EIGENVALUES OF THE LAPLACIAN ON THE GOLDBERG-COXETER CONSTRUCTIONS FOR 3- AND 4-VALENT GRAPHS

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ABSTRACT. We are concerned with spectral problems of the Goldberg-Coxeter construction for 3- and 4-valent finite graphs. The Goldberg-Coxeter construction $GC_{k,l}(X)$ of a finite 3- or 4-valent graph $X$ is a "subdivision" of $X$, whose number of vertices are increasing, nevertheless which has bounded girth. It is shown that the first several eigenvalues of the combinatorial Laplacian on $GC_{k,l}(X)$ can be estimated by the eigenvalues of $X$ itself, which implies that the first several eigenvalues of $GC_{k,l}(X)$ tend to 0 as $(k, l)$ goes to infinity. We also prove that the last several eigenvalues of $GC_{k,0}(X)$ tend to the natural upper bound as $k$ goes to infinity. These results reflect that the spectrum of $GC_{k,l}(X)$ is somehow related to that of $X$. However, it is shown that specific values always appear as eigenvalues of $GC_{2k,0}(X)$ almost independently to the structure of the initial graph $X$.

1. Introduction

The Goldberg-Coxeter construction is a subdivision of a 3- or 4-valent graph, and it is defined by Dutour-Deza [4] for a plane graph based on a simplicial subdivision of regular polytopes in [1, 7]. In [4], it is pointed out that it often appears in chemistry and architecture, and its combinatorial and algebraic structures are investigated. Goldberg-Coxeter constructions of regular polyhedra generate a class of Archimedean polyhedra, and infinite sequence of polyhedra, which are called Goldberg polyhedra. For example a Goldberg-Coxeter construction of a dodecahedron generates a truncated-icosahedron, which is known as a fullerene $C_{60}$ [10, 17]. Goldberg-Coxeter constructions are also applied to Mackay-like crystals, and explain large scale of spatial fullerenes [14, 16]. Mathematical modeling of self-assembly in nature is also widely studied in [1, 11]. Recently, Fujita et. al. have synthesized molecule structures with 4-valent Goldberg polyhedra, and they explain self-assembly from viewpoints of chemistry and biology [6].

On the other hand, the stability of a molecule is explained by eigenvalues of the finite graphs which express the molecule structure by Hückel method [2]. Hence, studies for eigenvalues of Goldberg-Coxeter constructions are worth trying. The Goldberg-Coxeter construction $GC_{k,l}(X)$ of a 3- or 4-valent graph $X$ has the parameters $k$ and $l$ both of which are integers and they are regarded to indicate a point in the triangular or square lattices, respectively. Then we are concerned with behavior of eigenvalues of $GC_{k,l}(X)$ when $k$ and $l$ tends to infinity.

Throughout this paper, unless otherwise indicated, a graph is always assumed to be connected, finite and simple. For a graph $X$, let us denote by $V(X)$ the set of vertices of $X$, and by $E(X)$ the set of undirected edges of $X$. For $p \in V(X)$, the set of its neighboring vertices is denoted by $N_X(p)$. The combinatorial Laplacian $\Delta_X$, simply called the Laplacian, of a graph $X$ acts on the set $\mathbb{R}^{V(X)}$ of functions on $V(X)$ and is defined as

$$(\Delta_X f)(p) := \deg(p)f(p) - \sum_{q \in N_X(p)} f(q) \quad \text{for } f \in \mathbb{R}^{V(X)} \text{ and } p \in V(X),$$

where $\deg(p) = 3$ or $4$ provided $X$ is respectively a 3- or 4-valent graph. As is well-known, the eigenvalues of $\Delta_X$ for a regular graph $X$ of degree $r$ necessarily lie in the interval $[0, 2r]$.

Key words and phrases. Goldberg-Coxeter constructions, graphs on surfaces, combinatorial Laplacian, spectra.
Theorem 1.3. The definition of the Goldberg-Coxeter constructions extends for general 3- or 4-valent graph \( X = (V(X), E(X)) \) equipped with an orientation at each vertex, in the sense that, for each \( p \in V(X) \), the set of edges emanating from \( p \) is ordered, and the following is proved.

**Theorem 1.1.** Let \( X = (V(X), E(X)) \) be a connected, finite and simple 3- or 4-valent graph equipped with an orientation at each vertex, \( X' = GC_{k,l}(X) \) be the Goldberg-Coxeter construction of \( X \), where \( k \geq l \geq 0 \) and \( k \neq 0 \) and

\[
0 = \lambda_1(X) < \lambda_2(X) \leq \cdots \leq \lambda_{|V(X)|}(X),
\]

\[
0 = \lambda_1(X') < \lambda_2(X') \leq \cdots \leq \lambda_{|V(X')|}(X')
\]

be the eigenvalues of their Laplacians \( \Delta_X, \Delta_{X'} \), respectively. Then there exist integers \( \mu(k,l) \) and \( \nu(k,l) \) depending only on \( k \) and \( l \) satisfying

\[
\lambda_i(X') \leq \begin{cases} 
\frac{\mu(k,l)}{k^2 + kl + l^2} \lambda_i(X), & \text{if } X \text{ is 3-valent,} \\
\frac{\nu(k,l)}{k^2 + l^2} \lambda_i(X), & \text{if } X \text{ is 4-valent,}
\end{cases}
\]

for \( i = 1, 2, \ldots, |V(X)| \). When \( X \) is 3-valent, \( \mu(k,l) \) satisfies

\[
\mu(k,l) = \begin{cases} 
k, & \text{if } l = 0, \\
3k, & \text{if } k = l > 0, \\
\leq 2k - 2, & \text{if } k > l > 0.
\end{cases}
\]

When \( X \) is 4-valent, \( \nu(k,l) \) satisfies

\[
\nu(k,l) = \begin{cases} 
k, & \text{if } l = 0, \\
2k, & \text{if } k = l > 0, \\
\leq 2k, & \text{if } k > l > 0.
\end{cases}
\]

As shall be explained later (cf. Proposition 2.2), if, in particular, \( X \) is “appropriately” embedded in an oriented surface, then \( X \) is endowed with a natural orientation at each vertex and \( GC_{k,l}(X) \) remains to be also embedded in the same surface. Thus (1.1) also gives an upper bound for such a graph \( X \).

There is a long line of works on upper bounds for the (especially, first nonzero) eigenvalues of general planar or genus \( g \) finite graphs (see [12][18] and the references therein). In [13], it is proved that the \( i \)-th eigenvalue of a graph embedded in an oriented surface of genus \( g \) is estimated from above by \( O((g + 1)\log^2(g + 1)i/n) \), where \( n \) is the number of the vertices. Our estimate (1.1) is different from their estimate on the point that (1.1) is independent of the genus.

On the other hand, as for the last several eigenvalues of \( GC_{k,0}(X) \) the following holds.

**Theorem 1.2.** Under the same assumptions as in Theorem 1.1 we have

\[
\lim_{k \to \infty} |V(GC_{k,0}(X))|^{-i+1}(GC_{k,0}(X)) = \begin{cases} 
6, & \text{if } X \text{ is 3-valent,} \\
8, & \text{if } X \text{ is 4-valent,}
\end{cases}
\]

for \( i = 1, 2, \ldots, |V(X)| \). If \( X \) is a bipartite 3-valent graph, then the convergence (1.3) remains valid also for arbitrary \( GC_{k,l}(X) \). Furthermore, for a fixed \( k \), the last \( |V(X)| \) eigenvalues of \( n \)-th iterated Goldberg-Coxeter constructions \( GC^n_{k,0}(X) \) converge to 6 or 8 exponentially fast as \( n \to \infty \).

As the following theorems show the Goldberg-Coxeter constructions have also steady eigenvalues.

**Theorem 1.3.** Let \( X \) be a connected, finite and simple 3-valent graph equipped with an orientation at each vertex, and \( GC_{2k,0}(X) \) be its Goldberg-Coxeter constructions for \( k \in \mathbb{N} \).

1. \( GC_{2,0}(X) \) has eigenvalue 4 and \( GC_{4,0}(X) \) has eigenvalue 2.
2. \( GC_{2k,0}(X) \) has eigenvalue 4, whose multiplicity is at least \( \lfloor k/2 \rfloor \).
(3) \(GC_{2k,0}(X)\) has eigenvalue 2, whose multiplicity is at least \(\lceil k/2 \rceil\).

*Here \([x]\) is the smallest integer \(\geq x\), and \(\lfloor x\rfloor\) is the largest integer \(\leq x\).*

**Theorem 1.4.** Let \(X\) be a connected, finite and simple 4-valent graph equipped with an orientation at each vertex, and \(GC_{2k,0}(X)\) be its Goldberg-Coxeter constructions for \(k \in \mathbb{N}\).

1. \(GC_{4,0}(X)\) has eigenvalue 4.
2. For \(k \geq 2\), \(GC_{2k,0}\) has eigenvalue 4, whose multiplicity is at least \([\lceil (k-1)/2 \rceil]\).

