Indirect measurement of temperature inside a furnace, ARX model identification

T Loussouarn$^{1,2}$, D Maillet$^1$, B Remy$^1$, V Schick$^1$ and D Dan$^2$

$^1$LEMTA, Université de Lorraine and CNRS, Vandoeuvre-Les-Nancy, 54504, France
$^2$Safran Aircraft Engines, Evry, 91019, France

E-mail: thomas.loussouarn@univ-lorraine.fr

Abstract. Thermal processing of ceramic parts at high temperature can be implemented in induction furnaces under vacuum conditions. In production, there is no direct access to these temperatures because of the high temperature level (higher than 1000°C) and also because the load cannot be equipped with temperature sensors. This paper deals with the indirect estimation of load temperatures. The estimation has to be made using transient point temperature measurements in the heating element. Indirect measurement of its temperature requires the calibration of a reference model that relates load temperatures to temperature in the susceptor first. A simplified reference furnace model, based on experimentations and thermal properties of its components, is derived. It is the reference model used to test the capacity of an identified AutoRegressive with eXogeneous variable model (ARX) to reproduce its output. This paper underlines the benefits of using ARX models instead of an analytical model as well as their close connection with convolutive products.

1. Introduction

Thermal processing of ceramic parts at high temperature can be implemented in induction furnaces under vacuum conditions. The furnaces used for this process are equipped with few sensors. The only information required for the control of the furnace during a manufacturing cycle are the control parameters. In our simplified configuration, the furnace has only one heating element. It is possible to study the complete system, from electrical power to temperature in the load and in the heating element.

The aim of this article is to get a reduced model of an induction furnace working under vacuum conditions, starting from the electrical power input and temperature outputs of the simulations that can be given by a detailed model, a Finite Element (FE) code. We have already studied the system with both the detailed Finite Element model and a 0D physical reduced model (white box method). This work has been published in a previous article [1]. The present article is the continuation, as we found that the 0D model was biased, because it could not replicate the outputs of the FE detailed model when one tried to estimate its parameters in a non-linear least square procedure, with some of their estimates taking non-physical values.

So, we have decided to investigate the capacity of ARX models to reproduce the outputs of a reference model that is not the FE model anymore but a 0.5 D model, whose structure will be given further on. The 0.5 D model is simpler than the FE model and it is also more pertinent than the 0D model in the heat transfer configuration of the studied furnace. The literature about physical detailed or reduced modeling of such furnaces is scarce. One can find
a description of a detailed model of a vacuum furnace in [2]. 0D and 1D modeling of a furnace, including radiation, is presented in [3], while a focus of different identification methods for a furnace equipped with burners can be found in [4].

ARX models have been used for very different applications. They have been used to predict temperatures of a slab in a slab reheating furnace [5], temperatures in aircraft powerplant systems [6] or temperatures in a building [7].

ARX (reduced) models have many advantages. We show in this article that ARX models (which are of the grey box type) are related to convolutive models, which have a structure that corresponds to the forced solution of a linear partial differential equation mathematical system with time independent coefficients. Convolutional models represent here transient heat diffusion in each solid subsystem of the furnace and linearized radiation between its surfaces (see: [8]). These convolutional models are based on transfer functions of the corresponding system and are definitely of the white box type since they characterize its forced response, even if explicit analytical forms are not available if the geometry is not 1D. Convolutional models can also be used in experimental inverse input problems [9].

ARX models have also other nice properties: they can model some non-zero initial conditions (relaxation of a non steady temperature field, before initial time here) and they can also accommodate some slight non-linearities. ARX models are useful also for the finality of our project because it also takes in account the bias of experimental measurements as they will be calibrated in production conditions to be used later on in production.

