A model for the $\mathbb{E}_3$ fusion-convolution product of constructible sheaves on the affine Grassmannian

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October 10, 2023

Abstract

Let $G$ be a complex reductive group. The spherical Hecke category of $G$ can be presented as the category of $G_O$-equivariant constructible sheaves on the affine Grassmannian $\text{Gr}_G$. This category admits a convolution product, extending the convolution product of equivariant perverse sheaves. In this paper, we upgrade the mentioned convolution product to an $\mathbb{E}_3$-monoidal structure in $\infty$-categories. The construction is intrinsic, in the sense that it is not performed by pullback from the spectral side of the Derived Satake Theorem by Bezrukavnikov and Finkelberg. We use classical properties of the Beilinson-Drinfeld Grassmannian, Lurie’s characterization of $\mathbb{E}_k$-algebras via the topological Ran space, Gaitsgory and Rozenblyum’s formalism of correspondences and the homotopy theory of stratified spaces.

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## 1 Introduction

The aim of this paper is to provide an extension of the convolution product of equivariant perverse sheaves on the affine Grassmannian to the $\infty$-category of $G_\mathcal{O}$-equivariant constructible sheaves on the affine Grassmannian, and upgrade this extension to an $\mathbb{E}_3$-algebra in $\infty$-categories. This upgrade is the derived avatar of Mirkovic and Vilonen’s commutativity constraint [MV07, Section 5]. The main results of the paper are thus the following.

Let $\mathcal{C}^{\infty}_R$ be the $\infty$-category of $R$-linear small $\infty$-categories (see Notation B.14) with its cartesian symmetric monoidal structure.

### Theorem 1.1 (Corollary 4.10)

Let $G$ be a complex reductive group and $R$ equal to a finite or profinite ring (e.g. $\mathbb{Z}/m\mathbb{Z}$ or to an algebraic extension of $\mathbb{Q}_\ell$ (e.g. $\mathbb{Q}_\ell$, $\overline{\mathbb{Q}}_\ell$). Then there exists an object

$$\text{Sph}(G)^{\text{loc.c.}} \in \text{Alg}_{\mathbb{E}_3}(\mathcal{C}^{\infty}_R)$$

whose underlying $\infty$-category is the small spherical Hecke category

$$\text{Cons}^{\text{fd}}_{G_\mathcal{O}}(\text{Gr}_G, R),$$

the bounded derived $\infty$-category of $G_\mathcal{O}$-equivariant constructible sheaves over the affine Grassmannian with finitely presented stalks.
1.1 Motivation: the Derived Satake Theorem

The restriction of this algebra structure to the abelian category of equivariant perverse sheaves coincides with the classical convolution product of perverse sheaves [MV07], up to the perverse truncation of the derived tensor product involved in the definition of the latter.

Let $\mathcal{P}_R$ be the symmetric monoidal $\infty$-category of $R$-linear presentable $\infty$-categories (and left adjoint functors between them) and let $\mathcal{P}_R^\otimes$ be the symmetric monoidal $\infty$-category of $R$-linear presentable $\infty$-categories (and right adjoint functors between them) (see Notation B.11, Remark B.13).

**Corollary 1.2** (Corollary 4.11). *In the same setting as Theorem 1.1, there is also an object*

$$\text{Sph}(G)^{\text{ren,}\otimes} \in \mathcal{A}_{E_3}(\mathcal{P}_R)$$

*whose underlying $\infty$-category is the renormalized spherical $\infty$-category*

$$\text{Sph}(G)^{\text{ren}} = \text{Ind}(\text{Sph}(G)^{\text{loc,}\otimes}).$$

Both Theorem 1.1 and Corollary 1.2 follow from an analogous statement in a purely “topological” setting, which is actually the main theorem of this paper.

**Theorem 1.3** (Theorem 4.9). *Let $G$ be a complex reductive group and $R$ an commutative ring. Then there exists an object*

$$\text{Sph}(G)^{\otimes} \in \mathcal{A}_{E_3}(\mathcal{P}_R)$$

*whose underlying $\infty$-category is*

$$\text{Cons}_{G^\text{an}}(\text{Gr}^\text{an}_G, R),$$

*the unbounded derived $\infty$-category of topological $G^\text{an}$-equivariant constructible sheaves over the complex space underlying the affine Grassmannian of $G$.*

In this section we provide some context and motivation for this result (Section 1.1) and an outline of the paper (Section 1.2).
sheaves \( \mathcal{Perv}_{GO}(\text{Gr}_G) \) and the category of finite dimensional representations of \( \check{G} \). The key new object here is the affine Grassmannian \( \text{Gr}_G \), an infinite dimensional algebro-geometric object with the property that \( \text{Gr}_G(\mathbb{C}) = G(\mathbb{C}(t))/G(\mathbb{C}[t]) \). The Grothendieck group of \( \mathcal{Perv}_{GO}(\text{Gr}_G) \) is exactly the Hecke algebra appearing in the original Satake theorem.

**Definition 1.4 (Affine Grassmannian).** Let \( G \) be a reductive group. The **arc group** \( G_{O} \) (also denoted by \( G[[t]] \) or \( L^+G \)) is defined as the functor

\[
\text{Aff}_{\mathbb{C}}^{\text{op}} \to \text{Set}
\]

\[
R \mapsto \text{Hom}(R[[t]], G).
\]

The **loop group** \( G_{X} \) (also denoted by \( G((t)) \) or \( LG \)) is defined as the ind-representable functor

\[
\text{Aff}_{\mathbb{C}}^{\text{op}} \to \text{Set}
\]

\[
R \mapsto \text{Hom}(R((t)), G).
\]

The **affine Grassmannian** \( \text{Gr}_G \) is the ind-scheme obtained as the fpqc quotient stack

\[
\text{Gr}_G = G_{X}/G_{O}.
\]

**Theorem 1.5 (Geometric Satake Equivalence, \([MV07, (13.1)]\)).** Fix a reductive algebraic group \( G \) over \( \mathbb{C} \), and a commutative ring \( R \), noetherian and of finite global dimension. There exists a symmetric monoidal structure \( \star \) on \( \mathcal{Perv}_{GO}(\text{Gr}_G, R) \), called convolution, and an equivalence of symmetric monoidal abelian categories

\[
(\mathcal{Perv}_{GO}(\text{Gr}_G, R), \star) \simeq (\text{Rep}^{\text{fd}}(\check{G}_{R}, R), \otimes)
\]

(1.1)

where \( \check{G}_R \) is the Langlands dual of \( G \) over \( R \) \([MV07, \text{Beginning of Sec. 12}]\) and \( \otimes \) denotes the standard tensor product of finite-dimensional \( \check{G}_R \)-representations with coefficients in \( R \).

We recall the meaning of this statement, together with various theoretical recollections necessary for this paper, in Appendix A. We refer the reader seeking for a complete survey to \([Zhu16]\) and \([BR18]\).

**Theorem 1.6 (Derived Satake Theorem, \([BF07, \text{Theorem 5}]\)).** Let \( G \) be a complex reductive group and \( k \) a field of coefficients of characteristic zero. There is a monoidal equivalence of triangulated categories\(^1\)

\[
\text{hCons}^{\text{fd}}_{G_{O}}(\text{Gr}_G, k) \simeq \text{hCoh}(\text{Spec Sym}(\check{g}_k^* \mathbb{Z}[1]))^{\check{G}_k}
\]

(1.2)

where \( \check{G}_k \) is the Langlands dual of \( G \) over \( k \) and \( \check{g}_k \) is its Lie algebra.

---

\(^1\)For the sake of coherence with the rest of the work, we adopt the notation h— in order to refer to “the homotopy category of a stable \( \infty \)-category”. Of course, in the original paper both sides are defined directly as triangulated categories.
Remark 1.7. The category $\text{Perv}_{G_o}(\text{Gr}_G, k)$ is the heart of a t-structure on $\text{Cons}^{fd}_{G_o}(\text{Gr}_G, k)$ (and hence on the homotopy category). Indeed, this t-structure is inherited from the presentation of the equivariant constructible category à la Bernstein-Lunts, see Remark 4.13. As explained in Remark 4.13, the Geometric Satake Theorem can be formally recovered from the Derived Satake Theorem by passing to the heart, up to a detail: a priori, the induced statement will only be a monoidal equivalence of monoidal abelian categories, and not a symmetric monoidal equivalence.

Both the left and right-hand side, as triangulated categories, carry a symmetric monoidal structure (for the left-hand-side, see [AR23, Section 3.3]; on the right-hand-side, it is the tensor product described in [BF07, 2.7]).

We now explain how our results fit into this framework. The notion of $E_k$-center of an $E_k$-$\infty$-category we are referring to is [Lur17, Definition 5.3.1.6, Example 5.3.1.13].

Folklore 1.8. The $\infty$-category $\text{Sph}^{loc,c}(G) = \text{Cons}^{fd}_{G_o}(\text{Gr})$, seen as a monoidal $\infty$-category, is equivalent to the $E_2$-center of the derived $\infty$-category of finite-dimensional representations

$$\text{DRep}^{fd}(\hat{G}, k)$$

seen as an $E_2$-$\infty$-category by forgetting its $E_\infty$-monoidal structure $\otimes$ along the map of operads $E_2 \to E_\infty$.

This result was announced by Gaitsgory and Lurie several years ago. One can make it follow from an $\infty$-categorical version of Theorem 1.6 and work of Ben-Zvi, Francis, Nadler and Preygel on centers [BZFN10], [BZNP17]; see [BZSV23, (17.1.2)] for a discussion on this matter. Folklore 1.8 formally implies the existence of an $E_3$-structure on the left-hand-side of Theorem 1.6 by pullback along the equivalence (1.2).

In the present paper, however, we prove the existence of such an $E_3$-monoidal structure (Theorem 1.1) independently of the existence of its counterpart on the spectral side. This is in the same spirit of the Tannakian reconstruction principle used in the proof of the Geometric Satake Theorem (Theorem 1.5), where the existence of a symmetric monoidal structure on $\text{Perv}_{G_o}(\text{Gr})$ is a part of the structure needed to apply the reconstruction machinery, and only a posteriori it is interpreted as corresponding to the tensor product in $\text{Rep}(\hat{G})$.

Remark 1.9. In recent work, Campbell and Raskin prove that, up to taking Ind-completion of both sides, Theorem 1.6 can be promoted to an equivalence of $\text{factorizable} \, \infty$-categories [CR23, Theorem 6.6.1]. The notion of “factorizability” corresponds, to a certain extent, to the “$E_2$ part” of the $E_3$ structure appearing in Corollary 1.2, the “$E_1$ part” being the monoidal structure already contained in Theorem 1.6. In particular, their work provides an incarnation of the $\infty$-categorical version of the Derived Satake Theorem mentioned before.

1.2 Outline of the work

The proof of Theorem 1.3 will be carried out as follows.
Let $X$ be any smooth complex curve. In Section 2 we recall that all vertices of the convolution diagram (A.2) admit a “global” (or “Beilinson-Drinfeld”) version which naturally lives over the Ran presheaf of $X$. In other words, the classical Beilinson-Drinfeld Grassmannian [MV07, Section 5] can be represented as a prestack $\text{Gr}_{\text{Ran}(X)}$, an approach already used in [Zhu16] and [GL]. We also show that the mentioned convolution diagram can be encoded as a 2-Segal semisimplicial object in a certain category $\text{StrStk}_C$ of algebro-geometric objects, and hence as a nonunital algebra structure in correspondences in the same category (Corollary 2.62). This category $\text{StrStk}_C$ (Definition B.6) is a suitable pro-completion and free cocompletion of the category of stratified stacks, and it is the right environment for our constructions since we want to keep track of the fact that our objects can be approximated by finite-dimensional objects at various levels. In particular, this will allow later to define categories of constructible sheaves in the right way (e.g. as a colimit along the coweight filtration over the affine Grassmannian).

In Section 3 we apply the stratified analytification functor $(-)^\text{an}$ of Construction B.4 to the objects constructed in the previous section, thus gaining the possibility of applying the machinery of topological factorization algebras. Thanks to the factorization property of the Beilinson-Drinfeld Grassmannian and to the fact that the analytification of $\text{Gr}_{\text{Ran}(\mathbb{A}^1_\mathbb{C})}$ is homotopy invariant under dilation of coordinates, we can apply [Lur17, Theorem 5.5.4.10] and build a 2-dimensional factorization algebra structure on the analytification of $\text{Gr}_{\text{Ran}(\mathbb{A}^1_\mathbb{C})}$, up to stratified homotopy. This structure is compatible with the convolution structure described in the previous section by construction.

The goal of Section 4 is to transfer both structures to the level of constructible sheaves. This involves extending the functor $\text{Cons}(\!-\!)\text{an}$ to a category of correspondences, which is is done in Appendix B by using the formalism of [GR17, Part III]. Proving functoriality and symmetric monoidality of this construction relies on the formalism of exit paths as developed in [Lur17, Appendix A], and therefore requires that our spaces (and maps between them) satisfy certain conditions at the level of topology and stratifications. We prove that such conditions are satisfied in Section 3.2: the main properties used are that the stratification in Schubert cells of the affine Grassmannian, when viewed in the complex-analytic topology, satisfies Whitney’s conditions, and that the left leg of the convolution diagram is a pro-smooth map. Invariance under stratified homotopy of constructible sheaves (Remark B.22) takes on a prominent role as well. All this results in an $\mathbb{E}_2^\text{nu} \times \mathbb{E}_1^\text{nu}$-monoidal structure on $\text{Cons}_{G_0^{\text{an}}}^{\text{an}}(\text{Gr}_{\text{Ran}(\mathbb{A}^1_\mathbb{C})})$, see Theorem 3.14.

Section 4.2 is devoted to transfer our two algebra structures from $\text{Cons}_{G_0^{\text{an}}}^{\text{an}}(\text{Gr}_{\text{Ran}(\mathbb{A}^1_\mathbb{C})})$ to $\text{Cons}_{G_0^{\text{an}}}(\text{Gr}_{G}^{\text{an}})$, which is done by specializing to any chosen point $x \in \mathbb{A}^1_\mathbb{C}$. Here we use, among other facts, that the right leg of the convolution diagram is ind-proper. We prove that at this level the two structures gain unitality, and therefore the Dunn-Lurie Additivity Theorem [Lur17, Theorem 5.1.2.2] can be applied, thus yielding the sought $\mathbb{E}_3$-structure on $\text{Cons}_{G_0^{\text{an}}}(\text{Gr}_{G}^{\text{an}})$.

Remark 1.10. It is worth stressing that the complex topology takes on a prominent role in our proofs. When taking constructible sheaves, we always look at the underlying complex-analytic
topological space $\text{Gr}^a_G$ of $\text{Gr}_G$, with its complex stratification in Schubert cells. However, by Corollary A.29, the small spherical category is the same both from the algebraic and the complex-analytic point of view and therefore our result works in both settings.

Note also that, with the complex-analytic topology, $\text{Gr}^a_G$ is homotopy equivalent to $\Omega G^a \simeq \Omega^2 \mathcal{B}(G^a)$, which carries a canonical $E_2$-algebra structure in spaces. This is not sufficient to derive the $E_2$-algebra structure on the spherical category though, because the functor taking constructible sheaves is stratified homotopy invariant but not homotopy invariant (for example, constructible sheaves on $\mathbb{R}$ and on the point are not the same).

The application of Lurie’s [Lur17, Theorem 5.5.4.10] to the affine Grassmannian also appears in [HY19], though in that paper the authors are interested in a purely topological problem and do not take constructible sheaves. Up to our knowledge, the formalism of constructible sheaves via exit paths and exodromy has never been applied to the study of the affine Grassmannian and the spherical Hecke category. Here we use it in order to take advantage of stratified homotopy invariance of the constructible sheaves functor, and to establish the existence of certain adjoints (see Corollary B.18).

Acknowledgments

This paper constitutes the core chapter of my thesis as a graduate student at Scuola Normale Superiore di Pisa and Université de Strasbourg (2018-2022) under the supervision of Mauro Porta and Gabriele Vezzosi.

During the last phase of revision I was supported by the ERC Starting Grant Foundations of motivic real K-theory held by Yonatan Harpaz (2020-2025).

I also wish to thank Pramod Achar, Pierre Baumann, Dario Beraldo, Roman Bezrukavnikov, Robert Cass, Ivan Di Liberti, Andrea Gagna, Dennis Gaitsgory, Jeremy Hahn, Yonatan Harpaz, Vasily Krylov, Jacob Lurie, Andrea Maffei, David Nadler, Emanuele Pavia, Michele Pernice, Sam Raskin, Simon Riche, James Tao and Angelo Vistoli and Allen Yuan for their suggestions and explanations.

I especially thank Peter Haine, Mark Macerato, Morena Porzio and Marco Volpe for the extensive and fruitful discussions carried out with them during various phases of the work, and Roman Bezrukavnikov for hosting me at MIT during April and May 2022.

2 Convolution over the Ran space

Throughout this whole work, $G$ will be a complex reductive group, $X$ a complex smooth curve, and $x$ a closed point in it.

Notation 2.1. When defining a presheaf over the category of complex affine schemes, we will usually drop the dependance on $\text{Spec} \mathbb{R}$ when it does not cause confusion. For example, a point
$x \in X(R)$ will just be denoted as $x \in X$, and its graph in $X \times \text{Spec} R$ by $\Gamma_x$ or just $x$ when this does not cause confusion.

Two $R$-points of $X$ will be declared “equal” if they coincide as maps $\text{Spec} R \to X$, “distinct” if they do not coincide (but their graphs may intersect nontrivially inside $X_R$), and “disjoint” if their graphs do not intersect.

For a finite set $I$ and a sequence $x_I = (x_{i_1}, \ldots, x_{i_{|I|}}) \in X(R)^I$, the expression $\Gamma_{x_I}$ (or just $x_I$ when this does not cause confusion) will mean $\sum_{i=1}^{\mid I \mid} \Gamma_{x_i}$ (the divisor of $X \times \text{Spec} R$ given by the sum of the graphs of the points, seen as a closed - possibly unreduced - scheme). The expressions $\hat{X}_{x_I}$ and $X_{x_I}$ will denote respectively the affine formal completion of $X_R$ at the closed subscheme $\bigcup_{k=1}^{\mid I \mid} \Gamma_{x_k}$, and its punctured counterpart (see Remark A.4).

Finally, a $G$-torsor $\mathcal{F} \in \text{Bun}_G(X_R)$ will just be denoted by $\mathcal{F} \in \text{Bun}_G(X)$, and the trivial $G$-bundle over $X_R$ will be denoted by $\mathcal{T}$.

2.1 The Beilinson-Drinfeld setting

The following definitions also appear, in various forms, in [Rei12], [Ric14] and [CvdHS22], and are natural generalizations of the characterizations recalled in Proposition A.9.

**Definition 2.2.** Let $I, I_1, I_2$ be nonempty finite sets. We recall the following definitions:

- the Beilinson-Drinfeld arc group
  
  
  $G_{O,I} = \{ x_I \in X^I, g \text{ automorphism of the trivial } G\text{-bundle on } \hat{X}_{x_I} \}$,

- the Beilinson-Drinfeld loop group
  
  $G_{X,I} = \{ x_I \in X^I, \mathcal{F} \in \text{Bun}_G(X),$
  
  $\alpha$ trivialization of $\mathcal{F}$ on $X \setminus x_I, \mu$ trivialization of $\mathcal{F}$ on $\hat{X}_{x_I} \}$
  
  with its “decoupled version”

  $G_{X,I_1,I_2} = \{ x_{I_1} \in X^{I_1}, x_{I_2} \in X^{I_2}, \mathcal{F} \in \text{Bun}_G(X),$
  
  $\alpha$ trivialization of $\mathcal{F}$ on $X \setminus x_{I_1}, \mu$ trivialization of $\mathcal{F}$ on $\hat{X}_{x_{I_2}} \}$,

- the Beilinson-Drinfeld Grassmannian

  $\text{Gr}_I = G_{X,I} / G_{O,I} \cong \{ x_I \in X^I, \mathcal{F} \in \text{Bun}_G(X), \alpha$ trivialization of $\mathcal{F}$ on $X \setminus x_I \}$,

  where the quotient is the fpqc quotient stack relative to $X^I$.

**Remark 2.3.** All the previous objects are ind-schemes over $X^I$. The object $G_{O,I}$ is representable, and has the structure of an infinite-dimensional group scheme relative to $X^I$. The object $\text{Gr}_I$ is often denoted by $\text{Gr}_{X^I}$ ([MV07], [Zhu16]).
Proposition 2.4 (Translational invariance). Let $X = \mathbb{A}^1_{\mathbb{C}}$. Then any choice of a closed point $x \in \mathbb{A}^1_{\mathbb{C}}$ induces splittings

$$\text{Gr}_{\{1\}} \simeq \text{Gr} \times \mathbb{A}^1_{\mathbb{C}}$$

$$G_{O,\{1\}} \simeq G_O \times \mathbb{A}^1_{\mathbb{C}}.$$

Proof. The case of $\text{Gr}$ follows from [Zhu16, (3.1.10)]. The case of $G_O$ is straightforward from the definition. \qed

Remark 2.5. Fix $I \in \text{Fin}_{\text{surj}, \geq 1}$. The group $G_{O,I}$ acts on $\text{Gr}_{G,I}$ over $X_I$ as follows:

$$(x_I, g)(x_I, F, \alpha) := (x_I, F|_{x_I} \circ \alpha|_{x_I}).$$

Warning 2.6. Such splittings only hold for $I = \{1\}$.

Notation 2.7. Let $I, J \in \text{Fin}_{\text{surj}, \geq 1}$ and $[\phi : I \rightarrow J]$ a $J$-partition of $I$, i.e. the equivalence class of a surjection $\phi : I \rightarrow J$ modulo autobijections of $J$. Following [CvdHS22, (4.2)], let

$$X^\phi = \{ x_I \in (x_1, \ldots, x_{|I|}) \in X^I \mid \phi(i) = \phi(i') \Rightarrow x_i = x_{i'}, \quad \phi(i) \neq \phi(i') \Rightarrow \Gamma_{x_i} \cap \Gamma_{x_{i'}} = \emptyset, i, i' \in I \} \subset X^I.$$

Proposition 2.8 (Factorization property). With the above notation, there is an isomorphism

$$\text{Gr}_I|_{X^\phi} \simeq (\prod_J \text{Gr}_X) \times_{X^I} X^\phi$$

where the map $\prod_J \text{Gr}_X \rightarrow X^I$ is induced by the diagonal $X^I \rightarrow X^I$ associated to $\phi$.

Proof. See [CvdHS22, Proposition 4.6] (which refers directly to [Zhu16, Proposition 3.1.13]). To be precise, the proof there is performed for $X = \mathbb{A}^1_{\mathbb{C}}$ (see Corollary 2.29), but it is literally the same in the general case. \qed

Remark 2.9. The stratification in Schubert cells of $\text{Gr}$ (Remark A.1) naturally induces a stratification on $\text{Gr}_X$ with the same stratifying poset $\mathbb{X}_\bullet(T)^+$, described in [Zhu16, (3.1.11)]. If $(x, F, \alpha) \in \text{Gr}_X$, denote by

$$\text{Inv}_x(F, \alpha) \in \mathbb{X}_\bullet(T)^+$$

the associated coweight (we will often abbreviate this by $\text{Inv}_x(\alpha)$). We also denote the stratum with coweight $\mu$ by

$$\text{Gr}_{X,\mu},$$

and set

$$\text{Gr}_{X,\leq \mu} = \bigcup_{\nu \leq \mu} \text{Gr}_{X,\nu}.$$
Remark 2.10. The notion of $\text{Inv}_x(\alpha)$ admits the following generalization. Let $\mathcal{F}_0, \mathcal{F}_1 \in \text{Bun}_G(X)$, and let $\eta : \mathcal{F}_1|_{X \setminus x} \cong \mathcal{F}_0|_{X \setminus x}$. Fix a trivialization $\lambda$ of $\mathcal{F}_0$ on the formal neighbourhood of $x$. Such a trivialization always exists because all $G$-torsors are trivial on $\text{Spec} \mathbb{C}[t]$. Then one can compute

$$\text{Inv}_x(\lambda|_{\mathcal{F}_0} \circ \eta|_{\mathcal{F}_1}) \in \mathbb{X}_+(T)^+$$

and check, by uniqueness of the Cartan decomposition, that this coweight is independent of the choice of $\lambda$. We denote it by $\text{Inv}_x(\eta)$.

Construction 2.11. There is a well-defined stratification on $\text{Gr}_I$ whose explicit description is provided in [CvdHS22, Definition 4.18]. We recall it here, just to fix notations for the generalization to the convolution Grassmannian. The indexing poset of the stratification is

$$\{ [\phi : I \rightarrow J] \text{ partition of } I, \mu_J = (\mu_1, \ldots, \mu_{|J|}) \in (\mathbb{X}_+^+(T))^J \}$$

where the order relation is given by:

$$(\phi, \mu_J) \leq (\phi', \mu_J')$$

if and only if there exists a refinement $[\psi : J' \rightarrow J]$ such that for every $h \in J$

$$\mu_h \leq \sum_{h' \in J', \psi(h') = h} \mu_{h'}.$$

The stratification is then defined by setting

$$\text{Gr}_{I, \phi, \mu_J} = \prod_{h \in J} \text{Gr}_{X, \mu_h} \times X^\phi \hookrightarrow \text{Gr}_I$$

where the embedding is induced by Proposition 2.8.

For $\mu \in \mathbb{X}_+(T)^+$, we denote

$$\text{Gr}_{I, \leq \mu} = \bigcup_{[\phi : I \rightarrow J], \sum_{i \in J} \eta_i \leq \mu} \text{Gr}_{I, \phi, \eta_J}.$$

Recall now from Remark A.1 that the action of $G_0$ on $\text{Gr}$ restricts to $\text{Gr}_{\leq \mu}$ for each $\mu \in \mathbb{X}_+(T)^+$ and that this restriction factors through a quotient

$$G_0 \twoheadrightarrow G_{0,(j_\mu)}$$

for a sufficiently large natural number $j_\mu$, where

$$G_{0,(j_\mu)} := G(\mathbb{C}[t]/t^{j_\mu}) \cong G(\mathbb{C}[t]/t^{j_\mu})$$

is now a group scheme of finite type.

For $j$ a natural number, and $Z \subset X$ a closed subscheme, let $Z(j)$ denote the $j$-thickening of $Z$, i.e. the scheme $(Z, \mathcal{O}_X/j_Z^j)$. 

**Definition 2.12.** Let $j$ be a natural number, and $I \in \text{Fin}_{\text{surj}, \geq 1}$. We define

$$G_{0, I, (j)}$$

as the functor classifying

$$\{ x_I \in X^I, g \in \text{Res}_{(\Gamma_I)_0} (G_{0, I}) \}$$

where $\text{Res}$ denotes the Weil restriction. Note that the scheme $\Gamma_I$ is not necessarily reduced.

**Remark 2.13.** By the proof of [Ric14, Lemma 2.5], we have that

$$G_{0, I} \simeq \lim_j G_{0, I, (j)}.$$ 

The functor $G_{0, I, (j)}$ is a smooth group scheme of finite type over $X_I$, again by the proof of [Ric14, Lemma 2.5.(ii)]. Smoothness may seem a bit counter-intuitive, since the fiber of $G_{0, I}$ (say $I = \{1, 2\}$) over a point in the diagonal $X \subset X^2$ is given by a copy of $G_0$, while for instance the fiber over a point in the disjoint locus of $X^2$ is given by $G_0 \times G_0$. However, one cannot argue that this contradicts flatness, in that we are dealing with infinite-dimensional objects. And in fact, when one truncates to $G_{0, I, (j)}$, the following happens. The fiber of $G_{0, I}$ at a point $(x, x)$ on the diagonal is, by definition,

$$G(\mathbb{C}[t]/t^{2j}),$$

since we need to perform the Weil restriction to the $j$-thickening of the sum of the divisor $x$ with itself. On the other hand, the fiber at a point $(x_1, x_2)$ outside the diagonal is

$$G_{0, (j)} \times G_{0, (j)}.$$ 

Therefore, the dimensions of these fibers are

$$\dim G(\mathbb{C}[t]/t^{2j}) = (\dim G)^{2j}$$

and

$$\dim(G(\mathbb{C}[t]/t^j))^2 = (\dim G)^{2j}.$$ 

**Remark 2.14.** Let $I \in \text{Fin}_{\text{surj}, \geq 1}, \mu \in X_\bullet (T)^+$. The action in Remark 2.5 restricts to each $\text{Gr}_{I, \leq \mu}$. By [Ric14, Corollary 2.7], the restriction factors through a quotient $G_{I, 0, (j_\mu)}$ for a sufficiently large natural number $j_\mu$.

