SUFFICIENCY OF NON-ISOLATED SINGULARITIES

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Abstract. We give, in terms of the Łojasiewicz inequality, a sufficient condition for $C^k$-mappings germs of non-isolated singularity at zero to be isotopical.

1. Introduction and results

Let $F: (\mathbb{R}^n, a) \to \mathbb{R}^m$ denote a mapping defined in a neighbourhood of $a \in \mathbb{R}^n$ with values in $\mathbb{R}^m$. If $F(a) = b$, we put $F: (\mathbb{R}^n, a) \to (\mathbb{R}^m, b)$. By $\nabla f$ we denote the gradient of a $C^1$-function $f: (\mathbb{R}^n, a) \to \mathbb{R}$. By $| \cdot |$ we denote a norm in $\mathbb{R}^n$ and $\text{dist}(x, V)$ - the distance of a point $x \in \mathbb{R}^n$ to a set $V \subset \mathbb{R}^n$ (or $\text{dist}(x, V) = 1$ if $V = \emptyset$).

By a $k$-jet at $a \in \mathbb{R}^n$ in the $C^l$ class we mean a family of $C^l$ functions $(\mathbb{R}^n, a) \to \mathbb{R}$, called $C^l$-realisations of this jet, possessing the same Taylor polynomial of degree $k$ at $a$. The $k$-jet is said to be $C^r$-sufficient (respectively $C^r$-v-sufficient) in the $C^l$ class, if for every of his $C^l$-realisations $f$ and $g$ there exists a $C^r$ diffeomorphism $\varphi: (\mathbb{R}^n, a) \to (\mathbb{R}^n, a)$, such that $f \circ \varphi = g$ (respectively $f^{-1}(0) = \varphi(g^{-1}(0))$) in a neighbourhood of $a$ (R. Thom [23]).

In the paper we will consider the $k$-jets in the class $C^k$ and write shortly $k$-jets.

The classical result in the subject sufficiency of jets is the following:

**Theorem 1** (Kuiper, Kuo, Bochnak-Łojasiewicz). Let $w$ be a $k$-jet at $0 \in \mathbb{R}^n$ and let $f$ be its $C^k$ realisation. If $f(0) = 0$ then the following conditions are equivalent:

(a) $w$ is $C^0$-sufficient in the $C^k$ class,

(b) $w$ is $C^0$-v-sufficient in the $C^k$ class,

(c) $|\nabla f(x)| \geq C|x|^{k-1}$ as $x \to 0$ for some constant $C > 0$.

The implication (c)$\Rightarrow$(a) was proved by N. H. Kuiper [10] and T. C. Kuo [11], (b)$\Rightarrow$(c) - by J. Bochnak and S. Łojasiewicz [2], and the implication (a)$\Rightarrow$(b) is obvious (see also [13], [20]). Analogous result in the complex case was proved by S. H. Chang and Y. C. Lu [4], B. Teissier [22] and J. Bochnak and W. Kucharz [1]. Similar considerations as above are carried out for functions in a neighbourhood of infinity (see [3], [19], [16]).

Theorem 1 concerns the isolated singularity of $f$ at $0$, i.e. the point $0$ is an isolated zero of $\nabla f$. The case of non-isolated singularities of real functions was investigated...
by many authors, for instance by J. Damon and T. Gaffney [3], T. Fukui and E. Yoshinaga [7], V. Grandjean [8], Xu Xu [24] and for complex functions - by D. Siersma [17, 18] and R. Pellikaan [14].

The purposes of this article are generalisations of the above results for a $C^k$ mappings in a neighbourhood of zero with non-isolated singularity at zero. Recall the definition of $k$-Z-jet in the class of functions with non-isolated singularity at zero (cf. [24]).

The set of $C^k$ mappings $(\mathbb{R}^n, a) \to \mathbb{R}^m$ we denote by $\mathcal{C}_k^n(n, m)$. For a function $f \in \mathcal{C}_a^n(n, 1)$, by $j^k f(a)$ we denote the $k$-jet at $a$ (in the $C^k$-class) determined by $f$. For a mapping $F = (f_1, \ldots, f_m) \in \mathcal{C}_a^n(n, m)$ we put $j^k F(a) = (j^k f_1(a), \ldots, j^k f_m(a))$.

