XYZ - SU3 Breakings from Laplace Sum Rule at Higher Orders: Summary

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Abstract
This talk reviews and summarizes some of our results in [1] on XYZ SU3 Breakings obtained from QCD Laplace Sum Rules (LSR) at next-to-next-leading order (N2LO) of perturbative (PT) theory and including next-to-leading order (NLO) SU3 breaking corrections and leading order (LO) contributions of dimensions \(d \leq (6 - 8)\) non-perturbative condensates. We conclude that the observed X states are good candidates for being 1++ and 0++ molecules states. We find that the SU3 breakings are relatively small for the masses (\(\leq 10\) (resp. 3 \%) for the charm (resp. bottom) channels while they are large (\(\leq 20\) \%) for the couplings. Like in the chiral limit case, the couplings decrease faster: \(1/m_c^{3/2}\) than \(1/m_b^{1/2}\) of HQET. Our approach cannot clearly separate (within the errors) some molecule states from the four-quark ones with the same quantum numbers.

Keywords: QCD Spectral Sum Rules, Perturbative and Non-perturbative QCD, Exotic hadrons, Masses and Decay constants.

1. Introduction
In recent papers [1–5], we have used QCD spectral (Laplace [6–9] and FESR [10]) sum rules [11, 12, 14] to improve some previous LO results for the masses and decay constants of the XYZ exotic heavy-light and charmonium-like mesons obtained in the chiral limit [5, 15–17]. In so doing, we include the SU3 NLO PT corrections into the N2LO PT factorizable chiral limit corrections to the heavy-light exotic correlators. To these higher order (HO) PT contributions, we add the LO contribution of condensates having a dimension \(d \leq 6\). Like in the chiral limit case [2], we do not include into the analysis contributions of condensates of higher dimension \((d \geq 8)\) but only consider their effects as a source of systematic errors due to the truncation of the Operator Product Expansion (OPE).

Recent measurements of the \(J/\psi\phi\) invariant masses from \(B^+ \rightarrow J/\psi\phi K^+\) decays by the LHCb collaboration [18] confirmed the existence of the \(X(4147)\) and \(X(4273)\) with quantum numbers 1++ found earlier by
the CDF [19, 20], the CMS [21] and the D0 [22] collaborations. In the same time, the LHCb collaboration has reported the existence of the $0^{++}$ states in the analogous $J/\psi\phi$ invariant masses.

2. Molecules and four-quark two point functions

We shall work with the transverse part $\Pi_{mol}^{(1)}$ of the two-point spectral functions $^{2}$:

$$\Pi_{mol}^{(1)}(q) \equiv i \int d^3x \ e^{iq\cdot x} \langle 0 | T[O_{mol}(x)O_{mol}^\dagger(0)] | 0 \rangle$$

$$= -\Pi_{mol}^{(1)}(q^2) = \left( x^2 - q^2 \right) + \Pi_{mol}^{(0)}(q^2) \left( x^2 - q^2 \right)^2,$$

for the spin 1 states while for the spin 0 ones, we shall use the two-point functions $\psi_{mol}(q^2)$ built directly from the (pseudo)scalar currents:

$$\psi_{mol}(q^2) = i \int d^3x e^{iq\cdot x} \langle 0 | T[O_{mol}(x)O_{mol}(0)] | 0 \rangle,$$

which is related to $\Pi_{mol}^{(0)}$ appearing in Eq. (1) via Ward identities [11, 12, 23]. Thanks to their analyticity properties $\Pi_{mol}^{(1)}$ and $\psi_{mol}$ obey the dispersion relation:

$$\Pi_{mol}^{(1,0)}(q^2); \psi_{mol}(q^2) = \frac{1}{\pi} \int_0^\infty \frac{dt}{t-q^2-i\epsilon} \left( \text{Im}\Pi_{mol}^{(1,0)}(t); \text{Im}\psi_{mol}(t) \right),$$

where $\text{Im}\Pi_{mol}^{(1,0)}(t)$ and $\text{Im}\psi_{mol}(t)$ are the spectral functions.