2. Presentation of the furnace

2.1. The furnace configuration

In order to understand the global behaviour of the system, a simplified configuration has been chosen, see figure 1. The furnace operates under vacuum conditions and it has only one inductor. We assume in the whole study that the surface temperature of the inductor is constant and equal to 473 K. The heating element, also known as a susceptor, is heated up by induction phenomena. Thermal radiation exchanges occur between insulator and inductor. Thermocouples are not represented in the figure. They will be considered as measurement points inside the susceptor here. In the heating module, the base of the cylindrical load, a mould here, rests on a cooling copper base, which is called the hearth. The surface temperature of the hearth is 293 K. Thermal contact between load and hearth is modeled by a low exchange coefficient $h_h$ corresponding to poor contact conditions. The load is heated up by radiation from the susceptor. A measurement campaign has been made. It has shown that radiation between load and susceptor can be linearized for the operating conditions considered. The upper and lower closings of the heating chamber are considered as perfect insulators. The system has a rotational symmetry relative to the vertical axis. That is why it will be represented in 2D axisymmetric coordinates here.
2.2. Simplified modeling of heat transfer in the heating module

The present modeling corresponds to the system made up of the load and of the heating element, see a schematic representation in figure 2. The susceptor is considered as a 0D lumped body at temperature $T(t)$ because of a low Biot number (in a first approximation $Bi \approx 0.014$ and a previous detailed modeling has shown that the susceptor has a uniform temperature. The susceptor is heated up by a volumetric power density ($\eta P$) where $P$ is the electrical power consumed by the device and $\eta$ is the induction yield. It exchanges heat with the load through radiation transfer that is linearized here to be written as $(hS_{\text{sus}}(T(t) - \tilde{T}_m(t)))$ with $h$ the linearized radiative exchange coefficient, $S_{\text{sus}}$ the cylindrical inner surface of the susceptor in front of the load at a spatial mean temperature $\tilde{T}_m(t)$. It losses heat towards the outside $(\frac{1}{R_{\text{out}}}(T(t) - T_{\text{out}}))$, where $R_{\text{out}}$ represents the conductive resistance of the insulator and the linearized radiation to the outside environment at a temperature $T_{\text{out}}$. The temperature of the susceptor $T$ is driven by the heat equation (1):

$$\rho CV \frac{dT}{dt}(t) = \eta P - hS_{\text{sus}}(T(t) - \tilde{T}_m(t)) - \frac{1}{R_{\text{out}}}(T(t) - T_{\text{out}})$$

(1)

with $\rho$ the specific mass, $C$ the specific heat and $V$ the volume of the susceptor. $\tilde{T}_m$, the spatial mean temperature of the load has been defined as:

$$\tilde{T}_m(t) = \frac{1}{H} \int_0^H T_m(t, z) \, dz$$

(2)

The load is considered as a thermal fin. We make the assumption that there is no exchange through the inner surface because of the symmetry of the system. There is also no exchange through the upper surface of the load because of the small fin cross-section. The temperature in the load is uniform at each elevation (fin approximation), so the temperature field in the load is defined by $T_m(t, z)$. Only linearized radiative transfer with the susceptor and thermal contact with the cooled hearth are considered. The heat equation of the load (3) is therefore:

$$\frac{\partial^2 T_m(t, z)}{\partial z^2} + \frac{h_m \lambda_m}{S_m} (T(t) - T_m(t, z)) = \frac{1}{a_m} \frac{\partial T_m(t, z)}{\partial t}$$

(3)
with \( \lambda_m \) the thermal conductivity and \( a_m \) the thermal diffusivity of the mould, and

\[
m = 2\pi r_{\text{out}} \quad \text{and} \quad S_m = \pi (r_{\text{out}}^2 - r_{\text{in}}^2)
\]

and the boundary conditions are

\[
\begin{align*}
\lambda_m \frac{\partial T_m}{\partial z}(t, z = 0) &= h_h(T_m(t, z = 0) - T_h) \\
\frac{\partial T_m}{\partial z}(t, z = H) &= 0
\end{align*}
\]  

(4a)  

(4b)

The initial temperature in the load is not uniform:

\[
T_m(t = 0, z) = T_{m,0}(z)
\]  

(5)

Figure 2. Schematic representation of the modeling of the heating module where \( H = 0.276 m \), \( r_{\text{in}} = 0.03 m \) and \( r_{\text{out}} = 0.037 m \). \( T(t) \) is the temperature of the susceptor considered as a 0D solid, \( T_m(t, z) \) is the temperature field in the load, considered as a 1D thermal fin, \( R_{\text{out}} \) is thermal resistance between the susceptor and outside environment at temperature \( T_{\text{out}} \). It integrates the resistance of the insulator. \( h_h \) is the unit area conductance between the basis of the load and the hearth at temperature \( T_h \). \( Q(z, t) \) represents the radiative exchange between load and susceptor that is linearized through a uniform radiative heat transfer coefficient.