Let $Z \subset \mathbb{R}^n$ be a set such that $0 \in Z$ and let $k \in \mathbb{Z}$, $k > 0$. By $k$-Z-jet in the class $\mathcal{C}_0^n(n, m)$, or shortly $k$-Z-jet, we mean an equivalence class $w \in C_0^n(n, m)$ of the equivalence relation $\sim$: $F \sim G$ iff for some neighbourhood $U \subset \mathbb{R}^n$ of the origin, $j^k F(a) = j^k G(a)$ for $a \in Z \cap U$ (cf. [24]). The mappings $F \in w$ we call $C^k$-Z-realisations of the jet $w$ and we write $w = j^k F$. The set of all jets $j^k F$ we denote by $J^k_2(n, m)$.

The $k$-Z-jet $w \in J^k_2(n, m)$ is said to be $C^r$-$Z$-sufficient (resp. $C^r$-$Z$-v-sufficient) in the $C^k$ class, if for every of its $C^k$-$Z$-realisations $f$ and $g$ there exists a $C^r$ diffeomorphism $\varphi: (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$, such that $f \circ \varphi = g$ (resp. $f^{-1}(0) = \varphi(g^{-1}(0))$) in a neighbourhood $U$ of 0 and $\varphi(x) = x$ for $x \in Z \cap U$.

The following Kuiper and Kuo criterion (Theorem II(c) ⇒ (a)) for jets with non-isolated singularity was proved by Xu Xu [24].

**Theorem 2.** Let $Z \subset \mathbb{R}^n$ be a closed set such that $0 \in Z$. If $f \in C^k(n, 1)$ such that $\nabla f(x) = 0$ for $x \in Z$, satisfies the condition

$$|\nabla f(x)| \geq C \text{ dist}(x, Z)^{k-1}$$

as $x \to 0$ for some constant $C > 0$,

then the $k$-Z-jet of $f$ is $C^0$-$Z$-sufficient.

The main result of this paper is Theorem 3 below. It is a generalisation of the Theorem 2 to the case of mapping jets. Let us start with some definition. Let $X, Y$ be Banach spaces over $\mathbb{R}$. Let $L(X, Y)$ denote the Banach space of linear continuous mappings from $X$ to $Y$. For $A \in L(X, Y)$, $A^*$ stands for the adjoint operator in $L(Y^*, X^*)$, where $X^*$ is the dual space of $X$. For $A \in L(X, Y)$ we put

$$\nu(A) = \inf \{ \| A^* \varphi \| : \varphi \in Y^*, \| \varphi \| = 1 \},$$

where $\| A \|$ is the norm of linear mapping $A$ (see [15]). In the case $f \in C_0^k(n, 1)$ we have $\nu(df) = |\nabla f|$, where $df$ is the differential of $f$.

**Theorem 3.** Let $f: (\mathbb{R}^n, 0) \to (\mathbb{R}^m, 0)$, where $m \leq n$, be a $C^k$-$Z$-realisation of a $k$-Z-jet $w \in J^k_2(n, m)$, where $k > 1$ and $Z = \{ x \in \mathbb{R}^n : \nu(df(x)) = 0 \}$, $0 \in Z$. Assume that for a positive constant $C$,

$$\nu(df(x)) \geq C \text{ dist}(x, Z)^{k-1}$$

as $x \to 0$.

Then the jet $w$ is $C^0$-$Z$-sufficient in the class $C^k$. Moreover for any $C^k$-$Z$-realisations $f_1, f_2$ of $w$, the deformation $f_1 + t(f_2 - f_1)$, $t \in \mathbb{R}$ is topologically trivial along $[0, 1]$. In particular the mappings $f_1$ and $f_2$ are isotopical at zero.
For the definition of isotopy and topological triviality see Subsection 2.3. By Lemmas 2 and 3 in Section 2, Theorem 3 is also true for holomorphic mappings. It is not clear to the authors if the inverse to Theorem 3 holds. In the proof of Theorem 3, given in Section 2, we use a method of the proof of Theorem 1 in [10].

