Unbounded entropy in spacetimes with positive cosmological constant

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ABSTRACT: In theories of gravity with a positive cosmological constant, we consider product solutions with flux, of the form \((A)dS_p \times S^q\). Most solutions are shown to be perturbatively unstable, including all uncharged \(dS_p \times S^q\) spacetimes. For dimensions greater than four, the stable class includes universes whose entropy exceeds that of de Sitter space, in violation of the conjectured “\(N\)-bound”. Hence, if quantum gravity theories with finite-dimensional Hilbert space exist, the specification of a positive cosmological constant will not suffice to characterize the class of spacetimes they describe.

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1. Introduction

String theory has had some success in defining quantum gravity in certain asymptotically flat and asymptotically Anti-de Sitter (AdS) spacetimes. It is natural, therefore, to seek a fundamental description of asymptotically de Sitter spacetimes. This is particularly interesting because our own universe may belong to this class [1,2]. Moreover, de Sitter space is a simple arena in which one may confront conceptual problems that arise in the description of any cosmological spacetime.

Although some progress has been reported (see, e.g., Refs. [3-5]), there is presently no fully satisfactory embedding of de Sitter space into string theory.\(^1\) It is important to understand whether this is only a technical problem, or whether significant new developments (comparable, e.g., to the discovery of D-branes [10]) will be required for progress. To this end, one may deduce properties of de Sitter quantum gravity by

\(^1\) Some recent approaches to quantum gravity in de Sitter space do not use string theory as a starting point (e.g., Refs. [6,7]; see also further references in Refs. [8,9]).
applying the general consequences we expect from a complete quantum theory, namely semi-classical gravity and the holographic principle [11–14]. One can then ask whether string theory may display such properties, and specific challenges may be pinpointed or obstructions identified.

One recent line of thought, initiated by Banks and Fischler [15–17], centers on the semi-classical result that de Sitter space has a finite entropy $S_0$ [18], inversely related to the cosmological constant. This approach reasons that de Sitter space should be described by a theory with a finite number $e^{S_0}$ of independent quantum states. Any additional states would be superfluous, as they would render the theory more complex than the phenomena it describes—an uneconomical and arbitrary excess.\(^2\) Hence, it is asserted, one must construct a quantum gravity theory with a finite dimensional Hilbert space in order to describe de Sitter space.

In its strongest form, this reasoning leads to a new perspective on the origin of vacuum energy [17]: the “Λ-N correspondence” [19]. The cosmological constant Λ should be understood as a direct consequence of the finite number of states $e^N$ in the Hilbert space describing the world. Λ effectively provides a cutoff on observable entropy, ensuring that the theory need never describe phenomena requiring a larger number of states; the smaller the Hilbert space, the larger the cosmological constant.

Originally these observations were made in the context of spacetimes that asymptote to de Sitter space in the future [15–17]. However, for reasons first noted in Ref. [19] and further elaborated below, it would be unnatural for any particular theory to describe only spacetimes of this type. A positive cosmological constant does not guarantee the presence of de Sitter asymptotic regions; worse, their existence can be affected by small deformations of Cauchy data. Consequently, a larger class of spacetimes must be identified as the “gravity dual” if theories with finite Hilbert spaces are to play a role in quantum gravity.

Once asymptotic conditions are abandoned, the finiteness of entropy is no longer obvious. A necessary condition for a (coarse-grained) spacetime to be described by a theory with $e^N$ states is that the entropy accessible to any observer in that spacetime must obey

$$S \leq N. \quad (1.1)$$

It may be possible to characterize various different suitable classes of spacetimes containing the asymptotically de Sitter spacetimes as a subset. However, in view of the “Λ-N correspondence”, the simplest proposal is to consider the set $\text{all}(\Lambda(N))$: the class

\(^2\)Such an excess should be distinguished from the situation of an ordinary gauge symmetry, where the variables are redundant, but a quotient nonetheless produces a minimal physical Hilbert space.
of spacetimes with positive cosmological constant $\Lambda(N)$, irrespective of asymptotic conditions and types of matter present [19].

Generally, one expects a good quantum gravity theory to predict specific matter content. However, of course, we do not yet have a quantum gravity theory with a finite number of states, and so we are unable to determine which matter fields should be allowed. This is a drawback, because restrictions on the matter content might enforce the condition (1.1) when it would otherwise be violated. However, the covariant entropy bound [13] limits entropy in terms of a geometric feature (area), assuming nothing about the matter content except that fairly generic energy conditions hold. Thus, if the geometries in a certain class of spacetimes obey suitable restrictions, Eq. (1.1) may follow from the holographic principle with few or no detailed assumptions about matter (see Ref. [20] for a review).

By applying the condition (1.1) to the candidate set all($\Lambda(N)$), one obtains a consistency requirement, the “$N$-bound”, which we state here for $D = 4$ [19]: In any universe with a positive cosmological constant $\Lambda$ (as well as arbitrary additional matter that may well dominate at all times) the observable entropy $S$ is bounded by $N=3\pi/\Lambda$. (We use Planck units throughout.) For the special case of central observers in spherically symmetric spacetimes with $\Lambda(N)$, the $N$-bound was proven [19] using the covariant entropy bound [13, 20]. This proof relies on general geometric properties of spacetimes with positive cosmological constant, and on a bound on matter entropy in de Sitter space [21]. It applies for all $D \geq 4$.

In this paper, however, we show that counterexamples to the $N$-bound exist in $D > 4$. They are not counterexamples to the covariant entropy bound, but they evade the proof of Ref. [19] by violating the assumption of spherical symmetry. Their key novel ingredient is flux, which is used to stabilize a product metric with one large or non-compact factor.

This shows, in particular, that the mere specification of a positive cosmological constant does not suffice to guarantee finite observable entropy. $\Lambda$ cannot be in correspondence with $N$ unless some additional conditions hold that exclude our counterexamples. In particular, conditions on matter content may be required. This leaves open the question of whether finite Hilbert space theories can have a gravity dual.

In Sec. 2, we discuss the classification and entropy of solutions related to de Sitter space. We formulate conditions that the gravity dual of a finite Hilbert space theory should satisfy (if it exists).

In Sec. 3, we consider a $D = p + q$ dimensional theory with positive cosmological constant and a $q$-form field strength. In addition to the dS$_{p+q}$ solution, we present all solutions of the product form $(A)dS_p \times S^q$ with or without $q$-form flux on the $S^q$. We note that these solutions can be generalized to include a black hole; this will later
permit us to introduce entropy into the AdS\(_p \times S^q\) solutions.

In Sec. 4, we investigate the stability of the product solutions we have described. Stability is essential if we wish to interpret their horizon areas as entropy. Our results apply for \(p > 2\). We find that stable dS\(_p \times S^q\) solutions exist for \(2 \leq q \leq 4\) and certain values of flux. Furthermore, stable AdS\(_p \times S^q\) solutions exist for \(q = 2, 3\) with any flux, and for other even \(q\) with flux sufficiently large. We further argue that if an AdS\(_p \times S^q\) solution is stable, all Schwarzschild-AdS\(_p \times S^q\) solutions with the same asymptotic conditions will also be stable.

