DERIVED EQUIVALENCE BETWEEN SHIODA’S FOURFOLD AND CM MUMFORD FOURFOLD

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Abstract. In [2] Shioda proved that the Jacobian $A_S$ of the curve $y^2 = x^9 - 1$ is a 4-dimensional CM abelian variety with codimension 2 Hodge cycles not generated by divisors. It was noted by Shioda that this behavior resembles the abelian varieties constructed by Mumford in [1]. We prove that Shioda’s fourfold $A_S$ is isogenous to an abelian variety that is derived equivalent to a CM Mumford fourfold $A_M$, but it cannot be realized as a special case of Mumford’s construction.

1. Introduction

In [1] Mumford constructed families of abelian fourfolds with exceptional Hodge cycles in their self-products (see [8] Example 5.9). Although the families constructed by Mumford are one-dimensional, whether they intersect with the moduli space of Jacobians of genus 4 curves is unknown (Problem 1 in [4]). It was noted by Shioda in [2] (Section 2) that, up to isogeny, the Jacobian $A_S$ of the genus 4 curve $y^2 = x^9 - 1$ demonstrates similar behavior as Mumford’s construction, which naturally leads to the question as to whether it can be realized as a special point of Mumford’s loci. The story of CM Mumford fourfolds remains mysterious, with some results proven over local places by Noot in [3].

In this paper we will introduce the Mumford-Tate group of CM abelian varieties, some basic facts about Mumford-Tate groups regarding products of abelian varieties and dual abelian varieties, and calculate the corresponding representations of CM Mumford fourfold (denoted $A_M$) and $A_S$ respectively. We will use Orlov’s theorem that two abelian varieties are derived equivalent if and only if their products with their dual yield to isomorphic abelian varieties, and translate the problem into a representation-theoretic one. We will then compare the weights of the representations of interest and their symmetric power, and arrive at the following theorem:

**Theorem 1.1.** $A_S$ itself is not a special case of Mumford’s construction. However, up to isogeny $A_S$ is derived equivalent to a CM Mumford fourfold $A_M$ equipped with complex multiplication by $\sqrt{-3}$.

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2. General facts about Mumford-Tate groups of CM abelian varieties

2.1. Mumford-Tate representations. The main reference for this section is [3] chapter 3.

We begin with the classical story: a Hodge structure on a rational vector space \( V \) of weight 1 is a homomorphism of real algebraic groups:

\[
h : \mathbb{C}^x \to GL(V \otimes \mathbb{R})
\]

such that \( V \otimes \mathbb{C} \) admits a decomposition into \( V^{1,0} \oplus V^{0,1} \) satisfying:

1. \( V^{1,0} \) is the complex conjugate of \( V^{0,1} \).
2. \( h(z) \) acts by multiplication by \( z^{-1} \) (resp. \( \overline{z}^{-1} \)) on \( V^{1,0} \) (resp. \( V^{0,1} \)).

Such a pair \((V, h)\) is called a \( \mathbb{Q} \)-rational Hodge structure of weight 1. If we restrict \( h \) to the set of norm 1 complex numbers (denoted \( U \)), then the Mumford-Tate group \( MT(V) \) of \((V, h)\) is defined to be the smallest algebraic subgroup of \( GL(V) \) (over \( \mathbb{Q} \)) such that \( h|_U : U \to GL(V \otimes \mathbb{R}) \) factors through \( MT(V) \otimes \mathbb{R} \). This yields a rational representation that will be denoted as \( \rho : MT(V) \hookrightarrow GL(V) \). We will use the term Mumford-Tate representation for the pair \((G, \rho)\).

Remark. This definition only works for Hodge structures of weight 1. Moreover, here we define what is usually called special Mumford-Tate group, as opposed to general Mumford-Tate group in [3].

Moreover, given a weight 1 Mumford-Tate representation \((G, \rho)\), we can recover \( V \) by looking at the target space of \( \rho \); similarly we can recover \( h \) by restricting \( \rho \) to the maximal compact torus of \( G_{\mathbb{R}} \). Since \( h \) decomposes \( V_{\mathbb{C}} \) into eigenspaces of weight \((1, 0)\) and \((0, 1)\), this gives us a way of assigning Hodge numbers to the tensor construction of \( V_{\mathbb{C}} \), similarly for \( G \)-invariant subspaces of such constructions. In fact, we can take \( G \)-invariant subspaces in \( V^{\otimes n} \) to be the definition of \( \mathbb{Q} \)-Hodge substructures in \( V^{\otimes n} \).