The following definitions are inspired by [AR23, 2.4.3]:

**Definition 2.15.** Let $I \in \text{Fin}_{\text{surj}, \geq 1}, \mu \in X_\bullet (T)^+$. We define

$$\text{Hck}_{I, \leq \mu, (j)} = G_{0, I, (j)} \backslash \text{Gr}_{I, \leq \mu}$$

as the fpqc quotient stack in the category $\text{Stk}_{/X^I}$. We also define

$$\text{Hck}_{I, \leq \mu} = \lim_{j \geq j_\mu} G_{0, I, (j)} \backslash \text{Gr}_{I, \leq \mu}.$$
Since the action of $G_{0,I(j)}$ respects the stratification of $\text{Gr}_{I,\leq \mu}$, each
\[ \text{Hck}_{I,\leq \mu(j)} \]
is a stratified stack in the sense of Remark B.5, and
\[ \text{Hck}_{I,\mu} \]
is a stratified pro-stack, hence an object of the category $\text{StrStk}_C$ from in Definition B.6.

**Remark 2.16.** The actual limit
\[ \text{Hck}_{\longleftarrow I,\leq \mu} = \lim_{j \geq j_{\mu}} \text{Hck}_{I,\leq \mu(j)} \]
can be described as the functor
\[ \text{Aff}_C^{op} \rightarrow \text{Grpd} \]
parametrizing the groupoid whose class of objects is
\[ \{ [\phi : I \rightarrow J], x_j \in X^j, \mathcal{F}_0, \mathcal{F}_1 \in \text{Bun}_G(X), \eta : \mathcal{F}_0|_{X \setminus \Gamma_j} \simeq \mathcal{F}_1|_{X \setminus \Gamma_j}, \sum_{i \in I} \text{Inv}_{x_i}(\eta) \leq \mu \} \]
and whose morphisms are induced by isomorphisms of pairs $(\mathcal{F}_0, \mathcal{F}_1)$ commuting with the $\eta$'s. This is what is usually called the (Beilinson-Drinfeld version of the) Hecke stack. However, we will need the pro-version in order to take constructible sheaves over it.

### 2.2 The convolution Grassmannian

Our goal now is to transfer the convolution diagram from Remark A.13 to the “Beilinson-Drinfeld” setting.

**Definition 2.17.** Given $I_1, I_2 \in \text{Fin}_{\text{surj}, \geq 1}, \mu \in \mathcal{X}_*^+(\mathcal{T})^+, j \geq j_{\mu}$, we define
\[ G_{X,I_1,I_2,\leq \mu(j)} = \{ [\phi : I_1 \rightarrow J], x_j \in X^j, x_{I_1} \in X^{I_1}, \mathcal{F} \in \text{Bun}_G(X), \alpha : \mathcal{F}|_{X \setminus \Gamma_j} \simeq \mathcal{F}|_{X \setminus \Gamma_{x_j}}, \]
\[ \mu : \mathcal{F}_0|_{(I_{x_j})_{\alpha(j)}} \simeq \mathcal{F}_1|_{(I_{x_j})_{\alpha(j)}}, \sum_{i \in I} \text{Inv}_{x_i}(\alpha) \leq \mu \} \]
and
\[ G_{X,I_1,I_2,\leq \mu} = \{ [\phi : I_1 \rightarrow J], x_j \in X^j, x_{I_2} \in X^{I_2}, \mathcal{F} \in \text{Bun}_G(X), \alpha : \mathcal{F}|_{X \setminus \Gamma_j} \simeq \mathcal{F}|_{X \setminus \Gamma_{x_j}}, \]
\[ \mu : \mathcal{F}_0|_{I_{x_{I_2}}} \simeq \mathcal{F}_1|_{I_{x_{I_2}}}, \sum_{i \in I} \text{Inv}_{x_i}(\alpha) \leq \mu \} \]
Remark 2.18. We have that
\[ G_{\mathcal{X}, I_1, I_2} \simeq \lim_{j \geq j_\mu} G_{\mathcal{X}, I_1, I_2, \leq \mu(j)} \]
and
\[ G_{\mathcal{X}, I_2} \simeq \text{colim}_{\mu \in \mathcal{X}_*(T)^+} G_{\mathcal{X}, I_2, \leq \mu}. \]

Definition 2.19. Let \( k \geq 1, I_1, \ldots, I_k \in \text{Fin}_{\text{surj} \geq 1}, \mu \in (\mathcal{X}_*(T)^+) \). We define the scheme
\[ \text{Conv}_{I_1, \ldots, I_k, \leq \mu} = G_{\mathcal{X}, I_1, I_2, \leq \mu} \times G_{\mathcal{X}, I_2, I_3, \leq \mu} \times \cdots \times G_{\mathcal{X}, I_k, \leq \mu} \times \text{Gr}_{I_k} \]
(note similar to Construction A.8, with the difference that now all actions are over \( X^I \). In particular, this is an fpqc quotient, but it is also a schematic quotient).

Remark 2.20. For any \( j \geq j_\mu \), the expression above can be rewritten as
\[ G_{\mathcal{X}, I_1, I_2} \times G_{\mathcal{X}, I_2, I_3} \times \cdots \times G_{\mathcal{X}, I_k} \times \text{Gr}_{I_k} \simeq \text{colim}_{\mu \in \mathcal{X}_*(T)^+} \text{Conv}_{I_1, \ldots, I_k, \leq \mu}. \]
(see [Zhu16, Discussion after Lemma 5.2.3]).

Definition 2.21. We define the convolution Grassmannian as
\[ \text{Conv}_{I_1, \ldots, I_k} = G_{\mathcal{X}, I_1, I_2, \leq \mu} \times G_{\mathcal{X}, I_2, I_3} \times \cdots \times G_{\mathcal{X}, I_k} \times \text{Gr}_{I_k} \simeq \text{colim}_{\mu \in \mathcal{X}_*(T)^+} \text{Conv}_{I_1, \ldots, I_k, \leq \mu}. \]

Remark 2.22. This colimit is filtered, hence the convolution Grassmannian is an ind-scheme.

Remark 2.23. The convolution Grassmannian classifies the datum
\[ \{ (x_1, \ldots, x_k), x_j \in X^{I_j} \text{ for each } j = 1, \ldots, k, \tau_1, \ldots, \tau_k, \}
\[ \alpha : \tau_1|_{X \setminus x_{I_1}} \simeq \tau|_{X \setminus x_{I_1}, \eta_j : \tau_j|_{X \setminus x_{I_j}} \simeq \tau_{j-1}|_{X \setminus x_{I_j}}, j = 2, \ldots, k}. \]
Suppose now fixed \( k \geq 1, I_1, \ldots, I_k, \mu \in \text{Fin}_{\text{surj} \geq 1}, \) and a partition \( [\phi : I_1 \sqcup \cdots \sqcup I_k \to J] \). Define \( X^\phi \) as in Notation 2.7, with \( I = I_1 \sqcup \cdots \sqcup I_k \).

Notation 2.24. Let us fix the following notation. A general element of \( \text{Conv}_{I_1, \ldots, I_k} \) will be denoted by
\[ \left( \tau_1 \xleftarrow{\alpha} \tau_2 \xleftarrow{\eta_1} \tau_3 \xleftarrow{\eta_2} \cdots \xleftarrow{\eta_{k-1}} \tau_k \right). \]
Of course this is just a symbolic notation, in that each of the arrows drawn here is defined over a (potentially) different open set.

Proposition 2.25. There is an isomorphism
\[ \text{Conv}_{I_1, \ldots, I_k}|_{X^\phi} \simeq \prod_{j \in J} \text{Conv}_{I_j, \mu_j} \times X^{I_j \setminus \mu_j} \times X^\phi \]
where:
• \( m_j = \# \{ b \mid 1 \leq b \leq k, \phi^{-1}(j) \cap I_b \neq \emptyset \} \)

• \( \text{Conv}_{\Delta, m_j} := \text{Conv}_{X \times \ldots \times m_j \times x \times \ldots} X \) (the map from \( X \) to \( X^m_j \), being the diagonal)

• the map \( \prod_{j \in J} \text{Conv}_{\Delta, m_j} \to X^{I_1 \cup \ldots \cup I_k} \) is induced by the diagonal map \( X^j \to X^{I_1 \cup \ldots \cup I_k} \) associated to \( \phi \).

Proof. (Sketch). This proof has been suggested to us by Robert Cass. We treat the case \( k = 3, I = I_1 = I_2 = I_3 = 1 \). We have three essentially distinct cases:

• \( J = \{1, 2, 3\}, \phi = \text{id} \). The isomorphism (adopting Notation 2.24) is given by

\[
\text{Conv}_{I, I_2, I_3} \times X^3 X^\phi \simeq (\text{Gr}_X \times \text{Gr}_X \times \text{Gr}_X) \times X^3 X^\phi
\]

\[
( \mathcal{T} \xleftarrow{\xi_1} \mathcal{X}_{\{x\}} \xleftarrow{\eta_2} \mathcal{X}_{\{x\}} \xleftarrow{\eta_3} \mathcal{X}_{\{x\}} ) \rightarrow
\]

\[
( \mathcal{T} \xleftarrow{\xi_1} \mathcal{X}_{\{x\}}, \mathcal{T} \xleftarrow{\xi_2 \circ \eta_2 \circ \xi_2} \mathcal{X}_{\{x\}}, \mathcal{T} \xleftarrow{\xi_3 \circ \eta_3 \circ \xi_3} \mathcal{X}_{\{x\}} )
\]

whose inverse is given by gluing sheaves (which can be done since the points are distinct).

• \( J = \{1, 2\}, \phi(1) = \phi(3) = 1, \phi(2) = 2 \) (we treat this case and not the case \( \phi(1) = \phi(2) = 1, \phi(3) = 3 \) since we want to show that our argument works even when the two equal coordinates are not adjacent to one another). The isomorphism is given by

\[
\text{Conv}_{I_1, I_2, I_3} \times X^3 X^\phi \simeq (\text{Conv}_{\Delta, 2} \times \text{Gr}_X) \times X^3 X^\phi
\]

\[
( \mathcal{T} \xleftarrow{\xi_1} \mathcal{X}_{\{x\}} \xleftarrow{\eta_2} \mathcal{X}_{\{x\}} \xleftarrow{\eta_2} \mathcal{X}_{\{x\}} ) \rightarrow
\]

\[
( \mathcal{T} \xleftarrow{\xi_1} \mathcal{X}_{\{x\}}, \mathcal{T} \xleftarrow{\xi_2 \circ \eta_2 \circ \xi_2} \mathcal{X}_{\{x\}}, \mathcal{T} \xleftarrow{\xi_2 \circ \eta_2 \circ \xi_2} \mathcal{X}_{\{x\}} )
\]

with inverse

\[
( \mathcal{T} \xleftarrow{\xi_1} \mathcal{X}_{\{x\}} \xleftarrow{\eta_2} \mathcal{X}_{\{x\}} \xleftarrow{\eta_2} \mathcal{X}_{\{x\}} ) \rightarrow
\]

\[
( \mathcal{T} \xleftarrow{\xi_1} \mathcal{X}_{\{x\}}, \mathcal{T} \xleftarrow{\xi_2 \circ \eta_2 \circ \xi_2} \mathcal{X}_{\{x\}}, \mathcal{T} \xleftarrow{\xi_2 \circ \eta_2 \circ \xi_2} \mathcal{X}_{\{x\}} )
\]

• \( J = \{1\} \). The isomorphism is the identity.

\[ \square \]

Definition 2.26. Let \( x \in X \) be a closed point. We define

\[ \text{Conv}_{x, k} = \text{Conv}_{\Delta, k} \times X \{x\}. \]
Remark 2.27. The object $\text{Conv}_{x,k}$ is isomorphic to
\[
\overbrace{G_X \times G_O \times \cdots \times G_O}^{k-1} G_X \times G_O \text{Gr}
\]
(notation as in Construction A.8).

In a similar fashion as Proposition 2.4, one can prove:

Proposition 2.28. Let $X = \mathbb{A}^1_\mathbb{C}, k \geq 1$. With the notations of Proposition 2.25, the choice of a point $x \in X$ induces a splitting
\[
\text{Conv}_{\Delta,k} \simeq \text{Conv}_{x,k} \times \mathbb{A}^1_\mathbb{C}.
\]

Corollary 2.29. In the case when $X = \mathbb{A}^1_\mathbb{C}$, Proposition 2.25 specializes to
\[
\text{Gr}_I |_{\psi} \simeq (\prod_I \text{Gr}) \times X^\psi
\]
for any chosen point $x \in X$.

Proof. It suffices to apply Proposition 2.4 and Proposition 2.28 respectively. \(\square\)

Remark 2.30. One can define a stratification of $\text{Conv}_{\Delta,k,\leq \mu}$ (the case $k = 1$ being $\text{Gr}_{X,\leq \mu}$), as follows. The stratifying poset is $(\mathbb{Y}_\bullet(T)^+)^k$. First of all, $G_{X,\Delta,\leq \mu}$ is stratified over $\mathbb{Y}_\bullet(T)^+$ by looking at $\text{Inv}_x(\alpha)$. This induces a stratification on the product
\[
G_{X,\Delta,\leq \mu} \times_X \cdots \times G_{X,\Delta,\leq \mu} \times_X \text{Gr}_{X,\leq \mu}
\]
and one can check that this stratification passes to the multiple quotient
\[
G_{X,\Delta,\leq \mu} \times_{G_{O,X}} \cdots \times_{G_{O,X}} \text{Gr}_{X,\leq \mu}.
\]
Equivalently, the stratum
\[
\text{Conv}_{\Delta,k,\mu_1,\ldots,\mu_k}
\]
is the sub-ind-scheme of $\text{Conv}_{\Delta,k}$ where one imposes the condition that $\text{Inv}_x(\alpha_i) = \mu_i$ for every $i$.

Let $x \in X$ be a closed point, and let $\text{Conv}_{x,k}$ be the pullback of $\text{Conv}_{\Delta,k}$ to $x$. Then each stratum of $\text{Conv}_{x,k}$ can be identified with $G_{X,x} \times_{G_O} \cdots \times_{G_O} \text{Gr}_x$, where $G_{X,x}$ is the preimage of $\text{Gr}_x$ along the quotient map. This definition is the direct generalization from $k = 2$ to arbitrary $k$ of [MV07, after Lemma 4.3].

These strata are not orbits for a group action, but they are smooth.
Proposition 2.31. Each stratum

\[ G_{X,v_1} \times G_{O} \cdots \times G_{O} \text{Gr}_{v_k} \]

is a smooth locally closed subscheme of Conv\(_{k}\).

**Proof.** The following proof has been suggested to us by Mark Macerato. First of all, we note that for any \( j \geq j_i \forall i \), then

\[ G_{X,v_1} \times G_{O} \cdots \times G_{O} \text{Gr}_{v_k} \simeq G_{X,v_1,(j)} \times G_{O,(j)} \cdots \times G_{O,(j)} \text{Gr}_{v_k} \]

(with the same argument as Remark 2.20). Now,

\[ G_{X,v_1,(j)} \times \cdots \times \text{Gr}_{v_k} \rightarrow \text{Gr}_{v_1} \times \cdots \times \text{Gr}_{v_k} \]

is a torsor with fiber \( G_{O,(j)}^{x_{O}-1} \), which is a smooth group scheme. Since the base is smooth ([Zhu16, Proposition 2.15 (1)]), the total space is smooth as well. Now, the map \( G_{X,v_1,(j)} \times \cdots \times \text{Gr}_{v_k} \rightarrow G_{X,v_1,(j)} \times G_{O,(j)} \cdots \times G_{O,(j)} \text{Gr}_{v_k} \) is the fpqc schematic quotient of a smooth scheme with respect to the group \( G_{O,(j)}^{x_{O}-1} \). In particular, it is an fpqc covering, and therefore by [SPA, Tag 02VL] we conclude.

**Remark 2.32.** We can now define a stratification for Conv\(_{I_1,\ldots,I_k}\). Recall the notation in Proposition 2.25. The stratifying poset will be

\[ \text{Cw}_{I_1,\ldots,I_k} = \{ [\phi : I_{1} \sqcup \cdots \sqcup I_{k} \rightarrow J], \mu_j = (\mu_{j_1}^{h_{j_1}}, \ldots, \mu_{j_k}^{h_{j_k}}) \in \prod_{j \in J} (X_{v}(T)^{+})^{m_j} \} \]

where “Cw” stays for “coweight” and, for each \( j \in J \), the \( h_{j}^{i} \)'s are those indexes for which \( \phi^{-1}(j) \cap I_{h_{j}^{i}} \neq \emptyset \). The stratification for Conv\(_{I_1,\ldots,I_k}\) is then defined as

\[ \text{Conv}_{I_1,\ldots,I_k,\phi,\mu} = \prod_{j \in J} \text{Conv}_{\Delta, m_{j_{1}}^{h_{j_{1}}}, \ldots, m_{j_{k}}^{h_{j_{k}}}, \phi_{j_{1}}, \ldots, \phi_{j_{k}}} \times X_{v} \xrightarrow{\phi} \text{Conv}_{I_1,\ldots,I_k} \]

where the embedding is induced by Proposition 2.25.

**Definition 2.33.** Let \( k \geq 1, I_1, \ldots, I_k \in \text{Fin}_{\text{surj}_{\geq 1}}, I = I_{1} \sqcup \cdots \sqcup I_{k}, \mu \in X_{v}(T)^{+}, j \geq j_{\mu} \). We define

\[ \text{Hck}_{I_1,\ldots,I_k,\leq \mu(j)} = G_{O,I,(k_j)} \backslash \text{Conv}_{I_1,\ldots,I_k,\leq \mu} \]

where the value of the action of

\[ (x_{I}, g) \in G_{O,I,(k_j)} \]

on

\[ (T_{1} \xleftarrow{\alpha} X_{v_1}, T_{2} \xleftarrow{\eta_2} X_{v_2}, \ldots, T_{k} \xleftarrow{\eta_k} X_{v_k}) \in \text{Conv}_{I_1,\ldots,I_k,\leq \mu} \]

is defined by modifying \( \alpha \) by \( g \) at every point (we stress that here \( \alpha \) must be understood as a trivialization over the whole \( X_{v_1} \), and not just over the punctured formal neighbourhood).
Remark 2.34. Let us inspect the behaviour of this action on different strata. For simplicity, we look at \( k = 2, I_1 = I_2 = \{1\} \), and we distinguish the two cases of equal points \( x_1 = x_2 = x \) and of two distinct points \( x, y \). In the first case, the action is just the action of \( G_{0,2(j)} \) on the first component of \( G_{X, \leq \mu, (j)} \times G_{0} \text{ Gr}_{x, \leq \mu} \). In the second case, the modification of \( \alpha \) at \( y \) propagates to \( \eta \) through factorization: more precisely, up to the isomorphism \( \text{Conv}_{x,y} \simeq \text{Gr}_{x} \times \text{Gr}_{y} \) induced by Proposition 2.25 the action splits as the canonical componentwise left action

\[
G_{0, X,(j)} \times G_{0, Y,(j)} \circ \text{Gr}_{x, \leq \mu} \times \text{Gr}_{y, \leq \mu}.
\]

This follows directly from inspecting the proof of Proposition 2.25.

Definition 2.35. We also define

\[
H_{ck_{I_1, \ldots, I_k}} := \colim_{\mu \in X(T)} \text{“ lim } \text{” } H_{ck_{I_1, \ldots, I_k, \leq \mu(j)}}
\]

where the quotient is by the action on the “\( x_{I_1}, \mathcal{J}_1, \alpha \)” part of the datum (in analogy to Definition A.10).

Remark 2.36. There is a version

\[
H_{ck_{\leftarrow I_1, \ldots, I_k}} := \colim_{\mu \in X(T)} \text{ lim } H_{ck_{I_1, \ldots, I_k, \leq \mu(j)}}
\]

where the inner limit is realized (and not taken as an abstract pro-object). This object parametrizes the groupoid having as class of objects

\[
\{(x_{I_1}, \ldots, x_{I_k}), \mathcal{F}_0, \ldots, \mathcal{F}_k, \eta_i : \mathcal{F}_j |_{X \setminus x_i} \simeq \mathcal{F}_{j-1} |_{X \setminus x_i}, i = 1, \ldots, k\}
\]

and as automorphisms automorphisms of \( \mathcal{F}_0 \) over \( \hat{X}_{x_{I_1}} \).

Remark 2.37. We can provide a “moduli interpretation” for \( H_{ck_{\leftarrow I_1, \ldots, I_k}} \) as well. It parametrizes the groupoid having as class of objects

\[
\{[\phi : I_1 \to J_1], x_{I_1}, \ldots, [\phi_k : I_k \to J_k], x_{I_k}, \mathcal{F}_0, \ldots, \mathcal{F}_k, \eta_b : \mathcal{F}_b |_{X \setminus x_{I_b}} \simeq \mathcal{F}_{b-1} |_{X \setminus x_{I_b}}, b = 1, \ldots, k \mid \sum_{i \in I_b} \text{Inv}_{x_i}(\eta_b) \leq \mu \forall b = 1, \ldots, k\}
\]

and as automorphisms automorphisms of \( \mathcal{F}_0 \) over \( (\Gamma_{x_{I_i}})_{(j)} \).

Warning 2.38. For simplicity, in some parts of the paper we will give some constructions by treating the pro-stack \( H_{ck_{\leftarrow I_1, \ldots, I_k}} \) as if it were the “actual realization” described in Remark 2.36. We will contextually remark that a similar construction holds for the pro-version. The only instance where the presentation as pro-stack really makes a difference is in Corollary 3.18, Proposition 3.19, Construction 4.1, where we will carefully avoid such confusion.
**Definition 2.39.** Let $I \in \text{Fin}_{\text{surj}}$, $\mu \in \mathcal{X}_*(T)^+$, $j \geq j_\mu$. We define

$$H_{\text{ck}}^\Delta_{\leq \mu(j)}$$

as the quotient of $\text{Conv}^\Delta_{\leq \mu}$ by the action of $G_0 X_{i(j)}$ (relative to $X$) which modifies the trivialization of the first torsor. We also define, for $x \in X$,

$$H_{\text{ck}}_{x, i, k, \leq \mu(j)} = H_{\text{ck}}^\Delta_{\leq \mu(j)} \times_X \{x\}.$$ 

These objects are organized into filtered pro-stratified stacks

$$H_{\text{ck}}^{\Delta, k} = \text{colim}_{\mu \in \mathcal{X}_*(T)^+} \lim_{j \geq j_\mu} H_{\text{ck}}^{\Delta, k, \leq \mu(j)} \in \text{StrStk}_{/C}.$$ 

**Remark 2.40.** As for the case of $H_{\text{ck}} X$, all these objects naturally inherit a stratification from the respective versions of the convolution Grassmannian.

The results regarding the convolution Grassmannian imply the following list of properties:

**Proposition 2.41.** *In the notation of Proposition 2.25, we have equivalences*

$$H_{\text{ck}} |_{X^\phi} \simeq \prod_{j \in J} H_{\text{ck}}^{\Delta, M_j} \times_{X^{\phi_{i(j)}}} X^\phi.$$ 

*Finally, if $X = \mathbb{A}^1_C$ we have*

$$H_{\text{ck}}^{\Delta, k} \simeq H_{\text{ck}}_{x, k} \times \mathbb{A}^1_C$$

$$H_{\text{ck}} |_{X^\phi} \simeq (\prod_{j \in J} H_{\text{ck}}^{x, M_j}) \times X^\phi.$$ 

### 2.3 The BD-convolution diagram as a 2-Segal object

**Remark 2.42.** The convolution diagram from Remark A.13 admits a version in the Beilinson-Drinfeld setting, namely the diagram

$$H_{\text{ck}} I_i \times H_{\text{ck}} I_j \xrightarrow{\overline{p}_{I_i, I_j}} H_{\text{ck}} I_i I_j \xleftarrow{\overline{p}_{I_i, I_j}} H_{\text{ck}} I_i \cup I_j$$

where the map $\overline{p}_{I_i, I_j}$ sends

$$(x_{I_i}, x_{I_j}, F_0, F_1, F_2, \eta_1 : F_2 |_{X \setminus x_{I_i}} \simeq F_0 |_{X \setminus x_{I_i}}, \eta : F_2 |_{X \setminus x_{I_j}} \simeq F_1 |_{X \setminus x_{I_j}})$$

to

$$( (x_{I_i}, F_0, F_1, \eta_1), (x_{I_j}, F_1, F_2, \eta_2) )$$
and the map \( \overline{m}_{I_i, I_2} \) sends the same datum to
\[
(x_{I_1} + x_{I_2}, \mathcal{T}_0, \mathcal{T}_2, \eta_1 \chi(x_{I_1} + x_{I_2}) \circ \eta_2 \chi(x_{I_1} + x_{I_2})),
\]
x_{I_1} + x_{I_2} being the sequence in \( X^{I_1 + I_2} \) given by the points of \( x_{I_1} \) followed by the points of \( x_{I_2} \), with repetitions if necessary.

This diagram is the exact analogue of Remark A.13 in the Beilinson-Drinfeld setting.

**Remark 2.43.** Note that \( \overline{p}_{I_i, I_2} \) can be described as follows. Let
\[
p_{I_i, I_2} : G_{\mathcal{X}(T)} \times X^{I_2} \rightarrow Gr_{I_1} \times Gr_{I_2}
\]
given by the right quotient by \( G_{0, I_2} \) in the first component. Then \( \overline{p}_{I_i, I_2} \) is the map obtained by passing to the left quotient by \( G_{0, I_1} \times G_{0, I_2} \) on both sides of \( p_{I_i, I_2} \).

**Remark 2.44.** Moreover, if we fix \( \mu \in \mathcal{X}(T)^+ \), \( j \geq j_{\mu} \), there is an “approximated” version
\[
\begin{array}{ccc}
\text{Hck}_{I_1, I_2 \leq \mu_{(j)}} & \xrightarrow{\overline{p}_{I_i, I_2 \leq \mu_{(j)}}} & \text{Hck}_{I_1, I_2 \leq \mu_{(j)}} \\
\text{Hck}_{I_1 \leq \mu_1, (j)} \times \text{Hck}_{I_2 \leq \mu_{(j)}} & & \text{Hck}_{I_1 \leq \mu_1, (j)} \times \text{Hck}_{I_2 \leq \mu_{(j)}} \\
\text{Hck}_{I_1 \leq \mu_{(j)}} & \langle & \text{Hck}_{I_2 \leq \mu_{(j)}}
\end{array}
\]
where the only difference in the definition of the maps is that now all quotients are with respect to the actions of the relevant \( (j) \)-Weil restrictions of \( G_{0, I_1}, G_{0, I_2}, G_{I_1, I_2} \).

