Activation Functions in Artificial Neural Networks:  
A Systematic Overview

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Abstract

Activation functions shape the outputs of artificial neurons and, therefore, 
are integral parts of neural networks in general and deep learning in particular. 
Some activation functions, such as logistic and relu, have been used for many 
decades. But with deep learning becoming a mainstream research topic, new 
activation functions have mushroomed, leading to confusion in both theory and 
practice. This paper provides an analytic yet up-to-date overview of popular 
activation functions and their properties, which makes it a timely resource for 
anyone who studies or applies neural networks.

Keywords: neural network; deep learning; activation function; transfer function.

1 Introduction

Artificial neural networks were introduced as mathematical models for biological 
neural networks [McCulloch and Pitts, 1943, Rosenblatt, 1958]. Modern artificial 
neural networks still reflect this biological motivation, even though they are often 
applied in completely different contexts. Given parameters $\beta \in \mathbb{R}$ and $\theta \in \mathbb{R}^d$ and a 
real function $f : \mathbb{R} \to \mathbb{R}$, the (artificial) neuron \( n_{\beta,\theta,f} \) (we omit the word “artificial” 
when the context is clear) is the function

\[
n_{\beta,\theta,f} : \mathbb{R}^d \to \mathbb{R}; \quad x \mapsto \left[ f[\beta + \theta^\top x] = f\left[ \beta + \sum_{j=1}^{d} \theta_j x_j \right] \right].
\]

See the left panel of Figure 1 for an illustration. The weights $\theta_1, \ldots, \theta_d$ can be 
interpreted as the neuron’s sensitivities to the different inputs and the bias $\beta$ as 
the neuron’s overall sensitivity; the function \( f \) can be interpreted as the neuron’s 
activation pattern and, therefore, is called the activation function\(^1\) [Bishop, 1995].

\(^1\)An alternative name that relates to the terminology in electrical engineering is transfer function.
Hence, artificial neurons resemble biological neurons in the way they translate multiple input signals into a single output signal [Martin et al., 2020].

Just as biological neurons are the basic units of biological neural networks, artificial neurons are the basic units of artificial neural networks. The key observation is that artificial neurons can be added and concatenated into new functions \( \mathbb{R}^d \to \mathbb{R} \): given parameters \( \zeta \in \mathbb{R}, \gamma \in \mathbb{R}^w \) and \( \beta^1, \ldots, \beta^w \in \mathbb{R}, \theta^1, \ldots, \theta^w \in \mathbb{R}^d \) and activation functions \( g, f^1, \ldots, f^w : \mathbb{R} \to \mathbb{R} \), the function

\[
\mathbb{R}^d \to \mathbb{R} \\
\mathbf{x} \mapsto n_{\zeta, \gamma, g} = n_{\beta^1, \theta^1, f^1} [\mathbf{x}] + \cdots + n_{\beta^w, \theta^w, f^w} [\mathbf{x}]
\]

is well-defined; see the right panel of Figure 1 for an illustration. Such functions can then be added and concatenated further with one another, eventually leading to very complex functions \( \mathbb{R}^d \to \mathbb{R} \). We call these functions, and functions that are derived from or motivated by them, (artificial) neural networks. We call the use of neural networks that have many layers of concatenated neurons deep learning [Goodfellow et al., 2016, LeCun et al., 2015, Schmidhuber, 2015].

Since a neural network is made of neurons, its characteristics are governed by the neurons’ parameters and activation functions. The parameters are usually fitted to training data; in contrast, the activation functions are usually chosen before looking at any data and remain fixed. The literature contains an entire zoo of different activation functions, and arguments for and against specific activation functions are often based on heuristics and anecdotal evidence. We instead attempt a systematic and objective overview of common activation functions. In particular, we examine the mathematical particularities of each function and discuss their practical effects, making our paper a useful resource for theorists and practitioners alike.

**Overview** The activation functions and their properties are discussed in Section 2, and practical implications are discussed in Section 3. Detailed proofs of the mathematical statements are established in the Appendix. The \( \mathbb{R} \) code for the plots and further visualizations are given on github.com/LedererLab/ActivationFunctions.
2 Common Activation Functions and Their Properties

We discuss a wide range of activation functions, with \textit{logistic}, \textit{tanh}, and \textit{relu} as popular examples. We study the functions’ derivatives as well as the functions themselves. The activation functions themselves influence the network’s expressivity, that is, the network’s ability to approximate target functions. For example, we will show that \textit{linear}-activation networks are always linear and, therefore, can only approximate linear functions. The activation functions together with their first derivatives are key factors in the network’s computational complexity, that is, the computational costs of parameter optimization. The reason is that the optimization steps of popular algorithms for parameter optimization, such as stochastic-gradient descent [Bubeck, 2015, Chapter 6], are based on the gradients or generalized gradients of the network with respect to the parameters, which means—recall the chain rule—that each optimization step requires many evaluations of the activation functions and their first derivatives. The second derivatives of the activation functions are finally important for certain mathematical theories about neural networks.

The first and second derivatives of an activation function $f$ are denoted by $\mathring{f}$ and $\mathring{\mathring{f}}$, respectively, and the first directional derivative of $f$ in direction $v$ by $d_v f$. The natural logarithm is denoted by \textit{log}. Proofs, details on directional derivatives, and other mathematical background are provided in the Appendix.

2.1 Sigmoid Functions

A \textit{sigmoid function} is a bounded and differentiable function that is nondecreasing and has exactly one inflection point. Roughly speaking, a sigmoid function is a smooth, “S-shaped” curve. Because sigmoid functions “squash” the real values into a bounded interval, they are sometimes called \textit{squashing functions}. Sigmoid activation has a long-standing tradition in the theory and practice of neural networks. One motivation has been the sigmoid-shaped activation patterns observed in neuroscience—see, for example, Lipetz [1969, Figure 3]. Another motivation has been the interpretation of sigmoid functions as mathematically tractable approximations of the step functions—see below.

In the following, we discuss four sigmoid functions: \textit{logistic}, \textit{arctan}, \textit{tanh}, and \textit{softsign}. An overview is provided in Figure 2 on the next page.

2.1.1 \textit{logistic}

The function

$$f_{\log} : \mathbb{R} \to (0, 1)$$

$$z \mapsto \frac{1}{1 + e^{-z}}$$

is called \textit{logistic (sigmoid) function} (\textit{logistic}).

\textit{logistic} activation can be thought of as a smooth version of \textit{binary activation}, which is the basis of the \textit{perceptron} [Rosenblatt, 1958], an early example of a neural network. The \textit{binary function} (or \textit{step function}) is
Figure 2: logistic (dashed), arctan (dotted), tanh (dotdashed), and softsign (solid) and their derivatives. The functions mainly differ in their output range. (The y-axis is scaled up to highlight the differences.)

\[ f_{\text{binary}} : \mathbb{R} \rightarrow \{0, 1\} \]
\[ z \mapsto \begin{cases} 
1 & \text{if } z \geq 0; \\
0 & \text{otherwise}.
\end{cases} \]

logistic approximates the binary function for large function values. However, while the binary function is not differentiable at 0, logistic is infinitely many times differentiable on its entire domain with first and second derivatives (see Lemma A.2)

\[ \dot{f}_{\log}[z] = f_{\log}[z](1 - f_{\log}[z]) \in (0, 1/4] \]
\[ \text{and} \quad \ddot{f}_{\log}[z] = f_{\log}[z](1 - f_{\log}[z])(1 - 2f_{\log}[z]) \in (-c, c) \]

for all \( z \in \mathbb{R} \) and \( c \approx 0.0962 \). Hence, logistic is a smooth approximation of the binary function—see Figure 3.

Figure 3: logistic (red, dashed) is a smooth version of the binary function (black, solid)
2.1.2 \texttt{arctan}

The function
\[ f_{\text{arctan}} : \mathbb{R} \to \left( -\frac{\pi}{2}, \frac{\pi}{2} \right) \]
\[ z \mapsto \arctan[z] \]
is called \textit{arcus tangens (arctan)}. Recall the tangent function
\[ \tan : \left( -\frac{\pi}{2}, \frac{\pi}{2} \right) \to \mathbb{R} \]
\[ z \mapsto \frac{\sin[z]}{\cos[z]} \]
from basic trigonometry. The value \( \arctan[z] \) is defined as the inverse function of the tangent function, that is, \( \tan[\arctan[z]] = z \) for all \( z \in \mathbb{R} \). This definition specifies the value of \texttt{arctan} uniquely, because the tangent function is strictly increasing.

An alternative notation for \( \text{arctan}[z] \) is \( \tan^{-1}[z] \), but this can cause confusion with the reciprocal (the multiplicative inverse): \( \tan[z] \cdot (\tan[z])^{-1} = 1 \) for all \( z \in (-\pi/2, \pi/2) \) but \( \tan[z] \cdot \tan^{-1}[z] = \tan[z] \cdot \text{arctan}[z] = 1 \) only for two values \( z \in (-\pi/2, \pi/2) \).

The output range \((0, 1)\) of \texttt{logistic} can be motivated biologically (“degree of neuron activation”). The output range \((-\pi/2, \pi/2)\) of \texttt{arctan} (and similarly the output ranges of \texttt{tanh} and \texttt{softsign} below) deviates from this motivation, but the additional symmetry \( f_{\text{arctan}}[z] = -f_{\text{arctan}}[-z] \) with respect to the origin is convenient from a mathematical and algorithmic perspective.

\texttt{arctan} is infinitely many times differentiable on its entire domain. The first and second derivatives are (see Lemma A.3)
\[ \dot{f}_{\text{arctan}}[z] = \frac{1}{1 + z^2} \in [0, \infty) \quad \text{and} \quad \ddot{f}_{\text{arctan}}[z] = -\frac{2z}{(1 + z^2)^2} \in \mathbb{R} \]
for all \( z \in \mathbb{R} \). The derivatives cannot be expressed by the original function, but they are still comparably easy to compute.

2.1.3 \texttt{tanh}

The function
\[ f_{\text{tanh}} : \mathbb{R} \to (-1, 1) \]
\[ z \mapsto \tanh[z] := \frac{e^z - e^{-z}}{e^z + e^{-z}} \]
is called \textit{hyperbolic tangent function (tanh)}.

\texttt{tanh} is infinitely many times differentiable with first and second derivatives (see Lemma A.4)
\[ \dot{f}_{\text{tanh}}[z] = 1 - (f_{\text{tanh}}[z])^2 \in (0, 1) \quad \text{and} \quad \ddot{f}_{\text{tanh}}[z] = -2f_{\text{tanh}}[z] \left(1 - (f_{\text{tanh}}[z])^2\right) \in (-c, c) \]
for all $z \in \mathbb{R}$ and $c \approx 0.770$.

Moreover (see Lemma A.6),

$$f_{\text{tanh}}[z] = 2f_{\log}[2z] - 1$$

for all $z \in \mathbb{R}$,

and the Taylor series of $\text{tanh}$ and $\arctan$ agree up to the fourth order. Thus, we can think of $\text{tanh}$ as a shifted and scaled version of $\logistic$ or as an approximation of $\arctan$ (see Figure 4). In particular, $\text{tanh}$ combines two popular features of $\logistic$ and $\arctan$: the derivative of $\text{tanh}$ is a simple expression of the original function (cf. $\logistic$), and it is centered around zero (cf. $\arctan$).

2.1.4 softsign

The function

$$f_{\text{soft}} : \mathbb{R} \rightarrow (-1, 1)$$

$$z \mapsto \frac{z}{1 + |z|}$$

is called $\text{elliottsig}$ or $\text{softsign}$. $\text{softsign}$ activation was introduced in Elliott [1993].

$\text{softsign}$ is one time differentiable on its entire domain with derivative (see Lemma A.5)

$$\hat{f}_{\text{soft}}[z] = (1 - |f_{\text{soft}}[z]|)^2 \in (0, 1)$$

for all $z \in \mathbb{R}$.

$\text{softsign}$ is infinitely many times differentiable at all points except for $z = 0$ with second derivative

$$\hat{\hat{f}}_{\text{soft}}[z] = -2\text{sign}[z](1 - |f_{\text{soft}}[z]|)^3 \in (-2, 2)$$

for all $z \in \mathbb{R} \setminus \{0\}$.

Like $\text{tanh}$, $\text{softsign}$ is centered around zero and has derivatives that are simple expressions of the original function. A computational advantage of $\text{softsign}$ over $\text{tanh}$ is its simplicity, especially the absence of exponential functions, which makes the evaluation of $f_{\text{soft}}$ and $\hat{f}_{\text{soft}}$ particularly cheap. Hence, $\text{softsign}$ is a particularly interesting candidate in the class of sigmoid functions.
2.2 Piecewise-Linear Functions

A piecewise-linear function is composed of line segments. Similarly as sigmoid activation, piecewise-linear activation has been used for many decades. It has initially been motivated by neurobiological observations; for example, the inhibiting effect of the activity of a visual-receptor unit on the activity of the neighboring units can be modeled by two line segments [Hartline and Ratliff, 1957, Figure 2]. The current popularity of piecewise-linear activation functions, however, originates in their computational properties: First, the functions and their derivatives (or directional derivatives—see Section A.1) are computationally inexpensive; in particular, in contrast to most of the sigmoid functions discussed in the previous section, no exponential or trigonometric functions are involved. Second, the derivatives of the functions discussed below do not converge to zero in the limit of infinitely large arguments (but they can be zero in the limit of infinitely small arguments), which might alleviate the vanishing-gradient problem of sigmoid activation.

