Arbitrary polynomial chaos expansion and its application to power flow analysis-Fast approximation of probability distribution by arbitrary polynomial expansion

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Abstract. This paper introduces an arbitrary polynomial chaos expansion method for performing probabilistic power flow analysis in power systems. The proposed method is used for uncertainty analysis, expressing the uncertainty of a system as random variables with an arbitrary output distribution based on orthogonal polynomial expansion. This method is advantageous because of its calculation speed and accuracy. This study expresses probabilistic power flow in a power system with many uncertain power sources using linear combination polynomial expansion. The orthogonal polynomial system employed is generated by moment analysis from renewable energy output data, with the polynomial coefficients derived from a collocation method. Simulation of probabilistic power flow using the proposed method is applied to a 29-bus transmission network model including three renewable energies, and the calculation speed and accuracy are evaluated by changing the expansion order of the polynomial. In addition, the influence on the polynomial coefficient is assessed when the system topology is changed due to a line fault. Therefore, since the arbitrary polynomial chaos expansion method can represent complex networks by linear combination of orthogonal polynomial sets, calculation based on it is several hundred times faster than the conventional Monte Carlo method. The results demonstrate that the proposed method is very useful for analyzing the probabilistic power distribution and that third-order expansion is practically appropriate.

1. Introduction
In recent years, the interconnection of renewable energy source (RES) to the power system has increased. Therefore, evaluating the effects of uncertainties on the system operation and planning is necessary. Because of the effects of uncertainty, probabilistic power flow (PPF) analysis treats the power flow in a power system as a probability value. As the PPF analysis method, the Monte Carlo (MC) method calculates a probability value from tens of thousands of random inputs, with popular methods proposed by Borkowska [1] and Dopazo et al. [2]. However, the many input required for the MC method is undesirable because of lengthy calculation times. Moreover, the methods proposed by Borkowska and Dopazo et al. assume that random variables with fluctuation follow a normal
distribution and thus are unsuitable for uncertainty evaluation of an RES with its arbitrary probability distribution.

Recently, a polynomial chaos (PC) expansion method is attracting attention as an uncertainty analysis method. This PC expansion method is a technique applied in many fields for approximating uncertainty and is advantageous because of its calculation speed and accuracy. This study applies an arbitrary PC (aPC) expansion method, extending the PC expansion method for PPF analysis involving random variables with arbitrary probability distributions. In 2018, Laowanitwattana [3] applied the aPC method to PPF analysis and verified its effectiveness. In this study, in addition to the application of the aPC method for PPF analysis, we elucidate its application in an actual system operation and evaluate the characteristics of the polynomial system.

The PC expansion method originates from the Wiener–Hermite chaos theory [4] proposed by Norbert Wiener. In [4], Wiener showed that any stochastic model of an event with uncertainty can be represented by a sum of Hermitian polynomials, one of the orthogonal polynomials. The Hermite polynomial used by Wiener as an example is most suitable for uncertainty approximating a standard normal distribution. So, if the random variables follow a different distribution, the expansion order of the polynomial may increase significantly. Therefore, Xiu et al. presented the Wiener–Askey chaos theory [5], enabling approximation of lower-order expansion orders to general probability distributions using polynomials orthogonal to non-normal distributions from hypergeometric orthogonal polynomials. For example, the approximation by the Laguerre polynomial is optimal for gamma distribution, whereas the Jacobi polynomial is optimal for beta distribution. The polynomial expansion approximation method predicated on the Wiener–Hermite chaos theory or the Wiener–Askey chaos theory is generally referred to as a general PC (gPC) expansion method. Conversely, a method approximating a stochastic model including a random variable with an arbitrary probability distribution by low-order PC expansion exists. For a finite number of moments, any probability distribution involves a series of orthonormal polynomials. Therefore, a data-driven orthogonal polynomial from an arbitrary probability distribution of a random variable input to the stochastic model can be generated, and the probabilistic model can be approximated by low-order polynomial expansion [6]. Such a PC method is referred to as an aPC expansion method, with PC expansion methods roughly divided into gPC and aPC methods. Since the output of the RES in this study changes naturally, application of the aPC method is considered optimal.