**Remark 2.45.** Note that \( \overline{p}_{I_i, I_2 \leq \mu_{(j)}} \) can be described as follows. Recall Definition 2.17 and define
\[
p_{I_i, I_2 \leq \mu_{(j)}} : G_{\mathcal{X}(T)} \times X^{I_2} \rightarrow Gr_{I_1} \times Gr_{I_2}
\]
given by the right quotient by \( G_{0, I_2(j)} \) in the first component. Then \( \overline{p}_{I_i, I_2} \) is the map obtained by passing to the left quotient by \( G_{0, I_1(j)} \times G_{0, I_2(j)} \) on both sides of \( p_{I_i, I_2} \). The fact that performing the quotient on the source yields \( \text{Hck}_{I_1, I_2 \leq \mu_{(j)}} \) is not immediate from the definition, because in principle \( G_{\mathcal{X}(T)} \times X^{I_2} \leq \mu_{(j)} \) should be replaced by \( G_{\mathcal{X}(T), I_2 \leq \mu} \) and the antidiagonal action of \( G_{0, I_2(j)} \) by that of \( G_{0, I_2} \). But indeed, just like in Remark 2.30, we have that
\[
G_{\mathcal{X}(T)} \times G_{0, I_2} \simeq G_{\mathcal{X}(T), I_2 \leq \mu} \simeq \text{Conv}_{I_1, I_2 \leq \mu}
\]
from which the desired result immediately follows.

**Definition 2.46.** The Ran presheaf of \( X \) is the colimit
\[
\text{Ran}(X) = \text{colim}_{I \in \text{Fin}_{\text{surj}}^{\text{op}}} X^I
\]
in the category \( \text{Fun}(\text{Aff}_{\mathcal{C}}^{\text{op}}, \text{Set}) \), where \( \text{Fin}_{\text{surj}, I \geq 1} \) is the category of nonempty finite sets, with surjective maps between them, and the diagram is the one that associates to a map \( I \rightarrow I \) the induced diagonal map \( X^I \rightarrow X^I \).

\(^2\)Note that \( G_{\mathcal{X}(T), I_2 \leq \mu} \simeq G_{I_1} \times X^{I_2} \).
Remark 2.47. The functor of points of \( \text{Ran}(X) \) can be described as
\[
\text{Ran}(X)(\text{Spec } R) = \{ S \subset X(R) \text{ nonempty unordered finite subset} \}.
\]

Definition 2.48. We define objects in \( \text{Stk}_C \)
\[
\begin{align*}
G_{O, \text{Ran}} &= \text{colim}_{I \in \text{Fin}_{\text{surj}}^{\text{op}}} G_{O, X^I} \\
G_{\text{Ran}} &= \text{colim}_{I \in \text{Fin}_{\text{surj}}^{\text{op}}} G_{X^I} \\
\text{Conv}_{\text{Ran}, k} &= \text{colim}_{I_1, \ldots, I_k \in \text{Fin}_{\text{surj}}^{\text{op}}} \text{Conv}_{I_1, \ldots, I_k} \\
\text{Hck}_{\text{Ran}, k} &= \text{colim}_{I_1, \ldots, I_k \in \text{Fin}_{\text{surj}}^{\text{op}}} \text{Hck}_{I_1, \ldots, I_k} \\
\text{Hck}_{\rightarrow \text{Ran}, k} &= \text{colim}_{I_1, \ldots, I_k \in \text{Fin}_{\text{surj}}^{\text{op}}} \text{Hck}_{I_1, \ldots, I_k}
\end{align*}
\]
where the colimits are taking along the diagrams consisting of the natural “diagonal” maps associated to surjections \( I \rightarrow f \).

The formation of these colimits loses any kind of descent\(^3\).

Remark 2.49. Let \( k \geq 1 \). We can describe the functors of points of the above objects as:
\[
\begin{align*}
\text{Gr}_{\text{Ran}} &= \{ S \subset X \text{ finite and nonempty }, \ g : \mathcal{T}|_{\widehat{X}_S} \simeq \mathcal{T}|_{\widehat{X}_S} \} \\
\text{Gr}_{\text{Ran}} &= \{ S \subset X \text{ finite and nonempty }, \ F \in \text{Bun}_G(X), \ \alpha : \mathcal{F}|_{X\setminus S} \simeq \mathcal{F}|_{X\setminus S} \} \\
\text{Conv}_{\text{Ran}, k} &= \{ S_1, \ldots, S_k \subset X \text{ finite and nonempty }, \mathcal{F}_1, \ldots, \mathcal{F}_k \in \text{Bun}_G(X), \ \\
&\quad \alpha : \mathcal{F}_1|_{X\setminus S_1} \simeq \mathcal{F}_1|_{X\setminus S_1}, \eta_j : \mathcal{F}_j|_{X\setminus S_j} \simeq \mathcal{F}_{j-1}|_{X\setminus S_j}, j = 2, \ldots, k \} \\
\text{Hck}_{\rightarrow \text{Ran}, k} &= \{ S_1, \ldots, S_k \subset X \text{ finite and nonempty }, \mathcal{F}_0, \ldots, \mathcal{F}_k \in \text{Bun}_G(X), \ \\
&\quad \eta_j : \mathcal{F}_j|_{X\setminus S_j} \simeq \mathcal{F}_{j-1}|_{X\setminus S_j}, j = 1, \ldots, k \}.
\end{align*}
\]
The last one is the only one with nontrivial automorphisms, given by automorphisms of the first \( G \)-torsor over \( \widehat{X}_{S_1} \). An usual, an analogue description holds for \( \text{Hck}_{\rightarrow \text{Ran}, k} \), only the automorphisms being different (i.e. defined over the \( j \)-th thickening of the relevant graph).

Remark 2.50. The stratifications on each of the diagrams of which we take the colimit in Definition 2.48 pass to the colimit formally (in the sense that, by definition, a stratified prestack is a colimit of stratified representable objects, etc., see Definition B.6). We will always implicitly see \( \text{Gr}_{\text{Ran}}, \text{Conv}_{\text{Ran}, k} \) and \( \text{Hck}_{\rightarrow \text{Ran}, k} \) as elements of \( \text{StrStk}_C \). For simplicity, we will sometimes give some constructions in the unstratified setting and then remark that they can be refined to the stratified setting.

\(^3\)See [GL, Warning 2.4.4]. There, \( \text{Ran}(X) \) is a more complex object, namely nontrivial automorphisms are taken into account. However, the same proof shows that our version does not satisfy étale descent.
Definition 2.51. We define \((\text{Ran}(X) \times \text{Ran}(X))_{\text{disj}}\) as the open subfunctor of \(\text{Ran}(X) \times \text{Ran}(X)\) parametrizing

\[
\{(S, T) \in \text{Ran}(X) \times \text{Ran}(X) \mid \Gamma_S \cap \Gamma_T = \emptyset\}.
\]

More generally, we define \(\text{Ran}(X)^{2k}_{\text{disj}} = \{S_1, \ldots, S_k, T_1, \ldots, T_k \in \text{Ran}(X) \mid \Gamma_{S_i} \cap \Gamma_{T_i} = \emptyset \forall i = 1, \ldots, k\} \subset \text{Ran}(X)^{2k}\).

Remark 2.52. Recall that we have natural maps from \(G_{\text{O, Ran}, k}, \text{Conv}_{\text{Ran}, k}\) and \(\text{Hck}_{\text{Ran}, k}\) to \(\text{Ran}(X)^k\) (the ones that only “remember” the systems of points).

Proposition 2.53. The maps

\[
(\text{Ran}(X) \times \text{Ran}(X))_{\text{disj}} \to \text{Ran}(X)
\]

\[
(S, T) \mapsto S \sqcup T
\]

and more generally

\[
\text{Ran}(X)^{2k}_{\text{disj}} \to \text{Ran}(X)^k
\]

\[
(S_1, \ldots, S_k, T_1, \ldots, T_k) \mapsto (S_1 \sqcup T_1, \ldots, S_k \sqcup T_k)
\]

induce isomorphisms

\[
\text{Gr}_{\text{Ran}} \times_{\text{Ran}(X)} (\text{Ran}(X) \times \text{Ran}(X))_{\text{disj}} \simeq (\text{Gr}_{\text{Ran}} \times \text{Gr}_{\text{Ran}}) \times_{\text{Ran}(X) \times \text{Ran}(X)} (\text{Ran}(X) \times \text{Ran}(X))_{\text{disj}},
\]

\[
G_{\text{O, Ran}, k} \times_{\text{Ran}(X)^k} \text{Ran}(X)^{2k}_{\text{disj}} \simeq (G_{\text{O, Ran}, k} \times G_{\text{O, Ran}, k}) \times_{\text{Ran}(X)^{2k}} \text{Ran}(X)^{2k}_{\text{disj}},
\]

\[
\text{Conv}_{\text{Ran}, k} \times_{\text{Ran}(X)^k} \text{Ran}(X)^{2k}_{\text{disj}} \simeq (\text{Conv}_{\text{Ran}, k} \times \text{Conv}_{\text{Ran}, k}) \times_{\text{Ran}(X)^{2k}} \text{Ran}(X)^{2k}_{\text{disj}},
\]

\[
\text{Hck}_{\text{Ran}, k} \times_{\text{Ran}(X)^k} \text{Ran}(X)^{2k}_{\text{disj}} \simeq (\text{Hck}_{\text{Ran}, k} \times \text{Hck}_{\text{Ran}, k}) \times_{\text{Ran}(X)^{2k}} \text{Ran}(X)^{2k}_{\text{disj}}.
\]

Proof. The proof follows from Proposition 2.8 and Proposition 2.25 (by decomposing and re-assemblying the suitable sets of indexes) and passing to colimits.

Notation 2.54. We will abbreviate the right-hand-sides of the formulas in Proposition 2.53 as

\[
(\text{Gr}_{\text{Ran}} \times \text{Gr}_{\text{Ran}})_{\text{disj}}
\]

\[
(G_{\text{O, Ran}, k} \times G_{\text{O, Ran}, k})_{\text{disj}}
\]

\[
(\text{Conv}_{\text{Ran}, k} \times \text{Conv}_{\text{Ran}, k})_{\text{disj}}
\]

\[
(\text{Hck}_{\text{Ran}, k} \times \text{Hck}_{\text{Ran}, k})_{\text{disj}}
\]

respectively.

We will now establish a semisimplicial structure on the collection of the \(\text{Hck}_{\text{Ran}, k}\), in order to prove associativity of the operation provided by Remark 2.42. We will later observe that it can be refined to one on \(\text{Hck}_{\text{Ran}, k}\).
Definition 2.55. We define $H^C_{k\leftarrow 0} = \text{Spec } \mathbb{C}$ (and $H^C_0 = \text{Spec } \mathbb{C}$ for later).

Definition 2.56. Let $k \geq 1$, $0 \leq i \leq k$, and $\partial_i$ be the face map from $[k-1]$ to $[k]$ omitting $i$. We define the corresponding map $\delta_i : H^C_{k\leftarrow \text{Ran},k} \rightarrow H^C_{k\leftarrow \text{Ran},k-1}$ as follows. A tuple $(S_1, \ldots, S_k, F_0, \ldots, F_k, \eta_1, \ldots, \eta_k)$ is sent to:

- the tuple $(S_2, \ldots, S_k, F_1, \ldots, F_k, \eta_2, \ldots, \eta_k)$, if $i = 0$;
- the tuple $(S_1, \ldots, S_{k-1}, F_0, \ldots, F_{k-1}, \eta_1, \ldots, \eta_{k-1})$, if $i = k$;
- the tuple $(S_1, \ldots, S_i \cup S_{i+1}, \ldots, S_k, F_0, \ldots, F_{i-1}, F_{i+1}, \ldots, F_k, \eta_1, \ldots, \eta_i \circ \eta_{i+1}, \ldots, \eta_k)$, otherwise.

In particular, all face maps $[0] \rightarrow [k]$ are sent to the constant map $H^C_{k\leftarrow \text{Ran},k} \rightarrow \text{Spec } \mathbb{C}$.

Remark 2.57. For $k = 2$, and taking the fiber over any point $x \in X(\mathbb{C})$, the three maps induce exactly the diagram described in Section A.2 ($\overline{\varphi} \simeq \delta_2 \times \delta_0$ and $\overline{m} \simeq \delta_1$), thus ensuring the consistency of our definition with what happens over the point.

Proposition 2.58. This construction defines a semisimplicial object

$$H^C_{k\leftarrow \text{Ran},k} : \Delta^{op} \rightarrow \hat{\text{Stk}}_{\mathbb{C}}$$

because the given maps satisfy the simplicial identities.

Proof. We prove the case $0 < j - 1 < j < k$ and leave the others to the reader:

$$\delta_j \circ \delta_{j-1} = \delta_{j-1} \circ \delta_{j-1}.$$

Indeed,

$$\delta_j \circ \delta_{j-1}(S_1, \ldots, S_k, F_0, \ldots, F_k, \eta_1, \ldots, \eta_k) =$$

$$\delta_{j-1}(S_1, \ldots, S_j \cup S_{j+1}, \ldots, S_k, F_0, \ldots, F_{j-1}, F_{j+1}, \ldots, F_k, \eta_1, \ldots, \eta_j \circ \eta_{j+1}, \ldots, \eta_k) =$$

$$(S_1, \ldots, S_j \cup S_{j+1}, \ldots, S_k, F_0, \ldots, F_j, \ldots, F_k, \eta_1, \ldots, \eta_j \circ \eta_{j+1}, \ldots, \eta_k),$$

whereas

$$\delta_{j-1} \circ \delta_{j-1}(S_1, \ldots, S_k, F_0, \ldots, F_k, \eta_1, \ldots, \eta_k) =$$
\[ \delta_{j-1}(S_1, \ldots, S_{j-1} \cup S_j, \ldots, S_k, \mathcal{T}_0, \ldots, \mathcal{T}_{j-2}, \mathcal{T}_j, \ldots, \mathcal{T}_k, \eta_1, \ldots, \eta_{j-1} \circ \eta_j, \ldots, \eta_k) = (S_1, \ldots, S_{j-1} \cup S_j, \ldots, S_k, \mathcal{T}_0, \ldots, \mathcal{T}_{j-2}, \mathcal{T}_{j+1}, \ldots, \mathcal{T}_k, \eta_1, \ldots, \eta_{j-1} \circ \eta_j \circ \eta_{j+1}, \ldots, \eta_k). \]

The crucial property of this structure, in order to encode the associativity of the convolution product, is the following:

**Proposition 2.59.** For any \( \text{Spec} \, R \in \text{Aff}_C \), the semisimplicial set \( \text{Hck}_{\text{Ran}, \ast}(\text{Spec} \, R) \) enjoys the 2-Segal property, that is the equivalent conditions of [DK19, Proposition 2.3.2].

**Proof.** We do the “basic” case \( n = 3, i = 0, j = 2 \). We adopt again Notation 2.24. For the first case, we have to prove that the map of groupoids

\[ \Phi: \{ \mathcal{T}_0 \leftarrow \mathcal{T}_1 \leftarrow \mathcal{T}_2 \leftarrow \mathcal{T}_3 \} \to \{ G_0 \leftarrow G_1 \leftarrow G_2 \leftarrow G_3, \; \phi_0 : G_0 \simeq G_0', \phi_1 : G_2 \simeq G_2' \} \]

given by

\[ \mathcal{T}_0 \leftarrow \mathcal{T}_1 \leftarrow \mathcal{T}_2 \leftarrow \mathcal{T}_3 \]

\[ \to ( \mathcal{T}_0 \leftarrow \mathcal{T}_1 \leftarrow \mathcal{T}_2 \leftarrow \mathcal{T}_3, \; \mathcal{T}_0 \simeq \mathcal{T}_0', \text{id} : \mathcal{T}_2 \simeq \mathcal{T}_2' ) \]

is an equivalence.

The pseudoinverse is given by

\[ \Psi: \{ G_0 \leftarrow G_1 \leftarrow G_2 \leftarrow G_3, \; \phi_0 : G_0 \simeq G_0', \phi_1 : G_2 \simeq G_2' \} \]

\[ \to ( G_0 \leftarrow G_1 \leftarrow G_2 \leftarrow G_3, \; \text{id} : G_0 \simeq G_0', \phi_1 : G_2 \simeq G_2' ) \]

The fact that \( \Psi \circ \Phi \simeq \text{id} \) is straightforward, while the equivalence \( \Phi \circ \Psi \simeq \text{id} \) is built out of the initial data \( \phi_0, \phi_1 \) (i.e. they themselves serve as part of the desired isomorphism inside the target groupoid, when suitably restricted to formal neighbourhoods). \( \Box \)

**Theorem 2.60.** Let \( G \) be a complex reductive group and \( X \) a complex smooth curve. The object

\[ \text{Hck}_{\text{Ran}} \in \text{Stk}_C \]
parametrizing
\[ \{ S \subset X \text{ finite nonempty subset}, \mathcal{F}_0, \mathcal{F}_1 \in \text{Bun}_G(X), \eta : \mathcal{F}_0|_{X \setminus S} \sim \mathcal{F}_1|_{X \setminus S} \} \]
carries a nonunital associative algebra structure in \( \text{Corr}(\text{Stk}_C) \times \), extending the convolution diagram (2.42).

The fiber of this algebra structure at any singleton \( \{ x \} \in \text{Ran}(X) \) encodes the convolution diagram for the Hecke stack (Remark A.13) and its associativity.

**Proof.** It suffices to apply (a nonunital version of) [DK19, Theorem 11.1.6] to the 2-Segal object \( \text{Hck}_{\text{Ran}, \bullet} \) (Definition 2.56, Proposition 2.59). 

**Remark 2.61.** The maps \( \delta_i \) defined in Definition 2.56 can be presented as colimits of maps \( \delta_{I_1, \ldots, I_k} : \text{Hck}_{I_1, \ldots, I_k} \rightarrow \text{Hck}_{I_1, \ldots, I_{i-1}, I_i \uplus I_{i+1}, I_{i+2}, \ldots, I_k} \). One can see that these \( \delta_{I_1, \ldots, I_k} \)'s agree with the stratifications from Remark 2.32, in the following sense.

Let \( I_1, \ldots, I_k \in \text{Fin}_{\text{surj}}, \geq 1 \). Let also \( \partial_i : [k-1] \rightarrow [k] \) the ordered map missing the index \( i \).

There is an induced map of posets (with the notations of Remark 2.32)
\[
\text{Cw}_i : \text{Cw}_{I_1, \ldots, I_k} \rightarrow \text{Cw}_{I_1, \ldots, I_{i-1}, I_i \uplus I_{i+1}, \ldots, I_k}
\]
which, in the case \( i \neq 0, k \) sends
\[
([\phi : [k] \rightarrow J] \text{ partition }, (\mu_1^j, \ldots, \mu_m^j) \in (\mathcal{X}_*(T)^+)_m, j \in J)
\]
to the following: fix \( j \in J \); if \( \phi^{-1}(j) \) intersects nontrivially both \( I_i \) and \( I_{i+1} \), then the “new” \( m_j \) for \( j \) is one less than the original one, and we choose the sequence of coweights
\[
(\mu_1^j, \ldots, \mu_i^j + \mu_{i+1}^j, \ldots, \mu_m^j).
\]
Otherwise the \( m_j \) remains the same and we keep the original sequence intact. In the “extremal” cases \( i = 0, i = k \) the map forgets data instead of summing them.

We now show that the map \( \delta_i \) from Definition 2.56 makes the diagram
\[
\begin{array}{ccc}
\text{Hck} & \xrightarrow{\delta_i} & \text{Hck} \\
\downarrow & & \downarrow \\
\text{Cw}_{I_1, \ldots, I_k} & \xrightarrow{\text{Cw}} & \text{Cw}_{I_1, \ldots, I_i \uplus I_{i+1}, \ldots, I_k}
\end{array}
\]
commute in \( \text{Stk}_C \). This can be reduced to the following statement: the map
\[
\text{Hck} \xrightarrow{x, m_j} \text{Hck} \xleftarrow{x, m_j-1}
\]
which multiplies the data in the places \( i, i+1, i \leq m_j \), sums coweights accordingly. This is implied by [MV07, Lemma 4.4].
The previous proof can be combined with Remark 2.44 to prove that not just the $\text{Hck}_{\text{Ran}}$’s, but also the $\text{Hck}_{\text{Ran},k}$’s can be organized into a semisimplicial object

$$\text{Hck}_{\text{Ran},\bullet} : \Delta^{\text{op}} \to \text{StrStk}_C.$$  

The proof that this semisimplicial object satisfies the 2-Segal property is analogous: the only additional care must be taken in choosing the appropriate $\mu$’s when checking the property, and in dealing with automorphisms (which differ between the “realized” version and the “truncated” version).

With the same proof of Theorem 2.60, we thus have the main result of this Section.

**Corollary 2.62.** Let $G$ be a complex reductive group and $X$ a complex smooth curve. The object

$$\text{Hck}_{\text{Ran},1} \in \text{StrStk}_C$$

is the underlying object of an algebra structure $\mathbb{E}_1^{\text{nu}} \to \text{Corr}(\text{StrStk}_C)^{\times}$, extending the convolution diagram (2.44).

## 3 Topological factorization for the Hecke stack

### 3.1 The factorizing cosheaf

**Theorem 3.1** ([Rey71, Théorème et définition 1.1]). Let $X$ be a scheme locally of finite type over $\mathbb{C}$. Then there exists an associated complex-analytic space $X^{\text{an}}$, whose underlying set is $X(\mathbb{C})$ and which represents the functor

$$\{\text{complex-analytic spaces}\} \to \text{Set}$$

$$Y \mapsto \text{Hom}_{\text{ringed spaces}}(Y, X).$$

Since we will never need to remember the sheaf of holomorphic functions of $X^{\text{an}}$, we will abuse notation and denote by $X^{\text{an}}$ the underlying topological space of the complex-analytic space associated to $X$.

The functor

$$(-)^{\text{an}} : \text{Sch}_{\text{lf}}^{\text{C}} \to \text{Top}$$

that we have just described preserves finite limits ([Rey71, 1.2]). It can be extended to a finite-limit-preserving functor $(-)^{\text{an}} : \text{Stk}_{\text{lf}}^{\text{C}} \to \text{TStk}$, and admits a stratified version, see Construction B.4.

**Definition 3.2.** Let $M$ be a topological manifold of dimension $m$. The **Ran space** of $M$ is the set of nonempty finite subsets of $M$, endowed with its so-called **metric topology**, i.e. the topology induced by the following base:

$$\left\{ \bigcap_{i=1}^k \text{Ran}(D_i) \mid \{D_i\} \text{ finite family of pairwise disjoint disks in } M \right\}.$$
Note that the given family is actually a family of subsets of \( \text{Ran}(M) \), in the sense that the union map \( \prod_{i=1}^{k} \text{Ran}(D_i) \rightarrow \text{Ran}(M) \) is injective whenever the disks are pairwise disjoint. We denote by \( \star_i \text{Ran}(D_i) \) the image of this map.

**Remark 3.3.** The set \( \text{Ran}(M) \) can be equivalently presented as the \( \text{colim}_{I \in \text{Fin}_{\text{surj}}} M^I \) in \( \text{Set} \). This carries a natural colimit topology, which is finer than the one presented above. For a thorough comparison between those, and many further considerations, see [CL21] and the very recent [DL23]. When unspecified, by \( \text{Ran}(M) \) we will always mean the Ran space with its metric topology.

One main feature of the Ran space is to encode the notion of factorization algebra in a particularly efficient way. Actually, a theorem by Jacob Lurie says even more, showing that a family over the Ran space with some conditions induces a “sheaf of \( \mathbb{E}_m \)-algebras” over \( M \) (\( m \) being the dimension of \( M \)).

**Theorem 3.4** (part of [Lur17, Theorem 5.5.4.10]). Let \( M \) be a topological manifold of dimension \( m \), and \( \mathcal{C}^\otimes \) a symmetric monoidal \( \infty \)-category where the tensor product preserves colimits separately in both variables. Any constructible factorizable cosheaf over \( \text{Ran}(M) \) with values in \( \mathcal{C} \) induces a nonunital \( \mathbb{E}_m \)-algebra structure on its stalk at any singleton \( \{x\} \in \text{Ran}(M) \).

**Recall 3.5.** We refer to [Lur17, Section 5.5.4] for the definitions. Here we just recall that:

- two open subsets \( U, V \) of \( \text{Ran}(M) \) are declared to be **independent** if for every \( S \in U, T \in V \), then \( S \cap T = \emptyset \). For example, for any two open subsets \( D, D' \) of \( M \), \( \text{Ran}(D) \) and \( \text{Ran}(D') \) are independent if and only if \( D \cap D' = \emptyset \). For \( U, V \) independent open subsets of \( \text{Ran}(M) \), one denotes \( U \star V = \{ S \cup T | S \in U, T \in V \} \). Again, this is homeomorphic to \( U \times V \).

- there is an operadic structure \( \text{Fact}(M)^\otimes \) over the category of open subsets of \( \text{Ran}(M) \), encoding the “partial operation” \( U \star V \) for independent \( U, V \).

- a **factorizable cosheaf** over \( \text{Ran}(M) \) is a map of operads

\[
\mathcal{A}^\otimes : \text{Fact}(M)^\otimes \rightarrow \mathcal{C}^\otimes
\]

satisfying the following conditions:

- the underlying functor \( A : \text{Open}(\text{Ran}(M)) \rightarrow \mathcal{C} \) is a cosheaf;
- for any two independent open subsets \( U, V \) of \( \text{Ran}(M) \), the map \( A(U)^\otimes A(V) \rightarrow A(U \star V) \) induced by the fact that \( A \) is a map of operads is an equivalence in \( \mathcal{C} \).

- such a functor is **constructible** if, as a cosheaf over \( \text{Ran}(M) \) with values in \( \mathcal{C} \), it is hypercomplete and sends the map

\[
\text{Ran}(D) \subset \text{Ran}(D')
\]

induced by an inclusion of open disks \( D \subset D' \) in \( M \) to an equivalence of objects in \( \mathcal{C} \) ([Lur17, Proposition 5.5.1.14]).
As observed above, there is a functor

\[ (-)^{an} : \text{StrStk}^{\text{ft}}_C \to \text{TStk} \]

preserving colimits and finite limits. We note that \( \text{Hck}_{\text{Ran}, \bullet} \) takes values in \( \text{StrStk}^{\text{ft}}_C \subset \text{StrStk}_C \), and therefore by applying the analytification functor we obtain a semisimplicial object \( \text{Hck}_{\text{Ran}, \bullet}^{an} \), which is again 2-Segal by finite-limit-preservation of \( (-)^{an} \). This object comes with a natural map towards \( \text{Ran}(X)^{an} \) with the colimit topology (by functoriality) and therefore towards \( \text{Ran}(X)^{an} \) with the metric topology described in Definition 3.2. For instance, if we consider \( \text{Hck}_{\text{Ran}, 1}^{an} \), this is given by

\[ \text{colim}_{I \in \text{Fin}_{\text{surj}}, \geq 1} \text{colim}_{\mu \in \text{X}_{\text{bbb}}} \text{lim}_{j \geq j_{\mu}} G_{0, I, \leq \mu, (j)}^{an} \text{Gr}_{G, I, \leq \mu} \]

Note that the single terms appearing in this formula are quotients, understood as stratified topological stacks (Remark B.5).

From now on, when proving a property for \( \text{Hck}_{\text{Ran}, k}^{an} \) we will always argue as “we prove the property for each term and the property is stable under the relevant colimits”.

**Remark 3.6.** By functoriality of analytification, each \( G_{0, I}^{an}, \text{Conv}_{\text{Ran}, k}^{an}, \text{Hck}_{\text{Ran}, k}^{an} \) admits a map towards \( (\text{Ran}(X)^{an})^k \) (in the category \( \text{StrTStk} \)). This latter admits a map towards \( \text{Ran}(X)^{an} \), induced by the fact that each \( (X^I)^{an} \) maps into \( \text{Ran}(X)^{an} \) by Remark 3.3. We stress that this latter is understood as a plain topological space with its metric topology. In practice, we are just saying that each \( G_{0, I}^{an}(\text{Conv}_{I_1, \ldots, I_k, \leq \mu})^{an} \) (with the notation of Proposition 2.25) naturally lives over \( \text{Ran}(X)^{an} \).