- **Interpolating currents**

  The interpolating current $O_{mol}$ for the molecules and $O_{eq}$ for four-quark states are given in Table 1 and Table 2.

- **Spectral function within MDA**

  We shall use the Minimal Duality Ansatz (MDA) given in Eq.4 for parametrizing the spectral function:

$$\frac{1}{\pi} \text{Im}\Pi_{mol}(t) = f_{HI}^2 M_H^6 \delta(t - M_H^2) + \text{“QCD continuum”} \theta(t - \tau_c),$$

where $f_{HI}$ is the decay constant defined as:

$$\langle 0 | O_{mol}^\dagger(H) = f_{HI} M_H^6 \epsilon_\mu, \langle 0 | O_{mol}^\dagger(H) = f_{HI} M_H^6 \epsilon_\mu,$$

respectively for spin 0 and 1 states with $\epsilon_\mu$ the (axial-)vector polarization. The higher order states contributions are smeared by the “QCD continuum” coming from the discontinuity of the QCD diagrams and starting from a constant threshold $\tau_c$.

$^{2}$Hereafter, similar expressions will be obtained for the four-quark states by replacing the sub-index mol by eq.

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**Table 1:** Interpolating currents $O_{mol}$ with a definite C-parity describing the molecule-like states. $Q \equiv c$ (resp. $b$) for the $D_s D_s$ (resp. $B_s B_s$)-like molecules.

| States | $J^{PC}$ | Molecule current $O_{mol}(x)$ |
|--------|----------|-------------------------------|
| Scalar | 0$^{++}$ | $D_J D_J$, $B_s B_s$, $D_J^0 D_J^+$, $B_s^0 B_s^+$, $D_J^0 D_J^+$, $B_s^0 B_s^+$ |
| Axial-vector | 0$^{-+}$ | $D_J D_J^*$, $B_s B_s^*$, $D_J^0 D_J^*$, $B_s^0 B_s^*$ |
| Pseudoscalar | 0$^{++}$ | $D_J^0 D_J^*$, $B_s^0 B_s^*$ |
| Vector | 1$^{-+}$ | $D_J^0 D_J^*$, $B_s^0 B_s^*$ |

**Table 2:** Interpolating currents describing the four-quark states. $Q \equiv c$ (resp. $b$). $k$ is an arbitrary current mixing where the optimal value is found to be $k = 0$ from [16].

| States | $J^{PC}$ | Four-quark current $O_{eq}(x)$ |
|--------|----------|-------------------------------|
| Scalar | 0$^{++}$ | $c_s c_s c_s^* c_s^*$, $c_s c_s c_s CQ^2, c_s C Q_s C Q_s$ |
| Axial-vector | 1$^{-+}$ | $c_s c_s c_s CQ^2, c_s C Q_s C Q_s$ |
| Pseudoscalar | 0$^{++}$ | $c_s c_s c_s CQ^2, c_s C Q_s C Q_s$ |
| Vector | 1$^{-+}$ | $c_s c_s c_s CQ^2, c_s C Q_s C Q_s$ |

- **NLO and 2LO PT corrections using factorization**

  Assuming a factorization of the four-quark interpolating current as a natural consequence of the molecular definition of the state, we can write the corresponding spectral function as a convolution of the spectral functions associated to quark bilinear current. In this way, we obtain[24] for the $DD$ and $D_s D_s^*$-like spin 1 states:

$$\frac{1}{\pi} \text{Im}\Pi_{mol}^{(1)}(t) = \theta(t - 4M_Q^2) \left[ \frac{1}{4\pi} \right]^2 \int_{M_Q^2}^{\sqrt{-M_Q^2}^2} \left( \sqrt{-t} - \sqrt{-t_1} \right) \int_{M_Q^2}^{\sqrt{-M_Q^2}^2} dt_1 dt_2 \times \sqrt{1/2} \sqrt{1/2} \text{Im}\Pi_{mol}^{(1)}(t_1) \text{Im}\psi^{(s,p)}(t_2).$$