This paper is organized as follows: Section 2 describes the aPC method. Section 3 describes the PPF analysis by the aPC and MC methods. In Section 4, the proposed method is applied to an actual network and its effectiveness assessed. Section 5 simulates transmission line accidents, considering realistic system operation problems. Finally, conclusions and future work are presented.

2. aPC expansion

2.1. PC expansion

In the PC expansion method, the probabilistic output $Y$ is expressed by an equation where $n$ independent input parameter random variables are $\xi = \{\xi_1, \xi_2, ..., \xi_n\}$ expressed as

$$Y(\xi_1, \xi_2, ..., \xi_n) \approx \sum_{i=0}^{P-1} \Psi_i(\xi_1, \xi_2, ..., \xi_n).$$

where $\Psi_i$ is the $i$th multivariate orthogonal polynomial basis, $c_i$ represents the $i$th multivariate orthogonal polynomial coefficient, and $P$ denotes the number of polynomials.

The parameter $P$ is determined by Eq. 2 based on the number $n$ of input parameters and the maximum degree $d$ of the polynomial.

$$P = \frac{(n + d)!}{n!d!}.$$  

The probabilistic output by the PC expansion method is approximated by MC simulation using Eq. 1. Therefore, crucial in the PC expansion method is deriving the PC expansion equation of Eq. 1. To
derive this equation, $P$ multivariate orthogonal polynomial bases $\Psi_i$ and coefficients $c_i$ are obtained. Sections 2.2 and 2.3 contain explanations for finding the bases and coefficients.

2.2. Multivariate orthogonal polynomial basis

The multivariate orthogonal polynomial $\Psi_i$ is expressed by a product thereof, using a univariate polynomial basis $\psi$ for expressing each input parameter expressed in Eq. 3 as

$$
\Psi_i(\xi_1, \xi_2, \ldots, \xi_n) = \prod_{j=1}^{n} \psi^{(k_j)}(\xi_j),
$$

$$
i = 0, 1, \ldots, P - 1; \quad k_j = 0, 1, \ldots, d; \quad \sum_{j=1}^{n} k_j \leq d. \quad \#(3)
$$

Here, $k_j$ is a variable indicating the expansion order of the univariate polynomial basis. That is, $\psi^{(k_j)}(\xi_j)$ is a univariate polynomial basis of order $k_j$ by the $j$th random variable $\xi_j$. Therefore, $\psi^{(k_j)}(\xi_j)$ is calculated through Eq. 4 as

$$
\psi^{(k_j)}(\xi_j) = \sum_{i=0}^{k_j} g_{j,i}^{(k_j)} \xi_j^i \quad \#(4)
$$

where $\xi_j^i$ is a term of the univariate polynomial basis and $g_{j,i}^{(k_j)}$ is a coefficient of the term.

In the gPC method, the Hermitian polynomials, Laguerre polynomials, and others are used as univariate polynomial bases. Therefore, the coefficients of the terms of the polynomial are uniquely determined by the order of the polynomial. Conversely, in the aPC method, the coefficients of the terms of the polynomial are determined by moment analysis [7] of actual data.

If the univariate polynomial $\psi(\xi)$ omitting the symbol $j$ by the random variable number is orthogonal, then Eq. 5 is satisfied.

$$
\langle \psi^0, \psi^0 \rangle = \int_{\Xi} \psi^0(\xi) \psi^0(\xi) w(\xi) d\xi
$$

$$
= \int_{\Xi} \left( \sum_{i=0}^{k} g_{i}^{(k)} \xi^i \right) \left( \sum_{j=0}^{k} g_{j}^{(k)} \xi^j \right) w(\xi) d\xi = \delta_{ij}, \quad \#(5)
$$

Here, $w(\xi)$ is a weight function of the random variable $\xi$, and $\delta_{ij}$ is Kronecker’s $\delta$.

