EMBEDDING OF THE RANK 1 DAHA INTO $\text{Mat}(2, \mathbb{T}_q)$ AND ITS AUTOMORPHISMS.

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Abstract. In this review paper we show how the Cherednik algebra of type $\hat{C}_1 C_1$ appears naturally as quantisation of the group algebra of the monodromy group associated to the sixth Painlevé equation. This fact naturally leads to an embedding of the Cherednik algebra of type $\hat{C}_1 C_1$ into $\text{Mat}(2, \mathbb{T}_q)$, i.e. $2 \times 2$ matrices with entries in the quantum torus. For $q = 1$ this result is equivalent to say that the Cherednik algebra of type $\hat{C}_1 C_1$ is Azumaya of degree $2$ [31]. By quantising the action of the braid group and of the Okamoto transformations on the monodromy group associated to the sixth Painlevé equation we study the automorphisms of the Cherednik algebra of type $\hat{C}_1 C_1$ and conjecture the existence of a new automorphism. Inspired by the confluences of the Painlevé equations, we produce similar embeddings for the confluent Cherednik algebras $\mathcal{H}_V, \mathcal{H}_{IV}, \mathcal{H}_{III}, \mathcal{H}_{II}$ and $\mathcal{H}_I$, defined in [27].

For the 60th birthday of Masatoshi Noumi

1. Introduction

The Painlevé sixth equation [16, 33, 17] describes the monodromy preserving deformations of a rank 2 Fuchsian system with four simple poles $a_1, a_2, a_3$ and $\infty$. The solution of this Fuchsian system is in general a multi-valued analytic vector–function in the punctured Riemann sphere $\mathbb{P}^1 \setminus \{a_1, a_2, a_3, \infty\}$ and its multivaluedness is described by the so-called monodromy group, i.e. a subgroup of $\text{SL}_2(\mathbb{C})$ generated by the images $M_1, M_2, M_3$ of the generators of the fundamental group under the anti-homomorphism:

$$\rho : \pi_1 (\mathbb{P}^1 \setminus \{a_1, a_2, a_3, \infty\}, \lambda_0) \to \text{SL}_2(\mathbb{C}).$$

In this paper, we introduce flat coordinates on a large open sub-set of the set of all possible monodromy groups obtained in this way (see Theorem 3.1). We then obtain a quantisation of the group algebra of the monodromy group by introducing a canonical quantisation for these flat coordinates. This quantum algebra is isomorphic to the Cherednik algebra of type $\hat{C}_1 C_1$, i.e. the algebra $\mathcal{H}$ generated by four elements $V_0, V_1, \hat{V}_0, \hat{V}_1$ which satisfy the following relations [17, 32, 30, 34]:

\begin{align}
(1.1) & \quad (V_0 - k_0)(V_0 + k_0^{-1}) = 0 \\
(1.2) & \quad (V_1 - k_1)(V_1 + k_1^{-1}) = 0 \\
(1.3) & \quad (V_0 - u_0)(V_0 + u_0^{-1}) = 0 \\
(1.4) & \quad (\hat{V}_1 - u_1)(\hat{V}_1 + u_1^{-1}) = 0 \\
(1.5) & \quad \hat{V}_1 V_1 V_0 \hat{V}_0 = q^{-1/2},
\end{align}

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where $k_0, k_1, u_0, u_1, q \in \mathbb{C}^*$, such that $q^m \neq 1$, $m \in \mathbb{Z}_{>0}$.

As a consequence we obtain an embedding of the Cherednik algebra of type $\tilde{C}_1C_1$ into $Mat(2, \mathbb{T}_q)$, i.e. $2 \times 2$ matrices with entries in the quantum torus:

**Theorem 1.1.** The map:

\[
V_0 \to \begin{pmatrix} k_0 - k_0^{-1} - ie^{-s_3} & -ie^{-s_3} \\ k_0^{-1} - k_0 + ie^{-s_3} + ie^{s_3} & ie^{s_3} \end{pmatrix}
\]

\[
V_1 \to \begin{pmatrix} k_1 - k_1^{-1} - ie^{s_2} & k_1 - k_1^{-1} - ie^{-s_2} - ie^{s_2} \\ ie^{s_2} & ie^{s_2} \end{pmatrix}
\]

\[
\tilde{V}_1 \to \begin{pmatrix} 0 & -ie^{s_1} \\ ie^{-s_1} & u_1 - u_1^{-1} \end{pmatrix}
\]

\[
\tilde{V}_0 \to \begin{pmatrix} u_0 & 0 \\ q^2s & -\frac{1}{u_0} \end{pmatrix},
\]

where $S_1, S_2, S_3$ satisfy the following commutation relations:

\[
[S_1, S_2] = [S_2, S_3] = [S_3, S_1] = i\pi \hbar, \quad u_0 = -ie^{-s_1} - s_2 - s_1,
\]

for $q = e^{i\pi \hbar}$ and

\[
s = k_0e^{-s_1} - s_2 + k_1e^{-s_1 + s_3} + u_1e^{s_2 + s_3} + ie^{-s_1} - s_3 + ie^{s_1} - s_3 + ie^{-s_1} + s_2 + s_3 - u_0,
\]

gives an embedding of $\mathcal{H}$ into $Mat(2, \mathbb{T}_q)$. In particular, the images of $V_0, V_0, V_1, \tilde{V}_1$ in $GL(2, \mathbb{T}_q)$ satisfy the relations (1.7) . . . (1.7) and (1.7), in the quantum ordering is dictated by the matrix product ordering $\star$.