Finally, in Sec. 5 we show that some of the stable solutions we have described violate the \(N\)-bound. For \(p > 2\), the stable AdS\(_p \times S^q\) solutions permit Schwarzschild black holes of arbitrarily large entropy. Also for \(p > 2\), the cosmological horizon area of the dS\(_p \times S^q\) spacetimes can exceed that of dS\(_{p+q}\). For \(D > 4\), this demonstrates that the observable entropy can be arbitrarily large at fixed positive \(\Lambda\).

We present our conclusions in Sec. 6.

2. Classification and properties of \(\Lambda > 0\) spacetimes

2.1 Asymptotically de Sitter spacetimes

To say that a spacetime is asymptotically de Sitter is an ambiguous statement. Unlike the flat and AdS cases, de Sitter space has two disconnected infinities, one in the past and one in the future. It is useful to distinguish which of these infinities are present. (For a brief summary of the de Sitter geometry, see the appendix of Ref. [19].)

For a given cosmological constant \(\Lambda > 0\), let us call a spacetime dS\(_\pm\) if it possesses both infinities. An example is a small perturbation about de Sitter space, or the Schwarzschild-de Sitter solution. The dS\(_\pm\)(\(\Lambda\)) set of spacetimes is the intersection of the dS\(_+\) and dS\(_-\) sets, i.e., the spacetimes that asymptote to de Sitter space\(^3\) in the future or in the past, respectively. For example, if our universe started from a big bang and is evolving to a de Sitter state, it belongs to the dS\(_+\) category. Finally, we may define an even larger set, all(\(\Lambda\)), by specifying only the cosmological constant \(\Lambda\) (defined here to be the lowest attainable vacuum energy in the theory), without demanding any

\(^3\)It is useful to characterize such spacetimes without making reference to all of \(\mathcal{I}^+\). As a working definition of dS\(_+\), we propose the following. Recall that the causal diamond \(C(p, q)\) is the intersection of the future of \(p\) with the past of \(q\). A spacetime is in dS\(_+\) if it contains at least one worldline \(\gamma\) with the following two properties. 1. Let \(\tau\) be the proper time on \(\gamma\). \(\gamma\) is geodesic after some finite time \(\tau_0\). 2. Let \(p, q\) be points on the world line such that \(\tau(p) < \tau(q)\). The geometry of the causal diamonds \(C(p, q)\) asymptotes to the static patch of de Sitter space as \(\tau(p) \to \infty, \tau(q) - \tau(p) \to \infty\). An analogous definition applies for dS\(_-\).
asymptotic conditions at all. For example, a closed recollapsing universe can have a positive vacuum energy as long as the matter energy always dominates.

Furthermore, small changes in the Cauchy data can take a spacetime in and out of any of the $dS^+$, $dS^-$, and $dS^\pm$ classes. This is in further contrast to asymptotically flat or AdS universes, which retain their asymptotic structure independently of the matter injected at the boundary (unless the cosmic censorship conjecture is violated), and independently of continuous deformations of Cauchy data.

For example, suppose we lived in a $dS^\pm$ universe. Outside our past light cone, a shell of matter might be collapsing toward us. When it reaches our causal domain, it will form a Schwarzschild-de Sitter black hole, which will evaporate and leave de Sitter space behind. However, black holes larger than the cosmological horizon cannot exist in de Sitter space. Consider a slightly subcritical shell, which forms a nearly-maximal black hole. If a tiny amount of matter is added to the shell initially, the collapse will result in a big crunch. A large enough black hole cannot form; instead, there will be contracting time slices on which the matter density dominates over the vacuum energy. Hence the universe collapses entirely, which places it in the $dS^-$ class.

A second example is that of a $\Lambda > 0$ closed universe filled with dust. Starting from a big bang, such a universe may expand indefinitely ($dS^+$), or it may recollapse, depending on the ratio of dust to vacuum energy. Thus, two universes with different fates can possess early time Cauchy surfaces with almost identical matter density and expansion rate.

### 2.2 $dS^+$ and finite entropy

In a $dS^+$ spacetime, any observer is surrounded by a cosmological event horizon, whose area is inversely related to the cosmological constant, $\Lambda$. Ordinary entropy is lost to the observer when matter crosses this horizon. Just as with a black hole horizon, in order to maintain a generalized second law of thermodynamics [22–24] the horizon must be assigned an entropy equal to a quarter of its area, $A_{\text{hor}}$ [18]:

$$S_{\text{hor}} = \frac{A_{\text{hor}}}{4}. \tag{2.1}$$

The horizon area generally varies with the amount of matter enclosed. With reasonable assumptions about the maximal entropy of matter [25, 26], one finds that the horizon area adjusts itself such that the combined entropy of matter and horizon is less than that of empty de Sitter space [21]:

$$S_{\text{matter}} + S_{\text{hor}} \leq S_0. \tag{2.2}$$
This result also follows immediately from the generalized second law. The final state of the system is empty de Sitter space, and earlier configurations may not exceed its entropy.

The entropy $S_0$ of empty de Sitter space is determined by the cosmological constant alone:

$$S_0 = \frac{\Omega_{D-2}}{4} \left[ \frac{(D-1)(D-2)}{2\Lambda} \right]^{\frac{D-2}{2}}, \quad (2.3)$$

where $\Omega_{D-2} = 2\pi^{(D-1)/2}/\Gamma((D-1)/2)$ is the area of a unit $(D-2)$-sphere. This is the largest entropy (counting both horizons and matter) observable in any of the dS$^+$ spacetimes.\(^4\)

Note, however, that $S_{\text{matter}}$ includes only entropy that is causally accessible to the observer, i.e., entropy contained in causal diamond regions. (Generally, a causal diamond is the intersection of the past and the future of an observer’s world line [19].) This is sensible because unobservable regions are operationally meaningless. Indeed, the unitary evolution of black holes suggests that causally disconnected observers cannot be simultaneously described without encountering a paradox [29]. The restriction to causal diamonds is crucial for the validity of Eq. (2.2), as the global entropy may be formally unbounded.\(^5\)

Analogously to the case of black holes, the entropy of dS$^+$ universes may be interpreted microscopically in terms of the number of accessible quantum states. (Though well motivated by the D-brane model of black hole entropy [30], we must stress that this interpretation is not inevitable.) By Eq. (2.2), it follows that $e^{S_0}$ states suffice to describe all dS$^+$ universes with cosmological constant $\Lambda$.

