Online TSP with Predictions

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Abstract

We initiate the study of online routing problems with predictions, inspired by recent exciting results in the area of learning-augmented algorithms. A learning-augmented online algorithm which incorporates predictions in a black-box manner to outperform existing algorithms if the predictions are accurate while otherwise maintaining theoretical guarantees is a popular framework for overcoming pessimistic worst-case competitive analysis.

In this study, we particularly begin investigating the classical online traveling salesman problem (OLTSP), where future requests are augmented with predictions. Unlike the prediction models in other previous studies, each actual request in the OLTSP, associated with its arrival time and position, may not coincide with the predicted ones, which, as imagined, leads to a troublesome situation. Our main result is to study different prediction models and design algorithms to improve the best-known results in the different settings. Moreover, we generalize the proposed results to the online dial-a-ride problem.

2012 ACM Subject Classification Theory of computation → Online algorithms; Theory of computation

Keywords and phrases traveling salesman problem, online algorithms, competitive ratio

Digital Object Identifier 10.4230/LIPIcs...00
Introduction

In many applications, people make decisions without knowing the future, and two approaches are widely used to address this issue: machine learning and online algorithms. While machine learning methods usually lack theoretical guarantees, online algorithms are often evaluated based on the ratio of the optimal offline cost to the cost achieved by online algorithms over worst-case instances (i.e., the competitive ratio).

In recent years, a rapidly growing field of research known as learning augmentation has attempted to merge the above two approaches: incorporating machine-learned predictions into online algorithms from a theoretical point of view. In this line of work, such an algorithm is given some type of prediction to the input, but the prediction is not always accurate. The goal of these studies is to design novel algorithms that have the following three properties: (1) achieve good performance when the given prediction is perfectly accurate, which is called consistency, (2) maintain worst-case bounds when the prediction is terribly wrong, which is called robustness, and (3) the performance should not degrade significantly when the prediction is slightly inaccurate, which is called smoothness. As the competitive ratio of a learning-augmented algorithm depends on errors, we define it as a function $c$ of error $\varepsilon$ using a given predictor such that $c(\varepsilon)$ represents the worst-case competitive ratio given that the error is at most $\varepsilon$. Note that the definition of $\varepsilon$ is subject to the problem setting (as shown in Section 2). Optimization problems that have been studied under this framework include online paging [15, 18, 22, 24], ski-rental [11, 12, 21, 25], scheduling [7, 13, 16, 21, 25], matching [4], bin packing [2], queueing [20], secretary [9, 10], covering [8], bidding [19], knapsack [14], facility location [11], and graph [6] problems.

However, in addition to the above online optimization problems, no existing studies extended the learning-augmented framework to online routing problems, accompanied by many real-world applications. For example, due to COVID-19, people have become more reliant on online food delivery platforms such as Seamless and Uber Eats. As a result, optimizing the delivery route is an important issue for the platform to win over such a huge market. Many platforms have used machine learning techniques to improve their services, but most studies did not provide worst-case guarantees. In this study, we look at how the learning-augmented framework can be applied to online routing problems. Note that previous research in online learning also studied the combination of online algorithms and machine learning for routing problems; readers may refer to the survey [9] for more details. However, our study aims at designing a route with predictions which learned from historical information rather than choosing the best possible strategy adaptively based on some given routes. In particular, we study the online TSP problem proposed by Ausiello et al. [5] and then extend to the online dial-a-ride problem.

Let the offline optimal cost and the cost achieved by an online algorithm be denoted by $Z^{OPT}$ and $Z^{ALG}$, respectively. To measure the performance of an learning-augmented online algorithm, we follow the definitions proposed in [18, 21].

▶ Definition 1 (Consistency). An algorithm is said to be $\alpha$-consistent if $c(0) = \alpha$.

▶ Definition 2 (Robustness). An algorithm is said to be $\beta$-robust if $c(\varepsilon) \leq \beta$ for any $\varepsilon$.

▶ Definition 3 (Smoothness). Given an $\alpha$-consistent algorithm, it is said to be $f(\varepsilon)$-smooth if $Z^{ALG} \leq \alpha \cdot Z^{OPT} + f(\varepsilon)$ holds on any input for some continuous function $f$ such that $f(0) = 0$.

The OLTSP. In this study, we first consider one of the classical online combinatorial optimization problems, the online traveling salesman problem (OLTSP), in which the input is
a sequence of requests that arrive in an online fashion. A salesman, the server, is out to visit every request after it arrives and eventually return to the origin such that the completion time is minimized. Ausiello et.al. [5] showed that the problem has a lower bound of 2 and presented an optimal 2-competitive algorithm, called Plan-At-Home (PAH). Meanwhile, the currently best polynomial-time algorithm is approximately 2.65-competitive [4].

The OLDARP. We also consider the (uncapacitated) online dial-a-ride problem (OLDARP), which is a generalization of the OLTSP and has the same lower bound of 2. The key difference between the two problems is that a server needs to transport each request from its pickup position to its delivery position. An optimal 2-competitive algorithm was proposed in [4].

The OLTSP with Predictions. The first question for developing learning-augmented algorithms is: “what kind of predictions are required for the problem?” We refer to the discussion in [16] to answer this question: a good prediction should be learnable; that is, it should be possible to learn the prediction from historical data with an appropriate sample complexity. For the OLTSP, it is natural to forecast its future requests. To show the learnability, one can set up a learning task and exploit some different features of historical data, e.g., position of the requests, arrival time of the requests, and the number of total requests, to learn a function; for instance, to minimize the mean square error (mse). In this study, we thus investigate three models with different types of predictions. Notice that the significant difference between these predictions is whether the input scale is known for learning. In the first two models, intuitively, we predict the whole sequence. Formally, each request in the prediction has its own predicted arrival time and position. We split into two cases: the sequence prediction without identity is a predicted sequence with an arbitrary size over requests, and the sequence prediction with identity is a predicted sequence that contains the exact same number of requests as the actual input. These two sequence predictions represent different learning models; the former one views the sequence of requests as a group, while the latter one examines each of the requests individually. Last, we consider a special prediction model, called the prediction of the last arrival time, where the server receives the least amount of information in the three models. The reason of choosing such a prediction is that the arrival time of the last request actually provides a lower bound for the optimal route and that this information is enough to help the server in a polynomial setting. In the following sections, we show how to design learning-augmented online algorithms by appropriately incorporating the predictions into online algorithms. As expected, predicting a sequence of requests provides more information about future events, certainly leading to a relatively higher cost than predicting a single request. That is, the choice of predictions can be read as a trade-off between the performance of online algorithms and the sample complexity.

Our Contributions. The key contribution is to develop three models for the OLTSP in each of which we present a learning-augmented algorithm. The results are as follows:

1. Consider the sequence prediction without identity. For this arbitrary sequence case, we design an algorithm that is \((1 + \lambda)\)-consistent and \((3 + 2/\lambda)\)-robust, where \(\lambda \in (0, 1]\) is a parameter describing the confidence level in the prediction.

2. Consider a restricted model in which the sequence prediction has the same size as the set of actual requests. We propose a \(\min\{3, 1 + (2\varepsilon_{\text{time}} + 4\varepsilon_{\text{pos}})/Z^{\text{OPT}}\}\)-competitive algorithm, where \(\varepsilon_{\text{time}} = \max_{i \in [n]} |t_i - \hat{t}_i|\) is the maximum time difference between the predicted and actual requests and \(\varepsilon_{\text{pos}} = \sum_i d(\hat{p}_i, p_i)\) is the accumulated distance between the predicted and actual positions.
3. Consider the prediction of the last arrival time. We design a \(\min\{4, 2.5 + |\epsilon_{last}|/Z^{OPT}\}\)-competitive polynomial-time algorithm, where \(\epsilon_{last} = t_n - t_n\) denotes the difference between the predicted and actual arrival time of the last request.

We also study the lower bounds for the OLTSP with predictions:

1. Consider the sequence prediction without identity. Any 1-consistent algorithm has no robustness. We show there is a trade-off between consistency and robustness.

2. Consider the sequence prediction with identity. Any 1-consistent algorithm has a robustness that is at least 2.

3. Consider the prediction of the last arrival time. We show that the consistency cannot be better than 2; that is, partial information is not enough for breaking the original lower bound of the OLTSP.

In addition, we extend our models to the online dial-a-ride problem.

2 Preliminaries

In this section, we first give the formal definition of the OLTSP, and revisit the PAH algorithm, a key ingredient for designing our algorithms. Finally, we define error measurement for the three models.

2.1 The OLTSP

Recall the metric OLTSP in which the input is a sequence of \(n\) requests \(x = (x_1, \ldots, x_n)\) in a metric space \(M\). Each request \(x_i \in x\) can be represented by \(x_i = (t_i, p_i)\), where \(t_i\) denotes the arrival time and \(p_i\) denotes the position of the request. In addition, we assume the requests are in non-decreasing order of arrival time; that is, \(t_i \leq t_j\) for any \(i < j\), and \(t_n\) is the arrival time of the last request in \(x\). A server is required to start at the origin \(o\) at time 0, visit each request \(x_i\) in \(x\) at position \(p_i\) no earlier than its arrival time \(t_i\), and at last come back to the origin \(o\), which is also called home. Assuming the server moves with unit speed, the goal is to find a route \(T_x\) such that the completion time, denoted by \(|T_x|\), is minimized.

The PAH Algorithm [5]. Before introducing the proposed online algorithms with predictions, we first revisit the online algorithm, PLAN-AT-HOME (PAH) for the OLTSP in a metric space, reported in [5], and see how we can attain the robustness of our algorithms. The greedy PAH algorithm achieves a competitive ratio of 2 when assuming the server can access an optimal solution to visit a set of released requests. Although the optimal route cannot be computed in polynomial time unless \(P = NP\), the PAH algorithm is 3-competitive by using Christofides’ heuristic [17] to approximate the optimal route. The details of the PAH algorithm can be referred to the Appendix A.

▶ Theorem 4 ([5], Theorem 4.2). The PAH algorithm is 2-competitive for the OLTSP in a metric space.

▶ Theorem 5 ([5], Theorem 5.3). The PAH algorithm is a 3-competitive polynomial-time algorithm for the OLTSP in a metric space when using Christofides’ heuristic.

The overview of the PAH algorithm has the following two operations: (1) find a route only when the server is at the origin \(o\), and (2) act differently in response to the requests that are relatively close to the origin and those that are relatively far. At moment \(t\) during the execution of the algorithm, let \(p(t)\) denote the position of the server and \(U_t\) denote the set comprising every unserved request that has been present so far. Precisely, if the server is at
the origin, i.e., \( p(t) = o \), it starts to follow an optimal (or approximate) route \( T_{U_t} \), which visits all released unerved requests in \( U_t \) and returns to the origin \( o \). Otherwise, when the server is on a route \( T_{U_t'} \), where \( t' < t \) and \( t' \) denotes the moment when the route was designed, and a new request \( x_i = (t_i, p_i) \) arrives, the server moves depending on the relationship between \( d(p_i, o) \) and \( d(p(t), o) \), where \( d(a, b) \) denotes the shortest distance between points \( a \) and \( b \). If \( d(p_i, o) > d(p(t), o) \), request \( x_i \) is considered far from the origin, and the server terminates the route \( T_{U_t'} \) and goes back to the origin \( o \) directly. Otherwise, request \( x_i \) is considered close to \( o \), and thus the server ignores \( x_i \) until it is back home and continues with the current route \( T_{U_t'} \).