In the case of nondegenerate analytic functions \( f, g \), a conditions for topological triviality of deformations \( f + tg, t \in [0, 1] \) in terms of Newton polyhedra was obtained by J. Damon and T. Gaffney [5], and for blow analytic triviality – T. Fukui and E. Yoshinaga [7] (see also [21], [25]).

From the proof of Theorem 3 we obtain a version of the theorem for functions of \( C^1 \) class with locally Lipschitz differentials.

**Corollary 1.** Let \( f, f_1 : (\mathbb{R}^n, 0) \to (\mathbb{R}^m, 0) \) be differentiable mappings with locally Lipschitz differentials \( df, df_1 : (\mathbb{R}^n, 0) \to L(\mathbb{R}^n, \mathbb{R}^m) \) let \( Z = \{ x \in \mathbb{R}^n : \nu(df(x)) = 0 \} \), and let \( 0 \in Z \). It

\[
(4) \quad \nu(df(x)) \geq C \operatorname{dist}(x, Z),
\]

\[
(5) \quad |f(x) - f_1(x)| \leq C_1 \nu(df(x))^2,
\]

\[
(6) \quad \|df(x) - df_1(x)\| \leq C_2 \nu(df(x))
\]
as \( x \to 0 \) for some constants \( C, C_1, C_2 > 0, C_2 < \frac{1}{2} \), then the deformation \( f + t(f_1 - f) \) is topologically trivial along \([0, 1]\). In particular \( f \) and \( f_1 \) are isotopical at zero.

The proof of the above corollary is given in Subsection 2.5.

In Section 3 we prove the following theorem type of Bochnak-Łojasiewicz (cf. implication \((b) \Rightarrow (c)\) in Theorem 1), that \( C^0. Z. v\)-sufficiency of a jets implies the \( \dot{\text{Ł}}ojasiewicz \) inequality, provided \( j^{k-1}f(0) = 0 \) for \( C^k. Z\)-realisations \( f \) of the jet. Namely, we will prove the following

**Theorem 4.** Let \( Z \subset \mathbb{R}^n \) be a set such that \( 0 \in Z \), let \( w \) be a \( k-Z\)-jet, \( k > 1 \), and let \( f \) be its \( C^k. Z\)-realisation. If \( w \) is \( C^0. Z. v\)-sufficient in \( C^k\)-class, \( j^{k-1}f(0) = 0 \) and \( V(\nabla f) \subset Z \), then

\[
(7) \quad |\nabla f(x)| \geq C \operatorname{dist}(x, Z)^{k-1} \quad \text{as} \quad x \to 0 \quad \text{for some constant} \quad C > 0.
\]

It is obvious that a \( C^0. Z\)-sufficient jet is also a \( C^0. Z. v\)-sufficient, so, Theorem 4 in a certain sense is an inverse of Theorem 2.

2. **Proof of Theorem 3**

2.1. **Differential equations.** Let us start from recalling the following

**Lemma 1.** Let \( G \subset \mathbb{R} \times \mathbb{R}^n \) be an open set, \( W : G \to \mathbb{R}^n \) be a continuous mapping and let \( V \subset \mathbb{R}^n \) be a closed set. If in \( G \setminus (\mathbb{R} \times V) \) system

\[
(8) \quad \frac{dy}{dt} = W(t, y)
\]

has a global unique solutions and there exist neighbourhood \( U \subset G \) of set \( (\mathbb{R} \times V) \cap G \) and a positive constant \( C \) such that

\[
(9) \quad |W(t, x)| \leq C \operatorname{dist}(x, V) \quad \text{for} \quad (t, x) \in U,
\]

then the system \((8)\) in \( G \) has a global unique solutions.
2.2. The Rabier function. Let $X, Y$ be Banach spaces over $\mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. Let $L(X, Y)$ denote the Banach space of linear continuous mappings from $X$ to $Y$. For $A \in L(X, Y)$, $A^*$ stands for the adjoint operator in $L(Y', X')$, where $X'$ is the dual space of $X$. We begin with recalling some properties of the Rabier function (cf. [16]).