Furthermore, the $k$th moment $\mu_k$ of the univariate random variable $\xi$ can be calculated through Eq. 6.

$$
\mu_k = \int_{\Xi} \xi^k w(\xi) d\xi = \sum_{i=1}^{M} \xi_i^k \psi(\xi_i). \quad \#(6)
$$

Considering the relationship in Eq. 5, the matrix in Eq. 7 is valid.

$$
\begin{bmatrix}
\mu_0 & \mu_1 & \cdots & \mu_k \\
\vdots & \ddots & \vdots \\
\mu_{k-1} & \mu_k & \cdots & \mu_{2k-1} \\
0 & 0 & 0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
g_{j,0}^{(k)} \\
g_{j,1}^{(k)} \\
\vdots \\
g_{j,k-1}^{(k)} \\
g_{j,k}^{(k)}
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
\vdots \\
0 \\
1
\end{bmatrix}. \quad \#(7)
$$

The coefficient $g$ of the term of the univariate polynomial is obtained by the inverse matrix calculation of Eq. 7. When the coefficient $g$ of the term of the univariate polynomial is obtained, the univariate polynomial basis $\Psi$ is derived from Eq. 4. Furthermore, from Eq. 3, $P$ multivariate orthogonal polynomial bases $\Psi$ of order $d$ or less are obtained.
2.3. Multivariate polynomial coefficients
This study introduces a method for estimating the multivariate orthogonal polynomial coefficient \( c \) using a collocation method \([8][9]\). This method provides several calculation points from a univariate polynomial basis based on each random variable and estimating a coefficient \( c \) deterministically using the points.

To select a calculation point, the root of a univariate polynomial of higher order than the expansion order \( d \) (order \( d + 1 \)) is usually utilized. That is, assuming \( n \) random variables \( \xi_j \), \( d + 1 \) calculation points \( \zeta_j \) are calculated. Furthermore, since the permutation of each calculation point \( \zeta_j \) is an actual input result, if the number of calculation points \( \zeta = (\zeta_1, \zeta_2, ..., \zeta_n) \) is \( N \), it is calculated by Eq. 8.

\[
N = (d + 1)^n. \#(8)
\]

Next, the multivariate orthogonal polynomial coefficient \( c \) is calculated using the calculation points \( \zeta \). When the construction of the multivariate orthogonal polynomial \( \Psi(\zeta) \) at the calculation point \( \zeta \) and the deterministic output \( Y(\zeta) \) are calculated, the following equation is established from the relationship in Eq. 1.

\[
\begin{bmatrix}
\Psi_0(\zeta_1) & \Psi_1(\zeta_1) & \cdots & \Psi_{p-1}(\zeta_1) \\
\vdots & \vdots & \ddots & \vdots \\
\Psi_0(\zeta_{m-1}) & \Psi_1(\zeta_{m-1}) & \cdots & \Psi_{p-1}(\zeta_{m-1}) \\
\Psi_0(\zeta_m) & \Psi_1(\zeta_m) & \cdots & \Psi_{p-1}(\zeta_m)
\end{bmatrix}
\begin{bmatrix}
c_0 \\
c_1 \\
\vdots \\
c_{p-1}
\end{bmatrix}
= 
\begin{bmatrix}
Y(\zeta_1) \\
Y(\zeta_p) \\
\vdots \\
Y(\zeta_m)
\end{bmatrix} \#(9)
\]

Assuming that the polynomial matrix in Eq. 9 is the \( H_1 \) matrix, the coefficient \( c \) is calculated using the transposed matrix \( H_1^T \) of the \( H_1 \) matrix or the inverse matrix \( H_2^{-1} \) of the \( H_2 \) matrix obtained by transforming the \( H_1 \) matrix into a square matrix. The calculation using the transposed matrix is given in Eq. 10.

\[
c = (H_1^T H_1)^{-1} H_1^T Y. \#(10)
\]