In addition, we can derive Corollary 6 from the proof of PAH in [5].

\[ \text{Corollary 6.} \quad \text{Assume an algorithm asks the server to wait at the origin} \ o \ \text{for time} \ t \ \text{before following the PAH algorithm where} \ t \leq t_n. \ \text{The algorithm is 2-competitive if the server has access to the offline optimal route to serve a set of released requests. Also, it is 3-competitive using Christofides’ algorithm.} \]

2.2 Prediction Models and Error Measure

In this section, we present three different models of prediction. First, we consider predicting a sequence of requests \( \hat{x} = (\hat{x}_1, \ldots, \hat{x}_m) \), where \( m \) denotes the number of predicted requests in the sequence. As the server is unaware of the size of \( x \) in the beginning, we assume that the learning model forecasts the number of requests before predicting the arrival time and positions of all requests. Formally, each request in the prediction is defined by \( \hat{x}_i = (\hat{t}_i, \hat{p}_i) \), where \( \hat{t}_i \) and \( \hat{p}_i \) are its predicted arrival time and position. As we need to compare two request sequences with different sizes, \( n \) and \( m \), apparently a proper definition of prediction errors does not exist. Consequently, we only care about whether any error exists. That is, we simply distinguish whether \( \hat{x} = x \) or not. In this model, we can design an algorithm that is consist and robust but not smooth.

Next, we consider a restricted version in which the number of requests \( n \) is given, implying that the prediction has the same size as the actual request sequence, i.e., \( |\hat{x}| = |x| = n \). We thus assume that there is a one-to-one correspondence between the requests in the two sequences \( \hat{x} \) and \( x \). That is, each predicted request \( \hat{x}_i \) is exactly paired with the actual request \( x_i \). Since the prediction gives us both the arrival time and position, we define errors with respect to the two parameters. We first define the time error, denoted by \( \varepsilon_{\text{time}} \), as the maximum difference between the arrival time of a predicted request and its corresponding actual request.

\[ \varepsilon_{\text{time}} := \max_{i \in [n]} |\hat{t}_i - t_i| \]

Then, we define the position error, denoted by \( \varepsilon_{\text{pos}} \), as the sum of distances between the positions of the predicted and actual requests.

\[ \varepsilon_{\text{pos}} := \sum_{i=1}^{n} d(\hat{p}_i, p_i) \]

In this restricted model, we extend the above algorithm to a new one which can improve the result and satisfy the smoothness requirement.

Last, we consider the prediction of the arrival time of the last request, denoted by \( \hat{t}_{\text{last}} \). We define the error of the prediction, denoted by \( \varepsilon_{\text{last}} \), to be the difference between the predicted and the actual arrival time of the last request:

\[ \varepsilon_{\text{last}} := \hat{t}_n - t_n. \]
Online TSP with Predictions

The motivation of making such a prediction is that the arrival time of the last request provides a guess of the lower bound for the optimal route. Later we show how to combine this prediction with the PAH algorithm and design a learning-augmented algorithm with consistency, robustness and smoothness. Furthermore, we also show the limit of this prediction, i.e., lower bound results.

Note that \( \hat{x} \) reveals much more information compared to \( \hat{t} \), so that the first two prediction models would be expected to help achieve better performance than the last model, assuming the prediction is perfect. However, when the prediction is not good, we must carefully control both \( \varepsilon_{\text{time}} \) and \( \varepsilon_{\text{pos}} \) to avoid paying too much extra cost.

Here we remark that very recently Azar et al. \cite{6} proposed a framework for designing learning-augmented online algorithms for some graph problems such as Steiner tree/forest, facility location, etc. They also presented a novel definition of prediction errors. However, in contrast, online routing has to carefully cope with the predictions of arrival time of future requests. Therefore, the notion reported in \cite{6} may not be directly applied to this study.

Predict a Sequence of Requests

Given a predicted sequence of requests \( \hat{x} \), we first discuss the intuition behind the two models: prediction without identity and prediction with identity. From the perspective of machine learning, the former model forecasts by viewing the sequence as a whole, while the latter one makes predictions based on the features of each request individually. Note that both models acquire the entire sequence prediction \( \hat{x} \) in the beginning. Next, we present two algorithms: LAR-NID and LAR-ID, respectively, where the former one has consistency and robustness, and the latter one can even additionally gain smoothness by knowing the number of requests.

We remark that, analogous to the discussion in \cite{5}, one can disregard the computational complexity of online algorithms and assume that the server has access to the offline optimal route for a sequence of released requests. This is similar to the assumption in \cite{5} that the server is given the optimal solution to visit requests without time constraints.

Sequence Prediction without Identity

In this model, the server makes a prediction \( \hat{x} \) comprising \( m \) predicted requests. We present the LAR-NID algorithm, which is \((1.5 + \lambda)\)-consistent and \((3 + 2/\lambda)\)-robust for \( \lambda \in (0, 1] \).

A naïve idea is that the server finds an optimal route \( \hat{T} \) for the prediction \( \hat{x} \) at time 0 and just follows the route \( \hat{T} \) directly. It is clearly a 1-consistent algorithm, which, however, may result in an arbitrarily bad robustness.

\[ \text{Theorem 7.} \quad \text{Given a sequence of predicted requests without identity, any 1-consistent algorithm has robustness of at least } 1/\delta \text{ for any } \delta \in (0, 1). \]

The brief idea of the LAR-NID algorithm is that the server finds an optimal route \( \hat{T} \) for the prediction \( \hat{x} \) at time 0 and follows the route based on a modified framework of PAH, which can guarantee a lower bound for \( Z^{\text{OPT}} \). Precisely, a server operated by LAR-NID is given a route \( \hat{T} \) which serves the requests in the prediction \( \hat{x} \) with the confidence level to the prediction, denoted by \( \lambda \), in the beginning. To obtain certain robustness, we set a condition to see whether the current moment \( t \) is earlier than \( \lambda|\hat{T}| \) or not. If \( t < \lambda|\hat{T}| \), we follow the PAH algorithm, unless the route \( T_{U_t} \) is too long (i.e., \( t + |T_{U_t}| > \lambda|\hat{T}| \)), where \( T_{U_t} \) denotes an optimal route that starts serving the set of released unserved requests \( U_t \) at time \( t \); that is, we adjust the route \( T_{U_t} \) by asking the server to return home and arrive at the origin \( o \) exactly at time \( \lambda|\hat{T}| \). By adding such a gadget, we can ensure good robustness. Otherwise, if
$t \geq \lambda |\hat{T}|$, the server gets to follow the predicted route $\hat{T}$ once there are unserved requests (i.e., $U_t \neq \emptyset$). Finally, we use the PAH algorithm to serve the remaining requests in $x \setminus \hat{x}$. Note that we tend to set the parameter $\lambda$ to a small value if we believe the quality of the prediction is good.

In particular, next, we show that LAR-NID can still be robust even when the prediction error is not properly defined. First, we prove it is feasible to add the gadget.

Lemma 8. Given that the server follows the LAR-NID algorithm, when time $t$ satisfies the condition: $t < \lambda |\hat{T}|$ and $t + |U_t| > \lambda |\hat{T}|$, there exist a moment $t_{back}$, $t < t_{back} < t + |U_t|$, such that $t_{back} + d(p(t_{back}), o) = \lambda |\hat{T}|$.

Algorithm 1 LEARNING-AUGMENTED ROUTING WITHOUT IDENTITY (LAR-NID)

Input: The current time $t$, a sequence prediction $\hat{x}$, the confidence level $\lambda \in (0, 1]$, and the set of current released unserved requests $U_t$.

1. First, compute an optimal route $\hat{T}$ to serve the requests in $\hat{x}$ and return to the origin $o$;
2. While $t < \lambda |\hat{T}|$ do
3. If the server is at the origin $o$ (i.e., $p(t) = o$.) then
4. Compute an optimal route $T_{U_t}$ to serve all the unserved requests in $U_t$ and return to the origin $o$;
5. If $t + |U_t| > \lambda |\hat{T}|$ then \footnote{Add a gadget}
6. Find the moment $t_{back}$ such that $t_{back} + d(p(t_{back}), o) = \lambda |\hat{T}|$;
7. Redesign a route $T_{U_t}^r$ by asking the server to go back to the origin $o$ at time $t_{back}$ along the shortest path;
8. Start to follow the route $T_{U_t}^r$;
9. Else
10. Start to follow the route $T_{U_t}$;
11. EndIf
12. ElseIf the server is currently moving along a route $T_{U_t'}$, for some $t' < t$ then
13. If a new request $x_i = (t_i, p_i)$ arrives \footnote{Similar to PAH}
14. If $d(p_i, o) > d(p(t), o)$ then
15. Go back to the origin $o$.
16. Else
17. Move ahead on the current route $T_{U_t'}$;
18. EndIf
19. EndIf
20. EndIf
21. EndWhile
22. While $t \geq \lambda |\hat{T}|$ then
23. Wait until $U_t \neq \emptyset$;
24. Follow the route $\hat{T}$ until the server is back to the origin $o$;
25. Follow PAH $(t, U_t)$. \footnote{Serve the remaining requests}
26. EndWhile

Lemma 9. The LAR-NID algorithm is $(1.5 + \lambda)$-consistent, where $\lambda \in (0, 1]$.

Lemma 10. The LAR-NID algorithm is $(3 + 2/\lambda)$-robust, where $\lambda \in (0, 1]$.

Theorem 11. The LAR-NID algorithm is $(1.5 + \lambda)$-consistent and $(3 + 2/\lambda)$-robust but not smooth, where $\lambda \in (0, 1]$ is the confidence level.
3.2 Sequence Prediction with Identity

In contrast to the previous model, we consider having access to the number of requests, i.e., \( n \). Given a sequence prediction, \( \hat{x} \) with the size of \( n \), we first show the limitation of this stronger prediction.

**Theorem 12.** Given a sequence of predicted requests with identity, any 1-consistent algorithm has robustness at least 2.

We first consider a naive algorithm, LAR-Trust, and show that it is consistent and smooth but not robust where the details can be founded in the Appendix B.2. However, we observe that the arrival time of the last request \( t_n \) gives a lower bound of the optimal solution, \( Z^{OPT} \). As a result, we modify the LAR-Trust algorithm and propose the LAR-ID algorithm, which is 1-consistent, 3-robust and \((2\epsilon_{\text{time}} + 4\epsilon_{\text{pos}})\)-smooth.

**Algorithm 2** Learning-Augmented Routing Trust (LAR-Trust)

**Input:** The current time \( t \), the number of requests \( n \), a sequence prediction \( \hat{x} \), and the set of current released unserved requests \( U_t \).

1. Compute an optimal route \( \hat{T} = (\hat{x}_{(1)}, \ldots, \hat{x}_{(n)}) \) to serve the requests in \( \hat{x} \) and return to the origin \( o \), where \( \hat{x}_{(i)} = (\hat{t}_{(i)}, \hat{p}_{(i)}) \) denotes the \( i^{th} \) predicted request in \( \hat{T} \).
2. Start to follow the route \( \hat{T} \).
3. For any \( i = 1, \ldots, n \) do
4. \[ \text{If } t = \hat{t}_{(i)} \text{ then} \]
5. \[ \text{Update the route } \hat{T} \text{ by adding the request } x_{(i)} \text{ after the predicted request } \hat{x}_{(i)}; \]
6. \[ \text{EndIf} \]
7. \[ \text{If } p(t) = \hat{p}_{(i)} \text{ and } t < \hat{t}_{(i)} \text{ then} \]
8. \[ \text{Wait at position } \hat{p}_{(i)} \text{ until time } \hat{t}_{(i)}. \] \[ \text{Wait until the request arrives} \]
9. \[ \text{EndIf} \]
10. EndFor

**Theorem 13.** The competitive ratio of the LAR-Trust algorithm is \( 1 + 2\epsilon_{\text{time}} + 4\epsilon_{\text{pos}} \). Thus, the algorithm is 1-consistent and \((2\epsilon_{\text{time}} + 4\epsilon_{\text{pos}})\)-smooth but not robust.

