Local Commutators and Deformations in Conformal Chiral Quantum Field Theories

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Abstract

We study the general form of Möbius covariant local commutation relations in conformal chiral quantum field theories and show that they are intrinsically determined up to structure constants, which are subject to an infinite system of constraints. The deformation theory of these commutators is controlled by a cohomology complex, whose cochain spaces consist of linear maps that are subject to a complicated symmetry property, a generalization of the anti-symmetry of the Lie algebra case.

1 Introduction

The theorem of Lüscher and Mack [Mack, 1988], which determined the commutation relations of the stress–energy tensor, is an inspiring example of how one can compute the commutators in conformal field theory just on the basis of the most general properties of a relativistic local quantum theory and conformal invariance. Using the same argument one can fix the commutators of the stress–energy tensor with an arbitrary primary field and one can almost fix the commutators of the stress–energy tensor with a quasiprimary field. We shall show that a similar strategy allows to determine the commutation relations between arbitrary conformal chiral fields (also known as “W-algebras”) up to some structure constants which we show to be subject to an infinite number of constraints, reflecting anti-symmetry of commutators and the Jacobi identity among smeared field operators. The solutions to these constraints carry information about the specific model considered.

The anti–symmetry of commutators produces a symmetry rule for the structure constants right away. However, the restrictions coming from the Jacobi identity are not visible at once, because the different terms there appear with different test functions and this does not allow us to obtain relations only among the structure constants. To do this, we study the effect of the commutator on the test function level and observe that it gives rise to local intertwiners of the $sl(2,\mathbb{R})$ action on the test function spaces. With the help of transformation matrices of local intertwiners we achieve a reduction of the field algebra, which means that we strip off the test functions. This reduced structure has the form of a bilinear bracket on a reduced field space. Apart from a mixed symmetry or anti–symmetry of this bracket, its Jacobi identity involves certain coefficient matrices multiplying the three terms of the Jacobi identity. These matrices are universal in the sense that

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they reflect only the underlying representation theory of $sl(2, \mathbb{R})$, but not the specific model. They are thus constitutive elements of a new generalized Lie–algebra-like bracket structure that can be used for the classification of $W$-algebras.

These new identities constitute an infinite number of quadratic constraints for the structure constants of $W$-algebras, not involving the test functions any more. The solutions of these constraints promote potential candidates for chiral conformal field theories. The idea to consider constraints in such form was cherished from [Bowcock, 1991], where a Jacobi identity among structure constants from commutators of Fourier modes of quasiprimary fields was considered.

We then study the deformation theory of the commutators of the reduced field algebra. The motivating example for us was [Hollands, 2008], where deformations in the setting of the OPE (operator product expansion) approach to quantum field theory on curved space–time were studied. We consider deformations in a sense of perturbative power series and work in a setting analogous to that in [Gerstenhaber, 1964], which is the prototype of deformation theory for algebraic structures. In all such theories the first step is to relate the deformation problem to a certain cochain complex. In the first examples of deformation theories of algebraic structures [Gerstenhaber, 1964, Nijenhuis & Richardson, 1967] the second step was to show that the first cohomology groups are directly related to the possibility to deform the algebraic structure considered. The more modern point of view is that the deformation theory in consideration is mastered by a differential graded Lie algebra (or in some cases a homotopy Lie algebra or $L_{\infty}$-algebra) which can be obtained from the cochain complex by constructing a bracket on this complex, which is skew symmetric with respect to the grading by dimension of the cochain spaces and satisfying a graded Jacobi identity [Nijenhuis & Richardson, 1964, Manetti, 1999, Borisov, 2005].

The cochain complex, which we constructed, consists of multilinear maps with a complicated permutation symmetry property — $Z^2$-symmetry (section 4.1). The origin of this symmetry can be traced back to the symmetry rules in the reduced algebra. We show that the first perturbations (also infinitesimal perturbations) of the reduced brackets are classes from the second cohomology group of our complex and we compute the obstruction operators to their integration. We expect that an explicit computation of the cohomology groups in the future will allow us to relate the first of these groups to the problem of rigidity of the bracket and the integrability of the first perturbations.

2 Preliminaries

The conformal group in a chiral theory is Diff($S^1$). It is represented by a unitary representation $U$ on the Hilbert space of the conformal field theory. A conformal chiral field $\Phi(z)$ on $S^1$ transforms under a diffeomorphism $\gamma$ as a covariant tensor of scaling dimension $d_{\Phi}$ if

$$U(\gamma)\Phi(z)U^{-1}(\gamma) = \left( \frac{d\gamma}{dz}\right)^{d_{\Phi}} \Phi(\gamma(z))$$

holds. For local fields, the scaling dimension is an integer. Fields which transform covariantly under the whole conformal group, are called primary. However, they do not exhaust the field content of a theory. For example, in every conformal quantum field theory is present the stress–energy tensor $T(x)$, which is responsible for infinitesimal conformal transformations. $T(x)$ transforms covariantly only under the Möbius subgroup SL(2, $\mathbb{R}$) of Diff($S^1$), and such fields are called quasiprimary. Furthermore, in the OPE of a primary field with $T(x)$ arise a series of other quasiprimary fields together with their derivatives, and such fields are called secondary.

In all that follows, we identify $\mathbb{R}$ with $S^1 \setminus \{-1\}$ by the Cayley transform, and regard the fields as distribution on $\mathbb{R}$. Since $A'(f) = -A(f')$, we don’t consider the derivatives of quasiprimary fields as independent fields. Hence a basis of the field algebra is an infinite set of quasiprimary fields. In a decent theory, e.g., such that $e^{-\beta L_0}$ is a trace-class operator, the number of quasiprimary fields of a given dimension is finite. We shall denote the basis of fields of scaling dimension $a$ by $W_a$, and assume without loss of generality that all $A \in W_a$ are hermitian fields.

The commutators of the stress–energy tensor in a chiral theory are intrinsically fixed:
Theorem 2.1 (Lüsher–Mack [Mack, 1988]). The stress–energy tensor in a chiral theory has the following commutation relations:

\[ i[T(x), T(y)] = T'(y)\delta(x - y) - 2T(y)\delta'(x - y) + \frac{c}{24}\delta'''(x - y) \]

With similar technique we find the commutator of \(T(x)\) with some arbitrary primary field \(\varphi(x)\):

\[ i[T(x), \varphi(y)] = \varphi'(y)\delta(x - y) - d_\varphi \varphi(y)\delta'(x - y) \]

and with some arbitrary quasiprimary field \(\phi(x)\):

\[ i[T(x), \phi(y)] = \phi'(y)\delta(x - y) - d_\phi \phi(y)\delta'(x - y) + \sum_{3 \leq k \leq h+1} \delta^{(k)}(x - y)\phi_k(y) \]

where \(\phi_k(x)\) are either quasiprimary fields or derivatives of quasiprimary fields of lower dimensions.

3 The general form of local commutation relations in conformal chiral field theories

In this section we will show that the commutation relations in conformal chiral field theories are intrinsically determined up to numerical factors ("structure constants") by locality, conformal invariance and Wightman positivity, and that the Lie algebra structure imposes further constraints on the possible values of the structure constants.

It will be enough to find just the commutators among the basis quasiprimary fields. Our strategy to understand the general structure of Möbius covariant commutators in chiral conformal field theories is similar to that of the Lüsher–Mack theorem:

Proposition 3.1. Locality, scale invariance and Wightman positivity imply the following general form of the commutator of two smeared quasiprimary field operators \(A(f)\) and \(B(g)\):

\[ -i [A(f), B(g)] = \sum_{c < a + b} \sum_{C \in W_c} F^C_{AB} C(\chi^c_{ab}(f, g)) , \]  

where \(a, b\) are the scaling dimensions of \(A\) and \(B\), the sum runs over a basis of quasiprimary fields of scaling dimension \(c < a + b\), \(F^C_{AB}\) are numerical coefficients, and

\[ \chi^c_{ab}(f, g) = \sum_{p, q \geq 0 \atop p + q = a + b - c - 1} \chi^c_{ab}(p, q) \partial^p f \cdot \partial^q g \]  

are bilinear maps on the test functions that preserve supports, i.e., \(\text{supp} \chi^c_{ab}(f, g) \subset \text{supp} f \cap \text{supp} g\). These maps depend only on the dimensions of the fields involved.

Proof. We present here the main steps of the proof:

1. Locality implies that the commutator \(-i [A(x), B(y)]\) has support on the line \(x = y\). Then follows that \(-i [A(x), B(y)] = \sum_{i=0}^{n} \delta^{(i)}(x - y)O_i(y)\), where \(O_i\) are linear combination of quasiprimary fields and their derivatives. This means that in the smeared version \(-i [A(f), B(g)]\) a quasiprimary field \(C\) must appear with the test function of the form \(\sum_{p, q \geq 0} d^{c}_{AB}(p, q) \partial^p f \cdot \partial^q g\). The coefficients \(d^{c}_{AB}(p, q)\) satisfy a recursion in \(p\) and \(q\), coming from Möbius invariance, and the solution of this recursion is fixed, up to a factor, only by the scaling dimensions of the fields \(A, B, C\). The numerical factor can be absorbed in the coefficients \(F^c_{AB}\).

2. Scaling invariance implies that \(p+q = n\) if \(C(y)\) is a local field of scaling dimension \(a+b-n-1\).

3. Wightman positivity implies that the scaling dimension of the fields on the theory must be non-negative (unitarity bound), hence \(c \in [0, a + b - 1]\).
Observation. The recursion for $\lambda^{c}_{ab}(p, q)$ coming from the Möbius invariance (for fixed $a, b \geq 1$ and positive $c$) is solved by:

$$\lambda^{c}_{ab}(p, q) = (-1)^q \frac{(c + b - a)_p}{p!} \frac{(c + a - b)_q}{q!} \delta_{p+q,a+b-c-1}$$

(3)

where $(x)_n$ denotes the Pochhammer symbol:

$$(x)_n := \frac{\Gamma(x + n)}{\Gamma(x)}.$$  (4)

In particular, the maps $\lambda^{c}_{ab}(f, g) = \sum_{p+q=a+b-c-1} \lambda^{c}_{ab}(p, q) \partial^p f \cdot \partial^q g$ enjoy the graded symmetry property

$$\lambda^{c}_{ab}(f, g) = (-1)^{a+b-c-1} \cdot \lambda^{c}_{ba}(g, f).$$  (5)

Note that this (anti)symmetry respects the $\mathbb{Z}_2$ grading of the source and range spaces, but the system of bilinear maps $\lambda^{c}_{ab}$ themselves don’t: there is no condition on $c$ apart from $c < a + b$.

