Neutrino masses and mixing from flavour antisymmetry

Anjan S. Joshipura\textsuperscript{1,}\footnote{anjan@prl.res.in}

\textsuperscript{1}Physical Research Laboratory, Navarangpura, Ahmedabad 380 009, India.

Abstract

We discuss consequences of assuming (i) that the (Majorana) neutrino mass matrix $M_\nu$ displays flavour antisymmetry, $S_\nu^T M_\nu S_\nu = -M_\nu$ with respect to some discrete symmetry $S_\nu$ contained in $SU(3)$ and (ii) $S_\nu$ together with a symmetry $T_l$ of the Hermitian combination $M_l M_l^\dagger$ of the charged lepton mass matrix forms a finite discrete subgroup $G_f$ of $SU(3)$ whose breaking generates these symmetries. Assumption (i) leads to at least one massless neutrino and allows only four textures for the neutrino mass matrix in a basis with a diagonal $S_\nu$ if it is assumed that the other two neutrinos are massive. Two of these textures contain a degenerate pair of neutrinos. Assumption (ii) can be used to determine the neutrino mixing patterns. We work out these patterns for two major group series $\Delta(3N^2)$ and $\Delta(6N^2)$ as $G_f$. It is found that all $\Delta(6N^2)$ and $\Delta(3N^2)$ groups with even $N$ contain some elements which can provide appropriate $S_\nu$. Mixing patterns can be determined analytically for these groups and it is found that only one of the four allowed neutrino mass textures is consistent with the observed values of the mixing angles $\theta_{13}$ and $\theta_{23}$. This texture corresponds to one massless and a degenerate pair of neutrinos which can provide the solar pair in the presence of some perturbations. The well-known groups $A_4$ and $S_4$ provide examples of the groups in respective series allowing correct $\theta_{13}$ and $\theta_{23}$. An explicit example based on $A_4$ and displaying a massless and two quasi degenerate neutrinos is discussed.
I. INTRODUCTION

Orderly pattern of neutrino mixing appears to hide some symmetry, discrete or continuous. It is possible to connect a given mixing pattern with some discrete symmetries of the leptonic mass matrices. Such symmetries may however be residual symmetries arising from a bigger symmetry in the underlying theory. One can obtain a possible larger picture by assuming that these symmetries are a part of a bigger group operating at the fundamental level whose breaking leads to the symmetries of the mass matrices. There is an extensive literature on study of possible residual symmetries of the mass matrices and of the groups which harbor them [1–15], see [16–18] for reviews and additional references.

Starting point in these approaches is to assume the existence of some symmetries \( S_\nu \) (usually a \( Z_2 \times Z_2 \)) and \( T_l \) (usually \( Z_N, N \geq 3 \)) of the (Majorana) neutrino and the charged lepton mass matrices

\[
T_l^\dagger M_l M_l^\dagger T_l = M_l M_l^\dagger, \tag{1}
\]
\[
S_\nu^T M_\nu S_\nu = M_\nu. \tag{2}
\]

Matrices diagonalizing the \( 3 \times 3 \) symmetry matrices \( S_\nu, T_l \) can be related to the mixing matrices in each sector. The structures of these matrices can also be independently fixed if one assume that \( S_\nu \) and \( T_l \) represent specific elements of some discrete group \( G_f \) in a given three dimensional representation. In this way, the leptonic mixing can be directly related to group theoretical structures. This reasoning has been used for the determination of the neutrino mixing angles in case of the three non-degenerate neutrinos [1–15], two or three degenerate neutrinos [19, 20] and one massless and two non-degenerate neutrinos [21, 22].

The residual symmetries may arise from spontaneous breaking of \( G_f \) if the vacuum expectation values of the Higgs fields responsible for generating leptonic masses break \( G_f \) but respect \( S_\nu, T_l \). We wish to study in this paper consequences of an alternative assumption that the spontaneous breaking of \( G_f \) leads to an \( M_\nu \) which displays antisymmetry instead of symmetry, i.e. assume that eq.(2) gets replaced by

\[
S_\nu^T M_\nu S_\nu = -M_\nu \tag{3}
\]

but (1) remains as it is. These assumptions prove to be quite powerful and are able to simultaneously restrict both the mass patterns and mixing angles when embedding of \( S_\nu, T_l \) into \( G_f \) is considered. We shall further assume that \( S_\nu, T_l \) belong to some finite discrete subgroup of \( SU(3) \) with \( \text{Det}(S_\nu, T_l) = \pm 1 \). Then the first consequence of imposing eq.(3) is that \( \text{Det}M_\nu = 0 \), i.e. at least one of the neutrinos remains massless. Since cases with two (or three !) massless neutrinos are not phenomenologically interesting, we shall restrict ourselves to cases with only one massless neutrino. Then as a second consequence of eq.(3), one can determine all the allowed forms of \( M_\nu \) in a given basis for all possible \( S_\nu \) contained in \( SU(3) \). There exist only four possible \( M_\nu \) (and their permutations) consistent with eq.(3) in a particular basis with a diagonal \( S_\nu \). Two of these give one massless and two non-degenerate
neutrinos and the other two give a massless and a degenerate pair of neutrinos which may be identified with the solar pair.

We determine all the allowed textures of the neutrino mass matrix in the next section. Subsequently, we discuss groups $\Delta(3N^2)$ and $\Delta(6N^2)$ and identify those which can give correct description of mixing using flavour antisymmetry. In section IV, we introduce $Z_2 \times Z_2$ as neutrino residual symmetry and present an example in which neutrino mass matrix gets fully determined group theoretically except for an overall scale. We discuss a realization of the basic idea with a simple example based on the $A_4$ group in section V. Section VI contains summary and comparison with earlier relevant works.

II. ALLOWED TEXTURES FOR NEUTRINO MASS MATRIX

We shall first consider the case of only one $S_\nu$ satisfying eq.(3) and subsequently generalize it to include two. The unitary matrix $S_\nu$ can be diagonalized by another unitary matrix $V_{S_\nu}$:

$$V_{S_\nu}^\dagger S_\nu V_{S_\nu} = \tilde{S}_\nu$$

where $\tilde{S}_\nu$ is a diagonal matrix having the form:

$$\tilde{S}_\nu = \text{diag.}(\lambda_1, \lambda_2, \lambda_3) , \quad (4)$$

Unitarity of $S_\nu$ implies that $\lambda_1, \lambda_2, \lambda_3$ are some roots of unity. They are related by the condition $\text{Det}S_\nu = +1$ which we assume without lose of generality. We now go to the basis with a diagonal $S_\nu$. Defining $\hat{M}_\nu = V_{S_\nu}^T M_\nu V_{S_\nu}$, eq.(3) can be rewritten as:

$$(\hat{M}_\nu)_{ij}(1 + \lambda_i \lambda_j) = 0 \quad (i,j \text{ not summed}) . \quad (5)$$

It follows that a given element $(\hat{M}_\nu)_{ij}$ is non-zero only if the factor in bracket multiplying it is zero. This cannot happen for an arbitrary set of $\lambda_i$ and one needs to impose specific relation among them to obtain a non-trivial $\hat{M}_\nu$. We now argue that only two possible forms of $\tilde{S}_\nu$ and their permutations lead to neutrino mass matrices with two massive neutrinos. The third mass will always be zero as a consequence of eq.(3) and the assumption that $S_\nu$ belongs to $SU(3)$. These forms of $\tilde{S}_\nu$ are given by:

$$\tilde{S}_1\nu = \text{diag.}(\lambda, -\lambda^*, -1) ,$$

$$\tilde{S}_2\nu = \text{diag.}(\pm i, \mp i, 1) . \quad (6)$$

$\lambda$ is an arbitrary root of unity. This can be argued as follows. Assume that at least one off-diagonal element of $\hat{M}_\nu$ is non-zero which we take as the 12 element for definiteness. In this case, eq.(5) immediately implies the first of eq.(6) as a necessary condition. One can distinguish three separate cases of this condition:

(I) $\lambda = 1$ (II) $\lambda = \pm i$ and (III) $\lambda \neq \pm 1, \pm i$.