For the $DD$ spin 0 state, one has:

$$\frac{1}{\pi} \text{Im}\psi^{(s)}(t) = \theta(4M_Q^2) \left[ \frac{1}{4\pi} \right]^2 \int_{M_Q^2}^{\sqrt{-M_Q^2}^2} \left( \sqrt{-t} - \sqrt{-t_1} \right) \int_{M_Q^2}^{\sqrt{-M_Q^2}^2} dt_1 dt_2 \times \sqrt{1/2} \left( \frac{t_1 + t_2}{t} - 1 \right)^2 \frac{1}{\pi} \text{Im}\psi^{(s,p)}(t_1) \frac{1}{\pi} \text{Im}\psi^{(s,p)}(t_2).$$
and for the $\tilde{D}^*D^*$ spin 0 state:
\[
\frac{1}{\pi} \text{Im} \Pi^{(s)}_{\text{mol}}(t) = \theta(t - 4M_0^2) \left( \frac{1}{4\pi^2} \right)^2 \left( \int_0^{\tau} \frac{dt_1}{M_0^2} \int_0^{\tau} \frac{dt_2}{M_0^2} \right) \times \lambda^{1/2} \left( \frac{t_1}{t_2} + \frac{t_2}{t_1} - 1 \right) + \frac{8\tau t_2}{t^3} \times \frac{1}{\pi} \text{Im} \Pi^{(1)}(t_2) \text{Im} \Pi^{(1)}(t_2),
\]
\[
\lambda = \left( 1 - \frac{\sqrt{\tau_1} - \sqrt{\tau_2}}{t} \right)^2 \left( 1 - \frac{\sqrt{\tau_1} + \sqrt{\tau_2}}{t} \right)^2,
\]
is the phase space factor and $M_0$ is the on-shell heavy quark mass. $\text{Im} \Pi^{(1)}(t)$ is the spectral function associated to the bilinear $\bar{c}i(\gamma_5)q$ (axial-)vector current, while $\text{Im} \Pi^{(s)}(t)$ is associated to the $\bar{c}i(\gamma_5)q$ (pseudo)scalar current. We shall assume that a such factorization also holds for four-quark states.

- **The inverse Laplace transform sum rule (LSR)**

The LSR and its ratio read:
\[
\mathcal{L}_{\text{mol}}(\tau, t, \mu) = \frac{1}{\pi} \int_{4M_0^2}^{\tau} dt \ e^{-\tau t} \text{Im} \Pi^{(1)}(t_1) \Pi^{(1)}(t_2),
\]
\[
\mathcal{R}_{\text{mol}}(\tau, t, \mu) = \left( \int_{4M_0^2}^{\tau} \frac{dt_1}{M_0^2} \int_{4M_0^2}^{\tau} \frac{dt_2}{M_0^2} \right) \approx M_0^2,
\]
where $\mu$ is the subtraction point which appears in the approximate QCD series when radiative corrections are included and $\tau$ is the sum rule variable replacing $\tau$. Double ratios of inverse Laplace transform sum rule

Double Ratios of Sum Rules (DRSR) [11, 12, 25–30] are also useful for extracting the SU3 breaking effects on couplings and mass ratios. They read:
\[
\mathcal{L}_{\text{mol}}^{sd}(\tau, t, \mu) = \frac{\mathcal{L}_{\text{mol}}(\tau, t, \mu)}{\mathcal{R}_{\text{mol}}(\tau, t, \mu)},
\]
\[
\mathcal{R}_{\text{mol}}^{sd}(\tau, t, \mu) = \frac{\mathcal{R}_{\text{mol}}(\tau, t, \mu)}{\mathcal{R}_{\text{mol}}(\tau, t, \mu)},
\]
the upper indices $s, d$ indicates the $s$ and $d$ quark channels. These DRSR can be used when each sum rule optimizes at the same values of the parameters $(\tau, t, \mu)$.

