \( \Gamma(\sigma, X) \cong P_5 \). Hence \( K_2 \oplus K_2 \leq \Gamma(\sigma, X) \). Thus, there exists distinct \( x, x' \in B_q \) and distinct \( y, y' \in D_q \) such that \( \{x, y\}, \{x', y'\} \in E(\Gamma(\sigma, X)) \) and \( \{x, y'\}, \{x', y\} \notin E(\Gamma(\sigma, X)) \). It follows from Fact 47 that \( B_q, D_q \in q^*(\sigma, X) \), which contradicts \( q^*(\sigma, X) \neq \emptyset \). Consequently, Statement (S3) does not hold. Hence, there exists \( Y \subseteq X \) such that \( |Y| = 3 \) and \( \sigma[X \cup Y] \) is prime, which completes the proof because \( |X| = 5 \). \( \square \)

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