Thanks to this modeling, the temperature of the load can be modeled at each vertical location. It allows the construction of virtual sensors that explains the temperature at each height of the load by the temperature of the susceptor.

2.3. Steady state solutions and upgrading of the transient energy equations

2.3.1. Steady state solution

The two heat equations have been given above with their boundary conditions and with their respective initial conditions. All the variables are noted with a subscript \( \cdot_{\text{ss}} \) for the steady state resolution problem. The heat equation (1) of the susceptor becomes:

\[
0 = \eta P_{\text{ss}} - h S_{\text{sus}} (T_{ss} - \bar{T}_{m,ss}) - \frac{1}{R_{\text{out}}}(T_{ss} - T_{\text{out}})
\]  

(6)
and the heat equation for the load (3) becomes:

$$\frac{d^2 T_{m,ss}(z)}{dz^2} + \frac{hm}{\lambda_m S_m} (T_{ss} - T_{m,ss}(z)) = 0$$ (7)

So, the steady state system can be solved with the help of boundary conditions (4a) and (4b) and the temperature ($T_{m,ss}$, $T_{m,ss}$ and $T_{ss}$) solutions can be found. Finally, the temperature of the load is:

$$T_{m,ss}(z) = T_{ss} - h_{h}(T_{ss} - T_{h}) \frac{\cosh(\alpha(H - z))}{h_{h} \cosh(\alpha H) + \lambda_m \alpha \sinh(\alpha H)}$$ (8)

with

$$\alpha = \sqrt{\frac{hm}{\lambda_m S_m}}$$

The expression of the mean spatial temperature of the load is:

$$\bar{T}_{m,ss} = T_{ss} - \frac{1}{H\alpha} (T_{ss} - T_{h}) \frac{\tanh(\alpha H)}{1 + \frac{2\lambda_m}{h_{h}} \tanh(\alpha H)}$$ (9)

And finally, we get the value of the steady state temperature of the susceptor:

$$T_{ss} = \frac{\eta P_{ss} + \frac{T_{out}}{R_{out}} + h_{S_s} \frac{\tanh(\alpha H)}{1 + \frac{2\lambda_m}{h_{h}} \tanh(\alpha H)} T_{h}}{\frac{1}{R_{out}} + h_{S_S} \frac{\tanh(\alpha H)}{1 + \frac{2\lambda_m}{h_{h}} \tanh(\alpha H)}}$$ (10)

2.3.2. Rewriting of the transient equations

The system is considered to be in steady state at initial time. So, the initial state equations can be subtracted from the transient equations. Our objective is getting variables with a zero value at initial time. For this purpose, we subtract equation (6) from equation (1):

$$\rho CV \frac{d\theta}{dt}(t) = \eta P' - h_{S_{sus}}(\theta(t) - \bar{\theta}_m(t)) - \frac{1}{R_{out}} \theta(t)$$ (11)

with

$$\theta(t) = T(t) - T_{ss}, \bar{\theta}_m(t) = \bar{T}_m(t) - \bar{T}_{m,ss}, \theta_m(z,t) = T_m(z,t) - T_{m,ss}(z)$$ and $P'(t) = P(t) - P_{ss}$

The heat equation of the load becomes

$$\frac{\partial^2 \theta_m(z,t)}{\partial z^2} + \frac{hm}{\lambda_m S_m} (\theta(t) - \theta_m(z,t)) = \frac{1}{\alpha_m} \frac{\partial \theta_m(z,t)}{\partial t}$$ (12)

and the associated boundary conditions are

$$\lambda_m \frac{\partial \theta_m}{\partial z}(z = 0,t) = h_{h} \theta_m(z = 0,t)$$ (13a)

$$\frac{\partial \theta_m}{\partial z}(z = H,t) = 0$$ (13b)