The intuition of the LAR-ID algorithm is to keep the impact of errors under control. Specifically, the server finds an optimal route \( \hat{T} \) to serve the requests in \( \hat{x} \) in the beginning. Before time \( t_n \), the server follows the route \( \hat{T} \) and adjusts it when necessary. To explain how we make adjustments, we describe the route \( \hat{T} \) as a sequence of requests that are in order of priority. That is, \( \hat{T} := (\hat{x}_{(1)}, \ldots, \hat{x}_{(n)}) \) where \( \hat{x}_{(i)} = (\hat{t}_{(i)}, \hat{p}_{(i)}) \) denotes the \( i^{th} \) request to be served in \( \hat{T} \). If a request \( x_{(i)} \) arrives, the server modifies the route \( \hat{T} \) by inserting the request \( x_{(i)} \) into the sequence \( \hat{T} \) after the request \( \hat{x}_{(i)} \). By updating the route, the server can visit each request \( x_{(i)} \) with the adjusted version of the route \( \hat{T} \). When the last request \( x_n \) arrives (i.e., \( t = t_n \)), we compute the length of our two possible routes: (1) the remaining distance of \( \hat{T} \) after updates, denoted by \( r_1 \), and (2) the distance to go back to the origin and follow the final route \( T_{r_2} \) to visit the requests in \( U_{t_n} \), denoted by \( r_2 \). Then, the server chooses the shorter one to visit the remaining requests in \( x \). For (1), we can rewrite it as the naive algorithm by updating the route sequentially, where the details can be founded in the Appendix B.2. We first show that the cost of the LAR-Trust algorithm is associated with the error defined in Section 2. Note that the server does not change the order to serve the requests in \( x \) unless the server gives up the route \( \hat{T} \) at time \( t_n \).
Algorithm 3 LEARNING-AUGMENTED ROUTING WITH IDENTITY (LAR-ID)

Input: The current time \( t \), the number of requests \( n \), a sequence prediction \( \hat{x} \), and the set of current released unserved requests \( U_t \).

1: \( F = 0 \); \( \triangleright \) Initialize \( F = 0 \) to indicate that we trust the prediction.
2: First, compute an optimal route \( \hat{T} = (\hat{x}(1), \ldots, \hat{x}(n)) \) to serve the requests in \( \hat{x} \) and return to the origin \( o \), where \( \hat{x}(i) = (\hat{t}(i), \hat{p}(i)) \) is the \( i \)-th request to serve in \( \hat{T} \);
3: Start to follow the route \( \hat{T} \);
4: While \( F = 0 \) do \( \triangleright \) Trust the prediction
5: If \( t = t_i \) then
6: \( \hat{T} = (\hat{x}(1), \ldots, \hat{x}(i), x(i), \hat{x}(i+1), \ldots, \hat{x}(n)) \), for \( i = 1, \ldots, n \); \( \triangleright \) Update the route
7: If \( t = t_n \) then \( \triangleright \) Find the shorter route
8: \( r_1 \leftarrow \) the remaining distance of following \( \hat{T} \);
9: Compute a route \( T_{U_{tn}} \) to start and finish at the origin \( o \) and serve the requests in \( U_{tn} \);
10: \( r_2 \leftarrow d(p(t), o) + |T_{U_{tn}}| \);
11: If \( r_1 > r_2 \) then
12: Go back to the origin \( o \); \( \triangleright \) Give up the predicted route
13: \( F = 1 \).
14: Else
15: Move ahead on the current route \( \hat{T} \).
16: EndIf
17: Else
18: Move ahead on the current route \( \hat{T} \).
19: EndIf
20: EndIf
21: If \( p(t) = \hat{p}(i) \) and \( t < t_i \) then
22: Wait at position \( \hat{p}(i) \) until time \( t_i \). \( \triangleright \) Wait until the request arrives
23: EndIf
24: EndWhile
25: While \( F = 1 \) do \( \triangleright \) Do not trust the prediction
26: Start to follow the route \( T_{U_{tn}} \) to serve the requests in \( U_{tn} \).
27: EndWhile

\( \triangleright \) Theorem 14. The competitive ratio of the LAR-ID algorithm is \( \min \{ 3, 1 + (2\varepsilon_{time} + 4\varepsilon_{pos})/Z_{OPT} \} \). Thus, the algorithm is 1-consistent, 3-robust and \( (2\varepsilon_{time} + 4\varepsilon_{pos}) \)-smooth.

\( \triangleright \) Corollary 15. The LAR-ID algorithm is a \( \min \{ 3.5, 2.5 + (3.5\varepsilon_{time} + 7\varepsilon_{pos})/Z_{OPT} \} \)-competitive polynomial-time algorithm using Christofides’ heuristic.

4 Predict the Last Arrival Time

One can observe that Corollary 6 provides a special insight about predictions. That is, when the last request arrives, if the server is at the origin, we can apply an offline algorithm for the TSP to serve all the remaining unserved requests without dealing with the release time of the requests. Based on the insight, it is fair to consider another simpler model by predicting the arrival time of the last requests \( \hat{t}_n \) only, rather than predicting a whole sequence of future requests. Though we first show the restricted power of the model due to the limited information of predictions. Next, compared to PAH, we introduce a pure online algorithm,
REDESIGN, and present a polynomial-time learning-augmented algorithm for the restricted prediction model.

Intuitively, if the arrival time of the last request is given, the server can just wait until the last request arrives before starting a route. That is, the completion time is $Z_{ALG} \leq t_n + |T_x| \leq 2Z_{OPT}$. Precisely, even when the prediction is correct, the lower bound of 2, i.e., Theorem 3.2 in [5] still holds and so does the corollary.

▶ Theorem 16 ([5], Theorem 3.2). There is no $c$-competitive algorithm for the OLTSP with $c < 2$.

▶ Corollary 17. Given the arrival time of the last request $\hat{t}_n$, there is no $c$-competitive algorithm with $c < 2$.

In order to design a learning-augmented online algorithm with robustness, we present a simpler online algorithm, REDESIGN, than PAH, and it can achieve the same competitive ratio of 3 in polynomial time. Compared to PAH, the REDESIGN algorithm always goes back to the origin and redesign an optimal (or approximate) route for $U_t$, whenever a new request $x_i$ arrives. The following two lemmas show that the REDESIGN algorithm is 3-competitive.

Algorithm 4 REDESIGN

Input: The current time $t$ and the set of current released unserved requests $U_t$

1. While $U_t \neq \emptyset$ do
2. If the server is at the origin $o$ then
3. Start to follow a route $T_U$ to serve the requests in $U_t$ and return to the origin $o$.
4. ElseIf the server is currently moving along a route $T_{U_{t'}}$, for some $t' < t$ then
5. If a new request $x_i = (t_i, p_i)$ arrives then
6. Go back to the origin $o$.
7. EndIf
8. EndIf
9. EndWhile

▶ Lemma 18. Given that $p(t)$ is the current position of a server operated by the REDESIGN algorithm, we have $d(p(t), o) \leq \frac{1}{2}Z_{OPT}$, for any $t$.

▶ Lemma 19. The REDESIGN algorithm is a 3-competitive polynomial-time algorithm for the OLTSP in a metric space using Christofides’ heuristic.

4.1 The Algorithm

In this section, we present the LAR-LAST algorithm and show that it is min{$4, 2.5 + |\epsilon_{last}|/Z_{OPT}$}-competitive. The idea of the algorithm is to ensure the server to arrive at the origin $o$ at time $\hat{t}_n$ and follow the REDESIGN algorithm, for the rest of the execution.

It is motivated by the desire to enable the server to approach the origin at time $\hat{t}_n$, when the last request arrives. An intuitive strategy is to wait at the origin $o$ until the predicted last arrival time $\hat{t}_n$ and then follow a route $T_{U_{\hat{t}_n}}$ that serves whatever is in hands. It seems to beat the previous result easily when the prediction is perfectly accurate, i.e., $\hat{t}_n = t_n$. However, waiting at the origin is potentially costly when the quality of predictions is undisclosed. If the expected time $\hat{t}_n$ is too late, the server might postpone serving requests for a long time.
In consequence, the competitive ratio might extend to infinity so that the algorithm has no robustness.

We propose a learning-augmented algorithm, called LAR-Last. Basically, we incorporate the predicted time \( \hat{t}_n \) into the Redesign algorithm by forcing the server to return to the origin at time \( \hat{t}_n \) in order to hedge against the possible loss. At any time \( t \), if the server is at the origin \( o \), it finds an approximate route \( T_U_t \) to visit all released unserved requests in \( U_t \) by using Christofides’ heuristic. Similar to LAR-NID, we add a gadget here: if the route is too long (i.e., \( t < \hat{t}_n < t + |T_U_t| \)), the server should find the moment \( t_{back} \) to stop the route \( T_U_t \) and start moving towards the origin so that the server can arrive at the origin exactly at time \( \hat{t}_n \). On the other hand, if the server is moving along a route \( T_U'_t \) and a new request \( x_i \) arrives, the server goes back to the origin \( o \) immediately at time \( t_i \), as Redesign performs.

**Algorithm 5 Learning-Augmented Routing With Last Arrival Time (LAR-LAST)**

**Input:** The current time \( t \), the predicted last arrival time \( \hat{t}_n \), and the set of current released unserved requests \( U_t \).

1. While \( U_t \neq \emptyset \) do
2.   If the server is at the origin \( o \) (i.e., \( p(t) = o \)) then
3.     Compute an approximate route \( T_U_t \) to serve all the requests in \( U_t \) and return to the origin \( o \);
4.     If \( t < \hat{t}_n \) and \( t + |T_U_t| > \hat{t}_n \) then \( \triangleright \) Add a gadget
5.     Find the moment \( t_{back} \) such that \( t_{back} + d(p(t_{back}), o) = \hat{t}_n \);
6.     Redesign a route \( T_U'_t \) by asking the server to go back to the origin \( o \) at time \( t_{back} \) along the shortest path;
7.     Start to follow the route \( T_U'_t \);
8.   Else
9.     Start to follow the route \( T_U_t \);
10. EndIf
11. ElseIf the server is currently moving along a route \( T_U'_t \), for some \( t' < t \) then
12.     If a new request \( x_i = (t_i, p_i) \) arrives \( \triangleright \) Similar to Redesign
13.     Go back to the origin \( o \).
14. EndIf
15. EndIf
16. EndWhile

Next, we prove the main result of the LAR-Last algorithm. Note that the existence of \( t_{back} \) can be similarly derived from Lemma 8.

**Theorem 20.** The LAR-Last algorithm is a \( \min\{4, 2.5 + |\varepsilon_{last}|/Z^{OPT}\} \)-competitive polynomial-time algorithm, where \( \varepsilon_{last} = \hat{t}_n - t_n \). Therefore, the LAR-Last algorithm is 2.5-consistent, 4-robust and \( |\varepsilon_{last}| \)-smooth.

## 5 Extension to The Dial-a-Ride Problem

As we found the OLDARP has some properties similar to the OLTSP, we extend our models and algorithms to the dial-a-ride problem with unlimited capacity. We only mention the necessary changes for adapting the proposed algorithms to this problem as well as the results we obtain, and the details are presented in the Appendix D. The main difference is that the learning-augmented algorithms for the OLDARP are based on the Redesign algorithm, which yields a competitive ratio of 2.5 when ignoring the computational issue. Note that although
the SmartStart algorithm [1] has a better competitive ratio than the Redesign strategy, it is also more complicated and thus more difficult to be integrated with predictions.

First, we develop the LADAR-NID algorithm by substituting the Redesign strategy for the PAH algorithm in LAR-NID. Note that if the server gets to go back to the origin when carrying some requests, it brings all of them to the origin. This algorithm is $(1.5 + \lambda)$-consistent and $(3.5 + 2.5/\lambda)$-robust but not smooth. Next, we consider the LADAR-ID algorithm. Note that a route for the OLDARP can be described as a series of pickup and delivery positions instead of requests. Accordingly, when a request becomes known, the server needs to update two positions. Then, the algorithm is 1-consistent, 3-robust, and $(2\varepsilon_{\text{time}} + 4\varepsilon_{\text{pos}})$-smooth. Last, we use the LADAR-Last algorithm for the prediction of last arrival time, which replaces Christofides’ heuristic with the optimal solutions for the offline DARP without release time. This change makes the algorithm 2-consistent, 3.5-robust, and $|\varepsilon_{\text{last}}|$-smooth.

