Approximate controllability for nonlinear degenerate parabolic problems with bilinear control✩

Giuseppe Floridia
Dipartimento di Matematica, Università di Roma “Tor Vergata”, I-00161 Roma, Italy

Abstract
In this paper, we study the global approximate multiplicative controllability for nonlinear degenerate parabolic Cauchy-Neumann problems. First, we will obtain embedding results for weighted Sobolev spaces, that have proved decisive in reaching well-posedness for nonlinear degenerate problems. Then, we show that the above systems can be steered in $L^2$ from any nonzero, nonnegative initial state into any neighborhood of any desirable nonnegative target-state by bilinear piecewise static controls. Moreover, we extend the above result relaxing the sign constraint on the initial date.

Keywords: approximate controllability, bilinear control, semilinear equations, degenerate parabolic equations, weighted Sobolev spaces

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1. Introduction
This paper is concerned with the analysis of semilinear parabolic control systems in one space dimension, governed in the bounded domain $(-1,1)$ by means of the bilinear control $\alpha(t,x)$, of the form

\begin{align}
\begin{cases}
  u_t - (a(x)u_x)_x = \alpha(t,x)u + f(t,x,u) & \text{in } Q_T := (0,T) \times (-1,1) \\
  a(x)u_x(t,x)|_{x=\pm 1} = 0 & t \in (0,T) \\
  u(0,x) = u_0(x) & x \in (-1,1) 
\end{cases}
\end{align}

The equation in the Cauchy-Neumann problem above is a degenerate parabolic equation, because the diffusion coefficient, positive on $(-1,1)$, is allowed to vanish at the extreme points of $[-1,1]$.

The main physical motivations for studying degenerate parabolic problems with the above structure come from mathematical models in climate science as we explain below.

1.1. Physical motivations: Climate models and degenerate parabolic equations
Climate depends on various parameters such as temperature, humidity, wind intensity, the effect of greenhouse gases, and so on. It is also affected by a complex set of interactions in the atmosphere, oceans

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*Post Doc Istituto Nazionale di Alta Matematica (INdAM) “F. Severi”, Roma.

Email address: floridia@mat.uniroma2.it (Giuseppe Floridia)
and continents, that involve physical, chemical, geological and biological processes. One of the first attempts to model the effects of the interaction between large ice masses and solar radiation on climate is the one due, independently, to Budyko [3], [10], and Sellers [41] (see also [24], [23], [30], [42], [4], [43] and the references therein). The Budyko-Sellers model is an energy balance model, which studies the role played by continental and oceanic areas of ice on climate change. The effect of solar radiation on climate can be summarized in the following:

\[ \text{Heat variation} = R_a - R_e + D, \]

where \( R_a \) is the absorbed energy, \( R_e \) is the emitted energy and \( D \) is the diffusion part.

The general formulation of the Budyko-Sellers model on a compact surface \( \mathcal{M} \) without boundary is as follows

\[ u_t - \Delta_{\mathcal{M}} u = R_a(t, X, u) - R_e(t, X, u), \]

where \( u(t, X) \) is the distribution of temperature, \( \Delta_{\mathcal{M}} \) is the classical Laplace-Beltrami operator, \( R_a(t, X, u) = Q(t, X)\beta(X, u) \). In the above, \( Q \) is the insolation function, that is, the incident solar radiation at the top of the atmosphere. In annual models, when the time scale is long enough, one may assume that the insolation function doesn’t depend on time \( t \), i.e. \( Q = Q(X) \). But, when the time scale is smaller, as in seasonal models, one uses a more realistic description of the incoming solar flux by assuming that \( Q \) depends on \( t \), i.e. \( Q = Q(t, X) \). \( \beta \) is the coalbedo function, that is, 1-albedo function. Albedo is the reflecting power of a surface. It is defined as the ratio of reflected radiation from the surface to incident radiation upon it. It may also be expressed as a percentage, and is measured on a scale from zero, for no reflecting power of a perfectly black surface, to 1, for perfect reflection of a white surface.

On \( \mathcal{M} = \Sigma^2 \) the Laplace-Beltrami operator is

\[ \Delta_{\mathcal{M}} u = \frac{1}{\sin \phi} \left\{ \frac{\partial}{\partial \phi} \left( \sin \phi \frac{\partial u}{\partial \phi} \right) + \frac{1}{\sin \phi} \frac{\partial^2 u}{\partial X^2} \right\}, \]

where \( \phi \) is the colatitude and \( \lambda \) is the longitude. In the one-dimensional Budyko-Sellers we take the average of the temperature at \( x = \cos \phi \), where \( \phi \) is the colatitude. In such a model, the sea level mean zonally averaged temperature \( u(t, x) \) on the Earth, where \( t \) denotes time, satisfies the following Cauchy-Neumann degenerate problem in the bounded domain \((-1, 1)\)

\[
\begin{aligned}
&u_t - ((1 - x^2)u_x)_x = g(t, x) h(u) + f(t, x, u), & x \in (-1, 1), \\
&(1 - x^2)u(t, x)|_{x=\pm1} = 0, & t \in (0, T),
\end{aligned}
\]

where the meaning of this boundary condition will be clarified in Section 3.

1.2. Mathematical motivations, contents and structure

Interest in degenerate parabolic equations dates back by almost a century. Significant contributions are due to Fichera’s and Oleinik’s studies (see e.g., respectively, [27] and [39]).

In control theory, boundary and interior locally distributed controls are usually employed (see, e.g., [16]–[18], [25], [26], [29], [5] and [6]). These controls are additive terms in the equation and have localized support. However, such models are unfit to study several interesting applied problems such as chemical reactions controlled by catalysts, and also smart materials, which are able to change their principal parameters under certain conditions.

Additive control problems for the Budyko-Sellers model have been studied by J.I. Diaz, in the work [21] (see also the interesting papers [20], [22] and [23]).

In the present work, the control action would take the form of a bilinear control, that is, a control given by a multiplicative coefficient. General references for multiplicative controllability are, e.g., [31], [32] and [3]. Our approach is inspired by [33] and [15]. In [33], A.Y. Khapalov studied the global nonnegative approximate controllability of the one dimensional non-degenerate semilinear convection-diffusion-reaction
equation governed in a bounded domain via bilinear control. In [15], P. Cannarsa and A.Y. Khapalov derived the same approximate controllability property in suitable classes of functions that change sign.

Then, I considered, in collaboration with P. Cannarsa, the linear degenerate problem associated to (1.1) (i.e. when \( f \equiv 0 \)) in two distinct kinds of set-up. Namely, first, in [12] and [28] we considered the weakly degenerate problems (WD), that is, when \( \frac{1}{a} \in L^1(-1,1) \); then, in [11] and [28] we considered the strongly degenerate problems (SD), that is, when \( \frac{1}{a} \notin L^1(-1,1) \). Observe that the Budyko-Sellers model is an example of SD operator.

The WD case is somewhat similar to the uniformly parabolic case. Indeed, it turns out that all functions in the domain of the corresponding differential operator possess a trace on the boundary, in spite of the fact that the operator degenerates at such points. In the WD case, we are able to study a Cauchy-Robin boundary problem, and we obtain a result of global nonnegative approximate multiplicative controllability in \( L^2(-1,1) \). So, we show that the above system can be steered, in the space of square-summable functions, from any nonzero, nonnegative initial state into any neighborhood of any desirable nonnegative target-state by bilinear static controls. Moreover, we extend the above result relaxing the sign constraint on the initial-state.

On the other hand, in the SD case one is forced to restrict to the Neumann type boundary conditions (as in the Budyko-Sellers model). Even in this case (SD linear case), we establish the global nonnegative approximate multiplicative controllability in \( L^2(-1,1) \), after proving the compact embedding in \( L^2(-1,1) \) of the weighted Sobolev space \( H^1_a(-1,1) \) (\( H^1_a(-1,1) \) is the space of all functions \( u \in L^2(-1,1) \) such that \( u \) is locally absolutely continuous in \((-1,1)\) and \( \sqrt{a} u_x \in L^2(-1,1) \)) under the assumption \( \xi_a \in L^1(-1,1) \), where \( \xi_a(x) = \int_0^x \frac{a(s)}{a(s)} ds \).

In this paper we focus just on semilinear strongly degenerate problems, and we obtain the global nonnegative approximate multiplicative controllability of (1.1) by bilinear piecewise static controls with initial state \( u_0 \in L^2(-1,1) \). The technique of this paper is inspired by A.Y. Khapalov in [33], for uniformly parabolic equations. The main technical difficulty to overcome with respect to the uniformly parabolic case, is the fact that functions in \( H^1_a(-1,1) \) need not be necessarily bounded when the operator is strongly degenerate. Thus, some embedding results for weighted Sobolev spaces obtained in this article have proved decisive in reaching the desired controllability. In particular, using the above embedding results and some results found in [28], we obtain the well-posedness of (1.1) with initial state in \( L^2(-1,1) \).

In [28], we established the existence and uniqueness of solution to (1.1) with initial data in \( H^1_a(-1,1) \) and, in order to obtain this result, we followed the classical method which consists in obtaining a local result by fixed point arguments, and then show that the solution is global in time by proving an a priori estimate (see [28] and also Appendix B). In fact, first, the nonlinear system (1.1) has been addressed in [28], assuming sufficient regularity on the initial data, that is, \( u_0 \in H^1_a(-1,1) \) and obtaining an approximate controllability result in large time. Such a regularity was necessary to develop the approach of [28], that was confined to strict solutions of (1.1) (see Section 3 for the definition of strict solution). On the other hand, the above procedure has some drawbacks, such as the restriction of the admissible target states to functions \( u_d \in H^1_a(-1,1) \) satisfying \( \langle u_0, u_d \rangle_{1,a} > 0 \). The main purpose of this paper is to extend the analysis of [28], relaxing the regularity assumptions on \( u_0, u_d \) to \( u_0, u_d \in L^2(-1,1) \) and \( u_0, u_d \geq 0 \), with \( u_0 \neq 0 \).

The structure of this paper is the following. Section 2 deals with the problem formulation and gives the main results. Section 3 deals with well-posedness for semilinear equations with initial state in \( L^2(-1,1) \), and includes some new embedding results for weighted Sobolev spaces. In Section 4, we prove the global nonnegative approximate controllability of (1.1) via bilinear controls. Moreover, in Appendix A we recall the proof of a result for singular Sturm-Liouville problems obtained in [11] (this result is used in the proofs of the main results) and we remind a classical regularity result of the positive and negative part of a given function. In Appendix B, we recall the proofs of the existence and uniqueness results for problem (1.1) with initial state in \( H^1_a(-1,1) \), previously obtained by the author in [28].

Now, let us consider some open questions pertaining to this paper. First of all, in the future we intend to investigate similar problems in higher space dimensions on domains with specific geometries, first in the uniformly parabolic case (see, e.g., the preprint [12]), then in the degenerate parabolic case. Finally, once the above two issues have been addressed, we would like to extend our approach to other nonlinear systems.
of parabolic type, such as the systems of fluid dynamics (see, e.g., [14]).

2. Problem formulation and main results

This section gives the problem formulation and the main results of controllability of the system (1.1).

2.1. Problem formulation

In this paper, we consider the problem

\[
\begin{aligned}
\begin{cases}
  u_t - (a(x)u_x)_x = \alpha(t, x)u + f(t, x, u) & \text{in } Q_T := (0, T) \times (-1, 1) \\
  a(x)u_x(t, x)|_{x=\pm 1} = 0 & t \in (0, T) \\
  u(0, x) = u_0(x) & x \in (-1, 1),
\end{cases}
\end{aligned}
\]

under the following assumptions:

(A.1) \( u_0 \in L^2(-1, 1) \);

(A.2) \( a \in L^\infty(Q_T) \);

(A.3) \( f : Q_T \times \mathbb{R} \to \mathbb{R} \) is such that

- \( (t, x) \mapsto f(t, x, u) \) is measurable \( \forall u \in \mathbb{R} \),
- \( u \mapsto f(t, x, u) \) is locally absolutely continuous for a.e. \( (t, x) \in Q_T \),
- \( t \mapsto f(t, x, u) \) is locally absolutely continuous for a.e. \( x \in (-1, 1) \), \( \forall u \in \mathbb{R} \),
- there exist constants \( \gamma_0 \geq 0, \vartheta \in (1, 3) \) and \( \nu \geq 0 \) such that
  \[
  |f(t, x, u)| \leq \gamma_0 |u|^\vartheta, \quad \text{for a.e. } (t, x) \in Q_T, \forall u \in \mathbb{R};
  \]
  \[
  - \nu(1 + |u|^{\vartheta-1}) \leq f_u(t, x, u) \leq \nu, \quad \text{for a.e. } (t, x) \in Q_T, \forall u \in \mathbb{R};
  \]
  \[
  f_t(t, x, u) u \geq -\nu u^2, \quad \text{for a.e. } (t, x) \in Q_T, \forall u \in \mathbb{R};
  \]

(A.4) \( a \in C^1([-1, 1]) \) is such that

\[
 a(x) > 0, \quad \forall x \in (-1, 1), \quad a(-1) = a(1) = 0,
\]

and, the function \( \xi_a(x) = \int_0^x \frac{\vartheta}{a(q)} dq \) satisfies the following

\[
 \xi_a \in L^{q_\vartheta}(-1, 1),
\]

where

\[
 q_\vartheta = \max\left\{ \frac{1+\vartheta}{3-\vartheta}, 2\vartheta-1 \right\}.
\]

Remark 2.1. The inequalities (2.2), in assumption (A.3), imply the following conditions on the function \( f \)

\[
 |f_u(t, x, u)| \leq \nu(1 + |u|^{\vartheta-1}), \quad \text{for a.e. } (t, x) \in Q_T, \forall u, v \in \mathbb{R};
\]

\[
 (f(t, x, u) - f(t, x, v))(u - v) \leq \nu(u - v)^2, \quad \text{for a.e. } (t, x) \in Q_T, \forall u, v \in \mathbb{R};
\]

\[
 |f(t, x, u) - f(t, x, v)| \leq \nu(1 + |u|^{\vartheta-1} + |v|^{\vartheta-1})|u - v|, \quad \text{for a.e. } (t, x) \in Q_T, \forall u, v \in \mathbb{R}.
\]

1 This assumption is used only for well-posedness, see Appendix B.
2 Since, for a.e. \( (t, x) \in Q_T, f(t, x, u) \) is locally absolutely continuous respect to \( u \), we have

\[
 (f(t, x, u) - f(t, x, v))(u - v) = (u - v) \int_v^u f_u(t, x, \xi)d\xi \leq (u - v) \int_v^u \nu d\xi \leq \nu(u - v)^2,
\]

\[
 |f(t, x, u) - f(t, x, v)| \leq \int_{\min(u,v)}^{\max(u,v)} |f_u(t, x, \xi)|d\xi \leq \nu \int_{\min(u,v)}^{\max(u,v)} (1 + |\xi|^{\vartheta-1})d\xi \leq \nu(1 + |u|^{\vartheta-1} + |v|^{\vartheta-1})|u - v|,
\]

for a.e. \( (t, x) \in Q_T, \) for every \( u, v \in \mathbb{R} \).
Remark 2.2. We note that all the results of this paper hold true replacing the assumption that \( u \mapsto f(t, x, u) \) is locally absolutely continuously by the mere continuity of such a function, for a.e. \((t, x) \in Q_T\). That is, in (A.3), it suffices to assume that:

- \((t, x, u) \mapsto f(t, x, u)\) is a Carathéodory function on \(Q_T \times \mathbb{R}\),
- \(t \mapsto f(t, x, u)\) is locally absolutely continuous for a.e. \(x \in (-1, 1)\), \(\forall u \in \mathbb{R}\),

and to substitute inequality (2.2) by the two more general inequalities (2.4) and (2.5).