It is noteworthy to recognize that $\lambda^{c}_{ab}$ coincide with the Rankin–Cohen brackets arising in the theory of modular forms. The latter are bilinear differential maps $[f, g]_n : M_{2k} \times M_{2l} \rightarrow M_{2k+2l+2n}$ on the spaces of modular forms of weights $2k, 2l$ ([Rankin, 1956; Cohen, 1975; Cohen et al., 1996]). In this context, of course, the test functions have to be replaced by modular forms, and the emphasis is on the discrete subgroup SL(2, Z) of SL(2, $\mathbb{R}$), under which modular forms are invariant. The precise relation is (with notations as in [Cohen et al., 1996])

$$\lambda^{c}_{ab}(f, g) \equiv [f, g]_{(k=1-a, l=1-b)}.$$  (6)

We will give some more comments in App. A.

It becomes clear that the overall structure of the commutators in conformal chiral field theories is to a great extent fixed – we know fields of which dimensions contribute to the commutator of any pair of fields and with which test functions these fields are smeared. The only unknown ingredients are the structure constants $F_{AB}^C$, which are numbers. We shall now investigate further restrictions of the structure constants due to the Lie algebra structure relations of the commutator.

Observation. The anti–symmetry of commutators together with the symmetry property (3) of $\lambda^{c}_{ab}$ implies the following symmetry rule for the structure constants:

$$F_{AB}^C = (-1)^{a+b-c} F_{BA}^C.$$  (7)

Taking adjoints, and recalling that the basis consists of hermitian fields, one finds that $F_{AB}^{C*}$ are real numbers.

Further restrictions for the structure constants $F_{AB}^C$ arise from the Jacobi identity for commutators of smeared field operators, as we will see in section 5.4. We cannot derive these restrictions directly, because the Jacobi identity in its original form would produce constraints burdened with test functions. A reduction of the field algebra, performed in section 3.3, will allow us to strip off the test functions from the Jacobi identity and to achieve a reduced Jacobi identity involving only the structure constants $F_{AB}^{C*}$.

The $F_{AB}^{C*}$ are also related to the amplitudes of 2- and 3-point functions as we will elaborate in section 5.3.

3.1 $\lambda^{c}_{ab}$ are intertwiners

Quasiprimary fields of scaling dimension $a$ extend to a larger test function space than just the Schwartz functions, namely to the space $\pi_a$ of smooth functions on $\mathbb{R}$ for which $x^{2-2a} f(x^{-1})$ extends smoothly to $x = 0$. We regard this space as a representation of $sl(2, \mathbb{R})$ with generators $p, d,$ and $k$ such that:

$$(pf)(x) = i\partial f(x), \quad (df)(x) = i(x\partial + 1 - a)f(x), \quad (kf)(x) = i(x^2 \partial + 2(1-a)x)f(x)$$  (8)

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We must remark that \( \pi_a \) is neither irreducible nor unitary. In particular, the inner product induced by the 2-point function annihilates the \((2a-1)\)-dimensional subspace of polynomials of order \(2a-2\).

The direct product \( \pi_a \times \pi_b \) equals \( \pi_a \otimes \pi_b \) as a space and carries the representation \((\pi_a \otimes \pi_b) \circ \Delta\), where the \( \Delta \) is the Lie algebra coproduct.

Then the maps
\[
\lambda_{ab}^c : \pi_a \times \pi_b \to \pi_c, \quad f \otimes g \mapsto \lambda_{ab}^c(f, g) = \sum_{p+q=a+b-c-1} \lambda_{ab}^c(p, q) \partial^p f \cdot \partial^q g
\]
intertwine the corresponding \(sl(2, \mathbb{R})\) actions on the spaces of test functions. Their distinguishing feature among all such intertwiners is that they preserve supports (see above), for which we call them local intertwiners. The constructive argument in the proof of Prop. 3.1 means that they are actually the unique local intertwiners of the \(sl(2, \mathbb{R})\) action. Therefore, our task will be to understand the category of representations \(\pi_a\) of \(sl(2, \mathbb{R})\) equipped with the local intertwiners.

### 3.2 Bases for the intertwiner spaces

One important observation is that the bound \(c < a+b\) for \(\lambda_{ab}^c\) guarantees that the intertwiner spaces \(\pi_{a_1} \times \pi_{a_2} \times ... \times \pi_{a_n} \to \pi_c\), where \(c < \sum_{i=1}^n a_i - n\), are finite-dimensional. In this subsection we will construct bases for the intertwiner spaces and describe the relevant matrices for a switch between bases.

Our “default” choice of basis, adapted to the structures which appear in our calculations (nested commutators), will be the following:

**Definition 3.2 (Default basis for intertwiners \(\pi_{a_1} \times \pi_{a_2} \times ... \times \pi_{a_n} \to \pi_c\)).** We define the operators:
\[
\left( T_{\mathbb{R}} \right)^{m_{a_1} - 1} = \lambda_{a_1, \varepsilon_1} \circ \left( 1_{a_1} \times \lambda_{a_2, \varepsilon_2} \circ \left( 1_{a_1} \times 1_{a_2} \times \lambda_{a_3, \varepsilon_3} \circ \left( ... \circ \left( 1_{a_1} \times ... \times 1_{a_{n-2}} \times \lambda_{a_{n-1}, \varepsilon_{n-1}} \right) ... \right) \right) \right).
\]

Here \(x \) stands for \(n\)-tuples \((x_1, ..., x_n)\), \(\omega_a\) is the \(n\)-tuple of scaling dimensions \(a_i\) and the indices \(m_i \in \mathbb{N}_0\) are related to the scaling dimensions as:
\[
m_{n-1} := a_{n-1} + a_n - \varepsilon_{n-2} - 1, \quad m_1 = a_1 + \varepsilon_1 - e - 1, \quad m_i := a_i + \varepsilon_i - \varepsilon_{i-1} - 1 \quad \text{for} \quad i = 2, ..., n.
\]

Then the set of operators \(\left( T_{\mathbb{R}} \right)^{m_{a_1} - 1}\), such that \(m_1 + ... + m_{n-1} = \sum_{i=1}^n a_i - e - n + 1\), constitute a basis for the intertwiner space \(\pi_{a_1} \times \pi_{a_2} \times ... \times \pi_{a_n} \to \pi_c\).

**Observation.** The \(n-1\)-tuple \(m_{a_1} - 1\) determines the values of the scaling dimensions \(\varepsilon_i\) and \(e\) of the intermediate and final representations:
\[
\varepsilon_i = \sum_{s=i+1}^n a_s - \sum_{t=i+1}^{n-1} m_t - n + i + 1, \quad e = \sum_{s=1}^n a_s - \sum_{t=1}^{n-1} m_t - n + 1
\]

They are subject to restrictions, originating from the bound \(c < a+b\) for \(\lambda_{ab}^c\):
\[
\varepsilon_{n-2} \leq a_{n-1} + a_n - 1, \quad \varepsilon_1 \geq e - a_1 + 1, \quad \varepsilon_i \leq \sum_{k=i+1}^n a_k - n + i + 1 \quad \text{for} \quad i = 1, ..., n-3
\]

It should be noted that some of the dimensions \(\varepsilon_i\) may be negative. We shall ignore the unitarity bound (admitting only nonnegative dimensions) at this point. It will be imposed later (Sect. 3.6).

**Remark.** The operators \(\left( T_{\mathbb{R}} \right)^{m_{a_1} - 1}\) are multilinear maps on functions \((f_1, ..., f_n)\) such that \(f_i \in \pi_{a_i}\). The images \(\left( T_{\mathbb{R}} \right)^{m_{a_1} - 1}(f_1, ..., f_n)\) are test functions belonging to the space \(\pi_c\) (as in (12)).

Occasionally it will be necessary to consider nested brackets in different order.
Example. An alternative basis for the intertwiner space $\pi_a \times \pi_b \times \pi_c \rightarrow \pi_e$ is:

$$(T_{S,abc})_{m_1m_2}^{m_1m_2} := \lambda_{ce}^c \circ (\lambda_{ab}^b \times 1_c), \quad m_1 + m_2 = M(a, b, c; e) = a + b + c - e - 2$$

(14)

In the general case, one may specify a “bracket scheme” $B$ and denote the corresponding basis of intertwiners by $(T_{B,abc})_{m_1m_2}^{m_1m_2}$.

3.2.1 Transformation matrices

From (5) one immediately has

$$(T_{abc})_{m_1m_2}^{m_1m_2}(f, g, h) = (-1)^{m_1}(T_{S,bca})_{m_1m_2}^{m_1m_2}(g, h, f) = (-1)^{m_2}(T_{acb})_{m_1m_2}^{m_1m_2}(f, h, g).$$

(15)

For the analysis of the Jacobi identity, however, we shall need relations among $(T_{abc})_{m_1m_2}^{m_1m_2}(f, g, h)$ and $(T_{bca})_{m_1m_2}^{m_1m_2}(g, h, f)$ and $(T_{cab})_{m_1m_2}^{m_1m_2}(h, f, g)$, not covered by (15). In this subsection we introduce the transformation matrices for general permutations and re-bracketings.

Definition 3.3 (The matrix $(Z_{B_1B_2\gamma_2,\sigma_{\lambda_2}})_{\mu_{\lambda_2}^{-1}}$). Let us define the matrix $(Z_{B_1B_2\gamma_2,\sigma_{\lambda_2}})_{\mu_{\lambda_2}^{-1}}$ which relates two bases $T_{B_1}$ and $T_{B_2}$ with permuted arguments:

$$(T_{B_1,\sigma_{\lambda_2}(\lambda_2)})_{\mu_{\lambda_2}^{-1}} \circ \tau_{\lambda_2} = (Z_{B_1B_2\gamma_2,\sigma_{\lambda_2}})_{\mu_{\lambda_2}^{-1}} (T_{B_2,\gamma_2})_{\mu_{\lambda_2}^{-1}}$$

(16)

where $\sigma_{\lambda}$ is the permutation of labels $(x_1, ..., x_n) \mapsto (y_1, ..., y_n)$ and $\tau_{\lambda_2} : (f_1, ..., f_n) \mapsto (g_1, ..., g_n)$ the corresponding permutation on $\pi_{a_1} \times \cdots \times \pi_{a_n}$. In other words, permutations act on intertwiner spaces $\pi_{a_1} \times \cdots \times \pi_{a_n} \rightarrow \pi_e$ by permutation of the factors, $\sigma(T) := T \circ \tau_{\sigma}$, and $(Z_{B_1B_2\gamma_2,\sigma_{\lambda_2}})_{\mu_{\lambda_2}^{-1}}$ are the matrix elements of these linear maps between intertwiner spaces in various bases of the latter.