The two heat equations are solved through integral transformations in the Laplace domain. Equation (11) is transformed in the Laplace domain as

$$(\rho CV s + h_{S_{sus}} + \frac{1}{R_{out}}) \bar{\theta} = \eta \bar{P}' + h_{S_{sus}} \bar{\theta}_m$$ (14)

where the upper bar notation designates the Laplace transform of the corresponding quantity and where $s$ is the Laplace parameter. Equation (12) is transformed in Laplace domain as

$$\frac{\partial^2 \bar{\theta}_m}{\partial z^2}(z,s) - (\frac{s}{\alpha_m} + \frac{hm}{\lambda_m S_m}) \bar{\theta}_m(z,s) + \frac{hm}{\lambda_m S_m} \bar{\theta}(s) = 0$$ (15)
2.4. Output from the physical model

In the Laplace domain, the relative load temperatures \( \overline{\theta_m}(z, s) \) and relative susceptor temperature \( \overline{\theta} \) are function of the relative input \( \overline{P} \). The heat equation of the load (15) is solved through the thermal quadrupoles method [10]. The temperature of the load is then expressed as a function of the susceptor temperature with a transfer function \( W_z \) called a transmissivity here:

\[
\overline{\theta_m}(z, s) = \overline{W_z}(s)\overline{\theta}(s)
\]

where

\[
\overline{W_z}(s) = \frac{hm}{\lambda_m S_m \gamma^2} \left[ 1 - \frac{h_m \cosh(\gamma (H - z))}{h_m \cosh(\alpha H) + \lambda_m \gamma \sinh(\gamma H)} \right]
\]

and

\[
\gamma = \sqrt{\frac{s}{\delta_m} + \frac{hm}{\lambda_m s_m}}
\]

The mean temperature of the load is obtained by integration of the previous expression of \( \overline{W_z} \):

\[
\overline{\theta_m}(s) = \overline{W}(s)\overline{\theta}(s) = \frac{hm}{\lambda_m S_m \gamma^2} \left[ 1 - \frac{1}{H \gamma} \frac{\tanh(\gamma H)}{1 + \frac{\lambda_m}{h_m} \tanh(\gamma H)} \right] \overline{\theta}(s)
\]

Finally, equations (14) and (17) are combined to get the expression of the susceptor temperature that is a function of the relative heating power and of impedance \( Z \):

\[
\overline{\theta}(s) = \overline{Z}(s)\overline{P}(s) = \frac{1}{\rho CV_s + \frac{1}{R_{\text{out}}} + h S_{\text{sus}}(1 - \overline{W}(s))} \overline{P}(s)
\]

The load temperature is a function of the relative heating power and of the associated impedance \( \overline{Z_m}(z, s) \).

\[
\overline{\theta_m}(z, s) = \overline{Z_m}(z, s)\overline{P}(s)
\]

Equations (16) and (18) yield the following relationship between these impedances and the transmissivity:

\[
\overline{W_z}(s) = \frac{\overline{Z_m}(z, s)}{\overline{Z}(s)}
\]

Our aim is to obtain the impulse responses, that is the original of the transfer functions in the time domain, so we apply an inverse Laplace transformation to the transmissivity \( \overline{W_z}(s) \) and to impedances \( \overline{Z}(s) \) and \( \overline{Z_m}(z, s) \) using a numerical inversion algorithm [11]. Since a product in the Laplace domain corresponds to a convolution product in the time domain, the temperature variation of the susceptor or at each point in the load in the time domain is a convolution product between the corresponding impulse response (a time impedance or a time transmissivity) and the relative power or the variation of temperature of the susceptor:

\[
\theta_m(z, t) = W_z(t) * \theta(t), \quad \theta(t) = Z(t) * \overline{P}(s), \quad \theta_m(z, t) = Z_m(z, t) * \overline{P}(s)
\]

where * stands for this convolution product. This can be written in continuous time or in discrete time, for \( u(t) \) the input, \( H(t) \) the impulse response and \( y(t) \) the output:

\[
y(t) = H(t) * u(t) = \int_0^t H(t - v) u(v) \, dv \Rightarrow y_i = y(t_i) = \Delta t \sum_{j=1}^{i} H_{i-j+1} u_j
\]

with \( H_j = H(t_j) \), \( u_j = u(t_j) \) and \( t_j = j \Delta t \) for \( j = 1 \) to \( i \).
2.5. ARX modeling