### 2.3 Defining gravity sectors with finite entropy

The de Sitter horizon differs from the horizon of a black hole in an important respect. The mass of a Schwarzschild black hole determines the area of its horizon and thus the size of the Hilbert space describing the black hole; similarly, the cosmological constant sets the area of the de Sitter horizon. However, the black hole mass is a variable parameter of the solution, which can even be time dependent; whereas the cosmological

\(^4\)A future event horizon is present in all expanding homogeneous isotropic universes with equation of state $p = w\rho$, $-1 \leq w < 1/3$ (for $D = 4$) (see, e.g., Refs. [27, 28]). However, this horizon has finite maximal area only in the de Sitter case ($w = -1$). For the remaining values of $w$ the horizon area grows without bound. Hence there is no bound on observable entropy in “quintessence” models, and they cannot be dual to a theory with a finite number of states.

\(^5\)With the stronger assumption that the spacetime is dS$^\pm$, Eq. (2.3) is a bound on the matter entropy on initial global slices [20].
constant $\Lambda$ (defined as the lowest accessible vacuum energy) can be interpreted as a fundamental characteristic of the theory.

In the introduction we argued that a theory should not contain more states than the phenomena it describes. This suggests that the dS$^+$ class of universes is in fact fully described by a quantum gravity theory with a Hilbert space $\mathcal{H}$ of finite dimension \[15–17\]

$$\dim \mathcal{H} = e^{S_0}. \tag{2.4}$$

For any quantum system, let us define $N$, the *number of degrees of freedom*, to be the logarithm of the dimension of the Hilbert space. (With this definition, degrees of freedom are spin-like, rather than fields or harmonic oscillators.) Then we may restate our conclusion as follows. A quantum description of the dS$^+$ universes requires only a finite number of degrees of freedom,

$$N = S_0. \tag{2.5}$$

Of course, it is quite possible that a theory of dS$^+$ universes will make use of a larger, or infinite-dimensional Hilbert space. It is remarkable, however, that a finite Hilbert space is in principle sufficient. By contrast, such a Hilbert space would necessarily be insufficient to describe asymptotically flat or AdS spacetimes, which can contain arbitrarily large entropy (for example, in the form of large black holes).

This result may point at a new class of theories, distinct from those for asymptotically flat or AdS spacetimes. The finiteness of $N$ may be a crucial qualitative feature underlying a successful description of dS$^+$ spacetimes.

Suppose, however, that we were given a theory $T$ with a finite number of degrees of freedom, $N$, and that this theory described all dS$^+$ universes with the corresponding value of $\Lambda$,

$$\Lambda(N) = \frac{(D - 1)(D - 2)}{2} \left( \frac{\Omega_{D-2}}{4N} \right)^{\frac{D-2}{D-2}}. \tag{2.6}$$

We discussed earlier that the difference in initial conditions between a spacetime $\mathcal{M}_1$ in the dS$^+$ class, and a spacetime $\mathcal{M}_2$ with no asymptotic de Sitter regions, can be arbitrarily small. It would be unnatural for a small deformation to invalidate the description of the spacetime by the theory $T$. Yet, in the absence of a future asymptotic region, the second law argument leading to Eq. (2.2) cannot be completed.\(^6\) It is not

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\(^6\)There are additional reasons why reference to an asymptotic region will not be ultimately satisfactory. Classically, a dS$^+$ spacetime may contain observers who do not get to $\mathcal{I}^+$—for example, if it contains black holes. Still worse, at the semi-classical level it becomes clear that no observer gets to $\mathcal{I}^+$. No system can withstand the thermal radiation \[18\] permeating de Sitter space indefinitely;
clear if there is an upper bound to the entropy in $\mathcal{M}_2$. Hence, it is unclear if $\mathcal{T}$ contains enough degrees of freedom to fully describe $\mathcal{M}_2$.

This poses a challenge. For a given value of $N$, one would like to identify a set of spacetimes $\mathcal{C}(N)$ that could be described by a theory $\mathcal{T}$ whose Hilbert space has dimension $N$.\footnote{Neither $\mathcal{T}$ nor $\mathcal{C}(N)$ need to be unique for given $N$. We do not assume this, nor do we assume that a consistent gravity theory exists for every value of $N$. Moreover, we must stress that the same low energy Lagrangian may contain different sectors of solutions corresponding to different values of $N$, including $N = \infty$. Each of these sectors would be described by a different theory $\mathcal{T}$. (This is familiar from the AdS/CFT correspondence. Solutions differing only by the number $n$ of units of flux are described by different $SU(n)$ gauge theories.)} Such a class should possess the following three properties:

1. $\mathcal{C}(N)$ should include, but not be limited to, the dS$^+$ solutions with $\Lambda = \Lambda(N)$:

   $$\mathcal{C}(N) \supset \text{dS}^+(\Lambda(N)).$$

2. $\mathcal{C}(N)$ should be closed under smooth deformations of Cauchy data.

3. $\mathcal{C}(N)$ must not include any universes with observable entropy greater than $N$.

Note that only the last condition is strictly necessary. As discussed in Sec. 1, one expects a unified theory to predict a particular matter content. This will limit the range of spacetimes defined by the first two conditions, and thus it may prevent violations of the third which a more ignorant analysis would produce. It may even remove the need for the second condition altogether.

For now, we have no information about matter content and will assume only that reasonable energy conditions are satisfied. Then the above conditions are useful for a preliminary assessment of the potential role of finite Hilbert space theories in quantum gravity.

Neither the existence nor the uniqueness of $\mathcal{C}(N)$ is guaranteed \textit{a priori}. One could exclude its existence, e.g., by showing that dS$^+$ Cauchy data can be smoothly deformed to yield a universe with observable entropy greater than $N$. This would render the case for finite Hilbert space theories related to quantum gravity in de Sitter rather, it will be thermalized in a finite time. Another aspect of this problem comes from noting that black holes will also form spontaneously in the thermal radiation [31]. It follows that eventually, after a long but finite time, any observer will be swallowed up by a black hole, with a finite probability (approaching one). In the case of a black hole of nearly maximal size, this process is indistinguishable from a big crunch. The creation rate for such black holes is roughly $e^{-S_0/3}$ [32]. Note that this is even greater than the estimated rate of Poincaré recurrences, $e^{-S_0}$ [33], which independently limits observer lifetimes in de Sitter space. More stringent upper bounds would be desirable but are not needed here.
space less compelling. If a suitable set is found, however, it will enhance the motivation for constructing such theories.

A natural proposal \[19\] would be to choose \( C(N) = \text{all}(\Lambda(N)) \), the set of all universes with positive cosmological constant \( \Lambda(N) \) given by Eq. (2.4). Since the cosmological constant is not part of the variable Cauchy data, this choice manifestly satisfies the first two properties. However, we find that the third property is violated by some spacetimes in \( \text{all}(\Lambda(N)) \). We now turn to a description of these solutions.