Remark 2.3. The equation in the Cauchy-Neumann problem (1.1) is a degenerate parabolic equation because the diffusion coefficient, positive on \((-1, 1)\), is allowed to vanish at the extreme points of \([-1, 1]\). In particular, since \(\frac{1}{2} \notin L^1(-1, 1)\), this problem is strongly degenerate. A sufficient condition for this is that \(a'(\pm 1) \neq 0\) (if \(a \in C^2([-1, 1])\) the above condition is also necessary).

The principal part of the operator in (1.1) coincides with that of the Budyko-Sellers model for \(a(x) = 1 - x^2\). In this case, \(\xi_a(x) = \frac{1}{2} \ln \left(\frac{1-x}{1+x}\right)\), so \(\xi_a \in L^p(-1, 1)\), for every \(p \geq 1\).

Remark 2.4. The assumption (2.4) is more general than the classical sign assumption \(\int_{-1}^{1} f(t, x, u) du dx \leq 0\), indeed the last condition is equivalent to \(f(t, x, u) du \leq 0\), for a.e. \((t, x) \in Q_T, \forall u \in \mathbb{R}\).

Example 2.1. An example of function \(f\) that satisfies the assumptions (A.3) is the following

\[ f(t, x, u) = c(t, x) \min\{|u|^{\alpha-1}, 1\} u - |u|^{\alpha-1} u, \]

where \(c\) is a Lipschitz continuous function.

2.2 Main results

We are interested in studying the nonnegative multiplicative controllability of (1.1) by the bilinear control \(\alpha(t, x)\). Let us start with the following definitions.

Definition 2.1. We say that a function \(\alpha \in L^\infty(Q_T)\) is piecewise static, if there exist \(n \in \mathbb{N}\), \(c_i(x) \in L^\infty(-1, 1)\) and \(t_i \in (0, T), t_{i-1} < t_i, i = 1, \ldots, n\) with \(t_0 = 0\) and \(t_n = T\), such that

\[ \alpha(t, x) = c_1(x) \chi_{[t_0, t_1]}(t) + \sum_{i=2}^{n} c_i(x) \chi_{(t_{i-1}, t_i]}(t), \]

where \(\chi_{[t_0, t_1]}\) and \(\chi_{(t_{i-1}, t_i]}\) are the indicator function of \([t_0, t_1]\) and \((t_{i-1}, t_i]\), respectively.

Definition 2.2. We say that the system (1.1) is nonnegatively globally approximately controllable in \(L^2(-1, 1)\), if for every \(\varepsilon > 0\) and for any nonnegative \(u_0, u_d \in L^2(-1, 1)\), with \(u_0 \neq 0\) there are a \(T = T(\varepsilon, u_0, u_d) > 0\) and a bilinear control \(\alpha = \alpha(t, x), \alpha \in L^\infty(Q_T)\) such that for the corresponding strong solution \(u(t, x)\) of (1.1) we obtain

\[ \|u(T, \cdot) - u_d\|_{L^2(-1, 1)} \leq \varepsilon. \]

The nonnegative global approximate controllability results are obtained for the semilinear system (1.1) in the following theorem.

Theorem 2.1. The semilinear system (1.1) is nonnegatively globally approximately controllable in \(L^2(-1, 1)\), by means of piecewise static bilinear controls \(\alpha\). Moreover, the corresponding strong solution \(^4\) to (1.1) remains nonnegative a.e. in \(Q_T\).

\(^3\)This integral condition is used in [32], in the uniformly parabolic case, but also there it can be generalized by a condition similar to (2.3).

\(^4\)See Definition 2.4 for the precise definition of strong solutions.
Moreover, we obtain the following result.

**Theorem 2.2.** For any \( u_d \in L^2(-1, 1), u_d \geq 0 \) and any \( u_0 \in L^2(-1, 1) \) such that

\[
(u_0, u_d)_{L^2(-1, 1)} > 0,
\]

(2.6)

for every \( \varepsilon > 0 \), there are \( T = T(\varepsilon, u_0, u_d) \geq 0 \) and a piecewise static bilinear control \( \alpha = \alpha(t, x), \alpha \in L^\infty(Q_T) \) such that

\[
\| u(T, \cdot) - u_d \|_{L^2(-1, 1)} \leq \varepsilon,
\]

where \( u \) is the strong solution \(^5\) to (1).

## 3. Well-posedness for nonlinear problems

In this section, first we obtain embedding results for weighted Sobolev spaces (Section 3.2), then we prove the existence and uniqueness of the strong solution to nonlinear problem \( \text{(1)} \) (Section 3.5).

### 3.1. The function spaces \( \mathcal{B}(Q_T) \) and \( \mathcal{H}(Q_T) \)

In order to deal with the well-posedness of nonlinear degenerate problem \( \text{(1)} \), it is necessary to introduce the weighted Sobolev spaces \( H^1_\alpha(-1, 1) \) and \( H^2_\alpha(-1, 1) \) (see also [11] and [28]).

We define

\[
H^1_\alpha(-1, 1) := \{ u \in L^2(-1, 1) \mid u \text{ is locally absolutely continuous in } (-1, 1) \text{ and } \sqrt{\alpha} u_x \in L^2(-1, 1) \},
\]

\[
H^2_\alpha(-1, 1) := \{ u \in H^1_\alpha(-1, 1) \mid au_x \in H^1(-1, 1) \},
\]

respectively with the following norms

\[
\| u \|_{1, \alpha}^2 := \| u \|_{L^2(-1, 1)}^2 + |u|_{1, a}^2 \quad \text{and} \quad \| u \|_{2, \alpha}^2 := \| u \|_{1, \alpha}^2 + \| (au_x)_x \|_{L^2(-1, 1)}^2,
\]

where \( |u|_{1, a}^2 := \| \sqrt{\alpha} u_x \|_{L^2(-1, 1)}^2 \) is a seminorm.

In [10], see Proposition 2.1 and the Appendix, the authors prove the following result (see also Lemma 2.5 in [10]).

**Proposition 3.1.** For every \( u \in H^2_\alpha(-1, 1) \) we have

\[
\lim_{x \to \pm 1} a(x) u_x(x) = 0 \quad \text{and} \quad au \in H^1_\alpha(-1, 1).
\]

\( H^1_\alpha(-1, 1) \) and \( H^2_\alpha(-1, 1) \) are Hilbert spaces with their natural scalar products, and we denote with \( \langle \cdot, \cdot \rangle_{1, \alpha} \) the scalar product of \( H^1_\alpha(-1, 1) \).

In the following, we will sometimes use \( \| \cdot \|, \langle \cdot, \cdot \rangle \) instead of \( \| \cdot \|_{L^2(-1, 1)}, \langle \cdot, \cdot \rangle_{L^2(-1, 1)} \), respectively, and \( \| \cdot \|_{\infty} \) instead of \( \| \cdot \|_{L^\infty(Q_T)} \).

Given \( T > 0 \), let us define the function spaces:

\[
\mathcal{B}(Q_T) := C([0, T]; L^2(-1, 1)) \cap L^2(0, T; H^1_\alpha(-1, 1))
\]

with the following norm

\[
\| u \|_{\mathcal{B}(Q_T)}^2 = \sup_{t \in [0, T]} \| u(t, \cdot) \|^2 + 2 \int_0^T \int_{-1}^1 a(x) u_x(x) dx \, dt,
\]

\(^5\) \( H^0_\alpha(-1, 1) = \{ u \in L^2(-1, 1) \mid au \in L^2(-1, 1) \text{ and } u(\pm 1) = 0 \} \).
and
\[ \mathcal{H}(Q_T) := L^2(0, T; H^2(\mathbb{R})) \cap H^1(0, T; L^2(\mathbb{R})) \cap C([0, T]; H^1(\mathbb{R})) \]
with the following norm
\[ \|u\|_{\mathcal{H}(Q_T)} = \sup_{[0, T]} \left( \|u\|^2 + \|\sqrt{\alpha}u_x\|^2 \right) + \int_0^T \left( \|u_t\|^2 + \|\alpha u_x\|^2 \right) dt. \]

**Remark 3.1.** We observe that \( B(Q_T) \) and \( \mathcal{H}(Q_T) \) are Banach spaces (see, e.g., [24]).

### 3.2. Some embedding theorems for weighted Sobolev spaces

Let \( \xi_a(x) = \int_0^x \frac{1}{a(s)} ds \), then we have the following

**Lemma 3.2.** If \( \xi_a \in L^p(-1, 1) \), for some \( p \geq 1 \), then
\[ H^1_a(-1, 1) \hookrightarrow L^{2p}(-1, 1). \]

Moreover,
\[ \|u\|_{L^{2p}(-1, 1)} \leq c \|u\|_{1,a} \]
where \( c \) is a positive constant.

**Proof.** Let \( u \in H^1_a(-1, 1) \). First, for every \( x \in (-1, 1) \), we have the following estimate
\[ \|u(x) - u(0)\| = \left| \int_0^x u'(s) ds \right| \leq \left| \int_0^x a(s)|u'(s)|^2 ds \right|^{\frac{1}{2}} \left| \int_0^x \frac{1}{a(s)} ds \right|^{\frac{1}{2}} \leq \sqrt{\xi_a(x)} \|u\|_{1,a}. \tag{3.1} \]

Moreover, keeping in mind that \( \xi_a \in L^p(-1, 1) \), we have
\[ \int_{-1}^1 |u(0)| dx \leq \int_{-1}^1 |u(x) - u(0)| dx + \int_{-1}^1 |u(x)| dx \leq |u|_{1,a} \int_{-1}^1 \sqrt{\xi_a(x)} dx + \sqrt{2} \|u\|. \]

Thus,
\[ |u(0)| \leq c_a |u|_{1,a} + \frac{\sqrt{2}}{2} \|u\| \leq \max \left\{ c_a, \frac{\sqrt{2}}{2} \right\} \|u\|_{1,a}, \quad \text{where } c_a = \frac{1}{2} \int_{-1}^1 \sqrt{\xi_a(x)} dx. \tag{3.2} \]

Finally, by (3.1) and (3.2) we have
\[ \int_{-1}^1 |u(x)|^{2p} dx \leq 2^{2p-1} \int_{-1}^1 (|u(x) - u(0)|^{2p} + |u(0)|^{2p}) dx \]
\[ \leq 2^{2p-1} |u|_{1,a}^{2p} \int_{-1}^1 \xi_a(x)^p dx + 2^{2p} \left( \max \left\{ c_a, \frac{\sqrt{2}}{2} \right\} \right)^{2p} \|u\|_{1,a}^{2p}. \]

Since \( \xi_a \in L^p(-1, 1) \), applying Hölder inequality, we deduce
\[ \int_{-1}^1 |u(x)|^{2p} dx \leq 2^{2p} c_a^{2p} |u|_{1,a}^{2p} + 2^{2p} \left( \max \left\{ c_a, \frac{\sqrt{2}}{2} \right\} \right)^{2p} \|u\|_{1,a}^{2p} \leq 2^{2p} \left( \max \left\{ c_a, \frac{\sqrt{2}}{2} \right\} \right)^{2p} \|u\|_{1,a}^{2p}. \]

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6 It’s well known that this norm is equivalent to the Hilbert norm
\[ \|u\|_{\mathcal{H}(Q_T)} = \int_0^T \left( \|u\|^2 + \|\sqrt{\alpha}u_x\|^2 + \|\alpha u_x\|^2 \right) dt. \]

7 We remember that, for every \( a, b \in [0, +\infty) \), the following numerical inequality holds true:
\[ (a + b)^q \leq 2^{q-1} (a^q + b^q), \quad \text{for every } q \geq 1. \]

8 We note that \( \int_{-1}^1 |\xi_a(x)|^p dx \leq \left( \int_{-1}^1 dx \right)^{1-2p} \left( \int_{-1}^1 |\xi_a(x)|^\frac{p}{2} dx \right)^{2p} = 2^{1-2p} (2c_a)^{2p} = 2c_a^{2p}. \)
Lemma 3.3. Let \( T > 0 \). If \( \xi_{\alpha} \in L^{\frac{1+\theta}{\theta}}(-1, 1) \) for some \( p \in [1, 2) \), then
\[
L^2(0, T; H_{\alpha}^1(-1, 1)) \cap L^\infty(0, T; L^2(-1, 1)) \subset L^2p(Q_T) \tag{9}
\]
and
\[
\|u\|_{L^2p(Q_T)} \leq c T^{\frac{1-\frac{2}{p}}{\theta}} \|u\|_{B(Q_T)} ,
\]
where \( c \) is a positive constant.

Proof. For every \( u \in L^2(0, T; H_{\alpha}^1(-1, 1)) \) we have
\[
\int_{Q_T} |u|^{2p} \, dx \, dt = \int_0^T \int_{-1}^1 |u|^p |u|^p \, dx \, dt \leq \int_0^T \left( \int_{-1}^1 |u|^{2} \, dx \right)^{\frac{p}{2}} \left( \int_{-1}^1 |u|^{\frac{2p}{2-p}} \, dx \right)^{\frac{2-p}{2}} dt.
\]
Recalling that \( u \in L^\infty(0, T; L^2(-1, 1)) \), by Lemma 3.2 we obtain
\[
\int_{Q_T} |u|^{2p} \, dx \, dt \leq \|u\|_{L^\infty(0, T; L^2(-1, 1))} \|u\|_{L^2p(Q_T)} \leq c \|u\|_{L^\infty(0, T; L^2(-1, 1))}^p \|u\|_{B(Q_T)}^p dt .
\]
Moreover, using Hölder’s inequality, we have
\[
\int_0^T \|u\|_{H_{\alpha}^1(-1, 1)}^{p} dt \leq \left( \int_0^T dt \right)^{1-\frac{2}{p}} \left( \int_0^T \|u\|_{1, a}^2 dt \right)^{\frac{2}{p}} \leq T^{1-\frac{2}{p}} \|u\|_{L^2p(0, T; H_{\alpha}^1(-1, 1))}^p .
\]
From the last two inequalities, it follows that
\[
\int_{Q_T} |u|^{2p} \, dx \, dt \leq c T^{1-\frac{2}{p}} \|u\|_{L^2p(0, T; H_{\alpha}^1(-1, 1))}^p \|u\|_{L^\infty(0, T; L^2(-1, 1))}^p \leq c T^{1-\frac{2}{p}} \|u\|_{B(Q_T)}^{2p} .
\]
Taking \( p = \frac{\theta+1}{2}, \ 1 \leq \theta < 3 \), in the previous lemma, we obtain the following corollary.