**Definition 3.7.** For any \( U \subset \text{Ran}(X)^{an} \) open subset, and \( k \geq 1 \), let

\[ \text{Conv}_{U, k} = \text{Conv}_{\text{Ran}, k}^{an} \times_{\text{Ran}(X)^{an}} U^k \in \text{StrTStk} \]
\[ \text{Hck}_{U, k} = \text{Hck}_{\text{Ran}, k}^{an} \times_{\text{Ran}(X)^{an}} U^k \in \text{StrTStk}. \]

We define the functors

\[ \text{Conv}_{\bullet}^{\text{fact}} : \text{Open}(\text{Ran}(M)) \times \Delta_{\text{inj}}^{\text{op}} \to \text{StrTStk} \]
\[ (U, [k]) \mapsto \text{Conv}_{U, k} \]

and

\[ \text{Hck}_{\bullet}^{\text{fact}} : \text{Open}(\text{Ran}(M)) \times \Delta_{\text{inj}}^{\text{op}} \to \text{StrTStk} \]
\[ (U, [k]) \mapsto \text{Hck}_{U, k} \]

on objects, and in the natural way on morphisms.
Indeed, an inclusion $U \subset V$ naturally induces embeddings

$$\text{Conv}_{U,k} \hookrightarrow \text{Conv}_{V,k}$$

$$\text{Hck}_{U,k} \hookrightarrow \text{Hck}_{V,k}$$

which are clearly functorial in $k$.

**Remark 3.8.** Note that by definition

$$\text{Conv}_{U,k} \cong \text{colim}_{I_1,\ldots, I_k} \text{Conv}_{I_1,\ldots, I_k} \times \text{Ran}(X^\text{an})^U.$$  

resp.

$$\text{Hck}_{U,k} \cong \text{colim}_{I_1,\ldots, I_k} \text{Hck}_{I_1,\ldots, I_k} \times \text{Ran}(X^\text{an})^U.$$  

We denote the terms appearing in the colimit by $\text{Conv}_{U,I_1,\ldots, I_k} \cong \text{colim}_{I_1,\ldots, I_k} \text{Conv}_{I_1,\ldots, I_k} \times \text{Ran}(X^\text{an})^U.$

Definition 3.7 is the first step in the direction of building a factorization algebra out of $\text{Hck}_{\text{Ran}, \cdot}^\text{an}$. To complete it, we need to upgrade $\text{Hck}_{\text{Ran}, \cdot}^\text{fact}$ to a functor

$$\text{Fact}(M)^\otimes \times \Delta^{\text{op}}_{\text{inj}} \rightarrow \widehat{\text{StrTStk}}$$

suitably encoding the “factorization property” of the Beilinson-Drinfeld Grassmannian.

**Notation 3.9.** Let $M$ be a topological manifold. In analogy to Definition 2.51 we denote by

$$(\text{Ran}(M)^k \times \text{Ran}(M)^k)_{\text{disj}}$$

the topological space

$$\{S_1,\ldots, S_k, T_1,\ldots, T_k \mid S_i \cap T_i = \emptyset, i = 1,\ldots, k\} \subset \text{Ran}(M)^k \times \text{Ran}(M)^k.$$

**Remark 3.10.** Consider two independent open subsets $U$ and $V$ of $\text{Ran}(X^\text{an})$. We have the following diagram

$$\begin{array}{ccc}
\text{Hck}_{U,k} \times \text{Hck}_{V,k} & \longrightarrow & (\text{Hck}_{\text{Ran},k} \times \text{Hck}_{\text{Ran},k})^\text{an}_{\text{disj}} \longrightarrow & \text{Hck}_{\text{Ran},k} \\
\downarrow \pi & & & & \downarrow \\
U^k \times V^k & \subseteq & (\text{Ran}(X^\text{an})^k \times \text{Ran}(X^\text{an})^k)_{\text{disj}} \longrightarrow & \text{Ran}(X^\text{an})^k,
\end{array}$$  

(3.2)
where the left hand square is a by definition a pullback in $\text{StrTStk}$, and the right top horizontal map is induced by Proposition 2.53 by applying $(-)^{an}$ (which preserves finite limits). Note also that the bottom composition coincides with $U \times V \to U \star V \hookrightarrow \text{Ran}(X^{an})$, the first map being the one taking unions of systems of points; hence, by the universal property of the fibered product, $\text{Hck}_{U,k} \times \text{Hck}_{V,k}$ admits a map towards $\text{Hck}_{U \star V,k}$, which we call $\chi_{U,V,k}$.

**Proposition 3.11.** Let $k \in \mathbb{N}$. There is a well-defined map of operads

$$\text{Hck}^{\text{fact}}_k : \text{Fact}(M)^\otimes \to \text{StrTStk}$$

assigning

$$(U_1, \ldots, U_m) \mapsto \text{Hck}_{U_1,k} \times \cdots \times \text{Hck}_{U_m,k}.$$

$$(U \subset V) \mapsto \text{Hck}_{U,k} \hookrightarrow \text{Hck}_{V,k}$$

$$((U, V) \to (U \star V)) \mapsto \chi_{U,V,k} : \text{Hck}_{U,k} \times \text{Hck}_{V,k} \to \text{Hck}_{U \star V,k}.$$

**Proof.** Since the inclusions $U \hookrightarrow V$ of open sets in $\text{Ran}(X)$ induce inclusions $\text{Conv}_{U,k} \hookrightarrow \text{Conv}_{V,k}$ and do not alter the “datum” (they just “notice” that the systems of points live in a bigger set), it suffices to prove that the diagram

$$
\begin{array}{ccc}
\text{Conv}_{U \star V,k} \times \text{Conv}_{W,k} & \to & \text{Conv}_{U \star W,k} \\
\chi_{U,V,k} \times \text{id} & \downarrow & \chi_{U,V,W,k} \\
\text{Conv}_{U,k} \times \text{Conv}_{V,k} \times \text{Conv}_{W,k} & \to & \text{Conv}_{U \star V \star W,k} \\
\text{id} \times \chi_{V,W,k} & \downarrow & \chi_{U \star V \star W,k} \\
\text{Conv}_{U,k} \times \text{Conv}_{V,k} & \to & \text{Conv}_{U \star V,k}
\end{array}
$$

(notations as in Definition 3.7) commutes in $\text{StrTStk}$. Now this is true because the operation of gluing is associative, as it is easily checked by means of the defining property of the gluing of sheaves.

Finally, to prove that the sought functor $\text{Hck}^{\text{fact}}_k$ is a map of operads, we use the characterization of inert morphisms in a Cartesian structure provided by [Lur17, Proposition 2.4.1.5]. Note that:

- An inert morphism in $\text{Fact}(M)^\otimes$ is a morphism of the form

  $$(U_1, \ldots, U_m) \to (U_{\phi^{-1}(1)}, \ldots, U_{\phi^{-1}(n)})$$

  covering some inert arrow $\phi : \langle m \rangle \to \langle n \rangle$ where every $i \in \langle n \rangle^\circ$ has exactly one preimage $\phi^{-1}(i)$.

- An inert morphism in $\text{StrTStk}^\times$ is a morphism of functors $\Phi$ between $f : \mathcal{P}(\langle m \rangle^\circ)^\text{op} \to \text{StrTStk}$ and $g : \mathcal{P}(\langle n \rangle^\circ)^\text{op} \to \text{StrTStk}$, covering some $\alpha : \langle m \rangle \to \langle n \rangle$, and such that, for any $S \subset \langle n \rangle$, the map induced by $\Phi$ from $f(\alpha^{-1}S) \to g(S)$ is an equivalence in $\text{StrTStk}$.  

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By definition, $\text{Hck}^\text{fact}_k(U_1, \ldots, U_m)$ is the functor $f$ assigning
$$T \subset \langle m \rangle^\circ \to \prod_{j \in T} \text{Hck}_{U_j,k},$$
and analogously $\text{Hck}^\text{fact}_k(U_{\phi^{-1}(1)}, \ldots, U_{\phi^{-1}(m)})$ is the functor $g$ assigning
$$S \subset \langle n \rangle^\circ \to \prod_{i \in S} \text{Hck}_{U_{\phi^{-1}(i)},k}.$$
But now, if $\alpha = \phi$ and $T = \phi^{-1}(S)$, we have the desired equivalence.

Recall the notion of stratified homotopy equivalence from Definition B.28.

**Proposition 3.12.** Let $k \in \mathbb{N}$. The functor $\text{Hck}^\text{fact}_k$ from Proposition 3.11 satisfies the factorization property (see Recall 3.5) and sends maps of the form (3.1) to stratified homotopy equivalences in $\text{StrTStk}$. 

**Proof.** The application of the analytification functor (which, we recall, preserves finite limits) to the equivalence in Proposition 2.53, together with the consideration that the diagram

$$
\begin{array}{ccc}
(Ran(X)_{an} \times Ran(X)_{an})_{\text{disj}} & \longrightarrow & Ran(X)_{an} \\
\downarrow & & \downarrow \\
(Ran(X)_{an}) \times (Ran(X)_{an})_{\text{disj}} & \longrightarrow & Ran(X)_{an}
\end{array}
$$

(where in the first row we are considering the colimit topology and in the second row we are considering the metric topology) is Cartesian, achieves the factorization property by pulling back everything along $U^k \times V^k \to Ran(X)_{an}^{2k}_{\text{disj}}$.

As for the second property, by using factorization it suffices to prove that for any inclusion of disks $i : D' \subset D$ in $X_{an}$, the induced map

$$\text{Hck}_{\text{Ran}(D'),k} \to \text{Hck}_{\text{Ran}(D),k}$$

is a stratified homotopy equivalence in $\text{StrTStk}$. That is, we want to prove that for each $I_1, \ldots, I_k \in \text{Fin}_{\text{surj} \geq 1}$, $\mu \in X_{\ast}(T)^{\ast}$, the maps

$$\text{Conv}_{(D')^{I_1}, \ldots, (D')^{I_k}, \leq \mu}^\text{an} \to \text{Conv}_{D^{I_1}, \ldots, D^{I_k}, \leq \mu}^\text{an}
$$

$$G_{0,(D')^{I_1}} \to G_{0,D^{I_1}}$$

are resp. stratified homotopy equivalences and homotopy equivalences, and in a compatible way (i.e. the homotopy inverse of the first map is equivariant with respect to the second one, and one can choose homotopies which are compatible with respect to the actions on source and target). A sketch of the proof for the first one, in the case $k = 1$, is given in [HY19, Proposition 3.17]. A full proof of the complete statement will appear in [NP].

\[\square\]
3.1 The Factorizing Cosheaf

Remark 3.13. The constructions performed in the proof of Proposition 3.11 are compatible with the face maps of $\text{Hck}_{\text{Ran}, \bullet}^\text{an}$, since the square

\[
\begin{array}{ccc}
  (\text{Hck}_{\text{Ran}, k} \times \text{Hck}_{\text{Ran}, k})_{\text{disj}} & \longrightarrow & \text{Hck}_{\text{Ran}, k} \\
  \downarrow_{\delta_i \times \delta_i} & & \downarrow_{\delta_i} \\
  (\text{Hck}_{\text{Ran}, k-1} \times \text{Hck}_{\text{Ran}, k-1})_{\text{disj}} & \longrightarrow & \text{Hck}_{\text{Ran}, k-1}
\end{array}
\]

commutes.

Therefore:

Theorem 3.14. Proposition 3.11 induces a well-defined map of operads

\[\text{Hck}^{\text{fact}} : \text{Fact}(M)^\otimes \times \mathbb{E}^{\text{nu}}_1 \rightarrow \text{Corr}^{\times}(\text{Str}\text{TStk})\]

such that:

- in the first variable, it satisfies the factorization property in the sense of Recall 3.5, and sends maps of the form (3.1) to stratified homotopy equivalences;

- for every open $U \subseteq \text{Ran}(X^{\text{an}})$, the restriction to $\{U\} \times \mathbb{E}^{\text{nu}}_1$ yields the map of operads defined in Corollary 2.62, analytified and pulled back from $\text{Ran}(X^{\text{an}})$ to $U$.

Proof. Remark 3.13 yields a functor

\[\text{Fact}(M)^\otimes \times \Delta^{\text{op}}_{\text{inj}} \rightarrow \text{Str}\text{TStk}^{\times}\]

which is a map of operads in the first variable. This in turn induces a functor\(^4\)

\[\Delta^{\text{op}}_{\text{inj}} \rightarrow \text{Map}_{\text{Op}_\infty}(\text{Fact}(M)^\otimes, \text{Str}\text{TStk}^{\times})\]

and by arguing as in the proof of Corollary 2.62, we obtain a map of operads

\[\mathbb{E}^{\text{nu}}_1 \rightarrow \text{Corr}(\text{Map}_{\text{Op}_\infty}^{\times}(\text{Fact}(M)^\otimes, \text{Str}\text{TStk}^{\times}))^{\times}\]

Note that the target admits a map of operads to

\[\text{Map}_{\text{Op}_\infty}(\text{Fact}(M)^\otimes, \text{Corr}(\text{Str}\text{TStk}^{\times}))^{\times}\]

(it is easy to provide a map towards the category of functors, and then one can check that it actually takes values in the full subcategory spanned by maps of operads). Finally, this is the same as a functor

\[\text{Fact}(M)^\otimes \times \mathbb{E}^{\text{nu}}_1 \rightarrow \text{Corr}(\text{Str}\text{TStk})^{\times}\]

which is a map of operads separately in both variables. The verification that the claimed properties hold is straightforward by restricting to the two separate variables.\(\square\)

\(^4\)The notation $\text{Map}_{\text{Op}_\infty}$ follows [Lur17] and stays for “maps of $\infty$-operads”.

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3.2 Setup for taking constructible sheaves

In the next section we will take constructible sheaves over the geometric objects introduced up to now and prove that this induces the sought $E_3$-algebra structure on $\text{Sph}(G)$. The first step, carried out in the present subsection, will be to check that the mentioned geometric objects (and maps between them) indeed belong to the source category of Theorem B.39.

Remark 3.15. First of all, notice that for $k \geq 1, I_1, \ldots, I_k \in \text{Fin}_{\text{surjective}}$, $\mu \in X_+(T)_+^\ast$ fixed, the stratifying poset of $\text{Conv}_{I_1, \ldots, I_k, \leq \mu}$ is given, with the notations of Remark 2.32, by

$$
\left\{ I_1, [\phi : I_1 \sqcup \cdots \sqcup I_k \to f], (y_i^j)_{j \in I_i, i = 1, \ldots, m_i} \in \prod_{j \in I_i} (X_+(T)_+^\ast)^{m_i} \mid \sum_{j, \phi^{-1}(j) \notin I_i} v_i^j \leq \mu \forall b = 1, \ldots, k \right\}
$$

which is finite, the bounds being induced by $\mu$ and the cardinality of $I_1 \sqcup \cdots \sqcup I_k$.

Proposition 3.16. The stratified space $\text{Gr}^\ast_{I, \leq \mu}$ belongs to $\text{StrTop}_\text{con}$.

Proof. It suffices to prove that the stratification is Whitney. Indeed, this implies that the stratification is conical, and since strata are smooth manifolds and possess tubular neighbourhoods, we also obtain that the conical neighbourhoods of each point can be chosen to be contractible: hence, the space is local of singular shape. The existence of tubular neighbourhoods has been proven by Mather [Mat70]. Together with Marco Volpe, we provided an explicit reformulation in the language of conically stratified spaces [NV23, Construction 3.4].

The following proof has been suggested to us by David Nadler [Nad]. First of all, recall that the stratification of $\text{Gr}^\ast$ in Schubert cells is Whitney (Remark A.1). Note that all strata of $\text{Gr}^\ast_{I, \leq \mu}$ are smooth manifolds by the combination of [Zhu16, Proposition 2.1.5 (1)], Proposition 2.8 and Example B.31. Consider now two strata $W$ and $Y$. We want to prove that they satisfy Whitney’s condition B; that is:

- when sequences $(w_i) \subset W$ and $(y_i) \subset Y$ tend to $y$, the secant lines $w_i y_i$ tend to a line $v$, and $T_{w_i} W$ tends to some vector space $\tau$, then $v \subset \tau$ (Whitney’s Condition B for $W, Y, (w_i), (y_i), y$).

The only case of interest is when $\overline{W \cap Y}$ is nonempty. Observe that, when the limit point $y \in Y$ appearing in Whitney’s condition B is fixed, condition B is local in $Y$, i.e. we can restrict our stratum $Y$ to a neighbourhood $U$ of the projection of $y$ in $\text{Ran}(M)$. Also, $Y$ lives over some stratum $\text{Ran}_n(M)$ of $\text{Ran}(M)$. Using the factorization property, which splits components and tangent spaces, we can suppose that $n = 1$. That is, $Y$ lives over the “cardinality 1” component of $\text{Ran}(M)$, i.e. $M$ itself. By the locality of condition B explained before, we can suppose that $X = \mathbb{A}^1_{\mathbb{C}}$, i.e. $M = \mathbb{C}$. By translational invariance (Proposition 2.4), we can suppose our stratum $Y$ concentrated over a fixed point $0 \in \mathbb{A}^1$, that is: both $y$ and the $y_i$ can simultaneously be seen inside $\text{Gr}_0 \subset \text{Gr}_{\mathbb{A}^1}$, by means of linear transformations (which therefore do not change inclusions of limits of secant lines in tangent spaces).

Now, by [Kal05, Theorem 2] we know that there exists at least a point $y \in \overline{X \cap Y}$ such that, for
any \((w_i),(y_i)\) → \(y\) as in the hypothesis of Whitney’s Condition B, Condition B is satisfied. In other words, the space

\[
\text{Sing}(X, Y) = \{ y \in \overline{X} \cap Y \mid y \text{ does not satisfy Whitney’s Condition B for some choice of } (w_i),(y_i) \}
\]
does not coincide with the whole \(\overline{X} \cap Y\). Let \(\pi : \text{Gr}^\text{an}_{I_{\leq \mu}} \to \text{Ran}(M)\) be the natural map. Note that \(W\) and \(Y\) are acted upon by \(G_{0,\text{Ran}} \times_{\text{Ran}(M)} \pi(W)\) and \(G_{0,\text{Ran}} \times_{\text{Ran}(M)} \pi(Y) \cong G_0\) respectively (recall that \(Y\) is identified to a subset of \(\text{Gr} = \text{Gr}_0 \subset \text{Gr}_{\text{an}}\)), and this action is transitive on the fibers over any point of \(\text{Ran}(M)\). Now, these actions take Whitney-regular points with respect to \(Y\) to Whitney-regular points with respect to \(Y\), since they preserve all strata. Let now \(y'\) be any other point in \(Y\), and suppose given sequences \((w_i),(y_i)\) → \(y'\) which do not satisfy Whitney’s condition B. There is an element \(g \in G_0\) s.t. \(g.y' = y\). Let \(g.w_i\) be the element of \(W\) obtained by applying \(g\) to each component of \(w_i\) (unlike \(y\) and the \(y_i\)’s, the \(w_i\)’s may live over higher-cardinality-strata of \(\text{Ran}(M)\)). Now, the sequences \((g.w_i)\) and \((g.y_i)\) converge to \(y\), the tangent spaces \(T_{g.w_i}\) converge to a vector space \(g.\tau\) and the secant lines \(\overline{g.w_i} \overline{g.y_i}\) converge to a line \(g.\nu\). Therefore, since \(y\) is Whitney-regular, \(g.\nu \subset g.\tau\), and hence \(\nu \subset \tau\).

**Proposition 3.17.** The stratified space \(\text{Conv}_{I_{1,\ldots,I_{k,\leq \mu}}}^\text{an}\) belongs to \(\text{StrTop}_{\text{con}}\).

**Proof.** The following proof has been suggested to us by Robert Cass.

First of all, strata are smooth by factorization (Proposition 2.25) and Proposition 2.31. Note then that \(\text{Gr}^\text{an}_{I_{1,\leq \mu}} \times \cdots \times \text{Gr}^\text{an}_{I_{k,\leq \mu}}\) and \(\text{Conv}_{I_{1,\ldots,I_{k,\leq \mu}}}^\text{an}\) admit a common smooth cover

\[
\text{Gr}^\text{an}_{X_{I_{1,\leq \mu}},1} \times_X \text{Gr}^\text{an}_{X_{I_{2,\leq \mu},1},1} \times_X \cdots \text{Gr}^\text{an}_{X_{I_{k,\leq \mu},1},1} \times_X \text{Gr}^\text{an}_{X_{I_{k+1,\leq \mu}},1}
\]

which projects onto each of them as a smooth bundle having as fiber the smooth unstratified group \(G_{0,\text{Ran}}(j_{\mu}) \times \cdots \times G_{0,\text{Ran}}(j_{\mu})\). It suffices to prove that the condition of being Whitney is stable under smooth bundles with unstratified fiber (in both directions). Since the Whitney condition is local, we reduce the problem to proving this in the case of a product of a space \(X \subset \mathbb{R}^N\) with a stratification \(s\) and a euclidean space \(\mathbb{R}^n\), with projection \(\pi : X \times \mathbb{R}^n \to X\). If \((X,s)\) is Whitney, then the product \(X \times \mathbb{R}^n\) with the trivial stratification of \(\mathbb{R}^n\) is Whitney. Conversely, suppose that \(X \times \mathbb{R}^n\) is Whitney with the trivial stratification on \(\mathbb{R}^n\), and pick two sequences \((w_i) \subset W \subset X,(y_i) \subset Y \subset X\) both converging to \(y \in Y\), such that \(w_i.y_i \to \nu, T_{w_i} W \to \tau\). We can define liftings of the \(w_i,y_i\)'s by \((w_i,0),(y_i,0)\), where \(0\) is the origin of \(\mathbb{R}^n\). The sequences of the secants and of the tangent spaces of these new points converge respectively to \(\nu \times 0\) and \(\tau \times 0\): therefore, we obtain that \(\nu \subset \tau\) by applying the hypothesis that \(X \times \mathbb{R}^n\) is Whitney.

**Corollary 3.18.** For any \(k\), the object \(\text{Hck}_{\text{Ran},k}^\text{an}\) lies in \(\text{StrTStk}_{\text{con}}\).

**Proof.** We need to prove that each term in the formula

\[
\colim_{l_1,\ldots,l_k \in \text{Fin}_{w_{\mu_1} \geq 1}} \colim_{\mu \in (X_{\mu}(T))^{\mu_1 \cdots \mu_k}} \lim_{j_{\geq l_\mu}} \colim_{[\mu]} \Delta_{\text{op}}(G_{0,\text{Ran}}(j_{\mu}))^\times \times \text{Conv}_{I_{1,\ldots,I_{k,\leq \mu}}}^\text{an}
\]

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(where the inner colimit is the colimit along the usual simplicial diagram encoding the action) belongs to $\text{StrTop}_{\text{con}}$. This statement is a formal consequence of the previous ones, since each $G^{an}_{O,J,i,j}$ is the analytification of a smooth group scheme (see Remark 2.13), hence a smooth topological group by Example B.31.

Recall now Definition B.30. We need to prove that:

**Proposition 3.19.** The functor $\text{Hck}^{\text{fact}}$ from Theorem 3.14 takes values in the subcategory

$$\overline{\text{Corr}(\text{StrT}_{\text{stk}}_{\text{con}})^{\times}}_{\text{all,subm}^*}$$

that is, that we have the right class of horizontal morphisms in order to apply Theorem B.39.

**Proof.** We need to describe the image of an arbitrary morphism $(x, \phi) : (U_1, \ldots, U_n, \langle k \rangle) \to (V_1, \ldots, V_m, \langle b \rangle)$ in $\text{Fact}(M)^{\circ} \times \mathbb{E}_2$ under the functor $\text{Hck}^{\text{fact}}$. Suppose first that $\phi$ is of the form $\phi : \langle 2 \rangle \to \langle 1 \rangle$ (either one of the two “projections” or the “multiplication” map). Since the image of $(x, \phi)$ can be factored, by definition, as the composition of the image of $(x, \text{id})$ under $\text{Hck}^{\text{fact}}(-, \langle 2 \rangle)$ and the image of $(\text{id}, \phi)$ under $\text{Hck}^{\text{fact}}((V_1, \ldots, V_n), -)$, let us inspect what happens on each component.

- any pair $(x, \text{id})$ will be sent to a vertical morphism.
- suppose that $\phi$ is inert. Then the pair $(\text{id}_{U_1, \ldots, U_n}, \phi)$ is sent to a vertical morphism, namely one of the two projections

$$(\text{Hck}_{U_1} \times \cdots \times \text{Hck}_{U_n}) \times (\text{Hck}_{U_1} \times \cdots \times \text{Hck}_{U_m}) \to \text{Hck}_{U_1} \times \cdots \times \text{Hck}_{U_m}.$$

- suppose finally that $\phi$ is the active morphism. By simplicity, suppose that $m = 1$ (the general case can be directly recovered from this). Then $(\text{id}, \phi)$ is sent to the correspondence

$$\text{Hck}_{U_2} \to \text{Hck}_{U_1} \times \text{Hck}_{U} \xrightarrow{H \times \text{Hck}_{U}} \text{Hck}_{U}.$$

It suffices to prove the claim for $U = \text{Ran}(X^{\text{an}})$, and by definition of the class subm we reduce to proving that for every $I_1, I_2 \in \text{Fin}_{\text{surj}_{| \geq 1}}, I = I_1 \sqcup I_2, \mu \in (\mathbb{X}_2(T)^+), j \geq j_{\mu}$

$$\overline{P}_{I_1, I_2, \langle j \rangle} : \text{Hck}_{I_1 \leq \mu, \langle j \rangle} \to \text{Hck}_{I_1 \leq \mu, \langle j \rangle} \times \text{Hck}_{I_2 \leq \mu, \langle j \rangle}$$

(notations as in Section 2.2) is sent in subm under analytification. Note however, by Remark 2.45, that this map can be obtained from the map of schemes

$$P_{I_1, I_2, \langle j \rangle} : G^\infty_{X, I_1 \leq \mu, \langle j \rangle} \times X^2 \text{Gr}_{I_2 \leq \mu} \to \text{Gr}_{I_1 \leq \mu} \times \text{Gr}_{I_2 \leq \mu}$$

which takes the right $G^0_{I_1, I_2, \langle j \rangle}^{\text{quotient}}$ in the first component and is the identity in the second one: indeed, $\overline{P}_{I_1, I_2, \langle j \rangle}$ is obtained from $\overline{P}_{I_1, I_2, \langle j \rangle}$ by taking the $G^0_{I_1, I_2, \langle j \rangle}^{\text{quotient}}$ on both sides with respect to the actions.
- on the source, the first copy of the group acts on the first component by left multiplication, and the second copy of the group acts “antidiagonally” (i.e. it can be described on $X^{I_2}$-fibers as $b.(g, \gamma) = (g b^{-1}, b \gamma)$ for $g \in G_{x, (j)}, h \in G_{O,(j)}, \gamma \in \text{Gr}$).

- on the target, the first copy of the group acts on the first component by left multiplication, and the second copy of the group acts on the second component by left multiplication.

This in particular proves that $\overline{P_{t_1,t_2,(j)}}$ is representable. Now, the map $P_{t_1,t_2,(j)}$ is a $G_{O,t_2,(j)}$-torsor (either by definition or by [Ric14, Lemma 2.5 (iii)]). This achieves the proof again by Remark 2.13 and Example B.31.

Since the object $Hck_{\text{Ran},*}$ is 2-Segal any other case can be recovered via pullback from the cases treated above (in the same spirit, see the definition before [DK19, Proposition 8.1.7]), and since our classes of vertical and horizontal morphisms are stable under pullbacks we conclude.