3. Product spacetimes with flux

We consider solutions to the following action in \( D = p + q \) dimensions:

\[
I = \frac{1}{16\pi} \int d^{p+q}x \sqrt{-g} \left( R - 2\Lambda - \frac{1}{2q!}F_q^2 \right).
\]

(3.1)

(Recall that we are working in Planck units with \( G_D = 1 \).) This describes Einstein gravity with a cosmological constant \( \Lambda \), taken to be positive, coupled to a \( q \)-form field strength, \( F_q = dA_{q-1} \). The equations of motion may be written as

\[
R_{MN} = \frac{1}{2(q-1)!} F_{MP_2...P_q} F_N^{P_2...P_q} - \frac{(q-1)}{2(D-2)q!} g_{MN} F_q^2 + \frac{2}{D-2} g_{MN}\Lambda, \quad (3.2)
\]

\[
d \ast F_q = 0. \quad (3.3)
\]

The most symmetric solution is \( d \)-dimensional de Sitter space, with \( F = 0 \) and metric

\[
d s^2 = -V(r) \, dt^2 + \frac{1}{V(r)} \, dr^2 + r^2 d\Omega_{D-2}^2, \quad (3.4)
\]

where

\[
V(r) = 1 - \frac{r^2}{L^2}. \quad (3.5)
\]

The radius of curvature is given in terms of the cosmological constant by

\[
L^2 = \frac{(D-1)(D-2)}{2\Lambda}. \quad (3.6)
\]

Next, we find product solutions of the form \( K_p \times M_q \), where \( K_p \) is Lorentzian with coordinates \( x^\mu \), \( M_q \) is Riemannian with coordinates \( y^\alpha \), and both factors are Einstein:

\[
R_{\mu\nu} = \frac{p-1}{L^2}g_{\mu\nu}, \quad R_{\alpha\beta} = \frac{q-1}{R^2}g_{\alpha\beta}. \quad (3.7)
\]

Additionally, we take the field strength \( F_q \) to be proportional to the volume form on \( M_q \):

\[
F_q = c \, \text{vol}_{M_q}, \quad (3.8)
\]
where $\text{vol}_{S_q}$ is normalized so that $\oint \text{vol}_{S_q} = R^q \Omega_q$. This field strength automatically satisfies the Maxwell equation (3.3) and the Bianchi identity $dF = 0$. Einstein’s equations now permit a family of solutions, parametrized by the dimensionless flux

$$ F \equiv \frac{c^2}{4\Lambda}. $$

(3.9)

The curvature radii $L, R$ satisfy

$$ \frac{p-1}{L^2} = \frac{2\Lambda}{D-2} [1 - (q-1)F], \quad \frac{q-1}{R^2} = \frac{2\Lambda}{D-2} [1 + (p-1)F]. $$

(3.10)

Since $F > 0$ by definition and we assume $\Lambda > 0$, equation (3.10) requires $R^2 > 0$ as well. Hence, $M_q$ must have positive curvature. We will generally take $M_q = S^q$, the $q$-dimensional sphere. As we discuss in the next section, this is the choice most likely to be stable.

On the other hand, we see that $L^2$ has indefinite sign. For small $F$, one finds $L^2 > 0$. This means that $K_p$ is positively curved and can be taken to be dS$_p$. At the value

$$ F = F_m \equiv \frac{1}{q - 1}, $$

(3.11)

the curvature radius diverges. In this case $K_p$ is flat; it can be taken to be $p$-dimensional Minkowski space, for example. For $F > F_m$, $L^2$ becomes negative. This corresponds to a change of sign of the Ricci scalar. The Lorentzian factor will be negatively curved, and it is useful to define $\tilde{L}^2 \equiv -L^2$. We can take $K_p$ to be $p$-dimensional anti-de Sitter space (with real curvature radius $\tilde{L}$) in this case. Note that $\tilde{L}$ satisfies

$$ \frac{p-1}{\tilde{L}^2} = \frac{2\Lambda}{D-2} [(q-1)F - 1]. $$

(3.12)

We observe that as $\Lambda \to 0$ (with $c$ fixed), $R$ and $\tilde{L}$ remain finite and our solutions reproduce the usual Freund-Rubin compactification [34] with geometry AdS$_p \times S^q$.

Independently of the sign of $L^2$, all metrics we consider for $K_p$ are described by Eq. (3.4), with $D$ replaced by $p$. Solving Einstein’s equations does not require $K_p$ to be maximally symmetric, rather it must simply satisfy (3.7). Hence, for $p > 2$, $K_p$ can be taken to be a $p$-dimensional Schwarzschild-(anti)-de Sitter solution with:

$$ V(r) = 1 - \frac{\mu}{r^{p-3}} - \frac{r^2}{\tilde{L}^2}. $$

(3.13)

This introduces an additional parameter, the “mass” $\mu$, into the space of solutions. We will ignore this freedom in the $L^2 > 0$ case, where we set $\mu = 0$ because empty dS$_p \times S^q$ has the largest horizon area in that family. However, in the $L^2 < 0$ case, we will find that black holes offer a convenient way of adding unlimited entropy without affecting the stability of an asymptotically AdS$_p \times S^q$ solution.
4. Stability

Naively, many of the solutions discussed in the previous section appear to violate the $N$-bound. However, to be confident that we have identified a true violation, we must show that the solutions are in fact stable. Hence, we now examine the perturbative stability of these product solutions. We consider the AdS$_p$ cases first, generalizing analogous work for $\Lambda = 0$ [35]. The computations involve performing a Kaluza-Klein decomposition of fluctuations on the compact space, and comparing the effective masses of the resulting modes to the Breitenlohner-Freedman stability bound. We argue that a black hole can be introduced into the AdS$_p$ without modifying the stability of the solutions. Then we consider the dS$_p$ cases, obtained by continuing the AdS$_p$ results, and identify instabilities as negative mass-squared modes. Finally we comment briefly on the fate of the unstable solutions.

There is a long history of Kaluza-Klein analysis on Freund-Rubin-type backgrounds, including notably the studies of fluctuations around the maximally supersymmetric solutions of supergravity, e.g. Refs. [36–38]; for a review, see Ref. [39]. The fluctuations on the full space are viewed as towers of modes on $K_p$, after performing the Kaluza-Klein reduction on $M_q$. Our present analysis relies heavily on that presented in Ref. [35], where the complete fluctuation spectrum of modes in $K_q \times M_q$ backgrounds (3.7), (3.8) of Einstein-Maxwell theory (3.1) with $\Lambda = 0$ were obtained; the Lorentzian factor in that case is necessarily negatively curved and was taken to be anti-de Sitter space. The form of the fluctuation equations did not depend on this choice, but the criterion for stability in principle may: it is well-known that in AdS$_p$, scalar fluctuations may have negative mass-squared without generating an instability as long as the masses do not violate the Breitenlohner-Freedman bound [40]:

$$m^2 \tilde{L}^2 \geq -\frac{(p-1)^2}{4}. \tag{4.1}$$

The existence of this stability window for naively tachyonic modes may be thought of as a consequence of the negative contribution to the energy from the negative mass-squared being overwhelmed by the positive contribution from the spatial variation.

