Categorified quantum sl(2) and
equivariant cohomology of iterated flag varieties

Aaron D. Lauda
Department of Mathematics,
Columbia University, New York, NY 10027, USA
email: lauda@math.columbia.edu

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Abstract

A 2-category was introduced in math.QA/0803.3652 that categorifies Lusztig’s integral
version of quantum sl(2). Here we construct for each positive integer \(N\) a representation
of this 2-category using the equivariant cohomology of iterated flag varieties. This rep-
resentation categorifies the irreducible \((N + 1)\)-dimensional representation of quantum
sl(2).

1 Introduction

The quantum group \(\mathbf{U}_q(\mathfrak{sl}_2)\) is the associative algebra (with unit) over \(\mathbb{Q}(q)\) with generators
\(E, F, K, K^{-1}\) and relations

\[
\begin{align*}
KK^{-1} &= 1 = K^{-1}K, \quad (1.1) \\
KE &= q^2EK, \quad (1.2) \\
KF &= q^{-2}FK, \quad (1.3) \\
EF - FE &= \frac{K - K^{-1}}{q - q^{-1}}. \quad (1.4)
\end{align*}
\]

Lusztig’s \(\hat{\mathbf{U}}\) is the nonunital algebra obtained from \(\mathbf{U}_q(\mathfrak{sl}_2)\) by adjoining a collection of orthogonal idempotents \(1_n\) indexed by the weight lattice of \(\mathfrak{sl}_2\) (see for example [10]). This decomposes the algebra \(\hat{\mathbf{U}}\) into a direct sum \(\bigoplus_{n,m \in \mathbb{Z}} 1_m \hat{\mathbf{U}} 1_n\). In \(\hat{\mathbf{U}}\) the elements \(E\) and \(F\) of \(\mathbf{U}_q(\mathfrak{sl}_2)\) become the collection of elements \(1_n + 2E1_n\) and \(1_nF1_{n+2}\), for \(n \in \mathbb{Z}\). We will simplify notation and write \(E1_n\) and \(F1_{n+2}\), with it understood that \(E\) increases the subscript by 2 and \(F\) decreases the subscript by 2 going from right to left. The integral form \(\mathcal{A} \hat{\mathbf{U}}\) of \(\hat{\mathbf{U}}\) is the \(\mathbb{Z}[q, q^{-1}]\)-subalgebra of \(\hat{\mathbf{U}}\) generated by products of divided powers \(E^{(a)} 1_n := \frac{E^n}{[a]!} 1_n\) and \(F^{(a)} 1_n := \frac{F^n}{[a]!} 1_n\), where \([a]!\) denotes the quantum factorial \([a]! = [a][a - 1] \ldots [2][1]\) with \([m] := \frac{q^m - q^{-m}}{q - q^{-1}}\). Algebra \(\mathcal{A} \hat{\mathbf{U}}\) has a canonical basis in which the structure constants are in \(\mathbb{N}[q, q^{-1}]\).

In a recent paper [9], the author categorified \(\mathcal{A} \hat{\mathbf{U}}\). A 2-category \(\hat{\mathcal{U}}\) was introduced whose split Grothendieck ring \(K_0(\mathcal{U})\) is isomorphic to the algebra \(\mathcal{A} \hat{\mathbf{U}}\). Objects of \(\hat{\mathcal{U}}\) correspond to
weights of $\mathfrak{sl}_2$. Morphisms from $n$ to $m$ consist of formal direct sums of 1-morphisms of the form

$$1_m E^{a_1} E^{\beta_1} E^{a_2} \cdots E^{\beta_k-1} E^{a_k} E^{\beta_k} 1_n \{s\}$$

where $m = n + 2(\sum \alpha_i - \sum \beta_i)$, and $s \in \mathbb{Z}$. The shift $\{s\}$ lifts the $\mathbb{Z}[q, q^{-1}]$-module structure of $\mathcal{A}\mathcal{U}$, that is, for a 1-morphism $x$ of $\mathcal{U}$ we have $[x\{s\}] = q^s[x]$ in $K_0(\mathcal{U})$. It was shown in [9] that isomorphism classes of indecomposable 1-morphisms of $\mathcal{U}$ bijectively correspond to elements in Lusztig’s canonical basis, so that every morphism in $\mathcal{U}$ decomposes into a direct sum of shifted copies of morphisms lifting Lusztig’s canonical basis. The idea for such a categorification goes back to the work of Crane and Frenkel [5, 6].

The 2-category $\tilde{\mathcal{U}}$ was shown to have a number of other desirable properties. For example, all morphisms in $\tilde{\mathcal{U}}$ have both left and right adjoints, and there is a natural enriched hom that associates a graded abelian group to each pair of morphisms in $\tilde{\mathcal{U}}$; the graded rank of this abelian group categorifies a semilinear form defined on $\mathcal{A}\mathcal{U}$. Furthermore, the morphisms $E^{a_1} E^{\beta_1} E^{a_2} \cdots E^{\beta_k-1} E^{a_k} E^{\beta_k}$ lift the elements $E^{a_1} E^{\beta_1} E^{a_2} \cdots E^{\beta_k-1} E^{a_k} E^{\beta_k}$ in $\mathcal{U}$ admit an action by the nilHecke algebra $N\mathcal{H}_a$. By this we mean that the abelian group of 2-morphisms $\tilde{\mathcal{U}}(E^{a_1} E^{\beta_1} E^{a_2} \cdots E^{\beta_k-1} E^{a_k} E^{\beta_k}, E^{a_1} E^{\beta_1} E^{a_2} \cdots E^{\beta_k-1} E^{a_k} E^{\beta_k})$ is a polynomial algebra $\mathbb{k}[\chi_1, \chi_2, \ldots, \chi_a]$. The 2-morphisms in $\tilde{\mathcal{U}}$ that are acted on by these $\chi_i$ are denoted by $\chi_i$. The subscripts are needed for compatibility with $E^{a_1} E^{\beta_1} E^{a_2} \cdots E^{\beta_k-1} E^{a_k} E^{\beta_k}$ in $\mathcal{U}$ increasing the subscript by 2. Using the ordinary cohomology rings of iterated flag varieties, a representation $\Gamma_N : \tilde{\mathcal{U}} \to \text{Flag}_N$ was also constructed in [9]. The representation $\Gamma_N$ is a 2-functor mapping $\tilde{\mathcal{U}}$ into a 2-category $\text{Flag}_N$. The 2-category $\text{Flag}_N$ has as objects the graded cohomology rings of certain Grassmannian varieties. The morphisms are graded bimodules obtained from the cohomology rings of partial flag varieties (iterated flag varieties), and the 2-morphisms are degree zero bimodule maps. It was already known that these iterated flag varieties categorify irreducible representations of $U_q(\mathfrak{sl}_2)$ [1, 4, 7]. In [9] the author showed that the 2-category $\tilde{\mathcal{U}}$, categorifying Lusztig’s $\tilde{\mathcal{U}}$, acts on the 2-category $\text{Flag}_N$, revealing new structure not previously observed.

The cohomology rings of Grassmannians and iterated flag varieties are described in terms of Chern classes of naturally associated tautological bundles. For example, fixing a complex vector space $W$ of dimension $N$, the tautological $k$-dimensional complex vector bundle $U_{k,N}$ on the Grassmannian $Gr(k, N)$ of complex $k$-planes in $W$ has total space consisting of pairs $(V, x)$ with $x \in V$ and $V \in Gr(k, N)$. Choose a hermitian metric on $W$. From the orthogonal complements of the fibres $V$ of the bundle $U_{k,N}$ we can construct an $(N-k)$-dimensional complex vector bundle $U_{N-k,N}$ with the property that

$$U_{k,N} \oplus U_{N-k,N} \cong I_N,$$

with $I$ the trivial rank 1 bundle. The Chern classes, $x_i := x_i(U_{k,N}) \in H^{2i}(Gr(k, N))$ for $1 \leq i \leq k$, and $y_j := y_j(U_{N-k,N}) \in H^{2j}(Gr(N-k, N))$ for $1 \leq j \leq N-k$, then satisfy

$$\left(1 + x_1 t + x_2 t^2 \cdots + x_k t^k \right) \left(1 + y_1 t + y_2 t^2 + \cdots + y_{N-k} t^{N-k} \right) = 1$$ (1.6)
by the Whitney sum axiom for Chern classes. Above, \( t \) is a formal variable used to keep track of homogeneous elements. Borel [3] showed that the cohomology ring of this variety is given by
\[
H^*(Gr(k, N)) = \mathbb{Q}[x_1, \ldots, x_k, y_1, \ldots, y_{N-k}]/I_{k,N}
\]
where \( I_{k,N} \) is the ideal generated by the homogeneous terms in (1.6).

Here we define new representations of \( \mathcal{U} \) using the equivariant cohomology of iterated flag varieties. Just like the ordinary cohomology rings, the equivariant cohomology rings are generated by Chern classes of naturally associated bundles. But there are no relations on the equivariant cohomology rings; the equivariant cohomology of the Grassmannian variety \( Gr(k, N) \) described above is given by
\[
H^*_{GL(N)}(Gr(k, N)) = \mathbb{Q}[x_1, \ldots, x_k, y_1, \ldots, y_{N-k}]/I_{k,N}
\]
the relations given by the homogeneous terms of (1.6) are no longer imposed. From the equivariant cohomology of iterated flag varieties we construct a 2-category \( \mathbf{EqFlag}_N \) and a representation \( \Gamma^G_N : \mathcal{U} \rightarrow \mathbf{EqFlag}_N \) for each positive integer \( N \). In Theorem 4.13 we show that \( \Gamma^G_N \) also categorifies the irreducible \((N+1)\)-dimensional representation of \( \mathcal{U} \).

The nilHecke algebra \( \mathcal{N} \mathcal{H}_a \) is an infinite dimensional algebra. However, the representations \( \Gamma_N : \mathcal{U} \rightarrow \mathbf{Flag}_N \) constructed from ordinary cohomology rings map \( \mathcal{N} \mathcal{H}_a \) to finite-dimensional quotients. The 2-morphism \( z_{n+2(a-i)} \) acted on by the polynomial subalgebra \( k[\chi_1, \chi_2, \ldots, \chi_a] \) of \( \mathcal{N} \mathcal{H}_a \) are mapped by \( \Gamma_N \) to nilpotent 2-morphisms in \( \mathbf{Flag}_N \). In particular, \( \Gamma_N(z_{n+2(a-i)})^N = 0 \) for all \( 1 \leq i \leq a \). A similar nilpotent action is built into the definition of the \( \mathfrak{sl}_2 \)-categorifications introduced by Chuang and Rouquier [4]. By restricting their attention to \( \mathfrak{sl}_2 \)-categorifications that admit an action by the affine or degenerate affine Hecke algebra they are able to obtain a beautiful classification for such categorified representations of \( \mathfrak{sl}_2 \).