By general Tannakian formalism (Corollary 4.5 in [3]), we have an equivalence of categories between \( \text{Rep}_{\mathbb{Q}}(MT(V)) \) and all \( \mathbb{Q} \)-Hodge substructures obtained by tensor operations on \( V \).

From this the following lemma is immediate:

**Lemma 2.1.** Given a reductive \( \mathbb{Q} \) group \( G \) and two representations \( \rho_1, \rho_2 \) into \( GL(V) \), if there exists a \( \rho_i \)-invariant subspace \( W \) \((i = 1, 2)\) in \( V^{\otimes n} \) such that its Hodge numbers are different under \( \rho_1 \) and \( \rho_2 \), then the two representations cannot be equivalent.

Another classical theorem states that abelian varieties are determined up to isogeny by their weight 1 Hodge structure. Therefore, in this paper we shall only consider the case when \((V, h)\) is Hodge structure of weight 1. Tannakian formalism states that this is equivalent to studying \((G, \rho)\) that can be realized as a Mumford-Tate representation of weight 1.

It is a general fact that for any representation \( \rho \) of a reductive group we can associate it with a dual representation \( \rho^\vee \). If we consider the maximal torus in \( G \) containing the image of \( h \), the cocharacter of \( \rho^\vee \) with respect to that torus equals to the negative cocharacter of \( \rho \). Since taking complex conjugation on \( U \) is the same thing as taking inverses, we have the following lemma:

**Lemma 2.2.** If \( A \) is an abelian variety given by the Mumford-Tate representation \((G, \rho)\) up to isogeny, then the dual abelian variety \( A^\vee \) of \( A \) is given by \((G, \rho^\vee)\).
Remark. The Mumford-Tate group of a weight 1 polarized Hodge structure always lies inside a symplectic group, whose representations are always self-dual. Hence we recover the classical fact that the dual abelian variety is isogenous to the original abelian variety.

We also need another general fact for Mumford-Tate groups:

**Proposition 2.3.** (Properties 2.1.4 in [9]) Let $H_1, H_2$ be two $\mathbb{Q}$-Hodge structures with Mumford-Tate groups $G_1, G_2$. Then the Mumford-Tate group of $H_1 \oplus H_2$ is an algebraic subgroup of $G_1 \times G_2$ and it admits surjective projections onto $G_1$ and $G_2$ respectively.

Combining the proposition with the previous lemma we have the following:

**Lemma 2.4.** If $A$ is an abelian variety given by the Mumford-Tate representation $(G, \rho)$, then $A \times A^\vee$ is given by the Mumford-Tate representation $(G, \rho \oplus \rho^\vee)$.

**Proof.** From the above proposition we know the Mumford-Tate group of $A \times A^\vee$ admits a surjection onto $G$. We claim $G$ itself is the group, for the representation $\rho \oplus \rho^\vee$ will have the correct embedding of $U$ because of the definition of dual representation. □

2.2. The case of CM abelian varieties. In this section we recall Deligne’s construction of CM abelian varieties for which the Mumford-Tate group is contained in an algebraic torus (see [3] Example 3.7).

By CM-field we mean a quadratic totally imaginary extension of a totally real field; a CM-algebra is a finite product of CM-fields. Let $E = E_1 \times \ldots \times E_n$ be such an algebra. Then there exists an involution $\iota$ acting by complex conjugation on each of the factors of $E$. Let $F = F_1 \times \ldots \times F_n$ denote the subalgebra in $E$ fixed by $\iota$.

We denote $S$ for the set $\text{Hom}_{\mathbb{Q}}(E, \mathbb{C})$. Then a CM-type for $E$ is a subset $\Sigma \subset S$ such that $S = \Sigma \uplus \iota \Sigma$.

The complex structure is given by the following decomposition:

$$E \otimes_{\mathbb{Q}} \mathbb{C} \cong \mathbb{C}^S \oplus \mathbb{C}^{\iota \Sigma}.$$ 

Let $\mathbb{C}^{\Sigma}$ be the $V^{1,0}$ space and $\mathbb{C}^{\iota \Sigma}$ its complex conjugate. Then we can view $E$ as $H_1(A, \mathbb{Q})$ of some CM abelian variety $A$, for example, given by $A(\mathbb{C}) = \mathbb{C}^S/\Sigma(O_E)$. One can see that the $H_1$ with $\mathbb{Q}$-coefficients of any CM abelian varieties arises from this construction.