Defining the fluctuations (valid for $p > 2$)

$$\delta g_{\mu\nu} = h_{\mu\nu} = H_{\mu\nu} - \frac{1}{p-2} g_{\mu\nu} h^\alpha_{\alpha},$$

$$\delta g_{\mu\alpha} = h_{\mu\alpha} , \quad \delta g_{\alpha\beta} = h_{\alpha\beta} , \quad \delta A_{q-1} = a_{q-1} , \quad \delta F_q \equiv f_q = da_{q-1} , \tag{4.2}$$

$$H_{\mu\nu} = H_{(\mu\nu)} + \frac{1}{p} g_{\mu\nu} H^\rho_{\rho} , \quad h_{\alpha\beta} = h_{(\alpha\beta)} + \frac{1}{q} g_{\alpha\beta} h^\gamma_{\gamma},$$

with $g^{\mu\nu} H_{(\mu\nu)} \equiv 0 \equiv g^{\alpha\beta} h_{(\alpha\beta)}$, it was found for $\Lambda = 0$ [35] that all fluctuation equations but one (the one for the modes $h_{(\alpha\beta)}$) depend on the choice of Einstein spaces solely
through the parameters $p$ and $q$. Moreover, these modes never violate the Breitenlohner-Freedman bound. Among these stable modes are the diagonal fluctuations of the metric and the fluctuations of the $q$-form along $M_q$, which form a coupled system that must be diagonalized. It was found that the minimum mass $m^2(\lambda)$ of the Kaluza-Klein modes on the lower branch of this coupled system exactly saturates the Breitenlohner-Freedman bound \([4.1]\) without violating it, as a function of (minus) the eigenvalue of the Laplacian on the compact space $\lambda$.

In contrast, the fluctuation equation for $h_{(\alpha\beta)}$ depends on the details of the choice of $M_q$. It was shown that for $M_q = S^q$ this field produces only positive-mass modes, and hence a generic AdS$_p \times S^q$ Freund-Rubin spacetime is perturbatively stable. In contrast, for the choice $M_q = S^n \times S^{q-n}$ with $q < 9$ an instability is generated in anti-de Sitter space, and other choices of $M_q$ may lead to instabilities as well.

Our analysis for nonzero $\Lambda$ uses the same expansion in spherical harmonics and gauge choices as Ref. \([35]\) and proceeds along the same lines. Furthermore, the dS$_p$ case can be obtained simply from the AdS$_p$ case by continuing $\tilde{L}^2 \to -L^2$. It is not difficult to show that most fluctuation equations are unchanged from when $\Lambda = 0$; this includes the $h_{(\alpha\beta)}$ mode, so once more only positive-mass modes will appear in this channel as long as the compact space is $S^q$. Since we are looking for stable solutions, we will generally take $S^q$ in what follows. Then the only potential instability lies in the modes that are affected by $\Lambda \neq 0$, namely the coupled system mentioned previously,

$$h_{\alpha}^I(x, y) = \sum_I \pi^I(x) Y^I(y), \quad H_\mu^I(x, y) = \sum_I H^I(x) Y^I(y), \quad (4.3)$$

$$a_{\beta_1...\beta_{q-1}}(x, y) = \sum_I b^I(x) \epsilon_\beta_{\beta_1...\beta_{q-1}} \nabla_\alpha Y^I(y),$$

which constitutes a fluctuation of the fields active in the background. One still finds (for most cases; see below) that $H^I$ may be algebraically eliminated in favor of $\pi^I$,

$$H^I = -\frac{2}{q} \frac{(D-2)}{(p-2)} \pi^I,$$  \(4.4\)

leaving the coupled system of equations

$$\tilde{L}^2 \square_x \left( \begin{array}{c} cb^I \\ \pi^I \end{array} \right) = \left( \begin{array}{cc} \tilde{\lambda}^I & \frac{q}{q} \frac{(p-1)(D-2)}{(D-2)} \\ \frac{2q(p-1)}{(D-2)} \tilde{\lambda}^I & 2q + 2(p-2) \tilde{\alpha} + 2(p-1)^2 \end{array} \right) \left( \begin{array}{c} cb^I \\ \pi^I \end{array} \right).$$  \(4.5\)

Here, $\tilde{\alpha} \equiv 2\Lambda \tilde{L}^2$ is related to the dimensionless flux $\mathcal{F}$ by

$$\tilde{\alpha} = \frac{(D-2)(p-1)}{(q-1)\mathcal{F} - 1},$$  \(4.6\)
and $\tilde{\lambda}'$ is the rescaled eigenvalue of the compact Laplacian:

$$-\tilde{\mathcal{L}}^2 \square_y Y^I \equiv \tilde{\lambda}' Y^I = \left( \frac{\tilde{\mathcal{L}}^2}{R^2} \right) \lambda' Y^I. \quad (4.7)$$

Diagonalization of the mass matrix (4.5) gives the spectrum of masses $m^2(\lambda)$ for various values of $\tilde{\alpha}$, equivalent to fluxes $\mathcal{F} > \mathcal{F}_m = 1/(q - 1)$, and continuation $\tilde{L}^2 \rightarrow -L^2$ gives the spectrum for the de Sitter values $0 \leq \mathcal{F} < \mathcal{F}_m$. We examine the stability of these fluctuations in turn.

### 4.1 Stability of AdS$_p \times S^q$

The mass spectrum for negatively curved $K_p$ is

$$m^2(\tilde{\lambda})\tilde{L}^2 = \tilde{\lambda} + \tilde{A} \pm \left[ \tilde{A}^2 + 4\tilde{\lambda}\tilde{B} \right]^{1/2}, \quad (4.8)$$

with the constants

$$\tilde{A} = \frac{p - 2}{D - 2} \tilde{\alpha} + (p - 1)^2, \quad \tilde{B} = \frac{p - 1}{D - 2} \tilde{\alpha} + (p - 1)^2. \quad (4.9)$$

When reducing on $S^q$, $\lambda$ takes values $k(k + q - 1)$ with $k \in \mathbb{Z}_+$. It turns out that the modes at $k = 0, 1$ are special cases for which there is only one physical perturbation, while the other is a spurious gauge mode related to the existence of conformal Killing vectors on the sphere. Separate analysis of these special cases leads to the conclusion that the “minus” branch of (4.8) is truncated to $k \geq 2$, whereas the “plus” branch is valid for all $k \geq 0$.

One may check that for $\tilde{\alpha} \rightarrow 0$ our result (4.8) coincides with that of Ref. [35]. Namely, the minimum occurs on the negative branch for $\tilde{\lambda}_{\text{min}} = (3/4)(p - 1)^2$, and at this minimum

$$m^2(\tilde{\lambda}_{\text{min}})\tilde{L}^2 = -\frac{1}{4}(p - 1)^2, \quad (\Lambda = 0) \quad (4.10)$$

which precisely matches the Breitenlohner-Freedman bound for fluctuations in AdS$_p$.