Of particular interest for us will be the matrix $(Y_{bca})_{\mu_1\mu_2}$ which describes the cyclic permutations of $(T_{abc})_{m_1m_2}^{m_1m_2}(f, g, h)$:

$$(T_{bca})_{\mu_1\mu_2}^{\mu_1\mu_2}(g, h, f) = (Y_{bca})_{\mu_1\mu_2}^{\mu_1\mu_2}(T_{abc})_{m_1m_2}^{m_1m_2}(f, g, h)$$

(17)

By (15), the transposition of the last two entries is described by the diagonal matrix

$$(T_{bca})_{\mu_1\mu_2}^{\mu_1\mu_2}(g, h, f) = (X_{abc})_{\mu_1\mu_2}^{\mu_1\mu_2}(T_{S,bca})(f, g, h).$$

(18)

From the definition follows directly that $Y_{abc} \cdot Y_{cab} \cdot Y_{bca} = 1$ and $Y_{abc} \cdot I \cdot Y_{cba} \cdot I = 1$, i.e., the matrices $Y$ and $I$ generate a representation of $S_3$. In particular, we have

$$T_{bca}(g, h, f) = IT_{bca}(g, h, f) = Y_{bca}T_{abc}(f, g, h).$$

(19)

A calculation and explicit expression for the (quite complicated) matrix elements $(Y_{abc})_{\mu_1\mu_2}^{\mu_1\mu_2}$ can be found in the App. A.

This matrix is closely related to the matrix that describes the passage from the basis $(T_{abc})_{m_1m_2}^{m_1m_2}$ to the basis $(T_{S,abc})_{m_1m_2}^{m_1m_2}$ without a permutation (“re-bracketing”):

$$(T_{abc})_{m_1m_2}^{m_1m_2}(f, g, h) = (X_{abc})_{\mu_1\mu_2}^{\mu_1\mu_2}(T_{S,abc})_{\mu_1\mu_2}^{\mu_1\mu_2}(f, g, h).$$

(20)

Namely, by (15), one has

$$X_{abc} = (1)^M I T_{cab}(h, f, g) = (1)^M Y_{abc}^{-1} T_{abc}(f, g, h),$$

where $M = M(a, b, c; e) = a + b + c - e - 2 = m_1 + m_2$, hence

$$X_{abc} = (1)^{M(a, b, c; e)}(Y_{abc})I.$$
We claim that the matrix elements \((Y_{abc})^{m_1m_2}_{m_3m_4}\) are the building blocks of every matrix element \((Z_{BC,DA})_{m_3m_4}^{m_2m_3}\). Namely, one can achieve every bracket scheme from the default bracket scheme \((10)\) by a sequence of applications of \((15)\) ("flips"), at the price of a permutation of the arguments. The flips will produce signs \((-1)^{m_i}\), where the label \(m_i\) refers to the flipped intertwiner. Now, the permutations can be undone by a sequence of transpositions without changing the bracket scheme. One sees from \((10)\) that in the default basis \((10)\) the transposition \(f \leftrightarrow g\) is represented by the map \(\phi: A \otimes f \rightarrow A(f), \; A \in V_a, \; f \in \pi_a\).

3.3 Reduction of the field algebra

The field algebra, which we will denote with \(\mathcal{V}\), decomposes as a linear space into a direct sum of representations via commutators of \(sl(2, \mathbb{R})\), which is a subalgebra of \(\mathcal{V}\):

\[
V = \bigoplus_{a \in \mathbb{N}} V_a
\]

(22)

Every subspace \(V_a\) is a span of (finitely many) quasiprimary fields with the same integer scaling dimension \(a > 0\) and is isomorphic to \(V_a \otimes \pi_a\). As in subsection 3.2, \(\pi_a\) is a test function space, which is a representation space for \(sl(2, \mathbb{R})\). \(V_a\) is a finite–dimensional multiplicity space with basis \(W_a\), which accounts for the number of fields with scaling dimension \(a\). The isomorphism above is realized by the map \(\phi_a\) which acts as:

\[
\phi_a : A \otimes f \mapsto A(f), \; A \in V_a, \; f \in \pi_a.
\]

(23)

We leave out the identity operator \(I\) (of dimension \(a = 0\)) from the reduced space for several reasons: first, \((23)\) fails to be an isomorphism in this case because \(I(f) = (\int f(x)dx) \cdot 1\) depends only on the integral of \(f\). Second, the unit operator is central in the field algebra, so its commutator with other fields contains no information. Third, the contribution of the unit operator to the commutator of two fields is completely determined by the 2-point function, which we shall treat as an independent structure element in Sect. 3.5.

Definition 3.4 (The reduced space \(V\)). The direct sum of all multiplicity spaces \(V = \bigoplus_{a \in \mathbb{N}} V_a\) will be called the reduced space \(V\).

In the following we will show that the Lie algebra structure of \(\mathcal{V}\) is enciphered into multi–component structures on the reduced space \(V\).

Definition 3.5 (The reduced Lie bracket \(\Gamma^*(\cdot, \cdot)_m\)). On the reduced space \(V = \bigoplus_{a \in \mathbb{N}} V_a\) the commutator \([\cdot, \cdot]\) in \(\mathcal{V}\) is represented by the multi–component \(*\)-bracket \([\cdot, \cdot]^*_m\) or \(\Gamma^*(\cdot, \cdot)_m : V_a \times V_b \rightarrow V_{a+b-1-m}, \; m \geq 0\):

\[
\Gamma^*(A, B)_m := \sum_{C \in W_{a+b-1-m}} F^C_{AB} C.
\]

(24)

Indeed, if we rewrite the Lie commutators \((11)\) using \((23)\) we find (suppressing the detailed form of the contribution from the unit operator)

\[
- i[\phi_a (A \otimes f), \phi_b (B \otimes g)] = \sum_{c < a+b} \phi_c \left( \sum_{C \in W_c} F^C_{AB} C \otimes \chi^c_{ab}(f, g) \right) + \text{(unit operator)}
\]

\[
= \sum_{c < a+b} \phi_c \left( \Gamma^*(A, B)_{m=a+b-1-c} \otimes \chi^c_{ab}(f, g) \right) + \text{(unit operator)}. \quad (25)
\]

Observation. The anti–symmetry property of the commutator is encoded in the graded symmetry property of the \(*\)-bracket:

\[
\Gamma^*(X_1, X_2)_m = (-1)^{m+1} \Gamma^*(X_1, X_2)_{m}. \quad (26)
\]

(26) actually reproduces the graded symmetry of the structure constants \(F^C_{AB}\) \((7)\).
Remark. The reduction of the algebra may be interpreted as disentangling the \(sl(2,\mathbb{R})\) “kinematic” representation details from the structure constants \(F^C_{AB}\). The former are completely dictated by the conformal symmetry, whereas the latter specify the model (together with the dimensions \(\text{dim } V_a\)).

In order to perform a complete reduction of the field algebra \(V\) we must also “reduce” the Jacobi identity and this will be done in the next section.

### 3.4 The reduced Jacobi identity and further constraints on \(F^C_{AB}\)

In this section we will examine what becomes of the Jacobi identity of commutators under the “space reduction”. In this way we will complete the reduction of the field algebra and we will find further restrictions on the coefficients \(F^C_{AB}\).

The Jacobi identity in its full form between three quasiprimary fields \(A(f) \in V_a, B(g) \in V_b\) and \(C(h) \in V_c\) is:

\[
[A(f), [B(g), C(h)]] + [B(g), [C(h), A(f)]] + [C(h), [A(f), B(g)]] = 0
\]

Now let us concentrate on the first term. As in \((25)\), we want to detach the test function contribution from the operator part. Using the construction of intertwiners for multiple products of representations \((10)\) and the relation \((23)\) we write:

\[
[A(f), [B(g), C(h)]] = \sum_{m_1 m_2} \Gamma^*(A, \Gamma^*(B, C))_{m_1 m_2} \otimes (T_{abc})_{m_1 m_2} (f, g, h)
\]

\[
\Gamma^*(A, \Gamma^*(B, C))_{m_1 m_2} = \sum_{E_1 \in W_{a_1} = W_{a_1} + C_{m_1 - m_2 - 2}} \sum_{E_2 \in W_{a_2} = W_{a_2} + C_{m_2 - 1}} F_{A E_1}^E_1 F_{B C E_1}^E
\]

Here and everywhere in the rest of this section the relation between the \(e\)’s and the \(m\)’s are as in section \(3.2\).

