Singular Hochschild cohomology via the singularity category
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Abstract. We show that the singular Hochschild cohomology (=Tate–Hochschild cohomology) of an algebra $A$ is isomorphic, as a graded algebra, to the Hochschild cohomology of the differential graded enhancement of the singularity category of $A$. The existence of such an isomorphism is suggested by recent work of Zhengfang Wang.

1. Introduction

Let $k$ be a commutative ring. We write $\otimes$ for $\otimes_k$. Let $A$ be a right noetherian (non commutative) $k$-algebra projective over $k$. The stable derived category or singularity category of $A$ is defined as the Verdier quotient

$$Sg(A) = D^b(\text{mod } A) / \text{per}(A)$$

of the bounded derived category of finitely generated (right) $A$-modules by the perfect derived category $\text{per}(A)$, i.e. the full subcategory of complexes quasi-isomorphic to bounded complexes of finitely generated projective modules. It was introduced by Buchweitz in an unpublished manuscript [4] in 1986 and rediscovered, in its scheme-theoretic variant, by Orlov in 2003 [24]. Notice that it vanishes when $A$ is of finite global dimension and thus measures the degree to which $A$ is ‘singular’, a view confirmed by the results of [24].

Let us suppose that the enveloping algebra $A^e = A \otimes A^{op}$ is also right noetherian. In analogy with Hochschild cohomology, in view of Buchweitz’ theory, it is natural to define the Tate–Hochschild cohomology or singular Hochschild cohomology of $A$ to be the graded algebra with components

$$HH^*_{sg}(A, A) = \text{Hom}_{Sg(A^e)}(A, \Sigma^n A), \ n \in \mathbb{Z},$$

where $\Sigma$ denotes the suspension (shift) functor. It was studied for example in [10, 2, 23] and more recently in [31, 32, 30, 33, 29, 5]. Wang showed in [31] that, like Hochschild cohomology [11], singular Hochschild cohomology carries a structure of Gerstenhaber algebra. Now recall that the Gerstenhaber algebra structure on Hochschild cohomology is a small part of much richer higher structure on the Hochschild cochain complex $C(A,A)$ itself, namely the structure of a $B_\infty$-algebra in the sense of Getzler–Jones [12, 5.2] given by the brace operations [1, 16]. In [29], Wang improves on [31] by defining a singular Hochschild cochain complex $C_{sg}(A,A)$ and endowing it with a $B_\infty$-structure which in particular yields the Gerstenhaber algebra structure on $HH^*_sg(A,A)$.

Using [17] Lowen–Van den Bergh showed in [21, Theorem 4.4.1] that the Hochschild cohomology of $A$ is isomorphic to the Hochschild cohomology of the canonical differential graded (=dg) enhancement of the (bounded or unbounded) derived category of $A$ and that the isomorphism lifts to the $B_\infty$-level (cf. Corollary 7.6 of [26] for a related statement). Together with the complete structural analogy between Hochschild and singular
Hochschild cohomology described above, this suggests the question whether the singular Hochschild cohomology of $A$ is isomorphic to the Hochschild cohomology of the canonical dg enhancement $\mathcal{S}_{gdg}(A)$ of the singularity category $\mathcal{S}_g(A)$ (note that such an enhancement exists by the construction of $\mathcal{S}_g(A)$ as a Verdier quotient [19, 6]). Chen–Li–Wang show in [5] that this does hold at the level of Gerstenhaber algebras when $A$ is the radical square zero algebra associated with a finite quiver without sources or sinks. Our main result is the following.

**Theorem 1.1.** There is a canonical isomorphism of graded algebras between the singular Hochschild cohomology of $A$ and the Hochschild cohomology of the dg singularity category $\mathcal{S}_{gdg}(A)$.

**Conjecture 1.2.** The isomorphism of the theorem lifts to an isomorphism

$$C_{sg}(A, A) \xrightarrow{\sim} C(\mathcal{S}_{gdg}(A), \mathcal{S}_{gdg}(A))$$

in the homotopy category of $B_\infty$-algebras.