**Lemma 2 ([12]).** Let $\Sigma$ be the set of operators $A \in L(X, Y)$ such that $A(X) \subset Y$. We have

$$\nu(A) = \text{dist}(A, \Sigma), \quad A \in L(X, Y).$$

**Lemma 3 ([15]).** Let $A, B \in L(X, Y)$. Then $|\nu(A) - \nu(B)| \leq \|A - B\|$. In particular $\nu : L(X, Y) \to \mathbb{R}$ is Lipschitz.

From Lemma 3 we have

**Lemma 4.** If $A, B \in L(X, Y)$ then

$$\nu(A + B) \geq \nu(A) - \|B\|.$$ 

**Definition 1 ([9]).** Let $a = [a_{ij}]$ be the matrix of $A \in L(\mathbb{K}^n, \mathbb{K}^m)$, $n \geq m$. By $M_I(A)$, where $I = (i_1, \ldots, i_m)$ is any subsequence of $(1, \ldots, n)$, we denote an $m \times m$ minor of $a$ given by columns indexed by $I$. Moreover, if $J = (j_1, \ldots, j_{m-1})$ is any subsequence of $(1, \ldots, n)$ and $j \in (1, \ldots, m)$, then by $M_J(j)(A)$ we denote an $(m-1) \times (m-1)$ minor of $a$ given by columns indexed by $J$ and with deleted $j$th row (if $m = 1$ we put $M_J(j)(A) = 1$). Let

$$h_l(A) = \max \{|M_J(j)(A)| : J \subset I, j = 1, \ldots, m\},$$

$$g'(A) = \max_j \frac{|M_J(A)|}{h_l(A)}.$$

Here we put $0/0 = 0$. If $m = n$, we put $h_l = h$.

**Lemma 5 ([9]).** There exist $C_1, C_2 > 0$, such that for any $A \in L(\mathbb{K}^n, \mathbb{K}^m)$ we have

$$C_1 g'(A) \leq \nu(A) \leq C_2 g'(A).$$

**Corollary 2 ([16]).** The function $g'$ is continuous.

**Lemma 6 ([12]).** Assume that $X, Y$ are complex Banach spaces. Let $\Sigma_\mathbb{C}$ (resp. $\Sigma_\mathbb{R}$) be the set of nonsurjective $\mathbb{C}$-linear (resp. $\mathbb{R}$-linear) continuous maps from $X$ to $Y$. Then for any continuous $\mathbb{C}$-linear map $A : X \to Y$,

$$\text{dist}(A, \Sigma_\mathbb{C}) = \text{dist}(A, \Sigma_\mathbb{R}).$$

2.3. Isotopy and triviality. Let $\Omega \subset \mathbb{R}^n$ be a neighbourhood of $0 \in \mathbb{R}^n$ and let $Z \subset \mathbb{R}^n$ be a set such that $0 \in Z$.

We will say, that a continuous mapping $H : \Omega \times [0, 1] \to \mathbb{R}^n$ is an **isotopy near $Z$ at zero** if

(a) $H_0(x) = x$ for $x \in \Omega$ and $H_t(x) = x$ for $t \in [0, 1]$ and $x \in \Omega \cap Z$,

(b) for any $t$ the mapping $H_t$ is a homeomorphism onto $H_t(\Omega)$,

where the mapping $H_t : \Omega \to \mathbb{R}^n$ is defined by $H_t(x) = H(x, t)$ for $x \in \Omega$, $t \in [0, 1]$.