Note that this result was already proved in [24] (using a different presentation for $\mathcal{H}$). The purpose of the current paper is to explain this result in the Painlevé context and to draw parallels between the theory of the Painlevé equations and the theory of the Cherednik algebra of type $\tilde{C}_1C_1$.

In particular, we prove that all the known automorphisms of the Cherednik algebra of type $\tilde{C}_1C_1$ are a quantisation of the action of the braid group on monodromy matrices proposed in [9, 26] to describe the analytic continuation of the solutions to the sixth Painlevé equation.

Next we deal with the Okamoto transformations of the sixth Painlevé equation and their action on the monodromy group. By quantisation we conjecture the existence of an automorphism of the Cherednik algebra of type $\tilde{C}_1C_1$ which acts as follows on the parameters:

\[
(u_1, u_0, k_1, k_0) \to \left( \frac{u_1}{\sqrt{u_1u_0k_1k_0}}, \frac{u_0}{\sqrt{u_1u_0k_1k_0}}, \frac{k_1}{\sqrt{u_1u_0k_1k_0}}, \frac{k_0}{\sqrt{u_1u_0k_1k_0}} \right)
\]

We postpone the computation of the action this automorphism on $V_0, V_1, \tilde{V}_0, \tilde{V}_1$ to a subsequent publication.

Finally, in [27], the author introduced confluent versions of the Cherednik algebra of type $\tilde{C}_1C_1$ by using a concatenation of Whittaker-type limits similar to those introduced in [8]. In this paper we explain the origin of these confluent Cherednik algebras from the point of view of the Painlevé theory. In [14] the confluence scheme

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1By this we mean that the product $AB$ of two matrices $A, B$ whose entries are in $\mathbb{T}_q$ is computed by keeping the entries of $A$ on the left matrix of the entries of $B$. 

[3.7]
Embedding of the rank 1 DAHA into Mat\((2, \mathbb{T}_q)\) and its automorphisms.

of the Painlevé differential equations was explained in terms of certain geometric operations giving rise to specific asymptotic limits in the classical coordinates \(s_1, s_2, s_3\) and parameters. Here, we quantise these asymptotic limits to obtain asymptotic limits for the quantum coordinates \(S_1, S_2, S_3\) and of the parameters \(k_0, k_1, u_0, u_1\) (see Fig. 1). By taking these limits in the matrices (1.6, ..., 1.9), we produce new matrices which turn out to provide embeddings for the confluent Cherednik algebras \(H_{\mathcal{V}}, H_{\mathcal{IV}}, H_{\mathcal{II}}, H_{\mathcal{II}}\) and \(H_{\mathcal{I}}\) in Mat\((2, \mathbb{T}_q)\).

Figure 1. The [6] confluence scheme for the Painlevé equations denoted here by \(PVI, PV, PIV, PIII, PIII_{D_7}, PIII_{D_8}, PII, PI\) and the corresponding rescaling of the quantum shifted shear co-ordinates \(S_1, S_2, S_3\) such that \(\lim_{\hbar \to 0} S_i = s_i + \frac{p_i}{2}, i = 1, 2, 3\).

2. Flat coordinates for the monodromy group of the sixth Painlevé equation

2.1. Sixth Painlevé equation as isomonodromic deformation equation.

We start by recalling without proof some very well known facts about the sixth Painlevé equation and its relation to the monodromy preserving deformations equations [22, 29].

The sixth Painlevé equation [16, 33, 17],

\[
y_{tt} = \frac{1}{2} \left( \frac{1}{y} + \frac{1}{y - 1} + \frac{1}{y - t} \right) y^2 - \left( \frac{1}{t} + \frac{1}{t - 1} + \frac{1}{y - t} \right) y_t + \frac{y(y-1)(y-t)}{t^2(t-1)^2} \left[ \alpha + \beta \frac{t}{y^2} + \gamma \frac{t-1}{(y-1)^2} + \delta \frac{t(t-1)}{(y-t)^2} \right],
\]

(2.11)

describes the monodromy preserving deformations of a rank 2 meromorphic connection over \(\mathbb{P}^1\) with four simple poles \(a_1, a_2, a_3\) and \(\infty\) (for example we may choose \(a_1 = 0, a_2 = t, a_3 = 1\)):

\[
\frac{d\Phi}{d\lambda} = \sum_{k=1}^{3} \frac{A_k(t)}{\lambda - a_k} \Phi,
\]

(2.12)

where

\[
eigen(A_i) = \pm \frac{\theta_i}{2}, \quad \text{for } i = 1, 2, 3, \quad A_{\infty} := -A_1 - A_2 - A_3
\]

\(\text{For simplicity sake, we are recalling here the main facts about the isomonodromic approach in the case when the parameters } \theta_1, \theta_2, \theta_3 \text{ and } \theta_{\infty} \text{ are not integers. This is just a technical restriction, all the results proved in the paper are actually valid also when we lift such restriction.} \)
(2.14) \[ A_\infty = \begin{pmatrix} \frac{\theta_3}{2} & 0 \\ 0 & -\frac{\theta_3}{2} \end{pmatrix}, \]

and the parameters \( \theta_i, \ i = 1, 2, 3, \infty \) are related to the PVI parameters by

\[ \alpha = \frac{(\theta_3 - 1)2}{2}, \quad \beta = -\frac{\theta_3^2}{2}, \quad \gamma = \frac{\theta_3^2}{2}, \quad \delta = 1 - \frac{\theta_3^2}{2}. \]

The precise dependence of the matrices \( A_1, A_2, A_3 \) on the PVI solution \( y(t) \) and its first derivative \( y_1(t) \) can be found in [29].

