Research Article

Comparative Study on Numerical Methods for Singularly Perturbed Advanced-Delay Differential Equations

P. Hammachukiattikul 1, E. Sekar 1, A. Tamilselvan 2, R. Vadivel 1, N. Gunasekaran 4, and Praveen Agarwal 5

1Department of Mathematics, Phuket Rajabhat University, 83000 Phuket, Thailand
2Department of Mathematics, SASTRA Deemed to be University, Thanjavur, Tamilnadu 613401, India
3Department of Mathematics, Bharathidasan University, Tiruchirappalli-620 024, Tamilnadu, India
4Department of Mathematical Sciences, Shibaura Institute of Technology, Saitama 337-8570, Japan
5Department of Mathematics, Anand International College of Engineering, Jaipur, India

Correspondence should be addressed to E. Sekar; sekar@maths.sastra.ac.in

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In this paper, we consider a class of singularly perturbed advanced-delay differential equations of convection-diffusion type. We use finite and hybrid difference schemes to solve the problem on piecewise Shishkin mesh. We have established almost first- and second-order convergence with respect to finite difference and hybrid difference methods. An error estimate is derived with the discrete norm. In the end, numerical examples are given to show the advantages of the proposed results (Mathematics Subject Classification: 65L11, 65L12, and 65L20).

1. Introduction

Differential equations depend both on past and future values (mixed delay) called functional differential equations. It attains many application problems such as optimal control problems [1], nerve conduction theory [2], economic dynamics [3], traveling waves in a spatial lattice [4] and has discussed both linear and nonlinear functional differential equations.

The functional differential equation has been multiplied by small parameter \(0 < \epsilon < 1\) in the highest order derivative term called the singularly perturbed mixed delay differential equations. The main determination for such a problem is the study of biological science, epidemics, and population [5–10].

The authors in [11] have considered functional differential equation in singularly perturbed problems, such as

\[
\left( \frac{\sigma^2}{2} \right) y''(x) + (\mu - x)y'(x) + \lambda_E y(x + a_E) + \lambda_I y(x - a_I) - (\lambda_E + \lambda_I) y(x) = -1,
\]

and considered the problem of determining the expected time for the generation of action potentials in nerve cells by random synaptic inputs in the dendrites. The general linear second-order functional differential equation with the boundary-value problem arises in the modeling of neuron activation, where \(\sigma\) and \(\mu\) are the variance and drift parameters and \(y\) is the expected first-exit time. The first-order derivative term \(-xy'(x)\) corresponds to exponential decay between synaptic and inputs. The undifferentiated terms correspond to excitatory and inhibitory synaptic inputs modeled as a Poisson process with mean rates \(\lambda_E\) and \(\lambda_I\); they produce jumps in the membrane potential of amounts \(a_E\) and \(a_I\).
and \( a_1 \), which are small quantities and could depend on the voltage. The boundary condition is
\[
y(x) = 0, \quad \forall x \notin (x_1, x_2),
\]
where the values \( x = x_1 \) and \( x = x_2 \) correspond to inhibitory reversal potential and the threshold value of membrane potential for action potential generation. This biological problem motivates the investigation of boundary-value problems for differential-difference equations with mixed shifts. In this biological model, using the Taylor series for the small delay term, provided the delay is of order \( \varepsilon \), the small delay problem has oscillatory solution that has been discussed in [12]. The same authors discussed the signal transmission problem in [13].

The authors in [14, 15] have considered the singularly perturbed problem with derivative depending on small delay term such as
\[
\varepsilon y''(t) + a(t)y'(t) + b(t)y(t) + c(t)y(t - \tau) + d(t)y(t + \tau) = f(t), \quad 0 < \tau \ll 1,
\]
to solve the boundary-value problem using the following numerical method such as the finite difference scheme [14, 16], fitted mesh B-spline collocation method [17], and hybrid difference scheme [15].

The authors in [18, 19] investigated various concepts of singularly perturbed differential equation with derivative depending on both past and future small variables,
\[
\varepsilon y''(t) + a(t)y'(t) + b(t)y(t) + c(t)y(t - \tau) + d(t)y(t + \tau) = f(t), \quad 0 < \tau \ll 1,
\]
also proposed a finite difference scheme to solve singular perturbation problems in [18, 20, 21].

The authors in [19] have been proposed to solve the singular perturbation problem with mixed small shifts using the fitted operator method. In recent years, the authors in [22–25] considered singular perturbation problem with derivative depending on large delay \( \tau \) variable, such as
\[
\varepsilon y''(t) + a(t)y'(t) + b(t)y(t) + c(t)y(t - \tau) = f(t),
\]
has been developed various numerical schemes are finite and hybrid difference method [22], iterative scheme [26], finite element method [27, 28]. The study in [23] proposed solving singularly perturbed delay differential equation with integral boundary condition using finite difference method.

Throughout the literature, the researcher concentrates on solving the singular perturbation problem with a small delay or mixed small delay or large delay using finite or hybrid or finite element methods on uniform meshes or nonuniform mesh. To the best of the author’s knowledge, up to now, no theoretical results are given for comparative study on numerical methods for singularly perturbed advanced-delay differential equations. Moreover, we proposed two numerical methods such as the finite and hybrid difference scheme on nonuniform meshes, to solve the singular perturbation problem with mixed large delay using the finite difference scheme and hybrid difference scheme on Shishkin mesh.

This paper is structured as follows: Section 2 describes the problem statement. Section 3 proves the maximum principle and stability result. Moreover, it introduces the terminology for Shishkin decomposition and proves many inequalities. In Section 4, we introduce the numerical methods to discretize the continuous problem. Error analysis for finite and hybrid difference scheme approximate solution is given in Sections 5 and 6. Finally, Section 7 presents numerical results.

Throughout our analysis, we use the following notations:
\[
\bar{\Gamma} = [0, 3], \quad \Gamma = (0, 3), \quad \Gamma_1 = (0, 1), \quad \Gamma_2 = (1, 2), \quad \Gamma_3 = (2, 3), \quad \Gamma^* = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3, \quad \Gamma_{1N} = [0, 1, 2, \ldots, 3N], \quad \Gamma_{12N} = [1, 2, \ldots, N - 1], \quad \Gamma_{2N} = [N + 1, \ldots, 2N - 1], \quad \Gamma_{3N} = [2N + 1, \ldots, 3N - 1].
\]
The parameter \( \varepsilon \) and mesh points \( 3N \) are independent of \( C \) and \( C_1 \) are positive constants. The norm is \( \|y\|_1 = \sup_{r \in \Gamma} |y(r)| \).

2. Statement of the Problem
Consider the following singularly perturbed mixed delay differential equation:
\[
\begin{cases}
\mathcal{X} y(r) = - \varepsilon y''(r) + a(r)y'(r) + b(r)y(r) + c(r)y(r - 1) + d(r)y(r + 1) = f(r), & r \in \Gamma,
\end{cases}
\]
\[
y(r) = \phi(r), \quad y(r) = \varphi(r),
\]
where \( \phi(r) \) and \( \varphi(r) \) are history function on \([-1, 0]\) and \([3, 4]\). Assume that \( a(r) \geq a_1 > 0, \quad b(r) \geq \beta > 0, \quad \gamma \leq c(r) \leq 0, \quad d(r) \geq \eta > 0, \quad \beta + \gamma \geq \beta_1 > 0 \), and the coefficients are smooth function on \( r \in \bar{\Gamma} \). The above problem solution satisfies \( y(r) \in G = C^0(\bar{\Gamma}) \cap C^1(\Gamma^*) \cap C^2(\Gamma^*) \). Problem (1) is rewritten as \( \mathcal{X} y(r) = g(r) \), where
3. Analytical Results