Notice that the $B_\infty$-structure on Hochschild cohomology of dg categories is preserved (up to quasi-isomorphism) under Morita equivalences, cf. [17].

Let us mention an application of Theorem 1.1 obtained in joint work with Zheng Hua. Suppose that $k$ is algebraically closed of characteristic 0 and let $P$ the power series algebra $k[[x_1, \ldots, x_n]]$.

**Theorem 1.3** ([15]). Suppose that $Q \in P$ has an isolated singularity at the origin and $A = P/(Q)$. Then $A$ is determined up to isomorphism by its dimension and the dg singularity category $\mathcal{S}_{gdg}(A)$.

In [8, Theorem 8.1], Efimov proves a related but different reconstruction theorem: He shows that if $Q$ is a polynomial, it is determined, up to a formal change of variables, by the differential $\mathbb{Z}/2$-graded endomorphism algebra $E$ of the residue field in the differential $\mathbb{Z}/2$-graded singularity category together with a fixed isomorphism between $H^*B$ and the exterior algebra $\Lambda(k^n)$.

In section 2, we generalize Theorem 1.1 to the non noetherian setting and prove the generalized statement. We comment on a possible lift of this proof to the $B_\infty$-level in section 3. We prove Theorem 1.3 in section 4.

2. **Generalization and proof**

2.1. **Generalization to the non noetherian case.** We assume that $A$ is an arbitrary $k$-algebra projective as a $k$-module. Its singularity category $\mathcal{S}_g(A)$ is defined as the Verdier quotient $\mathcal{H}^{-b}(\text{proj } A)/\mathcal{H}^b(\text{proj } A)$ of the homotopy category of right bounded complexes of finitely generated projective $A$-modules by its full subcategory of bounded complexes of finitely generated projective $A$-modules. Notice that when $A$ is right noetherian, this is equivalent to the definition given in the introduction.

The (partially) completed singularity category $\widehat{\mathcal{S}}_g(A)$ is defined as the Verdier quotient of the bounded derived category $\mathcal{D}^b(\text{Mod } A)$ of all right $A$-modules by its full subcategory consisting of all complexes quasi-isomorphic to bounded complexes of arbitrary projective modules.

**Lemma 2.2.** The canonical functor $\mathcal{S}_g(A) \to \widehat{\mathcal{S}}_g(A)$ is fully faithful.

**Proof.** Let $M$ be a right bounded complex of finitely generated projective modules with bounded homology and $P$ a bounded complex of arbitrary projective modules. Since the components of $M$ are finitely generated, each morphism $M \to P$ in the derived category...
factors through a bounded complex $P'$ with finitely generated projective components. This yields the claim.

Since we do not assume that $A^e$ is noetherian, the $A$-bimodule $A$ will not, in general, belong to the singularity category $\text{Sg}(A^e)$. But it always belongs to the completed singularity category $\hat{\text{Sg}}(A^e)$. We define the singular Hochschild cohomology of $A$ to be the graded algebra with components

$$HH^0_{\text{sg}}(A, A) = \text{Hom}_{\text{Sg}(A^e)}(A, \Sigma^n A), n \in \mathbb{Z}.$$  

**Theorem 2.3.** Even if $A^e$ is non noetherian, there is a canonical isomorphism of graded algebras between the singular Hochschild cohomology of $A$ and the Hochschild cohomology of the dg singularity category $\hat{\text{Sg}}(A^e)$.