Since the mass curve (4.8) with $\Lambda = 0$ saturates the Breitenlohner-Freedman bound, one might suspect that turning on $\Lambda$ will push the curve over the edge. For a fixed AdS$_p$ mass scale $1/\tilde{L}^2$, a positive value of $\Lambda$ causes the compact space to curve more strongly, and so one might think that the Kaluza-Klein scale $1/R^2$ set by $M_q$ will become more extreme relative to $1/\tilde{L}^2$.

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8If the compact space is not $S^q$, there is only one spurious mode ($Y^I$ constant), and the minus branch is defined for $k \geq 1$. 

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Indeed this is what we find. The mass curve $m^2(\bar{\lambda})$ has a minimum at

$$\bar{\lambda}_{\text{min}} = \bar{B} - \frac{\bar{A}^2}{4\bar{B}},$$

(4.11)

where it takes the value

$$m^2(\bar{\lambda}_{\text{min}})\bar{L}^2 = -\frac{(2\bar{B} - \bar{A})^2}{4\bar{B}} = -\frac{\bar{\alpha}}{D - 2} - \frac{1}{4(p - 1)} \left(\frac{(p - 1)^2}{(p - 1)} + \frac{p - 1}{D - 2}\right)^2 \frac{\bar{\alpha}^2}{D - 2}. \quad (4.12)$$

For infinitesimal $\bar{\alpha}$ (very large flux $F$), this becomes

$$m^2(\bar{\lambda}_{\text{min}})\bar{L}^2 \approx -\frac{1}{4}(p - 1)^2 - \frac{p + 1}{4(D - 2)} \bar{\alpha}. \quad (4.13)$$

Hence the minimum of the mass curve dips immediately below the Breitenlohner-Freedman stability bound as a positive cosmological constant is turned on. One may also show that $m^2(\bar{\lambda}_{\text{min}})\bar{L}^2$ is monotonically decreasing (so the minimum mass does not rise above the bound for any larger $\Lambda$), and that it approaches $-\infty$ as $\Lambda \to \infty$.

This analysis suggests that all AdS$_p$ x $S^q$ solutions with $\Lambda > 0$ are unstable to small fluctuations. However, the story is more subtle. Recall that only discrete values of $\bar{\lambda}$ actually arise as eigenvalues of the Laplacian on $M_q$, indexed by $k$. As discussed in Ref. [35], for $q$ odd there is always an integer $k = (q - 1)/2$ such that $k(k + q - 1)$ is $\lambda_{\text{min}}$, but for $q$ even there is not; the lowest-mass states sit on either side of the minimum of the curve. In the former case, infinitesimal $\bar{\alpha}$ will push the lowest state below the instability bound; this is true even though for $\Lambda > 0$ the physical state $k = (q - 1)/2$ no longer sits precisely at the minimum of the curve. In the latter case, however, there is some range with the cosmological constant sufficiently small in which the lowest-mass state has not yet moved below the bound. Hence, for $q$ even and $F$ sufficiently large, there is a window of stability.

Furthermore, for $q = 2$ and $q = 3$, the mode nearest to the minimum would be at $k = 0$ or $k = 1$; however, these modes are absent from the lower branch. Consequently, one must check whether the physical modes with $k \geq 2$ become tachyonic; it turns out that they do not, and as a result there are no instabilities whatsoever in these cases.

Hence a more intricate pattern of instabilities emerges. For $q = 2$ or $q = 3$, the spacetimes are stable regardless of $p$, and for all values of $F \geq F_m$. For $q \geq 4$, a perturbative tachyon always exists for $q$ odd, but for even $q$ there is a stable window for $F$ sufficiently large.

Let us consider the generalization of the stability analysis of empty anti-de Sitter space to the case of AdS-Schwarzschild solutions. As the Kaluza-Klein reduction

\footnote{The black hole itself is taken to have horizon radius much larger than the AdS$_p$ length $\bar{L}$, and hence is classically and semiclassically stable.}
only involves the compact space $S^q$, it can be carried out equally well in the presence of a black hole, and one obtains fluctuations with the same masses as in the case of ordinary $AdS_p$ above. Therefore, the spacetime would only be jeopardized if the criterion for stability—that is, the Breitenlohner-Freedman bound (4.1)—was changed by the introduction of the black hole.

The black hole spacetime asymptotically approaches $AdS$, and the boundary conditions on fluctuations required by energy conservation will consequently be unchanged. Since imposing these conditions require the mass-squared to exceed the usual Breitenlohner-Freedman bound, we see that the presence of the black hole cannot stabilize previously unstable modes.

Asymptotically $AdS$ spacetimes have been shown to have non-negative energy, with zero energy coinciding with pure $AdS$, for supersymmetric theories with tachyonic scalars satisfying the Breitenlohner-Freedman bound [41, 42]. This was generalized to non-supersymmetric theories for the case of a single scalar in Ref. [43], and it is natural to think that the proof will hold for the case of multiple scalars. These results do not, however, rule out the possibility that the fluctuations of the scalars (which violate the dominant energy condition [41]) could destabilize the black hole. The naive extension of Breitenlohner and Freedman’s derivation to the black hole spacetime, with a regularity condition at the horizon, fails to prove positivity of the energy. Gubser [44] has conjectured a criterion generalizing the Breitenlohner-Freedman bound in arbitrary asymptotically $AdS$ spacetimes.

Here we simply note that a destabilization would be problematic from the point of view of the $AdS$/CFT correspondence, in which a large $AdS$ black hole is described by a finite-temperature field theory, with no known characteristic instability. If the Breitenlohner-Freedman bound was raised even infinitesimally with the introduction of a black hole, the black hole solutions of $AdS_5 \times S^5$ in Type IIB supergravity and $AdS_4 \times S^7$ in eleven-dimensional supergravity, and any other solution with $q$ odd, would decay via the mode that saturates the Breitenlohner-Freedman bound in the case of empty $AdS$. Consequently we do not expect the introduction of a black hole to produce new instabilities.

4.2 Stability of $dS_p \times S^q$

By continuing $\tilde{L}^2 \to -L^2$, we find the mass spectrum for the coupled system (4.3) in the de Sitter case,

$$m^2_\pm(\lambda)L^2 = \tilde{\lambda} + A \pm \sqrt{A^2 + 4B\lambda},$$  \hspace{1cm} (4.14)
where

\[
A = \frac{p - 2}{D - 2} + (p - 1)^2 = \frac{(p - 1)^2(q - 1)}{1 - (q - 1)F} (F - F_s),
\]

\[
B = \frac{p - 1}{D - 2} + (p - 1)^2 = \frac{(p - 1)^2(q - 1)}{1 - (q - 1)F} F,
\]

\[
F_s \equiv \frac{1}{(p - 1)(q - 1)}.
\]

and \(\alpha \equiv 2\Lambda L^2, \bar{\lambda} = (L^2/R^2)\lambda\). As the dSp solutions exist over the range of values \(0 \leq F < F_m = 1/(q - 1)\), \(B\) is always positive-definite, while \(A\) flips sign at \(F = F_s\).