The solution \( \Phi(\lambda) \) of the system (2.12) is a multi-valued analytic function in the punctured Riemann sphere \( \mathbb{P}^1 \setminus \{a_1, a_2, a_3, \infty\} \) and its multivaluedness is described by the so-called monodromy matrices, i.e. the images of the generators of the fundamental group under the anti-homomorphism

\[ \rho : \pi_1(\mathbb{P}^1 \setminus \{a_1, a_2, a_3, \infty\}, \lambda_0) \to GL_2(\mathbb{C}). \]

In this paper we fix the base point \( \lambda_0 \) at infinity and the generators of the fundamental group to be \( l_1, l_2, l_3 \), where each \( l_i, \ i = 1, 2, 3, \) encircles only the pole \( a_i \) once and \( l_1, l_2, l_3 \) are oriented in such a way that

(2.15) \[ M_1M_2M_3M_\infty = 1, \]

where \( M_\infty = \exp(2\pi i A_\infty) \).

2.2. Riemann-Hilbert correspondence and monodromy manifold. Let us denote by \( \mathcal{F}(\theta_1, \theta_2, \theta_3, \theta_\infty) \) the moduli space of rank 2 meromorphic connection over \( \mathbb{P}^1 \) with four simple poles \( a_1, a_2, a_3, \infty \) of the form (2.12). Let \( \mathcal{M}(G_1, G_2, G_3, G_\infty) \) denote the moduli of monodromy representations \( \rho \) up to Jordan equivalence, with the local monodromy data of \( G_i \)'s prescribed by

\[ G_i := \Tr(M_i) = 2 \cos(\pi \theta_i), \quad i = 1, 2, 3, \infty. \]

Then the Riemann-Hilbert correspondence

\[ \mathcal{F}(\theta_1, \theta_2, \theta_3, \theta_\infty)/\Gamma \to \mathcal{M}(G_1, G_2, G_3, G_\infty)/GL_2(\mathbb{C}), \]

where \( \Gamma \) is the gauge group [2], is defined by associating to each Fuchsian system its monodromy representation class. The representation space \( \mathcal{M}(G_1, G_2, G_3, G_\infty)/GL_2(\mathbb{C}) \) is realised as an affine cubic surface (see [21])

(2.16) \[ G_{12}^2 + G_{23}^2 + G_{31}^2 + G_{12}G_{23}G_{31} - \omega_3G_{12} - \omega_1G_{23} - \omega_2G_{31} + \omega_\infty = 0, \]

where \( G_{12}, G_{23}, G_{31} \) defined as:

\[ G_{ij} = \Tr(M_iM_j), \quad i, j = 1, 2, 3, \]

and

\[ \omega_{ij} := G_iG_j + G_kG_\infty, \quad k \neq i, j, \quad \omega_\infty = G_0^2 + G_1^2 + G_\infty^2 + G_0G_1G_\infty - 4. \]

This cubic surface is called monodromy manifold of the sixth Painlevé equation and it is equipped with the following Poisson bracket:

\[ \{G_{12}, G_{23}\} = G_{12}G_{23} + 2G_{31} - \omega_2 \]

\[ \{G_{23}, G_{31}\} = G_{23}G_{31} + 2G_{12} - \omega_3 \]

\[ \{G_{31}, G_{12}\} = G_{31}G_{12} + 2G_{23} - \omega_1 \]

In [20], Iwasaki proved that the triple \( (G_{12}, G_{23}, G_{31}) \) satisfying the cubic relation (2.16) provides a set of coordinates on a large open subset \( S \subset \mathcal{M}(G_1, G_2, G_3, G_\infty) \). In the following sub-section, we restrict to such open set.
2.3. Teichmüller theory of the 4-holed Riemann sphere. The real slice of moduli space \( \mathcal{F}(\theta_1, \theta_2, \theta_3, \theta_\infty) \) of rank 2 meromorphic connections over \( \mathbb{P}^1 \) with four simple poles \( a_1, a_2, a_3, \infty \) can be obtained as a quotient of the Teichmüller space of the 4–holed Riemann sphere by the mapping class group. This fact allowed us to use the combinatorial description of the Teichmüller space of the 4–holed Riemann sphere in terms of fat-graphs to produce a good parameterisation of the monodromy manifold of PVI [4]. In this sub-section we recall the main ingredients of this construction.

We recall that according to Fock [13] [14], the fat graph associated to a Riemann surface \( \Sigma_{g,n} \) of genus \( g \) and with \( n \) holes is a connected three–valent graph drawn without self-intersections on \( \Sigma_{g,n} \) with a prescribed cyclic ordering of labelled edges entering each vertex; it must be a maximal graph in the sense that its complement on the Riemann surface is a set of disjoint polygons (faces), each polygon containing exactly one hole (and becoming simply connected after gluing this hole). In the case of a Riemann sphere \( \Sigma_{0,4} \) with 4 holes, the fat–graph is represented in Fig.2 (the fourth hole is the outside of the graph).