For direct calculation using an inverse matrix, the \( H_1 \) matrix is an \( N \times P \) (\( P \leq N \)) matrix, so that calculation by the inverse matrix is not possible. Therefore, \( P \) linearly independent rows with a rank of \( P \) are selected from the \( H_1 \) matrix. If the resulting \( P \times P \) matrix is an \( H_2 \) matrix, the coefficient \( c \) is calculated by inverse matrix calculation as in Eq. 11.

\[
c = H_2^{-1} Y. \#(11)
\]

For the calculation using the \( H_2 \) matrix, the calculation of the deterministic output \( Y(\zeta) \) should be performed at \( P \) calculation points selected in the linear independent rows. Therefore, the calculated output \( Y \) can be reduced from \( N \) to \( P \).

3. PPF analysis

3.1. Deterministic power flow analysis
Deterministic power flow (DPF) analysis determines electrical quantities in a power system by solving a power flow equation. As in Eq. 12, the power flow equation is a non-linear equation, and its solution is estimated by iterative calculation using the Newton–Raphson (NR) method.

\[
P_{ij} = Y_{ij} V_i V_j \cos(\delta_i - \delta_j + \theta_{ij}),
\]

\[
Q_{ij} = Y_{ij} V_i V_j \sin(\delta_i - \delta_j + \theta_{ij}). \#(12)
\]

Here, \( i \) and \( j \) are bus numbers, \( Y \) is the admittance of the line, \( V \) is the absolute value of the voltage, \( \delta \) is the phase angle of the voltage, and \( \theta \) is the argument of the admittance.

3.2. PPF analysis by aPC method
The PPF analysis using the aPC method is according to the procedure described in Section 2. From estimating the multivariate orthogonal polynomial coefficients in Section 2.3, the DPF calculation described in Section 3.1 is performed. The MC simulation based on Eq. 1 is performed using random inputs of tens of thousands of uncertain power sources. Thereby, the probability distribution of the
power flow in the power system is obtained. A flowchart of the PPF analysis using the aPC method is displayed in figure 1.

![Flowchart of PPF analysis by aPC method](image1)

**Figure 1.** Flowchart of PPF analysis by aPC method.

### 3.3. PPF analysis by MC method

The PPF analysis by the MC method in this paper refers to the method of obtaining the probability distribution of the power flow by performing the DPF analysis in Section 3.1 tens of thousands of times. That is, using tens of thousands of fixed input and output bus data and random inputs, calculation is performed by estimating the solution by the NR method for each point. The flowchart of PPF analysis by MC method is shown in figure 2.

![Flowchart of PPF analysis by MC method](image2)

**Figure 2.** Flowchart of PPF analysis by MC method.

### 4. Application to power system

In this section, the effectiveness of the PPF analysis by the aPC method is verified by comparing the PPF analysis result by the proposed aPC method with the PPF analysis result from the MC method.

#### 4.1. Test conditions

The 29-bus network in figure 3 was used for the test system, with actual 500 and 275 kV networks modeled. The transmission line impedance is calculated assuming that the 500 kV line is ACSR410 and the 275 kV line is ACSR330, constituting two lines, with the line lengths obtained through
geographic information system from the environmental assessment database. Wind farms (WFs), which are uncertain power sources, are connected to buses 22, 27, and 29 in figure 3. We assumed that there is no correlation between the WFs and the power factor is output at 0.9. The other generator and load buses were assumed to have fixed outputs and loads. A year’s actual output data for WFs was used to generate the multivariate orthogonal polynomials of the aPC method, with their probability distributions depicted in figure 4. The unit method was applied for calculation using a reference capacity of 1000 MVA.

The calculation was performed by a program built in C language using a personal computer equipped with an Intel Core i7-7700 CPU @ 3.60 GHz and 8.00 GB main memory.