Corollary 3.4. Let \( T > 0 \). If \( \xi_{\alpha} \in L^{\frac{1+\theta}{\theta}}(-1, 1) \) for some \( \theta \in [1, 3) \), then
\[
B(Q_T) \subset L^{1+\theta}(Q_T)
\]
and
\[
\|u\|_{L^{1+\theta}(Q_T)} \leq c T^{\frac{\theta}{\theta+1}} \|u\|_{B(Q_T)} ,
\]
where \( c \) is a positive constant.

Lemma 3.5. Let \( T > 0 \), \( p \geq 1 \). If \( \xi_{\alpha} \in L^{2p-1}(-1, 1) \), then
\[
H^1(0, T; L^2(-1, 1)) \cap L^\infty(0, T; H_{\alpha}^1(-1, 1)) \subset L^{2p}(Q_T) \tag{10}
\]
and
\[
\|u\|_{L^{2p}(Q_T)} \leq c T^{\frac{1}{2p}} \|u\|_{H^1(0, T; L^2(-1, 1))} \|u\|_{L^\infty(0, T; H_{\alpha}^1(-1, 1))}^{1-\frac{1}{2p}} ,
\]
where \( c \) is a positive constant.

\[9 \quad \|u\|_{L^2(0, T; H_{\alpha}^1(-1, 1))}^2 = \int_0^T (\|u\|^2 + \|\sqrt{\alpha} u_x\|^2) \, dt \quad \text{and} \quad \|u\|_{L^\infty(0, T; L^2(-1, 1))}^2 = \sup_{[0, T]} \|u\|^2 .
\]
\[10 \quad \|u\|_{H^1(0, T; L^2(-1, 1))}^2 = \sup_{[0, T]} \|u\|^2 + \int_0^T \|u_t\|^2 \, dt \quad \text{and} \quad \|u\|_{L^\infty(0, T; H_{\alpha}^1(-1, 1))}^2 = \sup_{[0, T]} (\|u\|^2 + \|\sqrt{\alpha} u_x\|^2) .
\]
Proof. For every $u \in H^1(0, T; L^2(-1, 1)) \cap L^\infty(0, T; H^1_0(-1, 1))$ we have
\[
\int_{Q_T} |u|^2 p \, dx \, dt = \int_0^T \int_{-1}^1 |u|^2 p -1 \, dx \, dt \leq \int_0^T \left( \int_{-1}^1 |u|^2 \, dx \right) \frac{p}{2} \left( \int_{-1}^1 |u|^{4p-2} \, dx \right) \frac{1}{4p-2} \, dt .
\]
Recalling that $u \in H^1(0, T; L^2(-1, 1))$, by the Lemma 3.3 and since $\xi^a \in L^{2p-1}(-1, 1)$, we obtain
\[
\int_{Q_T} |u|^2 p \, dx \, dt \leq \|u\|_{H^1(0, T; L^2(-1, 1))} \int_0^T \|u\|_{L^2p-1}^{2p-1} \, dt \leq c \|u\|_{H^1(0, T; L^2(-1, 1))} \int_0^T \|u\|_{L^2p-1}^{2p-1} \, dt .
\]
From the last inequality, it follows that
\[
\int_{Q_T} |u|^2 p \, dx \, dt \leq c T \|u\|_{H^1(0, T; L^2(-1, 1))} \|u\|_{L^2p(0, T; H^1_0(-1, 1))} .
\]
By Lemma 3.3 one directly obtains the following.

Corollary 3.6. Let $T > 0$, $\vartheta \geq 1$. If $\xi^a \in L^{2p-1}(-1, 1)$, then
\[\mathcal{H}(Q_T) \subset L^{2\vartheta}(Q_T)\]
and
\[\|u\|_{L^{2\vartheta}(Q_T)} \leq c T^{\frac{1}{2\vartheta}} \|u\|_{\mathcal{H}(Q_T)},\]
where $c$ is a positive constant.

3.3. Existence and uniqueness of solutions of linear problems

First, we recall an existence uniqueness result for the linear problems corresponding to (3.4), obtained in [10] (see also [1] and [28]), defined by
\[
\begin{aligned}
D(A_0) &= H^2(-1, 1) \\
A_0 u &= (au_x)_x, \quad \forall u \in D(A_0) .
\end{aligned}
\tag{3.3}
\]
For the following linear results it is sufficient that the diffusion coefficient $a(\cdot)$ satisfy the assumption (A.4) with $\xi \in L^1(-1, 1)$, instead of the condition (2.2). Next, given $\alpha \in L^\infty(-1, 1)$, let us introduce the operator
\[
\begin{aligned}
D(A) &= D(A_0) \\
A &= A_0 + \alpha I .
\end{aligned}
\tag{3.4}
\]
We consider the following linear problem in the Hilbert space $L^2(-1, 1)$
\[
\begin{aligned}
u'(t) &= A u(t) + g(t), \quad t > 0 \\
u(0) &= u_0 ,
\end{aligned}
\tag{3.5}
\]
where $A$ is the operator in (3.3), $g \in L^1(0, T; L^2(-1, 1))$, $u_0 \in L^2(-1, 1)$.

We recall that a weak solution of (3.5) is a function $u \in C^0([0, T]; L^2(-1, 1))$ such that for every $v \in D(A^*)$ ($A^*$ denotes the adjoint of $A$) the function $\langle u(t), v \rangle$ is absolutely continuous on $[0, T]$ and
\[
\frac{d}{dt} \langle u(t), v \rangle = \langle u(t), A^* v \rangle + \langle g(t), v \rangle ,
\]
where $u(0) = u_0$ and $v(0) = v_0$.
for almost all $t \in [0, T]$ (see [2]). For every $\alpha \in L^\infty(-1, 1)$ [1] and every $u_0 \in L^2(-1, 1)$, there exists a unique weak solution of (3.3), which is given by the following representation $e^{tA}u_0 + \int_0^t e^{(t-s)A}g(s) \, ds$, $t \in [0, T]$ (see also [11]).

Now, using a maximal regularity result in the Hilbert space $L^2(-1, 1)$ [12], by Theorem 3.1 in Section 3.6.3 of [6], pp. 79–82, we derive the following result (see also [19] and [28]).

Proposition 3.7. Given $T > 0$ and $g \in L^2(0, T; L^2(-1, 1))$ [3]. For every $\alpha \in L^\infty(-1, 1)(11)$ and every $u_0 \in H^1_0(-1, 1)$, there exists a unique solution $u \in \mathcal{H}(Q_T)$ of (3.3). Moreover, a positive constant $C_0(T)$ exists (nondecreasing in $T$), such that the following inequality holds

$$
\|u\|_{\mathcal{H}(Q_T)} \leq C_0(T) \left[\|u_0\|_{1, \alpha} + \|g\|_{L^2(Q_T)}\right].
$$

3.4. Some results for singular Sturm-Liouville problems

In [11], in collaboration with P. Cannarsa, we prove the following results (see also [28]).

Proposition 3.8. Assume that $\xi_\alpha \in L^1(-1, 1)$, where $\xi_\alpha(x) = \int_0^x \frac{ds}{1 + |s|}$. Then,

$$
H^1_0(-1, 1) \rightarrow L^2(-1, 1) \quad \text{with compact embedding}.
$$

Let $A = A_0 + \alpha I$, where the operator $A_0$ is defined in (3.3) and $\alpha \in L^\infty(-1, 1)$. Since $A$ is self-adjoint and $D(A) \rightarrow L^2(-1, 1)$ is compact (see Proposition 3.8), we have the following (see also [7]).

Lemma 3.9. There exists an increasing sequence $\{\lambda_k\}_{k \in \mathbb{N}}$, with $\lambda_k \rightarrow +\infty$ as $k \rightarrow \infty$, such that the eigenvalues of $A$ are given by $-\lambda_k$, $k \in \mathbb{N}$, and the corresponding eigenfunctions $\{\omega_k\}_{k \in \mathbb{N}}$ form a complete orthonormal system in $L^2(-1, 1)$.

Remark 3.2. In the case $a(x) = 1 - x^2$, so that $A_0 = (1 - x^2)u_x$, then the orthonormal eigenfunctions of $A_0$ are reduced to Legendre’s polynomials $P_k(x)$, and the eigenvalues are $\mu_k = (k - 1) / k, k \in \mathbb{N}$. $P_k(x)$ is equal to $\sqrt{\frac{2}{2k - 1}} L_k(x)$, where $L_k(x)$ is assigned by Rodrigues’s formula:

$$
L_k(x) = \frac{1}{2^{k-1}(k-1)!} \frac{d}{dx} \left( x^2 - 1 \right)^{k-1} \quad (k \geq 1).
$$

In [11] (see also [28]) we obtain the following result.

Lemma 3.10. Let $v \in C^\infty([-1, 1]), v > 0$ on $[-1, 1]$, let $\alpha_* = -\frac{a(x)v(x)}{v(x)}$, $x \in (-1, 1)$. Let $A$ be the operator defined in (3.4) with $\alpha = \alpha_*$. Then

$$
\begin{align*}
D(A) &= H^2_0(-1, 1) \\
A &= A_0 + \alpha_* I,
\end{align*}
$$

and let $\{\lambda_k\}, \{\omega_k\}$ be the eigenvalues and eigenfunctions of $A$, respectively, given by Lemma 3.9. Then

$$
\lambda_1 = 0 \quad \text{and} \quad |\omega_1| = \frac{v}{\|v\|}.
$$

Moreover, $\frac{\omega}{\|v\|}$ and $-\frac{\omega}{\|v\|}$ are the only eigenfunctions of $A$ with norm 1 that do not change sign in $(-1, 1)$.

Remark 3.3. This problem is equivalent to the following singular Sturm-Liouville problem

$$
\begin{align*}
(a(x)\omega_x)_x + \alpha_*(x)\omega + \lambda \omega &= 0 \quad \text{in} \quad (-1, 1) \\
(a(x)\omega_x)|_{x=\pm 1} &= 0.
\end{align*}
$$

The proof of Lemma 3.10 is recalled in Appendix A.1.

---

11 By repeated applications of this result, one can obtain an existence and uniqueness result when $\alpha$ is piecewise static (see Definition 2.1). The same result holds for $\alpha \in L^\infty(Q_T)$, but for the purposes of the present paper the piecewise static case will suffice.

12 By maximal regularity we mean that $u'$ and $Au$ have the same regularity of $g$.

13 We observe that $L^2(0, T; L^2(-1, 1)) = L^2(Q_T)$. 

10
3.5. Existence and uniqueness of solutions of semilinear problems

Observe that the nonlinear problem \[ u'(t) = A u(t) + \phi(u), \quad t > 0 \]
\[ u(0) = u_0, \] (3.7)
where $A$ is the operator defined in \[ A \alpha \in L^\infty(-1,1), \] \[ u_0 \in L^2(-1,1), \]
and, for every $u \in B(Q_T)$,
\[
\phi(u)(t, x) := f(t, x, u(t, x)), \quad \forall (t, x) \in Q_T. \quad (3.8)
\]

By the next lemmas (Lemma 3.13 and Lemma 3.12) we will deduce the following theorem.

**Theorem 3.11.** Let $T > 0$, $0 \leq \vartheta < 3$, $\xi_0 \in L^{q_0}(-1,1)$, where $q_0 = \max \left\{ \frac{3}{3\vartheta}, 2\vartheta - 1 \right\}$. Let $f : Q_T \times \mathbb{R} \to \mathbb{R}$ be a function that satisfies assumption (A.3), then $\phi : B(Q_T) \to L^{1+\frac{2}{\vartheta}}(Q_T)$ is a locally Lipschitz continuous map and $\phi(\mathcal{H}(Q_T)) \subseteq L^2(Q_T)$.

We start with the following lemma.

**Lemma 3.12.** Let $T > 0$, $0 \leq \vartheta < 3$, $\xi_0 \in L^{2\vartheta-1}(-1,1)$, and let $u \in \mathcal{H}(Q_T)$. Let $f : Q_T \times \mathbb{R} \to \mathbb{R}$ be a function that satisfies assumption (A.3) (\[ \xi_0 \in L^{2\vartheta-1}(-1,1) \]). Then, the function $(t, x) \mapsto f(t, x, u(t, x))$ belongs to $L^2(Q_T)$ and the following estimate holds
\[
\int_{Q_T} |f(t, x, u(t, x))|^2 \, dx \, dt \leq cT \|u\|^2_{H^1(0; T; L^2(-1,1))} \|u\|^{2\vartheta-1}_{L^{\infty}(0; T; H^1_0(-1,1))},
\]
for some positive constant $c$.

**Proof.** By Lemma 3.3 since $\xi_0 \in L^{2\vartheta-1}(-1,1)$ then $u \in L^{2\vartheta}(Q_T)$. By \[ A \xi_0 \in L^{2\vartheta-1}(-1,1) \] (see assumption (A.3)) we obtain
\[
\int_{Q_T} |f(t, x, u(t, x))|^2 \, dx \, dt \leq \gamma^2_0 \int_{Q_T} |u|^{2\vartheta} \, dx \, dt \leq kT \|u\|^{2\vartheta}_{H^1(0; T; L^2(-1,1))} \|u\|^{2\vartheta-1}_{L^{\infty}(0; T; H^1_0(-1,1))} < +\infty,
\]
from which the conclusion follows.

**Corollary 3.13.** Let $T > 0$, $0 \leq \vartheta < 3$, $\xi_0 \in L^{2\vartheta-1}(-1,1)$, and let $u \in \mathcal{H}(Q_T)$. Let $f : Q_T \times \mathbb{R} \to \mathbb{R}$ be a function that satisfies assumption (A.3) (\[ \xi_0 \in L^{2\vartheta-1}(-1,1) \]). Then, we have the following estimate
\[
\int_{Q_T} |f(t, x, u(t, x))|^2 \, dx \, dt \leq cT \|u\|^{2\vartheta}_{H^1(0; T; L^2(-1,1))},
\]
for some positive constant $c$.