6 Conclusion

In this study, we have investigated two well-known online routing problems based on the learning-augmented framework. The proposed prediction models and results raise some interesting questions to further explore. First, it would be of interest to improve the current competitive results; especially coping with waiting strategies is usually helpful to the OLTSP and the OLDARP. Note that the currently best results for these two problems exploited waiting strategies, which leaves us an open problem to improve our learning-augmented online algorithms by incorporating a waiting strategy. It would be also worthwhile to design learning-augmented online algorithms for other variants of online routing problems with limited capacity or multiple servers.
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A  Detail of the PAH Algorithm

Here we present the details of PAH.

Algorithm 6 PAH

Input: The current time $t$ and the set of current released unserved requests $U_t$

1: While $U_t \neq \emptyset$ do
2:     If the server is at the origin (i.e., $p(t) = o$) then
3:         Start to follow an optimal (or approximate) route $T_{U_t}$ passing through each request in $U_t$.
4:     ElseIf the server is currently moving along a route $T_{U_{t'}}$, for some $t' < t$ then
5:         If a request $x_i = (t_i, p_i)$ arrives then
6:             If $d(p_i, o) > d(p(t), o)$ then
7:                 Go back to the origin $o$.
8:         Else
9:             Move ahead on the current route $T_{U_{t'}}$.
10:     EndIf
11:     EndIf
12: EndWhile

B  Missing Proofs and Details in Section 3

B.1 Missing Proofs in Section 3.1

Theorem 7. Given a sequence of predicted requests without identity, any 1-consistent algorithm has robustness of at least $1/\delta$ for any $\delta \in (0, 1)$.

Proof. Consider the OLTSP on the real line, a special metric space. Let the prediction be $\hat{x} = \{x_1 = (\delta, \delta), x_2 = (1, 1)\}$, where $\delta \in (0, 1)$. First, suppose the actual input is $x = \hat{x}$. In this case, the offline optimal solution is to start moving at time 0, visit $x_1$ at time $\delta$ and $x_2$ at time 1, and return to the origin at time 2. To obtain the consistency of 1, the server must follow the exact same route as the optimal one. On the other hand, if the actual input is $x' = \{x_1' = (\delta, \delta)\}$, i.e., only one request. Since the number of requests is unknown, the algorithm recognizes the error at time 1 and cannot go back home until time 2. However, the offline optimal solution finishes at time $2\delta$. Thus, the competitive ratio is $Z_{ALG}/Z_{OPT} \geq 2/(2\delta) = 1/\delta$.

Lemma 8. Given that the server follows the LAR-NID algorithm, when time $t$ satisfies the condition: $t < \lambda|\hat{T}|$ and $t + |T_{U_t}| > \lambda|\hat{T}|$, there exist a moment $t_{back}$, $t < t_{back} < t + |T_{U_t}|$, such that $t_{back} + d(p(t_{back}), o) = \lambda|\hat{T}|$.

Proof. Define the function $f : [t, t + |T_{U_t}|] \rightarrow [t, t + |T_{U_t}|]$ to be the moment when the server arrives at the origin again; that is, the server stops following route $T_{U_t}$ and goes back to the origin at a particular moment $t_{back}$, i.e., $f(t_{back}) = t_{back} + d(p(t_{back}), o)$. Then, we let the function $g(x) := f(x) - \lambda|\hat{T}|$, and it has two properties: $g(t) = t - \lambda|\hat{T}| < 0$ and $g(t + |T_{U_t}|) = t + |T_{U_t}| - \lambda|\hat{T}| > 0$. Since $g$ is continuous, there must be at least one point $t_{back} \in (t, t + |T_{U_t}|)$ such that $g(t_{back}) = 0$ and thus $t_{back} + d(p(t_{back}), o) = \lambda|\hat{T}|$. Without loss of generality, we choose the last one satisfying $g(t_{back}) = 0$.

▼
Lemma 9. The LAR-NID algorithm is \((1.5 + \lambda)\)-consistent, where \(\lambda \in (0, 1]\).

Proof. The server gets to follow the modified framework of the PAH algorithm at time 0 and returns to the origin at exactly time \(\lambda|\hat{T}|\). Then, at time \(\lambda|\hat{T}|\), the server follows the predicted route \(\hat{T}\) if \(U_{\lambda|\hat{T}|} \neq \emptyset\); otherwise, it waits until the next request arrives. Assume the prediction is perfect, i.e., \(\hat{x} = x\) and \(|\hat{T}| = Z^{OPT}\). Since \(\hat{T}\) is an optimal route, the server completes after following \(\hat{T}\). We consider the following two cases:

1. \(U_{\lambda|\hat{T}|} \neq \emptyset\). In this case, the server gets to follow route \(\hat{T}\) at time \(\lambda|\hat{T}|\), and the completion time of the algorithm is \(Z^{ALG} = \lambda|\hat{T}| + |\hat{T}| \leq (1 + \lambda)Z^{OPT}\).

2. \(U_{\lambda|\hat{T}|} = \emptyset\). In this case, after time \(\lambda|\hat{T}|\), the server waits for the next request to move. Let \(x_j = (t_j, p_j)\) denote this request, which implies \(t_j = \min_t\{t > \lambda|\hat{T}| : U_t \neq \emptyset\}\). The completion time of the algorithm is \(Z^{ALG} = \lambda|\hat{T}| + (t_j - \lambda|\hat{T}|) + |\hat{T}|\). Note that \(t_j - \lambda|\hat{T}|\) is the time length that the server waits at the origin. We split the duration \(t_j - \lambda|\hat{T}|\) into two time intervals according to the operations of the offline optimal server, i.e., OPT: \(t_{move}\) and \(t_{idle}\). The former denotes how much time the OPT is moving, and the latter denotes the time length the OPT just waits at the origin. That is, \(t_{move} + t_{idle} = t_j - \lambda|\hat{T}|\). Note that LAR-NID is not visiting any request during \(t_{idle}\), either. Because the case \(t_{idle} \neq 0\) favors the algorithm, we simply consider the worst case \(t_{idle} = 0\) and thus \(t_{move} = t_j - \lambda|\hat{T}|\).

Intuitively, the OPT gains the advantage by moving toward the next request during \(t_{move}\). Moreover, the time/distance the OPT saves by moving earlier is at most \(\frac{1}{2}Z^{OPT}\) due to the triangle inequality. Since the server moves at unit speed, we have \(t_{move} \leq \frac{1}{2}Z^{OPT}\). To sum up, the completion time is \(Z^{ALG} \leq \lambda|\hat{T}| + t_{move} + |\hat{T}| \leq (1.5 + \lambda)Z^{OPT}\).

\(\triangleright\)

Lemma 10. The LAR-NID algorithm is \((3 + 2/\lambda)\)-robust, where \(\lambda \in (0, 1]\).

Proof. Note that \(t_i \leq Z^{OPT}\) for any \(i\) and \(|T_{U_i}| \leq Z^{OPT}\) for any \(t\). To show the robustness, we consider the following cases:

1. Assume the server finishes before \(\lambda|\hat{T}|\). In this case, the server acts exactly like the PAH algorithm and does not follow route \(\hat{T}\). Thus, by Theorem 4, the server obtains the bound \(Z^{ALG} \leq 2Z^{OPT}\).

2. The server cannot finish before \(\lambda|\hat{T}|\). In this case, there are some unserved requests at time \(\lambda|\hat{T}|\) or some request arriving after time \(\lambda|\hat{T}|\). In both cases, the algorithm first follows \(\hat{T}\) and then switches to the PAH algorithm if there remain some unserved requests. Also, since the algorithm cannot serve all of the requests before time \(\lambda|\hat{T}|\), it implies that \(Z^{OPT} \geq \lambda|\hat{T}|/2\); otherwise, it contradicts Theorem 4. To analyze the performance of LAR-NID in this case, we divide the case into the following four subcases:

- 2-1. \(U_{\lambda|\hat{T}|} \neq \emptyset\) and \(t_n > (1 + \lambda)|\hat{T}|\). Since the LAR-NID algorithm follows PAH after time \((1 + \lambda)|\hat{T}|\), we have \(Z^{ALG} \leq 2Z^{OPT}\) by Corollary 6.

- 2-2. \(U_{\lambda|\hat{T}|} \neq \emptyset\) and \(t_n \leq (1 + \lambda)|\hat{T}|\). By using \(\lambda|\hat{T}|/2 \leq Z^{OPT}\), the completion time of the algorithm can be derived as follows:

\[
Z^{ALG} \leq \lambda|\hat{T}| + |\hat{T}| + |T_{U_{(1+\lambda)|\hat{T}|}}| \\
\leq 2Z^{OPT} + (2/\lambda)Z^{OPT} + Z^{OPT} \\
\leq (3 + 2/\lambda)Z^{OPT}
\]

- 2-3. \(U_{\lambda|\hat{T}|} = \emptyset\) and \(t_n > t_j + |\hat{T}|\). This case is similar to the case 2-1. The LAR-NID algorithm follows PAH after time \(t_j + |\hat{T}|\). Thus, we have \(Z^{ALG} \leq 2Z^{OPT}\) by Corollary 6.
Thus, the algorithm is robust but not smooth, where \( \lambda \in (0, 1] \) is the confidence level.

**B.2 Missing Proofs and Details in Section 3.2**

**Theorem 11.** The LAR-NID algorithm is \((1.5 + \lambda)\)-consistent and \((3 + 2/\lambda)\)-robust but not smooth, where \( \lambda \in (0, 1] \) is the confidence level.

Proof. Consider the OLTSP on the real line, similar to the proof of Theorem 7 and assume the prediction is \( \hat{x} = \{x_1 = (0.5, 0.5), x_2 = (1, 1)\} \). On one hand, suppose the prediction is perfect, i.e., \( x = \hat{x} \). The OPT gets to serve \( x_2 \) at time 1 and returns to the origin at time 2. In order to reach the consistency of 1, the server must follow this route. On the other hand, we assume the actual input is \( x' = \{x'_1 = (0.5, 0.5), x'_2 = (1, 0)\} \). As the server can recognize the prediction error once upon the arrival of the second request \( x_2 \), it is at position 1 at time 1 and cannot return to the origin until time 2. However, the completion time of the OPT is \( Z^{OPT} = 1 \). Thus, the competitive ratio is \( Z^{ALG}/Z^{OPT} = 2 \).

The LAR-Trust Algorithm We first recall the LAR-Trust algorithm that the server follows a route for the prediction and only adjust the predicted route when an error is discovered. Formally, the server gets an optimal route \( \hat{T} \) to visit the requests in \( \hat{x} \) in the beginning and starts following \( \hat{T} \) immediately. As the requests arrive, the server modifies the route \( \hat{T} \) in two ways: (1) if the server moves to a request’s position \( \hat{p}_i \) but the request \( x_i \) has not arrived, the server waits for it to move; (2) if a request \( x_i \) arrives, the server adjusts the route \( \hat{T} \) by inserting the actual request \( x_i \) after the corresponding predicted one \( \hat{x}_i \). In this case, the server can visit all requests in \( x \) by following the adjusted route of \( \hat{T} \). Note that \( \hat{T} \) decides the order when the server visits these requests, and the server does not change the order even when the prediction errors are large. We analyze the performance of the LAR-Trust algorithm in the following paragraph.

To analyze the performance of the LAR-Trust algorithm, we show that the error term can be used to bound the cost of two different routes. Then, to compare the cost of two different routes, we first consider a restricted condition where the position of each request of the routes is the same but with a different arrival time.

**Theorem 13.** The competitive ratio of the LAR-Trust algorithm is \( 1 + 2\varepsilon_{time} + 4\varepsilon_{pos} \). Thus, the algorithm is \((1 + \lambda)\)-consistent and \((2 + 2/\lambda)\)-smooth but not robust.