Let $P$ be a right bounded complex of projective $A^e$-modules. For $q \in \mathbb{Z}$, let $\sigma_{>q}P$ and $\sigma_{\leq q}P$ denote its stupid truncations:

$$\sigma_{>q}P : \cdots \rightarrow 0 \rightarrow P_{q+1} \rightarrow P_{q+1} \rightarrow \cdots$$

$$\sigma_{\leq q}P : \cdots \rightarrow P_{q-1} \rightarrow P_q \rightarrow 0 \rightarrow \cdots$$

so that we have a triangle

$$\sigma_{>q}P \rightarrow P \rightarrow \sigma_{\leq q}P \rightarrow \Sigma \sigma_{>q}P.$$  

We have a direct system

$$P = \sigma_{\leq 0}P \rightarrow \sigma_{\leq -1}P \rightarrow \sigma_{\leq -2}P \rightarrow \cdots \rightarrow P_{\leq q} \rightarrow \cdots.$$  

**Lemma 2.4.** Let $L \in D^b(\text{Mod} A^e)$. We have a canonical isomorphism

$$\text{colim} \text{Hom}_{D \text{A}^e}(L, \sigma_{\leq q}P) \cong \text{Hom}_{\hat{\text{Sg}}(A^e)}(L, P).$$

In particular, if $P$ is a projective resolution of $A$ over $A^e$, we have

$$\text{colim} \text{Hom}_{D \text{A}^e}(A, \Sigma^n \sigma_{\leq q}P) \cong \text{Hom}_{\hat{\text{Sg}}(A^e)}(A, \Sigma^n A), n \in \mathbb{Z}.$$  

**Proof.** Clearly, if $Q$ is a bounded complex of projective modules, each morphism $Q \rightarrow P$ in the derived category $D \text{A}^e$ factors through $\sigma_{>q}P \rightarrow P$ for some $q \ll 0$. This shows that the morphisms $P \rightarrow \sigma_{\leq q}P$ form a cofinal subcategory in the category of morphisms $P \rightarrow P'$ whose cylinder is a bounded complex of projective modules. Whence the claim.  

**2.5. Proof of Theorem 2.3.** We refer to [18, 20, 27] for foundational material on dg categories. We will follow the terminology of [20] and use the model category structure on the category of dg categories constructed in [25]. For a dg category $A$, denote by $X \mapsto Y(X)$ the dg Yoneda functor and by $D A$ the derived category. We write $A^e$ for the enveloping dg category $A \overset{L}{\otimes}_k A^{op}$ and $I_A$ for the identity bimodule

$$I_A : (X, Y) \mapsto A(X, Y).$$  

By definition, the Hochschild cohomology of $A$ is the graded endomorphism algebra of $I_A$ in the derived category $D(A^e)$. In the case of the algebra $A$, the identity bimodule is the $A$-bimodule $A$. Recall that if $F : A \rightarrow B$ is a fully faithful dg functor, the restriction $F_* : D B \rightarrow D A$ is a localization functor admitting fully faithful left and right adjoint functors $F^*$ and $F^!$, given respectively by

$$F^* : M \mapsto M \otimes_A F B \quad \text{and} \quad F^! : N \mapsto \text{RHom}_A(BF, N),$$

where $F B = B(?, F -)$ and $BF = B(F ?, -)$. 

Let $\mathcal{M}_0 = C_{dg}^{\geq 0}(\text{proj} \ A)$ denote the dg category of right bounded complexes of finitely generated projective $A$-modules with bounded homology. Notice that the morphism complexes of $\mathcal{M}_0$ have terms which involve infinite products of projective $A$-modules so that in general, the morphism complexes of $\mathcal{M}_0$ will not be cofibrant over $k$. Let $\mathcal{M} \rightarrow \mathcal{M}_0$ be a cofibrant resolution of $\mathcal{M}_0$. We assume, as we may, that the quasi-equivalence $\mathcal{M} \rightarrow \mathcal{M}_0$ is the identity on objects. Notice that the morphism complexes of $\mathcal{M}$ are cofibrant over $k$ so that we have $\mathcal{M} \otimes_k \mathcal{M}^{\text{op}} \rightarrow \mathcal{M} \otimes \mathcal{M}^{\text{op}}$. Let $\mathcal{P} \subset \mathcal{M}$ be the full dg subcategory of $\mathcal{M}$ formed by the bounded complexes of finitely generated projective $A$-modules. Let $\mathcal{S}$ denote the dg quotient $\mathcal{M}/\mathcal{P}$. We assume, as we may, that $\mathcal{S}$ is cofibrant. In the homotopy category of dg categories, we have an isomorphism between $\mathcal{S}_{dg}(A)$ and $\mathcal{S} = \mathcal{M}/\mathcal{P}$. Let $B$ be the dg endomorphism algebra of $A$ considered as an object of $\mathcal{P} \subset \mathcal{M}$. Notice that we have a quasi-isomorphism $B \rightarrow A$ and that both $B$ and $A$ are cofibrant over $k$. We view $B$ as a dg category with one object whose endomorphism algebra is $B$. We have the obvious inclusion and projection dg functors