**Lemma 1** (maximum principle). If \( y(r) \in G \) such that \( y(0) \geq 0, y(3) \geq 0, \mathcal{K}_1y(r) \geq 0, \forall r \in \Gamma_1, \mathcal{K}_2y(r) \geq 0, \forall r \in \Gamma_2, \mathcal{K}_3y(r) \geq 0, \forall r \in \Gamma_3 \), then \( y(\cdot) \) is a solution of problems \((7)--(9)\), then

\[
y(y(r)) = \begin{cases} 
0, & r \in [0, 1], \\
\frac{1}{12} + \frac{r}{4}, & r \in [1, 2], \\
\frac{2}{12} + \frac{r}{6}, & r \in [2, 3].
\end{cases}
\]

Proof. Let

\[
s(r) = \begin{cases} 
\frac{1}{12} + \frac{r}{4}, & r \in [0, 1], \\
\frac{2}{12} + \frac{r}{6}, & r \in [1, 2], \\
\frac{4}{12} + \frac{r}{12}, & r \in [2, 3].
\end{cases}
\]

Clearly, \( s(r) > 0, \forall r \in \Gamma, \mathcal{K}s(r) > 0, \forall r \in \Gamma^*, s(0) > 0, s(3) > 0, [s'](1) < 0, \) and \([s'](2) < 0\). Consider that \( \mu = \max\{(-\psi(r))/s(r)) : r \in \Gamma\} \); then, there exists \( r_0 \in \Gamma^* \) such that \( \psi(r_0) + \mu s(r_0) = 0 \) and \( \psi(r) + \mu s(r) \geq 0, \forall r \in \Gamma \) implies that \( (\psi + \mu s) \) obtain minimum at \( t = r_0 \). If \( \mu < 0 \), then \( \psi(r) \geq 0 \).

If \( \mu > 0 \), then the function \( \psi(r) \) nonnegative is not possible. The following cases are easy to prove the contradiction if \( \mu > 0 \).

Case (i): \( r_0 = 0 \):

\[
0 \leq (\psi + \mu s), \quad (0) = (0) = \psi(0) + \mu s(0) = 0.
\]

Case (ii): \( r_0 \in \Gamma_1 \):

\[
0 < \mathcal{K}_1(\psi + \mu s)(r_0) \leq 0.
\]

Case (iii): \( r_0 = 1 \):

\[
0 \leq [(\psi + \mu s)'(1)] = [(\psi')(1) + \mu [s'](1)] < 0.
\]

Case (iv): \( r_0 \in \Gamma_2 \):

\[
0 < \mathcal{K}_2(\psi + \mu s)(r_0) \leq 0.
\]

Case (v): \( r_0 = 2 \):

\[
0 \leq [(\psi + \mu s')(2)] = [\psi'(2) + \mu s'(2)] < 0.
\]

Case (vi): \( r_0 \in \Gamma_3 \):

\[
0 < \mathcal{K}_3(\psi + \mu s)(r_0) \leq 0.
\]

Case (vii): \( r_0 = 3 \):

\[
0 \leq (\psi + \mu s)(3) = (\psi)(3) + \mu (s)(3) = 0.
\]

All the cases are contradiction.

**Lemma 2** (stability result). If \( y(r) \) is a solution of problems \((7)--(9)\), then

\[
|y(r)| \leq C \max\{|y(0)|, |y(3)|, \sup_{r \in \Gamma} |\mathcal{K}y(r)|\}, \quad r \in \Gamma.
\]

Proof. First, to prove \( y(r) \) is bound on \( \Gamma_1 \),

\[
\mathcal{K}_1y(r) = -\varepsilon y''(r) + a(r) y'(r) + b(r) y(r) + d(r) y(r + 1)
\]

\[
+ d(r) y(r + 1) = f(r) - c(r) \phi(r - 1).
\]

Integrating the above equation on both sides, we have

\[
\varepsilon y'(r) - y'(0) = -[a(r) y(r) - a(0) u(0)]
\]

\[
+ \int_0^r a'(t) y(t) \, dt - \int_0^r b(t) y(t) \, dt
\]

\[
- \int_0^r d'(t) y(t + 1) \, dt
\]

\[
+ \int_0^r [f(t) - c(t) \phi(t - 1)] \, dt.
\]

Therefore,

\[
\varepsilon y'(r) = \varepsilon y'(r) - [a(r) y(r) - a(0) y(0)]
\]

\[
+ \int_0^r a'(t) y(t) \, dt - \int_0^r b(t) y(t) \, dt
\]

\[
- \int_0^r d(t) y(t + 1) \, dt + \int_0^r [f(t) - c(t) \phi(t - 1)] \, dt.
\]
Using Mean Value Theorem, then $|\varepsilon y'(x)| \leq C(\|y(r)\|, \|f(r)\|, \|\phi\|_{[-1,0]})$, for some $x \in (0, \varepsilon)$ and $|\varepsilon y'(0)| \leq C(\|y(r)\| + \|f(r)\| + \|\phi(r)\|)$. Then, we have $|\varepsilon y'(r)| \leq C \max(\|y(r)\|, \|f(r)\|, \|\phi\|), r \in \Gamma_1$.

To prove $y'(r)$ is bound on $\Gamma_2$,
\[ \mathcal{X}_2 y(r) = -\varepsilon y''(r) + a(r)y'(r) + b(r)y(r) + c(r)y(r-1) + d(r)y(r+1) = f(r). \] (23)

Integrating the above equation on both sides, we have
\[ -\varepsilon (y'(r) - y'(0)) = -[a(r)y(r) - a(0)u(0)] + \int_0^r a'(t)y(t) dt - \int_0^r b(t)y(t) dt - \int_0^r c(t)y(t-1) dt \]
\[ - \int_0^r d(t)y(t+1) dt + \int_0^r f(t) dt. \] (24)

Therefore,
\[ \varepsilon y'(0) = \varepsilon y'(r) - [a(r)y(r) - a(0)u(0)] + \int_0^r a'(t)y(t) dt - \int_0^r b(t)y(t) dt - \int_0^r c(t)y(t-1) dt \]
\[ - \int_0^r d(t)y(t+1) dt + \int_0^r f(t) dt. \] (25)

Using Mean Value Theorem, then $|\varepsilon y'(x)| \leq C(\|y(r)\|, \|f(r)\|, \|\phi\|_{[-1,0]})$, for some $x \in (0, \varepsilon)$ and $|\varepsilon y'(0)| \leq C(\|y(r)\| + \|f(r)\| + \|\phi(r)\|)$. Then, we have $|\varepsilon y'(r)| \leq C \max(\|y(r)\|, \|f(r)\|, \|\phi\|), r \in \Gamma_2$.

Next, to prove $y'(r)$ is bound on $\Gamma_3$,
\[ \mathcal{X}_3 y(r) = -\varepsilon y''(r) + a(r)y'(r) + b(r)y(r) + c(r)y(r-1) = f(r) - d(r)\phi(r+1). \] (26)

Integrating the above equation on both sides, we have
\[ -\varepsilon (y'(r) - y'(0)) = -[a(r)y(r) - a(0)u(0)] + \int_0^r a'(t)y(t) dt - \int_0^r b(t)y(t) dt - \int_0^r c(t)y(t-1) dt \]
\[ - \int_0^r d(t)y(t+1) dt + \int_0^r f(t) dt. \] (27)

Therefore,
\[ \varepsilon y'(0) = \varepsilon y'(r) - [a(r)y(r) - a(0)u(0)] + \int_0^r a'(t)y(t) dt - \int_0^r b(t)y(t) dt - \int_0^r c(t)y(t-1) dt \]
\[ - \int_0^r d(t)y(t+1) dt + \int_0^r f(t) dt. \] (28)

Using Mean Value Theorem, then $|\varepsilon y'(x)| \leq C(\|y(r)\|, \|f(r)\|, \|\phi\|_{[-1,0]})$, for some $x \in (0, \varepsilon)$ and $|\varepsilon y'(0)| \leq C(\|y(r)\| + \|f(r)\| + \|\phi(r)\|)$. Then, we have $|\varepsilon y'(r)| \leq C \max(\|y(r)\|, \|f(r)\|, \|\phi\|), r \in \Gamma_3$.