Proof. Let \( T_x \) be the optimal route for the requests in \( x \) and \( T_{\hat{x}} \) be the optimal route to visit the requests in \( \hat{x} \). As mentioned in Section 3.2, a route can be regarded as the order to serve the requests. We consider the total completion time of the following routes: (1) let \( Z^{OPT} \) be the completion time of \( T_x \); (2) let \( Z^{ALG}_{\hat{x}} \) be the completion time if the server visits
the requests in \( \hat{x} \) by following the order of the route \( T_{\hat{x}} \). (3) let \( Z^*_x \) be the completion time of \( T_x \), and (4) let \( Z^{ALG} \) be the completion time of LAR-Trust, which visits the requests in \( x \) by following the order of the route \( T_x \). To compare \( Z^{ALG} \) with \( Z^{OPT} \), the proof is split into the following two parts: we first show that the completion time \( Z^*_x \) of the predicted route \( \hat{T} \) can be bounded in terms of \( Z^{OPT} \), and then we bound \( Z^{ALG} \) using \( Z^*_x \). Finally, we obtain the desired result by combining the results of the two parts.

\[ \text{Lemma 21. } Z^*_x \leq Z^* \leq Z^{OPT} + \varepsilon_{\text{time}} + 2\varepsilon_{\text{pos}}. \]

**Proof.** Consider the route \( Z^*_x \): (1) the server waits at the origin \( o \) for time \( \varepsilon_{\text{time}} \) before following the route \( T_x \), and (2) after the server arrives at \( p_i \), it moves to \( \hat{p}_i \) and goes back to \( p_i \) to continue the route \( T_x \). We argue that the server can visit all of requests in \( \hat{x} \) and \( x \) then return to the origin at time \( Z^{OPT} + \varepsilon_{\text{time}} + 2\varepsilon_{\text{pos}} \). For (1), it guarantees that the server does not have to pay any extra waiting cost compared to the offline optimal route plus the waiting time \( \varepsilon_{\text{time}} \). For (2), it takes \( 2d(p_i, \hat{p}_i) \) to serve request \( \hat{x}_i \) then go back to \( x_i \). Combing (1) and (2) leads to that the server can visit all of requests in \( \hat{x} \) and \( x \) then return to the origin at time \( Z^*_x \leq Z^{OPT} + \varepsilon_{\text{time}} + 2\varepsilon_{\text{pos}} \). Finally, since \( T_\hat{x} \) is the optimal route for \( \hat{x} \), we thus have \( Z^*_x \leq Z^*_\hat{x} \). Note that the result implies that compared with \( T_x \), the server might find a longer route \( \hat{T} \) if there is some request expected to arrive later, or if the prediction error in positions exists.

\[ \text{Lemma 22. } Z^{ALG} \leq Z^*_x + \varepsilon_{\text{time}} + 2\varepsilon_{\text{pos}}. \]

**Proof.** Consider the following route \( Z'_x \): (1) the server waits at the origin \( o \) for time \( \varepsilon_{\text{time}} \) before following the route \( \hat{T} \), and (2) after the server arrives at \( \hat{p}_i \), it moves to \( p_i \) and goes back to \( \hat{p}_i \) to continue the route \( \hat{T} \). We argue that the server can visit all of requests in \( \hat{x} \) and \( x \), and then returns to the origin at time \( Z^*_x + \varepsilon_{\text{time}} + 2\varepsilon_{\text{pos}} \). For (1), it ensures that \( x_i \) has arrived at \( p_i \) when the server is at \( \hat{p}_i \). For (2), it takes \( 2d(\hat{p}_i, p_i) \) to serve request \( x_i \) then go back to \( \hat{x}_i \), which is exactly the operation of the LAR-Trust algorithm. Therefore, we can derive that the server can visit all of requests in \( \hat{x} \) and \( x \), and then returns to the origin at time \( Z^{ALG} \leq Z^*_x + \varepsilon_{\text{time}} + 2\varepsilon_{\text{pos}} \). Notice that the server might need to adjust the route \( \hat{T} \) if some request arrives later than expected, or if some position error exists.

Finally, combining the results of Lemma 21 and Lemma 22 can obtain

\[ Z^{ALG} \leq Z^{OPT} + 2\varepsilon_{\text{time}} + 4\varepsilon_{\text{pos}}. \]

**Theorem 14.** The competitive ratio of the LAR-ID algorithm is \( \min\{3, 1 + (2\varepsilon_{\text{time}} + 4\varepsilon_{\text{pos}})/Z^{OPT}\} \). Thus, the algorithm is \( 1 \)-consistent, \( 3 \)-robust and \( (2\varepsilon_{\text{time}} + 4\varepsilon_{\text{pos}}) \)-smooth.

**Proof.** Initially, the server follows an optimal route \( \hat{T} \) for the requests in \( \hat{x} \), which implies it is \( 1 \)-consistent when the predictions are accurate, i.e. \( x = \hat{x} \). The server makes adjustments if needed (i.e., \( p_i \neq \hat{p}_i \) for some \( i \)) before and when the last request arrives. Finally, at time \( t_n \), it decides whether the server continue on the current route or not. Therefore, we discuss the following two cases depending on the relationship of \( r_1 \) and \( r_2 \). Note that \( t_n \leq Z^{OPT} \) and \( |T_{U_i}| \leq Z^{OPT} \) for any \( t \).
1. \( r_1 \leq r_2 \). In this case, the server knows that continuing on the route \( \hat{T} \) takes less time than redesigning a new one, which implies the errors are small. Thus, it follows the route \( \hat{T} \) from the beginning to the end. By Theorem 13, the completion time of the algorithm can be bounded by \( Z^{ALG} \leq Z^{OPT} + 2\varepsilon_{time} + 4\varepsilon_{pos} \).

2. \( r_1 > r_2 \). In this case, it is better to go back to the origin and design a new route. The server returns to the origin \( o \) at time \( t_n + d(p(t_n), o) \) and starts to follow a new route \( T_{U_t_n} \), which visits all unserved requests in \( U_t_n \). Because there is no more request, the server completes all of the requests at time \( Z^{ALG} = t_n + d(p(t_n), o) + |T_{U_t_n}| \). In addition, since the server moves with unit speed, \( d(p(t_n), o) = d(o, p(t_n)) \leq t_n \). Combining the two inequalities leads to the result \( Z^{ALG} \leq 3Z^{OPT} \).

As the server chooses the faster route among the two possible options, the competitive ratio can be stated as \( \min\{3, 1 + (2\varepsilon_{time} + 4\varepsilon_{pos})/Z^{OPT}\} \).

**Corollary 15.** The LAR-ID algorithm is a \( \min\{3, 5, 2.5 + (3.5\varepsilon_{time} + 7\varepsilon_{pos})/Z^{OPT}\} \)-competitive polynomial-time algorithm using Christofides’ heuristic.

**Proof.** We first consider the case that the server follows the optimal route \( \hat{T} \) from the beginning to the end. Let \( T_\hat{x} \) be the optimal route for the requests in \( x \) and \( T_\hat{x} \) be the approximate route to visit the requests in \( \hat{x} \). We consider the total completion time of the following routes: (1) let \( Z^{OPT} \) be the completion time of \( T_\hat{x} \), (2) let \( Z'_x \) be the completion time if the server visits the requests in \( \hat{x} \) by following the order of the route \( T_\hat{x} \), (3) let \( Z'_x \) be the completion time of \( T_\hat{x} \), and (4) let \( Z^{ALG} \) be the completion time of LAR-TRUST, which visits the requests in \( x \) by following the order of the route \( T_\hat{x} \).

**Lemma 23.** \( Z^{ALG} \leq 2.5Z^{OPT} + 3.5\varepsilon_{time} + 7\varepsilon_{pos} \).

**Proof.** We split into the following three parts:

1. \( Z'_x \leq Z^{OPT} + \varepsilon_{time} + 2\varepsilon_{pos} \). Consider the route \( Z'_x \) such that the server waits at the origin \( o \) for time \( \varepsilon_{time} \) before following the route \( T_\hat{x} \) and moves from \( p_i \) to \( \hat{p}_t \) to serve requests in \( \hat{x} \). The completion time of the route is \( Z'_x \leq Z^{OPT} + \varepsilon_{time} + 2\varepsilon_{pos} \).

2. \( Z'_x \leq 2.5Z^{ALG} \). Given that Christofides’ algorithm is 1.5-approximate, we find an upper bound for the approximation ratio of the offline TSP with release time; assuming the server waits at the origin \( o \) until the last requests arrive, i.e., \( t = t_n \), and then follows an approximate route \( T \) found by Christofides’ algorithm, the server can complete serving all the requests at time \( t_n + |T| \leq 2.5Z^{OPT} \). That is, the completion time of the approximate route \( Z^{ALG} \) can be bounded by 2.5 times the completion time of any other route for the requests in \( x \).

3. \( Z^{ALG} \leq Z'_x + \varepsilon_{time} + 2\varepsilon_{pos} \). Note that the route \( \hat{T} \) can visit all of the requests in \( \hat{x} \). Consider the route that the server waits at the origin \( o \) for \( \varepsilon_{time} \) and moves from \( \hat{p}_t \) to \( p_i \) to visit request \( x_i \). The server can complete serving the requests in \( x \) at time \( Z^{OPT} + \varepsilon_{time} + 2\varepsilon_{pos} \). Combining the above three inequalities, the result is as follows:

\[
Z^{ALG} \leq Z'_x + \varepsilon_{time} + 2\varepsilon_{pos} \\
\leq 2.5Z^{ALG} + \varepsilon_{time} + 2\varepsilon_{pos} \\
\leq 2.5(Z^{OPT} + \varepsilon_{time} + 2\varepsilon_{pos}) + \varepsilon_{time} + 2\varepsilon_{pos} \\
\leq 2.5Z^{OPT} + 3.5\varepsilon_{time} + 7\varepsilon_{pos}
\]

Then, we discuss the two following routes:
Proof. A simple explanation is that (1) at any time $t$ during the execution of the algorithm, the distance between the online server and the origin, i.e. $d(p(t), o)$ is at most the length between the farthest request and the origin, and (2) the offline optimal route must be longer than twice the distance between the farthest request and the origin. Now, we prove the lemma in a more rigorous way by specifying the state of the server at time $t$.

1. When the server is at the origin $o$ at time $t$, trivially, we get $d(p(t), o) = 0 \leq \frac{1}{2}Z^{OPT}$.

2. Otherwise, the server is not at the origin $o$ at time $t$. In this case, we know that the server is either on a route $T_{U_i}$, found by Christofides’ algorithm, where $t’ < t$, or already on its way home due to the arrival of some earlier request. Let $T^*$ denote an optimal route to visit the set of requests for which the route $T_{U_i}$ is planned. Note that neither route $T_{U_i}$ nor route $T^*$ considers the release time of the requests.

2-1. The server is on the route $T_{U_i}$. This implies that the server is traveling between a pair of points, denoted as $a$ and $b$, where a point must be the position of a request $x_i$ or the origin $o$. Since route $T^*$ contains at least point $a$, point $b$, and the origin $o$, we know $|T^*| \geq d(a, a) + d(a, b) + d(b, o)$. Then, we have $d(p(t), o) \leq \min\{d(p(t), a) + d(a, o), d(p(t), b) + d(a, o)\} \leq \frac{1}{2}|T^*| \leq \frac{1}{2}Z^{OPT}$.

2-2. The server is on its way back home. This implies that the server has terminated a route $T_{U_i}$. We know that some request $x_i = (t_i, p_i)$ arrives between the time $t$ and time $t'$. As we can obtain $d(p(t_i), o) \leq \frac{1}{2}Z^{OPT}$ by the above argument, we also have $d(p(t), o) \leq d(p(t_i), o) \leq \frac{1}{2}Z^{OPT}$.

Lemma 19. The Redesign algorithm is a 3-competitive polynomial-time algorithm for the OLTSP in a metric space using Christofides’ heuristic.