An AutoRegressive with eXogenous input model (ARX) has the structure presented in Eq. (23). The solution \( y(i) \) is a combination of the previous outputs, its previous input values and of a white noise \( \epsilon(i) \). This type of ARX model, which belongs to what is called a "grey box" model, has been extensively studied by Ljung [12].

\[
y(i) = -\sum_{j=1}^{n_a} a_j y(i-j) + \sum_{j=1}^{n_b} b_j e(i-j-n_d) + \epsilon(i)
\]  

(23)

Here \( y(i) \) and \( e(i) \) are respectively the output and the input at time \( t_i \). The order of this model is defined by the triplet \( (n_a, n_b, n_d) \). In this paper, we will not delay the input, so \( n_d = 0 \) and there will be no stochastic zero mean input so \( \epsilon(i) = 0 \). The output \( y \) and the input \( e \) correspond respectively to the load temperature \( \theta_m \) and to the susceptor temperature \( \theta \). Equation (23) can also be written in terms of matrices with \( A \) and \( B \) lower triangular matrices of size \( (n,n) \), where \( n \) is the number of observations, as

\[
Ay = Be
\]  

(24)

where

\[
A = \begin{pmatrix}
1 & 0 & \cdots & \cdots & \cdots & 0 \\
a_1 & \ddots & & & & \\
& \ddots & \ddots & & & \\
& & \ddots & \ddots & & \\
0 & a_1 & \cdots & & & 1 \\
& & \cdots & \ddots & \ddots & \\
& & & \ddots & \ddots & \ddots \\
& & & & \ddots & \ddots \\
& & & & & \ddots \\
0 & \cdots & a_{n_a} & \cdots & a_1 & 1
\end{pmatrix}
\]

and

\[
B = \begin{pmatrix}
b_1 & 0 & \cdots & \cdots & \cdots & 0 \\
& \ddots & & & & \\
& & \ddots & \ddots & & \\
& & & \ddots & \ddots & & \\
& & & & \ddots & \ddots \\
& & & & & \ddots \\
& & & & & \ddots \\
& & & & & b_1 \\
0 & \cdots & & \cdots & b_{n_b} & \cdots & \\
& & \cdots & \ddots & & \\
& & & \ddots & \ddots & \ddots \\
& & & & \ddots & \ddots \\
& & & & & \ddots & \ddots \\
0 & \cdots & & \cdots & b_{n_b} & \cdots & 1
\end{pmatrix}
\]

with \( y = [y_1 \ y_2 \ \cdots \ y_n]^T \) and \( e = [e_1 \ e_2 \ \cdots \ e_n]^T \).

The parameters \( a_i \) and \( b_i \) have to be estimated. We need two different input/output sets. The first set is called the calibration set and the second one the validation set. The calibration couple is used to estimate \( a_i \) and \( b_i \) as they are the unknowns of the least-squares estimation problem of system (23). These are estimated with a QR factorization in Matlab\(^\text{®} \). After this calibration, the estimated parameters and the corresponding \( (n_a, n_b, 0) \) ARX structure are validated with the validation set. The model expressed in equation (24), can be written in the following form

\[
y = Ce = A^{-1}Be
\]  

(25)

We can notice that if \( n_a \) is null and \( n_b = n \) then the ARX model is similar to a convolutive model presented in section 2.4.

Once the ARX model and the analytic physical model defined, we are now going to see the behaviour of both of them and their ability to fit each other.