The same considerations for the second and third terms yield similar expressions but with \(T_{abc}(g, h, f)\) and \(T_{cab}(h, f, g)\) in the last tensor factor. We then use \((17)\) to write them in the same form \(T_{abc}(f, g, h)\) as the first term. Then, by bilinearity of the tensor product, the Jacobi identity reads:

\[
\sum_{m_1 m_2} \left\{ \Gamma^*(A, \Gamma^*(B, C))_{m_1 m_2} + \Gamma^*(B, \Gamma^*(C, A))_{m_1 m_2} (Y_{bca})_{m_1 m_2} + \Gamma^*(C, \Gamma^*(A, B))_{m_1 m_2} (Y_{cab})_{m_1 m_2} \right\} \otimes (T_{abc})_{m_1 m_2} (f, g, h) = 0.
\]

Having in mind that the basis components \((T_{abc})_{m_1 m_2} (f, g, h)\) for different values of \(m_1\) and \(m_2\) are linearly independent functionals of the test functions, and the test functions are arbitrary, we conclude for any fixed pair \((m_1, m_2)\):

\[
\Gamma^*(A, \Gamma^*(B, C))_{m_1 m_2} + \Gamma^*(B, \Gamma^*(C, A))_{m_1 m_2} (Y_{bca})_{m_1 m_2} + \Gamma^*(C, \Gamma^*(A, B))_{m_1 m_2} (Y_{cab})_{m_1 m_2} = 0.
\]

Let us denote the left-hand-side of the reduced Jacobi identity \((31)\) with \(\text{RJI}(A, B, C)_{m_1 m_2}\). Clearly, because \(Y_{abc} \cdot Y_{cab} \cdot Y_{bca} = 1\), one has the following symmetry rule:

\[
\text{RJI}(A, B, C)_{m_1 m_2} = \text{RJI}(B, C, A)_{m_1 m_2} (Y_{bca})_{m_1 m_2} = \text{RJI}(C, A, B)_{m_1 m_2} (Y_{cab})_{m_1 m_2}.
\]

i.e., the vanishing of \(\text{RJI}(A, B, C)_{m_1 m_2}\) is invariant under cyclic permutations, as it should. If we use the explicit expressions for the nested \((\Gamma^*)'s\) from above, the reduced Jacobi identity becomes...
for every quadruple of quasiprimary fields $A, B, C$ and $E$ and for every pair $m_1, m_2$ such that $m_1 + m_2 = a + b + c - e - 2$:

$$
\sum_{E_2 \in W_{e_2}} F_{AE_2}^E F_{E_2 BC}^{E_2} + \sum_{E_2 \in W_{e_2}} F_{BE_2}^E F_{E_2 CA}^{E_2} \left( Y_{bca} \right)_{m_1 m_2} + \\
\sum_{E_2 \in W_{e_2}} F_{CE_2}^E F_{E_2 AB}^{E_2} \left( Y_{cab} \right)_{m_1 m_2} = 0
$$

Observation. The reduced form of the Jacobi identity gives an infinite set of constraints on the structure constants $F_{AB}^C$. Every solution of this set of constraints promotes a candidate for the commutator algebra of a local chiral conformal field theory.

As noted in App. A, the matrix elements of $Y_{abc}$ can have vanishing denominators, that have to be regularized (e.g., by giving small imaginary parts to the dimensions). As it turns out, in these singularities will not be suppressed in general by the vanishing of structure constants involving negative scaling dimensions. To make sense of the singular Jacobi identities, one has to multiply with the singular denominators and then remove the regulators. The effect will be that only one or two of the three terms of the Jacobi identity may survive, so that the general appearance of the Jacobi identity may be quite different from the usual “three-term” form. Notice that anyway, due to the multi-component structure of the bracket, each of the three terms is in general a sum over different “intermediate” representations.

3.5 Relation between $F_{BC}^A$ and 2- and 3-point amplitudes

The 2-point function of two hermitian fields $A(x)$ and $B(x)$ has the form:

$$
\langle \langle x_1 | B(x_2) \rangle \rangle = \langle AB \rangle \frac{-i}{x_{12} - i\varepsilon} \equiv \frac{\langle AB \rangle}{(ix_{12})^{2a}}.
$$

The map $A, B \mapsto \langle AB \rangle$ is a real bilinear map on the reduced space which

- is symmetric: $\langle AB \rangle = \langle BA \rangle$,
- respects the grading: $\langle AB \rangle = 0$ if the scaling dimensions $a \neq b$,
- is positive definite: $\langle AA \rangle > 0$ unless $A = 0$.

The first property reflects locality of the QFT, the second is a consequence of Möbius invariance, and the last one is Wightman positivity, i.e., the positive-definiteness of the Hilbert space inner product.

Similarly, the 3-point function has the following form:

$$
\langle \langle x | B(y)C(z) \rangle \rangle = \langle ABC \rangle \frac{(-i)^{a+b+c}}{(x - y - i\varepsilon)^{a+b-c}(y - z - i\varepsilon)^{b+c-a}(x - z - i\varepsilon)^{a+c-b}}
$$

and by locality its amplitude must satisfy:

$$
\langle BAC \rangle = (-1)^{a+b-c}\langle ABC \rangle.
$$

We will show that the amplitudes of the 2- and the 3-point functions are not independent on each other. For this purpose, let us consider the 3-point function $\langle \langle A, B | C \rangle \rangle$. We can find it as $\langle \langle A, B | C \rangle \rangle = \langle ABC \rangle - \langle BAC \rangle$. Using

$$
(-1)^n n! \left( \frac{1}{(x - i\varepsilon)^{n+1}} - \frac{1}{(x + i\varepsilon)^{n+1}} \right) = 2\pi i \delta^{(n)}(x)
$$
we obtain:

\[
\langle [A(x), B(y)] C(z) \rangle = \frac{(-i)^{a+b+c} \langle ABC \rangle}{(2c)a+b-c-1} 2\pi i (-1)^m \frac{m!}{m!} \delta^{(m)}(x-y) + \text{lower derivatives of } \delta
\]

On the other hand, taking into consideration \(-i[A, B] = \sum F_{AB}^C C\) and using the translation formula \(C(\partial^p f \cdot \partial^q g) \rightarrow (-1)^{p+q} \partial^p g(x-y) \cdot C(y)\) we end up with:

\[
\langle [A(x), B(y)] C(z) \rangle = \sum_{C' \in W_c} F_{AB}^{C'} (-1)^m \frac{(2c)m^m}{m!} \delta^{(m)}(x-y) \langle C' C \rangle \left( \frac{-i}{x-z-i\varepsilon} \right)^{2c} + \text{lower derivatives of } \delta
\]

Comparing (38) and (39) we obtain:

\[
\frac{(-i)^{a+b+c}}{(2c)a+b-c-1} 2\pi i \langle ABC \rangle = \sum_{C' \in W_c} F_{AB}^{C'} \langle C' C \rangle (-i)^{2c}.
\]

With the same considerations for \(\langle A[B, C] \rangle\) we obtain:

\[
\frac{(-i)^{a+b+c}}{2\pi i (a+b+c-1) 2\pi i \langle ABC \rangle} = \sum_{A' \in W_a} F_{BC}^{A'} \langle A' A \rangle (-i)^{2a}.
\]

The last two formulae allow us to find a new condition on the structure constants \(F_{AB}^{C}\) involving only 2-point amplitudes:

\[
(-1)^c (2c)a+b-c-1 \sum_{C' \in W_c} F_{AB}^{C'} \langle C' C \rangle = (-1)^a (2a)a-b+c-1 \sum_{A' \in W_a} F_{BC}^{A'} \langle A' A \rangle.
\]

or

\[
(-1)^c (2c)a+b-c-1 \langle \Gamma^a (A, B)_{a+b-c-1}, C \rangle = (-1)^a (2a)b+c-a-1 \langle A, \Gamma^b (B, C)_{b+c-a-1} \rangle.
\]

There are two ways how to look at this condition: either one assumes a given quadratic form \(\langle \cdot, \cdot \rangle\), which amounts to fixing bases of the finite-dimensional reduced field spaces \(V_a\): then (42) is indeed an additional constraint on the structure constants \(F_{AB}^{C}\). Or one regards the reduced algebra (24) subject to the structure relations (7) and (33) as the primary structure: then (43) is an invariance condition on the quadratic form, in the same way as the invariance condition \(g([X,Y], Z) = g(X, [Y,Z])\) on a quadratic form on a Lie algebra. This invariant quadratic form on the reduced Lie algebra corresponds to the vacuum expectation functional on the original commutator algebra.

### 3.6 Axiomatization of chiral conformal QFT

The upshot of the previous analysis is a new axiomatization of chiral conformal quantum field theory. It consists of the three data:

- a graded reduced space of fields \(V = \bigoplus_{a \in \mathbb{N}} V_a\),
- a generalized Lie bracket \(\Gamma^* = \sum_{m \geq 0} \Gamma_m^* : V \times V \rightarrow V\),
- and a quadratic form \(\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{R}\).

These data should enjoy the features outlined before: \(V_a\) are real linear spaces; the bracket is filtered: \(\Gamma^* (V_a \times V_b) \subset \bigoplus_{m \geq 0} V_{a+b-1-m}\), and satisfies the graded symmetry (23) and generalized Jacobi identity (31): the quadratic form is symmetric, positive definite, respects the grading, and is invariant (33) with respect to the bracket.

Notice that the unitarity bound (absence of negative scaling dimensions) has been imposed through the specification of the reduced space \(V\). Although the local intertwiner bases, and therefore also the coefficient matrices \(Y\) in the Jacobi identity do involve “intermediate” representations...
of negative dimensions \((a + b - 1 - m)\) may be \(< 0\), these do not contribute to the present axiomatization because they multiply non-existent structure constants. Recall also that the possibly singular instances of the Jacobi identity have to be understood as explained in the end of Sect. 3.3.

One may impose further physically motivated constraints, e.g., the existence of a stress-energy tensor as a distinguished field \(T \in V_2\) whose structure constants \(F^A_{TA}\) take canonical values; or the generation of the entire reduced space by iterated brackets of a finite set of fields, formulated as a surjectivity property of the bracket.

As a simple example, one may consider the constraints on the structure constants for the commutator of two fields \(A, B\) of dimension one. The only possibility in this case is \(\dim C = 1\). The generalized Jacobi identity just reduces to the classical Jacobi identity for the structure constants of some Lie algebra \(g\). Likewise, the invariance property of the quadratic form becomes the classical \(g\)-invariance of the quadratic form \(h(A, B) = \langle AB \rangle\) on \(g\). The positivity condition on the quadratic form implies that \(g\) must be compact, and that \(h\) is a multiple of the Cartan-Killing metric. In other words: one obtains precisely the Kac-Moody algebras as solutions to this part of the constraints. The quantization of the level is expected to arise by the interplay between the positivity condition with the higher generalized Jacobi identities.

Other approaches [Zamolodchikov, 1986; Bouwknegt, 1988; Blumenhagen et al. 1991] to the classification of \(W\)-algebras have, of course, exploited essentially the same consistency relations for a set of generating fields. Our focus here is, however, on the entire structure including all “composite” fields, and the possibility to formulate a deformation theory, to which we turn now.

### 4 Cohomology of the reduced Lie algebra

In this section we will develop the cohomology of the reduced Lie algebra as a prerequisite for the deformation theory in Sect. 5. The description is intrinsic in the sense that it does not refer to the commutator algebra of field operators is was derived from. We generalize the lines of the cohomology theory of Lie algebras [Chevalley & Eilenberg 1948], but the maps, which build the cochain spaces of our cochain complex, will possess a more complicated symmetry property, which we define now.