**Lemma 2.5.** The Mumford-Tate group $G$ of $A$ given by the CM-type $(E, \Sigma)$ lies inside $E^\times$. In particular, it is a torus that lies inside the $\mathbb{Q}$ group $U_E = \{ x \in E^\times | \text{Nm}_{E/F}(x) = 1 \}$.

Note that, as we shall see in a minute, this does not imply $G$ is the whole $U_E$. However, the proposition from the previous section says that $G$ admits surjection onto each $U_{E_i}$.

A general fact about $U(1)^n$ states that any map $U(1)^n \to U(1)$ is of the form $(g_1, \ldots, g_n) \mapsto g_1^{i_1} g_2^{i_2} \ldots g_n^{i_n}$. We will record this information by a vector $(i_1, i_2, \ldots, i_n)$. Similarly, for a map

$$\rho : U(1)^n \to U(1)^m$$

$$(g_1, \ldots, g_n) \mapsto (g_1^{i_{1,1}} \ldots g_1^{i_{1,n}}, \ldots, g_n^{i_{m,1}} \ldots g_n^{i_{m,n}})$$

We write

$$\rho = \begin{pmatrix} i_{1,1} & \cdots & i_{1,n} \\ \vdots & \ddots & \vdots \\ i_{m,1} & \cdots & i_{m,n} \end{pmatrix}$$

**Remark.** The rows in the above notation are interchangeable.
3. Describing \( A_S \) and \( A_M \) via Mumford-Tate representations

3.1. Shioda’s fourfold. It is established in Shioda’s paper [2] (Example 6.1) that the Jacobian \( A_S \) of the hyperelliptic curve \( C_9 : y^2 = x^9 - 1 \) is not simple, namely \( A_S \) is a product of an abelian threefold with CM field \( \mathbb{Q}(\zeta_9) \) and an elliptic curve with CM field \( \mathbb{Q}(\zeta_3) \).

By the previous chapter, we can describe this abelian fourfold via the Artinian ring \( E := \mathbb{Q}[x]/(x^8 + x^7 + \ldots + x + 1) \cong \mathbb{Q}(\zeta_9) \times \mathbb{Q}(\zeta_3) = H_1(A, \mathbb{Q}); \ H_1(A, \mathbb{Z}) \) can be obtained by considering the embedding of products of fractional ideals in \( \mathbb{Z}(\zeta_9) \) and \( \mathbb{Z}(\zeta_3) \) into \( H_1(A, \mathbb{Q}) \) up to a positivity condition which is induced by the Riemann condition. Therefore \( A_S \) is isogeneous to the abelian variety obtained by \( E/\mathcal{O}_E \) with the complex structure given by \( E_C = \mathbb{C}^{\Sigma} \oplus \mathbb{C}^{\Sigma} \oplus \mathbb{C}^r \oplus \mathbb{C}^r \) where \( \Sigma \) denotes the set of embeddings \( \mathbb{Q}(\zeta_9) \hookrightarrow \mathbb{C} \) such that their restriction onto \( \mathbb{Q}(\zeta_3) \) is identical.

**Lemma 3.1.** The Mumford-Tate group of \( A_S \) is

\[
G = \{ x \in \mathbb{Q}(\zeta_9) | \text{Nm}_{\mathbb{Q}(\zeta_9)/\mathbb{Q}(\zeta_9+\zeta_9)}(x) = 1 \}
\]

In particular, \( G_\mathbb{R} \cong U(1)^3 \) and the Galois group acts by cyclic group \( A_3 \) amongst the factors.

**Proof.** From Deligne’s construction we see \( G \) lies inside a rank 4 torus, and it admits surjection onto the Mumford-Tate group of the abelian threefold and the elliptic curve, which implies its rank is at least 3. To see that it’s actually rank 3, we calculate the number of trivial representations in \( H^4 \) for \( U(1)^4 \). First we embed \( U(1)^4 \) into \( SU(2)^4 \). Then

\[
H^1 = W_{1,0,0,0} \oplus W_{0,1,0,0} \oplus W_{0,0,1,0} \oplus W_{0,0,0,1}
\]
as an \( SU(2)^4 \) representation. One can compute

\[
H^4 = \mathbb{C} \oplus W_{1,0,0,0} \oplus W_{0,1,0,0} \oplus \ldots
\]