**Lemma 3.14.** Let $T > 0$, $0 \leq \vartheta < 3$, $\xi_0 \in L^{\frac{2\vartheta}{\vartheta-1}}(-1,1)$. Let $f : Q_T \times \mathbb{R} \to \mathbb{R}$ be a function that satisfies assumption (A.3). Then,

1. for every $u \in B(Q_T)$, the function $(t, x) \mapsto f(t, x, u(t, x))$ belongs to $L^{1+\frac{2}{\vartheta}}(Q_T)$ and the following estimate holds
\[
\int_{Q_T} |f(t, x, u(t, x))|^{1+\frac{2}{\vartheta}} \, dx \, dt \leq cT^{\frac{2\vartheta}{\vartheta-1}} \|u\|^{\frac{2}{\vartheta-1}}_{B(Q_T)}.
\]

\[ \text{We observe that the assumption (H.3) of Appendix B would be sufficient to place of (A.3).} \]
2. \( \phi : B(Q_T) \rightarrow L^{1+\frac{1}{2}}(Q_T) \) is a locally Lipschitz continuous map and, for every \( R > 0 \), the following estimate holds
\[
\|\phi(u) - \phi(v)\|_{L^{1+\frac{1}{2}}(Q_T)} \leq C_R(T)\|u - v\|_{B(Q_T)}, \quad \forall u, v \in B(Q_T), \|u\|_{B(Q_T)} \leq R, \|v\|_{B(Q_T)} \leq R, \quad (3.9)
\]
where \( C_R(T) \) is a positive constant increasing in \( T \).

Proof. By Corollary 5.4 since \( \xi \in L^{1+\vartheta}(-1,1) \), then \( u \in L^{1+\vartheta}(Q_T) \). By 2.1 (see assumption (A.3)) we obtain
\[
\int_{Q_T} |f(t,x,u(t,x))|^{1+\frac{1}{2}} dx dt \leq \gamma_0^{1+\frac{1}{2}} \int_{Q_T} |u|^{\vartheta(1+\frac{1}{2})} dx dt \leq k T^{\frac{3-\vartheta}{2}}\|u\|_{B(Q_T)}^{\vartheta+1} < +\infty,
\]
from which the point 1.) follows.

By 2.5 (see Remark 2.1), applying Corollary 5.4, we have
\[
\|\phi(u) - \phi(v)\|_{L^{1+\frac{1}{2}}(Q_T)}^{1+\frac{1}{2}} = \int_{Q_T} |f(t,x,u) - f(t,x,v)|^{1+\frac{1}{2}} dx dt \leq c \int_{Q_T} (1 + |u|^{\vartheta+1} + |v|^{\vartheta+1})|u - v|^{1+\frac{1}{2}} dx dt
\]
\[
\leq c \left( \int_{Q_T} (1 + |u|^{\vartheta+1} + |v|^{\vartheta+1}) dx dt \right)^{\frac{1}{2}} \left( \int_{Q_T} |u - v|^{\vartheta+1} dx dt \right)^{\frac{1}{2}}
\]
\[
\leq c T^{\frac{1}{2}}(1 + |u|^{\vartheta+1} + |v|^{\vartheta+1})\|u - v\|_{B(Q_T)}^{\vartheta+1} \leq c T^{\frac{3-\vartheta}{2}}(1 + |u|^{\vartheta+1} + |v|^{\vartheta+1})\|u - v\|_{B(Q_T)}^{\vartheta+1},
\]
for every \( u, v \in B(Q_T) \).

By the last inequalities we obtain the estimate 3.9.

We assume, for the following of this section, that assumptions (A.2), (A.4) are enforced, moreover we assume that assumption (A.3) is enforced with \( \vartheta \in [1,3) \) instead of \( \vartheta \in (1,3) \).

For the sequel, the next definitions are necessary.

Definition 3.1. If \( u_0 \in H^{\vartheta}_0(-1,1) \), \( u \) is a strict solution of problem (1.1), if \( u \in \mathcal{H}(Q_T) \) and
\[
\begin{align*}
& \quad \left\{ \begin{array}{ll}
& \quad u_t - (a(x)u_x)_x = \alpha(t,x)u + \phi(u) \quad \text{a.e. in } Q_T := (0,T) \times (-1,1) \\
& \quad a(x)u_x(t,x) = 0 \quad \text{a.e. } t \in (0,T) \\
& \quad u(0,x) = u_0(x) \quad x \in (-1,1). 
\end{array} \right. \\
& \quad \quad (5)
\end{align*}
\]

In the Ph.D. Thesis 28 we prove, in more general assumption on \( f \) of (A.3) (see, in Appendix B, the assumption (H.3)), the following result.

Theorem 3.15. For all \( u_0 \in H^{\vartheta}_0(-1,1) \) there exists a unique strict solution \( u \in \mathcal{H}(Q_T) \) to (1.1).

The lemmas and the complete proofs of the results that allow us to get the previous theorem can be found in Appendix B.

The following notion of “strong solutions” is classical in PDEs theory, see, for instance, 3, pp. 62-64.
Definition 3.2. Let \( u_0 \in L^2(-1,1) \). We say that \( u \in B(Q_T) \) is a strong solution to problem (1.1), if \( u(0, \cdot) = u_0 \) and there exists a sequence \( \{u_k\}_{k \in \mathbb{N}} \) in \( H(Q_T) \) such that, as \( k \to \infty \), \( u_k \rightharpoonup u \) in \( B(Q_T) \) and, for every \( \text{Lemma 3.12 and condition (2.4) (see Remark 2.1)} \), we obtain
\[
\begin{aligned}
\begin{cases}
  u_{kt} - (u_{kxt} x = \alpha(t,x)u_k + \phi(u_k) & \text{a.e. in } Q_T := (0,T) \times (-1,1) \\
  a(x)u_{kx}(t,x)|_{x=\pm 1} = 0 & \text{a.e. in } (0,T),
\end{cases}
\end{aligned}
\]
with initial datum \( u_k(0, x) \).

Remark 3.4. We note that, thanks to the definition of the \( B(Q_T) \)-norm (see Section 3.1), by the fact that, as \( k \to \infty \), \( u_k \rightharpoonup u \) in \( B(Q_T) \), from the Definition 3.2 we deduce that \( u_k(0, \cdot) \rightharpoonup u_0 \) in \( L^2(-1,1) \).

Moreover, since \( \phi \) is locally Lipschitz continuous (see Theorem 5.11),
\[
\phi(u_k) \rightharpoonup \phi(u), \quad \text{in } L^{1+\frac{1}{4}}(-1,1).
\]

Proposition 3.16. Let \( T > 0, u_0, v_0 \in L^2(-1,1) \). \( u, v \) are strong solutions of system (1.1), with initial date \( u_0, v_0 \) respectively. Then, we have
\[
\|u - v\|_{B(Q_T)} \leq \nu_T e^{\|\alpha\|_\infty T} \|u_0 - v_0\|_{L^2(-1,1)},
\]
where \( \alpha^+ \) denotes the positive part of \( \alpha \) \(^{17}\) and \( \nu_T := e^{\nu T} \).

Proof. Let us consider two strong solutions, \( u, v \in B(Q_T) \), of the problem (1.1). Then, there exist \( \{u_k\}_{k \in \mathbb{N}}, \{v_k\}_{k \in \mathbb{N}} \subseteq H(Q_T) \), sequences of strict solutions, such that, as \( k \to \infty \),
\[
u_T \to \nu \rightharpoonup u \rightharpoonup v \rightharpoonup v \text{ in } B(Q_T),
\]
and, for every \( k \in \mathbb{N} \),
\[
u_T - (u_{kxt} x = \alpha(t,x)u_k + \phi(u_k), \quad v_{kt} - (a(x)v_{kx} x = -\alpha(t,x)u_k = \phi(v_k).
\]

So, for every \( k \in \mathbb{N} \), by definition of \( u_k, v_k \) strict solutions, we obtain
\[
u_T - (u_{kxt} x - \alpha(t,x)u_k = \phi(u_k), \quad v_{kt} - (a(x)v_{kx} x - \alpha(t,x)v_k = \phi(v_k),
\]
and multiplying by \( u_k - v_k \) both members of the previous equation and integrating on \((-1,1)\) and applying Lemma 5.12 and condition (2.4) (see Remark 2.4) we obtain
\[
\frac{1}{2} \frac{d}{dt} \int_{-1}^{1} (u_k - v_k)^2 dx + \int_{-1}^{1} a(x)(u_k - v_k)^2 dx
\]
\[
= \int_{-1}^{1} \alpha(t,x)(u_k - v_k)^2 dx + \int_{-1}^{1} (f(t,x,u_k) - f(t,x,v_k))(u_k - v_k) dx
\]
\[
\leq \int_{-1}^{1} \alpha^+(t,x)(u_k - v_k)^2 + \nu \int_{-1}^{1} (u_k - v_k)^2 dx.
\]

Integrating on \((0, t)\), we have
\[
\frac{1}{2} \int_{0}^{t} \|u_k(t, \cdot) - v_k(t, \cdot)\|_{L^2(-1,1)}^2 + \int_{0}^{t} a(x)(u_k - v_k)^2(s, x) dx ds
\]
\[
\leq \frac{1}{2} \int_{0}^{t} \|u_k(0, \cdot) - v_k(0, \cdot)\|_{L^2(-1,1)}^2 + \|\alpha^+\|_\infty \int_{0}^{t} \|u_k(s, \cdot) - v_k(s, \cdot)\|_{L^2(-1,1)}^2 ds + \nu \int_{0}^{t} \|u_k(s, \cdot) - v_k(s, \cdot)\|_{L^2(-1,1)}^2 ds.
\]

\(^{17}\) \( \alpha^+(t,x) := \max\{\alpha(t,x),0\}, \forall (t,x) \in Q_T \), see also Appendix A.2.
Then we obtain
\[
\|u_k(t, \cdot) - v_k(t, \cdot)\|^2_{L^2(-1, 1)} + 2 \int_0^t \int_{-1}^1 a(x)(u_k - v_k)(s, x) dx ds \\
\leq \|u(0, \cdot) - v(0, \cdot)\|^2_{L^2(-1, 1)} + 2 \int_0^t \int_{-1}^1 2 (\|\alpha^+\|_{\infty} + \nu) \|u_k(s, \cdot) - v_k(s, \cdot)\|^2_{L^2(-1, 1)} ds \\
+ \int_0^t 2 (\|\alpha^+\|_{\infty} + \nu) \left(\|u_k(s, \cdot) - v_k(s, \cdot)\|^2_{L^2(-1, 1)} + 2 \int_0^s \int_{-1}^1 a(x)(u_k - v_k)(\tau, x) dx d\tau \right) ds, \forall t \in [0, T].
\]

Applying Grönwall’s lemma we have
\[
\|u_k(t, \cdot) - v_k(t, \cdot)\|^2_{L^2(-1, 1)} + 2 \int_0^t \int_{-1}^1 a(x)(u_k - v_k)(s, x) dx ds \\
\leq e^{2\|\alpha^+\|_{\infty} t + 2\nu t} \|u(0, \cdot) - v(0, \cdot)\|^2_{L^2(-1, 1)}.
\]

Therefore
\[
\|u - v\|^2_{B(Q_T)} \leq \nu_T e^{2\|\alpha^+\|_{\infty} T} \|u(0, \cdot) - v(0, \cdot)\|^2_{L^2(-1, 1)}.
\]

Passing to the limit, as \(k \to \infty\), we obtain
\[
\|u - v\|^2_{B(Q_T)} \leq \nu_T e^{2\|\alpha^+\|_{\infty} T} \|u_0 - v_0\|^2_{L^2(-1, 1)}.
\]

By the previous lemma, applying the inequality (2.1) (see assumptions (A.3)), we obtain the following Corollary 3.17.

**Corollary 3.17.** Let \(T > 0\). A strong solution \(u \in \mathcal{B}(Q_T)\) of system (1.1) satisfies the following a priori estimate
\[
\|u\|_{\mathcal{B}(Q_T)} \leq \nu_T e^{\|\alpha^+\|_{\infty} T} \|u_0\|_{L^2(-1, 1)},
\]

where \(\alpha^+\) denotes the positive part of \(\alpha\) (6) and \(\nu_T := e^{\nu T}\).

**Remark 3.5.** We note that Proposition 3.16 and Corollary 3.17 hold for strict solutions, independently of the notion of strong solution. Indeed, we proved the inequality (3.10) first, for strict solutions, then for strong solutions by approximation.

In this paper, we obtain the result of existence and uniqueness of solutions to (1.1) with initial state in \(L^2(-1, 1)\).

**Theorem 3.18.** For all \(u_0 \in L^2(-1, 1)\) there exists a unique strong solution \(u \in \mathcal{B}(Q_T)\) to (1.1).

**Proof.** Let \(u_0 \in L^2(-1, 1)\). There exists \(\{u^0_k\}_{k \in \mathbb{N}} \subseteq H^1_0(-1, 1)\) such that, as \(k \to \infty\), \(u^0_k \to u_0\) in \(L^2(-1, 1)\). For each \(k \in \mathbb{N}\), we consider the following problem
\[
\begin{align*}
    u_{kt} - (a(x)u_{kx})_x = & \alpha(t, x)u_k + f(t, x, u_k) \quad \text{a.e. in } Q_T := (0, T) \times (-1, 1) \\
    a(x)u_{kx}(t, x)|_{x = \pm 1} = & 0 \quad \text{a.e. in } (0, T) \\
    u_k(0, x) = & u^0_k(x) \quad \text{in } x \in (-1, 1).
\end{align*}
\]

For every \(k \in \mathbb{N}\), by the uniqueness and existence of the strict solution to system (3.11) (see Theorem 3.15), exists a unique \(u_k \in \mathcal{H}(Q_T)\) strict solution to (3.11). Then, we consider the sequence \(\{u_k\}_{k \in \mathbb{N}} \subseteq \mathcal{H}(Q_T)\) and by direct application of the Proposition 3.16 (see Remark 3.5) we prove that \(\{u_k\}_{k \in \mathbb{N}}\) is a Cauchy sequence in the Banach space \(\mathcal{B}(Q_T)\). Then, there exists \(u \in \mathcal{B}(Q_T)\) such that, as \(k \to \infty\), \(u_k \to u\) in \(\mathcal{B}(Q_T)\) and \(u(0, \cdot) \overset{L^2}{\underset{k \to \infty}{\lim}} u_k(0, \cdot) \equiv u_0\). So, \(u \in \mathcal{B}(Q_T)\) is a strong solution.

The uniqueness of the strong solution to (1.1) is trivial, applying Proposition 3.16. 

\(\square\)
4. Controllability of nonlinear problems

In this section we study the global non-negative approximate multiplicative controllability for semilinear degenerate parabolic Cauchy-Neumann problems. Given \( T > 0 \), let us consider the control system \((1.1)\) (strongly degenerate boundary problem in divergence form, governed in the bounded domain \((-1,1)\) by means of the bilinear control \(\alpha(t,x)\))

\[
\begin{aligned}
&v_t - (a(x)v_x)_x = \alpha(t,x)v + f(t,x,u) \quad \text{in} \quad Q_T := (0,T) \times (-1,1) \\
&a(x)v_x(t,x)|_{x = \pm 1} = 0 \quad \text{in} \quad (0,T) \\
&v(0,x) = v_0(x) \quad \text{in} \quad (-1,1),
\end{aligned}
\]

under the assumptions \((A.1) - (A.4)\).

We will show that this system can be steered in \(L^2(-1,1)\) from any nonzero, nonnegative initial state \(u_0 \in L^2(-1,1)\) into any neighborhood of any desirable nonnegative target-state \(u_d \in L^2(-1,1)\), by bilinear controls. Moreover, we extend the above result relaxing the sign constraint on \(u_0\).