Another interesting feature of the functions discussed below is that they are nonnegative homogenous: \( f[a z] = a f[z] \) for all \( a \in [0, \infty) \) and \( z \in \mathbb{R} \). This feature has turned out useful in deep-learning theory [Neyshabur et al., 2015, Taheri et al., 2020] and methodology [Hebiri and Lederer, 2020].

In the following, we discuss three piecewise-linear activation functions: linear, relu, and leakyrelu.

2.2.1 linear

In the context of neural-network activation, linear is the name for the identity function

\[
\text{linear} : \mathbb{R} \to (-\infty, \infty); \quad z \mapsto z.
\]

Hence, linear activation leaves the inputs unchanged. Of course, \( \text{linear} \) is differentiable on its entire domain with derivatives

\[
\frac{d}{dz} \text{linear}[z] = 1 \quad \text{and} \quad \frac{d^2}{dz^2} \text{linear}[z] = 0 \quad \text{for all } z \in \mathbb{R}.
\]

It is often claimed that the constant derivatives lead to uninformative updates in the optimization, but this neglects that objective functions in deep learning comprise not only the networks but also a loss function (such as cross-entropy, least-squares, or robust versions of it [Lederer, 2020]). An actual drawback of linear activation is its low expressivity; in particular, networks where all activations are linear are always linear functions—see Example A.1. Hence, linear activation is typically only applied to certain layers, such as output layers in regression settings.

However, linear-activation networks can also be used as toy models for analyzing and testing optimization algorithms; indeed, such networks are simple from a modeling perspective but highly intricate from an optimization perspective [Arora et al., 2019]. Consequently, one might hope that networks with linear activation convey general principles of deep-learning optimization.
2.2.2 relu

The function

$$f_{\text{relu}} : \mathbb{R} \rightarrow [0, \infty)$$
$$z \mapsto \max\{0, z\}$$

is called the positive-part function or ramp function. The positive-part function is the identity function (linear with slope 1) for positive arguments and the constant function with value zero otherwise.

In electrical engineering, a rectifier is a device that converts alternating current to direct current. Similarly, the positive-part function lets positive inputs pass unaltered but cuts negative inputs, that is, it transforms negative and nonnegative inputs into nonnegative outputs. Therefore, a neuron equipped with a positive-part function as the activation function is often called a rectifier linear unit (relu), and the positive-part function itself is often called relu in the context of neural networks.

A clear benefit of relu is that both the function itself and its derivatives are easy to implement and computationally inexpensive. relu is infinitely many times differentiable at $$z \in \mathbb{R} \setminus \{0\}$$ with first and second derivatives (see Lemma A.7)

$$\dot{f}_{\text{relu}}[z] = \begin{cases} 1 & \text{for all } z \in (0, \infty) \\ 0 & \text{for all } z \in (-\infty, 0) \end{cases} \quad \text{and} \quad \ddot{f}_{\text{relu}}[z] = 0 \text{ for all } z \in \mathbb{R} \setminus \{0\}.$$

relu is not differentiable at $$z = 0$$, but it is “almost” differentiable; for example, the directional derivatives exist for all points $$z \in \mathbb{R}$$ and directions $$v \in \mathbb{R}$$ and equal (see Lemma A.7)

$$d_v f_{\text{relu}}[z] = \begin{cases} v & \text{for all } v \in \mathbb{R} \text{ and } z \in (0, \infty) \text{ and for all } v \in [0, \infty) \text{ and } z = 0; \\ 0 & \text{otherwise}. \end{cases}$$

This result motivates using gradient-descent-type approaches in practice with the derivatives at zero (which do not exist) replaced by a fixed value between 0 and 1.

relu activation can be subject to the dying-relu phenomenon, which is a version of the vanishing-gradient problem. The dying-relu phenomenon indicates a situation where many relu nodes are inactive during much of the training process—see Example A.2—which can prevent the algorithms from learning complex models. Potential remedies are to choose lower learning rates or to replace relu by leakyrelu—see the next section. However, the relevance of the dying-relu phenomenon in practice, as well as the effectiveness of the mentioned remedies, remain unclear.

2.2.3 leakyrelu

Given a parameter $$a \in [0, \infty)$$, we call leakyrelu the function

$$f_{\text{leakyrelu},a} : \mathbb{R} \rightarrow [0, \infty)$$
$$z \mapsto \max\{0, z\} + \min\{0, az\}.$$ 

leakyrelu activation was introduced in Maas et al. [2013].
The idea behind leakyrelu is to mimic relu but to avoid the dying-relu phenomenon. leakyrelu equals relu in the case $a = 0$; for positive parameters $a$, however, the functions differ for negative inputs, most notably in their derivatives. leakyrelu is infinitely many times differentiable at $z \in \mathbb{R} \setminus \{0\}$ with first and second derivatives (see Lemma A.7)

$$\dot{\text{leakyrelu}}_{a}[z] = \begin{cases} 1 & \text{for all } z \in (0, \infty) \\ a & \text{for all } z \in (-\infty, 0) \end{cases}$$

and

$$\ddot{\text{leakyrelu}}_{a}[z] = 0 \text{ for all } z \in \mathbb{R} \setminus \{0\} .$$

leakyrelu is not differentiable at $z = 0$, but the directional derivatives exist for all points $z \in \mathbb{R}$ and directions $v \in \mathbb{R}$ and equal (see Lemma A.7)

$$d_{v}\text{leakyrelu}_{a}[z] = \begin{cases} v & \text{for all } v \in \mathbb{R} \text{ and } z \in (0, \infty) \text{ and for all } v \in [0, \infty) \text{ and } z = 0 \\ av & \text{otherwise} \end{cases}.$$

The key difference to relu is that $\dot{\text{leakyrelu}}_{a}[z] > 0$ for all $z \in \mathbb{R} \setminus \{0\}$ and $a > 0$. This property can be seen as an approach to avoid the problem of vanishing gradients.

A practical challenge inflicted by leakyrelu is choosing the parameter $a$. As we have just seen, $a$ is the slope of leakyrelu for negative inputs. It is usually chosen between 0 (where leakyrelu equals relu) and 1 (where leakyrelu equals linear), but there is no further consensus: the original paper sets the parameter to $a = 0.01$; [Xu et al., 2015] introduces randomized leaky rectifier linear unit (rrelu), which replaces the fixed parameter with a stochastic one, and [He et al., 2015] suggests to train the parameter (see Section 2.3.4). The benefits of these choices as compared to each other and as compared to vanilla relu still need to be explored.

### 2.3 Other Functions

Since each of the discussed activations has certain shortcomings, proposals of new activations have mushroomed. Most of these functions lack empirical or mathematical support, but there are notable exceptions. In the following, we discuss recently proposed activations that are popular or include novel ideas. This includes softplus, elu and selu, swish, and activations with parameters that are learned during training.

#### 2.3.1 softplus

softplus is defined as

$$\text{softplus} : \mathbb{R} \to [0, \infty) ;$$

$$z \mapsto \log(1 + e^{z}) .$$

Softplus activation was introduced in Dugas et al. [2001].

softplus is infinitely many times differentiable on its entire domain, and its first and second derivatives are (see Lemma A.8)

$$\dot{\text{softplus}}_{z} = \ddot{\log}[z] \in (0, 1) \quad \text{and} \quad \dddot{\text{softplus}}_{z} = \dddot{\log}[z] \in (0, 1/4]$$

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for all $z \in \mathbb{R}$. Hence, softplus is a primitive of logistic.

However, $f_{\text{soft+}}[z] \approx \log[1] = 0 = f_{\text{relu}}[z]$ for all $z \ll 0$ and $f_{\text{soft+}}[z] \approx \log[e^z] = z = f_{\text{relu}}[z]$ for all $z \gg 0$. So rather than comparing softplus to logistic, we should consider it a smooth version of relu—see Figure 5. The differentiability at $z = 0$ is a mathematical convenience of softplus as compared to relu, but the practical relevance of this feature is unclear: practical implementations seem to work with the derivative of relu in $z = 0$ just set to zero (cf. the discussion of the directional derivatives of relu in Section 2.2.2). The strict positivity of the first derivative of softplus can be seen as a measure against the dying-relu phenomenon, but the practical effect of this measure is again unclear: the derivatives are still small for small arguments, that is, $\dot{f}_{\text{soft+}}[z] \approx 0$ for all $z \ll 0$. A disadvantage of softplus is that the function and its derivative are both computationally more costly than their relu counterparts. Thus, in lack of clear evidence in favor of softplus, and given relu’s computational simplicity, relu is currently favored over softplus.

### 2.3.2 elu and selu

Given a parameter $a \in [0, \infty)$, the function

$$f_{\text{elu}},a : \mathbb{R} \to (-a, \infty)$$

$$z \mapsto \begin{cases} 
  z & \text{for all } z \in [0, \infty) \\
  a(e^z - 1) & \text{for all } z \in (-\infty, 0)
\end{cases}$$

is called—in analogy with relu—exponential linear unit (elu). (See Lemma A.9 for a calculation of the output ranges.) In fact, relu is a special case of elu: $f_{\text{elu}} = f_{\text{elu}},0$.

elu activation was introduced in Clevert et al. [2016].

elu is infinitely many times differentiable on $\mathbb{R} \setminus \{0\}$ with first and second derivatives

$$\dot{f}_{\text{elu}},a[z] = \begin{cases} 
  1 & \text{for all } z \in (0, \infty) \\
  f_{\text{elu}},a[z] + a & \text{for all } z \in (-\infty, 0)
\end{cases}$$

and

$$\ddot{f}_{\text{elu}},a[z] = \begin{cases} 
  0 & \text{for all } z \in (0, \infty) ; \\
  f_{\text{elu}},a[z] + a & \text{for all } z \in (-\infty, 0).
\end{cases}$$

The first directional derivatives exist on the entire real line and are equal to

$$d_v f_{\text{elu}},a[z] = \begin{cases} 
  v & \text{for all } v \in \mathbb{R} \text{ and } z \in (0, \infty) \text{ or } v \in [0, \infty) \text{ and } z = 0 ; \\
  v(f_{\text{elu}},a[z] + a) & \text{otherwise}.
\end{cases}$$
Figure 6: \texttt{relu}, that is, \texttt{elu} with $a = 0$ (blue, dashed) and \texttt{elu} with $a = 1$ (black, solid), $a = 2$ (black, dotted), and $a = 3$ (black, dashed) coincide for positive values but differ for negative values. (Best seen in color.)

Similarly as discussed for \texttt{softplus} in the previous section, a key difference of \texttt{elu} to \texttt{relu} is the fact that the derivatives of \texttt{elu} with $a \neq 0$ are strictly positive (where they exist). But again, the practical effect of this property is unclear. A difference to both \texttt{softplus} and \texttt{relu} is that \texttt{elu} is somewhat “centered” around zero—see Figure 6.

The mathematically most convenient parameter is $a = 1$, because this is the only choice that makes \texttt{elu} one time differentiable on the entire real line (but not twice differentiable). But except for this observation, there is little insight into how to choose $a$ in practice.

Observe also that the first derivatives of \texttt{elu} can be computed easily from the original functions, but the original functions involve an exponential function and, therefore, are more costly to compute than \texttt{relu}. Hence, in view of the unclear practical benefits and the computational disadvantages of \texttt{elu}, \texttt{relu} is currently preferred.

A variant of \texttt{elu} is \textit{scaled exponential linear unit (selu)} introduced in Klambauer et al. [2017]. \texttt{selu} is (see Section A.8.3 for further details)

\[
f_{\text{selu}} : \mathbb{R} \rightarrow (-b_0, \infty)
\]

\[
z \mapsto \begin{cases} 
a_0 z & \text{for all } z \in [0, \infty) 

b_0 (e^z - 1) & \text{for all } z \in (-\infty, 0)
\end{cases}
\]

for fixed parameters $a_0 \approx 1.05$ and $b_0 \approx 1.76$. \texttt{selu} is very similar to \texttt{elu} in general, but the specific choice of the parameters leads to an additional self-scaling property. Consider the map

\[
c : \mathbb{R}^2 \rightarrow \mathbb{R}^2;
\]

\[
(\mu, \nu) \mapsto \left( E[f_{\text{selu}}[r_{\mu, \nu}]], E[(f_{\text{selu}}[r_{\mu, \nu}])^2] \right),
\]

where $r_{\mu, \nu} \sim \mathcal{N}[\mu, \sqrt{\nu}]$ is a univariate Gaussian random variable with mean $\mu$ and variance $\nu$, and $E$ is the corresponding expectation. In other words, the function $c$ captures the mean and variance of a normal random variable after \texttt{selu} activation.
It is easy to show that \( c \) is a contraction with fixed point \((\mu, \nu) = (0, 1)\), which can be interpreted as a normalization property: broadly speaking, a Gaussian input is transformed into an output with mean closer to 0 and variance closer to 1. One can argue that this normalization property might alleviate vanishing and exploding gradients especially in networks that are wide (where the central-limit theorem might indeed ensure Gaussian-like inputs) and deep (where many gradients are multiplied together). However, the practical benefits, especially in view of batch normalization and other normalization schemes that are often used in deep-learning pipelines, are currently unclear.