\footnote{\(\lambda = -1\) case corresponds to permutation of the case with $\lambda = 1$.}
The entire structures of $\tilde{M}_\nu$ get determined in these cases from condition eq.(5) as follows:

Texture I : $\tilde{S}_{1\nu} = (1, -1, -1)$; $\tilde{M}_\nu = m_0 \begin{pmatrix} 0 & c \sin \beta e^{i\beta} \\ c & 0 & 0 \\ \sin \beta e^{i\beta} & 0 & 0 \end{pmatrix}$, \hspace{1cm} (7)

where $c = \cos \theta, s = \sin \theta$. This structure implies one massless and two degenerate neutrinos with a mass $|m_0|$. In case of (II),

Texture II : $\tilde{S}_{1\nu} = (\pm i, \pm i, -1)$; $\tilde{M}_\nu = \begin{pmatrix} x_1 & y & 0 \\ y & x_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. \hspace{1cm} (8)

This case corresponds to one massless and two non-degenerate neutrinos. In the third case one gets

Texture III : $\tilde{S}_{1\nu} = (\lambda, -\lambda^*, -1)$; $\tilde{M}_\nu = m_0 \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, \hspace{1cm} (\lambda \neq \pm 1, \pm i) \hspace{1cm} (9)

which implies a massless and a pair of degenerate neutrinos.

The cases (I,III) lead to the same mass spectrum but different mixing patterns. $\tilde{M}_\nu$ in eq.(7) is diagonalized as $V_{\nu}^T \tilde{M}_\nu V_{\nu} = \text{diag}(m_0, m_0, 0)$ with

$$V_{\nu} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ c \frac{1}{\sqrt{2}} & -i \frac{1}{\sqrt{2}} & s \\ \frac{e^{-i\beta}}{\sqrt{2}} & \frac{i e^{-i\beta}}{\sqrt{2}} & c e^{-i\beta} \end{pmatrix} \begin{pmatrix} \cos \phi & -\sin \phi & 0 \\ \sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{pmatrix}. \hspace{1cm} (10)$$

The arbitrary rotation by an angle $\phi$ originates due to degeneracy in masses. The texture II, eq.(8) is diagonalized by a unitary rotation in the 12 plane while the one in eq.(9) by a similar matrix with the angle $\frac{\pi}{4}$.

The permutations of entries in $\tilde{M}_\nu$ give equivalent structures and are obtained by permuting entries in $\tilde{S}_{1\nu}$. The case which is not equivalent to above textures follows with a starting assumption that one of the diagonal elements of $\tilde{M}_\nu \neq 0$ say, $(\tilde{M}_\nu)_{11} \neq 0$. In this case one requires $\tilde{S}_\nu = \text{diag}(\pm i, \lambda', \mp i \lambda'^*)$ with $|\lambda'| = 1$. The case with $\lambda' = \pm i$ gives $\tilde{S}_{1\nu}$ which is already covered. $\lambda' = \mp i$ implies the condition $\tilde{S}_{2\nu}$ in $[5]$. This leads to a new texture

Texture IV : $\tilde{S}_{\nu} = (i, -i, 1)$; $\tilde{M}_\nu = \begin{pmatrix} x_1 & 0 & 0 \\ 0 & x_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. \hspace{1cm} (11)

For $\lambda' = \pm 1$ one gets permutation of $\tilde{S}_{1\nu}$ or $\tilde{S}_{2\nu}$ and for $\lambda' \neq \pm 1, \pm i$ only 11 element of $\tilde{M}_\nu$ is non zero and two neutrinos remain massless. Thus conditions eq.(6) and their permutations exhaust all possible textures of $\tilde{M}_\nu$ consistent with the antisymmetry of $M_\nu$, eq.(3) and two
massive neutrinos. Any $G_f$ admitting an element with these sets of eigenvalues will give a viable choice for flavour antisymmetry group. Note that texture III (IV) can be obtained from I(II) by putting $s(y)$ to zero. But the residual symmetries in all four cases are different. Because of this, the embedding groups $G_f$ can also be different. We therefore discuss all these cases separately.

The mixing matrix in texture I contains two unknowns $\theta$ and $\beta$ apart from an overall complex scale $m_0$. This is a reflection of the fact that the corresponding $S_\nu$ is a $Z_2$ symmetry and contains two degenerate eigenvalues $-1$. These unknown can be fixed by imposing another residual $Z_2$ symmetry commuting with $S_\nu$ and satisfying eq. (2) or (3). We shall discuss such choices in section IV.

III. GROUP THEORETICAL DETERMINATION OF MIXING

The physical neutrino mixing matrix $U_{PMNS} \equiv U$ depends on the structure of $M_\nu$ and $M_lM_l^\dagger$. The latter can be determined if the symmetry $T_l$ as in eq. (1) is known. We now make an assumption that $S_\nu$ satisfying eq. (3) and $T_l$ as in eq. (1) are elements of some discrete subgroup (DSG) of $SU(3)$ denoted by $G_f$. The DSG of $SU(3)$ have been classified in [23–25]. They are further studied in [26–36]. These can be written in terms of few $3 \times 3$ presentation matrices whose multiple products generate various DSG. Two main groups series called $C$ and $D$ [35] constitute bulk of the DSG of $SU(3)$. Of these, we shall explicitly study two infinite groups series $\Delta(3N^2)$ and $\Delta(6N^2)$ which are examples of the type $C$ and $D$ respectively. See [37–39] for earlier studies of neutrino mixing using the groups $\Delta(3N^2)$ and $\Delta(6N^2)$ and neutrino symmetry rather than antisymmetry.

Eq. (1) implies that $T_l$ commutes with $M_lM_l^\dagger$. Thus, the matrix $U_l$ diagonalizing the former also diagonalizes $M_lM_l^\dagger$ and corresponds to the mixing matrix among the left handed charged leptons. Similarly, the matrix $U_\nu$ diagonalizing $M_\nu$ gets related to the structure of $S_\nu$. In this way, the knowledge of $S_\nu$ and $T_l$ can be used to determine the mixing matrix

$$U \equiv U_{PMNS} = U_l^\dagger U_\nu \ .$$

This is the strategy followed in the general approach and we shall also use this to determine all possible mixing pattern for a given $G_f$ consistent with eqs. (1) and (3).

Not all the groups $G_f$ can admit an $S_\nu$ which will provide a legitimate antisymmetry operator $S_\nu$, i.e. an element with eigenvalues specified by eq. (6). Our strategy would be to determine a class of groups which will have one or more allowed $S_\nu$ and then look for all viable $T_l$ within these groups. There would be different mixing patterns associated with each choice of $S_\nu, T_l$ and it is possible to determine all of them analytically for $\Delta(3N^2)$ and $\Delta(6N^2)$ groups.
A. $\Delta(3N^2)$

The $\Delta(3N^2)$ groups are isomorphic to $Z_N \times Z_N \times Z_3$, where $\times$ denotes the semi-direct product. The group theoretical details for $\Delta(3N^2)$ are discussed in [27, 40]. For our purpose, it is sufficient to note that all the elements of the group are generated from the multiple product of two basic generators defined as:

$$F = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \eta & 0 \\ 0 & 0 & \eta^2 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$ (13)

with $\eta = e^{\frac{2\pi i}{N}}$. Here $F$ generates one of the $Z_N$ groups and $E$ generates $Z_3$ in the semi-direct product $Z_N \times Z_N \rtimes Z_3$. The other $Z_N$ group is generated by $EFE^{-1}$. The above explicit matrices provide a faithful three dimensional irreducible representation of the group and multiple products of these matrices therefore generate the entire group whose elements can be labeled as:

$$W \equiv W(N, p, q) = \begin{pmatrix} \eta^p & 0 & 0 \\ 0 & \eta^q & 0 \\ 0 & 0 & \eta^{-p-q} \end{pmatrix}, \quad R \equiv R(N, p, q) = \begin{pmatrix} 0 & 0 & \eta^p \\ \eta^q & 0 & 0 \\ 0 & \eta^{-p-q} & 0 \end{pmatrix},$$

$$V \equiv V(N, p, q) = \begin{pmatrix} 0 & \eta^p & 0 \\ 0 & 0 & \eta^q \\ \eta^{-p-q} & 0 & 0 \end{pmatrix}.$$ (14)

All elements of $\Delta(3N^2)$ are obtained by varying $p, q$ over the allowed range $p, q = 0, 1, 2, ..., N-1$ in the above equation. Thus each matrices $W, R, V$ have $N^2$ elements giving in total $3N^2$ elements corresponding to the order of $\Delta(3N^2)$. The eigenvalue equation for the $2N^2$ non-diagonal elements $R$ and $V$ is simply given by $\lambda^3 = 1$. These elements therefore have eigenvalues $(1, \omega, \omega^2)$ with $\omega = e^{\frac{2\pi i}{3}}$. These are not in the form of eq.(6) required to get the neutrino antisymmetry operator $S_\nu$. Thus $S_\nu$ has to come from the $N^2$ diagonal elements. This requires that $N, p, q$ should be such that $W(N, p, q) = \text{diag} (\eta^p, \eta^q, \eta^{-p-q})$ matches the required eigenvalues $\tilde{S}_\nu$ of $S_\nu$ given by eq.(6) or their permutations. This cannot happen for all the values of variables and one can easily identify the viable cases. It is found that

- $W$ can match any of $\tilde{S}_\nu$ only for even $N$. Thus only $\Delta(12k^2)$ groups with $k = 1, 2, ..., N$ contain neutrino antisymmetry operator $S_\nu$.