Proof. Note that the algorithm finishes when the server receives the last request $x_n$, goes back to the origin $o$, and follows the last route $T_{U_{t_n}}$ found by Christofides’ algorithm. Namely, the completion time is $Z^{ALG} = t_n + d(p(t_n), o) + |T_{U_{t_n}}|$. By the two natural bounds $t_n \leq Z^{OPT}$ and $|T_{U_{t_n}}| \leq 1.5Z^{OPT}$, we know the completion time is bounded by $Z^{ALG} \leq 2.5Z^{OPT} + d(p(t_n), o)$. Meanwhile, we have $d(p(t_n), o) \leq 0.5Z^{OPT}$ by Lemma 18. Combining them, we get $Z^{ALG} \leq 3Z^{OPT}$.
C.1 Missing Proofs in Section 4.1

Theorem 20. The LAR-LAST algorithm is a min\{4, 2.5 + |\varepsilon_{last}/Z_{OPT}\}-competitive polynomial-time algorithm, where \varepsilon_{last} = \hat{t}_n - t_n. Therefore, the LAR-LAST algorithm is 2.5-consistent, 4-robust and |\varepsilon_{last}|-smooth.

Proof. The most important thing is the server’s position when the last request \(x_n\) arrives, and the ideal situation is that the server is at the origin \(o\) at time \(t_n\), i.e., \(p(t_n) = o\). We distinguish between two main cases depending on the relationship between the predicted and the ideal situation is that the server is at the origin \(o\) at time \(t_n\), i.e., \(p(t_n) = o\). We distinguish between two main cases depending on the relationship between the predicted and actual last arrival time, \(\hat{t}_n\) and \(t_n\). Note that we get \(t_n \leq Z_{OPT}\) and \(|T_{U_n}| \leq 1.5Z_{OPT}\) for any \(t\) trivially.

1. \(\hat{t}_n \leq t_n\). The server is at the origin \(o\) at time \(\hat{t}_n\). After then, it receives some request and returns to the origin only when a request arrives or a route is completed. Therefore, the completion time is \(Z_{ALG} = t_n + d(p(t_n), o) + |T_{U_n}|\). By the above two inequalities, this leaves us \(Z_{ALG} \leq 2.5Z_{OPT} + d(p(t_n), o)\) to discuss.

1-1. By Lemma 18, the distance from the server to the origin is bounded at any moment. We get \(d(p(t_n), o) \leq 0.5Z_{OPT}\) and our first bound \(Z_{ALG} \leq 3Z_{OPT}\).

1-2. Although the server might have left the origin at time \(t_n\), it cannot be too far if the error \(|\varepsilon_{last}|\) is small. Seeing that the server moves with unit speed, the distance it can travel between \(\hat{t}_n\) and \(t_n\) is at most \(t_n - \hat{t}_n\). As the server is at the origin \(o\) at time \(\hat{t}_n\), we obtain \(d(p(t_n), o) \leq t_n - \hat{t}_n = -\varepsilon_{last}\). Thus, our second bound for case 1 is \(Z_{ALG} \leq 2.5Z_{OPT} + (-\varepsilon_{last})\), where \(\varepsilon_{last} \leq 0\).

2. \(\hat{t}_n > t_n\). Since no request arrives after time \(\hat{t}_n\) and the server could start its last route \(T_{U_n}\) at \(\hat{t}_n\) (if needed), we have \(Z_{ALG} \leq \hat{t}_n + |T_{U_n}|\). Let \(t_L\) be the last time before \(\hat{t}_n\) that a route is planned and \(T^L\) be the original route planned at time \(t_L\). Let \(T^{\prime L}\) be the new route if \(T^L\) is adjusted and equal to \(T^L\) if not. Note that \(t_L + |T^{\prime L}| \leq t_n\) due to the choice of \(t_{back}\).

2-1. \(t_L + |T_L| \leq \hat{t}_n\). The server finishes serving all requests in \(x\) before time \(\hat{t}_n\) and thus \(Z_{ALG} \leq \hat{t}_n\). Note that none of the routes the server has followed is too long and needs to be adjusted. In this case, since the algorithm runs in the same way as the Redesign algorithm, we know that \(Z_{ALG} \leq 3Z_{OPT}\) still holds by Lemma 18.

2-2. \(t_L + |T_L^L| > \hat{t}_n\) and \(t_n \leq t_L\). The route \(T^L\) is too long, and the algorithm finds a time \(t_{back}\) and a substitute route \(T^{\prime L}\) so that the server can be at the origin \(o\) at time \(\hat{t}_n\). Since every request in \(x\) has arrived before \(t_L\), the server could have finished visiting all requests by following the original route \(T^L\). However, the server chooses the adjusted route \(T^{\prime L}\) instead. The additional cost of moving back to the origin is at most \(2d(p(t_{back}), o)\). Thus, by Lemma 18 and Lemma 19, the completion time is \(Z_{ALG} \leq 3Z_{OPT} + 2d(p(t_{back}), o) \leq 4Z_{OPT}\).

2-3. \(t_L + |T^L| > \hat{t}_n\) and \(t_n > t_L\). Again, the route \(T^L\) is too long, and at least one request is unknown at time \(t_L\). Also, the server cannot visit all requests in \(x\) before \(\hat{t}_n\) and has to start a new route at that moment, which is the last route since all requests have arrived by then. Noting the route \(T^{\prime L}\) also satisfies \(|T^{\prime L}| \leq 1.5Z_{OPT}\), the completion time is

\[
Z_{ALG} \leq \hat{t}_n + |T_{U_n}| = t_L + |T^{\prime L}| + |T_{U_n}| \\
\leq t_n + |T^{\prime L}| + |T_{U_n}| \\
\leq 4Z_{OPT}
\]
Combining with the results in 2-1 and 2-2, we have our first bound $Z_{ALG} \leq 4Z_{OPT}$.

2-4. If the error $|\varepsilon_{last}|$ is not zero, the server might have not reached the origin at time $t_n$. However, if the error is small, it must be close to the origin since it has planned to reach there at time $\hat{t}_n$. Formally, we have $Z_{ALG} \leq t_n + \varepsilon_{last} + |T_{\hat{t}_n}|$ by replacing $\hat{t}_n$ with $t_n + \varepsilon_{last}$, which leads to our second bound $Z_{ALG} \leq 2.5Z_{OPT} + \varepsilon_{last}$.

Both cases show $Z_{ALG} \leq \min\{4Z_{OPT}, 2.5Z_{OPT} + |\varepsilon_{last}|\}$, which completes the proof.

D The Learning-Augmented Dial-a-Ride Problem

Here we discuss how we extend the three models and algorithms to the OLDARP.

D.1 Problem Setting

With a slight abuse of notation, we use the same notations as the TSP but change their definitions when the context is clear. The input of the OLDARP is a sequence denoted by $x$ and with a size of $n$, the number of requests in a metric space. Each request in $x$ is denoted by $x_i = (t_i, a_i, b_i)$ where $t_i$ is the time when the request becomes known, $a_i$ is the pickup position, and $b_i$ is the delivery position. A server starts and ends at the origin $o$, and it has to move each request $x_i$ from $a_i$ to $b_i$. Note that the server cannot collect a request $x_i$ before its arrival time $t_i$ and let $c$ denote the maximum amount of requests the server can carry at a time. The goal is to minimize the completion time $|T_x|$ of a route $T_x$. In this work, we consider the server with unlimited capacity (i.e., $c = \infty$) and the non-preemptive version: once the server picks up a request $x_i$ at position $a_i$, it cannot drop it anywhere else except position $b_i$. Regarding the previous results, Shmoys et al. [23] proposed the non-polynomial Ignore algorithm with a competitive ratio of 2.5 by ignoring newly arrived requests if the server is already following a schedule, while the Replan algorithm achieved the same result by redesigning a route whenever a request becomes known. Ascheuer et al. [4] improved the ratio to 2, which meets the lower bound.

D.2 Models and Errors

We consider the OLDARP using the three types of predictions discussed in the previous sections. First, for the model with sequence prediction without identity $\hat{x}$, let $m$ denote the predicted size of the sequence and $\hat{x}_i = (\hat{t}_i, \hat{a}_i, \hat{b}_i)$ denote each predicted request for $i = 1, \ldots, m$. We do not quantify the error but only consider whether $\hat{x} = x$ or not. Next, we consider the sequence prediction with identity, $\hat{x}$ with size of $n$. As in the TSP model, the time error is defined as $\varepsilon_{time} := \max_{i \in [n]} |\hat{t}_i - t_i|$. However, unlike the OLTSP, the OLDARP involves two positions. Thus, we modify the position error $\varepsilon_{pos}$ and define it as the extra distance the server has to travel to deliver the actual requests, compared with the predicted ones.

$$\varepsilon_{pos} := \sum_i \left[ d(\hat{a}_i, a_i) + d(\hat{b}_i, b_i) \right]$$

Last, we discuss the prediction of last arrival time $\hat{t}_n$, and the prediction error is defined by $\varepsilon_{last} := \hat{t}_n - t_n$, as in the TSP.

D.3 The LADAR-Trust Algorithm

We introduce some properties of the Redesign algorithm and a naïve algorithm, LADAR-Trust, that we need before showing the details. First, recall that the Redesign algorithm is
2.5-competitive when ignoring the computational issue, while it is 3-competitive in polynomial time. As Corollary\(^{[4]}\) for the PAH algorithm, we show that postponing the time to start the REDesign algorithm does not affect the competitive ratio by Corollary\(^{[25]}\).

**Theorem 24** (\(^{[4]}\) Theorem 3). The REDesign algorithm is 2.5-competitive for the OLDARP in a metric space.

**Corollary 25.** Assume the server operated by an algorithm waits at the origin o for time \(t\) before following the REDesign algorithm and \(t \leq t_n\). This algorithm is still 2.5-competitive.

Then, we present the LADAR-Trust algorithm for the DARP. Since each request \(x_i = (\hat{t}_i, \hat{a}_i, \hat{b}_i)\) involves two positions, we describe a route as a sequence of positions \(\hat{T} = (\hat{p}(1), \ldots, \hat{p}(2n))\), where \(\hat{p}(i) \in \{\hat{a}_1, \ldots, \hat{a}_n, \hat{b}_1, \ldots, \hat{b}_n\}\) is the \(i^{th}\) position the server reaches in the route \(\hat{T}\). Thus, the route \(\hat{T}\) can show when the server pick up and deliver a request \(\hat{x}_i\). In addition, the server adjusts the predicted route \(\hat{T}\) when the actual requests in \(x\) arrive; when a request \(x_i = (t_i, a_i, b_i)\) is released, we insert \(a_i\) after \(\hat{a}_i\) and \(b_i\) after \(\hat{b}_i\).

**Algorithm 7** LEARNING-AUGMENTED DIAL-A-RIDE TRUST (LADAR-Trust)

**Input:** The current time \(t\), the number of requests \(n\), a sequence prediction \(\hat{x}\), and the set of current released unserved requests \(U_t\).

1: First, compute an optimal route \(\hat{T} = (\hat{p}(1), \ldots, \hat{p}(2n))\) to serve the requests in \(\hat{x}\) and return to the origin o, where \(\hat{p}(i) \in \{\hat{a}_1, \ldots, \hat{a}_n, \hat{b}_1, \ldots, \hat{b}_n\}\) is the \(i^{th}\) position the server reaches in \(\hat{T}\).

2: For any \(i = 1, \ldots, n\) do

3: If \(t = t(i)\) for any \(i\), where \(x(i) = (t(i), a(i), b(i))\) then

4: Update the route \(\hat{T}\) by adding positions \(a(i)\) and \(b(i)\) after their corresponding positions \(\hat{p}(i)\) and \(\hat{p}(j)\);

5: EndIf

6: If \(p(t) = \hat{p}(i)\) and \(t < t(i)\) then

7: Wait at position \(\hat{p}(i)\) until time \(t(i)\). \(\triangleright\) Wait until the request arrives

8: EndIf

9: EndFor

We can extend Theorem\(^{[13]}\) to the OLDARP by replacing the definition of \(\varepsilon_{\text{pos}} = \sum_{i=1}^{n} d(\hat{p}_i, p_i)\) with \(\varepsilon_{\text{pos}} = \sum_{i=1}^{n} [d(\hat{a}_i, a_i) + d(\hat{b}_i, b_i)]\) and obtain the following result.