Let $f : \Omega_1 \to \mathbb{R}^n$, $g : \Omega_2 \to \mathbb{R}^m$ where $\Omega_1, \Omega_2 \subset \mathbb{R}^n$ are neighbourhoods of $0 \in \mathbb{R}^n$ and let $Z \subset \mathbb{R}^n$ be a set such that $0 \in Z$. We call $f$ and $g$ **isotopical near $Z$ at zero**
if there exists an isotopy near \( Z \) at zero \( H : \Omega \times [0, 1] \to \mathbb{R}^n, \Omega \subset \Omega_1 \cap \Omega_2, \) such that \( f(H_1(x)) = g(x), x \in \Omega. \)

Let \( h : \Omega_3 \to \mathbb{R}^m \), where \( \Omega_3 \subset \mathbb{R}^n \) is a neighbourhood of \( 0 \in \mathbb{R}^n \). We say that a deformation \( f + th \), is topologically trivial near \( Z \) along \([0, 1]\) if there exists an isotopy near \( Z \) at zero \( H : \Omega \times [0, 1] \to \mathbb{R}^m, \Omega \subset \Omega_1 \cap \Omega_2, \) such that \( f(H(t, x)) + th(H(t, x)) \) do not depend on \( t \).

2.4. **Proof of Theorem** 3

By \( dP \) we denote the differential of \( P \) and \( dP(x) \) – the differential of \( P \) at the point \( x \). By \( d_x P \) we denote the differential of \( P \) with respect to the system of variables \( x \).

Let \( f, f_1 \in w \) and let \( P = f_1 - f = (P_1, \ldots, P_m) \). Then we have \( j^k P(a) = 0 \) for \( a \in Z \cap U \) for some neighbourhood \( U \subset \mathbb{R}^n \) of \( 0 \). In consequence, decreasing if necessary \( U \), we may assume that

\[
|P(x)| \leq \frac{C}{3} \text{dist}(x, Z)^k \quad \text{and} \quad \|dP(x)\| \leq \frac{C}{3} \text{dist}(x, Z)^{k-1}
\]

for \( x \in U \).

Consider the mapping \( F : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^m, \)

\[
F(\xi, x) = f(x) + \xi P(x).
\]

Let us fix \( \xi \in (-2, 2) \). By \( 10 \) and Lemma \( 3 \) we get

\[
\nu(d_x F(\xi, x)) \geq \nu(df(x)) - \|\xi\|\|dP(x)\| \geq \frac{C}{3} \text{dist}(x, Z)^{k-1}, \quad x \in U.
\]

Thus by Lemma \( 3 \) there exists \( C' > 0 \) such that

\[
g'(d_x F(\xi, x)) \geq C' \text{dist}(x, Z)^{k-1}, \quad \xi \in (-2, 2), \quad x \in U.
\]

Set \( G = \{(\xi, x) \in \mathbb{R} \times U : |\xi| < 2\} \). In the notation of Definition \( 10 \) we put

\[
A_I = \left\{(\xi, x) \in G : \frac{|M_I(d_x F(\xi, x))|}{h_I(d_x F(\xi, x))} \leq \frac{C'}{2} \text{dist}(x, Z)^{k-1} \right\}.
\]

By Corollary \( 2 \) the sets \( A_I \) are closed in \( G \) and \((\mathbb{R} \times Z) \cap G \subset A_I \). From \( 11 \) we see that \( \{G \cap A_I : I\} \) is an open covering of \( G \setminus (\mathbb{R} \times Z) \). Let \( \{\delta_I : I\} \) be a \( C^\infty \) partition of unity associated to this covering.

Let us consider the following system of linear equations

\[
(d_x F(\xi, x))W(\xi, x)^T = -P(x)^T
\]

with indeterminates \( W(\xi, x) = (W_1(\xi, x), \ldots, W_n(\xi, x)) \) and parameters \((\xi, x) \in G \). Let us take any subsequence \( I = (i_1, \ldots, i_m) \) of the sequence \((1, \ldots, n)\). For simplicity of notation we assume that \( I = (1, \ldots, m) \). For all \((\xi, x) \in G \) such that \( M_I(d_x F(\xi, x)) \neq 0 \) we put

\[
W_I^l(\xi, x) = \sum_{j=1}^{m} (-P_j(x))(-1)^{l+j} M_{I \setminus l}(j) \frac{M_I(j) (d_x F(\xi, x))}{M_I(d_x F(\xi, x))}, \quad l = 1, \ldots, m,
\]