- The eigenvalue set $\tilde{S}_\nu = (1, -1, -1)$ is always contained as a diagonal generator for all $\Delta(12k^2)$ groups and can be chosen as $S_\nu = W(2k, 0, k)$. Hence the texture I with two degenerate and one massless neutrino can follow in any $\Delta(12k^2)$. The smallest such group is $\Delta(12) = A_4$ which is one of the most studied flavour symmetry from other points of view [41–50].

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• The set $\tilde{S}_\nu = (\pm i, \pm i, -1)$ arises only for $N$ multiple of 4, i.e. in case of groups $\Delta(48l^2)$, $l = 1, 2, \ldots$. These groups also contain a $\tilde{S}_\nu$ satisfying the second of eq. (6). Thus textures $I, II, IV$ are possible for all $\Delta(48l^2)$ groups.

• The set $\tilde{S}_\nu = (\lambda, -\lambda^*, -1)$ with $\lambda \neq \pm 1, \pm i$ and the associated texture III is viable in $\Delta(12k^2)$ with $k \geq 3$

Let us now turn to the mixing pattern allowed within the $\Delta(12k^2)$ groups. $S_\nu$ has to be a diagonal operator identified above. Then $T_i$ can be any other diagonal operator $W(2k, p, q)$ or any of $R(2k, p, q)$ or $V(2k, p, q)$. In the former case, $U_i = 1$, where 1 denotes a $3 \times 3$ identity matrix. The neutrino mixing in this case coincides with $V_\nu$ diagonalizing any of the four textures of $\tilde{M}_\nu$ giving $U_{PMNS} = V_\nu$. None of the allowed $V_\nu$ are suitable to give the correct mixing pattern with a non-zero $\theta_{13}$. Thus, $T_i$ needs to be any of the non-diagonal element $R, V$. The matrices $V_{R,V}$ diagonalizing $R, V$ are given by

$$V_R(N, p, q) = \text{diag}.(1, \eta^p, \eta^{-p})U_\omega,$$

$$V_V(N, p, q) = \text{diag}.(1, \eta^{-p}, \eta^{-p-q})U_\omega^*,$$

where,

$$U_\omega = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{pmatrix}.\tag{16}$$

The final mixing matrix depends upon the choice of specific texture for $\tilde{M}_\nu$. Consider the texture I which arises within all the $\Delta(12k^2)$ groups. $U_\nu = V_\nu$ in this case is given by eq. (10) and $U_{PMNS} = V_{R,V}^TV_\nu$. Since a neutrino pair is degenerate, the solar mixing angle $\theta_{12}$ remains undetermined in the symmetry limit. This is reflected by the presence of an unknown angle $\phi$ in eq. (10). In this case, the neutrino mass hierarchy is inverted and the third column of $U_{PMNS} \equiv U$ needs to be identified with the massless state. It is independent of the angle $\phi$.

We get for $T_i = R(N, p, q)$,

$$U_{i3} = \frac{1}{\sqrt{3}} \left(ce^{-i\beta}\eta^{p+q} - s, c\omega e^{-i\beta} \eta^{q+p} - s\omega^2, c\omega^2 e^{-i\beta} \eta^{q+p} - s\omega \right)^T\tag{17}$$

with $\eta = e^{\frac{2\pi i}{12k}}$ for the group $\Delta(12k^2)$. $p, q$ take discrete values $0, \ldots, 2k - 1$ in above equation while $\beta$ and $\theta$ are unknown quantities appearing in the neutrino mixing matrix eq. (10).

The entries in $U_{i3}$ can be permuted by reordering the eigenvalues of $T_i$. We will identify the minimum of $|U_{i3}|^2$ with $s_{13}^2$. If the minimum of the remaining two is identified with $c_{13}^2s_{23}^2$ then one will get a solution with the atmospheric mixing angle $\theta_{23} \leq 45^\circ$. In the converse case, one will get a solution $\geq 45^\circ$. The experimental values of the leptonic angles are determined through fits to neutrino oscillation data [51,53]. Throughout, we shall specifically use the fits presented in [51] for definiteness. The texture I corresponds to the inverted hierarchy.
and the best fit values and $3\sigma$ ranges appropriate for this case are given \[51\] by:

$$\sin^2 \theta_{12} = 0.308 \ (0.259 - 0.359) ,$$
$$\sin^2 \theta_{23} = 0.455 \ (0.380 - 0.641) ,$$
$$\sin^2 \theta_{13} = 0.0240 \ (0.0178 - 0.0298) .$$

(18)

Let us mention salient features of results following from eq.(17)

- It is always possible to obtain correct $\theta_{13}, \theta_{23}$ by choosing unknown quantities $\theta$ and $\beta$ of $\tilde{M}_\nu$. This should be contrasted with situation found in \[20\] which used neutrino symmetry instead of antisymmetry to obtain a degenerate pair of neutrinos. As discussed there, none of the $\Delta(3N^2)$ groups could simultaneously account for the values of $\theta_{13}, \theta_{23}$ within $3\sigma$.

- It is possible to obtain more definite predictions by choosing specific values of $\theta$ and or $\beta$. In contrast to $\theta$ and $\beta$ which are unknown, the choice of $p,q$ is dictated by the choice of $T_l$ and it is possible to consider any specific choice of $p,q$ in the range $0,...,N-1$. Consider a very specific choice of real $\tilde{M}_\nu$, i.e. $\beta = 0$ and a residual symmetry $T_l = E^2$ corresponding to putting $p = q = 0$ in eq.(17). This equation in this case gives a prediction $|U_{23}| = |U_{33}|$ which holds for all values of $\theta$. This relation is equivalent to a maximal $\theta_{23}$ which lies within the $1\sigma$ range of the global fits \[51\]. $\theta$ then can be chosen to get the correct $\theta_{13}$. Since the specific choice $p = q = 0$ is allowed within all the $\Delta(12k^2)$ groups, all of them can predict the maximal $\theta_{23}$ and can accommodate correct $\theta_{13}$.

- The relation $|U_{23}| = |U_{33}|$ does not hold for a complex $\eta^{p+q}$ even if $\beta = 0$. Such choices of $T_l$ give departures from maximality in $\theta_{23}$. It is then possible to reproduce both the angles correctly by choosing $\theta$. This is non-trivial since a single unknown $\theta$ determines both $\theta_{13}$ and $\theta_{23}$ for a specific choice of group (i.e. $N$) and a residual symmetry $T_l$ (i.e. $p$ and $q$). The resulting prediction can be worked out numerically by varying $p, q, N$ over the allowed integer values and $\theta$ over continuous range from 0 to $2\pi$. Values of $s_{23}^2$ obtained this way are depicted in Fig.(1). This is obtained by requiring that $s_{13}^2$ lies within the allowed $1\sigma$ range. The phase $\beta$ is put to zero. It is seen form the Figure that all the $\Delta(12k^2)$ groups always allow maximal $\theta_{23}$ as already discussed. But solutions away from maximal are also possible for $k \geq 4$. The minimal group capable of doing this is $\Delta(192)$. The next group $\Delta(300)$ can lead to near to the best fit values of the parameters. Specifically the choice $T_l = R(10,0,7)$, $S_\nu = W(10,0,5)$ within the group and $\theta \sim 54.3^\circ$ gives $s_{13}^2 \sim 0.024$ and $s_{23}^2 \sim 0.442$ to be compared with the best fit values 0.024 and 0.455 in \[51\].