### 2.3.3 swish

Given a parameter \( a \in [0, \infty) \), the function

\[
\text{swish}_{a} : \mathbb{R} \rightarrow [c_a, \infty)
\]

\[
z \mapsto z \cdot \frac{1}{1 + e^{-az}}
\]

is called swish. The minimum of the function is \( c_a \approx -0.278/a \) for \( a \neq 0 \) and \( c_a = -\infty \) for \( a = 0 \) (see Lemma A.11). swish activation with \( a = 1 \) was as introduced in Elfwing et al. [2018] under the name \( \text{sil} \); the parametrized version \( \text{sil} \)

![Figure 7](image)

**Figure 7**: swish with \( a = 0.2 \) (black, dotdashed), \( a = 1 \) (black, dotted), and \( a = 5 \) (blue, dashed) as well as relu (blue, solid) and their derivatives. The larger the function parameter \( a \), the more swish resembles relu; the smaller the function parameter, the more swish resembles (a scaled version of) linear.
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with a potentially a learnable parameter (see the following section) and the name “swish” were introduced later in Ramachandran et al. [2017].

Swish can be seen as an interpolation between relu and (a scaled version of) linear: $f_{swish,a} \approx f_{relu}$ for $a \gg 1$ and $f_{swish,a} \approx f_{linear}/2$ for $a \ll 1$—see Figure 7.

Swish is infinitely many times differentiable on its entire domain with first and second derivatives (see Lemmas A.10, A.12, and A.13)

$$\dot{f}_{swish,a}[z] = af_{swish,a}[z] + f_{log}[az](1 - af_{swish,a}[z]) \in (1 - c', c')$$

and (see Lemma A.10)

$$\ddot{f}_{swish,a}[z] = a\left(af_{swish,a}[z] + 2f_{log}[az](1 - af_{swish,a}[z])\right)(1 - f_{log}[az]) \in (c''_a, c''_a)$$

for all $a \in [0, \infty)$ and $z \in \mathbb{R}$, and for $c' \approx 1.098$, $c''_a \approx -0.0369a$, and $c''_a = 0.5a$. We observe that the first and second derivatives of swish, as well as swish itself, are simple combinations of $f_{log}[az]$ and $zf_{log}[az]$. Hence, swish activation is comparable to logistic activation in terms of basic computational complexity.

Several variants of swish have been suggested recently, such as elish [Basirat and Roth, 2019] and mish [Misra, 2019]. A difference of swish and its variants to all other activation functions discussed so far is that swish is not monotone. But the practical benefits of this property, as well as the features of these activations more generally, are currently unclear. An interesting aspect of swish is, however, that it is the first activation function that originate from an automated search.

2.3.4 Learning Activation Functions

Activations usually remain fixed during the entire process of training and application of a neural network. But activation functions can also be fitted during training by selecting within predefined sets of activation functions [Liu and Yao, 1996, White and Ligomenides, 1993] or by fitting the parameters of preselected activation functions [Augusteijn and Harrington, 2004, Duch and Jankowski, 2001]. Recent implementations of this idea include fitting polynomial activations with the Taylor coefficients of standard activation functions as initial values for the parameters [Chung et al., 2016], parametric rectified linear unit (prelu) [He et al., 2015], which fits the parameter of leakyrelu, parametric exponential linear unit (pelu) [Trottier et al., 2017], which fits the parameters of a version of elu, and the aforementioned swish with its parameter fitted to the training data [Ramachandran et al., 2017].

A related concept is maxout introduced in Goodfellow et al. [2013]. Maxout replaces the neuron in (1) by

$$\tilde{n}_{\beta, \theta, k} : \mathbb{R}^d \rightarrow \mathbb{R};$$

$$\mathbf{x} \mapsto \max \left\{ \beta^1 + \sum_{j=1}^{d} \theta_j x_j, \beta^2 + \sum_{j=1}^{d} \theta_{j+d} x_j, \ldots, \beta^k \sum_{j=1}^{d} \theta_{j+(k-1)d} x_j \right\},$$

where $\beta \in \mathbb{R}^k$ and $\theta \in \mathbb{R}^{kd}$ are vector-valued parameters and $k \in \{1, 2, \ldots\}$ a fixed number. Maxout was designed to improve the performance of dropout (a technique that tries to alleviate overfitting by randomly excluding nodes when updating weights
with stochastic-gradient descent [Srivastava et al., 2014]), and it can be seen as a max pooling over neurons with linear activations (max pooling is a data-aggregation scheme that is popular especially for convolutional neural networks [Scherer et al., 2010]). \textit{maxout} is not an activation function, but it can be interpreted in the spirit of activation functions in two ways: First, as a generalization of \texttt{relu}: in the special case $k = 1$, \textit{maxout} simply yields a \texttt{relu} neuron, that is, $\tilde{\alpha}_{\beta, \theta, 1} = \alpha_{\beta, \theta, \text{relu}}$; this interpretation connects \textit{maxout} with our Section 2.2.2. Second, as a method for fitting piecewise-linear activation functions; this interpretation connects \textit{maxout} with the above-stated learning approaches.

\textit{maxout} hinges on $k$, which can be difficult to choose in practice, and it augments the parameters space from $d + 1$ to $k(d + 1)$ dimensions, which bears computational challenges and the risk of overfitting; in particular, when comparing \textit{maxout} networks to other networks, one should account for the increased parameter space [Castaneda et al., 2019]. More generally, theoretical and empirical support for \textit{maxout} as well as of the other learning schemes stated above is currently very limited.

3 Practical Implications

Figure 8 on the next page displays the main activation functions discussed in Section 2 in a single graph. Our findings in Section 2 have three practical implications:

First, our findings support \texttt{softsign} activation. Activation functions that have similar graphs presumably give similar empirical results when the network parameters are “sufficiently optimized” (see, for example, Basirat and Roth [2019, Section 5]). This suggests that choices among similar activation functions should be based on how much computational effort it takes to optimize the network parameters, that is, on how simple the activation functions are from a computational perspective. A practical implication of our comparison of sigmoid activations in Section 2.1 is, therefore, that \texttt{softsign}, a particularly simple sigmoid function, should receive much more attention (see Section 2.1.4).

Second, our findings support \texttt{relu} activation in a similar way: there is limited empirical and theoretical evidence for its competitors, and \texttt{relu} stands out in view of its computational simplicity (see Section 2.2.2). A practical implication of Section 2 is, therefore, that \texttt{softsign} and \texttt{relu} should—at least for now—be the standard activations in practice.

Third, our findings highlight three promising approaches for improving on these choices: theoretical considerations (such as in the context of \texttt{selu}, see Section 2.3.2), automated searches (such as in the context of \texttt{swish}, see Section 2.3.3), and data-adaptive selection schemes (see Section 2.3.4).

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Figure 8: Overview of sigmoid functions (red, see Section 2.1), piecewise-linear functions (blue, see Section 2.2), and other functions (green, see Section 2.3). (Best seen in color.)
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A Mathematical Details

In the following, we prove our mathematical claims. Our derivations require minimal prerequisites, and we recall mathematical background where needed.

A.1 Derivatives and Directional Derivatives

We first recall the notions of derivatives and directional derivatives. The basis of derivatives are (two-sided) limits. For all $\bar{t} \in \mathbb{R} \cup \{\pm \infty\}$ and $a \in (0, \infty)$, we define “balls” around $\bar{t}$ as follows:

- if $\bar{t} \in \mathbb{R}$, define $\mathcal{B}_a[\bar{t}] := \{c \in \mathbb{R} : |\bar{t} - c| < a\}$;
- if $\bar{t} = \infty$, define $\mathcal{B}_a[\bar{t}] := \{c \in \mathbb{R} : c > 1/a\}$;
- if $\bar{t} = -\infty$, define $\mathcal{B}_a[\bar{t}] := \{c \in \mathbb{R} : c < -1/a\}$.

Consider now a function $f : \mathbb{R} \to \mathbb{R}$. We say that the (two-sided) limit of $f$ at $\bar{t}$ exists (in $\mathbb{R} \cup \{\pm \infty\}$) if and only if there is an $l \in \mathbb{R} \cup \{\pm \infty\}$ such that the following holds: for every $a \in (0, \infty)$, there is a $b \in (0, \infty)$ such that $f[\bar{t}] \in \mathcal{B}_a[l]$ for all $t \in \mathcal{B}_b[\bar{t}]$. We write $\lim_{t \to \bar{t}} f[\bar{t}] := l$.

The (usual) derivatives of a function $f : \mathbb{R} \to \mathbb{R}$ are then
derive

$$f[\bar{z}] := \frac{\partial}{\partial \bar{z}} f[\bar{z}] := \lim_{t \to 0} \frac{f[\bar{z} + t] - f[\bar{z}]}{t}, \quad \hat{f}[\bar{z}] := \frac{\partial}{\partial \bar{z}} \hat{f}[\bar{z}] := \lim_{t \to 0} \frac{\hat{f}[\bar{z} + t] - \hat{f}[\bar{z}]}{t}, \quad \ldots$$

for all $\bar{z} \in \mathbb{R}$ where the limits exist. If needed for clarity, the distinction between the function variable (say $w$) and the point at which the derivative is evaluated (say $w_0$) is made explicit: for example,

$$\left.\frac{\partial}{\partial w} f[w]\right|_{w = w_0} := \lim_{t \to 0} \frac{f[w_0 + t] - f[w_0]}{t} \quad \text{(if the limit exists)}$$

is the derivative of $w \mapsto f[w]$ with respect to $w$ at point $w_0$.

Directional derivatives generalize standard derivatives: directional derivatives exist at every differentiable point of a function, and they also exist for some non-differentiable points, such as for the point $z = 0$ of $f_{\text{relu}} : z \mapsto \max\{z, 0\}$ from Section 2.2.2.

The basis of directional derivatives are right-sided limits. For all $\bar{t} \in \mathbb{R}$ and $a \in (0, \infty)$, we define “directed balls” around $\bar{t}$ as follows:

- $\mathcal{B}_a^+ [\bar{t}] := \{c \in \langle \bar{t}, \infty \rangle : |\bar{t} - c| \leq a\}$.

Consider again a function $f : \mathbb{R} \to \mathbb{R}$. We say that the right-sided limit of $f$ at $\bar{t}$ exists (in $\mathbb{R} \cup \{\pm \infty\}$) if and only if there is an $l \in \mathbb{R} \cup \{\pm \infty\}$ such that the following holds: for every $a \in (0, \infty)$, there is a $b \in (0, \infty)$ such that $f[\bar{t}] \in \mathcal{B}_a[l]$ for all $t \in \mathcal{B}_b^+ [\bar{t}]$. We then write $\lim_{t \to \bar{t}^+} f[\bar{t}] := l$.

The directional derivatives are then:
Definition A.1 (Directional derivatives). Consider a function $f : \mathbb{R} \rightarrow \mathbb{R}$ and two values $v, z \in \mathbb{R}$. The quantity
\[
    d_v f[z] := \lim_{t \to 0^+} \frac{f[z + tv] - f[z]}{t},
\]
if the limit exists, is called the first directional derivative of $f$ at $z$ in the direction $v$.

If $f$ is differentiable at $z$, it holds that $d_v f[z] = v \dot{f}[z]$. Conversely, if for a given $z \in \mathbb{R}$, it holds that $d_v f[z] = d_{-v} f[z]$ for all $v \in \mathbb{R}$, then $f$ is differentiable at $z$ with derivative that satisfies $d_v f[z] = v \dot{f}[z]$.

A.2 l’Hôpital’s Rule

l’Hôpital’s rule is a standard technique for evaluating limits. We state a version that fits our needs.

Lemma A.1 (l’Hôpital’s rule). Consider two functions $f, g : \mathbb{R} \rightarrow \mathbb{R}$ that are differentiable on an open interval $I \subset \mathbb{R}$, and consider a point $\bar{t} \in \mathbb{R}$.

1. If $w \in I$, $g[\bar{t}] \neq 0$ for all $t \in I \setminus \{\bar{t}\}$, and
   - $\lim_{t \to \bar{t}} f[t] = \lim_{t \to \bar{t}} g[t] = 0$ or
   - $\lim_{t \to \bar{t}} f[t] = \lim_{t \to \bar{t}} g[t] = \infty$,
   it holds that
     \[
     \lim_{t \to \bar{t}} \frac{f[t]}{g[t]} = \lim_{t \to \bar{t}} \frac{\dot{f}[t]}{\dot{g}[t]}
     \]
   —as long as the second limit exists.

2. If $I = (\bar{t}, \infty)$, $g[\bar{t}] \neq 0$ for all $t \in I$, and
   - $\lim_{t \to \bar{t}^-} f[t] = \lim_{t \to \bar{t}^-} g[t] = 0$ or
   - $\lim_{t \to \bar{t}^-} f[t] = \lim_{t \to \bar{t}^-} g[t] = \infty$,
   it holds that
     \[
     \lim_{t \to \bar{t}^-} \frac{f[t]}{g[t]} = \lim_{t \to \bar{t}^-} \frac{\dot{f}[t]}{\dot{g}[t]}
     \]
   —as long as the second limit exists.

