E-cordial and product e-cordial labeling for the extended duplicate graph of splitting graph of path

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Abstract. Based on the works of Yilmaz and Cahit on E-cordial labeling, we prove the existence of E-cordial labeling, total E-cordial labeling, product E-cordial labeling and total product E-cordial labeling for the extended duplicate graph of splitting graph of path \( P_m \).

1. Introduction
One of the most famous and productive labeling of graph theory is cordial labeling. This labeling was introduced by Cahit in the year 1987. For an extensive survey on graph labeling, we refer to Gallian [5]. E.Samphthkumar [3,4] introduced the concept of duplicate graph and splitting graph. In 1997, Yilmaz and Cahit have introduced a weaker version of edge-graceful called E-cordial[8]. B.Selvam, K.Thirusangu, and P.P. Ulaganathan have introduced the idea of extended duplicate graph of twig graphs and they proved that the EDG of twig graphs is E-cordial and product E-cordial [1,9]. E.Bala and K.Thirusangu have studied on E-Cordial labeling for Competition graph[2]. R.Avudainayaki, et.al., have proved that the Prime Cordial and Signed Product Cordial Labeling for the Extended Duplicate Graph of Arrow Graph [7]. Vaidya and Barasara proposed edge product cordial labeling [10]. P. Lawrence Rozario Raj and S. Koilraj have proved that cordial labeling for the splitting graph of some standard graphs [6].

2. Preliminaries
First, we will give brief summary of definitions which are useful for the present investigations.

Definition 1: each vertex \( v \) of a graph \( G \), take a new vertex \( v' \). Join \( v' \) to all the vertices of \( G \) adjacent to \( v \). The graph \( \text{Spl}(G) \) thus obtained is called splitting graph of \( G \).

Definition 2: Let \( G (V,E) \) be a graph. A duplicate graph of \( G \) is \( DG(V_1, E_1) \) where the vertex set \( V_1 = V \cup V' \) and \( V \cap V' = \phi \) and \( f : V \rightarrow V' \) is bijective and the edge set \( E_1 \) of \( DG \) is defined as the edge \( ab \) is in \( E \) if and only if both \( ab' \) and \( a'b \) are edges in \( E_1 \).
Definition 3: Extended duplicate graph of splitting graph of path is obtained by adding the edge \(v_2 v_2'\) to the duplicate graph. It is denoted by \(EDG(Spl(P_m))\) and it has 4m vertices and 6m-5 edges, where \(m \geq 2\) is the number of length.

Illustration 1: EDG of Splitting graph of Path \(P_6\)

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Definition 4: Let \(G\) be a graph and \(f : E \rightarrow \{0,1\}\). The induced function \(f^*\) on \(V\) by defining \(f^*(v) = \{\sum f(v_j)\} \mod 2\) where \(v, v_j \in E\). Let \(m_i(f) = \{e \in E / f(e) = i\}\) and \(n_i(f) = \{v \in V / f^*(v) = i\}\). The function \(f\) is called an \(E\)-cordial labeling of \(G\) if \(|m_i(f) - m_j(f)| \leq 1\) and \(|n_i(f) - n_j(f)| \leq 1\), \((i \neq j)\).

Definition 5: An \(E\)-cordial labeling \(f\) is called total \(E\)-cordial labeling of \(G\) if \(|\{m_i(f) + n_i(f)\} - \{m_j(f) + n_j(f)\}| \leq 1\).

Definition 6: Let \(G\) be a graph and \(f : E \rightarrow \{0,1\}\). The induced function \(f^*\) on \(V\) by defining \(f^*(v) = \{\sum f(v_j)\} \mod 2\) where \(v, v_j \in E\). Let \(m_i(f) = \{e \in E / f(e) = i\}\) and \(n_i(f) = \{v \in V / f^*(v) = i\}\). The function \(f\) is called an Product \(E\)-cordial labeling of \(G\) if \(|m_i(f) - m_j(f)| \leq 1\) and \(|n_i(f) - n_j(f)| \leq 1\), \((i \neq j)\).

Definition 7: An \(E\)-cordial labeling \(f\) is called total product \(E\)-cordial labeling of \(G\) if \(|\{m_i(f) + n_i(f)\} - \{m_j(f) + n_j(f)\}| \leq 1\).