- $p,q$ can only be zero or 1 and $\eta$ is real for the smallest group $\Delta(12) = A_4$. In this case, one immediately gets the prediction $\theta_{23} = \frac{\pi}{4}$ for $\beta = 0$. $\mu-\tau$ symmetry is often used to predict the maximal $\theta_{23}$. This is not even contained in $A_4$ which has only even
permutations of four objects. Still the use of antisymmetry rather than symmetry allows one to get the maximal $\theta_{23}$ and it also accommodates a non-zero $\theta_{13}$ within $A_4$. This should be contrasted with the situation obtained in case of the use of symmetry condition eq.(2) instead of (3). It is known that in this case $A_4$ group gives democratic value $\frac{1}{3}$ for $s_{13}^2$, see for example [6].

![Figure 1](image)

**FIG. 1.** Predictions for $\sin^2 \theta_{23}$ for the groups $\Delta(12k^2)$ as a function of $k$ when $\sin^2 \theta_{13}$ is allowed to vary within the $1\sigma$ range as obtained through global fits in [51]. Horizontal lines show $1\sigma$ limits on $\sin^2 \theta_{23}$.

We now argue that the other three textures though possible within $\Delta(12k^2)$ groups do not give the the correct mixing pattern. Texture II has one massless and in general two non-degenerate neutrinos. This texture can give both the normal and the inverted hierarchy. The mixing matrix $V_{\nu}$ is block-diagonal with a $2 \times 2$ matrix giving mixing among two massive states. Given this form for $V_{\nu}$ and a general $U_l$ as given in eq.(15), one finds that the case with inverted hierarchy leads to the prediction $\sin^2 \theta_{13} = \frac{1}{3}$ while the normal hierarchy gives instead $\cos^2 \theta_{13} \cos^2 \theta_{12} = \frac{1}{3}$. Neither of them come close to their experimental values.

The texture (III) having degenerate pair corresponds to the inverted hierarchy. $V_{\nu}$ in this case is block diagonal with an unknown solar angle. Given the most general form, eq.(15) for $V_l$ one obtains once again the wrong prediction $\sin^2 \theta_{13} = \frac{1}{3}$ ruling out this texture as well. Likewise, texture IV also gets ruled out. This corresponds to a diagonal $\tilde{M}_\nu$ with $V_{\nu} = 1$ and $|U_{\text{PMNS}}| = |U_l|$ has the universal structure $|U| = \frac{1}{3}1$.

To sum up, all the groups $\Delta(12k^2)$ contain a neutrino antisymmetry operator $S_\nu$ and allow a neutrino mass spectrum with two degenerate and one massless neutrino and can reproduce correctly two of the mixing angles $\theta_{13}, \theta_{23}$. The values for the solar angle and the solar scale have to be generated by small perturbations within these group. We shall study an example based on the minimal group $A_4 = \Delta(12)$ in this category in section V.
B. $\Delta(6N^2)$ groups

$\Delta(6N^2)$ groups are isomorphic to $Z_N \times Z_N \rtimes S_3$ with $N = 1, 2, 3, \ldots$. The $S_3$ group in the semi-direct product is generated by $E$ in eq. (13) and a matrix

$$
G = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}.
$$

(19)

The matrices $E, F, G$ provide a faithful irreducible representation of $\Delta(6N^2)$ [28] and generate the entire group with $6N^2$ elements. $3N^2$ elements generated by $E, F$ give the $\Delta(3N^2)$ subgroup. The additional $3N^2$ elements are generated from the multiple products of $G$ with elements of $\Delta(3N^2)$. These new elements can be parameterized by:

$$
S \equiv S(N, m, n) = \begin{pmatrix}
\eta^m & 0 & 0 \\
0 & \eta^m & 0 \\
0 & 0 & \eta^{m-n}
\end{pmatrix},
$$

$$
T \equiv T(n, m, n) = \begin{pmatrix}
0 & 0 & \eta^m \\
0 & \eta^m & 0 \\
\eta^{-m-n} & 0 & 0
\end{pmatrix},
$$

$$
U \equiv U(n, m, n) = \begin{pmatrix}
0 & \eta^n & 0 \\
\eta^{-m-n} & 0 & 0 \\
0 & 0 & \eta^m
\end{pmatrix}.
$$

(20)

Here $0 \leq (m, n) < N - 1$. Since $\Delta(3N^2)$ is a subgroup of $\Delta(6N^2)$, the neutrino mass and mixing patterns derived in the earlier section can also be obtained here. But the new elements $S, T, U$ allow more possibilities now. In particular, they allow more elements which can be used as neutrino antisymmetry $S_\nu$. To see this, note that the eigenvalues of $S, T, U$ are given by $(\eta^{-m/2}, -\eta^{-m/2}, -\eta^m)$. This can have the required form, eq.(6) when $m = 0$ or $m = N/2$. The eigenvalues in respective cases are $(1, -1, -1)$ or $(-i, i, 1)$ and one gets the textures I or IV by using any of $S, T, U$ as neutrino antisymmetry with $m = 0$ and $m = N/2$ respectively. Similarly, possible choices of the charged lepton symmetry $T_l$ also increases. It can be any of the six types of elements: $W, R, V$ as before or $S, T, U$. Important difference compared to $\Delta(3N^2)$ is that the texture I can now be obtained for both odd and even values of $N$ by choosing any of the $S, T, U$ with $m = 0$ as neutrino antisymmetry. Texture IV still requires $m = N/2$ and hence even $N$ for its realization. We determine mixing matrix $U$ for each of these textures and discuss them in turn.

1. Texture I

The residual anti symmetries which lead to texture I can be either (1) $S_\nu = W(2k, 0, k)$ or (2) $S_\nu = P(N, 0, n)$ where $P = S, T, U$. The residual symmetry $T_l$ of $M_lM_l^\dagger$ can be any elements in the group which we divide in three classes: (A) $T_l = W(N, p, q)$, (B) $T_l = P(N, p, q)$ and (C) $T_l = Q(N, p, q)$. Here and in the following, we use symbols $P$ and $Q$ to collectively denote $P = S, T, U$ and $Q = R, V$. We use the basis as specified
in eqs. (20) for $S_{\nu}, T_{l}$. Then the neutrino mixing matrix is given by $U_{\nu} = V_{\nu}$ in case (1) while it is given by $U_{\nu} = V_{P}(N, 0, n)V_{\nu}$ in case (2). This follows by noting that the texture $\tilde{M}_{\nu}$ given in eq. (7) holds in a basis with diagonal $S_{\nu}$ but $S_{\nu}$ in the chosen group basis of eq. (20) is non-diagonal in case (2). The neutrino mass matrix in this basis is thus given by $M_{\nu} = V_{P}\tilde{M}_{\nu}V_{P}^{\dagger}$ where $V_{P}$ diagonalizes $P(N, 0, n)$. The matrix $U_{\nu}$ which diagonalizes $M_{\nu}$ is then given by $U_{\nu} = V_{P}(N, 0, n)V_{\nu}$ where $V_{\nu}$ diagonalizes $\tilde{M}_{\nu}$. Explicitly, $V_{P}^{\dagger}(N, p, q)P(N, p, q)V_{P}(N, p, q) = \text{diag}(\eta^{-p/2}, -\eta^{-p/2}, -\eta^{p})$ with

$$V_{S}(N, p, q) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & \sqrt{2} \\ 1 & \eta^{q+p/2} & 0 \\ -\eta^{-q-p/2} & 1 & 0 \end{pmatrix}; \quad V_{U}(N, p, q) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & \eta^{q+p/2} & 0 \\ -\eta^{-q-p/2} & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix},$$

$$V_{T}(N, p, q) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & \eta^{q+p/2} & 0 \\ 0 & 0 & \sqrt{2} \\ -\eta^{-q-p/2} & 1 & 0 \end{pmatrix}. \quad (21)$$

We have chosen the ordering of columns of $V_{P}$ in such a way that the first column always corresponds to the eigenvalue $\eta^{-p/2}$. With this ordering one gets the texture I given in eq. (7) when $P(N, 0, n)$ is used as neutrino antisymmetry.