In the following, we will sometimes use \(\| \cdot \|, \langle \cdot, \cdot \rangle\) instead of \(\| \cdot \|_{L^2(-1,1)}, \langle \cdot, \cdot \rangle_{L^2(-1,1)}\), respectively, and \(\| \cdot \|_\infty\) instead of \(\| \cdot \|_{L^\infty(Q_T)}\).

4.1. Some useful lemmas

In Section 4.1, we consider the semilinear system \((1.1)\) and the associated linear system

\[
\begin{aligned}
v_t - (a(x)v_x)_x = \alpha(t,x)v \quad &\text{in} \quad Q_T := (0,T) \times (-1,1) \\
a(x)v_x(t,x)|_{x = \pm 1} = 0 \quad &\text{in} \quad (0,T) \\
v(0,x) = v_0(x) \quad &\text{in} \quad (-1,1),
\end{aligned}
\]  \( (4.1) \)

where \(v_0 \in L^2(-1,1)\), and the coefficients \(a(x)\) and \(\alpha(t,x)\) are the same as the semilinear system \((1.1)\).

In this Section 4.1, we obtain some useful results for the proofs of the main theorems.

Lemma 4.1. Let \( T > 0 \), let \(u_0 \in L^2(-1,1)\) and let \(u \in B(Q_T)\) be the strong solution of \((1.1)\) and \(v \in B(Q_T)\) be the weak solution of \((4.1)\) with initial state \(v_0 = u_0\). Then, the difference \(u - v\) belongs to \(B(Q_T)\) and satisfies

\[
\|u - v\|_{B(Q_T)} \leq C T^\rho e^{KT} \|u_0\|_{L^2(-1,1)},
\]

where \(C\) is a positive constant, \(\rho = \frac{3-\vartheta}{4}\) and \(K = (2 + \vartheta)\|\alpha^+\|_\infty + \vartheta \nu\) (\(\alpha^+\) denotes the positive part of \(\alpha\)).

Proof. Let \(\{u_k\}_{k \in \mathbb{N}} \subseteq \mathcal{H}(Q_T)\) be a approximating sequence of the strong solution \(u\). For every fixed \(k \in \mathbb{N}\), let \(v_k \in \mathcal{H}(Q_T)\) be the solution to \((4.1)\) with initial state \(u_k(0,x)\) \(\Box\). Setting, for simplicity of notation, \(w(t,x) := u_k(t,x) - v_k(t,x)\) in \(Q_T\), we have that \(w \in \mathcal{H}(Q_T)\) is strict solution of the following system

\[
\begin{aligned}
w_t - (aw_x)_x = \alpha w + f(t,x,u_k) \quad &\text{in} \quad Q_T \\
a(x)w_x(t,x)|_{x = \pm 1} = 0 \quad &\text{in} \quad (0,T) \\
w(0,x) = 0 \quad &\text{in} \quad (-1,1),
\end{aligned}
\]  \( (4.2) \)

\(18\) For existence, uniqueness and regularity of solutions of linear problem \((3.1)\) see Section 3.3.
So, for every $t \in (0, T)$ and integrating on $(-1, 1)$, we deduce that

\[
\frac{1}{2} \frac{d}{dt} \int_{-1}^{1} w^2 \, dx + \int_{-1}^{1} aw^2 \, dx = \int_{-1}^{1} \alpha w^2 \, dx + \int_{-1}^{1} f(t, x, u_k) w \, dx
\]

and therefore, integrating on $(-1, 1)$, we obtain

\[
\int_{-1}^{1} \alpha^+ w^2 \, dx + \int_{-1}^{1} |f(t, x, u_k)| |w| \, dx \leq \|w\|_{\infty} \int_{-1}^{1} \alpha^+ w^2 \, dx + \int_{-1}^{1} |f(t, x, u_k)| |w| \, dx.
\]

Fixing $t \in (0, T)$ and integrating on $(0, t)$, we obtain

\[
\|w(t, \cdot)\|_{L^2((t-1, 1))}^2 + 2 \int_{0}^{t} \int_{-1}^{1} aw^2 \, dx \leq 2 \|\alpha^+\|_{\infty} \int_{0}^{t} \|w(s, \cdot)\|_{L^2((-1, 1))}^2 \, ds + 2 \int_{0}^{t} \int_{-1}^{1} |f(s, x, u_k)| |w| \, dx \cdot\cdot\cdot\cdot
\]

Since $u_k, v_k \in H(Q_T)$ and therefore $w = u_k - v_k \in H(Q_T) \subseteq B(Q_T)$, by (2.31) and Hölder’s inequality, we have

\[
\int_{0}^{t} \int_{-1}^{1} |f(s, x, u_k)| |w| \, dx \leq \gamma_0 \int_{0}^{t} \int_{-1}^{1} |u_k|^\theta |w| \, dx \leq \gamma_0 \|u_k\|_{L^\theta+1(Q_T)}^\theta \|w\|_{L^\theta+1(Q_T)}.
\]

Thanks to the assumption (A.4) $\xi \in L^{\frac{2}{\theta-2}}((-1, 1),)$, then we can apply the Corollary 3.7, so, applying also Young’s inequality, we obtain

\[
\int_{0}^{t} \int_{-1}^{1} |f(t, x, u_k)| |w| \, dx \leq \gamma_0 \|u_k\|_{L^\theta+1(Q_T)}^\theta \|w\|_{L^\theta+1(Q_T)} \leq c t^{\frac{\theta-2}{2}} \|u_k\|_{B(Q_T)}^\theta \|w\|_{B(Q_T)} + \frac{1}{4} \|w\|_{B(Q_T)}^2.
\]

So, for every $t \in (0, T)$, we obtain

\[
\|w(t, \cdot)\|_{L^2((-1, 1))}^2 + 2 \int_{0}^{t} \int_{-1}^{1} aw^2 \, dx \leq 2 \|\alpha^+\|_{\infty} \int_{0}^{t} \|w(s, \cdot)\|_{L^2((-1, 1))}^2 \, ds + c t^{\frac{\theta-2}{2}} \|u_k\|_{B(Q_T)}^\theta + \frac{1}{2} \|w\|_{B(Q_T)}^2 \cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\·
Lemma 4.2. Let $T > 0$, let $u_0 \in L^2(-1,1)$, $u_0(x) \geq 0$ a.e. $x \in (-1,1)$ and let $u \in B(Q_T)$ be the strong solution to the semilinear system (1.1). Then

$$u(t,x) \geq 0, \quad \text{for a.e. } (t,x) \in Q_T.$$ 

Proof. Since $u_0 \in L^2(-1,1)$, $u_0 \geq 0$ a.e. $x \in (-1,1)$, there exists $\{u_k^0\}_{k \in \mathbb{N}} \subseteq C^\infty([-1,1]), u_k^0 \geq 0$ on $(-1,1)$ for every $k \in \mathbb{N}$, such that $u_k^0 \rightharpoonup u_0$ in $L^2(-1,1)$, as $k \to \infty$. For every $k \in \mathbb{N}$, we consider $u_k \in \mathcal{H}(Q_T)$ the strict solution to the semilinear system (1.1) with initial date $u_k^0$. Keeping in mind that $u(0,\cdot) = u_0$ and applying Proposition 5.16, we can observe that $u_k \rightharpoonup u$ in $B(Q_T)$, as $k \to \infty$.

First, we prove that $u_k(t,x) \equiv 0$ in $Q_T$.\footnote{We denote with $u_k^+, u_k^-$ the positive and negative part of $u_k$, respectively (see Appendix A.2).}

Multiplying both members of the equation

$$u_{kt} - (a(x)u_{kx})_x = \alpha u_k + f(t,x,u_k)$$

by $u_k^-$ and integrating on $(-1,1)$ we obtain

\[
\int_{-1}^{1} [u_{kt} u_k^- - (a(x)u_{kx})_x u_k^-] \, dx = \int_{-1}^{1} [\alpha u_k u_k^- + f(t,x,u_k) u_k^-] \, dx. \tag{4.3}
\]

Recalling the definition of $u^+$ and $u^-$ (see Appendix A.2), we have

\[
\int_{-1}^{1} u_k a u_k^- dx = \int_{-1}^{1} (u_k^+ - u_k^-) a u_k^- dx = - \int_{-1}^{1} (u_k^-) a u_k^- dx = - \frac{1}{2} \frac{d}{dt} \int (u_k^-)^2 \, dx.
\]

Integrating by parts and recalling that $u_k^-(t,\cdot) \in H^1_a(-1,1)$, for every $t \in (0,T)$, we obtain the following equality (see Appendix A.2)

\[
\int_{-1}^{1} a(x)u_{kx} \, dx = [a(x)u_{kx}u_k^-]_{-1}^{1} - \int_{-1}^{1} a(x)u_{kx}(-u_k^-)_x \, dx = \int_{-1}^{1} a(x)u_{k}^2 \, dx.
\]

We also have

\[
\int_{-1}^{1} \alpha u_k u_k^- dx = - \int_{-1}^{1} \alpha (u_k^-)^2 \, dx.
\]

Moreover, using (2.4), we have

\[
\int_{-1}^{1} f(t,x,u_k) u_k^- dx = \int_{-1}^{1} f(t,x,u_k^+ - u_k^-) u_k^- dx = \int_{-1}^{1} f(t,x,-u_k^-) u_k^- dx
\]

\[
= - \int_{-1}^{1} f(t,x,-u_k^-) (-u_k^-)_x \, dx \geq - \int_{-1}^{1} \nu (-u_k^-)^2 \, dx = - \int_{-1}^{1} \nu (u_k^-)^2 \, dx
\]

and therefore (4.3) becomes

\[
- \frac{1}{2} \frac{d}{dt} \int_{-1}^{1} (u_k^-)^2 \, dx + \int_{-1}^{1} (a_k^-)^2 \, dx + \int_{-1}^{1} \nu (u_k^-)^2 \, dx \geq \int_{-1}^{1} a(x)u_{k}^2 \, dx \geq 0,
\]

from which

\[
\frac{d}{dt} \int_{-1}^{1} (u_k^-)^2 \, dx \leq 2 \int_{-1}^{1} (\alpha(t,x) + \nu) (u_k^-)^2 \, dx \leq 2 (\|\alpha\|_\infty + \nu) \int_{-1}^{1} (u_k^-)^2 \, dx.
\]

From the above inequality, applying Gronwall’s inequality we obtain

\[
\int_{-1}^{1} (u_k^-(t,x))^2 \, dx \leq \nu_2^2 e^{2\|\alpha\|_\infty t} \int_{-1}^{1} (u_k^-(0,x))^2 \, dx, \quad \forall t \in (0,T).
\]
Since \( u_k(0, x) = u_k^0(x) \geq 0 \), we have \( u_k^-(0, x) = 0 \). Therefore,
\[
    u_k^-(t, x) = 0, \quad \forall (t, x) \in Q_T.
\]
From this, for every \( k \in \mathbb{N} \), it follows that
\[
    u_k(t, x) = u_k^+(t, x) \geq 0, \quad \forall (t, x) \in Q_T.
\]
(4.4)

Since \( u_k \to u \) in \( B(Q_T) \), as \( k \to \infty \), there exists \( \{u_{k_n}\}_{k \in \mathbb{N}} \subseteq \{u_k\}_{k \in \mathbb{N}} \) such that, as \( h \to \infty \),
\[
    u_{k_n}(t, x) \to u(t, x), \quad \text{a.e. } (t, x) \in Q_T.
\]
(4.5)

Applying (4.4) and (4.5), we obtain
\[
    u(t, x) \geq 0, \quad \text{a.e. } (t, x) \in Q_T.
\]
\[\square\]

4.2. Proofs of main results

Proof. (of Theorem 2.1) To prove Theorem 2.1 it is sufficient to consider the set of target states
\[
    u_d \in C^\infty([-1, 1]), \quad u_d > 0 \text{ on } [-1, 1].
\]
Indeed, every function \( u_d \in L^2(-1, 1), u_d \geq 0 \) can be approximated by a sequence of strictly positive functions of class \( C^\infty([-1, 1]) \).

Then, let us consider any \( u_0 \in L^2(-1, 1) \) and any \( u_d \in C^\infty([-1, 1]) \) such that \( u_0 \geq 0, u_d > 0 \) and \( u_0 \neq 0 \).

**STEP 1** We denote with \( \{-\mu_k\}_{k \in \mathbb{N}} \) and \( \{P_k\}_{k \in \mathbb{N}} \), respectively, the eigenvalues and orthonormal eigenfunctions of the spectral problem \( A_0\omega = \mu\omega \), with \( A_0 \) defined as in (3.3) (20) (see Lemma 3.9). Set
\[
    z(t, x) := \sum_{k=1}^{\infty} e^{-\mu_k t} \langle u_0, P_k \rangle P_k(x).
\]
Since \( z \in B(Q_T) \), we can observe that
\[
    z(t, x) = \sum_{k=1}^{\infty} (e^{-\mu_k t} - 1) \langle u_0, P_k \rangle P_k(x) + u_0(x) \xrightarrow{L^2} u_0(x), \quad \text{as } t \to 0.
\]
Fix any \( s \in (0, 1) \), thus
\[
    \exists \ t^*(s) > 0 \text{ such that } \|z(t, \cdot) - u_0\| \leq \frac{s}{2}, \quad \forall t \leq t^*(s).
\]
Moreover,
\[
    \exists \ \bar{t}(s) > 0 \text{ such that } t^* e^{K t} \leq \frac{s^2}{2C\|u_0\|^2}, \quad \forall t \leq \bar{t}(s),
\]
(4.7)
where \( \rho, C, K \) are the positive constants of Lemma 4.1. Now, set
\[
    t_1(s) = \min\{t^*(s), \bar{t}(s), 1\},
\]
we can observe that \( t_1(s) \to 0 \), as \( s \to 0 \).