**Theorem 26.** The competitive ratio of the LADAR-Trust algorithm is \(1 + 2\varepsilon_{\text{time}} + 4\varepsilon_{\text{pos}}\). Thus, it is \(1\)-consistent and \((2\varepsilon_{\text{time}} + 4\varepsilon_{\text{pos}})\)-smooth but not robust.

**Proof.** As mentioned above, a route can be regarded as the order to reach certain positions. As in Theorem\(^{[13]}\) let \(T_x\) be the actual optimal route for the requests in \(x\) and \(T_x^\prime\) be the predicted optimal route to visit the requests in \(\hat{x}\). We consider the total completion time of the following routes: (1) let \(Z^{OPT}\) be the completion time of \(T_x\), (2) let \(Z_x^\prime\) be the completion time if the server visits the requests in \(\hat{x}\) by following the order of the route \(T_x\), (3) let \(Z_x^\prime\) be the completion time of \(T_x^\prime\), and (4) let \(Z^{ALG}\) be the completion time of LADAR-Trust, which visits the requests in \(x\) by following the order of the route \(T_x\). To relate \(Z^{ALG}\) and \(Z^{OPT}\), the proof is split into the following two parts:

**Lemma 27.** \(Z_x^\prime \leq Z_x^\prime \leq Z^{OPT} + \varepsilon_{\text{time}} + 2\varepsilon_{\text{pos}}\).
To relate $Z'_x$ and $Z^{OPT}$, we consider the following route $Z'_x$ of server: (1) the server waits at the origin $o$ for time $\varepsilon_{time}$ before following the route $T_x$, and (2) after the server arrives $a_i$ (resp. $b_i$), it moves to $\hat{a}_i$ (resp. $\hat{b}_i$) and goes back to $a_i$ (resp. $b_i$) to continue the route $T_x$. We argue that the server can visit all of requests in \( \hat{x} \) and \( x \) then return to the origin at time $Z^{OPT} + \varepsilon_{time} + 2\varepsilon_{pos}$. For (1), it guarantees that the server does not have to pay any extra waiting cost compared to the optimal route. For (2), it takes $2d(a_i, \hat{a}_i) + 2d(\hat{b}_i, b_i)$ to serve pick up or deliver request $\hat{x}_i$, then go back to the route $T_x$. Combining (1) and (2), we can derive that the server can visit all of requests in \( \hat{x} \) and \( x \) then return to the origin at time $Z'_x \leq Z^{OPT} + \varepsilon_{time} + 2\varepsilon_{pos}$. Finally, since $T_x$ is the optimal route for \( \hat{x} \), we must have $Z'_x \leq Z_x$.

\begin{itemize}
\item \textbf{Lemma 28.} $Z^{ALG} \leq Z'_x + \varepsilon_{time} + 2\varepsilon_{pos}$.
\end{itemize}

\textbf{Proof.} To relate $Z^{ALG}$ and $Z'_x$, we consider the following route of server: (1) the server waits at the origin $o$ for time $\varepsilon_{time}$ before following the route $\hat{T}$, and (2) after the server arrives $\hat{a}_i$ (resp. $\hat{b}_i$), it moves to $a_i$ (resp. $b_i$) and goes back to $\hat{a}_i$ (resp. $\hat{b}_i$) to continue the route $\hat{T}$. We argue that the server can visit all of requests in \( \hat{x} \) and \( x \) then return to the origin at time $Z'_x + \varepsilon_{time} + 2\varepsilon_{pos}$. For (1), it ensures that $x_i$ has arrived $p_i$ when the server is at $\hat{p}_i$. For (2), it takes $2d(\hat{a}_i, a_i) + 2d(\hat{b}_i, b_i)$ to serve request $x_i$ then go back to $\hat{x}_i$, which is exactly the operation of the LADAR-TRUST algorithm. Therefore, we can derive that the server can visit all of requests in \( \hat{x} \) and \( x \) then return to the origin at time $Z^{ALG} \leq Z'_x + \varepsilon_{time} + 2\varepsilon_{pos}$. The implication is that the server might need to adjust the route $\hat{T}$ if some request arrives later than expected, or if the position error exists.

Finally, combining the results of Lemma 27 and Lemma 28, we thus can obtain

$Z^{ALG} \leq Z^{OPT} + 2\varepsilon_{time} + 4\varepsilon_{pos}$

\begin{itemize}
\item \textbf{Lemma 29.} The LADAR-NID algorithm is $(1.5 + \lambda)$-consistent, where $\lambda \in (0, 1]$.
\end{itemize}

\textbf{Proof.} The server follows the adjusted version of the REDESIGN algorithm at time $t = 0$ and returns to the origin at time $\lambda[T]$. At time $\lambda[T]$, the server might follows the predicted route immediately if $U_{\lambda[T]} \neq \emptyset$ or waits until the next request arrives. Assuming the prediction is perfect, i.e., $\hat{x} = x$ and $|T| = Z^{OPT}$, the server completes serving requests after following $\hat{T}$. Then, we consider the following cases:
Algorithm 8 Learning-Augmented Dial-a-Ride Without Identity (LADAR-NID)

Input: The current time \( t \), a sequence prediction \( \hat{x} \), the confidence level \( \lambda \in (0, 1] \), and the set of current released unserved requests \( U_t \).

1: First, compute an optimal route \( \hat{T} \) to serve the requests in \( \hat{x} \) and return to the origin \( o \);
2: \textbf{While} \( t < \lambda |\hat{T}| \) \textbf{do}
3: \hspace{1em} If the server is at the origin \( o \) (i.e., \( p(t) = o \)) \textbf{then}
4: \hspace{2em} Compute an optimal route \( T_{U_t} \) to serve all the unserved requests in \( U_t \) and return to the origin \( o \);
5: \hspace{2em} \textbf{If} \( t + |T_{U_t}| > \lambda |\hat{T}| \) \textbf{then} \hspace{6em} \( \triangleright \) Add a gadget
6: \hspace{3em} Find the moment \( t_{\text{back}} \) such that \( t_{\text{back}} + d(p(t_{\text{back}}), o) = \lambda |\hat{T}| \);
7: \hspace{3em} Redesign a route \( T'_{U_t} \) by asking the server to go back to the origin \( o \) at time \( t_{\text{back}} \) along the shortest path;
8: \hspace{3em} Start to follow the route \( T'_{U_t} \).
9: \hspace{2em} Else
10: \hspace{3em} Start to follow the route \( T_{U_t} \).
11: \hspace{1em} \textbf{EndIf}
12: \textbf{ElseIf} the server is currently moving along a route \( T'_{U_{t'}} \), for some \( t' < t \) \textbf{then}
13: \hspace{1em} \textbf{If} a new request \( x_i = (t_i, a_i, b_i) \) arrives \textbf{then} \hspace{6em} \( \triangleright \) Similar to \texttt{REDESIGN}
14: \hspace{2em} Go back to the origin \( o \).
15: \hspace{1em} \textbf{EndIf}
16: \textbf{EndIf}
17: \textbf{EndWhile}
18: \textbf{While} \( t \geq \lambda |\hat{T}| \) \textbf{then}
19: \hspace{1em} Wait until \( U_t \neq \emptyset \);
20: \hspace{1em} Follow the route \( \hat{T} \) until the server is back to the origin \( o \);
21: \hspace{1em} Follow \texttt{REDESIGN} \( (U_t) \). \hspace{6em} \( \triangleright \) Serve the remaining requests
22: \textbf{EndWhile}
1. $U_{\lambda|\hat{T}|} \neq \emptyset$. The server starts to follow route $\hat{T}$ at time $\lambda|\hat{T}|$, and the completion time is $Z^{ALG} = \lambda|\hat{T}| + |\hat{T}| \leq (1 + \lambda)Z^{OPT}$.

2. $U_{\lambda|\hat{T}|} = \emptyset$. After time $\lambda|\hat{T}|$, the server waits until the next request to come. Let $x_j = (t_j, p_j)$ denote this request. The completion time can be stated as $Z^{ALG} = \lambda|\hat{T}| + (t_j - \lambda|\hat{T}|) + |\hat{T}|$. Note that the online server waits at the origin $o$ for $t_j - \lambda|\hat{T}|$. Then, we consider the duration $t_j - \lambda|\hat{T}|$ as the sum of the two time intervals according to the operations of the offline optimal server, i.e., $\text{OPT}$: $t_{move}$ denotes how much time the OPT is moving, and $t_{idle}$ denotes the length of time the OPT has nothing to do and can only wait. Since the server operated by the algorithm is also not visiting any request during $t_{idle}$, the case $t_{idle} \neq 0$ favors the algorithm. Thus, we can simply assume the worst case $t_{idle} = 0$. In addition, we know the distance the OPT saves during $t_{move}$ by moving earlier than the online server cannot be longer than $\frac{1}{\lambda}Z^{OPT}$. Then, we have $t_{move} \leq \frac{1}{\lambda}Z^{OPT}$, and the completion time is $Z^{ALG} \leq \lambda|\hat{T}| + t_{move} + |\hat{T}| \leq (1.5 + \lambda)Z^{OPT}$.

\begin{lemma}
The LADAR-NID algorithm is $(3.5 + 2.5/\lambda)$-robust, where $\lambda \in (0, 1]$.
\end{lemma}

\textbf{Proof.} We show the robustness is $3.5 + 2.5/\lambda$ by the following cases.

- 1. The server finishes before $\lambda|\hat{T}|$. The server does what the original REDesign algorithm would do. Thus, by Theorem 24, the server obtains the bound $Z^{ALG} \leq 2.5Z^{OPT}$.

- 2. The server cannot finish before $\lambda|\hat{T}|$. After following $|\hat{T}|$, we cannot guarantee all requests in $x$ have been served. Therefore, the server switches to the REDesign algorithm to visit the remaining requests.
  - 2-1. $U_{\lambda|\hat{T}|} \neq \emptyset$ and $t_n > (1 + \lambda)|\hat{T}|$. We have $Z^{ALG} \leq 2.5Z^{OPT}$ by Corollary 25.
  - 2-2. $U_{\lambda|\hat{T}|} \neq \emptyset$ and $t_n \leq (1 + \lambda)|\hat{T}|$. The completion time is
    \[
    Z^{ALG} \leq \lambda|\hat{T}| + |\hat{T}| + |T_{U_{(1+\lambda)|\hat{T}|}}| \\
    \leq 2.5Z^{OPT} + (2.5/\lambda)Z^{OPT} + Z^{OPT} \\
    \leq (3.5 + 2.5/\lambda)Z^{OPT}
    \]

- 2-3. $U_{\lambda|\hat{T}|} = \emptyset$ and $t_n > t_j + |\hat{T}|$. Similarly, we have $Z^{ALG} \leq 2.5Z^{OPT}$ by Corollary 25.