The matrices $U_{l}$ diagonalizing $T_{l}$ in three cases above are given in the same basis by $U_{l} = 1, V_{P}(N, p, q), V_{Q}(N, p, q)$ in cases (A), (B), (C) respectively where $V_{Q}$ are given in eq. (15). Thus we have six (four) different choices for $U_{l} (U_{\nu})$ giving in all 24 leptonic mixing matrices $U_{\text{PMNS}}$. We list these choices and the corresponding $U_{\text{PMNS}}$ matrices in Table I.

| Case | $S_{\nu}$ | $T_{l}$ | $U_{l}$ | $U_{\nu}$ | $U_{\text{PMNS}}$ |
|------|-----------|---------|---------|-----------|------------------|
| 1A   | $W(2k, 0, k)$ | $W(N, p, q)$ | 1        | $V_{\nu}$ | $V_{\nu}$       |
| 1B   | $W(2k, 0, k)$ | $P(N, p, q)$ | $V_{P}(N, p, q)$ | $V_{\nu}$ | $V_{P}(N, p, q)V_{\nu}$ |
| 1C   | $W(2k, 0, k)$ | $Q(N, p, q)$ | $V_{Q}(N, p, q)$ | $V_{\nu}$ | $V_{Q}(N, p, q)V_{\nu}$ |
| 2A   | $P(N, 0, n)$ | $W(N, p, q)$ | 1        | $V_{P}(N, 0, n)V_{\nu}$ | $V_{P}(N, 0, n)V_{\nu}$ |
| 2B   | $P(N, 0, n)$ | $P'(N, p, q)$ | $V_{P}(N, 0, n)V_{\nu}$ | $V_{P}(N, 0, n)V_{\nu}$ | $V_{P}(N, 0, n)V_{\nu}$ |
| 2C   | $P(N, 0, n)$ | $Q(N, p, q)$ | $V_{Q}(N, p, q)$ | $V_{P}(N, 0, n)V_{\nu}$ | $V_{P}(N, 0, n)V_{\nu}$ |

**TABLE I.** All possible choices of the residual symmetries $S_{\nu}$ and $T_{l}$ within $\Delta(6N^{2})$ groups and the corresponding PMNS mixing matrices. $P, P'$ collectively denote any of $S, T, U$ defined in the text. $Q$ denotes $R$ and $V$. The mixing matrices $V_{P}, V_{Q}$ and $V_{\nu}$ appearing above are given in eq. (21), eq. (15) and eq. (10) respectively.

Not all of 24 mixing matrices listed in Table I give independent predictions for the third column of $U_{\text{PMNS}}$ which determines $s_{13}$ and $s_{23}$. We discuss the independent ones below.

The choice (1A) giving $U_{\text{PMNS}} = V_{\nu}$ has one of the entries zero and thus cannot lead to correct $\theta_{13}$ or $\theta_{23}$. The choice (1C) involves only elements belonging to the $\Delta(3N^{2})$ subgroup.
and its predictions are already discussed in the previous section. The remaining choices give new predictions.

The case (1B) leads to three different $U_{PMNS}$. One obtained with $T_l = S(N, p, q)$ contain a zero entry in the third column and can be used only as a zeroeth order choice. One gets the following result in (1B) if $T_l = T(N, p, q)$

$$|U_{23}|^2 = \frac{c^2}{2}, \quad |U_{33}|^2 = \frac{c^2}{2}, \quad |U_{13}|^2 = s^2.$$  \hspace{1cm} (22)

The ordering of the entries $|U_{i3}|^2$ can be changed by rearranging the eigenvectors of $T_l$ appearing in $U_l$. We have chosen here and below an ordering which is consistent with the values of the parameters $s^2_{13}, s^2_{23}$ when $U$ is equated with the standard form of the mixing matrix. The result in the third case with $T_l = U(N, p, q)$ can be obtained from above by the replacement $s \leftrightarrow c$. All the three entries above follow for all the choices of $p, q$ and the phase $\beta$. The case (1B) in this way gives a universal prediction. Two of the $|U_{i3}|^2$ are equal within this choice and they correspond to $c^2_{13}s_{23}$ and $c^2_{13}s_{23}$. Equality of the two then implies a $\theta$ independent prediction $\theta_{23} = \frac{\pi}{4}$. $s^2_{13}$ in the above case is then given by $s^2$ and can match the experimental value with appropriate choice of the unknown $\theta$. Since the choice of $S_\nu$ within (1B) is possible only for even $N$ it follows that all the groups $\Delta(24k^2)$ lead to a prediction of the maximal atmospheric mixing angle and can accommodate the correct $\theta_{13}$.

The choice (2A) also gives the same result for $|U_{i3}|^2$ as (1B) with an important difference. The neutrino residual symmetry used in this choice is allowed for all $N$ and not necessarily $N = 2k$. Thus one gets a universal prediction of the maximal $\theta_{23}$ for all $p, q, \theta, \beta$ within all $\Delta(6N^2)$ groups. The smallest group in this category is the permutation group $S_4 = \Delta(24)$ which contain symmetries appropriate for both the cases (1B) and (2A).

There are two independent structures within nine possible choices contained in case (2B). The example of the first one is provided by the choice $S_\nu = S(N, 0, n)$ and $T_l = S(N, p, q)$. The elements in the third column of mixing matrix are given in this case by

$$|U_{23}|^2 = \frac{1}{4} s^2 |\eta^n - \eta^{q+p/2}|^2,$$
$$|U_{33}|^2 = \frac{1}{4} s^2 |\eta^{-n} + \eta^{-q-p/2}|^2,$$
$$|U_{13}|^2 = c^2.$$  \hspace{1cm} (23)

While this choice does not give universal prediction as in the case (1B) discussed above it still leads to a prediction for $\theta_{23}$ which is independent of the unknown angle $\theta$ and phase $\beta$:

$$\tan^2 \theta_{23} \text{ or } \cot^2 \theta_{23} = \frac{|\eta^n - \eta^{q+p/2}|^2}{|\eta^{-n} + \eta^{-q-p/2}|^2}$$

This follows from eq.(23) when $|U_{13}|^2$ is identified with $s^2_{13}$. The predicted $\theta_{23}$ now depends only on the group theoretical factors $N, p, q, n$.

Unlike (1B), both the maximal and non-maximal values are allowed for $\theta_{23}$ in this case. The former occurs whenever $\cos \frac{2\pi(n-q-p/2)}{N} = 0$. The latter occurs for other choices. It is
possible to find values of parameters which lead to a non-maximal \( \theta_{23} \) within the experimental limits. The minimal such choice occurs for \( N = 7 \), i.e. the group \( \Delta(294) \) which leads as shown in Table II to a \( \sin^2 \theta_{23} \) within the \( 2\sigma \) range as given in [51]. The next example of the group \( \Delta(486) \) fairs slightly better.

The other prediction of the case (2B) is obtained with \( S_{\nu} = S(N, 0, n) \) and \( T_i = T(N, p, q) \). One obtains in this case

\[
|U_{23}|^2 = \frac{1}{4} \left| \sqrt{2} ce^{-i\beta} + s \eta^{p/2} \right|^2 ,
\]
\[
|U_{33}|^2 = \frac{1}{4} \left| \sqrt{2} ce^{-i\beta} - s \eta^{-q/2} \right|^2 ,
\]
\[
|U_{13}|^2 = \frac{1}{2} s^2 . \tag{24}
\]

In this case, \( \theta_{23} \) is necessary non-maximal if \( \theta_{13} \) is to be small but non-zero. We may identify, \( |U_{13}|^2 \) with \( s_{13}^2 \) and fix \( s^2 = 2s_{13}^2 \). This determines the other two entries of \( |U_{i3}|^2 \) for a given \( p, q, \beta \). For \( p = q = \beta = 0 \) one obtains \( \sin^2 \theta_{23} \) either 0.345 or 0.655. Thus all the \( \Delta(6N^2) \) groups with this specific choice give results close to the \( 3\sigma \) range in the global fits. This prediction can be improved by turning on \( \beta \) or choosing different \( T_i \). An example based on the group \( \Delta(150) \) giving \( \sin^2 \theta_{23} \) close to the best fit value [51] is shown in the table.