Hence, $|y^{(k)}(r)| \leq C \varepsilon^{-k}$, where $k = 2, 3, 4$. □

3.1. Shishkin Decomposition. The solution $y(r)$ is decomposed into $v(r)$ smooth component and $w(r)$-layer component. Furthermore, $v(r) = v_0(r) + e v_1(r) + e^2 v_2(r)$, where $v_0(r), v_1(r)$, and $v_2(r)$ are solutions of the following differential equations.

Obtain reduced problem solution $v_0(r) \in X$ such that

\[ \begin{cases} a(r)v_0'(r) + b(r)v_0(r) + c(r)v_0(r-1) = f(r), & r \in \Gamma \cap (\Gamma_1 \cup \Gamma_2), \\ v_0(r) = \phi(r), & r \in [-1,0]. \end{cases} \] (29a)

\[ \begin{cases} a(r)v_0'(r) + b(r)v_0(r) + d(r)v_0(r+1) = f(r), & r \in \Gamma \cap (\Gamma_2 \cup \Gamma_3) \cup [3], \\ v_0(r) = \phi(r), \end{cases} \] (29b)

If $v_1(r) \in C^0(\overline{\Gamma}) \cap C^1(\Gamma^* \cup [3])$,

\[ \begin{cases} a(r)v_1'(r) + b(r)v_1(r) + c(r)v_1(r-1) = v_0'(r), & r \in \Gamma \cap (\Gamma_1 \cup \Gamma_2), \\ v_1(r) = 0, & r \in [-1,0]. \end{cases} \] (30a)

\[ \begin{cases} a(r)v_1'(r) + b(r)v_1(r) + d(r)v_1(r+1) = v_0'(r), & r \in \Gamma \cap (\Gamma_2 \cup \Gamma_3) \cup [3], \\ v_1(r) = 0, & r \in [3,4]. \end{cases} \] (30b)
If $v_2(r) \in X$,  

$$
\begin{cases}
-\varepsilon v''_2(r) + a(r)v''_2(r) + b(r)v_2(r) + c(r)v_2(r - 1) + d(r)v_2(r + 1) = v'_1(r), & r \in \Gamma^*, \\
v_2(r) = 0, & r \in [-1, 0], \\
v_2(r) = 0, & r \in [3, 4].
\end{cases}
$$

(31)

If $v(r) \in C^0(\bar{T}) \cap C^2(\Gamma^*)$,  

$$
\begin{cases}
\mathcal{A}v(r) = -\varepsilon v''(r) + a(r)v'(r) + b(r)v(r) + c(r)v(r - 1) + d(r)v(r + 1) = f(r), & r \in \Gamma^*, \\
v(r) = \phi(r), & r \in [-1, 0], \\
v(1) = v_0(1) + \varepsilon v_1(1) + \varepsilon^2 v_2(1), \\
v(2) = v_0(2) + \varepsilon v_1(2) + \varepsilon^2 v_2(2), \\
v(r) = 0, & r \in [3, 4].
\end{cases}
$$

(32)

Also, $w(r)$ satisfies the following problem: if the singular component $w(r) \in C^0(\bar{T}) \cap C^2(\Gamma^*)$,  

$$
\begin{cases}
\mathcal{A}w(r) = -\varepsilon w''(r) + a(r)w'(r) + b(r)w(r) + c(r)w(r - 1) + d(r)w(r + 1) = 0, & r \in \Gamma^*, \\
w(r) = 0, & r \in [-1, 0], \\
[w'](1) = -[v'](1), \\
[w'](2) = -[v'](2), \\
w(r) = \varphi(r), & r \in [3, 4].
\end{cases}
$$

(33)

Furthermore, we decompose $w(r)$ as $w(r) = w_B(r) + w_{I_1}(r) + w_{I_2}(r)$, where the function $w_B(r)$ is boundary layer component and $w_{I_1}(r), w_{I_2}(r)$ are interior layer components.  

If the boundary layer $w_B(r) \in X$,  

$$
\begin{cases}
\mathcal{A}w_B(r) = -\varepsilon w''_B(r) + a(r)w'_B(r) + b(r)w_B(r) + c(r)w_B(r - 1) + d(r)w_B(r + 1) = 0, & r \in [-1, 0], \\
w_B(r) = 0, & r \in [3, 4].
\end{cases}
$$

(34)

If the first interior layer $w_{I_1}(r) \in C^0(\bar{T}) \cap C^2(\Gamma^*)$,  

$$
\begin{cases}
\mathcal{A}w_{I_1}(r) = -\varepsilon w''_{I_1}(r) + a(r)w'_{I_1}(r) + b(r)w_{I_1}(r) + c(r)w_{I_1}(r - 1) + d(r)w_{I_1}(r + 1) = 0, & r \in [-1, 0], \\
w_{I_1}(r) = 0, & r \in [3, 4].
\end{cases}
$$

(35)

If the second interior layer $w_{I_2}(r) \in C^0(\bar{T}) \cap C^2(\Gamma^*)$,  

$$
\begin{cases}
\mathcal{A}w_{I_2}(r) = -\varepsilon w''_{I_2}(r) + a(r)w'_{I_2}(r) + b(r)w_{I_2}(r) + c(r)w_{I_2}(r - 1) + d(r)w_{I_2}(r + 1) = 0, & r \in [-1, 0], \\
w_{I_2}(r) = 0, & r \in [3, 4].
\end{cases}
$$
\[ \mathcal{X}w_i(r) = -\varepsilon w_i(r) + a(r)w_i'(r) + b(r)w_i(r) + c(r)w_i(r-1) + d(r)w_i(r+1) = 0, \]
\[ w_i(r) = 0, \quad r \in [-1, 0], \]
\[ \left[ w_i' \right](2) = -[v'](2), \quad r \in [3, 4]. \]

**Theorem 1.** If \( y(r) \) and \( v_0(r) \) are solutions of problems (7)-(9) and (29a)-(29b), then
\[ |y(r) - v_0(r)| \leq C_1 \left( \varepsilon + \exp \left( \frac{-\alpha(3-r)}{\varepsilon} \right) \right), \quad r \in \Gamma. \]  

**Proof.** Consider
\[ \mathcal{X}_1 \Theta^+(r) = C_1 \left[ \frac{\alpha}{\varepsilon} (a(r) - a) + b(r) + d(r) \exp \left( \frac{-\alpha(3-r)}{\varepsilon} \right) + \varepsilon (a(r)s'(r) + b(r)s(r) + d(r)s(r+1)) \right] + \varepsilon v_0''(r), \]
\[ \geq C_1 \left[ \frac{\alpha}{\varepsilon} (a_1 - a) + \beta + \eta \exp \left( \frac{\alpha}{\varepsilon} \right) \exp \left( \frac{-\alpha(3-r)}{\varepsilon} \right) + \eta \varepsilon v_0'(r) + \varepsilon v_0(r) \right] + C\varepsilon \geq 0. \]

If \( r \in \Gamma_2, \)
\[ \mathcal{X}_2 \Theta^+(r) = C_1 \left[ \frac{\alpha}{\varepsilon} (a(r) - a) + b(r) + d(r) \exp \left( \frac{-\alpha(3-r)}{\varepsilon} \right) \right] + \varepsilon v_0''(r), \]
\[ \geq C_1 \left[ \frac{\alpha}{\varepsilon} (a_1 - a) + \beta + \eta \exp \left( \frac{\alpha}{\varepsilon} \right) \exp \left( \frac{-\alpha(3-r)}{\varepsilon} \right) + \varepsilon v_0'(r) + \varepsilon v_0(r) \right] + C\varepsilon \geq 0. \]

Following the same process, we have \( \mathcal{X}_i \Theta^+(r) \geq 0. \)

Using Lemma 1, then \( \Theta^+(r) \geq 0, r \in \Gamma. \) Therefore,
\[ |y(r) - v_0(r)| \leq C_1 \left( \varepsilon + \exp \left( \frac{-\alpha(3-r)}{\varepsilon} \right) \right). \]