The geodesic length functions, which are traces of hyperbolic elements in the Fuchsian group \( \Delta_{g,n} \) such that in Poincaré uniformisation:

\[
\Sigma_{g,n} \sim \mathbb{H}/\Delta_{g,n},
\]

are obtained by decomposing each hyperbolic matrix \( \gamma \in \Delta_{g,n} \) into a product of the so–called right, left and edge matrices: [13] [14]

\[
R := \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}, \quad L := \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}, \quad E_{s_i} := \begin{pmatrix} 0 & -\exp\left(-\frac{s_i^2}{2}\right) \\ \exp\left(-\frac{s_i^2}{2}\right) & 0 \end{pmatrix}.
\]

Let us consider the closed geodesics \( \gamma_{ij} \) encircling the i-th and j-th holes without self intersections (for example \( \gamma_{12} \) is drawn in Fig.1), then their geodesic length functions can be obtained as:

\[
G_{23} = -\text{Tr}(RE_{s_2}RE_{p_2}RE_{s_2}RE_{s_3}RE_{p_3}RE_{s_3}R),
\]

\[
G_{31} = -\text{Tr}(LE_{s_3}RE_{p_3}RE_{s_3}RE_{s_1}RE_{p_1}RE_{s_1}),
\]

\[
G_{12} = -\text{Tr}(E_{s_1}RE_{p_1}RE_{s_1}RE_{s_2}RE_{p_2}RE_{s_2}L),
\]

(2.18)
which leads to

\[
\begin{align*}
G_{23} &= -e^{s_2+s_3} - e^{-s_2-s_3} - e^{-s_2+s_3} - G_2 e^{s_3} - G_3 e^{-s_2} \\
G_{31} &= -e^{s_3+s_1} - e^{-s_3-s_1} - e^{-s_3+s_1} - G_3 e^{s_1} - G_1 e^{-s_3}, \\
G_{12} &= -e^{s_1+s_2} - e^{-s_1-s_2} - e^{-s_1+s_2} - G_1 e^{s_2} - G_2 e^{-s_1},
\end{align*}
\]

(2.19)

where

\[
G_i = e^{\pi i} + e^{-\pi i}, \quad i = 1, 2, 3,
\]

and

\[
G_{30} = e^{s_1+s_2+s_3} + e^{-s_1-s_2-s_3}.
\]

Due to the classical result that each conjugacy class in the fundamental group \(\mathbb{P}^1 \setminus \{a_1, a_2, a_3, \infty\}\) can be represented by a unique closed geodesic, we can make the following identification:

\[
G_{ij} := \text{Tr} (M_i M_j),
\]

(2.20)

and indeed it is a straightforward computation to show that \(G_{12}, G_{23}, G_{31}\) defined as in (2.19) indeed lie on the cubic (2.16).

Moreover, the Poisson algebra structure (2.17) is induced by the Poisson algebras of geodesic length functions constructed in [3] by postulating the Poisson relations on the level of the shear coordinates \(s_\alpha\) of the Teichmüller space. In our case these are:

\[
\{s_1, s_2\} = \{s_2, s_3\} = \{s_3, s_1\} = 1,
\]

while the perimeters \(p_1, p_2, p_3\) are assumed to be Casimirs.

Since the parameterisation (2.19) is analytic in \(s_1, s_2, s_3\), we can complexify \(s_1, s_2, s_3\) and \(G_1, G_2, G_3, G_{30}\) to claim that \(s_1, s_2, s_3 \in \mathbb{C}\) provide a system of flat coordinates on the Fricke cubic (2.19).

2.4. Parameterisation of the monodromy group. In the case of monodromy matrices, Korotkin and Samtleben in [25] proposed an \(r\)-matrix structure of the Fock–Rosly type [15] which did not however satisfy Jacobi relations on monodromy matrices themselves but became consistent on the level of adjoint invariant elements. Therefore the problem of quantising the monodromy group remained open.

In this section we show that thanks to the identification (2.20), we can impose

\[
\begin{align*}
M_1 &= E_{s_1} R E_{p_1} R E_{s_1}, \\
M_2 &= -R E_{s_2} R E_{p_2} R E_{s_2} L, \\
M_3 &= -L E_{s_3} R E_{p_3} R E_{s_3} R,
\end{align*}
\]

(2.21)

so that \(s_1, s_2, s_3 \in \mathbb{C}\) provide a system of flat coordinates on the monodromy group, rather than only the monodromy manifold. This will allow us to quantise as we shall see in subsection 3.

**Theorem 2.1.** Given any quadruple of diagonalisable matrices \(M_1, M_2, M_3, M_\infty \in SL_2(\mathbb{C})\), such that \(M_1 M_2 M_3 M_\infty = I\), the group \(\langle M_1, M_2, M_3 \rangle\) is irreducible and none of the matrices \(M_1, M_2, M_3, M_\infty\) is a multiple of the identity, we can find \(s_1, s_2, s_3, p_1, p_2, p_3 \in \mathbb{C}\) such that the following relations hold true (up to global conjugation and cyclic permutation):

\[
M_1 = \begin{pmatrix}
0 & -e^{s_1} \\
e^{-s_1} & e^{s_1} - e^{-p_1} - e^{-p_2}
\end{pmatrix},
\]

3Note that for simplicity we have actually shifted the shear coordinates \(s_i \rightarrow s_i + \frac{\pi i}{2}, i = 1, 2, 3\).
and its quantum version is defined as
\[ \text{U} \]
responding to geodesics going around exactly two holes was constructed and it was
satisfactory to replace those exponents by their quantum version,
\[ \text{G} \]
while the central elements, i.e. perimeters remain non-deformed.
\[ (3.22) \]
In this parameterisation
\[ \text{eigen}(M_j) = -e^{\pm \frac{\pi}{2}}, \quad j = 1, 2, 3, \]
so that Tr\( (M_i) = G_i = e^{\pm \frac{\pi}{2}}, \quad i = 1, 2, 3, \)
and \( M_\infty = (M_1 M_2 M_3)^{-1} \) is not diagonal but has eigenvalues \( e^\pm (s_1 + s_2 + s_3) \).

3. Quantisation

In the proper quantum ordering for a special class of geodesic functions corresponding to geodesics going around exactly two holes was constructed and it was proved that for each such geodesic, the matrix entries of the corresponding elements in the Fuchsian group satisfy a deformed version of the quantum universal enveloping algebra \( U_q(\mathfrak{sl}_2) \) relations.