3. Analog statements hold for $\bar{t} \in \{\pm \infty\}$.

A.3 Derivatives of the Sigmoid Functions

A.3.1 Derivatives of logistic

Lemma A.2 (Derivatives of logistic). The first and second derivatives of logistic are
\[
    \dot{f}_{\text{log}}[z] = \frac{e^{-z}}{(1 + e^{-z})^2} = f_{\text{log}}[z](1 - f_{\text{log}}[z]) \in (0, 1/4)
\]
and
\[
    \ddot{f}_{\text{log}}[z] = \frac{e^{-z}(e^{-z} - 1)}{(1 + e^{-z})^3} = f_{\text{log}}[z](1 - 2f_{\text{log}}[z]) = \dot{f}_{\text{log}}[z](1 - 2\dot{f}_{\text{log}}[z]) (1 - 2f_{\text{log}}[z]) \in (-c, c)
\]
for all $z \in \mathbb{R}$ and $c \approx 0.0962$. 


Proof of Lemma A.2. The first derivatives follow more or less directly from the definition of \( f_{\log} \) and the sum and chain rules:

\[
\dot{f}_{\log}[z] = \frac{\partial}{\partial z} \frac{1}{1 + e^{-z}} = \frac{-1}{(1 + e^{-z})^2} \frac{\partial}{\partial z} (1 + e^{-z}) = \frac{-1}{(1 + e^{-z})^2} \cdot (0 - e^{-z}) = e^{-z} \frac{e^{-z}}{(1 + e^{-z})^2}
\]

and further

\[
\dot{f}_{\log}[z] = \frac{e^{-z}}{(1 + e^{-z})^2} \quad \text{previous display}
\]

\[
= \frac{1}{1 + e^{-z}} \cdot \frac{e^{-z}}{1 + e^{-z}} \quad \text{splitting the fraction into two parts}
\]

\[
= \frac{1}{1 + e^{-z}} \cdot \left( \frac{1 + e^{-z}}{1 + e^{-z}} - \frac{1}{1 + e^{-z}} \right) \quad \text{splitting the second factor into two parts}
\]

\[
= f_{\log}[z] \left( 1 - \dot{f}_{\log}[z] \right), \quad \text{definition of } f_{\log} \text{ and simplification}
\]

as desired.

This result can then be used together with the sum and product rules to find the second derivatives:

\[
\ddot{f}_{\log}[z] = \frac{\partial}{\partial z} \dot{f}_{\log}[z]
\]

\[
= \frac{\partial}{\partial z} \left( f_{\log}[z] \left( 1 - \dot{f}_{\log}[z] \right) \right)
\]

\[
= \left( \frac{\partial}{\partial z} f_{\log}[z] \right) \left( 1 - \dot{f}_{\log}[z] \right) + f_{\log}[z] \frac{\partial}{\partial z} \left( 1 - \dot{f}_{\log}[z] \right)
\]

\[
= f_{\log}[z] \left( 1 - \dot{f}_{\log}[z] \right) + f_{\log}[z] \left( 0 - \dot{f}_{\log}[z] \right) \quad 1' = 0; \text{definition of } \dot{f}_{\log}; \text{sum rule}
\]

\[
= f_{\log}[z] \left( 1 - 2 \dot{f}_{\log}[z] \right)
\]

and further, using the result \( \dot{f}_{\log} = \dot{f}_{\log}[z] \left( 1 - \dot{f}_{\log}[z] \right) \),

\[
\ddot{f}_{\log}[z] = \dot{f}_{\log}[z] \left( 1 - \dot{f}_{\log}[z] \right) \left( 1 - 2 \dot{f}_{\log}[z] \right)
\]

We can then finally plug in the definition of \( \dot{f}_{\log} \) to find

\[
\ddot{f}_{\log}[z] = \dot{f}_{\log}[z] \left( 1 - 2 \dot{f}_{\log}[z] \right)
\]

\[
= \frac{e^{-z}}{(1 + e^{-z})^2} \left( 1 - \frac{2}{1 + e^{-z}} \right) \quad \text{above result for } \dot{f}_{\log}; \text{definition of } f_{\log}
\]

\[
= \frac{e^{-z}}{(1 + e^{-z})^2} \frac{1 + e^{-z} - 2}{1 + e^{-z}} \quad \text{summarizing the second factor}
\]

\[
= \frac{e^{-z}(e^{-z} - 1)}{(1 + e^{-z})^3}, \quad \text{consolidating}
\]
Thus, the output range of the derivative is indeed \((0, \infty)\) which implies the fact that \(\dot{\log} \cdot \log\) above. We first compute the third derivatives of \(\dot{\log}\) differentiated (see above), so that we can find its maximum by setting its derivatives to zero. We find

\[
\dot{\log} \cdot \log[z] = \frac{e^{-z}}{(1 + e^{-z})^2},
\]

which implies the fact that \(\dot{\log} \cdot \log[z] > 0\) for all \(z \in \mathbb{R}\) and \(\dot{\log} \cdot \log[z] \to 0\) for \(z \to \pm\infty\). Using this and the continuity of the derivative, we can conclude that the output range is \((0, c]\) for some \(c \in (0, \infty)\). To determine \(c\), we use that \(\dot{\log} \cdot \log\) can be continuously differentiated (see above), so that we can find its maximum by setting its derivatives to zero. We find

The above-derived equality for \(\ddot{\log} \cdot \log\) and the positivity of \(\dot{\log} \cdot \log\) and then the above-derived equality for \(\dddot{\log} \cdot \log\) yield

\[
\dddot{\log} \cdot \log[z] = 0 \Rightarrow \dot{\log} \cdot \log[z] \cdot (1 - 2\dot{\log} \cdot \log[z]) = 0 \Rightarrow 1 - 2\dot{\log} \cdot \log[z] = 0 \Rightarrow \dot{\log} \cdot \log[z] = \frac{1}{2}
\]

rearranging the terms

\[
\dddot{\log} \cdot \log[z] = 0 \Rightarrow \dddot{\log} \cdot \log[z] = \frac{1}{4}
\]

Thus, the output range of the derivative is indeed \((0, 1/4]\), as desired.

We can identify the output range of the second derivative with similar arguments above. We first compute the third derivatives of \(\log\) as desired. Detail: Alternatively, one could derive

\[
\dot{\log} \cdot \log[z] = \frac{\partial}{\partial z} \dot{\log} \cdot \log[z]
\]

definition of the second derivative

\[
= \frac{\partial}{\partial z} \left( \dot{\log} \cdot \log[z] \right)
\]

result for \(\dot{\log} \cdot \log\)

\[
= \frac{\partial}{\partial z} \left( \dot{\log} \cdot \log[z] \right)
\]

quotient rule

\[
= \frac{\partial}{\partial z} \left( \dot{\log} \cdot \log[z] \right)
\]

chain rule

\[
= -e^{-z}(1 + e^{-z})^2 - e^{-z} \cdot 2(1 + e^{-z}) \frac{\partial}{\partial z} \left( \dot{\log} \cdot \log[z] \right) + e^{-z} \frac{\partial}{\partial z} \left( \dot{\log} \cdot \log[z] \right)
\]

sum rule

\[
= \frac{e^{-z} \left( e^{-z} - 1 \right)}{(1 + e^{-z})^3}
\]

factoring out \(e^{-z}\) in the numerator

To identify the output range of the first derivatives, we use the above-derived equality

\[
\dot{\log} \cdot \log[z] = \frac{e^{-z}}{(1 + e^{-z})^2},
\]

and then the above results for \(\dot{\log} \cdot \log\), as desired.
\[
\dot{f}_\log[z] = \dot{f}_\log[z](1 - 2f_\log[z]) + \dot{f}_\log[z](0 - 2\dot{f}_\log[z])
\]
\text{definition of } \dot{f}_\log; \quad 1' = 0; \quad (-2\dot{f}_\log)' = -2\dot{f}_\log; \quad \text{sum rule}
\[
= \dot{f}_\log[z](1 - 2f_\log[z])(1 - 2f_\log[z]) - 2(\dot{f}_\log[z])^2
\]
\text{above equality for } \dot{f}_\log; \quad \text{consolidating the second term}
\[
= \dot{f}_\log[z] \left(1 - 4f_\log[z] + 4(f_\log[z])^2 - 2(\dot{f}_\log[z])\right)
\]
\text{summarizing the two terms}
\[
= \dot{f}_\log[z] \left(1 + e^{-z}\right)^2 - 4(1 + e^{-z}) + 4 - 2e^{-z}
\]
\text{definition of } f_\log; \quad \text{above-derived equality for } \dot{f}_\log
\[
= \dot{f}_\log[z] \left\{1 + 2e^{-z} + e^{-2z} - 4 - 4e^{-z} + 4 - 2e^{-z}\right\}
\]
\text{expanding the terms}
\[
= \dot{f}_\log[z] \left\{1 - 4e^{-z} + e^{-2z}\right\}
\]
\text{consolidating}
\[
= \dot{f}_\log[z] \left(1 - 2e^{-z}\right)^2 - 3(e^{-z})^2
\]
\text{rearranging the terms}
\]
\]
\text{which is equal to zero (both } \dot{f}_\log \text{ and } (1 + e^{-z})^2 \text{ are positive) if and only if } (1 - 2e^{-z})^2 = 3(e^{-z})^2. \quad \text{The claim then follows similarly as in the case of the first derivative (we omit some details) from}
\[
(1 - 2e^{-z})^2 = 3(e^{-z})^2
\]
\Rightarrow \quad 1 - 2e^{-z} = \pm \sqrt{3}e^{-z}
\text{taking square roots on both sides}
\Rightarrow \quad (-2 \pm \sqrt{3})e^{-z} = -1
\text{factoring out } e^{-z}
\Rightarrow \quad e^{-z} = 1/(2 \pm \sqrt{3})
\text{dividing both sides by } (-2 \pm \sqrt{3}) = -(2 \pm \sqrt{3}) \neq 0
\Rightarrow \quad -z = \log[1/(2 \pm \sqrt{3})]
\text{taking logarithms on both sides}
\Rightarrow \quad -z = -\log[2 \pm \sqrt{3}]
\text{log}[1/b] = -\log[b]
\Rightarrow \quad z = \log[2 \pm \sqrt{3}]
\text{multiplying both sides by } -1
\Rightarrow \quad z = \pm \log[2 \pm \sqrt{3}],
\text{see below}
\]
where the last step follows from
\[
\log[1] = 0 \quad \text{basic property of the logarithm}
\Rightarrow \quad \log[(2 + \sqrt{3})(2 - \sqrt{3})] = 0 \quad (2 + \sqrt{3})(2 - \sqrt{3}) = 1
\Rightarrow \quad \log[2 + \sqrt{3}] + \log[2 - \sqrt{3}] = 0 \quad \log[ab] = \log[a] + \log[b]
\Rightarrow \quad \log[2 + \sqrt{3}] = -\log[2 - \sqrt{3}] \quad \text{subtracting } \log[2 - \sqrt{3}] \text{ on both sides}
\]
\text{Numerical evaluation then yields the desired output range:}
\[
\pm c = \dot{f}_\log[\pm \log[2 + \sqrt{3}]] \approx \pm 0.0962.
\]
A.3.2 Derivatives of arctan

Lemma A.3 (Derivatives of arctan). The first and second derivatives of arctan are

\[
\dot{f}_{\text{arctan}}[z] = \frac{1}{1 + z^2} \in [0, \infty) \quad \text{and} \quad \ddot{f}_{\text{arctan}}[z] = -\frac{2z}{(1 + z^2)^2} = -\frac{2z}{(\dot{f}_{\text{arctan}}[z])^2} \in \mathbb{R}
\]

for all \( z \in \mathbb{R} \).

Proof of Lemma A.3. By the definition of arctan via \( \tan[\text{arctan}[z]] = z \), it holds that

\[
\frac{\partial}{\partial z} \tan[\text{arctan}[z]] = \frac{\partial}{\partial z} z = 1.
\]

On the other hand, the chain rule ensures that

\[
\frac{\partial}{\partial z} \tan[\text{arctan}[z]] = \left( \frac{\partial}{\partial w} \tan[w] \bigg|_{w=\text{arctan}[z]} \right) \frac{\partial}{\partial z} \text{arctan}[z].
\]

Combining these two inequalities yields

\[
\left( \frac{\partial}{\partial w} \tan[w] \bigg|_{w=\text{arctan}[z]} \right) \frac{\partial}{\partial z} \text{arctan}[z] = 1.
\]

This equation allows us to calculate the derivative of arctan through the derivative of tan.

The derivative of tan is

\[
\frac{\partial}{\partial w} \tan[w] \bigg|_{w=\text{arctan}[z]} = \frac{\partial}{\partial w} \left( \frac{\sin[w]}{\cos[w]} \right) \bigg|_{w=\text{arctan}[z]} = \frac{\cos[w]}{\cos^2[w]} - \sin[w] \left( \frac{\partial}{\partial w} \cos[w] \right) \bigg|_{w=\text{arctan}[z]} = \frac{\cos[w]^2 + \sin[w]^2}{\cos[w]^2} \bigg|_{w=\text{arctan}[z]} = 1 + \left( \frac{\sin[w]}{\cos[w]} \right)^2 \bigg|_{w=\text{arctan}[z]} = 1 + \tan^2[\text{arctan}[z]] = 1 + z^2.
\]

Plugging this back into the above display yields

\[
(1 + z^2) \frac{\partial}{\partial z} \text{arctan}[z] = 1,
\]
which yields after dividing both sides by $1 + z^2$ (observe that $1 + z^2 > 0$)

$$\dot{\arctan}[z] = \frac{\partial}{\partial z} \arctan[z] = \frac{1}{1 + z^2},$$

which is the desired first derivative.