4 The $\mathbb{E}_2$-structure

4.1 Spherical category over the Ran space

Construction 4.1. We are now able to consider the composition

$$\text{Sph}(G)^{\text{fact}} : \text{Fact}(M)^{\otimes} \times \mathbb{E}_1^{\text{nu}} \xrightarrow{Hck^{\text{fact}}} \text{Corr}(\text{StrTStk}_{\text{con}})^{\times} \xrightarrow{\text{Cons}^{\text{corr}}_{\otimes}} \mathcal{P}_R^{\otimes}$$

where the first functor is the one from Theorem 3.14 and the second functor is the one constructed in Theorem B.39. Note that $\text{Sph}(G)^{\text{fact}}$ sends a morphism of the form $(\text{Ran}(D) \to \text{Ran}(D'), \text{id}_{(j)})$ to an equivalence of $\infty$-categories by Theorem 3.14 and Proposition B.29. Moreover, it satisfies the factorization property from Recall 3.5 since $Hck^{\text{fact}}$ does and $\text{Cons}^{\otimes}$ is symmetric monoidal. That is, by Theorem B.39 the functor

$$\text{Cons}(Hck_U) \otimes \text{Cons}(Hck_V) \to \text{Cons}(Hck_U \times Hck_V)$$

is an equivalence for each independent $U, V \subset \text{Ran}(X^{\text{an}})$. Finally, it is a cosheaf in the first variable since $Hck^{\text{fact}}$ is and by the combination of the stratified Seifert-Van Kampen theorem [Lur17, Theorem A.7.1] and the definition of $\text{Cons}_R$ in Theorem B.39. Therefore we can apply [Lur17, Theorem 5.5.4.10] and obtain an $\mathbb{E}_M^{\text{nu}}$-algebra, where $M = X^{\text{an}}$, a 2-dimensional real manifold. Its restriction to any disk $D \subset M$ has the same value for different $D$’s (up to equivalence in $\mathcal{P}_R^{\otimes}$), since an $\mathbb{E}_M^{\text{nu}}$-algebra is in particular locally constant, and by [Lur17, Example 5.4.5.3], this restriction is naturally an $\mathbb{E}_2^{\text{nu}}$-algebra. This means that the stalk of $\text{Sph}(G)^{\text{fact}}$ in the first variable at any point $\{x\} \in \text{Ran}(X^{\text{an}})$, which is just its value at any $D$ centered at $x$, induces a map of operads

$$\mathbb{E}_2^{\text{nu}} \times \mathbb{E}_1^{\text{nu}} \to \mathcal{P}_R^{\otimes}.$$
Its underlying presentable ∞-category is

\[ \text{Cons}(\text{Hck}^\text{an}_{\text{Ran}(D)^*} R) \] (4.1)

for any (irrelevant) choice of a small disk \( D \) around \( x \). Also the choice of \( x \) is irrelevant, in the sense that different choices give noncanonically equivalent ∞-categories with equivalent algebra structures.

We will refer to the operation implicit in the \( \mathbb{E}_2 \)-component as fusion and to the one implicit in the \( \mathbb{E}_1 \)-component as convolution.

**Lemma 4.2.** Let \( G \) be a topological group, \( H < G \) a normal subgroup which is contractible, and \( (X, s) \) a conically stratified topological space with a \( G \)-action such that the restriction to \( H \) is trivial. Then the pullback map

\[ \text{Cons}_{G/H}(X, s) \to \text{Cons}_G(X, s) \]

is an equivalence.

**Proof.** We have a diagram

where all squares of the form

are pullback squares. Note that, at each term, the map between the \( (H, X) \) row and the \( (\ast, X) \) row is a stratified homotopy equivalence, and the homotopies witnessing this can be chosen to be compatible with the rest of the diagram, since the action of \( H \) on \( X \) is trivial. Also, \( G \to G/H \) is an (unstratified) Serre fibration, hence also the map between the \( (G, X) \) row and the \( (G/H, X) \) row is a stratified homotopy equivalence. Therefore, it induces a stratified homotopy equivalence of stacks on the colimits of the two rows, and therefore an equivalence at the level of constructible sheaves by Remark B.22. \( \square \)
4.1 SPHERICAL CATEGORY OVER THE RAN SPACE

**Proposition 4.3.** Fix $\mu \in \mathcal{X}_*(T)^+$, and $j_\mu$ an index such that the action of $G_\circ$ over $\text{Gr}_{\leq \mu}$ factors through $G_{0,(j_\mu)}$. Then for every $j \geq j_\mu$ the pullback map

$$\text{Cons}_{G_{0,(j_\mu)}}(\text{Gr}_{\leq \mu}) \to \text{Cons}_{G_{0,(j)}}(\text{Gr}_{\leq \mu})$$

is an equivalence.

**Proof.** This follows from Lemma 4.2 and the fact that the kernel of $G_{0,(j)} \to G_{0,(j_\mu)}$ is unipotent, hence isomorphic (as a scheme) to some affine space $\mathbb{A}^N$ (this can be proven by induction by using Lazard’s characterization of unipotent groups: namely, every connected unipotent group has a filtration by closed subgroups with quotients $\mathbb{C}_\alpha$). Therefore, its underlying complex space is contractible. 

**Remark 4.4.** Note that the $\infty$-category (4.1) is, by the construction in Theorem B.39, equivalent to

$$\text{colim}_{\mathcal{P}r} \text{colim}_{\mathcal{P}r} \text{colim}_{\mathcal{P}r} \text{colim}_{\mathcal{P}r} \text{colim}_{\mathcal{P}r} \text{colim}_{\mathcal{P}r} \text{colim}_{\mathcal{P}r} \text{cons}(\text{Hck}_{D^I,(\leq \mu,j)}(\mathbb{R})).$$

The transition functors in $I$ are given by $*$-pushforward along the closed embeddings $H_{\text{ck}}^{D^I,(\leq \mu,j)} \to H_{\text{ck}}^{D^I,(\leq \mu,j)}$ induced by $J \to I$. This colimit is not filtered. As a consequence of [Lur09, Theorem 5.5.3.13], it corresponds to the limit in $\hat{\text{Cat}}_{\infty,\mathbb{R}}$ taken with $*$-pullback functoriality.

The two inner colimits, put together, are the $\ldots$-colimit of the diagram

\[
\begin{array}{ccccccccc}
\ldots & \longrightarrow & H_{\text{ck}}^{D^I,(\leq \mu,(j))} & \longrightarrow & H_{\text{ck}}^{D^I,(\leq \nu,(j))} & \longrightarrow & H_{\text{ck}}^{D^I,(\leq \eta,(j))} & \longrightarrow & \ldots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\ldots & \longrightarrow & \ldots & \longrightarrow & \ldots & \longrightarrow & \ldots & \longrightarrow & \ldots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\ldots & \longrightarrow & H_{\text{ck}}^{D^I,(\leq \mu,(j+1))} & \longrightarrow & H_{\text{ck}}^{D^I,(\leq \nu,(j+1))} & \longrightarrow & \ldots \\
\downarrow & & \downarrow & & \downarrow & & \\
\ldots & \longrightarrow & H_{\text{ck}}^{D^I,(\leq \mu,(j))} & \longrightarrow & H_{\text{ck}}^{D^I,(\leq \nu,(j))} & \longrightarrow & \ldots \\
\downarrow & & \downarrow & & \downarrow & & \\
\ldots & \longrightarrow & \ldots & \longrightarrow & \ldots & \longrightarrow & \ldots \\
\downarrow & & \downarrow & & \downarrow & & \\
\ldots & \longrightarrow & H_{\text{ck}}^{D^I,(\leq \mu,(j+1))} & \longrightarrow & \ldots \\
\downarrow & & \downarrow & & \\
\ldots & \longrightarrow & H_{\text{ck}}^{D^I,(\leq \mu,(j))} & \longrightarrow & \ldots \\
\end{array}
\]

\[\text{Recall that these are closed embeddings of unions of strata, hence $*$-pushforward preserves constructible sheaves.}\]

\[^{6}\text{This is the $\infty$-category of large $\mathbb{R}$-linear $\infty$-categories, defined similarly to Notation B.14.}\]
Now, all vertical maps induce equivalences at the level of constructible sheaves by Proposition 4.3. Therefore, we can rewrite the whole expression as

\[
\lim_{I \in \text{Fin}_{\text{surj}}(\geq 1)} \text{colim}_{\mu \in X_+^{(T)}} \text{Cons}(\text{Hck}_{D', \leq \mu(j)}, R)
\]

(cf. Remark B.25). Note that the colimit in \( \mu \) is filtered, and hence it is the actual colimit in \( \text{Cat}_{\infty} \), and since the transition maps are closed embedding of unions of strata, the transition functors are just \( \ast \)-pushforwards (as indicated in the superscript). The fact that \( j_\mu \) is just “any sufficiently large number” is not a problem here, precisely because any other larger \( j \) will give the same result.

The reason why we chose to adopt the pro-object perspective is that now, with the possibility of increasing \( j \), it is clear what the transition maps are (a priori, there is no significant map between \( \text{Hck}_{D', \leq \mu(j)} \) and \( \text{Hck}_{D', \leq \nu(j)} \)).

Finally, each term \( \text{Cons}(\text{Hck}_{D', \leq \mu(j)} \times \text{Gr}_{D', \leq \mu}) \) is computed, again by construction, à la Bernstein-Lunts, i.e. as

\[
\text{colim}_{[n] \in \Delta} \text{Cons}(G_{0,D', \langle j \rangle} \times \text{Gr}_{D', \leq \mu}) \simeq \lim_{[n] \in \Delta} \text{Cons}(G_{0,D', \langle j \rangle} \times \text{Gr}_{D', \leq \mu})
\]

where the limit is taken along the simplicial diagram encoding the action of \( G_{0,D', \langle j \rangle} \) on \( \text{Gr}_{D', \leq \mu} \) with pullback functoriality.

### 4.2 Specialization to a point

Our goal in this subsection is to transfer the algebra structure established in Construction 4.1 to the \( \infty \)-category

\[
\text{Cons}_{G_{D}}(\text{Gr}^\text{an}, R).
\]

**Remark 4.5.** Let \( D \hookrightarrow \text{Ran}(D) \) denote the closed embedding corresponding to the cardinality 1 stratum. Let then \( \text{Gr}_{D} = \text{Gr}_{X_{\text{an}}} \times_X D \simeq \text{Gr}^\text{an} \times D \) and \( G_{0,D} = G_{0,X_{\text{an}}} \times_X D \simeq G_{0}^\text{an} \times D \). Let \( \text{Hck}_{D} = \text{Hck}_{X_{\text{an}}} \times_X D \in \text{Set}_{\text{TStk}} \).

First of all, notice that for any choice of a disk \( D \) in \( X_{\text{an}} \) there is an equivalence of \( \infty \)-categories

\[
\text{Cons}_{G_{D}}(\text{Gr}^\text{an}, R) \simeq \text{Cons}_{G_{0,D}}(\text{Gr}^\text{an}_{D}, R)
\]

induced by pulling back along the projection \( \text{Gr}^\text{an}_{D} \to \text{Gr}^\text{an} \) (the inverse functor is given by pulling back along the embedding \( \text{Gr}^\text{an} \times \{x_0\} \to \text{Gr}^\text{an} \times D \) for any choice of a point \( x_0 \) in \( D \)). This is true by Remark B.22. Therefore, to give an \( \mathbb{E}_3 \)-algebra structure on \( \text{Cons}_{G_{D}}(\text{Gr}^\text{an}, R) \) is the same as giving an \( \mathbb{E}_3 \)-algebra structure on \( \text{Cons}_{G_{0,D}}(\text{Gr}_{D}, R) \). By the very nature of this identification, the equivalence does not actually depend on the choice of \( x, D \).

Now, there is a pullback diagram in \( \text{Set}_{\text{TStk}} \)

\[
\begin{array}{ccc}
\text{Hck}_{D} & \xleftarrow{i} & \text{Hck}_{\text{Ran}(D)} \\
\downarrow & & \downarrow \\
D & \xleftarrow{j} & \text{Ran}(D)
\end{array}
\]
and an adjunction

\[ \begin{align*}
\text{Cons}_{G,0,D}(\text{Gr}_D,R) & \leftrightarrow \text{Cons}_{G,0,Ran(D)}(\text{Gr}_{Ran(D)},R) \\
\downarrow i^* & \downarrow_{i_*}
\end{align*} \tag{4.2} \]

where \( i_* \) is fully faithful because \( i \) is an equivariant closed embedding of a union of strata\(^7\). In order to transfer the \( E^\mu_2 \times E^\nu_1 \)-algebra structure to \( \text{Cons}_{G,0,D}(\text{Gr}_D) \), we want to apply [Lur17, Proposition 2.2.1.9] with \( \mathcal{O}^\otimes = \text{Cons}_{G,0,Ran(D)}(\text{Gr}_{Ran(D)},R) \) and each \( L_X \) induced by \( i^* \). As remarked in Construction 4.1, there are two compatible product structures on \( \text{Cons}_{G,0,Ran(D)}(\text{Gr}_{Ran(D)},R) \), which we call \( \star \) (convolution, the one parametrized by the \( E^\mu_2 \)-variable) and \( \otimes \) (fusion, the one parametrized by the \( E^\nu_1 \)-variable). In order to apply Lurie’s result, we need to verify that for any \( A, A', B \in \text{Cons}_{G,0,Ran(D)}(\text{Gr}_{Ran(D)},R) \), and a morphism

\[ f : A \to A' \]

such that \( i^* f \) is an equivalence in \( \text{Cons}_{G,0,D}(\text{Gr}_D,R) \), the natural maps

\[ i^*(A \star B) \to i^*(A' \star B) \]

and

\[ i^*(A \otimes B) \to i^*(A \otimes B) \]

are equivalences.

First of all, by the description in Construction 4.1, we can fix \( I_1, I_2 \) and assume that \( A, A' \in \text{Cons}(\text{Hck}_{D_{I_1}}), B \in \text{Cons}(\text{Hck}_{D_{I_2}}) \) (actually, we could even assume \( I_1 = I_2 \), but it is instructive to see what happens in the general case).

We need to prove two things:

* For the convolution case, we need to prove that

\[ i^*(m_{I_1,I_2}^* p_{I_1,I_2}^* (A \boxtimes B)) \to i^*(m_{I_1,I_2}^* p_{I_1,I_2}^* (A' \boxtimes B)) \]

is an equivalence, where the notations are as in the following diagram:

\[
\begin{array}{c}
\text{Hck}_D \times \text{Hck}_D & \xleftarrow{p} & \text{Hck}_{D,D} & \xrightarrow{m} & \text{Hck}_{D^2} & \xleftarrow{d} & \text{Hck}_D \\
\downarrow{j} & & \downarrow{i} & & \downarrow{j} & & \downarrow{i} \\
\text{Hck}_{D_{I_1}} \times \text{Hck}_{D_{I_2}} & \xleftarrow{p_{I_1,I_2}} & \text{Hck}_{D_{I_1,D_{I_2}}} & \xrightarrow{m_{I_1,I_2}} & \text{Hck}_{D_{I_1 \sqcup I_2}} \\
\end{array}
\]

Here \( i \) stays for the map \( i \) read at the \( I_1 \sqcup I_2 \)-level, and \( d \) for the map pulled back from the diagonal of \( D^2 \). Note that both squares and the triangle commute, the second square is a

---

\(^7\)This meaning that each piece \( G_{G,0,D(I)}^{*,\mu} \times \text{Gr}_{D,\leq \mu} \to G_{G,0,D(I)}^{*,\mu} \times \text{Gr}_{D,\leq \mu} \) is a closed embedding of a union of strata, for each \( n, I, \mu, j \geq j_{\mu} \).
pullback and the map $m_{t_1,t_2}$ is proper\textsuperscript{8}. Moreover, we have equivalences $m_s \simeq m_t$, $m_{t_1,t_2}$ \textsuperscript{9}. To see this, it suffices to see that $m_s$ and $m_{t_1,t_2}$ preserve constructible sheaves. By properness of $m, m_{t_1,t_2}$ and proper base change, it suffices to check this after pullback to each stratum of $X_{t_1 \cup t_2}$. There, the map can be realized as a product of copies of the multiplication map\textsuperscript{9} $H_{ck,x_2} \to H_{ck}$ for some point $x$. This latter map descends from a $G_0$-equivariant map Conv\textsubscript{x_2} $\to \Gr_x$, therefore, pushforward along it preserves equivariant sheaves which are constructible with respect to some stratification. But equivariant constructible sheaves over $\Gr_x$ with respect to some stratification are automatically constructible with respect with to the stratification by Schubert cells by Proposition A.24.

Therefore, we can apply proper base change and conclude that

\[ i^*(m_{t_1,t_2} p_{t_1,t_2}^*(A \boxtimes B)) \simeq d^* j^* m_{t_1,t_2} p_{t_1,t_2}^*(A \boxtimes B) \simeq d^* m_s j^* p_{t_1,t_2}^*(A \boxtimes B). \]

Note now that $j'$ corresponds to the map $i \times i$ read at the $(I_1, I_2)$-level, and therefore the last expression equals

\[ d^* m_s p^*(i^*A \boxtimes i^*B). \]

If we read this construction functorially in $A$, we conclude that the map $i^*(A \otimes B) \to i^*(A \otimes B)$ induced by $f$ is an equivalence.

\(\bigcirc\) For the fusion product, the product law is depicted by the diagram

\[ \begin{array}{ccc} Hck_D \times Hck_D & \xleftarrow{\sim} & Hck_{D_1} \times Hck_{D_2} \xleftarrow{u_{t_1} \circ u_{t_2}} Hck_{D_2} \xleftarrow{d} Hck_{D} \\ \downarrow & & \downarrow \\ Hck_{D_1} \times Hck_{D_2} & \xleftarrow{\sim} & Hck_{D_1} \times Hck_{D_2} \xleftarrow{u_{t_1} \circ u_{t_2}} Hck_{D_1 \cup D_2} \end{array} \] (4.3)

Note first of all that $d_{t_1 \cup t_2} = d j$ corresponds to the map $i$ read at the $I_1 \cup I_2$-level. Up to identifying the first column with the second one, $A \otimes B$ corresponds to $(u_{t_1} \circ u_{t_2})(A \boxtimes B)$ and

\[ i^*(A \otimes B) \simeq d^* j^*(u_{t_1} \circ u_{t_2})(A \boxtimes B). \]

Therefore, it suffices to show that the base-change map

\[ j^*(u_{t_1} \circ u_{t_2}) \to u_r j^* \]

induces an equivalence

\[ d_{t_1 \cup t_2}^*(u_{t_1} \circ u_{t_2})(A \boxtimes B) \simeq d^* j^*(u_{t_1} \circ u_{t_2})(A \boxtimes B) \simeq d^* u_r j^*(A \boxtimes B). \]

\textsuperscript{8}In the sense that each piece $m_{t_1,t_2 \in \mu(I)}$ is proper.

\textsuperscript{9}In the notation of Definition 2.39, and later of Definition 2.26.
Let us consider the diagram

\[
\begin{array}{ccc}
\text{Hck}_{D_1} \times \text{Hck}_{D_2} & \xrightarrow{u_{1,1}} & \text{Hck}_{D_2} \\
\downarrow & & \downarrow \rho \Delta \\
\text{Hck}_{D_1} \times \text{Hck}_{D_2} & \xrightarrow{\tilde{u}_{1,1}} & \text{Hck}_{D,D} \\
\downarrow & & \downarrow \rho \Delta \\
\text{Hck}_{D_1} \times \text{Hck}_{D_2} & \xrightarrow{u'_{1,1 \text{ s.h.e.}}} & \text{Hck}_D \times \text{Hck}_D \\
\downarrow & & \downarrow \rho' \Delta \\
\text{Hck}_{D_1} \times \text{Hck}_{D_2} & \xrightarrow{\tilde{u}'_{1,1 \text{ s.h.e.}}} & \text{Hck}_{D_l,D_r} \\
\downarrow & & \downarrow \rho' \Delta \\
\text{Hck}_{D_1} \times \text{Hck}_{D_2} & \xrightarrow{\tilde{u}_{1,1}} & \text{Hck}_{D_1 \times D_2} \\
\downarrow & & \downarrow \rho' \Delta \\
\text{Hck}_{D_1} \times \text{Hck}_{D_2} & \xrightarrow{u_{1,2}} & \text{Hck}_{D_1 \times D_2} \\
\downarrow & & \downarrow \rho' \Delta \\
\text{Hck}_{D_1} \times \text{Hck}_{D_2} & \xrightarrow{\tilde{u}_{1,2}} & \text{Hck}_{D_1 \times D_2} \\
\downarrow & & \downarrow \rho' \Delta \\
\text{Hck}_{D_1} \times \text{Hck}_{D_2} & \xrightarrow{u_{1,2}} & \text{Hck}_{D_1 \times D_2} \\
\end{array}
\]

(4.5)

where the maps are defined as follows:

- \( u_{1,1} \) and \( u_{1,2} \) are defined in (4.3) and come from the factorization property for \( \text{Hck}_{D_1 \sqcup D_2} \).
- \( \tilde{u}_{1,1} \) and \( \tilde{u}_{1,2} \) come from Proposition 2.25 and Remark 2.34.
- \( u'_{1,1} \) and \( u'_{1,2} \) are the open embeddings induced by the inclusions \( D_1 \subset D \), \( D_2 \subset D \).
- \( d \) and \( d_{1,2} \) are defined in (4.3).
- \( d' \) and \( d'_{1,2} \) are the maps induced by the diagonal inclusion \( D \subset D^2 \), \( D \subset D_{l_1} \times D_{l_2} \).
- \( \tilde{d} \) and \( \tilde{d}_{1,2} \) are the maps induced by the diagonal inclusion \( D \subset D^2 \), \( D \subset D_{l_1} \times D_{l_2} \) and by Proposition 2.25.
- \( j' \) comes from (4.3).
- \( \delta \) is the map induced by the diagonal inclusions \( D \subset D_{l_1} \), \( D_{l_2} \).
- \( \rho', m', \rho' \Delta, m' \Delta, \rho'_{l_1,l_2}, m'_{l_1,l_2} \) are (or are induced by) the maps in Remark 2.44.

The goal is to prove that if we start with a sheaf \( \mathcal{A} \boxtimes \mathcal{B} \) over \( \text{Hck}_{D_1} \times \text{Hck}_{D_2} \), then pulling back along the first column and performing push-pull along the first row is the same as performing push-pull along the last row. Note that all squares are pullback squares. By properness of the maps \( m' \Delta, m'_{l_1,l_2} \), we have equivalences

\[
d'_{1,2}(u_{1,2})_\ast \simeq d'_{1,2}(m'_{l_1,l_2})_\ast(\tilde{u}_{1,2})_\ast \simeq (m' \Delta)_\ast \tilde{d}'_{1,2}(\tilde{u}_{1,2})_\ast.
\]
We want to check that the map \( \mathcal{E}_2 \) as by the discussion in the convolution case). Then, by smoothness\(^{10}\) of \( \pi^* \), we have

\[
(\mathcal{E}_2), \pi^* \tilde{g}, (\pi^* \tilde{h}). \to (\mathcal{E}_2), \mathcal{E}_2 \pi^* \tilde{g} \pi^* \tilde{h} \to (\mathcal{E}_2), \mathcal{E}_2 \pi^* \tilde{g} \pi^* \tilde{h}.
\]

But now, since \( u_{1,1}' \) and \( u_{1,1} \) are stratified homotopy equivalences, the last expression is equivalent to

\[
(\mathcal{E}_2), \mathcal{E}_2 \pi^* \tilde{g} \pi^* \tilde{h} \to (\mathcal{E}_2), \mathcal{E}_2 \pi^* \tilde{g} \pi^* \tilde{h}.
\]

Then by applying again smooth and then proper base change on the squares on the top half of the diagram we conclude that the original expression is equivalent to

\[
d^\pi(u_{1,1}), f^\pi(A \Box B).
\]

Like in the convolution case, we can deduce from this that the map \( f \) induces an equivalence as desired.

By summing up:

**Corollary 4.6.** For any \( D \) disk in \( X \), the algebra structure from Construction 4.1 induces, after localization, a map of operads \( Sph(G) \otimes : \mathbb{E}_2^n \times \mathbb{E}_1 \to \mathcal{P}_R^{\otimes} \) whose underlying \( \infty \)-category is the topological spherical Hecke category \( Sph(G) \) (Definition A.30). This structure is independent of the choice of \( D \) and \( x \).

### 4.3 Units and t-exactness

By convenience, given three \( \infty \)-operads \( \mathcal{O}, \mathcal{O}', \mathcal{O}'' \), we will call “bilinear” maps those maps \( \mathcal{O} \times \mathcal{O}' \to \mathcal{O}'' \) which are maps of operads separately in each variable.

**Proposition 4.7.** Let \( x \) be any point in \( X \). The bilinear map \( Sph(G) \otimes : \mathbb{E}_2^n \times \mathbb{E}_1 \to \mathcal{P}_R^{\otimes} \) extends to a bilinear map \( \mathbb{E}_2 \times \mathbb{E}_1 \to \mathcal{P}_R^{\otimes} \), again denoted by \( Sph(G) \otimes \).

**Proof.** We can apply [Lur17, Theorem 5.4.4.5]: that is, it suffices to exhibit a quasi-unit for any \( Sph(G) \otimes (\text{–}, \{k\}) \), functorial in \( \{k\} \in \mathbb{E}_1 \). Consider the map (natural in \( k \)) \( \text{Spec} \mathbb{C} \to \text{Hck}_{x,k} \)\(^{11} \) represented by the sequence \( (\mathcal{X}, \ldots, \mathcal{X}, \text{id}_{X \setminus \{x\}}, \ldots, \text{id}_{X \setminus \{x\}}) \in \text{Hck}_{x,k} \). Note now that this induces a map of spaces

\[
e_k : * \to \text{Hck}_{x,k}.
\]

We want to check that the map \( e_k \) is a quasi-unit in the sense of [Lur17, Definition 5.4.3.1.] for any \( k \), functorially in \( k \). But both maps \( \text{Hck}_{x,k} \to \text{Hck}_{x,k} \times \text{Hck}_{x,k} \to \text{Hck}_{x,k} \) induced by \( e_k \) (by

---

\(^{10}\)Here, as usual, we take advantage of the definition of our categories of constructible sheaves as colimits of categories of sheaves over “truncated” objects of the form \( \text{Hck}_{x,\leq \mu, \{x\}} \), which allows us to work with smooth maps instead of pro-smooth, cf. Proposition 3.19.

\(^{11}\)Notation as in Definition 2.39.
targeting respectively the first or the second factor of the product) are the identity, since gluing with the trivial $G$-torsor along with its trivial trivialization does not change the original torsor. At the level of constructible sheaves, the unit is given by the pushforward of the constant sheaf $R$ along $e_k$.

**Proposition 4.8.** The bilinear map $\text{Sph}(G)^\otimes : \mathbb{E}_2 \times \mathbb{E}_1 \nuu \rightarrow \mathcal{P}^R_{R}$ extends to a bilinear map $\text{Sph}(G)^\otimes : \mathbb{E}_2 \times \mathbb{E}_1 \nuu \rightarrow \mathcal{P}^R_{R}$.

**Proof.** Again, it suffices to exhibit a quasi-unit. Let us denote by $1$ the pushforward along the trivial section $t : \ast \rightarrow \text{Hck}_{x,1}^\an$, $t(*) = (T, T, \text{id}|_{X \setminus x})$, of the constant sheaf with value $R$. The proof is given in [Rei12, Proposition IV.3.5]. We just rewrite it in our notation. We drop the superscript $(-)\an$ everywhere for simplicity. Let us assume that the entry in the variable $\mathbb{E}_2$ is $\langle 1 \rangle$ by simplicity (the general case is just “a direct power” of this one). We denote by $\ast$ the $\mathbb{E}_1$-product of equivariant constructible sheaves on $\text{Gr}$, described by $\text{Sph}(G)^\otimes (\langle 1 \rangle, -)$. For any $F \in \text{Cons}_{G_0}(\text{Gr}, R)$ we can compute the product via the convolution diagram

![Diagram](image)

where $j$ is the closed embedding $(\mathcal{F}, \alpha) \mapsto (\mathcal{F}, \text{id}|_{X \setminus x}, \mathcal{F}, \alpha)$ whose image is canonically identified with $\text{Gr}$. Let $F \in \text{Cons}_{G_0}(\text{Gr}, R)$. We want to prove that $1 \ast F \simeq j_* (R \boxtimes F)$, i.e. that

$$q^* j_* (R \boxtimes F) \simeq p^* (t \times \text{id})_* (R \boxtimes F).$$

Note that because of the consideration about the image of $j$ the support of both sides lies in $G_0 \times \text{Gr} \subset G_x \times \text{Gr}$, and this yields a restricted diagram

![Restricted Diagram](image)

This proves the claim. By applying $m_\ast$ we obtain

$$1 \ast F \simeq m_\ast (j_* (R \boxtimes F)) = R \boxtimes F = F$$

since $mj = \text{id.}$
Thanks to these results, our functor $\text{Sph}(G)^{\otimes}$ from Corollary 4.6 is finally promoted to a bilinear map $\mathbb{E}_2 \times \mathbb{E}_1 \to \mathcal{P}r^L_{R}$. By the Additivity Theorem ([Lur17, Theorem 5.1.2.2]), this is the same as an $\mathbb{E}_3$-algebra object in $\mathcal{P}r^L_{R}$.