#### 4.1 \(Z^n_B\)-symmetry

The reduced bracket \((\mathcal{B})\) obeys the symmetry rule \((\mathcal{B})\). The reduced Jacobi identity \((\mathcal{B})\) obeys the symmetry rule \((\mathcal{B})\). A symmetry rule, generalizing the last two rules for structures with more arguments, will be the following:

**Definition 4.1 (\(Z^n_B\)-symmetry).** Let \(V\) be the reduced space as in section 3.3 and let us consider the maps \(\omega^m_B(\cdot, \cdot, \cdot, \cdot)|_{m_1, \ldots, m_{n-1}} : V \times \cdots \times V \rightarrow V\). Let \(\mathbf{a}_n\) be an \(n\)-tuple of scaling dimensions \(a_i\), let \(X_i \in V_{a_i}\), and let \(m_{n-1}\) will be the \(n\)-tuple \((m_1, \ldots, m_{n-1})\). Let \(\omega^m_B(X_1, \ldots, X_n)|_{m_{n-1}}\) be non-zero only for \(m_i \leq \sum_{s=1}^n a_s - \sum_{t=i+1}^{n-1} m_t - n + i\). We will say that \(\omega^m_B(X_1, \ldots, X_n)|_{m_{n-1}}\) are \(Z^n_B\)-symmetric if for every permutation in \(S_n\)

\[
\omega^n_B(X_1, \ldots, X_n)|_{m_{n-1}} = \omega^n_B(X_{i_1}, \ldots, X_{i_n})|_{m_{n-1}} (Z^B_{BB,\mathbf{a}_n,\sigma_{\mathbf{a}_n}})|_{m_{n-1}} \tag{44}
\]

where \((Z^B_{BB,\mathbf{a}_n,\sigma_{\mathbf{a}_n}})|_{m_{n-1}} = \varepsilon_{i_1, \ldots, i_n} (Z^B_{BB,\mathbf{a}_n,\sigma_{\mathbf{a}_n}})|_{m_{n-1}}\), with \((Z^B_{BB,\mathbf{a}_n,\sigma_{\mathbf{a}_n}})|_{m_{n-1}}\) the matrix representation of \(S_n\) as in definition 3.3 and \(\sigma_{\mathbf{a}_n}\) the permutation \(\{i_1, \ldots, i_n\}\) of the indices \(\{1, \ldots, n\}\).

This definition is motivated by the following proposition:

**Proposition 4.2.** The \(Z^n_B\)-symmetry of \(\omega^m_B(X_1, \ldots, X_n)|_{m_{n-1}}\) ensures that the function

\[
\omega^n(X_1(f_1), \ldots, X_n(f_n)) := \sum_{\sum m_i \leq \sum a_k - n + 1} \omega^m_B(X_1, \ldots, X_n)|_{m_{n-1}} \otimes (T^B_{\mathbf{a}_n})|_{m_{n-1}}(f_1, \ldots, f_n) \tag{45}
\]
is completely anti–symmetric in the arguments \( X_i(f_i) \).

**Proof.** It follows directly from the definitions that:

\[
\omega^n(X_1(f_1), \ldots, X_n(f_n)) = \sum \omega^n_B(X_1, \ldots, X_n) \overline{m}_{n-1} \otimes (T_B) \overline{m}_{n-1} (f_1, \ldots, f_n) \\
= \sum \omega^n_B(X_1, \ldots, X_n) \overline{m}_{n-1} \otimes (Z_B) \overline{m}_{n-1} (f_1, \ldots, f_n) \\
= \sum \omega^n_B(X_1, \ldots, X_n) \overline{m}_{n-1} \otimes (Z_B) \overline{m}_{n-1} (f_1, \ldots, f_n) \\
= \varepsilon_{i_1 \ldots i_n} \omega^{*n}(X_{i_1}(f_{i_1}), \ldots, X_{i_n}(f_{i_n})) \\
\]

which proves the proposition.

**Notation.** We will be interested in those \( Z^n \)-symmetric maps for which \( B \) is the default basis \( T \) as in section 4.2. We will call such maps \( Z^n \)-symmetric.

**Example.** The two natural examples for \( Z^n \)-symmetric maps are the reduced bracket and the reduced Jacobi identity.

### 4.2 Reduced Lie algebra cohomology

In this section we will introduce the reduced Lie algebra cohomology complex:

**Definition 4.3 (reduced Lie algebra cohomology).** We define the reduced Lie algebra cohomology as:

- **Cochain complex:**
  1. **Cochain spaces \( C^n(V) \) of dimension \( n \):** The \( n \)-cochains in the cochain complex are the tensor–valued (i.e., multi-component) \( Z^n \)-symmetric maps \( \omega^{*n}(\varepsilon_{i_1 \ldots i_n}) \overline{m}_{n-1} \). The spaces \( C^n(V) \) of all \( Z^n \)-symmetric \( \omega^{*n} \)'s for a fixed \( n \) compose the cochain sequence \( C := (C^n(V))_{n \in \mathbb{N}_0} \).
  2. **Coboundary operators \( b^n \):** We define the coboundary operator \( b^n : C^n(V) \rightarrow C^{n+1}(V) \) through the following component–wise action, provided that \( m_i \leq \sum_{k=1}^n a_k = \sum_{t=i+1}^n m_t - n + i \):

\[
[b^n \omega^{*n}](\overline{X}_{n+1})_{\overline{m}_n} := (-1)^n \frac{1}{n!} \sum_{\overline{m}_{n+1}} \Gamma^*(X_{i_1}, \omega^{*n}(X_{i_2}, \ldots, X_{i_{n+1}})) (Z^*_{\overline{m}_{n+1}} \sigma^{\overline{m}_n}) + \\
+ \frac{1}{2(n-1)!} \sum_{\overline{m}_{n+1}} \omega^{*n}(X_{j_1}, \ldots, X_{j_{n+1}}, \Gamma^*(X_{j_{n+1}}, X_{j_{n+2}})) (Z^*_{\overline{m}_{n+1}} \sigma^{\overline{m}_n}) \\
\]

or equivalently (because \( \omega^{*n} \) is \( Z^n \)-symmetric):

\[
[b^n \omega^{*n}](\overline{X}_{n+1})_{\overline{m}_n} := (-1)^n \sum_{i=1}^{n+1} \Gamma^*(X_{i_1}, \omega^{*n}(X_{i_1}, \ldots, X_{i_{n+1}})) (Z^*_{\overline{m}_{n+1}} \sigma_{i}) + \\
+ \sum_{k \geq j \geq 1} \omega^{*n}(X_{j_1}, \ldots, X_{j_{k}}, \ldots, X_{n+1}, \Gamma^*(X_{j_{k}}, X_{j_{k+1}})) (Z^*_{\overline{m}_{n+1}} \sigma_{j}) \\
\]

where \( \sigma_{\overline{i}} \in S_{n+1} \) is the permutation \( \{i, 1, \ldots, \hat{i}, \ldots, n+1\} \) and \( \sigma_{j\overline{k}} \) is the permutation \( \{1, \ldots, \hat{j}, \ldots, k, \ldots, n+1, j, k\} \).
Here and below we write the sum over permutations as \( \sum_{\Delta_{i+1}} := \sum_{i_1 \neq \ldots \neq i_{n+1}} \).

For those \( n \)-tuples \((m_1, \ldots, m_n)\), for which the condition \( m_i \leq \sum_{s=1}^{n} a_s - \sum_{t=i+1}^{n+1} m_t - n + i \) does not hold, \([b^n \omega^n](X_1, \ldots, X_{n+1})]_{m_n}\) will be set to 0.

We will show below that \( b^{n+1} \circ b^n = 0 \).

- **Cohomology group:**
  
  **Proof.**

  We define:
  
  \[
  Z^n(V) := \text{Ker}(b^n) = \left\{ \omega^n \in C^n(V) \mid [b^n \omega^n](X_1, \ldots, X_{n+1})]_{m_n} = 0, \forall m_n \in \mathbb{N}_0^n \right\} \\
  B^n(V) := \text{Im}(b^n) = \left\{ \omega^n \in C^n(V) \mid \omega^n = b^{n-1} \omega^{n-1}, \omega^{n-1} \in C^{n-1}(V) \right\} 
  \]

  \( b^{n+1} \circ b^n = 0 \) implies \( B^n(V) \subseteq Z^n(V) \). Then we define the \( n^{th} \) reduced Lie algebra cohomology group as the quotient:

  \[
  RLH^n(V) = Z^n(V)/B^n(V). 
  \]

In writing \( Z^n(V), B^n(V) \) and \( RLH^n(V) \), it is understood that \( V \) is equipped with a bracket \( \Gamma^* \), on which these spaces clearly depend.

We have to prove that \( b^n \) are differentials, i.e.,

**Proposition 4.4.** \( b^{n+1} \circ b^n = 0 \) applied to any map \( \omega^n \) from the cochain complex.

**Proof.** The proof proceeds in perfect analogy with the cohomology of Lie algebras, wherein the various terms obtained by evaluating \( b^{n+1} \circ b^n \) can be seen to cancel each other by virtue of the antisymmetry of the Lie bracket, the Jacobi identity, and the antisymmetry of the co-chains. However, there arises one salient complication:

Due to the \( Z^2 \)-symmetrization, there arise in the second line of \((47)\) terms of the structure

\[
\omega(X, \ldots, X, \Gamma(X, X)_{\mu_n}, X, \ldots, X)_{\mu_{n-1}}. 
\]

\( \sum_{\Delta_{i+1}} \).