Problems on eigenvalues of combinatorial Laplacian on regular graphs are extensively investigated. In particular, an explicit formula of a limit density of eigenvalue distributions of certain sequences of regular graphs was obtained in [15], and its geometric proof using a trace formula is given in [9] (see also [3]). One of points in these works is that the sequence \(\{X_n\}\) of \(q\)-regular graphs with number of vertices \(|X_n| \to \infty\) as \(n \to \infty\) is assumed to have large girths \(g(X_n) \to \infty\) as \(n \to \infty\). From this assumption, the graphs \(X_n\) get similar, as \(n \to \infty\), to a universal covering graph, namely a \(q\)-regular tree at least locally, and then a trace formula becomes able to apply. The girths of the Goldberg-Coxeter constructions \(GC_{k,l}(X)\) with an initial graph \(X\) are uniformly bounded with respect to the parameters \(k\) and \(l\), and hence it would not be so straightforward to apply a trace formula to obtain a limit distribution of the eigenvalue distributions. Indeed, from several numerical results it is considered that the limit distributions of eigenvalue distributions of Goldberg-Coxeter constructions is not quite universal. This speculation is also supported by the following results.

**Theorem 1.5.** Let \(X\) be a connected, finite and simple 3-valent graph which is embedded in a plane. Assume that the number of edges surrounding each face is divisible by 3. Then the following hold.

1. The multiplicity of eigenvalue 4 of \(GC_{2,0}(X)\) is at least 3.
2. For any \(k \in \mathbb{N}\), both \(GC_{k,0}(X)\) and \(GC_{k,3}(X)\) have eigenvalue 4 (resp. 2), whose multiplicity is at least \([k/2]\) (resp. \([k/2]\)).

This paper is organized as follows. In Section 2 after giving the precise definition of the Goldberg-Coxeter constructions \(GC_{k,l}(X)\), let us study their structure which is related with the spectral problems. In Section 3 we obtain some comparisons of the eigenvalues of \(X\) and \(GC_{k,l}(X)\) to prove Theorem 1.4. In Section 4, we first present proofs of Theorem 1.3 and 1.4. At the end of this paper, we shall give a few criteria for a 3-valent plane graph \(X\) so that some \(GC_{k,0}(X)\)'s have eigenvalues 2 or 4, which proves Theorem 1.5.

### 2. Goldberg-Coxeter constructions

This section studies the structure of Goldberg-Coxeter constructions, which shall be necessary in the subsequent sections.

The notion of Goldberg-Coxeter constructions is defined, due to Deza-Dutour [4][5], for a plane graph. The definition can extend for a nonplanar graph \(X\); indeed, \(X\) has only to be equipped with an “orientation at each vertex”, and if, in particular, \(X\) is “appropriately” embedded on an oriented surface, then the constructions can be done on the surface (see Proposition 2.4). Let us give the precise definitions. To make description clear, we use the ring \(\mathbb{Z}[\omega]\) of Eisenstein integers and the ring \(\mathbb{Z}[i]\) of Gaussian integers, where \(\omega = e^{\pi i/3}\) and \(i = \sqrt{-1}\). \(\mathbb{Z}[\omega]\) gives the triangular lattice on \(\mathbb{C}\) having 0, 1 and \(\omega\) as its fundamental triangle, while \(\mathbb{Z}[i]\) gives the square lattice on \(\mathbb{C}\) having 0, 1, 1 + \(i\) and \(i\) as its fundamental square.

**Definition 2.1** (cf. Deza-Dutour [4][5]). Let \(X\) be a connected, finite and simple 3- or 4-valent (abstract) graph equipped with an orientation at each vertex in the sense that, for each \(p \in V(X)\), the set of edges emanating from \(p\) is ordered. For \((k, l) \in \mathbb{Z}^2\), \((k, l) \neq (0, 0)\), the Goldberg-Coxeter construction of \(X\) with parameters \(k\) and \(l\) is defined through the following steps.
(i) Let us first consider the equilateral triangle $\triangle = \triangle(0, z, \omega z)$ in $\mathbb{Z}[\omega]$ having the vertices $0, z = k + l\omega$ and $\omega z$ (resp. the square $\square = \square(0, z, (1 + i)z, iz)$ in $\mathbb{Z}[i]$ having the vertices $0, z = k + li, (1 + i)z$ and $iz$.

(ii) Let us take all the small triangles in $\mathbb{Z}[\omega]$ (resp. squares in $\mathbb{Z}[i]$) intersecting with $\triangle$ (resp. $\square$) in its interior and join the barycenters of the neighboring small triangles (resp. squares) to obtain a graph, which is, as an associated (abstract) graph with $p$ for each $p \in V(X)$, denoted by $\triangle(p)$ (resp. $\square(p)$). Let us assign each of the edges emanating from $p$ to exactly one edge of the triangle (resp. square) so that the orientation at $p$ coincides with the standard orientation of $\triangle$ in $\mathbb{Z}[\omega]$ (resp. $\square$ in $\mathbb{Z}[i]$). Note that $\triangle(p)$ (resp. $\square(p)$) has the $2\pi/3$-rotational symmetry (resp. the $\pi/2$-rotational symmetry).

(iii) For each $e \in E(X)$ with endpoints $p$ and $q$, we can glue $\triangle(p)$ and $\triangle(q)$ (resp. $\square(p)$ and $\square(q)$) similarly as in the original definitions as follows:

(iii-1) $\triangle(p)$ (resp. $\square(p)$) is identified, preserving the orientation, with the graph on $\triangle$ (resp. $\square$) so that $e$ is assigned to the edge $z, \omega z$ (resp. $z, (1 + i)z$);

(iii-2) $\triangle(q)$ (resp. $\square(q)$) is identified, preserving the orientation, with the graph on $\triangle(z, (1 + \omega)z, \omega z)$ (resp. $\square(z, 2z, (2 + i)z, iz)$) so that $e$ is assigned to the edge $\omega z, z$ (resp. $(1 + i)z, z$);

(iii-3) then let us glue $\triangle(p)$ and $\triangle(q)$ (resp. $\square(p)$ and $\square(q)$) by identifying all the vertices and edges overlapping with each other.

The obtained (abstract) graph is again a 3-valent (resp. 4-valent) graph, which is denoted by $GC_{k,l}(X) = GC_{k,l}(X)$, where $z = k + l\omega$ (resp. $z = k + li$).

**Proposition 2.2.** Let $X$ be a connected, finite and simple 3- or 4-valent graph which is embedded in an oriented surface $M$ in such a way that the closure of each face is simply connected. Then for $(k, l) \in \mathbb{Z}^2$, $(k, l) \neq (0, 0)$, $GC_{k,l}(X)$ is well-defined and is also embedded in $M$.

**Proof.** The oriented tangent plane to $M$ at $p \in V(X)$ defines the orientation at $p$, and $GC_{k,l}(X)$ is defined. The notion of faces is also well-defined. Since each face of $X$ is simply connected, we can take a dual graph $D_X$ of $X$ in $M$, all of whose faces are simply connected triangles (resp. rectangles) for the 3-valent case (resp. 4-valent case). The dividing step (ii) and the gluing step (iii) in Definition 2.1 are well done in $M$ via respective appropriate local charts. □

A Goldberg-Coxeter construction $GC_{k,l}(X)$ for 3-valent (resp. 4-valent) graph $X$ inserts some hexagons (resp. squares), according to its parameter $k$ and $l$, between each pair of original faces of $X$. The most famous example is a fullerene $C_{60}$, called also a buckminsterfullerene or a buckyball, which is nothing but $GC_{1,1}$(Dodecahedron). This construction owes its name to the pioneering work \[7\] due to M. Goldberg, where a so-called Goldberg polyhedron (a convex polyhedron whose 1-skeleton is a 3-valent graph, consisting of hexagons and pentagons with rotational icosahedral symmetry 3-valent graph as its 1-skeleton) is studied and proved to be of the form $GC_{k,l}(Dodecahedron)$ for some $k$ and $l$. A Goldberg-Coxeter construction for 3- or 4-valent plane graphs occurs in many other context; see \[4\] and the references therein. Several examples of Goldberg-Coxeter constructions for nonplanar 3-valent (infinite or finite quotient) graphs, such as for carbon nanotubes and Mackay-like crystals, are provided in \[14\].

The following proposition summarizes a few fundamental properties of Goldberg-Coxeter constructions.

**Proposition 2.3** (Deza-Dutour \[4\] \[5\]). Let $X = (V(X), E(X))$ be a 3-valent (resp. 4-valent) graph equipped with an orientation at each vertex. Then the following hold.

1. If $X$ is embedded in an oriented surface in such a way that the closure of each face is simply connected, and the orientation at each vertex coincides with the one of the surface, then $GC_z(GC_{z'}(X)) = GC_{z'}(X)$, for any $z, z' \in \mathbb{Z}[\omega]$ (resp. $z, z' \in \mathbb{Z}[i]$).
2. For any $(k, l) \in \mathbb{Z}^2$, $(k, l) \neq (0, 0)$, we have the following graph isomorphisms:

$$GC_{k,l}(X) \cong GC_{-l,k+l}(X) \cong GC_{-k-l}(X) \cong GC_{l,-k}(X) \cong GC_{k,l,-k}(X),$$

4
Each vertex \( x \).

Let a bipartition of Proposition 2.5.