Summing up:

**Theorem 4.9** (Spherical Hecke category). Let $G$ be a complex reductive group and $R$ be a commutative ring. There is an object $\text{Sph}(G)^{\otimes} \in \text{Alg}_{\mathbb{E}_3}(\mathcal{P}r^R_{R})$ having as underlying object the topological spherical Hecke category

$$\text{Sph}(G)^{\text{top}} = \text{Cons}_{G^{\text{an}}}(\text{Gr}_{G^{\text{an}}}, R)$$

(see Definition A.30).

**Corollary 4.10** (Small spherical Hecke category). In the same setting, there is an induced $\mathbb{E}_3$-monoidal structure in $\mathcal{C}at_{\infty,R}$ on

$$\text{Cons}^{\text{fd}}_{G^{\text{an}}}(\text{Gr}_{G^{\text{an}}}, R).$$

If $R$ is a finite or profinite ring (e.g. $\mathbb{Z}/m$, $\mathbb{Z}_l$) or an algebraic extension of $\mathbb{Q}_l$ (e.g. $\mathbb{Q}_l$, $\overline{\mathbb{Q}}_l$), then the latter category agrees with its algebraic counterpart, the small spherical Hecke category

$$\text{Sph}(G)^{\text{loc.c}} = \text{Cons}^{\text{fd}}_{G^{\text{an}}}(\text{Gr}_{G^{\text{an}}}, R)$$

(see Definition A.30). On objects, the restriction of this product to equivariant perverse sheaves coincides with the classical (commutative) convolution product of perverse sheaves [MV07], up to the perverse truncation of the derived tensor product appearing in the definition of the latter.

*Proof.* The inclusion $\mathcal{P}r^R_{R} \to \mathcal{C}at_{\infty,R}$ (the $\infty$-category of large $R$-linear $\infty$-categories) is lax monoidal, i.e. it is a map of operads. Therefore, $\text{Cons}^{\text{fd}}_{G^{\text{an}}}(\text{Gr}_{G^{\text{an}}}, R)$ has an induced $\mathbb{E}_3$-algebra structure in $\mathcal{C}at_{\infty,R}$. By using the convolution formula, one sees that the convolution product restricts to $\text{Cons}^{\text{fd}}_{G^{\text{an}}}(\text{Gr}_{G^{\text{an}}}, R)$. Since the inclusion $\mathcal{C}at_{\infty,R} \subset \mathcal{C}at_{\infty,R}$ is strong symmetric monoidal, we obtain the first part of the statement.

The claim regarding perverse sheaves follows from what observed in Remark A.13. 

**Corollary 4.11** (Renormalization). In the same setting of Corollary 4.10, there is an induced $\mathbb{E}_3$-monoidal structure in $\mathcal{P}r^L_{R}$ on

$$\text{Sph}(G,R)^{\text{ren}} = \text{Ind}(\text{Cons}^{\text{fd}}_{G^{\text{an}}}(\text{Gr}, R))$$

(see Definition A.31).

*Proof.* The proof follows from the fact that the functor $\text{Ind} : \mathcal{C}at_{\infty,R} \to \mathcal{P}r^L_{R}$ is symmetric monoidal, which in turn follows from [Lur17, Remark 4.8.1.8] with $\mathcal{K} = \emptyset$ and

$$\mathcal{K}' = \{ x\text{-filtered simplicial sets}, \text{for some regular cardinal } x \}. \qedhere$$
Remark 4.12. Let $\mathcal{C}^\otimes$ be an $E_k$-algebra in $\text{Cat}_{\infty}^\circ$, the $\infty$-category of stable $\infty$-categories and exact functors between them. Suppose given a t-structure on $\mathcal{C}$ which is compatible with the algebra structure, in the sense of [Lur17, Example 2.2.1.3] (intuitively, the subcategory $\mathcal{C}_{\geq 0}$ should be closed under tensor). Then by [Lur17, Proposition 2.2.1.8, Proposition 2.2.1.9] the heart $\mathcal{C}^\heartsuit$ of the t-structure canonically inherits an $E_k$-algebra structure (the proof goes along the same lines of [Lur17, Example 2.2.1.10], which deals with the case $E_\infty$).

Note that this “induced $E_k$-structure” procedure is functorial: given a stable-exact and t-left exact (in the sense of [Lur17, Definition 1.3.3.1]) $E_k$-monoidal functor $\mathcal{C}^\otimes \rightarrow \mathcal{D}^\otimes$, one obtains an additive functor $\mathcal{C}^\heartsuit \rightarrow \mathcal{D}^\heartsuit$ between abelian categories, which can be viewed as the composition of the $E_k$-monoidal functors

$$\mathcal{C}^\heartsuit \hookrightarrow \mathcal{C}^\leq_{\leq 0} \xrightarrow{F|_{\leq 0}} \mathcal{D}^\leq_{\leq 0} \xrightarrow{\tau_{\leq 0}} \mathcal{D}^\heartsuit$$

(notice that $F$ restricts to the coconnective parts by left-t-exactness).

Remark 4.13. The small spherical Hecke category carries a canonical t-structure inherited from the perverse t-structure on bounded categories of (finite-dimensional) constructible sheaves. Indeed, the Bernstein-Lunts presentation

$$\text{Sph}(G)^{\text{loc.c}} \simeq \lim_n \mathcal{D}_c^{\text{fd}}(G^\times_n \times \text{Gr})$$

of Definition A.20 establishes a canonical t-structure on the limit by [BL94, Section 2.5].

The convolution product on $\text{Sph}(G)^{\text{loc.c}}$ is left t-exact. This follows from the fact that:

- the derived tensor product is always left t-exact for the perverse t-structure (with coefficients in a field, it is actually t-exact).

- the map $m$ in the convolution diagram is stratified semi-small, [BR18, Lemma 6.4].

- (proper) pushforward along stratified semi-small maps is left t-exact (see the proof of [BR18, Proposition 6.1], the part about left t-exactness of $f_!$).

Therefore, one can apply Remark 4.12 to Theorem 1.6 with $k = 1$, and formally deduce the Geometric Satake Equivalence (only as a monoidal equivalence, not symmetric).

Note also that, as mentioned in Remark 1.7, if one applies Remark 4.12 to $\text{Cons}_G^{\text{fd}}(\text{Gr}, R)$ with $k = 3$ one recovers the full commutativity constraint for the convolution product of perverse sheaves.

A.1 The Satake category

Let us resume from the definition of affine Grassmannian, recalled in Definition 1.4.
Remark A.1 (see [Zhu16, Theorem 1.1.3]). There is a natural action of $G_0$ on $G_G$ by left multiplication, whose orbits define an algebraic stratification of $G_G$ over the poset $X_*(T)^+$ of dominant coweights of the Cartan group $T$ of $G$. When viewed from the point of view of the complex-analytic topology on $G_G$, this stratification satisfies the so-called Whitney conditions (for a proof, see [Str14]). One can characterize the stratification as follows. The Cartan decomposition\footnote{The symbol $\bigsqcup$ should only be understood as a set-theoretical decomposition, not a topological or scheme-theoretic one.} $\bigcup_{\mu \in X_*(T)^+} G_0 t^\mu G_0$ induces a partition $\bigcup_{\mu \in X_*(T)^+} G_0 t^\mu$. For an element $g$ of $G_X$, the associated $\mu$ is denoted by $\text{Inv}(g)$ and is the same for every $g'$ in the same right $G_0$-class, i.e. $\text{Inv}(\cdot)$ factors through $G_X \to G_G$. Note that $\text{Inv}(g \cdot g') = \text{Inv}(g) + \text{Inv}(g')$.

If $G = GL_n$, $X_*(T)^+ \cong \mathbb{N}^n$, $t^\mu$ is exactly the diagonal matrix $\text{diag}(t^{-\mu_i})$.

For $\mu \in X_*(T)^+$, one defines $Gr_\mu = \{ \Lambda \in Gr | \text{Inv}(\Lambda) = \mu \}$.

Ind-representability of $Gr$ is given by the natural filtration of $Gr$ in finite-dimensional projective schemes $Gr_{\leq \mu} = \bigsqcup_{\nu \leq \mu} Gr_\nu$, usually called the lattice filtration. Moreover, the action $G_0 \circ Gr_G$ preserves $Gr_{\leq \mu}$, and actually each $G_0 \circ Gr_{\leq \mu}$ factors through the quotient $G_0 \to G(C[[t]]/t^{j_\mu} C[[t]])$ for a sufficiently large $j_\mu$ ([Rei12, Lemma IV.1.4]). This is actually the reason why the orbits form a stratification in the first place ([Str14]).

Definition A.2. Let $j \geq 1$. We define $G_{0,(j)} = G(C[[t]]/t^{j_\mu} C[[t]])$.

Definition A.3. Let $R$ be a ring. The category of $G_0$-equivariant perverse sheaves on $Gr_G$ (or Satake category) with values in $R$-modules is $\mathcal{Perv}_{G_0}(Gr, k) := \text{colim}_{\mu \in X_*(T)^+} \mathcal{Perv}_{G_{0,(j_\mu)}}(Gr_{\leq \mu}, R)$ (see [Zhu16, 5.1 and A.1]).

The definition of each term is independent of $j_\mu$ because of [Zhu16, Lemma A.1.4].
Remark A.4. Let $X$ be a smooth complex curve, $R$ a complex ring, and $x \in X(R)$ be a point. There is a well-defined formal completion of $\Gamma_x$, the graph of $x$ in $X_R$, which is a formal scheme whose structure ring is $\hat{\mathcal{O}}_{\Gamma_x}$. We consider the **affine formal neighbourhood** of $x$, defined as the natural map

$$\text{Spec} \hat{\mathcal{O}}_{\Gamma_x} \rightarrow X_R.$$ 

Note that the source of this map is always isomorphic (noncanonically, since the isomorphism depends on the map $x$) to $\text{Spec} R[[t]]$. We can therefore say, in families, that each point $x$ of $X$ admits an affine formal neighbourhood

$$\tilde{X}_x \cong \text{Spec} \mathbb{C}[t] \rightarrow X.$$ 

There is a pullback square

$$\begin{array}{ccc}
(\hat{X}_R)_x \cong \text{Spec} R((t)) & \longrightarrow & \text{Spec} R[[t]] \\
\downarrow & & \downarrow \\
X_R \setminus \Gamma_x & \longrightarrow & X_R.
\end{array}$$

Again, in families, we say that a point $x \in X$ admits a **punctured affine formal neighbourhood**

$$\tilde{X}_x \cong \text{Spec} \mathbb{C}(t) \rightarrow X.$$ 

Definition A.5. Let $\text{Bun}_G$ be the moduli stack of $G$-torsors over $\mathbb{C}$. If a scheme $Z$ over $\mathbb{C}$ is given, we define the relative version

$$\text{Bun}_Z^G : \text{Alg}_C \rightarrow \text{Grpd}$$

$$R \mapsto \{ G\text{-torsors over } Z \times \text{Spec } R \} = \text{Bun}_G(Z_R).$$

In the language of mapping stacks, we can write

$$\text{Bun}_Z^G \cong \text{Map}_{\text{Stk}/C}(Z, \text{Bun}_G).$$

Proposition A.6. For any closed point $x$ of a smooth projective complex curve $X$, the functor $\text{Gr}_G$ is equivalent to the following:

$$\text{Gr}^{\text{loc}}_G : R \rightarrow \{ \mathcal{F} \in \text{Bun}_G(\tilde{X}_x \times \text{Spec } R), \alpha : \mathcal{F}|_{\tilde{X}_x \times \text{Spec } R} \sim \mathcal{F}|_{\tilde{X}_x \times \text{Spec } R} \} \quad (\text{A.1})$$

where $\mathcal{F}$ is the trivial $G$-torsor on $\tilde{X}_x \times \text{Spec } R$. In other words, $\text{Gr}_G$ is equivalent to the fiber at the trivial $G$-torsor of the functor $\text{Bun}_G^{\tilde{X}_x} \rightarrow \text{Bun}_G^{\tilde{X}_x}$. 

Proof. The proof is explained for instance in [Zhu16, Proposition 1.3.6].
Construction A.7. Define $\text{Gr}_{G}^{\text{glob}}$ as the fiber of the restriction map $\text{Bun}_{G}^{X} \to \text{Bun}_{G}^{X \setminus \{x\}}$, i.e. as the stack

$$R \to \{ \mathcal{F}, \alpha : \mathcal{F}|_{(X \setminus \{x\}) \times \text{Spec } R} \sim (\mathcal{T})|_{R}|_{(X \setminus \{x\}) \times \text{Spec } R} \}. $$

Indeed, in the diagram

$$
\begin{array}{ccc}
\text{Gr}_{G}^{\text{glob}} & \longrightarrow & \text{Bun}_{G}^{X} \\
\downarrow & & \downarrow \\
\{ \mathcal{T}|_{X \setminus \{x\}} \} & \longrightarrow & \text{Bun}_{G}^{X \setminus \{x\}} \\
\end{array}
$$

the right-hand square is cartesian by the Formal Gluing Theorem ([BD05, Remark 2.3.7], or in more generality [HPV16]), extending the theorem of Beauville and Laszlo [BL95] to families. Since the left-hand square is cartesian by definition, the outer square is cartesian. Therefore, $\text{Gr}_{G}^{\text{glob}}$ is isomorphic to the fiber of the restriction map $\text{Bun}_{G}^{X} \to \text{Bun}_{G}^{X \setminus \{x\}}$, which is exactly $\text{Gr}_{G}^{\text{loc}}$. For more details, see [Zhu16, Theorem 1.4.2].

Construction A.8. We recall now the tensor structure given by convolution product on $\mathcal{Perv}_{G_{0}}(\text{Gr}_{G})$. A more detailed account is given in [Zhu16, Section 1, Section 5.1, 5.4]. Consider the diagram

$$
\begin{array}{ccc}
\text{Gr}_{G} \times \text{Gr}_{G} & \longrightarrow & \text{Gr}_{G} \times \text{Gr}_{G}^{0} \\
\downarrow & & \downarrow \\
\text{Gr}_{G} \times \text{Gr}_{G} & \longrightarrow & \text{Gr}_{G}
\end{array}
$$

where $\text{Gr}_{G} \times \text{Gr}_{G}^{0}$ (or $\text{Conv}_{G}$) is the stack quotient of the product $\text{Gr}_{G} \times \text{Gr}_{G}$ with respect to the “anti-diagonal” left action of $G_{0}$ defined by $\gamma \cdot (g, [h]) = (g^{-1} \gamma, [\gamma h])$. The map $p$ is the projection to the quotient on the first factor and the identity on the second one, the map $q$ is the projection to the quotient by the “anti-diagonal” action of $G_{0}$, and the map $m$ is the multiplication map $(g, [h]) \mapsto [gh]$.

Note that the left multiplication action of $G_{0}$ on $\text{Gr}_{G}$ and on $\text{Gr}_{G}$ induces a left action of $G_{0} \times G_{0}$ on $\text{Gr}_{G} \times \text{Gr}_{G}$. It also induces an action of $G_{0}$ on $\text{Gr}_{G} \times \text{Gr}_{G}$ given by (left multiplication, id) which canonically projects to an action of $G_{0}$ on $\text{Gr}_{G} \times G_{0} \times \text{Gr}_{G}$. Note that $p, q$ and $m$ are equivariant with respect to these actions.

Now if $\mathcal{A}_{1}, \mathcal{A}_{2}$ are two $G_{0}$-equivariant perverse sheaves on $\text{Gr}_{G}$, one can define a convolution product

$$\mathcal{A}_{1} * \mathcal{A}_{2} = m_{*} \hat{\mathcal{A}}$$

where $m_{*}$ is the derived direct image functor, and $\hat{\mathcal{A}}$ is a perverse sheaf on $\text{Gr}_{G} \times G_{0} \times \text{Gr}_{G}$ which is equivariant with respect to the left action of $G_{0}$ and such that $q^{*} \hat{\mathcal{A}} = p^{*}(p^{0})^{G_{0}}(\mathcal{A}_{1} \boxtimes \mathcal{A}_{2})$. 13 Note 13The tensor product must be understood as a derived tensor product in the derived category. If the stalks of the two sheaves are flat, e.g. when the ring of coefficients is a field, the external tensor product is already perverse. In general, one needs to consider the perverse truncation, as in the formula.
A.2 The convolution product via quotient stacks

that such an $\tilde{A}$ exists because $q$ is the projection to the quotient and $A_2$ is $G_0$-equivariant, and one can prove that $\tilde{A}$ is again perverse.

Note that $m_*$ carries perverse sheaves to perverse sheaves: indeed, it can be proven that $m_*$ is ind-proper, i.e. it can be represented by a filtered colimit of proper maps of schemes compatibly with the lattice filtration. By [KW01, Lemma III.7.5], and the definition of $\mathcal{P}erv_{G_0}(\text{Gr}_G, k)$ as a filtered colimit, this ensures that $m_*$ carries perverse sheaves to perverse sheaves.

It is important to stress that, at every step, we are implicitly assuming our sheaves supported on some $\text{Gr}_\leq \mu$, and we are considering equivariant structures with respect to the truncations $G_0(j_\mu)$.

Observations similar to Construction A.7 prove the following:

**Proposition A.9.** We have the following equivalences of schemes or ind-schemes:

$$G_0(R) \simeq \text{Aut}_R(\tilde{X}_\times \times \text{Spec } R, \mathcal{I})$$

$$G_\mathcal{X}(R) \simeq \{ \mathcal{F} \in \text{Bun}_G(X_R), \alpha : \mathcal{F}|_{(X\setminus\{x\})\times \text{Spec } R} \simeq \mathcal{I}|_{(X\setminus\{x\})\times \text{Spec } R}, \mu : \mathcal{F}|_{\tilde{X}_\times \times \text{Spec } R} \simeq \mathcal{I}|_{\tilde{X}_\times \times \text{Spec } R} \}$$

$$(G_\mathcal{X} \times \text{Gr}_G)(R) \simeq \{ \mathcal{F} \in \text{Bun}_G(X_R), \alpha : \mathcal{F}|_{(X\setminus\{x\})\times \text{Spec } R} \simeq \mathcal{I}|_{(X\setminus\{x\})\times \text{Spec } R}, \mu : \mathcal{F}|_{\tilde{X}_\times \times \text{Spec } R} \simeq \mathcal{I}|_{\tilde{X}_\times \times \text{Spec } R}, \mathcal{G} \in \text{Bun}_G(X_R), \beta : \mathcal{F}|_{(X\setminus\{x\})\times \text{Spec } R} \simeq \mathcal{G}|_{(X\setminus\{x\})\times \text{Spec } R} \}$$

$$\text{Conv}(R) \simeq \{ \mathcal{F}, \alpha : \mathcal{F}|_{(X\setminus\{x\})\times \text{Spec } R} \simeq \mathcal{I}|_{(X\setminus\{x\})\times \text{Spec } R}, \mathcal{G} \in \text{Bun}_G(X_R), \eta : \mathcal{F}|_{(X\setminus\{x\})\times \text{Spec } R} \simeq \mathcal{G}|_{(X\setminus\{x\})\times \text{Spec } R} \}$$

**A.2 The convolution product via quotient stacks**

**Definition A.10.** We define the complex stack

$$\text{Hck} \leftarrow G_0 \backslash \text{Gr}$$

as the fpqc quotient stack of $\text{Gr}$ by the left action of $G_0$. We also define

$$\text{Hck}_2 \leftarrow G_0 \backslash \text{Conv}$$

as the fpqc quotient stack of $\text{Conv} = G_\mathcal{X} \times G_0 \backslash \text{Gr}$ by the left action of $G_0$ on the first factor $G_\mathcal{X}$.

The above notation is chosen in order to agree with Remark 2.36.

**Notation A.11.** Let us denote by $\hat{D}$ the affine formal disk $\text{Spec } \mathbb{C}[[t]]$ and by $\hat{D}$ the punctured affine formal disk $\text{Spec } \mathbb{C}((t))$. 

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Proposition A.12. There is an equivalence of stacks in groupoids

\[ \mathcal{G}_0/\text{Gr} = \{ \mathcal{F}_0, \mathcal{F}_1 \in \text{Bun}_G(\hat{D}), \eta : \mathcal{F}_0\vert_D \sim \mathcal{F}_1\vert_D \}. \]

Proof. There is a map from \( \text{Gr} \) to the moduli space appearing in the statement, described as follows. Let \( \mathcal{T} \) be the trivial \( G \)-bundle on \( \hat{D} \); then \( \pi(\mathcal{T}, \alpha) : (\mathcal{T}, \alpha) \mapsto (\mathcal{T}, \mathcal{F}, \alpha^{-1}) \). Let us show that this map is essentially surjective. For any triple \( (\mathcal{F}_0, \mathcal{F}_1, \eta) \) in \( \text{Hck} \) we can choose \( \alpha : \mathcal{F}_1\vert_D \sim \mathcal{T}\vert_D \) and \( \mu : \mathcal{F}_0\vert_D \sim \mathcal{T}\vert_D \) such that \( \eta = \alpha^{-1} \circ \mu\vert_D \). Then the triple \( (\mathcal{T}, \mathcal{F}, \alpha^{-1}) \) is equivalent to \( (\mathcal{F}_0, \mathcal{F}_1, \eta) \) by means of the isomorphism \( (\mu, \text{id}) : (\mathcal{F}_0, \mathcal{F}_1, \eta) \to (\mathcal{T}, \mathcal{F}, \alpha) \).

To conclude the proof, it suffices to prove that the fiber of \( \pi \) at each \( S \)-point of \( \text{Hck} \), for \( S \) a complex scheme, is \( \mathcal{G}_0 \times_{G} S \). But the fiber at \( (\mathcal{F}_0, \mathcal{F}_1, \eta) \) is the set of those \( (\alpha, \mu) \), \( \alpha : \mathcal{F}_1\vert_D \sim \mathcal{T}\vert_D \), \( \mu : \mathcal{F}_0\vert_D \sim \mathcal{T}\vert_D \) such that \( \eta = \alpha^{-1} \circ \mu\vert_D \), which in turn amounts to the set of \( \mu \)'s. But this is \( \mathcal{G}_0 \), since any two trivializations on \( \hat{D} \) are connected by a unique automorphism of \( \mathcal{T} \) on \( \hat{D} \).

In a similar way, one can prove that

\[ \mathcal{G}_0/\text{Conv} = \mathcal{Hck} \sim \{ \mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2 \in \text{Bun}_G(\hat{D}), \eta_1 : \mathcal{F}_0\vert_D \sim \mathcal{F}_1\vert_D, \eta_2 : \mathcal{F}_0\vert_D \sim \mathcal{F}_2\vert_D \}. \]

Remark A.13. Consider the diagram of stacks

\[
\begin{array}{ccc}
\mathcal{G}_0/\text{Gr} \times \mathcal{G}_0/\text{Gr} & \longrightarrow & \mathcal{G}_0/(G_{\mathcal{X}} \times G_0/\text{Gr}) \\
\mathcal{G}_0/(G_{\mathcal{X}} \times G_0/\text{Gr}) & \longrightarrow & \mathcal{G}_0/\text{Gr} \\
\mathcal{G}_0/\text{Gr} & \longrightarrow & \mathcal{G}_0/\text{Gr}
\end{array}
\]

where the action of the second \( G_0 \) factor on \( G_{\mathcal{X}} \times G_0 \) is the antidiagonal one described in (A.2), all other actions are induced by the left multiplication action of \( G_0 \) on \( G_{\mathcal{X}} \) and \( r \) is defined in such a way that the diagram commutes. Then:

- the horizontal map is an equivalence;
- a \( G_0 \)-equivariant perverse sheaf on \( \text{Gr} \) is the same thing as a perverse sheaf on \( \mathcal{G}_0/\text{Gr} \);
- under the identification at the previous point, the convolution product is equivalently described (up to the perverse truncations of the derived tensor product) by

\[ A_1 \ast A_2 = \mathcal{M}_s(r^*(A_1 \boxtimes A_2)). \]

In particular, we can say that the diagram of stacks

\[
\begin{array}{ccc}
\mathcal{Hck} \times \mathcal{Hck} & \longrightarrow & \mathcal{Hck} \\
\mathcal{Hck} & \longrightarrow & \mathcal{Hck}
\end{array}
\]
correctly models the convolution product of $G_0$-equivariant perverse sheaves over the affine Grassmannian (up to the perverse truncation of the derived tensor product appearing in the original definition).

**Lemma A.14.** The map $r$ can be described as

$$r(F_0, F_1, F_2, \eta_1, \eta_2) = ((F_0, F_1, \eta_1), (F_1, F_2, \eta_2)).$$

**Proof.** A priori, $r$ works as follows: choose

$$\mu_0 : F_0 \sim \to T \quad \mu_1 : F_1 \sim \to T \quad \alpha_1 : F_1|_\delta \sim \to T|_\delta \quad \alpha_2 : F_2|_\delta \sim \to T|_\delta$$

such that $\alpha_1^{-1}\mu_0|_\delta = \eta_1, \alpha_1^{-1} \circ \mu_1|_\delta \simeq \eta_2$.

Then by definition

$$r(F_0, F_1, F_2, \eta_1, \eta_2) = ((T, F_1, \alpha_1^{-1}), (T, F_2, \alpha_2^{-1})).$$

But now, there are squares of isomorphisms on $\hat{D}$

\[
\begin{array}{ccc}
\hat{T} & \overset{\alpha_1^{-1}}{\to} & F_1 \\
\mu_0|_\delta & \uparrow & \text{id} \\
F_0 & \overset{\eta_1}{\to} & F_1 \\
\end{array}
\quad
\begin{array}{ccc}
\hat{T} & \overset{\alpha_2^{-1}}{\to} & F_2 \\
\mu_1|_\delta & \uparrow & \text{id} \\
F_1 & \overset{\eta_2}{\to} & F_2 \\
\end{array}
\]

where the vertical maps are induced by the isomorphisms on $\hat{D}$, and we conclude.

**Remark A.15.** Note that this map does not coincide with the map induced by the isomorphism $(id, m) : \text{Conv} \to \text{Gr} \times \text{Gr}$ (cf. [Zhu16, (1.2.14)]), which is instead described by

$$(F_0, F_1, F_2, \eta_1, \eta_2) \mapsto ((F_0, F_1, \eta_1), (F_0, F_2, \eta_1 \circ \eta_2)).$$

Indeed, to prove that this is the same map we would need to build a commuting square

\[
\begin{array}{ccc}
\hat{T} & \overset{\alpha_2^{-1}}{\to} & F_2 \\
\uparrow & \text{id} & \uparrow \\
F_0 & \overset{\eta_2 \eta_1}{\to} & F_2 \\
\end{array}
\]

but here there is no reason why $\alpha_2 \eta_2 \eta_1$ would extend to the complete disk: we know that there exist $\hat{\mu}, \hat{\alpha}$ such that $\eta_2 \eta_1 = \hat{\alpha}^{-1}\hat{\mu}|_\delta$, which in general lies in $G_\infty$ and not in $G_0$. 