In this section we use the same quantum ordering for quantising matrix elements of the monodromy group.

In the quantum Painlevé cubic can be obtained by introducing the Hermitian operators \( S_2, S_3 \) subject to the commutation inherited from the Poisson bracket of \( s_1 \):
\[ [S_i, S_{i+1}] = i\pi h \{ s_i, s_{i+1} \} = i\pi h, \quad i = 1, 2, 3, \quad i + 3 \equiv i, \]
while the central elements, i.e. perimeters \( p_1, p_2, p_3 \) and \( S_1 + S_2 + S_3 \) remain non-deformed, so that the constants \( \omega_i^{(d)} \) remain non-deformed.

The Hermitian operators \( G_{23}^h, G_{31}^h, G_{12}^h \) corresponding to \( G_{23}, G_{31}, G_{12} \) are introduced as follows: consider the classical expressions for \( G_{23}, G_{31}, G_{12} \) is terms of \( s_1, s_2, s_3 \) and \( p_1, p_2, p_3 \). Write each product of exponential terms as the exponential of the sum of the exponents and replace those exponents by their quantum version, for example the classical \( G_{23} \) is
\[ G_{23} = -e^{s_2 + s_3} - e^{-s_2 - s_3} - e^{-s_2 + s_3} - G_2 e^{s_3} - G_3 e^{-s_2}, \]
and its quantum version is defined as
\[ G_{23}^h = -e^{S_2 + S_3} - e^{-S_2 - S_3} - e^{-S_2 + S_3} - G_2 e^{S_3} - G_3 e^{-S_2}. \]

Then \( G_{23}, G_{31}, G_{12}^h \) satisfy the following quantum algebra [4]:
\[ q^{-1/2} G_{12}^h G_{23}^h - q^{1/2} G_{23}^h G_{12}^h = (q^{-1} - q) G_{13}^h + (q^{-1/2} - q^{1/2}) \omega_2 \]
\[ q^{-1/2} G_{23}^h G_{13}^h - q^{1/2} G_{13}^h G_{23}^h = (q^{-1} - q) G_{12}^h + (q^{-1/2} - q^{1/2}) \omega_3 \]
\[ q^{-1/2} G_{13}^h G_{12}^h - q^{1/2} G_{12}^h G_{13}^h = (q^{-1} - q) G_{23}^h + (q^{-1/2} - q^{1/2}) \omega_1 \]
and satisfy the following quantum cubic relations:
\[ C^h = q^{-1/2} G_{12}^h G_{23}^h G_{13}^h - q^{-1} (G_{12}^h)^2 - q (G_{23}^h)^2 - q^{-1} (G_{13}^h)^2 \]
\[ -q^{-1/2} \omega_3 G^h_{12} - q^{1/2} \omega_1 G^h_{23} - q^{-1/2} \omega_2 G^h_{13}, \]

where \( C^h \) is a central element is the quantum algebra \( \text{(3.22)} \).

We now quantise the monodromy matrices in the same way:

**Theorem 3.1.** The following matrices

\[
V_1 = \begin{pmatrix}
0 & -e^{S_1} \\
e^{-S_1} & -e^{S_1} - e^{-p_2} - e^{p_2}
\end{pmatrix},
M_2^h = \begin{pmatrix}
e^{-p_2} - e^{-p_2} - e^{S_2} - e^{-p_2} - e^{S_2} - e^{-S_2} \\
e^{S_2}
\end{pmatrix},
M_3^h = \begin{pmatrix}
e^{-p_2} - e^{-p_2} - e^{S_3} - e^{-S_3} \\
e^{S_3}
\end{pmatrix},
M_\infty^h = \begin{pmatrix}
0 \\
s_{\infty}^h e^{-S_1 - S_2 - S_3} - e^{S_1 + S_2 + S_3}
\end{pmatrix},
\]

where

\[
s_{\infty}^h = \begin{pmatrix}
(e^{-p_2} + e^{p_2}) e^{-S_1 - S_2} + (e^{p_2} - e^{-p_2}) e^{S_1 + S_2} + (e^{p_2} + e^{-p_2}) e^{S_2} + e^{-S_1 - S_2 - S_3} + e^{-S_1 - S_2 + S_3} + e^{-S_1 + S_2 + S_3},
\end{pmatrix}
\]

are elements of \( SL(2, \mathbb{T}_q) \) and satisfy the following relations:

\[
(M_1^h + e^{p_2} \mathbb{I})(M_1^h + e^{-p_2} \mathbb{I}) = 0,
(M_2^h + e^{p_2} \mathbb{I})(M_2^h + e^{-p_2} \mathbb{I}) = 0,
(M_3^h + e^{p_2} \mathbb{I})(M_3^h + e^{-p_2} \mathbb{I}) = 0,
(M_\infty^h + e^{S_1 + S_2 + S_3} \mathbb{I})(M_\infty^h + e^{-S_1 - S_2 - S_3} \mathbb{I}) = 0,
M_\infty^h M_1^h M_2^h M_3^h = q^{-1} \mathbb{I},
\]

where \( \mathbb{I} \) is the 2 \( \times \) 2 identity matrix.