The importance of the equivariant representations \( \Gamma^G_N : \mathcal{U} \rightarrow \mathbf{EqFlag}_N \) introduced here is that they provide \( U_q(\mathfrak{sl}_2) \)-categorifications where the 2-morphisms \( z_{n+2(a-i)} : \mathcal{E}^a \mathbf{1}_n \rightarrow \mathcal{E}^a \mathbf{1}_n \) do not act nilpotently. These representations admit a faithful action of the nilHecke algebra, while still categorifying the irreducible \((N+1)\)-dimensional representation of \( U_q(\mathfrak{sl}_2) \). Thus, the representations \( \mathbf{EqFlag}_N \) provide new insights about what to expect from the categorified representation theory of \( U_q(\mathfrak{sl}_2) \).

The existence of these new 2-categorical representations \( \mathbf{EqFlag}_N \) also serves to reinforce the choice of axioms for the 2-category \( \mathcal{U} \). The 2-category \( \mathbf{Flag}_N \) built from the cohomology of iterated flag varieties, and the 2-category \( \mathbf{EqFlag}_N \) built from equivariant cohomology of these varieties, both satisfy all the relations on 2-morphisms required for a representation of \( \mathcal{U} \). This demonstrates the robustness of the 2-category \( \mathcal{U} \).

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2 The 2-category \( \mathcal{U} \)

The 2-category \( \mathcal{U} \) was constructed in [9], where it was shown that the split Grothendieck ring \( K_0(\mathcal{U}) \) is isomorphic to the algebra \( \mathcal{A} \mathcal{U} \) — Lusztig’s integral version of \( U_q(\mathfrak{sl}_2) \). The
2-category \( \mathcal{U}^* \) is the Karoubian envelope of a 2-category \( \mathcal{U} \) that is the restriction of a graded additive 2-category \( \mathcal{U}^* \) to the degree preserving 2-morphisms. We recall the definition here. The reader is referred to [9] for more details.

2.1 The 2-category \( \mathcal{U}^* \)

The graded additive 2-category with translation \( \mathcal{U}^* \) consists of

- objects: \( n \) for \( n \in \mathbb{Z} \),
- 1-morphisms: formal direct sums of composites of

\[
1_m \mathcal{E}^{\alpha_1} \mathcal{E}^{\beta_1} \mathcal{E}^{\alpha_2} \cdots \mathcal{E}^{\beta_{k-1}} \mathcal{E}^{\alpha_k} \mathcal{E}^{\beta_k} 1_n \{s\}
\]

where some of the \( \alpha_i \) and \( \beta_j \) may be zero, \( m = n + 2(\sum \alpha_i - \sum \beta_i) \), and \( s \in \mathbb{Z} \).
- graded 2-morphisms\(^1\)

![Diagram of graded 2-morphisms](image)

Together with identity 2-morphisms and isomorphisms \( x \simeq x\{s\} \) for each 1-morphism \( x \), such that:

- \( 1_{n+2} \mathcal{E} 1_n \) and \( 1_n \mathcal{F} 1_{n+2} \) are biadjoints with units and counits given by the pairs \( (\eta_n, \varepsilon_{n+2}) \) and \( (\tilde{\eta}_n, \tilde{\varepsilon}_{n-2}) \).
- All 2-morphisms are cyclic with respect to the above biadjoint structure. This is ensured by the relations,

\[
\begin{align*}
\begin{array}{c}
\begin{array}{c}
\text{deg } n+1 \\
\text{deg } n+1
\end{array}
\end{array}
\end{align*}
\]

\[
\left(2.1\right)
\]

\(^1\)Diagrams are read from bottom to top and from right to left. The morphism \( \mathcal{E} \) is represented by an upward pointing arrow, and \( \mathcal{F} \) by a downward pointing arrow. For example, \( U_n : \mathcal{E} \mathcal{E} 1_n \Rightarrow \mathcal{E} \mathcal{E} 1_n : n \to n + 4 \).
The cyclic conditions on 2-morphisms implies that boundary preserving planar isotopies of diagrams result in the same 2-morphism.

- All dotted closed bubbles of negative degree are zero. That is,

\[
\begin{align*}
\begin{array}{c}
\circlearrowright_n \\
\circlearrowright_m
\end{array}
&= 0, \quad \text{if } m < n - 1, \\
\begin{array}{c}
\circlearrowleft_n \\
\circlearrowright_m
\end{array}
&= 0, \quad \text{if } m < -n - 1,
\end{align*}
\]

for all \(m \in \mathbb{Z}_+\) and \(n \in \mathbb{Z}\). Above we use the shorthand

\[
n^m + 2 \begin{array}{c} n \\
\uparrow
\end{array} = \begin{pmatrix} n \\
\uparrow
\end{pmatrix}^m
\]

to denote iterated composites of the 2-morphisms \(z_n\).

- The nilHecke algebra \(NH_a\) acts on \(U^a(\mathcal{E}^a1_n, \mathcal{E}^a1_n)\) and \(U^a(\mathcal{F}^a1_n, \mathcal{F}^a1_n)\) for all \(n \in \mathbb{Z}\).

\[
\begin{align*}
\begin{array}{c}
\circlearrowright_n \\
\circlearrowright_n
\end{array}
&= 0 \\
\begin{array}{c}
\circlearrowleft_n \\
\circlearrowleft_n
\end{array}
&= 0
\end{align*}
\]  \hspace{1cm} (2.3)

\[
\begin{align*}
\begin{array}{c}
\circlearrowright_n
\end{array} - \begin{array}{c}
\circlearrowleft_n
\end{array}
&= 0 \\
\begin{array}{c}
\circlearrowright_n
\end{array} - \begin{array}{c}
\circlearrowleft_n
\end{array}
&= 0
\end{align*}
\]  \hspace{1cm} (2.4)

\[
\begin{align*}
\begin{array}{c}
\circlearrowright_n \\
\circlearrowleft_n
\end{array}
&= 0 \\
\begin{array}{c}
\circlearrowleft_n \\
\circlearrowright_n
\end{array}
&= 0
\end{align*}
\]  \hspace{1cm} (2.5)

for all values of \(n \in \mathbb{Z}\). The action on \(U^a(\mathcal{F}^a1_n, \mathcal{F}^a1_n)\) is obtained from the above relations using biadjointness.
• The 1-morphisms $\mathcal{E}$ and $\mathcal{F}$ lift the relations of $E$ and $F$ in $U_q(\mathfrak{sl}_2)$. This is ensured by
requiring the equalities

$$
\begin{align*}
\begin{array}{c}
\begin{tikzpicture}[baseline={([yshift=-.5ex]current bounding box.center)}]
\draw (0,0) circle (1);
\draw (0,0) -- (0,1);\node at (0,1.5) {$n$};
\end{tikzpicture}
\end{array}
&= -\sum_{\ell=0}^{n-1} \begin{array}{c}
\begin{tikzpicture}[baseline={([yshift=-.5ex]current bounding box.center)}]
\draw (0,0) circle (1);
\draw (0,0) -- (-1,1);\draw (0,0) -- (1,1);\node at (0,1.5) {$n$};
\end{tikzpicture}
\end{array} + \begin{array}{c}
\begin{tikzpicture}[baseline={([yshift=-.5ex]current bounding box.center)}]
\draw (0,0) circle (1);
\draw (0,0) .. controls (-1,1) .. (0,1);\draw (0,0) .. controls (1,1) .. (0,1);
\end{tikzpicture}
\end{array}
\begin{array}{c}
\begin{tikzpicture}[baseline={([yshift=-.5ex]current bounding box.center)}]
\draw (0,0) circle (1);
\draw (0,0) -- (0,1);\node at (0,1.5) {$n-1+\ell$};
\end{tikzpicture}
\end{array}
\end{align*}
\end{align}
(2.6)

for all $n \in \mathbb{Z}$, where the label next to the bullet indicates the number of composites of the 2-morphisms $z_n$ and $\tilde{z}_n$. In the above equations, and throughout this paper, all summations are increasing sums, or else they are taken to be zero. This means that in (2.6) the right-hand-side of the first equation is zero when $n > 0$, and the right-hand-side of the second equation is zero when $n < 0$.

Notice that for some values of $n$ the dotted bubbles appearing in (2.6) and (2.7) have negative labels. A composite of $z_n$ or $\tilde{z}_n$ with itself a negative number of times does not make sense. These dotted bubbles with negative labels, called fake bubbles, are formal symbols inductively defined by the equations

$$
\begin{align*}
\begin{array}{c}
\begin{tikzpicture}[baseline={([yshift=-.5ex]current bounding box.center)}]
\draw (0,0) circle (1);
\draw (0,0) -- (0,1);\node at (0,1.5) {$n$};
\end{tikzpicture}
\end{array}
\begin{array}{c}
\begin{tikzpicture}[baseline={([yshift=-.5ex]current bounding box.center)}]
\draw (0,0) circle (1);
\draw (0,0) -- (0,1);\node at (0,1.5) {$n$};
\end{tikzpicture}
\end{array}
\begin{array}{c}
\begin{tikzpicture}[baseline={([yshift=-.5ex]current bounding box.center)}]
\draw (0,0) circle (1);
\draw (0,0) -- (0,1);\node at (0,1.5) {$n$};
\end{tikzpicture}
\end{array}
\end{align*}
&= 1,
\end{align*}
\end{align}
(2.7)

Although the labels are negative, one can check that the overall degree of each fake bubble is still positive, so that these fake bubbles do not violate the positivity of dotted bubble axiom. See [9] for more details.

2.2 Karoubi envelope and $\hat{U}$

The Karoubi envelope $Kar(\mathcal{C})$ of a category $\mathcal{C}$ is an enlargement of the category $\mathcal{C}$ in which all idempotents split. There is a fully faithful functor $\mathcal{C} \rightarrow Kar(\mathcal{C})$ that is universal with respect to functors that split idempotents in $\mathcal{C}$. This means that if $F: \mathcal{C} \rightarrow \mathcal{D}$ is any functor where all idempotents split in $\mathcal{D}$, then $F$ extends uniquely (up to isomorphism) to a functor $\tilde{F}: Kar(\mathcal{C}) \rightarrow \mathcal{D}$ (see for example [2], Proposition 6.5.9). Furthermore, for any other functor $G: \mathcal{C} \rightarrow \mathcal{D}$ and a natural transformation $\alpha: F \Rightarrow G$, $\alpha$ extends uniquely to a natural transformation $\tilde{\alpha}: \tilde{F} \Rightarrow \tilde{G}$.