Then for any \( k \geq l \geq 0 \), \( k \neq 0 \) gives a system of representatives of graph isomorphism classes.

(3) The number of vertices of \( GC(X) \) is given as \( |V(GC(X))| = |z|^2|V(X)| = (k^2 + kl + l^2)|V(X)| \), where \( z = k + lw \).

In consideration of Proposition 2.3 (2), in the rest of this paper, we assume that \( k \) is a positive integer and \( l \) is a nonnegative integer satisfying \( k \geq l \geq 0 \) and \( k \neq 0 \).

2.1. Clusters for Goldberg-Coxeter constructions. A cluster is the central notion in this paper. Its definitions shall be given below in two different cases: where \( X \) is 3-valent and where \( X \) is 4-valent.

2.1.1. The case where \( X \) is 3-valent. For each \( p \in V(X) \), let us construct a subgraph \( X(p) = (V(p), E(p)) \) of \( \Delta(p) \subseteq GC_k,l(X) \), called the \((k, l)\)-cluster, so as to have \( k^2 + kl + l^2 \) vertices and the \( 2\pi/3 \)-rotational symmetry of \( \Delta(p) \). For this, we just have to define \( V(p) \) by the set of vertices \( x \) of \( \Delta(p) \) (considered as the graph on \( \Delta \subseteq \mathbb{Z}[\omega] \)) satisfying one of the following conditions:

(i) \( x \in \Delta(p) \) corresponds to a triangle in \( \mathbb{Z}[\omega] \) whose barycenter lies in the interior of \( \Delta = \Delta(0, z, \omega z) \), where \( z = k + lw \);

(ii) \( x \in \Delta(p) \) corresponds to an upward triangle in \( \mathbb{Z}[\omega] \) whose barycenter lies on an edge of \( \Delta \).

Here we mean an upward triangle \( \Delta(a, b) \) by the triangle in \( \mathbb{Z}[\omega] \) with vertices \( a + b\omega, a + 1 + b\omega \) and \( a + (b + 1)\omega \) for \( a, b \in \mathbb{Z} \) (see Figure 1). We also denote by \( \nabla(a, b) \), called downward triangle, the triangle with vertices \( a + b\omega, a + (b + 1)\omega \) and \( a - 1 + (b + 1)\omega \).

In the case that \( l = 0 \), \( X(p) \) is nothing but \( \Delta(p) \) itself, has \( k^2 \) vertices and has the dihedral symmetry \( D_3 \) (of order 6) (see Figure 1).

In the case that \( k = l > 0 \), it is easily seen that there are \( 2m = 2 \gcd(k, l) \) barycenters. Among these \( 2m \) vertices, exactly \( m \) vertices corresponding to upward triangles have just two adjacent triangles with barycenters lying in \( \Delta \). The combined \( 3m \) vertices on the three edges of \( \Delta \) are located in symmetric position with the rotation by \( 2\pi/3 \) of \( \Delta \).

Lemma 2.4 shows that the subgraph \( X(p) \) has \( (k - l)^2 + 3kl = k^2 + kl + l^2 \) vertices and also has the \( 2\pi/3 \)-rotational symmetry in the remaining case that \( k > l > 0 \).

Here we can prove the following proposition, which guarantees that the bipartiteness is kept after a Goldberg-Coxeter construction.

**Proposition 2.5.** Let \( X \) be a 3-valent bipartite graph equipped with an orientation at each vertex. Then for any \((k, l) \in \mathbb{Z}^2, (k, l) \neq (0, 0), GC_{k,l}(X) \) is also bipartite. So the spectrum of \( GC_{k,l}(X) \) is symmetric with respect to \( 3 \).

**Proof.** Let a bipartition of \( X \) be given and either black or white be assigned to each vertex \( p \in V(X) \). Each vertex \( x \) of each \((k, l)\)-cluster \( X(p) \) can be colored according to a rule that if \( p \) is white, then

- paint \( x \) black, provided the triangle in \( \mathbb{Z}[\omega] \) corresponding to \( x \) is upward;
- paint \( x \) white, provided the triangle in \( \mathbb{Z}[\omega] \) corresponding to \( x \) is downward;
and if \( p \) is black, then exchange black and white above. A white vertex is adjacent only to black vertices in \( X \), and two adjacent clusters \( X(p) \) and \( X(q) \) are positioned, in \( \mathbb{Z}[\omega] \), at \( \pi \)-rotation around the midpoint of an edge of \( \Delta \), which switches upward and downward triangles. So, the rule above gives a bipartition of \( \text{GC}_{k,l}(X) \).

2.1.2. The case where \( X \) is 4-valent. Similarly as in the 3-valent case, we construct for each \( p \in V(X) \) an appropriate subgraph \( X(p) = (V(p), E(p)) \) of \( \square(p) \), still called the \((k,l)\)-cluster, so as to have \( k^2 + l^2 \) vertices. To this end, we need to clarify the cases where a barycenter of a small square in \( \mathbb{Z}[i] \) lies on an edge of \( \square = \square(0, z, (1 + i)z, iz) \), where \( z = k + li \).

For \( a, b \in \mathbb{Z} \), we denote by \( \square(a,b) \) the small square in \( \mathbb{Z}[i] \) with vertices \( a + bi, (a + 1) + bi, (a + 1) + (b + 1)i, a + (b + 1)i \), whose barycenter is given as \( a + 1/2 + (b + 1/2)i \).

**Lemma 2.6.** Let \( k \geq 1 \geq l \geq 0, k \neq 0, m := \gcd(k,l) \), \( k_1 := k/m \) and \( l_1 := l/m \). An edge of the square \( \square = \square(0, z, (1 + i)z, iz) \), where \( z = k + li \) in \( \mathbb{Z}[i] \) passes through a barycenter of a small square in \( \mathbb{Z}[i] \) if and only if

\[
\text{(2.2)} \quad k_1 \not\equiv 0 \pmod{2}, \quad k_1 \equiv l_1 \pmod{2}
\]

Moreover, if this is the case, each edge of \( \square \) passes through exactly \( m \) barycenters.

Unlike the 3-valent case, we cannot choose a cluster \( X(p) \) with \( k^2 + l^2 \) vertices to have the \( \pi/2 \)-rotational symmetry in the case where \( k_1 \not\equiv 0 \pmod{2} \), \( k_1 \equiv l_1 \pmod{2} \) and \( m \not\equiv 0 \pmod{2} \) because no vertex of \( \square(p) \) is positioned at the barycenter of \( \square \) and \( k^2 + l^2 = m^2((k_1 - l_1)^2 + 2k_1l_1) \) is not divided by 4. Even in such cases, \( X(p) \) only has to have the same number of outward edges among the four directions to every adjacent cluster.

**Lemma 2.7.** Let \( X \) be a 4-valent graph equipped with an orientation at each vertex. Then there exists an Euler circuit \( e \) of \( X \) which turns either left or right at every vertex of \( X \).

**Proof.** As is well-known, any 4-valent graph \( X \) has an Euler circuit, which is by definition a closed path in \( X \) which visits every edge exactly once. Let us take an Euler circuit \( e \) of \( X \) and suppose that \( e \) goes straight ahead at a vertex \( p \in V(X) \). The circuit \( e \) comes back to \( p \) again from one of the other directions after it straight ahead at \( p \) (because \( X \) is 4-valent). By following the interval in opposite directions, the obtained circuit goes straight ahead one time fewer than \( e \). This proves Lemma 2.7.

The Euler circuit \( e \) obtained in Lemma 2.7 assigns a direction to each edge of \( X \) such that the direction alternates between inward and outward at each vertex of \( X \).

Now we can clearly define \( V(p) \) by the set of vertices \( x \) of \( \square(p) \) satisfying one of the following conditions:

(i) \( x \) corresponds to a square in \( \mathbb{Z}[i] \) whose barycenter lies in the interior of \( \square \);

(ii) \( x \) corresponds to a barycenter lying on the two edges of \( \square \) with opposite sides which correspond to the outward edges of \( X \) with respect to the Euler circuit \( e \) in Lemma 2.7.
2.2. **Structure of Goldberg-Coxeter constructions for the 3-valent case.** In this subsection a graph $X$ is assumed to be 3-valent and *embedded in a plane*, and we study some structure of Goldberg-Coxeter constructions of $X$. Let us consider the following four conditions on $X$, which is, as shall be seen in Section 4.3, related to the multiplicities of certain eigenvalues of Goldberg-Coxeter constructions.

**F** The number of edges surrounding each face is divisible by 3.

**CN** For each vertex $p \in V(X)$, the numbers 1, 2 and 3 are assigned in this order, with respect to the positive orientation, to the three edges of $X$ with $p$ as the common endpoint.

**N** There exists a vertex numbering $V(GC_{2,0}(X)) \to \{1, 2, 3\}$ with the following properties:

- **(N-i)** The number 0 is assigned to the center of each $(2,0)$-cluster $V(p)$ ($p \in X$);
- **(N-ii)** the number assigned to $x \in V(GC_{2,0}(X))$ is different from those of the adjacent vertices of $x$.