This theorem shows that we can interpret the Cherednik algebra as quantisation of the group algebra of the monodromy group of the sixth Painlevé equation, in fact the matrices defined by \( \text{(1.6)}, \text{(1.7)}, \text{(1.8)}, \text{(1.9)} \) are simply obtained as \( i M_3^h, i M_2^h, i M_1^h \) and \( i M_\infty^h \) respectively so that Theorem \( \text{(1.1)} \) can be stated as follows:

**Theorem 3.2.** The map:

\[
V_0 \to i M_3^h, \quad V_1 \to i M_2^h, \quad V_\infty \to i M_1^h, \quad V_0 \to i M_\infty^h,
\]

where \( M_1^h, M_2^h, M_3^h, M_\infty^h \) are defined as in \( \text{(3.24)} \), gives an embedding of \( \mathcal{H} \) into \( \text{Mat}(2, \mathbb{T}_q) \). In other words, the matrices \( i M_3^h, i M_2^h, i M_1^h \) and \( i M_\infty^h \) in \( GL(2, \mathbb{T}_q) \) satisfy the relations \( \text{(1.6)}, \text{(1.7)}, \text{(1.8)} \) and \( \text{(1.9)} \), in which the quantum ordering is dictated by the matrix product ordering and

\[
u_1 = -i e^{-p_2}, \quad k_0 = -i e^{-p_2}, \quad k_1 = -i e^{-p_2}, \quad u_0 = -i e^{-S_1 - S_2 - S_3}.
\]

**Proof of Theorem \( \text{(1.1)} \)** To prove this theorem, we use the fact that the algebras \( \mathcal{H} \) is the algebra generated by five elements \( T^{\pm 1}, X^{\pm 1}, Y^{\pm 1} \) with the following relations:

\[
XW = WX = 1,
YZ = ZY = 1,
XT + ab T^{-1} W + a + b = 0,
\]
We then use Theorem 5.2 from [27] to prove formulae (1.6,1.7,1.8,1.9).

In this section we prove that this extended braid group action gives rise to the automorphisms of the Cherednik algebra of type \( \tilde{\mathfrak{c}}_1 \mathfrak{c}_1 \) which were studied in [30, 31]. Here we list them as they appear in [31]:

\[
\begin{align*}
\beta_1(M_1, M_2, M_3, M_\infty) &= (M_1 M_2 M_1^{-1}, M_1, M_3, M_\infty), \\
\beta_2(M_1, M_2, M_3, M_\infty) &= (M_1, M_2 M_3 M_2^{-1}, M_2, M_\infty).
\end{align*}
\]

Indeed by quantising (4.32) and (4.33) we obtain

\[
\begin{align*}
\beta^h_1(V_1, V_0, V_0) &= (V_1 V_1 V_1^{-1}, V_1, V_0, V_0), \\
\beta^h_2(V_1, V_0, V_0) &= (V_1 V_1 V_0 V_1^{-1}, V_1, V_0), \\
r^h(V_1, V_1, V_0, V_0) &= (V_0 V_0 V_0^{-1}, V_1^{-1}, V_1^{-1}, V_0^{-1}).
\end{align*}
\]

It is not hard to check that \( \sigma = \beta^h_2 \beta^h_1 \beta^h_2 \), \( \tau = \pi^2 \beta^h_1 \pi^{-2} \) and \( \eta = r^h \), so that we can claim that the automorphisms of the Cherednik algebra of type \( \tilde{\mathfrak{c}}_1 \mathfrak{c}_1 \) are indeed the quantisation of the extended modular group action described in [31].

The Painlevé sixth equation admits also an affine group of bi-rational transformations as described in table 1:
We postpone the computation of the action this automorphism on $V$ to a subsequent publication, this will involve a quantum version of the middle convolution operation discussed in [12]. On the parameters $(u_1, u_0, k_1, k_0)$ this transformation acts as follows:

$$(u_1, u_0, k_1, k_0) \rightarrow \left( \frac{u_1}{\sqrt{u_1 u_0 k_1 k_0}}, \frac{u_0}{\sqrt{u_1 u_0 k_1 k_0}}, \frac{k_1}{\sqrt{u_1 u_0 k_1 k_0}}, \frac{k_0}{\sqrt{u_1 u_0 k_1 k_0}} \right)$$

We note that $w_1, w_2, w_3, w_\infty$ have no effect on the monodromy matrices and therefore on the generators of $H$, while $r_1, r_2, r_3$ act as combinations of cyclic permutations and the transformation $\pi$, while the permutations $\pi_{13}, \pi_{1\infty}, \pi_{12}$ correspond to a simple re-labelling of the parameters. The only transformation which does not have a simple explanation in terms of monodromy matrices $M_1, M_2, M_3, M_\infty$ is $w_\rho$, which was explained in terms of isomonodromic deformations of a $3 \times 3$ linear system with one irregular singularity and one simple pole in [28]. On the parameters $(u_1, u_0, k_0)$ this transformation acts as follows:

$$(u_1, u_0, k_0) \rightarrow \left( \frac{u_1}{\sqrt{u_1 u_0 k_0}}, \frac{u_0}{\sqrt{u_1 u_0 k_0}}, \frac{k_0}{\sqrt{u_1 u_0 k_0}} \right)$$

We postpone the computation of the action this automorphism on $V_0, V_1, \tilde{V}_0, \tilde{V}_1$ to a subsequent publication, this will involve a quantum version of the middle convolution operation discussed in [12].