Let $\hat{U}$ denote the restriction of $U^*$ to those 2-morphisms that are degree preserving.
Definition 2.1. Define the 2-category \( \mathcal{U} \) to have the same objects as \( \mathcal{U} \) and \( \mathcal{U}(n, m) = \text{Kar}(\mathcal{U}(n, m)) \). The fully-faithful functors \( \mathcal{U}(n, m) \to \mathcal{U}(n, m) \) combine to form a 2-functor \( \mathcal{U} \to \mathcal{U} \) universal with respect to splitting idempotents in the hom categories \( \mathcal{U}(n, m) \). The composition functor \( \mathcal{U}(n, m) \times \mathcal{U}(m, p) \to \mathcal{U}(n, p) \) is induced by the universal property of the Karoubi envelope from the composition functor for \( \mathcal{U} \).

Let \( \mathcal{K} \) be a 2-category in which idempotents split. Any 2-functor \( \Psi : \mathcal{U}^* \to \mathcal{K} \) extends uniquely (up to isomorphism) to a representation \( \overline{\Psi} : \mathcal{U} \to \mathcal{K} \). The 2-functor \( \overline{\Psi} \) is obtained from \( \Psi \) by restricting to the degree preserving 2-morphisms of \( \mathcal{U}^* \) and using the universal property of the Karoubi envelope.

Theorem 2.2 (Theorem 9.1.3 [9]). The split Grothendieck ring \( K_0(\mathcal{U}) \) is isomorphic to Lusztig’s \( \mathcal{A}\mathcal{U} \).

3 Iterated flag varieties

For details on the equivariant cohomology of Flag varieties we recommend the lecture notes of Fulton [8].

Let \( X \) be a space equipped with an action on the left by a Lie group \( G \). The equivariant cohomology of \( X \) with respect to \( G \) is the ordinary cohomology of \( EG \times^G X \):

\[
H^*(EG \times^G X).
\]

The equivariant cohomology of a point is then

\[
H^*_G(pt) = H^*(BG, \mathbb{Q}).
\]

The map \( X \to pt \) induces on cohomology a map \( H^*_G(pt) \to H^*_G(X) \) giving the equivariant cohomology of \( X \) the structure of an algebra over \( H^*_G(pt) \).

Here we are interested in \( G = GL(N) \) equivariant cohomology of various partial flag varieties. For \( GL(N) \) the equivariant cohomology ring of a point is generated by Chern classes \( x_1, y_i \) of degree \( 2i \), modulo an ideal \( I_{N,\infty} \),

\[
H^*_{\text{GL}(N)}(pt) = H^*(Gr(N, \infty)) = \mathbb{Q}[x_1, x_2, \ldots, x_N, y_1, y_2, \ldots y_j, \ldots]/I_{N,\infty},
\]

where \( I_{N,\infty} \) is the ideal generated by the homogeneous terms in

\[
(1 + x_1 t + x_2 t^2 + \cdots x_N t^N) (1 + y_1 t + y_2 t^2 + \cdots y_j t^j + \cdots) = 1.
\]

Thus, \( H^*_{\text{GL}(N)}(pt) \) is isomorphic to the polynomial ring

\[
H^*_{\text{GL}(N)}(pt) \cong \mathbb{Q}[x_1, x_2, \ldots, x_{N-1}, x_N]
\]

with \( x_i \) in degree \( 2i \).

Choose a positive integer \( N \) and let \( n = 2k - N \). Fix a complex vector space \( W \) of dimension \( N \). For \( 0 \leq k \leq N \), let \( G_k \) denote the variety of complex \( k \)-planes in \( W \). In this notation we suppress the explicit dependence on \( N \). If we wish to make this dependence explicit we use the notation \( Gr(k, N) \). The equivariant cohomology ring of \( G_k \) has a natural structure of a \( \mathbb{Z} \)-graded algebra,

\[
H^*_{\text{GL}(N)}(G_k, \mathbb{Q}) = \bigoplus_i H^i_{\text{GL}(N)}(G_k, \mathbb{Q}).
\]
For simplicity we sometimes write $H_k^G := H_{GL(N)}^*(G_k, \mathbb{Q})$.

$GL(N)$ acts transitively on $G_k$, so the equivariant cohomology of $G_k$ is

$$H_{GL(N)}^*(Gr(k,N)) = H_{Stab(pt)}^*(pt \in Gr(k,N)),$$

where the stabilizer of a point $(\mathbb{C}^k \subset \mathbb{C}^N)$ in $G_k$ is the group of invertible block $(k \times (N-k))$

upper–triangular matrices. This group is contractible onto its subgroup $GL(k) \times GL(N-k)$.

Hence,

$$H_k^G \cong H_{GL(k) \times GL(N-k)}^*(pt) \cong H_{GL(k)}^*(pt) \otimes H_{GL(N-k)}^*(pt) \cong \mathbb{Q}[x_1, x_2, \cdots, x_k] \otimes \mathbb{Q}[y_1, y_2, \cdots, y_{N-k}],$$

where we have introduced a parameter $n = 2k - N$ as an extra subscript in the last equation, with $\deg x_{i,j} = 2j$ and $\deg y_{i,j} = 2j$. The convenience of the additional subscript is that the equivariant cohomology ring of the variety $G_{k+1}$ of complex $(k+1)$-planes in $W$ is given by

$$H_{k+1}^G = \mathbb{Q}[x_1, x_2, \cdots, x_{k+1}, x_{k+1}+1, \cdots, y_{N-k-1}, y_{N-k}],$$

since $2(k+1) - N = n + 2$. Thus, these two rings are easily distinguished.

For $0 \leq k < m \leq N$ let $G_{k,m}$ be the variety of partial flags

$$\{(L_k, L_m) | 0 \subset L_k \subset L_m \subset W, \dim_{\mathbb{C}} L_k = k, \dim_{\mathbb{C}} L_m = m\}.$$ We also denote this same variety by $G_{m,k}$. Let $H_{k,m}^G$ be the equivariant cohomology algebra of $G_{k,m}$. Forgetful maps

$$G_k \xleftarrow{p_1} G_{k,m} \xrightarrow{p_2} G_m$$

induce maps of equivariant cohomology rings

$$H_k^G \xleftarrow{p_1^*} H_{k,m}^G \xrightarrow{p_2^*} H_m^G$$

which make the cohomology ring $H_{k,m}^G$ into a $H_k^G \otimes H_m^G$-module. Since the algebra $H_m^G$ is commutative, we can turn a left $H_m^G$-module into a right $H_m^G$-module. Hence, we can make $H_{k,m}^G$ into a $(H_k^G, H_m^G)$-bimodule.

Let $k_1, \ldots, k_m$ be a sequence of integers with $0 \leq k_i \leq N$ for all $i$. Form the $(H_{k_1}^G, H_{k_m}^G)$-bimodule

$$H_{k_1,\ldots,k_m}^G = H_{k_1,k_2}^G \otimes H_{k_2,k_3}^G \otimes H_{k_3,k_4}^G \cdots \otimes H_{k_{m-1},k_m}^G.$$

Consider the partial flag variety $G_{k_1,\ldots,k_m}$ which consists of sequences $(W_1, \ldots, W_m)$ of linear subspaces of $W$ such that the dimension of $W_i$ is $k_i$ and $W_i \subset W_{i+1}$ if $k_i \leq k_{i+1}$ and $W_i \supset W_{i+1}$ if $k_{i+1} \geq k_i$. The forgetful maps

$$G_{k_1} \xleftarrow{p_1} G_{k_1,\ldots,k_m} \xrightarrow{p_2} G_{k_m}$$

induce maps of cohomology rings

$$H_{k_1}^G \xleftarrow{p_1^*} H^G(G_{k_1,\ldots,k_m}, \mathbb{Q}) \xrightarrow{p_2^*} H_{k_m}^G$$

which make the cohomology ring $H^G(G_{k_1,\ldots,k_m}, \mathbb{Q})$ into a graded $(H_{k_1}^G, H_{k_m}^G)$-bimodule. As one might expect, there is an isomorphism

$$H_{GL(N)}^*(G_{k_1,\ldots,k_m}, \mathbb{Q}) \cong H_{k_1,\ldots,k_m}^G$$

of graded $(H_{k_1}^G, H_{k_m}^G)$-bimodules.
3.1 One step iterated flag varieties

A special role is played in our theory by the one step iterated flag varieties

\[ G_{k,k+1} = \{ (W_k, W_{k+1}) | \dim W_k = k, \dim W_{k+1} = (k+1), 0 \subset W_k \subset W_{k+1} \subset W \} \, . \]

Again, since GL(N) acts transitively on \( G_{k,k+1} \), we have

\[ H^*_\text{GL}(N)(Gr(k, k+1)) = H^*_\text{Stab}(pt) (pt \in G_{k,k+1}) \, . \]  

(3.9)

The stabilizer of a partial flag \( (\mathbb{C}^k \subset \mathbb{C}^{k+1} \subset \mathbb{C}^N) \in G_{k,k+1} \) is the group of invertible block \( (k \times 1 \times (N-k-1)) \) upper–triangular matrices. This group is contractible onto its subgroup \( GL(k) \times GL(1) \times GL(N-k-1) \). Therefore, the equivariant cohomology ring of \( G_{k,k+1} \) is

\[ H^G_{k,k+1} \cong H^G_{GL(k)}(pt) \otimes H^G_{GL(1)}(pt) \otimes H^G_{GL(N-k-1)}(pt) \]

\[ \cong \mathbb{Q}[w_1, w_2, \ldots, w_k; \xi; z_1, z_2, \ldots, z_{N-k-1}] \, . \]  

(3.10)

Although for our purposes it is useful to identify the generators \( w_j \) and \( z_j \) with their preimages under the inclusions:

\[ H^G_k \xrightarrow{p_1} H^G_{k,k+1} \xrightarrow{p_2} H^G_{k+1} \, . \]

These inclusions making \( H^G_{k,k+1} \) an \( (H^G_k, H^G_{k+1}) \)-bimodule are explicitly given as follows:

\[ \begin{align*}
H^G_k & \hookrightarrow H^G_{k,k+1} \\
x_j,n & \mapsto w_j \quad \text{for} \ 1 \leq j \leq k \\
y_1,n & \mapsto \xi + z_1 \\
y_{\ell,n} & \mapsto \xi \cdot z_{\ell-1} + z_\ell \quad \text{for} \ 1 < \ell < N - k \\
y_{N-k,n} & \mapsto \xi \cdot z_{N-k-1}
\end{align*} \]

and

\[ \begin{align*}
H^G_{k+1} & \hookrightarrow H^G_{k,k+1} \\
x_{1,n+2} & \mapsto \xi + w_1 \\
x_{j,n+2} & \mapsto \xi \cdot w_{j-1} + w_j \quad \text{for} \ 1 < j < k + 1 \\
x_{k+1,n+2} & \mapsto \xi \cdot w_k \\
y_{\ell,n+2} & \mapsto z_\ell \quad \text{for} \ 1 \leq \ell \leq N - k - 1.
\end{align*} \]

Using these inclusions we identify certain generators of \( H^G_k \) and \( H^G_{k+1} \) with their images in the bimodule \( H^G_{k,k+1} \), so that

\[ H^G_{k,k+1} \cong \mathbb{Q}[x_{1,n}, x_{2,n}, \ldots, x_{k,n}; \xi; y_{1,n+2}, y_{2,n+2}, \ldots, y_{N-k-1,n+2}] \, . \]  

(3.11)

We will refer to this set of generators as the canonical generators of the ring \( H^G_{k,k+1} \) in what follows.