2.6. Comparison of the double convolutive analytical model with the ARX model

Using the results obtained in part 2.4, equations (21a) and (21b) can be combined to eliminate \( P'(t) \). They can be written in terms of lower triangular Toeplitz matrices:

\[
M(Z(t))\theta_m(z,t) = M(Z_m(z,t))\theta(t)
\]  

(26)
where for example $M(Z(t))$ is equal to:

$$M(Z(t)) = \begin{pmatrix} Z_1 & 0 & \cdots & 0 \\ Z_2 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ Z_n & \cdots & \cdots & Z_1 \end{pmatrix}$$  \hspace{1cm} (27)$$

From equation (26), the expression of the load temperature for each time $t_i$ (for $i = 2$ to $n$) can be written as a function of the previous outputs and the previous input values including the current one:

$$\theta_m(z,i) = -\sum_{j=1}^{i-1} a_j \theta_m(z,i-j) + \sum_{j=0}^{i-1} b_j \theta(i-j)$$  \hspace{1cm} (28)$$

In fact, we can define $a_j$ and $b_j$ coefficients as

$$a_j = \frac{Z_{j+i}}{Z_1} \text{ for } (j = 1 \text{ to } (i - 1)) \text{ and } b_j = \frac{Z_{m,j+i}}{Z_1} \text{ (} j = 0 \text{ to } (i - 1))$$

The analogy with the ARX definition (23) is therefore clear, if $b_0$ is null. Let us notice that if $n$ observations of $\theta_m$ and $\theta$ are available, a higher number $(2n - 1)$ of coefficients $a$ or $b$ are present in equation (23). So it is impossible to estimate all of them. The maximum number of parameters that can be identified is $n_a + n_b = n$.

3. Results for one heating power input
3.1. Outputs from the analytic model

Here, the simulation starts from an equilibrium state at 1173 K for the susceptor and the load has a non uniform temperature. The temperature distribution in the load is presented in figure 3. The gradient is due to the cooled hearth. The gradient is high in the first millimeters because of the low conductivity of the load.

![Figure 3. Temperature of the load function of z for a susceptor in a steady state at 1173 K](image)

From this initial equilibrium, we are going to apply two profiles of relative electrical power. They are given in figure 4. The first one is the calibration input, that consists of different steps.
to get as much information as possible. The second input profile is the validation input. The relative electrical power can be negative because we built the model for $P'(t) = P(t) - P_{\text{init}}$. So, if the electrical power is stopped, we get $P'(t) = -P_{\text{init}} < 0$.

![Figure 4](image_url)

**Figure 4.** Profiles of electrical power as inputs for model calibration and validation

The analytical temperature responses for these inputs are presented in figure 5. We can clearly see the gradient in the first millimeter of the load, $T_m(0.0029, t) < T_m(0.0145, t)$. The load temperature is stable for higher elevations, and $T_m(0.087, t)$ and $T_m(0.276, t)$ almost overlap. We can also give the time profiles of the transmissivity $W_{0.087}(t)\Delta t$ at $z = 0.087m$ and the impedances $Z(t)\Delta t$, $Z_m(0.087, t)\Delta t$, for $\Delta t = 10s$, see figure 6. We can notice that $Z(t)\Delta t$ starts at a non-zero value because the susceptor is a lumped body and it reacts instantly to the heating power. On the contrary $Z_m(0.087, t)\Delta t$ starts at a zero value because the load has a non-null time response.

From the definition of $a_j$ and $b_j$ coefficients in part 2.6 and the value of the impedances, the $a_j$ and $b_j$ coefficients are estimated and presented in figure 7. Their values result from the ordinary least square estimate of the following model:

$$y = D(y)\beta$$

where

$$\beta = \begin{bmatrix} a_1 \\ \vdots \\ a_n \\ b_1 \\ \vdots \\ b_m \end{bmatrix}$$

and

$$y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$$

(29)

where $n$ observations $y_i$ have been made for the output and where sensitivity matrix $D$ can be derived from (23) and depends on $y$. 