\[
W_I^l(\xi, x) = 0, \quad l = m + 1, \ldots, n,
\]

where \( I \setminus l = (1, \ldots, l-1, l+1, \ldots, m) \) for \( l = 1, \ldots, m \). Cramer’s rule implies

\[
(d_x F(\xi, x))W^l(\xi, x)^T = -P(x)^T.
\]
Since $k > 1$, then $\delta_{t}W^{I}$ is a $C^{1}$ mapping on $G \setminus (R \times Z)$ (after suitable extension). Hence $W = \sum_{t} \delta_{t}W^{I}$ is also $C^{1}$ mapping on $G \setminus (R \times Z)$. We put $W(\xi, x) = 0$ for $(\xi, x) \in (R \times Z) \cap G$. It is easy to see, that $W$ satisfies the equation \((12)\).

Observe that

\[
\|W(\xi, x)\| \leq C'' \text{dist}(x, Z), \quad \xi \in (-2, 2), \quad x \in U,
\]

where $C'' = 2mC\sqrt{n}/(3C')$. Indeed from \((10)\) the definitions of $A_{t}$, the choice of $P$ and the above construction we get

\[
\|W(\xi, x)\| \leq \sum_{(t, \delta_{t}(\xi, x) \neq 0)} \delta_{t}(\xi, x)\|W^{I}(\xi, x)\| \\
\leq \sum_{(t, \delta_{t}(\xi, x) \neq 0)} \delta_{t}(\xi, x)\sqrt{n} \max_{i=1}^{m} |W^{I}_{i}(\xi, x)| \\
\leq \sum_{(t, \delta_{t}(\xi, x) \neq 0)} \delta_{t}(\xi, x)\sqrt{n} \sum_{j \in I} |P_{j}(\xi, x)| \frac{h_{t}(d_{z}F(\xi, x))}{|M_{t}(d_{z}F(\xi, x))|} \\
\leq \sum_{(t, \delta_{t}(\xi, x) \neq 0)} \delta_{t}(\xi, x)\sqrt{n} \sum_{j \in I} C \frac{2}{3} \text{dist}(x, Z)^{k} \frac{1}{C'} \frac{1}{\text{dist}(x, Z)^{k-1}} \\
= m\sqrt{n}C \frac{2}{3} \frac{2}{C'} \text{dist}(x, Z).
\]

Let us consider the following system of differential equations

\[
y' = W(t, y).
\]

Since $W$ is at least of class $C^{1}$ on $G \setminus (R \times Z)$, so it is a locally lipschitzian vector field. As a consequence, the above system has a uniqueness of solutions property in $G \setminus (R \times Z)$. Hence, inequality \((13)\) and Lemma \([11]\) implies the global uniqueness of solutions of the system \((14)\) in $G$.

Choose $(\xi, x) \in G$ and define $\varphi(\xi, x)$ to be the maximal solution of \((14)\) such that $\varphi(\xi, x)(\xi) = x$. Set $\Omega_{0} = \{x \in R^{n} : \|x\| < r_{0}\}$, $\Omega_{1} = \{x \in R^{n} : \|x\| < r_{1}\}$, where $r_{0}, r_{1} > 0$. Since $0 \in Z$, the mapping $\varphi(0) = 0$, $\xi \in R$ is a solution of \((14)\). Hence for sufficiently small $r_{0}, r_{1}$, for any $x \in \Omega_{0}$, the solution $\varphi_{0}(x)$ is defined on $[0, 1]$ and $\varphi_{0}(0)(t) \in \Omega_{1}$, if $t \in [0, 1]$ and for any $x \in \Omega_{0}$, the solution $\varphi_{0}(x)$ is also defined on $[0, 1]$. Let $H, \tilde{H} : \Omega_{0} \times [0, 1] \to \Omega_{1}$ be given by

\[
H(x, t) = \varphi_{0}(x)(t), \quad \tilde{H}(y, t) = \varphi_{0}(t, y)(0).
\]