The generators of \( H^G_k \) and \( H^G_{k+1} \) that are not mapped to canonical generators can be expressed in terms of canonical generators as follows:

\[ x_{\alpha,n+2} = x_{\alpha,n} + x_{\alpha-1,n} \cdot \xi \, , \quad y_{\alpha,n} = y_{\alpha,n+2} + y_{\alpha-1,n+2} \cdot \xi \, . \]  

(3.12)
Similarly, the canonical generators can be expressed using non-canonical generators:

\[
x_{\alpha,n} = \sum_{\ell=0}^{\alpha} (-1)^{\ell} x_{\alpha-\ell,n+2} \cdot \xi^{\ell}, \quad y_{\alpha,n+2} = \sum_{j=0}^{\alpha} (-1)^{j} y_{\alpha-j,n} \cdot \xi^{j}.
\]

(3.13)

In what follows we will often write equations as above, where we define \(x_{0,n} = 1, y_{0,n} = 1\) and

\[
x_{j,n} = 0 \quad \text{for } j \text{ not in the range } 0 \leq j \leq k,
\]

(3.14)

\[
y_{\ell,n} = 0 \quad \text{for } \ell \text{ not in the range } 0 \leq \ell \leq N - k.
\]

(3.15)

### 3.1.1 Inclusions of infinite Grassmannians

We introduce special elements of \(H^G_k\) obtained from the inclusions \(H^G_{GL(k)}(pt) = H^*(Gr(k, \infty)) \rightarrow H^G_k\) and \(H^G_{GL(N-k)}(pt) = H^*(Gr(N-k, \infty)) \rightarrow H^G_k\).

**Definition 3.1.** For \(\alpha, \beta \in \mathbb{Z},\) let \(X_{\alpha,n} = Y_{\beta,n} = 0\) for \(\alpha, \beta < 0\) and inductively define the elements \(X_{\alpha,n}, Y_{\beta,n} \in H^G_k\) by setting \(X_{0,n} = Y_{0,n} = 1,\) and

\[
X_{\alpha,n} := -\sum_{j=1}^{\alpha} y_{j,n} X_{\alpha-j,n}, \quad Y_{\beta,n} := -\sum_{j=1}^{\beta} x_{j,n} Y_{\beta-j,n}.
\]

(3.16)

For example,

\[
Y_{0,n} = 1, \quad Y_{1,n} = -x_{1,n}, \quad Y_{2,n} = x_{1,n}^2 - x_{2,n}, \quad Y_{3,n} = -x_{3,2} + 2x_{1,n}x_{2,n} - x_{1,n}^3.
\]

(3.17)

For a given choice of \(N,\) with \(n = 2k - N,\) we have by convention that \(x_{j,n} = 0\) for \(j > k.\) However, each element \(Y_{\alpha,n}\), for \(\alpha \in \mathbb{Z}_+,\) contains some power of \(x_{1,n}\) which for \(N > 0\) is never zero since there are no relations on the generators of \(H^G_k.\)

The collection of elements \(X_{j,n}\) and \(Y_{j,n}\) also satisfy the equations

\[
\left(1 + x_{1,n} + x_{2,n}^2 + \cdots + x_{k,n} t^k\right) \left(1 + Y_{1,n} + Y_{2,n} t^2 + \cdots + Y_{j,n} t^j + \cdots\right) = 1,
\]

\[
\left(1 + X_{1,n} + X_{2,n} t^2 + \cdots + X_{j,n} t^j + \cdots\right) \left(1 + y_{1,n} t + y_{2,n} t^2 + \cdots + y_{N-k,n} t^{N-k}\right) = 1.
\]

(3.18)

This fact follows immediately from the definition since the homogeneous terms in the above equations are given by:

\[
\sum_{j=0}^{\beta} x_{j,n} Y_{\beta-j,n} = \delta_{\beta,0}, \quad \sum_{j=0}^{\alpha} y_{j,n} X_{\alpha-j,n} = \delta_{\alpha,0}.
\]

(3.19)

The homogeneous terms of these expressions give the defining relations for the cohomology rings \(H^*(k, \infty)\) and \(H^*(N-k, \infty),\) respectively. The ring \(H^*(k, \infty)\) is given by

\[
H^*(k, \infty) = \mathbb{Q}[x_1, x_2, \cdots x_k; y_1, y_2, \cdots, y_j, \cdots]/I_{k,\infty}
\]

(3.20)

where \(I_{k,\infty}\) is the ideal generated by the homogeneous elements in

\[
\left(1 + x_1 t + x_2 t^2 + \cdots + x_k t^k\right) \left(1 + y_1 t + y_2 t^2 + \cdots + y_j t^j + \cdots\right) = 1.
\]

(3.21)
Hence, we have an inclusion of $H^*(Gr(k, \infty))$ into the equivariant cohomology ring $H^G_k$ given by
\[
H^*(Gr(k, \infty)) \longrightarrow H^G_k
\]
\[
x_j \mapsto x_{j,n} \quad (3.23)
\]
\[
y_{\ell} \mapsto Y_{\ell,n} \quad (3.24)
\]

Similarly, the cohomology ring $H^*(Gr(N - k, \infty))$, also generated by Chern classes, is
\[
H^*(k, \infty) = Q[x_1, x_2, \cdots x_j, \cdots; y_1, y_2, \cdots, y_{N-k}] / I_{N-k,\infty} \quad (3.25)
\]
where $I_{N-k,\infty}$ is the ideal generated by the homogeneous terms in
\[
(1 + x_1 t + x_2 t^2 + \cdots + x_j t^j + \cdots) (1 + y_1 t + y_2 t^2 + \cdots + y_{N-k} t^{N-k}) = 1. \quad (3.26)
\]
Thus, we have an inclusion
\[
H^*(Gr(N - k, \infty)) \longrightarrow H^G_k \quad (3.27)
\]
\[
x_j \mapsto X_{j,n} \quad (3.28)
\]
\[
y_{\ell} \mapsto y_{j,n} \quad (3.29)
\]

Just as we have identified the canonical generators of $H^G_k$ and $H^G_{k+1}$ with generators or sums of generators in $H^G_{k,k+1}$, we can do the same with the elements $X_{j,n}$ and $Y_{k,n}$. Here we collect some identities involving these elements in $H^G_{k,k+1}$. The reader is encouraged to convert these identities into a graphical form like the one used for the ordinary cohomology rings in [9].

**Proposition 3.2.** For all $\alpha \in \mathbb{Z}_+$, the equations
\[
X_{\alpha,n} = \sum_{\ell=0}^{\alpha} (-1)^{\ell} x_{\alpha-\ell,n+2} \cdot \xi^\ell \quad (3.30)
\]
\[
Y_{\alpha,n+2} = \sum_{\ell=0}^{\alpha} (-1)^{\ell} y_{\alpha-\ell,n} \cdot \xi^\ell \quad (3.31)
\]
\[
\xi^\alpha = (-1)^{\alpha} \sum_{j=0}^{\alpha} x_{\alpha-j,n} y_{j,n+2} \quad (3.32)
\]
\[
\xi^\alpha = (-1)^{\alpha} \sum_{j=0}^{\alpha} X_{\alpha-j,n} Y_{j,n+2} \quad (3.33)
\]
hold in $H^G_{k,k+1}$.

**Proof.** These relations follow from (3.12) and (3.13) together with the definitions of $X_{\alpha,n}$ and $Y_{\alpha,n}$. \qed
3.2 Iterated flag varieties

The $a$-step iterated flag variety $G_{k,k+1,...,k+a}$ consists of $a + 1$-tuples

$$\{(W_k, ..., W_{k+a}) | \dim_C W_{k+j} = (k+j), 0 \subset W_k \subset W_{k+1} \subset \cdots \subset W_{k+a} \subset W\},$$

where $0 \leq k \leq k+a \leq N$. Using that $GL(N)$ acts transitively on this variety, the cohomology ring of this variety is given by

$$H^G_{k,k+1,...,k+a} \cong \mathbb{Q}[x_{1,n}, ... , x_{k,n}; \xi_1, \xi_2, ..., \xi_a; y_{1,n+2a}, ..., y_{N-k-a,n+2a}].$$

(3.34)

However, this ring can also be equivalently described by

$$H^G_{k,k+1,...,k+a} \cong H^G_{k,k+1} \otimes H^G_{k+1,k+2} \otimes H^G_{k+2,k+3} \cdots \otimes H^G_{k+a-1,k+a} H^G_{k+a-1,k+a}.$$

(3.35)

For example, the cohomology ring $H^G_{k,k+1,k+2}$ can be written as

$$H^G_{k+1,k+2} \cong H^G_{k+1,k+1} \otimes H^G_{k+1,k+2},$$

(3.36)

but using the isomorphism $H^G_{k,k+1,k+2} \cong H^G_{k,k+1} \otimes H^G_{k+1,k+2}$ we will write

$$H^G_{k,k+1,k+2} \to H^G_{k,k+1} \otimes H^G_{k+1,k+2}.$$

(3.37)

Notice that in the tensor product anything with the second label equal to $n+2$ can be moved from one tensor factor to the other, so that $x_{j,n}x_{s,n+2} \otimes y_{\ell,n+4} = x_{j,n} \otimes x_{s,n+2}y_{\ell,n+4}$ and $y_{\ell,n+4} = x_{j,n} \otimes y_{s,n+2}+2y_{\ell,n+4}.$

This convention extends to other iterated flag varieties as well. The generators of the equivariant cohomology ring $H^G_{k,k+1,k}$ of the variety $G_{k,k+1,k}$ can be written as $x_{j,n} \otimes 1, 1 \otimes x_{j,n}, \xi \otimes 1, 1 \otimes \xi$, and $y_{s,n+2} \otimes 1 = 1 \otimes y_{s,n+2}$. However, the tensor product over the action of $H^G_{k+1}$ induces some relations on these generators. From (3.12) we have that

$$x_{\alpha,n} \otimes 1 + x_{\alpha-1,n} \cdot \xi \otimes 1 = x_{\alpha,n+2} \otimes 1 = 1 \otimes x_{\alpha,n+2} = 1 \otimes x_{\alpha,n} + 1 \otimes x_{\alpha-1,n} \cdot \xi.$$

(3.38)

Relations among other elements of $H^G_{k+1,k+1} \otimes H^G_{k+1,k}$ can also be derived using (3.13).