3. Main Results

3.1. E- cordial labeling

Here, we prove the existence of \(E\)-cordial and total \(E\)-cordial for \(EDG(Spl(P_m))\), \(m \geq 2\).

Theorem 1: For \(m \geq 2\), \(EDG(Spl(P_m))\) is \(E\)-cordial for \(m\) is even.

Proof: Let \(Spl(P_m)\), \(m \geq 2\) be a splitting path graph. Let \(EDG(Spl(P_m))\), \(m \geq 2\) be an extended duplicate graph of splitting path graph. To label the edges \(f : E \rightarrow \{0,1\}\) as given algorithm 1.

- The edges \(e_1, e_2, e_3, e_1', e_2', e_3\) and \(e_{3m+2}\) receive label ‘0’, ‘1’, ‘0’, ‘1’, ‘0’ and ‘1’ respectively;
- The edges \(e_{4+6i+j}\) receive label ‘0’ for \(0 \leq i \leq [(m-3)/2]\) and \(0 \leq j \leq 2\);
- The edges \(e_{7+6i+j}\) receive label ‘1’ for \(0 \leq i \leq [(m-4)/2]\) and \(0 \leq j \leq 2\);
- The edges \(e_3', e_{6+i}\) receive label ‘1’ for \(0 \leq i \leq [(m-2)/2]\);
- The edges \(e'_{6+i}\) receive label ‘0’ for \(0 \leq i \leq [(m-3)/2]\);
The edges $e'_{4+6i+j}$ receive label ‘1’ for $0 \leq i \leq \lfloor (m-3)/2 \rfloor$ and $0 \leq j \leq 1$;
The edges $e'_{7+6i+j}$ receive label ‘0’ for $0 \leq i \leq \lfloor (m-4)/2 \rfloor$ and $0 \leq j \leq 1$.
When $m$ is odd, 3m-2 edges receive label ‘0’ and 3m-3 edges receive label ‘1’ and when $m$ is even, 3m-2 edges receive label ‘1’ and 3m-3 edges receive label ‘0’.
In both the cases differ by at most one and satisfies the required condition.
Thus the entire 6m-5 edges are labeled.
The induced function $f^*$ on $V$ defined by
$$f^*(v) = \{ \sum f(u,v) \mid uv \in E \} \pmod{2}.$$  
The vertices are labeled as follows:
The vertices $v_1, v_2, v_1'$ and $v_2'$ receive label ‘1’ and the vertices $v_2', v_3$ and $v_3'$ receive label ‘0’;
When $m$ is even, the vertices $v_4,v_6,\ldots,v_{2m}$ receive label ‘0’, the vertices $v_5,v_7,\ldots,v_{2m-1}$ receive label ‘1’,
the vertices $v_6,v_8,\ldots,v_{2m}$’ receive label ‘1’, the vertices $v_5',v_7',\ldots,v_{2m-1}'$ receive label ‘0’.
When $m$ is odd, the vertices $v_4,v_6,\ldots,v_{2m-2}$ receive label ‘0’; the vertices $v_5,v_7,\ldots,v_{2m-1}$ receive label ‘1’;
the vertices $v_2,v_4,\ldots,v_{2m-2}$ receive label ‘1’; the vertices $v_3,v_5,\ldots,v_{2m-1}'$ receive label ‘0’ and the vertices $v_{2m-1}'$ receive label ‘0’.
The entire 4m vertices are labeled such that 2m vertices receive label ‘1’ and 2m vertices receive label ‘0’ differ by at most one and satisfies the required condition. Thus EDG(Spl(P,m)), $m \geq 2$ admits E- cordial labeling for $m$ is even.

**Theorem 2:** For $m \geq 2$, EDG(Spl(P,m)) is total E- cordial for $m$ is even.

**Proof:** In theorem 1, 2m vertices are allotted ‘0’ and 2m vertices are allotted the label ‘1’. When $m$ is odd, 3m-2 edges labeled with ‘0’ and 3m-3 edges labeled with ‘1’ and when $m$ is even, 3m-2 edges labeled with ‘1’ and 3m-3 edges labeled with ‘0’. In both cases, we see that the number of vertices and edges labeled with ‘1’ is $2m + 3m-3 = 5m-3$ and the number of vertices and edges labeled with ‘0’ is $2m+3m-2 = 5m-2$ differ by one and satisfies the required condition. Thus EDG (Spl(P,m)), $m \geq 2$ admits total E- cordial labeling for $m$ is even.