The PPF analysis by the aPC method employed a $10^5$ random input of WFs. In the PPF analysis by the MC method, $10^5$ deterministic inputs and random input of WFs were used. The evaluation index is the calculation time and the root-mean-square error (RMSE) between the probability distribution of the active power. The RMSE is calculated by Eq. 13, with $y_{aPC}$ as the output by the aPC method and $y_{MC}$ as the output by the MC method.

$$RMSE = \sqrt{\frac{1}{M} \sum_{i=1}^{M} (y_{MC}(i) - y_{aPC}(i))^2}$$

Here, since the input data are $10^5$ points, $M = 100000$.

![Figure 3. Actual 29-bus network.](image)

![Figure 4. WFs actual data.](image)

4.2. Results and discussions
The calculation time $t_{MC}$ by the MC method is 561 s, whereas that by the aPC method is presented in table 1. The calculation time by the aPC method is defined as $t_{aPC-T}$, for the result obtained by derivation of the polynomial coefficient $c$ using the transposed matrix, and $t_{aPC-I}$, for result from the direct solution of the inverse matrix. In the $t_{aPC}$ result, the time required for Steps 1–8 in figure 1 and the time required for Steps 6–8 are separately evaluated. This is because in the PPF analysis by the aPC method, when the topology of the system changes, the calculation may be performed again from step 6.

| Order | $t_{aPC-T}$ [s] | $t_{aPC-I}$ [s] |
|-------|-----------------|-----------------|
| 1     | 0.900           | 0.846           |
| 2     | 1.242           | 1.185           |

Table 1. Calculation time by aPC method showing steps for both approaches.
Table 2 shows the results of the RMSE of the probability distributions for T8 and T16. The result of the RMSE by the derivation method of the polynomial coefficient \( c \) using the transposed matrix is RMSE-T, whereas the result for the direct solution of the inverse matrix is RMSE-I.

| Order | RMSE-T | RMSE-I |
|-------|--------|--------|
|       | T8     | T16    | T8     | T16    |
| 1     | 1.52E-5 | 1.03E-6 | 3.34E-5 | 1.87E-6 |
| 2     | 3.59E-7 | 9.20E-8 | 5.39E-7 | 2.51E-7 |
| 3     | 4.68E-8 | 1.18E-8 | 3.81E-8 | 8.94E-9 |
| 4     | 7.07E-9 | 0     | 3.69E-8 | 7.75E-9 |

The calculation time (Table 1) confirms that the aPC method provides results in a few seconds compared with 561 s for the MC method. Clearly, the aPC method provides results several hundred times faster than the MC method. The MC method performs repeated calculations using the NR method with all \( 10^5 \) input points, whereas the aPC method linearly approximates the power flow equation without requiring repeated calculations, highlighting that high-speed calculation is possible. Comparing \( \tau_{\text{aPC-T}} \) and \( \tau_{\text{aPC-I}} \), with the fewer DPF calculation points produced faster calculation times. Furthermore, as the expansion order increases, the calculation points also increase, causing a difference between the various calculation times. If more uncertain power sources are added to the system and the number of random variables increases, the calculation points explosively increase, so the solution from the inverse matrix is regarded as superior. Also, considering the calculation time for Steps 1–8 and that for Steps 6–8, the direct solution from the inverse matrix is evidently superior for the system’s operation. Comparing the calculation time for Steps 1–8 with the calculation time for Steps 6–8, almost no time is needed to derive the multivariate orthogonal polynomial basis \( \Psi \) in Steps 1–5. That is, in the calculation for a system including many uncertain power sources, the aPC method is advantageous compared with the PPF analysis by the MC method.

For the calculation accuracy (Table 2), the PPF analysis by the aPC method is confirmed to yield considerable calculation accuracy even at a low order. The higher the order, the higher the accuracy, but the third-order expansion is sufficient. By comparing the RMSE-T and RMSE-I, we confirmed that the RMSE-T with more calculation points exhibits higher accuracy. For low-order expansion, the solution using the transposed matrix shows higher accuracy and is, therefore, advantageous.

5. Application to grid operation

In this section, the aPC method is applied in a real grid operation and a method is proposed for analyzing PPF values instantaneously using the aPC method, when the topology changes because of a system failure. In addition, when the power flow value changes because of topology change, the multivariate polynomial coefficient variation of the aPC method is evaluated.