- **Stability criteria and some phenomenological tests**

The variables $\tau, \mu$ and $t_0$ are, in principle, free external parameters. We shall use stability criteria (if any) with respect to these free 3 parameters, for extracting the optimal results. In the standard MDA given in Eq.4 for parametrizing the spectral function, the “QCD continuum” threshold $t_0$ is constant and is independent on the subtraction point $\mu$. One should notice that this standard MDA with constant $t_0$ describes quite well the properties of the lowest ground state as explicitly demonstrate in [31] and in various examples [11, 12] after confronting the integrated spectral function within this simple parametrization with the full data measurements. It has been also successfully tested in the large $N_c$ limit of QCD in [32]. Though it is difficult to estimate with a good precision the systematic error related to this simple model for reproducing accurately the data, we expect that the same feature is reproduced for the case of the XYZ discussed here where complete data are still lacking.

3. QCD input parameters

The QCD parameters which shall appear in the following analysis will be the charm and bottom quark masses $m_c, m_b$, the strange quark mass $m_s$ (we shall neglect the light quark masses $m_{u,d}$), the light quark condensate $\langle q\bar{q} \rangle$ ( $q \equiv u, d$), the gluon condensates $\langle \alpha_s G^2 \rangle$ and $\langle \alpha_s G^5 \rangle$ and $\langle g^4 G^3 \rangle = \langle g^4 f_{abc} G^a G^b G^c \rangle$, the mixed condensate $\langle q\bar{g} Gq \rangle = \langle q\bar{g} G q \rangle (\lambda/2) G^3 q = M_0^2 (q\bar{g} G q)$ and the four-quark condensate $\rho_{\alpha\beta}(q\bar{q} q)$, where $\rho = 3 – 4$ indicates the deviation from the four-quark vacuum saturation. Their values are given in Table 3 and more recently confirmed in [33]. The original errors on $\kappa = \langle \bar{s}s \rangle/\langle \bar{d}d \rangle$ have been enlarged to take into account the lattice result [34] which needs to be checked by some other groups. We shall work with the running light quark condensates, which read to leading order in $\alpha_s$:
\[
\langle \bar{q}q \rangle(\tau) = -\tilde{\mu}_{\bar{q}}^2 (-\beta_1 a_1)^{2/\beta_1},
\]
\[
\langle q\bar{g} G q \rangle(\tau) = -M_0^2 \tilde{\mu}_{\bar{q}}^2 (-\beta_1 a_1)^{1/\beta_1},
\]
and the running quark mass to NLO (for the number of flavours $n_f = 3$)
\[
\bar{m}_s(\tau) = \bar{m}_s (-\beta_1 a_1)^{2/\beta_1}(1 + 0.8951 a_1),
\]
where $\beta_1 = -(1/2)(11 - 2n_f/3)$ is the first coefficient of the $\beta$ function for $n_f$ flavours; $a_1 = \alpha_s(\tau) / \pi$; $\tilde{\mu}_{\bar{q}}$ and $\bar{m}_s$ is the spontaneous RGI light quark condensate [35] and strange quark mass.
Table 3: QCD input parameters: the original errors for \((\alpha_s G^2), (q^2 G^2)\) and \(\rho(\bar{q}q)^2\) have been multiplied by about a factor 3 for a conservative estimate of the errors (see also the text).

| Parameters \((M_T)\) | Values | Ref. |
|----------------------|--------|------|
| \(\alpha_s(M_T)\)   | 0.325(8) | [36–39] |
| \(m_s\)             | \(0.114 \pm 0.006\) GeV | [11, 36, 40–44] |
| \(\bar{m}_b(m_b)\)  | 1261(12) MeV | average [45–51] |
| \(\bar{m}_d(m_b)\)  | 4177(11) MeV | average [45–49] |
| \(\bar{\rho}_q\)    | (253 \pm 6) MeV | [11, 40–44] |
| \(\kappa \equiv (\bar{s}t)/(\bar{d}t)\) | \((0.74 \pm 0.13)\) | [11, 28, 29] |
| \(M_b^2\)           | (0.8 \pm 0.2) GeV\(^2\) | [52–57] |
| \((\alpha_s G^2)\)  | \((7 \pm 3) \times 10^{-2}\) GeV\(^4\) | [36, 46–48, 58–63] |
| \((q^2 G^2)\)       | \((8.2 \pm 2.0)\) GeV\(^2\) \times \((\alpha_s G^2)\) | [46–48] |
| \(\rho_{s\bar{s},(\bar{q}q)^2}\) | \((5.8 \pm 1.8) \times 10^{-4}\) GeV\(^6\) | [36, 52–54, 58] |