Predictions of the case (2C) can also be similarly worked out. Six different \( U_{PMNS} \) are associated with this choice but not all give different predictions for the third column. One of the independent structures corresponds to choosing \( T_i = R(N, p, q) \) and \( S_{\nu} = S(N, 0, n) \). The \( U_{PMNS} = V_R^T S V_{\nu} \) gives

\[
|U_{13}|^2 = \frac{1}{6} \left[ -s(\eta^p + \eta^{n-q}) + \sqrt{2} ce^{-i\beta} \right]^2 ,
\]
\[
|U_{23}|^2 = \frac{1}{6} \left[ -s(\eta^p \omega + \eta^{n-q} \omega^2) + \sqrt{2} ce^{-i\beta} \right]^2 ,
\]
\[
|U_{33}|^2 = \frac{1}{6} \left[ -s(\eta^p \omega^2 + \eta^{n-q} \omega) + \sqrt{2} ce^{-i\beta} \right]^2 . \tag{25}
\]

Rest of the choices within (2C) differ from the above only in the powers of \( \eta \). Their predictions can be obtained from the above by choosing different values of \( p, q, n \).

Eq. (25) gives \( s_{13}, s_{23} \) withing \( 3\sigma \) range for a suitable choice of \( N, p, q, \theta, \beta \). In particular, one predicts a maximal \( \theta_{23} \) if \( p = q = \beta = 0 \) as in the earlier cases. But now the maximal value of \( \theta \) also becomes a viable choice for all the groups \( \Delta(6N^2) \). This makes the choice in this class particularly interesting since such value of \( \theta \) can be forced by some additional symmetry. With the choice \( T_i = E^2 \) corresponding to \( p = q = 0 \), eq.(25) gives for \( n = \beta = 0, \)

\[
\theta_{23} = \frac{\pi}{4} , \ s_{13}^2 = \frac{1}{3} \left| c - \sqrt{2} s \right|^2 .
\]

The maximal value of \( \theta \) then leads to \( s_{13}^2 \sim 0.029 \) which is close to \( 2\sigma \) range as obtained in [51]. One can obtain a better solution with a different choice for \( p \) and \( q \) and \( \theta \). One particular solution based on the group \( \Delta(150) \) is shown in the table.
| Group | $T_i$ | $S_\nu$ | Predictions |
|-------|-------|--------|-------------|
| $\Delta(24k^2)$ | $T(2k, p, q)$ | $W(2k, 0, k)$ | Maximal $\theta_{23}$ for all $\beta, p, q, n$ $\sin^2 \theta_{13} = \sin^2 \theta$ |
| $\Delta(6N^2)$ | $W(N, p, q)$ | $P(N, 0, n)$ | Maximal $\theta_{23}$ for all $\beta, p, q, n$ $\sin^2 \theta_{13} = \cos^2 \theta$ |
| $\Delta(6N^2)$ | $S(N, p, q)$ | $S(N, 0, n)$ | Maximal $\theta_{23}$ for all $\beta, \frac{|n-q-p/2|}{N} = \frac{(2l+1)}{4}$ $\sin^2 \theta_{13} = \cos^2 \theta$ |
| $\Delta(294)$ | $S(7, 0, 2)$ | $S(7, 0, 0)$ | $s_{23}^2 = 0.39$ or 0.61 for all $\theta, \beta$ $\cos^2 \theta = s_{13}^2$ |
| $\Delta(486)$ | $S(9, 0, 2)$ | $S(9, 0, 0)$ | $s_{23}^2 = 0.41$ or 0.59 for all $\theta, \beta$ $\cos^2 \theta = s_{13}^2$ |
| $\Delta(6N^2)$ | $S(N, 0, 0)$ | $T(N, 0, n)$ | $\sin^2 \theta = 2s_{13}^2$ $s_{23}^2 = 0.345$ for $\beta = 0$ and the best fit $s_{13}^2$ |
| $\Delta(150)$ | $S(5, 2, 3)$ | $T(5, 0, 0)$ | $\sin^2 \theta = 2s_{13}^2$ $s_{23}^2 = 0.452$ for the best fit $\theta_{13}$ |
| $\Delta(6N^2)$ | $R(N, 0, 0)$ | $S(N, 0, 0)$ | $s_{23}^2 = 1/2$ for $\beta = 0$ and all $\theta$ $s_{13}^2 = 0.028$ for $\beta = 0$, maximal $\theta$ |
| $\Delta(150)$ | $R(5, 3, 1)$ | $S(5, 0, 0)$ | $s_{23}^2 = 0.484, s_{13}^2 = 0.022$ for $\beta = 0$ and $\theta \sim 80^\circ$ |

| TABLE II. Some illustrative predictions of the mixing angles $\sin^2 \theta_{13}$ and $\sin^2 \theta_{23}$ using $\Delta(6N^2)$ groups as flavour symmetry. $\sin^2 \theta_{12}$ remains undetermined due to degeneracy in two of the masses in all these cases. |

2. **Texture IV**

The diagonal texture IV given in eq.\([11]\) can be realized in $\Delta(6N^2)$ for even $N$ with the choice $S_\nu = P(2k, k, n)$. This texture has two non-degenerate and one massless neutrino. Thus both the normal and the inverted hierarchies are possible. The massless state has to be identified with the third (first) column of the mixing matrix for the inverted (normal) hierarchy. The neutrino mixing matrix $U_\nu$ in this case is given by the matrix which diagonalizes $P(2k, k, n)$. This is given for the inverted hierarchy by $V_P(2k, k, n)$ as defined in eq.\([21]\). For the normal hierarchy, one instead gets $U_\nu = V_P(2k, k, n)Z_{13}$ where $Z_{13}$ exchanges the first and the third column of of the mixing matrix obtained in case of the inverted hierarchy. Possible choice of $T_i$ can be any of the six types of generators and corresponding mixing matrices $U_i$ are the same as given in Table I with the choice (2A),(2B),(2C). It is then straightforward to work out the final mixing matrices $U_{PMNS}$. As the massless state in the basis with diagonal $S_\nu$ is given by $(1, 0, 0)^T$ and its cyclic permutation for $S_\nu = S, T, U$, the third column of the $U_{PMNS}$ is given by $(U_{PMNS})_{i3} = (U_i^\dagger)_{i1}, (U_i^\dagger)_{i2}, (U_i^\dagger)_{i3}$ when $S_\nu = S, T, U$. |
It follows from the structure of $U_1$ that the third column has either one or two zero entries or all elements have equal magnitudes. The same applies to the first column of $U_{PMNS}$ in case of the normal hierarchy. In either case, the texture IV cannot give phenomenologically consistent result at the zeroeth order.

**IV. MORE PREDICTIVE SCENARIO: $Z_2 \times Z_2$ SYMMETRY**

We have assumed so far that the flavour antisymmetry $S_\nu$ is the only invariance of the neutrino mass matrix. This fails in determining $\tilde{M}_\nu$ completely in case of the texture I which still has two unknown quantities $\theta$ and $\beta$. We give here example of an enlarged residual symmetry of $M_\nu$ which serves to determine $\tilde{M}_\nu$ completely apart from an overall complex mass scale. We use an additional symmetry $S'_\nu$ commuting with $S_\nu$ for this purpose. It should be such that $S_\nu, S'_\nu$ and $T_l$ together are contained in some $G_f$. $M_\nu$ may be antisymmetric with respect to transformation by $S'_\nu$ also. In this case, it will be symmetric with respect to the product $S_\nu S'_\nu$. Instead we assume that $S'_\nu$ is a symmetry of $M_\nu$, i.e.

$$S'^T_\nu M_\nu S'_\nu = M_\nu .$$

We can transform above equation to the basis with a diagonal $S_\nu$ by defining $\tilde{S}'_\nu \equiv V^\dagger S_\nu S'_\nu V S_\nu$.