Their multi-indices \( \mu_n = (\mu_{i+1}, \ldots, \mu_n) \) correspond to non-default bracket scheme \( B_{\kappa} \) (where \( \kappa = 1, \ldots, n \)) is the position of the insertion) with intertwiner basis of the structure

\[
\Gamma(1 \times \ldots \times 1 \times \lambda \times 1 \times \ldots \times 1).
\]

The necessary change of basis can be included into the matrix \( Z^n \) by virtue of

\[
Z_{B_{\kappa}, \sigma(\mu_n)} = Z_{\mu_n}. 
\]

One can then re-write \((47)\) as

\[
[b^n \omega^n](\Delta_{n+1})_{\mu_n} = \frac{(-1)^n}{n!} \sum_{\Delta_{i+1}} \left[ \Gamma^*(X_{i_1}, \omega^n(X_{i_2}, \ldots, X_{n+1})) \right](Z^n_{\Delta_{i+1}, \sigma(\mu_n)})_{\mu_n} + \]

\[
+ \frac{1}{2(n-1)!} \sum_{\kappa=1}^{n} \sum_{\Delta_{i+1}} \left[ \omega^n(X_{j_1}, \ldots, X_{j_{n-1}}, \Gamma^*(X_{j_n}, X_{j_{n+1}})) \right](Z^n_{\Delta_{i+1}, \sigma(\mu_n)})_{\mu_n}.
\]

(54)

where the term \( \left[ \omega^n(X_{j_1}, \ldots, X_{j_{n-1}}, \Gamma^*(X_{j_n}, X_{j_{n+1}})) \right]_{B_{\kappa}, \sigma(\mu_n)} \) collects all contributions, where \( \Gamma^*(X_{j_n}, X_{j_{n+1}}) \) was inserted in the \( \kappa^{th} \) position.

Then, when composing \( b^{n+1} \circ b^n \), one will encounter also bracket schemes \( B_{\kappa_1, \kappa_2} \) and \( \tilde{B}_{\kappa} \), \( \tilde{B}_{\kappa} \) corresponding to intertwiner bases of the structure

\[
T_{\Delta_{i+1}} \circ (1 \times \ldots \times 1 \times \lambda \times 1 \times \ldots \times 1),
\]

\[
T_{\mu_{n-1}} \circ (1 \times \ldots \times 1 \times (\lambda \circ (1 \times \lambda)) \times \ldots \times 1),
\]

\[
T_{\mu_{n-1}} \circ (1 \times \ldots \times 1 \times (\lambda \circ (1 \times \lambda)) \times \ldots \times 1),
\]

(55)

respectively, where \( \kappa \) and \( \kappa_1 \) stand for the positions of the insertions. Even if essentially straightforward, the precise details are quite cumbersome and will not be presented here, see [Kukhtina, 2011].
The result can then be written in the form

\[(b^{n+1} \circ b^n)(X_1, \ldots, X_{n+2})_{m+1} =
\]

(I) \[\frac{-(n+1)}{(n+1)!} \sum_{i=1}^{n+2} \left[ \omega^n(X_{i_1}, X_{i_2}, \ldots, X_{i_{n+2}}) \right] \frac{m}{m+1} \sum_{i=1}^{n+2} \left[ \Gamma^*(X_{i_1}, X_{i_2}, \ldots, X_{i_{n+2}}) \right] \]

(II) \[\frac{(-1)^{n+1} (n+1)n}{2(n+1)!} \sum_{i=1}^{n+2} \left[ \omega^n(X_{i_1}, X_{i_2}, \ldots, X_{i_{n+2}}) \right] \frac{m}{m+1} \sum_{i=1}^{n+2} \left[ \Gamma^*(X_{i_1}, X_{i_2}, \ldots, X_{i_{n+2}}) \right] \]

(III) \[\frac{(-1)^n}{2(n!)} \sum_{i=1}^{n+2} \left[ \omega^n(X_{i_1}, X_{i_2}, \ldots, X_{i_{n+2}}) \right] \frac{m}{m+1} \sum_{i=1}^{n+2} \left[ \Gamma^*(X_{i_1}, X_{i_2}, \ldots, X_{i_{n+2}}) \right] \]

(IV) \[\frac{1}{2(n!)} \sum_{i=1}^{n+2} \left[ \Gamma^*(X_{i_1}, X_{i_2}, \ldots, X_{i_{n+2}}) \right] \frac{m}{m+1} \sum_{i=1}^{n+2} \left[ \Gamma^*(X_{i_1}, X_{i_2}, \ldots, X_{i_{n+2}}) \right] \]

(V) \[\frac{n(n-1)}{4(n!)} \sum_{\kappa_1 \neq \kappa_2=1}^{n+2} \left[ \omega^n(X_{i_1}, \ldots, X_{i_{n+2}}) \right] \frac{m}{m+1} \sum_{\kappa_1 \neq \kappa_2=1}^{n+2} \left[ \Gamma^*(X_{i_1}, \ldots, X_{i_{n+2}}) \right] \]

(VI) \[\frac{n}{2(n!)} \sum_{\kappa_1 \neq \kappa_2=1}^{n+2} \left[ \omega^n(X_{i_1}, \ldots, X_{i_{n+2}}) \right] \frac{m}{m+1} \sum_{\kappa_1 \neq \kappa_2=1}^{n+2} \left[ \Gamma^*(X_{i_1}, \ldots, X_{i_{n+2}}) \right] \]

(56)

The terms with the bracket scheme \(\tilde{B}_\kappa\) are equal to those with the bracket scheme \(\tilde{B}_\kappa\) by virtue of the symmetry of \(\Gamma^*\), and are included in the term \(\text{(VI)}\).

Due to symmetry properties of \(\omega^n, \Gamma^*\) and \(Z^*_B\)-matrices one then realizes that:

- (II) + (III) = 0 term by term.
- (I) + (IV) combine to a Jacobi identity between \(X_{i_1}, X_{i_2}\) and \(\omega^n(X_{i_1}, \ldots, X_{i_{n+2}})\). This cancels them.
- the terms in (V) can be rewritten as a sum of pairs of terms (with \(\kappa_1\) and \(\kappa_2\) exchanged) which cancel each other.
- for each \(\kappa_1\) (VI) can be grouped in triples (by cyclic permutations of \(i_n, i_{n+1}, i_{n+2}\)) that contain a Jacobi identity which cancels them.

Then the sum of all terms is 0 and this proves the proposition.

5 Deformations of the reduced Lie algebra

We now consider formal deformations of the bracket of a reduced Lie algebra, which are defined as a perturbative series, such that the reduced Jacobi identity is respected. Our approach generalizes the cohomological analysis of deformations of associative algebras \cite{Gerstenhaber1964}.

**Definition 5.1 (Formal deformations of the reduced bracket).** A formal deformation of the bracket \(\Gamma^* : V \otimes V \rightarrow V\) is defined as a one-parameter family of brackets \(\Gamma^*(A, B, \lambda)_m = \Gamma^*(A, B)_m \) with \(\lambda \in \mathbb{R}\) and \(\Gamma^*(A, B, 0)_m \cong \Gamma^*(A, B)_m\). The deformed bracket is defined as a formal power series:

\[\Gamma^*(A, B, \lambda)_m := \sum_{i=0}^{\infty} \Gamma_i^*(A, B)_m \lambda^i\]  

and the \(i^{th}\) order perturbations of the bracket is:

\[\Gamma_i^*(A, B)_m := \frac{1}{i!} \frac{d^i}{d\lambda^i} \Gamma^*(\lambda)(A, B)_m\]  

(57)

Here \(\Gamma_0^*(A, B)_m \equiv \Gamma^*(A, B)_m\).
We are interested only in those deformations which are consistent with the generalized Jacobi identity \((31)\). This leads to a number of constraints which single out the admissible perturbations.

The first order perturbations \(\Gamma^+_1(A, B)_m\) must obey:

\[
\begin{align*}
\Gamma^+_0(A, \Gamma^+_1(B, C))_{m_1 m_2} &+ \Gamma^+_0(B, \Gamma^+_1(C, A))_{m_1 m_2} + \Gamma^+_0(C, \Gamma^+_1(A, B))_{m_1 m_2} \\
&+ \Gamma^+_1(A, \Gamma^+_0(B, C))_{m_1 m_2} + \Gamma^+_1(B, \Gamma^+_0(C, A))_{m_1 m_2} + \Gamma^+_1(C, \Gamma^+_0(A, B))_{m_1 m_2} = 0
\end{align*}
\]  

The higher order perturbations must satisfy the following condition:

\[
\sum_{k=0}^{n} \left( \Gamma^+_k(A, \Gamma^+_{n-k}(B, C))_{m_1 m_2} + \Gamma^+_k(B, \Gamma^+_{n-k}(C, A))_{m_1 m_2} + \right. \\
\left. + \Gamma^+_k(C, \Gamma^+_{n-k}(A, B))_{m_1 m_2} \right) = 0
\]  

(59)

We want to exclude from our considerations the “trivial” deformations, i.e., the simple \(\lambda\)-dependent changes of the basis \(Q^*: V \to V\), such that:

\[
\Gamma^*(A, B, \lambda)_m = Q^{*-1}(\Gamma^*(Q^* A, Q^* B)_m), \quad Q^* = 1 + \lambda q^*_1 + \lambda^2 q^*_2 + ...
\]  

(61)

Written in a series over \(\lambda\) up to first order, the deformed bracket becomes:

\[
\begin{align*}
Q^{*-1}(\Gamma^*(Q^* A, Q^* B)_m) &= (1 - \lambda q^*_1) \Gamma^+_0(A, (1 + \lambda q^*_1)B)_m + 0(\lambda^2) \\
&= \Gamma^+_0(A, B)_m + \lambda \Gamma^+_1(A, B)_m + 0(\lambda^2)
\end{align*}
\]  

(62)

\[
\Gamma^+_1(A, B)_m = \Gamma^+_0(A, q^*_1 B)_m + \Gamma^+_0(q^*_1 A, B)_m - q^*_1 \Gamma^+_0(A, B)_m
\]  

(63)

So, we have to “factorize” the set of admissible deformations over the set of trivial deformations. In the case of associative algebra such a factorization gave the opportunity to relate the deformations and the conditions for the \(i^{th}\)-order perturbation to a Hochschild cohomology complex. We will show that also in our case the deformations are described in terms of a cohomology complex, namely the reduced Lie algebra complex from the previous section.

In the following we formulate in cohomological language some of the formulas above:

**Observation (1).** The Jacobi identity for \(\Gamma^+_0\) can be rewritten in the compact form:

\[
(b^2 \Gamma^+_0)(A, B, C)_{m_1 m_2} = 0.
\]  

(64)

Here, and in the following, the differentials \(b^\nu\) of the chain complex \(\mathcal{C}\) are defined with \(\Gamma^+_0\).

**Observation (2).** When we insert the formal power series \(57\) into the Jacobi identity for the deformed bracket \(\Gamma^+\), we get in first order the following restriction on the first order perturbation:

\[
(b^2 \Gamma^+_1)(A, B, C)_{m_1 m_2} = 0,
\]  

(65)

i.e., \(\Gamma^+_1 \in Z^2(V)\). The first trivial perturbation is:

\[
\Gamma^+_1(A, B)_m = (b^1 q^*) (A, B)_m
\]  

(66)

which means \(\Gamma^+_1 \in B^2(V)\). Then it follows, that the non-trivial first order perturbations correspond to non-trivial classes \(\Gamma^+_1 \in RLH^2(V)\).