4. QCD expressions of the spectral functions

In our works [1, 2, 4], we provide new compact integrated expressions of QCD spectral functions at LO of PT QCD and including non-perturbative (NP) condensates having dimensions \(d \leq 6 – 8\). NLO and N2LO corrections are introduced using the convolution integrals in Eq. 6. The expressions of QCD spectral functions of heavy-light bilinear currents are known to order \(\alpha_s\) (NLO) from [64] and to order \(\alpha_s^2\) (N2LO) in the chiral limit \(m_q = 0\) from [65, 66] which are available as a Mathematica program named Rvs. We shall use the SU3 breaking PT corrections at NLO [67] from the two-point function formed by bilinear currents. N3LO corrections are estimated from a geometric growth of the QCD PT series [68] as a source of PT errors, which we expect to give a good approximation of the uncalculated higher order terms dual to the \(1/q^2\) contribution of a tachyonic gluon mass [69, 70].

In our analysis, we replace the on-shell (pole) mass appearing in the spectral functions with the running mass using the relation, to order \(\alpha_s^2\) [11–13]:

\[
M_Q = \bar{m}_Q(\mu) \left[ 1 + \frac{4}{3} \alpha_s + (16.2163 - 1.0414 n_l) \alpha_s^2 \right. \\
+ \log \left( \frac{\mu}{M_Q} \right) \left[ \alpha_s + (8.8472 - 0.3611 n_l) \alpha_s^2 \right] \\
\left. + \log^2 \left( \frac{\mu}{M_Q} \right) \left[ (1.7917 - 0.0833 n_l) \alpha_s^2 - \ldots \right] \right],
\]

for \(n_l\) light flavours where \(\mu\) is the arbitrary subtraction point and \(\alpha_s \equiv \alpha_s / \pi\).

5. Tests of the Factorization Assumption

- **Factorization test for PT@NP contributions at LO**

From our previous work [2, 3], we have noticed that assuming a factorization of the PT at LO and including NP contributions induces an effect about 2.2% for the decay constant and 0.5% for the mass, which is quite tiny. However, to avoid this (small) effect, we shall work in the following with the full non-factorized PT@NP of the LO expressions.

- **Test at NLO of PT from the \(B^0\bar{B}^0\) four-quark correlator**

For extracting the PT \(\alpha_s^0\) corrections to the correlator and due to the technical complexity of the calculations, we shall assume that these radiative corrections are dominated by the ones from factorized diagrams while we neglect the ones from non-factorized diagrams. This fact has been proven explicitly by [71, 72] in the case of \(B^0\bar{B}^0\) systems (very similar correlator as the ones discussed in the following) where the non-factorized \(\alpha_s\) corrections do not exceed 10% of the total \(\alpha_s\) contributions.

- **Conclusions of the factorization tests**

We expect from the previous LO examples that the masses of the molecules are known with a good accuracy while, for the coupling, we shall have in mind the systematics induced by the radiative corrections estimated by keeping only the factorized diagrams. The
contributions of the factorized diagrams will be extracted from the convolution integrals given in Eq. 5. Here, the suppression of the NLO corrections will be more pronounced for the extraction of the meson masses from the ratio of sum rules compared to the case of the $B^0\bar{B}^0$ systems.

6. The $0^{++}$ and $1^{±}$ Molecule States

We shall study the charm channel and their beauty analogue. Noticing that the qualitative behaviours of the curves in these channels are very similar, we shall illustrate the analysis in the case of $D_s D_s$.