**Lemma 4.** If \( v(r) \) and \( w(r) \) are the solution of regular and singular component problems (32) and (33), then
\[ |v^k(r)|_{\Gamma} \leq C(1 + \varepsilon^{2-k}), \quad \text{for} \ k = 0, 1, 2, 3, 4, \]
\[ |w_b^k(r)| \leq C\varepsilon^k \exp \left( \frac{-\alpha(3-r)}{\varepsilon} \right), \quad r \in \Gamma^*, \]
\[ |w_i^k(r)| \leq C \begin{cases} \varepsilon^{1-k} \exp \left( \frac{-\alpha(1-r)}{\varepsilon} \right), & r \in \Gamma_1, \\ \varepsilon^{1-k}, & r \in \Gamma_2, r \in \Gamma_3, \end{cases} \]
where \( k = 0, 1, 2, 3, 4. \)

Proof. The smooth component derivative bound is easy to prove by using stability result and integrating (30a), (30b), and (31). Next, to prove (42), consider that

\[
\mathcal{K}\Phi^+(r) \geq C \left[ \frac{\alpha}{\varepsilon} (\alpha_1 - \alpha) + \beta + \gamma \exp \left( \frac{\alpha}{\varepsilon} \right) \right] \exp \left( -\frac{\alpha (3 - r)}{\varepsilon} \right) \pm \mathcal{K} w_b(r) \geq 0. \quad (46)
\]

By Lemma 1,

\[
|w_b(r)| \leq C \exp \left( -\frac{\alpha (3 - r)}{\varepsilon} \right). \quad (47)
\]

Integration of (34) yields the estimates of \( |w^b_b(r)|. \) From the differential equations (33), one can derive the rest of the derivative estimates (42).

Inequalities (43) and (44) can be proved, using Theorem 1 and maximum principle for the barrier functions:

\[
\Phi^+(r) = C \left[ \exp \left( -\frac{\alpha (1 - r)}{\varepsilon} \right) \right] \pm w_{l_1}(r), \quad r \in \Gamma_1,
\]

\[
\Phi^+(r) = C \left[ \exp \left( -\frac{\alpha (2 - r)}{\varepsilon} \right) \right] \pm w_{l_2}(r), \quad r \in \Gamma_2,
\]

\[
\Phi^+(r) = C \left[ \exp \left( -\frac{\alpha (3 - r)}{\varepsilon} \right) \right] \pm w_{l_3}(r), \quad r \in \Gamma_3. \quad (48)
\]

Hence, it is proved.

Remark. The following inequalities are easy to prove, using Theorem 1 and Lemma 4:

\[
|y(r) - v(r)| \leq C \left[ \exp \left( -\frac{\alpha (1 - r)}{\varepsilon} \right) + \exp \left( -\frac{\alpha (3 - r)}{\varepsilon} \right) \right], \quad r \in \Gamma_1,
\]

\[
|y(r) - v(r)| \leq C \left[ \exp \left( -\frac{\alpha (2 - r)}{\varepsilon} \right) + \exp \left( -\frac{\alpha (3 - r)}{\varepsilon} \right) \right], \quad r \in \Gamma_2,
\]

\[
|y(r) - v(r)| \leq C \left[ \exp \left( -\frac{\alpha (3 - r)}{\varepsilon} \right) \right], \quad r \in \Gamma_3. \quad (49)
\]

4. The Discrete Problem

4.1. Shishkin Mesh. Problems (7)–(9) are convection-diffusion type containing delay term. Then, the layers occur in boundary at \( t = 3 \) and interior at \( t = 1 \) and \( t = 2. \)

The intervals \([0, 1], [1, 2], \) and \([2, 3] \) are partitioned into \([0, 1 - \sigma], [1 - \sigma, 1], [1, 2 - \sigma], [2 - \sigma, 2], [2, 3 - \sigma], \) and \([3 - \sigma, 3] \) for each interval \((N/2) \) mesh points and \( \sigma = \min(1/2), 2(\varepsilon/\alpha) \ln N \) is transition parameter.
The interior of points is denoted by \( \Gamma^{3N} = \{ r_0, r_1, \ldots, r_{3N} \} \). Then, the mesh widths are

\[
h(r_i) = \begin{cases} 
H = \frac{2(1 - \sigma)}{N} & \text{for } i = 1 \text{ to } \frac{N}{2}, \ i = N + 1 \text{ to } \frac{3N}{2}, \ i = 2N + 1 \text{ to } \frac{5N}{2} \\
h = \frac{2\sigma}{N} & \text{for } i = \frac{N}{2} + 1 \text{ to } N, \ i = \frac{3N}{2} + 1 \text{ to } 2N, \ i = \frac{5N}{2} \text{ to } 3N.
\end{cases}
\] (50)

4.2. Finite Difference Method. The discrete scheme corresponding to the original problems (7)-(9) is as follows:

\[
\begin{align*}
\mathcal{N}^N Y(r_i) &= \begin{cases} 
-\varepsilon \delta^2 Y(r_i) + a(r_i)D^{-} Y(r_i) + b(r_i)Y_{r_i} + d(r_i)Y_{r_i(N)} = f_i - c_i \phi_{i-N}, & r_i \in \Gamma^N_1, \\
-\varepsilon \delta^2 Y(r_i) + a(r_i)D^{-} Y(r_i) + b(r_i)Y_{r_i} + d(r_i)Y_{r_i(N)} = f_i, & r_i \in \Gamma^N_2, \\
-\varepsilon \delta^2 Y(r_i) + a(r_i)D^{-} Y(r_i) + b(r_i)Y_{r_i} + c(r_i)Y_{r_i(N)} + d(r_i)Y_{r_i(N)} = f_i, & r_i \in \Gamma^N_3,
\end{cases}
\end{align*}
\] (51)

with

\[
\begin{align*}
Y(r(0)) &= \phi_0, \\
D^{-} U_N &= D^{-} U_{2N}, \\
D^{-} U_{2N} &= D^{-} U_{3N}, \\
Y(r(3)) &= \varphi_{3N}.
\end{align*}
\] (52)

4.3. Hybrid Difference Scheme. The hybrid scheme corresponding to the original problems (7)-(9) is as follows:

\[
\begin{align*}
\mathcal{N}^N Y(r_i) &= \begin{cases} 
-\varepsilon \delta^2 Y(r_i) + a_{i-(1/2)}D^{-} Y(r_i) + b(r_i)\bar{U}(r_i) + d(r_i)\bar{U}(r_{i+N}) = f_i - c_i \bar{\phi}_{i-N}, & i = 1 \text{ to } \frac{N}{2}, \\
-\varepsilon \delta^2 Y(r_i) + a(r_i)D^{-} Y(r_i) + b(r_i)Y_{r_i} + d(r_i)Y_{r_i(N)} = f_i - c_i \phi_{i-N}, & i = \frac{N}{2} + 1 \text{ to } N - 1, \\
-\varepsilon \delta^2 Y(r_i) + a_{i-(1/2)}D^{-} Y(r_i) + b(r_i)\bar{U}(r_i) + c(r_i)\bar{U}(r_{i-N}) + d(r_i)\bar{U}(r_{i+N}) = f_i, & i = N + 1 \text{ to } \frac{3N}{2}, \\
-\varepsilon \delta^2 Y(r_i) + a(r_i)D^{-} Y(r_i) + b(r_i)Y_{r_i} + c(r_i)Y_{r_i(N)} + d(r_i)Y_{r_i(N)} = f_i, & i = \frac{3N}{2} + 1 \text{ to } 2N - 1,
\end{cases}
\] (53)