**C** $V(X)$ can be colored by two colors, say black and white, with the following properties:

- **(C-i)** A black vertex is adjacent to three white vertices;
- **(C-ii)** a white vertex is adjacent to exactly one black vertex, so the other two adjacent vertices are white;
- **(C-iii)** for any pair of black vertices $x, y \in V(X)$ which are three vertices away from each other, there is a path from $x$ to $y$ either turning left twice or turning right twice.

The *coherent edge numbering* (CN) implies the condition (N); indeed, let $p \in V(X)$ and let $e_1, e_2$ and $e_3$ be three edges of $X$ emanating from $p$. We assign 0 to $p$ regarded as a vertex of $GC_{2,0}(X)$, and, for $i = 1, 2$ and 3, assign $i$ to the vertex of $GC_{2,0}(X)$ positioned at the “opposite-side” to $e_i$. The resulting numbering of vertices of $GC_{2,0}(X)$ satisfies (N-i) and (N-ii) (see Figure 3). Moreover, as is easily proved, (N) implies the condition (F). So the following proposition shows that (F), (CN) and (N) are mutually equivalent.

**Proposition 2.8.** Let $X$ be a 3-valent plane graph satisfying (F). Then $X$ has a coherent edge numbering $E(X) \to \{1, 2, 3\}$ satisfying (CN).

**Proof.** For a sequence $e_1, e_2, \ldots, e_k \in E(X)$ of adjacent edges in $X$ (namely $e_i \neq e_{i+1}$ and they have a common endpoint), we denote by $(e_1, e_2, \ldots, e_k)$ the path along the edges starting from $m(e_1)$, the midpoint of $e_1$, and ending with $m(e_k)$. To get a desired numbering $\nu: E(X) \to \{1, 2, 3\}$, we fix an edge $e_0 \in E(X)$, and assign 3 to $e_0$. For adjacent edges $e, e' \in E(X)$ with the common endpoint $p$, let us define $\tau(e, e')$ as

$\tau(e, e') := \begin{cases} +1, & \text{if the path } (e, e') \text{ turns right at } p, \\ -1, & \text{if the path } (e, e') \text{ turns left at } p. \end{cases}$
Note that \( \tau(e, e') = -\tau(e', e) \). We then define \( \nu: E(X) \to \{1, 2, 3\} \) for \( e \in E(X) \) as

\[
\nu(e) := \sum_{i=1}^{n} \tau(e_{i-1}, e_i) \pmod{3}
\]

by choosing a path \( \gamma = (e_0, e_1, \ldots, e_{n-1}, e_n = e) \) from \( e_0 \) to \( e \). What we have to prove is that \( \nu(e) \) is independent of the choice of \( \gamma \). To this end, let \( \mathcal{P}(X, e_0) \) be the set of sequences of adjacent edges in \( X \) beginning with \( e_0 \) and let \( \varphi: \mathcal{P}(X, e_0) \to \{1, 2, 3\} \) be a map defined as

\[
\varphi(\gamma) := \sum_{i=1}^{n} \tau(e_{i-1}, e_i) \pmod{3}, \quad \text{for } \gamma = (e_0, e_1, \ldots, e_n) \in \mathcal{P}(X, e_0).
\]

Note that for any \( \gamma = (e_0, e_1, \ldots, e_n), \gamma' = (e_0, e'_1, \ldots, e'_{m-1}, e_n) \in \mathcal{P}(X, e_0) \), the joined (closed) path \( \gamma' \cdot \gamma = (e_0, e_1, \ldots, e_n, e'_{m-1}, \ldots, e'_1, e_0) \) satisfies

\[
\varphi(\gamma' \cdot \gamma) \equiv \varphi(\gamma) - \varphi(\gamma') \pmod{3}.
\]

Thus to prove that the map \( \nu \) defined by (2.4) is well-defined, it suffices to see that \( \varphi(\gamma) = 3 \) for any closed path \( \gamma = (e_0, e_1, \ldots, e_n = e_0) \). Notice that \( \varphi \) has the same image after removing a “backtracking” part, that is, if \( \gamma = (e_0, \ldots, e_{i-1}, e_i, e_{i+1}, \ldots, e_0) \) contains a triplet of mutually adjacent edges \( e_{i-1}, e_i \) and \( e_{i+1} \), then

\[
\varphi(\gamma) \equiv \varphi(e_0, \ldots, e_{i-2}, e_{i+2}, \ldots, e_0) \pmod{3},
\]

and if \( \gamma = (e_0, \ldots, e_{i-1}, e_i, e_{i+1}, \ldots, e_0) \) satisfies \( e_{i-1} = e_{i+1} \), then

\[
\varphi(\gamma) \equiv \varphi(e_0, \ldots, e_{i-1}, e_{i+2}, \ldots, e_0) \pmod{3}.
\]

Therefore the restriction \( \varphi: \mathcal{CP}(X, e_0) \to \{1, 2, 3\} \) of \( \varphi \) to \( \mathcal{CP}(X, e_0) \), the set of closed paths with base edge \( e_0 \), descends to a homomorphism

\[
\bar{\varphi}: \pi_1(X, m(e_0)) \to \{1, 2, 3\},
\]

where \( \pi_1(X, m(e_0)) \) is the fundamental group of \( X \) with base point \( m(e_0) \). Since \( \{1, 2, 3\} = \mathbb{Z}/3\mathbb{Z} \) is an abelian group, \( \bar{\varphi} \) further descends to a homomorphism

\[
\bar{\varphi}: \mathcal{H}_1(X, \mathbb{Z}) \to \{1, 2, 3\},
\]

where \( \mathcal{H}_1(X, \mathbb{Z}) \) is the 1-dimensional homology group of \( X \). Now any \( \gamma \in \mathcal{H}_1(X, \mathbb{Z}) \) can be written as \( \gamma = \sum_{f \text{ face of } X} a_f \partial f \), where \( a_f \in \mathbb{Z} \) and \( \partial f \) is the cycle consisting of edges around \( f \). Our assumption implies that \( \bar{\varphi}(\partial f) = 3 \) for any face \( f \) of \( X \). Hence we conclude that \( \bar{\varphi} \equiv 3 \), which implies that \( \varphi \equiv 3 \) on \( \mathcal{CP}(X, e_0) \).

A relation between (F) and (C) is stated as follows.

**Proposition 2.9.** Let \( X \) be a 3-valent plane graph satisfying (F). Then \( X \) has a vertex coherent coloring satisfying (C-i), (C-ii) and (C-iii).
**Proof.** Let \( p_0 \in V(X) \) be an arbitrary fixed vertex and color it black. Every vertex which is accessible by either turning left twice or turning right twice from a black vertex is, one after another, colored in black until no more vertices can be colored in black. The remaining vertices are colored in white. Now we have to check that (C-i) and (C-ii) are satisfied (while (C-iii) is necessarily satisfied). It is easily seen that a white vertex is adjacent to at least one black vertex; otherwise, all vertices of \( X \) must be white. It is also easily checked that if a white vertex is adjacent to two or more black vertices, then two other black vertices are necessarily adjacent somewhere else. So, it suffices to show that any pair of black vertices cannot be adjacent. Suppose that there is a pair of adjacent black vertices, say \( p, q \in V(X) \). From our way of the coloring, there is a path \( \gamma \) from \( p \) to \( q \) which is a sequence of either twice turning left or twice turning right between black vertices. Then \( \gamma \cup (q, p) \) is a closed path, which surrounds a finitely many faces, say \( f_1, f_2, \ldots, f_n \), after removing back-trackings. Now if \( n = 1 \), then \( \gamma \) consists of a circuit on the boundary \( \partial f_1 \) of a face \( f_1 \) and of some back-trackings with black base points on \( \partial f_1 \), which is a contradiction because the total of \( \tau \) defined by (2.3) is 0 (mod 3) after the crossing just prior to a lap of \( \gamma \cup (q, p) \). So assume that \( n \geq 2 \). There are just two possibilities of paths along the boundary of \( \bigcup_{i=1}^{n} f_i \) connecting a pair of black vertices with distance 3, as indicated in Figure 4. In either case, we can replace \( \gamma \cup (q, p) \) by a closed path which does not surround a face \( f_i \) (by ignoring back-trackings), and is still a sequence of either twice turning left or twice turning right between black vertices. Therefore the conclusion for the case where \( n \geq 2 \) can be deduced from the discussion given for the case \( n = 1 \). □

**Examples 2.10.**

1. The tetrahedron and any of its Goldberg-Coxeter constructions satisfy all the conditions above.
2. GC\(_{2,0}(X)\) for any 3-valent plane graph \( X \) always satisfies (C-i), (C-ii) and (C-iii); indeed, we just have to color only the “center” of each (2, 0)-cluster black, and the others white.
3. GC\(_{1,1}(X)\) for any 3-valent plane graph \( X \) also always satisfies (C-i), (C-ii) and (C-iii); indeed, we just have to color in accordance with the rule shown in Figure 6.
The case where \( k \geq l \geq 0 \) and \( k \neq 0 \) in consideration of Proposition 2.3 (2).