- $\tau$ and $t_c$ stabilities

We study the behaviour of the coupling $f_{D_s D_s}$ (resp. mass $M_{D_s D_s}$) and their SU3 ratios $f_{D_s D_s}^{SU3}$ (resp. $r_{D_s D_s}^{SU3}$) in terms of LSR variable $\tau$ at different values of $t_c$ at NLO as shown in Fig.3 and Fig.4. We consider as an optimal estimate the mean value of coupling, mass and their SU3 ratios obtained at the minimum or inflexion point for the common range of $t_c$-values ($\sqrt{\tau} ≃ 4.8 + 2m_s$, GeV) corresponding to the starting of the $\tau$-stability ($f_{D_s D_s}^{SU3}$) and the one where (almost) $t_c$-stability ($\sqrt{\tau} ≃ 6.7 + 2m_s$, GeV) is reached for $\tau = (0.38 ± 0.02)$ GeV$^{-2}$. In this stability regions, the requirement that the pole contribution is larger than the one of the continuum is automatically satisfied.

- $\mu$ stability

The analysis of the $\mu$ subtraction point behaviour of the $D_s D_s$ coupling and mass is very similar to the chiral limit case discussed in detail in [2]. We use the optimal choice obtained there: $\mu = (4.5 ± 0.5)$ GeV.

Figure 3: a) The coupling $f_{D_s D_s}$ at NLO as function of $\tau$ for different values of $t_c$, for $\mu = 4.5$ GeV and for the QCD parameters in Tables 3; b) The same as a) but for the mass $M_{D_s D_s}$.

Figure 5: a) The mass $M_{D_s D_s}$ at NLO as function of $\tau$ for different values of $t_c$, for $\mu = 4.5$ GeV; b) The same as a) but for the SU3 ratio of couplings $f_{D_s D_s}^{SU3}$.

7. The $0^{-+}$ and $1^{±}$ Molecule States

The behaviours of curves in these channels are very similar, we shall illustrate the analysis in the cases of $D_s^0 D_s^0$ and $D_s^+ D_s^-$.

- $D_s^0 D_s^0$ molecule state

Using the optimal choice of $\mu = 4.5 ± 0.5$ GeV obtained in [2], the mass and SU3 ratios of couplings present minima for $\tau = 0.18$ (resp.0.25) GeV$^{-2}$ and $\tau = 0.22$ (resp.0.24) GeV$^{-2}$, as shown in Fig.5, within the range of $t_c$ corresponding to the beginning of the $\tau$-stability for $\sqrt{\tau} = 5.8 + 2m_s$ GeV (resp. $\sqrt{\tau} = 7.3 + 2m_s$ GeV) the one where $t_c$ stability starts to be reached. We deduce from these regions:

\begin{align*}
M_{D_s^0 D_s^0} &\approx 5604(201)\mu(17)\tau ...\mu \text{MeV}, \\
f_{D_s^0 D_s}^{SU3} &\approx 0.938(41)\mu(2)\tau ... ,
\end{align*}

(16)

using the values of coupling and mass $f_{D_s^0 D_s}^{SU3} = 240(16)$ keV and $M_{D_s^0 D_s} = 5800(115)$ MeV, from chiral limit [2], we get at NLO:

\begin{align*}
f_{D_s^0 D_s}^{SU3} &\approx 0.97(2)\mu(5)\mu(2)\tau ... , \\
f_{D_s^0 D_s}^{SU3} &\approx 225(15)\mu(10)\mu(1)\tau ... \text{keV},
\end{align*}

(17)
from the coupling also presents and we shall not show them here. But for this case, the coupling also presents $τ$-stabilities as shown in Fig.6 from $\sqrt{τ} = 6.0 + 2\sqrt{m}$ GeV (resp. $\sqrt{τ} = 7.3 + 2\sqrt{m}$ GeV) and for $τ=0.14$ (resp. 0.21) GeV$^{-2}$. Within the same range of $t_r$, the ratio of couplings presents stability at $τ=0.22$ (resp. 0.24) GeV$^{-2}$ while the minima for the mass occur at $τ=0.19$ (resp. 0.25) GeV$^{-2}$.