\[
\begin{align*}
\mathcal{N}^{2N} Y(r_i) &= \begin{cases} 
-\varepsilon \delta^2 Y(r_i) + a_{i-(1/2)}D^{-} Y(r_i) + b(r_i)\bar{U}(r_i) + c(r_i)\bar{U}(r_{i-N}) + d(r_i)\bar{U}(r_{i+N}) = f_i, & i = 1 \text{ to } \frac{N}{2}, \\
-\varepsilon \delta^2 Y(r_i) + a(r_i)D^{-} Y(r_i) + b(r_i)Y_{r_i} + c(r_i)Y_{r_i(N)} + d(r_i)Y_{r_i(N)} = f_i, & i = \frac{N}{2} + 1 \text{ to } N - 1, \\
-\varepsilon \delta^2 Y(r_i) + a_{i-(1/2)}D^{-} Y(r_i) + b(r_i)\bar{U}(r_i) + c(r_i)\bar{U}(r_{i-N}) + d(r_i)\bar{U}(r_{i+N}) = f_i - d_i \bar{\phi}_{i+N}, & i = N + 1 \text{ to } \frac{3N}{2}, \\
-\varepsilon \delta^2 Y(r_i) + a(r_i)D^{-} Y(r_i) + b(r_i)Y_{r_i} + c(r_i)Y_{r_i(N)} + d(r_i)Y_{r_i(N)} = f_i - d_i \phi_{i+N}, & i = \frac{3N}{2} + 1 \text{ to } 2N - 1,
\end{cases}
\] (54)

\[
\begin{align*}
\mathcal{N}^{3N} Y(r_i) &= \begin{cases} 
-\varepsilon \delta^2 Y(r_i) + a_{i-(1/2)}D^{-} Y(r_i) + b(r_i)\bar{U}(r_i) + c(r_i)\bar{U}(r_{i-N}) + d(r_i)\bar{U}(r_{i+N}) = f_i, & i = 1 \text{ to } \frac{N}{2}, \\
-\varepsilon \delta^2 Y(r_i) + a(r_i)D^{-} Y(r_i) + b(r_i)Y_{r_i} + c(r_i)Y_{r_i(N)} + d(r_i)Y_{r_i(N)} = f_i - d_i \bar{\phi}_{i+N}, & i = \frac{N}{2} + 1 \text{ to } N - 1, \\
-\varepsilon \delta^2 Y(r_i) + a_{i-(1/2)}D^{-} Y(r_i) + b(r_i)\bar{U}(r_i) + c(r_i)\bar{U}(r_{i-N}) + d(r_i)\bar{U}(r_{i+N}) = f_i - d_i \phi_{i+N}, & i = N + 1 \text{ to } \frac{3N}{2}, \\
-\varepsilon \delta^2 Y(r_i) + a(r_i)D^{-} Y(r_i) + b(r_i)Y_{r_i} + c(r_i)Y_{r_i(N)} + d(r_i)Y_{r_i(N)} = f_i - d_i \phi_{i+N}, & i = \frac{3N}{2} + 1 \text{ to } 2N - 1,
\end{cases}
\] (55)

\[
\mathcal{N}^N Y(r_i) = \frac{Y_{i-2} - 4Y_{i-1} + 3Y_i}{2h} - \frac{Y_{i+2} - 4Y_{i+1} - 3Y_i}{2H}, \quad i = N, 2N,
\] (56)

where
\[ \delta^2 Y(r_i) = \frac{2}{h_i + h_{i+1}} \left( \frac{Y(r_{i+1}) - Y(r_i)}{h_{i+1}} - \frac{Y(r_i) - Y(r_{i-1})}{h_i} \right), \]

\[ \bar{U}(r_i) = \frac{Y(r_i) + Y(r_{i-1})}{2}, \]

\[ D^0 Y(r_i) = \frac{Y(r_{i+1}) - Y(r_{i-1})}{h_i + h_{i+1}}, \]

\[ D^0 Y(r_i) = \frac{Y(r_i) - Y(r_{i-1})}{h_i}, \]

\[ a_{i-(1/2)} = a \left( \frac{r_{i-1} + r_i}{2} \right). \]

5. Numerical Estimates for the Finite Difference Method

Lemma 5 (discrete maximum principle). If \( U(r_i) \) satisfies \( U(r_0) \geq 0, U(r_{2N}) \geq 0, \mathcal{K} N U(r_i) \geq 0, \mathcal{K} N U(r_i) \geq 0, \mathcal{K} N U(r_i) \geq 0, D^0(U(r_{i+1})) - D^0(U(r_{i-1})) \leq 0 \), then \( U(r_i) \geq 0, \forall r_i \in \Gamma^N. \)

Proof.

Define \( S(r_i) = \begin{cases} \frac{1}{12} + \frac{r_i}{4}, & r_i \in [0,1] \cap \Gamma^N, \\ \frac{2}{12} + \frac{r_i}{6}, & r_i \in [1,2] \cap \Gamma^N, \\ \frac{4}{12} - \frac{r_i}{12}, & r_i \in [2,3] \cap \Gamma^N. \end{cases} \) (58)

It is easy to see that \( s(r_i) > 0, \forall r_i \in \Gamma^N, \mathcal{A} s(r_i) > 0, \forall r_i \in \Gamma^N \cup \Gamma^N \cup \Gamma^N, s(r_0) > 0, s(r_{2N}) > 0, D^0(s(r_{N})) - D^0(s(r_{N})) < 0, \) and \( D^0(s(r_{2N})) - D^0(s(r_{2N})) < 0. \) Let \( \mu = \min \left\{ (-s(r_i))/s(r_i) : r_i \in \Gamma^N \right\}. \)

Then, there exists \( r_k \in \Gamma^N \) such that \( \psi(r_k) + \mu s(r_k) = 0 \) and \( \psi(r_i) + \mu s(r_i) \geq 0, \forall r_i \in \Gamma^N. \) Then, \( (\psi + \mu s) \) attains its maximum at \( r_j = r_k. \) If \( \mu < 0, \) then \( \psi \geq 0. \) Suppose \( \mu > 0. \)

Case (i): \( r_k = r_0: \)

\[ 0 < (\psi + \mu s)(r_0) = 0. \] (59)

Case (ii): \( r_k \in \Gamma^N: \)

\[ 0 < \mathcal{A} \psi(r_k) = -\epsilon \delta^2 (\psi + \mu s)(r_k) \]

\[ + a(r_k)D^-(\psi + \mu s)(r_k) + b(r_k)(\psi + \mu s)(r_k) \]

\[ + c(r_k)(\psi + \mu s)(r_{k-N}) \leq 0. \] (60)

Case (iii): \( r_k = r_N: \)

\[ 0 \leq [D](\psi + \mu s)(r_N) < 0. \] (61)

Case (iv): \( r_k \in \Gamma^N: \)

\[ 0 < \mathcal{A} \psi(r_k) = -\epsilon \delta^2 (\psi + \mu s)(r_k) \]

\[ + a(r_k)D^-(\psi + \mu s)(r_k) + b(r_k)(\psi + \mu s)(r_k) \]

\[ + c(r_k)(\psi + \mu s)(r_{k-N}) + d(r_k)(\psi + \mu s)(r_{k-N}) \leq 0. \] (62)

Case (v): \( r_k = r_{2N}: \)

\[ 0 \leq [D](\psi + \mu s)r_{2N} < 0. \] (63)

Case (vi): \( r_k \in \Gamma^N: \)

\[ 0 < \mathcal{A} \psi(r_k) = -\epsilon \delta^2 (\psi + \mu s)(r_k) \]

\[ + a(r_k)D^-(\psi + \mu s)(r_k) + b(r_k)(\psi + \mu s)(r_k) \]

\[ + c(r_k)(\psi + \mu s)(r_{k-N}) \leq 0. \] (64)

Case (vii): \( r_k = r_{3N}: \)

\[ 0 < (\psi + \mu s)r_{3N} = 0. \] (65)

All the cases are a contradiction. \( \square \)

Lemma 6. The discrete solution of (51) and (52) is bounded:

\[ [U(r_i)] \leq C \max \left\{ |U(r_0)|, |U(r_{2N})|, \max_{r_i \in \Gamma^N} |\mathcal{A}^N U(r_i)| \right\}. \] (66)

Proof. Consider \( \psi^+(r_i) = \text{CMs}(r_i) \pm U(r_i), 0 \leq i \leq 3N, \) where \( M = \max \left\{ |U(r_0)|, |U(r_{2N})|, \max_{r_i \in \Gamma^N} |\mathcal{A}^N U(r_i)| \right\}. \)

Observe \( \psi^+(r_0) \geq 0 \) and \( \psi^+(r_{2N}) \geq 0: \)
Theorem 2. If $Y(r_i)$ and $V(r_i)$ are a solution of discretization problem (51), (52), and (68), then $|Y(r_i) - V(r_i)| \leq CN^{-1}$.