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**Remark A.16.** Analogue considerations can be made for the “approximate” versions

\[
\text{Hck} = G_{\Omega(j)} \backslash \text{Gr}_{\leq \mu}
\]

(see Definition A.2) and

\[
\text{Hck}_2 = G_{\Omega(j)} \backslash \text{Conv}_{\leq \mu}
\]

(defined analogously, cfr. Definition 2.33 and Definition 2.39).

### A.3 Equivariant constructible sheaves and the spherical Hecke category

We now review the notion of equivariant constructible sheaves on the affine Grassmannian. All schemes will be complex schemes.

**Definition A.17.** Let \((Y, s)\) be a stratified scheme (see Definition B.3) of finite dimension, and \(R\) a commutative ring. Its \(\infty\)-category of constructible sheaves

\[
\text{Cons}^{\text{id}}(Y, s, R)
\]

is defined as the full subcategory of

\[
\text{Shv}_{\text{et}}(Y, \text{Mod}_R)
\]

spanned by those sheaves whose image in the homotopy category is represented by a bounded complex of sheaves of \(R\)-modules, with finitely presented stalks, and whose cohomology sheaves are constructible with respect to \(s\).

**Definition A.18.** Let \((Y, s)\) be a stratified scheme of finite dimension, \(H\) a group scheme of finite type acting on \(Y\) compatibly with the stratification, \(R\) a commutative ring. The \(\infty\)-category \(\text{Cons}_{H}^{\text{id}}(Y, s, R)\) of \(H\)-equivariant constructible sheaves on \(Y\) with respect to the stratification \(s\) is defined as the limit of the diagram (induced by pullback of sheaves)

\[
\ldots \xleftarrow{i} \text{Cons}^{\text{id}}(H \times H \times Y, s_3, R) \xleftarrow{i} \text{Cons}^{\text{id}}(H \times Y, s_2, R) \xleftarrow{i} \text{Cons}^{\text{id}}(Y, s_1, R), \quad (A.5)
\]

where \(s_i\) is the stratification on \(H \times \cdots \times H \times Y\) which is trivial on the group factors and \(s\) on the last factor, and the diagram is the simplicial diagram encoding the action of \(H\) on \(Y\).

There is also the notion of constructible sheaves with respect to some stratification (instead of a fixed one).

**Definition A.19** ([BL94, 4.1]). Let \(Y\) be a scheme locally of finite type, and \(R\) a commutative ring. We define

\[
\text{D}^{\text{id}}_{\mathcal{C}}(Y, R)
\]

as the full sub-\(\infty\)-category of

\[
\text{Shv}_{\text{et}}(Y, \text{Mod}_R)
\]
A.3 Equivariant constructible sheaves and the spherical Hecke category

spanned by those sheaves whose image in the homotopy category is represented by a bounded complex of sheaves of $R$-modules, with finitely presented stalks, and whose cohomology sheaves are constructible with respect to some algebraic stratification of $Y$.

Note that

$$D_c(Y,R) = \colim_{\text{algebraic stratification of } Y} \text{Cons}(Y,\mathcal{F},k).$$

The colimit (A.6) is filtered, although this is not a trivial result in that it depends on the possibility of refining two algebraic stratification by a common one (see [KRe20]).

**Definition A.20.** Let $Y$ be a finite-dimensional scheme, $H$ a group scheme of finite type acting on $Y$, $R$ a commutative ring. The $\infty$-category $D_{c,H}^{fd}(Y,R)$ of $H$-equivariant constructible sheaves on $Y$ is the limit of the diagram (induced by pullback of sheaves)

$$\cdots \xrightarrow{\times} D_c^{fd}(H \times H \times Y,R) \xrightarrow{\times} D_c^{fd}(H \times Y,R) \xrightarrow{\times} D_c^{fd}(Y,R).$$

(A.7)

In the setting of Definition A.20, if one assumes that there are finitely many orbits of $H$, these orbits form a stratification ([Str14]), which we denote by $s$. Let $R$ be a ring. We have a pullback square of stable $\infty$-categories

$$\begin{array}{ccc}
\text{Cons}_{H}^{\text{fd}}(Y,s,R) & \longrightarrow & \text{Cons}^{\text{fd}}(Y,s,R) \\
\downarrow & & \downarrow \\
D_{c,H}^{\text{fd}}(Y,R) & \longrightarrow & D_c^{\text{fd}}(Y,R).
\end{array}$$

(A.8)

Now, the horizontal arrows in (A.8) are not equivalences. For instance, in the case of the affine Grassmannian, the forgetful functor

$$\text{Perv}_{G_0}(\text{Gr},\mathcal{F},R) \to \text{Perv}(\text{Gr},\mathcal{F},k)$$

is an equivalence, at least for good $R$’s (see for example [BR18, Section 4.4]), but

$$\text{Cons}_{G_0}^{\text{fd}}(\text{Gr},\mathcal{F},R) \to \text{Cons}^{\text{fd}}(\text{Gr},\mathcal{F},R)$$

is not: its essential image only generates the target as a triangulated category ([Ric]).

On the contrary:

**Lemma A.21.** Let $H$ be a group scheme acting on a finite-dimensional scheme $Y$, and suppose that there are finitely many orbits, forming a stratification $s$ of $Y$. Then the functor $\text{Cons}_{H}^{\text{fd}}(Y,s) \to D_{c,H}^{\text{fd}}(Y)$ is an equivalence.

**Proof.** The functor is fully faithful because the transition maps in the colimit (A.6) are, and the colimit is filtered.

Let us now consider an equivariant constructible sheaf $\mathcal{F}$ with respect to some stratification, and let us prove that it is constructible with respect to the orbit stratification. Let us consider
the maximal open subset $U$ of $Y$ where the sheaf is locally constant: this is nonempty since we know that $\mathcal{F}$ is constructible with respect to some stratification, and any stratification of a finite-dimensional scheme has an open stratum. Then $U$ is $H$-stable by equivariantly of $\mathcal{F}$ and maximality of $U$ itself, and thus its complementary is $H$-stable as well and we can apply Noetherian induction.

Let $\mathcal{F}$ be the stratification in Schubert cells of the affine Grassmannian (Remark A.1). For $\mu \in X_*(T)^+$, let $\mathcal{F}_{\leq \mu}$ be its restriction to $\text{Gr}_{\leq \mu}$. The following definition is inspired and motivated by the contents of [Zhu16, Appendix A.1].

**Definition A.22.** Let $G$ be a reductive group over $\mathbb{C}$. We define

$$\text{Cons}^{\text{fd}}_{G_0} (\text{Gr}_{\leq \mu}, \mathcal{F}_{\leq \mu}, R)$$

as

$$\text{Cons}^{\text{fd}}_{G_0, j_\mu} (\text{Gr}_{\leq \mu}, \mathcal{F}_{\leq \mu}, R)$$

where $G_0, j_\mu$ is defined in Definition A.2 and $j_\mu$ is some integer such that the action of $G_0$ on $\text{Gr}_{\leq \mu}$ factors through $G_0 \to G_0, j_\mu)$ (cf. Remark A.1).\(^{14}\) We also define

$$\text{Cons}^{\text{fd}}_{G_0} (\text{Gr}_G, \mathcal{F}, R) = \text{colim}_{\mu \in X_*(T)^+} \text{Cons}^{\text{fd}}_{G_0} (\text{Gr}_{\leq \mu}, \mathcal{F}_{\leq \mu}, R).$$

**Definition A.23.** Let $G$ be a reductive group over $\mathbb{C}$. We define

$$\mathcal{D}^{\text{fd}}_{c, G_0} (\text{Gr}_{\leq \mu}, R)$$

as

$$\mathcal{D}^{\text{fd}}_{c, G_0, j_\mu} (\text{Gr}_{\leq \mu}, R).$$

We also define

$$\mathcal{D}^{\text{fd}}_{c, G_0} (\text{Gr}, R) = \text{colim}_{\mu \in X_*(T)^+} \mathcal{D}^{\text{fd}}_{c, G_0} (\text{Gr}_{\leq \mu}, R).$$

**Proposition A.24.** The map

$$\text{Cons}^{\text{fd}}_{G_0} (\text{Gr}, \mathcal{F}, R) \to \mathcal{D}^{\text{fd}}_{c, G_0} (\text{Gr}, R)$$

is an equivalence.

**Proof.** By definition, the claim can be checked on each $\text{Gr}_{\leq \mu}$, where it is true by Lemma A.21. \(\square\)

**Definition A.25.** Let $(Y, s)$ be a stratified topological space and $R$ a commutative ring. We define the $\infty$-category

$$\text{Cons}(Y, s)$$

as the full subcategory of $\text{Sh}(Y, \text{Mod}_R)$ spanned by those sheaves whose image in the homotopy category are constructible with respect to $s$.

\(^{14}\)By [BL94, § 2.6.2.] this definition is independent of $j_\mu$. 

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A.3 Equivariant constructible sheaves and the spherical Hecke category

**Definition A.26.** Let \((Y, s)\) be a stratified topological space and \(H\) a topological group acting on \(Y\) compatibly with \(s\). We define the \(\infty\)-category

\[
\text{Cons}_H(Y, s, R)
\]

as the limit of the diagram (induced by pullback of sheaves)

\[
\ldots \xrightarrow{\varepsilon} \text{Cons}(H \times H \times Y, s_3, R) \xrightarrow{\varepsilon} \text{Cons}(H \times Y, s_2, R) \xrightarrow{\varepsilon} \text{Cons}(Y, s_1, R), \tag{A.9}
\]

where \(s_i\) is the stratification on \(H \times \cdots \times H \times Y\) which is trivial on the group factors and \(\mathcal{S}\) on the last factor, and the diagram is the simplicial diagram encoding the action of \(H\) on \(Y\).

Note that here we did not assume that our objects are bounded finitely presented. The version where these two conditions are assumed will be denoted by

\[
\text{Cons}^\text{fd}_H(Y, s, R).
\]

Recall the stratified analytification functor \((-)^\text{an}\) from Construction B.4.

**Proposition A.27.** Let \((Y, s)\) be a finite-dimensional scheme, \(H\) a group scheme of finite type acting on \((Y, s)\), and \(R\) equal to a finite or profinite ring (e.g. \(\mathbb{Z}/m\mathbb{Z}\) or to an algebraic extension of \(\mathbb{Q}_\ell\) (e.g. \(\overline{\mathbb{Q}}_\ell\)). Then there is an equivalence of \(\infty\)-categories

\[
\text{Cons}^\text{fd}_H(Y, s, R) \simeq \text{Cons}^\text{fd}_H(Y^\text{an}, s^\text{an}, R).
\]

**Proof.** By the proof [BGH20, Proposition 12.6.4] (in turn building upon [AGV72, Théorème XVI. 4.1]), there is an equivalence of \(\infty\)-categories

\[
\text{Cons}^\text{fd}_H(Y, s, R) \simeq \text{Cons}^\text{fd}_H(Y^\text{an}, s^\text{an}, R).
\]

This proof only works with the mentioned coefficients, in that it relies on the Exodromy theorem both on the algebraic and topological side. Now, whereas the theorem has been proven in great generality on the topological side ([PT22]), the only fully established version on the algebraic side is (to the knowledge of the author) [BGH20, Theorem 13.7.8] (for the finite/profinite case) and [BGH20, Theorem 13.8.8] (for algebraic extensions of \(\mathbb{Q}_\ell\)).

Finally, the construction which adds the equivariant structure is the same on both sides (the analytification functor commutes with colimits and finite limits).

**Definition A.28.** Let \(G\) be a complex reductive group. We define

\[
\text{Cons}_{G^\text{an}}(\text{Gr}^\text{an}, \mathcal{S}^\text{an}, R) = \text{Cons}_{G^\text{an,0}}(\text{Gr}^\text{an}, \mathcal{S}^\text{an}, R)
\]

and

\[
\text{Cons}_{G^\text{an}}(\text{Gr}^\text{an}, \mathcal{S}^\text{an}, R) = \text{colim}_{\mu \in \mathcal{X}_G(T)} \text{Cons}_{G^\text{an}}(\text{Gr}^\text{an}, R).
\]

\[\text{Although the statement is given for categories of constructible sheaves with respect to some algebraic stratification (and its analytic counterpart) the proof actually proceeds by establishing the equivalence when the stratification is fixed, and then passes to the colimit.}\]
The first definition is independent of \( j_\mu \) by Proposition 4.3. The finite-dimensional version is defined analogously and denoted by

\[ \text{Cons}^{\text{fd}}_{G_\text{an}}(\text{Gr}^{\text{an}}, \mathcal{S}^{\text{an}}, R). \]

Proposition A.24 and Proposition A.27 imply the following:

**Corollary A.29.** Let \( G \) be a complex reductive group and \( R \) a profinite ring or an algebraic extension of \( \mathbb{Q}_\ell \). There is an equivalence of \( \infty \)-categories

\[ \mathcal{T}^{\text{fd}}_{c,G_\text{an}}(\text{Gr}, R) \simeq \text{Cons}^{\text{fd}}_{G_\text{an}}(\text{Gr}, \mathcal{S}, R) \simeq \text{Cons}^{\text{fd}}_{G_\text{an}}(\text{Gr}^{\text{an}}, \mathcal{S}^{\text{an}}, R). \]

**Definition A.30.** Let \( G \) be a complex reductive group and \( R \) a commutative ring. The *topological spherical Hecke category* of \( G \) with coefficients in \( R \) is

\[ \text{Sph}(G, R)^{\text{top}} = \text{Cons}^{\text{an}}_{G_\text{an}}(\text{Gr}^{\text{an}}, R) \]

(without assumptions of boundedness and finite presentability of stalks). If \( R \) is a profinite ring (e.g. \( \mathbb{Z}/m, \mathbb{Z}_\ell \)) or an algebraic extension of \( \mathbb{Q}_\ell \) (e.g. \( \mathbb{Q}_\ell, \overline{\mathbb{Q}}_\ell \)) we define the *small spherical Hecke category* of \( G \) with coefficients in \( R \) as

\[ \text{Sph}(G, R)^{\text{loc.c.}} = \text{Cons}^{\text{fd}}_{G_\text{an}}(\text{Gr}^{\text{an}}, R) \simeq \text{Cons}^{\text{fd}}_{G_\text{an}}(\text{Gr}, R). \]

In most recent works in the Geometric Langlands Program, a renormalization of \( \text{Sph}(G) \) is used:

**Definition A.31.** Let \( G \) be a complex reductive group and \( R \) be a profinite ring or an algebraic extension of \( \mathbb{Q}_\ell \). The *renormalized spherical Hecke category* of \( G \) with coefficients in \( R \) is

\[ \text{Sph}(G, R)^{\text{ren}} = \text{Ind}(\text{Sph}(G)^{\text{loc.c}}). \]

Note that, with the mentioned coefficients, the small and the renormalized spherical Hecke category do not distinguish between the algebraic and the analytic setting. Therefore, the same is true for the main result of our paper (Corollary 4.10), although the proof uses features of the analytic setting.

### B Recollections and complements in stratified homotopy theory

Our goal in this appendix is to prove:

**Theorem B.1** (Theorem B.39). *Let \( R \) be a commutative ring, \( \text{StrTStk}_{\text{con}}^{\text{all,subm}} \) be the class of maps from Definition B.33. Then there is a well-defined symmetric monoidal functor

\[ \text{Cons}^{\text{corr,} \otimes}_R : \text{Corr}(\text{StrTStk}_{\text{con}}^{\text{all,subm}}) \to \mathcal{P}_R^{\text{Rr,} \otimes} \]

which on representable objects is given by taking constructible sheaves with respect to a fixed stratification:

\[ (X, s) \mapsto \text{Cons}(X, s, R). \]
We will now give a series of definitions necessary to make sense of this theorem, and then prove it.

B.1 Stratified analytification

The following definitions are particular cases of [BGH20, 8.2.1] and ff.

**Definition B.2.** Let $\text{Top}$ be the 1-category of topological spaces. The category of **stratified topological spaces** is defined as

$$\text{StrTop} = \text{Fun}(\Delta^1, \text{Top}) \times_{\text{Top}} \text{Poset},$$

where the map $\text{Fun}(\Delta^1, \text{Top}) \to \text{Top}$ is the evaluation at 1, and $\text{Alex} : \text{Poset} \to \text{Top}$ assigns to each poset $P$ its underlying set with the so-called Alexandrov topology (see [BGH20, Definition 1.1.1]).

Note that $\text{StrTop}$ is complete and cocomplete, because $\text{Top}$, $\text{Fun}(\Delta^1, \text{Top})$ and $\text{Poset}$ are.

**Definition B.3.** The category of **stratified schemes** is defined as $\text{StrSch} = \mathbf{Sch} \times_{\text{Top}} \text{StrTop}$, where the map $\mathbf{Sch} \to \text{Top}$ sends a scheme $X$ to its underlying Zariski topological space, and the other map is the evaluation at $[0]$. Let $\text{StrAff}$ be the full subcategory of stratified affine schemes.

Analogously, one can define stratified **complex** schemes $\text{StrSch}^\text{C}$.

There is an analytification functor $\mathbf{Sch}_C^{\text{lf}} \to \mathbf{An}_C$, the source being the category of complex schemes locally of finite type, and the target being the category of complex analytic spaces. This is defined in [Rey71, Théorème et définition 1.1]. We can forget the structure sheaf of holomorphic functions and recover an underlying Hausdorff topological space (which corresponds to the operation denoted by $|-|$ in [Rey71]). We thus obtain a functor which by abuse of notation we denote by

$$(-)^\text{an} : \mathbf{Sch}_C^{\text{lf}} \to \text{Top}$$

(instead of $|(-)^\text{an}|$).

**Construction B.4.** Let now

$$\text{StrSch}_C^{\text{lf}} = \mathbf{Sch}_C^{\text{lf}} \times_{\text{Top, ev}_0} \text{StrTop}$$

$$\text{StrAn} = \mathbf{An} \times_{\text{Top, ev}_0} \text{StrTop}. $$

There is a natural stratified version of the functor $(-)^\text{an}$, namely the one induced by the map of ringed spaces $\nu : S^\text{an} \to S$:

$$\text{StrSch}_C^{\text{lf}} \to \text{StrAn} \to \text{StrTop}$$

$$(S, s) \mapsto (S^\text{an}, s \circ \nu).$$
Remark B.5. In our setting, there are two specially relevant Grothendieck topologies to consider: the étale topology on the algebraic side and the topology of local homeomorphisms on the topological side (which has however the same sheaves as the topology of open embeddings). We have thus sites $(\text{Aff}_C, \text{et})$ and $(\text{Top}, \text{loc})$. We can therefore consider the following $(2, 1)$-categories:

- $\text{Shv}_{\text{et}}(\text{Aff}_C, \text{Grpd})$, which we denote by $\text{Stk}_C$.
- $\text{Shv}_{\text{et}}(\text{Aff}^{\text{lf}}_C, \text{Grpd})$, which we denote by $\text{Stk}^{\text{lf}}_C$.
- $\text{Shv}_{\text{loc}}(\text{Top}, \text{Grpd})$, which we denote by $\text{TStk}$.

The analytification functor sends étale coverings to coverings in the local homeomorphism topology and therefore induces a functor from the former to the latter.

Now, there are analogs of both topologies in the stratified setting. Namely, we can define $\text{stret}$ as the topology whose coverings are étale coverings whose stratification is induced by that on the base, and $\text{strloc}$ as the topology whose coverings are jointly surjective families of local homeomorphisms such that, again, the stratification on the total space is induced by that on the base. Therefore, we have well-defined stratified analogues:

- $\text{Sh}_{\text{stret}}(\text{StrAff}_C, \text{Grpd})$, which we denote by $\text{StrStk}_C$;
- $\text{Sh}_{\text{stret}}(\text{StrSch}^{\text{lf}}_C, \text{Grpd})$, which we denote by $\text{StrStk}^{\text{lf}}_C$;
- $\text{Sh}_{\text{strloc}}(\text{StrTop}, \text{Grpd})$, which we denote by $\text{StrTStk}$.

The stratified analytification functor $(-)^{\text{an}} : \text{StrSch}^{\text{lf}}_C \rightarrow \text{StrTop}$ sends stratified étale coverings to stratified coverings in the topology of local homeomorphism, and thus induces a functor

$$\text{StrStk}^{\text{lf}}_C \rightarrow \text{StrTStk}.$$ 

Definition B.6. We define $(2, 1)$-categories

- $\underline{\text{Stk}}_C = \text{Fun}((\text{Pro}(\text{Stk}_C))^{\text{op}}, \text{Grpd})$
- $\underline{\text{Stk}}^{\text{lf}}_C = \text{Fun}((\text{Pro}(\text{Stk}^{\text{lf}}_C))^{\text{op}}, \text{Grpd})$
- $\underline{\text{TStk}} = \text{Fun}((\text{Pro}(\text{TStk}))^{\text{op}}, \text{Grpd})$

- $\underline{\text{StrStk}}_C = \text{Fun}((\text{Pro}(\text{StrStk}_C))^{\text{op}}, \text{Grpd})$
- $\underline{\text{StrStk}}^{\text{lf}}_C = \text{Fun}((\text{Pro}(\text{StrStk}^{\text{lf}}_C))^{\text{op}}, \text{Grpd})$
- $\underline{\text{StrTStk}} = \text{Fun}((\text{Pro}(\text{StrTStk}))^{\text{op}}, \text{Grpd})$.

We have a naturally induced analytification functor

$$(-)^{\text{an}} : \underline{\text{StrStk}}^{\text{lf}}_C \rightarrow \underline{\text{StrTStk}}.$$  (B.1)
B.2 Constructible sheaves functors

Fix a commutative ring $R$.

Recall B.7. Let $(X, s)$ be a stratified topological space. The notion of conical stratification is introduced [Lur17, Definition A.5.5] and amounts to asking that around each point of the stratified space $(X, s)$ under examination there exists a neighbourhood which is stratified homomorphic to $Z \times C(Y)$ where $Z$ is an unstratified space and $C(Y)$ is the stratified open cone of a stratified space $Y$. Being conical is the main condition required to a stratified space in order to make the so-called Exodromy Theorem (Recall B.9) true. The other two are that $X$ is “locally of singular shape”, which is implied for example by being locally contractible, and that the poset satisfies the ascending chain condition, which is implied for example by the poset being finite.

For simplicity, in the present paper a stratified space is called conical if satisfies all three conditions above:

Definition B.8. The category $\text{StrTop}_{\text{con}}$ is the full subcategory of $\text{StrTop}$ spanned by those stratified spaces $(X, s : X \to P)$ such that:

- $X$ is locally of singular shape ([Lur17, Definition A.4.15]);
- $P$ satisfies the ascending chain condition;
- the stratification is conical in the sense of [Lur17, Definition A.5.5].

This category admits finite products, essentially because the product of two cones is the cone of the join space. Therefore, there is a well-defined symmetric monoidal Cartesian structure $\text{StrTop}_{\text{con}}$.

Recall B.9. Let $(X, s)$ be a conical stratified space. By [Lur17, Theorem A.9.3] (nowadays known as the Exodromy Theorem in topology), the $\infty$-category $\text{Cons}(X, s, R)$ of constructible sheaves (see Definition A.25) is equivalent to the $\infty$-category

$$\text{Fun}(\text{Exit}(X, s), \text{Mod}_R).$$

Here $\text{Exit}(X, s)$ is the $\infty$-category of exit paths on $(X, s)$ (see [Lur17, Definition A.6.2], where it is denoted by $\text{Sing}^A(X)$, $A$ being the poset associated to the stratification).

Also, the equivalence restricts to an equivalence

$$\text{Cons}^\text{fd}(X, s, R) \simeq \text{Fun}(\text{Exit}(Y, s), \text{Mod}^\text{fd}_R)$$

where $\text{Mod}^\text{fd}_R$ denotes the full sub-$\infty$-category of bounded $R$-modules with finitely presented stalks.

We will often write $\text{Cons}(\_\_)$ instead of $\text{Cons}(\_\_\_, R)$. 

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Recall B.10. The (very large) $\infty$-category of presentable $\infty$-categories and left adjoint functors is denoted by $\mathcal{P}_R^L$. The $\infty$-category of presentable $\infty$-categories and right adjoint functors is denoted by $\mathcal{P}_R^R$.

**Notation B.11.** Let $R$ be a commutative ring. We denote by $\mathcal{P}_R^L$ be the $\infty$-category of presentable stable $R$-linear $\infty$-categories, i.e. the $\infty$-category $\text{Mod}_{\text{Mod}_R}(\mathcal{P}_R^{L,\otimes})$ (it is denoted by $\text{LinCat}_R$ in [Lur11, Definition 6.2]). Recall that $\mathcal{P}_R^L$ carries a natural symmetric monoidal structure inherited from $\mathcal{P}_R^{L,\otimes}$, which we denote by $\mathcal{P}_R^{L,\otimes}$.

**Notation B.12.** Let $\mathcal{C}$ be a symmetric monoidal $\infty$-category. We denote by $\mathcal{C}^{\otimes,\text{op}}$ the “pointwise-opposite” symmetric monoidal structure on $\mathcal{C}^{\text{op}}$ ([Bea], [BGN18]). For example, if $\mathcal{C}$ has finite products, $\mathcal{C}^{\times,\text{op}} \simeq (\mathcal{C}^{\text{op}})^{\Pi}$.

**Remark B.13.** Let $\mathcal{P}_R^R$ be the $\infty$-category of presentable $\infty$-categories with right adjoint functors between them. We denote by $\mathcal{P}_R^{R,\otimes}$ the symmetric monoidal structure induced by the equivalence $$(\mathcal{P}_R^{L})^{\text{op}} \simeq \mathcal{P}_R^{R}$$ i.e. $$\mathcal{P}_R^{R,\otimes} = \mathcal{P}_R^{L,\otimes,\text{op}}.$$ As usual, we also have the $R$-linear variant $\mathcal{P}_R^{R,\otimes}$.

The subcategory $\mathcal{P}_R^{LR} \subset \mathcal{P}_R^L$ spanned by all objects and those functors which are both continuous and cocontinuous also admits a symmetric monoidal structure, which is auto-dual and makes the inclusion functors $\mathcal{P}_R^{LR,\otimes} \subset \mathcal{P}_R^{L,\otimes}$ and $\mathcal{P}_R^{LR,\otimes} \subset \mathcal{P}_R^{R,\otimes}$ both symmetric monoidal.

**Notation B.14.** Let $R$ be a commutative ring. We denote by $\text{Cat}_{\infty, R}$ the $\infty$-category of small stable $R$-linear $\infty$-categories, or in other words, the $\infty$-category of small stable $\infty$-categories $\mathcal{C}$ together with an $R$-linear structure on $h\mathcal{C}$, and exact $R$-linear functors. We denote the Cartesian monoidal structure on this $\infty$-category by $\text{Cat}_{\infty, R}^{\times}$.

Recall B.9 implies that

**Corollary B.15.** Let $(X, s) \in \text{StrTop}_{\text{con}}$, and $R$ be a ring. Then $\text{Cons}(X, s, R)$ is a presentable stable $R$-linear $\infty$-category and $\text{Cons}^d(X, s, R)$ is a small $R$-linear $\infty$-category.