**STEP 2** We select the following negative constant bilinear control
\[
    \alpha(t, x) = \alpha_1(s) := \frac{\ln s}{t_1(s)} < 0, \quad \forall t \in [0, t_1(s)], \forall x \in (-1, 1),
\]
20In the case \( a(x) = 1 - x^2 \), that is, where the principal part of the operator is that the Budyko-Sellers model, the orthonormal eigenfunctions are reduced to Legendre polynomials, and the eigenvalues are \( \mu_k = (k - 1/2)^2 \), \( k \geq 1 \) (see also Remark 4.3).
that is, \( \alpha_1(s) \) is such that \( e^{\alpha_1(s) t_1(s)} = s \). On the interval \([0, t_1(s))\), we apply the negative constant control \( \alpha(t, x) = \alpha_1(s), \forall x \in (-1, 1) \). Now, we consider the linear problem \( (4.1) \) with \( \alpha(t, x) \equiv \alpha_1(s), \forall t \in [0, t_1(s)], \forall x \in (-1, 1) \), and initial state \( v_0 = u_0 \). For \( t = t_1(s) \), the weak solution \( v(t, x) \) of \( (4.1) \) (18) has the following representation in Fourier series

\[
v(t_1(s), x) = e^{\alpha_1(s) t_1(s)} \sum_{k=1}^{\infty} e^{-\mu_k t_1(s)} (u_0, P_k(x)) s z(t_1(s), x), \forall x \in (-1, 1).
\]

Therefore, by (4.10), we obtain

\[
\|v(t_1(s), \cdot) - su_0\| = s \|z(t_1(s), \cdot) - u_0\| \leq \frac{s^2}{2}.
\]

Let \( u \) be the strong solution to \( (1.1) \) with bilinear control \( \alpha(t, x) \equiv \alpha_1(s), t > 0, x \in (-1, 1) \), and initial state \( u_0 \). By Lemma 4.1 the choice of \( t_1(s) \) and (4.7) we have

\[
\|u(t_1(s), \cdot) - v(t_1(s), \cdot)\| \leq C (t_1(s))^{\rho} e^{K t_1(s)} \|u_0\|^\theta \leq \frac{s^2}{2},
\]

where \( \rho, C, K \) are the positive constants of Lemma 4.1. From (4.8) and (4.9) we obtain

\[
\|u(t_1(s), \cdot) - su_0\| \leq \|u(t_1(s), \cdot) - v(t_1(s), \cdot)\| + \|v(t_1(s), \cdot) - su_0\| \leq s^2.
\]

Let us define

\[
\delta_s(x) := u(t_1(s), x) - su_0(x), \forall x \in (-1, 1),
\]

and we observe that, in view of (4.11),

\[
\frac{\|\delta_s(\cdot)\|}{s} \longrightarrow 0, \quad \text{as } s \to 0.
\]

In this way, we have steered the nonlinear system \( (1.1) \) from the initial state \( u_0 \) to the target state \( su_0 + \delta_s \), at time \( t_1(s) \).

**STEP 2** Let us fix \( \eta \in (0, \theta - 1) \). We will steer the system from the initial state \( u(t_1(s), x) = su_0(x) + \delta_s(x), x \in (-1, 1) \), to an arbitrarily small neighborhood of the target state

\[
s^{1+\eta} u_d,
\]

at some time \( t_2(s) \). For this purpose, define

\[
\alpha_2(x) := \alpha_s(x) + \beta, \forall x \in (-1, 1),
\]

with \( \alpha_s(x) = -\frac{(u(x)u_{dd}(x))_x}{u_{dd}(x)}, x \in (-1, 1) \), and

\[
\beta = \min \left\{ -\| \alpha_s \|_{L^\infty((-1, 1))}, -\frac{\eta K}{\theta - 1 - \eta} \right\} - 1,
\]

where \( K \) is the positive constant of Lemma 4.1. We denote by \( \{-\lambda_k\}_{k \in \mathbb{N}} \) and \( \{\omega_k\}_{k \in \mathbb{N}} \), respectively, the eigenvalues and orthonormal eigenfunctions of the spectral problem \( A \omega = \lambda \omega \), with \( A = A_0 + \alpha_* I \) and \( D(A) = H_2^2((-1, 1)) \) \( (A_0 \) is the operator defined in (3.3), see also Lemma 3.9). Applying Lemma 3.10 we have that

\[
\lambda_1 = 0 \quad \text{and} \quad \omega_1(x) = \frac{u_d(x)}{\|u_d\|} > 0, \forall x \in (-1, 1).
\]

Set

\[
u_k(s) := \langle u(t_1(s), \cdot), \omega_k \rangle, \forall k \in \mathbb{N}.
\]
Thus,
\[ u_k(s) = s z_k(s), \text{ where } z_k(s) := \left(u_0 + \frac{\delta_s}{s}, \omega_k\right), \forall k \in \mathbb{N}. \]
Then, by (4.11) and (4.13), we can observe that
\[ z_1(s) \rightarrow \frac{1}{\| u_d \|} (u_0, u_d) > 0, \text{ as } s \rightarrow 0. \] (4.14)

The weak solution of linear problem (4.1), with \( \alpha(t, x) = \alpha_\ast(x) + \beta, \ t > t_1(s), \ x \in (-1, 1), \) and initial state \( v(t_1(s), \cdot) = s u_0(\cdot) + \delta_s(\cdot), \) has the following representation in Fourier series (23)
\[ v(t, x) = \sum_{k=1}^{\infty} e^{(-\lambda_k + \beta)(t-t_1(s))} u_k(s) \omega_k(x) = e^{\beta(t-t_1(s))} u_1(s) \omega_1(x) + \sum_{k>1} e^{(-\lambda_k + \beta)(t-t_1(s))} u_k(s) \omega_k(x). \]

Let
\[ r_s(t, x) = \sum_{k>1} e^{(-\lambda_k + \beta)(t-t_1(s))} u_k(s) \omega_k(x) \]
where \(-\lambda_k < -\lambda_1 = 0, \) for every \( k \in \mathbb{N}, k > 1 \) (see Lemma 3.9). Owing to (4.13),
\[ \| v(t, \cdot) - s^{1+\eta} u_d \| \leq \left\| e^{\beta(t-t_1(s))} u_1(s) \omega_1 - s^{1+\eta} u_d \omega_1 \right\| + \| r_s(t, x) \| = \left\| e^{\beta(t-t_1(s))} u_1(s) - s^{1+\eta} \| u_d \| \right\| + \| r_s(t, x) \|. \]

Since \(-\lambda_k < -\lambda_2, \) for every \( k \in \mathbb{N}, k > 2 \) (see Lemma 3.9), applying Parseval’s equality we have
\[ \| r_s(t, x) \|^2 \leq e^{2(-\lambda_2 + \beta)(t-t_1(s))} \sum_{k>1} |u_k(s)|^2 \| \omega_k \|^2 = e^{2(-\lambda_2 + \beta)(t-t_1(s))} \| s u_0 + \delta_s \|^2. \]

By (4.14) we obtain
\[ \exists \ s^* \in (0, 1) : u_1(s) = (s u_0 + \delta_s, \omega_1) > 0, \forall s \in (0, s^*). \] (4.15)

Then, we choose \( t_2(s), \ t_2(s) > t_1(s) \) such that
\[ e^{\beta(t_2(s)-t_1(s))} u_1(s) = s^{1+\eta} \| u_d \|, \] (4.16)
that is, since \( \omega_1 = \frac{u_d}{\| u_d \|}, \)
\[ t_2(s) = t_1(s) + \frac{1}{\beta} \ln \left( \frac{s^{\eta} \| u_d \|^2}{\langle u_0 + \frac{\delta_s}{s}, u_d \rangle} \right). \] (4.17)

So, by (4.16) and the above estimates for \( \| v(t_2(s), \cdot) - s^{1+\eta} u_d(\cdot) \| \) and \( \| r_s(t_2(s), \cdot) \| \) we conclude that
\[ \| v(t_2(s), \cdot) - s^{1+\eta} u_d(\cdot) \| \leq e^{(-\lambda_2 + \beta)(t_2(s)-t_1(s))} \| su_0 + \delta_s \| = e^{-\lambda_2(t_2(s)-t_1(s))} s^{1+\eta} \| u_d \| \frac{\| u_d \|}{\| u_1(s) \|} \| u_0 + \frac{\delta_s}{s} \| s^{1+\eta}. \] (4.18)

Thus, by (4.16) and by (4.15), we deduce that there exists \( s_0 \in (0, s^*) \) such that
\[ e^{-\lambda_2(t_2(s)-t_1(s))} \frac{\| u_d \| \| u_0 + \frac{\delta_s}{s} \|}{\| u_1(s) \|} = \left( s^{\eta} \| u_d \| \frac{s^{\eta} \| u_d \|^2}{\langle u_0 + \frac{\delta_s}{s}, u_d \rangle} \right)^{\frac{1}{\beta}} \| u_d \| \frac{\| u_0 + \frac{\delta_s}{s} \|}{\| u_1(s) \|} \leq c s, \forall s \in (0, s_0). \]

\[ \text{We observe that adding } \beta \in \mathbb{R} \text{ to the coefficient } \alpha_\ast(x) \text{ there is a shift of the eigenvalues corresponding to } \alpha_\ast \text{ from } \{ -\lambda_k \}_{k \in \mathbb{N}} \text{ to } \{ -\lambda_k + \beta \}_{k \in \mathbb{N}}, \text{ but the eigenfunctions remain the same for } \alpha_\ast \text{ and } \alpha_\ast + \beta. \]
From the above, the inequality (4.18) becomes
\[ \|v(t_2(s), \cdot) - s^{1+\eta}u_d(\cdot)\| \leq cs^{\frac{-\eta}{2}}s^{1+\eta}, \quad \forall s \in (0,s_0), \] (4.19)
where c is a positive constant.

Then, by (4.12), we observe that
\[ \alpha_2(t,x) = \alpha_3(x) + \beta < 0, \quad \forall t \in [t_1(s), t_2(s)], \forall x \in (-1,1). \]

Let u be the strong solution to (4.1) with \( \alpha(t,x) = \alpha_3(x) + \beta, \; t > t_1(s), \; x \in (-1,1) \), and initial state \( u(t_1(s), \cdot) = su_0(\cdot) + \delta_s(\cdot) \). Thus, by Lemma 4.1, we deduce the following estimate
\[ \|u(t_2(s), \cdot) - v(t_2(s), \cdot)\| \leq C (t_2(s) - t_1(s))^\rho e^{K(t_2(s) - t_1(s))} \|su_0 + \delta_s\|^\theta, \] (4.20)
where \( \rho, C, K \) are the positive constants of Lemma 4.1. Then, by (4.17), we deduce that
\[ e^{K(t_2(s) - t_1(s))} = \left( \frac{s^\eta \|u_d\|}{z_1(s)} \right)^{\frac{1}{\beta}} \leq c' s^{\frac{\rho}{1+\eta}}, \quad \forall s \in (0,s_0). \] (4.21)

Then, by (4.19) - (4.21), we have the following estimate
\[
\begin{align*}
\|u(t_2(s), \cdot) - s^{1+\eta}u_d(\cdot)\| &\leq \|u(t_2(s), \cdot) - v(t_2(s), \cdot)\| + \|v(t_2(s), \cdot) - s^{1+\eta}u_d(\cdot)\| \\
&\leq C (t_2(s) - t_1(s))^\rho e^{K(t_2(s) - t_1(s))} \|su_0 + \delta_s\|^\theta + cs^{\frac{-\eta}{2}}s^{1+\eta} \\
&\leq C (t_2(s) - t_1(s))^\rho c' s^{\frac{\rho}{1+\eta}} s^\theta \left\| u_0 + \frac{s \delta_s}{s} \right\|^\theta + cs^{\frac{-\eta}{2}}s^{1+\eta} \\
&\leq k \left( (t_2(s) - t_1(s))^\rho s^{\frac{\rho}{1+\eta}} s^\theta \left\| u_0 + \frac{s \delta_s}{s} \right\|^\theta + s^{\frac{-\eta}{2}} \right) s^{1+\eta}, \quad \forall s \in (0,s_0),
\end{align*}
\] (4.22)
where k is a positive constant. Now, we have
\[ t_2(s) - t_1(s) = \frac{1}{\beta} \ln \left( \frac{s^\eta \|u_d\|^2}{\langle u_0 + \frac{s \delta_s}{s}, u_d \rangle} \right) \longrightarrow +\infty, \quad \text{as} \; s \to 0^+. \]

Since \( \frac{nK}{2} + \theta - 1 - \eta > 0 \), by the choice of \( \beta \) (see (4.12)), we have
\[ (t_2(s) - t_1(s))^\rho s^{\frac{nK}{2} + \theta - 1 - \eta} = \left( \frac{1}{\beta} \ln \left( \frac{s^\eta \|u_d\|^2}{\langle u_0 + \frac{s \delta_s}{s}, u_d \rangle} \right) \right)^\rho s^{\frac{nK}{2} + \theta - 1 - \eta} \longrightarrow 0, \]
as \( s \to 0^+ \). Defining
\[ \delta_{s^{1+\eta}}(x) := u(t_2(s), \cdot) - s^{1+\eta}u_d(\cdot) \quad x \in (-1,1), \]
estimate (4.22) yields
\[ \frac{\|\delta_{s^{1+\eta}}(\cdot)\|}{s^{1+\eta}} \longrightarrow 0, \quad \text{as} \; s \to 0^+. \] (4.23)

**STEP 3** Let \( \tau > 0 \). On the interval \( (t_2(s), T(s)) \), with \( T(s) = t_2(s) + \tau \), we apply a positive constant control \( \alpha_3(x) \equiv \alpha_3 \) (its value will be chosen below).

We can represent the weak solution of the linear problem (4.1), with \( \alpha(t,x) \equiv \alpha_3 \) and initial state \( v(t_2(s), \cdot) = u(t_2(s), \cdot) = s^{1+\eta}u_d + \delta_{s^{1+\eta}} \), by Fourier series in the following way
\[ v(t_2(s) + \tau, x) = e^{\alpha_3 \tau} \sum_{k=1}^{\infty} e^{-\mu_k \tau} \langle u(t_2(s), \cdot), P_k \rangle P_k(x), \]
where \( \mu_k \) are the eigenvalues of the operator \( A \).
Let us consider
\[ z(\tau, x) := \sum_{k=1}^{\infty} e^{-\mu_k \tau} \langle u(t_2(s), \cdot), P_k \rangle P_k(x), \]
then,
\[ z(\tau, x) = \sum_{k=1}^{\infty} \left( e^{-\mu_k \tau} - 1 \right) \langle u(t_2(s), \cdot), P_k \rangle P_k(x) + s^{1+\eta} u_d(x) + \delta_s^{1+\eta}(x) \xrightarrow{L^2} s^{1+\eta} u_d + \delta_s^{1+\eta}, \quad \text{as } \tau \to 0^+. \]

(4.24)

Now, for every \( 0 < \varepsilon < 1 \), by (4.25), we have

\[ \exists s_\varepsilon \in (0, s_0) \text{ such that} \quad \frac{\| \delta_s^{1+\eta} \|}{s_\varepsilon^{1+\eta}} < \frac{\varepsilon}{4}. \]  

(4.25)

So, by (4.23),

\[ \exists \tau_\varepsilon = \tau(s_\varepsilon) > 0 \text{ such that} \]

\[ C^2 \varepsilon^{2+\theta} \| s_\varepsilon^{2(1+\eta)} \| (\| u_d \| + 1)^{\theta} \leq \frac{\varepsilon}{2} \quad \text{and} \quad \| u(t_2(s), \cdot) - (s_\varepsilon^{1+\eta} u_d + \delta_s^{1+\eta}) \| \leq \frac{\varepsilon}{4} s_\varepsilon^{1+\eta}, \]

(4.26)

where \( \rho, C \) are the positive constants of Lemma 4.1.