$$B \xrightarrow{i} \mathcal{M} \xrightarrow{p} \mathcal{S}.$$ Consider the fully faithful dg functors

$$B \otimes B^{\text{op}} \xrightarrow{1 \otimes i} B \otimes B^{\text{op}} \xrightarrow{1 \otimes 1} \mathcal{M} \otimes \mathcal{M}^{\text{op}}.$$

The restriction along $G = 1 \otimes i$ admits the left adjoint $G^*$ given by

$$G^* : X \mapsto \mathcal{M} L_i \otimes_B X,$$

and the restriction along $F = i \otimes 1$ admits the fully faithful left and right adjoints $F^*$ and $F^!$ given by

$$F^* : Y \mapsto Y L_i \otimes_B i \mathcal{M} \quad \text{and} \quad F^! : Y \mapsto \text{RHom}_B(M_i, Y).$$

Since $F^*$ and $F^!$ are the two adjoints of a localization functor, we have a canonical morphism $F^* \rightarrow F^!$.

**Lemma 2.6.** If $P$ is an arbitrary sum of copies of $B^e$, the morphism

$$F^* G^*(P) \rightarrow F^! G^*(P)$$

is invertible.

**Proof.** Let $P$ be the direct sum of copies of $B^e$ indexed by a set $J$. Since $F^*$ and $G^*$ commute with (arbitrary) coproducts, the left hand side is the dg module

$$\bigoplus_J \mathcal{M}(?, -) L_B (B \otimes B) L_B \mathcal{M}(?, i-) = \bigoplus_J \mathcal{M}(B, -) \otimes \mathcal{M}(?, B),$$

The right hand side is the dg module

$$\text{RHom}_B(M_i, \bigoplus_J B \otimes B) = \text{RHom}_B(M_i, \bigoplus_J (B, -) \otimes B).$$

Let us evaluate the canonical morphism at $(M, L) \in \mathcal{M} \otimes \mathcal{M}^{\text{op}}$. We find the canonical morphism

$$\bigoplus_J \mathcal{M}(B, L) \otimes \mathcal{M}(M, B) \rightarrow \text{RHom}_B(M(B, M), \bigoplus_J \mathcal{M}(B, L) \otimes B).$$

We have quasi-isomorphisms

$$\mathcal{M}(B, L) \otimes \mathcal{M}(M, B) \rightarrow \mathcal{M}_0(A, L) \otimes \mathcal{M}(M, B) \rightarrow L \otimes \mathcal{M}(M, B) \rightarrow L \otimes \text{Hom}_A(M, A).$$
because \(M(M, B)\) and \(L\) are cofibrant over \(k\). Now the equivalence \(\mathcal{D}(B) \xrightarrow{\sim} \mathcal{D}(A)\) takes \(M(B, L) \otimes B\) to \(M(B, L) \otimes A \xrightarrow{\sim} L \otimes A\). We have an quasi-isomorphism of dg \(B\)-modules \(M(B, M) \xrightarrow{\sim} M_0(A, M) = M\) and so the equivalence \(\mathcal{D}(B) \xrightarrow{\sim} \mathcal{D}(A)\) takes \(M(B, M)\) to \(M\). Whence an isomorphism
\[
\text{RHom}_B(M(B, M), \bigoplus_j M(B, L) \otimes B) \xrightarrow{\sim} \text{RHom}_A(M, \bigoplus_j L \otimes A) = \text{Hom}_A(M, \bigoplus_j L \otimes A).
\]
Thus, we have to show that the canonical morphism
\[
\bigoplus_j L \otimes \text{Hom}_A(M, A) \to \text{Hom}_A(M, \bigoplus_j L \otimes A)
\]
is a quasi-isomorphism. Recall that \(L\) and \(M\) are right bounded complexes of finitely generated projective modules with bounded homology. We fix \(M\) and consider the morphism as a morphism of triangle functors with argument \(L \in \mathcal{D}^b(\text{Mod} \, A)\). Then we are reduced to the case where \(L\) is in \(\text{Mod} \, A\). In this case, the morphism becomes an isomorphism of complexes because the components of \(M\) are finitely generated projective. \(\checkmark\)