In this basis, we get

$$\tilde{S}'_\nu^T \tilde{M}_\nu \tilde{S}'_\nu = \tilde{M}_\nu .$$

As before, we demand $\tilde{S}'_\nu$ to be contained in $SU(3)$. If it is diagonal, then $\tilde{S}'_\nu = \text{diag} (\lambda_1, \lambda_2, \lambda_1^* \lambda_2^*)$ with $\lambda_{1,2}$ being roots of unity. Then eq.(27) when applied to $\tilde{M}_\nu$ in eq.(10) implies that either $\tilde{S}'_\nu$ is proportional to identity or $s = 0$ or $c = 0$. A non-trivial prediction can be obtained if $\tilde{S}'_\nu$ is non-diagonal. Since $\tilde{S}_\nu = \text{diag} (1, -1, -1)$, a general $\tilde{S}'_\nu$ commuting with $\tilde{S}_\nu$ should have a block diagonal structure with the lower $2 \times 2$ block non-trivial. This block gets further restricted from the requirement that $S_\nu, S'_\nu, T_l$ are elements of some discrete group $G_f$. These requirements can be met within the already considered groups $\Delta(6N^2)$.

Consider the group $\Delta(12k^2)$. The choice $S_\nu = \tilde{S}_\nu = W(2k, 0, k) = \text{diag} (1, -1, -1)$ within it leads to texture I as already discussed. This commutes with all the discrete symmetries having a general form $S(M, m, n)$ as in eq.(20). Thus a viable choice for $S'_\nu$ is provided by $S'_\nu = S(M, m, n)$. Note that since $S_\nu$ is already diagonal, $\tilde{S}'_\nu = S'_\nu = S(M, m, n)$. Then eq.(27) and the form of $\tilde{M}_\nu$ implies a restriction:

$$m = 0 , \quad \theta = \pm \frac{\pi}{4} , \quad \beta = \frac{2\pi n}{M}$$

which fixes the unknown angle $\theta$ and phase $\beta$. Mixing pattern can be determined by choosing appropriate $T_l$ and let us choose $T_l = R(N, p, q)$. Since both $T_l$ and $S_\nu$ are contained in $\Delta(12k^2)$ mixing pattern is determined by the corresponding eq. (17) but now with $\theta$ and $\beta$ satisfying eq.(28) which follows from the inclusion of $S'_\nu$ as a residual symmetry. We can vary $p, q, M, N, n$ in eq(17) and look for a viable choice. Consider $M = N$ in which case
$T_l, S_{\nu}, S'_{\nu}$ are contained in $\Delta(6N^2)$. By varying $p, q, N$ one finds that the minimum group giving acceptable $\theta_{13}, \theta_{23}$ is $\Delta(600)$ corresponding to $N = 10$. One possible set of residual symmetries within $\Delta(600)$ is given by

$$S_{\nu} = W(10, 0, 5), \quad T_l = R(10, 4, 0), \quad S'_{\nu} = S(10, 0, 0).$$

With this choice, the $S'_{\nu}$ coincides with the $\mu$-$\tau$ symmetry and eqs.(17, 28) give a prediction

$$s_{13}^2 \approx 0.029, \quad s_{23}^2 \approx 0.38 \text{ or } 0.62$$

to be compared with the $3\sigma$ region given in eq.(18).

V. AN $A_4$ MODEL WITH FLAVOUR ANTISYMMETRY

Our discussion so far has been at the group theoretical level. We now present an explicit realization of flavour antisymmetric neutrino mass matrix using $A_4$ as an example. $A_4$ has been extensively used for several different purposes, for obtaining degenerate neutrinos \[41, 42\], to realize tri-bimaximal mixing \[43, 46\] for obtaining maximal CP phase \[44, 45, 47, 49, 50\] or to obtain texture zeros \[48\] in the leptonic mass matrices. As we discuss here, it also provides a viable alternative to get a massless and two quasi degenerate neutrinos with correct mixing pattern. In the following, we discuss the required symmetry, Higgs content and identify the vacuum needed to obtain antisymmetry. We also discuss possible perturbations which can split the degenerate pair and lead to the solar scale and mixing angle. The aim is not to construct a detailed model but to illustrate how the basic proposal of the paper can be used for construction of a model.

The group theory of $A_4$ is discussed extensively in many papers. We shall not elaborate on it. We follow the basis choice as given for example in \[46\]. In this basis, all the 12 elements of $A_4 = \Delta(12)$ can be generated from the two elements $E$ and $F$ defined in eq.(13) with $\eta = -1$.

We consider supersymmetric model with MSSM extended by a triplet Higgs field $\Delta$. The standard doublets $H_u, H_d$ and $\Delta$ are $A_4$ singlets. We use two flavon fields $\chi_{\nu}$ and $\chi_e$ to break $A_4$ and generate the flavour structures. Both these fields as well as the three generations of the leptonic doublets $l_L$ transform as triplets of $A_4$. The right handed charged leptons transform as triplets of $A_4$. The right handed charged leptons transform as $(e_R, \mu_R, \tau_R) \sim (1, 1')$ representation of $A_4$. We also impose an additional $Z_3$ symmetry under which $(l_L, \chi_{\nu}) \rightarrow \omega(l_L, \chi_{\nu})$ and $\chi_e \rightarrow \omega^2 \chi_e$. All other fields are assumed singlet under the $Z_3$. The charged lepton masses arise from the following superpotential

$$W_l = \frac{H_d}{M} \left( h_e(l_L\chi_e)_{1}e_R + h_{\mu}(l_L\chi_e)_{1'}\mu_R + h_{\tau}(l_L\chi_e)_{1''}\tau_R \right). \quad (29)$$

and the neutrino masses follow from

$$W_\nu = \frac{h_\nu}{2M}(t^T_L C \Delta l_L)_{33} \chi_{\nu}. \quad (30)$$
Here, $C$ is the charge conjugation matrix. The subscript $a$ in $(..)_a$ labels the $A_4$ representation according to which the quantity $(..)$ transforms. The scale $M$ and the flavon vacuum expectation values generate the effective Yukawa couplings in the model. The additional symmetry $Z_3$ introduced here serves two purposes. It prevents $\chi_e(\chi_\nu)$ couplings in $W_\nu(W_l)$ at the leading order. It also forbids a singlet term $(l_L^T C \Delta l_L)_1$ allowed by the $A_4$ symmetry.

The residual symmetries of the leptonic mass matrices are determined from the above superpotential by the flavon vacuum expectation values (vev). We shall choose these symmetries in accordance with the discussion given in section II. The symmetry $T_l$ of $M_lM_l^T$ is chosen as $E$ or $E^2$. This is realized when vev $< \chi_e >$ of $\chi_e$ satisfies $E < \chi_e >= < \chi_e >$. This requires equal vev for all the three components of $\chi_e$ and leads to a well-known form of $M_l$ considered in many models based on $A_4$, see for example [46]. The charged lepton mixing is determined in this case by $U_l = U_\nu$, defined in eq. (16). Only possible choice within $A_4$ for $S_\nu$ leading to flavour antisymmetry is given by $F$ or its cyclic permutations. In order to realize this antisymmetry as residual invariance, we impose $S_\nu < \chi_\nu >= - < \chi_\nu >$ with $S_\nu = F$. This leads to the configuration $< \chi_1 >= 0$ and $< \chi_{2,3} >= v_{2,3}$. Here $v_{2,3}$ are complex parameters in general. It is seen that eq. (30) then leads to the flavour antisymmetric mass matrix given in eq. (7) with $\tan \theta = \frac{|v_2|}{|v_1|}$, $\beta = \text{Arg}(v_2v_3^*)$ and appropriately defined $m_0$. The neutrino mixing matrix in this case is $U_\nu = V_\nu$. The resulting mixing pattern is a special case of eq. (17) obtained for $\Delta(12k^2)$ with $T_l = R(2k, p, q)$ and $S_\nu = W(2k, 0, k)$. In the present case, the residual symmetry $T_l = E^2$ coincides with $R(2, 0, 0)$ and $S_\nu = F = W(2, 0, 1)$. Thus $U_{i3}$ can be obtained by putting $p = q = 0$ and $\eta = -1$ in eq. (17). As already discussed, this leads to a prediction $\theta_{23} = \frac{\pi}{4}$ for $\beta = 0$ independent of the choice of $\theta$. The latter can be chosen to give the correct $\theta_{13}$ while the solar angle and scale remain unpredicted at this stage due to degeneracy in mass. We now discuss possible perturbations which can generate them.