Set \( T_\varepsilon = T(s_\varepsilon) = t_2(s_\varepsilon) + \tau_\varepsilon \). Let us define

\[ \alpha(t, x) = \alpha_3(s_\varepsilon) := -\frac{1 + \eta}{\tau_\varepsilon} \ln s_\varepsilon, \quad \forall t \in [t_2(s_\varepsilon), T_\varepsilon], \forall x \in (-1, 1). \]

(4.27)

Let \( u \) be the strong solution to (1.1) with bilinear control \( \alpha(t, x) \equiv \alpha_3, \quad t > t_2(s_\varepsilon), x \in (-1, 1) \), and initial state \( u(t_2(s_\varepsilon), \cdot) = s_\varepsilon^{1+\eta} u_d + \delta_s^{1+\eta} \). By Lemma 4.1 taking in mind that in our case the positive constant \( K \) of Lemma 4.1 is \( K = (2+\theta)\alpha_3(s_\varepsilon) + \theta \nu \), and by (4.25) and (4.27), since \( \varepsilon < 1 \), we obtain

\[ \| u(t_2(s), \tau_\varepsilon, \cdot) - u(t_2(s), \tau_\varepsilon, \cdot) \| \leq C^2 \varepsilon^{2+\theta} \| s_\varepsilon^{1+\eta} \| u_d + \delta_s^{1+\eta} \| \leq C^2 \varepsilon^{2+\theta} \| s_\varepsilon^{1+\eta} \| \leq \frac{\varepsilon}{2}. \]

Moreover, by (4.25) and (4.27), we deduce that

\[ \| v(T_\varepsilon, x) - u_d \| = \| e^{\nu \tau_\varepsilon} \varepsilon \tau_\varepsilon z(\tau_\varepsilon, \cdot) - u_d \| = s_\varepsilon^{1+\eta}\| z(\tau_\varepsilon, \cdot) - s_\varepsilon^{1+\eta} u_d \| \leq s_\varepsilon^{-1}(\| z(\tau_\varepsilon, \cdot) - (s_\varepsilon^{1+\eta} u_d + \delta_s^{1+\eta}) \| + \| \delta_s^{1+\eta} \|) \leq s_\varepsilon^{-1}(\| z(\tau_\varepsilon, \cdot) - (s_\varepsilon^{1+\eta} u_d + \delta_s^{1+\eta}) \| + \| \delta_s^{1+\eta} \|) \leq \frac{\varepsilon}{2}. \]

Therefore, by the last two inequalities we have

\[ \| u(T_\varepsilon, x) - u_d \| \leq \| u(T_\varepsilon, x) - u(T_\varepsilon, x) \| + \| v(T_\varepsilon, x) - u_d \| \leq \varepsilon, \]

from which the conclusion, keeping also in mind the Lemma 4.2.

\[ \square \]

**Proof.** (of Theorem 2.2) The proof of Theorem 2.1 can be adapted to Theorem 2.2 keeping in mind that in Step 2 of the previous proof, the inequality in (4.14) continues to hold in this new setting. In fact we have

\[ \int_{-1}^{1} u_0(x) \omega_1(x) \, dx = \int_{-1}^{1} u_0(x) \| u_d \| \, dx = \frac{1}{\| u_d \|} \int_{-1}^{1} u_0 u_0 \, dx > 0, \]

by assumption (2.6).

From this point on, one can proceed as in the proof of Theorem 2.1.

\[ \square \]
Appendix A.

Appendix A.1. Proof of a singular Sturm-Liouville result
In this section, we recall the proof of Lemma 3.10 (see also [11] and [28]).

Proof. (of Lemma 3.10). We denote by \{-\lambda_k\}_{k \in \mathbb{N}} and \{\omega_k\}_{k \in \mathbb{N}}, respectively, the eigenvalues and orthonormal eigenfunctions of the operator defined in (3.6) (see Lemma 3.9). Therefore,

\[
\langle \omega_k, \omega_h \rangle = \int_{-1}^{1} \omega_k(x)\omega_h(x)dx = 0, \quad \text{if } h \neq k.
\]

We can see, by easy calculations, that an eigenfunction of the operator defined in (3.6) is the function \(\frac{v(x)}{\|v\|}\), associated with the eigenvalue \(\lambda = 0\). Taking into account the above and considering that \(v(x) > 0, \forall x \in (-1,1)\)

\[
\exists k_* \in \mathbb{N} : \omega_{k_*}(x) = \frac{v(x)}{\|v\|} > 0 \quad \text{or} \quad \omega_{k_*}(x) = -\frac{v(x)}{\|v\|} < 0, \forall x \in (-1,1).
\]

Keeping in mind that \(\int_{-1}^{1} \omega_{k_*}(x)\omega_{h}(x)dx = 0\), if \(h \neq k_*\) and \(\omega_{k_*} > 0\) or \(\omega_{k_*} < 0\) in \((-1,1)\), we observe that \(\omega_{k_*}\) is the only eigenfunction of the operator defined in (3.6) that doesn't change sign in \((-1,1)\).

Let us now prove that \(k_* = 1\), that is, \(\lambda_1 = 0\).

By a well-known variational characterization of the first eigenvalue, we have

\[
\lambda_1 = \inf_{u \in H^1_{x}(-1,1)} \frac{\int_{-1}^{1} (a u_x^2 - \alpha u^2) dx}{\int_{-1}^{1} u^2 dx}.
\]

By Lemma 3.9 since \(\lambda_{k_*} = 0\), it is sufficient to prove that \(\lambda_1 \geq 0\), or

\[
\int_{-1}^{1} \alpha_* u_x^2 dx \leq \int_{-1}^{1} a u_x^2 dx, \quad \forall u \in H^1_{x}(-1,1).
\]

Integrating by parts, we obtain the desired inequality

\[
\int_{-1}^{1} \alpha_* u_x^2 dx = -\int_{-1}^{1} \left(\frac{a v_x}{v}\right) u_x^2 dx = \int_{-1}^{1} a v_x \left(\frac{u_x^2}{v^2}\right) dx
\]

\[
= \int_{-1}^{1} a v_x \left(\frac{2u u_x}{v}\right) dx - \int_{-1}^{1} a v_x \left(\frac{u^2}{v^2}\right) dx = 2 \int_{-1}^{1} \sqrt{a} \frac{u}{v} a v_x dx - 2 \int_{-1}^{1} \sqrt{a} \frac{u^2}{v^2} dx \leq \int_{-1}^{1} a \left(\frac{v_x u}{v}\right)^2 dx + \int_{-1}^{1} a u_x^2 dx - \int_{-1}^{1} a v_x^2 \left(\frac{u^2}{v^2}\right) dx = \int_{-1}^{1} a u_x^2 dx.
\]

Appendix A.2. Positive and negative part

In this section, we recall a useful regularity property of positive and negative part of a given function.

Given \(\Omega \subseteq \mathbb{R}^n\), \(v : \Omega \rightarrow \mathbb{R}\) we consider the positive-part function

\[
v^+(x) := \max \{v(x), 0\}, \quad \forall x \in \Omega,
\]

and the negative-part function

\[
v^-(x) := \max \{0, -v(x)\}, \quad \forall x \in \Omega.
\]

Then we have the following equality

\[
v = v^+ - v^- \quad \text{in} \ \Omega.
\]

For the functions \(v^+\) and \(v^-\) the following result of regularity in Sobolev’s spaces will be useful (see [37], Appendix A).
Proposition Appendix A.1. Let $\Omega \subset \mathbb{R}^n$, $u : \Omega \to \mathbb{R}$, $u \in H^{1,s}(\Omega)$, $1 \leq s \leq \infty$. Then $u^+, u^- \in H^{1,s}(\Omega)$ and, for $1 \leq i \leq n$,

$$(u^+)_{x_i} = \begin{cases} u_{x_i} & \text{in } \{x \in \Omega : u(x) > 0\} \\ 0 & \text{in } \{x \in \Omega : u(x) \leq 0\} \end{cases},$$

and

$$(u^-)_{x_i} = \begin{cases} -u_{x_i} & \text{in } \{x \in \Omega : u(x) < 0\} \\ 0 & \text{in } \{x \in \Omega : u(x) \geq 0\} \end{cases}.$$ 

Remark Appendix A.1. The previous result holds true replacing the Sobolev spaces $H^{1,s}(\Omega)$ by the weighted Sobolev space $H^{1,1}_a(\Omega)$, in the case $n = 1$ and $\Omega = (-1, 1)$.

Appendix B. Existence and uniqueness of strict solutions

This appendix contains the proof of Theorem 3.15 obtained in the Ph.D. Thesis 28, that is, we prove that there exists a unique strict solution $u \in \mathcal{H}(Q_T)$ to (1.1), for all initial datum $u_0 \in H^1_a(-1, 1)$.

We prove this theorem under the following assumptions (H.1)-(H.4):

(H.1) $u_0 \in H^1_a(-1, 1)$;

(H.2) $\alpha \in L^\infty(Q_T)$;

(H.3) $f : Q_T \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function (i.e. $f$ is Lebesgue measurable in $(t,x)$ for every $u \in \mathbb{R}$, and continuous in $u$ for a.e.$(t,x) \in Q_T$);

Moreover,

- there exist $\vartheta \geq 1$, $\gamma_0 \geq 0$ and $\gamma_1 \geq 0$ such that
  
  $$|f(t,x,u)| \leq \gamma_0 |u|^\vartheta, \text{ for a.e. } (t,x) \in Q_T, \forall u \in \mathbb{R},$$  

  $$(B.1)$$

  $$|f(t,x,u) - f(t,x,v)| \leq \gamma_1 (1 + |u|^\vartheta - 1 + |v|^\vartheta - 1)|u - v|, \text{ for a.e. } (t,x) \in Q_T, \forall u,v \in \mathbb{R};$$  

  $$(B.2)$$

- there exists a constant $\nu \geq 0$ such that
  
  $$f(t,x,u) \leq \nu u^2, \text{ for a.e. } (t,x) \in Q_T, \forall u \in \mathbb{R},$$  

  $$(B.3)$$

  $$f(t,x,u) \geq -\nu u^2, \text{ for a.e. } (t,x) \in Q_T, \forall u \in \mathbb{R},$$  

  $$(B.4)$$

below we will put $\nu_T = e^{\nu T}$;

(H.4) $a \in C^1([-1,1])$ is such that

$$a(x) > 0, \forall x \in (-1,1), \quad a(-1) = a(1) = 0,$$

and, the function $\xi_a(x) = \int_0^x \frac{da}{a(x)}$ satisfies the following

$$\xi_a \in L^{2\vartheta - 1}(-1,1).$$

Remark Appendix B.1. We observe that assumptions (H.3), (H.4) are more general than the assumptions (A.3), (A.4) (see also Remark 2.1 and Remark 2.2).

The proof of Theorem 3.15 follows from the next two lemmas. Firstly, the following Lemma 4 Appendix B.1 assures the local existence and uniqueness of the strict solution to (1.1).

---

22 By $H^{1,s}(\Omega)$ we denote the usual Sobolev spaces.
Lemma Appendix B.1. For every $R > 0$, there is $R > 0$ such that for all $\alpha \in L^\infty(-1, 1)$ and all $u_0 \in H^1_0(-1, 1)$ with $\|u_0\|_{1, \alpha} \leq R$ there is a unique strict solution $u \in \mathcal{H}(Q_{T_R})$ to (1.1).

Proof. Let us fix $R > 0$, $u_0 \in H^1_0(-1, 1)$ such that $\|u_0\|_{1, \alpha} \leq R$. Let $0 < T \leq 1$ (further constraints on $T$ will be imposed below). We define

$$
\mathcal{H}_R(Q_T) := \{u \in \mathcal{H}(Q_T) : \|u\|_{\mathcal{H}(Q_T)} \leq 2C_0(1)R\},
$$

where $C_0(1)$ is the constant $C_0(T)$ (nondecreasing in $T$) defined in Proposition 3.7 and valued in 1. Then, let us define the following map

$$
\Lambda : \mathcal{H}(Q_T) \rightarrow \mathcal{H}(Q_T),
$$

such that

$$
\Lambda (u)(t) := e^{tA}u_0 + \int_0^t e^{(t-s)A}\phi(s, u(s))\, ds, \quad \forall t \in [0, T].
$$

STEP 1. We prove that the map $\Lambda$ is well defined for some $T$. Fix $u \in \mathcal{H}_R(Q_T)$. Let us consider $U(t, x) := \Lambda (u)(t, x)$, then $U$ is solution of the following linear problem

$$
\begin{aligned}
U_t - (aU_x)_x &= f(t, x, u) \quad \text{in } Q_T \\
(a(x)U_x(t, x)|_{x=\pm1} &= 0 \\
U(0, x) &= u_0
\end{aligned}
$$

(B.5)

By Lemma 3.12, $f(\cdot, \cdot, u) \in L^2(Q_T) = L^2(0, T; L^2(-1, 1))$, then applying Proposition 3.7 we deduce that a unique solution $U \in \mathcal{H}(Q_T)$ of (B.5) exists and we have

$$
\|U\|_{\mathcal{H}(Q_T)} \leq C_0(T) \left(\|f(\cdot, \cdot, u)\|_{L^2(Q_T)} + \|u_0\|_{1, \alpha}\right).
$$

Thus, keeping in mind that $C_0(T) \leq C_0(1)$, by our choice of $T$, and applying Corollary 3.13 we obtain

$$
\|U\|_{\mathcal{H}(Q_T)} \leq C_0(1) \left(\|f(\cdot, \cdot, u)\|_{L^2(Q_T)} + \|u_0\|_{1, \alpha}\right)
\leq C_0(1) \left(\gamma_0 \|u\|_{2^*(Q_T)}^2 + \|u_0\|_{1, \alpha}\right)
\leq C_0(1) \left(c T^\frac{1}{2}\|u\|_{\mathcal{H}(Q_T)}^2 + \|u_0\|_{1, \alpha}\right)
\leq C_0(1) \left(c T^\frac{1}{2}(2C_0(1)R)^2 + R\right)
\leq C_0(1) \left(c C_0^0(1)R^2 T^\frac{1}{2} + R\right).
$$

Now, we fix $T_0(R) = \min \left\{\frac{1}{c C_0^0(1)R^2 T^\frac{1}{2}}, 1\right\}$. Then we have

$$
\|\Lambda (u)\|_{\mathcal{H}(Q_T)} \leq C_0(1) \left(c C_0^0(1)R^2 T^\frac{1}{2} + R\right) \leq 2C_0(1)R, \quad \forall T \in [0, T_0(R)].
$$

Thus, $\Lambda u \in \mathcal{H}_R(Q_T)$, $\forall T \in [0, T_0(R)]$.

STEP 2. We prove that exists $T_R \leq T_0(R)$ such that the map $\Lambda$ is a contraction.