We are interested in identifying potentially unstable modes in these spacetimes. There is no Breitenlohner-Freedman bound for dSp spacetimes, but rather we must simply require that no tachyonic modes arise in the Kaluza-Klein reduction, as in flat space. To see this, consider the local energy density measured by an observer. Of course, the effective cosmological constant contributes a fixed energy density. The modes associated with tachyonic fluctuations are found to be exponentially growing or decaying as one approaches the future or past boundary—see, for example, Refs. [45,46].

A short calculation shows that the contribution of the growing modes to the local energy density increases without bound in these regions of the spacetime, and hence quickly overwhelms that of the cosmological constant. Consequently they cannot consistently be treated as perturbations of the dS background. On the other hand, the same analysis shows there is no such instability for modes with positive mass-squared, even in the range \(0 < m^2 < (p - 1)^2/4L^2\), as the vacuum energy density dominates the perturbation at all times. For an alternate discussion of dS stability, see Ref. [47].

Just as in the AdSp case, the Kaluza-Klein reduction on \(S^q\) produces only one mode at each \(k = 0\) and \(k = 1\), and only for \(k \geq 2\) are both branches of (4.14) sampled. Interestingly, because \(A\) can change sign, whether the \(k = 0\) mode is on the plus or minus branch changes as \(F\) is varied. Performing the analysis for the \(k = 0\) mode, we find the mass

\[
m_0^2 L^2 = 2A.
\]

When \(F > F_s\), this is on the upper branch, while for \(F < F_s\) it moves to the lower branch; the two branches meet at zero mass for \(F = F_s\) (but there is still only one massless state at \(k = 0\)). We notice immediately that for flux smaller than \(F_s\), there is always a tachyonic mode constant on the \(S^q\). In particular, the case with no flux \((F = 0)\) is always unstable.\(^{10}\)

\(^{10}\)Our result (and, for \(p = 2\), Ref. [48]) refutes the conjecture made in Ref. [49] that the cosmological horizon in dSp \(\times S^q\) spacetimes without flux is classically stable.
One can show that $k = 1$ remains on the upper branch, and that the upper branch is a monotonically increasing function on the range $\bar{\lambda} \geq 0$ taking only positive values. Therefore, any instabilities in the region $\mathcal{F} > \mathcal{F}_s$ must be produced by the $k \geq 2$ modes on the minus branch. We find that $m^2_\perp$ has a minimum at

$$\bar{\lambda}_{\text{min}} = B - \frac{A^2}{4B},$$

(4.18)

at which point

$$m^2_\perp L^2 = -\frac{(2B - A)^2}{4B}.$$  

(4.19)

Naively it seems that this branch always produces tachyonic modes and hence all of the corresponding solutions should be unstable. However, this need not be so, since $\bar{\lambda}_{\text{min}}$ need not correspond to any physical excitation. One must check that there is some integer $k \geq 2$ that actually produces a tachyon.

One may show that on the range of positive $\bar{\lambda}$, $m^2_\perp L^2$ is negative for the region

$$0 < \bar{\lambda} < \bar{\lambda}_0 \equiv \frac{2(p - 1)^2(q - 1)}{1 - \mathcal{F}(q - 1)} \left(\mathcal{F} + \mathcal{F}_s\right).$$

(4.20)

Since $\bar{\lambda} \geq 0$, one can see that if a mode with $k \geq 3$ is tachyonic, the $k = 2$ mode will be a unstable as well, so the question of whether there are instabilities in the higher modes reduces the question of whether $\bar{\lambda}(k = 2)$ lies in the range (4.20). One finds that this inequality is only satisfied if $q \geq 4$ and

$$\mathcal{F} > \mathcal{F}_t \equiv \frac{2}{q(q - 3)(p - 1)}.$$ 

(4.21)

The condition for a stable dS$_p \times S^q$ spacetime, incorporating both the $k = 0$ tachyon and unstable modes from higher $k$, is thus

$$\mathcal{F}_s \leq \mathcal{F} \leq \mathcal{F}_t.$$ 

(4.22)

Since no instabilities arise from higher-$k$ modes for $q = 2$ or $q = 3$, spacetimes with a compact $S^2$ or $S^3$ are unstable for fluxes $\mathcal{F} < \mathcal{F}_s$ only. The stable region continues through the Minkowski point at $\mathcal{F} = \mathcal{F}_m$ and into the AdS region, which is also free of instabilities, as remarked in the last subsection.$^{11}$

For $q = 4$, (4.22) implies a window of stability, $1/[3(p - 1)] \leq \mathcal{F} \leq 1/[2(p - 1)]$. The instability at $1/[2(p - 1)] < \mathcal{F}$ continues through the Minkowski region into the AdS region.

$^{11}$There have been suggestions that the stability criterion for fluctuations in de Sitter space is more strict than we assume, in particular that there may be an “inverse BF bound” $m^2 L^2 \geq (p - 1)^2/4$ for stability $^{50}$. We note that even assuming this less favorable criterion, we find stable solutions.
For $q \geq 5$, we find $F_t < F_s$. Consequently the two regions of instability overlap, and instabilities occur for all values of $F$. Hence spacetimes with $q \geq 5$ can only be stable in the AdS region, and then only for $q$ even and very large flux.

One may additionally ask what can be said about the fate of the unstable solutions. It would be interesting to apply the analysis of Ref. [51] to this question. However, this is complicated by the absence of an asymptotically flat region. In cosmological spacetimes, the existence of an apparent horizon does not guarantee that an event horizon will be present at the initial time where perturbations are introduced. Hence it is not clear that the formation of a dS event horizon would require the type of topological transition that the Horowitz-Maeda argument excludes. Thus the ultimate fate of the unstable solutions remains an open question.

5. Entropy

In this section we show that some of the solutions we found have entropy greater than $N(\Lambda)$. Hence, they violate the $N$-bound.

Our solutions contain no entropy in the form of ordinary matter systems—all potential contributions to entropy come from the Bekenstein-Hawking entropy of event horizons. In order to interpret the horizon areas as entropy, furthermore, we must consider only the stable solutions. Event horizons are determined by the global structure of a spacetime, not by its shape at an instant of time. If we were to attribute entropy (and hence, temperature) to one of the unstable product solutions, we would run into a contradiction: the thermal fluctuations would destabilize the spacetime, and the far future (including any event horizon) would differ from the assumed unstable solution.

Let us first consider the stable product solutions with $F > F_m$. At fixed $F$, there exist AdS$_p \times S^q$ solutions with a black hole of arbitrarily large horizon area ($\mu \to \infty$), except if $p = 2$. Hence, there will be a solution with entropy greater than $N(\Lambda)$. Thus, the $N$-bound is violated for all stable Schwarzschild-AdS$_p \times S^q$ solutions which also satisfy $p > 2$.