**Observation (3).** The terms in the Jacobi identity \(60\) involving \(\Gamma^+_n\), i.e., those with \(k = 0, n\), precisely equal \(b^2 \Gamma^+_n(A, B, C)_{m_1 m_2}\). One can therefore write \(60\) as an equation for \(\Gamma^+_n\):

\[
\begin{align*}
&b^2 \Gamma^+_n(A, B, C)_{m_1 m_2} = - \sum_{k=1}^{n-1} \left\{ \Gamma^+_k(A, \Gamma^+_{n-k}(B, C))_{m_1 m_2} + \\
&+ \Gamma^+_k(B, \Gamma^+_{n-k}(C, A))_{\tilde{m}_1 \tilde{m}_2} + \Gamma^+_k(C, \Gamma^+_{n-k}(A, B))_{\tilde{m}_1 \tilde{m}_2} \right\}
\end{align*}
\]  

(67)
The interesting question is whether every first order perturbation in $RLH^2(V)$ is integrable, i.e., whether every $\tilde{\Gamma}_1 \in Z^2(V)$ serves as the first order perturbation for some one-parameter family of deformations of $\Gamma_0$. One has to decide whether the equations \eqref{eq:67} can be solved recursively with a given $\tilde{\Gamma}_1$.

Thus suppose that for some $n \geq 2$, candidates $\tilde{\Gamma}_1^* \in C^2(V)$ ($2 \leq j < n$) for the coefficients of a perturbative expansion have been found solving \eqref{eq:67} for $n' < n$. Then $\tilde{\Gamma}_n^*$ must solve

$$b^2 \tilde{\Gamma}_n^*{(A, B, C)}_{m_1 m_2} = G^n[\tilde{\Gamma}_1^*, \ldots, \tilde{\Gamma}_{n-1}^*](A, B, C)_{m_1 m_2}$$  \hspace{1cm} \quad (68)$$

where $G^n[\tilde{\Gamma}_1^*, \ldots, \tilde{\Gamma}_{n-1}^*](A, B, C)_{m_1 m_2}$ is the r.h.s. of \eqref{eq:67} evaluated on the lower order perturbations $\tilde{\Gamma}_j$:

$$G^n[\tilde{\Gamma}_1^*, \ldots, \tilde{\Gamma}_{n-1}^*](A, B, C)_{m_1 m_2} := - \sum_{k=1}^{n-1} \left\{ \tilde{\Gamma}_k^*{(A, \tilde{\Gamma}_n^*{B}(B, C))}_{m_1 m_2} + + \tilde{\Gamma}_k^*{(B, \tilde{\Gamma}_n^*{C}(B, A))}_{m_1 m_2} (Y_{ca})_{m_1 m_2} \tilde{\Gamma}_k^*{(C, \tilde{\Gamma}_n^*{A}(B, C))}_{m_1 m_2} (Y_{bc})_{m_1 m_2} \right\}$$  \hspace{1cm} \quad (69)$$

This clearly exhibits the role of $G^n$ as “obstruction operators”: namely the equation \eqref{eq:68} for $\tilde{\Gamma}_n^*$ is consistent only if

$$G^n[\tilde{\Gamma}_1^*, \ldots, \tilde{\Gamma}_{n-1}^*] \in B^3(V), \quad \quad \quad (70)$$

in which case $\tilde{\Gamma}_n^*$ is determined up to an element of $Z^2(V)$ (that one may absorb into $\tilde{\Gamma}_1$ by a redefinition of the perturbation parameter $\lambda$).

In analogy to deformation theory of associative algebras, we expect that for $\tilde{\Gamma}_1^*, \ldots, \tilde{\Gamma}_{n-1}^* \in C^2(V)$ solving \eqref{eq:67} for all $n' < n$, in particular $\tilde{\Gamma}_1^* \in Z^2(V)$, one always has

$$b^3 G^n[\tilde{\Gamma}_1^*, \ldots, \tilde{\Gamma}_{n-1}^*] = 0, \quad \text{i.e.,} \quad G^n[\tilde{\Gamma}_1^*, \ldots, \tilde{\Gamma}_{n-1}^*] \in Z^3(V).$$  \hspace{1cm} \quad (71)$$

The cohomology class of $G^n[\tilde{\Gamma}_1^*, \ldots, \tilde{\Gamma}_{n-1}^*]$ in $RLH^3(V)$ is referred to as an obstruction. For \eqref{eq:68} to have a solution, the obstruction must be zero. If it happens that $RLH^3(V)$ is trivial, no obstructions can occur in any order, and all $\tilde{\Gamma}_n^*$ can be found recursively from a given $\tilde{\Gamma}_1$ in $Z^2(V)$, i.e., every first-order perturbation is integrable. If $RLH^3(V)$ is nontrivial, then some first-order perturbations may still be integrable, but \eqref{eq:70} might impose further restrictions.

To address and decide the possibility of (formal) continuous deformations of a given reduced Lie algebra, one therefore has to compute its second and third cohomologies (provided \eqref{eq:71} can be established). This is outside the scope of this article.

6 Outlook

We showed that the commutation relations among quasiprimary fields in conformal chiral field theories ($W$-algebras) are fixed up to structure constants that are related to the 3-point amplitudes. We then explicitly exhibited an infinite number of constraints on the structure constants, which warrant anti-symmetry and Jacobi identity for the commutator of field operators, and positivity of the Hilbert space inner product. Their solutions can therefore be used as a new axiomatization of chiral CFT. It is not a surprise that in the easiest case, the solution of the constraints on the structure constants for fields of dimension 1 reproduces the well-known Kac-Moody algebras, including the necessary compactness of the underlying Lie algebra. It remains to analyze these constraints more carefully in the general case.

In more abstract language, the structure constants define a bracket on a reduced field space. We proceeded to explore the rigidity of this bracket under formal deformations, in other words to check whether there exist models in the neighborhood of a given model. Following the general strategy, we constructed a cohomology complex related to the deformation problem, and showed that the cohomology groups $RLH^2(V)$ and $RLH^3(V)$ determine the existence and integrability
of deformations (with the proviso that (71) was not yet proven). We have, though, not been able to actually compute the cohomology groups associated to this complex and this has to be done before a more complete deformation theory can be developed. Another option would be to try to construct a differential graded Lie algebra out of the cohomology complex, whose deformation theory would be tightly related to the deformation theory of the reduced bracket. For this purpose, one has to construct a bracket in this complex, such that it is skew symmetric with respect to the grading by dimension of the cochain spaces and satisfies a graded Jacobi identity.

As an example what the deformation theory would produce when fully worked out, one may think of the theory generated by the stress-energy tensor, which has the central charge $c$ as a free parameter. The number of composite fields of a given dimension is determined by a well-known character formula, so this would fix the multiplicities and hence the reduced space. But for $c < 1$ the presence of zero-norm vectors in the Verma module reduces the multiplicities. We therefore expect that for $c > 1$, the second cohomology $RLH^2(V)$ is nontrivial, admitting an infinitesimal change of $c$, and $RLH^3(V)$ could be trivial as there is no obstruction against finite variations of $c$; on the other hand, for $c < 1$ and with the reduced multiplicities, the second cohomology is expected to be trivial. Of course, presently we cannot establish these claims “from scratch”.

In previous approaches [Zamolodchikov, 1986; Bouwknegt, 1988; Blumenhagen et al., 1991], $W$-algebras were analysed in terms of finite sets of fields, which generate the infinite space of quasiprimary fields under the OPE. Indeed, the consistency of the commutation relations can be studied at the level of the generating fields; but – as the example of the stress-energy tensor shows – to address the issue of positivity, one has to include all their composite fields. In our approach, no distinguished role is assigned to the generating fields, except that they could possibly be a practical tool for solving the constraints in an inductive way.

The cohomological nature of the deformation theory was recognized previously and turned into a constructive tool, e.g., for perturbations of free fields [Hollands, 2008]; for the classification of $W$-algebras it was not explored yet.

### A Appendix: The matrix $(Y_{abc})^{m_1 m_2}_{\tilde{m}_1 \tilde{m}_2}$

In this subsection we will explain how we determined the matrix $(Y_{abc})^{m_1 m_2}_{\tilde{m}_1 \tilde{m}_2}$ which transforms $(T_{cab})^{\tilde{m}_1 \tilde{m}_2}_{m_1 m_2} (h, f, g)$ into $(T_{abc})^{m_1 m_2}_{\tilde{m}_1 \tilde{m}_2} (f, g, h)$, and which is the essential ingredient of the reduced Jacobi identity (31).

An elegant method would have been to exploit the associativity of a nontrivial one-parameter family of products on $\bigoplus_k M_{2k}$ defined in terms of Rankin–Cohen brackets [Cohen et al., 1996], generalizing an unpublished observation by Eholzer. Varying the parameter, one obtains linear relations between $\lambda_{bc}^d (\lambda_{ab}^c \times 1_c)$ and $\lambda_{bc}^d (1_a \times \lambda_{ac}^d)$ for every fixed $a, b, c, e$, from which one would read off the matrix $X_{abc}$ of Sect. 3.2.1 that describes the re-bracketing, and then by (21) the matrix $Y_{abc}$. Unfortunately, due to a symmetry with respect to the parameter, varying the parameter gives only one half of the necessary relations. This is a bit of a surprise since one would have naively expected that an associative product rather encodes twice as much information that a (generalized) commutator.

Instead, we have to adopt a much more down-to-earth linear algebra approach. By applying the intertwiners to test functions and comparing the resulting coefficients of products of derivatives, allowed us to derive a recursion formula for the entries of $(Y_{abc})^{m_1 m_2}_{\tilde{m}_1 \tilde{m}_2}$, which we were able to solve afterwards.