Let $T$, $0 < T \leq T_0(R)$ (will be fixed below). Fix $u, v \in \mathcal{H}_R(Q_T)$ and set $W := \Lambda (u) - \Lambda (v)$, $W$ is solution of the following problem

$$
\begin{aligned}
W_t - (aW_x)_x &= \alpha W + f(t, x, u) - f(t, x, v) \quad \text{in } Q_T \\
(a(x)W_x(t, x)|_{x=\pm1} &= 0 \\
W(0, x) &= 0
\end{aligned}
$$

(B.6)

By Lemma 3.12, $f(\cdot, \cdot, u) \in L^2(Q_T)$ and applying Proposition 3.7 we deduce that a unique solution $W \in \mathcal{H}(Q_T)$ of (B.6) exists and we have

$$
\|W\|_{\mathcal{H}(Q_T)} \leq C_0(T) \|f(\cdot, \cdot, u) - f(\cdot, \cdot, v)\|_{L^2(Q_T)}.
$$

(B.7)
Moreover, applying the inequality (B.2) (see assumptions (H.3)) and Hölder inequality we obtain

\[
\int_{Q_T} |f(t,x,u) - f(t,x,v)|^2 \, dx \, dt \leq \gamma_1^2 \int_{Q_T} (1 + |x|^\alpha + |v|^{\alpha - 1})^2 |u - v|^2 \, dx \, dt
\]

\[
\leq c \left( \int_{Q_T} (1 + |x|^{2(\alpha - 1)} + |v|^{2(\alpha - 1)}) \, dx \, dt \right)^{\frac{1}{\alpha}} \left( \int_{Q_T} |u - v|^{2\theta} \, dx \, dt \right)^{\frac{\theta}{\alpha}}
\]

\[
\leq c \left( T^{1 - \frac{\theta}{\alpha}} + \|u\|_{L^{2\theta}(Q_T)}^{2(\alpha - 1)} + \|v\|_{L^{2\theta}(Q_T)}^{2(\alpha - 1)} \right) \|u - v\|_{L^{2\theta}(Q_T)}^{2}.
\] (B.8)

Then, by (B.7) and (B.8), applying Corollary 3.6 we have

\[
\|\Lambda(u) - \Lambda(v)\|^2_{H(Q_T)} \leq c \left( T^{1 - \frac{\theta}{\alpha}} + \|u\|_{H(Q_T)}^{2(\alpha - 1)} + \|v\|_{H(Q_T)}^{2(\alpha - 1)} \right) T^{\frac{\theta}{\alpha}} \|u - v\|_{H(Q_T)}^{2}
\]

\[
\leq c \left( 1 + \|u\|_{H(Q_T)}^{2(\alpha - 1)} + \|v\|_{H(Q_T)}^{2(\alpha - 1)} \right) T^{\frac{\theta}{\alpha}} \|u - v\|_{H(Q_T)}^{2} \leq c \left[ 1 + 2(2C_0(1)R)^{2(\alpha - 1)} \right] T^{\frac{\theta}{\alpha}} \|u - v\|_{H(Q_T)}^{2}.
\]

Let \( T_1(R) = \left( \frac{1}{2(1 + 2C_0(1)R)^{2(\alpha - 1)}} \right)^{\frac{\theta}{\alpha}} \), and we define \( T_R = \min\{T_0(R), T_1(R)\} \). Then, \( \Lambda \) is a contraction map. Therefore, \( \Lambda \) has a unique fix point in \( H_R(Q_{T_R}) \), from which the conclusion follows.

Now, thanks to a classical result (see, e.g., [38] and [41]), the following Lemma (Appendix B.2) assures the global existence of the strict solution to (1.1), so we obtain the complete proof of Theorem 3.15.

**Lemma Appendix B.2.** Let \( T > 0 \), \( u_0 \in H^1_1(-1,1) \) and let \( \alpha \in L^{\infty}(-1,1) \). The strict solution \( u \in H(Q_T) \) of system (1.1) satisfies the following estimate

\[
\|u\|_{H(Q_T)} \leq C(\|u_0\|_{1,\alpha}) e^{kT} \|u_0\|_{1,\alpha},
\]

where \( C(\|u_0\|_{1,\alpha}) = h \left( 1 + \|u_0\|_{1,\alpha}^{\alpha - 1} \right)^{1 + \frac{\theta}{\alpha}} \), \( h \) and \( k \) are positive constants.

**Proof.** Multiplying by \( u \) both members of the equation in (1.1) and integrating on \((-1,1)\) we obtain

\[
\int_{-1}^{1} u_t^2(t,x) \, dx - \int_{-1}^{1} (a(x)u_x(t,x))_x \, u_t(t,x) \, dx = \int_{-1}^{1} a(x)u(t,x) u_t(t,x) \, dx + \int_{-1}^{1} f(t,x,u) u_t(t,x) \, dx,
\]

thus,

\[
\int_{-1}^{1} u_t^2(t,x) \, dx + \frac{1}{2} \frac{d}{dt} \int_{-1}^{1} a(x)u_x^2(t,x) \, dx = \frac{1}{2} \frac{d}{dt} \int_{-1}^{1} a(x)u_x^2(t,x) \, dx + \int_{-1}^{1} f(t,x,u) u_t(t,x) \, dx.
\]

Now, let us consider the following function \( F : Q_T \times \mathbb{R} \rightarrow \mathbb{R} \),

\[
F(t,x,u) := \int_0^u f(t,x,\zeta) \, d\zeta, \quad \forall (t,x,u) \in Q_T \times \mathbb{R}.
\]

Then, we observe that

\[
\frac{\partial F(t,x,u(t,x))}{\partial t} = f(t,x,u(t,x)) u_t(t,x) + \int_0^u f_t(t,x,\zeta) \, d\zeta, \quad \forall (t,x) \in Q_T.
\] (B.9)

Moreover, by (B.1) (see assumptions (H.3)), we have

\[
F(0,x,u_0(x)) = \int_{-1}^{u_0} f(0,x,\zeta) \, d\zeta \leq \gamma_0 \int_0^{u_0} |\zeta|^\theta \, d\zeta = \frac{\gamma_0}{\theta + 1} |u_0|^{\theta + 1}, \quad \forall x \in (-1,1).
\]
Thus, by (B.10)

\[ \int_{-1}^{1} |F(0,x,u_0(x))| \, dx \leq \frac{\gamma_0}{\nu+1} \| u_0 \|_{L^{\nu+1}((-1,1))}^{\nu+1} \leq c \| u_0 \|_{L^{\nu+1}((-1,1))}^{\nu+1}. \]  

(B.10)

Now, we observe the following property of the function $F$:
keeping in mind that, by (B.3), for almost every $(t,x) \in Q_T$, we obtain

- $f(t,x,\zeta) \leq \nu \zeta$, for every $\zeta \in \mathbb{R}$, $\zeta \geq 0$
- $f(t,x,\zeta) \geq \nu \zeta$, for every $\zeta \in \mathbb{R}$, $\zeta < 0$,

then, for almost every $(t,x) \in Q_T$, we have

- for every $u \in \mathbb{R}$, $u \geq 0$, $F(t,x,u) = \int_{0}^{u} f(t,x,\zeta) \, d\zeta \leq \nu \int_{0}^{u} \zeta \, d\zeta = \frac{\nu}{2} u^2$
- for every $u \in \mathbb{R}$, $u < 0$, $F(t,x,u) = -\int_{u}^{0} f(t,x,\zeta) \, d\zeta \leq -\nu \int_{u}^{0} \zeta \, d\zeta = -\frac{\nu}{2} u^2$.

Then,

\[ F(t,x,u) \leq \frac{\nu}{2} u^2, \quad \forall (t,x,u) \in Q_T \times \mathbb{R}. \]  

(B.11)

Now, by (B.4), proceeding similarly to (B.11), we obtain

\[ \int_{0}^{u} f_t(t,x,\zeta) \, d\zeta \geq -\frac{\nu}{2} u^2, \quad \forall (t,x,u) \in Q_T \times \mathbb{R}. \]  

(B.12)

In effect, by (B.3), for almost every $(t,x) \in Q_T$, we deduce that

- $f_t(t,x,\zeta) \geq -\nu \zeta$, for every $\zeta \in \mathbb{R}$, $\zeta \geq 0$
- $f_t(t,x,\zeta) \leq -\nu \zeta$, for every $\zeta \in \mathbb{R}$, $\zeta < 0$,

then, for almost every $(t,x) \in Q_T$, we obtain

- for every $u \in \mathbb{R}$, $u \geq 0$, $\int_{0}^{u} f_t(t,x,\zeta) \, d\zeta \geq -\nu \int_{0}^{u} \zeta \, d\zeta = -\frac{\nu}{2} u^2$
- for every $u \in \mathbb{R}$, $u < 0$, $\int_{0}^{u} f_t(t,x,\zeta) \, d\zeta = -\int_{u}^{0} f_t(t,x,\zeta) \, d\zeta \geq \nu \int_{u}^{0} \zeta \, d\zeta = -\frac{\nu}{2} u^2$.

By (B.3), we deduce

\[ \int_{-1}^{1} u_0^2(t,x) \, dx + \frac{1}{2} \frac{d}{dt} \int_{-1}^{1} \left\{ a(x)u_0^2(t,x) - \alpha(x)u^2(t,x) - 2F(t,x,u) \right\} \, dx + \int_{-1}^{1} \int_{0}^{u} f_t(t,x,\zeta) \, d\zeta \, dx = 0. \]

Fix $t \in (0,T)$ and integrate on $(0,t)$, we have

\[ \int_{0}^{t} \int_{-1}^{1} u_0^2(s,x) \, ds \, dx + \frac{1}{2} \int_{-1}^{1} \left\{ a(x)u_0^2(t,x) - \alpha(x)u^2(t,x) \right\} \, dx = \int_{-1}^{1} F(t,x,u(t,x)) \, dx + \frac{1}{2} \int_{-1}^{1} \left\{ a(x)u_t^2(x) - \alpha(x)u_0^2(x) \right\} \, dx 
- \int_{-1}^{1} F(0,x,u_0(x)) \, dx - \int_{0}^{t} \int_{-1}^{1} f_t(t,x,\zeta) \, d\zeta \, dx \, dt. \]

Thus, by (B.11) – (B.12), we obtain

\[ \int_{0}^{t} \| u_t(s,\cdot) \|^2 \, ds + \| \sqrt{a(x)}(t,\cdot) \|^2 \leq \left( (\| \alpha^+ \|_\infty + \nu) \| u(t,\cdot) \|^2 + \| \sqrt{a(x)} \|^2 \right) + \| \alpha^- \|_\infty \| u_0 \|^2 + 2 \int_{-1}^{1} |F(0,x,u_0(x))| \, dx + \nu \int_{0}^{t} \| u(s,\cdot) \|^2 \, ds 
\leq \left( (\| \alpha^+ \|_\infty + \nu) \| u(t,\cdot) \|^2 + |u_0|_{1,a}^2 + \| \alpha^- \|_\infty \| u_0 \|^2 + c \| u_0 \|^2 \| u \|^2 \right) + \nu \int_{0}^{t} \| u(s,\cdot) \|^2 \, ds, \]

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where we denote with $\alpha^+$, $\alpha^-$ the positive and negative part of $\alpha$, respectively (see Appendix A.2).

Let us consider for simplicity $\chi_T := e^{(\nu+||\alpha^+||_\infty)T}$. By Corollary 3.17 (see also Remark 3.15), we deduce

$$
\|u(t,\cdot)\|^2 + \sqrt{\alpha} u(t,\cdot)\|^2 + \int_0^t \|u(t,\cdot)\|^2 \, ds \\
\leq \left((\|\alpha^+\|_\infty + \nu + 1) \|u(t,\cdot)\|^2 + \|u_0\|_{1,a}^2 + \|\alpha^-\|_\infty \|u_0\|^2 + c \|u_0\|_{1,a}^{2\nu+1} + \nu \int_0^t \|u(s, \cdot)\|^2 \, ds \right)
$$

Moreover, by the equation in (1.1), we have

$$(a(x)u_x(t, x))_x = u_t(t, x) - \alpha(x)u(t, x) - f(t, x, u),$$

then, for every $t \in (0, T)$, we obtain

$$
\int_0^t \|\alpha(\cdot)u_x(s, \cdot)\|_x^2 \, ds \leq 2 \int_0^t \|u_t(s, \cdot)\|^2 \, ds + 2\|\alpha^+\|_\infty \int_0^t \|u(s, \cdot)\|^2 \, ds + 2 \int_{Q_t} |f(s, x, u)|^2 \, dx \, ds
$$

By Lemma 3.12 we deduce

$$
\int_{Q_t} |f(s, x, u)|^2 \, dx \, ds \leq \gamma_0^2 \int_{Q_t} |u|_x^{2\theta} \, dx \, ds \leq c T \|\alpha\|_{H^{1,0}(0,T;L^2(-1,1))} \|u\|_{L^\infty(0,t;H_x^2(-1,1))}^{2\nu-1}
$$

$$
\leq c T \left(\frac{t}{\int_0^t \|u_t(s, \cdot)\|^2 \, ds}\right)^{1/2} \left(\sup_{t \in [0,T]} \|u(t, \cdot)\|_{1,a}\right)^{2\nu-1}
$$

$$
\leq c T \left((1 + \|u_0\|_{1,a}^{2\nu-1}) \|u_0\|_{1,a}^{2\nu-1}\right)^{1/2} \left(1 + \|u_0\|_{1,a}^{2\nu-1}\right)^{1/2} \|u_0\|_{1,a}^{2\nu-1}
$$

$$
\leq c T (1 + \|u_0\|_{1,a}^{2\nu-1})^{1/2} \|u_0\|_{1,a}^{2\nu-1} \leq c e^{(1+\nu)T} \|u_0\|_{1,a}^{2\nu-1} \|u_0\|_{1,a}^{2\nu-1}.
$$

From which the conclusion

$$
\|u\|^2_{H(Q_T)} \leq c \left[e^{\nu T} \left(1 + \|u_0\|_{1,a}^{2\nu-1}\right) \|u_0\|_{1,a}^{2\nu-1} + c^{(1+\nu)T} \frac{\gamma_0^2}{\|u_0\|_{1,a}^{2\nu-1}} \left(\sup_{t \in [0,T]} \|u(t, \cdot)\|_{1,a}\right)^{2\nu-1}\right]
$$

$$
\leq c e^{(1+\nu)T} \left(1 + \|u_0\|_{1,a}^{2\nu-1}\right)^{1/2} \left(1 + \|u_0\|_{1,a}^{2\nu-1}\right)^{1/2} \|u_0\|_{1,a}^{2\nu-1}
$$

$$
\leq c e^{(1+\nu)T} \left(1 + \|u_0\|_{1,a}^{2\nu-1}\right)^{1/2} \left(1 + \|u_0\|_{1,a}^{2\nu-1}\right)^{1/2} \|u_0\|_{1,a}^{2\nu-1}
$$

$$
\leq c e^{2(1+\nu) + \|\alpha\|_\infty T} (1 + \|u_0\|_{1,a}^{2\nu+1}) \|u_0\|_{1,a}^{2\nu+1}.
$$

\[\Box\]

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