5. Embedding of the confluent Cherednik algebras into $Mat(2, T_q)$

The confluent limits of the Cherednik algebra of type $\tilde{C}_1 C_1$ were introduced in [27], in terms of a different presentation which is equivalent to the following (see Theorem 3.2 in [27]):

**Definition 5.1.** Let $k_1, u_0, u_1, q \in \mathbb{C}^*$, such that $q^m \neq 1$, $m \in \mathbb{Z}_{>0}$. The confluent Cherednik algebras $H_V, H_{IV}, H_{III}, H_{II}, H_I$ are the algebras generated by four elements $V_0, V_1, \tilde{V}_0, \tilde{V}_1$ satisfying the following relations respectively:

- $H_V$:
  
  \begin{align}
  (5.38) & \quad V_0^2 + V_0 = 0, \\
  (5.39) & \quad (V_1 - k_1)(V_1 + k_1^{-1}) = 0, \\
  (5.40) & \quad \tilde{V}_0^2 + u_0^{-1}\tilde{V}_0 = 0, \\
  (5.41) & \quad (\tilde{V}_1 - u_1)(\tilde{V}_1 + u_1^{-1}) = 0, \\
  (5.42) & \quad q^{1/2}V_1 V_0 = V_0 + u_0^{-1}, \\
  (5.43) & \quad q^{1/2}\tilde{V}_0 \tilde{V}_1 = V_0 + 1.
  \end{align}
EMBEDDING OF THE RANK 1 DAHA INTO $\text{Mat}(2, \mathbb{T}_q)$ AND ITS AUTOMORPHISMS.

- $\mathcal{H}_{IV}$:
  \begin{align}
  V_0^2 + V_0 &= 0, \\
  V_1^2 + V_1 &= 0, \\
  \hat{V}_0^2 + \frac{1}{u_0} \hat{V}_0 &= 0, \\
  (\hat{V}_1 - u_1)(\hat{V}_1 + u_1^{-1}) &= 0, \\
  q^{1/2} \hat{V}_1 V_0 &= \hat{V}_0 + u_0^{-1}, \\
  \hat{V}_0 \hat{V}_1 V_1 &= 0, \\
  \hat{V}_0 \hat{V}_0 &= 0.
  \end{align}

- $\mathcal{H}_{III}$:
  \begin{align}
  V_0^2 &= 0, \\
  (V_1 - k_1)(V_1 + k_1^{-1}) &= 0, \\
  \hat{V}_0^2 + \frac{1}{\sqrt{q}} \hat{V}_0 &= 0, \\
  (\hat{V}_1 - u_1)(\hat{V}_1 + u_1^{-1}) &= 0, \\
  q^{1/2} \hat{V}_1 V_0 &= \hat{V}_0 + \frac{1}{\sqrt{q}}, \\
  q^{1/2} \hat{V}_0 \hat{V}_1 V_1 &= V_0.
  \end{align}

- $\mathcal{H}_{II}$:
  \begin{align}
  V_0^2 + V_0 &= 0, \\
  V_1^2 &= 0, \\
  \hat{V}_0^2 + \hat{V}_0 &= 0, \\
  (\hat{V}_1 - u_1)(\hat{V}_1 + u_1^{-1}) &= 0, \\
  q^{1/2} \hat{V}_1 V_0 &= \hat{V}_0 + 1, \\
  \hat{V}_0 \hat{V}_1 V_1 &= 0, \\
  \hat{V}_0 \hat{V}_0 &= 0.
  \end{align}

- $\mathcal{H}_{I}$:
  \begin{align}
  V_0^2 + V_0 &= 0, \\
  V_1^2 &= 0, \\
  \hat{V}_0^2 + \hat{V}_0 &= 0, \\
  \hat{V}_1^2 + \hat{V}_1 &= 0, \\
  q^{1/2} \hat{V}_1 V_0 &= \hat{V}_0 + 1, \\
  \hat{V}_0 \hat{V}_1 &= 0, \\
  \hat{V}_0 \hat{V}_0 &= 0.
  \end{align}

All these algebras $\mathcal{H}_V, \mathcal{H}_{IV}, \mathcal{H}_{III}, \mathcal{H}_{II}, \mathcal{H}_I$ admit embeddings in $\text{Mat}(2, \mathbb{T}_q)$ (see Theorems 5.2, 5.3, 5.4, 5.5 and 5.5 in [27]). Here we report these embeddings for the generators $\hat{V}_1, V_1, V_0, \hat{V}_0$ in order to clarify the confluence scheme in accordance with Figure 1. Note that in Figure 1, we also have the algebras $\mathcal{H}^{D_7}_{III}$ and $\mathcal{H}^{D_8}_{III}$ for
which we don’t have a Noumi Stokman \cite{30} representation and for which we can’t prove the embedding into $\text{Mat}(2, T_q)$, so that the geometric explanation behind these algebras remains conjectural.

**Theorem 5.2.** The map:

\begin{align*}
V_0 &\rightarrow \left(\begin{array}{cc}
-1 & 0 \\
1 + i e^{S_3} & 0
\end{array}\right) \\
V_1 &\rightarrow \left(\begin{array}{cc}
k_1 - k_1^{-1} - i e^{S_2} & k_1 - k_1^{-1} - i e^{-S_2} - i e^{S_2} \\
\frac{k_1}{i e^{S_2}} & \frac{k_1}{i e^{S_2}}
\end{array}\right) \\
\tilde{V}_1 &\rightarrow \left(\begin{array}{cc}
0 & -i e^{S_1} \\
i e^{S_1} & u_1 - u_1^{-1}
\end{array}\right) \\
\tilde{V}_0 &\rightarrow \left(\begin{array}{cc}
0 & 0 \\
q^2 s & \frac{1}{u_0}
\end{array}\right),
\end{align*}

where

\begin{equation*}
s = e^{-S_1} + \left(\frac{1}{k_1} - k_1\right)e^{-S_1} - S_3 + \left(\frac{1}{u_1} - u_1\right)e^{S_2} + S_3 + i e^{-S_1} - S_2 + S_3 + i e^{-S_1} + S_2 + S_3.
\end{equation*}

gives and embedding of $\mathcal{H}_V$ into $\text{Mat}(2, T_q)$. The images of $V_0, V_0, V_1, \tilde{V}_1$ in $\text{Mat}(2, T_q)$ satisfy the relations (5.38), (5.39), (5.40), (5.41), (5.42), (5.43) in which the quantum ordering is dictated by the matrix product ordering.