3. Solutions of the eigenvalues of \( \mathbb{R}^V(X) \) and \( \mathbb{C} \mathbb{R}^l(X) \)

This section is devoted to the proof of Theorem 1.1 and 1.2. Throughout this section, let \( k \) and \( l \) be integers satisfying \( k \geq l \geq 0 \) and \( k \neq 0 \) in consideration of Proposition 2.3 (2).

3.1. The case where \( X \) is 3-valent. Let \( p \in V(X) \), \( q \in N_X(p) \) and set

\[
V_0(p) := \{ x \in V(p) \mid N_X(x) \subseteq V(q) \},
\]

\[
V_1(p) := \{ x \in V(p) \mid |N_X(x) \cap V(q)| = i \} \quad (i = 1, 2).
\]

Note that, for any \( x \in V(p) \) and \( q \in N_X(p) \), there are at most two edges emanating from \( x \) to \( V(q) \).

Proof of Theorem 1.1. Since there is nothing to discuss when \( (k, l) = (1, 0) \), we only consider the other cases. Let \( c = 1/\sqrt{\|V(p)\|} = 1/\sqrt{\sqrt{k^2 + kl + l^2}} \) and define a linear map \( Q : \mathbb{R}^V(X) \to \mathbb{R}^V(X) \) for \( f \in \mathbb{R}^V(X) \) and for \( x \in V(p) \) by

\[
(Qf)(x) := cf(p).
\]

The transpose \( ^tQ : \mathbb{R}^V(X) \to \mathbb{R}^V(X) \) of \( Q \) is then written as

\[
(^tQg)(p) = c \sum_{x \in V(p)} g(x)
\]

for \( g \in \mathbb{R}^V(X) \) and \( p \in V(X) \). It then follows that for any \( f \in \mathbb{R}^V(X) \) and for any \( p \in V(X) \),

\[
(^tQQf)(p) = c \sum_{x \in V(p)} (Qf)(x) = c^2 \sum_{x \in V(p)} f(p) = f(p),
\]

that is \( ^tQQ = \text{id}_{\mathbb{R}^V(X)} \). Also, for arbitrary \( f \in \mathbb{R}^V(X) \),

\[
(^tQ\Delta_X f)(p) = c \sum_{x \in V(p)} (\Delta_X f)(x)
\]

\[
= c \sum_{x \in V(p)} \left\{ 3(Qf)(x) - \sum_{y \in N_X(x)} (Qf)(y) \right\}
\]

\[
= 3c^2|V(p)|f(p) - c \sum_{x \in V(p)} \sum_{y \in N_X(x)} (Qf)(y) - c \sum_{x \in V(p) \setminus V_0(p)} \sum_{y \in N_X(x)} (Qf)(y).
\]

The second term equals \(-3c^2|V_0(p)|f(p)\) and the third term is computed as

\[
c \sum_{x \in V(p) \setminus V_0(p)} \sum_{y \in N_X(x)} (Qf)(y) = c \sum_{q \in N_X(p) \setminus V_0(p)} \sum_{x \in V_1(p) \setminus N_X(x)} (Qf)(y) + c \sum_{q \in N_X(p) \setminus V_0(p)} \sum_{y \in N_X(x)} (Qf)(y)
\]
where the last equality follows from the symmetry of $X(p)$. Therefore we obtain

$$
\lambda Q \Delta X \cdot Q f(p) = c^2 (|V_1^d(p)| + 2|V_2^d(p)|) \mu(k, l) \Delta_X f(p),
$$

where $\mu(k, l)$ is the number of edges in $X'$ connecting two clusters and depends only on $k$ and $l$. Theorem 1.1 now immediately follows from the following.

**Theorem 3.1** (Interlacing property, see for example [2]). Let $Q$ be a real $n \times m$ matrix satisfying $^t QQ = I_m$ and $A$ be a real symmetric $n \times n$ matrix. If the eigenvalues of $A$ and $^t QAQ$ are 

$$
\nu_1(A) \leq \nu_2(A) \leq \cdots \leq \nu_p(A), \quad \nu_1(^t QAQ) \leq \nu_2(^t QAQ) \leq \cdots \leq \nu_m(^t QAQ),
$$

respectively, then

$$
\nu_j(A) \leq \nu_j(^t QAQ) \leq \nu_{n-m+j}(A) \quad (j = 1, 2, \ldots, m).
$$

The equality (1.2) for $l = 0$ or $k = l > 0$ are easily proved. Let us estimate the number of edges crossing the edge $E = 0z$ when $k > l > 0$. Notice first that there is at most one crossing edge emanating from an upward triangle $\Delta(a, b)$, and that there are at most two crossing edge emanating from a downward triangle $\Delta(a, b)$. For each $c \in \mathbb{Z}$, “the zigzag path” which is obtained by joining the barycenters of $\Delta(a, b)$ and $\Delta(a + 1, b - 1)$ for all $a, b \in \mathbb{Z}$ with $a + b = c$ crosses the edge $E = 0z$ exactly once provided $0 \leq c \leq k + l - 1$ and does not cross $E$ otherwise. Also, the line passing through $a \in \mathbb{Z}$ with slant $1 + \omega$ crosses $E$ exactly once provided $0 \leq a \leq k - l$ and does not cross $E$ otherwise. Therefore the number of edges crossing $E$ is at most $k + l + (k - l - 2) = 2k - 2$. (See Figure 7 for an example.)

![Figure 7](image)

**Figure 7.** $(k, l) = (9, 3)$. 15 edges cross the edge $E = 0z$, $(z = 9 + 3\omega)$.

The assertion in Theorem 1.2 for a bipartite graph $X$ is an immediate consequence from Theorem 1.1 and Proposition 2.5. The former one in Theorem 1.2 follows from the following.

**Theorem 3.2.** Let $X = (V(X), E(X))$ be a connected, finite and simple $3$-valent graph equipped with an orientation at each vertex. Then

$$
3 + \sqrt{5 + 4 \cos \frac{2\pi}{k}} \leq \lambda_{i\{V(GC_{k,0}(X))\}} \lambda_{i+1\{GC_{k,0}(X)\}}
$$

for $i = 1, 2, \ldots, |V(X)|$.

**Definition 3.3.** $\lambda \geq 0$ is called a $D_3$-invariant eigenvalue for a $(k, 0)$-cluster $X(p)$ if there exists a non-zero function $u: V(p) \rightarrow \mathbb{R}$, called a $D_3$-invariant eigenfunction, with the following properties.

(i) $u$ solves $(\Delta_X(p)) u(x) = \lambda u(x)$ for $x \in V(p)$

(ii) $u(\sigma x) = u(x)$ for $x \in V(p)$, where $\sigma: X(p) \rightarrow X(p)$ is an element of the dihedral group $D_3$. 

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The following remark shall be repeatedly used in the sequel: by assigning the same function $u$ to the other clusters, we have a global function $u: \text{GC}_{k,0}(X) \to \mathbb{R}$, which is an eigenfunction of $\Delta_{\text{GC}_{k,0}(X)}$ with eigenvalue $\lambda$; indeed, (1) $\Delta_{(p)} u = \lambda u$ is equivalent to a Neumann problem:

\begin{equation}
\begin{cases}
\Delta_{\text{GC}_{k,0}(X)} u(x) = \lambda u(x), & \text{for } x \in V(p), \\
u(y) - u(x) = 0, & \text{for } x \in V(p) \setminus V_0(p) \text{ and } y \in N_{\text{GC}_{k,0}(X)}(x) \setminus V(p).
\end{cases}
\end{equation}

Theorem 3.2 is an immediate consequence from the the following Lemmata 3.4 and 3.5.

**Lemma 3.4.** Let $X = (V(X), E(X))$ be a connected, finite and simple 3-valent graph equipped with an orientation at each vertex. If $\lambda \geq 0$ is a $D_3$-invariant eigenvalue for $(k,0)$-cluster, then

\begin{equation}
\lambda \leq \lambda_{\text{GC}_{k,0}(X)} - i + 1(\text{GC}_{k,0}(X))
\end{equation}

for $i = 1, 2, \ldots, |V(X)|$.

**Proof.** Let us replace $c$ in (3.2) by $u: \text{GC}_{k,0}(X) \to \mathbb{R}$ which is obtained from a $D_3$-invariant eigenfunction on the $(k,0)$-cluster. We may assume that $\sum_{x \in V(p)} u(x)^2 = 1/|V(X)|$, so that $\text{Q}_Q Q = \text{id}_{\text{R}^{|V(X)|}}$. After a straightforward computation using (i) and (ii) in Definition 3.3 for $u$, we can obtain the following equality:

\begin{equation}
(\text{Q}_Q \Delta_X \text{Q}f)(p) = \left\{ \sum_{x \in V_1'(p)} u(x)^2 + 2 \sum_{x \in V_2'(p)} u(x)^2 \right\} (\Delta_X f)(p) + \lambda f(p)
\end{equation}

for any $f \in \mathbb{R}^{V(X)}$ and any $p \in V(X)$, where $q \in N_X(p)$ is an adjacent vertex to $p$. Again from Theorem 3.1, the desired inequality is proved.