This implies that the number of \((2,2)\)-Hodge classes would be 6, as opposed to Shioda’s result, which is 8 (Example 6.1 in [2]). Thus completes the proof. \( \Box \)

**Proposition 3.2.** (Sutherland et al.) The representation of \( G_\mathbb{R} \) on \( E_\mathbb{R} \) is given by

\[
G_\mathbb{R} \cong U(1)^3 \hookrightarrow SU(2)^3 \hookrightarrow SU(2)^4 \\
(u, v, w) \mapsto (u, v, w, wu)
\]

Using our notation,

\[
\rho_{S,\mathbb{R}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}
\]

**Proof.** The general theory of Mumford-Tate groups states that \( E_\mathbb{R} \cong \mathbb{R}^8 \) splits into two irreducible, Galois invariant subrepresentations of \( SU(2)^3 \), one is 6 dimensional and the other is 2 dimensional. By general representation theory, we know that these representations must be given by direct sums and exterior tensors of representations of \( SU(2) \) of highest weight 1 (because we are working with weight 1 space, and \( SL_2(\mathbb{R}) \) representations are all self-dual). Moreover, the list of weight vectors in each sub-representation should be invariant under the Galois group action, in other words, permuting the factors of exterior tensors.

It remains to show that \( \Sigma \) is the set we described. In [2] Shioda has already fixed a canonical basis for \( H^{1,0} \), namely the holomorphic 1-forms given by \( \eta_v = x^{v-1}dx/y, v = \ldots \)
1, 2, 3, 4. If we perform CM by \( \zeta \) on \( y^2 = x^9 - 1 \), we see that each \( \eta \) is an eigenvector with eigenvalue \( \zeta^e \). This implies that \( \Sigma \cup \{ \tau \} = \{ \zeta_9, \zeta_3^3, \zeta_3, \zeta_9^3 \} \), therefore determining the representation. \( \square \)

3.2. **Mumford fourfold with CM structure.** A description of the construction of general Mumford fourfolds can be found in [II], which gives rise to an isogeny class of abelian fourfolds. We will describe the Mumford-Tate representation \((G, \rho)\) when such a fourfold comes with a CM structure.

To begin with, the map \( h : U \to GL(\mathbb{R}^8) \) is the same as the general case:

\[
h(e^{i\theta}) = \text{Id}_2 \otimes \text{Id}_2 \otimes \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}
\]

The Mumford-Tate group \( G \) in this case will be a torus defined over a totally real cubic field \( K \) with \( \mathbb{Q} \)-dimension 3. In other words, \( G \otimes \mathbb{Q} \cong U(1) \times U(1) \times U(1) \), each factor is given by the different embeddings \( \sigma_i : K \to \mathbb{R} \) where \( i = 1, 2, 3 \).

The representation \( \rho_C \) is given by tensoring three copies of 2-dimensional representations of \( U(1) \), the standard notation of which is \( W_{1,1,1} \). If we write out the basis of \( W_{1,1,1} \) by \( \{ v_{i,j,k} | i, j, k = \pm 1 \} \) with each subindex remembering the weight, then we can also write down the symplectic form inducing the polarization

\[
\omega = v_{-1,-1,1} \wedge v_{1,1,-1} - v_{1,-1,1} \wedge v_{-1,1,-1} - v_{-1,1,1} \wedge v_{1,-1,-1} + v_{1,1,1} \wedge v_{-1,-1,-1}
\]

The form is written explicitly in a way such that it is a sum of \((1,0)\) form wedged with \((0,1)\) form. Since \( G \) must preserve this form, and by its construction it cannot take one summand to another, we have another way of writing down \( \rho_{M,\mathbb{R}} \), namely

\[
\rho_{M,\mathbb{R}} : U(1)^3 \to U(1)^4
\]

\[
(e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3}) \mapsto (e^{i(\theta_1+\theta_2+\theta_3)}, e^{i(-\theta_1+\theta_2+\theta_3)}, e^{i(-\theta_1-\theta_2+\theta_3)}, e^{i(-\theta_1+\theta_2+\theta_3)})
\]

The Galois group of \( K \) will shuffle the \( \theta_i \)'s.