**Proposition 3.3.** In the ring $H^G_{k,k+1} \otimes H^G_{k+1,k}$ we have

$$\sum_{j=0}^{\alpha} (-1)^j x_{j,n} \otimes \xi^{\alpha-j} = \sum_{\ell=0}^{\alpha} (-1)^\ell \xi^{\alpha-\ell} \otimes x_{\ell,n}$$

(3.39)

for all $\alpha \in \mathbb{Z}_+$. Similarly, in $H^G_{k,k-1} \otimes H^G_{k-1,k}$ we have

$$\sum_{j=0}^{\alpha} (-1)^j y_{j,n} \otimes \xi^{\alpha-j} = \sum_{\ell=0}^{\alpha} (-1)^\ell \xi^{\alpha-\ell} \otimes y_{\ell,n}$$

(3.40)

for all $\alpha \in \mathbb{Z}_+$. 

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Proof. The first identity is proven as follows:

\[ \sum_{j=0}^{\alpha} (-1)^j x_{j,n} \otimes \xi^{\alpha-j} = \sum_{j=0}^{\alpha} \sum_{\ell=0}^{\alpha-j} (-1)^j x_{j,n} \otimes (-1)^{\alpha-j} Y_{\alpha-j-\ell,n+2} x_{\ell,n} \tag{3.32} \]

\[ = \sum_{j=0}^{\alpha} \sum_{\ell=0}^{\alpha-j} (-1)^j x_{j,n} Y_{\alpha-j-\ell,n+2} \otimes x_{\ell,n} \tag{3.41} \]

\[ = \sum_{\ell=0}^{\alpha} (-1)^{\alpha-\ell} \sum_{j=0}^{\alpha-j} x_{j,n} Y_{(\alpha-\ell)-j,n+2} \otimes x_{\ell,n}. \tag{3.42} \]

Using (3.32) once more completes the proof. The second identity is proven similarly.

**Proposition 3.4** (Dot slide). The equations

\[ \sum_{\ell=0}^{N-k} (-1)^{\ell} \xi^{k-\ell+1} \otimes x_{\ell,n} = \sum_{\ell=0}^{k} (-1)^{\ell} \xi^{k-\ell} \otimes x_{\ell,n} \cdot \xi \tag{3.44} \]

\[ \sum_{j=0}^{N-k} (-1)^{j} y_{j,n} \cdot \xi \otimes \xi^{k-j} = \sum_{j=0}^{N-k} (-1)^{j} y_{j,n} \otimes \xi^{k-j+1} \tag{3.45} \]

hold in the rings \( H^G_{k,k+1} \otimes H^G_{k+1,k} \) and \( H^G_{k,k-1} \otimes H^G_{k-1,k} \), respectively.

This Proposition says that multiplication by \( \xi \) on one of the tensor factors in the above sums is equivalent to multiplication by \( \xi \) on the other tensor factor. We will see in the next section that this proposition is used to show that a dot can be slid from one side of a cup to the other (Lemma 4.6).

**Proof.** We have

\[ \sum_{\ell=0}^{k} (-1)^{\ell} \xi^{k-\ell+1} \otimes x_{\ell,n} = \sum_{\ell=0}^{k} \sum_{j=0}^{k+1-\ell} (-1)^{k+1-\ell} \cdot x_{j,n} Y_{k+1-\ell-j,n+2} \otimes x_{\ell,n} \tag{3.46} \]

\[ = \sum_{j=0}^{k+1} (-1)^{k+1} x_{j,n} \otimes \sum_{\ell=0}^{k+1-j} Y_{(k+1-j)-\ell,n+2} x_{\ell,n} \tag{3.47} \]

\[ = \sum_{j=0}^{k+1} (-1)^{j} x_{j,n} \otimes \xi^{k+1-j+1} \tag{3.48} \]

were in the second equality we have used that \( x_{k+1,n} = 0 \) to increase the \( \ell \)-summation index and then switched the summation order. The first claim now follows from (3.33). The second claim is proven similarly.

**Corollary 3.5.** The assignments

\[ H^G_k \rightarrow \left( H^G_{k,k+1} \otimes H^G_{k+1,k} \right) \]

\[ 1 \rightarrow \sum_{j=0}^{k} (-1)^{j} x_{j,n} \otimes \xi^{k-j} = \sum_{\ell=0}^{k} (-1)^{\ell} \xi^{k-\ell} \otimes x_{\ell,n} \]
and

\[ H^G_k \rightarrow \left( H^G_{k,k-1} \otimes H^G_{k-1,k} \right) \]
\[ 1 \mapsto \sum_{j=0}^{N-k} (-1)^j y_{j,n} \otimes \xi^{N-k-j} = \sum_{\ell=0}^{N-k} (-1)^\ell \xi^{N-k-\ell} \otimes y_{\ell,n} \]

define morphisms of graded bimodules of degree 2 and 2, respectively.

**Proof.** We prove the first claim, the second is left to the reader. It suffices to check that the left action of \( H^G_k \) on the image of 1 in \( H^G_k \) in \( \left( H^G_{k,k+1} \otimes H^G_{k+1,k} \right) \) is equal to right action of \( H^G_k \). The Corollary follows since \( x_{j,n} \) can be moved across tensor products by \( (3.12) \) and \( (3.13) \) at the cost of introducing powers of \( \xi \) on one of the tensor factors. By Proposition 3.4, factors of \( \xi \) can be slid across tensor factors in the above sums.

\( \Box \)

### 3.3 Defining the 2-category \( \text{EqFlag}^*_N \)

Recall the additive 2-category \( \text{Bim} \) whose objects are graded rings, morphisms are graded bimodules, and the 2-morphisms are degree-preserving bimodule maps. Idempotent bimodule homomorphisms split in \( \text{Bim} \). Denote by \( \text{Bim}^\ast \) the graded additive 2-category with the same objects and 1-morphisms as \( \text{Bim} \), and whose 2-morphisms consist of all bimodule maps, each of which is the sum of its homogeneous components.

Fix a positive integer \( N \) and let \( n = 2k - N \) for \( 0 \leq k \leq N \).

**Definition 3.6.** The 2-category \( \text{EqFlag}^*_N \) is the idempotent completion inside of \( \text{Bim} \) (see [9, Section 6.1.5]) of the 2-category consisting of

- objects: the graded equivariant cohomology rings \( H^G_k \) for each \( 0 \leq k \leq N \),
- morphisms: generated by the graded \( (H^G_k,H^G_k) \)-bimodule \( H^G_k \), the graded \( (H^G_k,H^G_{k+1}) \)-bimodule \( H^G_{k,k+1} \) and the graded \( (H^G_{k+1},H^G_k) \)-bimodule \( H^G_{k+1,k} \), together with their shifts \( H^G_k \{ s \} \), \( H^G_{k,k+1} \{ s \} \), and \( H^G_{k+1,k} \{ s \} \) for \( s \in \mathbb{Z} \). The bimodules \( H^G_k = H^G_k \{ 0 \} \) are the identity 1-morphisms. Thus, a generic morphism from \( H^G_{k_1} \) to \( H^G_{k_m} \) is a direct sum of graded \( (H^G_{k_1},H^G_{k_m}) \)-bimodules of the form

\[ H^G_{k_1,k_2} \otimes H^G_{k_2,k_3} \otimes H^G_{k_3,k_4} \cdots \otimes H^G_{k_{m-1},k_m} \{ s \} \]  \hspace{1cm} (3.49)

where \( k_{i+1} = k_i \pm 1 \) for \( 1 < i < m \).
- 2-morphisms: degree-preserving bimodule maps

There is a 2-subcategory \( \text{EqFlag}^*_N \) of \( \text{Bim}^\ast \) with the same objects and morphisms as \( \text{EqFlag}^*_N \), and with 2-morphisms

\[ \text{EqFlag}^*_N(x,y) := \bigoplus_{s \in \mathbb{Z}} \text{EqFlag}_N(x\{ s \},y). \]  \hspace{1cm} (3.50)

Given two \( (H^G_{k_1},H^G_{k_m}) \)-bimodules \( M_1 \) and \( M_2 \) of the form \( (3.49) \), there are canonical inclusions of \( H^G_{k_1} \) into the first factor of the tensor product and canonical inclusion of \( H^G_{k_m} \) into the last factor. Let \( n_1 = 2k_1 - N \) and \( n_m = 2k_m - N \). Under these inclusions, the left action of elements \( x_{j,n_1}, y_{\ell,n_1} \in H^G_{k_1} \) can be written as

\[ x_{j,n_1} \otimes 1 \otimes \cdots \otimes 1, \quad y_{\ell,n_1} \otimes 1 \otimes \cdots \otimes 1 \]
in either bimodule $M_1$ or $M_2$. Likewise, the right action of $x_{j,n_m}, y_{\ell,n_m} \in H_{k_m}^G$ can be written as
\[
1 \otimes \cdots \otimes 1 \otimes x_{j,n_m}, \quad 1 \otimes \cdots \otimes 1 \otimes y_{\ell,n_m},
\]
in $M_1$ or $M_2$. As in [9], to define an $(H_{k_1}^G, H_{k_m}^G)$-bimodule homomorphism from $M_1$ to $M_2$ we specify the map on elements of the form $\xi^{a_1} \otimes \xi^{a_2} \otimes \cdots \otimes \xi^{a_m}$ with it understood that preservation of left action of $H_{k_1}$, and the right action of $H_{k_m}$, require that elements
\[
x_{j,n_1} \otimes 1 \otimes \cdots \otimes 1, \quad y_{\ell,n_1} \otimes 1 \otimes \cdots \otimes 1, \quad 1 \otimes \cdots \otimes 1 \otimes x_{j,n_m}, \quad 1 \otimes \cdots \otimes 1 \otimes y_{\ell,n_m},
\]
of $M_1$ are mapped to the corresponding elements of $M_2$.