**Lemma B.16.** The functor $$\text{Exit} : \text{StrTop}_{\text{con}} \to \text{Cat}_{\infty}$$ $$(X, s : X \to P) \mapsto \text{Exit}(X, s)$$ carries a symmetric monoidal structure when we endow both source and target with the Cartesian symmetric monoidal structure. In other words, $\text{Exit}$ preserves finite products.
**Proof.** Given two stratified topological spaces $X, s : X \to P, Y, t : Y \to Q,$ in the notations of [Lur17, A.6], consider the commutative diagram of simplicial sets

\[
\begin{array}{ccc}
\text{Sing}^P(X \times Y) & \longrightarrow & \text{Sing}(X \times Y) \\
\downarrow & & \downarrow \\
N(P \times Q) & \longrightarrow & \text{Sing}(P \times Q) \\
\sim & & \sim \\
N(P) \times N(Q) & \longrightarrow & \text{Sing}(P) \times \text{Sing}(Q).
\end{array}
\]

The inner diagram is Cartesian by definition. Therefore the outer diagram is Cartesian, and we conclude that $\text{Sing}^P(X \times Y)$ is canonically equivalent to $\text{Sing}^P(X) \times \text{Sing}^Q(Y).$ Since $\text{Sing}^P(X)$ models the $\infty$-category of exit paths of $X$ with respect to $s,$ and similarly for the other spaces, we conclude. \qed

**Lemma B.17** ([Lur17, Remark 4.8.1.8 and Proposition 4.8.1.15]). There exist symmetric monoidal functors

\[
\begin{align*}
P^*: \text{Cat}_{\infty}^{\times \text{op}} & \to \text{Pr}^{\text{LR}, \otimes} \\
P_{(-)}: \text{Cat}_{\infty}^* & \to \text{Pr}^{L^*, \otimes} \\
P_{(\cdot)}: \text{Cat}_{\infty}^* & \to \text{Pr}^{R^*, \otimes}
\end{align*}
\]

sending an $\infty$-category $\mathcal{C}$ to the $\infty$-category of $S$-valued\(^{16}\) presheaves $\mathcal{P}(\mathcal{C}),$ and a functor $F : \mathcal{C} \to \mathcal{D}$ to the functor

\[
F^* : \mathcal{P}(\mathcal{D}) \to \mathcal{P}(\mathcal{C})
\]

induced by precomposition by $F,$ resp. to the functor

\[
F_4 : \mathcal{P}(\mathcal{C}) \to \mathcal{P}(\mathcal{D})
\]

induced by left Kan extension along $F,$ resp. to the functor

\[
F_\_ : \mathcal{P}(\mathcal{C}) \to \mathcal{P}(\mathcal{D})
\]

induced by right Kan extension along $F.$

**Proof.** The functor $P_{(-)}$ takes values in $\text{Pr}^L$ since for each $F : \mathcal{C} \to \mathcal{D}$ the functor $F_4 : \mathcal{P}(\mathcal{C}) \to \mathcal{P}(\mathcal{D})$ admits a right adjoint given by $F^*.$ The existence of the (strong) monoidal structure on it follows from [Lur17, Proposition 4.8.1.8]. Indeed, if we take $\mathcal{K} = \emptyset, \mathcal{K}' = \{\text{all simplicial sets}\}$ in loc.cit., the functor $\mathcal{C} \to \mathcal{P}_{\mathcal{K}}^\mathcal{C}(\mathcal{C})$ is what we call $P_{(-)},$ since it is the functor sending $F : \mathcal{C} \to \mathcal{D}$ to the left Kan extension of the Yoneda embedding $\mathcal{C} \to \mathcal{P}(\mathcal{C})$ along $\mathcal{C} \to \mathcal{F} \to \mathcal{D} \to \mathcal{P}(\mathcal{D})$ (see the proof of [Lur09, Proposition 5.3.6.2]).

\(^{16}\)S is the $\infty$-category of spaces.
As for \( \mathcal{P}(\ast) \) and \( \mathcal{P}(\triangleright) \), the claim follows from the fact that we have a symmetric monoidal equivalence (Notation B.12).

\[ \mathcal{P}_{R, \otimes} \sim \mathcal{P}_{L, \otimes}. \]

\[ \square \]

**Corollary B.18.** Let \( R \) be a ring. There are well-defined symmetric monoidal functors

- \( \text{Cons}_{\ast, \otimes} : \text{StrTop}_{\ast} \rightarrow \mathcal{P}_{R, \otimes} \)
  \[ (X, s) \mapsto \text{Cons}(X, s, R) \]
  \[ f \mapsto f^\ast = \circ \text{Exit}(f). \]

- \( \text{Cons}_{\triangleright, \otimes} : \text{StrTop}_{\triangleright} \rightarrow \mathcal{P}_{L, \otimes} \)
  \[ (X, s) \mapsto \text{Cons}(X, s, R) \]
  \[ f \mapsto f^- := \text{Lan}_{\text{Exit}(f)}. \]

- \( \text{Cons}_{(\ast, \triangleright), \otimes} : \text{StrTop}_{(\ast, \triangleright)} \rightarrow \mathcal{P}_{R, \otimes} \)
  \[ (X, s) \mapsto \text{Cons}(X, s, R) \]
  \[ f \mapsto f^- := \text{Ran}_{\text{Exit}(f)}. \]

**Proof.** Let us deal with the first case first. The previous constructions provide us with a symmetric monoidal functor

\[ \text{StrTop}_{\ast} \overset{\text{Exit}(\ast)}{\longrightarrow} \text{Cat}_{\infty} \overset{\text{Sp}}{\longrightarrow} \mathcal{P}_{R, \otimes} \]

sending

\[ (X, s) \mapsto \text{Fun}(\text{Exit}(X, s), S), \]
\[ f \mapsto f^\ast. \]

But now, with the notations of [Lur17, Subsection 1.4.2], for any \( \infty \)-category \( \mathcal{C} \) we have

\[ \text{Fun}(\mathcal{C}, \text{Sp}) = \text{Sp}(\text{Fun}(\mathcal{C}, S)). \]

Then we can apply [Rob14, Remark 4.2.16] and finally [Rob14, Theorem 4.2.5], which establish a symmetric monoidal structure for the functor \( \text{Sp}(\ast) : \mathcal{P}_{R, \otimes} \rightarrow \mathcal{P}_{R, \otimes} \). The upgrade from \( \text{Sp} \) to \( \text{Mod}_R \) is straightforward and produces a last functor \( \mathcal{P}_{R, \otimes}^{\ast, \otimes} \rightarrow \mathcal{P}_{R, \otimes}^{\ast, \otimes} \) (the \( \infty \)-category of presentable stable \( R \)-linear categories with left adjoint functors). Finally, one notices that the functor factors through the symmetric monoidal inclusion \( \mathcal{P}_{R, \otimes}^{\ast, \otimes} \subset \mathcal{P}_{R, \otimes} \). The proof for the other two cases is similar. \[ \square \]
From now on, we will often omit the “linear” part of the matter and prove statements about the relevant functor \( \text{StrTop}_{\text{con}}^X \to \mathcal{P}_R \) or \( \mathcal{P}_L \), because the passage to the stable \( R \)-linear setting is symmetric monoidal.

**Definition B.19.** Let \( \alpha, \beta : (X, s) \to (Y, t) \) be two stratified maps between stratified topological spaces. Let \( \tilde{s} \) be the stratification of \([0, 1] \times X\) induced by the projection \([0, 1] \times X \to X\). A stratified homotopy between \( \alpha \) and \( \beta \) is a stratified map

\[
H : ([0, 1] \times X, \tilde{s}) \to (Y, t)
\]

such that \( H(0, -) = f, H(1, -) = g \).

**Definition B.20.** A stratified homotopy equivalence is a stratified map \( f : (X, s) \to (Y, t) \) such that there exist a stratified map \( g : (Y, t) \to (X, s) \) and stratified homotopies \( f g \simeq \text{id}_Y, g f \simeq \text{id}_X \).

**Remark B.21.** A stratified homotopy equivalence induces an isomorphism at the level of posets, and homotopy equivalences on each stratum.

**Remark B.22.** The functors constructed in Corollary B.18 take stratified homotopy equivalences to equivalences of infinite-categories. Indeed, consider \( H : [0, 1] \times Y \to Y \) as above. This map has the property that the compositions \( \{0\} \times Y \to [0, 1] \times Y \to Y \) is \( f \circ g \) and \( \{1\} \times Y \to [0, 1] \times Y \to Y \) is \( \text{id}_Y \). Since the functor \( \text{Exit}(-) \) preserves products (Lemma B.16), one gets a map \( \text{Exit}([0, 1]) \times \text{Exit}(Y) \to \text{Exit}(Y) \) such that

\[
\text{Exit}(\{0\}) \times \text{Exit}(Y) \to \text{Exit}([0, 1]) \times \text{Exit}(Y) \to \text{Exit}(Y)
\]

is \( \text{Exit}(f) \circ \text{Exit}(g) \) and

\[
\text{Exit}(\{1\}) \times \text{Exit}(Y) \to \text{Exit}([0, 1]) \times \text{Exit}(Y) \to \text{Exit}(Y)
\]

is \( \text{Exit}(\text{id}_Y) \). But since \([0, 1]\) is contractible and unstratified, \( \text{Exit}([0, 1]) \) is equivalent to the terminal infinite-category \(*\), and the two compositions are equivalent as functors \( \text{Exit}(Y) \to \text{Exit}(Y) \). Therefore \( \text{Exit}(f) \) is a left inverse to \( \text{Exit}(g) \). By repeating the argument on \( K \) one obtains that \( \text{Exit}(g) \) is a left inverse to \( \text{Exit}(f) \).

**Definition B.23.** We define

\[
\text{StrTStk}_{\text{con}} = \text{Shv}_{\text{strloc}}(\text{StrTop}_{\text{con}}, \text{Grpd})
\]

and

\[
\text{StrTStk}_{\text{con}} = \text{Fun}((\text{Pro}(\text{StrTStk}_{\text{con}}))^{\text{op}}, \text{Grpd}).
\]

**Construction B.24.** There is a right Kan extension diagram

\[
\begin{array}{ccc}
\text{StrTop}_{\text{con}}^{\text{op}} & \xrightarrow{\text{Cons}^{(\text{op})}} & \mathcal{P}_R^R \\
\downarrow & & \downarrow \\
\text{StrTStk}_{\text{con}}^{\text{op}} & \text{cofib} & .
\end{array}
\]
or equivalently, a left Kan extension diagram

\[
\begin{array}{ccc}
\text{StrTop}_\text{con} & \xrightarrow{\text{Cons}_{\downarrow}} & \mathcal{P}_R^L \\
\downarrow & & \Rightarrow \\
\text{StrTStk}_\text{con} & \xrightarrow{} & \\
\end{array}
\]

which we still denote by Cons_{\downarrow}.

**Remark B.25.** Performing a left Kan extension on a category of pro-objects may sound like a poor choice, and indeed it is in abstract, since for example, if the indexing category is \( \mathbb{Z}_{\leq 0} \) then the colimit will just be the evaluation at the terminal object of the diagram. However, in practice, all transition functors in our diagrams will be equivalences, hence there will be no difference between limit and colimit. In formulas, if

\[
\text{"lim"}_0 \leftarrow c_{-1} \leftarrow \ldots
\]

is a sequential pro-object in a category \( \mathcal{C} \), and

\[
F : \mathcal{C} \to \mathcal{P}_R^L
\]

is a functor which sends all \( c_i \to c_{i+1} \) to equivalences, then

\[
\text{LKE}_{\mathcal{C} \to \text{pro}(\mathcal{C})}(F : \mathcal{C} \to \mathcal{P}_R^L)(\text{"lim"}_0 \leftarrow c_{-1} \leftarrow \ldots) = \text{colim}_{c_0 \leftarrow c_{-1} \leftarrow \ldots} F(c_i) \simeq F(c_0) \simeq \text{lim}_{c_0 \leftarrow c_{-1} \leftarrow \ldots} F(c_i)
\]

which is the formula that usually makes sense for pro-objects. The last equivalence is motivated by the fact that \( F \) sends all transition maps to equivalences.

**Remark B.26.** Defining the "correct" extension for pro-objects (i.e. the right Kan extension with covariant functoriality) would be possible, but it would unnecessarily complicate our constructions because of its interaction with the formalism of correspondences. Since in our paper we are in the situation depicted Remark B.25 (see Remark 4.4), we decided to stick to this treatment. One could argue that, at this point, using pro-object is pointless, since it would be sufficient to evaluate at the terminal object of the diagram. Remark 4.4 addresses this point, explaining how mixing our pro-objects with the addition of certain filtered colimits makes the situation nontrivial, and our approach meaningful.

**Definition B.27.** An object \( Z \in \widehat{\text{StrTStk}}_\text{con} \) is **representable** if it belongs to the essential image of the fully faithful functor \( \text{StrTop}_\text{con} \to \text{StrTStk}_\text{con} \).

**Definition B.28.** A **stratified homotopy equivalence** in \( \widehat{\text{StrTStk}}_\text{con} \) is a morphism \( f : \mathcal{X} \to \mathcal{Y} \) such that, for any \( Z \) representable and a morphism \( X \to \mathcal{Y} \) in \( \text{StrTStk}_\text{con} \), the pullback \( Z \times_{\mathcal{Y}} \mathcal{X} \) is representable and the resulting map \( Z \times_{\mathcal{Y}} \mathcal{X} \to Z \) is equivalent to a stratified homotopy equivalence in \( \text{StrTop}_\text{con} \).
Proposition B.29. The extended functors Cons\(^{(s)}\), Cons\(^{(\dashv)}\) from Construction B.24 send stratified homotopy equivalences to equivalences of \(\infty\)-categories.

Proof. This follows from Remark B.22 and by construction. \(\square\)

B.3 Constructible sheaves and correspondences

Definition B.30. Let \(f : (X, s) \to (Y, t)\) be a morphism in StrTop\(_{\text{con}}^\wedge\). We say that \(f\) is a smooth stratified submersion if, locally in the topology of stratified local homeomorphisms on \(X\), it is of the form of a projection \((Y, t) \times \mathbb{R}^N \to (Y, t)\) for some \(N\) (where \(\mathbb{R}^N\) is seen as a trivially stratified space).

We denote this class by subm. When both \(X\) and \(Y\) are unstratified, we will just speak about smooth topological submersion.

Example B.31 (Smooth algebraic maps). Let \(X, Y\) be complex schemes, locally of finite type, and \(f : X \to Y\) be a smooth morphism in the sense of algebraic geometry. Then \(f^{an}\) is a smooth topological submersion. In particular, the analytification of a smooth scheme locally of finite type is a topological manifold.

Proof. This is [Vak22, Exercise 13.6.A]. \(\square\)

Example B.32 (Stratified torsors with smooth fiber). Let \(X, Y\) be complex schemes, locally of finite type, and \(f : X \to Y\) be a torsor whose fiber is a smooth unstratified scheme. Then \(f^{an}\) is a smooth stratified submersion.

Proof. By hypothesis, locally in \(Y\) the map is of the form \(Z \times X \to Y\), where \(Z\) is a smooth unstratified scheme, locally of finite type. By Example B.31, \(Z\) is a topological manifold, hence the claim. \(\square\)

Definition B.33. We define the class of representable morphisms in StrStk\(_{\text{con}}^\wedge\) to be the smallest class of morphisms \(f : \mathcal{X} \to \mathcal{Y}\) in StrStk\(_{\text{con}}^\wedge\) such that for every \(Z \in \text{StrTop}_{\text{con}}\) and a map \(Z \to Y\), the pullback \(Z \times_Y X\) in StrStk\(_{\text{con}}^\wedge\) belongs to StrTop\(_{\text{con}}\).

By slightly abusing notation, we define the class

\[
\text{subm} \subseteq \text{Mor}(\text{StrStk}_{\text{con}}^\wedge)
\]

to be the class of morphisms \(f : \mathcal{X} \to \mathcal{Y}\) which are representable and such that under any map \(Z \to Y\) with \(Z\) representable (i.e. in StrTop\(_{\text{con}}\)), \(f\) pulls back to a map which is a stratified smooth submersion in the sense of Definition B.30.

We now turn to a some essential recalls of the theory of Gaitsgory and Rozenblyum’s correspondences.
Let $\mathcal{C}$ be an $(\infty, 1)$-category, and $\text{vert, horiz, adm}$ classes of morphisms with the properties [GR17, Chapter 7, 1.1.1]. The $(\infty, 2)$-category of correspondences $\text{Corr}(\mathcal{C})^{\text{adm}}_{\text{vert, horiz}}$ is defined in [GR17, Chapter 7, 1.2.5].

**Remark B.34** ([GR17, Chapter 7, 1.3.3]). If $\text{adm} = \text{isom}$, then $\text{Corr}(\mathcal{C})^{\text{adm}}_{\text{vert, horiz}}$ is an $(\infty, 1)$-category, which we denote simply by $\text{Corr}(\mathcal{C})_{\text{vert, horiz}}$. We will always be in this situation. However, the results from [GR17] used in the proof of Theorem B.39 rely on $(\infty, 2)$-categorical constructions.

**Remark B.35.** Let $\mathcal{C}_{\text{vert}}$ and $\mathcal{C}_{\text{horiz}}$ the subcategories of $\mathcal{C}$ spanned by all objects and vertical or horizontal morphisms, respectively. There are embeddings

$$\mathcal{C}_{\text{vert}} \to \text{Corr}(\mathcal{C})^{\text{adm}}_{\text{vert, horiz}}$$

$$\mathcal{C}_{\text{horiz}}^{\text{op}} \to \text{Corr}(\mathcal{C})^{\text{adm}}_{\text{vert, horiz}}.$$

**Remark B.36** ([GR17, Chapter 9, 2.1.3]). If $\mathcal{C}^\otimes$ is a symmetric monoidal $\infty$-category and the classes $\text{vert, horiz, adm}$ are closed under tensor product, then there is a symmetric monoidal structure on $\text{Corr}(\mathcal{C})^{\text{adm}}_{\text{vert, horiz}}$ which we denote by $\text{Corr}(\mathcal{C})^{\text{adm, } \otimes}_{\text{vert, horiz}}$.

In our specific case, we will always consider Cartesian symmetric monoidal structures. Note that the class $\text{subm}$ from Definition B.33 is closed under cartesian product and pullback.

**Notation B.37.** If $\mathcal{C}$ is an $(\infty, 1)$-category, we denote by all the class of all morphisms in $\mathcal{C}$.

We will need the following lemma:

**Lemma B.38.** Let $\mathcal{C}^\times$ be a Cartesian symmetric monoidal $\infty$-category. Let $X, Y \in \mathcal{C}$ be objects. Then the map

$$\mathcal{C}_{/X} \times \mathcal{C}_{/Y} \to \mathcal{C}_{/X \times Y}$$

is cofinal.

If $F : \mathcal{C}^\times \to \mathcal{D}^\times$ is a symmetric monoidal functor between symmetric monoidal structures, and $X, Y \in \mathcal{D}$ are objects, then the map $F_{/X} \times F_{/Y} \to F_{/X \times Y}$ is cofinal.

**Proof.** By Quillen’s Theorem A, it suffices to prove that fibers are contractible. This can be straightforwardly checked by considering, for $Z \to X \times Y$, $Z$ representable, the canonical factorization

$$\Delta \to Z \times Z \to X \times Y$$

and proving that it is initial amongst all factorizations

$$\Delta \to X \times Y \to X \times Y$$

induced by maps $X \to X, Y \to Y, Z \to X, Z \to Y$, with $X, Y$ representable. □

---

17Note that we indeed need a definition that works at least for $\mathcal{C}$ a $(2, 1)$-category, since for example $\text{StrTStk}_{\text{con}}$ is such.
Theorem B.39. Let $R$ be a ring of coefficients. There is a symmetric monoidal functor of $\infty$-categories

$$\text{Cons}^\text{corr} : \text{Corr}(\text{StrT}_{\text{Stk}}_{\text{con}}_{\text{all,subm}}) \rightarrow \mathcal{P}_{R}^{L,R}$$

extending the functor

$$\text{Cons}^{(\ast)} : \text{StrTop}_{\text{con}}^{\text{op}} \rightarrow \mathcal{P}_{R}^{L,R}$$

from Corollary B.18.

Proof. Let us start with the functor $\text{Cons}^{(\ast)} : \text{StrTop}_{\text{con}}^{\text{op}} \rightarrow \mathcal{P}_{R}^{L,R}$ defined in Corollary B.18. Let us prove that this functor satisfies the right Beck-Chevelley condition [GR17, Chapter 7, Definition 3.1.5] with respect to the classes $\text{vert} = \text{subm}, \text{horiz} = \text{all}, \text{adm} = \text{isom}$. First for all, for each $\beta : (Y', s') \rightarrow (Y, s)$ in $\text{subm} \subset \text{Mor}(\text{StrTop}_{\text{con}})$, the functor $\beta^{\ast} : \text{Cons}(Y, s) \rightarrow \text{Cons}(Y', s')$ admits a left adjoint $\beta_{\ast}$. Then, consider a cartesian diagram

$$
\begin{array}{ccc}
(X', t') & \xrightarrow{\alpha_{0}} & (X, t) \\
\downarrow \beta_{1} & & \downarrow \beta_{0} \\
(Y', s') & \xrightarrow{\alpha_{1}} & (Y, s)
\end{array}
$$

where $\alpha_{0}, \alpha_{1} \in \text{all}, \beta_{0}, \beta_{1}' \in \text{subm}$. Then we want to prove that the base-change map

$$
\beta_{1} \cdot \alpha_{0} \rightarrow \alpha_{1} \ast \beta_{0}
$$

is an equivalence of functors $\text{Cons}(X, t) \rightarrow \text{Cons}(Y', s')$. Note that, since all functors are left adjoints, this formula is local in $X$ and $X'$, and therefore we may assume that $\beta_{0}$ is of the form $(X, t) \times \mathbb{R}^{N} \rightarrow (X, t)$ for some $N$, and analogously for $\beta_{1}$. But in this case $\beta_{0}'$ and $\beta_{1}'$ are stratified homotopy equivalences, which implies that $\beta_{0,\ast}, \beta_{1,\ast}$ are equivalences. Note incidentally that a similar argument (reduction to the local case and the fact that in the local case $\beta_{j}'$ is an equivalence on constructible sheaves) proves that $\beta_{1,\ast}$ coincides with $\beta_{1,\#}$ from [Vol21, Lemma 3.23]. We will however not use this identification.

We may now apply [GR17, Chapter 7, Theorem 3.2.2. (b)] and obtain an extension at the level of $(\infty, 2)$-categories

$$\begin{array}{ccc}
\text{StrTop}_{\text{con}}^{\text{op}} & \xrightarrow{\mathcal{P}_{R}^{L,R}} & \text{Corr}(\text{StrTop}_{\text{con}}^{\text{subm,all}}) \\
\downarrow & & \downarrow \text{Corr}(\text{StrTop}_{\text{con}}^{\text{subm,all}}) \\
\text{Corr}(\text{StrTop}_{\text{con}}^{\text{subm,all}}) & \rightarrow & \mathcal{P}_{R}^{L,R}
\end{array}$$

and one can check that this restricts to a functor of $(\infty, 1)$-categories

$$\text{Corr}(\text{StrTop}_{\text{con}}^{\text{subm,all}}) \rightarrow \mathcal{P}_{R}^{L,R} \quad (B.2)$$

Here we use the existence of $\alpha_{0,\ast}, \alpha_{1,\ast}$ implied by Corollary B.18, i.e. that $\text{Cons}^{(\ast)}$ lands in $\mathcal{P}_{R}^{L,R}$ and not just in $\mathcal{P}_{R}^{R}$. 

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which encodes \((-^*,-)_\text{functoriality}.\)

Let us now consider the embedding \(F : \text{StrTop}_{\text{con}} \to \text{StrTStk}_{\text{con}}\). This functor satisfies the conditions of [GR17, Chapter 8, 6.1.1 and Theorem 6.1.5], (with \(C = \text{StrTop}_{\text{con}}, D = \text{StrTStk}_{\text{con}}, \text{vert} = \text{subm}, \text{horiz} = \text{all}, \text{adm} = \text{isom}\)), because:

- it preserves finite limits;
- it sends subm to subm by definition of subm \(\subset \text{Mor} (\text{StrTStk}_{\text{con}})\);
- for each \((Y,s) \in \text{StrTop}_{\text{con}}\), the functor \(((\text{StrTop}_{\text{con}})_{\text{subm}})/(Y,s) \to ((\text{StrTStk}_{\text{con}})_{\text{subm}})/F(Y,s)\) is an equivalence. Indeed, since \(F\) is fully faithful, it suffices to prove that for any \(Y \in \text{StrTStk}_{\text{con}}, \phi : Y \to F(Y,s)\) in \(\text{StrTStk}_{\text{con}}\), then \(Y\) belongs to the essential image of \(F\). But this is true since any morphism in subm \(\subset \text{Mor}(\text{StrTStk}_{\text{con}})\) is assumed to be representable, cf. Definition B.33.

Therefore, we may apply [GR17, Theorem 6.1.5] to (B.2) and obtain a horizontal right Kan extension (in the terminology of that theorem)

\[
\begin{array}{ccc}
\text{Corr}(\text{StrTop}_{\text{con}})_{\text{subm,all}} & \xrightarrow{\text{Pr}_R^L} & \text{Pr}_R^L \\
\downarrow & & \downarrow \text{Pr}_R^L \\
\text{Corr}(\text{StrTStk}_{\text{con}})_{\text{subm,all}} & \xrightarrow{(.\otimes)} & \text{Pr}_R^L \otimes \\
\end{array}
\]

(B.3)

Note that, when restricted to horizontal morphisms, this extension agrees with Construction B.24.

Let us now establish a symmetric monoidal structure on this functor. Note first of all that the class subm is closed under products. Then we can apply the combination of [GR17, Proposition 3.1.5] with \(\text{vert} = \text{subm}, \text{horiz} = \text{adm} = \text{all}, \text{co-adm} = \text{isom}, \) and [GR17, Proposition 3.2.4] with the same classes, and obtain a right-lax monoidal functor

\[
\text{Corr}(\text{StrTStk}_{\text{con}})_{\text{subm,all}}^\times \to \text{Pr}_R^L \otimes 
\]

Let us prove that it is indeed strongly monoidal. It suffices to prove that, for every \(X, Y \in \text{StrTStk}_{\text{con}}\), the map

\[
\text{Cons}(X) \otimes \text{Cons}(Y) \to \text{Cons}(X \times Y)
\]

is an equivalence. But this map can be presented as the chain of equivalences

\[
\text{Cons}(X) \otimes \text{Cons}(Y) \simeq \text{colim}_{\text{Pr}_R^L(\text{StrTop}_{\text{con}})\times X} \text{Cons}(X) \otimes \text{colim}_{\text{Pr}_R^L(\text{StrTop}_{\text{con}})\times Y} \text{Cons}(Y) \simeq \\
\text{colim}_{\text{Pr}_R^L(\text{StrTop}_{\text{con}})\times X \times \text{Pr}_R^L(\text{StrTop}_{\text{con}})\times Y} \text{Cons}(X) \otimes \text{Cons}(Y) \simeq \text{colim}_{\text{Pr}_R^L(\text{StrTop}_{\text{con}})\times X \times \text{Pr}_R^L(\text{StrTop}_{\text{con}})\times Y} \text{Cons}(X \times Y) \simeq \\
\text{colim}_{Z \in \text{Pr}_R^L(\text{StrTop}_{\text{con}})\times X \times Y} \text{Cons}(Z) \simeq \text{Cons}(X \times Y)
\]

Therefore, we may apply [GR17, Theorem 6.1.5] to (B.2) and obtain a horizontal right Kan extension (in the terminology of that theorem)

\[
\begin{array}{ccc}
\text{Corr}(\text{StrTop}_{\text{con}})_{\text{subm,all}} & \xrightarrow{\text{Pr}_R^L} & \text{Pr}_R^L \\
\downarrow & & \downarrow \text{Pr}_R^L \\
\text{Corr}(\text{StrTStk}_{\text{con}})_{\text{subm,all}} & \xrightarrow{(.\otimes)} & \text{Pr}_R^L \otimes \\
\end{array}
\]

(B.3)

Note that, when restricted to horizontal morphisms, this extension agrees with Construction B.24.
where the second-to-last equivalence holds by Lemma B.38.

From this, by passing to opposite categories (and opposite monoidal structures), we obtain the sought symmetric monoidal functor of \((\infty, 1)\)-categories

\[
\text{Cons}_{R}^{\text{corr.} \otimes} : \text{Corr}(\text{StrTStk}_{\text{con}}_{\text{all,subm}})_{\times} \to \mathcal{P}^{R}_{R} \tag{B.4}
\]

which encodes \((-^{\ast}, -_{\ast})\)-functoriality. \(\square\)

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