One source of perturbation that we consider comes from the $A_4 \times Z_3$ invariant non-leading corrections to $W_\nu$, eq. (30). These are given by

$$W'_\nu = \frac{1}{2M^2} \left( f_1(l_L^T C \Delta l_L)_{33}(\chi_e\chi_e)_{33} + f_2(l_L^T C \Delta l_L)_{11}(\chi_e\chi_e)_{11} + f_3(l_L^T C \Delta l_L)_{1'}(\chi_e\chi_e)_{1'} + f_4(l_L^T C \Delta l_L)_{1''}(\chi_e\chi_e)_{1''} \right).$$

Only the first two terms give non-zero contribution when all the components of $\chi_e$ acquire equal vev. The neutrino mass matrix including these correction is given by

$$\tilde{M}_\nu = m_0 \begin{pmatrix} \epsilon_1 & c & se^{i\beta} \\ c & \epsilon_1 & \epsilon_2 \\ se^{i\beta} & \epsilon_2 & \epsilon_1 \end{pmatrix},$$

where $\epsilon_{1,2} \ll 1$. Eq. (31) corrects the leading order non-zero elements $(\tilde{M}_\nu)_{12}$ and $(\tilde{M}_\nu)_{13}$. We have absorbed these corrections through redefinition of $\theta$, $\beta$ and $m_0$. $\epsilon_{1,2}$ are new corrections arising from eq. (31). Eq. (31) does not exhaust all the non-leading corrections. One could also write similar corrections to $W_l$ quadratic in $\chi_\nu$. These will correct the charged
lepton mixing. Similarly, a more elaborate realistic model leading to the assumed vacuum configuration will also contain non-leading corrections which may change the leading order vev assumed here. All these corrections will add more parameters to the model and we assume their contribution to be small. Here we show that two parameters $\epsilon_1, \epsilon_2$ introduced by eq.(31) in eq.(32) are sufficient to reproduce the neutrino mixing and scales correctly. They split the degeneracy and generate the solar scale and angle correctly. For example,

$$(\theta, \beta, \epsilon_1 \epsilon_2) = (0.5904, -0.1818 - 0.0579, 0.1186)$$

(33)

give the following values of the observables

$$\sin^2 \theta_{13} \sim 0.024 \ , \ \sin^2 \theta_{23} \sim 0.455 \ , \ \sin^2 \theta_{12} \sim 0.307 \ , \ \frac{\Delta_{\odot}}{\Delta_{\text{atm}}} \sim 0.0317$$

(34)

which corresponds to (nearly) best fit values obtained for example with a global fits in [51].

VI. SUMMARY

The bottom up approach of finding discrete symmetry groups starting with possible symmetries of the residual mass matrices has been successfully used in last several years to predict leptonic mixing angles. The residual symmetry assumed in these works leaves the neutrino mass matrix $M_\nu$ invariant. We have proposed here a different possibility in which $M_\nu$ displays antisymmetry as defined in eq.(3) under a residual symmetry. Just like symmetry, the antisymmetry can also come from breaking of some discrete group $G_f$ as we have illustrated with an example based on $A_4$. The use of antisymmetry is found to be more predictive than symmetry. It is able to restrict both neutrino masses and mixing angles unlike all the previous works in this category which [1–15] could predict only mixing angles. Moreover, the antisymmetry condition by itself is sufficient for determining all possible discrete residual antisymmetry operators $S_\nu$ residing in $SU(3)$. This in turn leads to very specific textures of the neutrino mass matrix satisfying antisymmetry condition. These are given by eqs.(7,8,9,11).

We studied the mixing angle predictions in the specific context of the groups $\Delta(3N^2)$ and $\Delta(6N^2)$. The main results obtained are:

- Only the groups $\Delta(12k^2)$ with $k = 1, 2...$ and all $\Delta(6N^2)$ groups contain the residual antisymmetry operator.

- Of the four possible neutrino mass textures allowed by antisymmetry, only texture I having one massless and two degenerate neutrinos can lead to correct mixing pattern. This case provides a very good zeroeth order approximation to reality if the neutrino mass hierarchy is inverted.

- There always exists within these groups residual symmetries of $M_l M_l^\dagger$ and $M_\nu$ such that the atmospheric neutrino mixing angle is maximal. Correct value of $\theta_{13}$ can be
accommodated by choosing the unknown angle in eq.(7) appropriately. There also exists other choices of residual symmetries which for some groups allow non-maximal values of the atmospheric neutrino mixing angle as well. The results of various cases are summarized in Fig. 1 and Table II.

- The successful texture I still has two free parameters apart from an overall mass scale. But as we have shown here, predicted atmospheric mixing angle in many cases is independent of these unknowns. The reactor angle \( \theta_{13} \) depends on it but it is possible to determine these unknowns also by enlarging the residual symmetry and we have given an example of a \( Z_2 \times Z_2 \) residual symmetry which can determine the complete neutrino mass matrix up to an overall scale in terms of group theoretical parameters alone and have identified \( \Delta(600) \) as a possible group which can give correct \( \theta_{13} \) and \( \theta_{23} \) with this symmetry.

We end this section with a comparison of the present work with some earlier relevant works.

- The texture I, eq.(7) has been extensively studied since long in the context of \( L_e - L_\mu - L_\tau \) global symmetry which implies it, see for example [54] and references therein. Imposition of this symmetry on the charged lepton mass matrix \( M_l \) makes it diagonal after redefinition of \( \theta \) appearing in (7). Thus the matrix \( V_\nu \) as given in eq.(10) corresponds to the final mixing matrix which is now not allowed by the present experimental constraints. This is not the case here since the \( M_l M_l^\dagger \) is non-trivial with the imposed discrete symmetry.

- Neutrino mass matrix displaying a specific flavour antisymmetry namely, \( \mu-\tau \) antisymmetry was studied in [55]. This antisymmetry was assumed there to hold in the neutrino flavour basis. In our terminology, this would correspond to study of a specific example within the choice (2A) discussed in section IIIB. The structure of the neutrino mass matrix and the mixing angle predictions obtained here for this choice agrees with ref.[55] after suitable basis change. The study presented here is not limited to the \( \mu-\tau \) antisymmetry but encompasses all possible antisymmetry operators within \( SU(3) \) and leads to many new phenomenological predictions.

- The antisymmetry condition, eq.(3), can be converted to the usually assumed symmetry condition by redefining the operator \( S_\nu \to iS_\nu \). The new operator does not however have unit determinant and would belong to a \( U(3) \) group. The occurrence of massless state within such group with condition, eq.(2) was discussed in [21, 22]. The residual symmetry operators used there had eigenvalues \( (\eta, 1, -1) \) (or its permutations) with \( \eta \neq \pm 1 \). This coincides with eigenvalues of \( iS_\nu \) for texture IV when \( \eta = i \). Only texture IV was considered in [21, 22] and it was shown there that a large class of DSG of \( U(3) \) imply \( \sin^2 \theta_{13} \) to be either 0 or \( \frac{1}{3} \) with condition (2). The same conclusion is found to be true here with eq.(3) and texture IV in case of the group series \( \Delta(3N^2) \) and \( \Delta(6N^2) \).
• It is possible to obtain a degenerate pair of neutrinos using symmetry condition, eq. (2) and DSG of $SU(3)$. This was studied for the finite von-Dyck groups in [19] and for all DSG of $SU(3)$ having three dimensional IR in [20]. Here, the third state is not implied to be massless. The case of one massless and two degenerate neutrinos can follow from the symmetry condition if DSG of $U(3)$ are used. This was also discussed in [20]. The successful examples found in these two works are different from here because of the difference in the assumed residual symmetries. The cases studied in the context of DSG of $SU(3)$ and $U(3)$ [20] have texture similar to the texture II in the present terminology. It was found there that this texture can give non-trivial values of $s_{13}^2$, $s_{23}^2$ in several $\Delta(6N^2)$ groups when symmetry condition (2) is used. This does not happen with the antisymmetry condition in case of texture II as argued here. On the other hand, one can obtain correct values for $\theta_{13}$ and $\theta_{23}$ in all the $\Delta(3N^2)$ groups with texture I when antisymmetry condition is employed. Thus symmetry and antisymmetry conditions appear complementary to each other and allow more possibilities for flavour symmetries $G_f$.

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