## 4 Equivariant representations of $\mathcal{U}$

### 4.1 Defining the 2-functor $\Gamma_N^G$

On objects the 2-functor $\Gamma_N^G : \mathcal{U}^* \to \mathbf{EqFlag}_N^*$ sends $n$ to the ring $H_n^G$ whenever $n$ and $k$ are compatible:
\[
\Gamma_N^G : \mathcal{U}^* \to \mathbf{EqFlag}_N^* \quad \begin{cases} H_k^G & \text{with } n = 2k - N \text{ and } 0 \leq k \leq N \\ 0 & \text{otherwise.} \end{cases} \quad (4.1)
\]

Morphisms of $\mathcal{U}^*$ get mapped by $\Gamma_N^G$ to graded bimodules:
\[
\Gamma_N^G : \mathcal{U}^* \to \mathbf{EqFlag}_N^* \quad \begin{aligned}
1_n\{s\} & \mapsto \begin{cases} H_k^G\{s\} & \text{with } n = 2k - N \text{ and } 0 \leq k \leq N \\ 0 & \text{otherwise.} \end{cases} \quad (4.2) \\
\mathcal{E}1_n\{s\} & \mapsto \begin{cases} H_{k,k+1}^G\{s+1-N+k\} & \text{with } n = 2k - N \text{ and } 0 \leq k < N \\ 0 & \text{otherwise.} \end{cases} \quad (4.3) \\
\mathcal{F}1_n\{s\} & \mapsto \begin{cases} H_{k,k+1}^G\{s+1-k\} & \text{with } n = 2k - N \text{ and } 0 \leq k < N \\ 0 & \text{otherwise.} \end{cases} \quad (4.4)
\end{aligned}
\]

Here $H_{k,k+1}^G\{s+1-k\}$ is the bimodule $H_{k,k+1}^G$ with the grading shifted by $s + 1 - k$ so that
\[
(H_{k,k+1}^G\{s+1-k\})_j = (H_{k,k+1}^G)_{j+s+1-k}.
\]

More generally, we have
\[
\mathcal{E}^\alpha 1_n\{s\} \mapsto H_{k,k+1,k+2,\ldots,k+(\alpha-1),k+\alpha}^G\{s+\alpha(-N+k)+\alpha\} \\
\mathcal{F}^\beta 1_n\{s\} \mapsto H_{k,k-1,k-2,\ldots,k-(\beta-1),k-\beta}^G\{s-\beta k+2-\beta\}
\]
so that
\[
\mathcal{E}^{a_1} \mathcal{F}^{\beta_1} \mathcal{E}^{a_2} \ldots \mathcal{F}^{\beta_{k-1}} \mathcal{E}^{a_k} \mathcal{F}^{\beta_k} 1_n\{s\} \cong \mathcal{E}^{a_1} 1_{n-\sum(\beta_j-\alpha_j)} \circ \ldots \circ \mathcal{E}^{a_k} 1_{n-2\beta_k} \circ \mathcal{F}^{\beta_k} 1_n\{s\} \quad (4.5)
\]
is mapped to the graded bimodule
\[
H_{k,k-1,\ldots,k-\beta_k,\ldots,k-\beta_{k+1}+\ldots,k-\beta_{k+\alpha_k},k-\beta_k+\alpha_k+1,\ldots,k-\sum_j(\beta_j+\alpha_j)}^G
\]

\[
\begin{align*}
\mathcal{E}^{a_1} \mathcal{F}^{\beta_1} \mathcal{E}^{a_2} \ldots \mathcal{F}^{\beta_{k-1}} \mathcal{E}^{a_k} \mathcal{F}^{\beta_k} 1_n\{s\} 
\end{align*}
\]
with grading shift \([s + s']\), where \(s'\) is the sum of the grading shifts for each terms of the composition in \((3.5)\). Formal direct sums of morphisms of the above form are mapped to
direct sums of the corresponding bimodules.

It follows from \((3.8)\) that \(\Gamma^G_N\) preserves composites up to isomorphism. Hence, the 2-functor \(\Gamma^G_N\) is a weak 2-functor or bifunctor. In what follows we will often simplify our notation and write \(\Gamma^G\) instead of \(\Gamma^G_N\). We now proceed to define \(\Gamma^G\) on 2-morphisms.

### 4.1.1 Biadjointness

**Definition 4.1.** The 2-morphisms generating biadjointness in \(\mathcal{U}^*\) are mapped by \(\Gamma^G\) to the following bimodule maps.

\[
\Gamma^G \left( \begin{array}{c} \varepsilon \\ n \end{array} \right) : \begin{array}{c} H_k^G \\ \{1 - N\} \end{array} \rightarrow \begin{array}{c} \left( H_k^{G+1} \otimes H_{k+1}^G \right) \{1 - N\} \\ \{1 - N\} \end{array}
\]

\[
\Gamma^G \left( \begin{array}{c} \varepsilon \\ n \end{array} \right) : \begin{array}{c} H_k^G \\ \{1 - N\} \end{array} \rightarrow \begin{array}{c} \left( H_k^{G+1} \otimes H_{k+1}^G \right) \{1 - N\} \\ \{1 - N\} \end{array}
\]

\[
\Gamma^G \left( \begin{array}{c} \varepsilon \\ n \end{array} \right) : \begin{array}{c} H_k^G \\ \{1 - N\} \end{array} \rightarrow \begin{array}{c} \left( H_k^{G+1} \otimes H_{k+1}^G \right) \{1 - N\} \\ \{1 - N\} \end{array}
\]

Corollary 3.5 shows that the caps above are well-defined bimodule maps. It is clear that the caps are bimodule maps since the image of \(\xi_1^{m_1} \otimes \xi_2^{m_2}\) only depends on the sum \(m_1 + m_2\).

These definitions preserve the degree of the 2-morphisms of \(\mathcal{U}^*\) defined in Section 2.1. In \((1.0)\) the element 1 is in degree zero and is mapped to a sum of elements in degree 2\(k\) that have been shifted by \(\{1 - N\}\) for a total degree \(2k + 1 - N = 1 + n\). The degree in \((4.7)\) is \(2(N - k) + (1 - N) = 1 - n\). Similarly, in \((4.8)\) a degree 2\((m_1 + m_2)\) element shifted by \(1 - N\) is mapped to a degree 2\((m_1 + m_2 + k - N - 1)\) element, so the bimodule map has total degree \(1 + n\). One can easily check that the bimodule map defined in \((4.9)\) is of degree \(1 - n\).

Notice the similarity between the definitions \((4.6)\)–\((4.9)\) defining the 2-functor \(\Gamma^G_N\) and the corresponding definitions for the 2-functor \(\Gamma_N\) defined in [9]. In fact, in \(\text{Flag}_N\) the definitions agree. That is, in the presence of the extra relations imposed on the ordinary cohomology rings, the above assignments for \(\Gamma^G_N\) are equal to the assignments made by \(\Gamma_N\).

### 4.1.2 NilHecke generators

We show that the nilHecke algebra \(\mathcal{N}H_a\) acts on \(\text{End}(H^G_{k+1,k+1},\ldots,k+a)\) with \(\chi_i\) acting by multiplication by \(\xi_i\) and \(u_i\) acting by the divided differences \(\partial_i\) operator on the variables \(\xi_i\).
Definition 4.2. The 2-morphisms $z_n$ and $\hat{z}_n$ in $U^*$ are mapped by $\Gamma_N^G$ to the graded bimodule maps:

\[
\Gamma^G_{n+2} \begin{array}{c} \scriptstyle n+2 \vline \scriptstyle n \end{array} : \begin{cases} H_{k,k+1}^G\{1 - N + k\} \rightarrow H_{k,k+1}^G\{1 - N + k\} \\ \xi^m \mapsto \xi^{m+1} \end{cases} \tag{4.10}
\]

\[
\Gamma^G_{n} \begin{array}{c} \scriptstyle n \vline \scriptstyle n+2 \end{array} : \begin{cases} H_{k+1,k}^G\{1 - k\} \rightarrow H_{k+1,k}^G\{1 - k\} \\ \xi^m \mapsto \xi^{m+1} \end{cases} \tag{4.11}
\]

Note that these assignment are degree preserving since these bimodule maps are degree 2.

The nilCoxeter generator $U_n$ is mapped to the bimodule map which acts as the divided difference operator in the variables $\xi_j$ for $1 \leq j \leq a$.

Definition 4.3. Define bimodule maps by

\[
\Gamma^G_{n} \begin{array}{c} \scriptstyle n \vline \scriptstyle n+4 \end{array} : \begin{cases} H_{k,k+1,k+2}^G\{1 - N\} \rightarrow H_{k,k+1,k+2}^G\{1 - N\} \\ \xi_1^{m_1} \otimes \xi_2^{m_2} \mapsto \sum_{j=0}^{m_1-1} \xi_1^{m_1+m_2-1-j} \otimes \xi_2^j - \sum_{j=0}^{m_2-1} \xi_1^{m_1} \otimes \xi_2^{m_2-1-j} \otimes \xi_2^j \end{cases} \tag{4.12}
\]

The value of these maps on any other generator is determined from the rules above together with the requirement that the maps preserve the actions of $H_k^G$ and $H_{k+2}^G$. The maps $\Gamma^G(U_n)$ and $\Gamma^G(\hat{U}_n)$, as defined above, have degree $-2$ so that $\Gamma^G$ preserves the degree of $U_n$ and $\hat{U}_n$.

Remark 4.4. $\Gamma^G(U_n)(\xi_1^{m_1} \otimes \xi_2^{m_2})$ is zero when $m_1 = m_2$. This is clear from the definition. When the number of dots the upward oriented lines is equal the two sums above cancel.

4.2 Checking the relations of $\hat{U}$

In this section we show that the relations of Section 2.1 for $U^*$ are satisfied in $\text{EqFlag}_N^\ast$, thus establishing that $\Gamma_N^G$ is a 2-functor. The proof is analogous to the proof given in [9] that $\Gamma_N : U^* \rightarrow \text{Flag}_N^\ast$ is a 2-functor. Here we must be careful that we never use the additional relations from the ordinary cohomology rings. From the definitions in the previous section it is clear that $\Gamma_N^G$ preserves the degree associated to generators. For this reason, we often simplify our notation in this section by omitting the grading shifts when no confusion is likely to arise.

Lemma 4.5 (Biadjointness). In $\text{EqFlag}_N^\ast$ the following identities are satisfied

\[
\Gamma^G_{n+2} \begin{array}{c} \scriptstyle n+2 \vline \scriptstyle n \end{array} = \Gamma^G_{n+2} \begin{array}{c} \scriptstyle n+2 \vline \scriptstyle n \end{array} \quad \Gamma^G_{n} \begin{array}{c} \scriptstyle n+2 \vline \scriptstyle n \end{array} = \Gamma^G_{n} \begin{array}{c} \scriptstyle n+2 \vline \scriptstyle n \end{array} \quad \Gamma^G_{n-2} \begin{array}{c} \scriptstyle n \vline \scriptstyle n \end{array} = \Gamma^G_{n-2} \begin{array}{c} \scriptstyle n \vline \scriptstyle n \end{array}
\]
Checking the relations of $\dot{U}$

\[
\Gamma^G \left( \begin{array}{c}
\uparrow \\
n+2 \\
n
\end{array} \right) = \Gamma^G \left( \begin{array}{c}
\uparrow \\
n+2 \\
n
\end{array} \right) \quad \Gamma^G \left( \begin{array}{c}
\uparrow \\
n-2 \\
n
\end{array} \right) = \Gamma^G \left( \begin{array}{c}
\uparrow \\
n-2 \\
n
\end{array} \right)
\]

for all $n \in \mathbb{Z}$.