**Lemma 3.5.** Let $k$ be an integer with $k \geq 2$. Then

\begin{equation}
\lambda = 3 + \sqrt{5 + 4 \cos \frac{2\pi}{k}}
\end{equation}

is the largest $D_3$-invariant eigenvalue for the $(k,0)$-cluster.

**Proof.** Let us first construct all the eigenfunctions on a hexagonal lattice with toroidal boundary condition. If we set $m := (1 + \omega)/3$, where $\omega = e^{2\pi i/3}$, then the discrete set

\begin{equation}
[a + b\omega \mid a, b \in \mathbb{Z}] \cup [m + a + b\omega \mid a, b \in \mathbb{Z}]
\end{equation}

is naturally regarded as a hexagonal lattice. For a fixed $k \in \mathbb{N}$, let us consider the equations

\begin{equation}
\begin{align*}
3v(a + b\omega) - v(m + a + b\omega) - v(m + a - 1 + b\omega) - v(m + a + (b - 1)\omega) &= \lambda v(a + b\omega), \\
3v(m + a + b\omega) - v(a + b\omega) - v(a + 1 + b\omega) - v(a + (b + 1)\omega) &= \lambda v(m + a + b\omega)
\end{align*}
\end{equation}

for a function $v$ on the parallelogram

\begin{equation}
P(k) := \{a + b\omega \mid 0 \leq a, b \leq k - 1\} \cup \{m + a + b\omega \mid 0 \leq a, b \leq k - 1\},
\end{equation}

where $a$ and $b$ in (3.7) are considered modulo $k$, such as

\begin{equation}
3v(0) - v(m) - v(m + k - 1) - v(m + (k - 1)\omega) = \lambda v(0)
\end{equation}

for the former equation of (3.7) with $a = b = 0$. So if $v$ solves (3.7), then it gives an eigenfunction with eigenvalue $\lambda$ on the finite 3-valent graph $T(k)$ with $2k^2$ vertices obtained by adding edges between $a$ and $m + a + (k - 1)\omega$, and between $b\omega$ and $m + k - 1 + b\omega$ for each $a, b = 0, 1, \ldots, k - 1$.

A simple computation shows that all the eigenvalues are of the form

\begin{equation}
\lambda_{x,t}^* = \lambda_{x,t}^*(k) = 3 \pm \sqrt{3 + 2 \cos \frac{2\pi s}{k} + 2 \cos \frac{2\pi t}{k} + 2 \cos \frac{2\pi (s-t)}{k}},
\end{equation}
whose corresponding eigenfunction is given as

\[ v_{s,t}^\pm(a + b\omega) = e^{2\pi i(a+b)/k}, \]

\[ v_{s,t}^\pm(m + a + b\omega) = \frac{1}{3 - \Lambda_{s,t}^\pm} v_{s,t}^\pm(a + b\omega) \left(1 + e^{2\pi i/k} + e^{2\pi i/k}\right) \]

\((a + b\omega, m + a + b\omega \in P(k))\) for \(s, t = 0, 1, \ldots, k - 1\).

We now claim that (the real part of) the average \( u := \sum_{\sigma \in \mathcal{D}_6} \sigma v_{1,0}^+ \) under an action of \( D_6 \) on \( T(k) \) gives a function on the \((k, 0)\)-cluster \([a+b\omega \in P(k) \mid a+b \leq k-1] \cup [m+a+b\omega \in P(k) \mid a+b \leq k-2]\) satisfying (i) and (ii) in Definition 3.3 with \( \lambda = \lambda_{1,0}^+ \). Here \( D_6 \) is a dihedral group of order 12 generated by the three automorphisms on \( T(k) \) induced from

- the rotation by \( 2\pi/3 \):
  \[ \begin{cases} a + b\omega &\mapsto k - a - b - 1 + a\omega, \\ m + a + b\omega &\mapsto k - a - b - 2 + a\omega, \end{cases} \]

- the reflection along a diagonal line of the parallelogram:
  \[ \begin{cases} a + b\omega &\mapsto b + a\omega, \\ m + a + b\omega &\mapsto m + b + a\omega, \end{cases} \]

- and the reflection along the other one:
  \[ \begin{cases} a + b\omega &\mapsto m + k - b - 1 + (k - a - 1)\omega, \\ m + a + b\omega &\mapsto k - b - 1 + (k - a - 1)\omega, \end{cases} \]

where \( a \) and \( b \) are again considered modulo \( k \) and these maps are considered as \( P(k) \to P(k) \). Since the Laplacian on a graph is equivariant under the action of an automorphism and the Neumann boundary condition as in (3.3) is satisfied by the definition of \( u \), \( u \) satisfies \( \Delta u = \lambda_{1,0}^+ u \). Moreover it is easily checked by computing the total sum of \( v_{1,0}^+ \) along the “boundary” of \( P(k) \) and the “diagonal line between \((m+)k - 1 \) and \((m+)k - 1)\omega" of \( P(k) \) that \( u \) is not identically zero (except in the case \( k = 1 \)). This proves that \( \lambda_{1,0}^+ \) is a \( D_3 \)-invariant eigenvalue for the \((k, 0)\)-cluster \((k \geq 2)\). Since \( \lambda_{s,t}^+ = 6 \) if and only if \((s, t) = (0, 0)\) and the sign is positive, and since a \( D_3 \)-invariant eigenvalue is necessarily an eigenvalue of \( T(k), \lambda_{1,0}^+ \) above is the largest \( D_3 \)-invariant eigenvalue for the \((k, 0)\)-cluster.

\[ \square \]

Remark 3.6. Another \( \lambda_{s,t}^+ \) for \( s, t = 0, 1, \ldots, k - 1 \) is also a \( D_3 \)-invariant eigenvalue for the \((k, 0)\)-cluster, so that it is an eigenvalue of \( GC_{k,0}(X) \) unless the average \( \sum_{\sigma \in \mathcal{D}_6} \sigma v_{s,t}^\pm \) is identically zero on \( X(p) \). If \( s = t = 0 \), then \( \lambda_{0,0}^+ = 0 \) and \( \lambda_{1,0}^+ = 6 \), and the average for eigenfunctions corresponding to \( \lambda_{0,0}^+ \)'s is constant (= 12), while the one for eigenfunctions corresponding to \( \lambda_{1,0}^+ \)'s is identically zero.

3.2. The case where \( X \) is 4-valent. The same notation as (3.1) is used also in the 4-valent case. The notion of \( D_4 \)-invariant eigenvalue is also defined exactly in the same way as in the 3-valent case. Since the proof of Theorem 1.4 for the 4-valent case is almost similar as that for the 3-valent case, we omit it. Moreover, (3.4) is valid also for a 4-valent graph; indeed, the same equality as in (3.5) holds, whose proof is also omitted. A corresponding result to Lemma 3.5 is stated as follows (its proof is omitted again).

Lemma 3.7. Let \( k \) be an integer with \( k \geq 2 \). If \( k \) is even (resp. odd), then

\[ \lambda = 4 + 4 \cos \frac{2\pi}{k} \left( \text{resp.} 4 + 4 \cos \frac{\pi}{k} \right) \]

is the second largest (resp. the largest) \( D_4 \)-invariant eigenvalue for the \((k, 0)\)-cluster and converges to 8 as \( k \) tends to infinity.

4. On the eigenvalues 2 and 4 for Goldberg-Coxeter constructions

This section provides proofs of the theorems on multiplicities of eigenvalues 2, 4 stated in Section 1. In the first two subsections, we shall prove Theorems 1.3 and 1.4. As is seen below, a reason for large multiplicities of eigenvalues 2 or 4 of \( GC_{2k,0}(X) \) is that the \((2k, 0)\)-clusters also have large multiplicities of eigenvalues 2 or 4. On the other hand, it is considered that the structure of an initial graph \( X \) would affect the eigenvalue distribution of its Goldberg-Coxeter constructions. A
few remarkable examples shall be provided in Section 4.3 where a proof of Theorem 1.5 is also included.

4.1. The case where $X$ is 3-valent.

**Lemma 4.1.** The (2,0)-cluster (resp. (4,0)-cluster) has $D_3$-invariant eigenvalue 4 (resp. 2).

*Proof.* The function given in Figure 8(a) where $\alpha \in \mathbb{R}$ is arbitrary, is a $D_3$-invariant eigenfunction with eigenvalue 4 for the (2,0)-cluster. Also, the function given in Figure 8(b) is a $D_3$-invariant eigenfunction with eigenvalue 2 for the (4,0)-cluster.

\[\begin{array}{c}
\alpha \\
\alpha \\
\alpha \\
\end{array}\]

(a) with eigenvalue 4

\[\begin{array}{c}
\alpha \\
\alpha \\
\alpha \\
\end{array}\]

(b) with eigenvalue 2

Figure 8. $D_3$-invariant eigenfunctions

By induction starting with Lemma 4.1 we shall prove the following.