The explicit formulae thus obtained below are meromorphic functions which may have poles at real positive values of the dimensions $a, b, c$. In other words, the intertwiner bases may become degenerate at these points. These singularities can be regularized, e.g., by letting the scaling dimensions have small positive imaginary parts, while keeping the summation indices $p, q$ in (2) and $m$ in (11) a fixed integer. While the representation theory of $SL(2, \mathbb{R})$ is perfectly meaningful for complex $a, b, c$, the physical dimensions are of course positive integers. For the removal of the regularization in QFT, see Sect. 3.3.
Using (3) we write the explicit expression for the composite intertwiners:

\[
\left( T_{abc} \right)^{m_1 m_2} (f, g, h) = \lambda_{abc}^1 \circ (1_a \times \lambda_{abc}^2)(f, g, h) \\
= \sum_{p+q+m_1 = m_2 \atop s+t = m_2} (-1)^{q+t} \frac{(2b + 2c - m_1 - 2m_2 - 3)_p (2a - m_1 - 1)_q}{p! q!} \times \\
\times \frac{(2h - m_2 - 1)_t (2b - m_2 - 1)_t}{s! t!} \partial^p f \partial^q g \partial^r h,
\]

which can be expanded as

\[
\left( T_{abc} \right)^{m_1 m_2} (f, g, h) = \sum_{r_1 + r_2 + r_3 = m_1 + m_2} \left( T_{abc} \right)^{m_1 m_2}_{r_1 r_2 r_3} \partial^{r_1} f \partial^{r_2} g \partial^{r_3} h
\]

with \( \left( T_{abc} \right)^{m_1 m_2}_{r_1 r_2 r_3} \) numerical coefficients, and similar for \( \left( T_{abc} \right)^{\tilde{m}_1 \tilde{m}_2} (h, f, g) \). We therefore have to solve, for any fixed triple \((r_1, r_2, r_3)\), the equation

\[
\left( Y_{abc} \right)^{m_1 m_2}_{\tilde{m}_1 \tilde{m}_2} \left( T_{abc} \right)^{m_1 m_2}_{r_1 r_2 r_3} = \left( T_{abc} \right)^{m_1 m_2}_{r_1 r_2 r_3}.
\]

The following two observations now simplify the problem.

1. Because \( r_1 + r_2 + r_3 = m_1 + m_2 \), we must have \( m_1 + m_2 = \tilde{m}_1 + \tilde{m}_2 \). The matrix \( \left( Y_{abc} \right)^{m_1 m_2}_{\tilde{m}_1 \tilde{m}_2} \) therefore has a block form, reflecting the fact, that it is not possible to decompose \( \left( T_{abc} \right)^{m_1 m_2} \) in the basis of \( \left( T_{abc} \right)^{\tilde{m}_1 \tilde{m}_2} \) if they map to representations with different scaling dimensions. Then we can relax two of the indices of \( \left( Y_{abc} \right)^{m_1 m_2}_{\tilde{m}_1 \tilde{m}_2} \):

\[
\left( Y_{abc} \right)^{m_1 m_2}_{\tilde{m}_1 \tilde{m}_2} = \delta_{m_1 + m_2 - \tilde{m}_1 - \tilde{m}_2} \left( Y_{abc}(n) \right)^{m_2}_{\tilde{m}_2}, \quad n := m_1 + m_2
\]

(We could as well relax the indices \( m_2 \) and \( \tilde{m}_2 \) instead of \( m_1 \) and \( \tilde{m}_1 \), it is just a matter of choice.) Then, we have to solve for any fixed triple \((r_1, r_2, r_3)\)

\[
\left( Y_{abc}(n) \right)^{m_2}_{\tilde{m}_2} \left( T_{abc} \right)^{n - \tilde{m}_2, \tilde{m}_2}_{r_1 r_2 r_3} = \left( T_{abc} \right)^{n - \tilde{m}_2, \tilde{m}_2}_{r_1 r_2 r_3}.
\]

2. \( \ell \) taken for \( n + 1 \) triples \((r_1, r_2, r_3)\) and \( m_2, \tilde{m}_2 \in [0, n]\) gives a system of \((n + 1) \times (n + 1)\) equations for \((n + 1) \times (n + 1)\) unknown quantities, and if these equations are linearly independent it is enough to fix all the entries of \( \left( Y_{abc}(n) \right)^{m_2}_{\tilde{m}_2} \). A most convenient choice are the triples \((k, 0, n - k)\) with \( k \in [0, n]\), because the coefficients \( \left( T_{abc} \right)^{n - \tilde{m}_2, \tilde{m}_2}_{k,0,n-k} \), read off from (72), are zero if \( \tilde{m}_2 > k \). This allows to establish a recursion, such that the component \( \left( Y_{abc}(n) \right)^{m_2}_{\tilde{m}_2} \) is obtained recursively from the components \( \left( Y_{abc}(n) \right)^{m_2}_{\tilde{m}_2} \) with \( \tilde{m}_2 < \tilde{m}_2 \).

**Proposition A.1.** The entries of \( \left( Y_{abc}(n) \right)^{m_2}_{\tilde{m}_2} \) satisfy the recursion formula:

\[
\left( Y_{abc}(n) \right)^{m_2}_{\tilde{m}_2} = \left( T_{abc} \right)^{n - \tilde{m}_2, m_2}_{\tilde{m}_2, 0, n - \tilde{m}_2} - \sum_{j=0}^{\tilde{m}_2 - 1} \left( Y_{abc}(n) \right)^{m_2}_{\tilde{m}_2} \left( T_{abc} \right)^{n - j, j}_{\tilde{m}_2, 0, n - \tilde{m}_2}
\]

To solve this recursion, we “insert repeatedly this formula into itself” and obtain:

\[
\left( Y_{abc}(n) \right)^{m_2}_{\tilde{m}_2} = \sum_{s=0}^{\tilde{m}_2} \left( T_{abc} \right)^{n - m_2, m_2}_{s, 0, n - s} \left[ -1 \right]^{m_2 - s} \left( n - s \right)! \frac{(2c - n - s - 1)_{m_2 - s}}{(2a + 2b - 2m_2 - 3)_{m_2 - 2s}} \times \\
\times \sum_{(j_1)_{\tilde{m}_2}} \left( -1 \right)^{j_1 - 1} \prod_{j \in (j_1)} \frac{(2a + 2b - 2j + 1 - 3)(j + 1 - j)}{(j + 1 - j)!}
\]

\[
(78)
\]
where \( \{ j_i \}_s^m \) are the possible sets \( \{ j_i = s, j_i < j_{k+1}, j_i = m \} \), including \( \{ s, m \} \).

Claim A.2. Extrapolating from calculations for small \( l \), we found an explicit identity to perform the multiple sum:

\[
\sum_{(j_i)_1^m} (-1)^{j_1} \prod_{j_r \in (j_i)_1^m} \frac{(2a + 2b - 2j_{r+1} - 3)_{j_{r+1} - j_r}}{(j_{r+1} - j_r)!} = (-1)^{m-s} \frac{(2a + 2b - 2m - 3)(2a + 2b - m - s - 2)_{m-s-1}}{(m-s)!}
\]

Using this identity (with \( m = \tilde{m}_2 \)) we can reduce (78) to a single sum. We finally obtain

**Proposition A.3.** The matrix \((Y_{abc}(n))_{\tilde{m}_2}^{m_2}\) is given by the following expression:

\[
(Y_{abc}(n))_{\tilde{m}_2}^{m_2} = (-1)^{n-\tilde{m}_2} \frac{n!}{(n-m)!} \sum_{s=0}^{\tilde{m}_2} \frac{(n-m-2s)(n+m-2s-2-2c+4)_s}{(n-m-2-2c+2)_s} \frac{1}{(2a+2b-2\tilde{m}_2-2)_s} (2a+2b-2m-3)_{m-\tilde{m}_2-1}
\]

This expression presumably cannot be further simplified, since the sum does not factorize in general as a rational function of the dimensions.

We observed the following interesting property of the matrix \((Y_{abc}(n))_{\tilde{m}_2}^{m_2}\):

**Proposition A.4.** The entries from an arbitrary column of the matrix \((Y_{abc}(n))_{\tilde{m}_2}^{m_2}\) sum to \((-1)^{n+\tilde{m}_2}\), where \( \tilde{m}_2 \) is the number of the column.

**Proof.** We will prove this statement by induction on the number of the column.

Let us first consider the column \( m_2 = 0 \). The entries from this column are expressed as:

\[
(Y_{abc}(n))_{0}^{m_2} = (-1)^n \frac{(2-2a)_n (2-2b)_n}{(n-m)!} \frac{n!}{(n-m)!} \frac{(2a+2b-2m-3)_m}{(2a+2b-2m)_m}
\]

Then, using the property \( \frac{(a+b)_n}{n!} = \frac{1}{n!} \frac{(a)_n (b)_n}{(n)_n} \), we compute \( \sum_{m_2=0}^n (Y_{abc}(n))_{0}^{m_2} = (-1)^n \), i.e., the statement of the proposition holds for \( \tilde{m}_2 = 0 \).

Now let us assume that \( \sum_{m_2=0}^n (Y_{abc}(n))_{0}^{m_2} = (-1)^{n+j} \) is true for every \( j \leq k - 1 \). We will prove that then \( \sum_{m_2=0}^n (Y_{abc}(n))_{k}^{m_2} = (-1)^{n+k} \). We start from formula (77) and obtain:

\[
(Y_{abc}(n))_{k}^{m_2} = \frac{1}{(T_{cab})_{k,0,n-k}} \sum_{m_2=0}^n \left( \sum_{m=0}^{n-k} (T_{abc})_{k,0,n-k}^{n-m,m_2,m_2} - \sum_{j=0}^k (-1)^{n+j} (T_{cab})_{k,0,n-k}^{n-j,j} \right) + (-1)^{n+k}
\]

Hence, we have to prove that the expression in the brackets vanishes. Let us write this expression explicitly:

\[
\sum_{m=0}^{n-k} (T_{abc})_{k,0,n-k}^{n-m,m_2,m_2} = (-1)^{n+k} \sum_{m=0}^{n-k} \frac{(2a+2b-2m-3)_{m_2}}{m!} \frac{(2b-m-1)_m}{(n-m-k)!}
\]

With the identities \( \binom{a+b}{k} = \sum_{j=0}^n \binom{a+j+1}{j} \binom{b+j+1}{j} \) and \( (a)_m = (a)_m (a+m)_m \) one can prove that the first sum is equal to the second sum in (83), hence the bracket in (82) vanishes:

\[
\sum_{m_2=0}^n (Y_{abc}(n))_{k}^{m_2} = (-1)^{n+k}
\]

This proves the induction hypothesis and the proposition.
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