The mappings $H, \tilde{H}$ are well defined. Moreover one can extend these mappings to continuous mappings on some open neighbourhood of $\Omega_{0} \times [0, 1]$. Put $\Omega = \Omega_{1}$, $\Omega' = \{y \in R^{n} : \tilde{H}(y, t) \in \Omega_{1}, t \in [0, 1]\}$. By uniqueness solutions of \((14)\) for any $t$ we have $H(H(x, t), t) = x$, $H(x, 0) = x$, $x \in \Omega$, and $H(\tilde{H}(y, t)) = y$, $y \in \Omega'$. Moreover there exists a neighbourhood $\Omega' \subset R^{n}$ of $0$ such that $\Omega' \subset \Omega'$ for any $t$.

Finally, by \((12)\) we have

\[
\frac{d}{dt}F(t, \varphi_{0}(x)(t)) = P(x)^{T} + (d_{z}F)(t, \varphi_{0}(x)(t))W(t, \varphi_{0}(x)(t))^{T} = 0,
\]

so, $F(t, \varphi_{0}(x)(t)) = f(x)$ and consequently $f(H(x, 1)) + tP(H(x, 1)) = f(x)$ for $t \in [0, 1]$ and $x \in \Omega'$. This ends the proof. \(\square\)
2.5. Proof of Corollary 1. Under notations of the proof of Theorem 3 by (11), (13) and Lemma 3 we obtain \( \nu(d_{df}(\xi, x)) = (d_{df}^k(x) + \xi d_{P}(x)) \geq \nu(d_{df}(x)) - |\xi||d_{P}(x)|| \geq C(1 - 2C_2) \text{dist}(x, Z), \; x \in U. \) Obviously \( C(1 - 2C_2) > 0. \) Then there exists \( C' > 0 \) such that

\[
(15) \quad g'(d_{df}(\xi, x)) \geq C' \text{dist}(x, Z), \quad \xi \in (-2, 2), \quad x \in U.
\]

So, we will use (15) instead of (11). By (5) we obtain (13). Moreover, the assumption that \( d_{f} \) and \( d_{f_1} \) are locally Lipschitz mappings implies that the mapping \( W \) is locally Lipschitz outside \((-2, 2) \times Z.\) Then, by the same argument as in the proof of Theorem 3 we deduce the assertion.

3. PROOF OF THEOREM 4

We will use the idea from [2]. It suffices to prove of the Theorem for \( Z = V(\nabla f). \) Suppose to the contrary that for any neighbourhood \( U \) of 0 and for any constant \( C > 0 \) there exist \( x \in U \) such that

\[
|\nabla f(x)| < C \text{dist}(x, Z)^{k-1}.
\]

Then for some sequence \( (a_{\nu}) \subset \mathbb{R}^n \setminus Z \) such that \( a_{\nu} \to 0 \) when \( \nu \to \infty \) we have

\[
(16) \quad |\nabla f(a_{\nu})| \leq \frac{1}{\nu} \text{dist}(a_{\nu}, Z)^{k-1} \quad \text{for} \; \nu \in \mathbb{N}.
\]

Choosing a subsequence of \( (a_{\nu}) \), if necessary, we can assume that

\[
\text{dist}(a_{\nu+1}, Z) < \frac{1}{2} \text{dist}(a_{\nu}, Z), \quad \text{for} \; \nu \in \mathbb{N}.
\]

Then

\[
B_{\nu} = \{ x \in \mathbb{R}^n : |x - a_{\nu}| \leq \frac{1}{4} \text{dist}(a_{\nu}, Z) \}, \quad \nu \in \mathbb{N},
\]

is family of pairwise disjoint balls.