With the optimal results deduced from these regions and using the values of coupling and mass $f_{D^*_sD^*}$=490(25) keV and $M_{D^*_sD^*}$=5898(89) MeV, from chiral limit [2], we get at NLO:

$$
M_{D^*_sD^*} = 5724(176)_{h_s}(14)_{h_c}...MeV,
$$

$$
n_{D^*_sD^*} = 0.97(1.5)_{M(5)}_{h_s}(0)_{h_c}...,
$$

$$
f_{D^*_sD^*} = 455(22)_{M(5)}...keV,
$$

$$
f_{D^*_sD^*} = 0.93(1)....
$$

(18)

8. The Molecule States masses and couplings

The results are given in Table 4 (resp. Table 5) for the charm (resp. bottom) channel. The errors come from the QCD parameters and from the range of $τ, t_r$ and $μ$ where the optimal results are extracted. We find that the SU3 breakings are relatively small for the masses ($≤ 10$ (resp. 3) %) for the charm (resp. bottom) channels while they are large ($≤ 20$ %) for the couplings. Like in the chiral limit case, the couplings decrease faster: $1/m_b^{1/2}$ than $1/m_b^{1/2}$ of HQET.

9. Four-quark states masses and couplings

The behaviours of the corresponding curves are very similar to the previous molecule ones. The results are given in Table 6 (resp. Table 7) for the charm (resp. bottom) channel. The sources of errors are the same as in the molecules case. Our conclusion is similar to the previous case of molecule states.

10. Confrontation with some LO results and data

- **Comparison with some previous LO QSSR results**

The comparison is only informative as it is known that the LO results suffer from the ill-defined definition of the quark mass used in the analysis at this order. Most of the authors (see e.g [73–77]) use the running mass which is not justified when one implicitly uses the QCD expression obtained within the on-shell scheme. The difference between some results is also due to the way for extracting the optimal information from the analysis. Here we use well-defined stability criteria verified from the example of the harmonic oscillator in quantum mechanics and from different well-known hadronic channels.

- **Confrontation with experiments**

We conclude from the previous analysis that:

- The $0^{++}$ X(4700) experimental candidate might be identified with a $D^*_sD_s$ molecule ground state.
- The interpretation of the $0^{++}$ candidates as pure four-quark ground states is not favoured by our result.
- The $1^{++}$ X(4147) and X(4273) are compatible within the error with the one of the $D^*_sD_s$ molecule state and with the one of the axial-vector $A_c$ four-quark state.
- Our predictions suggest the presence of $0^{++} D^*_sD_s$ and $D^*_sD^*_s$ molecule states in the range (4121 ~ 4396) MeV and a $D^*_sD^-D^-_s$ state around 4841 MeV.
- We also present new predictions for the $0^{-+}, 1^{-+}$ and for different beauty states which can be tested in future experiments.

11. Conclusion

We have summarized our results for SU3 breaking at NLO and N2LO of PT [1] for molecule and four-quark states (see Table 4 to Table 7). They are important for further building of an effective theory for these exotic states and can be tested by lattice calculations. We plan to extend this analysis for the estimate of the meson widths.

Acknowledgements

We thank A. Rabemananjara for participating at the early stage of this work.
Table 4: $\bar{D}D$-like molecules couplings, masses and their corresponding SU3 ratios from LSR within stability criteria at NLO to N2LO of PT. The * indicates that the value does not come from a direct determination.

| Channels | $\beta_\alpha$ | $\beta_\alpha$ | $\beta_\alpha$ | $\beta_\alpha$ |
|----------|----------------|----------------|----------------|----------------|
| $\bar{D}_S D_0$ | 0.95(3) | 0.98(4) | 156(17) | 167(18) | 1.00(4) | 1.07(4) | 4169(48) | 4169(48) |
| $\bar{D}_S D_0$ | 0.93(3) | 0.95(3) | 265(11) | 284(14) | 1.06(3) | 1.07(3) | 4192(200) | 4196(200) |
| $\bar{D}_S D_0$ | 0.88(6) | 0.89(6) | 85(12) | 102(14) | 1.06(69) | 1.08(68) | 4277(134) | 4225(132) |
| $\bar{D}_S D_0$ | 0.90(63) | 0.90(34) | 209(28) | 229(31) | 1.09(7) | 1.09(7) | 4187(62) | 4124(61) |
| $\bar{D}_S D_0$ | – | – | 97(15) | 114(18) | – | – | 4003(227) | 3954(224) |
| $\bar{D}_S D_0$ | – | – | 236(32) | 274(37) | – | – | 3838(57) | 3784(56) |
| $\bar{D}_S D_0$ | 0.93(3) | 0.97(3) | 143(16) | 156(17) | 1.07(4) | 1.07(4) | 4174(67) | 4188(67) |
| $\bar{D}_S D_0$ | 0.90(1) | 0.82(1) | 87(14) | 110(18) | 1.11(24) | 1.10(24) | 4269(205) | 4275(206) |
| $\bar{D}_S D_0$ | – | – | 96(15) | 112(17) | – | – | 3849(182) | 3854(182) |