The second derivative then follows essentially from the chain rule:

$$\ddot{\arctan}[z] = \frac{\partial}{\partial z} \dot{\arctan}[z]$$

definition of the second derivative

$$= \frac{\partial}{\partial z} \frac{1}{1 + z^2}$$

above result

$$= \frac{\partial}{\partial z} \frac{1}{(1 + z^2)}$$

chain rule

$$= -\frac{2z}{(1 + z^2)^2}$$

$1' = 0; (z^2)' = 2z$

$$= -\frac{2z}{(\dot{\arctan}[z])^2},$$

above derivations

which is the desired second derivative.

The output ranges then follow readily.

\[\square\]

### A.3.3 Derivatives of $\tanh$

**Lemma A.4** (Derivatives of $\tanh$). The first and second derivatives of $\tanh$ are

$$\dot{\tanh}[z] = 1 - (\tanh[z])^2 \in (0, 1)$$

and

$$\ddot{\tanh}[z] = -2\tanh[z] \left(1 - (\tanh[z])^2\right) \in (-c, c)$$

for all $z \in \mathbb{R}$ and $c \approx 0.770$.

**Proof of Lemma A.4.** The claims can be derived by using elementary differential calculus:

$$\dot{\tanh}[z] = \frac{\partial}{\partial z} \frac{e^z - e^{-z}}{e^z + e^{-z}}$$

definition of $\tanh$

$$= \frac{\left(\frac{\partial}{\partial z} (e^z - e^{-z})\right)(e^z + e^{-z}) - (e^z - e^{-z})\left(\frac{\partial}{\partial z} (e^z + e^{-z})\right)}{(e^z + e^{-z})^2}$$

quotient rule

$$= \frac{(e^z + e^{-z})(e^z + e^{-z}) - (e^z - e^{-z})(e^z - e^{-z})}{(e^z + e^{-z})^2}$$

$$= 1 - \left(\frac{e^z - e^{-z}}{e^z + e^{-z}}\right)^2$$

splitting the fraction up and simplifying

$$= 1 - (\tanh[z])^2$$

definition of $\tanh$

and

$$\ddot{\tanh}[z] = \frac{\partial}{\partial z} \dot{\tanh}[z]$$

definition of the second derivative
\[
\frac{\partial}{\partial z} \left( 1 - \left( f_{\text{tanh}}[z] \right)^2 \right) = \frac{-2 f_{\text{tanh}}[z] f'_{\text{tanh}}[z]}{1 - \left( f_{\text{tanh}}[z] \right)^2},
\]

as desired.

The output range of the first derivative follows directly from the output range of the original function and of the form of the derivatives.

The output range of the second derivative is the output range of the following function (see the form of the second derivative):

\[
g : (-1, 1) \rightarrow \mathbb{R}
\]

\[
w \mapsto -2w(1 - w^2).
\]

The function \(g\) is smooth and \(\lim_{w \to \pm 1} g[w] = 0\). Hence, the output range is determined by the minima and maxima of the function, which must satisfy \(g[w] = 0\). Now,

\[
\frac{\partial}{\partial w} g[w] = 0
\]

\[
\Rightarrow \frac{\partial}{\partial w} \left( -2w(1 - w^2) \right) = 0
\]

\[
\Rightarrow -2 \frac{\partial}{\partial w} \left( w(1 - w^2) \right) = 0
\]

\[
\Rightarrow \frac{\partial}{\partial w} \left( w(1 - w^2) \right) = 0
\]

\[
\Rightarrow 1 \cdot (1 - w^2) + w \cdot (0 - 2w) = 0
\]

\[
w' = 1; 1' = 0; -(w^2)' = -2w; \text{ product and sum rules}
\]

\[
\Rightarrow 1 - 3w^2 = 0
\]

\[
\Rightarrow w^2 = \frac{1}{3}
\]

\[
\Rightarrow z = \pm \frac{1}{\sqrt{3}}.
\]

One can check readily that these points are indeed the minimum and maximum of \(g\).

Plugging the points back into the function yields via basic algebra

\[
g \left[ \pm \frac{1}{\sqrt{3}} \right] = -2 \left( \pm \frac{1}{\sqrt{3}} \right) \left( 1 - \left( \pm \frac{1}{\sqrt{3}} \right)^2 \right) = \mp \frac{2}{\sqrt{3}} \left( 1 - \frac{1}{3} \right) = \mp \frac{4}{\sqrt{27}} \approx \mp 0.770,
\]

as desired. \(\square\)

### A.3.4 Derivatives of softsign

**Lemma A.5** (Derivatives of softsign). The first derivative of softsign is

\[
f'_{\text{soft}}[z] = \frac{1}{(1 + |z|)^2} = \left( 1 - |f_{\text{soft}}[z]| \right)^2 \in (0, 1) \quad \text{for all } z \in \mathbb{R}.
\]
The second derivative of \texttt{softsign} is

\[ \ddot{f}_{\text{soft}}[z] = \frac{-2 \text{sign}[z]}{(1 + |z|)^3} = -2 \text{sign}[z] (\dot{f}_{\text{soft}}[z])^{3/2} \]

\[ = -2 \text{sign}[z] (1 - |f_{\text{soft}}[z]|)^3 \in (-2, 2) \]

for all \( z \in \mathbb{R} \setminus \{0\} \).

\textbf{Proof of Lemma A.5.} We establish the first and second derivatives in order. The only small difficulty is that the case \( z = 0 \) needs special attention. The output ranges of the derivatives follow almost directly from the explicit forms of the derivatives—we omit the details.

We start with the first derivative. For \( z \neq 0 \), we find that

\[ \frac{\partial}{\partial z} f_{\text{soft}}[z] = \frac{\partial}{\partial z} \frac{z}{1 + |z|} \]

\[ = \left( \frac{\partial}{\partial z} \frac{z}{1 + |z|} \right) \left(1 + |z|\right)^2 \]

\[ = \frac{1 \cdot (1 + |z|) - z \cdot (0 + \text{sign}[z])}{(1 + |z|)^2} \]

\[ = \frac{1 + |z| - z \text{sign}[z]}{(1 + |z|)^2} \]

\[ = \frac{1}{(1 + |z|)^2}. \]

This expression can be related to the original function:

\[ \frac{\partial}{\partial z} f_{\text{soft}}[z] = \frac{1}{(1 + |z|)^2} \]

\[ = \left( \frac{1 + |z| - |z|}{1 + |z|} \right)^2 \]

\[ = \left( \frac{1 - |z|}{1 + |z|} \right)^2 \]

\[ = \left( \frac{1 - |z|}{1 + |z|} \right)^2 \]

\[ = \left( 1 - |f_{\text{soft}}[z]| \right)^2 \]

\texttt{as desired.}

The case \( z = 0 \) can be treated via the basic definition of derivatives:

\[ \left. \frac{\partial}{\partial z} f_{\text{soft}}[z] \right|_{z=0} = \lim_{t \to 0} \frac{f_{\text{soft}}[0 + t] - f_{\text{soft}}[0]}{t} \]

\texttt{definition of derivatives}
\[
\lim_{t \to 0} \frac{t}{|t|} = 0 \quad \text{definition of } f_{\text{soft}}
\]

\[
\lim_{t \to 0} \frac{1}{1 + |t|} = 1. \quad \text{simplification and evaluation of the limit}
\]

Again, we can formulate this in terms of the original function:

\[
\frac{\partial}{\partial z} f_{\text{soft}}[z] \bigg|_{z=0} = 1 \quad \text{previous display}
\]

\[
= (1 - 0)^2 \quad \text{basic calculus}
\]

\[
= (1 - |f_{\text{soft}}[z]|)^2, \quad f_{\text{soft}}[0] = 0
\]

as desired.

The second derivative at \( z \neq 0 \) now follows readily from the first derivative:

\[
\ddot{f}_{\text{soft}}[z] = \frac{\partial}{\partial z} \dot{f}_{\text{soft}}[z] \quad \text{definition of second derivative}
\]

\[
= \frac{\partial}{\partial z} (1 - |f_{\text{soft}}[z]|)^2 \quad \text{above results}
\]

\[
= 2(1 - |f_{\text{soft}}[z]|) \frac{\partial}{\partial z} (1 - |f_{\text{soft}}[z]|) \quad \text{chain rule}
\]

\[
= 2(1 - |f_{\text{soft}}[z]|) \frac{\partial}{\partial z} (1 - \text{sign}[z]f_{\text{soft}}[z]) \quad |b| = \text{sign}[b]b
\]

\[
= 2(1 - |f_{\text{soft}}[z]|) (0 - \text{sign}[z]f_{\text{soft}}[z] - 0f_{\text{soft}}[z]) \quad 1' = 0; \text{sign}[z]' = 0 \text{ at } z \neq 0; \text{sum and product rules}
\]

\[
= -2 \text{sign}[z] (1 - |f_{\text{soft}}[z]|) \dot{f}_{\text{soft}}[z] \quad \text{consolidation}
\]

\[
= -2 \text{sign}[z] (\dot{f}_{\text{soft}}[z])^{3/2} \quad \text{above results; } \dot{f}_{\text{soft}} > 0
\]

as desired. In explicit terms, we find (see again the above results)

\[
\ddot{f}_{\text{soft}}[z] = -2 \text{sign}[z] (\dot{f}_{\text{soft}}[z])^{3/2} = -\frac{2 \text{sign}[z]}{(1 + |z|)^{3/2}},
\]

as desired.

Detail: The second derivative can also be established from the explicit form of the first derivative:

\[
\dot{f}_{\text{soft}}[z] = \frac{\partial}{\partial z} f_{\text{soft}}[z] \quad \text{definition of second derivative}
\]

\[
= \frac{\partial}{\partial z} \frac{1}{(1 + |z|)^2} \quad \text{above results}
\]

\[
= -2 \frac{0 + \text{sign}[z]}{(1 + |z|)^3} \quad 1' = 0; |z'| = \text{sign}[z] \text{ for } z \neq 0; \text{sum rule}
\]

\[
= -2 \frac{\text{sign}[z]}{(1 + |z|)^3} \quad \text{consolidation}
\]
A.4 Properties of tanh

**Lemma A.6** (Properties of tanh). It holds that

\[ f_{tanh}[z] = 2f_{\log}[2z] - 1 \quad \text{for all } z \in \mathbb{R}. \]

and

\[ f_{tanh}[0] = f_{\arctan}[0], \quad \dot{f}_{tanh}[0] = \dot{f}_{\arctan}[0], \quad \ddot{f}_{tanh}[0] = \ddot{f}_{\arctan}[0], \quad \text{and} \quad \dddot{f}_{tanh}[0] = \dddot{f}_{\arctan}[0]. \]

**Proof of Lemma A.6.** The first claim follows from elementary differential calculus:

\[
\begin{align*}
\dot{f}_{tanh}[z] &= \frac{e^z - e^{-z}}{e^z + e^{-z}} \quad \text{definition of } f_{tanh} \\
&= \frac{e^z + e^z - e^z - e^{-z}}{e^z + e^{-z}} \quad \text{adding a zero-valued term} \\
&= \frac{2e^z}{e^z + e^{-z}} - 1 \quad \text{splitting the fraction up and simplifying} \\
&= \frac{2e^z}{e^z(1 + e^{-2z})} - 1 \quad \text{factoring out an } e^z \\
&= \frac{2}{1 + e^{-2z}} - 1 \quad \text{consolidating} \\
&= 2f_{\log}[2z] - 1, \quad \text{definition of } f_{\log}
\end{align*}
\]

as desired.

We leave the second part to the reader. \( \square \)

A.5 Derivatives of the Piecewise-Linear Functions

A.5.1 Derivatives of linear, relu, and leakyrelu

**Lemma A.7** (Derivatives of leakyrelu). It holds that

\[ \dot{f}_{\text{relu},a}[z] = \begin{cases} 
1 & \text{for all } z \in (0, \infty) \\
a & \text{for all } z \in (-\infty, 0)
\end{cases} \quad \text{and} \quad \ddot{f}_{\text{relu},a}[z] = 0 \quad \text{for all } z \in \mathbb{R} \setminus \{0\} \]

and

\[ d_v f_{\text{relu},a}[z] = \begin{cases} 
v & \text{for all } v \in \mathbb{R} \text{ and } z \in (0, \infty) \text{ and for all } v \in [0, \infty) \text{ and } z = 0; \\
v & \text{otherwise.}
\end{cases} \]

As special cases, the lemma entails first and second derivatives and first directional derivatives also for **linear** \((a = 1)\) and **relu** \((a = 0)\).