**Proof.** Consider the bimodule map

\[
\Gamma^G \left( \begin{array}{c}
\uparrow \\
n+2 \\
n
\end{array} \right) : \xi^\alpha \to \sum_{j=0}^{k} (-1)^j (-1)^{\alpha + k - j - k} Y_{\alpha - j, n+2} \cdot x_{j, n} \quad (4.13)
\]

After simplifying, the image of $\xi^\alpha$ is

\[
\sum_{j=0}^{k} (-1)^\alpha Y_{\alpha - j, n+2} \cdot x_{j, n} = \xi^\alpha \quad (4.14)
\]

which, depending on whether $\alpha \leq k$ or $\alpha > k$, uses that $Y_{\ell, n+2} = 0$ for $\ell < 0$, or $x_{\ell, n} = 0$ for $\ell > k$. Since $\xi^\alpha$ is fixed by $\Gamma^G \left( \begin{array}{c}
\uparrow \\
n+2 \\
n
\end{array} \right)$ we have established the first identity. The others are proven similarly using (3.19). \qed

**Lemma 4.6** (Duality for $z_n$). The equations

\[
\Gamma^G \left( \begin{array}{c}
\uparrow \\
n+4 \\
n
\end{array} \right) = \Gamma^G \left( \begin{array}{c}
\uparrow \\
n+4 \\
n
\end{array} \right) \quad \Gamma^G \left( \begin{array}{c}
\uparrow \\
n+4 \\
n
\end{array} \right) = \Gamma^G \left( \begin{array}{c}
\uparrow \\
n+4 \\
n
\end{array} \right) \quad (4.15)
\]

of bimodule maps hold in $\text{EqFlag}^*_N$ for all $n \in \mathbb{Z}$.

**Proof.** Since we have already established that $\Gamma^G$ preserves the biadjoint structure of $U^*$ in Lemma 4.3 the above (4.15) is equivalent to proving

\[
\Gamma^G \left( \begin{array}{c}
\uparrow \\
n \\
n+4 \\
n
\end{array} \right) = \Gamma^G \left( \begin{array}{c}
\uparrow \\
n \\
n+4 \\
n
\end{array} \right) \quad \Gamma^G \left( \begin{array}{c}
\uparrow \\
n \\
n+4 \\
n
\end{array} \right) = \Gamma^G \left( \begin{array}{c}
\uparrow \\
n \\
n+4 \\
n
\end{array} \right)
\]

for all $n \in \mathbb{Z}$. Up to a grading shift, this is precisely the content of Proposition 3.4. \qed

**Lemma 4.7** (Duality for $U_n$). The equation

\[
\Gamma^G \left( \begin{array}{c}
\uparrow \\
n+4 \\
n
\end{array} \right) = \Gamma^G \left( \begin{array}{c}
\uparrow \\
n+4 \\
n
\end{array} \right) \quad \Gamma^G \left( \begin{array}{c}
\uparrow \\
n+4 \\
n
\end{array} \right) = \Gamma^G \left( \begin{array}{c}
\uparrow \\
n+4 \\
n
\end{array} \right) \quad (4.16)
\]

holds in $\text{EqFlag}^*_N$ for all $n \in \mathbb{Z}$. 18
Proof. It suffices to consider the element $1 \otimes \xi^\alpha$. We have

$$\Gamma^G \left( \begin{array}{c} n \\ n+4 \end{array} \right) (1 \otimes \xi^\alpha) = \sum_{j=0}^{k} (-1)^{\alpha-1} x_{j,n} \otimes Y_{\alpha-1-j,n+4}. \quad (4.17)$$

We can write this as a sum from $j = 0$ to $j = \alpha - 1$ using that $x_{j,n} = 0$ for $j > k$ and $Y_{\ell,n+4} = 0$ for $\ell < 0$. Now using (3.31) we have

$$= \sum_{j=0}^{\alpha-1} \sum_{p=0}^{\alpha-1-p} (-1)^{\alpha-1-p} x_{j,n} \otimes Y_{\alpha-1-j-p,n+4} \xi^p \quad (4.18)$$

This is the same as $\Gamma^G(\hat{U}_n)$ defined in Definition 4.3. The other identity is proved similarly. 

For the remaining identities it is helpful to compute

$$\Gamma^G \left( \begin{array}{c} n \\ n-1+\alpha \end{array} \right) : 1 \rightarrow (-1)^{\alpha} \sum_{\ell=0}^{N-k} y_{\ell,n} Y_{\alpha-\ell,n}, \quad \Gamma^G \left( \begin{array}{c} n \\ -n-1+\alpha \end{array} \right) : 1 \rightarrow (-1)^{\alpha} \sum_{\ell=0}^{k} x_{\ell,n} X_{\alpha-\ell,n}$$

which follow immediately from Definition 4.1.

**Lemma 4.8** (Positive degree of closed bubbles). For all $m \geq 0$ we have

$$\Gamma^G \left( \begin{array}{c} n \\ m \end{array} \right) = 0 \text{ if } m < n - 1, \quad \Gamma^G \left( \begin{array}{c} n \\ m \end{array} \right) = 0 \text{ if } m < -n - 1,$$

for all $n \in \mathbb{Z}$.

**Proof.** This is clear from the definitions above and the positive degree of $x_{j,n}$, $y_{j,n}$, $X_{j,n}$, and $Y_{j,n}$. 

**Lemma 4.9** (Reduction to bubbles). The equations

$$\Gamma^G \left( \begin{array}{c} n \\ -n \end{array} \right) = \Gamma^G \left( \begin{array}{c} -n \\ -n+\ell \end{array} \right) \quad \Gamma^G \left( \begin{array}{c} n \\ n \end{array} \right) = \Gamma^G \left( \begin{array}{c} n \\ j \end{array} \right) \quad \Gamma^G \left( \begin{array}{c} n \\ n+4 \end{array} \right)$$

of bimodule maps hold in $\text{EqFlag}_N^*$ for all $n \in \mathbb{Z}$. 

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Proof. Consider the first equality on $1 \in H_{k,k+1}^G$. We have

$$\Gamma^G \begin{pmatrix} n \end{pmatrix}: 1 \rightarrow - \sum_{\ell=0}^{N-k} \sum_{j=0}^{N-k-\ell-1} (-1)^{-n-j} \xi_j^{\ell,1_n} Y_{-n-\ell,j,n}. \quad (4.20)$$

But the $Y_{-n-\ell,j,n}$ are only nonzero when $j \leq -n - \ell = N - 2k - \ell$. For $k > 0$ this implies $-n - \ell \leq N - k - \ell - 1$. After changing the $j$-summation to reflect this fact, so that $0 \leq j \leq -n - \ell$, the above is equal to the image of $1 \in H_{k,k+1}^G$ under the bimodule map on the right hand side of the first equality above. The second equality is proven similarly. \qed

Lemma 4.10 (NilHecke action). The equations

$$\Gamma^G \begin{pmatrix} n \end{pmatrix} = 0, \quad \Gamma^G \begin{pmatrix} n \end{pmatrix} = \Gamma^G \begin{pmatrix} n \end{pmatrix},$$

$$\Gamma^G \begin{pmatrix} n \end{pmatrix} = \Gamma^G \begin{pmatrix} n \end{pmatrix} - \Gamma^G \begin{pmatrix} n \end{pmatrix} = \Gamma^G \begin{pmatrix} n \end{pmatrix} - \Gamma^G \begin{pmatrix} n \end{pmatrix}$$

hold in EqFlag$_N$ for all $n \in \mathbb{Z}$.

Proof. The proof is identical to the proof of the same relation for the 2-functor $\Gamma$ in [9]. That proof never made use of the relations imposed on the canonical generators for the ordinary cohomology rings of the iterated flag varieties. \qed

Lemma 4.11 (Identity decomposition). The equations

$$\Gamma^G \begin{pmatrix} n \end{pmatrix} = \Gamma^G \begin{pmatrix} n \end{pmatrix} + \Gamma^G \begin{pmatrix} n \end{pmatrix} \quad \Gamma^G \begin{pmatrix} n \end{pmatrix} = \Gamma^G \begin{pmatrix} n \end{pmatrix} + \Gamma^G \begin{pmatrix} n \end{pmatrix}$$

hold for all $n \in \mathbb{Z}$. 
Proof. Using Lemma 4.5 proving the first identity is equivalent to proving
\[
\Gamma\left(\begin{array}{c}
\alpha \\
n - 2
\end{array}\right) = \Gamma\left(\begin{array}{c}
\alpha \\
n - 2
\end{array}\right) + \Gamma\left(\sum_{\ell=0}^{n-1} \sum_{j=0}^{\ell} \sum_{p=0}^{n-\ell} \xi^{s,n-\ell-j} x_{s,n-2}\right)
\]

We compute the above maps on the elements $1 \otimes \xi^\alpha$ since these elements together with relations of Proposition 3.2 and the bimodule property determine the image on all other elements.