- 2-4. $U_{\lambda|\hat{T}|} = \emptyset$ and $t_n \leq t_j + |\hat{T}|$. The server completes visiting the requests at
    \[
    Z^{ALG} \leq t_j + |\hat{T}| + |T_{U_{t_j+|\hat{T}|}}| \\
    \leq Z^{OPT} + (1/\lambda)Z^{OPT} + Z^{OPT} \\
    \leq (2 + 1/\lambda)Z^{OPT}
    \]

\begin{theorem}
The LADAR-NID algorithm is $(1.5 + \lambda)$-consistent and $(3.5 + 2.5/\lambda)$-robust but not smooth, where $\lambda \in (0, 1]$ is the confidence level.
\end{theorem}

\textbf{Proof.} We completes the proof by combining Lemma 29 and Lemma 30.
D.5 Sequence Prediction with Identity

The server gets a sequence prediction \( \hat{x} \), which has the same size \( n \) as the actual sequence \( x \). Since the arrival time of the last request provides a lower bound of the optimal completion time, the server can follow the route \( \hat{T} \) earlier than in the previous model and still achieve robustness. Note that the route \( \hat{T} \) can be described as a sequence of positions including \( \hat{a}_i \) and \( \hat{b}_i \) for any \( i \). Before time \( t_n \), the server follows the route \( \hat{T} \) while making necessary adjustments: (1) if the server has reached the predicted position \( \hat{p}_i \) but the request \( x_i \) has not arrived, the server waits at \( \hat{p}_i \) until \( x_i \) comes, (2) when the request \( x_i = (t_i, a_i, b_i) \) arrives, the server inserts position \( a_i \) after \( \hat{a}_i \) and position \( b_i \) after \( \hat{b}_i \) in the route \( \hat{T} \). When the request \( x_n \) arrives, the server knows it is the last one since the number of requests \( n \) is given. Thus, the server weighs its two options: one is to continue on the route \( \hat{T} \), whose distance is denoted as \( r_1 \), and the other is to go back to the origin and design a new route, denoted as \( r_2 \). Then, the server chooses the shorter one.

**Theorem 32.** The competitive ratio of the LADAR-ID algorithm is \( \min\{3, 1 + (2\varepsilon_{\text{time}} + 4\varepsilon_{\text{pos}})/2Z^{\text{OPT}}\} \). Thus, the algorithm is 1-consistent, 3-robust, and \( (2\varepsilon_{\text{time}} + 4\varepsilon_{\text{pos}}) \)-smooth.

**Proof.** The server follows an optimal route \( \hat{T} \) before time \( t_n \) while making adjustments. Then, at time \( t_n \), it consider two available options. Note that \( t_n \leq Z^{\text{OPT}} \) and \( |T_{U_1}| \leq Z^{\text{OPT}} \) for any \( t \).

1. \( r_1 \). The server continues on the route \( \hat{T} \). This implies that the server follows the route \( \hat{T} \) from beginning to end. By Theorem 26, the completion time of the algorithm can be bounded by \( Z^{\text{ALG}} \leq \min\{3, 1 + (2\varepsilon_{\text{time}} + 4\varepsilon_{\text{pos}})/2Z^{\text{OPT}}\} \).

2. \( r_2 \). The server goes back to the origin \( o \) at time \( t_n + d(p(t_n), o) \) and starts to follow a new route \( T \), which visits all unserved requests in \( U \). Since no more request arrives after \( t_n \), the server completes at time \( Z^{\text{ALG}} = t_n + d(p(t_n), o) + |T_{U_1}| \). In addition, the distance between the server and the origin is bounded by \( d(p(t_n), o) \leq t_n \). Combined with the two inequalities mentioned above, we obtain the result \( Z^{\text{ALG}} \leq 3Z^{\text{OPT}} \).

As the server choose the shorter route among \( r_1 \) and \( r_2 \), the competitive ratio can be stated as \( \min\{3, 1 + (2\varepsilon_{\text{time}} + 4\varepsilon_{\text{pos}})/2Z^{\text{OPT}}\} \).

D.6 Prediction of the Last Arrival Time

The idea is to be at the origin \( o \) and starts the final route when the last request \( x_n \) arrives. Thus, the server return to the origin \( o \) at the predicted last arrival time \( \hat{t}_n \) and follow the REDesign algorithm for the rest of the time. Whenever the server is at the origin \( o \), it finds a route \( T_{U_1} \) to visit the requests in \( U_1 \). If the route \( T_{U_1} \) is too long, the server designs a shorter route \( T'_{U_1} \) which would allow itself to be at the origin at time \( \hat{t}_n \); otherwise, the server follows the route \( T_{U_1} \) directly. If any request \( x_i = (t_i, a_i, b_i) \) arrives, the server goes back to the origin \( o \) and finds a new route.

**Theorem 33.** The LADAR-LAST algorithm is \( \min\{3, 5, 2 + |\varepsilon_{\text{last}}|/Z^{\text{OPT}}\} \)-competitive algorithm, where \( \varepsilon_{\text{last}} = \hat{t}_n - t_n \). Therefore, this algorithm is 2-consistent, 3.5-robust, and \( |\varepsilon_{\text{last}}| \)-smooth.

**Proof.** We consider the server’s position when the last request \( x_n \) arrives. We distinguish between two main cases depending on the predicted and actual last arrival time, \( \hat{t}_n \) and \( t_n \). Note that we get \( t_n \leq Z^{\text{OPT}} \) trivially.
Algorithm 9: Learning-Augmented Dial-a-Ride With Identity (LADAR-ID)

**Input:** The current time \( t \), the number of requests \( n \), a sequence prediction \( \hat{x} \), and the set of current released unserved requests \( U_t \).

1. \( F = 0; \quad \triangleright \) Initialize \( F = 0 \) to indicate that we trust the prediction at first
2. First, compute an optimal route \( \hat{T} = (\hat{p}(1), \ldots, \hat{p}(2n)) \) to serve the requests in \( \hat{x} \) and return to the origin \( o \), where \( \hat{p}(i) \in \{\hat{a}_1, \ldots, \hat{a}_n, \hat{b}_1, \ldots, \hat{b}_n\} \) is the \( i^{th} \) position the server reaches in \( \hat{T} \).
3. Start to follow the route \( \hat{T} \);
4. While \( F = 0 \) do \quad \triangleright \) Trust the prediction
5. If \( t = t(i) \) for any \( i \), where \( x(i) = (t(i), a(i), b(j)) \) then
6. Update the route \( \hat{T} \) by adding positions \( a(i) \) and \( b(j) \) after their corresponding positions \( \hat{p}(i) \) and \( \hat{p}(j) \); \quad \triangleright \) Update the route
7. If \( t = t_n \) then \quad \triangleright \) Find the shorter route
8. \( r_1 \leftarrow \) the remaining distance of following \( \hat{T} \);
9. Compute a route \( T_{U_{t_n}} \) to start and finish at the origin \( o \) and serve the requests in \( U_{t_n} \);
10. \( r_2 \leftarrow d(p(t), o) + |T_{U_{t_n}}| \);
11. If \( r_1 > r_2 \) then
12. Go back to the origin \( o \); \quad \triangleright \) Give up the predicted route
13. \( F = 1. \)
14. Else
15. Move ahead on the current route \( \hat{T} \).
16. EndIf
17. Else
18. Move ahead on the current route \( \hat{T} \).
19. EndIf
20. EndIf
21. If \( p(t) = \hat{p}(i) \) and \( t < t(i) \) then
22. Wait at position \( \hat{p}(i) \) until time \( t(i) \). \quad \triangleright \) Wait until the request arrives
23. EndIf
24. EndWhile
25. While \( F = 1 \) do \quad \triangleright \) Do not trust the prediction
26. Start to follow the route \( T_{U_{t_n}} \) to serve the requests in \( U_{t_n} \).
27. EndWhile
The current time $t$, the predicted last arrival time $\hat{t}_n$, and the set of current released unserved requests $U_t$.

1. While $U_t \neq \emptyset$ do

2. If the server is at the origin $o$ (i.e., $p(t) = o$) then

3. Compute an optimal route $T_{U_t}$ to serve the requests in $U_t$ and return to the origin $o$;

4. If $t < t_n$ and $t + |T_{U_t}| > t_n$ then \(\triangleright\) Add a gadget

5. Find the moment $t_{back}$ such that $t_{back} + d(p(t_{back}), o) = \hat{t}_n$;

6. Redesign a route $T'_{U_t}$ by asking the server to go back to the origin $o$ at time $t_{back}$ along the shortest path;

7. Start to follow the route $T'_{U_t}$.

8. Else

9. Start to follow the route $T_{U_t}$.

10. EndIf

11. ElseIf the server is currently moving along a route $T_{U_{i'}}$, for some $t' < t$ then

12. If a new request $x_i = (t_i, a_i, b_i)$ arrives then \(\triangleright\) Similar to Redesign

13. Go back to the origin $o$.

14. EndIf

15. EndIf

16. EndWhile

\begin{itemize}
\item 1. $\hat{t}_n \leq t_n$. The server is at the origin at time $\hat{t}_n$. After then, it receives some request and returns to the origin only when a request arrives or a route is completed. Therefore, the completion time is $Z^{ALG} = t_n + d(p(t_n), o) + |T_{U_t}|$, which leads to $Z^{ALG} \leq 2Z^{OPT} + d(p(t_n), o)$. Note that we can extend Lemma [18] to the REDISEN algorithm on the DARP.

\item 1-1. By Lemma [18] we get $d(p(t_n), o) \leq 0.5Z^{OPT}$ and our first bound $Z^{ALG} \leq 2.5Z^{OPT}$.

\item 1-2. The farthest position the server can be at time $t_n$ is $d(p(t_n), o) < t_n - \hat{t}_n = |\epsilon_{last}|$, since the server it at the origin at time $\hat{t}_n$. Our second bound for case 1 is $Z^{ALG} \leq 2Z^{OPT} + |\epsilon_{last}|$.

\item 2. $\hat{t}_n > t_n$. Since the last request arrives before time $\hat{t}_n$ and the server could start its final route at $\hat{t}_n$. (if needed), we have $Z^{ALG} \leq \hat{t}_n + T_{U_t}$. Let $t_L$ be the last time before $\hat{t}_n$ that a route is planned and $T^L$ be the original route planned at time $t_L$. Let $T^L$ be the new route if $T^L$ is adjusted and equal to $T^L$ if not. Note that $t_L + |T^L| \leq \hat{t}_n$ due to the choice of $t_{back}$.

\item 2-1. $t_L + |T^L| < \hat{t}_n$. The server finishes serving all requests in $x$ before time $\hat{t}_n$ and thus $Z^{ALG} \leq \hat{t}_n$. In this case, since the algorithm runs in the same way as the REDISEN algorithm, we know that $Z^{ALG} \leq 2.5Z^{OPT}$ by Theorem [24] still holds.

\item 2-2. $t_L + |T^L| \geq \hat{t}_n$ and $t_n \leq t_L$. The route $T^L$ is too long, and the algorithm find a new route $T^{L'}$ to let the server be at the origin $o$ at time $\hat{t}_n$. Since every request in $x$ arrives before $t_L$, the server could have finished visiting the requests by following the original route $T^L$. However, the server chooses the adjusted route $T'^L$ instead. The additional cost of moving back to the origin is at most $2d(p(t_{back}), o)$. Thus, by Lemma [18] and Lemma [19] the completion time is $Z^{ALG} \leq 2.5Z^{OPT} + 2d(p(t_{back}), o) \leq 3.5Z^{OPT}$.

\item 2-3. $t_L + |T^L| \geq \hat{t}_n$ and $t_n > t_L$. The route $T^L$ is too long, and at least one request is unknown at time $t_L$. The server has to start a new route at $\hat{t}_n$, which is the last
route since all requests have arrived by then. Noting the route $T^L$ also satisfies $|T^L| \leq Z^{OPT}$, the completion time is

$$Z^{ALG} \leq \hat{t}_n + |T_{U_{\hat{t}_n}}|$$

$$= t_L + |T^L| + |T_{U_{\hat{t}_n}}|$$

$$\leq t_n + |T^L| + Z^{OPT}$$

$$\leq 3Z^{OPT}$$

Combining with the results in 2-1 and 2-2, we have our first bound $Z^{ALG} \leq 3.5Z^{OPT}$. 2-4. The server might have not reached the origin at time $t_n$. However, if the error $\epsilon_{last}$ is small, it must be close to the origin $o$ because of $t_{back}$. Formally, we have $Z^{ALG} \leq t_n + \epsilon_{last} + |T_{U_{\hat{t}_n}}|$ by replacing $\hat{t}_n$ with $t_n + \epsilon_{last}$, which leads to our second bound $Z^{ALG} \leq 2Z^{OPT} + \epsilon_{last}$. Both cases have shown $Z^{ALG} \leq \min\{3.5Z^{OPT}, 2Z^{OPT} + |\epsilon_{last}|\}$. ◀