Let us take sequence \( (\lambda_{\nu}) \subset \mathbb{R} \) such that \( \lambda_{\nu} > 0 \) for any \( \nu \in \mathbb{N} \) and

\[
(17) \quad \frac{\lambda_{\nu}}{\text{dist}(a_{\nu}, Z)^{k-2}} \to 0, \quad \nu \to \infty.
\]

Since \( k > 1 \), we may assume that

\[
(18) \quad \lambda_{\nu} \text{ is not eigenvalue of matrix } \left[ \frac{\partial^2 f}{\partial x_i \partial x_j}(a_{\nu}) \right].
\]

Let \( \alpha : \mathbb{R}^n \to \mathbb{R} \) be function of \( C^\infty \)-class such that \( \alpha(x) = 0 \) for \( |x| \geq \frac{1}{4} \) and \( \alpha(x) = 1 \) in some neighbourhood of 0. By \( \langle \cdot, \cdot \rangle \) we denote the standard inner product in \( \mathbb{R}^n \). Consider function \( F : \mathbb{R}^n \to \mathbb{R} \) defined by the formulas

\[
F(x) = \alpha \left( \frac{x - a_{\nu}}{\text{dist}(a_{\nu}, Z)} \right) \left( f(a_{\nu}) + \langle \nabla f(a_{\nu}), x - a_{\nu} \rangle + \frac{1}{2} \lambda_{\nu} |x - a_{\nu}|^2 \right),
\]

for \( x \in B_{\nu} \) and \( F(x) = 0 \) for \( x \notin \bigcup_{\nu=1}^{\infty} B_{\nu}. \) Then \( F \) is a \( C^k \)-function and \( F(0) = 0. \) Moreover \( f(a_{\nu}) = F(a_{\nu}) \) and \( \nabla f(a_{\nu}) = \nabla F(a_{\nu}) \) so

\[
(19) \quad (f - F)(a_{\nu}) = 0 \quad \text{and} \quad \nabla (f - F)(a_{\nu}) = 0, \quad \nu \in \mathbb{N}.
\]
Let $M > 0$ be such that $|\alpha(x)| \leq M$ for $x \in \mathbb{R}^n$. Then for $x \in B_\nu$ we have
\[
\frac{|F(x)|}{\text{dist}(x, Z)^k} \leq M \frac{|f(a_\nu) + (\nabla f(a_\nu), x - a_\nu) + \frac{1}{2} \lambda_\nu |x - a_\nu|^2|}{\text{dist}(x, Z)^k} 
\leq 2^k M \frac{|f(a_\nu)| + |\nabla f(a_\nu)| \text{dist}(a_\nu, Z) + \frac{1}{2} |\lambda_\nu| \text{dist}(a_\nu, Z)^2}{\text{dist}(a_\nu, Z)^k}.
\]
Since $j^{k-1}f(0) = 0$, then
\[
\frac{|f(a_\nu)|}{|a_\nu|^{k-1}} \to 0, \quad \text{when } \nu \to \infty.
\]
Hence, from the above, and from (16) and (17) we obtain
\[
\frac{|F(x)|}{\text{dist}(x, Z)^k} \to 0, \quad \text{when } x \to 0,
\]
so
\[
\frac{|F(x)|}{|x|^k} \to 0, \quad \text{when } x \to 0.
\]
Therefore $f - F$ is $C^k$-$Z$-realisation of $k$-$Z$-jet $w$ (recall that for any $x \in Z \setminus \{0\}$ the function $F$ vanishes in a neighbourhood of $x$). From (19) we have that $(f - F)$ has zeros outside the set $Z$, so by our assumption, $f$ has zeros outside the set $Z$. By the implicit function theorem for some neighbourhood $U$ of $0 \in \mathbb{R}^n$, we obtain that $f^{-1}(0) \cap (U \setminus Z)$ is $(n-1)$-dimensional topological manifold. Therefore $(f - F)^{-1}(0) \cap (U_1 \setminus Z)$ is also $(n-1)$-dimensional topological manifold for some neighbourhood $U_1$ of $0 \in \mathbb{R}^n$. On the other hand (18) gives
\[
\det \left[ \frac{\partial^2 f}{\partial x_i \partial x_j}(a_\nu) \right] \neq 0, \quad \text{dla } \nu \in \mathbb{N},
\]
hence and from (19) $(f - F)$ has Morse singularities in points $a_\nu$, so, $(f - F)^{-1}(0)$ is not $(n-1)$-dimensional topological manifold in any neighbourhood of point $a_\nu$. This contradiction completes the proof of Theorem 4.

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