We begin by computing the image of the element $1 \otimes \xi^\alpha$ under the maps on the right hand side of (4.21).

\[
\Gamma\left(\begin{array}{c}
\alpha \\
n - 2
\end{array}\right) (1 \otimes \xi^\alpha) = \sum_{s=0}^{k-1} \sum_{j=0}^{\alpha-1} \sum_{p=0}^{k-s-1} (-1)^{a+n-p-j} \xi^{p} X_{n+\alpha-s-p-j+1,n} \otimes \xi^{\alpha} x_{s,n-2}
\]

Removing terms that are zero and shifting the second term by letting $s' = s - 1$ we have

\[
(-1)^{a+n} \left( \sum_{s=0}^{k-1} \sum_{j=0}^{\alpha-1} \sum_{p=0}^{k-s-1} (-1)^{a+n-p-j} \xi^{p} X_{n+\alpha-s-p-j+1,n} \otimes \xi^{\alpha} x_{s,n-2} \right)
\]

\[
- \sum_{s'=0}^{k-1} \sum_{j=0}^{\alpha-1} \sum_{p=0}^{k-s'-2} (-1)^{a+n-p-j} \xi^{p} X_{n+\alpha-s'-p-j+1,n} \otimes \xi^{\alpha} x_{s',n-2}
\]

Now the only terms that do not cancel are the $p = k - s - 1$ term of the first factor and the $j = \alpha$ term of the second factor:

\[
(-1)^{a+k-N} \sum_{s=0}^{k-1} \sum_{j=0}^{\alpha-1} (-1)^{-j+1} \xi^{k-s-1} X_{\alpha+k-N-j,n} \otimes \xi^{\alpha} x_{s,n-2}
\]

\[
- \sum_{s'=0}^{k-2} \sum_{p=0}^{k-s'-2} (-1)^{n-p} \xi^{p} X_{n-s'-p-1,n} \otimes \xi^{\alpha} x_{s',n-2}.
\]

In the first term note that $\alpha - 1 \geq \alpha - (N - k)$ since $N - k \geq 1$, otherwise $k = N$. So we change the upper limit of the $j$ summand to $\alpha - (N - k)$ and apply (3.30). In the second summand we note that we must have $n - s' - p - 1 \geq 0$ and that $k - s - 2 \geq n - s - 1$ if $N - k \geq 1$. Hence, the $s'$ summation goes only as high as $n - 1$, and the $p$ summation only as high as $n - 1 - s$ yielding

\[
(-1)^{a+k-N+1} \sum_{s=0}^{k-1} (-1)^{s} \xi^{k-s-1} \otimes X_{\alpha+k-N,n-2} x_{s,n-2} - \sum_{s'=0}^{n-1} \sum_{p=0}^{n-1-s'} (-1)^{n-p} \xi^{p} X_{n-s-1,n} \otimes \xi^{\alpha} x_{s,n-2}.
\]
Now we compute the other maps involved in (4.21). The second map on the right hand side is
\[
\Gamma \left( \sum_{\ell=0}^{n-1} \sum_{j=0}^{\ell} \sum_{s=0}^{\ell-j} (-1)^j \xi^{n-1-\ell} x_{j-s,n} \otimes \xi^{\ell-j+\alpha} \right) = \sum_{\ell=0}^{n-1} \sum_{j=0}^{\ell} \sum_{s=0}^{\ell-j} (-1)^j \xi^{n-1-\ell} x_{j-s,n} \otimes \xi^{\ell-j+\alpha}. 
\]
Notice that \( j \leq k \) since \( j \leq n-1 = 2k - N - 1 = k + (k - N - 1) \) and \( k - N - 1 < 0 \). Thus, we remove the \( \min(j,k) \) in the summation and use (3.13) to change the non-canonical generator \( x_{s,n} \) into canonical generators,
\[
\sum_{\ell=0}^{n-1} \sum_{j=0}^{\ell} \sum_{s=0}^{\ell-j} (-1)^j \xi^{n-1-\ell} x_{j-s,n} \otimes \xi^{\ell-j+\alpha} = \sum_{\ell=0}^{n-1} \sum_{j=0}^{\ell} \sum_{s=0}^{\ell-j} (-1)^j \xi^{n-1-\ell} x_{j-s,n} \otimes \xi^{\ell-j+\alpha+1} x_{s-1,n-2}.
\]
Now shift the indices of the second term by letting \( s' = s - 1 \) and \( j' = j - 1 \) so that the two terms cancel leaving only the \( j = \ell \) term of the first term,
\[
\sum_{\ell=0}^{n-1} \sum_{s=0}^{\ell} (-1)^\ell \xi^{n-1-\ell} x_{\ell-s,n} \otimes \xi^{\ell} x_{s,n-2}.
\]
If we let \( p = n - 1 - \ell \) and \( s = s' \) this term cancels with the second term in (4.23). One can check that minus the first term in (4.23) is equal to the image of \( 1 \otimes \xi^\alpha \) under the left hand side of (4.21).

**Theorem 4.12.** The assignments given in subsection 4.1 define a graded additive 2-functor \( \Gamma^G_N : \mathcal{U}^* \to \text{EqFlag}^N \). By restricting to degree preserving 2-morphisms we also get an additive 2-functor \( \Gamma^G_N : \mathcal{U} \to \text{EqFlag}^N \).

**Proof.** We have already seen that \( \Gamma^G_N \) preserves composites of 1-morphisms up to isomorphism. The lemmas above show that \( \Gamma^G_N \) preserve the defining relations of the 2-morphisms in \( \mathcal{U}^* \). Therefore, since degrees and direct sums are also preserved, \( \Gamma^G_N \) is a graded additive 2-functor. It is clear that restricting to the degree preserving maps gives the restricted additive 2-functor \( \Gamma^G_N : \mathcal{U} \to \text{EqFlag}^N \). \qed

**4.3 Categorification of \( V_N \)**

**Theorem 4.13.** The representation \( \Gamma^G_N : \mathcal{U} \to \text{EqFlag}^N \) yields a representation \( \hat{\Gamma}^G_N : \hat{\mathcal{U}} \to \text{EqFlag}^N \). This representation categorifies the irreducible \((N+1)\)-dimensional representation \( V_N \) of \( \hat{\mathcal{U}} \).

**Proof.** Idempotent 2-morphisms split in \( \text{EqFlag}^N \) so, by the universal property of the Karoubi envelope, we have
\[
\begin{array}{ccc}
\mathcal{U} & \to & \hat{\mathcal{U}} \\
\Gamma^G_N & \downarrow & \hat{\Gamma}^G_N \\
\text{EqFlag}^N & \downarrow & 
\end{array}
\]
showing that \( \Gamma^G_N \) induces a representation \( \hat{\Gamma}^G_N \) of \( \hat{\mathcal{U}} \).
The rings $H_k^G$ forming the objects of $\text{EqFlag}_N^*$ are (positively, even) graded local rings with degree zero part isomorphic to $\mathbb{Q}$. Thus, every graded finitely-generated projective module is free, and $H_k^G$ has (up to isomorphism and grading shift) a unique graded indecomposable projective module. Let $H_k^G\text{-pmod}$ denote the category of finitely generated graded projective $H_k^G$-modules. The split Grothendieck group of the category $\bigoplus_{j=0}^{N} H_j^G\text{-pmod}$ is then a free $\mathbb{Z}[q, q^{-1}]$-module of rank $N + 1$, freely generated by the indecomposable projective modules, where $q^i$ acts by shifting the grading degree by $i$. Thus, we have

$$K_0\left(\bigoplus_{k=0}^{N} H_k\text{-pmod}\right) \cong \mathcal{A}(V_N), \quad K_0\left(\bigoplus_{k=0}^{N} H_k\text{-pmod}\right) \otimes_{\mathbb{Z}[q, q^{-1}]} \mathbb{Q}(q) \cong V_N,$$  

(4.25)

as $\mathbb{Z}[q, q^{-1}]$-modules, respectively $\mathbb{Q}(q)$-modules, where $\mathcal{A}(V_N)$ is a representation of $\mathcal{A}\hat{U}$, an integral form of the representation $V_N$ of $\hat{U}$.

The bimodules $\Gamma_N^G(1_n)$, $\Gamma_N^G(E1_n)$ and $\Gamma_N^G(F1_n)$ induce, by tensor product, functors on the graded module categories. More precisely, consider the restriction functors

$$\text{Res}_{k,k+1}^k : H_{k,k+1}^G\text{-pmod} \to H_k^G\text{-pmod}$$

$$\text{Res}_{k,k+1}^k : H_{k,k+1}^G\text{-pmod} \to H_{k+1}^G\text{-pmod}$$

given by the inclusions $H_k^G \to H_{k,k+1}^G$ and $H_k^G \to H_{k,k+1}^G$. For $0 \leq k \leq N$ and $n = 2k - N$ define functors

$$1_n := H_k^G \otimes_{H_k^G} H_k^G : H_k\text{-pmod} \to H_k^G\text{-pmod}$$

$$E1_n := \text{Res}_{k+1,k}^{k+1} H_{k+1,k}^G \otimes_{H_k^G} \{1 - N + k\} : H_k^G\text{-pmod} \to H_{k+1}^G\text{-pmod}$$

$$F1_{n+2} := \text{Res}_{k}^{k+1} H_{k,k+1}^G \otimes_{H_k^G} \{-k\} : H_{k+1}^G\text{-pmod} \to H_k^G\text{-pmod}.$$  

(4.26)

(4.27)

(4.28)

These functors have both left and right adjoints and commute with the shift functor, so they will induce $\mathbb{Z}[q, q^{-1}]$-module maps on Grothendieck groups. Furthermore, the 2-functor $\hat{\Gamma}_N^G$ must preserve the relations of $\hat{U}$, so by Theorem 2.2 these functors satisfy relations lifting those of $\hat{U}$.

References

[1] A. A. BEILINSON, G. LUSZTIG and R. MACPHERSON: A geometric setting for the quantum deformation of $GL_n$. Duke Math. J. 61, No. 2 (1990) 655–677, MR 1074310.

[2] F. BORCEUX: Handbook of categorical algebra. 1, volume 50 of Encyclopedia of Mathematics and its Applications. Cambridge University Press, Cambridge, 1994. MR 1291599.

[3] A. BOREL: Sur la cohomologie des espaces fibrés principaux et des espaces homogènes de groupes de Lie compacts. Ann. of Math. (2) 57 (1953) 115–207, MR 0051508.

[4] J. CHUANG and R. ROUQUIER: Derived equivalences for symmetric groups and $sl_2$-categorification. Ann. of Math. 167 (2008) 245–298, math.RT/0407205.
[5] L. Crane and I. Frenkel: Four-dimensional topological quantum field theory, Hopf categories, and the canonical bases. *J. Math. Phys.* **35**, No. 10 (1994) 5136–5154, MR 1295461, hep-th/9405183. Topology and physics.

[6] I. Frenkel: Unpublished Notes, circulation year 1993.

[7] I. Frenkel, M. Khovanov and C. Stroppel: A categorification of finite-dimensional irreducible representations of quantum $sl(2)$ and their tensor products. *Selecta Math. (N.S.)* **12**, No. 3-4 (2006) 379–431, MR 2305608, math.QA/0511467.

[8] W. Fulton: Equivariant cohomology in algebraic geometry. Eilenberg lectures, Columbia University, Spring 2007. Notes by Dave Answer, available at http://www.math.lsa.umich.edu/~dandersn/eilenberg/ (2007).

[9] A. D. Lauda: A categorification of quantum $sl(2)$ (2008), math.QA/0803.3652.

[10] G. Lusztig: Introduction to quantum groups, volume 110 of *Progress in Mathematics*. Birkhäuser Boston Inc., Boston, MA, 1993. ISBN 0-8176-3712-5, MR 1227098.