**Lemma 3.3.** Suppose \( K \) is Galois over \( \mathbb{Q} \) and its Galois group is given by \( \{ 1, \sigma, \sigma^2 \} \) such that for \((g_1, g_2, g_3) \in G_{\overline{\mathbb{R}}} \), \( \sigma(g_1, g_2, g_3) = (g_2, g_3, g_1) \). If \( g \) is an element in \( G \) and we denote \( g' \) to be \( g^{-1}g^*g'^2 \), then we can write \( \rho_M \) into

\[
\rho_{M,\mathbb{R}} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}
\]

**Remark.** There is a factor in the above imbedding that stands out as Galois invariant: \( e^{i(\theta_1+\theta_2+\theta_3)} \). When \( K \) is Galois over \( \mathbb{Q} \), one could read off the CM condition imposed on the Mumford fourfold.

4. **Derived equivalence of \( A_S \) with some \( A_M \)**

We now show that \( A_S \) up to isogeny is derived equivalent to some \( A_M \) and give some arithmetic data of \( A_M \). As the previous remark has noted, if such a thing is possible, then the CM field of \( A_M \) should be given by the CM field of the elliptic curve component of \( A_S \), namely \( \mathbb{Q}(\sqrt{-3}) \). First we prove the following:
Proposition 4.1. $A_S$ is not one of Mumford fourfolds.

Proof. To see why, we recall the result by Galluzzi in [5] which shows that, for a general Mumford fourfold, there exists a Hodge substructure of $K3$ type in $\text{Sym}^2(H^1(A_M))$ with Hodge number $(1, 7, 1)$, with the representation is given by $W_{2,0,0} \oplus W_{0,2,0} \oplus W_{0,0,2}$. When the fourfold has CM, the Hodge substructure specializes since each summand now has a trivial representation of $U(1)^3$. Therefore the Hodge substructure decomposes into a summand having Hodge number $(1, 4, 1)$, and three extra Hodge cycles.

We run the same calculation for $\text{Sym}^2(H^1(A_S))$. Writing $H^1(A_S) = V_1 \oplus V_2$ with $V_1 = W_{1,0,0} \oplus W_{0,1,0} \oplus W_{0,0,1}$ and $\text{dim}(V_2) = 2$, we have

$$\text{Sym}^2(H^1(A_S)) = (\text{Sym}^2(V_2)) \oplus (V_1 \otimes V_2) \oplus (W_{2,0,0} \oplus W_{0,2,0} \oplus W_{0,0,2}) \oplus \cdots$$

In this case, however, the only Hodge substructure of dimension 6 lies inside $W_{2,0,0} \oplus W_{0,2,0} \oplus W_{0,0,2}$, and it has Hodge number $(3, 0, 3)$. By the Tannakian formalism, we know that $(G, \rho_S)$ is not $(G, \rho_M)$. □

For the derived equivalence, we begin by recalling a result of Orlov:

Proposition 4.2. (Theorem 2.10 in [7]) Two abelian varieties $A_1, A_2$ are derived equivalent over $\mathbb{C}$ if and only if $A_1 \times A_1^\vee$ is isomorphic to $A_2 \times A_2^\vee$.

Corollary 4.2.1. $A_1$ is derived equivalent to an abelian variety isogenous to $A_2$ if and only if the Mumford-Tate representations of $A_1 \times A_1^\vee$ and $A_2 \times A_2^\vee$ are $\mathbb{Q}$-equivalent.

Now consider a $A_M$ such that its Mumford-Tate group $G_M$ is defined over the totally real cubic subfield of $\mathbb{Q}(\zeta_9)$, and it decomposes into a product of CM threefold and an elliptic curve with CM field $\mathbb{Q}(\zeta_3)$. We know from our previous discussion that $A_M \times A_M^\vee$ is given by the Mumford-Tate representation $(G_M, \rho_M \oplus \rho_M^\vee)$. Comparing this representation with $(G_S, \rho_S \oplus \rho_S^\vee)$ coming from Shioda’s fourfold $A_S \times A_S^\vee$, we see that up to a group isomorphism

$$\phi: \quad G_M \rightarrow G_S$$

$$g \quad \mapsto \quad g^{-1}g^*g'^2$$

The two representations coincide:

$$\rho_S = \rho_M = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 1 & 1 \\
-1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & -1 \\
-1 & -1 & -1
\end{pmatrix}$$

Therefore we have proven the following theorem:

Theorem 4.3. The Shioda’s fourfold $A_S$ is derived equivalent to some CM Mumford fourfold $A_M$ defined over the totally real cubic subfield of $\mathbb{Q}(\zeta_9)$ with a CM action of $\sqrt{-3}$. 
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