3.2. ARX results

3.2.1. Study without noise  Different orders of ARX are studied in this section. We focus here on autoregressive models with $n_a = n_b$ and convolutive models with $n_a = 0$. The identification results, corresponding to inversion of equation (29) are presented in table 1 for the calibration and validation phases in terms of the square mean root estimation error ($E_{RMS}$) with

$$E_{RMS} = \|\theta_m(z, t) - y\| = \sqrt{\frac{1}{n} \sum_{j=1}^{n} (\theta_m(z, j) - y(j))^2}$$  \hspace{1cm} (30)

where $y$ is the output from the ARX model. This ARX model provides excellent fits even for small orders like (1,1,0). Table 1 shows clearly the interest of ARX models since the number of parameters to estimate is much smaller for an autoregressive model than for a convolutive model in order to get an equivalent result in terms of $E_{RMS}$.

The parameters $a_j$ and $b_j$ values are presented, as a function of the their order $j$, for different ARX orders ($n_a, n_b, 0$), in figure 8. For the convolutive problem, the parameters $b_j$ seem to converge to a stable solution when the ARX order increase. In the case of an autoregressive model, the $a_j$’s and $b_j$’s coefficients oscillate and do not seem to vary in a continuous way with their order, in contrast with their values derived from the analytical model shown in figure 7.

In fact, the $b_j$ coefficients of the convolutive model and the matrix coefficients $c_j$ of the $C$ in equation (25) converge to the value of $W_{0.087}(t)\Delta t$ when their maximum order is increased, see figure 9.
Figure 6. Time profiles of the different transfer functions multiplied by the time step $W_{0.087}(t)\Delta t$, $Z(t)\Delta t$ and $Z_m(0.087,t)\Delta t$

Figure 7. Profiles of $a_j$ and $b_j$ coefficients estimated from the analytical model, equation (28)

3.2.2. Study with noise  The addition of a white noise with a 5 K standard deviation, which is the worst case in practice, does not impact the results. In this case we use Truncated Singular Values Decomposition (TSVD) to estimate the $n_a + n_b$ coefficients, if this number is large with, as a consequence, an ill-posed inversion of (29). This is made using Morozov’s discrepancy principle. The results for the calibration and the validation phases are presented in figure 10. The $E_{RMS}$ for the calibration phase is 6.48 K and 6.39 K for the validation phase which is equivalent to the standard deviation.
Table 1. Ability of the ARX model to fit the calibration and validation sequences at \( z = 0.087m \) in the load

| ARX order   | Calibration (\( E_{RMS} \) in K) | Validation (\( E_{RMS} \) in K) |
|-------------|----------------------------------|----------------------------------|
| Autoregressive models |
| (1,1,0)               | 1.73 \( 10^{-6} \)               | 2.52 \( 10^{-6} \)               |
| (4,4,0)               | 2.38 \( 10^{-7} \)               | 3.35 \( 10^{-7} \)               |
| (50,50,0)             | 9.14 \( 10^{-10} \)              | 1.32 \( 10^{-9} \)               |
| Convolutive models   |
| (0,2,0)               | 1.20                             | 1.71                             |
| (0,8,0)               | 0.36                             | 0.52                             |
| (0,70,0)              | 2.03 \( 10^{-6} \)               | 2.88 \( 10^{-6} \)               |
| (0,100,0)             | 4.65 \( 10^{-9} \)               | 6.64 \( 10^{-9} \)               |

Figure 8. Estimated values of the parameters \( a_j \) and \( b_j \) function of the order \( j \) for different ARX order \((n_a,n_b,0)\)

4. Conclusion and perspectives
The ARX model has shown its capacity to fit the analytical model even with a significant white noise and it has been validated in a synthetic experiment different from the calibration one (with noise). The transfer functions of the analytic model based on a double convolution product (a “white box” model) can be recovered from the ARX coefficients \( a_j \) and \( b_j \). Furthermore, the autoregressive models are efficient to represent the analytic model with a fewer number of parameters than the convolutive model.

The next step will be to explain experimental data for a more complex system with several susceptors at different temperatures with the help of ARX models.
Figure 9. Estimated values of the $b_j$ coefficients for ARX order $(0,100,0)$, estimated values of the $c_j$ coefficients for ARX order $(4,4,0)$ and the transmissivity $W(0.087,j) \times dt$ value function of the occurrence $j$.

Figure 10. Time Profiles of $\theta_m(z = 0.087,t)$ and corresponding ARX outputs for the calibration and validation phases.

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