**Proof of Lemma A.7.** Observe first that \( d_v f_{\text{relu},a}[z] = d_{-v} f_{\text{relu},a}[z] \) for all \( v \in \mathbb{R} \) and \( z \in \mathbb{R} \setminus \{0\} \); hence, in view of our comments after Definition A.1, the first derivatives on \( \mathbb{R} \setminus \{0\} \) follow from the directional derivatives. The second derivatives \( \mathbb{R} \setminus \{0\} \) follow readily from the first derivatives.

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What is left to prove is that the directional derivatives are as claimed. We separate this proof into four cases.

Case 1: $v \in \mathbb{R}, z \in (0, \infty)$

It holds that $z > 0$ by assumption and, therefore, that $z + tv > 0$ for small enough $t \in (0, \infty)$. These two observations combined with the definition of leakyrelu ensure that $f[z] = z$ and $f[z + tv] = z + tv$ for small enough $t \in (0, \infty)$. Using these two equalities and basic algebra, as well as Definition A.1 about directional derivatives, yields

$$d_v f[z] = \lim_{t \to 0^+} \frac{f[z + tv] - f[z]}{t} = \lim_{t \to 0^+} \frac{z + tv - z}{t} = \lim_{t \to 0^+} v = v,$$

as desired.

Case 2: $v \in [0, \infty), z = 0$

It holds that $z = 0$ and $v > 0$ by assumption and, therefore, that $z + tv > 0$ for small enough $t \in (0, \infty)$. These observations combined with the definition of leakyrelu ensure that $f[z] = 0$ and $f[z + tv] = tv$ for small enough $t \in (0, \infty)$. Therefore, similarly as above,

$$d_v f[z] = \lim_{t \to 0^+} \frac{f[z + tv] - f[z]}{t} = \lim_{t \to 0^+} \frac{tv - 0}{t} = \lim_{t \to 0^+} v = v,$$

as desired.

Case 3: $v \in \mathbb{R}, z \in (-\infty, 0)$

It holds that $z < 0$ by assumption and, therefore, that $z + tv < 0$ for small enough $t \in (0, \infty)$. These two observations combined with the definition of leakyrelu ensure that $f[z] = az$ and $f[z + tv] = a(z + tv)$ for small enough $t \in (0, \infty)$. Using these equalities and basic algebra yields similarly as before

$$d_v f[z] = \lim_{t \to 0^+} \frac{f[z + tv] - f[z]}{t} = \lim_{t \to 0^+} \frac{a(z + tv) - az}{t} = \lim_{t \to 0^+} av = av,$$

as desired.

Case 4: $v \in (-\infty, 0), z = 0$

It holds that $z = 0$ and $v < 0$ by assumption and, therefore, that $z + tv < 0$ for small enough $t \in (0, \infty)$. These two observations combined with the definition of leakyrelu ensure that $f[z] = 0$ and $f[z + tv] = atv$ for small enough $t \in (0, \infty)$. Using these equalities and basic algebra yields

$$d_v f[z] = \lim_{t \to 0^+} \frac{f[z + tv] - f[z]}{t} = \lim_{t \to 0^+} \frac{atv - 0}{t} = \lim_{t \to 0^+} av = av,$$

as desired.

\[\square\]

A.6 Expressivities of the Piecewise-Linear Functions

Example A.1 (Expressivities of linear, relu, and leakyrelu). The expressivity of a class of networks is its capability to approximate different functions. In this example, we illustrate the limited expressivity of linear networks as compared to
ReLU networks. The concatenated neurons on Page 2 simplify in the case of purely linear activation to

\[
\begin{bmatrix}
  n_{\beta^1, \theta^1; \text{linear}}[x] \\
  \vdots \\
  n_{\beta^n, \theta^n; \text{linear}}[x]
\end{bmatrix} = \zeta + \sum_{k=1}^{w} \gamma_k \left( \beta^k + \sum_{j=1}^{d} (\theta^k)_j x_j \right)
\]

\[
= \zeta + \sum_{k=1}^{w} \gamma_k \beta^k + \sum_{j=1}^{d} \left( \sum_{k=1}^{w} \gamma_k (\theta^k)_j \right) x_j
\]

\[
= n_{\kappa, \eta; \text{linear}}[x],
\]

which is a single linear neuron. Thus, subsequent layers with linear activation collapse into a linear layer, which means that fitting data-generating processes that are nonlinear requires the inclusion of other activations somewhere in the network.

In contrast, ReLU layers (and one can expect that leakyrelu behaves very similarly as ReLU in terms of expressivity) only collapse if the weights are nonnegative; see Hebiri and Lederer [2020, Theorem 1] for a precise formulation of this feature and Hebiri and Lederer [2020, Section 2.3] for a description of how it can be leveraged for regulating layer depths. More generally, one can show that ReLU networks can approximate a range of nonlinear functions; see Corlay et al. [2019] and references therein.

### A.7 Dying-relu Phenomenon

**Example A.2** (Dying-relu phenomenon). We illustrate the dying-relu and revitalization phenomena in a toy model. We consider networks that have real-valued inputs and outputs and that consist of a ReLU and a tanh layer that each has one neuron and no bias term; in other words, we consider the functions

\[ \mathbb{R} \rightarrow \mathbb{R} \]
\[ x \mapsto f_{\text{relu}}[\gamma f_{\text{tanh}}[\theta x]] \]

parametrized by \(\gamma, \theta \in \mathbb{R}\). Given data \((x_1, y_1), \ldots, (x_n, y_n) \in \mathbb{R} \times \mathbb{R}\), we want to fit the parameters by optimizing the usual least-squares loss

\[ (\gamma, \theta) \mapsto \sum_{i=1}^{n} \left( y_i - f_{\text{relu}}[\gamma f_{\text{tanh}}[\theta x_i]] \right)^2. \]

A standard method for such optimizations is stochastic-gradient descent. The \(i\)th updates for \(\gamma\) and \(\theta\) of stochastic-gradient descent with step size \(s_i \in (0, \infty)\) at \((\gamma_i, \theta_i) \in \mathbb{R} \times \mathbb{R}\) are (assuming for simplicity that \(\theta x_i \neq 0\) to ensure differentiability)

\[- s_i \frac{\partial}{\partial \gamma} \bigg|_{(\gamma, \theta) = (\gamma_i, \theta_i)} \left( y_i - f_{\text{relu}}[\gamma f_{\text{tanh}}[\theta x_i]] \right)^2 \]

and

\[- s_i \frac{\partial}{\partial \theta} \bigg|_{(\gamma, \theta) = (\gamma_i, \theta_i)} \left( y_i - f_{\text{relu}}[\gamma f_{\text{tanh}}[\theta x_i]] \right)^2. \]
One can verify readily (use Lemmas A.4 and A.7 and the chain rule) that

\[-s_i \frac{\partial}{\partial \gamma} \bigg|_{(\gamma, \theta) = (\gamma_i, \theta_i)} \left( y_i - f_{\text{relu}}[\gamma f_{\tanh}[\theta x_i]] \right)^2 = 0 \]

and

\[-s_i \frac{\partial}{\partial \theta} \bigg|_{(\gamma, \theta) = (\gamma_i, \theta_i)} \left( y_i - f_{\text{relu}}[\gamma f_{\tanh}[\theta x_i]] \right)^2 = 0 \]

if the optimization is in a state with \( \gamma_i, \theta_i > 0 \) and \( x_i < 0 \). Hence, the parameters remain unchanged. The underlying reason is that under the stated conditions, the function \( \eta \mapsto f_{\text{relu}}[\eta] \) is constant and equal to zero in an environment around \( \eta_i := \gamma_i f_{\tanh}[\theta_i x_i] \), that is, the relu node does not transmit information; we could that the relu neuron is inactive.

But the neurons can become active again. Assume that \( x_{i+1} > 0 \). Then, one can verify readily (use again Lemmas A.4 and A.7 and the chain rule) that

\[-s_{i+1} \frac{\partial}{\partial \gamma} \bigg|_{(\gamma, \theta) = (\gamma_{i+1}, \theta_{i+1})} \left( y_{i+1} - f_{\text{relu}}[\gamma f_{\tanh}[\theta x_{i+1}]] \right)^2 \neq 0 \]

and

\[-s_{i+1} \frac{\partial}{\partial \theta} \bigg|_{(\gamma, \theta) = (\gamma_{i+1}, \theta_{i+1})} \left( y_{i+1} - f_{\text{relu}}[\gamma f_{\tanh}[\theta x_{i+1}]] \right)^2 \neq 0 \]

as long as \( f_{\text{relu}}[\gamma_{i+1} f_{\tanh}[\theta_{i+1} x_{i+1}]] \neq y_{i+1} \). Hence, the parameters are updated in a nontrivial way, and we can say that the relu node is active again.

These observations indicate 1. that dead relu nodes are common in optimization steps but also 2. that relu nodes rarely stay dead during the entire optimization. In special cases, or simply if there are many relu nodes, some nodes can be inactive for all or almost all of the training process; we then say that those nodes are dead. Dead relu nodes can be undesired if they are abundant, because then, they can avoid learning complex models. We then speak of the dying-relu phenomenon.

### A.8 Properties of the Other Functions

#### A.8.1 Derivatives of softplus

**Lemma A.8** (Derivatives of softplus). The first and second derivatives of softplus are

\[ f_{\text{soft+}}[z] = \frac{1}{1 + e^{-z}} = f_{\log}[z] \in (0, 1) \]

and

\[ f_{\text{soft+}}'[z] = \frac{e^{-z}}{(1 + e^{-z})^2} = f_{\log}[z] = f_{\log}[z] (1 - f_{\log}[z]) \in (0, 1/4] , \]

respectively, for all \( z \in \mathbb{R} \).

**Proof of Lemma A.8.** Observe that

\[
\frac{\partial}{\partial z} f_{\text{soft+}}[z] = \frac{\partial}{\partial z} \log[1 + e^z]
\]

\[
= \frac{\partial}{\partial z} \left( 1 + e^z \right) \left( \log z \right)' = 1/z \text{ for } z > 0; \text{ chain rule}
\]
= \frac{e^z}{1 + e^z} \quad 1' = 0; (e^z)' = e^z; \text{ sum rule}

= \frac{1}{1 + e^{-z}} \quad \text{multiplying numerator and denominator by } e^{-z}

= f_{\log}[z]. \quad \text{definition of } f_{\log}

The rest follows from Section 2.1.1 and Lemma A.2 on logistic.

A.8.2 Output Ranges and Derivatives of $elu$ and $selu$

Lemma A.9 (Output ranges and derivatives of $elu$). The graphs of $elu$ satisfy

$$\lim_{z \to -\infty} f_{elu,a}[z] = \inf_{z \in \mathbb{R}} f_{elu,a}[z] = -a \quad \text{and} \quad f_{elu,a}[z] > -a \quad \text{for all } z \in \mathbb{R}$$

for all $a \in [0, \infty)$. The first and second derivatives of $elu$ are

$$f_{elu,a}[z] = \begin{cases} 1 & \text{for all } z \in (0, \infty) \\ ae^z = f_{elu,a}[z] + a & \text{for all } z \in (-\infty, 0) \end{cases}$$

and

$$\dot{f}_{elu,a}[z] = \begin{cases} 0 & \text{for all } z \in (0, \infty) \\ ae^z = f_{elu,a}[z] + a & \text{for all } z \in (-\infty, 0) \end{cases}$$

for all $a \in [0, \infty)$. The first directional derivatives of $elu$ are

$$d_v f_{elu,a}[z] = \begin{cases} \frac{v}{v} & \text{for all } v \in \mathbb{R} \text{ and } z \in (0, \infty) \text{ or } v \in [0, \infty) \text{ and } z = 0 \\ v[f_{elu,a}[z] + a] & \text{otherwise} \end{cases}$$

for all $a \in [0, \infty)$. The corresponding properties of $selu$ can be derived along the same lines. Observe that

$$\lim_{z \to -\infty} f_{elu,1}[z] = \lim_{z \to -\infty} \dot{f}_{elu,1}[-z]$$

that is, $elu$ with $z = 1$ is one time differentiable on the entire real line.

Proof of Lemma A.9. The first claim follows from the observations that $elu$ is strictly increasing on the real line (both $z \mapsto z$ and $z \mapsto a(e^{-z} - 1)$ are strictly increasing and $f_{elu,a}[z_1] < 0 < f_{elu,a}[z_2]$ for all $z_1 \in (-\infty, 0), z_2 \in (0, \infty)$) and that

$$\lim_{z \to -\infty} f_{elu,a}[z] = \lim_{z \to -\infty} a(e^{-z} - 1) \quad f_{elu,a}[z] = a(e^{-z} - 1) \quad \text{for } z \text{ small enough}$$

$$e^{-z} \to 0 \quad \text{for } z \to -\infty$$

The first and second derivatives follow from standard differential calculus.

We illustrate the derivations of the directional derivatives in the case $z = 0$ and $v \in (-\infty, 0)$—the other cases can be treated in the same way. Given any such $z$ and $v$ as well as an arbitrary $a$, we can make three simple observations:

Observation (i): $f_{elu,a}[0] = 0$.

This observation follows directly from the definition of $f_{elu,a}$.

Observation (ii): $f_{elu,a}[0 + tv] = a(e^tv - 1)$.
This observation follows from the definition of $f_{elu,a}$ and the fact that $tv < 0$ for all $t \in (0, \infty)$ and $v \in (-\infty, 0)$

Observation (iii): $(ae^{tv} - a)/t \to av$ for $t \to 0^+$.