Next, consider the stable product solutions with $F < F_m$. At fixed $F$, the entropy $S(\Lambda, F)$ of the corresponding dS$_p \times S^q$ solution is finite. It is given by a quarter of the dS$_p$ horizon, times the volume of the $S^q$:

$$S(\Lambda, F) = \frac{1}{4} \Omega_{p-2} \Omega_q \left( \frac{D-2}{2\Lambda} \right)^{\frac{D-2}{2}} \left( \frac{p-1}{1-(q-1)F} \right)^{\frac{p-2}{2}} \left( \frac{q-1}{1+(p-1)F} \right)^{\frac{q}{2}}. \quad (5.1)$$

This entropy cannot be increased by introducing black holes, because the cosmological horizon area overcompensates by shrinking.
If the ratio between (5.1) and the entropy of the $d$-dimensional de Sitter horizon (2.3),
\[
\frac{S(F)}{S_0} = \frac{\Omega_{p-2} \Omega_q}{\Omega_{D-2}} \left( \frac{p - 1}{1 - (q - 1)F} \right)^{p-2} \left( \frac{q - 1}{1 + (p - 1)F} \right)^q,
\]
(5.2)
is greater than unity, the $N$-bound (2.5) will be violated. The ratio is less than unity for $F = 0$, and as $F$ increases, the ratio further decreases until $F = F_s$. These values do not correspond to stable solutions in any case. Above $F_s$ the ratio begins to increase. It crosses unity at some value $F_{\text{crit}}$ and actually diverges at $F = F_m$. Hence for a range of values, the ratio is actually larger than one. For $q = 2$ and $q = 3$, this entire range corresponds to stable spacetimes, while for $q = 4$, stable spacetimes occur for $F_s \leq F \leq F_t$. Consequently there also exist stable $\text{dS}_p \times S^q$ spacetimes that violate the $N$-bound.

Finally, we should remark on the case $p = 2$. The AdS$_2 \times S^q$ solutions contain no entropy as such, and we have not found a way of introducing entropy greater than $N$. Indeed, we do not expect this to be possible. In the vacuum solution, one finds that the area of the $S^q$ is less than $4N$ independently of the flux. Introducing excitations will destroy the asymptotic behavior of the spacetime at least on one of the two disconnected components of the boundary $\mathcal{I}$ [52]. Let us consider a solution in which $\mathcal{I}$ is foliated by a single copy of $S^q$, with area equal to the $S^q$ area of the vacuum solution. Applying the covariant entropy bound to this sphere, one concludes that the entropy on the corresponding light-sheet is less than $N$. Such light-sheets foliate both the causal past and the causal future of $\mathcal{I}$. Hence, entropy that would violate the $N$-bound would have to be causally disconnected from infinity.

The $\text{dS}_2 \times S^q$ spacetimes are known as charged Nariai solutions. They correspond (for each value of $F$) to the largest charged black hole in de Sitter space. The entropy of such solutions is less than that of de Sitter space [53]. The solutions are classically unstable and develop an asymptotically $\text{dS}_{p+2}$ region only upon perturbation. Their instabilities have been noted in Refs. [18, 32] and studied in detail in Refs. [48, 54–57].

6. Conclusions

We have shown that some spacetimes with positive cosmological constant $\Lambda(N)$ contain observable entropy greater than $N$. Hence they cannot be described by a theory with a Hilbert space of finite dimension $e^N$. We have found no such spacetimes for $D = 4$. The significance of this exception is not clear.

It would be important to know whether quantum gravity theories with finite-dimensional Hilbert space exist. In order to assess this prospect, one would like to
characterize the spacetimes that such theories would describe. We have given a number of conditions that one would expect a suitable class of dual spacetimes, $\mathcal{C}(N)$, to satisfy. A candidate for $\mathcal{C}(N)$ was proposed in Ref. [21]: the set of spacetimes with positive cosmological constant, $\text{all}(\Lambda(N))$.

The following conclusions apply for $D > 4$:

The product spacetimes we have analyzed demonstrate that the set $\text{all}(\Lambda(N))$ is not suitable. It fails to satisfy the $N$-bound, which demands that none of its spacetimes exhibit observable entropy greater than $N$.

It follows that the specification of a positive cosmological constant alone does not suffice to characterize the spacetimes dual to a finite Hilbert space theory. There cannot be a straightforward correspondence between $\Lambda$, and the number of degrees of freedom, $N$ (a $\Lambda$-$N$ correspondence).

This does not exclude the possibility that a positive cosmological constant should be properly regarded as a consequence of a finite-dimensional Hilbert space. However it cannot be the sole consequence.

Although we have demonstrated that $\text{all}(\Lambda(N))$ is not an adequate choice for $\mathcal{C}(N)$, we have not shown that some other suitable $\mathcal{C}(N)$ cannot exist. All the problematic examples we found are of product form, and hence cannot be obtained from dS$^+$ Cauchy data by smooth deformation. Consequently it is possible that there is a $\mathcal{C}(N)$ that excludes these solutions, yet contains the dS$^+$ solutions and is still closed under deformations of Cauchy data, without violating the $N$-bound. Its characterization remains unclear.

Experience with non-perturbative definitions of string theory has taught us that the same low energy Lagrangian (in fact, the same fundamental theory) can have different ‘superselection’ sectors described by different dual theories. For example, in the AdS/CFT correspondence, the size of the gauge group of the CFT dual to a ($\Lambda = 0$) Freund-Rubin compactification depends on the flux. In the present context then, it could be that a particular dual theory $\mathcal{T}$ with a specific number of degrees of freedom $N$ will only capture the spacetime physics of a certain sector of a given low energy theory. In this case, one approach to defining the corresponding $\mathcal{C}$ may be to include fluxes alongside the cosmological constant$^{12}$ among the parameters necessary to be specified in order to ensure that the entropy will not exceed $N$. This appears to produce rather complicated conditions whose sufficiency is not obvious. Moreover, this approach fails to capitalize on the need to deal only with smooth deformations of dS$^+$ spacetimes.

$^{12}$The specification of flux will identify an isolated sector only if flux-changing instantons [58, 59] are completely suppressed. This is the case for the $\Lambda = 0$ product solutions studied in the AdS/CFT context, but one might expect an onset of non-perturbative instabilities if $\Lambda > 0$. 
We have argued that a restriction to dS+ spacetimes, where the $N$-bound is satisfied, is unnatural because of the Cauchy data deformation problem. However, it is conceivable that what appear to be smooth, infinitesimal deformations from the point of view of the solution space to the effective Lagrangian may in fact be ruled out in the full non-perturbative quantum theory.

In $D = 4$, the set $\text{all}(\Lambda(N))$ is still a viable candidate for $\mathcal{C}(N)$. In general it remains a challenge either to find, or to prove the non-existence of, a class of spacetimes that could represent the classical limit of a quantum gravity theory with a finite number of degrees of freedom.

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