**Theorem 5.3.** The map:

\begin{align*}
V_0 &\rightarrow \left(\begin{array}{cc}
-1 & 0 \\
1 + i e^{S_3} & 0
\end{array}\right) \\
V_1 &\rightarrow \left(\begin{array}{cc}
-1 - i e^{S_2} & -1 - i e^{S_2} \\
i e^{S_2} & i e^{S_2}
\end{array}\right) \\
\tilde{V}_1 &\rightarrow \left(\begin{array}{cc}
0 & -i e^{S_1} \\
i e^{S_1} & u_1 - u_1^{-1}
\end{array}\right) \\
\tilde{V}_0 &\rightarrow \left(\begin{array}{cc}
0 & 0 \\
q^2 s & \frac{1}{u_0}
\end{array}\right),
\end{align*}

where

\begin{equation*}
s = e^{-S_1} + S_3 + \left(\frac{1}{u_1} - u_1\right)e^{S_2} + S_3 + i e^{-S_1} + S_2 + S_3.
\end{equation*}

gives and embedding of $\mathcal{H}_I V$ into $\text{Mat}(2, T_q)$. The images of $V_0, \tilde{V}_0, V_1, \tilde{V}_1$ in $\text{Mat}(2, T_q)$ satisfy the relations (5.44), (5.45), (5.46), (5.47), (5.48), (5.49), in which the quantum ordering is dictated by the matrix product ordering.

**Theorem 5.4.** The map:

\begin{align*}
V_0 &\rightarrow \left(\begin{array}{cc}
0 & 0 \\
i e^{S_3} & 0
\end{array}\right) \\
V_1 &\rightarrow \left(\begin{array}{cc}
k_1 - k_1^{-1} - i e^{S_2} & k_1 - k_1^{-1} - i e^{-S_2} - i e^{S_2} \\
k_1^{-1} & \frac{k_1}{i e^{S_2}}
\end{array}\right)
\end{align*}
EMBEDDING OF THE RANK 1 DAHA INTO Mat(2, T_q) AND ITS AUTOMORPHISMS.

\[ V_1 \to \begin{pmatrix} 0 & -ie^{S_1} \\ ie^{-S_1} & u_1 - u_1^{-1} \end{pmatrix} \]

\[ V_0 \to \begin{pmatrix} 0 & 0 \\ q^2 s & -1 \end{pmatrix}, \]

where

\[ s = e^{-S_1-S_2} + \left( \frac{1}{k_1} - k_1 \right) e^{-S_1+S_3} + \left( \frac{1}{u_1} - u_1 \right) e^{S_2+S_3} + ie^{-S_1-S_2+S_3} + i e^{-S_1+S_2+S_3}. \]

gives an embedding of \( \mathcal{H}_{II} \) into \( \text{Mat}(2, T_q) \). The images of \( V_0, \tilde{V}_0, V_1, \tilde{V}_1 \) in \( \text{Mat}(2, T_q) \) satisfy the relations (5.51), (5.52), (5.53), (5.54), (5.55), (5.56), in which the quantum ordering is dictated by the matrix product ordering.

**Theorem 5.5.** The map:

\[ V_0 \to \begin{pmatrix} -1 & 0 \\ 1 + ie^{S_3} & 0 \end{pmatrix} \]

\[ V_1 \to \begin{pmatrix} -ie^{S_2} & -ie^{S_2} \\ ie^{S_2} & ie^{S_2} \end{pmatrix} \]

\[ \tilde{V}_1 \to \begin{pmatrix} 0 & -ie^{S_1} \\ 0 & -1 \end{pmatrix} \]

\[ \tilde{V}_0 \to \begin{pmatrix} 0 & 0 \\ i\sqrt{q} e^{-S_1} e^{S_2} e^{-S_3} - \sqrt{q} \left( u_1 - \frac{1}{u_1} \right) e^{S_2} e^{S_3} & 0 \\ 0 & 1 \end{pmatrix}, \]

gives an embedding of \( \mathcal{H}_{II} \) into \( \text{Mat}(2, T_q) \). The images of \( V_0, \tilde{V}_0, V_1, \tilde{V}_1 \) in \( \text{Mat}(2, T_q) \) satisfy the relations (5.57), (5.58), (5.59), (5.61), (5.62), (5.63), in which the quantum ordering is dictated by the matrix product ordering.

**Theorem 5.6.** The map:

\[ V_0 \to \begin{pmatrix} -1 & 0 \\ 1 + ie^{S_3} & 0 \end{pmatrix} \]

\[ V_1 \to \begin{pmatrix} -ie^{S_2} & -ie^{S_2} \\ ie^{S_2} & ie^{S_2} \end{pmatrix} \]

\[ \tilde{V}_1 \to \begin{pmatrix} 0 & -ie^{S_1} \\ 0 & -1 \end{pmatrix} \]

\[ \tilde{V}_0 \to \begin{pmatrix} 0 & 0 \\ q^2 e^{S_2} e^{S_3} & 0 \\ 0 & 1 \end{pmatrix}, \]

gives an embedding of \( \mathcal{H}_{II} \) into \( \text{Mat}(2, T_q) \). The images of \( V_0, \tilde{V}_0, V_1, \tilde{V}_1 \) in \( \text{Mat}(2, T_q) \) satisfy the relations (5.64), (5.65), (5.66), (5.67), (5.68), (5.70), in which the quantum ordering is dictated by the matrix product ordering.

**Proof.** The proof of this Theorem is very similar to the proof of Theorem 5.5 and is therefore omitted.

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