Proof. Consider
\[
\theta^i(r_j) = CN^{-1}s(r_j) \pm (Y(r_j) - V(r_j)), \quad \forall r_j \in \Gamma^{3N}.
\]
(70)

Note that $\theta^i(r_0) \geq 0$ and $\theta^i(r_{3N}) \geq 0$:
\[
\begin{align*}
\mathcal{X}_1^{N}\theta^i(r_j) & \geq 0, \quad \text{for all } i \in \{1, 2, \ldots, N - 1\}, \\
\mathcal{X}_2^{N}\theta^i(r_j) & \geq 0, \quad \text{for all } i \in \{N + 1, \ldots, 2N - 1\}, \\
\mathcal{X}_3^{N}\theta^i(r_j) & \geq 0, \quad \text{for all } i \in \{2N + 1, \ldots, 3N - 1\},
\end{align*}
\]
(71)

Using Lemma 5, $\psi^i(r_j) \geq 0$, $\forall r_j \in \Gamma^{3N}$.

Using Lemma 6, then
\[
|\nu(r_j) - V(r_j)| \leq CN^{-1}, \quad i \in \Gamma^{3N} \cup \Gamma_2^{3N} \cup \Gamma_3^{3N}.
\]
(73)

Theorem 3. The error estimates for smooth components bounded by $CN^{-1}$:
\[
|\nu(r_j) - V(r_j)| \leq CN^{-1}, \quad r_j \in \Gamma^{3N}.
\]
(72)

Proof. The proof of Theorem 3 has the same idea in [29]:

Using Lemma 5, $\psi^i(r_j) \geq 0$, $\forall r_j \in \Gamma^{3N}$.

To decompose numerical solution $Y(r_i)$ into $V(r_i)$ and $W(r_i)$ satisfy the following equations, respectively:

Using Lemma 5, $\psi^i(r_j) \geq 0$, $\forall r_j \in \Gamma^{3N}$.
\[ |w(r_i) - W(r_i)| \leq |Y(r_i) - V(r_i)| + 2|v(r_i) - V(r_i)| + |y(r_i) - v(r_i)| \]
\[ \leq C_1N^{-1} + C_1 \exp\left(-\frac{a(3 - r)}{\varepsilon}\right) + \varepsilon \]
\[ \leq C_1N^{-1} + C_1 \exp\left(-\frac{a(3 - r)}{\varepsilon}\right) + C_1N^{-1} \]
\[ \leq C_1 \exp\left(-\frac{a\sigma}{\varepsilon}\right) + C_1N^{-1} \]
\[ \leq CN^{-1}, \quad i = 0 \text{ to } 5N \]

Consider the mesh functions:
\[ \Phi^\pm (r_i) = C_1N^{-1}s(r_i) + C_1N^{-1}\frac{\sigma}{\varepsilon} (r_i - (3 - \sigma)) \]

(79)
\[ \pm (w(r_i) - W(r_i)), \quad r_i \in [3 - \sigma, 3] \cap \Gamma^N. \]

Observe that \( \Phi^\pm (r_{i(5N/2)}) \geq 0 \) and \( \Phi^\pm (r_{3N}) \geq 0 \), and \( \mathcal{N}\Phi^\pm (r_i) \geq 0 \).

Then, by the Lemma 5, we have \( \Phi^\pm (r_i) \geq 0, \forall r_i \in \Gamma^N \).

Therefore,
\[ |w(r_i) - W(r_i)| \leq CN^{-1}\log^2 N, \quad r_i \in \Gamma^N. \]

(80)

Theorem 5. If \( y(r_i) \) and \( Y(r_i) \) are a solution of (7)-(9) and (51), (52),
\[ |y(r_i) - Y(r_i)| \leq CN^{-1}\log^2 N, \quad r_i \in \Gamma^N. \]

(81)

That is, the order of convergence is almost one.

Proof. The proof of Theorem 5 follows from \( y_k = v_k + w_k, \)
\( Y_k = V_k + W_k \), and Theorems 3 and 4.

6. Numerical Estimates for the Hybrid Difference Method

Assume the following inequality:
\[ \frac{N}{\ln N} \geq 2\frac{a_1}{\alpha} \]

(82)

Lemma 7. Assume (78) holds true. Let \( \Psi(r_i) \) satisfy \( \Psi(r_0) \geq 0, \Psi(r_{3N}) \geq 0; \) the operator \( \mathcal{N} \) defined by (53)–(55) satisfies \( \mathcal{N} \Psi(r_i) \geq 0, \mathcal{N}^2 \Psi(r_i) \geq 0, \mathcal{N}^3 \Psi(r_i) \geq 0; \) and then \( \Psi(r_i) \geq 0, \forall r_i \in \Gamma^N \).

Lemma 8. If \( \Psi(r_i) \) is discrete solution of problems (53)–(55), then
\[ |\Psi(r_i)| \leq C\max \left\{|\Psi(r_0)|, |\Psi(r_{3N})|, \max_{i \in \Gamma^N \cup \Gamma^N \cup \Gamma^N} |\mathcal{N} \Psi(r_i)| \right\}. \]

(83)

6.1. Error Estimate. To decompose the numerical solution \( Y(r_i) \) into \( V(r_i) \) and \( W(r_i) \), satisfy the following equations, respectively:

\[ \mathcal{N}^0 V(r_i) = \begin{cases} -\varepsilon \delta^2 V(r_i) + a_{i-1(1/2)} D^2 V(r_i) + b(r_i) V(r_i) + c(r_i) V(r_i) + d(r_i) V(r_i) = f_{i-1(1/2)} \end{cases} \]

(84)

\[ \mathcal{N}^0 W(r_i) = \begin{cases} -\varepsilon \delta^2 W(r_i) + a_{i-1(1/2)} D^2 W(r_i) + b(r_i) W(r_i) + c(r_i) W(r_i) + d(r_i) W(r_i) = 0 \end{cases} \]

(85)

Lemma 9. Derive the error estimation of discretization original problems (53)–(56) and regular problem (84) solutions:
\[ |Y(r_i) - V(r_i)| \leq CN^{-2}. \]

(86)

Proof. The proof of Lemma 9 has the same idea in Lemma 7:
\[ \theta^i (r_i) = CN^{-2} s(r_i) \pm (Y(r_i) - V(r_i)), \quad \forall r_i \in \Gamma^N. \]

(87)
Lemma 10. The error estimates for smooth components are bounded by $CN^{-2}$:
\[ |V(r_i) - v(r_i)| \leq CN^{-2}, \quad r_i \in \Gamma^{3N} . \quad (88) \]

Proof. Utilizing the method adopted in [30],
\[ |\mathcal{K}_n (Y - y)(r_i)| \leq C_n \begin{cases} \varepsilon H \| y^{(3)} \| + H^2 \left( | y^{(3)} | + | y^{(2)} | \right), & i = 1 \text{ to } \frac{N}{2}, \\ \varepsilon h^2 \| y^{(4)} \| + h^2 \| a_i \| \| y^{(3)} \|, & i = \frac{N}{2} + 1 \text{ to } N - 1. \end{cases} \quad (89) \]

Using $\varepsilon \leq CN^{-1}$ and the above equation, the bounds on the derivatives of $v$ can be written as
\[ |\mathcal{K}_n (V - v)(r_i)| \leq C_n \begin{cases} \varepsilon H \| v^{(3)} \| + H^2 \left( | v^{(3)} | + | v^{(2)} | \right), & i = 1 \text{ to } \frac{N}{2}, \\ \varepsilon h^2 \| v^{(4)} \| + h^2 \| a_i \| \| v^{(3)} \| + h^2 \| a_i \| \| v^{(3)} \|, & i = \frac{N}{2} + 1 \text{ to } N - 1. \end{cases} \quad (90) \]