**Lemma 4.2.** A (2k,0)-cluster has $D_3$-invariant eigenvalue 4, whose multiplicity is exactly $\lfloor k/2 \rfloor$.

Theorem 1.3 (2) is an immediate consequence of Lemma 4.2 from what was mentioned right after Definition 3.3

*Proof of Lemma 4.2.* We introduce the coordinate in (3.6) on the vertex set $V(p)$ of the (2k,0)-cluster $X(p)$. For each positive integer $n$, we set

$$V_n := \{ a + b\omega \in \mathbb{Z}[\omega] \mid a, b \geq 0, a + b \leq n - 1 \}$$

$$\cup \{ m + c + d\omega \in \mathbb{Z}[\omega] \mid c, d \geq 0, c + d \leq n - 2 \},$$

which is considered as the vertex set of an $(n,0)$-cluster. The subgraph of the hexagonal lattice induced by $V_n$ is denoted by $X_n$, which is identified with an $(n,0)$-cluster. Also, for each $a + b\omega, m + c + d\omega \in V_n$, we label for the corresponding equation of $\Delta_{X(p)}u = 4u$ as follows:

$$E(a + b\omega) : u(a + b\omega) + u(m + a + b\omega) + u(m + a - 1 + b\omega) + u(m + a + (b - 1)\omega) = 0,$$

$$E(m + c + d\omega) : u(m + c + d\omega) + u(c + (d + 1)\omega) + u(c + d\omega) + u(c + 1 + d\omega) = 0,$$

where $u$ is supposed to take the same value at a vertex outside $V_n$ as at the unique adjacent vertex of $V_n$, such as

$$E(0) : u(0) + u(m) + u(0) + u(0) = 0,$$

$$E(1) : u(1) + u(m + 1) + u(m) + u(1) = 0.$$

Let us first discuss the solvability of the following families of equations and the $T_1$-invariance of the solutions, where $T_1 : \mathbb{C} \rightarrow \mathbb{C}$ is defined by $T_1(z) := \omega \bar{z}$.

(1-a) $\{ E(a + b\omega) \mid a + b = l, m \leq a \leq l - m \}$; assume that $u$ is defined on

$$\{ a + b\omega \in V_{2k} \mid a + b \leq l \} \cup \{ m + c + d\omega \in V_{2k} \mid c + d \leq l - 1 \},$$

and that $u$ is invariant under $T_1$ on this set.
(1-b) \( \{ E(m + a + b\omega) \mid a + b = l, \ m \leq a \leq l - m \} \); assume that \( u \) is defined on

\[
\{ a + b\omega \in V_{2k} \mid a + b \leq l \} \cup \{ m + c + d\omega \in V_{2k} \mid c + d \leq l \} \\
\cup \{ m + (l - m + 1)\omega, \ l - m + 1 + m\omega \},
\]

and that \( u \) is invariant under \( T_1 \) on this set.

It is easily proved that (1-a) for any \( l \) and (1-b) for \( l \) odd are uniquely solvable and that each \( u \) of the solutions is invariant under \( T_1 \) on the set where \( u \) is newly defined. In the case where \( l \) is even, it is also proved from the \( T_1 \)-invariance of \( u \) that (1-b) is uniquely solvable if and only if \( u(l) = u(l+1) \), and that the solution is invariant under \( T_1 \).

Let us define \( T_2 : \mathbb{C} \to \mathbb{C} \) by \( T_2(z) = T_2^1(z) = (1 - \omega) \bar{z} + (2k - 1)\omega \) and \( T_3 : \mathbb{C} \to \mathbb{C} \) by \( T_3(z) = T_3^1(z) = -\bar{z} + 2k - 1 \). We denote by (2-a) and (2-b) the families of equations transferred via \( T_1 \) from (1-a) and (1-b) respectively, and by (3-a) and (3-b) via \( T_2 \). Then similar arguments as above (or simply symmetry of \( V_{2k} \)) show the solvability and the \( T_2 \)-invariance (resp. \( T_3 \)-invariance) of the solutions of (2-a) and (2-b) (resp. (3-a) and (3-b)) provided \( u(2k-l-2+(l+1)\omega) = u(2k-l-1+l\omega) \) (resp. \( u((2k-l-1)\omega) = u((2k-l-2)\omega) \)). We shall finish the proof by using the solvability and the symmetry of the solutions of (1-a)–(3-b) in an appropriate order.

Let us consider only the case where \( k \) is odd and we set \( k = 2s + 1 \). The case where \( k \) is even can be similarly discussed using, for example, a \( D_3 \)-invariant eigenfunction on \( X_{4s-2} \), whose existence is about to be proved. Suppose that we are given a \( D_3 \)-invariant eigenfunction \( w \) on \( X_{4s-2} \) \( (s \geq 1) \) with eigenvalue 4 and an arbitrary real number \( a \). We further assume that \( w(2a) = w(2a+1) \) for \( a = 0, 1, \ldots, s - 1 \). We set

\[
\begin{align*}
  u(a + b\omega) := & \begin{cases} 
    w(a + b\omega), & \text{for } a + b \leq 2s - 1, \\
    w(a - 4 + b\omega), & \text{for } a \geq 2s + 2, \\
    w(a + (b - 4)\omega), & \text{for } b \geq 2s + 2,
  \end{cases} \\
  u(m + c + d\omega) := & \begin{cases} 
    w(m + c + d\omega), & \text{for } c + d \leq 2s - 1, \\
    w(m + c - 4 + d\omega), & \text{for } c \geq 2s + 1, \\
    w(m + c + (d - 4)\omega), & \text{for } d \geq 2s + 1,
  \end{cases}
\end{align*}
\]

(4.1)

where \( a, b, c \) and \( d \) are supposed to satisfy \( a + b\omega, m + c + d\omega \in V_{4s+2} \) in either case above (see Figure 9). (When \( s = 1 \), let \( w(m + c + d\omega) = w(c + d\omega) \) for \( c + d = 2s - 1 \).) We then set

\[
\begin{align*}
  u(2s) = u(2s+1) := & \alpha, \\
  u(2s\omega) = u((2s+1)\omega) := & \alpha, \\
  u(2s + (2s+1)\omega) = u(2s+1 + 2s\omega) := & \alpha
\end{align*}
\]

(4.2)

to satisfy \( u(2a) = u(2a+1) \) and \( u(2a\omega) = u((2a+1)\omega) \) for \( a = 0, 1, \ldots, 2s \). We shall show that (4.1) and (4.2) uniquely determine an \( D_3 \)-invariant eigenfunction \( u \) of \( \Delta_{X_{4s+2}} \) by solving the remaining equations toward the center of \( V_{4s+2} \).

Let us start by (i-b) with \( l = 2s - 1, m = 0 \) and with (4.2) for each \( i = 1, 2, 3 \). So far \( u \) is defined on

\[
\begin{align*}
  \{ a + b\omega \in V_{4s+2} \mid a + b \leq 2s + j \} \cup \{ m + c + d\omega \in V_{4s+2} \mid c + d \leq 2s - 1 + j \} \\
  \cup \{ a + b\omega \in V_{4s+2} \mid a \geq 2s + 1 + j \} \cup \{ m + c + d\omega \in V_{4s+2} \mid c \geq 2s + 1 + j \} \\
  \cup \{ a + b\omega \in V_{4s+2} \mid b \geq 2s + 1 + j \} \cup \{ m + c + d\omega \in V_{4s+2} \mid d \geq 2s + 1 + j \}
\end{align*}
\]

(4.3)

with \( j = 0 \), on which \( u \) is invariant under the \( D_3 \)-action on \( V_{4s+2} \), and on which \( E \) is satisfied except on “the inside boundary”:

\[
\begin{align*}
  \{ a + b\omega \in V_{4s+2} \mid a + b = 2s + j, \ 2j \leq a \leq 2s - j \} \\
  \cup \{ a + b\omega \in V_{4s+2} \mid a = 2s + 1 - j, \ 2j \leq b \leq 2s - j \} \\
  \cup \{ a + b\omega \in V_{4s+2} \mid b = 2s + 1 - j, \ 2j \leq a \leq 2s - j \}
\end{align*}
\]

(4.4)
Figure 9. GC_{10,0}(X) (s = 2); u takes the same value on the white vertices as w.

We assign \( \alpha \) to the six large black vertices.

with \( j = 0 \). Now we assume that on (4.3) with \( j \) replaced by \( j - 1 \) \((j \geq 1)\), \( u \) is defined and is invariant under \( D_3 \)-action on \( V_{4s+2} \) and \( E \) is satisfied except on the inside boundary (4.4) with \( j \) replaced by \( j - 1 \). Then, by symmetry we can solve \((i-a)\) with \( l = 2s - 1 + j, m = 2(j - 1) \) for \( i = 1, 2, 3 \). What we have to see is the solvability of

\[
E (2s + 1 - j + (2j - 1)\omega), \quad E (2s + 1 - j + (2s + 1 - j)\omega), \quad \text{and} \quad E (2j - 1 + (2s + 1 - j)\omega),
\]

which is valid because of \( u(2a) = u(2a + 1) \) and \( u(2a\omega) = u((2a + 1)\omega) \) for \( a = 0, 1, \ldots, 2s \). Now we conclude that \( u \) is defined on (4.3), where \( u \) is invariant under \( D_3 \)-action on \( V_{4s+2} \) and \( E \) is satisfied except on the inside boundary (4.4).