Using Lemma A.1 (l’Hôpital’s rule) and basic algebra yields

$$\lim_{t \to 0^+} \frac{ae^{tv} - a}{t} = \lim_{t \to 0^+} \frac{(ae^{tv} - a)'}{(t)'}$$

2. in Lemma A.1

$$= \lim_{t \to 0^+} \frac{ave^{tv} - 0}{1}$$

sum and chain rules for differentiation

$$= \lim_{t \to 0^+} ave^{tv}$$

consolidation

$$= av$$

$e^{0}v = 1$

(where we have used l’Hôpital’s rule with $f : t \mapsto ae^{tv} - a$, $g : t \mapsto t$, and $\bar{t} = 0$), as desired.

Using these three observations together with the definition of directional derivatives on Page 20 yields for all $a \in [0, \infty)$ that

$$d_v f_{elu,a}[0] = \lim_{t \to 0^+} \frac{f_{elu,a}[0 + tv] - f_{elu,a}[0]}{t}$$

Definition A.1 (directional derivatives)

$$= \lim_{t \to 0^+} \frac{a(e^{tv} - 1) - 0}{t}$$

Observations (i) and (ii)

$$= \lim_{t \to 0^+} \frac{ae^{tv} - a}{t}$$

consolidation

$$= av$$

Observation (iii)

$$= v(0 + a)$$

adding a zero-valued term

$$= v(f_{elu,a}[0] + a)$$,

$f_{elu,a}[0] = 0$ by definition

as desired.

\[ \square \]

A.8.3 Further Details on $selu$

Our formulation of $selu$ in Section 2.3.2 differs slightly from the original formulation Klambauer et al. [2017, Equation (1)] in that we set $a_0 := \lambda$ and $b_0 := \lambda \alpha$ for conciseness.

More precise values for the constants are $a_0 \approx 1.05070098$ and $b_0 \approx 1.7580993261$. Analytical expressions are stated in Klambauer et al. [2017, Equation (8) in the Supplementary Material].

A precise version of the statements about $c$ is that $\|c[(\mu, \nu)] - (\overline{\mu}, \overline{\nu})\|_2 < \|((\mu, \nu) - (\overline{\mu}, \overline{\nu}))\|$ and $c[(\overline{\mu}, \overline{\nu})] = (\overline{\mu}, \overline{\nu})$ for all $(\mu, \nu) \in \mathbb{R}^2$.

A.8.4 Derivatives of $swish$

Lemma A.10 (Derivatives of $swish$). The first and second derivatives of $swish$ are

$$\dot{f}_{swish,a}[z] = \frac{1 + (1 + az)e^{-az}}{(1 + e^{-az})^2} = af_{swish,a}[z] + f_{\log}(az)(1 - af_{swish,a}[z])$$

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and

\[ \dot{f}_{\text{swish}, a}[z] = ae^{az} \cdot \frac{2 - az + (2 + az)e^{-az}}{(1 + e^{-az})^3} \]

\[ = a\left(a\dot{f}_{\text{swish}, a}[z] + (2 + az)f_{\text{log}}[az] \left(1 - a\dot{f}_{\text{swish}, a}[z]\right)\right) \left(1 - f_{\text{log}}[az]\right) \]

for all \( z \in \mathbb{R} \) and \( a \in [0, \infty) \).

**Proof of Lemma A.10.** Using the basic rules for differentiation and the definitions of \( \dot{f}_{\text{swish}, a} \) and \( f_{\text{log}} \), we find for the first derivative

\[ \dot{f}_{\text{swish}, a}[z] = \frac{\partial}{\partial z} \frac{z}{1 + e^{-az}} \]

\[ = \frac{\left(\frac{\partial}{\partial z} z\right)(1 + e^{-az}) - z \frac{\partial}{\partial z} (1 + e^{-az})}{(1 + e^{-az})^2} \]

\[ = \frac{1 \cdot (1 + e^{-az}) - z \cdot (0 - ae^{-az})}{(1 + e^{-az})^2} \]

\[ = \frac{1 + (1 + az)e^{-az}}{(1 + e^{-az})^2}, \]

which proves the first part of the equality of the derivative. We then find further

\[ \dot{f}_{\text{swish}, a}[z] = \frac{1 + (1 + az)e^{-az}}{(1 + e^{-az})^2} \]

\[ = az(1 + e^{-az}) - az + 1 + e^{-az} \]

\[ = az(1 + e^{-az}) \frac{1}{1 + e^{-az}} - az + 1 + e^{-az} \]

\[ = a \cdot \frac{z}{1 + e^{-az}} + \frac{1}{1 + e^{-az}} \cdot \frac{1 + e^{-az} - az}{1 + e^{-az}} \]

\[ = a \cdot \frac{z}{1 + e^{-az}} + \frac{1}{1 + e^{-az}} \cdot \left(1 - a \cdot \frac{z}{1 + e^{-az}}\right) \]

\[ = a\dot{f}_{\text{swish}, a}[z] + f_{\text{log}}[az] \left(1 - a\dot{f}_{\text{swish}, a}[z]\right), \]

definitions of \( \text{swish} \) and \( \text{logistic} \)

as desired.

**Detail:** The second equality can also be established directly via the composite form of \( \text{swish} \):

\[ \dot{f}_{\text{swish}, a}[z] = \frac{\partial}{\partial z} \left(z \cdot f_{\text{log}}[az]\right) \]

\[ = \left(\frac{\partial}{\partial z}\right)f_{\text{log}}[az] + z \left(\frac{\partial}{\partial az} f_{\text{log}}[az]\right) \]

\[ = f_{\text{log}}[az] + z \cdot a \left(\frac{\partial}{\partial w} f_{\text{log}}[w]\right)_{w=az} \]

\[ = f_{\text{log}}[az] + z \cdot f_{\text{log}}[az] \left(1 - f_{\text{log}}[az]\right) \]

\[ = f_{\text{log}}[az] + a\dot{f}_{\text{swish}, a}[z] \left(1 - f_{\text{log}}[az]\right) \]

\[ = a\dot{f}_{\text{swish}, a}[z] + f_{\text{log}}[az] \left(1 - a\dot{f}_{\text{swish}, a}[z]\right). \]

Definition of \( \text{swish} \).
Similarly, the second derivative of \textit{swish} can be calculated based on the explicit form or the composite form of the first derivative; we opt for the latter:

\[
\ddot{j}_{\text{swish},a}[z] = \frac{\partial}{\partial z} \dot{j}_{\text{swish},a}[z]
\]

\[
= \frac{\partial}{\partial z} \left( a \dot{j}_{\text{swish},a}[z] + f_{\log}[az](1 - a \dot{j}_{\text{swish},a}[z]) \right)
\]

\[
= \left( \frac{\partial}{\partial z} \left( a \dot{j}_{\text{swish},a}[z] \right) \right)
+ \left( \frac{\partial}{\partial z} f_{\log}[az] \right) (1 - a \dot{j}_{\text{swish},a}[z])
+ f_{\log}[az] \left( \frac{\partial}{\partial z} (1 - a \dot{j}_{\text{swish},a}[z]) \right)
\]

\[
= a \left( \frac{\partial}{\partial z} \dot{j}_{\text{swish},a}[z] \right)
+ a \left( \frac{\partial}{\partial w} f_{\log}[w] \right) \bigg|_{w=az} (1 - a \dot{j}_{\text{swish},a}[z])
+ f_{\log}[az] \left( \frac{\partial}{\partial z} 1 \right) - a f_{\log}[az] \left( \frac{\partial}{\partial z} \dot{j}_{\text{swish},a}[z] \right)
\]

\[
= a \left( a \dot{j}_{\text{swish},a}[z] + f_{\log}[az](1 - a \dot{j}_{\text{swish},a}[z]) \right)
+ a \dot{j}_{\log}[az] \left( 1 - f_{\log}[az] \right) \left( 1 - a \dot{j}_{\text{swish},a}[z] \right)
\]

Lemma A.2 (derivative of \textit{logistic})

\[
= a \left( a \dot{j}_{\text{swish},a}[z] + 2 f_{\log}[az] - 2 \left( f_{\log}[az] \right)^2 - 3 a f_{\log}[az] \dot{j}_{\text{swish},a}[z] \right)
\]

\[
+ 2 a \left( f_{\log}[az] \right)^2 \dot{j}_{\text{swish},a}[z]
\]

\[
= a \left( a \dot{j}_{\text{swish},a}[z] (1 - f_{\log}[az]) \right)
+ 2 f_{\log}[az] \left( 1 - a \dot{j}_{\text{swish},a}[z] \right)
\]

\[
- 2 \left( f_{\log}[az] \right)^2 \left( 1 - a \dot{j}_{\text{swish},a}[z] \right)
\]

\[
= a \left( a \dot{j}_{\text{swish},a}[z] (1 - f_{\log}[az]) \right)
+ 2 f_{\log}[az] \left( 1 - f_{\log}[az] \right) \left( 1 - a \dot{j}_{\text{swish},a}[z] \right)
\]

\[
= a \left( a \dot{j}_{\text{swish},a}[z] + 2 f_{\log}[az] \left( 1 - a \dot{j}_{\text{swish},a}[z] \right) \right) \left( 1 - f_{\log}[az] \right)
\]

summarizing the terms

We can then find further

\[
\ddot{j}_{\text{swish},a}[z] = a \left( 2 \dot{j}_{\text{swish},a}[z] - a \dot{j}_{\text{swish},a}[z] \right) \left( 1 - f_{\log}[az] \right)
\]

above-derived result for \( \ddot{j}_{\text{swish},a} \)
\[ = a \left( \frac{2 \cdot (1 + (1 + az)e^{-az})}{(1 + e^{-az})^2} - a \cdot \frac{ze^{-az}}{1 + e^{-az}} \right) \left( 1 - \frac{1}{1 + e^{-az}} \right) \]

plugging in the explicit forms of \( \hat{f}_{\text{swish},a}, \hat{f}_{\text{swish},a}, \hat{f}_{\text{log}} \)

\[ = a \cdot \frac{2 + 2(1 + az)e^{-az} - az(1 + e^{-az})}{(1 + e^{-az})^2}, \]

combining the summands

\[ = a \cdot \frac{2 - az + (2 + az)e^{-az}}{(1 + e^{-az})^2}, \]

consolidating

\[ = ae^{-az} : \frac{2 - az + (2 + az)e^{-az}}{(1 + e^{-az})^3}, \]

consolidating further

as desired. Detail: The explicit derivation looks as follows:

\[
\hat{f}_{\text{swish},a}[z] = \frac{\partial}{\partial z} \hat{f}_{\text{swish},a}[z]
\]

definition of the second derivative

\[
= \frac{\partial}{\partial z} \left( \frac{1 + (1 + az)e^{-az}}{1 + e^{-az}} \right)
\]

above results

\[
= \frac{\left( \frac{\partial}{\partial z} (1 + az) \right)(1 + e^{-az})^2 - (1 + az)(\frac{\partial}{\partial z} e^{-az}) \left( \frac{\partial}{\partial z} (1 + e^{-az})^2 \right)}{(1 + e^{-az})^4}
\]

quotient rule

\[
= \left( \frac{\partial}{\partial z} (1 + az) \right)e^{-az} \right) \right) \right) (1 + e^{-az})^2 - (1 + az)(\frac{\partial}{\partial z} e^{-az}) \cdot 2(1 + e^{-az})(-ae^{-az})
\]

sum and chain rules

\[
= \frac{(0 + az)e^{-az} + (1 + az)(-ae^{-az}) \right) \right) (1 + e^{-az})^2 + 2ae^{-az}(1 + az)e^{-az})}{(1 + e^{-az})^3} \]

product rule

\[
= (a) = a; \ (\hat{e}^{-az})' = -ae^{-az}; \ sum \ rule
\]

\[
= ae^{-az} : \frac{(1 - (1 + az))(1 + e^{-az}) + 2(1 + az)e^{-az}}{(1 + e^{-az})^3}, \]

consolidating

\[
= ae^{-az} : \frac{-az - 2az + 2ae^{-az} + 2ae^{-az}}{(1 + e^{-az})^3}, \]

consolidating further and expanding

\[
= ae^{-az} : \frac{2 - az + (2 + az)e^{-az}}{(1 + e^{-az})^3}, \]

consolidating

\[ \blacksquare \]

A.8.5 Output Ranges of swish and Its Derivatives

**Lemma A.11** (Output ranges of swish). swish diverges in the limit of large arguments:

\[ \lim_{z \to +\infty} \hat{f}_{\text{swish},a}[z] = \infty \quad \text{for all } a \in [0, \infty). \]
Moreover, the graphs of \swish\ satisfy
\[
\lim_{z \to -\infty} f_{\swish,0}[z] = -\infty \quad \text{and} \quad \lim_{z \to -\infty} f_{\swish,a}[z] = 0
\]
as well as
\[
\arg\min_{z \in \mathbb{R}} f_{\swish,a}[z] = \frac{z_1}{a} \quad \text{and} \quad \min_{z \in \mathbb{R}} f_{\swish,a}[z] = \frac{f_{\swish,1}[z_1]}{a} = \frac{1 + z_1}{a}
\]
for all \( a \in (0, \infty) \) and \( z_1 \in \mathbb{R} \) the unique value that fulfills \( 1 + (1 + z_1)e^{-z_1} = 0 \).