Then, we have $|\mathcal{K}_n (V - v)(r_i)| \leq CN^{-2}$. Similarly, $|\mathcal{K}_n (V - v)(r_i)| \leq CN^{-2}, j = 2, 3, |\mathcal{K}_n (V - v)(r_i)| \leq CN^{-2}, j = 1, 2, 3, i \in \Gamma^{3N} \setminus \{0, 2N, 3N\}$, and, by Lemma 8, we have
\[ |V(r_i) - v(r_i)| \leq CN^{-2}, \quad r_i \in \Gamma^{3N} . \quad (91) \]

Lemma 11. Derive the error estimates for singular components bounded by $CN^{-2} \log^2 N$:
\[ |w(r_i) - W(r_i)| \leq |Y(r_i) - V(r_i)| + 2|v(r_i) - V(r_i)| + |y(r_i) - v(r_i)|, \]
\[ \leq C_1 \exp \left( -\frac{\alpha \sigma}{\varepsilon} \right) + C_1 N^{-2} \leq CN^{-2}, \quad i = 0 \text{ to } \frac{5N}{2}. \quad (94) \]

Consider the mesh functions
\[ \Phi^s(r_i) = C_1 N^{-2}s(r_i) + C_1 N^{-2} \frac{\sigma}{\varepsilon} (r_i - (3 - \sigma)) \]
\[ \pm (w(r_i) - W(r_i)), \quad r_i \in [3 - \sigma, 3] \cap \Gamma^{3N} . \quad (95) \]

Clearly, $\Phi^s(r_{\{3N/2\}}) \geq 0$ and $\Phi^s(r_{3N}) \geq 0$, for a suitable choice of $C_1 > 0$.
\[ \mathcal{K}^{-N} \Phi^s(r_i) \geq 0. \quad (96) \]

Then, by Lemma 7, we have $\Phi^s(r_i) \geq 0, r_i \in \Gamma^{3N}$. Therefore,
\[ |w(r_i) - W(r_i)| \leq CN^{-2} \log^2 N, \quad r_i \in \Gamma^{3N} . \quad (97) \]

Theorem 6. If $y(r_i)$ and $Y(r_i)$ are the solution of (7)–(9) and (53)–(56), then
\[ |y(r_i) - Y(r_i)| \leq CN^{-2} \log^2 N, \quad r_i \in \Gamma^{3N} . \quad (98) \]
### Table 1: Computed $P^N$ rate of convergence and $D^N$ maximum errors for Example 1.

| $\varepsilon$ | 16  | 32  | 64  | 128 | 256 | 512 | 1024 |
|----------------|-----|-----|-----|-----|-----|-----|-------|
| $P^N$          |     |     |     |     |     |     |       |
| Finite difference method |
| $10^{-3}$      | 2.8530e-03 | 1.6423e-03 | 8.6257e-04 | 4.3424e-04 | 2.1521e-04 | 1.0694e-04 | 5.4208e-05 |
| $10^{-4}$      | 3.0817e-03 | 1.8315e-03 | 9.9232e-04 | 5.1418e-04 | 2.6100e-04 | 1.3150e-04 | 6.6324e-05 |
| $10^{-5}$      | 3.1526e-03 | 1.8900e-03 | 1.0321e-03 | 5.3860e-04 | 2.7489e-04 | 1.3887e-04 | 6.9901e-05 |
| $10^{-6}$      | 3.1750e-03 | 1.9083e-03 | 1.0446e-03 | 5.4624e-04 | 2.7922e-04 | 1.4116e-04 | 7.1080e-05 |
| $10^{-7}$      | 3.1820e-03 | 1.9141e-03 | 1.0486e-03 | 5.4864e-04 | 2.8058e-04 | 1.4188e-04 | 7.1355e-05 |
| $10^{-8}$      | 3.1842e-03 | 1.9159e-03 | 1.0498e-03 | 5.4940e-04 | 2.8101e-04 | 1.4211e-04 | 7.1465e-05 |
| $10^{-9}$      | 3.1849e-03 | 1.9165e-03 | 1.0502e-03 | 5.4965e-04 | 2.8115e-04 | 1.4218e-04 | 7.1499e-05 |
| $10^{-10}$     | 3.1851e-03 | 1.9167e-03 | 1.0503e-03 | 5.4972e-04 | 2.8119e-04 | 1.4220e-04 | 7.1510e-05 |
| $D^N$          | 3.1851e-03 | 1.9167e-03 | 1.0503e-03 | 5.4972e-04 | 2.8119e-04 | 1.4220e-04 | 7.1510e-05 |

### Table 2: Computed $P^N$ rate of convergence and $D^N$ maximum errors for Example 2.

| $\varepsilon$ | 16  | 32  | 64  | 128 | 256 | 512 | 1024 |
|----------------|-----|-----|-----|-----|-----|-----|-------|
| $P^N$          |     |     |     |     |     |     |       |
| Finite difference method |
| $10^{-3}$      | 5.5384e-03 | 2.5272e-03 | 1.1887e-03 | 5.6672e-04 | 2.7241e-04 | 1.3205e-04 | 6.4844e-05 |
| $10^{-4}$      | 5.8841e-03 | 2.7585e-03 | 1.3350e-03 | 6.5461e-04 | 3.2299e-04 | 1.6003e-04 | 7.9627e-05 |
| $10^{-5}$      | 5.9911e-03 | 2.8296e-03 | 1.3798e-03 | 6.8149e-04 | 3.3841e-04 | 1.6851e-04 | 8.4085e-05 |
| $10^{-6}$      | 6.0248e-03 | 2.8519e-03 | 1.3939e-03 | 6.8989e-04 | 3.4321e-04 | 1.7116e-04 | 8.5469e-05 |
| $10^{-7}$      | 6.0354e-03 | 2.8590e-03 | 1.3983e-03 | 6.9253e-04 | 3.4473e-04 | 1.7199e-04 | 8.5904e-05 |
| $10^{-8}$      | 6.0387e-03 | 2.8612e-03 | 1.3997e-03 | 6.9337e-04 | 3.4521e-04 | 1.7225e-04 | 8.6041e-05 |
| $10^{-9}$      | 6.0398e-03 | 2.8619e-03 | 1.4001e-03 | 6.9363e-04 | 3.4536e-04 | 1.7233e-04 | 8.6085e-05 |
| $10^{-10}$     | 6.0401e-03 | 2.8621e-03 | 1.4003e-03 | 6.9371e-04 | 3.4541e-04 | 1.7236e-04 | 8.6098e-05 |
| $D^N$          | 6.0401e-03 | 2.8621e-03 | 1.4003e-03 | 6.9371e-04 | 3.4541e-04 | 1.7236e-04 | 8.6098e-05 |

### Table 1: Computed $P^N$ rate of convergence and $D^N$ maximum errors for Example 1.

### Table 2: Computed $P^N$ rate of convergence and $D^N$ maximum errors for Example 2.
7. Numerical Experiments

In this section, consider two examples for constant and variable coefficient problems and apply both of the numerical methods to find error and rate of convergence. The exact solution is not easy to find in these problems. Therefore, we use the double mesh principle:

\[
D_i^N = \max_{0 \leq i \leq 3N} |U_i^N - U_{2i}^N|.
\]

(99)

We compute the uniform error and the rate of convergence as

\[
D^N = \max_i D_i^N,
\]

\[
\rho^N = \log_2 \left( \frac{D^N}{D^{2N}} \right).
\]

(100)

To solve the following numerical examples, we use two computational methods such as finite and hybrid difference scheme on the nonuniform mesh.