Table 5: $\bar{B}B$-like molecules couplings, masses and their corresponding SU3 ratios from LSR within stability criteria at NLO to N2LO of PT. The * indicates that the value does not come from a direct determination.

| Channels | $\beta_\alpha$ | $\beta_\alpha$ | $\beta_\alpha$ |
|----------|----------------|----------------|----------------|
| $\bar{B}_S D_0$ | 0.93(4) | 0.94(0) | 55(35) | 58(38) | 0.97(50) | 0.94(40) | 5684(223) | 5385(214) |
| $\bar{B}_S D_0$ | 0.93(4) | 0.90(4) | 455(34) | 506(38) | 0.97(50) | 0.97(34) | 5724(195) | 5632(192) |
| $\bar{B}_S D_0$ | 0.87(4) | 0.86(4) | 208(11) | 216(11) | 0.98(33) | 0.95(32) | 5708(184) | 5571(180) |
| $\bar{B}_S D_0$ | 0.97(3) | 0.93(3) | 202(12) | 213(13) | 0.97(33) | 0.95(31) | 5499(122) | 5272(120) |
| $\bar{B}_S D_0$ | 0.96(5) | 0.92(5) | 219(17) | 231(18) | 0.96(32) | 0.94(32) | 5699(184) | 5528(179) |
| $\bar{B}_S D_0$ | 0.92(3) | 0.88(3) | 195(13) | 212(14) | 0.95(34) | 0.95(34) | 5599(155) | 5487(152) |

Table 6: 4-quark couplings, masses and their corresponding SU3 ratios from LSR within stability criteria at NLO and N2LO of PT. The * indicates that the value does not come from a direct determination.

| Channels | $\beta_\alpha$ | $\beta_\alpha$ | $\beta_\alpha$ |
|----------|----------------|----------------|----------------|
| $\bar{B}_S D_0$ | 1.10(4) | 1.15(4) | 17(2) | 20(2) | 1.02(74) | 1.02(74) | 10884(74) | 10906(74) |
| $\bar{B}_S D_0$ | 1.00(3) | 1.12(3) | 31(5) | 36(6) | 1.02(85) | 1.02(85) | 10944(134) | 10956(134) |
| $\bar{B}_S D_0$ | 1.11(5) | 1.07(5) | 13(3) | 17(4) | 1.05(11) | 1.03(11) | 11182(227) | 11014(224) |
| $\bar{B}_S D_0$ | 1.197(73) | 1.214(74) | 24(5) | 29(6) | 1.04(2) | 1.03(2) | 10959(170) | 10882(169) |
| $\bar{B}_S D_0$ | – | – | 20(3) | 28(6.4) | – | – | 10514(149) | 10514(149) |
| $\bar{B}_S D_0$ | 1.01(3) | 1.18(4) | 16(7) | 20(2) | 1.02(84) | 1.03(84) | 10972(195) | 10972(195) |
| $\bar{B}_S D_0$ | 0.808(4) | 0.79(4) | 9.1(2.2) | 10.7(2.6) | 1.05(24) | 1.03(24) | 11234(208) | 11021(204) |

Table 7: 4-quark couplings, masses and their corresponding SU3 ratios from LSR within stability criteria at NLO and N2LO of PT. The * indicates that the value does not come from a direct determination.