5.1. Application of sensitivity method

One of the methods for assessing the effects of an assumed power system accident is sensitivity method [10]. This method utilizes the sensitivity indicating the effect of an accident on the power flow when a transmission line in the system is disconnected and when the power output of a generator in the system changes. This is a method for calculating the power flow after the accident. The sensitivity to transmission line accidents is known as the line outage distribution factor (LODF), whereas the sensitivity to power generation changes is referred to as the generation shift factor. In this study, to consider transmission line faults, a program for calculating the power flow after each power line fault using the LODF is incorporated into the aPC method program, and a PPF analysis of the line where the power flow changes significantly after the fault is performed.
Furthermore, to evaluate the influence of the polynomial coefficient of the aPC method on the probability distribution change, this change is observed by altering the polynomials $P$ of the aPC method. This enables characteristic evaluation of the polynomial coefficient.

5.2. Test conditions
The conditions are the same as in Section 4. Assuming all power line accidents in the PPF analysis by the aPC method before the accident, the relative change of the average power flow of the lines other than the transmission line in the accident is calculated. From the results, the pattern with the highest relative change is selected and PPF analysis by the aPC method is performed.

Evaluation of the polynomial coefficients is performed by arranging them in ascending order from one.

5.3. Results and discussions
On the basis of the calculation by the sensitivity method, outage of transmission line T36 generates the highest power flow change in T14. Therefore, a PPF analysis of T14 was performed by a fourth-order expansion aPC method, and the change in the probability distribution is shown in figure 5.

![Figure 5. Probability distribution of power flow of T14 before and after the accident.](image)

The results suggest that analyses of severe accidents are instantaneously possible by introducing the sensitivity method to the PPF analysis by the aPC method in a real system operation. The high-speed and high-accuracy PPF analysis by the aPC method is considered useful for a detailed analysis of a line accident.

Next, the result of the characteristic evaluation of the polynomial coefficient is described. figure 6 shows the change in the probability distribution of T14 after the line T36 accident due to the increase in the polynomials $P$.

![Figure 6. Change in the probability distribution due to the increase in the number $P$ of polynomials](image)

For $P$ of 1, the power flow is undistributed but approximately represents an average value. This is because the zero-order polynomial coefficient is characterized by an average value. As the number of polynomials $P$ increases, the distribution expands. At $P$ of 4, a distribution similar to when the value of $P$ is unaltered ($P = 35$) is obtained. From these facts, it is considered that the approximate distribution of the polynomial is determined by the low-order polynomial, with the higher-order polynomial only contributing to the deformation of the harmonic. That is, the change in the low-order polynomial coefficient is considered to significantly affect changes in the power flow and its...
distribution. If the relationship between the change in the polynomial coefficient and the change in the distribution of the power flow is quantitatively evaluated, a possibility exists that the approximate distribution in the system can be instantaneously known.

6. Conclusions
This study applied the aPC method to PPF analysis, with its effectiveness verified via the analysis of a 29-bus actual network. Consequently, the power flow analysis by the aPC method was proven to highly reduce the computational load by expressing a complicated system linearly. In addition, since the calculation accuracy is equivalent to that of the MC method, the proposed method is considered very useful in system analysis involving more uncertain power sources.

Furthermore, it was demonstrated that applying the sensitivity method to PPF analysis by the aPC method produced a detailed analysis of the transmission lines at risk of instantaneous overloaded when there is a transmission accident.

Evaluation of the influence of the polynomial coefficient approximating the probability distribution of the PPF confirmed that the low-order polynomial coefficient largely affects the probability distribution. In the future, by quantitatively evaluating the coefficient’s change, applications for the system’s state estimation and others can be expected.

As a future work, we envisage to establish a PPF analysis based on the aPC method considering the correlation of WFs and to handle uncertain power sources with various distributions.

Acknowledgments
This work was supported by JST SICORP Grant Number JPMJSC17E1, Japan.

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