**Illustration 2:** E-CORDIAL LABELING FOR EDG OF SPLITTING GRAPH OF PATH
3.2. Product e-cordial

Here, we prove the existence of product E-cordial and total product E-cordial for \( EDG(\text{Spl}(P_m)) \), \( m \geq 2 \).

**Theorem 3**: For \( m \geq 2 \), \( EDG(\text{Spl}(P_m)) \) is product E-cordial.

**Proof**: Let \( \text{Spl}(P_m) \), \( m \geq 2 \) be a splitting path graph. Let \( EDG(\text{Spl}(P_m)) \), \( m \geq 2 \) be a extended duplicate graph of splitting path graph. To label the edges \( f : E \rightarrow \{0,1\} \) as given algorithm 2.

- The edge \( e = 3m-2 \) receive label ‘1’;
- The edges \( e_{1+6i+j} \) receive label ‘1’ and the edges \( e'_{1+6i+j} \) receive label ‘0’ for \( 0 \leq i \leq [(m-2)/2] \) and \( 0 \leq j \leq 2 \);
- The edges \( e_{4+6i+j} \) receive label ‘0’ and the edges \( e'_{4+6i+j} \) receive label ‘1’ for \( 0 \leq i \leq [(m-3)/2] \) and \( 0 \leq j \leq 2 \).

When \( m \) is odd, \( 3m-2 \) edges receive label ‘0’ and \( 3m-3 \) edges receive label ‘1’ which differ by at most one and when \( m \) is even, \( 3m-2 \) edges receive label ‘1’ and \( 3m-3 \) edges receive label ‘0’.

In both the cases which differ by at most one and satisfies the required condition. Thus the entire 6m-5 edges are labeled.

The induced function \( f^* \) on \( V \) defined by

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f^*(v) = \{ \prod f(u,v) \mid uv \in E \} \pmod{2}.
\]

The vertices are labeled as follows:

- When \( m \) is even, the vertices \( v_3, v_7, v_{11}, \ldots, v_{2m-1} \) and \( v_4, v_8, v_{12}, \ldots, v_{2m} \) receive label ‘0’; the vertices \( v_1, v_5, v_9, \ldots, v_{2m-3} \) and the vertices \( v_2, v_6, v_{10}, \ldots, v_{2m-2} \) receive label ‘1’.
- When \( m \) is odd, the vertices \( v_4, v_6, \ldots, v_{2m-2} \) receive label ‘0’; the vertices \( v_5, v_7, \ldots, v_{2m-1} \) receive label ‘1’; the vertices \( v_4', v_6', \ldots, v_{2m-2}' \) receive label ‘1’; the vertices \( v_5', v_7', \ldots, v_{2m-1}' \) receive label ‘0’; the vertices \( v_2m \) receive label ‘1’.

Thus the entire 4m vertices are labeled such that 2m vertices receive label ‘1’ and 2m vertices receive label ‘0’ differ by at most one and satisfies the required condition.

Hence \( EDG(\text{Spl}(P_m)) \), \( m \geq 2 \) admits product E-cordial labeling.
Illustration 3 : PRODUCT E-CORDIAL LABELING FOR EDG OF SPLITTING GRAPH OF PATH

Theorem 4 : For m ≥ 2 , EDG(Spl(P_m)), m ≥ 2  is total product E-cordial.

Proof : In theorem 3, 2m vertices are allotted ‘0’ and 2m vertices are allotted ‘1’. When m is odd, number of edges labeled with ‘0’ is 3m-2 and number of edges labeled with ‘1’ is 3m-3. When m is even, number of edges labeled with ‘1’ is 3m-2 and number of edges labeled with ‘0’ is 3m-3. In both cases, we see that the number of vertices and edges labeled with ‘1’ is 2m + 3m-3 = 5m-3 and the number of vertices and edges labeled with ‘0’ is 2m+(3m-2) = 5m-2, differ by one and satisfies the required condition. Thus EDG (Spl(P_m)), m ≥ 2 admits total product E-cordial labeling.

4. Conclusion
We have shown that EDG (Spl(P_m)) , m ≥ 2 is E-cordial, total E-cordial, product E-cordial and total product E-cordial. In future it would be interesting to extend the different type of graphs and its possible labeling for EDG (Spl(P_m)).

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