We can continue the inductive step until we reach the center of \( V_{4s+2} \), which is classified by \( 4s + 2 \) modulo 3 (see Figure 10). In the cases that \( 4s + 2 \equiv 0, 1 \pmod{3} \) we finish the proof in

Figure 10. The terminating steps; the white vertices are determined at the end.

the inductive arguments above. We only have to see that \( E(m + 4s/3 + 4s\omega/3) \) is satisfied in the case that \( 4s + 2 \equiv 2 \pmod{3} \), i.e. \( s \equiv 0 \pmod{3} \). This is valid again because of the induction hypothesis \( u(8s/3) = u(8s/3 + 1) \). Now we complete the proof in the case where \( k \) is odd and the proof of Lemma 4.2.

A very similar proof works for eigenvalue 2 and the following is obtained.

Lemma 4.3. A \((2k, 0)\)-cluster has \( D_3 \)-invariant eigenvalue 2, whose multiplicity is exactly \( \lfloor k/2 \rfloor \).
Proof. Since the proof is almost similar to that of Lemma 4.2 let us explain different parts. The corresponding equations $E(a + b\omega)$ and $E(m + c + d\omega)$ to $\Delta_{1\pi}(P)u = 2u$ can be written as

$$E(a + b\omega) : u(a + b\omega) - u(m + a + b\omega) - u(m + a - 1 + b\omega) - u(m + a + (b - 1)\omega) = 0,$$
$$E(m + c + d\omega) : u(m + c + d\omega) - u(c + (d + 1)\omega) - u(c + d\omega) - u(c + 1 + d\omega) = 0,$$

for $a + b\omega, m + c + d\omega \in V_n$, where $u$ is again supposed to take the same value at a vertex outside $V_n$ as at the unique adjacent vertex of $V_n$. We can prove the solvability and the symmetry of the solutions of (1-a)–(3-b) in which only the equalities $u(2a) = u(2a + 1)$, $u(2k - 2 - 2a) = u(2k - 2 - 2a)$ and $u((2k - 2 - 2a)\omega) = u((2k - 2 - 2a)\omega)$ are replaced with

$$u(2a) = -u(2a + 1),$$
$$u(2k - 2 - 2a) = -u(2k - 2 - 2a),$$
$$u((2k - 2 - 2a)\omega) = -u((2k - 2 - 2a)\omega),$$

respectively. The remaining proof is almost the same as that for Lemma 4.2 and let us omit it. □

Remark 4.4. A $(2k + 1, 0)$-cluster has no $D_3$-invariant eigenvalue 2 or 4. For, a $D_3$-invariant eigenvalue for a cluster is necessarily eigenvalue of $P(2k + 1)$ with toroidal boundary condition (see Section 3), while $\lambda_{k,1}^3(k)$ of (3.8) cannot be 2 or 4 provided $k$ is odd.

4.2. The case where $X$ is 4-valent. The results corresponding to Lemma 4.1, 4.2 and 4.3 are the following, which immediately imply Theorem 1.4

**Lemma 4.5.** In the case of 4-valent, the following hold.

1. The $(4, 0)$-cluster has $D_4$-invariant eigenvalue 4.
2. For $k \geq 2$, $s(2k, 0)$-cluster has $D_4$-invariant eigenvalue 4, whose multiplicity is exactly $\lceil (k - 1)/2 \rceil$.

Proof. (1) The function given in Figure 11(a) where $\alpha \in \mathbb{R}$ is arbitrary, is a $D_4$-invariant eigenfunction with eigenvalue 4.

**Figure 11.** $D_4$-invariant eigenfunctions with eigenvalue 4
(2) Since the proof is again almost similar as that of Theorem \(1.3\), let us explain part of the differences. We introduce the same coordinate \(\mathbb{Z}[i]\) as before on the vertex set \(V(p)\) of the \((2k,0)\)-cluster \(X(p)\). For each positive integer \(n\), we set
\[
V_n := \{a + bi \in \mathbb{Z}[i] \mid 0 \leq a \leq n - 1, \ 0 \leq b \leq n - 1\}.
\]
The induced subgraph by \(V_n\) is denoted by \(X_n\). Also, for each \(a + bi \in V_n\), we label for the corresponding equation of \(\Delta_{X(p)} u = 4u\) as follows:
\[
E(a + bi): \ u(a + 1 + bi) + u(a + (b + 1)i) + u(a - 1 + bi) + u(a + (b - 1)i) = 0,
\]
where \(u\) is supposed to take the same value at a vertex outside \(V_n\) as at the unique adjacent vertex of \(V_n\). As in the proof of Theorem \(1.3\), we can construct a \(D_4\)-invariant eigenfunction \(u\) on \(X_{2k}\) satisfying \(u(2a) = -u(2a + 1)\) by an inductive argument. Let us omit the remaining proof. (See Figure 11(b) for an example.)

4.3. Dependence on the structure of \(X\) for 3-valent case.

This subsection provides the proofs of Theorems \(1.5\) which describes relations between the conditions (F), (CN), (N) and (C) in Section 2.2 and eigenvalues 2 and 4 of some \(GC_{k,0}(X)\)'s.

**Proof of (1) of Theorem 1.5** Let us take a vertex numbering \(V(GC_{2,0}(X)) \rightarrow \{1, 2, 3\}\) satisfying (N), whose existence is guaranteed by Proposition 2.8. Let \(a_0, a_1, a_2\) and \(a_3\) be real numbers satisfying \(a_0 + a_1 + a_2 + a_3 = 0\). Then it is easy to see that the function \(v: V(GC_{2,0}(X)) \rightarrow \mathbb{R}\) which maps a vertex with number \(i\) to \(a_i\) is an eigenfunction of \(\Delta_{GC_{2,0}(X)}\) with eigenvalue 4. By this means we can find two more eigenfunctions which are linearly independent with \(u = u_1\) which was obtained in Theorem \(1.3\); in fact, set \((a_0, a_1, a_2, a_3) = (0, 1, -1, 0), (0, 1, 1, -2)\) for example. \(\square\)

Let us next consider the condition (C). Our assertions are summarized as follows.

**Proposition 4.6.** Let \(X\) be a 3-valent plane graph.

1. If \(X\) has a vertex coloring satisfying (C-i) and (C-ii), then for any \(s \in \mathbb{N}\), \(GC_{2s-1,0}(X)\) has eigenvalue 4.

2. If \(X\) has a vertex coloring satisfying (C-i), (C-ii) and (C-iii), then for any \(k \in \mathbb{N}\), both \(GC_{k,0}(X)\) and \(GC_{k,k}(X)\) have eigenvalue 4 (resp. 2), whose multiplicity is at least \([k/2]\) (resp. \([k/2]\)).

**Proof.** (1) The function \(u: V(X) \rightarrow \mathbb{R}\) which maps a black vertex to -3 and a white one to 1 is an eigenfunction of \(\Delta_X\) with eigenvalue 4, which proves (1) for \(s = 1\).

For \(s \geq 2\), a quadruplet \(\{X(p), X(q_1), X(q_2), X(q_3)\}\) of \((2s - 1, 0)\)-clusters, where \(p\) is black and \(N_X(p) = \{q_1, q_2, q_3\}\) are all white, can be glued with each other to be identified with a \((4s - 2, 0)\)-cluster. On the other hand, it follows from the proof of Lemma 4.2 that there is a \(D_3\)-invariant eigenfunction \(u\) on the \((4s - 2, 0)\)-cluster \(X_{4s-2}\) with eigenvalue 4 with constant boundary values, or \(u(a) = u(a + 1)\) for \(a = 0, 1, \ldots, 4s - 3\). Therefore \(u\) defines an eigenfunction on \(GC_{2s-1,0}(X)\) with eigenvalue 4, proving (1).

(Note here that \(X_{4s+2}\) has no nontrivial \(D_3\)-invariant eigenfunction with eigenvalue 2 satisfying \(u(a) = -u(a + 1)\) for \(a = 0, 1, \ldots\) because \(u(2s + 1) = u(2s + 2) = 0\).)

(2) In the argument above to prove the existence \(u\) on \(X_{4s-2}\), if (C-iii) is further satisfied, then any \(D_3\)-invariant eigenfunction on \(X_{4s-2}\) with eigenvalue 4 (resp. eigenvalue 2) gives an eigenfunction on \(GC_{2s-1,0}(X)\) with eigenvalue 4 (resp. eigenvalue 2). For exactly the same reason, any \(D_3\)-invariant eigenfunction on \(X_{15}\) with eigenvalue 4 (resp. eigenvalue 2) gives an eigenfunction \(GC_{25,0}(X)\) with eigenvalue 4 (resp. eigenvalue 2). This and (3) in Examples 2.10 now prove (2). \(\square\)

**Proof of (2) of Theorem 1.5** The assertion is an immediate consequence of Proposition 4.6 and Proposition 2.9.

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