Let us put \(H = F^!G^*: \mathcal{D}(B^e) \to \mathcal{D}(M^e)\). Let us compute the image of the identity bimodule \(B\) under \(H\). We have
\[
H(B) = F^!(M_i \otimes_B B) = F^!(M_i) = \text{RHom}_B(M_i, M_i)
\]
and when we evaluate at \(L, M\) in \(M\), we find
\[
H(B)(L, M) = \text{RHom}_B(M(i?, L), M(i?, M)) = \text{RHom}_B(M(B, L), M(B, M)).
\]
We have seen in the above proof that the equivalence \(\mathcal{D}(B) \xrightarrow{\sim} \mathcal{D}(A)\) takes \(M(B, L)\) to \(L\). Whence quasi-isomorphisms
\[
H(B)(L, M) = \text{RHom}_B(M(B, L), M(B, M)) \xrightarrow{\sim} \text{RHom}_A(L, M) = \text{Hom}(L, M)
\]
\[
\leftarrow M(L, M).
\]
Thus, the functor \(H\) takes the identity bimodule \(B\) to the identity bimodule \(I_M\). Since \(F^!\) and \(G^!\) are fully faithful so is \(H\). Denote by \(N\) the image under the composition of \(H\) with \(\mathcal{D}(A^e) \xrightarrow{\sim} \mathcal{D}(B^e)\) of the closure of \(\text{Proj} \, A^e\) under finite extensions. Then \(H\) yields a fully faithful functor
\[
\hat{S}_g(A^e) \to \mathcal{D}(M^e)/N
\]
taking the bimodule \(A\) to the identity bimodule \(I_M\). Now notice that we have a Morita morphism of dg categories
\[
S^e \xleftarrow{\sim} \frac{M \otimes M^{\text{op}}}{\mathcal{P} \otimes M^{\text{op}} + M \otimes \mathcal{P}^{\text{op}}}.
\]
The functor \(p^* : \mathcal{D}(M^e) \to \mathcal{D}(S^e)\) induces the quotient functor
\[
\frac{\mathcal{D}(M \otimes M^{\text{op}})}{N} \xrightarrow{\sim} \frac{\mathcal{D}(M \otimes M^{\text{op}})}{\mathcal{D}(\mathcal{P} \otimes M^{\text{op}} + M \otimes \mathcal{P}^{\text{op}})} = \mathcal{D}(S^e).
\]
Since \(p : M \to S\) is a localization, the image \(p^*(I_M)\) is isomorphic to \(I_S\). It suffices to show that \(p^*\) induces bijections in the morphism spaces with target \(I_M\)
\[
\text{Hom}_{\mathcal{D}(S^e)}(?, I_M) \xrightarrow{\sim} \text{Hom}_{\mathcal{D}(S^e)}(p^*(?), p^*(I_M)).
\]
For this, it suffices to show that \(I_M\) is right orthogonal in \(\mathcal{D}(M^e)/N\) on the images under the Yoneda functor of the objects in \(\mathcal{P} \otimes M^{\text{op}} + M \otimes \mathcal{P}^{\text{op}}\). To show that \(I_M\) is right orthogonal
on $Y(M \otimes \mathcal{P}^{op})$, it suffices to show that it is right orthogonal to an object $Y(M, B)$, $M \in M$. Now a morphism in $\mathcal{D}(M^e)/N$ is given by a diagram of $\mathcal{D}(M^e)$ representing a left fraction