**Example 1**

\[-\varepsilon y''(r) + 5y'(r) + 2y(r) - y(r-1) + y(r+1) = 1, \quad \text{for } r \in \Gamma^*,
\]

\[y(r) = 1, \quad \text{for } r \in [-1, 0], \]

\[y(r) = 2, \quad \text{for } r \in [3, 4].
\]

(101)

**Example 2**

\[-\varepsilon y''(r) + (r + 5)y'(r) + 2y(r) - y(r-1) + x^2 y(r+1) = \varepsilon^4, \quad \text{for } r \in \Gamma^*,
\]

\[y(r) = 1, \quad \text{for } r \in [-1, 0], \]

\[y(r) = 2, \quad \text{for } r \in [3, 4].
\]

(102)

We proved that the error is of order \(O(N^{-1} \ln N)\) and \(O(N^{-2} \ln^2 N)\). The theory has been validated with two examples; referring to these numerical results, it can be observed that the proposed method has been effective and applicable.

8. Discussion

In the literature, many authors have considered singular perturbation problem mixed delay \((\tau \ll 1)\) differential equation. In this paper, we consider a singular perturbation problem with mixed delay \((\tau = 1)\) differential equation. We suggested two computational methods such as finite and hybrid difference scheme. We proved that the error is of order \(O(N^{-1} \ln N)\) and \(O(N^{-2} \ln^2 N)\). Finally, two numerical examples are also presented to validate the theoretical results of this study. Maximum pointwise errors and order of convergence of Examples 1 and 2 are given in Tables 1 and 2, respectively.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

[1] A. Rustichini, “Functional differential equations of mixed type: the linear autonomous case,” *Journal of Dynamics and Differential Equations*, vol. 1, no. 2, pp. 121–143, 1989.

[2] H. Chi, J. Bell, and B. Hassard, “Numerical solution of a nonlinear advance-delay-differential equation from nerve conduction theory,” *Journal of Mathematical Biology*, vol. 24, no. 5, pp. 583–601, 1986.

[3] A. Rustichini, "Hopf bifurcation for functional differential equations of mixed type," *Journal of Dynamics and Differential Equations*, vol. 1, no. 2, pp. 145–177, 1989.

[4] K. A. Abell, C. E. Elmer, A. R. Humphries, and E. S. Van Vleck, “Computation of mixed type functional differential boundary value problems,” *SIAM Journal on Applied Dynamical Systems*, vol. 4, no. 3, pp. 755–781, 2005.

[5] Y. Kuang, *Delay Differential Equations with Applications in Population Dynamics*, Academic Press, Cambridge, MA, USA, 1993.

[6] R. B. Stein, “A theoretical analysis of neuronal variability,” *Biophysical Journal*, vol. 5, no. 2, pp. 173–194, 1965.

[7] R. B. Stein, “Some models of neuronal variability,” *Biophysical Journal*, vol. 7, no. 1, pp. 37–68, 1967.

[8] A. Rahmoune, D. Ouchenane, S. Boulaaras, and P. Agarwal, “Growth of solutions for a coupled nonlinear klein–gordon system with strong damping, source, and distributed delay terms,” *Advances in Difference Equations*, vol. 1, pp. 1–15, 2020.

[9] M. Abbas, “Existence results and the ulam stability for fractional differential equations with hybrid proportional-caputo derivatives,” *Journal of Nonlinear Functional Analysis ID*, vol. 48, 2020.

[10] M. Timoumi, “Ininitely many homoclinic solutions for fourth-order differential equations with locally defined potentials,” *Journal of Nonlinear and Variational Analysis*, vol. 3, no. 305–316, 2019.

[11] C. Lange and R. Miura, “Singular perturbation analysis of boundary-value problems for differential-difference equations (v) small shifts with layer behavior,” *SIAM Journal of Applied Mathematics*, vol. 54, pp. 249–272, 1994.

[12] C. G. Lange and R. M. Miura, “Singular perturbation analysis of boundary-value problems for differential-difference equations. VI. Small shifts with rapid oscillations,” *SIAM Journal on Applied Mathematics*, vol. 54, no. 1, pp. 273–283, 1994.

[13] C. G. Lange and R. M. Miura, “Singular perturbation analysis of boundary value problems for differential-difference equations,” *SIAM Journal on Applied Mathematics*, vol. 42, no. 3, pp. 502–531, 1982.

[14] M. Kadbalbajoo and K. Sharma, “Numerical treatment of boundary value problems for second order singularly perturbed delay differential equations,” *Journal of Computational and Applied Mathematics*, vol. 24, no. 2, pp. 151–172, 2005.
[15] F. Erdogan and Z. Cen, “A uniformly almost second order convergent numerical method for singularly perturbed delay differential equations,” Journal of Computational and Applied Mathematics, vol. 333, pp. 382–394, 2018.

[16] M. Kadalbajoo and K. Sharma, “Parameter-uniform fitted mesh method for singularly perturbed delay differential equations with layer behavior,” Electronic Transactions on Numerical Analysis, vol. 23, pp. 180–201, 2006.

[17] M. K. Kadalbajoo and D. Kumar, “Fitted mesh B-spline collocation method for singularly perturbed differential-difference equations with small delay,” Applied Mathematics and Computation, vol. 204, no. 1, pp. 90–98, 2008.

[18] M. K. Kadalbajoo and K. K. Sharma, “Numerical analysis of boundary-value problems for singularly-perturbed differential-difference equations with small shifts of mixed type,” Journal of Optimization Theory and Applications, vol. 115, no. 1, pp. 145–163, 2002.

[19] K. C. Patidar and K. K. Sharma, “Uniformly convergent non-standard finite difference methods for singularly perturbed differential-difference equations with delay and advance,” International Journal for Numerical Methods in Engineering, vol. 66, no. 2, pp. 272–296, 2006.

[20] M. K. Kadalbajoo and K. K. Sharma, “ε-uniform fitted mesh method for singularly perturbed differential-difference equations: mixed type of shifts with layer behavior,” International Journal of Computer Mathematics, vol. 81, no. 1, pp. 49–62, 2004.

[21] M. K. Kadalbajoo and K. K. Sharma, “Numerical treatment of a mathematical model arising from a model of neuronal variability,” Journal of Mathematical Analysis and Applications, vol. 307, no. 2, pp. 606–627, 2005.

[22] V. Subburayan and N. Ramanujam, “An initial value technique for singularly perturbed reaction-diffusion problems with a negative shift,” Novi Sad Journal of Mathematics, vol. 43, no. 2, pp. 67–80, 2013.

[23] E. Sekar and A. Tamilselvan, “Singularly perturbed delay differential equations of convection-diffusion type with integral boundary condition,” Journal of Applied Mathematics and Computing, vol. 59, no. 1-2, pp. 701–722, 2019.

[24] K. Kumar, P. P. Chakravarthy, H. Ramos, and J. Vigo-Aguiar, “A stable finite difference scheme and error estimates for parabolic singularly perturbed PDEs with shift parameters,” Journal of Computational and Applied Mathematics, Article ID 113050, 2020.

[25] D. Kumar and P. Kumari, “Parameter-uniform numerical treatment of singularly perturbed initial-boundary value problems with large delay,” Applied Numerical Mathematics, vol. 153, pp. 412–429, 2020.

[26] P. Selvi and N. Ramanujam, “An iterative numerical method for singularly perturbed reaction–diffusion equations with negative shift,” Journal of Computational and Applied Mathematics, vol. 296, pp. 10–23, 2016.

[27] S. Nicaise and C. Xenophontos, “Robust approximation of singularly perturbed delay differential equations by the hp finite element method,” Computational Methods in Applied Mathematics, vol. 13, no. 1, pp. 21–37, 2013.

[28] N. A. Shah, P. Agarwal, J. D. Chung, E. R. El-Zahar, and Y. S. Hamed, “Analysis of optical solitons for nonlinear schrödinger equation with detuning term by iterative transform method,” Symmetry, vol. 12, no. 11, p. 1850, 2020.

[29] J. Miller, E. O’Riordan, and G. Shishkin, Fitted Numerical Methods for Singular Perturbation Problems, World Scientific Publishing Co., Singapore, 1996.