These results highlight that the graph of \swish\ is unbounded from below if the parameter \( a \) equals zero but is bounded from below otherwise. One can verify numerically that \( z_1 \approx -1.278 \) and \( f_{\swish,1}[z_1] \approx -0.278 \).

**Proof of Lemma A.11.** The first three claims follow almost directly from the definition of \swish: for every \( a \in (0, \infty) \), we find
\[
\lim_{z \to +\infty} f_{\swish,a}[z] = \lim_{z \to +\infty} \frac{z}{1 + e^{-az}} = \infty \quad \text{definition of } f_{\swish,a}
\]
and, similarly (use Lemma A.1 in the last line),
\[
\lim_{z \to -\infty} f_{\swish,0}[z] = \lim_{z \to -\infty} \frac{z}{1 + e^{0z}} = -\infty,
\]
and
\[
\lim_{z \to -\infty} f_{\swish,a}[z] = \lim_{z \to -\infty} \frac{z}{1 + e^{-az}} = \lim_{z \to -\infty} \frac{1}{-ae^{-az}} = 0.
\]

Motivated by the fact that \swish\ is twice differentiable (see Lemma A.10), we try to find the minimizers and minima of \( f_{\swish,a} \) with \( a \in (0, \infty) \) by setting its derivatives equal to zero. In other words, we consider the candidates \( z_a \in \{ z \in \mathbb{R} : \dot{f}_{\swish,a}[z] = 0 \} \). We find that
\[
\dot{f}_{\swish,a}[z_a] = 0
\]
\[
\Rightarrow \quad \frac{1 + (1 + az_a)e^{-az_a}}{(1 + e^{-az_a})^2} = 0 \quad \text{Lemma A.10 on the derivatives of } \swish
\]
\[
\Rightarrow \quad 1 + (1 + az_a)e^{-az_a} = 0. \quad (1 + e^{-az_a})^2 > 0
\]
Reparameterizing then yields \( z_a = z_1/a \) (recall the assumption that \( a \neq 0 \)) with
\[
1 + (1 + z_1)e^{-z_1} = 0.
\]
One can verify readily the fact that exactly one such \( z_1 \) exists, that is, \( z_a \) is unique.

We now calculate the function values for \( z_a \). We find for all \( a \in (0, \infty) \)
\[
f_{\swish,a}[z_a] = f_{\swish,a}[z_1/a] \quad \text{previous derivations}
\]
\[
= \frac{z_1/a}{1 + e^{-a(z_1/a)}} \quad \text{definition of } \swish
\]
\[
= \frac{1}{a} \cdot \frac{z_1}{1 + e^{-1/z_1}} \quad \text{simplification}
\]
We find further
\[ f_{\text{swish},1}[z_1] = \frac{z_1}{1 + e^{-z_1}}. \] definition of \textit{swish}
\[ = \frac{z_1}{1 - 1/(1 + z_1)} \quad \text{multiplying numerator and denominator with } 1 + z_1 \]
\[ = \frac{1 + z_1}{1 + z_1 - 1} \quad \text{consolidating} \]
\[ = \frac{z_1}{1 + z_1}. \quad \text{consolidating further} \]

We finally have to verify that \( z_a \) is indeed a minimizer of \( f_{\text{swish},a} \). Since \( f_{\text{swish},a} \) tends to \( \infty \) and 0 in the limits \( z \to \infty \) and \( z \to -\infty \), respectively (see above), it is sufficient to show that \( f_{\text{swish},a}[z_a] < 0 \). In view of the above display, this is equivalent to \( f_{\text{swish},1}[z_1] < 0 \), and this fact can be easily confirmed numerically.

\begin{lemma} (Output ranges of \textit{swish}'s first derivatives) \end{lemma}

The first derivative of \textit{swish} with parameter \( a = 1 \) is constant:
\[ \dot{f}_{\text{swish},0}[z] = \frac{1}{2} \quad \text{for all } z \in \mathbb{R}. \]
In contrast, for all \( a \in (0, \infty) \), the graphs of the first derivatives of \textit{swish} satisfy
\[ \lim_{z \to +\infty} \dot{f}_{\text{swish},a}[z] = 1 \quad \text{and} \quad \lim_{z \to -\infty} \dot{f}_{\text{swish},a}[z] = 0 \]
as well as
\[ \arg \max_{z \in \mathbb{R}} \dot{f}_{\text{swish},a}[z] = - \arg \min_{z \in \mathbb{R}} \dot{f}_{\text{swish},a}[z] = \frac{z_1'}{a} \]
and
\[ \max_{z \in \mathbb{R}} \dot{f}_{\text{swish},a}[z] = 1 - \min_{z \in \mathbb{R}} \dot{f}_{\text{swish},a}[z] = \dot{f}_{\text{swish},1}[z_1'] \]
for \( z_1' \in [0, \infty) \) the unique (nonnegative) value that fulfills \( 2 - z_1' + (2 + z_1')e^{-z_1'} = 0 \).

A numerical evaluation yields \( z_1' \approx 2.218 \) and \( \dot{f}_{\text{swish},1}[z_1'] \approx 1.098 \). The second derivatives can be treated very similarly.

\textit{Proof of Lemma A.12.} The first claim follows readily from Lemma A.10:
\[ \dot{f}_{\text{swish},0}[z] = \frac{1 + (1 + 0 \cdot z)e^{-0 \cdot z}}{(1 + e^{-0 \cdot z})^2} \quad \text{equality for } \dot{f}_{\text{swish},1} \text{ from Lemma A.10} \]
\[ = \frac{1 + (1 + 0) \cdot 1}{(1 + 1)^2} = \frac{1}{2}, \quad \text{simplification} \]
as desired.
The second and third claims follow similarly: for all \( a \in (0, \infty) \), we find
\[
\lim_{z \to +\infty} \dot{\text{swish}}_{\text{swish},a}[z] = \lim_{z \to +\infty} \frac{1+(1+az)e^{-az}}{(1+e^{-az})^2} = \lim_{z \to +\infty} \frac{1+0}{(1+0)^2} = 1 \quad a > 0; (1+az)e^{-az} \to 0; e^{-az} \to 0 \text{ for } z \to +\infty
\]
and
\[
\lim_{z \to -\infty} \dot{\text{swish}}_{\text{swish},a}[z] = \lim_{z \to -\infty} \frac{1+(1+az)e^{-az}}{(1+e^{-az})^2} = \lim_{z \to -\infty} \frac{ae^{-az}}{(e^{-az})^2} = a > 0; (1+az)e^{-az} \ll 1; e^{-az} \gg 1
\]
as desired.

For the last claims, we proceed similarly as in the proof of Lemma A.11. The critical points \( z'_a \in \{z \in \mathbb{R} : \dot{\text{swish}}_{\text{swish},a}[z] = 0\} \) satisfy
\[
\dot{\text{swish}}_{\text{swish},a}[z'_a] = 0 \quad \Rightarrow \quad \frac{ae^{-az'_a}2 - az'_a + (2 + az'_a)e^{-az'_a}}{(1 + e^{-az'_a})^3} = 0 \quad \text{equality for } \dot{\text{swish}}_{\text{swish},a} \text{ from Lemma A.10}
\]
\[
\Rightarrow \quad 2 - az'_a + (2 + az'_a)e^{-az'_a} = 0. \quad \text{equality for } \dot{\text{swish}}_{\text{swish},a} \text{ from Lemma A.10}
\]
Hence, \( z'_a = z'_1/a \) with \( z'_1 \) the unique value that satisfies
\[
2 - z'_1 + (2 + z'_1)e^{-z'_1} = 0.
\]

Observe then that
\[
\min_{z \in \mathbb{R}} \dot{\text{swish}}_{\text{swish},a}[z] = \dot{\text{swish}}_{\text{swish},a}[z'_1/a] = \frac{1+(1+az'_1)/a)e^{-a(z'_1/a)}}{(1+e^{-a(z'_1/a))2}} - 1 \quad \text{derivatives of } \text{swish} \text{ from Lemma A.10}
\]
\[
= 1 + (1 + 1 \cdot z'_1)e^{-1-z'_1} \quad \text{derivatives of } \text{swish} \text{ from Lemma A.10}
\]
\[
= \dot{\text{swish}}_{\text{swish},1}[z'_1].
\]
We find further
\[
\dot{\text{swish}}_{\text{swish},1}[z'_1] = \frac{1+(1+z'_1)e^{-z'_1}}{(1+e^{-z'_1})^2} = \frac{1+(1+z'_1)(z'_1-2)/(2+z'_1)}{(1+(z'_1-2)/(2+z'_1))^2} \quad 2 - z'_1 + (2 + z'_1)e^{-z'_1} = 0 \Rightarrow e^{-z'_1} = (z'_1 - 2)/(2 + z'_1)
\]
\[
= (2 + z'_1)^2 + (2 + z'_1)(1 + z'_1)(z'_1 - 2) \quad \text{multiplying numerator and denominator by } (2 + z'_1)^2
\]
\[
= \frac{(2 + z'_1)^2 + (2 + z'_1)(1 + z'_1)(z'_1 - 2)}{(2 + z'_1)^2 + (z'_1 - 2)^2}
\]
\[
(2 + z_1')^2 + ((z_1')^2 - 4)(1 + z_1') \\
= \frac{4 + 2z_1' + (z_1')^2 + (z_1')^3 + (z_1')^2 - 4z_1' - 4}{4(z_1')^2}
\]

expanding the terms in the numerator

\[
= \frac{-2z_1' + 2(z_1')^2 + (z_1')^3}{4(z_1')^2}
\]

consolidating the numerator

\[
= \frac{-2 + 2z_1' + (z_1')^2}{4z_1'}.
\]

simplifying

We can then proceed similarly as in the proof of Lemma A.11 and use Lemma A.13 about the symmetries of \( \hat{f}_{swish,a} \) to conclude

**Lemma A.13** (Symmetry properties of \( \text{swish} \)'s first derivatives). It holds for all \( z \in \mathbb{R} \) that

\[
2 - z + (2 + z)e^{-z} = 0 \quad \Rightarrow \quad 2 - (-z) + (2 + (-z))e^{-(z)} = 0.
\]

Moreover, it holds for all \( z \in \mathbb{R} \) that

\[
\hat{f}_{swish,a}[-z] = 1 - \hat{f}_{swish,a}[z].
\]

The first statement illustrates that the solution set \( 2 - z_1' + (2 + z_1')e^{-z_1'} = 0 \) over the entire real line is symmetric. The second statement illustrates that the first derivatives of \( \text{swish} \) are symmetric around the function value 0.5. We can conclude, for example, that \( \min_{z \in \mathbb{R}} \hat{f}_{swish,a}[z] = \hat{f}_{swish,1}[-z_1'] \).

**Proof of Lemma A.13.** For the first claim, we find

\[
2 - z + (2 + z)e^{-z} = 0
\]

\[
\Rightarrow \quad (2 + z)e^{-z} = -2 + z \quad \text{adding } -2 + z \text{ on both sides of the equation}
\]

\[
\Rightarrow \quad e^{-z} = -2 + z \quad \text{adding } -2 + z \text{ on both sides of the equation}
\]

\[
\Rightarrow \quad e^z = \frac{2 + z}{-2 + z}
\]

\[
\Rightarrow \quad (-2 + z)e^z = 2 + z \quad \text{using the reciprocals (verify that } z = -2 \text{ is not a solution of the above equality either)}
\]

\[
\Rightarrow \quad -2 - z + (-2 + z)e^z = 0 \quad \text{adding } -2 - z \text{ to both sides}
\]

\[
\Rightarrow \quad 2 + z + (2 - z)e^z = 0 \quad \text{multiplying both sides by } -1
\]

\[
\Rightarrow \quad 2 - (z) + (2 + (z))e^{-(z)} = 0, \quad z = -(-z)
\]

as desired.

For the second claim, we find

\[
\hat{f}_{swish,a}[-z] = \frac{1 + (1 + a(-z))e^{-a(-z)}}{(1 + e^{-a(-z)})^2} \quad \text{Lemma A.10 for } \hat{f}_{swish,a}
\]
\[ \begin{align*}
    &= \frac{1 + (1 - az)e^{az}}{(1 + e^{az})^2} \\
    &= \frac{(1 + e^{az})^2 - (1 + e^{az})^2 + 1 + (1 - az)e^{az}}{(1 + e^{az})^2} \\
    &= 1 - \frac{(1 + e^{az})^2 - 1 - (1 - az)e^{az}}{(1 + e^{az})^2} \\
    &= 1 - \frac{1 + 2e^{az} + e^{2az} - 1 - e^{az} + az e^{az}}{(1 + e^{az})^2} \\
    &= 1 - \frac{e^{2az} + (1 + az)e^{az}}{(1 + e^{az})^2} \\
    &= 1 - \frac{1 + (1 + az)e^{-az}}{e^{-2az}(1 + e^{az})^2} \\
    &= 1 - \frac{1 + (1 + az)e^{-az}}{(1 + e^{-az})^2} \\
    &= 1 - \hat{f}_{\text{swish, } a}[z],
\end{align*} \]

as desired. \( \square \)