$$Y(M, B) \longrightarrow I_M^I \longleftarrow I_M$$

where the cone over $I_M \rightarrow I_M^I$ lies in $N$. For each object $X$ of $\mathcal{D}M^e$, we have canonical isomorphisms

$$\text{Hom}_{\mathcal{D}M^e}(Y(M, B), X) \cong H^0(X(M, B)) = \text{Hom}_{\mathcal{D}M}(Y(M), X(?), B)).$$

Thus, the given fraction corresponds to a diagram in $\mathcal{D}(M)$ of the form

$$Y(M) \longrightarrow I_M(?), B \longleftarrow I_M(?), B = M(?), B,$$

where the cone over $I_M(?), B \rightarrow I_M^I(?), B$ is the image under $\mathcal{D}A \rightarrow \mathcal{D}B \rightarrow \mathcal{D}M$ of a bounded complex with projective components. Thus, the object $I_M^I(?), B$ is a direct factor of a finite extension of shifts of arbitrary coproducts $B$. Since $Y(M)$ is compact, the given morphism $Y(M) \rightarrow I_M^I(?), M$ must then factor through $Y(Q)$ for an object $Q$ of $\mathcal{P}$. This means that the given morphism $Y(M, B) \rightarrow I_M^I$ factors through $Y(Q, B)$, which lies in $N$. Thus, the given fraction represents the zero morphism of $\mathcal{D}(M^e)/N$, as was to be shown. The case of an object in $Y(\mathcal{P} \otimes \mathcal{M}^{op})$ is analogous. In summary, we have shown that the maps

$$\mathcal{S}g(A^e)(A, \Sigma^n A) \xrightarrow{H} \mathcal{D}(M^e)/N(I_M, \Sigma^n I_M) \xrightarrow{p^*} \mathcal{D}(S^e)(I_S, \Sigma^n I_S)$$

are bijective, which implies the assertion on Hochschild cohomology.

3. REMARK ON A POSSIBLE LIFT TO THE $B_\infty$-LEVEL

Let $P \rightarrow A$ be a resolution of $A$ by projective $A$-$A$-bimodules. Let us assume for simplicity that $k$ is a field so that we can take $M = M_0$ and $B = A$. The proof in section 2 produces in fact isomorphisms in the derived category of $k$-modules

$$\colim \text{RHom}_{A^e}(A, \sigma_{\leq q} P) \rightarrow \colim \text{RHom}_{M^e}(I_M, H\sigma_{\leq q} P)$$

$$\rightarrow \colim \text{RHom}_{S^e}(I_S, p^* H\sigma_{\leq q} P)$$

$$= \text{RHom}_{S^e}(I_S, I_S).$$

For the bar resolution $P$, the truncation $\sigma_{\leq -q} P$ is canonically isomorphic to $\Sigma^q \Omega^q A$ so that the first complex carries a canonical $B_\infty$-structure constructed by Wang [29]. As explained in the introduction, it is classical that the last complex carries a canonical $B_\infty$-structure. It is not obvious to make the intermediate complexes explicit because the functor $H$, being a composition of a right adjoint with a left adjoint to a restriction functor, does not take cofibrant objects to cofibrant objects.

4. PROOF OF THEOREM 1.3

By the Weierstrass preparation theorem, we may assume that $Q$ is a polynomial. Let $P_0 = k[x_1, \ldots, x_n]$ and $S = P_0/(Q)$. Then $S$ has isolated singularities but may have singularities other than the origin. Let $m$ be the maximal ideal of $P_0$ generated by the $x_i$ and let $R$ be the localization of $S$ at $m$. Now $R$ is local with an isolated singularity at $m$ and $A$ is isomorphic to the completion $\hat{R}$. By Theorem 3.2.7 of [14], in sufficiently high degrees $r$, the Hochschild cohomology of $S$ is isomorphic to the homology in degree $r$ of the complex

$$k[u] \otimes K(S, \partial_1 Q, \ldots, \partial_n Q),$$
where \( u \) is of degree 2 and \( K \) denotes the Koszul complex. Now \( S \) is isomorphic to \( K(P_0, Q) \) and so \( K(S, \partial_1 Q, \ldots, \partial_n Q) \) is isomorphic to
\[
K(P_0, Q, \partial_1 Q, \ldots, \partial_n Q).
\]
Since \( Q \) has isolated singularities, the \( \partial_i Q \) form a regular sequence in \( P_0 \). So
\[
K(P_0, Q, \partial_1 Q, \ldots, \partial_n Q)
\]
is quasi-isomorphic to \( K(M, Q) \), where \( M = P_0/(\partial_1 Q, \ldots, \partial_n Q) \). Therefore, in high even degrees \( 2r \), the Hochschild cohomology of \( S \) is isomorphic to
\[
T = k[x_1, \ldots, x_n]/(Q, \partial_1 Q, \ldots, \partial_n Q)
\]
as an \( S \)-module. Since \( S \) and \( S^e \) are noetherian, this implies that the Hochschild cohomology of \( R \) in high even degrees is isomorphic to the localisation \( T_m \). Since \( R \otimes R \) is noetherian and Gorenstein (cf. Theorem 1.6 of [28]), by Theorem 6.3.4 of [4], the singular Hochschild cohomology of \( R \) coincides with Hochschild cohomology in sufficiently high degrees. By Theorem 1.1, the Hochschild cohomology of \( S_{gd}(R) \) is isomorphic to the singular Hochschild cohomology of \( R \) and thus isomorphic to \( T_m \) in high even degrees. Since \( R \) is a hypersurface, the dg category \( S_{gd}(R) \) is isomorphic, in the homotopy category of dg categories, to the underlying differential \( \mathbb{Z} \)-graded category of the differential \( \mathbb{Z}/2 \)-graded category of matrix factorizations of \( Q \), cf. [9], [24] and Theorem 2.49 of [3]. Thus, it is 2-periodic and so is its Hochschild cohomology. It follows that the zeroth Hochschild cohomology of \( S_{gd}(R) \) is isomorphic to \( T_m \) as an algebra. The completion functor \( \hat{\otimes}_R \hat{R} \) yields an embedding \( S_{gd}(R) \to S_{gd}(A) \) through which \( S_{gd}(A) \) identifies with the idempotent completion of the triangulated category \( S_{gd}(R) \), cf. Theorem 5.7 of [7]. Therefore, the corresponding dg functor \( S_{gd}(R) \to S_{gd}(A) \) induces an equivalence in the derived categories and an isomorphism in Hochschild cohomology. So we find an isomorphism
\[
HH^0(S_{gd}(A), S_{gd}(A)) \cong T_m.
\]
Since \( Q \in k[x_1, \ldots, x_n]_m \) has an isolated singularity at the origin, we have an isomorphism
\[
T_m \cong k[[x_1, \ldots, x_n]]/(Q, \partial_1 Q, \ldots, \partial_n Q)
\]
with the Tyurina algebra of \( A = P/(Q) \). Now by the Mather–Yau theorem [22], more precisely by its formal version [13, Prop. 2.1], in a fixed dimension, the Tyurina algebra determines \( A \) up to isomorphism.

Notice that the Hochschild cohomology of the dg category of matrix factorizations considered as a differential \( \mathbb{Z}/2 \)-graded category is different: As shown by Dyckerhoff [7], it is isomorphic to the Milnor algebra \( P/(\partial_1 Q, \ldots, \partial_n Q) \) in even degree and vanishes in odd degree.

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